Phase Transitions of Euler-Heisenberg AdS Black Hole Under the high-order QED Correction and Quintessence Dark Energy Context

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Nov. 2021

Abstract. Two phase transitions are found in the Euler-Heisenberg-AdS (EHAdS) black hole (BH) in the extended phase space. Their physical origins are an unclear. By considering the high-order quantum electrodynamics (QED) correction and introducing the quintessence dark energy, we find that the phase transitions do not violate the first thermodynamics law, the first phase transition disappears when the high-order QED correction is considered, and the second phase transition shows similar characteristics to the phase transition of the charged AdS BH under the influence of the quintessence dark energy. These results suggest that the first phase transition of the EHAdS BH in the extended phase space is induced by the nonlinear electric field, and the second phase transition corresponds to a gravitational phase transition, which follows the Maxwell behavior.

Keywords: Euler-Heisenberg AdS black hole; High-order QED correction; Quintessence dark energy

1. Introduction

Black hole (BH) thermodynamics is considered as a bridge connecting general relativity, quantum mechanics, and classical thermodynamics. Taking the AdS BH as a thermodynamic system, Hawking and Page found that the thermodynamic characteristics of BH in an asymptotically AdS space-time are similar to a classical system [1]. Considering the cosmological constant as the thermodynamic pressure, Dolan et al. proposed a scheme in which the BH thermodynamics can be discussed in the extended phase [2,3]. Recently, many kinds of researches showed that the phase transition of the charged AdS BH in the extended phase space is similar to that of the van der Waals (vdW) system. By investigating the $P - v$ critical behavior of the phase transition for the charged AdS BH in the extended phase space, Kubiznak et al. found that the critical exponents of this phase transition is precisely the same as the vdW system [4]. Johnson found that the charged AdS BH in the extended phase
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space can be built as a heat engine model just like the vdW fluid, and the heat engine efficiency can be calculated similarly [5]. Aydıner et al. investigated the Joule-Thomson expansion of the charged AdS BH in the extended phase space and obtained the inversion and isenthalpic curves and the heating-cooling regions in the $T - P$ plane [6]. Similar conclusions are also mentioned in the literature [7–12]. It is interesting to note that the Maxwell’s equal area law of the vdW system can be extended to the phase transition of the charged AdS BH in the extended phase space. It can describe the BH transition structure and ensure that the thermodynamic first law is not broken [13–15].

Based on Dirac’s positron theory, Euler and Heisenberg proposed a new approach to describe the electromagnetic field. They derived the effective Lagrangian density of the electromagnetic field by revising Maxwell’s equations in the vacuum. This new effective Lagrangian density has the high-order terms for the nonlinear electromagnetic field [16]. Schwinger reformulated this non-perturbative one-loop effective Lagrangian density under the quantum electrodynamics (QED) framework [17]. By considering the one-loop effective Lagrangian density coupled with the Einstein field equation, Yajima et al. obtained the Euler-Heisenberg (EH) BH [18]. In addition, the one-loop effective Lagrangian density carries the main characteristics of the EH nonlinear electromagnetic field, that is, the vacuum behaves as a polarizable medium by this electromagnetic field [19]. When the field strength exceeds the critical value ($m^2c^3/\hbar$), the electromagnetic field creates pairs of particles in the vacuum, the so-called “QED effect” [20]. Ruffini et al. studied the QED effect mainly occupied the region of the EH BH and calculated the minimum mass of that BH [21]. Especially, Magos et al. obtained the EH AdS BH and discussed the thermodynamic of this BH in the extended phase space. It is found that the two phase transitions of the EHAdS BH and the critical exponents at one of the phase transitions are similar to the vdW system [22].

Nevertheless, the physical origin of the two phase transitions of the EHAdS BH is still an open question. We explore the issue in this paper. By introducing the quintessence dark energy (gravitational properties) and the high-order QED correction (nonlinear properties), we attempt to investigate the differentiation of two phase transitions of the EHAdS BH from the critical behaviors and the phase transition structures. The dark energy is one of the possible sources driving the accelerated expansion of the universe. The regular scalar field can describe the simple feature of dark energy [23]. Ratra et al. found that the non-relativistic gravitational fields cannot perturb the energy density in the cosmic scalar field [24]. Caldwell et al. showed that the scalar field model has the same characteristics as the cosmological constant, the so-called “quintessence” [25]. Considered a scenario that the charged AdS BH is surrounded by quintessence dark energy in the extended phase space, Liu et al. proposed a BH heat engine model and derived the heat engine efficiency [26]. By investigating the $P - v$ critical behavior in the extended phase space, Li obtained that the phase transition of the charged AdS BH surrounded by quintessence dark energy is still similar to the vdW system, showing that the quintessence dark energy does not influence the phase transition structure for the charged AdS BH [27].
The paper is organized as follows: In Sec. 2, we review the thermodynamics of the EHAdS BH in the extended phase space, including the $P - \nu$ critical behavior and the Maxwell’s equal area law. Sec. 3 discusses the thermodynamics of the EHAdS BH with high-order QED correction, paying attention to the change of the $P - \nu$ critical behavior and the Maxwell’s equal area law regions of the two phase transitions. In Sec. 4, the case of the EHAdS BH surrounded by quintessence dark energy is examined in the same way. Sec. 5 ends up with our conclusions and discussion.

2. General EHAdS BH

In the framework of the EHAdS BH, the line element is given by \[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega^2, \] (1)

where $f(r)$ is the metric potential, which can be written as

\[ f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{aQ^4}{20r^6} - \frac{\Lambda r^2}{3}, \] (2)

in which $M$ and $Q$ are the mass and charge of the BH, $a$ is the EH parameter, $\Lambda$ is defined as the thermodynamic pressure in the extended phase space, satisfying $\Lambda = -8\pi P$ \[2,3\].

The event horizon radius $r_+$ of the BH is derived from the largest root of equation $f(r_+) = 0$. Hence, we obtain the BH mass as

\[ M = \frac{r_+}{2} + \frac{Q^2}{2r_+} - \frac{aQ^4}{40r_+^5} - \frac{\Lambda r_+^2}{6}. \] (3)

According to the first law of BH thermodynamics, the EHAdS BH temperature as

\[ T = \frac{1}{4\pi r_+} \left( 1 - \frac{Q^2}{r_+^2} + \frac{aQ^4}{4r_+^6} - \Lambda r_+^2 \right). \] (4)

Using the condition of the extended phase space $\Lambda = -8\pi P$ and Eq. (4), the equation of the state for the general EHAdS BH can be expressed by

\[ P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} + \frac{Q^2}{8\pi r_+^4} - \frac{aQ^4}{32\pi r_+^6}. \] (5)

According to critical conditions \[ \left( \frac{\partial P}{\partial r_+} \right) = \left( \frac{\partial^2 P}{\partial r_+^2} \right) = 0 \] \[4\], the critical thermodynamic quantities of the general EHAdS BH are obtained by using Eq. (5),

\[ T_{gc} = \frac{1}{2\pi r_{gc}} \left( 1 - \frac{2Q^2}{r_{gc}^2} + \frac{aQ^4}{r_{gc}^6} \right), \] (6)

\[ P_{gc} = \frac{1}{8\pi r_{gc}^2} \left( 1 - \frac{3Q^2}{r_{gc}^2} + \frac{7aQ^4}{4r_{gc}^6} \right), \] (7)

\[ r_{gc}^2 = 2Q^2 \left[ 2 \arccos \left( \frac{1}{3} \arccos \left( 1 - \frac{7a}{16Q^2} \right) - \frac{2\pi k}{3} \right) + 1 \right], \quad k = 0; 1; 2, \] (8)

where $k = 2, r_{gc}^2 > 0$ does not hold. $k = 0$ and $k = 1$ correspond to two critical points, respectively. The critical thermodynamic quantities universal constant is defined as $\varepsilon \equiv P_c\nu_c/T_c$ in which $\nu_c$ is
specific volume, satisfying $v_c = 2r_c$ [27]. For the vdW system, this universal constant is $\varepsilon_{vdW} = 0.375$. For the EHAdS BH, we obtain the universal constant is $\varepsilon_g \simeq -0.787$ at first critical point ($k = 1$), and $\varepsilon_g \simeq 0.37$ at second critical point ($k = 0$) when the EH parameter $a = 1$ and the charge $Q = 0.8$. The result implies that the second critical point of the two phase transitions is similar to the critical point of vdW system.

Using critical condition and the equations of state, one can obtain that the equation of horizon radius, i.e.

$$g(r_+) \equiv r_+^6 - 6Q^2r_+^4 + 7aQ_+^4 = 0.$$ (9)

g($r_+$) as a function of $r_+$ are plotted in Figure 1. The $g(r_+)$ function curve intersects the horizontal axis twice (blue solid points), indicating that the general EHAdS BH has two critical points. The left point corresponds to $k = 1$, which we call the first critical point, and the right point corresponds to $k = 0$, which we call the second critical point.

![Fig 1. $g(r_+)$ as a function of $r_+$. The Euler-Heisenberg parameter is taken as $a = 1$, the BH charge is $Q = 0.8$.](image)

The $P - \nu$ and $T - S$ types Maxwell’s equal area law can be given by [15]

$$P_i(v_2 - v_1) = \int_{v_1}^{v_2} Pdv, \quad (10)$$

$$T_i(S_2 - S_1) = \int_{S_1}^{S_2} TdS, \quad (11)$$

where $P_i$ ($T_i$) represents the pressure (temperature) of isobar (isotherm), $\nu$ is specific volume, $S$ is entropy, the subscripts 1 (2) stands the start (end) phase of isobaric process (isothermal) process.

According to Eqs. (5) and (10), we construct the $P - \nu$ type equal area law for the general EHAdS BH case,

$$P_{gi} = \frac{T_g}{2r_{g1}} - \frac{1}{8\pi r_{g1}^2} + \frac{Q^2}{8\pi r_{g1}^4} - \frac{aQ^4}{32\pi r_{g1}^8},$$ (12)

$$P_{gi} = \frac{T_g}{2r_{g2}} - \frac{1}{8\pi r_{g2}^2} + \frac{Q^2}{8\pi r_{g2}^4} - \frac{aQ^4}{32\pi r_{g2}^8},$$ (13)
where \( T \) the phase transition temperature revealed by the equal area law of the first and second phase transitions. One can observe the subscripts \( c \) general, the quintessence, and the high-order QED correction cases, respectively, and the phase transition temperature (pressure) at the horizonal red and blue lines mark the phase transition process (\( \nu \)).

We define a parameter \( \gamma \) to measure the different phase transition temperature (pressure) at the \( P - \nu \) diagram, where \( Z \) is the phase transition temperature \( T \) or pressure \( P \), \( n \) is set as \( g, q, \) and \( Q \) for the general, the quintessence, and the high-order QED correction cases, respectively, and the subscripts \( c1 \) and \( c2 \) denote the two critical points. The isobaric (isothermal) curves of the general EHAdS BH on \( P - \nu \) plane are shown in Figure 2. The solid horizontal red and blue lines mark the phase transition process (\( \nu \) ranges) or pressure \( P \) (S ranges) revealed by the equal area law of the first and second phase transitions. One can observe that each phase transitions satisfies the equal area law, but they do not satisfy the the equal area law simultaneously. The \( \nu \) range (S range) revealed by the equal area law of the first phase transition increases with the temperature (pressure), but it decreases with the temperature (pressure) for the second phase transition.

The increase of temperature leads to the increase of kinetic energy of micro molecules. For a classical vdw system, the molecular potential difference before and

\[
2P_g = \frac{3T_g(1 + x_g)}{2r_g2(\sum_{i=0}^{2} x_i)} - \frac{3}{4\pi r_g^2(\sum_{i=0}^{2} x_i)} + \frac{3Q^2}{4\pi r_g^4 x_g(\sum_{i=0}^{2} x_i)} - \frac{3aQ^4(\sum_{i=0}^{2} x_i)}{80\pi r_g^8 x_g^2(\sum_{i=0}^{2} x_i)}.
\]

(14)

where \( x_g \equiv r_g1/r_g2 \) (\( 0 < x_g < 1 \)). Based on Eqs. (12) and (13), one can get

\[
T_g = \frac{1 + x_g}{4\pi r_g x_g} + \frac{aQ^4(\sum_{i=0}^{2} x_i)}{16\pi r_g^2 x_g^2} - \frac{Q^2(\sum_{i=0}^{2} x_i)}{4\pi r_g^3 x_g^3},
\]

(15)

\[
2P_g = \frac{T_g(1 + x_g)}{2r_g2 x_g} - \frac{1 + x_g^2}{8\pi r_g^2 x_g^2} + \frac{Q^2(1 + x_g^4)}{8\pi r_g^4 x_g^4} - \frac{aQ^4(1 + x_g^8)}{32\pi r_g^8 x_g}. \tag{16}
\]

Using the Eqs. (14)-(16), \( r_g2 \) can be obtained,

\[
x_g^2 = \frac{1 + 4x_g + x_g^2}{6x_g^2} B \tag{17}
\]

where \( B \equiv 2Q^2[1 + 2\cos(\arccos(1 - (27aC)/[40Q^2]))/3 - 2k\pi/3] \) and \( C \equiv (5 + 20x_g + 29x_g^2 + 32x_g^3 + 29x_g^4 + 20x_g^5 + 5x_g^6)/(1 + 4x_g + x_g^2)^3 \). The phase transition temperature can be derived as

\[
T_g = \sqrt[3]{6(1 + x_g)} \frac{3\sqrt[6]{6Q^2(1 + x_g)(1 + x_g^2)}}{2\pi B^{7/2}(1 + 4x_g + x_g^2)^{7/2}} + \frac{27\sqrt[6]{6aQ^4(1 + x_g)(1 + x_g^2)(1 + x_g^4)}}{2\pi B^{7/2}(1 + 4x_g + x_g^2)^{7/2}}. \tag{18}
\]

We also construct the \( T - S \) type equal area law for the general EHAdS BH case with Eqs. (5) and (11). The phase transition pressure can be written as

\[
P_g = \frac{3x_g}{4\pi B(1 + 4x_g + x_g^2)} - \frac{9Q^2 x_g(1 + x_g + x_g^2)}{2\pi B^2(1 + 4x_g + x_g^2)^2} + \frac{81aQ^4 x_g(\sum_{i=0}^{2} x_i)}{2\pi B^4(1 + 4x_g + x_g^2)^4}. \tag{19}
\]
after phase transition does not change significantly with temperature. Thus, molecules with higher kinetic energy are easier to cross the molecular potential barrier, making the $\nu$ range of the phase transition decrease with temperature. For the general EHAdS BH, the $\nu$ range of the second phase transition decreases with temperature, being consistent with the vdW system. Instead, the $\nu$ range of first phase transition increases with temperature, being different from the vdW system. This implies the existence of a new molecular potential difference, which is related to the temperature. Similarly, the increase of pressure leads to the increase of collision probability of micro molecules. The $S$ range of first phase transition increases with pressure, which implies the new molecular potential difference is also related to the pressure.

![Graph](image.png)

**Fig 2.** The equal area law of the $P - \nu$ diagram (left) and $T - S$ diagram (right). The red dash-dotted (blue dotted) curves show that the Maxwell's equal area law is satisfied at the first (second) phase transition. We take $a = 1$ and $Q = 0.8$.

### 3. EHAdS BH with high-order QED correction

The Coulomb’s law correction at $r \to \infty$ can be regarded as resisting the coupling of a nonlinear electric field to BH. Considering this resist to the general EHAdS BH, the metric potential $f(r)$ is given by

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{aQ^4}{20r^5} - \frac{\Lambda r^2}{3} + \frac{\beta a^2 Q^6}{36r^{10}},$$

where $\beta$ ($0 < \beta < 1$) is the dissipation parameter. In this scenario, the BH mass, temperature, and the equation of the state can be written as

$$M = \frac{r_+}{2} + \frac{Q^2}{2r_+} - \frac{aQ^4}{40r_+^5} - \frac{\Lambda r_+^2}{6} + \frac{\beta a^2 Q^6}{72r_+^{10}},$$

$$T = \frac{1}{4\pi r_+} \left( 1 - \frac{Q^2}{r_+^2} + \frac{aQ^4}{4r_+^6} - \frac{\Lambda r_+^2}{4r_+^{10}} - \frac{\beta a^2 Q^6}{4r_+^{12}} \right),$$

$$P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} + \frac{Q^2}{8\pi r_+^4} - \frac{aQ^4}{32\pi r_+^8} + \frac{\beta a^2 Q^6}{32\pi r_+^{12}}.$$
According to Eq. (23) and the critical conditions, the critical thermodynamic quantities of the EHAdS BH with high-order QED correction are obtained, i.e.,

\[
T_{Qc} = -\frac{3a^2 Q^6 \beta}{4\pi r_{q_c}^{11}} + \frac{aQ^4}{2\pi r_{q_c}^{7}} - \frac{Q^2}{\pi r_{q_c}^{3}} + \frac{1}{2\pi r_{q_c}},
\]

(24)

\[
P_{Qc} = -\frac{11a^2 Q^6 \beta}{32\pi r_{q_c}^{12}} + \frac{7aQ^4}{32\pi r_{q_c}^{8}} - \frac{3Q^2}{8\pi r_{q_c}^{4}} + \frac{1}{8\pi r_{q_c}^{2}},
\]

(25)

\[
r_{q_c}^2 = r_{gc}^2 + (-1)^k \frac{\sqrt{66a^2Q^6(7aQ^4 - 36Q^2r_{gc}^4 + 10r_{gc}^6)\beta + A^2 - A}}{14aQ^4 - 72Q^2r_{gc}^4 + 20r_{gc}^6},
\]

(26)

where \( A \equiv 14aQ^4r_{gc}^2 - 24Q^2r_{gc}^6 + 5r_{gc}^8 \). The numerical results of the critical thermodynamic quantities for the different \( \beta \) are also listed in Table 1.

| \( \beta \) | \( r_{Qc1}(k = 1) \) | \( P_{Qc1} \) | \( T_{Qc1} \) | \( \varepsilon_{Q1} \) | \( r_{Qc2}(k = 0) \) | \( P_{Qc2} \) | \( T_{Qc2} \) | \( \varepsilon_{Q2} \) |
|---|---|---|---|---|---|---|---|---|
| 1  | -  | -  | -  | -  | 1.9103 | 0.005315 | 0.05474 | 0.371 |
| 0.8| -  | -  | -  | -  | 1.9088 | 0.005318 | 0.05475 | 0.371 |
| 0.6| -  | -  | -  | -  | 1.9074 | 0.005320 | 0.05476 | 0.371 |
| 0.4| -  | -  | -  | -  | 1.9060 | 0.005323 | 0.05477 | 0.371 |
| 0.2| -  | -  | -  | -  | 1.9045 | 0.005325 | 0.05478 | 0.370 |
| 0  | 1.0029 | -0.008088 | 0.02062 | -0.787 | 1.9030 | 0.005328 | 0.05479 | 0.370 |

We found the first critical point disappears when \( \beta \) is not zero, implying that the high-order QED correction leads to the disappearance of the phase transition of the nonlinear property of the EHAdS BH. When the \( \beta = 0 \), the critical physical quantities returns to the general EHAdS BH. Note that the change of \( \beta \) is insensitive to the second critical point. It means that the high-order QED correction does not influence to the second critical point. For the critical thermodynamic quantities universal constant, \( \varepsilon_{Q1} \) disappears with the first critical point, while \( \varepsilon_{Q2} \) is approximatively equal to 0.370. It is similar with the VdW system.

Using the critical condition and the equations of state in this scenario, one can obtain the horizon radius equation,

\[
Q(r_+) \equiv r_+^6 - 6Q^2r_+^4 + 7aQ^4 - \frac{33a^2 Q^6 \beta}{2r_+^4} = 0.
\]

(27)

\( Q(r_+) \) as a function of \( r_+ \) is plotted in Figure 3. We can see that the second phase transition point still sustains but the first phase transition point disappears for different \( \beta \) values, implying that the physical origin of the first critical point is induced by the nonlinear electric field and the second critical point is not affected by the nonlinear electric field.
Fig 3. $Q(r_+)$ as a function of $r_+$ under different values of $\beta$ for $a = 1$ and $Q = 0.8$.

The $P - v$ type Maxwells equal area law in this scenario is constructed according to Eqs. (10) and (29), i.e.,

$$P_{Qi} = \frac{T_Q}{2r_Q} - \frac{1}{8\pi r_Q^2} + \frac{Q^2}{8\pi r_Q^4} - \frac{aQ^4}{32\pi r_Q^6} + \frac{\beta a^2 Q^6}{32\pi r_Q^{12}},$$  \tag{28}

$$P_{Qi} = \frac{T_Q}{2r_Q} - \frac{1}{8\pi r_Q^2} + \frac{Q^2}{8\pi r_Q^4} - \frac{aQ^4}{32\pi r_Q^6} + \frac{\beta a^2 Q^6}{32\pi r_Q^{12}},$$  \tag{29}

$$2P_{Qi} = \frac{3T_Q(1 + x_Q)}{2r_Q(1 + x_Q + x_Q^2)} - \frac{3Q^2}{4\pi r_Q^3(1 + x_Q + x_Q^2)} + \frac{a^2 Q^6(\sum_{i=0}^{4} x_Q^i)}{48\pi r_Q^8 x_Q(1 + x_Q + x_Q^2)} - \frac{3aQ^4(\sum_{i=0}^{4} x_Q^i)}{80\pi r_Q^8 x_Q(1 + x_Q + x_Q^2)},$$  \tag{30}

where $x_Q \equiv r_Q/2$ (0 < $x_Q$ < 1). Based on Eqs. (28) and (29), one can get

$$T_Q = \frac{1 + x_Q}{4\pi r_Q^2 x_Q} - \frac{Q^2(\sum_{i=0}^{4} x_Q^i)}{4\pi r_Q^3 x_Q^2} + \frac{aQ^4(\sum_{i=0}^{4} x_Q^i)}{16\pi r_Q^7 x_Q^2} - \frac{a^2 Q^6(\sum_{i=0}^{4} x_Q^i)}{16\pi r_Q^{11} x_Q^2},$$  \tag{31}

$$2P_{Qi} = \frac{T_Q(1 + x_Q)}{2r_Q x_Q} - \frac{1 + x_Q^2}{8\pi r_Q^4 x_Q^2} + \frac{Q^2(1 + x_Q^4)}{8\pi r_Q^4 x_Q^2} - \frac{aQ^4(1 + x_Q^4)}{32\pi r_Q^8 x_Q^4} + \frac{a^2 Q^6(1 + x_Q^{12})}{32\pi r_Q^{12} x_Q^2}.$$  \tag{32}

Using Eqs. (30)-(32), $r_Q^2$ can be obtained,

$$r_Q^2 = r_g^2 + \frac{2Dr_g^2 + 4E r_g^6 + 5r_g^8}{2(D + 6E r_g^4 + 10r_g^6)} + \frac{(-1)^k}{2(D + 12E r_g^4 + 20r_g^6)} \sqrt{\left(2Dr_g^2 + 4E r_g^6 + 5r_g^8\right)^2 - 4F(D + 6E r_g^4 + 10r_g^6)\beta},$$  \tag{33}

in which $D \equiv aQ^4(5 + 20x_Q + 29x_Q^2 + 32x_Q^3 + 29x_Q^4 + 20x_Q^5 + x_Q^6)/(20x_Q^6)$, $E \equiv -Q^2(1 + 4x_Q + x_Q^2)/x_Q^8$, and $F \equiv -aQ^6(3 + 12x_Q + 19x_Q^2 + 24x_Q^3 + 27x_Q^4 + 28x_Q^5 + 27x_Q^6 + 24x_Q^7 + 19x_Q^8)/(12x_Q^{10}) - a^2 Q^6(4 + x_Q)/(4x_Q)$. For keeping $r_Q^2$ as positive, the $k$ value should be zero.
The $T - S$ type equal area law is derived from Eqs. (11) and (23), i.e.

$$P_Q = \frac{1}{8\pi r_Q^2} \frac{Q^2}{x^1_{Q}^2} + \frac{aQ^4}{8\pi r_Q^2 x^2_{Q}} - \frac{a^2Q^6}{32\pi r_Q^2 x^3_{Q}} - \frac{a^2Q^6}{32\pi r_Q^2 x^1_{Q}}. \quad (34)$$

Using Eq. (31), (33), and (34), the isobaric (isothermal) curves of the EHAdS BH with high-order QED correction on the $P - \upsilon$ and $T - S$ planes are shown in Figure 4. It is observed that the second phase transition satisfies the Maxwell’s equal area law, and the $\upsilon$ range (or $S$ range) of the second phase transition decreases with the temperature (or pressure), being similar to that of the vdW system and the charged AdS BH. These results indicate that the first phase transition is resulted from the nonlinear electric field.

![Fig 4. The Maxwells equal area law of the EHAdS BH with high-order QED correction. We take $a = 1$, $\beta = 1$, $Q = 0.8$.](image)

**4. EHAdS BH surrounded by quintessence dark energy**

Considering that the quintessence dark energy has gravitational properties, the general EHAdS BH solution can be extended to the quintessence dark energy solution. The metric potential $f(r)$ is modified as

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{aQ^4}{20r^6} - \frac{\Lambda r^3}{3} - \frac{b}{r^{3\omega_q + 1}}, \quad (35)$$

where $\omega_q (-1 < \omega_q < -1/3)$ is the quintessence dark energy state parameter, $b$ is the normalization parameter [27]. In this situation, the BH mass, temperature and the equation of the state are

$$M = \frac{r_+}{2} + \frac{Q^2}{2r_+} - \frac{aQ^4}{40r_+^5} - \frac{\Lambda r_+^3}{6} - \frac{b}{2r_+^{3\omega_q}}; \quad (36)$$

$$T = \frac{1}{4\pi r_+} \left( 1 - \frac{Q^2}{r_+^2} + \frac{aQ^4}{4r_+^6} - \frac{\Lambda r_+^3}{3} + \frac{3\omega_q b}{r_+^{3\omega_q + 1}} \right); \quad (37)$$

$$P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} + \frac{Q^2}{8\pi r_+^4} - \frac{aQ^4}{32\pi r_+^8} - \frac{3\omega_q b}{8\pi r_+^{3(\omega_q + 1)}} \quad (38)$$
According to the critical conditions and Eq. (38), the critical thermodynamic quantities of the EHAdS BH surrounded by quintessence dark energy can be written as

\[ T_{qc} = \frac{1}{2\pi r_{qc}} \left( 1 - \frac{2Q^2}{r_{qc}^2} + \frac{aQ^4}{r_{qc}^6} + \frac{9\omega_q(\omega_q + 1)b}{2r_{qc}^{3\omega_q+1}} \right), \]  

(39)

\[ P_{qc} = \frac{1}{8\pi r_{qc}^2} \left( 1 - \frac{3Q^2}{r_{qc}^2} + \frac{7aQ^4}{4r_{qc}^6} + \frac{3\omega_q(3\omega_q + 2)b}{r_{qc}^{3\omega_q+1}} \right). \]  

(40)

The equation of horizon radius is defined as

\[ q(r_+) \equiv r_+^6 - 6Q^2r_+^4 + 7aQ^4 + \frac{9}{2}b\omega_q(\omega_q + 1)(3\omega_q + 2)r_+^{3-3\omega_q}. \]  

(41)

Figure 5 shows that \( q(r_+) \) as a function of \( r_+ \). The critical radius \( r_{qc} \) is given by \( q(r_+) = 0 \). Two critical points can be seen in the \( q(r_+) \) curve (left blue solid point and right red solid points) for the different \( \omega_q \) values. The values of the \( \omega_q \) do not influence to the location of the first critical point, but it obviously affects the second critical point. The results suggest that the second critical point corresponds to a gravitational phase transition, which follows the Maxwell behavior.

![Fig 5. \( q(r_+) \) as a function of \( r_+ \) under different values of \( \omega_q \) for \( a = 1, b = 0.1, \) and \( Q = 0.8. \)](image)

The critical temperature \( T_{qc} \) and the universal constant \( \varepsilon_q \) as a function of \( \omega_q \) are shown in Figure 6. The left panel shows that the critical temperature decreased with the increase of \( \omega_q \) at the first critical point. For the second critical point, the critical temperature first decreases and then increases with the increase of \( \omega_q \), which has the same characteristics as the charged AdS BH surrounded by quintessence dark energy [27]. From the right panel, we can see that the \( \varepsilon_{q1} \) is negative and decreases gradually with the increase of \( \omega_q \). It is totally out of line with the characteristics of the vdW system. While \( \varepsilon_{q2} \) gradually decreases from 1 to about 0.375, which is similar to the VdW system and the charged AdS BH. Thus, the results suggest that the physical characteristics of the second critical point of the EHAdS BH is consistent with the charged AdS BH on the quintessence dark energy background.
According to Eqs. (11) and (23), we also have

\[ \begin{align*}
T_{q} & = \frac{T}{2r_{q1}} - \frac{1}{8\pi r_{q1}} + \frac{Q^2}{8\pi r_{q1}^4} - \frac{aQ^4}{32\pi r_{q1}^8} - \frac{3\omega qb}{8\pi r_{q1}^{3(\omega_q + 1)}}, \\
P_{q} & = \frac{T}{2r_{q2}} - \frac{1}{8\pi r_{q2}} + \frac{Q^2}{8\pi r_{q2}^4} - \frac{aQ^4}{32\pi r_{q2}^8} - \frac{3\omega qb}{8\pi r_{q2}^{3(\omega_q + 1)}},
\end{align*} \]

\[ \begin{align*}
2P_{q} & = \frac{3T_q(1 + x_q)}{2r_{q2}(1 + x_q + x_q^2)} - \frac{4r_{q2}^2(1 + x_q + x_q^2)}{4r_{q2}^2(1 + x_q + x_q^2)} + \frac{3Q^2}{4r_{q2}^2(1 + x_q + x_q^2)}, \\
& - \frac{3aQ^4(\sum_{i=0}^{4} x_q^i)}{80\pi r_{q2}^8 x_q^5(1 + x_q + x_q^2)} - \frac{3(1 - x_q^{3\omega q})b}{4r_{q2}^{3(\omega_q + 1)} x_q^{3\omega q}(1 - x_q^3)},
\end{align*} \]

where \( x_q \equiv r_{q1}/r_{q2} \) (0 < \( x_q < 1 \)). Based on Eqs. (42) and (43), one can get

\[ \begin{align*}
T_q & = \frac{1 + x_q}{4r_{q2}^2 x_q} - \frac{Q^2(\sum_{i=0}^{3} x_q^i)}{4r_{q2}^2 x_q^3 x_q^2} - \frac{aQ^4(\sum_{i=0}^{7} x_q^i)}{16r_{q2}^2 x_q^3 x_q^2} \\
& + \frac{3\omega_q(x_q^{3(\omega_q + 1)} - 1)b}{4r_{q2}^{3\omega_q + 2} x_q^{3\omega_q + 2}(x_q - 1)},
\end{align*} \]

\[ \begin{align*}
2P_q & = \frac{T_q(1 + x_q)}{2r_{q2}^2 x_q} - \frac{1 + x_q^2}{8r_{q2}^2 x_q^2} + \frac{Q^2(1 + x_q^4)}{8r_{q2}^2 x_q^4} - \frac{aQ^4(1 + x_q^5)}{32r_{q2}^8 x_q^5 x_q^2} \\
& - \frac{3\omega_q(x_q^{3(\omega_q + 1)} - 1)b}{8r_{q2}^{3(\omega_q + 1)} x_q^{3\omega_q + 2}(x_q - 1)}.
\end{align*} \]
Based on Eqs. (44), (45), and (47), we plot the isobaric (isothermal) curves on the $P - \nu$ ($T - S$) plane for this case in Figure 7 and Figure 8. One can see that the two critical points do not satisfy the equal area law simultaneously. The $\nu$ range ($S$ range) is insensitive to the change of $\omega_q$ for the first phase transition, while the ranges decrease with the $\omega_q$ for the second phase transition. It means that the quintessence dark energy only affects the second phase transition. As mentioned above, the quintessence dark energy has the gravitational properties, which is consistent with the phase transition of the charged AdS BH.

**Fig 7** The $P - \nu$ type of the Maxwells equal area law for the EHAdS BH surrounded by quintessence dark energy. The left and right panels correspond to $\omega_q = -0.7$ and $\omega_q = -0.4$. We take $a = 1$, $b = 0.1$, and $Q = 0.8$.

**Fig 8** The $T - S$ type of the Maxwells equal area law for the EHAdS BH surrounded by quintessence dark energy. The left and right panels correspond to $\omega_q = -0.7$ and $\omega_q = -0.4$. We adopt $a = 1$, $b = 0.1$, and $Q = 0.8$.

5. Discussion and conclusion

The critical behaviors and the phase transition structures of the EHAdS BH two phase transitions are revealed in this analysis. For the general EHAdS BH, the critical thermodynamic quantities show two different behaviors. We found that the universal
constant is negative at the first critical point, but it is 0.37 for the second critical point, implying that the second critical point of the two phase transitions is similar to the critical point of vdW system. By constructing the Maxwell’s equal area law of the two phase transitions for the general EHAdS BH, we found that each phase transitions satisfies the equal area law, but they do not satisfy the equal area law simultaneously. The \( v \) range (\( S \) range) revealed by the equal area law of the first phase transition increases with the temperature (pressure), but it decreases with the temperature (pressure) for the second phase transition.

For the EHAdS BH with high-order QED correction, we found that the high-order QED correction leads to the disappearance of the phase transition of the nonlinear property of the EHAdS BH, and it is not influence to the second critical point. The results showed that the first critical point is induced by the nonlinear electric field. The phase transition structures of the second phase transition is similar to that of the vdW system and the charged AdS BH. Based on the quintessence dark energy with gravity attribute, we found that the \( \omega_q \) significantly influences to the second critical point, suggesting that the second critical point corresponds to a gravitational phase transition, which follows the Maxwell behaviour.

Furthermore, we verify our conclusion by constructing isothermal characteristic triangles, as shown in Figure 9. The area of the characteristic triangle describes the phase transition structure at the two phase transition points. The characteristic triangle \( tri_1 \) corresponds the first phase transition and \( tri_2 \) represents the second phase transition. For the high-order QED correction case, the area of \( tri_1 \) is zero and the area of \( tri_2 \) is not changed. For the EHAdS BH surrounded by the quintessence dark energy, the area of \( tri_1 \) keeps unchange for different values of \( \omega_q \), but the \( tri_2 \) area changes obviously. These results are self-consistent with our analysis above.

Fig 9. \( P - v \) isothermal characteristic triangles. We take \( a = 1 \) and \( Q = 0.8 \) in our calculations.
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Acknowledgments

This work is supported by the National Natural Science Foundation of China (Grant No.12133003, 11851304, and U1731239), by the Guangxi Science Foundation and special funding for Guangxi distinguished professors (2017AD22006).

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