Multi-impurity effects on the entanglement of anisotropic Heisenberg ring XXZ under a homogeneous magnetic field

Gao Wen-Bin, Yang Guo-Hui, and Zhou Ling
Department of Physics, Dalian University of Technology, Dalian 116024, PR China

The effects of multi-impurity on the entanglement of anisotropic Heisenberg ring XXZ under a homogeneous magnetic field have been studied. The impurities make the equal pairwise entanglement in a ring compete with each other so that the pairwise entanglement exhibits oscillation. If the impurities are of larger couplings, both the critical temperature and pairwise entanglement can be improved.

PACS numbers: 75.10.Jm, 03.67.Mn

Keywords:

I. INTRODUCTION

Entanglement is not only the fabulous feature of quantum mechanics but also very important to the quantum information processing (QIP).[1] In the studies of quantum entanglement, solid state system with Heisenberg model interaction is the simple and applicable candidates for the realization of quantum information. Therefore, there are many works focusing mainly on the different kinds of Heisenberg models[2−21] such as spin ring etc..

The impurities often exist in solid system and plays a very obvious and important part in condensed matter physics. As a candidate of QIP, solid system with impurity is also one of our important study object. In the previous researches, the impurity effects on the quantum entanglement have been studied in a three-spin system [22,23] and a large spin systems under zero temperature.[24] However, in these works, they have just studied single impurity.

In this paper, we will focus on studying the effects of multi-impurity on the pairwise thermal entanglement in a ring chain. We find the impurities make the equal pairwise entanglement in a ring compete with each other. If impurities are of large couplings, the critical temperature and the pairwise entanglement which coupled to the impurities can be improved. Our studying results not only provide a standard to judge impurities but also provide a way to enhance entanglement and critical temperature.

II. NON-NEAREST NEIGHBORING IMPURITY EFFECT

Firstly, we investigate the multi-impurity effect when the impurities are non-nearest neighbors shown as the Fig. 1. In the two figures, square represents impurity qubit and round stands for normal qubit.

For the case of Fig. 1a, the Hamiltonian can be written as

\[
H = \frac{1}{2} \sum_{i=1}^{2} \left[ J (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + J_z \sigma_i^z \sigma_{i+1}^z \right] \\
+ \frac{1}{2} \sum_{i=7}^{N} \left[ J (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + J_z \sigma_i^z \sigma_{i+1}^z \right] \\
+ \frac{1}{2} \sum_{i=3}^{6} \left[ J' (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + J'_z \sigma_i^z \sigma_{i+1}^z \right] \\
+ \frac{1}{2} \sum_{i=1}^{N} B (\sigma_i^z + \sigma_{i+1}^z),
\] (1)

where \((\sigma_i^x, \sigma_i^y, \sigma_i^z)\) are the vector of Pauli matrices; \(J\) and \(J_z\) are the real coupling coefficients of arbitrary nearest neighboring two qubits. We restrict the \(B \geq 0\) along z direction and \(N + 1 = 1\). We choose the parameters \(B, J, J_z\) and \(T\) are dimensionless and assume the coupling coefficients between normal qubit and impurity one has the relation
where \( \alpha \) characterizes the relative strength of the extra coupling between the impurity and its nearest neighboring qubits.\(^{24}\) For the case of Fig. 1b, one can write it easily following Eq.(1). We do not give it here any more.

As we know, for a system in equilibrium at temperature \( T \), the density operator is \( \rho = \frac{1}{Z} \exp(-H/k_BT) \), where \( Z = \text{Tr}[\exp(-H/k_BT)] \) is the partition function and \( k_B \) is Boltzman’s constant. For simplicity, we write \( k_B = 1 \). The value of entanglement between two qubits can be measured by Concurrence \( C \) which is written as

\[
C = \max(0, 2 \max \lambda_i - \sum_{i=1}^{4} \lambda_i)
\]

\(^{25,26,27,28}\) in which \( \lambda_i \) is the square roots of the eigenvalues of the matrix

\[
R = \rho(\sigma_i^x \otimes \sigma_j^y)\rho^* (\sigma_i^x \otimes \sigma_j^y),
\]

where \( \rho \) is the density matrix and the symbol \( * \) stands for complex conjugate. The Concurrence can be calculated no matter whether \( \rho \) is pure or mixed. In the following, we just take the pairwise entanglement into account. We will trace over the qubits and study the reduced density matrix of the two qubits which we are interested in.

\[ J' = \alpha \ast J, J'_z = \alpha \ast J_z, \]

\[ C = \max(0, 2 \max \lambda_i - \sum_{i=1}^{4} \lambda_i) \]

\[^{24}\] FIG. 3: Nearest neighboring concurrences as a function of \( \alpha \) for the three-impurity model (4th, 6th and 8th are three identical impurities) \( \alpha=0.1 \) (a), \( \alpha=2 \) (b). \( T=1, B=0.4, J=1, J_z=0.65 \).

Because here we study multi-impurity, we can not obtain analytic expression of the system. We will directly numerical calculate and plot entanglement. In Fig. 2, we plot the pairwise entanglement as a function of \( \alpha \), corresponding to Fig. 1a and Fig. 1b, respectively. For both of the two cases, in the regions of far away from impurities, the entanglement, for example \( C_{12} \) and \( C_{101} \), are slightly affected by the various values of \( \alpha \). Within the impurities regions, we observe that there is the almost same threshold value of \( \alpha \), after which a qubit and its nearest impurity start entangling such as \( C_{34}, C_{45} \) etc.. In Fig. 3, we show clearly that the pairwise entanglement versus site \( i \). If \( \alpha \) is small shown in Fig.3a, the case equal to cutting at 4th and 8th, thus the chain 9-10-1-2-3 is similar to the open chain\(^{24}\) while the part 4-5-6-7-8 chain have no entanglement because of the weak couplings. If \( \alpha > 1 \) such as \( \alpha = 2 \) shown in Fig. 3b, we still can cut the chain into two parts because \( J' > J \). Within the pure regions, entanglement will compete while in containing impurity part pairwise entanglement still compete each other.

Fig. 4 shows the influence of temperature and the values of \( \alpha \) on the entanglement in three-impurity model. From this figure, we can judge again that the second and the third qubit are pure qubits while the third and the forth contain one impurity. Usually, it is difficult to adjust the coupling coefficients, which means we will meet with difficult if we directly use the behavior of Fig. 3 to judge which one is impurity. But we still can do it by measuring the Concurrence changing with temperature (Refs \(^{29,30}\) proposed that Concurrence can be measured), because changing the temperature is very easy. On the other hand, we find that \( \alpha \) can effectively enhance the Concurrence and critical temperature if \( \alpha > 1 \) which is show in Fig. 3 and 4 clearly. By introduc-
FIG. 4: Nearest neighboring concurrence $C_{23}$ and $C_{34}$ versus $\alpha$ and $T$ for the three-impurity model (4th, 6th and 8th are three identical impurities). $B = 0.4, J = 1, J_z = 0.65$.

![Diagram of spin ring configurations](image)

FIG. 5: Two configurations of spin ring with nearest-neighbor impurity. (a): qubit ring formed with 10 qubits. The 5th and 6th are two identical impurities. (b): qubit ring formed with 10 qubits. The 4th, 7th and 8th are three identical impurities.

In this section, we study the nearest neighboring impurity effect on entanglement. We study the rings with a structure of Fig. 5. According to the Fig. 5a,

the Hamiltonian is

$$H = \frac{1}{2} \sum_{i=1}^{3} [J (\sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1}) + J_z \sigma^z_i \sigma^z_{i+1}]$$

$$+ \frac{1}{2} \sum_{i=7}^{N} [J (\sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1}) + J_z \sigma^z_i \sigma^z_{i+1}]$$

$$+ \frac{1}{2} \sum_{i=4,6} [J' (\sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1}) + J'_z \sigma^z_i \sigma^z_{i+1}]$$

$$+ \frac{1}{2} \sum_{i=5}^{N} [J'' (\sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1}) + J''_z \sigma^z_i \sigma^z_{i+1}]$$

$$+ \frac{1}{2} \sum_{i=1}^{N} B (\sigma^z_i + \sigma^z_{i+1}),$$

with

$$J'' = \beta \star J, J''_z = \beta \star J_z,$$

where $\beta$ characterizes the relative strength of the extra coupling between the two nearest neighbor impurities and $J', J'_z$ still has the relation of Eq.(2). Similarly, one can write the Hamiltonian corresponding to Fig. 5b. We plot the pairwise Concurrence near the two-nearest-impurity qubits area as a function of $\beta$ which is shown in Fig. 6. From this figure, we can see easily that the nearest neighboring impurities coupling only affect the nearest two-impurity and the others which couple with the impurities. For example, in Fig. 6a, the nearest neighboring impurities $C_{56}$ has a threshold value of $\beta$, affected heavily while $C_{45}$ also will decrease as a results of the competition between neighbor qubits. For the case of Fig. 5b, although we have more impurities, the nearest

FIG. 6: Nearest neighboring concurrences versus $\beta$ for the two-nearest-impurity model (5th and 6th are two identical impurities) (a) and the three-nearest-impurity model (4th, 7th and 8th are three identical impurities) (b). $B = 0.4, J = 1, J_z = 0.65, \alpha = 0.8$. 

III. NEAREST NEIGHBORING IMPURITY EFFECT

In this section, we study the nearest neighboring impurity effect on entanglement. We study the rings with a structure of Fig. 5. According to the Fig. 5a,
neighbor coupling only affect entanglement of themselves $C_{78}$ and that coupling with the impurities $C_{67}, C_{89}$; and all the others pairwise entanglement almost can not be affected.

**IV. CONCLUSION**

In conclusion, for a Heisenberg XXZ ring under a homogeneous magnetic field, we have studied entanglement in two-impurity and three-impurity under the two case of non-nearest-impurity and nearest-impurity. We find that the introducing of impurities make the originally equal pairwise entanglement compete with each other. For the weak and strong $\alpha$, we can cut the ring chain into different open chain and then use the open chain property to explain the competition. For the case with nearest neighbor qubits, the change of the relative coupling $\beta$ can only affect the qubits which couple to the impurities. If introducing impurity with large $\alpha$ and $\beta$, the pairwise entanglement, which couple with the impurities directly, can be enhanced and critical temperature also will be improved.

This work was supported by Natural Science Foundation of China under Grant No. 10575017 and Natural Science Foundation of Liaoning Province of China under No. 20031073.

**V. REFERENCES**

[1] Bennett C H and DiVincenzo D P Nature 2000 404 247
[2] Nielsen M A Phys. Rev. A 2001 63 022114
[3] Kamta G L and Starace A F Phys. Rev. Lett. 2002 88 107901
[4] O’Connor K M and Wootters W K Phys. Rev. A 2001 63 052302
[5] Deng L L and Man S L Chin. Phys. 2002 11 383
[6] Yeo Y Phys. Rev. A 2002 66 062312
[7] Sun J R, Wei Y N and Pu F K Chin. Phys. 1995 4 542
[8] Zhang G F, Li S S and Liang J Q Opt. Commun. 2005 245 457
[9] Zhou L, Song H S, Guo Y Q and Li C Phys. Rev. A 2003 68 024301
[10] Wang X G Phys. Rev. A 2001 64 012313
[11] Gunlycke D, Kendon V M and Vedral V Phys. Rev. A 2001 64 042302
[12] Arnesen M C, Bose S and Vedral V Phys. Rev. Lett. 2001 87 017901
[13] Canosa N and Rossignoli R Phys. Rev. A 2004 69 052306
[14] Khvveshchenko D V Phys. Rev. B 2003 68 193307
[15] Zhang Y M and Xu B W Chin. Phys. 1995 4 842
[16] Zhang T, Hui X Q and Yue R H Acta Phys. Sin. 2004 53 2755
[17] Shang Y M and Yao K L Chin. Phys. 1998 7 864
[18] Sun Y, Chen Y G and Chen H Phys. Rev. A 2003 68 044301
[19] Shao Y Z, Lan T and Lin G M Acta Phys. Sin. 2001 50 948
[20] Dong Z H and Feng S P Chin. Phys. 1998 7 348
[21] Hui X Q, Chen W X, Liu Q and Yue R H Acta Phys. Sin. 2006 55 3026
[22] Fu H, Solomon A I and Wang X J. Phys. A 2002 35 4293
[23] Xi X Q, Chen W X, Hao S R and Yue R H Phys. Lett. A 2002 297 291
[24] Wang X G Phys. Rev. E 2004 69 066118
[25] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K Phys. Rev. A 1996 54 3824
[26] Wootters W K Phys. Rev. Lett. 1998 80 2254
[27] Hill S and Wootters W K Phys. Rev. Lett. 1997 78 5022
[28] Anteneodo C and Souza A M C J. of Opt. B:Quantum Semiclassical Opt. 2003 5 73
[29] Sancho J M G and Huelga S F Phys. Rev. A 2000 61 042303
[30] Davidovich L "Entanglement As An Observable" 2006 reported in Texas A&M University