LIFSHITZ ASYMPTOTICS FOR HAMILTONIANS MONOTONE IN THE RANDOMNESS

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ABSTRACT. This is a note for the report on the Oberwolfach Mini-Workshop: Multiscale and Variational Methods in Material Science and Quantum Theory of Solids, published in [MW].

In various aspects of the spectral analysis of random Schrödinger operators monotonicity with respect to the randomness plays a key role. In particular, both the continuity properties and the low energy behaviour of the integrated density of states (IDS) are much better understood if such a monotonicity is present in the model than if not.

In this note we present Lifshitz-type bounds on the IDS for two classes of random potentials. One of them is a slight generalization of a model for which a Lifshitz bound was derived in a recent joint paper with Werner Kirsch [KV]. The second one is a breather type potential which is a sum of characteristic functions of intervals. Although the second model is very simple, it seems that it cannot be treated by the methods of [KV]. The models and the proofs are motivated by well-established methods developed for so called alloy type potentials. The basic notions of random Schrödinger operators and the IDS can be inferred e.g. from [CL90, PF92, Sto01, KM07, Ves06].

§1 Random Schrödinger operators and the IDS. We consider Schrödinger operators on $L^2(\mathbb{R}^d)$ with a random, $\mathbb{Z}^d$-ergodic potential. More precisely, the random potential $W_\omega: \mathbb{R}^d \to \mathbb{R}$ is determined by an i.i.d. family of non-trivial, bounded random variables $\lambda_k: \Omega \to [\lambda_-, \lambda_+] =: J$ indexed by $k \in \mathbb{Z}^d$ and distributed according to the measure $\mu$, and a jointly measurable single site potential $u: J \times \mathbb{R}^d \to \mathbb{R}$.

We assume that $\lambda_- \in \text{supp } \mu$ and that $\sup_{\lambda \in J} |u(\lambda, \cdot)| \in \ell^1(L^p)$, $p > \max(2, d/2)$.

Under these assumptions the random potential

$$W_\omega(x) = \sum_{k \in \mathbb{Z}^d} u(\lambda_k(\omega), x - k)$$

is relatively bounded with respect to the Laplacian with relative bound zero, uniformly in $\omega$. Consequently, for a bounded $\mathbb{Z}^d$-periodic potential $W_{\text{per}}$ the operators $H_{\text{per}} := -\Delta + W_{\text{per}}$ and $H_\omega := H_{\text{per}} + W_\omega$ are selfadjoint on the domain of $\Delta$ and lower bounded uniformly in $\omega$. Moreover, $(H_\omega)_{\omega}$ forms an ergodic family of operators. Hence there exist a closed $\Sigma \subset \mathbb{R}$ and an $\Omega' \subset \Omega$ of full measure, such that for all $\omega \in \Omega'$ the spectrum of $H_\omega$ coincides with $\Sigma$. For $\Lambda_L := [-L/2, L/2]^d, L \in \mathbb{N}$ define the distribution function $N(E) := L^{-d} \mathbb{E} \{ \text{Tr}[\chi_{-\infty, E}(H_\omega) \chi_{\Lambda_L}] \}$. This function is independent of $L$ and is called IDS or spectral distribution function. The support of the associated measure coincides with $\Sigma$. The IDS can be approximated in the sense of distribution functions by its finite volume analogs $N_\omega(E) := L^{-d} \{ \text{eigenvalues of } H_\omega^L \leq E \}$ almost surely. Here $H_\omega^L$ denotes the restriction of $H_\omega$
to $\Lambda_L$ with Neumann boundary conditions. For many types of random Hamiltonians the IDS is expected to be very "thin" near the spectral minimum $E_0 := \min \Sigma$. More precisely I. M. Lifšic conjectured in [Lif63, Lif64] an asymptotic behaviour of the form $N(E) \sim e^{-\tilde{c}(E-E_0)^{-d/2}}$ for $E - E_0$ small and positive, where $c, \tilde{c}$ denote some positive constants. The spectrum near $E_0$ corresponds to very rare configurations of the randomness and $E_0$ is consequently called a fluctuation boundary.

§2 A class of potentials monotone in the randomness. Here we present a slight extension of the main result in [KV]. Assume that the potentials $u$ and $W_{\text{per}}$ satisfy the following

**Hypothesis A.** For any $\lambda \in J$ we have supp $u(\lambda, \cdot) \subset \Lambda_1$ as well as $u(\lambda, x) \geq u(\lambda_-, x)$ for all $x \in \mathbb{R}^d$. There exist $\epsilon_1, \epsilon_2 > 0$ such that for all $\lambda \in [\lambda_-, \lambda_+ + \epsilon_2]$

$$\int_{\mathbb{R}^d} d\lambda u(\lambda, x) \geq \epsilon_1 (\lambda - \lambda_-) + \int_{\mathbb{R}^d} d\lambda u(\lambda_-, x)$$

and for all $\lambda \in [\lambda_- + \epsilon_2, \lambda_+]$

$$\int_{\mathbb{R}^d} d\lambda u(\lambda, x) \geq \int_{\mathbb{R}^d} d\lambda u(\lambda_- + \epsilon_2, x)$$

hold. The function $\lambda \mapsto u(\lambda, x)$ is Lipschitz continuous at $\lambda_-$. More precisely, for some $\kappa$, all $x \in \Lambda_1$ and all $\lambda \in [\lambda_-, \lambda_+ + \epsilon_2]$ we have $u(\lambda, x) - u(\lambda_-, x) \leq \kappa(\lambda - \lambda_-)$. If $d \geq 2$, then for any $\lambda \in J$ the functions $u(\lambda, \cdot)$ and $W_{\text{per}}$ are reflection symmetric with respect to all $d$ coordinate axes.

Typical examples of potentials $u$ satisfying Hypothesis A are: an alloy type potential, i.e. $u(\lambda, x) = \lambda f(x)$ with $L^\infty(\Lambda_1) \ni f \geq 0$, and a breather type potential, i.e. $u(\lambda, x) = f(x/\lambda)$ with supp $f \subset \Lambda_{\lambda_-}, \lambda_- > 0, f \in C^1(\mathbb{R}^d \setminus \{0\})$ and $L^\infty(\mathbb{R}^d) \ni g(x) := -x \cdot (\nabla f)(x) \geq 0$.

**Theorem B.** (Lifshitz bound) Under the Hypothesis A the IDS of the Schrödinger operator $H_\omega := -\Delta + W_{\text{per}} + W_\omega$ satisfies

$$\lim_{E \searrow E_0} \frac{\log |\log N(E)|}{\log(E-E_0)} = -\frac{d}{2}$$

Thus for $E - E_0$ small and positive, asymptotically the bound $0 < N(E) \leq e^{-\tilde{c}(E-E_0)^{-d/2}}$ holds. The proof is essentially the same as in [KV].

§3 Breather potentials with characteristic functions of intervals. We consider a very explicit class of random potentials on $\mathbb{R}$. Let $(\lambda_k)_{k \in \mathbb{Z}}$ be as before with $\lambda_- = 0$, $\lambda_+ = 1$. The breather type potential

$$W_\omega(x) = \sum_{k \in \mathbb{Z}} u(\lambda_k(\omega), x - k), \quad u(\lambda, x) = \chi_{[0, \lambda]}(x)$$

does not satisfy the Lipschitz condition in Hypothesis A. Nevertheless we have

**Theorem C.** The IDS of the Schrödinger operator $H_\omega := -\Delta + W_\omega$, where $W_\omega$ is as in (3), satisfies the Lifshitz bound (2). Note that $E_0 = 0$ for this model.

It seems that the reason why the method of [KV] is not applicable to the potential (3) is the use of Temple’s inequality [Tem28]. For Temple’s inequality to yield an efficient estimate, the second moment $\langle H_\omega^2 \psi, H_\omega^2 \psi \rangle$ in an well chosen state $\psi$ has to be much smaller than the first moment $\langle \psi, H_\omega^2 \psi \rangle$. For the current application the best choice of $\psi$ seems to be the periodic, positive ground state of $H_{\text{per}}$. However for such $\psi$ and for the potential (3), the first and second moment coincide! It turns out that Thirring’s inequality [Thi94, 3.5.32] is better adapted to the model under consideration. It was used before in [KM83] in a similar context.
Sketch of proof: As before the superscript \( L \) denotes the Neumann b. c. restriction to \([-L/2, L/2] \). Since \( N(E) \leq L^{-1}\text{Tr}|\mathcal{A}|_{-\infty,E_1}(-\Delta_L)|\mathbb{P}(\omega \mid E_1(H^L_0) \leq E)\) for any \( L \in \mathbb{N} \), it is sufficient to derive an exponential bound on the probability that the first eigenvalue \( E_1 \) of \( H^L_0 \) does not exceed \( E \).

We set \( I_L := A_L \cap \mathbb{Z}, H_0 := -\Delta - \alpha/4L^2, \psi = L^{-1/2} \chi_{A_L} \) and \( V_\omega(x) = \alpha/4L^2 + W_\omega(x) \). Then \( E_1(H^L_0) = -\alpha/4L^2 \) and \( E_2(H^L_0) \geq 3\alpha/4L^2 \), cf. [KS87]. Since \( V_\omega \) does not vanish, \( V_\omega^{-1} \) is well-defined and we calculate \( L(\int_{A_L} V_\omega(x)^{-1}dx)^{-1} = \frac{\alpha}{4L^2 - 4\pi^2} \) for averages, \( \tilde{\lambda}_k := \min(\lambda_k, 1/2) \) for cut-off random variables and similarly \( \tilde{V}_\omega, \tilde{S}_L \) for the potential and the averages. Then \( E_1(H^L_0) + \langle \psi, \tilde{V}_\omega^{-1}\psi \rangle^{-1} \leq \alpha/4L^2 < E_2(H^L_0) \), thus Thirring's inequality is applicable and yields

\[
E_1(H^L_0) \geq E_1(\tilde{H}^L_0) \geq E_1(H^L_0) + \langle \psi, \tilde{V}_\omega^{-1}\psi \rangle^{-1} \geq \frac{\alpha \tilde{S}_L}{5L^2}
\]

as soon as \( L^2 \geq \alpha \). For a given \( E > 0 \) choose \( L := \lceil \beta E^{-1/2} \rceil \), then \( \mathbb{P}(E_1(H^L_0) \leq E) \leq \mathbb{P}(\alpha \tilde{S}_L/5L^2 \leq E) \leq \mathbb{P}(\alpha \tilde{S}_L/5 \leq \beta^2) \). Since \( 0 < \mathbb{E} \{ \tilde{S}_L \} = \mathbb{E} \{ \tilde{\lambda}_k \} \leq 1/2 \)

it is possible to choose \( 0 < \beta \leq \sqrt{\alpha \mathbb{E} \{ \tilde{\lambda}_k \}/10} \). With this choice we have \( \mathbb{P}(\tilde{S}_L \leq 5\beta^2/\alpha) \leq \mathbb{P}(\tilde{S}_L \leq \mathbb{E} \{ \tilde{S}_L \}/2) \). A large deviation estimate bounds this probability by \( e^{-\tilde{L}^2/\tilde{c}^2} \) for some positive constants \( c, \tilde{c} \). This completes the proof.

The higher dimensional analog of this model is currently under study.

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