On noncommutative Nahm transform.

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March 28, 2022

Motivated by the recently observed relation between the physics of $D$-branes in the background of $B$-field and the noncommutative geometry we study the analogue of Nahm transform for the instantons on the noncommutative torus.

1 Introduction

It is shown in [C-D-S] that noncommutative geometry can be successfully applied to the analysis of M(atrix) theory. In particular, it is proven that one can compactify M(atrix) theory on a noncommutative torus; later, compactifications of this kind were studied in numerous papers. In present paper, we analyze instantons in noncommutative toroidal compactifications. This question is important because instantons can be considered as BPS states of compactified M(atrix) model. Instantons on a noncommutative $\mathbb{R}^4$ were considered earlier in [N-S]. It is shown in [N-S] that these instantons give

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some insight in the structure of (2,0) super-conformal six dimensional theory; the instantons on a noncommutative torus also should be useful in this relation. The main mathematical tool used in [N-S] is the noncommutative analogue of ADHM construction of instantons. The present paper is devoted to the noncommutative analogue of Nahm transform (recall that the Nahm transform can be regarded as some kind of generalization of ADHM construction). We prove that some of important properties of Nahm transform remain correct in noncommutative case.

2 Preliminaries.

In this section we recall several notions related to the theory of noncommutative tori. We roughly discuss the ideas behind the noncommutative Nahm transform and formulate our main results. A more formal approach to the noncommutative Nahm transform is taken in the next section.

**Definition 2.1** An $n$-dimensional noncommutative torus $\mathcal{A}_\theta$ is a $C^*$-algebra having unitary generators $U_i$, $i \in \mathbb{Z}^n$ obeying

$$U_i U_j = e^{\frac{i}{2} \theta(i,j)} U_{i+j}, \quad U_0 = 1; \quad (1)$$

where $\theta(\cdot, \cdot)$ is a skew-symmetric bilinear form on $\mathbb{Z}^n$.

We can naturally consider $\theta$ as a skew-symmetric bilinear form on $\mathbb{R}^n$. Any element of $\mathcal{A}_\theta$ can be uniquely represented as a sum $\sum_{i \in \mathbb{Z}^n} c_i U_i$, where $c_i$ are complex numbers. Let $e_k$, $1 \leq k \leq n$ be the natural base in $\mathbb{Z}^n$. The transformations $\delta_l U_{e_k} = \delta_{l,k} U_{e_k}$, $1 \leq k, l \leq n$ generate an abelian Lie algebra $L_\theta$ of infinitesimal automorphisms of $\mathcal{A}_\theta$. We use $L_\theta$ to define the notion of connection in a $\mathcal{A}_\theta$-module following [Con1] (we do not need the general notion of connection [Con2]).

Any element from $\mathcal{A}_\theta$ can be considered as a function on the $n$-dimensional torus whose Fourier coefficients are $c_i$ (see above). The space of smooth functions on $T^n$ forms a subalgebra of $\mathcal{A}_\theta$. We denote it by $\mathcal{A}_\theta^{\text{smooth}}$ and call it the smooth part of $\mathcal{A}_\theta$. If $E$ is a projective finitely generated $\mathcal{A}_\theta$ module one can define its smooth part $E^{\text{smooth}}$ in a similar manner (see [Rf1]). Now we can define the notion of $\mathcal{A}_\theta$ connection.
Definition 2.2 \( A_\theta \) connection on a right \( A_\theta \) module \( E \) is a linear map \( \nabla \) from \( L_\theta \) to the space \( \text{End}_{A_\theta}E \) of endomorphisms of \( E^{\text{smooth}} \), satisfying the condition
\[
\nabla_\delta(ea) = (\nabla_\delta e)a + e(\delta(a)),
\]
where \( e \in E^{\text{smooth}}, \ a \in A_\theta^{\text{smooth}}, \) and \( \delta \in L_\theta \). The curvature \( F_{\mu,\nu} = [\nabla_\mu, \nabla_\nu] \) of connection \( \nabla \) is considered as a two-form on \( L_\theta \) with values in endomorphisms of \( E \).

We always consider Hermitian modules and Hermitian connections. This means that if \( E \) is a right \( A_\theta \) module it is equipped with \( A_\theta \) valued Hermitian inner product \( \langle \cdot, \cdot \rangle \) (for the detailed list of properties see [Bl]) and all connections that we will consider should be compatible with this inner product.

If \( E \) is endowed with a \( A_\theta \)-connection, then one can define a Chern character
\[
\text{ch}(E) = \sum_{k=0}^{\infty} \frac{\hat{\tau}(F^k)}{(2\pi i)^k k!}.
\]
where \( F \) is a curvature of a connection on \( E \), and \( \hat{\tau} \) is the canonical trace on \( \hat{A} = \text{End}_{A_\theta}(E) \) (we use that \( A_\theta \) is equipped with a canonical trace \( \tau = c_0 \)).

One can consider \( \text{ch}(E) \) as an element in the Grassmann algebra \( \Lambda(L_\theta^*) \).

However, it is convenient to use the fact that this Grassmann algebra can be identified with cohomology \( H(T, \mathbb{C}) \) where \( T \) stands for the Lie group of automorphisms of the algebra \( A_\theta \) corresponding to the Lie algebra \( L_\theta \). (In other words \( T = L_\theta/D \) where \( D \) is a lattice.) In the commutative case \( \text{ch}(E) \) is an integral cohomology class. In noncommutative case this is wrong, but there exists an integral cohomology class \( \mu(E) \in H(T, \mathbb{Z}) \) related to \( \text{ch}(E) \) by the following formula (see [Ell], [Rf1])
\[
\text{ch}(E) = e^{\iota(\theta)}\mu(E),
\]
where \( \iota(\theta) \) stands for the operation of contraction with \( \theta \) considered as an element of two-dimensional homology group of \( T \). In particular, formula (3) means that \( e^{-\iota(\theta)}\text{ch}(E) \in H(T, \mathbb{Z}) \).

One can regard \( \mu(E) \) as a collection of integer quantum numbers characterizing topological class of a gauge field on noncommutative torus (or from mathematical viewpoint as a K-theory class of projective module \( E \)).
The formula (3) is familiar to physicists [D-WZ] in the following (T-dual) form:

\[ L_{WZ} = \int_X \nu \wedge \tilde{C}, \quad \nu = \text{ch}(E)e^{-\frac{B}{2\pi}}\sqrt{\mathcal{A}(X)} \]  

The element \( \nu \) of the cohomology group \( H^{\text{even}}(X, \mathbb{R}) \) is called the generalized Mukai vector. Here \( L_{WZ} \) describes the so-called Wess-Zumino couplings on the worldvolume \( X \) of a \( D \)-brane (more precisely of a stack of \( D \)-branes), \( E \) is the Chan-Paton bundle (or sheaf), \( \tilde{C} \) is the collection of all Ramond-Ramond potentials, \( \mathcal{A}(X) \) is the A-roof genus and \( B \) is the Neveu-Schwarz B-field. The formula (3) captures the effect of the non-trivial topology of the Chan-Paton bundle on the \( D \)-brane charges induced on the brane. Since in our case \( X \) is a torus with flat metric then \( \mathcal{A}(X) = 1 \) and we arrive at (4) provided that we performed a \( T \)-duality transformation which maps \( B \)-field two form into a bivector \( \theta \) and also exchanges \( \text{ch}_k \) with \( \text{ch}_{2-k} \).

**Definition 2.3** An instanton is a connection such that the self-dual part of its curvature is a scalar operator, i.e., \( F_+^{\alpha\beta} \) is a multiplication operator by the scalar that we denote \( -\omega_+^{\alpha\beta} \).

We are interested in instantons on four-dimensional noncommutative torus. In the framework of supersymmetric gauge theory they can be interpreted as BPS-fields. Notice that in the definition of Hodge dual \( \ast F \) we need an inner product on the Lie algebra \( L_\theta \); we fix such a product.

As in commutative case we can prove that the minimum of euclidean action \( A(E, \nabla) = \frac{1}{8\pi^2} \ast (F \wedge \ast F)[T] \) for connections in module \( E \) (i.e. for gauge fields with given topological numbers) is achieved on instantons or anti-instantons (connections where antiselfdual part of curvature is a scalar). Here \( [T] \) stands for the fundamental homology class of \( T \).

**Claim 2.1** The expression of the instanton action in terms of \( \mu \) and \( \theta \) has the following form:

\[ A(E, \nabla) = \left( \mu_2(E) - \frac{i}{2\pi} \frac{\left( \mu_1(E)^+ + (\iota(\theta)\mu_2(E))^+ \right)^2}{\mu_0(E) + \iota(\theta)\mu_1(E) + \frac{1}{2}\iota(\theta)^2\mu_2(E)} \right)[T], \]  

where \( \mu_k(E) \) is \( 2k \)-dimensional component of \( \mu(E) \).
To prove this formula we notice that if \((E, \nabla)\) is an instanton (\(i.e. F^+ + \omega^+ = 0\)) the we can express \(*F\) as follows
\[
*F = -F - 2\omega^+.
\] (6)

Using (6) we obtain
\[
\hat{\tau}(F \wedge *F) = \hat{\tau}(F \wedge [-F - 2\omega^+]) = -\hat{\tau}(F \wedge F) - 2\hat{\tau}(F) \wedge \omega^+,
\]

since \(\omega\) is complex valued two-form hence
\[
\hat{\tau}(F \wedge \omega^+) = \hat{\tau}(F) \wedge \omega^+ = \hat{\tau}(1) \omega^+ \wedge \omega^+.
\]

One can easily obtain from the formula (3) that
\[
\hat{\tau}(F \wedge F) = -8\pi^2 \mu_2(E),
\]
\[
\hat{\tau}(F) = 2\pi i (\mu_1(E) + \iota(\theta) \mu_2(E)).
\]
\[\blacksquare\]

We will construct a generalization of Nahm’s transform \([Na], [D-K]\) relating connections on \(A_{\theta}-\text{modules with connections on } A_{\hat{\theta}}-\text{modules.} \) (Here \(A_{\theta}\) and \(A_{\hat{\theta}}\) are two four-dimensional noncommutative tori.) To define a noncommutative generalization of Nahm’s transform we need a \((A_{\theta}, A_{\hat{\theta}})\)-module \(P\) with \(A_{\theta}\)-connection \(\nabla\) and \(A_{\hat{\theta}}\)-connection \(\hat{\nabla}\). Speaking about \((A_{\theta}, A_{\hat{\theta}})\)-module \(P\) we have in mind that \(P\) is a left \(A_{\theta}\)-module and a right \(A_{\hat{\theta}}\)-module; \(\\) and \((ax)b = a(xb)\) for \(a \in A_{\theta}, \; b \in A_{\hat{\theta}}, \; x \in P\). We assume that the commutators \([\nabla_\alpha, \nabla_\beta], \; [\hat{\nabla}_\mu, \hat{\nabla}_\nu], \; [\nabla_\alpha, \hat{\nabla}_\mu]\) are scalars:
\[
[\nabla_\alpha, \nabla_\beta] = \omega_{\alpha\beta}, \; [\hat{\nabla}_\mu, \hat{\nabla}_\nu] = \hat{\omega}_{\mu\nu}, \; [\nabla_\alpha, \hat{\nabla}_\mu] = \sigma_{\alpha\mu}.
\]

One more assumption is that \(\nabla_\alpha\) commutes with the multiplication by the elements of \(A_{\hat{\theta}}\) and \(\hat{\nabla}_\alpha\) commutes with the multiplication by the elements of \(A_{\theta}\). One can reformulate the above conditions saying that \(P\) is an \(A_{\theta\hat{\theta}}\)-module, and \(\nabla, \hat{\nabla}\) give us a constant curvature connection on it. For every right \(A_{\theta}\)-module \(R\) with connection \(\nabla^R\) we consider Dirac operator \(D = \Gamma^a(\nabla^R_\alpha + \nabla_\alpha)\) acting on the tensor product
\[
(R \otimes_{A_{\theta}} P) \otimes S
\]
(or more precisely on its smooth part). To define \(\Gamma\)-matrices we introduce an inner product in \(L_{\theta}\). \(S\) is a \(\mathbb{Z}/2\mathbb{Z}\) graded vector space \(S = S^+ \oplus S^-\) and Dirac operator is an odd operator. Thus, we can consider
\[
D^+: (R \otimes_{A_{\theta}} P) \otimes S^+ \to (R \otimes_{A_{\theta}} P) \otimes S^-, \; D^-: (R \otimes_{A_{\theta}} P) \otimes S^- \to (R \otimes_{A_{\theta}} P) \otimes S^+.
\] (7)
The Dirac operator commutes with the multiplication by the elements of $A_\tilde{\theta}$, therefore the space of zero modes of $D$ can be regarded as $\mathbb{Z}/2\mathbb{Z}$-graded $A_\tilde{\theta}$-module; we denote it by $\hat{R}$. We will see later that sometimes it is reasonable to modify this definition of $\hat{R}$.

Next we would like to get a connection on $\hat{R}$. Using $A_\theta$ Hermitian inner product on $R$, $A_\theta \times A_\tilde{\theta}$ Hermitian inner product on $\mathcal{P}$, and a canonical trace $\tau$ on $A_\theta$ we can define an $A_\tilde{\theta}$ Hermitian inner product on $(R \otimes A_\theta \mathcal{P}) \otimes S$. We assume that we have an orthogonal projection (with respect to the $A_\tilde{\theta}$ Hermitian inner product) $P$ of $(R \otimes A_\theta \mathcal{P}) \otimes S$ onto $\hat{R}$. We will prove its existence in the next section under some additional conditions. The connection $\hat{\nabla}$ induces a connection $\nabla\hat{\nabla}'$ on $(R \otimes A_\theta \mathcal{P}) \otimes S$; we obtain a connection $\nabla\hat{\nabla}$ on $\hat{R}$ as $P\nabla\hat{\nabla}'$.

The above construction can be regarded as a generalized Nahm’s transform. To prove that its properties are similar to the properties of standard Nahm’s transform we should impose additional conditions on module $\mathcal{P}$ and connections $\nabla$, $\hat{\nabla}$. We assume that $\mathcal{P}$ is a projective Hermitian module over $A_{\theta \oplus (-\tilde{\theta})}$ and that $\sigma_{\alpha\mu}$ determines a non-degenerate pairing between $L_{\theta}$ and $L_{\tilde{\theta}}$. Then we can use this pairing to define an inner product in $L_{\tilde{\theta}}$; this allows us to talk about instantons on $A_\tilde{\theta}$. Under certain conditions we prove that the Nahm transform of an instanton is again an instanton. More precisely, if $F_\alpha^+ + \omega_\alpha^+ = 0$ where $\omega_\alpha^+$ stands for the self-dual part of $\omega_\alpha$, then $\hat{F}_\alpha^+ - \hat{\omega}_\alpha^+ = 0$. Here $F_{\alpha\beta}$ (correspondingly $\hat{F}_{\alpha\beta}$) is the curvature of the connection $\nabla\hat{\nabla}$ (correspondingly $\nabla\hat{\nabla}'$) on $R$ (correspondingly $\hat{R}$) and $+$ stands for the self-dual part of it.

Notice that by taking the trace of the curvature of the connection $\nabla \oplus \hat{\nabla}$ on $\mathcal{P}$ we can express $\omega$ in terms of topological quantum numbers:

$$\omega = 2\pi i \frac{\frac{1}{3}t(\tilde{\theta})^3 \mu_4(\mathcal{P}) + \frac{1}{2}t(\tilde{\theta})^2 \mu_3(\mathcal{P}) + t(\tilde{\theta}) \mu_2(\mathcal{P}) + \mu_1(\mathcal{P})}{\frac{1}{2}t(\tilde{\theta})^4 \mu_4(\mathcal{P}) + \frac{1}{6}t(\tilde{\theta})^3 \mu_3(\mathcal{P}) + \frac{1}{2}t(\tilde{\theta})^2 \mu_2(\mathcal{P}) + t(\tilde{\theta}) \mu_1(\mathcal{P}) + \mu_0(\mathcal{P})} |_{L_\theta}.$$ 

Here we use the notation $\tilde{\theta}$ for $\theta \oplus (-\tilde{\theta})$, $\mu_k(\mathcal{P})$ for $2k$-dimensional component of $\mu(\mathcal{P})$.

In general, the Nahm transform defined above is not bijective (even in commutative case, i.e. when $\theta = \tilde{\theta} = 0$). However the commutative Nahm
transform is bijective if \( P \) is “Poincare module” (the module corresponding to the Poincare line bundle). Strictly speaking, the term “Nahm transform” is used only in this situation. It is natural to define the Nahm transform in noncommutative case using an \( \mathcal{A}_{\theta \bar{\theta}(-\hat{\theta})} \) module having the same topological numbers as Poincare module. (We will prove in Section 5 that the deformed Poincare module can be equipped with constant curvature connection and hence it can be used to define the Nahm transform.) One should expect that in noncommutative case the Nahm transform is a bijection (and its square is \((-1)^*\)). We can give only heuristic proof of this conjecture.

**Remark.** The relation between topological quantum numbers of \( \hat{R} \) and \( R \) in the case when \( P \) is deformed Poincare module is the same as in commutative case (the relation between discrete quantities cannot change under continuous deformation):

\[
\mu(R) = p + \frac{1}{2}a_{ij} \alpha^i \wedge \alpha^j + q \alpha^1 \wedge \alpha^2 \wedge \alpha^3 \wedge \alpha^4,
\]
\[
\mu(\hat{R}) = q - \frac{1}{4} \varepsilon^{ijkl} a_{ij} \beta_k \wedge \beta_l + p \beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \beta_4
\]

where \( \varepsilon^{ijkl} \) is completely antisymmetric tensor and the bases \( \alpha^i \) and \( \beta_i \) are the standard dual bases of \( H^2(T, \mathbb{Z}) \) and \( H^2(\hat{T}, \mathbb{Z}) \).

We now present the formulae which relate \( \omega, \hat{\omega}, \theta, \hat{\theta}, \) etc. Recall that in the commutative case \( L_{\theta} \approx L'_{\hat{\theta}} \). The integral class \( \mu(P) = \exp \sum_k \alpha^k \wedge \beta_k \) is the same as in the commutative case and it allows to identify \( L_{\theta}' \approx L_{\hat{\theta}} \) in the noncommutative case too. It is convenient to think of the forms \( \omega, \hat{\omega}, \sigma \) as of the operators:

\[
\omega : L_{\theta} \rightarrow L'_{\theta}, \hat{\omega} : L_{\hat{\theta}} \rightarrow L_{\theta}, \sigma : L_{\hat{\theta}} \rightarrow L_{\hat{\theta}}.
\]

In the same way the bivectors \( \theta, \hat{\theta} \) are viewed as skew-symmetric operators: \( \theta : L_{\hat{\theta}} \rightarrow L_{\theta}, \hat{\theta} : L_{\theta} \rightarrow L_{\hat{\theta}} \). From (3) using the Wick theorem we get:

\[
\dim \mathcal{P} = \sqrt{\text{Det}(1 - \theta \hat{\theta})},
\]
\[
\omega = \hat{\theta}(1 - \theta \hat{\theta})^{-1}
\]
\[
\hat{\omega} = -\theta(1 - \theta \hat{\theta})^{-1}
\]
\[
\sigma = (1 - \theta \hat{\theta})^{-1}
\]

(9)
3 Definitions.

We assume that all modules are Hermitian modules, all connections are Hermitian connections, and noncommutative tori \( A_\theta \) and \( \hat{A}_\theta \) are four dimensional. Let \( (\mathcal{P}, \nabla, \hat{\nabla}) \) be a finitely generated projective \( A_{\theta;\hat{\theta}} \) module equipped with constant curvature connection \( \nabla \oplus \hat{\nabla} \). The curvature of \( \nabla \oplus \hat{\nabla} \) is an element of \( \wedge^2 (L_\theta)' \oplus (L_\theta \otimes L_{\hat{\theta}})' \oplus \wedge^2 (L_{\hat{\theta}})' \). We denote by \( \omega_{\alpha\beta} \) the \( \wedge^2 (L_\theta)' \) part of the curvature of \( \nabla \oplus \hat{\nabla} \), by \( \sigma_{\alpha\mu} \) the \( (L_\theta \otimes L_{\hat{\theta}})' \) part, and by \( \hat{\omega}_{\mu\nu} \) the \( \wedge^2 (L_{\hat{\theta}})' \) part. We fix an inner product on \( L_\theta \). The inner product on \( L_{\hat{\theta}} \) is obtained from the inner product on \( L_\theta \) via the pairing \( \sigma_{\alpha\mu} \).

**Definition 3.1** A connection \( \nabla^R \) on \( A_\theta \)-module \( R \) is called \( \mathcal{P} \)-irreducible iff there exists an inverse operator \( G \) to the Laplacian \( \Delta = \sum_i (\nabla^R_{\alpha_i} + \nabla_{\alpha_i}) (\nabla^R_{\alpha_i} + \nabla_{\alpha_i}) \) and \( G \) is bounded operator.

**Lemma 3.1** If \( \nabla^R \) is \( \mathcal{P} \)-irreducible connection on \( R \) and \( F^+ + \omega^+ \cdot 1 = 0 \) then \( \ker(D^+) = 0 \), and

\[
D^-D^+ = \Delta.
\]  

**Proof:** The proof is the same as in the commutative case. \( \Box \)

Now we can define a noncommutative Nahm transform. Let \( R \) be a projective Hermitian module over \( A_\theta \) with \( \mathcal{P} \)-irreducible connection \( \nabla^R \) such that its curvature \( F \) satisfies the condition \( F^+ + \omega^+ \cdot 1 = 0 \). Denote by \( \hat{R} \) the closure of the kernel of \( D^- : (R \otimes A_\theta \mathcal{P}) \otimes S^- \rightarrow (R \otimes A_\theta \mathcal{P}) \otimes S^+ \). Notice that \( (R \otimes A_\theta \mathcal{P}) \otimes S^- \) is Hermitian \( A_{\theta} \) module and that \( D^- \) commutes with \( A_{\theta} \) action. Therefore, \( \hat{R} \) is a Hermitian \( A_{\hat{\theta}} \) module (submodule of \( (R \otimes A_\theta \mathcal{P}) \otimes S^- \)). Let us denote by \( P \) the projection operator (with respect to the \( A_{\hat{\theta}} \) Hermitian inner product) from \( (R \otimes A_\theta \mathcal{P}) \otimes S^- \) onto \( \hat{R} \). In other words, \( P \) is Hermitian, \( P^2 = P \), and \( \text{Im } P = \hat{R} \). Its existence is proven in the theorem [3,4] below. We denote by \( \nabla^R \) the composition \( P \circ \hat{\nabla} \).

**Theorem 3.1** \( \hat{R} \) is a finitely generated projective Hermitian \( A_{\hat{\theta}} \) module and \( \nabla^R \) is a Hermitian \( A_{\hat{\theta}} \) connection on \( \hat{R} \).
Proof: The projection operator on the kernel of $D^-$ can be defined by the following explicit formula $P = 1 - D^+GD^-$. We can check that $P$ is hermitian, $P^2 = P$, and $\text{Im} P = \text{Ker} D^-$ by means of formal algebraic manipulations using $D^-D^+ = \Delta$. We claim that $P$ is “compact” operator over $\mathcal{A}\hat{\theta}$ (that is a limit of the operators of the type $\sum_i x_i(y_i, \cdot)$). This follows immediately as usual from the fact that $D^+$ admits a parametrix (over $\mathcal{A}\hat{\theta}$, see Appendix A) and $GD^-$ is left inverse to $D^+$. Since $P$ is “compact” over $\mathcal{A}\hat{\theta}$ it follows from the general theory (see Appendix A or [3]) that $P = \sum_j u_j(v_j, \cdot)$ is a projection on a finitely generated projective $\mathcal{A}\hat{\theta}$ module which proves the first statement. $\hat{R}$ inherits $\mathcal{A}\hat{\theta}$ valued inner product from $(R \otimes \mathcal{A}_\theta \mathcal{P}) \otimes S^-$. The second statement follows immediately from the fact that $P$ commutes with the action of $\mathcal{A}\hat{\theta}$ (since $D^+, D^-, G$ commute with $\mathcal{A}\hat{\theta}$ action). □

Definition 3.2 The noncommutative Nahm transform $\mathcal{N}$ of $(R, \nabla^R)$ is a pair $(\hat{R}, \nabla^{\hat{R}})$ of projective Hermitian $\mathcal{A}\hat{\theta}$ module $\hat{R}$ and connection $\nabla^{\hat{R}}$.

4 Properties of noncommutative Nahm transform.

Now we can find a formula for the curvature of $\nabla^{\hat{R}}$.

Proposition 4.1 We have the following formula for the curvature $\hat{F}$ of connection $\nabla^{\hat{R}}$

$$\hat{F}_{\mu\nu} = \hat{\omega}_{\mu\nu} + PG\left( \sum_{\alpha, \beta} (\Gamma^\alpha \Gamma^\beta - \Gamma^\beta \Gamma^\alpha) \sigma_{\alpha\mu} \sigma_{\beta\nu} \right).$$

(11)

Remark 4.1 It follows from (10) that $D^-D^+$ commutes with $\Gamma^\alpha$. Therefore $\Gamma^\alpha$ commutes with $G$. We will use this in the proof.

Proof: Since the Lie algebra $L\hat{\theta}$ is commutative the curvature $\hat{F}_{\mu\nu} = [\nabla^{\hat{R}}_\mu, \nabla^{\hat{R}}_\nu]$. The definition of $\nabla^{\hat{R}}$ gives us that $\hat{F}_{\mu\nu} = P\nabla^\mu P\nabla^\nu - P\nabla^\nu P\nabla^\mu$. 

9
Let us simplify this expression using the definition of $P = 1 - D^+GD^-$. If we replace the middle $P$ by this expression we obtain

$$\hat{F}_{\mu\nu} = P[\hat{\nabla}_\mu, \hat{\nabla}_\nu] + P(\hat{\nabla}_\nu D^+ GD^- \hat{\nabla}_\mu - \hat{\nabla}_\mu D^+ GD^- \hat{\nabla}_\nu) = \hat{\omega}_{\mu\nu} + P(\hat{\nabla}_\nu D^+ GD^- \hat{\nabla}_\mu - \hat{\nabla}_\mu D^+ GD^- \hat{\nabla}_\nu).$$  \hfill (12)

Next, let us notice that $PD^+ = 0$ and $D^-$ equals zero on $\hat{R}$. Therefore, we can rewrite (12) as

$$\hat{F}_{\mu\nu} = \hat{\omega}_{\mu\nu} + P([\hat{\nabla}_\nu, D^+]G[D^-, \hat{\nabla}_\mu] - [\hat{\nabla}_\mu, D^+]G[D^-, \hat{\nabla}_\nu]).$$  \hfill (13)

To proceed further we need the following commutation relation

$$[D, \hat{\nabla}_\lambda] = \sum_\alpha \Gamma^\alpha \sigma_{\alpha\lambda}.$$  \hfill (14)

(The proof is straightforward calculation using the definition of $D = \sum_\alpha \Gamma^\alpha(\nabla^R_\alpha + \nabla_\alpha)$ and the fact that $\nabla^R$ commutes with $\hat{\nabla}$.)

The immediate consequence of (14) is that

$$\hat{F}_{\mu\nu} = \hat{\omega}_{\mu\nu} + P\left(-\{\sum_\beta \Gamma^\beta \sigma_{\beta\nu}\}G[\sum_\alpha \Gamma^\alpha \sigma_{\alpha\mu}] + \{\sum_\alpha \Gamma^\alpha \sigma_{\alpha\mu}\}G[\sum_\beta \Gamma^\beta \sigma_{\beta\nu}]\right).$$

Now the theorem follows from the remark that $\Gamma^\lambda$ commutes with $G$ and very simple formal manipulations. □

Next, we would like to compute $\hat{F}^+$ (the self-dual part of the curvature). Let us remind that the inner product on $L_\hat{\theta}$ came from a non-degenerate pairing between $L_\theta$ and $L_\hat{\theta}$ given by $\sigma_{\alpha\mu}$.

**Lemma 4.1** The selfdual part of $(\sum_{\alpha,\beta}(\Gamma^\alpha \Gamma^\beta - \Gamma^\beta \Gamma^\alpha)\sigma_{\alpha\mu} \sigma_{\beta\nu})$ on $S^-$ is equal to zero.

**Proof:** Let us choose an orthonormal basis $\{\alpha_i\}$ in $L_\theta$. Let $\{\mu_j\}$ be the dual basis in $L_\hat{\theta}$ with respect to the pairing given by $\sigma_{\alpha\mu}$. Then, the element

$$\left(\sum_{i,j}(\Gamma^{\alpha_i} \Gamma^{\alpha_j} - \Gamma^{\alpha_j} \Gamma^{\alpha_i})\sigma_{\alpha_i,\mu_k} \sigma_{\alpha_j,\mu_l}\right) = \Gamma^{\alpha_k} \Gamma^{\alpha_l} - \Gamma^{\alpha_l} \Gamma^{\alpha_k}$$

10
operator. It is well-known that this element is antiselfdual on $S^-$. Thus, the selfdual part of it is equal to zero. □

As an immediate corollary we obtain that noncommutative Nahm transform is similar to a commutative Nahm transform in the following relation.

**Theorem 4.1** Let $(R, \nabla^R)$ be a projective Hermitian module over $A_{\theta}$ with $\mathcal{P}$-irreducible connection $\nabla^R$ which satisfy the condition $F^+ + \omega^+ \cdot 1 = 0$, where $F$ is the curvature of $\nabla^R$. Let $(\hat{R}, \hat{\nabla}^\hat{R})$ be a noncommutative Nahm transform of $(R, \nabla^R)$. Then the curvature $\hat{F}$ of $\nabla^\hat{R}$ satisfies the equation $\hat{F}^+ - \hat{\omega}^+ \cdot 1 = 0$.

**Proof:** The statement immediately follows from Proposition 4.1 and the previous lemma. □

Let $\hat{R}^*$ be the left $A_{\hat{\theta}}$ module dual to $\hat{R}$, i.e., $\hat{R}^* = \text{Hom}_{A_{\hat{\theta}}}(\hat{R}, A_{\theta})$. Notice that as a vector space $\hat{R}^*$ is isomorphic to $\hat{R}$ since $\hat{R}$ is a projective Hermitian $A_{\hat{\theta}}$ module. Consider the tensor product $\hat{R} \otimes_{A_{\hat{\theta}}} \hat{R}^*$. Since $\hat{R}$ is a finitely generated projective $A_{\hat{\theta}}$ module the algebra $\hat{R} \otimes_{A_{\hat{\theta}}} \hat{R}^*$ is naturally isomorphic to the algebra $\text{End}_{A_{\hat{\theta}}}(\hat{R})$. Let $e$ be an identity element in $\text{End}_{A_{\hat{\theta}}}(\hat{R})$. By abuse of notation we denote its image in $\hat{R} \otimes_{A_{\hat{\theta}}} \hat{R}^*$ by the same letter $e$.

**Remark 4.2** Notice that $e$ is a finite sum $\sum_i x_i \otimes y_i$, where $x_i \in \hat{R}$ and $y_i \in \hat{R}^*$, because $\hat{R}$ is a finitely generated projective Hermitian module over $A_{\hat{\theta}}$.

The module $\hat{R}$ was defined as a submodule of $(R \otimes_{A_{\theta}} \mathcal{P}) \otimes S^-$. Therefore, we have a canonical embedding $\hat{R} \otimes_{A_{\hat{\theta}}} \hat{R}^*$ into $((R \otimes_{A_{\theta}} \mathcal{P}) \otimes S^-) \otimes_{A_{\hat{\theta}}} \hat{R}^*$. Denote by $\Psi$ the image of $e$ in $((R \otimes_{A_{\theta}} \mathcal{P}) \otimes S^-) \otimes_{A_{\hat{\theta}}} \hat{R}^*$. Let us notice that the module $((R \otimes_{A_{\theta}} \mathcal{P}) \otimes S^-) \otimes_{A_{\hat{\theta}}} \hat{R}^*$ is canonically isomorphic to $(R \otimes_{A_{\theta}} \mathcal{P} \otimes_{A_{\hat{\theta}}} \hat{R}^* \otimes S^-)$. We use the latter one everywhere instead of the former.

Denote by $R^*$ the left $A_{\theta}$ module dual to $R$, $R^* = \text{Hom}_{A_{\theta}}(R, A_{\theta})$. Any element $f \in R^*$ gives us a map from $(R \otimes_{A_{\theta}} \mathcal{P} \otimes_{A_{\hat{\theta}}} \hat{R}^* \otimes S^-)$ to $(\mathcal{P} \otimes_{A_{\hat{\theta}}} \hat{R}^*) \otimes S^-$. Let $x \otimes p \otimes y \otimes s \mapsto (f(x)p \otimes y) \otimes s \in (\mathcal{P} \otimes_{A_{\hat{\theta}}} \hat{R}^*) \otimes S^-$. 

11
where \( x \in R, \ p \in P, \ y \in \hat{R}^{*}, \) and \( s \in S^{-}. \) We denote this map by \( f. \)

Notice, that \( G \) (inverse to \( D^{-D^{+}} \)) commutes with the action of \( \mathcal{A}_{\hat{\vartheta}}. \) Therefore, it acts on \((R \otimes \mathcal{A}_{\theta} \ P \otimes \mathcal{A}_{\hat{\vartheta}} \ \hat{R}^{*}) \otimes S^{-}. \) We would like to consider a canonical element \( G\Psi \in (R \otimes \mathcal{A}_{\theta} \ P \otimes \mathcal{A}_{\hat{\vartheta}} \ \hat{R}^{*}) \otimes S^{-}. \) Strictly speaking the spinor spaces in the definitions of \( D_R \) and \( D_{\hat{R}^{*}} \) are different (one of them is constructed using \( L_{\theta}, \) another one using \( L_{\hat{\vartheta}}). \) However, we may use \( \sigma_{\alpha \mu} \) to identify Euclidean spaces \( L_{\theta} \) and \( L_{\hat{\vartheta}} \) and hence the corresponding spinor spaces. Thus we can consider the Dirac operator \( D_{\hat{R}^{*}} \) as a map from \((P \otimes \mathcal{A}_{\hat{\vartheta}} \ \hat{R}^{*}) \otimes S^{-} \) to \((P \otimes \mathcal{A}_{\hat{\vartheta}} \ \hat{R}^{*}) \otimes S^{+}. \) Notice that this identification does not respect the duality used in (9).

**Proposition 4.2** For any element \( f \in R^{*} \) the element \( f(G\Psi) \in (P \otimes \mathcal{A}_{\hat{\vartheta}} \ \hat{R}^{*}) \otimes S^{-} \) lies in the kernel of \( D_{\hat{R}^{*}}. \)

**Proof:** First, we need two lemmas.

**Lemma 4.2** Let \( \{\alpha_i\} \) be an orthonormal basis in \( L_{\theta} \) and \( \{\mu_i\} \) be the dual basis (also orthonormal) in \( L_{\hat{\vartheta}} \) (the pairing between \( L_{\theta} \) and \( L_{\hat{\vartheta}} \) is given by \( \sigma_{\alpha \mu}). \) Then we have

\[
[\hat{\nabla}_{\mu_i}, G] = 2G(\nabla_{\alpha_i} + \nabla_{\alpha_i})G.
\]  

**Proof:** Recall that \( G \) was defined as an inverse operator to \( D^{-D^{+}}. \) From Lemma 3.1 we know that \( D^{-D^{+}} = \sum_i(\nabla_{\alpha_i} + \nabla_{\alpha_i})^*(\nabla_{\alpha_i} + \nabla_{\alpha_i}) = \sum_i(\nabla_{\alpha_i}^R + \nabla_{\alpha_i})(\nabla_{\alpha_i}^R + \nabla_{\alpha_i}), \) since all connections are by definition Hermitian, i.e., self-adjoint. From (14) we obtain

\[
[D^{-D^{+}}, \hat{\nabla}_{\mu_j}] = \left[ \sum_i(\nabla_{\alpha_i}^R + \nabla_{\alpha_i})(\nabla_{\alpha_i}^R + \nabla_{\alpha_i}), \hat{\nabla}_{\mu_j} \right] = 2(\nabla_{\alpha_j}^R + \nabla_{\alpha_j}),
\]  

since \( \sigma_{\alpha_k, \mu_l} = \delta_{k,l}. \) Multiplying the formula (17) by \( G \) from the left and by \( G \) from the right and using the fact that \( G \) is inverse to \( D^{-D^{+}}, \) we obtain (15). \( \square \)

Let \( e = \sum_i x_i \otimes y_i \) be an element of \( \hat{R} \otimes \mathcal{A}_{\hat{\vartheta}} \ \hat{R}^{*} \) as in Remark 4.2. Then, any element \( z \in \hat{R}^{*} \) can be written as

\[
z = \sum_i (z, x_i) y_i
\]
Lemma 4.3  For any smooth element $z \in \hat{R}^*$ we have

$$\left(\nabla^R_{\mu_i}\right)^* z = - \sum_{k,l} (z, \nabla_{\mu_i} x_k - \Gamma^{\alpha_i} \Gamma_{\alpha_i}^\alpha (\nabla^{R}_{\alpha_i} + \nabla_{\alpha_i}) G x_k) y_k,$$

where the basises $\{\alpha_i\}$ and $\{\mu_i\}$ are chosen as in Lemma 4.2.

**Proof:** The proof is the following tedious trivial calculation.

$$\left(\nabla^R_{\mu_i}\right)^* z = \sum_k ((\nabla^R_{\mu_i})^* z, x_k) y_k = - \sum_k (z, \nabla^R_{\mu_i} x_k) y_k = \sum_k (z, P \nabla_{\mu_i} x_k) y_k = - \sum_k (z, (1 - D^+ GD^-) \nabla_{\mu_i} x_k) y_k = - \sum_k (z, \hat{\nabla}_{\mu_i} x_k - D^+ G [D^-, \hat{\nabla}_{\mu_i}] x_k) y_k,$$

since $D^- x_k = 0$. From (4) and the choice of the basises $\{\alpha_i\}$ and $\{\mu_i\}$ it follows that $[D^-, \hat{\nabla}_{\mu_i}] = \Gamma^{\alpha_i}$. Therefore, we obtain

$$\left(\nabla^R_{\mu_i}\right)^* z = - \sum_k (z, \nabla_{\mu_i} x_k - D^+ G \Gamma^{\alpha_i} x_k) y_k =$$

$$= - \sum_{k,l} (z, \nabla_{\mu_i} x_k - \Gamma^{\alpha_i} \Gamma_{\alpha_i}^\alpha (\nabla^{R}_{\alpha_i} + \nabla_{\alpha_i}) G x_k) y_k,$$

since $G$ commutes with $\Gamma^\alpha$ (and we replaced $D^+$ by its definition $D^+ = \sum_i \Gamma^{\alpha_i} (\nabla^{R}_{\alpha_i} + \nabla_{\alpha_i})$).

Now we prove the proposition. Let us choose the bases $\{\alpha_i\}$ and $\{\mu_i\}$ as in lemma 4.2. First, recall that $D^-_{\hat{R}^*} = \sum_j \Gamma^{\mu_j} \left( \hat{\nabla}_{\mu_j} + (\nabla^R_{\mu_j})^* \right)$. Since we identified the spinors for $A_\hat{g}$ with the spinors for $A_\Theta \Gamma^{\mu_j} = \Gamma^{\alpha_j}$ and $D^-_{\hat{R}^*} = \sum_j \Gamma^{\alpha_j} \left( \hat{\nabla}_{\mu_j} + (\nabla^R_{\mu_j})^* \right)$. Second, recall that $\Psi = \sum_k x_k \otimes y_k$, therefore

$$f(G\Psi) = f(G \sum_k x_k \otimes y_k) = \sum_k f(G x_k \otimes y_k),$$

since $G$ acts only on the first argument. Third, notice that $D^-_{\hat{R}^*}$ commutes with $f$ therefore we obtain

$$D^-_{\hat{R}^*} (f(G\Psi)) = f \left[ \sum_{j,k} \Gamma^{\alpha_j} \left( \hat{\nabla}_{\mu_j} + (\nabla^R_{\mu_j})^* \right) (G x_k \otimes y_k) \right].$$

13
We continue our manipulations

\[ D^{-}_{R^{*}}(f(G\Psi)) = f \left( \sum_{j,k} \Gamma^{\alpha_{j}} \left( \nabla_{\mu_{j}} Gx_{k} \otimes y_{k} + Gx_{k} \otimes (\nabla^{R}_{\mu_{j}})^{*} y_{k} \right) \right). \]  

(18)

Using lemma 4.3 we can rewrite

\[ \sum_{k} Gx_{k} \otimes (\nabla^{R}_{\mu_{j}})^{*} y_{k} = \]

\[ = \sum_{k} Gx_{k} \otimes (-\sum_{l,m}(y_{k}, \nabla_{\mu_{j}} x_{m} - \Gamma^{\alpha_{l}} \Gamma^{\alpha_{j}} \Gamma^{\alpha_{l}} G(\nabla^{R}_{\alpha_{l}} + \nabla_{\alpha_{j}}) Gx_{m})y_{m}) = \]

\[ = -\sum_{k,l,m} Gx_{k}(y_{k}, \nabla_{\mu_{j}} x_{m} - \Gamma^{\alpha_{l}} \Gamma^{\alpha_{j}} \Gamma^{\alpha_{l}} G(\nabla^{R}_{\alpha_{l}} + \nabla_{\alpha_{j}}) Gx_{m}) \otimes y_{m} = \]

\[ = -\sum_{k,l} \left( G\nabla_{\mu_{j}} x_{k} - \Gamma^{\alpha_{l}} \Gamma^{\alpha_{j}} G(\nabla^{R}_{\alpha_{l}} + \nabla_{\alpha_{j}}) Gx_{k} \right) \otimes y_{k}, \]

in the last line we replaced everywhere \( m \) by \( k \). Substituting this into the formula (18) we get

\[ D^{-}_{R^{*}}(f(G\Psi)) = f \]

\[ \left[ \sum_{j,k} \Gamma^{\alpha_{j}} \left( \nabla_{\mu_{j}} Gx_{k} \otimes y_{k} - \sum_{k,l} \left( G\nabla_{\mu_{j}} x_{k} - \Gamma^{\alpha_{l}} \Gamma^{\alpha_{j}} G(\nabla^{R}_{\alpha_{l}} + \nabla_{\alpha_{j}}) Gx_{k} \right) \otimes y_{k} \right) \right] = \]

\[ f \left[ \sum_{k} \left( \sum_{j} \Gamma^{\alpha_{j}} \nabla_{\mu_{j}} G + \sum_{j,l} \Gamma^{\alpha_{j}} \Gamma^{\alpha_{l}} G(\nabla^{R}_{\alpha_{l}} + \nabla_{\alpha_{j}}) Gx_{k} \right) \right] \].

(19)

Using lemma 4.2 we substitute \( 2G(\nabla^{R}_{\alpha_{j}} + \nabla_{\alpha_{j}}) G \) instead of \( [\nabla_{\mu_{j}}, G] \) and obtain

\[ D^{-}_{R^{*}}(f(G\Psi)) = f \left[ \sum_{k,l} \left( 2\Gamma^{\alpha_{l}} + \sum_{j} \Gamma^{\alpha_{j}} \Gamma^{\alpha_{l}} \Gamma^{\alpha_{j}} \right) G(\nabla^{R}_{\alpha_{l}} + \nabla_{\alpha_{j}}) Gx_{k} \otimes y_{k} \right]. \]

(20)

The proposition follows from the fact that \( 2\Gamma^{\alpha_{l}} + \sum_{j} \Gamma^{\alpha_{j}} \Gamma^{\alpha_{l}} \Gamma^{\alpha_{j}} \) equals zero for all \( l \), since the dimension is four. \( \square \)

We interpret this proposition as existence of \( A_{q} \) homomorphism from \( R^{*} \) to the kernel of \( D^{-}_{R^{*}} \). If we knew that \( \hat{R}^{*} \) is \( \mathcal{P} \)-irreducible we could say that we have a \( A_{q} \) homomorphism from \( R^{*} \) to \( \hat{R}^{*} \).
5 Examples.

In our previous discussion we assumed that we have a module $\mathcal{P}$ over $\mathcal{A}_{\theta \oplus (-\hat{\theta})}$ with some properties. In this section we give examples of such modules. Although the modules are quite simple the constructions of connections with desired properties are quite technical, and tedious.

At first we consider so called elementary finitely generated projective modules over $\mathcal{A}_{\theta \oplus (-\hat{\theta})}$ (see [RF1]). The second example shows how to deform the commutative Poincare module.Essentially we will show that the commutative Poincare module can be viewed as an elementary finitely generated projective module (in the language of [RF1]). This allows us to deform Poincare module and constant curvature connection on it using the constructions of the first part of the section.

In the first part we follow mainly the exposition of paper [RF1] with some minor modifications. We can think about $\theta$ and $\hat{\theta}$ as skew-symmetric bilinear forms on $\mathbb{Z}^4$. We embed $\mathbb{Z}^4$ in a usual way into $\mathbb{R}^4$ and extend the forms $\theta$ and $\hat{\theta}$ to be skew-symmetric bilinear forms on $\mathbb{R}^4$. For simplicity we assume that $\theta$ and $\hat{\theta}$ are non-degenerate symplectic forms. We can take their direct sum $\theta \oplus (-\hat{\theta})$ and consider it as a two-form on $\mathbb{R}^8$. Now take another $\mathbb{R}^8 = \mathbb{R}^4 \oplus (\mathbb{R}^4)^\prime$ with a canonical skew-symmetric bilinear form $\Omega$ given by $\Omega ((x_1, y_1), (x_2, y_2)) = \langle y_2, x_1 \rangle - \langle y_1, x_2 \rangle$, where $x_1, x_2 \in \mathbb{R}^4$ and $y_1, y_2 \in (\mathbb{R}^4)^\prime$. Let $\eta$ be an arbitrary integral two form (by definition, it takes integer values on the lattice $\mathbb{Z}^8$) on $\mathbb{R}^8$. Let $T$ be a linear map from $\mathbb{R}^8$ to $\mathbb{R}^4 \oplus (\mathbb{R}^4)^\prime$ such that $\theta \oplus (-\hat{\theta}) + \eta = T^\ast (\Omega)$. Such map always exists but it is never unique (any two maps of this kind are conjugate by an element from a symplectic group). Now we can describe some examples of modules $\mathcal{P}$.

We can realize $\mathcal{P}$ as a space of functions on $\mathbb{R}^4$; the smooth part $\mathcal{P}^{\text{smooth}} = S(\mathbb{R}^4)$ is the space of Schwartz functions on $\mathbb{R}^4$. Let us describe the left action of $\mathcal{A}_\theta$ and the right action of $\mathcal{A}_{\hat{\theta}}$ (which commute with each other). First, notice that $\mathbb{R}^4 \oplus (\mathbb{R}^4)^\prime$ acts on functions as follows: $((x, y). f)(z) = e^{2\pi i (y,z)} f(z + x)$. The left action of an element $U_\nu \in \mathcal{A}_\theta$, $\nu \in \mathbb{Z}_4$ is given by

$$(U_\nu f)(z) = (T(i_1(\nu)) f)(z),$$

where $i_1$ is the canonical inclusion of $\mathbb{Z}^4 \hookrightarrow \mathbb{R}^4 \hookrightarrow \mathbb{R}^4 \oplus \mathbb{R}^4 = \mathbb{R}^8$ in the first $\mathbb{R}^4$, i.e., $i_1(\nu) = (\nu, 0)$. Similarly, we define $i_2$ as the canonical inclusion of $\mathbb{Z}^4$ in the second $\mathbb{R}^4$ in $\mathbb{R}^8$, i.e., $i_2(\nu) = (0, \nu)$. We define the right action

15
of $\hat{U}_\nu \in \mathcal{A}_\hat{\theta}$, $\nu \in \mathbb{Z}^4$ as

$$(\hat{U}_\nu f)(z) = (T(i_2(\nu))f)(z).$$

A straightforward calculation shows that in this way we actually get an $(\mathcal{A}_\theta, \mathcal{A}_\hat{\theta})$ module. For the proof that $\mathcal{P}$ is projective $\mathcal{A}_\theta \otimes (-\hat{\theta})$ module see [R].

Next we define connections $\nabla$ and $\hat{\nabla}$ on $\mathcal{P}$. Notice that we can identify the Lie algebra $L_\theta$ with $(\mathbb{R}^4)'$. If $\alpha \in (\mathbb{R}^4)'$ then the corresponding derivation $\delta_\alpha$ acts on $U_\nu$, $\nu \in \mathbb{Z}^4$ as a multiplication by $2\pi i \langle \alpha, \nu \rangle$. Similarly, we can identify the Lie algebra $L_\hat{\theta}$ with $(\mathbb{R}^4)'$. For $\alpha \in (\mathbb{R}^4)'$, we have $\delta_\alpha(\hat{U}_\nu) = 2\pi i \langle \alpha, \nu \rangle \hat{U}_\nu$.

Now, let us define for any $(x, y) \in \mathbb{R}^4 \oplus (\mathbb{R}^4)'$ an operator $Q_{(x,y)}$ on the smooth functions on $\mathbb{R}^4$ as follows

$$(Q_{(x,y)}f)(z) = 2\pi i \langle y, z \rangle f(z) + \frac{d(f(z + tx))}{dt} |_{t=0}.$$ 

The straightforward calculation shows that

$$[Q_{(x,y)}, U_\nu] = 2\pi i \Omega((x,y), T((\nu, 0))) U_\nu,$$

$$[Q_{(x,y)}, \hat{U}_\nu] = 2\pi i \Omega((x,y), T((0, \nu))) \hat{U}_\nu.$$ 

We define connection $\nabla_\alpha$ as $Q_{V_1(\alpha)}$, where $V_1$ is the unique map from $L_\theta = (\mathbb{R}^4)'$ such that

$$\Omega(V_1(\alpha), T((\nu, \mu))) = \langle \alpha, \nu \rangle,$$

where $\alpha \in L_\theta = (\mathbb{R}^4)'$, and $\nu, \mu \in \mathbb{R}^4$. Since $\Omega$ is a non-degenerate bilinear form, $T$ is an isomorphism and the existence and uniqueness of $V_1$ is obvious. From this definition it is clear that $\nabla$ commutes with $\mathcal{A}_\theta$. Similarly, we define $\hat{\nabla}_\alpha$ as $Q_{V_2(\alpha)}$, where $V_2$ is the unique map from $L_\hat{\theta} = (\mathbb{R}^4)'$ such that

$$\Omega(V_2(\alpha), T((\nu, \mu))) = \langle \alpha, \mu \rangle,$$

where $\alpha \in L_\hat{\theta} = (\mathbb{R}^4)'$, and $\nu, \mu \in \mathbb{R}^4$. From this definition we immediately see that $\hat{\nabla}$ commutes with $\mathcal{A}_\theta$.

To compute the curvature of the connections $\nabla$ and $\hat{\nabla}$ and to compute the commutators $[\nabla_\alpha, \hat{\nabla}_\beta]$ we need to know the commutator $[Q_{(x_1,y_1)}, Q_{(x_2,y_2)}]$. It is easy to see that

$$[Q_{(x_1,y_1)}, Q_{(x_2,y_2)}] = 2\pi i \Omega((x_1, y_1), (x_2, y_2)) \text{Id}. $$

16
As an immediate corollary we obtain that connections $\nabla$ and $\hat{\nabla}$ have constant curvature and that $[\nabla_\alpha, \hat{\nabla}_\beta] = 2\pi i \Omega(V_1(\alpha), V_2(\beta))$. It is not hard to check that generically this pairing will be non-degenerate. Therefore, for generic $\theta$ and $\hat{\theta}$ we get many non-trivial examples of module $\mathcal{P}$.

The most interesting example of module $\mathcal{P}$ can be obtained by deforming the “Poincare module” - the space of sections of Poincare line bundle (see [B-Br], [D-K]).

Let us remind one of the possible definitions of the Poincare module. Let $\mathbb{Z}^4$ be a lattice in $\mathbb{R}^4$. Let $(\mathbb{Z}^4)'$ be the dual lattice in $(\mathbb{R}^4)'$. The Poincare module $\mathcal{P}$ consists of functions $f(x, \hat{x})$ on $\mathbb{R}^4 \oplus (\mathbb{R}^4)'$ which satisfy the following condition:

$$f(x + \lambda, \hat{x} + \hat{\lambda}) = e^{-2\pi i \langle \hat{\lambda}, x \rangle} f(x, \hat{x}),$$

(21)

for all $\lambda \in \mathbb{Z}^4$ and $\hat{\lambda} \in (\mathbb{Z}^4)'$. The algebra of functions on the torus considered as the algebra of periodic functions on $\mathbb{R}^4 \oplus (\mathbb{R}^4)'$; it acts on the module by multiplication.

We will use another construction of Poincare module (this construction is similar to the construction of modules over two-dimensional torus given in [H]).

Notice that for fixed $\hat{x}$ the function $f(x, \hat{x})$ is a periodic function in $x$ which can be written as a Fourier series with $\hat{x}$-dependent coefficients:

$$f(x, \hat{x}) = \sum_{\mu \in (\mathbb{Z}^4)'} e^{2\pi i \langle \mu, x \rangle} c\mu(\hat{x}).$$

(22)

The coefficients $c\mu(\hat{x})$ are smooth functions of $\hat{x}$ if the original function was smooth since they are given by the integrals over torus (which is compact). For convenience we denote $c_0(\hat{x})$ by $\phi(\hat{x})$. The property (21) of function $f(x, \hat{x})$ gives us that

$$c\mu(\hat{x}) = c_0(\hat{x} + \mu).$$

Therefore, we can rewrite (22) as

$$f(x, \hat{x}) = \sum_{\mu \in (\mathbb{Z}^4)'} e^{2\pi i \langle \mu, x \rangle} \phi(\hat{x} + \mu).$$

(23)

Moreover, it is not hard to see that function $f(x, \hat{x})$ given by the formula (23) is smooth if and only if the function $\phi(\hat{x})$ belongs to the Schwartz space.
The algebra \( \mathcal{A}_0 \) is generated by the elements \( U_\mu = e^{2\pi i (\mu, x)} \) and \( \hat{U}_\nu = e^{2\pi i (\nu, \hat{x})} \), where \( \nu \in \mathbb{Z}^4 \) and \( \mu \in (\mathbb{Z}^4)' \). We can rewrite the action of the operators \( U_\mu \) and \( \hat{U}_\nu \) in terms of their action on \( \phi(\hat{x}) \). One easily obtains
\[
\hat{U}_\nu(\phi(\hat{x})) = e^{2\pi i (\nu, \hat{x})} \phi(\hat{x}) \quad \text{and} \quad U_\mu(\phi(\hat{x})) = \phi(\hat{x} - \mu). \tag{24}
\]

Thus, we realized the Poincare module as the space of functions on \((\mathbb{R}^4)'\) with the action given by the formula (24).

The constant curvature connection on the Poincare line bundle is given by
\[
\nabla = d + 2\pi i \langle \hat{x}, dx \rangle
\]
in the first realization of Poincare module. We can easily rewrite it in terms of its action on the function \( \phi(\hat{x}) \). If \( y \in \mathbb{R}^4 \) and \( \hat{y} \in (\mathbb{R}^4)' \) then
\[
\nabla_y \phi(\hat{x}) = 2\pi i \langle y, \hat{x} \rangle \phi(\hat{x}) \quad \text{and} \quad \nabla_{\hat{y}} \phi(\hat{x}) = \frac{d \phi(\hat{x} + t\hat{y})}{dt}\bigg|_{t=0}. \tag{25}
\]

One can also easily rewrite the Hermitian inner product \( \langle f(x, \hat{x}), \tilde{f}(x, \hat{x}) \rangle \) in terms of \( \phi(\hat{x}) \) and \( \tilde{\phi}(\hat{x}) \). In particular, the trace of the inner product of \( f(x, \hat{x}) \) and \( \tilde{f}(x, \hat{x}) \) equals
\[
\tau(\langle f(x, \hat{x}), \tilde{f}(x, \hat{x}) \rangle) = \int_{-\infty}^{\infty} \phi(\hat{x}) \tilde{\phi}(\hat{x}) d\hat{x}. \tag{26}
\]

From the above discussion we see that Poincare module fits very well in the picture in the first example. More precisely, the map \( T \) from the first example is
\[
T(x, y) = (x, y).
\]

Here, we identified the dual space \((\mathbb{R}^4)'\) with \(\mathbb{R}^4\) using some fixed standard integral bases. We can easily calculate \( \theta, \tilde{\theta} \), and \( \eta \) and we obtain that \( \theta = \tilde{\theta} = 0 \) and \( \eta \) is the two form on \(\mathbb{R}^8\) given by
\[
\eta((x_1, y_1), (x_2, y_2)) = \langle y_2, x_1 \rangle - \langle y_1, x_2 \rangle,
\]
where \( x_i \in \mathbb{R}^4 \) and \( y_i \in \mathbb{R}^4 \) for \( i = 1, 2 \). The inner product \( \langle \cdot, \cdot \rangle \) comes from the above identification of \(\mathbb{R}^4\) with \((\mathbb{R}^4)'\). It is obvious that \( \eta \) is integral two form in this case.
We can easily deform the map $T$ so that the form $\eta$ will be preserved but the forms $\theta$ and $\hat{\theta}$ will be deformed. In this way we obtain a deformation of the commutative Poincare module. One can easily check that this deformation is always possible for small $\theta$ and $\hat{\theta}$. Using the above construction we obtain a constant curvature connection in the deformed Poincare module.

Explicitly, the equation (24) is deformed to:

\[ \hat{U}_\nu(\phi(\hat{x})) = e^{2\pi i (C_\nu, \hat{x})} \phi(\hat{x} - A \nu) \quad \text{and} \quad U_\mu(\phi(\hat{x})) = e^{2\pi i (B_\mu, \hat{x})} \phi(\hat{x} - D \mu). \]

(27)

where $A, B, C, D$ are the operators: $\mathbb{R}^4 \to (\mathbb{R}^4)'$ such that:

\[ B^tD - D^tB = \frac{\theta}{2\pi i}, \quad C^tA - A^tC = \frac{\hat{\theta}}{2\pi i}, \quad B^tA - D^tC \in \text{GL}(4, \mathbb{Z}) \]

6 Appendix A.

In this section we will discuss under which conditions on the module $\mathcal{P}$ the kernel of $D^-$ (we assume that the kernel of $D^+$ is trivial) is a finitely generated projective module over $A_{\hat{\theta}}$. First, we will explain how to construct a parametrix to $D^- (D)$. More precisely, what we want to construct is a “compact” operator $Q$ commuting with the action of $A_{\hat{\theta}}$ such that $DQ = 1 + K_1$ and $QD = 1 + K_2$, and $K_1, K_2, DK_1, DK_2, K_1D, K_2D$ are “compact” operators. This is enough for all our purposes.

We would like to construct a parametrix to $D : E \otimes_{A_\theta} \mathcal{P} \otimes S \to E \otimes_{A_\theta} \mathcal{P} \otimes S$, where $E$ is a finitely generated projective module over $A_\theta$ and $\mathcal{P}$ is a finitely generated projective module over $A_\theta \times A_{\hat{\theta}}$. First, we can reduce the problem to the case when $E$ is a free $A_\theta$ module. Indeed, since $E$ is a finitely generated projective $A_\theta$ module it is a direct summand in $A_\theta^k$ for some $k$. We have: $A_\theta^k = E \oplus \hat{E}$. Let $P_E$ be an orthogonal projection on $E$. Choose any $A_\theta$ connection on $\hat{E}$. Consider the Dirac operator $\hat{D} : (E \oplus \hat{E}) \otimes_{A_\theta} \mathcal{P} \otimes S \to (E \oplus \hat{E}) \otimes_{A_\theta} \mathcal{P} \otimes S$. Notice that $\hat{D} = D \oplus \hat{D}$, where $\hat{D}$ is the Dirac operator on $\hat{E} \otimes_{A_\theta} \mathcal{P} \otimes S$. Moreover, $P_E \hat{D} = \hat{D} P_E$. Therefore, if $\hat{Q}$ is a parametrix to $\hat{D}$ then it is easy to see that $Q = P_E \hat{Q} P_E$ is a parametrix to $D$. More precisely, if $\hat{Q} \hat{D} = 1 + \hat{K}_1$ and $D \hat{Q} = 1 + \hat{K}_2$, then $QD = 1 + K_1$ and $DQ = 1 + K_2$, where $K_1 = P_E \hat{K}_1 P_E$ and $K_2 = P_E \hat{K}_2 P_E$. Notice that $K_1$ and $K_2$ preserve the properties of $\hat{K}_1$ and $\hat{K}_2$. 

19
Similarly, we can assume that $\mathcal{P}$ is a free module over $A_\theta \times A_\hat{\theta}$. Therefore, we need to construct parametrix only in the case of Dirac operator $D$ on $(A_\theta \times A_\hat{\theta})^k \otimes S$. One can consider $(A_\theta \times A_\hat{\theta})$ as the space of functions on the product of tori $T^n \times T^n$. The Dirac operator $D$ in this case becomes a sum of a usual commutative Dirac operator $D_c$ along the first torus plus a bounded operator $A$ (preserving the space of smooth functions), that is $D = D_c + A$. Let $Q_c$ be a parametrix to $D_c$. Then, $Q_c D = Q_c D_c + Q_c A$. It follows immediately from the properties of $Q_c$ that $Q_c A$ is bounded operator from $H_{(0)}$ to $H_{(1)}$, where $H_{(l)}$ are Sobolev spaces of functions along the first torus. Thus, for any given natural number $l$ we can easily construct an operator $Q$ (maps $H_{(0)}$ to $H_{(1)}$) such that $QD = 1 + K_1$ and $K_1$ maps $H_{(0)}$ to $H_{(l)}$. Similarly, for any given natural number $l$ we can construct an operator $Q$ (maps $H_{(0)}$ to $H_{(1)}$) such that $DQ = 1 + K_2$ and $K_2$ maps $H_{(0)}$ to $H_{(l)}$. Moreover, one can see that $Q$ can be chosen to be the same. Such an operator $Q$ is enough for all our purposes. We call it parametrix here.

Second, let us prove the following lemma.

**Lemma 6.1** Let $P$ be an operator on $L^2(\mathbb{R}) \times A_\hat{\theta}$ commuting with the action of $A_\hat{\theta}$. Assume that $P^* = P$, $P^2 = P$ and $P$ is “compact” (i.e., $P \in \mathcal{K} \hat{\otimes} A_\hat{\theta}$, where $\mathcal{K}$ is the algebra of compact operators on $L^2(\mathbb{R})$). Then $P$ is a projection on a finitely generated projective $A_\hat{\theta}$ module.

**Proof:** First, we can approximate $P$ by a self-adjoint operator $A = \sum_j \langle f_j, \cdot \rangle g_j \times a_j$, where $f_j, g_j \in L^2(\mathbb{R})$ and $a_j \in A_\hat{\theta}$ (since $P$ is self-adjoint). That is, $P = A + \epsilon$, where $\epsilon$ is a self-adjoint operator of norm less then $\frac{1}{100}$. It is obvious that $P$ is a projection on the kernel of $1 - P$. We can write $1 - P = 1 - \epsilon - A = (1 - A(1 - \epsilon)^{-1})(1 - \epsilon)$. If $(1 - P)x = 0$ it means that $(1 - A(1 - \epsilon)^{-1})(1 - \epsilon)x = 0$. Denote by $y = (1 - \epsilon)x$. Then $(1 - A(1 - \epsilon)^{-1})y = 0$. From this we see that $y = \sum_j \langle f_j, (1 - \epsilon)^{-1}y \rangle g_j \times a_j$. In particular this means that $y \in \sum_j g_j \times A_\hat{\theta}$. The module $\sum_j g_j \times A_\hat{\theta}$ is a free finitely generated $A_\hat{\theta}$ submodule of $L^2(\mathbb{R}) \times A_\hat{\theta}$. We denote it by $U$. Denote by $M$ the set $(1 - \epsilon)^{-1}U$. Since $(1 - \epsilon)^{-1}$ is invertible operator commuting with $A_\hat{\theta}$ we see that $M$ is $A_\hat{\theta}$ module. It is not hard to check (using the fact that $(1 - \epsilon)^{-1}$ is self-adjoint and close to identity) that $M$ is a free hermitian finitely generated $A_\hat{\theta}$ module.

From the above it is obvious that if $(1 - P)x = 0$ then $x \in M$. Therefore, we can restrict projection $P$ on $M$ and we will get the same image. What is left to check is that $P|_M$ is a projection operator. We see that if $m \in M$
then \((1 - P)Pm = Pm - P^2m = Pm - Pm = 0\). This implies that \(P|_M\) is projection operator. Now, since \(M\) is finitely generated free \(A_\theta\) module and \(P : M \to M\) is a projection lemma is proved. \(\square\)

We cannot directly apply the above lemma to the module \(E \otimes_{A_\theta} \mathcal{P} \otimes S\). But since \(E \otimes_{A_\theta} \mathcal{P} \otimes S\) can be considered a direct summand in \((A_\theta \times A_\hat{\theta})^k \otimes S\) apply Lemma 6.1 to \((A_\theta \times A_\hat{\theta})^k \otimes S\).

7 Acknowledgments.

We are grateful to G. Elliott, D. Fuchs, A. Gorokhovsky, G. Kuperberg, A. Lawrence, A. Losev, M. Rieffel, S. Shatashvili and I. Zakharevich for discussions. The research of N. N. was supported by Harvard Society of Fellows, partially by NSF under grant PHY-98-02-709, partially by RFFI under grant 98-01-00327 and partially by grant 96-15-96455 for scientific schools. The research of A. S. was partially supported by NSF under grant DMS-9500704.

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21
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