Isomorphic Well-Posedness of the Final Value Problem for the Heat Equation with the Homogeneous Neumann Condition

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Abstract. This paper concerns the final value problem for the heat equation subjected to the homogeneous Neumann condition on the boundary of a smooth open set in Euclidean space. The problem is here shown to be isomorphically well posed in the sense that there exists a linear homeomorphism between suitably chosen Hilbert spaces containing the solutions and the data, respectively. This improves a recent work of the author, in which the same problem was proven well-posed in the original sense of Hadamard under an additional assumption of Hölder continuity of the source term. Like for its predecessor, the point of departure is an abstract analysis in spaces of vector distributions of final value problems generated by coercive Lax–Milgram operators, now yielding isomorphic well-posedness for such problems. Hereby the data space is the graph normed domain of an unbounded operator that maps final states to the corresponding initial states, resulting in a non-local compatibility condition on the data. As a novelty, a stronger version of the compatibility condition is introduced with the purpose of characterising the data that yield solutions having the regularity property of being square integrable in the generator’s graph norm (instead of in the form domain norm). This result allows a direct application to the class 2 boundary condition in the considered inverse Neumann heat problem.

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1. Introduction

The purpose of the present paper is to show rigorously that the heat conduction final value problem with the homogeneous Neumann condition is isomorphically well-posed, in the sense that there exists an isomorphism between suitably chosen spaces for the data and the corresponding solutions.
This result is obtained below by improving the recent results in [27] by application of classical regularity properties in spaces of low regularity.

The central theme below is to characterise the \( u(t,x) \) that in a fixed bounded open set \( \Omega \subset \mathbb{R}^n \) \( (n \geq 1) \) with \( C^\infty \)-smooth boundary \( \Gamma = \partial \Omega \) fulfil the following equations, whereby \( \Delta = \partial^2_{x_1} + \cdots + \partial^2_{x_n} \) denotes the Laplace operator and \( \nu(x) \) stands for the exterior unit normal vector field at \( \Gamma \):

\[
\begin{align*}
\partial_t u(t,x) - \Delta u(t,x) &= f(t,x) \quad \text{for } t \in ]0,T[, x \in \Omega, \\
(\nu \cdot \text{grad}) u(t,x) &= 0 \quad \text{for } t \in ]0,T[, x \in \Gamma, \\
u(T,x) &= u_T(x) \quad \text{for } t = T, x \in \Omega.
\end{align*}
\tag{1.1}
\]

In view of the final value condition at \( t = T \), this problem is also called the inverse Neumann heat equation. One area of interest of this could be a nuclear power plant hit by power failure at \( t = 0 \): when at \( t = T > 0 \) power is regained and the reactor temperatures \( u_T(x) \) are measured, a backwards calculation could possibly settle whether at some \( t < T \) the temperatures \( u(t,x) \) were high enough to cause damage to the fuel rods.

Here it should be noted that the Neumann condition, which controls the heat flux through the boundary, is more natural from a physical point of view than the Dirichlet condition \( u|_\Gamma = g \), in which the value itself is prescribed at the boundary in terms of a given function \( g(t,x) \) on \( ]0,T[ \times \Gamma \). Other boundary conditions may also be natural, but are for the sake of simplicity not treated here.

Previously final value problems for the heat equation with the Dirichlet condition were shown to be well posed in a joint work of the author [6], with more concise expositions in [7,26]. For this problem, the obtained well-posedness was in the original sense of Hadamard, namely, there is existence, uniqueness and stability of a solution \( u \in X \) for given data \( (f,g,u_T) \in Y \), in certain Hilbertable spaces \( X, Y \) that were described explicitly. Hereby the data space \( Y \) was defined in terms of a special compatibility condition on the triples \( (f,g,u_T) \), which was introduced for the purpose in [6].

This development seemingly closed a gap that had remained in the understanding since the 1950s, even though well-posedness is crucial for the interpretation and accuracy of numerical schemes for the problem (the work of John [24] was pioneering, but e.g. also Eldén [11] could be mentioned). Briefly phrased, the results are obtained via a suitable structure on the reachable set for parabolic evolution equations.

In the present paper the intention is not just to focus on the more relevant Neumann condition, but also to go an important step further by introducing solution and data spaces \( X_1 \) and \( Y_1 \) that are so chosen that the operator \( P(u) = (u' - \Delta u, u(T)) \) is an isomorphism, that is, a linear homeomorphism

\[
X_1 \xrightarrow{P} Y_1.
\tag{1.2}
\]

It is also proposed to indicate this strong form of well-posedness by terming (1.1) isomorphically well posed; cf. the title of the paper.
More specifically, using the full yield of the source term, which is the vector \( y_f = \int_0^T e^{(T-t)\Delta_N} f(t) \, dt \), the isomorphic well-posedness of (1.1) is obtained below for the spaces
\[
X_1 = L_2(0, T; H^2(\Omega)) \bigcap C([0, T]; H^1(\Omega)) \bigcap H^1(0, T; L_2(\Omega)),
\]
\[
Y_1 = \left\{ (f, u_T) \in L_2(0, T; L_2(\Omega)) \oplus H^1(\Omega) \left| u_T - y_f \in e^{T\Delta_N} (H^1(\Omega)) \right. \right\},
\]
which are both shown to be Banach spaces under the norms
\[
\|u\|_{X_1} = \left( \int_0^T \left( \|u(t)\|_{H^2(\Omega)}^2 + \|\partial_t u(t)\|_{L_2(\Omega)}^2 \right) \, dt \right)^{1/2},
\]
\[
\|(f, u_T)\|_{Y_1} = \left( \int_0^T \|f(t)\|_{L_2(\Omega)}^2 \, dt \right)^{1/2}
+ \int_{\Omega} \sum_{|\alpha| \leq 1} \left( |\partial_x^\alpha u_T(x)|^2 + |\partial_x^\alpha e^{-T\Delta_N}(u_T - y_f)(x)|^2 \right) \, dx \right)^{1/2}.
\]
Furthermore, it is also established that the solution is given for \( 0 \leq t \leq T \) by the following variant of the Duhamel formula,
\[
u(t) = e^{t\Delta_N} e^{-T\Delta_N} \left( u_T - \int_0^T e^{(T-t)\Delta_N} f(t) \, dt \right) + \int_0^t e^{(t-s)\Delta_N} f(s) \, ds.
\]
For brevity the reader is referred to the details below in Theorem 5.4 and Corollary 5.5 on the isomorphic well-posedness of (1.1). These results substantially improve the Hadamard well-posedness in [27].

However, it is noteworthy here, that both formula (1.7) as well as the definition of \( Y_1 \) and its norm in (1.4), (1.6) make use of the fact that the Neumann realisation \( \Delta_N \) of the Laplace operator generates an analytic semigroup \( e^{T\Delta_N} \) in \( L_2(\Omega) \), which is invertible in the class of closed operators in \( L_2(\Omega) \), so that one can set
\[
e^{-T\Delta_N} = (e^{T\Delta_N})^{-1}.
\]
Section 2 below reviews the fact that every analytic semigroup consists entirely of injective operators, which has (1.8) as a special case. The exposition there is close to the account in [6], but it has been included not only to make the present paper reasonably self-contained, but also to make it precise that the semigroups need not be uniformly bounded (following up on the indications made in [26,27]) and to add a local version and some historical remarks.
As prerequisites, Sect. 3 recalls a few known extensions of the analysis of initial value problems in the classical treatise of Lions and Magenes [30]. Some of this was detailed in [27], like solvability theory of problems generated by $V$-coercive Lax–Milgram operators, estimates of the solution operator and the resulting Duhamel formula. But for the Neumann problem it is indispensable to include a regularity result in order to treat the class 2 boundary condition in (1.1) and the above $H^2$-spaces, cf. (1.3). Moreover, in addition to the sufficiency of a certain data regularity shown explicitly in [30], its necessity is important for the purpose of obtaining an isomorphism, so Sect. 3 accounts for this.

Section 4 analyses final value problems in a framework of Lax–Milgram operators $A$ that are $V$-coercive in a Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$. This extends the $V$-elliptic case covered in [6,7,26], as also done in [27], but as a new feature final value problems are treated in the more regular spaces $D(A) \hookrightarrow [D(A),H]_{1/2} \hookrightarrow H$. At this level isomorphic well-posedness is obtained too; cf. Corollary 4.5.

Section 5 accounts for the treatment of the final value problem (1.1) using the general results in Sect. 4; cf. Theorem 5.2, Theorem 5.4 and Corollary 5.5 there. Some final remarks are gathered in Sect. 6.

2. Preliminaries: Injectivity of Analytic Semigroups

For analysis of final value problems it is crucial that analytic semigroups of operators always consist of injections. This shows up both technically and conceptually, that is, both in the proofs and in the objects entering the theorems.

Some aspects of semigroup theory in a complex Banach space $B$ are therefore recalled. Besides classical references by Davies [8], Pazy [34], Tanabe [39] or Yosida [42], more recent accounts are given by Engel and Nagel [12], Arendt [4] or in [1].

The generator is $A x = \lim_{t \to 0^+} \frac{1}{t} (e^{tA} x - x)$, where $x$ belongs to the domain $D(A)$ when the limit exists. $A$ is a densely defined, closed linear operator in $B$ that for some $\omega \in \mathbb{R}, M \geq 1$ satisfies the resolvent estimates

$$\| (A - \lambda I)^{-n} \|_{B(B)} \leq M/|\lambda - \omega|^n$$

for $\lambda > \omega, n \in \mathbb{N}$.

The associated $C_0$-semigroup of operators $e^{tA} \in \mathcal{B}(B)$ of type $(M,\omega)$: it fulfils $e^{tA}e^{sA} = e^{(s+t)A}$ for $s, t \geq 0, e^{0A} = I$ (the identity) and for $x \in B$ that $\lim_{t \to 0^+} e^{tA} x = x$, whilst there is an estimate

$$\|e^{tA}\|_{\mathcal{B}(B)} \leq M e^{\omega t} \quad \text{for} \quad 0 \leq t < \infty.$$  \hspace{1cm} (2.1)

There is also a well-known translation trick, which is used repeatedly throughout, namely one has

$$e^{tA} = e^{t\mu} e^{t(A-\mu I)} \quad \text{for every} \quad \mu \in \mathbb{C}.$$ \hspace{1cm} (2.2)

Indeed, the right-hand side is a $C_0$-semigroup having $A$ as its generator (since $e^{t\mu} = 1 + t\mu + o(t)$), so the formula results from the injectivity of $e^{tA} \mapsto A$. More explicitly, by the proof of the Hille–Yosida theorem, there is a bijection
of the semigroups of type \((M, \omega)\) onto (the resolvents of) the stated class of
generators given by the Laplace transformation formula
\[
(\lambda I - A)^{-1} = \int_0^\infty e^{-t\lambda} e^{tA} \, dt = \int_0^\infty e^{t(A - \lambda I)} \, dt, \quad \text{for } \Re \lambda > \omega. \quad (2.3)
\]
This formula also follows from the Fundamental Theorem for vector functions.

Now, as the evaluation map \(E_x T = T x\) is bounded \(B(B) \to B\) for \(x \in B\), the
Bochner identity implies that for \(\Re z > \omega\),
\[
\left( \int_0^\infty e^{t(A-zI)} \, dt \right) x = (zI - A)^{-1} x = \int_0^\infty e^{t(A-zI)} x \, dt. \quad (2.4)
\]

As for the injectivity, recall that if \(e^{tA}\) is analytic, then \(u' = Au, u(0) = u_0\) is for every \(u_0 \in B \) uniquely solved by \(u(t) = e^{tA}u_0\). So injectivity of \(e^{tA}\) is
obviously equivalent to the geometric property that the trajectories of two
solutions \(e^{tA}v, e^{tA}w\) of \(u' = Au\) have no confluence point in \(B\) for \(v \neq w\).

Despite this, the literature seems to have focused on examples of non-
invertible \(e^{tA}\), like [34, Ex. 2.2.1]; these necessarily concern non-analytic
cases. The well-known result gives a criterion for \(A\) to generate a
\(C_0\)-semigroup \(e^{zA}\) that is defined and analytic for \(z\) in the open sector
\[
S_\theta = \{ z \in \mathbb{C} \mid z \neq 0, \ | \arg z | < \theta \}. \quad (2.5)
\]
This notation is also used for the spectral sector in property (i) in the result:

**Proposition 2.1.** If \(A\) generates a \(C_0\)-semigroup of type \((M, \omega)\) and \(\omega \in \rho(A)\),
the following properties are equivalent for each \(\theta \in ]0, \pi/2[\):

(i) The resolvent set \(\rho(A)\) contains \(\{ \omega \} \cup (\omega + S_{\theta+\pi/2})\) and
\[
\sup \{ |\lambda - \omega| \cdot \|(\lambda I - A)^{-1}\|_{B(B)} \mid \lambda \in \omega + S_{\theta+\pi/2} \} < \infty. \quad (2.6)
\]

(ii) The semigroup \(e^{tA}\) extends to an analytic semigroup \(e^{zA}\) defined for
\(z \in S_\theta\) with
\[
\sup \{ e^{-z\omega} \|e^{zA}\|_{B(B)} \mid z \in S_{\theta'} \} < \infty \quad \text{whenever } 0 < \theta' < \theta. \quad (2.7)
\]

In the affirmative case, \(e^{tA}\) is differentiable in \(B(B)\) with \((e^{tA})' = Ae^{tA}\) for\( t > 0\), and
\[
\sup_{t > 0} e^{-t\eta} \|e^{tA}\|_{B(B)} + \sup_{t > 0} te^{-t\eta} \|Ae^{tA}\|_{B(B)} < \infty \quad (2.8)
\]
for every \(\eta > \alpha(A)\), whereby \(\alpha(A) = \sup \Re \sigma(A)\) denotes the spectral abscissa
of \(A\).

If \(\omega = 0\), the equivalence (i) \(\iff\) (ii) is contained in Theorem 2.5.2 in
[34]. This extends to \(\omega > 0\), using for both implications that \(2.2\) holds with
\(\mu = \omega\) for complex \(t\) in \(S_\theta\) by unique analytic extension.

The first part of \(2.8\) holds since analyticity implies \(\alpha(A) = \omega_0\), where
the growth bound \(\omega_0\) is the infimum of the \(\omega\) such that \(\|e^{tA}\|_{B(B)} \leq M e^{t\omega}\) for
some \(M\) (so \(\eta = \omega\) is possible); cf. [12, Cor. IV.3.12]. For the last part we have
\(\omega > \alpha(A) = \omega_0\) (as \(\omega \in \rho(A)\)) and may hence consider \(\alpha(A) < \eta' < \eta,\)
insert \(A = \eta'I + (A - \eta'I)\) and invoke the classical uniform bound of \(t\|Ae^{tA}\|_{B(B)}\)
from the case \(\omega = 0\).
The purpose of stating Proposition 2.1 for general type \((M, \omega)\) semigroups is to emphasize that cases with \(\omega > 0\) only have other estimates in the closed subsectors \(\overline{S}_\theta\), whereas the mere analyticity in \(S_\theta\) is unaffected by the translation by \(\omega I\). This lead to the following sharpening of [6, Prop. 1]:

**Proposition 2.2.** [27] If a \(C_0\)-semigroup \(e^{tA}\) of type \((M, \omega)\) on a complex Banach space \(B\) has an analytic extension \(e^{zA}\) to \(S_\theta\) for some \(\theta > 0\), then \(e^{zA}\) is an injective operator for every \(z \in S_\theta\).

**Proof.** Let \(e^{z_0A}u_0 = 0\) hold for \(u_0 \in B\) and \(z_0 \in S_\theta\). By the differential calculus in Banach spaces, analyticity of \(e^{zA}\) in \(S_\theta\) carries over \(f(z) = e^{zA}u_0\). So for \(z\) in a suitable open ball \(B(z_0, r) \subset S_\theta\), a Taylor expansion and the identity \(f^{(n)}(z_0) = A^ne^{z_0A}u_0\) for analytic semigroups (cf. [34, Lem. 2.4.2]) give

\[
f(z) = \sum_{n=0}^{\infty} \frac{1}{n!}(z - z_0)^nf^{(n)}(z_0) = \sum_{n=0}^{\infty} \frac{1}{n!}(z - z_0)^nA^n e^{z_0A}u_0 = 0.
\]

(2.9)

Therefore \(f \equiv 0\) holds on \(S_\theta\) by unique analytic extension, hence \(u_0 = \lim_{t \to 0^+} e^{tA}u_0 = \lim_{t \to 0^+} f(t) = 0\). Thus the null space of \(e^{z_0A}\) is trivial for every \(z_0 \in S_\theta\). \(\square\)

As a corollary to the proof, in case \(B = L_p(\Omega)\) for \(1 \leq p < \infty\) and some open set \(\Omega \subset \mathbb{R}^n\), it is seen that if \(u = e^{tA}u_0\) fulfills \(u(t_0, \cdot) = 0\) in an open subset \(\Omega_0 \subset \Omega\) for a given \(t_0 > 0\), the partial sums of the above power series converge to \(u \in L_p(\Omega)\) for \(z = t, z_0 = t_0\) with \(|t - t_0| < t_0 \sin \theta\); so for each such \(t\), a subsequence converges pointwise to \(u(t, \cdot)\) a.e. in \(\Omega_0\), which simplifies to 0 a.e. in \(\Omega_0\) if \(A\) preserves support in \(\Omega\), so that \(A^n u(t, \cdot) = 0\) in \(\Omega_0\). As an iteration will cover all \(t > 0\), one has the local result:

**Proposition 2.3.** If, in addition to the hypothesis in Proposition 2.2, the Banach space is given as \(B = L_p(\Omega)\) for \(1 \leq p < \infty\) and an open set \(\Omega \subset \mathbb{R}^n\) in which \(A\) preserves support, when a solution of \(u' = Au\) fulfills \(u(t_0, \cdot) = 0\) in an open subset \(\Omega_0 \subset \Omega\) for some \(t_0 > 0\), then \(u(t, x) = 0\) for all \(t > 0\) and a.e. \(x \in \Omega_0\).

**Remark 2.4.** Not surprisingly, Proposition 2.3 has a forerunner in work of Yosida [41], who gave the above argument for \(p = 2\) under the extra assumption that \(A\) is a strongly elliptic differential operator in \(\Omega\). The concise conclusion in Proposition 2.2 was not reached in [41] (although \(\Omega_0 = \Omega\) is a possibility in Proposition 2.3), and it seems not to have appeared in the semigroup literature during the following decades, until it was shown for \(\omega = 0\) in [6,7]. Proposition 2.2 was anticipated for \(z > 0, \theta \leq \pi/4\) and \(B\) a Hilbert space in [38], but not quite obtained; cf. details in [6, Rem. 1] and [25, Rem. 3]. Masuda [32] used the unique continuation property to obtain the stronger result that \(u = 0\) extends from \(\{t_0\} \times \Omega_0\) to \(\mathbb{R}_+ \times \Omega\); Rauch [35, Cor. 4.3.9] gave a version for \(\Delta\) on \(\mathbb{R}^n\).

As a result of the above injectivity, for an analytic semigroup \(e^{tA}\) we may consider its inverse that, like when \(e^{tA}\) forms a group in \(\mathbb{B}(B)\), may be denoted for \(t > 0\) by \(e^{-tA} = (e^{tA})^{-1}\). Clearly \(e^{-tA}\) maps its domain
$D(e^{-tA}) = R(e^{tA})$ bijectively onto $H$, and it is an unbounded, but closed operator in $B$.

Specialising to a Hilbert space $B = H$, then also $(e^{tA})^* = e^{tA^*}$ is analytic, so $Z(e^{tA^*}) = \{0\}$ holds for its null space by Proposition 2.2; whence $D(e^{-tA})$ is dense in $H$. Some further basic properties from [6, Prop. 2] are:

**Proposition 2.5.** The inverses $e^{-tA}$ form a semigroup of unbounded operators in $H$,

$$e^{-sA}e^{-tA} = e^{-(s+t)A} \quad \text{for } t, s \geq 0. \quad (2.10)$$

This extends to $(s, t) \in \mathbb{R} \times ]-\infty, 0]$, whereby $e^{-(t+s)A}$ may be unbounded for $t + s > 0$. Moreover, as unbounded operators the $e^{-tA}$ commute with $e^{sA} \in \mathcal{B}(H)$, that is, $e^{sA}e^{-tA} \subset e^{-tA}e^{sA}$ for $t, s \geq 0$, and have a descending chain of domains, $H \supset D(e^{-tA}) \supset D(e^{-t'A})$ for $0 < t < t'$.

**Remark 2.6.** The domains $D(e^{-tA})$ of the inverses have been introduced independently in the literature on the regularisation of final value problems (albeit for $t > T$). A very recent example is [14].

### 3. Evolution Equations Revisited

This section outlines the prerequisites on evolution equations needed in Sects. 4 and 5. The material is entirely classical, and the treatise of Lions and Magenes [30] is chosen as the main source, although the below Theorem 3.5 was not stated there (it is known nowadays in a more general setting).

The basic analysis is made for a Lax–Milgram operator $A$ defined in $H$ from a $V$-coercive sesquilinear form $a(\cdot, \cdot)$ in a Gelfand triple, i.e., three separable, densely injected Hilbert spaces $V \hookrightarrow H \hookrightarrow V^*$ having the norms $\| \cdot \|$, $| \cdot |$ and $\| \cdot \|_1$, respectively. Hereby $V$ is the form domain of $a$; and $V^*$ the antidual of $V$. Specifically there are constants $C_j > 0$ and $k \in \mathbb{R}$ such that all $u, v \in V$ satisfy $\|v\|_1 \leq C_1|v| \leq C_2\|v\|$ and

$$|a(u, v)| \leq C_3\|u\|\|v\| \quad \text{Re}a(v, v) \geq C_4\|v\|^2 - k|u|^2. \quad (3.1)$$

In fact, $D(A)$ consists of the $u \in V$ for which $a(u, v) = (f | v)$ for some $f \in H$ and all $v \in V$; then $Au = f$. Hereby $(u | v)$ denotes the inner product in $H$. There is also an extension $A \in \mathcal{B}(V, V^*)$ defined by the identity $\langle Au, v \rangle = a(u, v)$ for all $u, v \in V$. This is uniquely determined as $D(A)$ is dense in $V$.

Both $a$ and $A$ are referred to as $V$-elliptic if the above holds for $k = 0$; then $A \in \mathcal{B}(V, V^*)$ is a bijection. E.g. [18], [20] or [6] give more details on the set-up and basic properties of the unbounded, but closed operator $A$ in $H$. Throughout $D(A)$ is endowed with the graph norm, which is complete.

The operator $A$ is self-adjoint if and only if $a(v, w) = \overline{a(w, v)}$, which is not assumed. $A$ may also be nonnormal in general. For a non-trivial example based on the advection–diffusion operators $-\partial_x^2 \pm \partial_x$ with mixed Dirichlet, Neumann and Robin conditions on an interval $]0, \beta[$, both $D(A^*) \setminus D(A) \neq \emptyset$ and $D(A) \setminus D(A^*) \neq \emptyset$ are shown to hold in [25, Ex. 1].
In this framework, the general Cauchy problem is, for given data \( f \in L_2(0,T;V^*) \) and \( u_0 \in H \), to determine the \( u \in \mathcal{D}'(0,T;V^*) \) (i.e. the space of continuous linear maps \( C_0^\infty([0,T]) \to V \), cf. [37]) satisfying

\[
\begin{align*}
\partial_t u + Au &= f \quad \text{in } \mathcal{D}'(0,T;V^*), \\
 u(0) &= u_0 \quad \text{in } H.
\end{align*}
\]

By definition of Schwartz’ vector distribution space \( \mathcal{D}'(0,T;V^*) \), the first equation above means that for every scalar test function \( \varphi \in C_0^\infty([0,T]) \) the identity \( \langle u, -\varphi' \rangle + \langle Au, \varphi \rangle = \langle f, \varphi \rangle \) holds in \( V^* \).

As is well known, a wealth of parabolic Cauchy problems with homogeneous boundary conditions have been treated via triples \((H,V,a)\) and the \( \mathcal{D}'(0,T;V^*) \) framework in (3.2); cf. the work of Lions and Magenes [30], Tanabe [39], Temam [40], Amann [3] etc.

For problem (3.2), it is classical to seek solutions \( u \) in the Banach space

\[
X = L_2(0,T;V) \bigcap C([0,T];H) \bigcap H^1(0,T;V^*),
\]

\[
\|u\|_X = \left( \int_0^T \|u(t)\|^2 dt + \sup_{0 \leq t \leq T} |u(t)|^2 + \int_0^T (\|u(t)\|_{V^*}^2 + \|u'(t)\|_{V^*}^2) dt \right)^{1/2}.
\]

However, to point out a redundancy, note that the Banach space \( X \) can have its norm rewritten—using the Sobolev space \( H^1(0,T;V^*) = \{ u \in L_2(0,T;V^*) \mid \partial_t u \in L_2(0,T;V^*) \} \)—in the form

\[
\|u\|_X = \left( \|u\|_{L_2(0,T;V)}^2 + \sup_{0 \leq t \leq T} |u(t)|^2 + \|u\|_{H^1(0,T;V^*)}^2 \right)^{1/2},
\]

(3.4)

Here there is a well-known inclusion \( L_2(0,T;V) \cap H^1(0,T;V^*) \subset C([0,T];H) \) and an associated Sobolev inequality for vector functions \( \sup_{0 \leq t \leq T} |u(t)|^2 \leq (1 + \frac{C_2^2}{C_1^2}) \int_0^T \|u(t)\|^2 dt + \int_0^T \|u'(t)\|_{V^*}^2 dt \) (cf. [6]). Hence one can safely omit the space \( C([0,T];H) \) in (3.3) and remove \( \sup_{0 \leq t \leq T} \cdot \) from \( \| \cdot \|_X \). Similarly \( \int_0^T \|u(t)\|_{V^*}^2 dt \) is redundant in (3.3) because \( \| \cdot \|_{V^*} \leq C_2 \| \cdot \| \), so an equivalent norm on \( X \) is given by

\[
\|u\|_X = \left( \int_0^T \|u(t)\|^2 dt + \int_0^T \|u'(t)\|_{V^*}^2 dt \right)^{1/2}.
\]

(3.5)

Thus \( X \) is more precisely a Hilbertable space, as \( V^* \) is so. But the form given in (3.3) serves the purpose of emphasizing the properties of the solutions.

For (3.2) the following result is known from the work of Lions and Magenes [30]:

**Proposition 3.1.** Let \( V \) be a separable Hilbert space with \( V \subset H \) algebraically, topologically and densely, and let \( A \) denote the Lax–Milgram operator induced by a \( V \)-coercive, bounded sesquilinear form \( a(\cdot, \cdot) \) on \( V \), as well as its extension \( A \in \mathbb{B}(V, V^*) \). To given \( u_0 \in H \) and \( f \in L_2(0,T;V^*) \) there exists a
uniquely determined solution $u$ belonging to $X$, cf. (3.3), of the Cauchy problem (3.2).

The solution operator $R: (f, u_0) \mapsto u$ is bounded $L_2(0, T; V^*) \oplus H \rightarrow X$, and problem (3.2) is well posed.

The existence and uniqueness statements in Proposition 3.1 were mentioned after [30, Sect. 3.4.4], where they indicated an extension to the coercive case by means of a translation trick as in (2.2). (Details may e.g. be found in [27] together with an explicit proof of boundedness of $R$.)

As a note on the equation $u' + Au = f$ with $u \in X$, the continuous function $u: [0, T] \rightarrow H$ fulfills $u(t) \in V$ for a.e. $t \in [0, T]$, so the extension $A \in \mathcal{B}(V, V^*)$ applies for a.e. $t$. Hence $Au(t)$ belongs to $L_2(0, T; V^*)$.

Remark 3.2. It is recalled that $A = -A$ generates an analytic semigroup $e^{-zA}$ in both $\mathcal{B}(H)$ and $\mathcal{B}(V^*)$, each defined in $S_\theta$ for $\theta = \arccot(C_3/C_4) > 0$, cf. (3.1). For $V$-elliptic $A$, these known generation results were concisely shown in [6, Lem. 4] e.g. In the $V$-coercive case, this applies to $A = -(A+kI)$, so (2.2) gives $e^{-zA} = e^{kz} e^{-z(A+kI)}$ for $z \geq 0$; which then defines $e^{-zA}$ by the right-hand side for every $z \in S_\theta$. (An involved argument was given in [34, Thm. 7.2.7] for uniformly strongly elliptic differential operators.)

In addition to the existence and uniqueness stated in Proposition 3.1, it is useful to have an expression for the solution $u \in X$. In view of Remark 3.2, the canonical candidate is the Duhamel formula,

$$u(t) = e^{-tA} u_0 + \int_0^t e^{-(t-s)A} f(s) \, ds \quad \text{for } 0 \leq t \leq T. \tag{3.6}$$

This is known from analytic semigroup theory to produce a classical solution, cf. the books of Pazy [34, Cor. 4.3.3], Amann [3, Rem. 2.1.2] or Arendt [4], provided that the $H$-valued source term $f(t)$ is Hölder continuous of some order $\sigma \in ]0, 1[$,

$$\sup \{ |f(t) - f(s)| \cdot |t - s|^{-\sigma} \mid 0 \leq s < t \leq T \} < \infty. \tag{3.7}$$

This was exploited for the Hadamard well-posedness of (1.1) shown in [27].

But in the present framework, Duhamel’s formula (3.6) needs another justification, because the $u \in X$ are solutions in the distribution sense, and the data $f \in L_2(0, T; V^*)$ need not be Hölder continuous.

However, by virtue of Remark 3.2 and Proposition 2.2, the operators $e^{-tA}$ are always injective, even when $A$ is merely $V$-coercive, and this allows an easy extension of the classical integration factor technique, yielding a proof of (3.6) for all $u_0 \in H$ and $f \in L_2(0, T; V^*)$; cf. [27]. Thus one has

**Proposition 3.3.** The unique solution $u$ in $X$ provided by Proposition 3.1 is given by Duhamel’s formula (3.6), where each of the three terms belongs to $X$.

For the treatment of the heat equation further below, it is decisive to know that $u(t) \in D(A)$ in order to make sense of the Neumann boundary condition. The well-known way to obtain this is to work with the more regular
solutions that belong to the subspace $X_1 \subset X$ given by

$$X_1 = L_2(0,T;D(A)) \cap H^1(0,T;H),$$  \hspace{1cm} (3.8)

$$\|u\|_{X_1} = \left( \int_0^T \|u(t)\|^2_{D(A)} \, dt + \int_0^T |\partial_t u(t)|^2 \, dt \right)^{1/2}. \hspace{1cm} (3.9)$$

Here it is classical that the intersection $L_2(0,T;D(A)) \cap H^1(0,T;H)$ is embedded into $C([0,T];U)$ for the interpolation space $U = [D(A),H]_{1/2}$; cf. Theorem 3.1 in [30]. For the norm on this space, the reader is referred to the formula given in [30], which exploits the spectral decomposition of self-adjoint operators in terms of direct Hilbert integrals.

Along with the said inclusion, there is the estimate

$$\sup_{t \in [0,T]} \|u(t)\|_{[D(A),H]_{1/2}} \leq c \left( \|u\|_{L_2(0,T;D(A))} + \|u\|_{H^1(0,T;H)} \right), \hspace{1cm} (3.10)$$

which yields an equivalent norm on $X_1$ when added to $\|u\|_{X_1}$. Thus one obtains the well-known additional regularity property of solutions in $X_1$,

$$u \in C \left([0,T];[D(A),H]_{1/2}\right). \hspace{1cm} (3.11)$$

Furthermore, the initial data $u_0$ must therefore be given in $[D(A),H]_{1/2}$. That this space is compatible with the requirements that $u \in X_1$ is seen from (i) $\iff$ (ii) by taking $X = D(A), Y = H, \theta = 1/2$ and $a = u_0$ in the clarifying result [30, Thm. I.10.1]—although this had an unfortunate lack of quantifiers that may be remedied thus:

**Theorem 3.4.** [30] Let $X,Y$ be arbitrary complex Hilbert spaces with a continuous dense injection $X \hookrightarrow Y$. Then the following properties are equivalent for each $\theta \in ]0,1[$ and $a \in Y$:

(i) $a \in [X,Y]_{\theta}$;

(ii) $a = u(0)$ for some $u : \mathbb{R}_+ \to Y$

such that $t^{\theta - \frac{1}{2}} u \in L_2(\mathbb{R}_+;X), t^{\theta - \frac{1}{2}} u' \in L_2(\mathbb{R}_+;Y)$;

(iii) $t^{\theta - \frac{1}{2}} e^{tA} a - a \in L_2(\mathbb{R}_+;Y)$ for some generator $A$ in $Y$ of a $C_0$-semigroup satisfying $D(A) = X$ with equivalent norms.

In the affirmative case, (iii) is valid for every $C_0$-semigroup of the mentioned kind, with equivalent norms

$$\|a\|_{[X,Y]_{\theta}} \quad \text{and} \quad \left( \|a\|_Y^2 + \int_0^\infty t^{2(\theta - \frac{1}{2})} \left| \frac{e^{tA_0} a - a}{t} \right|^2 \, dt \right)^{1/2} \hspace{1cm} (3.12)$$

whenever $A_0$ is a selfadjoint generator fulfilling (iii) (these induce the interpolation spaces $[X,Y]_{\theta}$).

Indeed, one may readily see that the proof given in [30] (where $\alpha := \theta - \frac{1}{2}$) achieves via the Hardy–Littlewood–Polya inequality that (ii) $\implies$ (iii) for arbitrary $C_0$-semigroups fulfilling the criteria in (iii). Then the proof there shows that when $A$ satisfies (iii), then (ii) holds for some specific function $u$. Finally equivalence of (i) and (iii) is verified if one takes for $A_0$ any of the generators that can be utilised in the definition of $[X,Y]_{\theta}$, whereby the equivalence of the norms also is obtained.
Thus prepared, it is now possible to formulate the following regularity result, which is well known among experts, but here given in a form suitable for the purposes in Sects. 4 and 5:

**Theorem 3.5.** For the unique solution \( u \in X \) of the Cauchy problem (3.2) for given data \((f, u_0) \in Y\), the following additional properties are equivalent:

(i) \( f \in L_2(0, T; H) \), \( u_0 \in [D(A), H]_{1/2} \)

(ii) \( u \in L_2(0, T; D(A)) \cap C([0, T]; [D(A), H]_{1/2}) \cap H^1(0, T; H) \).

In the affirmative case all terms in Duhamel’s formula (3.6) belong to the space in (ii), and there is a constant \( c > 0 \) such that each solution \( u \) corresponding to data \((f, u_0)\) satisfying (i) will fulfil

\[
\int_0^T (|u(t)|^2 + |Au(t)|^2 + |u'(t)|^2) \, dt 
\leq c \left( \int_0^T |f(t)|^2 \, dt + \|u_0\|^2_{[D(A), H]_{1/2}} \right),
\]

(3.13)

If \( A^* = A \) in \( H \), the form domain \( V \) identifies with the space \([D(A), H]_{1/2}\) in (i), (ii) and (3.13) above.

This result was not formulated by Lions and Magenes [30] but undoubtedly known to them. For a brief discussion, note that (ii) \( \implies \) (i) is obvious from the mapping properties of \( \mathcal{P}(u) = (u' + Au, u(T)) \). As for (i) \( \implies \) (ii), it is clear from Theorem 3.4 that it suffices to treat the case \( u_0 = 0 \) for arbitrary \( f \) in \( L_2(0, T; H) \); this was done in [30, Sect. IV.3] using the general theory of Laplace transformation of vector distributions and convolutions \( \mathcal{G} \ast (A + \frac{\partial}{\partial t}) \) exposed in [36] or [37, Ch. VIII]. (One can also make a version based only on the semigroup \( e^{-tA} \).) Consequently also the last term \( e^{-tA}u_0 \) in (3.6) belongs to the space \( X_1 \) in (ii). The estimate is seen from the Closed Graph Theorem, since \( \mathcal{R} \) is the inverse of \( \mathcal{P} \), which is bounded from \( X_1 \) to \( L_2(0, T; H) \oplus [D(A), H]_{1/2} \).

The final fact that \( V = [D(A), H]_{1/2} \) holds is also standard when \( A^* = A \geq 0 \) (both spaces equal \( D(A^{1/2}) \) then, by spectral theory), and this applies in the general self-adjoint \( V \)-coercive case to \( A + k'I \) for \( k' > k \), which has the same domain and form domain as \( A \).

**Remark 3.6.** The estimate in Theorem 3.5 has been known for decades by experts in evolution equations. More generally, with \( H \) replaced by a UMD Banach space \( B \), and \( H^1(0, T; H) \) replaced by the \( L_p \)-Sobolev space \( W^1_p(0, T; B) \), \( 1 < p < \infty \), it has been a major theme, known as maximal regularity, since the 1980s to establish that \( u \mapsto u' + Au \) as a map \( W^1_p(0, T; B) \cap L_p(0, T; D(A)) \to L_p(0, T; B) \) has a bounded inverse. This was first shown by Dore and Venni [10] under suitable assumptions, but this generality is not needed here (though for \( p = 2 \) the choices \( B = V^* \) and \( B = H \) give back Proposition 3.1 and Theorem 3.5 with \( u_0 = 0 \)). The reader may consult [3,9], or for a survey of maximal regularity also [4, Sec. 5].
To complete this review of linear evolution equations, it is natural to introduce a notation for the full yield of the source term $f: [0, T] \rightarrow V^*$, namely the following vector that a priori belongs to $V^*$

$$y_f = \int_0^T e^{-(T-t)A} f(t) \, dt. \quad (3.14)$$

In fact $y_f \in H$ as by Duhamel’s formula it equals the final state of a solution in $C([0, T]; H)$ of a Cauchy problem having $u_0 = 0$. Moreover, Theorem 3.5 shows that $y_f \in [D(A), H]_{1/2}$ whenever $f \in L^2(0, T; H)$.

For $t = T$ formula (3.6) now obviously yields a bijection $u(0) \leftrightarrow u(T)$ between the initial and terminal states (for fixed $f$), as one can solve for $u_0$ by means of the inverse $e^{TA}$. Indeed, all terms in (3.6) belong to $C([0, T]; H)$, so evaluation at $t = T$ gives $u(T) = e^{-TA}u(0) + y_f$; cf. (3.14). This is a flow map

$$u(0) \mapsto u(T). \quad (3.15)$$

Invoking injectivity of $e^{-TA}$ once again, and that (3.6) implies $u(T) - y_f = e^{-TA}u(0)$, which clearly belongs to $D(e^{TA})$, the flow is inverted by

$$u(0) = e^{TA}(u(T) - y_f). \quad (3.16)$$

In other words, not only are the solutions in $X$ to $u' + Au = f$ parametrised by the initial states $u(0)$ in $H$ (for fixed $f$) according to Proposition 3.1, but also the final states $u(T)$ are parametrised by the $u(0)$.

For one thing, this means that the differential equation $u' + Au = f$ has the backward uniqueness property regardless of whether $A$ itself is injective or not: that is, $u(t) = 0$ holds in $H$ for all $t \in [0, T]$ if $u(T) = 0$. This property has been studied for decades in various situations, cf. Remark 2.4 and Remark 6.3 below.

Secondly, the remarks on the above flow lead to the isomorphic well-posedness in the next section.

4. Final Value Problems with Coercive Generators

In the framework of Sect. 3, the general final value problem is, for given data $f \in L^2(0, T; V^*)$ and $u_T \in H$, to determine the $u \in \mathcal{D}'(0, T; V)$ satisfying

$$\begin{align*}
\partial_t u + Au &= f \quad \text{in } \mathcal{D}'(0, T; V^*), \\
u(T) &= u_T \quad \text{in } H.
\end{align*} \quad (4.1)$$

The point of departure for this is to make a comparison of (4.1) with the corresponding Cauchy problem for the equation $u' + Au = f$, cf. (3.2). Thus it would be natural to seek solutions $u$ in the same space $X$ in (3.3). As shown first for the $V$-elliptic case in [6], this is possible only for data $(f, u_T)$ subjected to certain compatibility conditions, which have a special form for final value problems.

The compatibility condition is formulated by means of the inverse $e^{tA}$ that enters the theory through its domain $D(e^{tA})$, to which Proposition 2.5 applies. Although this identifies with the range $R(e^{-tA})$ in the algebraic sense,
it has the virtue of being a Hilbert space under the graph norm \( \|u\| = (|u|^2 + |e^{tA}u|^2)^{1/2} \).

The remarks on \( y_f \) made after (3.14) make it clear that in the following general result the difference in (4.3) is a priori a member of \( H \). The theorem relaxes the assumption of \( V \)-ellipticity in [6,7] to \( V \)-coercivity. Because of its relative novelty, it is given here with details for the reader’s sake.

**Theorem 4.1.** [27] Let \( A \) be a \( V \)-coercive Lax–Milgram operator defined from a triple \((H,V,a)\) as above. The abstract final value problem (4.1) then has a solution \( u(t) \) belonging to the space \( X \) in (3.3) if, and only if, the data \((f,u_T)\) belong to the Banach space \( Y \), which is the subspace

\[
Y \subset L_2(0,T;V^*) \oplus H
\]

(4.2)
defined by the compatibility condition

\[
u_T - \int_0^T e^{-(T-t)A} f(t) \, dt \in D(e^{TA}).\]

(4.3)

In the affirmative case, the solution \( u \) is uniquely determined in \( X \) and

\[
\|u\|_X \leq c \left( |u_T|^2 + \int_0^T \|f(t)\|^2 \, dt \right)
\]

\[
+ \left| e^{TA} \left( u_T - \int_0^T e^{-(T-t)A} f(t) \, dt \right) \right|^2 \right)^{1/2} =: c \|(f,u_T)\|_Y,
\]

(4.4)

whence the solution operator \((f,u_T) \mapsto u\) is continuous \( Y \to X \). Moreover,

\[
u(t) = e^{-tA} e^{TA} \left( u_T - \int_0^T e^{-(T-t)A} f(t) \, dt \right) + \int_0^t e^{-(t-s)A} f(s) \, ds,
\]

(4.5)

where all terms belong to \( X \) as functions of \( t \in [0,T] \), and the difference in (4.3) equals \( e^{-TA}u(0) \) in \( H \).

**Proof.** If (4.1) is solved by \( u \in X \), then \( u(T) = u_T \) is reached from the unique initial state \( u(0) \) in (3.16). But the argument for (3.16) showed that \( u_T - y_f = e^{-TA}u(0) \in D(e^{TA}) \), so (4.3) is necessary.

Given data \((f,u_T)\) fulfilling (4.3), then \( u_0 = e^{TA}(u_T - y_f) \) is a well-defined vector in \( H \), so Proposition 3.1 yields a function \( u \in X \) solving \( u' + Au = f \) and \( u(0) = u_0 \). By (3.15), this \( u(t) \) clearly has final state \( u(T) = e^{-TA} \left( u_T - y_f \right) + y_f = u_T \), hence satisfies both equations in (4.1). Thus (4.3) suffices for solvability.

In the affirmative case, (4.5) results for any solution \( u \in X \) by inserting formula (3.16) for \( u(0) \) into (3.6). Uniqueness of \( u \) in \( X \) is seen from the right-hand side of (4.5), where all terms depend only on the given \( f,u_T,A \) and \( T > 0 \). That each term in (4.5) is a function belonging to \( X \) was seen in Proposition 3.3.
Moreover, the solution $u$ can be estimated in $X$ by substituting the expression (3.16) for $u_0$ into the inequality that expresses the boundedness of $R$ in Proposition 3.1,

$$
\|u\|_X^2 \leq c \left( |u_0|^2 + \int_0^T \|f(s)\|_*^2 \, ds \right) \leq c (|e^{TA}(u_T - y_f)|^2 + \|f\|_{L^2(0,T;V^*)}^2).
$$

(4.6)

Here one may add $|u_T|^2$ on the right-hand side to arrive at the expression for $\|(f, u_T)\|_Y^2$ in (4.4).

\[\square\]

Remark 4.2. Clearly $\mathcal{P}u = (u' + Au, u(T))$ is bounded $X \to Y$; cf. the proofs and definitions. The statement in Theorem 4.1 means that the solution operator $R(f, u_T) = u$ is well defined, bounded and satisfies $PR = I$; but by the uniqueness also $RP = I$ holds. Hence $R$ is a linear homeomorphism $Y \to X$.

The norm on the data space $Y$ in (4.4) is seen at once to be the graph norm of the composite map

$$(f, u_T) \mapsto u_T - y_f \mapsto e^{TA}(u_T - y_f)$$

(4.7)

that in terms of the first part $\Phi(f, u_T) = u_T - y_f$ is the operator

$$L_2(0,T;V^*) \oplus H \xrightarrow{\Phi} H \xrightarrow{e^{TA}} H.$$  

(4.8)

In fact, the solvability criterion (4.3) is met if and only if $e^{TA}\Phi$ is defined at $(f, u_T)$, so the data space $Y$ is its domain. Being an inverse, $e^{TA}$ is a closed operator in $H$; hence $e^{TA}\Phi$ is closed, and $Y = D(e^{TA}\Phi)$ is complete. Now, since in (4.4) the Banach space $V^*$ is Hilbertable, so is $Y$.

In control theoretic terms, the role of $e^{TA}\Phi$ is also for $V$-coercive $A$ to provide the unique initial state $u(0) = e^{TA}\Phi(f, u_T) = e^{TA}(u_T - y_f)$ steered by $f$ to the final state $u(T) = u_T$ at time $T$; cf. the Duhamel formula (3.6).

Criterion (4.3) is a generalised compatibility condition on data $(f, u_T)$; such conditions have long been known in the theory of parabolic problems, cf. Remark 4.6. The presence of $e^{-(T-t)A}$ and the integral over $[0,T]$ makes (4.3) non-local in both space and time. This aspect is further complicated by the reference to the abstract domain $D(e^{TA})$, which for larger final times $T$ typically gives increasingly stricter conditions:

Proposition 4.3. If the spectrum $\sigma(A)$ of $A$ is not contained in the strip $\{ z \in \mathbb{C} \mid -k \leq \Re z \leq k \}$, whereby $k$ is the constant from (3.1), then the domains $D(e^{tA})$ form a strictly descending chain, that is,

$$H \supseteq D(e^{tA}) \supseteq D(e^{t'A}) \quad \text{for } 0 < t < t'.$$

(4.9)

This results from the injectivity of $e^{-tA}$ via known facts for semigroups reviewed in [6, Thm. 11] (with reference to [34]), and the arguments given for $k = 0$ in [6, Prop. 11] apply mutatis mutandis.

The regularity result in Theorem 3.5 gives rise to a well-posedness result further below, which concerns some more regular data and solution spaces.
For convenience we shall for an unbounded operator $S: X \rightarrow Y$ between two general Banach spaces $X, Y$ and any given subspace $U \subset Y$ adopt the notation

$$D(S; U) = \{ x \in D(S) \mid Sx \in U \} = D(S) \cap S^{-1}(U). \quad (4.10)$$

This is the domain of the composite map $I_US: D(S) \rightarrow Y$ whereby $I_U$ denotes the inclusion map $U \rightarrow Y$. When $S$ has closed graph and $I_U$ is bounded with respect to a complete norm $\| \cdot \|_U$ on $U$, then $D(S; U)$ is complete with respect to the modified graph norm $\|x\|_{D(S; U)} = \|x\|_X + \|Sx\|_U$.

This applies especially to the inverse operator $e^{TA}$, for which we have $D(e^{TA}; U) = e^{-TA}(U)$ when $U \subset H$.

As a more regular solution space for (4.1) one may use (3.8) ff.,

$$X_1 = L_2(0, T; D(A)) \cap C([0, T]; [D(A), H]_{1/2}) \cap \overline{H^1}(0, T; H), \quad (4.11)$$

$$\|u\|_{X_1}^2 = \int_0^T \|u(t)\|^2_{D(A)} dt + \sup_{t \in [0, T]} \|u(t)\|^2_{[D(A), H]_{1/2}} + \int_0^T |\partial_t u(t)|^2 dt. \quad (4.12)$$

The corresponding data space $Y_1$ is given as

$$Y_1 = \left\{ (f, u_T) \in L_2(0, T; H) \oplus [D(A), H]_{1/2} \bigg| \right.\left. u_T - y_f \in D(e^{TA}; [D(A), H]_{1/2}) \right\}, \quad (4.13)$$

$$\|(f, u_T)\|_{Y_1}^2 = \int_0^T |f(t)|^2 dt + \|u_T\|^2_{[D(A), H]_{1/2}} + \|e^{TA}(u_T - y_f)\|^2_{[D(A), H]_{1/2}}. \quad (4.14)$$

It was noted above that the yield of the source term $y_f = \int_0^T e^{-(T-t)A} f(t) dt$ a priori belongs to $[D(A), H]_{1/2}$ for the stipulated $f$ in $L_2(0, T; H)$.

It is an exercise to show that $Y_1$ is a Banach space, for if $(f_n, u_{T,n})$ is a Cauchy sequence in $Y_1$, then $f_n, u_{T,n}$ and $e^{TA}(u_{T,n} - y_{f_n})$ converge to some $f$ in $L_2(0, T; H)$ and $u_T, v$ in $[D(A), H]_{1/2}$, respectively; for reasons of continuity, $y_{f_n} \rightarrow y_f$ so that $u_{T,n} - y_{f_n} \rightarrow u_T - y_f$ in $H$ for $n \rightarrow \infty$; as $e^{TA}$ is closed in $H$, it follows that $u_T - y_f$ belongs to $D(e^{TA})$ with $e^{TA}(u_T - y_f) = v$; finally, as $v \in [D(A), H]_{1/2}$, the vector $u_T - y_f$ fulfills the condition in (4.13). Hence $(f_n, u_{T,n})$ converges in the norm of $Y_1$ to the element $(f, u_T)$ in $Y_1$, as desired.

To compare with Theorem 4.1, note that there clearly are continuous embeddings

$$X_1 \hookrightarrow X, \quad Y_1 \hookrightarrow Y. \quad (4.15)$$

Thus prepared, it is now possible to give a concise proof of the following novel result, which is a companion to Theorem 4.1 in which the solutions have regularity properties that (instead of relating to the extension $A \in \mathcal{B}(V, V^*)$ as in Theorem 4.1) are more closely connected to the unbounded operator $A$ in $H$:
Theorem 4.4. Let $A$ be a $V$-coercive Lax–Milgram operator defined from a triple $(H, V, a)$ as above, and let $(f, u_T) \in L^2(0, T; H) \oplus [D(A), H]_{1/2}$ be given. Then the abstract final value problem (4.1) has a solution $u(t)$ belonging to the space $X_1$ in (4.11), if and only if the data $(f, u_T)$ belong to the subspace $Y_1$, that is,

$$u_T - y_f \in D(e^{TA}; [D(A), H]_{1/2}).$$

(4.16)

In the affirmative case, the solution $u$ is uniquely determined in $X_1$ and it fulfills $\|u\|_{X_1} \leq c\|(f, u_T)\|_{Y_1}$, whence the solution operator $R_1: (f, u_T) \mapsto u$ is continuous $Y_1 \to X_1$. Moreover,

$$u(t) = e^{-tA}e^{TA} \left( u_T - \int_0^T e^{-(T-t)A} f(t) \, dt \right) + \int_0^t e^{-(t-s)A} f(s) \, ds,$$

(4.17)

where all terms belong to $X_1$ as functions of $t \in [0, T]$, and the difference in (4.16) equals $e^{-T_A}u(0)$, which belongs to $D(e^{TA}; [D(A), H]_{1/2}) = e^{-T_A}([D(A), H]_{1/2})$.

Proof. If (4.1) is solved by $u \in X_1$, then $u(T) = u_T$ is by (3.16) reached from the unique initial state $u(0)$, which is in $[D(A), H]_{1/2}$ since $u$ as a member of $X_1$ is continuous $[0, T] \mapsto [D(A), H]_{1/2}$. But then we have $u_T - y_f = e^{-T_A}u(0) \in e^{-T_A}([D(A), H]_{1/2})$, whence necessity of (4.16) and the last claim is covered.

Given $(f, u_T) \in Y_1$, there is first of all by Theorem 4.1 and (4.15) a unique solution $u \in X$ satisfying (4.17). Secondly, (4.16) entails that the implicit initial state $u(0)$ belongs to $[D(A), H]_{1/2}$, and since $f$ is given in $L^2(0, T; H)$, Theorem 3.5 then yields the stronger conclusion that $u$ as well as all terms in (4.17) belong to the subspace $X_1 \subset X$. In view of (3.10), the stated estimate $\|u\|_{X_1} \leq c\|(f, u_T)\|_{Y_1}$ may be deduced from the one in Theorem 3.5 by replacing $u_0$ by the expression for $u(0)$, and adding $\|u_T\|_{[D(A), H]_{1/2}}^2$.

It is noted that the above proof of existence of a solution in $X_1$ was conducted via an a posteriori estimate of the solution $u$ belonging to the larger space $X$. Therefore Theorem 4.4 is basically a regularity result.

To extend the control and operator theoretic remarks from $Y$ to $Y_1$ in (4.13), one may as a variant of (4.8) consider the unbounded composite operator

$$L^2(0, T; H) \oplus [D(A), H]_{1/2} \xrightarrow{\Phi} [D(A), H]_{1/2} \overset{e^{TA}}{\longrightarrow} [D(A), H]_{1/2}.$$

(4.18)

This map $e^{TA} \Phi$ is defined at $(f, u_T)$ if and only if (4.16) is met, so $Y_1$ is its domain; and clearly $\|\cdot\|_{Y_1}$ is its graph norm. To obtain completeness of $Y_1$ in this set-up, it therefore suffices to show that the above $e^{TA} \Phi$ is closed; which by the continuity of $\Phi(f, u_T) = u_T - y_f$ results from closedness of the restriction in (4.18) of $e^{TA}$ to an operator in $[D(A), H]_{1/2}$, but that
follows at once from its closedness in $H$. Since $[D(A),H]_{1/2}$ has a Hilbert space structure (being the graph normed domain $D(\Lambda^{1/2})$ for a (non-unique) positive selfadjoint operator $\Lambda$ in $H$, cf. [30, Sect. I.2]), the data space $Y_1$ is also Hilbertable.

In analogy with Remark 4.2, it is seen that $u \mapsto (u' + Au, u(T))$ also gives a bounded operator $\mathcal{P}_1 : X_1 \to Y_1$ and that the solution operator $\mathcal{R}_1 : Y_1 \to X_1$ is everywhere defined and bounded according to Theorem 4.4; and moreover that $\mathcal{P}_1\mathcal{R}_1 = I$ and $\mathcal{R}_1\mathcal{P}_1 = I$ hold on $Y_1$ and $X_1$, respectively. This can be summed up thus:

**Corollary 4.5.** The final value problem (4.1) generated by the Lax–Milgram operator $A$, defined from a $V$-coercive triple $(H,V,a)$, is isomorphically well posed in the pair of spaces $(X,Y)$ as well as in $(X_1,Y_1)$.

In addition to the above isomorphic well-posedness, it is remarked that the Duhamel formula (3.6) also shows that $u(T)$ has two radically different contributions, even if $A$ has nice properties.

First, for $t = T$ the integral in (3.6) amounts to $y_f$, which can be anywhere in $H$, as $f \mapsto y_f$ is a surjection $y_f : L_2(0,T;V^*) \to H$. This was shown for $k = 0$ via the Closed Range Theorem in [6, Prop. 5, and more generally the surjectivity follows from this case since $e^{-(T-s)A}f(s) = e^{-(T-s)(A+kI)}e^{(T-s)k}f(s)$ in the integrand, whereby $A+kI$ is $V$-elliptic and $f \mapsto e^{(T-s)k}f$ is a bijection on $L_2(0,T;V^*)$.

Secondly, in (3.6) the first term $e^{-tA}u(0)$ solves $u' + Au = 0$, and for $u(0) \neq 0$ there is for $V$-elliptic $A$ the precise property in non-selfadjoint dynamics that the “height” function $h(t) = \lVert e^{-tA}u(0) \rVert$ is

- strictly positive ($h > 0$),
- strictly decreasing ($h' < 0$), and
- strictly convex ($\Leftarrow h'' > 0$). \hfill (4.19)

Whilst this holds if $A$ is self-adjoint or normal, it was emphasized in [6] that it suffices that $A$ is just hyponormal (i.e., $D(A) \subset D(A^*)$ and $|Ax| \geq |A^*x|$ for $x \in D(A)$, cf. Janas [23]). Recently this was followed up in [25,28], where the (in the context) stronger logarithmic convexity of $h(t)$ was proved equivalent to the property weaker than hyponormality of $A$ that for $x \in D(A^2)$,

$$2(\Re(Ax|x))^2 \leq \Re(A^2x|x)^2 + |Ax|^2|x|^2.$$ \hfill (4.20)

For $V$-coercive $A$ the strict positivity $h > 0$ also holds, by injectivity of $e^{-tA}$. But the strict decay need not extend to such $A$ (e.g. $h$ is constant if $u_0$ is a constant function in $\Omega$ for $A = -\Delta_N$; cf. Sect. 5). However, most conveniently, the strict convexity in (4.19) can simply be replaced by log-convexity for coercive $A$.

Indeed, the characterisation in [25, Lem. 2.2] or [28] of the log-convex $C^2$-functions $f(t)$ on $[0,\infty[$ as the solutions of the differential inequality $f'' \cdot f \geq (f')^2$ and the resulting criterion for $A$ in (4.20) apply verbatim to the coercive case. Hereby the differential calculus in Banach spaces is exploited.
in a classical derivation of the formulae for \( u(t) = e^{-tA}u(0) \),

\[
\begin{align*}
  h'(t) &= -\frac{\Re(Au(t) | u(t))}{|u(t)|}, \\
  h''(t) &= \frac{\Re(A^2 u(t) | u(t)) + |Au(t)|^2}{|u(t)|} - \frac{(\Re(Au(t) | u(t)))^2}{|u(t)|^3}.
\end{align*}
\] (4.21)

But it is due to the strict positivity \(|u(t)| > 0\) for \( t \geq 0 \) in the denominators that the expressions make sense, so Proposition 2.2 is also crucial here. The singularity of \(|\cdot|\) at the origin likewise poses no problems for differentiation of \( h(t) \). So perhaps the natural formulae for \( h', h'' \) were first made rigorous in [25].

However, the stiffness intrinsic to strict convexity corresponds well (when applicable) with the fact that \( u(T) = e^{-TA}u(0) \) also for coercive \( A \) is confined to a very small dense space, as by the analyticity

\[ u(T) \in \bigcap_{n \in \mathbb{N}} D(A^n). \] (4.22)

For \( u' + Au = f \neq 0 \), the possible \( u_T \) will hence be a sum of an arbitrary \( y_f \in H \) and the stiff term \( e^{-TA}u(0) \). Thus \( u_T \) can be prescribed in the affine space \( y_f + D(e^{TA}) \). As the vector \( y_f \neq 0 \) will shift \( D(e^{TA}) \subset H \) in an arbitrary direction, \( u(T) \) can be expected anywhere in \( H \) (unless \( y_f \in D(e^{TA}) \) is known). So neither (4.22) nor \( u(T) \in D(e^{TA}) \) can be expected to hold if \( y_f \neq 0 \)—not even if \(|y_f|\) is much smaller than \(|e^{-TA}u(0)|\). Thus it seems fruitful for final value problems to consider inhomogeneous equations from the outset.

Remark 4.6. Grubb and Solonnikov treated a large class of initial-boundary problems of parabolic pseudo-differential equations in [19] and worked out compatibility conditions characterising the well-posedness in full scales of anisotropic \( L_2 \)-Sobolev spaces (such conditions have a long history in the differential operator case, cf. work of Lions and Magenes [30] and Ladyzhenskaya, Solonnikov and Ural’ceva [31]). Their conditions are local at the curved corner \( \{0\} \times \Gamma \), except for half-integer values of the smoothness \( s \) that were shown to require coincidence, which is expressed via integrals over the Cartesian product of the two boundaries \( \{0\} \times \Omega \) and \( \{0\} \times T' \times \Gamma \). While in [19] the conditions address the regularity, condition (4.3) in Theorem 4.1 and (4.16) in Theorem 4.4 pertain to the existence of solutions in the respective spaces.

Remark 4.7. Recently Almog, Grebenkov, Helffer, Henry [2, 15, 16] studied the complex Airy operator \(-\Delta + i x_1\) via triples \((H, V, a)\), leading to Dirichlet, Neumann, Robin and transmission boundary conditions, in bounded and unbounded regions. To improve earlier remarks, Theorem 4.1 is expected to apply to their Dirichlet realisations while Theorem 4.4 would pertain to the Neumann and Robin realisations, leading to final value problems for those of their realisations that satisfy the (strong) coercivity in (3.1). As \(-\Delta + i x_1\) has empty spectrum on \( \mathbb{R}^n \), as shown by Herbst [21], it remains to be clarified for which of the regions in [2, 15, 16] there is a strictly descending chain of domains as in (4.9).
5. The Heat Problem with the Neumann Condition

In the sequel \(\Omega\) stands for a \(C^\infty\) smooth, open bounded set in \(\mathbb{R}^n, n \geq 1\) as described in [18, App. C]. In particular \(\Omega\) is locally one side of its boundary \(\Gamma = \partial \Omega\). The problem is then to characterise the \(u(t, x)\) such that

\[
\begin{aligned}
\partial_t u(t, x) - \Delta u(t, x) &= f(t, x) & \text{in } & ]0, T[ \times \Omega, \\
\gamma_1 u(t, x) &= 0 & \text{on } & ]0, T[ \times \Gamma, \\
r_T u(x) &= u_T(x) & \text{at } & \{T\} \times \Omega.
\end{aligned}
\] (5.1)

While \(r_T u(x) = u(T, x)\), the Neumann trace on \(\Gamma\) is written in the operator notation \(\gamma_1 u = (\nu \cdot \nabla u)|_{\Gamma}\), whereby \(\nu\) is the unit outward pointing normal vector field. Similarly \(\gamma_1\) is used for traces on \(]0, T[ \times \Gamma\).

Moreover, \(H^m(\Omega)\) denotes for \(m \in \mathbb{N}_0\) the Sobolev space normed by

\[
\|u\|_m = \left(\sum_{|\alpha| \leq m} \int_{\Omega} |\partial^\alpha u|^2 \, dx\right)^{1/2},
\]

which up to equivalent norms equals the space \(H^m(\Omega)\) of restrictions to \(\Omega\) of \(H^m(\mathbb{R}^n)\) endowed with the infimum norm, that also is denoted by \(\| \cdot \|_m\). This is useful since the dual space of \(H^m(\Omega)\) has an identification with the closed subspace of \(H^{-m}(\mathbb{R}^n)\) that is given by the support condition in

\[
H_0^m(\Omega) = \left\{ u \in H^{-m}(\mathbb{R}^n) \mid \text{supp } u \subset \overline{\Omega}\right\}. \quad (5.2)
\]

For these standard facts in functional analysis the reader may consult [22, App. B.2]. Chapter 6 and (9.25) in [18] could also be references for this and basics on boundary value problems; cf. also [13, 35].

The main result in Theorem 4.1 applies to (5.1) for \(V = H^1(\Omega), H = L_2(\Omega)\) and \(V^* \simeq H_0^{-1}(\Omega)\), for which there are inclusions

\[
H^1(\Omega) \subset L_2(\Omega) \subset H_0^{-1}(\Omega), \quad (5.3)
\]

when \(g \in L_2(\Omega)\) via extension by zero outside of \(\Omega\), denoted by \(e_\Omega\), is identified with \(e_\Omega g\) belonging to \(H_0^{-1}(\Omega)\). (This of course modifies the usual identification \(L_2(\Omega)^* \simeq L_2(\Omega)\) slightly, but \(e_\Omega g\) is the function on \(\mathbb{R}^n\) at which the infimum in the norm \(\|g\|_0\) is attained.) The Dirichlet form

\[
s(u, v) = \sum_{j=1}^n \int_{\Omega} \partial_j u \partial_j v \, dx \quad (5.4)
\]

fulfils \(|s(v, w)| \leq \|v\|_1 \|w\|_1\), and the coercivity (3.1) holds for \(C_4 = 1, k = 1\) since \(s(v, v) = \|v\|_1^2 - \|v\|_0^2\).

The induced Lax–Milgram operator is the Neumann realisation \(-\Delta_N\), which is selfadjoint due to the symmetry of \(s\) and has its domain given by \(D(\Delta_N) = \left\{ u \in H^2(\Omega) \mid \gamma_1 u = 0 \right\}\). This is a non-trivial classical result (cf. the remarks prior to Theorem 4.28 in [18], or Section 11.3 ff. there; or [35]). Thus the homogeneous boundary condition is imposed via the condition \(u(t) \in D(\Delta_N)\) for \(t\) in \(]0, T[\) a.e.

By the \(H^1\)-coercivity, \(-A = \Delta_N\) generates an analytic semigroup of injections \(e^{z\Delta_N}\) in \(\mathcal{B}(L_2(\Omega))\), cf. Proposition 2.2 and Remark 3.2, and like before \(e^{-t\Delta_N} := (e^{t\Delta_N})^{-1}\). The extension \(\hat{\Delta} \in \mathcal{B}(H^1(\Omega), H_0^{-1}(\Omega))\) induces
the analytic semigroup $e^{z\hat{\Delta}}$ defined for $z \in S_{\pi/4}$ on $H_{-1}^0(\Omega)$, and as observed in [27], if not before, it can be explicitly described:

**Lemma 5.1.** The action of the bounded extension $\hat{\Delta} : H^1(\Omega) \to H_{-1}^0(\Omega)$ of $\Delta_N$ is given by

\[
\hat{\Delta} u = \text{div}(e_\Omega \text{grad} u) \quad \text{for } u \in H^1(\Omega),
\]

\[
\hat{\Delta} u = e_\Omega (\Delta u) - (\gamma_1 u) dS \quad \text{for } u \in H^2(\Omega),
\]

whereby $dS \in D'(\mathbb{R}^n)$ denotes the surface measure at $\Gamma$.

**Proof.** When $w \in H^1(\mathbb{R}^n)$ coincides with $v$ in $\Omega$, for given $u, v \in H^1(\Omega)$, then (5.4) gives (5.5) as follows,

\[
\langle -\hat{\Delta} u, v \rangle = s(u, v) = \sum_{j=1}^n \int_{\mathbb{R}^n} e_\Omega (\partial_j u) \cdot \partial_j w \, dx
\]

\[
= \sum_{j=1}^n \langle -\partial_j (e_\Omega \partial_j u), w \rangle_{H^{-1}(\mathbb{R}^n) \times H^1(\mathbb{R}^n)}
\]

\[
= \langle -\text{div}(e_\Omega \text{grad} u), v \rangle_{H_{-1}^0(\Omega) \times H^1(\Omega)}.
\]

To show (5.6), one may recall that $\partial_j (u \chi_\Omega) = (\partial_j u) \chi_\Omega - \nu_j (\gamma_0 u) dS$ holds for $u \in C^1(\mathbb{R}^n)$ when $\chi_\Omega$ denotes the characteristic function of $\Omega$, and $\gamma_0$ stands for the restriction to $\Gamma$; cf. the proof in [22, Thm. 3.1.9]. Replacing $u$ by $\partial_j u$ for some $u \in C^2(\Omega)$, and using that $\nu(x)$ is a smooth vector field around $\Gamma$, we get $\partial_j (e_\Omega \partial_j u) = e_\Omega (\partial^2_j u) - (\gamma_0 \nu_j \partial_j u) dS$, which after summation over $j$ yields (5.6) for such $u$ in view of (5.5). The formula then extends to all $u \in H^2(\Omega)$ by continuity and density of $C^2$. \qed

In (5.6) the last term vanishes for $u \in D(\Delta_N)$ as $\gamma_1 u = 0$; whence for such $u$, clearly $\hat{\Delta} u = \text{div}(e_\Omega \text{grad} u)$ identifies in $\Omega$ with the $L_2$-function $\Delta u$. However, for general $u$ in the form domain $H^1(\Omega)$, the terms on the right hand side of (5.6) do not make sense.

To account for the consequences of Theorem 4.1 for (5.1), note that (3.3) gives rise to the solution space

\[
X_0 = L_2(0, T; H^1(\Omega)) \cap C([0, T]; L_2(\Omega)) \cap H^1(0, T; H_{-1}^0(\Omega)),
\]

\[
\|u\|_{X_0} = \left( \int_0^T \|u(t)\|_{H^1(\Omega)}^2 \, dt \right)^{1/2}
\]

\[+ \sup_{t \in [0, T]} \int_\Omega |u(x, t)|^2 \, dx + \int_0^T \|\partial_t u(t)\|^2_{H_{-1}^0(\Omega)} \, dt \right)^{1/2}.
\]

(5.8)
The corresponding data space is here given in terms of the vector \( y_f = \int_0^T e^{(T-t)} \Delta f(t) \, dt \) from (3.14) as

\[
Y_0 = \left\{ (f, u_T) \in L_2(0, T; H_0^1(\Omega)) \oplus L_2(\Omega) \left| \begin{array}{c} u_T - y_f \in D(e^{-T\Delta_N}) \end{array} \right. \right\},
\]

\[
\| (f, u_T) \|_{Y_0} = \left( \int_0^T \| f(t) \|_{H_0^1(\Omega)}^2 \, dt \right)^{1/2} + \int_{\Omega} \left( |u_T(x)|^2 + |e^{-T\Delta_N}(u_T - y_f)(x)|^2 \right) \, dx.
\]

(5.9)

Using this framework, as in [27, Thm. 4.1], the above Theorem 4.1 at once gives the following (partial) result for (5.1), which further below may serve as a reference point for the reader:

**Theorem 5.2.** Let \( A = -\Delta_N \) be the Neumann realization of the Laplacian in \( L_2(\Omega) \) and having its bounded extension \( H^1(\Omega) \to H_0^1(\Omega) \) given by \( \Delta = -\text{div}(e_\Omega \text{grad } \cdot) \). Whenever \( f \in L_2(0, T; H_0^1(\Omega)) \), \( u_T \in L_2(\Omega) \), there exists a solution \( u \) in \( X_0 \) of

\[
\partial_t u - \text{div}(e_\Omega \text{grad } u) = f, \quad r_T u = u_T
\]

(5.10)

if and only if the data \( (f, u_T) \) belong to \( Y_0 \), i.e. if and only if

\[
u_T - \int_0^T e^{(T-s)} \Delta f(s) \, ds \quad \text{belongs to} \quad D(e^{-T\Delta_N}) = R(e^{T\Delta_N}).
\]

(5.11)

In the affirmative case, \( u \) is uniquely determined in \( X_0 \) and satisfies the estimate \( \| u \|_{X_0} \leq c \| (f, u_T) \|_{Y_0} \). It is given by the formula, in which all terms belong to \( X_0 \),

\[
u(t) = e^{t\Delta_N} e^{-T\Delta_N} \left( u_T - \int_0^T e^{(T-t)} \Delta f(t) \, dt \right) + \int_0^t e^{(t-s)} \Delta f(s) \, ds.
\]

(5.12)

Moreover, in (5.11) the difference equals \( e^{T\Delta_N} u(0) \) in \( L_2(\Omega) \).

Besides the deplorable fact that \( \Delta = \text{div}(e_\Omega \text{grad } \cdot) \) appears in the differential equation, instead of \( \Delta \), there is also no information on the boundary condition. However, if in addition (3.7) is fulfilled, the Hölder continuity yields \( u(t) \in D(\Delta_N) \) for \( t > 0 \), so \( \gamma_1 u = 0 \) is fulfilled and \( \Delta u \) identifies with \( \Delta u \); whence one has

**Corollary 5.3.** [27] If \( u_T \in L_2(\Omega) \) and \( f : [0, T] \to L_2(\Omega) \) is Hölder continuous of order \( \sigma \in [0, 1[ \), and if \( u_T - y_f \) fulfills (5.11), then the homogeneous Neumann heat conduction final value problem (5.1) has a uniquely determined solution \( u \) in \( X_0 \), satisfying \( u(t) \in \{ u \in H^2(\Omega) \left| \gamma_1 u = 0 \right. \} \) for \( t > 0 \), and depending continuously on \( (f, u_T) \) in \( Y_0 \). Hence problem (5.1) is well posed in the sense of Hadamard.
This result is less than ideal, of course, since Hölder continuity is not available for the general source terms \( f \) in \( Y_0 \), and so the corollary pertains only to some dense, but non-closed subspace of \( Y_0 \), which is unsatisfying when, as stated, stability only refers to the norm on the full data space \( Y_0 \).

It is therefore the purpose of this paper to obtain isomorphic well-posedness of (5.1) in other, more suitable spaces \( X_1, Y_1 \). The point of departure is the general well-posedness result in Theorem 4.4, whereby the interpolation space satisfies \([D(\Delta_N), L_2(\Omega)]_{1/2} = V = H^1(\Omega)\) here, as \( \Delta_N \) is self-adjoint in \( L_2(\Omega) \).

In view of this, (4.10) yields for the inverse \( e^{-T\Delta_N} \) the simplification \( D(e^{-T\Delta_N}; H^1(\Omega)) = e^{T\Delta_N}(H^1(\Omega)) \). The data space \( Y_1 \) in (4.13) is therefore taken, in terms of \( y_f = \int_0^T e^{(T-t)\Delta_N} f(t) \, dt \) belonging to \( H^1(\Omega) \), as

\[
Y_1 = \left\{ (f, u_T) \in L_2(0, T; L_2(\Omega)) \oplus H^1(\Omega) \mid u_T - y_f \in D(e^{-T\Delta_N}; H^1(\Omega)) \right\},
\]

\[
\| (f, u_T) \|^2_{Y_1} = \int_0^T \| f(t) \|_{L_2(\Omega)}^2 \, dt + \int_{\Omega} \sum_{|\alpha| \leq 1} (|\partial_x^\alpha u_T(x)|^2 + |\partial_T^\alpha e^{-T\Delta_N}(u_T - y_f)(x)|^2) \, dx.
\]

Correspondingly the solution space in (4.11) amounts to

\[
X_1 = L_2(0, T; H^2(\Omega)) \cap C([0, T]; H^1(\Omega)) \cap H^1(0, T; L_2(\Omega)),
\]

\[
\| u \|^2_{X_1} = \int_0^T \| u(t) \|^2_{H^2(\Omega)} \, dt + \sup_{t \in [0, T]} \sum_{|\alpha| \leq 1} \int_{\Omega} |\partial_x^\alpha u(x, t)|^2 \, dx + \int_0^T \| \partial_t u(t) \|^2_{L^2(\Omega)} \, dt.
\]

There are, of course, also continuous embeddings \( X_1 \hookrightarrow X_0 \) and \( Y_1 \hookrightarrow Y_0 \) among these spaces.

Within this framework, the stronger Theorem 4.4 at once gives the following novelty for the classical inverse heat conduction problem with the homogeneous Neumann condition in (5.1):

**Theorem 5.4.** Let \( A = -\Delta_N \) be the Neumann realization of the Laplacian in \( L_2(\Omega) \). If \( f \in L_2(0, T; L_2(\Omega)) \) and \( u_T \in H^1(\Omega) \), there exists in the Banach space \( X_1 \) in (5.14) a solution \( u \) of the final value problem (5.1), namely

\[
\partial_t u - \Delta_N u = f, \quad \gamma_1 u = 0, \quad r_T u = u_T,
\]

if and only if the data \( (f, u_T) \) are given in the Banach space \( Y_1 \) in (5.13), i.e. if and only if \( (f, u_T) \) satisfy the compatibility condition:

\[
u_T - \int_0^T e^{(T-s)\Delta_N} f(s) \, ds \quad \text{belongs to} \quad D(e^{-T\Delta_N}; H^1(\Omega)).
\]

In the affirmative case, the solution \( u \) is uniquely determined in \( X_1 \) and for some constant \( c > 0 \) independent of \( (f, u_T) \) it satisfies \( \| u \|_{X_1} \leq c \|(f, u_T)\|_{Y_1} \).
It is given by the formula, in which all terms belong to $X_1$,

$$u(t) = e^{t\Delta_N} e^{-T\Delta_N} \left( u_T - \int_0^T e^{(T-t)\Delta_N} f(t) \, dt \right) + \int_0^t e^{(t-s)\Delta_N} f(s) \, ds.$$  

(5.17)

Furthermore the difference in (5.16) equals $e^{T\Delta_N} u(0)$, which belongs to the space $D(e^{-T\Delta_N}, H^1(\Omega)) = e^{T\Delta_N} (H^1(\Omega))$.

To emphasize the complex nature of the above inverse Neumann heat equation, it might serve a purpose to write out the inequality that according to Theorem 5.4 is satisfied by the solution $u$ in $X_1$:

$$\int_0^T \int_\Omega \left( |\partial_t u(t, x)|^2 + \sum_{|\alpha| \leq 2} |\partial^\alpha_x u(t, x)|^2 \right) \, dx \, dt$$

$$+ \sup_{t \in [0,T]} \sum_{|\alpha| \leq 1} \int_\Omega |\partial^\alpha_x u(x, t)|^2 \, dx$$

$$\leq c \int_0^T \int_\Omega |f(t, x)|^2 \, dx \, dt + c \sum_{|\alpha| \leq 1} \int_\Omega \left( |\partial^\alpha_x u_T(x)|^2 \right.$$  

$$+ |\partial^\alpha_x e^{-T\Delta_N} (u_T - \int_0^T e^{(T-t)\Delta_N} f(t) \, dt)(x)|^2 \right) \, dx.$$  

(5.18)

As a particular case of the comments after Theorem 4.4, $\mathcal{P} = (\partial_t - \Delta_N, r_T)$ is a linear homeomorphism $X_1 \to Y_1$ between Hilbertable spaces $X_1, Y_1$. Hence there is the following new result on a classical problem:

**Corollary 5.5.** The final value problem (5.1) for the homogeneous Neumann heat equation in the smooth open bounded set $\Omega$ is isomorphically well posed in the spaces $X_1$ and $Y_1$ in (5.14) and (5.13).

It is left for the future to develop a theory for the final value heat conduction problem subjected to the inhomogeneous Neumann condition $\gamma_1 u = (\nu \cdot \nabla u)|_\Gamma = g$ at the curved boundary. It is envisaged that the techniques used in [6] for the inhomogeneous Dirichlet condition can be adapted to the set-up above.

**Remark 5.6.** To give some background, it is recalled that there is a phenomenon of $L_2$-instability in case $f = 0$ in (1.1). This was perhaps first described by Miranker [33], who addressed the homogeneous Dirichlet condition at the boundary. The instability is found via the Dirichlet realization of the Laplacian, $-\Delta_D$, and its $L_2(\Omega)$-orthonormal basis $e_1(x), e_2(x), \ldots$ of eigenfunctions associated to the usual ordering of its eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ counted with multiplicities. A similar notation applies to the Neumann realization $-\Delta_N$ studied above, although $\lambda_1 = 0$ in this case. It has been a major classical theme (with a too rich history to recall here) that Weyl’s law for the counting function $N(\lambda) = \# \{ j \mid 0 \leq \lambda_j \leq \lambda \}$ in terms of the measures
Hereby $-\,+$ refers to the Dirichlet and Neumann boundary conditions, respectively, but as the leading term is the same, a classical inversion gives the same crude eigenvalue asymptotics for both conditions,

$$
\lambda_j = O(j^{2/n}) \quad \text{for } j \to \infty.
$$

Hence there is also $L_2$-instability for the homogeneous Neumann problem: The eigenfunction basis $e_1(x), e_2(x), \ldots$ gives rise to a sequence of final value data $u_{T,j}(x) = e_j(x)$ lying on the unit sphere in $L_2(\Omega)$ as $\|u_{T,j}\| = \|e_j\| = 1$ for $j \in \mathbb{N}$. But the corresponding solutions to $u' - \Delta u = 0$, i.e. $u_j(t, x) = e^{(T-t)\lambda_j} e_j(x)$, have initial states $u(0, x)$ with $L_2$-norms that because of (5.20) grow rapidly for $j \to \infty$,

$$
\|u_j(0, \cdot)\| = e^{T\lambda_j} \|e_j\| = e^{T\lambda_j} \to \infty.
$$

However, this $L_2$-instability only indicates that the $L_2(\Omega)$-norm is an insensitive choice for problem (1.1). The task is hence to obtain a norm on $u_T$ giving better control over the backward calculations of $u(t, x)$—for the homogeneous Neumann heat problem (1.1), an account of this was given in Theorem 5.4 ff.

6. Final Remarks

Remark 6.1. Since the Neumann condition $\gamma_1 u = 0$ is given in terms of a trace operator effectively of class 2 (as $\gamma_1$ is defined on $H^2$ but not on $H^1$), it is not surprising that the well-posedness for (5.1) is obtained in the more regular spaces $X_1, Y_1$ in Theorem 5.4, whereas Theorem 5.2 is somewhat inconclusive. However, since Theorem 5.4 can be seen as a regularity result adjoined to Theorem 5.2, it is an important clarification that the additional assumption $f \in L_2(0, T; L_2(\Omega))$ does not alone suffice for the regularity needed to ascertain that $u$ belongs to $X_1$: to avoid a singularity at $t = 0$ in the $L_2$-norm of $\Delta_N e^{t\Delta_N} u(0)$, the implicit initial state $u(0)$ must be stipulated to belong to the interpolation space $[D(\Delta_N), L_2]_{1/2} = H^1(\Omega)$; cf. the equivalent conditions in Theorem 3.4. In particular this gave rise to the compatibility condition (5.16) using the modified domain $D(e^{-T\Delta_N}, H^1(\Omega))$, which is a new and non-trivial element of the theory.

Remark 6.2. It is envisaged that the isomorphic well-posedness in Theorem 5.4 and Corollary 5.5 can be carried over to the inhomogeneous Neumann condition and to the Robin condition and other class 2 problems. More generally the present results should extend to final value problems for differential equations with boundary conditions that define parabolic Cauchy problems belonging to the pseudo-differential boundary operator calculus, cf. [17,19]. This is left for the future—the main purpose of the present paper...
is to show how the compatibility conditions should be modified in order to cover a problem of a high class.

Remark 6.3. Injectivity of the linear map $u(0) \mapsto u(T)$ for the homogeneous equation $u' + Au = 0$, i.e. its backwards uniqueness, was proved 60 years ago by Lions and Malgrange [29] for problems with $t$-dependent sesquilinear forms $a(t; u, v)$. Besides some $C^1$-properties in $t$, they assumed that (the principal part of) $a(t; u, v)$ is symmetric and uniformly $V$-coercive in the sense that $a(t; v, v) + \lambda \|v\|^2 \geq \alpha \|v\|^2$ for fixed $\lambda \in \mathbb{R}$, $\alpha > 0$ and all $v \in V$. (Bardos and Tartar [5] relaxed these $C^1$-assumptions and made some non-linear extensions.) In Problem 3.4 of [29], the authors asked if backward uniqueness can be shown under the general non-symmetric hypothesis of strong $V$-coercivity $\Re a(t; v, v) + \lambda \|v\|^2 \geq \alpha \|v\|^2$. The above Proposition 2.2 gives an affirmative answer for the $t$-independent case of their problem.

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