On Fabry’s Quotient Theorem

Lev Buhovsky

Abstract. We present a short proof of the Fabry quotient theorem, which states that for a complex power series with unit radius of convergence, if the quotient of its consecutive coefficients tends to $s$, then the point $z = s$ is a singular point of the series. This proof only uses material from undergraduate university studies.

1. INTRODUCTION. Fabry’s celebrated theorems detect singular points of power series on the boundary of the disk of convergence and provide large classes of Taylor series that cannot be analytically continued through any arc of the boundary circle. The reader will find these theorems in the Bieberbach and Dienes treatises [4, Chapter 2], [6, Sections 93–94]. For more recent results and references, see, for instance, papers by Arakelian and Martirosyan [2, 3], by Arakelian, Luh, and Müller [1], and by Eremenko [9, 10].

Fabry’s theorems are known for their formidable formulations and complicated proofs. The original proofs of Fabry’s theorems were quite ingenious and long but used only basic properties of Taylor series [4, Section 2.1]. Faber and then Pólya developed another approach to general Fabry’s theorems which is based on the interpolation of the coefficients of the Taylor series by an entire function and on the connection between the growth of entire functions and the distribution of their zeros. This approach is well explained in the above-mentioned books by Dienes and Bieberbach.

Probably the most well-known consequence of the general Fabry theorems is Fabry’s gap theorem, which has numerous connections with other areas of analysis and has attracted the attention of many prominent mathematicians, see, for instance, Bieberbach [4, Chapter 2], Ehrenpreis [7, Chapter XIII], Havin and Jöricke [11, Sections II.4.6–II.4.10], and Paley and Wiener [13, Section 32].

Another remarkable consequence of general Fabry’s theorems is Fabry’s quotient theorem, whose elegant formulation deserves to be included in courses of basic complex analysis.

Theorem 1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a complex power series with unit radius of convergence. Assume that $\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = s$. Then $z = s$ is a singular point of $f$.

My friend and colleague Misha Sodin recently mentioned to me that he is not aware of a proof of this theorem that could be explained in a first course in complex analysis or included in textbooks. In this short article, we provide such a proof, which is based on an idea from harmonic analysis. A somewhat similar approach was used by Wiener for proving a version of Fabry’s gap theorem that was weaker than the original one: see [13, Section 32], [15], as well as in other instances; see also Erdős et al. [8, Lemma], Montgomery [12, Chapter 7, Section 3] (and the Notes to this Chapter), and Shapiro [14].

It could be that our proof of Theorem 1 (or another short and elementary proof) might be known to experts, or even published, although we did not find any evidence.

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2. PROOFS.

**Definition 2.** Let \( F(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \) be a formal Fourier series with complex coefficients, and let \( N \geq 2 \) be an integer. We say that \( F \) is \( N \)-good if for every \( n \in \mathbb{Z} \) and \( 0 \leq k \leq N - 1 \) we have \( \Re((a_n a_{n+k})^*) \geq 0 \).

**Remark 3.** An equivalent way to say that a Fourier series \( F(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \) is \( N \)-good, is that for any \( n \in \mathbb{Z} \), the complex coefficients \( a_n, a_{n+1}, \ldots, a_{n+N-1} \) lie inside an angle of size \( \frac{\pi}{2} \) centered at the origin (i.e., a closed quadrant rotated around the origin).

For a trigonometric polynomial \( P(\theta) = \sum_{k=-M}^{M} c_k e^{ik\theta} \), we will say that \( P \) is symmetric with nonnegative coefficients if \( c_k = c_{-k} \geq 0 \) for each \( 0 \leq k \leq M \).

**Observation 4.** Let \( F(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \) be an \( N \)-good \( L^2[-\pi, \pi] \) Fourier series, and let \( P(\theta) = \sum_{k=-N+1}^{N-1} c_k e^{ik\theta} \) be symmetric with nonnegative coefficients. Then
\[
\int_{-\pi}^{\pi} |F(\theta)|^2 P(\theta) \, d\theta = 2\pi \sum_{n=-\infty}^{\infty} \sum_{k=-N+1}^{N-1} a_n a_{n+k} c_k
\]
\[
= 2\pi c_0 \sum_{n=-\infty}^{\infty} |a_n|^2 + 2\pi \sum_{n=-\infty}^{\infty} \sum_{k=1}^{N-1} c_k (a_n a_{n+k} + a_{n+k} a_n)
\]
\[
= 2\pi c_0 \sum_{n=-\infty}^{\infty} |a_n|^2 + 4\pi \sum_{n=-\infty}^{\infty} \sum_{k=1}^{N-1} c_k \Re(a_n a_{n+k})
\]
\[
\geq 2\pi c_0 \sum_{n=-\infty}^{\infty} |a_n|^2 \geq 0.
\]

**Lemma 5.** For every integer \( N \geq 2 \) there exists a trigonometric polynomial \( P(\theta) = \sum_{k=-N+1}^{N-1} c_k e^{ik\theta} \), symmetric with nonnegative coefficients, such that we have \( P(\theta) < 0 \) for \( \theta \in [-\pi, \pi] \setminus (-\frac{4\pi}{N}, \frac{4\pi}{N}) \).

We believe that Lemma 5 (or its sharper versions) is well known to experts. We postpone our proof of the lemma to the end of the section. The combination of Observation 4 and Lemma 5 implies the following corollary:

**Corollary 6.** Let \( F(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \) be an \( N \)-good \( L^2[-\pi, \pi] \) Fourier series. Then
\[
\int_{-\pi}^{\pi} |F(\theta)|^2 \, d\theta \leq C \int_{-\frac{4\pi}{N}}^{\frac{4\pi}{N}} |F(\theta)|^2 \, d\theta,
\]
where \( C = C(N) \).

Indeed, if \( P(\theta) \) is as in Lemma 5, then by continuity of \( P(\theta) \) we conclude that there exist positive constants \( c_1, C_1 > 0 \) that depend only on \( N \), such that we have \( P(\theta) \leq -c_1 \) for \( \theta \in [-\pi, \pi] \setminus (-\frac{4\pi}{N}, \frac{4\pi}{N}) \), and \( P(\theta) \leq C_1 \) for \( \theta \in [-\frac{4\pi}{N}, \frac{4\pi}{N}] \). Then Observation 4 yields
\[ \int_{-\pi}^{\pi} |F(\theta)|^2 d\theta = \left( \int_{\pi}^{\pi} + \int_{[\pi,\pi] \setminus (-\pi, \pi)} \right) |F(\theta)|^2 d\theta \]
\[ \leq \int_{-\pi}^{\pi} |F(\theta)|^2 d\theta - \frac{1}{c_1} \int_{[\pi,\pi] \setminus (-\pi, \pi)} |F(\theta)|^2 P(\theta) d\theta \]
\[ \leq \int_{-\pi}^{\pi} |F(\theta)|^2 d\theta + \frac{1}{c_1} \int_{-\pi}^{\pi} |F(\theta)|^2 P(\theta) d\theta \leq \left( 1 + \frac{C_1}{c_1} \right) \int_{-\pi}^{\pi} |F(\theta)|^2 d\theta . \]

**Remark 7.** An inequality analogous to (1) was obtained by N. Wiener [15] for Fourier series having “uniform gaps,” with a sharp growth rate of the corresponding coefficient \( C \) (see Theorem I therein). In addition, Wiener showed an inequality analogous to (1) for Fourier series with nonnegative coefficients (see Theorem 2.1 in [14]), again with a sharp growth rate of the corresponding coefficient. It would also be interesting to find the sharp growth rate of \( C(N) \) in terms of \( N \) in Corollary 6. Our proof gives \( C(N) = O(N^2) \).

Our proof of Theorem 1 uses the following simple lemma:

**Lemma 8.** If \( v : \mathbb{R} \to \mathbb{C} \) is a \( 2\pi \)-periodic continuously differentiable function having zero mean on \([-\pi, \pi]\) (i.e., \( \int_{-\pi}^{\pi} v(t) dt = 0 \)), then
\[
\max_{t \in [-\pi, \pi]} |v(t)| \leq \sqrt{\frac{\pi}{6}} \int_{-\pi}^{\pi} |v'(t)|^2 dt.
\]

**Proof.** Let us show that \( |v(s)| \leq \sqrt{\frac{\pi}{6}} \int_{-\pi}^{\pi} |v'(t)|^2 dt \), for any \( s \in [-\pi, \pi] \). It is enough to consider only the case \( s = \pi \) since for any other \( s \in [-\pi, \pi] \) one can reduce to this case by applying it to the function \( v_s(t) = v(t + s - \pi) \).

Integration by parts gives us
\[
\int_{-\pi}^{\pi} tv'(t) dt = tv(t)|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} v(t) dt = \pi v(\pi) + \pi v(-\pi) = 2\pi v(\pi).
\]

Then, by the Cauchy–Schwarz inequality, we get
\[
4\pi^2 |v(\pi)|^2 = \left| \int_{-\pi}^{\pi} tv'(t) dt \right|^2 \leq \left( \int_{-\pi}^{\pi} t^2 dt \right) \left( \int_{-\pi}^{\pi} |v'(t)|^2 dt \right) = \frac{2\pi^3}{3} \int_{-\pi}^{\pi} |v'(t)|^2 dt,
\]

and we conclude \( |v(\pi)| \leq \sqrt{\frac{\pi}{6}} \int_{-\pi}^{\pi} |v'(t)|^2 dt \). \( \blacksquare \)

We now proceed to the proof of Theorem 1.

**Proof of Theorem 1.** We will in fact prove a more general quantitative statement: If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is an \( N \)-good Taylor series with radius of convergence \( R = 1 \), then
f cannot be extended analytically through the arc \( C = \{ e^{i\theta} \mid \theta \in [-\frac{4\pi}{N}, \frac{4\pi}{N}] \} \), i.e., \( f \) is not analytic on \([|z| < 1] \cup C\).

To deduce the theorem from this statement, first notice that in the theorem one can without loss of generality assume that \( s = 1 \), by the change of variables \( w = z/s \). Then, the assumption of the theorem gives us \( \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = 1 \). Hence, for every \( k \in \mathbb{N} \), we get

\[
\mathfrak{R}(a_n a_{n+k}) = \mathfrak{R}\left( |a_{n+k}|^2 \frac{a_n}{a_{n+k}} \right) \geq 0
\]

when \( n \) is large enough. As a result, we get that for every integer \( N \geq 2 \) there exists \( m \in \mathbb{N} \) such that \( f_m(z) := \sum_{n=m}^{\infty} a_n z^n \) is \( N \)-good and has radius of convergence \( R = 1 \). Therefore, assuming the statement, we conclude that \( f_m \), and hence \( f \) as well, cannot be extended analytically through the arc \( C = \{ e^{i\theta} \mid \theta \in [-\frac{4\pi}{N}, \frac{4\pi}{N}] \} \). Since this holds for every integer \( N \geq 2 \), the theorem follows.

Let us now pass to our proof of the above quantitative statement. The proof is by contradiction, so assume that we have an \( N \)-good Taylor series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) with radius of convergence \( R = 1 \), such that \( f \) is holomorphic on \([|z| < 1] \cup C\), where \( C = \{ e^{i\theta} \mid \theta \in [-\frac{4\pi}{N}, \frac{4\pi}{N}] \} \).

Consider the \( 2\pi \)-periodic complex function \( F(w) := f(e^{iw}) \). Since \( f \) is analytic in a neighborhood of \([|z| < 1] \cup C\) (where \( C = \{ e^{i\theta} \mid \theta \in [-\frac{4\pi}{N}, \frac{4\pi}{N}] \} \) as before), we conclude that \( F \) is analytic in some neighborhood \( \Omega \) of \( \{|w| > 0\} \cup [-\frac{4\pi}{N}, \frac{4\pi}{N}] \). For \( w \in [-\frac{4\pi}{N}, \frac{4\pi}{N}] \times [0, \infty) \subset \Omega \), the distance from \( w \) to the boundary of \( \Omega \) is bounded from below by a positive constant. Therefore, by the Cauchy estimates, we have \( |F(w)| \leq C_1 A^\ell \ell! \) for every \( w \in [-\frac{4\pi}{N}, \frac{4\pi}{N}] \times [0, \infty) \) and \( \ell \geq 0 \). Now, for \( \tau > 0 \), define \( F_{\tau}(\theta) := F(\theta + i\tau) = f(e^{-\tau+i\theta}), \theta \in \mathbb{R} \). We conclude

\[
|F_{\tau}(\theta)| = |F(\theta + i\tau)| \leq C_1 A^\ell \ell!
\]

for \( \theta \in [-\frac{4\pi}{N}, \frac{4\pi}{N}] \) and \( \tau > 0 \).

For any given \( \tau > 0 \), the function \( F_{\tau} \) admits the Fourier series \( F_{\tau}(\theta) = f(e^{-\tau+i\theta}) = \sum_{n=0}^{\infty} a_n e^{-\tau n} e^{i n \theta} \), which is \( N \)-good. For any integer \( \ell \geq 1 \), we apply Corollary 6 to the Fourier series \( F_{\tau}(\theta) = i\ell \sum_{n=m}^{\infty} a_n e^{-\tau n} e^{i n \theta} \) (which is \( N \)-good as well). Hence \( F_{\tau}(\theta) \) satisfies inequality (1). By the estimate (2), we conclude

\[
\int_{-\pi}^{\pi} |F_{\tau}(\theta)|^2 d\theta \leq C \int_{-\frac{4\pi}{N}}^{\frac{4\pi}{N}} |F_{\tau}(\theta)|^2 d\theta \leq (C_2 A^\ell \ell!)^2
\]

for \( \tau > 0 \). Since \( F_{\tau}(\theta) \) has a zero mean on \([-\pi, \pi] \), by applying Lemma 8 with \( v = F_{\tau}(\theta) \), we get

\[
\max_{\theta \in [-\pi, \pi]} |F_{\tau}(\theta)| \leq \sqrt{\frac{\pi}{6}} \int_{-\pi}^{\pi} |F_{\tau}(\theta)|^2 d\theta \leq \sqrt{\frac{\pi}{6}} C_2 A^\ell \ell! (\ell + 1)!
\]

for every \( \tau > 0 \). In other words, we have an estimate

\[
|F(\theta)| \leq C_3 A^\ell (\ell + 1)!
\]

for \( \Im w > 0 \) and \( \ell \geq 1 \).
The estimate (3) implies that the Taylor series of $F$ at any point from the upper half-plane has radius of convergence at least $1/A$ (which is independent of the point). This provides the function $F$ an analytic continuation from the upper half-plane $\{\Im w > 0\}$ through its boundary $\partial\{\Im w > 0\} = \mathbb{R}$. Let us explain that. For any open disk $B$ of radius $1/A$ that is not contained inside the upper half-plane but whose center $a$ lies in the upper half-plane, the Taylor series of $F$ at $a$ defines an analytic function $F_B : B \to \mathbb{C}$ which by the Cauchy theorem agrees with $F$ on a small disk around $a$. Hence by the uniqueness theorem, $F_B$ agrees with $F$ on the whole $B \cap \{\Im w > 0\}$. Moreover, we claim that for any two such disks $B_1$ and $B_2$ that intersect, we have $F_{B_1} = F_{B_2}$ on $B_1 \cap B_2$. To see this, note that since $B_1 \cap B_2 \neq \emptyset$, and since the centers of both disks lie in the upper half-plane, it follows that $B_1 \cap B_2 \cap \{\Im w > 0\} \neq \emptyset$, and then since $F_{B_1} = F_{B_2}$ on $B_1 \cap B_2 \cap \{\Im w > 0\}$ (both functions agree there with $F$, as we have just checked), again by the uniqueness theorem we conclude that $F_{B_1}$ agrees with $F_{B_2}$ on $B_1 \cap B_2$. Therefore, the function $F$ together with all the functions $F_B$ agree pairwise on mutual intersections of their domains of definition, which provides $F$ an analytic continuation to the open set $\{\Im w > -1/A\}$ (which equals the union of the upper half-plane with all such disks $B$).

We have shown that the function $F$ admits an analytic continuation to a neighborhood of the closed upper half-plane $\{\Im w \geq 0\}$. But this means that $f$ admits an analytic continuation to a neighborhood of the closed disk $\{|z| \leq 1\}$, a contradiction. □

**Remark 9.** In the quantitative statement appearing at the beginning of the proof of Theorem 1, one can clearly weaken the assumption of $N$-goodness for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ to an “eventual $N$-goodness,” i.e., assuming that we have $\Im(a_n a_{n+k}) \geq 0$ for every $1 \leq k \leq N - 1$ and $a$ sufficiently large $n$. Moreover, the quantitative statement is not sharp, and we expect that under the assumption that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has unit radius of convergence and is $N$-good, one can conclude that $f$ cannot be extended analytically through the shorter arc $C' = \{e^{i\theta} \mid \theta \in [-\pi/2, \pi/2]\}$.

In fact, it is natural to consider the following more general situation. Let $a \in (0, \pi)$, and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a Taylor series with radius of convergence $R = 1$, such that for a sufficiently large $n$, the complex coefficients $a_n, a_{n+1}, \ldots, a_{n+N-1}$ lie inside an angle of size $a$ centered at the origin. Then we expect that $f$ cannot be extended analytically through the arc $\{e^{i\theta} \mid \theta \in [-a/N, a/N]\}$ (cf. Delange’s theorem [4, Theorem 2.3.3], [5]).

It remains to prove Lemma 5.

**Proof of Lemma 5.** First of all, we can without loss of generality assume that $N \geq 8$ (for $2 \leq N \leq 7$, just take $P(\theta) = \cos \theta = \frac{1}{2} e^{-i\theta} + \frac{1}{2} e^{i\theta}$). Consider the Fejér kernel

$$F_N(\theta) = \sum_{k=-N+1}^{N-1} \left(1 - \frac{|k|}{N}\right) e^{ik\theta} = \frac{1}{N} \left(\frac{\sin \frac{N\theta}{2}}{\sin \frac{\theta}{2}}\right)^2.$$

Define

$$G_N(\theta) := F_N(\theta + \frac{\pi}{2N}) + F_N(\theta - \frac{\pi}{2N}) = \sum_{k=-N+1}^{N-1} \left(2 - \frac{2|k|}{N}\right) \left(\cos \frac{k\pi}{2N}\right) e^{ik\theta}.$$
We have
\[ G_N(\theta) = \frac{1}{N} \left( \frac{\sin\left( \frac{N\theta}{2} - \frac{\pi}{4} \right)}{\sin\left( \frac{\theta}{2} - \frac{\pi}{4N} \right)} \right)^2 + \left( \frac{\sin\left( \frac{N\theta}{2} + \frac{\pi}{4} \right)}{\sin\left( \frac{\theta}{2} + \frac{\pi}{4N} \right)} \right)^2. \]

Hence
\[ A_N(\theta) := N^{-1} \left( \max \left( \left| \sin\left( \frac{\theta}{2} - \frac{\pi}{4N} \right) \right|, \left| \sin\left( \frac{\theta}{2} + \frac{\pi}{4N} \right) \right| \right) \right)^{-2} \leq G_N(\theta) \leq B_N(\theta) := N^{-1} \left( \min \left( \left| \sin\left( \frac{\theta}{2} - \frac{\pi}{4N} \right) \right|, \left| \sin\left( \frac{\theta}{2} + \frac{\pi}{4N} \right) \right| \right) \right)^{-2} \]

because \( \sin^2\left( \frac{N\theta}{2} - \frac{\pi}{4} \right) + \sin^2\left( \frac{N\theta}{2} + \frac{\pi}{4} \right) = 1. \)

Let us show that \( G_N(\theta) < G_{\left\lceil \frac{\pi}{N} \right\rceil}(\theta) \) for \( \theta \in [-\pi, \pi] \setminus \left( -\frac{4\pi}{N}, \frac{4\pi}{N} \right). \) For this, it is enough to show that \( B_N(\theta) < A_{\left\lceil \frac{\pi}{N} \right\rceil}(\theta) \) for \( \theta \in \left[ \frac{4\pi}{N}, \pi \right]. \) The latter inequality immediately follows from
\begin{align*}
C_N(\theta) &:= 2 \min \left( \left| \sin\left( \frac{\theta}{2} - \frac{\pi}{4N} \right) \right|, \left| \sin\left( \frac{\theta}{2} + \frac{\pi}{4N} \right) \right| \right) \\
&> D_N(\theta) := \max \left( \sin\left( \frac{\theta}{2} - \frac{\pi}{4\left\lceil \frac{N}{4} \right\rceil} \right), \sin\left( \frac{\theta}{2} + \frac{\pi}{4\left\lceil \frac{N}{4} \right\rceil} \right) \right). \tag{4}
\end{align*}

Let us show the inequality (4) for \( \theta \in \left[ \frac{4\pi}{N}, \pi \right]. \) Assume first that \( \theta \in \left[ \frac{\pi}{2}, \pi \right]. \) Then for \( N \geq 8 \) we have
\[ \frac{\pi}{6} < \frac{\pi}{4} - \frac{\pi}{32} \leq \frac{\theta}{2} - \frac{\pi}{4N} < \frac{\theta}{2} + \frac{\pi}{4N} \leq \frac{\pi}{2} + \frac{\pi}{32} < \frac{5\pi}{6}, \]
and consequently, \( C_N(\theta) > 1 \geq D_N(\theta). \) It now remains to verify the case \( \theta \in \left[ \frac{4\pi}{N}, \frac{\pi}{2} \right]. \)

In that case, again for \( N \geq 8 \) we get
\[ \theta - \frac{\pi}{2N} = \frac{\theta}{2} + \left( \frac{\theta}{2} - \frac{\pi}{2N} \right) \geq \frac{\theta}{2} + \frac{3\pi}{2N} \geq \frac{\theta}{2} + \frac{\pi}{4\left\lceil \frac{N}{4} \right\rceil} \]
and
\[ \frac{\theta}{2} - \frac{\pi}{4\left\lceil \frac{N}{4} \right\rceil} > \frac{2\pi}{N} - \frac{\pi}{N - 4} \geq 0; \]

hence we conclude
\[ 0 < \frac{\theta}{2} - \frac{\pi}{4\left\lceil \frac{N}{4} \right\rceil} < \frac{\theta}{2} - \frac{\pi}{4N} < \frac{\theta}{2} + \frac{\pi}{4N} < \frac{\theta}{2} + \frac{\pi}{4\left\lceil \frac{N}{4} \right\rceil} \leq \theta - \frac{\pi}{2N} < \frac{\pi}{2}. \]
We wish to emphasize the following parts of the latter chain of inequalities:

\[ 0 < \frac{\theta}{2} - \frac{\pi}{4N} < \frac{\theta}{2} + \frac{\pi}{4N} < \frac{\pi}{2}, \tag{5} \]

\[ 0 < \frac{\theta}{2} + \frac{\pi}{4 \left\lfloor \frac{N}{4} \right\rfloor} \leq \frac{\theta}{2} - \frac{\pi}{2N} < \frac{\pi}{2}, \tag{6} \]

and

\[ 0 < \frac{\theta}{2} - \frac{\pi}{4 \left\lfloor \frac{N}{4} \right\rfloor} < \frac{\theta}{2} + \frac{\pi}{4 \left\lfloor \frac{N}{4} \right\rfloor} < \frac{\pi}{2}. \tag{7} \]

We can now conclude

\[ C_N(\theta) = 2 \sin \left( \frac{\theta}{2} - \frac{\pi}{4N} \right) > 2 \sin \left( \frac{\theta}{2} - \frac{\pi}{4N} \right) \cos \left( \frac{\theta}{2} - \frac{\pi}{4N} \right) \]

\[ = \sin \left( \frac{\theta}{2} - \frac{\pi}{2N} \right) = \sin \left( \frac{\theta}{2} + \frac{\pi}{4 \left\lfloor \frac{N}{4} \right\rfloor} \right) = D_N(\theta), \]

where the first equality and the consequent inequality follow from (5), the remaining inequality follows from (6), and the last equality follows from (7).

Now put \( P(\theta) := G_N(\theta) - G_{\lfloor N/4 \rfloor}(\theta) \). Then \( P(\theta) \) is a trigonometric polynomial of degree \( N - 1 \). We have verified that \( P(\theta) < 0 \) for \( \theta \in [-\pi, \pi] \setminus (-4\pi N, 4\pi N) \). Clearly, \( P(\theta) \) is symmetric. It therefore remains to check that its coefficients are nonnegative. Denote \( M := \left\lfloor \frac{N}{4} \right\rfloor < N \), and let \(-N + 1 \leq k \leq N - 1\) be an integer. If \(|k| \leq M - 1\), then the \( k \)th coefficient of \( P(\theta) = G_N(\theta) - G_M(\theta) \) is equal to

\[ \left( 2 - \frac{2|k|}{N} \right) \cos \frac{k\pi}{2N} - \left( 2 - \frac{2|k|}{M} \right) \cos \frac{k\pi}{2M} \]

and hence is nonnegative (since \(|k| < M < N\)). But if \( M \leq |k| \leq N - 1 \), then the \( k \)th coefficient of \( P(\theta) \) is

\[ \left( 2 - \frac{2|k|}{N} \right) \cos \frac{k\pi}{2N} \]

and is consequently nonnegative as well. We have checked that the trigonometric polynomial \( P(\theta) \) has all the required properties (also note that \( P(\theta) \) has a zero mean on \([-\pi, \pi]\)).

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LEV BUHOVSKY is a professor at Tel Aviv University. He received his Ph.D. in Mathematics from Tel Aviv University, and later completed postdocs at MSRI and the University of Chicago. His research interests are symplectic geometry and analysis.
School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 6997801, Israel
levbuh@tauex.tau.ac.il

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