ON SINGULARITY FORMATION FOR THE TWO DIMENSIONAL UNSTEADY PRANDTL’S SYSTEM

CHARLES COLLOT, TEJ-EDDINE GHOUL, SLIM IBRAHIM, AND NADER MASMOUDI

Abstract. We consider the two dimensional unsteady Prandtl’s system. For a special class of outer Euler flows and solutions of the Prandtl system, the trace of the tangential derivative along the transversal axis solves a closed one dimensional equation. We give a precise description of singular solutions for this reduced problem. A stable blow-up pattern and a countable family of other unstable solutions are found. The blow-up point is ejected to infinity in finite time, and the solutions form a plateau with growing length. The proof uses modulation techniques and different energy estimates in the various zones of interest.

1. Introduction

We consider the two dimensional unsteady Prandtl boundary layer equations:

\[\begin{align*}
    u_t - u_{yy} + uu_x + v u_y &= -p^E_x, \\
    u_x + v_y &= 0, \\
    u|_{y=0} = v|_{y=0} &= 0, \\
    u|_{y=\infty} &= u^E,
\end{align*}\]

where \(\vec{u} = (u, v)\) is the velocity field, \(u^E\) and \(p^E\) are the trace at the boundary of the tangential component of the underlying inviscid velocity field and the pressure. Prandtl introduced this model to describe the behaviour of a fluid close to a physical boundary for high Reynolds numbers. He obtained this model as a formal limit of the Navier-Stokes equation when the viscosity goes to zero. He proposed the appearance of a boundary layer where the viscosity is still effective, describing the solution between the boundary and the interior part where the dynamics is inviscid. The leading order term in the expansion in the boundary layer solves (1.1), see for exemple [28, 29, 22] for more on the derivation of the system.

1.1. On singularity formation for the 2-dimensional Prandtl’s equations

In this paper we are interested by the formation of singularity in the Prandtl system. Indeed, the fact that a singularity can appear in this system is actually a physical phenomenon that is called the unsteady separation. Van Dommelen and Shen [31] obtained the first reliable numerical result, and explained how the separation is linked to the formation of singularity. They described the singularity as being a consequence of particles squashed in the streamwise direction, with a compensating expansion in the normal direction of the boundary. We refer to [5, 27, 11, 18] and references therein for additional numerical results on the singularity formation.

The precise description of the formation of singularity is still an open problem. However, E and Engquist [7] proved that blow-up can happen. They make some symmetry assumptions and consider a trivial inviscid flow in the outer region \((u^E = p^E = 0)\). In this case, the trace of the

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tangential derivative along the transversal axis solves a closed one dimensional equation (1.3). They proved existence of blow-up for this reduced problem. Their approach is by contradiction and do not provide any information about the mechanism that leads to the singularity. For a more general class of non-trivial inviscid flows in the outer region \((u^E, p^E)\) but still with a suitable assumption of symmetry, this reduction is still possible. The corresponding one dimensional problem still admits blow-up solutions [20]. The authors of [20] also use a convexity argument which does not give details about the singularity.

In this paper, we give a complete description of the mechanism that leads to the singularity for the reduced one dimensional problem, including the case of nontrivial inviscid flows in the outer region. In particular, we prove the existence of a stable blow-up pattern, and other unstable ones.

Our approach is inspired by the description of the so-called ODE blow-up for the semi-linear heat equation, see [14, 2, 17, 24] in particular. Note that the incompressibility condition generates difficulties through the appearance of a nonlocal nonlinear transport term. Actually, this nonlocal term will induce two new effects, the singular point is ejected to infinity in finite time, and the solution forms a plateau with a growing length. Another difficulty comes from the boundary. Indeed, the blow-up is not localised near a single point but happens on a large zone. We perform a careful treatment near the boundary to show that the solution stays bounded in its vicinity.

The reduced one dimensional problem (1.3) with a different domain and boundary conditions also appears in a special class of infinite energy solutions to the Navier-Stokes equations [10]. The authors proved the existence of a similar stable blow-up pattern as the one we describe here, for a particular class of solutions. Their approach is based on parabolic methods and maximum principles, allowing for a non-perturbative argument, but requires many special assumptions. In particular, their argument does not apply to the problem that we consider in the present paper. In addition, our approach based on energy methods is more robust, since it allows us to prove the stability of the fundamental profile, to construct unstable blow-ups and to derive weighted estimates.

One can wonder how the one dimensional reduction is related to the full two dimensional problem. From the numerics in [11] it seems that for certain solutions with symmetries the blow-up indeed happens on the vertical axis. However, for other solutions, such as the singularity considered by Van Dommelen and Shen, still from the numerics another singularity appears before the one on the vertical axis. In [4] we treated a two dimensional Burgers model with transverse viscosity. This corresponds to a simplified version of the Prandtl system with a trivial flow at infinity \(u^E = p^E = 0\) and no vertical velocity \(v = 0\). A similar one dimensional reduction can be made. More interestingly we were able to prove that the one dimensional problem captures the main features of the two dimensional singularity. As a result we obtained a complete description of the mechanism that leads to singularity for the two dimensional problem.

In the present work, we show that the viscosity is asymptotically negligible during the singularity formation. This indicates that the full 2-d blow-up could correspond to leading order to that of the inviscid Prandtl’s equations. This has been proposed for the Van Dommelen and Shen singularity in [30, 8, 3]. In a forthcoming paper, we study the self-similar blow-up profiles of the
inviscid 2-d Prandtl’s equations. In particular, we show that there exists one of the form

\[ u(t, x, y) = (T - t)^{\frac{1}{2}} \Theta \left( \frac{x}{(T - t)^{\frac{1}{2}}}, \frac{y}{(T - t)^{\frac{1}{2}}} \right) \]

where \( T \) is the blow-up time, and where the profile \( \Theta(X, Y) \) satisfies

\[ \partial_X \Theta(0, Y) = -\sin^2(Y/2) \mathbb{I}_{0 \leq Y \leq 2\pi}. \]

Our main result in Theorem 1 shows that this is precisely the profile of the reduced one-dimensional equation. Therefore our result can be understood as a partial stability result for the profile \( \Theta \). In a further step we hope to treat the complete two dimensional Prandtl’s system.

Singularity formation is one problem out of many others regarding the Prandtl’s boundary layer system. The system is locally well-posed in the analytical setting \([28, 21, 19]\). Under monotonicity assumptions, well-posedness holds in Sobolev regularity \([26, 23, 1]\) and global weak solutions also exist globally \([34]\). Note that the solutions we consider here do not satisfy the monotonicity assumption. In this case, the equation can be ill-posed in Sobolev regularity \([12]\). Similar instabilities prevent the Prandtl’s system from being a good approximation of the Navier-Stokes equations at high Reynolds number in certain cases \([15]\). Indeed, monotonicity and/or Gevrey regularity in the tangential \( x \)-variable are necessary to insure that this approximation holds. We refer to \([28, 13]\) and the references therein. Finally, let us mention that the Goldstein singularity in the steady case has been recently constructed in \([6]\).

1.2. Statement of the result

Without loss of generality, consider a trivial vanishing outer flow \( u^E = p^E = 0 \). Our result adapts straightforwardly to more general outer flows, as they just generates additional lower order terms, see comments below. Consider an initial datum \( u_0(x, y) \) of the horizontal component of the velocity field for the Prandtl equation that is odd in \( x \). Consequently, the corresponding solution \( u(t, x, y) \) is also odd in \( x \) and

\[ u(t, 0, y) = u_{xx}(t, 0, y) = 0, \]

this allows one to consider only the dynamic of the tangential derivative of \( u \) along the \( y \)-axis. To do so, we set

\[ \xi(t, y) = -u_x(t, 0, y), \]  

which obey the following equation for \( y \in [0, +\infty) \):

\[ \begin{cases} 
\xi_t - \xi_{yy} - \xi^2 + \left( \int_0^y \xi \right) \xi_y = 0, \\
\xi(t, 0) = 0, \quad \xi(0, y) = \xi_0(y). 
\end{cases} \]

The local well-posedness for the above equation is standard, see for example Proposition \(6\) which adapts the result of \([33]\). In particular, solutions for initial data in \( L^1([0, +\infty)) \) exist, are instantaneously regularised and there holds the following blow-up criterion. If the maximal time of existence of the solution is finite, then

\[ \limsup_{t \uparrow T} \|\xi(t, \cdot)\|_{L^\infty([0, +\infty))]} = +\infty. \]  

Our main result is the precise description of the singularity formation for the reduced one-dimensional problem \((1.3)\).

**Theorem 1** (Stable blow-up for Equation \((1.3)\)). There exists \( \lambda_0^* \gg 1 \) such that for all \( \lambda_0 \geq \lambda_0^* \), an \( \epsilon(\lambda_0) > 0 \) exists with the following property. Consider for \( \lambda_0 \geq \lambda_0^* \) an initial datum of the
form:

\[ \xi_0(y) = \lambda_0^2 \cos^2 \left( \frac{y - \lambda_0 \pi}{2 \lambda_0} \right) \mathbb{1}_{0 \leq y \leq 2 \lambda_0 \pi} + \hat{\xi}_0(y), \quad \text{with} \quad \|\hat{\xi}_0\|_{L^1([0, +\infty))} \leq \epsilon(\lambda_0). \quad (1.5) \]

Then the solution to (1.3) blows up at some time \( T > 0 \), with \( T \to 0 \) as \( \lambda_0 \to +\infty \), with:

\[ \xi(t, y) = \lambda^2(t) \cos^2 \left( \frac{y - y^*(t)}{2 \lambda(t) \mu(t)} \right) \mathbb{1}_{-\pi \leq \frac{y-y^*(t)}{\lambda \mu} \leq \pi} + \hat{\xi}, \]

where, for some \( \mu_\infty > 0 \):

\[ \lambda(t) = \frac{1}{\sqrt{T-t}} + O((T-t)^{3/2}), \quad \mu(t) = \mu_\infty + O((T-t)), \quad y^*(t) = \frac{\mu_\infty \pi}{\sqrt{T-t}} + O((T-t)^{-1/4}), \quad (1.6) \]

and

\[ \|\hat{\xi}\|_{L^\infty} \leq (T-t)^{-1+\frac{3}{4}}. \quad (1.7) \]

Moreover, on any compact set, the solution remains uniformly regular up to time \( T \), so that for any \( y \in [0, +\infty) \), the limit \( \lim_{t \uparrow T} \xi(t, y) = \xi^*(y) \) exists and satisfies:

\[ \xi^*(y) \sim \frac{y^2}{4 \mu_\infty^2} \quad \text{as} \quad y \to +\infty. \quad (1.8) \]

There also exist a countable family of other unstable blow-up scenarios. The solution also forms a bump-like profile, with a support that is bigger than in the stable blow-up case. The instability corresponds to the appearance of unstable and localised eigenmodes at the linearised level which are not linked to the symmetries of the equation.

**Theorem 2** (Instable blow-ups for Equation (1.3)). For any \( k \in \mathbb{N} \), with \( k \geq 2 \), there exists a solution to (1.3) blowing up at time \( T > 0 \), with:

\[ \xi(t, y) = \lambda^{2k/(2k-1)} G_k \left( \frac{y-y^*(t)}{\lambda(t) \mu(t)} \right) \mathbb{1}_{-a_k \leq \frac{y-y^*(t)}{\lambda \mu} \leq a_k} + \hat{\xi}, \]

where \( a_k = \pi/(2k \sin(\pi/2k)) \), \( G_k \) is defined in Proposition 4, and with, for some \( \mu_\infty, \nu > 0 \):

\[ \lambda(t) = \frac{1}{(T-t)^{1-\frac{2}{4k}}}(1+O((T-t)^{\nu})), \quad \mu(t) = \mu_\infty + O((T-t)^{\nu}), \quad y^*(t) = \frac{\mu_\infty a_k}{(T-t)^{1-\frac{2}{4k}}}(1+O((T-t)^{\nu})), \]

and

\[ \|\hat{\xi}\|_{L^\infty} \leq (T-t)^{-1+\nu}. \]

Moreover, on any compact set, the solution remains uniformly regular up to time \( T \), so that for any \( y \in [0, +\infty) \), the limit \( \lim_{t \uparrow T} \xi(t, y) = \xi^*(y) \) exists and satisfies:

\[ \xi^*(y) \sim \left( \frac{2k-1}{\mu_\infty} \right)^{2k/(2k-1)} \frac{y^{2k/(2k-1)}}{y^{1/(2k-1)}} \quad \text{as} \quad y \to +\infty. \]

Let us make the following comments on the results of Theorem 1 and 2.

1. **On the implication for the Prandtl’s boundary layer.** Our result shows that the blow-up does not happen at the boundary, nor at a finite distance from it, but the singularity is ejected to infinity. This fact is rarely emphasised, but can be seen on numerical results, see [11] for example. This suggests that the boundary layer should interact with the outer Euler flow. Moreover, Prandtl’s equations are derived neglecting the viscosity effects in the horizontal direction \( x \). Since the \( x \)-derivative becomes unbounded in our result, the approximation of the Navier-Stokes equations by the Prandtl’s system is not valid just before the the singularity formation.
2. On symmetry assumptions and the stable singularity formation. The reduction to the one-dimensional problem (1.3) breaks down in the general case without symmetry assumptions. Hence our stability result in Theorem 1 should be understood within the symmetry class of odd solutions. Actually, the stable 2-d singularity is expected to be a non-symmetrical one from [31, 30, 8, 3]. In particular, the blow-up scales in the transversal $y$ direction are different from the one of Theorem 1, see [11].

3. On more general outer flows. Our results can be extended to other non-trivial outer flows satisfying suitable symmetry assumptions (e.g. $u^E$ odd and $p^E$ even in $x$). Indeed, this will just induce the presence of new terms that are of lower order asymptotically during singularity formation, and will not perturb the blow-up mechanism. Hence the statements of Theorems 1 and 2 remain true. This is the case, for example, of the impulsively started cylinder [31] $u^E = \kappa \sin x$ and $p^E = (\kappa^2 / 4) \cos(2x)$, for which the reduced equation (1.3) becomes:

$$
\begin{aligned}
\xi_t - \xi_{yy} - \xi^2 + \left( \int_0^y \xi \right) \xi_y &= -\kappa^2, \\
\xi(t,0) &= 0, \quad \xi(t,y) \underset{y \to +\infty}{\to} -\kappa.
\end{aligned}
$$

1.3. Strategy of the proof and organisation of the paper

The proof relies on a perturbative bootstrap argument around the blow-up profile. The maximum of the solution is the most sensitive location, where the viscosity effects are non negligible at the parabolic scale. There, the dynamics is given by an elliptic operator with compact resolvent (3.1) in a suitable weighted space, as in [14, 2, 17, 24]. A decomposition of the solution onto the eigenmodes allows to derive modulation equations for the parameters and decay for the remainder due to a spectral gap. In the midrange zone, away from the maximum but still on the support of the blow-up profile, the viscosity is negligible and we face a singularly perturbed problem (4.38). We use a new Lyapunov functional with an adapted weight and take derivatives with a suitable vector field, which are the main technical novelties of the present paper. Finally, the solution is studied near the boundary via a no blow-up argument inspired from [14, 16, 25].

The paper is organised as follows. In Section 3, we give a heuristic argument for the derivation of the blow-up profiles and some of their properties in Proposition 4. Then we prove Theorem 1 for which Section 4 is the heart of the paper. The bootstrap argument is described in Subsection 4.3 and Proposition 12 states the perturbative result in renormalised variables. The analysis near the maximum is in Subsection 4.4, the modulation equations and the interior Lyapunov functional are established in Lemmas 14 and 15. The midrange zone is analysed in Subsection 4.5, the exterior Lyapunov functionals are established in Lemmas 17 and 18. The solution is studied on compact sets in the original variable in Lemma 20. Proposition 12 and Theorem 1 are then proved in Subsection 4.7. Finally, we explain how the proof adapts to show Theorem 2 in Section 5.

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2. Notations

Let the measure
\[ \rho(Y) = \frac{1}{2} \sqrt{\frac{3}{\pi}} e^{-\frac{3y^2}{4}}. \]
For a function \( h \) defined on some half line \([Y_0, +\infty)\), we will write with an abuse of notation:
\[ \|h\|_{L^2}^2 = \int_{Y_0}^{+\infty} h^2(Y) \rho(Y) dY, \quad \|h\|^2_{H^s_{\rho}} = \int_{Y_0}^{+\infty} (h^2(Y) + \|\partial_Y h(Y)\|)^2 \rho(Y) dY, \quad (2.1) \]
and the value of \( Y_0 \), being the image of the origin in original variables \( y \) by a change of variable, due to the boundary condition in (1.3), will always be clear from the context. We denote the primitive of a function integrated from the origin by
\[ \partial_y^{-1} h(y) = \int_0^y h(\tilde{y}) d\tilde{y}, \quad \partial_Y^{-1} h(Y) = \int_0^Y h(\tilde{Y}) d\tilde{Y}, \quad \partial_Z^{-1} h(Z) = \int_0^Z h(\tilde{Z}) d\tilde{Z}, \]
the integration being with respect to the variables \( y, Y \) or \( Z \) to be defined later on. Note that the origin will not be preserved by the change of variables: \( y = 0 \) does not correspond to \( Y = 0 \) and the integrals do not start from the same point. Consider the Hermite polynomials:
\[ h_0 = 1, \quad h_1 = \sqrt{3} Y, \quad h_2 = 3Y^2 - 2. \quad (2.2) \]
The heat kernel will be denoted by:
\[ K_\tau(x) = \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-\frac{x^2}{4\tau}}. \]
We write \( A \leq CB \) if \( A, B \geq 0 \) and if the constant \( C \) is independent of the other parameters, or which is independent of the initial renormalised time \( s_0 \), and its value will change from one line to another. We write \( A \lesssim B \) if \( A \leq CB \), and \( O(B) \) means a quantity that is \( \lesssim B \). We write \( C(K) \) for example to precise that the constant depends only on some parameter \( K \). We write \( A \approx B \) if \( A \lesssim B \) and \( B \lesssim A \).

3. Formal analysis and blow-up profiles

In this section we derive formally the blow-up profile for (1.3). This approach relying on matched asymptotics is inspired by [32, 9, 5, 24, 17, 10]. Let us first perform a formal computation for the effect of the viscosity near the maximum of the solution, and for the obtention of the suitable self-similar variables. Assume that the solution to (1.3) blows up at time \( T \), with its maximum at a point \( y^*(t) \), and that the speed of this point is given by the transport part of the equation: \( y^*_t = \partial_y^{-1} \xi(y^*) \). We then use parabolic self-similar variables:
\[ Y = \frac{y - y^*}{\sqrt{T - t}}, \quad s = -\log(T - t), \quad f(s, Y) = (T - t)\xi(t, y) \]
and find that \( f \) solves, assuming that one can neglect the boundary condition,
\[ f_s + f + \frac{Y}{2} \partial_Y f - f^2 + \partial_Y^{-1} f \partial_Y f - \partial_Y Y f = 0. \]
An obvious solution of the above equation is the constant in space-time solution \( f = 1 \), which corresponds to \( \phi = 1/(T - t) \) in original variables (which solves (1.3) but does not satisfy the boundary condition). Assuming that 1 is a good approximation of the solution for some large zone in the variable \( Y \), we compute the evolution of the correction \( \varepsilon = f - 1 \):
\[ \varepsilon_s + \mathcal{L} \varepsilon = NL, \quad \mathcal{L} \varepsilon := -\varepsilon + \frac{3}{2} Y \partial_Y \varepsilon - \varepsilon_{yy}, \quad NL = \varepsilon^2 - \partial_Y^{-1} \varepsilon \partial_Y \varepsilon. \quad (3.1) \]
The linearised operator $\mathcal{L}$ is well known.

**Proposition 3.** The operator $\mathcal{L} : L^2_\rho \to L^2_\rho$ is essentially self-adjoint with compact resolvent. Its spectrum is $\{-1 + 3i/2, \ i = 0, 1, 2, \ldots\}$, with associated eigenfunctions

$$h_i(Y) = H_i \left( \sqrt{3}Y \right) = \sum_{j=0}^{[i/2]} \frac{i!}{j!(i-2j)!} 3^{i-2j/2} (-1)^j Y^{i-2j}$$

where $H_j$ is a Hermite polynomial.

**Proof.** Changing variables and setting $u(Y) = w(z)$, $z = \sqrt{3}Y$ gives $\mathcal{L}u = 3(\tilde{\mathcal{L}}w)(z)$ where $\tilde{\mathcal{L}} := \partial_{zz} - 1/2\partial_z + 1/3$ and the result follows from the corresponding result on $\tilde{\mathcal{L}}$ whose eigenbasis consists on Hermite polynomials, see [24].

From Proposition 3 one sees that the linearised dynamics possesses one instability direction, and an infinite number of stable modes. The unstable direction corresponds to the constant in space mode 1, and is related to a symmetry of the equation: the invariance by time translation. One can assume that the blow-up time has been chosen well, so that this mode is not excited. Neglecting the nonlinear effects, one can assume from Proposition 3 that one mode dominates:

$$\varepsilon(s, Y) \approx C e^{(1 - \frac{3}{2})s} h_i(Y), \ i \geq 1.$$ 

From the behaviour at infinity of the polynomials $h_i$, the fact that $1 + \varepsilon$ is maximal near the origin implies that $C = -c < 0$ and that $i = 2k$ is an even positive integer (the modes associated to odd integers are related to another symmetry of the equation: the invariance by space translation). Therefore, $\varepsilon(s, Y) \approx -ce^{(1 - 3k)s} h_{2k}(Y) \approx -ce^{(1 - 3k)s} Y^{2k}$ for $Y$ large. The correction $\varepsilon$ then starts to be of the same size as the leading order term 1 in the zone

$$|Y| \sim e^{(\frac{4}{3} - \frac{3k}{4})s}, \ i.e. \ y - y^* \sim (T - t)^{-1 + \frac{1}{4k}}.$$

This suggests to introduce the new variables:

$$Z := \frac{Y}{e^{(\frac{4}{3} - \frac{3k}{4})s}} = (T - t)^{1 - \frac{1}{4k}} (y - y^*), \ F(s, Z) := f(s, Y)$$

and $F$ solves

$$F_s + F - F^2 + \left( - \left(1 - \frac{1}{2k}\right) Z + \int_0^Z F(s, \tilde{Z}) d\tilde{Z} \right) \partial_Z F - e^{-(3 - t)s} \partial_{ZZ} F = 0.$$ 

Assuming that $F$ is the correct rescaled unknown, the viscosity is asymptotically negligible and $F$ should converge to a stationary solution of the self-similar inviscid equation

$$F - F^2 + \left( - \left(1 - \frac{1}{2k}\right) Z + \int_0^Z F(\tilde{Z}) d\tilde{Z} \right) \frac{d}{d\tilde{Z}} F = 0. \quad (3.2) \tag{eq:F1}$$

In other words, $F$ should tend in renormalized variables to a self-similar solution of (1.3) without viscosity and boundary which is:

$$\psi_t - \psi^2 + \left( \int_{-\infty}^y \psi \right) \partial_y \psi = 0. \quad (3.3) \tag{eq:inviscidp}$$

This equation admits a four-parameters group of symmetries: invariance by space and time translation and a two-dimensional scaling group. Namely, if $\psi(t, x)$ is a solution then so is

$$\frac{1}{\lambda} \psi \left( \frac{t - t_0}{\lambda}, \frac{y - y_0}{\mu} \right), \ (t_0, y_0, \mu, \lambda) \in \mathbb{R}^2 \times (0, +\infty)^2.$$
Proposition 4. Let $k \in \mathbb{N}$. Equation (3.2) admits a one-parameter family of solutions

$$G_k \left( \frac{Z}{\mu} \right), \quad \mu > 0,$$

(3.4)

where $G_k$ is even, compactly supported on $[-a_k, a_k]$ with $a_k = \pi/(2k \sin(\pi/2k))$, positive and increasing on $(-a_k, 0)$, of class $C^{1+1/(2k-1)}$ on $\mathbb{R}$, and satisfies the asymptotic expansions

$$G_k(Z) = (2k-1)^{1+k/2k-1}(Z + a_k)^{1+k/2k-1} \quad \text{as} \quad Z \to -a_k, \quad G_k(Z) = 1 - Z^{2k} \quad \text{as} \quad Z \to 0.$$

For $k = 1$ one has $a = \pi$ and the explicit formula up to the rescaling (3.4):

$$G_1(Z) = \cos^2 \left( \frac{Z}{2} \right) \mathbb{1}_{-\pi \leq Z \leq \pi}.$$

(3.5)

Remark 5. As is clear from the proof of Proposition 3 provided in below, we have $\int_0^a F(Z) dZ = (1 - 1/(2k))a_k$. Using this fact, one sees that equation (3.2) admits other solutions of the form $G_k((Z - \mu a_k)/\mu)$. It also admits the trivial solutions 0 and 1. We claim that all other bounded solutions of (3.2) can be obtained by gluing a finite or an infinite number of these solutions, when they attain 1 or 0. For example, the function:

$$F(Z) = \begin{cases} 1 & \text{for } Z \leq 0, \\ G_k(Z) & \text{for } 0 \leq Z \leq a_k, \\ G_k \left( \frac{Z - \mu a_k - a_k}{\mu} \right) & \text{for } a_k \leq Z \end{cases}$$

is also a solution with the same regularity.

The solutions $G_k$ of (4.5) are also well defined for $k > 0$ and $k \notin \mathbb{N}$. There is then a continuum of blow-up speeds for equation (3.3), but adding viscosity selects only the smooth blow-up profiles.

Proof. The formula for $k = 1$ is a direct computation. We perform the change of variables on $[0, +\infty)$:

$$\frac{d\xi}{dZ} = \frac{\xi}{-(1 - 1/2k) Z + \int_0^Z G(\tilde{Z})d\tilde{Z}}, \quad H(\xi) := G(Z).$$

So that the equation (3.2) becomes

$$H - H^2 + \xi \partial_{\xi} H = 0$$

whose solution is $H = (1 + \xi)^{-1}$ (renormalising the constant of integration). Unwinding the transformation one finds

$$\frac{dZ}{d\xi} = \frac{1}{\xi} \left[ - \left( 1 - \frac{1}{2k} \right) Z + \int_0^Z F(\tilde{Z})d\tilde{Z} \right]$$

which gives

$$\frac{d^2 Z}{d\xi^2} = -\frac{1}{\xi} \frac{dZ}{d\xi} - \left( 1 - \frac{1}{2k} \right) \frac{1}{\xi} \frac{dZ}{d\xi} + \frac{1}{\xi} \frac{dZ}{d\xi} F(Z) = \frac{dZ}{d\xi} \left[ - \left( 2 - \frac{1}{2k} \right) \frac{1}{\xi} + \frac{1}{\xi + \xi^2} \right]$$

and hence

$$\frac{d}{dZ} \left( \log \frac{dZ}{d\xi} \right) = - \left( 2 - \frac{1}{2k} \right) \frac{1}{\xi} + \frac{1}{\xi + \xi^2}$$
that after integration yields
\[ \log \frac{dZ}{d\xi} = C + \log \left( \xi^{-\left(\frac{1}{2k}\right)} \right) + \log \xi - \log(\xi + 1) \]
where \( C \) is an integration constant. From this one deduces that, taking \( C \) to get a constant equal to 1 in the expression below
\[ \frac{dZ}{d\xi} = \frac{\xi^{-\left(\frac{1}{2k}\right)}}{1 + \xi}, \quad Z(0) = 0. \]
Since \( Z(0) = 0 \), one deduces that
\[ \lim_{\xi \to +\infty} Z(\xi) = \int_0^{+\infty} \frac{\xi^{-\left(\frac{1}{2k}\right)}}{1 + \xi} d\xi = \frac{\pi}{\sin\left(\frac{\pi}{2k}\right)}. \]
and that as \( \xi \to 0 \),
\[ Z = 2k\xi^{\frac{1}{2k}} (1 + O(\xi)) \]
and that as \( \xi \to +\infty \):
\[ Z = \frac{\pi}{\sin\left(\frac{\pi}{2k}\right)} - \frac{\xi^{-1+\frac{1}{2k}}}{1 - \frac{1}{2k}} (1 + O(\xi^{-1})). \]
This yields near the origin
\[ \xi = \left( \frac{Z}{2k} \right)^{2k} (1 + O(Z^{2k})) \]
and at infinity:
\[ \xi = \left( 1 - \frac{1}{2k} \right)^{-\frac{2k}{2k-1}} \left( \frac{\pi}{\sin\left(\frac{\pi}{2k}\right)} - Z \right)^{-\frac{2k}{2k-1}} \left( 1 + O\left( \frac{\pi}{\sin\left(\frac{\pi}{2k}\right)} - Z \right)^{\frac{2k}{2k-1}} \right). \]
Therefore near the origin \( G(Z) = 1 - (2k)^{-2k} Z^{2k} + O(Z^{4k}) \) and near \( a_k = \pi / \sin(\pi/(2k)) \)
\[ G(Z) = \left( 1 - \frac{1}{2k} \right)^{\frac{2k}{2k-1}} (a_k - Z)^{\frac{2k}{2k-1}} (1 + O((a_k - Z)^{\frac{2k}{2k-1}})) \]
and the result follows.

From Proposition 4 and Remark 5, equation (3.3) then admits a family of backward self-similar profiles for \( k \in \mathbb{N} \) which are smooth on their support:
\[ \psi(t, y) = \frac{1}{T - t} G_k \left( (y - y^*(t)) \frac{(T - t)^{1 - \frac{1}{2k}}}{\mu} \right), \quad y^*(t) = \frac{\mu a_k}{(T - t)^{1 - \frac{1}{2k}}} + y_0^*, \quad \mu > 0. \]
They blow up in finite time and their support, which is \( y \in [y_0^*, y_0^* + 2a_k/(\mu(T - t)^{1 - 1/2k})] \), is growing to infinity. The formal analysis we just performed indicates that they could be at the heart of the blow-up phenomenon.
4. Proof of Theorem 1

First, let us give the following local well-posedness result which is an adaptation of [33]. Note that if $\xi$ solves (1.3), then $\lambda^2 \xi(\lambda^2 t, \lambda y)$ is also a solution. The scaling transformation $h \mapsto \lambda^2 h(\lambda y)$ is an isometry on $L^{1/2}([0, +\infty))$ and (1.3) is then said to be $L^{1/2}$-critical.

**Proposition 6** (Local well-posedness). Let $\xi_0 \in L^1([0, +\infty))$. There exists $T(\|\xi_0\|_{L^1}) > 0$ and a unique solution to (1.3) in Duhamel formulation such that $\xi \in C([0, T], L^1([0, +\infty)))$, $\xi(0, \cdot) = \xi_0(\cdot)$ and $\|\partial_y \xi(t)\|_{L^1} \lesssim t^{-1/2}$. Moreover, $\xi \in C^\infty((0, T] \times [0, +\infty))$ and for each $k \in \mathbb{R}$, $\partial^k_y \xi \in C((0, T], L^1([0, +\infty)))$. For any $k \in \mathbb{N}$ and $0 < T_1 \leq T$, the solution map is locally uniformly continuous from $L^1$ into $C([T_1, T], W^{k, 1}[0, +\infty))$.

Solutions associated to initial data of the form (1.5) are thus well-defined and we now turn to the proof of Theorem 1. We will use sometimes alternative formula for the profile:

$$G_1(Z) = \cos^2\left(\frac{Z}{2}\right) \mathbb{1}_{-\pi \leq Z \leq \pi} = \left(\frac{1}{2} + \frac{1}{2} \cos(Z)\right) \mathbb{1}_{-\pi \leq Z \leq \pi} \quad (4.1)$$

$$= 1 - \frac{Z^2}{4} + \frac{Z^4}{48} + O(|Z|^6) \text{ as } Z \to 0.$$  

The proof of Theorem 1 relies on a bootstrap argument performed near the blow-up profile. First we explain how to suitably decompose a solution near the blow-up profile and then set up the bootstrap procedure. The fact that such solutions satisfy the properties of Theorem 1 is then showed at the end of this section.

4.1. Adapted geometrical decomposition and renormalised flow

The following lemma states that in a suitable neighbourhood of the set of self-similar profiles, there exists a unique way to project the solution onto this set using adapted orthogonality conditions.

**Lemma 7** (Geometrical decomposition). There exist $\lambda^*, \delta, K > 0$ such that for all $\lambda_0 \geq \lambda^*$ and $Y_0 \leq -\lambda_0^2$, for any $\epsilon \in B_{L^2}(\delta \lambda_0^{-4})$ with $\epsilon(Y_0) = -G_1(Y_0/\lambda_0^2)$, there exist $(\lambda, \mu, Y_0) \in (0, +\infty)^2 \times \mathbb{R}$, such that the following decomposition holds

$$G_1\left(\frac{Y}{\lambda_0^2}\right) + \epsilon(Y) = \lambda^2 G_1\left(\frac{Y - Y_0}{\lambda_2 \mu}\right) + \bar{\epsilon}(Y - Y_0) \text{ with } \bar{\epsilon} \perp h_0, h_1, h_2 \text{ in } L^2_p.$$  

Moreover, these are the only such parameters satisfying $|\lambda - 1| \lambda_0^2 + |\mu| + |\bar{Y}_0| \leq K$. This defines a mapping $\epsilon \mapsto (\lambda, \mu, Y_0)$, which is of class $C^1$ in $L^2_p$.

**Remark 8.** One has to keep track of the free boundary in the $Y$ variable, and we made a slight abuse of notations in Lemma 7. Indeed, note that the space $L^2_p$ in which $\epsilon$ belongs is given by (2.1) with boundary at $Y_0$, whereas the space $L^2_p$ in which $\bar{\epsilon}$ belongs, and in which it enjoys orthogonality condition is (2.1) with boundary at $Y_0 - \bar{Y}_0$.

The proof of the above lemma is a standard combination of the implicit function theorem and a Taylor expansion of $G_1$ near the origin. It is relegated to Appendix B.

For a function $\xi : [0, T) \times [0, +\infty) \rightarrow \mathbb{R}$, given parameters $(\lambda, \mu, y^*) \in C^1([0, T), \mathbb{R})$, we define two renormalisations. The first one is the parabolic self-similar renormalisation close to the blow-up point:

$$s = s_0 + \int_{t_0}^t \lambda^2(\tilde{t}) d\tilde{t}, \quad Y = \lambda(y - y^*), \quad f(s, Y) = \frac{1}{\lambda^2} \xi(t, y). \quad (4.2)$$
The second one is the renormalisation associated to the leading order profile:
\[
Z = \frac{y - y^*}{\lambda \mu} = \frac{Y}{\lambda^2 \mu}, \quad F(s, Z) = \frac{1}{\lambda^2} \xi(t, y) = f(s, Y).
\] (4.3) \textbf{def:renormalisation}

The function \(\xi\) solves (1.3) if and only if the functions \(f\) and \(F\) solve the equations
\[
\begin{align*}
f_s + \frac{2}{\lambda}(2 + Y \partial_y) f - f^2 + \partial_Y^{-1} f \partial_Y f + \left(f_0^0 - \lambda y_s^*\right) \partial_Y f - \partial_Y f = 0, \\
f(s, -(\pi + a)\lambda^2 \mu) = 0,
\end{align*}
\] (4.4) \textbf{eq:f}

and
\[
\begin{align*}
F_s + \frac{2}{\lambda}(2 - Z \partial_Z) F - F^2 + \partial_Z^{-1} F \partial_Z F + \left(f_0^0 - \frac{2}{\lambda \mu} \right) \partial_Z f - \frac{1}{\lambda^2 \mu^2} \partial_Z F = 0, \\
F(s, -(\pi + a)) = 0.
\end{align*}
\] (4.5) \textbf{eq:F}

Since \(\lambda\) will behave like \((T - t)^{-1/2}\), and the blow-up point will behave like \(\pi \mu (T - t)^{-1/2}\), we introduce the correction \(a\):
\[
y^* = \lambda \mu (\pi + a).
\] (4.6) \textbf{def:a}

We take the following notation for the remainder:
\[
f(s, Y) = G_1(Z) + \varepsilon(s, Y), \quad F(s, Z) = G_1(Z) + u(s, Z), \quad \text{so that } \varepsilon(s, Y) = u(s, Z).
\] (4.7) \textbf{id:decomposition}

\subsection*{4.2. The weighted norm and derivative outside the blow-up point}

To control the solution, we need a special weight and a special vector field to take derivatives, both adapted to the linearized operator in the \(Z\) variable. We refer to Subsection 4.5 and Lemma 16 for more information regarding these choices. Let \(q : \mathbb{R} \to [0, +\infty)\) be an even function satisfying the following properties. \(q \in C^2((0, +\infty)),\) \(q(0) = 0, q' > 0\) on \((0, \pi)\) with a limit on the right of the origin that exists and satisfies \(\lim_{Z \to 0} q'(Z) > 0, q''(\pi) = 0, q''''(\pi) < 0,\) and \(q(Z) = q(\pi) = 1\) for \(Z \geq \pi\). Define the weight \(w\) on \((0, +\infty) \times \mathbb{R}^*\) by:
\[
w(s, Z) := \begin{cases} 
\frac{1 - \cos Z}{1 - \cos Z} \frac{1}{\sin^2 Z} \frac{1}{\sin Z} \left(\frac{1}{\sin Z}ight)^3 \frac{1}{\sin Z} & \text{if } Z \in (-\pi, 0), \\
\frac{1 - \cos Z}{1 - \cos Z} \frac{1}{\sin^2 Z} \frac{1}{\sin Z} \left(\frac{1}{\sin Z}ight)^3 \frac{1}{\sin Z} & \text{if } Z \in (0, \pi), \\
\frac{1}{s}, & \text{if } |Z| \geq \pi.
\end{cases}
\] (4.8) \textbf{eq:def w}

Note that the weight \(w(s, \cdot)\) is even, of class \(C^1\) on \((0, +\infty)\), and \(C^2\) on \((0, \pi)\) and \((\pi, +\infty)\). To take derivatives in a suitable way, we will use the vector field \(A \partial_Z\), where:
\[
A(Z) := \begin{cases} 
-1 & \text{for } Z \leq -\frac{\pi}{2}, \\
\sin Z & \text{for } -\frac{\pi}{2} \leq Z \leq \frac{\pi}{2}, \\
1 & \text{for } \frac{\pi}{2} \leq Z.
\end{cases}
\] (4.9) \textbf{eq:def A}

Note that one has the following sizes for \(s > 0\) and \(Z \in [-\pi, \pi]\
\[
w \approx \frac{1}{|Z|^s q(Z)}, \quad |A| \approx |Z|.
\] (4.10) \textbf{bd:w}

\subsection*{4.3. The bootstrap regime}

The solution we will consider will be close to the blow-up profile in the following sense. At initial time we require the following bounds, involving parameters which will be fixed later on. Note that Lemma 7 implies the uniqueness of the decomposition used below.
Definition 9 (Initial closeness). Let $M \gg 1$, $s_0 \gg 1$, $0 < \nu \ll 1$ and $\xi(0) \in C^\infty([0, +\infty), \mathbb{R})$ with $\xi(0) = \partial_y \xi(0) = 0$. We say that $\xi(0)$ is initially close to the blow-up profile if there exists $\lambda_0 > 0$, $a_0 \in \mathbb{R}$ and $\mu_0 > 0$ such that the following properties are verified. In the variables (4.2) one has:

$$f(s, Y) = G_1 \left( \frac{Y}{\lambda_0^2 h_0} \right) + \varepsilon_0, \quad \varepsilon_0 \perp (h_0, h_1, h_2),$$

and the remainder and the parameters satisfy:

(i) Initial values of the modulation parameters:

$$\frac{1}{2} \varepsilon_0^2 < \lambda_0 < 2 \varepsilon_0^2, \quad \frac{1}{2} < \mu_0 < 2, \quad |a_0| < e^{-\frac{1}{2} s_0}.$$  \hspace{1cm} (4.12)

(ii) Initial smallness of the remainder in parabolic variables:

$$\|\varepsilon_0\|_{L_\mu^3} < e^{-\frac{7}{2} s_0}, \quad \|\varepsilon_0\|_{H^3(|Y| \leq M^3)} < e^{-\frac{7}{2} s_0}.$$  \hspace{1cm} (4.13)

(iii) Initial smallness of the remainder in inviscid self-similar variables:

$$\int_{-\pi-a}^{-M e^{-s_0}} u^2 w dZ + \int_{M e^{-s_0}}^{+\infty} u^2 w dZ < e^{-2(\frac{7}{2} - \nu) s_0}, \quad \int_{-\pi-a}^{-M e^{-s_0}} |A \partial_Z u|^2 w dZ + \int_{M e^{-s_0}}^{+\infty} |A \partial_Z u|^2 w dZ < e^{2\nu s_0}.$$  \hspace{1cm} (4.14)

(iv) Initial regularity close to the origin in original variables:

$$\|\xi_0\|_{W^{1, \infty}([0, 2])} < 1.$$  \hspace{1cm} (4.15)

We aim at proving that solutions which are initially close to the blow-up profile in the sense of Definition 9 will stay close to this blow-up profile up to modulation. The proximity at later times is defined as follows.

Definition 10 (Trapped solutions). Let $K \gg 1$ and $0 < \nu' \ll \nu$. We say that a solution is trapped on $[s_0, s^*]$ if it satisfies the properties of Definition 9 at time $s_0$, and if it can be decomposed according to (4.7) and (4.11) for all $s \in [s_0, s^*]$ with:

(i) Values of the modulation parameters:

$$\frac{1}{K} e^{\frac{7}{2} - s} < \lambda < K e^{\frac{7}{2}}, \quad \frac{1}{K} < \mu < K, \quad |a| < K e^{-(\frac{7}{2} - 2\nu) s}.$$  \hspace{1cm} (4.16)

(ii) Smallness of the remainder in parabolic variables:

$$\|\varepsilon\|_{L_\mu^3} < K e^{-\frac{7}{2} s}, \quad \|\varepsilon\|_{H^3(|Y| \leq M^2)} < K e^{-(\frac{7}{2} - \nu') s}.$$  \hspace{1cm} (4.17)

(iii) Smallness of the remainder in the inviscid self-similar variables:

$$\int_{-\pi-a}^{-M e^{-s}} u^2 w dZ + \int_{M e^{-s}}^{+\infty} u^2 w dZ < K^2 e^{-2(\frac{7}{2} - \nu) s}, \quad \int_{-\pi-a}^{-M e^{-s}} |A \partial_Z u|^2 w dZ + \int_{M e^{-s}}^{+\infty} |A \partial_Z u|^2 w dZ < K^2 e^{2\nu s}.$$  \hspace{1cm} (4.18)

Remark 11. Lemma 7 and the regularity of the flow, Proposition 6, imply that the parameters of Definition 10 are uniquely determined and in $C^1([s_0, s_1])$. In particular, the renormalisation (4.2) and (4.3) is indeed well-defined.

The heart of the paper is the following bootstrap proposition.

Proposition 12. There exist universal constants $K, M, s_0^* \gg 1$ and $0 < \nu' \ll \nu \ll 1$ such that the following holds for any $s_0 \geq s_0^*$. Any solution which is initially close to the blow-up profile in the sense of Definition 9 is trapped on $[s_0, +\infty)$ in the sense of Definition 10.
Lemma 7, and a standard continuity argument, imply that for $s_0$ large enough, any solution which is initially close to the blow-up profile in the sense of Definition 9 is trapped in the sense of Definition 10 on some interval $[s_0, s_1]$ with $s_1 > s_0$. Let $s^* > s_0$ be the supremum of times $s_1 \geq s_0$ such that the solution is trapped on $[s_0, s_1]$. The strategy is now to show that $s^* = +\infty$ by studying the trapped regime in several following several lemmas and showing that the solutions cannot escape from the open set defined by Definition 10. The proof of Proposition 12 is then given at the end of this section.

Note that the constants $K, M, s_0^*, \nu^*, \nu$ and $\eta$ will be adjusted during the proof: we will always be able to conclude the proof of the various lemmas by choosing $M$ large enough depending on $K$ and then $s_0^*$ large enough. First, note that one has pointwise control of the remainder for trapped solutions.

**Lemma 13.** Let $u$ be trapped on $[s_0, s_1]$. Then for $s_0$ large enough there holds for all $s \in [s_0, s_1]$:

$$
\|\varepsilon\|_{L^\infty} = \|u\|_{L^\infty} \lesssim Ke^{-\left(\frac{1}{4} - \nu\right)s}.
$$

**Proof.** First, from (4.17), Sobolev embedding implies:

$$
\|\varepsilon\|_{L^\infty([Z] \leq M^2)} \lesssim Ke^{-\frac{1}{2}s}.
$$

Let $E := \{-\pi - a \leq Z \leq -M e^{-s}\} \cup \{M e^{-s} \leq Z\}$. Then, one notices from (4.10) that $w \gtrsim s^{-1}$ and $|A|w \gtrsim s^{-1}$ on $E$, implying:

$$
\|u\|_{L^2(E)}^2 \lesssim s \int_{-\pi - a}^{-Me^{-s}} u^2 w + s \int_{-Me^{-s}}^{+\infty} u^2 w, \quad \|\partial_Z u\|_{L^2(E)}^2 \lesssim s \int_{-\pi - a}^{-Me^{-s}} |A\partial_Z u|^2 w + s \int_{Me^{-s}}^{+\infty} |A\partial_Z u|^2 w.
$$

Therefore, using Agmon’s inequality and (4.18) gives:

$$
\|u\|_{L^\infty(E)} \lesssim \|u\|_{L^2(E)}^{\frac{1}{2}} \|\partial_Z u\|_{L^2(E)}^{\frac{1}{2}} \lesssim K s^{\frac{1}{2}} e^{-\left(\frac{1}{4} - \nu\right)s}.
$$

Hence, as for $M$ large enough depending on $K$ the two zones $|Z| \geq Me^{-s}$ and $|Y| \leq M^2$ cover the entire space, there holds $\|u\|_{L^\infty} \lesssim Ke^{-\left(\frac{1}{4} - \nu\right)s} + Ke^{-\frac{1}{2}s} \lesssim K s e^{-\left(\frac{1}{4} - \nu\right)s}$ for $s_0$ large enough. \(\square\)

### 4.4. Analysis near the blow-up point

This subsection is devoted to the study of the solution near $y^*$ in parabolic variables (4.2). This is the most sensible zone, in which the blow-up parameters are selected. The remainder is dissipated away from this point, until it reaches the outside region $|Z| \geq 1$ where another dynamics takes place (see next subsection). The analysis near the blow-up point is the consequence of the blow-up profile structure, the linear structure Proposition 3 and the orthogonality conditions (4.11). The measure $\rho = ce^{-3Y^2/4}$ decreases very fast because of the transport part of the operator $L$ which is unbounded and pushes the characteristics away from the origin. Therefore, the analysis here is poorly affected by the exterior dynamics. From (4.4), (4.7), (3.2) and (4.6) we infer that $\varepsilon$ solves

$$
\left\{ \begin{array}{l}
\varepsilon_s + \mathcal{L}\varepsilon + \tilde{\mathcal{L}}\varepsilon + \text{Mod} + NL - \frac{1}{\lambda^2 \mu} \partial_Z G_1(Z) = 0, \\
\varepsilon(s, -(\pi + a)\lambda^2 \mu) = -G_1(-\pi - a).
\end{array} \right.
$$

(4.20)

where $\mathcal{L}$ is defined by (3.1), the small linear term, the modulation term and the nonlinear term are

$$
\tilde{\mathcal{L}} \varepsilon := 2(1 - G_1(Z)) \varepsilon + (\lambda^2 \mu \partial_Z^{-1} G_1(Z) - Y) \partial_Y \varepsilon + \frac{1}{\lambda^2 \mu} \partial_Z G_1(Z) \partial_Y^{-1} \varepsilon,
$$

(4.21)
The small linear term is evaluated as follows. First, one computes using Cauchy-Schwarz that,

$$\int_{-\pi + a}^0 f dY - \lambda y^*_m \left( \frac{1}{\lambda^2 \mu} \partial_Z G_1 + \partial_Y \varepsilon \right),$$

which is a direct and standard computation using the definition of the geometrical decomposition. The parameters evolve according to the following dynamics.

Lemma 14 (Modulation equations). Let a solution be trapped on $[s_0, s^*]$. Then one has for $s_0$ large enough:

$$\int_{-\lambda y^*}^{s_0} f dY - \lambda y^*_m \lesssim e^{-e^4} + \|\varepsilon\|_{L^2} + \lambda^4 \|\partial_Y \varepsilon\|_{L^2} + \lambda^4 \|\varepsilon\|_{L^\infty} \|\partial_Y \varepsilon\|_{L^2},$$

$$\left| \int_{-\lambda y^*}^{s_0} f dY - \lambda y^*_m \right| \lesssim \lambda^{-8} + \|\varepsilon\|_{L^2} + \lambda^4 \|\partial_Y \varepsilon\|_{L^2} + \lambda^4 \|\varepsilon\|_{L^\infty} \|\partial_Y \varepsilon\|_{L^2},$$

$$\left| \int_{-\lambda y^*}^{s_0} f dY - \lambda y^*_m \right| \lesssim e^{-\left(\frac{4}{3} + \frac{1}{4}\right)s} + e^{\left(\frac{4}{3} - \frac{1}{4}\right)s} \|\partial_Y \varepsilon\|_{L^2},$$

Proof. This is a direct and standard computation using the definition of the geometrical decomposition and the spectral structure of the linearised dynamics. To ease notations we introduce $m_1 = \lambda s / \lambda - 1/2$, $m_2 = \mu s / \mu$ and $m_3 = \int_{-\lambda y^*}^{s_0} f dY - \lambda y^*_m$. $m_1$ is the difference between the evolution of $\lambda$ and the expected self-similar law. $m_3$ is the difference between the speed of the blow-up point and the value of the transport part of the equation at this point. First we differentiate the orthogonality conditions (4.11) for $i = 0, 1, 2$ using the boundary condition (4.20):

$$0 = \frac{d}{ds} \left( \int_{-\lambda y^*}^{\pm \infty} \varepsilon h_i \rho dY \right) = \frac{d}{ds} (\lambda y^*_m)(h_i \rho)(-\lambda y^*)G_1(-\pi - a) + \int_{-\lambda y^*}^{\pm \infty} \varepsilon h_i \rho dY.$$

Since $\lambda y^*_m \gtrsim e^4$ from (4.16), one has that $|\rho(\lambda y^*_m)| \lesssim e^{-\frac{4}{3} s}$ for $s_0$ large enough. Therefore, as $|\int_{-\lambda y^*}^{s_0} f dY| \lesssim \lambda^2 \mu \lesssim e^4$ from (4.7) and (4.19), the above identity can be rewritten as:

$$\int_{-\lambda y^*}^{\pm \infty} \varepsilon h_i \rho dY = O(e^{-e^4}(1 + |m_1| + |m_3|)).$$

We now estimate the contribution of each term when injecting (4.20) in the above identity.

Step 1 The linear and small linear terms. Performing integration by parts and thanks to the orthogonality (4.11) and Proposition 3, using the boundary condition (4.20) and (4.77):

$$\int_{-\lambda y^*}^{\pm \infty} h_i \partial_Y \varepsilon \rho dY = (\partial_Y \varepsilon \rho h_i)(-\lambda y^*) - (\varepsilon \rho \partial_Y h_i)(-\lambda y^*) + \int_{-\lambda y^*}^{\pm \infty} \partial_Y h_i \varepsilon \rho dY$$

$$= (\partial_Y \varepsilon \rho h_i)(-\lambda y^*) + (\rho \partial_Y h_i)(-\lambda y^*)G_1(-\pi - a) = O(e^{-e^4}(1 + |\partial_Y \varepsilon(-\lambda y^*)|)) = O(e^{-e^4}).$$

The small linear term is evaluated as follows. First, one computes using Cauchy-Schwarz that, since $|1 - G_1(Z)| \lesssim Z^2 \lesssim \lambda^{-4} Y^2$:

$$\int_{-\lambda y^*}^{\pm \infty} h_i(1 - G_1(Z)) \varepsilon \rho dY \lesssim \lambda^{-4}\|\varepsilon\|_{L^2}.$$

Similarly, since $|\lambda^2 \mu \partial_Y^{-1} G_1(Z) - Y| + |Y| \|\partial_Y(\lambda^2 \mu \partial_Y^{-1} G_1(Z) - Y)\| \lesssim \lambda^{-4} |Y|^3$:

$$\int_{-\lambda y^*}^{\pm \infty} \lambda^2 \mu \partial_Y^{-1} G_1(Z) \partial_Y \varepsilon \rho dY \lesssim \lambda^{-4}\|\varepsilon\|_{L^2}.$$
Using Cauchy-Schwarz one estimates that
\[
\int_0^Y \varepsilon(s, \tilde{Y}) d\tilde{Y} \leq \|\varepsilon\|_{L^2_\rho} \left( \int_0^Y \varepsilon^{\frac{2}{r^2}} d\tilde{Y} \right)^{\frac{1}{2}} \lesssim \|\varepsilon\|_{L^2_\rho} \left( \frac{e^{\frac{3y^2}{2}}}{(1 + |Y|)^{\frac{5}{2}}} \right)
\]
(4.25) \eq{controlenonlocal}
which implies the bound, since |\partial_Z G_1(Z)| \lesssim \lambda^{-4}|Y|:
\[
\left| \int_{-\lambda y}^{+\infty} h_i \frac{1}{\lambda^2 \mu} \partial_Z G_1(Z) \partial_Y^{-1} \varepsilon \rho dY \right| \lesssim \lambda^{-4} \|\varepsilon\|_{L^2_\rho}.
\]
From (4.21) this gives the bound for the small linear term:
\[
\left| \int_{-\lambda y}^{+\infty} h_i \tilde{Z} \varepsilon \rho dY \right| \lesssim \lambda^{-4} \|\varepsilon\|_{L^2_\rho}.
\]
(4.26) \bd{mod12}

**Step 2 The modulation term.** We first rewrite it performing a Taylor expansion on \( G_1 \) from (4.1) near the origin and using (2.2):
\[
\text{Mod}(Y) = m_2 \left( \frac{1}{\lambda^4 \mu^2} \left( \frac{1}{6} h_2(Y) + \frac{1}{3} h_0(Y) \right) + \mu^{-4} \lambda^{-8} r_2(Y) \right)
+ m_1 \left( 2h_0(Y) + \mu^{-4} \lambda^{-8} r_1(Y) + (2 + Y \partial_Y) \varepsilon \right)
+ m_3 \left( -\frac{1}{\lambda^4 \mu^2} \frac{1}{2\sqrt{3}} h_1(Y) + \mu^{-4} \lambda^{-8} r_3(Y) + \partial_Y \varepsilon \right)
\]
(4.28)
where \( r_1(Y) = \mu^4 \lambda^8 ((2 - Z \partial_Z) G_1 - 2) \) and \( r_2 = -\mu^4 \lambda^8 Z (\partial_Z G_1 + Z/2) \) are even functions which are \( O(Y^4) \), and \( r_3 = \mu^3 \lambda^6 (\partial_Z G_1 + Z/2) \) is an odd function that is \( O(Y^3) \). We recall that \( h_2 \) and \( h_{2i+1} \) are even and odd functions and form an almost orthogonal family: \( \int_{-\lambda y}^{+\infty} h_i h_j \rho = -\int_{-\lambda y}^{+\infty} h_i h_j \rho = O(e^{-3s/2}) \). From (4.11) and (2.2), one has \( \int_{-\lambda y}^{+\infty} p \varepsilon = 0 \) for any polynomial \( p \) of degree 2. Using this (4.11) and the boundary condition (4.20) we obtain that
\[
\int_{-\lambda y}^{+\infty} h_0 \text{Mod} \rho = m_2 \left( \frac{\|h_0\|_{L^2_\rho}^2 + O(\lambda^{-4})}{3\lambda^4 \mu^2} \right) + m_1 \left( 2\|h_0\|_{L^2_\rho}^2 + O(\lambda^{-8}) + (Y \rho)(-\lambda y^*) G_1(-\pi - a) \right)
+ m_3 \left( -\mu^{-4} \lambda^{-8} \int_{-\lambda y}^{+\infty} r_3 \rho + \rho(-\lambda y^*) G_1(-\pi - a) \right)
= m_2 \left( \frac{\|h_0\|_{L^2_\rho}^2}{3\lambda^4 \mu^2} + O(\lambda^{-8}) \right) + m_1 \left( 2\|h_0\|_{L^2_\rho}^2 + O(\lambda^{-8}) \right) + m_3 O(e^{-e^*})
\]
(4.29)
where for the last bound we used the fact that \( \lambda y^* \gtrsim e^* \) and \( \rho = Ce^{-\frac{3y^2}{2}} \); similarly
\[
\int_{-\lambda y}^{+\infty} h_1 \text{Mod} \rho
= m_1 \left( O(e^{-e^*}) - \mu^{-4} \lambda^{-8} \int_{-\infty}^{-\lambda y^*} h_1 r_1 \rho + \frac{(Y^2 \rho)}{\sqrt{3}} (-\lambda y^*) G_1(-\pi - a) + O(\|\varepsilon\|_{L^2_\rho}) \right)
+ m_2 \left( O(e^{-e^*}) - \mu^{-4} \lambda^{-8} \int_{-\infty}^{-\lambda y^*} h_1 \rho \right)
+ m_3 \left( -\frac{\|h_1\|_{L^2_\rho}^2 + O(\lambda^{-4})}{2\sqrt{3} \lambda^4 \mu^2} - \frac{(Y \rho)}{\sqrt{3}} (-\lambda y^*) G_1(-\pi - a) \right)
= m_2 O(e^{-e^*}) + m_1 O(e^{-e^*} + \|\varepsilon\|_{L^2_\rho}) - \frac{m_3 \|h_1\|_{L^2_\rho}^2 + O(\lambda^{-4})}{2\sqrt{3} \lambda^2 \mu}.
\]
(4.30)
and
\[
\int_{-\lambda y^*}^{+\infty} h_2 \mathrm{Mod} \rho \\
= m_2 \left( \frac{\|h_2\|_{L^2_{\rho}}^2}{6\lambda^4 \mu^2} + O(\lambda^{-8}) \right) + m_1 \left( O(\lambda^{-8}) + (Y h_2 \rho)(-\lambda y^*) G_1(-\pi - a) + O(\|\varepsilon\|_{L^2_{\rho}}) \right) \\
+ m_3 \left( O(e^{-\varepsilon^*}) - \mu^{-4} \lambda^{-8} \int_{-\infty}^{-\lambda y^*} h_2 r_3 \rho + (h_2 \rho)(-\lambda y^*) G_1(-\pi - a) + O(\|\varepsilon\|_{L^2_{\rho}}) \right) \\
= m_2 \frac{\|h_2\|_{L^2_{\rho}}^2}{6\lambda^4 \mu^2} + m_1 O(\lambda^{-8} + \|\varepsilon\|_{L^2_{\rho}}) + m_3 O(e^{-\varepsilon^*} + \|\varepsilon\|_{L^2_{\rho}}).
\]

**Step 3** The nonlinear term. Since \(|h_i| \lesssim (1 + Y^2)| for \(i = 0, 1, 2\) we estimate using the Poincaré inequality (A.1):
\[
\left| \int_{-\lambda y^*}^{+\infty} \varepsilon^2 h_i \rho \partial Y \right| \lesssim \|\varepsilon\|_{H^1_{\rho}}^2.
\]

Using a direct \(L^\infty\) bound one estimates that
\[
\left| \int_{-\lambda y^*}^{+\infty} h_i \partial Y \varepsilon \partial Y^{-1} \varepsilon \rho \partial Y \right| \lesssim \|\varepsilon\|_{L^\infty} \|\partial Y \varepsilon\|_{L^2_{\rho}}.
\]

Therefore, for \(i = 0, 1, 2\):
\[
\left| \int_{-\lambda y^*}^{+\infty} h_i N L \partial \rho \partial Y \right| \lesssim \|\varepsilon\|_{H^1_{\rho}}^2 + \|\varepsilon\|_{L^\infty} \|\partial Y \varepsilon\|_{L^2_{\rho}}.
\]

**Step 4** The error term. Finally, using a Taylor expansion:
\[
\frac{1}{\lambda^4 \mu^2} \partial_{ZZ} G_1(Z) = \frac{1}{\lambda^4 \mu^2} \left( -\left( \frac{1}{2} - \frac{1}{6\lambda^4 \mu^2} \right) h_0 + \frac{1}{12\lambda^4 \mu^2} h_2 + (\partial_{ZZ} G_1 + \frac{1}{2} - \frac{Z^2}{4}) \right).
\]

This gives (since this term is an even function and \(h_1\) is an odd function):
\[
\int_{-\lambda y^*}^{+\infty} \frac{1}{\lambda^4 \mu^2} \partial_{ZZ} G_1(Z) h_i \rho \partial Y = \begin{cases} \\
\frac{1}{\lambda^4 \mu^2} \left( \frac{1}{2} - \frac{1}{6\lambda^4 \mu^2} \right) \|h_0\|_{L^2_{\rho}}^2 + O(\lambda^{-12}) & \text{if } i = 0, \\
O(e^{-\varepsilon^*}) & \text{if } i = 1, \\
\frac{1}{12\lambda^4 \mu^2} \|h_2\|_{L^2_{\rho}}^2 + O(\lambda^{-12}) & \text{if } i = 2.
\end{cases}
\]

**Step 5** End of the proof. We collect the above estimates (4.24), (4.26), (4.29), (4.30), (4.31), (4.32) and (4.34) and inject them in (4.23) using (4.20). One obtains:
\[
m_2 \frac{1 + O(\lambda^{-4})}{3\lambda^4 \mu^2} + m_1 \left( 2 + O(\lambda^{-8}) \right) + m_3 O(e^{-\varepsilon^*}) \\
= -\frac{1}{\lambda^4 \mu^2} \left( \frac{1}{2} - \frac{1}{6\lambda^4 \mu^2} \right) + O(\lambda^{-12}) + O(\lambda^{-4} \|\varepsilon\|_{L^2_{\rho}} + \|\varepsilon\|_{H^1_{\rho}}^2 + \|\varepsilon\|_{L^\infty} \|\partial Y \varepsilon\|_{L^2_{\rho}}),
\]
\[
m_2 O(e^{-\varepsilon^*}) + m_1 O(e^{-\varepsilon^*} + \|\varepsilon\|_{L^2_{\rho}}) - \frac{1 + O(\lambda^{-4})}{\lambda^4 \mu^2} m_3 \\
= O(e^{-\varepsilon^*}) + O(\lambda^{-4} \|\varepsilon\|_{L^2_{\rho}} + \|\varepsilon\|_{H^1_{\rho}}^2 + \|\varepsilon\|_{L^\infty} \|\partial Y \varepsilon\|_{L^2_{\rho}}),
\]
and
\[ m_2 \frac{1 + O(\lambda^{-4})}{6\lambda^4 \mu^2} + m_1 O(\lambda^{-8} + \|\varepsilon\|_{L^2_\rho}) + m_3 O(e^{-\varepsilon^s} + \|\varepsilon\|_{L^2_\rho}) \]
\[ = \frac{1}{12\lambda^8 \mu^4} + O(\lambda^{-12}) + O(\lambda^{-4}\|\varepsilon\|_{L^2_\rho} + \|\varepsilon\|_{H^1_\rho} + \|\varepsilon\|_{L^\infty} \|\partial_Y \varepsilon\|_{L^2_\rho}). \]

These three estimates, together with the fact that \( \|\varepsilon\|_{L^2_\rho} \lesssim e^{-T_\rho s/2} \) and \( \lambda \approx e^{s/2} \) obtained from (4.17) and (4.16), imply the first two modulation relations in (4.22). The third one is obtained as consequence of the first two, using in addition the following identity from (4.19) since \( \nu \) is small: \( \|\varepsilon\|_{L^\infty} \|\partial_Y \varepsilon\|_{L^2_\rho} \lesssim e^{-(7/2+1/4)s} + e^{(7/2-1/8)s} \|\partial_Y \varepsilon\|_{L^2_\rho}^2 \). Finally, the last inequality in (4.22) is obtained from the three others, since from (4.6) and \( \int_{-\pi}^0 G_1(\pi/2) \):

\[ \int_{-\lambda y^*}^0 f dY - \lambda y^* = \lambda^2 \mu \left[ \int_{-\pi}^{-\pi-a} G_1 dZ + \frac{1}{\lambda^2 \mu} \int_{-(\pi+a)\lambda^2 \mu}^0 \varepsilon dY - a s - \frac{a}{2} - (m_1 + m_2)(\pi + a) \right]. \]

The decay of the remainder \( \varepsilon \) is encoded by the following Lyapunov functional.

Lemma 15 (Interior Lyapunov functional and energy dissipation). There exists a universal \( C > 0 \), such that for any \( 0 < \eta \ll 1 \) small enough independent of the other constants, for \( s_0 \) large enough, for a trapped solution one has

\[ \frac{d}{ds} \left( \frac{1}{2} \|\varepsilon\|_{L^2_\rho}^2 \right) + \left( \frac{7}{2} - C e^{-\eta s} \right) \|\varepsilon\|_{L^2_\rho} + e^{-\eta s} \|\partial_Y \varepsilon\|_{L^2_\rho}^2 \leq C \|\varepsilon\|_{L^2_\rho} \lambda^{-12} + C e^{-\varepsilon^s}. \]

Proof. This is a direct computation relying on the spectral gap that absorbs the nonlinear effects, the modulation equations established previously, and the rapid decay of the measure \( \rho \). One first computes from (4.20) and (4.22) that

\[ \frac{d}{ds} \left( \frac{1}{2} \|\varepsilon\|_{L^2_\rho}^2 \right) = \frac{1}{2} \frac{d}{ds} \int_{-\lambda y^*}^{+\infty} \varepsilon^2 p dY \]
\[ = \frac{1}{2} (\varepsilon^2 \rho)(-\lambda y^*) \frac{d}{ds} (-\lambda y^*) + \int_{-\lambda y^*}^{+\infty} \left( -\mathcal{L} \varepsilon - \tilde{\mathcal{L}} \varepsilon - \text{Mod} - \text{NL} + \frac{1}{\lambda^4 \mu^2} \partial_{ZZ} G_1 \right) \varepsilon p dY \]
\[ = O(e^{-\varepsilon^s} (1 + \|\partial_Y \varepsilon\|_{L^2_\rho}^2)) + \int_{-\lambda y^*}^{+\infty} \left( -\mathcal{L} \varepsilon - \tilde{\mathcal{L}} \varepsilon - \text{Mod} - \text{NL} + \frac{1}{\lambda^4 \mu^2} \partial_{ZZ} G_1 \right) \varepsilon p dY. \]

Step 1 The linear term. We first claim the dissipative spectral gap estimate

\[ \int_{-\lambda y^*}^{+\infty} |\partial_Y \varepsilon|^2 p dY \geq \frac{9}{4} (1 - C e^{-\eta s}) \int_{-\lambda y^*}^{+\infty} \varepsilon^2 p dY + 2 e^{-\eta s} \int_{-\lambda y^*}^{+\infty} |\partial_Y \varepsilon|^2 p dY - C e^{-\varepsilon^s}. \]

for some universal constant \( C > 0 \). We use analytical results on the whole space \( \mathbb{R} \), with scalar product \( \langle u, v \rangle = \int_{\mathbb{R}} u v p \) (only for the few next lines). Define the extension

\[ \bar{\varepsilon} := \left\{ \begin{array}{ll} \varepsilon(\lambda y^*) & \text{for } Y \leq -\lambda y^*, \\ \varepsilon(Y) & \text{for } Y \geq -\lambda y^*. \end{array} \right. \]

Then \( \bar{\varepsilon} \in H^1_\rho \). Define the projection on higher modes

\[ \bar{\varepsilon} := \bar{\varepsilon} - \frac{\langle \bar{\varepsilon}, h_0 \rangle}{\|h_0\|_{L^2_\rho}^2} h_0 - \frac{\langle \bar{\varepsilon}, h_1 \rangle}{\|h_1\|_{L^2_\rho}^2} h_1 - \frac{\langle \bar{\varepsilon}, h_2 \rangle}{\|h_2\|_{L^2_\rho}^2} h_2. \]
Then from the orthogonality (4.11), since $\varepsilon(-\lambda y^*) = -G_1(-\pi - a)$ from the Dirichlet boundary condition, one infers that

$$\langle \tilde{\varepsilon}, h_i \rangle = -\int_{-\infty}^{-\lambda y^*} h_i G_1(-\pi - a)e^{-\frac{3}{4}y^2}dY = O(e^{-e^s})$$

as $\lambda y^* \gtrsim e^s$. This implies that

$$\int_{-\lambda y^*}^{+\infty} \varepsilon^2 \rho dY \leq \|\tilde{\varepsilon}\|^2_{L^2_\rho} + C e^{-e^s}, \quad \int_{-\lambda y^*}^{+\infty} |\partial_Y \varepsilon|^2 \rho dY \geq \|\partial_Y \tilde{\varepsilon}\|^2_{L^2_\rho} - C e^{-e^s}.$$ 

As $\tilde{\varepsilon} \in H^1_\rho$ with $\tilde{\varepsilon} \perp h_i$ for $i = 0, 1, 2$, one has the spectral gap estimate from (3):

$$\|\partial_Y \tilde{\varepsilon}\|^2_{L^2_\rho} \geq \frac{9}{2} \|\tilde{\varepsilon}\|^2_{L^2_\rho}.$$ 

The two above estimates imply (4.37). Therefore, the linear term gives from the boundary condition (4.20) and (4.77):

$$-\int_{-\lambda y^*}^{+\infty} \mathcal{L}\varepsilon \rho dY = \int_{-\lambda y^*}^{+\infty} \varepsilon^2 \rho dY - \int_{-\lambda y^*}^{+\infty} |\partial_Y \varepsilon|^2 \rho dY + (\partial_Y \varepsilon \rho)(-\lambda y^*)G_1(-\pi - a)$$

$$\leq -\left(\frac{7}{2} - C e^{-e^s}\right) \int_{-\lambda y^*}^{+\infty} \varepsilon^2 \rho dY - 2e^{-e^s} \int_{-\lambda y^*}^{+\infty} |\partial_Y \varepsilon|^2 \rho dY + C e^{-e^s}.$$ 

**Step 2 The small linear term.** Recall (4.21). One computes using Poincare (A.1) and the fact that $|G_1(Z) - 1| \lesssim \lambda^{-1} Y^2$:

$$\left| \int_{-\lambda y^*}^{+\infty} (1 - G_1(Z)) \varepsilon^2 \rho dY \right| \leq C \lambda^{-4} \|\varepsilon\|^2_{H^1_\rho}.$$ 

Next, one performs an integration by parts and use the boundary condition (4.20) since $\lambda y^* \gtrsim e^s$:

$$\int_{-\lambda y^*}^{+\infty} \varepsilon \left(\lambda^2 \mu \partial_Z^{-1}G_1(Z) - Y\right) \partial_Y \varepsilon \rho dY$$

$$= -\left(\lambda^2 \mu \partial_Z^{-1}G_1(Z) - Y\right) \frac{\varepsilon^2}{2} \rho \left(-\lambda y^*\right) - \frac{1}{2} \int_{-\lambda y^*}^{+\infty} \varepsilon^2 \partial_Y \left((\lambda^2 \mu \partial_Z^{-1}G_1(Z) - Y)\rho\right) dY$$

$$= O(e^{-e^s}) - \frac{1}{2} \int_{-\lambda y^*}^{+\infty} \varepsilon^2 \partial_Y \left((\lambda^2 \mu \partial_Z^{-1}G_1(Z) - Y)\rho\right) dY$$

Then, one notices that for $|Y| \leq e^{3s/4}$ there holds:

$$|\partial_Y \left((\lambda^2 \mu \partial_Z^{-1}G_1(Z) - Y)\rho\right)| \lesssim \lambda^{-4} |Y|^2 (1 + |Y|)^2 \rho \lesssim e^{-\frac{3}{2}} |Y|^2 \rho.$$ 

Hence, applying (4.19), (A.1), and splitting in two zones $E = \{ |Y| \leq e^{3s/4} \}$ and $E' = [-\lambda y^*, +\infty) \setminus E$:

$$\left| \int_{-\lambda y^*}^{+\infty} \varepsilon \left(\lambda^2 \mu \partial_Z^{-1}G_1(Z) - Y\right) \partial_Y \varepsilon \rho \right| \lesssim e^{-e^s} + \left| \int_{E} \varepsilon^2 \partial_Y \left((\lambda^2 \mu \partial_Z^{-1}G_1(Z) - Y)\rho\right) \right|$$

$$+ \left| \int_{E'} \varepsilon^2 \partial_Y \left((\lambda^2 \mu \partial_Z^{-1}G_1(Z) - Y)\rho\right) \right| \lesssim e^{-e^s} + e^{-\frac{3}{2}} \|\varepsilon\|^2_{H^1_\rho}.$$
For the last term, using (4.25), since $\frac{1}{\lambda^4 \mu} |\partial_Z G_1(Z)| \lesssim \lambda^{-4}|Y|$ one has:
\[
\left| \int_{-\lambda_y^*}^{+\infty} \varepsilon \lambda^4 \mu \partial_Z G_1(Z) \partial_Y^{-1} \varepsilon \rho dY \right| \lesssim \|\varepsilon\|_{L^2_\rho} \lambda^{-4} \int_{-\lambda_y^*}^{+\infty} \|\varepsilon\|_E \frac{e^{-\frac{\lambda^2 y^2}{8}}}{(1 + |Y|)^{\frac{7}{2}}} dY
\]
\[
\lesssim \|\varepsilon\|_{L^2_\rho} \lambda^{-4} \int_{E} \|\varepsilon\|_E \frac{e^{-\frac{\lambda^2 y^2}{8}}}{(1 + |Y|)^{\frac{7}{2}}} dY + \|\varepsilon\|_{L^2_\rho} \lambda^{-4} \int_{E'} \|\varepsilon\|_E \frac{e^{-\frac{\lambda^2 y^2}{8}}}{(1 + |Y|)^{\frac{7}{2}}} dY
\]
\[
\lesssim \|\varepsilon\|_{L^2_\rho} \lambda^{-4} \|\varepsilon\|_{L^2_\rho} \left( \int_{|Y| \leq e^{s/4}} \frac{dY}{1 + |Y|} \right)^{\frac{1}{2}} + O(e^{-e^s}) \lesssim \|\varepsilon\|_{H^2_\rho} \lambda^{-3} + O(e^{-e^s})
\]
where we used (A.1) and (4.19). Therefore, putting all the above estimates together, as $\lambda \approx e^{s/2}$:
\[
\left| \int_{-\lambda_y^*}^{+\infty} \varepsilon \mathcal{L} \varepsilon_\rho dY \right| \lesssim e^{-e^s} + e^{-\frac{s^2}{4}} \|\varepsilon\|_{H^1_\rho}^2.
\]
**Step 3** The modulation term. We use the decomposition (4.27) and the orthogonality (4.11) to obtain first:
\[
\int_{-\lambda_y^*}^{+\infty} \varepsilon \text{Mod}\rho dY = -\frac{\mu_s}{\mu} \int_{-\lambda_y^*}^{+\infty} \varepsilon Z(\partial_Z G_1 + Z) \rho dY
\]
\[
+ \left( \frac{\lambda_y^*}{\lambda} - \frac{1}{2} \right) \int_{-\lambda_y^*}^{+\infty} ((2 - Z \partial Y G_1 - 2) + (2 + Y \partial Y)\varepsilon) \varepsilon \rho dY
\]
\[
+ \left( \int_{-\lambda_y^*}^{0} f dY - \lambda_y^* \right) \int_{-\lambda_y^*}^{+\infty} \varepsilon \left( \frac{1}{\lambda^2 \mu} (\partial_Z G_1 + Z) + \partial_Y \varepsilon \right) \rho dY.
\]
For the first term, using the modulation estimate (4.22) and the fact that $Z |\partial_Z G_1 + Z| \lesssim \lambda^{-8}|Y|^4$:
\[
\frac{\mu_s}{\mu} \int_{-\lambda_y^*}^{+\infty} \varepsilon Z(\partial_Z G_1 + Z) \rho dY \lesssim \|\varepsilon\|_{L^2_\rho} \left( \lambda^{-12} + \lambda^{-4} \|\varepsilon\|_{L^2_\rho} + \lambda^{-4} \|\partial_Y \varepsilon\|_{L^2_\rho} + \lambda^{-4} \|\varepsilon\|_{L^\infty} \|\partial_Y \varepsilon\|_{L^2_\rho} \right)
\]
\[
\lesssim \|\varepsilon\|_{L^2_\rho} \lambda^{-12} + e^{-s} ||\varepsilon||_{H^1_\rho}^2.
\]
We claim that the other terms can be estimated the same way by the same upper bound, yielding:
\[
\left| \int_{-\lambda_y^*}^{+\infty} \varepsilon \text{Mod}\rho dY \right| \lesssim \|\varepsilon\|_{L^2_\rho} \lambda^{-12} + e^{-s} \|\varepsilon\|_{H^1_\rho}^2.
\]
**Step 4** The nonlinear term. A direct $L^\infty$ estimate gives:
\[
\left| \int_{-\lambda_y^*}^{+\infty} \varepsilon^3 \rho dY \right| \lesssim \|\varepsilon\|_{L^\infty} \|\varepsilon\|_{L^2_\rho}^2.
\]
For the other nonlinear term one first performs an integration by parts, then a brute force bound for the boundary term, the same estimate as above for the second term, and (A.1):
\[
\int_{-\lambda_y^*}^{+\infty} \varepsilon \partial_Y^2 \partial_Y^{-1} \varepsilon \rho dY = -\frac{1}{2}(\varepsilon^3 \partial_Y^{-1} \rho)(-\lambda_y^*) - \int_{-\lambda_y^*}^{+\infty} \frac{\varepsilon^3}{2} \rho dY - \frac{1}{2} \int_{-\lambda_y^*}^{+\infty} \varepsilon^2 \partial_Y^{-1} \varepsilon \rho dY
\]
\[
= O(e^{-e^s}) + O(||\varepsilon||_{L^\infty} ||\varepsilon||_{H^1_\rho}^2).
\]
**Step 5** The error term. Using the decomposition (4.33), the orthogonality (4.11) and $|\partial_Z G_1 + \frac{1}{2} - Z^2/4| \lesssim Z^4 \approx \lambda^{-8} Y^4$ one obtains that:
\[
\left| \int_{-\lambda_y^*}^{+\infty} \frac{1}{\lambda^4 \mu^2} \partial_Z G_1(Z) \rho dY \right| = \frac{1}{\lambda^4 \mu^2} \left| \int_{-\lambda_y^*}^{+\infty} \varepsilon (\partial_Z G_1 + \frac{1}{2} - Z^2/4) \rho dY \right| \lesssim \lambda^{-12} \|\varepsilon\|_{L^2}.
Step 6 End of the proof. Collecting all the estimates in Step 1, 2, 3, 4 and 5 one finally obtains from (4.36) that:

\[
\frac{d}{ds} \left( \frac{1}{2} \| \varepsilon \|_{L^2_2}^2 \right) \leq - \left( \frac{7}{2} - C e^{-\eta s} \right) \| \varepsilon \|_{L^2_2}^2 - 2 e^{-\eta s} \| \partial_Y \varepsilon \|_{L^2_2}^2 + C(e^{-\frac{s}{7}} + \| \varepsilon \|_{L^\infty}) \| \varepsilon \|_{H^1_2}^2 + C \| \varepsilon \|_{L^2_2} \lambda^{-12} + C e^{-\varepsilon s},
\]

if \( \eta \) has been chosen small enough and \( s_0 \) large enough, where we used (4.19).

\[
\square
\]

4.5. Analysis outside the blow-up point in the inviscid self-similar zone

This subsection is devoted to the study of the solution outside the blow-up point \( y^*(t) \) and we switch to the \( Z \) variable (4.3). The aim is to find decay for \( u \), which receives information from the boundaries \( Z = -\pi - a \) and \( Z = 0 \). We first explain the linear estimate which explains the choice of the weight \( w \) and then prove full energy estimates. In view of the decomposition (4.7), the modulation equations (4.22) and (3.2), we rewrite (4.5) as:

\[
\begin{cases}
  u_s + \mathcal{H} u - \frac{1}{\lambda^2} \partial Z u + \tilde{H} u + NL + \Psi = 0, \\
  u(s, -(\pi + a)) = -G_1(-(\pi + a)),
\end{cases}
\]  

where the leading order linearised operator is

\[
\mathcal{H} := T \partial_Z u + V u + \partial_Z^{-1} u \partial_Z G_1,
\]

with the transport and the potential term being defined by

\[
T(Z) := \left( -\frac{Z}{2} + \partial_Z^{-1} G_1 \right) = \begin{cases}
  -\left( \frac{Z}{2} + \frac{\pi}{2} \right) & \text{for } Z \leq -\pi, \\
  \frac{1}{2} \sin Z & \text{for } -\pi \leq Z \leq \pi, \\
  -\left( \frac{Z}{2} - \frac{\pi}{2} \right) & \text{for } \pi \leq Z,
\end{cases}
\]

\[
V(Z) := 1 - 2G_1(Z) = \begin{cases}
  1 & \text{for } Z \leq -\pi, \\
  -\cos Z & \text{for } -\pi \leq Z \leq \pi, \\
  1 & \text{for } \pi \leq Z,
\end{cases}
\]

the small linear, the nonlinear term and the error term are given by:

\[
\tilde{H} u := d(s) \partial_Z u + \left( \frac{\lambda_s}{\lambda} - \frac{1}{2} \right) (2 - Z \partial_Z) u, \quad d(s) := \left( \int_{-(\pi + a)}^{0} F dZ - \frac{y^*_Z}{\lambda \mu} \right),
\]

\[
NL := -u^2 + \partial_Z^{-1} u \partial_Z u,
\]

\[
\psi(s, Z) := -\frac{1}{\lambda^4 \mu^2} \partial_{ZZ} G_1(Z) + \left( \frac{\lambda_s}{\lambda} - \frac{1}{2} \right) (2 - Z \partial_Z) G_1(Z) + d(s) \partial_Z G_1(Z).
\]

Thanks to (4.22) the parameter \( d \) satisfy:

\[
|d(s)| \lesssim e^{-\left( \frac{2}{7} + \frac{s}{\lambda} \right) s} + e^{\left( \frac{2}{7} - \frac{s}{\lambda} \right) s} \| \partial_Y \varepsilon \|_{L^2_2}^2.
\]
4.5.1. Linear analysis

We claim that the dynamics of Equation (4.38) is driven to leading order by the transport and potential terms, and that the nonlocal, viscosity and nonlinear terms are negligible. From a direct check, the eigenvalue problem:

\[ T \partial_Z \phi_{\beta} + V \phi_{\beta} = \beta \phi_{\beta} \]

admits a solution for all \( \beta \in \mathbb{R} \) under the form:

\[ \phi_{\beta}(Z) := \begin{cases} \phi^{\text{int}}_{\beta}(Z) & \text{for } Z \in (-\pi, \pi) \setminus \{0\}, \\ \phi^{\text{ext}}_{\beta}(Z) & \text{for } Z \in (-\infty, -\pi) \cup (\pi, +\infty), \end{cases} \quad V \phi_{\beta} + T \partial_Z \phi_{\beta} = \beta \phi_{\beta}. \quad (4.44) \]

where

\[ \phi^{\text{int}}_{\beta} = \left( \frac{1 - \cos(Z)}{1 + \cos(Z)} \right)^{\beta} \left( \sin Z \right)^2, \quad \phi^{\text{ext}}_{\beta}(Z) = \begin{cases} (-(Z + \pi))^{2(1-\beta)} & \text{for } Z < -\pi, \\ (Z - \pi)^{2(1-\beta)} & \text{for } Z > \pi. \end{cases} \]

Note that one has \( \phi_{\beta}(Z) \sim Z^{2(1+\beta)} \) as \( Z \to 0 \) and \( \phi_{\beta}(\pi + Z) \sim |Z|^{2(1-\beta)} \) as \( Z \to 0 \). The reduced operator satisfy the following comparison-type \( L^\infty \) weighted bound:

\[ \left\| e^{-s(T\partial_Z + V)} u_0 \right\|_{L^\infty} \leq e^{-\beta s} \left\| u_0 \right\|_{L^\infty} \]

which can be showed by differentiating along the characteristics. The above bound shows how cancellations near the origin for \( u_0 \) are crucial for decay since \( \phi_{\beta} \) cancels at the origin for positive \( \beta \). Our aim for the full linear problem is to perform a weighted Sobolev energy estimate which mimics the above estimate. We will modify the weight \( 1/\phi_{\beta} \) according to three principles: 1) any multiplication by a weight which is decreasing along the underlying vector field preserves the spectral gap estimate, 2) the nonlocal part is “slaved” by the transport and potential parts see the Appendix, Section (C) for an explicit computation on a very similar equation, 3) the viscosity is negligible if one is sufficiently away from the origin. These are the reasons behind the specific choice of \( w \) in (4.8). The underlying eigenfunction is \( \phi_{1/2} \) to optimise the decay with the one that holds near the maximum (4.17). We claim the following decay, at the linear level, of a Lyapunov functional with weight \( w \). We state it on the left of the origin but the analogue holds true on the right as well.

**Lemma 16.** Let \( M \gg 1, \lambda, \mu \) and \( a \) satisfy (4.16) and \( \nu > 0 \). Assume that \( u \) solves

\[ u_t + Hu - \frac{1}{\lambda^4 \mu^2} \partial_Z Z u = 0. \quad (4.45) \]

Let \( Z_1 := -(\pi + a) \) and \( Z_2 := -Me^{-s} \). Then one has the identity for \( M \) and \( s_0 \) large enough:

\begin{align*}
\frac{d}{ds} \left( \frac{1}{2} \int_{Z_1}^{Z_2} u^2 wdZ \right) &+ \left( \frac{1}{2} - \frac{\nu}{4} \right) \int_{Z_1}^{Z_2} u^2 wdZ + \frac{1}{\lambda^4 \mu^2} \int_{Z_1}^{Z_2} |\partial_Z u|^2 wdZ \\
&\leq Ce^{6s} u^2(Z_2) + Ce^{4s} |\partial_Z u|^2(Z_2) + Cu^2(Z_1) \left( e^{-(\frac{\nu}{4})s} + |a_s| \right) \\
&+ C|\partial_Z u|^2(Z_1) e^{-2s} + \frac{Ce^{2s}}{M^2} \left( \int_{Z_2}^{0} |u| dZ \right) \left( \int_{Z_1}^{Z_2} u^2 wdZ \right)^{\frac{1}{2}}.
\end{align*}

**Proof.** One computes first the following identity

\[ \frac{d}{ds} \left( \frac{1}{2} \int_{Z_1}^{Z_2} u^2 wdZ \right) = \int_{Z_1}^{Z_2} uu_t wdZ + \frac{1}{2} \int_{Z_1}^{Z_2} u^2 w_{tt} dZ + \frac{a_s}{2} (u^2 w)(Z_1) + Me^{-s} (u^2 w)(Z_2). \quad (4.46) \]
We now turn to the dissipative effects. Integrating by parts one finds:

\[
- \int_{Z_1}^{Z_2} u T \partial_Z u \, w \, dZ = \frac{1}{2} (u^2 T w)(Z_1) - \frac{1}{2} (u^2 T w)(Z_2) + \int_{Z_1}^{Z_2} u^2 \, \partial_Z (T w) \, dZ.
\]

One then computes that for \( -\pi < Z < \pi \), from (4.40) and (4.44):

\[
\frac{1}{2} \partial_Z (T w) = \frac{1}{4} \partial_Z \left( -\frac{1 + \cos Z}{(1 - \cos Z) \sin^4(Z)} \frac{4(\pi + Z)^3}{s^q(Z)} \right) = \frac{1}{4} \partial_Z \left( -\frac{1}{\phi^{\frac{3}{2}}} \frac{4(\pi + Z)^3}{s^q(Z)} \right).
\]

Therefore, one has from (4.44) the inequality behind the inviscid spectral gap:

\[
-Vu^2 w + u^2 \frac{1}{2} \partial_Z (T w) \leq -u^2 w \frac{1}{\phi^{\frac{3}{2}}} \left( V \phi^{\frac{3}{2}} + T \partial_Z \phi^{\frac{3}{2}} \right) = -\frac{1}{2} u^2 w,
\]

and on \((-\infty, -\pi]\) one has from (4.41):

\[
-Vu^2 w + u^2 \frac{1}{2} \partial_Z (T w) = -u^2 w - \frac{1}{4} w u^2 = -\frac{5}{4} w u^2.
\]

Therefore, from the two inequalities above, on the whole ray \((-\infty, 0]\) there holds:

\[
-Vu^2 w + u^2 \frac{1}{2} \partial_Z (T w) \leq -\frac{1}{2} w u^2.
\]

That is why one has for the part involving the operator \(T \partial_Z + V\):

\[
\int_{Z_1}^{Z_2} u (-Vu - T \partial_Z u) \, w \, dZ \leq -\frac{1}{2} \int_{Z_1}^{Z_2} u^2 w \, dZ + \frac{1}{2} (u^2 T w)(Z_1) - \frac{1}{2} (u^2 T w)(Z_2).
\]

We now turn to the dissipative effects. Integrating by parts one finds:

\[
\int_{Z_1}^{Z_2} u \partial_Z u \, w \, dZ = \left( \frac{1}{2} u^2 \partial_Z w - u \partial_Z u w \right)(Z_1) - \left( \frac{1}{2} u^2 \partial_Z w - u \partial_Z u w \right)(Z_2) - \int_{Z_1}^{Z_2} |\partial_Z u|^2 \, w + \frac{1}{2} \int_{Z_1}^{Z_2} u^2 \, dZ.
\]

The function \(\partial_Z u\), from (4.8), is supported in \((-\pi, 0]\) where one has the bound:

\[
|\partial_Z u| \lesssim |Z|^{-7} \partial_Z (s^{-q(Z)}) + |Z|^{-8} \partial_Z (s^{-q(Z)}) + |Z|^{-9} s^{-q(Z)} \lesssim |Z|^{-9} s^{-q(Z)} (1 + Z^2 \ln^2(s) + |Z| \ln(s))
\]

so that for \(s\) large enough depending on \(M\), for \(Z \leq -Me^{-s}\):

\[
|e^{-2s} \partial_Z u| \lesssim \frac{w}{M^2}.
\]
From (4.16), the above identity becomes for $s$ large enough (since $\partial Z w \geq 0$ near the origin):
\[
\frac{1}{\lambda^4 \mu^2} \int_{Z_1}^{Z_2} u \partial Z w u w dZ \leq C e^{-2s} \left| \frac{1}{2} u^2 \partial Z w - u \partial Z w |(Z_1) + C e^{-2s} |u\partial Z w|(Z_2) \right.
\]
\[
- \frac{1}{\lambda^4 \mu^2} \int_{Z_1}^{Z_2} |\partial Z w|^2 w dZ + \frac{C(K)}{M^2} \int_{Z_1}^{Z_2} u^2 w dZ.
\]
At this point one has proved that for the operator $T \partial Z + V - \lambda^{-4} \mu^{-2} \partial Z Z$:
\[
\int_{Z_1}^{Z_2} u \left( -V u - T \partial Z u + \frac{1}{\lambda^4 \mu^2} \partial Z Z u w \right) w + \frac{1}{2} \int_{Z_1}^{Z_2} u^2 w s + \frac{a_s}{2} (u^2 w)(Z_1) + M e^{-s} (u^2 w)(Z_2)
\]
\[
\leq - \frac{1}{2} \int_{Z_1}^{Z_2} u^2 w - \frac{1}{\lambda^4 \mu^2} \int_{Z_1}^{Z_2} |\partial Z w|^2 w + \frac{C(K)}{M^2} \int_{Z_1}^{Z_2} u^2 w + \frac{a_s}{2} (u^2 w)(Z_1) + M e^{-s} (u^2 w)(Z_2)
\]
\[
+ \frac{1}{2} (u^2 T w)(Z_1) - \frac{1}{2} (u^2 T w)(Z_2) + C e^{-2s} \left| \frac{1}{2} u^2 \partial Z w - u \partial Z w |(Z_1) + C e^{-2s} |u\partial Z w|(Z_2) \right.
\]
\[
\leq - \frac{1}{2} \int_{Z_1}^{Z_2} u^2 w - \frac{1}{\lambda^4 \mu^2} \int_{Z_1}^{Z_2} |\partial Z w|^2 w + \frac{C(K)}{M^2} \int_{Z_1}^{Z_2} u^2 w + C e^{6s} u^2 (Z_2) + C e^{4s} |\partial Z w|^2 (Z_2)
\]
\[
+ C u^2 (Z_1) \left( e^{-\left( \frac{1}{2} - \nu \right) s} + |a_s| \right) + C |\partial Z w|^2 (Z_1) e^{-2s},
\] (4.48)
where we used (4.16), the fact that $|w(Z_2)| \lesssim Z_2^{-7} \lesssim e^{7s}$, $|w(Z_1)| \lesssim 1$, $|T(Z_1)| \lesssim |\pi + Z_1| \lesssim |a| \lesssim e^{-1/2 - \nu s}$, $|T(Z_2)| \lesssim |Z_2| \lesssim e^{-s}$, $|\partial Z w(Z_1)| \lesssim 1$.

**Step 2** The nonlocal term. Using Cauchy-Schwarz one has for $Z \in (-\pi, 0)$:
\[
\left| \int_Z^Z u(s, \tilde{Z}) d\tilde{Z} \right| \leq \int_Z^Z |u(s)| dZ + \left( \int_Z^Z u^2 w dZ \right)^{1/2} \left( \int_Z^Z w^{-1}(s, \tilde{Z}) d\tilde{Z} \right)^{1/2}.
\] (4.49)
One computes that for $Z \in (-\pi, 0)$:
\[
|w^{-1}(s, Z)| \lesssim |Z|^{-q(Z)} = |Z|^{-q(Z) \ln s}
\]
from what we infer from the assumptions on $q$ in Subsection 4.2:
\[
\int_Z^Z w^{-1}(s, \tilde{Z}) d\tilde{Z} \lesssim \int_Z^Z |\tilde{Z}|^{-q(Z) \ln s} d\tilde{Z} \lesssim |Z|^{-q(Z) \ln s} \int_Z^Z \frac{1}{\ln s} \partial Z \frac{d}{dZ} (e^{q(Z) \ln s}) \lesssim \frac{|Z|^{-q(Z)} e^{q(Z) \ln s}}{|\pi + Z| \ln s}.
\] (4.50)
Therefore:
\[
\left( \int_Z^Z w^{-1}(s, \tilde{Z}) d\tilde{Z} \right)^{1/2} \lesssim \frac{|Z|^{-q(Z) \ln s}}{\ln s} \sin Z
\]
which produces
\[
\int_{Z_1}^{Z_2} \left( \int_Z^Z w^{-1}(s, \tilde{Z}) d\tilde{Z} \right)^{1/2} \sin^2 Z 1_{-\pi \leq Z \leq 0} w dZ \lesssim \int_{-\pi}^0 |Z|^{-q(Z) \ln s} \sin Z dZ \lesssim \frac{1}{\ln s}.
\]
One also has
\[
\int_{Z_2}^{Z_1} \sin^2 Z 1_{0 \leq Z \leq \pi} w dZ \approx \int_{Z_2}^{Z_1} \frac{dZ}{|Z|^{q(Z)}} \lesssim \frac{e^{4s}}{M^4}.
\]
Thus the contribution of the nonlocal term is estimated as follows:

\[
\begin{align*}
\int_{Z_2}^{Z_1} u \left( \int_{0}^{Z} u(s, \tilde{Z})d\tilde{Z} \right) \sin Z \mathbb{1}_{-\pi \leq \tilde{Z} \leq 0}wdZ = \\
\lesssim \int_{Z_2}^{Z_1} u \left( \int_{Z_2}^{0} u \right) \sin Z \mathbb{1}_{-\pi \leq \tilde{Z} \leq 0}wdZ + \int_{Z_2}^{Z_1} u \left( \int_{Z_2}^{Z} u \right) \sin Z \mathbb{1}_{-\pi \leq \tilde{Z} \leq 0}wdZ \\
\lesssim \frac{1}{\ln s} \int_{Z_2}^{Z_1} u^2wdZ + \frac{e^{2s}}{M^2} \left( \int_{Z_2}^{0} |u|dZ \right) \left( \int_{Z_2}^{Z_1} u^2wdZ \right)^{\frac{1}{2}}.
\end{align*}
\]

**Step 3** End of the proof. The above identity, (4.48), and (4.46) finally yield

\[
\frac{d}{ds} \left( \frac{1}{2} \int_{Z_2}^{Z_1} u^2w \right) \leq \frac{1}{2} \int_{Z_1}^{Z_2} u^2w - \frac{1}{\lambda^4 \mu^2} \int_{Z_1}^{Z_2} \partial_Z u^2 w + \frac{C(K)}{M^2} \int_{Z_1}^{Z_2} u^2 w + Ce^{6s} u^2(Z_2) + Ce^{4s} |\partial_Z u|^2(Z_2) \\
+ Cu^2(Z_1) \left( e^{-\left(\frac{1}{2} - \nu\right)s} + |a_s| \right) + C|\partial_Z u|^2(Z_1) \left( e^{-2s} + \frac{1}{\ln s} \int_{Z_2}^{Z_1} u^2w + \frac{e^{2s}}{M^2} \left( \int_{Z_2}^{0} |u| \right) \left( \int_{Z_2}^{Z_1} u^2wdZ \right)^{\frac{1}{2}} \right),
\]

\[
\leq \left( \frac{1}{2} + \frac{C(K)}{M^2} + \frac{C}{\ln s} \right) \int_{Z_1}^{Z_2} u^2w + Ce^{6s} u^2(Z_2) + Ce^{4s} |\partial_Z u|^2(Z_2) + Cu^2(Z_1) \left( e^{-\left(\frac{1}{2} - \nu\right)s} + |a_s| \right) \\
+ C|\partial_Z u|^2(Z_1) \left( e^{-2s} + \frac{e^{2s}}{M^2} \left( \int_{Z_2}^{0} |u| \right) \left( \int_{Z_1}^{Z_2} u^2wdZ \right)^{\frac{1}{2}} \right) - \frac{1}{\lambda^4 \mu^2} \int_{Z_1}^{Z_2} |\partial_Z u|^2 w,
\]

which ends the proof of the Lemma for M and s_0 large enough.

\[
\square
\]

### 4.5.2. Exterior Lyapunov estimates

We now study the functional (16) for the the full problem. First, let us estimate the function at the boundaries, Z_1 = -\pi - \alpha and Z_2 = -Me^{-s}. From (4.17) and Sobolev near the maximum:

\[
u^2(Z_2) = e^2(-M\lambda^2 \mu e^{-s}) \leq C||\epsilon||_{H^2(\Omega) \leq M^2} \leq Ce^{-(7-2\nu)s}, \quad (\partial_Z u)^2(Z_2) \leq Ce^{-(5-2\nu)s}.
\]

From the boundary condition (4.5), the decomposition (4.7), (3.5) and (4.16), at the origin in original variables:

\[
u^2(Z_1) = G_1^2(-\pi - \alpha) \leq Ca^4 \leq Ce^{-\left(2-8\nu\right)s}.
\]

Finally, from (4.7), (4.7) and (4.16):

\[
|\partial_Z u(Z_1)| \leq |\partial Z F(Z_1)| + |\partial_Z G_1(Z_1)| \leq \lambda^{-1} \mu |\partial_y \xi(0)| + C|\alpha| \leq Ce^{-\left(\frac{1}{2} - 2\nu\right)s}.
\]

One has the following energy estimate for the function in Z variable outside the maximum.

**Lemma 17** (Exterior Lyapunov functional on the left). For M and s_0 large enough, there exists C > 0 such that if F is trapped on [s_0, s^*):

\[
\frac{d}{ds} \left( \frac{1}{2} \int_{Z_1}^{Z_2} u^2wdZ \right) + \left( \frac{1}{2} - \frac{\nu}{2} \right) \int_{Z_1}^{Z_2} u^2wdZ \leq C \left( e^{6s} u^2(Z_2) + e^{4s} |\partial_Z u|^2(Z_2) + \left( \int_{Z_1}^{Z_2} u^2wdZ \right)^{\frac{1}{2}} e^{-\left(\frac{1}{2} + \frac{1}{4}\right)s} + e^{-(2+\frac{1}{4})s} + e^{(6-\frac{1}{6})\|\partial_Z \epsilon\|_L^2} \right)
\]

(4.54)
Proof. One first computes from (4.38) the identity
\[
\frac{d}{ds} \left( \frac{1}{2} \int_{Z_1}^{Z_2} u^2 w \right) = \int_{Z_1}^{Z_2} u (-\mathcal{H}u + \frac{\partial ZZ u}{\lambda^2 \mu^2} - \tilde{\Psi}u - NL - \Psi) w + \int_{Z_1}^{Z_2} \frac{u^2}{2} w_s + a_s (u^2 w)(Z_1) + \frac{M e^{-s}}{2} (u^2 w)(Z_2). 
\]  
(4.55)  

**Step 1** The leading order linear terms. From (4.52), (4.87) and (4.53):
\[
u^2(Z_1) \left( e^{-\left( \frac{2}{5} - \nu \right) s} \right) + |\partial_Z u|^2(Z_1) e^{-2s} \lesssim e^{\left( \frac{4}{5} - 10\nu \right) s} + e^{\left( \frac{2}{5} - \frac{8}{3} \nu \right) s} \|\partial_Y \varepsilon\|_{L^2_\rho}^2.
\]
From (4.3), (4.16) and (4.17), as \(\lambda^2 e^{-s} \mu M \approx 1\):
\[e^{2s} \int_{Z_2}^{Z_1} |u| dZ = e^{2s} \int_{Z_2}^{Z_1} \|\varepsilon\|_{L^2_\rho} \lesssim e^{\frac{5}{2}s}.\]

We now apply Lemma 16 and inject the two above inequalities:
\[
\int_{Z_1}^{Z_2} u (-\mathcal{H}u + \frac{1}{\lambda^2 \mu^2} \partial ZZ u) w + \frac{1}{2} \int_{Z_1}^{Z_2} u^2 w_s + a_s (u^2 w)(Z_1) + \frac{M e^{-s}}{2} (u^2 w)(Z_2)
\]
\[\leq \left( -\frac{1}{2} + \frac{\nu}{4} \right) \int_{Z_1}^{Z_2} u^2 w + C e^{6s} u^2(Z_2) + C e^{4s} |\partial_Z u|^2(Z_2) + e^{-\left( \frac{4}{5} - 10\nu \right) s} + e^{\left( \frac{2}{5} - \frac{8}{3} \nu \right) s} \|\partial_Y \varepsilon\|_{L^2_\rho}^2
\]
\[+ e^{-\frac{5}{2}s} \left( \int_{Z_1}^{Z_2} u^2 w \right)^{\frac{1}{2}} - \frac{1}{\lambda^4 \mu^2} \int_{Z_1}^{Z_2} |\partial_Z u|^2 w.\]
(4.56)

**Step 2** The small linear term. Recall (4.42), then
\[ - \int_{Z_1}^{Z_2} u \tilde{\mathcal{H}} u w dZ = - \int_{Z_1}^{Z_2} u \left( d(s) \partial_Z u + \left( \frac{\lambda_s}{\lambda} - \frac{1}{2} \right) \right) (2 - Z \partial_Z u) w dZ.\]
Integrating by parts, one has:
\[\int_{Z_1}^{Z_2} u \partial_Z u w dZ = \frac{1}{2} (u^2 w)(Z_2) - \frac{1}{2} (u^2 w)(Z_1) - \frac{1}{2} \int_{Z_1}^{Z_2} u^2 \partial_Z w dZ.\]
One has that \(\partial_Z w\) is supported on \((-\pi, 0)\), with for \(|Z| \gtrsim e^{-s}\):
\[|\partial_Z w| \lesssim |Z|^8 s^9(Z) (1 + |Z| \ln s) \lesssim e^{8s}.\]
Therefore, since \(w(Z_1) \lesssim 1\) and \(w(Z_2) \lesssim e^{7s}\):
\[\left| \int_{Z_1}^{Z_2} u d(s) \partial_Z u w dZ \right| \lesssim e^s |d| e^{6s} u^2(Z_2) + |d| u^2(Z_1) + e^s |d| \int_{Z_1}^{Z_2} u^2 w dZ.\]
The very same strategy applies for the other term, and since \(|\partial_Z(Z w)| \lesssim e^{8/2} w\), this gives:
\[\left| \int_{Z_1}^{Z_2} u \left( \frac{\lambda_s}{\lambda} - \frac{1}{2} \right) ((2 - Z \partial_Z u) w) \right| \lesssim \left| \frac{\lambda_s}{\lambda} - \frac{1}{2} \right| e^{6s} u^2(Z_2) + \left| \frac{\lambda_s}{\lambda} - \frac{1}{2} \right| u^2(Z_1) + e^\frac{s}{2} \left| \frac{\lambda_s}{\lambda} - \frac{1}{2} \right| \int_{Z_1}^{Z_2} u^2 w.\]
In conclusion one has for the small linear term, using (4.43), (4.22), (4.52), (4.18) and (4.51) as 0 < \(\nu' \ll \nu\):
\[\left| \int_{Z_1}^{Z_2} u \tilde{\mathcal{H}} u w dZ \right| \lesssim \left( \frac{\lambda_s}{\lambda} - \frac{1}{2} \right) e^s |d| e^{6s} u^2(Z_2) + \left( \frac{\lambda_s}{\lambda} - \frac{1}{2} \right) |d| u^2(Z_1)
\]
\[+ \left( e^\frac{s}{2} \left| \frac{\lambda_s}{\lambda} - \frac{1}{2} \right| + e^s |d| \right) \int_{Z_1}^{Z_2} u^2 w dZ
\]
\[\lesssim e^{-\left( \frac{2}{5} + \frac{4}{3} \nu' \right) s} e^{6s} u^2(Z_2) + e^{-\left( \frac{2}{5} + \frac{4}{3} - 2\nu \right) s} + e^{\left( \frac{2}{5} - \frac{8}{3} \nu' \right) s} \|\partial_Y \varepsilon\|_{L^2_\rho}^2.\]
(4.57)
Step 3 The nonlinear term. For the nonlinear term one recalls the identity:
\[
\int_{Z_1}^{Z_2} uNL \, w \, dZ = \int_{Z_1}^{Z_2} u(-u^2 + \partial_Z^{-1}u \partial_Z u) \, w \, dZ.
\]
The first term is estimated in brute force:
\[
\left| \int_{Z_1}^{Z_2} u^3 \, w \, dZ \right| \leq \|u\|_{L^\infty} \int_{Z_1}^{Z_2} u^2 \, w \, dZ.
\]
For the second, we integrate by parts and use the brute force estimate
\[
\int_{Z_1}^{Z_2} u \partial_Z^{-1}u \partial_Z w = \frac{1}{2}(u^2 \partial_Z^{-1}u)w(Z_2) - \frac{1}{2}(u^2 \partial_Z^{-1}u)w(Z_1) - \frac{1}{2} \int_{Z_1}^{Z_2} u^3 \, w \, dZ \leq \frac{1}{2} \int_{Z_1}^{Z_2} \partial_Z^{-1}u \partial_Z wu^2
\]
since \(|\partial_Z w| \lesssim \ln(s)w\). In conclusion, the contribution of the nonlinear term is, using (4.19) and (4.52) for \(s_0\) large enough:
\[
\left| \int_{Z_1}^{Z_2} uNL \, w \, dZ \right| \lesssim \|u\|_{L^\infty} e^{6s} u^2(Z_2) + \|u\|_{L^\infty} u^2(Z_1) + \ln(s) \|u\|_{L^\infty} \int_{Z_1}^{Z_2} wu^2 \, dZ
\]
\[
\lesssim e^{(6-\frac{1}{2})s} u^2(Z_2) + e^{-(2+\frac{1}{2}-9\nu)s} + \nu \frac{1}{4} \int_{Z_1}^{Z_2} wu^2 \, dZ. \quad (4.58)
\]
Step 4 The error term. One has that \(\psi\) is supported on \([-\pi, 0]\), with the estimate from (4.1)
\[
|\psi(s, Z)| = \left| -\frac{1}{\lambda^4 \mu^2} \partial_{ZZ} G_1(Z) + \left(\frac{\lambda_s}{\lambda} - \frac{1}{2}\right)(2 - Z \partial_Z) G_1(Z) + d(s) \partial_Z G_1(Z) \right|
\]
\[
\lesssim \left| \frac{\lambda_s}{\lambda} - \frac{1}{2} + \frac{1}{4 \lambda^4 \mu^2} \right| + Z^2 \left(\frac{1}{\lambda^2} + \left| \frac{\lambda_s}{\lambda} - \frac{1}{2}\right| \right) + |d||Z|.
\]
Since \(w \lesssim |Z|^{-7}\) one has, using (4.22), (4.43), (4.16) and (4.17):
\[
\int_{Z_1}^{Z_2} \psi^2 \, w \, dZ \lesssim e^{6s} \left(\frac{\lambda_s}{\lambda} - \frac{1}{2} + \frac{1}{4 \lambda^4 \mu^2} \right)^2 + e^{2s} \left(\frac{1}{\lambda^2} + \left| \frac{\lambda_s}{\lambda} - \frac{1}{2}\right| \right)^2 + e^{4s} \, d^2
\]
\[
\lesssim e^{-(1+\frac{1}{2})s} + e^{(13-\frac{1}{2})s} \|\partial_Y \|_L^2.
\]
By Cauchy Schwarz and (4.18), one has proved that for the error term:
\[
\left| \int_{Z_1}^{Z_2} \psi^2 \, w \, dZ \right| \lesssim \left( \int_{Z_1}^{Z_2} u^2 \, w \right)^{\frac{1}{2}} e^{-(\frac{1}{2}+\frac{1}{2})s} + e^{(\frac{13}{2}-\frac{1}{2})s} \|\partial_Y \|_L^2 \left( \int_{Z_1}^{Z_2} u^2 \, w \right)^{\frac{1}{2}} \lesssim \left( \int_{Z_1}^{Z_2} u^2 \, w \right)^{\frac{1}{2}} e^{-(\frac{1}{2}+\frac{1}{2})s} + e^{(6+\nu-\frac{1}{2})s} \|\partial_Y \|_L^2. \quad (4.59)
\]
Step 5 End of the proof. Collecting the previous estimates (4.56), (4.57), (4.58) and (4.59) and injecting them in (4.55) yields the desired energy estimate (4.54).
Since $A = 1$ near $-\pi$, from (4.53):

$$|A\partial_Z u|^2(Z_1) \leq C|\partial_Z u|^2(Z_1) \leq Ce^{-(1-4\nu)s}.$$  \hspace{1cm} (4.62)

Now we write $\partial_Z(A\partial_Z u) = A\partial_Z^2 u$ since $|\partial_Z A(-\pi - a)| = 0$. Since $\partial_y \xi(0) = 0$ from the boundary condition in the equation (1.3), (4.7) and (4.16) imply:

$$|\partial_Z(A\partial_Z u)(Z_1)| = |\partial_Z^2 u(Z_1)| = |\partial_Z^2 (F - G_1)(Z_1)| \leq |\lambda^2 \mu^2 \partial_y \xi(0)| + |\partial_Z^2 G_1|(-\pi - a) \leq \frac{1}{2}. $$  \hspace{1cm} (4.63)

We perform the same weighted energy estimate outside the maximum for $A\partial_Z u$ as we did for $u$.

**Lemma 18** (Exterior Lyapunov functional on the left for the derivative). Let $Z_1 = -\pi - a$, $Z_2 = -Me^{-s}$ and $v = A\partial_Z u$. There exists a constant $C$ independent of the bootstrap constants, such that if $F$ is trapped on $[s_0, s^*]$ one has

$$\frac{d}{ds} \left( \frac{1}{2} \int_{Z_1}^{Z_2} v^2 w dZ \right) - \nu \int_{Z_1}^{Z_2} v^2 w dZ + \frac{1}{2\lambda^4 \mu^2} \int_{Z_1}^{Z_2} |\partial_Z v|^2 w dZ \leq Ce^{-4s} + Ce^{\lambda^4 - \frac{1}{2} \mu^2 s}\|\partial_Y \xi\|^2_{L^2}.$$  \hspace{1cm} (4.64)

**Proof.** One first computes the evolution equation for $v = A\partial_Z u$ from (4.38):

$$0 = v_s + (T\partial_Z + V)v + \frac{A\partial_Z T - T\partial_Z A}{A}v - \frac{1}{\lambda^4 \mu^2} (\partial_Z^2 v + [A\partial_Z, \partial_Z^2]u) + \tilde{H}v + [A\partial_Z, \tilde{H}]u, $$
$$+ NL + A\partial_Z \Psi + A\partial_Z G_1 + \partial_Z^{-1}u A\partial_Z G_1$$  \hspace{1cm} (4.65)

where

$$NL = - \left( u + \partial_Z^{-1}u \frac{\partial_Z A}{A} \right)v + \partial_Z^{-1}u \partial_Z v.$$  

One first has the following identity for the energy estimate:

$$\frac{d}{ds} \left( \frac{1}{2} \int_{Z_1}^{Z_2} v^2 w dZ \right) = \int_{Z_1}^{Z_2} v w_s w dZ + \frac{1}{2} \int_{Z_1}^{Z_2} v^2 w_s dZ + \frac{a_s}{2} (v^2 w)(Z_1) + \frac{Me^{-s}}{2} (v^2 w)(Z_2).$$  \hspace{1cm} (4.66)

**Step 1** The leading order linear terms. From (4.48), injecting (4.60), (4.61), (4.62), (4.63):

$$\int_{Z_1}^{Z_2} v \left( -v v - T\partial_Z v + \frac{1}{\lambda^4 \mu^2} \partial_Z^2 v \right) w + \frac{1}{2} \int_{Z_1}^{Z_2} u^2 w_s + \frac{a_s}{2} (v^2 w)(Z_1) + \frac{Me^{-s}}{2} (v^2 w)(Z_2)$$
$$\leq -\frac{1}{2} \int_{Z_1}^{Z_2} v^2 w - \frac{1}{\lambda^4 \mu^2} \int_{Z_1}^{Z_2} |\partial_Z v|^2 w + \nu \int_{Z_1}^{Z_2} v^2 w + Ce^{6s} v(Z_2) + Ce^{2s} |\partial_Z v|^2(Z_2)$$
$$+ C v^2(Z_1)e^{-(\frac{1}{2} - \nu)s} + C|\partial_Z v|^2(Z_1)e^{-2s}$$
$$\leq -\frac{1}{2} \int_{Z_1}^{Z_2} v^2 w - \frac{1}{\lambda^4 \mu^2} \int_{Z_1}^{Z_2} |\partial_Z v|^2 w + \nu \int_{Z_1}^{Z_2} v^2 w + Ce^{-(1-2\nu)s}.$$  \hspace{1cm} (4.67)

Then, for the commutator with $A$ and the transport $T$, a direct computation shows, since $A = 2T$ for $|Z| \leq \pi/2$, and $A = -1$ for $Z \leq -\pi/2$, that for all for $Z \leq 0$:

$$\frac{A\partial_Z T - T\partial_Z A}{A} = \partial_Z T\mathbb{1}_{Z \leq -\frac{\pi}{2}} \leq -\frac{1}{2} \mathbb{1}_{Z \leq -\frac{\pi}{2}}$$

which implies:

$$-\int_{Z_1}^{Z_2} v \frac{A\partial_Z T - T\partial_Z A}{A} w \leq \frac{1}{2} \int_{Z_1}^{Z_2} v^2 w.$$  \hspace{1cm} (4.68)
Step 2 The small linear term and other commutators. For the small linear term, from (4.57), inserting (4.22), (4.43), (4.60), (4.61) and (4.18):

\[
\left| \int_{Z_1}^{Z_2} v \tilde{H} v w dZ \right| \lesssim \left( \left| \frac{\lambda_s}{\lambda} - \frac{1}{2} \right| + e^s |d| \right) e^{6s} v^2(Z_2) + \left( \left| \frac{\lambda_s}{\lambda} - \frac{1}{2} \right| + |d| \right) v^2(Z_1)
+ \left( e^s \left| \frac{\lambda_s}{\lambda} - \frac{1}{2} \right| + e^s |d| \right) \int_{Z_1}^{Z_2} v^2 w dZ
\lesssim e^{-\left( \frac{4}{3} - 2\nu \right)s} + e^{\left( \frac{1}{2} - \frac{1}{2} + 2\nu \right)s} \| \partial_Y \varepsilon \|^2_{L^2}.
\]  

(4.69)

Next, we turn to the commutator with the dissipative term. one has

\[ [A \partial_Z, \partial_{ZZ}]u = \left( -\frac{\partial_{ZZ} A}{A} + \frac{2(\partial_Z A)^2}{A^2} \right) v - 2 \frac{\partial_Z A}{A} \partial_Z v. \]

Since, for \( Z \geq Me^{-s} \):

\[ \left| \frac{\partial_{ZZ} A}{A} \right| + \left| \frac{(\partial_Z A)^2}{A^2} \right| \leq \frac{C e^{2s}}{M^2}, \]

one has for the first term that:

\[ \left| \frac{1}{\lambda^4 \mu^2} \int_{Z_1}^{Z_2} v^2 \left( -\frac{\partial_{ZZ} A}{A} v + \frac{2(\partial_Z A)^2}{A^2} \right) w dZ \right| \leq \frac{C(K)}{M^2} \int_{Z_1}^{Z_2} v^2 w dZ. \]

For the second term, one first integrates by parts:

\[ -\int_{Z_1}^{Z_2} 2v \frac{\partial_Z A}{A} \partial_Z v w dZ = \left( v^2 \frac{\partial_Z A}{A} w \right)(Z_1) - \left( v^2 \frac{\partial_Z A}{A} w \right)(Z_2) + \int_{Z_1}^{Z_2} v^2 \partial_Z \left( \frac{\partial_Z A}{A} w \right) dZ \]
\[ = -\left( v^2 \frac{\partial_Z A}{A} w \right)(Z_2) + \int_{Z_1}^{Z_2} v^2 \partial_Z \left( \frac{\partial_Z A}{A} w \right) dZ \]

since \( \partial_Z A(Z_1) = 0 \). From a direct inspection:

\[ \left| \partial_Z \left( \frac{\partial_Z A}{A} w \right) \right| \leq \frac{C w}{Z^2} \leq \frac{C e^{2s}}{M^2} w. \]

Therefore:

\[ \left| \frac{1}{\lambda^4 \mu^2} \int_{Z_1}^{Z_2} 2v \frac{\partial_Z A}{A} \partial_Z v w dZ \right| \leq C e^{6s} v^2(Z_2) + \frac{C(K)}{M^2} \int_{Z_1}^{Z_2} v^2 w dZ. \]

One has proved that for the commutator with the dissipative term, for \( M \) large enough depending on \( K \), using (4.60):

\[ \left| \frac{1}{\lambda^4 \mu^2} \int_{Z_1}^{Z_2} v [A \partial_Z, \partial_{ZZ}]u \right| \leq C e^{6s} v^2(Z_2) + \frac{C(K)}{M^2} \int_{Z_1}^{Z_2} v^2 w \leq C e^{-\left( 1 - 2\nu \right)s} + \frac{v}{8} \int_{Z_1}^{Z_2} v^2 w. \]  

(4.70)

Next, one computes that the commutator with the small linear term is:

\[ [A \partial_Z, \tilde{H}]u = \left( -d \frac{\partial_Z A}{A} - \left( \frac{\lambda_s}{\lambda} - \frac{1}{2} \right) \left( 1 - \frac{Z \partial_Z A}{A} \right) \right) v. \]

Since \( |\partial_Z A/A| \lesssim 1/Z \lesssim e^s \) for \( |Z| \geq Me^{-s} \), this implies using (4.22), (4.43) and (4.18):

\[ \left| \int_{Z_1}^{Z_2} v [A \partial_Z, \tilde{H}]u \right| \lesssim \left( \left| \frac{\lambda_s}{\lambda} - \frac{1}{2} \right| + e^s |d| \right) \int_{Z_1}^{Z_2} v^2 w \lesssim e^{-\left( \frac{5}{3} + 2\nu \right)s} + e^{\left( \frac{1}{2} - \frac{1}{2} + 2\nu \right)s} \| \partial_Y \varepsilon \|^2_{L^2}. \]  

(4.71)
Step 3 The nonlinear term. Since $|\partial_Z A| \lesssim 1/Z$ one has:

$$\left| \int_{Z_1}^{Z_2} v \left( u + \partial_Z^{-1} \frac{\partial Z A}{A} \right) v w dZ \right| \lesssim \|u\|_{L^\infty} \int_{Z_1}^{Z_2} v^2 w dZ.$$  

For the other term, an integration by parts gives:

$$\left| \int_{Z_1}^{Z_2} v \partial_Z^{-1} u \partial_Z v w dZ \right| = \left| \frac{1}{2} (\partial_Z^{-1} w)^2 (Z_1) - \frac{1}{2} (\partial_Z^{-1} w)^2 (Z_2) + \int_{Z_1}^{Z_2} v^2 \partial_Z (\partial_Z^{-1} u) w dZ \right|$$

$$\lesssim \|u\|_{L^\infty} v^2 (Z_1) + \|u\|_{L^\infty} e^{6s} v^2 (Z_2) + \log(s) \|u\|_{L^\infty} \int_{Z_1}^{Z_2} v^2 w dZ,$$

where we used the fact that $|\partial_Z w| \lesssim \log(s) Z^{-1} w$. One has then showed that for the nonlinear term, using (4.19), (4.60) and (4.62), as $0 < \nu' \ll \nu$:

$$\left| \int_{Z_1}^{Z_2} v \partial_Z \psi (s, Z) \right| \lesssim \|u\|_{L^\infty} v^2 (Z_1) + \|u\|_{L^\infty} e^{6s} v^2 (Z_2) + \log(s) \|u\|_{L^\infty} \int_{Z_1}^{Z_2} v^2 w dZ$$

$$\lesssim e^{-(1 + \frac{1}{4} - 5\nu)s} + \nu \int_{Z_1}^{Z_2} v^2 w dZ. \tag{4.72}$$

Step 4 The error term. For the error one first computes, since $|A| \lesssim |Z|$ for $|Z| \leq \pi$ with $A(\pi) = -1$, and since $\partial_{ZZ} G_1$ has limit 0 and 1/2 on the left and on the right of $-\pi$ respectively:

$$A \partial_Z \psi (s, Z) = A \partial_Z \left( -\frac{1}{\lambda^4} \partial_{ZZ} G_1 (Z) + \frac{\lambda s}{\lambda} - \frac{1}{2} \right) (2 - Z \partial_Z) G_1 (Z) + d(s) \partial_Z G_1 (Z)$$

$$= \frac{1}{2} \delta (Z - \pi) + O \left( Z^2 \left( \frac{1}{\lambda^4} + \frac{\lambda s}{\lambda} - \frac{1}{2} \right) \right).$$

Since $w \lesssim |Z|^7$ one has:

$$\int_{Z_1}^{Z_2} \left| O \left( Z^2 \left( \frac{1}{\lambda^4} + \left| \frac{\lambda s}{\lambda} - \frac{1}{2} \right| \right) + |d| Z \right) \right|^2 w dZ \lesssim e^{2s} \left( \frac{1}{\lambda^4} + \left| \frac{\lambda s}{\lambda} - \frac{1}{2} \right| \right)^2 + e^{4s} d^2.$$  

For the Dirac term, either one has $a < 0$ and then $-\pi < Z_1$ in which case there is nothing to estimate since

$$\int_{Z_1}^{Z_2} v \delta (Z - \pi) dZ = 0.$$  

Otherwise, if $Z_1 \leq -\pi$, we use Sobolev embedding (since $w \approx s^{-1}$ near $-\pi$) to find:

$$\frac{1}{\lambda^4} \int_{Z_1}^{Z_2} v \delta (Z - \pi) dZ = \frac{1}{\lambda^4} \int_{Z_1}^{Z_2} v (\pi) v (-\pi) \leq \frac{C}{\lambda^4} \left( \left( \int_{Z_1}^{Z_2} v^2 w \right)^{\frac{1}{2}} + \left( \int_{Z_1}^{Z_2} (\partial_Z v)^2 w \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}$$

$$\leq \frac{C}{\lambda^4} \left( \int_{Z_1}^{Z_2} v^2 w \right)^{\frac{1}{2}} + \frac{C K}{\lambda^4} \int_{Z_1}^{Z_2} (\partial_Z v)^2 w + \frac{C}{\kappa \lambda^4}.$$  

Using Cauchy-Schwarz, one has then showed that for the error term, in both cases $Z_1 \leq \pi$ or $Z_1 > \pi$, for $\kappa$ small enough:

$$\int_{Z_1}^{Z_2} v A \partial_Z \psi w \leq \left( e^{s} \left( \frac{1}{\lambda^4} + \left| \frac{\lambda s}{\lambda} - \frac{1}{2} \right| \right) + e^{2s} |d| \right) \left( \int_{Z_1}^{Z_2} v^2 w \right)^{\frac{1}{2}} + \frac{1}{2 \lambda^4} \int_{Z_1}^{Z_2} (\partial_Z v)^2 w + \frac{C}{\kappa \lambda^4}$$

$$\lesssim e^{-(1 + \frac{1}{4} - \nu) s} + e^{(\frac{1}{4} - \frac{1}{8} + \nu) s} ||\partial_Y \psi||_p^2 + \frac{1}{2 \lambda^4} \int_{Z_1}^{Z_2} (\partial_Z v)^2 w \tag{4.73}$$

where we used (4.16), (4.22), (4.43) and (4.18) for the last inequality.
Step 5 The remaining lower order terms. One has from (4.18) that for the first one:

\[
\left| \int_{Z_1}^{Z_2} v A u \partial_Z G_1 w dZ \right| \lesssim \left( \int_{Z_1}^{Z_2} u^2 w dZ \right)^{\frac{1}{2}} \left( \int_{Z_1}^{Z_2} v^2 w dZ \right)^{\frac{1}{2}} \lesssim e^{-\left\lfloor \frac{1}{2} - 2\nu \right\rfloor s} \tag{4.74}
\]

since \( A \partial_Z G_1 \) is bounded. For the last term, from (4.49) one has:

\[
|\partial_Z^{-1} u A \partial_Z Z G_1| \lesssim \left( \int_{Z_2}^{0} |u| d\tilde{Z} \right) |Z| 1_{-\pi \leq Z \leq 0} + \left( \int_{Z_1}^{Z_2} u^2 w d\tilde{Z} \right)^{\frac{1}{2}} \left( \int_{Z_1}^{0} w^{-1} d\tilde{Z} \right)^{\frac{1}{2}} |Z| 1_{-\pi \leq Z \leq 0}
\]

\[
\lesssim \left( \int_{Z_2}^{0} |u| d\tilde{Z} \right) |Z| 1_{-\pi \leq Z \leq 0} + \sqrt{s} \left( \int_{Z_1}^{Z_2} u^2 w d\tilde{Z} \right)^{\frac{1}{2}} |Z|^5 1_{0 \leq Z \leq \pi}
\]

where we used the fact that \( w \approx |Z|^{-7} s^{-q(Z)} \) for \(-\pi \leq Z < 0\), and that \( q \) is maximal at \(-\pi\) with \( q(-\pi) = 1 \). One then computes that

\[
\int_{Z_1}^{Z_2} Z^2 w dZ \lesssim \int_{Z_1}^{Z_2} Z^{-5} dZ \lesssim e^{4s}, \quad \int_{Z_1}^{Z_2} Z^{10} w dZ \lesssim 1.
\]

Therefore:

\[
\int_{Z_1}^{Z_2} |\partial_Z^{-1} u A \partial_Z Z G_1|^2 w dZ \lesssim e^{4s} \left( \int_{Z_2}^{0} |u| d\tilde{Z} \right)^2 + \int_{Z_1}^{Z_2} u^2 w dZ
\]

which, by Cauchy-Schwarz, gives for the lower last order term, using (4.17) and (4.18):

\[
\left| \int_{Z_1}^{Z_2} v \partial_Z^{-1} u A \partial_Z Z G_1 w dZ \right| \lesssim \left( \int_{Z_1}^{Z_2} v^2 w dZ \right)^{\frac{1}{2}} \left( \int_{Z_1}^{1} |u| d\tilde{Z} \right) + s \left( \int_{Z_1}^{Z_2} u^2 w dZ \right)^{\frac{1}{2}} \lesssim e^{\nu s} \left( e^{s} e^{-\frac{7}{2} s} + s e^{-(\frac{3}{2} - \nu) s} \right) \lesssim e^{-(\frac{3}{2} - 2\nu) s}.
\]

Step 6 End of the proof. In conclusion, from the identities (4.65), (4.66), collecting the estimates (4.67), (4.68), (4.70), (4.69), (4.71), (4.72), (4.73), (4.74) and the above inequality:

\[
\frac{d}{ds} \left( \frac{1}{2} \int_{Z_1}^{Z_2} v^2 w dZ \right) \leq -\frac{1}{2} \int_{Z_1}^{Z_2} v^2 w dZ - \frac{1}{\lambda^4 \mu^2} \int_{Z_1}^{Z_2} |\partial_Z v|^2 w dZ + \frac{\nu}{4} \int_{Z_1}^{Z_2} v^2 w dZ + C e^{-(1-2\nu)s} + \frac{1}{2} \int_{Z_1}^{Z_2} v^2 w dZ + C e^{-(\frac{3}{2} - 2\nu)s} + C e^{\left(\frac{1}{2} - \frac{3}{4} + 2\nu\right)s} \||\partial_Y \varepsilon\|^2_{L^2_p} + C e^{-(\frac{3}{2} + \frac{1}{4} - 2\nu)s}
\]

\[
+ C e^{\left(\frac{1}{2} - \frac{3}{4} + 2\nu\right)s} \||\partial_Y \varepsilon\|^2_{L^2_p} + C e^{-(1+\frac{1}{4} - 5\nu)s} + \frac{\nu}{8} \int_{Z_1}^{Z_2} v^2 w - C e^{-(\frac{1}{2} + \frac{1}{4} - \nu)s} + C e^{\left(\frac{1}{4} - \frac{1}{2} + \nu\right)s} \||\partial_Y \varepsilon\|^2_{L^2_p} + \frac{1}{2} \lambda^4 \mu^2 \int_{Z_1}^{Z_2} (\partial_Z v)^2 w dZ + C e^{-(\frac{3}{2} - 2\nu)s}
\]

\[
\leq \frac{\nu}{2} \int_{Z_1}^{Z_2} v^2 w dZ - \frac{1}{2 \lambda^4 \mu^2} \int_{Z_1}^{Z_2} |\partial_Z v|^2 w dZ + C e^{-(\frac{3}{2} - 2\nu)s} + C e^{\left(\frac{1}{4} - \frac{1}{2} + \nu\right)s} \||\partial_Y \varepsilon\|^2_{L^2_p}
\]

which is the desired differential inequality (4.64). \( \square \)

The very same analysis can be done at the right of the origin. The analogues of Lemmas 17 and 18 hold and their proofs are exactly the same.
Lemma 19 (Exterior Lyapunov functionals on the right). Let $Z_3 = M e^{-s}$. For $M$ and $s_0$ large enough, there exists $C > 0$ such that if $F$ is trapped on $[s_0, s^*]$ one has, where $v = A \partial_z u$:

$$
\frac{d}{ds} \left( \frac{1}{2} \int_{Z_3}^{+\infty} u^2 w dZ \right) + \left( \frac{1}{2} - \frac{\nu}{2} \right) \int_{Z_3}^{+\infty} u^2 w dZ 
\leq C \left( e^{6s} u^2(Z_3) + e^{4s} |\partial_z u|^2(Z_3) + e^{-(2 + \frac{1}{4})s} + \left( \int_{Z_3}^{+\infty} u^2 w dZ \right)^{\frac{1}{2}} e^{-\left( \frac{1}{4} + \frac{1}{4} \right)s} + e^{\left( \frac{1}{4} - \frac{1}{4} \right)s} \right) \| \partial_y \zeta \|_{L^2}^2
$$

(4.75)

Proof. The proof of Lemma 19 follows exactly the same lines as the proof of Lemmas 17 and 18, since everything is symmetric except the boundary condition, and we safely skip it. The only difference is that in this case the only boundary terms are coming from $Z_3$.

4.6. Analysis close to the origin

This subsection is devoted to the analysis of the solution in original variables, on compact sets and in particular close to the origin. Since the blow-up happens at infinity the nonlinear effects become eventually weak and the solution stays regular. We state it in a perturbative way and track precisely the constants, so that this can be used both to derive uniform estimates at the origin, and to derive the asymptotics (1.8) for the profile at blow-up time.

Lemma 20 (No blow-up on compact sets). Let $0 \leq s_0 \leq s_1$, $b = O(1)$, $N, L, L' \geq 1$, $q \in 2\mathbb{N}$. Assume that $s$ is given by (4.2) with $\lambda$ satisfying (4.16). Let $\xi$ solve (1.3) on $[0, t(s_1)] \times [0, 2N]$, with $\xi \in C^3([0, t(s_1)] \times [0, 2N])$, and such that the following properties hold:

$$
\xi_0(t(s_0)) = b y^2 + \xi(t(s_0)), \quad \|\xi(t(s_0))\|_{L^\infty([0, 2N])} \leq L, \quad \|\partial_y \xi(t(s_0))\|_{L^2([0, 2N])} \leq L'.
$$

and for all $t \in [t(s_0), t(s_1)]$:

$$
\|\xi\|_{L^\infty([0, 2N])} \leq e^{\left(1 - \frac{1}{8}\right)s}, \quad \|\partial_y \xi(t(s_0))\|_{L^2([0, 2N])} \leq e^s,
$$

then, writing $\xi = b y^2 + \tilde{\xi}$, for all $t \in [t(s_0), t(s_1)]$:

$$
\|\tilde{\xi}\|_{L^q([0, N])} \lesssim L N^{\frac{1}{q}} + N^{2 + \frac{1}{4}} e^{-\frac{q}{16}}, \quad \|\partial_y \tilde{\xi}\|_{L^2([0, N])} \lesssim L' + N^{\frac{3}{2}} e^{-\frac{q}{16}}.
$$

Corollary 21. There exists a constant $C > 0$ independent of the other parameters such that for a trapped solution, for all $t \in [0, t(s_1)]$:

$$
\|\xi\|_{W^{1, \infty}([0, 1/2])} \leq C.
$$

(4.77)

Proof of Corollary 21. From (4.19), (4.16) and (4.3) we infer that for $s_0$ large enough one has for all $s \in [s_0, s_1]$:

$$
\|\xi\|_{L^\infty([0, 2])} \leq e^{-\left(1 - \frac{1}{8}\right)s}.
$$

Hence one obtains from Lemma 20, using (4.15), that for all $t \in [0, t(s_1)]$:

$$
\|\xi\|_{L^q([0, 1])} \lesssim 1, \quad \|\partial_y \xi\|_{L^2([0, 1])} \leq 1.
$$

The desired bound (4.77) then follows from a standard parabolic regularity result. We do not prove it here and refer to the proof of Lemma 23 for a similar strategy.
Proof of Lemma 20. The proof relies on a standard localised bootstrap argument similar to [14]. The fact that we performed such an argument close to the anticipated profile at blow-up time is inspired by [16, 25].

**Step 1** The bootstrap procedure. Let \( 1 < \alpha_1 < 2, 0 < \kappa < 1 \) with \( \kappa \neq 1 - 1/(16q) \), \( L_1 = LN^{\frac{1}{\alpha}} + N^{2 \frac{1}{\alpha}} e^{-\frac{40}{\alpha}} \) and assume that for \( t \in [t(s_0), t(s_1)] \) one has the bound

\[
\int_{y \leq 2N} \tilde{\xi}^q dy \leq L_1^q e^{q(1-\kappa)s}.
\]

We claim that then for all \( t \in [0, t(s_1)] \) one has the bound:

\[
\int_{y \leq \alpha_1 N} \tilde{\xi}^q dy \lesssim \begin{cases} L_1^q e^{q(1-\kappa - \frac{1}{16q})s} & \text{if } \kappa < 1 - 1/(8q) \\ L_1^q & \text{if } 1 - 1/(8q) < \kappa. \end{cases}
\]

We now prove this claim. We write \( \xi = y^2 + \tilde{\xi}. \) Then \( \tilde{\xi} \) solves:

\[
\tilde{\xi}_t - \partial_y \tilde{\xi} + \partial_y^{-1} \partial_y \tilde{\xi} - \tilde{\xi}^2 + 2b \partial_y^{-1} \xi y - 2b = 0, \quad \tilde{\xi}(t, 0) = 0.
\]

Let \( 0 < \alpha \ll 1 \) and \( \chi \) be a smooth cut-off function, with \( \chi(y) = 1 \) for \( y \leq 1 + \alpha \) and \( \chi(y) = 0 \) for \( y \geq 1 + 2\alpha \), set \( \chi_1 = \chi \left( \frac{y}{\alpha_1 N} \right) \) and let \( v := \chi_1 \xi. \) Then \( v \) solves:

\[
v_t - \partial_{yy} v + \partial_y^{-1} \xi \partial_y v + 2\partial_y \chi_1 \partial_y \tilde{\xi} - \chi_1 \xi^2 + 2b \partial_y^{-1} \xi \chi_1 y - 2b \chi_1 + \partial_{yy} \chi_1 \tilde{\xi} - \partial_y^{-1} \xi \partial_y \chi_1 \tilde{\xi} = 0.
\]

One then has the following identity for an \( L^q \) energy estimate:

\[
0 = \frac{d}{dt} \left( \frac{1}{q} \int v^q dy \right) + (q - 1) \int v^{q-2} |\partial_y v|^2 dy
+ \int v^{q-1} \left( \partial_y^{-1} \xi \partial_y v + 2\partial_y \chi_1 \partial_y \tilde{\xi} - \chi_1 \xi^2 + 2b \partial_y^{-1} \xi \chi_1 y - 2b \chi_1 + \partial_{yy} \chi_1 \tilde{\xi} - \partial_y^{-1} \xi \partial_y \chi_1 \tilde{\xi} \right) dy.
\]

We now estimate all terms. For the first one, an integration by parts gives, using \( |v| \lesssim |\tilde{\xi}| \):

\[
\left| \int v^{q-1} \partial_y^{-1} \xi \partial_y v dy \right| \leq \frac{1}{q} \int v^q \xi dy \lesssim \|\xi\|_{L^{\infty}([0,2N])} \int_{y \leq 2N} |\tilde{\xi}|^q dy \lesssim L_1^q e^{q(1-\kappa + 1 - \frac{1}{q})s}.
\]

For the second one, integrating by parts, applying Hölder and Young inequality and \( |v| \lesssim |\tilde{\xi}| \):

\[
\left| \int v^{q-1} \partial_y \chi_1 \partial_y \tilde{\xi} \right| \leq \frac{1}{2} \int |\partial_y v|^2 v^{q-2} + C \int_{y \leq 2N} |\tilde{\xi}|^q \leq \frac{1}{2} \int |\partial_y v|^2 v^{q-2} + CL_1^q e^{q(1-\kappa)s}.
\]

For the third term, since \( |v| \lesssim \tilde{\xi} \) and \( \xi^2 \lesssim |\tilde{\xi}| (|\tilde{\xi}| + y^2) \) there holds from Hölder and (4.78):

\[
\left| \int v^{q-1} \chi_1 \xi^2 \right| \lesssim \|\xi\|_{L^{\infty}([0,2N])} \int_{y \leq 2N} \tilde{\xi}^q + \|\xi\|_{L^{\infty}([0,2N])} \left( \int_{y \leq 2N} y^{2q} dy \right)^{\frac{1}{q}} \left( \int_{y \leq 2N} |\tilde{\xi}|^q \right)^{1 - \frac{1}{q}}
\lesssim L_1^q e^{(q(1-\kappa) + 1 - \frac{1}{q})s} e^{(1 - \frac{1}{q})s} N^{2 + \frac{1}{q}} + e^{(1 - \frac{1}{q})s} L_1^{q-1} e^{(q-1)(1-\kappa)s} \lesssim L_1^q e^{(q(1-\kappa) + 1 - \frac{1}{16})s}.
\]

since \( e^{-\frac{1}{16}} N^{2 + \frac{1}{q}} \leq L_1 \). For the fourth term, since \( |\partial_y^{-1} \xi y| \leq \|\xi\|_{L^{\infty}([0,2N])} y^2 \) and \( |v| \lesssim |\tilde{\xi}| \):

\[
\left| \int v^{q-1} \partial_y^{-1} \xi \chi_1 y \right| \leq \|\xi\|_{L^{\infty}([0,2N])} \left( \int_{y \leq 2N} y^{2q} dy \right)^{\frac{1}{q}} \left( \int_{y \leq 2N} |\tilde{\xi}|^q \right)^{1 - \frac{1}{q}} \lesssim L_1^q e^{(q(1-\kappa) + 1 - \frac{1}{16})s}.
\]

For the the next two terms:

\[
\int v^{q-1} \left( -2b \chi_1 + \partial_{yy} \chi_1 \tilde{\xi} \right) dy \lesssim \int_{y \leq 2N} \tilde{\xi}^q dy \lesssim L_1^q e^{q(1-\kappa)s}.
\]
Finally, for the last term, as \( \partial_y \chi_1 \lesssim N^{-1} \), one has \(|\partial_y^{-1} \xi \partial_y \chi_1| \lesssim \|\xi\|_{L^\infty((0,2N])}\) and:

\[
\int v^q \partial_y^{-1} \partial_y^2 \xi \partial_y \chi_1 \zeta dy \leq \|\xi\|_{L^\infty((0,2N])} \int_{y \leq 2N} \dot{\zeta}^q dy \lesssim L_1^q e^{(q(1-\kappa)+1-\frac{1}{4q})s}.
\]

Collecting all the above estimates gives:

\[
\frac{d}{dt} \left( \int v^q dy \right) \lesssim L_1^q e^{(q(1-\kappa)+1-\frac{1}{4q})s}.
\]

We reintegrate with time the above identity, using the relation \( ds/dt = \lambda^2 \approx e^s \) from (4.16):

\[
\int v^q \lesssim \int |\dot{\xi}(s_0)|^q + \left( \int_{s_0}^s e^{(q(1-\kappa)-\frac{1}{4q})s'} ds' \right) \lesssim \begin{cases} L_1^q e^{(q(1-\kappa)-\frac{1}{4q})s} & \text{if } \kappa < 1 - \frac{1}{16q}, \\ L_1^q N + L_1^q e^{(q(1-\kappa)-\frac{1}{4q})s_0} & \text{if } \kappa > 1 - \frac{1}{16q}, \end{cases}
\]

since \( L_1 = LN^\frac{q}{2} + N^{2+\frac{q}{4}} e^{-\frac{3q}{2q}} \) (the case \( \kappa = 1 - 1/(16q) \) produces a harmless log which can be avoided by choosing slightly different parameters without affecting the result). This ends the proof of (4.79) and of the claim.

**Step 2 Uniform in time \( L^q \) bound.** We iterate Step 1 for a sequence of intervals \([0, \alpha_1 N], \ldots, [0, \alpha_k N]\) and parameter \( \kappa_1, \ldots, \kappa_k \). Note that this is possible from the initial bounds. At each iteration, if one is not in the second case the gain in (4.79) is \( \kappa_i = \kappa_{i-1} + 1/(16q) \). Hence we only need a finite number of iterations depending on the choice of \( q \) to reach the second case, yielding:

\[
\int_{y \leq N} |\dot{\xi}|^q dy \lesssim L_1^q = L_1^q N + N^{2+1+1} e^{-\frac{3q}{2q}}.
\]

**Step 3 The bootstrap procedure for the derivative.** Let \( 1 < \alpha_1 < 2, 0 \leq \kappa < 2 \) with \( \kappa \neq 2-1/8 \), \( L_1 = L' + N^{3/2} e^{-\frac{3q}{2q}} \) and assume that for \( t \in [t(s_0), t(s_1)] \) one has the bound

\[
\int_{y \leq 2N} |\partial_y \xi|^2 dy \lesssim L_1^q e^{(2-\kappa)s}.
\]

We claim that then for all \( t \in [0, t(s_1)] \) one has the bound:

\[
\int_{y \leq \alpha_1 N} |\dot{\xi}|^2 dy \lesssim \begin{cases} L_1^q e^{(2-\kappa-\frac{1}{4q})s} & \text{if } \kappa < 2 - 1/8 \\ L_1^q & \text{if } 2 - 1/8 < \kappa. \end{cases}
\]

We now prove this claim. Let \( \zeta := \partial_y \xi \). Then it solves:

\[
\zeta_t - \zeta \chi_1 + \partial_y^{-1} \zeta \partial_y \zeta - \partial_y \zeta = 0.
\]

We write \( \zeta = h + \tilde{\zeta} \) with \( h \) smooth such that \( h = 2by \) for \( y \geq 1, h(0) = h'(0) = h''(0) = 0 \). Then \( \tilde{\zeta} \) solves:

\[
\tilde{\zeta}_t - \partial_y \tilde{\zeta} + \partial_y^{-1} \zeta \partial_y \tilde{\zeta} - \zeta \chi_1 \zeta + \partial_y^{-1} \zeta \partial_y h - \partial_y \zeta h = 0, \quad \partial_y \zeta(t,0) = 0.
\]

Let \( 0 < \alpha \ll 1 \) and \( \chi \) be a smooth cut-off function, with \( \chi(y) = 1 \) for \( y \leq 1 + \alpha \) and \( \chi(y) = 0 \) for \( y \geq 1 + 2\alpha \), set \( \chi_1 = \chi \left( \frac{y}{\alpha_1 N} \right) \) and let \( v := \chi \tilde{\zeta} \). Then \( v \) solves:

\[
v_t - \partial_y v + \partial_y^{-1} \xi \partial_y v + 2\partial_y \chi_1 \partial_y \tilde{\zeta} - \chi_1 \xi \zeta + \partial_y^{-1} \xi \partial_y h - 2b \partial_y h + \partial_y \chi_1 \tilde{\zeta} - \partial_y^{-1} \xi \partial_y \chi_1 \tilde{\zeta} = 0.
\]

An \( L^2 \) energy estimate then writes:

\[
\frac{d}{dt} \left( \frac{1}{2} \int v^q \right) + \int |\partial_y v|^2 + \int v \left( \partial_y^{-1} \xi \partial_y v + 2\partial_y \chi_1 \partial_y \tilde{\zeta} - \chi_1 \xi \zeta + \partial_y^{-1} \xi \partial_y h - 2b \partial_y h + \partial_y \chi_1 \tilde{\zeta} - \partial_y^{-1} \xi \partial_y \chi_1 \tilde{\zeta} \right) = 0.
\]
We now estimate all terms. For the first one, an integration by parts gives, using \(|v| \lesssim |\tilde{\chi}|:
\[
\int v \partial_y^{-1} \xi \partial_y v dy = \frac{1}{2} \int v^2 \xi dy \lesssim \|\xi\|_{L^\infty([0,2N])} \int_{y \leq 2N} |\tilde{\chi}|^2 dy \lesssim L_1^2 e^{(2-\kappa+1-\frac{1}{8})s}.
\]

For the second one, integrating by parts, applying Hölder, Young inequality and \(|v| \lesssim |\tilde{\chi}|:
\[
\left| \int v \partial_y \chi_1 \partial_y \tilde{\chi} dy \right| \leq \frac{1}{2} \int |\partial_y v|^2 dy + C \int_{y \leq 2N} |\tilde{\chi}|^2 dy \leq \frac{1}{2} \int |\partial_y v|^2 dy + CL_1^2 e^{(2-\kappa)s}.
\]

For the third term, since \(|v\xi| \lesssim |\tilde{\chi}|^2|\xi| + y|\xi|\) there holds:
\[
\left| \int v \chi_1 \xi \tilde{\chi} \right| \lesssim \|\xi\|_{L^\infty([0,2N])} \int_{y \leq 2N} \tilde{\chi}^2 + \|\xi\|_{L^\infty([0,2N])} \int_{y \leq 2N} y^2 \lesssim L_1^2 e^{(2-\kappa+1-\frac{1}{8})s} + N^3 e^{(1-\frac{1}{8})s}.
\]

Similarly for the forth term, since \(|\partial_y^{-1} \xi \partial_y h| \leq \|\xi\|_{L^\infty([0,2N])}y\) and \(|v| \lesssim |\tilde{\chi}|:
\[
\left| \int v \partial_y^{-1} \xi \partial_y h \right| \leq \|\xi\|_{L^\infty([0,2N])} \int_{y \leq 2N} \tilde{\chi}^2 + \|\xi\|_{L^\infty([0,2N])} \int_{y \leq 2N} y^2 \lesssim L_1^2 e^{(2-\kappa+1-\frac{1}{8})s} + N^3 e^{(1-\frac{1}{8})s}.
\]

Finally, for the next two terms:
\[
\left| \int v \left( -\partial_{yy} h \chi_1 + \partial_{yy} \chi_1 \tilde{\chi} \right) \right| \lesssim \int_{y \leq 2N} \tilde{\chi}^2 dy \lesssim L_1^2 e^{(2-\kappa)s}.
\]

Finally, for the last term, as \(\partial_y \chi_1 \lesssim N^{-1}\), one has \(|\partial_y^{-1} \xi \partial_y \chi_1| \lesssim \|\xi\|_{L^\infty([0,2N])}\) and:
\[
\left| \int v \partial_y^{-1} \xi \partial_y \tilde{\chi} \right| \lesssim \|\xi\|_{L^\infty([0,2N])} \int_{y \leq 2N} \tilde{\chi}^2 + \|\xi\|_{L^\infty([0,2N])} \int_{y \leq 2N} y^2 dy \lesssim L_1^2 e^{(2-\kappa+1-\frac{1}{8})s} + N^3 e^{(1-\frac{1}{8})s}.
\]

Collecting all the above estimates gives:
\[
\frac{d}{dt} \left( \int v^2 dy \right) \lesssim L_1^2 e^{(2-\kappa+1-\frac{1}{8})s} + N^3 e^{(1-\frac{1}{8})s}.
\]

We reintegrate with time the above identity, using the relation \(ds/dt = \lambda^2 \approx e^s\) from (4.16):
\[
\int v^2 \lesssim \int |\tilde{\chi}(s_0)|^2 + L_1^2 \int_{s_0}^s e^{(2-\kappa-\frac{1}{8})s} ds' + N^3 \int_{s_0}^s e^{-\frac{1}{8}s} ds' \leq \begin{cases} L_1^2 e^{(2-\kappa-\frac{1}{8})s} + N^3 e^{-\frac{1}{8}s} & \text{if } \kappa < 2 - 1/8, \\ L_1^2 + L_1^2 e^{(2-\kappa-\frac{1}{8})s_0} + N^3 e^{-\frac{1}{8}s_0} & \text{if } \kappa > 2 - 1/8, \end{cases}
\]

since \(L_1 = L' + N^3 e^{-\frac{1}{8}}\). This ends the proof of (4.81) and of the claim.

**Step 4 Uniform in time \(L^2\) bound for the derivative.** Again, as in Step 2, we iterate Step 3 for a finite sequence of intervals \([0, \alpha_1 N],...,[0, \alpha_k N]\) and finally obtain:
\[
\int_{y \leq N} |\partial_y \tilde{\chi}|^2 dy \lesssim L_1^2 = L^2 + N^3 e^{-\frac{20}{\alpha}}.
\]
4.7. End of the proof of Proposition 12 and proof of Theorem 1

In this subsection we reintegrate over time the modulation equations and the various energy estimates, to show that the various upper bounds describing the bootstrap cannot be saturated. We first reintegrate the modulation equations and Lyapunov functionals.

Lemma 22. Let $0 < \eta \ll \nu'$. For a solution which is trapped on $[s_0, s_1]$ there holds for $s_0$ large enough, at time $s \in [s_0, s_1]$:

$$
\|\varepsilon\|_{L^2_\rho}^2 \leq 2e^{-\frac{7}{4}s}, \quad \int_{s_0}^{s} e^{(7-\eta)s} \|\partial_Y \varepsilon(\bar{s})\|_{L^2_\rho}^2 \, d\bar{s} \leq 2,
$$

(4.82)

$$
\frac{1}{2e} \leq \mu \leq 2e, \quad \frac{1}{4} e^{\frac{7}{2}} \leq \lambda \leq \frac{9}{4} e^{\frac{7}{2}}, \quad |a| \leq 2e^{-(\frac{1}{2} - 2\nu)s},
$$

(4.83)

$$
\mu = \mu_\infty (1 + O(e^{-s})), \quad \lambda = e^{\frac{7}{2}} \lambda_\infty (1 + O(e^{-s})),
$$

(4.84)

$$
\int_{Z_1} Z_{2} u^2 \, w Z d + \int_{Z_2} Z_{3} u^2 \, w Z d \leq 4e^{-(1-2\nu)s}, \quad \int_{Z_1} |A \partial_Z u|^2 \, w Z d + \int_{Z_2} |A \partial_Z u|^2 \, w Z d \leq 4e^{2\nu s}.
$$

(4.85)

Proof. Step 1 Interior Lyapunov functional and energy dissipation. We rewrite (4.35) as:

$$
\frac{d}{ds} e^{7s} \|\varepsilon\|_{L^2_\rho}^2 = e^{(7-\eta)s} \|\partial_Y \varepsilon(\bar{s})\|_{L^2_\rho}^2 \leq Ce^{(7-\eta)s} \|\varepsilon\|_{L^2_\rho}^2 + Ce^{7s} \|\varepsilon\|_{L^2_\rho}^2 \lambda^{-12} + Ce^{7s-e^{-s}}.
$$

Injecting the bounds (4.16) and (4.17) and integrating in time using (4.13) gives:

$$
e^{7s} \|\varepsilon\|_{L^2_\rho}^2 - 1 + \int_{s_0}^{s} e^{(7-\eta)s} \|\partial_Y \varepsilon(\bar{s})\|_{L^2_\rho}^2 \leq \int_{s_0}^{s} \left( Ce^{-\eta\bar{s}} + C(K)e^{-\frac{7}{2}\bar{s}} + Ce^{7\bar{s}-e^{-\bar{s}}} \right) \, d\bar{s} \leq 1
$$

for $s_0$ large enough, which implies the desired estimate (4.82).

Step 2 Law for $\mu$. We inject in the inequality for $\mu$ in (4.22) the bounds (4.16):

$$
\left| \frac{\mu}{\lambda} \right| \lesssim e^{-(\frac{3}{2} + \frac{1}{4})s} + e^{(\frac{11}{2} - \frac{1}{4})s} \|\partial_Y \varepsilon\|_{L^2_\rho}^2
$$

When reintegrated over time using (4.82), this applies for $s_0$ large enough:

$$
|\log \mu(s) - \log(\mu(s_0))| \leq C \int_{s_0}^{s} e^{-(\frac{3}{2} + \frac{1}{4})\bar{s}} \, d\bar{s} + \int_{s_0}^{s} e^{-(\frac{3}{2} - \frac{1}{4} + \eta)\bar{s}} e^{(7-\eta)\bar{s}} \|\partial_Y \varepsilon(\bar{s})\|_{L^2_\rho}^2 \, d\bar{s} \leq 1
$$

which using (4.12) gives indeed $(2e)^{-1} \leq \mu \leq 2e$ and if the solution is trapped for all times:

$$
\mu(s) = \mu(s_0) \exp \left( \int_{s_0}^{s} O(e^{-(\frac{3}{2} + \frac{1}{4})\bar{s}}) \, d\bar{s} + \int_{s_0}^{s} O(e^{-(\frac{3}{2} - \frac{1}{4} + \eta)\bar{s}} e^{(7-\eta)\bar{s}} \|\partial_Y \varepsilon(\bar{s})\|_{L^2_\rho}^2) \, d\bar{s} \right) = \mu_\infty (1 + O(e^{-s})).
$$

Step 3 Law for $\lambda$. We rewrite as in Step 2 the equation for $\lambda$ in (4.22) using (4.16):

$$
\left| \frac{\lambda}{\frac{1}{2}} \right| \lesssim Ce^{-2s} + Ce^{7\bar{s} - \frac{1}{16}s} \|\partial_Y \varepsilon\|_{L^2_\rho}^2.
$$

(4.86)

This can be written alternatively as (since $\lambda \leq Ke^{s/2}$ in the bootstrap):

$$
\left| \frac{d}{ds} e^{-\frac{7}{2}\lambda} \right| \leq Ce^{-2s} + Ce^{(\frac{3}{2} - \frac{1}{4})s} \|\partial_Y \varepsilon\|_{L^2_\rho}^2.
$$

When reintegrated over time using (4.82) and (4.12), this implies for $s_0$ large enough:

$$
|e^{-\frac{7}{2}\lambda} - e^{-\frac{7}{2}\lambda(s_0)}| \leq C \int_{s_0}^{s} e^{-2s\bar{s}} \, d\bar{s} + C \int_{s_0}^{s} e^{-\frac{7}{4}(\frac{1}{2} - \frac{1}{4})\bar{s}} e^{(7-\eta)\bar{s}} \|\partial_Y \varepsilon(\bar{s})\|_{L^2_\rho}^2 \, d\bar{s} \leq \frac{1}{4},
$$
which with (4.12) yields $1/4 \leq e^{-s/2} \lambda \leq 9/4$ which implies the bound for $\lambda$ in (4.83). If the solution is trapped for all times this gives:

$$\lambda = e^{\frac{s}{2}} \left( e^{-\frac{2s}{7}} \lambda_0 + \int_{s_0}^{s} O(e^{-2s}) + \int_{s_0}^{s} O(e^{-(4+\frac{1}{s})s} e^{(7-\eta)s} ||\partial_Y\varepsilon(\tilde{s})||_{L^2_{\rho}}^2 d\tilde{s}) \right) = e^{\frac{s}{2}} \lambda_\infty (1 + O(e^{-2s}))$$

**Step 4** Law for $a$. One has $||\varepsilon||_{L^\infty} ||\partial_Y\varepsilon||_{L^2_{\rho}} \lesssim e^{-s((1/2)+s)} + e^{(7/2-1/8)s} ||\partial_Y\varepsilon||_{L^2_{\rho}}^2$ from (4.19) and (4.17), and $|G_1| \lesssim |\pi + Z|^2$ near $-\pi$. We rewrite the equation for $a$ in (4.22) and inject the bounds (4.16), (4.17) and (4.19):

$$\left| \frac{d}{ds} (e^{\frac{s}{2}} a) \right| \lesssim e^{\frac{s}{2}} \left( \int_{-\pi}^{-\pi} G_1 dZ + \int_{-\pi}^{0} u dZ \right) + e^{-s(\frac{3}{2} + 1)} s + e^{(1+1/s)s} ||\partial_Y\varepsilon||_{L^2_{\rho}}^2$$

$$\lesssim e^{\frac{s}{2}} |a|^3 + e^{\frac{s}{2}} \left( \int_{-\pi}^{-\pi} u dZ + \int_{-\pi}^{0} u dZ \right) + e^{-s(1+1/s)s} + e^{(6-s)s} ||\partial_Y\varepsilon||_{L^2_{\rho}}^2$$

$$\lesssim e^{-(1-6\nu)s} e^{(6-s)s} ||\partial_Y\varepsilon||_{L^2_{\rho}}^2 + e^{\frac{s}{2}} \left( \int_{-\pi}^{-\pi} wu^2 dZ \right)^{\frac{1}{2}} + e^{s} \int_{CM} ||\varepsilon|| dY$$

This implies the following bound for $a_s$ using (4.16):

$$|a_s| \lesssim e^{-(1-2\nu)s} + e^{(6-s)s} ||\partial_Y\varepsilon||_{L^2_{\rho}}^2$$

Reintegrating over time the first estimate gives using (4.12) and (4.82):

$$|a| = e^{-\frac{s}{2}} \left( a_0 e^{\frac{s}{2}} + \int_{s_0}^{s} O(e^{s} \varepsilon(\tilde{s})) + \int_{s_0}^{s} O(e^{(6-s)\tilde{s}} ||\partial_Y\varepsilon||_{L^2_{\rho}}^2) d\tilde{s} \right) \leq 2e^{(\frac{s}{2}-2\nu)s}$$

**Step 5** Exterior energy functionals. We inject in (4.54) the bounds (4.18) and (4.51):

$$\frac{d}{ds} \left( e^{(1-\nu)s} \int_{Z_1}^{Z_2} u^2 w \right) + \left( \frac{1}{2} - \frac{\nu}{2} \right) \int_{Z_1}^{Z_2} u^2 w$$

$$\lesssim e^{(1-\nu)s} \left( e^{6s} u^2(Z_2) + e^{4s} ||\partial_Z w||^2(Z_2) + e^{-2(\frac{3}{2})s} \left( \int_{Z_1}^{Z_2} u^2 w \right)^{\frac{1}{2}} \right) + e^{-s(\frac{1}{2}+1)s} + e^{(6-\frac{1}{s})s} ||\partial_Y\varepsilon||_{L^2_{\rho}}^2$$

$$\lesssim Ce^{(2\nu'-\nu)s} + Ce^{-(1+\frac{1}{s})(\nu'-\nu)s} + Ce^{(\frac{3}{2}+2\nu)s} + e^{(6-\frac{1}{s})s} ||\partial_Y\varepsilon||_{L^2_{\rho}}^2$$

where the $e^{-(\nu'-\nu)s}$ is the worst term, due to the boundary condition at $Z_2$. Indeed, we optimized the weight $w$ to match the exterior decay with the interior decay, hence the choice of $\beta = 1/2$ for the eigenfunction (4.44) in the weight (4.8). Reintegrating in time the above identity using (4.82) and (4.14) yields since $0 < \eta < \nu' < \nu < 1$:

$$\int_{Z_1}^{Z_2} u^2 w \leq e^{-s(\nu-s)} \left( e^{(1-\nu)s} \int_{Z_1}^{Z_2} u^2 w + C \int_{s_0}^{s} e^{-(\nu-2\nu)s} + C \int_{s_0}^{s} e^{(7-\eta)s} ||\partial_Y\varepsilon||_{L^2_{\rho}}^2 d\tilde{s} \right)$$

$$\leq e^{-(1-2\nu)s} \left( e^{\nu(s_0-s)} + e^{-\nu s} C \right) \leq 2e^{-(1-2\nu)s}.$$
The differential inequality on the right (4.75) can be reintegrated with time the very same way, giving \( \int_{Z_2}^{\infty} u^2 w \leq 2e^{-(1-2\nu)s} \). These two bounds imply the first bound in (4.85). We now turn to the derivative. We write (4.64) as:

\[
\left| \frac{d}{ds} \left( e^{\nu s} \int_{Z_1}^{Z_2} |A\partial_Z u|^2 w dz \right) \right| \leq Ce^{-\frac{4s}{2}} + Ce^{\left( -\frac{1}{2} - \frac{1}{10^2} \right) s} e^{(7-\eta)s} \| \partial_T \epsilon \|_{L^2}^2
\]

Note that compared to the differential inequality for \( u \), the above identity for \( A\partial_Z u \) is better. Indeed the fact that \( A \sim Z \) near the origin improves the control of the boundary term at \( Z_2 \), and \( A\partial_Z \) kills the worst component of the error near the origin. Reintegrating in time the above identity using (4.82) and (4.14) yields:

\[
\int_{Z_1}^{Z_2} |A\partial_Z u|^2 \leq e^{2\nu s} \left( e^{\nu s} \int_{Z_1(s_0)}^{Z_2(s_0)} |A\partial_Z u(s_0)|^2 + Ce^{\nu s} \int_{s_0}^{s} e^{-\frac{4s}{2}} ds + Ce^{\nu s} \int_{s_0}^{s} e^{(7-\eta)s} \| \partial_T \epsilon \|_{L^2}^2 \right) ds \leq e^{2\nu s} (\nu(s_0-s) + Ce^{-\nu s}) \leq 2e^{2\nu s}.
\]

The same bound can also be proved the same way for the derivative at the right of the origin, implying the last bound in (4.85).

\[ \square \]

We now bootstrap the last bound and control \( \epsilon \) on \([-M^2, M^2]\) using parabolic regularity.

**Lemma 23.** For a solution that is trapped on \([s_0, s_1]\), for \( 0 < \nu' \ll \nu \), there holds at time \( s_1 \):

\[
\| \epsilon \|_{H^3(Y | \leq M^2)} \leq 10e^{-(7-\nu')s_1}.
\]

**Proof.** The proof is a classical use of parabolic regularity: \( \epsilon \) evolves according to a parabolic equation, its size and the size of the forcing terms are precisely \( e^{-7s/2} \), hence this bound propagates for higher order derivatives due to the smoothing effect of the heat kernel. We rewrite (4.20) as:

\[
\epsilon_s - \partial_T \epsilon + \tilde{V} \epsilon + \tilde{T} \partial_T \epsilon = \mathcal{F},
\]

where

\[
\tilde{V} := \left( 2\frac{\lambda s}{\lambda} - 2G_1 - \varepsilon \right), \quad \tilde{T} := \frac{\lambda s}{\lambda} Y + \int_{-(\pi-a)\lambda^2}^{0} f - \lambda y_s^* + \lambda^2 \mu Z^{-1} G_1 + \partial_Y^{-1} \epsilon,
\]

\[
\mathcal{F} := \left( \frac{\mu s}{\mu} - \frac{1}{2\lambda^2} \right) Z\partial_Z G_1 + \left( \frac{\lambda s}{\lambda} - \frac{1}{2} \right) \int_{-(\pi+a)\lambda^2}^{0} f - \lambda y_s^* \right) \frac{1}{\lambda^2 \mu} \partial_Z G_1
\]

\[
- \frac{1}{\lambda^4 \mu^2} \left( \partial_{ZZ} G_1 + \frac{1}{4} \partial Z \partial G_1 + \frac{1}{2} G_1 \right).
\]

Note that from (4.22) one has:

\[
\| \tilde{T} \|_{W^{1,\infty}(Y | \leq M^3)} + \| \tilde{V} \|_{W^{1,\infty}(Y | \leq M^3)} \leq C + C\| \partial_Y \epsilon \|_{L^2}^2.
\]

We now let \( \epsilon_1 := \partial_T \epsilon \). It solves:

\[
\epsilon_1 - \partial_T \epsilon + (\tilde{V} + \partial_T \tilde{T}) \epsilon + \tilde{T} \partial_T \epsilon = -\partial_T \tilde{V} \epsilon + \mathcal{F}.
\]

Let \( M^2 < M_1 < M_2 < M^3 \), and \( \chi \) be a cut-off function with \( \chi = 1 \) for \( Y \leq M_1 \) and \( \chi = 0 \) for \( Y \geq M_2 \) and let \( v = \chi \epsilon \). Then \( v \) solves:

\[
v - \partial_T \gamma \v + (\tilde{V} + \partial_T \tilde{T}) \v + \tilde{T} \partial_T \v = -\partial_T \gamma \chi \v - 2 \partial_T \gamma \partial_T \gamma \v - \tilde{T} \partial_T \gamma \v - \chi \partial_T \gamma + \gamma + \chi \mathcal{F}.
\]
We then perform a standard energy estimate:

\[
\frac{d}{ds} \left( \frac{1}{2} \int v^2 dY \right) + \int |\partial_Y v|^2 dY = \int \left( -\partial_Y \chi \varepsilon^1 - 2\partial_Y \chi \partial_Y \varepsilon^1 - \tilde{T} \partial_Y \chi \varepsilon^1 + \chi \tilde{F} \right) v dY
\]

\[-\int \left( (\tilde{V} + \partial_Y \tilde{T}) v + \tilde{T} \partial_Y v \right) v dY.
\]

Let \(0 < \kappa \ll 1\), integrating by parts and using Young inequality one finds since \(|v| \lesssim \varepsilon^1\):

\[
\left| \int \left( -\partial_Y \chi \varepsilon^1 - 2\partial_Y \chi \partial_Y \varepsilon^1 \right) v dY \right| \leq \frac{C}{\kappa} \int_{|Y| \leq M_3} |\varepsilon^1|^2 + \kappa C \int |\partial_Y v|^2 dY \leq C\|\partial_Y \varepsilon\|^2_{L^p} + \frac{1}{4} \int |\partial_Y v|^2 dY
\]

for \(\kappa\) small enough. Similarly, integrating by parts, using Young inequality, (4.17) and (4.89):

\[
\left| \int \tilde{T} \partial_Y \chi v \right| = \left| \int \tilde{T} \partial_Y \chi \partial_Y v \right| \leq \frac{1}{4} \int |\partial_Y v|^2 + C(1 + \|\partial_Y \varepsilon\|^2_{L^2}) \leq C\|\partial_Y \varepsilon\|^2_{L^p} + C\|\partial_Y \varepsilon\|^2_{L^p} \leq \frac{1}{4} \int |\partial_Y v|^2 + C e^{-7s} + C\|\partial_Y \varepsilon\|^2_{L^p}.
\]

Next, from Cauchy-Schwarz, (4.17) and (4.89):

\[
\left| \int \chi \partial_Y \tilde{V} v \right| \leq C\|\partial_Y \tilde{V}\|_{L^{\infty}(|Y| \leq M^3)} \|v\|_{L^2} \|\varepsilon\|_{L^2} \leq C e^{-7s} + C\|\partial_Y \varepsilon\|^2_{L^p} + C\|\partial_Y \varepsilon\|^2_{L^p} \|v\|^2_{L^2}.
\]

For the error, we recall the cancellation \(\partial_Z Z G_1 + \frac{1}{4} Z \partial_Z G_1 + \frac{1}{2} G_1 = O(|Z|^4)\) and \(|\partial_Z G_1| = O(|Z|)\) as \(Z \to 0\), which implies using (4.22) that:

\[
\int \chi^2 F^2 dY \leq C e^{-11s} + C\|\varepsilon\|_{L^{\infty}}^2 \|\partial_Y \varepsilon\|^2_{L^p} + C\|\partial_Y \varepsilon\|^4_{L^p}
\]

which by Cauchy-Schwarz yields:

\[
\left| \int \chi \mathcal{F} v dY \right| \leq C \left( e^{-\frac{1}{2}s} + C\|\varepsilon\|_{L^{\infty}} \|\partial_Y \varepsilon\|_{L^p} + C\|\partial_Y \varepsilon\|^2_{L^p} \right) \|v\|_{L^2}.
\]

Performing an integration by parts and using (4.89):

\[
\left| \int \left( (\tilde{V} + \partial_Y \tilde{T}) v + \tilde{T} \partial_Y v \right) v \right| \leq \|v\|^2_{L^2} (\|\tilde{V}\|_{W^{1,\infty}(|Y| \leq M^3)} + \|\tilde{T}\|_{W^{1,\infty}(|Y| \leq M^3)}) \leq \|\partial_Y \varepsilon\|^2_{L^p} + \|\partial_Y \varepsilon\|^2_{L^p} \|v\|^2_{L^2}.
\]

Let \(0 < \eta \ll \nu_1 \ll \nu'\). Collecting all the estimates above, and since \(|v| \lesssim \varepsilon^1\) one has the energy identity:

\[
\frac{d}{ds} \left( e^{(7-\nu_1)s} \int v^2 \right) + \frac{1}{2} e^{(7-\nu_1)s} \int |\partial_Y v|^2 \leq C e^{(7-\nu_1)s} \|\partial_Y \varepsilon\|^2_{L^p} + C e^{(7-\nu_1)s} + C\|\partial_Y \varepsilon\|^2_{L^p} \|v\|^2_{L^2} e^{(7-\nu_1)s}
\]

\[+ C \left( e^{-\frac{1}{2}s} + C\|\varepsilon\|_{L^{\infty}} \|\partial_Y \varepsilon\|_{L^p} + C\|\partial_Y \varepsilon\|^2_{L^p} \right) \|v\|_{L^2} e^{(7-\nu_1)s}.
\]

Let now \(\tilde{s} \in [s_0, s_1]\) be the supremum of times \(s \geq s_0\) such that \(|v|_{L^2} \leq 10 e^{-\left(\frac{7}{4} - \nu_1\right)s'}\) for all \(s' \in [s_0, s]\). From (4.13) and a continuum argument one has \(\tilde{s} > s_0\). We claim that \(\tilde{s} = s_1\). Indeed, on \([s_0, \tilde{s}]\) the above differential inequality gives:

\[
\frac{d}{ds} \left( e^{(7-\nu_1)s} \int v^2 \right) + \frac{1}{2} e^{(7-\nu_1)s} \int |\partial_Y v|^2 \leq C e^{(7-\nu_1)s} \|\partial_Y \varepsilon\|^2_{L^p} + C e^{(7-\nu_1)s} + C s e^{-\left(\frac{7}{4} - \nu_1\right)\frac{s}{2}} e^{\frac{1}{2}(7-\eta)s} \|\partial_Y \varepsilon\|_{L^p}
\]

Reintegrated with time, using (4.82) this gives for \(s_0\) large enough:

\[
e^{(7-\nu_1)s_0} \int v^2 dY + \frac{1}{2} \int_{s_0}^{s} e^{(7-\nu_1)s'} \int |\partial_Y v|^2 dY ds' \leq e^{(7-\nu_1)s_0} \int v_0^2 dY + 1 \leq 2.
\]
Therefore, \( \|v(\tilde{s})\|_{L^2} \leq 2e^{-(\tilde{\tilde{\nu}} - \nu_1)\tilde{s}} \). Hence a continuity argument implies \( \tilde{s} = s_1 \). One has then proved the following pointwise bound for \( \partial_Y \varepsilon \) and integrated bound for \( \partial_Y Y \varepsilon \):

\[
\forall s \in [s_0, s_1], \quad \int_{|Y| \leq M_1} |\partial_Y \varepsilon|^2 dY \leq 10e^{-(\tilde{\tilde{\nu}} - \nu_1)s}, \quad \text{and} \quad \int_{s_0}^{s} e^{(T - \nu_1)s'} \int_{|Y| \leq M_1} |\partial_Y Y \varepsilon|^2 dY ds' \leq 2.
\]

Let now \( M^2 < M_4 < M_3 < M_1 \). We claim that we can differentiate equation (4.90) and, with the exact same arguments, obtain the analogue of the above estimates for \( \partial_Y Y \varepsilon \), with an exponent \( \nu_2 \) such that \( \nu_1 \ll \nu_2 \ll \nu' \). Indeed, the only crucial arguments to derive the above bounds were the pointwise in time boundedness (4.17) of \( \|\varepsilon\|_{L^2} \) and the dissipation estimate (4.82) for \( \|\partial_Y \varepsilon\|_{L^2} \), and we just obtained the analogues for \( \partial_Y \varepsilon \) so that the same strategy can be applied.

Then, another iteration yields the analogue of the above bounds for \( \partial_Y^{(3)} \varepsilon \) for \( |Y| \leq M_4 \) for an exponent \( \nu_2 \ll \nu_3 \ll \nu' \), which ends the proof of the Lemma.

All the bounds of the bootstrap and the modulation equations have been investigated previously. We can now end the proof of Proposition 12.

**Proof of Proposition 12.** Let an initial datum satisfy the properties of Definition 9 at time \( s_0 \). Let \( \tilde{s} \) be the supremum of times such that the solution is trapped on \( [s_0, \tilde{s}] \). Assume by contradiction that \( \tilde{s} < +\infty \). Then from the local well-posedness Proposition 6 and the blow-up criterion (1.4), the solution can be extended beyond the time \( \tilde{s} \). Hence, from the definition of \( \tilde{s} \) and Definition 10 and a continuity argument, one of the inequalities (4.16), (4.17) or (4.18) must be an equality at time \( \tilde{s} \). This is however impossible for \( K \) large enough from (4.82), (4.83), (4.85) and (4.88), which is desired contradiction. Hence \( \tilde{s} = +\infty \) which proves Proposition 12.

Theorem 1 is a direct consequence of Proposition 12 and we can now give its proof.

**Proof of Theorem 1.** For an initial datum of the form (1.5), let \( s_0 = 2 \ln(\lambda_0^2) \). Then for \( \varepsilon(\lambda_0) > 0 \) small enough, thanks to the smoothing effect of the equation, see Proposition 6, \( \tilde{\xi}_0 \) is instantaneously regularised, and \( \xi(t^*) \) is initially trapped in the sense of Definition 9. Applying Proposition 12, the solution is then trapped for all times in the sense of Definition 10. Since \( ds/dt = \lambda^2 \) and \( \lambda \) satisfies (4.84):

\[
\frac{dt}{ds} = e^{-s} \lambda_{\infty}^{-2}(1 + O(e^{-2s})).
\]

Reintegrating the above equation, there exists \( T > 0 \) such that:

\[
T - t = e^{-s} \lambda_{\infty}^{-2}(1 + O(e^{-2s})).
\]

This implies \( e^{-s} = \lambda_{\infty}^2(T - t) + O((T - t)^3) \). The identities (1.6) are then consequences of (4.84). From (4.19), \( \tilde{x}(t, y) = u(s, Z) \) and (4.7) one infers:

\[
\|\tilde{\xi}\|_{L^\infty} = \lambda^2 \|u\|_{L^\infty} \lesssim e^{-s}e^{-\left(\frac{1}{4} - \nu\right)s} \leq C(T - t)^{1-\frac{1}{8}}
\]

which proves (1.7). We now investigate the existence and asymptotic behaviour of the blow-up profile at time \( T \). The existence of a limit \( \xi(t, y) \rightarrow \xi^*(y) \) as \( t \uparrow T \) follows from Lemma 20 and a standard parabolic bootstrap argument. We now use more carefully Lemma 20 to find the asymptotic of the profile at blow-up time. For \( y^* \geq e^{(\frac{1}{2} - \frac{1}{16})s_0} \) we define the following adapted time, which now depends on the point that we consider:

\[
s_0(y^*) = \left(\frac{1}{2} - \frac{1}{16}\right)^{-1} \log(y) = \log(y^0), \quad \alpha := \left(\frac{1}{2} - \frac{1}{16}\right)^{-1} = \frac{16}{7}, \quad \text{so that} \quad y^* = e^{(\frac{1}{2} - \frac{1}{16})s_0(y)}.
\]
For \( s \geq s_0(y) \), for \( y \in [0, 2y^*] \), one has
\[
Z(y) = \frac{y - y^*}{\lambda \mu} = -\pi - a + \frac{y}{\lambda \mu} = -\pi + O(e^{-\frac{a_0}{16}}).
\]

Therefore one can apply the Taylor expansion of \( G_1 \) near the origin for \( s_0 \) large enough. Using (4.83), (4.84) and (4.19), for \( s \geq s_0(y^*) \):
\[
\lambda^2(s_0)G_1(Z(y)) = \frac{1}{4} \left( -a + \frac{y}{\lambda \mu} \right)^2 \lambda^2 + \lambda^2 O \left( -a + \frac{y}{\lambda \mu} \right) = \frac{y^2}{4\mu_2^2} + O(y^{s-\frac{1}{16}}) \leq e^{1-\frac{1}{8}}s_0 \leq e^{1-\frac{1}{8}}s,
\]
\[
|\lambda^2(s_0)u(s_0, Z(y))| \leq C\lambda^2(s_0)e^{-\frac{a_0}{16}} \leq C e^{1-\frac{1}{8}}s_0 = Cy^{\frac{a_0}{6}} = Cy^{2-\frac{a}{16}},
\]
The two above identities imply that, writing \( \xi = \frac{y^2}{4\mu_2^2} + \tilde{\xi} \), at time \( s_0^* \) on \([0, y^*]\):
\[
\xi(t(s_0(y)), y) = \frac{y^2}{4\mu_2^2} + O(y^{s-\frac{1}{16}}), \quad \text{i.e.} \quad \|\tilde{\xi}(s_0(y^*))\|_{L^{\infty}([0, 2y^*])} \leq Cy^{2-\frac{a}{16}},
\]
and that for \( s \geq s_0(y^*) \):
\[
\|\xi\|_{L^{\infty}([0, 2y^*])} \lesssim e^{1-\frac{1}{8}}s.
\]

Moreover, from (4.18), changing variables:
\[
\|\partial_y(\lambda^2 u(s, Z(y)))\|_{L^2([0, 2y^*])} \lesssim \lambda^2 \|\partial_Z u(s, Z)\|_{L^2([0, 2y^*/\lambda])} \lesssim e^{\frac{s}{8}}e^{2\nu} \leq e^s,
\]
\[
\|\partial_y(\lambda^2 G_1(s, Z(y)))\|_{L^2([0, 2y^*])} \lesssim \lambda^2 \|\partial_Z G_1(s, Z)\|_{L^2([0, 2y^*/\lambda])} \lesssim e^{\frac{s}{8}} \leq e^s,
\]
\[
\|\partial_y(y^2)\|_{L^2([0, 2y^*])} \lesssim \lambda^2 \|\partial_Z G_1(s, Z)\|_{L^2([0, 2y^*/\lambda])} \lesssim y^{\frac{s}{8}} \leq e^s,
\]
for \( s_0 \) large enough, so that for \( s \geq s_0(y^*) \):
\[
\|\partial_y \tilde{\xi}\|_{L^2([0, 2y^*])} \leq e^s.
\]

We apply Lemma 20 and obtain that for all \( t \geq t(s_0(y^*)) \):
\[
\|\tilde{\xi}\|_{L^q([0, y^*])} \lesssim y^{s-\frac{1}{16}}y^{\frac{1}{2}y^*} + y^{s-\frac{1}{16}}y^{\frac{1}{2}y^*} \lesssim y^{s-\frac{1}{16}+\frac{1}{q}}
\]
and for some fixed constant \( c > 0 \):
\[
\|\partial_y \tilde{\xi}\|_{L^2([0, y^*])} \lesssim y^{s\alpha} + y^{\frac{s}{2}}e^{-\frac{a_0}{8}} \lesssim y^{sc}.
\]

We apply the following interpolated Sobolev inequality:
\[
\|h\|_{L^\infty} \lesssim \|h\|_{L^q}^{\frac{1}{q} - \frac{2}{q+2}} \|\partial_y h\|_{L^2}^{\frac{2}{q+2}},
\]
yielding that for all \( t \geq t(s_0(y^*)) \):
\[
\|\tilde{\xi}\|_{L^\infty([0, y^*])} \lesssim y^{s-\frac{1}{16}+\frac{c}{q}} \lesssim y^{s-\frac{1}{16}}
\]
for \( q \) large enough. Therefore, since this remains true at the limit at time \( T \) one has showed that for \( y^* \geq e^{(\frac{1}{2} - \frac{1}{16})s_0} \):
\[
\xi(y^*) = \frac{y^{s^2}}{4\mu_2^2} + O(y^{s^2-\frac{1}{16}})
\]
which ends the proof of (1.8).
5. Proof of Theorem 2

We only gave a detailed construction for the stable blow-up profile corresponding to $k = 1$ in Proposition 4, which is Theorem 1. The explicit formula indeed simplifies notations and makes the proof reader-friendly. However, the arguments never rely on these explicit computations and could be propagated easily to $k \geq 2$. We now sketch how to adapt the argument.

Indeed, close to the maximum, one can still use the spectral structure as in Subsection 4.4. The $k$-th unstable blow-up corresponds to an excitation of the $2k$-th mode in Proposition 3. One can then decompose the perturbation onto all modes $h_j$ for $0 \leq j \leq 2k$. The component of the solution on the mode $h_j$ for $1 \leq j \leq 2k-2$ then does not decay fast enough at the linear level and is not linked to an invariance of the flow. This generates instabilities and the control of these modes can nevertheless be obtained using a topological argument such as Brouwer fixed point theorem. As a result, these modes remain under control provided the initial datum lies within a manifold with codimension $2k-2$. The remainder of the perturbation located on higher order modes $j \geq 2k+1$ then decays due to the spectral gap. For an implementation of this strategy we refer to [4] Subsection 4.1 for the case of the unstable ODE profiles of the semi-linear heat equation.

Outside the maximum, the analogue of the linear analysis performed in Subsection 4.5.1 can be performed. This part indeed also does not rely on explicit formulas, but solely on the behaviour of $G_k$ near the origin obtained in Proposition 4. The analogue of the weight $w$ and of the vector field $A\partial_Z$ can therefore be constructed. The analogue of the weighted exterior Lyapunov functionals can be derived. On compact sets and close to the origin, nothing new happens and the very same analysis can be applied.

A. Functional analysis

Lemma 24. Let $Y_0 \in \mathbb{R}$ and $\varepsilon : [Y_0, +\infty) \rightarrow \mathbb{R}$ with $\varepsilon \in H^1_{\text{loc}}((Y_0, +\infty))$. Then:

$$\int_{Y_0}^{+\infty} Y^2 \varepsilon^2 e^{-\frac{Y^2}{4}} dY \lesssim \|\varepsilon\|_{H^1_{\rho}}^2 \quad (A.1)$$

Proof. Assume $Y_0 = -\infty$. Integrating by parts one finds the identity

$$4 \int \varepsilon \partial_Y \varepsilon Y e^{-\frac{Y^2}{4}} dY + 2 \int \varepsilon^2 e^{-\frac{Y^2}{4}} dY = \int \varepsilon^2 Y^2 e^{-\frac{Y^2}{4}} dY.$$

From Cauchy-Schwarz and Young inequalities, $4 \int \varepsilon \partial_Y \varepsilon Y e^{-\frac{Y^2}{4}} dY \leq 1/2 \int Y^2 \varepsilon^2 e^{-\frac{Y^2}{4}} dY + 8 \int |\partial_Y \varepsilon|^2 e^{-\frac{Y^2}{4}} dY$ and we infer from the above identity that:

$$\int \varepsilon^2 Y^2 e^{-\frac{Y^2}{4}} dY \leq 4 \int \varepsilon^2 e^{-\frac{Y^2}{4}} dY + 16 \int |\partial_Y \varepsilon|^2 e^{-\frac{Y^2}{4}} dY$$

which proves (A.1). If $Y_0 > -\infty$, then extending $\varepsilon$ by even reflection on $(-\infty, Y_0)$ and applying the above inequality to the extension yields the desired result (we recall the convention $\|\varepsilon\|_{H^1_{\rho}}^2 = \int_{Y_0}^{+\infty} (\varepsilon^2 + |\partial_Y \varepsilon|^2) \rho$ in that case).

B. Geometrical decomposition

Proof of Lemma 7. The proof relies on a classical use of the implicit function theorem, preceded by a renormalisation procedure to obtain a result which is uniformly valid for all $\lambda$ large enough.
Define the mapping  
\[ \Phi : (\varepsilon, \lambda, \bar{Y}_0) \mapsto \lambda_0^4 (\langle \varepsilon, h_0 \rangle_\rho, \langle \bar{\varepsilon}, h_1 \rangle_\rho, \langle \bar{\varepsilon}, h_2 \rangle_\rho) , \]
where  \( \langle u, v \rangle_\rho = \int_{Y_0 - \bar{Y}_0} uv \rho \) and, for  \( Y \geq Y_0 - \bar{Y}_0 \):

\[ \bar{\varepsilon}(Y) = G_1 \left( \frac{Y + \bar{Y}_0}{\lambda_0^2} \right) - (1 + \lambda_0^{-1})^2 G_1 \left( \frac{Y}{\lambda_0^2(1 + \lambda_0^{-1})^2} \right) + \frac{\varepsilon}{\lambda_0^2}(Y + \bar{Y}_0) . \]

\( \Phi \) is a  \( C^2 \) mapping on  \( L^2_\rho \times (-\lambda_0^4, +\infty) \times (0, +\infty) \times \mathbb{R} \). Moreover, one computes that its differential at  \( (0,0,1,0) \) is, where  \( \langle u, v \rangle = \int_{Y \geq Y_0} uv \rho \):

\[
J \Phi(0,0,1,0) + O(e^{-\lambda_0^3}) = \begin{pmatrix}
\langle \cdot, h_0 \rangle & -2G_1 \left( \frac{Y}{\lambda_0^2} + 2 \frac{Y^2}{\lambda_0^4} \partial \bar{z} \middle| \bar{z} \right) + O \left( \frac{Y^4}{\lambda_0^8} \right) - \frac{1}{6}h_0(Y) - \frac{1}{3}h_0(Y) + O \left( \frac{Y^8}{\lambda_0^8} \right) \\
\langle \cdot, h_1 \rangle & \lambda_0 \partial \bar{z} \middle| \bar{z} \right) + O \left( \frac{Y^4}{\lambda_0^8} \right) - \frac{1}{6}h_2(Y) - \frac{1}{3}h_0(Y) + O \left( \frac{Y^4}{\lambda_0^8} \right) \\
\langle \cdot, h_2 \rangle & -2G_1 \left( \frac{Y}{\lambda_0^2} + 2 \frac{Y^2}{\lambda_0^4} \partial \bar{z} \middle| \bar{z} \right) + O \left( \frac{Y^4}{\lambda_0^8} \right) - \frac{1}{6}h_2(Y) - \frac{1}{3}h_0(Y) + O \left( \frac{Y^8}{\lambda_0^8} \right)
\end{pmatrix},
\]

where  \( O(e^{-\lambda_0^3}) \) comes from the boundary terms. Using the Taylor expansion of  \( G_1 \) one has:

\[
-2G_1 \left( \frac{Y}{\lambda_0^2} + 2 \frac{Y^2}{\lambda_0^4} \partial \bar{z} \middle| \bar{z} \right) + O \left( \frac{Y^4}{\lambda_0^8} \right) - \frac{1}{6}h_0(Y) - \frac{1}{3}h_0(Y) + O \left( \frac{Y^8}{\lambda_0^8} \right) .
\]

Therefore:

\[
J \Phi(0,0,1,0) = \begin{pmatrix}
\langle \cdot, h_0 \rangle & -2\|h_0\|_{L^2_\rho}^2 + O(\lambda_0^{-4}) - \frac{1}{3}h_0(Y)^2 + O(\lambda_0^{-4}) - \frac{1}{2}\|h_1\|_{L^2_\rho}^2 + O(\lambda_0^{-4})
\\
\langle \cdot, h_1 \rangle & O(\lambda_0^{-4}) - \frac{1}{3}h_0(Y)^2 + O(\lambda_0^{-4}) - \frac{1}{2}\|h_1\|_{L^2_\rho}^2 + O(\lambda_0^{-4})
\\
\langle \cdot, h_2 \rangle & O(\lambda_0^{-4}) - \frac{1}{3}h_0(Y)^2 + O(\lambda_0^{-4}) - \frac{1}{2}\|h_1\|_{L^2_\rho}^2 + O(\lambda_0^{-4})
\end{pmatrix} .
\]

This implies that the restriction of the differential to  \( \{0\} \times \mathbb{R}^3 \) is invertible for  \( \lambda_0 \) large enough, with a uniform size. Moreover, one can also check similarly that the second differential of  \( \Phi \) is bounded near  \( (0,0,1,0) \), and this uniformly for large  \( \lambda \). Therefore the implicit function theorem applies uniformly for all  \( \lambda_0 \geq \lambda^* \) large enough and  \( Y_0 \leq -\lambda_0^4 \). There exists  \( \delta, K > 0 \) such that for each  \( \varepsilon \in L^2_\rho \) with  \( \|\varepsilon\|_{L^2_\rho} \leq \delta \), there exist unique parameters  \( (\lambda, \mu, \bar{Y}_0) \) with  \( |\lambda| + |\mu - 1| + |\bar{Y}_0| \leq K \) such that  \( \Phi(\varepsilon, \lambda, \mu, \bar{Y}_0) = 0 \). Moreover, they define  \( C^1 \) functions with respect to the  \( L^2_\rho \) topology.

Let  \( \lambda_0 \geq \lambda^* \) and  \( \|\varepsilon\|_{L^2_\rho} \leq \delta \lambda_0^{-4} \). The above discussion yields the existence, uniqueness, and differentiability of  \( (\lambda, \mu, Y_0) \) such that  \( \Phi(\lambda_0^4 \varepsilon, \lambda, \mu, Y_0) = 0 \). Let  \( (\tilde{\lambda}, \tilde{\mu}, \tilde{Y}_0) = (1 + \lambda_0^{-1} \lambda, \mu, Y_0) \). Then they produce indeed

\[ G_1 \left( \frac{Y}{\lambda_0^2} \right) + \bar{\varepsilon}(Y) = \tilde{\lambda}^2 G_1 \left( \frac{Y - \bar{Y}_0}{\lambda_0^2} \right) + \bar{\varepsilon}(Y - \bar{Y}_0) \text{ with } \bar{\varepsilon} \perp h_0, h_1, h_2 \text{ in } L^2_\rho \]

and one has  \( |\tilde{\lambda} - 1| \leq K \lambda_0^{-4} \) and  \( |\mu - 1| + |Y_0| \leq K \). The uniqueness when requiring these bounds follows similarly, and implies the smoothness from the above discussion. This ends the proof.
C. On the leading order linearised dynamics in inviscid self-similar variables

In this section, our aim is to explain with exact computations on an example how in a suitable evolution problem similar to
\[ u_t + \mathcal{H}u = 0, \]
where \( \mathcal{H} \) is defined by (4.39), the nonlocal term is slaved by the dynamics of the first two terms. First, We consider the following purely non-local problem for \( t, x \geq 0 \):
\[ \partial_t u(t, x) - \int_0^x u(t, \tilde{x}) d\tilde{x} = 0. \]  
(C.1)  
\( \text{eq:nonlocal} \)

Note that if \( u(t, x) \) is a solution, then \( u(\lambda t, x/\lambda) \) is also a solution. We look for a Green solution
\[ \partial_t K(t, x) - \int_0^x K(t, \tilde{x}) d\tilde{x} = 0, \quad K(0, x) = \begin{cases} \ 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases} \]

\( \text{K has to be invariant with respect to the scaling transformation. Hence } K(t, x) = k(tx). \) The function \( k \) must then solve
\[ yk'(y) - \int_0^y k(\tilde{y}) d\tilde{y} = 0 \text{ for } y > 0, \quad k(0) = 1, \quad k(y) = 0 \text{ for } y < 0. \]

One checks that the following entire series solves the above equation:
\[ k(y) = \sum_{n=0}^{+\infty} \frac{y^{n}}{(n!)^2} \text{ for } y \geq 0, \quad k(y) = 0 \text{ for } y < 0. \]

Any solution to (C.1) can then be written in a convolution form using the kernel \( K \):
\[ u(t, x) = u_0(x) + t \int_0^x u_0(y)k'(t(x - y)) dy = u_0(0)k(tx) + \int_0^x \partial_y u_0(y)k(t(x - y)) dy. \]

Let \( k^{(0)}(x) = k(x) \) and \( k^{(\ell)}(x) = \int_0^x k^{(-i-1)}(y) dy \) be the \( \ell \)-th primitive of \( k \) with 0 as the origin of integration. Integrating by parts yields for any integer \( \ell \):
\[ u(t, x) = \sum_{i=0}^{\ell} u_0^{(i)}(0)\ell^{-i}k^{(-i)}(tx) + t^{-\ell} \int_0^{tx} u_0^{(\ell+1)}(y)k^{(-\ell)}(tx - y) dy. \]

The growth of \( k \) can be compared to:
\[ \lim_{y \to +\infty} \frac{k(y)}{e^{\sqrt{y}}} = +\infty, \quad \lim_{y \to +\infty} \frac{k(y)}{e^{y^\alpha}} = 0 \text{ for } \alpha > 1/2. \]

Hence the growth for Equation (C.1) is sublinear, and cancellations near the origin improve the decay. Indeed, one gets that \( k^{(-i-i)}(x) \leq x^i \) for \( x \leq 1 \) and \( k^{(-i)}(x) \leq x^{-1-2i}e^{\frac{i}{2} + \frac{i}{2}} \) for any \( \epsilon > 0 \).
If \( u_0 \) is such that \( u_0^{(i)}(0) = 0 \) for \( i = 0, \ldots, \ell - 1, \) \( u_0^{(\ell)}(0) \neq 0 \) and \( \| u_0^{(\ell+1)} \|_{L^\infty} < +\infty \), then:
\[ u(t) \lesssim \begin{cases} x^k & \text{for } x \leq t^{-1}, \\ \left( t^{-\frac{\ell+1}{2}-\frac{1}{2}-\epsilon} x^\frac{\ell+1}{2}-\epsilon + t^{-\frac{\ell+1}{2}-\epsilon} x^\frac{\ell+1}{2}-\epsilon \right) e^{(tx)^{\frac{1}{2}+\epsilon}} & \text{for } x \geq t^{-1}. \end{cases} \]

We now consider the following nonlocal problem with a transport part:
\[ v_t - \partial_x^{-1} v + x \partial_x v = 0. \]  
(C.2)  
\( \text{eq:nonlocal2} \)

Note that this problem corresponds to (4.45) without viscosity to leading order near the origin, which is the zone dictating the decay. We change variables \( s = e^t, \ y = xe^{-t} \) and \( u(s, y) = v(t, x). \)
Then $u$ solves the previous equation (C.1). Therefore, assuming that $v_0$ is such that $v_0^{(i)}(0) = 0$ for $i = 0, \ldots, \ell - 1$, $v_0^{(\ell)}(0) \neq 0$ and $\|v^{(\ell+1)}\|_{L^\infty} < +\infty$ one has for any $\epsilon > 0$:

$$v(t) \leq e^{-\epsilon t} \begin{cases} x^\ell \left( \frac{1}{2} - \epsilon e^{-\ell x^2} + e^{-\frac{1}{2}(\ell + 1)x} \right) e^{x^2 + \frac{1}{2} - \epsilon} & \text{for } x \leq (e^\ell - 1)^{-1}, \\
\ell e^{x - e - \epsilon} & \text{for } x \geq (e^\ell - 1)^{-1}.
\end{cases}$$

In particular, one sees that on compact sets there holds the same decay as for the transport equation without the nonlocal term. Indeed, if $v_0$ is such that $v_0^{(i)}(0) = 0$ for $i = 0, \ldots, \ell - 1$, $v_0^{(\ell)}(0) \neq 0$ and $\|v^{(\ell+1)}\|_{L^\infty} < +\infty$ one has $|e^{-tx^2}v| \leq C_Le^{-\epsilon t}$ for any compact $0 \leq x \leq L$. 

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