DUAL CANONICAL BASES FOR UNIPOTENT GROUPS AND BASE AFFINE SPACES

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Abstract. Denote by $N \subset SL_k$ the subgroup of unipotent upper triangular matrices. In this paper, we show that the dual canonical basis of $\mathbb{C}[N]$ (and base affine spaces) can be parameterized by semi-standard Young tableaux. Moreover, we give an explicit formula for every element in the dual canonical basis using the data of the corresponding semistandard Young tableau. We apply our results to study cluster variables in $\mathbb{C}[N]$.

Contents

1. Introduction 1
Acknowledgements 5
2. Preliminary 5
2.1. Cluster algebras 5
2.2. Cluster structure on $\mathbb{C}[N]$ and $\mathbb{C}[SL_k]^{N^-}$ 6
2.3. Monoidal categorification of the cluster algebra structure on $\mathbb{C}[N]$ 7
3. The monoid of semi-standard Young tableaux 8
4. Isomorphisms of monoids $P_{k,\Delta}^+$ and SSYT($k-1,[k],\sim$) 10
4.1. Factorization of a tableau as a product of fundamental tableaux 10
4.2. Weights on semi-standard tableaux and on products of flag minors 11
4.3. Isomorphism of monoids 11
5. Formula for elements in the dual canonical basis 13
5.1. Formula for $ch(T)$ 13
5.2. Proof of Theorem 5.3 14
6. Mutation of tableaux 16
7. Application to classification of cluster variables in $\mathbb{C}[N]$ 19
References 21

1. Introduction

Quantum groups (or quantized universal enveloping algebras) were introduced independently by Drinfeld [20] and Jimbo [37] around 1985.

Let $\mathfrak{g}$ be a simple complex Lie algebra of type $A, D, E$. Denote by $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ a triangular decomposition of $\mathfrak{g}$. Let $q$ be an indeterminate and let $U_q(\mathfrak{g}) = U_q(\mathfrak{n}) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^-)$ be the Drinfeld-Jimbo quantum group over $\mathbb{C}(q)$. Inspired by a seminal work of Ringel [57], Lusztig introduced a canonical basis $\mathbf{B}$ of $U_q(\mathfrak{n})$ with remarkable properties in [50, 51]. In [39], Kashiwara found an alternative approach to the canonical basis of [50] which made sense in the more general context of Kac-Moody Lie algebras.
The quantum algebra $U_q(n)$ is endowed with a distinguished scalar product. Let $B^*$ be the basis of $U_q(n)$ adjoint to the canonical basis $B$ with respect to this scalar product. The dual canonical basis is defined to be image of the basis $B^*$ under the identification of the graded dual of $U_q(n)$ with $A_q(n)$. The graded dual $A_q(n)$ of $U_q(n)$ can be regarded as the quantum coordinate ring of the unipotent group $N$ with Lie algebra $n$ (see e.g. [33, 36]). When $q \to 1$, the basis $B^*$ specializes to a basis of the coordinate ring $\mathbb{C}[N]$ and it is called the dual canonical basis of $\mathbb{C}[N]$.

Canonical basis and dual canonical basis (in particular, the dual canonical basis of $\mathbb{C}[N]$) has been studied intensively in the literature using different methods and many important results are obtained, see e.g. [5, 6, 7, 8, 25, 26, 27, 29, 30, 31, 32, 33, 36, 38, 42, 45, 46, 47, 55, 56, 58, 60].

On the other hand, more work is needed to give a full description of the dual canonical basis, see e.g. the paragraph before the last paragraph of Section 2 in [31].

The aim of this paper is to give an explicit description of the dual canonical basis of $\mathbb{C}[N]$ in the case that $N \subset SL_k$ is the subgroup of unipotent upper triangular matrices, and the dual canonical basis of $\mathbb{C}[SL_k]^{N^-}$ which is closely related to $\mathbb{C}[N]$.

Let $N^- \subset G = SL_k$ be the subgroup of unipotent lower triangular matrices. The group $N^-$ acts on $G$ by left multiplication. Denote by $\mathbb{C}[SL_k]^{N^-}$ the ring of $N^-$-invariant regular functions on $SL_k$. Explicit description of the dual canonical basis of $\mathbb{C}[SL_k]^{N^-}$ is still an open problem, see e.g. the end of Section 6.5 in [24].

The algebra $\mathbb{C}[SL_k]^{N^-}$ is of high importance because it carries exactly one copy of each polynomial $GL_k$ representation exactly once. Thus, this paper is addressing the dual canonical basis for all $GL_k$ representations at once.

Our main result is to give an explicit formula of dual canonical basis elements using the data of semistandard Young tableaux. The dual canonical basis of $\mathbb{C}[N]$ is studied using geometric method in [1, 15, 28, 62, 63]. The description of the dual canonical basis using semistandard Young tableaux is useful in studying the dual canonical basis combinatorially. For example, it is useful in classifying cluster variables in the dual canonical basis, see Section 7. We also give a description of mutations in the cluster algebra $\mathbb{C}[N]$ using tableaux. This description agrees with a recent work [4, Section 7.2] of Bai, Dranowski, and Kamnitzer.

Brundan [13] gave a formula for the entries of the unitriangular transition matrices between the standard monomial and dual canonical bases of the irreducible polynomial representations of $U_q(\mathfrak{sl}_n)$ in terms of Kazhdan–Lusztig polynomials. The main difference between Brundan’s result and our result is that we work directly on $\mathbb{C}[N]$ and $\mathbb{C}[SL_k]^{N^-}$. Brundan’s methods used quantum Schur-Weyl duality and our method is to apply categorifications of cluster algebras using finite dimensional representations of type A quantum affine algebras.

The ring $\mathbb{C}[N]$ has a cluster algebra structure which can be obtained from a cluster algebra structure on $\mathbb{C}[SL_k]^{N^-}$ by identifying leading principal minors with 1 [24]. Denote by $\mathbb{C}[SL_k]^{N^-}$ the quotient of $\mathbb{C}[SL_k]^{N^-}$ by identifying the leading principal minors with 1. The algebras $\mathbb{C}[N]$ and $\mathbb{C}[SL_k]^{N^-}$ have the same cluster algebra structure (cf. Section 2.2).

Denote by $SSYT(k-1, [k], \sim)$ a certain quotient of the monoid $SSYT(k-1, [k])$ of semistandard tableaux with at most $k-1$ rows and with entries in $[k]$ (cf. Section 3). Our main result is the following.
Theorem 1.1 (Theorems 4.8 and 5.3). For a tableau $T$, we define in Section 5.1, a tableau $T'$ each of whose columns is a fundamental tableau with $m$ many columns, a permutation $w_T$ on $m$ letters, and a monomial $\Delta_{u,T'}$ for all permutations $u$ on $m$ letters.

The set \( \{ \text{ch}_{C[N]}(T) : T \in \text{SSYT}(k-1, \{k\}, \sim) \} \) (respectively, \( \{ \text{ch}_{C[SL_k]}(T) : T \in \text{SSYT}(k-1, \{k\}, \sim) \} \)) is the dual canonical basis of $C[N]$ (respectively, $C[SL_k]$),
\[
\text{ch}_{C[N]}(T) = \sum_{u \in S_m} (-1)^{\ell(uT)} p_{uw_0, wTyw_0}(1) \Delta_{u,T'} \in C[N],
\]
\[
\text{ch}_{C[SL_k]}(T) = \sum_{u \in S_m} (-1)^{\ell(uT)} p_{uw_0, wTyw_0}(1) \Delta_{u,T'} \in C[SL_k],
\]
where $w_0$ is the longest permutation and $p_{y,y'}(t)$ is a Kazhdan-Lusztig polynomial [44].

The difference between the formulas for $\text{ch}_{C[N]}(T)$ and $\text{ch}_{C[SL_k]}(T)$ is that the flag minors in the formula for $\text{ch}_{C[N]}(T)$ are flag minors in $C[N]$ while the flag minors in the formula for $\text{ch}_{C[SL_k]}(T)$ are flag minors in $C[SL_k]$. We write $\text{ch}_{C[N]}(T)$ (respectively, $\text{ch}_{C[SL_k]}(T)$) as $\text{ch}(T)$ if there is no confusion.

The basic approach of this paper including the proof technique of Theorem 1.1 is very similar to the approach in [19]. On the other hand, there are differences between the results in this paper and the results in [19]. The formula in Theorem 5.8 in [19] involves only rectangular semistandard tableaux while Theorem 1.1 involves semistandard tableaux of any shape.

To prove Theorem 1.1, we applied Hernandez-Leclerc’s monoidal categorification of $C[N]$ [36], a $q$-character formula in [19, Theorem 1.3] which is obtained from a result due to Arakawa-Suzuki [2] (see also Section 10.1 in [49], and [3, 34]) and from the quantum affine Schur-Weyl duality [17], and the following theorem.

Theorem 1.2 (Theorem 4.6). There is an isomorphism $\mathcal{P}_{k,\Delta}^+ \rightarrow \text{SSYT}(k-1, \{k\}, \sim)$ of monoids.

Here $\mathcal{P}_{k,\Delta}^+$ is a certain submonoid of the monoid of dominant monomials (cf. Section 2.3).

Remark 1.3. Though our combinatorial results bear a similarity with [19], they are different: there, the monoid was free on small gap tableau (which happen to correspond to fundamental l-weights) while here they are free on one-column tableaux whose entries are of the form $1, 2, \ldots, p - 1, p + k - i$, where $1 \leq p \leq i \leq k - 1$. Although both of small gap tableaux in [19] and fundamental tableaux in this paper correspond to fundamental l-weights, the form of fundamental tableaux in this paper is different than the form of small gap tableaux in [19].

By Theorem 1.1, the dual canonical basis of $C[N]$ (respectively, $C[SL_k]$) is parametrized by semi-standard tableaux in $\text{SSYT}(k-1, \{k\}, \sim)$ and every dual canonical basis element is of the form $\text{ch}(T)$ for some $T \in \text{SSYT}(k-1, \{k\}, \sim)$. In [42, 56], it is shown that cluster monomials in $C[N]$ (respectively, $C[SL_k]$) belong to the dual canonical basis. Therefore every cluster variable in $C[N]$ (respectively, $C[SL_k]$) is also of the form $\text{ch}(T)$.

Denote by $\Delta_j(x) = \Delta_{\{1, \ldots, |J|\}, J}(x)$ the minor of a matrix $x$ which takes rows $1, \ldots, |J|$ and columns $J$. 
Example 1.4. The cluster variables (not including frozen variables) of $\mathbb{C}[N]$, $N \subset SL_4$, (respectively, $\mathbb{C}[SL_4]^N$) are indexed by the following tableaux:

\[
\begin{align*}
T_1 &= \begin{array}{c} 2 \\ 3 \end{array}, & T_2 &= \begin{array}{c} 1 \\ 3 \end{array}, & T_3 &= \begin{array}{c} 4 \\ 1 \end{array}, & T_4 &= \begin{array}{c} 2 \\ 3 \end{array}, & T_5 &= \begin{array}{c} 1 \\ 2 \end{array}, & T_6 &= \begin{array}{c} 2 \\ 3 \end{array}, & T_7 &= \begin{array}{c} 1 \\ 3 \end{array}, & T_8 &= \begin{array}{c} 4 \\ 2 \end{array}, & T_9 &= \begin{array}{c} 4 \\ 3 \end{array}.
\end{align*}
\]

In $\mathbb{C}[SL_4]^N$ and $\mathbb{C}[N]$, we have that

\[
\begin{align*}
\text{ch}(T_1) &= \Delta_2, & \text{ch}(T_2) &= \Delta_3, & \text{ch}(T_3) &= \Delta_{13}, & \text{ch}(T_4) &= \Delta_{14}, & \text{ch}(T_5) &= \Delta_{23}, \\
\text{ch}(T_6) &= \Delta_{24}, & \text{ch}(T_7) &= \Delta_{124}, & \text{ch}(T_8) &= \Delta_{134}, & \text{ch}(T_9) &= \Delta_{3}\Delta_{124} - \Delta_{4}\Delta_{123}.
\end{align*}
\]

In both of $\mathbb{C}[N]$ and $\mathbb{C}[SL_4]^N$, all flag minors are cluster variables or frozen variables. On the other hand, in both $\mathbb{C}[N]$ and $\mathbb{C}[SL_4]^N$, there is some matrix minor (not flag minor) which is not a cluster variable. In $\mathbb{C}[N]$, the matrix minor $\Delta_{13,24} = x_{12}x_{34}$ is not a cluster variable. In $\mathbb{C}[SL_4]^N$, the matrix minor $\Delta_{13,24} = x_{12}x_{34} - x_{14}x_{32}$ is also not a cluster variable.

In $\mathbb{C}[N]$, we have that $\text{ch}(T_9) = \Delta_3\Delta_{124} - \Delta_4\Delta_{123} = x_{13}x_{34} - x_{14} = \Delta_{13,34}$. Therefore all cluster variables and frozen variables in $\mathbb{C}[N]$ are matrix minors.

In $\mathbb{C}[SL_4]^N$, the cluster variable $\text{ch}(T_9) = \Delta_3\Delta_{124} - \Delta_4\Delta_{123}$ is not a matrix minor.

Every tableau $T$ in SSYT$(k - 1, [k])$ can be written as $T = T'' \cup T'$ where "$\cup$" is the multiplication in the monoid SSYT$(k - 1, [k])$ (cf. Section 3). $T''$ is a tableau whose columns are fundamental tableaux and $T''$ is a fraction of two trivial tableaux (cf. Section 3).

For a tableau $T$ with columns $T_1, \ldots, T_r$, we denote by $\Delta_T = \Delta_{T_1} \cdots \Delta_{T_r}$ the standard monomial of $T$. For a fraction $ST^{-1}$ of two tableaux $S, T$, we denote $\Delta_{ST^{-1}} = \Delta_S\Delta_T^{-1}$ (cf. Section 4.2).

For $T \in \text{SSYT}(k - 1, [k])$, we define $\text{ch}'(T) = \Delta_T^{-1}\text{ch}_{C[SL_4]^N}^{-1}(T')$. We conjecture that $\{\text{ch}'(T) : T \in \text{SSYT}(k - 1, [k])\}$ is the dual canonical basis of $C[SL_4]^N$, see Conjecture 5.6.

We also apply our results to classification of cluster variables in $C[N], N \subset SL_6$, up to 4-column tableaux, cf. Section 7.

We showed that the numbers of rank 1, 2, 3, 4 tableaux (not including frozen variables) which are cluster variables in $C[N], N \subset SL_6$, are 52, 118, 170, 212 respectively. Moreover, we found the simplest non-real tableau $T = \begin{array}{c} 1 \\ 2 \\ 4 \\ 6 \\ 3 \\ 5 \end{array}$. It corresponds to a prime element $\text{ch}(T)$ in the dual canonical basis of $C[N]$ which is not a cluster variable, see (7.1). This tableau corresponds to the simple $U_q(\hat{sl}_6)$-module $L(Y_{3,-1}Y_{4,-4}Y_{4,2}Y_{5,-1})$, see Theorem 4.6. This module is very similar to the non-real $U_q(\hat{sl}_6)$-module $L(Y_{2,-2}Y_{3,-5}Y_{3,1}Y_{4,-2})$ (after translating to the language of dominant monomials, see Section 7) in Section 2.7 in [48]. This module is also similar to the non-real $U_q(\hat{sl}_4)$-module $L(Y_{1,4}Y_{2,1}Y_{2,7}Y_{3,4})$ in Section 13.6 in [35].

We will study the problem of classification of cluster variables in $C[N]$ systematically in another work.
The paper is organized as follows. In Section 2, we give some background on cluster algebras, quantum affine algebras, cluster structure on \( \mathbb{C}[N] \) and \( \mathbb{C}[SL_k]^{N-} \), and Hernandez-Leclerc’s monoidal categorification of \( \mathbb{C}[N] \). In Section 3, we describe the monoid of semi-standard Young tableaux. In Section 4, we show that a certain submonoid of the monoid of dominant monomials is isomorphic to the monoid of semi-standard tableaux. In Section 5, we give a formula for every element in the dual canonical basis of dominant monomials is isomorphic to the monoid of semi-standard tableaux. In Section 6, we describe the mutation rule in \( \mathbb{C}[N] \) (respectively, \( \mathbb{C}[SL_k]^{N-} \)) in terms of tableaux. In Section 7, we apply our results to classification of cluster variables in \( \mathbb{C}[N], N \subset SL_6 \), up to 4-column tableaux.

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2. Preliminary

2.1. Cluster algebras. Fomin and Zelevinsky introduced cluster algebras [26] in order to understand in a concrete and combinatorial way the theory of total positivity (cf. [52, 53]) and canonical bases in quantum groups (cf. [50, 51, 39]). We recall the definition of cluster algebras.

A quiver \( Q \) is an oriented graph given by a set of vertices \( Q_0 \), a set of arrows \( Q_1 \), and two maps \( s, t : Q_1 \to Q_0 \) taking an arrow to its source and target, respectively.

Let \( Q \) be a finite quiver without loops or 2-cycles. For a vertex \( k \in Q_0 \), the mutated quiver \( \mu_k(Q) \) is a quiver with the same set of vertices as \( Q \), and its set of arrows is obtained by the following procedure:

(i) add a new arrow \( i \to j \) for every existing pair of arrows \( i \to k, k \to j \);
(ii) reverse the orientation of every arrow with target or source equal to \( k \);
(iii) erase every pair of opposite arrows possibly created by (i).

Let \( m \geq n \) be positive integers and let \( \mathcal{F} \) be an ambient field of rational functions in \( n \) independent variables over \( \mathbb{Q}(x_{n+1}, \ldots, x_m) \). A seed in \( \mathcal{F} \) is a pair \((\mathbf{x}, Q)\), where \( \mathbf{x} = (x_1, \ldots, x_m) \) is a free generating set of \( \mathcal{F} \), and \( Q \) is a quiver (without loops or 2-cycles) with vertices \([m]\) whose vertices \(1, \ldots, n\) are called mutable and whose vertices \(n+1, \ldots, m\) are called frozen. For a seed \((\mathbf{x}, Q)\) in \( \mathcal{F} \) and \( k \in [n] \), the mutated seed \( \mu_k(\mathbf{x}, Q) \) in direction \( k \) is \((\mathbf{x}', \mu_k(Q))\), where \( \mathbf{x}' = (x'_1, \ldots, x'_m) \) with \( x'_j = x_j \) for \( j \neq k \) and \( x'_k \in \mathcal{F} \) is determined by the exchange relation:

\[
x'_k x_k = \prod_{\alpha \in Q_1, s(\alpha) = k} x_{t(\alpha)} + \prod_{\alpha \in Q_1, t(\alpha) = k} x_{s(\alpha)}.
\]

The mutation class of a seed \((\mathbf{x}, Q)\) is the set of all seeds obtained from \((\mathbf{x}, Q)\) by a finite sequence of mutations. For every seed \((x'_1, \ldots, x'_n, x_{n+1}, \ldots, x_m, Q')\) in the mutation class, the set \(\{x'_1, \ldots, x'_n, x_{n+1}, \ldots, x_m\}\) is called a cluster, \(x'_1, \ldots, x'_n\) are called cluster variables, and \(x_{n+1}, \ldots, x_m\) are called frozen variables. The cluster algebra \( \mathcal{A}(\mathbf{x}, Q) \) is the \(\mathbb{Z}[x_{n+1}, \ldots, x_m]\)-subalgebra of \( \mathcal{F} \) generated by all cluster variables. A cluster monomial is a product of non-negative powers of cluster variables belonging to the same cluster.
2.2. Cluster structure on $\mathbb{C}[N]$ and $\mathbb{C}[SL_k]^{N^-}$. In this subsection, we recall the cluster structure on $\mathbb{C}[N]$ and $\mathbb{C}[SL_k]^{N^-}$, cf. [7, 8, 25, 27, 31].

Let $V \cong \mathbb{C}^k$ be a $k$-dimensional complex vector space. By choosing a basis in $V$, one can identify $G = SL_k$ with the special linear group $SL(V)$ complex matrices with determinant 1. The subgroup $N^- \subset G$ of unipotent lower triangular matrices acts on $G$ by left multiplication. This action induces an action of $N^-$ on the coordinate ring $\mathbb{C}[G]$. Denote by $\mathbb{C}[G]^{N^-}$ the ring of $N^-$-invariant regular functions on $G$. The ring $\mathbb{C}[SL_k]^{N^-}$ has a cluster algebra structure whose initial cluster is given as follows.

For a $n \times n$ matrix $z$ and $J', J \subset [n]$ ($|J'| = |J|$), denote by $\Delta_{J,J'}(z)$ the determinant of the submatrix of $z$ with rows labeled by $J'$ and columns labeled by $J$. In the case that $J' = \{1, 2, \ldots, |J|\}$, we write $\Delta_{J,J'} = \Delta_{J,J'}$ and it is called a flag minor.

Let $I = [k - 1]$ be the set of the vertices of the Dynkin diagram of $\mathfrak{sl}_k$. Let $Q_{k,\Delta}$ be a quiver with the vertex set $V_{k,\Delta} = \{(i,p) : i \in I \cup \{k\}, p \in [i]\} \setminus \{(k,k)\}$ and with edge set:

$$(i, p) \rightarrow (i + 1, p + 1), \quad (i, p) \rightarrow (i, p - 1), \quad (i, p) \rightarrow (i - 1, p),$$

see Figure 1. The vertices $(i, i)$, $i \in I$ and $(k, p)$, $p \in I$ are frozen.

For $i \in I$, $p \in [i]$, denote $\Delta_{(i,p)} = \Delta_{J}$, where $J = \{1, 2, \ldots, p - 1, p + k - i\}$. Attach to the vertex $(i, p)$ the flag minor $\Delta_{(i,p)}$, $i \in I$, $p \in [i]$. An initial cluster of $\mathbb{C}[SL_k]^{N^-}$ consists of the initial quiver $Q_{k,\Delta}$ and initial cluster variables $\Delta_{(i,p)}$, $i \in I$, $p \in [i]$. Figure 1 is the initial cluster for $\mathbb{C}[SL_k]^{N^-}$ ($k = 5$) if we replace $\Delta_{1,...,i} = 1$ by $\Delta_{1,...,i}$, $i \in [k - 1]$.

In Figure 1,

$$\Delta_5, \Delta_4, \Delta_3, \Delta_2, \Delta_1, \Delta_{45}, \Delta_{34}, \Delta_{23}, \Delta_{12}, \Delta_{345}, \Delta_{234}, \Delta_{123}, \Delta_{2345}, \Delta_{1234},$$

sit at the vertices

$$(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (2, 2), (3, 2), (4, 2), (5, 2), (3, 3), (4, 3), (5, 3), (4, 4), (5, 4),$$

respectively.
Denote by $C[SL_k]^{N^-}$ the quotient of $C[SL_k]^{N^+}$ by identifying the leading principal minors $\Delta_{1,...,i}$ ($i \in [k-1]$) with 1. The cluster algebra structure on $C[SL_k]^{N^-}$ induces a cluster algebra structure on $C[SL_k]^{N^+}$.

Denote by $N \subset SL_k$ the subgroup of unipotent upper triangular matrices. The ring map $C[SL_k]^{N^-} \to C[N]$ defined by restricting $N^-$-invariant functions on $SL_k$ to the subgroup $N$. This map is onto and transforms the above described cluster structure on $C[SL_k]^{N^-}$ into a cluster structure on $C[N]$ (cf. [24]). This cluster structure on $C[N]$ has an initial cluster consisting of the initial quiver $Q_{k,\Delta}$ and initial cluster variables $\Delta^{(i,p)}$, $i \in I$, $p \in [i]$, see Figure 1.

2.3. Monoidal categorification of the cluster algebra structure on $C[N]$. Hernandez and Leclerc introduced the notion of a monoidal categorification of a cluster algebra in [35, 40]. For a monoidal category $(C, \otimes)$, a simple object $S$ of $C$ is called real if $S \otimes S$ is simple. A simple object $S$ is called prime if there exists no non-trivial factorization $S \cong S_1 \otimes S_2$. The monoidal category $C$ is called a monoidal categorification of a cluster algebra $\mathcal{A}$ if the Grothendieck ring of $C$ is isomorphic to $\mathcal{A}$ and if (1) any cluster monomial of $\mathcal{A}$ corresponds to the class of a real simple object of $C$, and (2) any cluster variable of $\mathcal{A}$ corresponds to the class of a real simple prime object of $C$.

Let $Q$ be an orientation of the Dynkin diagram of $\mathfrak{g}$. Hernandez and Leclerc [36] constructed a tensor category $C_Q$ and showed that $C_Q$ is a monoidal categorification of the ring $C[N]$ and its dual canonical basis. To our purpose, we use a special case $C_{k,\Delta}$ of $C_Q$. We recall the definition of $C_{k,\Delta}$ in the following.

Let $\mathfrak{g}$ be a simple Lie algebra and $I$ the set of the vertices of the Dynkin diagram of $\mathfrak{g}$. Denote by $P$ the weight lattice of $\mathfrak{g}$ and by $Q \subset P$ the root lattice of $\mathfrak{g}$. There is a partial order on $P$ given by $\lambda \leq \lambda'$ if and only if $\lambda' - \lambda$ is equal to a non-negative integer linear combination of positive roots.

In this paper, we take $q$ to be a non-zero complex number which is not a root of unity, $\mathfrak{g} = \mathfrak{sl}_k$, and $I = [k-1]$ be the set of vertices of the Dynkin diagram of $\mathfrak{g}$. The quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ is a Hopf algebra that is a $q$-deformation of the universal enveloping algebra of $\mathfrak{g}$ [20, 21, 37].

We fix $a \in \mathbb{C}^\times$ and denote $Y_{i,s} = Y_{i,aq^s}$, $i \in I$, $s \in \mathbb{Z}$. Denote by $\mathcal{P}$ the free abelian group generated by $Y_{i,s}^\pm$, $i \in I$, $s \in \mathbb{Z}$, denote by $\mathcal{P}^+$ the submonoid of $\mathcal{P}$ generated by $Y_{i,s}$, $i \in I$, $s \in \mathbb{Z}$, and denote by $\mathcal{P}_{k,\Delta}^\pm$ the submonoid of $\mathcal{P}^+$ generated by $Y_{i,i-2p}$, $i \in I$, $p \in [i]$. An object $V$ in $C_{k,\Delta}$ is a finite dimensional $U_q(\widehat{\mathfrak{sl}_k})$-module which satisfies the condition: for every composition factor $S$ of $V$, the highest $l$-weight of $S$ is a monomial in $Y_{i,i-2p}$, $i \in I$, $p \in [i]$. Simple modules in $C_{k,\Delta}$ are of the form $L(M)$ (cf. [16], [35]), where $M \in \mathcal{P}_{k,\Delta}^+$ and $M$ is called the highest $l$-weight of $L(M)$. The elements in $\mathcal{P}^+$ are called dominant monomials. Denote by $K(C_{k,\Delta})$ the Grothendieck ring of $C_{k,\Delta}$.

Let $\mathbb{Z}\mathcal{P} = \mathbb{Z}[Y_{i,s}^\pm]_{i,s \in \mathbb{Z}}$ be the group ring of $\mathcal{P}$. The $q$-character of a $U_q(\widehat{\mathfrak{g}})$-module $V$ is given by (cf. [23])

$$\chi_q(V) = \sum_{m \in \mathcal{P}} \dim(V_m)m \in \mathbb{Z}\mathcal{P},$$

where $V_m$ is the $l$-weight space with $l$-weight $m$ ($l$-weights of $V$ are identified with monomials in $\mathcal{P}$). It is shown in [23] that $q$-characters characterize simple $U_q(\widehat{\mathfrak{g}})$-modules up to isomorphism.
Denote by \( \text{wt} : \mathcal{P} \to \mathcal{P} \) the group homomorphism defined by sending \( Y_{i,k}^\pm \mapsto \pm \omega_i, \ i \in I \), where \( \omega_i \)'s are fundamental weights of \( \mathfrak{g} \). For a finite dimensional simple \( U_q(\mathfrak{g}) \)-module \( L(M) \), we write \( \text{wt}(L(M)) = \text{wt}(M) \) and call it the highest weight of \( L(M) \).

Let \( \mathcal{Q}^+ \) be the monoid generated (in the case that \( \mathfrak{g} = \mathfrak{sl}_k \)) by
\[
2.1 \quad A_{i,s} = Y_{i,s+1} Y_{i,s-1} \prod_{j \in I, |j-i|=1} Y_{j,i}^{-1}, \ i \in I, \ s \in \mathbb{Z}.
\]

There is a partial order \( \leq \) on \( \mathcal{P} \) (cf. [22, 54]) defined by
\[
2.2 \quad M \leq M' \text{ if and only if } M'M^{-1} \in \mathcal{Q}^+.
\]

For \( i \in I, \ s \in \mathbb{Z}, \ k \in \mathbb{Z}_{\geq 1} \), the modules \( L(X_{i,k}^{(s)}) \), where \( X_{i,k}^{(s)} = Y_{i,s} Y_{i,s+2} \cdots Y_{i,s+2k-2} \), are called Kirillov-Reshetikhin modules. The modules \( L(X_{i,1}^{(s)}) = L(Y_{i,s}) \) are called fundamental modules.

Hernandez and Leclerc [36] proved that the tensor category \( \mathcal{C}_{k,\Delta} \) is a monoidal categorification of the ring \( \mathbb{C}[N] \) and its dual canonical basis. The Grothendieck ring \( K(\mathcal{C}_{k,\Delta}) \) has a cluster algebra structure with an initial seed consisting of the initial quiver \( Q_{k,\Delta} \) and initial cluster variables \( X_{i,p}^{(i-2p)} \), \( i \in I, \ p \in [i] \), where \( X_{i,p}^{(i-2p)} \) sits at the position \((i,p)\) of the quiver \( Q_{k,\Delta} \); see Figure 2. We put trivial modules \( \mathbb{C} \) at the positions \((k,i)\), \( i \in [k-1] \), in order to compare with the quiver in Figure 1.

Recall that in Section 2.2, for \( i \in I, \ p \in [i] \), we denote \( \Delta^{(i,p)} = \Delta_J \), where \( J = \{1,2,\ldots,p-1,p+k-i\} \).

**Theorem 2.1** ([36, Theorems 1.1, 1.2, and 6.1]). The assignments \( L(Y_{i,i-2p}) \mapsto \Delta^{(i,p)} \), \( i \in I, \ p \in [i] \), induce an algebraic isomorphism \( \Phi_{\mathbb{C}[N]} : K(\mathcal{C}_{k,\Delta}) \to \mathbb{C}[N] \).

The assignments \( L(Y_{i,i-2p}) \mapsto \Delta^{(i,p)} \), \( i \in I, \ p \in [i] \), induce an algebraic isomorphism \( \Phi_{\mathcal{C}[SL_k]^N} : K(\mathcal{C}_{k,\Delta}) \to \mathbb{C}[SL_k]^N \).

We usually write \( \Phi_{\mathbb{C}[N]} \) (respectively, \( \Phi_{\mathcal{C}[SL_k]^N} \)) as \( \Phi \) if there is no confusion.

### 3. The monoid of semi-standard Young tableaux

In this section, we show that the set of semi-standard Young tableaux with at most \( k \) rows and with entries in a set \([m]\) form a monoid under certain product “\( \cup \)”. For \( k, m \in \mathbb{Z}_{\geq 1} \), denote by \( \text{SSYT}(k,[m]) \) the set of all semi-standard Young tableaux (including the empty tableau denoted by \( \mathbf{1} \)) with less or equal to \( k \) rows and with entries in \([m]\). For a tableau \( T \in \text{SSYT}(k,[m]) \) with \( k' \) (\( k' \leq k \)) rows, we say the \( i \)th (\( i > k' \)) row of \( T \), we understand that the \( i \)th row is empty.

For \( T, T' \in \text{SSYT}(k,[m]) \), we denote by \( T \cup T' \) the row-increasing tableau whose \( i \)th row is the union of the \( i \)th rows of \( T \) and \( T' \) (as multisets).

**Example 3.1.** In \( \text{SSYT}(5,[6]) \), we have that
\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & & \\
5 & & \\
\end{array}
\cup
\begin{array}{ccc}
2 & 4 & \\
& 3 & \\
& & 5 \\
\end{array}
= \begin{array}{ccc}
1 & 2 & 3 \\
2 & 4 & 5 \\
\end{array}
\]
For $S, T \in \text{SSYT}(k, [m])$, we say that $S$ is a factor of $T$ (denoted by $S \subset T$) if for every $i \in [k]$, the $i$th row of $S$ is contained in the $i$th row of $T$ (as multisets). For a factor $S$ of $T$, we define $T_S = S^{-1}T = TS^{-1}$ to be the row-increasing tableau whose elements in the $i$th row are the elements in the multiset-difference of $i$th row of $T$ and the $i$th row of $S$, for every $i \in [k]$.

We call a tableau $T \in \text{SSYT}(k, [m])$ trivial if it is a one-column tableau with entries $\{1, \ldots, p\}$ for some $p \in [k]$. For any $T \in \text{SSYT}(k, [m])$, we denote by $T_{\text{red}} \subset T$ the semi-standard tableau obtained by removing a maximal trivial factor from $T$. For $S, T \in \text{SSYT}(k, [m])$, define $S \sim T$ if $S_{\text{red}} = T_{\text{red}}$. Note that if $T \sim T'$, then $T, T'$ have the same number of rows. It is clear that “$\sim$” is an equivalence relation. We denote by $\text{SSYT}(k, [m], \sim)$ the set of $\sim$-equivalence classes in $\text{SSYT}(k, [m])$. With a slight abuse of notation, we write $T \in \text{SSYT}(k, [m], \sim)$ instead of $[T] \in \text{SSYT}(k, [m], \sim)$.

In [19, Lemma 3.6], we proved that the set of all semi-standard Young tableaux of rectangular shape with $k$ rows and with entries in $[m]$ is a monoid with the multiplication “$\cup$”. Similarly, we have the following result.

**Lemma 3.2.** The set $\text{SSYT}(k, [m])$ (respectively, $\text{SSYT}(k, [m], \sim)$) form a commutative cancellative monoid with the multiplication “$\cup$”.

**Proof.** It is clear that the set $\text{SSYT}(k, [m])$ form a commutative cancellative monoid implies that the set $\text{SSYT}(k, [m], \sim)$ form a commutative cancellative monoid. Therefore it suffices to prove the result for $\text{SSYT}(k, [m])$.

By definition, “$\cup$” is commutative and associative. Suppose that $A, T, T' \in \text{SSYT}(k, [m])$ and $A \cup T = A \cup T'$. For every $i \in [k]$, the $i$th row of $T$ (respectively, $T'$) is obtained from the $i$th row of $A \cup T$ (respectively, $A \cup T'$) by removing elements in the $i$th row of $A$ (as multisets). Since $A \cup T = A \cup T'$, we have that the $i$th rows of $T, T'$ are the same for every $i \in [k]$. Therefore $T = T'$.
We now prove that for \( T, T' \in \text{SSYT}(k, [m]) \), we have \( T \cup T' \in \text{SSYT}(k, [m]) \). Denote by \( S(i) \) the \( i \)th row of a tableau \( S \). We need to prove that for any \( i < j \), the 2-row tableau with the first row \( T(i) \cup T'(i) \) and the second row \( T(j) \cup T'(j) \) is semi-standard. It suffices to prove this in the case that \( T' \) has one column. Let \( i, j \) rows of \( T \) be

\[
\begin{array}{cccc}
  a_1 & a_2 & \cdots & a_{r_1} \\
  b_1 & b_2 & \cdots & b_{r_2},
\end{array}
\]

for some \( r_1 \geq r_2 \). We have the following cases.

**Case 1.** \( T' \) does not have entry in rows \( i \) and \( j \). In this case, the result is trivial.

**Case 2.** \( T' \) has an entry \( a' \) in row \( i \) and the row \( j \) is empty. There exists \( k \in [0, r_1] \) such that

\[
a_1 \leq \cdots \leq a_k \leq a' \leq a_{k+1} \leq \cdots \leq a_{r_1}.
\]

The \( i, j \) rows of \( T \cup T' \) are

\[
\begin{array}{cccc}
  a_1 & a_2 & \cdots & a_k & a' & a_{k+1} & \cdots & a_{r_1} \\
  b_1 & b_2 & \cdots & b_k & b_{k+1} & b_{k+2} & \cdots & b_{r_2},
\end{array}
\]

We have that \( a' \leq a_{k+1} < b_{k+1} \) and for all \( d \in [k+1, r_2-1] \), \( a_d < b_d \leq b_{d+1} \). Therefore the \( i, j \) rows of \( T \cup T' \) form a 2-row semi-standard tableau.

**Case 3.** \( T' \) has entries \( a' \) and \( b' \) in rows \( i \) and \( j \). There are \( k \in [0, r_1] \), \( l \in [0, r_2] \) such that

\[
a_1 \leq \cdots \leq a_k \leq a' \leq a_{k+1} \leq \cdots \leq a_{r_1} \text{ and } b_1 \leq \cdots \leq b_l \leq b' \leq b_{k+1} \leq \cdots \leq b_{r_2}.
\]

If \( k = l \), then the \( i, j \) rows of \( T \cup T' \) form a 2-row semi-standard tableau. If \( k > l \), then the \( i, j \) rows of \( T \cup T' \) are

\[
\begin{array}{cccc}
  a_1 & a_2 & \cdots & a_l & a_{l+1} & a_{l+2} & \cdots & a_k & a' & a_{k+1} & \cdots & a_{r_1} \\
  b_1 & b_2 & \cdots & b_l & b_{l+1} & b_{l+2} & \cdots & b_k & b_{k+1} & b_{k+2} & \cdots & b_{r_2},
\end{array}
\]

We have \( a' < b' \leq b_k, a_{l+1} \leq a' < b' \), and for all \( d \in [l+2, k] \), \( a_d < a' < b' \leq b_{d-1} \). Therefore the \( i, j \) rows of \( T \cup T' \) form a 2-row semi-standard tableau.

If \( k < l \), then the \( i, j \) rows of \( T \cup T' \) are

\[
\begin{array}{cccc}
  a_1 & a_2 & \cdots & a_k & a' & a_{k+1} & \cdots & a_{l-1} & a_l & a_{l+1} & \cdots & a_{r_1} \\
  b_1 & b_2 & \cdots & b_k & b_{k+1} & b_{k+2} & \cdots & b_l & b' & b_{l+1} & \cdots & b_{r_2},
\end{array}
\]

We have \( a' \leq a_{k+1} < b_{k+1}, a_l \leq b_l \leq b' \), and for all \( d \in [k+1, l-1] \), \( a_d < b_d \leq b_{d+1} \). Therefore the \( i, j \) rows of \( T \cup T' \) form a 2-row semi-standard tableau.

\[\square\]

4. **Isomorphisms of Monoids \( \mathcal{P}_{k,\triangle}^+ \) and \( \text{SSYT}(k - 1, [k], \sim) \)**

In this section, we show that the monoids \( \mathcal{P}_{k,\triangle}^+ \) and \( \text{SSYT}(k - 1, [k], \sim) \) are isomorphic.

4.1. **Factorization of a tableau as a product of fundamental tableaux.** For \( i \in I, \ p \in [\bar{s}] \), denote by \( T^{(i,p)} \) the one-column tableau with entries \( \{1, 2, \ldots, p - 1, p + k - i\} \). We call the tableau \( T^{(i,p)} \) a fundamental tableau. We also use \( T_{(l,a)} \) to denote a fundamental tableau with \( l \) rows and whose last entry \( a \). We have that \( T_{(l,a)} = T^{(l+k-a,l)} \).

There is a total order on the set of one-column fundamental tableaux in \( \text{SSYT}(k, [m]) \): for two one column fundamental tableaux \( T = T_{(l,a)}, T' = T_{(l',a')}, T \leq T' \) if either \( l > l' \) or \( l = l', \ a \leq a' \). For example,

\[
\begin{array}{cccc}
  1 & 1 \\
  2 & 2 < 1 & 3 < 1 & 4 < 2.
\end{array}
\]
If the columns $T_1, \ldots, T_r$ ($T_i$ is the $i$th column of $T$) of a tableau $T \in \text{SSYT}(k, [m])$ are all fundamental tableaux, then $T_1 \preceq T_2 \preceq \cdots \preceq T_r$ in the above described total order.

**Lemma 4.1.** For $k, m \in \mathbb{Z}$, every $T \in \text{SSYT}(k, [m], ~)$ can be uniquely factorized as a $\cup$-product of fundamental tableaux and there is a unique $T' \in \text{SSYT}(k, [m], ~)$ such that $T' ~ T$ and the columns of $T'$ are fundamental tableaux.

**Proof.** First we prove the existence. It suffices to prove the existence in the case that $T$ is a one-column tableau. Denote by $T_0$ the fundamental tableaux, then

$$T = T_0 \cup T_1 \cup \cdots \cup T_r,$$

where $T_0$ and the columns of $T_1, \ldots, T_r$ are fundamental tableaux. Then $T' = T_0 \cup T_1 \cup \cdots \cup T_r \cup T''$, where $T'$ is the union of the fundamental tableaux $T(j, i)$, where the entries of $T(j, i)$ are \{1, 2, \ldots, $j - 1, i\}$, $j \in [2, r]$. If $i_1 > 1$, then $T \sim T'$, where $T'$ is the union of the fundamental tableaux $T(j, i)$, $j \in [r]$.

Now we prove uniqueness. Suppose that $T \sim T'$, $T \sim T''$, and the columns of $T', T''$ are fundamental tableaux. Then $T' \sim T''$. It follows that there are trivial tableaux $A, B$ such that $A \cup T' = B \cup T''$. Since the columns of $A, B$ are trivial tableaux and the columns of $T', T''$ are fundamental tableaux, we have that $A = B$. It follows that $T' = T''$ since $\text{SSYT}(k, [m], ~)$ is cancellative by Lemma 3.2. □

**Example 4.2.** In $\text{SSYT}(5, [6], ~)$, we have that

\[
\begin{array}{ccc|ccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 2 & 4 & 5 & 6 & 3 \\
\hline
1 & 2 & 5 & 6 & 3 & 4
\end{array}
\sim
\begin{array}{ccc|ccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 2 & 4 & 5 & 6 & 3 \\
\hline
1 & 2 & 5 & 6 & 3 & 4
\end{array} =
\begin{array}{ccc|ccc}
1 & 1 & 1 & 1 & 1 & 2 \\
\hline
2 & 2 & 3 & 4 & 5 & 6
\end{array}.
\]

### 4.2. Weights on semi-standard tableaux and on products of flag minors.

There is a bijection between the set of one-column semi-standard tableaux in $\text{SSYT}(k-1, [k], ~)$ and the set of (non-trivial) flag minors of $\mathbb{C}[N]$ sending the one-column tableau with entries in $J \subset [k]$ to the flag minor $\Delta_J$. Denote by $T_{\Delta}$ the tableau corresponding to a flag minor $\Delta$ and $\Delta_T$ the flag minor corresponding to a one-column tableau $T$. For a tableau $T$ with columns $T_1, \ldots, T_r$, we denote by $\Delta_T = \Delta_{T_1} \cdot \Delta_{T_r}$ the standard monomial of $T$. For a fraction $ST^{-1}$ of two tableaux $S, T$, we denote $\Delta_{ST^{-1}} = \Delta_S \Delta_T^{-1}$.

**Definition 4.3.** For a fundamental tableau $T^{(i, p)} \in \text{SSYT}(k-1, [k], ~)$, $i \in I$, $p \in [i]$, we define the weight of the tableau as $\omega(T^{(i, p)}) = \omega_i \in P$, where $\omega_i$ is a fundamental weight of $g$. We define $\omega(1) = 0$.

For a tableau $T \in \text{SSYT}(k-1, [k], ~)$, we define the weight of $T$ as $\omega(T) = \sum_j \omega(T^{(i)})$, where $T = \cup_j T^{(i)}$ is the unique factorization of the tableau $T$ into fundamental tableaux.

**Definition 4.4.** For a flag minor $\Delta \in \mathbb{C}[N]$, we define the weight of $\Delta$ as $\omega(T_{\Delta})$. For a product $\prod_j \Delta^{(j)}$ of flag minors, we define $\omega(\prod_j \Delta^{(j)}) = \sum_j \omega(\Delta^{(j)})$.

### 4.3. Isomorphism of monoids.

By Theorem 2.1, $\{\Delta_T : T \in \text{SSYT}(k-1, [k], ~)\}$ is an additive basis of $\mathbb{C}[N]$, $N \subset SL_k$. Therefore for any module $[L(M)] \in K(\mathcal{C}_{k, \Delta})$,

\[
\Phi([L(M)]) = \sum_{T \in \text{SSYT}(k-1, [k], ~)} c_T \Delta_T \in \mathbb{C}[N],
\]

for some $c_T \in \mathbb{C}^\times$.

Define $\text{Top}(\Phi([L(M)]))$ to be the tableau which appears on the right hand side of (4.1) with the highest weight. By the same proof as the proof of Lemma 3.22 in [19] using $q$-character theory, we have that $\text{Top}(\Phi(L(M)))$ exists for every $L(M) \in K(\mathcal{C}_{k, \Delta})$. Moreover, $\omega(L(M)) = \omega(\text{Top}(\Phi([L(M)])))$. 

We define a map
\begin{equation}
\tilde{\Phi} : \mathcal{P}^+_{k,\Delta} \rightarrow \text{SSYT}(k-1, [k], \sim), \quad M \mapsto \text{Top}(\Phi(L(M))),
\end{equation}
and denote $T_M = \tilde{\Phi}(M)$.

Recall that for $i \in I$, $p \in [i]$, $T^{(i,p)}$ is the one-column tableau with entries $\{1, 2, \ldots, p-1, p+1\}$. The following lemma follows from Theorem 2.1 and the definition of $\tilde{\Phi}$.

**Lemma 4.5.** For fundamental modules $L(Y_{i,-2p}) \in \mathcal{C}_{k,\Delta}$, $i \in I$, $p \in [i]$, we have that $\tilde{\Phi}(Y_{i,-2p}) = T^{(i,p)}$ and $\text{wt}(Y_{i,-2p}) = \text{wt}(T^{(i,p)}) = \omega_i$.

Recall that $T_{(l,a)}$ is a one-column fundamental tableau with $l$ rows and whose last entry is $a$, and $T_{(l,a)} = T^{(l+k-a,l)}$.

By Lemma 4.1, every $T \in \text{SSYT}(k-1, [k], \sim)$ has a unique factorization $T \sim \bigcup_{i=1}^r T_{(l,a)}$. We define
\begin{equation}
\Psi : \text{SSYT}(k-1, [k], \sim) \rightarrow \mathcal{P}^+_{k,\Delta}, \quad T \mapsto \prod_{i=1}^r Y_{l+k-a,i-k-a-i},
\end{equation}
and denote $M_T = \Psi(T)$. We will show that $\Psi$ is the inverse of $\tilde{\Phi}$.

**Theorem 4.6.** The map $\tilde{\Phi} : \mathcal{P}^+_{k,\Delta} \rightarrow \text{SSYT}(k-1, [k], \sim)$ is an isomorphism of monoids and its inverse is $\Psi$.

*Proof.* We first show that $\tilde{\Phi}$ is a homomorphism of monoids. By the theory of $q$-characters, for any $M, M' \in \mathcal{P}^+_{k,\Delta}$, we have that
\begin{equation}
[L(M)][L(M')] = [L(MM')] + \sum_{\tilde{M}, \text{wt}(\tilde{M}) < \text{wt}(MM')} c_{\tilde{M}}[L(\tilde{M})],
\end{equation}
for some $c_{\tilde{M}} \in \mathbb{Z}_{\geq 0}$. Since $\Phi : \mathcal{K}(\mathcal{C}_{k,\Delta}) \rightarrow \mathbb{C}[N]$ is an algebra isomorphism, we have that
\[
\Phi(L(M))\Phi(L(M')) = \Phi(L(MM')) + \sum_{\tilde{M}, \text{wt}(\tilde{M}) < \text{wt}(MM')} c_{\tilde{M}} \Phi(L(\tilde{M})).
\]
It follows that $\text{Top}(\Phi(L(M))\Phi(L(M'))) = \text{Top}(\Phi(L(MM'))).$ Therefore $\tilde{\Phi}(MM') = \tilde{\Phi}(M) \cup \tilde{\Phi}(M')$.

We now show that $\Psi$ is a homomorphism of monoids. Since $\Psi(T)$ only depends on the equivalence class of $T$, it suffices to check that $\Psi(T)\Psi(T') = \Psi(T \cup T')$ when $T, T'$ are tableaux whose columns are fundamental tableaux. It is clear that the columns of the product $T \cup T'$ are also fundamental tableaux. By definition, the value of $\Psi$ on a tableau whose columns are fundamental tableaux is product of the values of $\Psi$ on every column of the tableau. It follows that $\Psi(T)\Psi(T') = \Psi(T \cup T')$.

We now check that both composites $\Psi \tilde{\Phi}$ and $\tilde{\Phi} \Psi$ are the identity map. It suffices to check this on generators. For any $i \in I$, $p \in [i]$, by Lemma 4.5 and the definition of $\Psi$, we have
\[
\Psi \tilde{\Phi}(Y_{i,-2p}) = \Psi(T^{(i,p)}) = \Psi(T^{(p,k+p-i)}) = Y_{i,i-2p}.
\]
Every fundamental tableau in $\text{SSYT}(k-1, [k], \sim)$ is a one-column tableau of the form $T_{(l,a)}$ for some $a \in [2, k]$ and $l \in [a-1]$. We have
\[
\tilde{\Phi} \Psi(T_{(l,a)}) = \tilde{\Phi}(Y_{l+k-a,k-a-l}) = T^{(l+k-a,l)} = T_{(l,a)}.
\]
\hfill \square
Table 1. Correspondence between fundamental monomials and fundamental tableaux in SSYT(4,[5],∼). Since all tableaux in the table are one-column tableaux, we represent them by their entries.

| module | tableau |
|--------|---------|
| L(1,1) | {5}     |
| L(1,0) | {3}     |
| L(1,1) | {2}     |
| L(1,2) | {1,4}   |
| L(1,2) | {1,5}   |
| L(2,0) | {2}     |
| L(2,1) | {1,3}   |
| L(2,2) | {1,2,4} |
| L(3,0) | {1,2,3,5} |

In Table 1, the first column consists of all fundamental modules in $C_{5,\triangle}$ and the second column consists of the corresponding fundamental tableaux in SSYT(4,[5],∼).

**Definition 4.7.** For a tableau $T \in \text{SSYT}(k-1,[k],\sim)$, we define an element $\mu_{C[N]}(T) \in \mathbb{C}[N]$ (resp., $\mu_{C[SL_k]}(T) \in \mathbb{C}[SL_k]$) to be the $\Phi_{C[N]}([L(M_T)])$ (resp., $\Phi_{C[SL_k]}([L(M_T)])$).

Usually we write $\mu_{C[N]}(T)$ (respectively, $\mu_{C[SL_k]}(T)$) as $\mu(T)$ when we know that we are working on $\mathbb{C}[N]$ (respectively, $\mathbb{C}[SL_k]$).

By Theorems 1.1, 1.2, and 6.1 in [36] and Theorem 4.6, we have that following.

**Theorem 4.8.** The set $\{\mu_{C[N]}(T) : T \in \text{SSYT}(k-1,[k],\sim)\}$ (respectively, $\{\mu_{C[SL_k]}(T) : T \in \text{SSYT}(k-1,[k],\sim)\}$) is the dual canonical basis of $\mathbb{C}[N]$ (respectively, $\mathbb{C}[SL_k]$).

5. **Formula for elements in the dual canonical basis**

In this section, we give an explicit formula for every element $\mu_{C[N]}(T)$ (respectively, $\mu_{C[SL_k]}(T)$) in the dual canonical basis of $\mathbb{C}[N]$ (respectively, $\mathbb{C}[SL_k]$).

5.1. **Formula for $\mu(T)$**. Let $T \in \text{SSYT}(k-1,[k],\sim)$ be a tableau which is $\sim$-equivalent to a tableaux $T'$ whose columns are fundamental tableaux and which has $m$ columns. We have that the columns of $T'$ are $T'_{(i,j)}$, $i = 1, \ldots, m$, for some $a_1, \ldots, a_m \in [k-1], b_1, \ldots, b_m \in [k]$. Denote $p_T = \{(a_i,b_i) : i \in [m]\}$ (as a multi-set). We define $i_T = (i_1, \ldots, i_m)$ and $j_T = (j_1, \ldots, j_m)$, where $i_1 \leq \cdots \leq i_m$ are $a_1, \ldots, a_m$ written in weakly increasing order and $j_1 \leq \cdots \leq j_m$ are the elements $b_1, \ldots, b_m$ written in weakly increasing order. For $c = (c_1, \ldots, c_m), d = (d_1, \ldots, d_m) \in \mathbb{Z}^m$, we denote $p_{c,d} = \{(c_i,d_i) : i \in [m]\}$ (as a multi-set).

Let $S_m$ be the symmetric group on $[m]$. Denote by $\ell(w)$ the length of $w \in S_m$ and denote by $w_0 \in S_m$ be the longest permutation. For $i = (i_1, \ldots, i_m) \in \mathbb{Z}^m$, denote by $S_i$ the subgroup of $S_m$ consisting of elements $\sigma$ such that $i_{\sigma(j)} = i_j$, $j \in [m]$. It is clear that for $i,j \in \mathbb{Z}^m,$
Example 5.2. Let \( \Delta \) and \( \Delta u \) be a non-archimedean local field. Complex, smooth representations of \( GL_n(F) \) of finite length are parameterized by multisegments \( [a, b] \), where \( a, b \in \mathbb{Z}, a \leq b \).

By quantum Schur-Weyl duality [17, Section 7.6], there is a correspondence between multisegments and dominant monomials

\[
[a, b] \mapsto Y_{b-a+1,a+b-1}, \quad Y_{i,s} \mapsto \left[ \frac{s - i + 2}{2}, \frac{s + i}{2} \right].
\]

Denote by \( M_m \) the monomial corresponding to a multisegment \( m \) and \( m_M \) the multisegment corresponding to a monomial \( M \).

We interpret \( M_{[a, a-1]} \) as the trivial monomial \( 1 \in \mathcal{P}^+ \) and interpret \( M_{[a, b]} \) with \( b < a - 1 \) as 0. For any \( m \)-tuples \( (\mu, \lambda) \in \mathbb{Z}^m \times \mathbb{Z}^m \), we define a multi-set:

\[
\text{Fund}_M(\mu, \lambda) = \{ M_{[\mu_i, \lambda_i]} : i \in [m] \}.
\]

For \( \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{Z}^m \), denote by \( S_\lambda \) the subgroup of \( S_m \) consisting of elements \( \sigma \) such that \( \lambda_{\sigma(i)} = \lambda_i \). For \( \mu = (\mu_1, \ldots, \mu_m) \), \( \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{Z}^m \), we denote \( m_{\mu, \lambda} = \sum_{i=1}^{m} [\mu_i, \lambda_i] \).
For a multisegment $\mathbf{m}$ with $m$ terms, there exist unique weakly decreasing tuples $\mu_{\mathbf{m}}, \lambda_{\mathbf{m}} \in \mathbb{Z}^m$ and unique permutation of maximal length $w_{\mathbf{m}} \in S_m$ such that $\mathbf{m} = \mathbf{w}_{\mathbf{m}} \mu_{\mathbf{m}}, \lambda_{\mathbf{m}}$ ([11, Sections 2.4, 2.5], [41, Proposition 2.3], and [10, Proposition 2.7]). Note that for any $w, w' \in S_m$ and any $\mu, \lambda \in \mathbb{Z}^m$, $\mathbf{m}_{w' \mu, \lambda} = \mathbf{m}_{w \mu, \lambda}$ if and only if $w' \in S_{\lambda} w S_{\mu}$. The element $w_{\mathbf{m}} \in S_m$ is also the unique permutation of maximal length in $S_{\lambda_{\mathbf{m}}} w_{\mathbf{m}} S_{\mu_{\mathbf{m}}}$. We write $\lambda_{\mathbf{m}} = \lambda_M, \mu_{\mathbf{m}} = \mu_M, w_{\mathbf{m}} = w_M$ for $M = M_{\mathbf{m}}$.

**Proof of Theorem 5.3.** We will prove the formula (5.1) for $\text{ch} \widetilde{C}_{N^-}(T)$. The proof of the formula (5.2) for $\text{ch} \widetilde{C}_{\mathbb{C}[SL_k]^N}(T)$ is the same.

For every finite dimensional $U_q(\mathfrak{sl}_k)$-module $L(M)$, we have that

\[
\chi_q(L(M)) = \sum_{w \in S_m} (-1)^{\ell(w_{\mathbf{m}})} p_{\mathbf{w}_{\mathbf{m}} w_{\mathbf{m}} w_0}(1) \prod_{M' \in \text{Fund}_M(u_{\mu_{\mathbf{m}}, \lambda_{\mathbf{m}}})} \chi_q(L(M')).
\]

This formula (see Section 5.2 in [19]) is obtained from a result due to Arakawa-Suzuki [2] (see also Section 10.1 in [49], and [3, 34]) and from the quantum affine Schur-Weyl duality [17]. In (5.4), we interpret $\chi_q(L(M_{[a, a-1]})) = 1$ and $\chi_q(L(M_{[a, b]})) = 0$ if $b < a - 1$.

By (5.3) and Theorem 4.6, there is a correspondence between multisegments and tableaux induced by the following correspondence between segments and fundamental tableaux:

\[
[\mu, \lambda] \mapsto T_{(1-\mu, k-\lambda)}, \quad T_{(l, a)} \mapsto [1-l, k-a],
\]

where $T_{(1-\mu, k-\lambda)}$ is the one-column tableau with entries $\{1, 2, \ldots, -\mu, k - \lambda\}$. Denote by $T_m$ the tableau corresponding to the multisegment $\mathbf{m}$ and denote by $\mathbf{m}_T$ the multisegment corresponding to the tableau $T$.

Denote $\mathbf{i}_T = (i_1, \ldots, i_m)$, $\mathbf{j}_T = (j_1, \ldots, j_m)$. By (5.5), we have that $i_a = 1 - \mu_a, j_a = k - \lambda_a$ for $a \in [k]$. Therefore $w_T$ defined in Subsection 5.1 and $w_{\mathbf{m}_T}$ defined in this subsection are the same.

Apply the isomorphism $\Phi_{\mathbb{C}[N]}$ in Theorem 2.1 and the isomorphism $\widetilde{\Phi}$ in Theorem 4.6 to the formula (5.4), we obtain the formula (5.1).

\[\square\]

**Remark 5.4.** The difference between the formulas for $\text{ch} \mathbb{C}[N](T)$ and $\text{ch} \widetilde{C}_{\mathbb{C}[SL_k]^N}(T)$ is that the flag minors in (5.1) are flag minors in $\mathbb{C}[N]$ while the flag minors in (5.2) are flag minors in $\mathbb{C}[SL_k]^N$.

For example, in $\mathbb{C}[SL_4]^N$ and $\mathbb{C}[N]$, we have that

\[
\text{ch}(\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}) = \Delta_4 \Delta_{124} - \Delta_4 \Delta_{123}.
\]

On the other hand, in $\mathbb{C}[N]$, this is equal to $x_{15} x_{34} - x_{14} = \Delta_{13, 34}$.

We give an example of a computation of $\text{ch}(T)$.
Example 5.5. We take $T = \begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{array} \in \text{SSYT}(5, [6], \sim)$ as in Example 5.2. Then $i_T = (1, 2, 2, 3, 3), i_T = (2, 6, 4, 5, 6), \text{and } w_T = s_2 s_4$. By Theorem 5.3, we have that
\[
ch(T) = \Delta_2 \Delta_4 \Delta_1 \Delta_2 \Delta_2 \Delta_2 + \Delta_3 \Delta_5 \Delta_2 \Delta_2 \Delta_2 \Delta_2 + \Delta_2 \Delta_3 \Delta_4 \Delta_2 \Delta_2 \Delta_2 \\
+ \Delta_5 \Delta_4 \Delta_2 \Delta_2 \Delta_2 \Delta_2 + \Delta_4 \Delta_2 \Delta_2 \Delta_2 \Delta_2 \Delta_2 - \Delta_3 \Delta_2 \Delta_2 \Delta_2 \Delta_2 \Delta_2 \\
- \Delta_2 \Delta_2 \Delta_2 \Delta_2 \Delta_2 \Delta_2 - \Delta_2 \Delta_2 \Delta_2 \Delta_2 \Delta_2 \Delta_2 - \Delta_2 \Delta_2 \Delta_2 \Delta_2 \Delta_2 \Delta_2.
\] (5.6)

Recall that in Section 4.2, for a fraction $ST^{-1}$ of two tableaux $S, T$, we denote $\Delta_{ST^{-1}} = \Delta_S \Delta_T^{-1}$. For $T \in \text{SSYT}(k - 1, [k])$, we have that $T = T'' \cup T'$, where $T'$ is a tableau whose columns are fundamental tableaux and $T''$ is a fraction of two trivial tableaux. Define $\text{ch}'(T) = \Delta_{T'} \text{ch}_{C[SL_k]}^{-1}(T')$. We have the following conjecture.

Conjecture 5.6. For every $T \in \text{SSYT}(k - 1, [k]), \text{ch}'(T) \in \mathbb{C}[SL_k]^{N^-}$. Moreover, $\{\text{ch}'(T) : T \in \text{SSYT}(k - 1, [k])\}$ is the dual canonical basis of $\mathbb{C}[SL_k]^{N^-}$.

We give an example to explain Conjecture 5.6.

Example 5.7. We take $T = \begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{array} \in \text{SSYT}(5, [6])$. Then $T = T'' \cup T'$, where $T' = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 5 & 6 \end{array}$, $T'' = \begin{array}{cc} 1 & 1 \\ 1 & 2 \\ 2 & 2 \end{array}$. We have that

$\text{ch}'(T) = \Delta_{T'} \text{ch}_{C[SL_k]}^{-1}(T') = \Delta_{136} \Delta_{245} - \Delta_{126} \Delta_{345} \in \mathbb{C}[SL_k]^{N^-}$,

where $\text{ch}(T')$ is equal to (5.6).

6. Mutation of Tableaux

In this section, we give a mutation rule for the cluster algebra $\mathbb{C}[N]$ (respectively, $\mathbb{C}[SL_k]^{N^-}$) using tableaux.

A finite dimensional $U_q(\mathfrak{g})$-module is called prime if it is not isomorphic to a tensor product of two nontrivial $U_q(\mathfrak{g})$-modules (cf. [18]). A simple $U_q(\mathfrak{g})$-module $M$ is real if $M \otimes M$ is simple (cf. [48]). We say that a tableau $T \in \text{SSYT}(k - 1, [k], \sim)$ is real (respectively, prime) if $M_T$ is real (respectively, prime).

By Theorem 4.8, every element in the dual canonical basis of $\mathbb{C}[N]$ (respectively, $\mathbb{C}[SL_k]^{N^-}$) is of the form $\text{ch}(T), T \in \text{SSYT}(k - 1, [k], \sim)$. In [42, 56], it is shown that cluster monomials in $\mathbb{C}[N]$ (respectively, $\mathbb{C}[SL_k]^{N^-}$) belong to the dual canonical basis and they correspond to real modules in $C_k, \Delta$. The cluster variables in $\mathbb{C}[N]$ (respectively, $\mathbb{C}[SL_k]^{N^-}$) correspond to real prime modules in $C_k, \Delta$. Therefore cluster monomials (respectively, cluster variables) in
\( \mathbb{C}[N] \) (respectively, \( \mathbb{C}[SL_k]^{-} \)) are also of the form \( \text{ch}(T) \), where \( T \) is a real (respectively, real prime) tableau in \( \text{SSYT}(k-1, [k], \sim) \).

In [19, Section 4], it is shown that the mutation rule in Grassmannian cluster algebras can be described using semi-standard Young tableaux of rectangular shape. Similarly, we now show that the mutation rule in \( \mathbb{C}[N] \) (respectively, \( \mathbb{C}[SL_k]^{-} \)) can be described using semi-standard Young tableaux.

Starting from the initial seed of \( \mathbb{C}[N] \) (respectively, \( \mathbb{C}[SL_k]^{-} \)), each time we perform a mutation at a cluster variable \( \text{ch}(T_r) \), we obtain a new cluster variable \( \text{ch}(T'_r) \) defined recursively by

\[
\text{ch}(T'_r)\text{ch}(T_r) = \prod_{i \rightarrow r} \text{ch}(T_i) + \prod_{r \rightarrow i} \text{ch}(T_i),
\]

where \( \text{ch}(T_i) \) the cluster variable at the vertex \( i \). On the other hand, by Theorem 2.1 and the formula (4.4), we have that

\[
(6.1) \quad \text{ch}(T_r)\text{ch}(T'_r) = \text{ch}(T_r \cup T'_r) + \sum_{T''} c_{T''} \text{ch}(T'')
\]

for some \( T'' \in \text{SSYT}(k-1, [k], \sim) \), \( \text{wt}(T'') < \text{wt}(T_r \cup T'_r) \), \( c_{T''} \in \mathbb{Z}_{\geq 0} \). Therefore one of the two tableaux \( \cup_{i \rightarrow r} T_i \) or \( \cup_{r \rightarrow i} T_i \) has strictly greater weight than the other, and moreover the one with higher weight is equal to \( T_r \cup T'_r \) in \( \text{SSYT}(k-1, [k], \sim) \). Denote by \( \max\{\cup_{i \rightarrow r} T_i, \cup_{r \rightarrow i} T_i\} \) this higher weight tableau. Then

\[
(6.2) \quad T'_r = T_r^{-1} \max\{\cup_{i \rightarrow r} T_i, \cup_{r \rightarrow i} T_i\}.
\]

**Remark 6.1.** There is a partial order called *dominance order* in the set of semi-standard Young tableaux (cf. [12, Section 5.5]).

Let \( \lambda = (\lambda_1, \ldots, \lambda_{\ell}) \), \( \mu = (\mu_1, \ldots, \mu_{\ell}) \), with \( \lambda_1 \geq \cdots \geq \lambda_{\ell} \geq 0 \), \( \mu_1 \geq \cdots \geq \mu_{\ell} \geq 0 \), be partitions. Then \( \lambda \leq_{\text{dom}} \mu \) in the dominance order if \( \sum_{j \leq i} \lambda_j \leq \sum_{j \leq i} \mu_j \) for \( i = 1, \ldots, \ell \).

For a semi-standard tableau \( T \) in \( \text{SSYT}(k, [m]) \) and \( i \in [m] \), denote by \( T[i] \) the sub-tableau obtained from \( T \) by restriction to the entries in \( [i] \). For a tableau \( T \), let \( \text{sh}(T) \) denote the shape of \( T \). For \( T, T' \in \text{SSYT}(k, [m]) \) of the same shape, \( T \leq_{\text{dom}} T' \) in the dominance order if for every \( i \in [i] \), \( \text{sh}(T[i]) \leq_{\text{dom}} \text{sh}(T'[i]) \) in the dominance order on partitions.

The *content* of a tableau \( T \in \text{SSYT}(k, [m]) \) is the vector \( (\nu_1, \ldots, \nu_m) \in \mathbb{Z}^m \), where \( \nu_i \) is the number of \( i \)-filled boxes in \( T \). By a similar proof as the proof of Proposition 3.28 in [19], for \( T, T' \in \text{SSYT}(k-1, [k]) \) with the same content and with the same shape, \( T \leq_{\text{dom}} T' \) in the dominance order if and only if \( M_T \leq M_{T'} \in \mathbb{P}^+ \) in the monomial order in (2.2).

In the mutation described above, if we use tableaux in \( \text{SSYT}(k-1, [k]) \) (not other tableau representatives of equivalence classes in \( \text{SSYT}(k-1, [k], \sim) \)), then in every step, \( \cup_{i \rightarrow r} T_i \) and \( \cup_{r \rightarrow i} T_i \) have the same shape and the same content. Therefore in the mutations, one can also use tableaux in \( \text{SSYT}(k-1, [k]) \) and use the dominance order on tableaux to compute \( \max\{\cup_{i \rightarrow r} T_i, \cup_{r \rightarrow i} T_i\} \) in (6.2).

**Example 6.2.** The following are some examples of exchange relations in \( \mathbb{C}[N] \), \( N \subset SL_6 \), (respectively, \( \mathbb{C}[SL_6]^{-} \)): \( \text{ch}(T_1)\text{ch}(T_2) = \text{ch}(T_3)\text{ch}(T_4)\text{ch}(T_5) + \text{ch}(T_6)\text{ch}(T_7)\text{ch}(T_8) \), where \( T_i \)'s
are the following tableaux respectively

for some \( i_1 < \cdots < i_m \) is the set of entries in \( T \). For tableaux above with two or more columns, we have
that
\[
\begin{align*}
\text{ch}(T_1) &= \Delta_{124} \Delta_3 - \Delta_{123} \Delta_4, \\
\text{ch}(T_2) &= \Delta_{125} \Delta_3 - \Delta_{123} \Delta_5, \\
\text{ch}(T_3) &= \Delta_{125} \Delta_4 - \Delta_{123} \Delta_4, \\
\text{ch}(T_4) &= \Delta_{125} \Delta_{13} \Delta_4 + \Delta_{123} \Delta_{14} \Delta_5 - \Delta_{123} \Delta_{15} \Delta_4 - \Delta_{124} \Delta_{13} \Delta_5, \\
\text{ch}(T_5) &= \Delta_{123} \Delta_1 \Delta_2 \Delta_5 + \Delta_{12} \Delta_{124} \Delta_3 \Delta_5 + \Delta_{125} \Delta_{13} \Delta_2 \Delta_4 \\
&\quad - \Delta_{123} \Delta_{15} \Delta_2 \Delta_4 - \Delta_{12} \Delta_{125} \Delta_3 \Delta_4 - \Delta_{124} \Delta_{13} \Delta_2 \Delta_5, \\
\text{ch}(T_6) &= \Delta_{12} \Delta_{124} \Delta_1 \Delta_5 \Delta_3 - \Delta_{12} \Delta_{123} \Delta_1 \Delta_5 \Delta_4 - \Delta_{124} \Delta_1 \Delta_3 \Delta_5 \Delta_2 \\
&\quad - \Delta_{12} \Delta_{125} \Delta_1 \Delta_4 \Delta_3 + \Delta_{12} \Delta_{123} \Delta_1 \Delta_5 \Delta_5 + \Delta_{125} \Delta_1 \Delta_3 \Delta_4 \Delta_2, \\
\text{ch}(T_7) &= \Delta_{123} \Delta_{14} \Delta_5 - \Delta_{123} \Delta_{15} \Delta_4 - \Delta_{124} \Delta_{13} \Delta_5 + \Delta_{124} \Delta_{15} \Delta_3, \\
\text{ch}(T_8) &= \Delta_{1235} \Delta_{124} \Delta_4 + \Delta_{123} \Delta_{1234} \Delta_5 - \Delta_{1234} \Delta_{125} \Delta_3 - \Delta_{123} \Delta_{1235} \Delta_4, \\
\text{ch}(T_9) &= \Delta_{1235} \Delta_4 - \Delta_{1234} \Delta_5, \\
\text{ch}(T_{10}) &= \Delta_{1235} \Delta_{14} - \Delta_{1234} \Delta_{15}, \\
\text{ch}(T_{11}) &= \Delta_{1235} \Delta_{14} \Delta_2 + \Delta_{12} \Delta_{1234} \Delta_5 - \Delta_{1234} \Delta_1 \Delta_5 \Delta_2 - \Delta_{12} \Delta_{1235} \Delta_4, \\
\text{ch}(T_{12}) &= \Delta_{1235} \Delta_{14} \Delta_3 + \Delta_{1234} \Delta_1 \Delta_5 \Delta_3 - \Delta_{1234} \Delta_{13} \Delta_4, \\
\text{ch}(T_{13}) &= \Delta_{1234} \Delta_{125} \Delta_{13} \Delta_4 - \Delta_{123} \Delta_{1234} \Delta_1 \Delta_5 \Delta_4 - \Delta_{1234} \Delta_{125} \Delta_1 \Delta_5 \Delta_3 \\
&\quad + \Delta_{12} \Delta_{1234} \Delta_1 \Delta_5 \Delta_5 - \Delta_{1235} \Delta_{124} \Delta_1 \Delta_3 \Delta_4 + \Delta_{1235} \Delta_{124} \Delta_1 \Delta_4 \Delta_3, \\
\text{ch}(T_{14}) &= \Delta_{1235} \Delta_{124} \Delta_{15} \Delta_3 + \Delta_{1234} \Delta_{125} \Delta_{13} \Delta_5 + \Delta_{123} \Delta_{1235} \Delta_{14} \Delta_5 \\
&\quad - \Delta_{123} \Delta_{1235} \Delta_{15} \Delta_4 - \Delta_{1234} \Delta_{125} \Delta_{15} \Delta_3 - \Delta_{1235} \Delta_{124} \Delta_{13} \Delta_5.
\end{align*}
\]

7. Application to Classification of Cluster Variables in $\mathbb{C}[N]$

In this section, we apply the results in previous sections to classify cluster variables in $\mathbb{C}[N]$, in the case of $N \subset SL_6$, up to 4-column tableaux.

We say that a tableau is of rank $r$ if the tableau has $r$ columns. We say that a tableau $T$ is a cluster variable if $\text{ch}(T)$ is a cluster variable. We say that a tableau $T$ is real if $\text{ch}(T)$ satisfies $\text{ch}(T \cup T') = \text{ch}(T)\text{ch}(T')$. That is, $T$ is real if and only if the corresponding $U_q(\widehat{\mathfrak{sl}}_k)$-module $L(M_T)$ is real.

For $r \in \mathbb{Z}_{\geq 2}$, we call $T_1, \ldots, T_r \in \text{SSYT}(k, [n])$ compatible if $\text{ch}(T_1) \cdots \text{ch}(T_r) = \text{ch}(T_1 \cup \cdots \cup T_r)$.

By [42] and [56], all cluster variables are real and prime. Therefore we first classify real prime tableaux in $\text{SSYT}(k - 1, [k])$.

All rank 1 tableaux are cluster variables. There are 52 rank 1 cluster variables (not including frozen variables) in $\mathbb{C}[N]$, $N \subset SL_6$. These tableaux are in $\text{SSYT}(5, [6])$.

There are 1652 semistandard tableaux of rank 2 in $\text{SSYT}(6, [6])$. There are 1533 compatible pairs of rank 1 tableaux in $\text{SSYT}(6, [6])$. Therefore there are 119 prime tableaux of rank 2 in $\text{SSYT}(6, [6])$. These tableaux are all in $\text{SSYT}(5, [6])$. Among them 118 tableaux are cluster
Table 2. Correspondence among fundamental monomials, positive roots, and good Lyndon words.

\[
\begin{array}{cccccccccccc}
Y_{5,3} & Y_{5,1} & Y_{5,-3} & Y_{5,-5} & Y_{4,2} & Y_{4,-2} & Y_{4,3} & Y_{3,1} & Y_{3,-3} & Y_{2,0} & Y_{2,-2} & Y_{1,-1} \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_{12} & \alpha_{23} & \alpha_{34} & \alpha_{45} & \alpha_{13} & \alpha_{24} & \alpha_{14} \\
1 & 2 & 3 & 4 & 5 & 12 & 23 & 34 & 45 & 13 & 24 & 14 \\
\end{array}
\]

variables. The only rank 2 prime tableau which is not cluster variable is \( T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 0 \end{pmatrix} \). We have that

\[
\text{ch}(T) = -2 \Delta_{123} \Delta_{1234} \Delta_{16} \Delta_5 - \Delta_{1235} \Delta_{124} \Delta_{16} \Delta_3 - \Delta_{123} \Delta_{1236} \Delta_{15} \Delta_4 - \Delta_{1234} \Delta_{125} \Delta_{13} \Delta_6 - \Delta_{1234} \Delta_{126} \Delta_{15} \Delta_3 - \Delta_{1234} \Delta_{1235} \Delta_{14} \Delta_6 - \Delta_{1236} \Delta_{124} \Delta_{13} \Delta_5 + 2 \Delta_{123} \Delta_{1234} \Delta_{15} \Delta_6 + \Delta_{1236} \Delta_{124} \Delta_{15} \Delta_3 + \Delta_{123} \Delta_{1235} \Delta_{16} \Delta_4 + \Delta_{1234} \Delta_{126} \Delta_{13} \Delta_5 + \Delta_{1234} \Delta_{125} \Delta_{16} \Delta_3 + \Delta_{123} \Delta_{1236} \Delta_{14} \Delta_5 + \Delta_{1235} \Delta_{124} \Delta_{13} \Delta_6.
\]

and

\[
\text{ch}(T)\text{ch}(T) = \text{ch}(T \cup T) + \text{ch}(\begin{pmatrix} 3 & 5 \\ 4 & 6 \\ 3 & 5 \\ 6 \end{pmatrix}).
\]

The tableau \( T \) corresponds to the simple \( U_q(\widehat{\mathfrak{sl}_6}) \)-module \( L(Y_{3,-1}Y_{4,-4}Y_{4,2}Y_{5,-1}) \), see Theorem 4.6.

A non-real element in the canonical basis of \( U_q(\mathfrak{n}) \), \( \mathfrak{n} \) is a maximal nilpotent subalgebra of \( \mathfrak{sl}_6 \), is given in Section 2.7 in [48]. Using the generalized quantum affine Schur-Weyl duality [17, 43], and compare the initial quivers in Figure 1, Figure 2, and the figure in Theorem 1.2 in [14], we have the following correspondence among fundamental l-weights, good Lyndon words, positive roots in Table 2, where in the table \( \alpha_{ij} = \alpha_i + \cdots + \alpha_j \), \( \alpha_i \)'s are simple roots. Therefore the vector \( b \) in Section 2.7 in [48] corresponds to the non-real \( U_q(\widehat{\mathfrak{sl}_6}) \)-module \( L(Y_{2,-2}Y_{3,-5}Y_{3,1}Y_{4,-2}) \). This module is very similar to \( L(Y_{3,-1}Y_{4,-4}Y_{4,2}Y_{5,-1}) \) that is obtained from the tableau above.

There are 25740 semistandard tableaux of rank 3 in SSYT(6, [6]). There are 21657 compatible triples of rank 1 tableaux in SSYT(6, [6]). There are 3913 compatible pairs of a rank 2 prime tableau and a rank 1 tableau in SSYT(6, [6]). Therefore there are 170 prime tableaux of rank 3 in SSYT(6, [6]). These tableaux are all in SSYT(5, [6]). All of these tableaux are cluster variables in \( \mathbb{C}[N] \).

There are 279279 semistandard tableaux of rank 4 in SSYT(6, [6]). There are 212127 compatible 4-tuples of rank 1 tableaux in SSYT(6, [6]). There are 60966 compatible triples of two rank 1 tableaux and a rank 2 prime tableau in SSYT(6, [6]). There are 4322 compatible pairs of a rank 1 tableau and a rank 3 prime tableau in SSYT(6, [6]). There are 1649 compatible pairs of a rank 2 prime tableau and a rank 2 prime tableau in SSYT(6, [6]). Therefore there are 215 prime tableaux of rank 4 in SSYT(6, [6]). These tableaux are all in SSYT(5, [6]). Among the 215 prime tableaux, 214 of them are cluster variables in \( \mathbb{C}[N] \). We
conjecture that the remaining one tableau

\[
\begin{array}{cccc}
1 & 1 & 2 & 4 \\
2 & 3 & 4 & \\
3 & 5 & 6 & \\
5 & & & \\
6 & & & \\
\end{array}
\]

is not real.

The computations in this section use SageMath 9.6 [59]. The codes and data are available on the webpage: https://sites.google.com/view/jianrong-li.

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