Discrete Painlevé equations and random matrix averages

P.J. Forrester and N.S. Witte

Department of Mathematics and Statistics (and School of Physics), University of Melbourne, Victoria 3010, Australia; email: p.forrester@ms.unimelb.edu.au; n.witte@ms.unimelb.edu.au

The τ-function theory of Painlevé systems is used to derive recurrences in the rank \( n \) of certain random matrix averages over \( U(n) \). These recurrences involve auxiliary quantities which satisfy discrete Painlevé equations. The random matrix averages include cases which can be interpreted as eigenvalue distributions at the hard edge and in the bulk of matrix ensembles with unitary symmetry. The recurrences are illustrated by computing the value of a sequence of these distributions as \( n \) varies, and demonstrating convergence to the value of the appropriate limiting distribution.

1 Introduction

1.1 Motivations and objectives

In a recent series of papers \[15\]–\[18\] we have shown how the Okamoto τ-function theory of Painlevé systems can be applied to rederive known evaluations of certain random matrix averages in terms of Painlevé transcendent. Moreover it was shown how this theory could similarly be used to evaluate random matrix averages not known from previous studies, and to also yield recurrences of the discrete Painlevé type for the shift by unity of a parameter or parameters in the same random matrix averages. Subsequent to our works \[15, 16\] two different major theories — one on the discrete Riemann-Hilbert problem due to Borodin \[8, 7\], and the other based on the integrable Toeplitz lattice due to Adler and van Moerbeke \[2\] — were applied in \[5, 6\] and \[1\] respectively to also provide recurrences for random matrix averages with respect to a shift by unity of a parameter. The averages considered were with respect to the unitary group \( U(n) \), and the shift performed in the rank \( n \) of the matrices. As with our own work, the average itself is related to an auxiliary quantity or quantities, and it is the latter which satisfy the primary coupled recurrences.

It is our objective in this work to further develop the Okamoto τ-function theory as it relates to specifying recurrences for random matrix averages. Whereas in our earlier works recurrences were obtained mostly with respect to an otherwise continuous parameter within the average, in the present work, as with the works by Borodin, and Adler and van Moerbeke, our attention will be focussed on obtaining recurrences with respect to the rank of the random matrix and thus the dimension of the average itself (the averages under consideration couple only to the eigenvalues of the matrix).

Typical of the results of this paper is the recurrence obtained in our work \[15\] for the particular PIV τ-function

\[
\tau^{IV}[n](t; \mu) = \frac{1}{C} \int_{-\infty}^{t} dx_1 \cdots \int_{-\infty}^{t} dx_n \left\{ \prod_{j=1}^{n} e^{-x_j^2/2} (t-x_j)^\mu \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \right\}
\]

\[
= \left\langle \prod_{j=1}^{n} \chi^{(j)}_{(-\infty,t)}(t-x_j)^\mu \right\rangle_{\text{GUE}_n}.
\]
Here GUE\textsubscript{n} refers to the probability density function
\[
\frac{1}{C} \prod_{j=1}^{n} e^{-x_j^2} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2, \tag{1.2}
\]
with \(C\) denoting the normalization, realized by the eigenvalues of Hermitian matrices with certain complex Gaussian entries (see e.g. [11]) and \(\chi_j^{(j)} = 1\) for \(x_j \in J\), \(\chi_j^{(j)} = 0\) otherwise. From equations (2.86),(2.4),(2.75) we have that for an appropriate \(C\) in the literature as the discrete Painlevé I equation. Specification of these coupled recurrences were shown to be equivalent to a single second order difference equation known below, the more general PIV\(\tau\)-function
\[
\chi \text{ complex Gaussian entries (see e.g. [11]) and }\]
\[
\mu, \nu \tau \text{ below) uniquely determines } \{f_0[n]\}_{n=1,2,...}, \{f_2[n]\}_{n=1,2,...} \text{ and } \{\tau IV[n]\}_{n=2,3,...}. \text{ As noted in Section 2 below, the more general PIV } \tau \text{-function}
\]
\[
\tau IV[n](t; \mu; \xi) = \left( \prod_{j=1}^{n} (1 - \xi \chi(1,\infty)) (t - x_j)^\mu \right)_{\text{GUE}_n} \tag{1.6}
\]
also satisfies the system \(1.3 - 1.5\).

Recurrences with respect to the dimension of the random matrix will also be given for three averages over the unitary group \(U(N)\), known from our earlier work to be \(\tau\)-functions for certain Painlevé systems. With \(z_l := e^{i\theta_l}\) these are
\[
\tau IV'[N](t; \mu) := \left( \prod_{l=1}^{N} z_l^\mu e^{\frac{1}{2} \sqrt{N} (z_l - z_l^{-1})} \right)_{U(N)} \tag{1.7}
\]
\[
\tau V'[N](t; \mu, \nu) := \left( \prod_{l=1}^{N} (1 + z_l)^\mu (1 + 1/z_l)^\nu e^{t z_l} \right)_{U(N)} \tag{1.8}
\]
\[
\tau V'[N](t; \mu, w_1, w_2; \xi) := \left( \prod_{l=1}^{N} (1 - \xi \chi(l,\infty)) e^{w_2 \theta_l} |1 + z_l|^{2w_1} \left( \frac{1}{t z_l} \right)^\mu (1 + t z_l)^{2\mu} \right)_{U(N)}, \tag{1.9}
\]
where \(U(N)\) refers to the probability density function
\[
\frac{1}{(2\pi)^N N!} \prod_{1 \leq j < k \leq N} |z_k - z_j|^2, \quad (-\pi \leq \theta_j \leq \pi, \; j = 1, \ldots, N). \tag{1.10}
\]
In the case of (1.7) we only take the \(U(N)\) average as the definition for \(\mu \in \mathbb{Z}\); for general \(\mu\) the \(\tau\)-function \(\tau IV'[N](t; \mu)\) is to be defined as the Toeplitz determinant given in (2.11) below. Also, as written \(1.8\) is only well defined for \(\mu, \nu \in \mathbb{Z}_{\geq 0}\). However with \(z = e^{i\theta}\), use of the identity
\[
(1 + z)^\mu (1 + 1/z)^\nu = z^{(\mu-\nu)/2} |1 + z|^{\mu+\nu} \tag{1.11}
\]

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gives
\[ \tau^V[N](t; \mu, \nu) := \left\langle \prod_{l=1}^{n} e^{(\mu-\nu)/2|1 + z_i|^{\mu+\nu} e^{\ell z_i}} \right\rangle_{U(N)}, \] (1.12)
which is well defined for \( \text{Re}(\mu + \nu) > -1 \).

We also indicate how the PV \( \tau \)-function \[16\]
\[ \tilde{\tau}^V[n](t; \mu, a; \xi) := \left\langle \prod_{j=1}^{n} (1 - \xi \chi_{(0,1)}(x_j - t)^\mu) \right\rangle_{\text{LUE}_n} \] (1.13)
and the PVI \( \tau \)-function \[17\]
\[ \tilde{\tau}^{VI}[n](t; \mu, a, b; \xi) := \left\langle \prod_{j=1}^{n} (1 - \xi \chi_{(t,1)}(t - x_j)^\mu) \right\rangle_{\text{JUE}_n} \] (1.14)
can be characterized by recurrences. Here \( \text{LUE}_n \) refers to the probability density function
\[ \frac{1}{I_N(a)} \prod_{j=1}^{n} \chi_{(0,\infty)}(x_j) x_j^a e^{-x_j} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \] (1.15)
while \( \text{JUE}_n \) refers to the probability density function
\[ \frac{1}{J_N(a, b)} \prod_{j=1}^{n} \chi_{(0,1)}(x_j) (1 - x_j)^b \prod_{1 \leq j < k \leq n} (x_k - x_j)^2. \] (1.16)

The normalizations in \ref{1.15} and \ref{1.16} are
\[ I_N(a) := \int_{0}^{\infty} dx_1 \cdots \int_{0}^{\infty} dx_N \prod_{i=1}^{N} x_i^a e^{-x_i} \prod_{1 \leq j < k \leq N} (x_k - x_j)^2 = \prod_{j=0}^{N-1} \Gamma(2 + j) \Gamma(a + 1 + j) \] (1.17)
and
\[ J_N(a, b) := \int_{0}^{1} dx_1 x_1^a (1 - x_1)^b \cdots \int_{0}^{1} dx_N x_N^a (1 - x_N)^b \prod_{1 \leq j < k \leq N} (x_k - x_j)^2 \]
\[ = \prod_{j=0}^{N-1} \frac{\Gamma(a + 1 + j) \Gamma(b + 1 + j) \Gamma(2 + j)}{\Gamma(a + b + 1 + N + j)}. \] (1.18)

We remark that both \ref{1.15} and \ref{1.16} can be realized as the eigenvalue probability density function for certain ensembles of random matrices (see e.g. \[11\]). We have not been able to derive recurrences for \ref{1.15} and \ref{1.16} in \( n \) only; rather the recurrences to be indicated also act on the parameter \( a \).

### 1.2 Strategy

The Okamoto theory is based on a Hamiltonian formulation of the Painlevé equations, which in turn can be traced back to Malmquist \[22\]. Corresponding to each of the Painlevé equations PII–PVI is a Hamiltonian \( H \), which is itself a function of the conjugate variables \( p \) and \( q \), the independent variable \( t \), and a number of parameters. The conjugate variables \( p \) and \( q \) are also dependent on the independent variable \( t \) and the parameters. By eliminating \( p \) in the Hamilton equations
\[ q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q} \] (1.19)
the Painlevé equation in $q$ results, although we have no explicit use for this defining feature of $H$ below. A particular parameter $n$ is distinguished and we write $H = H_n$, $p = p_n$, $q = q_n$. Our primary concern is in so called Schlesinger transformations, which relate the Hamiltonian system with parameter $n + 1$ to the Hamiltonian system with parameter $n$.

One introduces a $\tau$-function $\tau_n$, a function of the independent variable $t$ and the parameters, by the requirement that

$$H_n = \frac{d}{dt} \log \tau_n. \quad (1.20)$$

¿From the Okamoto theory it is known that

$$\frac{\tau_{n-1} \tau_{n+1}}{(\tau_n)^2} = f(p_n, q_n, t) \quad (1.21)$$

for some explicit polynomial function $f$, typically related to the time derivative of $H_n$. Furthermore, the Schlesinger transformation theory gives that $\{p_n, q_n\}$ satisfy coupled first order recurrences

$$p_{n+1} = g_1(p_n, q_n), \quad q_{n+1} = g_2(p_n, q_n) \quad (1.22)$$

for some explicit rational functions $g_1, g_2$. Thus once $p_0, q_0$ have been specified $\{p_n, q_n\}_{n=1,2,...}$ can be generated from (1.22). With this information, and knowledge of $\tau_0, \tau_1$, (1.21) can be iterated to specify $\{\tau_2, \tau_3, \ldots\}$.

1.3 Paper outline

We will devote separate sections to each of the $\tau$-functions (1.6)–(1.9), with (1.13) and (1.14) considered during the discussion of (1.8) and (1.9) respectively. In the cases of (1.6), (1.7) and (1.8) the Schlesinger transformations which increment the dimension of respective random matrix averages are known from our earlier works [15, 16]. The formulation of the recurrences is then a straightforward application of the strategy outlined above. However in the case of (1.9) there is some complication as one must first change variables to obtain a random matrix average for which the standard Schlesinger transformation increments the dimension of the random matrix average. In the final section some uses of our recurrences for the computation of the random matrix averages as they occur in applied problems will be discussed.

2 The $\tau$-function sequence $\{\tau^{IV}[n](t; \mu, \xi)\}_{n=0,1,...}$

The Hamiltonian for the PIV system is given by [25]

$$H^{IV} = (2p - q - 2t)pq - 2\alpha_1 p - \alpha_2 q. \quad (2.1)$$

Let

$$(\alpha_1, \alpha_2) = (-\mu, -n) \quad (2.2)$$

and write $H^{IV} = H^{IV}_n$ thus distinguishing the parameter $\alpha_2 = -n$. It was shown in [15 Prop. 22] that corresponding to the sequence of Hamiltonians $\{H^{IV}_n\}_{n=0,1,...}$ is the sequence of $\tau$-functions $\{\tau^{IV}[n](t; \mu)\}_{n=0,1,...}$ as specified by (1.1). Moreover, combining the result of [15 Prop. 6] with the workings leading to [15 Prop. 7 and Prop. 22] it follows that more generally $\tau^{IV}[n](t; \mu, \xi)$ is a $\tau$-function for $H^{IV}_n$.

The significance of this latter fact is that with

$$f_0[n] := 2t + q_n - 2p_n, \quad f_2[n] := 2p_n \quad (2.3)$$

...
we know from [15, eq. (2.75)] that the recurrences \((1.3)–(1.5)\) hold, and these recurrences fully determine \(\{\tau_{IV}^n(t; \mu, \xi)\}_{n=2,3,\ldots}\) once we specify \(f_0[0], f_1[0]\) in \((1.4), (1.5)\), and \(C_n, \tau_{IV}[0], \tau_{IV}[1]\) in \([15]\). To determine \(C_n\) we require the fact \([15, \text{eqs. (2.41), (2.42)}]\) that with
\[
C_n = \frac{\gamma_{n+1} \gamma_{n-1}}{\gamma_n^2}, \quad \gamma_n e^{i^2 n \tau_{IV}^n} \rightarrow \sigma_{IV}^n[n], \quad \tau_{IV}[0] = \sigma_{IV}[0] = 1
\] (2.4)
the function \(\sigma_{IV}^n[n]\) has the explicit double Wronskian form
\[
\sigma_{IV}^n[n] = \det \left[ \frac{d^{i+j}}{dt^{i+j}} \sigma_{IV}^1[1] \right]_{j,k=0,\ldots,n-1}.
\] (2.5)

Now it follows from \([15, \text{eq. (2.41), Prop. 6}\]) that up to a proportionality constant, which we are free to choose to be unity,
\[
\sigma_{IV}^1 = e^{i^2} \left( \int_{-\infty}^{\infty} -\xi \int_{-\infty}^{0} \right) (t-x)^\mu e^{-x^2} dx.
\] (2.6)
Noting that \(2.6\) can be written
\[
\sigma_{IV}^1 = \left( \int_{-\infty}^{\infty} \xi \int_{0}^{\infty} \right) (-x)^\mu e^{-x^2 - 2tx} dx
\]
the differentiation required by \(2.5\) becomes simple to perform and we obtain
\[
\frac{d^{i+j}}{dt^{i+j}} \sigma_{IV}^1[1] = 2^{i+j} e^{i^2} \left( \int_{-\infty}^{\infty} -\xi \int_{0}^{\infty} \right) (t-x)^{\mu+i+j} e^{-x^2} dx.
\] (2.7)
Substituting this in \(2.5\) and recalling the workings of \([15, \text{proof of Prop. 21}\]) we see that
\[
\sigma_{IV}^n[n] = \frac{2^{n(n-1)}}{n!} e^{i^2 n} \left( \int_{-\infty}^{\infty} -\xi \int_{-\infty}^{\infty} \right) dx_1 \cdots \left( \int_{-\infty}^{\infty} -\xi \int_{0}^{\infty} \right) dx_n \prod_{j=1}^{n} e^{-x_j^2 (t-x_j)^\mu}
\times \prod_{1 \leq j < k \leq n} (x_k - x_j)^2.
\] (2.8)
It is well known (see e.g. [11]) that the normalization \(C\) in the definition \([12]\) of the \(GUE_n\) probability density function has the explicit form
\[
C = n! 2^{-(n-1)n/2} \pi^{n/2} \prod_{l=0}^{n-1} l!
\]
so \(2.8\) can be written
\[
\sigma_{IV}^n[n] = 2^{n(n-1)/2} \pi^{n/2} \prod_{l=0}^{n-1} l! e^{i^2 n} \left[ \prod_{l=1}^{n} (1 - \xi \chi^l(t, \infty)) (t-x)^\mu \right]_{GUE_n}
\]
\[
= 2^{n(n-1)/2} \pi^{n/2} \prod_{l=0}^{n-1} l! e^{i^2 n \tau_{IV}^n[n]}(t; \mu; \xi).
\] (2.9)
Recalling \(2.3\) we thus have
\[
\gamma_n = 2^{n(n-1)/2} \pi^{n/2} \prod_{l=0}^{n-1} l!
\]
and this in turn implies
\[
C_n = 2n.
\] (2.11)
Regarding the initial conditions for \( \text{(1.4)} \) and \( \text{(1.5)} \), we require the facts [15, proof of Prop. 6] that

\[
p_0 = 0, \quad q_0 = \frac{d}{dt} \log \tau^{IV}[1].
\]

Thus recalling (2.3) and (1.6) we have

\[
f_0[0] = 2t + \frac{d}{dt} \log \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} (t - \xi)^{n} e^{-x^2} \, dx \right) \right), \quad f_2[0] = 0.
\] (2.12)

The initial conditions for (1.3) are by definition

\[
\tau^{IV}[0] = 1, \quad \tau^{IV}[1] = \frac{1}{\sqrt{\pi}} \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} (t - \xi)^{n} e^{-x^2} \, dx \right) \right).
\] (2.13)

In summary, we thus have that the following result holds.

**Proposition 1.** Let \( \tau^{IV}[n] = \tau^{IV}[n](t; \mu; \xi) \) as specified by (1.4). Let \( p_n, q_n \) denote the conjugate variables in the Hamiltonian (2.1) with parameters given by (2.2), and define \( f_0[n] \) and \( f_2[n] \) in terms of these variables by (2.3). We have that \( \{f_0[n]\}_{n=1,2,...} \), \( \{f_2[n]\}_{n=1,2,...} \) and \( \{\tau^{IV}[n]\}_{n=2,3,...} \) are determined by the recurrences (2.12), (2.13) subject to the initial conditions (2.14), (2.15).

We remark that in the special case \( \xi = 0, \mu \in \mathbb{Z}_{\geq 0}, \text{(1.6)} \) is a polynomial in \( t \), which in view of (2.4), (2.5), (2.6) and the integral representation

\[
\int_{-\infty}^{\infty} (t - ix)^{n} e^{-x^2} \, dx = \sqrt{\pi} 2^{-n} H_n(t)
\]

has the explicit form

\[
\tau^{IV}[n] = \frac{1}{\gamma_n} \det \left( (2i)^{-\mu} i^{-(j+k)} H_{\mu+j+k}(it) \right)_{j,k=0,...,n-1}.
\] (2.14)

In this case (2.12) and (2.13) can be written

\[
f_0[0] = 2t + \frac{2(i)^{\mu} H_{\mu-1}(it)}{H_{\mu}(it)}, \quad f_2[0] = 0
\]

\[
\tau^{IV}[0] = 1, \quad \tau^{IV}[1] = \frac{(2i)^{-\mu}}{\gamma_1} H_{\mu}(it).
\] (2.15)

It is of interest to recall the duality formula [15, eq. (4.37)]

\[
\tau^{IV}[n](t; \mu, 0) = i^{-n} \tau^{IV}[\mu](it; n, 0) = \frac{1}{\gamma_n} \det \left[ 2^{-n} i^{-(j+k)} H_{n+j+k}(t) \right]_{j,k=0,...,\mu-1}
\] (2.16)

thus giving (2.14) for \( n = 0, 1, \ldots \) as a sequence of \( \mu \times \mu \) determinants.

Another point of interest is that with the initial conditions (2.14) a closed form solution of the coupled recurrences (2.4) and (1.5) can be given. Thus it follows from (2.4) eq. (4.8) that

\[
f_0[n] = 2 \frac{\tau^{IV}[n](t; \mu, \xi) \tau^{IV}[n+1](t; \mu + 1, \xi)}{\tau^{IV}[n](t; \mu + 1, \xi) \tau^{IV}[n+1](t; \mu, \xi)} \tau^{IV}[n+1](t; \mu + 1, \xi), \quad f_2[n] = n \frac{\tau^{IV}[n+1](t; \mu, \xi) \tau^{IV}[n-1](t; \mu + 1, \xi)}{\tau^{IV}[n](t; \mu, \xi) \tau^{IV}[n](t; \mu + 1, \xi)}
\] (2.17)

(the proportionality constants cannot be read off from [24]; these are determined by considering the \( t \to \infty \) behaviour of (1.3)–(1.5)).
3 The \( \tau \)-function sequence \( \{ \tau^{III'}[N](t; \mu) \}_{N=0,1,...} \)

Although (1.7) is well defined for all complex \( \mu \), only for \( \mu \in \mathbb{Z} \) will we take the \( U(N) \) average as the definition. For general \( \mu \) we will make use of a Toeplitz determinant form, obtained by applying the well known identity

\[
\left\langle \prod_{l=1}^{N} w(z_l) \right\rangle _{U(N)} = \det \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} w(z) z^{j-k} \, d\theta \right]_{j,k=1,...,N}.
\]  

(3.1)

This gives

\[
\left\langle \prod_{l=1}^{N} z_l^\mu e^{\sqrt{\tau} (z_l + z_l^{-1})} \right\rangle _{U(N)} = \det \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(\mu+j-k)\theta} e^{\sqrt{\tau} \cos \theta} \, d\theta \right]_{j,k=1,...,N} = \det [I_{\mu+j-k}(\sqrt{\tau})]_{j,k=1,...,N}.
\]  

(3.2)

The second equality of (3.2) follows from an integral formula for \( I_{\nu}(z) \), valid for \( \nu \in \mathbb{Z} \). We take this latter determinant as the meaning of \( \tau^{III'}[N] \) for general \( \mu \).

Now the Hamiltonian for the PIII' system is given by (2.8)

\[ tH^{III'} = q^2 p^2 - (q^2 + v_1 q - t)p + \frac{1}{2}(v_1 + v_2)q. \]  

(3.3)

We showed in (16, Section 4.3) that with

\[(v_1, v_2) = (\mu + N, -\mu + N) \]  

(3.4)

the quantity

\[ t^{-N\mu/2} \det [I_{\mu+j-k}(\sqrt{\tau})]_{j,k=1,...,N} \bigg|_{t \to 4t} \]  

(3.5)

is a \( \tau \)-function for the corresponding sequence of Hamiltonians \( \{ tH^{III'}_N \}_{N=0,1,...} \) (it is still valid to call (1.7) a \( \tau \)-function for a PIII' system as the extra factor \( t^{-N\mu/2} \) is equivalent to the addition of a constant to the Hamiltonian (3.3), which of course does not alter the Hamilton equations). From the working in (17) summarizing the Okamoto theory of PIII', we can deduce the following recurrences for (1.7).

**Proposition 2.** Let \( \tau^{III'}[N] = \tau^{III'}[N](t; \mu) \) as given by (1.7), and let \( p_N \), \( q_N \) denote the conjugate variables in the Hamiltonian (3.3) with parameters (3.4). The sequences \( \{ \tau^{III'}[N] \}_{N=0,1,...} \), \( \{ p_N \}_{N=0,1,...} \), \( \{ q_N \}_{N=0,1,...} \) satisfy the coupled recurrences

\[
\frac{\tau^{III'}[N+1]\tau^{III'}[N-1]}{(\tau^{III'}[N])^2} \bigg|_{t \to 4t} = p_N \quad (N = 1, 2, \ldots)
\]  

(3.6)

\[
p_{N+1} = \frac{q_N}{t}(p_N - 1) - \frac{\mu q_N}{t} + 1 \quad (N = 0, 1, \ldots)
\]  

(3.7)

\[
q_{N+1} = -\frac{t}{q_N} + \frac{(1 + N)t}{q_N(q_N(p_N - 1) - \mu) + t} \quad (N = 0, 1, \ldots)
\]  

(3.8)

subject to the initial conditions

\[
p_0 = 0, \quad q_0 = t \frac{d}{dt} \log t^{-\mu/2} I_{\mu}(2\sqrt{\tau})
\]  

(3.9)

\[
\tau^{III'}[0] = 1, \quad \tau^{III'}[1] = I_{\mu}(2\sqrt{\tau}).
\]  

(3.10)

Proof. The working of (16, proof of Prop. 4.2), which in turn is based on (21), together with (16) eqs. (4.13), (4.20),(4.21) tell us that

\[
\frac{\tau^{III'}[N+1]\tau^{III'}[N-1]}{(\tau^{III'}[N])^2} \bigg|_{t \to 4t} = \frac{\partial}{\partial t}(tH^{III'}). \]  

(3.11)
Proposition 3. Hilbert approach distinct from that of Borodin and van Moerbeke \cite{1} from their theory of the Toeplitz lattice, and by Baik \cite{3} who used a Riemann-Hilbert approach, and subsequently obtained by Adler by Borodin \cite{5} from a discrete Riemann-Hilbert approach, and subsequently obtained by Adler.

subject to the initial conditions

\[ \{v_0, v_1, v_2\} = (v_1^{(0)} + n, v_2^{(0)} + n) \]

\[ p_{n+1} = \frac{q_n^2}{t}(p_n - 1) - \frac{q_n}{2t}(v_1^{(0)} - v_2^{(0)}) + 1 \] (3.11)

\[ q_{n+1} = -\frac{t}{q_n} + \frac{1}{2}(v_1^{(0)} + v_2^{(0)} + 2 + 2n)t - \frac{1}{2}(v_1^{(0)} - v_2^{(0)}) + t. \] (3.12)

Setting \( v_1^{(0)} = -v_2^{(0)} = \mu \) gives \((3.7)\) and \((3.8)\). The initial conditions \((3.9)\) follow from \[(16, eqs. (4.40),(4.41))\].

Setting \( v_0 = 0 \) the sequence \( \{v_n\}_{n=0,1,...} \) satisfies the particular discrete Painlevé II equation

\[ \frac{1 + N}{q_N q_{N+1} + t} + \frac{N}{q_{N-1} q_N + t} = \frac{1}{q_N} - \frac{q_N}{t} + \frac{N - \mu}{t}, \quad N = 0, 1, ... \] (3.13)

In the special case \( \mu = 0 \) the sequence \( \{p_n\}_{n=0,1,...} \) itself is also determined by a particular discrete Painlevé II equation. To see this, note that \((3.7)\) with \( \mu = 0 \) gives

\[ q_N^2 = \frac{t}{1 - p_N}. \] (3.14)

Setting

\[ q_N = \sqrt{\frac{P_{N+1}}{P_N}} \]

we see that \((3.13)\) has the unique solution

\[ p_N = 1 - P_N^2. \] (3.16)

Making use of this in Proposition \ref{prop:2} we obtain the following recurrence scheme for \( \{\tau^{III}[N](t, 0)\} \), first derived by Borodin \cite{5} from a discrete Riemann-Hilbert approach, and subsequently obtained by Adler and van Moerbeke \cite{1} from their theory of the Toeplitz lattice, and by Baik \cite{3} who used a Riemann-Hilbert approach distinct from that of Borodin.

Proposition 3. We have

\[ 1 - P_n^2 = \frac{\tau^{III}[N + 1] \tau^{III}[N - 1]}{\tau^{III}[N]^2} \mid_{t \to \mu} \]

where \( \{P_n\}_{n=1,2,...} \) satisfies the particular discrete Painlevé II equation

\[ P_{n+1} + P_{n-1} = \frac{n P_n}{\sqrt{(1 - P_n^2)}}, \quad n = 1, 2, ... \]

subject to the initial conditions

\[ P_0 = 1, \quad P_1 = \frac{I_1(2\sqrt{t})}{I_0(2\sqrt{t})}. \]

Let us show how \( q_N \), like \( p_N \), can be written in terms of the \( \tau \)-functions. Put

\[ \langle t^{-N\mu/2} \tau^{III}[N](t; \mu) \rangle \mid_{t \to \mu} =: \tau_N^\mu \]

and denote the corresponding Hamiltonian by \( H_N^\mu \). Denote by \( T_1 \) (\( T_2 \)) the Schlesinger operators with the action on the parameters \((v_1, v_2) \mapsto (v_1 + 1, v_2 + 1) \) \(((v_1, v_2) \mapsto (v_1 + 1, v_2 - 1))\). It is known \cite{28} that

\[ T_1 tH_N^\mu = tH_{N+1}^\mu = tH_N^\mu + q_N(1 - p_N), \quad T_2 tH_N^\mu = tH_{N+1}^\mu = tH_N^\mu - q_Np_N \] (3.17)

where \( q_N := q_N^\mu, p_N := p_N^\mu \). Using \((3.17)\) together with \((3.11), (3.12)\) the sought formula can be deduced.
Proposition 4. We have
\[ q_N = (-1)^N \sqrt{\frac{\tau_{III}^* [N] (4t, \mu) \tau_{III}^* [N + 1] (4t, \mu + 1)}{\tau_{III}^*[N + 1] (4t, \mu) \tau_{III}^*[N] (4t, \mu + 1)}}. \] (3.18)

Proof. We have
\[ t \frac{d}{dt} \log \left( \frac{\tau_{n+1}^* \tau_n^{\mu+1}}{\tau_n^* \tau_{n+1}^{\mu}} \right) = -t(H_{n+1}^{\mu+1} - H_n^{\mu+1}) + t(H_n^{\mu+1} - H_n^{\mu}) = q_{n+1} p_{n+1} - q_n p_n. \]

According to (3.11), (3.12)
\[ q_{n+1} p_{n+1} = -q_n (p_n - 1) + v_1 + 1 - \frac{t}{q_n}. \]

Thus
\[ t \frac{d}{dt} \log \left( \frac{\tau_{n+1}^* \tau_n^{\mu+1}}{\tau_n^* \tau_{n+1}^{\mu}} \right) = -\frac{1}{q_n} \left( 2q_n^2 p_n - q_n^2 - (v_1 + 1)q_n + t \right) \]
\[ = -\frac{1}{q_n} \left( \frac{\partial H_{III}^*}{\partial p_n} - q_n \right) = -t \frac{d}{dt} \log(q_n/t) \]
where to obtain the final equality, use has been made of the first of the Hamilton equations. This implies (3.18) up to a proportionality constant. To determine the proportionality, \( c_N \) say, we use the asymptotic formula \([16, \text{proof of Cor. 4.5}]\)
\[ \det[I_{j-k+N}(\sqrt{t})]_{j,k=0,\ldots,n-1} \sim e^{\sqrt{t}-(n^2/4) \log t+O(1)} \]
which in light of (3.2) and (3.6) implies
\[ p_N \sim (4t)^{-1/2} \quad (N \neq 0) \] (3.19)
while (3.18) (with the proportionality still unknown) implies
\[ q_N \sim c_N \sqrt{t}. \] (3.20)

Substituting in (3.8) and taking into consideration (3.9) implies \( c_N = (-1)^N. \) \( \square \)

4 The \( \tau \)-function sequence \( \{\tau^V[N](t; \mu, \nu)\} \)

The definition (1.12) of \( \tau^V[N](t; \mu, \nu) \) is well defined for Re(\( \mu + \nu \)) > -1. This domain can be extended by using (3.1) to rewrite (1.12) as a Toeplitz determinant and evaluating the integral,
\[ \tau^V[N](t; \mu, \nu) = \det \left[ \frac{1}{2\pi \int_{-\pi}^{\pi} z^{-k} e^{(\mu-\nu)/2} (1 + z)^{\mu+\nu} e^{tz} d\theta} \right]_{j,k=1,\ldots,N} \]
\[ = \det \left[ \frac{\Gamma(\mu + \nu + 1)}{\Gamma(\mu + j - k + 1) \Gamma(\nu - j + k + 1)} F_1(-\nu + j - k, \mu + 1 + j - k; -t) \right]_{j,k=1,\ldots,N}. \] (4.1)

Here the integral evaluation, which is well defined for general complex \( \mu, \nu \), follows by expanding the exponential in the first Toeplitz determinant and evaluating the resulting integrals using the formula
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} z^{(a-b)/2} (1 + z)^{a+b} d\theta = \frac{\Gamma(a + b + 1)}{\Gamma(a + 1) \Gamma(b + 1)}. \] (4.2)
In [16] display eq. below proof of Prop. 3.6, \( \{\tau^V[N](t;\mu,\nu)\}_{N=0,1,...} \) has been identified as the \( \tau \)-function sequence corresponding to a particular Schlesinger operator for the PV system. However some technical details of the derivation given there leads to complication for the present purposes, which in fact can be avoided by revising some of the workings in [16]. Let us then undertake such a program.

We will construct a \( \tau \)-function sequence relating to the Hamiltonian \[ \{22,21\} \]

\[ tH^{V^*} = q(q - 1)(p + t) - (v_2 - v_1 + v_3 - v_4)qp + (v_2 - v_1)p + (v_1 - v_3)tq \]  (4.3)

for which eliminating \( p \) in the Hamilton equations shows \( 1 + 1/(q - 1) \) satisfies the PV equation. Since eliminating \( p \) does not give the Painlevé equation in \( q \) itself, we refer to this as the PV* system. (In [16] we made use of the mapping between the PV and PV* systems [32], which is in fact unnecessary and is what leads to complications for the present purposes.) Our interest is in the particular Schlesinger transformation with action on the parameters

\[ T_0^{-1}(v_1, v_2, v_3, v_4) = (v_1 - \frac{1}{4}, v_2 - \frac{1}{4}, v_3 - \frac{1}{4}, v_4 + \frac{3}{4}). \]  (4.4)

We know from [16] eq. (2.16) that

\[ T_0^{-1}H^{V^*} = H^{V^*}|_{v \rightarrow T_0^{-1}v}. \]  (4.5)

This motivates introducing the sequence of Hamiltonians

\[ H_n^{V^*} := H_0^{V^*}|_{v \rightarrow (v_1^{(0)} - n/4,v_2^{(0)} - n/4,v_3^{(0)} - n/4,v_4^{(0)} + 3n/4)} \]

and the corresponding sequence of \( \tau \)-functions \( \tau_n^{V^*} \) specified so that

\[ H_n^{V^*} = \frac{d}{dt} \log \tau_n^{V^*}. \]  (4.6)

Following [27], the seed initializing the sequence of \( \tau \)-functions is a classical solution to the PV* system.

**Proposition 5.** Let \( v_3^{(0)} - v_4^{(0)} = 0 \). Then the PV* system admits the solution

\[ q_0 = 1, \quad tH_0^{V^*} = (v_1^{(0)} - v_3^{(0)})t, \quad \tau_0 = e^{(v_1^{(0)} - v_3^{(0)})t}, \quad p_0 = t\frac{d}{dt} \log \tau_1^{V^*} + (v_3^{(0)} - v_1^{(0)})t \]  (4.7)

where \( e^{-(v_1^{(0)} - v_3^{(0)})t} \tau_1^{V^*} \) satisfies the confluent hypergeometric differential equation

\[ ty'' + (v_1^{(0)} - v_2^{(0)} + 1 + t)y' + (v_1^{(0)} - v_3^{(0)})y = 0. \]  (4.8)

Proof. Direct substitution of \( q_0 = 1, v_3^{(0)} - v_4^{(0)} = 0 \) into [16] gives the stated value of \( tH_0^{V^*} \). The final equation in [16] follows from [16, 13] and [16] which together give

\[ T_0^{-1}tH_0^{V^*} := tH_1^{V^*} = t\frac{d}{dt} \log \tau_1^{V^*} = tH_0^{V^*} + p_0 = (v_1^{(0)} - v_3^{(0)})t + p_0. \]

Now that the final equation in [16] is established, [16] can be derived from the second of the Hamilton equations [16]

\[ tp_0 = -\frac{\partial H_0^{V^*}}{\partial q} \bigg|_{v_3^{(0)} - v_4^{(0)} = 0} = -\left( p_0(t_0 + t) - (v_2^{(0)} - v_1^{(0)})p_0 + (v_1^{(0)} - v_3^{(0)})t\right), \]  (4.9)

by substituting the former equation for \( p_0 \) throughout. \( \square \)
According to [16] proof of Prop. 2.2, with
\[ \tilde{\tau}_n := n^{\mu/2} e^{(v_4^{(0)} - v_1^{(0)} + n) t} \tau_n \]
and \( \tau_n^{V^*} \) as in [4.7] so that \( \tilde{\tau}_0 = 1 \) (recall that in [4.7] we require \( v_3^{(0)} - v_4^{(0)} = 0 \)), the sequence \( \{\tilde{\tau}_n\}_{n=2,3,...} \) is specified by the determinant formula
\[ \tilde{\tau}_n = \det[\delta^{j+k}\tilde{\tau}_1]_{j,k=0,...,n-1}. \]

For an appropriate choice of the solution of (4.8) and thus of \( \tilde{\tau}_1 \), (4.11) can be related to the \( \tau \)-function sequence (4.8).

**Proposition 6.** Of the two linearly independent solutions to (4.8), choose the solution analytic at the origin,
\[ e^{-(\nu(0) - \nu(0))} \tau_n^{V^*} = l F_1(v_1^{(0)} - v_3^{(0)} - v_2^{(0)} + 1; -t) \]
with \( v_1^{(0)} - v_3^{(0)} = -\nu, \quad v_1^{(0)} - v_2^{(0)} = \mu. \)

Then (4.10) and (4.11) give
\[ e^{\nu t} \tau_n^{V^*} = \left( \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \nu + 1)} \right)^n \prod_{l=0}^{n-1} \Gamma(\nu + l + 1) \tau_n^{V^{[n]}(t; \mu, \nu)}. \]

**Proof.** Choosing \( \tau_1^{V^*} \) as in (4.12) and the parameters as in (4.13) gives, upon comparing with (4.11)
\[ e^{\nu t} \tau_1^{V^*} = \frac{\Gamma(\mu + 1)\Gamma(\nu + 1)}{\Gamma(\mu + \nu + 1)} \tau_1^{V^{[1]}(t; \mu, \nu)}. \]
Furthermore, use of (4.10) and (4.11) shows
\[ t^{\nu/2} e^{(\nu+n)} \tau_n^{V^*} = \left( \frac{\Gamma(\mu + 1)\Gamma(\nu + 1)}{2\pi \Gamma(\mu + \nu + 1)} \right)^n \det \left[ \delta^{j+k} t^{1/2} e^t \int_{-\pi}^{\pi} (\mu - \nu)/2 |1 + z|^{\nu + \nu} e^{\mu t} z d\theta \right]_{j,k=0,...,n-1}. \]

We know [16] proof of Prop. 3.1] that (4.11) is equivalent to
\[ t^{\nu/2} e^{(\nu+n)} \tau_n^{V^*} = \det[\delta^{j+k} t^{1/2} e^t]_{j,k=0,...,n-1} \]
for any \( c \), and thus choosing \( c = -1/2 \) the factor of \( t^{1/2} \) in the determinant is cancelled, while the left hand side of (4.16) is multiplied by \( t^{-n/2} \). Now simple manipulation involving integration by parts shows
\[ \delta \left( e^t \int_{-\pi}^{\pi} (1 + z)^{\mu(1 + 1/z)^{\nu}} e^{\mu t} \right) \]
\[ = e^t \left( (\nu + 1) \int_{-\pi}^{\pi} (1 + z)^{\mu(1 + 1/z)^{\nu}} e^{\mu t} \right) \]
so making use of (4.11) and applying (4.17) to column \( k \) and subtracting \( (\mu + \nu + 1) \) times column \( k - 1 \) for \( k = n - 1, n - 2, \ldots, 1 \) in order shows (4.16) can be reduced to
\[ e^{(n-1)/2} e^{(\nu+n)} \tau_n^{V^*} = \left( \frac{\Gamma(\mu + 1)\Gamma(\nu + 1)}{2\pi \Gamma(\mu + \nu + 1)} \right)^n (\nu + 1)^{n-1} \]
\[ \times \det \left[ \delta^{j+k} \int_{-\pi}^{\pi} (1 + z)^{\mu(1 + 1/z)^{\nu}} e^{\mu t} \delta^{j+k-1} \int_{-\pi}^{\pi} (1 + z)^{\mu(1 + 1/z)^{\nu}} e^{\mu t} \right]_{j=0,...,n-1}. \]
Repeating this procedure for columns $k = n - 1, n - 2, \ldots, 2$ and so on shows

\[
\tau^n_{n-1/2} e^{(\nu+n)t} \mathcal{V}^* \mathcal{V}^{-1} = \left( \frac{\Gamma(\mu + 1)}{2 \pi \Gamma(\mu + \nu + 1)} \right)^n \prod_{i=0}^{n-1} \Gamma(\nu + l + 1) \det \left[ \delta^i e^t \int_{-\pi}^{\pi} (1 + z)^{\mu} (1 + 1/z)^{\nu+k} e^{t z} \, d\theta \right]_{j,k=0,\ldots,n-1}^n.
\]

and application of the general identities \[\text{[16]}\]

then gives

\[
\det \left[ \delta^j (u(t) f_k(t)) \right]_{j,k=0,\ldots,n-1} = (u(t))^n \det \left[ \delta^j f_k(t) \right]_{j,k=0,\ldots,n-1} = (u(t))^n \det \left[ \frac{d^j}{dt^j} f_k(t) \right]_{j,k=0,\ldots,n-1},
\]

and

\[
e^{\nu t} \tau^n_{n-1} = \left( \frac{\Gamma(\mu + 1)}{2 \pi \Gamma(\mu + \nu + 1)} \right)^n \prod_{i=0}^{n-1} \Gamma(\nu + l + 1) \det \left[ \int_{-\pi}^{\pi} (1 + z)^{\mu} (1 + 1/z)^{\nu+k} z^j e^{t z} \, d\theta \right]_{j,k=0,\ldots,n-1}.
\]

The stated result now follows after noting that the factor $(1 + 1/z)^k$ in the integral can be replaced by $(1/z)^k$ without changing the value of the determinant.

Knowledge of \[\text{[4.18]}\] and key recurrences from the Okamoto theory of PV as detailed in \[\text{[16]}\] allows the following recurrence for $\tau^V[N](t; \mu, \nu)$ to be deduced.

**Proposition 7.** Let $\tau^V[N] = \tau^V[N](t; \mu, \nu)$ as given by \[\text{[4.8]}\] or more generally \[\text{[4.14]}\]. Let $p_N, q_N$ denote the conjugate variables in the Hamiltonian \[\text{[4.5]}\] with parameters

\[
v_1 - v_3 = -\nu, \quad v_1 - v_2 = \mu, \quad v_4 - v_3 = N
\]

and define

\[
x_N := (p_N + t)q_N - \frac{1}{2}(v_2 - v_1), \quad y_N = \frac{1}{q_N}.
\]

The sequences $\{\tau^V[N]\}_{N=0,1,\ldots}$, $\{x_N\}_{N=0,1,\ldots}$, $\{y_N\}_{N=0,1,\ldots}$ satisfy the coupled recurrences

\[
(N + \nu) \frac{\tau^V[N + 1] \tau^V[N - 1]}{(\tau^V[N])^2} = \left( x_N - \frac{t}{y_N} - \nu - \frac{\mu}{2} \right) \left( \frac{1}{y_N} - 1 \right) + N
\]

\[
x_N + x_{N-1} = \frac{t}{y_N} - \frac{N}{1 - y_N}
\]

\[
y_N y_{N+1} = \frac{x_N + v + \mu/2 + N + 1}{x_N^2 - (\mu/2)^2}
\]

subject to the initial conditions

\[
x_0 = t + \mu/2 + t \frac{d}{dt} \log_1 F_1(-\nu, \mu + 1; -t), \quad y_0 = 1,
\]

\[
\tau^V[0] = 1, \quad \tau^V[1] = \frac{\Gamma(\mu + \nu + 1)}{\Gamma(\mu) \Gamma(\nu + 1)} F_1(-\nu, \mu + 1; -t).
\]

Proof. According to \[\text{[16]}\] proof of Prop. 2.2

\[
\frac{\tau^V_{n+1} \tau^V_{n-1}}{(\tau^V_n)^2} = \frac{\partial}{\partial t} \left( \frac{t + \nu + (v_4^{(0)} - v_1^{(0)} + n)t}{|H|} \right)
\]

(4.26)
so taking into consideration \ref{1.14}, \ref{1.15} and \ref{1.16} we arrive at \ref{1.22}, \ref{1.23}. Furthermore in \cite{16} Prop. 2.4 it was shown that with \(x_N, y_N\) specified in terms of \(p_N, q_N\) by \ref{1.20}, \(\{x_N, y_N\}\) satisfy the discrete Painlevé IV recurrences

\[
x_N + x_{N-1} = \frac{t}{y_N} + \frac{v_3^{(0)} - v_4^{(0)}}{1 - y_N}
\]

\[
y_N y_{N+1} = \frac{t x_N - \frac{1}{2} (v_1^{(0)} + v_2^{(0)}) + 1 + v_4^{(0)} + N}{x_N^2 - \frac{1}{4} (v_2^{(0)} - v_1^{(0)})^2}.
\]

Making use of \ref{4.19} then gives \ref{4.22} and \ref{4.23}. The initial conditions follow from \ref{4.20}, \ref{4.7} and \ref{4.12}.

For \(n \in \mathbb{Z}_{\geq 0}\), the confluent hypergeometric function \(1F_1(-n;c;-t)\) is proportional to a Laguerre polynomial,

\[
1F_1(-n;\alpha + 1;-t) = \frac{\Gamma(\alpha + 1) \Gamma(n + 1)}{\Gamma(n + \alpha + 1)} L_n^\alpha(-t).
\]

Thus for \(\nu \in \mathbb{Z}_{\geq 0}\) it follows from \ref{4.11} that

\[
t^\nu \left[ N \right](t; \mu, \nu) = \text{det} \left[ L_{\nu+k-j}^\mu(-t) \right]_{j, k = 1, \ldots, N}
\]

(note that \(L_n^\mu(-t) := 0\) for \(n < 0\)). According to \ref{4.24}, \ref{4.25} we also have

\[
x_0 = t + \nu + \mu/2 - (\nu + \mu)L_{\nu+k-j}^\mu(-t), \quad y_0 = 1, \quad \tau^V[0] = 1, \quad \tau^V[1] = L_n^\mu(-t).
\]

As in the PIV theory, in the case \(\nu \in \mathbb{Z}_{\geq 0}\) the \(N \times N\) determinant for \(\tau^V[N]\) can also be expressed as a \(\nu \times \nu\) determinant. Thus we know from \cite{16} Props. 3.6.3.7 that for \(\nu \in \mathbb{Z}_{\geq 0}\)

\[
t^\nu \left[ N \right](t; \mu, \nu) \propto \text{det} \left[ \frac{d^j}{dt^j} L_{\nu+k}^\mu(-t) \right]_{j, k = 0, 1, \ldots, \nu - 1}.
\]

Using the Laguerre polynomial identities

\[
L_n^{\mu - 1}(x) = L_n^\mu(x) - L_{n-1}^\mu(x), \quad \frac{d}{dx} L_n^\mu(x) = -L_{n-1}^{\mu+1}(x)
\]

this is equivalent to

\[
t^\nu \left[ N \right](t; \mu, \nu) \propto \text{det} \left[ L_{\nu+k-j}^{\mu+1}(-t) \right]_{j, k = 1, \ldots, \nu}
\]

and thus

\[
t^\nu \left[ N \right](t; \mu, \nu) \propto \tau^V[\nu](t; \mu, N).
\]

To determine the proportionality, we use the fact, following from \ref{1.1} and \ref{1.11}, that

\[
t^\nu \left[ N \right](0, \mu, \nu) = \frac{1}{N!} M_N(\mu, \nu)
\]

where

\[
M_N(a, b) := \int_{-1/2}^{1/2} dx_1 \cdots \int_{-1/2}^{1/2} dx_N \prod_{i=1}^{N} u_i^{(a-b)/2} [1 + u_i]^{a+b} \prod_{1 \leq j < k \leq N} |u_k - u_j|^2, \quad u_i := e^{2\pi i x_i}.
\]

\[
= \prod_{j=0}^{N-1} \frac{\Gamma(a + b + 1 + j) \Gamma(2 + j)}{\Gamma(a + 1 + j) \Gamma(b + 1 + j)}.
\]

\[
(4.31)
\]
Hence for $\nu \in \mathbb{Z}_{\geq 0}$

$$\tau^V[N](t; \mu, \nu) = \frac{\nu!}{N!} M_N(\mu, \nu) \tau^V[N](t; \mu, N).$$  \hspace{1cm} (4.32)

Another point of interest is the explicit $\tau$-function form of the sequences $\{x_N\}_{N=0,1,...}$, $\{y_N\}_{N=0,1,...}$ generated by the discrete Painlevé IV coupled recurrences \(\text{(4.22), (4.23)}\) with initial conditions \(\text{(4.24)}\)\). We find

$$x_N - \mu/2 = \frac{\tau[N + 1](t; \mu + 1, \nu) \tau[N](t; \mu - 1, \nu + 1)}{\tau[N + 1](t; \mu, \nu) \tau[N](t; \mu, \nu + 1)} \tau[N](t; \mu, \nu + 1).$$

$$y_N = \frac{\tau[N](t; \mu, \nu + 1) \tau[N](t; \mu, \nu)}{\tau[N](t; \mu - 1, \nu + 1) \tau[N](t; \mu + 1, \nu)}. \hspace{1cm} (4.33)$$

These formulas can be established by proceeding in an analogous fashion to the proof of Proposition 8. Let us now turn our attention to the PV $\tau$-function $\tilde{\tau}^V$ as specified by (4.13). In some special cases this is intimately related to $\tau^V$ as specified by (4.12). Thus with $I_N(a)$ specified by (1.17) we have [16, Prop. 3.7]

$$\frac{I_N(a)}{I_N(a + \mu)} \tilde{\tau}^V[N](t; \mu, a; 0) = \frac{M_0(0, 0)}{M_0(\mu, N)} \tau^V[\mu](t; a, N), \quad \mu \in \mathbb{Z}_{\geq 0} \hspace{1cm} (4.34)$$

$$\frac{I_N(a)}{I_N(a + \mu)} \tilde{\tau}^V[N](t; \mu, a; 1) = \frac{M_0(0, 0)}{M_0(\mu, N)} \tau^V[a](t; \mu, N), \quad a \in \mathbb{Z}_{\geq 0} \hspace{1cm} (4.35)$$

The identities (4.34) and (4.35) allow those special cases of $\tilde{\tau}^V[n]$ to be computed by the recurrences of Proposition 7, however the recurrences will no longer be with respect to the dimension of the average $\tau$. Thus we have $\tilde{\tau}^V[n]$ can also be given, but these recurrences alter both $n$ and the parameter $a$. This comes about as a consequence of the following analogue of Proposition 8.

**Proposition 8.** Write the general solution of the confluent hypergeometric equation (4.38) in the integral form

$$e^{-(v_1^{(0)} - v_3^{(0)})} \tau^V[\gamma] = \left( \int_0^\infty -\xi \int_0^1 e^{-tu} (u - 1)^{v_2^{(0)} - v_3^{(0)}} v_1^{(0)} - v_2^{(0)} - 1 \right) du,$$

and set

$$\alpha := v_2^{(0)} - v_3^{(0)} + 1, \quad \gamma := v_1^{(0)} - v_3^{(0)} + 1.$$

We have

$$\tau^V[\gamma] = e^{(\gamma-1)\xi} \Gamma(\gamma - \alpha) \prod_{l=0}^{n-1} \Gamma(l+1) \tilde{\tau}^V[n](t; \alpha - 1, \gamma - \alpha - n; \xi). \hspace{1cm} (4.36)$$

Proof. With

$$F(\alpha, \gamma; t) := e^{t \left( \int_0^\infty -\xi \int_0^1 e^{-tu} (u - 1)^{\alpha - 1} u^{\gamma - \alpha - 1} \right) du}$$

we see from (4.10), (4.11) and (4.16) that

$$\tau^V[\gamma] = \det[\delta^{j+k} F(\alpha, \gamma; t)]_{j,k=0,\ldots,n-1}.$$

Analogous to (4.17) we can show that

$$\delta F(\alpha, \gamma; t) = -\alpha F(\alpha, \gamma; t) - (\gamma - \alpha - 1) F(\alpha + 1, \gamma; t).$$
Then proceeding as in the derivation of (4.18) we deduce
\[ e^{-(\gamma - 1)t} \tau_i = \frac{\left( \Gamma(\gamma - \alpha) \right)^{-1}}{\prod_{k=1}^{n-1} \Gamma(\gamma - \alpha - l)} \det[e^{-t} F(\alpha + k, \gamma + j; t)]_{j,k=0,\ldots,n-1}. \]
The method of the proof of \[16\] proof of Prop. 3.1 allows this determinant to be written as a multiple integral. Making use too of (4.17) then gives (4.36).

\[ \square \]

Because (4.26) and (4.27) hold for any \( \tau \)-function sequence with the property (4.4), we see that (4.36) allows us to specify the analogue of Proposition \[4\] for \( \{\tau^V[n]|t; \alpha - 1, \gamma - \alpha - n; \xi\}_{n=0,1,\ldots} \), although we stop short of writing it down.

5 The \( \tau \)-function sequence \( \{\tau^V[n]|t; \mu, w_1, w_2; \xi\}_{n=0,1,\ldots} \)

As written \[14\] requires \(-\pi < \phi \leq \pi \) to make sense, however we can readily extend this definition to general complex \( t := e^{i\phi} \) (5.1)

First we make use of (3.1) to obtain
\[ \tau^V[N] = \det \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - \xi \chi(\pi - \phi, \pi)) e^{izw_1} |1 + z|^2 w_1 |e^{i(\pi - \phi)} - z|^{2w_2 - k} d\theta \right]_{j,k=0,\ldots,N}. \] (5.2)

The integral in (5.2) naturally breaks into two. Introducing \( t \) according to (5.1) and setting \( \mu \in \mathbb{Z}_{\geq 0} \) the first portion reads
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{izw_1} |1 + z|^2 w_1 \left( \frac{1}{tz} \right)^\mu (1 + tz)^{2w_2 - k} d\theta = \frac{\Gamma(a + b + 1)}{\Gamma(a + 1) \Gamma(b + 1)} t^{-\mu} F_1(-2\mu, -b; a + 1; t) \] (5.3)
where, with \( w := w_1 + iw_2 \),
\[ a = \bar{w} - \mu + j - k, \quad b = w + \mu - j + k \]
(the integral evaluation follows upon making use of (4.22)). For the second portion, taking both \( w_1, \mu \in \mathbb{Z}_{\geq 0} \), and writing in terms of \( dz \) instead of \( d\theta \) we have
\[ -\frac{\xi t^{-\mu}}{2\pi i} \int_{C(-1/t,-1)} z^{-iw_2 + j - k - w_1} (1 + z)^{2w_1} (1 + tz)^{2\mu} \frac{dz}{z} \] (5.4)
where \( C(-1/t,-1) \) is a simple closed contour starting at \( z = -1/t \) and finishing at \( z = -1 \). Making the successive transformations \( z \mapsto -z/t, z \mapsto -z + 1, z \mapsto (1 - t)z \), then making use of the integral formula
\[ \int_0^1 x^{\lambda_1}(1 - x)^{\lambda_2}(1 - tx)^{-\tau} dx = \frac{\Gamma(\lambda_1 + 1) \Gamma(\lambda_2 + 1)}{\Gamma(\lambda_1 + \lambda_2 + 2)} F_1(\tau, \lambda_1 + 1, \lambda_1 + \lambda_2 + 2; t) \]
shows (5.4) is equal to
\[ \frac{\xi t^{-\mu} e^{\pm i\pi(k - j + \mu - \bar{w})}}{2\pi i} \frac{\Gamma(2\mu + 1) \Gamma(2w_1 + 1)}{\Gamma(2\mu + 2w_1 + 2)} j^{k - j + \mu - \bar{w}} (1 - t)^{2\mu + 2w_1 + 1} \times F_1(2\mu + 1, 1 + k - j + \mu + w; 2\mu + 2w_1 + 2; 1 - t) \] (5.5)
where the \( \pm \) sign is taken accordingly as \( \text{Im}(\mu) \leq 0 \). Substituting for the integral in (5.2) the sum of the hypergeometric functions (5.3) and (5.5) gives meaning to \( \tau^V[N] \) for general complex values of (5.1).
We know from \[16\] that the CUE$_N$ average \((1.9)\) can be written as an average over the generalized Cauchy unitary ensemble \[33, 10\] specified by the p.d.f.

\[
\frac{1}{C} \prod_{l=1}^{N} \frac{1}{(1 + i x_l)^{\eta}(1 - i x_l)^{\bar{\eta}}} \prod_{1 \leq j < k \leq N} (x_k - x_j)^2, \quad C = 2^{-N(N-1)\pi N} M_N(\bar{\eta} - N, \eta - N) \tag{5.6}
\]

where \(M_N\) is given by \((4.31)\). Thus making the change of variables \(z_l = \frac{1 + i x_l}{1 - i x_l}\) in \((1.9)\) shows \[17, eq. (1.19) with \(\mu \mapsto 2\mu\]}

\[
\frac{N!}{M_N(\mu + \bar{w}, \mu + w)} \tau^{VI}[N](e^{i\phi}; \mu, w_1, w_2; \xi^*) = \frac{1}{(1 + s^2)^{N\mu}} \left\langle \prod_{l=1}^{N} (1 - \xi \chi^{(l)}(s, \infty))(s - x_l)^{2\mu} \right\rangle_{C_{YUE_N}} \left| \eta \mapsto \eta + N, \bar{\eta} \mapsto \bar{\eta} - N \right. \tag{5.7}
\]

where

\[
\xi^* := 1 - (1 - \xi) e^{-\pi i \mu} \tag{5.8}
\]

Furthermore, we know from \[17\] that the CyUE average in \((5.7)\) as a function of \(s\) is also the \(\tau\)-function for a particular PVI system. Moreover, unlike the situation with \((1.9)\), the Schlesinger transformation studied in \[17\] increments \(N\) in this average and leaves the other parameters unchanged (in \((1.9)\) this same Schlesinger transformation increments \(N\) but also decrements \(\mu\)).

To make these statements more explicit, we recall \[26\] that the Hamiltonian for the PVI system is given by

\[
t(t-1)H^{VI} = q(q-1)(q-t)p^2 - \left((v_1 + v_2)(q-1)(q-t)\right.
\]

\[
+ (v_1 - v_2)(q-t) + (v_3 + v_4)(q-1))p + (v_1 + v_4)(v_1 + v_3)(q-t). \tag{5.9}
\]

Introduce the Schlesinger operator \(T_3\) with action on the parameters

\[
T_3 v = (v_1, v_2, v_3 + 1, v_4),
\]

and with appropriate actions on the conjugate variables \(p, q\). We know \[17, eq. (2.27)\] that

\[
T_3^0 H_0^{VI} =: H_n^{VI} = H_0^{VI} \mid_{v \mapsto (v_1^{(0)}, v_2^{(0)}, v_3^{(0)} + n, v_4^{(0)})}.
\]

From this we introduce a sequence of \(\tau\)-functions \(\tau_n^{VI}\) specified so that

\[
H_n^{VI} = \frac{d}{dt} \log \tau_n^{VI}.
\]

We know from \[17\] Prop. 15 that

\[
\left\langle \prod_{l=1}^{N} (1 - \xi \chi^{(l)}(s, \infty))(s - x_l)^{\mu} \right\rangle_{C_{YUE_N}} \left| \eta \mapsto \eta + N, \bar{\eta} \mapsto \bar{\eta} - N \right. \propto \tau_N^{VI} \left( \frac{i s + 1}{2}; \nu \right) \tag{5.10}
\]

where with \(\eta := \eta_1 + i \eta_2\)

\[
\nu = (-\eta_1, i \eta_2, \eta_1 + N, -\mu + \eta_1). \tag{5.11}
\]

For our present purposes the proportionality constant in \((5.10)\), not calculated in \[17\], is of importance. To obtain its value we must recall some of the results from \[17\].
We know from [17] eqs. (2.30), (2.37) that with
\[ \tau_n^{VI} := (t(t-1))^{(n+e_3^0) + v_3^0 + v_4^0)/(n+e_3^0 + v_3^0 + v_4^0)/2 \tau_n \]  
we have
\[ \hat{\tau}_n^{VI} = \det[\delta^{j+k} \tau_1^{VI}]_{j,k=0,...,n-1}, \quad \delta := t(t-1) \frac{d}{dt}. \]  
We also know that \( \tau_1^{VI} \) satisfies the Gauss hypergeometric differential equation
\[ t(1-t)y'' + (c - (a + b + 1)t) y' - aby = 0 \]  
with
\[ a = v_4^0 - v_3^0, \quad b = 1 + v_3^0 + v_4^0, \quad c = 1 + v_2^0 + v_4^0 \]  
and that a general solution of (5.14) is given by [17, eqs. (2.67)]
\[ \theta = \left( \int_{-\infty}^{\infty} -\xi \int_{t}^{\infty} u^{a-c} (1-u)^{c-b-1}(t-u)^{-a} \, du. \right) \]  
Noting that in (5.11) we have \( v_1^0 = v_3^0 = 0 \), it thus follows from (5.12), (5.13), (5.16) and (5.17) that
\[ \hat{\tau}_n^{VI} = \det[\delta^{j+k}(t(t-1))^{b/2} F(a, b, c; t)]_{j,k=0,...,n-1}. \]  
We know from [17] proof of Prop. 6] that \( F \) satisfies the differential-difference relations
\[ t \frac{d}{dt} F(a, b, c; t) = -(c-b-1) F(a, b+1, c; t) - b F(a, b, c; t) \]  
\[ t(1-t) \frac{d}{dt} F(a, b, c; t) = (a-c) F(a-1, b, c; t) + (a-c+bt) F(a, b, c; t). \]  
It follows that
\[ \delta ((t(t-1))^{b/2} F(a, b, c; t)) = \frac{b}{2} (t(t-1))^{b/2} F(a, b, c; t) + (b+1-c) t^{b/2}(t-1)^{b/2+1} F(a, b+1, c; t) \]  
\[ \delta ((t(t-1))^{b/2} F(a, b, c; t)) = (c-a-b/2)(t(t-1))^{b/2} F(a, b, c; t) + (c-a)(t(t-1))^{b/2} F(a-1, b, c; t). \]  
\[ \text{From these latter relations the working of the proof of [17] Prop. 4] gives} \]
\[ \tau_n^{VI} = \prod_{j=1}^{n-1} (b + 1-c) (c-a) F^{b/2}((t(t-1))^{b/2+n(n-1)/2} \det \left[ F(a - j, b + k, c; t) \right]_{j,k=0,...,n-1}. \]  
The method of the proof of [17] Prop. 5] allows this to be rewritten as a multiple integral, which when substituted in (5.12) and after substitution of the parameters according to (5.15), (5.11) shows
\[ \tau_n^{VI} = \left( \frac{-1}{n!} \prod_{j=1}^{n-1} (1 + \bar{\eta}_j)(1 + \eta_j) \left( \int_{-\infty}^{\infty} -\xi \int_{t}^{\infty} \right) du_1 \cdots \left( \int_{-\infty}^{\infty} -\xi \int_{t}^{\infty} \right) du_n \right) \times \prod_{j=1}^{n} u_j^{-\eta+n} (1-u_j)^{-(\eta+n)} (t-u_j)^{\mu} \prod_{1 \leq j < k \leq n} (u_k - u_j)^2. \]  
Replacing \( b \) by \((is+1)/2\), changing variables in the integrations \( u_j \mapsto (iv_j + 1)/2 \) and making use of (5.6) we deduce that the proportionality constant in (5.11) (taken for convenience to be on the left hand side) is equal to
\[ \left( \frac{1}{N!} \prod_{j=1}^{n-1} (1 + \bar{\eta}_j)(1 + \eta_j) \right)^{i(s+1)N} 2^{-(\eta+2n-1)N} \pi^n M_N(\bar{\eta}, \eta). \]
Let Proposition 9. Conjugate variables in the Hamiltonian (5.9) with parameters (5.11). We have

$$\left. \frac{1}{1 + s^2} \pi^N \left( \frac{i \theta}{2}, v \right) \right|_{s = \frac{\theta}{2}, \mu = 2 \mu_j} = \prod_{j=1}^{N-1} (1 + \bar{w} + \mu_j)(1 + w + \mu_j) \chi_{(\mu + 1)N} 2^{-(2\mu_1 + 1)N} \pi^N \chi_{\mu, w}(\mu, w_1, w_2; \xi).$$ \hspace{1cm} (5.23)

Consequently, after recalling (5.7), we conclude

$$q_n = \frac{q_n}{q_n - 1}, \quad f_n = q_n(q_n - 1) p_n + (1 + n - \alpha_2^{(0)} - \alpha_4^{(0)}) (q_n - 1) - \alpha_3^{(0)} q_n - (\alpha_0^{(0)} + n) \frac{q_n(q_n - 1)}{q_n - t}. \hspace{1cm} (5.25)$$

where

$$\alpha_0^{(0)} = v_3^{(0)} + v_4^{(0)} + 1, \quad \alpha_1^{(0)} = v_3^{(0)} - v_4^{(0)}, \quad \alpha_2^{(0)} = -(v_1^{(0)} + v_3^{(0)}), \quad \alpha_3^{(0)} = v_1^{(0)} - v_2^{(0)}, \quad \alpha_4^{(0)} = v_1^{(0)} + v_2^{(0)}. \hspace{1cm} (5.26)$$

Then we have \[\text{Prop. 10}\]

$$g_{n+1} = \frac{f_n + 1 + n - \alpha_2^{(0)} - \alpha_4^{(0)} (f_n + 1 + n - \alpha_2^{(0)} - \alpha_4^{(0)})}{f_n \alpha_3^{(0)}} \quad \text{and} \quad f_n + f_{n-1} = -\alpha_3^{(0)} + \frac{\alpha_2^{(0)} + n}{g_n - 1} + \frac{(\alpha_0^{(0)} + n) t}{t(g_n - 1) - g_n}, \hspace{1cm} (5.27)$$

which are a version of the discrete Painlevé V equations [30]. To use these recurrences to determine \{p_n\}, \{q_n\} given \(f_0, g_0\) we first iterate \[\text{Prop. 11}\] to determine \{f_n\}, \{g_n\}. According to the first equation in \[\text{Prop. 20}\] \(q_n\) can then be calculated in terms of \(g_n\),

$$q_n = \frac{g_n}{g_n - 1}. \hspace{1cm} (5.28)$$

Now that \(q_n\) is known the second equation in \[\text{Prop. 20}\] allows \(p_n\) to be calculated in terms of \(f_n, q_n\),

$$p_n = \frac{1}{q_n(q_n - 1)} \left( f_n + (1 + n - \alpha_2^{(0)} - \alpha_4^{(0)}) (1 - q_n) + \alpha_3^{(0)} q_n + (\alpha_0^{(0)} + n) \frac{q_n(q_n - 1)}{q_n - t} \right). \hspace{1cm} (5.29)$$

To calculate \(\tau^{(v)}(\mu, w_1, w_2; \xi)\) the following recurrence scheme can therefore be given.

**Proposition 9.** Let \(\tau^{(v)}(\mu, w_1, w_2; \xi)\) as specified by \[\text{Prop. 11}\] and let \(p_N, q_N\) denote the conjugate variables in the Hamiltonian \[\text{(5.24)}\] with parameters \[\text{(5.11)}\]. We have

$$q_n = \frac{g_n}{g_n - 1}, \quad p_n = \frac{1}{q_n(q_n - 1)} \left( f_n + (1 + n + \bar{w} + \mu)(1 - q_n) - (w + \mu) q_n + (1 + n + 2w_1) \frac{q_n(q_n - 1)}{q_n - (1 - e^{i\phi})^{-1}} \right). \hspace{1cm} (5.30)$$
where \( \{f_n\}_{n=0,1,...} \), \( \{g_n\}_{n=0,1,...} \) are determined by the recurrences

\[
g_{n+1}g_n = e^{-i\phi}(f_n + 1 + n)(f_n + 1 + n + \bar{w} + \mu)\]

\[
f_n + f_{n-1} = w + \mu + \frac{2\mu + n}{g_n - 1} - \frac{(1 + n + 2w_1)}{1 - e^{i\phi}g_n}
\]

subject to the initial conditions

\[
g_0 = \frac{q_0}{q_0 - 1}, \quad f_0 = (1 + \bar{w} + \mu)(q_0 - 1) + (w + \mu)q_0 - (2w_1 + 1) \frac{q_0(q_0 - 1)}{q_0 - (1 - e^{i\phi})^{-1}}
\]

with

\[
q_0 = \frac{1}{2}(1 + \frac{i}{\mu} \frac{d}{d\phi} \log \tau V I[1]).
\]

Given \( \tau V I[0] = 1 \), and \( \tau V I[1] \) as the element of the determinant in (5.32) with \( j - k = 0 \), we have that \( \{\tau V I[N]\}_{N=2,3,...} \) can be computed in terms of \( \{p_N\}_{N=1,2,...} \) and \( \{q_N\}_{N=1,2,...} \) by the recurrence

\[
(N + \bar{w} + \mu)(N + w + \mu) \frac{\tau V I[N + 1] \tau V I[N - 1]}{(\tau V I[N])^2} = q_N(1 - q_N)p_N^2 - 2(w_1 + \mu)q_Np_N + (\bar{w} + \mu)p_N + N(2w_1 + 2\mu).
\]

Proof. The only remaining point to require explanation is the initial conditions [5.32], [5.33]. These come about because the PVI system admits the solution [17, Prop. 3]

\[
p_0 = 0, \quad t(t - 1) \frac{d}{dt} \log \tau V I(t) = -\mu(q_0 - t).
\]

According to [5.28] we require \( t = \frac{1}{2}(is + 1), s = \cot \phi/2 \) and so \( t = 1/(1 - e^{i\phi}) \). Hence the second equation in [5.28], together with [5.28] in the case \( N = 1 \), gives [5.33]. Also, setting \( n = 0 \) in the second equation of [5.30] and equating the right hand side to zero gives the second initial condition in [5.32]. The first initial condition in [5.32] follows immediately from the first equation in [5.30].

\[
\text{From the Okamoto theory [26]} \text{ we know } p = p(t; \alpha), q = q(t; \alpha) \text{ must satisfy a number of transformation formulas with respect to } t \text{ and } \alpha. \text{ Thus with}
\]

\[
\alpha^1 := (\alpha_0, \alpha_1, \alpha_2, \alpha_4, \alpha_3) \quad \alpha^2 := (\alpha_0, \alpha_4, \alpha_2, \alpha_3, \alpha_1) \quad \alpha^3 := (\alpha_4, \alpha_1, \alpha_2, \alpha_3, \alpha_0)
\]

one has

\[
p(1 - t, \alpha^1) = -p(t; \alpha), \quad q(1 - t, \alpha^1) = 1 - q(t; \alpha)
\]

\[
p\left(\frac{1}{t}; \alpha^2\right) = -\alpha_2q(t; \alpha) - q^2(t; \alpha)p(t; \alpha), \quad q\left(\frac{1}{t}; \alpha^2\right) = \frac{1}{q(t; \alpha)}
\]

\[
p\left(\frac{t}{t - 1}; \alpha^3\right) = -(t - 1)p(t; \alpha), \quad q\left(\frac{t}{t - 1}; \alpha^3\right) = \frac{t - q(t; \alpha)}{t - 1}.
\]

Setting \( t = 1/(1 - e^{i\phi}) \) and inverting these formulas we could, if required, write down a variant of Proposition 3 in each of these cases which implicitly involves the variables \( 1 - t, 1/t \) and \( t/(t - 1) \) respectively, but at the expense of permuting the \( \alpha \)'s. Consider in particular the first mapping involving \( t \mapsto 1 - t, \alpha \mapsto \alpha^1 \). With \( t = 1/(1 - e^{i\phi}) \) and \( \alpha_3 = -w - \mu, \alpha_4 = -\bar{w} - \mu \) this corresponds to simply taking the complex conjugate. Indeed making the replacements \( p \mapsto -P, q \mapsto 1 - Q \) we see that the right hand side of [5.34] formally becomes equal to its complex conjugate, provided we identify \( P, Q \) with \( \bar{p}, \bar{q} \) respectively.
Let us now turn our attention to the special case $2\mu \in \mathbb{Z}_{\geq 0}$ and $\xi = 0$. Then according to (5.3) with $t$ given by (5.4), $t^{N\mu} \tau^{VI}[N]$ is a polynomial in $t$,

$$
 t^{N\mu} \tau^{VI}[N] = \text{det} \left[ \frac{\Gamma(2w_1 + 1)}{\Gamma(\bar{w} - \mu + j - k + 1)\Gamma(w + \mu - j + k + 1)} \times _2F_1(-2\mu, -w - \mu + j - k; \bar{w} - \mu + j - k + 1; t) \right]_{j,k=1,\ldots,N}.
$$

Furthermore in this situation (5.33) reduces to

$$
 q_0 = 1 - \frac{t}{2\mu} \log_2 _2F_1(-2\mu, -w - \mu; \bar{w} - \mu + 1; t) = \frac{2\mu - 1, -w - \mu; \bar{w} - \mu + 1; t}{2\mu - 1, -w - \mu; \bar{w} - \mu + 1; t}.
$$

Also for $2\mu \in \mathbb{Z}_{\geq 0}$ we have the duality type relation between averages (3.42)]

$$
 \left\langle \prod_{l=1}^{N} |z|^{n-\bar{n}/2} |z|^{1+\bar{z}}(1 + t\bar{z})^{2\mu} \right\rangle_{U(N)} \propto \left\langle \prod_{l=1}^{2\mu} |z|^{n+\bar{z}/2} |z|^{1+(1-t)\bar{z}} \right\rangle_{U(2\mu)}.
$$

Recalling (1.9) we thus have

$$
 \tau^{VI}[N](t; \mu, w_1, w_2; 0) \propto \tau^{VI}[2\mu](1 - t; N/2, \bar{w} - \mu, \bar{w} + 1; 1/2(t + 2w + \mu + N); 0),
$$

which is the PVI analogue of (4.32) and (4.36). Setting $t = 0$ shows the proportionality constant to be equal to

$$
 \frac{(2\mu)!}{N!} \frac{M_N(-\mu, \bar{w}, \mu - w)}{M_{2\mu}(2w_1 + N, -\mu - w)}.
$$

Each of the quantities $q_n, p_n, f_n, g_n$ in Proposition 4 can be written as a ratio of $\tau$-functions. Introducing for convenience

$$
 \hat{\tau}^{VI}[n](t; \mu, w, \bar{w}, \xi) := t^{N\mu} \tau^{VI}[N](t; \mu, w_1, w_2; \xi),
$$

on the basis of exact tabulations with initial condition (5.38) we are led to the formulas

$$
\begin{align*}
 q_n &= \frac{\hat{\tau}^{VI}[n + 1](t; \mu - 1/2, w + 1/2, \bar{w} - 1/2; \xi)}{\hat{\tau}^{VI}[n + 1](t; \mu, w, \bar{w}; \xi)} \frac{\hat{\tau}^{VI}[n](t; \mu, w, \bar{w} + 1; \xi)}{\hat{\tau}^{VI}[n](t; \mu - 1/2, w + 1/2, \bar{w} + 1/2; \xi)} \quad (5.42) \\
p_n &= 2\mu(t - 1) \frac{\hat{\tau}^{VI}[n + 1](t; \mu, w, \bar{w}; \xi)}{\hat{\tau}^{VI}[n](t; \mu, w, \bar{w}; \xi)} \times \frac{\hat{\tau}^{VI}[n](t; \mu, w, \bar{w} + 1; \xi)}{\hat{\tau}^{VI}[n](t; \mu, w + 1, \bar{w}; \xi)} \frac{\hat{\tau}^{VI}[n - 1](t; \mu + 1/2, w + 1/2, \bar{w} + 1/2; \xi)}{\hat{\tau}^{VI}[n](t; \mu + 1, \bar{w}; \xi)} \quad (5.43) \\
f_n &= -(n + 1) t \frac{\hat{\tau}^{VI}[n + 1](t; \mu - 1/2, w - 1/2, \bar{w} + 1/2; \xi)}{\hat{\tau}^{VI}[n + 1](t; \mu, w, \bar{w}; \xi)} \frac{\hat{\tau}^{VI}[n](t; \mu, w + 1, \bar{w}; \xi)}{\hat{\tau}^{VI}[n](t; \mu + 1, \bar{w}; \xi)} \quad (5.44) \\
g_n &= -\frac{1}{t} \frac{\hat{\tau}^{VI}[n + 1](t; \mu - 1/2, w + 1/2, \bar{w} - 1/2; \xi)}{\hat{\tau}^{VI}[n + 1](t; \mu, w, \bar{w}; \xi)} \frac{\hat{\tau}^{VI}[n](t; \mu, w, \bar{w} + 1; \xi)}{\hat{\tau}^{VI}[n](t; \mu + 1/2, w + 1/2, \bar{w}; \xi)} \quad (5.45)
\end{align*}
$$

We have not completed a proof of these relations. However, as with the formulas (5.38), an outline of how one goes about proving (5.42)–(5.45) is provided by the proof of Proposition 4. Here the matter is complicated by there being four fundamental Schlesinger operators instead of the two in PIII theory, and the fact that $p_n, q_n$ are functions of $1/(1 - t)$ rather than $t$. Some details of dealing with the first of these complications is given in (23), while use of the transformation identities (5.36) is required to deal with the second. Such arguing can only be used to establish (5.42)–(5.45) up to proportionality constants. To
determine the latter, we proceed on the assumption that the proportionality is independent of $\xi$, allowing us to set $\xi = 0$. Then according to (5.34) and the formulas of Proposition 9 we must have

\[
q_n \sim \frac{1}{t^a}, \quad g_n \sim q_n, \quad f_n \sim -(n+1), \quad p_n \sim \frac{2\mu n}{w-\mu + n}.
\]

On the other hand it follows from (6.41) and the definition of $M_N(a, b)$ in (6.31) that with $\xi = 0$

\[
\hat{\tau}^{VI}[n] \sim \frac{1}{t^a N} M_n(w - \mu, w + \mu), \quad \hat{\tau}^{VI}[n] \sim t^{2n\mu} \frac{1}{n!} M_n(w + \mu, w - \mu).
\]

Now using the evaluation formula in (6.31) we deduce the proportionality in (5.42)–(5.45).

Thus we know that the working leading to (5.21) can be carried through with

\[
\left( \int_{-\infty}^{-\xi} \int_{t}^{\infty} \right) \mapsto \left( \int_{0}^{1} -\xi \int_{t}^{1} \right),
\]

and hence the PVI system admits a $\tau$-function sequence

\[
\tau_n^{VI}(t; \nu(0)) = \frac{(-1)^n(n-1)/2}{n!} \times \prod_{j=1}^{n-1} (1 + v_3^{(0)} - v_2^{(0)}) \times \cdots \times \prod_{i=1}^{n} v_i^{(-v_3^{(0)}+n)} (1 - u_i) v_2^{(0) - (v_3^{(0)}+n)} (t - u_i)^{-v_3^{(0)}} \prod_{1 \leq j < k \leq n} (u_k - u_j)^2.
\]

Comparison with (6.41) shows

\[
\tau_n^{VI}\left(t; \left(\frac{1}{2}(a + b), \frac{1}{2}(a - b), -\frac{1}{2}(a + b), -\frac{1}{2}(a + b) - \mu\right)\right) = (-1)^{N(N-1)/2} \frac{J_N(a - N, b - N)}{N!} \prod_{j=1}^{N} (1 - a_j) (1 - b_j) \hat{\tau}^{VI}[N](t; \mu, a - N, b - N; \xi) \tag{5.46}
\]

where

\[
J_N(a, b) := \int_{0}^{1} dx_1 x_1^a (1 - x_1)^b \cdots \int_{0}^{1} dx_N x_N^a (1 - x_N)^b \prod_{1 \leq j < k \leq N} (x_k - x_j)^2.
\]

It follows from this and (5.24) that we can compute \{\hat{\tau}^{VI}[N](t; \mu, a - N, b - N; \xi)\}_{N=0, 1, \ldots} by a recurrence analogous to that in Proposition 9 although we do not pursue the details.

6 Applications

In this section we will present results from the numerical evaluation of examples of the $\tau$-functions (1.7) and (1.30) based on the recurrences of Propositions 9 and 9 respectively. Consider first the $\tau$-function (1.7). The cases $\mu = 0$ and $\mu = 2$ have particular significance. Thus let $E_{N,2}(0, (0, s); x^a e^{-x})$ denote the probability that there are no eigenvalues in the interval $(0, s)$ of the LUE$_N$ as specified by the eigenvalue probability density function (1.14), and let $p_{N,2}(0, (0, s); x^a e^{-x})$ denote the probability density of the
The smallest eigenvalue in the same ensemble. These two quantities are inter-related by a single differentiation,

\[ p_{N,2}(0, (0, s); x^a e^{-x}) = \frac{d}{ds} E_{N,2}(0, (0, s); x^a e^{-x}). \]  

(6.1)

To make contact with consider the scaled limit of these quantities,

\[ E^\text{hard}_2(0, (0, t)) := \lim_{N \to \infty} E_{N,2}(0, (0, t/4N); x^a e^{-x}) \]
\[ p^\text{hard}_2(0, t) := \lim_{N \to \infty} \frac{1}{4N} p_{N,2}(0, t/4N) \]

(6.2)

(the reason for the superscripts “hard” is that the neighbourhood of the origin in the Laguerre ensemble is referred to as the hard edge; see e.g. [12]). Now we know from [13] that

\[ E^\text{hard}_2(0, (0, t)) = e^{-t/4} \det[I_{j-k}(\sqrt{t})]_{j,k=1,...,a} = e^{-t/4} \tau^{III'}[a](t; 0) \]
\[ p^\text{hard}_2(0, t) = \frac{1}{4} e^{-t/4} \det[I_{2+j-k}(\sqrt{t})]_{j,k=1,...,a} = \frac{1}{4} e^{-t/4} \tau^{III'}[a](t; 2) \]

(6.3)

where in both cases the second equality follows from (6.2). Analogous to (6.1) we have

\[ E^\text{hard}_2(0, (0, t)) = \frac{d}{dt} E^\text{hard}_2(0, (0, t)). \]

The large \( a \) limit of (6.2), (6.3) is particularly interesting. Thus according to the Baik-Deift-Johansson theorem [4] (see [9] for a recent simplified proof)

\[ \lim_{a \to \infty} E^\text{hard}_2(0, (0, a^2 - 2a/a(2)^{1/3} s)) = E^\text{soft}_2(0, (s, \infty)) \]

(6.4)

where \( E^\text{soft}_2(0, (s, \infty)) \) denotes the scaled probability of no eigenvalues in the neighbourhood of infinity, and similarly

\[ \lim_{a \to \infty} (2a(a/2)^{1/3}s)p_2(0, a^2 - 2a(a/2)^{1/3}s) = p^\text{soft}_2(0, s) \]

where \( p^\text{soft}_2(0, s) \) denotes the scaled distribution of the largest eigenvalue. Let us then address the task of computing

\[ E^\text{hard}_2(0, (0, a^2 - 2a(a/2)^{1/3}s)) = g^\text{hard}(a; s) \]

(6.5)

using (6.2) and the recurrence scheme of Proposition [3] First it is clear that for large \( a \) and \( s \) of order unity the sequence

\[ \{ e^{-t/4} \tau^{III'}[n](t; 0) \}_{a^2 - 2(a/2)^{1/3}s} \]

consists initially of numbers very small in magnitude. Hence it is necessary to work with high precision arithmetic throughout the calculation to ensure an accurate final result for the final member, which is equal to \( g^\text{hard}(a; s) \). This sequence in turn is calculated in terms of the sequence \( \{P_n\}_{n=0,1,...,a-1} \) as specified by the recurrence in Proposition [3] with \( t \) replaced by \( t/4 \). For the specific value \( s = 0.5 \) the results of Table [4] are thereby obtained.

The data fits well the extrapolation

\[ g^\text{hard}(a; 0.5) = g_0 + \frac{g_1}{a^{2/3}} + \frac{g_2}{a} \]

giving \( g_0 = 0.990543 \) and thus from (6.2) predicting

\[ E^\text{soft}_2(0, (0.5, \infty)) = 0.990543. \]

(6.6)

In fact \( E^\text{soft}_2(0, (s, \infty)) \) is known in terms of a particular Painlevé II transcendent \( q(s) \) [31]. High precision data by way of the values of \( E^\text{soft}_2(0, (0, \infty)), q(0), q'(0) \) to 50 decimals have recently been given [29].
Table 1: Tabulation of $g^{\text{hard}}(a; 0.5)$ as specified by (6.5) in the case $s = 0.5$.

| $a$ | $g^{\text{hard}}(a; 0.5)$ |
|-----|-------------------|
| 60  | 0.991338737       |
| 80  | 0.991201326       |
| 100 | 0.991111203       |
| 120 | 0.991046762       |
| 140 | 0.990997995       |
| 160 | 0.990959574       |

allowing for accurate determination of $E^{\text{soft}}_2$ for general $s$. One finds $E^{\text{soft}}_2(0, (0.5, \infty)) = 0.990544...$, showing us that (6.6) is accurate to $1$ part in $10^6$.

We now turn our attention to a particular example of the $\tau$-function (1.9). Let $p_{N-2,0}(\theta)$ denote the probability density function for the spacing between consecutive eigenvalues in the CUE$_N$ or equivalently $U(N)$. Then as noted in [17], it follows from the definitions that

$$
\left(\frac{2\pi}{N}\right)p_{N-2}^{\text{CUE}}\left(\frac{2\pi X}{N}\right) = \frac{1}{3}(N^2 - 1) \sin^2 \frac{\pi X}{N} \frac{\tau^{VI}[N - 2](e^{2\pi i X/N}; 1, 1, 0; 1)}{\tau^{VI}[N - 2](1; 1, 1, 0; 1)}.
$$

Use of (5.3) shows

$$
\tau^{VI}[N](1; 1, 1, 0; 1) = \frac{(N + 2)^2(N + 1)(N + 3)}{12}
$$

and thus

$$
\left(\frac{2\pi}{N}\right)p_{N-2}^{\text{CUE}}\left(\frac{2\pi X}{N}\right) = \frac{4}{N^2} \sin^2 \frac{\pi X}{N} \tau^{VI}[N - 2](e^{2\pi i X/N}; 1, 1, 0; 1).
$$

According to Proposition 9 the key quantity in computing $\{\tau^{VI}[n]\}$ by recurrence is $\tau^{VI}[1]$. Now, from (5.3) and (5.5)

$$
\tau^{VI}(e^{i\phi}; 1, 1, 0; 1) = e^{-i\phi} F_1(-2, -2; 1; e^{i\phi}) + \frac{1}{60\pi^3} e^{-i\phi}(1 - e^{i\phi})^5 F_1(3, 3; 6; 1 - e^{i\phi})
$$

(identities for the $F_1$ function can be used to check that this quantity is real. Using this in Proposition 9 and again using high precision computing, for the specific value $X = 1/10$ and a sequence of $N$ values, we evaluated (6.8), obtaining the data listed in Table 2.

The limiting distribution

$$
p_2^{\text{bulk}}(X) = \lim_{N \to \infty} \frac{2\pi}{N} p_{N-2}^{\text{CUE}}\left(\frac{2\pi X}{N}\right)
$$

(6.10)

can itself be expressed in terms of a Painlevé transcendent [20] [14] [17]. Moreover its power series about $X = 0$ is known to high accuracy [19], and from this we can compute

$$
p_2^{\text{bulk}}(X) \bigg|_{X=1/10} = 0.032468767196387...
$$

Extrapolating the data of Table 2 using the ansatz

$$
\frac{2\pi}{N} p_{N-2}^{\text{CUE}}\left(\frac{2\pi X}{N}\right) = s_0 + \frac{s_1}{N^2} + \frac{s_2}{N^4}
$$
gives $s_0 = 0.032468767193...$ which agrees with (6.11) to 3 parts in $10^{12}$. 

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\[
\begin{array}{|c|c|}
\hline
N & \frac{2\pi}{N} P_{N-2}(\frac{2\pi X}{N}) \bigg|_{X=1/10} \\
\hline
10 & 0.03215040321 \\
30 & 0.03243339939 \\
50 & 0.03245603495 \\
70 & 0.03246227118 \\
90 & 0.03246483751 \\
\hline
\end{array}
\]

Table 2: Tabulation of the scaled probability density at \( X = 0.1 \) for the spacing between consecutive eigenvalues in the CUE\(_N\).

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