Whole-grain Petri Nets and Processes

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We present a formalism for Petri nets based on polynomial-style finite-set configurations and etale maps. The formalism supports both a geometric semantics in the style of Goltz and Reisig (processes are etale maps from graphs) and an algebraic semantics in the style of Meseguer and Montanari, in terms of free coloured props, and allows the following unification: for $P$ a Petri net, the Segal space of $P$-processes is shown to be the free coloured prop-in-groupoids on $P$. There is also an unfolding semantics à la Winskel, which bypasses the classical symmetry problems: with the new formalism, every Petri net admits a universal unfolding, which in turn has associated an event structure and a Scott domain. Since everything is encoded with explicit sets, Petri nets and their processes have elements. In particular, individual-token semantics is native. (Collective-token semantics emerges from rather drastic quotient constructions à la Best–Devillers, involving taking $\pi_0$ of the groupoids of states.)

CCS Concepts: • Software and its engineering → Petri nets; • Theory of computation → Concurrency; Process calculi; Operational semantics; Categorical semantics; Algebraic semantics; Denotational semantics;

Additional Key Words and Phrases: Petri nets, processes, operational semantics, categorical semantics, unfolding, graphs, hypergraphs

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0 INTRODUCTION
Background
Petri nets are an important framework for describing networks in which resources interact and transform, such as chemical reaction networks and population dynamics, including compartmental models in epidemiology. In computer science they serve as a widely used model of concurrency.

A Petri net (cf. Section 2) has places holding tokens, and transitions describing how tokens flow and transform. Arcs connect places and transitions to express the interactions. A Petri net is thus a kind of graph and may look like the following figure: places are always pictured as circles, and transitions as squares. Tokens are drawn as bullets:

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The operational semantics of Petri nets is the mathematical formalisation of their behaviour, expressed informally in the *token game*, whereby firing of transitions move tokens around between places. Two main classes of operational semantics can be called geometric and algebraic, although this terminology is not standard in the literature. Geometric semantics take as starting point the notion of nonsequential processes, formalised as certain morphisms into a given Petri net from *causal nets*, certain posets or graphs. This viewpoint was pioneered by Petri himself [61], and found a very clean formulation in the seminal 1983 paper of Goltz and Reisig [32], leading to many developments and insights regarding reachability, density, safeness, as expressed for example in the 1988 monograph of Best and Fernández [14]. Algebraic semantics, on the other hand, focus on the way the transitions of a Petri net generate a formal algebraic system. After earlier work of Reisig [62] and Winskel [70], a breakthrough was the 1990 work of Meseguer and Montanari [54], who observed that Petri nets can be defined as monoids in certain directed graphs, and that they freely generate symmetric monoidal categories of certain kinds, equivalently, free props. The morphisms in such a symmetric monoidal category are thus built up from the transitions of a given Petri net by serial and parallel connection (composition and the tensor product), and are thus certain equivalence classes of firing sequences.

In view of the well-known graphical calculus for monoidal categories in terms of string diagrams [43], one could expect a unification of the geometric and algebraic semantics. This has turned out to be a difficult problem, though: in the ‘geometric’ frameworks, there is no way to compose processes in the sense of category theory, and in the algebraic frameworks there is no synthetic description of processes—they rely solely on formal inductive descriptions. Over the past four decades, a lot of research has been dedicated to this problem. Already in 1987, Best and Devillers [13] searched for an algebraic description of the Goltz–Reisig processes, finding that it was necessary to impose drastic equivalence relations on processes. Starting from the Meseguer–Montanari formalism, many people, including in particular Bruni, Meseguer, Montanari, and Sassone [16, 17], tried to avoid these quotientings by introducing further bookkeeping on Petri nets. They arrived at the notion of pre-net, which are Petri nets with numberings. For the notion of pre-net the semantics problems can be solved, but pre-nets are not quite a satisfactory substitute for Petri nets. In particular, they do not allow enough morphisms. The final section of this paper contains some further discussion of these issues.

Closely related to the geometric-algebraic dichotomy is the distinction between individual-token and collective-token philosophies, which roughly asks whether the tokens of a state of a Petri net execution are individual elements in a set or if they are only the expression of a quantity, a natural number. (See [17, 18, 29–31] for more thorough discussion). The geometric semantics favour the individual-token philosophy, since processes are explicit maps from other nets, and in particular involves mappings of sets. In contrast, the algebraic semantics are more geared towards the collective-token philosophy, as both their states and the pre- and post-conditions of a transition are given by multisets, meaning sets with multiplicities. In its full version (see Van Glabbeek [29]) the individual-token philosophy requires keeping track of all tokens at all times, which is done by annotating each token occurrence with the complete history of the transitions that produced it from a given initial state. (This involves knowing which tokens enter which input slots of a given transition, and in particular it is necessary to be able to distinguish such slots, something that is essentially impossible in the Petri-nets-as-monoids formalism).
A third main approach to operational semantics is based on the idea of unfolding, pioneered in the work of Winskel [67] and Nielsen, Plotkin, and Winskel [58]. It is important because it establishes connections to domain theory and denotational semantics. The key ingredient, the universal unfolding of a Petri net, was established for safe Petri nets by Nielsen, Plotkin, and Winskel, but there are symmetry issues preventing the existence of a universal unfolding for general Petri nets. Montanari, Meseguer, and Sassone [55, 56] worked around the symmetry problems by considering certain decorated unfoldings, but their construction was not universal in the categorical sense. Baldan, Bruni, and Montanari [7] established instead the existence of a universal unfolding for pre-nets, where the symmetries do not come up. Finally, Hayman and Winskel [36, 37] came to grips with the symmetries themselves and established a weaker form of universal property, through a notion of adjunction up to symmetry in a precise technical sense.

Contributions of this Paper

The present paper proposes a rather natural solution to these problems, by using some elementary homotopy theory to overcome the symmetry issues. The novelty essentially boils down to one small modification at the foundational level: to abolish the traditional notion of multisets in favour of the representable analogue: instead of sets of multiplicity functions $S \to \mathbb{N}$, we consider groupoids of $S$-coloured finite sets $A \to S$; instead of assigning to each $s \in S$ a multiplicity, there is now assigned an actual set, namely the pre-image $A_s$. The isomorphism classes of $S$-coloured sets are the traditional multisets on $S$. The benefit is that $S$-coloured sets have elements, which can be accessed individually, giving full control over symmetries.

The multiset modification is incorporated in the very definition of Petri net, following the lead of the analogous formalism for directed graphs [46] (in turn motivated by the polynomial formalism in the theory of operads [45]): we define a Petri net to be a diagram of finite sets

$$S \leftarrow I \rightarrow T \leftarrow O \rightarrow S.$$ 

Here $T$ is the set of transitions, $S$ is the set of places, and $I$ and $O$ are the sets of incoming and outgoing arcs of transitions. In particular, for a transition $t \in T$, the fibres (i.e., pre-image sets) $I_t$ and $O_t$ are explicit sets, and they are not necessarily subsets of $S$. In this paper, the new Petri nets are called whole-grain Petri nets, to distinguish them from the traditional notion of Petri net. A whole-grain Petri net is thus precisely what we see in the pictures. It is only a slight modification of the usual definition, but it has important implications, and the theory develops quite neatly from it, exploiting some recent insights from algebraic combinatorics [20, 21, 23, 24].

An etale map is a diagram

$$(S' \leftarrow I' \rightarrow T' \leftarrow O' \rightarrow S')$$

(the middle squares being pullbacks). A graph is a SITOS diagram where the outer maps $S \leftarrow I$ and $O \rightarrow S$ are injective [46]. A process is an etale map from an acyclic graph. The processes of a Petri net $P$ assemble naturally into a simplicial groupoid $X_\bullet$, shown to be a symmetric monoidal Segal space. (Segal spaces (recalled below in §4) are a homotopy version of categories. Appendix B provides some background). The following is the first main result of this work.

**Theorem** (Cf. 6.9). $X_\bullet$ is the free prop-in-groupoids on $P$.

It expresses the reconciliation of the geometric and algebraic semantics for whole-grain Petri nets. A 1-categorical analogue of this result is also extracted (7.2). Making sense of the free prop on a (whole-grain) Petri net depends on an explicit fully faithfully embedding $\text{Petri} \to \text{PrSh}(\text{elGr})$.
of the category of (whole-grain) Petri nets into the presheaf category on elementary graphs (5.2, 5.3). Such presheaves, the structure underlying coloured props, are called digraphical species\(^1\) (or sometimes bicollections) [46].

The second main result shows the benefit of whole-grain Petri nets for unfolding:

**Theorem** (Cf. 9.6). Any (grounded) whole-grain Petri net admits a universal unfolding.

No safety assumptions are required here, in contrast to the classical result of Nielsen–Plotkin–Winskel [58]. It should be stressed that the proof of this theorem follows the ideas and arguments employed by Winskel in the safe case—the only novelty is the new whole-grain setting, where the arguments go through even in the non-safe case.

The key point of the new formalism, as exploited in the proofs of the main theorems, is that whole-grain Petri nets are configurations of sets, and that these sets have elements that can be accessed, giving total control over symmetries. With the systematic use of groupoids, these symmetries are kept around as they are, giving some advantage over the traditional formalisms of Petri nets, where the symmetries are difficult to control.

The general importance of groupoids in combinatorics was discovered and advocated by Joyal [39] and Baez–Dolan [3]. The specific insight of representing algebraic structures by groupoids of configurations of sets has been found useful in algebraic topology (mainly in the theory of operads) [9, 25, 40, 45, 46] and in algebraic combinatorics (in connection with incidence bialgebras and Möbius inversion) [20, 21, 24]. The mathematical tools are thus already available. The whole-grain Petri nets of the present paper can be seen as an intermediate notion between traditional Petri nets and pre-nets [16, 17], in the same way as polynomial monads are intermediate between symmetric operads and non-symmetric operads [45].

The basic ideas and results are elementary, relying solely on manipulations with finite sets, mostly pullbacks and pushouts. Some of the theoretical results and justifications (Sections 4–7) require some more category theory and some elementary homotopy theory.

The core mathematical content could be given more succinctly, but the text has grown longer for three reasons. Firstly, many examples, figures, and explanations have been included to illustrate the formalism, where different from that of the traditional approaches to Petri nets. Secondly, many arguments of homotopical nature have been accompanied by intuitive explanations and background material, hopefully making them accessible also to readers without background in homotopy theory. Finally, an effort has been made to try to point out origins of ideas and provide comparison with related developments.

**Outline of the Paper**

We begin in Section 1 with a brief summary of the formalism of directed graphs from Kock [46]. This is where the semantics lives, both in geometric and algebraic form, and most of the work will take place at this level.

In Section 2 we define the whole-grain Petri nets and their etale maps, and in Section 3, we define their processes to be etale maps from (acyclic) graphs. Section 4 sets up the symmetric monoidal Segal space \(X\) of processes of a fixed Petri net \(P\).

In Section 5 we fully faithfully embed the category of Petri nets into the presheaf category on elementary graphs—these presheaves are called digraphical species—and characterise the image as the flat digraphical species. This embedding is used to define the free prop on a Petri net in

\(^1\)Warning: The word ‘species’ is used here as in combinatorics [10]. It has nothing to do with the use of the word to mean ‘places,’ as occurs sometimes in Petri net theory motivated by chemical reaction networks (see for example, Baez and Pollard [6] and the references therein).
Section 6. We show that the symmetric monoidal Segal space $X_\bullet$ associated to $P$ is the free prop-in-groupoids on $P$. This is the first main theorem of the paper, Theorem 6.9. In Section 7 we show how to trim down the symmetric monoidal Segal space to a symmetric monoidal category, and discuss the homotopy issues involved.

In Section 8 we set up some definitions and results about certain classes of hypergraphs. This is preparation for Section 9 where we come to the second main theorem of the paper, Theorem 9.6, establishing the existence of universal unfoldings for general Petri nets.

In Section 10 we look into fancier notions of morphisms of Petri nets than the basic etale maps, and establish functoriality of $X_\bullet$ in these more general maps. Modulo the difference in setup, this covers the notion of morphisms of Meseguer and Montanari [54] (monoid homomorphisms), as well as the more general notions given in terms of multi-relations, studied by Winskel [70].

The closing Section 11 makes an attempt at situating this work in a bigger picture, and in particular provide comparison with the pre-nets of Bruni, Meseguer, Montanari, and Sassone [17]. There are two short appendices: one with a few basic facts about groupoids and homotopy pullbacks, and one with simplicial groupoids and Segal spaces.

**Related Work and Outlook**

After this work was first released in preprint form (May 2020), Baez, Genovese, Master, and Shulman [4] have made an important contribution to the theory, which complements the material in Sections 5–7 in various ways, and constitutes a nice overall picture of monoidal-category theoretic operational semantics. First of all, they propose to work directly with digraphical species (which they call Σ-nets), as a substitute for Petri nets. This is a fascinating idea, going beyond the strong graphical aspect of Petri nets. Second, they give slick categorical proofs of key adjunctions, including a nice alternative proof of the adjunction described below in Section 6.4.

Although the field of Petri net theory is very much driven by applications—in computer science, natural sciences, and industry—the present work is motivated by theoretical interest. Up to the point developed here, I think the theoretical picture is quite satisfactory and clean, at the price of some homotopy overhead. It is my opinion that Petri-net theorists and end-users can bargain against this price in various ways (see Reference [26] for some explicit bargaining) which I think amounts to falling back on previous approaches. It is my hope nevertheless that the account given here will resonate with modern trends in theoretical computer science, and with homotopy type theory [65] in particular, where specified bijections are part of the whole setup in the form of terms of identity types.

At the same time, the whole-grain formalism in itself can be appreciated without any homotopy overhead. The main contributions, the definitions in Sections 1–3, have no other prerequisites than pullbacks of finite sets. The formalism aligns very well with the way Petri nets are employed in applications, the main feature being that whole-grain Petri nets are just configurations of finite sets, lending themselves to software implementation. Recently, Baas, Fairbanks, Halter, and Patterson chose the whole-grain formalism as the basis for an implementation in Julia [15, 44], of a Petri net library [1] (see Patterson, Lynch and Fairbanks [59] for theoretical background), and used it in epidemiology, first to give elegant modelling of COVID-19 data from the UK [34], and later for a more general compositional approach to epidemiology, cf. Libkind–Baas–Halter–Patterson–Fairbanks [50].

1 **GRAPHS (ACCORDING TO [46])**

The following formalism for directed graphs (and all the results in this section) are from Kock [46], which in turn was heavily inspired by the polynomial formalism for trees [45] and by the formalism for Feynman graphs of Joyal–Kock [40]. Some further comparison is provided in Section 11. The
graphs will play the role of what are also called causal nets in Petri-net theory (see for example, Goltz and Reisig [32]), their purpose being to define processes.

1.1 Graphs (AINOA Style). A graph (meaning directed graph admitting open-ended edges) is a diagram of finite sets

\[
A \xleftarrow{I} N \xleftarrow{O} A
\]

where the outermost maps are injective. Here \(A\) is the set of edges, and \(N\) is the set of nodes. The set \(I\) expresses the incidence of edges and nodes from the viewpoint of edges incoming to nodes, and \(O\) the same for outgoing edges. The injectivity condition says that an edge is incoming (or outgoing) for at most one node. The elements in \(A\) not in the image of \(I\) or not in the image of \(O\) are the open-ended edges. We shall only consider acyclic graphs, meaning having no directed cycles (see [46] and 1.12 below).

1.2 Example. The graph

\[
\{a, b, c, d, e\} \xleftarrow{\{b \rightarrow y, c \rightarrow z, d \rightarrow z\}} \{x, y, z\} \xleftarrow{\{c, d, e\}} \{a, b, c, d, e\}
\]

can be pictured like this:

![Graph Diagram]

The picture is a full rendition of the data of the AINOA diagram, except that the sets \(I\) and \(O\) are not explicit. They can be derived from the picture as subsets of \(A\).

1.3 Etale Maps. An etale map of graphs is a diagram

\[
A' \xleftarrow{I'} N' \xleftarrow{O'} A'
\]

where the middle squares are pullbacks. The pullback condition expresses that arities of nodes must be respected; in other words the map is a 'homeomorphism' locally at each node. An open map is an etale map that is furthermore injective on nodes and edges.

1.4 Remark. The notion of etale map has a clear intuitive content. It also fits into the axiomatic notion of classes of etale maps of Joyal–Moerdijk [41]; see also [36, 42]. There are other useful notions of morphisms of AINOA graphs, some of which are used in Kock [46] (see also Kock [47]), but for the present purposes only etale maps are relevant.

1.5 Sums and Connectedness. The category \(\text{Gr}\) of acyclic graphs and etale maps has categorical sums given by disjoint union of graphs. These are calculated pointwise (i.e., on \(A\), \(N\), \(I\), and \(O\) separately). The empty graph is neutral for sum. A graph is connected if it is non-empty and cannot be written as a sum of smaller non-empty graphs.\(^2\)

\(^2\)Warning: In [46] the symbol \(\text{Gr}\) denotes the category of connected graphs.
1.6 Pullbacks. The category $Gr$ has pullbacks, computed pointwise (i.e., for the $A, I, N, O$ components separately).

1.7 Unit Graph and Edges. The unit graph is the graph $10001$. An edge in a graph $G = AINOA$

$$
\begin{array}{cccc}
1 & \rightarrow & 0 & \leftarrow \emptyset \\
\downarrow & & \downarrow & \downarrow \\
A & \rightarrow & I & \leftarrow N \\
\alpha & & \downarrow & \downarrow \\
& & O & \rightarrow A
\end{array}
$$

is incoming to $G$ if the right-most square is a pullback, and outgoing if the left-most square is a pullback (in addition to the standing requirement that the two middle squares are pullbacks). Accordingly, the in-boundary of $G$ is defined to be the set $\text{in}(G)$ of edges in the complement of $O \rightarrow A$ (that is, the complement of the image of this injective map), and the out-boundary $\text{out}(G)$ is similarly defined as the complement of $A \leftarrow I$. An edge is isolated if it belongs to both the in-boundary and the out-boundary. An edge is inner if it is outgoing to some node and incoming to some node. The set of inner edges is thus the intersection of the (images of the) two maps $O \rightarrow A \leftarrow I$. We write $x \lessdot y$ if there is an inner edge from $x$ to $y$.

In Example 1.2, the in-boundary is $\{a, b\}$, the out-boundary is $\{a, e\}$, and the set of inner edges is $\{c, d\}$.

1.8 Elementary Graphs. An elementary graph is a connected graph with no inner edges. An elementary graph is thus either a unit graph or a corolla, which means one of the form

$$
m + n \leftarrow m \rightarrow 1 \leftarrow n \rightarrow m + n.
$$

Here $m$ is the set of incoming edges and $n$ is the set of outgoing edges.

Let $elGr \subset Gr$ denote the full subcategory of elementary graphs and etale maps—in fact, it is practical to work rather with a skeleton of this category, as we do henceforth. We denote by $[\bullet]$ the unit graph, and by $[\begin{array}{c}m \\ n \end{array}]$ the corolla with $m$ incoming and $n$ outgoing edges. These are pictured, respectively, as

$$
\begin{array}{c}
\bullet \\
\downarrow \\
\begin{array}{c}m \\ n \end{array}
\end{array}
$$

Note that the only non-invertible maps in $elGr$ are the inclusions of $[\bullet]$ into a corolla $[\begin{array}{c}m \\ n \end{array}]$ (of which there are $m+n$). In addition there are the symmetries of the corollas: there are $m!n!$ invertible maps $[\begin{array}{c}m \\ n \end{array}] \rightarrow [\begin{array}{c}n \\ m \end{array}]$. The etale condition precludes non-invertible maps between corollas. Denote by $Cor$ the full subcategory consisting of the corollas; it is thus a groupoid.

1.9 Colimits and Gluing. The category $Gr$ admits enough colimits to account neatly for gluing [46]. In particular, if $M$ is a node-less graph (disjoint union of unit graphs), which embeds into the out-boundary of a graph $G_1$ and embeds into the in-boundary of a graph $G_2$, then the pushout exists in the category $Gr$, and it is calculated pointwise (i.e., for the $A, I, N, O$ components separately). See [46] for details. The finite-set pushouts are along injections and can be computed very
explicitly. It gives a clean formalisation of the intuitive idea of gluing parts of the out-boundary of one graph to parts of the in-boundary of another, as exemplified in this picture:

The reader is encouraged to write down these graphs in AINOA diagrams and actually compute the pushout.

**Lemma 1.10. ([46])** *Every graph is canonically the colimit of its elementary subgraphs:*

\[ G \simeq \colim_{E \in \text{el}(G)} E. \]

The colimit is over the *category of elements* of the graph \( G \), defined as the comma category

\[ \text{el}(G) := \text{elGr} \downarrow G, \]

in turn defined by the natural transformation diagram (lax pullback)

\[ \begin{array}{ccc}
\text{elGr} \downarrow G & \longrightarrow & 1 \\
\downarrow & \Rightarrow & \downarrow \text{“} G^{\uparrow} \text{”} \\
\text{elGr} & \longrightarrow & \text{Gr.}
\end{array} \]

The objects of \( \text{el}(G) \) are thus the elementary subgraphs of \( G \) (or more precisely, etale maps from elementary graphs). For details, see [46]. In concrete terms, the diagram consists of all the edges and all the nodes, with each edge mapping into the corollas it is incident to, as exemplified in this colimit decomposition of a graph \( G \):

**1.11 Dynamics of Graphs.** It is fruitful to view a graph as an interpolation between the in-boundary and the out-boundary, an evolution—not with respect to any absolute notion of time, but reflecting the fact that both \( N \) and \( A \) carry a preorder structure. A graph is *acyclic* if the
preorder $N$ is actually a poset (i.e., is an anti-symmetric relation).³ Further notions of time can be imposed in terms of the following concept:

1.12 Level Functions. A level function of a graph $G = AINOA$ is a monotone map $f : N \rightarrow k$, where $k := \{1, 2, \ldots, k\}$ for some $k \in \mathbb{N}$. Monotone means that for every edge from node $x$ to node $y$, we have $f(x) \leq f(y)$. A level function is strict if the inequality is strict (that is, $x < y \Rightarrow f(x) < f(y)$). A graph is acyclic iff it admits a strict level function.

1.13 Layers, Cuts, and Pushout Decompositions. Level functions serve in particular to split graphs into layers. For example, a level function $f : N \rightarrow 2$ will partition the node set into two sets $N = N_1 + N_2$, namely the pre-images $N_1 := f^{-1}(1)$ and $N_2 := f^{-1}(2)$. This in turn will induce two open subgraphs $G_1$ and $G_2$ called layers, defined as follows: $G_1$ is the open subgraph containing all the nodes in $N_1$ and all their incident edges, and also the in-boundary of $G$. Precisely, to obtain $G_1$, first take pullbacks as indicated with dotted arrows:

Then add the dashed arrows, which are just the inclusions. $G_2$ is constructed similarly from $N_2$, but with $\text{out}(G)$ instead of $\text{in}(G)$. The out-boundary of $G_1$ will coincide with the in-boundary of $G_2$ (as subsets of $A$). Note that isolated edges will belong to both $G_1$ and $G_2$. The intersection $M := G_1 \cap G_2$ (that is, pullback of the inclusion maps) constitutes a disjoint union of units graphs, called a cut. We have

$$\text{in}(G) = \text{in}(G_1), \quad \text{out}(G_1) = M = \text{in}(G_2), \quad \text{out}(G_2) = \text{out}(G).$$

The diagram

$$\begin{array}{ccc}
M & \rightarrow & G_2 \\
\downarrow & & \downarrow \\
G_1 & \rightarrow & G
\end{array}$$

is not just a pullback but also a pushout, as in Section 1.9.

More generally, a level function $N \rightarrow k$ provides $k - 1$ compatible cuts, splitting $G$ into $k$ subgraphs, giving an iterated-pushout formula

$$G = G_1 \sqcup M_1 \sqcup G_2 \sqcup M_2 \cdots \sqcup M_{k-1} \sqcup G_k.$$  

We stress again that all these constructions take place in the category of finite sets. They are at the same time elementary and rigourous.

1.14 Digraphical Species [46]. A digraphical species⁴ is a presheaf $\mathcal{F} : \text{elGr}^\text{op} \rightarrow \text{Set}$. An etale map of graphs is called a cover if it is surjective on nodes and edges. This defines the etale topology on $\text{Gr}$. A sheaf on $\text{Gr}$ with respect to the etale topology is a presheaf $\mathcal{F} : \text{Gr}^\text{op} \rightarrow \text{Set}$ whose value is determined by its values on any cover. Since every graph is the colimit of its elementary subgraphs (see 1.10), and, since these elementary graphs constitute a canonical cover, to give a sheaf it is enough to give its values on elementary graphs. Altogether, sheaves on $\text{Gr}$ are equivalent to presheaves on $\text{elGr}$:

$$\text{PrSh}(\text{elGr}) \simeq \text{Sh}_{\text{et}}(\text{Gr}).$$  

³One can also put $N$ and $A$ together in a single poset, whose Hasse diagram is then bipartite. Such posets form the substrate of the classical poset semantics of Petri nets [27, 32, 61]; see for example the monograph Best and Fernández [14].

⁴The notion is from Joyal and Kock [40], in the setting of undirected graphs.
1.15 Local Structures on Graphs. Digraphical species $\mathcal{F}$ serve to impose or specify local structure or property on graphs, by considering comma categories $\mathbf{Gr} \downarrow \mathcal{F}$, whose objects $G \to \mathcal{F}$ are called $\mathcal{F}$-graphs. This comma category is defined as the lax pullback

$$
\begin{array}{ccc}
\mathbf{Gr} \downarrow \mathcal{F} & \longrightarrow & 1 \\
\downarrow & \Rightarrow & \downarrow \mathcal{F}^* \\
\mathbf{Gr} \downarrow \text{Hom}_{\mathbf{Gr}}(-, G) & \longrightarrow & \text{PrSh}(\text{elGr})
\end{array}
$$

An $\mathcal{F}$-graph is thus a graph whose edges are decorated with elements in $\mathcal{F}[\star]$, and whose $(m, n)$-nodes are decorated with elements in $\mathcal{F}[\binom{n}{m}]$, compatibly with the edge decorations and the projections $\mathcal{F}[\binom{n}{m}] \to \mathcal{F}[\star]$.

For example: $k$-regular, polarised, bipartite graphs are $\mathcal{F}$-graphs for suitable $\mathcal{F}$ (whereas non-local notions such as connected, strongly regular, distance regular, and so on, cannot be encoded with digraphical species). We shall see shortly that each Petri net $P$ defines a digraphical species $P$, and then define its processes to be $\mathcal{F}$-graphs.

The local nature means that the key feature of graphs holds for $\mathcal{F}$-graphs too: Every $\mathcal{F}$-graph is canonically the colimit of its elementary sub-$\mathcal{F}$-graphs, in the category $\mathbf{Gr} \downarrow \mathcal{F}$.

2 WHOLE-GRAIN PETRI NETS

The following definition is possibly the main contribution of this work.

2.1 Petri Nets (SITOS Style). A Petri net $P$ is defined to be a diagram of finite sets

$$
\begin{array}{cccc}
S & \longleftarrow & I & \longrightarrow & T & \longleftarrow & O & \longrightarrow & S
\end{array}
$$

without any conditions. $T$ is now the set of transitions (pictured as small squares) and $S$ is the set of places (pictured as circles). The sets $I$ and $O$ are the sets of arcs, expressing the incidences between transitions and places. For $t \in T$, the fibre (i.e., pre-image set) $I_t$ is called the pre-set of $t$, and the fibre $O_t$ is called the post-set of $t$. (Note that these are not necessarily subsets of $S$, so there can be parallel arcs).

This notion of Petri net is the only one used in this work. When contrast with ‘traditional’ definitions is required, we shall refer to the present notion as whole-grain Petri nets.

2.2 Example. The Petri net

$$
\begin{array}{cccc}
\{s_1, s_2, s_3\} & \longleftarrow & \{i_1, i_2, i_3, i_4\} & \longrightarrow & \{t_1, t_2\} & \longleftarrow & \{o_1, o_2, o_3, o_4\} & \longrightarrow & \{s_1, s_2, s_3\}
\end{array}
$$

is pictured in the usual way as

$$
\begin{array}{cccc}
s_1 & \longleftarrow & i_1 & \longrightarrow & t_1 & \longleftarrow & o_1 & \longrightarrow & s_2 \\
s_2 & \longleftarrow & i_2 & \longrightarrow & t_1 & \longleftarrow & o_2 & \longrightarrow & s_3 \\
s_3 & \longleftarrow & i_3 & \longrightarrow & t_2 & \longleftarrow & o_3 & \longrightarrow & s_1 \\
s_3 & \longleftarrow & i_4 & \longrightarrow & t_2 & \longleftarrow & o_4 & \longrightarrow & s_1
\end{array}
$$

For example to obtain bipartite graphs, consider the digraphical species $\text{elGr}^{op} \to \text{Set}$ with $\binom{n}{m} \mapsto \{r, b\}$ (stipulating that there are two kinds of nodes, red and blue) and $\star \mapsto \{rb, br\}$ (stipulating that there are two kinds of edges, red-to-blue and blue-to-red). Now we should say where the arrows of $\text{elGr}$ go: We declare of course that for all $\star \mapsto \binom{n}{m}$ that hit an incoming edge the map should be $\{r, b\} \mapsto \{rb, br\}$ sending $r$ to $br$ and $b$ to $rb$ (and the other way around for output).
There is a one-to-one correspondence between the elements in the SITOS diagram and the elements of the picture. Also, the maps of the diagram can be read off the picture.

2.3 Comparison with Traditional Definitions of Petri Nets. Traditionally, instead of allowing parallel arcs, arcs can have multiplicities. This is usually formalised by equipping two incidence relations \( I \subset S \times T \) and \( O \subset T \times S \) with multiplicity functions. An elegant formulation is due to Meseguer and Montanari \([54]\), who define a Petri net to be a pair of maps \( T \to C(S) \), where \( C(S) \) is the free commutative monoid on \( S \).

It is quite curious that the difference between parallel arcs and multiplicities does not seem to have been exploited before. Apparently, parallel arcs and multiplicities have always been regarded as the same thing.\(^6\)

The SITOS diagrams of the whole-grain formalism could be written more economically, avoiding the repetition of the set \( S \):

\[
\begin{array}{ccc}
T & \overset{I}{\leftarrow} & O \\
\downarrow & & \downarrow \\
S, & & S
\end{array}
\]

emphasising that it is the data of two parallel spans, in analogy with the Meseguer–Montanari definition \( T \to C(S) \). For further comparison, see Sections 11.5–11.6. The SITOS arrangement of the diagrams, duplicating the set \( S \) and putting it at the ends (as we shall always do), turns out to be the most practical in connection with morphisms. It is also an important design choice that the diagrams extend the polynomial formalism for trees \([45]\) (trees and forests are the special case \( A \leftarrow M \to N = N \to A \)) as well as graphs and hypergraphs \([46]\), as will play a crucial role in this work.

2.4 Morphisms of Petri Nets. A Petri net is thought of as a configuration of interacting transitions, in turn prescriptions for computation steps of certain kinds. The main characteristic of a transition is its interface, its input-output typing. Morphisms of Petri nets should respect these characteristics (in addition to respecting the incidences expressed by the interaction). For this reason, it is natural to stipulate (for the moment) that a morphism of Petri nets is an etale map of diagrams

\[
\begin{array}{ccccccc}
S' & \overset{a}{\leftarrow} & I' & \overset{\alpha}{\to} & T' & \overset{\omega}{\leftarrow} & O' & \overset{\gamma}{\to} & S' \\
S & \downarrow & I & \downarrow & T & \downarrow & O & \downarrow & S.
\end{array}
\]

Let \textbf{Petri} denote the category of Petri nets and etale maps. The etale maps correspond to what Winskel \([68]\) calls folding maps in the setting of traditional Petri nets (they are also the maps

\(^6\)An early example is Hack (1975) \([33]\), who describes ‘generalised Petri nets’ as having ‘bundles of arcs,’ but when he formalises the notion he uses multiplicity functions; see also the influential 1977 survey by Peterson \([60]\).
employed in [4, 5, 8]). Other authors (including Winskel [70] and Meseguer–Montanari [54]) work with more general morphisms. We shall come back to these more general morphisms in Section 10.

2.5 Graphs. Graphs (in the sense of Section 1) are clearly Petri nets, interpreting nodes as transitions and edges as places. They are the Petri nets with the special property that every place is in the pre-set of at most one transition and in the post-set of at most one transition. The interplay between graphs and Petri nets is a main ingredient in the theory. In analogy with the category of graphs, we find:

**Lemma 2.6.** The category Petri has pullbacks, and they are calculated pointwise (i.e., separately on the S, I, T, O components).

**Lemma 2.7.** The category Petri has pushouts and coequalisers over unit graphs, and they are calculated pointwise. Every Petri net is the colimit of elementary graphs over unit graphs (transitions glued along places).

2.8 Example. The Petri net P (from Example 2.2) is the colimit of the graphs arranged in the grey diagram:

![Diagram](image)

2.9 Marking (or State). A marking (also called a state\(^7\)) of a Petri net \(P = S I T O S\) is a map of finite sets \(M \to S\), regarded as an etale map from \(M000M\) to \(P\). The elements of \(M\) are called tokens. Throughout we shall use the letter \(M\) for a set of tokens, and tacitly write \(M = M000M\) for the corresponding nodeless graph.

The markings of \(P\) naturally form a category \(\text{FinSet} \downarrow P \simeq \text{FinSet} \downarrow S\), but we shall be more interested in the corresponding groupoid \(\mathcal{B} \downarrow P \simeq \mathcal{B} \downarrow S\) (where \(\mathcal{B}\) denotes the groupoid of finite sets and bijections), defined by the lax pullback

\[
\begin{array}{ccc}
\mathcal{B} \downarrow P & \xrightarrow{1} & 1 \\
\downarrow & & \downarrow \sim P \\
\mathcal{B} & \xrightarrow{M \to M000M} & \text{Petri}.
\end{array}
\]

\(^7\)The tendency is to use the word ‘marking’ as long as the Petri net is considered just a combinatorial or pictorial gadget, whereas ‘state’ is used when the operational interpretation is in focus.
The objects are thus the markings, and an isomorphism between two markings is a bijection of sets $M \sim M'$ compatible with the maps to $P$. Two markings are isomorphic in this groupoid if and only if they have the same number of tokens for each place. So the isomorphism classes of markings are precisely the classical multisets on $S$. More categorically speaking, the groupoid $\mathbb{B} \downarrow S$ is the free symmetric monoidal category on $S$, and its set of isomorphism classes is $\pi_0(\mathbb{B} \downarrow S) = C(S)$, the set of multisets on $S$. In traditional Petri net theory, that is the definition of marking.

2.10 Example. To specify a marking (of the Petri net from Example 2.2) such as

$$\{m_1, m_2, m_3, m'_4\} \leftarrow \emptyset \rightarrow \emptyset \leftarrow \emptyset \rightarrow \{m_1, m_2, m_3, m'_4\}$$

$$\{s_1, s_2, s_3\} \leftarrow \{i_1, i_2, i_3, i_4\} \rightarrow \{t_1, t_2\} \leftarrow \{o_1, o_2, o_3, o_4\} \rightarrow \{s_1, s_2, s_3\}$$

it is necessary to say where each token lands, as indicated here on the left-most vertical arrow (recall that the right-most vertical arrow is the same map). Since any finite set $M$ can appear in a marking, there is generally no way to avoid specifying the map $M \rightarrow S$ element by element. However, we may always choose to work with an isomorphic marking, ‘renaming’ the tokens for convenience. (In this example, the token ‘names’ were chosen so the map can be read off the subscripts). In the drawing

the left-hand picture contains the exact same information as the diagram. The right-hand picture contains only the information of the isomorphism class of the marked Petri net: This contains all the information about the maps in the diagram, except the choices of ‘names’ of elements. In practice, it may often be the case that the abstract picture (the isomorphism class) exists first, for example as a result of human thought. The act of choosing an explicit representative of the iso-class to work with is effectively to choose the names for the elements in the picture and use those elements as constituents of the sets $S, I, T, O, M$. It is thus important that some choice of representing sets is made, so there are explicit sets to work with, but it is not important which choice is made.

2.11 Individual vs. Collective Tokens. (See [17, 18, 29–31] for more thorough discussion). Classical Petri-net theory favours the collective-tokens philosophy, according to which a state is just a multiplicity function $S \rightarrow \mathbb{N}$, that is, a multiset on $S$. In the present formalism the tokens of a state form an explicit set, and in particular, each token is an individual element in a set (as in the individual-tokens philosophy). However, since the states form a groupoid, both the viewpoints are encoded simultaneously: From the groupoid, one can pass to the individual tokens by considering the underlying set of the groupoid, and one can pass to the indistinguishable-tokens viewpoint by passing to the set of isomorphism classes of the groupoid. At this point the difference is not so big. The real difference arises when it comes to tracing tokens around in processes, as we shall see.
2.12 Initial State. In many applications, Petri nets have an initial state, and it is common to include this in the very definition of Petri net. We do not do so here. When required, initial states are given as diagrams $M \to P$, and morphisms are then required to respect this. The relevant categories are then coslice categories. We shall come back to this in Section 9 in connection with unfolding.

3 PROCESSES

3.1 Simple Firing. A firing of a transition $t \in T$ in a Petri net $P = \text{SITOS}$ in a given state $M \to P$ intuitively consumes a token from each of the places ingoing to $t$ and produces a token in each of the outgoing places. More precisely, it consumes a token for each element in the pre-set $I_t$ (the restriction $S \leftarrow I_t$ tells where the tokens are taken from) and produces a token for each element in the post-set $O_t$ (the restriction $O_t \to S$ then tells where the new tokens are put).

The minimal state in which the firing of $t \in T$ can occur is

\[
\begin{array}{cccccccc}
I_t & \leftarrow & \emptyset & \rightarrow & \emptyset & \leftarrow & \emptyset & \rightarrow & I_t \\
S & \leftarrow & I & \rightarrow & T & \leftarrow & O & \rightarrow & S, \\
\end{array}
\]

and the state after the transition has fired will then be

\[
\begin{array}{cccccccc}
O_t & \leftarrow & \emptyset & \rightarrow & \emptyset & \leftarrow & \emptyset & \rightarrow & O_t \\
S & \leftarrow & I & \rightarrow & T & \leftarrow & O & \rightarrow & S. \\
\end{array}
\]

The whole firing (in the minimal state enabling it) is encoded geometrically by a single etale map $C \to P$ from a corolla, namely

\[
\begin{array}{cccccccc}
I_t + O_t & \leftarrow & I_t & \rightarrow & \{t\} & \leftarrow & O_t & \rightarrow & I_t + O_t \\
S & \leftarrow & I & \rightarrow & T & \leftarrow & O & \rightarrow & S. \\
\end{array}
\]

(Note that any other isomorphic sets could take the place of $I_t, O_t$ to constitute a corolla). When reading such a $C \to P$ as a firing, the initial state is that given by the in-boundary of $C$ and the final state is that given by the out-boundary of $C$ (cf. 1.7 and 1.11). (In the displayed case, these are $I_t$ and $O_t$).

A firing of the transition $t \in T$ in a general state is encoded by just adding more tokens. From the viewpoint of the corolla $C$, this is to add a bunch of isolated edges, so the general firing of $t \in T$ has the form

\[
\begin{array}{cccccccc}
I_t + O_t + M & \leftarrow & I_t & \rightarrow & \{t\} & \leftarrow & O_t & \rightarrow & I_t + O_t + M \\
S & \leftarrow & I & \rightarrow & T & \leftarrow & O & \rightarrow & S \\
\end{array}
\]

for some set $M$. The domain is then no longer a corolla, but it is still a graph. The interpretation of initial and final state in terms of in-boundary and out-boundary of the graph is still valid, since the isolated edges, corresponding to the tokens not changed by the firing, belong to both the in- and the out-boundary (cf. 1.7).
3.2 Example. Here is a minimal firing $p : C \rightarrow P$ of the transition $t_1$ in the Petri net $P$ from Example 2.2:

```
{a_1, a_2, b_2, b_3} \leftarrow \{a_1, a_2\} \rightarrow \{x_1\} \leftarrow \{b_2, b_3\} \rightarrow \{a_1, a_2, b_2, b_3\}
```

The top row is the corolla $C$; the bottom row is the Petri net $P$. The vertical maps, constituting the etale map $p$, are specified (in blue) simply by telling where each individual element goes. In the picture

![Diagram](image)

this information is conveyed by the ‘postage stamp.’

3.3 Executions — Preliminary Discussion. An execution of a Petri net $P = SITOS$ in a state $M \rightarrow P$ is supposed to be just a bunch of firings taking place in sequence or concurrently. The fully parallel situation is given by an etale map $p : G \rightarrow P$, where $G$ is a disjoint union of elementary graphs: The corollas in $G$ then express the simultaneous firing, and the isolated edges are just dead weight contributing to the state. Again, the initial state is the (restriction of $p$ to the) in-boundary of the graph $G$, and the final state is the (restriction of $p$ to the) out-boundary of $G$. Causal relationships between firings are expressed with more general graphs:

3.4 Processes = P-Graphs. A process of a Petri net $P = SITOS$ is an etale map $p : G \rightarrow P$ where $G$ is an acyclic graph. This is also called a P-graph.

We have previously used $AINOA$ notation for graphs. From now on, the symbols $I$ and $O$ are reserved for the incidence sets of Petri nets, so for graphs we now use the notation $A_N$ and $N_A$ for the subsets of $A$ consisting of the edges that are incoming to some node and outgoing of some node, respectively. A process for $P$ is thus a diagram

```
A \leftarrow A_N \rightarrow N \leftarrow N_A \rightarrow A
```

where the top row is an acyclic graph.

From the viewpoint of graphs, a process of $P$ is an acyclic graph $G$ where each edge is decorated by a place of $P$ and each node is decorated with a transition of $P$ of matching interface. Furthermore, the elements expressing the incidence relations of the graph $G$ must be mapped to arcs of $P$. We stress that the maps $A_N \rightarrow I$ and $N_A \rightarrow O$ must be specified too—they are not implied from
the maps \( N \to T \) and \( A \to S \). All these assignments should be compatible, as expressed by the commutativity of the diagram. The initial state of the process is the (restriction of the etale map to the) in-boundary of \( G \) and the final state is the (restriction to the) out-boundary of \( G \).

3.5 The Category of Processes. Define the category of processes of \( P \) to be the comma category

\[
\mathsf{Proc}(P) := \mathsf{Gr} \downarrow P.
\]

The morphisms are thus commutative triangles (of etale maps)

\[
\begin{array}{ccc}
G & \longrightarrow & G' \\
\downarrow p & & \downarrow p' \\
\downarrow P & & \\
\end{array}
\]

The most important maps in \( \mathsf{Proc}(P) \) are the invertible maps: They allow us to use the sets we like as constituents of the graphs, and allow us to replace these sets by isomorphic ones, effectively ‘renaming’ elements to our liking. The invertible maps keep track of these ‘renamings’, and ensure the ability to distinguish elements in these sets. There may also be invertible maps from one graph to itself—its symmetries.

The category \( \mathsf{Proc}(P) \) also has non-invertible maps, such as in particular sub-graph inclusions. From the viewpoint of processes as computations, these represent shorter (or partial) computations. In particular it is important that the inclusion of the in- or out-boundary of a process is a map in \( \mathsf{Proc}(P) \) (representing the trivial computation, just a state): Our goal will be to define serial composition of processes, which will be achieved (in Section 4) in terms of gluings along graph boundaries, in turn described as pushouts as in Section 1.9. For these pushouts (and other colimits) to make sense, it is essential to have non-invertible maps in \( \mathsf{Proc}(P) \).

Proposition 3.6. The assignment \( P \mapsto \mathsf{Proc}(P) \) is functorial in etale maps of Petri nets: an etale map of nets \( f : P' \to P \) induces canonically a functor \( f_! : \mathsf{Proc}(P') \to \mathsf{Proc}(P) \) simply by post-composition of etale maps. Altogether, this defines a functor \( \mathsf{Proc} : \mathsf{Petri} \to \mathsf{Cat} \).

3.7 Comparison with Traditional Notions of Processes. Modulo the differences in the definitions of Petri net and graphs, the above definition of process is precisely that of Goltz and Reisig [32]. The idea of modelling processes of a Petri net with maps from graphs or certain posets goes back to Petri himself [61] (1977).

Despite this close similarity, the passage from traditional Petri nets to whole-grain Petri nets implies some essential differences for processes, as illustrated in the examples, and as will be important for the main theorems. To indicate the point briefly, consider a transition \( t \): in a traditional Petri net, \( \text{pre}(t) \) and \( \text{post}(t) \) are mere numbers, and the ‘etale’ condition on a map from a graph says that it can receive only nodes \( x \) with that many incoming and outgoing edges, but there is no way to control those incoming and outgoing edges, and in particular there are situations where they can be permuted. In the SITOS formalism, \( \text{pre}(t) \) and \( \text{post}(t) \) are actual sets, and being an etale map involves explicit bijections with these sets, \( \text{in}(x) \xrightarrow{\sim} \text{pre}(t) \) and \( \text{out}(x) \xrightarrow{\sim} \text{post}(t) \), giving a good handle on symmetries.

3.8 Scheduling. A scheduling of a process \( \rho : G \to P \), also called a run, is just a strict level function on \( G \) (cf. 1.12). The scheduling is sequential if the level function is bijective, and the scheduling is then called a firing sequence. A strict level function \( f : G \to k \) prescribes a colimit decomposition

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of the graph as a sequence of pushouts over nodeless graphs as in Section 1.13,
\[ G \simeq G_1 \sqcup_{M_1} \cdots \sqcup_{M_{k-1}} G_k, \]
where each layer \( G_i \) is a disjoint union of elementary graphs, and where the \( M_i \) are nodeless graphs. Therefore, each \( M_i \to P \) is a state and each \( G_i \) is a simultaneous firing \( G_i \to P \) with initial state \( M_{i-1} \) and final state \( M_i \) (here, we include \( M_0 \) defined as the in-boundary of \( G_1 \) (which is also the in-boundary of \( G \)) and \( M_k \) defined as the out-boundary of \( G_k \) (which is also the out-boundary of \( G \)). If the strict level function is bijective, then each firing \( G_i \to P \) is a simple firing (in the sense of Section 3.1).

3.9 Example. The main purpose of this example is to show how closely tokens are kept track of in a process. The following diagram is a process \( p : G \to P \).

The bottom row is the Petri net \( P \) from Example 2.2. In the top row (the graph \( G \)), the constituent arrows have not been indicated, but they can be read off the following picture, where the graph \( G \) is on the left. To specify the etale map (the vertical arrows), it is necessary to tell where each individual element goes, as done with the blue annotations. For the outermost map \( A \to S \), the assignments \( a_1 \mapsto s_1, a_2 \mapsto s_2, a_3 \mapsto s_3, b_1 \mapsto s_1, b_2 \mapsto s_2, b_3 \mapsto s_3, c_1 \mapsto s_1, c_2 \mapsto s_2, c_3 \mapsto s_3 \) have been suppressed from the diagram to avoid clutter, since they can be inferred from the maps at the \( I \) and \( O \) levels. (Only the element \( d_3 \) is not accounted for like this, since it is an isolated edge.) The reason for bothering with this ‘economy’ of annotation is that it is natural from the graphical viewpoint, where it is clear that the mapping information of \( p \) is fully specified by telling where the local interfaces of each node are sent (and then annotating the isolated edge separately):

A choice of level function has been indicated with the red dashed lines.

Note that the maps at the \( I \) and \( O \) levels contain very precise information about token flow: we see for example that when \( t_2 \) fires in the process \( p \), its \( i_3 \) input slot consumes the token that was previously produced by the \( t_1 \)-firing (namely, \( b_3 \)) whereas the \( i_4 \) input slot of \( t_2 \) consumes the token \( a_3 \) that was already there from the start. If the etale map were modified to have \( a_3 \mapsto i_5 \) instead of \( a_3 \mapsto i_4 \), and \( b_3 \mapsto i_4 \) instead of \( b_3 \mapsto i_3 \) (corresponding to interchanging the two
decorations $i_3$ and $i_4$ in the picture), then a different (and non-isomorphic) process would result, with a different flow of tokens.

A more fundamentally different $P$-process $q$ (with a level function) is pictured here:

The decorations specify precisely how the graph maps to $P$, but the graph itself has only been specified up to isomorphism: it remains to choose representative sets (‘names of elements’) for the graph. Since we are generally only interested in structural properties of Petri nets and processes, it is quite reasonable to indicate only the isomorphism class, leaving the implementation details such as ‘element names’ to the reader. We shall freely do this throughout the paper.

In process $p$, there is one token $d_3$ that does not participate other than staying put in $s_3$ all the time. In $q$, all tokens participate actively. This process $q$ also has the property that it could have been scheduled differently: by choosing a different level function one could fire $t_2$ before firing $t_1$. This shows that $p$ and $q$ do not have the same causal structure. In fact, it is clear that $p$ and $q$ are not isomorphic: they are not even isomorphic as underlying graphs. However, with the level functions indicated in the pictures with dashed red lines, the two processes both correspond to the following sequence of isomorphism classes of firings.

This shows that the processes cannot be reconstructed, even up to isomorphism, from knowledge of the isomorphism classes of the steps in the firing sequence: The isomorphism classes do not retain enough information about the tokens to be able to compose by gluing.\(^8\)

4 THE SYMMETRIC MONOIDAL SEGAL SPACE OF PROCESSES

From this point on (and up to and including Section 7), a little bit of elementary homotopy theory of groupoids is involved. This is necessary since we are interested in objects up to isomorphism, but still want to keep track of their symmetries. A few definitions and basic facts are recollected in Appendix A. The goal is to assemble processes (for a fixed Petri net $P$) into a kind of category. This will require some simplicial structures, explained along the way; some background is given in Appendix B.

\(^8\)The fact that different graphs/posets can correspond to the same firing sequence was stressed in the Best–Fernández book [14]. Drastic quotients are required to make the viewpoints match up [13], as we shall briefly comment on in Section 7.3.
4.1 Towards Composition of Processes. Fix a Petri net \( P \). Let \( X_0 \) denote the groupoid of states of \( P \). Let \( X_1 = \operatorname{Proc}_{\text{iso}}(P) \) denote the groupoid of processes of \( P \). Every state is also a process, so we have a canonical map \( s_0 : X_0 \to X_1 \). Every process has an initial state and a final state, so we also have maps

\[
X_0 \xrightarrow{d_0} X_1 \quad \text{and} \quad X_1 \xrightarrow{d_1} X_0.
\]

Here \( d_0 \) returns the final state and \( d_1 \) returns the initial state, not the other way around. Although this convention may look a bit counter-intuitive, the indexing is dictated by the standard simplicial formalism we shall exploit, where an index always indicates the vertex that was deleted, as explained in B.3. We think of a process \( p \) as an arrow in a category, from its initial state \( d_1(p) \) to its final state \( d_0(p) \).

Since states form a groupoid, whose morphisms express renaming of tokens, we may be more interested in composing processes that only match up to a specified isomorphism: Given two processes \( p_1 : G_1 \to P \) and \( p_2 : G_2 \to P \) (that is, \( p_1 \in X_1 \) and \( p_2 \in X_1 \)), we consider the situation where the final state of \( p_1 \) is isomorphic to the initial state of \( p_2 \), with a specified isomorphism \( \sigma : d_0(p_1) \sim \rightarrow d_1(p_2) \). This means precisely that the triple \((p_1, p_2, \sigma)\) is an element in the standard homotopy pullback of groupoids (see A.2)

\[
\begin{array}{ccc}
X_1 \times_{X_0}^h X_1 & \xrightarrow{\text{proj}} & X_1 \\
\downarrow \text{proj}_1 & \sim & \downarrow d_1 \\
X_1 & \xrightarrow{d_0} & X_0.
\end{array}
\]

We wish to define a weak composition law

\[
X_1 \times_{X_0}^h X_1 \xrightarrow{\text{comp}} X_1,
\]

allowing to compose processes via connecting isomorphisms of states.\(^9\) To formalise this idea, we first fit \( X_0 \) and \( X_1 \) into the structure of a simplicial groupoid \( X_\bullet : \Delta^{\text{op}} \to \text{Grpd} \); then we show that this simplicial groupoid is a Segal space, which is the standard way to encode weak categories (see Appendix B for a bit of background and motivation).

4.2 The Simplicial Groupoid of Processes. Fix a Petri net \( P \). For each \( k \geq 0 \), denote by \( X_k \) the groupoid of \( P \)-graphs \( p : G \to P \) equipped with a level function \( G \to k \), i.e., processes \( G \to P \) equipped with \( k - 1 \) compatible cuts, as in 1.13. The morphisms in this groupoid are isomorphisms of graphs compatible with both the etale map to \( P \) and the level function to \( k \). In particular, \( X_1 \) is just the groupoid of processes (since every \( P \)-graph has a unique 1-level function), and \( X_0 \) is the groupoid of states (because a 0-level function can exist only for \( P \)-graphs with no nodes). Next, \( X_2 \) is the groupoid whose objects are processes \( G \to P \) equipped with a cut (as in 1.13).

The groupoids \( X_k \) assemble into a strict simplicial groupoid (see B.2 and B.5)

\[
X_\bullet : \Delta^{\text{op}} \to \text{Grpd}
\]

\[
[k] \quad \mapsto \quad X_k.
\]

For this, we need to describe the face and degeneracy maps (cf. B.1).

The degeneracy maps \( s_i : X_k \to X_{k+1} \) (for \( 0 \leq i \leq k \)) insert an empty layer. Formally, this can be described as composing level functions \( G \to k \) with injective monotone maps \( k \to k+1 \). We shall

\(^9\)See Sassone [63] for a version of this viewpoint.
not go into details with the degeneracy maps. Although they are required to define a simplicial object, they do not play any role for the Segal condition.

The inner face maps $X_{k-1} \xrightarrow{d_i} X_k$ ($0 < i < k$) join adjacent layers, or equivalently, delete cuts. Formally this is to postcompose level functions $G \to k$ with surjective monotone maps $k \to k-1$.

So far, only the level functions are affected, whereas the underlying $P$-graph is not changed. Finally, the outer face maps $X_{k-1} \xleftarrow{d_0} X_k$ project away the first or the last layer; this obviously changes the underlying $P$-graph.

Having defined the face and degeneracy maps, we should now verify the simplicial identities as listed in B.1. The way the face and degeneracy maps have been defined, this is straightforward. For example, the face-map identities state that if two cuts are deleted, then it does not matter in which order, or that throwing away the first two layers is the same as first joining them and then throwing away the joined layer in one go. The following example illustrates that throwing away the first layer and then the last is the same as throwing away the last and then the first.

4.3 Example. The following is a picture of a 2-layered process $p \in X_2$ (of the Petri net in Examples 2.2 and 3.9) and the effect of the three face maps.

\begin{align*}
d_0 \text{ applied to } p \text{ erases the first layer;} & \\
d_1 \text{ joins the two layers; } & \\
d_2 \text{ erases the last layer. The further face maps applied give the commutative square on the left, which illustrates the simplicial identity } & \\
d_0d_2 = d_1d_0. & 
\end{align*}

4.4 Remark. Constructions of simplicial groupoids like this are not uncommon in combinatorics. In fact, this particular simplicial groupoid $X_\bullet$ is almost the same as that of the directed restriction species of directed graphs explained in Example 7.13 of [23], where substantial details of the construction can be found, although in a much more general setting. The difference (apart from the $P$-decorations) concerns isolated edges, excluded in the setting of restriction species, but clearly essential to keep for the present purposes.

4.5 Segal Spaces. For a simplicial groupoid to work as a (weak) category, it should be a Segal space (see B.6). First, this means that the canonical map

\[(d_2, d_0) : X_2 \to X_1 \times_{X_0} X_1\]
should be an equivalence of groupoids. More generally, it is required that for all \( k \geq 1 \) the canonical map that returns the \( k \) layers

\[
X_k \longrightarrow X_1 \times_h X_0 \cdots \times_h X_0 X_1
\]

is an equivalence of groupoids.

To establish this, we first establish the following important fibrancy condition (see A.3), which allows to use ordinary strict pullbacks instead of homotopy pullbacks.

**Lemma 4.6.** The face map \( d_0 : X_1 \to X_0 \) (which given a process returns its final state) is a fibration of groupoids (cf. A.3).\(^{10}\)

**Proof.** Given a process \( p : G \to P \) with final state \( d_0(p) = z : M \to P \) (meaning that \( M = \text{out}(G) \)), and given an isomorphism with another state \( z' : M' \to S \), amounting to a commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{u} & M' \\
\downarrow{z} & & \downarrow{z'} \\
S & & \\
\end{array}
\]

the statement is that there exists a lift of \( u \) to \( p \), namely an isomorphism of processes \( \bar{u} : p \simeq p' \) (for some other process \( p' \)), whose restriction to the out-boundaries reproduces the bijection \( u \). To construct this, suppose the graph \( G \) is given by \( A \leftarrow A_N \to N \leftarrow N A \to A \), with out-boundary \( \text{out}(G) = M \). By the definition of out-boundary (1.7), the set of edges thus splits into two disjoint subsets

\[ A = M + A_N. \]

Now define the new graph \( G' \) and the morphism \( \bar{u} \) by modifying as little as possible—only the out-boundary:

\[
\begin{array}{ccc}
G & \xrightarrow{\bar{u}} & M + A_N \leftarrow A_N \to N \leftarrow N A \to M + A_N \\
\downarrow{p} & & \downarrow{\bar{u} + \text{id}} \\
G' & \xrightarrow{p'} & M' + A_N \leftarrow A_N \to N \leftarrow N A \to M' + A_N \\
\end{array}
\]

The new graph \( G' \) is naturally a \( P \)-graph as indicated, and by construction the new map \( \bar{u} \) is a morphism of \( P \)-graphs.

The benefit of the fibrancy condition is the general fact that homotopy pullbacks along fibrations can be computed as strict pullbacks (see A.2–A.3).

**Proposition 4.7.** The simplicial groupoid \( X_* \) is a Rezk-complete Segal space.

**Proof.** The Segal condition states that the map \( X_k \to X_1 \times_{X_0} \cdots \times_{X_0} X_1 \) in (4) is an equivalence of groupoids. For notational convenience, we do the case \( k = 2 \) only.\(^{11}\) Since this homotopy pullback (which is (3)) is along the fibration \( d_0 \), we can prove instead that

\[
X_2 \xrightarrow{g} X_1 \times_{X_0} \text{strict } X_1
\]

\(^{10}\)By the same argument, also \( d_1 : X_1 \to X_0 \) (initial state) is a fibration, but we shall not need that.

\(^{11}\)For higher \( k \), one can use induction, reformulating the Segal condition as \( X_k \simeq X_{k-1} \times_{X_0} X_1 \).
is an equivalence of groupoids. An element in this strict pullback is a pair \((p_1, p_2)\) of processes such that the final state \(d_0(p_1)\) of the first is literally equal to the initial state \(d_1(p_2)\) of the second.

The map \(g\) is bijective at the level of \(\pi_0\), because the pushout construction of 1.9 provides an up-to-isomorphism inverse: given processes \(p_1 : G_1 \to P\) and \(p_2 : G_2 \to P\) with \(\text{out}(G_1) = M = \text{in}(G_2)\), the pushout produces a single \(P\)-graph \(G_1 \sqcup_M G_2\) with the obvious 2-level function; it is clear that this construction is inverse to \(g\) up to isomorphism. To establish that \(g\) is an equivalence, it remains to check that it is also an isomorphism on automorphism groups (cf. A.1). The automorphisms of a 2-levelled \(P\)-graph

\[
\begin{array}{ccc}
G & \xrightarrow{f} & 2 \\
\downarrow{p} & & \\
P
\end{array}
\]

are the automorphisms of \(G\) compatible with both \(p\) and \(f\). This amounts to giving an automorphism of each layer \(G_1\) and \(G_2\), whose restrictions to the cut \(M\) agree. But this is precisely the description of the automorphism group of the corresponding object \((G_1, G_2)\) in \(X_1 \times_{X_0}^{\text{strict}} X_1\).

Finally, Rezk completeness (B.6) is the statement that the only invertible processes in \(X_1\) are the degenerate ones, i.e., in the image of \(s_0 : X_0 \to X_1\). A 1-simplex \(p\) in a Segal space is called invertible if composition with it from either side defines an equivalence \(X_1 \to X_1\). In the present case, \(p\) is a \(P\)-graph and composition is gluing. But composition with a graph having a node will increase the number of nodes, so the only invertible processes are the node-less graphs. These are precisely those in the image of \(s_0\), as required.

**Lemma 4.8.** \(X_\bullet\) is a symmetric monoidal Segal space under disjoint union \(+\). That is, the simplicial maps

\[
\begin{array}{ccc}
1_\bullet & \xleftarrow{\otimes_0} & X_\bullet \\
& & \leftarrow\downarrow{\times} \\
& & X_\bullet \times X_\bullet
\end{array}
\]

satisfy the standard associative, unital, and symmetry axioms.

Here \(1_\bullet\) is the constant simplicial groupoid on the terminal groupoid, and the map in degree \(k\) picks out the empty \(k\)-levelled \(P\)-graph. The required coherence constraints, given separately in each simplicial degree, follow from the universal properties of \(\emptyset\) and \(+\) as initial object and categorical sum.

**4.9 Remark.** The symmetric monoidal Segal space \(X_\bullet\) is very easy to set up, since it is defined in terms of \emph{decomposition} instead of composition. This phenomenon, that decomposition is easier to achieve than composition, is quite general, and is one starting point of the recent theory of \emph{decomposition spaces} [21–24]. For the symmetric monoidal structure, it must be appreciated that \(X_\bullet\) is Rezk complete. This implies that functor categories can be dealt with pointwise, i.e., in each simplicial degree separately. The symmetric monoidal structure in each simplicial degree is obvious.

**Proposition 4.10.** An etale map of Petri nets \(e : P \to P'\) induces a symmetric monoidal functor of Segal spaces

\[
e! : X_\bullet \to X'_\bullet
\]

by sending a layered process \(G \to P\) to the composite \(G \to P \to P'\) with the same layering.

Indeed, the map in simplicial degree 1 is a special case of Proposition 3.6. The remaining simplicial degrees are a matter of level functions and refer only to the underlying graph \(G\); they are not affected by \(P\)- or \(P'\)-structure.
5 DIGRAPHICAL SPECIES FROM PETRI NETS

Recall that a digraphical species is a presheaf \( \mathcal{F} : \text{elGr}^{\text{op}} \to \text{Set} \) on the category of elementary graphs, and that we call \( \mathcal{F}[\star] \) the set of colours. The set \( \mathcal{F}[\frac{n}{m}] \) is called the set of \((m,n)\)-ary operations.\(^{12}\)

5.1 Digraphical Species of a Petri Net. A Petri net \( P \) defines a digraphical species

\[
P : \text{elGr}^{\text{op}} \longrightarrow \text{Set}
\]

\( E \mapsto \text{Hom}(E, P), \)

which is simply the restricted Yoneda embedding along the full inclusions \( \text{elGr} \subset \text{Gr} \subset \text{Petri} \). The set of colours of \( P \) is thus the set of places \( S \), and the operations are the transitions \( T \), symmetrised: Indeed, to give an etale map from a corolla to \( P \) is first to say where the unique node of the corolla goes, say \( \mapsto t \in T \). If the transition \( t \) has \( m \) incoming arcs and \( n \) outgoing arcs, then the domain of the map has to be the \([\frac{n}{m}]\)-corolla, by etaleness. There are now are \( m!n! \) different etale maps \([\frac{n}{m}] \to P \) hitting \( t \). All of these maps are valid, as there are no constraints on places. For each map, the corolla will acquire colours on edges according to the places they are mapped to.

From the decomposition of Petri nets into colimits of elementary graphs (Lemma 2.7), we get\(^{13}\):

**Lemma 5.2. (Density lemma.)** The functor

\[
\text{Petri} \longrightarrow \text{PrSh}(\text{elGr})
\]

\( P \mapsto \text{Hom}(\cdot, P) \)

is fully faithful.

The following result provides a characterisation of Petri nets by describing the image of this embedding.

**Proposition 5.3.** A digraphical species \( \mathcal{F} : \text{elGr}^{\text{op}} \to \text{Set} \) is a Petri net if and only if it is flat (meaning that the symmetric-group actions \((\mathbb{S}_m \times \mathbb{S}_n) \times \mathcal{F}[\frac{n}{m}] \to \mathcal{F}[\frac{n}{m}] \) are free) and it takes finite values and has finite support (meaning that there is an upper bound on the \( m \) and \( n \) for which \( \mathcal{F}[\frac{n}{m}] \) is non-empty).

**5.4 Remark.** This result is a variation of Theorem 2.4.10 of Kock [45], which characterises polynomial endofunctors among the presheaves on elementary trees. The terminology flat is from the theory of combinatorial species [10]. In the theory of operads, the same condition is called sigma-cofibrant. Note finally that the flat digraphical species are also essentially the same thing as the tensor schemes of Joyal and Street [43].

**Proof.** Starting with a Petri net \( P = \text{SITOS} \), the associated graphical species is \( \text{Hom}(\cdot, P) \):

\[
\text{elGr}^{\text{op}} \longrightarrow \text{Set}
\]

\( [\star] \mapsto \text{Hom}(\star, P) = S \)

\( [\frac{n}{m}] \mapsto \text{Hom}(\frac{n}{m}, P). \)

\(^{12}\)This terminology comes from the theory of operads.

\(^{13}\)by standard category theory [51, Ch. X, §6].
To specify an etale map \([ n \leftarrow m \rightarrow P]\) is to give a transition \(t \in T\) and bijections \(\beta_{\text{in}} : m \rightarrow \text{pre}(t)\) and \(\beta_{\text{out}} : n \rightarrow \text{post}(t)\). The natural \((\mathcal{E}_m \times \mathcal{E}_n)\)-action

\[
(\mathcal{E}_m \times \mathcal{E}_n) \times \text{Hom}(\left[ n \leftarrow m \rightarrow P\right], P) \rightarrow \text{Hom}(\left[ n \leftarrow m \rightarrow P\right], P)
\]

\[
((\sigma_{\text{in}}, \sigma_{\text{out}}), (t, \beta_{\text{in}}, \beta_{\text{out}})) \mapsto (t, \beta_{\text{in}} \circ \sigma_{\text{in}}, \beta_{\text{out}} \circ \sigma_{\text{out}})
\]

is clearly free. Since each transition \(t\) has finite \(\text{pre}(t)\) and \(\text{post}(t)\), and since there are only finitely many transitions, we find that all the sets \(\text{Hom}(\left[ n \leftarrow m \rightarrow P\right], P)\) are finite, and empty for big \(m\) and \(n\).

Conversely, given a flat digraphical species \(\mathcal{F} : \text{elGr}^{\text{op}} \rightarrow \text{Set}\), we construct a Petri net \(P = \text{SITOS}\) by putting \(S := \mathcal{F}[\star]\) and defining \(T\) to be the disjoint union

\[
T := \sum_{m,n} \mathcal{F}[\left[ n \leftarrow m \rightarrow \mathcal{F}\right]]/(\mathcal{E}_m \times \mathcal{E}_n).
\]

This is a finite set by the finiteness assumptions on \(\mathcal{F}\). It remains to define the sets \(I\) and \(O\) to have correct fibres over \(T\) and appropriate ‘colours.’ If \(t \in T\) is in the \((m,n)\)-summand, then the \(I\)-fibre should have \(m\) elements and the \(O\)-fibre should have \(n\) elements. Essentially, \(I\) and \(O\) should be defined as disjoint unions over all such data, and the projections to \(S\) should return the corresponding colour according to the \(\mathcal{F}\)-structure. To describe this in a natural way, we first fix \(m\) and \(n\) and note that \(\mathcal{F}[\left[ n \leftarrow m \rightarrow \mathcal{F}\right]]\) can be interpreted as the set \([\left[ n \leftarrow m \rightarrow \mathcal{F}\right]]\) via the Yoneda lemma. Consider now the two fancier sets: \([\star \rightarrow n \leftarrow m \rightarrow \mathcal{F}]\) where the first map is any of the \(m\) maps of elementary graphs that pick out an input edge, and \([\star \rightarrow \left[ n \leftarrow m \rightarrow \mathcal{F}\right]]\) where the first map is any of the \(n\) maps of elementary graphs that pick out an output edge. With the natural projections

\[
S = \{\star \rightarrow \mathcal{F}\} \quad \{\left[ n \leftarrow m \rightarrow \mathcal{F}\right] \rightarrow \mathcal{F}\} \quad \{\star \rightarrow \mathcal{F}\} = S
\]

we get the basic building blocks for the final SITOS diagram. It remains to divide out by the group actions, and sum up all \(m\) and \(n\). We already have the free \((\mathcal{E}_m \times \mathcal{E}_n)\)-action on the set \(\{\left[ n \leftarrow m \rightarrow \mathcal{F}\right]\}\). For the fancier sets, each group element \((\sigma_{\text{in}}, \sigma_{\text{out}}) \in \mathcal{E}_m \times \mathcal{E}_n\) acts by permuting the \(\left[ n \leftarrow m \rightarrow \mathcal{F}\right]\) in the middle, sending the upper composite in the diagram

\[
\begin{array}{c}
\star \\
\downarrow \sigma_{\text{in},\sigma_{\text{out}}} \\
\left[ n \leftarrow m \rightarrow \mathcal{F}\right]
\end{array}
\]

to the lower composite. The new map \([\star] \rightarrow \left[ n \leftarrow m \rightarrow \mathcal{F}\right]\) is defined as the old map followed by \((\sigma_{\text{in}}, \sigma_{\text{out}})^{-1}\), which means that the composite \([\star] \rightarrow \mathcal{F}\) is not changed by the action: It is still the same colour of \(\mathcal{F}\) picked out. This means that when we pass to the quotient, the maps to \(S\) are still well defined.

Taking now quotients and summing over \(m, n\), the final SITOS-diagram is altogether defined as

\[
\begin{align*}
\sum_{m,n} & \frac{\{\star \rightarrow \mathcal{F}\}}{\mathcal{E}_m \times \mathcal{E}_n} \\
\sum_{m,n} & \frac{\{\left[ n \leftarrow m \rightarrow \mathcal{F}\right] \rightarrow \mathcal{F}\}}{\mathcal{E}_m \times \mathcal{E}_n} \\
\sum_{m,n} & \frac{\{\star \rightarrow \mathcal{F}\}}{\mathcal{E}_m \times \mathcal{E}_n}
\end{align*}
\]

\[
S = \{\star \rightarrow \mathcal{F}\} \quad \sum_{m,n} \frac{\{\left[ n \leftarrow m \rightarrow \mathcal{F}\right] \rightarrow \mathcal{F}\}}{\mathcal{E}_m \times \mathcal{E}_n} \quad \{\star \rightarrow \mathcal{F}\} = S
\]
We omit the straightforward verification that the two constructions are inverse to each other, up to natural isomorphism.

**Lemma 5.5.** For a Petri net, P-graphs have no infinitesimal automorphisms, in the sense that if an automorphism fixes a node, then it is the identity on that connected component.

**Proof.** Let $f : G \to G$ be an automorphism of P-graphs that fixes a node $x$. Since $f$ preserves the incidences, it must map the set $\text{in}(x)$ of incoming edges of $x$ to itself and the set $\text{out}(x)$ of outgoing edges of $x$ to itself. The P-structure is an etale map $G \to P$, mapping $x$ to some transition $t$, and the etale condition gives bijections $\text{in}(x) \cong \text{pre}(t)$ and $\text{out}(x) \cong \text{post}(t)$. Since $f$ must commute with these bijections, it is forced to be the identity on $\text{in}(x)$ and $\text{out}(x)$. Therefore it must also fix any incident nodes, and so on, forcing altogether the whole connected component to be fixed. □

**5.6 Remark.** Even when all nodes of a P-graph are fixed, it is still possible to permute isolated edges that map to the same place in $P$. This is relevant, since isolated edges in a process correspond to unaffected tokens (as noted in 3.3), meaning tokens that belong to both the initial and the final marking.

**5.7 Example.** There are two possible etale maps corresponding to the two possible bijections $\text{in}(x) \cong \text{pre}(t)$. But these two processes are isomorphic by the graph isomorphism that interchanges the two edges. More generally, there are of course infinitely many graphs corresponding to the figure, depending on which 2-element set is chosen as set of edges, and so on, and therefore also infinitely many distinct processes of this shape. For any two such graphs $A\text{INOA}$ and $A'\text{INOA}'$, there are two possible isomorphisms between them (since there are always two distinct bijections $A \cong A'$ between 2-element sets). However, as processes there is precisely one possible bijection, because the process involves a bijection $\text{in}(x) \cong \text{pre}(t)$, and only one bijection $A \cong A'$ can be compatible with those. In conclusion, any two processes of the shape of the figure are uniquely isomorphic. This illustrates Lemma 5.5.

Here we see a crucial difference with traditional Petri net theory, where instead of parallel arcs there would be merely a multiplicity:

![Multiplicity](image)

The domain graph of a process looks the same in traditional Petri net theory, but instead of the etale condition involving explicit bijections, we now have the condition that the number 2 of incoming edges to the node must match the multiplicity 2 in the Petri net. Again it is the case that all possible processes of this shape are isomorphic, but they are no longer uniquely isomorphic: Even a fixed process admits a nontrivial automorphism, by interchanging the two edges of the graph. The multiplicity decoration cannot prevent this automorphism. This difference in the behaviour of symmetries is a key feature of whole-grain Petri nets compared to traditional Petri nets.

**6 FREE PROPS**

The following description of the free-prop monad is essentially from [46], except that there only the free-properad monad is described, considering only connected graphs.
6.1 Residue and \([\frac{n}{m}]\)-Graphs. For an (acyclic) graph \(G\) (either an \(\mathcal{F}\)-graph or naked (i.e., with no \(\mathcal{F}\)-structure)), the residue \(\text{res}(G)\) is the naked corolla formed by the in-boundary and the out-boundary (and a single node). For example,

\[
\text{res}(G) = \quad = 
\]

This defines a functor \(\text{res} : \text{Gr}_{\text{iso}} \to \text{Cor}\) (and for each \(\mathcal{F}\) a functor \(\text{res} : \text{Gr}_{\text{iso}} \downarrow \mathcal{F} \to \text{Cor}\)).

An \([\frac{n}{m}]\)-graph is an acyclic graph \(G\) equipped with an isomorphism \(\text{res}(G) \cong [\frac{n}{m}]\). It amounts to a numbering of the elements in the in-boundary and a separate numbering for the out-boundary (since we have defined the specific corollas \([\frac{n}{m}]\) in terms of natural numbers). More formally, the groupoid of \([\frac{n}{m}]\)-graphs \([\frac{n}{m}]\)-\text{Gr}_{\text{iso}}\) is the homotopy fibre (see A.5) of \(\text{res} : \text{Gr}_{\text{iso}} \to \text{Cor}\) over \([\frac{n}{m}]\). For \(\mathcal{F}\) a digraphical species, an \([\frac{n}{m}]\)-\(\mathcal{F}\)-graph is an \(\mathcal{F}\)-graph \(G\) equipped with an isomorphism \(\text{res}(G) \cong [\frac{n}{m}]\). These are the objects of the groupoid \([\frac{n}{m}]\)-\text{Gr}_{\text{iso}} \downarrow \mathcal{F}\), the homotopy fibre over \([\frac{n}{m}]\) of the functor \(\text{res} : \text{Gr}_{\text{iso}} \downarrow \mathcal{F} \to \text{Cor}\).

6.2 Remark. Note that we allow non-connected graphs (in contrast to [46]). In particular, a nodeless \(\mathcal{F}\)-graph \(U\) consisting of \(m\) isolated edges is a \([\frac{m}{m}]\)-graph in \(m!\) ways, depending on the possible bijections \(\text{in}(U) = m\) and \(\text{out}(U) = m\) (which are independent). This example also shows that \(\text{res} : \text{Gr}_{\text{iso}} \downarrow \mathcal{F} \to \text{Cor}\) is not a fibration: not all automorphisms of \([\frac{m}{m}]\) admit a lift to \(U\).

6.3 The Idea of Props and Free Props. A category has objects and arrows (forming an underlying graph) and then a prescription for composing arrows. The free category on a graph will have not only the edges as arrows, but will also promote the paths in the graph to be arrows. Composition in a free category is just concatenation of paths.

A (coloured) prop has colours and many-in/many-out operations (so as to have an underlying digraphical species) and then a prescription for composing operations. The free prop on a digraphical species \(\mathcal{F}\) will have not only the \(\mathcal{F}\)-corollas as operations, but will also promote all \(\mathcal{F}\)-graphs to be operations. Composition in a free prop is just gluing of graphs. The following discussion, although it is a little bit technical, is just a formalisation of this idea.

6.4 Free-Prop Monad. We describe the free prop\(^{14}\) on a digraphical species. Recall (from 1.14) that presheaves on \(\text{elGr}\) are naturally equivalent to sheaves on \(\text{Gr}\), so a presheaf on elementary graphs can be evaluated also on general (acyclic) graphs by the limit formula

\[
\mathcal{F}[G] \cong \lim_{E \in \text{el}(G)} \mathcal{F}[E].
\]

The limit is over the elementary subgraphs of \(G\) as in 1.10.

\(^{14}\)It should be remarked that the algebras for the prop monad described here are not exactly the same as the props of Mac Lane [51], and they should properly be called graphical props (see Batanin–Berger [9], Remark 10.5). The difference is with the \((0, 0)\)-operations: In a Mac Lane prop, \(\text{End}(1)\) is always a commutative monoid (by the Eckmann–Hilton argument). In a graphical prop, \(\text{End}(1)\) can be a noncommutative monoid. The difference does not affect us here, as we are only concerned with free graphical props, and these automatically have commutative \(\text{End}(1)\) and are therefore also props in the sense of Mac Lane.
The free prop monad

\[ \PrSh(\text{elGr}) \to \PrSh(\text{elGr}) \]

\[ \mathcal{F} \mapsto \overline{\mathcal{F}} \]

is given (at the level of its underlying endofunctor) by \( \overline{\mathcal{F}}[\star] := \mathcal{F}[\star] \) and

\[ \overline{\mathcal{F}}[\begin{smallmatrix} n \\ m \end{smallmatrix}] := \colim_{G \in \begin{smallmatrix} n \\ m \end{smallmatrix} - \text{Gr}_{\text{iso}}} \mathcal{F}[G] \]

\[ \approx \sum_{G \in \pi_0(\begin{smallmatrix} n \\ m \end{smallmatrix} - \text{Gr}_{\text{iso}})} \frac{\mathcal{F}[G]}{\text{Aut}_{\begin{smallmatrix} n \\ m \end{smallmatrix}}(G)} \]

\[ \approx \pi_0(\begin{smallmatrix} n \\ m \end{smallmatrix} - \text{Gr}_{\text{iso}} \downarrow \mathcal{F}). \]

Here, the first equation follows since \( \begin{smallmatrix} n \\ m \end{smallmatrix} - \text{Gr}_{\text{iso}} \) is just a groupoid: The sum is over isomorphism classes of \( \begin{smallmatrix} n \\ m \end{smallmatrix} \)-graphs, and \( \text{Aut}_{\begin{smallmatrix} n \\ m \end{smallmatrix}}(G) \) denotes the automorphism group of \( G \in \begin{smallmatrix} n \\ m \end{smallmatrix} - \text{Gr}_{\text{iso}} \). The \( (m,n) \)-operations of \( \overline{\mathcal{F}} \) are thus iso-classes of \( \begin{smallmatrix} n \\ m \end{smallmatrix} \)-\( \mathcal{F} \)-graphs.

(We omit description of the monad multiplication and unit (essentially gluing of graphs), although of course this is essential information. See Kock [46] for all details in the connected case, the free properad monad).

6.5 Free Prop on a Petri Net, and Underlying Symmetric Monoidal Category. The free prop on a Petri net \( P \) has as operations the iso-classes of processes \( G \to P \) with fixed boundaries. An element in \( \pi_0(\begin{smallmatrix} n \\ m \end{smallmatrix} - \text{Gr}_{\text{iso}} \downarrow P) \) is a morphism in the underlying symmetric monoidal category of the free prop. The objects are strings of elements in \( P[\star] \), that is, maps \( n \to P[\star] \). The domain of an element \( p : G \to P \) in \( \pi_0(\begin{smallmatrix} n \\ m \end{smallmatrix} - \text{Gr}_{\text{iso}} \downarrow P) \) is the composite \( m \approx \text{in}(G) \to G \to P \), and similarly the codomain is the composite \( n \approx \text{out}(G) \to G \to P \).

Composition is given by gluing (pushout in the category \( \text{Gr} \)). Formally this comes about from the monad multiplication, whose description we omitted.

6.6 Remark. It is easy to see that the digraphical species \( \overline{P} \) is flat again, since the groups \( \Xi_m \times \Xi_n \) act on \( \overline{P} \) only through the boundary isomorphisms. So except for the fact that \( \overline{P} \) takes infinite values, it could be regarded as a Petri net again, and the free-prop adjunction could then be interpreted directly on the category of (infinite) Petri nets. This viewpoint, although it has played an important role in Petri-net theory (from [54] to [17]), is not given emphasis in the present work, where we insist on Petri nets being finite. While \( \overline{P} \) is of interest as a prop or symmetric monoidal category, the corresponding (infinite) Petri net seems to defy the purposes of Petri nets and is hugely redundant. It has a transition for every process of \( P \)—this includes an ‘identity’ transition for every place \( s \in S \) and every \( n \in \mathbb{N} \), corresponding to the disjoint union of \( n \) \( s \)-labelled unit graphs.

6.7 Props in Groupoids. The free prop formula just described involves taking \( \pi_0 \). Avoiding this leads to a simpler construction, that of the free prop in groupoids. It works for digraphical species valued in groupoids, \( \mathcal{F} : \text{elGr}^{\text{op}} \to \text{Grpd} \). In particular, an ordinary graphical species, such as a Petri net, is a groupoid-valued graphical species via the inclusion functor \( \text{Set} \to \text{Grpd} \).

\[ \text{The fact that morphisms are } P \text{-graphs equipped with specified maps from its domain and codomain onto its boundaries makes it an example of a cospan category with boundaries living in ‘lower dimension’ than the apex. Cospans of this nature play also an important role in some recent approaches to open Petri nets; see for example [2, 5, 8].} \]

\[ \text{For an elegant proof of the adjunction between digraphical species and symmetric monoidal categories, see the recent Baez–Genovese–Master–Shulman [4].} \]
The free prop-in-groupoid on a digraphical species \( F : \mathbf{elGr}^{op} \to \mathbf{Grpd} \) is given by \( \overline{F}[\star] := F[\star] \) and

\[
\overline{F}[\begin{bmatrix} n \\ m \end{bmatrix}] := [\begin{bmatrix} n \\ m \end{bmatrix}] \mathbf{Gr}_\text{iso} \downarrow F,
\]

without taking \( \pi_0 \).

Apart from \( \overline{F}[\star] \) (where \( F \mapsto \overline{F} \) is the identity anyway), \( \overline{F} \) is simply the groupoid-valued presheaf corresponding to the projection \( \text{res} : \mathbf{Gr}_\text{iso} \downarrow F \to \mathbf{Cor} \). (It should be noted though, that since \( \text{res} \) is not a fibration, it is essential to use homotopy fibres (see A.5) rather than strict fibres to extract the value on a given \( [\begin{bmatrix} n \\ m \end{bmatrix}] \)).

6.8 Underlying Segal Space of \( \overline{F} \). The underlying (symmetric monoidal) Segal space \( Y_\bullet \) of the free prop-in-groupoids \( \overline{F} \) is given as follows: \( Y_1 \) is the homotopy sum (A.6) of all the \( \overline{F}[\begin{bmatrix} n \\ m \end{bmatrix}] \):

\[
Y_1 = \int [\begin{bmatrix} n \\ m \end{bmatrix}] \overline{F}[\begin{bmatrix} n \\ m \end{bmatrix}] \simeq \sum_{[\begin{bmatrix} n \\ m \end{bmatrix}] \in \pi_0(\mathbf{Cor})} \overline{F}[\begin{bmatrix} n \\ m \end{bmatrix}]//\text{Aut}(\begin{bmatrix} n \\ m \end{bmatrix})).
\]

(This is the appropriate homotopical way of describing the groupoid whose homotopy fibre over \( [\begin{bmatrix} n \\ m \end{bmatrix}] \) is \( \overline{F}[\begin{bmatrix} n \\ m \end{bmatrix}] \). The double bar is homotopy quotient (see A.6). In degree zero, we have \( Y_0 = \mathbb{B} \downarrow \mathbf{F}[\star] \), the free symmetric monoidal category on \( \mathbf{F}[\star] \). In higher simplicial degrees, we have \( Y_k = Y_1 \times_{Y_0} \cdots \times_{Y_0} Y_1 \). The face and degeneracy maps come from the monad structure: In particular \( Y_1 \leftarrow Y_2 \) is the monad multiplication, which allows to contract a 2-level graph to a single corolla. (For details, see [46], although that reference only covers the connected case, the case of properads).

**Theorem 6.9.** For \( P \) a Petri net, the symmetric monoidal Segal space \( X_\bullet \) of \( P \)-processes is the free prop-in-groupoids on the digraphical species \( P \).

**Proof.** We compare the mapping groupoids (B.7) of the two Segal spaces. Fix \( M \) and \( N \) two states of \( P \) with \( m \) and \( n \) tokens, respectively. Both for \( X_\bullet \) and for the free prop, the mapping groupoids consist of processes from \( M \) to \( N \), and in a sense the check is routine; we include it to showcase the calculus of homotopy pullbacks (A.2): The two groupoids will be identified by both satisfying the universal property of the same homotopy pullback.

The mapping groupoids appear in the following diagram of homotopy pullbacks:

\[
\begin{array}{ccc}
\text{Map}_{X_\bullet}(M, N) & \longrightarrow & \overline{P}[\begin{bmatrix} n \\ m \end{bmatrix}] \\
\downarrow \quad r_{(M,N)^\gamma} & & \downarrow \quad r_{(M,N)^\gamma} \\
1 & \longrightarrow & (X_0 \times X_0)[\begin{bmatrix} n \\ m \end{bmatrix}] \\
\downarrow \quad r_{(M,N)^\gamma} & & \downarrow \quad r_{(M,N)^\gamma} \\
1 & \longrightarrow & \mathbf{Cor}.
\end{array}
\]

The curved homotopy-pullback rectangle is the definition of \( \text{Map}_{X_\bullet}(M, N) \) (cf. B.7): homotopy pullback along the curved middle map \( r_{(M,N)^\gamma} \). The right-hand composite homotopy-pullback rectangle is the definition of \( \overline{P}[\begin{bmatrix} n \\ m \end{bmatrix}] \), realised in two steps. Note that \( \mathbf{Cor} \) is the groupoid of naked corollas, without reference to \( P \), whereas \( X_0 \times X_0 \) is the groupoid of pairs of \( P \)-states. Since \( M \) and \( N \) are assumed to have cardinality \( m \) and \( n \), the middle curved arrow factors through
(X₀ × X₀)_{n \leq m}, inducing first the lower dotted map by the universal property of the homotopy pullback (X₀ × X₀)_{n \leq m} and inducing next the upper dotted arrow by the universal property of the homotopy pullback \( \overline{P}[n/m] \). The closure properties of homotopy pullbacks \( \text{(A.4)} \) force the resulting upper left-hand square to be a homotopy pullback. This shows that \( \text{Map}_{X_\bullet}(M, N) \) is both the mapping groupoid in \( \overline{P} \) and in \( X_\bullet \). (It also exhibits \( \overline{P}[n/m] \) as the homotopy sum \( \text{(A.6)} \) of its homotopy fibres \( \text{(A.5)} \) over varying \( (M, N) \).

\[
\overline{P}[n/m] = \int_{(M, N) \in (X_0 \times X_0)_{n \leq m}} \text{Map}_{X_\bullet}(M, N).
\]

The remaining checks (higher simplicial degrees) follow from the Segal condition. The symmetric monoidal structure is clear, as it is just disjoint union. □

7 THE SYMMETRIC MONOIDAL CATEGORY OF PROCESSES

7.1 Segal Space vs. Category. The symmetric monoidal Segal space \( X_\bullet \) is nice to work with for its clean combinatorial description. The fact that there is a groupoid of processes, \( X_1 \), instead of just a set of them, should be seen as an advantage rather than a drawback. The general point is that since whole-grain Petri nets are configurations of finite sets, it is natural to admit that these configurations form groupoids, and continue to work with these groupoids rather than with their set quotients. Passing to sets of iso-classes too early can easily lead to unnecessary complications or even serious problems. Concretely, working in the groupoid of processes, we can glue processes, thanks to explicit bijections between their boundaries (states), but it is plainly not possible to glue isomorphism classes of processes along isomorphism classes of states.

Notwithstanding the previous paragraph, it is sometimes convenient to extract from \( X_\bullet \), a honest symmetric monoidal category \( (\mathcal{C}, +, \emptyset) \), which is then the free symmetric monoidal category on \( \overline{P} \). (The construction is the same for any digraphical species \( \mathcal{F} \).) The answer was already given: It is the symmetric monoidal category underlying the free prop on \( P \). However, this description is a bit roundabout, and in fact we cheated by not describing the monad multiplication for the free-prop monad (since all the details were given in [46]). Instead we can give a direct description from \( X_\bullet \), taking the opportunity to explain a few homotopy issues.

**Theorem 7.2.** The free symmetric monoidal category on \( P \), denoted \( \mathcal{C} \), is the homotopy category of \( X_\bullet \) (or more precisely, its codescent object\(^{17} \)). The objects of \( \mathcal{C} \) are the states \( M \to P \), and its hom sets are

\[
\text{Hom}_\mathcal{C}(M, N) = \pi_0(\text{Map}_{X_\bullet}(M, N)).
\]

The mapping groupoid (cf. B.7) already appeared in the proof of Theorem 6.9: \( \text{Map}_{X_\bullet}(M, N) \) is the groupoid whose objects are triples \((p, \sigma_1, \sigma_2)\) consisting of a process \( p : G \to P \), and isomorphisms \( \sigma_1 : M \simeq d_1(G) \) and \( \sigma_2 : N \simeq d_0(G) \). The arrows of the groupoid \( \text{Map}_{X_\bullet}(M, N) \) are isomorphisms of processes \( p = p' \) compatible with the boundary isos. In conclusion, in the free symmetric monoidal category \( \mathcal{C} \), the morphisms from \( M \) to \( N \) are iso-classes of processes (where the isomorphisms are required to fix the boundary).

Before explaining this (without reference to the free prop construction), it is instructive to consider some other attempts at obtaining an ordinary symmetric monoidal category out of \( X_\bullet \).

7.3 Brutally Applying \( \pi_0 \) and Trying to Cope… One could think of simply applying \( \pi_0 \) on \( X_\bullet \) degree-wise. This does not immediately work, because \( \pi_0 \) does not preserve homotopy pullbacks

\(^{17}\text{Codescent object means colimit of } X_\bullet \text{ in } \text{Cat} \text{ weighted by } \Delta \to \text{Cat}; \text{ see Lack [48] and Weber [66].} \)
and will destroy the Segal condition. Again, this is a well-known phenomenon, explained carefully in [24] in the very similar example of trees.

One may then try to correct the problem by taking quotients. Note first that \( \pi_0(X_0) \) is the free commutative monoid on the set of isomorphism classes of states, precisely as employed in the collective-tokens approach. To correct the problem that \( \pi_0(X_2) \rightarrow \pi_0(X_1) \times \pi_0(X_0) \) is not a bijection, some stuff must be quotiented out: One has to identify two processes \( G \) and \( G' \) if there exist cuts

\[
G = G_1 \sqcup M \sqcup M' \sqcup G_2 \quad \text{and} \quad G' = G'_1 \sqcup M' \sqcup M'' \sqcup G'_2
\]

such that \( M = M' \), \( G_1 = G'_1 \), and \( G_2 = G'_2 \)—all in \( \pi_0(X_0) \). In other words, the two processes satisfy the *swap property* of Best–Devillers [13]: One can cut and reconnect (a kind of surgery). For example, the two processes \( p \) and \( q \) of Example 3.9 are swap-equivalent, as they have the same layers, only glued together differently. This is non-trivial to control, and the final result is the Best–Devillers category of equivalence classes of processes [13]. In the end, the building blocks are the iso-classes of simple firings. Presumably, this is precisely the free symmetric monoidal category of iso-classes of transitions, of Meseguer and Montanari [54]. Note that it is in fact a *commutative* monoidal category.

This is not what we want here.

### 7.4 Weak-Double-Category Viewpoint

A second approach takes the viewpoint of double categories.\(^{18}\) Observe that a (symmetric monoidal) Segal space can be regarded as a special case of a (symmetric monoidal) weak double category, by choosing a pseudo-inverse to the equivalence \( X_2 \rightarrow X_1 \times X_0 \). In the present case, the pseudo-inverse is given by gluing graphs. By making a global choice of pushouts one gets a weak double category, which is a groupoid in the vertical direction: The objects are the states of \( P \), the vertical arrows are the isomorphisms of states, and the horizontal arrows are the \( P \)-processes.

From a weak double category, one can always extract its *horizontal bicategory*, namely by disregarding the vertical arrows. (Getting a bicategory is an important step towards getting an ordinary category, because now one can take \( \pi_0 \): this operation does preserve pullbacks over discrete objects). However, in the case of \( X_n \) this construction will destroy the monoidal structure! The reason is that the coherence constraints of the monoidal structure live in the vertical dimension, and when they are killed the associativity is lost.

It is worth mentioning this, because the horizontal-bicategory construction is a common way to construct symmetric monoidal bicategories (and then symmetric monoidal categories, by taking \( \pi_0 \)), as explained in great detail by Hansen and Shulman [35] and exploited in the context of Petri nets by Baez et al. [2, 5]. However, the Hansen–Shulman construction specifically requires the double category to be *fibrant*, which means that \( X_1 \xrightarrow{(d_1,d_0)} X_0 \times X_0 \) should be a fibration. This is not the case for our \( X_n \): Even though both \( d_0 \) and \( d_1 \) are fibrations individually (cf. Lemma 4.6), the map \( (d_1,d_0) \) is not a fibration. (For example, let \( p : G \rightarrow P \) be the trivial process consisting of \( n \) unit graphs mapping to the same place, then \( (d_1,d_0) \) sends \( p \) to the pair of two copies of that state. Now \( p \) has automorphism group \( \mathcal{E}_n \), whereas \( (d_1,d_0)(p) \) has automorphism group \( \mathcal{E}_n \times \mathcal{E}_n \). It is not possible to lift automorphisms outside the diagonal. (Compare with Remark 6.2.))

With these preliminary analyses, we are ready for the correct solution:

### 7.5 Change of Objects

Instead of getting a bicategory by throwing away the vertical maps (isomorphisms), the passage from double category to bicategory should incorporate the vertical

\(^{18}\)Double-categorical approaches to Petri nets have appeared recently in [5, 6].

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maps into the horizontal part. This is achieved as an instance of the standard change-of-objects construction:

Recall the change-of-objects construction for ordinary categories \( \mathcal{C} \): given a map of sets \( \phi : D \to \text{obj}(\mathcal{C}) \) one can obtain a new category \( \mathcal{C}^\phi \) with object set \( D \) and hom sets \( \text{Hom}_{\mathcal{C}^\phi}(x, y) := \text{Hom}_{\mathcal{C}}(\phi x, \phi y) \). Then by construction there is a fully faithful functor \( \mathcal{C}^\phi \to \mathcal{C} \), which is an equivalence if \( \phi \) is essentially surjective. The same construction works for Segal spaces as follows. Given a Segal space \( X_* \) and a map of groupoids \( \phi : D \to X_0 \), one gets a new Segal space \( X_\phi^* \) whose \( k \)-simplices are given by the homotopy pullback

\[
\begin{array}{ccc}
(X^\phi)_k & \to & X_k \\
\downarrow \phi & & \downarrow \\
D^{k+1} & \to & (X^0)_k^{k+1}.
\end{array}
\]

Now the bicategory associated to a Segal space \( X_* \) is obtained by applying this general construction to the case \( D = \text{obj}(X_0) \) with the inclusion map \( \phi : \text{obj}(X_0) \to X_0 \). (Note that if \( (d_1, d_0) \) is a fibration (as in the Hansen–Shulman situation), then the homotopy pullback can be taken to be a strict pullback, and \( X_\phi^* \) is precisely the horizontal bicategory).

Since \( X_\phi^* \) is equivalent to \( X_* \), the symmetric monoidal structure carries over. It should be noted that \( X_\phi^* \) is far from being Rezk-complete, and the resulting symmetric monoidal structure is not just pointwise. For example, in simplicial degree 0 it will appear that the monoidal product is not even associative, because there is no room for associators, but the associators (as well as the unit and symmetry coherence constraints) now live in simplicial degree 1! Looking into these subtleties makes one appreciate the simplicity of the Rezk complete \( X_* \) itself. But \( X_\phi^* \) is only an intermediate step towards the:

7.6 Codescent Object. Coming back to the Segal space of processes \( X_* \), we now have an (equivalent) Segal space \( X_\phi^* : \Delta^{\text{op}} \to \text{Grpd} \) with degree-0 discrete, and now the final step is simply to post-compose with \( \pi_0 : \text{Grpd} \to \text{Set} \). The functor \( \pi_0 \) does preserve homotopy pullbacks over discrete objects, and since we have arranged for \( X^\phi_0 \) to be discrete, the composite is again a Segal space—that is, a category. Since in any case \( \pi_0 \) preserves products, the symmetric monoidal structure is preserved.

The final resulting symmetric monoidal category \( \mathcal{C} \) is as stated in Theorem 7.2. In fact, \( \mathcal{C} \) is the codescent object of \( X_* \), a general construction that in other contexts serves to compute internal-algebra classifiers for monads, as well as the prop envelope of an operad [66]. As just explained, it is rather easy to compute for Segal spaces (weak double categories whose vertical category is just a groupoid); it is considerably more involved to compute for general double categories [66].

8 HYPERGRAPHS

We now change perspective a little bit, starting to work towards the unfolding theorem, proved in Section 9. While the process semantics of Sections 3–7 is concerned with gluing P-processes in-boundary to out-boundary, the relevant colimits will now be unions of bigger and bigger P-graphs (corresponding to longer and longer computations, accumulating more and more information), all starting from the same fixed initial state. Such colimits do not usually exist in the category of graphs, but they do exist in the bigger category of (directed) hypergraphs. In this section, we set up the necessary machinery. The main task is to characterise those hypergraphs that arise as colimits of graphs like this.
8.1 Hypergraphs [46]. A hypergraph \( H \) is a diagram of sets and maps
\[
A \leftarrow I \rightarrow N \leftarrow O \rightarrow A,
\]
where the two spans are relations. We now allow infinite sets, but insist that the maps \( I \to N \leftarrow O \) be only finite maps (that is, with finite fibres). The set \( A \) is now read as the set of hyper-edges. We picture hypergraphs as Petri nets, with nodes represented as small squares, and hyper-edges represented as small circles, like for example

![Hypergraph diagram](image)

(6)

The notion of etale map is as for graphs and Petri nets, and again an open sub-hypergraph is an etale map that is injective point-wise (i.e., on \( A, I, N, O \) components separately). Note that the arity of hyper-edges is not necessarily preserved by etale or open maps.

Denote by \( \text{Hgr} \) the category of hypergraphs and their etale maps.

8.2 Forward Hypergraphs. We shall only need hypergraphs where the right-hand part is graph-like, but allow the left-hand part to be general-hypergraph-like. We define a forward hypergraph to be a hypergraph where the last leg is injective:
\[
A \leftarrow I \rightarrow N \leftarrow O \rightarrow A.
\]

This is to say: Each hyper-edge is outgoing of at most one node. In pictures, only hyper-edges that locally look like

![Forward hypergraph diagram](image)

(7)
are allowed. So for example, the hypergraph above in (6) is not forward, whereas those of (10) are forward.

8.3 Various AINOA Structures. Except for finiteness conditions, the various AINOA-shape structures introduced can all be considered special cases of Petri nets. We have

\[
\text{acyc. graphs} \subset \text{occurrence hypergraphs} \subset \text{forward hypergraphs} \subset \text{hypergraphs} \subset \text{Petri nets}
\]

(The subtle notion of occurrence hypergraph will be introduced in 8.13 below). From this viewpoint, hypergraphs are the Petri nets where there are no parallel arcs (equivalently, in the traditional setting: all arc multiplicities are 1.) The forward condition prohibits what is called backward conflicts [58] but still allows the forward conflicts pictured in (7). Acyclic graphs are the conflict-free Petri nets, also called causal nets [32, 69].

While it is thus important that all these structures share the same AINOA shape, we stress this difference: Petri nets are required to be finite, but express infinite behaviour by not being required acyclic; hypergraphs are allowed to be infinite, but should instead be required to be (acyclic and) well-founded,\(^{19}\) which is the next condition we impose:

\(^{19}\)This contrast was stressed by Scott [64] in the context of flow diagrams.
8.4 Well-Founded Hypergraphs. We write \( x \preceq y \) for two nodes in a hypergraph \( H = AINOA \) if there exists a hyper-edge \( a \) that is outgoing for \( x \) and incoming for \( y \). More precisely, \( x \) and \( a \) are \( O \)-related and \( a \) and \( y \) are \( I \)-related. Note that in a forward hypergraph the set \( \{ x \mid x \preceq y \} \) is finite for every node \( y \) (as a consequence of the assumption that the map \( I \to N \) is finite). We denote by \( < \) the transitive closure of \( \preceq \), and by \( \leq \) the transitive and reflexive closure of \( \preceq \).

A hypergraph \( H = AINOA \) is well-founded when the relation \( < \) is anti-reflexive (meaning that there are no directed cycles) and for each node \( y \) the set \( \{ x \mid x \preceq y \} \) is finite. In particular, the relation \( \leq \) is then anti-symmetric, so \((N, \leq)\) is a poset. The following equivalent characterisation of well-foundedness is technically convenient: a forward hypergraph is well-founded when the ‘earliest-start-time’ function\(^{20} \)

\[
\begin{align*}
  f(y) &= 1 \quad \text{when } \not\exists x < y \\
  f(y) &= n + 1 \quad \text{when } f(x) \text{ is defined for all } x < y \text{ and } n = \max\{f(x) \mid x < y\}
\end{align*}
\]

is a total function. In words, \( f(y) \) is the length of the longest chain ending in \( y \) (plus one).

8.5 B-Hypergraphs. Since for a forward hypergraph \( H = AINOA \) the map \( O \to A \) is injective, it has a well-defined complement called the in-boundary of \( H \). Throughout we shall fix a finite set \( B \), with its corresponding node-less graph \( B = B\emptyset\emptyset\emptyset \), and only consider forward hypergraphs with fixed in-boundary \( B \). These are called \( B \)-hypergraphs; they are thus maps \( B \to H \) for which the square

\[
\begin{array}{c}
\emptyset \to B \\
\downarrow \\
O \to A
\end{array}
\]

is a pushout (and hence also a pullback). Note that by closure properties of pushouts, the \( B \)-maps \( H' \to H \) have the property that the right-most square is a pushout (and hence a pullback):

\[
\begin{array}{c}
A' \leftarrow I' \to N' \leftarrow O' \to A' \\
\downarrow \quad \downarrow \quad \quad \downarrow \quad \downarrow \quad \quad \downarrow \quad \downarrow \\
A \leftarrow I \to N \leftarrow O \to A
\end{array}
\]

A \( B \)-subgraph of \( H \) will be called a process of \( H \) (from \( B \)).

8.6 Lowersets. Recall that a lowerset\(^{21} \) of a poset \( N \) is a sub-poset \( L \subseteq N \) such that if \( x \leq y \) and \( y \in L \) then also \( x \in L \). A lowerset in a well-founded hypergraph \( H \) is an open sub-hypergraph (i.e., injective etale map) \( G \to H \) that is a poset lowerset on nodes. Finally, for a well-founded hypergraph \( H \) with in-boundary \( B \), we define \( B \)-lowerset to be a lowerset whose in-boundary is \( B \) again.

**Lemma 8.7.** For a well-founded \( B \)-hypergraph \( H = AINOA \), there is a bijection between lowersets \( L \subseteq N \) in the sense of posets, and \( B \)-lowersets \( G \subseteq H \) as \( B \)-hypergraphs.

**Proof.** From a \( B \)-lowerset \( G \subseteq H \), just return the poset of nodes in \( G \), which is a lowerset in \( N \) by assumption. Conversely, given a lowerset \( L \subseteq N \) in the poset of nodes of \( H \), define the hyperedges of \( G \) to be those incident to some node in \( L \), together with all the hyperedges in \( B \). \( \Box \)

\(^{20}\)Winskel [69] calls it depth.

\(^{21}\)Also called order ideal, or left-closed subset [69].
8.8 Principal Lowersets. For a node \( y \in N \) in a well-founded hypergraph \( H = A\text{INO}A \), we can consider the set of nodes \( N \downarrow y := \{ x \in N \mid x \leq y \} \), which is the lowerset in the sense of posets. This set of nodes spans an open sub-hypergraph of \( H \) denoted \( H \downarrow y \) called the principal lowerset of \( y \). Note that \( H \downarrow y \) contains also all the outgoing hyper-edges of all its nodes. For example,

\[
\begin{array}{c}
\text{\( a \)} \\
\text{\( \downarrow \)} \\
\text{\( \text{\( y \)} \)} \\
\end{array}
\begin{array}{c}
\text{\( \subset \)} \\
\text{\( \downarrow \)} \\
\text{\( \text{\( a \)} \)} \\
\end{array}
\]

(10)
is the inclusion of the principal lowerset \( H \downarrow y \) in the hypergraph \( H \). The hyperedges \( a \) and \( b \) have to be included even though they are not below the node \( y \), since otherwise there could not be an etale map.

For \( B \)-hypergraphs, we also need the notion of \( B \)-lowerset, where we insist on including the whole in-boundary \( B \) in \( H \downarrow y \), as in Lemma 8.7.

Note that in a well-founded forward hypergraph, all principal lowersets are finite hypergraphs, as a consequence of \( I \to N \) being finite.

Lemma 8.9. Let \( H = A\text{INO}A \) be a well-founded forward \( B \)-hypergraph, and let \( H' \to H \) be an injective etale map of \( B \)-hypergraphs (i.e., an open sub-\( B \)-hypergraph). Then for each node \( y \) in \( H' \) there is a natural identification of \( B \)-lowersets \( H' \downarrow y = H \downarrow y \).

Proof. In view of well-foundedness, it is enough to show that for \( x \ll y \) in \( H \), we have also \( x \ll y \) in \( H' \). The idea is that \( y \) has the same incoming hyper-edges in \( H' \) as in \( H \) because the map is etale, and that each such hyper-edge \( a \) has the same preceding nodes in \( H' \) and in \( H \) by the property of \( B \)-maps (8.5, Equation (9)). The formalisation of this idea is a pleasant exercise with pullbacks and \( A\text{INO}A \) diagrams: The map \( f : H' \to H \) is a diagram

\[
\begin{array}{ccccccc}
A' & \leftarrow & I' & \rightarrow & N' & \leftarrow & O' & \rightarrow & A' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \leftarrow & I & \rightarrow & N & \leftarrow & O & \rightarrow & A.
\end{array}
\]

(11)

and since it is a \( B \)-map, the last square is a pullback (cf. 8.5). The statement \( x \ll y \in N \) translates into the existence of a hyper-edge \( a \in A \) such that \( a_i y \) and \( x O a \), which in turn means we have elements \( i \in I \) and \( o \in O \) like this:

\[
a \leftarrow i \mapsto y \quad x \leftarrow o \mapsto a.
\]

Furthermore \( y = f(y') \). We need to construct

\[
a' \leftarrow i' \mapsto y' \quad x' \leftarrow o' \mapsto a'
\]

in the diagram for \( H' \), mapping to \( x, i, o, a \). Define \( i' \in I' \) to be the pullback of \( i \) and \( y' \), and let \( a' \) be its image in \( A' \); then \( a'I'y' \) by construction. Now define \( o' \in O' \) to be the pullback of \( o \) and \( a' \), and let \( x' \) be its image in \( N' \); then \( x'O'a' \) by construction. \( \Box \)

Corollary 8.10. Let \( H \) be a well-founded forward \( B \)-hypergraph, and let \( H' \to H \) be an open sub-\( B \)-hypergraph. Then \( H' \) is a \( B \)-lowerset in \( H \).

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Corollary 8.11. An injective etale $B$-map $H' \to H$ of well-founded forward $B$-hypergraphs automatically preserves the earliest-start-time function (of 8.4).

Proof. The value of the earliest-start-time function on a node $y$ of $H$ depends only on the principal lowerset $H \downarrow y$, so the result follows from Lemma 8.9. □

8.12 Remark/Example. Without the forward condition, it is not always true that $B$-sub-hypergraphs are lowersets, and it is not always true that the earliest-start-time functions agree, as exemplified by

8.13 Occurrence Hypergraphs. A well-founded forward hypergraph $H = AINOA$ is called an occurrence hypergraph if every principal lowerset is a graph. For example, the hypergraphs in (10) are occurrence hypergraphs, whereas the hypergraph of Example 8.16 below is not.

We denote by $\text{OccHgr}$ the category of occurrence hypergraphs and their etale maps, and by $B\text{-OccHgr}$ the category of occurrence $B$-hypergraphs and etale maps that induce a bijection on in-boundaries.

Proposition 8.14. The category of hypergraphs and etale maps has colimits of connected diagrams of injective maps, and they are calculated pointwise (i.e., separately on the $A, I, N, O$ components).

If the hypergraphs involved are

1. forward,
2. $B$-hypergraphs (and the maps are $B$-maps),
3. well-founded,
4. occurrence hypergraphs,

then the same holds for the colimit hypergraph.

Proof. General hypergraphs—We write down the colimit computed pointwise and check that the result is a hypergraph again and that the maps to it are etale and injective again. Connected colimits can be built from pushouts and filtered colimits, so we can deal with these two cases separately.

For pushouts, if $G \subset H'$ and $G \subset H''$ are injective etale maps, then the pointwise pushout

is a hypergraph again, and the dotted arrows are injective etale again. This is only a slight generalisation of Proposition 3.19 of [46], and the proof is essentially the same.

For filtered colimits, if $(H_\alpha)_\alpha$ is a filtered diagram of injective etale maps, then we need to show that the pointwise colimit $H := \operatorname{colim}_\alpha H_\alpha$ is again a hypergraph, and the induced maps $H_\alpha \to H$

22Compare Winskel [69, Def.3.3.1].
23This is analogous to the following characterisation of forests: A poset $P$ is a forest when for each element $y$ the principal lowerset $P \downarrow y$ is a linear order.
are injective and etale again. To check that the result is a hypergraph again, note first that the hypergraph condition (that the two spans are relations) can be expressed with finite limits (namely, that the pullback of $I \to A \times N$ with itself gives $I$ again). Now the statement follows from the fact that finite limits commute with filtered colimits in the category of sets. That the induced maps $H_\alpha \to H$ are injective and etale again can be checked separately on the $I \leftarrow N$ and $N \to O$ components. Here it is precisely the fact that filtered colimits of monomorphisms are Van Kampen in the category of sets [38]. Finally we check the finiteness condition: Every node $y$ in the colimit $H$ is a node in some $H_\alpha$, and since the inclusion is etale, it follows that the pre- and post-set of $y$ can be computed in $H_\alpha$. In particular they are finite, so altogether the maps $I \to N \leftarrow O$ are finite maps for $H$, as required.

Forward hypergraphs—The condition of being forward (injectivity of the map $O \to A$) is a finite-limit condition, so the same reasoning as above shows that if all the hypergraphs $H_\alpha$ are forward, then so is the colimit $H$.

$B$-hypergraphs—Suppose that all the hypergraphs $H_\alpha$ involved are $B$-hypergraphs and that all the maps are $B$-maps. This means that all the right-hand squares in the AINOA diagrams form pushout squares with $\emptyset \to B$, as in (8). But this property is preserved in the colimit, since colimits commute with pushouts.

Well-founded hypergraphs—Assume that all the hypergraphs $H_\alpha$ are well-founded; let $f_\alpha : H_\alpha \to \mathbb{N}$ be the earliest-start-time functions, as in 8.4. By Corollary 8.11 all the maps in the diagram are compatible with these functions, so we can define the earliest-start-time function on the colimit $H := \operatorname{colim}_\alpha H_\alpha$ by

$$f(y) := f_\alpha(y) \quad \text{if } y \in H_\alpha.$$  

So $H$ is well-founded again.

Occurrence hypergraphs—Assume that all the hypergraphs $H_\alpha$ are occurrence hypergraphs. Every node $y$ in the colimit $H := \operatorname{colim}_\alpha H_\alpha$ belongs to some $H_\alpha$, and by Lemma 8.9 the lowerset $H \downarrow y$ is identified with the lowerset $H_\alpha \downarrow y$, which is a graph by assumption. So $H$ too is an occurrence hypergraph.

**Proposition 8.15.** Every occurrence $B$-hypergraph $H = AINOA$ is canonically the colimit of its $B$-processes:

$$H = \operatorname{colim}_{B \in \mathcal{G}_H} G.$$

**Proof.** It is clear that $H$ is the colimit of its principal $B$-lowersets, $H = \operatorname{colim}_{y \in N} H \downarrow y$, and these lowerset are all graphs by the occurrence assumption. There are many more $B$-graphs than principal $B$-lowersets, but each $B$-graph in $H$ is itself a colimit of its principal $B$-lowersets, which are also principal $B$-lowersets in $H$ by Lemma 8.9, so the diagram of all principal $B$-lowersets in $H$ is cofinal in the diagram of all $B$-graphs in $H$.

**8.16 Example.** The result is not true without the occurrence condition, as exemplified by the forward hypergraph $H$

\[
\begin{tikzpicture}
  \node (v) at (0,0) {$v$};
  \node (a) at (-1,-1) {$a$};
  \node (b) at (1,-1) {$b$};
  \draw (v) -- (a);
  \draw (v) -- (b);
  \draw (a) -- (b);
\end{tikzpicture}
\]

where $H \downarrow y$ is not a graph, and $y$ is not contained in any $B$-subgraph.
9 UNFOLDING

An important discovery by Winskel is that the totality of all processes of a (marked) Petri net can be assembled into denotational structures; first an event structure, then a Scott domain. The key point is the existence of a universal unfolding of a (marked) Petri net. This was established for safe Petri nets by Winskel [67] and Nielsen, Plotkin, and Winskel [58] (see also [69]), but did not work for non-safe nets, due to problems with symmetries. Various workarounds were provided later [7, 36, 55, 56]; see 9.7 below for further discussion.

In this section, we show that in the whole-grain setting, the symmetry problems simply disappear: Winskel’s ideas, notions, and proof arguments in the safe case now work in full generality. After the main Theorem 9.6, we give some examples to illustrate the unfolding of some non-safe Petri nets to see how the symmetries are taken care of automatically. Finally we briefly comment on the subsequent constructions of event structures and domains.

9.1 Marked Petri Nets. Fix a finite set $B$; denote by $B$ the graph $B \emptyset$. A $B$-marked Petri net is a Petri net $P$ with a marking $B \to P$. A morphism of $B$-marked Petri nets is a commutative triangle of etale maps

$$
\begin{array}{ccc}
B & \rightarrow & P' \\
\downarrow & & \downarrow \\
B & \rightarrow & P \\
\end{array}
$$

The $B$-marked Petri nets thus form the coslice category $B$-Petri := $B \downarrow \text{Petri}$.

9.2 $B$-P-Processes ($= B$-P-Graphs = P-Processes from $B$). For a marked Petri net $(P, B)$, we are concerned with processes that start at the initial marking $B$, which is to say that we only consider processes $G \to P$ for which the in-boundary of $G$ agrees with $B$ (as for hypergraphs, this means that square (8) is a pushout). A $B$-P-process ($= P$-process from $B$) is thus a commutative triangle of etale maps

$$
\begin{array}{ccc}
B & \rightarrow & G \\
\downarrow & & \downarrow \\
B & \rightarrow & P \\
\end{array}
$$

where the map $B \to G$ is a bijection onto the in-boundary of $G$. A morphism of $B$-P-graphs, or $B$-P-map, is a morphism of $P$-graphs inducing a bijection of in-boundaries.

9.3 Grounded Petri Nets. A Petri net $P = \text{SITOS}$ is called grounded$^{24}$ if the map $I \to T$ is surjective. In other words, the pre-sets of transitions are not allowed to be empty. The condition is natural from the viewpoint of unfolding, because we are interested in the behaviour springing from the initial marking, and without the grounding assumption a transition could fire spontaneously and produce an arbitrary number of tokens that there would be no control over.

From a technical viewpoint, an important consequence is the following result.

**Lemma 9.4.** Let $P$ be a grounded Petri net $P$ with initial marking $B$.

1. Every morphism of $B$-P-processes is injective.$^{25}$
2. A $B$-P-process has no non-trivial automorphisms.

**Proof.** By the etale condition, any graph $G$ underlying a $P$-process is again grounded. Let $f : G \to H$ be a morphism of $B$-P-processes. If two nodes of $G$ have the same image in $H$, say

$^{24}$In the literature (see for example Van Glabbeek [29]), grounded Petri nets are sometimes called ‘standard Petri nets.’

$^{25}$This is analogous to the fact that a map of trees is injective [45] and that a grounded map of forests is injective.
If \( f(a) \) belongs to the in-boundary \( B \) of \( H \), then so does \( a \), because \( f \) is a bijection on \( B \), and this forces \( a = a' \), and thereby also \( y = y' \). If \( f(a) \) does not belong to \( B \), then neither do \( a \) or \( a' \). They are thus outgoing edges of some \( x \) and \( x' \), and since \( f(a) = f(a') \), we also have \( f(x) = f(x') \). Now repeat the argument. In a finite number of steps, we arrive at \( B \), where \( f \) is a bijection by assumption. This proves (1). Statement (2) follows from a similar argument: We already know from Lemma 5.5 that a \( P \)-process has no infinitesimal automorphisms, other than the identity. If an automorphism had \( f(y) \neq y \), then the same would be true for some incoming edge \( a \) (which exists by the grounding condition), and so on, until we arrive at \( B \), which is fixed by assumption, arguing as before. \( \square \)

9.5 Unfolding of Petri Nets. Let \( P \) be a grounded Petri net with initial state \( B \). We shall continue to require \( P \) to be finite, but to accommodate the notion of unfolding, it is practical temporarily to enlarge the category \( \text{Petri} \) to allow infinite Petri nets, although we still insist on demanding the two maps \( I \to T \leftarrow O \) to have finite fibres. An unfolding of \( P \) (from \( B \)) is a commutative triangle of etale maps

\[
\begin{array}{ccc}
B & \rightarrow & H \\
\downarrow & & \downarrow \\
& P,
\end{array}
\]

where \( B \to H \) is an occurrence \( B \)-hypergraph. Note that since \( P \) is assumed grounded, so is any unfolding (as an offending node in \( H \) would have nowhere in \( P \) to land).

**Theorem 9.6.** Fix a grounded Petri net \( P \), with an initial marking \( B \to P \) (given by a finite set \( B \)).

1. There is a universal unfolding of \( P \) from \( B \)

\[
B \to U_B P \xrightarrow{\epsilon} P,
\]

in the sense that for every unfolding \( H \to P \) from \( B \) there is a unique morphism of \( B \)-hypergraphs over \( P \):

\[
\begin{array}{ccc}
B & \rightarrow & U_B P \\
\downarrow & \nearrow & \downarrow \\
H & \rightarrow & P,
\end{array}
\]

2. The universal unfolding is constructed as the colimit in \( B \)-\text{Hgr} of all \( P \)-processes starting at \( B \):

\[
U_B P = \text{colim} \ G.
\]

The diagram takes place in the category \( \text{Petri} \) where we temporarily allow infinite Petri nets. The first statement can also be formulated like this: for any grounded marked Petri net \((P, B)\) and any occurrence \( B \)-hypergraph \( H \), there is a canonical bijection

\[
\text{Hom}_{B \text{-Petri}}(H, P) \cong \text{Hom}_{B \text{-OccHgr}}(H, U_B P).
\]

In other words, \( U_B P \) represents the functor

\[
B \text{-OccHgr}^{\text{op}} \to \text{Set}
\]

\[
H \mapsto \text{Hom}_{B \text{-Petri}}(H, P).
\]

Note also that by restricting to graphs, we get

\[
\text{Proc}_B(P) \cong \text{Proc}_B(U_B P).
\]
One could also adjust the categories involved to formulate the statement as an adjunction: for this, define $B$-$\text{Petri}$ to be the category of possibly infinite $B$-marked Petri nets, but only grounded ones, and let $B$-$\text{OccHgr}$ be the category of occurrence $B$-hypergraphs also required to be grounded. Then there is a forgetful functor $B$-$\text{OccHgr} \to B$-$\text{Petri}$, and the statement is that this functor has a right adjoint $U_B$.

**Proof.** The diagram we want to take colimit of consists of injective maps, since morphisms of $B$-$\text{P}$-processes are always injective by Lemma 9.4 (1). It is therefore of the kind that admits a colimit in the category of $B$-hypergraphs as in Proposition 8.14, and this colimit is an occurrence hypergraph by 8.14 (4). By the universal property of the colimit, it comes with an etale map to $P$.

Given an unfolding $B \to H \to P$, write $H$ as the colimit of its processes: $H = \text{colim}_{B \subset G \subset H} G$, as in Proposition 8.15. By postcomposition, each $G$ is also a process of $P$ (still from $B$), and therefore it maps into the union $\text{colim}_{B \subset G \subset P} G = U_B P$, and this map is unique, since $P$-processes under $B$ have no automorphisms (by Lemma 9.4 (2)). This collection of maps $G \to U_B P$ defines a unique map out of the colimit $H = \text{colim}_{B \subset G \subset H} G$. □

**9.7 Remarks.** In the setting of traditional Petri nets, Nielsen, Plotkin, and Winskel [58] proved Theorem 9.6 for safe Petri nets (see Winskel [69], Theorems 3.3.9 and 3.3.13). A marked Petri net is safe if all arcs have multiplicity 1, and the initial marking as well as all reachable markings are multiplicity-free. For general Petri nets in the traditional sense, the theorem breaks down, as the universal property is violated by symmetries, cf. Example 9.9 below, which is also given by Hayman and Winskel [36]. From the viewpoint of the present paper, one could say that the reason safe Petri nets do satisfy the theorem in the traditional case is that safe nets are already whole-grain.

Two lines of development led to further insight into unfolding of non-safe Petri nets. One way to tackle the problem was devised by Meseguer, Montanari, and Sassone [55, 56], exploiting certain decorated unfoldings, which prevented the symmetries. Baldan, Bruni, and Montanari [7] established the result for pre-nets, where the symmetries do not come up, due to the rather restrictive notion of morphism (see also Van Glabbeek [29]). Finally, Hayman and Winskel [36], rather than trying to avoid the symmetries, managed to embrace them, proving an unfolding theorem where the universal property is replaced by a universal property up to symmetry, a concept they formalised in terms of certain quasi-isomorphisms defined with spans of open maps [37].

It must be stressed that, since the theorem is a question of representability, it is a much easier question in the ‘representable’ SITOS setting than in the traditional setting. The whole-grain formalism bypasses the difficulties, not by any miracle or deep insight, but simply because the tokens are now elements in specific sets, and this gives complete control over their symmetries.

We go through a couple of examples to illustrate the workings of unfolding in the whole-grain setting. The first example is relatively straightforward (but it is not a safe net); the next examples are more subtle, involving multiple markings and parallel arcs.

**9.8 Example.** Consider the following Petri net $P$, with its marking $B \to P$ by a 3-element set $B$: 

![Petri net diagram](image_url)

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To compute its universal unfolding, we first list all its processes from \( B \) (up to isomorphism), arranged into a poset with the order given by \( B\)-\( P \)-maps. The small numbers in the graph pictures indicate the \( P \)-structure: For example, the number 1 on an edge indicates that it is mapped to place \( s_1 \), and so on. How the nodes map to transitions is unambiguous from the drawings, and since \( P \) has no parallel arcs, the effect of the etale maps on the sets of arcs is implied too. There are 9 possible processes from \( B \):

Note that the two zigzag-shaped graphs are isomorphic as abstract graphs, but not isomorphic as \( B \)-graphs.

To compute the colimit of this diagram in the category \( H_{gr} \), it is enough to compute the colimit of the subdiagram consisting of the three graphs pictured with black frame (technically: that subdiagram is cofinal), so it amounts to computing a single pushout:

In the figure, the letters on the nodes are only to indicate how the graphs embed into the colimit hypergraph, the unfolding. As before, the numbers indicate how the four hypergraphs are \( P \)-hypergraphs.

---

26 Note that the diagram of all \( B\)-\( P \)-graphs is hugely redundant and includes all isomorphisms of \( B\)-\( P \)-graphs. By picking one representative from each isomorphism class, we get a skeleton to work with instead. Since \( B\)-\( P \)-graphs have no nontrivial automorphisms and all maps are injective (cf. 9.4), this skeleton is a poset, and this is the poset to work with in practice.
Intuitively, the pushout is obtained by gluing the two 3-node graphs together along the overlap: The gluing locus consists of the four edges marked 1 and 2, so these become hyperedges (the blue circles) in the gluing, rather than plain edges. It should be stressed again that this computation is really formal and straightforward once the AINOA diagrams have been written down, and that it is nothing but computing a few pushouts in the category of sets. For example, all three graphs have 5 edges, so also the colimit hypergraph has 5 hyper-edges; in each of the two 3-node graphs there are 5 incoming arcs (the $I$-set in an AINOA diagram), and only 1 of them is in common (coming from the small graph), therefore, the pushout $U_B P$ has $5 + 5 - 1 = 9$ incoming arcs, as clearly seen in the figure; and so on. It is an instructive exercise now to write down the processes of $U_B P$ from $B$. One will find the same poset of graphs as for $P$ itself, as an illustration of Theorem 9.6.

9.9 Example. Consider the Petri net $P$,

$$\{s\} \leftarrow \{u\} \rightarrow \{t\} \leftarrow \emptyset \rightarrow \{s\},$$

with its marking $B \rightarrow P$ by a 2-element set $B = \{b_1, b_2\}$:

```
   t
  / \        =
 /   \ 
 /     /
 u --- s
```

Here is the poset of processes, and the colimit, which is the universal unfolding:

```
   b_1 b_2
  /   \
 /     /  
 b_1 b_2
   \   / \
    \ /  
     / \\
 b_1 b_2
```

The decorations with $b_1$ and $b_2$ indicate the $B$-structure of each graph. Note again that although the two one-node graphs are isomorphic as abstract graphs (and in fact isomorphic as $P$-graphs), they are not isomorphic as $B$-graphs. In this case, the poset already has a terminal object, which is then of course the colimit.

This very Petri net was indicated by Hayman and Winskel [36, Figure 1] as an example of a Petri that does not have a universal unfolding—in the traditional setting. The reason is that, in the traditional setting, this unfolding hypergraph (which happens to be a graph) has an automorphism spoiling the universal property. However, this automorphism is an artefact of thinking the marking as a mere multiplicity 2. In the whole-grain setting, the marking is the explicit 2-element set $B = \{b_1, b_2\}$, which fixes the automorphisms.
9.10 Example. Consider the Petri net $P$ with its marking $B \to P$ by a 2-element set $B = \{b_1, b_2\}$:

![Petri net diagram]

Here is the poset of processes $B \to G \to P$, and the colimit, the universal unfolding:

![Colimit diagram]

The decorations with $b_1$ and $b_2$ indicate the $B$-structure of each graph, whereas the decorations with $u$ and $v$ indicate the $P$-structure. The tilde symbols only serve to remind us that the two one-node graphs are not isomorphic (and how they map into the colimit): While they are isomorphic as abstract graphs, they are not isomorphic as $P$-graphs: the decorations with the letter $u$ and $v$ indicate the effect of the etale map on incoming arcs ($I$-sets). From an intuitive operational viewpoint, the difference between the two processes is which input arc receives which token, a distinction that would not be visible in traditional Petri nets, where instead of parallel arcs there would be just a multiplicity.

With the previous example understood, we are ready to look at the following interesting variation of Example 9.8.

9.11 Example. Consider the following Petri net $Q$, with its marking $B \to Q$ by a 3-element set $B$:

![Petri net diagram]

Compared to the Petri net $P$ from Example 9.8, here two places have been joined to one, with otherwise similar distribution of tokens and the same pre- and post-conditions. However, since the tokens are now 'all in the same pool' they can flow in many more ways. Here is the universal unfolding:
Note that after $t_1$ has fired, there are 4 tokens in $s_1$, and that above that point, the universal unfolding consists of $\binom{4}{2} = 6$ small figures, as in Example 9.10, corresponding to which 2 of the 4 tokens in $s_1$ are consumed (and how).

9.12 Event Structures and Scott Domains. The universal unfolding of a Petri net is the crucial step towards denotational semantics in terms of Scott domains. These discoveries and insights are due to Winskel [67] and Nielsen, Plotkin, and Winskel [58]. We briefly explain how the steps from unfoldings to event structures and domains look in the SITOS formalism. Recall (from Winskel [69]) that a prime event structure is a poset $(P, \sqsubseteq)$ equipped with a conflict relation $\#$: It is an symmetric, anti-reflexive relation compatible with the partial order $\sqsubseteq$ in the sense that if $x \# y$ and $y \sqsubseteq z$, then also $x \# z$. The elements of $P$ are called events, and $\sqsubseteq$ expresses causality. The compatibility axiom says that if two events are in conflict, then any event that causally depends on the second event will again be in conflict with the first (and conversely, by symmetry).

On the other hand, a finitary prime algebraic domain (the same thing as the dl-domains of Berry [12]) are certain Scott domains that are distributive and where each finite element has finite lowerset. We shall not need the precise definition here; we refer instead to Winskel [69], both for the definition and for the equivalence of the category of these domains with the category of prime event structures. It is sufficient here to say that the domain associated to a prime event structure is the poset of conflict-free lowersets.
For any occurrence hypergraph $H = AINOA$, consider the poset $N$ with $\sqsubseteq$ defined as the transitive and reflexive closure of the relation $\prec$. As in Winskel [69], the conflict relation is defined in two steps, whereby it comes to obey the conflict axiom by construction: We first declare two nodes $x$ and $y$ to be in immediate conflict, written $x \#_m y$, if $x \neq y$ and there exists a hyper-edge $a$ such that $alx$ and $aly$. Then we define # to be the closure of $\#_m$ under $\sqsubseteq$, meaning that $x \# z$ holds if there exists $y \leq z$ with $x \#_m y$. By construction, this is a prime event structure.

The associated domain has as its elements the conflict-free lowersets of this event structure [69]. Now the bijection in Lemma 8.7 yields a one-to-one correspondence between the conflict-free lowersets of the event structure $(N, \sqsubseteq, \#)$ and the conflict-free lowersets of the occurrence hypergraph that have the whole $B$ as in-boundary. In other words, the elements of the domain are precisely the (possibly infinite) $B$-processes of the occurrence hypergraph. So far the reasoning concerns any occurrence hypergraph. In the special case of the universal unfolding $U_B P$ of a Petri net $P$, the main Theorem 9.6 tells us that the (possibly infinite) processes of $U_B P$ from $B$ are in bijection with the (possibly infinite) processes of $P$ from $B$. It is clear that these bijections are also isomorphisms of posets (as in all three cases, the order relation is given by inclusion). All told, the domain associated to a Petri net $(P, B)$ is canonically identified with the set of its (possibly infinite) $B$-processes, with poset structure given by $B$-preserving inclusions.

10 RATIONAL MAPS (OF PETRI NETS)

We are going to introduce more general morphisms of Petri nets, corresponding to notions studied in the literature, generally motivated by bisimulation. Winskel’s papers [68, 70] seem to be the most complete studies. His notions of morphism are defined in terms of multi-relations. In the whole-grain setting, multi-relations become spans. The general morphisms of Petri nets we consider will be spans

$$
P \xleftarrow{\text{special}} P' \xrightarrow{\text{etale}} Q
$$

consisting of a backwards map of a certain special type, followed by the ‘main part,’ which is an etale map. Where the etale part preserves the interfaces of transitions, the special backwards maps will serve to make a kind of ‘place-respecting correction’ prior to the etale map. In the most general situation, ‘special’ will simply mean ‘place-etale,’ meaning respecting the interfaces of places. A place-etale map is thus a diagram of the form

$$
\begin{array}{c}
\begin{array}{c}
S' \leftarrow I' \rightarrow T' \leftarrow O' \rightarrow S'
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\beta
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
S \leftarrow I \rightarrow T \leftarrow O \rightarrow S,
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\beta
\end{array}
\end{array}
\end{array}
$$

and when combined with the etale maps as in (12), the general kind of morphism will thus be diagrams like the following, to be read from top to bottom:

$$
\begin{array}{c}
\begin{array}{c}
P \xrightarrow{\text{place-etale}} \quad S \leftarrow I \rightarrow T \leftarrow O \rightarrow S
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
P' \xrightarrow{\text{etale}} \quad S' \leftarrow I' \rightarrow T' \leftarrow O' \rightarrow S'
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
Q \leftarrow R \rightarrow J \rightarrow Z \leftarrow U \rightarrow R
\end{array}
\end{array}
\end{array}
$$

This idea does not come out of the blue: Composite squares with pullbacks like the ones from $T$ to $R$ are precisely the commutative squares in the (bi)category of spans (see [20, 21]); the pullback conditions express that there is a natural isomorphism between the composite of $T \leftarrow O \rightarrow S$
with \( S \leftarrow S' \rightarrow R \) and the composite of \( T \leftarrow T' \rightarrow Z \) with \( Z \leftarrow U \rightarrow R \) (and similarly with the left-hand composite square).

Spans are matrices of sets, and they are the categorical counterpart of linear maps, such as maps given by multi-relations [68] or maps of commutative monoids [54]. In the latter two notions, there is a matrix of natural-number multiplicities instead of a matrix of sets. This is analogous to the difference in the very definitions of Petri nets: In the traditional definition of Petri nets, instead of spans \( S \leftarrow I \rightarrow T \) and \( T \leftarrow O \rightarrow S \), there are multi-relations [32, 68] or—equivalently—Kleisli maps for the commutative-monoid monad:

\[
\begin{align*}
C(S) & \leftarrow T \quad \text{and} \quad T \rightarrow C(S),
\end{align*}
\]

as introduced by Meseguer and Montanari [54].

The general maps of (13) correspond to the most general multi-relation maps considered by Winskel [68, 70]. It may not be entirely clear what the Petri-net meaning of such a general map should be, and Winskel himself concentrates on two more restrictive notions of morphism of Petri nets. A first restriction is to require the transition-level multi-relation to be a partial map; these are simply called *morphisms* by Winskel [68]. In the present setting, this amounts to demanding the map \( T' \rightarrow T \) to be injective. A second restriction consists in actually demanding it to be a total function, which is to say that \( T' \rightarrow T \) is the identity. These are for Winskel [68] the *synchronous* maps, and they are (modulo the difference in definitions) precisely the Petri-net morphisms considered by Meseguer and Montanari [54], which in the free-commutative-monoid formalism take the form

\[
\begin{align*}
T \xrightarrow{f} C(S) \\
Z \xrightarrow{g} C(R)
\end{align*}
\]

(with notation as in (13)). A third restriction asks also the multi-relation at the place-level to be a total function; these are called *folding maps* of Winskel. In the monoid formalism, as in (14), this amounts to demanding \( g \) to be free on a set map. These correspond to the etale maps.

In the following, we could work with any of these three classes of special maps, yielding three classes of morphisms of Petri nets beyond the etale maps. For simplicity, we choose to cover only the case of `synchronous` maps, in the sense of Winskel [68], which are also the main class of morphisms studied by Meseguer and Montanari [54] (modulo the difference in the definition of Petri net). The definition will come in 10.5.

10.1 Cabling Maps. For (graphs or) Petri nets, a *cabling map* is a diagram of the form

\[
\begin{align*}
S' \xleftarrow{\delta} I' \xrightarrow{L} N \xleftarrow{\delta} S' \\
S \xleftarrow{L} I \xrightarrow{L} N \xleftarrow{\delta} O \xrightarrow{I} S.
\end{align*}
\]

They are thus bijective on transitions and arity preserving on places.

They can do two things:

1. They can add a place to a Petri net and connect that new place to existing transitions in any way.
2. Or they can take a set of parallel places (i.e., all having the same interface (pre-sets and post-sets) and all connecting to the same transitions) and `cable' them into a single place with that same interface with those same transitions.

Note that the interfaces of transitions are not preserved by these changes.
It may also be helpful to interpret the morphism in the opposite direction: The moves one can do are then (1) delete a place and all its arcs (but not the transitions); (2) pick a place and refine it into \( k \) parallel copies with the same arcs (a cable with \( k \) wires), linking in the same way to the same transitions.

This shows that given a Petri net \( P = SITOS \), the possible cabling maps to \( P \) are given by keeping \( T \) fixed and by pullback along any map \( \beta : S' \to S \) like this:

\[
\begin{array}{ccccccc}
S' & \rightarrow & T & \rightarrow & \cdots & \cdots & S' \\
\downarrow \beta & & \downarrow \beta & & \cdots & \cdots & \downarrow \beta \\
S & \leftarrow & I & \rightarrow & T & \leftarrow & O \rightarrow S.
\end{array}
\]

Since injections are stable under pullback, we get:

**Lemma 10.2.** Given a cabling map of Petri nets \( Q \to G \) where \( G \) is a graph, then also \( Q \) is a graph.

Let \( \text{Petri}_{\text{cab}} \) denote the category of Petri nets and cabling maps. There is a pseudo-functor

\[
\text{Petri}_{\text{cab}}^{\text{op}} \to \text{Cat}
\]

\[
P \mapsto \text{Proc}(P).
\]

It takes a cabling map \( b : P' \to P \) to the functor

\[
\text{Proc}(P) \to \text{Proc}(P')
\]

\[
p \mapsto b^*(p)
\]

defined by the pointwise pullback

\[
\begin{array}{ccc}
G' & \xrightarrow{\text{cabling}} & G \\
\downarrow \beta & & \downarrow \rho \\
P' & \xrightarrow{\text{cabling}} & P.
\end{array}
\]

It follows easily from closure properties of pullbacks that the map \( b^*p \) defined by this pointwise pullback is etale again, and that the map \( G' \to G \) is a cabling again. Therefore (by Lemma 10.2), \( G' \) is a graph, and altogether \( p' := b^*p \) is a process. The upshot is that \( \text{Proc} \) is contravariantly pseudo-functorial in cabling maps.

10.3 *Cablings of Graphs.* The notion of cabling maps is the same for graphs. It should also be noted that cabling interacts well with level functions. Given a level function \( f : G \to k \), and given a cabling map \( G' \to G \), there is induced an obvious level function \( f' : G' \to k \). Also, the induced colimit decompositions restrict along cabling maps (for simplicity, we only treat the case \( k = 2 \)): If

\[
G = G_1 \sqcup_M G_2
\]

is the colimit decomposition induced by a 2-level function \( G \to 2 \), and if \( b : G' \to G \) is a cabling map, then the colimit decomposition \( G' = G'_1 \sqcup_M G'_2 \) resulting from the level function \( G' \to G \to 2 \) can also be described as the pullback of the original decomposition. In other words, \( b \) restricts to cabling maps between the subgraphs involved. From these observations, the next result follows readily:

**Lemma 10.4.** A cabling map of Petri nets \( b : P' \to P \) induces a symmetric monoidal simplicial map \( X_\bullet \to X'_\bullet \).
10.5 Rational Maps of Petri Nets. A rational map of Petri nets from \( P \) to \( Q \) is by definition a diagram
\[
P \xleftrightarrow{\text{cabling}} P' \xrightarrow{\text{etale}} Q
\]
—a cabling backwards followed by an etale map. Spelled out, these are thus diagrams
\[
\begin{array}{ccc}
P & \xleftarrow{\text{cabling}} & S \\
\downarrow & & \downarrow b \\
P' & \xleftarrow{\text{etale}} & I' \\
\downarrow & & \downarrow r \\
O' & \xleftarrow{\text{etale}} & T \\
\downarrow & & \downarrow \gamma \\
S' & \xleftarrow{\text{etale}} & O \\
\downarrow & & \downarrow b \\
Q & \xleftarrow{\text{etale}} & R \\
\end{array}
\]

Rational maps are composed by pointwise span composition, i.e., by pullback in the category of sets. It is a routine exercise in pullbacks to see that in the result the required squares are pullbacks. Pullback is only defined up to isomorphism, so the composition law is only weak, and the result is not a category but a bicategory. The 2-cells are isomorphisms of spans
\[
\begin{array}{ccc}
P & \xleftarrow{\text{cabling}} & P' \\
\downarrow & & \downarrow \cong \\
P'' & \xleftarrow{\text{etale}} & Q.
\end{array}
\]

The universal property of pullbacks ensures all the required coherences in the usual way.\(^{27}\)

All these constructions restrict to the subcategory of graphs. The only thing to check is that the injectivity condition is preserved under these pullback manipulations, which is because injective maps are stable under pullback.

10.6 Functoriality in Rational Maps. If \( P' \to P \) is a rational map of Petri nets (backward cabling followed by forward etale), then there is induced a functor
\[
\text{Proc}(P') \to \text{Proc}(P)
\]
from the category of processes of \( P' \) to the category of processes of \( P \). (Recall that \( \text{Proc}(P) = \text{Gr} \downarrow P \) is the category of etale maps from graphs.) This is just a question of combining the functorialities already established. This works by refactorisation: A process is itself an etale map \( G' \to P' \). So now we have altogether an etale map followed by a backward cabling, followed by an etale map
\[
G' \xrightarrow{\text{etale}} P' \xrightarrow{\text{cabling}} Q \xrightarrow{\text{etale}} P.
\]
Just refactor the first pair into backward cabling followed by etale:
\[
G' \xrightarrow{\text{cabling}} G \xrightarrow{\text{etale}} Q \xrightarrow{\text{etale}} P.
\]
The middle object \( G \) appearing here is a graph by Lemma 10.2. Then throw away \( G' \).

In other words, the process will have a different underlying graph, but with the same nodes. The way \( G' \) is modified is the minimal way to ensure an etale map to \( P \).

Combining the functorialities already established (4.10 and 10.4), we find

\(^{27}\)All this is very similar to the situation for polynomial endofunctors [25]: There, the cartesian natural transformations play the role of etale maps, whereas general natural transformations uniquely decompose into something ‘backward’ followed by cartesian.
Proposition 10.7. The construction of the symmetric monoidal Segal space \( X_\bullet \) is functorial in rational maps.

11 NOTES

11.1 Geometric vs. Algebraic Process Semantics. The geometric process semantics was pioneered by Goltz and Reisig [32] in the language of posets. As noted in Section 3, this cannot immediately be linked with firing sequences and the idea of a categorical composition law, but Best and Devillers [13] actually figured out the equivalence relations required both on processes and on ‘occurrence sequences’ (certain free categories) to make them match up. Meanwhile, purely algebraic approaches were introduced by Winskel [70] and Meseguer–Montanari [54], giving symmetric monoidal categories, but without clear connection to the Goltz–Reisig processes.

It took a longer journey, begun perhaps by Degano, Meseguer, and Montanari [19], to find a reconciliation of the two viewpoints; this inevitably involved a gradual shift from collective- to individual-tokens viewpoints. Roughly, the ideas were to introduce book-keeping devices in ingenious ways, in terms of numbering schemes, to compensate for the slack of multisets, to get enough control over the symmetries of states and processes to be able to compose them in a meaningful way. The short survey of Meseguer, Montanari and Sassone [57] describes this: first *concatenable processes* [19], introducing numberings of subsets of in- and out-boundaries of processes mapping to the same place; then *strongly concatenable processes* [63], with full numbering of in- and out-boundaries, in terms of ‘strings as explicit representatives of multisets.’ Finally, these developments culminated with the clean idea of pre-nets of Bruni, Meseguer, Montanari, and Sassone [16, 17], where ‘everything’ is numbered: here multisets are finally discarded and replaced by lists: a pre-net is the data of

\[ T \Rightarrow M(S) \]

where \( M \) is the free-monoid monad, instead of the free-commutative-monoid monad \( C \) of the definition of Meseguer and Montanari [54]. In combination with the idea [7] of working with states as words (or lists) of places instead of multisets of places, this gives finally full-blown individual-tokens semantics. The individuality of a token in a given state is encoded as its position in a word, and the individuality of token occurrences is encoded through its history, as in unfolding semantics.

11.2 Symmetry Problems in Unfolding Semantics. The unfolding semantics is not directly affected by the dichotomy geometric/algebraic, as it is not concerned with composition of processes end to end. It may be considered geometric in that the processes are defined as maps from ‘causal nets,’ a viewpoint that goes back to Petri himself [61]. Nevertheless, the symmetry problems turn out to be of very similar nature, and as we have seen, the problems can be given the diagnosis that traditional Petri nets are not structured enough to control symmetries of processes.

Although the developments from concatenable processes to pre-nets were motivated by monoidal-category semantics, the ideas were also applied to unfolding, where again the rigidity imposed by decorations and numberings served to work around the symmetry problems. Important contributions in this direction were Meseguer, Montanari, and Sassone [55, 56] and Baldan, Bruni, and Montanari [7].

11.3 Pre-Nets vs. Whole-Grain Petri Nets. Some of the successive key insights of the Bruni-Meseguer–Montanari–Sassone line of work are subsumed in the SITOS formalism. Note first that traditional Petri nets too, induce digraphical species, but not generally flat ones. The strongly concatenable-processes insight of numbering boundaries is subsumed in the general idea of the free-prop monad, and more particularly in the notion of \( [ n ] \)-graph, an inherent ingredient in the
theory of symmetric monoidal categories, present since the notion of tensor scheme of Joyal and Street [43].

The ultimate consequence of those developments, replacing Petri nets with pre-nets for the sake of handling symmetries, is a question of imposing structure enough to land in the subcategory of flat digraphical species. Indeed, Every (finite) flat digraphical species is the symmetrisation of a pre-net—simply, because finite sets (as in SITOS diagrams) admit a linear ordering. However, the functor from pre-nets to digraphical species is not full. Its full image is equivalent to the category of whole-grain Petri nets (by 5.2 and 5.3). If, to correct this deficiency, morphisms of pre-nets were defined to be morphisms of their associated flat digraphical species, then in the end it would amount essentially to working with whole-grain Petri nets, but with the numberings imposed on top.

Compared to the Bruni–Meseguer–Montanari–Sassone line of development, the present work takes a different approach. Instead of numbering schemes and linear orders, the individuality of elements is encoded simply by taking seriously the graphical rendition of Petri nets. The elements of the sets

\[ S \leftarrow I \rightarrow T \leftarrow O \rightarrow S \]

are precisely the elements seen in a picture of a Petri net: places, transitions, and arcs; the notion is very close to the traditional definition of Petri nets and does not involve numberings. But having sets of arcs instead of just numbers of arcs makes a big difference, leading to the representability feature formalised through the identification (Proposition 5.3) of SITOS Petri nets with flat digraphical species.

The SITOS Petri nets contain the pre-nets, as the special case where the maps

\[ I \rightarrow T \leftarrow O \]

are forced to be of the special kind

\[ \sum_{t \in T} m_t \rightarrow T \leftarrow \sum_{t \in T} n_t \]

where all \( m_t \) and \( n_t \) are standard linear orders. But while every Petri net is isomorphic to one of this kind, by appropriate choices of representing sets, the general morphisms will not preserve these choices.

11.4 Comparison Functors. Baez, Genovese, Master, and Shulman [4] have furthered the theory of whole-grain Petri nets and digraphical species (which they call \( \Sigma \)-nets) and provide in particular the following system of adjunctions, which clarifies the relationship between notions of net:

\[ \text{PreNet} \leftarrow \text{PreSh(elGr)} \rightarrow \text{TradPetri} \]

Note that the right-hand triangle from whole-grain to traditional to digraphical species does not commute—a key difference between whole-grain Petri nets and traditional Petri nets is the fact that they embed differently into digraphical species. Note that in all these categories, the morphisms are only the étale maps. Further work will be required to perform similar comparison with fancier notions of morphisms, as in Section 10, and in particular it is not clear if embedding into digraphical species would work.

11.5 Which Monad? Classical Petri nets are diagrams \( T \Rightarrow C(S) \) for \( C \) the free-commutative-monoid monad. Pre-nets are diagrams \( T \Rightarrow M(S) \) for \( M \) the free-monoid monad. Master [52] has
begun the study of notions of Petri nets relative to other monads than these two, in fact relative to Lawvere theories. This generality covers also other interesting flavours of Petri nets.

The free-commutative-monoid monad $C$ does not provide sufficient grip on the symmetries. The free-monoid monad $M$ introduces artificial linear orders that are not present in the intuitive picture we have of Petri nets, and it also does not lead to reasonable notions of morphism. In between the options $C$ and $M$ there is the symmetric-monoidal-category monad. It has the commutative flavour of the free-commutative-monoid monad (which indeed is its $\pi_0$), but at the same time has the important property shared with the free-monoid monad that it is cartesian. The only issue with the free-symmetric-monoidal-category monad—at first sight a serious blow to the idea—is that it does not exist on the category of sets! It requires at least groupoids: The free symmetric monoidal category on a set gives a groupoid. Sassone, in the outlook section of [63], suggests that the symmetry issues related to the strongly concatenable-processes formalism should be overcome by stepping up to 2-categories. The free-symmetric-monoidal-category monad on $\text{Grpd}$ fits this idea perfectly. The short expository paper by Bruni, Meseguer, Montanari, and Sassone [57] also hints at this, describing the role of the 2-structure being to carry information about multisets, thus making an explicit quotient construction unnecessary. This viewpoint, however, did not find its way into the final paper [17], and apparently it was not pursued. The symmetric monoidal Segal space $X_\bullet$ can be seen as fleshing out this idea, while insisting that Petri nets are just configurations of sets.

11.6 SITOS Formalism and Symmetric-Monoidal-Category Monad. The difference between traditional multisets and representable multisets constitutes the passage from the free-commutative-monoid monad to the free-symmetric-monoidal-category monad, in the following sense, giving an interpretation of the SITOS formalism in terms of the free-symmetric-monoidal-category monad.

In its strictest version, the free-symmetric-monoidal-category monad sends a set $S$ (regarded as a discrete groupoid) to the groupoids whose underlying set is the set of words in $S$, and whose morphisms are the permutations. A word is just a set map $n \to S$ and a permutation is a commutative diagram

$$
\begin{array}{ccc}
n & \longrightarrow & n \\
& \downarrow & \\
& S.
\end{array}
$$

A more invariant presentation uses arbitrary finite sets instead of insisting on the sets $n$. The groupoid is then $\text{B} \downarrow S$. In this view on multisets, to assign to every transition $t$ its post-set is to specify a map $O_t \to S$. All these maps can be conveniently bundled into the configuration $T \leftarrow O \rightarrow S$ (and similarly of course with pre-sets and $S \leftarrow I \rightarrow T$). The SITOS formalism is therefore almost unavoidable.

11.7 Symmetric Operads vs. Non-Symmetric Operads vs. Polynomial Monads. The use of set-configurations instead of numberings was advocated in the polynomial formalism in operad theory [45]. Symmetric operads can be seen as the many-in/one-out case of props. Their symmetry issues are subtle, as expressed by the fact that the corresponding monads on the category of sets are generally not cartesian but only weakly cartesian. Every non-symmetric operad defines a symmetric operad by symmetrisation, and the result is sigma-cofibrant by construction. Every sigma-cofibrant operad admits the structure of non-symmetric operad, but does not come with one. It turns out that sigma-cofibrant operads are the same thing as finitary polynomial monads.

---

28The distinction is subtle, and it is the origin of some mistakes in the literature on operads and Lawvere theories, as explained in detail in Leinster’s book [49].
This hinges on the crucial representability feature of polynomial endofunctors [25]: They are represented by diagrams

$$I \leftarrow E \rightarrow B \rightarrow I.$$ The theory of polynomial functors [25, 45] is the original inspiration for both [46] and the present work.

It is interesting to note that the distinction between (finitary) polynomial monads and operads, which is purely a question of symmetries, goes away in the world of $\infty$-categories, by a theorem of Gepner, Haugseng, and Kock [28] (in fact a groupoid-enriched setting is enough). By the same arguments, the distinction between (whole-grain) Petri nets and digraphical species would vanish if Petri nets were allowed to have a groupoid of transitions instead of just a set of them. While this might seem wild from the perspectives of current Petri-net theory, it would look natural from the viewpoint of implementing Petri nets in (intensional) dependent type theory [65], with places and transitions being types instead of sets. The recent work of Baez, Genovese, Master, and Shulman [4] can also be seen to point in that direction, their definition of digraphical species (there called $\Sigma$-net) having very much the flavour of a dependent type.

APPENDICES

A GROUPOIDS AND HOMOTOPY PULLBACKS

We recall here a few basic notions regarding groupoids.

A.1 Groupoids. A groupoid is a category in which all arrows are invertible; a map of groupoids is just a functor. For their use in combinatorics, it is rather their topological-space aspects that are important. If $X$ is a groupoid, then $\pi_0(X)$ denotes the set of connected components, i.e., the set of iso-classes. For each $x \in X$, we denote by $\pi_1(X,x) = \text{Aut}_X(x)$ the group of automorphisms. A map of groupoids is an equivalence iff it is bijective on $\pi_0$ and invertible on each $\pi_1$ (Whitehead’s theorem).

We are interested in groupoids up to equivalence, and since the ordinary categorical constructions with groupoids—such as pullbacks and fibres—are not invariant under equivalence, they should be replaced by their homotopy analogues, which are characterised by universal properties up to equivalence. If just these homotopy notions are used consistently, then they behave very much like the ordinary notions do for sets.

A.2 Homotopy Pullbacks. A homotopy pullback is an up-to-isomorphism commutative square

$$\begin{array}{ccc}
P & \longrightarrow & Y \\
\downarrow & \simeq & \downarrow q \\
X & \longrightarrow & S
\end{array}$$

satisfying a universal property among all such squares with common $p$ and $q$. As such it is determined uniquely up to equivalence. There are different (but equivalent) models for homotopy pullback. The standard homotopy pullback $P = X \times^h_S Y$ has as objects triples $(x, y, \sigma)$ consisting of $x \in X, y \in Y$, and $\sigma : px \rightarrow qy$ in $S$; its arrows $(x, y, \sigma) \rightarrow (x', y', \sigma')$ are pairs $(\phi, \psi) \in \text{Hom}_X(x, x') \times \text{Hom}_Y(y, y')$ such that $\sigma' \circ p(\phi) = q(\psi) \circ \sigma$. (From the viewpoint of category theory, the standard homotopy pullback is thus the comma category $p \downarrow q$.) While the standard homotopy pullback is always correct, it can be a bit cumbersome to use. It is sometimes possible to be more economical by exploiting fibrations.
A.3 Fibrations. A map of groupoids $p : X \to B$ is a fibration when it satisfies the path lifting property: For each $x \in X$ and $\beta : p(x) \sim b$ in $B$, there exists an arrow $\phi : x \to x'$ such that $p(\phi) = \beta$. The benefit of this notion is that an ordinary (strict) pullback

$$
\begin{array}{ccc}
X \times^p_{Y} Y & \longrightarrow & Y \\
\downarrow & & \downarrow q \\
X & \longrightarrow & S
\end{array}
$$

is also a homotopy pullback whenever one of the two maps $p$ and $q$ is a fibration.

**Lemma A.4.** (Prism Lemma.) Given a prism diagram of groupoids

$$
\begin{array}{ccc}
X'' & \longrightarrow & X' & \longrightarrow & X \\
\downarrow & \simeq & \downarrow j & \simeq & \downarrow \\
Y'' & \longrightarrow & Y' & \longrightarrow & Y
\end{array}
$$

in which the right-hand square is a homotopy pullback, then the outer rectangle is a homotopy pullback if and only if the left-hand square is a homotopy pullback.

A.5 Homotopy Fibre. Given a map of groupoids $p : X \to S$ and an element $s \in S$, the homotopy fibre $X_s$ of $p$ over $s$ is the homotopy pullback

$$
\begin{array}{ccc}
X_s & \longrightarrow & X \\
\downarrow j & \simeq & \downarrow p \\
1 & \longrightarrow & S.
\end{array}
$$

A.6 Homotopy Quotient and Homotopy Sum. Given a group action $G \times X \to X$ (for $G$ a group and $X$ a set or groupoid), instead of the naive quotient (set of orbits), the homotopy quotient $X//G$ is obtained from $X$ by sewing in a path from $x$ to $g.x$ for each $x \in X$ and $g \in G$. Note that it is a groupoid, not a set. (See Baez–Dolan [3] for a very enjoyable account of this.)

A homotopy sum is a (homotopy) colimit indexed by a groupoid, just like an ordinary sum is a colimit indexed by a set. It can be computed as an ordinary sum of homotopy quotients: If $B$ is a groupoid and $F : B \to \text{Grpd}$ is a diagram of groupoids indexed by $B$, then the homotopy sum is

$$
\int_{b \in B} F(b) = \sum_{b \in \pi_0(B)} F(b)//\text{Aut}(b).
$$

B SIMPLICIAL GROUPOIDS AND SEGAL SPACES

Simplicial methods are fundamental in topology and algebra. Further information can be found in May’s little 1967 book *Simplicial Methods in Algebraic Topology* [53]. In this paper, we use simplicial groupoids to encode ‘weak categories,’ a standard idea in higher category theory [11, 49]. For their role in combinatorics, see [24].

B.1 The Idea of Simplicial Sets. Where a directed graph $X_0 \subseteq X_1$ (in the ordinary sense, without the open-ended edges of the graphs in Section 1) has vertices and edges, a simplicial set $X_\bullet$ has (directed) cells in all dimensions$^{29}$: After vertices and edges come a set $X_2$ of triangles, a set $X_3$ of tetrahedra, a set $X_4$ of 4-dimensional simplices, and so on. Just like an edge has two endpoints that

$^{29}$The bullet in $X_\bullet$ is there to remind that it is not a single set, but a whole configuration of sets.
are vertices, a triangle has three vertices, numbered 0, 1, 2, but it also has three sides that are edges, so there should be three maps $X_1 \xleftarrow{d_1} X_0 \xrightarrow{d_0} X_2$. Similarly, a tetrahedron has four vertices, numbered 0, 1, 2, 3, and four faces that are triangles (to have four maps from $X_4$ to $X_3$), and so on. Altogether, a simplicial set $X_\bullet$ has a sequence of sets $X_0, X_1, X_2, X_3, X_4, \ldots$ with face maps

$$
X_0 \xleftarrow{d_1} X_1 \xrightarrow{d_0} X_2 \xleftarrow{d_1} X_3 \xrightarrow{d_0} X_4 \ldots
$$

subject to the identities given below. The most systematic way of numbering the sides of a triangle, the faces of a tetrahedron, and so on, is to record the vertex removed to get it. So, for example, face 2 of a tetrahedron is the triangle opposite vertex 2. It is thus important to index the face maps such that $d_i$ removes vertex number $i$ to get a simplex of one dimension lower. With these conventions (which have proven optimal through many decades of work in topology and algebra), the face map identities are

$$
d_i d_j = d_{j-1} d_i \quad \text{for all } i < j, \text{ and in all dimensions},$$

which essentially say that for two vertices, it does not matter in which order they are removed. (It turns out to be harmless to use the same name for face maps in all dimensions, although of course they are different maps.)

Furthermore, a simplicial set should have degeneracy maps, whereby a vertex can be regarded as a degenerate edge, and an edge can be regarded as a degenerate triangle (in two ways), and so on. The degeneracy maps are always denoted $s_i$. With this further structure, we arrive at the full picture of a simplicial set: It is a diagram

$$
X_0 \xleftarrow{d_1} X_1 \xrightarrow{d_0} X_2 \xleftarrow{d_1} X_3 \xrightarrow{d_0} X_4 \ldots
$$

subject to the simplicial identities:

$$
d_i s_i = d_{i+1} s_i = 1, \quad (\forall i)$$

$$
d_i d_j = d_{j-1} d_i, \quad d_{j+1} s_i = s_1 d_j, \quad d_i s_j = s_{j+1} d_i, \quad s_j s_i = s_i s_{j-1}, \quad (i < j).$$

While these identities express clear geometric intuition, it takes some practice to get familiar with the indices. Luckily, there is a categorical way to describe the structure more conceptually:

**B.2 Definition (of Simplicial Set).** Let $\Delta$ denote the category whose objects are the nonempty finite total orders

$$[k] := \{0 \leq 1 \leq \cdots \leq k\}$$

and whose arrows are the monotone maps. A simplicial set is a functor $X_\bullet : \Delta^\text{op} \to \text{Set}$. The value of $X_\bullet$ on $[k]$ is denoted $X_k$, and its elements are called $k$-simplices.

A simplicial map $X_\bullet \to Y_\bullet$ between simplicial sets is a natural transformation of functors $\Delta^\text{op} \to \text{Set}$. The simplicial identities are now accounted for by noting that the category $\Delta$ is generated by the injections $d^i : [k-1] \to [k]$ that skip the value $i$ and the surjections $s^i : [k+1] \to [k]$ that repeat $i$. The relations holding between these maps are obvious and induce the simplicial identities by functoriality.
The reason simplicial sets are fundamental objects in topology is that they provide a combinatorial model for spaces up to homotopy [53]. In the present context they are important for their role in category theory, which is via the nerve construction:

**B.3 The Nerve of a Small Category.** The nerve of a small category \( \mathcal{C} \) is the simplicial set

\[
N\mathcal{C} : \Delta^{op} \to \text{Set}
\]

whose set of \( k \)-simplices is the set of sequences of \( k \) composable arrows in \( \mathcal{C} \) (allowing identity arrows).

- The degeneracy maps are given by inserting an identity map in the sequence.
- The inner face maps are given by composing adjacent arrows.
- The outer face maps discard arrows at the beginning or the end of the sequence.

By regarding the total order \([k]\) as a category, we see that a sequence of \( k \) composable arrows in \( \mathcal{C} \) is the same thing as a functor \([k] \to \mathcal{C}\), and more formally the set of \( k \)-simplices can be described as

\[
(N\mathcal{C})_k = \text{Fun}([k], \mathcal{C}).
\]

Note that \( d_0 : X_1 \to X_0 \) assigns to an arrow its codomain, and \( d_1 : X_1 \to X_0 \) assigns to an arrow its domain. Note further that \( d_1 : X_2 \to X_1 \) is the composition map. Also, \( d_0 : X_2 \to X_1 \) assigns to a composable pair the second arrow, and \( d_2 : X_2 \to X_1 \) assigns to a composable pair the first arrow (cf. the standard index convention of B.1).

**B.4 Nerve Theorem.** The nerve functor is fully faithful, and its image is characterised by the Segal condition, which states that the canonical projection map that returns the edges of a \( k \)-simplex

\[
X_k \longrightarrow X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1
\]

is a bijection for each \( k \geq 1 \). The fibre products express that the edges match up at the vertices. Extracting the edges of a simplex can be done with iterated use of face maps. The easiest and most important case of the Segal condition is \( k = 2 \). Here the condition says that the map \((d_2, d_0) : X_2 \to X_1 \times_{X_0} X_1 \) is a bijection, which is equivalent to saying that this square is a pullback:

\[
\begin{array}{ccc}
X_2 & \xrightarrow{d_0} & X_1 \\
\downarrow_{d_1} & & \downarrow_{d_1} \\
X_1 & \xrightarrow{d_0} & X_0.
\end{array}
\]

The condition thus says that the 2-simplices are precisely the composable pairs.

**B.5 Simplicial Groupoids.** One can talk about simplicial objects in any category \( \mathcal{E} \): They are functors \( X_* : \Delta^{op} \to \mathcal{E} \). In particular, a simplicial groupoid is a functor

\[
X_* : \Delta^{op} \to \text{Grpd},
\]

and it thus amounts to a sequence of groupoids \( X_k, (k \geq 0) \), and face and degeneracy map \( d_i : X_k \to X_{k-1}, s_i : X_k \to X_{k+1}, (0 \leq i \leq k) \), subject to the simplicial identities above.

From the viewpoint of simplicial groupoids, it is now possible (and natural) to weaken the notion of category by relaxing the requirement that the maps (17) should be isomorphisms to just demanding they be equivalences of groupoids, with the caveat that we should now refer to the homotopy pullback instead of the strict pullback. Additionally, there is one more technical condition called Rezk completeness:

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B.6 Rezk-Complete Segal Spaces. A simplicial groupoid is a Segal space\(^{30}\) if \(X_2 \cong X_1 \times_{X_0}^h X_1\), and in general the canonical projection map

\[
X_k \longrightarrow X_1 \times_{X_0}^h X_1 \times_{X_0}^h \cdots \times_{X_0}^h X_1
\]

is an equivalence of groupoids for each \(k \geq 1\).

Composition in a Segal space is up to equivalence: Given a pair of arrows that are composable up to equivalence, their composite is given by

\[
X_1 \times_{X_0}^h X_1 \xrightarrow{(d_2, d_0)} X_2 \xrightarrow{d_1} X_1,
\]

where we first have to go backwards along the equivalence to \(X_2\): The Segal condition tells us that a composable pair is equivalent to a whole 2-simplex, and now we can return its ‘long edge,’ using \(d_1\). (Note that \(d_1\) deletes the middle vertex of a triangle to leave us with the ‘long edge’ from vertex 0 to vertex 2).

An arrow \(f \in X_1\) is called invertible if composition with it from either side defines an equivalence of groupoids \(X_1 \rightarrow X_1\). A Segal space \(X_\bullet\) is Rezk complete if \(s_0 : X_0 \rightarrow X_1\) is fully faithful and has as its essential image the invertible arrows.

B.7 Mapping Groupoids. Just as an ordinary category has hom sets, a Segal space has mapping groupoids (often called mapping spaces). Given two objects \(x, y\) in a Segal space \(X_\bullet\), the mapping groupoid Map_{\(X_\bullet\)}(\(x, y\)) is defined as the homotopy pullback

\[
\begin{array}{ccc}
\text{Map}_{X_\bullet}(x, y) & \xrightarrow{J} & X_1 \\
\downarrow & \Downarrow{\cong} & \downarrow{(d_2, d_0)} \\
1 & \xleftarrow{\tau(x, y)^\gamma} & X_0 \times X_0,
\end{array}
\]

which is the appropriate way of picking out those arrows that start in \(x\) and end in \(y\)—up to homotopy.

From the viewpoint of homotopy theory, it is quite natural to consider that if some combinatorial structure with symmetries is hard to assemble into a category, then it should perhaps be a weak category instead—a (Rezk-complete) Segal space. This is what happens in Section 4.

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REFERENCES

[1] Andrew Baas, James Fairbanks, Micah Halter, and Evan Patterson. 2020. Algebraic petri. In Julia Library for Building Petri Net Models Compositionally. Retrieved from https://github.com/AlgebraicJulia/AlgebraicPetri.jl.
[2] John C. Baez and Kenny Courser. 2020. Structured cospans. Theor. Appl. Categ. 35 (2020), 1771–1822. ArXiv:1911.04630.
[3] John C. Baez and James Dolan. 2001. From finite sets to Feynman diagrams. In Mathematics Unlimited—2001 and Beyond, B. Engquist and W. Schmid (Eds.). Springer-Verlag, Berlin, 29–50.

\(^{30}\)The full-blown notion used in homotopy theory and \(\infty\)-category theory uses \(\infty\)-groupoids instead of just ordinary groupoids. Rezk-complete Segal spaces are an important model for weak categories in higher category theory and homotopy theory; see Bergner [11] for a survey.
[4] John C. Baez, Fabrizio Genovese, Jade Master, and Michael Shulman. 2021. Categories of nets. In Proceedings of the 36th Annual ACM/IEEE Symposium on Logic in Computer Science. ACM, New York, NY. ArXiv:2101.04238.
[5] John C. Baez and Jade Master. 2020. Open Petri nets. Math. Struct. Comput. Sci. 30 (2020), 314–341. ArXiv:1808.05415.
[6] John C. Baez and Blake S. Pollard. 2017. A compositional framework for reaction networks. Rev. Math. Phys. 29 (2017), 1750028, 41. ArXiv:1704.02051.
[7] Paolo Baldan, Roberto Bruni, and Ugo Montanari. 2002. Pre-nets, read arcs and unfolding: A functorial presentation. In Recent Trends in Algebraic Development Techniques, 16th International Workshop, WADT'02, Frauenchiemsee, Germany, September 24–27, 2002, Revised Selected Papers, M. Wirsing, D. Pattinson, R. Hennicker (Eds.). Lecture Notes in Computer Science, Vol. 2755, Springer, 145–164.
[8] Paolo Baldan, Andrea Corradini, Hartmut Ehrig, and Reiko Heckel. 2005. Compositional semantics for open Petri nets based on deterministic processes. Math. Struct. Comput. Sci. 15 (2005), 1–35.
[9] Michael Batanin and Clemens Berger. 2017. Homotopy theory for algebras over polynomial monads. Theor. Appl. Categ. 32 (2017), 148–253.
[10] François Bergeron, Gilbert Labelle, and Pierre Leroux. 1998. Combinatorial species and tree-like structures. In Encyclopedia of Mathematics and Its Applications, Vol. 67, Cambridge University Press, Cambridge.
[11] Julia E. Bergner. 2010. A survey of (∞, 1)-categories. In Towards Higher Categories, Vol. 152 of IMA Vol. Math. Appl., Springer, New York, 69–83. ArXiv:math.AT/0610239.
[12] Gérard Berry. 1979. Modèles Complètement Adeéquats et Stables Des Lambda-calculs Typés. PhD thesis, Thèse de Doctorat d’Etat, Université Paris VII.
[13] Eike Best and Raymond Devillers. 1987. Sequential and concurrent behaviour in Petri net theory. Theor. Comput. Sci. 55 (1987), 87–136.
[14] Eike Best and César Fernández. 1988. Nonsequential Processes—A Petri Net View, EATCS Monographs on Theoretical Computer Science, Vol. 13. Springer-Verlag, Berlin.
[15] Jeff Bezanson, Stefan Karpinski, Viral B. Shah, and Alan Edelman. 2012. Julia: A Fast Dynamic Language for Technical Computing. Technical report. ArXiv:1209.5145.
[16] Roberto Bruni, José Meseguer, Ugo Montanari, and Vladimiro Sassone. 1999. Functorial semantics for Petri nets under the individual token philosophy. Electron. Notes Theor. Comput. Sci. 29, 29003 (1999), 18pp.
[17] Roberto Bruni, José Meseguer, Ugo Montanari, and Vladimiro Sassone. 2001. Functorial models for Petri nets. Inf. Comput. 170 (2001), 207–236.
[18] Roberto Bruni and Ugo Montanari. 2000. Zero-safe nets: Comparing the collective and individual token approaches. Inf. Comput. 156 (2000), 46–89.
[19] Pierpaolo Degano, José Meseguer, and Ugo Montanari. 1996. Axiomatizing the algebra of net computations and processes. Acta Inform. 33 (1996), 641–667.
[20] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. 2018. Homotopy linear algebra. Proc. Royal Soc. Edinburgh Sec. A 148 (2018), 293–325. ArXiv:1602.05082.
[21] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. 2018. Decomposition spaces, incidence algebras and Möbius inversion I: Basic theory. Adv. Math. 331 (2018), 952–1015. ArXiv:1512.07573.
[22] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. 2018. Decomposition spaces, incidence algebras and Möbius inversion II: Completeness, length filtration, and finiteness. Adv. Math. 333 (2018), 1242–1292. ArXiv:1512.07577.
[23] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. 2020. Decomposition spaces and restriction species. Int. Math. Res. Not. 2020 (2020), 7558–7616. ArXiv:1708.02570.
[24] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. 2016. Decomposition Spaces in Combinatorics. Preprint, arXiv:1612.09225.
[25] Nicola Gambino and Joachim Kock. 2013. Polynomial functors and polynomial monads. Math. Proc. Camb. Philos. Soc. 154 (2013), 153–192. ArXiv:0906.4931.
[26] Fabrizio Genovese, Alex Gryzlov, Jelle Herold, Marco Perone, Erik Post, and André Videla. 2019. Computational Petri Nets: Adjunctions Considered Harmful. Preprint, arXiv:1904.12974.
[27] Hartmann J. Genrich and Ewa Stankiewicz-Wiechno. 1980. A dictionary of some basic notions of net theory. In Net Theory and Applications (Proc. Adv. Course General Net Theory Processes Systems, Hamburg, 1979), Lecture Notes in Computer Science, Vol. 84, Springer, Berlin-New York, 519–535.
[28] David Gepner, Rune Haugseng, and Joachim Kock. 2022. ∞-operads as analytic monads. Int. Math. Res. Not. (2022), 12516–12624.
[29] Robert Jan van Glabbeek. 2005. The individual and collective token interpretations of Petri nets. In Proceedings of the Conference onConcurrency Theory, Lecture Notes in Computer Science, Vol. 3653. Springer, Berlin, 323–337.
[30] Rob J. van Glabbeek and Gordon D. Plotkin. 1995. Configuration structures. In Proceedings of the 10th Annual IEEE Symposium on Logic in Computer Science. IEEE Computer Society, 199–209.
[31] Rob J. van Glabbeek and Gordon D. Plotkin. 2009. Configuration structures, event structures and Petri nets. *Theoret. Comput. Sci.* 410 (2009), 4111–4159.

[32] Ursula Goltz and Wolfgang Reisig. 1983. The nonsequential behaviour of Petri nets. *Inf. Contr.* 57 (1983), 125–147.

[33] Michel Hack. 1976. *Decidability Questions for Petri Nets.* PhD thesis, Massachusetts Institute of Technology, Cambridge, MA.

[34] Micah Halter and Evan Patterson. 2020. *Compositional Epidemiological Modeling Using Structured Cospans.* Oct. 2020. Retrieved from [https://www.algebraicjulia.org/blog/post/2020/10/structured-cospans](https://www.algebraicjulia.org/blog/post/2020/10/structured-cospans).

[35] Linde Wester Hansen and Michael Shulman. 2019. Constructing Symmetric Monoidal Bicategories Functorially. Preprint, *arXiv*:1910.09240.

[36] Jonathan Hayman and Glynn Winskel. 2008. The unfolding of general Petri nets. In *Proceedings of the IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science*, LIPIcs. Leibniz Int. Proc. Inform., Vol. 2, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 223–234.

[37] Jonathan Hayman and Glynn Winskel. 2009. Symmetry in Petri nets. In *Perspectives in Concurrency Theory*. University Press, Hyderabad, 231–263.

[38] Tobias Heindel and Pawel Sobociński. 2011. Being Van Kampen is a universal property. *Log. Meth. Comput. Sci.* 7 (2011). ArXiv:1101.4594.

[39] André Joyal. 1981. Une théorie combinatoire des séries formelles. *Adv. Math.* 42 (1981), 1–82.

[40] André Joyal and Joachim Kock. 2011. Feynman graphs, and nerve theorem for compact symmetric multicategories (extended abstract). *Electron. Notes Theor. Comput. Sci.* 270 (2011), 105–113.

[41] André Joyal and Ieke Moerdijk. 1994. A completeness theorem for open maps. *Ann. Pure Appl. Logic* 70 (1994), 51–86.

[42] André Joyal, Mogens Nielsen, and Glynn Winskel. 1996. Bisimulation from open maps. *Inf. Comput.* 127 (1996), 164–185.

[43] André Joyal and Ross Street. 1991. The geometry of tensor calculus. I. *Adv. Math.* 88 (1991), 55–112.

[44] *The Julia Programming Language*. Retrieved from [https://julialang.org/](https://julialang.org/).

[45] Joachim Kock. 2011. Polynomial functors and trees. *Int. Math. Res. Not.* (2011), 609–673. ArXiv:0807.2874.

[46] Joachim Kock. 2016. Graphs, hypergraphs, and properads. *Adv. Math.* 280 (2015), 155–190. ArXiv:1407.3744.

[47] Joachim Kock. 2018. Cospan construction of the graph category of Borisov and Manin. *Publ. Mat.* 62 (2018), 331–353.

[48] Stephen Lack. 2002. Codescent objects and coherence. *J. Pure Appl. Algebra.* 175 (2002), 223–241. ArXiv:1510.08925.

[49] Tom Leinster. 2004. Higher operads, higher categories. *Trans. Amer. Math. Soc.* 356 (2004), 1535–1570. ArXiv:math.CT/0305049.

[50] Sophie Libkind, Andrew Baas, Evan Patterson, and James Fairbanks. 2020. An algebraic framework for structured epidemic modeling. *Philos. Trans. A Math. Phys. Eng. Sci.* 380 (2020), 20210309. DOI: 10.1098/rsta.2021.0309, arXiv:2203.16345.

[51] Saunders Mac Lane. 1998. *Categories for the Working Mathematician*, Second Edition. Graduate Texts in Mathematics, No. 5. Springer-Verlag, New York.

[52] Jade Master. 2020. Petri nets based on Lawvere theories. *Math. Struct. Comput. Sci.* 30 (2020), 833–864. ArXiv:1904.09091.

[53] J. Peter May. 1967. Simplicial objects in algebraic topology. *Van Nostrand Mathematical Studies*, No. 11. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London.

[54] José Meseguer and Ugo Montanari. 1990. Petri nets are monoids. *Inf. Comput.* 88 (1990), 105–155.

[55] José Meseguer, Ugo Montanari, and Vladimiro Sassone. 1996. Process versus unfolding semantics for place/transition Petri nets. *Theor. Comput. Sci.* 153 (1996), 171–210.

[56] José Meseguer, Ugo Montanari, and Vladimiro Sassone. 1997. On the semantics of place/transition Petri nets. *Math. Struct. Comput. Sci.* 7 (1997), 359–397.

[57] José Meseguer, Ugo Montanari, and Vladimiro Sassone. 1997. Representation theorems for Petri nets. In *Foundations of Computer Science: Potential - Theory - Cognition*, to Wilfried Brauer on the Occasion of His 60th birthday, C. Freksa, M. Jantzen, R. Valk (Eds.). Lecture Notes in Computer Science, Vol. 1337. Springer, 239–249.

[58] Mogens Nielsen, Gordon Plotkin, and Glynn Winskel. 1983. Non-Sequential Processes. *Inf. Contr.* 57 (1983), also: Gesellschaft für Mathematik und Datenverarbeitung Bonn, ISF-77-6, 3. revidierte und ergänzte Auflage. Translation *Non-Sequential Processes* by P. Krause, J. Low. Gesellschaft für Mathematik und Datenverarbeitung Bonn, ISF-77-5, 1977.
[62] Wolfgang Reisig. 1985. *Petri Nets: An Introduction*, Vol. 4 of *EATCS Monographs on Theoretical Computer Science*. Springer, (1985).

[63] Vladimiro Sassone. 1998. An axiomatization of the category of Petri net computations. *Math. Struct. Comput. Sci.* 8 (1998), 117–151.

[64] Dana S. Scott. 1971. The lattice of flow diagrams. In *Proceedings of the Symposium on Semantics of Algorithmic Languages*, E. Engeler (Ed.), *Lecture Notes in Mathematics*, Vol. 188. Springer, 311–366.

[65] The Univalent Foundations Program. 2013. *Homotopy Type Theory-Univalent Foundations of Mathematics*. The Univalent Foundations Program, Princeton, NJ; Institute for Advanced Study (IAS), Princeton, NJ. Retrieved from [http://homotopytypetheory.org/book](http://homotopytypetheory.org/book).

[66] Mark Weber. 2015. Internal algebra classifiers as codescent objects of crossed internal categories. *Theor. Appl. Categ.* 30 (2015), 1713–1792.

[67] Glynn Winskel. 1980. *Events in Computation*. PhD thesis, Department of Computer Science, University of Edinburgh.

[68] Glynn Winskel. 1984. A new definition of morphism on Petri nets. In *Proceedings of the Symposium on Theoretical Aspects of Computer Science, Lecture Notes in Computer Science*, Vol. 166. Springer, Berlin, 140–150.

[69] Glynn Winskel. 1986. Event structures. In *Petri Nets: Central Models and Their Properties, Advances in Petri Nets, Part II, Proceedings of an Advanced Course*, Bad Honnef, Germany, 8-19 September (1986), W. Brauer, W. Reisig, G. Rozenberg (Eds.). *Lecture Notes in Computer Science*, Vol. 255. Springer, 325–392.

[70] Glynn Winskel. 1987. Petri nets, algebras, morphisms, and compositionality. *Inf. Comput.* 72 (1987), 197–238.

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