Naked singularities in three-dimensions.

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Abstract

We study an analytical solution to the Einstein’s equations in 2 + 1-dimensions, representing the self-similar collapse of a circularly symmetric, minimally coupled, massless, scalar field. Depending on the value of certain parameters, this solution represents the formation of naked singularities. Since our solution is asymptotically flat, these naked singularities may be relevant for the weak cosmic censorship conjecture in 2 + 1-dimensions.

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Since the work of M. W. Choptuik on the gravitational collapse of a massless scalar field \[1\], many physicists have focused their attentions on the issue of gravitational collapse. An important arena where one can study the gravitational collapse is general relativity in 2 + 1-dimensions. The great appeal of this theory comes from the fact that it retains many of the properties of general relativity in 3 + 1-dimensions, but the field equations are greatly simplified \[2\].

Presently, several black hole solutions in 2 + 1-dimensional general relativity are known \[3\]. Including the first one to be discovered, the so-called BTZ black hole \[4\]. All of them have an important property in common: the presence of a negative cosmological constant, which makes them asymptotically anti-de Sitter.

Indeed, in a recent work it was demonstrated that a three-dimensional solution to the Einstein’s equations, with a positive cosmological constant (\(\Lambda\)), such that the stress-energy tensor satisfies the dominant energy condition, contains no apparent horizons \[5\]. The same result applies to the case \(\Lambda = 0\) in the presence of matter fields. Therefore, this result explains the necessity of a negative cosmological constant in order to a black hole to form, in three-dimensional general relativity.

Based on \[5\], we can say that the gravitational collapse of ordinary matter, without a negative cosmological constant, will never form a black hole in 2 + 1-dimensional general relativity. On the other hand, one can not exclude the possible formation of naked singularities as the result of the gravitational collapse, without a cosmological constant.

In fact, it has already been shown that the collapse of a disk of pressureless dust in 2 + 1-dimensions, without a cosmological constant, has as one of its possible end states a naked singularity \[6\]. The singularity is space-like and it is a scalar polynomial singularity \[7\], in other words, scalar polynomials constructed from the Riemann tensor are unbounded there. For an observer far from the collapse region the space-time is flat and conical. Since the naked singularity defines a region of non-zero measure in the parameter space of solutions and is asymptotically flat, it may be considered as a counter-example to the weak cosmic censorship conjecture \[8\] in 2 + 1-dimensions.
In the present paper we would like to present a solution to the Einstein’s equation, without a cosmological constant, representing the self-similar, circularly symmetric, collapse of a minimally coupled, massless, scalar field, in $2+1$-dimensions. As we shall see this solution, depending on the value of certain parameters, represents the formation of naked singularities as the result of the collapse process.

We shall start by writing down the ansatz for the space-time metric. As we have mentioned before, we would like to consider the circularly symmetric, self-similar, collapse of a massless scalar field in $2+1$-dimensions. Therefore, we shall write our metric ansatz as,

$$ds^2 = -2e^{2\sigma(u,v)} du dv + r^2(u,v)d\theta^2,$$

where $\sigma(u,v)$ and $r(u,v)$ are two arbitrary functions to be determined by the field equations, $(u,v)$ is a pair of null coordinates varying in the range $(-\infty, \infty)$, and $\theta$ is an angular coordinate taking values in the usual domain $[0, 2\pi]$.

The scalar field $\Phi$ will be a function only of the two null coordinates and the expression for its stress-energy tensor $T_{\alpha\beta}$ is given by [9],

$$T_{\alpha\beta} = \Phi_{,\alpha} \Phi_{,\beta} - \frac{1}{2} g_{\alpha\beta} \Phi_{,\lambda} \Phi^{,\lambda}.$$  

(2)

where $\,$ denotes partial differentiation.

Now, combining Eqs. (1) and (2) we may obtain the Einstein’s equations which in the units of Ref. [9] and after re-scaling the scalar field, so that it absorbs the appropriate numerical factor, take the following form,

$$2\sigma_{,u} r_{,u} - r_{,uu} = r(\Phi_{,u})^2,$$

(3)

$$2\sigma_{,v} r_{,v} - r_{,vv} = r(\Phi_{,v})^2,$$

(4)

$$2r\sigma_{,uv} + r_{,uv} = -r(\Phi_{,u} \Phi_{,v}),$$

(5)

$$r_{,uv} = 0,$$

(6)
The equation of motion for the scalar field, in these coordinates, is

\[ 2r \Phi_{,uv} + \Phi_{,v} r_u + \Phi_{,u} r_v = 0. \]  

(7)

The above system of non-linear, second-order, coupled, partial differential equations (3)-(7) has an analytical solution if we impose that it is continuously self-similar. More precisely, the solution assumes the existence of an homothetic Killing vector of the form,

\[ \xi = u \frac{\partial}{\partial u} + \alpha v \frac{\partial}{\partial v}, \]

(8)

where \( \alpha \) is a non-negative, real number associated with the type of kinematic self-similarity. Following Coley [10], \( \alpha = 1 \) characterizes a self-similarity of the first kind and for all other values of \( \alpha \) we shall have a self-similarity of the second kind. Here, we shall restrict our attention to \( 0 < \alpha \leq 1 \). We can express the solution in terms of the variable \( z = (\alpha v)^{1/\alpha}/u \).

Under these conditions our solution will be given by [11],

\[ r(u, v) = \beta (\alpha v)^{1/\alpha} + \gamma u, \]

(9)

\[ \sigma(u, v) = \left( \frac{1 - \alpha}{2} \right) \ln \left( \frac{r}{u} \right) + \sigma_0, \]

(10)

and the scalar field has the following values,

\[ \Phi(u, v) = 2(\alpha - 1)^{1/2} \arctan \sqrt{\frac{\beta (\alpha v)^{1/\alpha}}{\gamma u}}, \]

(11)

for \( \gamma/\beta > 0 \) and

\[ \Phi(u, v) = (1 - \alpha)^{1/2} \ln \left[ \frac{\sqrt{\gamma/\beta} u - i \sqrt{\alpha v^{1/\alpha}}}{\sqrt{\gamma/\beta} u + i \sqrt{\alpha v^{1/\alpha}}} \right], \]

(12)

for \( \gamma/\beta < 0 \). Where \( \gamma \) and \( \beta \) are real, integration constants and we shall restrict our attention to the principal value of the complex logarithm function in Eq. (12). Based on [12], we shall assume that \( \Phi(u, v) \equiv 0 \) for \( v < 0 \).

In terms of \( r(u, v) \) Eq. (9), and \( \sigma(u, v) \) Eq. (10), the line element Eq. (1) becomes,

\[ ds^2 = -2 e^{2\sigma_0} \left( \frac{r}{u} \right)^{(1-\alpha)} du dv + r^2 d\theta^2. \]

(13)
One may notice from Eqs. (9-12), that for different values of $\alpha$, $\beta$ and $\gamma$, one has different space-times.

Observing Eq. (13), we notice that these space-times have a singularity at $r = 0$. It is a physical singularity as can be seen directly from the curvature scalar $R$.

In order to show this result we start writing down the Ricci tensor that, in the present case, has the following expression \[13\],

$$R_{\alpha\beta} = \Phi_{,\alpha} \Phi_{,\beta}. \tag{14}$$

From it, we may compute $R$ straightforwardly with the aid of Eqs. (9)-(13), finding,

$$R = -2(1 - \alpha)\gamma/\beta e^{-2\sigma_0} [((\alpha v)^{1/\alpha})^{(1-\alpha)}]^{r(3-\alpha)}. \tag{15}$$

Finally, taking the limit $r \to 0$ in $R$ Eq. (15), we find that this quantity diverges at $r = 0$. There is no other physical singularity for these space-times because $R$ is well defined outside $r = 0$. In particular, $u = 0$ is just an apparent singularity and a new coordinate system can be found where it disappears.

Another important property we can learn from $R$ is the asymptotic behavior of our solution. If we take the limit $r \to \infty$ of $R$ Eq. (15), we find that $R \to 0$. Therefore, we conclude that the space-times under investigation are asymptotically flat.

Now, we would like to select the values of $\alpha$, $\beta$, $\gamma$, such that, our solution (9-12) represents the formation of naked singularities. There are some conditions to be satisfied. Initially, $r$ Eq. (9) has to be a real, positive function. Also, $r = \text{constant}$ must be a set of time-like surfaces for different constants. On the other hand, observing the scalar field expressions Eqs. (11) and (12) we notice that the free parameters will have to be chosen in a way that $\Phi$ becomes real.

It is clear from Eqs. (13) and (15) that $\alpha = 1$ is, locally, the three-dimensional Minkowski space-time. Therefore, we shall restrict our attention to space-times with self-similarity of the second kind. Since $\Phi(u, v) \equiv 0$, for $v < 0$, it is appropriate to consider the influx of scalar field to be turned on at the advanced time $v = 0$. So that for $v < 0$ and $|\gamma| = 1$
the space-time is Minkowskian and the metric is therefore $C^1$ at the surface $v = 0$. If we let $|\gamma| \neq 1$ the space-time for $v < 0$ would be Minkowskian only locally. It could develop a globally non-trivial structure. For example, if $0 < |\gamma| < 1$ the space-time for $v < 0$ would be conical with $0 < \theta < 2\pi\gamma$.

The naked singularity space-times are obtained for $\gamma = -1$, $\beta < 0$ and $\alpha = l/m$, where $l$ and $m$ are integer numbers. $l$ being even and $m$ odd. It means that, the scalar field expression relevant to these space-times is Eq. (11). Figure 1 shows a conformal diagram for a typical space-time in this case. We may see that the space-time is divided, naturally, in two distinct regions. The first one is the Minkowskian region where $v < 0$ (I). Then, we have the collapse region where $v > 0$ and $u < 0$ (II).

The scalar field Eq. (11) starts collapsing from $v = 0$, in the collapse region. From Eq. (11), it is not difficult to see that it is real and may be written as,

$$
\Phi(u, v) = 2(1 - \alpha)^{1/2} \tanh^{-1} \sqrt{\frac{\beta(\alpha v)^{1/\alpha}}{u}},
$$

in this region. $\Phi = 0$ at $v = 0$ and increases with $v$ until it blows up at the singularity $r = 0$.

In order to identify the nature of the surfaces $r = \text{constant}$ we shall have to compute,

$$
2g^{uv} r_{,u} r_{,v}.
$$

For the above space-times, this equation (17) takes the following form when we introduce the appropriate information from Eqs. (9) and (13),

$$
2e^{-2\sigma_0} \beta \left[ \frac{(\alpha v)^{1/\alpha} u}{r} \right]^{(1-\alpha)}.
$$

We can see from Eq. (18) that the surfaces $r = \text{constant}$ will be time-like in the collapse region because with our choice of $l$ and $m$ the numerator of $1 - \alpha = (m - l)/m$, is an odd integer number. Therefore, the singularity $r = 0$ will be a time-like one. Once this singularity is not hidden by any horizon, we may consider these space-times representing naked singularities.

An important property of the naked singularities formed in this process comes from the fact that they are asymptotically flat. If we add to this the fact that they define a
region of non-zero measure in the parameter space of solutions, we can say that the naked
singularities found here, in the same way as the ones found in [3], may be considered as
counter-examples to the weak cosmic censorship conjecture. On the other hand, due to the
fact that the $r = 0$ singularity is time-like in the present case, differently from the ones in [3],
the naked singularities are also counter-examples to the strong cosmic censorship conjecture
[14].

We may also describe our solution with the aid of the time coordinate,

$$ t(u, v) = -\beta(\alpha v)^{1/\alpha} + \gamma u, \quad (19) $$
in terms of which, the line element Eq. (13) and the scalar fields Eqs. (11) and (12) become,
respectively,

$$ ds^2 = -(2)^{(1-2\alpha)} e^{2\sigma_0} \left( \frac{r}{r^2 - t^2} \right)^{(1-\alpha)} (-dt^2 + dr^2) + r^2 d\theta^2, \quad (20) $$

$$ \Phi(u, v) = 2(\alpha - 1)^{1/2} \arctan \sqrt{\frac{r - t}{r + t}} \quad (21) $$

and

$$ \Phi(r, t) = (1 - \alpha)^{1/2} \ln \left[ \frac{\sqrt{r + t} - i\sqrt{r - t}}{\sqrt{r + t} + i\sqrt{r - t}} \right]. \quad (22) $$

Where we restrict our attention to the principal value of the complex logarithm function in
Eq. (22).

We end this article by noting that recently another continuously self-similar solution
(CSS), to the same problem treated here, was found in [15]. There, it was shown to cor-
rectly describes the critical solution, found numerically in [16], to the collapse of a circularly
symmetric, minimally coupled, massless, real scalar field with a negative cosmological con-
stant in $2 + 1$-dimensional general relativity. Although, the presence of a cosmological
constant prevents the equations to have the CSS symmetry it was shown in [10] that the
critical solution has this symmetry. Indeed, the analytical solution found in [15] has a CSS
of the first kind [10].
The critical solution separates two sets of end states for the scalar field collapse. In the first one, the scalar field collapses, interacts and disperses leaving behind the anti-de Sitter space-time. In the second, black holes with exterior regions settling down to a BTZ form are formed. The critical solution cannot have an apparent horizon, therefore the singularity formed in the collapse should be naked. The solution found in [16] has no apparent horizons, the scalar field is real and it has a null singularity \( u = 0 \) in our coordinates, in the region where it was compared with the numerical solution of [16].

It is clear from the above that our solution describes a different physical situation than the one found in [15].

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FIG. 1. Conformal diagram for a typical naked singularity solution. The null surface $v = 0$ separates the Minkowskian region (I) from the collapse region (II) where lies the time-like singularity at $r = 0$. 
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