A characterization of $Q$-polynomial association schemes

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Abstract

We prove a necessary and sufficient condition for a symmetric association scheme to be a $Q$-polynomial scheme.

Key words: $Q$-polynomial association scheme, $s$-distance set.

1 Introduction

A symmetric association scheme of class $d$ is a pair $X = (X, \{R_i\}_{i=0}^d)$, where $X$ is a finite set and each $R_i$ is a nonempty subset of $X \times X$ satisfying the following:

1. $R_0 = \{(x, x) \mid x \in X\}$,
2. $X \times X = \bigcup_{i=0}^d R_i$ and $R_i \cap R_j$ is empty if $i \neq j$,
3. $\forall R_i$ for any $i \in \{0, 1, \ldots, d\}$, where $\forall R_i = \{(y, x) \mid (x, y) \in R_i\}$,
4. for all $i, j, k \in \{0, 1, \ldots, d\}$, there exist integers $p_{ij}^k$ such that for all $x, y \in X$ with $(x, y) \in R_k$,

$$p_{ij}^k = |\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}|.$$

The integers $p_{ij}^k$ are called the intersection numbers.

Let $X$ be a symmetric association scheme. The $i$-th adjacency matrix $A_i$ of $X$ is the matrix with rows and columns indexed by $X$ such that the $(x, y)$-entry is $1$ if $(x, y) \in R_i$ or $0$ otherwise. The Bose–Mesner algebra of $X$ is the algebra generated by the adjacency matrices $\{A_i\}_{i=0}^d$ over the complex field $\mathbb{C}$. Then $\{A_i\}_{i=0}^d$ is a natural basis of the Bose–Mesner algebra. By [2] page 59, the Bose–Mesner algebra has a second basis $\{E_i\}_{i=0}^d$ such that

1. $E_0 = |X|^{-1} J$, where $J$ is the all-ones matrix,
2. $I = \sum_{i=0}^d E_i$, where $I$ is the identity matrix,
3. $E_i E_j = \delta_{ij} E_i$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

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The basis $\{E_i\}_{i=0}^d$ is called the primitive idempotents of $\mathfrak{X}$. We have the following equations:

\[ A_i = \sum_{j=0}^d p_i(j) E_j, \quad (1.1) \]
\[ E_i = \frac{1}{|X|} \sum_{j=0}^d q_i(j) A_j, \quad (1.2) \]
\[ A_i A_j = \sum_{k=0}^d q_{ij}^k A_k, \quad (1.3) \]
\[ E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^d q_{ij}^k E_k, \quad (1.4) \]

where $\circ$ denotes the Hadamard product, that is, the entry-wise matrix product. The matrices $P = (p_i(i))_{i,j=0}^d$ and $Q = (q_i(i))_{i,j=0}^d$ are called the first and second eigenmatrices, respectively. The numbers $q_{ij}^k$ are called the Krein parameters. The Krein parameters are nonnegative real numbers (the Krein condition) \[11\] [2] page 69.

A symmetric association scheme is called a $P$-polynomial scheme (or a metric scheme) with respect to the ordering $\{A_i\}_{i=0}^d$ if for each $i \in \{0, 1, \ldots, d\}$, there exists a polynomial $v_i$ of degree $i$ such that $p_i(j) = v_i(p_1(j))$ for any $j \in \{0, 1, \ldots, d\}$. We say a symmetric association scheme is a $P$-polynomial scheme with respect to $A_1$ if it has the $P$-polynomial property with respect to some ordering $A_0, A_1, A_2, A_3, \ldots, A_d$. Dually a symmetric association scheme is called a $Q$-polynomial scheme (or a cometric scheme) with respect to the ordering $\{E_i\}_{i=0}^d$ if for each $i \in \{0, 1, \ldots, d\}$, there exists a polynomial $v^*_i$ of degree $i$ such that $q_i(j) = v^*_i(q_1(j))$ for any $j \in \{0, 1, \ldots, d\}$. Moreover a symmetric association scheme is called a $Q$-polynomial scheme with respect to $E_1$ if it has the $Q$-polynomial property with respect to some ordering $E_0, E_1, E_2, E_3, \ldots, E_d$. Note that both $\{v_i\}_{i=0}^d$ and $\{v^*_i\}_{i=0}^d$ form systems of orthogonal polynomials.

Throughout this paper, we use the notation $m_i = q_i(0)$ and $\theta^*_i = q_i(i)$ for $0 \leq i \leq d$. If an association scheme is $Q$-polynomial, then $\{\theta^*_i\}_{i=0}^d$ are mutually distinct because the second eigenmatrix $Q = (v^*_i(\theta^*_j))_{i,j=0}^d$ is non-singular. For a univariate polynomial $f$ and a matrix $M$, we denote by $f(M^2)$ the matrix obtained by substituting $M$ into $f$ with multiplication the Hadamard product. We introduce known equivalent conditions of the $Q$-polynomial property of symmetric association schemes [2] page 193. The following are equivalent:

1. $\mathfrak{X}$ is a $Q$-polynomial scheme with respect to the ordering $\{E_i\}_{i=0}^d$.
2. $\{q^i_{i,j}\}_{i,j=0}^d$ is an irreducible tridiagonal matrix.
3. For each $i \in \{0, 1, \ldots, d\}$, there exists a polynomial $f_i$ of degree $i$ such that $E_i = f_i(E^d_1)$.

In the present paper, we prove a new necessary and sufficient condition for a symmetric association scheme to be $Q$-polynomial. Since the $Q$-polynomial property of a symmetric association scheme of class 1 is trivial, we assume that $d$ is greater than 1.
Theorem 1.1. Let $X$ be a symmetric association scheme of class $d \geq 2$. Suppose that $\{\theta_j^*\}_{j=0}^d$ are mutually distinct. Then the following are equivalent:

1. $X$ is a $Q$-polynomial scheme with respect to $E_1$.
2. There exists $l \in \{2, 3, \ldots, d\}$ such that for any $i \in \{1, 2, \ldots, d\}$,
   $$\prod_{j \neq i}^d \frac{\theta_0^* - \theta_j^*}{\theta_i^* - \theta_j^*} = -p_i(l).$$

Moreover if (2) holds, then $l = i_d$.

Remark 1.2. We call a finite set $X$ in $\mathbb{R}^m$ a $d$-distance set if the number of the Euclidean distances between distinct two points in $X$ is equal to $d$. Larman–Rogers–Seidel [7] proved that if the size of a two-distance set with the distances $a, b$ ($a < b$) is greater than $2m + 3$, then there exists a positive integer $k$ such that $a^2/b^2 = (k - 1)/k$, i.e. $k = b^2/(b^2 - a^2)$. Bannai–Bannai [1] proved that the ratio $k$ of the spherical embedding of a primitive association scheme of class 2 coincides with $-p_i(2)$. The research of the present paper is motivated by [1]. For a symmetric association scheme satisfying that $\{\theta_j^*\}_{j=0}^d$ are mutually distinct, the values $K_i := \prod_{j \neq i}^d (\theta_0^* - \theta_j^*)(\theta_i^* - \theta_j^*)^{-1} (1 \leq i \leq d)$ are the generalized Larman–Rogers–Seidel’s ratios [10] of the spherical embedding of this association scheme with respect to $E_1$. Theorem 1.1 is an extension of Bannai–Bannai’s result to $Q$-polynomial schemes of any class. Furthermore Theorem 1.1 is a new characterization of the $Q$-polynomial property on the spherical embedding of a symmetric association scheme.

At the end of this paper, we give some sufficient conditions for the integrality of $K_i$.

2 Proof of Theorem 1.1

First we give several lemmas that will be needed to prove Theorem 1.1.

Lemma 2.1. For any mutually distinct real numbers $\beta_1, \beta_2, \ldots, \beta_s$, the following identity holds.

$$\sum_{i=1}^s \beta_i^j \prod_{k \neq i}^s \frac{x - \beta_k}{\beta_i - \beta_k} = x^j$$

for any $j \in \{0, 1, \ldots, s - 1\}$, where $x$ is a variable.

Proof. For each $j \in \{0, 1, \ldots, s - 1\}$, the polynomial

$$L_j(x) := \sum_{i=1}^s \beta_i^j \prod_{k \neq i}^s \frac{x - \beta_k}{\beta_i - \beta_k}$$

of degree at most $s - 1$ is known as the interpolation polynomial in the Lagrange form (see [3]). Namely, the property $L_j(\beta_i) = \beta_i^j$ holds for any $i \in \{1, 2, \ldots, s\}$. Therefore $L_j(x) = x^j$, and the lemma follows.
We say $E_j$ is a component of an element $M$ of the Bose–Mesner algebra if $E_jM 
eq 0$. Let $N_h$ denote the set of indices $j$ such that $E_j$ is a component of $E_1^{\alpha}$ but not of $E_1^\alpha (0 \leq l \leq h - 1)$. Note that $N_0 = \{0\}$ and $N_1 = \{1\}$.

**Lemma 2.2.** Suppose $X$ is a symmetric association scheme of class $d \geq 2$. Then the following are equivalent.

(1) $X$ is a $Q$-polynomial scheme with respect to $E_1$.

(2) The cardinality of $N_d$ is equal to 1.

(3) $N_d$ is nonempty.

**Proof.** (2) $\Rightarrow$ (3): Clear.

(1) $\Rightarrow$ (2): Without loss of generality, we assume that $X$ is a $Q$-polynomial scheme with respect to $\{E_i\}_{i=0}^d$. By noting that $\{\beta_i^1\}_{i=0}^d$ are mutually distinct, $\{E_1^\alpha\}_{i=0}^d$ are linearly independent, and a basis of the Bose–Mesner algebra. We have

$$E_i = f_i(E_1^\alpha) = \sum_{j=0}^i \alpha_{i,j} E_1'^j,$$

where $\alpha_{i,j} \in \mathbb{R}$ are the coefficients of a polynomial $f_i$ of degree $i$. The upper triangular matrix $(\alpha_{i,j})_{i,j=0}^d$ is non-singular because $\alpha_{i,i} \neq 0$ for each $i$. Since the inverse matrix $(\alpha'_{i,j})_{i,j=0}^d$ of $(\alpha_{i,j})_{i,j=0}^d$ is also an upper triangular matrix with $\alpha'_{i,i} \neq 0$ for each $i$, we can express

$$E_1'^i = \sum_{j=0}^i \alpha'_{i,j} E_j.$$

Therefore (2) follows.

(3) $\Rightarrow$ (1): First we prove that if $N_i$ is empty for some $i \in \{1, 2, \ldots, d - 1\}$, then $N_{i+1}$ is also empty. Let $I = \cup_{j=0}^{i-1} N_j$. We consider the expression $\sum_{j=0}^{i-1} E_1'^j = \sum_{j \in I} \beta_j E_j$. Note that $\beta_j > 0$ for any $j \in I$ by the Krein condition. Then we have

$$E_1 \circ (\sum_{h=0}^{i-1} E_1^{\alpha h}) = \sum_{j \in I} \beta_j \sum_{k=0}^d q_{1,j}^k E_k = \sum_{k=0}^d \sum_{j \in I} \beta_j q_{1,j}^k E_k.$$

If $N_i$ is empty, then

$$q_{1,j}^k = 0 \text{ for any } j \in I \text{ and any } k \not\in I \tag{2.1}$$

because $\beta_j > 0$ holds for any $j \in I$. We can express $E_1'^i = \sum_{j \in I} \beta_j^i E_j$, where $\beta_j^i$ are non-negative integers for any $j \in I$. By (2.1) and the equalities

$$E_1'^{i+1} = E_1 \circ E_1'^i = E_1 \circ \sum_{j \in I} \beta_j^i E_j = \sum_{k=0}^d \sum_{j \in I} \beta_j^i q_{1,j}^k E_k,$$

we obtain $\sum_{j \in I} \beta_j^i q_{1,j}^k = 0$ for $k \not\in I$. Hence $N_{i+1}$ is also empty. This means that if $N_d$ is not empty, then the cardinalities of $N_h$ is equal to 1 for any $h \in \{0, 1, \ldots, d\}$. Put $N_h = \{i_h\}$ and order $E_0, E_1, E_{i_2}, E_{i_3}, \ldots, E_{i_d}$. Then we can construct polynomials $f_h$ of degree $h$ such that $f_h(E_1^1) = E_{i_h}$ for any $h \in \{0, 1, \ldots, d\}$. Hence (1) follows. \(\square\)
Now we prove Theorem 1.1.

**Proof of Theorem 1.1.**

(1) \(\Rightarrow\) (2): Without loss of generality, we assume that \(X\) is a \(Q\)-polynomial scheme with respect to \(\{E_i\}_{i=0}^d\). For each \(i \in \{1, 2, \ldots, d\}\), we define the polynomial

\[
F_i(t) := \prod_{j=1, j \neq i}^d |X| t - \theta_i^* \theta_j^* - \theta_j^* - \theta_i^*
\]

of degree \(d - 1\). Set \(M_i = F_i(E_1)\). Then \(|X| E_1 = \sum_{j=0}^d \theta_j^* A_j\) yields that the \((x, y)\)-entries of \(M_i\) are

\[
M_i(x, y) = \begin{cases} 
K_i & \text{if } (x, y) \in R_0, \\
1 & \text{if } (x, y) \in R_i, \\
0 & \text{otherwise},
\end{cases}
\]

where \(K_i := \prod_{j=1, j \neq i}^d (\theta_0^* - \theta_j^*) (\theta_i^* - \theta_j^*)^{-1}\). Since \(F_i\) is a polynomial of degree \(d - 1\), the matrix \(M_i\) is a linear combination of \(\{E_i\}_{i=0}^{d-1}\). This means that \(M_i E_d = 0\). By (1.1),

\[
0 = M_i E_d = (K_i I + A_i) E_d = (K_i + p_i(d)) E_d
\]

for any \(i \in \{1, 2, \ldots, d\}\). Therefore the desired result follows.

(2) \(\Rightarrow\) (1): From the equation \(A_i = \sum_{j=0}^d p_i(j) E_j\) and our assumptions, we have

\[
A_i E_l = p_i(l) E_l = -K_i E_l.
\]

By Lemma 2.1,

\[
(|X| E_1)^{\otimes j} E_l = (\theta_0^*)^j I + \sum_{i=1}^d (\theta_i^*)^j A_i E_l = (\theta_0^*)^j - \sum_{i=1}^d (\theta_i^*)^j K_i E_l = 0
\]

for any \(j \leq d - 1\). This means that \(l\) is not an element of \(N_j\) for any \(j \leq d - 1\).

Note that the following equality holds:

\[
\prod_{j=1}^d |X| E_1 - \theta_i^* E_1 = I,
\]

where the multiplication is the Hadamard product. Obviously, \(I\) has \(E_1\) as a component. Since \(l \notin N_i\) for any \(i \in \{0, 1, \ldots, d - 1\}\), we have \(l \in N_d\). By Lemma 2.2 the desired result follows.

### 3 Integrality of \(K_i\)

In this section, we consider when \(K_i = -p_i(d)\) is an integer for each \(i \in \{1, 2, \ldots, d\}\) for a \(Q\)-polynomial scheme. The following theorem is important in this section.
Theorem 3.1 (Suzuki [12]). Let $X$ with $m_1 > 2$ be a $Q$-polynomial scheme with respect to the ordering $\{E_i^d\}_{i=0}^d$. Suppose $X$ is $Q$-polynomial with respect to another ordering. Then the new ordering is one of the following:

1. $E_0, E_2, E_4, E_6, \ldots, E_5, E_3, E_1$,
2. $E_0, E_d, E_1, E_{d-1}, E_2, E_{d-2}, E_3, E_{d-3}, \ldots$,
3. $E_0, E_d, E_2, E_{d-2}, E_4, E_{d-4}, \ldots, E_{d-5}, E_5, E_{d-3}, E_3, E_{d-1}, E_1$,
4. $E_0, E_{d-1}, E_2, E_{d-3}, E_4, E_{d-5}, \ldots, E_5, E_{d-4}, E_3, E_{d-2}, E_1, E_d$,
5. $d = 5$ and $E_0, E_5, E_3, E_2, E_4, E_1$.

Note that $Q$-polynomial schemes with $m_1 = 2$ are the ordinary $n$-gons as distance-regular graphs.

Proposition 3.2. Let $X$ with $m_1 > 2$ be a $Q$-polynomial association scheme with respect to the ordering $\{E_i^d\}_{i=0}^d$. If there exists $t$ such that $t \leq d/2$, $t \equiv 1 \pmod{2}$ and $m_t \neq m_{d-t+1}$, then $K_j$ is an integer for any $j$.

Proof. Let $F$ be the splitting field of the scheme, generated by the entries of the first eigenvector $P$. Then $F$ is a Galois extension of the rational field. Let $G$ be the Galois group $\text{Gal}(F/\mathbb{Q})$. We consider the action of $G$ on the primitive idempotents $E_i$, where elements of $G$ are applied entry-wise. Then the action of $G$ on $\{E_i^d\}_{i=0}^d$ is faithful and $|G| \leq 2$ [8].

Suppose $K_j$ is not an integer for some $j$. Since $-K_j = p_j(d)$ is an eigenvalue of $A_j$, $K_j$ is an algebraic integer. By the basic number theory, $K_j$ is irrational. Therefore $|G| \neq 1$ and hence $|G| = 2$. Let $\sigma$ be the non-identity element of $G$. From the definition of $K_j$, $E_1$ must have an irrational entry, and $E_1^\sigma \neq E_1$.

Suppose $\{E_i^d\}_{i=0}^d$ is another $Q$-polynomial ordering with the same polynomials $f_i$. Hence $\{E_i^d\}_{i=0}^d$ coincides with one of (1)-(5) in Theorem 3.1.

For $d = 2$, it is known that $K_j$ is an integer for each $i = 1, 2$ if $m_1 \neq m_2$ [4]. For (1) and (2) with $d > 2$, $(E_i^d)^\sigma \neq E_1$, this contradicts that $\sigma^2$ is the identity. Since $p_j(d)$ is irrational and $A_j E_d = p_j(d) E_d$, $E_d$ has an irrational entry. Therefore $E_d^\sigma \neq E_d$. For (4), $\sigma$ fixes $E_d$, a contradiction. Therefore the ordering $\{E_i^d\}_{i=0}^d$ coincides with (3) or (5).

Suppose that there exists $t$ such that $t \leq d/2$, $t \equiv 1 \pmod{2}$ and $m_t \neq m_{d-t+1}$. Since $E_t \circ I = (m_t/|X|)I$, we have $E_t^\sigma \circ I^\sigma = (m_t/|X|)I^\sigma$ and hence $E_t^\sigma \circ I = (m_t/|X|)I \neq (m_{d-t+1}/|X|)I$. Therefore $E_t^\sigma \neq E_{d-t+1}$. Thus, the ordering $\{E_i^d\}_{i=0}^d$ coincides with (3) for $d \geq 2$. If $d = 5$, then $m_3 \neq m_5$ and hence $E_3^5 \neq E_5$. Therefore $\{E_i^5\}_{i=0}^5$ does not coincide with (5). Thus the proposition follows.

Remark that the known $Q$-polynomial schemes with some irrational $K_i$ and $d > 2$ are the ordinary $n$-gons and the association scheme obtained from theicosahedron [5,8]. We can give a similar equivalent condition of the $P$-polynomial property of symmetric association schemes [4]. Let $\theta_i = p_1(i)$ for $0 \leq i \leq d$.

Theorem 3.3. Let $X$ be a symmetric association scheme of class $d \geq 2$. Suppose $\{\theta_i^d\}_{j=0}^d$ are mutually distinct. Then the following are equivalent:

1. $X$ is a $P$-polynomial association scheme with respect to $A_1$. 

(2) There exists \( l \in \{2, 3, \ldots, d\} \) such that for any \( i \in \{1, 2, \ldots, d\} \),

\[
\prod_{j \neq i}^{d} \frac{\theta_i - \theta_j}{\theta_i - \theta_j} = -q_i(l).
\]

Moreover if (2) holds, then \( l = i_d \).

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