COMPACTNESS OF HAMILTONIAN STATIONARY LAGRANGIAN SUBMANIFOLDS IN SYMPLECTIC MANIFOLD

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Abstract. In this work, we prove a compactness theorem on the space of all Hamiltonian stationary Lagrangian submanifolds in a compact symplectic manifold with uniform bounds on area and total extrinsic curvature. This generalizes the compactness theorems in \cite{8} and \cite{6}.

1. Introduction

Let \((M, \omega, h, J)\) be a \(2n\)-dimensional symplectic manifold with a symplectic 2-form \(\omega\), an almost complex structure \(J\) and a compatible metric \(h\). An immersed submanifold \(L\) in \(M\) is Lagrangian if \(\dim L = n\) and \(\omega|_L = 0\), and a Lagrangian immersion is Hamiltonian stationary (HSL) if it is a critical point of the volume functional among all Hamiltonian variations. The notion was first introduced in \cite{15}, \cite{16}. A compact and graded Lagrangian in a Calabi-Yau manifold is Hamiltonian stationary if and only if it is special \cite{19}. Thus HSL submanifolds are natural generalization of special Lagrangians submanifolds for general symplectic manifolds.

Examples of HSL submanifolds include the totally geodesic \(\mathbb{RP}^n\) in \(\mathbb{CP}^n\) and the flat tori \(S^1(a_1) \times \cdots \times S^1(a_n)\) in \(\mathbb{C}^n\) \cite{15}, \cite{16}. In two dimension, Schoen and Wolfson constructed in \cite{17} HSL surfaces with conical singularity in any Kähler surfaces, which are also area minimizer in its Lagrangian homology class. On the other hand, using perturbation methods, Joyce, Lee and Schoen proved in \cite{11} the existence of closed HSL submanifolds in every compact symplectic manifolds (see also the previous works \cite{4, 13} and a family version in \cite{14}). A large classes of examples of HSLs are constructed using techniques in integrable system. We refer the readers to the bibliography of \cite{6} for a more comprehensive list of references.

Using the regularity theory developed in \cite{7}, Chen and Warren obtain the smoothness estimates and small Willmore energy regularity in \cite{8} and prove a compactness theorem for HSL submanifolds in \(\mathbb{C}^n\) with uniformly bounded areas and total extrinsic curvatures in \(\mathbb{C}^n\). The regularity and compactness results in \cite{7} \cite{8} rely on the assumption that the ambient space is \(\mathbb{C}^n\) since it is used, in an essential way, that the Lagrangian phase angle \(\Theta\) can be written as \(\arctan \lambda_1 + \cdots + \arctan \lambda_n\) for the graphic representation \((x, Du)\), where \(\lambda_i\)'s are the eigenvalues of \(D^2u\).

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In another direction, using the bubble tree convergence for conformal mappings with uniformly bounded area and Willmore energy, we proved in [6] a compactness theorem on the space of compact HSL surfaces with bounded area, genus and Willmore energy in a Kähler surfaces. A sequence of such HSL surfaces would converges smoothly away from finitely many points to a branched HSL immersion. Unlike the case for $\mathbb{C}^n$ where one make use of the Lagrangian angle, in [6] we exploit the fact that the mean curvature one form of a HSL immersion in a Kähler manifold satisfies a first order elliptic system of "Hodge type"; this is also essential in the regularity results in [17] for a Lagrangian area minimizers in two dimensions.

Our main result is

**Theorem 1.1.** Let $(M, \omega, h)$ be a compact $2n$-dimensional symplectic manifold, where $h$ is a Hermitian metric compatible with $\omega$. Let $C_V, C_A$ be positive constants and let $(L_k)_{k=1}^\infty$ be a sequence of connected compact Hamiltonian stationary Lagrangian immersion in $M$ so that

\[
\text{Vol}(L_k) \leq C_V, \quad \|A_k\|_{L^\infty} \leq C_A, \quad \forall k \in \mathbb{N}.
\]

Then either $(L_k)$ converges to a point, or there is a finite set $S$ so that a subsequence $(L_{k_i})$ of $(L_k)$ converges smoothly locally graphically to a Hamiltonian stationary Lagrangian immersion $L$ on $M \setminus S$ and

\[
\text{Vol}(L) = \lim_{i \to \infty} \text{Vol}(L_{k_i}).
\]

Also, the closure $\overline{L}$ in $M$ is connected and admits a structure of a Lagrangian varifold, and is Hamiltonian stationary in the sense that the generalized mean curvature vector $\vec{H}$ of the varifold $(\overline{L}, \mu_L)$ satisfies

\[
\int_M h(J \nabla^M f, \vec{H}) \, d\mu_L = 0, \quad \forall f \in C^\infty_c(M).
\]

This generalizes the compactness theorems in [8], [6] to HSLs in any compact symplectic manifolds with metrics compatible with the symplectic structures and the almost complex structures. The main step in proving compactness is to derive a local a-priori estimates for HSLs with small total curvature. In [8], [6], this is done by representing the Hamiltonian stationary conditions as a coupled elliptic systems of lower order. This simplification is not available for HSLs in a general symplectic manifold. Therefore, the methods in [8], [6] do not directly yield a proof of Theorem 1.1. In a Darboux coordinates, if one represents the HSL as a gradient graph $(x, Du(x))$ of a function $u$, then the Hamiltonian stationary condition is captured by a fourth order PDE on $u$. In a recent work [2], Bhattacharya, Chen and Warren have studied a fourth order elliptic PDE of double divergence form and derive a regularity results. Together with the constructions of Darboux coordinates with estimates in [11] (See also subsection 2.2), it is proved that a $C^1$ HSL in a general symplectic manifolds is smooth. In this work, we apply the results in [2] to derive a-priori estimates (Theorem 4.2) and an $\epsilon$-regularity result (Theorem 4.7) for HSL submanifolds, which is essential to the proof of Theorem 1.1.
The paper is organized as follows. In section 2.1, we discuss some background in Lagrangian submanifolds, which includes the definition of HSL and the Darboux coordinates with estimates constructed in [11]. In section 3, we use the regularity results in [2] to derive local $C^k$ estimates for functions which satisfy a fourth order elliptic equation in double divergence form. In section 4, we prove a $\epsilon$-regularity theorem. Theorem 1.1 is proved in sections 5 and 6.

In the following, given any vector space $W$ with a metric $h$, we denote $|w|_h = \sqrt{h(w,w)}$ for $w \in W$. When $W$ is the Euclidean space, we write $|w| = |w|_{h_0}$, where $h_0$ is the Euclidean metric on $W$. Given any open subset $\Omega$ of $\mathbb{R}^n$ and any function $u : \Omega \rightarrow \mathbb{R}$ on $\Omega$, we use $\|u\|_{C^{k,\alpha}(\Omega)}$, $\|u\|_{L^p(\Omega)}$ and $\|u\|_{W^{k,p}(\Omega)}$ to denote respectively the $C^{k,\alpha}$, $L^p$ and $W^{k,p}$ norms of $u$. For any $k \in \mathbb{N}$, $x \in \mathbb{R}^k$ and $r > 0$, the open ball with radius $r$ and center $x$ in $\mathbb{R}^k$ is denoted $B^k_r(x)$. We also write $B^k_r$ for $B^k_r(0)$.

2. Background

2.1. Hamiltonian stationary Lagrangian submanifolds. Let $(M, h)$ be a Riemannian manifold. Unless otherwise specified, throughout this paper a submanifold of $M$ is an immersed submanifold, that is, a submanifold of $M$ and a proper map. Let $g$ be a symplectic manifold. Assume that $N$ is compact (resp. connected), if $N$ is compact (resp. connected), and is proper if $\iota$ is a proper map.

Assume that $M$ is compact. By the Nash embedding theorem, there is an isometric embedding of $(M, h)$ into $\mathbb{R}^K$ for some positive integer $K$. Fixing such an embedding, the definition of varifolds on $M$ is given in [1]. Let $U$ be an open subset of $M$, and let $L$ be a properly immersed $k$-dimensional submanifold of $U$ given by the immersion $\iota: N \rightarrow U$. Let $g = \iota^*h$ be the induced metric on $N$ and let $dV_g$ be the volume form. The volume of a submanifold $L$ is

$$\text{Vol}(L) = \int_N dV_g.$$  

(2.1)

Let $\pi_k : G_kM \rightarrow M$ be the Grassmann bundle on $M$, where each fiber at $x \in M$ is the Grassmann manifold of $k$-dimensional subspaces of $T_xM$. Given any $k$-dimensional immersed submanifold $L$ of $M$, let $G_{\iota} : N \rightarrow G_kM$ be the Gauss map given by $x \mapsto (\iota_*)_T T_xN$. Then $(L, \iota)$ is given a structure of a varifold $\mu_L$ by pushing forward: $|L| := (G_{\iota})_* dV_g$, or

$$|L|(f) := \int_N f \circ G_{\iota} \ dV_g, \ \forall f \in C(G_kM).$$

(2.2)

For any integral varifold $V$, the weight measure $\mu_V$ on $M$ is given by

$$\int_M \phi d\mu_V := \int_{G_kM} \phi(x) dV, \ \forall \phi \in C(M).$$

(2.3)

Let $\mu_L := mu_{|L|}$. Clearly $\mu_L$ and $|L|$ depend not only on $L$ but also on $\iota$. Let $(M, \omega, h, J)$ be a $2n$-dimensional symplectic manifold, where $\omega$ is the symplectic form, $J$ is an almost complex structure and $h$ is a Riemannian metric. We assume that
\[ h(X, Y) = \omega(X, JY), \quad h(JX, JY) = h(X, Y) \]

for all tangent vectors \( X, Y \) at the same point.

An \( n \)-dimensional immersed submanifold \( L \) in \( M \) given by an immersion \( \iota \) is called Lagrangian if \( \iota^*\omega = 0 \). Given an immersed Lagrangian submanifold \( L \), (2.4) implies that \( J \) maps tangent vectors of \( L \) to normal vectors. Thus the second fundamental form of \( L \) can be written as

\[ A : T_pL \times T_pL \times T_pL \to \mathbb{R}, \quad A(X, Y, Z) = h(\nabla_X \tilde{Y}, JZ) = \omega(Z, \nabla_X \tilde{Y}), \]

where \( \tilde{Y} \) is any local extension of \( Y \). The mean curvature one form of \( L \) is a one form on \( N \) defined by

\[ \alpha_x(X) = \omega_{\iota(x)}(\iota_*X, \tilde{H}(x)) \]

for any \( x \in N \) and \( X \in T_xN \). Here \( \tilde{H} \) is the mean curvature vector of the immersion \( \iota : N \to M \).

Next we give the definition of a Hamiltonian stationary Lagrangian immersion. As in [8], we first give the definition for general Lagrangian integral \( n \)-varifolds (An integral \( n \)-varifold \( V \) on \( (M, \omega, h) \) is called Lagrangian if \( \omega|_{T_xV} = 0 \) for \( V \)-almost every \( x \)).

**Definition 2.1.** A Lagrangian integral \( n \)-varifold \( V \) in \( U \) is called Hamiltonian stationary Lagrangian (HSL) in \( U \) if

\[ \int_U h(J\nabla^M f, \tilde{H})d\mu_V = 0, \quad \forall f \in C^\infty_c(U), \]

where \( \tilde{H} \) is the generalized mean curvature vector of the immersion \( \iota : N \to M \).

**Definition 2.2.** Let \( L \) be a Lagrangian immersion into an open subset \( U \) of \( M \) given by a proper immersion \( \iota \), and let \( |L| \) be the integral \( n \)-varifold associated with \( L \) as in (2.2). \( L \) is called a Hamiltonian stationary Lagrangian (HSL) immersion in \( U \) if \( |L| \) is HSL as in Definition 2.1.

Let \( \iota : N \to U \) be a proper Lagrangian immersion. For any \( f \in C^\infty_c(U) \), let \( \bar{f} = f \circ \iota \). By (2.3), (2.4) and (2.6), if the immersed submanifold \( (L, \iota) \) is HSL in \( U \), then

\[ \int_N (d\bar{f}, \alpha)_g dV_g = 0, \]

where \( \alpha \) is the mean curvature one form of \( L \) defined in (2.6). Let \( L \) be an embedded Lagrangian submanifold on \( M \). Since every smooth function with compact support \( \psi \) on \( L \) can be extended to \( f \in C^\infty_c(M) \), \( L \) is HSL if and only if

\[ \int_N (d\psi, \alpha)_g dV_g = 0, \quad \forall \psi \in C^\infty_c(N). \]

This implies that \( \alpha \) is co-closed. That is,

\[ d^*\alpha = 0, \]
COMPACTNESS OF HSL

where \( d^* = \pm * d * \) is the formal adjoint of the exterior derivative \( d \) with respect to the metric \( g \).

Let \( L \) be an immersed submanifold in \( M \) given by an immersion \( \iota : N \to M \) and let \( U \subset M \) be an open set in \( M \). Decompose \( \iota^{-1}(U) \) as connected components

\[
\iota^{-1}(U) = \bigsqcup_i V_i.
\]

(2.9)

If \( \iota|_{V_i} \) is an embedding for each \( i \), we say that each \( L_i = \iota(V_i) \) is an embedded connected component of \( U \cap L \) and that \( \iota \) splits into embedded connected components on \( U \).

As in [8, Proposition 2.2], if \( L \) is a proper immersion, for each \( p \in L \) there is an open neighborhood \( U \) of \( p \) so that \( L \cap U \) splits into finitely many embedded connected components. We will have a more precise statement in Proposition 4.6.

Proposition 2.3. Let \( L \) be a Lagrangian submanifold in an open subset \( U \) of \( M \) given by a proper immersion \( \iota : N \to U \). Then \( L \) is HSL as in Definition 2.2 if and only if its mean curvature one form \( \alpha \) is co-closed.

Proof. The proof is similar to that of [8, Proposition 2.5 (1)] and we only sketch the proof for the direction \((\Rightarrow)\). Let \( p \in L \). By [8, Proposition 2.2], let \( W \) be an open neighborhood of \( p \) in \( U \) so that \( \iota^{-1}(W) \) splits into finitely many connected components \( \iota|_{E_i} : E_i \to \Sigma_i \), where \( \Sigma_i = \iota(E_i) \). We recall that \( q \in W \cap L \) is called an embedded point of \( L \) if there is an open neighborhood \( V_q \) of \( q \) in \( W \) so that \( L \cap V_q \) is an embedded submanifold. Let \( \mathcal{E} \subset L \cap W \) be the set of all embedded points of \( L \). Using embeddedness, one can argue as in above that \( d^* \alpha = 0 \) on \( \iota^{-1}(\mathcal{E}) \) using Definition 2.2. For each \( i \), one can prove that \( \mathcal{E} \cap \Sigma_i \) is dense in \( \Sigma_i \). Hence \( d^* \alpha = 0 \) on \( \iota^{-1}(W) \). Since \( p \in L \) is arbitrary, \( \alpha \) is co-closed.

In particular, the Hamiltonian stationary condition holds on each embedded connected components.

Corollary 2.4. Let \( L \) be a properly immersed HSL in an open subset \( U \) of \( M \). Let \( W \) be an open subset of \( U \) such that \( W \cap L \) splits into embedded connected components \( \iota|_{E_i} : E_i \to \Sigma_i \). Then for each \( i \),

\[
\int_W h(J\nabla^M f, \tilde{H}_i) d\mu_{\Sigma_i} = 0, \quad \forall f \in C^\infty_c(W).
\]

(2.10)

Here \( \tilde{H}_i \) is the mean curvature vector of \( \Sigma_i \).

2.2. Darboux coordinates with estimates. In this subsection we recall the Darboux coordinates with estimates constructed in [11]. Let \((M, \omega)\) be a compact symplectic \( 2n \)-manifold, and let \( h \) be a Riemannian metric on \( M \) compatible with \( \omega \). Let \( \mathbb{U} \) be the \( U(n) \) frame bundle of \( M \): that is, for each \( p \in M \), the fiber \( \mathbb{U}_p \) consists of unitary linear mappings \( v : \mathbb{R}^{2n} \to T_p M \).

We recall Proposition 3.2 in [11]:

Proposition 2.5. For small \( \epsilon > 0 \), we can choose a family of embeddings \( \Upsilon_{p,v} : B^2_{\epsilon} \to M \) depending smoothly on \((p,v) \in \mathbb{U} \) such that for all \((p,v) \in \mathbb{U} \), we have
Definition 2.6. Let \( p \in M \) and \( 0 < r \leq 1 \). The Darboux ball at \( p \) with radius \( r \) is
\[
\mathcal{B}_r(p) := \mathcal{Y}_{p,v}(B^{2n}_r).
\]

By (ii) in Proposition 2.5, \( \mathcal{B}_r(p) \) is well-defined, independent of \( v \in U_p \). For each \( R \geq 1 \) and \( 0 < t \leq R^{-1} \), we define \( t : B^{2n}_R \to B^{2n}_1 \) as \( z \mapsto tz \). The following is proved in [11, Proposition 3.4].

Proposition 2.7. There exists \( K_0, K_1, K_2, \ldots \), depending only on \((M, \omega, h)\) such that for all \((p, v) \in U\) and \( t \in (0, R^{-1}]\), the metric \( h^t_{p,v} = t^{-2}(\mathcal{Y}_{p,v} \circ t)^*h \) satisfies on \( B^{2n}_R \)
\[
|h^t_{p,v} - h_0| \leq K_0t \quad \text{and} \quad |\partial_t h^t_{p,v}| \leq K_j t^j \quad \text{for} \quad j = 1, 2, \ldots.
\]

Again, by scaling \( \omega, h \) if necessary, we assume that
\[
\max\{K_0, K_1\} \leq \epsilon_1 < \max\{1/2, \epsilon_1(1/2, n)\},
\]
where \( \epsilon_1(1/2, n) \) is given in [2, Lemma 3.2]. By the first inequality in (2.12) and \( K_0 < 1/2 \),
\[
B^{M}_{r/2}(p) \subset \mathcal{B}_r(p) \subset B^M_s(p)
\]
for all \( p \in M \) and \( r \in (0,1) \), where \( B^M_s(p) \) is the metric ball with radius \( s \) in \( (M, h) \).

2.3. Convergence of immersed submanifolds. We recall the definition of smooth convergence as local graphs of a sequence of immersed submanifolds introduced in [8] (see also [9], [18] for similar definitions for sequences of embedded hypersurfaces).

Definition 2.8. Let \((\Sigma_k)_{k=1}^\infty\) be a sequence of properly immersed submanifolds in an open subset \( U \) of a smooth manifold \( M \) given by a proper immersion \( \iota_k : N_k \to U \) for each \( k \). Let \( S \) be an integral varifold in \((M, h)\) and \( m \in \mathbb{N} \cup \{\infty\} \). We say that \((\Sigma_k)_{k=1}^\infty\) converges graphically in \( U \) to \( S \) in \( C^m \)-topology, if there is \( I \in \mathbb{N} \cup \{0\} \) such that

- \( \Sigma_k \) splits into \( I \) embedded connected components \( \iota_k|_{E^i_k} : E^i_k \to \Sigma^i_k \), \( i = 1, \ldots, I \).
  
  Here \( E^i_k \) is open in \( N_k \) and \( \iota_k|_{E^i_k} \) is an embedding onto \( \Sigma^i_k \), and
- \( S \cap U = \sum^\cup \cdots \cup \sum^I \), where each \( \sum^i \), \( i = 1, \ldots, I \), is an embedded submanifold in \( U \).

Moreover, for each \( i = 1, \ldots, I \), there is \( p_i \in \sum^i \) such that \( \Sigma^i_k \), \( k \in \mathbb{N} \), and \( \sum^i \) are graphs of smooth mappings \( X^i_k \) and \( X^i \) respectively, where \( X^i_k \) and \( X^i \) are defined on some open subsets of \( T_{p_i} \sum^i \), such that \( X^i_k \) converges in \( C^m \) norm to \( X^i \) as \( k \to \infty \).

We remark when \( I = 0 \), it is understood that \( \Sigma_k = S \cap U = \emptyset \).

Definition 2.9. Let \((\Sigma_k)_{k=1}^\infty\) be a sequence of properly immersed submanifolds in \( M \), and let \( V \) be an open subset of \( M \). We say that \((\Sigma_k)_{k=1}^\infty\) converges in \( V \) smoothly as local graphs to an integral varifold \( S \), if for all \( q \in V \), there is an open neighborhood \( U \) of \( q \) contained in \( V \) such that \((\Sigma_k \cap U)_{k=1}^\infty\) converges graphically to \( S \cap U \) in \( C^\infty \)-topology.
3. Estimates for double divergence equation

In [2], the authors study the following fourth order equation in double divergence form

\[ \partial_x \partial_x F^{ij}(x, Du, D^2 u) = \partial_x a^k(x, Du, D^2 u) - b(x, Du, D^2 u) \]

where \( F^{ij}, a^k \) and \( b \) are smooth functions in \( x, Du, D^2 u \), defined in a convex neighborhood \( U \subset B^n_r \times \mathbb{R}^n \times S^{n \times n} \), where \( S^{n \times n} \) is the space of \( n \times n \) symmetric matrices. A \( W^{2,\infty} \) function \( u \) on \( B^n_r \) is called a weak solution to (3.1) if

\[ \int_{B^n_r} \left[ F^{ij} \eta_{ij} + a^k \eta_k + b \eta \right] dx = 0, \quad \text{for all } \eta \in C^\infty_c(B^n_r). \]

**Definition 3.1.** Let \( \Lambda > 0 \). We say that (3.1) is \( \Lambda \)-uniform on \( U \) if the standard Legendre ellipticity condition

\[ \frac{\partial F^{jl}}{\partial u_{ik}}(\xi) \sigma_{ij} \sigma_{kl} \geq \Lambda |\sigma|^2, \quad \forall \sigma \in S^{n \times n} \]

is satisfied for any \( \xi \in U \).

For each \( p = 1, \ldots, n \), let \( h_p = h e_p \),

\[ \eta^{-h_p}(x) := \frac{\eta(x - h_p) - \eta(x)}{h}, \]

and \( f := u^{h_p} \). Define

\[ \xi_0 = (x, Du(x), D^2 u(x)), \]
\[ \xi_h = (x + h_p, Du(x + h_p), D^2 u(x + h_p)), \]
\[ \bar{V} = \xi_h - \xi_0. \]

One can check that (3.2) implies

\[ \int_{B^n_r} \left( \beta^{ij,kl} f^{ij} \eta_{jl} + \gamma^{il} \eta_{jl} + \psi^{k,\eta_{-h_p}} + \zeta \eta^{-h_p} \right) dx = 0 \quad \forall \eta \in C^\infty_c(B^n_{r-h}), \]

where

\[ \beta^{ij,kl}(x) = \int_0^1 \frac{\partial F^{jl}}{\partial u_{ik}}(\xi_0 + tv) dt, \]
\[ \gamma^{jl}(x) = \int_0^1 \left( \frac{\partial F^{jl}}{\partial u_k}(\xi_0 + tv) f_k + \frac{\partial F^{jl}}{\partial x_p}(\xi_0 + tv) \right) dt, \]
\[ \psi^{k}(x) = a^k(x, Du, D^2 u), \]
\[ \zeta(x) = b(x, Du, D^2 u). \]

In the following, we assume that \( u \) is at least \( C^2 \) and satisfies (3.2). The following propositions are the key steps in proving the regularity results in [2].

**Proposition 3.2.** Assume that \( u \in C^2,\alpha(B^n_r) \) satisfies (3.4), then for all \( 0 < r' < r \) and \( 0 < \alpha' < \alpha \) we have \( u \in C^{3,\alpha'}(B^{n}_{r'}) \) and

\[ \| D^3 u \|_{C^{\alpha'}(B^{n}_{r'})} \leq C(\Lambda, \alpha, \alpha', r, r', \| u \|_{W^{2,\infty}(B^{n}_{r})}, \| \Psi \|_{C^{\alpha}(B^{n}_{r})}) \]
where for simplicity we write
$$\Psi = ((\beta_{ij,kl}^0)_i,j,k,l, (\gamma_{jl}^p)_j,l,p, (\psi^k)_k, \zeta).$$

and
\begin{align*}
\beta_{ij,kl}^0(x) &= \frac{\partial F_{jl}^i}{\partial u_{ik}}(x, Du, D^2 u), \\
\gamma_{jl}^p(x) &= \frac{\partial F_{jl}^i}{\partial u_{ik}}(x, Du, D^2 u)u_{pk} + \frac{\partial F_{jl}^i}{\partial x_p}(x, Du, D^2 u).
\end{align*}

\textbf{Proof.} We first assume $r' = 1/5$ and $r = 1$. Without rewriting the whole proof again, we merely point out that in the proof of [2, Proposition 2.3], the constants used have the following dependence:
\begin{align*}
C_3 &= \|\beta_{ij,kl}\|_{C^\alpha(B^n_1)}, \\
C_4 &= \|\beta_{ij,kl}\|_{C^\alpha(B^n_1)}, \\
C_5 &= C(n)\|\gamma_{jl}^i\|_{C^\alpha(B^n_1)}, \\
C_6 &= C(n)\|\psi^k\|_{C^\alpha(B^n_1)}, \\
C_7 &= C(n)\|\zeta\|_{C^\alpha(B^n_1)}, \\
C_2 &= C(\Lambda) \\
C_1 &= C_1(n) \\
\tilde{\alpha} &= 1 - \delta, \\
\tilde{q} &= \frac{n}{2 - 2\tilde{\alpha}}, \\
\tilde{K} &= C(n)(\|\gamma_{jl}^i\|^2_{L^{2\tilde{q}}(B^n_1)} + \|\psi^k\|^2_{L^{2\tilde{q}}(B^n_1)} + \|\zeta\|^2_{L^{2\tilde{q}}(B^n_1)}), \\
r_0 &= r_0(C_1, \Lambda, n, \delta), \\
C_8 &= C_8(C_1, n, \delta), \\
C_9 &= C_9(C_1, C_8, r_0, \Lambda, \|D^2 f\|_{L^2(B^n_{1/2})}), \\
C_{10} &= C_{10}(C_4, C_5, C_7, \Lambda, \alpha, \delta, r_0, \|D^2 f\|^2_{L^2(B^n_{1/2})}, \tilde{K})
\end{align*}

Thus by Proposition 2.1 and Proposition 2.3 in [2],
\begin{equation}
\|D^3 u\|_{C^{\alpha - \delta/2}(B^{1/5}_1)} \leq C(\Lambda, \alpha, \delta, \|u\|_{W^{2,\infty}(B^n_1)}, \|\Psi\|_{C^\alpha(B^n_1)}),
\end{equation}

In general, given $0 < r' < r \leq 1$ and $x_0 \in B^n_{r'}$, we scale the ball $B^n_{r-r'}(x_0)$ to radius 1 and obtain, via (3.9),
\begin{equation}
\|D^3 u\|_{C^{\alpha - \delta/2}(B^{1/5}_{r-r'})} \leq C(\Lambda, \alpha, \delta, r, r', \|u\|_{W^{2,\infty}(B^n_1)}, \|\Psi\|_{C^\alpha(B^n_1)}),
\end{equation}

Since $x_0 \in B^n_{r'}$ is arbitrary, (3.6) is proved. \qed

From the above proposition, we obtain
Proposition 3.3. For each $0 < r' < r \leq 1$, $0 < \alpha' < \alpha < 1$ and $k \geq 3$, assume that $u \in C^{2,\alpha}(B^n_r)$ satisfies (3.2) and $(F^2)$ satisfies the Legendre ellipticity condition (3.3), then $u$ is smooth in $B_r$ and

\begin{equation}
\|D^k u\|_{C^{\alpha'}(B^n_k)} \leq C(k, \Lambda, \alpha, \alpha', r, r', \|u\|_{W^{2,\infty}(B^n_r)}, \|(\tilde{F}, \tilde{a}, b)\|_{C^{k-3}(B^n_r)}).
\end{equation}

Proof. That $u$ is smooth is proved in [2]. Taking the limit $h \to 0$ in (3.5), we obtain (writing $f = u_p$)

\begin{equation}
\int_{B^n_r} \left[ (\beta_0^{ij,kl} f_{ij} + \gamma_j^l) \eta_{jl} + \partial_p \psi^k \eta_k + \partial_p \zeta \eta \right] dx = 0, \quad \forall \eta \in C_c^\infty(B^n_r).
\end{equation}

This is an equation of the form (3.2) with the leading term

\begin{equation}
F^{ij}_{(p)}(x, Df, D^2 f) = \beta_0^{ij,kl} f_{ij} + \gamma_j^l
\end{equation}

and in particular

\begin{equation}
\frac{\partial F^{ij}_{(p)}}{\partial f_{ik}} = \beta_0^{ij,kl}.
\end{equation}

Hence (3.11) also satisfies the $A$-uniform condition (3.3).

In general, one can inductively show that for any $p_1, \ldots, p_m$, the function $f = u_{p_1 \cdots p_m}$ satisfies

\begin{equation}
\int_{B^n_r} \left[ (\beta_0^{ij,kl} f_{ik} + \gamma_j^{l(p_1 \cdots p_m)}) \eta_{ij} + \psi_k^{p_1 \cdots p_m} \eta_k + \zeta_{p_1 \cdots p_m} \eta \right] dx = 0, \quad \forall \eta \in C_c^\infty(B^n_r),
\end{equation}

here $\gamma_j^{l(p_1 \cdots p_m)}$ is defined inductively by

\begin{equation}
\gamma_j^{l(p_1 \cdots p_m)} = (\gamma_j^{l(p_1 \cdots p_{m-1})})_{x_{pm}} + (\beta_0^{ij,kl})_{x_{pm}} u_{ikp_1 \cdots p_m}.
\end{equation}

By Proposition 3.2 we obtain

\begin{equation}
\|D^2 f\|_{C^{\alpha'}(B^n_r)} \leq C(\Lambda, \alpha, \alpha', r, r', \|f\|_{W^{2,\infty}(B^n_r)}, \|\Psi_{(p_1 \cdots p_m)}\|_{C^{\alpha}(B^n_r)}),
\end{equation}

where

\begin{equation}
\Psi_{(p_1 \cdots p_m)} = ((\beta_0^{ij,kl})_{i,j,k,l}, (\gamma_j^{l(p_1 \cdots p_m)})_{j,l,p_1 \cdots p_m}, (\partial_p \psi^k_{p_1 \cdots p_m})_{p_1 \cdots p_m}, \partial_{p_1 \cdots p_m} \zeta).
\end{equation}

Arguing inductively from (3.8), (3.15) that

\begin{equation}
\|\Psi_{p_1 \cdots p_m}\|_{C^{\alpha}(B^n_r)} \leq C(m, \|\tilde{a}\|_{C^m(B^n_r)}, \|\tilde{F}\|_{C^m(B^n_r)}, \|b\|_{C^m(B^n_r)}, \|u\|_{C^{m+2,\alpha}(B^n_r)}).
\end{equation}

Hence we have for any $0 < r' < r < 1$ and $0 < \alpha' < \alpha < 1$,

\begin{equation}
\|u\|_{C^{m+3,\alpha'}(B^n_r)} \leq C(m, \Lambda, r, r', \alpha, \alpha', \|(\tilde{F}, \tilde{a}, b)\|_{C^m(B^n_r)}, \|u\|_{C^{m+2,\alpha}(B^n_r)}).
\end{equation}

The proposition is then proved by choosing

\[ r' = r_m < r_{m-1} < \cdots < r_1 < r_0 = r, \]
\[ \alpha' = \alpha_m < \alpha_{m-1} < \cdots < \alpha_1 < \alpha_0 = \alpha \]

and applying (3.18) $m - 1$ times (with $r, r', \alpha, \alpha'$ replaced by $r_{j-1}, r_j, \alpha_{j-1}, \alpha_j$ respectively) and lastly Proposition 3.2. \qed
4. Local Calculations (Symplectic)

In this section, we represent locally a Lagrangian immersion as a gradient graph in a Darboux coordinates chart given in Proposition 2.5. We apply the estimates in the previous section. Under a smallness assumption on the $L^n$-norm of the second fundamental form, we derive a $\epsilon$-regularity result (Theorem 4.7).

Let $h = h_{p,\nu}^t$ be a Riemannian metric on $B_{R}^{2n}$ which satisfies (2.12) for some $t \in (0, R^{-1}]$ and $K_0, K_1$ satisfy (2.13).

Let $(x^1, \ldots, x^{2n})$ be the standard coordinates on $B_{R}^{2n}$ and $e_1, \ldots, e_{2n}$ be the standard basis of $\mathbb{R}^{2n}$. The Christoffel symbols for the Levi-Civita connection of $h$ is given by

$$\nabla_{e_i}e_j = \Gamma^k_{ij}e_k + \bar{\Gamma}_{ij}^k e_{\bar{k}},$$

where $\bar{k} := n + k$ and the repeated indices are summed from 1 to $n$.

From (2.12) we have

$$|\Gamma^k_{ij}| + |\bar{\Gamma}_{ij}^k| \leq K_1 R^{-1}, \quad \forall i, j, k.$$

Let $L$ be a Lagrangian submanifold in $(B_{R}^{2n}, \omega_0)$ given by a gradient graph

$$\Gamma(u) := \{\Gamma_u(x) : x \in U_R \subset B_{R}^{n} \subset \mathbb{R}^n\},$$

where

$$\Gamma_u(x) = (x, Du(x)).$$

for some smooth function $u : U_R \rightarrow \mathbb{R}$, where $U_R \subset B_{R}^{n}$. In this case, $L = \Gamma(u)$ satisfies (2.2) if it is a critical point of the area functional

$$\text{Vol}(u) = \int_{U_R} \sqrt{\det g_{ij}} dx,$$

where $g = \Gamma_u^* h$, among all Hamiltonian variations $t \mapsto u + t\eta$, where $\eta \in C_c^\infty(U_R)$. The Euler-Lagrange equation has been computed in [2, Lemma 3.1], which is

$$\int_{U_R} [F_{jl}\eta_{jl} + c_{k}\eta_k] dx = 0, \quad \forall \eta \in C_c^\infty(U_R)$$

where

$$F_{jl}(x, Du, D^2 u) = \sqrt{g} g^{ij} ((\delta^{kl} + B_{lk}) u_{ik} + C_{li}),$$

$$c_{k}(x, Du, D^2 u) = \frac{1}{2} \sqrt{g} g^{ij} (D_{yk} A_{ij} + 2 u_{im} D_{yk} C_{mj} + u_{im} u_{jl} D_{yk} B_{kl}),$$

here $A, B$ and $C$ are the components of $(h - I_{2n}) \circ \Gamma_u$ (see (3.2) in [2]). Note that $\sqrt{g}, g^{ij}$ can also be represented by $D^2 u$, $A, B$ and $C$. By (2.12) we obtain the following lemma.

**Lemma 4.1.** For each $m \in \mathbb{N}$, there is a constant $C = C(r, m, K_0, \ldots, K_m, \|D^2 u\|_{L^\infty(B_{R}^{n})})$ such that

$$\|F_{jl}\|_{C_m(B_{R}^{n})} \leq C,$$

$$\|c_{k}\|_{C^{m-1}(B_{R}^{n})} \leq C,$$

where $F_{jl}$ and $c_{k}$ are defined in (4.6), (4.7).
By (2.13) we have $|Dh| < \epsilon_1$. By Lemma 3.2, if
\[(4.10)\]
\[\|D^2u\|_{C^0(B_r^c)} \leq \epsilon_1,\]
then (4.15) is $\Lambda$-uniform with $\Lambda = 1/2$. Hence we can apply Proposition 3.3 and Lemma 4.1 to obtain

**Theorem 4.2.** Let $u : B_r^n \to \mathbb{R}$ be a smooth function which satisfies $u(0) = Du(0) = D^2u(0) = 0$ and (4.10). If the graph $\Gamma(u)$ is Hamiltonian stationary Lagrangian in $(B_r^{2n}, \omega, h)$, where $h$ satisfies (2.12) and $K_0, K_1$ satisfy (2.13). Then for each $k \geq 3$ and $x \in B^c_{r/2}$,
\[(4.11)\]
\[|D^k u(x)| \leq C(k, r, K_0, \cdots, K_{k-3}).\]

4.1. **Curvature estimates, $\epsilon$-regularity.** Next we use the result in the previous subsection to derive local curvature estimates for HSL immersions in $(M, \omega, J, h)$ with small $L^n$-norm of the second fundamental form.

Using the Darboux coordinates in subsection 2.2 one can think of $L \cap \mathbb{B}_r(p)$ as a Lagrangian immersion in $(B_r^{2n}, \omega_0)$. We first derive an inequality comparing the second fundamental form of $L$ computed using two different Riemannian metrics.

**Proposition 4.3.** Let $k, K \in \mathbb{N}$ and $k < K$. Let $L$ be a $k$-dimensional immersed submanifold in an open subset $U$ of $\mathbb{R}^K$ and let $h_1$ be a Riemannian metrics on $U$ so that $\|h_1 - I\|_{C^0(U)} + \|Dh_1\|_{C^0(U)} < \epsilon$ for some positive number $\epsilon < 1/2$. Let $A_1, A_0$ be the second fundamental form of $L$ calculated with respect to $h_1$ and the Euclidean metric $h_0 = I$ respectively. Then there is $C = C(K)$ so that
\[|A_0(X, X)| \leq (1 + C\epsilon)|A_1(X, X)|_{h_1} + C\epsilon|X|_{h_1}^2\]
for all $x \in L$ and $X \in T_xL$.

**Proof.** Let $\nabla^1, \nabla^0$ be the Levi-Civita connections of $h_1, h_0$ respectively. Let $\mathcal{C}$ be the tensor $\mathcal{C}(X, Y) = \nabla^1_X Y - \nabla^0_X Y$. Let $x \in L$ and let $\pi^\perp_\alpha$ be the orthogonal projection onto the orthogonal complement of $T_xL$ with respect to $h_\alpha$ for $\alpha = 1, 0$. Let $\{e_1, \cdots, e_K\}$ be a basis of $\mathbb{R}^K$ so that $\{e_1, \cdots, e_k\}$ forms a basis for $T_xL$. Then
\[\pi^\perp_\alpha(Z) = Z - \sum_{a, b=1}^k H^a_\alpha h_\alpha(Z, e_a)e_b, \quad \alpha = 1, 0.\]

Here $(H_\alpha)_{ab} := h_\alpha(e_a, e_b)$ and $(H^a_\alpha)$ is the inverse matrix of $(H_\alpha)_{ab}$. Hence
\[\sum_{a, b=1}^k (H^a_1 h_1(Z, e_a) - H^a_0 h_0(Z, e_a)) e_b\]
\[= \sum_{a, b=1}^k ((H^a_1 - H^a_0) h_1(Z, e_a) + H^a_0 (h_1(Z, e_a) - h_0(Z, e_a))) e_b.\]
Write \( e_a = (e_a^1, \cdots, e_a^K) \) and \( Z = (Z^1, \cdots, Z^K) \), then
\[
(H_{\alpha})_{ab} = \sum_{i,j=1}^K e_a^i e_b^j (h_{\alpha})_{ij}, \quad h_{\alpha}(Z, e_a) = \sum_{i,j=1}^K Z^i e_a^j (h_{\alpha})_{ij}.
\]

This implies
\[
|H_1^{ab} - H_0^{ab}| \leq C\|h_1 - h_0\|_{C^0(U)},
\]
\[
|h_1(Z, e_a) - h_0(Z, e_a)| \leq C\|h_1 - h_0\|_{C^0(U)}|Z|
\]
and
\[
(4.12) \quad |(\pi_1^1 - \pi_2^2)Z| \leq C\|h_1 - h_0\|_{C^0(U)}|Z|.
\]

for some \( C = C(K) \). Let \( X, Y \) be any tangent vector fields of \( L \) defined on some open neighborhood of \( x \). Then
\[
A_0(X, Y) = \pi_0^1 (\nabla_X^0 Y) = \pi_0^1 (-\mathcal{G}(X,Y) + \nabla_X^1 Y) = -\pi_0^1 \mathcal{G}(X,Y) + \pi_0^1 \nabla_X^1 Y + (\pi_0^1 - \pi_1^1) (\nabla_X^1 Y) = A_1(X,Y) - \pi_0^1 \mathcal{G}(X,Y) + (\pi_0^1 - \pi_1^1) (\nabla_X^1 Y).
\]

Now let \( X \) be chosen such that \( \nabla_X^1 X = A_1(X,X) \) at \( x \). Using (4.12), at \( x \) we have
\[
|A_0(X,X)| \leq |A_1(X,X)| + |\mathcal{G}(X,X)| + C\|h_1 - h_2\|_{C^0(U)}|A_1(X,X)| \leq (1 + C\epsilon)|A_1(X,X)|_{h_1} + |\mathcal{G}(X,X)|.
\]

For some \( C = C(K) \). To estimates \( \mathcal{G}(X,X) \), note that locally
\[
\mathcal{G}_{ij}^k = (\Gamma_1)_i^k - (\Gamma_0)_i^k = (\Gamma_1)_i^k,
\]
where \( (\Gamma_1)_i^k \) are the Christoffel symbols of \( h_1 \). Since \( \|Dh_1\|_{C^0(U)} < \epsilon \) by assumption and
\[
(\Gamma_1)_i^k = h_1^{-1} \ast Dh_1,
\]
\[
|\mathcal{G}(X,X)| \leq C\epsilon |X|_{h_1}^2
\]
for some \( C = C(K) \). This finishes the proof of Proposition 4.3. \( \square \)

**Lemma 4.4.** Let \( U \) be an open subset in \( \mathbb{R}^{2n} \) and let \( L \) be an immersed Lagrangian submanifold in \( (U, \omega_0, h, J) \), so that \( h \) satisfies (2.12) and the \( C^0 \)-norm of the second fundamental form of \( L \) is bounded by \( C_A \). Then there is \( C_1 \) depending on \( n, K_0, K_1 \) so that the following holds: for any \( \epsilon \in (0, 1] \), write \( r = \epsilon/(C_1(C_A + 1)) \). For any \( p \in L \cap U \) so that \( B_{r^n}(p) \subset U \), any embedded connected component \( L_i \) of \( L \cap B_{r^n}(p) \) containing \( p \) can be written as a gradient graph \( \Gamma_u \), where \( u \) is a function defined on \( U_i \subset T_pL_i \), \( B_{r/2} \subset U_i \) and \( u \) satisfies
\[
(4.13) \quad u(0) = Du(0) = D^2u(0) = 0
\]
and
\[
(4.14) \quad \|D^2u\|_{C^0(U_i)} \leq \epsilon.
\]
Proof. Apply Proposition \[4.3\] with \(h_1 = h\), we conclude that \(|A_0| \leq c_2(C_A + 1)\), where \(A_0\) is the second fundamental form of \(L\) in \((U, h_0)\) and \(c_2\) depends only on \(n, K_0, K_1\). By \[3\] Theorem 2.6 \(\text{(see also \[12\] Theorem 2.4)}\), there is a dimensional constant \(c\) so that if \(L'\) is an embedded connected component of \(L \cap B^2(\xi):(C_A + 1)^{-1}(p)\) containing \(p\), then \(L'\) is graphical: there is \(X_i : U_i \to \mathbb{R}^n\), where \(B^2_{\xi}(C_A + 1)^{-1}(p) \subseteq U_i\), so that

\[L' = \{(x, X_i(x)) : x \in U_i\}\]

with \(X_i(0) = DX_i(0) = 0\) and \(|X_i(x)| \leq \epsilon\) for all \(x \in U_i\). Since \(L'\) is Lagrangian, \(X_i = Du_i\) for some smooth function \(u_i : U_i \to \mathbb{R}\) with \(u_i(0) = 0\). This finishes the proof of the lemma, by choosing \(C_1 = 2c_2/c\).

The following lemma can be proved by a direct computation, using the bounds on \(D^2u, h - I\) and \(Dh\).

**Lemma 4.5.** With the same assumption as in Lemma \[4.4\], there is \(C_2 > 0\) depending only on \(n, K_0, K_1\) such that

\[(4.15) \quad |D^3u(x)| \leq C_2(|A(x)| + 1), \quad |A(x)| \leq C_2(|D^3u(x)| + 1)\]

whenever \(|x| \leq \epsilon_1(C_1(C_A + 1))^{-1}\).

From the above lemmas we obtain the following proposition.

**Proposition 4.6.** Let \(L\) be a properly immersed Lagrangian submanifold in \(M\). Let \(p \in M\) and let \(\delta, C_A\) be positive numbers. Assume that the \(C^0\)-norm of the second fundamental form \(A\) of \(L\) is bounded above by \(C_A\) in the Darboux ball \(\mathcal{B}_\delta(p)\). Then there is \(C_2 > 0\) depending only on \(n, K_0, K_1\) such that the following holds: for any \(\epsilon \in (0, \epsilon_1]\), \(L \cap \mathcal{B}_r(p)\) splits into embedded connected components, where \(r = \min\{\delta/2, \epsilon(C_2(C_A + 1))^{-1}\}\). Moreover, for each \(v \in U_p\), each embedded connected component \(L_i\) of \(L \cap \mathcal{B}_r(p)\), \(z \in \tilde{L}_i := \Upsilon_{p,v}(L_i)\), up to a unitary action, \(\tilde{L}_i\) is the gradient graph of a function \(u_i\) with \(z = (0, 0)\),

(i) \(u_i(0) = Du_i(0) = D^2u_i(0)\),

(ii) \(|D^2u_i(x)| \leq \epsilon\), and

(iii) \(|D^3u_i(x)| \leq C_1(C_A + 1)\).

Proof. For any \(r < \delta\) and \(v \in U_p\), let \(\tilde{L} := \Upsilon_{p,v}(L \cap \mathcal{B}_p(r))\). Then \(\tilde{L}\) is an immersed Lagrangian submanifold in \(B^{2n}_r\). For any \(\epsilon \in (0, \epsilon_1]\), let \(r_1(\epsilon) = \epsilon/(C_1(C_A + 1))\), where \(C_1\) is defined in Lemma \[4.4\] and let \(r_2 = r_1(\epsilon)/4\). Then for all \(q \in B^{2n}_{r_1}\) we have \(B^{2n}_{r_2} \subset B^{2n}_{r_1}(q) \subseteq B^{2n}_{r_2}\). By Lemma \[4.4\] when \(r = r_1(\epsilon)\), \(\tilde{L}\) splits into finitely many embedded connected components. Let \(L_1^i, \ldots, \tilde{L}_i^i\) be those components that intersects \(B^{2n}_{r_2}\). For any \(i = 1, \ldots, \tilde{L}_i\) and \(z \in \tilde{L}_i^i \cap B^{2n}_{r_2}\), apply Lemma \[4.4\] to the ball \(B^{2n}_{r_2}(z)\). Thus up to a unitary action, \(\tilde{L}_i^i \cap B^{2n}_{r_2}\) is the gradient graph of a function \(u_i\). (i), (ii), (iii) follows from Lemma \[4.4\] and Lemma \[4.5\]. Since \(B^{2n}_{r_2} \subset B^{2n}_{r_2}\), the Proposition is proved by choosing \(C_2 = 4C_1\). \(\square\)

The following \(\epsilon\)-regularity theorem is essential to the proof of Theorem \[4.4\].
Theorem 4.7. There are positive numbers $\epsilon_0, C_{\epsilon_0}$ depending only on $K_0, K_1, K_2$ such that the following holds: if $L$ is an $n$-dimensional properly immersed HSL submanifold in a symplectic manifold $M$, $p \in L$ and

$$ \int_{L \cap \mathcal{B}_{r_0}(p)} |A|^p d\mu_L \leq \epsilon_0, $$

where $r_0 \leq 1$. Then for all $0 < \sigma \leq r_0$ and $y \in \mathcal{B}_{r_0-\sigma}(p) \cap L$,

$$ \sigma|A(y)| \leq C_{\epsilon_0}. $$

Proof. Let $L$ be given by an immersion $\iota : N \to M$. Let $\Upsilon_{p,v} : B_{r_0}^{2n} \to M$ be given by Proposition 2.5, where $v \in U_p$. Let $z_0$ be the maximum of the function

$$ z \mapsto (r_0 - |z|)^2 \max_{s \in \iota^{-1}(\Upsilon_{p,v}(z))} |A(s)|^2 $$

defined on $\Upsilon_{p,v}^{-1}(L \cap \mathcal{B}_{r_0}(p))$. Note that maximum exists since $\iota^{-1}(\Upsilon_{p,v}(z))$ is finite for each $z$. We assume this maximum is positive, or otherwise the result is trivial. So $|z_0| < r_0$. Let $s_0 \in \iota^{-1}(\Upsilon_{p,v}(z_0))$ such that

$$ |A(s_0)|^2 = \max_{s \in \iota^{-1}(\Upsilon_{p,v}(z_0))} |A(s)|^2. $$

For all $z \in B_{\frac{r_0 - |z_0|}{2}}(z_0)$ and $s \in \iota^{-1}(\Upsilon_{p,v}(z))$,

$$ |A(s)|^2 \leq \frac{(r_0 - |z|)^2}{(r_0 - |z_0|)^2} |A(s_0)|^2 $$

$$ \leq \frac{(r_0 - |z|)^2}{\left(\frac{r_0 - |z_0|}{2}\right)^2} |A(s_0)|^2 $$

$$ \leq 4|A(s_0)|^2. $$

Let $p_0 = \Upsilon_{p,v}(z_0)$. By (2.14), $U := \mathcal{B}_{\frac{r_0 - |z_0|}{2}}(p_0) \subset \Upsilon_{p,v}(B_{\frac{r_0 - |z_0|}{2}}^{2n}(z_0))$. By Proposition 4.6, there is $C_1 > 0$ such that $L \cap U \cap \mathcal{B}_r(p_0)$ splits into embedded connected components, where $r = (C_1(2|A(s_0)| + 1))^{-1}\epsilon_1$ and each of the components is HSL by Corollary 2.4. Moreover, let $L_i$ be a connected component so that $\iota^{-1}(L_i)$ contains $s_0$. Choose $v_0 \in U_{p_0}$ so that $v_0(\mathbb{R}^n \times \{0\}) = T_{p_0}L_i$. Then $\Upsilon_{p_0,v_0}^{-1} L_i$ is the gradient graph of a smooth function $u$ with $u(0) = Du(0) = D^2(0) = 0$ and

$$ |D^2u(x)| \leq \epsilon_1, \quad |D^3u(x)| \leq C_1(2|A(s_0)| + 1) $$

for all $x$ in the domain of $u$.

Let

$$ R = \frac{r_0 - |z_0|}{8}|A(s_0)|, \quad t = \frac{1}{|A(s_0)|} $$

and let

$$ t : B_{8R}^{2n} \to B_{\frac{r_0 - |z_0|}{8}}^{2n} $$
be the scaling \( z \mapsto tz \). Then \( \tilde{L} = (\Upsilon_{p_0,v_0} \circ t)^{-1}(L_i) \) is an immersed HSL in \((B^{2n}_R, \omega_0, \tilde{h})\) given by
\[
\tilde{t} = t^{-1} \circ t : V \to B^{2n}_R
\]
with metric \( \tilde{h} = t^{-2}(\Upsilon_{p_0,v_0} \circ t)^*h \). By the choice of \( R \), the second fundamental form \( \tilde{A} \) of \( \tilde{L} \) satisfies \( \|\tilde{A}\|_0 \leq 2 \) and \( |\tilde{A}(0)| = 1 \). Also, \( \tilde{L} \) is a gradient graph of a smooth function \( \tilde{u} \) with
\[
|D^2 \tilde{u}(x)| \leq \epsilon_1, \quad |D^3 \tilde{u}(x)| \leq \frac{C_1(2|A(s_0)| + 1)}{|A(s_0)|} \leq 3C_1
\]
for all \( x \in B^{2n}_{(3C_1)^{-1}} \cap B^{2n}_R \). Moreover,
\[
(4.18) \quad \int_{L_i} |\tilde{A}|^n d\mu_{\tilde{L}} \leq \int_{B^{2n}_{\tilde{r}}} |A|^n d\mu < \epsilon_0
\]
since the quantity is scale-invariant. Using Proposition 3.3 with \( m = 4 \) and (4.1)
\[
|D^4 \tilde{u}(x)| \leq C(K_0, K_1, K_2)
\]
for all \( x \in B^{2n}_R \) with \( |x| \leq (4C_1)^{-1} \). This implies
\[
|\nabla \tilde{A}| \leq C(K_0, K_1, K_2)
\]
for all \( x \in B^{2n}_R \) with \( |x| \leq (4C_1)^{-1} \). Since \( |\tilde{A}(0)| = 1 \), one concludes
\[
|\tilde{A}(x)| \geq \frac{1}{2}
\]
for all \( x \in \tilde{L} \) with \( |x| \leq \tilde{r} \), where \( \tilde{r} = \tilde{r}(K_0, K_1, K_2) \). Now choose \( \epsilon_0 = \omega_n^{-1}(4\tilde{r})^n \). If \( R > \tilde{r} \), then
\[
\int_{B^{2n}_R} |\tilde{A}|^n d\mu_{\tilde{L}} \geq \int_{B^{2n}_{\tilde{r}}} |\tilde{A}|^n d\mu_{\tilde{L}} \geq \frac{1}{2^n} \mu_{\tilde{L}}(B^{2n}_{\tilde{r}}) \geq \frac{1}{\omega_n(4\tilde{r})^n} = \epsilon_0,
\]
which is a contradiction. Thus
\[
R = \frac{r_0 - |z_0|}{8} |A(s_0)| \leq \tilde{r},
\]
which implies for all \( z \in \B_r(p) \) and \( s \in t^{-1}(\Upsilon_{p,v}(z)) \),
\[
|A(s)| \leq \frac{8\tilde{r}}{r_0 - |z|}.
\]

5. PROOF OF THEOREM 1.1 FOR \( n = 1 \)

In this section we prove Theorem 1.1 for \( n = 1 \). The general case is proved in the next section. We remark that it is necessary to split the proof into two cases: when \( n \geq 2 \) we use [8, Corollary 4.5], which says that if \( L \) is a Hamiltonian stationary Lagrangian in \( M \setminus S \) and \( S \) is finite, then \( \tilde{L} \) is also Hamiltonian stationary Lagrangian varifold in \( M \). As pointed out in the introduction in [8], this does not hold when \( n = 1 \).
Proof of Theorem 1.1 for \( n = 1 \). In this case, \((L_k)\) is a sequence of compact connected immersed curves in an oriented Riemannian surface \((M, h, J)\) and the Lagrangian condition is automatically satisfied by any smooth curve. For each \( k \), \( L_k \) is given by an immersion \( \gamma_k : S^1 \to M \). We assume each \( \gamma_k \) to be parameterized proportional to arc length.

Since \( J \) is compatible to \( h \), one checks that an immersed curve \( \gamma : S^1 \to M \) in \((M, h)\) is Hamiltonian stationary if and only if its curvature is constant. We remark that if \( h \) is generic then \((M, h)\) admits infinitely many embedded circles with constant curvatures.

For each \( k \in \mathbb{N} \), let \( \kappa_k \) be the (constant) curvature of \( \gamma_k \). Let \( \text{Length}(L) \) denotes the length of the immersed curves \( L \) defined in (2.1). The condition (1.1) implies

\[
\text{Length}(\gamma_k) = \text{Length}(L_k) \leq C_V \tag{5.1}
\]

and

\[
|\kappa_k| \text{Length}(\gamma_k) = \int_{L_k} |\kappa_k| \leq C_A. \tag{5.2}
\]

Taking a subsequence of \((L_k)\) if necessary, we may assume that

\[
\lim_{k \to \infty} \text{Length}(\gamma_k) = \ell_c \in [0, C_V] \tag{5.3}
\]

and

\[
\lim_{k \to \infty} \kappa_k = \kappa_c \in [-\infty, \infty]. \tag{5.4}
\]

By compactness of \( M \), we may also assume that

\[
\lim_{k \to \infty} \gamma_k(0) = p, \quad \lim_{k \to \infty} \gamma_k'(0) = v
\]

for some \( p \in M \) and \( v \in T_p M \).

If \( \ell_c = 0 \), then \((\gamma_k)\) converges to the point \( p \). If \( \ell_c > 0 \), then \( \kappa_c \neq \pm \infty \) by (5.2). Using (5.1) and the smooth dependence of ODE on initial data and parameters [10, Theorem 4.1], \((\gamma_k)\) converges smoothly to an immersed curve \( \gamma_c \) with \( \gamma_c(0) = p, \gamma_c'(0) = v \) and constant curvature \( \kappa_c \). This finishes the proof of Theorem 1.1 when \( n = 1 \).

\[\square\]

6. Proof of Theorem 1.1 for \( n \geq 2 \)

First we prove a simple covering lemma by Darboux balls.

**Proposition 6.1.** Let \( \lambda \in (0, 1/4) \). There is \( b \) depending only on \( n \) and \( \lambda \) such that the following holds. For any \( r < 1 \), there is a finite open covering \( \mathcal{U} \) of \( M \) by Darboux balls \( \mathcal{B}_r(a) \) so that (i) each member of \( \mathcal{U} \) intersect with at most \( b - 1 \) other members of \( \mathcal{U} \), and (ii) the open cover \( \{ \mathcal{B}_r(p) : p \in \mathcal{U} \} \) still covers \( M \).

**Proof.** Let \( \mathcal{V} = \{ \mathcal{B}_{r/8}(p_1), \ldots, \mathcal{B}_{r/8}(p_N) \} \) be a maximal collection of disjoint Darboux balls of radius \( \lambda r/8 \) in \( M \). Then \( \{ \mathcal{B}_{r/8}(p_1), \ldots, \mathcal{B}_{r/8}(p_N) \} \) covers \( M \): to see this, assume
Since $\mathcal{V}$ is maximal, there is $i$ so that $B_\lambda(x)(p) \cap B_\lambda(x)(p_i) \text{ contains an element } y$. By (2.14) and triangle inequality,
\[
d(p, p_i) \leq d(p, y) + d(y, p_i) < \frac{\lambda r}{4} + \frac{\lambda r}{4} = \frac{\lambda r}{2}.
\]
Hence $p \in B_{\frac{\lambda r}{2}}(p_i) \subset B_\lambda(x)(p_i)$ by (2.14) again. Let $\mathcal{U}$ be the open cover
\[
\mathcal{U} = \{ B_r(p_1), \cdots, B_r(p_N) \}.
\]
Note that $\mathcal{U}$ satisfies (ii) by construction. To show (i), let $p_i$ be fixed and assume that $y_{ij}$ lies in the intersection of $B_r(p_i)$ and $B_r(p_{i_j})$, for some $j = 1, \cdots, k$. Then $d(p_i, p_{i_j}) \leq 4r$, or $p_{i_j} \in B_{8r}(p_i)$ for all $i_j$. However, since $\mathcal{V}$ is a collection of disjoint Darboux balls with radius $\lambda r/8$, one has $d(p_{i_j}, p_i) \geq \lambda r/8$. Let $v_i \in U_{p_i}$ be fixed and let $z_j = \tilde{r}_{p_i,v_i}(p_{i_j})$. Then $\{z_1, \cdots, z_k\}$ be a collection of points in $B_8^{2n}$ so that $|z_i - z_j| \geq \lambda r/16$ whenever $i \neq j$. This implies $k \leq b$, where $b$ depends only on $n$, $\lambda$ but not on $r$. This finishes the proof of (i).

**Proof of Theorem 1.1** for $n \geq 2$. Choose $\lambda = \frac{\epsilon_1}{2 n \tau C_1 C_0}$, where $\epsilon_1$, $C_2$ and $C_0$ are defined in (4.10), Proposition 4.6 and Theorem 4.7 respectively. By Proposition 6.1 there is a constant $b = b(n, \lambda)$ so that the following holds: for any $m \in \mathbb{N}$, there is a finite open cover
\[
\mathcal{U}_m = \{ B_{2^{-m}}(\tilde{p}_1^m), \cdots, B_{2^{-m}}(\tilde{p}_{j(m)}^m) \}
\]
of $M$ by Darboux balls of radius $2^{-m}$, so that
\begin{enumerate}[(i)]
  \item each elements in $\mathcal{U}$ intersects at most $b - 1$ other members in $\mathcal{U}$, and
  \item the collection
  \[
  \mathcal{U}_{m,\lambda} = \{ B_{\lambda 2^{-m}}(\tilde{p}_1^m), \cdots, B_{\lambda 2^{-m}}(\tilde{p}_{j(m)}^m) \}
  \]
  still covers $M$.
\end{enumerate}

The first condition implies that for each $k \in \mathbb{N}$,
\[
(6.1) \quad \sum_{j=1}^{j(m)} \int_{B_{2^{-m}}(\tilde{p}_j^m) \cap L_k} |A_k|^n d\mu_k < b \int_{L_k} |A_k|^n d\mu_k < b C_A,
\]
where $C_A$ is give in Theorem 1.1 and $\mu_k := \mu_{L_k}$. For any fixed $k$ and $m$, let $J_{k,m} = \{ p_{1,k}^m, \cdots, p_{\ell,k}^m \}$ be the subset of $\{ \tilde{p}_1^m, \cdots, \tilde{p}_{j(m)}^m \}$ so that
\[
\int_{B_{2^{-m}}(\tilde{p}_j^m) \cap L_k} |A_k|^n d\mu_k \geq \epsilon_0.
\]
Here $\epsilon_0$ is given in Theorem 4.7 and $\ell = \ell(m,k)$ is less than $b C_A / \epsilon_0$ by (6.1). Taking a diagonal subsequence of $(L_k)$ if necessary, we may assume that $\ell(k,m) = \ell(m)$ and $J_{k,m} = J_m$ are both independent of $k$ for all $m \in \mathbb{N}$. Write
\[
J_m = \{ p_1^m, \cdots, p_{\ell(m)}^m \}.
\]
Using \( \ell(m) < bC_A/\epsilon_0 \), there is a subsequence \((m_i)\) such that \( \ell(m_i) = \ell \) for all \( i \in \mathbb{N} \). Using the compactness of \( M \) and taking a further subsequence of \((m_i)\) if necessary, we may assume that

\[
\lim_{i \to \infty} p_j^{m_i} = p_j
\]

for all \( j = 1, \ldots, \ell \).

Let \( S = \{p_1, \ldots, p_\ell\} \) and \( r_i = 2^{-m_i} \). Fix any \( p \in \{\bar{p}_i^{m_i}, \ldots, \bar{p}_{j(m_i)}^{m_i}\} \setminus J_{m_i} \). By definition of \( J_{m_i} \) this implies

\[
\int_{\mathcal{B}_{r_i}(p) \cap L_k} \vert A_k \vert^n d\mu_k < \epsilon_0
\]

for all \( k \in \mathbb{N} \). By Theorem 4.7

\[
(6.2) \quad \vert A_k(q) \vert \leq 2C_{\epsilon_0} / r_i, \quad \forall k \in \mathbb{N} \text{ and } q \in \mathcal{B}_{r_i/2}(p).
\]

Setting \( C_A = 2C_{\epsilon_0} / r_i \) and note that

\[
\epsilon_1 / C_2(C_A + 1) = \frac{r_i \epsilon_1}{C_2(2C_{\epsilon_0} + r_i)} \geq \frac{\epsilon_1}{3C_2C_{\epsilon_0}} r_i = 9\lambda r_i.
\]

By Proposition 4.6 for each \( k \in \mathbb{N} \), \( \mathcal{B}_{8\lambda r_i}(p) \cap L_k \) splits into finitely many embedded connected components

\[
\mathcal{B}_{8\lambda r_i}(p) \cap L_k = L_k^1 \cup \cdots \cup L_k^k.
\]

Let \( v \in U_p \) be fixed and for each \( a = 1, \ldots, I_k \), denote \( \tilde{L}_k^a := \Upsilon_{p,v}^{-1}(L_k^a) \).

**Lemma 6.2.** For each \( i \in \mathbb{N} \), there is \( M_i \in \mathbb{N} \) depending on \( i, \lambda, n \) and \( C_V \) in (1.1) only so that \( \mathcal{B}_{8\lambda r_i}(p) \cap L_k \) has at most \( M_i \) embedded connected components which intersect \( \mathcal{B}_{\lambda r_i}(p) \).

**Proof of Lemma 6.2.** Let \( L_k^a \) be one of the components \( \mathcal{B}_{8\lambda r_i}(p) \cap L_k \) which intersects \( \mathcal{B}_{\lambda r_i}(p) \). Let \( z_k^a \in B_{2\lambda r_i} \cap \tilde{L}_k^a \). Then there is a unitary matrix \( A_k^a \) and a smooth function \( u_k^a \) defined on \( U_k^a \) so that

\[
(\Upsilon_k^a)^{-1}(\tilde{L}_k^a \cap B_{2\lambda r_i}(z_k^a)) = \{(x, Du_k^a(x)) : x \in U_k^a\},
\]

where \( \Upsilon_k(z) = A_k^a z + z_k^a \). Using (2.12), we obtain \( \mu(L_k^a) \geq C(n)(\lambda r_i)^n \) for some dimensional constant \( C(n) \). Together with the assumption on the area bound in (1.1), the lemma is proved. \( \square \)

By Lemma 6.2 and taking a diagonal subsequence of \( (L_k) \) if necessary, we may assume that for any \( i \in \mathbb{N} \) and \( p \in \{\bar{p}_i^{m_i}, \ldots, \bar{p}_{j(m_i)}^{m_i}\} \setminus J_{m_i} \), there is an integer \( n_i(p) \) such that \( \mathcal{B}_{8\lambda r_i}(p) \cap L_k \) has exactly \( n_i(p) \) connected components which intersect \( \mathcal{B}_{\lambda r_i}(p) \), where \( n_i(p) \leq M_i \). Let \( L_k^1, \ldots, L_k^{n_i(p)} \) be such connected components.

Let \( z_k^a, A_k^a, \Upsilon_k^a \) and \( u_k^a : U_k^a \to \mathbb{R} \) be defined as in the proof of Lemma 6.2 where \( u = u_k^a \) satisfies (4.13) and (4.14) with \( \epsilon = \epsilon_1 \). By Theorem 4.2 for any \( m \geq 3 \), there are \( C_m \) depending on \( r_i, \lambda, m, K_0, \ldots, K_{m-3} \) so that

\[
(6.3) \quad \vert D^m u_k^a(x) \vert \leq C_m, \quad \text{for all } x \in B_{4\lambda r_i}^a.
\]
Taking a diagonal subsequence of \((L^k)^\infty_{k=1}\) if necessary, we may assume that for each \(i \in \mathbb{N}\) and \(p\),

1. the sequence of points \((z^a)^\infty_{k=1}\) converges to \(z^a \in B^m_{\lambda r_i}\) as \(k \to \infty\),
2. the sequence of unitary matrices \((A^a)^\infty_{k=1}\) converges to some \(A^a \in U(n)\) as \(k \to \infty\), and
3. the sequence of functions \((u^a)^\infty_{k=1}\) converges in \(C^m(B^a_{3\lambda r_i})\) for all \(m \in \mathbb{N}\) to a smooth function \(u^a : B^a_r \to \mathbb{R}\). This is possible by the higher order estimates (6.3).

Then the sequence of immersions \((\Upsilon_k^a \circ \Gamma_{u_k^a})^k\) converges to the immersion \(\Upsilon^a \circ \Gamma_{u^a}\), where for any \(\rho > 0\) \(= (\int L^a \cap B^a_{2\lambda r_i})\). Vol\((L^a)\) converges to some \(\lambda r_i\) defined on \(B^a_{2\lambda r_i}\) so that \(\tilde{L}^a_{\lambda r_i}\) is locally given by the immersion \(\Upsilon^a \circ \Gamma_{u^a}\).

Next we define the Lagrangian integral \(n\)-variifold \(L \setminus S\). Let
\[
\mathcal{B} = \{ \mathcal{B}_{\lambda r_i}(p) : i \in \mathbb{N}, \; p \in \{ \tilde{p}_1^m, \cdots, \tilde{p}_j^m \} \} \setminus J_m\).
\]
Clearly \(\mathcal{B}\) is an open cover of \(M \setminus S\). For each \(K \in \mathbb{N}\) let
\[
\mathcal{B}_K = \{ \mathcal{B}_{\lambda r_i}(p) \in \mathcal{B} : \mathcal{B}_{\lambda r_i}(p) \cap S_{1/K} = \emptyset \},
\]
where for any \(\rho > 0\) we set \(S_{\rho} = \{ q \in M : d^m(q, s) < \rho \text{ for some } s \in S \}\). Then \(\tilde{\mathcal{B}} = \bigcup_K \mathcal{B}_K\) is an open cover of \(M \setminus S\). For each \(q \in M \setminus S\), then \(q \in \mathcal{B}_{\lambda r_i}(p)\) for some \(U = \mathcal{B}_{\lambda r_i}(p) \in \tilde{\mathcal{B}}\). Define \(|L|\) on \(\mathcal{B}_{\lambda r_i}(p)\) by
\[
|L| = |L^1| + \cdots + |L^n(p)|,
\]
where for each \(a = 1, \cdots, n_i(p)\), \(L^a\) is the limits of \((L^a_k \cap \mathcal{B}_{\lambda r_i}(p))^\infty_{k=1}\) constructed in the previous paragraph. Since \(z^a \in B^a_{2\lambda r_i}\) for each \(a = 1, \cdots, n_i(p)\), \(B^a_{2\lambda r_i} \subset B^a_{2\lambda r_i}(z^a)\). Hence \((L^k \cap \mathcal{B}_{\lambda r_i}(p))^\infty_{k=1}\) converges graphically in \(\mathcal{B}_{\lambda r_i}(p)\) to \(L \cap U\) in \(C^\infty\)-topology. Since \(g \in M \setminus S\) is arbitrary, \((L_k)^\infty_{k=1}\) converges in \(M \setminus S\) smoothly locally graphically to \(L\). Since the convergence is locally smooth and each \(L_k\) is HSL, \(L\) is also HSL on \(M \setminus S\). By [8 Corollary 4.4], \(\mathcal{T}\) admits a structure of an \(n\)-integral Lagrangian varifold so that (4.5) holds (although Theorem 4.3 and Corollary 4.4 in [8] are stated only for Lagrangian immersions in \(\mathbb{R}^{2n}\), by isometrically embed \((M, h)\) into some Euclidean space, the proofs work for immersions in \((M, \omega, J, h)\)). Connectedness of \(\mathcal{T}\) can be shown as in the proof of [8 Theorem 1.1].

Lastly, it remains to show (1.2). By the Nash embedding theorem, one can assume that \((M, h)\) is an embedded submanifold of \(\mathbb{R}^{n+K}\) for some natural number \(K\). Let \(L\) be an immersed submanifold \(M\) and let \(\tilde{A}, \tilde{H}\) be the second fundamental form and mean curvature vector of \(L\) respectively as a submanifold in \(\mathbb{R}^{n+K}\). Then \(\tilde{A} = A + A^M\), where \(A^M\) is the second fundamental form of \(M\) in \(\mathbb{R}^{n+K}\). Together with the fact that \(|A^M| \leq C(M)\) since \(M\) is compact, we obtain
\[
\left( \int_L |\tilde{H}|^n d\mu_L \right)^{1/n} \leq \left( \int_L |H|^n d\mu_L \right)^{1/n} + C(M)\mu(L).
\]
Thus we can apply [8 Proposition 4.1] to \(L = L_k\) and conclude that
\[
\text{Vol}(B_{\rho}(p) \cap L_k) \leq C(\left| \ln \rho \right| + 1)^n \rho^n
\]
for all \( p \in M \) and \( \rho > 0 \) small enough, here \( C \) depends only on \( M \) and \( C_V, C_A \) in Theorem 1.1. For any \( \rho > 0 \), by (6.4) we have

\[
\text{Vol}(L_k) - C(|\ln \rho| + 1)^n \rho^n \leq \text{Vol}(L_k \setminus S_\rho) \leq \text{Vol}(L_k).
\]

(6.5)

Since \((L_k)\) converges smoothly graphically to \( L \) away from \( S_\rho \), taking \( k \to \infty \) in (6.5) gives

\[
\lim_{k \to \infty} \text{Vol}(L_k) - C(|\ln \rho| + 1)^n \rho^n \leq \text{Vol}(L \setminus S_\rho) \leq \lim_{k \to \infty} \text{Vol}(L_k).
\]

Thus (1.2) is shown by taking \( \rho \to 0 \) and this finishes the proof of Theorem 1.1.

\[\square\]

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