New bounds of degree-based topological indices for some classes of $c$-cyclic graphs

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Abstract

Making use of a majorization technique for a suitable class of graphs, we derive upper and lower bounds for some topological indices depending on the degree sequence over all vertices, namely the first general Zagreb index and the first multiplicative Zagreb index. Specifically, after characterizing $c$-cyclic graphs ($0 \leq c \leq 6$) as those whose degree sequence belongs to particular subsets of $\mathbb{R}^n$, we identify the maximal and minimal vectors of these subsets with respect to the majorization order. This technique allows us to determine lower and upper bounds of the above indices recovering those existing in the literature as well obtaining new ones.

Key Words: Majorization, Schur-convex functions, $c$-cyclic graphs, Zagreb indices.

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1 Introduction

Many topological indices in Mathematical Chemistry are based on the degree sequence of a finite graph $G = (V,E)$ over all vertices. One of the most famous among these is the first Zagreb index defined as $M_1(G) = \sum_{i=1}^{n} d_i^2$ where $d_i$ ($i = 1, ..., n$) stands for the degree of the vertex $i$ and $n = |V|$ (see [11], [12], [20]). The notion of $M_1(G)$ was extended by Li and Zheng [16] as the first general Zagreb index $M_1^\alpha(G) = \sum_{i=1}^{n} d_i^\alpha$, for $\alpha$ an arbitrary real number different from 0 and 1, that coincides with the zeroth-order general Randić index (see [17]). For $\alpha = 2$ we recover the first Zagreb index while for $\alpha = -1$ we get the inverse degree $\rho(G) = M_1^{-1} = \sum_{j=1}^{n} \frac{1}{d_j}$ which has generated increased attention motivated by conjectures of the computer program Graffiti (see [9]).

In this paper we are concerned precisely with those indices depending on the degree sequence over all vertices of $G$, for which we adopt a unified approach aimed to determine new lower and upper bounds. This fruitful methodology, synthetically introduced in Section 2, is based on the majorization order and Schur-convexity ([19]), and has already been used by some of the authors ([1] and [10]) in other contexts, as well as for localizing some relevant topological indicators of a graph ([2], [3], [4] and [5]), which is also the aim of the present article. We restrict our attention to a particular class of graphs, the $c$-cyclic graphs for $0 \leq c \leq 6$, which contain exactly $c$ independent cycles (i.e., cycles that do not contain other cycles within themselves). In Section 3 we provide a new characterization of

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cyclic graphs, needed to determine their extremal degree sequences with respect to the majorization order discussed in Section 4. In Section 5 we determine upper and lower bounds for some degree-based topological indices. Section 6 concludes with a summary and some final comments.

2 Notations and preliminaries results on majorization

In this section we recall some basic notions on majorization, referring for more details to [2] and [19]. In the sequel we denote by \([x_1^n, x_2^n, \ldots, x_p^n]\) a vector in \(\mathbb{R}^n\) with \(\alpha_i\) components equal to \(x_i\), where \(\sum_{i=1}^p \alpha_i = n\). If \(\alpha_i = 1\) we use for convenience \(x_i^1\) instead of \(x_i\), while \(x_i^0\) means that the component \(x_i\) is not present.

**Definition 1.** Given two vectors \(y, z \in D = \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \ldots \geq x_n\}\), the majorization order \(y \preceq z\) means:

\[
\begin{align*}
\langle y, s^k \rangle &\leq \langle z, s^k \rangle, \quad k = 1, \ldots, (n-1) \\
\langle y, s^n \rangle &\leq \langle z, s^n \rangle
\end{align*}
\]

where \(\langle \cdot, \cdot \rangle\) is the inner product in \(\mathbb{R}^n\) and \(s^j = [1^j, 0^{n-j}], \quad j = 1, 2, \ldots, n\).

In what follows we will consider some subsets of

\[\Sigma_a = D \cap \{x \in \mathbb{R}^n_+ : \langle x, s^n \rangle = a\},\]

where \(a\) is a positive real number. Given a closed subset \(S \subseteq \Sigma_a\), a vector \(x^*(S) \in S\) is said to be maximal for \(S\) with respect to the majorization order if \(x \preceq x^*(S)\) for each \(x \in S\). Analogously, a vector \(x_*(S) \in S\) is said to be minimal for \(S\) with respect to the majorization order if \(x_*(S) \preceq x\) for each \(x \in S\). Notice that if \(S \subseteq T\), then \(x^*(S) \preceq x^*(T)\) and \(x_*(T) \preceq x_*(S)\). In [2] some of the authors derived the maximal and minimal elements, with respect to the majorization order, of the set

\[S_a = \Sigma_a \cap \{x \in \mathbb{R}^n : m_i \geq x_i \geq m_i, \quad i = 1, \ldots, n\},\tag{1}\]

where \(m = [m_1, m_2, \ldots, m_n]\) and \(M = [M_1, M_2, \ldots, M_n]\) are two fixed vectors arranged in nonincreasing order with \(0 \leq m_i \leq M_i\) for all \(i = 1, \ldots, n\), and \(a\) is a positive real number such that \(\langle m, s^n \rangle \leq a \leq \langle M, s^n \rangle\). For the sake of completeness we recall the main results we will use in Section 4. We start discussing the maximal element. Let \(v^j = [0^j, 1^{n-j}], \quad j = 0, \ldots, n\).

**Theorem 2.** Let \(k \geq 0\) be the smallest integer such that

\[\langle M, s^k \rangle + \langle m, v^k \rangle \leq a < \langle M, s^{k+1} \rangle + \langle m, v^{k+1} \rangle,\tag{2}\]

and \(\theta = a - \langle M, s^k \rangle - \langle m, v^{k+1} \rangle\). Then

\[x^*(S_a) = [M_1, M_2, \ldots, M_k, \theta, M_{k+2}, \ldots, M_n].\tag{3}\]

From this general result, the maximal element of particular subsets of \(S_a\) can be deduced. In what follows we will often focus on sets of the type

\[S_a^{[h]} = \Sigma_a \cap \{x \in \mathbb{R}^n : M_1 \geq x_1 \geq \ldots \geq x_h \geq m_1, \quad M_2 \geq x_{h+1} \geq \ldots \geq x_n \geq m_2\}\]

where \(1 \leq h \leq n, \quad 0 \leq m_2 \leq m_1, \quad 0 \leq M_2 \leq M_1, \quad m_i < M_i, i = 1, 2\) and

\[hm_1 + (n-h)m_2 \leq a \leq hM_1 + (n-h)M_2.\]

In this case, given \(a^* = hM_1 + (n-h)m_2\), let

\[k = \begin{cases} 
\frac{a-h(m_1-m_2)-nm_2}{M_1-m_1} & \text{if } a < a^* \\
\frac{a-h(M_1-M_2)-nm_2}{M_2-m_2} & \text{if } a \geq a^*
\end{cases}\]
where $\lfloor x \rfloor$ denote the integer part of the real number $x$. In Corollary 3 in [2] it has been shown that

$$x^*(S_a^{[h]}) = \begin{cases} 
\left[ M^k_1, \theta, m^{h-k-1}_1, m^{n-h}_2 \right] & \text{if } a < a^* \\
\left[ M^h_1, M^{k-h}_2, \theta, m^{n-k-1}_2 \right] & \text{if } a \geq a^*
\end{cases}$$

where $\theta$ is evaluated in order to entail $x^*(S_a^{[h]}) \in \Sigma_a$.

The computation of the minimal element of the set $S_a$ is more tangled. The minimal element of $\Sigma_a$ is $x_*(\Sigma_a) = \lfloor (\frac{a}{n})^n \rfloor$. If it belongs to $S_a$ then it is its minimal element, too. Otherwise we will use the following theorem

**Theorem 3.** Let $k \geq 0$ and $d \geq 0$ be the smallest integers such that

1) $k + d < n$

2) $m_{k+1} \leq \rho \leq M_{n-d}$ where $\rho = \frac{a - \langle m, s^k \rangle - \langle M, v^{n-d} \rangle}{n-k-d}$.

Then

$$x_*(S_a) = [m_1, \ldots, m_k, \rho^{n-d-k}, M_{n-d+1} \cdots, M_n].$$

For the set $S_a^{[h]}$ we can express the minimal element in a more accessible way (see Corollary 10 in [2]). We recall only the expression for the case $m_1 \leq M_2$, since it is the only one we need in the sequel:

$$x_*(S_a^{[h]}) = \begin{cases} 
\lfloor (\frac{a}{n})^n \rfloor & \text{if } m_1 \leq \frac{a}{n} \leq M_2 \\
\left[ m^h_1, \left( \frac{a-hm_1}{n-h} \right)^{n-h} \right] & \text{if } \frac{a}{n} < m_1 \\
\left[ \left( \frac{a-M_2(n-h)}{h} \right)^h, M^{n-h}_2 \right] & \text{if } \frac{a}{n} > M_2
\end{cases}$$

(4)

Notice that the minimal element of the set $S_a$ does not necessarily have integer components, while this is not the case for the maximal element. For our purposes it is crucial to find the minimal vector in $S_a$ with integer components which can be constructed by the following procedure (see Remark 12 in [2]). Let us consider, for instance, the vector $x_*(S_a^{[h]}) = \lfloor (\frac{a}{n})^n \rfloor$ which corresponds to the case $m_1 \leq \frac{a}{n} \leq M_2$. If $\frac{a}{n}$ is not an integer, we will find the index $k$, $1 \leq k \leq n$, such that

$$\left( \lfloor \frac{a}{n} \rfloor + 1 \right) k + \left\lfloor \frac{a}{n} \right\rfloor (n-k) = a$$

i.e., $k = a - \left\lfloor \frac{a}{n} \right\rfloor n$. The vector

$$x_1 = \left[ \left( \left\lfloor \frac{a}{n} \right\rfloor + 1 \right)^k, \left( \left\lfloor \frac{a}{n} \right\rfloor \right)^{n-k} \right]$$

is the minimal element of $S_a^{[h]}$ with integer components. With slight modifications the same procedure can be applied also to the other cases discussed in [4] of Theorem 3.

### 3 A new characterization of $c$-cyclic graphs

Let $G = (V, E)$ a simple, connected, undirected graph with order $|V| = n$ and size $|E| = m$. Denote by $\pi = (d_1, d_2, \cdots, d_n)$ the degree sequence of $G$, where $d_i$ is the degree of the vertex $i$, arranged in nonincreasing order $d_1 \geq d_2 \geq \cdots \geq d_n$. It is well known that

$$\sum d_i = 2m, \text{ and } d_1 \leq n-1 \leq m.$$
The cyclomatic number $c$ of a graph $G$ is given by $c = m - n + 1$. It corresponds to the number of independent cycles in $G$, i.e. cycles that do not contain other cycles (see [1]). In particular, graphs with cyclomatic number $c = 0$ are trees, graphs with cyclomatic number $c = 1$ are unicyclic graphs and, more generally, graphs with cyclomatic number $c$ are $c$-cyclic graphs. In this Section we will deal with graphs having a cyclomatic number $c \leq 6$.

Schocker in [21] gave a characterization of the degree sequences of $c$-cyclic graphs (see also [13], [22], [23], [24]). For the sake of completeness we recall his results for the case $0 \leq c \leq 6$ (see Theorem 2.4 and Corollary 2.5 in [21]).

Theorem 4. The integers $(n - 1) \geq d_1 \geq d_2 \geq \cdots \geq d_n$ are the vertex degree sequence of

i. a tree $(c = 0)$ if and only if $m \geq 1$, $n = m + 1$;

ii. a unicyclic graph $(c = 1)$ if and only if $m \geq 3$, $n = m$, $d_1 + d_2 \leq n + 1$;

iii. a bicyclic graph $(c = 2)$ if and only if $m \geq 5$, $n = m - 1$, $d_1 + d_2 \leq n + 2$, $d_1 + d_2 + d_3 \leq n + 4$;

iv. a tricyclic graph $(c = 3)$ if and only if $m \geq 6$, $n = m - 2$, $d_1 + d_2 \leq n + 3$ and $d_1 + d_2 + d_3 \leq n + 5$;

v. a tetracyclic graph $(c = 4)$ if and only if $m \geq 8$, $n = m - 3$, $d_1 + d_2 \leq n + 4$, $d_1 + d_2 + d_3 \leq n + 6$ and $d_1 + d_2 + d_3 + d_4 \leq n + 9$;

vi. a pentacyclic graph $(c = 5)$ if and only if $m \geq 9$, $n = m - 4$, $d_1 + d_2 \leq n + 5$, $d_1 + d_2 + d_3 \leq n + 7$, $d_1 + d_2 + d_3 + d_4 \leq n + 10$, $2d_1 + 2d_2 + d_3 + d_4 + d_5 \leq 2n + 16$.

vii. a hexacyclic graph $(c = 6)$ if and only if $m \geq 10$, $n = m - 5$, $d_1 + d_2 \leq n + 6$, $d_1 + d_2 + d_3 \leq n + 8$, $d_1 + d_2 + d_3 + d_4 \leq n + 11$, $2d_1 + 2d_2 + d_3 + d_4 + d_5 \leq 2n + 18$, $2d_1 + 2d_2 + d_3 + d_4 + d_5 + d_6 \leq 2n + 20$.

In the class of $c$-cyclic graphs we are interested in finding graphs associated to the maximal (minimal) degree sequence with respect to the majorization order.

To this aim, by Theorem 4 we derive a new characterization of $c$-cyclic graphs as those whose degree sequences belongs to particular subsets of $\Sigma_{2(n+c-1)}$. Next theorem gives the structure of the subset of $\Sigma_{2(n+c-1)}$ we will deal with. Notice that the cases $c \leq 2$ are well known (see [22]) while the case $c = 3$ was incorrectly discussed in Lemma 2.2 of [23].

Theorem 5. The integers $(n - 1) \geq d_1 \geq d_2 \geq \cdots \geq d_n$ are the vertex degree sequence of

i. a tree $(c = 0)$ if and only if $n \geq 2$, $\sum_{i=1}^{n} d_i = 2(n - 1)$;

ii. a unicyclic graph $(c = 1)$ if and only if $n \geq 3$, $\sum_{i=1}^{n} d_i = 2n$ and at least three of them are greater or equal to 2;

iii. a bicyclic graph $(c = 2)$ if and only if $n \geq 4$, $\sum_{i=1}^{n} d_i = 2(n + 1)$ and at least four of them are greater than or equal to 2;

iv. a tricyclic graph $(c = 3)$ if and only if $n \geq 4$, $\sum_{i=1}^{n} d_i = 2(n + 2)$, and one of the following conditions holds:

1. if $n \geq 5$, at least five of them are greater or equal to 2,
2. at least four of them are greater or equal to 3;

v. a tetracyclic graph $(c = 4)$ if and only if $n \geq 5$, $\sum_{i=1}^{n} d_i = 2(n + 3)$, and one of the following conditions holds:

1. if $n \geq 6$, at least six of them are greater than or equal to 2,
2. at least four of them are greater or equal to 3 and at least five of them are greater than or equal to 2;

vi. a pentacyclic graph $(c = 5)$ if and only if $n \geq 5$, $\sum_{i=1}^{n} d_i = 2(n + 4)$, and one of the following conditions holds:
1. if \( n \geq 7 \), at least seven of them are greater than or equal to 2,
2. if \( n \geq 6 \) at least six of them are greater than or equal to 2 and at least four of them are greater than or equal to 3,
3. at least five of them are greater than or equal to 3 and at least three of them are greater than or equal to 4.

vii. a hexacyclic graph (\( c = 6 \)) if and only if \( n \geq 5 \), \( \sum_{i=1}^{n} d_i = 2(n + 5) \), and one of the following conditions holds:

1. if \( n \geq 8 \), at least eight of them are greater than or equal to 2,
2. if \( n \geq 7 \) at least seven of them are greater than or equal to 2 and at least four of them are greater than or equal to 3,
3. if \( n \geq 6 \) at least six of them are greater than or equal to 2, at least five of them are greater than or equal to 3 and at least three of them are greater or equal to 4,
4. if \( n \geq 6 \) at least six of them are greater than or equal to 3,
5. at least five of them are greater than or equal to 4.

**Proof. Case c = 0:** well known.

**Case c = 1.** Necessary condition: it is evident that the vertices of the unique cycle have the degrees greater than or equal to 2 (see also Lemma 2.1 in [22]).

Sufficient condition: the condition \( d_1 + d_2 \leq n + 1 \) is equivalent to \( d_3 + \cdots + d_n \geq n - 1 \) and this inequality is satisfied if \( d_3 \geq 2 \).

**Case c = 2.** Necessary condition: let us assume by contradiction that \( d_4 < 2 \), i.e. \( d_4 = 1 \) which implies \( d_1 + d_2 + d_3 = n + 5 \), against the condition \( d_1 + d_2 + d_3 \leq n + 4 \) (see also Lemma 2.2 in [22]).

Sufficient condition: the condition \( d_1 + d_2 + d_3 \leq n + 4 \) is equivalent to \( d_4 + \cdots + d_n \geq n - 2 \) while the condition \( d_1 + d_2 \leq n + 2 \) is equivalent to \( d_3 + \cdots + d_n \geq n \). If \( d_4 \geq 2 \) they are both satisfied.

**Case c = 3.** For \( n = 4 \) there is only one tricyclic graph associated to the degree sequence \([3^4]\). Let us consider the case \( n \geq 5 \).

Necessary condition: assume by contradiction that \( d_5 = 1 \) and \( d_4 \leq 2 \). This implies \( d_1 + d_2 + d_3 + d_4 = n + 8 \) and \( d_1 + d_2 + d_3 \geq n + 6 \), against the condition \( d_1 + d_2 + d_3 \leq n + 5 \).

Sufficient condition: the condition \( d_1 + d_2 + d_3 \leq n + 5 \) is equivalent to \( d_4 + \cdots + d_n \geq n - 1 \) while the condition \( d_1 + d_2 \leq n + 3 \) is equivalent to \( d_3 + \cdots + d_n \geq n + 1 \). They are both satisfied if either \( d_5 \geq 2 \) or \( d_4 \geq 3 \).

**Case c = 4:** Let us consider the case \( n \geq 6 \).

Necessary condition: assume by contradiction that both the conditions are not fulfilled. This means that either \( d_5 = 1 \) or \( d_6 = 1 \) and \( d_1 \leq 2 \). In the first case \( d_1 + d_2 + d_3 + d_4 = n + 10 \) contradicts the necessary condition \( d_1 + d_2 + d_3 + d_4 \leq n + 9 \). In the second case, \( d_1 + d_2 + d_3 + d_4 + d_5 = n + 11 \) and \( d_1 + d_2 + d_3 \geq n + 7 \) contradict the necessary condition \( d_1 + d_2 + d_3 \leq n + 6 \).

Sufficient condition: the condition \( d_1 + d_2 + d_3 + d_4 \leq n + 9 \) is equivalent to \( d_5 + \cdots + d_n \geq n - 3 \) which is satisfied if \( d_5 \geq 2 \), thus in particular if either \( v.1 \) or \( v.2 \) hold.

The condition \( d_1 + d_2 + d_3 \leq n + 6 \) is equivalent to \( d_4 + \cdots + d_n \geq n \) while \( d_1 + d_2 \leq n + 4 \) is equivalent to \( d_3 + \cdots + d_n \geq n + 2 \). They are both satisfied if either \( v.1 \) or \( v.2 \) hold.

The case \( n = 5 \) can be proved following similar steps.

**Case c = 5:** for \( n = 5 \) there is only one pentacyclic graph associated to the degree sequence \([4^3, 3^2]\).

Let us consider the case \( n \geq 7 \).

Necessary condition: assume by contradiction that neither of the three conditions are fulfilled. Then one of the following cases should occur:

a) \( d_6 = 1 \) and \( d_5 \leq 2 \); b) \( d_6 = 1 \) and \( d_5 \leq 3 \); c) \( d_7 = 1 \) and \( d_4 \leq 2 \).

In cases a) and b) we have \( d_1 + d_2 + d_3 + d_4 + d_5 = n + 13 \). In case a) we get \( d_1 + d_2 + d_3 + d_4 \geq n + 11 \) which contradicts the necessary condition \( d_1 + d_2 + d_3 + d_4 \leq n + 10 \). In case b) \( d_1 + d_2 \geq n + 4 \) implies \( 2d_1 + 2d_2 \geq 2n + 8 \). By using the necessary condition \( 2d_1 + 2d_2 + d_3 + d_4 + d_5 \leq 2n + 16 \), we
get $d_3 + d_4 + d_5 \leq 8$ and thus $d_4 \leq 2$. But under this condition we get again $d_1 + d_2 + d_3 + d_4 \geq n + 11$ contradicting the necessary condition $d_1 + d_2 + d_3 + d_4 \leq n + 10$.

Finally in case c) we have $d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = n + 14$ and $d_1 + d_2 + d_3 \geq n + 8$ against the necessary condition $d_1 + d_2 + d_3 \leq n + 7$.

**Sufficient condition:** the condition $d_1 + d_2 + d_3 + d_4 \leq n + 10$ is equivalent to $d_5 + \cdots + d_n \geq n - 2$ which is satisfied if either $d_6 \geq 2$ or $d_5 \geq 3$, thus in particular if vi.1 or vi.2 or vi.3 hold.

The conditions $d_1 + d_2 + d_3 \leq n + 7$, equivalent to $d_4 + \cdots + d_n \geq n + 1$, and $d_1 + d_2 \leq n + 5$, equivalent to $d_3 + \cdots d_n \geq n + 3$, are both fulfilled if vi.1 or vi.2 or vi.3 hold.

The last condition $2d_1 + 2d_2 + d_3 + d_4 + d_5 \leq 2n + 16$ is equivalent to $d_6 + \cdots + d_n \geq d_1 + d_2 - 8$. Easy computations show that it is satisfied either for $d_7 \geq 2$ or for $d_6 \geq 2$ and $d_4 \geq 3$. Indeed, for $d_7 \geq 2$ we get

$$d_6 + \cdots + d_n \geq 4 + (n - 7) = (n + 5) - 8 \geq d_1 + d_2 - 8,$$

while for $d_6 \geq 2$ and $d_4 \geq 3$ we get

$$d_3 + d_4 + d_6 + \cdots + d_n \geq 6 + 2 + (n - 6) = (n + 10) - 8 \geq d_1 + d_2 + d_3 + d_4 - 8.$$

Finally, for $d_5 \geq 3$ and $d_3 \geq 4$ we get

$$d_3 + d_4 + d_6 + \cdots + d_n \geq 7 + (n - 5) = (n + 10) - 8 \geq d_1 + d_2 + d_3 + d_4 - 8.$$

The case $n = 6$ can be proved following similar steps.

**Case c = 6:** for $n = 5$ there is only one pentacyclic graph associated to the degree sequence $[4^5]$. Let us consider the case $n \geq 8$.

**Necessary condition:** let us assume by contradiction that neither of the conditions are fulfilled.

Then one of the following five cases should occur:

- a) $d_6 = 1$ and $d_5 \leq 3$;  
- b) $d_6 = 1$ and $d_4 \leq 2$;  
- c) $d_7 = 1$ and $d_5 \leq 2$;
- d) $d_7 = 1$, $d_6 \leq 2$ and $d_5 \leq 3$;  
- e) $d_8 = 1$ and $d_4 \leq 2$.

In cases a) and b) we have $d_1 + d_2 + d_3 + d_4 + d_5 = n + 15$. In case a) we get $d_1 + d_2 + d_3 + d_4 \geq n + 12$ against the necessary condition $d_1 + d_2 + d_3 \leq n + 11$. In case b) we get $d_1 + d_2 + d_3 \geq n + 11$ against the necessary condition $d_1 + d_2 + d_3 \leq n + 8$.

In cases c) and d) we have $d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = n + 16$. In case c) we get again $d_1 + d_2 + d_3 + d_4 \geq n + 12$ which contradicts the necessary condition $d_1 + d_2 + d_3 + d_4 \leq n + 11$. In case d) we get $d_1 + d_2 \geq n + 5$ which implies $2d_1 + 2d_2 \geq 2n + 10$. By using the necessary condition $2d_1 + 2d_2 + d_3 + d_4 + d_5 + d_6 \leq 2n + 20$, we get $d_3 + d_4 + d_5 + d_6 \leq 10$ and thus $d_5 \leq 2$ or $d_6 = 1$.

Under the condition $d_5 \leq 2$ we go back to case c) while for $d_6 = 1$ to case a).

Finally in case e) we have $d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = n + 17$ and we get $d_1 + d_2 + d_3 \geq n + 9$ contradicting the necessary condition $d_1 + d_2 + d_3 \leq n + 8$.

**Sufficient condition:** the condition $d_1 + d_2 + d_3 + d_4 \leq n + 11$ is equivalent to $d_5 + \cdots + d_n \geq n - 1$ which is satisfied if $d_7 \geq 2$ or $d_6 \geq 2$ and $d_5 \geq 3$ or $d_5 \geq 4$, thus in particular if vii.1-vii.5 hold.

The conditions $d_1 + d_2 + d_3 \leq n + 7$, equivalent to $d_4 + \cdots + d_n \geq n + 2$, and $d_1 + d_2 \leq n + 5$, equivalent to $d_3 + \cdots d_n \geq n + 4$, are both fulfilled if one of the conditions vii.1-vii.5 holds.

Let us now consider the last two conditions: $2d_1 + 2d_2 + d_3 + d_4 + d_5 \leq 2n + 16$ is equivalent to $d_6 + \cdots + d_n \geq d_1 + d_2 - 8$ while $2d_1 + 2d_2 + d_3 + d_4 + d_5 + d_6 \leq 2n + 20$ is equivalent to $d_7 + \cdots + d_n \geq d_1 + d_2 - 10$. First of all, notice that for $d_6 \geq 2$ the first inequality is satisfied if and only if the second one holds.

Easy computations show that the inequality $d_7 + \cdots + d_n \geq d_1 + d_2 - 10$ is satisfied either for $d_8 \geq 2$ or for $d_7 \geq 2$ and $d_4 \geq 3$. Indeed, for $d_8 \geq 2$ we get

$$d_7 + \cdots + d_n \geq 4 + (n - 8) = (n + 6) - 10 \geq d_1 + d_2 - 10,$$

while for $d_7 \geq 2$ and $d_4 \geq 3$ we get

$$d_3 + d_4 + d_6 + \cdots + d_n \geq 6 + 4 + (n - 7) = (n + 11) - 8 \geq d_1 + d_2 + d_3 + d_4 - 8.$$

In a similar way, for $d_6 \geq 2$, $d_5 \geq 3$ and $d_3 \geq 4$ we get

$$d_3 + d_4 + d_6 + \cdots + d_n \geq 4 + 3 + 2 + (n - 6) = (n + 11) - 8 \geq d_1 + d_2 + d_3 + d_4 - 8$$
while for \(d_6 \geq 3\)
\[
d_3 + d_4 + d_6 + \cdots + d_n \geq 9 + (n - 6) = (n + 11) - 8 \geq d_1 + d_2 + d_3 + d_4 - 8.
\]
Finally for \(d_5 \geq 4\) we get
\[
d_3 + d_4 + d_7 + \cdots + d_n \geq 8 + (n - 6) = (n + 12) - 10 \geq d_1 + d_2 + d_3 + d_4 - 10
\]
\[
d_3 + d_4 + d_6 + \cdots + d_n \geq 8 + (n - 5) = (n + 11) - 8 \geq d_1 + d_2 + d_3 + d_4 - 10
\]
The case \(n = 6\) and \(n = 7\) can be proved following similar steps. \(\square\)

## 4 Extremal degree sequences

Endowed with Theorem 5, we are now in a position to apply the results of Section 2 in order to get extremal degree sequences associated to \(c\)-cyclic graphs. This topic has been investigated also by Gutman who gave in [13] a characterization of maximal degree sequences by using partitions of an integer in unequal parts (see Lemma 2 in [13]). This approach is completely different from ours and provides only the maximal elements while we are also able to build minimal ones.

In the sequel, for any \(c\), the vector \(d = (d_1, d_2, \cdots, d_n)\) is arranged in non increasing order, \(d_1 \geq d_2 \geq \cdots \geq d_n\) and \(\sum_{i=1}^{n} d_i = 2(n + c - 1)\), i.e. \(d \in \Sigma_2(n+c-1)\).

- **Trees.** We face the set
\[
S_0 = \{d \in D \cap \Sigma_2(n-1) : 1 \leq d_i \leq (n - 1)\}
\]
The maximal element of \(S_0\) is given by:
\[
x^*(S_0) = [(n - 1), 1^{n-1}]
\]
which is the degree sequence of the star \(S_n\).

The minimal element of the set \(S_0\) is given by
\[
x_*(S_0) = \left[\left(\frac{2(n-1)}{n}\right)^n\right]. \quad (5)
\]
Since it has not integer components for \(n > 2\), we apply the procedure explained in Section 2.

Taking \(k = 2(n - 1) - \left\lfloor \frac{2(n - 1)}{n}\right\rfloor n = (n - 2)\) for \(n \geq 2\), the minimal element with integer component is
\[
x'_*(S_0) = [2^{n-2}, 1^2]
\]
which is the degree sequence of the path \(P_n\).

- **Unicyclic.** We face the set
\[
S_1 = \{d \in \Sigma_2n : 1 \leq d_n \leq \cdots \leq d_4 \leq (n - 1), \ 2 \leq d_3 \leq d_2 \leq d_1 \leq (n - 1)\}
\]
The maximal element of \(S_1\) is given by:
\[
x^*(S_1) = [(n - 1), 2^2, 1^{n-3}]
\]
which is the degree sequence of the graph obtained joining two leaves of the star \(S_n\). The minimal element of the set \(S_1\) is given by
\[
x_*(S_1) = [2^n] \quad (6)
\]
which is the degree sequence of the cycle.
• **Bicyclic.** We face the set

\[ S_2 = \{ d \in \Sigma_{2(n+1)} : 1 \leq d_n \leq \cdots \leq d_5 \leq (n-1), \ 2 \leq d_4 \leq \cdots \leq d_1 \leq (n-1) \} \]

The maximal element of \( S_2 \) is given by:

\[ x^*(S_2) = [(n-1), 3, 2^2, 1^{n-4}] \]

which is the degree sequence of the graph \( H^1_n \) in [24].

The minimal element of the set \( S_2 \) is given by

\[ x_*(S_2) = \left[ \left( \frac{2(n+1)}{n} \right)^n \right] \]  

(7)

Since it has not integer components, we apply the procedure explained in Section 2.

Taking \( k = 2(n+1) - \left\lfloor \frac{2(n+1)}{n} \right\rfloor \) \( n = 2 \), the minimal element with integer component is

\[ x'_*(S_2) = [3^2, 2^{n-2}] \]

which is the degree sequence of the class of graph \( G^1_n \) in [24]. They can be obtained joining two non adjacent nodes in a circle.

• **Tricyclic.** For \( n = 4 \) there is only one tricyclic graph associated to the sequence \([3^4]\). Assuming \( n \geq 5 \), unlike the previous cases, we should now consider two different sets:

\[ S_3^1 = \{ d \in \Sigma_{2(n+2)} : 1 \leq d_n \leq \cdots \leq d_6 \leq (n-1), \ 2 \leq d_5 \leq \cdots \leq d_1 \leq (n-1) \} \]

\[ S_3^2 = \{ d \in \Sigma_{2(n+2)} : 1 \leq d_n \leq \cdots \leq d_5 \leq (n-1), \ 3 \leq d_4 \leq \cdots \leq d_1 \leq (n-1) \} \]

The maximal elements are, respectively,

\[ x^*(S_3^1) = [(n-1), 4, 2^3, 1^{n-5}] \]

\[ x^*(S_3^2) = [(n-1), 3^2, 1^{n-4}] \]

Notice that the two degree sequences are not comparable in the majorization order. Thus in case of tricyclic graphs we have two maximal degree sequences which correspond to the graphs \( G^*(S_3^1) \) and \( G^*(S_3^2) \) in Figure 1 drawing for the sake of simplicity for \( n = 8 \) (the second one is the graph \( H_4 \) in [23].)

The minimal element of the set \( S_3^1 \) is given by

\[ x_*(S_3^1) = \left[ \left( \frac{2(n+2)}{n} \right)^n \right] . \]  

(8)

Since it has not integer components for \( n \geq 5 \), we apply the procedure given in Section 2. Taking \( k = 2(n+2) - \left\lfloor \frac{2(n+2)}{n} \right\rfloor \) \( n = 4 \), the minimal element with integer component is

\[ x'_*(S_3^1) = [3^4, 2^{n-4}] \]

The minimal element of the set \( S_3^2 \) can be computed via formula (4). In our case \( a_n = \frac{2(n+2)}{n} \), \( m_1 = 3 \) and \( h = 4 \). Since for \( n \geq 5 \), we are in the case \( \frac{2(n+2)}{n} < 3 \). Simple computations show that the minimal element of the set \( S_3^2 \) is the same as before, i.e. \( x'_*(S_3^2) = x_*(S_3^1) \).

Thus in case of tricyclic graphs we have one minimal degree sequence which corresponds to the graph \( G_*(S_3^1) \) in Figure 2 drawing for the sake of simplicity for \( n = 8 \).
• **Tetracyclic.** For \( n \geq 6 \) we should consider the two different sets:

\[
S_4^1 = \{ d \in \Sigma_{2(n+3)} : 1 \leq d_n \leq \cdots \leq d_7 \leq (n-1), \ 2 \leq d_6 \leq \cdots \leq d_1 \leq (n-1) \}
\]

\[
S_4^2 = \{ d \in \Sigma_{2(n+3)} : 1 \leq d_n \leq \cdots \leq d_6 \leq (n-1), \ 2 \leq d_5 \leq (n-1), \ 3 \leq d_4 \leq \cdots \leq d_1 \leq (n-1) \},
\]

while for \( n = 5 \) only the second one.

For \( n \geq 6 \), the maximal elements are respectively,

\[
x^*(S_4^1) = [(n-1), 5, 2^4, 1^{n-6}] \\
x^*(S_4^2) = [(n-1), 4, 3^2, 2, 1^{n-5}]
\]

Notice that the maximal element for \( S_4^2 \) has been computed via Theorem 2 providing \( k = 1 \).

The two degree sequences are not comparable in the majorization order. Thus also in case of tetracyclic graphs we have one minimal degree sequence which corresponds the graph \( G_{\ast}(S_4^1) \), while for \( n = 6 \), the minimal element with integer components is

\[
x_\ast(S_4^1) = \left[ \left( \frac{2(n+3)}{n} \right) \right].
\]  

(9)

For \( n = 6 \) we get \( x_\ast(S_4^1) = [3^6] \), while for \( n > 6 \) the procedure given in Section 2 must be applied.

Taking \( k = 2(n+3) - \left\lfloor \frac{2(n+3)}{n} \right\rfloor \) \( n = 6 \), the minimal element with integer components is

\[
x_\ast'(S_4^1) = [3^6, 2^{n-6}]
\]

The minimal element of the set \( S_4^2 \) for \( n \geq 6 \) is the same as before, i.e. \( x_\ast'(S_4^1) = x_\ast(S_4^2) \). This can be proved easily by direct computations. Indeed \( x_\ast'(S_4^1) \in S_4^2 \) and it is impossible to build an element of \( S_4^2 \) minorizing it.

Thus in case of tetracyclic graphs we have one minimal degree sequence which corresponds the graph \( G_{\ast}(S_4^1) \) in Figure 2 drawing for the sake of simplicity for \( n = 8 \).

Finally for \( n = 5 \) simple computations give as maximal element \([4^2, 3^2, 2]\) and as minimal \([4, 3^4]\).

• **Pentacyclic.** For \( n \geq 7 \) we should consider three different sets:

\[
S_5^1 = \{ d \in \Sigma_{2(n+4)} : 1 \leq d_n \leq \cdots \leq d_7 \leq (n-1), \ 2 \leq d_6 \leq \cdots \leq d_1 \leq (n-1) \}
\]

\[
S_5^2 = \{ d \in \Sigma_{2(n+4)} : 1 \leq d_n \leq \cdots \leq d_6 \leq (n-1), \ 2 \leq d_5 \leq (n-1), \ 3 \leq d_4 \leq \cdots \leq d_1 \leq (n-1) \}
\]

\[
S_5^3 = \{ d \in \Sigma_{2(n+4)} : 1 \leq d_n \leq \cdots \leq d_6 \leq (n-1), \ 3 \leq d_5 \leq d_4 \leq (n-1), \ 4 \leq d_3 \leq d_2 \leq d_1 \leq (n-1) \}
\]

while for \( n = 5 \) only the last one and for \( n = 6 \) the second and the third sets.

The maximal elements, for \( n \geq 7 \), are respectively,

\[
x^*(S_5^1) = [(n-1), 6, 2^5, 1^{n-7}]
\]

\[
x^*(S_5^2) = [(n-1), 5, 3^2, 2^2, 1^{n-6}]
\]
\[x^*(S_3^n) = [(n-1), 4^2, 3^2, 1^{n-5}]\]

Notice that the three degree sequences are not comparable in the majorization order. Thus in case of pentacyclic graphs we have three degree sequences which correspond to the graphs \(G^*(S_3^5), G^*(S_3^6)\) and \(G^*(S_3^7)\) in Figure 1 drawing for the sake of simplicity for \(n = 9\).

The minimal element of the set \(S_3^1\) is given by

\[x_*(S_3^1) = \left[\left(\frac{2(n+4)}{n}\right)^n\right].\] (10)

For \(n = 8\) it has integer components and thus \(x_*(S_3^1) = [3^8]\).

For \(n = 7\), by applying the procedure of Section 2, we find as minimal element the vector \([4, 3^6]\) while for \(n > 8\), taking \(k = 2(n+4) - \left[\frac{2(n+4)}{n}\right] = 8\), the minimal element with integer components is

\[x_*(S_3^1) = [3^8, 2^{n-8}].\]

Simple computations show that the minimal element of the set \(S_3^2\), for \(n \geq 7\) is the same as before. Indeed, the minimal element of the set \(S_3^2\) belongs to \(S_3^2\) and it is impossible to find an element of \(S_3^2\) minorizing it.

For the set \(S_3^3\) and \(n \geq 7\), we cannot apply the same argument as above since the minimal element of the set \(S_3^2\) does not belong to \(S_3^3\). By Theorem 3 we get \(k = 5, d = 0, \rho = 2\). Thus the minimal element is

\[x_*(S_3^3) = [4^3, 3^2, 2^{n-5}]\]

Notice that \([3^8, 2^{n-8}] \preceq [4^3, 3^2, 2^{n-5}]\).

Thus in case of pentacyclic graphs we have one minimal degree sequence which corresponds to the graph \(G_*(S_3^2)\) in Figure 2 drawing for the sake of simplicity for \(n = 9\).

Finally for \(n = 5\) there is only one pentacyclic graph associated to the degree sequence \([4^3, 3^2]\) while for \(n = 6\) we get as maximal elements of \(S_3^2\) and \(S_3^4\) the non comparable sequences \([5^2, 3^2, 2^5]\) and \([5, 4^2, 3^2, 1]\). The minimal elements are \([4^2, 3^4]\) and \([4^3, 3^2, 2]\) with \([4^2, 3^4] \preceq [4^3, 3^2, 2]\).

- **Hexacyclic.** For \(n \geq 8\) we should consider five different sets:

\[S_6^1 = \{d \in \Sigma_{2(n+5)} : 1 \leq d_n \leq \cdots \leq d_9 \leq (n-1), \ 2 \leq d_8 \leq \cdots \leq d_1 \leq (n-1)\}\]

\[S_6^2 = \{d \in \Sigma_{2(n+5)} : 1 \leq d_n \leq \cdots \leq d_8 \leq (n-1), \ 2 \leq d_7 \leq d_6 \leq d_5 \leq (n-1), \ 3 \leq d_4 \leq \cdots \leq d_1 \leq (n-1)\}\]

\[S_6^3 = \{d \in \Sigma_{2(n+5)} : 1 \leq d_n \leq \cdots \leq d_7 \leq (n-1), \ 2 \leq d_6 \leq (n-1), \ 3 \leq d_5 \leq d_4 \leq (n-1), \ 4 \leq d_3 \leq d_2 \leq d_1 \leq (n-1)\}\]

\[S_6^4 = \{d \in \Sigma_{2(n+5)} : 1 \leq d_n \leq \cdots \leq d_7 \leq (n-1), \ 3 \leq d_6 \leq \cdots \leq d_1 \leq (n-1)\}\]

\[S_6^5 = \{d \in \Sigma_{2(n+5)} : 1 \leq d_n \leq \cdots \leq d_6 \leq (n-1), \ 4 \leq d_5 \leq \cdots \leq d_1 \leq (n-1)\}\]

while for \(n = 5\) only the last one, for \(n = 6\) the last three sets and for \(n = 7\) the last four sets.

The maximal elements, for \(n \geq 8\), can be computed via Theorem 2 and are respectively,

\[x^*(S_6^1) = [(n-1), 7, 2^6, 1^{n-8}]\]
\[
\begin{align*}
\mathbf{x}^*(S_6^3) &= [(n-1), 6, 3^2, 2^3, 1^{n-7}] \\
\mathbf{x}^*(S_6^4) &= [(n-1), 5, 4, 3^2, 2, 1^{n-6}] \\
\mathbf{x}^*(S_6^5) &= [(n-1), 5, 3^4, 1^{n-6}] \\
\mathbf{x}^*(S_6^6) &= [(n-1), 4^2, 1^{n-5}] 
\end{align*}
\]

Notice that the first three degree sequences and the last one are not comparable in the majorization order, while \(\mathbf{x}^*(S_6^4) \preceq \mathbf{x}^*(S_6^5)\). Thus in case of hexacyclic graphs we have four degree sequences which correspond to the graphs \(G^*(S_6^1), G^*(S_6^2), G^*(S_6^3),\) and \(G^*(S_6^5)\) in Figure 1 drawing for the sake of simplicity for \(n = 11\).

The minimal element of the set \(S_6^1\) is given by

\[
\mathbf{x}_*(S_6^1) = \left(\frac{2(n+5)}{n}\right)^n. \tag{11}
\]

It has integer components for \(n = 10\) and the minimal element is \(\mathbf{x}_*(S_6^1) = [3^{10}]\).

For \(n = 8\) and \(n = 9\), by applying the procedure of Section 2, we find as minimal element the vectors \([4^2, 3^6]\) and \([4, 3^8]\), respectively.

Finally, for \(n > 10\), being \(k = 2(n+5) - \left\lfloor \frac{2(n+5)}{n} \right\rfloor n = 10\), the minimal element with integer components is

\[
\mathbf{x}'_*(S_6^1) = [3^{10}, 2^{n-10}].
\]

Simple computations show that the minimal elements of the sets \(S_6^2\) and \(S_6^3\), for \(n \geq 8\) are the same as before. Indeed, the minimal element of the set \(S_6^1\) belongs to both sets \(S_6^2\) and \(S_6^3\) and it is impossible to find an element of the sets minorizing it.

For the sets \(S_6^2\) and \(S_6^3\) and \(n \geq 8\), we cannot apply the same argument as above since the minimal element of the set \(S_6^1\) does not belong to \(S_6^2\) and \(S_6^3\).

For the first case, by Theorem 3 we get \(k = 5, d = 0, \rho = \left(\frac{2n-8}{n-8}\right)^{n-5}\). After computing the minimal vector with integer components we get

\[
\mathbf{x}_*(S_6^3) = [4^3, 3^4, 2^{n-7}]
\]

Notice that \([3^{10}, 2^{n-10}] \preceq [4^3, 3^4, 2^{n-7}]\).

For the second case, by Theorem 3 we get \(k = 6, d = 0, \rho = 2\). Thus the minimal element is

\[
\mathbf{x}_*(S_6^5) = [4^5, 2^{n-5}]
\]

Notice that \([3^{10}, 2^{n-10}] \preceq [4^5, 2^{n-5}]\).

Thus in case of hexacyclic graphs we have one minimal degree sequence which corresponds, for \(n = 10\), to the graph \(G_*(S_6^1)\) in Figure 2 drawing for the sake of simplicity for \(n = 11\).

Finally, for \(n = 5\) there is only one hexacyclic graph associated to the degree sequence \([4^5]\).

For \(n = 6\) we get as maximal elements of \(S_6^1\) and \(S_6^3\) the sequences \([5^2, 4, 3^2, 2]\) and \([5, 4^2, 1]\) which are non comparable with respect to the majorization order, while for \(S_6^3\) we get the sequence \([5^2, 3^2]\) majorized by \([5^2, 4, 3^2, 2]\).

The minimal element of \(S_6^3\) and \(S_6^1\) is \([4^4, 3^2]\) while the minimal element of \(S_6^5\) is \([4^6, 2]\) which is minorized by \([4^4, 3^2]\).

For \(n = 7\) we get as maximal elements of \(S_6^2\), \(S_6^4\) and \(S_6^5\) the sequences \([6^2, 3^2, 2^3]\), \([6, 5, 4, 3^2, 2, 1]\) and \([6, 4^4, 1, 1]\) which are non comparable with respect to the majorization order, while for \(S_6^4\) we get the sequence \([6, 5, 3^4]\) majorized by \([6, 5, 4, 3^2, 2, 1]\).

The minimal element of \(S_6^2\), \(S_6^3\) and \(S_6^5\) is \([4^3, 3^2]\) while the minimal element of \(S_6^5\) is \([4^5, 2^2]\) which is minorized by \([4^3, 3^2]\).
In the following table we summarize the maximal and minimal degree sequences of $c$-cyclic graphs for $0 \leq c \leq 6$ and $n \geq c + 2$.

| $c$ | Maximal degree sequences | Minimal degree sequences |
|-----|--------------------------|--------------------------|
| 0   | $[(n - 1), 1^{n-1}]$     | $[2^{n-2}, 1^2]$         |
| 1   | $[(n - 1), 2^2, 1^{n-3}]$ | $[2^n]$                  |
| 2   | $[(n - 1), 3, 2^2, 1^{n-4}]$ | $[3^2, 2^{n-2}]$         |
| 3   | $[(n - 1), 4, 3^2, 1^{n-5}], [(n - 1), 3^3, 1^{n-4}]$ | $[3^4, 2^{n-4}]$         |
| 4   | $[(n - 1), 5, 2^4, 1^{n-6}], [(n - 1), 4, 3^2, 2, 1^{n-5}]$ | $[3^6, 2^{n-6}]$         |
| 5   | $[(n - 1), 6, 2^5, 1^{n-7}], [(n - 1), 5, 3^2, 2^2, 1^{n-6}], [(n - 1), 2^6, 3^2, 1^{n-5}]$ | $[4, 3^6] (n = 7), [3^8, 2^{n-8}] (n \geq 8)$ |
| 6   | $[(n - 1), 7, 2^6, 1^{n-8}], [(n - 1), 6, 3^2, 2^3, 1^{n-7}], [(n - 1), 2^7, 3^2, 2, 1^{n-6}], [(n - 1), 5, 4, 3^2, 2, 1^{n-6}], [(n - 1), 4^4, 1^{n-5}]$ | $[4^2, 3^6] (n = 8), [4, 3^8] (n = 9), [3^{10}, 2^{n-10}] (n \geq 10)$ |

Table 1: Extremal degree sequences

**Remark 6.** We observe that the maximal and minimal degree sequences of $c$-cyclic graphs, for $0 \leq c \leq 6$ and $n \geq c + 2$, can be expressed in terms of the cyclomatic number $c$. In fact

- For every $c \geq 0$ there is the maximal element
  $$[(n - 1), (c + 1), 2^c, 1^{n-c-2}]$$

- For $c \geq 3$ there is a second maximal element non comparable with the previous one:
  $$[(n - 1), c, 3^2, 2^{c-3}, 1^{n-c-1}]$$

- For $c \geq 5$ there is a third maximal element non comparable with the previous ones:
  $$[(n - 1), c - 1, 4, 3^2, 2^{c-5}, 1^{n-c}]$$

- For every $c$ such that $2c - 2 \leq n$, the minimal element is
  $$[3^{2c-2}, 2^{n-2c+2}]$$

One may conjecture that these properties hold also for $c > 6$.

## 5 Upper and lower bounds of degree-based topological indices.

In this Section we discuss upper and lower bounds for degree-base topological indices for $c$-cyclic graphs, $0 \leq c \leq 6$, via majorization techniques. To this aim, we will apply the characterizations of $c$-cyclic graphs provided in Theorem 5 and the extremal elements computed in Section 4.

We recall that a symmetric function $\phi : A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$, is said to be Schur-convex on $A$ if $x \preceq y$ implies $\phi(x) \leq \phi(y)$. Given an interval $I \subseteq \mathbb{R}$, and a convex function $g : I \to \mathbb{R}$, the function $\phi(x) = \sum_{i=1}^n g(x_i)$ is Schur-convex on $I^n = \prod_{i=1}^n I$ times.

If $\phi$ is a Schur-convex function and $S$ is a subset of $\Sigma_a$ which admits maximal and minimal elements with respect to the majorization order, the solutions of the constrained optimization problems

$$\begin{cases}
\max \ (\min) \ \phi(x) \\
\text{subject to } x \in S
\end{cases}$$

(P)
are the maximal element $x^*(S)$ and the minimal element $x_*(S)$, respectively. On the other hand, when $\phi$ is a Schur-concave function, the solutions to problem $(P)$ are the minimal element $x_*(S)$ and the maximal element $x^*(S)$, respectively.

Let now $F(d_1, d_2, \ldots, d_n)$ be any topological index which is a Schur-convex function of its arguments, defined on a subset $S \subseteq \Sigma_a$. Then by the order preserving property of the Schur-convex functions, we get

$$F(x_*(S)) \leq F(d_1, d_2, \ldots, d_n) \leq F(x^*(S))$$

Analogously, if $F$ is a Schur-concave function, then

$$F(x^*(S)) \leq F(d_1, d_2, \ldots, d_n) \leq F(x_*(S)).$$

A class of topological indices, of particular interest found in the literature and depending on the degree sequence of a graph over all vertices, are the first general Zagreb indices defined as

$$M_1^\alpha = \sum_{i=1}^{n} d_i^{\alpha}$$

where $\alpha$ is an arbitrary real number except 0 and 1 (see [16]). For $\alpha = 2$ we get the first Zagreb index while for $\alpha = -1$ the inverse degree. Notice that $M_1^\alpha$ is a Schur convex function of the degree sequence either for $\alpha < 0$ or $\alpha > 1$ while it is a Schur concave function of the degree sequence for $0 < \alpha < 1$. Also the first multiplicative Zagreb index

$$\ln M_1 = 2 \sum_{i=1}^{n} \ln(d_i)$$

introduced by Gutman in [14], is a Schur concave function of the degree sequence.

For $c$-cyclic graphs, $0 \leq c \leq 6$, the upper and lower bounds of indices (12), (13), and in general of any index which is a Schur convex/concave function of the degree sequence, can be found throughout the solution of $(P)$ where $S$ in any of the subsets $S_j$ described in Section 4. From these arguments the upper and lower bounds in Table 2 follow. For convenience we restrict our analysis to the first general Zagreb index with either $\alpha < 0$ or $\alpha > 1$ and $n \geq c + 2$. In the case $0 < \alpha < 1$ the upper and lower bounds are turned over. Bounds for the first multiplicative Zagreb index can be obtained analogously.

Note that in the case of bicyclic graphs, for the bounds of $M_1^\alpha$ we recover the same results as in [24], Theorems 1, 5, 7 and 8.

Notice that when more maximal elements are identified, the best choice depends on $\alpha$. We discuss in detail the case $\alpha = -1$, which gives the inverse degree $\rho = M_1^{-1} = \sum \frac{1}{d_i}$ (see [18], [7] and [8]).

Let $G$ be a $c$–cyclic graph ($0 \leq c \leq 6$).

- Tree ($c = 0$):

$$\frac{n + 2}{2} \leq \rho \leq (n - 1) + \frac{1}{n - 1},$$

where the lower and the upper bounds are attained if and only if $G = P_n$ and $G = S_n$ respectively, being $P_n$ and $S_n$ the path and the star with $n$ vertices.

- Unicyclic ($c = 1$):

$$\frac{n}{2} \leq \rho \leq (n - 2) + \frac{1}{n - 1},$$

where the lower and the upper bounds are attained if and only if $G = C_n$ and $G = S_n^+$ respectively, being $C_n$ the cycle and $S_n^+$ the graph obtained from $S_n$ joining two pendant vertices with an edge.

It is worth to point out that for trees and unicyclic graphs we recover the same results as in [18], Theorems 3 and 4, respectively.
Table 2: Bounds for $M_1^\alpha$ ($\alpha < 0 \lor \alpha > 1$)

| c   | Lower bounds                        | Upper bounds                        |
|-----|-------------------------------------|-------------------------------------|
| 1   | $(n-1)^\alpha + 2^{\alpha+1} + (n-3)$ | $n(2^{\alpha})$                    |
| 2   | $(n-1)^\alpha + 3^\alpha + 2^{\alpha+1} + (n-4)$ | $2(3^{\alpha}) + (2^{\alpha})(n-2)$ |
| 3   | $4(3^{\alpha}) + (2^{\alpha})(n-4)$ | $(n-1)^\alpha + 4^\alpha + 3(2^{\alpha}) + (n-5)$ |
|     |                                     | $(n-1)^\alpha + 3^{\alpha+1} + (n-4)$ |
| 4   | $2(3^{\alpha+1}) + (2^{\alpha})(n-6)$ | $(n-1)^\alpha + 5^\alpha + 2^{\alpha+2} + (n-6)$ |
|     |                                     | $(n-1)^\alpha + 2^\alpha(2^{\alpha} + 1) + 2(3^{\alpha}) + (n-5)$ |
| 5   | $8(3^{\alpha}) + (n-8)2^\alpha$     | $(n-1)^\alpha + 6^\alpha + 5(2^{\alpha}) + (n-7)$ |
|     |                                     | $(n-1)^\alpha + 5^\alpha + 2(3^{\alpha}) + 2(2^{\alpha}) + (n-6)$ |
|     |                                     | $(n-1)^\alpha + 2^{2\alpha+1} + 2(3^{\alpha}) + (n-8)$ |
| 6   | $10(3^{\alpha}) + (n-10)2^\alpha$   | $(n-1)^\alpha + 7 + 6(2^{\alpha})(n-8)$ |
|     |                                     | $(n-1)^\alpha + 6 + 2(3^{\alpha}) + 3(2^{\alpha}) + (n-7)$ |
|     |                                     | $(n-1)^\alpha + 5 + 4 + 2(3^{\alpha}) + 2 + (n-6)$ |
|     |                                     | $(n-1)^\alpha + 2^{2\alpha+2} + (n-5)$ |

• Bicyclic ($c = 2$):

$$\frac{n-2}{2} + \frac{2}{3} \leq \rho \leq (n-3) + \frac{1}{n-1} + \frac{1}{3},$$

where the upper bound is attained by the graphs $H^1_n$ in [24], the lower by the graph in the class $G^1_n$ in [24].

For $c \geq 3$ we take into account the upper bounds corresponding to the largest values that can be attained. We consider the case $n \geq c + 2$.

• Tricyclic ($c = 3$):

$$\frac{n-4}{2} + \frac{4}{3} \leq \rho \leq (n-3) + \frac{1}{n-1}.$$

The graphs $G_s(S^1_3)$ in Figure 2 and $G^*(S^2_3)$ in Figure 1 achieve, for $n = 8$, the lower and the upper bounds, respectively.

• Tetracyclic ($c = 4$):

$$\frac{n-2}{2} \leq \rho \leq (n-5) + \frac{1}{n-1} + \frac{17}{12}.$$

The graphs $G_s(S^1_4)$ in Figure 2 and $G^*(S^2_4)$ in Figure 1 achieve, for $n = 8$, the lower and the upper bounds, respectively.

• Pentacyclic ($c = 5$):

$$\frac{n-8}{2} + \frac{8}{3} \leq \rho \leq (n-5) + \frac{1}{n-1} + \frac{7}{6}.$$

The graphs $G_s(S^1_5)$ in Figure 2 and $G^*(S^2_5)$ in Figure 1 achieve, for $n = 9$, the lower and the upper bounds, respectively.
• Hexacyclic \((c = 6)\):

\[
\frac{n - 10}{2} + \frac{10}{3} \leq \rho \leq (n - 4) + \frac{1}{n - 1}.
\]

The graphs \(G_s(S_6^1)\) in Figure 2 and \(G^*(S_6^5)\) in Figure 1 achieve, for \(n = 11\), the lower and the upper bounds, respectively.

We observe that for \(c \geq 3\) if the additional information \(d_{c+2} \geq 2\) is known, the upper bound can be improved yielding:

\[
\rho \leq (n - c) + \frac{1}{n - 1} + \frac{c^2 - 3c - 2}{2(c + 1)}
\]

which is attained by the graphs \(G^*(S_c^1)\).

6 Conclusion

In this paper we focus on lower and upper bounds of some relevant graph topological indices, based on the degree sequence over all vertices of the graph. We get our results through new characterizations of \(c\)-cyclic graphs aimed to identify extremal vectors with respect to the majorization order of particular subset of \(\mathbb{R}^n\). We have shown that classical bounds can be recovered and new ones can be obtained. Our results suggest that as well as the generalization of Theorem 5 to \(c > 6\) can be found, our theoretical approach can be extended also to a larger class of \(c\)-cyclic graphs. Finally, other topological indices of \(c\)-cyclic graphs can be provided whenever they can be expressed as Schur-convex or Schur-concave functions of the degree sequence of the graph.
Figure 1: Maximal degree sequence graphs

(a) $G^*(S_1^1) = [7, 4, 2^3, 1^3]$  
(b) $G^*(S_2^2) = [7, 3^3, 1^4]$  
(c) $G^*(S_1^3) = [7, 5, 2^4, 1^2]$  

(d) $G^*(S_3^3) = [7, 4, 3^2, 2, 1^3]$  
(e) $G^*(S_2^4) = [8, 6, 2^5, 1^2,]$  
(f) $G^*(S_3^5) = [8, 5, 3^2, 2^2, 1^3]$  

(g) $G^*(S_4^5) = [8, 4^2, 3^2, 1^4]$  
(h) $G^*(S_5^6) = [10, 7, 2^6, 1^3]$  
(i) $G^*(S_2^7) = [10, 6, 3^2, 2^3, 1^4]$  

(j) $G^*(S_3^8) = [10, 5, 4, 3^2, 2, 1^5]$  
(k) $G^*(S_4^9) = [10, 4^4, 1^6]$
Figure 2: Minimal degree sequence graphs

(a) $G_*(\mathcal{S}_1^1) = [3^4, 2^4]
(b) G_*(\mathcal{S}_2^1) = [3^6, 2^2]
(c) G_*(\mathcal{S}_3^1) = [3^8, 2]
(d) G_*(\mathcal{S}_4^1) = [3^{10}, 2]$
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