Some Aspect of Certain two Subclass of Analytic Functions with Negative Coefficients Defined by Rafid Operator

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Abstract

In this paper, we recall the subclasses \( R^\delta_{\mu,p}(\alpha; A, B) \) and \( P^\delta_{\mu,p}(\alpha; A, B) \) of analytic functions in the open unit disc. Then the neighborhood properties, integral means inequalities and some results concerning the partial sums of the functions were discussed.

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1. Introduction

Let \( T(p) \) be the class of all p-valent functions of the from

\[
f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \quad (a_k \geq 0, p \in \mathbb{N} = \{1, 2, 3...\}),
\]

which are analytic in the open unit disc \( U = \{z \in \mathbb{C} : |z| < 1\} \). A function \( f \in T(p) \) is called p-valent starlike of order \( \alpha(0 \leq \alpha < p) \), if and only if

\[
\text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < p; z \in U),
\]

we denote by \( T^*(p, \alpha) \) the class of all p-valent starlike functions of order \( \alpha \). Also a function \( f \in T(p) \) is called p-valent convex of order \( \alpha(0 \leq \alpha < p) \), if and only if

\[
\text{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (0 \leq \alpha < p; z \in U),
\]

we denote by \( C(p, \alpha) \) the class of all p-valently convex functions of order \( \alpha \). For more informations about the subclasses \( T^*(p, \alpha) \) and \( C(p, \alpha) \), see [12].

Motivated by Atshan and Rafid see [2], we introduce the following p-valent analogue \( R^\delta_{\mu,p} : T(p) \rightarrow \)
\[ T(p) : \]
For \(0 \leq \mu < 1\) and \(0 \leq \delta \leq 1\),
\[
R_{\mu,p}^\delta f(z) = \frac{1}{\Gamma(p + \delta)(1 - \mu)^p} \int_0^\infty t^{\delta-1} e^{-\left(\frac{t}{1 - \mu}\right)} f(zt) dt
\]  
(1.4)

Then it is easily to deduce the series representation of the function \(R_{\mu,p}^\delta f(z)\) as following:
\[
R_{\mu,p}^\delta (z) = z^\mu - \sum_{k=p+1}^\infty \frac{\Gamma(k + \delta)(1 - \mu)^{k-p}}{\Gamma(p + \delta)} a_k z^k,
\]
where \(\Gamma\) stands for Euler’s Gamma function (which is valid for all complex numbers except the non-positive integers). More operators on the spaces of functions, see [3], [4] and [9].

Let \(f\) and \(g\) be analytic in \(U\). Then we say that the function \(g\) is subordinate to \(f\) if there exists an analytic function in \(U\) such that \(|w(z)| < 1\) (\(\forall z \in U\)) and \(g(z) = f(w(z))\). For this subordination, the symbol \(g(z) \prec f(z)\) is used. In case \(f(z)\) is univalent in \(U\), the subordination \(g(z) \prec f(z)\) is equivalent to \(g(0) = f(0)\) and \(g(U) \subset f(U)\) (see Miller and Mocanu [8]).

For \(-1 \leq B < A \leq 1\) and \(0 \leq \alpha < p\), let \(R_{\mu,p}^\delta(\alpha; A, B)\) be the subclass of functions \(f \in T(p)\) for which:
\[
\frac{z(R_{\mu,p}^\delta f(z))'}{R_{\mu,p}^\delta f(z)} \prec (p - \alpha) \frac{1 + A z}{1 + B z} + \alpha,
\]  
(1.5)

that is, that
\[
R_{\mu,p}^\delta(\alpha; A, B) = \left\{ f \in T(p) : \left| \frac{z(R_{\mu,p}^\delta f(z))'}{R_{\mu,p}^\delta f(z)} - p \right| < 1, z \in U \right\}.
\]

Note that \(\text{Re} \left\{ (p - \alpha) \frac{1 + A z}{1 + B z} + \alpha \right\} > \frac{1 - A + \alpha(A - B)}{1 - B}\).

Also, for \(-1 \leq B < A \leq 1\) and \(0 \leq \alpha < p\), let \(P_{\mu,p}^\delta(\alpha; A, B)\) be the subclass of functions \(f \in T(p)\) for which:
\[
1 + \frac{z(R_{\mu,p}^\delta f(z))''}{(R_{\mu,p}^\delta f(z))'} \prec (p - \alpha) \frac{1 + A z}{1 + B z} + \alpha,
\]  
(1.6)

For (1.5) and (1.6) it is clear that
\[
f(z) \in P_{\mu,p}^\delta(\alpha; A, B) \iff \frac{zf'(z)}{p} \in R_{\mu,p}^\delta(\alpha; A, B)
\]  
(1.7)

The object of the present paper is to investigate the coefficients bounds, neighborhood properties, integral means inequalities and some results concerning partial sums for functions belonging to the subclasses \(R_{\mu,p}^\delta(\alpha; A, B)\) and \(P_{\mu,p}^\delta(\alpha; A, B)\).
2. Neighborhood Results

We assume in the reminder of this paper that, 0 ≤ α < p, 0 ≤ μ < 1, 0 ≤ δ ≤ 1, −1 ≤ B < A ≤ 1, p ∈ N and z ∈ U. Also, we shall need the following two lemmas.

Lemma 1 (see [5]). Let the function f(z) be given by (1.1). Then f ∈ R_{μ,p}^δ(α; A, B), if and only if

\[ \sum_{k=p+1}^{∞} [(1 - B)(k - p) + (A - B)(p - α)] (1 - μ)^{k-p} \frac{Γ(k+δ)}{Γ(p+δ)} a_k ≤ (A - B)(p - α). \] (2.1)

Lemma 2 (see [5]). Let the function f(z) be given by (1.1). Then f ∈ P_{μ,p}^δ(α; A, B), if and only if

\[ \sum_{k=p+1}^{∞} k[(1 - B)(k - p) + (A - B)(p - α)] (1 - μ)^{k-p} \frac{Γ(k+δ)}{Γ(p+δ)} a_k ≤ p(A - B)(p - α). \] (2.2)

Following the earlier investigations of Goodman [6] and Ruscheweyh [10], we recall the ε—neighborhood of a function f of the form (1.1) as following:

\[ N_ε(f) = \left\{ g ∈ T(p) : g(z) = z^p - \sum_{k=p+1}^{∞} b_k z^k, \sum_{k=p+1}^{∞} k |a_k - b_k| ≤ ε \right\}. \] (2.3)

For the identity function e(z) = z, we immediately have

\[ N_ε(e) = \left\{ g ∈ T(P) : g(z) = z^p - \sum_{k=p+1}^{∞} b_k z^k, \sum_{k=p+1}^{∞} k b_k ≤ ε \right\}. \] (2.4)

Theorem 1. If the function f(z) defined by (1.1) is in the class R_{μ,p}^δ(α; A, B), then R_{μ,p}^δ(α; A, B) ⊆ N_ε(e), where

\[ ε = \frac{(p + 1)(A - B)(p - α)}{[(1 - B) + (A - B)(p - α)] (1 - μ)(p + δ)}. \]

Proof. Since f ∈ R_{μ,p}^δ(α; A, B), by using Lemma 1, we find

\[ \frac{[(1 - B) + (A - B)(p - α)] (1 - μ)^{p+δ+1}}{Γ(p+δ)} \sum_{k=p+1}^{∞} ka_k ≤ \sum_{k=p+1}^{∞} [(1 - B)(k - p) + (A - B)(p - α)] (1 - μ)^{k-p} \frac{Γ(k+δ)}{Γ(p+δ)} a_k ≤ (A - B)(p - α). \]

Then, it is clear that

\[ \sum_{k=p+1}^{∞} ka_k ≤ \frac{(p + 1)(A - B)(p - α)}{[(1 - B) + (A - B)(p - α)] (1 - μ)(p + δ)} = ε. \]

This completes the proof.

Theorem 2. If
\[ \epsilon = \frac{\Gamma(p + \delta)p(p + 1)(A - B)(p - \alpha)}{[(1 - B) + (A - B)(p - \alpha)](1 - \mu)\Gamma(p + \delta + 1)} \]

then \( P_{\mu,p}^{\delta}(\alpha, A, B) \subseteq N_\epsilon(\epsilon) \).

**Proof.** For function \( f \in P_{\mu,p}^{\delta}(\alpha, A, B) \), of the form (1.1), from Lemma 2, we find

\[
[(1 - B) + (A - B)(p - \alpha)](1 - \mu)\frac{\Gamma(p + \delta + 1)}{\Gamma(p + \delta)} \sum_{k=p+1}^{\infty} ka_k \leq \sum_{k=p+1}^{\infty} [(k - p)(1 - B) + (A - B)(p - \alpha)](1 - \mu)k^p\frac{\Gamma(k + \delta)}{\Gamma(p + \delta)} ka_k \leq p(A - B)(p - \alpha),
\]

then

\[
\sum_{k=p+1}^{\infty} ka_k \leq \frac{p(A - B)(p - \alpha)}{[(1 - B) + (A - B)(p - \alpha)](1 - \mu)(p + \delta)} = \epsilon,
\]

and the proof is completed.

Moreover, we will determine the neighborhood properties for each of the following (slightly modified) function classes \( R_{\mu,p}^{\delta,\rho}(\alpha, A, B) \) and \( P_{\mu,p}^{\delta,\rho}(\alpha, A, B) \).

A functions \( f \in T(p) \) is said to be in the class \( R_{\mu,p}^{\delta,\rho}(\alpha, A, B) \) if there exists a function \( g \in R_{\mu,p}^{\delta}(\alpha, A, B) \) such that

\[
\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \rho \quad (z \in U; 0 \leq \rho < 1).
\]

Analogously, a function \( f \in T(p) \) is said to be in the class \( P_{\mu,p}^{\delta,\rho}(\alpha, A, B) \) if there exists a function \( g \in P_{\mu,p}^{\delta}(\alpha, A, B) \) such that the inequality (2.5) holds true.

Now, using the same technique of Altintas et al., the neighborhood properties of the subclasses \( R_{\mu,p}^{\delta}(\alpha, A, B) \) and \( P_{\mu,p}^{\delta}(\alpha, A, B) \) are given.

**Theorem 3.** Let \( g \in R_{\mu,p}^{\delta}(\alpha, A, B) \), Suppose also that

\[
\rho_1 = 1 - \frac{\epsilon[(1-B)+(A-B)(p-\alpha)](1-\mu)(p+\delta)}{(p+1)[(1-B)+(A-B)(p-\alpha)](1-\mu)(p+\delta)-(A-B)(p-\alpha)}.
\]

then

\[
N_\epsilon(g) \subseteq P_{\mu,p}^{\delta,\rho_1}(\alpha, A, B).
\]

**Proof.** Assume that \( f \in N_\epsilon(g) \). Then we find from (2.3) we get

\[
\sum_{k=p+1}^{\infty} ka_k \leq \epsilon,
\]

since \( g \in P_{\mu,p}^{\delta}(\alpha, A, B) \), then we have

\[
\sum_{k=p+1}^{\infty} b_k \leq \frac{(A-B)(p-\alpha)}{[(1-B)+(A-B)(p-\alpha)](1-\mu)(p+\delta)},
\]

so that

\[
\left| \frac{f(z)}{g(z)} - 1 \right| \leq \frac{\sum_{k=p+1}^{\infty} |a_k - b_k|}{1 - \sum_{k=p+1}^{\infty} b_k} \leq \frac{\epsilon [(1-B)+(A-B)(p-\alpha)](1-\mu)(p+\delta)}{(p+1)[(1-B)+(A-B)(p-\alpha)](1-\mu)(p+\delta)-(A-B)(p-\alpha)} = 1 - \rho_1,
\]

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provided that \( \rho_1 \) is given precisely by (2.6). Thus, by definition, \( f \in R^{\delta, \rho_1}(\alpha; A, B) \) for \( \rho_1 \) given by (2.6). This evidently completes the proof of Theorem 3.

Another result regarding the subclass \( P^{\delta, p}(\alpha; A, B) \) is given below and the proof is omitted.

**Theorem 4.** If \( g \in P^{\delta, p}(\alpha; A, B) \) and

\[
\rho_2 = 1 - \frac{\epsilon[(1-B)+(A-B)(p-\alpha)](1-\mu)(p+\delta)}{[(1-B)+(A-B)(p-\alpha)](1-\mu)(p+\delta)|1-(A-B)(p-\alpha)},
\]

then

\[
N_\epsilon(g) \subset P^{\delta, \rho_2}(\alpha; A, B).
\]

Now, a third neighborhood result is discussed, for this purpose we define the subclass \( H^{\delta, \alpha, \varphi; A, B} \) which is related to the main subclass \( R^{\delta, \alpha; A, B} \), as following:

A function \( f \in T(p) \) is said to be in the class \( H^{\delta, \alpha, \varphi; A, B} \) if it satisfies the following nonhomogeneous Cauchy-Euler differential equation:

\[
z^2 \frac{d^2 f}{dz^2} + 2(\varphi + 1)z \frac{df}{dz} + \varphi(\varphi + 1)f(z) = (p + \varphi)(p + \varphi + 1)g(z)
\]

\((g \in R^{\delta, \alpha; A, B}; \varphi > -p; )\).

**Theorem 5.** If \( f \in T(p) \) is in the subclass \( H^{\delta, \alpha, \varphi; A, B} \), then

\[
H^{\delta, \alpha, \varphi; A, B} \subset N_\epsilon(g),
\]

where

\[
\epsilon = \frac{(p+1)(A-B)(p-\alpha)}{[(1-B)+(A-B)(p-\alpha)](1-\mu)(p+\delta)} \left( \frac{2(p+\varphi+1)}{p+\varphi+2} \right)
\]

**Proof.** Suppose that \( f \in H^{\delta, \alpha, \varphi; A, B} \) and \( f \) is given by (1.1), then from (2.8) we deduce that

\[
a_k = \frac{(p+\varphi)(p+\varphi + 1)}{(k+\varphi)(k+\varphi + 1)} b_k \quad (k \geq p + 1).
\]

Moreover,

\[
\sum_{k=p+1}^{\infty} k |b_k - a_k| \leq \sum_{k=p+1}^{\infty} k b_k + \sum_{k=p+1}^{\infty} k a_k \quad (a_k \geq 0, b_k \geq 0).
\]

by using (2.10), then (2.11) can be rewritten as following

\[
\sum_{k=p+1}^{\infty} k |b_k - a_k| \leq \sum_{k=p+1}^{\infty} k b_k + \sum_{k=p+1}^{\infty} \frac{(p+\varphi)(p+\varphi + 1)}{(k+\varphi)(k+\varphi + 1)} k b_k.
\]

Next, since \( g \in R^{\delta, \alpha; A, B} \), then assertion (2.1) of the Lemma 1 yields

\[
\sum_{k=p+1}^{\infty} k b_k \leq \frac{(p+1)(A-B)(p-\alpha)}{[(1-B)+(A-B)(p-\alpha)](1-\mu)(p+\delta)}.
\]

Finally, by making use of (2.13) on the right-hand side of (2.12), we find that
Lemma 3 \((7)\). In this section, we shall need the subordination lemma of Littlewood \([7]\).

\[ \sum_{k=2}^{\infty} k |b_k - a_k| \leq \frac{(p + 1)(A - B)(p - \alpha)}{[(1 - B) + (A - B)(p - \alpha)](1 - \mu)(p + \delta)} \left( 1 + \frac{p + \varphi}{p + \varphi + 2} \right) \]
\[ = \frac{(p + 1)(A - B)(p - \alpha)}{[(1 - B) + (A - B)(p - \alpha)](1 - \mu)(p + \delta)} \left( 2(p + \varphi + 1) \right) = \epsilon \]

Thus, \( f \in N_e(g) \). This, evidently, completes the proof of Theorem 5.

A similar result regarding the class \( P_{\mu,p}^\delta(\alpha; A, B) \) can be achieved using the same techniques as performed in Theorem 5, thus it is omitted.

### 3. Integral Means Inequalities

In this section, we shall need the subordination lemma of Littlewood \([7]\).

**Lemma 3 \((7)\).** If the functions \( f \) and \( g \) are analytic in \( U \) with \( g(z) < f(z) \) then

\[
\int_0^{2\pi} |g(re^{i\theta})|^\tau \, d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^\tau \, d\theta \quad (\tau > 0; 0 < r < 1).
\]  

**Theorem 6.** Let \( f \in P_{\mu,p}^\delta(\alpha; A, B) \), and suppose that

\[
f_{p+1}(z) = z^p \frac{(p + 1)(A - B)(p - \alpha)}{[(1 - B) + (A - B)(p - \alpha)](1 - \mu)(p + \delta)} \equiv \phi(z).
\]

then for we have

\[
\int_0^{2\pi} |f(re^{i\theta})|^\tau \, d\theta \leq \int_0^{2\pi} |f_{p+1}(re^{i\theta})|^\tau \, d\theta
\]

\[
(\tau > 0, z = re^{i\theta}, 0 < r < 1))
\]

**Proof.** From lemma 3, it would suffice to show that

\[
1 - \sum_{k=p+1}^{\infty} a_k z^{k-p} < 1 - \frac{(A - B)(p - \alpha)}{[(1 - B) + (A - B)(p - \alpha)](1 - \mu)(p + \delta)} z.
\]

By setting

\[
1 - \sum_{k=p+1}^{\infty} a_k z^{k-p} = 1 - \frac{(A - B)(p - \alpha)}{[(1 - B) + (A - B)(p - \alpha)](1 - \mu)(p + \delta)} w(z).
\]

Then we find that

\[
|w(z)| = \left| \sum_{k=p+1}^{\infty} \frac{[(1 - B) + (A - B)(p - \alpha)](1 - \mu)(p + \delta)}{(A - B)(p - \alpha)} a_k z^{k-p} \right|
\]
\[
\leq |z| \sum_{k=p+1}^{\infty} \frac{[(1 - B) + (A - B)(p - \alpha)](1 - \mu)(p + \delta)}{(A - B)(p - \alpha)} a_k
\]
\[
\leq |z| \leq 1,
\]

by using (2.1). Hence \( f(z) < f_{p+1}(z) \) which readily yields the integral means inequality (3.3).
4. Partial Sums

In this section we will study the ratio of a function of the form (1.1) to its sequence of partial sums defined by $f_1(z) = z$ and $f_m(z) = z^p - \sum_{k=p+1}^{m} a_k z^k$ when the coefficients of $f(z)$ are satisfy the condition (2.1). We will determine sharp lower bounds of $\text{Re} \left( \frac{f(z)}{f_m(z)} \right)$, $\text{Re} \left( \frac{f_m(z)}{f(z)} \right)$, $\text{Re} \left( \frac{f'(z)}{f_m'(z)} \right)$ and $\text{Re} \left( \frac{f_m'(z)}{f'(z)} \right)$.

In what follows, we will use the well-known result

$$\text{Re} \left( \frac{1 - w(z)}{1 + w(z)} \right) > 0 \quad (z \in U),$$

if and only if

$$w(z) = \sum_{k=1}^{\infty} c_k z^k,$$

satisfies the inequality $|w(z)| \leq |z|$.

**Theorem 7.** Let $f \in R^8_{\mu, p}(\alpha; A, B)$, then

$$\text{Re} \left( \frac{f(z)}{f_m(z)} \right) \geq 1 - \frac{1}{c_{m+1}} \quad (z \in U, m \in \mathbb{N}), \quad (4.1)$$

and

$$\text{Re} \left( \frac{f_m(z)}{f(z)} \right) \geq \frac{c_{m+1}}{1 + c_{m+1}} \quad (z \in U, m \in \mathbb{N}), \quad (4.2)$$

where

$$c_k = \frac{[(1 - B)(k - p) + (A - B)(p - \alpha)](1 - \mu)\Gamma(p + \delta + 1)}{\Gamma(p + \delta)(A - B)(p - \alpha)}. \quad (4.3)$$

The estimates in (4.1) and (4.2) are sharp.

**Proof.** Employing the same technique used by Silverman [11]. The function $f \in R^8_{\mu, p}(\alpha; A, B)$ if and only if $\sum_{k=1}^{\infty} c_k z^k \leq 1$. It is easy to verify that $c_{k+1} > c_k > 1$. Thus

$$\sum_{k=p}^{m} a_k + c_{m+1} \sum_{k=m+1}^{\infty} a_k \leq \sum_{k=p+1}^{\infty} c_k a_k < 1. \quad (4.4)$$

Now, setting

$$c_{m+1} \left\{ \frac{f(z)}{f_m(z)} - \left( 1 - \frac{1}{c_{m+1}} \right) \right\} = \frac{1 - \sum_{k=p}^{m} a_k z^{k-p} - c_{m+1} \sum_{k=m+1}^{\infty} a_k z^{k-p}}{1 - \sum_{k=p}^{m} a_k z^{k-p}} = \frac{1 + D(z)}{1 + E(z)},$$

and

$$\frac{1 + D(z)}{1 + E(z)} = \frac{1 - w(z)}{1 + w(z)},$$

then we have

$$w(z) = \frac{E(z) - D(z)}{2 + D(z) + E(z)} = \frac{c_{m+1} \sum_{k=m+1}^{\infty} a_k z^{k-p}}{2 - 2 \sum_{k=p}^{m} a_k z^{k-p} - c_{m+1} \sum_{k=m+1}^{\infty} a_k z^{k-p}}$$
which implies
\[ |w(z)| \leq \frac{c_{m+1} \sum_{k=m+1}^{\infty} a_k}{2 - 2 \sum_{k=p}^{m} a_k - c_{m+1} \sum_{k=m+1}^{\infty} a_k}. \]

Hence \( |w(z)| \leq 1 \), if and only if
\[ \sum_{k=p}^{m} a_k + c_{m+1} \sum_{k=m+1}^{\infty} a_k \leq 1 \]
which is true by (4.4). This readily yields (4.1).

Now consider the function
\[ f(z) = 1 - \frac{z^{m+1}}{c_{m+1}} \]
Thus
\[ \frac{f_m(z)}{f(z)} \frac{f_m'(z)}{f'(z)} = \frac{1 - \sum_{k=p}^{m} a_k z^{k-p} + c_{m+1} \sum_{k=m+1}^{\infty} a_k z^{k-p}}{1 - \sum_{k=p}^{m} a_k z^{k-p}} = \frac{1 - w(z)}{1 + w(z)}, \]
where
\[ |w(z)| \leq \frac{(1 + c_{m+1}) \sum_{k=m+1}^{\infty} a_k}{2 - 2 \sum_{k=p}^{m} a_k + (1 - c_{m+1}) \sum_{k=m+1}^{\infty} a_k}. \]

Now \( |w(z)| \leq 1 \), if and only if
\[ \sum_{k=p}^{m} a_k + c_{m+1} \sum_{k=m+1}^{\infty} a_k \leq 1, \]
which readily implies the assertion (4.2). The estimate in (4.2) is sharp with the extremal function \( f(z) \) given by (4.5). This completes the proof of the theorem 7.

Following similar steps to that followed in Theorem 7, we can state the following theorem
**Theorem 8.** Let \( f \in \mathcal{R}_{\mu,p}(\alpha; A, B) \), then
\[ \text{Re} \left( \frac{f'(z)}{f_m'(z)} \right) \geq 1 - \frac{m + 1}{c_{m+1}} \quad (z \in U, m \in \mathbb{N}), \quad (4.6) \]
and
\[ \text{Re} \left( \frac{f_m'(z)}{f'(z)} \right) \geq \frac{c_{m+1}}{m + 1 + c_{m+1}} \quad (z \in U, m \in \mathbb{N}), \quad (4.7) \]
In both cases, the extremal function \( f(z) \) is as defined in (4.5).
**Proof.** We prove only (4.6), which is similar in spirit to the proof of theorem 7 and similarly we proof (4.7). We write
\[
\frac{c_{m+1}}{m+1} \left\{ \frac{f'(z)}{f_m'(z)} - \left( 1 - \frac{m+1}{c_{m+1}} \right) \right\} = \frac{1 - \sum_{k=p}^{m} a_k z^{k-1} - \frac{c_{m+1}}{m+1} \sum_{k=m+p}^{\infty} a_k z^{k-1}}{1 - \sum_{k=p}^{m} a_k z^{k-1}} = \frac{1 + D(z)}{1 + E(z)}
\]

and \( \frac{1+D(z)}{1+E(z)} = \frac{1-w(z)}{1+w(z)} \), then we have

\[
w(z) = \frac{E(z) - D(z)}{2 + D(z) + E(z)} = \frac{\frac{c_{m+1}}{m+1} \sum_{k=m+p}^{\infty} a_k z^{k-1}}{2 - 2 \sum_{k=p}^{m} a_k z^{k-1} - \frac{c_{m+1}}{m+1} \sum_{k=m+p}^{\infty} a_k z^{k-1}}
\]

which implies

\[
|w(z)| \leq \frac{\frac{c_{m+1}}{m+1} \sum_{k=m+p}^{\infty} a_k}{2 - 2 \sum_{k=p}^{m} a_k - \frac{c_{m+1}}{m+1} \sum_{k=m+p}^{\infty} a_k}.
\]

Hence \( |w(z)| \leq 1 \), if and only if

\[
\sum_{k=p}^{m} a_k + \frac{c_{m+1}}{m+1} \sum_{k=m+p}^{\infty} a_k \leq 1.
\]

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