EXISTENCE AND NONEXISTENCE OF LEAST ENERGY SOLUTIONS OF THE NEUMANN PROBLEM FOR A SEMILINEAR ELLIPTIC EQUATION WITH CRITICAL SOBOLEV EXPONENT AND A CRITICAL LOWER-ORDER PERTURBATION

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Abstract. Let Ω be a smooth bounded domain in $\mathbb{R}^N$, with $N \geq 5$, $a > 0$, $\alpha \geq 0$ and $2^* = \frac{2N}{N-2}$ be the critical exponent for the Sobolev embedding $H^1(\Omega) \subset L^{2^*}(\Omega)$ and $2^# = \frac{2(N-1)}{N-2}$. We consider the problem

$$
\begin{cases}
-\Delta u + au = u^{2^*-1} - \alpha u^{q-1} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
$$

Namely, we prove that when $q = \frac{2(N-1)}{N-2}$ there exists an $\alpha_0 > 0$ such that the problem has a least energy solution if $\alpha < \alpha_0$ and has no least energy solution if $\alpha > \alpha_0$.

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, with $N \geq 5$, $a > 0$ and $\alpha \geq 0$. Let $2^* = \frac{2N}{N-2}$ be the critical exponent for the Sobolev embedding $H^1(\Omega) \subset L^{2^*}(\Omega)$ and $2^# = \frac{2(N-1)}{N-2}$. We consider the problem

$$
\begin{cases}
-\Delta u + au = u^{2^*-1} - \alpha u^{q-1} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
$$

We regard $a$ as fixed and $\alpha$ as a parameter. From Theorem 3.2 of [18], due to X.J. Wang, we know that if $2 < q < 2^#$, then problem $(\mathcal{P}_{\alpha,q})$ has a least energy solution for all values of $\alpha \geq 0$. (Wang’s result actually holds for $N \geq 3$.) A question that naturally arises is the following: what happens for $q = 2^#$?

It is well known that the solutions of $(\mathcal{P}_{\alpha,q})$ correspond to critical points of the functional $\Phi_\alpha : H^1(\Omega) \to \mathbb{R}$, defined by

$$
\Phi_\alpha(u) := \frac{1}{2} |\nabla u|_2^2 + \frac{a}{2} |u|_2^2 + \frac{\alpha}{q} |u|^q - \frac{1}{2^*} |u|^{2^*},
$$

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where $|u|_p$ denotes the $L^p$ norm of $u$ in $\Omega$. We recall that a least energy solution is a function $u \in H^1(\Omega)$ such that
\[
\Phi_\alpha(u) = \inf_{N} \Phi_\alpha.
\]

The set $\mathcal{N}$ is the Nehari manifold, $\mathcal{N} := \{ u \in H^1(\Omega) : \Phi'_\alpha(u)u = 0, \ u \neq 0 \}$. It is interesting to note that when $q = 2^*$ it is possible to determine explicitly the function $\Phi_\alpha|_{\mathcal{N}}$ by solving a quadratic equation. We take full advantage of this fact.

We recall that the infimum
\[
S := \inf \left\{ \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{(\int_{\mathbb{R}^N} |u|^2)^{2/2}} \mid u \in L^2(\mathbb{R}^N), \nabla u \in L^2(\mathbb{R}^N), \ u \neq 0 \right\}
\]
is achieved by the Talenti instanton $U(x) := \left( \frac{N(N-2)}{N(N-2)+|x|^2} \right)^{\frac{N-2}{2}}$. For $\varepsilon > 0$ and $y \in \mathbb{R}^N$, we define $U_{\varepsilon,y} := e^{-\frac{N-2}{4N}\varepsilon^2}U \left( \frac{x-y}{\varepsilon} \right)$.

Heuristically, we can summarize the main idea behind the analysis of problem $(P_{\alpha,q})$, when $q = 2^*$, as follows. There exists an $\alpha_0 \in ]0, +\infty]$ such that $\inf_{\mathcal{N}} \Phi_\alpha < \frac{S^*}{2N}$ for $\alpha < \alpha_0$, and $\inf_{\mathcal{N}} \Phi_\alpha = \frac{S^*}{2N}$ for $\alpha \geq \alpha_0$. If $\alpha < \alpha_0$, then $(P_{\alpha,2^*})$ has a least energy solution whereas, if $\alpha > \alpha_0$, then $(P_{\alpha,2^*})$ does not have a least energy solution. Suppose $\alpha_0 = +\infty$, so that there exist least energy solutions for all $\alpha \geq 0$. We choose a sequence $\alpha_k \to +\infty$ as $k \to +\infty$ and denote by $u_k$ a corresponding sequence of least energy solutions. Then there would exist a sequence of positive numbers $\varepsilon_k$ converging to zero, and a sequence of points $P_k \in \partial \Omega$, such that, modulo a subsequence, $P_k \to P$ and $|\nabla(u_k - U_{\varepsilon_k,P_k})|_2 \to 0$, as $k \to +\infty$. We can use $\Phi_{\alpha_k}(U_{\varepsilon_k,P_k})$ to estimate $\Phi_{\alpha_k}(u_k)$ from below with an error that is $o(\alpha_k \varepsilon_k)$.

However, from Adimurthi and Mancini [1] and X.J. Wang [18], we have the estimate
\[
\Phi_{\alpha_k}(U_{\varepsilon_k,P_k}) = \frac{S^*}{2N} - \frac{S^*}{2} H(P_k) A(N) \varepsilon_k + \frac{1}{2} B(N) \alpha_k \varepsilon_k + o(\alpha_k \varepsilon_k),
\]
where $A(N)$ and $B(N)$ are positive constants that only depend on $N$, and $H(P_k)$ is the mean curvature of $\partial \Omega$ at $P_k$ with respect to the unit outward normal. This lower bound is greater than $\frac{S^*}{2N}$, for large $k$. This contradicts the hypothesis that $\alpha_0 = +\infty$.

It is somewhat delicate to justify the use of $\Phi_{\alpha_k}(U_{\varepsilon_k,P_k})$ to estimate $\Phi_{\alpha_k}(u_k)$ from below. This was first done by Adimurthi, Pacella and Yadava in [2], who treated the case where $\alpha = 0$. The argument involves an expansion to second order of the energy at $U_{\varepsilon_k,P_k}$ and a comparison of the eigenvalues of the linearized problem at $U_{\varepsilon_k,P_k}$ with the eigenvalues of a limiting problem.

The present analysis builds on the work [2] of Adimurthi, Pacella and Yadava, which we will frequently refer to as [APY]. Of course, the works of Talenti [17], Brézis and Nirenberg [9], P.L. Lions [15], Adimurthi and Mancini [1], and X.J. Wang [18] are also of major importance for our study.

Our main result is the following
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Theorem. Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$, with $N \geq 5$, $a > 0$ and $\alpha \geq 0$. There exists a positive real number $\alpha_0 = \alpha_0(a, \Omega)$ such that

(i) if $\alpha < \alpha_0$, then problem $(P_{\alpha, 2\#})$ has a least energy solution;
(ii) if $\alpha > \alpha_0$, then problem $(P_{\alpha, 2\#})$ has no least energy solution.

We remark that this result contrasts with Theorem 3.2 of [18], referred to above. Also, from this theorem we deduce an inequality (see (15)) which implies Aubin’s inequality (16) (see [6] and Cherrier [11]).

We should mention that for any pair $(a, \alpha)$ (with $a > 0$ and $\alpha \geq 0$) problem $(P_{\alpha, 2\#})$ has the constant solution

$$u = \kappa := \left(\frac{a + \sqrt{\alpha^2 + 4a}}{2}\right)^{N-2}.$$

The energy of this solution is

$$\Phi_\alpha(\kappa) = \frac{|\Omega|}{2\#N}\left[\left(\frac{a + \sqrt{\alpha^2 + 4a}}{2}\right)^N + \frac{2^*}{2^*}a \left(\frac{a + \sqrt{\alpha^2 + 4a}}{2}\right)^{N-2}\right],$$

where $|\Omega|$ denotes the $N$-dimensional Lebesgue measure of $\Omega$. It follows that for $a > 0$ and $\alpha \geq 0$ sufficiently small, namely for $a \leq S/(2|\Omega|)^{\frac{2}{N}}$ and $\alpha$ such that $\Phi_\alpha(\kappa) \leq S^2/(2N)$, then the least energy solutions might be constant.

When the domain $\Omega$ is a ball and $a$ is small, Adimurthi and Yadava [3] proved that $(P_{0, 2\#})$ has more than one solution for $N = 4, 5$ and 6. However, when $N = 3$ a uniqueness result was proved by M. Zhu in [21] for convex domains, $\alpha = 0$ and small $a$.

Other works in the spirit of ours are those of Brézis and Lieb [8], Adimurthi and Yadava [4], M. Zhu [20], Z.Q. Wang [19] and Chabrowski and Willem [10].

The organization of this work is as follows. In Section 2 we give the setup of our work and the statement of the main result. In Section 3 we prove existence of least energy solutions. We then assume that the value $\alpha_0$ is infinite and analyze the asymptotic behavior of the least energy solutions as $\alpha \to +\infty$. In Section 4 we prove nonexistence of least energy solutions. In Section 5 we give a lower bound for $\alpha_0$ and, using the ideas of Chabrowski and Willem [10], give partial results concerning existence of least energy solutions for $\alpha = \alpha_0$. In Appendix A we check that the Nehari set $\mathcal{N}$ is a manifold and a natural constraint for $\Phi_\alpha$, we derive expressions for $\Phi_\alpha|\mathcal{N}$, and we derive upper and lower bounds for $\Phi_\alpha|\mathcal{N}$. Finally, in Appendix B we prove a technical estimate, used in our study, similar to those in Adimurthi and Mancini [1].

Motivated by this work, in [13] the second author has proved an inequality which improves inequality (15). In [14] he proves a family of inequalities which contains, as special cases, an inequality in Zhu’s work [20] and the inequality in [13].

2. The setup and statement of the main result

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$, with $N \geq 5$. Let $2^* = \frac{2N}{N-2}$ be the critical exponent for the Sobolev embedding $H^1(\Omega) \subset L^{2^*}(\Omega)$ and $2\# = \frac{2(N-1)}{N-2}$. Finally, let $a > 0$ and $\alpha \geq 0$. We are concerned with the
problem of existence of a least energy solution of
\[
\begin{cases}
-\Delta u + au = u^{2^* - 1} - \alpha u^{2^* - 1} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\]
(1a)

Solutions of (1a) correspond to critical points of the functional \(\Phi_{\alpha} : H^1(\Omega) \to \mathbb{R}\) defined by
\[
\Phi_{\alpha}(u) := \frac{1}{2} ||u||^2 + \frac{\alpha}{2^{2^*}} |u|^{2^*} - \frac{1}{2^{2^*}} |u|^{2^{2^*}}.
\]
(2)

We use the notations
\[
|u|^p := (\int |u|^p)^{\frac{1}{p}} \quad \text{and} \quad ||u|| := \left( |\nabla u|_2^2 + a |u|_2^2 \right)^{\frac{1}{2}}.
\]

Unless otherwise indicated, integrals are over \(\Omega\).

We recall that the Nehari manifold is
\[
N := \{ u \in H^1(\Omega) : \Phi_{\alpha}'(u) = 0, u \neq 0 \}.
\]

For any \(u \in H^1(\Omega) \setminus \{0\}\), there exists a unique \(t(u) > 0\) such that \(t(u)u \in N\); the value of \(t(u)\) is given in expression (62) of Appendix A. We define \(\Psi_{\alpha} : H^1(\Omega) \setminus \{0\} \to \mathbb{R}\) by
\[
\Psi_{\alpha}(u) := \Phi_{\alpha}(t(u)u).
\]

As can be checked in Appendix B,
\[
\Psi_{\alpha} := \frac{1}{N} \frac{1}{2} \frac{1}{2^N} \left[ \left( \gamma + \sqrt{\gamma^2 + 4\beta} \right)^N + 2 \cdot 2^* \beta \left( \gamma + \sqrt{\gamma^2 + 4\beta} \right)^{N-2} \right],
\]
(3)

where \(\beta, \gamma : H^1(\Omega) \setminus \{0\} \to \mathbb{R}\) are defined by
\[
\beta(u) := \frac{||u||^2}{|u|_2^{2^*}} \quad \text{and} \quad \gamma(u) = \gamma_{\alpha}(u) := \alpha \frac{|u|_2^{2^*}}{|u|_2^{2^*}}.
\]
(4)

Equivalently,
\[
\Psi_{\alpha} = \frac{1}{N} \frac{1}{2} \frac{1}{2^N} \left[ \left( \delta + \sqrt{\delta^2 + 1} \right)^N + 2 \cdot 2^* \left( \delta + \sqrt{\delta^2 + 1} \right)^{N-2} \right],
\]
(6)

with \(\beta\) as above and \(\delta : H^1(\Omega) \setminus \{0\} \to \mathbb{R}\) defined by
\[
\delta(u) = \delta_{\alpha}(u) := \frac{\gamma(u)}{2\sqrt{\beta(u)}} = \frac{1}{2} \frac{\alpha |u|_2^{2^*}}{||u|| \cdot |u|_2^{2^*}},
\]
(7)

Obviously, every nonzero critical point of \(\Phi_{\alpha}\) is a critical point of \(\Psi_{\alpha}\). Since the Nehari manifold is a natural constraint for \(\Phi_{\alpha}\), if \(u\) is a critical point of \(\Psi_{\alpha}\), then \(t(u)u\) is a critical point of \(\Phi_{\alpha}\).

As is usual, we say that \(u \neq 0\) is a ground state critical point of \(\Phi_{\alpha}\), or a least energy solution of (1a), if
\[
\Phi_{\alpha}(u) = \inf_{N} \Phi_{\alpha} = \inf_{H^1(\Omega) \setminus \{0\}} \Psi_{\alpha}.
\]
Our aim is to establish existence and nonexistence of least energy solutions of \((1_\alpha)\). We will consider the minimization problem corresponding to
\[
S_\alpha := \inf \left\{ I_\alpha(u) | u \in H^1(\Omega) \setminus \{0\} \right\},
\]
where \(I_\alpha : H^1(\Omega) \setminus \{0\} \to \mathbb{R}\) is defined by
\[
I_\alpha := (N\Psi_\alpha)^{\frac{2}{N}}.
\]
From \((3)\) and \((6)\) we obtain
\[
I_\alpha = \frac{1}{4(2^*)^\frac{N}{2}} \left[ \left( \gamma + \sqrt{\gamma^2 + 4\beta} \right)^N + 2 \cdot 2^* \beta \left( \gamma + \sqrt{\gamma^2 + 4\beta} \right)^{N-2} \right]^{\frac{2}{N}}, \tag{9}
\]
and
\[
I_\alpha = \frac{\beta}{(2^*)^\frac{N}{2}} \left[ \left( \delta + \sqrt{\delta^2 + 1} \right)^N + \frac{2^*}{2} \left( \delta + \sqrt{\delta^2 + 1} \right)^{N-2} \right]^{\frac{2}{N}}. \tag{10}
\]
We observe that
\[
I_\alpha \geq \beta, \tag{11}
\]
since
\[
\frac{1}{2^*} \left( 1 + \frac{2^*}{2} \right) = 1.
\]
Before stating our main result, we recall that the infimum
\[
S := \inf \left\{ \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left( \int_{\mathbb{R}^N} |u|^{2^*} \right)^{2/2^*}} | u \in L^{2^*}(\mathbb{R}^N), \nabla u \in L^2(\mathbb{R}^N), u \neq 0 \right\},
\]
which depends on \(N\), is achieved by the Talenti instanton
\[
U(x) := \left( \frac{N(N-2)}{N(N-2) + |x|^2} \right)^{\frac{N-2}{2}}.
\]
This instanton \(U\) satisfies
\[
- \Delta U = U^{2^*-1}, \tag{12}
\]
so that
\[
\int_{\mathbb{R}^N} |\nabla U|^2 = \int_{\mathbb{R}^N} U^{2^*} = S^{\frac{2}{N}}. \tag{13}
\]
Let \(\varepsilon > 0\) and \(y \in \mathbb{R}^N\). For later use, we define the rescaled instanton
\[
U_{\varepsilon, y} := \varepsilon^{\frac{N-2}{2}} U \left( \frac{x - y}{\varepsilon} \right), \tag{14}
\]
which also satisfies \((12)\) and \((13)\).

Our main result is

**Theorem 2.1.** Let \(\Omega\) be a smooth bounded domain in \(\mathbb{R}^N\), with \(N \geq 5\), \(a > 0\) and \(\alpha \geq 0\). There exists a positive real number \(\alpha_0 = \alpha_0(a, \Omega)\) such that

1. if \(\alpha < \alpha_0\), then \((1_\alpha)\) has a least energy solution;
(ii) if $\alpha > \alpha_0$, then $(1_\alpha)$ does not have a least energy solution and

$$\frac{S}{2^N} \leq \frac{\beta}{(2^\#)^N} \left[ \left( \delta + \sqrt{\delta^2 + 1} \right)^N + \frac{2^\#}{2} \left( \delta + \sqrt{\delta^2 + 1} \right)^{N-2} \right] \frac{1}{2^\#} \tag{15}$$

in $H^1(\Omega) \setminus \{0\}$, where $\beta$ and $\delta = \delta_\alpha$ are defined in (4) and (7), respectively. The constant on the left hand side of (15) is sharp.

**Corollary 2.2 (Aubin’s inequality).** Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$, with $N \geq 5$. For every $\varsigma > 0$, there exists a $C(\varsigma, \Omega) > 0$ such that

$$\frac{S}{2^N} - \varsigma \leq \frac{\| \nabla u \|_2^2 + C(\varsigma, \Omega)\| u \|_2^2}{\| u \|_2^2}, \tag{16}$$

for all $u \in H^1(\Omega) \setminus \{0\}$.

**Proof.** From Lemma 5.1, there exists a constant $\bar{c} > 0$ such that the right hand side of (15) is less than or equal to

$$\beta \left( 1 + \frac{4}{2^\#} \delta + \bar{c} \delta^2 \right)$$

and from Hölder’s inequality $\| u \|_2^2 \leq \| u \|_2 \| u \|_2^{2/2}$. Hence $\delta(u) \leq \frac{\delta}{2 \| u \|_2}$. Let $\epsilon > 0$. For all $u \in H^1(\Omega)$,

$$\frac{S}{2^N} \| u \|_2^2 \leq \| u \|^{2} \left( 1 + \frac{2}{2^\#} \alpha_0 \frac{\| u \|_2}{\| u \|} + \epsilon \frac{\alpha_0^2}{4} \frac{\| u \|_2^2}{\| u \|_2} \right)$$

$$= \| u \|^{2} + \frac{2}{2^\#} \alpha_0 \| u \| \| u \|_2 + \epsilon \frac{\alpha_0^2}{4} \| u \|_2^2$$

$$\leq (1 + \epsilon)\| \nabla u \|_2^2 + \left( \frac{\alpha_0^2}{(2^\#)^2 \epsilon} + \alpha \epsilon + \frac{\alpha_0^2}{4} \right) \| u \|_2^2. \tag{17}$$

**Remark 2.3.** Let $\kappa > 0$. By scaling, we easily check that

$$\alpha_0 \left( \kappa^2 a, \frac{\Omega}{\kappa} \right) = \kappa \alpha_0(a, \Omega).$$

**3. Existence of least energy solutions and their asymptotic behavior**

In this section we start by proving the basic properties of the map $\alpha \mapsto S_\alpha$ and assertion (i) of Theorem 2.1. We then assume that the value $\alpha_0$ in Theorem 2.1 is infinite and analyze the asymptotic behavior of the least energy solutions as $\alpha \to +\infty$.

As explained in the previous section, we consider the minimization problem corresponding to

$$S_\alpha := \inf \{ I_\alpha(u) | u \in H^1(\Omega), u \neq 0 \}.$$

From Adimurthi and Mancini [1] and X.J. Wang [18], we know that

$$0 < S_0 < \frac{S}{2^N} \tag{17}$$
(see (42) and (46) ahead). Obviously, $S_\alpha$ is nondecreasing as $\alpha$ increases. Choose any point $P \in \partial \Omega$. By testing $I_\alpha$ with $U_{\varepsilon,P}$ and letting $\varepsilon \to 0$, we conclude that $S_\alpha \leq \frac{S}{2\pi}$ for all $\alpha \geq 0$.

**Lemma 3.1.** If $S_\alpha < \frac{S}{2\pi}$, then $S_\alpha$ is achieved.

**Proof.** Let $u_k$ be a minimizing sequence with $|u_k|^2 = 1$. Since $\beta \leq I_\alpha$, from (11), $(u_k)$ is bounded in $H^1(\Omega)$. We can assume that $u_k \rightharpoonup u$ in $H^1(\Omega)$, $u_k \to u$ a.e. on $\Omega$, and $|\nabla(u_k - u)|^2 \to \mu$ and $|u_k - u|^2 \to \nu$ in the sense of measures on $\Omega$. Modulo a subsequence, the concentration-compactness lemma implies that

$$\lim_{k \to \infty} |\nabla u_k|^2 = |\nabla u|^2 + ||\mu||$$

and

$$\lim_{k \to \infty} |u_k|^2 = |u|^2 + ||\nu|| = 1,$$

where

$$\frac{S}{2\pi} ||\nu||^2 \leq ||\mu||.$$

This last inequality is an immediate consequence of inequality (16). For $S_\alpha = \lim_{k \to \infty} I_\alpha(u_k)$, we obtain that $S_\alpha$ equals

$$\frac{1}{4(2\pi)\frac{2}{N}} \left[ \left( \gamma_\infty + \sqrt{\gamma_\infty^2 + 4\beta_\infty} \right)^N + 2 \cdot 2^* \beta_\infty \left( \gamma_\infty + \sqrt{\gamma_\infty^2 + 4\beta_\infty} \right)^{N-2} \right]^{\frac{2}{N}},$$

with

$$\beta_\infty = ||u||^2 + ||\mu|| = \frac{||u||^2 + ||\mu||}{(|u|^2 + ||\nu||)^{\frac{2}{N}}}$$

and

$$\gamma_\infty = \alpha |u|^2 = \alpha \frac{||u||^2}{(|u|^2 + ||\nu||)^{\frac{2}{N}}}.$$

If $u = 0$, then

$$\beta_\infty = \frac{||\mu||}{||\nu||^2} \geq \frac{S}{2\pi},$$

a contradiction. So $u \neq 0$.

We claim that $||\mu|| = 0$. We argue by contradiction and suppose that $||\mu|| \neq 0$. If $||\nu|| = 0$, then $S_\alpha > I_\alpha(u)$, which is impossible. So $||\nu|| \neq 0$.

Let $x_0 := |u|^2$, so that $1 - x_0 = ||\nu||$. We define $f$, $g$ and $h : [0, 1] \to \mathbb{R}$ by

$$f(x) := \gamma x^{2\#} + \sqrt{\gamma x^{2\#}} + 4\beta x^{2\#} + 4\frac{||\mu||}{||\nu||^2}(1 - x)^{\frac{2}{N}},$$

$$g(x) := \beta x^{2\#} + \frac{||\mu||}{||\nu||^2}(1 - x)^{\frac{2}{N}} + 4|\nu|^2,$$

and

$$h := f^N + 2 \cdot 2^* f^{N-2} g,$$

for $\beta = \beta(u)$ and $\gamma = \gamma(u)$. The value $S_\alpha$ is

$$S_\alpha = \frac{1}{4(2\pi)\frac{2}{N}} [h(x_0)]^{\frac{2}{N}} .$$
We wish to prove that the minimum of \( h \) occurs at 0 or 1. The former case corresponds to \( u = 0 \) and the latter to \( \| \nu \| = 0 \). In either case we are led to a contradiction. This will prove that \( \| \mu \| = 0 \), thereby establishing the claim.

The derivative of \( h \) is

\[
h' = f^{N-3}[N(f^2 + 4g)f' + 2 \cdot 2^* fg'].
\]

Since

\[
f^2 + 4g = 2f \sqrt{\gamma^2 x^{2^*} + 4g},
\]

we can write

\[
h' = 2f^{N-2} \left[ N \sqrt{\gamma^2 x^{2^*} + 4g} f' + 2^* g' \right].
\]

The expression for \( h' \) can be further simplified by computing \( f' \):

\[
\sqrt{\gamma^2 x^{2^*} + 4g} f' = \frac{1}{2^*} \left[2^* \gamma x^{2^* - 1} \sqrt{\gamma^2 x^{2^*} + 4g} + 2^* \gamma^2 x^{2^* - 1} + 4g' \right]
\]

\[
= \frac{2^*}{2^*} \left[ \gamma x^{2^* - 1} f + 2^* \gamma \frac{2^*}{2^*} g' \right].
\]

This yields

\[
h' = 2(N - 1)f^{N-2} \left[ \gamma x^{2^* - 1} f + 2^* g' \right]
\]

\[
= 2(N - 1)f^{N-2} \left[ \gamma x^{2^* - 1} f + 2^* \gamma \frac{2^*}{2^*} g' \right].
\]

We notice that \( h'(0) = +\infty \) and \( h'(1) = -\infty \); at a zero of \( h' \), \( g' < 0 \).

At a point of minimum of \( h \) in the interior of \([0, 1] \), \( h' = 0 \) and

\[
\sqrt{\gamma^2 x^{2^*} + 4g} f' = \frac{2^*}{2^*} \left[ \gamma x^{2^* - 1} f + 2^* g' - 2^* \left( 1 - \frac{2^*}{2^*} \right) g' \right]
\]

\[
= -(2^* - 2)g';
\]

we notice that at a zero of \( h' \), \( f' > 0 \).

We consider

\[
\kappa := -2^* x^{1-\frac{2^*}{2^*}} g',
\]

whose derivative is

\[
\kappa' = -(2^* - 2^*) x^{-\frac{2^*}{2^*}} g' - 2^* x^{2^* - \frac{2^*}{2^*}} g''
\]

\[
> -(2^* - 2^*) x^{-\frac{2^*}{2^*}} g'
\]

\[
= x^{-\frac{2^*}{2^*}} \sqrt{\gamma^2 x^{2^*} + 4g} f' \quad \text{for} \quad h' = 0
\]

\[
> \gamma f'.
\]

The zeros of \( h' \) occur when \( \gamma f = \kappa \). We just proved that \( \kappa' > \gamma f' \) at the zeros of \( h' \). This implies that the graphs of \( \gamma f \) and \( \kappa \) can cross at most once, and that \( h' \) has at most one zero. If the function \( h \) were to have a minimum in the interior of \([0, 1] \), then \( h' \) would have at least three zeros because \( h'(0) = +\infty \) and \( h'(1) = -\infty \). We conclude that \( h \) has no minimum
inside $[0,1]$. (The conditions on the derivative of $h$ at the end points of the interval, or the fact that the graphs of $\gamma f$ and $\kappa$ cross, imply that $h'$ does vanish inside $[0,1]$, at a point of maximum of $h$.) Therefore the minimum of $h$ occurs either at 0 or 1 and we have proved our claim.

Since $|\mu|=0$, the function $u$ is a minimizer for $I_\alpha$. \hfill \Box

**Lemma 3.2.** The map $\alpha \mapsto S_\alpha$ is continuous on $[0, +\infty]$.\hfill \Box

**Proof.** Let $\bar{\alpha} \in [0, +\infty[$. First we prove that $\alpha \mapsto S_\alpha$ is continuous from the right at $\bar{\alpha}$. If $S_{\bar{\alpha}} = \frac{S}{2^{\frac{N}{2}}}$, then continuity from the right at $\bar{\alpha}$ is obvious. If $S_{\bar{\alpha}} < \frac{S}{2^{\frac{N}{2}}}$, let $u_{\bar{\alpha}}$ be a minimizer of $I_{\bar{\alpha}}$, which exists by the previous lemma. If $\alpha > \bar{\alpha}$, then $S_{\bar{\alpha}} \leq S_\alpha \leq I_\alpha(u_{\bar{\alpha}}) \to S_{\bar{\alpha}}$ as $\alpha \searrow \bar{\alpha}$. This proves continuity from the right at $\bar{\alpha}$.

To prove continuity from the left we show that $\lim_{\alpha \searrow \bar{\alpha}} S_\alpha = S_{\bar{\alpha}}$. If the value of the limit on the left hand side is $\frac{S}{2^{\frac{N}{2}}}$, then this equality is obvious. So suppose $\lim_{\alpha \searrow \bar{\alpha}} S_\alpha < \frac{S}{2^{\frac{N}{2}}}$. Choose a sequence $\alpha_k \searrow \bar{\alpha}$ and $u_k \in H^1(\Omega)$, with $|u_k|^2 = 1$, such that $I_{\alpha_k}(u_k) = S_{\alpha_k}$. By (11), the sequence $(u_k)$ is bounded in $H^1(\Omega)$ and we can assume that $u_k \rightharpoonup u$ in $H^1(\Omega)$. An application of the concentration-compactness principle, as in the previous lemma, shows that $u \neq 0$ and

$$\lim_{k \to \infty} I_{\alpha_k}(u_k) \geq I_{\bar{\alpha}}(u).$$

So,

$$S_{\bar{\alpha}} \leq I_{\bar{\alpha}}(u) \leq \lim_{k \to \infty} I_{\alpha_k}(u_k) = \lim_{\alpha \searrow \bar{\alpha}} S_\alpha$$

and $S_{\bar{\alpha}} = \lim_{\alpha \searrow \bar{\alpha}} S_\alpha$. \hfill \Box

By the previous lemma, the value

$$\alpha_0 := \begin{cases} +\infty, &\text{if } S_\alpha < \frac{S}{2^{\frac{N}{2}}} \text{ for all } \alpha \in [0, +\infty[, \\ \min \left\{ \alpha \in [0, +\infty] \left| S_\alpha = \frac{S}{2^{\frac{N}{2}}} \right. \right\}, &\text{otherwise.} \end{cases} \tag{18}$$

is well defined. By (17) it is not zero. Lemma 3.1 implies the following two corollaries:

**Corollary 3.3.** The map $\alpha \mapsto S_\alpha$ is strictly increasing on $[0, \alpha_0]$.\hfill \Box

**Corollary 3.4.** If $\alpha \in [0, \alpha_0[$, then $(1_\alpha)$ has a least energy solution $u_\alpha$. If $\alpha \in ]\alpha_0, +\infty[$, then $(1_\alpha)$ does not have a least energy solution.

This proves (i) of Theorem 2.1. Assertion (ii) of Theorem 2.1 will also follow once we establish that $\alpha_0$ is finite.

**Lemma 3.5.** If $S_\alpha < \frac{S}{2^{\frac{N}{2}}}$ for all $\alpha \geq 0$, then

$$\lim_{\alpha \to +\infty} S_\alpha = \frac{S}{2^{\frac{N}{2}}}. \tag{19}$$

Suppose $\alpha_k \to +\infty$ as $k \to +\infty$ and $u_k$ is a minimizer for $I_{\alpha_k}$ satisfying $(1_{\alpha_k})$. Then $u_k \rightharpoonup 0$ in $H^1(\Omega)$ and

$$M_k := \max_{\Omega} u_k$$

converges to $+\infty$, as $k \to \infty$.\hfill \Box
Proof. Suppose $S_\alpha < \frac{S}{2^{\frac{\alpha}{2}\pi}}$ for all $\alpha \geq 0$ and choose any sequence $\alpha_k \to +\infty$ as $k \to +\infty$. Let $u_k$ be a minimizer for $I_{\alpha_k}$ satisfying $(1_{\alpha_k})$, which necessarily exists by Lemma 3.1 and rescaling. We claim that $u_k$ is bounded in $H^1(\Omega)$. Indeed, by $(9)$,

$$\frac{1}{(2^\#)^\alpha} \gamma^2(u_k) \leq I_\alpha(u_k) \leq \frac{S}{2^{\frac{\alpha}{2}\pi}}.$$ 

So,

$$\alpha_k |u_k|_2^{2^\#} \leq \left( \frac{2^\#}{2} \right)^\frac{1}{\alpha} S^\frac{1}{2} |u_k|_2^{2^\#}.$$ 

By $(1_{\alpha_k})$,

$$|u_k|_2^{2^\#} = |u_k|^2 + \alpha_k |u_k|_2^{2^\#}.$$ 

Together,

$$|u_k|_2^{2^\# - 2} \leq \beta(u_k) + \left( \frac{2^\#}{2} \right)^\frac{1}{\alpha} S^\frac{1}{2} |u_k|_2^{2^\# - 2} \leq \frac{S}{2^{\frac{\alpha}{2}\pi}} + \left( \frac{2^\#}{2} \right)^\frac{1}{\alpha} S^\frac{1}{2} |u_k|_2^{2^\# - 2},$$

since, by $(11)$, $\beta(u_k) \leq I_{\alpha_k}(u_k) \leq \frac{S}{2^{\frac{\alpha}{2}\pi}}$. So $|u_k|_2$ is bounded. Recalling that $\beta(u_k) = \frac{|u_k|^2}{|u_k|_2^{2^\#}}$, we conclude that $u_k$ is bounded in $H^1(\Omega)$.

From $(20)$, we conclude that $u_k \to 0$ in $H^1(\Omega)$. We can assume that $u_k \to 0$ a.e. on $\Omega$, and $|\nabla u_k|^2 \to \mu$ and $|u_k|_2^2 \to \nu$ in the sense of measures on $\Omega$. Then

$$\lim_{k \to \infty} |\nabla u_k|_2^2 = ||\mu||$$ 

and

$$\lim_{k \to \infty} |u_k|_2^{2^\#} = ||\nu||,$$

where

$$\frac{S}{2^{\frac{\alpha}{2}\pi}} ||\nu||_2^{\frac{\alpha}{2}} \leq ||\mu||.$$ 

Thus

$$\frac{S}{2^{\frac{\alpha}{2}\pi}} \geq \lim_{k \to \infty} S_{\alpha_k} = \lim_{k \to \infty} I_{\alpha_k}(u_k) \geq \frac{||\mu||}{||\nu||_2^{\frac{\alpha}{2}}} \geq \frac{S}{2^{\frac{\alpha}{2}\pi}},$$

and the inequalities in $(24)$ are equalities. This proves $(19)$.

From $(1_{\alpha_k})$, the values $M_k$ satisfy

$$a + \alpha_k M_k^{2^\# - 2} \leq M_k^{2^\# - 2}$$

and consequently $M_k \to +\infty$ as $k \to +\infty$. \hfill \Box

Lemma 3.6. Let $S_{\alpha_k} < \frac{S}{2^{\frac{\alpha}{2}\pi}}$ and $S_{\alpha_k} \to \frac{S}{2^{\frac{\alpha}{2}\pi}}$ as $\alpha_k \to \alpha_0 \in ]0, +\infty]$. Denote by $u_k \in H^1(\Omega)$ a minimizer for $I_{\alpha_k}$ satisfying $(1_{\alpha_k})$. In case $\alpha_0 < +\infty$ suppose that $u_k \to 0$. Then

$$\lim_{k \to \infty} |\nabla u_k|_2^2 = \lim_{k \to \infty} |u_k|_2^{2^\#} = \frac{S^{\frac{\alpha}{2}}}{2}.$$ 

(26)
Moreover, if \( \alpha_0 = +\infty \), or if \( \alpha_0 < +\infty \) and we further assume that \( \lim_{\alpha_k \to \alpha_0} M_k = +\infty \), then we also have

\[
\lim_{k \to \infty} \alpha_k \delta_k = 0,
\]

\[
\lim_{k \to \infty} |\nabla u_k - \nabla U_{\delta_k} p_k|_2 = 0
\]

and \( P_k \in \partial \Omega \), for large \( k \). Here, we are denoting

\[
\delta_k := \frac{M - 2N - \delta}{\delta_k},
\]

and \( P_k \) is such that \( M_k = u_k(P_k) \).

**Note.** If \( \alpha_0 = +\infty \), Lemma 3.5 guarantees the conditions \( S_{\alpha_k} \to S_{2^*} \), \( u_k \rightharpoonup 0 \) and \( M_k \to +\infty \) are satisfied.

**Proof.** By (1.1), \( u_k \) satisfies (20). Since \( u_k \to 0 \) in \( H^1(\Omega) \), \( u_k \) is bounded in \( H^1(\Omega) \). Therefore, (21), (22), (23) and (24) hold, with equalities in (24). Hence, \( \beta(u_k) \to \frac{S}{2^*} \). From (10), \( \delta(u_k) \to 0 \) and

\[
\lim_{k \to \infty} \alpha_k |u_k|^{2^*_\#}_\# = 0,
\]

as \( u_k \) is bounded in \( H^1(\Omega) \). Taking limits in (20) as \( k \to \infty \),

\[
||\nu|| = ||\mu||.
\]

Combining (24) and (29), equalities (26) follow.

We now use the Gidas and Spruck blow up technique [12]. Let \( v_k(x) := \frac{\delta_k}{\delta_k(x + P_k)} u_k(\delta_k x + P_k) \) for \( x \in \Omega_k := (\Omega - P_k)/\delta_k \), so that

\[
\begin{cases}
-\Delta v_k + a\delta_k^2 v_k + \alpha_k \delta_k v_k^{2^*-1} = v_k^{2^*-1} & \text{in } \Omega_k, \\
0 < v_k \leq v_k(0) = 1 & \text{in } \Omega_k, \\
\frac{\partial v_k}{\partial \nu} = 0 & \text{on } \partial \Omega_k.
\end{cases}
\]

Rewriting (25) in terms of the \( \delta_k \),

\[
a_0 \delta_k^2 + \alpha_k \delta_k \leq 1.
\]

So, we can assume that \( P_k \to P_0 \),

\[
\text{dist}(P_k, \partial \Omega)/\delta_k \to L \in [0, +\infty],
\]

\[
\Omega_k \to \Omega_\infty := \{(\hat{x}, x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > -L\}
\]

and \( \alpha_k \delta_k \to \bar{\alpha} \). By the elliptic estimates in [5],

\[
v_k \to v \text{ in } C^2_{\text{loc}}(\Omega_\infty)
\]

where \( v \) satisfies

\[
\begin{cases}
-\Delta v + \bar{\alpha} v^{2^*-1} = v^{2^*-1} & \text{in } \Omega_\infty, \\
0 < v \leq v(0) = 1 & \text{in } \Omega_\infty, \\
\frac{\partial v}{\partial \nu} = 0 & \text{on } \partial \Omega_\infty
\end{cases}
\]

as \( a \delta_k^2 \to 0 \). By lower semicontINUITY of the norm, \( v \in L^{2^*}(\Omega_\infty) \) and \( \nabla v \in L^2(\Omega_\infty) \). So, we can apply Pohozaev’s identity and get \( \bar{\alpha} = 0 \), and thus \( v = U \).
If \( L = +\infty \), then \( \Omega_\infty = \mathbb{R}^N \). From (26),
\[
S^N = \int_{\mathbb{R}^N} |\nabla U|^2 \leq \lim_{k \to \infty} |\nabla u_k|^2 = \frac{S^N}{2},
\]
which is impossible.

So \( L \) is finite. This implies that \( P_0 \in \partial \Omega \). In fact, \( L \) has to be zero since \( v \leq v(0) \). Using a diffeomorphism to straighten a boundary portion of \( \Omega \), the argument in Lemma 2.2 of [APY] shows that \( P_k \in \partial \Omega \) for large \( k \). Finally, from (26), (30) and
\[
\int_{\mathbb{R}^N_+} |\nabla U|^2 = \frac{S^N}{2},
\]
we deduce (28). \( \square \)

As in \([2]\) and \([7]\), let
\[
\mathcal{M} := \{CU_{\varepsilon,y}, C \in \mathbb{R}, \varepsilon > 0, y \in \partial \Omega\}
\]
and \( d(u, \mathcal{M}) := \inf \{||\nabla(u - V)||_2, V \in \mathcal{M}\} \). The set \( \mathcal{M} \setminus \{0\} \) is a manifold of dimension \( N + 1 \). The tangent space \( T_{C_l,\varepsilon_l,y_l}(\mathcal{M}) \) at \( C_l U_{\varepsilon_l,y_l} \) is given by
\[
T_{C_l,\varepsilon_l,y_l}(\mathcal{M}) = \text{span} \left\{ U_{\varepsilon_l,y_l} C \frac{\partial}{\partial \varepsilon} U_{\varepsilon_l,y_l}, C \frac{\partial}{\partial \tau_i} U_{\varepsilon_l,y_l}, 1 \leq i \leq N - 1 \right\}_{(C_l,\varepsilon_l,y_l)}
\]
where \( T_x(\partial \Omega) = \text{span}\{\tau_1, \ldots, \tau_{N-1}\} \).

For large \( k \), the infimum \( d(u_k, \mathcal{M}) \) is achieved:
\[
d(u_k, \mathcal{M}) = ||\nabla(u_k - C_k U_{\varepsilon_k,y_k})||_2 \text{ for } C_k U_{\varepsilon_k,y_k} \in \mathcal{M} \quad (31)
\]
Furthermore,
\[
C_k = 1 + o(1) \quad (32)
\]
y_k \to P_0 and \( \varepsilon_k/\delta_k \to 1 \) (see Lemma 1 of \([7]\) and Lemma 2.3 of \([2]\)). From (27),
\[
\alpha_k \varepsilon_k \to 0. \quad (33)
\]

We define
\[
w_k := u_k - C_k U_{\varepsilon_k,y_k},
\]
so that
\[
\int \nabla U_{\varepsilon_k,y_k} \cdot \nabla w_k = 0. \quad (34)
\]
Now, on the one hand, from (28),
\[
\lim_{k \to \infty} ||\nabla(u_k - C_k U_{\varepsilon_k,y_k})||_2 = 0.
\]
On the other hand, from Poincaré’s inequality, and the fact that both the average of \( u_k \) and the average of \( C_k U_{\varepsilon_k,y_k} \), in \( \Omega \), converge to zero,
\[
\lim_{k \to \infty} ||u_k - C_k U_{\varepsilon_k,y_k}||_{2^*} = 0.
\]
Together,
\[
\lim_{k \to \infty} ||w_k|| = 0. \quad (35)
\]

Our next objective is the upper bound in Lemma 3.11 for \( \int U_{\varepsilon_k,y_k}^{2^*-2} w_k^2 \) in terms of \( ||\nabla w_k||^2 + (2^* - 1)\alpha_k \int U_{\varepsilon_k,y_k}^{2^*-2} w_k^2 \). This will be crucial in the lower bound estimates for the energy in Section 4.
The eigenvalue problems arising from the linearization of \((1_{\alpha_k})\) at \(U_{\varepsilon_k,y_k}\) are related to the eigenvalue problem in

**Lemma 3.7** (Bianchi and Egnell [7], Rey [16]). The eigenvalue problem

\[
\begin{cases}
-\Delta \varphi = \mu U^{2^{*} - 2}\varphi & \text{in } \mathbb{R}^N_+,
\
\frac{\partial \varphi}{\partial \nu} = 0 & \text{on } \partial \mathbb{R}^N_+,
\
\int_{\mathbb{R}^N_+} U^{2^{*} - 2}\varphi^2 < \infty
\end{cases}
\]

admits a discrete spectrum \(\mu_1 < \mu_2 \leq \mu_3 \leq \ldots\) such that \(\mu_1 = 1\), \(\mu_2 = \mu_3 = \ldots = \mu_N = 2^{*} - 1\) and \(\mu_{N+1} > 2^{*} - 1\). The eigenspaces \(V_1\) and \(V_{(2^{*} - 1)}\), corresponding to 1 and \((2^{*} - 1)\), are given by

\[
V_1 = \text{span } U,
\]

\[
V_{(2^{*} - 1)} = \text{span } \left\{ \frac{\partial U_{i,\varepsilon}}{\partial \nu_{\varepsilon}}, \text{ for } 1 \leq i \leq N - 1 \right\} .
\]

We will consider the eigenvalue problems arising from the linearization of \((1_{\alpha_k})\) at \(U_{\varepsilon_k,y_k}\). Let \(\varepsilon > 0\), \(\nu > 0\), and \(y_\varepsilon \in \partial \Omega\) with \(\lim_{\varepsilon \to 0} y_\varepsilon = y_0\). Let \(\{\varphi_{i,\varepsilon}\}_{i=1}^{\infty}\) be a complete set of orthogonal eigenfunctions with eigenvalues \(\mu_{1,\varepsilon} < \mu_{2,\varepsilon} \leq \mu_{3,\varepsilon} \leq \ldots\) for the weighted eigenvalue problem

\[
\begin{cases}
-\Delta \varphi + \nu \varepsilon U^{2^{*}_\varepsilon - 2}\varphi = \mu U^{2^{*}_\varepsilon - 2}\varphi & \text{in } \Omega,
\
\frac{\partial \varphi}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

with \(\varphi_{1,\varepsilon} > 0\) and

\[
\int_{\Omega} U^{2^{*}_\varepsilon - 2}\varphi_{i,\varepsilon}\varphi_{j,\varepsilon} = \delta_{i,j} .
\]

Let

\[
\Omega_\varepsilon := (\Omega - y_\varepsilon)/\varepsilon.
\]

The sets \(\Omega_\varepsilon\) converge to a half space as \(\varepsilon \to 0\). For a function \(v\) on \(\Omega\), we define \(\tilde{v}\) on \(\Omega_\varepsilon\) by

\[
\tilde{v}(x) := \varepsilon^{\frac{N-2}{2}} v(\varepsilon x + y_\varepsilon).
\]

The relation between these eigenvalue problems and the one considered in Lemma 3.7 is given in

**Lemma 3.8.** Suppose \(y_\varepsilon \in \partial \Omega\), \(\lim_{\varepsilon \to 0} y_\varepsilon = y_0\), \(\lim_{\varepsilon \to 0} (\varepsilon \nu_\varepsilon) = 0\) and the sets \(\Omega_\varepsilon\) converge to \(\mathbb{R}^N_+\). Then, up to a subsequence,

\[
\lim_{\varepsilon \to 0} \mu_{i,\varepsilon} = \mu_i
\]

and

\[
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} U^{2^{*}_\varepsilon - 2}(\tilde{\varphi}_{i,\varepsilon} - \tilde{\varphi}_i)^2 = 0 .
\]

The \(\mu_i\) and \(\tilde{\varphi}_i\) satisfy

\[
\begin{cases}
-\Delta \tilde{\varphi}_i = \mu_i U^{2^{*}_\varepsilon - 2}\tilde{\varphi}_i & \text{in } \mathbb{R}^N_+,
\
\frac{\partial \tilde{\varphi}_i}{\partial \nu} = 0 & \text{on } \partial \mathbb{R}^N_+,
\
\int_{\mathbb{R}^N_+} U^{2^{*}_\varepsilon - 2}\tilde{\varphi}_i^2 = 1,
\end{cases}
\]

and the functions \(\tilde{\varphi}_i\) are supposed extended to \(\mathbb{R}^N\) by reflection. In particular, from the previous lemma, \(\mu_1 = 1\), \(\tilde{\varphi}_1 = CU\) for some constant \(C > 0\), \(\mu_i = 2^{*} - 1\) for \(2 \leq i \leq N\) and \(\mu_{N+1} > 2^{*} - 1\). Also, \(\{\tilde{\varphi}_i\}_{i=2}^N\) is in the span of \(\{\partial U_{1,y}/\partial y_i\}_{y=0}, \text{ for } 1 \leq i \leq N - 1\}.\)

The proof of Lemma 3.8 is a consequence of the arguments in the proof of Lemma 3.3 of [APY], of Lemma 3.9 and of Remark 3.10. For the details we refer to the proof of Lemma 5.5 of [14] for parameter \(s\) there equal to one.

**Lemma 3.9.** Suppose \(y_\epsilon \in \bar{\Omega}, \varphi_\epsilon \in H^1(\Omega),\)

\[
\begin{align*}
\int U_{\epsilon,y_\epsilon}^{2-2}\varphi_\epsilon^2 &\to 0, \\
\int \lvert \nabla \varphi_\epsilon \rvert^2 &\to 0,
\end{align*}
\]

as \(\epsilon \to 0\). Then

\[
\int U_{\epsilon,y_\epsilon}^{2}\varphi_\epsilon^2 \to 0,
\]

as \(\epsilon \to 0\).

**Proof.** We denote the average of \(\varphi_\epsilon\) in \(\Omega\) by \(\bar{\varphi}_\epsilon\). By Poincaré’s inequality,

\[
\lvert \varphi_\epsilon - \bar{\varphi}_\epsilon \rvert_{2^*} \to 0.
\]

The limits in this proof are taken as \(\epsilon \to 0\). So we can write \(\varphi_\epsilon = \bar{\varphi}_\epsilon + \eta_\epsilon\), with \(\eta_\epsilon \to 0\) in \(L^{2^*}\). We know that

\[
\int U_{\epsilon,y_\epsilon}^{2}(\bar{\varphi}_\epsilon^2 + 2\bar{\varphi}_\epsilon \eta_\epsilon + \eta_\epsilon^2) = o(1).
\]

We have the following estimates for the three terms on the left hand side:

\[
\int U_{\epsilon,y_\epsilon}^{2}\bar{\varphi}_\epsilon^2 \geq b\bar{\varphi}_\epsilon^2,
\]

for some \(b > 0\), and

\[
\int U_{\epsilon,y_\epsilon}^{2}\eta_\epsilon \bar{\varphi}_\epsilon \leq \lvert \eta_\epsilon \rvert_{2^*} \lvert \bar{\varphi}_\epsilon \rvert \left( \int U_{\epsilon,y_\epsilon}^{\frac{N}{2}} \right)^{\frac{N+2}{2N}} \leq C\lvert \eta_\epsilon \rvert_{2^*} \lvert \bar{\varphi}_\epsilon \rvert \epsilon,
\]

by (39); and

\[
\int U_{\epsilon,y_\epsilon}^{2}\eta_\epsilon^2 \leq \lvert \eta_\epsilon \rvert_{2^*}^2 \left( \int U_{\epsilon,y_\epsilon}^{\frac{N}{2}} \right)^{\frac{2}{N}} \leq C\lvert \eta_\epsilon \rvert_{2^*}^2 \lvert \bar{\varphi}_\epsilon \rvert \log \epsilon \epsilon^{\frac{2N}{N+2}},
\]

by (40). (Inequalities (39), (40) and (41) are in the beginning of the next section.) Thus

\[
b \bar{\varphi}_\epsilon^2 \epsilon \leq C\lvert \bar{\varphi}_\epsilon \rvert \epsilon + o(1).
\]

This shows that \(\bar{\varphi}_\epsilon \sqrt{\epsilon}\) is bounded. But if \(\bar{\varphi}_\epsilon \sqrt{\epsilon}\) is bounded this shows that

\[
\bar{\varphi}_\epsilon \sqrt{\epsilon} \to 0.
\]

(37)

We want to prove that

\[
\int U_{\epsilon,y_\epsilon}^{2}(\bar{\varphi}_\epsilon^2 + 2\bar{\varphi}_\epsilon \eta_\epsilon + \eta_\epsilon^2) = o(1).
\]

For the first term on the left hand side we have, by (39) and then (37),

\[
\int U_{\epsilon,y_\epsilon}^{2}\bar{\varphi}_\epsilon^2 \leq C\bar{\varphi}_\epsilon^2 \epsilon \to 0.
\]
For the third term we have
\[ \int U_{\varepsilon,y_\varepsilon}^{2^* - 2} \varphi_\varepsilon \eta_\varepsilon \leq C |\eta_\varepsilon|_{2^*}^2 \to 0. \]

We claim that the remaining term also converges to zero. This will prove the lemma. For the second term we have the estimate
\[ \zeta_\varepsilon := \left| \int U_{\varepsilon,y_\varepsilon}^{2^* - 2} \varphi_\varepsilon \eta_\varepsilon \right| \leq |\eta_\varepsilon|_{2^*} |\varphi_\varepsilon| \left( \int U_{\varepsilon,y_\varepsilon}^{\frac{N}{N-2}} \frac{8}{N+2} \right)^{\frac{N-2}{N}}. \]

If \( N = 5 \), by (41),
\[ \zeta_\varepsilon \leq C |\eta_\varepsilon|_{2^*} |\varphi_\varepsilon| \varepsilon^{N(1 - \frac{1}{N-2})} \frac{N-2}{N+2} \leq C |\eta_\varepsilon|_{2^*} |\varphi_\varepsilon| \varepsilon^{\frac{N-2}{2}} = C |\eta_\varepsilon|_{2^*} |\varphi_\varepsilon| \varepsilon^{\frac{N}{2}}. \]

If \( N = 6 \), by (40),
\[ \zeta_\varepsilon \leq C |\eta_\varepsilon|_{2^*} |\varphi_\varepsilon| \varepsilon^2 \log \varepsilon \varepsilon^{\frac{N}{2}}. \]

Finally, if \( N \geq 7 \), by (39),
\[ \zeta_\varepsilon \leq C |\eta_\varepsilon|_{2^*} |\varphi_\varepsilon| \varepsilon^2. \]

In all cases, (37) implies that \( \zeta_\varepsilon \to 0. \)

**Remark 3.10.** If in the previous lemma, instead of assuming \( \int |\nabla \varphi_\varepsilon|^2 \to 0 \), we assume that \( \int |\nabla \varphi_\varepsilon|^2 \) is bounded, then we can still conclude \( \varphi_\varepsilon \varepsilon^{-1} \to 0 \) and \( \int U_{\varepsilon,y_\varepsilon}^{2^* - 2} (\varphi_\varepsilon^2 + 2 \varphi_\varepsilon \eta_\varepsilon) = \int_\Omega U^{2^* - 2} (\varphi_\varepsilon^2 + 2 \varphi_\varepsilon \eta_\varepsilon) \to 0, \) as \( \varepsilon \to 0. \)

Using Lemma 3.8 and the arguments in the proof of Lemma 3.4 of [APY], we deduce

**Lemma 3.11.** Suppose \( y_\varepsilon \in \partial \Omega, \lim_{\varepsilon \to 0} y_\varepsilon = y_0 \) and \( \lim_{\varepsilon \to 0} (\varepsilon \nu_\varepsilon) = 0. \) There exists a constant \( \gamma_1 > 0 \) such that, for sufficiently small \( \varepsilon, \)
\[ |\nabla w|^2 + \nu_\varepsilon \int U_{\varepsilon,y_\varepsilon}^{2^* - 2} w^2 \geq (2^* - 1 + \gamma_1) \int U_{\varepsilon,y_\varepsilon}^{2^* - 2} w^2 + O(\varepsilon^2 ||w||^2) \]
for \( w \) orthogonal to \( T_{1,\varepsilon,y_\varepsilon}(\mathcal{M}). \)

**4. Nonexistence of Least Energy Solutions**

In this section we prove (ii) of Theorem 2.1. The idea of the proof is to obtain a lower bound for \( I_\alpha \) and show that if \( \alpha_0, \) defined in (18), is infinite, then the least energy solutions \( u_k \) of \((1_\alpha)\) have energy \( I_{\alpha_k}(u_k) > \frac{8}{2^*} \), for large \( \alpha_k. \) This is impossible. Therefore \( \alpha_0 \) is finite. By Corollary 3.4, (ii) of Theorem 2.1 follows.

Assume
\[ u_k = C_k U_{\varepsilon_k,y_k} + w_k, \]
(26), (31), (32), (33) and (35). From (10), \( I_\alpha \) has the lower bound
\[ I_\alpha \geq \beta \left( 1 + \frac{4}{2^*} \right) \]
(38)
(this is also checked in (63) of Appendix A). We will expand \( \beta \) and \( \delta \) to second order around \( U_{\varepsilon_k,y_k}. \) We start by deriving estimates for the terms that appear in this expansion.
We recall, from Brézis and Nirenberg [9], that, for \( y \in \bar{\Omega} \), there exist positive constants \( c_1 \) and \( c_2 \) such that:

if \( 1 \leq q < \frac{N}{N-2} \), then

\[
c_1 \varepsilon^{\frac{q(N-2)}{2}} \leq |U_{\varepsilon,y}|_q^q \leq c_2 \varepsilon^{\frac{q(N-2)}{2}};
\]  

(39)

if \( q = \frac{N}{N-2} \), then

\[
c_1 \varepsilon^{\frac{N}{2}} \log \varepsilon \leq |U_{\varepsilon,y}|_q^q \leq c_2 \varepsilon^{\frac{N}{2}} \log \varepsilon;
\]  

(40)

and if \( \frac{N}{N-2} < q \leq \frac{2N}{N-2} \), then

\[
c_1 \varepsilon^{N(1-\frac{q}{2})} \leq |U_{\varepsilon,y}|_q^q \leq c_2 \varepsilon^{N(1-\frac{q}{2})}.
\]  

(41)

For brevity, we shall write \( U_k := U_{\varepsilon_k,y_k} \).

**Estimate for \( |U_k|_{2^*}^2 \):** For \( N \geq 5 \), \( \frac{N}{N-2} < 2 \). From (41),

\[
|U_k|_{2^*}^2 = O(\varepsilon_{k})^2.
\]  

(42)

**Estimate for \( |U_k|_{2^*}^2 \):** Since \( y_k \in \partial \Omega \) and we are supposing that the domain is smooth,

\[
|U_k|_{2^*}^2 = \frac{2^* B(N)\varepsilon_k}{2} + o(\varepsilon_k),
\]  

(43)

with

\[
B(N) = \frac{1}{2^*} \int_{\mathbb{R}^N} u^{2^*} \quad \text{as proved in Appendix B.}
\]

Here \( \omega_N \) is the volume of the \( N-1 \) dimensional unit sphere.

**Estimate for \( |\nabla U_k|_2^2 \) and for \( |U_k|_{2^*}^2 \):** From Adimurthi and Mancini [1], since \( N \geq 5 \),

\[
|\nabla U_k|_2^2 = \frac{S^{\frac{N}{2}}}{2} - \bar{C}_1 \varepsilon_k + O(\varepsilon_k^2)
\]  

(44)

and

\[
|U_k|_{2^*}^2 = \frac{S^{\frac{N}{2}}}{2} - \bar{C}_2 \varepsilon_k + O(\varepsilon_k^2),
\]  

(45)

where

\[
\bar{C}_1 = H(y_k) \frac{\omega_{N-1}(N-2)^2 \Gamma\left(\frac{N+1}{2}\right) \Gamma\left(\frac{N-1}{2}\right)}{4 \Gamma(N)} [N(N-2)]^{-\frac{N-2}{2}}
\]

and

\[
\bar{C}_2 = H(y_k) \frac{\omega_{N-1}}{4} \frac{\Gamma\left(\frac{N+1}{2}\right) \Gamma\left(\frac{N-1}{2}\right)}{\Gamma(N)} [N(N-2)]^{-\frac{N}{2}}.
\]
Here $H(y_k)$ denotes the mean curvature of $\partial\Omega$ at $y_k$ with respect to the unit outward normal and, as above, $\omega_N$ is the volume of the $N - 1$ dimensional unit sphere. This yields
\[
\frac{|\nabla U_k|^2}{|U_k|^2} = \frac{S}{2^\frac{N}{2}} - 2^\frac{N-2}{N} S H(y_k) A(N) \varepsilon_k + O(\varepsilon_k^2) \tag{46}
\]
with
\[
A(N) = \frac{2}{N} \frac{\omega_{N-1}}{\omega_N} \frac{\Gamma \left( \frac{N+1}{2} \right) \Gamma \left( \frac{N-3}{2} \right)}{\Gamma \left( \frac{N}{2} \right) \Gamma \left( \frac{N-2}{2} \right)} = \frac{N-1}{N} \frac{1}{\sqrt{\pi}} \frac{\Gamma \left( \frac{N-3}{2} \right)}{\Gamma \left( \frac{N-2}{2} \right)}.
\tag{47}
\]
To justify the last equality we recall that if $\omega_N(r)$ is the volume of the $N - 1$ dimensional sphere with radius $r$, then
\[
\omega_N(r) = \int_0^\pi \omega_{N-1}(r \sin \varphi) r d\varphi = r^{N-1} \omega_{N-1}(1) \int_0^\pi \sin^{N-2} \varphi d\varphi,
\]
which yields
\[
\frac{\omega_{N-1}}{\omega_N} = \frac{\omega_{N-1}(1)}{\omega_N(1)} = \frac{1}{\sqrt{\pi}} \frac{\Gamma \left( \frac{N}{2} \right)}{\Gamma \left( \frac{N-1}{2} \right)}.
\]
We mention that the Talenti instanton we use does not coincide with the one in [1]. Denoting the Talenti instanton in Adimurthi and Mancini by $V$, $V(\cdot) = U((N(N-2))^{1/2} \cdot)$. 

Estimate for $\int U_k w_k$:

**Lemma 4.1.**

\[
\left| \int U_k w_k \right| \leq \begin{cases} O \left( \varepsilon_k^2 ||w_k|| \right) & \text{if } N = 5, \\ O \left( \varepsilon_k^2 \log \varepsilon_k \varepsilon_k^{\frac{2}{N-2}} ||w_k|| \right) & \text{if } N = 6, \\ O \left( \varepsilon_k^2 ||w_k|| \right) & \text{if } N \geq 7. \end{cases}
\tag{48}
\]

**Proof.**

\[
\left| \int U_k w_k \right| \leq |w_k|_{2^*} \left( \int U_k \frac{2N}{N+2} \right)^\frac{N+2}{2N}.
\]

If $N = 5$, then $\frac{2N}{N+2} < \frac{N}{N-2}$. By (39),
\[
\left| \int U_k w_k \right| \leq C ||w_k|| \varepsilon_k^{\frac{N-3}{2}} = O \left( \varepsilon_k^2 ||w_k|| \right).
\]

If $N = 6$, then $\frac{2N}{N+2} = \frac{N}{N-2}$. By (40),
\[
\left| \int U_k w_k \right| \leq C ||w_k|| \left( \varepsilon_k \log \varepsilon_k \varepsilon_k^{\frac{2}{N-2}} ||w_k|| \right)^\frac{N+2}{2N} = O \left( \varepsilon_k^2 \log \varepsilon_k \varepsilon_k^{\frac{2}{N-2}} ||w_k|| \right).
\]

If $N \geq 7$, then $\frac{N}{N-2} < \frac{2N}{N+2}$. By (41),
\[
\left| \int U_k w_k \right| \leq C ||w_k|| \left( \varepsilon_k^{N \left( 1 - \frac{2N}{N+2} \right) \frac{N-3}{2N-4}} \right)^\frac{N+2}{2N} = O \left( \varepsilon_k^2 ||w_k|| \right).
\]

$\square$
Estimate for $\int U_k^{2^* - 1} w_k$: From [APY], Equations (3.15), for $N \geq 5$,
\[ \int U_k^{2^* - 1} w_k = O(\varepsilon_k ||w_k||). \]  
(49)

Estimate for $\int U_k^{2^* - 1} |w_k|$: Since $\frac{2N}{N+2} > 1$,
\[ \int U_k^{2^* - 1} |w_k| \leq |w_k|\varepsilon_k \left( \int U_k^{\frac{N}{N-2}} \right)^{\frac{N+2}{2N}} \]
\[ \leq C|w_k|\varepsilon_k \left( 1 - \frac{N}{N+2} \right)^{\frac{N+2}{2}} \]
\[ = O(\varepsilon_k ||w_k||). \]  
(50)

Estimate for $\int U_k^{2^* - 2} w_k^2$: 
\[ \int U_k^{2^* - 2} w_k^2 = O(||w_k||^2). \]  
(51)

Now we will obtain a lower bound for $I_{\alpha_k}(u_k)$. Let $v_k = u_k/C_k = U_k + w_k = U_k + w_k/C_k$. Because of (32), the sequence $(v_k)$ satisfies (26) and the sequence $\tilde{w}_k$ satisfies (35). Of course, $d(v_k, M)$ is achieved by $U_k$. Because $I$ is homogeneous of degree zero, $I_{\alpha_k}(u_k) = I_{\alpha_k}(v_k)$. We will compute $I_{\alpha_k}(v_k)$ but we will still call $v_k$ by $u_k$, and $\tilde{w}_k$ by $w_k$.

Going back to (38), $I_{\alpha}(u_k)$ is bounded below by the sum of $\beta(u_k)$ and $\frac{1}{2} \beta(u_k)\delta(u_k)$. We start by obtaining lower bounds for $\beta(u_k)$ and $\frac{1}{2} \beta(u_k)\delta(u_k)$ separately. The expression for $\beta(u_k)$ involves two terms: $||u_k||^2$ and $|u_k|^{2^*}$. The first one is obviously
\[ ||u_k||^2 = ||U_k||^2 + 2(\int \nabla U_k \cdot \nabla w_k + a \int U_k w_k) + ||w_k||^2 \]  
(52)
\[ = A_1 + A_2 + A_3. \]

For the second term we use

Lemma 4.2 ([APY] Lemma 3.5). Let $q > 1$ and $L$ be a non negative integer with $L \leq q$. Let $V$ and $\omega$ be measurable functions on $\Omega$ with $V \geq 0$ and $V + \omega \geq 0$. Then
\[ \int (V + \omega)^q \leq \sum_{i=0}^L \frac{q(q-1)\ldots(q-i+1)}{i!} \int V^{q-i}\omega^i \]
\[ + O \left( \int |V^{q-r}|\omega^r + |\omega|^q \right), \]
where $r = \min\{L+1, q\}$.

Taking $L = 2$ and $q = 2^*$,
\[ |u_k|^{2^*} = |U_k|^{2^*} + 2^* \int U_k^{2^* - 1} w_k + \frac{2^*(2^*-1)}{2} \int U_k^{2^* - 2} w_k^2 + O(||w_k||^r), \]
(53)
where $r = \min\{2^*, 3\}$, i.e., $r = 3$ if $N = 5$, and $r = 2^*$ if $N > 5$. The inequality
\[ (1 + z)^{-\eta} \geq 1 - \eta z, \]  
(54)
for $\eta > 0$ and $z \geq -1$, implies
\[
|u_k|^2 \geq |U_k|^2 \left( 1 - \frac{2\int U_k^{2^*-1}w_k}{|U_k|^{2^*}} \right.
- \left( 2^*-1 \right) \int U_k^{2^*-2}w_k^2 \frac{|U_k|^{2^*}}{|U_k|^{2^*}} + O(||w_k||^r) \bigg) \right.
\]
\[
= B_1 + B_2 + B_3 + B_4. \tag{55}
\]

Let
\[
l := \frac{\|\nabla U_k\|}{|U_k|^{2^*}}.
\]

From (44) and (45),
\[
l = 1 + O(\varepsilon_k). \tag{56}
\]

Using (52) and (55), we can write,
\[
\beta(u_k) \geq \bar{I}_1 + \bar{I}_2 + \bar{I}_3 + \bar{I}_4,
\]
where
\[
\bar{I}_1 = \frac{|U_k|}{|U_k|^{2^*}} = A_1 B_1,
\]
\[
\bar{I}_2 = \frac{2}{|U_k|^{2^*}} \left[ \int \nabla U_k \cdot \nabla w_k + a \int U_k w_k - l \int U_k^{2^*-1}w_k \right]
\]
\[
= A_2 B_1 + A_1 B_2,
\]
\[
\bar{I}_3 = \frac{1}{|U_k|^{2^*}} \left[ ||w_k||^2 - l(2^*-1) \int U_k^{2^*-2}w_k^2 \right]
\]
\[
= A_3 B_1 + A_1 B_3
\]
and
\[
\bar{I}_4 = [(A_1 + A_3)B_4] + [A_2(B_2 + B_3 + B_4)] + A_3 B_2 + A_3 B_3
\]
\[
= E_1 + E_2 + E_3 + E_4.
\]

By (42) and (45),
\[
\bar{I}_1 = \frac{\|\nabla U_k\|}{|U_k|^{2^*}} + o(\varepsilon_k).
\]

We recall (35), $w_k \rightarrow 0$ in $H^1(\Omega)$.

By (34), the first of the four terms in $\bar{I}_2$ is zero; by Lemma 4.1 and by (49) the second and the third ones are $o(\varepsilon_k)$:
\[
\bar{I}_2 = o(\varepsilon_k).
\]

By (45), (51) and (56),
\[
\bar{I}_3 = 2^{\frac{N-2}{N}} S^{\frac{2-N}{2}} \left[ ||w_k||^2 - (2^*-1) \int U_k^{2^*-2}w_k^2 \right]
\]
\[
= E_1 + E_2 + E_3 + E_4.
\]

The term $E_1$ is $o(||w_k||^2)$ because $B_4$ is $o(||w_k||^2)$. The term $E_2$ is $o(\varepsilon_k)$ because, from Lemma 4.1, $A_2$ is $o(\varepsilon_k)$. The term $E_3$ is $o(\varepsilon_k)$ because, from
(49), $B_2$ is $o(\varepsilon_k)$. Finally, the term $E_4$ is $o(||w_k||^2)$ because both $A_3$ and $B_3$ are $O(||w_k||^2)$. Therefore,

$$I_4 = o(\varepsilon_k) + o(||w_k||^2).$$

Combining the expressions for $I_1$, $I_2$, $I_3$ and $I_4$,

$$\beta(u_k) = \frac{\nabla U_k^2}{|U_k|^{2^*}} + 2^{\frac{N-2}{N}} S \frac{2^{2-N}}{2} \left[ ||w_k||^2 - (2^* - 1) \int U_k^{2^*-2} w_k^2 \right] + o(\varepsilon_k) + o(||w_k||^2)$$

$$\geq \frac{\nabla U_k^2}{|U_k|^{2^*}} + 2^{\frac{N-2}{N}} S \frac{2^{2-N}}{2} \left[ \gamma_2 ||w_k||^2 - (2^* - 1) \int U_k^{2^*-2} w_k^2 \right] + o(\varepsilon_k),$$

for any fixed number $\gamma_2 < 1$, because $a > 0$. This is our lower bound for $\beta(u_k)$.

Now we turn to the term $\frac{4}{2^*} \beta(u_k) \delta(u_k)$ and write

$$\frac{4}{2^*} \beta(u_k) \delta(u_k) = \frac{2}{2^*} \frac{||u_k||}{|u_k|^{2^*/2}} \alpha_k ||w_k||^{2^*}$$

(57)

We obtain a lower bound for $||u_k||$ from (52). Using (34), (42), (44) and Lemma 4.1,

$$||u_k|| \geq \left( \frac{S^{2^*}}{2} \right)^{\frac{1}{2^*}} + o(\varepsilon_k) + O(||w_k||^2).$$

We obtain a lower bound for $|u_k|^{-(2+2^*/2)}$ from (53). Using (45), (49), (51) and (54),

$$|u_k|^{-(2+2^*/2)} \geq \left( \frac{S^{2^*}}{2} \right)^{-\frac{1}{2} - \frac{2}{2^*}} + o(\varepsilon_k) + O(||w_k||^2).$$

For the product we obtain the lower bound

$$\frac{||u_k||}{|u_k|^{2^*/2}} \geq \frac{2^{\frac{N-2}{N}} S^{\frac{2-N}{2}}}{2^*} + o(\varepsilon_k) + O(||w_k||^2)$$

(58)

$$= D_1 + D_2 + D_3.$$

To estimate the term $\alpha_k ||w_k||^{2^*}$ we do not use Lemma 4.2 because it would give rise to a term $O \left( \alpha_k ||w_k||^{2^*} \right)$, for which we do not have estimates. Instead we use this calculus

**Lemma 4.3.** Let $\eta > 2$. For any $z \geq -1$,

$$\frac{\eta(\eta - 1)}{2} z^2 - C|z| + 1 \leq (z + 1)^\eta,$$

(59)

where $\tilde{C} = 1 + \eta(\eta - 1)/2$.

**Proof.** The difference between the right hand side and the left hand side is zero for $z = -1$ and $z = 0$. It is increasing for $z > 0$ and concave for $-1 < z < 0$. $\square$
((59) also hold for \( \eta = 2 \), with equality for negative values of \( z \).) As a consequence of Lemma 4.3,
\[
|u_k|_{2^*}^2 \geq |U_k|_{2^*}^2 - 2^{2^*} \hat{C} \int U_k^{2^*-1} |w_k| + \frac{2^* (2^* - 1)}{2} \int U_k^{2^*-2} w_k^2,
\]
with
\[
\hat{C} := \frac{\hat{C}}{2^*} = \frac{1}{2^*} + \frac{2^* - 1}{2}.
\]
Using (43) and (50),
\[
\frac{2}{2^*} \alpha_k |u_k|_{2^*}^2 \geq B(N) \alpha_k \varepsilon_k + (2^* - 1) \alpha_k \int U_k^{2^*-2} w_k^2 + o(\alpha_k \varepsilon_k) \quad (60)
\]
\[
= F_1 + F_2 + F_3.
\]
We will now substitute (58) and (60) in (57). On the one hand,
\[
(D_1 + D_2 + D_3) F_3 = o(\alpha_k \varepsilon_k)
\]
and
\[
(D_2 + D_3) F_1 = o(\alpha_k \varepsilon_k).
\]
On the other hand, by (36),
\[
D_2 F_2 = O \left( \alpha_k \varepsilon_k^2 \log \varepsilon_k \| \frac{\partial}{\partial x} w_k \|_2^2 \right) = o(\alpha_k \varepsilon_k).
\]
So,
\[
\frac{4}{2^*} \beta(u_k) \delta(u_k) \geq 2^{\frac{N-2}{N-2^*}} S^{\frac{2-N}{2}} B(N) \alpha_k \varepsilon_k
\]
\[
+ 2^{\frac{N-2}{N-2^*}} S^{\frac{2-N}{2}} (2^*-1) \alpha_k \int U_k^{2^*-2} w_k^2 + o(\alpha_k \varepsilon_k)
\]
\[
\geq 2^{\frac{N-2}{N-2^*}} S^{\frac{2-N}{2}} \left[ B(N) \alpha_k \varepsilon_k + \gamma_2 (2^*-1) \alpha_k \int U_k^{2^*-2} w_k^2 \right]
\]
\[
+ o(\alpha_k \varepsilon_k),
\]
for any fixed number \( \gamma_2 < 1 \). This is our lower bound for \( \frac{4}{2^*} \beta(u_k) \delta(u_k) \).
Combining the lower bounds for \( \beta(u_k) \) and for \( \frac{4}{2^*} \beta(u_k) \delta(u_k) \),
\[
I_{\alpha_k}(u_k) \geq \frac{|\nabla U_k|^2}{|U_k|_{2^*}^2} + 2^{\frac{N-2}{N-2^*}} S^{\frac{2-N}{2}} B(N) \alpha_k \varepsilon_k
\]
\[
+ 2^{\frac{N-2}{N-2^*}} S^{\frac{2-N}{2}} \left[ \gamma_2 \| w_k \|_2^2 + \gamma_2 (2^*-1) \alpha_k \int U_k^{2^*-2} w_k^2
\]
\[
- (2^*-1) \int U_k^{2^*-2} w_k^2 \right] + o(\alpha_k \varepsilon_k).
\]
From Lemma 3.11, the term inside the square parenthesis is greater than
\[
\left[ \left( \gamma_2 - \frac{(2^*-1)}{(2^*-1)} \right) \left( \| w_k \|_2^2 + (2^*-1) \alpha_k \int U_k^{2^*-2} w_k^2 \right) + o(\varepsilon_k) \right].
\]
Choosing \( \gamma_2 \geq \frac{(2^*-1)}{(2^*-1)} \), yields that this term is greater than \( o(\varepsilon_k) \). Hence,
\[
I_{\alpha_k}(u_k) \geq \frac{|\nabla U_k|^2}{|U_k|_{2^*}^2} + 2^{\frac{N-2}{N-2^*}} S^{\frac{2-N}{2}} B(N) \alpha_k \varepsilon_k + o(\alpha_k \varepsilon_k).
\]
Substituting (46) into this expression, we obtain
\[
I_{\alpha_k}(u_k) \geq \frac{S}{2^{\frac{2N}{2^N}}} + 2 \frac{\alpha_k}{2^N} S \frac{N}{2^N} B(N) \alpha_k \xi_k \left[ 1 - \frac{S}{2^{\frac{2N}{2^N}}} \frac{A(N)}{B(N)} H(y_k) \frac{1}{\alpha_k} + o(1) \right]
\]
for large \( k \).

So assume \( \alpha_0 \), in (18), is \(+\infty\). Choose a sequence \( \alpha_k \to +\infty \) as \( k \to +\infty \) and denote by \( u_k \) a minimizer for \( I_{\alpha_k} \) satisfying \((1_{\alpha_k})\). From Lemmas 3.5 and 3.6, the conditions \((26), (31), (32), (33) \) and \((35) \) hold. Therefore \( S_{\alpha_k} = I_{\alpha_k}(u_k) > \frac{S}{2^{\frac{2N}{2^N}}} \), for large \( k \), which is impossible. By Corollary 3.4, this establishes (ii) of Theorem 2.1.

**Remark 4.4.** Since \( S_{\frac{N}{2^N}} = \int_{\mathbb{R}^N} U^{2^*} = \omega_N \frac{1}{2^N} \sqrt{\pi} \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N}{2} - 1)} [N(N - 2)]^{\frac{N}{2}} \), it follows that \( B(N) = S_{\frac{N}{2^N}} \).

Using
\[
\omega_N = \frac{2\pi \frac{N}{2^N}}{\Gamma \left( \frac{N}{2} \right)}.
\]
the common value is
\[
B(N) = S_{\frac{N}{2^N}} = \frac{\pi \frac{N+1}{2^{N-1}}}{\Gamma \left( \frac{N+1}{2} \right)} [N(N - 2)]^{\frac{N}{2}}.
\]

5. Least energy solutions of \((1_{\alpha_0})\)

In this section we give a lower bound for \( \alpha_0 = \min \left\{ \alpha \mid S_{\alpha} = S/2^N \right\} \), and give partial results concerning existence of least energy solutions of \((1_{\alpha_0})\).

From (10) we obtain

**Lemma 5.1.** There exists a constant \( \bar{c} > \frac{4}{(2^\#)^{2N}} \) such that
\[
I_{\alpha} \leq \beta \left( 1 + \frac{4}{2^\#} \delta + \bar{c} \delta^2 \right).
\]

**Proof.** Consider \( \Lambda : [0, +\infty[ \to \mathbb{R} \), defined by
\[
\Lambda(\delta) := \frac{1}{(2^\#)^{2N}} \left[ \left( \delta + \sqrt{\delta^2 + 1} \right)^N + \frac{2^\#}{2} \left( \delta + \sqrt{\delta^2 + 1} \right)^{N-2} \right]^{\frac{N}{2^N}}.
\]
Since \( \frac{\partial}{\partial \delta} \sqrt{\delta^2 + 1} \bigg|_{\delta = 0} = 0 \) and \( \frac{\partial}{\partial \delta} \frac{2^\#}{2} \left( \delta + \sqrt{\delta^2 + 1} \right)^{N-2} \bigg|_{\delta = 0} = 0 \), the first two derivatives of \( \Lambda \) at zero are
\[
\Lambda'(0) = \frac{1}{(2^\#)^{2N}} \frac{2^\#}{N (2^\#)^{\frac{N}{2^N}-1}} \left( N + \frac{2^\#}{2} (N - 2) \right) = \frac{4}{2^\#}
\]
and
\[
\Lambda''(0) = \frac{1}{(2^\#)^{2N}} \frac{2^\#}{N (2^\#)^{\frac{N}{2^N}-2} (2N)^2} + \frac{2}{2^\#} [N + (N - 2)] = \frac{4}{2^\#} \frac{2^{N-3}}{N-1}.
\]
Fix any number \( c_1 > \frac{2}{2^N-N-1} \). There exists an \( \epsilon > 0 \) such that (61) holds for \( \bar{c} = c_1 \) and \( 0 \leq \delta < \epsilon \).

Fix any number \( c_2 > \frac{4}{(2^N-N-1)^2} \). From (10), there exists an \( L > 0 \) such that (61) holds for \( \bar{c} = c_2 \) and \( \delta > L \).

The inequalities \( \frac{2}{2^N-N-1} < \frac{4}{(2^N-N-1)^2} < \frac{4}{(2^N-N-1)^2} \) show that max \( \left\{ \frac{2}{2^N-N-1}, \frac{4}{(2^N-N-1)^2} \right\} = \frac{4}{(2^N-N-1)^2} \).

By taking \( \bar{c} \geq \max\{c_1, c_2\} \), \( \bar{c} \) sufficiently large, we can guarantee (61) for all \( \delta \in [\epsilon, L] \). \( \square \)

**Lemma 5.2.** If \( \alpha < A(N) \max_{\partial \Omega} H \), then \( S_\alpha < \frac{S}{2^N} \).

*Proof.* Choose \( P \in \partial \Omega \) such that \( H(P) = \max_{\partial \Omega} H \). From (42) and (46),

\[ \beta(U_{\epsilon,P}) = \frac{S}{2^N} - \frac{2}{N} S H(P) A(N) \epsilon + o(\epsilon), \]

whereas, from (7) and (42)-(45),

\[ \delta(U_{\epsilon,P}) = \frac{2}{S^2} - \frac{2}{4} B(N) \alpha \epsilon + o(\epsilon). \]

The previous lemma implies that

\[ S_\alpha \leq \frac{I_\alpha(U_{\epsilon,P})}{\alpha} \leq \frac{S}{2^N} - \frac{2}{N} S \frac{2}{N} B(N) \alpha \epsilon \left[ S \frac{N}{2} A(N) \frac{H(P)}{B(N)} \frac{1}{\alpha} - 1 + o(1) \right] = \frac{S}{2^N} - \frac{2}{N} S \alpha \epsilon \left[ A(N) H(P) \frac{1}{\alpha} - 1 + o(1) \right] \]

as \( \epsilon \to 0 \). Since, by assumption, \( \alpha < A(N) \max_{\partial \Omega} H = A(N) H(P) \), \( S_\alpha < \frac{S}{2^N} \). \( \square \)

**Corollary 5.3.** The value \( \alpha_0 \) is greater than or equal to \( A(N) \max_{\partial \Omega} H \).

We let \( |\Omega| \) denote the Lebesgue measure of \( \Omega \). By testing \( I_\alpha \) with constant functions we obtain

**Lemma 5.4.** If \( a \leq \frac{S}{(2|\Omega|)^{\frac{N}{2}}} \), then \( \alpha_0 \geq \max \left\{ \alpha \in [0, +\infty [ \right| I_\alpha(1) \leq \frac{S}{2^N} \right\}. \)

*Note.* The value of \( I_\alpha(1) \) is

\[ I_\alpha(1) = \frac{|\Omega|^{\frac{N}{2}}}{(2^N)^{\frac{N}{2}}} \left[ \left( \frac{a + \sqrt{a^2 + 4\alpha}}{2} \right)^N + \frac{2a}{N} \right] \left( \frac{a + \sqrt{a^2 + 4\alpha}}{2} \right)^{N-2}. \]

We have not determined the exact value of \( \alpha_0 \). However, using the ideas of Chabrowski and Willem [10], we have the following proposition concerning existence of least energy solutions for \( \alpha = \alpha_0 \):

**Proposition 5.5.** If \( \alpha_0 > A(N) \max_{\partial \Omega} H \) then there exists a least energy solution of \( (1_{\alpha_0}) \).

*Proof.* Choose a sequence \( \alpha_k \nearrow \alpha_0 \). Let \( u_k \) be a minimizer of \( I_{\alpha_k} \) satisfying \( (1_{\alpha_k}) \). As in the proof of Lemma 3.5, we conclude that the sequence \( (u_k) \) is bounded in \( H^1(\Omega) \). So we can assume \( u_k \rightharpoonup u \).
We claim that \( u \neq 0 \). Suppose, by contradiction, that \( u = 0 \). If the norms \( |u_k|_{L^\infty(\Omega)} \) are uniformly bounded, then, from (26), \( |u|^2_k = \frac{S \alpha_k}{2^\frac{N}{2}} \), which contradicts \( u = 0 \). If \( |u_k|_{L^\infty(\Omega)} \to +\infty \), then Lemma 3.6 implies that we can repeat the argument of the previous sections to conclude that \( S_{\alpha_k} > \frac{S}{2^\frac{N}{2}} \), for large \( k \). This is also a contradiction. So \( u \neq 0 \).

Since \( u \neq 0 \), the argument in the proof of Lemmas 3.1 and 3.2 yields that \( u \) is a least energy solution of (1), indeed, with the notations in the proof of Lemma 3.1, \( x_0 \neq 0 \). If \([h(1)]^\frac{1}{N}/[4(2^\#)^\frac{1}{2}] > \frac{S}{2^\frac{N}{2}} \), then \( S_{\alpha_0} = [h(x_0)]^\frac{1}{N}/[4(2^\#)^\frac{1}{2}] > \frac{S}{2^\frac{N}{2}} \). Hence \( I_{\alpha_0}(u) = [h(1)]^\frac{1}{N}/[4(2^\#)^\frac{1}{2}] = \frac{S}{2^\frac{N}{2}} \). □

**Remark 5.6.** If \( a \) is sufficiently small and \( \Omega \) is a (unit) ball, then the lower bound for \( \alpha_0 \) in Corollary 5.3 is smaller than the lower bound for \( \alpha_0 \) in Lemma 5.4 so that the previous proposition applies.

**Proof.** The lower bound for \( \alpha_0 \) in Corollary 5.3 is \( A(N) \), given in (47). As \( a \to 0 \), the lower bound for \( \alpha_0 \) in Lemma 5.4 tends to

\[
\left( \frac{(2^\#)^\frac{1}{2}}{\Omega} \frac{S}{2^\frac{N}{2}} \right)^{\frac{1}{2}} = \left( \frac{2^\#}{2} \right) \frac{S^\frac{1}{2}}{\Omega^\frac{1}{2}} \left( \frac{1}{N^\frac{1}{2}} \right)^{\frac{N+1}{2}} \left[ N(N-2) \right]^{\frac{1}{2}} \left( \frac{1}{\pi} \right)^\frac{N+1}{2} \left( \frac{\Gamma \left( \frac{N+2}{2} \right)}{\Gamma \left( \frac{N+1}{2} \right)} \right)^\frac{1}{2} \left[ \frac{1}{2^N} \right]^{\frac{N-1}{2}} \left( \frac{\Gamma \left( \frac{N+2}{2} \right)}{\Gamma \left( \frac{N+1}{2} \right)} \right)^\frac{1}{2} \left[ N(N-2) \right]^{\frac{1}{2}}.
\]

Suppose now \( \alpha_0 = A(N) \max_{\partial \Omega} H \). Once again, choose a sequence \( \alpha_k \) such that \( \alpha_0 \) and let \( u_k \) be a minimizer of \( I_{\alpha_k} \) satisfying (1). The argument in the proof of the previous proposition shows that, modulo a subsequence, either \( u_k \to u \neq 0 \), or \( u_k \to 0 \) and \( |u_k|_{L^\infty(\Omega)} \to +\infty \). We have not determined which of these alternatives holds. In the first case \( u \) is a least energy solution of (1). In the second case let, as before, \( P_k \) be such that \( u_k(P_k) = |u_k|_{L^\infty(\Omega)} \). Any limit point of \( (P_k) \) is contained in the set of points of maximum mean curvature of \( \partial \Omega \). For if \( y_0 \) is a limit point of \( P_k \), then

\[
-2^{\frac{N-2}{N}} SH(y_k)A(N)\varepsilon_k = \left[ 2^{\frac{N-2}{N}} SH(y_0)A(N)\varepsilon_k - 2^{\frac{N-2}{N}} SH(y_k)A(N)\varepsilon_k \right]
= -2^{\frac{N-2}{N}} SH(y_0)A(N)\varepsilon_k + o(\varepsilon_k).
\]

If \( H(y_0) < \max_{\partial \Omega} H \), then the argument in the previous section shows that \( S_{\alpha_k} > \frac{S}{2^\frac{N}{2}} \) for large \( k \).

We summarize these observations in

**Proposition 5.7.** Suppose \( \alpha_0 = A(N) \max_{\partial \Omega} H \). Then

(i) either there exists a least energy solution of (1),
Since, i.e. \( J \), there exists a \( \lambda \). This yields \( u \). 

The set \( \{ u \in H^1(\Omega): u = 0, u \neq 0 \} \), is a manifold (called the Nehari manifold). Indeed, if \( u \in \mathcal{N} \), then \( J'(u) \neq 0 \), because if \( J_a(u) = 0 \) and \( J'_a(u)u = 0 \), then

\[
0 = 2^* J_a(u) - J'_a(u)u = (2^* - 2)||u||^2 + (2^* - 2^\#)\alpha|u|^{2^\#}.
\]

This yields \( u = 0 \). Furthermore, the Nehari manifold is a natural constraint for \( \Phi_a \), by which we mean that any critical point of \( \Phi_a|_{\mathcal{N}} \) is a critical point of \( \Phi_a \). Indeed, suppose that \( u \in \mathcal{N} \) is a critical point of \( \Phi_a|_{\mathcal{N}} \). Then there exists a \( \lambda \in \mathbb{R} \) such that \( \Phi'_a(u) = \lambda J'_a(u) \). Applying both sides to \( u \),

\[
0 = J_a(u) = \Phi'_a(u)u = \lambda J'_a(u)u.
\]

However, we just saw that \( J'_a(u)u \neq 0 \) if \( J_a(u) = 0 \) and \( u \neq 0 \). It follows that \( \lambda = 0 \) and \( u \) is a critical point of \( \Phi_a \).

For any \( u \in H^1(\Omega) \setminus \{0\} \) there exists a unique \( t(u) > 0 \) such that \( t(u)u \in \mathcal{N} \), i.e. \( \Phi'_a(t(u)u)\)\( t(u)u = 0 \). The value of \( t(u) \) is the solution of

\[
||u||^2 + \alpha|u|^{2^\#} |t(u)|^{2^\# - 2} - |u|^{2^\#} |t(u)|^{2^\# - 2} = 0.
\]

Since \( 2^\# - 2 = \frac{2}{N-2} \) is half of \( 2^* - 2 \), the equation

\[
a + bt^{2^* - 2} - ct^{2^* - 2} = 0.
\]

is quadratic in \( t^{\frac{2}{N-2}} \). Define the functionals \( a, b \) and \( c : H^1(\Omega) \setminus \{0\} \rightarrow \mathbb{R} \) by

\[
a(u) := ||u||^2, \quad b(u) := \alpha|u|^{2^\#} = \beta(u), \quad c(u) := |u|^{2^\#}.
\]

(Note that \( a \neq \beta \).) The value of \( t(u) \) is

\[
t(u) = \left( \frac{b + \sqrt{b^2 + 4ac}}{2c} \right)^{\frac{N-2}{2}} (u). \tag{62}
\]

The functional \( t : H^1(\Omega) \setminus \{0\} \rightarrow \mathbb{R} \) is obviously continuous and the map \( u \mapsto t(u)u \) defines a homeomorphism of the unit sphere in \( H^1(\Omega) \) with \( \mathcal{N} \). Its inverse is the retraction \( u \mapsto \frac{u}{||u||} \).
We define \( \Psi_\alpha : H^1(\Omega) \setminus \{0\} \to \mathbb{R} \) by
\[
\Psi_\alpha(u) := \Phi_\alpha(t(u)u).
\]
In terms of \( a, b, c \) and \( t \),
\[
\Psi_\alpha = \frac{1}{2}at^2 + \frac{1}{2#}bct^2 - \frac{1}{2^*}ct^2^*.
\]
Replacing (62) into this expression for \( \Psi_\alpha \), and simplifying, leads to
\[
\Psi_\alpha = \frac{1}{N}2^* \left[ \frac{b + \sqrt{b^2 + 4ac}}{2c} \right]^N c + 2^* \left( \frac{b + \sqrt{b^2 + 4ac}}{2c} \right)^{N-2} a.
\]
We now introduce the functionals \( \beta, \gamma : H^1(\Omega) \setminus \{0\} \to \mathbb{R} \), defined by
\[
\beta := a \frac{c}{N^2^*},
\]
and
\[
\gamma = \gamma_\alpha := b \frac{c}{N^2^*},
\]
as in expressions (4) and (5), respectively. In terms of \( \beta \) and \( \gamma \), the expression for \( \Psi_\alpha \) is
\[
\Psi_\alpha = \frac{1}{N}2^* \frac{1}{2N} \left[ \left( \gamma + \sqrt{\gamma^2 + 4\beta} \right)^N + 2 \cdot 2^* \beta \left( \gamma + \sqrt{\gamma^2 + 4\beta} \right)^{N-2} \right].
\]
This is (3). If we introduce still another functional \( \delta : H^1(\Omega) \setminus \{0\} \to \mathbb{R} \), defined by
\[
\delta = \delta_\alpha := \frac{\gamma}{2\sqrt{\beta}},
\]
as in expression (7), then we can write \( \Psi_\alpha \) as
\[
\Psi_\alpha = \frac{1}{N}2^* \frac{1}{2N} \left[ \left( \delta + \sqrt{\delta^2 + 1} \right)^N + 2 \cdot 2^* \left( \delta + \sqrt{\delta^2 + 1} \right)^{N-2} \right].
\]
This is (6).

We give an expression for \( I_\alpha = (N\Psi_\alpha)^\frac{1}{N} \), defined in (8), equivalent to (9) and to (10):
\[
I_\alpha = \beta \left( \delta + \sqrt{\delta^2 + 1} \right)^\frac{4}{N} \left( \frac{2}{2^*} \delta^2 + \frac{2}{2^*} \delta \sqrt{\delta^2 + 1} + 1 \right)^\frac{2}{N^*}.
\]
Since
\[
\frac{4}{2^*} + \frac{2}{2} = \frac{4}{2^*},
\]
\( I_\alpha \) has the lower bound
\[
I_\alpha \geq \beta \left( 1 + \frac{4}{2^*} \delta \right).
\]
For an upper bound for \( I_\alpha \) we refer to Lemma 5.1.
Appendix B. The estimate for $|U_k|_{2#}^2$

In this Appendix we use the ideas of Adimurthi and Mancini [1] to prove (43).

We wish to estimate $|U_{\varepsilon,y}|_{2#}^2$, where $U_{\varepsilon,y}$ is defined in (14) and $y \in \partial\Omega$. By a change of coordinates we can assume that $y = 0$, $B_R(0) \cap \Omega = \{(x', x_N) \in B_R(0) | x_N > \rho(x')\}$ and $B_R(0) \cap \partial\Omega = \{(x', x_N) \in B_R(0) | x_N = \rho(x')\}$, for some $R > 0$, where $x' = (x_1, \ldots, x_{N-1})$, $\rho(x') = \sum_{i=1}^{N-1} \lambda_i x_i^2 + O(|x'|^3)$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq N - 1$.

We begin by supposing all the $\lambda_i$’s are positive. Let $U_{\varepsilon} := U_{\varepsilon,0}$ and $\Sigma := \{(x', x_N) \in B_R(0) | 0 < x_N < \rho(x')\}$. Then

$$|U_{\varepsilon}|_{2#}^2 = \frac{1}{2} \int_{B_R(0)} U_{\varepsilon}^2 - \int_{\Sigma} U_{\varepsilon}^2 + \int_{B_R(0) \cap \partial\Omega} U_{\varepsilon}^2. \quad (64)$$

We will estimate each of the three terms on the right hand side of (64). For the third term we have

$$\int_{B_R(0) \cap \partial\Omega} U_{\varepsilon}^2 \leq \int_{B_R(0) \cap \partial\Omega} U_{\varepsilon}^2 = O\left( \int_{R/\varepsilon}^{r+\infty} \frac{r^{N-1}}{(1 + r^2)^{N-1}} dr \right) = O(\varepsilon \times \varepsilon^{N-2}) = O(\varepsilon^{N-1})$$

Using this estimate, for the first term on the right hand side of (64) we have

$$\frac{1}{2} \int_{B_R(0)} U_{\varepsilon}^2 = \frac{1}{2} \int_{\mathbb{R}^N} U_{\varepsilon}^2 + O(\varepsilon^{N-1}) = \frac{1}{2} \varepsilon \int_{\mathbb{R}^N} U_{\varepsilon}^2 + O(\varepsilon^{N-1}) = \frac{2^#}{2} B(N) \varepsilon + O(\varepsilon^{N-1}),$$
with
\[ B(N) := \frac{1}{2^#} \int_{\mathbb{R}^N} U^{2^#} \]
\[ = \frac{1}{2^#} \omega_N \int_0^{+\infty} \frac{r^{N-1}}{(1+r^2)^{N-1}} dr \times [N(N-2)]^{\frac{N}{2}} \]
\[ = \frac{N-2}{2(N-1)} \omega_N \times \frac{1}{2^{N-1} \sqrt{\pi}} \frac{\Gamma \left( \frac{N-2}{2} \right)}{\Gamma \left( \frac{N-1}{2} \right)} \times [N(N-2)]^{\frac{N}{2}} \]
\[ = \omega_N \frac{1}{2^{N} \sqrt{\pi}} \frac{\Gamma \left( \frac{N}{2} \right)}{\Gamma \left( \frac{N+1}{2} \right)} [N(N-2)]^{\frac{N}{2}} ; \]

here \( \omega_N \) is the volume of the \( N-1 \) dimensional unit sphere.

So we are left with the estimate of the second term on the right hand side of (64). Let \( \sigma > 0 \) be such that
\[ L_\sigma := \{ x \in \mathbb{R}^N \mid |x_i| < \sigma, 1 \leq i \leq N \} \subset B_\mathbb{R}(0) \]
and define
\[ \Delta_\sigma := \{ x' \mid |x_i| < \sigma, 1 \leq i \leq N-1 \} . \]

For the second term on the right hand side of (64),
\[ \int_\Sigma U^{2^#} = \int_{\Sigma \cap L_\sigma} U^{2^#} + O(\varepsilon^{N-1}) \]
\[ = \int_{\Delta_\sigma} \int_0^{\rho(x')} U^{2^#} \, dx_N \, dx' + O(\varepsilon^{N-1}) \]
\[ = O \left( \int_{\Delta_\sigma} \int_0^{\rho(x')} \frac{\varepsilon^{N-1}}{(\varepsilon^2 + |x'|^2)^{N-1}} \, dx_N \, dx' \right) + O(\varepsilon^{N-1}) ; \]

using the change of variables \( \sqrt{\varepsilon^2 + |x'|^2} y_N = x_N \),
\[ = O \left( \int_{\Delta_\sigma} \frac{1}{(\varepsilon^2 + |x'|^2)^{N-1}} \int_0^{\rho(x')} \frac{1}{(1 + y_N^2)^{N-1}} dy_N \, dx' \right) \]
\[ = O(\varepsilon^{N-1}) ; \]

since \( \int_0^s \frac{1}{(1+t^2)^{N-1}} dt = s + O(s^3) \),
\[ = O \left( \varepsilon^{-N-1} \int_{\Delta_\sigma} \frac{\sum \lambda_i x_i^2}{(\varepsilon^2 + |x'|^2)^{N-1}} \, dx' \right) \]
\[ + O \left( \varepsilon^{-N-1} \int_{\Delta_\sigma} \frac{|x'|^3}{(\varepsilon^2 + |x'|^2)^{N-1}} \, dx' \right) \]
\[ + O(\varepsilon^{N-1}) \]
\[ = O \left( \varepsilon^2 \int_{\Delta_\sigma/\varepsilon} \frac{|y'|^2}{(1 + |y'|^2)^{N-1}} \, dy' \right) \]
\[ + O \left( \varepsilon^3 \int_{\Delta_\sigma/\varepsilon} \frac{|y'|^3}{(1 + |y'|^2)^{N-1}} \, dy' \right) \]
\[ + O(\varepsilon^{N-1}) \]
\[ = O(\varepsilon^2) . \]
Combining the estimates for the three terms on the right hand side of (64),

$$|U_\varepsilon|^2_{2^*} = \frac{2^*}{2} B(N) \varepsilon + O(\varepsilon^2),$$  \hspace{1cm} (65)

if all the $\lambda_i$'s are positive. If all the $\lambda_i$'s are negative, then the minus sign on the right hand side of (64) turns into a plus sign, and (65) follows. From these two cases we deduce that (65) holds no matter what the sign of the $\lambda_i$'s is.

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