Penrose limits and non-relativistic geometries

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Abstract
For the AdS/CFT duality, considerations of the plane wave metric, which is obtained as the Penrose limit of $AdS_5 \times S^5$ have proved to be quite useful and interesting. In this work, we obtain Penrose limit metrics for Lifshitz, Schrödinger, hyperscaling violating Lifshitz and hyperscaling violating Schrödinger geometries. These geometries usually contain singularities for a certain range of parameters and we discuss how these singularities appear in the Penrose limit metric. For some cases, there are non-singular metrics possible for certain parameter values and the metric can be extended beyond the coordinate singularity, as discussed in many previous works. Corresponding Penrose limit metrics also display similar features. For the hyperscaling violating Schrödinger metric, we obtain metric extension for some cases.

Keywords: holography, Penrose limit, AdS/CMT

1. Introduction

Holographic duality has been one of the most fruitful recent ideas in theoretical physics. Dual gravitational description of certain strongly coupled field theories gives a new tool to tackle the otherwise difficult strong coupling regime. Strong coupling questions can sometimes be translated to weakly coupled gravitational problems in bulk. In the last few years, there have been rapid advances in applying it to cases far removed from the original (and still the best understood) example of $AdS_5/CFT_4$. In particular, after the pioneering work of [1–3] there has been a lot of work on gravitational duals of non-relativistic systems. Using gauge/gravity duality one expects to be able to probe strong coupling limit of these field theories. Lifshitz and Schrödinger geometries represent gravitational duals of systems with anisotropic scaling $(t, x^i) \rightarrow (\lambda t, \lambda x^i)$. In the case of $AdS/CFT$, it proved fruitful to consider the Penrose limit of
both sides of the correspondence. Since string theory on pp-wave spacetimes can be quantized, this allowed for a more thorough check of duality [4]. Based on these considerations, we obtain the Penrose limit of Lifshitz and Schrödinger type geometries. The field theory duals of these systems are not that well understood so we will have much less to say about the other side of duality. If non-relativistic duality is correct then string theory on the plane wave metrics that we obtain should be dual to some sector of states on the field theory side. Lifshitz metrics are known to have pp-curvature singularities (infinite tidal force for freely falling observers) [3, 5, 6]. Schrödinger geometries do not suffer from such singularities (for $z \geq 2$) and for such cases, smooth extensions were constructed in [7]. In the plane wave limits that we find, we will see these singularities in the Lifshitz case and their absence in the Schrödinger case. In addition to Lifshitz and Schrödinger spacetimes, we also consider Penrose limits of the hyperscaling violating geometries considered in [9, 10]. Among the class of hyperscaling violating geometries, some have smooth extensions while others have pp-curvature singularities [11]. The Penrose limit of these geometries also exhibits these features. In plane wave metrics obtained after taking the Penrose limit, all the information is contained in the profile function and singularities of the metric occur where the profile function is singular.

The Penrose limit has been discussed extensively since the original work of Penrose [12]. In [13] and many subsequent papers [14, 15], the process of obtaining the Penrose limit has been spelt out in great detail. We will use the method given in these works to obtain the Penrose limit.

Even though geometries like Lifshitz and Schrödinger give dual description of non-relativistic field theories and the Penrose limit is associated with observers in a highly boosted frame, we find that the limiting metrics we get are not pathological. Hence, as easily tractable limits, the plane wave spacetimes that we obtain are of particular interest.

This paper is organized as follows. In the next section, we briefly review some of the metrics dual to non-relativistic systems which have been discussed in recent years. In section 3, we obtain the Penrose limit of the Lifshitz metric. We derive the Penrose limit metric by two different methods. The traditional one is based on first converting the metric to Penrose adapted coordinates and then taking the Penrose limit. The metric one is in Rosen coordinates which may contain spurious coordinate singularities. It is usually better to work with Brinkmann coordinates in which the plane wave metric is characterised by a single symmetric matrix-valued function $A_{\mu\nu}(u)$. We perform the coordinate transformations to obtain the Penrose limit Lifshitz metric in Brinkmann form. In [14], a covariant characterization of the Penrose limit was given which directly gives us the Penrose limit metric in Brinkmann coordinates and hence wave profile $A_{\mu\nu}(u)$. In section 3.2, we derive the Penrose limit metric for the Lifshitz case using this covariant method and see that the wave profile matches that obtained by the traditional method. We also note that the Penrose limit of the Lifshitz metric gives the plane wave metric which is identical to that obtained in [6] by Horowitz and Way after taking the near singularity limit of the Lifshitz metric (in their paper, it is not presented as Penrose limit metric). In section 4, we find the Penrose limit of the Schrödinger metric and from the profile function $A_{\mu\nu}$ it is clear that the $z < 2$ case is singular while the $z \geq 2$ case is regular. In section 5, we find the Penrose limit of hyperscaling violating Lifshitz metrics. Here there are two parameters, the dynamical exponent and the hyperscaling violating parameter and hence the structure of the Penrose limit metric is more interesting. We can consider various limits and identify ranges where singularities occur. In section 6, we consider hyperscaling violating Schrödinger spacetimes. As in the previous case, there are two parameters and hence one has to consider various ranges for which singularities occur. In section 6.1, we derive the constraints from the null energy condition on dynamical exponent $z$ and hyperscaling violating parameter $k$. In section 6.2, we derive the
Penrose limit for this case and consider possible singularities. For non-singular cases, metric extensions for Schrödinger and hyperscaling violating Lifshitz metrics have been constructed in [7] and [11] respectively. In section 6.3, we construct a smooth coordinate system for the case when \( z \) and \( k \) are both equal to 2. In section 7, we conclude and discuss future directions. In appendix, we review some results about Penrose limits, drawing heavily on [16].

2. Non-relativistic field theories and their dual metrics

Many non-relativistic conformal field theories (CFT) have been found to be useful in condensed matter physics (e.g. [8]) and many other areas of physics. In general these systems have non-trivial scaling properties rather than just scale invariance. Dual gravitational descriptions of these have been explored actively in the past few years and the simplest of these are Lifshitz and Schrödinger metrics

\[
dx^2_{\text{Lif}} = \left( -r^{2z} dr^2 + \frac{dr^2}{r^2} + r^2 dx_i dx^i \right)
\]  

\[
dx^2_{\text{Sch}} = -r^{2z} dr^2 + r^2 \left( -2 dr d\zeta + dx_i dx^i \right) + \frac{dr^2}{r^2}.
\]

Here \( z \) is called the dynamical critical exponent and \( i = 1, \ldots, d \). Case \( z = 1 \) corresponds to the AdS metric in Poincaré coordinates with AdS radius unity. For condensed matter applications, in addition to these, hyperscaling violating Lifshitz and Schrödinger metrics have also been considered. Without hyperscaling violation, the entropy density of dual field theory scales as \( S \sim T^d/z \) where \( T \) is temperature and \( d \) is the number of spatial dimensions. Hyperscaling is violated in many condensed matter systems. Gravitational duals of these reflect this altered scaling. For the Lifshitz case

\[
dx^2 = \frac{1}{r^{2\theta/d}} \left( -r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 dx_i dx^i \right).
\]

Non-zero \( \theta \) parametrizes the deviation from naive scaling for entropy in field theory. Similarly, for the Schrödinger case, we have

\[
dx^2 = r^{-k} \left( -r^{2z} dt^2 + r^2 \left( -2 dr d\zeta + dx_i dx^i \right) + \frac{dr^2}{r^2} \right).
\]

Here \( k \) plays the same role as \( 2\theta/d \) in Lifshitz case above. These metrics are not scale invariant but transform with a constant conformal factor under scale transformation. For example, in the Lifshitz case, under a scaling \((t, x^i, r) \rightarrow (\lambda t, \lambda x^i, r/\lambda)\), the metric scales as \( ds^2 \rightarrow \lambda^{2\theta/d} ds^2 \). Roughly speaking, such theories with hyperscaling violation have the thermodynamic behaviour of a theory with scaling \( z \) but living in \( d - \theta \) dimensions as discussed in [10].

3. Penrose limit of the Lifshitz metric

In this section, we construct the Penrose limit of the Lifshitz metric. As is known [3, 6], the Lifshitz metric has a null singularity at \( r = 0 \). We will see that the corresponding Penrose limit reflects this. As an illustration, we will calculate the Penrose limit using two different methods: one using Penrose adapted coordinates and the second using a covariant method as
presented in [14]. The first method gives the plane wave metric in Rosen coordinates which we convert to Brinkmann form, while the covariant method gives the metric directly in Brinkmann form. For completeness, some details about the construction of the Penrose limit metric are given in appendix.

3.1. Penrose limit via adapted coordinates

We start with Lifshitz metric

\[
\text{ds}^2 = -r^2 \text{d}t^2 + \frac{\text{dr}^2}{r^2} + r^2 \text{d}x_i \text{d}x_i. \tag{3.1}
\]

In this method for obtaining the Penrose limit, as reviewed in appendix, we first transform the metric to Penrose adapted coordinates which are coordinates adapted to a chosen null geodesic, say \( \gamma \)

\[
\text{ds}^2 = 2dUdV + a(U, V, Y^k)dV^2 + 2b(U, V, Y^k)dVdY^k + g_{ij}(U, V, Y^i)dY^i dY^j. \tag{3.2}
\]

Null geodesics are given by

\[-r^2\dot{t}^2 + \frac{\dot{r}^2}{r^2} + r^2 \dot{x}_i \dot{x}_i = 0. \tag{3.3}
\]

Here the derivative is with respect to affine parameter \( \tau \). This Lagrangian gives \( i = -\frac{E}{r^2} \) and \( \dot{x}_i = \frac{p_i}{r^2} \). Imposing nullity yields

\[-\frac{E^2}{r^2} + \frac{\dot{r}^2}{r^2} + \frac{p^2}{r^2} = 0 \tag{3.4}
\]

where \( p^2 = \sum p_i p_i \). The radial coordinate thus satisfies

\[
\left( \frac{\text{d}r}{\text{d}\tau} \right)^2 = r^2 \left( \frac{E^2}{r^2} - \frac{p^2}{r^2} \right). \tag{3.5}
\]

For simplicity, we take only one of \( p_i \), say \( p_1 \) to be non-zero. To go to adapted coordinates, we choose \( \tau \) as new coordinate \( U \). As discussed in [16], we take the solution of the Hamilton–Jacobi (HJ) equation \( g^{\mu\nu} \partial_\mu S \partial_\nu S = 0 \) as another null coordinate \( V \), i.e. \( S = V \). Based on isometries, we substitute the ansatz \( S = -Et + p_1 x_1 + \rho(r) \). Putting this in the HJ equation, we get

\[
g^{\mu\nu} \left( \frac{\partial S}{\partial r} \right)^2 + g^{\nu\nu} \left( \frac{\partial S}{\partial x_i} \right) \left( \frac{\partial S}{\partial x_j} \right) + g^{rr} \rho'^2 = 0. \tag{3.6}
\]

This gives, using the null condition,

\[
\rho' = \frac{\text{d}\rho}{\text{d}r} = \frac{1}{r} \frac{\text{d}r}{\text{d}\tau}, \quad \frac{\text{d}\rho}{\text{d}r} = \frac{1}{r^2} \left( \frac{\text{dr}}{\text{d}\tau} \right)^2. \tag{3.7}
\]

Since \( x_1 = X + \int \rho \text{d}U \), we take integration constant \( X \) as the new coordinate. Using

\[
\text{d}r = \dot{r}(U) \text{d}U, \quad \text{d}x_1 = \text{d}X + \frac{p_1 \text{d}U}{r^2} \tag{3.8}
\]
\[ dV = dS = -Edt + p_t \left( dX + \frac{p_t dt}{r^2} \right) + dU \left( \frac{E^2}{r^{2z}} - \frac{p_t^2}{r^2} \right). \] (3.9)

Putting these in the Lifshitz metric, we get
\[ ds^2 = 2dUdV + 2\frac{p_t}{E^2} r^{2z}dVdX - \frac{r^{2z}}{E^2} dV^2 + r^2 \sum_{i=2}^d dx_i^2 + dX^2 \left( r^2 - r^{2z}\frac{p_t^2}{E^2} \right). \] (3.10)

Here \( r = r(U) \). Taking the Penrose limit, as described in the appendix, we get
\[ ds^2 = 2dUdV + \left[ \frac{r^2 - \frac{p_t^2}{E^2}}{r^{2z}} \right] dX^2 + r^2 \sum_{i=2}^d dx_i^2. \] (3.11)

Converting from Rosen to Brinkmann coordinates using A.8, we have

\[ A_{11} = \frac{\dot{r}}{r} = \frac{1}{r^2} \frac{d^2}{dU^2} \left( \frac{r}{r^2} \right) = \frac{z(1-z)p_t^2}{r^2} + \frac{(1-z)E^2}{r^{2z}} \] (3.12)

\[ A_{ii} = \frac{\dot{x}_i}{x_i} = \dot{x}_i = \frac{(1-z)E^2}{r^{2z}}. \] (3.13)

We see that for AdS (i.e. \( z = 1 \)) the Penrose limit metric is flat space as expected. In the next section we will derive these equations directly using the covariant method.

### 3.2. Penrose limit using the covariant method

In this section, we use the covariant characterization of the Penrose limit as given in [14]. In this approach, we directly arrive at the Penrose limit metric in Brinkmann coordinates and the plane wave is described by a wave profile \( A_{ij} \). Again, we start with the Lifshitz metric
\[ ds^2 = \left( -r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2 dx^2 \right) \] (3.14)

where \( \vec{x} \) represents a vector in the \( d \) dimensional flat subspace \( (dx_1^2 + \ldots + dx_d^2) \). So the total spacetime dimension is \( D = d + 2 \). The geodesic Lagrangian in this spacetime is given by
\[ L = \frac{1}{2} \left( -r^{2z}\dot{t}^2 + \frac{\dot{r}^2}{r^2} + r^2 \dot{x}^2 \right). \] (3.15)

Here, the derivative is with respect to affine parameter \( U \). We can choose the transverse coordinates \((\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_d)\) such that \( \hat{x}_1 = \hat{x}_2 = \cdots = \hat{x}_d = 0 \). The Lagrangian then becomes
\[ L = \frac{1}{2} \left( -r^{2z}\dot{t}^2 + \frac{\dot{r}^2}{r^2} + r^2 \dot{\hat{x}}^2 \right). \] (3.16)
The Euler–Lagrange equations give
\[ i = \frac{E}{r^2}, \quad \dot{x}_i = \frac{P}{r^2} \]  
where \( E \) is a constant and \( P \) is a constant (same as \( p_1 \)). Using the condition for the null geodesic (\( \mathcal{L} = 0 \)), we will get the equation for \( r \).
\[ \dot{r}^2 = \left( \frac{E^2}{r^2(z-1)} - P^2 \right). \]  
(3.18)

Now we need to find the Hamilton–Jacobi function, which we calculated in the previous section
\[ S = -Et + Px_1 + \rho(r). \]  
(3.19)

Using the earlier result
\[ \rho(r) = \int \frac{1}{r^2} \dot{r}^2 dU = \int \left( \frac{E^2}{r^2} - \frac{P^2}{r^2} \right) dU. \]  
(3.20)

From now we use \( u \) instead of \( U \) since final Brinkmann coordinate is conventionally \( u \). Next we need to write \( ds^2 \) in a parallel frame
\[ \dot{e}_a^2 = 2e^\gamma e_\gamma + \delta_{ab} e^a e^b. \]  
(3.21)

We construct the parallely propagated frame as follows:
\[ e_\gamma = i\partial_\gamma + r\partial_r + \dot{x}_i \partial_{x_i}, \quad e_{+\gamma} = \partial_\gamma. \]  
(3.22)

Since there is no evolution in the \( x_a \) directions, for \( a = 2, 3, \ldots, d \) we can have
\[ e_a = r dx_a. \]  
(3.23)

To calculate the wave profile \( A_{ab} \), we calculate \( B_{ab} \) as reviewed in appendix.
\[ B_{ab} = e_a^\mu e_b^\nu \nabla_\mu \partial_\nu S. \]  
(3.24)

For our case, we get
\[ B_{ab} = \delta_{ab} \partial_u \log r. \]  
(3.25)

By taking the trace of \( B_{ab} \), we also have the condition
\[ Tr(B) = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \dot{x}^\mu) = \partial_u \log (\dot{r} r^{d+z-1}). \]  
(3.26)

Hence
\[ B_{11} = \partial_u \log (r \dot{r}). \]  
(3.27)

It can easily be shown that \( B_{ab} \) is diagonal. So in calculating the profile function using
\[ A_{ab} = \frac{d}{dt} B_{ab} + (B_{ab})^2, \]  
(3.28)

we use \( B_{ab} = \delta_{ab} \partial_u \log K_a \) (no sum over \( a \)) to get
\[ A_{ab} = \delta_{ab} K_a^{-1} \partial_u^2 K_a \]  
(3.29)

where for \( a = 1, K_a = \dot{r} \dot{r} \) and for \( a = 2 \) to \( d, K_a = r \).
Using the geodesic equation for $r$ we get

$$A_{22} = A_{33} = \cdots = A_{dd} = \frac{\dot{r}}{r} = \frac{(1-z)E^2}{r^{2z}}$$

(3.30)

$$A_{11} = \frac{z(1-z)P^2}{r^2} + \frac{(1-z)E^2}{r^{2z}}.$$  

(3.31)

The case $z = 1$ gives the flat metric, expectedly, as the Penrose limit of AdS is flat. The wave profile diverges at $r(u) = 0$ which indicates the presence of a true singularity at that point, as initially the parallel null geodesic diverges at the point of divergence of the Penrose limit wave profile $A_{ab}$.

As $r \to 0$, we see that the second term in $A_{11}$ dominates (for $z \geq 1$) and also in the radial geodesic equation, we have

$$\dot{r} \approx \frac{E}{r^{z-1}}.$$  

(3.32)

Solving the above equation for $r(u)$ we will get

$$u = \frac{r^z}{Ez}.$$  

(3.33)

On substituting this in the expression of the wave profile $A_{ab}$, we get

$$A_{ab} = \frac{1 - z}{z u^2} \delta_{ab}.$$  

(3.34)

So in Brinkmann coordinates, the Penrose limit metric looks like

$$ds^2 = 2dudv + \frac{1 - z}{z^2 u^2} x^2 du^2 + dx^2$$

(3.35)

where $x^i$ are the transverse coordinates. This is consistent with the behaviour of the Penrose limit metric near the singularity as discussed in [15]. The dominant singularity is $u^{-2}$ even though there are subdominant pieces with weaker singularity $u^{-\alpha}$ with $\alpha < 2$. The null energy condition for the Lifshitz metric requires $z > 1$ and so we only need to consider this case. From the geodesic equation (3.18), one can see that we can not have $E = 0$ and $P = 0$. For the Lifshitz metric, the null energy condition requires $z > 1$ and so $A_{ab}$ are negative for cases satisfying the null energy condition.

In [6], Horowitz and Way take the near singularity limit of the Lifshitz metric and show that in this limit, they get a plane wave metric. They use this plane wave metric to show that string passing through such a singularity will be infinitely excited. The Penrose limit metric for the radial null geodesic that we have calculated, via different methods, turns out to be the same as the near singularity limit calculated by Horowitz and Way. So the near singularity limit of Horowitz and Way is the same as the Penrose limit metric. String propagation in this near singularity limit was considered by Horowitz and Way and the same results will hold in our case. Strings become infinitely excited and hence $r = 0$ is a real singularity in string theory as well.

4. Penrose limit of the Schrödinger metric

In this section, we derive the plane wave metric corresponding to the Schrödinger metric. Unlike the Lifshitz metric, the Schrödinger metric is non-singular at the horizon $r = 0$ for $z \geq 2$ and metric extensions have been constructed for these cases [7]. We will see that the
Penrose limit metric shares this and makes it manifest. We start with $d + 3$ dimensional Schrödinger metric

$$\text{d} s^2 = - r^{2z} \text{d} t^2 + r^2 \left( - 2 \text{d} r \text{d} \zeta + \text{d} x^2 \right) + \frac{\text{d} r^2}{r^2}$$

(4.1)

where $\vec{x}$ represents a vector in $d$ dimensional flat Euclidean space. In this case we will include $t, r, \zeta$ and one of the transverse coordinates $x_i$ in our geodesic. So the Lagrangian in this case will be

$$\mathcal{L} = \frac{1}{2} \left( - r^{2z} i^2 - 2 r^2 i^2 \dot{\zeta} + \frac{r^2}{r^2} + r^2 \dot{x}_i^2 \right).$$

(4.2)

Here, derivative is with respect to affine parameter $U$. Solving the Euler–Lagrange equations, we get

$$r^{2z} i + r^2 \dot{\zeta} = E, \quad i = \frac{M}{r^2}, \quad \dot{x}_i = \frac{P}{r^2}.$$  

(4.3)

We will use the null condition i.e. $\mathcal{L} = 0$ instead of the $r$ geodesic equation.

$$\dot{r}^2 = 2 M E - P^2 - M^2 r^{2(z-1)}.$$  

(4.4)

Proceeding in the same way as in the previous sections to get the Penrose limit, we get

$$\text{d} s^2 = 2 dU dV + \left( \frac{2 M E - P^2}{M^2} - r^{2z} \right) \text{d} T^2 + r^2 \text{d} X^2 + r^2 \text{d} x_i^2$$

(4.5)

$$\text{d} s^2 = 2 dU dV + \frac{r^2}{M^2} \text{d} T^2 + r^2 \text{d} X^2 + r^2 \text{d} x_i^2.$$  

(4.6)

Remembering that $r = r(U)$, we have the plane wave metric in Rosen coordinates. We now convert this to Brinkmann coordinates.

$$A_{00} = -2z (z - 1) r^{2(z-2)} M^2$$  

(4.7)

$$A_{ii} = -(z - 1) r^{2(z-2)} M^2.$$  

(4.8)

From the geodesic equation, one can see that, for $z \geq 1$, near $r \approx 0, r \propto u$ (here we note that in going from Rosen to Brinkmann coordinates we have $U = u$). So the profile function $A_{ab}(u)$ has the same behaviour. For $z = 2$, we get a constant wave profile and for $z > 2$, there are no divergences. Note that the null energy condition requires $z \geq 1$ (for positive $z$) and hence $A_{ab}$ are all negative.

**5. Hyperscaling violating Lifshitz spacetimes**

In this section, we consider hyperscaling violating Lifshitz geometry and its Penrose limit. Hyperscaling violating Lifshitz spacetime is given by

$$\text{d} s^2 = \frac{1}{r^{2d/4}} \left( - r^{2z} \text{d} t^2 + \frac{\text{d} r^2}{r^2} + r^2 \text{d} x_i^2 \right).$$

(5.1)

Since the slice of this spacetime at constant $r$ and $t$ is just a flat space, we can choose our coordinate system in such a way that only one of the $x$ vary along the geodesics. Lagrangian for this system then is
\[ \mathcal{L} = \frac{1}{2\ell^{2(\theta/d)}} \left( -r^2t^2 + \frac{r^2}{t^2} + r^2x_i^2 \right). \] (5.2)

Solving the Euler–Lagrange equations, we get
\[ i = \frac{E}{r^{2(z-\theta/d)}}, \quad j_i = \frac{P}{r^{2(1-\theta/d)}} \] (5.3)

where \(-E\) is the conserved quantity corresponding to the \(t\) coordinate and \(P\) is the momentum corresponding to the \(x_i\) coordinate. For the \(r\) equation, we can use the condition for the null geodesic, i.e. \(\mathcal{L} = 0\) which gives
\[ \dot{r}^2 = r^{4(\theta/d)} \left[ \frac{E^2}{r^{2(z-1)} - P^2} \right]. \] (5.4)

Taking the Penrose limit (covariant method) we find the profile function for the Penrose limit metric in these coordinates is
\[ A_{11} = \frac{1}{r^{2(z-\theta/d)}} \frac{\partial^2}{\partial r^2} \left( \frac{r^{2(\theta/d-1)}}{r^{2(z-\theta/d)}} \right) \]
\[ = E^2 \left( 1 - \frac{\theta}{d} \right) \left( 1 - \frac{\theta}{d} \right) r^{2(2\theta/d-z)} + P^2 \left[ z - \frac{z^2 - \theta}{d} + \frac{\theta^2}{d^2} \right] r^{2(2\theta/d-1)} \] (5.5)

and for \(i, j = 2, 3, ..., d\)
\[ A_{ij} = \delta_{ij} \frac{1}{r^{1-\theta/d}} \frac{\partial^2}{\partial r^2} \left( r^{1-\theta/d} \right) \]
\[ = E^2 \left( 1 - \frac{\theta}{d} \right) \left( 1 - \frac{\theta}{d} \right) r^{2(2\theta/d-z)} - P^2 \delta_{ij} \left( 1 - \frac{\theta}{d} \right) r^{2(2\theta/d-1)}. \] (5.6)

For singularity analysis, we will work in a slightly different coordinate system. In [5], possible singularities in hyperscaling violating Lifshitz spacetimes were exhaustively studied and constraints due to the null energy condition were analysed. To match with their conventions, let us perform the following coordinate transformation (in [5], they take \(d\) to be total spacetime dimension. Here, we use \(d\) for transverse directions only. So our total dimension is \(d + 2\) here)
\[ \frac{\theta}{d} = 1 - 1/n, \quad z = (m + n - 1)/n, \quad r = r'^n, \quad t = mt', \quad x = nx' \] (5.7)

we get the metric as
\[ dx^2 = n^2 \left( -r'^2 dt'^2 + \frac{dr'^2}{r'^2} + r'^2 dx'^2 \right). \] (5.8)

Case \(m = n = 1\) just corresponds to AdS. For allowed values of \(m, n\), see [5]. According to the null energy condition
\[ R_{\mu
u} l^\mu l^\nu \geq 0 \] (5.9)

(where \(l\) is a null vector) we have, following [5],
\[ m \geq n \] (5.10)
\[ (m - 1)(m + n + d - 1) \geq 0. \] (5.11)
The Lagrangian for this case can be written as
\[
\mathcal{L} = \frac{n^2}{2} \left[ -r^{2n} \dot{r}^2 + \frac{\dot{r}^2}{r^{2m}} + r^{2n-4} \right].
\] (5.12)

Here, the derivative is with respect to affine parameter \( U \). The geodesic equations are
\[
i = \frac{E}{r^{2m}}, \quad \dot{\chi} = \frac{P}{r^{2n}}.
\] (5.13)

Using the null geodesic condition \( \mathcal{L} = 0 \) we get
\[
\dot{r}^2 = r^{2(n-1)} \left[ \frac{E^2}{r^{2(m-1)}} - p^2 \right].
\] (5.14)

Taking the Penrose limit we get
\[
dx_i^2 = 2dUdV - \frac{\rho^2 m}{E^2} \left( \frac{E^2}{r^{2(m-1)}} - p^2 \right) dX^2 + \sum_{i=2}^d r^{2m} \ddot{x}_i^2.
\] (5.15)

or
\[
dx_i^2 = 2dUdV + \frac{\rho^2 r^{2(m-n+1)}}{E^2} dX^2 + \sum_{i=2}^d r^{2m} \ddot{x}_i^2.
\] (5.16)

The wave profile\(^2\) we get on converting the metric to the Brinkmann coordinate system is given by
\[
A_{11} = E^2 (n - m) r^{2(n-m-1)} - P^2 m (n + m - 2) r^{2(n-2)}
\] (5.17)

and for \( i > 1 \) we get
\[
A_{ii} = E^2 (n - m) r^{2(n-m-1)} - P^2 (n - 1) r^{2(n-2)}.
\] (5.18)

Near \( r' \approx 0 \), we see that dominant term in \( A_{ab} \) is (converting from \( r' \) to \( u \) using radial geodesic equation and taking \( n > 0 \))
\[
A_{ab} (u) \approx \delta_{ab} \frac{n - m}{(m - n + 1)^2 u^2}
\] (5.19)

where \( \left( U = u \right) \).

The null energy condition suggests that the Penrose limit wave profile of the hyperscaling violating metric will be negative at least in the near singularity limit. There are subdominant pieces and in the special case, \( n = m \) when a leading divergent piece is not there, we get weaker singularities which go like \( A_{ab} \propto u^{2(n-2)} \). For \( m = n \geq 2 \), we get the regular Penrose limit metric. This agrees with the analysis of [5]. Smooth extensions for this non-singular hyperscaling violating Lifshitz geometry were obtained in [11]. Even in these non-singular cases, it is possible that even though the metric is non-singular, derivatives of the metric may be singular. This will happen when \( 2(n - 2) \) is not an integer. Since wave profile \( A_{ab} \) of the Penrose limit metric captures the Riemann tensor of the original metric, non-integer powers of \( u \) can lead to divergences in derivatives of the Riemann tensor.

For \( m = n \) and \( 1 < n < 2 \), we have a singular Penrose limit metric but the singularity is weaker than \( 1/u^2 \). Such cases are called weak null singularities and were discussed in [17]. In this case, even though there are divergent tidal forces at \( u = 0 \), string propagation is smooth.

\(^2\) Note that since a rescaling (5.7) by \( n \) is involved between coordinates in (5.1) and coordinates in (5.8), we get a factor of \( n \) when converting from the profile function in one coordinates to another. We can absorb this in our definition of affine parameter.
and one can continue the metric beyond the singularity. Physically, the distortion suffered by a freely falling observer is finite and a falling string does not become infinitely excited while crossing $u = 0$. In [15], using the dominant energy condition, it was conjectured that there is a universal $u^{-2}$ dominant near singularity behaviour. In the plane wave spacetimes obtained here, we also got $u^{-2}$ behaviour for the near singularity region in the Lifshitz case and general $m \neq n$ hyperscaling violating Lifshitz spacetimes. But the class $m = n$ and $1 < n < 2$ seems to lead to weaker singularity for metrics satisfying the null energy condition. Since this case violates the dominant energy condition, this is not a counterexample to [15] but interesting nonetheless. We plan to investigate the issues regarding the dominant energy conditions in more detail later.

6. Hyperscaling violating Schrödinger spacetime

In this section, we study the Penrose limit metric for Schrödinger spacetimes with hyperscaling violation. In [9, 10, 18], various aspects of these spacetimes have been discussed.

$$ds^2 = r^{-k}\left\{-r^{2z}dt^2 + r^2\left(-2drd\zeta + d\nu^2\right) + \frac{dr^2}{r^2}\right\}$$ (6.1)

### 6.1. Null energy condition

The null energy condition can be used to constrain the parameters of this metric to physically acceptable values. Even though this has been discussed in [9, 10, 18], we present this in our conventions. For the null vector $l^a$ the condition becomes

$$R_{ab}l^a l^b \geq 0.$$ (6.2)

For the $(d + 3)$ dimensional spacetime, non-zero components of the Ricci tensor are

$$R_{tt} = \frac{1}{4}\left[8(z^2 + 1) + 4(d - 2)z - 2(d + 2)k + (d + 1)k(k - 2z)\right]r^{2z}$$ (6.3)

$$R_{rr} = (d + 2)\frac{k - 2}{2r^2}$$ (6.4)

$$R_{xx} = -\frac{1}{4}\left[(d + 1)k^2 - 2(2d + 3)k + 4(d + 2)\right]r^2$$ (6.5)

$$R_{\zeta\zeta} = R_{\nu\nu} = \frac{1}{4}\left[(d + 1)k^2 - 2(2d + 3)k + 4(d + 2)\right]r^2.$$ (6.6)

So the null energy condition becomes

$$-\frac{1}{4}\left[(d + 1)k^2 - 2(2d + 3)k + 4(d + 2)\right]\left[-r^{2z}(l^\nu)^2 + r^2\left(-2l^\nu l^\zeta + \left|l_{\nu}\right|^2\right) + \frac{(l^\nu)^2}{r^2}\right] + \frac{1}{4}\left[8z^2 + 4(d - 2)z - 2k(d + 1)(z - 1) - 4d\right]r^{2z}(l^\nu)^2 + \frac{1}{4}(d + 1)(k^2 - 2k)r^2(l^\nu)^2.$$ (6.7)
Since the vector $l$ is null, we have the null energy condition as
\[ (z - 1)[8z + 4d - 2k(d + 1)] \geq 0 \]
\[ k(k - 2) \geq 0 \]  
where for $z \geq 1$ we get
\[ k \leq \frac{2(2z + d)}{d + 1}. \]  

### 6.2. Penrose limit

To obtain the Penrose limit, we start with the Lagrangian given by
\[ \mathcal{L} = \frac{1}{2} \left( -r^{-k+2} \dot{\zeta}^2 - 2r^{-k+2} \dot{\xi}_i^2 + r^{-k+2} \dot{x}_1^2 + r^{-2(k+1)} \right). \]  

Equations of motion are
\[ i = Mr^{(k-2)} \]
\[ \dot{\zeta} = Er^{(k-2)} - Mr^{2z+k-4} \]
\[ \dot{x}_1 = Pr^{(k-2)}. \]

Using the null geodesic condition i.e. $\mathcal{L} = 0$ we get
\[ \dot{r}^2 = \left( 2ME - P^2 \right) r^{2k} - M^2 r^{2(1+k)}. \]

Proceeding as before we get the Penrose limit metric as
\[ ds^2 = 2dUdV + \left( \frac{2ME - P^2}{M^2} r^{k-2} - r^{2z-k} \right) dT^2 + r^{-k+2} d\xi_i^2 + r^{-k+2} \sum_{i=2}^d dx_i^2. \]

After renaming the rest of $x_i$ as $X_i$ and using the geodesic equation, we get
\[ ds^2 = 2dUdV + \frac{r^{-3k+2}_\mu}{M^2} dT^2 + r^{-k+2} \sum_{i=1}^d dX_i^2. \]

For the metric of this type, the wave profile we get on converting it to the Brinkmann coordinate system is
\[ A_{00} = \frac{k}{2} \left( 1 - \frac{k}{2} \right) \left( 2ME - P^2 \right) r^{2(k-1)} \]
\[ - \left\{ 2z \left( z - 1 - \frac{k}{4} \right) + k \left( 1 - \frac{k}{4} \right) \right\} M^2 r^{2(1+k)}. \]

and for $i \neq 0$
\[ A_{i\mu} = \frac{k}{2} \left( 1 - \frac{k}{2} \right) \left( 2ME - P^2 \right) r^{3(k-1)} - \left\{ z \left( 1 - \frac{k}{2} \right) - \left( 1 - \frac{k}{2} \right)^2 \right\} M^2 r^{2(1+k)}. \]

For $k = 0$ i.e. Schrödinger spacetime without hyperscaling violation, a non-singular metric for $z \geq 2$ and metric extension was done in [7]. We obtained the Penrose limits for these cases earlier. With hyperscaling violation i.e. non-zero $k$, for the wave profile to be finite at $r = 0$ (i.e. Penrose limit metric to be non-singular) we need to have
According to the null energy condition either \( k \leq 0 \) or \( k \geq 2 \). So for the plane wave metric to be regular at origin we need to have \( k \geq 2 \).

For \( z \geq 1 \) and \( k > 2 \), in the near \( r = 0 \) limit we have

\[
A_{00} \approx \frac{k}{2} \left(1 - \frac{k}{2}\right) \left(2ME - P^2\right) r^{2(k-1)}
\]

\[
A_{ii} \approx \frac{k}{2} \left(1 - \frac{k}{2}\right) \left(2ME - P^2\right) r^{2(k-1)}.
\]

Using

\[
r^2 \approx \left(2ME - P^2\right) r^{2k}
\]

i.e.

\[
r^{k-1} = \frac{1}{(k-1)\left(2ME - P^2\right)^{1/2}U}.
\]

The wave profile becomes

\[
A_{00} = \frac{k(2-k)}{4(k-1)^2} u^{-2}
\]

\[
A_{ii} = \frac{k(2-k)}{4(k-1)^2} u^{-2}.
\]

Here we note that in going from Rosen to Brinkmann coordinates, we have used \( U = u \).

Since \( r \) and \( u \) are inversely related, \( r \to 0 \) corresponds to \( u \to \infty \) and hence the wave profile is not singular as \( r \to 0 \). For \( k = 2 \) and \( z > 1 \), we see that the Penrose limit metric is quite simple, with the only non-zero component of \( A_{ab} \) being \( A_{00} = -(z-1)(2z-1)M r^2 r^2 \).

### 6.3. \( k = 2 \) hyperscaling violating Schrödinger geometry

For \( k = 2 \), the hyperscaling violating Schrödinger metric is

\[
d\mathbf{s}^2 = -r^{2z-2} dr^2 + \left( -2 dr d\zeta + d\overline{r}^2 \right) + \frac{dr^2}{r^2}.
\]

For \( z = 1 \), we just get part of flat space which can be extended to the whole of flat space. This is expected since the \( z = 1 \) Schrödinger metric corresponds to \( AdS \) which is conformal to flat space. The hyperscaling violating factor just cancels this conformal factor. For \( z = 2 \), we can extend this metric beyond \( r = 0 \) by following coordinate transformations, which are simplified forms of transformations used to go to Penrose adapted coordinates

\[
t = MU + T
\]

\[
\zeta = \int \left( E - Mr^2 \right) dU + \lambda
\]

\[
r = \int \dot{r} dU
\]

where dot means derivative with respect to \( U \) and we use (6.15) with \( P = 0 \) to convert from \( r \) to \( U \). In terms of these coordinates \( r \to 0 \) corresponds to \( U \to -\infty \). The metric becomes
In these coordinates, the metric can be extended beyond \( r = 0 \) as metric components remain smooth there and the determinant of the metric is non-zero as \( r \to 0 \).

7. Conclusions

We have obtained plane wave metrics corresponding to several geometries which have been used recently in giving dual descriptions of many non-relativistic field theories. We also showed that the near singularity limit considered by Horowitz and Way [6] is the same as taking the Penrose limit. One feature of the Penrose limit metrics of these geometries is that singularities in the original metrics become manifest in the plane wave limit. This is to be expected since pp-curvature singularities are given by divergences in components of the Riemann tensor in the parallely propagated frame and the wave profile in Brinkmann coordinates is directly related to the Riemann tensor in the parallely propagated frame. We have not considered dual field theory of these Penrose limit metrics in this paper. But based on our experience with AdS, we would expect the Penrose limit to describe a certain sector of states in field theory. It would be interesting to flesh out the details of the analogue of the BMN limit for these cases. Since plane wave spacetimes are quite tractable for detailed string theory and field theory considerations, one can do a lot more in this limit than in general non-relativistic metrics. We are currently working on various aspects of this correspondence, including string theory in these plane wave backgrounds and will report on them in the near future.

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Appendix. Review of the Penrose limit

In this section, we review the methods to obtain the Penrose limits, in particular the covariant characterization of the Penrose limit as discussed in detail in [14] and [16]. This plane wave depends upon the metric we start with and the choice of the null geodesic along which the observer is moving.

A.1. Penrose limit via adapted coordinates

To find the Penrose limit of a given metric for a given geodesic congruence we need to write the metric given to us in the Penrose adapted coordinates.

\[
\frac{ds^2}{\gamma} = 2dUdV + a\left( U, V, Y^k \right) dV^2 + 2b\left( U, V, Y^k \right) dVdY^k + g_{ij}\left( U, V, Y^k \right) dY^idY^j
\]

where \( U \) plays the role of the affine parameter along the chosen null geodesic. Then we have to perform a coordinate transformation which is the same as scaling the metric followed by a boost.

\[
\left( U, V, Y^k \right) \to \left( u, \lambda^2v, \lambda y^k \right) \lambda R.
\]
The Penrose limit metric is defined as

$$\lim_{\lambda \to 0} \lambda^2 ds^2_{\lambda}$$

(A.3)

where $ds^2_{\lambda}$ is the metric we get upon performing the above defined coordinate transformation. So after taking the Penrose limit we get the metric

$$ds^2 = 2dudv + g_{ij}(u)dy^idy^j.$$  

(A.4)

This is the familiar plane wave metric in Rosen coordinates. So the crucial step is to get the metric from the initial coordinates to Penrose adapted coordinates. Finding such coordinates is equivalent to embedding the original geodesic in a twist-free null geodesic congruence. To do this, we can use Hamilton–Jacobi construction as reviewed in [16]. Basically one solves for null geodesics and the Hamilton–Jacobi equation in the original coordinates. We then use an affine parameter along geodesic $U$ as one of the coordinates and use the solution to the Hamilton–Jacobi equation $g^{\mu\nu} \partial_\mu S \partial_\nu S = 0$ as another null coordinate $V$ i.e. $S = V$. Transverse coordinates are usually found by guesswork in simple cases that we consider but a more systematic approach is outlined in [16]. For most purposes, a second coordinate system, called Brinkmann coordinates, is better suited to representing plane waves. In this coordinate system

$$dv^2 = 2dudv + A_{ab}x^ax^bdud^2 + dx^2.$$  

(A.5)

Here $a, b = 1, \ldots n$ go over all transverse coordinates. In Brinkmann coordinates all the information of the wave is contained in the coefficient of $du^2$ which is $A_{ab}$. To convert from Rosen to Brinkmann coordinates, we use

$$A_{ab}(u) = e_a^\mu e_b^\nu g_{\mu\nu}$$

(A.6)

where $e_a^\mu$ are vielbein for $g_{ij}$ i.e.

$$g_{ij} = \delta_{ab} e^a_i e^b_j.$$  

(A.7)

For diagonal metrics encountered in this paper, we can write a simplified expression (no sum over $a$)

$$A_{ab}(u) = \frac{\dot{e}_a}{e_u} \delta_{ab}.$$  

(A.8)

In the next subsection, we will review the covariant method which gives metric in Brinkmann form directly. Once we have the given metric in Penrose adapted coordinates, we can see that the effect of taking the Penrose limit is the same as neglecting the $dV^2$ and $dVdY^k$ term from the metric and keeping the remaining term as it is. Also $g_{\gamma}(U)$ is the restriction of the $g_{\gamma}(U, V, Y^k)$ on the null geodesic $\gamma$. So on taking the Penrose limit we are actually observing an infinitesimal region of the spacetime which is near the null geodesic $\gamma$ and expanding it to form the entire spacetime.

**A.2. Covariant description of the Penrose limit**

On taking the Penrose limit for a given metric $(g_{\mu\nu})$ and null geodesic $(\gamma)$ congruence, what we are interested in is the wave profile $(A_{ab})$ of the limiting plane wave metric. In this section, we will see the covariant way of calculating the wave profile instead of going through the several coordinate transformation and rescaling of the metric. This method was presented in [14] and we refer to that and [16] for details.
One can see that the profile $A_{ab} = -R_{mab}$ in Brinkmann coordinates. In terms of the plane wave in Rosen coordinates, it is

$$A_{ab} = -e^i e^j R_{iju}$$

(A.9)

where $e^a$ are the vielbeins for the transverse metric in the Rosen coordinate $(g_{ij})$ and satisfy the symmetry condition

$$e^a e^b = e^b e^a.$$  

(A.10)

For the metric in Penrose adapted coordinates, the component of the Riemann tensor $R_{iju}$ is given as

$$R_{iju} = -(\partial_i \Gamma^a_j - \Gamma^a_{jk} \Gamma^k_i).$$

(A.11)

This is independent of $a$ and $b$, and depends only upon $g_{ij}$ and its $U$ derivative. (This can easily be seen by just expanding each of the Christoffel symbols in the above expression.) Hence, on taking the Penrose limit, we will have

$$\tilde{R}_{iju} = R_{iju}. |.$$ 

(A.12)

In order to look for a covariant description, we would like to write the metric given to us as

$$d\sigma^2 = 2e^+ e^- + \delta_{ab} e^ae^b$$

(A.13)

where $e^A$ are parallel along the null geodesic congruence i.e. $\nabla_l e^A = 0$. We choose one of the legs to be the tangent along the null geodesic $e^+ = \frac{\partial}{\partial U}$.

Finally we get our wave profile of the Penrose limit metric as $A_{ab} = \frac{d}{du} B_{ab} + B_{a b} B^c_c$ where $B_{ab} = e^a e^j \nabla_a \partial_b S$. Here $S$ is the solution to the Hamilton–Jacobi equation as discussed in the previous subsection. Derivation of these results is given in [14]. Although we have these results in Penrose adapted coordinates, what we have got at the end has only vielbein indices and all the coordinate indices have been summed up. So this quantity is a scalar. Hence we can use any coordinate system to compute these quantities and we will get the same result at the end.

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