Long time asymptotics for the defocusing mKdV equation with finite density initial data in different solitonic regions

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Abstract

We investigate the long time asymptotics for the Cauchy problem of the defocusing modified Kortweg-de Vries (mKdV) equation with finite density initial data in different solitonic regions

\[ q_t(x, t) - 6q^2(x, t)q_x(x, t) + q_{xxx}(x, t) = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \]
\[ q(x, 0) = q_0(x), \quad \lim_{x \to \pm \infty} q_0(x) = \pm 1, \]

where \( q_0 \mp 1 \in H^{4,4} (\mathbb{R}) \). Based on the spectral analysis of the Lax pair, we express the solution of the mKdV equation in terms of a Riemann-Hilbert problem. In our previous article, we have obtained long time asymptotics and soliton resolutions for the mKdV equation in the solitonic region \( \xi \in (-6, -2) \) with \( \xi = \frac{x}{t} \). In this paper, we calculate the asymptotic expansion of the solution \( q(x, t) \) for the solitonic region \( \xi \in (-\varpi, -6) \cup (-2, \varpi) \) with \( 6 < \varpi < \infty \) being an arbitrary constant. For \( -\varpi < \xi < -6 \), there exist four stationary phase points on jump contour, and the asymptotic approximations can be characterized with an \( N \)-soliton on discrete spectrums and a leading order term \( O(t^{-1/2}) \) on continuous spectrum up to a residual error order \( O(t^{-3/4}) \). For \( -2 < \xi < \varpi \), the leading term of asymptotic expansion is described by the soliton solution and the error order \( O(t^{-1}) \) comes from a \( \bar{\partial} \)-problem. Additionally, asymptotic stability can be obtained.

Keywords: Defocusing mKdV equation, Riemann-Hilbert problem, \( \bar{\partial} \) steepest descent method, Long time asymptotics, Asymptotic stability.

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1 Introduction

In this paper, we investigate the long-time asymptotic behavior for the Cauchy problem of the defocusing modified Kortweg-de Vries (mKdV) equation in different solitonic regions under nonzero boundary conditions

\[ q_t(x, t) - 6q^2(x, t)q_x(x, t) + q_{xxx}(x, t) = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \]  
\[ q(x, 0) = q_0(x), \quad \lim_{x \to \pm \infty} q_0(x) = \pm 1, \]  

where we assume that \( q_0 \pm 1 \in H^{4,4}([\mathbb{R}]). \) The mKdV equation arises in various of physical fields, such as acoustic wave and phonons in a certain anharmonic lattice [1, 2], Alfvén wave in a cold collision-free plasma [3, 4].

Since the inverse scattering transform (IST) technique, one of the most powerful tool to study solitons of nonlinear PDEs, was firstly presented to solve the initial-value problem for the KdV equation by Gardner, Greene, Kruskal and Miurra [5], there have been some results on the mKdV equation. For instance, Wadati studied the focusing mKdV equation with zero boundary conditions and obtained simple-pole, double-pole and triple-pole solutions [6, 7], after which the \( N \)-soliton solutions and breather solutions for the focusing mKdV was derived by Demontis [8]. Recently, Zhang and Yan presented the inverse scattering transform for the focusing and defocusing mKdV equation for nonzero boundary conditions in terms of the matrix-valued Riemann-Hilbert (RH) problem [9].

Besides exact solutions, the long time asymptotic behavior of solutions of the Cauchy problem for nonlinear dispersive PDEs has been studied extensively. The research on the long-time behavior of nonlinear wave equations was firstly investigated with IST technique by Manakov in 1974 [10]. By using the method, Zakharov and Manakov presented the first result on the large time asymptotic behavior of solutions for the nonlinear Schrödinger equation with decaying initial data [11]. In a later development, Segur and Ablowitz extended this method to derive the leading asymptotics for the solution of the mKdV, KdV and Sine-Gordon equations, including full information on the phase [12]. The most influential work to investigate the long-time behavior of integrable PDEs is the nonlinear steepest
descent method which was firstly proposed by Deift and Zhou (Deift-Zhou method) in 1993 [13]. In this work, Deift and Zhou developed rigorous analytic method to present the long-time asymptotic representation of the solution for defocusing mKdV equation by deforming contours to reduce the original RH problem to a model whose solution can be derived in terms of parabolic cylinder functions for Schwarz class initial data under zero boundary conditions. In 2016, Germain studied a full asymptotic stability for solitons of the Cauchy problem for the focusing mKdV equation [14]. And Griffiths made progress in the long-time asymptotics to the mKdV equation in the same year [15]. In [16], Lenells proved a nonlinear steepest descent theorem for RH problems with Carleson jump contours and jump matrices of low regularity and slow decay. Recently, Chen and Liu extended the asymptotics to the solutions with initial data in lower regularity spaces for focusing and defocusing mKdV equation [17, 18]. For generic initial data in $H^{1,1}(\mathbb{R})$, Wang and Ma studied the asymptotics for defocusing mKdV equation under the zero boundary conditions [19].

In this paper, we study the long-time behavior of the Cauchy problem for defocusing mKdV equation (1.1) with finite density type initial data. The main tool is the nonlinear $\bar{\partial}$ steepest approach introduced by McLaughlin-Miller, which was first applied to analyze asymptotic of orthogonal polynomials [20, 21], later applied to analyze long time asymptotics of integrable systems [22–24]. Here we outline our main results as follows.

1.1 Main results

The central results of this work are the following theorems that give the long-time asymptotic behavior of the solutions $q(x, t)$ of defocusing mKdV equation (1.1) in different regions respectively.

Theorem 1.1. Let $q(x, t)$ be the solution for the Cauchy problem (1.1) with generic data $q_0(x)\mp 1 \in H^{1,4}(\mathbb{R})$ and scattering data $\{r(z), \{\eta_n, c_n\}_{n=1}^{2N}\}$. Then there exits $T = T(\xi, \varpi) > 0$ such that for $\xi \in (-\varpi, -6) \cup (-2, \varpi)$, as $T < t \to +\infty$, $q(x, t)$ has the following asymptotic representation in different space-time regions

- In Region I ($-\varpi < \xi < -6$), in which there exist four phase points on real axis. We find the asymptotic expansion

  $$q(x, t) = q^{sol}(x, t) + t^{\frac{1}{2}} f + O(t^{-\frac{3}{4}}),$$

  where $q^{sol}(x, t)$ is defined by (4.119), $f$ is defined by (4.179).
Figure 1: The space-time cone for $\xi = \frac{x}{t}$. Region I: $-\omega < \xi < -6$; Region II: $-6 < \xi < -2$; Region III: $-2 < \xi < 6$; Region IV: $6 < \xi < \omega$

- In Region II ($-6 < \xi < -2$), in which there is no phase point on real axis. We get soliton resolution
  \[
  q(x, t) = -1 + \sum_{j=0}^{N} \text{sol}(z_j, x - x_j, t) + 1 + O(t^{-1}), \tag{1.3}
  \]
  for details, please refer our work [25].

- In Region III ($-2 < \xi < 6$), in which there are no phase points on real axis. We have the asymptotic expansion
  \[
  q(x, t) = q^{\text{sol}}(x, t) + O(t^{-1}),
  \]
  where $q^{\text{sol}}(x, t)$ is defined by (4.119).

- In Region IV ($6 < \xi < \omega$), in which there exist four phase points, which all distribute on imaginary axis, rather than on the jump contour, we still have asymptotic expansion
  \[
  q(x, t) = q^{\text{sol}}(x, t) + O(t^{-1}),
  \]
  where $q^{\text{sol}}(x, t)$ is defined by (4.119).

The other main result of this work refers to the asymptotic stability which is Theorem 6.2.

**Remark 1.1.** The smoothness and decay properties of the reflection coefficient are needed in our analysis. Here we point out that
• Proposition 2.6 shows that: $q_0 \mp 1 \in H^{3,3}(\mathbb{R}) \Rightarrow q_0 \mp 1 \in L^{1,2}(\mathbb{R}) \Rightarrow r(z) \in H^1(\mathbb{R})$.

• (2.40) shows $r(z) = \mathcal{O}(z^{-2})$ as $z \to \infty$, and we can obtain that $r(z)$ also belongs to $L^{2,1}(\mathbb{R})$. Moreover, $r(z) \in H^{1,1}(\mathbb{R}) = L^{2,1}(\mathbb{R}) \cap H^1(\mathbb{R})$. It’s the Corollary 2.1.

• $q_0 \mp 1 \in H^{4,4}(\mathbb{R})$ is needed to include all conditions to show that $r(z) \in H^1(\mathbb{R})$, which can help us bound the $\overline{\partial}$ derivatives of our extensions in Proposition 4.3, Proposition 4.7 and Proposition 5.2, etc.

Remark 1.2. We set $6 < \varpi < \infty$, which implies finite velocity $\xi = \frac{x}{t}$ and ensure the following issue.

• The saddle points $\xi_1, \xi_4$ defined by (3.8) are bounded;

• The estimates on $\Im \theta(z)$, jump matrix and $\overline{\partial}$ derivatives are reasonable, see Lemma 4.1, Lemma 4.3, Proposition 4.3, Proposition 4.4, Proposition 4.5, Proposition 4.7 and Proposition 5.2, Theorem 6.2, etc.

• The higher power term of the expansion $\theta(z)$ near saddle point could decay as $t \to \infty$, by which we can set up the local model. See Proposition 4.8 and Appendix B.

• For the region $\varpi < |x/t|$, we have $|t/x| \leq 1/\varpi$, we still can study the large-$x$ asymptotic behavior in a similar way to large-$t$ asymptotics.

1.2 Outline of this paper

The structure of this work is as follows.

In section 2, we recall the elementary results on the RH problem formulation of the Cauchy problem for the defocusing mKdV equation (1.1) obtained in [9], which is the basis to analyze the asymptotic behavior of the defocusing mKdV equation in our work.

In section 3, we present the distribution of phase points and depict the decay regions of $e^{2it\theta}$ by some pictures.

In section 4.1, we introduce a function $\delta(z)$ to make the first transformation $m(z) \rightarrow m^{(1)}(z)$, which admits different triangular decomposition of jump matrix in different oscillatory regions. In section 4.2, we introduce $R^{(2)}(z)$ to make continuous extension for the jump matrix to remove the jump from the real axis in such away that the new problem takes advantage of the decay of $\exp(\pm 2it\theta)$ for $z \notin \mathbb{R}$. In particular, a mixed $\overline{\partial}$-RH problem is set up in subsection 4.2.3. In subsection 4.2.4, we decompose the $m^{(2)}$ into a pure $m^{rhp}$ and a pure $\overline{\partial}$-Problem for $m^{(3)}(z)$. In this decomposition, $m^{sol}(z) = m^{rhp}_{V(z)=I}$ is affected
by discrete spectrum, which is shown in subsection 4.3.1. $m^{\text{mod}}(z)$ uses classical parabolic cylinder model to build a matrix to match the jump relation of $m^{\text{rhp}}$, which is shown in subsection 4.3.2 and Appendix A. $m^{\text{err}}(z)$ is an error function between $m^{\text{rhp}}$ and $m^{\text{sol}}$. And the solution of $m^{\text{err}}$ satisfies a small norm RH problem which is shown in subsection 4.3.3.

As for the $\bar{\partial}$-problem $m^{(3)} = m^{(2)}(m^{\text{rhp}})^{-1}$, we will present details in the Section 4.4.

Finally, in section 6, a relation formulae is constructed as

$$m(z) = m^{(3)}(z) m^{\text{err}}(z) m^{\text{sol}}(z) R^{(2)}(z)^{-1} \delta(z)^{-\sigma_3}, \quad \text{for} \quad -\varpi < \xi < -6,$$

$$m(z) = m^{(3)}(z) m^{\text{sol}}(z) R^{(2)}(z)^{-1} \delta(z)^{-\sigma_3}, \quad \text{for} \quad -2 < \xi < \varpi,$$

from which we obtain the long time asymptotic behavior for the defocusing mKdV equation (1.1) via a potential recovering formulae.

## 2 Direct and Inverse Scattering transform

### 2.1 Notations

We recall some notations. $\sigma_1, \sigma_2, \sigma_3$ are classical Pauli matrices as follows

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We introduce the Japanese bracket $\langle x \rangle := \sqrt{1 + |x|^2}$ and the normed spaces:

- A weighted $L^{p,s}(\mathbb{R})$ is defined by
  $$L^{p,s}(\mathbb{R}) = \{ u \in L^p(\mathbb{R}) : \langle x \rangle^s u(x) \in L^p(\mathbb{R}) \}.$$
  And $\| u \|_{L^{p,s}(\mathbb{R})} : = \| \langle x \rangle^s u \|_{L^p(\mathbb{R})}$.

- A Sobolev space is defined by
  $$W^{m,p}(\mathbb{R}) = \{ u \in L^p(\mathbb{R}) : \partial^j u(x) \in L^p(\mathbb{R}) \quad \text{for} \quad j = 0, 1, 2, \ldots, m \}.$$
  And $\| u \|_{W^{m,p}(\mathbb{R})} := \sum_{j=0}^m \| \partial^j u \|_{L^p(\mathbb{R})}$. Additionally, we are used to expressing $H^m(\mathbb{R}) := W^{m,2}(\mathbb{R})$.

- A weighted Sobolev space is defined by
  $$H^{m,s}(\mathbb{R}) := L^{2,s}(\mathbb{R}) \cap H^m(\mathbb{R})$$

In this paper, we use $a \lesssim b$ to express $\exists c = c(\xi, \varpi) > 0$, s.t. $a \leq cb$. We take $\Re$ and $\Im$ to express the real part and imaginary part of a complex variable respectively.
2.2 The Lax pair and spectral analysis

The defocusing mKdV equation (1.1) admits the following Lax pair [26]

\[ \Phi_x = X\Phi, \quad \Phi_t = T\Phi, \]  

(2.1)

where

\[ X = ik\sigma_3 + Q, \quad T = 4k^2X - 2ik\sigma_3(Q_x - Q^2) + 2Q^3 - Q_{xx}, \]

and \( k \in \mathbb{C} \) is a spectral parameter.

By using the boundary condition of \( q \), the Lax pair (2.1) becomes

\[ \Phi_{\pm,x} \sim X_{\pm}\Phi_{\pm}, \quad \Phi_{\pm,t} \sim T_{\pm}\Phi_{\pm}, \quad x \to \pm \infty, \]  

(2.2)

where

\[ X_{\pm} = ik\sigma_3 + Q_{\pm}, \quad T_{\pm} = (4k^2 + 2)X_{\pm}, \]

and

\[ Q_{\pm} = \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix} \]

The eigenvalues of the \( X_{\pm} \) are \( \pm i\lambda \), which satisfy the equality

\[ \lambda^2 = k^2 - 1. \]  

(2.3)

Since the eigenvalue \( \lambda \) is multi-valued, we introduce the following uniformization variable to ensure that our discussion is based on a complex plane rather than a Riemann surface

\[ z = k + \lambda, \]  

(2.4)

and obtain two single-valued functions

\[ \lambda(z) = \frac{1}{2}(z - \frac{1}{z}), \quad k(z) = \frac{1}{2}(z + \frac{1}{z}). \]  

(2.5)

We can define two domains \( D_+, D_- \) and their boundary \( \Sigma \) on \( z \)-plane by

\[ D_+ = \{ z \in \mathbb{C}, \Im \lambda(1 + \frac{1}{|z|^2}) > 0 \}, \]

\[ D_- = \{ z \in \mathbb{C}, \Im \lambda(1 + \frac{1}{|z|^2}) < 0 \}, \]

\[ \Sigma = \mathbb{R}\backslash\{0\}. \]
which are shown in the Figure 2.

Figure 2: The domains $D^+$, $D^-$ and boundary $\Sigma = \mathbb{R}\setminus\{0\}$

We derive the Jost solution of the asymptotic spectral problem (2.2)

$$\Phi^\pm \sim E^\pm(z)e^{i\lambda(z)x\sigma_3}, \quad (2.6)$$

where

$$E^\pm = \begin{bmatrix} 1 & \pm i \\ \mp i & 1 \end{bmatrix}.$$

We introduce the modified Jost solution

$$\mu^\pm = \Phi^\pm e^{-i\lambda(z)x\sigma_3}, \quad (2.7)$$

then we have

$$\mu^\pm \sim E^\pm, \quad \text{as} \quad x \to \pm\infty,$$

$$\det(\Phi^\pm) = \det(\mu^\pm) = \det(E^\pm) = 1 - \frac{1}{z^2},$$

and $\mu^\pm$ admit the Volterra type integral equations

$$\mu^\pm(x; z) = E^\pm(z) + \int_{\pm\infty}^{x} E^\pm(z)e^{i\lambda(z)(x-y)\sigma_3} \left[(E^{-1}_\pm(z)\Delta Q^\pm(y)\mu^\pm(y; z)\right] dy, \quad z \neq \pm 1,$$

$$\mu^\pm(x; z) = E^\pm(z) + \int_{\pm\infty}^{x} [I + (x-y)(Q^\pm \pm i\sigma_3)] \Delta Q^\pm(y)\mu^\pm(y; z) dy, \quad z = \pm 1, \quad (2.8)$$
Proposition 2.1. Given \( n \in \mathbb{N}_0 \), let \( q \equiv 1 \in L^{1,n+1}(\mathbb{R}) \), \( q' \in W^{1,1}(\mathbb{R}) \).

- For \( z \in \mathbb{C} \setminus \{0\} \), \( \mu_{+,1}(x,t;z) \) and \( \mu_{-,2}(x,t;z) \) can be analytically extended to \( \mathbb{C}^+ \) and continuously extended to \( \mathbb{C}^+ \cup \Sigma; \mu_{-,1}(x,t;z) \) and \( \mu_{+,2}(x,t;z) \) can be analytically extended to \( \mathbb{C}^- \) and continuously extended to \( \mathbb{C}^- \cup \Sigma \).

- The map \( q \rightarrow \frac{\partial q}{\partial x} \mu_{\pm,i}(z) \) (\( i = 1,2,n \geq 0 \)) are Lipschitz continuous, specifically, for any \( x_0 \in \mathbb{R} \), \( \mu_{-,1}(z) \) and \( \mu_{+,2}(z) \) are continuously differentiable mappings:

\[
\begin{align*}
\partial_x^n \mu_{-,1} : \mathbb{C}^- \setminus \{0\} &\rightarrow L^\infty_{loc}(\mathbb{C}^- \setminus \{0\}, C^1((-\infty,x_0], \mathbb{C}^2)) \cap W^{1,\infty}((-\infty,x_0], \mathbb{C}^2)), \\
\partial_x^n \mu_{+,2} : \mathbb{C}^- \setminus \{0\} &\rightarrow L^\infty_{loc}(\mathbb{C}^- \setminus \{0\}, C^1([x_0,\infty), \mathbb{C}^2)) \cap W^{1,\infty}([x_0,\infty), \mathbb{C}^2)).
\end{align*}
\]

(2.10) \( \mu_{+,1}(z) \) and \( \mu_{-,2}(z) \) are continuously differentiable mappings:

\[
\begin{align*}
\partial_x^n \mu_{+,1} : \mathbb{C}^+ \setminus \{0\} &\rightarrow L^\infty_{loc}(\mathbb{C}^+ \setminus \{0\}, C^1([x_0,\infty), \mathbb{C}^2)) \cap W^{1,\infty}([x_0,\infty), \mathbb{C}^2)), \\
\partial_x^n \mu_{-,2} : \mathbb{C}^+ \setminus \{0\} &\rightarrow L^\infty_{loc}(\mathbb{C}^+ \setminus \{0\}, C^1((-\infty,x_0], \mathbb{C}^2)) \cap W^{1,\infty}((-\infty,x_0], \mathbb{C}^2)).
\end{align*}
\]

(2.13)

- Let \( K \) be a compact neighborhood of \( \{-1,1\} \) in \( \mathbb{C}^+ \setminus \{0\} \). Set \( x^\pm = \max\{\pm x,0\} \), then there exists a \( C \) such that for \( z \in K \) we have

\[
|\mu_{+,1}(z) - (1, z^{-1})^T| \leq C(x^-)^{-1}e^{C \int_{x^-}^{x^+} |q^{-1}|^2 dy} \|q - \tilde{q}\|_{L^{1,1}(x,\infty)},
\]

(2.14)
i.e., the map \( z \rightarrow \mu_{+,1}(z) \) extends as a continuous map to the points \( \pm 1 \) with values in \( C^1([x_0,\infty), \mathbb{C}) \cap W^{1,n}([x_0,\infty), \mathbb{C}) \) for any preassigned \( x_0 \in \mathbb{R} \). Moreover, the map \( q \rightarrow \mu_+^i(z) \) is locally Lipschitz continuous from:

\[
L^{1,1}(\mathbb{R}) \rightarrow L^{\infty}(\mathbb{C}^+ \setminus \{0\}, C^1([x_0,\infty), \mathbb{C}) \cap W^{1,\infty}([x_0,\infty), \mathbb{C}).
\]

(2.15)

Analogous statements hold for \( \mu_{-,2} \) and for \( \mu_{-,1} \) \( (j = 1,2) \). Furthermore, the maps \( z \rightarrow \partial_x^n \mu_{+,1}(z) \) and \( q \rightarrow \partial_x^n \mu_{+,1}(z) \) also satisfy:

\[
|\partial_x^n \mu_{+,1}(z)| \leq F_n \left[ (1 + |x|)^{n+1} \|q - \tilde{q}\|_{L^{1,n+1}(x,\infty)} \right], \quad z \in K.
\]

(2.16)

It can be shown that the eigenfunction \( \mu_{\pm} \) admit the symmetry
**Proposition 2.2** (The symmetries of $\mu_\pm$, [9]). The modified Jost functions defined by (2.8) and (2.9) admit two reduction conditions on the $z$-plane:

The first symmetry reduction:

$$\mu_\pm(z) = \sigma_1 \bar{\mu}_\pm(\bar{z}) \sigma_1 = \bar{\mu}_\pm(-\bar{z}),$$

(2.17)

The second symmetry reduction:

$$\mu_\pm(z) = \frac{-1}{z} \mu_\pm(z^{-1}) \sigma_2.$$  

(2.18)

The asymptotic behavior of $\mu_{\pm,j}$, $j = 1, 2$ could be described by following proposition.

**Proposition 2.3.** Suppose that $q \mp 1 \in L^{1,n+1}(\mathbb{R})$ and $q' \in W^{1,1}(\mathbb{R})$. Then as $z \to \infty$, with $\Im z \geq 0$ we have

$$\mu_{+,1}(z) = e_1 + \frac{1}{z} \left( -i \int_x^\infty (q^2 - 1) dx \right) + O(z^{-2}),$$

(2.19)

$$\mu_{-,2}(z) = e_2 + \frac{1}{z} \left( i \int_x^\infty (q^2 - 1) dx \right) + O(z^{-2}),$$

(2.20)

and for $\Im z \leq 0$ as $z \to \infty$ we have

$$\mu_{-,1}(z) = e_1 + \frac{1}{z} \left( -i \int_x^\infty (q^2 - 1) dx \right) + O(z^{-2}),$$

(2.21)

$$\mu_{+,2}(z) = e_2 + \frac{1}{z} \left( i \int_x^\infty (q^2 - 1) dx \right) + O(z^{-2}).$$

(2.22)

For $z \in \mathbb{C}^+$, as $z \to 0$, we have

$$\mu_{+,1}(z) = -\frac{i}{z} e_2 + O(1), \quad \mu_{-,2}(z) = -\frac{i}{z} e_1 + O(1);$$

(2.23)

for $z \in \mathbb{C}^-$, as $z \to 0$, we have

$$\mu_{-,1}(z) = \frac{i}{z} e_2 + O(1), \quad \mu_{+,2}(z) = \frac{i}{z} e_1 + O(1);$$

(2.24)

where $e_1 = (1, 0)^T$, $e_2 = (0, 1)^T$.

The columns of $\Phi_+(z)$ and $\Phi_-(z)$ each form a solutions basis of (2.2) for $z \in \Sigma\setminus\{\pm 1\}$. It follows that the matrices must satisfy the linear relation

$$\Phi_+(x, t; z) = \Phi_-(x, t; z) S(z).$$

(2.25)

Owe to (2.7), $\Phi_\pm$ admit the following symmetry.
**Proposition 2.4** (The symmetries of $\Phi_\pm$). The first symmetry reduction:

$$\Phi_\pm(z) = \sigma_1\overline{\Phi_\pm(\bar{z})}\sigma_1 = \Phi_\pm(-\bar{z}),$$  \hspace{1cm} (2.26)

The second symmetry reduction:

$$\Phi_\pm(z) = \mp \frac{1}{z} \Phi_\pm(z^{-1})\sigma_2.$$  \hspace{1cm} (2.27)

Now we introduce the symmetries of $S(z)$

**Proposition 2.5** (The symmetries of $S(z)$). The scattering matrix $S(z)$ admits the symmetries as follows

$$S(z) = \sigma_1 \overline{S(\bar{z})}\sigma_1 = \overline{S(-\bar{z})} = -\sigma_2 S(z^{-1})\sigma_2.$$  \hspace{1cm} (2.28)

With the symmetries of $S(z)$, we can see that

$$S(z) = \begin{bmatrix} a(z) & b(z) \\ b(z) & a(z) \end{bmatrix}, \quad z \in \Sigma \backslash \{\pm 1\}.$$  \hspace{1cm} (2.29)

The scattering coefficients $a(z)$ and $b(z)$ define the reflection coefficient as well as transmission coefficient

$$r(z) := \frac{b(z)}{a(z)}, \quad \tilde{r}(z) := \frac{b(\bar{z})}{a(\bar{z})}.$$  \hspace{1cm} (2.30)

The following lemma records several properties of $a(z)$, $b(z)$ and $r(z)$

**Lemma 2.1.** Let $z \in \Sigma \backslash \{\pm 1\}$ and $a(z)$, $b(z)$ and $r(z)$ be the data as mentioned above. Then

1) The scattering coefficients can be expressed in terms of the Jost functions as

$$a(z) = \frac{\det(\Phi_{+,1}, \Phi_{-,2})}{1 - z^{-2}}, \quad b(z) = \frac{\det(\Phi_{-,1}, \Phi_{+,2})}{1 - z^{-2}}.$$  \hspace{1cm} (2.31)

2) For each $z \in \Sigma \backslash \{\pm 1\}$, we have

$$\det S(z) = \vert a(z) \vert^2 - \vert b(z) \vert^2 = 1, \quad \vert r(z) \vert^2 = 1 - \vert a(z) \vert^{-2} < 1.$$  \hspace{1cm} (2.32)

3) $a(z)$, $b(z)$ and the reflection coefficient $r(z)$ satisfy the symmetries

$$a(z) = \overline{a(-\bar{z})} = -\overline{a(z^{-1})}$$  \hspace{1cm} (2.33)

$$b(z) = \overline{b(-\bar{z})} = \overline{b(z^{-1})}$$  \hspace{1cm} (2.34)

$$r(z) = \overline{r(-\bar{z})} = -\overline{r(z^{-1})}.$$  \hspace{1cm} (2.35)
4) The scattering data have the asymptotics

\[
\begin{align*}
\lim_{z \to \infty} (a(z) - 1)z &= i \int_{\mathbb{R}} (q^2 - 1)dx, \quad z \in \mathbb{C}^+, \\
\lim_{z \to 0} (a(z) + 1)z^{-1} &= i \int_{\mathbb{R}} (q^2 - 1)dx, \quad z \in \mathbb{C}^+, \\
|b(z)| &= O(|z|^{-2}), \quad \text{as } |z| \to \infty, \quad z \in \mathbb{R}, \\
|b(z)| &= O(|z|^2), \quad \text{as } |z| \to 0, \quad z \in \mathbb{R}.
\end{align*}
\]

So that

\[
r(z) \sim z^{-2}, \quad |z| \to \infty; \quad r(z) \sim 0, \quad |z| \to 0.
\]

Proof. The first property follows from applying Cramer’s rule to (2.25). The second property can be obtained by direct calculation. The third property comes from the symmetries of \(S(z)\). The fourth property follows from some simple calculation.

Although \(a(z)\) and \(b(z)\) have singularities at points \(\pm 1\), we can claim that the reflection coefficient \(r(z)\) remains bounded at \(z = \pm 1\) and \(|r(\pm 1)| = 1\). In fact, by direct calculation, we obtain

\[
\begin{align*}
a(z) &= \frac{\pm S_{\pm}}{z \mp 1} + O(1), \\
b(z) &= \frac{\mp S_{\pm}}{z \mp 1} + O(1)
\end{align*}
\]

where \(S_{\pm} = \frac{1}{2} \text{det} [\mu_{\pm,1}(\pm 1, x), \mu_{\pm,2}(\pm 1, x)]\). Then we have

\[
\lim_{z \to \pm 1} r(z) = \mp i.
\]

Remark 2.1. The above discussions suggest that the Jost functions exhibit singular behavior for \(z\) near \(\pm 1, 0\). The singularities of these solutions at \(z = \pm 1\) can be removable, however, the singular behavior of these solutions at \(z = 0\) plays a non-trivial and unavoidable role in our analysis.

The next proposition shows that, given data \(q_0\) with sufficient smoothness and decay properties, the reflection coefficients will also be smooth and decaying.

Proposition 2.6. For given \(q, q' \in L^{1,2}(\mathbb{R}), q' \in W^{1,1}(\mathbb{R})\), we then have \(r(z) \in H^1(\mathbb{R})\).

Proof. The proof is the same with [25, Proposition 3.2]
Remark 2.2. $\|r\|_{H^1(\mathbb{R})}$ is always be used in the following estimation, such as Proposition 4.1, Proposition 4.3, Proposition 4.7, Proposition 5.2, etc.

In fact, we can claim that $r(z) \in H^{1,1}(\mathbb{R})$.

Corollary 2.1. For given $q \pm 1 \in L^{1,2}(\mathbb{R})$, $q' \in W^{1,1}(\mathbb{R})$, we then have $r(z) \in H^{1,1}(\mathbb{R})$.

Proof. Since $H^{1,1}(\mathbb{R}) = L^{2,1}(\mathbb{R}) \cap H^1(\mathbb{R})$, what we need to prove is $r \in L^{2,1}(\mathbb{R})$. With (2.40), we can see that

$$|z|^2 r^2(z) \sim |z|^{-2}, \quad |z| \to \infty$$

Thus

$$\int_{\mathbb{R}} |(z)r(z)|^2 < \infty,$$

which help us obtain the result.

In a similar way [25, 27], we can show that zeros of $a(z)$ are finite and simple, all of which are placed on $\{z : |z| = 1\}$.

Suppose that $a(z)$ has finite $N$ simple zeros $z_1, z_2, \ldots, z_N$ on $D_+ \cap \{z : |z| = 1, \Im z > 0, \Re z > 0\}$. The symmetries of $S(z)$ imply that

$$a(z_n) = 0 \iff a(\bar{z}_n) = 0 \iff a(-z_n) = 0 \iff a(-\bar{z}_n) = 0, \quad n = 1, \ldots, N.$$

Therefore we give the discrete spectrum as

$$\mathcal{Z} = \{z_n, \bar{z}_n - z_n, -z_n\}_{n=1}^N,$$

where $z_n$ satisfies that $|z_n| = 1, \Re z_n > 0, \Im z_n > 0$.

It is convenient to define that

$$\eta_n = \begin{cases} z_n, & n = 1, \ldots, N, \\ -\bar{z}_{n-N}, & n = N + 1, \ldots, 2N, \end{cases}$$

Finally we express the discrete spectrum as

$$\mathcal{Z} = \{\eta_n, \bar{\eta}_n\}_{n=1}^{2N}.$$

And the distribution of $\mathcal{Z}$ on the $z$-plane is shown in Figure 3.
As shown in [9], denoting norming constant \( c_n = b_n/a'(\eta_n) \). Then we have residue condition as follows

\[
\text{Res}_{z=\eta_n} \left[ \frac{\mu_{+1}(z)}{a(z)} \right] = c_n e^{-2i\lambda(\eta_n)x} \mu_{-2}(\eta_n), \quad (2.49)
\]

\[
\text{Res}_{z=\bar{\eta}_n} \left[ \frac{\mu_{+2}(z)}{a(\bar{z})} \right] = \bar{c}_n e^{2i\lambda(\bar{\eta}_n)x} \mu_{-1}(\bar{\eta}_n). \quad (2.50)
\]

And the collection \( \sigma_d = \{\eta_n, c_n\}_{n=1}^{2N} \) is called the scattering data.

Now we try to carry out the time evolution of the scattering data. If \( q \) also depends on time variable \( t \), we can obtain the functions \( a(z) \) and \( b(z) \) mentioned above for all times \( t \in \mathbb{R} \). This can be seen applying \( \partial_t \) to (2.1) and some standard arguments can be taken, for example [28, 29]. Then time dependence of scattering data which can be expressed in terms of the following replacement

\[
c(\eta_n) \to c(t, \eta_n) = c(0, \eta_n)e^{\lambda(\eta_n)(4k^2(\eta_n)+2)t}, \quad (2.51)
\]

\[
r(z) \to r(t, z) = r(0, z)e^{(4k^2+2)t} \quad (2.52)
\]

**Remark 2.3.** At time \( t = 0 \), the initial function \( q(x, 0) \) produces \( 4N \) simple zeroes of \( a(z, 0) \). If \( q \) evolves in terms of the (1.1), then \( q(x, t) \) will produce exactly the same \( 4N \) simple zeroes at time \( 0 \neq t \in \mathbb{R} \) for \( a(z, t) \). And the scattering data with time variable \( t \) can be given by

\[
\left\{ r(z)e^{(4k^2+2)t}, \{\eta_n, c(\eta_n)e^{\lambda(\eta_n)(4k^2(\eta_n)+2)t}\}_{n=1}^{2N} \right\}
\]
where \( \{ r(z), \{ \eta_n, c_n \}_{n=1}^{2N} \} \) are corresponded to initial data \( q_0(x) \).

Denote the phase function
\[
\theta(z) = \lambda(z) \left[ \frac{x}{l} + 4k^2(z) + \frac{2}{l} \right],
\]
(2.53)
and for convenience we write \( \theta_n := \theta(\eta_n) \).

Before we construct the matrix-valued RH problem, we conclude the asymptotic behaviors of the modified Jost solution and scattering matrix as \( z \to 0 \) and \( z \to \infty \).

**Proposition 2.7.** The modified Jost solutions \( \mu_{\pm} \) have the following asymptotic behaviors
\[
\mu_{\pm}(x, t; z) = I + O(z^{-1}), \quad z \to \infty,
\]
(2.54)
\[
\mu_{\pm}(x, t; z) = \frac{i}{z} \sigma_3 Q_{\pm} + O(1), \quad z \to 0.
\]
(2.55)
The scattering matrix \( S(z) \) admits asymptotic behaviors
\[
S(z) = I + O(z^{-1}), \quad z \to \infty,
\]
(2.56)
\[
S(z) = -I + O(z), \quad z \to 0.
\]
(2.57)
Further we have \( r(0), \tilde{r}(0) \to 0, \quad z \to 0 \).

Moreover, from trace formulae we have
\[
a(z) = \prod_{n=1}^{2N} \frac{z - \eta_n}{z - \bar{\eta}_n} \exp \left[ - \frac{1}{2\pi i} \int_{\Sigma} \frac{\log(1 - r(s)\tilde{r}(s))}{s - z} ds \right], \quad z \in \mathbb{C}^+.
\]
(2.58)

**Remark 2.4.** The following relations are always used in our paper.
\[
\lambda(z) = \overline{\lambda(\bar{z})} = -\lambda(-z) = -\lambda(z^{-1}),
\]
\[
\theta(z) = \overline{\theta(\bar{z})}, \quad \bar{\theta}_n = \theta(\bar{\eta}_n).
\]

**2.3 Set up of the Riemann-Hilbert problem**

Define a sectionally meromorphic matrix as follows
\[
m(z) = m(x, t; z) := \begin{cases} 
\left( \frac{\mu_{+, 1}(x, t; z)}{a(z)}, \frac{\mu_{+, 2}(x, t; z)}{a(z)} \right), & z \in \mathbb{C}^+ \\
\left( \frac{\mu_{-, 1}(x, t; z)}{a(\bar{z})}, \frac{\mu_{-, 2}(x, t; z)}{a(\bar{z})} \right), & z \in \mathbb{C}^-
\end{cases},
\]
(2.59)
which solves the following RH problem.
RHP 2.1. Find a $2 \times 2$ matrix-valued function $m(x,t; z)$ such that

* $m(z)$ is analytical in $\mathbb{C}\setminus(\Sigma \cup \mathcal{Z})$ and has simple poles in $\mathcal{Z} = \{ \eta_n, \bar{\eta}_n \}_{n=1}^{2N}$.

* $m(z) = \sigma_1 \overline{m(\bar{z})} \sigma_1 = m(-\bar{z}) = \mp z^{-1} m(z^{-1}) \sigma_2$.

* The non-tangential limits $m_{\pm}(z) = \lim_{s \to z^\pm} m(s), s \in \mathbb{C}^\pm$ exist for any $z \in \Sigma$ and satisfy the jump relation $m_+(z) = m_-(z) V(z)$ where

$$
V(z) = \begin{bmatrix}
1 - |r(z)|^2 & -r(z) e^{2i\theta} \\
r(z) e^{-2i\theta} & 1
\end{bmatrix}, \quad z \in \Sigma.
$$

* Asymptotic behavior

$$
m(x,t; z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty;
$$

$$
m(x,t; z) = \frac{i}{z} \sigma_3 Q_+ + \mathcal{O}(1), \quad z \to 0.
$$

* Residue conditions

$$
\text{Res } m(z) = \lim_{z \to \eta_n} m(z) \begin{bmatrix}
0 \\
0
\end{bmatrix},
$$

$$
\text{Res } m(z) = \lim_{z \to \bar{\eta}_n} m(z) \begin{bmatrix}
0 \\
0
\end{bmatrix}.
$$

Next proposition can help us construct a reconstruction formulae for defocusing mKdV equation.

**Proposition 2.8.** Assume $q \equiv 1 \in L^{1,2}(\mathbb{R})$ and $q^\prime(x) \in W^{1,1}(\mathbb{R})$, we have the following asymptotics of $m(z)$ as $z \to \infty$ and $z \to 0$:

$$
\lim_{z \to \infty} z(m(z) - I) = \begin{bmatrix}
- i \int_{x}^{\infty} (q^2 - 1) dx \\
iq \\
-q
\end{bmatrix},
$$

$$
\lim_{z \to 0} (m(z) - \frac{\sigma_2}{z}) = \begin{bmatrix}
iq \\
i \int_{x}^{\infty} (q^2 - 1) dx \\
-iq
\end{bmatrix}.
$$

By Proposition 2.8, the potential $q(x,t)$ is founded by the reconstruction formulae

$$
q(x,t) = -i (m_1)_{12} = -i \lim_{z \to \infty} (zm)_{12},
$$

where $m_1$ appears in the expansion of $m = I + z^{-1} m_1 + O(z^{-2})$ as $z \to \infty$. 

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3 Distribution of Saddle Points and Signature Table

We notice that the long-time asymptotic behavior of RHP 2.1 is influenced by the growth and decay of the exponential function

$$e^{\pm 2it\theta}, \quad \theta(z) = \frac{1}{2} \left( \frac{x}{t} + 2 + \left( 1 + \frac{1}{z} \right)^2 \right)$$

which not only appear in jump matrix \( V(z) \) but also in the residue condition. Based on this observation, we shall make analysis for the real part of \( \pm 2it\theta \) to ensure the exponential decaying property. Therefore, in this section, we introduce a new transformation \( m(z) \rightarrow m^{(1)}(z) \), which make that the \( m^{(1)}(z) \) is well behaved as \( t \rightarrow \infty \) along the characteristic line (or steepest line). Let \( \xi = \frac{x}{t} \). Aiming at obtaining the asymptotic behavior of \( e^{2it\theta} \), we present the real part of \( 2it\theta \):

$$\Re(2it\theta(z)) = -t \left[ (\xi + 3)\Re z - (\xi + 3)|z|^{-2}\Re \bar{z} + \Im(z^3) - |z|^{-6}\Im(z^3) \right].$$

To find the stationary phase points (or saddle points), we need the \( \theta'(z) \)

$$\theta'(z) = \frac{3}{2} z^2 + \frac{\xi + 3}{2z^2} + \frac{3}{2z^4} + \frac{\xi + 3}{2}.$$

**Proposition 3.1** (Distributions of saddle points). *Besides the two fixed saddle points \( i, -i \), there exist four saddle points which satisfy the following properties for different \( \xi \):

i. For \(-\infty < \xi < -6\), the four saddle points \( \xi_j, j = 1, 2, 3, 4 \) are located on the jump contour \( \Sigma = \mathbb{R}\backslash \{0\} \). Moreover, we have \( \xi_4 < -1 < \xi_3 < 0 < \xi_2 < 1 < \xi_1 \) and \( \xi_1 = \xi_2 = -\frac{1}{\xi_3} = -\xi_4 \);

ii. For \(-6 < \xi < \infty\), the four saddle points which are away from the coordinate axis (both real and imaginary axis);

iii. For \(6 < \xi < \infty\), the four saddle points which are all located on the imaginary axis. Moreover, we have \( \Im \xi_1 > 1 > \Im \xi_2 > 0 > \Im \xi_3 > -1 > \Im \xi_4 \) and \( \xi_1 \xi_2 = \xi_3 \xi_4 = -1 \).

**Proof.** From \( \theta'(z) = 0 \), we have

$$3z^6 + (\xi + 3)z^4 + (\xi + 3)z^2 + 3 = 0.$$

Taking use the factorization technique, we obtain

$$(1 + z^2) \left( 3z^4 + (\xi + 3)z^2 + 3 \right) = 0.$$
From the equality, we have two fixed saddle points $i, -i$. And we can solve that

$$ z^2 = -\frac{\xi + \sqrt{\xi^2 - 36}}{6}, \quad \text{or} \quad z^2 = -\frac{\xi - \sqrt{\xi^2 - 36}}{6}. \quad (3.7) $$

For $-\varpi < \xi < -6$, both $-\frac{\xi + \sqrt{\xi^2 - 36}}{6}$ and $-\frac{\xi - \sqrt{\xi^2 - 36}}{6}$ are greater than zero, we can easily check that there exist four real roots which are

$$ \xi_1 = \sqrt{-\frac{\xi - \sqrt{\xi^2 - 36}}{6}}, \quad \xi_4 = -\sqrt{-\frac{\xi - \sqrt{\xi^2 - 36}}{6}} \quad (3.8) $$

Moreover, we can know that $\xi_4 < -1 < \xi_3 < 0 < \xi_2 < 1 < \xi_1$ and $\xi_1 = \frac{1}{\xi_2} = -\frac{1}{\xi_3} = -\xi_4$.

For $-6 < \xi < 6$, the discriminant $\xi^2 - 36$ is less than zero, we can know that there exist four saddle points $\xi_j = \Re(\xi_j) + i\Im(\xi_j)$, where $\Re(\xi_j), \Im(\xi_j) \neq 0, j = 1, 2, 3, 4$.

For $6 < \xi < \varpi$, both $-\frac{\xi + \sqrt{\xi^2 - 36}}{6}$ and $-\frac{\xi - \sqrt{\xi^2 - 36}}{6}$ are less than zero, we can solve four pure imaginary saddle points

$$ \xi_1 = i\sqrt{-\frac{\xi + \sqrt{\xi^2 - 36}}{6}}, \quad \xi_4 = -i\sqrt{-\frac{\xi + \sqrt{\xi^2 - 36}}{6}} \quad (3.10) $$

Moreover, we can find that $\Im\xi_1 > 1 > \Im\xi_2 > 0 > \Im\xi_3 > -1 > \Im\xi_4$ and $\xi_1\xi_2 = \xi_3\xi_4 = -1$.

**Remark 3.1.** We can see that for example, $\xi_1, \xi_4$ defined by (3.8) can’t be $\infty$ because of the finite $\xi$. This is a point to understand why we should choose a $\varpi$ to restrict the velocity.

**Remark 3.2.** For $-\varpi < \xi < -6$, we can calculate $\theta''(\xi_j) > 0$, $j = 1, 3$, on the other side, $\theta''(\xi_j) < 0$, $j = 2, 4$ through simple computation, thus we define a sign function which will be used as follows

$$ \epsilon_j = \text{sgn}(\theta''(\xi_j)) = \begin{cases} 1, & j = 1, 3, \\ -1, & j = 2, 4, \end{cases} $$

where $\theta(z)$ is the phase function defined by (3.1).

The distributions of stationary phase points are shown in Figure 4.

The decaying regions of $\Im\theta$ are shown in Figure 5. According to the Figure 4 and Figure 5, we can divide our discussion into three cases, which based on the decay regions of $\Im\theta$ rather than the distributions of saddle points.
Figure 4: Plots of the distributions for saddle points: (a) $-\infty < \xi < -6$, (b) $-6 < \xi < 6$, (c) $6 < \xi < \infty$. The red curve shows the $\Re \theta'(z) = 0$, and the green curve shows the $\Im \theta'(z) = 0$. The intersection points are the saddle points which express $\theta'(z) = 0$.

Figure 5: Plots of the $\Im \theta$ with different $\xi = x/t$: (a) $-\infty < \xi < -6$, (b) $-6 < \xi < -2$, (c) $-2 < \xi < \infty$. The black curve is unit circle. In the purple region, $\Im \theta > 0$ ($|e^{2it\theta}| \to 0$ as $t \to \infty$). and $\Im \theta < 0$ ($|e^{-2it\theta}| \to 0$ as $t \to \infty$) in the white region. $\Im \theta = 0$ on the purple dotted curve.
• $-\varpi < \xi < -6$: In this case, there exist four stationary phase points besides $i, -i$, which are all distributed on the jump contour $\Sigma$ as shown in Figure 4(a). And decay regions are shown in Figure 5(a). Additionally, $z = 0$ is a singularity point. **We will mainly discuss this case in next Section 4.**

• $-6 < \xi < -2$: The distributions of phase points are shown in Figure 4(b) and decay regions are shown in Figure 5(b). As for this case, we have discussed in [25].

• $-2 < \xi < \varpi$: In this case, there exist four stationary phase points besides $i, -i$. When $-2 < \xi < 6$, the four saddle points are away from the coordinate axis (both real and imaginary axis), which is corresponded to Figure 4(b) and the decay regions are shown in Figure 5(c). When $6 < \xi < \varpi$, the four saddle points are all distributed on the imaginary axis as shown in Figure 4(c) and the decay regions are still shown in Figure 5(c). Because the four phase points which are not on the jump contour, the main contribution to RH problem comes from the jump contour and $z = 0$. **we will present a sketch of this case in Section 5.**

4 Deformation of the RH Problem for $-\varpi < \xi < -6$

4.1 Jump matrix factorizations

Now we use factorizations of the jump matrix along the real axis to deform the contours onto those on which the oscillatory jump on the real axis is traded for exponential decay. This step is aided by two well known factorizations of the jump matrix $V(z)$ in (2.60):

$$V(z) = \begin{bmatrix} 1 & -\overline{r(z)}e^{2it\theta} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ r(z)e^{-2it\theta} & 1 \end{bmatrix}, \quad z \in \tilde{\Gamma},$$

$$= \begin{bmatrix} \frac{1}{r(z)e^{-2it\theta}} & 0 \\ 1 - |r(z)|^2 & 1 \end{bmatrix} \left(1 - |r(z)|^2\right)^\sigma_3 \begin{bmatrix} 1 & -\frac{1}{1-|r(z)|^2} \\ 0 & \frac{1}{1-|r(z)|^2} \end{bmatrix}, \quad z \in \Gamma,$$

where

$$\Gamma := (-\infty, \xi_4) \cup (\xi_3, 0) \cup (0, \xi_2) \cup (\xi_1, +\infty),$$

$$\tilde{\Gamma} := (\xi_4, \xi_3) \cup (\xi_2, \xi_1).$$

The leftmost term is the factorization can be deformed into $\mathbb{C}^-$, the rightmost term can be deformed into $\mathbb{C}^+$, while any central terms remain on the real axis. These deformations are useful when they deformed the factors into regions in which the corresponding off-diagonal exponentials $e^{\pm 2it\theta}$ are decaying as $t \to \infty$. 

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Define the function\[ \delta(z) := \delta(z, \xi) = \exp \left( -\frac{1}{2\pi} \int_{\Gamma} \log (1 - |r(s)|^2) \frac{1}{s - z} ds \right). \] (4.3)

Taking \( \nu(z) = \frac{1}{2\pi} \log(1 - |r(z)|^2) < 0 \), then we can express \[ \delta(z) = \exp \left( i \int_{\Gamma} \nu(s) ds \right). \] (4.4)

In the above formulae, we choose the principal branch of power and logarithm functions.

**Proposition 4.1.** The function defined by (4.4) admits following properties:

i). \( \delta(z) \) is analytical in \( C \setminus \Gamma \);

ii). \( \delta_-(z, \xi) = \delta_+(z, \xi) (1 - |r(z)|^2), \ z \in \Gamma; \)

iii). \( \delta(z) = \delta(z)^{-1} \delta(z^{-1})^{-1}; \)

iv). \( \delta(\infty) := \lim_{z \to \infty} \delta(z) = 1. \) And \( \delta(z) \) is continuous at \( z = 0 \) with \( \delta(0) = \delta(\infty) = 1; \)

v). \( \delta(z) \) is uniformly bounded in \( \mathbb{C} \)

\[ (1 - \rho^2)^{1/2} \leq |\delta(z)| \leq (1 - \rho^2)^{-1/2}, \] (4.5)

where \(|r|_{L^\infty} \leq \rho < 1). \)

vi). As \( z \to \xi_j \) along any ray \( \xi_j + e^{i\phi} \mathbb{R}^+ \) with \( |\phi| < \pi \), we have

\[ ||\beta||_{L^\infty} \leq \frac{c||r||_{H^{1,0}}}{1 - \rho}, \] (4.6)

\[ |\beta(z, \xi) - \beta(\xi_j, \xi_j)| < \frac{c||r||_{H^{1,0}}}{1 - \rho} |z - \xi_j|^1 \frac{1}{2}, \] (4.7)

where

\[ \beta(z, \xi) = \int_{\Gamma} \frac{\nu(s) - \chi(s) \nu(\xi_j)}{s - z} ds - \epsilon_j \nu(\xi_j) \log(z - \xi_j + 1), \] (4.8)

\( \chi(s) \) is the characteristic function of \( (\xi_j - 1, \xi_j). \)

**Proof.** Properties of i), iv) can be obtained by simple calculation from the definition of \( \delta(z) \). The jump relation ii) follows from the Plemelj formulae. As for the property iii), the symmetry comes from the symmetry of \( r(z) \). We specially point out that, the third symmetry follows from the symmetry of \( r(z) \) as well as the following equality

\[ \exp \left( i \int_{\Gamma} \nu(s) \frac{1}{s - z} ds \right) = \exp \left[ i \int_{\Gamma} \nu(s) \left( \frac{1}{s - z} - \frac{1}{2s} \right) ds \right], \] (4.9)

for \( z \in \Gamma \). As for the property v) and vi), the analysis is similar to [22, Lemma 3.1].

\[ \square \]
Then we can rewrite

\[ \delta(z, \xi) = (z - \xi_j)^{i\nu(\xi_j)} \exp(i\beta(z, \xi)). \] (4.10)

**Remark 4.1.** We notice that all discrete spectrums \( \eta_n \in \mathbb{C}^+ \cap \{ z : |z| = 1 \} \) satisfy \( \Im(\eta_n) < 0 \), all discrete spectrums \( \bar{\eta}_n \in \mathbb{C}^- \cap \{ z : |z| = 1 \} \) satisfy \( \Im(\bar{\eta}_n) > 0 \). Owe to this good property, that’s why we do not classify the discrete spectrum by \( \delta(z, \xi) \).

By using \( \delta(z, \xi) \), the new matrix-valued function \( m^{(1)}(z) \) is defined as

\[ m^{(1)}(x, t; z) := m^{(1)}(z) = m(z)\delta(z)\sigma_3, \] (4.11)

which satisfies the following RH problem.

**RHP 4.1.** Find a \( 2 \times 2 \) matrix-valued function \( m^{(1)}(x, t; z) \) such that

* \( m^{(1)}(z) \) is meromorphic in \( \mathbb{C} \setminus \Sigma^{(1)} \) and has simple poles in \( \mathcal{Z} = \{ \eta_n, \bar{\eta}_n \}_{n=1}^{2N} \), where \( \Sigma^{(1)} = \Sigma \).

* \( m^{(1)}(z) = \sigma_1 m^{(1)}(\bar{z}) \sigma_1 = m^{(1)}(-\bar{z}) = \mp z^{-1} m^{(1)}(z^{-1}) \sigma_2 \).

* The non-tangential limits \( m^{(1)}(z) \) exist for any \( z \in \Sigma^{(1)} \) and satisfy the jump relation

\[ m^{(1)}_+ (z) = m^{(1)}_- (z) V^{(1)}(z) \] where

\[ V^{(1)}(z) = \begin{cases} \left[ \begin{array}{cc} 1 & -r(z)\delta(z)^{-2}e^{2it\theta} \\ 0 & 1 \end{array} \right], & z \in \tilde{\Gamma}, \end{cases} \] (4.12)

\[ = \left[ \begin{array}{cc} 1 & 0 \\ r(z)\delta^2(z)e^{-2it\theta} & 1 \end{array} \right], & z \in \Gamma. \] (4.13)

* Asymptotic behavior

\[ m^{(1)}(x, t; z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty, \] (4.14)

\[ m^{(1)}(x, t; z) = \frac{i}{z} \sigma_3 Q_- + \mathcal{O}(1), \quad z \to 0. \] (4.15)

* Residue conditions

\[ \text{Res} \left. m^{(1)}(z) \right|_{z=\eta_n} = \lim_{z \to \eta_n} m^{(1)}(z) \left[ \begin{array}{cc} 0 & 0 \\ c_n \delta^2(\eta_n)e^{-2it\theta_n} \end{array} \right], \] (4.16)

\[ \text{Res} \left. m^{(1)}(z) \right|_{z=\bar{\eta}_n} = \lim_{z \to \bar{\eta}_n} m^{(1)}(z) \left[ \begin{array}{cc} 0 & 0 \\ 0 & c_n \delta^{-2}(\bar{\eta}_n)e^{2it\theta_n} \end{array} \right], \] (4.17)
Proof. By simple computation, we can obtain the jump relation and residue condition from (4.11), (2.63), (2.64) as well as the jump relation of RHP 2.1. The analyticity and symmetry of \( m^{(1)}(z) \) directly follows from its definition, the symmetries of RHP 2.1 as well as (4.11). As for asymptotic behaviors, we notice that \( \delta(\infty) = \delta(0) = 1 \), thus \( m^{(1)}(z) \) has the same asymptotic behaviors as \( m(z) \).

4.2 Set up and decomposition of a mixed \( \partial \)-RH problem

Next we want to make continuous extension for the jump matrix \( V^{(1)}(z) \) to remove the jump from the real axis in such a way that the new problem takes advantage of the decay of \( \exp(\pm 2it\theta) \) for \( z \notin \mathbb{R} \).

4.2.1 Characteristic lines

Fix a small enough angle \( \theta_0 \) which satisfies that

i) \[
\left\{ z \in \mathbb{C} : \tan \theta_0 < \left| \frac{\Im z}{\Re z - \xi_j} \right| \right\}
\]

(4.18)

does not intersect the set \( \mathcal{Z} \), \( j = 2, 3 \), for any \( \xi \in (-\pi, -6) \);

ii) The following regions \( \Omega_{jk}, j = 2, 3, k = 3, 4 \) do not intersect discrete spectrums, which implies that

\[
\Upsilon(\xi) = \min \left\{ \theta_0, \frac{\pi}{4} \right\},
\]

(4.19)

iii) Recall the Proposition 3.1, we make

\[
d \in \left(0, \frac{\xi_2}{2 \cos \Upsilon}\right), \quad \tilde{d} \in \left(0, \frac{\xi_1 - \xi_2}{2 \cos \Upsilon}\right).
\]

(4.20)

Characteristic Lines at Saddle Points
Then for an angle \( \phi \) satisfies the above conditions (4.18), (4.19) and (4.20), We denote

\[
\Sigma_j = \begin{cases} 
\xi_j + e^{i[(j-1)\pi + (-1)^{j-1}\phi]} \Re^+, & j = 1, 4, \\
\xi_j + e^{i[(j-1)\pi + (-1)^{j-1}\phi]} d, & j = 2, 3,
\end{cases}
\]

(4.21)

\[
\Sigma_j = \begin{cases} 
\xi_j + e^{-i[(j-1)\pi + (-1)^{j-1}\phi]} \Re^+, & j = 1, 4, \\
\xi_j + e^{-i[(j-1)\pi + (-1)^{j-1}\phi]} d, & j = 2, 3,
\end{cases}
\]

(4.22)

\[
\Sigma_j = \begin{cases} 
\xi_j + e^{-i[j\pi + (-1)^j\phi]} \tilde{d}, & j = 1, 4, \\
\xi_j + e^{-i[j\pi + (-1)^j\phi]} \tilde{d}, & j = 2, 3,
\end{cases}
\]

(4.23)

\[
\Sigma_j = \begin{cases} 
\xi_j + e^{i[j\pi + (-1)^j\phi]} \tilde{d}, & j = 1, 4, \\
\xi_j + e^{i[j\pi + (-1)^j\phi]} \tilde{d}, & j = 2, 3,
\end{cases}
\]

(4.24)

**Characteristic Lines at** \( z = 0 \)

\[
\Sigma_{j1} = \begin{cases} 
e^{i\phi} d, & j = 0^+, \\
e^{i(\pi - \phi)} d, & j = 0^-,
\end{cases}
\]

(4.25)

\[
\Sigma_{j2} = \begin{cases} 
e^{-i\phi} d, & j = 0^+, \\
e^{-i(\pi - \phi)} d, & j = 0^-,
\end{cases}
\]

(4.26)

Meanwhile, there exist jumps on the vertical segments between \( \xi_j \) and \( \xi_{j+1}, j = 1, 3, \) as well as between \( z = 0 \) and \( \xi_j, j = 2, 3. \) These jumps can be expressed as follows

\[
\Sigma_{1/2}^j := \begin{cases} 
\frac{\xi_j + e^{i[(j-1)\pi + (-1)^{j-1}\phi]} \tilde{d}}{2} + d \sin \phi e^{\pm \frac{\pi}{2} i}, & j = 1, 4, \\
\frac{\xi_j}{2} + d \sin \phi e^{\pm \frac{\pi}{2} i}, & j = 2, 3,
\end{cases}
\]

(4.27)

The complex plane \( \mathbb{C} \) is separated by these contours, which is shown in Figure 6, Figure 7 and Figure 8. Let

\[
\Omega = \left( \bigcup_{j,k=1,2,3,4} \Omega_{jk} \right) \bigcup \left( \bigcup_{k=1,2} \Omega_{0^k} \right), \quad \Omega_{\pm} = \mathbb{C} \setminus \Omega,
\]

(4.28)

\[
\Sigma^{(2)} := \left( \bigcup_{j,k=1,2,3,4} \Sigma_{jk} \right) \bigcup \left( \bigcup_{k=1,2} \Omega_{0^k} \right) \bigcup \left( \bigcup_{j=1,2,3,4} \Sigma_{j^k}^1 \right).
\]

(4.29)

**4.2.2 Some estimates for** \( \Im \theta(z) \) **in different regions**

In this subsection, we try to give some estimates for \( \Im \theta(z) \) in different regions.
Figure 6: there are four stationary phase points $\xi_1, \ldots, \xi_4$ with $\xi_1 = -\xi_4 = 1/\xi_2 = -1/\xi_3$ as $-\omega < \xi < -6$. Open jump contour $\mathbb{R}\backslash\{0\}$ such that red and blue lines don’t intersect the discrete spectrum on the unite circle $|z| = 1$. Additionally, the blue “+” implies that $e^{2it\theta} \to 0$ as $t \to +\infty$, on the other side, the red “−” implies that $e^{-2it\theta} \to 0$ as $t \to +\infty$.

Figure 7: $\Sigma_{jk}$ separate complex $\mathbb{C}$ into some sectors denoted by $\Omega_{jk}$. The blue $\Omega_{jk}$ corresponds to $\Im \theta > 0$ (i.e. $|e^{2it\theta}| \to 0$). And The red $\Omega_{jk}$ corresponds to $\Im \theta < 0$ (i.e. $|e^{-2it\theta}| \to 0$). This figure describes the regions near saddle points.
In this graph, we enlarge the neighborhood of $z = 0$. $\Sigma_{jk}$ separate complex $\mathbb{C}$ into some sectors denoted by $\Omega_{jk}$. The blue $\Omega_{jk}$ corresponds to $\Im \theta > 0$ (i.e. $|e^{2it\theta}| \to 0$). And the red $\Omega_{jk}$ corresponds to $\Im \theta < 0$ (i.e. $|e^{-2it\theta}| \to 0$). This figure describes the regions of $z = 0$.

Lemma 4.1 (corresponding to $z = 0$). For a fixed small angle $\phi$ satisfies (4.18), (4.19) and (4.20), the imaginary part of phase function $\theta(z)$ defined by (3.1) has following estimations:

$$
\Im \theta(z) \geq c|\sin \phi|\sqrt{\alpha}, \quad \text{as} \quad z \in \Omega_{0 \pm 1},
$$

$$
\Im \theta(z) \leq -c|\sin \phi|\sqrt{\alpha}, \quad \text{as} \quad z \in \Omega_{0 \pm 2},
$$

where

$$
c = c(\xi, \varpi) > 0, \quad \alpha = 3 - \frac{\xi + 3}{1 + 2\cos(2\phi)}. \quad (4.32)
$$

Proof. We give a proof for $z \in \Omega_{0+1}$, the others are similar. We take that $z = le^{i\phi}$, then we can rewrite the (3.2) as

$$
\Im \theta(z) = \frac{1}{2}F(l)\sin \phi \left[ \xi - 6\cos(2\phi) + (2\cos(2\phi) + 1)F^2(l) \right],
$$

where $F(l) = l + l^{-1} \geq 2$. Firstly we calculate the critical situation $\Im \theta(z) = 0$. Take use (4.33), $F(l) \geq 2$ as well as $\sin \phi > 0$, we have

$$
\xi - 6\cos(2\phi) + [2\cos(2\phi) + 1]F^2(l) = 0. \quad (4.34)
$$

Thus

$$
F^2(l) = 3 - \frac{\xi + 3}{1 + 2\cos(2\phi)} =: \alpha > 4, \quad (4.35)
$$

moreover, from $F^2(l) = l + l^{-1}$, we have $l^2 - \sqrt{\alpha}l + 1 = 0$. Solve this quadratic equation, we obtain two roots

$$
l_1 = \frac{\sqrt{\alpha} - \sqrt{\alpha - 4}}{2} < l_2 = \frac{\sqrt{\alpha} + \sqrt{\alpha - 4}}{2}. \quad (4.36)
$$

We claim that: $\Im \theta(z) > 0$ as $l < l_1$ (corresponding to $z \in \Omega_{0+1}$). It’s easy to check that $h(l) := l^2 - \sqrt{\alpha}l + 1$ is monotonically increasing on the $(\sqrt{\alpha}/2, +\infty)$, while monotonically increasing on...
Thus we bring this proof to an end.

\( h \)

Take \( \tau \) the second step we used \( \Im \) Corollary 4.1. \( -\infty \) decreasing on the \((0, l_1)\). Thus we have \( h(l) > h(l_1) = 0 \) and \( F(l) > \sqrt{\alpha} \), which implies that

\[
\Im(\theta(z)) > \frac{1}{2}\sqrt{\alpha}\sin \phi [\xi_2 - 6\cos(2\phi) + (2\cos(2\phi) + 1)\alpha] = 0.
\]  

(4.37)

Thus we bring this proof to an end. \( \square \)

**Corollary 4.1.** \( \Im(\theta(z)) \) has following evaluation for \( z = le^{i\phi} := u_0 + iv \)

\[
\Im(\theta(z)) \geq c_v, \quad \text{as} \quad z \in \Omega_{0+1},
\]  

(4.38)

\[
\Im(\theta(z)) \leq -c_v, \quad \text{as} \quad z \in \Omega_{0+2}.
\]  

(4.39)

where \( c = c(\xi, \varpi) > 0 \).

**Lemma 4.2** (corresponding to \( z = \xi_j, j = 2, 3 \)). For a fixed small angle \( \phi \) (the same in Lemma 4.1) satisfies \( (4.18), (4.19) \) and \( (4.20) \), the imaginary part of phase function \( \theta(z) \) defined by \( (3.1) \) has following estimations:

\[
\Im(\theta(z)) \leq -c (1 + |z|^{-2}) v^2, \quad z \in \Omega_{jk}, \quad j = 2, 3, \quad k = 2, 4
\]  

(4.40)

\[
\Im(\theta(z)) \geq c (1 + |z|^{-2}) v^2, \quad z \in \Omega_{jk}, \quad j = 2, 3, \quad k = 1, 3,
\]  

(4.41)

where \( c = c(\xi, \varpi) > 0 \).

**Proof.** We take \( z \in \Omega_{24} \) as an example, and the proof for the other regions is similar. Denote \( z = \xi_2 + le^{i\phi} := \xi_2 + u_2 + iv \), then we can rewrite the \( (3.2) \) as

\[
\Im(\theta(z)) = \frac{\nu}{2} (1 + |z|^{-2}) [\xi + 3 + 3(|z|^2 + |z|^{-2} - 1) - 4v^2(1 + |z|^{-4} - |z|^{-2})]
\leq c (1 + |z|^{-2}) [\xi + 3 + 3(|z|^2 + |z|^{-2} - 1) - 4v^2(1 + |z|^{-4} - |z|^{-2})],
\]  

(4.42)

the second step we used \( v \leq \frac{\xi_1 - \xi_2}{2} \) and \( c \) varies from line to line.

we consider

\[
h(|z|^2) := [\xi + 3 + 3(|z|^2 + |z|^{-2} - 1) - 4v^2(1 + |z|^{-4} - |z|^{-2})].
\]  

(4.43)

Take \( \tau = |z|^2 \in (\xi_2^2, \xi_1^2) \), we obtain that

\[
h(\tau) = 3(\tau + \tau^{-1} - 1) - 4v^2(1 + \tau^{-2} - \tau^{-1}) + \xi + 3.
\]  

(4.44)

It is not difficult to verify that \( h'(\tau) < 0 \) for \( \tau \in (\xi_2^2, \xi_1^2) \), thus

\[
h(\tau) \leq h(\xi_2^2)
\]

\[
= 3 (\xi_2^2 + \xi_2^{-2} - 1) - 4v^2(1 + \xi_2^{-4} - \xi_2^{-2}) + \xi + 3
\]

\[
\xi_2 = \frac{1}{\xi_1} 3 (\xi_2^2 + \xi_1^{-2} - 1) - 4v^2(1 + \xi_1^4 - \xi_1^2) + \xi + 3.
\]  

(4.45)
Since $\xi_1$ is the saddle point, we have $\theta'(\xi_1) = 0$. Taking use $\xi_1\xi_2 = 1$ again, we can obtain the following relation from $\theta'(\xi_1) = 0$ that
\[
\xi + 3 = \frac{-3(\xi_2^2 + \xi_1^2)}{1 + \xi_2^2}.
\] (4.46)

With (4.46), we are lucky enough to find that
\[
3(\xi_2^2 + \xi_1^2 - 1) + \xi + 3 = 0.
\] (4.47)

Then we obtain
\[
h(\tau) \leq -4v^2 \left(1 + \xi_1^4 - \xi_1^2\right).
\] (4.48)

As a consequence,
\[
\Im \theta(z) \leq -c(\xi, \varpi)v^2(1 + |z|^{-2}) < 0.
\] (4.49)

**Lemma 4.3** (corresponding to $z = \xi_j, j = 1, 4$). For a fixed small angle $\phi$ (the same in Lemma 4.1) satisfies (4.18), (4.19) and (4.20), the imaginary part of phase function $\theta(z)$ defined by (3.1) has following estimations:
\[
\Im \theta(z) \geq cv|\Re z - \xi_j|, \quad z \in \Omega_{jk}, \quad j = 1, 4, \quad k = 1, 3 \quad (4.50)
\]
\[
\Im \theta(z) \leq -c|\Re z - \xi_j|, \quad z \in \Omega_{jk}, \quad j = 1, 4, \quad k = 2, 4, \quad (4.51)
\]
where $c = c(\xi, \varpi) > 0$.

**Proof.** We take the $z \in \Omega_{11}$ as an example to give the proof. Notice that we always open lens by the same sufficient small angle $\phi$, we can find that characteristic line, such as $\Sigma_{11}^0 + 1$ is parallel to $\Sigma_{11}$. Based on the observation, we can prove this lemma as the following two steps.

Firstly, as the same procedure of Lemma 4.1, we can obtain that $\Im \theta(z) \leq cv$ for $l > l_2$. And $l_2$ is defined by (4.36). It’s obvious that $z$ which satisfies $l > l_2$ is located in the region $\Omega_{11}$.

Secondly, the parallel properties between $\Sigma_{0+1}$ and $\Sigma_{11}$ bring the proof to an end. □

**Remark 4.2.** Based on the Lemma 4.1, Lemma 4.2 as well as Lemma 4.3, we know that the $v = v(\phi)$ is fixed for each fixed $\phi$ in the above estimates.
4.2.3 Opening \(\bar{\partial}\) lenses

The estimates mentioned above suggest that we should open lenses using the first factorization for \(z \in \tilde{\Gamma}\) and second factorization for \(z \in \Gamma\). To do so, we need to define extensions off the diagonal entries of jump matrix \(V^{(1)}\) off the real axis, which is the content of this subsection.

To be brief, we introduce following functions: for \(j = 0^\pm, 1, 2, 3, 4\)

\[
p_{j1}(z) := p_{j1}(z, \xi) = \frac{r(z)}{1 - |r(z)|^2}, \quad p_{j3}(z) := -r(z),
\]

\[
p_{j2}(z) := \frac{r(z)}{1 - |r(z)|^2}, \quad p_{j4}(z) := -r(z).
\]

We choose \(R^{(2)}(z) := R^{(2)}(z, \xi)\) as

\[
R^{(2)}(z) = \begin{cases}
\begin{bmatrix} 1 & f_{j1} e^{2i\theta} \\ 0 & 1 \end{bmatrix}, & z \in \Omega_{j1}, \ j = 0^\pm, 1, 2, 3, 4 \\
\begin{bmatrix} 1 & f_{j2} e^{-2i\theta} \\ 0 & 1 \end{bmatrix}, & z \in \Omega_{j2}, \ j = 0^\pm, 1, 2, 3, 4 \\
\begin{bmatrix} 1 & f_{j3} e^{2i\theta} \\ 0 & 1 \end{bmatrix}, & z \in \Omega_{j3}, \ j = 1, 2, 3, 4 \\
\begin{bmatrix} 1 & f_{j4} e^{-2i\theta} \\ 0 & 1 \end{bmatrix}, & z \in \Omega_{j4}, \ j = 1, 2, 3, 4 \\
I, & \text{elsewhere},
\end{cases}
\]

(4.54)

where the functions \(f_{jk}\) is defined as the following two propositions.

**Proposition 4.2** (Opening lens at \(z = 0\)). \(f_{jk} : \tilde{\Omega}_{jk} \to \mathbb{C}, \ j = 0^\pm, k = 1, 2\) are continuous on \(\tilde{\Omega}_{jk}, j = 0^\pm, k = 1, 2\) with boundary values:

\[
f_{0^\pm 1}(z) = \begin{cases}
p_{j1}(z) \delta^2_+(z), & z \in \left(\frac{\xi_3}{2}, 0\right) \cup \left(0, \frac{\xi_2}{2}\right), \\
0, & z \in \Sigma_{0^\pm 1}
\end{cases}
\]

(4.55)

\[
f_{0^\pm 2}(z) = \begin{cases}
p_{j3}(z) \delta^2_-(z), & z \in \left(\frac{\xi_3}{2}, 0\right) \cup \left(0, \frac{\xi_2}{2}\right), \\
0, & z \in \Sigma_{0^\pm 2}
\end{cases}
\]

(4.56)

And \(f_{jk}, j = 0^\pm, k = 1, 2\) have following property:

\[
|\bar{\partial} f_{jk}(z)| \lesssim |p'_{jk}(|z|)| + |z|^{-1/2}, \quad z \in \Omega_{jk}, \ j = 0^\pm, k = 1, 2.
\]

(4.57)

Moreover

\[
|\bar{\partial} f_{jk}(z)| \lesssim |p'_{jk}(|z|)| + |z|^{-1}, \quad z \in \Omega_{jk}, \ j = 0^\pm, k = 1, 2.
\]

(4.58)
Proof. We will give the details for \( f_{0+1}(z) \), which extension can be constructed by

\[
\begin{align*}
f_{0+1}(z) &= p_{0+1}(z) \delta_+^2(z) \cos(\kappa_0 \arg z), \quad \kappa_0 = \frac{\pi}{2\phi}. \tag{4.59}
\end{align*}
\]

Denote \( z = le^{i\varphi} \), then we have \( \bar{\partial} \)-derivative \( \bar{\partial} = \frac{1}{2} e^{i\varphi}(\partial_t + il^{-1}\partial_\varphi) \). Hence

\[
\bar{\partial} f_{0+1}(z) = \frac{e^{i\varphi}}{2} \delta_+^2(z) \left[ p_{0+1}'(l) \cos(\kappa_0 \varphi) - \frac{i}{l} \kappa_0 \sin(\kappa_0 \varphi) p_{0+1}(l) \right]. \tag{4.60}
\]

Using Cauchy-Schwarz inequality, we have

\[
|p_{0+1}(l)| = |p_{0+1}(l) - p_{0+1}(0)| = \left| \int_0^l p_{0+1}'(s) ds \right| \lesssim \|p_{0+1}'\|_{L^2} \lesssim 1^{1/2}. \tag{4.61}
\]

Meanwhile, the boundedness of \( \delta_+^2(z) \) is guaranteed by the property v) of Proposition 4.4. Thus (4.57) comes true. As for (4.58), we just notice \( p_{0+1}(l) \in L^\infty \).

**Proposition 4.3** (Opening lens at saddle points). \( f_{jk} : \bar{\Omega}_{jk} \to \mathbb{C}, j, k = 1, 2, 3, 4 \) are continuous on \( \bar{\Omega}_{jk}, j, k = 1, 2, 3, 4 \) with boundary values:

\[
\begin{align*}
f_{j1}(z) &= \begin{cases} p_{j1}(z) \delta_+^{-2}(z), & z \in I_{j1}, \\ p_{j1}(\xi_j)e^{-2i\beta(\xi_j, \xi)}(z - \xi_j)^{-2i\epsilon_j, \nu(\xi_j)}, & z \in \Sigma_{j1}, \end{cases} \tag{4.62} \\
f_{j2}(z) &= \begin{cases} p_{j2}(z) \delta_+^2(z), & z \in I_{j2}, \\ p_{j2}(\xi_j)e^{2i\beta(\xi_j, \xi)}(z - \xi_j)^{2i\epsilon_j, \nu(\xi_j)}, & z \in \Sigma_{j2}, \end{cases} \tag{4.63} \\
f_{j3}(z) &= \begin{cases} p_{j3}(z) \delta_+^{-2}(z), & z \in I_{j3}, \\ p_{j3}(\xi_j)e^{-2i\beta(\xi_j, \xi)}(z - \xi_j)^{-2i\epsilon_j, \nu(\xi_j)}, & z \in \Sigma_{j3}, \end{cases} \tag{4.64} \\
f_{j4}(z) &= \begin{cases} p_{j4}(z) \delta_+^2(z), & z \in I_{j4}, \\ p_{j4}(\xi_j)e^{2i\beta(\xi_j, \xi)}(z - \xi_j)^{2i\epsilon_j, \nu(\xi_j)}, & z \in \Sigma_{j4}. \end{cases} \tag{4.65}
\end{align*}
\]

where

\[
\begin{align*}
I_{11} &= I_{12} := (\xi_1, +\infty), \quad I_{21} = I_{22} := \left( \frac{\xi_2}{2}, \xi_2 \right), \\
I_{31} &= I_{32} := (\xi_3, \frac{\xi_3}{2}), \quad I_{41} = I_{42} := (-\infty, \xi_4), \\
I_{13} &= I_{14} := \left( \frac{\xi_2 + \xi_1}{2}, \xi_1 \right), \quad I_{23} = I_{24} := \left( \xi_2, \frac{\xi_2 + \xi_1}{2} \right), \\
I_{33} &= I_{34} := \left( \xi_4 + \frac{\xi_3}{2}, \xi_3 \right), \quad I_{43} = I_{44} := \left( \xi_4, \frac{\xi_4 + \xi_3}{2} \right). \tag{4.66} 
\end{align*}
\]

And \( f_{jk}, j, k = 1, 2, 3, 4 \) have following properties:

\[
\begin{align*}
|\bar{\partial} f_{jk}(z)| &\lesssim |p_{jk}'(\Re z)| + |z - \xi_j|^{-1/2}, \quad z \in \Omega_{jk}, \quad j, k = 1, 2, 3, 4, \\
|f_{jk}(z)| &\lesssim \sin^2(\kappa_0 \arg(z - \xi_j)) + (\Re z)^{-1}, \quad z \in \Omega_{jk}, \quad j, k = 1, 2, 3, 4. \tag{4.68}
\end{align*}
\]
Moreover, when $z \to 1$

$$
|\bar{\partial}f_{jk}(z)| \lesssim |p'_{jk}|z - 1|, \quad z \in \Omega_{24}, \Omega_{23}, \quad (4.70)
$$

$$
|\bar{\partial}f_{jk}(z)| \lesssim |p'_{jk}|z + 1|, \quad z \in \Omega_{34}, \Omega_{33}. \quad (4.71)
$$

**Proof.** We take the $f_{11}(z)$ and $f_{24}(z)$ as examples to present this proof.

The continuous extension of $f_{11}(z)$ on $\Omega_{11}$ can be constructed by

$$
f_{11}(z) = p_{11}(\xi_1)e^{-2i\beta(\xi_1, \xi_1)}(z - \xi_1)^{-2i\nu(\xi_1)} [1 - \cos (\kappa_0 \arg (z - \xi_1))] 
+ p_{11}(\Re z)\delta^{-2}(z)\cos (\kappa_0 \arg (z - \xi_1)), \quad (4.72)
$$

Where $\kappa_0 = \frac{\pi}{2\nu}$. Denote $z = \xi_1 + le^{i\varphi} := \xi_1 + u + iv$, where $l, \varphi, u, v \in \Re$. Firstly, we have $|p_{11}(\Re z)| = \frac{|r(\Re z)|}{1 - r(\Re z)} \lesssim (\Re z)^{-1}$, Recall the (4.6), we obtain (4.69). Then take the equality $\bar{\partial} = \frac{1}{2} e^{i\varphi}(\partial_t + il^{-1}\partial_\varphi)$, we have

$$
\bar{\partial}f_{11} = \left[ p_{11}(\xi_1) e^{-2i\beta(\xi_1, \xi_1)} (z - \xi_1)^{-2i\nu(\xi_1)} \right] \bar{\partial}\cos (\kappa_0 \varphi) 
+ \frac{1}{2} \delta^{-2}(z)p'_{11}(u, \xi)\cos (\kappa_0 \varphi). \quad (4.73)
$$

Recall the estimate (4.7), we get (4.68) at once.

For $f_{24}$, take the same method to $f_{11}$, we have

$$
\bar{\partial}f_{24} = \left[ p_{24}(\xi_2) e^{2i\beta(\xi_2, \xi_2)} (z - \xi_2)^{-2i\nu(\xi_2)} \right] \bar{\partial}\cos (\kappa_0 \varphi) 
+ \frac{1}{2} \delta^{-2}(z)p'_{24}(u, \xi)\cos (\kappa_0 \varphi). \quad (4.74)
$$

Finally $z$ near 1, we have $\varphi \to 0$, thus we obtain

$$
|\bar{\partial}f_{24}| \lesssim |p'_{24}|\cos (\kappa_0 \varphi) \lesssim |p'_{24}|z - 1|. \quad (4.75)
$$

Until now, we finish the proof.

We now use $R^{(2)}$ to define a new transformation

$$
m^{(2)}(z) := m^{(2)}(x, t; z) = m^{(1)}(z)R^{(2)}(z), \quad (4.76)
$$

which help us set up the mixed $\bar{\partial}$-RH problem as follows

**RHP 4.2.** Find a $2 \times 2$ matrix-valued function $m^{(2)}(x, t; z)$ such that

- $m^{(2)}(z)$ is continuous in $\mathbb{C}\setminus \Sigma^{(2)}$ and meromorphic out $\bar{\Omega}$ with simple poles $Z$,
* $m^{(2)}(z)$ takes continuous boundary values $m^{(2)}_{\pm}(z)$ on $\Sigma^{(2)}$ with jump relation

$$m^{(2)}_{+}(z) = m^{(2)}_{-}(z)V^{(2)}(z),$$  \hspace{1cm} (4.77)

where

$$V^{(2)}(z) = \begin{cases}
R^{(2)}(z)^{-1}|_{\Sigma_{j1}} & z \in \Sigma_{j1}, \quad j = 0\pm, 1, 2, 3, 4, \\
R^{(2)}(z)^{-1}|_{\Sigma_{j4}} & z \in \Sigma_{j4}, \quad j = 1, 2, 3, 4, \\
R^{(2)}(z)|_{\Sigma_{j2}} & z \in \Sigma_{j2}, \quad j = 0\pm, 1, 2, 3, 4, \\
R^{(2)}(z)|_{\Sigma_{j3}} & z \in \Sigma_{j3}, \quad j = 1, 2, 3, 4, \\
R^{(2)}(z)^{-1}|_{\Sigma_{j1}} & z \in \Sigma_{j1}, \quad j = 1, 3, \quad k = 3, 4, \\
R^{(2)}(z)^{-1}|_{\Sigma_{j4}} & z \in \Sigma_{j4}, \quad j = 1, 2, 3, 4, \\
R^{(2)}(z)^{-1}|_{\Sigma_{j2}} & z \in \Sigma_{j2}, \quad j = 1, 2, 3, 4, \\
R^{(2)}(z)^{-1}|_{\Sigma_{j3}} & z \in \Sigma_{j3}, \quad j = 1, 2, 3, 4.
\end{cases}$$ \hspace{1cm} (4.78)

* asymptotic behavior

$$m^{(2)}(x,t; z) = I + O(z^{-1}), \quad z \to \infty,$$  \hspace{1cm} (4.79)

$$m^{(2)}(x,t; z) = \frac{i}{z} \sigma_3 Q_- + O(1), \quad z \to 0.$$  \hspace{1cm} (4.80)

* for $z \in \mathbb{C}$, we have the $\bar{\partial}$-derivative equality

$$\bar{\partial}m^{(2)} = m^{(2)}\bar{\partial}R^{(2)},$$  \hspace{1cm} (4.81)

where

$$\bar{\partial}R^{(2)} = \begin{cases}
1 & z \in \Omega_{j1}, \quad j = 0\pm, 1, 2, 3, 4, \\
0 & z \in \Omega_{j2}, \quad j = 0\pm, 1, 2, 3, 4, \\
\bar{\partial}f_{j2}e^{-2it\theta} & z \in \Omega_{j3}, \quad j = 1, 2, 3, 4, \\
0 & z \in \Omega_{j4}, \quad j = 1, 2, 3, 4.
\end{cases}$$  \hspace{1cm} (4.82)

* Residue conditions

$$\text{Res} m^{(2)}(z) = \lim_{z \to \eta_n} m^{(2)}(z) \begin{pmatrix} 0 & 0 \\ c_n \delta^2(\eta_n) e^{-2it\theta_n} & 0 \end{pmatrix},$$  \hspace{1cm} (4.83)

$$\text{Res} m^{(2)}(z) = \lim_{z \to \eta_n} m^{(2)}(z) \begin{pmatrix} 0 & 0 \\ 0 & c_n \delta^{-2}(\eta_n) e^{2it\theta_n} \end{pmatrix},$$  \hspace{1cm} (4.84)
Remark 4.3. Notice that the settings of \( f_{jk}, j = 0^\pm, k = 1, 2 \) in (4.55), (4.56), we actually know that
\[
V^{(2)}(z) = I, \quad \text{for} \quad z \in \Sigma_{0^\pm k}, \quad k = 1, 2. \tag{4.85}
\]

4.2.4 Decomposition of the mixed \( \bar{\partial} \)-RH Problem

Aiming at solving mixed \( \bar{\partial} \)-RH problem, we decompose it a pure RH problem for \( M^{rhp} \) with \( \bar{\partial}R^{(2)} \equiv 0 \) as well as a pure \( \bar{\partial} \)-problem with nonzero \( \bar{\partial}R^{(2)} \) derivatives. This can be shown as the following structure
\[
m^{(2)} = m^{(3)} m^{rhp} \begin{cases}
\bar{\partial}R^{(2)} \equiv 0 & \rightarrow m^{rhp}, \\
\bar{\partial}R^{(2)} \neq 0 & \rightarrow m^{(3)} = m^{(2)} m^{rhp}^{-1}.
\end{cases} \tag{4.86}
\]

RHP 4.3. Find a \( 2 \times 2 \) matrix-valued function \( m^{rhp}(x, t; z) \) such that
* \( m^{rhp}(z) \) is meromorphic in \( \mathbb{C} \setminus \Sigma^{(2)} \) with simple poles \( Z \),
* \( m^{rhp}(z) \) takes continuous boundary values \( m^{rhp}_\pm(z) \) on \( \Sigma^{(2)} \) with jump relation
\[
m^{rhp}_+(z) = m^{rhp}_-(z) V^{(2)}(z), \tag{4.87}
\]
* asymptotic behavior
\[
m^{rhp}(x, t; z) = I + O(z^{-1}), \quad z \to \infty, \tag{4.88}
m^{rhp}(x, t; z) = \frac{i}{z} \sigma_3 Q_- + O(1), \quad z \to 0. \tag{4.89}
\]
* for \( z \in \mathbb{C}, \bar{\partial}R^{(2)} = 0 \)
* Residue conditions
\[
\text{Res}_{z=\eta_n} m^{rhp}(z) = \lim_{z \to \eta_n} m^{rhp}(z) \begin{bmatrix}
0 \\
c_n \delta^2(\eta_n) e^{-2it\theta_n} 0
\end{bmatrix}, \tag{4.90}
\]
\[
\text{Res}_{z=\bar{\eta}_n} m^{rhp}(z) = \lim_{z \to \bar{\eta}_n} m^{rhp}(z) \begin{bmatrix}
0 \\
\bar{c}_n \delta^{-2}(\bar{\eta}_n) e^{2it\theta_n}
\end{bmatrix}, \tag{4.91}
\]

Now we give a sign \( U(\xi) \) as the union set of neighborhood of saddle point \( \xi_j \) for \( j = 1, 2, 3, 4 \). In particular,
\[
U(\xi) = \bigcup_{j=1,2,3,4} U_{\xi_j}, \tag{4.92}
\]
\[
U_{\xi_j} = \{ z : |z - \xi_j| < \rho \}, \tag{4.93}
\]
where
\[ \varrho < \frac{1}{3} \min \left\{ \min \{ |\Im \eta_n| \}_{n=1}^N, \quad \min_{k \neq l} |\eta_l - \eta_k|, \quad \frac{1}{2} \min_{j=1,2,3,4} |\xi_j \pm 1|, \quad \frac{1}{2} \min_{j=1,2,3,4} |\xi_j| \right\} \, . \] (4.94)

**Remark 4.4.** The third and fourth restriction of (4.94) is to remove the singularity \( z = 0, \pm 1 \) from local model which will be discussed in Subsection 4.3.2.

The following two propositions can help us separate the contribution for \( m^{rhp} \).

**Proposition 4.5.** For \( 1 \leq p \leq +\infty \), there exists a constant \( h = h(p) > 0 \), such that the jump matrix \( V^{(2)} \) defined in (4.78) admit the following estimate as \( t \to +\infty \)
\[ \| V^{(2)} - I \|_{L^p(\Sigma_{ik} \setminus U_{\xi_j})} = \mathcal{O}(e^{-ht}), \quad \text{for} \quad j, k = 1, 2, 3, 4. \] (4.95)

**Proof.** We take \( z \in \Sigma_{24} \setminus U_{\xi_2} \) as an example, the other cases can be proved in a similar way. For \( z \in \Sigma_{24} \setminus U_{\xi_2} \), when \( 1 \leq p < +\infty \), by using (4.78) and (4.69), we have
\[ \| V^{(2)} - I \|_{L^p(\Sigma_{24} \setminus U_{\xi_2})} = \| p_{24}(\xi_2)e^{2i\beta(z_2, \xi)}(z - \xi_2)^{-2i\nu(z_2)}e^{-2it\theta} \|_{L^p(\Sigma_{24} \setminus U_{\xi_2})} \]
\[ \lesssim \| e^{-2it\theta} \|_{L^p(\Sigma_{24} \setminus U_{\xi_2})}. \] (4.96)
\[ \| e^{-2it\theta} \|_{L^p(\Sigma_{24} \setminus U_{\xi_2})} \leq \int_{\Sigma_{24} \setminus U_{\xi_2}} e^{-2tpe(1 + |z|^{-2})} \varrho^2 \, d\zeta \]
\[ = (1 + |z|^{-2} \geq 1) \lesssim \int_{\varrho}^{\infty} e^{-pcit} \, dl \]
\[ \lesssim t^{-\frac{1}{p}} e^{-ct}, \] (4.98)
where the value of \( c = c(\xi, \varpi) \) above changes from line to line. \( \square \)

**Proposition 4.5.** For \( 1 \leq p < +\infty \), there exists a constant \( h' = h'(p) > 0 \), such that the jump matrix \( V^{(2)} \) defined in (4.78) admit the following estimate as \( t \to +\infty \)
\[ \| V^{(2)} - I \|_{L^p(\Sigma_{1/2} \setminus U_{\xi_2})} = \mathcal{O}(e^{-h't}), \quad \text{for} \quad j = 1, 2, 3, 4. \] (4.99)

**Proof.** We only give the details for \( z \in \Sigma_{1/2} \).
\[ \| V^{(2)} - I \|_{L^p(\Sigma_{1/2} \setminus U_{\xi_2})} = \| (f_{24} - f_{14})e^{-2it\theta} \|_{L^p(\Sigma_{1/2} \setminus U_{\xi_2})} \]
\[ \lesssim \| e^{-2it\theta} \|_{L^p(\Sigma_{1/2} \setminus U_{\xi_2})} \]
\[ \lesssim t^{-\frac{1}{p}} e^{-ct}. \] (4.101)
For $z \in \Sigma_{1}^{j \pm}$, the proof is the same. For $z \in \Sigma_{2}^{j \pm}$, $j = 2, 3$, we shall additionally use (4.58) at the second step in the above equation.

The Proposition 4.4, Proposition 4.5 and Remark 4.3 tell the fact the jump $V^{(2)}$ uniformly goes to $I$ in terms of exponentially small error outside $U(\xi)$. Thus we can ignore the jump relation of $m^{rhp}(z)$ as $t \to \infty$. Inspired by this point, we decompose $m^{rhp}(z)$ as following structure

$$m^{rhp}(z) = \begin{cases} m^{err}(z)m^{sol}(z), & z \in \mathbb{C} \setminus U(\xi) \\ m^{err}(z)m^{rhp}(z)m^{mod}(z), & z \in U(\xi). \end{cases}$$

(4.102)

In this decomposition, $m^{sol}(z) = m^{rhp}|_{V^{(2)} = I}$ is affected by discrete spectrum, which is shown in Section 4.3.1. $m^{mod}(z)$ uses classical parabolic cylinder model to build a matrix to match the jump relation of $m^{rhp}$, which is shown in Section 4.3.2. $m^{err}(z)$ is an error function and a solution of a small norm Riemann-Hilbert problem which is shown in Section 4.3.3. As for the $\bar{\partial}$-problem $m^{(3)} = m^{(2)}(m^{rhp})^{-1}$, we will present details in the Section 4.4.

### 4.3 Analysis on a pure RH Problem

#### 4.3.1 Outside: A solvable model on discrete spectrum

In this section, we construct a reflectionless case of RHP 2.1 to show that its solution can be approximated by $m^{sol}(z)$. We keep all discrete spectrums after those transformations mentioned above, thus we find that these discrete spectrums which are all corresponded to the $m^{sol}$. The following RH problem constructs the $m^{sol}$.

**RHP 4.4.** Find a $2 \times 2$ matrix-valued function $m^{sol}(x, t; z)$ which satisfies

* $m^{sol}(z)$ is analytical in $\mathbb{C} \setminus \{\eta_n, \bar{\eta}_n\}_{n=1}^{2N}$,

* Asymptotic behavior

$$m^{sol}(x, t; z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty,$$

$$m^{sol}(x, t; z) = \frac{i}{z} \sigma_3 Q_+ + \mathcal{O}(1), \quad z \to 0.$$  

(4.103)

(4.104)

* Residue conditions

$$\text{Res} m^{sol}(z) = \lim_{z \to \eta_n} m^{sol}(z) \begin{bmatrix} 0 & c_n \delta^2(\eta_n) e^{-2it\theta_n} \\ c_n \delta^2(\eta_n) e^{-2it\theta_n} & 0 \end{bmatrix},$$

$$\text{Res} m^{sol}(z) = \lim_{z \to \bar{\eta}_n} m^{sol}(z) \begin{bmatrix} 0 & c_n \delta^{-2}(\bar{\eta}_n) e^{2it\theta_n} \\ c_n \delta^{-2}(\bar{\eta}_n) e^{2it\theta_n} & 0 \end{bmatrix}.$$  

(4.105)

(4.106)
Furthermore, we set a new RH problem \( N(z; \sigma_d) \), which is a reflectionless case \((r(z) = 0)\) of original RHP 2.1, and satisfies

**RHP 4.5.** Find a 2 × 2 matrix-valued function \( N(x, t; z; \sigma_d) \) which satisfies

* \( N(z; \sigma_d) \) is analytical in \( \mathbb{C} \setminus \mathcal{Z} \) and has simple poles in \( \mathcal{Z} = \{ \eta_n, \bar{\eta}_n \}_{n=1}^{2N} \).

* \( N(z; \sigma_d) = \sigma_1 \overline{N(\bar{z}; \sigma_d)\sigma_1} = \overline{N(-\bar{z}; \sigma_d)} = \mp z^{-1}N(z^{-1}; \sigma_d)\sigma_2 \).

* Asymptotic behavior

\[
N(x, t; z; \sigma_d) = I + O(z^{-1}), \quad z \to \infty, \quad (4.107)
\]

\[
N(x, t; z; \sigma_d) = \frac{i}{z} \sigma_3 Q_{-} + O(1), \quad z \to 0. \quad (4.108)
\]

* Residue conditions

\[
\text{Res}_{z = \eta_n} N(z; \sigma_d) = \lim_{z \to \eta_n} N(z; \sigma_d) \begin{bmatrix} 0 & 0 \\ c_n e^{-2i \theta_n} & 0 \end{bmatrix}, \quad (4.109)
\]

\[
\text{Res}_{z = \bar{\eta}_n} N(z; \sigma_d) = \lim_{z \to \bar{\eta}_n} N(z; \sigma_d) \begin{bmatrix} 0 & c_n e^{2i \theta_n} \\ 0 & 0 \end{bmatrix}, \quad (4.110)
\]

where the scattering data \( \sigma_d = \left\{ 0, \{ \eta_n, c_n \}_{n=1}^{2N} \right\} \).

Now we present the following proposition to explain that the solution under reflection case can be obtained from the solution reflectionless case.

**Proposition 4.6.** Given scattering data \( \sigma_d = \left\{ 0, \{ \eta_n, c_n \}_{n=1}^{2N} \right\} \), the RHP 4.5 exists an unique solution. Moreover, there exits an unique solution \( m_{sol} \) of RHP 4.4, which can be expressed as

\[
m_{sol}(z) = N(z; \tilde{\sigma}_d) = N(z, \sigma_d)\delta(z)^{\sigma_3}, \quad (4.111)
\]

where the modified scattering data \( \tilde{\sigma}_d = \left\{ 0, \{ \eta_n, c_n \delta^2(\eta_n) \}_{n=1}^{2N} \right\} \).

**Proof.** The uniqueness of solution follows from the Liouville’s theorem through an explicit transformation \( \delta(z) \). And

\[
q^{sol} = -i \lim_{z \to \infty} (zm^{sol}(z))_{12} = -i \lim_{z \to \infty} (zN(z; \sigma_d) \delta(z)^{\sigma_3})_{12} = q^{sol,N}. \quad (4.112)
\]

More details could be referred in [25, Proposition 5.6] and [23, Appendix A, B].

To calculate the \( m^{sol}(z) \), we give the following lemma.
Lemma 4.4. The RHP 4.4 exists an unique solution. Moreover, \( m^{\text{sol}}(z) \) is equivalent to a reflectionless solution of the original RHP 2.1 with scattering data \( \tilde{\sigma}_d = \{ \eta_n, c_n \delta^2(\eta_n) \}_{n=1}^{2N} \) as follows

\[
m^{\text{sol}}(z) = I + \frac{i}{z} \sigma_3 Q_z + \sum_{n=1}^{2N} \left[ \frac{\alpha_n}{z-\eta_n} - \frac{\bar{\beta}_n}{z-\eta_n} \sigma_1 \right],
\]

where \( \alpha_n = \alpha_n(x,t), \beta_n = \beta_n(x,t) \) with following relations:

\[
\alpha_n + \frac{i}{\eta_l} - \sum_{n=1}^{2N} \bar{\beta}_n (\eta_l - \bar{\eta}_n) = 0, \tag{4.114}
\]

\[
\beta_n - 1 - \sum_{n=1}^{2N} \bar{\alpha}_n (\eta_l - \bar{\eta}_n) = 0, \tag{4.115}
\]

where \( \gamma_{\eta_l} = c_l \delta^2(\eta_l) e^{-2i\theta_l} \), for \( l = 1, \cdots, 2N \) respectively.

Proof. The uniqueness of solution follows from the Liouville’s theorem.

Let us define a nilpotent matrix

\[
N_{\eta_n} = \begin{bmatrix} 0 & 0 \\ \gamma_{\eta_n} & 0 \end{bmatrix},
\]

then we can take use the symmetries of \( \delta(z), \theta(z) \) to rewrite residue conditions as

\[
\text{Res}_{z=\eta_n} m^{\text{sol}}(z) = \lim_{z \to \eta_n} m^{\text{sol}}(z) N_{\eta_n}, \tag{4.117}
\]

\[
\text{Res}_{z=\bar{\eta}_n} m^{\text{sol}}(z) = \lim_{z \to \bar{\eta}_n} m^{\text{sol}}(z) \sigma_1 \bar{N}_{\eta_n} \sigma_1, \tag{4.118}
\]

With this symmetry of residue condition, (4.113) follows from the Plemelj formulae with \( r(z) \equiv 0 \). To obtain the relation (4.114) and (4.115), we just substitute (4.113) into the residue conditions.

Corollary 4.2. Denote \( q^{\text{sol}} \) the soliton solution with scattering data \( \tilde{\sigma}_d = \{ \eta_n, c_n \delta^2(\eta_n) \}_{n=1}^{2N} \).

By potential recovering formulae (2.67), the soliton solution is given by

\[
q^{\text{sol}}(x,t) = -1 - i \sum_{n=1}^{2N} \bar{\beta}_n. \tag{4.119}
\]

4.3.2 Inside: A local solvable RH model near saddle points

Based on the Proposition 4.4, The RHP is localized in the small neighborhoods of those stationary phase points. Denote a new contour \( \Sigma^{\text{mod}} = (\cup_{j,k=1,2,3,4} \Sigma_{jk}) \cap U(\xi) \) in figure 9.

Then we consider the following RH problem.
RHP 4.6. Find a 2 × 2 matrix-valued function \( m^{\text{mod}}(x, t; z) \) such that

* \( m^{\text{mod}}(z) \) is analytical in \( \mathbb{C} \setminus \Sigma^{\text{mod}} \),

* \( m^{\text{mod}}(z) \) takes continuous boundary values \( m^{\text{mod}}_{\pm}(z) \) on \( \Sigma^{\text{mod}} \) with jump relation

\[
m^{\text{mod}}_{+}(z) = m^{\text{mod}}_{-}(z)V^{(2)}(z), \quad z \in \Sigma^{\text{mod}}
\] (4.120)

* Asymptotic behavior

\[
m^{\text{mod}}(x, t; z) = I + O(z^{-1}), \quad z \to \infty.
\] (4.121)

There exist jump relation but no poles in the RHP 4.6. The analysis of the RH problem is based on the so-called Beals-Coifman operator theory [30]. Now we use the Beals-Coifman theory to construct the relation between \( m^{\text{mod}} \) and \( \sum_{j=1}^{4} m^{\text{mod}, \xi_j} \), where \( m^{\text{mod}, \xi_j} \) can be constructed by parabolic cylinder equation.

For \( z \in \Sigma^{\text{mod}}_{jk} \), \( k = 1, 2, 3, 4 \) we define the following RH problem

RHP 4.7. Looking for 2 × 2 matrix-valued function \( N^{(1)}(z) \) such that

* \( N^{(1)}(z) \) is analytical off \( \Sigma^{\text{mod}}_{jk} \), \( k = 1, 2, 3, 4 \)

* \( N^{(1)}(z) \) takes continuous boundary values \( N^{(1)}_{\pm}(z) \) on \( \Sigma^{\text{mod}}_{jk} \), \( k = 1, 2, 3, 4 \) with jump relation

\[
N^{(1)}_{+}(z) = N^{(1)}_{-}(z)V^{(2)}_{N^{(1)}}(z), \quad z \in \Sigma_{jk},
\] (4.122)

where \( V^{(2)}_{N^{(1)}}(z) = V^{(2)}(z), z \in \Sigma_{jk}, \) for \( k = 1, 2, 3, 4 \)

* Asymptotic behavior

\[
N^{(1)}(x, t; z) = I + O(z^{-1}), \quad z \to \infty,
\] (4.123)
$V_{N(1)}(z)$ enjoys a factorization

$$(I - w_{jk}^-)^{-1} (I + w_{jk}^+), \quad (4.124)$$

$$w_{jk}^- = I - V_{N(1)}^{-1} = V_{N(1)} - I, \quad w_{jk}^+ = 0, \quad (4.125)$$

and the superscript $\pm$ indicate the analyticity in the positive/negative neighborhood of the contour.

Recall the Cauchy projection operator $C_{\pm}$ on $\Sigma_{jk}^{\text{mod}}$, $k = 1, 2, 3, 4$

$$C_{\pm} f(z) = \lim_{z \to s \in \Sigma_{jk}^{\text{mod}}_{\pm}} \frac{1}{2\pi i} \int_{\Sigma_{jk}^{\text{mod}}} \frac{f(s)}{s - z} ds, \quad (4.126)$$

we can define the Beals-Coifman operator on $\Sigma_{jk}^{\text{mod}}$, $k = 1, 2, 3, 4$ as follows

$$C_{w_{jk}}(f) := C_+(fw_{jk}^-) + C_-(fw_{jk}^+). \quad (4.127)$$

Then we define

$$w_j = \sum_{k=1}^{4} w_{jk}, \quad \Sigma_j^{\text{mod}} = \bigcup_{k=1,2,3,4} \Sigma_{jk}^{\text{mod}}, \quad (4.128)$$

$$w = \sum_{j=1}^{4} w_j = \sum_{j,k=1}^{4} w_{jk}, \quad \Sigma^{\text{mod}} = \bigcup_{j=1,2,3,4} \Sigma_j^{\text{mod}} = \bigcup_{j,k=1,2,3,4} \Sigma_{jk}^{\text{mod}}, \quad (4.129)$$

then we obtain $C_w = \sum_{j=1}^{4} C_{w_j} = \sum_{j,k=1}^{4} C_{w_{jk}}$. Now we introduce the following theorem, which plays a vital role in the steepest method

**Theorem 4.1.** If $\mu \in I + L^2(\Sigma)$ is the solution of the singular integral equation

$$\mu = I + C_w(\mu), \quad (4.130)$$

Then there exists unique solution to the RHP for $m_{\text{mod}}$ written as

$$m_{\text{mod}} = I + C(\mu w). \quad (4.131)$$

Based on the above discussions, we now try to construct the Beals-Coifman solution of $m_{\text{mod}}$. We start with the following lemma

**Lemma 4.5.** The matrix functions $w_{jk}$ defined in (4.125) admit the following estimation

$$\|w_{jk}\|_{L^2(\Sigma_{jk}^{\text{mod}})} = O(t^{-1/2}). \quad (4.132)$$
This lemma implies that $1-C_w$, $1-C_{wj}$ and $1-C_{wk}$ exist. Moreover, with the Theorem 4.1, the Beals-Cofiman solution for $m^{\text{mod}}$ exist unique as

$$m^{\text{mod}} = I + \frac{1}{2\pi i} \int_{\Sigma^{\text{mod}}} \frac{(1 - C_w)^{-1}Iw}{s - z} ds. \quad (4.133)$$

However, the integral $I + \frac{1}{2\pi i} \int_{\Sigma^{\text{mod}}} \frac{(1 - C_w)^{-1}Iw}{s - z} ds$ is still hard to compute. Follow the standard procedure of Deift-Zhou [28], we can separate the contributions from each saddle point. Before executing this procedure, we need the following lemma.

**Lemma 4.6.** As $t \to +\infty$, for $j \neq k$

$$\|C_{wj}C_{wk}\|_{L^2(\Sigma^{\text{mod}})} = O(t^{-1}), \quad \|C_{wj}C_{wk}\|_{L^\infty(\Sigma^{\text{mod}}) \to L^2(\Sigma^{\text{mod}})} = O(t^{-1}) \quad (4.134)$$

**Proof.** Thanks to the observation of Varzugin [31], we have

$$1 - \sum_{j \neq k} C_{wj}C_{wk} (1 - C_{wk})^{-1} = (1 - C_w) \left(1 + \sum_{j=1}^4 C_{wj} (1 - C_{wj})^{-1}\right), \quad (4.135)$$

$$1 - \sum_{j \neq k} (1 - C_{wk})^{-1} C_{wj}C_{wk} = \left(1 + \sum_{j=1}^4 C_{wj} (1 - C_{wj})^{-1}\right) (1 - C_w). \quad (4.136)$$

Take use of Lemma 4.5, we prove this lemma. \qed

Now we can separate the contribution of Beals-Cofiman solution for $m^{\text{mod}}$ from each saddle point, which is expressed by the following proposition.

**Proposition 4.7.** As $t \to +\infty$

$$\int_{\Sigma^{\text{mod}}} \frac{(1 - C_w)^{-1}Iw}{s - z} ds = \sum_{j=1}^4 \int_{\Sigma^{\text{mod}}} \frac{(1 - C_{wj})^{-1}Iw_j}{s - z} ds + O(t^{-1}). \quad (4.137)$$

**Proof.** Firstly, we can decompose the resolvent $(1 - C_w)^{-1}I$ as

$$(1 - C_w)^{-1}I = I + \sum_{j=1}^4 C_{wj} (1 - C_{wj})^{-1}I + QPRI, \quad (4.138)$$

where

$$Q := 1 + \sum_{j=1}^4 C_{wj} (1 - C_{wj})^{-1}, \quad (4.139)$$

$$P := \left(1 - \sum_{j \neq k} C_{wj}C_{wk} (1 - C_{wk})^{-1}\right)^{-1} \quad (4.140)$$

$$R := \sum_{j \neq k} C_{wj}C_{wk} (1 - C_{wk})^{-1} \quad (4.141)$$
By Cauchy-Schwarz inequality
\[
| \int QPRIw | \leq \| Q \|_{L^2(\Sigma^\mod_j)} \| P \|_{L^2(\Sigma^\mod_j)} \| R \|_{L^2(\Sigma^\mod_j)} \| w \|_{L^2} \lesssim t^{-1}. \tag{4.142}
\]
The rest of the proof is trivial. \hfill \Box

Based on the proposition, we reduce the RHP 4.6 to a model RHP whose solution can be expressed in terms of parabolic cylinder functions (or Webber equation) on each contour \(\Sigma^\mod_j, j = 1, 2, 3, 4\). For \(z\) near \(\xi_j\), we rewrite phase function as
\[
\theta(z) = \theta(\xi_j) + \frac{\theta''(\xi_j)}{2}(z - \xi_j)^2 + O(|z - \xi_j|^3), \quad z \to \xi_j. \tag{4.143}
\]

**Proposition 4.8.** For the phase reduction, the error generated by reducing the phase function \(\theta(z)\) to \(\theta(\xi_j) + \frac{\theta''(\xi_j)}{2}(z - \xi_j)^2\) is bounded by \(O(t^{-1})\) as \(t \to +\infty\).

**Proof.** The rigorous analysis is similar to [32, Section 8.2] or [33, Lemma 3.35]. Here we present a simple explanation in Appendix B. \hfill \Box

Now we consider the following RH problem near the saddle point \(\xi_j, j = 1, 2, 3, 4\).

**RHP 4.8.** Find a \(2 \times 2\) matrix-valued function \(m^{\mod,j}(x,t;z)\) such that

* \(m^{\mod,j}(z) := m^{\mod,j}(\xi_j(z))\) is analytical in \(\mathbb{C} \setminus \Sigma^\mod_j\),

* \(m^{\mod,j}(z)\) takes continuous boundary values \(m^{\mod,j}_\pm(z)\) on \(\Sigma^\mod_j\) with jump relation
\[
m^{\mod,j}_+(z) = m^{\mod,j}_-(z)V^{\mod,j}(z), \quad z \in \Sigma^\mod_j \tag{4.144}
\]

where
\[
V^{\mod,j}(z) = \begin{cases} 
    \begin{bmatrix} 
        1 & \frac{r(\xi_j)}{1-|r(\xi_j)|^2} e^{2i\beta(\xi_j,\xi)}(z - \xi_j) e^{-2i\nu(\xi_j)t} \\
        0 & 1 
    \end{bmatrix}, & z \in \Sigma^\mod_{j1}, \\
    \begin{bmatrix} 
        1 & 0 \\
        \frac{r(\xi_j)}{1-|r(\xi_j)|^2} e^{-2i\beta(\xi_j,\xi)}(z - \xi_j) e^{2i\nu(\xi_j)t} & 1 
    \end{bmatrix}, & z \in \Sigma^\mod_{j2}, \\
    \begin{bmatrix} 
        1 & 0 \\
        -\frac{r(\xi_j)}{1-|r(\xi_j)|^2} e^{-2i\beta(\xi_j,\xi)}(z - \xi_j) e^{2i\nu(\xi_j)t} & 1 
    \end{bmatrix}, & z \in \Sigma^\mod_{j3}, \\
    \begin{bmatrix} 
        1 & 0 \\
        r(\xi_j) e^{2i\beta(\xi_j,\xi)}(z - \xi_j) e^{-2i\nu(\xi_j)t} & 1 
    \end{bmatrix}, & z \in \Sigma^\mod_{j4}. 
\end{cases} \tag{4.145}
\]

* asymptotic behavior
\[
m^{\mod,j}(x,t;z) = I + O(z^{-1}), \quad z \to \infty. \tag{4.146}
\]
\[ \begin{pmatrix} \frac{1}{r(\xi_1) + 2i\beta(\xi_1, \xi_1)}(z - \xi_1) e^{-2i\nu(\xi_1) x} - 2i e^{-2i\beta(\xi_1, \xi_1)(z - \xi_1) - 2i\nu(\xi_1) x} \frac{r(\xi_1)}{1 - r(\xi_1)} e^{-2i\beta(\xi_1, \xi_1)(z - \xi_1) - 2i\nu(\xi_1) x} \\ 0 \end{pmatrix} \]

\[ \begin{pmatrix} 1 - \frac{r(\xi_1)}{1 - r(\xi_1)} e^{-2i\beta(\xi_1, \xi_1)(z - \xi_1) - 2i\nu(\xi_1) x} \frac{r(\xi_1)}{1 - r(\xi_1)} e^{-2i\beta(\xi_1, \xi_1)(z - \xi_1) - 2i\nu(\xi_1) x} \\ 1 \end{pmatrix} \]

Figure 10: The contour \( \Sigma_1^{\text{mod}} \) and the jump matrix on it.

\[ \begin{pmatrix} 1 - \frac{r(\xi_2)}{1 - r(\xi_2)} e^{-2i\beta(\xi_2, \xi_2)(z - \xi_2) - 2i\nu(\xi_2) x} \frac{r(\xi_2)}{1 - r(\xi_2)} e^{-2i\beta(\xi_2, \xi_2)(z - \xi_2) - 2i\nu(\xi_2) x} \\ 0 \end{pmatrix} \]

\[ \begin{pmatrix} \frac{r(\xi_2)}{1 - r(\xi_2)} e^{2i\beta(\xi_2, \xi_2)(z - \xi_2) - 2i\nu(\xi_2) x} - 2i e^{2i\beta(\xi_2, \xi_2)(z - \xi_2) - 2i\nu(\xi_2) x} \frac{r(\xi_2)}{1 - r(\xi_2)} e^{2i\beta(\xi_2, \xi_2)(z - \xi_2) - 2i\nu(\xi_2) x} \\ 0 \end{pmatrix} \]

Figure 11: The contour \( \Sigma_2^{\text{mod}} \) and the jump matrix on it.
In order to motivate the model, let \( \zeta = \zeta(z) \) be the rescaled local variable

\[
\zeta(z) = (2t\epsilon_j \theta''(\xi_j))^{\frac{1}{2}} (z - \xi_j).
\]  

(4.147)

And this scaling defined by \( N_{\xi_j} \) admits the following mapping

\[
N_{\xi_j} : U_{\xi_j} \rightarrow U_0,
\]

\[
z \mapsto \zeta.
\]  

(4.148)

Additionally, we let the free variable

\[
r_{\xi_j} = r(\xi_j) e^{2i\beta_j(\xi_j) - 2it\theta(\xi_j)} \exp \left[ -i\epsilon_j \nu(\xi_j) \log(2t\epsilon_j \theta''(\xi_j)) \right],
\]

(4.150)

with the equality \( |r(\xi_j)|^2 = |r_{\xi_j}|^2 \).

In the above expression, the complex powers are defined by choosing the branch of the logarithm with \( 0 < \arg \zeta < 2\pi \) near \( \xi_1, \xi_3 \), and the branch of the logarithm with \( -\pi < \arg \zeta < \pi \) near \( \xi_2, \xi_4 \). Through this scaling of variable, the jump \( V_{\text{mod},j}(z) \) approximates the jump of a parabolic cylinder model which can be checked in Appendix A.

Follow the procedure in [28] or [34] and use the results (A.27), (A.53) of Appendix in our paper, we know that for each \( \xi_j \)

\[
m_{\text{mod},j}(z) = I + \frac{(2t\epsilon_j \theta''(\xi_j))^{-\frac{1}{2}}}{z - \xi_j} \begin{pmatrix}
0 & \epsilon_j i\beta_{12}^j \\
-\epsilon_j i\beta_{21}^j & 0
\end{pmatrix} + O(t^{-1}).
\]  

(4.151)

Thus the solution \( m_{\text{mod}} \) can be expressed as the following proposition as \( t \to +\infty \).

**Proposition 4.9.** As \( t \to +\infty \), we have

\[
m_{\text{mod}}(z) = I + t^{-\frac{1}{2}} \sum_{j=1}^{4} \frac{it\epsilon_j A_{j}^{\text{mat}}}{(2\epsilon_j \theta''(\xi_j))^{\frac{1}{2}} (z - \xi_j)} + O(t^{-1}),
\]

(4.152)

where

\[
A_{j}^{\text{mat}} = \begin{pmatrix}
0 & \beta_{12}^j \\
-\beta_{21}^j & 0
\end{pmatrix}
\]

(4.153)

For \( j = 1, 3 \)

\[
\beta_{12}^j = \frac{\sqrt{2\pi e^{-\frac{i\pi}{4}} e^{\frac{3\pi i}{4}(\xi_j)}}}{r_{\xi_j} \Gamma(i\nu(\xi_j))},
\]

(4.154)

\[
\beta_{12}^j \beta_{21}^j = \nu(\xi_j),
\]

(4.155)

\[
\arg \beta_{21}^j = -\frac{5\pi}{4} - \arg r_{\xi_j} - \arg \Gamma(i\nu(\xi_j))
\]

\[
= -\frac{5\pi}{4} - 2\beta(\xi_j, \xi) + 2t\theta(\xi_j) + \nu(\xi_j)\log(2t\epsilon_j \theta''(\xi_j)) - \arg \Gamma(i\nu(\xi_j)).
\]

(4.156)
For \( j = 2, 4 \)

\[
\beta_{12}^{\xi_j} = \frac{\sqrt{2\pi e^{i\pi/4}}}{r_{\xi_j} \Gamma(-i\nu(\xi_j))},
\]

\( \beta_{12}^{\xi_j} \beta_{21}^{\xi_j} = \nu(\xi_j), \) \hspace{1cm} (4.157)

\[
\arg(\beta_{12}^{\xi_j}) = \frac{\pi}{4} - \arg(r_{\xi_j}) - \arg(\Gamma(-i\nu(\xi_j)))
\]

\( = \frac{\pi}{4} - 2\beta(\xi_j, \xi) + 2t\theta(\xi_j) - \nu(\xi_j)\log(-2t\theta''(\xi_j)) - \arg(\Gamma(-i\nu(\xi_j))). \) \hspace{1cm} (4.159)

**Corollary 4.3.** As \( t \to \infty \), we have

\[
|m_{\text{mod}} - I| \lesssim t^{-\frac{1}{2}}, \quad \|m_{\text{mod}}\|_{\infty} < 1.
\]

(4.160)

**Proof.** The first one directly follows from (4.152), and the second one is similar to [35, Lemma D.1]. \( \Box \)

### 4.3.3 Error: A small norm RH problem

In this section, we consider the error function \( m_{\text{err}} \) which include the jump outside \( U(\xi) \). In fact, \( m_{\text{err}} \) satisfies the following RHP

**RHP 4.9.** Find a \( 2 \times 2 \) matrix-valued function \( m_{\text{err}}(z) \) such that

* \( m_{\text{err}} \) is analytical in \( \mathbb{C} \setminus \Sigma_{\text{err}} \), where

\[
\Sigma_{\text{err}} = \partial U(\xi) \cup \left( \Sigma^{(2)} \setminus \partial U(\xi) \right);
\]

(4.161)

* \( m_{\text{err}} \) takes continuous boundary values \( m_{\text{err}}^\pm(z) \) on \( \Sigma_{\text{err}} \) and

\[
m_{\text{err}}^+(z) = m_{\text{err}}^-(z)V_{\text{err}}(z),
\]

(4.162)

where

\[
V_{\text{err}}(z) = \begin{cases} 
    m_{\text{sol}}(z)V^{(2)}(z)m_{\text{sol}}^{-1}(z), & z \in \Sigma^{(2)} \setminus \partial U(\xi), \\
    m_{\text{mod}}(z)m_{\text{sol}}(z)m_{\text{sol}}^{-1}(z), & z \in \partial U(\xi).
\end{cases}
\]

(4.163)

* asymptotic behavior

\[
m_{\text{err}} = I + \mathcal{O}(z^{-1}), \quad z \to \infty.
\]

(4.164)
Then, $m^{err}$ follows from the small norm theory for RH problem. Recall Proposition 4.4 and 4.5, we have
\[
\|V^{err} - I\|_{L^2} \lesssim \begin{cases} 
  e^{-ht}, & z \in \Sigma_{jk} \setminus U(\xi) \\
  e^{-\eta t'}, & z \in \Sigma_{j \pm}^{1/2}.
\end{cases} \tag{4.165}
\]
For $z \in \partial U(\xi)$, $m^{sol}(z)$ is bounded, we obtain that
\[
|V^{err} - I| = |m^{sol}(z)m^{mod}(z)m^{sol}(z)^{-1} - I| = |m^{sol}(z)(m^{mod}(z) - I)m^{sol}(z)^{-1}|
\]
\[
= O(t^{-1/2}) \tag{4.152}
\]
According to Beals-Cofiman theory, the solution for RHP 4.9 can be given by
\[
m^{err} = I + \frac{1}{2\pi i} \int_{\Sigma^{err}} \frac{(I + \mu(s))(V^{err}(s) - I)}{s - z} ds, \tag{4.167}
\]
where $\mu \in L^2(\Sigma^{err})$ is the unique solution of $(1 - C_{V^{err}})\mu = C_{V^{err}}I$. And $C_{V^{err}}: L^2(\Sigma^{err}) \rightarrow L^2(\Sigma^{err})$ is the Cauchy projection operator on $\Sigma^{err}$
\[
C_{V^{err}}(f)(z) = C_{-}f(V^{err} - I) = \lim_{s \to z} \int_{\Sigma^{err}} \frac{f(s)(V^{err}(s) - I)}{s - z} ds. \tag{4.168}
\]
Existence and uniqueness of $\mu$ follows from the boundedness of the Cauchy projection operator $C_{-}$, which implies
\[
\|C_{V^{err}}\| \lesssim \|C_{-}\|_{L^2 \rightarrow L^2}\|V^{err} - I\|_{L^2} \lesssim O(t^{-1/2}), \tag{4.169}
\]
moreover
\[
\|\mu\|_{L^2(\Sigma^{err})} \lesssim \frac{\|C_{V^{err}}\|}{1 - \|C_{V^{err}}\|} \lesssim t^{-1/2}. \tag{4.170}
\]
Base on the discussions above for $m^{err}$, we have the following proposition

**Proposition 4.10.** As $t \to \infty$, we have
\[
m^{err}_1 = i\epsilon_j \sum_{j=1}^4 (2t\epsilon_j \theta''(\xi_j))^{-\frac{1}{2}} m^{sol}(\xi_j) A^{mat}_j m^{sol}(\xi_j)^{-1} + O(t^{-1}). \tag{4.171}
\]
where $m^{err}_1$ comes from the asymptotic expansion of $m^{err}$
\[
m^{err} = I + z^{-1}m^{err}_1 + O(z^{-2}). \tag{4.172}
\]
Proof. Recall (4.167), we know that
\[ m_{1}^{err} = -\frac{1}{2\pi i} \int_{\Sigma^{err}} (I + \mu(s)) (V^{err}(s) - I) ds := I_{1} + I_{2} + I_{3}. \]  
(4.173)

where

\[ I_{1} = -\frac{1}{2\pi i} \oint_{\partial U(\xi)} (V^{err}(s) - I) ds, \]  
(4.174)

\[ I_{2} = -\frac{1}{2\pi i} \int_{\Sigma^{err} \setminus U(\xi)} (V^{err}(s) - I) ds, \]  
(4.175)

\[ I_{3} = -\frac{1}{2\pi i} \int_{\Sigma^{err}} \mu(s)(V^{err}(s) - I) ds. \]  
(4.176)

Take use Proposition 4.4 and 4.5, we can obtain \(|I_{2}| = O(t^{-1})\).

As for \(I_{3}\), we take use (4.170) and (4.166)
\[ |I_{3}| \lesssim \|\mu\|_{L^{2}} \|V^{err} - I\|_{L^{2}} \lesssim t^{-1}. \]  
(4.177)

Finally, we deal with \(I_{1}\)
\[ I_{1} = \frac{1}{2\pi i} \sum_{j=1}^{4} \oint_{\partial U_{\epsilon_{j}}} m^{sol}(s) (m^{mod,j}(s) - I) m^{sol}(s)^{-1} ds \]
\[ = \frac{1}{2\pi i} \sum_{j=1}^{4} \oint_{\partial U_{\epsilon_{j}}} \frac{i\epsilon_{j}}{2t^{2} \xi_{j}^{2}} m^{sol}(s) A^{\text{mat}}_{j} m^{sol}(s) m^{sol}(s)^{-1} ds \]
(residue theorem) \[ = i\epsilon_{j} \sum_{j=1}^{4} (2t^{2}\theta''(\xi_{j}))^{-\frac{1}{2}} m^{sol}(\xi_{j}) A^{\text{mat}}_{j} m^{sol}(\xi_{j})^{-1}. \]  
(4.178)

\[ \square \]

**Corollary 4.4.** Define \(-i (m_{1}^{err})_{12} := t^{-\frac{1}{2}} f\), we have
\[ f = \epsilon_{j} \sum_{j=1}^{4} (1 - \xi_{j}^{-2})^{-1} (2\epsilon_{j} \theta''(\xi_{j}))^{-\frac{1}{2}} \left[ \beta_{12}^{\xi_{j}} m_{11}^{sol}(\xi_{j})^{2} + \beta_{21}^{\xi_{j}} m_{12}^{sol}(\xi_{j})^{2} \right]. \]  
(4.179)

And the \(\beta_{12}^{\xi_{j}}, \beta_{21}^{\xi_{j}}\) follows from Proposition 4.9.

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Proof. From Proposition 4.10, we can know that
\[ m_1^{cterr} = i \epsilon \sum_{j=1}^{4} \left( 2 t \epsilon \theta''(\xi_j) \right)^{-\frac{1}{2}} m^{sol}(\xi_j) A_j^{mat}(\xi_j)^{-1} + O(t^{-1}) \]
\[ = i \epsilon \sum_{j=1}^{4} \left( 2 t \epsilon \theta''(\xi_j) \right)^{-\frac{1}{2}} m^{sol}(\xi_j) A_j^{mat} (1 - \xi_j^{-2})^{-1} \sigma_2 m^{sol}(\xi_j)^{-1} + O(t^{-1}) \]
\[ = i \epsilon \sum_{j=1}^{4} (1 - \xi_j^{-2})^{-1} \left( 2 \epsilon \theta''(\xi_j) \right)^{-\frac{1}{2}} t^{-\frac{1}{2}} m^{sol}(\xi_j) A_j^{mat} \sigma_2 m^{sol}(\xi_j)^{-1} + O(t^{-1}). \] (4.180)

and
\[ (m_1^{cterr})_{12} = i \epsilon \sum_{j=1}^{4} (1 - \xi_j^{-2})^{-1} \left( 2 \epsilon \theta''(\xi_j) \right)^{-\frac{1}{2}} t^{-\frac{1}{2}} \left[ \beta_{12}^2 m_{12}^{sol}(\xi_j)^2 + \beta_{21}^2 m_{12}^{sol}(\xi_j)^2 \right]. \] (4.181)

By the potential recover formulae (2.67), we obtain our result (4.179). □

4.4 Analysis on a continuous component pure $\bar{\partial}$-Problem

Now we define the function
\[ m^{(3)}(z) = m^{(2)}(z)(m^{rhp}(z))^{-1}. \] (4.182)

Then $m^{(3)}$ satisfies the following $\bar{\partial}$-Problem.

$\bar{\partial}$-Problem 4.1. Find a $2 \times 2$ matrix-valued function $m^{(3)}(z)$ such that
* $m^{(3)}(z)$ is continuous in $\mathbb{C}$ and analytic in $\mathbb{C}\setminus \overline{\Omega}$;
* asymptotic behavior
\[ m^{(3)}(z) = I + O(z^{-1}), \quad z \to \infty; \] (4.183)
* For $z \in \mathbb{C}$, we have
\[ \bar{\partial} m^{(3)}(z) = m^{(3)}(z) W^{(3)}(z); \] (4.184)

where $W^{(3)} = m^{rhp}(z) \bar{\partial} R^{(2)}(z)(m^{rhp}(z))^{-1}$.

Proof. The first property is obvious. Then we prove the following claims.

Claim 1: $m^{(3)}$ has no jumps;
Since $m^{(2)}$ and $m^{rhp}$ take the same jump matrix, we have
\[ m_-^{(3)}(z)^{-1} m_+^{(3)}(z) = m_-^{rhp}(m_-^{(2)})^{-1} m_+^{(2)}(m_+^{rhp})^{-1} \]
\[ = m_-^{rhp} V^{(2)}(m_+^{rhp})^{-1} = I. \] (4.185)

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Claim 2: \( m^{(3)} \) has no singularity at \( z = 0 \).

Near \( z = 0 \), we have

\[
(m^{rhp})^{-1} = \left( I + \frac{i}{z} \sigma_3 Q - \frac{1}{z^2} \right)^{-1} = \frac{\sigma_1 m^{rhp} \sigma_1}{1 - z^{-2}}. 
\] (4.186)

Thus

\[
\lim_{z \to 0} m^{(3)} = \lim_{z \to 0} \frac{m^{(2)}(z) \sigma_1 m^{rhp} \sigma_1}{1 - z^{-2}} = (\sigma_3 Q - \sigma_1)^2 = I = \mathcal{O}(1). 
\] (4.187)

Claim 3: \( m^{(3)} \) has no singularities at \( Z = \{ \eta_n, \bar{\eta}_n \} \).

Let \( \mathcal{N}_\eta \) denote the nilpotent matrix which appears in the residue condition of \( m^{(2)} \) and \( m^{rhp} \), we have Laurent expansions at \( z = \eta \)

\[
m^{(2)}(z) = a(\eta) \left( \frac{\mathcal{N}_\eta}{z - \eta} + I \right) + \mathcal{O}((z - \eta)),
\] (4.188)

\[
m^{rhp}(z) = A(\eta) \left( \frac{\mathcal{N}_\eta}{z - \eta} + I \right) + \mathcal{O}((z - \eta)),
\] (4.189)

then, we have

\[
m^{(3)} = m^{(2)}(m^{rhp})^{-1} = \left[ a(\eta) \left( \frac{\mathcal{N}_\eta}{z - \eta} + I \right) \right] \left[ \left( -\frac{\mathcal{N}_\eta}{z - \eta} + I \right) \sigma_2 A^T(\eta) \sigma_2 \right] = \mathcal{O}(1). 
\] (4.190)

Claim 4: \( m^{(3)} \) has no singularities at \( z = \pm 1 \).

This follows from observing that the symmetries of RH problem applied to the local expansion of \( m^{(2)}(z) \) and \( m^{rhp}(z) \) imply that

\[
V(z) = \begin{bmatrix} c & \mp ic \\
\pm ic & \bar{c} \end{bmatrix}, \quad m^{rhp}(z)^{-1} = \frac{\pm 1}{2(z \mp 1)} \sigma_1 \begin{bmatrix} \gamma & \pm i\bar{\gamma} \\
\mp i\gamma & \bar{\gamma} \end{bmatrix} \sigma_1 + \mathcal{O}(1), \tag{4.191}
\]

for some constants \( c \) and \( \gamma \). Thus we have \( \lim_{z \to \pm 1} m^{(3)}(z) = \mathcal{O}(1) \).

\[\square\]

Now we consider the long time asymptotic behavior of \( m^{(3)} \). The solution of \( \bar{\partial} \)-Problem 4.1 can be solved by the following integral equation

\[
m^{(3)}(z) = I - \frac{1}{\pi} \int_{\mathbb{C}} \frac{m^{(3)}(s)W^{(3)}(s)}{s - z} dA(s), \tag{4.192}
\]

where \( A(s) \) is the Lebesgue measure on \( \mathbb{C} \). Denote \( S \) as the Cauchy-Green integral operator

\[
S[f](z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{f(s)W^{(3)}(s)}{s - z} dA(s), \tag{4.193}
\]

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then (4.192) can be written as the following operator equation

\[(1 - S)m^{(3)}(z) = I.\]  

(4.194)

To prove the existence of the operator at large time, we present the following lemma

**Lemma 4.7.** Consider the operator \( S \) defined by (4.193), then we have \( S : L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C}) \cap C^0(\mathbb{C}) \) and

\[\|S\|_{L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C})} \lesssim t^{-\frac{1}{4}}.\]  

(4.195)

**Proof.** For any \( f \in L^\infty \), we have

\[\|Sf\|_{L^\infty} \leq \|f\|_{L^\infty} \frac{1}{\pi} \iint_\mathbb{C} \frac{|W^{(3)}(s)|}{|s - z|} dA(s).\]  

(4.196)

Recall the definition \( W^{(3)} = m_{rhp}(z)\overline{\partial R^{(2)}(z)(m_{rhp}(z))^{-1}} \) and \( \partial R^{(2)} \), we know that \( W^{(3)}(z) \equiv 0 \) for \( z \in \mathbb{C} \setminus \bar{\Omega} \). Besides, we only take into account that matrix functions have support in the sector \( \Omega_{jk} \). Based on these conditions, what we need to do is to control the boundedness of the integral \( \iint_\mathbb{C} \frac{|W^{(3)}(s)|}{|s - z|} dA(s) \) for \( z \in \Omega_{jk}, j = 0^{\pm}, 1, 2, 3, 4, k = 1, 2, 3, 4 \). We can take \( z \in \Omega_{0+1}, z \in \Omega_{24} \) and \( z \in \Omega_{11} \) as examples, the proofs for the rest regions are similar.

Since \( \det m_{rhp}(z) = 1 - z^{-2} \) and \( m_{rhp}(z)^{-1} = (1 - z^{-2})^{-1}\sigma_2m_{rhp}\sigma_2 \), we have

\[|W^{(3)}(s)| \leq |m_{rhp}(s)|^2|1 - s^{-2}|^{-1}|\partial R^{(2)}(s)|.\]  

(4.197)

Next, since \( m_{rhp} = m_{sol}(I + z^{-1}m_{err} + \mathcal{O}(z^{-2})) \), we can bound \( m_{rhp} \)

\[|m_{rhp}(s)| \lesssim 1 + |s|^{-1} = c\sqrt{1 + |s|^{-1}} \lesssim \sqrt{1 + |s|^{-2}} = |s|^{-1} \sqrt{1 + |s|^2} = |s|^{-1}\langle s \rangle.\]  

(4.198)

Since \( z = 1 \in \Omega_{24} \), we have

\[
\frac{|s|^{-2}\langle s \rangle^2}{|1 - s^{-2}|^2} = \frac{\langle s \rangle^2}{|1 - s^2|^2} = \begin{cases} \mathcal{O}(1), & z \in \Omega_{0+1}, \Omega_{11} \\ \langle s \rangle^2 \frac{|s|}{|s - 1|}, & z \in \Omega_{24}. \end{cases}
\]  

(4.199)

Here we can see that: For \( z \in \Omega_{0+1} \) and \( \Omega_{11} \), we can take the procedure under the zero boundary conditions like in [22]. However, for \( z \in \Omega_{24} \), the singularities \( z = \pm 1 \) should be treated by us in a more delicate way. To some extent, how to deal with the singularity points plays a core role in our analysis. From the easier to the more advance, we deal with \( \Omega_{0+1}, \Omega_{11} \) firstly, finally we face to the \( \Omega_{24} \).
we introduce an inequality which plays an vital role in our analysis. Make \( s = z_0 + ie^{i\phi} = z_0 + u + iv, \) \( z = x + iy, \) \( u, v, x, y > 0 \) we have the inequality

\[
\| \frac{1}{s - z} \|_{L^q(v, \infty)} \lesssim \left( \int_{\mathbb{R}^+} \left[ 1 + \left( \frac{u + z_0 - x}{v - y} \right)^2 \right]^{-\frac{q}{2}} (v - y)^{-q} du \right)^{\frac{1}{q}}
\]

\[
= |v - y|^{\frac{1}{q} - 1} \left( \int_{\mathbb{R}^+} \left[ 1 + \left( \frac{u + z_0 - x}{v - y} \right)^2 \right]^{-\frac{q}{2}} d \left( \frac{u + z_0 - x}{v - y} \right) \right)^{1/q}
\]

\[
\lesssim q > 1 \lesssim |v - y|^{1/q - 1}.
\]

For \( z \in \Omega_{0+1} \), we make \( z = x + iy, \) \( s = 0 + u + iv. \) Thanks to (4.199), we have

\[
\frac{1}{\pi} \int_{\Omega_{0+1}} \frac{|W^{(3)}(s)|}{|s - z|} dA(s) \lesssim \frac{1}{\pi} \int_{\Omega_{0+1}} \frac{|\partial f_{0+1} e^{2it\theta}|}{|s - z|} dA(s) = \frac{1}{\pi} \int_{\Omega_{0+1}} \frac{|\partial f_{0+1} e^{2it\theta}|}{|s - z|} dA(s).
\]

Recall the Proposition 4.2, we can divide the integral into two parts

\[
\frac{1}{\pi} \int_{\Omega_{0+1}} \frac{|\partial f_{0+1} e^{2it\theta}|}{|s - z|} dA(s) \lesssim I_1 + I_2,
\]

where

\[
I_1 = \int_{\Omega_{0+1}} \frac{|p'_{0+1}(s)|}{|s - z|} e^{-2it\theta} dA(s), \quad I_2 = \int_{\Omega_{0+1}} \frac{|s|^{-\frac{q}{2}} e^{-2it\theta}}{|s - z|} dA(s).
\]

Notice that \( \Omega_{0+1} \) is a bounded area, thus we find \( 0 < x, u < \xi_2/2 \) and \( 0 < y, v < \xi_2\Lambda\pi\phi \) \( < \xi_2/2 \)

Thus

\[
I_1 \lesssim \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{|p'_{0+1}|}{|s - z|} e^{-ctv} du dv
\]

\[
\lesssim \int_{\mathbb{R}^+} \| \frac{1}{s - z} \|_{L^2(\mathbb{R}^+)} \| p'_{0+1} \|_{L^2(\mathbb{R}^+)} e^{-ctv} dv
\]

\[
\lesssim \int_{\mathbb{R}^+} |v - y|^{-1/2} e^{-ctv} dv
\]

\[
\lesssim t^{-\frac{1}{2}}.
\]

Next, we introduce the following inequality for \( p > 2 \)

\[
\| |s|^{-\frac{1}{2}} \|_{L^p(v, +\infty)} = \left( \int_{v}^{+\infty} \left( \sqrt{u^2 + v^2} \right)^{-p/2} du \right)^{1/p}
\]

\[
l^2 = u^2 + v^2 \quad \left( \int_{v}^{+\infty} l^{-p/2} \cdot l \cdot u^{-1} dl \right)^{\frac{1}{p}}
\]

\[
\lesssim v^{-\frac{1}{2} + \frac{1}{p}}.
\]
Then by Hölder inequality
\[
I_2 \lesssim \int_{\mathbb{R}^+} \left\| \frac{1}{s-z} \right\|_{L^q(\mathbb{R}^+)} \|s\|^{-1/2} \|e^{-ctv}dv \leq \int_{\mathbb{R}^+} |v-y|^{1/q-1}v^{-1/2}e^{-ctv}dv \\
\lesssim t^{-\frac{3}{2}}. \tag{4.206}
\]

For \(z \in \Omega_{0^2k}, k = 1, 2\), we can conclude that \(\|S\|_{L^\infty(\mathbb{C}) \rightarrow L^\infty(\mathbb{C})} \lesssim t^{-\frac{3}{2}}\).

For \(z \in \Omega_{11}\), we make \(z = x + iy, s = \xi_1 + u + iv\). Thanks to (4.199), we have
\[
\frac{1}{\pi} \int \int_{\Omega_{11}} \frac{|W^{(3)}(s)|}{|s-z|} dA(s) \lesssim \frac{1}{\pi} \int \int_{\Omega_{11}} \frac{|\partial f_{11}|e^{-2i\theta}}{|s-z|} dA(s). \tag{4.207}
\]

Recall Proposition 4.3, we still divide the integral into two parts
\[
I_3 = \int \int_{\Omega_{11}} \frac{|p_{11}^{(c)}(s)|e^{-2i\theta}}{|s-z|} dA(s), \quad I_4 = \int \int_{\Omega_{11}} \frac{|s-\xi_1|^{-\frac{1}{2}}e^{-2i\theta}}{|s-z|} dA(s). \tag{4.208}
\]

For \(I_3\), we use Proposition 4.3
\[
I_3 = \int_0^\infty \int_0^\infty \frac{|p_{11}^{(c)}|e^{-ctv}}{|s-z|} dv \leq \int_0^\infty \int_0^\infty \frac{|p_{11}^{(c)}|e^{-ctv}}{|s-z|} dv \\
\lesssim \int_0^\infty \int_0^\infty \frac{|p_{11}^{(c)}|e^{-ctv}}{|s-z|} dv \\
\lesssim \int_0^\infty e^{-ctv} \left\| p_{11}^{(c)} \right\|_{L^2(v, +\infty)} \left\| \frac{1}{s-z} \right\|_{L^2(v, +\infty)} dv \\
\lesssim \int_0^\infty e^{-4ctv} |v-y|^{-1/2}dv \lesssim t^{-1/4}. \tag{4.209}
\]

For \(I_4\), we have
\[
I_4 \lesssim \int_0^\infty e^{-ctv^2} dv \int_0^\infty \frac{|s-\xi_2|^{-\frac{1}{2}}}{|s-z|} dv \\
\lesssim \int_0^\infty e^{-ctv^2} \left\| \frac{1}{s-z} \right\|_{L^q(\mathbb{R}^+)} \left\| |s-\xi_1|^{-\frac{1}{2}} \right\|_{L^p(\mathbb{R}^+)} dv \\
= \left( \int_0^g + \int_0^{+\infty} \right) v^{-\frac{1}{2}+\frac{1}{q}} |v-y|^{1/q-1} e^{-ctv^2} dv. \tag{4.210}
\]

For the first integral, we have
\[
\int_0^g v^{-\frac{1}{2}+\frac{1}{q}} |v-y|^{1/q-1} e^{-ctv^2} dv = \int_0^1 \sqrt{g} e^{-cty^2} w^{1/p-1/2} |1-w|^{1/q-1} \\
\lesssim t^{-\frac{3}{4}}, \tag{4.211}
\]

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For the second integral, take \( v = y + w \), we have
\[
\int_y^{+\infty} e^{-ctv^2} v^{\frac{1}{p} - \frac{1}{2}} |v - y|^{1/q - 1} dv = \int_0^{+\infty} e^{-t(y+w)^2} (y+w)^{1/p-1/2} w^{1/q-1} dw \\
\leq \int_0^{+\infty} e^{-ctw^2} w^{1/p-1/2+1/q-1} dw \\
= \int_0^{+\infty} e^{-tw^2} w^{-\frac{1}{2}} dw \lesssim t^{-\frac{1}{4}}. \tag{4.212}
\]
For \( z \in \Omega_{jk}, j = 1, 4, k = 1, 2, 3, 4 \), we can conclude that \( \|S\|_{L^\infty(C) \rightarrow L^\infty(C)} \lesssim t^{-\frac{1}{4}} \).

For \( z \in \Omega_{24} \), we make \( z = x + iy \), \( s = \xi_2 + u + iv \). Since \( \Omega_{24} \) is a bounded area we find that \( 0 < u < \frac{\xi_1 - \xi_2}{2}, 0 < v < (\xi_1 - \xi_2) \tan \phi (\xi_1 - \xi_2) \).

Owe to (4.199) and, we have
\[
\frac{1}{\pi} \iint_{\Omega_{24}} \frac{|W^{(3)}(s)|}{|s-z|} dA(s) \lesssim \frac{1}{\pi} \iint_{\Omega_{24}} \frac{(s)|\bar{\partial}f_{24}|e^{2it\bar{\Theta}}}{|s-z||s-1|} dA(s) = \frac{1}{\pi} \iint_{\Omega_{24}} \frac{(s)|\partial f_{24}|e^{2it\bar{\Theta}}}{|s-z||s-1|} dA(s). \tag{4.213}
\]
Further
\[
\frac{1}{\pi} \iint_{\Omega_{24}} \frac{(s)|\bar{\partial}f_{24}|e^{2it\bar{\Theta}}}{|s-z||s-1|} dA(s) \lesssim I_5 + I_6, \tag{4.214}
\]
where
\[
I_5 = \frac{1}{\pi} \iint_{\Omega_{24}} \frac{(s)|\bar{\partial}f_{24}|e^{2it\bar{\Theta}} \chi_{\{\xi_2, 1\}}(|s|)}{|s-z||s-1|} dA(s), \tag{4.215}
\]
\[
I_6 = \frac{1}{\pi} \iint_{\Omega_{24}} \frac{(s)|\bar{\partial}f_{24}|e^{2it\bar{\Theta}} \chi_{\{1, \xi_2 - \xi_1\}}(|s|)}{|s-z||s-1|} dA(s). \tag{4.216}
\]
where \( \chi_{\{\xi_2, 1\}}(|s|) + \chi_{\{1, \xi_2 - \xi_1\}}(|s|) \) is the partition of unity.

We consider \( I_5 \) firstly. Since \( |s| \in [\xi_2, 1] \), we have \( \frac{(s)}{|s-1|} = O(1) \). Recall Proposition 4.3, we can divide \( I_5 \) as
\[
I_5 \leq I_5^{(1)} + I_5^{(2)}, \tag{4.217}
\]
where
\[
I_5^{(1)} = \iint_{\Omega_{24}} \frac{|p'_{24}(s)|e^{2it\bar{\Theta}}}{|s-z|} dA(s), \quad I_5^{(2)} = \iint_{\Omega_{24}} \frac{|s-\xi_2|^2 e^{2it\bar{\Theta}}}{|s-z|} dA(s). \tag{4.218}
\]
Then Lemma 4.2 implies that
\[
I_5^{(1)} \leq \int_0^{+\infty} \int_0^{+\infty} \frac{|p_2(s)|}{|s - z|} e^{-ct(1+|s|^{-2})} v^2 \, du \, dv
\]
\[
\lesssim \int_{\mathbb{R}^+} \left\| \frac{1}{s - z} \right\|_{L^2(\mathbb{R}^+)} \| p_2' \|_{L^2(\mathbb{R}^+)} e^{-ctv^2} \, dv
\]
\[
\lesssim \left( \int_0^y + \int_y^{+\infty} \right) |v - y|^{-\frac{1}{2}} e^{-ctv^2} \, dv. \tag{4.219}
\]
For the first integral, we have
\[
\int_0^y (y - v)^{-\frac{1}{2}} e^{-ctv^2} \, dv \lesssim \int_0^y (y - v)^{-\frac{1}{2}} v^{-\frac{1}{2}} \, dv \cdot t^{-\frac{1}{4}}
\]
\[
\lesssim t^{-\frac{1}{4}}. \tag{4.220}
\]
The first step we use $e^{-z} \lesssim z^{-1/4}$. As for the second integral, we set $w = v - y$ thus we have
\[
\int_y^{+\infty} (v - y)^{-\frac{1}{2}} e^{-ctv^2} \, dv = \int_0^{+\infty} w^{-\frac{1}{2}} e^{-ct(w+y)^2} \, dw \lesssim e^{-cty^2} \tag{4.221}
\]
Then we discuss the boundedness for $I_5^{(2)}$. We set $p > 2$, and do the procedure which is similar to (4.206)
\[
I_5^{(2)} \leq \int_0^{+\infty} \left\| \frac{1}{s - z} \right\|_{L^q(\mathbb{R}^+)} \| s - \xi_2 \|^{-\frac{1}{2}}_{L^p(\mathbb{R}^+)} e^{-ctv^2} \, dv
\]
\[
\lesssim \int_0^{+\infty} v^{-\frac{1}{2} + \frac{1}{p}} |v - y|^{1/q - 1} e^{-ctv^2} \, dv
\]
\[
= \left( \int_0^y + \int_y^{+\infty} \right) v^{-\frac{1}{2} + \frac{1}{p}} |v - y|^{1/q - 1} e^{-ctv^2} \, dv. \tag{4.222}
\]
The methods for $I_5^{(1)}$ can still be applied to bound the two integrals above.
\[
\int_0^y (y - v)^{-\frac{1}{2} - 1} v^{-\frac{1}{2} + \frac{1}{p}} e^{-ctv^2} \, dv \lesssim t^{-\frac{1}{4}} \int_0^y (y - v)^{-\frac{1}{2}} \, dv
\]
\[
\lesssim t^{-\frac{1}{4}}. \tag{4.223}
\]
And
\[
\int_y^{+\infty} v^{-\frac{1}{2} + \frac{1}{p}} |v - y|^{1/q - 1} e^{-ctv^2} \, dv \lesssim \int_0^{+\infty} w^{-\frac{1}{2}} e^{-ct(w+y)^2} \, dw \lesssim e^{-cty^2} \tag{4.224}
\]
Next we consider $I_6$. Since $|s| \in \left[1, \frac{\xi_1 - \xi_2}{\sqrt{2}} \right)$, we have $\langle s \rangle \lesssim c(\xi, \varpi)$. The singularity $z = 1$ could be balanced by (4.70), we obtain

$$I_6 \lesssim c(\xi, \varpi) \frac{1}{\pi} \int_{\Omega_{24}} |p'_{24}| e^{2t\xi_3\theta} |s - z|^{-1} dA(s)$$

$$\lesssim \int_0^{+\infty} \int_{\mathbb{R}^+} |p'_{24}| e^{-ct(1+|s|^{-2})v^2} dudv$$

$$\lesssim \int_{\mathbb{R}^+} \frac{1}{|s - z|} \|p'_{24}\|_{L^2(\mathbb{R}^+)} e^{-ctv^2} dv$$

$$\lesssim \left( \int_0^y + \int_y^{+\infty} \right) |v - y|^{-\frac{1}{4}} e^{-ctv^2} dv.$$  (4.225)

From estimates of $I_5$ and $I_6$, we can conclude that $\|S\|_{L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C})} \lesssim t^{-\frac{1}{4}}$ for $z \in \Omega_{jk}$, $j = 2, 3$, $k = 1, 2, 3, 4$.

Based on the three cases we discuss, $\|S\|_{L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C})} \lesssim t^{-\frac{1}{4}}$ as $t \to \infty$.  \(\Box\)

**Corollary 4.5.** As $t \to \infty$, $(1 - S)^{-1}$ exists.

**Proof.** It is a direct result of Lemma 4.7.  \(\Box\)

Aim at finding the asymptotic behavior of $m^{(3)}$, we make the asymptotic expansion

$$m^{(3)} = I + z^{-1} m_1^{(3)}(x, t) + \mathcal{O}(z^{-2}), \text{ as } z \to \infty$$  (4.226)

where

$$m_1^{(3)}(x, t) = \frac{1}{\pi} \int_{\mathbb{C}} m^{(3)}(s) W^{(3)}(s) dA(s).$$  (4.227)

To recover the solution of defocusing mKdV (1.1), we shall discuss the asymptotic behavior of $m_1^{(3)}(x, t)$, thus we have the following proposition.

**Proposition 4.11.** As $t \to \infty$,

$$|m_1^{(3)}(x, t)| \lesssim t^{-\frac{3}{4}}.$$  (4.228)

**Proof.** Notice the boundedness of $m^{(3)}$ and $m^{(3)}$, we have

$$|m_1^{(3)}| \leq \int_{\Omega_{jk}} |m^{(3)}(x, t) - \mathcal{H}^{(2)}(m^{(3)})^{-1}| dA(s)$$

$$\lesssim \int_{\Omega_{jk}} \left| \int_{\mathbb{C}} m^{(3)}(s) W^{(3)}(s) dA(s) \right|$$

$$= \int_{\Omega_{jk}} \frac{s}{|s - 1|} |f_j e^{\pm 2it\varpi} \chi_{\Omega_{jk}} e^{\pm 2it\varpi} dA(s).$$  (4.229)
Similar to the proof of Lemma 4.7, we only take into account that matrix functions have support in the sector $\Omega_{jk}$. What we need to do is to control the boundedness of the integral $\int_{\Omega_{jk}} \frac{(s)}{|s|} |\bar{\partial}f_{jk}| e^{\mp 2t\Im \theta} dA(s)$ for $z \in \Omega_{jk}$, $j = 0^k, 1, 2, 3, 4$, $k = 1, 2, 3, 4$. We can take $z \in \Omega_{0+1}$, $z \in \Omega_{11}$ and $z \in \Omega_{24}$ as examples, the proofs for the rest regions are similar. We point out that the analysis to $\Omega_{24}$ is a bit different from $z \in \Omega_{0+1}, \Omega_{11}$, because we should deal with the singularity $z = 1$ as what we do in the Lemma 4.7. We still deal with them from easy to difficult.

For $z \in \Omega_{0+1}$, we make $z = x + iy$, $s = u + iv$ which satisfy $0 < x, u < \xi_2/2$, $0 < y, v < \frac{\xi_2 \tan \phi}{2}$ ($< \frac{\xi_2}{2}$). Owe to (4.199), $\langle s \rangle / |s - 1| = O(1)$ for $z \in \Omega_{0+1}$. And we can divide the integral into two parts

$$\int_{\Omega_{0+1}} |\bar{\partial}f_{0+1}(s)| e^{\mp 2t\Im \theta} dA(s) \lesssim I_1 + I_2,$$

where

$$I_1 = \int_{\Omega_{0+1}} |p_{0+1}'(s)| e^{\mp 2t\Im \theta} dA(s),$$

$$I_2 = \int_{\Omega_{0+1}} |s|^{-\frac{1}{2}} e^{\mp 2t\Im \theta} dA(s).$$

Since $\|p_{0+1}'\|_{L^2}$ is bounded, we bound $I_1$ by Cauchy-Schwarz inequality

$$|I_1| \leq \int_{\Omega_{0+1}} |p_{0+1}'| e^{-2tcuv} dA(s)$$

$$\leq \int_0^{\xi_2 \tan \phi} \|p_{0+1}'\|_{L^2(v, \xi_2/2)} \left( \int_v^{\xi_2/2} e^{-4tcuv} du \right)^{1/2} dv$$

$$\lesssim \int_0^{+\infty} v^{\frac{1}{2}} e^{-2ctv} dv$$

$$w = \frac{v}{2} 2 \int_0^{+\infty} w^2 e^{-2ctw^2} dw \lesssim t^{-\frac{3}{2}}.$$

As for $I_2$, we use H"{o}lder equality and (4.205)

$$|I_2| \lesssim \int_{\Omega_{0+1}} |s|^{-\frac{1}{2}} \|\cdot\|_{L^p(v, \xi_2/2)} \left( \int_v^{\xi_2/2} e^{-2ctqv} du \right)^{1/q} dv$$

$$\lesssim \int_0^{+\infty} v^{-\frac{1}{2} + \frac{1}{p} v^\frac{1}{2}} e^{-2ctv} dv$$

$$\int_0^{+\infty} v^\frac{1}{2} e^{-2ctv} dv$$

$$\lesssim t^{-\frac{3}{2}}.$$
For \( z \in \Omega_{0z_1}, k = 1, 2 \), we can conclude that \(|m^{(3)}_1| \lesssim t^{-\frac{3}{2}}\).

For \( z \in \Omega_{11}, (s)/|s - 1| = O(1) \). And we make \( z = x + iy, s = \xi_1 + u + iv \).

\[
\int_{\Omega_{11}} \left| \tilde{f}_{11}(s) \right| e^{-2i\xi_3\theta} dA(s) \lesssim I_3 + I_4, \tag{4.235}
\]

where

\[
I_3 = \int_{\Omega_{11}} |p'_{11}(s)| e^{-2i\xi_3\theta} dA(s), \tag{4.236}
\]

\[
I_4 = \int_{\Omega_{11}} |s - \xi_1|^{-\frac{1}{2}} e^{-2i\xi_3\theta} dA(s). \tag{4.237}
\]

With the help of Lemma 4.3, we can bound \( I_3, I_4 \). Take use Cauchy-Schwarz inequality

\[
|I_3| \leq \int_{\Omega_{11}} |p'_{11}(s)| e^{-2tuv(s-\xi_1)} dA(s)
= \int_{\Omega_{11}} |p'_{11}(s)| e^{-2tuv} dA(s)
\]

\[
r \in H^1 \ni \int_0^{+\infty} ||p'_{11}(s)||_{L^2(v+\xi_1,\infty)} \left( \int_0^{+\infty} e^{-4tuv} du \right)^{\frac{1}{2}} dv
\]

\[
\lesssim t^{-\frac{1}{2}} \int_0^{+\infty} v^{-\frac{1}{2}} e^{-2tv^2} \lesssim t^{-\frac{3}{4}}. \tag{4.238}
\]

As for \( I_4 \), we take the advantage of H"older inequality and (4.205) again

\[
|I_4| \lesssim \int_0^{+\infty} v^{-\frac{1}{2} + \frac{1}{p}} \left( \int_0^{+\infty} e^{-2tuv} dv \right)^{\frac{1}{2}} du
\]

\[
\lesssim \int_0^{+\infty} v^{-\frac{1}{2} + \frac{1}{p}} (qtv)^{-\frac{1}{2}} e^{-2tv^2} dv
\]

\[
= t^{-\frac{3}{4}} \int_0^{+\infty} w^{\frac{2}{p} - \frac{3}{2}} e^{-2aw^2} dw
\]

\[
\lesssim t^{-\frac{3}{4}}, \tag{4.239}
\]

where we have used the substitution \( w = t^{1/2}v \) and the fact that \(-1 < 2/p - 3/2 < -1/2\) to make sure the convergence of generalized integral.

For \( z \in \Omega_{jkc}, j = 1, 4, k = 1, 2, 3, 4 \), we can conclude that \(|m^{(3)}_1| \lesssim t^{-\frac{3}{4}}\).

For \( z \in \Omega_{24}, \) we make \( z = x + iy, s = \xi_2 + u + iv \) which satisfy \( 0 < u < \frac{\xi_1 - \xi_2}{2}, 0 < v < \frac{\xi_1 - \xi_2}{2\tan\phi(< \frac{\xi_1 - \xi_2}{2})}\).

\[
\int_{\Omega_{24}} \left| \tilde{f}_{24}(s) \right| e^{2i\xi_3\theta} dA(s) \lesssim I_5 + I_6, \tag{4.240}
\]
where

\[ I_5 = \frac{1}{\pi} \int \int_{\Omega_{24}} \frac{\langle s \rangle |\bar{f}_{24}| e^{2\nu \theta} \chi_{[\xi_2, 1)}(\langle s \rangle)}{|s - 1|} dA(s), \quad (4.241) \]

\[ I_6 = \frac{1}{\pi} \int \int_{\Omega_{24}} \frac{\langle s \rangle |\bar{f}_{24}| e^{2\nu \theta} \chi_{[1, \xi_2-\xi_2]}(\langle s \rangle)}{|s - 1|} dA(s), \quad (4.242) \]

and \( \chi_{[\xi_2, 1)}(|s|) + \chi_{[1, \xi_2-\xi_2]}(|s|) \) is the partition of unity.

Notice \( \langle s \rangle / |s - 1| = \mathcal{O}(1) \) for \( |s| \in [\xi_2, 1) \). Combine Proposition 4.3, we can divide \( I_5 \) into two parts

\[ I_5 \lesssim I_5^{(1)} + I_5^{(2)}, \quad (4.243) \]

where

\[ I_5^{(1)} = \int \int_{\Omega_{24}} |p_{24}'(s)| e^{2\nu \theta} dA(s), \quad (4.244) \]

\[ I_5^{(2)} = \int \int_{\Omega_{24}} |s - \xi_2|^{-\frac{1}{2}} e^{2\nu \theta} dA(s). \quad (4.245) \]

With the help of Lemma 4.2, we can bound \( I_5^{(1)}, I_5^{(2)} \).

We bound \( I_5^{(1)} \) by Cauchy-Schwarz inequality

\[ |I_5^{(1)}| \leq \int \int_{\Omega_{24}} |p_{24}'(s)| e^{-2c(1+|z|^{-2})tv^3} dA(s) \]

\[ \leq \int \int_{\Omega_{24}} |p_{24}'(s)| e^{-2ctv^2} dA(s) \]

\[ \leq \int_0^{\xi_2} \|p_{24}(s)\|_{L^2(v+\xi_2, \xi_2-\xi_2)} \left( \int_0^{\xi_2} e^{-4ctv^2} dv \right)^{1/2} \]

\[ \approx \int_0^{+\infty} v^{\frac{1}{2}} e^{-2ctv^2} dv \]

\[ w=tv^2 \quad t^{-\frac{3}{2}} \int_0^{+\infty} w^2 e^{-2cw^4} dw \]

\[ \lesssim t^{-\frac{3}{4}}. \quad (4.246) \]
As for $I_{5}^{(2)}$, the Hölder inequality and (4.205) still be used to obtain

\[
|I_{5}^{(2)}| \lesssim \int_{0}^{+\infty} v^{-\frac{1}{2} + \frac{1}{p}} v^{\frac{1}{2}} e^{-2ctv^{2}} dv
\]

\[
\lesssim t^{-\frac{3}{4}}.
\]  

(4.247)

We finally deal with $I_{6}$. Thanks to the (4.70), the singularity $z = 1$ can be balanced. Additionally, for $|s| \in \left[1, \sqrt{\frac{1}{\omega}}\right)$, $\langle s \rangle \leq c(\xi, \omega)$. As a consequence, we obtain

\[
|I_{6}| \leq c(\xi, \omega) \int_{\Omega_{24}} |p_{24}'|e^{2t\chi d} dA(s)
\]

\[
\lesssim \int_{0}^{+\infty} \int_{-\infty}^{+\infty} |p_{24}'|e^{-ct(1+|s|^{-2})} dv^{2} du dv
\]

\[
\lesssim \int_{0}^{+\infty} |p_{24}'|_{L^{2}(\mathbb{R}^{+})} \left( \int_{v}^{+\infty} e^{-2ctv^{2}} dv \right)^{\frac{1}{2}} dv
\]

\[
\lesssim \int_{0}^{+\infty} v^{\frac{1}{2}} e^{-ctv^{2}} dv \lesssim t^{-3/4}.
\]  

(4.248)

Summarize the estimates $I_{5}$ and $I_{6}$, we can conclude that $|m_{1}^{(3)}| \lesssim t^{-\frac{3}{4}}$ for $z \in \Omega_{jk}$, $j = 2, 3$, $k = 1, 2, 3, 4$.

Based on the computation for the three cases, we obtain that

\[
|m_{1}^{(3)}| \lesssim t^{-\frac{3}{4}} + t^{-\frac{3}{4}} + t^{-\frac{3}{4}} \lesssim t^{-\frac{3}{4}}.
\]  

(4.249)

Remark 4.5. (4.249) shows that: the main contribution for $m_{1}^{(3)}$ comes from the saddle points $\xi_{j}$, $j = 1, 2, 3, 4$.

5 Deformation of the RH Problem for $-2 < \xi < \omega$

In this section, we will discuss some results for the case $-2 < \xi < \omega$. The original RH problem still satisfies the RHP 2.1. Then we choose

\[
\delta(z) := \delta(z, \xi) = \exp \left[ -\frac{1}{2\pi i} \int_{\Sigma} \log (1 - |r(s)|^{2}) \left( \frac{1}{s - z} \right) ds \right].
\]  

(5.1)
to make the transformation $m^{(1)}(z) = m(z)\delta(z)^{\sigma_z}$, which implies that $m^{(1)}(z)$ still satisfies the RHP 4.1 formally.

**Remark 5.1.** The difference between the $\delta(z)$ defined by (4.4) and the new $\delta(z)$ is the integral interval. The interval of the former is $\Gamma := (\xi_4, \infty) \cup (0, \xi_2) \cup (\xi_1, +\infty)$, however, the latter is $\Sigma$.

### 5.1 Opening $\bar{\partial}$ lenses

Be similar to the case $-\varpi < \xi < -6$, we still want to remove the jump from the real axis in such a way that the new problem takes advantage of the growth of decay of $e^{\pm 2i(t\theta(z))}$ for $z \notin \mathbb{R}$. We can still find a small sufficiently angle $\phi : \phi < \theta_0$ and define a new region

$$
\Omega = \bigcup_{j=1,2,3,4} \Omega_j,
$$

where

$$
\begin{align*}
\Omega_1 &= \{ z \in \mathbb{C} : 0 < \arg z < \phi \}, \quad \Omega_2 = \{ z \in \mathbb{C} : \pi - \phi < \arg z < \phi \}, \\
\Omega_3 &= \{ z \in \mathbb{C} : -\pi < \arg z < -\pi + \phi \}, \quad \Omega_4 = \{ z \in \mathbb{C} : -\phi < \arg z < 0 \}.
\end{align*}
$$

Finally, denote by

$$
\begin{align*}
\Sigma_1 &= e^{i\phi} \mathbb{R}^+, \quad \Sigma_2 = e^{i(\pi - \phi)} \mathbb{R}^+, \\
\Sigma_3 &= e^{-i(\pi - \phi)} \mathbb{R}^+, \quad \Sigma_4 = e^{-i\phi} \mathbb{R}^+.
\end{align*}
$$

the left-to-right oriented boundaries of $\Omega$, see Figure 12.

**Proposition 5.1.** For $-2 < \xi < -\varpi$, $z = le^{i\phi}$, and $F(l) = l + l^{-1}$, the phase function $\theta(z)$ defined by (3.1) satisfies

$$
\begin{align*}
\Im \theta(z) &\geq \frac{1}{2} F(l) \sin \phi \left[ (2F^2(l) - 6) \cos (2\phi) + \xi + F^2(l) \right], \quad z \in \Omega_j, \quad j = 1, 2 \\
\Im \theta(z) &\leq -\frac{1}{2} F(l) \sin \phi \left[ \xi + F^2(l) \right], \quad z \in \Omega_j, \quad j = 3, 4.
\end{align*}
$$

**Proof.** Take the $\Omega_1$ as an example, the proof for the other regions is similar. Recall (4.33), we have

$$
\begin{align*}
\Im \theta(z) &= \frac{1}{2} F(l) \sin \phi \left[ (2F^2(l) - 6) \cos (2\phi) + \xi + F^2(l) \right] \\
&\geq \frac{1}{2} F(l) \sin \phi \left[ \xi + F^2(l) \right] > 0.
\end{align*}
$$

$\square$
Figure 12: Without stationary phase points on the jump contour corresponds the cases $-2 < \xi < \varpi$. The purple regions imply that $e^{2it\theta} \to 0$, however, the yellow regions imply that $e^{-2it\theta} \to 0$.

We choose $R^{(2)}(z)$ as

$$R^{(2)}(z) = \begin{cases} \begin{bmatrix} 1 & f_je^{2it\theta} \\ 0 & 1 \end{bmatrix}, & z \in \Omega_j, \ j = 1, 2 \\ \begin{bmatrix} 1 & 0 \\ f_je^{-2it\theta} & 1 \end{bmatrix}, & z \in \Omega_j, \ j = 3, 4 \\ I, & \text{elsewhere}, \end{cases} \tag{5.9}$$

where $f_j$ is defined by the following proposition

**Proposition 5.2** (Opening lens at $z = 0$ for $-2 < \xi < \varpi$). $f_j : \tilde{\Omega}_j \to \mathbb{C}$, $j = 1, 2, 3, 4$ are continuous on $\tilde{\Omega}_j$ with boundary values:

$$f_j(z) = \begin{cases} \frac{r(z)}{1 - |r(z)|^2} \delta^{+2}(z), & z \in \mathbb{R}, \\ 0, & z \in \Sigma_j, \ j = 1, 2 \end{cases} \tag{5.10}$$

$$f_j(z) = \begin{cases} \frac{r(z)}{1 - |r(z)|^2} \delta^{-2}(z), & z \in \mathbb{R}, \\ 0, & z \in \Sigma_j, \ j = 3, 4. \end{cases} \tag{5.11}$$

And $f_j, j = 1, 2, 3, 4$ have following property:

$$|\bar{\partial}f_j(z)| \leq c|z|^{-\frac{1}{2}} + c|r'(|z|)| + c\varphi(|z|), \ j = 1, 2, 3, 4, \tag{5.12}$$

where $\varphi \in C_0^\infty(\mathbb{R}, [0, 1])$ is a cutoff function with small support near 1.
Moreover

\[ |\partial f_j(z)| \leq c|z - 1|, \quad z \in \Omega_j, \quad j = 1, 4, \]  
\[ |\partial f_j(z)| \leq c|z + 1|, \quad z \in \Omega_j, \quad j = 2, 3. \]  

(5.13)  
(5.14)

Proof. The proof is the analogue of [25, Proposition 5.5] or [23, Lemma 6.5]. Here we only present a sketch for \( f_1 \). As observed in (2.43), we know that \( |r(z)| \to 1 \) as \( z \to \pm 1 \). This implies that \( f_1(z) \) is singular at \( z = 1 \). However, the singular behavior is exactly balanced by the factor \( \delta^2(z) \). With the help of (2.30)-(2.32), we have

\[ \frac{r(z)}{1 - |r(z)|^2} \delta^2_{p}(z) = \frac{b(z)}{a(z)} \left( \frac{a(z)}{\delta_+(z)} \right)^2 = \frac{J_b(z)}{J_a(z)} \left( \frac{a(z)}{\delta_+(z)} \right)^2, \]  

(5.15)

where \( J_a(z) = \det(\Phi_{-,1}, \Phi_{-,2}) \), \( J_b(z) = \det(\Phi_{-,1}, \Phi_{+,1}) \). It’s not difficult to know that the denominator of each factor in the r.h.s of (5.15) is nonzero and analytic in \( \Omega_1 \), with a well defined nonzero limit on \( \partial \Omega_1 \). Notice also that in \( \Omega_1 \) away from the point \( z = 1 \) the factors in the l.h.s of (5.15) are well behaved.

We introduce the cutoff functions \( \chi_0, \chi_1 \in C_0^\infty(\mathbb{R}, [0, 1]) \) with small support near \( z = 0 \) and \( z = 1 \) respectively, such that for any sufficiently small \( s \), \( \chi_0(s) = \chi_1(s + 1) = 1 \). Additionally, we impose the condition \( \chi_1(s) = \chi_1(s^{-1}) \) to preserve symmetry. Then we can rewrite the function \( f_1(z) \) in \( \mathbb{R}_+ \) as \( f_1(z) = f_1^{(1)}(z) + f_1^{(2)}(z) \), where

\[ f_1^{(1)}(z) = (1 - \chi_1(z)) \frac{r(z)}{1 - |r(z)|^2} \delta^2_{p}(z), \quad f_1^{(2)}(z) = \chi_1(z) \frac{J_b(z)}{J_a(z)} \left( \frac{a(z)}{\delta_+(z)} \right)^2. \]  

(5.16)

The aim of (5.16) is to balance the effect raised by the singularity \( z = 1 \) due to \( |r(1)| = 1 \). Fix a small \( \kappa_0 > 0 \), we extend \( f_1^{(1)}(z) \) and \( f_1^{(2)}(z) \) in \( \Omega_1 \) by

\[ f_1^{(1)}(z) = (1 - \chi_1(|z|)) \frac{r(|z|)}{1 - |r(|z|)|^2} \delta^2_{p}(z) \cos(\kappa |z|). \]  

(5.17)

\[ f_1^{(2)}(z) = h(|z|)g(z)\cos(\kappa |z|) + \frac{|z|}{\kappa} \chi_0 \left( \frac{\arg z}{\kappa_0} \right) h'(|z|)g(z)\sin(\kappa |z|), \]  

(5.18)

where

\[ \kappa := \frac{\pi}{2\theta_0}, \quad h(z) := \chi_1(z) \frac{J_b(z)}{J_a(z)}, \quad g(z) := \left( \frac{a(z)}{\delta(z)} \right)^2. \]  

(5.19)

Notice that the definition of \( f_1 \) preserves the symmetry \( f_1(s) = -f_1(s^{-1}) \).

Firstly we bound the \( f_1^{(1)} \).

\[ \bar{\partial}f_1^{(1)}(z) = -\frac{\bar{\partial} \chi_1(|z|)}{\delta^2(z)} \frac{r(|z|)\cos(\kappa |z|)}{1 - |r(|z|)|^2} + \frac{1 - \chi_1(|z|)}{\delta^2(z)} \bar{\partial} \left( \frac{r(|z|)\cos(\kappa |z|)}{1 - |r(|z|)|^2} \right). \]  

(5.20)
We know that $1 - |r(z)|^2 > c > 0$ as $z \in \text{supp}(1 - \chi_1(|z|))$ and $\delta^{-2}(z)$ is bounded as $z \in \Omega_1 \cap \text{supp}(1 - \chi_1(|z|))$. Take $z = le^{i\gamma} := u + iv$, we still have the equality $\partial = \frac{e^{i\gamma}}{2} (\partial_l + il^{-1}\partial_r)$ and apply it to the first term of (5.20)

$$\left| \frac{\partial \chi_1(|z|) r(|z|)\cos(k\arg z)}{\delta^2(z)} \right| = \left| \frac{e^{i\gamma} \chi_2 \cos(k\gamma)}{2\delta^2(z)(1 - |r(|z|)|^2)} \right| \lesssim \varphi(|z|) \quad (5.21)$$

for a appropriate $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}, [0, 1])$ with a small support near 1 and with $\varphi = 1$ on $\text{supp}\chi_1$. As $r(0) = 0$ and $r(z) \in H^1(\mathbb{R})$ it follows that $|r(|z|)| \lesssim |z|^{1/2} \|r'\|_{L^2(\mathbb{R})}$, we have

$$\left| \frac{1 - \chi_1(|z|)}{\delta^2(z)} \partial \left( \frac{r(|z|)\cos(k\arg z)}{1 - |r(|z|)|^2} \right) \right| \lesssim |r'(z)| + \frac{|r(z)|}{|z|} \lesssim |r'(z)| + |z|^{-\frac{1}{2}}. \quad (5.22)$$

So we can conclude that

$$|f^{(1)}_1(z)| \lesssim \varphi(|z|) + |r'(z)| + |z|^{-\frac{1}{2}}. \quad (5.23)$$

Next we bound $\partial f^{(2)}_1$,

$$\partial f^{(2)}_1 = \frac{1}{2} e^{i\gamma} g(z) \left[ h'\cos(k\gamma) \left( 1 - \chi_0 \left( \frac{\gamma}{\kappa_0} \right) \right) - \frac{i\kappa h(l)}{l} \sin(k\gamma) + \frac{i}{\kappa} (lh'(l)) \sin(k\gamma) \chi_0 \left( \frac{\gamma}{\kappa_0} \right) + \frac{i}{\kappa} h'(l) \sin(k\gamma) \chi_0' \left( \frac{\gamma}{\kappa_0} \right) \right] \quad (5.24)$$

in which $g(z)$ is bounded, $q \in L^{1,2}(\mathbb{R})$ and $q' \in W^{1,1}(\mathbb{R})$. So we claim $\partial f^{(2)}_1(z) \lesssim \varphi(|z|)$ for a $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}, [0, 1])$ with a small support near 1, thus yielding (5.12).

Finally $z \sim 1$, we have

$$|\partial f^{(2)}_1(z)| = \mathcal{O}(\gamma) \quad (5.25)$$

from which (5.13) follows immediately. 

We now use $R^{(2)}$ to define transformation $m^{(2)} = m^{(1)}R^{(2)}$, which help us set up the following mixed $\partial$-RH problem for the case $-2 < \xi < \omega$

**RHP 5.1.** Find a $2 \times 2$ matrix-valued function $m^{(2)}(z)$ such that

* $m^{(2)}(z)$ is continuous in $\mathbb{C}\setminus \Sigma^{(2)}$ and meromorphic out $\tilde{\Omega}$ with simple poles $\mathcal{Z}$,

* $m^{(2)}(z)$ takes continuous boundary values $m^{(2)}_\pm(z)$ on $\Sigma^{(2)}$ with jump relation

$$m^{(2)}_+(z) = m^{(2)}_- (z)V^{(2)}(z), \quad (5.26)$$

where $V^{(2)} = I$
asymptotic behavior

\[ m^{(2)}(x, t; z) = I + O(z^{-1}), \quad z \to \infty, \quad (5.27) \]

\[ m^{(2)}(x, t; z) = \frac{i}{z} \sigma_3 Q - + O(1), \quad z \to 0. \quad (5.28) \]

for \( z \in \mathbb{C} \), we have the \( \bar{\partial} \)-derivative equality

\[ \bar{\partial} m^{(2)} = m^{(2)} \bar{\partial} R^{(2)}, \quad (5.29) \]

where

\[ \bar{\partial} R^{(2)} = \begin{cases} 
1 \quad & \text{if } \Omega_j \quad j = 1, 2 \\
0 & \text{elsewhere.} 
\end{cases} \]

* Residue conditions

\[ \text{Res}_{z = \eta_n} m^{(2)}(z) = \lim_{z \to \eta_n} m^{(2)}(z) \begin{bmatrix} 1 \quad & 0 \\
0 & 1 \end{bmatrix}, \quad (5.31) \]

\[ \text{Res}_{z = \bar{\eta}_n} m^{(2)}(z) = \lim_{z \to \bar{\eta}_n} m^{(2)}(z) \begin{bmatrix} 0 \quad & \tilde{c}_n \delta^{-2}(\tilde{\eta}_n)(e^{2it\bar{\theta}}) \\
0 & 1 \end{bmatrix}, \quad (5.32) \]

5.2 Decomposition of mixed \( \bar{\partial} \)-RH problem

Like the case \(-\varpi < \xi < -6\), we still divide the RHP 5.1 into a pure RH problem \( m^{\text{sol}}(z) \) with \( \bar{\partial} R^{(2)} = 0 \) as well as a pure \( \bar{\partial} \)-problem with nonzero \( \bar{\partial} \)-derivatives.

\[ \begin{cases} \bar{\partial} R^{(2)} \equiv 0 \to m^{\text{sol}}(z), \\
\bar{\partial} R^{(2)} \neq 0 \to m^{(3)}(z) = m^{(2)}(z)m^{\text{sol}}^{-1}(z). \end{cases} \quad (5.33) \]

Remark 5.2. In this case, we replace the \( m^{rhp}(z) \), which appeared in the case \(-\varpi < \xi < -6\), by \( m^{\text{sol}}(z) \), because there exist no errors between \( m^{rhp}(z) \) and \( m^{\text{sol}}(z) \) in the case \(-2 < \xi < \varpi\).

Remark 5.3. For the case \(-2 < \xi < \varpi\), because all discrete spectrums are preserved, the \( m^{\text{sol}}(z) \) is the same with the RHP 4.4 formally in the subsection 4.3.1. Because the \( \delta \) has changed, the \( \tilde{\beta}_n \) defined in the form (4.119) changes for the case \(-2 < \xi < \varpi\). In the final result, we still preserve the form as (4.119).
5.3 Analysis on the pure $\bar{\partial}$-problem

Similar to the Section 4.1, we focus our insights on the estimates for the Cauchy-Green operator $S$ defined by (4.193) and $m_1^{(3)}(x,t)$ defined by (4.227). Then we have the following two estimations

**Lemma 5.1.** Consider the operator $S$ defined in (4.193), then we have $S : L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C}) \cap C^0(\mathbb{C})$ and

$$\|S\|_{L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C})} \lesssim t^{-\frac{1}{4}}. \quad (5.34)$$

**Proof.** The proof is the analogue of Lemma 4.7 or [25, Proposition 5.11]. □

**Proposition 5.3.** For $-2 < \xi < \pi$, 

$$|m_1^{(3)}(x,t)| \lesssim t^{-1}, \quad \text{as} \quad t \to \infty. \quad (5.35)$$

**Proof.** We present the details for $f_1$. By the standard procedure, like Proposition 4.11, we have

$$|m_1^{(3)}(x,t)| \lesssim I_1 + I_2 + I_3, \quad (5.36)$$

where

$$I_1 = \int_{\Omega_t} \int \frac{\langle s \rangle |\bar{\partial}f_1|e^{-2iy\theta}X_{[0,1]}(|s|)}{s-1} dA(s), \quad (5.37)$$

$$I_2 = \int_{\Omega_t} \int \frac{\langle s \rangle |\bar{\partial}f_1|e^{-2iy\theta}X_{[1,2]}(|s|)}{s-1} dA(s), \quad (5.38)$$

$$I_3 = \int_{\Omega_t} \int \frac{\langle s \rangle |\bar{\partial}f_1|e^{-2iy\theta}X_{[2,\infty]}(|s|)}{s-1} dA(s). \quad (5.39)$$

For the term with $X_{[2,\infty]}(|s|)$ the factor $\langle s \rangle |s-1|^{-1} = O(1)$, and fixing a $p > 2, q \in (1,2)$ we obtain the superabound for $I_3$

$$I_3 \lesssim \int \int R^+ \left[ |r^iv^s| + \varphi(|s|) + |s|^{-\frac{1}{2}} \right] e^{-2iy\theta}X_{[1,\infty]}(|s|) dA(s)$$

$$\lesssim \int R^+ \|e^{-ctuv}L^2(\max\{v,1/\sqrt{2}\},\infty) + \|e^{-ctuv}L^2(\max\{v,1/\sqrt{2}\},\infty)\|_{L^q(\infty)}dA(s)$$

$$\lesssim \int e^{-ctv} \left( tv^{-1/2} + t^{-1/p + 1/q - 1/2} \right) dv \lesssim t^{-1}. \quad (5.40)$$

For $s \in [0,2)$, $\langle s \rangle \leq 5$, so it will be omitted from the remaining estimates. For the $X_{[1,2]}(|s|)$, we use (5.12) to obtain that $I_2 \lesssim t^{-1}$ at once. For the $X_{[0,1]}(|s|)$, the changes of variables $w = z^{-1}$ and $r(s) = -r(s^{-1})$ imply that

$$I_1 = \int \int \int_{\Omega_1} |\bar{\partial}f_1|e^{-2iy\theta}|w| - 1|^{-1} \chi_{[1,\infty]}(|w|)|w|^{-1} dA(s) \lesssim t^{-1}. \quad (5.41)$$
6 Long time asymptotics for defocusing mKdV equation

6.1 The Case $-\varpi < \xi < -6$

Now we start to construct the asymptotic solution for the defocusing mKdV equation (1.1) for the case $-\varpi < \xi < -6$. Recall all the transformations (4.11), (4.76), (4.102) and (4.182) as follows:

$$m^{(1)} = m^{\varpi \sigma_3}, \quad m^{(2)} = m^{(1)} R^{(2)}, \quad m^{(3)} = m^{(2)} (m^{r\text{hp}})^{-1}, \quad m^{r\text{hp}} = m^{\text{err}} m^{\text{sol}}. \quad (6.1)$$

we have

$$m(z) = m^{(3)}(z) m^{\text{err}}(z) m^{\text{sol}}(z) R^{(2)}(z)^{-1} \delta(z)^{-\sigma_3}, \quad z \in \mathbb{C} \setminus U(\xi). \quad (6.2)$$

Take $z \to \infty$ out $\bar{\Omega}$ that means $R^{(2)} = I$, we obtain

$$m = \left( I + z^{-1} m^{(3)}_1 + \cdots \right) \left( I + z^{-1} m^{\text{err}}_1 + \cdots \right) \left( I + z^{-1} m^{\text{sol}}_1 + \cdots \right) \left( I - z^{-1} \delta_1 \sigma_3 + \cdots \right), \quad (6.3)$$

thus

$$m_1 = m^{\text{sol}}_1 + m^{\text{err}}_1 + m^{(3)}_1 - \delta_1 \sigma_3. \quad (6.4)$$

Take use the potential recovering formulae (2.67), we have

$$q(x,t) = -i \left( m^{\text{sol}}_1 \right)_{12} - i \left( m^{\text{err}}_1 \right)_{12} + \mathcal{O}\left(t^{-\frac{3}{2}}\right). \quad (6.5)$$

Remark 6.1. Recall the Section 4.3.1, we know that $-i \left( m^{\text{sol}}_1 \right)_{12} = q^{\text{sol}}(x,t)$ which is defined in (4.119).

Recall the Corollary 4.4 in Section 4.3.3, we know that $-i \left( m^{\text{err}}_1 \right)_{12} = t^{-1/2} f$, where

$$f = \epsilon_j \sum_{j=1}^{4} (1 - \xi_j^{-2})^{-1} (2 \epsilon_j \theta''(\xi_j))^{-\frac{1}{2}} \left[ \beta_{12}^{(1)} m^{\text{sol}}_1(\xi_j)^2 + \beta_{21}^{(1)} m^{\text{sol}}_1(\xi_j)^2 \right].$$

The contribution $\mathcal{O}(t^{-\frac{3}{2}})$ comes from the pure $\tilde{\partial}$-Problem of saddle points $\xi_j, \ j = 1,2,3,4$. $\mathcal{O}(t^{-\frac{3}{2}})$ comes from the pure $\tilde{\partial}$-Problem of $z = 0$, which can be absorbed by $\mathcal{O}(t^{-\frac{3}{2}})$ as $t \to +\infty$. 

Finally, we get the desired estimate.
6.2 The Case $-2 < \xi < \varpi$

In this subsection, we construct the asymptotic solution for the defocusing mKdV equation (1.1) for the case $-2 < \xi < \varpi$. Recall all the transformations in the case $-2 < \xi < \varpi$:

$$m^{(1)} = m\delta^3, \quad m^{(2)} = m^{(1)} R^{(2)}, \quad m^{(3)} = m^{(2)} (m\text{sol}^{-1}),$$

(6.6)

we have

$$m(z) = m^{(3)}(z) m\text{sol}(z) R^{(2)}(z)^{-1}(z)^{-\delta},$$

(6.7)

thus

$$m^{(3)}_1 = m^{(3)}_1 + m^{(3)}_1 - \delta \sigma_3.$$

(6.8)

Take use the potential recovering formulae (2.67), we have

$$q(x,t) = -i (m^{(3)}_1)^{12} + O(t^{-1}) = q^{\text{sol}} + O(t^{-1}).$$

(6.9)

**Remark 6.2.** $O(t^{-1})$ comes from the pure $\bar{\partial}$-Problem of $z = 0$.

Summarize all the above facts, we get our destination, in fact the Theorem 1.1, we repeat it again.

**Theorem 6.1.** Let $q(x,t)$ be the solution for the Cauchy problem (1.1) with generic data $q_0(x) \mp 1 \in H^{4,4}(\mathbb{R})$ and scattering data $\{r(z), \{\eta_n, c_n\}_{n=1}^{2N}\}$. Then there exits $T = T(\xi, \varpi) > 0$, $\xi : \xi \in (-\varpi, -6) \cup (-2, \varpi)$, $\forall t: T < t \to +\infty$, $q(x,t)$ has the following asymptotic representation in different space-time regions

- **In Region I** (i.e. $-\varpi < \xi < -6$)

  $$q(x,t) = q^{\text{sol}}(x,t) + t^{-\frac{1}{2}} f + O(t^{-\frac{3}{4}}),$$

  (6.10)

  where $q^{\text{sol}}$ is defined by (4.119), $f$ is defined by (4.179).

- **In Region III and Region IV** (i.e. $-2 < \xi < \varpi$)

  $$q(x,t) = q^{\text{sol}}(x,t) + O(t^{-1}),$$

  (6.11)

  where $q^{\text{sol}}$ is defined by (4.119).

**Theorem 6.2 (Asymptotic Stability).** Consider an $H$-soliton $q^{\text{sol},H}(x,t)$ satisfying both boundary conditions in (1.2) and let $\{0, \{\eta_n, c_n\}_{n=1}^{N}\}$ denote its reflectionless scattering data.
There exist \( \varepsilon_0 \) and \( C > 0 \) such that for any initial datum \( q_0 \) of Cauchy problem (1.1)-(1.2) with
\[
\varepsilon \equiv \| q_0 - q^{\text{sol},H}(x,0) \|_{H^{1,4}} < \varepsilon_0 \quad (6.12)
\]
the initial data \( q_0 \) generates scattering data \( \{ \tilde{r}, \{ \tilde{\eta}_n, \tilde{c}_j \}_{n=1}^N \} \) for some \( N \geq H \) (\( N \) is finite). The distribution of the poles is: \( H \) poles are close to the discrete data of \( q^{\text{sol},H} \), and the other additional \( N - H \) poles are close to either 1 or \(-1\), where the poles have been reordered as \( \Re \eta_1 < \Re \eta_2 < \cdots < \Re \eta_n \). In particular, there exists an integer \( Y \): \( 1 \leq Y \leq N \) such that
\[
Y + H \leq N \quad \text{for which we have}
\]
\[
\max_{1 \leq n \leq H} (|\eta_n - \tilde{\eta}_{n+Y}| + |\tilde{c}_n - \tilde{c}_{n+Y}|) + \max_{N > n > H + Y} (|1 + \tilde{\eta}_n| + \max_{1 < n < Y} |1 - \tilde{\eta}_n|) < C \varepsilon. \quad (6.13)
\]
Furthermore, \( q_0 \) has reflection coefficient \( r' \in H^1(\mathbb{R}) \). Let \( \xi = \frac{\xi}{t} \in (-\varpi, -6) \) or \( \xi \in (-2, \varpi) \). Then there exists a \( T_0 := T_0(q_0, \xi, \varpi) > 0 \), \( C := C_0(q_0, \xi, \varpi) > 0 \) such that for \( t > T_0 \), the following inequalities come true:
\[
|q(x,t) - q^{\text{sol},H}(x,t)| \lesssim t^{-\frac{1}{2}}, \quad \xi \in (-\varpi, -6), \quad (6.14)
\]
\[
|q(x,t) - q^{\text{sol},H}(x,t)| \lesssim t^{-1}, \quad \xi \in (-2, \varpi) \quad (6.15)
\]
Proof. Owe to the Lipschitz continuity of map such (2.10)-(2.12) in Proposition 2.1, the solutions of the corresponding matrix-valued functions \( m^0(x,t;z) \) for the initial value \( q_0(x) \) as well as \( m(x,t;z) \) for the initial value \( q^{\text{sol},H}(x,0) \) satisfy
\[
|m^0(x,t;z) - m(x,t;z)| \lesssim \|q_0(x) - q^{\text{sol},H}(x,0)\|_{L^1(x,\infty)} \lesssim \varepsilon_0. \quad (6.16)
\]
In particular, for the \( N \)-soliton solution \( q^{\text{sol}}(x,t) \) decided by the initial value \( q_0 \), it admits that
\[
|q^{\text{sol}}(x,t) - q^{\text{sol},H}(x,t)| \lesssim \varepsilon_0. \quad (6.17)
\]
At last, we have
\[
|q(x,t) - q^{\text{sol},H}(x,t)| \leq |q(x,t) - q^{\text{sol}}(x,t)| + |q^{\text{sol}}(x,t) - q^{\text{sol},H}(x,t)|
\]
\[
\begin{cases}
\mathcal{O}(t^{-\frac{1}{2}}), & \xi \in (-\varpi, -6), \\
\mathcal{O}(t^{-1}), & \xi \in (-2, \varpi).
\end{cases} \quad (6.18)
\]
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A Parabolic Cylinder Model Near $\xi_j$, $j = 1, 2, 3, 4$

A.1 Local Model Near $\xi_j$, $j = 1, 3$

We take $\xi_1$ as an example to present this standard model.

RHP A.1. Find a matrix-valued function $m^{pc, \xi_1}(\zeta) := m^{pc, \xi_1}(\zeta; \xi)$ with following properties:

* $m^{pc, \xi_1}(\zeta; \xi)$ is analytical in $\mathbb{C}\setminus \Sigma^{pc}$ with $\Sigma^{pc} = \{\Re e^{i\phi}\} \cup \{\Re e^{i(\pi - \phi)}\}$ shown in Figure A1;

* $m^{pc, \xi_1}$ has continuous boundary values $m^{pc, \xi_1 \pm}$ on $\Sigma^{pc}$ and

$$m^{pc, \xi_1}_+(\zeta) = m^{pc, \xi_1}_-(\zeta)V^{pc}(\zeta), \quad \zeta \in \Sigma^{pc},$$

where

$$V^{pc}(\zeta) = \begin{cases} \zeta^{-i\nu} e^{\frac{i\nu}{2} \sigma_3} \begin{pmatrix} 1 & -\bar{r}_{\xi_1} \frac{1}{1-|r_{\xi_1}|^2} \\ 0 & 1 \end{pmatrix}, & \zeta \in \mathbb{R}^+ e^{i\phi}, \\
\zeta^{-i\nu} e^{\frac{i\nu}{2} \sigma_3} \begin{pmatrix} 1 & 0 \\ \bar{r}_{\xi_1} \frac{1}{1-|r_{\xi_1}|^2} & 1 \end{pmatrix}, & \zeta \in \mathbb{R}^+ e^{(2\pi - \phi)i}, \\
\zeta^{-i\nu} e^{\frac{i\nu}{2} \sigma_3} \begin{pmatrix} 1 & -\bar{r}_{\xi_1} \frac{1}{1-|r_{\xi_1}|^2} \\ 0 & 1 \end{pmatrix}, & \zeta \in \mathbb{R}^+ e^{(\pi - \phi)i}, \\
\zeta^{-i\nu} e^{\frac{i\nu}{2} \sigma_3} \begin{pmatrix} 1 & 0 \\ r_{\xi_1} & 1 \end{pmatrix}, & \zeta \in \mathbb{R}^+ e^{(\pi + \phi)i}, \end{cases}$$ (A.2)

and $\nu = \nu(\xi_1)$;

* asymptotic behaviors:

$$m^{pc, \xi_1}(\zeta) = I + m^{pc, \xi_1}_1(\zeta) + O(\zeta^{-2}), \quad \zeta \to \infty.$$ (A.3)

The RHP A.1 has an explicit solution, which can be expressed in terms of Webber equation $(\frac{d^2}{dz^2} + (\frac{1}{2} - \frac{z^2}{2} + a))D_\alpha(z) = 0$. Taking the transformation

$$m^{pc, \xi_1} = \psi(\zeta) \mathcal{P} e^{-i\nu \sigma_3} e^{-\frac{i\nu}{2} \sigma_3},$$ (A.4)
where

\[
\mathcal{P}(\xi) = \begin{cases} 
\begin{pmatrix} 1 & \bar{r}_{\xi_1} \\ 0 & 1 - |r_{\xi_1}|^2 \end{pmatrix}, & \text{arg} \, \zeta \in (0, \phi), \\
\begin{pmatrix} 1 & 0 \\ -r_{\xi_1} & 1 \end{pmatrix}, & \text{arg} \, \zeta \in (\pi - \phi, \pi), \\
\begin{pmatrix} 1 & -\bar{r}_{\xi_1} \\ 0 & 1 \end{pmatrix}, & \text{arg} \, \zeta \in (\pi, \pi + \phi), \\
I, & \text{else.}
\end{cases}
\]  

(A.5)

The function \( \psi \) satisfies the following properties

**RHP A.2.** Find a 2 \times 2 matrix-valued function \( \psi(\zeta) \) such that

* \( \psi \) is analytical in \( \mathbb{C} \setminus \mathbb{R} \);

* Due to the branch cut along \( \mathbb{R}^+ \), \( \psi(\zeta) \) takes continuous boundary values \( \psi_{\pm} \) on \( \mathbb{R} \) and

\[
\psi_{\pm}(\zeta) = \psi_{-}(\zeta)V^\psi, \quad \zeta \in \mathbb{R},
\]  

(A.6)

where

\[
V^\psi(\xi) = \begin{pmatrix} 1 - |r_{\xi_1}|^2 & -\bar{r}_{\xi_1} \\ r_{\xi_1} & 1 \end{pmatrix}.
\]  

(A.7)

* asymptotic behavior:

\[
\psi = \zeta^{-i\sigma_3}e^{\frac{i}{2}\sigma_3}e^{\frac{i}{2}\sigma_3} \left( I + m_{1}^{pc_{\xi_1}}\zeta^{-1} + \mathcal{O}(\zeta^{-2}) \right), \quad \zeta \to \infty.
\]  

(A.8)

Differentiating (A.6) with respect to \( \zeta \), and combining \( \frac{\partial}{\partial \zeta} \sigma_3 \psi_+ = \frac{i\zeta}{2} \sigma_3 \psi_+ V^\psi \), we obtain

\[
\left( \frac{d\psi}{d\zeta} - \frac{i\zeta}{2} \sigma_3 \psi \right)_+ = \left( \frac{d\psi}{d\zeta} - \frac{i\zeta}{2} \sigma_3 \psi \right)_- V^\psi.
\]  

(A.9)
Notice that \( \det V^\psi = 1 \), thus we have \( \det \psi_+ = \det \psi_- \). Moreover, we can know that \( \det \psi \) is holomorphic in \( \mathbb{C} \) by Painlevé analytic continuation theorem. It follows \( \psi^{-1} \) exists and is bounded. The matrix function \( \left( \frac{d\psi}{d\zeta} - \frac{i\zeta}{2} \sigma_3 \psi \right) \psi^{-1} \) has no jump along the real axis and is an entire function with respect to \( \zeta \). Combine to (A.4), we can directly compute that

\[
\left( \frac{d\psi}{d\zeta} - \frac{i\zeta}{2} \sigma_3 \psi \right) \psi^{-1} = \left( \frac{dm_{pc,\xi_1}}{d\zeta} - \frac{i\zeta}{2} \sigma_3 \right) \left( \frac{m_{pc,\xi_1}}{\zeta} \right)^{-1} + \frac{i\zeta}{2} \frac{m_{pc,\xi_1}}{\sigma_3} \left( \frac{m_{pc,\xi_1}}{\zeta} \right)^{-1},
\]

(A.10)

The first term in the R.H.S of (A.10) tends to zero as \( \zeta \to \infty \). We use \( m_{pc,\xi_1}(\zeta) = I + m_{12} \zeta^{-1} + \mathcal{O}(\zeta^{-2}) \) as well as Liouville theorem to obtain that there exists a constant matrix \( \beta_{10} \) such that

\[
\left( \begin{array}{cc} 0 & \beta_{12} \\ \beta_{21} & 0 \end{array} \right) = \beta_{10} = \frac{i}{2} \left[ m_{pc,\xi_1}, \sigma_3 \right] = \left( \begin{array}{cc} 0 & -i[m_{12}]_{12} \\ i[m_{12}]_{21} & 0 \end{array} \right),
\]

(A.11)

which implies that \( [m_{12}]_{12} = i\beta_{12}, [m_{12}]_{21} = -i\beta_{21} \). Use Liouville theorem again, we have

\[
\left( \frac{d\psi}{d\zeta} - \frac{i\zeta}{2} \sigma_3 \psi \right) = \beta_{10} \psi.
\]

(A.12)

We rewrite the above equality to the following ODE system

\[
\frac{d\psi_{11}}{d\zeta} - \frac{i\zeta}{2} \psi_{11} = \beta_{12} \psi_{21},
\]

(A.13)

\[
\frac{d\psi_{21}}{d\zeta} + \frac{i\zeta}{2} \psi_{21} = \beta_{21} \psi_{11},
\]

(A.14)

as well as

\[
\frac{d\psi_{12}}{d\zeta} - \frac{i\zeta}{2} \psi_{12} = \beta_{12} \psi_{22},
\]

(A.15)

\[
\frac{d\psi_{22}}{d\zeta} + \frac{i\zeta}{2} \psi_{22} = \beta_{21} \psi_{12},
\]

(A.16)

From (A.13) to (A.16), we can solve that

\[
\frac{d^2\psi_{11}}{d\zeta^2} + \left( -\frac{i}{2} + \frac{\zeta^2}{4} - \beta_{12} \beta_{21} \right) \psi_{11} = 0, \quad \frac{d^2\psi_{21}}{d\zeta^2} + \left( \frac{i}{2} + \frac{\zeta^2}{4} - \beta_{12} \beta_{21} \right) \psi_{21} = 0,
\]

(A.17)

\[
\frac{d^2\psi_{12}}{d\zeta^2} + \left( -\frac{i}{2} + \frac{\zeta^2}{4} - \beta_{12} \beta_{21} \right) \psi_{12} = 0, \quad \frac{d^2\psi_{22}}{d\zeta^2} + \left( \frac{i}{2} + \frac{\zeta^2}{4} - \beta_{12} \beta_{21} \right) \psi_{22} = 0.
\]

(A.18)

The Webber equation is

\[
y'' + \left( \frac{1}{2} - \frac{\zeta^2}{4} + a \right) y = 0.
\]

(A.19)
The parabolic cylinder functions $D_a(z)$, $D_a(-z)$, $D_{-a-1}(iz)$, $D_{-a-1}(-iz)$ all satisfy (A.19) and are entire $\forall a$. The large-$z$ behavior of $D_a(z)$ can be uniquely given by the following formulae.

$$
D_a(z) = \begin{cases} 
    z^a e^{-z^2/4} (1 + \mathcal{O}(z^{-2})), & |\arg z| < \frac{3\pi}{4}, \\
    z^a e^{-z^2/4} (1 + \mathcal{O}(z^{-2})) - \frac{\sqrt{2\pi}}{\Gamma(-a)} e^{i\pi z-a-1} e^{z^2/4} (1 + \mathcal{O}(z^{-2})), & \frac{\pi}{4} < |\arg z| < \frac{5\pi}{4} , \\
    z^a e^{-z^2/4} (1 + \mathcal{O}(z^{-2})) - \frac{\sqrt{2\pi}}{\Gamma(-a)} e^{-i\pi z-a-1} e^{z^2/4} (1 + \mathcal{O}(z^{-2})), & \frac{-5\pi}{4} < |\arg z| < -\frac{\pi}{4}.
\end{cases}
$$

We set $\nu = \frac{\beta_1}{12} \frac{\beta_2}{21}$. For $\psi_{11}, \Im \zeta > 0$, we introduce the new variable $\eta = \zeta e^{-\frac{\pi i}{4}}$, and the first equation of (A.17) becomes

$$
\frac{d^2 \psi_{11}}{d\eta^2} + \left( \frac{1}{2} - \frac{\eta^2}{4} - i\nu \right) \psi_{11} = 0.
$$

(A.20)

For $\zeta \in \mathbb{C}^+, 0 < \text{Arg}\zeta < \pi, -\frac{\pi}{4} < \text{Arg}\eta < \frac{3\pi}{4}$. We have $\psi_{11} = e^{\frac{i\pi}{4} \nu(\xi)} D_{-\nu(\xi)}(e^{-\frac{i\pi}{4} \zeta}) \sim \zeta^{-i\nu} e^{\frac{i\pi}{4} \xi^2}$ corresponding to the $(1,1)$-entry of (A.8).

To save the space, we present the other results for $\psi$ below. The unique solution to RHP A.2 is

when $\zeta \in \mathbb{C}^+$,

$$
\psi(\zeta) = \begin{pmatrix} e^{\frac{i\pi}{4} \nu(\xi)} D_{-\nu(\xi)}(e^{-\frac{i\pi}{4} \zeta}) & \frac{i\nu(\xi)}{\beta_{21}} e^{-\frac{3\pi}{4} \nu(\xi) + i} D_{\nu(\xi)-1}(e^{-\frac{3\pi}{4} \zeta}) \\
-\frac{i\nu(\xi)}{\beta_{12}} e^{\frac{3\pi}{4} \nu(\xi) - i} D_{-\nu(\xi)-1}(e^{-\frac{3\pi}{4} \zeta}) & e^{-\frac{3\pi}{4} \nu(\xi)} D_{\nu(\xi)}(e^{-\frac{3\pi}{4} \zeta}) \end{pmatrix},
$$

(A.21)

when $\zeta \in \mathbb{C}^-$,

$$
\psi(\zeta) = \begin{pmatrix} e^{\frac{3\pi}{4} \nu(\xi)} D_{-\nu(\xi)}(e^{-\frac{3\pi}{4} \zeta}) & \frac{i\nu(\xi)}{\beta_{21}} e^{-\frac{3\pi}{4} \nu(\xi) + i} D_{\nu(\xi)-1}(e^{-\frac{3\pi}{4} \zeta}) \\
-\frac{i\nu(\xi)}{\beta_{12}} e^{\frac{3\pi}{4} \nu(\xi) - i} D_{-\nu(\xi)-1}(e^{-\frac{3\pi}{4} \zeta}) & e^{-\frac{3\pi}{4} \nu(\xi)} D_{\nu(\xi)}(e^{-\frac{3\pi}{4} \zeta}) \end{pmatrix},
$$

(A.22)

Which is similar to [35, Appendix C.3].
From (A.6), we know that \((\psi_-)^{-1}\psi_+ = V^\psi\) and

\[
\begin{align*}
r_{\xi_1} &= \psi_{-11}\psi_{+21} - \psi_{-21}\psi_{+11} \\
&= e^{\frac{\pi}{4} \nu(\xi_1)} D_{-i\nu(\xi_1)}(e^{-\frac{\pi}{4} i} \zeta) \cdot e^{\frac{\nu(\xi_1)}{4}} \frac{e^{\frac{\nu(\xi_1)}{4}}}{\beta_{12}^{\xi_1}} \left[ \partial_\xi (D_{-i\nu(\xi_1)}(e^{-\frac{\pi}{4} i} \zeta)) - \frac{i\zeta}{2} D_{-i\nu(\xi_1)}(e^{-\frac{\pi}{4} i} \zeta) \right] \\
&\quad - e^{\frac{\pi}{4} \nu(\xi_1)} D_{-i\nu(\xi_1)}(e^{-\frac{\pi}{4} i} \zeta) \cdot e^{\frac{\nu(\xi_1)}{4}} \frac{e^{\frac{\nu(\xi_1)}{4}}}{\beta_{12}^{\xi_1}} \left[ \partial_\xi (D_{-i\nu(\xi_1)}(e^{-\frac{\pi}{4} i} \zeta)) - \frac{i\zeta}{2} D_{-i\nu(\xi_1)}(e^{-\frac{\pi}{4} i} \zeta) \right] \\
&= \frac{e^{\frac{3\pi}{4} \nu(\xi_1)}}{\beta_{12}^{\xi_1}} \text{Wr} \left( D_{-i\nu(\xi_1)}(e^{-\frac{3\pi}{4} i} \zeta), D_{-i\nu(\xi_1)}(e^{-\frac{7\pi}{4} i} \zeta) \right) \\
&= \frac{e^{\frac{3\pi}{4} \nu(\xi_1)}}{\beta_{12}^{\xi_1}} \sqrt{2\pi} e^{-\frac{3\pi}{4} i} \Gamma(i\nu(\xi_1)). \quad \text{(A.23)}
\end{align*}
\]

The second "=" we used the equality \(D'_a(z) + \frac{i}{2} D_a(z) = a D_{a-1}(z)\). And the last "=" we used the Wronskian identity \(\text{Wr}(D_a(z), D_a(-z)) = \sqrt{\frac{2\pi}{1-a}}\).

And

\[
\begin{align*}
\beta_{12}^{\xi_1} &= \sqrt{2\pi} e^{-\frac{3\pi}{4} i} e^{\frac{3\nu(\xi_1)}{4}} \\
\frac{\beta_{12}^{\xi_1}}{\beta_{21}^{\xi_1}} &= \nu(\xi_1), \quad \text{(A.24)} \\
\arg\beta_{21}^{\xi_1} &= -\frac{5\pi}{4} - \arg\xi_1 - \arg\Gamma(i\nu(\xi_1)) \\
&= -\frac{5\pi}{4} - 2 \beta(\xi_1, \xi) + 2 \theta(\xi_1) + \nu(\xi_1) \log(2\theta''(\xi_1)) - \arg\Gamma(i\nu(\xi_1)). \quad \text{(A.25)}
\end{align*}
\]

And we have

\[
m_{p^c,\xi_1} = I + \frac{1}{\zeta} \left( \begin{array}{cc} 0 & i\beta_{21}^{\xi_1} \\ -i\beta_{21}^{\xi_1} & 0 \end{array} \right) + O(\zeta^{-2}). \quad \text{(A.27)}
\]

Finally, we point out the results of Appendix A.1 also can be applied to the local model near \(\xi_3\).

### A.2 Local Model Near \(\xi_j, \ j = 2, 4\)

**RHP A.3.** Find a matrix-valued function \(m_{p^c,\xi_j}(\zeta) := m_{p^c,\xi_j}(\zeta; \xi)\) with following properties:

* \(m_{p^c,\xi_j}(\zeta; \xi)\) is analytical in \(\mathbb{C}\backslash\Sigma_{p^c}\) with \(\Sigma_{p^c} = \{Re^{i\phi}\} \cup \{Re^{i(\pi - \phi)}\}\) shown in Figure A.2;

* \(m_{p^c}\) has continuous boundary values \(m_{p^c,\xi_j}\) on \(\Sigma_{p^c}\) and

\[
m_{p^c,\xi_j}(\zeta) = m_{p^c,\xi_j}(\zeta) V_{p^c}(\zeta), \quad \zeta \in \Sigma_{p^c},
\]

\[
\text{(A.28)}
\]

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where

\[
V_{pc}(\zeta) = \begin{cases} 
\zeta^i\hat{\sigma}_3 e^{-\frac{i\zeta^2}{2}\hat{\sigma}_3} \left( \begin{array}{cc} 1 & 0 \\ r_{\xi_2} & 1 \end{array} \right), & \zeta \in \mathbb{R}^+ e^{\phi i}, \\
\zeta^i\hat{\sigma}_3 e^{-\frac{i\zeta^2}{2}\hat{\sigma}_3} \left( \begin{array}{cc} 1 & -\bar{r}_{\xi_2} \\ 0 & 1 \end{array} \right), & \zeta \in \mathbb{R}^+ e^{-\phi i}, \\
\zeta^i\hat{\sigma}_3 e^{-\frac{i\zeta^2}{2}\hat{\sigma}_3} \left( \begin{array}{cc} 1 & 0 \\ \frac{r_{\xi_2}}{1-|r_{\xi_2}|^2} & 1 \end{array} \right), & \zeta \in \mathbb{R}^+ e^{(\phi-\pi)i}, \\
\zeta^i\hat{\sigma}_3 e^{-\frac{i\zeta^2}{2}\hat{\sigma}_3} \left( \begin{array}{cc} 1 & -\frac{\bar{r}_{\xi_2}}{1-|r_{\xi_2}|^2} \\ 0 & 1 \end{array} \right), & \zeta \in \mathbb{R}^+ e^{(-\phi+\pi)i}.
\end{cases}
\]  

(A.29)

and \( \nu = \nu(\xi_2) \).

Asymptotic behaviors:

\[
m_{pc,\xi_2}(\zeta) = I + m_{1,pc,\xi_2}^{pc,\xi_2} \zeta^{-1} + \mathcal{O}(\zeta^{-2}), \quad \zeta \to \infty.
\]  

(A.30)

The RHP A.3 has an explicit solution, which can be expressed in terms of Webber equation \((\frac{d^2}{dz^2} + (\frac{1}{2} - \frac{z^2}{2} + a))D_a(z) = 0\). Taking the transformation

\[
m_{pc,\xi_2}^{pc} = \psi(\zeta) P\zeta^{-i\nu\sigma_3} e^\frac{i\zeta^2}{2}\sigma_3,
\]  

(A.31)
where

\[
\mathcal{P}(\xi) = \begin{cases} 
\begin{pmatrix}
1 & 0 \\
-r\xi_2 & 1
\end{pmatrix}, & \arg \zeta \in (0, \phi), \\
\begin{pmatrix}
1 & -\bar{r}\xi_2 \\
0 & 1
\end{pmatrix}, & \arg \zeta \in (-\phi, 0), \\
\begin{pmatrix}
1 & 0 \\
\frac{1}{1-|r\xi_2|^2} & 1
\end{pmatrix}, & \arg \zeta \in (\phi - \pi, -\pi), \\
\begin{pmatrix}
1 & \frac{\bar{r}\xi_2}{1-|r\xi_2|^2} \\
0 & 1
\end{pmatrix}, & \arg \zeta \in (-\phi + \pi, \pi), \\
I, & \text{else.}
\end{cases}
\]  

(A.32)

The function \(\psi\) satisfies the following properties

**RHP A.4.** Find a \(2 \times 2\) matrix-valued function \(\psi(\zeta)\) such that

* \(\psi\) is analytical in \(\mathbb{C}\backslash \mathbb{R}\);

* Due to the branch cut along \(\mathbb{R}^+\), \(\psi(\zeta)\) takes continuous boundary values \(\psi_{\pm}\) on \(\mathbb{R}\) and

\[
\psi_+(\zeta) = \psi_-(\zeta)V^\psi, \quad \zeta \in \mathbb{R},
\]  

where

\[
V^\psi(\xi) = \begin{pmatrix} 1 - |r\xi_2|^2 & -\bar{r}\xi_2 \\ r\xi_2 & 1 \end{pmatrix}.
\]  

(A.33)

* asymptotic behavior:

\[
\psi = \zeta\nu_{\sigma_3}e^{-\frac{i}{4}\zeta^2\sigma_3}\left(I + m_{1}^{pc,\xi_2}\zeta^{-1} + \mathcal{O}(\zeta^{-2})\right), \quad \text{as} \quad \zeta \to \infty.
\]  

(A.35)

Differentiating (A.33) with respect to \(\zeta\), and combining \(\frac{i\zeta}{2}\sigma_3\psi_+ = \frac{i\zeta}{2}\sigma_3\psi_--V^\psi\), we obtain

\[
\left(\frac{d\psi}{d\zeta} + \frac{i\zeta}{2}\sigma_3\psi\right)_+ = \left(\frac{d\psi}{d\zeta} + \frac{i\zeta}{2}\sigma_3\psi\right)_-V^\psi.
\]  

(A.36)

Since the same reasons presented in Appendix A.1, the matrix function \(\left(\frac{d\psi}{d\zeta} + \frac{i\zeta}{2}\sigma_3\psi\right)\psi^{-1}\) has no jump along the real axis and is an entire function with respect to \(\zeta\). Combine (A.31), we can directly compute that

\[
\left(\frac{d\psi}{d\zeta} + \frac{i\zeta}{2}\sigma_3\psi\right)\psi^{-1} = \left[m_{1}^{pc,\xi_2}\frac{d}{d\zeta} + m^{pc,\xi_2}\frac{i\nu}{\zeta}\sigma_3\right](m_{1}^{pc,\xi_2})^{-1} + \frac{i\zeta}{2}[\sigma_3, m^{pc,\xi_2}](m_{1}^{pc,\xi_2})^{-1},
\]  

(A.37)
The first term in the R.H.S of (A.37) tends to zero as $\zeta \to \infty$. We use $m_{pc,\xi_2}(\zeta) = I + m_{pc,\xi_2}^1 \zeta^{-1} + O(\zeta^{-2})$ as well as Liouville theorem to obtain that there exists a constant matrix $\beta_2^{\text{mat}}$ such that

$$
\begin{pmatrix}
0 & \beta_2^{\xi_2} \\
\beta_2^{\xi_2} & 0
\end{pmatrix} = \beta_2^{\text{mat}} = \frac{i}{2} \left[ \sigma_3, m_{pc,\xi_2}^1 \right] = \left( \begin{array}{cc}
0 & i [m_{pc,\xi_2}^1]_{21} \\
-i [m_{pc,\xi_2}^1]_{21} & 0
\end{array} \right),
$$

(A.38)

which implies that $[m_{pc,\xi_2}^1]_{21} = -i \beta_1^{\xi_2}$, $[m_{pc,\xi_2}^1]_{21} = i \beta_1^{\xi_2}$. Use Liouville theorem again, we have

$$
\left( \frac{d\psi}{d\zeta} + \frac{i\zeta}{2} \sigma_3 \psi \right) = \beta_2^{\text{mat}} \psi.
$$

(A.39)

We rewrite the above equality to the following ODE system

$$
\frac{d\psi_{11}}{d\zeta} + \frac{i\zeta}{2} \psi_{11} = \beta_{12}^{\xi_2} \psi_{21},
$$

(A.40)

$$
\frac{d\psi_{21}}{d\zeta} - \frac{i\zeta}{2} \psi_{21} = \beta_{12}^{\xi_2} \psi_{11},
$$

(A.41)

as well as

$$
\frac{d\psi_{12}}{d\zeta} + \frac{i\zeta}{2} \psi_{12} = \beta_{12}^{\xi_2} \psi_{22},
$$

(A.42)

$$
\frac{d\psi_{22}}{d\zeta} - \frac{i\zeta}{2} \psi_{22} = \beta_{12}^{\xi_2} \psi_{12}.
$$

(A.43)

From (A.40) to (A.43), we can solve that

$$
\frac{d^2 \psi_{11}}{d\zeta^2} + \left( \frac{i}{2} + \frac{\zeta^2}{4} - \beta_{12}^{\xi_2} \beta_{21}^{\xi_2} \right) \psi_{11} = 0,
$$

(A.44)

$$
\frac{d^2 \psi_{12}}{d\zeta^2} + \left( \frac{i}{2} + \frac{\zeta^2}{4} - \beta_{12}^{\xi_2} \beta_{21}^{\xi_2} \right) \psi_{12} = 0,
$$

(A.45)

We set $\nu = \beta_{12}^{\xi_2} \beta_{21}^{\xi_2}$. For $\psi_{11}, \Im \zeta > 0$ we introduce the new variable $\eta = \zeta e^{-\frac{3\pi}{4}}$, and the first equation of (A.44) becomes

$$
\frac{d^2 \psi_{11}}{d\eta^2} + \left( \frac{1}{2} - \frac{\eta^2}{4} + i\nu \right) \psi_{11} = 0.
$$

(A.46)

For $\zeta \in \mathbb{C}^+$, $0 < \arg \zeta < \pi$, $-\frac{3\pi}{4} < \arg \eta < \frac{\pi}{4}$. We have $\psi_{11} = e^{-\frac{3\pi}{4} \nu(\zeta)} D_{\nu(\zeta)}(e^{-\frac{3\pi}{4} \zeta}) \sim \zeta^{\nu} e^{-\frac{3\pi}{4} \zeta}$ corresponding to the (1,1)-entry of (A.35). To save the space, we present the other results for $\psi$ below.

The unique solution to RHP A.4 is when $\zeta \in \mathbb{C}^+$,

$$
\psi(\zeta) = \begin{pmatrix}
\frac{e^{-\frac{3\pi}{4} \nu(\zeta)} D_{\nu(\zeta)}(e^{-\frac{3\pi}{4} i \zeta})}{\beta_{12}^{\xi_2}} e^{-\frac{3\pi}{4} \nu(\zeta) + i} D_{\nu(\zeta) - 1}(e^{-\frac{3\pi}{4} \zeta}) & -\frac{i \nu(\zeta)}{\beta_{12}^{\xi_2}} e^{\frac{3\pi}{4} \nu(\zeta) - i} D_{-\nu(\zeta) - 1}(e^{-\frac{3\pi}{4} \zeta}) \\
\frac{i \nu(\zeta)}{\beta_{12}^{\xi_2}} e^{\frac{3\pi}{4} \nu(\zeta) + i} D_{\nu(\zeta) - 1}(e^{-\frac{3\pi}{4} \zeta}) & e^{\frac{3\pi}{4} \nu(\zeta)} D_{-\nu(\zeta)}(e^{-\frac{3\pi}{4} i \zeta})
\end{pmatrix}.
$$

(A.47)
when $\zeta \in \mathbb{C}^-$,
\[
\psi(\zeta) = \left( \begin{array}{c}
\frac{\nu(\xi_2)}{\beta_{12}^2} e^{\nu(\xi_2)} D_{iv(\xi_2)}(e^{\frac{\pi i}{4}}) \\
\frac{\nu(\xi_2)}{\beta_{21}^2} e^{\nu(\xi_2)+i} D_{iv(\xi_2)} - 1(e^{\frac{3\pi i}{4}}) \\
-\frac{\nu(\xi_2)}{\beta_{21}^2} e^{-\frac{3\pi i}{4}} D_{iv(\xi_2)} - 1(e^{\frac{3\pi i}{4}}) \\
-\frac{\nu(\xi_2)}{\beta_{12}^2} e^{\nu(\xi_2)-i} D_{iv(\xi_2)} - 1(e^{\frac{3\pi i}{4}}) \\
\end{array} \right),
\]
(A.48)

Which is derived in [13, Section 4] and verified in [35, Proposition 5.5].

From (A.33), we know that $\left(\psi_+\right)^{-1} \psi_- = V \psi$ and
\[
|_{\xi_2} = \psi_{-11} \psi_{+,21} - \psi_{-,21} \psi_{+,11} \\
= e^{\frac{\pi i}{4}} D_{iv(\xi_2)}(e^{\frac{\pi i}{4}}) \cdot \frac{\nu(\xi_2)}{\beta_{12}^2} \left[ \partial_{\xi} \left(D_{iv(\xi_2)}(e^{\frac{3\pi i}{4}}) \right) + \frac{i}{2} D_{iv(\xi_2)}(e^{\frac{3\pi i}{4}}) \right] \\
- e^{-\frac{3\pi i}{4}} D_{iv(\xi_2)}(e^{-\frac{3\pi i}{4}}) \cdot \frac{\nu(\xi_2)}{\beta_{21}^2} \left[ \partial_{\xi} \left(D_{iv(\xi_2)}(e^{\frac{3\pi i}{4}}) \right) + \frac{i}{2} D_{iv(\xi_2)}(e^{\frac{3\pi i}{4}}) \right] \\
= \frac{e^{-\frac{\pi i}{4}}}{\beta_{12}^2} \psi_{\xi_2} \cdot \frac{e^{\frac{\pi i}{4}}}{\beta_{21}^2} \left( D_{iv(\xi_2)}(e^{\frac{3\pi i}{4}}), D_{iv(\xi_2)}(e^{\frac{3\pi i}{4}}) \right) \frac{1}{\Gamma(-i\nu(\xi_2))},
\]
(A.49)

And
\[
\beta_{12}^2 = \frac{\sqrt{2\pi e^{\frac{\pi i}{4}}} e^{-\frac{\nu(\xi_2)}{2}}}{r_{\xi_2} \Gamma(-i\nu(\xi_2))},
\]
(A.50)
\[
\beta_{21}^2 = \nu(\xi_2),
\]
(A.51)
\[
\arg \beta_{12}^2 = \frac{\pi}{4} - \arg r_{\xi_2} - \arg \Gamma(-i\nu(\xi_2)) \\
= \frac{\pi}{4} - 2\beta(\xi_2, \zeta_1 + 2t\theta(\xi_2) - \nu(\xi_2) \log(-2t\theta''(\xi_2)) - \arg \Gamma(-i\nu(\xi_2)).
\]
(A.52)

And we have
\[
m^{pc,\xi_2} = I + \frac{1}{\zeta} \left( \begin{array}{cc}
0 & -i\beta_{12}^2 \\
i\beta_{21}^2 & 0 \end{array} \right) + O(\zeta^{-2}).
\]
(A.53)

Finally, we point out the results of Appendix A.2 also can be applied to the local model near $\xi_1$.

**B A Proof for Proposition 4.8**

Without loss of generality, we take the neighborhood of $\xi_1$ as an example. We expand
\[
\theta(z) = \theta(\xi_1) + \frac{\theta''(\xi_1)}{2}(z - \xi_1)^2 + \theta_c(z - \xi_1)^3,
\]
(B.1)
where $\theta_c = \frac{\theta''(\kappa \xi_1 + (1-\kappa)z)}{3}$, $\kappa \in (0, 1)$ is the coefficient of Lagrangian remainder.

Owe to the scaling (4.147), we have the following transformation

$$N : g \rightarrow (Ng)(\zeta) := g \left( \left( 2t \theta''(\xi_1) \right)^{-\frac{1}{2}} \zeta + \xi_1 \right), \quad (B.2)$$

which acts on $e^{2it\theta(z)}$

$$e^{2it\theta(z)} = e^{2it(N\theta)(\zeta)} \left( \left. \frac{4.147}{e^{2it\theta(\xi_1)} \cdot e^{\frac{1}{2} \xi^2} \cdot e^{2it\theta_c(2\theta''(\xi_1))^{-\frac{1}{2}} \zeta^3} \right. \right) \quad \text{B.3}$$

The first and second term of R.H.S of (B.3) are used to match parabolic cylinder model as presented Subsection 4.3.2. What we care about is the third term.

Since $\zeta \in U_0$, the neighborhood of zero, we can set $\zeta = u + iv$, $|u| < \varepsilon$, $|v| < \varepsilon$. Thus

$$\left| e^{2it\theta_c(2\theta''(\xi_1))^{-\frac{1}{2}} \zeta^3} \right| = \left| e^{2it\theta_c(2\theta''(\xi_1))^{-\frac{1}{2}} (u+iv)^3} \right|
\begin{align*}
&= \exp \left( \left( 2t \theta''(\xi_1) \right)^{-\frac{1}{2}} \Re \left( 2it (\Re(\theta_c) + i \Im(\theta_c)) \cdot (u + iv)^3 \right) \right) \\
&= \exp \left[ -(2t)^{-\frac{1}{2}} \left( \theta''(\xi_1) \right)^{-\frac{1}{2}} \left( \Re(\theta_c) \cdot (3u^2v - v^3) + \Im(\theta_c) \cdot (u^3 - 3uv^2) \right) \right] \\
&\rightarrow 1 \quad \text{as} \quad t \rightarrow +\infty, \quad (B.4)
\end{align*}$$

by which the effects of the higher power could be ignored. The premise of this result is that

$$\Re(\theta_c) \cdot (3u^2v - v^3) + \Im(\theta_c) \cdot (u^3 - 3uv^2)$$

is finite, which follows from finite $\xi$.

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