DEFORMATIONS OF FREE BOUNDARY CMC HYPERSURFACES AND APPLICATIONS

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ABSTRACT. We study deformations of free boundary constant mean curvature hypersurfaces up to ambient isometries, allowing to simultaneously vary the value of the mean curvature and the ambient metric, provided that the infinitesimal ambient symmetries vary smoothly. We discuss applications to free boundary CMC disks and Delaunay annuli in the unit ball of a space form, as well as realizing an equivariantly nondegenerate interface of a phase transition as the limit of nodal sets of solutions to the Allen-Cahn equation.

1. INTRODUCTION

Hypersurfaces with constant mean curvature (CMC) are among the most classical objects in Geometry. These hypersurfaces have critical area among those that enclose a fixed volume, and are used to model interfaces between different media, such as soap films, having profuse applications in areas including Architecture, Chemistry, Physics and Engineering. Free boundary CMC hypersurfaces are particularly important in this regard, as they realize the natural physical constraint of allowing the interface to move freely about the boundary of the considered region.

There are many natural deformation questions regarding a given free boundary CMC hypersurface $\Sigma_0$ inside a manifold with boundary $M$. For instance, consider the problems of deforming $\Sigma_0$ through other free boundary CMC hypersurfaces in $M$ varying the value of its mean curvature, or deforming the ambient metric on $M$ while preserving the existence of a free boundary CMC hypersurface. Any answer to these deformation questions carries an intrinsic ambiguity due to ambient isometries in $M$, and should hence be considered up to such symmetries. In the absence of boundary conditions, these problems have been addressed in [5, 6]. The main result of the present paper is a simultaneous answer to both deformation questions above for free boundary CMC hypersurfaces. The key assumption to ensure that such deformations are feasible is that $\Sigma_0$ is equivariantly nondegenerate, meaning that the only Jacobi fields along $\Sigma_0$ originate from ambient symmetries (see Definition 2.2).

Under this hypothesis, we have the following deformation result:

Main Theorem. Let $\Sigma_0$ be an equivariantly nondegenerate free boundary CMC hypersurface of $(M, g_{\lambda_0})$ with mean curvature $h_0$. Assume that $g_{\lambda}$ is a smooth family of Riemannian metrics on $M$ whose Killing fields vary smoothly with $\lambda$. Then, $\Sigma_0$ is...
the member $\Sigma_{(h_0,\lambda_0)}$ of a smooth family $\Sigma_{(h,\lambda)}$ of free boundary CMC hypersurfaces in $(M,g_\lambda)$ with mean curvature $h$, which is locally unique up to ambient isometries.

Further technical details on the above statement are discussed in Theorem 5.1. The core of the proof is a convenient formulation of the CMC problem (5.1), that allows to combine the Implicit Function Theorem and a flux of Killing fields argument to obtain the desired equivariant deformation. This is inspired on an idea of Kapouleas [16, 17], see also [6], which we adapt to the free boundary framework with varying ambient metric.

In certain cases, up to using ambient isometries, the family $\Sigma_{(h,\lambda_0)}$ determines a foliation by free boundary CMC hypersurfaces near $\Sigma_{(h_0,\lambda_0)}$. Sufficient conditions for this are given in Propositions 6.1 and 6.2. For instance, spherical caps are free boundary CMC hypersurfaces that clearly foliate the unit ball $B^{n+1}_\lambda$ in a space form $M^{n+1}_\lambda$, see Figure 1. This fact is reobtained as a consequence of our main result applied to a flat disk $D^n \subset B^{n+1}_0$, which is equivariantly nondegenerate.

A more complex situation arises by considering CMC annuli instead of disks, which can be seen as a deformation of the so-called critical catenoid. This is the portion of a catenoid that meets the boundary of the unit ball $B^3_\lambda$ orthogonally, and has been the object of several recent investigations by Fraser and Schoen [10, 11, 12]. We establish the equivariant nondegeneracy of the critical catenoid (Lemma 8.1), allowing us to deform it through other free boundary CMC annuli in the unit ball of a space form. These surfaces can be recognized as compact portions of Delaunay surfaces, see Section 8. In the particular case of fixing the Euclidean ambient metric and only varying the mean curvature, this family of Delaunay annuli containing the critical catenoid is formed by potions of unduloids and nodoids. These are rotationally symmetric surfaces that intersect the unit ball orthogonally and whose profile curve is the roulette of a conic section, that is, the path traced by one of the foci of a conic as it rolls without slipping along a straight line (see Figure 5). The critical catenoid corresponds to rolling a parabola, which is the limiting case of rolling ellipses or hyperbolas with eccentricity close to 1, that respectively give rise to the unduloids and nodoids mentioned above.

Another application concerns realizing equivariantly nondegenerate minimal free boundary hypersurfaces $\Sigma_0$ as the limit of the nodal sets of solutions to the Allen-Cahn equation (9.2) on a manifold with boundary. This problem had been previously studied by Pacard and Ritoré [25], under the assumption that $\Sigma_0$ is nondegenerate, which we are able to relax using the above deformation result.

This paper is organized as follows. The basic framework for the free boundary CMC problem is recalled in Section 2, and preliminary computations regarding the flux of Killing fields are made in Section 3. In Section 4, we discuss the structure of the set of hypersurfaces with fixed diffeomorphism type in a manifold with boundary $M$ that meet the boundary $\partial M$ orthogonally. The detailed statement and proof of our main deformation result (Theorem 5.1) is given in Section 5. Issues regarding foliations by free boundary CMC hypersurfaces are discussed in Section 6. Sections 7 and 8 respectively contain applications to free boundary CMC disks and annuli in the unit ball of a space form. Finally, Section 9 explains how equivariantly nondegenerate free boundary CMC hypersurfaces can be realized as interfaces in phase transitions, by extending the results of Pacard and Ritoré [25].

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2. Free boundary CMC hypersurfaces

2.1. Basic definitions. Let \((M, g)\) be an \((n + 1)\)-dimensional Riemannian manifold with boundary. We denote by \(\vec{n}_\partial M\) the outer unit normal vector field to the boundary of \(M\). This choice is used to define the second fundamental form

\[
\Pi^\partial M(X, Y) := g(\nabla_X Y, \vec{n}_\partial M).
\]

The main objects studied in this paper are hypersurfaces \(\Sigma\) of such a manifold \(M\).

In order to simplify notation for these objects, we identify embeddings \(x: \Sigma \hookrightarrow M\) with their image \(x(\Sigma) \subset M\).

**Definition 2.1.** We say that a hypersurface \(\Sigma \subset M\) is admissible if:

(a) \(\Sigma \cap \partial M = \partial \Sigma\);
(b) \(M \setminus \Sigma = \Omega_1 \cup \Omega_2\), with \(\Omega_1\) compact and \(\Omega_1 \cap \Omega_2 = \emptyset\).

If, in addition, \(\vec{n}_\partial M(p) \in T_p \Sigma\) for all \(p \in \partial \Sigma\), we say \(\Sigma\) is normal.

Note that if \(\Sigma\) is normal, then \(\Sigma\) and \(\partial M\) are transverse submanifolds.

Due to (b), admissible hypersurfaces are compact and have orientable normal bundle. We denote by \(\vec{n}_\Sigma\) the unit normal vector field along \(\Sigma\) pointing inside the bounded region \(\Omega_1\). This choice is used to define second fundamental form

\[
\Pi^\Sigma(X, Y) := g(\nabla_X Y, \vec{n}_\Sigma),
\]

whose trace is called the mean curvature function \(H_\Sigma\). The mean curvature vector of \(\Sigma\) is defined as \(\vec{H}_\Sigma := H_\Sigma \vec{n}_\Sigma\). If \(H_\Sigma\) is constant, \(\Sigma\) is called a constant mean curvature hypersurface (or \(\text{CMC hypersurface}\)), and a minimal hypersurface if this constant is zero.

2.2. Isoperimetric approach. It is well-known that CMC hypersurfaces of \(M\) are those that have critical area among hypersurfaces \(\Sigma\) enclosing a fixed volume. A natural extension to the case in which \(\partial \Sigma\) is nonempty is the free boundary CMC problem, in which \(\partial \Sigma\) is allowed to move freely along \(\partial M\). Solutions of this problem, called free boundary CMC hypersurfaces, are normal CMC hypersurfaces. More precisely, an admissible hypersurface \(x_0: \Sigma \hookrightarrow M\) is a normal hypersurface with constant mean curvature equal to \(H\) if and only if it is a critical point of

\[
f_H(x) := \text{Area}(x) - H \text{Vol}(x),
\]

where \(\text{Area}(x) := \int_\Sigma \text{vol}_{x^*(g)}\) is the \(n\)-volume of \(\Sigma \subset M\) and \(\text{Vol}(x) := \int_{\Omega_1} \text{vol}_g\) is the \((n + 1)\)-volume of the bounded region \(\Omega_1\) enclosed by \(\Sigma\). When the above Lagrange multiplier \(H\) is nonzero, this means that \(x_0: \Sigma \hookrightarrow M\) is critical if and only if it has stationary area among admissible hypersurfaces \(x: \Sigma \hookrightarrow M\) with same enclosed volume \(\text{Vol}(x) = \text{Vol}(x_0)\).

2.3. Jacobi fields. Given a free boundary CMC hypersurface \(x_0: \Sigma \hookrightarrow M\), denote by \(J_{x_0}\) its Jacobi operator, which is the second-order linear differential operator

\[
J_{x_0}(\psi) := \Delta_\Sigma \psi - \left(\|\Pi^\Sigma\|^2 + \text{Ric}_g(\vec{n}_\Sigma, \vec{n}_\Sigma)\right)\psi,
\]

defined on the space of \(C^2\) functions \(\psi: \Sigma \to \mathbb{R}\). In the above, \(\Delta_\Sigma\) is the (nonnegative) Laplacian of \((\Sigma, x_0^*(g))\) and \(\|\Pi^\Sigma\|^2\) is the squared Hilbert-Schmidt norm of the second fundamental form of \(x_0: \Sigma \hookrightarrow M\). A Jacobi field along \(x_0\) is a smooth function \(\psi: \Sigma \to \mathbb{R}\) such that \(J_{x_0}(\psi) = 0\).
The Jacobi operator $J_{x_0}$ represents the second variation $d^2f_H(x_0)$ of the functional (2.1) at the critical point $x_0$, with respect to the $L^2$ inner product. Let $x_s: \Sigma \hookrightarrow M$, $s \in (-\varepsilon, \varepsilon)$, be a smooth variation of $x_0$, where each $x_s$ is a CMC embedding with mean curvature $H_s$. Let $V = \frac{d}{ds}|_{s=0}x_s$ be the corresponding variational vector field. Then $\psi_V := g(V, \tilde{n}_\Sigma)$ satisfies
\begin{equation}
J_{x_0}(\psi_V) = \frac{d}{ds}|_{s=0}H_s,
\end{equation}
so that $\psi_V$ is a Jacobi field exactly when $H_s \equiv H_0$ to first order as $s \to 0$. Furthermore, if each $x_s$ is a free boundary CMC hypersurface, then $\psi = \psi_V$ satisfies the linearized free boundary condition
\begin{equation}
g(\nabla \psi, \tilde{n}_\partial M) + \Pi_{\partial M}(\tilde{n}_\Sigma, \tilde{n}_\Sigma) \psi = 0.
\end{equation}

If the functional (2.1) is considered on the space of $C^{k,\alpha}$ embeddings (see Section 4), the Jacobi operator (2.2) acts on the corresponding tangent space at $x_0$, which can be identified with $C^{k,\alpha}(\Sigma)$. The restriction of $J_{x_0}$ to the closed subspace
\begin{equation}
C_{\partial}^{k,\alpha}(\Sigma) := \{ \psi \in C^{k,\alpha}(\Sigma) : g(\nabla \psi, \tilde{n}_\partial M) + \Pi_{\partial M}(\tilde{n}_\Sigma, \tilde{n}_\Sigma) \psi = 0 \}
\end{equation}
is a Fredholm operator of index zero that takes values in $C^{k-2,\alpha}(\Sigma)$, see [20, Sec 2]. The free boundary CMC hypersurface $x_0: \Sigma \hookrightarrow M$ is called nondegenerate if ker $J_{x_0} \cap C_{\partial}^{k,\alpha}(\Sigma) = \{0\}$, that is, if $J_{x_0}: C_{\partial}^{k,\alpha}(\Sigma) \to C^{k-2,\alpha}(\Sigma)$ is an isomorphism of Banach spaces. The dimension of ker $J_{x_0} \cap C_{\partial}^{k,\alpha}(\Sigma)$ is called the nullity of $x_0$.

2.4. Killing-Jacobi fields. A (local) vector field $K$ on $M$ is a (local) Killing vector field if $K_p \in T_p(\partial M)$ for all $p \in \partial M$ in the domain of $K$, and if $\nabla K$ is a skew-symmetric $(1,1)$-tensor, that is, $\mathcal{L}_Kg = 0$.

Given a free boundary CMC hypersurface $x_0: \Sigma \hookrightarrow M$, every local Killing vector field $K$ whose domain contains $x_0(\Sigma)$ induces a Jacobi field
\begin{equation}
\psi_K := g(K, \tilde{n}_\Sigma)
\end{equation}
along $x_0$, with $\psi_K \in C_{\partial}^{k,\alpha}(\Sigma)$. Indeed, denote by $I_s$, $s \in (-\varepsilon, \varepsilon)$, the local flow of $K$. Since each $I_s$ is a local isometry that preserves $\partial M$, the embeddings $x_s = I_s \circ x_0$ are free boundary CMC hypersurfaces with the same mean curvature as $x_0$. Recalling formula (2.3), this implies that $\psi_K$ is a Jacobi field in $C_{\partial}^{k,\alpha}(\Sigma)$. Such Jacobi fields of the form (2.6) are called Killing-Jacobi fields.

Definition 2.2. The free boundary CMC hypersurface $x_0: \Sigma \hookrightarrow M$ is called equivariantly nondegenerate if ker $J_{x_0} \cap C_{\partial}^{k,\alpha}(\Sigma)$ consists only of Killing-Jacobi fields.

Given a free boundary CMC hypersurface $x: \Sigma \to M$ and a local isometry $I$ of $(M, g)$ whose domain contains $x(\Sigma)$, then $I \circ x$ is another free boundary CMC hypersurface. Local isometries of $(M, g)$ preserve $\partial M$, and therefore also the orthogonality condition. Free boundary CMC embeddings $x, x': \Sigma \hookrightarrow M$ are said to be congruent if there exists a local isometry $I$ of $(M, g)$ such that $I(x(\Sigma)) = x'(\Sigma)$.

3. Flux of Killing fields

In preparation for the proof of our main result, we study the flux of Killing vector fields across hypersurfaces. This provides a convenient approach to study the equation $H_{\Sigma} \equiv \text{const.}$ modulo ambient isometries, which originates from the work of Kapouleas [16, 17], see also [6].
Proposition 3.1. Let $\Sigma \subset M$ be an admissible hypersurface and $K$ be a Killing vector field on $M$. Then

\[(3.1) \int_{\Sigma} g(K, \vec{n}_\Sigma) = 0.\]

Moreover, if $\Sigma$ is normal, then

\[(3.2) \int_{\Sigma} g(K, \vec{H}_\Sigma) = 0.\]

Proof. Since $\partial \Omega_1 = \Sigma \cup (\partial \Omega_1 \cap \partial M)$, and $K$ is tangent to $\partial M$, we have that

\[\int_{\Sigma} g(K, \vec{n}_\Sigma) = \int_{\partial \Omega_1} g(K, \vec{n}_{\partial \Omega_1}) = \int_{\Omega_1} \text{div}_g(K) = 0,\]

where the second equality follows from Stokes’ theorem, and the third from the fact that Killing vector fields have zero divergence.

In order to prove (3.2), denote by $K_\Sigma$ the component of $K$ tangent to $\Sigma$. Since $\Sigma$ is normal, $K$ is orthogonal to $\partial \Sigma$ along $\partial \Sigma$, hence $K_\Sigma \equiv 0$ on $\partial \Sigma$. An easy computation gives $\text{div}_\Sigma(K_\Sigma) = g(K, \vec{H}_\Sigma)$. Hence, by Stokes’ theorem,$\int_{\Sigma} g(K, \vec{H}_\Sigma) = 0.$

□

Corollary 3.2. Let $\Sigma$ be a normal hypersurface of $M$, and let $K_1, \ldots, K_r$ be Killing vector fields in $M$ such that the corresponding Killing-Jacobi fields $\psi_i: \Sigma \rightarrow \mathbb{R},$ $\psi_i = g(K_i, \vec{n}_\Sigma), \quad i = 1, \ldots, r,$ are linearly independent. For any $\alpha \in \mathbb{R}^r$, the function $H_\Sigma + \sum_{i=1}^r \alpha_i \psi_i$ is constant if and only if $H_\Sigma$ is constant and $\alpha = 0$.

Proof. Suppose $H_\Sigma + \sum_{i=1}^r \alpha_i \psi_i \equiv h$ is constant. Multiplying both sides by $\sum_{i=1}^r \alpha_i \psi_i$ and integrating over $\Sigma$, we have

\[\int_{\Sigma} \left( \sum_{i=1}^r \alpha_i \psi_i \right)^2 = h \sum_{i=1}^r \int_{\Sigma} \alpha_i \psi_i - \sum_{i=1}^r \int_{\Sigma} \alpha_i H_\Sigma \psi_i \]

\[= h \sum_{i=1}^r \alpha_i \left( \int_{\Sigma} g(K_i, \vec{n}_\Sigma) \right) - \sum_{i=1}^r \alpha_i \left( \int_{\Sigma} g(K_i, \vec{H}_\Sigma) \right) = 0,\]

by (3.1) and (3.2). Thus, $\sum_{i=1}^r \alpha_i \psi_i = 0$. Since the $\psi_i$ are linearly independent, it follows that $\alpha_i = 0$, $i = 1, \ldots, r$, and $H_\Sigma = h$. The converse is obvious. □

We remark that all of the above results remain valid if Killing vector fields are replaced with local Killing vector fields defined on a neighborhood of $\Omega_1$.

4. THE MANIFOLD OF NORMAL HYPERSURFACES

In this section, we give details on the existence of a natural smooth structure on the set of normal hypersurfaces of a Riemannian manifold $(M, g)$ that are close to a given compact normal hypersurface $\Sigma$ in some suitable topology. As usual, perturbations of $\Sigma$ are parametrized as graphs over $\Sigma$ of sufficiently small smooth functions $f: \Sigma \rightarrow \mathbb{R}$. The property of being a normal hypersurface (see Definition 2.1) may not preserved under the normal exponential flow of $\Sigma$, unless $\partial M$ is totally geodesic in $M$. However, up to using the exponential flow of an auxiliary metric for which $\partial M$ is totally geodesic and has the same normal bundle, we
may assume without loss of generality for the remainder of this section that \( \partial M \) is totally geodesic in \((M, g)\).

A detailed description of the smooth structure on the set of submanifolds with fixed diffeomorphism type can be found in the series of papers by Michor [22, 23, 24], in the \( C^\infty \) case. In order to fulfill the appropriate technical requirements of the Implicit Function Theorem, one has to go beyond the \( C^\infty \) realm, using embeddings of Hölder class \( C^{k,\alpha} \). There are several issues concerning the regularity of the set of unparametrized embeddings in this low regularity setting, which are extensively discussed in [1]. In this paper, however, we are only interested in the smooth structure near a given smooth unparametrized embedding, which avoids all the subtleties involved in the lack of regularity of the change of coordinates.

### 4.1. Unparametrized embeddings

The appropriate setup for studying the set of submanifolds of a given diffeomorphism type is obtained by considering the notion of unparametrized embeddings. Given a compact manifold \( \Sigma \), two embeddings \( x_1, x_2 : \Sigma \rightarrow M \) are equivalent if there exists a diffeomorphism \( \phi : \Sigma \rightarrow \Sigma \) such that \( x_1 = x_2 \circ \phi \). Equivalence classes of embeddings are called unparametrized embeddings of \( \Sigma \) in \( M \).

We denote by \( \mathcal{E}(\Sigma, M) \) the space of \( C^{k,\alpha} \)-unparametrized embeddings of \( \Sigma \) in \( M \), and by \( \mathcal{E}_0(\Sigma, M) \) the subset of \( \mathcal{E}(\Sigma, M) \) consisting of unparametrized embeddings \( x : \Sigma \rightarrow M \) such that \( x(\partial \Sigma) \subset \partial M \). Given an embedding \( x : \Sigma \rightarrow M \), we will denote by \([x] \in \mathcal{E}(\Sigma, M)\) the unparametrized embedding defined by \( x \). Note that \([x]\) is uniquely determined by the image \( x(\Sigma) \). The notions of admissible and normal hypersurfaces (Definition 2.1) extend naturally to unparametrized embeddings. Let \( \mathcal{E}_\partial^+(\Sigma, M) \) denote the subset of \( \mathcal{E}_0(\Sigma, M) \) consisting of unparametrized normal embeddings; observe that the admissible embeddings form an open subset of \( \mathcal{E}_\partial^+(\Sigma, M) \).

**Proposition 4.1.** Let \( \Sigma \) be a compact manifold with boundary and \( x_0 : \Sigma \rightarrow M \) be an admissible smooth normal embedding. A sufficiently small neighborhood of \([x_0] \) in \( \mathcal{E}_\partial^+(\Sigma, M) \) can be identified with an infinite-dimensional smooth submanifold \( \mathcal{N} \) of the Banach space \( C^{k,\alpha}(\Sigma) \), with \( 0 \in \mathcal{N} \) corresponding to \([x_0] \), such that \( T_0\mathcal{N} = C^{k,\alpha}_\partial(\Sigma) \), see (2.5).

Moreover, assume that \( K_1, \ldots, K_r \) is a family of local Killing vector fields defined in a neighborhood of \( x_0(\Sigma) \), and consider the functions \( \psi_i = g(K_i, \tilde{n}_0) \) in \( C^{k,\alpha}_\partial(\Sigma) \). Then, the pseudo-group of local isometries generated by the \( K_i \) has a continuous local action on \( \mathcal{N} \), and the orbit of 0 under this action is a smooth submanifold of \( \mathcal{N} \), of dimension greater than or equal to the dimension of the span of \( \psi_i \).

**Proof.** To each sufficiently small \( C^{k,\alpha} \)-map \( f : \Sigma \rightarrow \mathbb{R} \), we associate the embedding

\[
(4.1) \quad x_f : \Sigma \rightarrow M, \quad x_f(p) := \exp_{x_0(p)} (f(p) \tilde{n}_0),
\]

where \( \tilde{n}_0 \) is the unit normal vector field along \( x_0 : \Sigma \rightarrow M \). Since \( x_0 \) is a normal embedding, \( \tilde{n}_0(p) \in T_p(\partial M) \) for all \( p \in x_0(\Sigma) \cap \partial M \), and hence \( x_f(\partial \Sigma) \subset \partial M \), because \( \partial M \) is totally geodesic. Clearly, if \( f \equiv 0 \), then \( x_f = x_0 \). In order to simplify notation, we denote the image of \( x_f : \Sigma \rightarrow M \) by \( \Sigma_f \); in particular, \( \Sigma_0 \) is the image of the original embedding \( x_0 : \Sigma \rightarrow M \).

The above correspondence \( f \mapsto \Sigma_f \) gives a bijection from a sufficiently small neighborhood \( \mathcal{U} \) of \( 0 \in C^{k,\alpha}(\Sigma) \) to a neighborhood \( \mathcal{V} \) of \([x_0] \in \mathcal{E}_\partial(\Sigma, M)\). Denote by \( \mathcal{N} \) the subset of \( \mathcal{U} \) consisting of maps \( f \) such that \( x_f \) is a normal embedding.
Claim 4.2. \(\mathcal{N}\) is a submanifold of \(C^{k,\alpha}(\Sigma)\).

In order to prove this claim, let \(\varepsilon > 0\) be small and consider the the diffeomorphism 
\[
\Phi: [0, \varepsilon) \times \Sigma \rightarrow \mathcal{A}, \quad \Phi(t, p) := \exp_{\pi_0(p)} \left( t \, \bar{n}_0(p) \right),
\]
where \(\mathcal{A}\) is an open subset of \(M\) containing \(\Sigma_0\). Note that \(\Phi(0, p) = \pi_0(p)\) is the original normal hypersurface \(\Sigma_0\), and that \(d\Phi_{(t, p)}\) is an isomorphism for \(t > 0\) sufficiently small. In fact, 
\[
d\Phi_{(0, p)}(\tau, v) = \tau \, n_0(p) + v,
\]
for all \(\tau \in \mathbb{R}\) and \(v \in T_p \Sigma_0\). Set 
\[
(\zeta(t, p), X(t, p)) := d\Phi_{(t, p)}^{-1}(\bar{n}_\partial M(\Phi(t, p))) \in \mathbb{R} \times T_p \Sigma.
\]
For \(f \in \mathcal{U}\), \(x_f(p) = \Phi(f(p), p)\), and so \(T_{x_f(p)} \Sigma_f = \text{Im} \left[ d\Phi(f(p), p) \circ (df(p), \text{Id}) \right] \). Thus, 
\[
d\Phi_{(f, p)}^{-1} \left[ T_{x_f(p)} \Sigma_f \right] = \text{Im} \left( df(p), \text{Id} \right) = \text{Graph} (df(p)).
\]
The embedding \(x_f\) is normal if and only if \(\bar{n}_\partial M(x_f(p)) \in T_{x_f(p)} \Sigma_f\) for all \(p \in \partial \Sigma\). From (4.3) and (4.4), this condition reads
\[
\zeta(f(p), p) = df(p) \cdot X(f(p), p) \quad \text{for all } p \in \partial \Sigma.
\]
In other words, the set of normal embeddings \(x_f\) is identified with the inverse image \(\eta^{-1}(0)\), where \(\eta: C^{k,\alpha}(\Sigma) \rightarrow C^{k-1,\alpha}(\partial \Sigma)\) is the smooth map defined by 
\[
\eta(f) = df(p) \cdot X(f(p), p) - \zeta(f, p), \quad p \in \partial \Sigma.
\]
We prove that \(\mathcal{N}\) is a smooth submanifold of \(C^{k,\alpha}(\Sigma)\) by showing that \(\eta\) is a submersion at \(f = 0\). The linearization of \(\eta\) at 0 applied to \(\psi \in C^{k,\alpha}(\Sigma)\) reads 
\[
d\eta_0(\psi)(p) = df(p) \cdot X(0, p) - \frac{\partial \zeta}{\partial t}(0, p) \psi(p), \quad p \in \partial \Sigma.
\]
Claim 4.3. For all \(p \in \partial \Sigma\), 
\[
\frac{\partial \zeta}{\partial t}(0, p) = -\Pi^{\partial M} (\bar{n}_0(p), \bar{n}_0(p)).
\]
Let us first assume the above claim and complete the proof of the theorem. Showing that \(\eta: C^{k,\alpha}(\Sigma) \rightarrow C^{k-1,\alpha}(\partial \Sigma)\) is a submersion at \(f = 0\) amounts to showing that \(d\eta_0\) is surjective and \(\ker d\eta_0\) is complemented in \(C^{k,\alpha}(\Sigma)\). Recall that a bounded linear map between Banach spaces is surjective and has complemented kernel if and only if it admits a bounded linear right-inverse. Consider an open subset \(U \subset \Sigma\) containing \(\partial \Sigma\), and a diffeomorphism \(^1\)(s, \pi): U \rightarrow [0, \varepsilon) \times \partial \Sigma\) carrying \(\partial \Sigma\) onto \(\{0\} \times \partial \Sigma\), satisfying \(\bar{n}_\partial M(s) \equiv 1\) and \(d\pi_p(\bar{n}_\partial M) = 0\) on \(\partial \Sigma\). Let \(\chi\) be a smooth cut-off function with support contained in \(U\), such that \(\chi \equiv 1\) in a neighborhood of \(\partial \Sigma\). It is easy to check that \(T: C^{k-1,\alpha}(\partial \Sigma) \rightarrow C^{k,\alpha}(\Sigma)\), given by 
\[
T(\phi) = \chi \cdot (\phi \circ \pi) \cdot s,
\]
is a bounded right-inverse for \(d\eta_0: C^{k,\alpha}(\Sigma) \rightarrow C^{k-1,\alpha}(\partial \Sigma)\). Finally, \(T_0 \mathcal{N} = C^{k,\alpha}_p(\Sigma)\) follows from \(T_0 \mathcal{N} = \ker d\eta_0\), using (4.5), (4.6), and the identity (4.8) below.

As to the natural action of the (pseudo-)group of isometries generated by a family of (local) Killing fields on the set of \(C^{k,\alpha}\)-unparametrized embeddings, note that this restricts to an action on the set of unparametrized normal embeddings of \(\Sigma\).

\(^1\)Such diffeomorphism can be defined near \(\partial \Sigma\), e.g., as the inverse of \((t, p) \mapsto \exp_p \left( t \bar{n}_\partial M(p) \right)\).
in $M$, i.e., a local action on $N$. This action is only continuous, but the orbit of any smooth map is a smooth submanifold [1, Prop 5.1]. It is easy to see that the tangent space to the orbit through 0 contains the functions $\psi_i = g(K_i, \tilde{n}_0)$. □

**Proof of Claim 4.3.** Rewrite equation (4.3) as

$$d\Phi_{t,p}(\zeta(t,p), X(t,p)) = \tilde{n}_{\partial M}(\Phi(t,p)).$$

Using (4.2), (4.7), and that $\tilde{n}_{\partial M}(p) \in T_p\Sigma_0$ for $p \in \partial \Sigma$, we have:

$$\zeta(0,p) = 0, \quad \text{and} \quad X(0,p) = \tilde{n}_{\partial M}(p).$$

Fix $p \in \partial \Sigma$, and consider (4.7) as an equality between vector fields along the geodesic $t \mapsto \Phi(t,p)$. Let us covariantly differentiate (with respect to the Levi-Civita connection $\nabla$) equality (4.7) at $t = 0$. For $s, t \in \mathbb{R}$ small, set:

$$\rho(t,s) := d\Phi_{t,p}(\zeta(s,p), X(s,p)),$$

so that the left-hand side of (4.7) is given by $\rho(t,t)$. Denoting covariant derivatives with respect to the Levi-Civita connection $\nabla$ of $g$ by $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$, we have:

$$\frac{\partial}{\partial t}|_{t=0} \rho(t,t) = \frac{\partial}{\partial t}|_{t=0} \rho(t,0) + \frac{\partial}{\partial s}|_{s=0} \rho(0,s).$$

Moreover,

$$\rho(t,0) = (4.8) \frac{\partial}{\partial s}|_{s=0} d\Phi_{t,p}(0, \tilde{n}_{\partial M}(p)), \quad \rho(0,s) = (4.2) \zeta(s,p) \tilde{n}_0(p) + X(s,p).$$

In order to differentiate $\rho(t,0)$, let us consider a smooth curve $u \mapsto p(u) \in \Sigma_0$ with $p(0) = p$ and $p'(0) = \tilde{n}_{\partial M}(p)$, so that $\rho(t,0) = \frac{\partial}{\partial u}|_{u=0} \Phi(t,p(u))$. Then

$$\frac{\partial}{\partial t}|_{t=0} \rho(t,0) = \frac{\partial}{\partial t}|_{t=0} \frac{\partial}{\partial u}|_{u=0} \Phi(t,p(u)) = \frac{\partial}{\partial u}|_{u=0} \nabla_{\tilde{n}_{\partial M}(p)} \tilde{n}_0.$$ 

The derivative of $\rho(0,s)$ is given by

$$\frac{\partial}{\partial s}|_{s=0} \rho(0,s) = \frac{\partial \zeta}{\partial t}(0,p) \tilde{n}_0(p) + \frac{\partial X}{\partial t}(0,p).$$

Note that, for $p$ fixed, $t \mapsto X(t,p)$ is a curve in the fixed vector space $T_p\Sigma_0$, and $\frac{\partial X}{\partial t}(0,p) \in T_p\Sigma_0$ is the standard derivative of this curve at $t = 0$. Finally, the derivative of the right-hand side of (4.7) is given by

$$\frac{\partial}{\partial t}|_{t=0} \tilde{n}_{\partial M}(\Phi(t,p)) = \nabla_{\tilde{n}_0(p)} \tilde{n}_{\partial M}.$$ 

Using (4.9), (4.10) and (4.11), we obtain that the covariant derivative of equation (4.7) at $t = 0$ reads (cf. [29, p.174]):

$$\nabla_{\tilde{n}_{\partial M}(p)} \tilde{n}_0 + \frac{\partial \zeta}{\partial t}(0,p) \tilde{n}_0(p) + \frac{\partial X}{\partial t}(0,p) = \nabla_{\tilde{n}_0(p)} \tilde{n}_{\partial M}.$$ 

Note that the first and the third terms in the left-hand side of the above are tangent to $\Sigma_0$. Thus, multiplying both sides by the unit normal vector $\tilde{n}_0(p)$ yields

$$\frac{\partial \zeta}{\partial t}(0,p) = g(\nabla_{\tilde{n}_0(p)} \tilde{n}_{\partial M}, \tilde{n}_0(p)) = -\mathbb{II}_{\partial M}(\tilde{n}_0(p), \tilde{n}_0(p)).$$ □
5. Deformations of free boundary CMC hypersurfaces

In this section, we give the detailed statement and proof of our main deformation result for free boundary CMC hypersurfaces.

**Theorem 5.1.** Let \( \Sigma_0 \subset M \) be an admissible hypersurface and \( g_\lambda \) be a smooth family of Riemannian metrics on \( M \) that define the same normal bundle \( T(\partial M)^\perp \). Assume that \( \Sigma_0 \) is an equivariantly nondegenerate free boundary CMC embedding in \((M, g_{\lambda_0})\) with mean curvature \( h_0 \). Furthermore, assume that each Killing-Jacobi field \( K_{\lambda_0} \) along \( \Sigma_0 \) belongs to a smooth family \( K_\lambda \) of local vector fields on \( M \) such that \( K_\lambda \) is Killing for \( g_\lambda \). Then, there exists a smooth family \((h, \lambda) \mapsto \Sigma_{(h, \lambda)} \subset M\) of admissible hypersurfaces, defined for \((h, \lambda) \) near \((h_0, \lambda_0)\), such that:

- \( \Sigma_{(h, \lambda)} \) is a free boundary CMC hypersurface in \((M, g_\lambda)\) diffeomorphic to \( \Sigma_0 \) and with mean curvature \( h \);
- \( \Sigma_{(h_0, \lambda_0)} = \Sigma_0 \).

Moreover, the family \( \Sigma_{(h, \lambda)} \) is unique modulo congruence, i.e., if \( \mathcal{N} \) is sufficiently close to \( \lambda_0 \) and \( \Sigma' \) is a free boundary CMC hypersurface of \((M, g_{\lambda'})\) sufficiently close to \( \Sigma_0 \), with constant mean curvature \( h' \), then \( \Sigma' \) is congruent to \( \Sigma_{(h', \lambda')} \).

**Proof.** The strategy is to use an Implicit Function Theorem on the space of unparametrized normal embeddings\(^2\) \( \mathcal{E}_\partial^+(\Sigma, M) \) discussed in Section 4, together with a gauge argument using the flux of Killing fields discussed in Section 3.

Denote by \( x_0 : \Sigma \hookrightarrow M \) the smooth embedding with image \( \Sigma_0 \), and by \( \bar{n}_0 \) the inward unit normal along \( \Sigma_0 \subset (M, g_{\lambda_0}) \). A sufficiently \( C^{k, \alpha} \)-small neighborhood of \([x_0]\) in \( \mathcal{E}_\partial^+(\Sigma, M) \) is identified with a smooth submanifold \( \mathcal{N} \subset C^{k, \alpha}(\Sigma) \), as in Proposition 4.1. For \( f \in \mathcal{N} \), we denote by \( x_f : \Sigma \hookrightarrow M \) the corresponding embedding and by \( \Sigma_f \) its image. Furthermore, let \( \bar{n}_f^i \) be the inward unit normal along \( \Sigma_f \subset (M, g_\lambda) \), and \( H_f^i \) be the mean curvature function of \( \Sigma_f \subset (M, g_\lambda) \).

Consider the smooth function

\[
\mathcal{H} : \mathcal{N} \times \mathbb{R}^r \times \Lambda \longrightarrow C^{k-2, \alpha}(\Sigma) \times \Lambda
\]

\[
\mathcal{H}(f, \alpha, \lambda) := \left( H_f^i + \sum_{i=1}^r \alpha_i g_\lambda(K_{\lambda_0}^i, \bar{n}_f^i), \lambda \right),
\]

where \( K_{\lambda_0}^i, i = 1, \ldots, r \), are the local Killing vector fields in \((M, g_\lambda)\) that extend the Killing-Jacobi fields along \( \Sigma_0 \). Corollary 3.2 implies that if \( h \) is a constant and \( \mathcal{H}(f, \alpha, \lambda) = (h, \lambda) \), then \( \alpha = 0 \) and \( \Sigma_f \subset (M, g_\lambda) \) has constant mean curvature \( h \).

**Claim 5.2.** \( \mathcal{H} \) is a submersion near \((0, 0, \lambda_0) \in \mathcal{N} \times \mathbb{R}^r \times \Lambda \).

The result of differentiating (5.1) at \((0, 0, \lambda_0)\) in the direction \((\psi, \beta, \mu)\), with \( \psi \in C^{k, \alpha}_\partial(\Sigma) \), \( \beta \in \mathbb{R}^r \) and \( \mu \in T_{\lambda_0} \Lambda \), is

\[
d\mathcal{H}(0, 0, \lambda_0)[\psi, \beta, \mu] = \left( J_{\lambda_0} \psi + \sum_{i=1}^r \beta_i g_\lambda(K_{\lambda_0}^i, \bar{n}_{\lambda_0}^i) + P(\mu), \mu \right),
\]

where \( J_{\lambda_0} \) is the Jacobi operator of the CMC hypersurface \( \Sigma_0 \subset (M, g_{\lambda_0}) \), and \( P : T_{\lambda_0} \Lambda \to C^{k-2, \alpha}(\Sigma) \) is a linear operator given by the derivative of the map.

\(^2\)Since \( g_\lambda \) define the same \( T(\partial M)^\perp \), the notion of normal embedding is independent on \( \lambda \).
\( \Lambda \ni \lambda \mapsto H_\lambda^0 \in C^{k-2,\alpha}(\Sigma) \) at \( \lambda_0 \). Note that the linear map
\[
C_{\partial}^{k,\alpha}(\Sigma) \times \mathbb{R}^r \ni (\psi, \beta) \mapsto J_{\lambda_0} \psi + \sum_{i=1}^r \beta_i g_{\lambda_0}(K^i_{\lambda_0}, \bar{m}_{\lambda_0}^0) \in C^{k-2,\alpha}(\Sigma)
\]
is surjective. This follows from the fact that \( J_{\lambda_0} : C_{\partial}^{k,\alpha}(\Sigma) \to C^{k-2,\alpha}(\Sigma) \) is an \( L^2 \)-symmetric Fredholm operator of index 0, and the functions \( g_{\lambda_0}(K^i_{\lambda_0}, \bar{m}_{\lambda_0}^0) \) span its kernel. From (5.2), this implies that \( d\mathcal{H}(0,0,\lambda_0) \) is surjective. It also follows readily from (5.2) that:
\[
\ker d\mathcal{H}(0,0,\lambda_0) = \ker J_{\lambda_0} \times \{0\} \times \{0\},
\]
which is a subspace of dimension \( r \), and therefore complemented, in \( C_{\partial}^{k,\alpha}(\Sigma) \times \mathbb{R}^r \times T_{\lambda_0} \Lambda \). This completes the proof of Claim 5.2.

Since submersions admit local sections, there exists a neighborhood \( \mathcal{U} \) of \( (h_0, \lambda_0) \) in \( \mathbb{R} \times \Lambda \) and a smooth map \( (h, \lambda) \mapsto (f(h, \lambda), \alpha(h, \lambda), \lambda) \in \mathcal{N} \times \mathbb{R}^r \times \Lambda \) such that:
\[
\begin{align*}
(a) & f(h_0, \lambda_0) = 0 \\
(b) & \mathcal{H}(f(h, \lambda), \alpha(h, \lambda), \lambda) = (h, \lambda), \text{ for all } (h, \lambda) \in \mathcal{U}.
\end{align*}
\]
As we observed above, (b) implies that \( \alpha(h, \lambda) = 0 \) for all \( (h, \lambda) \in \mathcal{U} \), and hence \( x_{f(h, \lambda)} \) is a \( g_\lambda \)-CMC normal embedding with mean curvature \( h \). Moreover, by (a), \( x_{f(h_0, \lambda_0)} = x_0 \). The desired smooth family is given by \( \Sigma_{(h, \lambda)} := x_{f(h, \lambda)}(\Sigma) \).

As to the uniqueness modulo congruence, observe that for \( (h, \lambda) \) near \( (h_0, \lambda_0) \), the inverse image \( \mathcal{H}^{-1}(h, \lambda) \) is a smooth submanifold of \( \mathcal{N} \times \{0\} \times \{\lambda\} \) with dimension \( r = \dim \ker d\mathcal{H}(0,0,\lambda_0) \). On the other hand, for \( (h, \lambda) \) sufficiently close to \( (h_0, \lambda_0) \), the functions \( \psi^k_{\lambda} = g_\lambda(K^k_{\lambda}, \bar{m}_{\lambda}^0) \) are linearly independent by continuity. This implies that the \( f(h, \lambda) \)-orbit\(^3\) of the local action on \( \mathcal{N} \) by the pseudo-group of local \( g_\lambda \)-isometries generated by the \( K^k_{\lambda} \) is a smooth manifold of dimension \( \geq r \), see Proposition 4.1. Clearly, if \( f' \) belongs to this orbit, then \( (f', 0, \lambda) \in \mathcal{H}^{-1}(h, \lambda) \). This implies that the \( f(h, \lambda) \)-orbit has in fact dimension equal to \( r \), and that a sufficiently small neighborhood of \( (f(h, \lambda), 0, \lambda) \) in \( \mathcal{H}^{-1}(h, \lambda) \) contains only elements of the \( f(h, \lambda) \)-orbit. In other words, any free boundary CMC hypersurface \( \Sigma' \), with mean curvature equal to \( h \), and sufficiently close to \( \Sigma_0 \), must be congruent to \( \Sigma_{(h, \lambda)} \). \( \square \)

### 6. Foliations by CMC Hypersurfaces

A direct consequence of Theorem 5.1 is that an equivariantly nondegenerate free boundary CMC hypersurface \( \Sigma_0 \) with mean curvature \( h_0 \) is part of a 1-parameter family of free boundary CMC hypersurfaces \( \{\Sigma_s\}_{s \in (-\varepsilon, \varepsilon)} \), with mean curvature \( h_0 + s \). A very natural geometric question is to determine whether a tubular neighborhood of \( \Sigma_0 \) has a foliation by free boundary CMC hypersurfaces. In particular, since \( \Sigma_s \) are unique modulo congruence, this amounts to determining whether there exist ambient isometries \( I_s \), with \( I_0 = \text{id} \), such that \( \{I_s(\Sigma_s)\}_{s \in (-\varepsilon, \varepsilon)} \) is a foliation.

Foliations by CMC hypersurfaces are important geometric objects, with deep ramifications in mathematical physics, see [14, 33]. Deep contributions regarding the problem of foliating a tubular neighborhood of a submanifold with CMC hypersurfaces have been given by Ye [32] and Mazzeo and Pacard [21]. In the former,\(^3\)Note that \( \mathcal{H}^{-1}(h, \lambda) \) is identified with the set of \( g_\lambda \)-free boundary CMC hypersurface of \( M \) near \( \Sigma_0 \) that are diffeomorphic to \( \Sigma_0 \) and with mean curvature equal to \( h \), while the \( f(h, \lambda) \)-orbit is identified with the set of hypersurfaces of \( M \) near \( \Sigma_0 \) that are congruent to \( \Sigma_{(h, \lambda)} \). Thus, a proof of uniqueness modulo congruence is obtained showing these two sets coincide near \( \Sigma_{(h, \lambda)} \).
it is proved that the foliation by geodesic spheres of small radius with center at a nondegenerate critical point of the scalar curvature function can be perturbed to a foliation by CMC spheres. The latter considered simple closed nondegenerate geodesics, proving that a tubular neighborhood can be partially foliated with CMC hypersurfaces obtained by perturbing tubes around the geodesic. The main difficulty in obtaining an actual foliation in this case, as well as in higher dimensions, is related to a bifurcation phenomena as the CMC hypersurfaces collapse, see [4, 19].

6.1. Nondegenerate \( \Sigma_0 \). Initially, let us make the simplifying assumption that \( \Sigma_0 \) is actually nondegenerate, i.e., there are no non-zero Jacobi fields along \( \Sigma_0 \) in \( C^{2,\alpha}_0(\Sigma_0) \). In this situation, the Jacobi operator \( J_0: C^{2,\alpha}_0(\Sigma_0) \to C^{0,\alpha}(\Sigma_0) \) is an isomorphism, hence there exists a unique \( \psi \in C^{2,\alpha}_0(\Sigma_0) \) satisfying

\[
J_0(\psi) \equiv 1.
\]

**Proposition 6.1.** If the solution \( \psi \) to (6.1) does not have zeros in \( \Sigma_0 \), then the 1-parameter family \( \{\Sigma_s\}_{s \in (-\varepsilon,\varepsilon)} \) above is a foliation of a neighborhood of \( \Sigma_0 \).

**Proof.** Let \( \phi_s \in C^{2,\alpha}_0(\Sigma_0) \) be such that \( \Sigma_s = \exp(\phi_s \vec{n}_{\Sigma_0}) \); in particular, \( \phi_0 = 0 \). Consider the induced variational field along \( \Sigma_0 \), given by \( V = \frac{d}{ds}|_{s=0} \phi_s \vec{n}_{\Sigma_0} \).

Moreover, let \( \psi : g(V,\vec{n}_{\Sigma_0}) = \frac{d}{ds}|_{s=0} \phi_s \). Then,

\[
J_0(\psi) = \frac{d}{ds}|_{s=0} (h_0 + s) = 1.
\]

By uniqueness, \( \psi = \psi \). Since \( \Sigma_0 \) is embedded, \( \Sigma_0 \times \mathbb{R} \owns (p,t) \mapsto \exp_p(t \vec{n}_{\Sigma_0}(p)) \in M \) gives a diffeomorphism from a neighborhood of \( \Sigma_0 \times \{0\} \) in \( \Sigma_0 \times \mathbb{R} \) onto a neighborhood of \( \Sigma_0 \) in \( M \). By composition, the map \( (p,s) \mapsto \exp_p(\phi(p) \vec{n}_{\Sigma_0}(p)) \) is a diffeomorphism from \( \Sigma_0 \times (-\varepsilon,\varepsilon) \) onto a neighborhood \( U \) of \( \Sigma_0 \) in \( M \), for \( \varepsilon > 0 \) sufficiently small. Under this diffeomorphism, the hypersurfaces \( \Sigma_s \) correspond to the slices \( \Sigma_0 \times \{s\} \), which form a foliation of \( U \).

6.2. Equivariantly nondegenerate \( \Sigma_0 \). Let us now consider the more general case in which \( \Sigma_0 \) is equivariantly nondegenerate (recall Definition 2.2). By Proposition 3.1, the constant function 1 is \( L^2 \)-orthogonal to every Killing-Jacobi field, and therefore belongs to the image of the Jacobi operator \( J_0 \). In particular, equation (6.1) has solutions in \( C^{2,\alpha}_0(\Sigma_0) \).

**Proposition 6.2.** If there exists a nonvanishing solution of (6.1), then there exists a neighborhood of \( \Sigma_0 \) foliated by a smooth 1-parameter family of free boundary CMC hypersurfaces \( \{\Sigma_s(\Sigma_s)\}_{s \in (-\varepsilon,\varepsilon)} \), as described above.

**Proof.** It suffices to show that, given a nonvanishing solution \( \psi \) to (6.1), there exists a smooth 1-parameter family of maps \( \varphi_s \in C^{2,\alpha}_0(\Sigma_0) \), with \( \varphi_0 = 0 \) and \( \frac{d}{ds}|_{s=0} \varphi_s = \psi \), such that the hypersurface \( \exp(\varphi_s \vec{n}_{\Sigma_0}) \) has constant mean curvature \( h_0 + s \), for \( |s| \) sufficiently small. Once the existence of such family \( \varphi_s \) is established, the proof follows exactly as in the nondegenerate case (Proposition 6.1).

Applying Theorem 5.1 with a fixed metric \( g_{\lambda_0} \), we get the existence of a smooth 1-parameter family of maps \( \phi_s \in C^{2,\alpha}_0(\Sigma_0) \), with \( \phi_0 = 0 \), such that \( \Sigma_s = \exp(\phi_s \vec{n}_{\Sigma_0}) \) has constant mean curvature \( h_0 + s \), for \( |s| \) sufficiently small. Note that the function

\[\text{The assumption that } \psi = \frac{d}{ds}|_{s=0} \phi_s \text{ does not vanish on } \Sigma_0 \text{ and compactness of } \Sigma_0 \text{ imply that the map } \Sigma_0 \times \mathbb{R} \owns (p,s) \mapsto (p, \phi_s(p)) \in \Sigma_0 \times \mathbb{R} \text{ is a diffeomorphism between two neighborhoods of } \Sigma_0 \times \{0\} \text{ in } \Sigma_0 \times \mathbb{R} \text{.}\]
\[ \tilde{\psi} = \frac{d}{ds} \bigg|_{s=0} \phi_s \] is another solution to (6.1). Thus, there exists a local Killing vector field \( K \) such that \( \psi = \tilde{\psi} + \psi_K \), where \( \psi_K \) is the Killing-Jacobi field (2.6). Denote by \( \{I_s\}_{s \in (-\varepsilon, \varepsilon)} \) the local flow of \( K \). Since \( K \) is Killing, \( I_s(\Sigma_0) \) is a smooth perturbation of \( \Sigma_0 \) with constant mean curvature \( h_0 + s \). The corresponding variational field is:
\[ (6.2) \quad \frac{d}{ds} \bigg|_{s=0} I_s(\exp(\phi_s \tilde{n}_\Sigma_0)) = \frac{d}{ds} \bigg|_{s=0} \left[ I_s(\exp(\phi_0 \tilde{n}_\Sigma_0)) \right] + \frac{d}{ds} \bigg|_{s=0} \left[ I_0(\exp(\phi_s \tilde{n}_\Sigma_0)) \right] = K + V, \]
where \( V = \frac{d}{ds} \bigg|_{s=0} (\exp(\phi_s \tilde{n}_\Sigma_0)) = \tilde{\psi} \tilde{n}_\Sigma_0 \) is the variational field corresponding to \( \Sigma_0 \). By Proposition 4.1, there exists a smooth 1-parameter family of maps \( \phi_s \in C^{2,\alpha}(\Sigma_0) \), with \( \phi_0 = 0 \), such that \( I_s(\Sigma_0) = \exp(\phi_s \tilde{n}_\Sigma_0) \). Finally, from (6.2):
\[ \frac{d}{ds} \bigg|_{s=0} \psi = g(K + V, \tilde{n}_\Sigma_0) = \psi_K + \tilde{\psi} = \psi. \]

7. **Free boundary disks in the unit ball**

The simplest free boundary minimal hypersurface in the unit ball \( B^{n+1} \subset \mathbb{R}^{n+1} \) is the flat disk \( D^n \) obtained by intersecting \( B^{n+1} \) with a codimension 1 linear subspace of \( \mathbb{R}^{n+1} \), which we may assume to be \( \mathbb{R}^n = \{e_{n+1}\} \). As a warm-up example, we apply Theorem 5.1 to verify that \( D^n \) can be deformed to free boundary CMC hypersurfaces inside the unit ball of other \((n+1)\)-dimensional space forms. We remark that stability issues for such surfaces have been studied in [26, 27, 28, 30].

7.1. **Equivariant nondegeneracy.** The Jacobi operator (2.2) of \( D^n \) is simply the (nonnegative) Laplacian of the flat metric, as \( D^n \subset B^{n+1} \) is totally geodesic and the ambient curvature vanishes. Moreover, the linearized free boundary condition (2.4) reads \( \langle \nabla \psi(x), x \rangle = \psi(x) \), since \( \partial B^{n+1} = S^n \) is the unit sphere in \( \mathbb{R}^{n+1} \). Altogether, Jacobi fields along \( D^n \) are harmonic functions \( \psi: D^n \rightarrow \mathbb{R} \) with Robin boundary conditions:
\[ (7.1) \quad \begin{cases} \Delta \psi = 0 & \text{in } D^n, \\ \frac{\partial \psi}{\partial n} = \psi & \text{on } \partial D^n. \end{cases} \]

The space of solutions to (7.1) is spanned by the coordinate functions:
\[ (7.2) \quad f_i(x) := \langle x, e_i \rangle, \quad i = 1, \ldots, n, \]
where \( \{e_i\} \) is an orthonormal basis of \( \mathbb{R}^n \). Indeed, expanding a solution \( \psi = \sum_k \psi_k \) as a sum of homogeneous polynomials \( \psi_k \) of degree \( k \) (e.g., using the Taylor series at \( x = 0 \)), we have that \( \frac{\partial \psi_k}{\partial n} = \sum_k \langle \nabla \psi_k(x), x \rangle = \sum_k k \psi_k \), which can only be equal to \( \psi \) provided \( \psi_k = 0 \) for all \( k \neq 1 \). In other words, \( \psi \) must be a homogeneous polynomial of degree 1, hence a linear combination of (7.2). In particular, the nullity of \( D^n \) is equal to \( n \).

The space of Killing fields of the flat ball \( B^{n+1} \) is identified with the orthogonal Lie algebra \( \mathfrak{o}(n) \), as each \( A \in \mathfrak{o}(n) \) corresponds to the Killing field \( K_A(x) = Ax \). Killing-Jacobi fields on \( D^n \) are given by the restriction to \( D^n \) of the linear map \( x \mapsto \langle K_A(x), e_{n+1} \rangle = \sum_{j=1}^n a_{n+1,j} x_j \), where \( A = (a_{ij}) \). Thus, there are \( n \) linearly independent Killing-Jacobi fields\(^5\) along \( D^n \), which is hence equivariantly nondegenerate. See Lemma 8.1 below for a generalization to other surfaces of revolution.

\(^5\)Geometrically, these are infinitesimal rotations of \( B^{n+1} \) with rotation axis tangent to \( D^n \).
7.2. Killing fields in space forms. In order to apply Theorem 5.1 to deform $D^n$, we still have to verify that the above Killing fields of the flat ambient metric extend to a smooth family of Killing fields of ambient metrics with constant curvature.

Let $M^{n+1}_\lambda$ be the simply-connected space form of constant curvature $\lambda$. We denote by $B^{n+1}_0$ the unit ball in $M^{n+1}_\lambda$, which is a Riemannian manifold with boundary, equipped with the induced metric $g_\lambda := dr^2 + sn^2(\lambda) ds^2_\lambda$, where $ds^2_\lambda$ is the standard metric on the unit sphere $S^n$ and

$$sn_\lambda(r) = \begin{cases} \frac{1}{\sqrt{\lambda}} \sin (r\sqrt{\lambda}) & \text{if } \lambda > 0, \\ r & \text{if } \lambda = 0, \\ \frac{1}{\sqrt{-\lambda}} \sinh (r\sqrt{-\lambda}) & \text{if } \lambda < 0. \end{cases}$$

Lemma 7.1. Every Killing field $K_0$ of $B^{n+1}_0$ extends to a smooth $1$-parameter family $K_\lambda$ of Killing fields of $B^{n+1}_\lambda$.

Proof. It is a well-known fact that the isometry groups $G_\lambda$ of $M^{n+1}_\lambda$ form a smooth bundle of Lie groups over $\mathbb{R}$, and the corresponding Lie algebras $g_\lambda$ form a smooth bundle of Lie subalgebras of $\mathfrak{gl}(n+1)$ over $\mathbb{R}$, see e.g. [31] for $n = 2$ and [5, Ex. 2.6] for $n \geq 2$. The isometry group of the corresponding unit ball $B^{n+1}_\lambda \subset M^{n+1}_\lambda$ is the subgroup $H_\lambda$ of $G_\lambda$ formed by isometries that fix the origin $0 \in B^{n+1}_\lambda$, and the corresponding subalgebras are $\mathfrak{h}_\lambda = \{ K_\lambda \in \mathfrak{g}_\lambda : K_\lambda(0) = 0 \}$. As $\mathfrak{h}_\lambda$ is a smooth bundle of Lie subalgebras of $\mathfrak{gl}(n+1)$ over $\mathbb{R}$, it follows that every Killing field $K_0 \in \mathfrak{h}_0$ of $B^{n+1}_0$ admits a smooth extension $K_\lambda \in \mathfrak{h}_\lambda$ to a Killing field of $B^{n+1}_\lambda$. Alternatively, the smooth dependence on $\lambda$ of solutions $K$ to $L_K(g_\lambda) = 0$ can be used to prove that the Killing field $K_0$ admits an extension $K_\lambda$. □

7.3. Free boundary CMC disks. As a result of the above discussion, Theorem 5.1 can be applied to the flat disk $D^n \subset B^{n+1}_0$ in the flat unit ball, yielding:

Proposition 7.2. The flat disk $D^n$ is the member $\Sigma_{(0,0)}$ of a $2$-parameter family $\Sigma_{(h,\lambda)}$ of free boundary disks with constant mean curvature $h$ inside the unit ball $B^{n+1}_0$ of constant curvature $\lambda$, which is unique modulo congruence.

These free boundary CMC disks can be easily identified as spherical caps inside $B^{n+1}_\lambda$ that meet the boundary orthogonally; see [27, 30] for stability issues. Furthermore, it is not difficult to see that the family of spherical caps $\Sigma_{(h,\lambda_0)}$ determines a foliation of $B^{n+1}_{\lambda_0}$ for each fixed $\lambda_0$, see Figure 1. In the case $\lambda_0 = 0$, this can be easily seen as a consequence of Proposition 6.2, since $\psi(x) = \frac{1}{2}(\|x\|^2 + 1)$ satisfies

$$\begin{cases} \Delta \psi = 1 & \text{in } B^{n+1}_0, \\ \frac{\partial \psi}{\partial n} = \psi & \text{on } \partial B^{n+1}_0, \end{cases}$$

and is hence a nonvanishing solution to (6.1) that belongs to $C^{2,\alpha}_0(D^n)$.

8. Free boundary Delaunay annuli in the unit ball

In this section, we apply our main result to prove that a well-studied free boundary minimal surface of the unit ball $B^3$ in Euclidean space, the critical catenoid, is

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6For simplicity, we only consider $\lambda < \pi^2$, so that the unit ball in $M^{n+1}_\lambda$ remains a manifold with boundary. Of course, higher values of $\lambda > \pi^2$ can be achieved by rescaling; considering instead of a unit ball, a ball of radius $\frac{\sqrt{\lambda}}{\pi}$.
a member of a 2-parameter family of free boundary CMC surfaces inside the unit ball in a space form. We then identify these surfaces in terms of Delaunay surfaces.

8.1. Critical catenoid. Recall that a catenoid is a minimal surface in $\mathbb{R}^3$ obtained by rotating a catenary curve

$$x = c \cosh \frac{z}{c}, \quad c > 0,$$

about its directrix $x = 0$. Besides affine planes, catenoids are the only minimal surfaces of revolution in $\mathbb{R}^3$. A simple argument shows that the portion of the catenoid generated by (8.1) with $|z| \leq z_c$, where $z_c$ is the positive solution to

$$z_c = c \coth \frac{z_c}{c},$$

is a normal minimal surface inside the ball of radius $r_c = \sqrt{z_c^2 + c^2 \cosh^2(z_c/c)}$ centered at the origin, see Figure 2. In particular, choosing $c > 0$ such that $r_c = 1$, the portion of the corresponding catenoid contained in the unit ball $B^3$ is a free boundary minimal surface. This surface is called the critical catenoid, and has been studied in great detail by Fraser and Schoen [10, 11, 12], in connection to the first Steklov eigenvalue.

As explained in [10, Lemma 2.2], $x: \Sigma \hookrightarrow B^{n+1}$ is a free boundary minimal surface in the unit ball $B^{n+1} \subset \mathbb{R}^{n+1}$ if and only if its coordinate functions $x_i: \Sigma \to \mathbb{R}$ are eigenfunctions with eigenvalue 1 of the Dirichlet-to-Neumann operator

$$L: C^\infty(\partial \Sigma) \to C^\infty(\partial \Sigma), \quad L(u) := \langle \nabla \hat{u}, \vec{n}_{\partial B^{n+1}} \rangle,$$

where $\hat{u} \in C^\infty(\Sigma)$ denotes the harmonic extension of $u$, i.e., the solution of

$$\begin{cases}
\Delta \hat{u} = 0 & \text{in } \Sigma, \\
\hat{u} = u & \text{on } \partial \Sigma.
\end{cases}$$

Indeed, the conditions $\Delta x_i = 0$ are equivalent to $\Sigma$ being minimal, and $L(x_i) = x_i$ are equivalent to $\Sigma$ being normal in $B^{n+1}$, cf. (7.1). Remarkably, the critical catenoid

Figure 1. Free boundary CMC spherical caps foliating the unit ball.
Figure 2. The catenary (8.1) meets the circle of radius $r_c$ orthogonally.

catenoid in $B^3$ is the unique free boundary minimal annulus in the unit ball, up to isometries, whose coordinate functions are first eigenfunctions of $L$ [12, Thm 1.2].

8.2. Equivariant nondegeneracy. The key step to apply Theorem 5.1 to deform the critical catenoid is to verify its equivariantly nondegeneracy (see Definition 2.2).

**Lemma 8.1.** Let $\Sigma$ be a CMC surface of revolution normal in $B^3$. If $\Sigma$ is symmetric with respect to a plane orthogonal to the rotation axis, then its nullity is either 2 or 3. Furthermore, the critical catenoid has nullity 2 and is equivariantly nondegenerate.

*Proof.* We may assume that the rotation axis is the line $x = y = 0$, and that the orthogonal plane is the coordinate plane $z = 0$. The space of Killing-Jacobi fields along $\Sigma$ is two-dimensional, and spanned by the functions

$$f_i := \langle E_i \times X, \vec{n}_\Sigma \rangle, \quad i = 1, 2,$$

where $E_1$ (resp., $E_2$) denotes the constant field in the direction of the $x$-axis (resp., of the $y$-axis), and $X$ the position vector. In other words, $f_1$ and $f_2$ are infinitesimal rotations around the $x$-axis and $y$-axis, respectively. Note that each of these Killing-Jacobi fields:

1. has exactly two nodal domains (see Figure 3);
2. is not rotationally invariant.

Let $[- \frac{1}{2} L_\gamma, \frac{1}{2} L_\gamma] \ni s \mapsto \gamma(s) = (x(s), z(s))$ be the unit speed parametrization of the generatrix of $\Sigma$ in the $xz$-plane. By the symmetry assumptions, $x > 0$ is an even function, and $z$ is odd. Parametric equations for $\Sigma$ are hence given by:

$$(x(s) \cos \theta, x(s) \sin \theta, z(s)), \quad (s, \theta) \in [- \frac{1}{2} L_\gamma, \frac{1}{2} L_\gamma] \times [0, 2\pi).$$

A unit normal $\vec{n}_\Sigma: \Sigma \to S^2$ is given by:

$$\vec{n}_\Sigma(s, \theta) = (\dot{x}(s) \cos \theta, \dot{x}(s) \sin \theta, -\dot{z}(s)).$$

The Laplacian $\Delta_\Sigma$ of the pull-back metric is easily seen to be the operator

$$\Delta_\Sigma = -1 \frac{\partial}{\partial s} \left( x \frac{\partial}{\partial s} \right) - \frac{1}{x^2} \frac{\partial^2}{\partial \theta^2},$$
Figure 3. Two nodal domains of the Jacobi fields \(f_1\) and \(f_2\).

hence the Jacobi operator (2.2) of \(\Sigma\) is given by:

\[
J_\Sigma = -\frac{1}{x} \frac{\partial}{\partial s} \left( x \frac{\partial}{\partial s} \right) - \frac{1}{x^2} \frac{\partial^2}{\partial \theta^2} - \|\Pi^\Sigma\|^2.
\]

By rotational symmetry, \(\|\Pi^\Sigma\|^2\) depends only on \(s\), and can be explicitly computed as the sum of the squares of the principal curvatures of \(\Sigma\),

\[
\|\Pi^\Sigma\|^2 = \|\gamma\|^2 + \frac{\dot{z}^2}{x^2} = (\dot{x}\ddot{z} - \ddot{x}\dot{z})^2 + \frac{\dot{z}^2}{x^2}.
\]

The linearized free boundary conditions (2.4) for a Jacobi field \(\psi\) read:

\[
\frac{\partial \psi}{\partial s} \left( -\frac{1}{2} L_\gamma, \theta \right) + \psi \left( -\frac{1}{2} L_\gamma, \theta \right) = 0,
\]

\[
\frac{\partial \psi}{\partial s} \left( \frac{1}{2} L_\gamma, \theta \right) - \psi \left( \frac{1}{2} L_\gamma, \theta \right) = 0.
\]

Separation of variables \(\psi = S(s) \Theta(\theta)\) for the above problem gives the following pair of linear ODEs on \(S\) and \(\Theta\):

\[
-(xS')' + \left( \frac{n^2}{x} - x\|\Pi^\Sigma\|^2 \right) S = 0
\]

\[
\Theta'' + \kappa \Theta = 0
\]

and boundary conditions

\[
S' \left( -\frac{1}{2} L_\gamma \right) + S \left( -\frac{1}{2} L_\gamma \right) = 0,
\]

\[
S' \left( \frac{1}{2} L_\gamma \right) - S \left( \frac{1}{2} L_\gamma \right) = 0.
\]

In (8.8) and (8.9), \(\kappa\) is an arbitrary real constant. Equation (8.9) admits periodic solutions only when \(\kappa = n^2\), for some nonnegative integer \(n\). Hence, a basis for \(\ker J_\Sigma\) is given by functions of the form \(S_n(s) \cdot \sin(n\theta)\) and \(S_n(s) \cdot \cos(n\theta)\), where \(n\) is some nonnegative integer for which there exists a nontrivial solution \(S_n\) of the boundary value problem

\[
-(xS')' + \left( \frac{n^2}{x} - x\|\Pi^\Sigma\|^2 \right) S = 0,
\]

with boundary conditions (8.10).

Using the properties (1) and (2) of the Killing-Jacobi fields \(f_i\), it is easy to see that \(f_1\) and \(f_2\) correspond respectively to \(S_1 \sin \theta\) and \(S_1 \cos \theta\), where \(S_1\) is the solution of (8.11) with \(n = 1\), satisfying (8.10). Since \(f_1\) and \(f_2\) have exactly two nodal domains, it follows that \(S_1\) has no zero in \([-\frac{1}{2}L_\gamma, \frac{1}{2}L_\gamma]\).
Consider (8.11) as a Sturm-Liouville equation
\[(8.12)\quad -(xS')' - x\|\mathbf{II}\|^2S = \lambda \cdot \frac{1}{2}S,\]
with \(\lambda = -n^2, n \in \mathbb{N}\). When \(n = 1\), the solution of (8.12) satisfying (8.10) is \(S_0\). Since it has no zero in \([-\frac{1}{2}L_{\gamma}, \frac{1}{2}L_{\gamma}]\), it follows from Sturm-Liouville theory that the first eigenvalue of (8.12) must be \(\lambda_1 = -1\). In particular, there is no nontrivial solution of (8.12) satisfying (8.10) when \(\lambda = -n^2, n > 1\). Thus, \(J_2\) is spanned by \(f_1, f_2\) and at most one more rotationally symmetric Jacobi field satisfying (8.10), corresponding to a solution of (8.12) when \(\lambda = 0\). This shows that the nullity of \(\Sigma\) is either 2 or 3.

Let us show that there cannot be a solution of (8.12) with \(\lambda = 0\) satisfying (8.10) when \(\Sigma\) is the critical catenoid. We claim that any such solution \(S_0\) must be either an even or an odd function of \(s\). Note that \(\tilde{S}_0(s) := S_0(-s)\) also solves (8.12) and (8.10). As the space of solutions has dimension \(\leq 1\), we have \(\tilde{S}_0 = \alpha S_0\), for some \(\alpha \in \mathbb{R}\). Iteration gives \(\alpha^2 = 1\), hence \(\alpha = \pm 1\), proving the claim. Furthermore, disregarding boundary conditions, (8.12) admits two linearly independent solutions; one even and one odd. The odd solution is the rotationally symmetric Jacobi field \(\nu_3\) along \(\Sigma\), given by the normal component of the Killing field \(E_3\) of translations parallel to the \(z\)-axis. The even solution is given by the support function \(q_\Sigma : \Sigma \to \mathbb{R}\), \(q_\Sigma = (X, \tilde{n}_\Sigma)\), where \(X\) is the position vector. From (8.6), we have that
\[(8.13)\quad \nu_3 = \dot{x} \quad \text{and} \quad q_\Sigma = x\dot{z} - z\dot{x}.\]
Note that \(q_\Sigma\) is a Jacobi field only along minimal surfaces,\(^7\) and it vanishes along \(\partial\Sigma\), since \(\Sigma\) is normal. Thus, the space of odd solutions of (8.12) has dimension 1, as well as the space of even solutions of (8.12). This implies that \(S_0\) must be a multiple of either \(\nu_3\) or \(q_\Sigma\), according to its parity. A direct computation shows that the unit speed parametrization \(\gamma(s) = (x(s), z(s))\) of the catenary (8.1) is
\[(8.14)\quad x(s) = c \sqrt{1 + \frac{s^2}{c^2}}, \quad z(s) = c \log \left(\frac{s}{c} + \sqrt{1 + \frac{s^2}{c^2}}\right),\]
and \(L_{\gamma} = 2c\sinh(z_c/c)\). It is easy to verify that neither of (8.13) satisfy the boundary conditions (8.10). Thus, there is no nontrivial solution of (8.12) with \(\lambda = 0\) satisfying (8.10), hence the critical catenoid is equivariantly nondegenerate. \(\square\)

8.3. Delaunay annuli. Since the critical catenoid is equivariantly nondegenerate, it can be deformed through other CMC annuli (which we later identify as Delaunay annuli) varying both the values of its mean curvature and of the ambient curvature.

**Proposition 8.2.** The critical catenoid is the member \(\Sigma_{(0,0)}\) of a 2-parameter family \(\Sigma_{(h,\lambda)}\) of free boundary annuli with constant mean curvature \(h\) inside the unit ball \(B^3\) of constant curvature \(\lambda\), which is unique modulo congruence.

**Proof.** From Lemma 8.1, the critical catenoid \(\Sigma_0 \subset B^3\) is equivariantly nondegenerate and each Killing-Jacobi field (8.5) fixes the origin. By Lemma 7.1, every such Killing field extends to a smooth 1-parameter family of Killing fields of \(B^3\). Thus, the desired conclusion follows from Theorem 5.1. \(\square\)

\(^7\)If \(X : \Sigma \to \mathbb{R}^3\) is a surface in \(\mathbb{R}^3\) with constant mean curvature \(H_\Sigma\) and support function \(q_\Sigma\), then \(J_2(q_\Sigma) = -2H_\Sigma\). In fact, \(q_\Sigma\) can be written as \(\frac{d}{dt}|_{t=1}(X_t, \tilde{n}_\Sigma)\), where \(X_t = tX\) is a CMC variation of \(X\) with mean curvature \(H(X_t) = \frac{1}{t}H_\Sigma\).
Constant mean curvature surfaces of revolution in $\mathbb{R}^3$ are commonly known as Delaunay surfaces, as they were classified by Delaunay [7] in 1841. Their genera-trix was ingeniously observed to be the roulette $\gamma$ of some conic section $\beta$, which is the curve traced by one of the foci of $\beta$ as it rolls without slipping along a line $\ell$. Parametrizing the line and the conic section as $\ell : \mathbb{R} \to \mathbb{C}$ and $\beta : \mathbb{R} \to \mathbb{C}$ respectively, with $\ell(0) = \beta(0)$, $\ell'(0) = \beta'(0)$, and $|\ell'(s)| = |\beta'(s)| = 1$, the parametrization $\gamma : \mathbb{R} \to \mathbb{C}$ for the corresponding roulette is easily derived to be

\begin{equation}
\gamma(s) = \ell(s) - \frac{\ell'(s)}{\beta'(s)}(\beta(s) - \gamma(0)), \quad s \in \mathbb{R},
\end{equation}

where the starting point $\gamma(0) \in \mathbb{C}$ is to be chosen as one of the foci of $\beta$.

Conic sections $\beta$ depend smoothly on two positive parameters, called eccentricity $e$ and focal parameter $p$. The conic is a circle, ellipse, parabola or hyperbola respectively when $e = 0$, $e \in (0, 1)$, $e = 1$ or $e \in (1, +\infty)$, while $p$ is the distance between one of the foci and the directrix, and serves as a rescaling parameter. In polar coordinates, a conic with eccentricity $e > 0$, focal parameter $p > 0$ and one of the foci at the origin is given by

\begin{equation}
r(\theta) = \frac{e p}{1 + e \cos \theta}.
\end{equation}

Reparametrizing $\beta(\theta) = r(\theta) \cos \theta + i r(\theta) \sin \theta$ by arc length, one obtains the conic $\beta : \mathbb{R} \to \mathbb{C}$ and hence the corresponding roulette $\gamma$ via (8.15), using $\gamma(0) = 0$ and $\ell(s) = \frac{ep}{1+e} + is$, see Figure 5. Since $\beta$ and $\ell$ depend smoothly on $p$ and $e$, the roulette $\gamma$ given by (8.15) and hence the corresponding Delaunay surface $D$ in $\mathbb{R}^3$ also depends smoothly on these 2 parameters. Thus, Delaunay surfaces are members of a smooth 2-parameter family $D_{(e,p)}$ of CMC surfaces of revolution, obtained by revolving the roulette $\gamma$ of a conic with eccentricity $e$ and focal parameter $p$ around

![Figure 4](image-url)
The mean curvature of $D(e,p)$ can be computed to be
\begin{equation}
H(e,p) = \frac{|e^2 - 1|}{ep}.
\end{equation}

The surface $D(e,p)$ is called an unduloid, catenoid or nodoid according to the cases $0 < e < 1$, $e = 1$ and $e > 1$ respectively, that is, depending on the originating conic $\beta$ being an ellipse, parabola or hyperbola, see Figure 6. The parameter $p > 0$ corresponds to ambient homotheties, as the rescaled surface $\alpha D(e,p)$ is congruent to $D(e,\alpha p)$. Together with round cylinders and round spheres, which correspond to limits $e = 0$ and $e = +\infty$, these are all the CMC surfaces of revolution in $\mathbb{R}^3$.

Up to ambient isometries, we may assume that the rotation axis of the Delaunay surface $D(e,p)$ is the $z$-axis, and that the roulette $\gamma$ is contained in the $xz$-plane. In this case, the portion of $D(e,p)$ contained in the ball $B^3(\rho) \subset \mathbb{R}^3$ of radius $\rho$ centered at the origin is a normal surface in $B^3(\rho)$ if and only if the tangent line to $\gamma$ at the point $\gamma(s_*)$, for which $|\gamma(s_*)| = \rho$, passes through the origin (cf. Figure 2). As $\gamma$ depends smoothly on the eccentricity $e$ and focal parameter $p$ of the corresponding rolling conic, the Implicit Function Theorem can be used to show that there exists a smooth function $\rho(e,p)$, linear in $p$, such that $D(e,p) \cap B^3(\rho(e,p))$ is a normal surface in $B^3(\rho(e,p))$. Thus, Delaunay surfaces provide a smooth 1-parameter family
\begin{equation}
\mathcal{A}_e := D(e,\rho(e,1)) \cap B^3(1)
\end{equation}
of free boundary annuli in the unit ball, with constant mean curvature $h(e) := H(e, \rho(e,1))$, see also [28]. From the above discussion, the surface $\mathcal{A}_1$ is the critical

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8 An explicit parametrization of $D(e,p)$ involves elliptic integrals (needed to parametrize $\beta$ by arc length), and can be found, e.g., in [3, 7, 18].
Figure 6. Compact portions of an unduloid, a catenoid and a nodoid in $\mathbb{R}^3$, and the corresponding half sections.

catenoid. By the uniqueness modulo congruence in Proposition 8.2, it follows that $\Sigma_{(h(\epsilon),0)}$ must be congruent to the Delaunay annulus $A_{\epsilon}$.

Delaunay surfaces also exist in the other space forms $S^3$ and $H^3$, and can be described as the rotationally symmetric CMC surfaces in such spaces, see [13, 15]. Just as in the Euclidean case, these Delaunay surfaces induce free boundary CMC surfaces in any unit ball with constant curvature, see [30]. By a uniqueness modulo congruence argument similar to the above, the surfaces in the family $\Sigma_{(h,\lambda)}$ from Proposition 8.2 can be identified as Delaunay annuli in $H^3(1/\sqrt{-\lambda})$ or $S^3(1/\sqrt{\lambda})$, according to $\lambda < 0$ or $\lambda > 0$.

9. Equivariantly nondegenerate interfaces in phase transitions

In this section, we generalize results of Pacard and Ritoré [25] on realizing a non-degenerate CMC hypersurface as the interface of a phase transition. Let $(M,g)$ be a $(n+1)$-dimensional compact Riemannian manifold with boundary, and $W: \mathbb{R} \to \mathbb{R}$ be a positive function away from $\pm 1$ such that $W(\pm 1) = 0$ and $W''(\pm 1) > 0$. For $\varepsilon > 0$, a function $u \in H^1(M)$ is a critical point of the energy

$$E_{\varepsilon}(u) = \int_M (|\nabla u|_g^2 + W(u)) \, \text{vol}_g,$$

if and only if it satisfies the Allen-Cahn equation

$$\begin{cases}
-\varepsilon^2 \Delta_g u + W'(u) = 0 & \text{in } M, \\
g(\nabla u, \bar{n}_{\partial M}) = 0 & \text{on } \partial M,
\end{cases}$$

9The results in this section trivially extend to the case without boundary, for which the free boundary condition is vacuous.
where \( \vec{n}_{\partial M} \) denotes the unit vector field normal to \( \partial M \). It is well-known that, as \( \varepsilon \searrow 0 \), the nodal sets of solutions to (9.2) converge to minimal hypersurfaces, called interfaces. An interesting problem raised in [25] regards the converse statement:

**Problem 9.1.** Can a given free boundary minimal hypersurface \( \Sigma \subset M \) be realized as an interface; that is, the limit as \( \varepsilon \searrow 0 \) of nodal sets of solutions \( u_{\varepsilon} \) to (9.2)?

Pacard and Ritoré [25, Thm 4.1] obtained an affirmative answer to the above problem under the assumption that \( \Sigma \) is nondegenerate. Furthermore, a few degenerate examples could be handled by restricting to spaces of functions equivariant with respect to certain finite group actions, see [25, p. 368-371]. Our Theorem 5.1 allows to extend this affirmative answer to the equivariantly nondegenerate case:

**Proposition 9.2.** Let \( \Sigma \subset M \) be an equivariantly nondegenerate free boundary minimal hypersurface. Then, for all \( \varepsilon > 0 \) sufficiently small, there exist solutions \( u_{\varepsilon} \) to (9.2) whose nodal sets converge to \( \Sigma \).

**Proof.** The crucial step in which nondegeneracy of \( \Sigma \) is required in the proof of [25, Thm 4.1] is to use the Implicit Function Theorem in [25, Lemma 12.1] to find a family of hypersurfaces, containing \( \Sigma \), with constant mean curvature near zero. Provided \( \Sigma \) is equivariantly nondegenerate, this is guaranteed by Theorem 5.1. \( \square \)

Another situation considered by Pacard and Ritoré [25] is to introduce the volume constraint \( \int_M u \operatorname{vol}_g = c_0 \operatorname{Vol}(M) \) for the energy functional (9.1). In this case, critical points are solutions to

\[
\begin{cases}
-\varepsilon^2 \Delta_g u + W'(u) = \varepsilon \lambda & \text{in } M, \\
g(\nabla u, \vec{n}_{\partial M}) = 0 & \text{on } \partial M,
\end{cases}
\]

where \( \varepsilon \lambda \) corresponds to the Lagrange multiplier of the constraint. The analogue of Problem 9.1 is to realize a given free boundary CMC hypersurface as limit of nodal sets of solutions to (9.3). This is achieved by Pacard and Ritoré [25, Thm 4.2] under the assumption that \( \Sigma \) is volume-nondegenerate, that is, there are no nontrivial solutions \( (\psi, c) \in C^2_0(\Sigma) \times \mathbb{R} \) to \( J_\Sigma(\psi) = c \), also satisfying \( \int_\Sigma \psi \operatorname{vol}_g = 0 \).\(^{10}\)

Analogously to Proposition 9.2, this result can also be generalized by relaxing the assumption that the free boundary CMC hypersurface \( \Sigma \) is volume-nondegenerate to equivariantly nondegenerate.

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\(^{10}\)Note that if \( \Sigma \) is volume-nondegenerate, then the Implicit Function Theorem allows to find a hypersurface \( \Sigma' \) close to \( \Sigma \) whose mean curvature \( H_{\Sigma'} \) can be, up to a constant, prescribed close to \( H_\Sigma \) and the enclosed volume of \( \Sigma' \) is prescribed close to the enclosed volume of \( \Sigma \).
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