Quantum field theories in spaces with neutral signatures

Matej Pavšič
Jožef Stefan Institute, Jamova 39, SI-1000, Ljubljana, Slovenia
E-mail: matej.pavsic@ijs.si

Abstract. We point out that quantum field theories based on the concept of Clifford space and Clifford algebra valued-fields involve both positive and negative energies. This is a consequence of the indefinite signature \((p,q)\) of the Clifford space. When the signature is neutral, \(p = q\), then vacuum energy vanishes and there is no cosmological constant problem. A question of the stability of such theories in the presence of interactions arises. We investigate a toy model of the harmonic oscillator in the space \(M_{1,1}\). We have found that in the presence of certain interactions the amplitude of oscillations can remain finite. In general this is not the case and the amplitude grows to infinity, but only when the two frequencies are exactly the same. When they are even slightly different, the amplitude remains finite and the system is stable. We show how such oscillator comes from the Stueckelberg action in curved space, and how it can be generalized to field theories.

1. Introduction
Research in quantum gravity leads us to higher derivative theories that imply negative energies. They have turned out to be a stumbling block against further progress, because they imply vacuum instability. Negative energies also occur in spaces with non Lorentzian signature, \(M_{p,q}\). Therefore, such spaces are commonly considered as unsuitable for physics. In the literature it is usually stated that negative signatures imply negative probabilities or ghosts. In ref. ([1]–[5]) it has been pointed out that by an appropriate redefinition of vacuum, instead of negative probabilities and positive energies, one has positive probabilities and negative energies.

In this paper we investigate the issue of negative energies in some detail. We consider a classical interacting oscillator in the space \(M_{1,1}\). Solutions of its equations of motion can be obtained numerically. We have found that the solutions are not always runaway, unstable ones. In many cases they have oscillatory behavior, and do not escape into infinity. The same is true for a truncated quantum oscillator.

Then we consider field theories in field spaces with neutral signature, an example of which is the Clifford space, discussed in Sec. 2. The vacuum energy of such fields is zero, which resolves the notorious cosmological constant problem. But there remain the problem of “instantaneous” vacuum decay of an interacting field. We show in Sec. 6 how this problem can be circumvented, and the “explosion” of the vacuum stabilized.

2. Clifford space: An extension of spacetime
An extended object, \(O\), can be be sampled by a finite set of parameters, for instance, by the center of mass coordinates, and by the orientation of its axes of symmetries. Higher multipole
deformations, such as the dipole and the quadrupole ones, can also be taken into account. For practical reasons, only a finite number of multipoles can be taken into account. Instead of the infinite number of degrees of freedom, we consider only a finite number of degrees of freedom. We thus perform a mapping from an infinite dimensional configuration space, associated with the object \( O \), to a finite dimensional subspace.

Extended objects of particular interest for theoretical physics are strings and branes. They can be described by coordinate functions \( X^\mu(\xi^a), \mu = 0, 1, 2, ..., N - 1, a = 0, 1, 2, ..., n - 1 \), where \( n \leq N \). Such a description is infinite dimensional. In refs. [6] it was pointed out how one can employ a finite description in terms of a quenched mini superspace.

The idea has been further developed [7]–[11] by means of Clifford algebras, a very useful tool for description of geometry [12]. Here we are interested in description of spacetime, \( M_N \), and the objects embedded in \( M_N \). Therefore, let us start by considering the line element in \( M_N \):

\[
Q = ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \quad \mu, \nu = 0, 1, 2, ..., N - 1.
\]

If we take the square root, \( \sqrt{Q} \), we have the following possibilities:

i) \( \sqrt{Q} = \sqrt{g_{\mu\nu}}dx^\mu dx^\nu \) scalar

ii) \( \sqrt{Q} = \gamma_\mu dx^\mu \) vector

Here \( \gamma_\mu \) are generators of the Clifford algebra \( Cl(p, q) \), \( p + q = N \), satisfying

\[
\gamma_\mu \cdot \gamma_\nu = \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = g_{\mu\nu},
\]

where \( g_{\mu\nu} \) is the metric of \( M_N \).

The generators \( \gamma_\mu \) have the role of basis vectors of the spacetime \( M_N \). The symmetric product \( \gamma_\mu \cdot \gamma_\nu \) represents the inner product. The antisymmetric (wedge) product of two basis vectors gives a unit bivector:

\[
\gamma_\mu \wedge \gamma_\nu = \frac{1}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)
\]

and has thus the role of outer product. In analogous way we obtain 3-vectors, 4-vectors, etc.,

We assume that the signature of an \( N \)-dimensional spacetime is \((1, N - 1)\), i.e., \((+---...). In the case of the 4-dimensional spacetime we thus have the signature \((1, 3)\), i.e., \((+---). The corresponding Clifford algebra is \( Cl(1, 3) \).

The basis of \( Cl(1, N - 1) \) is

\[
\{1, \gamma_\mu, \gamma_\mu_1 \wedge \gamma_\mu_2, ..., \gamma_\mu_1 \wedge \gamma_\mu_2 \wedge ... \wedge \gamma_\mu_N \}.
\]

A generic element, \( X \in Cl(1, N - 1) \), is a superposition

\[
X = \sum_{r=0}^{N} \frac{1}{r!} X^{\mu_1 \mu_2 ... \mu_r} \gamma_\mu_1 \wedge \gamma_\mu_2 \wedge ... \wedge \gamma_\mu_r \equiv X^M \gamma_M,
\]

called a Clifford aggregate or polyvector.

In refs. [10, 11, 13] it has been demonstrated that \( r \)-vectors \( X^{\mu_1 \mu_2 ... \mu_r} \) can be associated with closed instantonic \((r - 1)\)-branes or open instantonic \( r \)-branes. A generic polyvector, \( X = X^M \gamma_M \), can be associated with a conglomerate of (instantonic) \( r \)-branes for various values of \( r = 0, 1, 2, ..., N \).
Our objects are instantonic r-branes, which means that they are localized in spacetime\(^1\). They generalize the concept of ‘event’, a spacetime point, \(x^\mu\), \(\mu = 0, 1, 2, 3\). Instead of an event, we have now an extended event, \(E\), described by coordinates \(X^{\mu_1\mu_2...\mu_r}\), \(r = 0, 1, 2, 3, 4\). The space of extended events is called Clifford space, \(C\). It is a manifold whose tangent space at any of its points is a Clifford algebra \(Cl(1,3)\). If \(C\) is a flat space, then it is isomorphic to the Clifford algebra \(Cl(1,3)\) with elements \(X = \sum_{r=0}^{4} \frac{1}{r!} X^{\mu_1\mu_2...\mu_r} \gamma_{\mu_1\mu_2...\mu_r} \equiv X^M \gamma_M\).

In flat \(C\)-space, the basis vectors are equal to the wedge product
\[
\gamma_M = \gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \ldots \wedge \gamma_{\mu_r} \tag{6}
\]
at every point \(E \in C\). This not true in curved \(C\)-space: if we (parallel) transport a polyvector \(A = A^M \gamma_M\) from a point \(E \in C\) along a closed path back to the original point, \(E\), then the orientation of the polyvector \(A\) after such transport will not coincide with the initial orientation of \(A\). After the transport along a closed path we will obtain a new polyvector \(A' = A'^M \gamma_M\). If, in particular, the initial polyvector is one of the Clifford algebra basis elements, \(A = \gamma_M\), i.e., an object with definite grade, then the final polyvector will be \(A' = A'^M \gamma_M\), which is an object with mixed grade. A consequence is that in curved Clifford space \(C\), basis vectors cannot have definite grade at all points of \(C\).

The situation in a curved Clifford space, \(C\), is analogous to that in a usual curved space, where after the (parallel) transport along a closed path, a vector changes its orientation. In Clifford space, a change of orientation in general implies a change of a polyvector’s grade, so that, e.g., a definite grade polyvector changes into a mixed grade polyvector.

However, if we impose a condition that, under parallel transport, the grade of a polyvector does not change, then one has a very special kind of curved Clifford space\(^{[14]}\). In such a space, after a parallel transport along a closed path, the vector part \((A)_1 = a^\mu \gamma_\mu\) changes into \((A')_1 = a'^\mu \gamma_\mu\), the bivector part \((A)_2 = a^\mu \gamma_\mu \wedge \gamma_\nu\) changes into \((A')_2 = a'^\mu \gamma_\mu \wedge \gamma_\nu\), etc., but one grade does not change into another grade. Such special Clifford space, in which the consequences of curvature manifest themselves within each of the subspaces with definite grade separately, but not between those subspaces, is very complicated. We will not consider such special Clifford spaces, because they are analogous to the usual curved spaces of the product form \(M = M_1 \times M_2 \times \ldots M_n\), where \(M_i \subset M\) is a curved lower dimensional subspace of \(M\), and where only those (parallel) transports are allowed that bring tangent vectors of \(M_i\) into another tangent vectors of the same subspace \(M_i\).

The squared line element in Clifford space, \(C\), is
\[
dS^2 = G_{MN} dx^M dx^N = X^{\dagger} * dX = (dX^{\dagger} dX). \tag{7}
\]
Here \(dX = dx^M \gamma_M\), and \(dX^{\dagger} = dx^M \gamma_M^{\dagger}\), where \(\dagger\) denotes the operation of inversion: \((\gamma_{\mu_1} \gamma_{\mu_2} \ldots \gamma_{\mu_r})^{\dagger} = \gamma_{\mu_r} \gamma_{\mu_{r-1}} \ldots \gamma_{\mu_1}\). The metric of \(C\) is
\[
G_{MN} = \gamma_M^{\dagger} * \gamma_N = (\gamma_N \gamma_M)^0, \tag{8}
\]

\(^1\) The usual \(p\)-branes are localized in space, but they are infinitely extended into a time-like direction, so that they are \((p + 1)\)-dimensional worldsheets in spacetime.
where $\langle \cdot \rangle_0$ means the scalar part. A Clifford space with such a metric has signature [11] (8, 8), i.e., $(+ + + + + + + + - - - - - - - - - -)$. This is ultrahyperbolic space with neutral signature.

In the following sections we will show that, contrary to the wide spread belief, the physics in spaces with signature $(n, n)$ makes sense.

3. A toy model: Harmonic oscillator in the pseudo Euclidean space $M_{r,s}$

3.1. Case $M_{1,1}$

In ref. [4] we considered the harmonic oscillator described by the Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 - \dot{y}^2) - \frac{1}{2}\omega^2(x^2 - y^2).$$

(9)

The change of sign in front of the $y$-terms has no influence on the equations of motion. A difference occurs when we calculate the canonical momenta

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x}, \quad p_y = \frac{\partial L}{\partial \dot{y}} = -\dot{y}.$$

(10)

The Hamiltonian

$$H = p_x \dot{x} + p_y \dot{y} - L = \frac{1}{2}(p_x^2 - p_y^2) + \frac{\omega^2}{2}(x^2 - y^2).$$

(11)

Figure 1. Examples of solutions, obtained by Mathematica, for various initial conditions, of the interactive pseudo-Euclidean oscillator for the potential $V = (1/2)(x^2 - y^2) + V_1$, with $V_1 = 0.1(x^4y^2 - x^2y^4)$. a) $\dot{x}(0) = 1$, $\dot{y}(0) = 0$, $x(0) = 0$, $y(0) = 1$; b) $\dot{x}(0) = 0.9$, $\dot{y}(0) = 0$, $x(0) = 0$, $y(0) = 1$; c) $\dot{x}(0) = 0.9$, $\dot{y}(0) = 0.2$, $x(0) = 0.6$, $y(0) = 1.5$; d) $\dot{x}(0) = 2$, $\dot{y}(0) = 1$, $x(0) = 0.3$, $y(0) = 1$; e) $\dot{x}(0) = 0.2$, $\dot{y}(0) = 0.2$, $x(0) = 0.3$, $y(0) = 1$. 

The Hamiltonian
can have positive or negative values, but this does not mean that the system is unstable. Namely, the equations of motion
\[ \ddot{x} = -\frac{\partial V}{\partial x}, \quad \ddot{y} = \frac{\partial V}{\partial y}, \quad V = \frac{\omega^2}{2}(x^2 - y^2) \] (12)
have different signs in front of the force terms. The criterion for stability of the \( y \)-degree of freedom is that the potential has to have a maximum in the \((y, V)\)-plane [5].

Stability could be destroyed if we include an extra interaction term into the potential \( V \). But for
\[ V = \frac{1}{2}(x^2 - y^2) + \lambda(x^4y^2 - x^2y^4) \] (13)
we find, by solving numerically the equations of motion, that the system is stable (Fig. 1). We find the same phenomenon also for other potentials (Fig. 2). Those figures show that the trajectories in the \((x, y)\) space remain confined, and do not escape into infinity, which reveals that the interacting pseudo-Euclidean oscillator is not automatically unstable. Later we will study this in more detail.

\[ V_1 = 0.1(x^4y^2 + x^2y^4) \]
\[ \dot{x}(0) = 0.9, \quad \dot{y}(0) = 0 \]
\[ x(0) = 0, \quad y(0) = 1 \]

\[ V_1 = 0.1(x^4y - x^3) \]
\[ \dot{x}(0) = 0.8, \quad \dot{y}(0) = 0.2 \]
\[ x(0) = 0.2, \quad y(0) = 0.9 \]

\[ V_1 = 0.1(x^4 + y^4) \]
\[ \dot{x}(0) = 1, \quad \dot{y}(0) = 0.2 \]
\[ x(0) = 0, \quad y(0) = 1 \]

**Figure 2.** Examples of solutions, for various initial conditions and potentials, of the interactive pseudo-Euclidean oscillator

In refs. [4, 5] we also studied the quantized non interacting system (9) by replacing the coordinates and momenta with the operators, satisfying \([x, p_x] = i\), \([y, p_y] = i\), \([x, y] = [p_x, p_y] = 0\), and by introducing the operators
\[ c_x = \frac{1}{\sqrt{2}}(\sqrt{\omega} x + \frac{i}{\sqrt{\omega}} p_x), \quad c_x^\dagger = \frac{1}{\sqrt{2}}(\sqrt{\omega} x - \frac{i}{\sqrt{\omega}} p_x) \] (14)
\[ c_y = \frac{1}{\sqrt{2}}(\sqrt{\omega} y + \frac{i}{\sqrt{\omega}} p_y), \quad c_y^\dagger = \frac{1}{\sqrt{2}}(\sqrt{\omega} y - \frac{i}{\sqrt{\omega}} p_y) \] (15)
satisfying the commutation relations
\[ [c_x, c_x^\dagger] = 1, \quad [c_y, c_y^\dagger] = 1, \quad [c_x, c_y] = [c_x^\dagger, c_y^\dagger] = 0 \] (16)
\[ [c_x, c_y] = [c_x^†, c_y^†] = 0 \]  \hspace{1cm} (17)

The Hamilton operator then reads
\[ H = \omega \left( c_x^† c_x + c_x c_x^† - c_y^† c_y - c_y c_y^† \right). \]  \hspace{1cm} (18)

Let us define vacuum so that it is annihilated by \( c_x \) and \( c_y \):
\[ c_x |0\rangle = 0, \quad c_y |0\rangle = 0. \]  \hspace{1cm} (19)

If we arrange the terms in the Hamiltonian so that the creation operators are on the left, we obtain
\[ H = \omega \left( c_x^† c_x - c_y^† c_y \right) \]  \hspace{1cm} (20)

which has vanishing vacuum expectation value,
\[ \langle 0 | H | 0 \rangle = 0. \]  \hspace{1cm} (21)

The excited states \( c_x^† |0\rangle, c_x c_x^† |0\rangle, \ldots \), have positive energies, whereas the states \( c_y^† |0\rangle, c_y c_y^† |0\rangle, \ldots \), have negative energies. But all those states have positive norms, because the commutators (16) are positive.

In the Schrödinger representation, \( x, y \in \mathbb{R}, p_x = -i \partial / \partial x, p_y = -i \partial / \partial y \), and \( \langle x, y | 0 \rangle \equiv \psi_0(x,y) \). The relations (19) become
\[ \frac{1}{2} \left( \sqrt{\omega} x + \frac{1}{\sqrt{\omega}} \frac{\partial}{\partial x} \right) \psi_0(x,y) = 0 \]  \hspace{1cm} (22)
\[ \frac{1}{2} \left( \sqrt{\omega} y + \frac{1}{\sqrt{\omega}} \frac{\partial}{\partial y} \right) \psi_0(x,y) = 0 \]  \hspace{1cm} (23)

and their solution is the following vacuum wave function:
\[ \psi_0 = A e^{\frac{1}{2} \omega (x^2 + y^2)}. \]  \hspace{1cm} (24)

It is normalized according to \( \int \psi_0^2 dx dy = 1 \) if \( A = \sqrt{\omega / \pi} \), and it satisfies the Schrödinger equation \( H \psi = E \psi \), with \( E = 0 \).

4. Generalization to \( M_{r,s} \)

The system (9) can be generalized [4] to a space of arbitrary dimension and signature, \( M_{r,s} \). The Lagrangian is then
\[ L = \frac{1}{2} \dot{x}^a \dot{x}_a - \frac{1}{2} \omega^2 x^a x_a. \]  \hspace{1cm} (25)

The corresponding Hamiltonian is
\[ \frac{1}{2} p^a p_a + \frac{1}{2} \omega^2 x^a x_a, \]  \hspace{1cm} (26)

where \( p_a = \partial L / \partial \dot{x}^a = \dot{x}_a = \eta_{ab} \dot{x}^b \) are the canonical momenta, and \( \eta_{ab} = \text{diag}(1,1,1,\ldots,-1,-1,-1) \) the metric tensor with signature \((r,s)\). Upon quantization, the classical variables \( x^a \), \( p_a \) are replaced by the operators, satisfying
\[ [x^a, p_b] = i \delta_a^b \quad \text{or} \quad [x^a, p^b] = i \eta^{ab} \]  \hspace{1cm} (27)
Because the momenta \( p_a \) are canonically conjugated to \( x^a \), a natural choice of creation and annihilation operators is

\[
\begin{align*}
  c^a &= \frac{1}{\sqrt{2}} \left( \sqrt{\omega} x^a + \frac{i}{\sqrt{\omega}} p_a \right) \\
  c^a \dagger &= \frac{1}{\sqrt{2}} \left( \sqrt{\omega} x^a - \frac{i}{\sqrt{\omega}} p_a \right)
\end{align*}
\] (28)

the commutation relations being

\[
[c^a, c^b \dagger] = \delta^{ab}, \quad [c^a, c^b] = [c^a \dagger, c^b \dagger] = 0.
\] (30)

The Hamiltonian then becomes

\[
H = \frac{1}{2} \omega (c^a \dagger c^a + c^a c^a \dagger).
\] (31)

Defining vacuum according to

\[
c^a |0\rangle = 0,
\] (32)

and using \( c^a c^a \dagger = \eta_{ab} c^a c^b \dagger = \eta_{ab} (c^b \dagger c^a + \delta^{ab}) = c^a \dagger c_a + r - s \), the Hamiltonian (31) can be rewritten as

\[
H = \omega (c^a \dagger c^a + \frac{r}{2} - \frac{s}{2}).
\] (33)

Vacuum expectation of the latter Hamiltonian is zero if the signature is neutral, i.e., if \( r = s \).

In eqs. (28),(29) we composed the creation and annihilation operators in terms of the \textit{contravariant} components \( x^a \) and the \textit{covariant} components \( p_a \). But in the literature, the following operators are used:

\[
\begin{align*}
  a^a &= \frac{1}{2} \left( \sqrt{\omega} x^a + \frac{i}{\sqrt{\omega}} p_a \right) \\
  a^a \dagger &= \frac{1}{2} \left( \sqrt{\omega} x^a - \frac{i}{\sqrt{\omega}} p_a \right)
\end{align*}
\] (34)

They are composed in terms of \( x^a \) and \( p^a = \eta^{ab} p_b \), and satisfy

\[
[a^a, a^b \dagger] = \delta^a_b, \quad [a^a, a^b] = \eta^{ab}.
\] (36)

The Hamiltonian is

\[
H = \frac{1}{2} \omega (a^a \dagger a_a + a_a a^a \dagger).
\] (37)

There are two possible definitions of the vacuum:

\textit{Possibility I}

\[
a^a |0\rangle = 0.
\] (38)

This is the usual definition. Rewriting the Hamiltonian in the form

\[
H = \omega (a^a \dagger a_a + \frac{r}{2} + \frac{s}{2}),
\] (39)

we see that the vacuum energy, \( \langle 0 | H | 0 \rangle = (r + s)/2 \), is positive. The eigenvalues of \( H \) are all positive. But there exist negative norm states, called ghosts.
Possibility II

\[ a^a|0\rangle = 0, \quad a^a|0\rangle = 0, \quad (40) \]

where we have split the index \( a \) according to

\[ \bar{a} = 1, 2, ..., r; \quad a = r + 1, r + 2, ..., r + s. \quad (41) \]

Such definition of vacuum (and its consequences) has been used in refs. [3, 1, 4]. Writing the Hamiltonian (37) in the form

\[ H = \omega (a^{\bar{a}} a_{\bar{a}} + a^a a^a + \frac{r}{2} - \frac{s}{2}), \quad (42) \]

we see that the energy of the vacuum (40) is \( E = \omega (r - s)/2. \) If \( r = s \), then the vacuum energy vanishes. The excitation states have positive or negative energies, \( E = \omega (m - n + r/2 - s/2) \), depending on the mode \((m, n)\) of excitation. There are no negative norm (ghost) states.

The vacuum definition (40) is equivalent to the definition (32), because the operators \( a^a, a^{a\dagger} \) are just rewritten operators \( c^a \) and \( c^{a\dagger} \). From the definition (28), (29) of the latter operators it follows that the vacuum defined according to (40) is covariant under the transformations of the group SO(r,s).

5. Non interacting quantum field theory

5.1. The scalar fields

In refs. [4] we considered the scalar field theory described by the action

\[ I[\varphi^a] = \frac{1}{2} \int d^4x \sqrt{-g}(g^{\mu\nu} \partial_\mu \varphi^a \partial_\nu \varphi^b - m^2 \varphi^a \varphi^b) \gamma_{ab}, \quad (43) \]

where \( \gamma_{ab} \) is a metric in the space of fields \( \varphi^a(x) \) at a fixed point \( x \equiv x^\mu \) of spacetime. We will consider the case \( \gamma_{ab} = \text{diag}(1,1,1,...,-1,-1,-1) = \gamma^{ab} \) and \( g_{\mu\nu} = \text{diag}(1,-1,-1,-1) = g^{\mu\nu}. \)

The canonical momenta are

\[ \pi_a = \frac{\partial L}{\partial \partial_0 \varphi^a} = \partial_\nu \varphi_a = \partial_0 \varphi_a \equiv \dot{\varphi}_a, \quad (44) \]

and the Hamiltonian is

\[ H = \frac{1}{2} \int d^3x (\dot{\varphi}^a \dot{\varphi}_b - \partial_\nu \varphi^a \partial^\nu \varphi^b + m^2 \varphi^a \varphi^b) \gamma_{ab}. \quad (45) \]

In the quantized theory, \( \varphi^a(x) \) and \( \pi_a(x) \) are operators satisfying

\[ [\varphi^a(x), \pi_b(x')] = i\delta^3(x - x')\delta^a_b. \quad (46) \]

Expanding the field according to

\[ \varphi^a = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} (a^a(k)e^{-ikx} + a^{a\dagger}(k)e^{ikx}), \quad (47) \]

where \( \omega_k = \sqrt{m^2 + k^2}, \) and

\[ [a^a(k), a^{a\dagger}(k')] = (2\pi)^3 2\omega_k \delta^3(k - k')\delta^a_b, \quad (48) \]
i.e.,
\[ [a^a(k), a^{b\dagger}(k')] = (2\pi)^3 2\omega_k \delta^3(k - k') \gamma^{ab}, \]  
the Hamiltonian (45) becomes
\[ H = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\omega_k}{2\omega_k} (a^{a\dagger}(k)a^b(k) + a^a(k)a^{b\dagger}(k))\gamma_{ab}. \]

Analogously to eq. (40), we define the vacuum as
\[ a^{\tilde{a}}(k)|0\rangle = 0, \quad a^{a\dagger}(k)|0\rangle = 0, \]
where we split the index \( a \) into the part \( \tilde{a} \), running over the positive, and the part \( a \), running over the negative signature components.

Using (49) we can rewrite the Hamiltonian (50) into the form
\[ H = \int \frac{d^3k}{(2\pi)^3} \frac{\omega_k}{2\omega_k} (a^{\tilde{a}\dagger}(k)a_\tilde{a}(k) + a_\tilde{a}(k)a^{\tilde{a}\dagger}(k)) + \frac{1}{2} \int d^3k \omega_k \delta^3(0)(r - s). \]  
If the signature has equal number of plus and minus signs, i.e., if \( r = s \), then the zero point energies cancel out from the Hamiltonian.

5.2. Generalization to Clifford space

5.2.1. The generalized Klein-Gordon equation
In ref. [8, 15, 16, 17] it has been proposed to consider the Clifford algebra valued field in \( C \)-space:
\[ \Phi(X) = \phi^A(X)\gamma_A, \]
where \( \gamma_A \equiv (1/r!)\gamma_{a_1} \wedge \gamma_{a_2} \wedge \ldots \wedge \gamma_{a_p}, \, r = 0, 1, 2, \ldots, n, \) is a basis of the Clifford algebra \( Cl(1, n-1) \) of the spacetime \( M_{1,n-1} \).

In general, the field depends on coordinates \( X^M \equiv X^{\mu_1 \ldots \mu_r} \) of \( C \)-space. In this paper we will consider the case in which the field depends on spacetime coordinates only. The action is then
\[ I = \frac{1}{2} \int d^nx \sqrt{-g} (g^{\mu\nu} \partial_\mu \phi^A \partial_\nu \phi^B - m^2 \phi^A \phi^B)G_{AB}, \]
where \( \mu = 0, 1, 2, \ldots, n - 1. \) The metric is defined as \( G_{AB} = \gamma_A^* \gamma_B \equiv \langle \gamma_A^* \gamma_B \rangle_0, \) where \( \langle \ldots \rangle_0 \) means the scalar part of the expression, and the operation \( \dagger \) reverses the order of vectors. It turns out [5] that such metric has signature \((R, S)\) with \( R = S, \) where \( R + S = 2^n \) is the dimension of \( Cl(1, n-1) \). Because the signature is neutral, the action (54) is just like the action (43). Following the same procedure as before, we obtain that the vacuum energy vanishes. Therefore, in such theory there is no cosmological constant problem. Recall that in Einstein’s equations there is a term with the cosmological constant, \( \lambda g_{\mu\nu}, \) and the term with the stress-energy tensor, \( 8\pi G T_{\mu\nu}. \) The vacuum of a quantum field gives \( T_{\mu\nu} = \rho g_{\mu\nu}, \) where \( \rho \) is the energy density. The two contributions sum to an effective cosmological constant \( \Lambda = \lambda + 8\pi G \rho. \) Here \( \lambda \) is a free parameter that can have in principle any values. Usual field theoretic calculations give infinite vacuum energy density \( \rho. \) When taking into account the cutoff at the Planck scale, it turns
out that $\rho$ is 120 orders of magnitude bigger than expected from observations. Therefore, the effective cosmological constant, $\Lambda$, is too big as well. This is the notorious cosmological constant problem. Here we have found a way, how $\rho$ can vanish. The cosmological constant problem is thus resolved, because what remains is $\Lambda = \lambda$, and there is no longer the annoying huge contribution coming from quantum vacuum. How $\lambda$ can be theoretically calculated remains, of course, a problem. But this is another sort of problem from the one, why the $\rho$ is so big. The small observed cosmological constant could be a residual effect of something else, e.g., of a spacetime filling brane [18, 19].

5.2.2. Generalized Dirac equation (Dirac-Kähler equation) Another possibility is to assume that the Clifford algebra valued field (53) satisfies the Dirac-Kähler equation [20]

$$\tag{55} (i\gamma^\mu \partial_\mu - m)\Phi = 0.$$ 

Instead of expanding $\Phi$ in terms of $\gamma_A = (1, \gamma_{a_1}, \gamma_{a_1} \wedge \gamma_{a_2}, ..., \gamma_{a_1} \wedge ... \wedge \gamma_{a_n})$, we can expand it in terms of the spinor basis, $\xi_A$ of $\text{Cl}(1, n-1)$, considered in refs. [21, 17, 22]. Then

$$\Phi = \psi^A \xi_A \equiv \psi^{\alpha i} \xi_{\alpha i}, \quad (56)$$

where $\alpha$ is the spinor index of a left minimal ideal of $\text{Cl}(1, n-1)$, whereas $i$ runs over $2^{n/2}$ left ideal of $\text{Cl}(1, n-1)$. From now on we will consider the case $n = 4$.

The $\gamma^\mu$ are abstract objects, the Clifford numbers, i.e., the elements of $\text{Cl}(1,3)$. They can be represented as matrices according to

$$\langle (\xi^A)^\dagger \gamma^\mu \xi_B \rangle_S \equiv (\gamma^\mu)^A_B,$$

where the subscript $S$ means the normalized scalar part of the expression (see [16]). Multiplying eq. (55) from the left by $(\xi^A)^\dagger$ and taking the scalar part, we obtain

$$\left( i (\gamma^\mu)^A_B \partial_\mu - m\delta^A_B \right) \psi^B = 0. \quad (58)$$

The $16 \times 16$ matrices, $(\gamma^\mu)^A_B$, can be factorized as $(\gamma^\mu)^A_B = (\gamma^\mu)^{\alpha}_\beta \delta^i_j$, and eq. (58) written in the form

$$\left( i \gamma^\mu \partial_\mu - m\delta^\alpha_\beta \right) \psi^{\beta i} = 0, \quad (59)$$

or shortly,

$$\left( i \gamma^\mu \partial_\mu - m \right) \psi^i = 0. \quad (60)$$

In the last equation, the spinor index, $\alpha$, has been omitted, and kept only the index $i = 1, 2, 3, 4$ denoting four left ideals. Bear in mind that the $i$ as an index (superscript or subscript) has different a role than the $i$ as a factor in front of $\gamma^\mu$, in which case it is the imaginary unit, $i^2 = -1$.

The action is

$$I = \int d^4x \bar{\psi}^i (i \gamma^\mu \partial_\mu - m) \psi^j z_{ij}. \quad (61)$$
where $z_{ij}$ is the metric in the space of ideals. It comes\(^2\) from the metric $z_{AB}$ of the 16-dimensional spinor space of $Cl(1,3)$:

$$(\xi^A)^{\dagger} \ast \xi^B = z_{AB} = z_{(\alpha i)(\beta j)} = z_{\alpha \beta} z_{ij}, \quad (62)$$

where

$$z_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad z_{\alpha \beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (63)$$

The fields $\psi^i \equiv \psi^{\alpha i}(x)$ and their conjugate momenta $\pi_{\alpha i}(x) = i\psi^{\ast \alpha i}(x)$ satisfy the equal time anticommutation relations

$$\{\psi^{\alpha i}(t, \mathbf{x}), \pi_{\beta j}(t, \mathbf{x}')\} = i \delta^\alpha_\beta \delta^i_j (\mathbf{x} - \mathbf{x}') \quad (64)$$

$$\{\psi^{\alpha i}(t, \mathbf{x}), \psi^{\beta j}(t, \mathbf{x}')\} = 0, \quad \{\pi^{\dagger}_{\alpha i}(t, \mathbf{x}), \pi^{\dagger}_{\beta j}(t, \mathbf{x}')\} = 0. \quad (65)$$

The Hamiltonian, belonging to the action (61) is

$$H = \int d^3x \bar{\psi} (\gamma^\mu \partial_\mu + m) \psi \ z_{ij}. \quad (66)$$

Expanding the $\psi^i$ in terms of the annihilation and creation operators, we obtain

$$H = \sum_{n=1}^{2} \frac{d^3 p}{(2\pi)^3} m \left( b_n^{\dagger}(\mathbf{p}) b_n^j(\mathbf{p}) - d_n^{\dagger}(\mathbf{p}) d_n^j(\mathbf{p}) \right) z_{ij}. \quad (67)$$

The index $i = 1, 2, 3, 4$ distinguishes the spinors of different left ideals of $Cl(1,3)$. The index $s = 1, 2$ is the usual one that distinguishes ‘spin up’ and ‘spin down’ states. We have the following anticommutation relations:

$$\{ b_n^{\dagger}(\mathbf{p}), b_{n'}^{\dagger}(\mathbf{p}') \} = (2\pi)^3 \frac{E}{M} \delta^{\dagger}(\mathbf{p} - \mathbf{p}') \delta_{ns'} z^{ij}, \quad (68)$$

$$\{ d_n^{\dagger}(\mathbf{p}), d_{n'}^{\dagger}(\mathbf{p}') \} = (2\pi)^3 \frac{E}{M} \delta^{\dagger}(\mathbf{p} - \mathbf{p}') \delta_{ns'} z^{ij}, \quad (69)$$

where $E = \sqrt{\mathbf{p}^2 + m^2}$.

Let us now split the index according to $i = (i, \bar{i})$, where $i = 1, 2$ and $\bar{i} = 3, 4$, and define vacuum as follows:

$$b_\bar{i}^\dagger |0\rangle = 0, \quad d_\bar{i}^\dagger |0\rangle = 0 \quad (70)$$

The terms in the Hamiltonian (67) can be arranged so that we obtain

$$H = \sum_{s=1}^{2} \frac{d^3 p}{(2\pi)^3} m \left[ b_s^{\dagger}(\mathbf{p}) b_s^j(\mathbf{p}) + d_s^{\dagger}(\mathbf{p}) d_s^j(\mathbf{p}) \right] z_{ij}. \quad (67)$$

\(^2\) For more details see ref.\([16]\).

\(^3\) They come from the action (61) written in the explicit form $I = \int d^4 x \psi^{\ast}_{\alpha i} \left( (\gamma^\mu)_{\alpha}^{\beta} \partial_\mu - m \right) \psi^{\beta}_i$, from which we obtain $\pi_C = \partial L/\partial \dot{\psi}^i = i\psi^{\ast}_{\alpha i}(\gamma^0)_{\alpha}^{\beta} \dot{\psi}^{\beta}_i = i\psi^{\ast}_{\alpha i}(\gamma^0)_{\alpha}^{\beta} \delta_j \ast = i\psi^{\ast}_{\alpha i}(\gamma^0)_{\alpha}^{\beta} = i\psi^{\ast}_{\alpha i} z_{\alpha \beta} z_{ij} (\gamma^0)_{\alpha}^{\beta}$. Because $z_{\alpha \beta} (\gamma^0)_{\alpha}^{\beta} = \delta_{\alpha \beta}$ (see ref.\([16]\)), we have $\pi_{ij} = i\psi^{\ast}_{\alpha i}$. \(\ast\)
\[
+ \left[ b_s^{ij}(p)b_s^{ji}(p) + d_s^{ij}(p)d_s^{ji}(p) \right] z_{ij}
\]
\[
+ \sum_{s=1}^{2} d_3^3 p E \delta(0)( -\delta_{ij} z_{ij} - \delta_{ij} z_{ij} )
\]

(71)

The term with \( z_{ij} \) gives positive, whereas the term with \( z_{ij} \) gives negative expectation values. In the term with \( \delta(0) \), the positive and negative contributions cancel out, because \( z_{ij} = \delta_{ij} \), and \( z_{ij} = -\delta_{ij} \). Therefore, the vacuum expectation value of this Hamiltonian is zero.

Each fermion \( \psi^i \) couples to the corresponding gauge field. The Casimir force between two metallic plates, consisting of \( \psi^i, i = 1 \), is not expected to vanish in this theory [4]. The vacuum expectation value \( \langle H \rangle = \langle T^{00} \rangle \) is the source of the gravitational field. Because \( \langle 0 | H | 0 \rangle = 0 \), the cosmological constant vanishes. There is no problem of the huge cosmological constant.

6. Presence of interactions

6.1. Classical oscillator

We will now add an interaction term to the Lagrangian for an oscillator in the pseudo Euclidean space \( M_{1,1} \):

\[
L = \frac{1}{2}(\dot{x}^2 - \dot{y}^2) - V, \quad V = \frac{\omega^2}{2}(x^2 - y^2) + V_1
\]

(72)

Equations of motions are

\[
\ddot{x} + \omega^2 x + \frac{\partial V_1}{\partial x} = 0,
\]

(73)

\[
\ddot{y} + \omega^2 y - \frac{\partial V_1}{\partial y} = 0,
\]

(74)

Figure 3. The trajectory in the \((x, y)\) space of the system described by eqs. (76),(77). Left: For the time period from \( t = 0 \) to \( t = 25 \). Right: For the time period from \( t = 0 \) to \( t = 100 \).

As an example, we will study the following interaction:

\[
V_1 = \frac{\lambda}{4}(x^2 - y^2)^2.
\]

(75)
So we have

\[ \ddot{x} + \omega^2 x + \lambda (x^2 - y^2) = 0, \]  
(76)

\[ \ddot{y} + \omega^2 y + \lambda (x^2 - y^2) = 0. \]  
(77)

**Figure 4.** Left: The kinetic energy \( \dot{x}^2/2 \) as function of time. The envelop of oscillations grows to infinity. Right: The total energy, \( E_{\text{tot}} \), remains constant within the numerical error.

These equations can be solved numerically by the program Mathematica. In Figs. 3, 4 we show the result for \( \omega = 1, \lambda = 0.1 \) and the initial conditions \( \dot{x}(0) = 1, \dot{y}(0) = 0, x(0) = 0, y(0) = 1 \). We see that the system is unstable in the sense that the amplitudes of \( x(t), \dot{x}(t) \) grow to infinity. The total energy, \( E_{\text{tot}} = (\dot{x}^2 - \dot{y}^2)/2 + V(x, y) \), remains constant (Fig. 4).

In Fig. 5 we have the solution for the initial conditions \( \dot{x}(0) = 1, \dot{y}(0) = -1.2, x(0) = 0, y(0) = 0.5 \). Here also the system is unstable.

**Figure 5.** The solution of the system described by eqs. (76),(77), for the initial conditions \( \dot{x}(0) = 1, \dot{y}(0) = -1.2, x(0) = 0, y(0) = 0.5 \).
Something fascinating happens, if instead of eqs. (76), (77), we take slightly different equations,

\[
\ddot{x} + \mu [\omega^2 x + \lambda x(x^2 - y^2)] = 0, \quad (78)
\]
\[
\ddot{y} + \nu [\omega^2 y + \lambda y(x^2 - y^2)] = 0. \quad (79)
\]

The solution for eqs. (78)(79) for constants \(\mu = 1.01\) and \(\nu = 1\).

The solution for \(\mu = 1.01\), \(\nu = 1\), and the same initial condition as in Figs. 3, 4 is shown in Fig. 6. We see that the trajectory in the \((x, y)\) space does not escape into infinity. Instead, a second arm is formed, and the trajectory remains confined within a star-like envelope. The kinetic energy \(\dot{x}^2/2\) now does not grow to infinity. Instead, the amplitude of the kinetic energy oscillations is modulated by a slowly oscillating envelope.

In Fig. 6 we also repeat the calculation for \(\mu = 1.0001\) and \(\nu = 1\). This is now very close to \(\mu = 1\), \(\nu = 1\), the case of eqs. (76), (77). Now the trajectory in the \((x, y)\) space goes much farther from the origin than in the case \(\mu = 1.01\). But again it does not go to infinity; instead, after some time, the trajectory start to move within a second arm. The envelope of the kinetic energy oscillations now consists of separated peaks. If \(\mu\) approaches 1, the height of the peaks becomes higher and higher, and their separation increases. In the limit \(\mu \to 1\), the height of the first peak is infinite, and there is no second or other peaks (because their positions recedes to infinity).

We have found a very interesting result that \(\mu = 1\), \(\nu = 1\) is a singular case, in which the system is unstable. But if \(\mu\) slightly differs from \(\nu\), then the system is stable; its trajectory
remains confined within a finite region of the \((x,y)\) space, and the kinetic energy remains finite as well.

We will now show that the stable system (78),(79) is a special case of a more general system, described by the action

\[
I = \frac{1}{2} \int d\tau g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu, \quad \mu, \nu = 0, 1, 2, ..., D - 1. \tag{80}
\]

This is the Stueckelberg action [23] for a point particle in a \(D\)-dimensional curved space with a metric \(g_{\mu\nu}\) [8] (see also [24]). Introducing

\[
\gamma_{ab} = g_{ab} - \frac{g_{0a}g_{0b}}{g_{00}}, \tag{81}
\]

we can split\(^4\) the quadratic form according to

\[
g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu = \gamma_{ab} \dot{X}^a \dot{X}^b + \frac{\dot{X}_0^2}{g_{00}}. \tag{82}
\]

Here \(a, b = 1, 2, ..., D - 1\). In general, the signature is \((r, s)\), with \(r + s = D - 1\). We will take \(r = s\). Using (81),(82), the action (80) then becomes

\[
I = \frac{1}{2} \int d\tau \left( \gamma_{ab} \dot{X}^a \dot{X}^b + \frac{\dot{X}_0^2}{g_{00}} \right). \tag{83}
\]

If \(g_{\mu\nu} = 0\), then \(\dot{X}_0\) is a constant of motion\(^5\). We will assume that this is the case.

Assuming that \(\gamma_{ab, c} \equiv \partial \gamma_{ab} / \partial X^c \gamma_{ab} = 0\), the equations of motion derived from (83) are

\[
\ddot{X}^a + \frac{1}{2} \frac{C}{g_{00}^b} \gamma_{ab} = 0, \tag{84}
\]

where \(C = \dot{X}_0^2\) is a constant.

Introducing

\[
V = -\frac{1}{2} \frac{\dot{X}_0^2}{g_{00}} = -\frac{1}{2} \frac{C}{g_{00}}, \tag{85}
\]

we have

\[
\ddot{X}^a + V_b \gamma_{ab} = 0. \tag{86}
\]

The latter equations correspond to eqs. (78),(79), if \(\gamma_{ab} = \text{diag}(\gamma_{11}, \gamma_{22}, ..., \gamma_{D-1,D-1})\), i.e., a diagonal metric. Then eqs. (86) become

\[
\ddot{X}^1 + V_1 \gamma_{11} = 0,
\]

\[
\ddot{X}^2 + V_2 \gamma_{22} = 0,
\]

\[
\vdots \tag{87}
\]

\(^4\) An analogous splitting is used in Kaluza-Klein theories.

\(^5\) Analogously, in Kaluza-Klein theories, \(\dot{X}_5\) is a constant of motion, if the \(5D\) metric does not depend on the fifth dimension.
If we consider the case $D - 1 = 2$, identify $X^1 \equiv x$, $X^2 \equiv y$, $\gamma^{11} \equiv \mu$, $\gamma^{22} \equiv \nu$, and take
$V = \frac{\omega^2}{2}(x^2 - y^2) + \frac{\lambda}{4}(x^2 - y^2)^2$, then the system (87) becomes the system (78),(79).

The $g_{\mu\nu}(x)$ in the action (80) need not be a fixed background metric field. It can be a dynamical field. Then one has to include into the action (80) a kinetic term for $g_{\mu\nu}$, e.g., the Einstein-Hilbert term with the curvature scalar. The dynamics of such a system is involved, and, in view of the results given in Fig.6, one cannot a priori claim that the system is unstable.

6.2. Collision of the oscillator with a free particle

Let us assume that in the surroundings of the oscillator, $O$, described by the Lagrangian (72), $P$, there is free particle, $P$. Depending on the initial conditions, it may happen that the oscillator hits the particle. Such combined system of $O$ and $P$ can be modeled by the Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 - \dot{y}^2) - \frac{1}{2}(x^2 - y^2) - \frac{\lambda}{2}(x^2 - y^2)^2 + \frac{1}{2}(\dot{u}^2 + \dot{v}^2) - \frac{\alpha/5}{[(u-x)^2 + (v-y)^2 + a]^2},$$

where $u, v$ are particle’s coordinates. The equations of motions are

$$\ddot{x} + x + \lambda x(x^2 - y^2) = \frac{\alpha(u-x)}{(u-x)^2 + (v-y)^2 + a} \gamma^x = 0$$
$$\ddot{y} + y + \lambda y(x^2 - y^2) = \frac{\alpha(u-y)}{(u-x)^2 + (v-y)^2 + a} \gamma^y = 0$$
$$\ddot{u} - \frac{\alpha(u-x)}{(u-x)^2 + (v-y)^2 + a} = 0$$
$$\ddot{v} - \frac{\alpha(u-y)}{(u-x)^2 + (v-y)^2 + a} = 0$$

Figure 7. Collision of the oscillator with a particle. Up: Particle’s position and velocity as functions of time. Low left: Oscillator’s kinetic energy $\dot{x}^2/2$ as function of time. Low left: The total energy of the the oscillator and the particle as function of time.

The numerical solution for the constants $\Lambda = 0.1$, $\alpha = 1$, and the initial conditions $\dot{x}(0) = 1$, $\dot{y}(0) = 0$, $\ddot{u}(0) = 0$, $\ddot{v}(0) = 0$, $x(0) = 0$, $y(0) = 1$, $u(0) = 12$, $v(0) = 11.5$ is shown in Fig.7. We see that the solution properly reproduces the fact that the particle $P$
is initially at rest, and after the interaction with the oscillator $O$, its moves with a constant velocity.

The positive component of the oscillator’s kinetic energy starts to increase, but at time around $t = 112$ it drops down to zero. After a while, the positive kinetic energy "recovers" and start to increase again. A further collision with some other particle would again drop down $\dot{x}^2/2$. Analogous holds for $\dot{y}^2/2$. We see that, according to this numerical solution, the surrounding particles stabilize the oscillator and prevent it to escape into the infinity. We expect that many such oscillators, immersed into a bath of particles would increase their average $\dot{u}^2/2$ and $\dot{v}^2/2$, i.e., the temperature of the bath. Further investigation of this interesting topics is beyond the scope of the present paper.

6.3. Quantum oscillator

The quantum oscillator is described by the wave function $\psi(t, x, y)$, satisfying the Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = H \psi,$$

with

$$H = \frac{1}{2} \left( -\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} \right) + V(x, y).$$

Let us expand the wave function in terms of the basis functions of the 2D harmonic pseudo-Euclidean oscillator, which are the same as those of the Euclidean oscillator:

$$\psi = \sum_{m,n=0}^{\infty} c_{mn}(t) \psi_{mn}.$$  

Here $\psi_{mn} = \frac{1}{\pi^{1/4} \sqrt{2^n n!}} \frac{1}{\pi^{1/4} \sqrt{2^m m!}} H_n(x) H_m(y) e^{-(x^2+y^2)/2}$ are orthonormal eigenfunctions (with positive or negative energies), of the Hamiltonian

$$H_0 = \frac{1}{2} \left( -\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} \right) + \frac{1}{2} (x^2 - y^2).$$

Figure 8. The potential used in the calculations of the quantum pseudo-Euclidean harmonic oscillator described by eqs. (90),(91).

Using the matrix elements

$$H_{mn;rs} = \int dx \, dy \, \psi^*_{mn} H \psi_{rs},$$
the Schrödinger equation (90) can be rewritten as

\[ i \dot{c}_{mn} = \sum_{rs} H_{mn;rs} c_{rs}, \]  

(95)

We will investigate the case of the potential (see Fig. 8)

\[ V(x, y) = \frac{1}{2} \epsilon \left( 1 - e^{-\epsilon (x^2 - y^2)} \right), \]  

(96)

where \( \epsilon = \text{sign} (x^2 - y^2) \), i.e., \( \epsilon = 1 \), if \( x^2 - y^2 > 0 \), \( \epsilon = -1 \), if \( x^2 - y^2 < 0 \), and \( \epsilon = 0 \), if \( x^2 - y^2 = 0 \). Such a potential is a 2D model of a potential that could eventually occur in a three or higher dimensional world, where potentials running into infinity are unrealistic.

Figure 9. The plot of \(|\psi(t, x, y)|^2\), calculated for the initial conditions \(c_{00}(0) = 1\), \(c_{01}(0) = C_{10}(0) = c_{11}(0) = ... = 0\), at different values of the time \(t\).

Instead of the system (95) of infinite number of first order differential equations, let us consider a finite system with \(m, n = 0, 1, 2, ..., N\). Then we can numerically solve the system and calculate the coefficients \(C_{mn}(t)\) for given initial conditions. So we obtain the wave functions

\[ \psi(t, x, y) = \sum_{m,n=0}^{N} c_{mn}(t) \psi_{mn}, \]  

(97)

and its absolute square, \(|\psi(t, x, y)|^2\).

In Fig. 9 we show the result for \(N = 4\), and the initial condition \(c_{00} = 1\), with the remaining coefficients being equal to zero. We see that with increasing time \(t\), the vacuum \(\psi(0) = \psi_{00} = \) gradually decays, and, after a while (at \(t \approx 5\)) occurs again. The wave function thus oscillates between the vacuum and a decayed vacuum.

In Fig. 10 we show the calculation of \(|\psi(t, x, y)|^2\) for the initial conditions \(c_{01}(0) = 1/\sqrt{2}\), \(c_{10}(0) = 1/\sqrt{2}\), \(c_{00}(0) = 0\), \(c_{11}(0) = 0\), \(c_{12}(0) = 0\), ..., \(c_{44}(0) = 0\). Initially, we have two
Figure 10. The plot of $|\psi(t, x, y)|^2$, calculated for the initial conditions $c_{01}(0) = C_{10}(0) = 1/\sqrt{2}$, $c_{00}(0) = c_{11}(0) = c_{12}(0) = ... = 0$, at different values of the time $t$. 
Figure 11. Up: The plot of $|\psi(t, x, y)|^2$, calculated for the initial conditions $c_{01}(0) = C_{10}(0) = 1/\sqrt{2}$, $c_{00}(0) = c_{11}(0) = c_{12}(0) = \ldots = 0$, at two very large values the time, $t = 500$ and $t = 700$. Low: The plot of $|\psi(t, x = 0, y = 50)|^2$, as a function of $t$, calculated for the same initial conditions as above.

peaks, one for the positive energy excitation, $c_{01}(0)$, and the other one for the negative energy excitations, $c_{01}(0)$. The system then oscillates as shown in Fig. 10 and Fig. 11.

Such a wave function (98), of course, is not a solution of the Schrödinger equation (90) for $H$ given in eqs. (91), (96). Since $c_{mn}(t)$, $m, n = 0, 1, 2, 3, 4$, are solution of the truncated system (95), also $\psi(t, x, y)$ of eq. (97) is a solution of a truncated system, presumably of the Schrödinger equation on the lattice. A realistic lattice is a discrete set of closely separated points corresponding to the system, not of the $5 \times 5$ equations (95), but of the $N \times N$ equations, with $N$ very large. The $N$ is related to the maximum absolute energy of the oscillator excitations that, in turn, is related to the minimum distance (cutoff). Regardless of how large is $N$, and how small is the cutoff distance, the examples of Figs. 9–11 indicate that there are oscillations, and that the system is thus stable. Only in the limit of infinitely small cutoff distance, implying infinite quantum numbers $m, n$ of positive and negative energy excitations, would the system be unstable. Then the peak in Fig. 9 would spread to infinity. However, according to common belief, the Planck length is the cutoff distance under which it is no longer possible to probe spacetime distances. Our spacetime is then like a huge lattice. Then also the corresponding Clifford space, $C$, is like a lattice. The space $M_{1,1}$ is a subspace of $C$. Instead of the infinite system of differential equations (95), we then have a finite, though very big, system of differential equations.

6.4. Interacting quantum fields

6.4.1. Scalar fields  As an example let us consider the scalar fields described by the action

$$I = \frac{1}{2} \int d^4x [g^{\mu\nu} \partial_\mu \varphi^a \partial_\nu \varphi^b \gamma_{ab} + V(\varphi)],$$

(98)
which is similar to the action (43), except that now $V(\varphi)$ is a more general potential with an interaction term, e.g., $V(\varphi) = m^2\varphi^a\varphi^b\gamma_{ab} + \varphi^a\varphi^b\varphi^c\varphi^d\lambda_{abcd}$.

Upon quantization, the system is described by a state vector,

$$|\Psi\rangle = \sum |P\rangle \langle P|\Psi\rangle,$$

that can be expanded in terms of the Fock space basis vectors $|P\rangle \equiv |p_1p_2...p_n\rangle$ that are eigenvectors of the free field Hamiltonian $H_0$ (without the interaction quartic term $\varphi^a\varphi^b\varphi^c\varphi^d\lambda_{abcd}$). Also the vacuum, $|0\rangle$, is an eigenstate $H_0$. If the metric $\gamma_{ab}$ has signature $(r,s)$, the states $|P\rangle$ can have positive or negative energies.

The time evolution of the state vectors is governed by the Hamilton operator $H$, corresponding to the field action (98):

$$|\Psi(t)\rangle = e^{-iH(t-t_0)}|\Psi(t_0)\rangle.$$

From (100) we have

$$\langle P|\Psi(t)\rangle = \sum_{P'} \langle P|e^{-iH(t-t_0)}|P'\rangle \langle P'|\Psi(t_0)\rangle.$$

If the initial state is the vacuum, $|\psi(t_0)\rangle = |0\rangle$, an eigenstate of the unperturbed Hamiltonian, $H_0$, then we have

$$\langle P|\Psi(t)\rangle = \langle P|e^{-iH(t-t_0)}|0\rangle.$$

Such transition is possible, because a state $|P\rangle$ contains particles with positive and negative energies. Therefore, the total energy in such transition is conserved and remains equal to the energy of the initial state $|0\rangle$, which is zero.

The vacuum thus decays into a superposition of many particle states:

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} |p_1p_2...p_n\rangle \langle p_1p_2...p_n|\Psi(t)\rangle.$$

Here $\langle p_1p_2...p_n|\Psi(t)\rangle$ is the amplitude that we will measure the multiparticle state $|p_1p_2...p_n\rangle$.

The probabilities that the vacuum decays into any of the states $|p_1\rangle$, $|p_1p_2\rangle$, $|p_1p_2...p_n\rangle$, ..., are not drastically different from each other, and they sum to 1:

$$\sum_{p_1} |\langle p_1|\Psi\rangle|^2 + \sum_{p_1,p_2} |\langle p_1,p_2|\Psi\rangle|^2 + \sum_{p_1,p_2,...,p_n} |\langle p_1,p_2,...,p_n|\Psi\rangle|^2 + ... = 1.$$

We see that the probability of vacuum decay into 2,4,6,8, or any finite number of particles, is negligible in comparison to the probability of the decay into infinite number of particles, because such configuration spaces occupy the vast majority of the terms in the sum (104). According to such reasoning, the vacuum instantly decays into infinitely many particles. It is then argued that because such instantaneous vacuum decay makes no sense in physics, ultrahyperbolic spaces are unphysical. In my opinion such conclusion is not unavoidable, for the reasons described below and in the remaining parts of the paper.

Let us consider a generalization of the field action (98). We can rewrite it in a more compact notation:

$$I = \frac{1}{2} \partial_{\mu} \varphi^{a(x)} \partial_{\nu} \varphi^{b(x')} \gamma^{\mu\nu}_a(x)b(x') - U[\varphi]$$
Here \( \varphi^a(x) \equiv \varphi^a(x) \), where \( x \) is the continuous index, denoting components of an infinite dimensional vector. In addition, for every \( x \), the components are also denoted by a discrete index \( a \). Altogether, vector components are denoted by the double index \( a(x) \). Alternative notation, often used in the literature, is \( \varphi^a(x) \equiv \varphi^{ax} \) or \( \varphi^a(x) \equiv \varphi^{(ax)} \).

The action (105) may be obtained from a higher dimensional action

\[
I_\phi = \frac{1}{2} \partial_\mu \phi^{A(x)} \partial^\nu \phi^{B(x')} G^{\mu\nu}_{A(x)B(x')},
\]

where \( A = (a, \bar{A}) \), and \( \phi^{A(x)} = (\phi^a(x), \phi^{\bar{a}}(x)) \). The higher dimensional metric is a functional of \( \phi^A(x) \). Performing the Kaluza-Klein split,

\[
G^{\mu\nu}_{AB} = \left( \begin{array}{cc} \gamma^{\mu\nu}_{ab} + A_a^A B^b_B G^{\mu\nu}_{AB}, & A_a^A B^b_B G^{\mu\nu}_{AB} \\ A_b^B G^{\mu\nu}_{AB}, & G^{\mu\nu}_{AB} \end{array} \right),
\]

where for simplicity we have omitted the index \( x \), we find,

\[
I_\phi = \frac{1}{2} \partial_\mu \phi^a(x) \partial^\nu \phi^b(x') \gamma^{\mu\nu}_{a(x)b(x')} + \frac{1}{2} \partial_\mu \phi^{\bar{a}}(x) \partial^\nu \phi^{\bar{b}}(x') G^{\mu\nu}(\bar{a}(x) \bar{b}(x)).
\]

Identifying \( \phi^a(x) \equiv \varphi^a(x) \), and denoting \( \frac{1}{2} \partial_\mu \phi^{a}(x) \partial^\nu \phi^{b}(x') G^{\mu\nu}(\bar{a}(x) \bar{b}(x)) = -U[\varphi] \), we obtain the action (105).

Thus, the field action (98) is embedded in a higher dimensional action with a metric \( G^{\mu\nu}_{A(x)B(x)} \) in field space. A question arises as to which field space metric to choose. The lesson from general relativity tells us that the metric itself is dynamical. Let us therefore assume that this is so in the case of field theory[8] as well. Then (106) must be completed by a kinetic term, \( I_G \), for the field space metric [8]. The total action is then

\[
I[\phi, G] = I_\phi + I_G.
\]

According to such dynamical principle, not only the field \( \phi \), but also the metric \( G^{\mu\nu}_{A(x)B(x)} \) changes with the evolution of the system. This implies that also the potential \( U[\varphi] \) of eq. (105) changes with evolution, and so does the potential \( V(\varphi) \), occurring in eq. (98).

Let us assume that the action (109) describes the whole universe\(^6\). Then \( |\psi(t)\rangle \) of eq. (103) contains everything in such universe, including observers. There is no external observer, \( O_{ext} \), according to whom the coefficients \( \langle p_1...p_n|\psi(t)\rangle \) in eq. (103) could be related to the probability densities \( |\langle p_1...p_n|\psi(t)\rangle|^2 \) of finding the system in an \( n \)-particle states with momenta \( p_1, ..., p_n, n = 0, 1, 2, 3, ..., \infty \). There are only inside observers, \( O \), incorporated within appropriate multiparticle states \( |p_1...p_n\rangle, n = 0, 1, 2, 3, ..., \infty \), of the “universal” state \( |\psi(t)\rangle \). According to the Everett interpretation of quantum mechanics, all states, \( |0\rangle, |p_1p_2|, ..., |p_1...p_n|, ..., n = 0, 1, 2, ..., \infty \), in the superposition (103) actually occur, each in a different world. The Everett interpretation is now getting increasing support among cosmologists (see, e.g., Ref. [25]). For such an inside observer, \( O \), there is no instantaneous vacuum decay into infinitely many

\(^6\) Of course, such a model universe is not realistic, because our universe contains fermions and accompanying gauge fields as well.
particles. For $O$, at a given time $t$, there exists a configuration $|P\rangle$ of $n$-particles (that includes $O$ himself), and a certain field potential $V(\varphi)$ (coming from $G^{\mu\nu}A(x)B(x)$). At some later time, $t + \Delta t$, there exists a slightly different configuration $|P'\rangle$ and potential $V'(\varphi)$, etc. Because $O$ nearly continuously measures the state $|\psi(t)\rangle$ of his universe, the evolution of the system is being “altered”, due to the notorious “watchdog effect” or “Zeno effect” [26] of quantum mechanics (besides being altered by the evolution of the potential $V(\varphi)$). The peculiar behavior of a quantum system between two measurements has also been investigated in Refs. [27].

6.4.2. The generalized Dirac field In Sec. 4.2.2 we considered the generalized Dirac field described by the Dirac-Kähler equation (55). In the spinor basis of $Cl(1,3)$, the field can be represented by a $4 \times 4$ matrix

$$
\psi = \begin{pmatrix}
\psi^{11} & \psi^{12} & \psi^{13} & \psi^{14} \\
\psi^{21} & \psi^{22} & \psi^{23} & \psi^{24} \\
\psi^{31} & \psi^{32} & \psi^{33} & \psi^{34} \\
\psi^{41} & \psi^{42} & \psi^{43} & \psi^{44}
\end{pmatrix},
$$

(110)

where each column represents a Dirac spinor. The components $\psi^{\alpha i}$ have either positive or negative energies, according to the following scheme:

$$
\text{Energy} = \begin{pmatrix}
+ & + & - & - \\
+ & + & - & - \\
- & - & + & + \\
- & - & + & +
\end{pmatrix},
$$

(111)

which is a consequence of the metric (62),(63).

If we add an interaction term to the action (61), then transitions between positive and negative energy states become possible. On the other hand, positive and energy states of the usual Dirac spinors do not mix in our Universe. Even if once they did mix, the evolution of the Universe must have led to the current situation with no mixing. This was not so clear when Dirac proposed his theory. The occurrence of negative energy states was puzzling at that time, and Fermi [28] wrote:

"It is well known that the most serious difficulty in Dirac’s relativistic wave equations lies in the fact that it yields besides the normal positive states also negative ones, which have no physical significance. This would do no harm if no transition between positive and negative state were possible (as are, e.g., transitions between states with symmetrical and antisymmetrical wave function). But this is unfortunately not the case: Klein has shown by a very simple example that electrons impinging against a very high potential barrier have a finite probability of going over in a negative state."

Thus, Dirac’s relativistic wave equation could have been put aside and ignored. Fortunately, this did not happen. The problem was resolved by the Dirac sea of negative energy states.

Now the situation is analogous as in 1932. Within the Clifford algebra framework, besides the negative energy states of the first left ideal of $Cl(1,3)$, we have also negative energy states of the second, third and fourth ideal. The existence of such states also should not be considered a priori as problematic. A deeper investigation is necessary before making any definite conclusion.

A column may represent the Weyl and Majorana spinors as well
6.4.3. Clifford algebra description of fermionic fields

It is known that spinors can be represented as Fock space like objects \([21, 17, 22]\) embedded in a Clifford algebra. An analogous procedure can be carried out for spinor fields \([8, 22]\). A vector \(\Psi\) in an infinite dimensional space, \(S\), can be represented as \([8, 22]\)

\[
\Psi = \psi^r(x) h_{r(x)}, \quad r = 1, 2; \quad x \in \mathbb{R}^3 \text{ or } x \in \mathbb{R}^{1,3}.
\]

Here \(\psi^r(x)\) are the vector components (see Sec. 6.4.1), and \(h_{r(x)}\) are basis vectors, represented as generators of the infinite dimensional Clifford Algebra, \(Cl(\infty)\), satisfying

\[
h_{r(x)} \cdot h_{r'(x')} = \frac{1}{2} (h_{r(x)} h_{r'(x')} + h_{r'(x')} h_{r(x)}) = \rho_{r(x)r'(x')},
\]

where \(\rho_{r(x)r'(x')}\) is the metric of \(S\). In particular, it can be \(\rho_{r(x)r'(x')} = \delta_{rr'}\delta(x - x')\). The 4-component spinor indices, \(\alpha = 1, 2, 3, 4\), and the index \(i = 1, 2, 3, 4\), denoting the ideals, is implicit in \(\psi^r(x) \equiv \psi^{\alpha i r}(x)\), and \(h_{r(x)} \equiv h_{\alpha i r}(x)\). In a new basis, called the Witt basis,

\[
h_{(x)} = \frac{1}{\sqrt{2}}(h_{1(x)} + ih_{2(x)}) \quad \text{(114)}
\]

\[
h_{*(x)} = \frac{1}{\sqrt{2}}(h_{1(x)} - ih_{2(x)}) \quad \text{(115)}
\]

instead of (113), we have\(^8\)

\[
h_{(x)} \cdot h_{*(x')} = \rho_{(x)*(x')}, \quad \text{(116)}
\]

\[
h_{(x)} \cdot h_{(x')} = h_{*(x)} \cdot h_{*(x')} = 0, \quad \text{(117)}
\]

which are fermionic anticommutation relations. A vector \(\Psi\) can be expanded as

\[
\Psi = \psi^{(x)} h_{(x)} + \psi^{*(x)} h_{*(x)}. \quad \text{(118)}
\]

The scalar product is

\[
\langle \Psi \Psi \rangle_S = \psi^{(x)} \rho_{(x)*(x')} \psi^{*(x')} + \psi^{*(x)} \rho_{*(x)}(x') \psi^{(x')} \quad \text{(119)}
\]

Let us introduce the object

\[
\Omega = \prod_x h_{*(x)} \quad \text{(120)}
\]

satisfying

\[
h_{*(x)} \Omega = 0. \quad \text{(121)}
\]

We see that \(\Omega\) has the role of vacuum, and \(h_{*(x)}\) are annihilation operators. The object \(h_{(x)}\) gives a state \(h_{(x)} \Omega \neq 0\), with a hole in vacuum. Therefore, \(h_{(x)}\) are creation operators (of holes in \(\Omega\)).

If we act with a vector \(\Psi\) on the vacuum \(\Omega\), we obtain

\[
\Psi \Omega = \psi^{(x)} h_{(x)} \Omega. \quad \text{(122)}
\]

\(^8\) Instead of the indices 1, 2, denoting real and imaginary component, we now use the space, \(\cdot\), and the star, \(\ast\).
This is a superposition of one particle (or better, one hole) states.

A generalization of eq. (122) is

\[ (\psi_0 + \psi(x)h(x) + \psi(x^\prime)h(x^\prime) + \ldots)\Omega. \]  

(123)

This is a superposition of zero, one two,...,many,..., particle states. Such a state is the infinite dimensional analog [22] of the spinor as an element of a left ideal\(^9\) of Clifford algebra [21, 22].

Besides (121), there are other possible vacuums, e.g.,

\[ \Omega = \prod_x h(x), \quad h(x)\Omega = 0 \]  

(124)

\[ \Omega = \left( \prod_{x \in R_1} h(x) \right) \left( \prod_{x \in R_2} h(x) \right), \quad h(x)\Omega = 0, \quad \text{if} \ x \in R_1 \]

\[ h(x)\Omega = 0, \quad \text{if} \ x \in R_2 \]  

(125)

eetc.

where\(^{10}\) \(R_1 \in \mathbb{R}^3, \ R_2 \in \mathbb{R}^3, \ R_1 \cup R_2 = \mathbb{R}^3\).

An analogous situation also holds in momentum representation, where the operators are \(c(p), c_*(p)\), with \(p = (p^0, \mathbf{p})\). The vacuum can then be defined as, e.g.,

\[ \Omega = \left( \prod_{p^0 > 0, \mathbf{p}} c_*(p^0, \mathbf{p}) \right) \left( \prod_{p^0 < 0, \mathbf{p}} c(p^0, \mathbf{p}) \right). \]  

(126)

Instead of the above notation, adapted to the Clifford algebraic description, we will now again use our notation of sec. 5.2.2, which is a straightforward generalization of the usual notation.

We have the following operators in momentum space:

\[ b_\dagger_i(s)(p), \quad d_\dagger_i(s)(p) \]  

(127)

\[ b_i(s)(p), \quad d_i(s)(p) \]  

(128)

which annihilate the vacuum

\[ \Omega = \left( \prod_{s, \mathbf{p}} b_\dagger_i(s)(p) \right) \left( \prod_{s, \mathbf{p}} d_\dagger_i(s)(p) \right) \left( \prod_{s, \mathbf{p}} b_i(s)(p) \right) \left( \prod_{s, \mathbf{p}} d_i(s)(p) \right). \]  

(129)

The Fock space states are

\[ b_\dagger_i(s)(p)\Omega, \quad d_\dagger_i(s)(p)\Omega, \quad b_\dagger_i\Omega(p), \quad d_\dagger_i\Omega(p), \ldots, \text{and all many particle states} \]

\[ \text{positive energies} \quad \text{negative energies} \]  

(130)

We see that, though considered nowadays as obsolete, the concept of the Dirac sea finds its revival with the Clifford algebra approach to field theory. Besides (129), other vacua can also be constructed by taking various combinations of the operators \(b_\dagger_i, b_i, d_\dagger_i, d_i\) (see ref. [22]).

\(^9\) In the case of a finite dimensional Clifford algebra \(Cl(p, q)\), a left ideal is a subspace of \(Cl(p, q)\) that is invariant under the left multiplication by any element of \(Cl(p, q)\).

\(^{10}\) If we consider the Stueckelberg theory [23], then instead of \(\mathbb{R}^3\) we have \(\mathbb{R}^{1,3}\).
In the absence of interactions, the vacuum (129) has vanishing energy, i.e., the expectation value of the Hamiltonian (67) is \( \langle \Omega^\dagger H \Omega \rangle_S = 0 \). In the presence of interactions that mix the positive and negative energy states, the vacuum \( \Omega \) decays into a superposition of positive and negative energy states. The finite state with infinitely many positive and negative energy particles, \( \{|p_1p_2...p_\infty\}\rangle \), is the state in which all the operators were removed from the vacuum \( \Omega \):

\[
\Psi(t) = b_{s_1}^\dagger(p_1)b_{s_2}^\dagger(p_2)...d_{s_1}^\dagger(p_1)d_{s_2}^\dagger(p_2)...\Omega = 1 \quad (131)
\]

The expectation value of the free Hamiltonian (67) in the latter state is also zero. The state (131) also is “unstable” and can evolve into another state that is a superposition of the basis states

\[
b_{s_1}^\dagger(p), \quad d_{s}^\dagger(p) \quad b_{s}^\dagger(p), \quad d_{s}^\dagger(p) , \quad \text{and all many operator states} \quad (132)
\]

In a finite (closed) universe, with a minimal (e.g., Planck) distance, there would be a maximal absolute value of the particle’s energy, and there would be a finite number of discrete momenta values. The \( p \) in eqs. (129) would be discrete. The unstable vacuum \( \Omega \) would then decay, in a finite time, into a finite number of positive and negative energy states. The latter state would then “decay” back into \( \Omega \) (just as in the simple model of the oscillator illustrated in Fig.9). So there would be oscillations between \( \Psi = \Omega \) and \( \Psi = 1 \). Such would be the situation for an external observer, \( O_{\text{ext}} \), who does not disturb the state \( \Psi(t) \). But for an observer who is a part of the universe described by the state \( \Psi(t) \), and who “nearly” continuously watches (at least a part of) his universe, the evolution of \( \Psi(t) \) is “altered” according to the watchdog effect of QM; it is no longer an undisturbed evolution. The discussion at the end of Sec. 6.3.1 is valid for fermionic fields as well.

7. Discussion

Clifford algebras and Clifford spaces appear as a promising framework for the unification of particles and (gauge) fields. Such spaces do not have Lorentzian signature, \( (+ - - - - ...) \), but a more general signature \( (+ + - - - ...) \). The Clifford space, \( C \), whose tangent space at any of its point is the Clifford algebra \( C(1,3) \), has the neutral signature \( (8, 8) \). We have investigated the stability of the classical and quantum harmonic oscillator in the space \( M_{1,1} \), which is a subspace of \( C \). It is known [1, 4] that the negative energies occurring in a space with neutral signature pose no problem, if there are no interactions. We have shown that even the presence of an interaction causing transitions between positive and negative energy states, need not be problematic, and that such a system may have stable solutions.

In the case of the classical oscillator, we have found that certain interactions prevent the runaway solutions, and make the system stable. Collisions of an otherwise unstable pseudo Euclidean oscillator with surrounding particles also stabilize the oscillator. After such collisions, the particles gain kinetic energy. A material made of such oscillators would thus increase the temperature of the surrounding medium, after being immersed into it. This is very hypothetical, but the history of physics teaches us that we can never be sure about what surprises are waiting ahead of us. Recall that quasicrystals, with crystallographically forbidden symmetries\(^\text{11}\), cannot

\(^\text{11}\) Schechtman’s discovery of quasicrystals [29] was ridiculed, because, in view of the established crystallographic
be explained in terms of local interactions in three dimensions. They can be explained as regular crystals in 6-dimensions, projected in 3-dimensions. The Clifford space, associated with objects in spacetime, has sixteen dimensions, and one can envisage that there exist in $C$ the crystals that, from the 3D point of view, appear as quasicrystals. If such an explanation of quasicrystals eventually turns out to be correct, then the next step would be to investigate whether the pseudo Euclidean oscillators, allowed by the physics in Clifford space, actually exist in nature.

In the case of the quantum oscillator in $M_{1,1}$, we have calculated numerical solutions for a truncated system. Because the truncated system is finite, the solution is oscillating, and does not run away into infinity. For practical reasons we considered the energy modes up to $m, n = 4$, so that all higher energies were cut off. We can envisage a cutoff at much higher energies. This corresponds to a small minimum cutoff distance$^{12}$ in the space $M_{1,1}$. Then, instead of continuous space $M_{1,1}$, we have a finite periodic lattice. An analogous situation occurs if we consider an oscillator in 16D Clifford space.

After having investigated how the theory works on the examples of the classical and quantum pseudo Euclidean oscillator, we considered quantum field theories. If the metric of field space is neutral, then there occur positive and negative energy states. An interaction causes transitions between those states. A vacuum therefore decays into a superposition of states with positive and negative energies. Because of the vast phase space of infinitely many particles, such a vacuum decay is instantaneous—for an external observer. But we, as a part of our universe, are not external observers. We are internal observers entangled with the “wave function” of the rest of our universe. According to the Everett interpretation of quantum mechanics, we find ourselves in one of the branches of the universal wave function. In the scenario with decaying vacuum, our branch can consist of a finite number of particles. Once being in such a branch, it is improbable that at the next moment we will find ourselves in a branch with infinitely many particles. For us, because of the “watchdog effect” of quantum mechanics, the evolution of the universal wave function relative to us is frozen to the extent that instantaneous vacuum decay is not possible. The Everett interpretation is gaining increasing support among quantum cosmologists [25].

We have also pointed out that the field potential $V(\varphi)$ need not be constant during the evolution of the universe. It may change, and thus alter a system’s dynamics from an unstable to a stable regime.

If there exists a minimal distance (e.g., at the Planck scale), and if the universe is closed, then it contains a finite, albeit very large, number of points; it is a sort of lattice. Such a finite system cannot run into infinity, because there is neither infinite energy or infinitely small distance, neither infinite number of particles. Therefore, such a system is stable even in the presence of negative energies. It oscillates with large, but finite, amplitudes. In Sec. 6.3 we showed how does oscillate a simple quantum finite oscillator. An analogous situation holds for quantum fields on a finite lattice.

In a field theory with neutral signature, there is an outburst of positive and negative energy

theory, it was considered as impossible. Fortunately, that was a simple experiment and it was not difficult for other labs to repeat it.

$^{12}$ It is generally believed that the Planck distance is the minimal distance, therefore the idea that space or Clifford space is in fact a lattice makes sense.
particles, like an explosion, that is eventually stabilized. This is reminiscent of the Big Bang. Our universe indeed emerged in an explosion. But in our universe we do not “see” equal number of positive and negative energy particles. Can then such a quasi unstable vacuum be an explanation for Big Bang?

Description of our universe requires fermions and accompanying gauge fields, including gravitation. According to the Clifford algebra generalized Dirac equation—Dirac-Kähler equation—there are four sorts of the 4-component spinors, with energy signs a shown in eq. (111). The vacuum of such field has vanishing energy and evolves into a superposition of positive and negative energy fermions, so that the total energy is conserved. A possible scenario is that the branch of the superposition in which we find ourselves, has the sea of negative energy states of the first and the second, and the sea positive energy states of the third and forth minimal left ideal of Cl(1, 3). According to ref. [17], the former states are associated with the familiar, weakly interacting particles, whereas the latter states are associated with mirror particles, coupled to mirror gauge fields, and thus invisible to us. According to the field theory based on the Dirac-Kähler equation, the unstable vacuum could be an explanation for Big Bang.

Introduction of spaces with neutral signature that imply the existence of negative energies, is potentially significant for further important progress of theoretical physics. The usual arguments—mostly related to stability—why such spaces are not suitable for physics, can be circumvented along the lines indicated in this paper.

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