GRAPH mixing

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Abstract

In the era of gravitational wave (GW) detection from astrophysical sources by LIGO/VIRGO, it is of great importance to take the quantum gravity effect of graviton-photon (GRAPH) mixing in the cosmic magnetic field to the next level. In this work, we study such effect and derive for the first time perturbative solutions of the linearized equations of motions of the GRAPH mixing in an expanding universe. In our formalism we take into account all known standard dispersive and coherence breaking effects of photons such as the Faraday effect, the Cotton-Mouton effect and plasma effects in the cosmic magnetic field. Our formalism, applies to cosmic magnetic field either uniform or slowly varying non-homogeneous field of spacetime coordinates with arbitrary field direction. For binary systems of astrophysical sources of GWs with chirp masses $M_{\text{CH}}$ of few solar masses, GW present-day frequencies $\nu_0 \simeq 50 - 700$ Hz and present-day cosmic magnetic field amplitudes $\bar{B}_0 \simeq 10^{-10} - 10^{-6}$ G, the power of electromagnetic radiation generated in the GRAPH mixing at present is substantial and in the range $P_\gamma \simeq 10^6 - 10^{15}$ (erg/s). On the other hand, the associated power flux $F_\gamma$ is quite faint depending on the source distance with respect to the Earth. Since in the GRAPH mixing the velocities of photons and gravitons are preserved and are equal, this effect is the only one known to us, whose certainty of the contemporary arrival of GWs and electromagnetic radiation at the detector is guaranteed.

1 Introduction

The detections of several GW events by the LIGO/VIRGO collaborations [1], have finally confirmed a long-standing problem, that indeed do exist spacetime perturbations that propagate with the speed of light and that are not an artifact prediction of the theory of General Relativity. The detection of GWs followed after several decades of intensive theoretical studies and experimental efforts that took a great push forward starting from the first detection of a GW source, namely the PSR B1913+16 binary system of neutron stars [2]. The LIGO/VIRGO detections apart from being important in many aspects of physics shed a new light in favor of the graviton, namely the quantized particle of spin two of the gravitational field. The GW events detected by the LIGO/VIRGO collaborations, so far, have confirmed with good accuracy that GWs propagate in the vacuum with the speed of light and if the graviton is a massive particle, its mass should be smaller than $m_g < 1.2 \times 10^{-22}$ eV, see Refs. [1] for details.

One of the key assumptions about the nature of GWs is that they weakly interact with matter and fields while propagating from the source to the detector and consequently their velocities and amplitudes are assumed to remain unaltered. This assumption is justifiable in most situations because being the interaction strength of GWs with matter and fields very small, one usually does not expect any loss or transformation of GWs propagating though cosmological distances. Even though this assumption is quite realistic in most cases, there might be some exceptions in the case when GWs interact with spatially extended electromagnetic fields comparable with astrophysical and cosmological distances. Indeed, as the theory of General Relativity teaches us, every form of non-stationary stress energy-tensor on the
right-hand side of the Einstein field equations produces spacetime perturbations or simply GWs. So, in principle non-stationary interactions among electromagnetic fields would produce GWs.

While non-stationary interactions among electromagnetic fields produce GWs such as for example the interaction of a plane electromagnetic wave with a static magnetic field, it is also possible that the interaction of GWs with external electromagnetic fields would produce electromagnetic radiation out of GWs. Therefore, the overall outcome is that GWs and electromagnetic waves mix with each other in the presence of external electromagnetic fields and this effect propagates in space throughout the region where the external electromagnetic field is spatially located, see Ref. [3] for an intuitive explanation. Based on this fundamental prediction of the theory of General Relativity, the possibility to generate GWs in the laboratory from the interaction of electromagnetic radiation with external prescribed static magnetic fields was initially proposed in Ref. [4].

Through the decades the possibility of mixing of GWs with electromagnetic waves and vice versa in constant external magnetic field has been studied by several authors [5] for some specific magnetic field configuration, which in most cases it has been taken to be perpendicular to the propagation of the incident GW and/or electromagnetic wave. In those cases where the field was taken not to be perpendicular with respect to the incident field propagation, important dispersive and coherence breaking effects such as the Faraday effect and the Cotton-Mouton effect have not been taken into account. In these studies [5], have been employed classical and field theory approaches to the mixing problem and some possibilities for applying this effect in cosmological scenarios have been proposed in Ref. [6].

In order for the GRAPH mixing to work, it is necessary an external electromagnetic field and in cosmological situations it can be possible in the presence of large-scale cosmic magnetic fields (for general concepts on cosmic magnetic fields see Ref. [7]). Indeed, as it is well known, the presence of large-scale magnetic field in galaxies and galaxy clusters has been experimentally verified, while it is still unclear if such field is present in the intergalactic space. In galaxy clusters, the measurements of the rotation angle of the received light due to the Faraday effect confirm the presence of a large-scale magnetic field inside them, with a magnitude of the order of few \( \mu \)G. On the other hand, in the intergalactic space recent studies by the Planck collaboration [8] would suggest a weaker large scale cosmic magnetic field with upper limit field strength \( \tilde{B}_0 < 3 - 1380 \) nG. The limit of the order of 1380 nG is set from the Faraday effect of the CMB, while the limit of \( \tilde{B}_0 < 3 \) nG is set from the CMB temperature anisotropy. In addition, from the non-observation of gamma ray emission from the intergalactic medium due to the injection of high energy particles by blazars [9], is inferred a lower value on the strength of the intergalactic magnetic field of the order \( \tilde{B}_0 \geq 10^{-16} - 10^{-15} \) G.

The detection of GWs from astrophysical binary systems gives us a rather unique opportunity to probe the GRAPH mixing effect in intergalactic and galactic magnetic fields. Some important questions which we can ask at this stage are; if large-scale magnetic fields do exist in intergalactic space, which is the probability of transformation of GWs into electromagnetic radiation? Which is the energy per unit time and/or the energy density received at the Earth? Which is the polarization of the electromagnetic radiation received? In this work, we address these questions by applying the GRAPH mixing to astrophysical binary systems and make predictions on the energy power and energy power flux of the electromagnetic radiation generated in the GRAPH mixing. With respect to other works where the GRAPH mixing was applied in a constant magnetic field [5], and in the early universe where the density matrix equations of motions were solved numerically [6], in this work we find analytic solutions of the field equations of motion for a slowly varying non-homogeneous magnetic. In addition, with respect to other studies [5]-[6], we allow the direction of the external magnetic field to be arbitrary with respect to the GW direction of propagation and take into account the Faraday and Cotton-Mouton effects in the magnetic field.

This paper is organized as follows: in Sec. 2 we derive the linearized field equations of motion in spatially and temporally non-homogeneous magnetic field with the field inhomogeneity scale bigger than the GW wavelength. In Sec. 3 we discuss all standard dispersive and coherence breaking electromagnetic wave effects in a magnetized plasma by writing explicitly the elements of the photon polarization tensor in
In order to describe the GRAPH mixing it is necessary first to start with the total action of the GRAPH mixing. In Sec. 2 we calculate the power and the power flux of the electromagnetic radiation generated in the GRAPH mixing. In Sec. 3 we find some analytic expressions of the integrals which do appear in the Stokes parameters. In Sec. 4 we find analytic solutions of the linearized equations of motion by using perturbation theory. In Sec. 5 we find the Stokes parameters of the electromagnetic radiation generated in the GRAPH mixing. In Sec. 6 we find the Stokes parameters of the electromagnetic radiation generated in the external magnetic field. In Sec. 7 we calculate the power and the power flux of the electromagnetic radiation generated in the GRAPH mixing. In Sec. 8 we conclude. In this work we use the values of the cosmological parameters found by the Planck collaboration [10] with $\Omega_\Lambda \simeq 0.68$, $\Omega_M \simeq 0.31$, $h_0 \simeq 0.67$ with zero spatial curvature $\Omega_\kappa = 0$.

2 Field mixing in external magnetic field

In order to describe the GRAPH mixing it is necessary first to start with the total action of the GRAPH mixing. In general, the action for a given Lagrangian density $L$ minimally coupled to gravity is $S = \int d^4x \sqrt{-g} L$ where $L$ describes the total Lagrangian density of matter and fields and their interactions. In our case, it is given by the sum of the following terms

$$L = L_{\text{gr}} + L_{\text{em}},$$

where $L_{\text{gr}}$ and $L_{\text{em}}$ are respectively the Lagrangian densities of gravitational and electromagnetic fields. These terms are respectively given by

$$L_{\text{gr}} = \frac{1}{\kappa^2} R, \quad L_{\text{em}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \int d^4x A_\mu(x) \Pi^{\mu\nu}(x, x') A_\nu(x').$$

Here $R$ is the Ricci scalar, $g$ is the metric determinant, $F_{\mu\nu}$ is the electromagnetic field tensor, $\kappa^2 = 16\pi G_N$ with $G_N$ being the Newtonian constant and $\Pi^{\mu\nu}$ is the photon polarization tensor in a magnetized medium.

By expanding the metric tensor around the flat Minkowski spacetime as $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} + \ldots$, we get the following expression for the total effective action

$$S_{\text{eff}} = \frac{1}{4} \int d^4x \left[ 2\partial_\mu h^{\mu\nu} \partial_\rho h_{\nu\rho} + \partial_\mu h \partial^\mu h - \partial_\mu h_{\alpha\beta} \partial^\alpha h^{\beta\gamma} - 2\partial_\mu h^{\mu\nu} \partial_\nu h \right] - \frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} + \frac{\kappa}{2} \int d^4x h_{\mu\nu} T_{\text{em}}^{\mu\nu}$$

$$- \frac{1}{2} \int d^4x \int d^4x' A_\mu(x) \Pi^{\mu\nu}(x, x') A_\nu(x') + O(\kappa \partial h^3) + O(\kappa h \Pi),$$

where $h_{\mu\nu}$ is the gravitational wave tensor with $h = \eta_{\mu\nu} h^{\mu\nu}$ and $T_{\text{em}}^{\mu\nu}$ is the electromagnetic field tensor.

Let us suppose that we have GWs propagating in vacuum and that after enter a region where does exist only an external magnetic field. We can put GWs in the TT gauge before entering the magnetic field region, namely $h_{0i} = 0, \partial^0 h_{ij} = 0, h_i^0 = 0$. The Euler-Lagrange equations of motion from (3) for the propagating photon and graviton fields components, $A^\mu$ and $h_{ij}$ propagating in the external magnetic field are given by

$$\nabla^2 A^0 = 0,$$

$$\Box A^i + \left( \int d^4x' \Pi^{ij}(x, x') A_j(x') \right) + \partial^0 \partial_\mu A^\mu = \kappa \partial_\mu [h^{\mu\beta} F_\beta^i - h^{i\beta} F_\beta^\mu],$$

$$\Box h_{ij} = -\kappa (B_i B_j - B_i B_j + B_i B_j).$$

1With the metric with signature $\eta_{\mu\nu} = \text{diag}[1, -1, -1, -1]$, the expressions for the spatial components of the electromagnetic stress-energy tensor are $T_{ij} = \varepsilon \varepsilon_j + B_i B_j - (1/2) \delta_{ij} (\varepsilon^2 + B^2)$ where $\varepsilon = \varepsilon_i + \varepsilon_j, B_i = B_i + B_i$ are respectively the components of the total electric and magnetic fields. The stress-energy tensor of the incident photon field tensor, $f_{\mu\nu}$, is not a source of GWs.
In obtaining the system of Eqs. [4], the electromagnetic field tensor has been written as the sum of the incident photon field tensor \( f_{\mu\nu} \) and of the external field tensor \( F_{\mu\nu} \), namely \( F_{\mu\nu} = f_{\mu\nu} + \tilde{F}_{\mu\nu} \). However, since we are assuming that does exist only an external magnetic field, we essentially have only that \( \tilde{F}_{ij} \neq 0 \). In addition, we assume that the external magnetic field varies in space on much larger scales than the incident GW wavelength, namely \( \lambda_B \gg \lambda_{gw} \). The latter assumption does not necessarily mean that the external magnetic field is only a uniform function of space coordinates where the condition \( \lambda_B \gg \lambda_{gw} \) is always satisfied. On the contrary, the magnetic field is assumed to be a slowly varying function of space coordinates, namely the field could be as well non-homogeneous in space and in time. The condition \( \lambda_B \gg \lambda_{gw} \) implies that \( |h\partial\tilde{F}| \ll |\tilde{F}\partial h| \), where for simplicity we suppressed the indexes in \( h_{ij} \) and \( \tilde{F}_{ij} \). By using these approximations, we can simplify the system (4) and write it in the form

\[
\nabla^2 A^0 = 0,
\]

\[
\Box A^i + \left( \int d^4x'\Pi^{ij}(x,x')A_j(x') \right) + \partial^i\partial_\mu A^\mu = -\kappa (\partial_j h^{jk})\tilde{F}^k,
\]

\[
\Box h_{ij} = -\kappa (B_i\bar{B}_j + \bar{B}_iB_j + \bar{B}_i\bar{B}_j),
\]

where we used the fact that \( F_{\mu\nu}\tilde{F}^{\mu\nu} = -4E \cdot B, \tilde{F}^{0i} = -B^i \) and the TT-gauge conditions.

In order to solve the system of equations[5], we must choose a gauge for the photon field which would simplify the equations a lot. In this work we employ the Coulomb gauge condition where \( \partial_i A^i = 0 \). In addition, from the first equation in the system [5] we can also choose \( A^0 = 0 \). Now by using the same method as shown in Ref. [11], namely we expand the fields \( A_i(x,t) \) and \( h_{ij}(x,t) \) in the form

\[
A^i(x,t) = \sum_{\lambda=x,+} e_{ij}^\lambda(\hat{n})A_\lambda(x,\omega)e^{-i\int\omega(t')dt'}, \quad h_{ij}(x,t) = \sum_{\lambda=x,+} h_\lambda(x,\omega)e_{ij}^\lambda(\hat{n})e^{-i\int\omega(t')dt'},
\]

where \( e_{ij}^\lambda \) is the photon polarization vector, \( e_{ij}^\lambda \) is the GW polarization tensor with \( \lambda \) indicating the polarization index or helicity state and \( \hat{n} = x/r \) with \( r = |x| \). Here \( \hat{n} \) is the direction of the propagation of the GW. Without any loss of generality, let us suppose now that the GW propagates in a given coordinate system along the \( z \) axis, namely \( \hat{n} = \hat{z} \). Since we are working in the Coulomb gauge where there is not a propagating longitudinal component for \( A^i \) and because \( x = rz \hat{z} \), we have that the third term on the left-hand side of the second equation in [5], namely \( \partial^i\partial_\mu A^\mu \) is zero because of the Coulomb gauge and because \( A^0 = 0 \). In the equation governing the GW evolution (the third equation in [5]), the last term \( B_i\bar{B}_j \), is a slowly varying function in space and time and can be neglected with respect to the interference terms \( B_i\bar{B}_j \) and \( \bar{B}_iB_j \).

Consider now the external magnetic field with components \( B(x,t) = [B_x(x,t), B_y(x,t), B_z(x,t)] \) and the vector potential with components \( A(x,t) = [A_x(x,t), A_y(x,t), A_z(x,t)] \). With the GW and electromagnetic wave propagating along the \( \hat{z} \) axis, \( h_{ij} = h_{ij}(r,t), A_i = A_i(r,t) \) and with the field
mass mixing matrix which is given by
\[ h = \omega \text{ expansion (9)} \text{ in the Eqs. (7) and (8), we get the following system of first order differential equations} \]
\[ \omega^2 + \partial_r^2 \ h_+(r, \omega) = -\kappa \left[ \partial_r A_x(r, \omega) \bar{B}_y + \partial_r A_y(r, \omega) \bar{B}_x \right], \]
\[ \omega^2 + \partial_r^2 \ h_+(r, \omega) = \kappa \left[ \partial_r A_x(r, \omega) \bar{B}_y - \partial_r A_y(r, \omega) \bar{B}_x \right], \]
where we used for the propagating electromagnetic wave \( B_x(r, t) = -\partial_r A_y(r, t), B_y(r, t) = \partial_r A_z(r, t), B_z(r, t) = 0 \) with \( \partial_r = \partial / \partial r \). In obtaining the Eqs. (7) we used the fact that the GW polarization tensor is symmetric and depends only on \( e^{ij}_X(z) \) and used the property \( e^{ij}_X e^{ij}_Y = 2 \delta_{XY} \).

In the case of equations of motion for the photon field \( A \) components in (7), we obtain
\[ \begin{align*}
[\omega^2 + \partial_r^2 - \Pi_{xx}(r, \omega)] A_x(r, \omega) - \Pi_{xy}(r, \omega) A_y(r, \omega) &- \Pi_{xz}(r, \omega) A_z(r, \omega) = \kappa \left[ \partial_r h_+(r, \omega) \bar{B}_y - \partial_r h_+(r, \omega) \bar{B}_x \right], \\
[\omega^2 + \partial_r^2 - \Pi_{yy}(r, \omega)] A_y(r, \omega) &- \Pi_{yx}(r, \omega) A_x(r, \omega) = \kappa \left[ \partial_r h_+(r, \omega) \bar{B}_y + \partial_r h_+(r, \omega) \bar{B}_x \right], \\
[\omega^2 \delta_{zz} - \Pi_{zz}(r, \omega)] A_z(r, \omega) & = 0,
\end{align*} \]
where in the Coulomb gauge there is no propagating longitudinal electromagnetic wave \( \partial_r A_z(r, t) = 0 \) and \( \Pi^{ij} = \Pi_{ij} = \Pi_{ij}(r, \omega) \) are the elements of the photon polarization tensor calculated in the adiabatic varying envelope approximation (SVEA) which is a WKB-like approximation, namely that \( \text{CF} \) is actually a constraint on \( A_z \).

It can be shown [23] that by solving this equation, namely by expressing \( A_z \) in terms of the transverse photon states \( A_x \) and \( A_y \) and then substituting it in the first two equations in (8), the components of \( \Pi_{ij} \) for \( i, j = x, y \) get a contribution from the longitudinal photon state. However, for the frequency range of the GWs and electromagnetic waves considered in this work, this extra contribution is very small and can safely be neglected.

The next step on solving Eqs. (7) and (8), is to look for solutions of field amplitudes of the form
\[ h_{+, x}(r, \omega) = \tilde{h}_{+, x}(r, \omega) e^{ikr}, \quad A_{x, y}(r, \omega) = \tilde{A}_{x, y}(r, \omega) e^{ikr}, \]
where \( k \) is the momentum of the fields corresponding to the mode \( k \). In addition, we work in the slowly varying envelope approximation (SVEA) which is a WKB-like approximation, namely that \( |\partial_r \tilde{h}_{+, x}| \ll |\tilde{h}_{+, x}| \) and \( |\partial_r \tilde{A}_{x, y}| \ll |\tilde{A}_{x, y}| \) with \( (\omega^2 + \partial_r^2)(\cdot) = (\omega - i\partial_r)(\omega + i\partial_r)(\cdot) = (\omega + k)(\partial_r + i\partial_r)(\cdot) \). By using the expansion (9) in the Eqs. (7) and (8), we get the following system of first order differential equations for the field amplitudes \( h_{+, x} \) and \( A_{x, y} \)
\[ (\omega + i\partial_r) \Psi(r, \omega) I + M(r, \omega) \Psi(r, \omega) = 0. \]
In (10) \( I \) is the unit matrix, \( \Psi(r, \omega) = (h_x, h_+, A_x, A_y)^T \) is a four component field and \( M(r, \omega) \) is the mass mixing matrix which is given by
\[ M = \begin{pmatrix}
0 & 0 & -iM^x_{x\gamma} & iM^y_{x\gamma} \\
0 & 0 & iM^x_{y\gamma} & iM^y_{y\gamma} \\
-iM^x_{x\gamma} & -iM^y_{x\gamma} & M_x & M_{CF} \\
-iM^x_{y\gamma} & -iM^y_{y\gamma} & M^*_{CF} & M_y
\end{pmatrix}, \]
where the elements of the mixing matrix \( M \) are given by \( M_{x\gamma} = \kappa k \bar{B}_z / (\omega + k) \), \( M_{y\gamma} = \kappa k \bar{B}_y / (\omega + k) \), \( M_x = \Pi_{xx} / (\omega + k) \), \( M_y = \Pi_{yy} / (\omega + k) \). Here \( M_{CF} = -\Pi_{xy} / (\omega + k) \) is a term which includes a combination of the Cotton-Mouton effect and the Faraday effect and which depends on the magnetic field direction with respect to the photon propagation. Here \( \omega \) is the total energy of the fields, namely \( \omega = \omega_{xx} \equiv \omega_x \). In this work all the particles participating in the mixing are assumed to be relativistic, namely \( \omega + k \approx 2k \).
3 Dispersive and coherence breaking effects in a magnetized plasma

In the previous section we have been able to reduce the equations of motion for the GRAPH mixing to a system of first order differential equations with variable coefficients. Before trying to look for a solution of the system [10], it is important to write the explicit expressions for $M_x$, $M_y$ and $M_{xy}$ which in turn depend on the elements of the photon polarization tensor in a magnetized medium. Here we present the explicit expressions for the elements $\Pi_{xx}, \Pi_{yy}$ and $\Pi_{xy}$ of the photon polarization tensor and for a detailed discussion and derivation of these expressions see Ref. [11]. The matrix elements $\Pi_{xx}$ and $\Pi_{yy}$ correspond to the modification of the dispersion and coherence breaking relations of the states $A_x$ and $A_y$ respectively, namely the momentum space Maxwell equations become, $\omega^2 - k_x^2 = \omega^2(1 - n_x^2) = \Pi_{xx,yy}$, where $n_{x,y}$ are the total indexes of refraction. The expressions for the elements $\Pi_{xx}$ and $\Pi_{yy}$ are given\(^2\) in Ref. [11]

$$
\Pi_{xx} = \frac{\omega^2\omega_{pl}^2}{\omega^2 - \omega_c^2} - \frac{\omega_{pl}^2 \omega_e^2 \cos^2(\Theta)}{\omega^2 - \omega_c^2}, \quad \Pi_{yy} = \frac{\omega^2\omega_{pl}^2}{\omega^2 - \omega_c^2} - \frac{\omega_{pl}^2 \omega_e^2 \sin^2(\Theta) \cos^2(\Phi)}{\omega^2 - \omega_c^2},
$$

where $\omega_{pl} = \sqrt{4\pi \alpha n_e/m_e}$ is the plasma frequency and $\omega_e = eB/m_e$ is the cyclotron frequency. Here $m_e$ is the electron mass, $e$ is the electron charge, $n_e$ is the number density of the free electrons in the plasma and $\vec{B}(\vec{x}, t) = |\vec{B}(\vec{x}, t)| \text{ is the external magnetic field strength}$. In addition, $\Theta$ is the polar angle of the external magnetic field with respect to the $x$ axis which points to the North and $\Phi$ is the azimuthal angle of the external magnetic field with respect to the $y$ axis which points outward. For this configuration, we can write $\vec{B}(\vec{x}, t) = [B_x(x, t), B_y(x, t), B_z(x, t)] = \vec{B}(x, t) [\cos(\Theta), \sin(\Theta) \cos(\Phi), \sin(\Theta) \sin(\Phi)]$.

The firsts terms in $\Pi_{xx}$ and $\Pi_{yy}$ in [12] correspond to the effect of only electronic plasma to the polarization tensor. The second terms in [12] correspond to the Cotton-Mouton effect in plasma since this effect is proportional to $\vec{B}^2$ (see Fig. [1]). On the other hand, the element $\Pi_{xy}$ is given by

$$
\Pi_{xy} = -\frac{\omega_{pl}^2 \omega_e^2 \sin(2\Theta) \cos(\Phi)}{2(\omega^2 - \omega_c^2)} - \frac{i \omega_{pl}^2 \omega_e \sin(\Theta) \sin(\Phi)}{\omega^2 - \omega_c^2}.
$$

The first term in [13] is due to the Cotton-Mouton effect while the second term corresponds to the Faraday effect in plasma. Since the second term is imaginary, it essentially means that the Faraday effect changes the intensity of each photon polarization state, namely a coherence breaking effect. Typically in the literature is used to get rid of the first term in $\Pi_{xy}$ by choosing $\Phi = \pi/2$, namely by choosing the external magnetic field $\vec{B}$ and the photon wave-vector $\vec{k}$ in the $xz$ plane. In such case $\Pi_{xy}$ is purely imaginary and it includes the Faraday effect only.

In many situations one can simplify the expressions of the elements of the photon polarization tensor by making some reasonable assumptions on the magnitude of the photon frequency with respect to the plasma and cyclotron frequencies. The numerical value of the angular plasma frequency can be written as $\omega_{pl} = 5.64 \times 10^4 \sqrt{n_e/cm^3}$ (rad/s) or $\nu_{pl} = \omega_{pl}/(2\pi) = 8976.33 \sqrt{n_e/cm^3}$ (Hz) for the frequency. On the other hand the numerical value of the cyclotron angular frequency is given by $\omega_c = 1.76 \times 10^7 (B/G)$ (rad/s). The cases when $\omega \gg \omega_{pl}$ and $\omega \gg \omega_c$ are of particular interest in many situations and especially in this work. As shown in the previous section, the quantities $\omega_c$ and $\omega_{pl}$ do not explicitly depend on the time $t$ but do explicitly depend on the distance $r$. However, in the case of photon propagation in an expanding universe, we can express the distance $r$ in terms of the cosmological time $t$ as $r = r(t)$. Consequently, each quantity that explicitly depends on $r$, does also implicitly depends on $t$ because of $r = r(t)$. Therefore, the conditions $\omega \gg \omega_{pl}$ and $\omega \gg \omega_c$ in an expanding universe, are respectively satisfied when

$$
\left(\frac{\nu_0}{\text{Hz}}\right) \gg 8976.33 \left(\frac{0.76n_B(t_0)X_e(t)}{\text{cm}^3}\right)^{1/2} \left(\frac{\alpha(t_0)}{\alpha(t)}\right)^{1/2} \quad \text{and} \quad \left(\frac{\nu_0}{\text{Hz}}\right) \gg 2.8 \times 10^6 \left(\frac{B_0}{G}\right) \left(\frac{\alpha(t_0)}{\alpha(t)}\right),
$$

\(^2\)All expressions for the photon polarization tensor elements are derived under the condition $\omega \neq \omega_c$ and $\omega > 0$. In addition, propagating electromagnetic waves exist only when $\omega > \omega_{pl}$ and $\omega > (\pm \omega_c + \sqrt{\omega_e^2 + \omega_{pl}^2/4})/2$. 

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where we expressed \( \nu(t) = \nu_0[a(t_0)/a(t)] \) with \( \nu_0 \) being the frequency of the electromagnetic radiation at the present time \( t = t_0 \), \( a(t) \) being the universe expansion scale factor and \( B_0 = B(t_0) \) is the magnetic field strength at the present time. Here we expressed the number density of free electrons as \( n_e(t) = n_e(t_0)/[a(t)/a(t_0)]^3 \) where \( n_B(t_0) \) is the total baryon number density at the present time and \( X_e(t) \) is the ionization fraction of the free electrons. The factor of 0.76 takes into account the contribution of hydrogen atoms to the free electrons at the post decoupling time.

By taking for example \( n_B(t_0) \approx 2.47 \times 10^{-7} \text{ cm}^{-3} \) as given by the Planck collaboration \[10\], and expressing \( a(t_0)/a(t) = 1 + z \) where \( z \) is the source redshift, we can write the conditions \[14\] as

\[
\left( \frac{\nu_0}{\text{Hz}} \right) \gg 3.88(1+z)^{1/2} \quad \text{and} \quad \left( \frac{\nu_0}{\text{Hz}} \right) \gg 2.8 \times 10^6 \left( \frac{B_0}{G} \right) (1+z),
\]

where at the post-decoupling epoch we can safely assume \( X_e(t) \approx 1 \). In most situations, photon frequencies that satisfy the first condition in \[15\], satisfy also the second condition in \[15\] for realistic values of \( B_0 \) and for redshifts \( z \lesssim 20 \). After these considerations, we can approximate the expressions of the elements of the photon polarization tensor as

\[
\Pi_{xx} \approx \omega_{pl}^2 \left[ 1 - \frac{\omega^2 \cos^2(\Theta)}{\omega^2} \right], \quad \Pi_{yy} \approx \omega_{pl}^2 \left[ 1 - \frac{\omega^2 \sin^2(\Theta) \cos^2(\Phi)}{\omega^2} \right],
\]

\[
\Pi_{xy} \approx -\frac{\omega_{pl}^2 \omega^2 \sin(2\Theta) \cos(\Phi)}{2\omega^2} - \frac{\omega^2 \omega_e \sin(\Theta) \sin(\Phi)}{\omega}.
\]

There is another fact about the expressions in \[16\] which is important to mention since now. The second terms in \( \Pi_{xx,yy} \), which essentially correspond to the Cotton-Mouton effect, are indeed very small quantities with respect to unity in the case \( \omega \gg \omega_e, \omega_{pl} \) and can be neglected in many cases. The only case when these quantities cannot be neglected is when we have to deal with the difference \( \Pi_{xx} - \Pi_{yy} \) or vice-versa. Regarding the term \( \Pi_{xy} \), we may note that in the cases when \( \sin(\Theta) \sin(\Phi) \neq 0 \), the magnitude of the imaginary term which essentially corresponds to the Faraday effect is much bigger than the magnitude of the real term which corresponds to the CM effect.

### 4 Perturbative solutions of the equations of motion

In this section we focus on perturbative solutions of the equations of motion \[10\]. The main reason to look for such solutions is because do not exist exact closed solutions except in some particular cases which are of no interest in this work. Here we employ a similar formalism as in quantum mechanics, namely similar to the time dependent perturbation theory, where usually one writes the total hamiltonian of the system as the sum of a “free” term plus a time dependent small interaction term. In our specific case the mass mixing matrix \( M \) plays the role of the total hamiltonian and which depends on the distance rather than the time. Consequently, in our case we may split the mass mixing matrix in the following way \( M(\omega, r) = M_0(\omega, r) + M_1(\omega, r) \) where \( M_0(\omega, r) \) is a matrix which would enter the equations of motion \[10\] in the case when there would not be present GWs and \( M_1(\omega, r) \) is a perturbation matrix which takes into account the interaction of GWs with the external magnetic field

\[
M_0(r) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & M_x & M_{\text{CF}} \\
0 & 0 & M_{\text{CF}} & M_y
\end{pmatrix}, \quad M_1(r) = \begin{pmatrix}
0 & 0 & -iM^x_{g\gamma} & iM^y_{g\gamma} \\
0 & 0 & iM^y_{g\gamma} & iM^x_{g\gamma} \\
iM^x_{g\gamma} & -iM^y_{g\gamma} & 0 & 0 \\
-iM^y_{g\gamma} & -iM^x_{g\gamma} & 0 & 0
\end{pmatrix}.
\]

In the case when GWs are missing, the matrix \( M_0 \) would enter the equation \[10\] in the form \((\omega + i\partial_r)\Psi(\omega, r)I + M_0(\omega, r)\Psi(\omega, r) = 0\) without the presence of the perturbation matrix \( M_1 \). However, even in the absence of the perturbation matrix \( M_1 \), it is not possible to find closed analytical solution for Eq. \[10\] since we are dealing with a first order system of differential equations with variable coefficients.
which analytic solutions are rare except in some particular cases. There is a possibility to solve analytically Eq. [10] for $M = M_0$ in the case when $M_x = M_y$. In fact, we may take from the expressions of $\Pi_{x,y}$ in [16], that in the case when $\omega \gg \omega_c$, the CM effect can be neglected with respect to the plasma effect. In this regime we may approximate $M_x \simeq M_y$ in $M_0$. In this case the commutator $[M_0(\omega, r), M_0(\omega, r')] = 0$ and the solution of Eq. [10] for $M = M_0$ is given by $\Psi(\omega, r) = U(r, r_i)\Psi(\omega, r_i)$ where $U$ is the usual unitary evolution operator which is given by $U(r, r_i) = \exp[-i \int_{r_i}^r dr' (\omega(r') I - M_0(r'))]$.

In the case when the interaction is present, namely when $M = M_0 + M_1$, in order to solve Eq. [10], it is convenient to move to the “interaction picture” by defining $\Psi_{\text{int}}(\omega, r) = U(\omega, r)\Psi(\omega, r)$ and $M_{\text{int}}(\omega, r) = U^\dagger(\omega, r_i)M_1(\omega, r)U(\omega, r_i)$. In the “interaction picture”, Eq. [10] becomes $i\partial_r \Psi_{\text{int}}(\omega, r) = M_{\text{int}}(\omega, r)\Psi_{\text{int}}(\omega, r)$. By using an iterative procedure, we find the following perturbative solution for $\Psi_{\text{int}}(\omega, r)$ to first and second order in the perturbation matrix $M_{\text{int}}(\omega, r)$

$$
\Psi_{\text{int}}^{(1)}(\omega, r) = -i \int_{r_i}^r dr' M_{\text{int}}(\omega, r')\Psi(r_i, \omega_i), \quad \Psi_{\text{int}}^{(2)}(\omega, r) = -i \int_{r_i}^r \int_{r_i}^{r'} dr'' M_{\text{int}}(\omega, r') M_{\text{int}}(\omega, r'')\Psi(r_i, \omega_i),
$$

where $\Psi_{\text{int}}^{(0)}(\omega, r) = \Psi(\omega_i, r_i)$, $\Psi_{\text{int}}(\omega, r) = \Psi_{\text{int}}^{(0)}(\omega, r) + \Psi_{\text{int}}^{(1)}(\omega, r) + \Psi_{\text{int}}^{(2)}(\omega, r) + \text{higher order terms}$. Since we have that the elements $|\int_{r_i}^r dr' M_{1, ij}(r')| \ll 1$ for reasonable values of the parameters, the series expansion converges rapidly and consequently is not necessary to go beyond the first order expansion. Therefore, performing several operations and by dropping for the moment the dependence of the fields on $\omega$, we get the following solutions for the field amplitudes up to the first order in perturbation theory

$$
\begin{align}
& e^{iM_x(r)} h_x(r) = h_x(r_i) - A_x(r_i) \int_{r_i}^r dr' \left( \cos \left[ \sqrt{M_{CF}(r')} \sqrt{M_{CF}^*(r')} \right] M_{CF}^x(\omega, r') - i C(\omega) \sin \left[ \sqrt{M_{CF}(r')} \sqrt{M_{CF}^*(r')} \right] \right) M_{CF}^x(\omega, r') e^{iM_1(r')}, \\
& e^{iM_+} h_+(r) = h_+(r_i) + A_x(r_i) \int_{r_i}^r dr' \left( \cos \left[ \sqrt{M_{CF}(r')} \sqrt{M_{CF}^*(r')} \right] M_{CF}^y(\omega, r') + i C(\omega) \sin \left[ \sqrt{M_{CF}(r')} \sqrt{M_{CF}^*(r')} \right] \right) M_{CF}^y(\omega, r') e^{iM_1(r')}, \\
& e^{iM_1} A_x(r) = A_x(r_i) + h_+ (r_i) \int_{r_i}^r dr' \left( \cos \left[ \sqrt{M_{CF}(r')} \sqrt{M_{CF}^*(r')} \right] M_{CF}^y(\omega, r') + i C(\omega) \sin \left[ \sqrt{M_{CF}(r')} \sqrt{M_{CF}^*(r')} \right] \right) M_{CF}^y(\omega, r') e^{iM_1(r')}, \\
& e^{iM_1} A_y(r) = A_y(r_i) - h_+ (r_i) \int_{r_i}^r dr' \left( \cos \left[ \sqrt{M_{CF}(r')} \sqrt{M_{CF}^*(r')} \right] M_{CF}^y(\omega, r') - i C(\omega) \sin \left[ \sqrt{M_{CF}(r')} \sqrt{M_{CF}^*(r')} \right] \right) M_{CF}^y(\omega, r') e^{iM_1(r')}.
\end{align}
$$

(19)
where we have defined

\[ \tilde{M}_{\{x,+x,y\}}(r) \equiv \int_{r_i}^{r} dr' \left[ -\omega - M_{\{x,+x,y\}}(r') \right], \quad M_{\{1,2\}}(r) \equiv \int_{r_i}^{r} dr' M_{\{x,y\}}(r'), \]

\[ M_{\text{CF}}(r) \equiv \int_{r_i}^{r} dr' M_{\text{CF}}(r'), \quad M_{\text{CF}}^*(r) \equiv \int_{r_i}^{r} dr' M_{\text{CF}}^*(r'), \quad C(r) \equiv \sqrt{M_{\text{CF}}^*(r)/M_{\text{CF}}(r)}, \]

with \( r_i \) being the initial distance. In obtaining the solutions \([19]\), we have assumed that \( M_{\text{CF}}^*(r) \neq 0, M_{\text{CF}}(r) \neq 0 \). In addition, since the gravitons are assumed to be exactly massless, we have that \( M_{x,+} = 0 \). As already mentioned above, on obtaining the solutions \([19]\) we have assumed that \( M_y \simeq M_x \) and therefore we have approximated \( M_2 \simeq M_1 \). We may also note from the solutions \([19]\) that in the expressions of \( h_{x,+}(r) \) do not appear \( h_{+,x}(r_i) \), namely there is no mixing between the states \( h_{x,+} \) at the first order in the perturbation theory. Such a mixing appears starting from the second order of iteration. Analog conclusions apply also for the photon states \( A_{x,y} \).

As it will be clear in what follows, it is very convenient in many calculations involving the photon amplitudes to write

\[ A_x(r) = I_1(r)h_x(r_i) - I_2(r)h_+(r_i) + A_x(r_i), \]
\[ A_y(r) = -I_3(r)h_x(r_i) - I_4(r)h_+(r_i) + A_y(r_i), \]  \( (20) \)

where we have defined

\[ I_1(r) \equiv \int_{r_i}^{r} dr' \left( \cos \left[ \sqrt{M_{\text{CF}}(r')} \sqrt{M_{\text{CF}}^*(r')} \right] M_{g\gamma}^x(r') + iC^{-1}(r') \sin \left[ \sqrt{M_{\text{CF}}(r')} \sqrt{M_{\text{CF}}^*(r')} \right] M_{g\gamma}^y(r') \right) e^{-iM_1(r')}, \]
\[ I_2(r) \equiv \int_{r_i}^{r} dr' \left( \cos \left[ \sqrt{M_{\text{CF}}(r')} \sqrt{M_{\text{CF}}^*(r')} \right] M_{g\gamma}^y(r') - iC^{-1}(r') \sin \left[ \sqrt{M_{\text{CF}}(r')} \sqrt{M_{\text{CF}}^*(r')} \right] M_{g\gamma}^x(r') \right) e^{-iM_1(r')}, \]
\[ I_3(r) \equiv \int_{r_i}^{r} dr' \left( \cos \left[ \sqrt{M_{\text{CF}}(r')} \sqrt{M_{\text{CF}}^*(r')} \right] M_{g\gamma}^y(r') + iC(r') \sin \left[ \sqrt{M_{\text{CF}}(r')} \sqrt{M_{\text{CF}}^*(r')} \right] M_{g\gamma}^x(r') \right) e^{-iM_1(r')}, \]
\[ I_4(r) \equiv \int_{r_i}^{r} dr' \left( \cos \left[ \sqrt{M_{\text{CF}}(r')} \sqrt{M_{\text{CF}}^*(r')} \right] M_{g\gamma}^x(r') - iC(r') \sin \left[ \sqrt{M_{\text{CF}}(r')} \sqrt{M_{\text{CF}}^*(r')} \right] M_{g\gamma}^y(r') \right) e^{-iM_1(r')}, \]  \( (21) \)

5 Generation of electromagnetic radiation and Stokes parameters

In this section we focus on our attention on the generation of the electromagnetic radiation for the GRAPH mixing. In particular, here we consider the situation of a source which emits GWs and we want to calculate useful quantities regarding the electromagnetic radiation such as the intensity, the power etc. In order to have a full picture of the generated electromagnetic radiation in the GRAPH mixing, is quite convenient to start with the Stokes parameters which give a complete description of the intensity and polarization state of the electromagnetic radiation. They are usually defined in terms of the transverse electric field amplitudes \( E_x \) and \( E_y \) \( (E(x,t) = [E_x(x,t), E_y(x,t)]) \) at a fixed point in space \( x \) as

\[ I_\gamma(x,t) \equiv |E_\gamma(x,t)|^2, \quad Q(x,t) \equiv |E_x(x,t)|^2 - |E_y(x,t)|^2, \]
\[ U(x,t) \equiv 2 \text{Re} \left\{ E_x(x,t) E_y^*(x,t) \right\}, \quad V(x,t) \equiv -2 \text{Im} \left\{ E_x(x,t) E_y^*(x,t) \right\}. \]  \( (22) \)

Consider now the situation where a given source emits GWs with polarization states \( h_{x,+} \) and initially there are not present photons. By re-introducing the dependence of the fields on \( \omega \) again, the amplitudes of the photon states \( A_{x,y} \) at the distance \( r \) from the source, given in expression \( (20) \), can be write as

\[ A_x(r,\omega) = I_1(r) h_x(r_i,\omega_i) - I_2(r) h_+(r_i,\omega_i), \]
\[ A_y(r,\omega) = -I_3(r) h_x(r_i,\omega_i) - I_4(r) h_+(r_i,\omega_i), \]  \( (23) \)
where the dependence of the fields on $\omega$ appears through the integrals $I_{1,2,3,4}$ which do depend on $\omega$ parametrically, $I_{1,2,3,4}(r; \omega)$. Let us concentrate on the calculation of the photon intensity $I_{\gamma}(r,t)$ and other Stokes parameters. In this case we need the explicit expressions for the electric field amplitudes $E_x$ and $E_y$ which are respectively given by $E_x(x, t) = -\partial_t A_x(x, t) - \nabla \cdot A^0(x, t)$ and $E_y(x, t) = -\partial_t A_y(x, t) - \nabla \cdot A^0(x, t)$. If the generated electromagnetic wave travels along the $z$ axis, then we have that at the distance $r$ from the source $A_{x,y}(r,t) = A_{x,y}(r,\omega)e^{-i\int \omega (t) dt}$. On the other hand, the expression for the scalar potential, $A^0(x,t) = 0$ by choice. After these considerations, we can write the expressions for the components of the electric field in the SVEA approximation, for an electromagnetic wave propagating along the $z$ axis at a distance $r$ from the source

$$E_{x,y}(r,t) \simeq -i \omega(t) A_{x,y}(r,\omega) e^{-i\int \omega (t) dt} = -i \omega(t) A_{x,y}(r,t). \quad (24)$$

With the expression for the electric field components given in (24), we can easily calculate the expression for the Stokes parameters for the generated electromagnetic field radiation, which are given by

$$I_\gamma(r,t) = \omega^2(t) \left[ |A_x(r,t)|^2 + |A_y(r,t)|^2 \right], \quad \frac{Q(r,t)}{I(r,t)} = \omega^2(t) \left[ |A_x(r,t)|^2 - |A_y(r,t)|^2 \right], \quad U(r,t) = 2 \omega^2(t) \text{Re} \left\{ A_x(r,t) A^*_y(r,t) \right\}, \quad V(r,t) = -2 \omega^2(t) \text{Im} \left\{ A_x(r,t) A^*_y(r,t) \right\}. \quad (25)$$

Now by using the expressions (23) in (25), we get

$$I_x(r,t) = \omega^2 \left[ (|I_1(r)|^2 + |I_3(r)|^2) |h_x(r_i)|^2 + (|I_2(r)|^2 + |I_4(r)|^2) |h_z(r_i)|^2 \right],$$

$$Q(r,t) = \omega^2 \left[ (|I_1(r)|^2 - |I_3(r)|^2) |h_x(r_i)|^2 + (|I_2(r)|^2 - |I_4(r)|^2) |h_z(r_i)|^2 \right] - 2 \omega \text{Re} \left\{ I_1(r) I^*_2(r) - I_3(r) I^*_4(r) \right\} h_x(r_i) h^*_z(r_i),$$

$$U(r,t) = 2 \omega^2 \text{Re} \left\{ -I_1(r) I^*_3(r) h_x(r_i) |I_2(r)|^2 + I_2(r) I^*_4(r) |h_x(r_i)|^2 - I_1(r) I^*_3(r) h_x(r_i) h^*_z(r_i) - I_2(r) I^*_4(r) h_x(r_i) h^*_z(r_i) \right\},$$

$$V(r,t) = 2 \omega^2 \text{Im} \left\{ I_1(r) I^*_3(r) h_x(r_i) |I_2(r)|^2 - I_2(r) I^*_4(r) |h_x(r_i)|^2 + I_1(r) I^*_3(r) h_x(r_i) h^*_z(r_i) + I_2(r) I^*_4(r) h_x(r_i) h^*_z(r_i) \right\}. \quad (26)$$

The expressions for the Stokes parameters given in (26) are one of the most important results in this work and will be the basis of our study of the generation of the electromagnetic radiation in the GRAPH mixing.

6 Evaluation of the integrals, $I_1$, $I_2$, $I_3$, and $I_4$

As we see from the expressions of the Stokes parameters given in (26), in order to calculate them, first we must calculate the integrals $I_1, I_2, I_3$ and $I_4$ which do appear in each of the parameters. The explicit expressions for the integrals $I_1, I_2, I_3$ and $I_4$ are given in (21). We may note that each of them contains to first order in perturbation theory, the integration over the distance of either $M^x$ or $M^y$ times trigonometric functions containing the CM and Faraday effects and also the exponential of plasma effects. Before evaluating the integrals first we must explicitly write all quantities which enter in each of them.

The explicit expressions for $M_1, M_2$ and $M_{CF}$, for $\omega \simeq k$, are given by

$$M_1(r) = \int^r_{r_1} dr' M_1(r') = -\int^{r}_{r_1} dr' \left( \frac{\Pi^{11}_1}{2\omega} \right) \simeq -\int^{r}_{r_1} dr' \frac{\omega^2_{pl}}{2\omega} \left[ 1 - \frac{\omega^2 \text{cos}^2(\Theta)}{\omega^2} \right],$$

$$M_2(r) = \int^r_{r_1} dr' M_2(r') = -\int^{r}_{r_1} dr' \left( \frac{\Pi^{12}_1}{2\omega} \right) \simeq -\int^{r}_{r_1} dr' \frac{\omega^2_{pl}}{2\omega} \left[ 1 - \frac{\omega^2 \text{sin}^2(\Theta) \text{cos}^2(\Phi)}{\omega^2} \right],$$

$$M_{CF}(r) = M_C(r) + i M_F(r) = -\frac{\Pi^{12}_2}{2\omega} \simeq \frac{\omega^2_{pl} \omega^2 \text{sin}(2\Theta) \text{cos}(\Phi)}{4 \omega^3} + i \frac{\omega^2_{pl} \omega^2 \text{sin}(\Theta) \text{sin}(\Phi)}{2 \omega^2}. \quad (27)$$
where we used the expressions for the elements of the photon polarization tensor given in (12). On the other hand, the explicit expressions for $M^x_{\gamma r}$ and $M^y_{\gamma r}$ are respectively given by $M^x_{\gamma r}(r) = kB \cos(\Theta)/2$ and $M^y_{\gamma r}(r) = kB \sin(\Theta) \cos(\Phi)/2$. The quantities $\omega_{\text{pl}}, \omega \equiv \omega_c$ and $B$ in (27) depend on the distance $r$ and implicitly depend on the time in an expanding universe, see below. Also the angles $\Theta$ and $\Phi$ may depend on the time but in this work we assume that the external magnetic field direction at a given point $x$ does not change in time, therefore $\Theta$ and $\Phi$ are assumed to be constant in time. In (27) we have expanded $M_{\text{CF}} = M_C + iM_F$ with $M_C$ being the term corresponding to the CM effect and $M_F$ being the term corresponding to the Faraday effect.

After the considerations made above, let us now focus on the calculations of the integrals $I_{1,2}$ and $I_{3,4}$. As we may note, the integrals $I_1$ and $I_2$ have the same structure, and therefore it will be sufficient to calculate only one of them. At this stage it is more useful to express each space dependent quantity as a function of the redshift $z$ since we are going to deal with electromagnetic radiation and GWs propagating in an expanding universe. For relativistic particles propagating in null geodesics we have that the line element $ds^2 = 0$ which implies that $dt = dr$ where $r$ is the light travelled distance and $t$ is the cosmological time. In this case the integration over the distance in each integral is replaced with the integration over the redshift $z$ by using the following prescription

$$\int_{r_i}^{r} dr'(...)= \int_{t_i}^{t} dt'(...) = \int_{z_i}^{z} \frac{dz'}{H_0(1+z')\sqrt{\Omega_A + \Omega_M(1+z')^3 + \Omega_R(1+z')^4}}, \quad (28)$$

where $\Omega_A \simeq 0.68$ is the present epoch density parameter of the vacuum energy, $\Omega_M \simeq 0.31$ is the present epoch density parameter of the non-relativistic matter and $\Omega_R \ll 1$ is the present epoch density parameter of the relativistic matter which essentially includes relativistic photons and neutrinos. Here we are assuming an universe with zero spatial curvature, namely $\Omega_k = 0$. In addition $r_i < r$ and $z < z_i$ where $z_i$ is the redshift of the GWs emitting source. In general for astrophysical sources of GWs which are located at relatively low redshifts, one can safely neglect the contribution of the relativistic matter to the total energy density. Moreover, in many cases it is quite accurate to approximate, $\sqrt{\Omega_A + \Omega_M(1+z)^3} \simeq \sqrt{\Omega_A}$ for $z < \left[(\Omega_A/\Omega_M)^{1/3} - 1\right] \approx 0.29$ or $\sqrt{\Omega_A + \Omega_M(1+z)^3} \simeq \sqrt{\Omega_M(1+z)^3}$ for $z \gg 0.29$.

In case when $\omega \gg \omega_c$, we may neglect the second terms proportional to the plasma frequency in $M_{1,2}$ in (27) and approximate

$$M_1(z) \simeq M_2(z) = -\int_{z_i}^{z} \frac{dz'}{H_0(1+z')\sqrt{\Omega_A + \Omega_M(1+z')^3}} \left(\frac{\omega_{\text{pl}}^2(z')}{2\omega(z')}\right),$$

$$\simeq \begin{cases} 
-2A_1\Omega_M^{-1/2}H_0^{-1}(\sqrt{1+z_i} - \sqrt{1+z}), & \text{for } z \gg \left[(\Omega_A/\Omega_M)^{1/3} - 1\right], \\
-(A_1/2)\Omega_M^{-1/2}H_0^{-1}[(z_i^2 - z^2) + 2(z_i - z)], & \text{for } z \ll \left[(\Omega_A/\Omega_M)^{1/3} - 1\right], 
\end{cases} \quad (29)$$

where we expressed the plasma and incident photon frequencies as a function of the redshift as shown in Sec. 3, namely $\omega_{\text{pl}}^2/(2\omega) = A_1(1+z)^2$ where $A_1 \equiv 3.12 \times 10^{-14}(\text{Hz}/\nu_0)$ (eV). Since in this work we focus on at the post-decoupling epoch, we assume that $X_i(z) \simeq 1$. Now in order to calculate the integrals in (21), let us write the amplitude of the external magnetic field as $B(z) = B_0(1+z)^2$, which is derived from the assumption that the magnetic flux in the cosmological plasma is a conserved quantity. The other quantities which we be useful in what follows are $M_C$ and $M_F$. The expression for $M_F$ can be calculated exactly and is given by

$$M_F(z) = \int_{r_i}^{r} dr' M_F(r') = \frac{2}{3\Omega_M} A_1A_2H_0^{-1}\sin(\Theta)\sin(\Phi) \left(\sqrt{\Omega_A + \Omega_M(1+z_i)^3} - \sqrt{\Omega_A + \Omega_M(1+z)^3}\right), \quad (30)$$

where $A_2 \equiv 2.8 \times 10^6(B_0/G)/(\text{Hz}/\nu_0)$. In the case of $M_C$ does not exist exact expressions for any $z$ but
only in some limiting cases

\[ \mathcal{M}_C(z) \equiv \int_{r_i}^r dr' M_C(r') \simeq \begin{cases} \sin(2\Theta) \cos(\Phi)(A_1/5) A_2^2 \Omega_M^{1/2} H_0^{-1} \left[ \sqrt{1 + z_i (1 + z_i (2 + z_i))} - \sqrt{1 + z (1 + z (2 + z))} \right] & \text{for } z \gg \left( \Omega_A/\Omega_M \right)^{1/3} - 1, \\
\sin(2\Theta) \cos(\Phi) (A_1/8) A_2^2 \Omega_A^{1/2} H_0^{-1} \left[ (z_i - z) (2 + z_i + z) (2 + z_i (2 + z_i) + z(2 + z)) \right] & \text{for } z \ll \left( \Omega_A/\Omega_M \right)^{1/3} - 1. 
\end{cases} \] (31)

6.1 The case when \( \Phi = \pi/2 \).

In this section we study the particular case when \( \Phi = \pi/2 \) which essentially corresponds to \( \mathcal{M}_C(z) = 0 \). In this case in and \( \mathcal{M}_{CF} \) is present only the Faraday effect term \( \mathcal{M}_F(z) \) which we assume to be different from zero, namely when \( \sin(\Theta) \neq 0 \). Indeed, if \( \sin(\Theta) \neq 0 \), the Faraday effect term is several orders of magnitude bigger than the CM effect without necessarily having the condition \( \Phi = \pi/2 \). Therefore the latter condition is a formal one as far as the Faraday effect term is different from zero. For \( \Phi = \pi/2 \), we have that \( C(r) = i \) and \( \mathcal{M}_{CM}(r) = \mathcal{M}_F(r) \).

With the above considerations, let us now concentrate on the calculation of the integral \( I_1 \) in (21), which for \( \Phi = \pi/2 \) becomes

\[ I_1(z) = \int_z^{z_i} \frac{dz'}{H_0(1 + z') \sqrt{\Omega_A + \Omega_M(1 + z')^3}} \cos \left[ \mathcal{M}_F(z') \right] M_{g\gamma}^x(z') e^{-i M_1(z')} \]

\[ = \int_z^{z_i} \frac{dz'}{H_0(1 + z') \sqrt{\Omega_A + \Omega_M(1 + z')^3}} \left( \cos \left[ \mathcal{M}_F(z') \right] \cos[M_1(z')] M_{g\gamma}^x(z') - i \cos \left[ \mathcal{M}_F(z') \right] \sin[M_1(z')] M_{g\gamma}^x(z') \right) \] (32)

Even though the integral \( I_1 \) has been significantly simplified for \( \Phi = \pi/2 \), it is still not possible to find an analytic expression because of the complexity of the integrands. Let us in addition assume that \( \Theta \to 0 \), which means that \( \mathcal{M}_F \ll 1 \), namely the external magnetic field is almost transverse with respect to the GW/electromagnetic wave propagation. In this regime, we can approximate \( \cos[M_1(z')] \simeq 1 \) in (33) and get

\[ I_1(z) \simeq \int_z^{z_i} \frac{dz'}{H_0(1 + z') \sqrt{\Omega_A + \Omega_M(1 + z')^3}} \left( \cos[M_1(z')] M_{g\gamma}^x(z') - i \sin[M_1(z')] M_{g\gamma}^x(z') \right) \] (33)

The integrals of the first and second terms in (33) can be calculated exactly and are given by

\[ \frac{\kappa}{2} \int_z^{z_i} \frac{dz'}{H_0(1 + z') \sqrt{\Omega_A + \Omega_M(1 + z')^3}} B(z') \cos[M_1(z')] = C \sin[M_1(z)], \]

\[ i \frac{\kappa}{2} \int_z^{z_i} \frac{dz'}{H_0(1 + z') \sqrt{\Omega_A + \Omega_M(1 + z')^3}} B(z') \sin[M_1(z')] = i C \left( 1 - \cos[M_1(z)] \right). \] (34)

where \( C = 9.75 \times 10^{-3} A^{-1} \kappa (\bar{B}_0/G) \text{ (eV}^2) \). Now we can use the expressions in (34) into the integral in (33) and obtain the final expression

\[ I_1(z) = -C \left( i - i \cos[M_1(z)] - \sin[M_1(z)] \right). \] (35)

We may also note that in the limits which we found \( I_1(z) \), we have that \( I_1(z) = I_2(z) \). Again in this limit we have from (21) that \( I_2(z) \simeq I_3(z) \). In addition, in the limits considered in this section, we have that \( |I_1(z)| \gg |I_2(z)| \) since the integrand in \( I_2 \) is proportional to \( \sin[M_F] M_{g\gamma}^x \simeq M_F M_{g\gamma}^y \ll 1 \).
7 Electromagnetic radiation from astrophysical binary systems

In the previous section we have been able to find analytic expressions for the integrals in (21) in the case when \( \mathcal{M}_F \ll 1 \) and \( \Phi = \pi/2 \). In more general cases it is not possible to find analytic solutions due to the complexity of the integrands in (21) and in these cases numerical results may be in order. In this section, we want to calculate some quantities related to the electromagnetic radiation in the GRAPH mixing such as the energy power \( P_\gamma \) and/or the energy power flux \( F_\gamma \). The latter quantity is simply given by \( I_s \) in (26) where by definition \( I_s \) represent the energy density of photons at a given point in space, while the former quantity can be easily calculated once we know the distance of the GW source.

The intensity of the generated electromagnetic radiation in the GRAPH mixing, in the case when \( \mathcal{M}_F \ll 1 \) and \( \Phi = \pi/2 \) and by using the results of the previous section, is given by

\[
I_s(r, t) \simeq \omega^2(t) |I_1(r)|^2 \left[ |h_\times(r, t_i)|^2 + |h_+ (r, t_i)|^2 \right],
\]

where we have neglected the term proportional to \( \text{Re}\{\ldots\} \) in \( I_s \) in (26) because it is a small quantity with respect to the other terms and used the fact that \( I_1 = I_4 \) in the limit \( \Theta \to 0 \) and \( \Phi = \pi/2 \). Therefore, in order to find the intensity of electromagnetic radiation or related quantities at given distance \( r \), we need the amplitudes of the GW at the distance \( r_i \) when GWs enter the region of magnetic field.

The amplitudes of GWs of binary systems of astrophysical sources are usually calculated starting from the multipole expansion of the stress energy-momentum tensor of the source. For binary systems, typically the quadrupole approximation of a quasi-circular orbit is a rather good approximation up to a maximum frequency \( \nu_{\text{max}} \) (see discussion section below), where beyond this frequency the strong gravity effects become dominant and the binary system coalesce. Therefore, let us assume that we have a binary system which emits GWs and which is undergoing to an inspiral phase of quasi-circular motion. The amplitudes of GWs at a distance \( r \) from the source in the quadrupole approximation and in the local wave zone are given by [14]

\[
\kappa h_+ (r, t_s) = h_c(t_s^{\text{ret}}) \left( \frac{1 + \cos^2(t_i)}{2} \right) \cos[\Psi(t_s^{\text{ret}})], \quad \kappa h_\times (r, t_s) = h_c(t_s^{\text{ret}}) \cos(t_i) \sin[\Psi(t_s^{\text{ret}})],
\]

\[
h_c(t_s^{\text{ret}}) = \frac{4}{r} (G \mathcal{M}_\text{CH})^{5/3} \left[ \pi \nu_s(t_s^{\text{ret}}) \right]^{2/3}, \quad \Psi(t_s^{\text{ret}}) \equiv \int^{t_s^{\text{ret}}} dt_s' \omega_s(t_s')
\]

where \( t_s \) is the time measured in the reference system of the GW source, \( t_s^{\text{ret}} = t_s - r \) is the retarded time, \( \mathcal{M}_\text{CH} = (m_1 m_2)^{3/5}(m_1 + m_2)^{-1/5} \) is the chirp mass of the source with \( m_{1,2} \) being the mass components of the binary system and \( t_i \) is angle of the normal of the binary system orbit with respect to the direction of observation. We may note the factor \( \kappa \) in (37) which we have introduced in order to conform with the notation used in Ref. [14], which uses the metric expansion \( g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \), while in our notations we use \( g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \).

The GWs amplitudes in (37) are expressed in terms of the source variables that are measured in the source reference system. Moreover, they do not take into account the universe expansion yet and have been calculated in the local wave zone, namely at distances \( r \gg d \) where \( d \) is the typical size of the binary system orbit. In order to make our treatment as much simple as possible, let us assume that at the initial distance \( r_i \), in the local wave zone, is present a large scale magnetic field. Let \( r_0 \) be the light travelled distance from the source until present epoch. Thus the effective distance travelled by GWs once they enter the region of large scale magnetic field is \( r_0 - r_i \simeq r_0 \) where \( r_0 \gg r_i \). It is more convenient for our purposes to express the amplitudes in (37) in terms of laboratory variables at present epoch. Consequently, we can write \( \nu_s(t_s^{\text{ret}}) = \nu_0(t_0^{\text{ret}})/(1 + z) \) where \( t_0^{\text{ret}} = (1 + z)t_s^{\text{ret}} \) is the observed retarded time in the laboratory reference system. One can also easily check that \( \Psi(t_0^{\text{ret}}) = \Psi(t_s^{\text{ret}}) \). Therefore, the initial GW amplitudes which enter the region of large scale magnetic field at the initial distance \( r_i \) from
the source, expressed in terms of present epoch variables, are given by
\[
\kappa h_+ (r_i, t_0) = h_c (t_0^\text{ret}) \left( \frac{1 + \cos^2 (\theta)}{2} \right) \cos [\Psi(t_0^\text{ret})], \quad \kappa h_\times (r_i, t_0) = h_c (t_0^\text{ret}) \cos (\theta) \sin [\Psi(t_0^\text{ret})],
\]
\[
h_c (t_0^\text{ret}) \equiv \frac{4}{r_i} (G_N M_{\text{CH}})^{5/3} \left[ \pi \nu_0 (t_0^\text{ret}) \right]^{2/3} (1 + z_i)^{2/3}.
\]

At this stage there are two important things to point out and which are of great importance in what follows. So far, we have considered the propagation of the GWs in a magnetized plasma and the equation of motion which we have derived in \([10]\), take into account the change of the initial GW amplitude in the \text{GRAPH} mixing only. However, for point sources of GWs, the amplitudes have an intrinsic decay with the distance of the form \(\propto 1/r\). Intentionally we did not look for solutions of the form \(h_{x+,}(r, t) \propto 1/r\) and \(A_{x,y}(r, t) \propto 1/r\) in Eqs. \([7]\) and \([8]\) in order to simplify our formalism as much as possible. So, in order to include the intrinsic decay of the amplitudes with the distance in the expression of the intensity given in \([36]\), we introduces the scaling \(I_\gamma \rightarrow I_\gamma (r_i/r)^2\). Another important thing to note is that Eqs. \([7]\) and \([8]\) have been derived in Minkowski space-time. However, our problem of \text{GRAPH} mixing essentially needs to be applied to the FRW metric in the case when GWs propagate in an expanding universe. As shown in Ref. \([12]\), the universe expansion is represented by the Hubble friction term \(-3H\partial_t\) and if one includes this term in the equations of motions, the amplitudes square of GWs \((h_{x,+})\) and of electromagnetic radiation \((A_{x,y})\), scale with the redshift as \(\propto (1 + z)^2\). Since \(I_\gamma \propto \omega^2 |A|^2\) represents the energy density of photons and because \(\omega^2 (z) \propto (1 + z)^2\), we have that \(I_\gamma (r_0, t_0) \propto (1 + z)^4\). Consequently, we have that the intensity of the electromagnetic radiation at present, \(t = t_0\) or \(z = 0\), is given by
\[
I_\gamma (r_0, t_0) \simeq \omega_0^2 |I_1(0)|^2 \left[ |h_\times (r_i, t_0)|^2 + |h_+ (r_i, t_0)|^2 \right] (1 + z_i)^4 (r_i/r_0)^2,
\]
where we remind that \(z_i\) is the redshift of the GW source at present epoch which is not related to \(r_i\). Here we are assuming that the redshift of the source \(z_i\) is approximately the same as the redshift when GWs enter the region of the large scale magnetic field.

The expression for the intensity in \([39]\) still is not in the final form because of the presence of \(\sin [\Psi]\) and \(\cos [\Psi]\) in the initial GW amplitudes and also the dependence on the angle \(\theta\). At this point it is more convenient to average the intensity \(I_\gamma\) over the phase \(0 \leq \Psi \leq 2\pi\) and \(0 \leq \theta \leq \pi\). By putting all together, we get
\[
\bar{I}_\gamma (r_0, t_0) \simeq \frac{9}{64} \left( \frac{\pi \nu_0^2}{\kappa r_0} \right)^2 \left[ r_i h_c (t_0^\text{obs}) \right]^2 |I_1(0)|^2 (1 + z_i)^4
\]
\[
= \frac{9}{16} \left( \frac{\pi \nu_0^2}{\kappa r_0} \right)^2 \left[ r_i h_c (t_0^\text{obs}) \right]^2 C^2 \sin^2 [M_1(0)/2] (1 + z_i)^4
\]
where \(\bar{I}_\gamma\) is the average value of the intensity on \(\Psi\) and \(\theta\) and not on \(\Phi\) and \(\Theta\). The energy per unit time (or the power \(P_\gamma\)) of the electromagnetic radiation, generated in the \text{GRAPH} mixing is given by
\[
P_\gamma (t_0) = 4\pi r_0^2 I_\gamma (r_0, t_0) = \frac{9\pi}{4} \left( \frac{\pi \nu_0^2}{\kappa} \right)^2 \left[ r_i h_c (t_0^\text{obs}) \right]^2 C^2 \sin^2 [M_1(0)/2] (1 + z_i)^4
\]
\[
\simeq 4.09 \times 10^{23} \left( \frac{M_{\text{CH}}}{M_{\odot}} \right)^{4/3} \left( \frac{\nu_0}{\text{Hz}} \right)^{4/3} \left( \frac{B_0}{\text{G}} \right)^2 \sin^2 [M_1(0)/2] (1 + z_i)^{16/3} \text{ (erg/s)},
\]
where \(M_{\odot}\) is the solar mass.

In Fig. 2 plots of the average power of electromagnetic radiation, given in \([11]\), generated in the \text{GRAPH} mixing are shown. In Fig. 2a the plots of the power as a function of the present day value of the cosmological magnetic field are shown. We may note that the power emitted at the frequencies \(\nu_0 = 150\ \text{Hz}\) and \(\nu_0 = 700\ \text{Hz}\) is larger than the power for \(\nu_0 = 500\ \text{Hz}\). The reason for this behaviour is because the power given in expression \([41]\) is proportional to \(\nu_0^{4/3}\) and also proportional to \(\sin^2 [M_1(0)/2]\).
Thus, even though for higher values of the frequencies $\nu_0^{4/3}$ increases, it is also true that $\sin^2[M_1(0)/2]$ is an extremely oscillating function of the frequency. Therefore, it happens that for $\nu_0 = 500$ Hz, the value of $\sin^2[M_1(0)/2]$ is much smaller than the values of $\sin^2[M_1(0)/2]$ at $\nu_0 = 150$ Hz and $\nu_0 = 700$ Hz. The fast oscillatory behaviour of the average power as a function of the frequency and redshift, due to the term $\sin^2[M_1(0)/2]$, are shown in Fig. 3.

![Figure 2](image1.png)

Figure 2: In (a) logarithmic scale plots of the power of the electromagnetic radiation $P_\gamma$ (erg/s) at present time as a function of the present day value of cosmic magnetic field $B_0$ (G), generated in the GRAPH mixing, for a typical binary system of neutron stars with equal masses $m_1 = m_2 = 1.4M_\odot$ and chirp mass $M_{\text{CH}} \simeq 1.21M_\odot$, for $z = 0.1$ and frequencies $\nu_0 = \{150, 500, 700\}$ Hz are shown. In (b) logarithmic scale plots of the power of the electromagnetic radiation $P_\gamma$ (erg/s) at present time as a function of the binary system chirp mass $M_{\text{CH}}$ (in units of the solar mass) for a binary system of equal masses, for $B_0 = 1$ nG, $z = 0.1$ and frequencies $\nu_0 = \{50, 100, 200\}$ Hz are shown.

![Figure 3](image2.png)

Figure 3: In (a) plot of the power of the electromagnetic radiation $P_\gamma$ (erg/s) at present time as a function of the GW source redshift $z \in [10^{-3}, 0.5]$, generated in the GRAPH mixing, for a typical binary system of neutron stars with equal masses $m_1 = m_2 = 1.4M_\odot$ and chirp mass $M_{\text{CH}} \simeq 1.21M_\odot$, for $B_0 = 1$ nG and frequency $\nu_0 = 500$ Hz is shown. In (b) plot of the power of the electromagnetic radiation $P_\gamma$ (erg/s) at present time as a function of the GW frequency $\nu_0 \in [50, 700]$ Hz for a binary system with equal masses $m_1 = m_2 = 1.4M_\odot$ and chirp mass $M_{\text{CH}} = 1.21M_\odot$, for $B_0 = 1$ nG and $z = 0.1$ is shown.
In (a) plots of the power fluxes of the electromagnetic radiation \( \gamma \) (erg cm\(^{-2} \) s\(^{-1} \)) at present time as a function of the present day value of GW frequency \( \nu_0 \in [10^{-10}, 10^{-6}] \) (Hz), generated in the GRAPH mixing, for a typical binary system of neutron stars with equal masses \( m_1 = m_2 = 1.4M_\odot \) and chirp mass \( M_{CH} \simeq 1.21M_\odot \), for a source located at redshift \( z = 0.1 \) and frequencies \( \nu_0 = \{300, 500, 700\} \) Hz are shown. In (b) similar plots as in (a) for source redshifts \( z = 0.1 \) and frequencies \( \nu_0 = \{50, 100, 150\} \) Hz are shown.

**Figure 4:** In (a) logarithmic scale plots of the power fluxes of the electromagnetic radiation \( \gamma \) (erg cm\(^{-2} \) s\(^{-1} \)) at present time as a function of the source chirp mass \( M_{CH} \in [1.21, 4]M_\odot \) for \( B_0 = 1 \) nG, source redshift \( z = 0.1 \) and frequencies \( \nu_0 = \{300, 500, 700\} \) Hz are shown. In (b) logarithmic scale plots of the power fluxes of the electromagnetic radiation \( \gamma \) (erg cm\(^{-2} \) s\(^{-1} \)) at present time as a function of the source chirp mass \( M_{CH} \in [1.21, 4]M_\odot \) for \( B_0 = 1 \) nG, source redshift \( z = 0.1 \) and frequencies \( \nu_0 = \{50, 100, 150\} \) Hz are shown.

**Figure 5:** In (a) plots of the power fluxes of the electromagnetic radiation \( \gamma \) (erg cm\(^{-2} \) s\(^{-1} \)) at present time as a function of the present day value of GW frequency \( \nu_0 \in [100, 500] \) (Hz), generated in the GRAPH mixing, for a typical binary system of neutron stars with equal masses \( m_1 = m_2 = 1.4M_\odot \) and chirp mass \( M_{CH} \simeq 1.21M_\odot \), for \( B_0 = 1 \) nG and source redshifts \( z = 0.1 \) and \( z = 0.5 \) are shown. In (b) similar plots as in (a) for source redshifts \( z = 1 \) and \( z = 1.5 \) and frequency interval \( \nu_0 \in [100, 300] \) (Hz) are shown.

In Figs. 4, 5, the average power fluxes of the electromagnetic radiation generated in the GRAPH mixing received at the Earth, given by expression (40), for a source of GWs located at redshift \( z_i \) are shown. As we can see, the average power fluxes received today are quite faint and they rapidly oscillate with the frequency \( \nu_0 \) and the redshift \( z \). The rapid oscillation of the received energy power flux is evident in Fig. 5 where plots of the energy power flux as a function of the frequency are shown. As already discussed above in the case of the energy power, higher values of the frequencies do not necessarily mean higher values of the power flux. Again, this behavior is due to the \( \sin^2 [M_1(0)/2] \) term in (40) which is an extremely fast oscillating function of the frequency. In addition, as we can see in Fig. 5b, there are cases
where the average energy power flux received from closer to the Earth binary systems or low redshift 
GW sources, is smaller than the average energy power flux received from far away binary systems of 
higher redshift. This behavior still is due to the factor $\sin^2[M_1(0)/2]$ which explicitly depends on 
the redshift and consequently is an extremely fast oscillating function of $z$ as well.

As already discussed above and shown in Figs.2 - 5, the energy power and energy power flux given 
respectively in expressions (11) and (10) are proportional to the term $\sin^2[M_1(0)/2]$ which is an extremely 
fast oscillating function of the parameters. It may be convenient for several reasons to average the energy 
power and energy power fluxes over a given observation frequency range. This might be for example the 
case of a detector which measures the energy power flux in a specific frequency range due to the detector 
characteristics. In this case, we have to average $\nu_0^{4/3} \sin^2[M_1(0)/2]$ on a frequency interval. However, 
since the integral of $\nu_0^{4/3} \sin^2[M_1(0)/2]$ is not an elementary one, for simplicity here we make the following 
observation; given a frequency interval with $0 < \nu_{0,1} \leq \nu_0 \leq \nu_{0,2}$ (where the frequency is expressed in 
units of Hz), we have that

$$\int_{\nu_{0,1}}^{\nu_{0,2}} d\nu' \nu_0^{4/3} \sin^2[M_1(0)/2] \leq \int_{\nu_{0,1}}^{\nu_{0,2}} d\nu' \nu_0^{4/3} \sin^2[M_1(0)/2] \leq \int_{\nu_{0,1}}^{\nu_{0,2}} d\nu' \nu_0^{4/3} = \frac{3}{7} \left( \nu_{0,2}^{7/3} - \nu_{0,1}^{7/3} \right).$$

Consequently, we have for example that the average energy power, is at maximum

$$\langle \bar{P}_\gamma(t_0) \rangle \simeq 4.09 \times 10^{23} \left( \frac{M_{\text{CH}}}{M_\odot} \right)^{4/3} \left( \frac{\bar{B}_0}{G} \right)^2 \left( \nu_0^{4/3} \sin^2[M_1(0)/2] \right) (1 + z_i)^{16/3} \ (\text{erg/s})$$

$$\leq 1.75 \times 10^{23} \left( \frac{M_{\text{CH}}}{M_\odot} \right)^{4/3} \left( \frac{\bar{B}_0}{G} \right)^2 (1 + z_i)^{16/3} \left( \nu_{0,2}^{7/3} - \nu_{0,1}^{7/3} \nu_{0,2} - \nu_{0,1} \right) \ (\text{erg/s}). \quad (42)$$

For example, if we consider a GW source with $M_{\text{CH}} = 1.21M_\odot$, $\bar{B}_0 = 1nG$ and $\nu_{0,1} = 50 \text{ Hz}$, $\nu_{0,2} = 600$ Hz, we get from (42)

$$\langle \bar{P}_\gamma(t_0) \rangle \leq 1.24 \times 10^9 \ (1 + z_i)^{16/3} \ (\text{erg/s}) \text{ for } z \ll 1. \quad (43)$$

At this point we can calculate from (42) - (43) the upper limit of the frequency averaged power flux 
which is given by $\langle \bar{F}_\gamma(r_0, t_0) \rangle = \langle \bar{P}_\gamma(t_0) \rangle / (4\pi r_0^2)$. We can calculate the light travelled distance $r_0$ from 
the expression (28). However, since does not exist an analytic expression for the integral in (28) for 
arbitrary $z$, let us consider for simplicity the case of low redshift GW sources, such as for example 
$z \ll [(\Omega_A/\Omega_M)^{1/3} - 1] \simeq 0.29$. In this case we find $r_0 - r_i \simeq r_0 \simeq 3.25 \times 10^{28} \ln(1 + z_i) \ (\text{cm})$. Therefore, the upper 
limit of the frequency averaged value of the power flux at $r_0$ and $t_0$ is given by

$$\langle \bar{F}_\gamma(r_0, t_0) \rangle \leq 1.31 \times 10^{-35} \left( \frac{M_{\text{CH}}}{M_\odot} \right)^{4/3} \left( \frac{\bar{B}_0}{G} \right)^2 \left( \frac{(1 + z_i)^{16/3}}{\ln^2(1 + z_i)} \right) \left( \frac{\nu_{0,2}^{7/3} - \nu_{0,1}^{7/3}}{\nu_{0,2} - \nu_{0,1}} \right) \ (\text{erg s}^{-1}\text{cm}^{-2}). \quad (44)$$

If we take again for example the same parameters as above, namely $M_{\text{CH}} = 1.21M_\odot$, $\bar{B}_0 = 1nG$ and 
$\nu_{0,1} = 50 \text{ Hz}$, $\nu_{0,2} = 600 \text{ Hz}$, we get from (44)

$$\langle \bar{F}_\gamma(r_0, t_0) \rangle \leq 7.21 \times 10^{-50} \left( \frac{(1 + z_i)^{16/3}}{\ln^2(1 + z_i)} \right) \ (\text{erg s}^{-1}\text{cm}^{-2}) \text{ for } z \ll 0.29. \quad (45)$$

8 Conclusions

In this work, we have studied the GRAPH mixing effect in large-scale cosmic magnetic field and have 
applied it to the case of astrophysical GW sources. This effect which has never been observed so far
might have an important contribution to the electromagnetic radiation received from a binary system of a GW source. In this work, we considered all standard effects that generate dispersion and coherence breaking of the electromagnetic radiation generated in the GRAPH mixing. In order to obtain the energy power and energy power fluxes, we had to solve a system of linear differential equations with variable coefficients. In order to solve the equations of motion, we used the perturbation theory where the terms related to the interaction of GWs with electromagnetic waves in the mixing matrix \( M \) have been considered as small perturbations with respect to dispersive and coherence breaking terms of the electromagnetic radiation.

From the technical point of view, even by using a perturbative approach to solve the equations of motion, the resulting final expressions for the Stokes parameters contain integrations on the redshift of complicated functions and in most cases is not possible to obtain analytic expressions of the integrals. Indeed, we have already seen this happen in Secs. 6-7 where we obtained analytic expressions for the integrals \( I_{1,2,3,4} \) only in the case when \( \Phi = \pi/2 \) and \( M_F \ll 1 \). In more general cases where the angles \( \Theta \) and \( \Phi \) are different from zero, the integrals appearing in the Stokes parameters do not have analytic expressions and in order to calculate the Stokes parameters, one must use numerical integration. In this work, we focused our attention on the \( L_\gamma \) Stokes parameter and did not study the evolution of the polarization parameters \( Q, U \) and \( V \). However, it is quite evident from the expressions \(^{26}\) that the generated electromagnetic radiation in the GRAPH mixing is elliptically polarized, namely, it has both linear and circular polarizations.

Our main goal in this work has been to obtain useful quantities such as the energy power and energy power fluxes of the electromagnetic radiation, which can be used in many contexts especially to confront with experimentally measurable quantities. In this regard, in Secs. \(^{47}\) we calculated the energy power \( P_\gamma(t_0) \) and the energy power flux \( F_\gamma(r_0,t_0) \) in the case of quasi-perpendicular external magnetic field with respect to the GW direction of propagation, namely the case where \( \Phi = \pi/2 \) and \( M_F \ll 1 \). In this regime, where analytic expressions do exist for \( P_\gamma(t_0) \) and \( F_\gamma(r_0,t_0) \), we have shown in Figs. 3-5 the power and power fluxes as a function of different quantities such as \( \dot{B}_0, \nu_0, M_{\text{CH}} \) and the redshift \( z \). The energy power \( P_\gamma \) generated in the GRAPH mixing effect is usually quite substantial and in the interval \( P_\gamma \simeq 10^6 - 10^{15} \text{ (erg/s)} \) for magnetic field amplitudes \( \dot{B}_0 \in [10^{-10}, 10^{-6}] \text{ G} \). On the other hand, the energy power flux received at the Earth is usually quite faint and it depends on the distance of the source if other parameters are fixed. One common feature of \( P_\gamma \) and \( F_\gamma \) is that they are extremely fast oscillating functions of the frequency \( \nu_0 \) and redshift \( z \). These features are respectively shown in Fig. 3 and Fig. 5. The high oscillatory feature of \( P_\gamma \) and \( F_\gamma \), often makes it quite difficult to numerically average them over \( \nu_0 \). Indeed, since the function \( \sin^2[M_1(0)/2] \) which does appear in \(^{40} \) and \(^{41} \) is a highly oscillating one, it is necessary in many cases to keep several digits of accuracy in the argument in order to minimize calculation errors. Different levels of accuracy in the argument of \( \sin^2[M_1(0)/2] \) may give slightly different values of \( P_\gamma \) and \( F_\gamma \) as functions of the parameters.

In the case when the direction of the cosmic magnetic field is arbitrary, it is not possible to find analytical expressions for \( P_\gamma \) and \( F_\gamma \) because of the complexity of the integrands in \( I_{1,2,3,4} \), which appear in the Stokes parameters. However, even though we did not calculate \( P_\gamma \) and \( F_\gamma \) in the general case, we can make some general discussions about their magnitudes. Indeed, by observing the integrands in the integrals in \(^{21} \), we may notice that in the general case when \( M_{\text{CF}} \neq 0 \), the integrals \( I_{1,2,3,4} \) contain as integrands \( \sin[M_{\text{CF}}] \) and \( \cos[M_{\text{CF}}] \) multiplied with \( M_{\gamma}^x y \sin[M_1] \) or \( M_{\gamma}^x y \cos[M_1] \). Due to the fact that the absolute value of trigonometric functions is between zero and one, we expect that in the general case where \( M_{\text{CF}} \neq 0 \), the magnitudes of \( P_\gamma \) and \( F_\gamma \), to be either smaller or at maximum the same as those found in the case when \( M_{\text{CF}} \to 0 \) as explicitly calculated in Sec. \(^{7} \). Of course, this fact should not surprise since the Faraday and Cotton-Motton effects, appearing in \( M_{\text{CF}} \), are coherence braking and dispersive phenomena, which tend to limit the GRAPH mixing with respect to the case when these effects are almost absent.

There are two important points which deserve special discussions. First, in this work, we considered GWs with observed frequencies roughly speaking above 50 Hz and below 700 Hz. The reasons for this
choice are strictly related to the approximations used in this work. In the lower frequency range, we considered GWs and electromagnetic waves with frequencies above the plasma frequency as discussed in Sec. 3. If the GW frequency is below the plasma frequency, the electromagnetic wave generated in the GRAPH mixing would much likely not propagate in the plasma and be absorbed by it. This essential fact makes the GRAPH mixing not much appealing for GWs with $\nu_0 \lesssim$ few Hz. However, the common statement that the electromagnetic radiation does not propagate when their frequencies are below the plasma frequency is based on the assumption that do not exist external currents that couple to photons such as GWs in the macroscopic Maxwell equations. But given the fact that such coupling is very small in general, we expect that the common statement that electromagnetic radiation with frequencies below that plasma frequency does not propagate, to remain still valid to the first order of approximation. On the other hand, we have chosen GWs emitted from binary systems in quasi-circular motion in the quadrupole approximation. This approximation, as discussed in details in Ref. [14], is valid up to a maximum separation distance of the binary system which corresponds a maximum frequency equal to the present day ISCO frequency $\nu_{\text{ISCO}}^\text{max} = (\nu_0)_{\text{ISCO}} \simeq 2.2 \times 10^5 (1 + z_i)^{-1} (M_\odot/M_T) \text{ Hz}$, where $M_T$ is the total mass of the binary system.

The second point is that in this work, we considered GWs generated in the quadrupole approximation which is valid for distances $r \gg d$ where $d$ is the typical size of the binary system. However, if $r < d$ and in the binary system already exist a magnetic field generated by internal process in the source, the GRAPH mixing effect can take place and generation of electromagnetic radiation might be substantial, given the fact that for binary systems of pulsars the magnetic field strength is very large and of the order of $B \simeq 10^{12}$ G. In any case, for $r < d$ the quadrupole approximation is not valid anymore and if a magnetic field exists at such distances, the effective GRAPH mixing strength is unknown because at such distances inside the GW source, usually there are five GW modes and not two as in the case of vacuum at distances $r \gg d$. The calculations of the GRAPH mixing strength for $r < d$ is beyond the purposes of this work.

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