A BOOTSTRAP APPROXIMATION TO $L_p$-STATISTIC OF KERNEL DENSITY ESTIMATOR IN LENGTH-BIASED MODEL

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Abstract. This article presents a bootstrap approximation to the error quantity $I_n(p) = \int_\mathbb{R} |f_n(x) - f(x)|^p d\mu(x)$, $1 \leq p < \infty$, where $f_n$ is the kernel density estimator proposed by Jones [11] for length-biased data. The article establishes one bootstrap central limit theorem for the corresponding bootstrap version of $I_n(p)$.

1. Introduction

Kernel methods are widely used to estimate probability density functions. Suppose $X_i \in \mathbb{R}$ are independent with a common density $f$. Then the kernel density estimate of $f$ is

$$f_n(x) = \frac{1}{nh_n} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h_n}\right),$$

where $K$ is a kernel function and $h_n$ is a sequence of (positive) "bandwidths" tending to zero as $n \to \infty$. (See Rosenblatt [15]). A standard and common measure of $f_n$ is given by the $L_p$ distance

$$I_n(p) = \int_\mathbb{R} |f_n(x) - f(x)|^p d\mu(x), 1 \leq p < \infty,$$

where, $\mu$ is a measure on the Borel sets of $\mathbb{R}$. The mean integrated square error, that is, $E(I_n(2))$, is a very popular measure of the distance of $f_n$ from $f$. The other well-investigated case is $p = 1$. In general, $I_n(p)$ can be used to carry out tests of hypothesis (asymptotic) for the density $f$. Csörgő and Horváth [4] obtained a central limit theorem for $L_p$ distances ($1 \leq p < \infty$) of kernel estimators based on complete samples. In the random censorship model, Csörgő et.al. [2] obtained central limit theorems for $L_p$ distances ($1 \leq p < \infty$) of kernel estimators. They

Key words and phrases. Bootstrap, Kernel, Length-biased.
tested their result in Monte Carlo trails and applied them for goodness of fit. Groeneboom et. al. [8] studied the asymptotic normality of a suitably rescaled version of the $L_1$ distance of the Grenander estimator, using properties of a jump process was introduced by Gronenboom [7]. In Length-biased setting, Fakoor and Zamini [6], proved a central limit theorem for $L_p$ distances ($1 \leq p < \infty$). Also they presented a central limit theorem for approximation of $I_n(p)$. Mojirsheibani [14], presented two approximations for $L_p$ distances ($1 \leq p < \infty$) on complete samples. He also defined two approximations for $I_n(p)$ ($1 \leq p < \infty$) based on bootstrap versions of $I_n(p)$ with central limit theorems for them.

In this paper a bootstrap version of $I_n(p)$ in Length-biased sampling is defined and a central limit theorem for it is defined. In biased sampling, the data are sampled from a distribution different from censoring sampling. In censoring, some of the observations are not completely observed, but are known only to belong to a set. The prototypical example is the time until an event. For an event that has not happened by time $t$, the value is known only to be in $(t, \infty)$. Truncation is a more severe distortion than censoring. Where censoring replaces a data value by a subset, truncation deletes that value from the sample if it would have been in a certain range. Truncation is an extreme form of biased sampling where certain data values are unobservable. Length-biased data appear naturally in many fields, and particularly in problems related to renewal processes. This special truncation model has been studied by e.g. Wicksel [17], McFadden [13], Cox [1], Vardi [16].

2. Main results

Let $Y_1, \ldots, Y_n$ be $n$ independent and identically distribute (i.i.d.) nonnegative random variables (r.v.) from a distribution $G$, defined on $\mathbb{R}^+ = [0, \infty)$. $G$ is called a length-biased distribution corresponding to a given distribution $F$ (also defined on $\mathbb{R}^+$), if

$$G(t) = \mu^{-1} \int_0^t x F(x), \quad \text{for every } y \in \mathbb{R}^+,$$

where $\mu = v^{-1} = \int_0^\infty x dF(x)$ is assumed to be finite. A simple calculation shows that

$$F(t) = \mu \int_0^t y^{-1} dG(y), \quad t > 0.$$
The empirical estimator of $F$ can be written by $G_n(t) = \frac{1}{n} \sum_{i=1}^{n} I(Y_i \leq t)$, namely

$$F_n(t) = \mu_n \int_{0}^{t} y^{-1} dG_n(y),$$

where $\mu_n^{-1} = v_n = \int_{0}^{\infty} y^{-1} dG_n(y)$. Using $F_n(t)$, the following estimator for density function of $f = F'$ in length-biased model is known,

$$f_n(t) = \frac{1}{h_n} \int_{\mathbb{R}} K\left(\frac{t-x}{h_n}\right) dF_n(x). \quad (2.1)$$

(For some references about this subject, see the references are given in Fakoor and Zamini [6].)

We start by stating the further notations. Assume that $T < \tau = \sup \{x, G(x) < 1\} < \infty$. Throughout this paper $N = N(0, 1)$ stands for a standard normal r.v. Let

$$m(p) = E|N|^p \left( \int_{\mathbb{R}} K^2(t) dt \right)^{p/2} \int_{0}^{T} f'^{p+2}(t) dt, \quad (2.2)$$

and

$$\sigma^2 = \sigma_1^2 \int_{0}^{T} f'^{p+2}(t) dt \left( \int_{\mathbb{R}} K^2(t) dt \right)^{p}, \quad (2.3)$$

where,

$$\sigma_1^2 = (2\pi)^{-1} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |xy|^p (1 - r^2(u))^{-1/2}. \exp \left( - \frac{1}{2(1-r^2(u))} (x^2 - 2xyr(u) + y^2) \right) dxdy - (E|N|^p)^2 \right) du,$$

with

$$r(t) = \frac{\int_{\mathbb{R}} K(u)K(t+u)du}{\int_{\mathbb{R}} K^2(u)du}.$$

Let $B(t, n)$ is a two-parameter Gaussian process with zero mean and covariance function

$$E[B(x, n)B(y, m)] = (mn)^{1/2}(mn)[G(x \wedge y)G(x)G(y)](a \wedge b = \min(a, b)).$$

Based on $B(x, n)$, Horváth [9], defined the mean zero Gaussian process

$$\Gamma(t, n) = \mu \int_{0}^{t} y^{-1} dB(y, n) - \mu F(t) \int_{0}^{\infty} y^{-1} dB(y, n), \quad (2.4)$$

with covariance function

$$E(\Gamma(x, n)\Gamma(y, m)) = (mn)^{-1/2}(m \wedge n) [\sigma(x \wedge y) - F(x)\sigma(y) - F(y)\sigma(x) + F(x)F(y)\sigma], \quad (2.5)$$
such that $\Gamma(t, n)$ approximates the empirical process $\alpha_n(t) = \sqrt{n}[F_n(t) - F(t)]$. In (2.5), $\sigma(t) = \mu^2 \int_0^t y^{-2} dG(y)$, and $\sigma = \lim_{t \to \infty} \sigma(t) = \mu^2 \int_0^\infty y^{-2} dG(y)$. Fakoor and Zamini [6] used the strong approximation defined in (2.4) and investigated asymptotic normality behavior

$$I_n(p) = \int_0^T |f_n(x) - f(x)|^p \left( \frac{x}{\mu} \right)^{p/2} dF(x),$$  \hspace{1cm} \text{(2.6)}$$

where $f_n$ is defined in (2.1). They showed that under some conditions on $h_n$,

$$(h_n \sigma^2(p))^{-1/2} \{ (nh_n)^{p/2} I_n(p) - m(p) \} \xrightarrow{D} N(0, 1).$$  \hspace{1cm} \text{(2.7)}$$

In this article we prove one bootstrap central limit theorem for the corresponding bootstrap version of $I_n(p)$ in (2.6).

Given the random sample, $Y_1, \ldots, Y_n$, let $Y_1^*, \ldots, Y_n^*$ be a bootstrap sample drawn from $Y_1, \ldots, Y_n$. That is, $Y_1^*, \ldots, Y_n^*$ be conditionally independent random variables with common distribution function $G_n(t) = \frac{1}{n} \sum_{i=1}^n I(Y_i^* \leq t)$. (See e.g. Csörgő et.al.[5]). Let

$$F_{n,n}(t) = \mu_{n,n} \int_0^t y^{-1} dG_{n,n}(y),$$

and

$$f_{n,n}(x) = h_n^{-1} \int_{\mathbb{R}} K \left( \frac{x - y}{h_n} \right) dF_{n,n}(y),$$  \hspace{1cm} \text{(2.8)}$$

where

$$\mu_{n,n}^{-1} = v_{n,n} = \int_0^T y^{-1} dG_{n,n}(y),$$

with $G_{n,n}(y) = \frac{1}{n} \sum_{i=1}^n I[Y_i^* \leq n]$. Using $f_n$ and $f_{n,n}(x)$ in (2.1) and (2.8) respectively, one can write the bootstrap version of $I_n(p)$ in (2.6) by

$$I_{n,n}(p) = \int_0^T \left| f_{n,n}(x) - f_n(x) \right|^p \left( \frac{x}{\mu_n} \right)^{p/2} f_n(x) dx.$$  \hspace{1cm} \text{(2.9)}$$

A result of Csörgő et al.[5], shows that there exists a sequence of Brownian bridges $\{B_{n,n}(t), 0 \leq t \leq 1 \}$ such that

$$\sup_{-\infty < x < \infty} \left| \beta_{n,n}(x) - B_{n,n}(G(x)) \right| = O(n^{-1/2} \log n) \ \text{a.s.,}$$  \hspace{1cm} \text{(2.10)}$$

where

$$\beta_{n,n}(x) = n^{1/4} \left[ G_{n,n}(x) - G_n(x) \right].$$  \hspace{1cm} \text{(2.11)}$$
Set
\[ \alpha_{n,n}(t) = \frac{n^{\frac{1}{2}}}{2} \left[ F_{n,n}(t) - F_n(t) \right]. \]  
(2.12)

Clearly
\[ \alpha_{n,n}(t) = \mu_n \int_0^t y^{-1} d\beta_{n,n}(y) - \left( \int_0^t y^{-1} dG_{n,n}(y) \right) \mu_n \mu_n \int_0^\tau y^{-1} d\beta_{n,n}(y). \]  
(2.13)

This form of \( \alpha_{n,n} \) suggests that the approximation processes for \( \alpha_{n,n} \) will be
\[ \Gamma_{n,n}(t) = \mu_n \int_0^t y^{-1} dB_{n,n}(G(y)) - \left( \int_0^t y^{-1} dG_{n,n}(y) \right) \mu_n \mu_n \int_0^\tau y^{-1} dB_{n,n}(G(y)). \]  
(2.14)

In this article we use the approximation defined in (2.14) for \( \alpha_{n,n} \) and investigate asymptotic normality behavior \( I_{n,n} \) in (2.9). The bootstrap is a widely used tool in statistics and, therefore, the properties of \( I_{n,n}(p) \) are of great interest in applied as well as in theoretical statistics.

Before stating our result, we list all assumptions used in this paper.

**Assumptions**

**C(1).** \( d\mu(t) = w(t) dt \), where \( w(t) \geq 0 \) and continuous on \([0, \tau]\), where \( T < \tau < \infty \) and \( \tau = \sup\{x, G(x) < 1\} \).

**K(1).** There is a finite interval such that \( K \) is continuous and bounded on it and vanishes outside of this interval.

**K(2).** \( \int_{\mathbb{R}} K^2(t) dt > 0 \).

**K(3).** \( K' \) exists and is bounded.

**K(4).** \( \int K(t) dt = 1 \).

**F(1).** \( \left| \frac{f(x)}{x^r} \right| \) and \( \left| \frac{f'(x)}{x^r} \right| \) is uniformly bounded (a.s.) on the \((0, \tau)\).

**G(1).** \( (G(x))^{\frac{1}{r}} x^{-2} \) is uniformly bounded (a.s.) on the \((0, \tau)\) for some \( r > 4 \).

Define \( \hat{m}(p) \) and \( \hat{\sigma}^2(p) \) to be the counterparts of \( m(p) \) and \( \sigma^2(p) \), after replacing \( f \) by \( f_n \) in (2.2) and (2.3) respectively, i.e.,
\[ \hat{m}(p) = E[N^p \left( \int_{\mathbb{R}} K^2(t) dt \right)^{\frac{p}{2}} \int_0^T f_n^{p+2}(t) dt], \]
\[ \hat{\sigma}^2(p) = \sigma^2_1 \int_0^T f_n^{p+2}(t) dt \left( \int_{\mathbb{R}} K^2(t) dt \right)^{p}. \]  
(2.15)

**Theorem 2.1.** Suppose that Assumptions **K(1)-K(4), C(1), F(1) and G(1)** hold. If, as \( n \to \infty \)
\[ h_n \to 0, \quad \frac{\log \log n}{n h_n^3} \to 0, \quad n^{-\frac{1}{2}} h_n^{-1} \to 0, \quad \frac{\log n}{n h_n^2} \to 0, \]
then for $\hat{m}(p)$ and $\hat{\sigma}(p)$ in (2.15), one has

$$(h_n\hat{\sigma}^2(p))^{-\frac{1}{2}}\left\{(nh_n)^{\frac{p}{2}}I_{n,n}(p) - \hat{m}(p)\right\} \xrightarrow{D} N(0,1), \quad 1 < p < \infty.$$  

**Proof.** At first, notice that with using the inequality

$$\left| |a(x)|^p - |b(x)|^p \right| \leq p2^{p-1}|a(x) - b(x)|^p + p2^{p-1}|b(x)|^{p-1}|a(x) - b(x)|,$$

(for $p \geq 1$), and (2.29), we may write

$$|v_n^{\frac{p}{2}} - v_\Gamma^{\frac{p}{2}}| \leq \frac{p}{2}2^{2p-1}|v_n - v|^{\frac{p}{2}} + \frac{p}{2}2^{2p-1}|v|^{\frac{p}{2}-1}|v_n - v|,$$

$$= O_p\left(\frac{\log \log n}{n}\right)^{\frac{1}{2}}. \quad (2.16)$$

Now, start by writing

$$(h_n\hat{\sigma}^2(p))^{-\frac{1}{2}}\left\{(nh_n)^{\frac{p}{2}}I_{n,n}(p) - \hat{m}(p)\right\} = (h_n\hat{\sigma}^2(p))^{-\frac{1}{2}}\left\{(nh_n)^{\frac{p}{2}}I_{n,n}(p) - m(p)\right\} + (h_n\hat{\sigma}^2(p))^{-\frac{1}{2}}(\hat{m}(p) - m(p))
\begin{equation}
:= Z_n + V_n. \quad (2.17)
\end{equation}$$

By Lemma 2.4

$$V_n = o(1) \quad \text{a.s.}$$

Let $\Gamma_{n,3}(t)$ be the term of defined in Lemma 2.3. Then

$$I_{n,n}(p) = \int_0^T \left|h_n^{-1} \int_\mathbb{R} K\left(\frac{t-x}{h_n}\right)d\left(F_{n,n}(x) - F_n(x)\right)\right|^p (tv_n)^{\frac{p}{2}} f_n(t) dt
\begin{equation}
= n^{-\frac{p}{2}}h_n^{-p} \int_0^T \left(\left| \int_\mathbb{R} K\left(\frac{t-x}{h_n}\right)d\alpha_{n,n}(x)\right|^p - |\Gamma_{n,3}(t)|^p\right) (tv_n)^{\frac{p}{2}} f_n(t) dt
\begin{equation}
+ n^{-\frac{p}{2}}h_n^{-p} \int_0^T |\Gamma_{n,3}(t)|^p (tv_n)^{\frac{p}{2}} f_n(t) dt
\begin{equation}
:= R_n + S_n. \quad (2.18)
\end{equation}$$

Using the inequality (2.26) we can write

$$n^{\frac{p}{2}}h_n^p R_n \leq p2^{p-1}\left\{ \int_0^T \left| \int_\mathbb{R} K\left(\frac{t-x}{h_n}\right)d\left(\alpha_{n,n}(x) - \Gamma_{n,n}(x)\right)\right|^p (tv_n)^{\frac{p}{2}} f_n(t) dt \right\}
\begin{equation}
+ p2^{p-1}\left( \int_0^T |\Gamma_{n,3}(t)|^p (tv_n)^{\frac{p}{2}} f_n(t) dt \right)^{\frac{p-1}{p}}
\begin{equation}
\times \left( \int_0^T \left| \int_\mathbb{R} K\left(\frac{t-x}{h_n}\right)d\left(\alpha_{n,n}(x) - \Gamma_{n,n}(x)\right)\right|^p (tv_n)^{\frac{p}{2}} f_n(t) dt \right)^{\frac{1}{p}}.
\begin{equation}
(2.19)
\end{equation}$$
Next, note that by Lemma 2.2 and the bounded variation assumption on $K$

\[ \left| \int_{\mathbb{R}} K\left(\frac{t-x}{h_n}\right)d\left(\alpha_{n,n}(x) - \Gamma_{n,n}(x)\right) \right| \leq \int_{\mathbb{R}} \left| \alpha_{n,n}(t-yh_n) - \Gamma_{n,n}(t-yh_n) \right| dK(y) \]

\[ \leq \sup_x |\alpha_{n,n}(x) - \Gamma_{n,n}(x)| \int_{\mathbb{R}} |dK(y)| = O_p(n^{-\frac{1}{p}}). \]

Therefore,

\[ \text{RHS of (2.19)} = O_p\left(n^{-\frac{p}{p}}\right) + O_p\left(n^{-\frac{1}{p}}\right) \left( \int_0^T |\Gamma_{n,3}(t)|^p (tv_n)^{\frac{p}{p}} f_n(t)dt \right)^{\frac{(p-1)}{p}}. \] (2.20)

On the other hand, by the proof of Lemma 2 of Fakoor and Zamini [6], one can write

\[ \Gamma_n^{(2)}(x) = P(x)\Gamma_n^{(1)}(x) + o_p(h_n), \]

where $\Gamma_n^{(2)}(x) = \int K\left(\frac{x-u}{h_n}\right)P(y)dW(y)$, with $P(x) = (\sigma'(x))^{1/2}$ and $\Gamma_n^{(1)}(x) = \int K\left(\frac{x-y}{h_n}\right)dW(y).$ But, $|\Gamma_n^{(1)}(x)| \overset{D}{=} |N|/n^{1/2} \left( \int K(u^2)du \right)^{1/2},$ hence by $F(1)$,

\[ \sup_{0<x<\tau} |\Gamma_n^{(2)}(x)| = o_p\left(h_n^{1/2}\right). \] (2.21)

and the proof of Lemma 2.3 conclude that

\[ \sup_{0<x<\tau} |\Gamma_{n,3}(t)| = O_p\left(h_n^{\frac{3}{p}}\right). \] (2.22)

Hence, by (2.51),(2.16) and (2.22)

\[ \left| \int_0^T |\Gamma_{n,3}(t)|^p (tv_n)^{\frac{p}{p}} f_n(t)dt - \int_0^T |\Gamma_{n,3}(t)|^p (tv)^{\frac{p}{p}} f(t)dt \right| \]

\[ \leq \int_0^T |\Gamma_{n,3}(t)|^p (tv_n)^{\frac{p}{p}} |f_n(t) - f(t)| dt 
+ \left| v_n^\frac{p}{p} - v^\frac{p}{p} \right| \int_0^T |\Gamma_{n,3}(t)|^p |t| \frac{p}{p} |f(t)| dt \]

\[ \leq (Tv_n)^{\frac{p}{p}} \sup_{x} |f_n(x) - f(x)| \int_0^T |\Gamma_{n,3}(t)|^p dt 
+ (Tv_n)^{\frac{p}{p}} \sup_{x} |f(x)| \left| v_n^\frac{p}{p} - v^\frac{p}{p} \right| \int_0^T |\Gamma_{n,3}(t)|^p dt \]

\[ = O_p\left((h_n)^{p/2}\right) \left( h_n^{-\frac{1}{p}} O_p\left(\log \log n \right) + O_p(h_n) \right). \]
This last result and Lemma 2.3 imply that

\[
(h_n^{p+1}\sigma^2(p))^{-\frac{1}{2}} \left\{ \int_0^T \left| \Gamma_{n,3}(t) \right|^p (tv_n)^{\frac{p}{2}} f_n(t)dt - h_n^{\frac{p}{2}} m(p) \right\} \xrightarrow{D} N(0, 1). \tag{2.23}
\]

Consequently

\[
\int_0^T \left| \Gamma_{n,3}(t) \right|^p (tv_n)^{\frac{p}{2}} f_n(t)dt = O_p(h_n^{\frac{p}{2}}). \tag{2.24}
\]

Putting together (2.19), (2.20) and (2.24), one finds

\[
R_n = n^{-\frac{p}{2}} h_n^{-p} O_p(n^{-\frac{p}{2}}) + n^{-\frac{p}{2}} h_n^{-p} O_p(n^{-\frac{1}{p}}) O_p(h_n^{\frac{p-1}{2}}). \tag{2.25}
\]

The term \( Z_n \) that appears in (2.17) can now be handled as follows

\[
Z_n = (h_n \hat{\sigma}^2(p))^{-\frac{1}{2}} \left\{ (nh_n)^{\frac{p}{2}} \left( R_n + S_n \right) - m(p) \right\} = (h_n \hat{\sigma}^2(p))^{-\frac{1}{2}} (nh_n)^{\frac{p}{2}} R_n + (h_n \hat{\sigma}^2(p))^{-\frac{1}{2}} \left\{ (nh_n)^{\frac{p}{2}} S_n - m(p) \right\} := Z_{n,1} + Z_{n,2}.
\]

The fact that \( \hat{\sigma}^2(p) / \sigma^2(p) = 1 + o(1) \) a.s., together with (2.23) imply

\[
Z_{n,2} = (h_n^{p+1} \hat{\sigma}^2(p))^{-\frac{1}{2}} \left\{ \int_0^T \left| \Gamma_{n,3}(t) \right|^p (tv_n)^{\frac{p}{2}} f_n(t)dt - h_n^{\frac{p}{2}} m(p) \right\} \xrightarrow{D} N(0, 1).
\]

As for \( Z_{n,1} \), the bound in (2.25) gives

\[
\left| Z_{n,1} \right| = h_n^{-\frac{p+1}{2}} n^{-\frac{p}{2}} o_p(1) + h_n^{-1} n^{-\frac{1}{p}} o_p(1) \leq h_n^{-p} n^{-\frac{p}{2}} o_p(1) + h_n^{-1} n^{-\frac{1}{p}} o_p(1) = o_p(1).
\]

This completes the proof of Theorem 2.1. \( \square \)

**Appendix**

In order to make the proof of the main result easier, some auxiliary results and notations are needed.
The following inequality will be useful later on. Let $1 \leq p < \infty$, then for functions $q$ and $u$ in $L_p$ we have
\[
\int_0^\infty \left[ |q(t)|^p - |u(t)|^p \right] d(\mu(t)) \leq p 2^{p-1} \int |q(t) - u(t)|^p d(\mu(t)) + p 2^{p-1} \left( \int_0^\infty |u(t)|^p d(\mu(t)) \right)^{1 - \frac{1}{p}} \times \left( \int_0^\infty |q(t) - u(t)|^p d(\mu(t)) \right)^{1/p} \tag{2.26}
\]

**Lemma 2.2.** Under Assumption G(1), one can write
\[
\sup_{0 < t < \tau} \left| \alpha_{n,n}(t) - \Gamma_{n,n}(t) \right| = O(n^{-1/r}) \quad a.s.
\]

**Proof.** Applying (2.13) and (2.14), one can obtain
\[
\sup_{0 < t < \tau} \left| \alpha_{n,n}(t) - \Gamma_{n,n}(t) \right| \leq 2v_n^{-1} \sup_{0 < t < \tau} \left| \frac{\beta_{n,n}(t) - B_{n,n}(G(t))}{t} \right| + 2v_n^{-1} \int_0^\tau y^{-2} \left| \beta_{n,n}(y) - B_{n,n}(G(y)) \right| dy
\]
\[
\leq 2v_n^{-1} \tau \sup_{0 < t < \tau} \left( \frac{G(t)^{1/r}}{t^2} \right) \sup_{0 < t < \tau} \left| \frac{\beta_{n,n}(t) - B_{n,n}(G(t))}{(G(t)(1 - G(t)))^{1/2 - 1/r}} \right| + 2v_n^{-1} \sup_{0 < t < \tau} \left| \frac{\beta_{n,n}(t) - B_{n,n}(G(t))}{(G(t)(1 - G(t)))^{1/2 - 1/r}} \right| \times \int_0^\tau y^{-2} (G(y))^{1/r} = O_p(n^{-1/r}) + O_p(n^{-1/r}) = O_p(n^{-1/r}),
\]
where last equality results from Assumption G(1) and Remark 2 of Csörgő and Mason [3]. \hfill \Box

**Lemma 2.3.** Let $\Gamma_{n,3}(t) = \int_\mathbb{R} K\left(\frac{t-x}{h_n}\right) d\Gamma_{n,n}(x)$. Suppose that Assumptions K(1)-K(3), C(1), G(1), F(1) and conditions
\[
h_n \to 0, \quad \frac{\log n}{nh_n^2} \to 0,
\]
hold, then we can write
\[
(h_n^{p+1}\sigma^2(p))^{-\frac{1}{2}} \left\{ \int_0^\tau |\Gamma_{n,3}(t)|^p (tv)^{1/2} f(t) dt - h_n^2 m(p) \right\} \xrightarrow{D} N(0, 1).
\]
Proof. At first, note that
\[ \Gamma_{n,3}(t) = \frac{1}{v_n} \int_{\mathbb{R}} K\left(\frac{t-x}{h_n}\right) x^{-1} dB_{n,n}(G(x)) \]
\[ - \left( \int_{\tau}^{y} y^{-1} dB_{n,n}(G(y)) \right) \int_{\mathbb{R}} K\left(\frac{t-x}{h_n}\right) x^{-1} dG_{n,n}(x) \]
\[ := B_{n}^{(1)}(t) + B_{n}^{(2)}(t). \tag{2.27} \]
The fact that \( \{B_{n,n}(z), 0 \leq z \leq 1\} \overset{D}{=} \{W(z) - zW(1), 0 \leq z \leq 1\} \) for all \( n \geq 1 \), implies that
\[ B_{n}^{(1)}(t) \overset{D}{=} \frac{1}{v_n} \int_{\mathbb{R}} K\left(\frac{t-x}{h_n}\right) x^{-1} d\left( W(G(x)) - G(x)W(1) \right) \]
\[ := B_{n}^{(1)}(t). \tag{2.28} \]
Also, with using Theorem of James \[10\] and \( G(1) \), one can obtain, for any \( 0 < \delta < -\frac{1}{2} < -\frac{1}{r} \)
\[ |v - v_n| = \left| \int_{0}^{\tau} y^{-1} d\left( G_n(y) - G(y) \right) \right| \]
\[ = n^{-\frac{1}{2}} \left| \int_{0}^{\tau} y^{-2} \beta_n(y) dy \right| \]
\[ \leq n^{-\frac{1}{2}} \sup_{0 < y < \tau} (G(y))^{\frac{\delta - \frac{1}{2}}{2}} |\beta_n(y)| \left| \int_{0}^{\tau} y^{-2}(G(y))^{\frac{1}{2} - \delta} dy \right| \]
\[ = O\left( \left( \frac{\log \log n}{n} \right)^{\frac{1}{2}} \right) \text{ a.s.} \tag{2.29} \]
(2.26), (2.29) and Lemma 2 of Fakoor and Zamin\[6\], imply that
\[ (h_{n}^{p+1} \sigma^2(p))^{-\frac{1}{2}} \left\{ \int_{0}^{T} \left| \frac{v}{v_n} \Gamma_{n}^{(2)}(t) \right|^{p} (tv)^{\frac{\delta}{2}} f(t) dt - h_{n}^{\frac{\delta}{2}} m(p) \right\} \overset{D}{\rightarrow} N(0,1), \]
where \( \Gamma_{n}^{(2)}(x) \) is introduced in Lemma 2 of Fakoor and Zamin\[6\].
Now, since
\[ H_{n}^{(1)}(t) := \frac{1}{v_n} \int_{\mathbb{R}} K\left(\frac{t-x}{h_n}\right) x^{-1} d\left( W(G(x)) \right) \overset{D}{=} \frac{v}{v_n} \Gamma_{n}^{(2)}(t), \]
one can write
\[ (h_{n}^{p+1} \sigma^2(p))^{-\frac{1}{2}} \left\{ \int_{0}^{T} \left| H_{n}^{(1)}(t) \right|^{p} (tv)^{\frac{\delta}{2}} f(t) dt - h_{n}^{\frac{\delta}{2}} m(p) \right\} \overset{D}{\rightarrow} N(0,1). \tag{2.30} \]
Also
\[ H_{n}^{(2)}(t) := -W(1)\frac{vh_{n}}{v_n} \int_{\mathbb{R}} K(u)f(t-uh_n) du, \]
A bootstrap approximation to $L_p$-statistic in length-biased $L_p$ is normally distributed with mean 0 and variance

$$\sum_{n,t} = \frac{v^2}{v_n^2} h_n^2 \left( \int_R K(u)f(t - uh_n)du \right)^2.$$

Therefore, for each $t$,

$$\left| H_n^{(2)}(t) \right| \stackrel{D}{=} |N| \left| \frac{v}{v_n} h_n \right| \int_R K(u)f(t - uh_n)du.$$

Since

$$\frac{v}{v_n} h_n \left| \int_R K(u)f(t - uh_n)du \right| \leq \frac{v}{v_n} h_n \sup_{0 < x < \tau} f(x) \int_{-1}^{+1} |K(u)|du = O_p(h_n),$$

hence

$$H_n^{(2)}(t) = O_p(h_n). \quad (2.31)$$

Now, by (2.26), (2.30) and (2.31), one can see

$$\left| \int_0^T B_n^{(1)}(t)^p (tv)^{\frac{p}{2}} f(t)dt \right| - \left| \int_0^T H_n^{(1)}(t)^p (tv)^{\frac{p}{2}} f(t)dt \right|$$

$$\leq p^{2p-1} \int_0^T \left| H_n^{(2)}(t)^p (tv)^{\frac{p}{2}} f(t)dt \right|$$

$$+ p^{2p-1} \left( \int_0^T \left| H_n^{(1)}(t)^p (tv)^{\frac{p}{2}} f(t)dt \right| \right)^{1-\frac{1}{p}}$$

$$\times \left( \int_0^T \left| H_n^{(2)}(t)^p (tv)^{\frac{p}{2}} f(t)dt \right|^p \right)^{\frac{1}{p}}$$

$$= O_p \left( h_n^{\frac{p+1}{2}} \right).$$

Last result with together (2.30) and (2.28) conclude that

$$\left( h_n^{p+1} \sigma^2(p) \right)^{-\frac{1}{2}} \left\{ \int_0^T B_n^{(1)}(t)^p (tv)^{\frac{p}{2}} f(t)dt - h_n^{\frac{p}{2}} m(p) \right\} \xrightarrow{D} N(0, 1). \quad (2.32)$$

The term $B_n^{(2)}(t)$ can be handled as follows.

Since $\{ B_{n,n}(z), 0 \leq z \leq 1 \} \stackrel{D}{=} \{ W(z) - zW(1), 0 \leq z \leq 1 \}$ for all $n \geq 1$, hence

$$\left| B_n^{(2)}(t) \right| \stackrel{D}{=} \left| \frac{1}{v_n} \int_0^T \left( 1 \right) y^{-1} d \left[ W(G(y)) - G(y)W(1) \right] \right| = \frac{1}{v_n} \int_0^T \left| K(\frac{t}{h_n}) x^{-1} dG_n(x) \right|. \quad (2.33)$$
But

\[
\left| \frac{1}{v_n} \int_0^\tau y^{-1}d\left(W(G(y)) - G(y)W(1)\right) \right| \leq \left( \frac{1}{v_n} \int_0^\tau y^{-1}d\left(W(G(y))\right) \right) + \frac{1}{v_n} \left| W(1) \int_0^\tau y^{-1}g(y)dy \right| \\
\times \left| \frac{1}{v_{nn}} \int_\mathbb{R} K\left(\frac{t - x}{h_n}\right)x^{-1}dG_{n,n}(x) \right| \\
:= \frac{1}{v_n} \left( A_1 + A_2 \right) \times A_3. \quad (2.34)
\]

Now, since

\[
|A_1| \overset{D}{=} |N| \left( \int_0^\tau y^{-2}g(y)dy \right)^{\frac{1}{2}},
\]

therefore,

\[
\frac{1}{v_n} A_1 = O_p(1). \quad (2.35)
\]

Also

\[
|A_2| \overset{D}{=} |N| \left( \int_0^\tau y^{-1}g(y)dy \right),
\]

hence

\[
\frac{1}{v_n} A_2 = O_p(1). \quad (2.36)
\]
Next, let \( \{B_{n,n}(x); 0 \leq x \leq 1\} \) be the sequence of Brownian bridges in (2.10). Now, one can write

\[
A_3 = \frac{1}{\nu_{n,n}} \left| \int_{\mathbb{R}} x^{-1} K\left(\frac{t-x}{h_n}\right) dG_{n,n}(x) \right|
\]

\[
\leq \tau \left| \int_{\mathbb{R}} x^{-1} K\left(\frac{t-x}{h_n}\right) dG_{n,n}(x) \right|
\]

\[
\leq n^{-1/2} \left| \int_{\mathbb{R}} x^{-1} K\left(\frac{t-x}{h_n}\right) d\left(\beta_{n,n}(x) - B_{n,n}(G(x))\right) \right|
\]

\[
+ n^{-1/2} \left| \int_{\mathbb{R}} x^{-1} K\left(\frac{t-x}{h_n}\right) d\left(B_{n,n}(G(x)) \right) \right|
\]

\[
+ n^{-1/2} \left| \int_{\mathbb{R}} x^{-1} K\left(\frac{t-x}{h_n}\right) d\left(\beta_{n}(x)\right) \right|
\]

\[
+ \left| \int_{\mathbb{R}} x^{-1} K\left(\frac{t-x}{h_n}\right) d\left(G(x) \right) \right|
\]

\[
:= L^{(1)}_n(t) + L^{(2)}_n(t) + L^{(3)}_n(t) + L^{(4)}_n(t).
\]  

(2.10) implies that

\[
L^{(1)}_n(t) = n^{-\frac{1}{2}} \int_{\mathbb{R}} x^{-1} K\left(\frac{t-x}{h_n}\right) d\left(\beta_{n,n}(x) - B_{n,n}(G(x))\right)
\]

\[
= n^{-\frac{1}{2}} \int_{\mathbb{R}} \left(\beta_{n,n}(t-uh_n) - B_{n,n}(G(t-uh_n))\right) d\left(\frac{K(u)}{t-uh_n}\right)
\]

\[
\leq n^{-\frac{1}{2}} \sup_{x>0} \left| \beta_{n,n}(x) - B_{n,n}(G(x)) \right| \int_{\mathbb{R}} d\left(\frac{K(u)}{t-uh_n}\right)
\]

\[
= O\left(n^{-1} \log n\right) \left( \int_{-1}^{+1} \frac{K'(u)}{t-uh_n} + \int_{-1}^{+1} \frac{h_n K(u)}{(t-uh_n)^2} \right)
\]

\[
\leq \max \left( \sup_x |K(x)|, \sup_x |K'(x)| \right) O\left(n^{-1} \log n\right)
\]

\[
\times \left( \frac{1}{h_n} \log \frac{t+h_n}{t-h_n} + \frac{2h_n}{t^2-h_n^2} \right)
\]

\[
= O\left(\frac{\log n}{nh_n}\right)(O(1) + O(1))
\]

\[
= O\left(\frac{\log n}{nh_n}\right).
\]  

(2.38)
To deal with $L^2_n(t)$, observe that

\[
L^2_n(t) \overset{D}{=} n^{-\frac{1}{2}} \left| \int_{\mathbb{R}} x^{-1} K\left(\frac{t-x}{h_n}\right) d\left( W(G(x)) - G(x)W(1) \right) \right|
\]

\[
\leq n^{-\frac{1}{2}} \left| \int_{\mathbb{R}} x^{-1} K\left(\frac{t-x}{h_n}\right) dW(G(x)) \right|
\]

\[
+ n^{-\frac{1}{2}} \left| \int_{\mathbb{R}} x^{-1} K\left(\frac{t-x}{h_n}\right) W(1) g(x) dx \right|
\]

\[
: = A_1'(t) + A_2'(t). \tag{2.39}
\]

Using the facts that

\[
\left| h_n^{-\frac{1}{2}} n^\frac{1}{2} A_1'(t) \right| \overset{D}{=} h_n^{-\frac{1}{2}} \left| \int_{\mathbb{R}} x^{-1} K\left(\frac{t-x}{h_n}\right) g^\frac{1}{2}(x) dW(x) \right|
\]

\[
\overset{D}{=} h_n^{-\frac{1}{2}} \left| N \left( \int_{\mathbb{R}} x^{-2} K^2\left(\frac{t-x}{h_n}\right) g(x) dx \right)^\frac{1}{2} \right|
\]

and

\[
h_n^{-\frac{1}{2}} \left( \int_{\mathbb{R}} x^{-2} K^2\left(\frac{t-x}{h_n}\right) g(x) dx \right)^\frac{1}{2} = \left( \int_{\mathbb{R}} v f(t - uh_n) K^2(u) du \right)^\frac{1}{2}
\]

\[
\longrightarrow \left( \frac{f(t)}{t} \right)^\frac{1}{2} \left( v \int_{-1}^{+1} K^2(u) du \right)^\frac{1}{2},
\]

it is not difficult to show that

\[
A_1'(t) = O_p\left( n^{-\frac{1}{2}} h_n^\frac{1}{2} \right) \quad \text{a.s.} \tag{2.40}
\]

Also

\[
\left| h_n^{-1} n^\frac{1}{2} A_2'(t) \right| \overset{D}{=} \left| N \left| v \int_{\mathbb{R}} K(u) f(t - uh_n) du \right| \right|
\]

\[
\longrightarrow \left| N \right| v f(t) \int_{\mathbb{R}} K(u) du
\]

\[
= O_p(1),
\]

therefore, it follows that

\[
A_2'(t) = O_p\left( h_n n^{-\frac{1}{2}} \right). \tag{2.41}
\]

(2.39), (2.40) and (2.41) imply that

\[
L_n^{(2)}(t) = O_p\left( n^{-\frac{1}{2}} h_n^\frac{1}{2} \right). \tag{2.42}
\]
A Bootstrap Approximation to $L_p$-Statistic in Length-Biased

From Komlós et al. [12], there exists a sequence of Brownian bridges
\$\{B_n(t), 0 \leq t \leq 1\}\$ such that
\[
\sup_{-\infty < x < \infty} \left| \beta_n(x) - B_n(G(x)) \right| = O_p\left( n^{-\frac{1}{2}} \log n \right). \tag{2.43}
\]
Consequently,
\[
L_n^{(3)}(t) = n^{-\frac{1}{2}} \int_{\mathbb{R}} x^{-1} K\left( \frac{t - x}{h_n} \right) d\beta_n(x)
\leq n^{-\frac{1}{2}} \int_{\mathbb{R}} x^{-1} K\left( \frac{t - x}{h_n} \right) d\left( \beta_n(x) - B_n(G(x)) \right)
+ n^{-\frac{1}{2}} \int_{\mathbb{R}} x^{-1} K\left( \frac{t - x}{h_n} \right) d\left( B_n(G(x)) \right). \tag{2.44}
\]
With using (2.43) and similar to the term $L_n^{(1)}(t)$, one gets
\[
\left| n^{-\frac{1}{2}} \int_{\mathbb{R}} x^{-1} K\left( \frac{t - x}{h_n} \right) d\left( \beta_n(x) - B_n(G(x)) \right) \right|
= O_p\left( \frac{\log n}{nh_n} \right). \tag{2.45}
\]
Similar to $L_n^{(2)}(t)$
\[
\left| n^{-\frac{1}{2}} \int_{\mathbb{R}} x^{-1} K\left( \frac{t - x}{h_n} \right) d\left( B_n(G(x)) \right) \right|
= O_p\left( n^{-\frac{1}{2}} h_n \right). \tag{2.46}
\]
(2.44)-(2.46) conclude that
\[
L_n^{(3)}(t) = O_p\left( \frac{\log n}{nh_n} \right) + O_p\left( n^{-\frac{1}{2}} h_n \right). \tag{2.47}
\]
To deal with the term of $L_n^{(4)}(t)$, observe that
\[
h_n^{-1} L_n^{(4)}(t) = h_n^{-1} \left| \int_{\mathbb{R}} x^{-1} K\left( \frac{t - x}{h_n} \right) g(x) \, dx \right|
= v \left| \int_{\mathbb{R}} K(u) f(t - uh_n) \, du \right|
\leq v \sup_{0 < x < \tau} f(x) \int_{-1}^{+1} K(u) \, du = O_p(1).
\]
Hence
\[
L_n^{(4)}(t) = O_p(h_n). \tag{2.48}
\]
Putting together (2.33)-(2.38), (2.42) and (2.47)-(2.48) one concludes that

\[ B_n^{(2)}(t) = O_p(h_n). \]  

(2.49) implies that

\[ \int_0^T |B_n^{(2)}(t)|^{p} (tv)^{\frac{p}{2}} f(t) dt = O_p(h_n^p). \]  

(2.26), (2.27), (2.32) and (2.50) follow

\[
\begin{align*}
& \left| \int_0^T |\Gamma_{n,3}(t)|^p (tv)^{\frac{p}{2}} f(t) dt - \int_0^T |B_n^{(1)}(t)|^p (tv)^{\frac{p}{2}} f(t) dt \right| \\
& \leq p^{2p-1} \int_0^T |B_n^{(2)}(t)|^p (tv)^{\frac{p}{2}} f(t) dt \\
& + p^{2p-1} \left( \int_0^T |B_n^{(1)}(t)|^p (tv)^{\frac{p}{2}} f(t) dt \right)^{1-\frac{1}{p}} \left( \int_0^T |B_n^{(2)}(t)|^p (tv)^{\frac{p}{2}} f(t) dt \right)^{\frac{1}{p}} \\
& = O_p(h_n^p) \\
& + O_p(h_n^{-\frac{1}{p}}) O_p(h_n).
\end{align*}
\]

This last result and (2.32) follow

\[
(h_n^{p+1} \sigma^2(p))^{-\frac{1}{2}} \left\{ \int_0^T |\Gamma_{n,3}(t)|^p (tv)^{\frac{p}{2}} f(t) dt - h_n^{\frac{p}{2}} m(p) \right\} \xrightarrow{D} N(0, 1).
\]

Lemma 2.4. Suppose \( f'(x) \) exists and is bounded (a.s.) on the (0, \( \tau \)). Then under \( K(1), K(3), K(4), G(1) \) and conditions

\[ h_n \to 0, \quad \frac{\log \log n}{nh_n^3} \to 0, \]

one can write

\[ h_n^{-\frac{1}{2}} |\hat{m}(p) - m(p)| \to 0 \ a.s. \quad \text{and} \quad \hat{\sigma}^2(p) \to \sigma^2(p) \ a.s. \]
Proof. By (2.26), one can see

\[ |\hat{m}(p) - m(p)| \leq C(K) \int_0^T \left| f_n^{\frac{p+2}{2}}(x) - f^{\frac{p+2}{2}}(x) \right| dx \]

\[ \leq C(K)(\frac{p+2}{2})2^{(\frac{p+2}{2})-1} \int_0^T \left| f_n(x) - f(x) \right|^{\frac{p+2}{2}} dx \]

\[ + C(K)(\frac{p+2}{2})2^{(\frac{p+2}{2})-1} \left( \int_0^T f^{\frac{(p+2)}{2}}(x) dx \right)^{1-\frac{2}{(p+2)}} \]

\[ \times \left( \int_0^T \left| f_n(x) - f(x) \right|^{\frac{(p+2)}{2}}(x) dx \right)^{\frac{2}{(p+2)}} , \]

where \( C(K) \) is a constant with respect to \( K \).

Now, in order to show \( h_n^{-\frac{1}{2}}|\hat{m}(p) - m(p)| \to 0 \) \( a.s. \), it is sufficient to show that

\[ h_n^{-\frac{1}{2}} \left( \int_0^T \left| f_n(x) - f(x) \right|^{\frac{(p+2)}{2}}(x) dx \right)^{\frac{2}{(p+2)}} \to 0 \] \( a.s. \).

But by Lemma 5 of Fakoor and Zamini [6], and Assumptions \( K(1), K(3), K(4), \) and \( G(1) \), one can get

\[ \sup_x \left| f_n(x) - f(x) \right| \leq \sup_x \left| h_n^{-1} \int_R K\left( \frac{x - y}{h_n} \right) dF_n(y) - h_n^{-1} \int_R K\left( \frac{x - y}{h_n} \right) dF(y) \right| \]

\[ + \sup_x \left| \int_R \left( f(x - h_nt) - f(x) \right) K(t) dt \right| \]

\[ \leq h_n^{-1} \sup_{0<x<r} \left| F_n(x) - F(x) \right| \int_R \left| dK(t) \right| \]

\[ + h_n \sup_x \left| f'(x) \right| \int_{-1}^{+1} \left| t \right| K(t) dt \]

\[ = h_n^{-1} O\left( \left( \log \log n \frac{3}{n} \right) \right) + O(h_n) \] \( a.s. \) \hspace{1cm} (2.51)

Clearly by (2.51), one has

\[ h_n^{-\frac{1}{2}} \left( \int_0^T \left| f_n(x) - f(x) \right|^{\frac{(p+2)}{2}} dx \right)^{\frac{2}{(p+2)}} = O\left( \left( \log \log n \frac{1}{n} \right) \right) \]

\[ + O\left( \frac{1}{h_n^2} \right) \] \( a.s. \)

Conditions \( h_n \to 0 \) and \( \frac{\log \log n}{nh_n^3} \to 0 \), conclude that

\[ h_n^{-\frac{1}{2}} |\hat{m}(p) - m(p)| \to 0 \] \( a.s. \).
The proof of $\hat{\sigma}^2(p) \rightarrow \sigma^2(p)$ \ a.s. is similar and will not be given. □

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