DEFORMATIONS OF NONCOMPACT COMPLEX CURVES
AND MEROMORPHIC ENVELOPES OF SPHERES

S. M. IVASHKOVICH AND V. V. SHEVCHISHIN

23/JAN/98

ABSTRACT. The paper is devoted to the properties of the envelopes of meromorphy of neighborhoods of symplectically immersed two-spheres in complex Kähler surfaces. The method used to study the envelopes of meromorphy is based on Gromov’s theory of pseudoholomorphic curves. The exposition includes a construction of a complete family of holomorphic deformations of a non-compact complex curve in a complex manifold, parametrized by a finite codimension analytic subset of a Banach ball. The existence of this family is used to prove a generalization of Levi’s continuity principle, which is applied to describe envelopes of meromorphy.

Bibliography: 15 titles.

0. Introduction

In the present paper we study the envelopes of meromorphy of neighborhoods of two-spheres immersed in complex surfaces.

Throughout the paper a complex surface means a (Hausdorff) connected complex two-dimensional manifold $X$ countable at infinity. Let $U$ be a domain in $X$. Its envelope of meromorphy $(\hat{U}, \pi)$ is the maximal domain over $X$ satisfying the following conditions:

(i) there exists a holomorphic embedding $i: U \to \hat{U}$ with $\pi \circ i = \text{Id}_U$;
(ii) each meromorphic function $f$ on $U$ extends to a meromorphic function $\hat{f}$ on $\hat{U}$, that is, $\hat{f} \circ i = f$.

The envelope of meromorphy exists for each domain $U$. This can be proved, for example, by applying the Cartan–Thullen method to the sheaf of meromorphic functions on $X$, see [1].

In the sequel we shall restrict ourselves to Kähler complex surfaces, that is, we assume that $X$ carries a strictly positive closed $(1, 1)$-form $\omega$.

Let $S$ be an oriented real surface.

Definition. A $C^1$-smooth immersion $u: S \to (X, \omega)$ is called symplectic if $u^*\omega$ does not vanish anywhere on $S$.

The aim of the present paper is the following result.

This research was carried out with the financial support of the ‘RiP’ program of the Mathematical Institute in Oberwolfach.
Main theorem. Let \( u : S^2 \to X \) be a symplectic immersion of the two-sphere \( S^2 \) in a disc-convex Kähler surface \( X \) such that \( M := u(S) \) has only positive double points. Assume that \( c_1(X)[M] > 0 \). Then the envelope of meromorphy \((\hat{U}, \pi)\) of an arbitrary neighborhood \( U \) of \( M \) contains a rational curve \( C \) with \( \pi^* c_1(X)[C] > 0 \).

The definition of disc-convexity is given in §4. At this point, we only observe that all compact manifolds are disc-convex. As usual, let \( c_1(X) \) be the first Chern class of \( X \).

This result is a considerable improvement of Theorem 1 in [2]. First of all, the ‘positivity’ condition for \( X \) is removed. Moreover, another condition from [2] is weakened, namely, the condition \( c_1(X)[M] \geq \delta \). In the case when \( X = \mathbb{CP}^2 \) this means that \( [M] \) may have arbitrary degree in \( H_2(\mathbb{CP}^2, \mathbb{Z}) \), whereas the condition \( c_1(X)[M] \geq \delta \) imposes the restriction \( \deg[M] \leq 8 \) (cf [2]; Corollary 1).

It may be helpful to observe (cf., for example, [3]) that a compact complex algebraic surface \( X \) contains a smooth rational curve \( C \) with \( c_1(X)[C] > 0 \) only in the following three cases:

1. \( c_1(X)[C] = 1 \Rightarrow [C]^2 = -1 \Rightarrow C \) is an exceptional curve of the first kind;
2. \( c_1(X)[C] = 2 \Rightarrow [C]^2 = 0 \Rightarrow X \) can be blown-down to a ruled surface;
3. \( c_1(X)[C] \geq 3 \Rightarrow [C]^2 \geq 1 \Rightarrow X \) is either \( \mathbb{CP}^2 \), or a Hirzebruch surface \( \Sigma_n \), or a modification of the latter.

In the last case \( X \setminus C \) is pseudoconcave and therefore, by a theorem of Grauert [4] the envelope \( \hat{U} \supset C \) coincides with \( X \).

Note also that for embedded real surfaces (not necessarily spheres) in \( \mathbb{CP}^2 \) the results of [2] have been recently improved by Nemirovski [5], who used a different approach based on the Seiberg–Witten theory.

The method used in the present paper to construct the envelopes of meromorphy was proposed in [2]. It is based on Gromov’s theory of pseudoholomorphic curves [6]. A considerable improvement of the results of [2] is achieved by the development of the transversality theory in the moduli space of pseudoholomorphic curves, see §2.

It should be pointed out that the proof of the continuity principle in [2]; Theorem 5.1.3 relies heavily on the existence of a complete holomorphic family of deformations of a non-compact complex curve in a complex manifold ([2]; Theorem 6.3.1). In the present paper we give a complete proof of this result, see Theorem 3.4.

1. Moduli space of pseudoholomorphic curves and the first variation of the \( \overline{\partial} \)-equation

In this section we recall briefly, in the form convenient for this paper, and also complete the results of [2] concerning certain basic ideas of the theory of pseudoholomorphic curves.

Consider the Teichmüller space \( \mathbb{T}_g \) of complex structures on a closed real oriented surface \( S \) of genus \( g \). This is a complex manifold of dimension

\[
\dim_{\mathbb{C}} \mathbb{T}_g = \begin{cases} 
0, & \text{if } g = 0; \\
1, & \text{if } g = 1; \\
3g - 3, & \text{if } g \geq 2,
\end{cases}
\]
uniquely characterized by the following property: the product $S \times \mathbb{T}_g$ admits a complex structure $J_{S \times \mathbb{T}_g}$ such that

(i) the natural projection $\pi|_T : S \times \mathbb{T}_g \to \mathbb{T}_g$ is holomorphic, and therefore for each $\tau \in \mathbb{T}_g$ the identification $S \cong S \times \{\tau\}$ defines a complex structure $J_S(\tau) := J_S \big|_{S \times \{\tau\}}$ on $S$;

(ii) for each complex structure $J_S$ on $S$ there exists a unique $\tau \in \mathbb{T}_g$ and a diffeomorphism $f : S \to S$ such that $J_S = f^*J_S(\tau)$ (that is, the map $f : (S,J_S) \to (S,J_S(\tau))$ is holomorphic) and $f$ is isotopic to the identity map $\text{Id}_S : S \to S$.

Let $G$ denote the automorphism group of $S \times \mathbb{T}_g$. Then

$$G = \begin{cases} 
\text{PGl}(2,\mathbb{C}) & \text{for } g = 0, \\
\text{Sl}(2,\mathbb{Z}) \ltimes T^2 & \text{for } g = 1, \\
\text{is discrete} & \text{for } g \geq 2.
\end{cases}$$

We shall use the following results on about $\mathbb{T}_g$.

If $g = 0$, then the surface $S$ is the Riemann sphere $S^2$ and all complex structures on $S^2$ are equivalent to the standard structure $S^2 \cong \mathbb{CP}^1$. Hence $\mathbb{T}_0$ consists of one point and $G = \text{PGl}(2,\mathbb{C})$ is the group of biholomorphisms of $\mathbb{CP}^1$.

If $g = 1$, then the surface $S$ is the torus $T^2$ and $\mathbb{T}_1$ is the upper half-plane $\mathbb{C}_+ = \{\tau \in \mathbb{C} : \Im(\tau) > 0\}$. Here the product $S \times \mathbb{T}_1$ can be identified with the quotient $(\mathbb{C} \times \mathbb{C}_+)/\mathbb{Z}^2$ under the action

$$(m,n) \in \mathbb{Z}^2 \times \mathbb{C} \to (m,n) \cdot (z,\tau) := (z + m + n\tau,\tau) \in \mathbb{C} \times \mathbb{C}_+.$$ 

In this case $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ is the identity component of $e \in G$; in particular, $T^2$ is normal. The group $G = \text{Sl}(2,\mathbb{Z}) \ltimes T^2$ is a semi-direct product and the action of $T^2$ on $(\mathbb{C} \times \mathbb{C}_+)/\mathbb{Z}^2$ is given by the formula

$$((t_1,t_2),([z],\tau)) \in T^2 \times (\mathbb{C} \times \mathbb{C}_+)/\mathbb{Z}^2 \mapsto [t_1,t_2] \cdot ([z],\tau) := ([z + t_1 + t_2\tau],\tau) \in (\mathbb{C} \times \mathbb{C}_+)/\mathbb{Z}^2.$$ 

The action of $G$ on $S \times \mathbb{T}_g$ is effective for all $g \geq 0$ and preserves the fibers of the projection $\pi|_T : S \times \mathbb{T}_g \to \mathbb{T}_g$. This gives us an action of $G$ on $\mathbb{T}_g$. Furthermore, for each $\tau \in \mathbb{T}_g$ and $f \in G$ there exists a unique diffeomorphism $\tilde{f}_\tau : S \to S$ such that

$$(1.1) \quad f \cdot (x,\tau) = (\tilde{f}_\tau(x),f \cdot \tau).$$

For each $\tau \in \mathbb{T}_g$ we have the natural isomorphisms $T_\tau \mathbb{T}_g \cong H^1(S,\mathcal{O}_\tau(TS))$ and $T_\tau G \cong H^0(S,\mathcal{O}_\tau(TS))$, where $\mathcal{O}_\tau(TS)$ denotes the sheaf of sections of $TS$ that are holomorphic with respect to $J_S(\tau)$.

Below we denote elements of $\mathbb{T}_g$ by $J_S$ and regard them as the corresponding complex structures on $S$.

Consider a symplectic manifold $(X,\omega)$ with a fixed almost complex structure $J_{st}$. Recall that $J_{st}$ is called $\omega$-tamed (see [6]) if for each non-zero vector $v \in T_x X$ we have $\omega(v,J_{st}v) > 0$. This is equivalent to the condition that the $(1,1)$-component of the form $\omega$ be positive with respect to $J_{st}$: $\omega^{(1,1)} > 0$. In our applications $\omega$ and $J_{st}$ will define a Kähler structure on $X$. 
Let $U$ be a relatively compact subdomain of $X$ that may coincide with $X$ if $X$ is compact. Let $S$ be a (fixed) compact oriented real surface of genus $g \geq 0$ and let $u_0: S \to X$ be a non-constant $C^1$-smooth map with $u_0(S) \subset U$.

We fix $p$, $2 < p < \infty$, and consider the Banach manifold $L^{1,p}(S,X)$ of all (continuous) maps $u: S \to X$ in the Sobolev class $L^{1,p}$. This is a smooth manifold, and its tangent space $T_u L^{1,p}(S,X)$ at the point $u$ is the Banach space $L^{1,p}(S,u^*TX)$ of $L^{1,p}$-smooth sections of the pulled-back tangent bundle $TX$. Let $S_U$ be the set of maps $u$ in $L^{1,p}(S,X)$ that are homotopic to $u_0$ and satisfy the condition $u(S) \cap U \neq \emptyset$. We fix $k \geq 1$ and denote by $\mathcal{J}^k_U$ the set of $C^k$-smooth almost complex structures $J$ on $X$ satisfying the following two conditions:

(i) $\{ x \in X : J(x) \neq J_{st}(x) \} \subset U$;

(ii) $J$ is $\omega$-tamed.

The map $ev:S \times S_U \times \mathbb{T} \times \mathcal{J}^k_U \to X$ given by the formula $ev(x,u,J_S,J) := u(x)$ defines a bundle $E := ev^*(TX)$ over $S \times S_U \times \mathbb{T} \times \mathcal{J}^k_U$. We equip $E$ with the natural complex structure, which is equal to $J(u(x))$ on each fiber $E_{(x,u,J_S,J)} \cong T_{u(x)}X$. Let $(E_u,J)$ be the restriction of $E$ to $S \times \{(u,J_S,J)\}$; it is isomorphic to $u^*TX$.

The bundle $E$ with complex structure $J$ defines two complex Banach bundles, $\hat{\mathcal{E}}$ and $\hat{\mathcal{E}}'$, over the product $S_U \times \mathbb{T} \times \mathcal{J}^k_U$. They have the fibers

$$\hat{\mathcal{E}}(u,J_S,J) := L^{1,p}(S,E_u) \quad \text{and} \quad \hat{\mathcal{E}}'(u,J_S,J) := L^p(S,E_u \otimes \Lambda^{0,1})S).$$

Here $S$ carries the complex structure $J_S$, $\Lambda^{0,1}S$ is the complex line bundle of $(0,1)$-forms on $S$, and $\otimes$ is the complex tensor product of corresponding complex vector bundles. Note that $\hat{\mathcal{E}}$ is the pull-back of the tangent bundle $TL^{1,p}(S,X)$ with respect to the projection $(u,J_S,J) \in S_U \times \mathbb{T} \times \mathcal{J}^k_U \mapsto u \in L^{1,p}(S,X)$.

In its turn, the bundle $\hat{\mathcal{E}}'$ is the range of the $\overline{\partial}$-operator on the manifold $L^{1,p}(S,X) \times \mathbb{T} \times \mathcal{J}^k_U$. Namely, the $\overline{\partial}$-operator defines a section $\sigma_{\overline{\partial}}$ of the bundle $\hat{\mathcal{E}}'$ by the formula

$$(1.2) \quad \sigma_{\overline{\partial}}(u,J_S,J) := \overline{\partial}_{J_S,J}u := \frac{1}{2}(du + J \circ du \circ J_S).$$

If $f$ is another section of $\hat{\mathcal{E}}'$ (defined, for example, by an explicit geometric construction), then we can consider the non-homogeneous $\overline{\partial}$-equation

$$\overline{\partial}_{J_S,J}u = f(u,J_S,J).$$

In the present paper we consider only the homogeneous case $f(u,J_S,J) \equiv 0$. We denote the corresponding set of solutions by

$$(1.3) \quad \mathcal{P} := \{(u,J_S,J) \in S_U \times \mathbb{T} \times \mathcal{J}^k_U : \overline{\partial}_{J_S,J}u = 0\}.$$

If $u \in L^{1,p}(S,X)$ satisfies the equation

$$(1.4) \quad \overline{\partial}_{J_S,J}u = 0$$

with appropriate $J_S \in \mathbb{T}_g$ and $J \in \mathcal{J}^k_U$, then we say that the map $u$ is pseudoholomorphic, or $J$-holomorphic, or $(J_S,J)$-holomorphic, and we call the image $M := u(S)$ a pseudoholomorphic (or $J$-holomorphic) curve.
ENVELOPES OF SPHERES

The operator $\overline{\partial}$ is elliptic with Cauchy–Riemann symbol. The theory of elliptic partial differential equations (see, for example, [7]) shows that $\mathcal{P}$ is closed in the space

$$\mathcal{X} := \{(u, J_S, J) \in S_U \times \mathbb{T} \times \mathcal{J}_U^k : u \in C^1(S, X)\}.$$ 

Note that $\mathcal{X}$ is a closed Banach manifold. We set

$$\mathcal{X}^* := \{(u, J_S, J) \in \mathcal{X} : \text{there exists a non-empty open subset } V \text{ of } S \text{ such that } u|_V \text{ is an embedding and } u(V) \cap u(S \setminus V) = \emptyset\},$$

and let $\mathcal{P} := \mathcal{P} \cap \mathcal{X}^*$. Then $\mathcal{X}^*$ is open in $\mathcal{X}$ and $\mathcal{P}^*$ is open in $\mathcal{P}$. We consider the following natural action of the group $G$ on $S_U \times \mathbb{T} \times \mathcal{J}_U^k$ by means of compositions:

$$(f, u, J_S, J) \in G \times S_U \times \mathbb{T} \times \mathcal{J}_U^k \mapsto f \cdot (u, J_S, J) := (u \circ f^{-1}, f \cdot J_S, J),$$

where $f: S \to S$ is the diffeomorphism induced by $f \in G$ and $J_S$, see (1.1). The inverse map $(f)^{-1}$ is introduced to make this action associative: $(g_1 \cdot g_2) \cdot (u, J_S, J) = g_1 \cdot (g_2 \cdot (u, J_S, J))$.

The sets $\mathcal{X}^*$, $\mathcal{P}$ and $\mathcal{P}^*$ are invariant with respect to the action of $G$. Let $\mathcal{M} := \mathcal{P}^*/G$ be the quotient space for the action of $G$ and $\pi_\mathcal{P}: \mathcal{P}^* \to \mathcal{M}$ the corresponding projection. Since $G$ does not act on $\mathcal{J}_U^k$, the projection $\pi_\mathcal{P}: \mathcal{M} \to \mathcal{J}_U^k$ is well-defined.

**Lemma 1.1.** The projections $\mathcal{X}^* \to \mathcal{X}^*/G$ and $\pi_\mathcal{P}: \mathcal{P}^* \to \mathcal{M}$ are principal $G$-bundles.

**Proof.** First, we consider the case $g \geq 2$. It is known that $G$ is discrete in this case and acts properly discontinuously on $T_g$ (see, for example, [8]). Hence the action of $G$ on $\mathcal{X}$ is also proper. Together with the definition of $\mathcal{X}^*$ this means that $G$ acts freely on $\mathcal{X}^*$. Thus, $\mathcal{X}^* \to \mathcal{X}^*/G$ is an unramified covering.

Now, we turn to the case $g = 0$. In this case $S = S^2$ and $T_0 = \{J_{st}\}$. We fix some $(u^0, J_{st}, J^0) \in \mathcal{X}^*$. Let $y_1$, $y_2$, and $y_3$ be three distinct points on $S^2$ such that $u^0$ is an embedding in a neighborhood of each $y_i$ and, in particular, $du^0$ does not vanish at $y_i$. We consider smooth submanifolds $Z_i$ of $X$ of codimension 2 that intersect $u^0(S^2)$ transversally at the points $u^0(x_i)$, respectively.

Let $V \ni (u^0, J_{st}, J^0)$ be an open subset of $\mathcal{X}^*$, $W$ its projection onto $\mathcal{X}^*/G$, and $G \cdot V := \{f \cdot (u, J_{st}, J) : f \in G, (u, J_{st}, J) \in V\}$ its $G$-saturation. We consider the set

$$\mathcal{Z} := \{(u, J_{st}, J) \in G \cdot V : u(y_i) \in Z_i\}.$$ 

If $V$ is sufficiently small, then $\mathcal{Z}$ is a smooth submanifold of $G \cdot V$, which intersects each orbit $G \cdot (u, J_{st}, J)$ transversally at a single point. This defines a $G$-invariant diffeomorphism $G \cdot V \cong G \times \mathcal{Z}$, and therefore $\mathcal{Z}$ is a local slice of the $G$-action at $(u^0, J_{st}, J^0)$.

The projection of these local slices of $\mathcal{Z}$ onto $\mathcal{X}^*/G$ defines the structure of a smooth Banach manifold on $\mathcal{X}^*/G$ and the structure of a principal $G$-bundle on the projection $\mathcal{X}^* \to \mathcal{X}^*/G$.

The case $g = 1$ can be regarded as a combination of the previous cases because for $g = 1$ the group $G$ has the ‘continuous’ part and the ‘discrete’ part, $T^2$ and $Sl(2, \mathbb{Z})$, respectively. We fix $(u^0, J^0_S, J^0) \in \mathcal{X}^*$ first and then a point $y$ on $S$ such that $u^0$
is an embedding in a neighborhood of \( y \) and, in particular, \( du^0 \) does not vanish at \( y \). Let \( Z \) be a smooth submanifold of \( X \) of codimension 2 that intersects \( u^0(S) \) transversally at \( u^0(y) \). As in the case \( g = 0 \), we fix a neighborhood \( V \ni (u^0, J^0_S, J^0) \) in \( X \) and consider the set

\[
Z := \{(u, J_S, J) \in G \cdot V : u(y) \in Z\},
\]

where \( G \cdot V := \{ f \cdot (u, J_{st}, J) : f \in G, (u, J_{st}, J) \in V \} \) is the \( G \)-saturation of \( V \). If \( V \) is sufficiently small, then \( Z \) is a slice of the action of the subgroup \( T^2 \) of \( G \). Hence \( X^* \to X^*/T^2 \) is a principle \( T^2 \)-bundle.

We consider now the action of \( \text{SL}(2, \mathbb{Z}) = G/T^2 \) on \( X^*/T^2 \). In the same way as in the case \( g \geq 2 \), one can show that \( X^*/T^2 \to X^*/G \) is a covering. Consequently, \( X^* \to X^*/G \) is a principal \( G \)-bundle.

Considering the projection \( P^* \to M \equiv P^*/G \) we observe that the natural inclusion \( M \hookrightarrow X^*/G \) is continuous and closed. This gives us a \( G \)-invariant homeomorphism \( P^* \cong M \times_{X^*/G} X^* \) and therefore a principal \( G \)-bundle structure on \( P^* \) with base \( M \).

**Remark.** It can be shown that if \( (u, J_S, J) \in P^* \), then \( u \) is an embedding in a neighborhood of each point of \( S \), except for a finitely many points. In particular, \( J_S \) is uniquely determined by \( u \) and \( J \). In a similar way, each class \( G \cdot (u, J_S, J) \in M \) is uniquely determined by \( J \in \mathcal{I}^0 \) and the pseudoholomorphic curve \( M := u(S) \). Using this observation we denote elements of \( M \) by \( (M, J) \). The motivation for this notation is that the objects that we shall study and use are pseudoholomorphic curves themselves, rather than their particular parametrizations. We hope that this formal inaccuracy will not lead to a misunderstanding.

This result and the construction in the proof of the lemma enable us to push down \( G \)-invariant objects from \( P^* \) to \( M \).

For example, there exists a (trivial) bundle over \( P \) with fiber \( S \), total space \( P \times S \), natural projection on \( P \), complex structure \( J_S \) in the fiber over \( (u, J_S, J) \), and map \( ev: P \times S \to X \) given by the formula \( ev(u, J_S, J; y) := u(y) \).

These structures are invariant under the action of \( G \) on \( P \times S \) extending the initial action of \( G \) on \( S \times \mathbb{T} \). This gives us a \( G \)-bundle \( \pi_\mathcal{E}: \mathcal{E} \to M \) with total space \( \mathcal{E} := P^* \times_G S \) and fiber \( S \) and also a map \( ev: \mathcal{E} \to X \). We shall regard \( M \) as the moduli space of all pseudoholomorphic curves in \( X \) with appropriate topological properties, \( \pi_\mathcal{E}: \mathcal{E} \to M \) as the corresponding ‘universal curve’ family, and the map \( ev: \mathcal{E} \to X \) as the realization of this ‘universal curve’ in \( X \). In particular, each fiber \( \mathcal{E}_{(M, J)} := \pi_\mathcal{E}^{-1}(M, J) \) over \( (M, J) \in M \) carries a natural complex structure \( J_{\mathcal{E}_{(M, J)}} = J_S \).

We can define Banach bundles \( \mathcal{E} \) and \( \mathcal{E}' \) over \( X^*/G \supseteq M \) in a similar way. To this end we observe that the inclusion \( X^* \hookrightarrow S_U \times \mathbb{D}_U^k \) is continuous and \( G \)-invariant. For \( (u, J_S, J) \in S_U \times \mathbb{T} \times \mathcal{I}^0 \) and \( f \in G \) we define a diffeomorphism \( \tilde{f}: S \to S \) by (1.1), so that \( f \cdot (u, J_S, J) = (u \circ \tilde{f}^{-1}, f \cdot J_S, J) \). We define the operator \( f_* \) by setting

\[
f_*: s \in \tilde{\mathcal{E}}_{(u, J_S, J)} \longmapsto (\tilde{f}^{-1})^* s \in \tilde{\mathcal{E}}_{f \cdot (u, J_S, J)}.
\]

This gives us a natural lifting of the \( G \)-action on \( S_U \times \mathbb{T} \times \mathcal{I}^0 \) to a \( G \)-action on \( \tilde{\mathcal{E}} \). A lifting of the \( G \)-action to \( \tilde{\mathcal{E}}' \) is defined in the same way. Since the projection \( X^* \to X^*/G \) admits local \( G \)-slices, there exist Banach bundles \( \mathcal{E} \) and \( \mathcal{E}' \) over \( X^*/G \).
such that their liftings to \(X^*\) are \(G\)-equivariantly isomorphic to the bundles \(\hat{E}\) and \(\hat{E}'\), respectively. In particular, if \((u,J_S,J) \in X^*\) represents the class \(G \cdot (u,J_S,J) \in X^*/G\), then we have the following natural isomorphisms:

\[
\mathcal{E}_G \cdot (u,J_S,J) \cong \hat{\mathcal{E}}(u,J_S,J) = L^{1,p}(S,E_u),
\]
\[
\mathcal{E}'_G \cdot (u,J_S,J) \cong \hat{\mathcal{E}}'(u,J_S,J) = L^p(S,E_u \otimes \Lambda^{(0,1)} S).
\]

We intend to study the problem of deformation of a fixed (compact) \(J_0\)-holomorphic curve \(M_0 = u_0(S)\) into a compact complex curve \(M_1\), which is holomorphic with respect to some given (for instance, integrable) almost complex structure \(J_{st}\) on \(X\), using the continuation method. The idea is to find a suitable homotopy \(\hat{h}(t) = J_t, t \in [0,1]\), of almost complex structures between \(J_0\) and \(J_{st} = J_1\) and to construct a continuous deformation \(u_t : S \to X\) of the map \(u_0\) into the map \(u_1\) such that \(u_t\) is \(J_t\)-holomorphic for all \(t \in [0,1]\). To this end we shall study the linearization of the equation \(\partial J_f u = 0\).

**Lemma 1.2.** Let \(X\) be a Banach manifold, let \(\mathcal{E} \to X\) and \(\mathcal{E}' \to X\) be \(C^1\)-smooth Banach bundles over \(X\), and let \(\nabla\) and \(\nabla'\) be connections in \(\mathcal{E}\) and \(\mathcal{E}'\), respectively. Let \(\sigma\) be a (local) \(C^1\)-section of \(\mathcal{E}\) and let \(D : \mathcal{E} \to \mathcal{E}'\) be a \(C^1\)-smooth bundle homomorphism.

(i) If \(\sigma(x) = 0\) for some \(x \in X\), then the map \(\nabla \sigma_x : T_x X \to \mathcal{E}_x\) does not depend on the choice of the connection \(\nabla\) in \(\mathcal{E}\).

(ii) Let \(K_x := \text{Ker}(D_x : \mathcal{E}_x \to \mathcal{E}'_x)\) and \(Q_x := \text{Coker}(D_x : \mathcal{E}_x \to \mathcal{E}'_x)\). Fix the corresponding inclusion \(i_x : K_x \to \mathcal{E}_x\) and the projection \(p_x : \mathcal{E}_x' \to Q_x\). Let \(\nabla'^{\text{hom}}\) be the connection in \(\mathcal{H}(\mathcal{E},\mathcal{E}')\) induced by \(\nabla\) and \(\nabla'\). Then the map

\[
p_x \circ (\nabla'^{\text{hom}} D_x) \circ i_x : T_x X \to \mathcal{H}(K_x, Q_x)
\]

does not depend on the choice of \(\nabla\) and \(\nabla'\).

**Remark.** In view of the results of the lemma, we shall use the following notation. For \(\sigma \in \Gamma(X,\mathcal{E})\), \(D \in \Gamma(X,\mathcal{H}(\mathcal{E},\mathcal{E}')\) and \(x \in X\) as in the lemma, we shall write \(\nabla \sigma_x : T_x X \to \mathcal{E}_x\) and \(\nabla D : T_x X \times \text{Ker} D_x \to \text{Coker} D_x\) to denote the corresponding operators, without specifying the connections used in their definitions.

**Proof.** (i) Let \(\tilde{\nabla}\) be another connection in \(\mathcal{E}\). Then \(\tilde{\nabla}\) has the form \(\tilde{\nabla} = \nabla + A\) for some \(A \in \Gamma(X,\mathcal{H}(T X,\text{End}(\mathcal{E})))\). Hence, for \(\xi \in T_x X\) we have \(\tilde{\nabla}_\xi \sigma - \nabla \sigma_x = A(\xi,\sigma(x)) = 0\).

(ii) In a similar way, let \(\tilde{\nabla}'\) be another connection in \(\mathcal{E}'\), and let \(\tilde{\nabla}'^{\text{hom}}\) be the connection in \(\mathcal{H}(\mathcal{E},\mathcal{E}')\) induced by \(\tilde{\nabla}\) and \(\tilde{\nabla}'\). Then \(\tilde{\nabla}'\) also has the form \(\tilde{\nabla}' = \nabla + A'\) with some \(A' \in \Gamma(X,\mathcal{H}(T X,\text{End}(\mathcal{E}'))\)). Thus, for \(\xi \in T_x X\) we obtain \(\nabla'^{\text{hom}} D - \tilde{\nabla}'^{\text{hom}} D = A'(\xi) \circ D_x - D_x \circ A(\xi)\). The result of the lemma follows now from the identities \(p_x \circ D_x = 0\) and \(D_x \circ i_x = 0\).

The space \(\mathcal{P}\) of pseudoholomorphic maps has been defined as the zero set of the section \(\sigma_\mathcal{E}: S_U \times \mathbb{T} \times g_U^k \to \mathcal{E}'\). Lemma 1.2 asserts that for each \((u,J_S,J) \in \mathcal{P}\) we have a well-defined operator

\[
\nabla \sigma_\mathcal{E} : T_u L^{1,p}(S,X) \oplus T_{J_S} \mathbb{T} \oplus T_J g_U^k \rightarrow \hat{\mathcal{E}}(u,J_S,J).
\]
This operator is the required linearization of the equation $\overline{\partial} J_{S,J} u = 0$. Covariant differentiation of (1.2) shows that it has the following form:

\begin{equation}
\nabla \sigma_\sigma: (v, J_S, J) \longmapsto D_{(u,J)} v + J \circ du \circ J_S + \hat{J} \circ du \circ J_S \in \tilde{E}'_{(u,J,S,J)}.
\end{equation}

Here $D = D_{(u,J)}$ is the Gromov operator defined on $T_u S \equiv \mathcal{E}_{(u,J)} \equiv L^{1,p}(S, u^* TX)$. To obtain an explicit formula for $D$ we first some connections $\nabla^S$ on $TS$ and $\nabla^X$ on $TX$. Note that we can choose $\nabla^S$ to be symmetric and $J_S$-complex for each fixed $J_S$, which means that $\nabla^S J_S = 0$. However, in general there exists no connection $\nabla^X$ with these two properties (that is, both symmetric and $J$-complex), and therefore we must choose one of them. In the present paper we prefer symmetric connections $\nabla^X$, because this enables us to fix one connection for all $J \in \mathfrak{g}^{U}$. In the sequel we use the same symbol $\nabla$ for connections in $TS$ and $TX$, and also in other bundles obtained from $TX$ and $TS$, such as $E_u = u^* TX$. Note that the connections $\nabla^S$ and $\nabla^X$ induce connections in the Banach bundles $\mathcal{E}$ and $\mathcal{E}'$. This yields the following formula for the Gromov operator $D_{u,J} : L^{1,p}(S, E_u) \rightarrow L^p(S, E_u \otimes \Lambda^{(0,1)} S)$:

\begin{equation}
D_{u,J}(v) = \nabla v + \frac{1}{2} J \circ \nabla v \circ J_S + \frac{1}{2} (\nabla_{\hat{J}} v) \circ du \circ J_S.
\end{equation}

Explicit calculations have been carried out in [2] (see also [9] for the case of a $J$-complex connection $\nabla^X$ in $TX$). The operator $D = D_{u,J}$ is $\mathbb{G}$-equivariant and therefore induces a bundle homomorphism $D : \mathcal{E} \rightarrow \mathcal{E}'$ over $\mathcal{M}$.

The following properties of $D$ will be used below. Let $(u, J_S, J) \in \mathcal{P}$.

**Lemma 1.3.** The Gromov operator $D_{u,J} : L^{1,p}(S, E_u) \rightarrow L^p(S, E_u \otimes \Lambda^{(0,1)} S)$ is an $\mathbb{R}$-linear differential operator order 1 with Cauchy–Riemann symbol. It can be split into the sum $D_{u,J} = \overline{\partial}_{u,J} + R$, where the $\mathbb{C}$-linear operator $\overline{\partial}_E := \overline{\partial}_{u,J}$ and the $\mathbb{C}$-antilinear operator $R$ have the following properties:

(i) there exists a unique holomorphic structure on $E_u$ such that $\overline{\partial}_E = \overline{\partial}_{u,J}$ is the corresponding Cauchy–Riemann operator;

(ii) if $J$ is $C^k$-smooth, then $R$ is a $C^{k-1}$-smooth $\mathbb{C}$-antilinear homomorphism from $E_u$ to $E_u \otimes \Lambda^{(0,1)} S$, that is, $R \in C^{k-1}(S, \text{Hom}_E(E_u, E_u \otimes \Lambda^{(0,1)} S))$; moreover, for $x \in S$, $v \in E_x = T_{u(x)} X$ and $\xi \in T_x S$ we have $R(v, \xi) = N_J(v, du(\xi))$, where $N_J$ is the Nijenhuis torsion tensor of the almost complex structure $J$ (see [10]);

(iii) the a priori continuous homomorphism $du : TS \rightarrow E_u$ is in fact holomorphic, that is, it satisfies the relation $du \circ \overline{\partial}_{TS} = \overline{\partial}_E \circ du$; furthermore, $R \circ du = 0$.

**Proof.** Explicit formula (1.4) shows that the Gromov operator $D$ is a first-order differential operator with Cauchy–Riemann symbol. Since this symbol is $\mathbb{C}$-linear, the $\mathbb{C}$-antilinear part $R$ of $D$ has order 0, that is, $R$ is a bundle homomorphism. The proof of the formula $R = N_J \circ du$ can be found in [2]; Lemma 2.2.1.

The $\mathbb{C}$-linear part $\overline{\partial}_{u,J}$ of $D$ is of order 1 and has the Cauchy–Riemann symbol. The fact that there exists a holomorphic structure on $E_u$ with Cauchy–Riemann operator $\overline{\partial}_{u,J}$ is standard.

Let $\xi \in L^{1,p}(S, TS)$, and let $\Phi_t : S \rightarrow S$ be the one-parametric diffeomorphism group generated by $\xi$. We set $u_t := u \circ \Phi_t$ and $J_S(t) := \Phi_{t-1}^* J_S$. Differentiating the
Remark. Part (i) of Corollary 1.4 was proved by McDuff for

\[ N \] is the quotient sheaf \( N = \mathcal{O}(u_0) / du(\mathcal{O}(TS)) \). This sheaf splits into the direct sum \( N_u = \mathcal{O}(N_u) \oplus N_u^{\text{sing}} \), where \( N_u \) is a holomorphic vector bundle and \( N_u^{\text{sing}} = \bigoplus_{i=1}^{P} C_{a_i} \). By \( C_{a_i} \) we mean the sheaf concentrated at a critical point \( a_i \in S \) of the map \( du \) with stalk \( C_{a_i} \), where \( n_i = \text{ord}_{a_i} du \) is the order of the zero of \( du \) at \( a_i \). We call \( N_u \) the normal sheaf of \( M = u(S) \), \( N_u \) the normal bundle of \( M \), and \( N_u^{\text{sing}} \) the ramification sheaf of \( M \).

Part (iii) of the lemma follows now by taking the \( \mathbb{C} \)-linear and the \( \mathbb{C} \)-antilinear parts of (1.7).

For a detailed proof of the lemma we refer to [2]; § 2.

Corollary 1.4. (i) The set of critical points of a \( J \)-holomorphic map \( u: (S,J_S) \to (X,J) \) is discrete in \( S \), provided that \( J \in C^1(X) \);

(ii) the order of zero of \( du \) at a point \( a \) is well defined for each \( a \in S \).

Remark. Part (i) of Corollary 1.4 was proved by McDuff for \( J \in C^\infty \) and by Sikorav for \( J \in C^1 \).

Another consequence of Lemma 1.3 is the following exact sequence of coherent sheaves:

\[ 0 \to \mathcal{O}(TS) \xrightarrow{du} \mathcal{O}(E_u) \to N_u \to 0. \]

Here \( N \) is the quotient sheaf \( \mathcal{O}(E)/du(\mathcal{O}(TS)) \). This sheaf splits into the direct sum \( N_u = \mathcal{O}(N_u) \oplus N_u^{\text{sing}} \), where \( N_u \) is a holomorphic vector bundle and \( N_u^{\text{sing}} = \bigoplus_{i=1}^{P} C_{a_i} \). By \( C_{a_i} \) we mean the sheaf concentrated at a critical point \( a_i \in S \) of the map \( du \) with stalk \( C_{a_i} \), where \( n_i = \text{ord}_{a_i} du \) is the order of the zero of \( du \) at \( a_i \). We call \( N_u \) the normal sheaf of \( M = u(S) \), \( N_u \) the normal bundle of \( M \), and \( N_u^{\text{sing}} \) the ramification sheaf of \( M \).

Let \( Z_{du} \) be the divisor \( \sum_{i=1}^{P} n_i[a_i] \), and let \( L(Z_{du}) \) be the sheaf of meromorphic functions on \( S \) with poles of order at most \( n_i \) at \( a_i \). The corresponding linear bundle will be also denoted by \( L(Z_{du}) \). Then (1.8) gives rise to the following exact sequence of holomorphic vector bundles:

\[ 0 \to TS \otimes L(Z_{du}) \xrightarrow{du} E \to N_u \to 0. \]

Let \( L^p_{(0,1)}(S,E) \) be the space of \( L^p \)-integrable \((0,1)\)-forms on \( S \) with coefficients in \( E \). Then, by Lemma 1.3 and equality (1.7) we obtain the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \to & L^1_p(S,TS \otimes L(Z_{du})) & \xrightarrow{du} & L^1_p(S,E) & \xrightarrow{pr} & L^1_p(S,N_u) & \to & 0 \\
 & & \downarrow & & \downarrow D_{u,J} & & \downarrow & \\
0 & \to & L^p_{(0,1)}(S,TS \otimes L(Z_{du})) & \xrightarrow{du} & L^p_{(0,1)}(S,E) & \to & L^p_{(0,1)}(S,N_u) & \to & 0.
\end{array}
\]

This gives us the operator

\[ D^N_{u,J}: L^1_p(S,N_u) \to L^p_{(0,1)}(S,N_u) \]

of the form \( D^N_{u,J} = \bar{\partial}_N + R_N \). As usual, \( \bar{\partial}_N \) is the \( \bar{\partial} \)-operator in \( N_u \) and \( R_N \in C^0(S,\text{Hom}_{\mathbb{R}}(N_u,\Lambda^{0,1} \otimes N_u)) \).
Definition Let $E$ be a holomorphic vector bundle over a compact Riemann surface $(S,J_S)$, and let $D: L^{1,p}(S,E) \to L^p_{(0,1)}(S,E)$ be an operator of the form $D = \bar{\partial} + R$, where $R \in L^p(S, \mathcal{Hom}_\mathbb{R}(E, \Lambda^{0,1}S \otimes E))$, $2 < p < \infty$. Then we set $\text{H}^0_D(S,E) := \text{Ker} D$ and $\text{H}^1_D(S,E) := \text{Coker} D$.

Remark. It is not difficult to show that for $E$, $R \in L^p$, $2 < p < \infty$, and $D = \bar{\partial} + R$ as above, the spaces $\text{H}^i_D(S,E)$ can be defined as the kernel and the cokernel of the operator

$$\bar{\partial} + R: L^{1,q}(S,E) \to L^q_{(0,1)}(S,E)$$

with arbitrary $q$, $1 < q \leq p$. Consequently, the $\text{H}^i_D(S,E)$ do not depend on the choice of the function spaces. Furthermore, the operator $D$ is Fredholm, of index

$$\text{ind}_\mathbb{R}(D) := \text{dim}_\mathbb{R}(\text{Ker}(D)) - \text{dim}_\mathbb{R}(\text{Coker}(D))$$

$$= \text{ind}_\mathbb{R}(\bar{\partial}) = 2(c_1(E)[S] + (1 - g) \cdot \text{rk}_\mathbb{C}(E)).$$

The proof of these facts is given in [2].

By the ‘snake lemma’ from homological algebra the diagram (1.10) gives us the following exact sequence of $D$-cohomology:

$$0 \longrightarrow \text{H}^0(S,TS \otimes \mathcal{L}(Z_{du})) \longrightarrow \text{H}^0_D(S,E) \longrightarrow \text{H}^0_D(S,N_u) \longrightarrow 0.$$ (1.11)

$$\delta \text{H}^1(S,TS \otimes \mathcal{L}(Z_{du})) \longrightarrow \text{H}^1_D(S,E) \longrightarrow \text{H}^1_D(S,N_u) \longrightarrow 0.$$

The proof of the next result can be found in [2]; Lemma 2.3.2.

Lemma 1.5 (Serre duality for $D$-cohomology). Let $E$ be a holomorphic vector bundle over a compact Riemann surface $(S,J_S)$, and let $D: L^{1,p}(S,E) \to L^p_{(0,1)}(S,E)$ be an operator of the form $D = \bar{\partial} + R$, where $R \in L^p(S, \mathcal{Hom}_\mathbb{R}(E, \Lambda^{0,1}S \otimes E))$, $2 < p < \infty$. Let $K_S := \Lambda^{1,0}S$ be the canonical bundle of $(S,J_S)$. Then there exists a naturally defined operator

$$D^* = \bar{\partial} - R^*: L^{1,p}(S,E^* \otimes K_S) \to L^p_{(0,1)}(S,E^* \otimes K_S)$$

with $R^* \in L^p(S, \mathcal{Hom}(E^* \otimes K_S, \Lambda^{0,1}S \otimes E^* \otimes K_S))$ and there exist natural isomorphisms

$$\text{H}^0_D(S,E)^* \cong \text{H}^1_{D^*}(S,E^* \otimes K_S),$$

$$\text{H}^1_D(S,E)^* \cong \text{H}^0_{D^*}(S,E^* \otimes K_S).$$

If, furthermore, $R$ is $\mathbb{C}$-antilinear, then $R^*$ is also $\mathbb{C}$-antilinear.

Remark. The proof of Lemma 1.5 uses the following identity, which that may be taken as the definition of $D^*$. For $\xi \in L^{1,p}(S,E)$ and $\eta \in L^{1,q}(S,E^* \otimes K_S)$ we have the equalities

$$\Re \int_S \langle \bar{\partial}\xi + R\xi, \eta \rangle = \Re \int_S \bar{\partial}\langle \xi, \eta \rangle + \Re \int_S \langle \xi, -(\bar{\partial} - R^*)\eta \rangle = \Re \int_S \langle \xi, -D^*\eta \rangle.$$ (1.12)

Hence the natural pairing $\langle \cdot, \cdot \rangle: L^p_{(0,1)}(S,E) \times \text{H}^0_{D^*}(S,E^* \otimes K_S) \to \mathbb{R}$, $\langle \phi, \eta \rangle := \Re \int_S \langle \phi, \eta \rangle$ vanishes on the range of $D$ and induces the pairing

$$\langle \cdot, \cdot \rangle: \text{H}^0_D(S,E) \times \text{H}^0_{D^*}(S,E^* \otimes K_S) \to \mathbb{R}.$$

The latter is the required duality.
Corollary 1.6 [6], [11] (Vanishing theorems for $D$-cohomology). Let $S$ be a Riemann surface of genus $g$ with complex structure $J_S$. Let $L$ be a holomorphic line bundle over $S$ with operator $D$: $L^{1,p}(S,L) \to L^p_{(0,1)}(S,L)$ of the form $D = \overline{\partial} + R$, where $R \in L^p(S,\text{Hom}_{\mathbb{C}}(L,L^0,1\otimes L))$, $p > 2$.

If $c_1(L) < 0$, then $H^0_D(S,L) = 0$, and if $c_1(L) > 2g - 2$, then $H^1_D(S,L) = 0$.

In particular, if $g = 0$, then either $\dim_{\mathbb{R}}(H^0_D(S,L)) = 0$ or $\dim_{\mathbb{R}}(H^1_D(S,L)) = 0$ and $\dim_{\mathbb{R}}(H^1_D(S,L))$ is positive and even.

Proof. Let $\xi$ be a non-trivial $L^{1,p}$-section of $L$ that satisfies the condition $D\xi = 0$. By Lemma 3.1.1 in [2] the section $\xi$ has only finitely many zeros $a_i \in S$, which have positive multiplicities $\mu_i$, so that $c_1(L) = \sum \mu_i = 0$. Consequently, $H^0_D(S,L)$ is trivial for $c_1(L) < 0$. The triviality of $H^1_D$ follows in a similar way by the Serre duality of Lemma 1.5.

In particular, if $g = 0$, then either $c_1(L) < 0$, or $c_1(L) > -2$. Hence, either $H^0_D(S,L)$ or $H^1_D(S,L)$ is trivial. The fact that $\dim_{\mathbb{R}}(H^1_D(S,L))$ is even follows from the equality $\text{ind}_{\mathbb{R}}(D) = \text{ind}_{\mathbb{C}}(\overline{D}) = 2\text{ind}_{\mathbb{C}}(\overline{D})$, which is a consequence of the index theorem.

Note that, by Lemma 1.3 we obtain the following commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
0 & \to & L^{1,p}(S,T^*S) & \overset{\partial_u}{\to} & L^{1,p}(S,E) & \overset{\overline{\partial}}{\to} & L^{1,p}(S,E)/\partial_u(L^{1,p}(S,T^*S)) & \to 0 \\
\downarrow \overline{\partial}_S & & \downarrow \partial & & \downarrow \overline{\partial} & & \\
0 & \to & L^p_{(0,1)}(S,T^*S) & \overset{\partial_u}{\to} & L^p_{(0,1)}(S,E) & \overset{\overline{\partial}}{\to} & L^p_{(0,1)}(S,E)/\partial_u(L^p_{(0,1)}(S,T^*S)) & \to 0,
\end{array}
$$

(1.13)

where $\overline{\partial}$ is induced by $D \equiv D_u,J$.

Lemma 1.7. For the operator $\overline{\partial}$ we have the natural isomorphisms $\text{Ker}\overline{\partial} = H^0_D(S,N_u) \oplus H^0(S,N_u^{\text{sing}})$ and $\text{Coker}\overline{\partial} = H^1_D(S,N_u)$.

For the proof see [2]; Lemma 2.4.1.

Corollary 1.8. The short exact sequence of sheaves (1.8) induces the following long exact sequence of $D$-cohomology:

$$
\begin{array}{c}
0 \to H^0(S,T^*S) \overset{\partial_u}{\to} H^0_D(S,E) \to H^0_D(S,N_u) \oplus H^0(S,N_u^{\text{sing}}) \\
\overset{\delta}{\to} H^1(S,T^*S) \overset{\partial_u}{\to} H^1_D(S,E) \to H^1_D(S,N_u) \to 0.
\end{array}
$$

(1.14)

§ 2. Transversality property of the moduli space

To deform a pseudoholomorphic curve $M_t$ along a given path of almost complex structures $J_t$, it is useful to know at which points $(u,J_S,J)$ the set of pseudoholomorphic maps $\mathcal{P}$ is a Banach manifold. Note that, by definition, $\mathcal{P}$ is the intersection of the zero section and the section $\sigma_{\overline{\partial}}$ of the bundle $\hat{\mathcal{E}}^t$ over $S \times \mathbb{T} \times \mathcal{E}_0$. Hence the problem reduces to the question of the transversality of these two sections.

Definition Let $X$, $Y$, and $Z$ be Banach manifolds, and let $f: Y \to X$ and $g: Z \to X$ be $C^k$-smooth maps, $k \geq 1$. We define the fibered product $Y \times_X Z$ by setting $Y \times_X Z := \{(y,z) \in Y \times Z : f(y) = g(z)\}$. The map $f$ is said to be transversal to $g$.
at a point \((y, z) \in Y \times_X Z\), where \(x := f(y) = g(z)\), and \((y, z)\) is called a **transversal point** if the map \(df_y \oplus -dg_z : T_y Y \oplus T_z Z \to T_x X\) is surjective and its kernel admits a closed complementary subspace. We denote the set of transversal points in \(Y \times_X Z\) by \(Y \times_X^\triangledown Z\), where \(\triangledown\) indicates the transversality condition.

In particular, if the map \(g : Z \to X\) is an embedding, then the fibered product \(Y \times_X Z\) is just the inverse image \(f^{-1}Z\) of the set \(Z \subset X\), and therefore each point \((y, z) \in Y \times_X Z\) is completely defined by \(y \in Y\). In this case we say that \(f : Y \to X\) is **transversal** to \(Z\) at \(y \in Y\) if \((y, f(y))\) is a transversal point in \(Y \times_X Z \cong f^{-1}Z\).

**Lemma 2.1.** The set \(Y \times_X^\triangledown Z\) is open in \(Y \times_X Z\). It is a \(C^k\)-smooth Banach manifold with tangent space

\[
T_{(y,z)} Y \times_X^\triangledown Z = \text{Ker} \left( df_y \oplus -dg_z : T_y Y \oplus T_z Z \to T_x X \right).
\]

**Proof.** We fix some \(w_0 := (y_0, z_0) \in Y \times_X^\triangledown Z\) and set

\[
K_0 := \text{Ker} \left( df_{y_0} \oplus -dg_{z_0} : T_{y_0} Y \oplus T_{z_0} Z \to T_x X \right).
\]

Let \(Q_0\) be a closed complement of \(K_0\). Then the map \(df_{y_0} \oplus -dg_{z_0} : Q_0 \to T_x X\) is an isomorphism.

Our choice of \(Q_0\) ensures that there exist a neighborhood \(V \subset Y \times Z\) of \((y_0, z_0)\) and \(C^k\)-maps \(w' : V \to K_0\) and \(w'' : V \to Q_0\) such that \(dw'_{w_0}\) (respectively, \(dw''_{w_0}\)) is the projection of \(T_{y_0} Y \oplus T_{z_0} Z\) onto \(K_0\) (respectively, onto \(Q_0\)). Hence \((w', w'')\) will are local variables in a (smaller) neighborhood \(V_1 \subset Y \times Z\) of \(w_0 = (y_0, z_0)\).

The lemma follows now by the implicit function theorem applied to the equation \(f(y) = g(z)\) in the new coordinates \((w', w'')\).

It is easy to see that the set \(\mathcal{P}\) is in fact the fibered product of the Banach manifold \(S_U \times T \times \partial_U^k\) by itself with respect to the maps \(\sigma_0\) and \(\sigma_{\mathcal{P}}\) into \(E'\). By Lemma 2.1, \(\mathcal{P}\) is a Banach manifold at points \((u, J_S, J) \in \mathcal{P}\) where \(\sigma_{\mathcal{P}}\) is transversal to \(\sigma_0\). However, at each point \((u, J_S, J; 0)\) on the zero section \(\sigma_0\) of the bundle \(\hat{E}'\) we have the following natural decomposition:

\[
T_{(u, J_S, J; 0)} \hat{E}' = d\sigma_0 \left( T_{(u, J_S, J)} (S_U \times \partial_S \times \partial_U^k) \right) \oplus \hat{E}'_{(u, J_S, J)},
\]

where the first component is the tangent space of the zero section of \(\hat{E}'\) and the second is the tangent space of the fiber \(\hat{E}'_{(u, J_S, J)}\). Let \(p_2\) be the projection onto the second component. Then the transversality of \(\sigma_{\mathcal{P}}\) and \(\sigma_0\) in \((u, J_S, J; 0)\) is equivalent to the surjectivity of the map \(p_2 \circ d\sigma_{\mathcal{P}} : T_{(u, J_S, J)} (S_U \times \partial_U^k) \to \hat{E}'_{(u, J_S, J)}\). However, by Lemma 1.2 this map is the linearization of \(\sigma_{\mathcal{P}}\) at \((u, J_S, J)\) and, therefore, has the form (1.3).

Thus, the transversality of \(\sigma_{\mathcal{P}}\) and \(\sigma_0\) at a point \((u, J_S, J) \in \mathcal{P}\) is equivalent to the surjectivity of the following operator:

\[
\nabla \sigma_{\mathcal{P}} : T_u L^{1,p}(S, X) \oplus T_{J_S} T_g \oplus T_J \partial_U^k \to \hat{E}'_{(u, J_S, J)}; \\
\nabla \sigma_{\mathcal{P}} : (v, J_S, J) \mapsto D_{(u, J)} v + J \circ du \circ J_S + J \circ du \circ J_S.
\]

By Definition 1.1 the quotient of \(\hat{E}'_{(u, J_S, J)} = L_{(0,1)}^p(S, E_u)\) by the image \(D_{u, J}\) is equal to \(H^1_D(S, E_u)\). The induced map \(J_S \in T_{J_S} T_g \mapsto [J \circ du \circ J_S] \in H^1_D(S, E_u)\)
is also easy to describe. It follows from (1.5) and from Corollary 1.8 that the image of $T_jS\mathbb{T}_g$ under this map coincides with the range of the homomorphism $du \circ J_S: H^1(S,TS) \cong T_jS\mathbb{T}_g \rightarrow H^1_D(S,E_u)$, and therefore its cokernel is $H^1_D(S,N_u)$.

Thus, it remains to understand the image of $T_j\mathfrak{J}_U^k$ in $H^1_D(S,N_u)$. For $(u,J_S,J) \in \mathcal{P}$ we define the map $\Psi = \Psi_{(u,J)}: T_j\mathfrak{J}_U^k \rightarrow \mathfrak{J}'_{(u,J_S,J)}$ by the formula $\Psi_{(u,J)}(J) := J \circ du \circ J_S$. Let $\nabla = \nabla_{(u,J)}: T_j\mathfrak{J}_U^k \rightarrow H^1_D(S,N_u)$ be induced by $\Psi$. Recall that if $(u,J_S,J) \in \mathcal{P}$, then $J_S$ is uniquely determined by $u$ and $J$.

**Lemma 2.2** (infinitesimal transversality). The operator $\nabla: T_j\mathfrak{J}_U^k \rightarrow H^1_D(S,N_u)$ is surjective for each $(u,J_S,J) \in \mathcal{P}^*$.

**Proof.** It is proved in [2] (see also [9]) that if $(u,J_S,J) \in \mathcal{P}^*$, then the map $u$ is an embedding in a neighborhood of each point $x \in S$, except for finitely many points. Hence there exists a non-empty open subset $V \subset S$ such that $u(V) \subset U$ and $u|_V$ is an embedding.

By Lemma 1.5 we have the isomorphism $H^0_D(S,N_u^* \otimes K_S) \cong H^1_D(S,N_u)^*$. The fact that an operator of the form $D = \bar{\partial} + R: L^1_p(S,E) \rightarrow L^1_{p,0}(S,E)$ on a compact Riemann surface $(S,J_S)$ is Fredholm shows that there exists a finite basis $\xi_1, \ldots, \xi_l$ of the space $H^0_D(S,N_u^* \otimes K_S)$. By Lemma 3.1.1 in [2] each $\xi \in H^0_D(S,N_u^* \otimes K_S)$ vanishes at no more than $c_1(N_u^* \otimes K_S)[S]$ points in $S$ (cf. the proof of Corollary 1.6). Hence there exist sections $\psi_i \in C^\infty_c(V,N \otimes \Lambda^{0,1}), i = 1, \ldots, l$, that make up an $\mathbb{R}$-basis of the space $H^1_D(S,N)$.

We consider an arbitrary $\psi_i \in C^\infty_c(V,N \otimes \Lambda^{0,1})$. This is a $\mathbb{C}$-antilinear $C^k$-smooth homomorphism from $TS|_V$ into $N|_V$ that vanishes outside a compact subset of $V$. Since $u|_V$ is a $C^k$-smooth embedding and $u(V) \subset U$, $\psi_i$ can be represented as the composition $\psi_i = \text{pr}_N \circ J \circ du \circ J_S$, where $J$ is a $J$-antilinear $C^k$-smooth endomorphism of the bundle $TX$ vanishing outside a compact subset of $U$. Hence $\dot{J} = T_j\mathfrak{J}_U^k$ and $\nabla \dot{J} = \psi_i$.

**Corollary 2.3.** Both $\mathcal{M}$ and $\mathcal{P}^*$ are $C^k$-smooth Banach manifolds, and the map $\pi_3: \mathcal{M} \rightarrow \mathfrak{J}_U^k$ is Fredholm. For each $(M,J)$ in $\mathcal{M}$ with $M = u(S)$ we have the following natural isomorphisms:

- $\text{Ker}(d\pi_3: T_{(M,J)}\mathcal{M} \rightarrow T_j\mathfrak{J}_U^k) \cong H^0_D(S,N_M)$,
- $\text{Coker}(d\pi_3: T_{(M,J)}\mathcal{M} \rightarrow T_j\mathfrak{J}_U^k) \cong H^1_D(S,N_M)$,

where $N_M = O(N_u) \oplus N_u^\text{sing}$ is the normal sheaf of $M$, $H^0_D(S,N_M)$ is the sum $H^0_D(S,N_u^\text{sing}) \oplus H^0_D(S,N_u^\text{sing})$, and $H^1_D(S,N_M)$ means $H^1_D(S,N_u)$. The index of the projection $\pi_3$ is described by the formula

$$\text{ind}_\mathbb{R}(\pi_3) | = \text{ind}_\mathbb{C}(N_M) := \text{dim}_\mathbb{R} H^0_D(S,N_M) - H^1_D(S,N_M)$$

and is equal to $2(c_1(X)[M] + (n-3)(1-g))$, $n := \text{dim}_\mathbb{C} X$.

**Proof.** It is easy to see that the section $\sigma_\mathfrak{J}$ is $C^k$-smooth for $J \in \mathfrak{J}_U^k$. Hence the assertion about $\mathcal{P}^*$ follows from Lemmas 2.1 and 2.2. Moreover, $\mathcal{P}^*$ is a $C^k$-smooth submanifold of $\mathfrak{X}^*$. It is not difficult to show that the slices of the $G$-action on $\mathfrak{X}^*$, constructed in the proof of Lemma 1.1, are $C^\infty$-smooth. Consequently, the projection $\pi_p: \mathcal{P}^* \rightarrow \mathcal{M}$ induces the structure of a $C^k$-manifold in $\mathcal{M}$.  

**ENVELOPES OF SPHERES**
We consider now the projection \( \pi : \mathcal{P}^* \to \mathcal{J}_U^k \). The tangent space \( T_{(u,J_S,J)} \mathcal{P}^* \) consists of the triples \((v, \dot{J}_S, \dot{J})\) that satisfy the condition
\[
D_{u,J}v + \frac{1}{2} J \circ du \circ \dot{J}_S + \frac{1}{2} \dot{J} \circ du \circ J_S = 0,
\]
and the differential \( d\pi : T_{(u,J_S,J)} \mathcal{P}^* \to T_{\dot{J}} \mathcal{J}_U^k \) has the form \((v, \dot{J}_S, \dot{J}) \in T_{(u,J_S,J)} \mathcal{P}^* \mapsto \dot{J} \in T_{\dot{J}} \mathcal{J}_U^k \).

Hence the kernel \( \text{Ker}(d\pi) \subset T_{(u,J_S,J)} \mathcal{P}^* \) is parametrized by the solutions of the equation
\[
D_{u,J}v + \frac{1}{2} J \circ du \circ \dot{J}_S = 0,
\]
where \( v \in \mathcal{E}_{(u,J_S,J)} \) and \( \dot{J}_S \in T_{J_S} \mathbb{T}_g \). Since the map \( \pi_\mathcal{P} : \mathcal{P}^* \to \mathcal{M} \) is a principle \( \mathbf{G} \)-bundle, \( \text{Ker}(d\pi_\mathcal{P} : T_{(u,J)} \mathcal{M} \to T_{\dot{J}} \mathcal{J}_U^k) \) is the quotient of \( \text{Ker}(d\pi) \) by the tangent space of the fiber \( \mathbf{G} \cdot (u,J_S,J) \). Since \( T_e \mathbf{G} \equiv H^0(S,TS) \) and \( \mathbf{G}_0 \) acts trivially on \( \mathbb{T}_g \) and on \( \mathcal{J}_U^k \), the tangent space of the fiber \( \mathbf{G} \cdot (u,J_S,J) \) in \((u,J_S,J)\) consists of the vectors of the form \((v,0,0), v \in du(H^0(S,TS)) \).

From the equalities
\[
H^0(S,TS) = \text{Ker}(\overline{\partial}_TS : L^{1,p}(S,TS) \to L^p(S,TS \otimes \Lambda^{0,1}(S))),
\]
\[
T_{J_S} \mathbb{T}_g \cong H^1(S,TS) = \text{Coker}(\overline{\partial}_TS),
\]
\[
du \circ \overline{\partial}_TS = D_{(u,J)} du
\]
we see that \( \text{Ker}(d\pi_\mathcal{P}) \) is isomorphic to the quotient
\[
\{ v \in L^{1,p}(S,E_u) : Du = du(\phi) \text{ for some } \phi \in L^p(S,TS \otimes \Lambda^{0,1}(S)) / du(L^{1,p}(S,TS)) \}.
\]
Hence \( \text{Ker}(d\pi_\mathcal{P} : T_{(M,J)} \mathcal{M} \to T_{\dot{J}} \mathcal{J}_U^k) \cong H^0_D(M,N_M) \) by Lemma 1.7. In particular, \( \text{Ker}(d\pi_\mathcal{P}) \) is finite dimensional.

In a similar way, the image \( d\pi_\mathcal{P} \) consists of \( \dot{J} \) such that the equation
\[
D_{u,J}v + \frac{1}{2} J \circ du \circ \dot{J}_S + \frac{1}{2} \dot{J} \circ du \circ J_S = 0
\]
has a solution \((v, \dot{J}_S)\). Consequently,
\[
\exists (d\pi_\mathcal{P}) = \text{Ker} \overline{\nabla} \quad \text{and} \quad \text{Coker}(d\pi_\mathcal{P}) \cong H^1_D(S,N_u).
\]
Hence \( d\pi_\mathcal{P} \) is Fredholm, and therefore \( \pi : \mathcal{P}^* \to \mathcal{J}_U^k \) is also Fredholm.

Corollary 1.8 yields the equality \( \text{ind}_R(N_M) = \text{ind}_R(E_u) - \text{ind}_R(TS) \). Using the Riemann–Roch theorem together with the equalities \( c_1(E) = c_1(X)[M] \) and \( c_1(TS) = 2 - 2g \) we obtain the required formula:
\[
\text{ind}_R(N) = 2(c_1(X)[M] + n(1 - g) - (3 - 3g)) = 2c_1(X)[M] + (n - 3)(1 - g).
\]

Before stating other results, let us introduce further notation.

Definition Let \( Y \) be a \( C^k \)-smooth finite-dimensional manifold that may have a non-empty \( C^k \)-smooth boundary \( \partial Y \), and let \( h : Y \to \mathcal{J}_U^k \) be a \( C^k \)-smooth map. We define the \textit{relative moduli space} as follows:
\[
\mathcal{M}_h := \{ (u,J_S,y) \in S_U \times \mathbb{T}_g \times Y : (u,J_S,h(y)) \in \mathcal{P}^* \} / \mathbf{G},
\]
and equip it with the natural projection \( \pi_h : \mathcal{M}_h \to Y \). In the particular case when \( Y = \{ J \} \hookrightarrow \mathcal{J}_U^k \) we obtain the moduli space of \( J \)-holomorphic curves \( \mathcal{M}_J := \pi_g^{-1}(J) \).

In general, the projection \( \pi_h \) has fibers \( \pi_h^{-1}(y) = \mathcal{M}_{h(y)} \). We shall denote the elements of \( \mathcal{M}_h \) by \((M,y)\), where \( M = u(S) \) and the map \( u : S \to X \) is \( h(y) \)-holomorphic.
Lemma 2.4. Let Y be a $C^k$-smooth finite dimensional manifold, and let $h: Y \to \mathcal{J}_U^k$ be a $C^k$-smooth map. Assume that for some $y_0 \in Y$ and $M_0 = u_0(S) \in \mathcal{M}_h(y_0)$ the map $\overline{\Psi} \circ dh: T_{y_0}Y \to H^1_D(S, Nu_0)$ is surjective. Then, in some neighborhood of $(M_0, y_0) \in \mathcal{M}_h$, the space $\mathcal{M}_h$ is a $C^k$-smooth manifold with tangent space

\[(2.1) \quad T_{(M,y)}\mathcal{M}_h = \text{Ker} \left( D \oplus \Psi \circ dh : \mathcal{E}_{u_0}(y) \oplus T_y Y \to \mathcal{E}_{u_0,h(y)} \right) / \text{du} \left( H^0(S, TS) \right). \]

Proof. Let $y \in Y$ and let $(u, J_S, h(y)) \in \mathcal{P}_*$. Then $(M, y) \in \mathcal{M}_h$. It follows from the proof of Corollary 2.3 that the range of the homomorphism $d\pi_3: T_{(M,y)}\mathcal{M} \to T_{h(y)}\mathcal{J}_U^k$ coincides with $\text{Ker} \left( \overline{\Psi}_{(u_0, h(y))} \right)$ and $\overline{\Psi}$ maps the cokernel $\text{Coker} \left( d\pi_3 \right)$ isomorphically onto $H^1_D(S, Nu_0)$. Hence the lemma follows from Lemma 2.1.

Definition. Let $Y$ be a compact manifold, let $h: Y \to \mathcal{J}_U^k$ be a $C^k$-smooth map, and let $\mathcal{M}_h \subset \mathcal{M} \times Y$ the corresponding moduli space and $(M_0, y_0) \in \mathcal{M}_h$ be a point in this space. Then the component through $(M_0, y_0)$ of the space $\mathcal{M}_h$ is the subset $\mathcal{M}_h(M_0, y_0)$ of $(M, y) \in \mathcal{M}_h$ such that for each neighborhood $W$ of the image $h(Y) \subset \mathcal{J}_U^k$ there exists a continuous path $\gamma: [0, 1] \to \mathcal{M}$ with the following properties:

- (a) $\gamma(0) = (M_0, h(y_0))$ and $\gamma(1) = (M, h(y))$, that is, $\gamma$ joins $(M_0, y_0)$ with $(M, y)$ in $\mathcal{M}$;
- (b) $J_t := \pi_3(\gamma(t)) \in W \subset \mathcal{J}_U^k$ for all $t \in [0, 1]$, that is, the corresponding path of almost complex structures $J_t$ lies in the fixed neighborhood $W$ of $h(Y)$.

Lemma 2.5. Assume that $h: Y \to \mathcal{J}_U^k$, let $(M_0, y_0) \in \mathcal{M}_h$, and $\mathcal{M}_h(M_0, y_0)$ are as in Definition 2.3. Then we have the following results:

- (i) $\mathcal{M}_h(M_0, y_0)$ is a closed subset of $\mathcal{M}_h$;
- (ii) if the component $\mathcal{M}_h(M_0, y_0)$ is compact, then there exists a subset $\mathcal{M}_h^0$ that contains $\mathcal{M}_h(M_0, y_0)$ and is compact and open in $\mathcal{M}_h$;
- (iii) if the component $\mathcal{M}_h(M_0, y_0)$ is non-compact, then there exists a continuous path $\gamma: [0, 1] \to \mathcal{M}_h$ with the following properties:
  - (a) $\gamma(0) = (M_0, h(y_0))$, that is, $\gamma$ starts at $(M_0, y_0)$;
  - (b) there exists a sequence $t_n \to 1$ such that the sequence $(M_n, J_n) := \gamma(t_n)$ lies in $\mathcal{M}_h$ and is discrete there, while the sequence $\{J_n\}$ converges in $C^k$ to some $J^* \in \mathcal{J}_U^k$.

Proof. (i) Let $(M', y') \in \overline{\mathcal{M}_h(M_0, y_0)} \subset \mathcal{M}_h$ and let $J' = h(y')$. Let $W$ be a neighborhood of $h(Y) \subset \mathcal{J}_U^k$ and let $\{\{M_n, y_n]\}\} be a sequence in $\mathcal{M}_h$ that converges to $(M', y')$. Then there exists a ball $B \supset (M', J')$ in $\mathcal{M}$ such that its projection to $\mathcal{J}_U^k$ lies in $W$. Since $(M_n, h(y_n))$ belongs to $B$ for $n$ sufficiently large, there exists a path $(M_t, J_t)$ in $\mathcal{M}$ joining $(M_0, h(y_0))$ and $(M', J')$ such that $J_t \in W$ for all $t \in [0, 1]$. Hence $\mathcal{M}_h(M_0, y_0)$ is closed.

(ii) Let $(M', y') \in \mathcal{M}_h$ and let $J' = h(y')$. We choose a finite-dimensional subspace $F \subset T_{J'}\mathcal{J}_U^k$ such that the map $D_{u', J'} \oplus \Psi: \mathcal{E}_{u', J'} \oplus F \to \mathcal{E}_{u', J'}$ is surjective. Let $B \supset 0$ be a ball in $F$. Then there exists a $C^k$-smooth map $H: \bar{Y} \times B$ such that $H(y, 0) = h(y)$ and $dH_{(y', 0)}: T_{(y', 0)}(Y \times B) \to T_{J'}\mathcal{J}_U^k$ induces an isomorphism $T_0B \xrightarrow{\cong} F \subset T_{J'}\mathcal{J}_U^k$. Thus, $\mathcal{M}_H$ contains a neighborhood $V \supset (M', y', 0)$ such that it is a manifold and $V \cap \mathcal{M}_h$ is closed in $V$. It follows that $\mathcal{M}_h$ is a locally compact topological space.

Since $\mathcal{M}_h(M_0, y_0)$ is a compact subset of $\mathcal{M}_h$, it has a neighborhood $V$ with compact closure $\mathcal{V} \subset \mathcal{M}_h$. Assume that $W_i \subset \mathcal{J}_U^k$ make up a fundamental system of
neighborhoods of $h(Y)$ and, in particular, $\bigcap_i W_i = h(Y)$. Let $V_i$ be the set of all $(M, y) \in M_h$ such that $(M, h(y))$ can be joined with $(M_0, h(y_0))$ by a path $(M_t, J_t)$ in $M$ with $J_t$ in $W_i$. Then we have $\bigcap_i V_i = M_h(M_0, y_0)$. The same argument as in part (i) shows that the $V_i$ are open and closed in $M_h$.

We claim that there exists a positive integer $N \in \mathbb{N}$ such that $(\bigcap_{i=1}^{N} V_i) \cap V \subset V$. For otherwise, for each $n \in \mathbb{N}$ there exists $(M_n, y_n) \in (\bigcap_{i=1}^{n} V_i) \cap V \setminus V$. Then, however, a subsequence of $\{(M_n, y_n)\}$ converges to some $(M^*, y^*) \in (\bigcap_{i=1}^{\infty} V_i) \cap V \setminus V$, which is impossible because $\bigcap_{i=1}^{\infty} V_i \subset V$.

For such $N \in \mathbb{N}$ the set

\begin{equation}
M_0^0 := \left(\bigcap_{i=1}^{N} V_i\right) \cap V = \left(\bigcap_{i=1}^{N} V_i\right) \cap V
\end{equation}

satisfies the assumptions of part (ii) of the lemma.

(iii) Assume that $M_h(M_0, y_0)$ is non-compact. Then there exists a discrete sequence $\{(M_n, y_n)\}$ in $M_h(M_0, y_0)$. Since $Y$ is compact, we may assume that the $y_n$ converge to some $y^*$. For each $n \in \mathbb{N}$ we fix a path $\gamma_n : [0, 1] \to M_h$ between $(M_{n-1}, h(y_{n-1}))$ and $(M_n, h(y_n))$. We set $t_n := 1 - 2^{-n}$. For $t \in [t_{n-1}, t_n]$ we set $\gamma(t) := \gamma_n(2^n(t - t_{n-1}))$. Then $\gamma : [0, 1] \to M_h$ and $t_n \nearrow 1$ are the required path and sequence.

**Theorem 2.6.** Let $(M_0, J_0) \in M$ and let $h : [0, 1] \to \mathcal{A}^k_U$ be a $C^k$-smooth path with $h(0) = J_0$. Assume that there exists a compact open subset $M_0^0$ of $M_h$ containing $(M_0, J_0)$. Assume further that the index $\text{ind}(\pi_0) = 2(c_1(X)[M_0] + (n-3)(1-g))$ is non-negative. Then $h$ can be $C^k$-approximated by smooth maps $h_n : [0, 1] \to \mathcal{A}^k_U$ with the following properties:

(i) each $M_n$ contains a component $M_0^0$ that is a $C^k$-smooth manifold of the expected dimension $\dim M_0^0 = \text{ind}(\pi_0) + 1$;

(ii) the sets $M_0^0_{h_n(0)} := \pi^{-1}_n(0) \cap M_0^0$ and $M_0^0_{h_n(1)} := \pi^{-1}_n(1) \cap M_0^0$ are also $C^k$-smooth manifolds of the expected dimension $\text{ind}(\pi_0)$ ($M_0^0_{h_n(1)}$ can be empty!), and $M_0^0_{h_n}$ is a $C^k$-smooth bordism between $M_0^0_{h_n(0)}$ and $M_0^0_{h_n(1)}$;

(iii) each $M_0^0_{h_n(0)}$ can be joined with $(M_0, J_0)$ by a path in $M$, that is, there exist $C^k$-smooth paths $\gamma_n : [0, 1] \to M$ with $\gamma_n(0) = (M_0, J_0)$ and $\gamma_n(1) \in M_0^0_{h_n(0)}$; in particular, all $M_0^0_{h_n(0)}$ are not empty;

(iv) for each element $(M, J)$ of $M_h$, we have the inequality $\dim \mathbb{R}^1 \cap D_{\mathcal{A}^k}^1(S, N_M) \leq 1$.

**Proof.** We denote $M_0^0$ by $K$. Let $\mathcal{E}_K$ and $\mathcal{E}'_K$ be the Banach bundles over $K$ induced by the bundles $\mathcal{E} \to M$ and $\mathcal{E}' \to M$, respectively. Let also $\mathcal{I} := h^*\mathcal{T}\mathcal{A}^k_U$ be the pull-back of $T\mathcal{A}^k_U$ to $[0, 1]$.

By Lemma 2.2, for each $(M, J) \in K$ with $M = u(S)$ there exists $m = (M, J)$ in $\mathbb{N}$ and a $C^k$-smooth homomorphism $P_{(M, J)} : F_{(M, J)} \to \mathcal{T}$ of the trivial vector bundle $F_{(M, J)}$ of rank $\text{rk}F_{(M, J)} = m(M, J)$ over $[0, 1]$ such that the operator

\begin{equation}
D_{(u, J)} \oplus \Psi_{(u, J)} \circ P_{(M, J)} : \mathcal{E}_{(M, J)} \oplus F_{(M, J)} \to \mathcal{E}'_{(M, J)}
\end{equation}

is surjective. Note that the operator (2.3) remains surjective for all $(M', J')$ from a neighborhood of $(M, J)$. Since $K$ is compact, we may choose finitely many pairs...
Thus, \( H(t,0) = h(t) \), that is, \( H \) is a deformation of \( h \) with parameter space \( B = B(0,r) \subset \mathbb{R}^n \); 

(i) \( D(y_0, u, J) = 0 \) for all \( J \in \mathcal{J}_U \).

(ii) \( D(y_0, u, J) = 0 \) for all \( J \in \mathcal{J}_U \).

(iii) \( D(y_0, u, J) = 0 \) for all \( J \in \mathcal{J}_U \).

We claim that there exists a smaller ball \( B \) such that \( (M, J) \in \mathcal{J}_U \).
converging to $0 \in B$ and set $h_n(t) := H(t, y_n)$. Let $M_{h_n}^0 := M_{h_n} \cap V$, so that $M_{h_n}^0 = p^{-1}(y_n)$. Then each $M_{h_n}^0$ is a $C^\infty$-smooth non-empty manifold that can be joined with $(M_0, J_0)$ by a path in $M$.

By Lemma 2.4 the tangent space to $V_1 \subset M_H$ in $(u, t, y)$ is canonically isomorphic to the quotient

$$\ker(D_{u,H(t,y)} \oplus \Psi \circ dH : \mathcal{E}_{u,H(t,y)} \oplus T(t,y)([0,1] \times B_1) \to \mathcal{E}_{u,H(t,y)}') / du(H^0(S,TS)).$$

Since $p : V_1 \to B_1$ is the projection $(u, t, y) \in V_1 \mapsto y \in B_1$, the differential $dp_{(u,t,y)}$ maps a tangent vector of the form $(\dot{u}, \dot{t}, \dot{y})$ to $\dot{y} \in T_yB_1$. This means that $dp_{(u,t,y)}$ is the restriction to $\ker(D_{u,H(t,y)} \oplus \Psi \circ dH)$ of the linear projection $p_B : \mathcal{E}_{u,H(t,y)} \oplus T(t,y)([0,1] \times B_1) \to T_yB_1$ defined by the formula $p_B(\dot{u}, \dot{t}, \dot{y}) = \dot{y}$. In particular, if $y = y_n$, then the map $dp_{(u,t,y)}$ is surjective, which means the surjectivity of the map

$$p_B : \ker(D_{u,H(t,y)} \oplus \Psi \circ dH) \to T_{y_n}B_1.$$

The last result is equivalent to the surjectivity of the map

$$D_{u,H(t,y_n)} \oplus \Psi \circ dH \oplus p_B : \mathcal{E}_{u,H(t,y_n)} \oplus T(t,y_n)([0,1] \times B_1) \to \mathcal{E}_{u,H(t,y_n)}' \oplus T_{y_n}B_1$$

and, therefore, of the map

$$(2.4) \quad D_{u,h_n(t)} \oplus \Psi \circ dh_n : \mathcal{E}_{u,h_n(t)} \oplus T_t[0,1] \to \mathcal{E}_{u,h_n(t)}'.$$

Consequently, $\dim_{\mathbb{R}} H^1_D(S, N_M) \leq 1$ for all $(M, t) \in M_{h_n} \cap V_1$.

**Corollary 2.7.** Under the hypothesis of Theorem 2.6 assume in addition that $S$ is the sphere $S^2$. Then for all $(M, t) \in M_{h_n}^0$ the associated $D_N$-operator is surjective, that is, $H^1_D(S^2, N_M) = 0$.

Moreover, $M_{h_n}$ is the trivial bordism: $M_{h_n} \cong M_{h_n(0)} \times [0,1]$. In particular, for each $h_n(0)$-holomorphic sphere $M_0 \in M_{h_n(0)}$ there exists a continuous family of $h_n(t)$-holomorphic spheres $M_{n,t} = u_{n,t}(S^2)$ with $M_{n,0} = M_0$.

**Proof.** Assume that $H^1_D(M, N_M) \neq 0$ for some $(M, t) \in M_{h_n}^0$. Then $H^1_D(M, N_M) = 1$ by Theorem 2.6. However, this contradicts the result of Corollary 1.6 for $S = S^2$ and $L := N_M$.

Let $(M, t) \in M_{h_n}^0$ satisfy $M = u(S)$ and $J = h_n(t)$. Let also $\dot{J} \neq 0 \in dh_n(T_t[0,1])$. Then, by Lemma 2.1 and Corollary 2.3 the tangent space $T_{(M,t)}M_{h_n}^0$ is canonically isomorphic to

$$\ker(D_{u,J} \oplus \Psi : \mathcal{E}_{u,J} \oplus \mathbb{R}\dot{J} \to \mathcal{E}_{u,J}') / du(H^0(S,TS)),$$

and the differential of the projection $d\pi_{h_n} : T_{(M,t)}M_{h_n}^0 \to T_t[0,1] \cong \mathbb{R}$ takes the form $d\pi_{h_n}[v,aJ] = a$. When $S = S^2$, the space $H^1(S,TS)$ is trivial and Corollary 1.8 shows that $D_{u,J} : \mathcal{E}_{u,J} \to \mathcal{E}_{u,J}'$ is surjective. Hence, for $a \neq 0 \in \mathbb{R}$ there exists $v \in \mathcal{E}_{u,J}$ such that $[v,aJ] \in T_{(M,t)}M_{h_n}^0$. This means that for each $(M, t) \in M_{h_n}^0$ the projection $d\pi_{h_n} : T_{(M,t)}M_{h_n}^0 \to T_t[0,1]$ is surjective. Since $M_{h_n}^0$ is compact, there exists a diffeomorphism $M_{h_n} \cong M_{h_n(0)} \times [0,1]$.
§ 3. GROMOV TOPOLOGY AND DEFORMATIONS OF NON-COMPACT HOLOMORPHIC CURVES

The techniques developed in the previous section enables us to construct local deformations of \( J_t \)-holomorphic spheres \( M_t = u_t(S^2) \) for appropriate families of almost complex structures \( J_t \). The obstruction to the existence of a ‘global family’ is that the sphere \( M_t \) can eventually ‘break down’ into several components. For our purposes here we must know the exact fashion of this break, and we must also learn to deform the reducible (that is, consisting of several components) curves produced by such a ‘breakdown’. We start with the indication of a suitable category of reducible curves.

**Definition** The complex analytic set \( A_0 := \{(z_1, z_2) \in \Delta^2 : z_1 \cdot z_2 = 0\} \) is called the *standard node*. A *nodal curve* is a connected reduced complex space \( C \) of dimension 1 having finitely many irreducible components and with singularities only at finitely many *nodal points* that have neighborhoods isomorphic to the standard node. Furthermore, we assume that the boundary \( \partial C \) of \( C \) consists of finitely many smooth circles and \( \overline{C} := C \cup \partial C \) is compact. The case of \( \partial C = \emptyset \) is not excluded.

**Definition** A smooth oriented real surface \( \Sigma \) with boundary \( \partial \Sigma \) parametrizes a nodal complex curve \( C \) if there exists a continuous map \( \sigma: \Sigma \rightarrow C \) with the following properties:

1. If \( a \in C \) is a nodal point, then \( \gamma_a := \sigma^{-1}(a) \) is a smooth embedded circle in \( \Sigma \);
2. \( \sigma: \Sigma \setminus \bigcup_{i=1}^{N} \gamma_{a_i} \rightarrow \overline{C} \setminus \{a_1, \ldots, a_N\} \) is a diffeomorphism, where \( \{a_1, \ldots, a_N\} \) is the set of all nodal points in \( C \).

The map \( \sigma \) is called a *parametrization* of \( C \).

The projection \( \sigma \) ‘contracts’ circles \( \gamma_1, \ldots, \gamma_5 \) into the nodal points \( a_1, \ldots, a_5 \).

Note that the parametrization \( \sigma \) is not unique: if \( g: \Sigma \rightarrow \Sigma \) is a diffeomorphism, then \( \sigma \circ g: \Sigma \rightarrow C \) is also a parametrization.

**Definition** A *stable curve over an almost complex manifold* \((X, J)\) is a pair \((C, u)\), where \( C \) is a nodal curve with boundary \( \partial C \) and \( u: C \rightarrow X \) is a \( J \)-holomorphic map with the following property: if the map \( u \) is constant on a compact irreducible component \( C_j \) of the curve \( C \), then the group of biholomorphic automorphisms of \( C_j \) preserving the nodal points of \( C_j \) is finite.

It is easy to see that this condition imposes only the following two restrictions:

1. If \( C_j \) is rational, that is, biholomorphic to \( \mathbb{CP}^1 \), then \( C_j \) contains at least three nodal points of \( C \);
2. If \( C_j \) is a torus, then \( C_j \) contains at least one nodal point of \( C \).
This definition was given by Deligne and Mumford in the case of abstract algebraic curves and was later generalized by Kontsevich to the case of maps into $\mathcal{X}$.

Stable curves occur in a natural way when one attempts to compactify the space of embedded or immersed curves in $\mathcal{X}$. Namely, one introduces the Gromov topology in the set of all stable curves over $\mathcal{X}$. We shall describe this topology by describing convergent sequences.

Let $\{J_n\}$ be a sequence of $C^k$-smooth almost complex structures on $\mathcal{X}$ convergent to some $J$ in $C^k$. Let $\{(C_n, u_n)\}$ be a sequence of $J_n$-holomorphic curves that are stable over $\mathcal{X}$ and are parametrized by the same real surface $\Sigma$.

**Definition** The sequence $\{(C_n, u_n)\}$ converges to a stable $J$-holomorphic curve $(C_\infty, u_\infty)$ over $\mathcal{X}$ if the following holds:

1. $C_\infty$ is also parametrized by $\Sigma$; moreover, there exist parametrizations $\sigma_n : \Sigma \to C_n$ and $\sigma_\infty : \Sigma \to C_\infty$ such that $u_n \circ \sigma_n$ converge to $u_\infty \circ \sigma_\infty$ in the $C^0$-topology on $\Sigma$ (that is, up to the boundary);
2. let $\{a_1, \ldots, a_N\}$ be the nodal points of $C_\infty$ and let $\gamma_i := \sigma_\infty^{-1}(a_i)$; let also $K$ be a compact subset of $\Sigma \setminus \bigcup_i \gamma_i$; then for all $n \geq n^*(K)$ the set $\sigma_n(K)$ does not contain nodal points of $C_n$, $u_n \circ \sigma_n$ converges to $u_\infty \circ \sigma_\infty$ in the $L^1$-topology on $K$, and the inverse images $\sigma_n^* J_{C_n}$ of the complex structures $J_{C_n}$ on the curves $C_n$ converge to the inverse image $\sigma_\infty^* J_{C_\infty}$ of the complex structure $J_{C_\infty}$ on $C_\infty$ in the $C^\infty$-topology on $K$.

**Definition** An annulus $A$ on a real surface or a complex curve is a domain diffeomorphic (respectively, biholomorphic) to the standard annulus $A_{r,R} := \{z \in \mathbb{C} : r < |z| < R\}$. A subdomain of a real surface or a complex curve that is diffeomorphic to a disc with two holes is called pants. In both cases we assume that the boundary of the domain consists of smooth embedded circles. An annulus $A$ is adjacent to a circle $\gamma$ if $\gamma$ is a component of its boundary $\partial A$.

**Theorem 3.1.** Let $\{(C_n, u_n)\}$ be a sequence of stable $J_n$-holomorphic curves over $\mathcal{X}$ that satisfy the following conditions:

1. the $J_n$ are $C^k$-smooth and converge to $J$ in $C^k$ for some $k \geq 1$;
2. $\text{area}[u_n(C_n)] \leq A$ for all $n$;
3. all $C_n$ are parametrized by the same real surface $\Sigma$;
(4) the sequence \((C_n, u_n)\) converges near the boundary; this means that there exist parametrizations \(\sigma_n: \Sigma \to C_n\) and annuli \(A_\alpha\) in \(\Sigma\) such that

(a) each \(A_\alpha\) is adjacent to a single component \(\gamma_\alpha\) of the boundary \(\partial \Sigma\), the images \(\sigma_n(A_\alpha)\) do not contain the nodal points of \(C_n\) and the complex structures \(\sigma_n^* J_{C_n}\) are constant on each annulus \(A_\alpha\);

(b) the maps \(u_n \circ \sigma_n\) converge in the \(L^{1,p}\)-topology on each \(A_\alpha\).

Then there exists a subsequence of stable curves, also denoted by \(\{(C_n, u_n)\}\), that converges in the Gromov topology to a curve \((C_\infty, u_\infty)\) stable over \(X\).

This theorem is due to Gromov [6]; proofs can be also found in [13] and [14]. We shall require a more precise description of Gromov convergency, which can be derived from [14].

**Proposition 3.2.** Under the assumptions of Theorem 3.1 one can choose parametrizations \(\sigma_n: \Sigma \to C_n\) and a finite cover of \(\Sigma\) by open subsets \(\{V_\alpha\}\) and \(\{V_{\alpha \beta}\}\) with the following properties (see Fig. 6):

(i) each \(V_\alpha\) is a disc, an annulus, or pants, and \(V_\alpha \cap V_\beta = \emptyset\) for all \(\alpha \neq \beta\);

(ii) for each component of the boundary of \(\Sigma\) there exists a single annulus \(V_\alpha\) adjacent to this component;

(iii) each \(V_{\alpha \beta}\) is an annulus intersecting \(V_\alpha\) and \(V_\beta\) by annuli \(W_{\alpha \beta} := V_{\alpha \beta} \cap V_\alpha\) and \(W_{\beta \alpha} := V_{\alpha \beta} \cap V_\beta\); furthermore, \(V_{\alpha \beta} = V_{\beta \alpha}\) and the annuli \(W_{\alpha \beta}\) and \(W_{\beta \alpha}\) are distinct and disjoint;

(iv) \(\sigma_n^* J_{C_n}|_{V_\alpha}\) does not depend on \(n\), and the sequence of \(u_n \circ \sigma_n|_{V_\alpha}\) converges in \(L^{1,p}\) to \(u_\infty \circ \sigma_\infty|_{V_\alpha}\);

(v) there exist \(L^{1,p}\)-smooth maps \(\phi_{\alpha \beta}^n: V_{\alpha \beta} \to \Delta^2\) and \(\phi_{\alpha \beta}^\infty: V_{\alpha \beta} \to \Delta^2\) that induce biholomorphisms \(\overline{\phi}_{\alpha \beta}^n: \sigma_n(V_{\alpha \beta}) \xrightarrow{\cong} \{(z_1, z_2) \in \Delta^2: z_1 \cdot z_2 = \lambda_{\alpha \beta}^n\}\)
and \(\overline{\phi}_{\alpha \beta}^\infty: \sigma_\infty(V_{\alpha \beta}) \xrightarrow{\cong} \{(z_1, z_2) \in \Delta^2: z_1 \cdot z_2 = \lambda_{\alpha \beta}^\infty\}\);

(vi) for the coordinate functions \(z_1\) and \(z_2\) on \(\Delta^2\) the compositions \(z_1 \circ \phi_{\alpha \beta}^n|_{W_{\alpha \beta}}\) and \(z_2 \circ \phi_{\alpha \beta}^n|_{W_{\beta \alpha}}\) do not depend on \(n\); moreover, the images \(z_1 \circ \phi_{\alpha \beta}^n(W_{\alpha \beta})\) and \(z_2 \circ \phi_{\alpha \beta}^n(W_{\beta \alpha})\) are some annuli \(A_{r_{\alpha \beta}1}\) and \(A_{r_{\beta \alpha}1}\), respectively;

(vii) for a fixed limiting curve \((C_\infty, u_\infty)\) the corresponding covering \(\{V_\alpha, V_{\alpha \beta}\}\) of the real surface \(\Sigma\) can be chosen to depend only on \((C_\infty, u_\infty)\), that is, to be the same for all sequences \(\{(C_n, u_n)\}\) converging to \((C_\infty, u_\infty)\).

Thus, Theorem 3.1 ensures that a deformation of a pseudoholomorphic curve can ‘break down’ only into pseudoholomorphic curves, which gives us a possibility to continue the process of local deformation.

We shall now consider local deformations of non-compact curves.

**Definition** A Banach ball is a ball in some complex Banach space. A subset \(M\) of the Banach ball \(B\) is called a Banach analytic set of finite codimension (a BASFC) if there exists a holomorphic map \(F: B \to \mathbb{C}^N, N < \infty\) such that \(M = \{x \in B: F(x) = 0\}\).

This concept is important because, unlike general Banach analytic sets, BASFC’s have properties similar to those of usual finite-dimensional analytic sets. Namely, we have the following result.

**Theorem 3.3** [15]. Let \(B\) be a ball in a Banach space \(\mathcal{F}\), let \(M \subset B\) be a BASFC,
and \(x_0\) a point in \(\mathcal{M}\). Then there exists a neighborhood \(U \ni x_0\) in \(B\) such that \(\mathcal{M} \cap U\) is a finite union of BASFC’s \(\mathcal{M}_j\), each of them irreducible at \(x_0\).

Moreover, each \(\mathcal{M}_j\) can be represented as a proper ramified covering over a domain in a closed linear subspace \(\mathcal{F}_j \subset \mathcal{F}\) of finite codimension.

Our aim in this section is to prove the existence of a complete family of holomorphic deformations of a stable curve over \(X\), parametrized by a BASFC. Before stating the result we introduce the following definition.

**Definition** Let \(C\) be a nodal curve, let \(E\) be a holomorphic vector bundle over \(C\), and let \(C = \bigcup_{i=1}^l C_i\) be the decomposition of \(C\) into irreducible components. Assume that \(E\) extends sufficiently smoothly up to the boundary \(\partial C\). We define an \(L^1-p\)-section \(v\) of the bundle \(E\) over \(C\) as a collection \((v_i)_{i=1}^l\) of \(v_i \in L^{1,p}(C_i,E)\) such that at each nodal point \(z \in C_i \cap C_j\) we have \(v_i(z) = v_j(z)\). We also define an \(E\)-valued \(L^p\)-integrable \((0,1)\)-form \(\xi\) on \(C\) as a collection \((\xi_i)_{i=1}^l\) of \((0,1)\)-forms \(\xi_i \in L^p(C_i,E \otimes \Lambda^{(0,1)})\). Let \(L^{1,p}(C,E)\) be the Banach space of \(L^{1,p}\)-sections of \(E\) over \(C\) and let \(L^p(C,E \otimes \Lambda^{(0,1)})\) be the Banach space of \(L^p\)-integrable \((0,1)\)-forms \(C\). We also denote by \(\mathcal{H}^{1,p}(C,E)\) the Banach space of holomorphic \(L^{1,p}\)-sections of \(E\) over \(C\).

In a similar way, for each complex manifold \(X\) we shall mean by \(L^{1,p}(C,X)\) the set of all collections \(u = (u_i)_{i=1}^l\) of maps \(u_i \in L^{1,p}(C_i,X)\) satisfying the equality \(u_i(z) = u_j(z)\) at each nodal point \(z \in C_i \cap C_j\). It is easy to see that \(L^{1,p}(C,X)\) is a Banach manifold with tangent space \(T_uL^{1,p}(C,X) = L^{1,p}(C,u^*TX)\). Further, we denote the space of all holomorphic \(L^{1,p}\)-maps from \(C\) to \(X\) by \(\mathcal{H}^{1,p}(C,X)\). Note that for each \(u \in \mathcal{H}^{1,p}(C,X)\) we have \(u(C) \subset u(\overline{C}) \Subset X\).

The aim of this section is the following result.

**Theorem 3.4.** Let \((X,J)\) be a complex manifold and \((C_0,u_0)\) a stable complex curve over \(X\) parametrized by a real surface \(\Sigma\). Then there exist BASFC’s \(\mathcal{M}\) and \(\mathcal{C}\) and holomorphic maps \(F: \mathcal{C} \to X\) and \(\pi: \mathcal{C} \to \mathcal{M}\) with the following properties:

(a) for each \(\lambda \in \mathcal{M}\) the fiber \(C_\lambda = \pi^{-1}(\lambda)\) is a nodal curve parametrized by \(\Sigma\), and \(C_{\lambda_0} \cong C_0\) for some \(\lambda_0\);
(b) for \(F_\lambda := F|_{C_\lambda}\) the pair \((F_\lambda,C_\lambda)\) is a stable curve over \(X\), and \(F_{\lambda_0} = u_0\);
(c) if \((C',u')\) is a stable curve over \(X\) that is sufficiently close to \((C_0,u_0)\) in the Gromov topology, then there exists \(\lambda' \in \mathcal{M}\) such that \((C',u') = (C_{\lambda'},F_{\lambda'})\);
(d) for an appropriate integer \(N \in \mathbb{N}\) and a small ball \(B\) in the Banach space \(\mathcal{H}^{1,p}(C_0,u_0^*TX) \oplus \mathbb{C}^N\) the BASFC \(\mathcal{M}\) can be realized as the zero set of a holomorphic map \(\Phi\) from \(B\) into the finite dimensional space \(H^{1}(C,u_0^*TX)\).

The proof relies on the construction of local deformations of stable curves and on the analysis of conditions for patching together local models. The following result on the solution of a Cousin-type problem plays an important role in this proof.

**Lemma 3.5.** Let \(C\) be a nodal curve and \(E\) a holomorphic vector bundle over \(C\) that is \(C^1\)-smooth up to the boundary. Let \(\{V_i\}_{i=1}^l\) be a finite cover of \(C\) by Stein domains with piecewise smooth boundaries. Set \(V_{ij} := V_i \cap V_j\) and assume that all triple intersections \(V_i \cap V_j \cap V_k\) with \(i \neq j \neq k \neq i\) are empty.

Then for all \(2 \leq p < \infty\) the Čech coboundary operator

\[
\delta: \sum_{i=1}^l \mathcal{H}^{1,p}(V_i,E) \longrightarrow \sum_{i<j} \mathcal{H}^{1,p}(V_{ij},E), \quad (v_i)_{i=1}^l \longmapsto \sum_{i<j} (v_i - v_j)
\]
has the following properties:

(i) the image $\Im(\delta)$ is closed and has finite codimension; moreover, $\text{Coker}(\delta) = H^1(C, E) = H^1(C_{\text{comp}}, E)$, where $C_{\text{comp}}$ is the union of all compact irreducible components of $C$;

(ii) the kernel $\text{Ker}(\delta)$ is isomorphic to $\mathcal{H}^{1-p}(C, E)$ and has a closed complementary subspace.

Proof. Before the Čech complex, let us discuss the corresponding $\overline{\partial}$-problem. We consider the following operator:

\[ \overline{\partial}: L^1(C, E) \rightarrow L^p(C, E \otimes \Lambda^{(0,1)}_C). \]

First, we shall prove that the properties of this operator are similar to those of (3.1), that is, $\text{Ker}(\overline{\partial})$ has a closed complementary subspace, $\Im(\overline{\partial})$ is closed and of finite codimension, and $\text{Coker}(\overline{\partial}) = H^1(C, E) = H^1(C_{\text{comp}}, E)$. Moreover, we shall construct explicit isomorphisms between the (co)kernels of (3.1) and (3.2).

Since the boundary of $C$ is smooth, there exist nodal curves $C^+$ and $C^{++}$ such that $C \subset C^+ \subset C^{++}$ and the difference $C^+ \setminus C$ (respectively, $C^{++} \setminus C^+$) consists of annuli $A^+_\alpha$ (respectively, $A^{++}_\alpha$) adjacent to the corresponding components $\gamma_\alpha$ (respectively, $\gamma^+_\alpha$) of the boundary $\partial C$ (respectively, $\partial C^+$, see Fig. 4). Then $E$ extends to a holomorphic vector bundle over $C^{++}$, which we also denote by $E$.

Fig.4. $C \subset C^+ \subset C^{++}$.

Boundaries of $C^{++}$, $C^+$ and $C$ are marked by solid, dotted, and dashed lines, respectively.

We consider the following sheaves on $C^{++}$:

\[ L^1_{\text{loc}}(\cdot, E): V \mapsto L^1_{\text{loc}}(V, E), \]
\[ L^p_{\text{loc}}(\cdot, E \otimes \Lambda^{(0,1)}_{C^{++}}): V \mapsto L^p_{\text{loc}}(V, E \otimes \Lambda^{(0,1)}_{C^{++}}) \]

with the sheaf homomorphism induced by the operator

\[ \overline{\partial}: L^1_{\text{loc}}(V, E) \rightarrow L^p_{\text{loc}}(V, E \otimes \Lambda^{(0,1)}_{C^{++}}). \]

The sheaves $L^1_{\text{loc}}(\cdot, E)$ and $L^p_{\text{loc}}(\cdot, E \otimes \Lambda^{(0,1)}_{C^{++}})$, together with the $\overline{\partial}$-homomorphism, form a fine resolution of the (coherent) sheaf $\mathcal{O}_E$ of holomorphic sections of $E$ over $C^{++}$. At smooth points of $C^{++}$ this follows from the $L^p$-regularity of the elliptic operator $\overline{\partial}$, while at nodal points we use the following argument.

Let $z \in C$ be a nodal point in the intersection of two irreducible components, $C_i$ and $C_j$, of $C$. Let $\xi_i$ (respectively, $\xi_j$) be $E$-valued $L^p_{\text{loc}}$-integrable $(0,1)$-forms, defined in a neighborhood of $z$ in $C_i$ (respectively, in $C_j$). We find an $L^1_{\text{loc}}$-solution $v_i$ (respectively, $v_j$) of $\overline{\partial}v_i = \xi_i$ (respectively, $\overline{\partial}v_j = \xi_j$). By adding a local holomorphic section of $E$ over $C_i$ we obtain that $v_i(z) = v_j(z)$. Now, the pair $(v_i, v_j)$ defines a section of $L^1_{\text{loc}}(\cdot, E)$ in a neighborhood of $z$. This shows that the Dolbeault lemma for the holomorphic bundle $E$ holds also in a neighborhood of a nodal point.
This gives us the following natural isomorphisms:

\[
\begin{align*}
\text{Ker}(\overline{\partial}: L_{\text{loc}}^{1,p}(C^{++}, E) & \rightarrow L_{\text{loc}}^{p}(C^{++}, E \otimes \Lambda_{C^{++}}^{(0,1)})) = H^{0}(C^{++}, E), \\
\text{Coker}(\overline{\partial}: L_{\text{loc}}^{1,p}(C^{++}, E) & \rightarrow L_{\text{loc}}^{p}(C^{++}, E \otimes \Lambda_{C^{++}}^{(0,1)})) = H^{1}(C^{++}, E).
\end{align*}
\]  

(3.3)

We point out that similar isomorphisms exist for \( C \) and \( C^{+} \). Note also that we have the natural isomorphism

\[
H^{1}(C^{++}, E) = H^{1}(C^{+}, E) = H^{1}(C, E) = H^{1}(C_{\text{comp}}, E)
\]
induced by the restrictions

\[
\begin{align*}
L_{\text{loc}}^{p}(C^{++}, E \otimes \Lambda_{C^{++}}^{(0,1)}) & \rightarrow L_{\text{loc}}^{p}(C^{+}, E \otimes \Lambda_{C^{+}}^{(0,1)}) \rightarrow L_{\text{loc}}^{p}(C, E \otimes \Lambda_{C}^{(0,1)}) \\
& \rightarrow L_{\text{loc}}^{p}(C_{\text{comp}}, E \otimes \Lambda_{C}^{(0,1)}).
\end{align*}
\]

(3.4)

Now, fix arbitrary \( \xi \in L^{2}(C^{+}, E \otimes \Lambda_{C^{+}}^{(0,1)}) \) with zero cohomology class in \( H^{1}(C^{+}, E) \). We can extend \( \xi \) by zero to an element \( \tilde{\xi} \in L_{\text{loc}}^{2}(C^{++}, E \otimes \Lambda_{C^{++}}^{(0,1)}) \). Since \([\tilde{\xi}]_{\overline{\partial}} = [\xi]_{\overline{\partial}} = 0\), there exists a section \( \tilde{v} \in L_{\text{loc}}^{1,2}(C^{++}, E) \) such that \( \partial \tilde{v} = \tilde{\xi} \). The restriction \( v := \tilde{v}|_{C^{+}} \) satisfies the relations \( \overline{\partial} v = \xi \) and \( v \in L^{1,2}(C^{+}, E) \). This shows that the range of the (continuous!) operator

\[
\overline{\partial}: L^{1,2}(C^{+}, E) \rightarrow L^{2}(C^{+}, E \otimes \Lambda_{C^{+}}^{(0,1)})
\]

has finite codimension. By Banach’s open mapping theorem this range is closed. Further, since \( L^{1,2}(C^{+}, E) \) is a Hilbert space, the kernel of (3.4) admits a direct complement \( Q \subset L^{1,2}(C^{+}, E) \). Moreover, the operator (3.4) maps \( Q \) isomorphically onto its image. Hence the operator (3.4) splits, that is, there exists a continuous operator

\[
T^{+}: L^{2}(C^{+}, E \otimes \Lambda_{C^{+}}^{(0,1)}) \rightarrow L^{1,2}(C^{+}, E)
\]

(3.5)

such that \( Q(T^{+}) = Q \) and for each \( \xi \in L^{2}(C^{+}, E \otimes \Lambda_{C^{+}}^{(0,1)}) \) with \([\xi]_{\overline{\partial}} = 0 \in H^{1}(C^{+}, E)\) we have \( \overline{\partial}(T^{+}(\xi)) = \xi \).

We define the operator \( T: L^{2}(C, E \otimes \Lambda_{C}^{(0,1)}) \rightarrow L^{1,2}(C, E) \) as follows. We extend each \( \xi \in L^{2}(C, E \otimes \Lambda_{C}^{(0,1)}) \) by zero to \( \tilde{\xi} \in L^{2}(C^{+}, E \otimes \Lambda_{C^{+}}^{(0,1)}) \) and set \( T(\xi) = T^{+}(\tilde{\xi})|_{C} \).

Then \( T \) is obviously continuous and, moreover,

\[
\|T^{+}(\tilde{\xi})\|_{L^{1,2}(C^{+})} \leq c \cdot \|\xi\|_{L^{2}(C)},
\]

where the constant \( c \) does not depend on \( \xi \). If \( 2 \leq p < \infty \) and \( v \in L_{\text{loc}}^{1,p}(C^{+}, E) \), then by the \( L^{p} \)-regularity of the elliptic \( \overline{\partial} \)-operator (see, for example, [7]) we obtain the following interior estimate:

\[
\|v\|_{L^{1,p}(C)} \leq c' \cdot (\|v\|_{L^{1,2}(C^{+})} + \|\overline{\partial} v\|_{L^{p}(C^{+})}),
\]

(3.6)
where the constant \( c' \) does not depend on \( v \). It follows that for \( 2 \leq p < \infty \) and \( \xi \in L^p(C, E \otimes \Lambda_C^{0,1}) \) with \( [\xi]_{\partial'} = 0 \in \mathbb{H}^1(C, E) \) we have the estimate

\[
\|T(\xi)\|_{L^1-C} \leq c'' \cdot \|\xi\|_{L^p(C)},
\]

where the constant \( c'' \) does not depend on \( \xi \). This means that the operator \( T \) is a splitting of (3.2). Hence the operator (3.2) has properties (i) and (ii) from Lemma 3.5.

We now return to the Čech coboundary operator (3.1). We fix a partition of unity \( 1 = \sum_{i=1}^l \phi_j \) subordinate to the cover \( \{V_i\}_{i=1}^l \) of the curve \( C \) and fix a cocycle \( w = (w_{ij}) \in \mathcal{H}^{1,p}(V_{ij}, E) \), where for \( i > j \) we set \( w_{ij} := -w_{ji} \). We also set \( f_i := \sum_j \phi_j w_{ij} \). Then \( f_i \in L^{1,p}(V_i, E) \) and \( f_i - f_j = w_{ij} \). Consequently, \( \overline{\partial} f_i \in L^{p}(V_i, E \otimes \Lambda_C^{0,1}) \) and \( \overline{\partial} f_i = \overline{\partial} f_j \) in \( V_{ij} \). Hence \( \overline{\partial} f_i = \xi |_{V_i} \) for some well-defined section \( \xi \in L^{p}(C, E \otimes \Lambda_C^{0,1}) \). Moreover, \( (w_{ij}) \) and \( \xi \) define the same cohomology class \( [w_{ij}] = [\xi] \) in \( \mathbb{H}^1(C, E) \).

Assume in addition that the induced cohomology class \( [w_{ij}] \) is trivial. We set \( f := T(\xi) \) and \( v_i = f_i - f \). Then \( v_i \in L^{1,p}(V_i, E) \), \( v_i - v_j = w_{ij} \), and \( \overline{\partial} v_i = \overline{\partial} f_i - \overline{\partial} f = 0 \). Hence \( v := (v_i) \in \sum_{i=1}^l \mathcal{H}^{1,p}(V_i, E) \) and \( \delta(v) = w \). It follows that the formula \( T_\delta \colon w \mapsto v \) defines an operator \( T_\delta \) that is a splitting of \( \delta \). This explicit construction shows that \( T_\delta \) is continuous, which proves Lemma 3.5.

**Lemma 3.6.** Let \( C \) be a Stein nodal curve with piecewise smooth boundary, and let \( X \) be a complex manifold. Then

(i) \( \mathcal{H}^{1,p}(C, X) \) has a natural complex manifold structure with tangent space \( T_u \mathcal{H}^{1,p}(C, X) = \mathcal{H}^{1,p}(C, u^*TX) \);

(ii) if \( C' \subset C \) is a nodal curve, then the restriction map \( \mathcal{H}^{1,p}(C, U) \to \mathcal{H}^{1,p}(C', U) \) is holomorphic and its differential at a point \( u \in \mathcal{H}^{1,p}(C, U) \) is also the restriction map \( \mathcal{H}^{1,p}(C, u^*TU) \to \mathcal{H}^{1,p}(C', u^*TU), v \mapsto v|_{C'} \).

**Proof.** This consists of several steps.

**Step 1.** Assume first that \( u(C) \) lies in a coordinate chart \( U \subset X \) with complex coordinates \( w = (w_1, \ldots, w_n) \colon U \xrightarrow{\Phi} U' \subset \mathbb{C}^n \). Then the set \( \mathcal{H}^{1,p}(C, U) \) is an open neighborhood of \( u \) in \( \mathcal{H}^{1,p}(C, X) \) and can be naturally identified with the set \( \mathcal{H}^{1,p}(C', U') \), which is an open subset of the Banach space \( \mathcal{H}^{1,p}(C, \mathbb{C}^n) \). This gives us a complex Banach manifold structure on \( \mathcal{H}^{1,p}(C, X) \) with tangent space \( T_u \mathcal{H}^{1,p}(C, X) = \mathcal{H}^{1,p}(C, u^*TU) \) at \( u \in \mathcal{H}^{1,p}(C, U) \).

Note that if \( u_t, t \in [0, 1] \), is a \( C^1 \)-curve in \( \mathcal{H}^{1,p}(C, U) \), then the tangent vector \( v \in \mathcal{H}^{1,p}(C, u^*TU) \) to \( u_t \) at \( u_0 \) is given by the formula \( v(z) = \frac{\partial u}{\partial t}(z) \in T_{u(z)}U \). This last formula does not depend on the choice of complex coordinates \( w = (w_1, \ldots, w_n) \colon U \to \mathbb{C}^n \) in \( U \). This has the following two consequences.

Firstly, the complex structure on \( \mathcal{H}^{1,p}(C, U) \) does not depend on the choice of complex coordinates \( w = (w_1, \ldots, w_n) \colon U \to \mathbb{C}^n \) in \( U \). Secondly, \( C \) has property (ii) from the statement of the lemma.

Thus, Lemma 3.6 is proved in the case when \( u(C) \) lies in a coordinate chart.

**Step 2.** Assume that \( u_0 \in \mathcal{H}^{1,p}(C, X) \) is fixed and there exists a finite cover \( \{V_i\}_{i=1}^l \) of \( C \) such that, firstly, the assumptions of Lemma 3.5 are satisfied and,
such that there exist biholomorphisms $\psi_B \subset H^1,p(V_{ij},u_0^*TX) \cong T_{u_0}H^1,p(V_{ij},X)$, such that there exist biholomorphisms $\psi_{ij} : B_{ij} \cong H^1,p(V_{ij},X)$ with $\psi_{ij}(0) = u_0|V_{ij}$ and $d\psi_{ij}(0) = \text{Id} : H^1,p(V_{ij},u_0^*TX) \to H^1,p(V_{ij},u_0^*TX)$. Then we choose balls $B_i \subset H^1,p(V_i,X)$ such that $u_0|V_i \in B_i$ and $u_i|V_{ij} \in B_{ij}$, for each $u_i \in B_i$

This defines holomorphic maps $\phi_{ij} : B_i \to B_{ij} \subset H^1,p(V_{ij},u_0^*TX)$ such that $\phi_{ij} : u_i \mapsto \psi_{ij}^{-1}(u_i|V_{ij})$, and a holomorphic map

$$\Phi : \prod_{i=1}^l B_i \to \sum_{i<j} H^1,p(V_{ij},u_0^*TX),$$

$$\Phi : (u_i)_{i=1}^l \mapsto \phi_{ij}(u_i) - \phi_{ji}(u_j).$$

It is easy to see that the map $\Phi$ gives us a condition for the compatibility of local holomorphic maps $u_i : V_i \to X$; namely, $(u_i)_{i=1}^l \in \prod_{i=1}^l B_i$ defines a holomorphic map $u : C \to X$ if and only if $\Phi(u_i) = 0$. Furthermore, the differential $d\Phi$ in $(u_0|V_i)$ is equal to the Čech coboundary operator (3.1). Since $C$ is Stein, it follows that $H^1(C,u_0^*TX) = 0$. Using Lemma 3.5 and the implicit function theorem we conclude that parts (i) and (ii) of Lemma 3.6 hold in a neighborhood of the map $u_0 \in H^1,p(C,X)$.

Step 3. Applying step 2 sufficiently many times one can show that for each Stein nodal curve $C$ and each $u \in H^1,p(C,X)$ parts (i) and (ii) of Lemma 3.6 hold in a neighborhood of the map $u$. For example, if $C$ is the annulus $A_{r,R}$, then we can cover it with narrow annuli $A_{r_i,R_i}$, $0 < R_i - r_i \ll 1$, and then cover each $A_{r_i,R_i}$ with sectors $V_{ij} = \{z = re^{i\theta} \in \mathbb{C} : r_i < r < R_i, \alpha_j < \theta < \beta_j\}$, where $0 < \beta_j - \alpha_j \ll 1$. We leave the details to the reader.

One of the difficulties in the construction of holomorphic families of stable curves is that the moduli space of holomorphic structures on a non-compact Riemann surface $\Sigma$ does not have a natural complex structure and, moreover, its real dimension may be odd. For example, if $\Sigma$ is an annulus, then it is biholomorphic to the standard annulus $A_{r,1}$ for some unique $r \in (0,1)$, and therefore the corresponding moduli space is the interval $(0,1)$. In general, if $\Sigma$ is of genus $g$ and has $k$ boundary components, then the real dimension of the moduli space is equal to $d = 6g - 6 + 3k$, except for the four cases when $\Sigma$ is either a sphere ($g = 0$, $k = 0$), or a torus ($g = 1$, $k = 0$), or a disc ($g = 0$, $k = 1$), or an annulus ($g = 0$, $k = 2$) (see, for example, [8]). Note that these are the only cases when the dimension of the group of holomorphic automorphisms of the corresponding complex curve $(\Sigma,J)$ is positive.

The problem may be fixed by an introduction of $k$ additional parameters, namely, by fixing $k$ marked points, one on each boundary component. Let $A$ be an annulus with boundary circles $\gamma_0$ and $\gamma_1$, and let $X$ be a complex manifold.

**Theorem 3.7.** There exist complex Banach manifolds $\mathcal{M}(A,X)$ and $\mathcal{C}(A,X)$, a holomorphic projection $\pi_0 : \mathcal{C}(A,X) \to \mathcal{M}(A,X)$, and holomorphic maps $\text{ev} : \mathcal{C}(A,X) \to X$, $z_1 : \mathcal{C}(A,X) \to \Delta$, $z_2 : \mathcal{C}(A,X) \to \Delta$ and $\lambda_M : \mathcal{M}(A,X) \to \Delta$ with the following properties:

(i) for each $y \in \mathcal{M}(A,X)$ the fiber $C_y := \pi_0^{-1}(y)$ is a nodal curve parametrized by the annulus $A$; moreover, the map $(z_1,z_2) : C_y \to \Delta^2$ is a biholomorphism
onto the curve \( \{(z_1, z_2) \in \Delta^2 : z_1 \cdot z_2 = \lambda_M(y)\} \); in particular, \( C_y \) is either a standard node (if \( \lambda_M(y) = 0 \)), or a holomorphic annulus \( \{|\lambda_M(y)| < |z_1| < 1| \) ;

(ii) the diagram

\[
\begin{array}{ccc}
\mathcal{C}(A, X) & \xrightarrow{\text{ev}_{z_1, z_2}} & X \times \Delta^2 \\
\downarrow \pi_c & & \downarrow \lambda = z_1 \cdot z_2 \\
M(A, X) & \xrightarrow{\lambda_M} & \Delta
\end{array}
\]

(3.7)

is commutative; moreover, for each \( y \in M(A, X) \) the restriction \( \text{ev}|_{C_y} \) belongs to \( \mathcal{H}^{1, p}(A_a, X) \) with \( a = \lambda_M(y) \) and the maps \( \text{ev}_1 : y \in M(A, X) \mapsto \text{ev}|_{C_y}(z_1^{-1}(1)) \) and \( \text{ev}_2 : y \in M(A, X) \mapsto \text{ev}|_{C_y}(z_2^{-1}(1)) \) are holomorphic;

(iii) let \( \mathcal{C} \) be an annulus or a node with smooth boundary \( \partial C = \gamma_1 \cup \gamma_2 \), \( p_i \in \gamma_i \) marked points, and \( u : C \to X \) a holomorphic map of class \( L^{1, p} \)\( ; \) then there exists unique \( y \in M(A, X) \) and a unique biholomorphism \( \phi : C \to C_y \) such that \( \text{ev} \circ \phi = u : C \to X \) and \( z_i \circ \phi(p_i) = 1 \in \overline{\Delta} \); in other words, \( M(A, X) \) parameterizes holomorphic maps into \( X \) of annuli and nodes with marked boundary points;

(iv) if the commutative diagram

\[
\begin{array}{ccc}
\mathcal{Z} & \xrightarrow{\text{ev}_{\mathcal{Z}, \mathcal{Z}_2}} & X \times \Delta^2 \\
\downarrow \pi_\mathcal{Z} & & \downarrow \lambda = \tilde{z}_1 \cdot \tilde{z}_2 \\
\mathcal{W} & \xrightarrow{\lambda_\mathcal{W}} & \Delta
\end{array}
\]

(3.8)

of complex spaces \( \mathcal{W} \) and \( \mathcal{Z} \) and holomorphic maps has properties (i) and (ii), in particular, if the fibers \( \mathcal{Z}_w := \pi^{-1}_\mathcal{Z}(w) \) are nodal curves with induced maps \( f_w := \text{ev}_\mathcal{Z}|_{\mathcal{Z}_w} \in \mathcal{H}^{1, p}(\mathcal{Z}_w, X) \), then the two diagrams (3.7) and (3.8) can be completed in a unique way to the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{Z} & \xrightarrow{\bar{F}} & \mathcal{C}(A, X) & \xrightarrow{\text{ev}_{z_1, z_2}} & X \times \Delta^2 \\
\downarrow \pi_\mathcal{Z} & & \downarrow \pi_c & & \downarrow \lambda = z_1 \cdot z_2 \\
\mathcal{W} & \xrightarrow{F} & M(A, X) & \xrightarrow{\lambda_M} & \Delta
\end{array}
\]

(3.9)

where \( \lambda_M \circ F = \lambda_\mathcal{W} \) and \( (\text{ev}, z_1, z_2) \circ \bar{F} = (\text{ev}_{\mathcal{Z}}, \tilde{z}_1, \tilde{z}_2) \);

(v) the differential \( d\lambda_M : T_y M(A, X) \to T_{\lambda_M(y)} \Delta \cong \mathbb{C} \) is non-degenerate at each point \( y \in M(A, X) \), and for each \( a \in \Delta \) the fiber \( \lambda_M^{-1}(a) \) is naturally isomorphic to the manifold \( \mathcal{H}^{1, p}(A_a, X) \), where \( A_a \) is the curve \( \{(z_1, z_2) \in \Delta^2 : z_1 \cdot z_2 = a\} \); in particular, for each \( y \in M(A, X) \) one obtains a biholomorphism \( C_y \cong A_{\lambda_M(y)} \) and the following natural exact sequence:

\[
0 \longrightarrow \mathcal{H}^{1, p}(C_y, u^*TX) \xrightarrow{t_y} T_y M(A, X) \xrightarrow{d\lambda_M(y)} \mathbb{C} \longrightarrow 0.
\]
Proof. Let \((A,p_1,p_2)\) be a smooth annulus with a marked point on each boundary component \(\gamma_i \cong S^1\), and let \(J\) be a complex structure on \(A\). We know that \((A,J)\) is biholomorphic to one of the annuli \(A_{r,1} = \{r < |z| < 1\}\). It is easy to see that there exists only one isomorphism \(\psi: (A,J) \to A_{r,1}\) that extends smoothly to a diffeomorphism \(\psi: \overline{A} \to \overline{A}_{r,1}\) with \(\phi(p_1) = 1\). Set \(a := \phi(p_2)\). It is now evident that there exists a unique biholomorphism \(\phi: (A,J) \to A_a := \{(z_1,z_2) \in \Delta^2 : z_1 \cdot z_2 = a\}\) such that \(\phi(p_1) = 1\) and \(\phi(p_2) = a\).

Thus, the map \(\lambda: \Delta^2 \to \Delta, \lambda(z_1,z_2) = z_1 \cdot z_2\), with fiber \(A_a\) over \(a \in \Delta\) forms the holomorphic moduli space of annuli with marked points on boundary components completed by the standard node at \(a = 0\). If \(a \neq 0\), then the coordinate functions \(z_i, i = 1,2\), define an embedding of each \(A_a\) in \(\mathbb{C}\) such that \(\gamma_i\) becomes the outer unit circle. As \(a \to 0\), the annuli \(A_a\) degenerate into the standard node, and each \(z_i\) becomes the standard coordinate function on the corresponding component of the node.

Remark. In what follows, we denote by \(A_a\) an annulus (or a node) with marked points on its boundary \(\partial A_a\) and with coordinate functions \(z_1\) and \(z_2\) defined as above.

Fix \(r, 0 < r < 1\). For \(|a| < r\) we define the maps \(\zeta^a_1, \zeta^a_2: A_{r,1} \to A_a\) by the formulae \(\zeta^a_1(z) := z\) and \(\zeta^a_2(z) := a/z\), so that the \(\zeta^a_i\) are the reciprocals of the coordinate functions \(z_i\). We consider the following map:

\[
\Psi_r: \prod_{|a| < r} \mathcal{H}^{1,p}(A_a,X) \to \mathcal{H}^{1,p}(A_{r,1},X) \times \mathcal{H}^{1,p}(A_{r,1},X) \times \Delta(r),
\]

\[
\Psi_r: u \in \mathcal{H}^{1,p}(A_a,X) \mapsto (u \circ \zeta^a_1, u \circ \zeta^a_2, a).
\]

It is easy to see that \(\Psi_r\) is holomorphic on each space \(\mathcal{H}^{1,p}(A_a,X)\) and the image of \(\prod_{|a| < r} \mathcal{H}^{1,p}(A_a,X)\) consists of triples \((u_1,u_2,a)\) such that each map \(u_i \in \mathcal{H}^{1,p}(A_{r,1},X)\) extends to a map \(u_i \in \mathcal{H}^{1,p}(A_{|a|,1},X)\) and \(u_2(z) = u_1(a/z)\). Hence \(\Psi_r\) is injective in \(\prod_{|a| < r} \mathcal{H}^{1,p}(A_a,X)\) and this image is closed. We consider the topology induced by the maps \(\Psi_r\) on the disjoint union \(\mathcal{M}(A,X) := \coprod_{a \in \Delta} \mathcal{H}^{1,p}(A_a,X)\). Clearly, it is compatible with the topology on each fiber \(\mathcal{H}^{1,p}(A_a,X)\).

Our aim is to construct an appropriate holomorphic structure on \(\mathcal{M}(A,X)\) compatible with the holomorphic structures on the fibers \(\mathcal{H}^{1,p}(A_a,X)\) and with the topology on \(\mathcal{M}(A,X)\) introduced above.

We start with the special case of \(X = \mathbb{C}^n\). It is easy to see that for \(a \neq 0\) each function \(f \in \mathcal{H}^{1,p}(A_a,\mathbb{C}^n)\) can be uniquely expanded in the Laurent series \(f(z_1) = \sum_{i=0}^{\infty} c_i z_1^i\). We set \(f^+(z_1) := \sum_{i=0}^{\infty} c_i z_1^i\) and \(f^-(z_1) := \sum_{i=-\infty}^{0} c_i z_1^i\). It is also convenient to regard \(f^+\) as a function of the variable \(z_2 = a/z_1\) with \(f^-(z_2) = \sum_{i=0}^{\infty} c_{-i}(z_2/a)^i\). We have \(f^+ \in \mathcal{H}^{1,p}(\{|z_1| < 1\},\mathbb{C}^n)\), \(f^- \in \mathcal{H}^{1,p}(\{|z_2| < 1\},\mathbb{C}^n)\), \(f^+(0) = f^-\), \(f^+(z_1) = f^+(z_2) = 0 = c_0\), and \(f(z_1) = f^+(z_1) + f^-(a/z_1) - c_0\), so that the pair \((f^+,f^-)\) defines a holomorphic function \(\hat{f} \in \mathcal{H}^{1,p}(A_0,\mathbb{C}^n)\). The resulting canonical isomorphisms \(\mathcal{H}^{1,p}(A_a,\mathbb{C}^n) \cong \mathcal{H}^{1,p}(A_0,\mathbb{C}^n)\) define a structure of a trivial Banach bundle over \(\{|a| < 1\}\), and therefore a structure of a Banach manifold, on \(\prod_{|a| < 1} \mathcal{H}^{1,p}(A_a,\mathbb{C}^n)\).

Now, the map \(\Psi_r: \prod_{|a| < r} \mathcal{H}^{1,p}(A_a,\mathbb{C}^n) \to \mathcal{H}^{1,p}(A_{r,1},\mathbb{C}^n) \times \Delta(r)\) is holomorphic. If \(U\) is an open subset of \(\mathbb{C}^n\), then \(\mathcal{M}(A,U)\) is also open in \(\mathcal{M}(A,\mathbb{C}^n)\) and therefore inherits a holomorphic structure. The natural projection \(\lambda_M: \mathcal{M}(A,U) \to \Delta\) is now holomorphic. It is easy to see that the differential \(d\lambda_M\) is non-degenerate. Hence we can define the universal family of curves \(\mathcal{C}(A,U)\) as the fibered product
\(M(A,U) \times _\Delta \Delta ^2\) with respect to the maps \(\lambda _M: M(A,U) \to \Delta\) and \(\lambda : \Delta ^2 \to \Delta\). This is a holomorphic Banach manifold because \(d\lambda _M\) is non-degenerate.

Let \(\pi _0: \mathcal{C}(A,U) \to M(A,U)\) be the natural projection. Then the fiber \(C_y\) over \(y \in M(A,U)\) is biholomorphic to \(A_0\) with \(a = \lambda _M(y)\). The natural projection of \(\mathcal{C}(A,U)\) onto \(\Delta ^2\) induces \(\Delta\)-valued holomorphic functions \(z_1\) and \(z_2\) on \(\mathcal{C}(A,U)\) that have property (i) from Theorem 3.7.

Assume now that \(a \neq 0\) in \(\Delta\) and let \(f \in \mathcal{H}^{1,p}(A_0,\mathbb{C}^n)\). We can represent \(f\) in the form \(f(z_1) = f^+(z_1) + f^-(a/z_1) - f_0\), where \(f^\pm \in \mathcal{H}^{1,p}(\Delta,\mathbb{C}^n)\) and \(f^+(0) = f^-(0) = f_0\). In a similar way, for each \(f \in \mathcal{H}^{1,p}(A_0,\mathbb{C}^n)\) we have \(f = (f^+,f^-)\), where we also have \(f^\pm \in \mathcal{H}^{1,p}(\Delta,\mathbb{C}^n)\) with \(f^+(0) = f^-(0) = f_0\). Consider the holomorphic function \(\tilde{f}(z_1,z_2) := f^+(z_1) + f^-(z_2) - f_0\), \(\tilde{f} \in \mathcal{H}^{1,p}(\Delta ^2,\mathbb{C}^n)\), and define the maps \(\mathcal{E}V_a: \mathcal{H}^{1,p}(A_0,\mathbb{C}^n) \times \Delta ^2 \to \mathbb{C}^n\) by the formula \(\mathcal{E}V_a(f,z_1,z_2) := \tilde{f}(z_1,z_2)\). It is easy to see that the \(\mathcal{E}V_a\) define a holomorphic map \(\mathcal{E}: M(A,\mathbb{C}^n) \times \Delta ^2 \to \mathbb{C}^n\). Let \(\mathcal{E}\) be the restriction of this map to \(\mathcal{C}(\mathbb{C}^n) \subset M(A,\mathbb{C}^n) \times \Delta ^2\). We leave it to the reader to verify that the assertions of Theorem 3.7 hold for \(M(A,\mathbb{C}^n), \mathcal{C}(\mathbb{C}^n)\), \(\mathcal{E}: \mathcal{C}(\mathbb{C}^n) \to \mathbb{C}^n\), and \(z_{1,2}: \mathcal{C}(\mathbb{C}^n) \to \Delta\).

It follows now that for \(U \subset \mathbb{C}^n\) we can define the map \(\mathcal{E}: \mathcal{C}(A,U) \to U\) as a mere restriction \(\mathcal{E}: \mathcal{C}(A,\mathbb{C}^n) \to \mathbb{C}^n\). The assertions of Theorem 3.7 hold again. In particular, if \(G: U \to U' \subset \mathbb{C}^n\) is biholomorphic, then the natural bijections \(M(A,U) \xrightarrow{\sim} M(A,U')\) and \(\mathcal{C}(A,U) \xrightarrow{\sim} \mathcal{C}(A,U')\) are biholomorphisms. This means that the holomorphic structure on \(M(A,U)\) does not depend on the embedding \(U \subset \mathbb{C}^n\).

Let \(C = A_{r,1}\) be an annulus and let \(u: C \to X\) be a holomorphic embedding. Then \(du: TC \to u^*TX\) is an embedding of holomorphic bundles over \(C\), which allows us to define the holomorphic normal bundle as the quotient bundle \(N_C := u^*TX/TC\). Since \(C\) is Stein, \(N_C\) is holomorphically trivial. We fix a holomorphic frame \(\sigma_1,\ldots,\sigma_{n-1} \in \mathcal{H}^{1,p}(C,N_C)\), \(n = \dim _CX\), and its lifting \(\tilde{\sigma}_1,\ldots,\tilde{\sigma}_{n-1} \in \mathcal{H}^{1,p}(C,u^*TX)\). Let \(B^{n-1}(r)\) be the ball of radius \(r\) in the space \(\mathbb{C}^{n-1}\) with coordinate functions \(w = (w_1,\ldots,w_{n-1})\). By Lemma 3.6 there exists a holomorphic map \(\Psi : C \times B^{n-1}(r) \to X\) such that \(\frac{\partial \Psi}{\partial w_i}\bigg|_{z \in C,w=0} = \sigma_i(z)\). Hence \(\Psi\) is biholomorphic in a neighborhood of \(C \equiv C \times \{0\}\). In particular, if \(r\) is sufficiently small, then the image \(U := \Psi(C \times B^{n-1}(r))\) is a local chart with coordinates \((z,w_1,\ldots,w_{n-1})\). Note that the image \(u(C')\) of each smaller annulus \(C' \subset C\) also lies in \(U\).

To complete the proof of the theorem it remains only to consider the general case when \(C \cong A_0\) is arbitrary and \(u: C \to X\) is a holomorphic map. If \(a = 0\), then \(C\) is a node and therefore there exists a neighborhood \(V_0\) of the nodal point such that \(u(V_0)\) lies in some coordinate chart in \(X\). If \(a \neq 0\) and \(u(C)\) does not lie in a coordinate chart in \(X\), then \(u\) is not constant. Hence the map \(u\) is an embedding in a neighborhood of the circle \(S^1_r := \{|z| = r\} \subset C \cong \{|a| < |z| < 1\}\) for some \(|a| < r < 1\).

In any case we obtain a cover \(\{V_0,V_1,V_2\}\) of the curve \(C = \{(z_1,z_2) \in \Delta ^2 : z_1z_2 = a\}\) that has the following form:

\[
V_1 = \{(z_1,z_2) \in C : r_1 < |z_1| < 1\},
V_2 = \{(z_1,z_2) \in C : r_2 < |z_2| < 1\},
V_0 = \{(z_1,z_2) \in C : |z_1| < R_1,|z_2| < R_2\}.
\]
Here $0 < r_1 < R_1 < 1$, $0 < r_2 < R_2 < 1$, $r_1 \cdot R_2 > |a| < r_2 \cdot R_1$ and $V_0$ has the following property: $u(\overrightarrow{V_0})$ lies in a coordinate chart $U$ in $X$.

The curve $A_a$ is drawn by solid line as a piece of a hyperbola, the elements of the covering $V_0$, $V_1$, and $V_2$ by punctured line.

We fix the coordinate function $z_1$ in $V_1$ and the coordinate function $z_2$ in $V_2$. Next we introduce the new coordinates $\tilde{z}_1 := z_1/R_1$ and $\tilde{z}_2 := z_2/R_2$ on $V_0$ and fix the marked points $\tilde{p}_1 := R_1$ and $\tilde{p}_2 := a/R_2$. We set $\tilde{a} := a/(R_1R_2)$. Then $V_0 \cong A_\tilde{a} = \{(\tilde{z}_1, \tilde{z}_2) \in \mathbb{D}^2 : \tilde{z}_1 \cdot \tilde{z}_2 = \tilde{a}\}$ and $\tilde{z}_i(\tilde{p}_1) = 1$. It is easy to see that changing the complex parameter $a = z_1z_2$ on $C$, which parameterizes the holomorphic structures on an annulus with marked points on the boundary, can be reduced to changing a similar parameter in $V_0$. Namely, let $C' \cong A_{\tilde{a}'} = \{(z_1, z_2) \in \mathbb{D}^2 : z_1 \cdot z_2 = a'\}$ be a result of a (small) deformation of $a$. We set $\tilde{a}' := a'/(R_1R_2)$ and regard $C'$ as a result of gluing together the complex curves $V_i'$, $V_1$ and $V_2$ defined in $C'$ by the same equations $V_i' = \{(z_1, z_2) \in C' : |z_1| < r_1, |z_2| < r_2\}$ and $V_i = \{r_i < |z_i| < 1\}$ for $i = 1, 2$, with coordinate functions $(z_1, z_2)$ satisfy now the new relation $z_1 \cdot z_2 = a'$, so that $V_0' \cong A_{\tilde{a}'}$.

Thus, in an appropriate small neighborhood $W \subset M(A, X)$ of the curve $(C, u)$ over $X$ we have the following holomorphic map:

$$\begin{align*}
\Theta: \quad W & \quad \rightarrow \quad M(A, U) \times \mathcal{H}^{1-p}(V_1, X) \times \mathcal{H}^{1-p}(V_2, X); \\
\Theta: \quad (C', u') & \quad \rightarrow \quad ((A_{\tilde{a}'}, u|_{V_0'}), u|_{V_1}, u|_{V_2}).
\end{align*}$$

On the other hand the collection of maps $u_0: V_0' \rightarrow X$, $u_1: V_1 \rightarrow X$ and $u_2: V_2 \rightarrow X$ defines a map $u': C' \rightarrow X$ if and only if $u_1$ coincides with $u_0$ on $W_1 := V_1 \cap V_0'$ and $u_2$ coincides with $u_0$ on $W_2 := V_2 \cap V_0'$. We point out that the domains $W_i \subset V_i$ do not change in the deformation of the complex structure of $C$. The construction of gluing from step 2 in the proof of Lemma 3.6 completes the proof of the theorem.

Now we can complete the proof of Theorem 3.4.

**Proof.** Proof of Theorem 3.4 Let $(C_0, u_0)$ be a stable curve over a complex manifold $X$ with parametrization $\sigma_0: \Sigma \rightarrow C_0$. We use Proposition 3.2 to fix a cover $\{V_0, V_{\alpha\beta}\}$ of $\Sigma$ with properties (i)–(vi). In particular, there exist biholomorphisms $\phi^0_{\alpha\beta}: \sigma_0(V_{\alpha\beta}) \rightarrow A_{\lambda^0_{\alpha\beta}}$. It follows from properties (i)–(vi) that for each collection $\lambda := (\lambda_{\alpha\beta})$ that is sufficiently close to $\lambda^0 := (\lambda^0_{\alpha\beta})$, there exists a nodal curve $C$ with parametrization $\sigma: \Sigma \rightarrow C$ such that the properties (i)–(vi) still hold and there exist biholomorphisms $\phi_{\alpha\beta}: \sigma(V_{\alpha\beta}) \rightarrow A_{\lambda_{\alpha\beta}}$. In particular, complex structures on each $\sigma(W_{\alpha\beta})$ do not change. Moreover, we can choose holomorphic coordinate functions $z_1$ and $z_2$ in $V_{\alpha\beta}$ such that $z_1 \cdot z_2 \equiv \lambda_{\alpha\beta}$, and both $z_1|_{W_{\alpha\beta}}$ and $z_2|_{W_{\alpha\beta}}$ do not change under a variation of $\lambda_{\alpha\beta}$. This means that the disc
\[ \Delta_{\alpha \beta} := \{ \lambda_{\alpha \beta} : |\lambda^0_{\alpha \beta} - \lambda_{\alpha \beta}| \leq \varepsilon \} \] parameterizes a holomorphic family of curves of the form \( \sigma(V_{\alpha \beta}) \).

The deformation of complex structure on \( V_{\alpha \beta} \) is shown in Fig. 6.

![Diagram](image)

Fig. 6.

On this picture one sees \( V_{\alpha \beta} \) together with adjoint \( V_{\alpha} \) and \( V_{\beta} \). One can suppose that, under varying of \( \lambda_{\alpha \beta} \), the complex structure varies only in shadowed domain.

Let \( N \in \mathbb{N} \) be the number of elements of the covering of the type \( V_{\alpha \beta} = V_{\beta \alpha} \). Then, for sufficiently small \( \varepsilon > 0 \) the polydisc

\[ \Delta^N_{\lambda} := \{ \lambda := (\lambda_{\alpha \beta}) : |\lambda^0_{\alpha \beta} - \lambda_{\alpha \beta}| \leq \varepsilon \} \]

parameterizes a holomorphic family of nodal curves \( \{C_{\lambda}\}_{\lambda \in \Delta^N_{\lambda}} \). Each curve \( C_{\lambda} = \sigma(\Sigma) \) is obtained by patching together the pieces \( \sigma(V_{\alpha}) \) and \( \sigma(V_{\alpha \beta}) \). We set \( M := \bigcup_{\lambda \in \Delta^N_{\lambda}} \mathcal{H}^{1,p}(C_{\lambda}, X) \). The pieces \( \sigma(V_{\alpha}) \) do not contain nodal points, and the complex structures on \( \sigma(V_{\alpha \beta}) \) are constant and independent of \( \lambda = (\lambda_{\alpha \beta}) \). Furthermore, we have the natural isomorphisms \( \sigma(V_{\alpha \beta}) \cong \mathcal{A}_{\lambda_{\alpha \beta}} \). Hence the following map is well defined:

\[
\begin{align*}
\Theta &: M \rightarrow \prod_{\alpha} \mathcal{H}^{1,p}(V_{\alpha}, X) \times \prod_{\alpha \beta} M(V_{\alpha \beta}, X); \\
\Theta &: (C_{\lambda}, u) \rightarrow (u|_{V_{\alpha}}, (\lambda_{\alpha \beta}, u|_{V_{\alpha \beta}})).
\end{align*}
\]

It is easy to see that a collection \( (u_{\alpha}, (\lambda_{\alpha \beta}, u_{\alpha \beta})) \in \prod_{\alpha} \mathcal{H}^{1,p}(V_{\alpha}, X) \times \prod_{\alpha \beta} M(V_{\alpha \beta}, X) \) belongs to \( \Theta(M) \) if and only if the gluing conditions \( u_{\alpha}|_{W_{\alpha \beta}} = u_{\alpha \beta}|_{W_{\alpha \beta}} \) are satisfied for all pairs \( (\alpha, \beta) \).

We shall repeat the gluing procedure of step 2 in Lemma 3.6. To this end we choose the balls

\[
\begin{align*}
B_{\alpha} \subset \mathcal{H}^{1,p}(V_{\alpha}, u_0^*TX) &\cong T_{u_0} \mathcal{H}^{1,p}(V_{\alpha}, X), \\
B_{\alpha \beta} \subset \mathcal{H}^{1,p}(V_{\alpha \beta}, u_0^*TX) &\oplus \mathbb{C} \cong T_{u_0} M(V_{\alpha \beta}, X), \\
B'_{\alpha \beta} \subset \mathcal{H}^{1,p}(W_{\alpha \beta}, u_0^*TX) &\cong T_{u_0} \mathcal{H}^{1,p}(W_{\alpha \beta}, X)
\end{align*}
\]

such that there exist biholomorphisms

\[
\begin{align*}
\psi_{\alpha} &: B_{\alpha} \xrightarrow{\sim} \psi_{\alpha}(B_{\alpha}) \subset \mathcal{H}^{1,p}(V_{\alpha}, X), \\
\psi_{\alpha \beta} &: B_{\alpha \beta} \xrightarrow{\sim} \psi_{\alpha \beta}(B_{\alpha \beta}) \subset M(V_{\alpha \beta}, X), \\
\psi'_{\alpha \beta} &: B'_{\alpha \beta} \xrightarrow{\sim} \psi'_{\alpha \beta}(B'_{\alpha \beta}) \subset \mathcal{H}^{1,p}(W_{\alpha \beta}, X)
\end{align*}
\]

with the following properties:

\[
\begin{align*}
\psi_{\alpha}(0) &= u_0|_{V_{\alpha}}, & d\psi_{\alpha}(0) &= \text{id} : T_{u_0} \mathcal{H}^{1,p}(V_{\alpha}, X) \to T_{u_0} \mathcal{H}^{1,p}(V_{\alpha}, X), \\
\psi_{\alpha \beta}(0) &= u_0|_{V_{\alpha \beta}}, & d\psi_{\alpha \beta}(0) &= \text{id} : T_{(\lambda^0_{\alpha \beta}, u_0)} M(V_{\alpha \beta}, X) \to T_{(\lambda^0_{\alpha \beta}, u_0)} M(V_{\alpha \beta}, X), \\
\psi'_{\alpha \beta}(0) &= u_0|_{W_{\alpha \beta}}, & d\psi'_{\alpha \beta}(0) &= \text{id} : T_{u_0} \mathcal{H}^{1,p}(W_{\alpha \beta}, X) \to T_{u_0} \mathcal{H}^{1,p}(W_{\alpha \beta}, X).
\end{align*}
\]
Shrinking the balls $B_\alpha$ and $B_{\alpha\beta}$ if necessary, we may assume that for all $\xi_\alpha \in B_\alpha$ and all $\xi_{\alpha\beta} \in B_{\alpha\beta}$ the restrictions of their images $\psi_\alpha(\xi_\alpha)|_{W_{\alpha,\beta}}$ and $\psi_{\alpha\beta}(\xi_{\alpha\beta})|_{W_{\alpha,\beta}}$ belong to the image $\psi'_{\alpha,\beta}(B'_{\alpha,\beta})$. We consider the following holomorphic map:

$$\Psi: \prod_\alpha B_\alpha \times \prod_{\alpha < \beta} B_{\alpha\beta} \rightarrow \sum_{\alpha,\beta} \mathcal{H}^{1,p}(W_{\alpha,\beta}, u_0^*TX);$$

$$\Psi: (v_\alpha, v_{\alpha\beta}) \mapsto \psi'_{\alpha,\beta}^{-1}(\psi_\alpha(v_\alpha)|_{W_{\alpha,\beta}} - \psi_{\alpha\beta}'^{-1}(\psi_{\alpha\beta}(v_{\alpha\beta})|_{W_{\alpha,\beta}}).$$

As in similar situations above, the map $\Psi$ gives us gluing conditions for local holomorphic maps $\psi_\alpha(v_\alpha): V_\alpha \to X$ and $\psi_{\alpha\beta}(v_{\alpha\beta}): V_{\alpha\beta} \to X$. Hence we can identify $\mathcal{M} \cap \prod \alpha B_\alpha \times \prod_{\alpha < \beta} B_{\alpha\beta}$ with the set $\Psi^{-1}(0)$.

We shall now study in greater detail the behavior of $\Psi$ at the point $y_0 \in \prod \alpha B_\alpha \times \prod_{\alpha < \beta} B_{\alpha\beta}$ that we are interested in: namely, $y_0 = (\psi_\alpha^{-1}(u_0|_{V_\alpha}), \psi_{\alpha\beta}^{-1}(u_0|_{V_{\alpha\beta}}))$, so that $\Psi(y_0) = 0 \in \sum_{\alpha,\beta} \mathcal{H}^{1,p}(W_{\alpha,\beta}, u_0^*TX)$. It is easy to see that the tangent space at $y_0$ is

$$T_{y_0}(\prod \alpha B_\alpha \times \prod_{\alpha < \beta} B_{\alpha\beta}) = \sum_{\alpha} \mathcal{H}^{1,p}(V_\alpha, u_0^*TX) \oplus \sum_{\alpha \beta} \mathcal{H}^{1,p}(V_{\alpha\beta}, u_0^*TX) \oplus \mathbb{C}^N$$

and the differential $d\Psi(y_0)$ coincides on the term $\sum \mathcal{H}^{1,p}(V_\alpha, u_0^*TX) \oplus \sum \mathcal{H}^{1,p}(V_{\alpha\beta}, u_0^*TX)$ with the Čech codifferential $(3.1)$ with respect to the cover $\{V_\alpha, V_{\alpha\beta}\}$ of the curve $C_0$. By Lemma 3.5 we can decompose $\sum \mathcal{H}^{1,p}(W_{\alpha\beta}, u_0^*TX)$ into the direct sum $\mathcal{W} \oplus \mathcal{Q}$, where $\mathcal{W} = \mathcal{Z}(d\Psi(y_0))$ and $\mathcal{Q}$ is isomorphic to $\mathcal{H}^1(C_0, u_0^*TX)$ and finite-dimensional. Let $\Psi_\mathcal{W}$ and $\Psi_\mathcal{Q}$ be the components of $\Psi = (\Psi_\mathcal{W}, \Psi_\mathcal{Q})$ with respect to this decomposition, and let $\bar{\mathcal{M}} := \Psi^{-1}_\mathcal{W}(0)$. It follows from Lemma 3.5 and the implicit function theorem that $\bar{\mathcal{M}}$ is a complex submanifold of $\prod \alpha B_\alpha \times \prod_{\alpha < \beta} B_{\alpha\beta}$ with tangent space $\mathcal{H}^{1,p}(C_0, u_0^*TX) \oplus \mathbb{C}^N$ at the point $y_0 \in \bar{\mathcal{M}}$, while $\mathcal{M}$ is defined in $\bar{\mathcal{M}}$ as the zero set of the holomorphic map $\Phi := \Psi|_{\bar{\mathcal{M}}}: \bar{\mathcal{M}} \to \mathcal{Q} \cong \mathcal{H}^1(C_0, u_0^*TX)$. This defines on $\mathcal{M}$ a structure of a Banach analytic set of finite codimension.

To complete the proof it remains to construct the corresponding family of nodal curves $\pi: \mathcal{C} \to \mathcal{M}$ and a holomorphic map $F: \mathcal{C} \to X$. Note that with each ball $B_\alpha$ we can associate in a natural way the trivial family $\pi_\alpha: \mathcal{C}_\alpha := B_\alpha \times V_\alpha \to B_\alpha$ and with each $B_{\alpha\beta}$ we can associate the holomorphic family $\pi_{\alpha\beta}: \mathcal{C}_{\alpha\beta} \to B_{\alpha\beta}$ with fiber $\pi_{\alpha\beta}^{-1}(v_{\alpha\beta}) = A_{\alpha\beta}$, where $A_{\alpha\beta}$ is uniquely determined by the relation $\psi_{\alpha\beta}(v_{\alpha\beta}) = (\lambda_{\alpha\beta}, u_{\alpha\beta}) \in \mathcal{M}(V_{\alpha\beta}, X)$.

We extend these families to the families $\bar{\pi}_\alpha: \bar{\mathcal{C}}_\alpha \to \prod B_\alpha \times \prod B_{\alpha\beta}$ and $\bar{\pi}_{\alpha\beta}: \bar{\mathcal{C}}_{\alpha\beta} \to \prod B_\alpha \times \prod B_{\alpha\beta}$. Clearly, $\mathcal{C}_{\alpha}$ and $\mathcal{C}_{\alpha\beta}$ can be glued together canonically, producing a global family of nodal curves $\bar{\pi}: \bar{\mathcal{C}} \to \prod B_\alpha \times \prod B_{\alpha\beta}$. Note that $\bar{\mathcal{C}}$ is a Banach manifold. Furthermore, we obtain well-defined holomorphic maps $F_\alpha: \bar{\mathcal{C}}_\alpha \to X$ and $F_{\alpha\beta}: \bar{\mathcal{C}}_{\alpha\beta} \to X$ such that $F(v_\alpha, v_{\alpha\beta}, z) := \psi_\alpha(v_\alpha)[z]$ for each $z \in V_\alpha$ and a similar relation holds for $F_{\alpha\beta}$.

We define $\bar{\mathcal{C}}$ to be the restriction $\bar{\mathcal{C}} := \bar{\mathcal{C}}|_{\mathcal{M}}$. We observe that the restriction of the trivial holomorphic family $\bar{\mathcal{C}}_\alpha = V_\alpha \times \prod B_\alpha \times \prod B_{\alpha\beta}$ to $\mathcal{M}$ is also a trivial holomorphic family. It follows that $\mathcal{C}$ is a BASFC in a neighborhood of each point $y \in \mathcal{C} \cap \bar{\mathcal{C}}_\alpha$. In a similar way, each holomorphic family of curves $\mathcal{C}_{\alpha\beta}$ can be distinguished in the trivial bundle $\Delta^2 \times \prod B_\alpha \times \prod B_{\alpha\beta} \cong \prod B_\alpha \times \prod B_{\alpha\beta}$ by the condition $z_1 \cdot z_2 - \lambda_{\alpha\beta} = 0$, where $\lambda_{\alpha\beta}: B_{\alpha\beta} \to \Delta$ is the holomorphic parameter of the deformation of the complex structure on $V_{\alpha\beta}$ and $(z_1, z_2)$ are the standard
coordinate functions on $\Delta^2$. Hence $\mathcal{C}$ is also a BASFC in a neighborhood of each point $y \in \mathcal{C} \cap \mathcal{C}_{\alpha\beta}$. Since $\mathcal{M}$ has been in fact defined by the condition that the local maps $F_\alpha$ and $F_{\alpha\beta}$ coincide, there exists a global holomorphic map from $\mathcal{C}$ to $F : \mathcal{C} \to X$.

Properties (a), (b) and (d) from Theorem 3.4 follow now for $\mathcal{M}$, $\mathcal{C}$, and $F$ directly from the construction, and (c) follows by Proposition 3.2.

The proof of the required variant of the continuity principle for meromorphic maps (see Theorem 4.2 in the next section or Theorem 5.1.3 in [2]) uses the following consequence of the main results in this section.

Assume that a sequence $(C_n, u_n)$ of irreducible stable curves over $X$ converges to a stable curve $(C_\infty, u_\infty)$.

**Lemma 3.8.** For some positive integer $N$ there exist a complex (maybe, singular) surface $Z$ and holomorphic maps $\pi_Z : Z \to \Delta$ and $F : Z \to X$ defining a holomorphic family of stable nodal curves over $X$ joining $(C_N, u_N)$ and $(C_\infty, u_\infty)$. More precisely, the following results hold:

1. for each $\lambda \in \Delta$ the fiber $C_\lambda = \pi_Z^{-1}(\lambda)$ is a connected nodal curve with boundary $\partial C_\lambda$ and the pair $(C_\lambda, u_\lambda)$ with $u_\lambda := F|_{C_\lambda}$ is a stable curve over $X$;
2. all $C_\lambda$, except for finitely many curves, are connected and smooth;
3. $(C_0, u_0)$ is equal to $(C_\infty, u_\infty)$ and there exists $\lambda_N \in \Delta$ such that $(C_{\lambda_N}, u_{\lambda_N}) = (C_N, u_N)$;
4. there exist open sets $V_1, \ldots, V_m$ in $Z$ such that each $V_j$ is biholomorphic to $\Delta \times A_j$ for some annulus $A_j$; moreover, the diagram

$$
\begin{array}{ccc}
V_j & \xrightarrow{\cong} & \Delta \times A_j \\
\downarrow & & \downarrow \\
\Delta & = & \Delta
\end{array}
$$

is commutative, each annulus $C_\lambda \cap V_j \cong \{\lambda\} \times A_j$ is adjacent to a unique boundary component of $\partial C_\lambda$, and the number $m$ of the domains $V_j$ is equal to the number of boundary components of each $C_\lambda$.

**Remark.** This lemma was stated without proof in Proposition 5.1.1 of [2].

**Proof.** Assume that $\pi_C : \mathcal{C} \to \mathcal{M}$ and $\text{ev} : \mathcal{C} \to X$ define the complete family of holomorphic deformations of the stable nodal curve $(C_\infty, u_\infty)$ over $X$ constructed in Theorem 3.4. Assume that $\lambda^* \in \mathcal{M}$ parameterizes the curve $(C_\infty, u_\infty)$. We consider a sequence $\lambda_n \to \lambda^*$ in $\mathcal{M}$ such that $(C_n, u_n) \cong (\pi_C^{-1}(\lambda_n), \text{ev}|_{\pi_C^{-1}(\lambda_n)})$ for all sufficiently large $n$. Since $(C_\infty, u_\infty) = (\pi_C^{-1}(\lambda^*), \text{ev}|_{\pi_C^{-1}(\lambda^*)})$ can be lifted in a neighborhood of the boundary of $\tilde{U}$, shrinking $\mathcal{M}$ we can assume that this is true for all $\lambda \in \mathcal{M}$.

By construction, the space $\mathcal{M}$ is a BASFC and therefore Theorem 3.3 applies. In particular, $\mathcal{M}$ has finitely many irreducible components at $\lambda^*$. Let $\mathcal{M}_1$ be a component of $\mathcal{M}$ that contains infinitely many $\lambda_n$. We can represent $\mathcal{M}_1$ in a neighborhood of $\lambda^*$ as a proper ramified cover $\pi_1 : \mathcal{M}_1 \to B_1$ over a Banach ball $B_1$. If $\Delta$ is an arbitrary embedded disc in $B_1$, then $\pi_1^{-1}(\Delta)$ is a one-dimensional analytic subset with irreducible components also parametrized by discs. Hence there exists
a holomorphic map \( \phi: \Delta \to M_1 \) passing through \( \lambda^* \) and \( \lambda_N \) for some \( N \gg 1 \). The inverse image of the family \( \pi_0: C \to M \) under \( \phi \) is a holomorphic family of stable nodal curves over \( X \), which has the total space \( \pi_Z: Z \to \Delta \) equipped with a map \( F: Z \to X \) and contains \((C_\infty, u_\infty)\) and \((C_N, u_N)\).

Since \( C_N \) is smooth, the general fiber \( C_\lambda = \pi_Z^{-1}(\lambda) \) is also smooth. Shrinking the disc if necessary, we may assume that the \( C_\lambda \) are singular only for finitely many \( \lambda \in \Delta \). The other properties from (1)–(4) follow by the construction of the family \( \pi_Z: Z \to \Delta \) and the map \( F: Z \to X \).

\[ \frac{\Delta}{\lambda} \] 4. CONTINUITY PRINCIPLE AND THE PROOF OF THE MAIN THEOREM

We shall now formulate the continuity principle from [2] required for the proof of the main theorem.

**Definition** A Hermitian complex manifold \((X, h)\) is said to be **disc-convex** if for each sequence \((C_n, u_n)\) of stable curves over \( X \) parametrized by the same real surface \( \Sigma \) and having the properties

1. the curves \( C_n \) are irreducible for all \( n \) and the boundaries \( \partial C_n \) are not empty;
2. the areas \( \text{area}_h[u_n(C_n)] \) are uniformly bounded;
3. the \( (C_n, u_n) \) converge in a neighborhood of their boundaries \( \partial C_n \),

there exists a compact subset \( K \) of \( X \) containing all \( u_n(C_n) \).

In particular, all compact manifolds are disc-convex. The property to be disc-convex remains invariant under the replacement of the Hermitian metric \( h \) by an equivalent metric \( h' \), \( c \cdot h \leq h' \leq C \cdot h \). We point out also that by Theorem 3.1, such a sequence \((C_n, u_n)\) contains a subsequence converging in the Gromov topology.

Let \( U \) be a domain in a complex manifold \( X \), and let \( Y \) be a complex space.

**Definition** The *envelope of meromorphy* of \( U \) with respect to \( Y \) is the maximal domain \((\hat{U}_Y, \hat{\pi})\) over \( X \) containing \( U \) (that is, there exists an embedding \( i: U \to \hat{U}_Y \) with \( \hat{\pi} \circ i = \text{Id} \)) such that each meromorphic map \( f: U \to Y \) extends to a meromorphic map \( \hat{f}: \hat{U} \to Y \).

The Cartan–Thullen construction (cf. [1]) applies here to prove the existence and the uniqueness of the envelope.

**Proposition 4.1.** For each domain \( U \) in a complex manifold \( X \) and each complex space \( Y \) there exists an envelope of meromorphy \((\hat{U}_Y, \hat{\pi})\) of \( U \) with respect to \( Y \).

The following result was proved in [2]; Theorem 5.1.3.

**Theorem 4.2.** For each domain \( U \subset X \) in a disc-convex complex Hermitian surface \((X, h)\) and each disc-convex Kähler space \( Y \) the envelope of meromorphy \((\hat{U}_Y, \hat{\pi})\) equipped with the Hermitian metric \( \hat{\pi}^*h \) is also disc-convex.

**Remark.** Let us explain the meaning of this theorem using the following example. Assume that \( X \) is disc-convex and consider a domain \( U \subset X \). Let \( f: U \to Y \) be a meromorphic map and let \( \{(C_n, u_n)\} \) be a sequence of stable holomorphic curves over \( X \) convergent to \((C_\infty, u_\infty)\) in the Gromov topology such that the images of the boundaries \( u_n(\partial C_n) \) and \( u_\infty(\partial C_\infty) \) lie in \( U \). Assume further that \( f \) extends along each \( u_n(C_n) \). This means that there exist a complex surface \( V_n \) containing \( C_n \) and a holomorphic, locally biholomorphic map \( u'_n: V_n \to X \) such that \( u'_n|_{C_n} = u_n \) and \( f \) extends meromorphically from \( u_n^{-1}U \) to the whole of \( V_n \).
The last assumption is equivalent to the condition that the curves \((C_n,u_n)\) can be lifted to the envelope \(\hat{U}\), that is, there exist holomorphic maps \(\hat{u}_n: C_n \to \hat{U}\) such that \(\hat{\pi} \circ \hat{u}_n = u_n\). In other words, we may take \(V_n\) to be a neighborhood of the lifting of \(C_n\) in \(\hat{U}\). It is easy to see that the lifted curves \((C_n,\hat{u}_n)\) are stable over \(\hat{U}\), have uniformly bounded area, and converge near the boundary \(\partial C_n\). By Theorem 4.2 and Gromov’s compactness theorem (Theorem 3.1) a subsequence of \((C_n,\hat{u}_n)\) converges to a \(\hat{U}\)-stable curve \((C_\infty,\hat{u}_\infty)\) such that \(\hat{\pi} \circ \hat{u}_\infty = u_\infty\). This means that \(f\) extends along \(u_\infty(C_\infty)\). Thus, Theorem 4.2 is a generalization of Levi’s continuity principle.

We shall actually prove a stronger result than the main theorem. Namely, in place of meromorphic functions (that is, meromorphic maps into Levi’s continuity principle. Let \(\pi^n\) a disc-convex Kähler surface \(\pi\) such that \(\pi^n\). Let \(Y\) consider the general case of meromorphic maps into an arbitrary disc-convex Kähler space \(Y\).

**Theorem 4.3.** Let \(u: S^2 \to X\) be a symplectic immersion of the sphere \(S^2\) into a disc-convex Kähler surface \(X\) such that \(M := u(S)\) has only positive double points. Assume that \(c_1(X)[M] > 0\). Then the envelope of meromorphy \((\hat{U}_Y, \hat{\pi})\) of a neighborhood \(U\) of \(M\) with respect to a disc-convex Kähler space \(Y\) contains a rational curve \(C\) with \(\pi^*c_1(X)[C] > 0\).

**Proof.** Let \(u: S^2 \to X\) be a symplectically immersed sphere with only positive self-intersections. Let \(U\) be a relatively compact subdomain of \(X\) containing \(M := u(S^2)\). We denote its envelope of meromorphy with respect to \(Y\) by \((\hat{U}_Y, \hat{\pi}_Y)\).

**Step 1.** There exists an \((\omega\text{-tamed})\) almost complex structure \(J_0 \in \mathcal{J}_U\) such that \(M\) is a \(J_0\)-holomorphic curve.

This has been proved in [2]; Lemma 1.1.2. Moreover, there exists a smooth homotopy \(h: [0,1] \to \mathcal{J}_U\) between \(J_0 = h(0)\) and \(J_{st} = h(1)\). We set \(M_0 := M\). As in Lemma 2.5, let \(\mathcal{M}_h(M_0,J_0)\) be the component of \(\mathcal{M}_h\) passing through \((M_0,J_0)\).

**Step 2.** Assume that the component \(\mathcal{M}_h(M_0,J_0)\) is non-compact.

Then part (iii) of Lemma 2.5 says that there exists a continuous curve \(\gamma: [0,1] \to \mathcal{M}_h\) starting at \((M_0,J_0)\) and having property (b) from part (iii) of Lemma 2.5. We consider now \(J_n\)-holomorphic spheres \(M_n\) that form a discrete set in \(\mathcal{M}_h(M_0,J_0)\), as \(J_n\) converges to \(J^* \in \mathcal{J}_U\).

If \(M_n \cap U = \emptyset\) for some \(n\), then \(M_n\) is the required rational curve because \(J_n = J_{st}\) on \(\hat{U} \setminus U\).

If \(M_n \cap U \neq \emptyset\) for all \(n\), then there exists a subsequence, which we still denote by \(M_n\), that converges in the Gromov topology to some (in general, reducible) curve \(M^{(1)}\). If \(M^{(1)}\) has an irreducible component \(M_0^{(1)}\) lying outside \(U\) and satisfying \(c_1(X)[M_0^{(1)}] > 0\), then \(M_0^{(1)}\) is the required rational curve.

Otherwise, there exists a component \(M_0^{(1)}\) of the limiting curve \(M^{(1)}\) intersecting \(U\) and such that \(c_1(X)[M_0^{(1)}] > 0\). We repeat step 2 with \(M_0^{(1)}\) in place of \(M_0\). Since the area of pseudoholomorphic curves is bounded from below (see [6]), after several repetitions of step 2 we shall either find a rational curve in the envelope \(\hat{U}_Y\), or arrive at the situation described below.

**Step 3.** \(\mathcal{M}_h(M_0,J_0)\) is compact.

By part (iii) of Theorem 2.6 and Corollary 2.7 we obtain continuous paths \((M^n_t,J^n_t)\) such that:
(1) $M_n^0 = M_0$ for all $n$;
(2) $J_0^n = J_0$ for all $n$;
(3) $J_1^n \to J_{st}$.

By Gromov’s compactness theorem a subsequence of $M_1^n$ converges to a $J_{st}$-holomorphic nodal curve $C^*$ over $\hat{U}_Y$. We choose an irreducible component $C$ of $C^*$ with $\pi^*c_1(X)[C] > 0$. Then $C$ is the required rational curve in $\hat{U}_Y$.

References

1. S. Ivashkovich, “Extension of analytic objects by the method of Cartan–Thullen”, Proc. of Conf. “Complex analysis and mathematical physics” (Divnogorsk), Krasnoyarsk 1988, pp. 53–61. (Russian)
2. S. Ivashkovich and V. Shevchishin, “Pseudo-holomorphic curves and envelopes of meromorphy of two-spheres in $\mathbb{C}P^2$”, Preprint. Bochum SFB-237, 1995, http://xxx.lanl.gov-9804014.
3. W. Barth, C. Peters, and A. Van de Ven, Compact complex surfaces, Springer-Verlag, Berlin 1984.
4. H. Grauert, “Über Modifikationen und exceptionelle analytische Mengen”, Math. Ann. 146 (1962), 331–368.
5. S. Nemirovski, “Holomorphic functions and embedded real surfaces”, Mat. Zametki 63 (1998), no. 4, 599–606 (Russian); English transl. in Math. Notes 63 (1998), no. 4.
6. M. Gromov, “Pseudo holomorphic curves in symplectic manifolds”, Invent. Math. 82 (1985), 307–347.
7. C. B. Morrey, Multiple integrals in the calculus of variations, Springer-Verlag, New-York 1966.
8. W. Abikoff, Real analytic theory of Teichmüller space, Springer Lecture Notes in Math., vol. 820, Springer-Verlag, Berlin 1980.
9. D. McDuff and D. Salamon, “$J$-holomorphic curves and quantum cohomology”, AMS Univ. Lecture Series. 6 (1994).
10. S. Kobayashi and K. Nomizu, Foundations of differential geometry, vol. II, Wiley–Interscience, New York 1969.
11. H. Hofer, V. Lizan, and J.-C. Sikorav, “On genericity for holomorphic curves in 4-dimensional almost-complex manifolds”, Preprint, 1994.
12. H. Federer H., Geometric measure theory, Grundlehren der mathematischen Wissenschaften, vol. 153, Springer-Verlag, Berlin 1969.
13. T. Parker and J. Wolfson, “Pseudo-holomorphic maps and bubble trees”, J. Geom. Anal. 3 (1993), 63–98.
14. Chr. Hummel, Gromov’s compactness theorem for pseudo-holomorphic curves, Birkhäuser, Basel 1997.
15. J.-P. Ramis, Sous-ensembles analytiques d’une variété banachique complexe, Springer-Verlag, Berlin 1970.