Counting of discrete Rossby/drift wave resonant triads

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Abstract

The purpose of this note is to remove the confusion about counting of resonant wave triads for Rossby and drift waves in the context of the Charney-Hasegawa-Mima equation. In particular, we aim to point out a major error of over-counting of triads in the paper \textit{Discrete exact and quasi-resonances of Rossby/drift waves on $\beta$-plane with periodic boundary conditions} [KK13].

1 Preliminaries

The Charney-Hasegawa-Mima (CHM) equation on a bi-periodic domain $[0, 2\pi)^2$ in physical space for an infinite deformation radius is

\[ \partial_t \Delta \psi + \beta \partial_x \psi + (\partial_x \psi) \partial_y \Delta \psi - (\partial_y \psi) \partial_x \Delta \psi = 0, \quad (1) \]

where $\psi = \psi(x, y, t)$ (a real-valued function) and $\beta$ is a constant. Let us introduce $\hat{\psi}_k$, the Fourier transform of $\psi(x, y, t)$:

\[ \psi(x, y, t) = \sum_{k \in \mathbb{Z}^2} \hat{\psi}_k(t) \exp(i \mathbf{k} \cdot \mathbf{x}). \quad (2) \]

The two-dimensional wavevectors are decomposed as $\mathbf{k} = (k_x, k_y)$. In the context of this discussion, we will restrict the allowed interacting modes to those which are not zonal ($k_x \neq 0$) and, more importantly, we restrict the discussion to exactly resonant triads only: Fourier wavevectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 \in \mathbb{Z}^2$ can interact if and only if

\[ \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 \quad \text{and} \quad \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = \omega(\mathbf{k}_3), \quad (3) \]

where we introduced the linear dispersion $\omega(\mathbf{k}) \equiv -\frac{\beta k_x}{|\mathbf{k}|^2}$. The set of non-zonal wavevectors satisfying equations (3) is called “resonant set” and is denoted by $R(\subset \mathbb{Z}^2)$. In this case it is

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well established (see [ZP88] and the book [N11], equations (7.8) and (6.11)) that the CHM equation can be cast in canonical form

\[ \text{i} \dot{a}_k = \omega_k a_k + \text{sign}(k_x) \sum_{k_1+k_2=k} V_{12}^{k} a_{k_1} a_{k_2}, \]  

(4)

with canonical variable

\[ a_k = -\frac{|k|^2}{\sqrt{\beta |k_x|}} \hat{\psi}_k. \]  

(5)

The nonlinear interaction coefficient is

\[ V_{12}^{k} \equiv V_{k_1,k_2}^{k} = \frac{i\sqrt{\beta}}{2} \sqrt{|k_x| k_{1x} k_{2x}|} \left( \frac{k_{1y}}{|k_1|^2} + \frac{k_{2y}}{|k_2|^2} - \frac{k_y}{|k|^2} \right), \]  

(6)

and the sum in equation (4) is restricted to the resonant set \( R \). In particular, \( k \in R \) in this equation.

2 Major error in [KK13]

The over-counting problem in [KK13] is easily shown. In equation (10) of that paper, six triads are listed. The authors state that these are six different triads, but in fact all six are physically identical and mathematically equivalent. The simplest way to see that there is redundancy is to compare the first and fifth triad in equation (10): the only difference is that all wavenumbers in the fifth triad are the negatives of the wavenumbers in the first. But as is well known (see our equation (2)), for a real field \( \psi(x,y,t) \) the two modes \( \hat{\psi}_k \) and \( \hat{\psi}_{-k} \) must occur with amplitudes which are equal in absolute value. They are not independent. Thus, the claim of [KK13] that they are listing six separate triads is wrong.

In fact the problem is greater than this: all six triads are equivalent, as we shall now show. [KK13] state that any exactly resonant triad \( k_1 + k_2 = k_3 \) interacts with another five triads: \( k_1 + (-k_3) = (-k_2), k_3 + (-k_2) = k_1, k_3 + (-k_1) = k_2, (-k_2) + (-k_1) = -k_3 \) and \( k_2 + (-k_3) = -k_1 \). Crucially, in [KK13] the modes with wavenumbers \( k_1, k_2, k_3, (-k_1), (-k_2) \) and \( (-k_3) \) are considered as six different modes. In a general setting (complex-valued \( \psi(x,y,t) \)) this would be true because equation (1) alone gives separate evolution equations for \( a_k \) and \( a_{-k} \). However, due to the fact that the underlying field \( \psi(x,y,t) \) is real, the extra condition \( a_{-k} = \overline{a_k} \) is imposed on the modes (overbars denote complex conjugation). This extra condition is preserved in time by the system of evolution equations (1), allowing for a reduction of the original variables \( \{a_k\}_{k \in R} \) to a smaller space \( \{a_k\}_{k \in \tilde{R}} \) where \( \tilde{R} \) is the restriction of the resonant set \( R \) to the half plane \( k_x > 0 \). Once this reduction is made, the six triads stated in [KK13] as different reduce to a single triad contributing with only one interaction term in the evolution equations. The six modes stated in [KK13] as different, reduce to three modes.

Detailed reduction. Let us look at the contributions from equations (1) to the evolution of the six modes’ amplitudes \( a_{k_1}, a_{k_2}, a_{k_3}, a_{-k_1}, a_{-k_2}, a_{-k_3} \), without using the extra condition

\(^1\)Equation (5) uses a slightly different notation as compared to equation (7.6) in [N11].
\(a_{-k} = \pi_k\). We obtain

\[
\begin{align*}
    i\dot{a}_{k_1} &= \omega_{k_1} a_{k_1} + 2 \text{sign}(k_{1x}) V^k_{-k_2,k_3} a_{-k_2} a_{k_3} + \ldots, \\
    i\dot{a}_{-k_2} &= \omega_{-k_2} a_{-k_2} + 2 \text{sign}(-k_{2x}) V^{-k}_{-k_2,k_1} a_{-k_2} a_{k_1} + \ldots, \\
    i\dot{a}_{k_3} &= \omega_{k_3} a_{k_3} + 2 \text{sign}(k_{3x}) V^k_{k_1,k_2} a_{k_1} a_{k_2} + \ldots, \\
    i\dot{a}_{-k_3} &= \omega_{-k_3} a_{-k_3} + 2 \text{sign}(-k_{3x}) V^{-k}_{-k_1,-k_2} a_{-k_1} a_{-k_2} + \ldots, \\
    i\dot{a}_{k_2} &= \omega_{k_2} a_{k_2} + 2 \text{sign}(k_{2x}) V^k_{k_3,-k_1} a_{k_3} a_{-k_1} + \ldots, \\
    i\dot{a}_{-k_1} &= \omega_{-k_1} a_{-k_1} + 2 \text{sign}(-k_{1x}) V^{-k}_{k_2,-k_3} a_{k_2} a_{-k_3} + \ldots, \\
\end{align*}
\]

where “...” denote terms involving modes beyond \(k_1, k_2, k_3, (-k_1), (-k_2)\) and \((-k_3)\). To derive these equations we used the symmetry of the interaction coefficient \(V^k_{k',k''} = V^k_{k',k''}\) stemming from equation (6).

Equations (7) appear like a system of six coupled equations for six complex variables. However, once the extra condition \(a_{-k} = \pi_k\) (stemming from the reality of the underlying field \(\psi(x,y,t)\)) is used, these six equations reduce to only three independent equations, for three independent variables \(a_{k_1}, a_{k_2}, a_{k_3}\). We show explicitly that the first and the last equations in (7) are equivalent. The remaining pairings can be shown in a similar way. The complex conjugate of the first equation gives

\[-i\dot{\bar{a}}_{k_1} = \omega_{k_1} \bar{a}_{k_1} + 2 \text{sign}(k_{1x}) \bar{V}^k_{-k_2,k_3} \bar{a}_{-k_2} \bar{a}_{k_3} + \ldots.\]

Using \(\pi_k = a_{-k}\) and \(\bar{V}^k_{-k_2,k_3} = -V^{-k}_{-k_2,k_3} = \bar{V}^k_{k_2,-k_3}\) we obtain

\[-i\dot{\bar{a}}_{-k_1} = \omega_{k_1} a_{-k_1} + 2 \text{sign}(k_{1x}) V^{-k}_{k_2,-k_3} a_{k_2} a_{-k_3} + \ldots,\]

but this is equivalent to the last equation in (7) after using the identity \(\omega_k = -\omega_{-k}\).

In summary, equations (7) reduce to the well-known triad system

\[
\begin{align*}
    i\dot{a}_{k_1} &= \omega_{k_1} a_{k_1} + 2 \text{sign}(k_{1x}) V^k_{k_3,-k_2} a_{k_3} \bar{a}_{k_2} + \ldots, \\
    i\dot{a}_{k_2} &= \omega_{k_2} a_{k_2} + 2 \text{sign}(k_{2x}) V^k_{k_3,-k_1} a_{k_3} \bar{a}_{k_1} + \ldots, \\
    i\dot{a}_{k_3} &= \omega_{k_3} a_{k_3} + 2 \text{sign}(k_{3x}) V^k_{k_1,k_2} a_{k_1} a_{k_2} + \ldots. \\
\end{align*}
\]

Therefore all six triads stated in \([KK13]\) as different are in fact only one physical triad.

For the domain defined by \(0 \leq |k_{x}|, |k_{y}| \leq 200\), \([KK13]\) report finding 828 triads while \([BH13]\) find a significantly lower number. The discrepancy is easily explained: since \([KK13]\) over-count by a factor of 6, they have in reality identified only \((828/6) = 138\) valid triads. \([BH13]\) confined their search to the interior of the domain \(0 \leq |k_{x}|, |k_{y}| \leq 200\) and found 136 triads. Two additional triads have a vertex sitting on the boundary of the domain: \((24, 88), (40, -200), (64, -112)\) and \((24, -88), (40, 200), (64, 112)\). This gives a total of 138 triads, precisely one sixth of the number reported by \([KK13]\).
3 False deduction by [KK13] from a Theorem in [YY13]

In [KK13] it is stated that there is a contradiction between the work of [BH13] and a mathematical theorem in [YY13]. However, [YY13] considered the limit of large $\beta$, whereas the results of [BH13] are for finite $\beta$. Moreover, the Theorem of [YY13] relies on the presence of viscosity. [KK13] have drawn an unjustified conclusion from the theorem in [YY13].

4 Quasi-resonances

[KK13] state that all quasi-resonances found in [BH13] in a given domain $0 \leq |k_x|, |k_y| \leq L$ are formed in the neighborhood of exact resonant triads. This is false. In fact, the majority of quasi-resonant triads found by the method of [BH13] are arbitrarily far from the exact resonant triads that are used to generate them, because these exact resonant triads are typically of wavenumbers much greater than $L$ (by orders of magnitude). More importantly, [BH13] produce triads close to the so-called resonant manifold, which is actually a set of curves on $\mathbb{R}^2$, typically with only a few integer points, if any. These integer points correspond to exact resonances. Incidentally, [KK13] attempt a similar method in their Section IV.

5 Conclusions

There are a number of other errors in [KK13]. However, the most egregious error is the 6-fold over-counting of triads.

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