Abstract. We study general equations modeling electrostatic MEMS devices
\[
\varphi(r, -u'(r)) = \lambda \int_0^r \frac{f(s)}{g(u(s))} \, ds, \quad r \in (0, 1),
\]
\[
0 < u(r) < 1, \quad r \in (0, 1),
\]
\[
u(1) = 0,
\]
where \(\varphi, g, f\) are some functions on \([0, 1]\) and \(\lambda > 0\) is a parameter. We obtain results on the existence and regularity of a touchdown solution to \((P_\lambda)\) and find upper and lower bounds on the respective pull-in voltage. In the particular case, when \(\varphi(r, v) = r^\alpha |v|^\beta v\), i.e., when the associated differential equation involves the operator \(r^{-\gamma}(r^\alpha |u'|^2 u')'\), we obtain an exact asymptotic behavior of the touchdown solution in a neighborhood of the origin.

Keywords: Touchdown solution, singular solution, pull-in voltage, sub- and supersolution, asymptotic behavior

2020 MSC: 34B15, 35A24, 35B09, 35B40, 35J75

1. Introduction

Microelectromechanical systems (MEMS) are microscopic devices consisting of electrical and mechanical components built together on a chip. This system is usually of a tiny dimension, between 1 and 100 micrometers. MEMS devices are indispensable in the modern technology such as telecommunications, biomedical engineering, exploring the space, etc. For more information on the applications of MEMS, we refer the reader to P. Esposito et al. [6] and to D. Bernstein, J. Pelesko [20].

Equations describing MEMS are written with respect to the displacement of a thin and deformable microplate (membrane) which is fixed along the boundary of a bounded domain (usually a unit ball) below a fixed rigid plate. The membrane deflects towards the rigid plate when one applies a voltage, represented here by \(\lambda\). It may happen that the membrane touches the rigid plate when the voltage reaches a certain critical value \(\lambda^*\), called the pull-in voltage. In the latter case, an instability is created within the system which may greatly affect the MEMS device. The corresponding solution for the model is then called a touchdown solution. Since one aims to achieve a better design for a MEMS device, studying touchdown solutions in different MEMS models is therefore of great interest and importance.
1.1 Motivation and related results

Recently, many authors proposed MEMS models generalizing the classical one, i.e., whose main equation is $-\Delta u = \frac{\lambda f(u)}{(1-u)^2}$. We refer the reader to [6, 20] for a survey of results on the classical MEMS model. Several authors, e.g., [1, 2, 14, 15], studied so-called $p$-MEMS models, that is, when the main MEMS equation contains a $p$-Laplacian instead of the Laplacian. Subsequently, MEMS problems on a ball involving a more general radial nonlinear operator $L(\alpha, \beta, \gamma)[u] = r^{-\gamma}(r^\alpha|u'|^\beta u')'$ received some attention in the literature [16, 17, 18]. Problems containing the operator $L(\alpha, \beta, \gamma)$ have been also considered in many other contexts [3, 12, 19]. As such, in [3], Clément et al. studied a Brezis-Nirenberg-type problem, in [12], J. Jacobsen and K. Schmitt studied a Liouville-Bratu-Gelfand problem. We refer the reader to [5, 13] for a survey of models containing the operator $L(\alpha, \beta, \gamma)$. On the other hand, touchdown issues in MEMS models received a lot of attention in the literature; see, e.g., [8, 9, 10, 11, 14].

Motivated by the aforementioned works and in the spirit of the pioneering work [4], we study here the very general class of MEMS problems described by its unique formulation $(P_\lambda)$. Our main motivation for the choice of the model $(P_\lambda)$ is to encompass the following important examples, where the function $h(r)$ stands for the spatially varying dielectric permittivity, which is assumed positive and integrable on $[0, 1]$.

$(E_1)$ $L(\alpha, \beta, \gamma) = r^{-\gamma}(r^\alpha|u'|^\beta u')'$ with $\alpha \leq \gamma$ and $\beta > -1$; the associated equation in $(P_\lambda)$ takes the form

$$-r^\alpha|u'|^\beta u' = \lambda \int_0^r \frac{s^\gamma h(s)}{g(u)} \, ds; \quad \varphi(r, v) = r^\alpha |v|^\beta v. \quad (\hat{P}_\lambda)$$

$(E_2)$ $\sum_{i=1}^n L(\alpha_i, \beta_i, \gamma)$ with $\alpha_i \leq \gamma$ and $\beta_i > -1$. In this case, the associated equation in $(P_\lambda)$ becomes

$$-\sum_{i=1}^n r^{\alpha_i}|u'|^\beta_i u' = \lambda \int_0^r \frac{s^\gamma h(s)}{g(u)} \, ds; \quad \varphi(r, v) = \sum_{i=1}^n r^{\alpha_i} |v|^\beta_i v.$$

$(E_3)$ Radial form of the $p(x)$-Laplacian. For this operator, the associated equation in $(P_\lambda)$ is

$$-r^{N-1}|u'|^{p(r)-2} u' = \lambda \int_0^r \frac{s^{N-1} h(s)}{g(u)} \, ds; \quad \varphi(r, v) = r^{N-1} |v|^{p(r)-2} v,$$

where $p(r)$ is a continuous function with values in $[1 + \epsilon, N]$, $\epsilon \in (0, 1)$, $N > 1$.

$(E_4)$ Laplace-Beltrami operator on an $N$-dimensional sphere of radius $\rho \geq 1$. Here, the equation in $(P_\lambda)$ is

$$-\rho \left( \sin \left( \frac{r}{\rho} \right) \right)^{N-1} u' = \lambda \int_0^r \frac{(\sin \left( \frac{r}{\rho} \right))^{N-1} h(s)}{g(u)} \, ds; \quad \varphi(r, v) = \rho \sin \left( \frac{r}{\rho} \right)^{N-1} v.$$

We remark that apart from examples $(E_1)$–$(E_4)$, our general formulation $(P_\lambda)$ can possibly be applied to other MEMS models.
Note that the operator $L(\alpha, \beta, \gamma)$ includes a $k$-Hessian and a $p$-Laplacian. Namely, the two latter cases are associated with the following values of $\alpha, \beta, \gamma$:

|       | $\alpha$ | $\beta$ | $\gamma$ | $\theta$ |
|-------|----------|----------|----------|----------|
| $k$-Hessian | $N-k$    | $k-1$    | $N-1$    | $2k$     |
| $p$-Laplacian | $N-1$   | $p-2$    | $N-1$    | $p$      |

The number $\theta = \gamma + 2 + \beta - \alpha$ will be used in Theorem 1.7, and $N$ stands for the spatial dimension of the associated problem.

Model $(E_1)$, in application to MEMS, has been previously studied in the literature by some authors [11, 17, 18]. The role of the parameters $\alpha, \beta, \gamma$ is to describe properties of the substance between the plates of the MEMS device. If this substance is inhomogeneous, i.e., its properties are changing from layer to layer, a better model is $(E_2)$. To describe properties of a substance changing along the radius of the ball, one uses model $(E_3)$. Finally, model $(E_4)$ was proposed in [16] to account for the specific profile of the deflected membrane.

### 1.2 Hypotheses

Throughout Sections 2–4, we assume

- $(H_1)$ $g \in C((-\infty, 1) \rightarrow [0, \infty))$ is strictly decreasing and $g(1) = 0$.
- $(H_2)$ $f : [0, 1] \rightarrow [0, +\infty)$ is a measurable function such that for its primitive function $F(r) = \int_0^r f(s) ds$, it holds that $F(1) < +\infty$ and $F(r) > 0$ for all $r \in (0, 1]$.
- $(H_3)$ $\varphi : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function with the following properties. For each $r \in (0, 1]$, the function $v \mapsto \varphi(r, v)$ is strictly increasing, continuous, and such that $\varphi(r, 0) = 0$, $\lim_{v \rightarrow +\infty} \varphi(r, v) = +\infty$, and $\int_0^1 \varphi^{-1}(s, F(s)) ds < +\infty$, where for each fixed $r \in [0, 1]$, $\varphi^{-1}(r, \cdot)$ denotes the inverse to $\varphi(r, \cdot)$.
- $(H_4)$ There exist a polynomial-like function $\mathcal{P}(v) = \sum_{i=1}^{m} c_i v^{d_i}$, where $c_i, d_i \in (0, +\infty)$, and a measurable function $a : [0, 1] \rightarrow (0, +\infty)$ with the property $\sup_{r \in [0, 1]} a(r) < +\infty$, such that for all $(r, v) \in [0, 1] \times [0, +\infty)$,

$$\varphi(r, v) \leq a(r) \mathcal{P}(v).$$

For the problem $(\hat{P}_3)$, our goal is to construct a touchdown solution and obtain its exact asymptotic behavior in a neighborhood of the origin. For this task, we assume

- $(A_1)$ $(H_1)$ holds;
- $(A_2)$ $g \in C^2(0, 1)$ and there exist constants $q \in (0, 1)$ and $A > 0$ such that $g'(u) = -Ag(u)^q(1 + o(1)) \quad (u \rightarrow 1-)$.
- $(A_3)$ $\beta > -1, \alpha > \beta + 1, \gamma \geq \alpha$.
- $(A_4)$ $h : \mathbb{R}_+ \rightarrow [c, b] \subset (0, +\infty)$ is a measurable function such that $h(r) = C(1 + o(1)) \quad (r \rightarrow 0_+)$, where $C > 0$ is a constant.
Remark 1.1. Note that $(A_1)$ and $(A_2)$ hold for the MEMS nonlinearity $g(u) = (1-u)^p$, $p > 1$. In this case, $A = p$ and $q = \frac{p-1}{p}$. Other examples may include the nonlinearity of the form $g(u) = \zeta(u)^p$, $p > 1$, where $\zeta \in C^2(0,1)$ satisfies $(A_1)$ and in a neighborhood of $u = 1$ solves the equation $c_1 \zeta(u)^{\nu+1} - \zeta(u) = c_2(u-1)$ with $c_1, c_2, \nu > 0$.

Remark 1.2. Note that Assumption $(A_2)$ is equivalent to the following

$$g'(g^{-1}(w)) = -A w^q(1 + o(1)) \quad (w \to 0_+).$$

1.3 Contributions of this work

In this paper, we address the question of existence of a pull-in voltage and a touchdown solution for the MEMS model $(P_\lambda)$ encompassing $(E_1)-(E_4)$. Furthermore, we address obtaining an exact asymptotic behavior, in a neighborhood of the origin, of the touchdown solution for model $(E_1)$.

Most of our results are obtained under hypotheses $(H_1)-(H_4)$. More specifically, we start by showing the existence of a finite pull-in voltage $\lambda^*$ employing the sub- and supersolution method. We then explicitly compute upper and lower bounds on $\lambda^*$. Since most of standard methods cannot be applied to our very general framework, we use a different approach, relying on the Zorn lemma, to show the existence and monotonicity (in $\lambda$) of the maximal and minimal branches of solutions to problem $(P_\lambda)$. Finally, assuming the existence of the limit $\lim_{\lambda \to \lambda^*} u_\lambda(0) = 1$ for the maximal or minimal branches, we obtain the existence and uniqueness of a touchdown solution to $(P_\lambda)$. In addition, we obtain the $C^2$-regularity of the touchdown solution under the assumption of the $C^1$-regularity of the function $\varphi$.

Our most challenging result is obtaining a sharp asymptotic representation (in a neighborhood of the origin) of the touchdown solution for the model $(E_1)$. A related result was obtained in [11]; however, only in the form $u^*(r) = G^{-1}\left[\frac{\theta}{\pi}(1 + o(1))\right]$ with some positive constants $\theta$ and $A$, and the function $G(u) = \int_0^u g(s)\frac{1}{s^{\nu+1}}ds$. An exact asymptotic behavior, similar to as it is announced in Theorem 1.7 of the present work, was only obtained in [11] for a few explicitly given functions $g$. In contrast, the result of our Theorem 1.7 does not rely on any specification of the function $g$; it is obtained for an arbitrary function $g$ satisfying $(A_1)$ and $(A_2)$.

We believe that some of our results, such as upper and lower bounds on the pull-in voltage and the sharp asymptotic behavior of the touchdown solution, can be used for studying MEMS by physicists, engineers, and numerical mathematicians. In addition, we present new methods, e.g., the use of the Zorn lemma to show the existence of the maximal and minimal branches (Theorem 1.4) or fitting unknown parameters in the representation of a singular solution to make a certain map have a fixed point (Theorem 1.7). Below, we announce the main results of this work.

**Theorem 1.3** (Finite pull-in voltage and its bounds). Assume $(H_1)-(H_4)$. Then, the pull-in voltage $\lambda^*$ for problem $(P_\lambda)$ exists; furthermore, it is positive and finite.
Moreover, we have the upper bound
\[ \lambda^* \leq p_{d,e}^{-1} \left( 2P_{d,e}(P(1) \sup_{r \in [0,1]} a(r))P_{d,e}(g(0)) [p_{d,e}(F(1/2))]^{-1} \right), \]  
where \( F(\cdot) \) is defined in (H2), \( P_{d,e}(r) = \max\{r^{\frac{1}{2}}, r^{\frac{1}{2}}\}, \) \( p_{d,e}(r) = \min\{r^{\frac{1}{2}}, r^{\frac{1}{2}}\} \) with \( d = \max_i d_i \) and \( e = \min_i d_i \). Furthermore, defining \( \Phi(\lambda) = \int_0^1 \varphi^{-1}(s, \lambda F(s))ds \), we have the lower bound
\[ \lambda^* \geq \begin{cases} 
   g(\Phi(1)) & \text{if } \Phi(1) < 1, \\
   \sup_{\delta \in [0,1]} \Phi^{-1}(\delta)g(\delta) & \text{if } \Phi(1) \geq 1.
\end{cases} \]  

**Theorem 1.4.** Assume (H1)–(H4). Then, there exist a unique maximal and a unique minimal branches of solutions for problem (P_\lambda), where \( \lambda \) varies over \((0, \lambda^*)\).

**Theorem 1.5** (Existence of a touchdown solution). Assume (H1)–(H4). Let \( \{u_\lambda\}_{\lambda \in (0, \lambda^*)} \) be the maximal or minimal branch of solutions to (P_\lambda) such that \( \lim_{\lambda \to \lambda^*} u_\lambda(0) = 1 \). Then, the pointwise limit \( u_{\lambda^*}(r) = \lim_{\lambda \to \lambda^*} u_\lambda(r) \) exists and it is an integral touchdown solution to (P_\lambda^*).

**Theorem 1.6** (Uniqueness of a touchdown solution). Assume (H1)–(H4). Then, an integral touchdown solution, whenever it exists, is unique and coincides with \( u_{\lambda^*} \), the pointwise limit of the maximal branch as \( \lambda \to \lambda^* \).

**Theorem 1.7** (Asymptotic behavior of the touchdown solution for problem (\( \tilde{P}_\lambda \))). Assume (A1)–(A4). Then, the pull-in voltage \( \lambda^* \) and the touchdown solution \( u^* \) for problem (\( \tilde{P}_\lambda \)) exist. Moreover, as \( r \to 0_+ \), it holds that
\[ u^*(r) = 1 - C r^{\beta x(1-q)} + o(r^{\beta x(1-q)}), \]  
where \( \beta = ((1-q)(\beta + 1))^{-1} \), \( \kappa = \left[ \frac{A}{\theta x} \right]^{(\beta + 1)\zeta}, \) \( C = \kappa^{1-q}(\lambda^*)^\zeta(1-q))A^{-1}(1 - q)^{-1} \) and \( C = \kappa^{1-q}(\lambda^*)^\zeta(1-q))A^{-1}(1 - q)^{-1} \).  

The idea of the proof of Theorem 1.7 is as follows. First, we transform problem (\( \tilde{P}_\lambda \)) to an equivalent one, but given in a neighborhood of infinity \([T, +\infty)\). We then find a certain normed space \( X_T \) and a compact continuous map on it to apply Schauder’s fixed point theorem. The constants \( x \) and \( \kappa \) in Theorem 1.7 are uniquely fixed to condition a fixed point map take values in \( X_T \).

We remark that our approach to obtaining the existence and asymptotics of the touchdown solution to (\( \tilde{P}_\lambda \)) differs significantly from the method used in [11]. It is somewhat in the spirit of the method of [19], although there is no technical similarities.

**1.4 Outline**

In Section 2, we present a sub- and supersolution method appropriate for our general framework. In the same section, we prove the existence of a pull-in voltage \( \lambda^* \) for the problem (P_\lambda) and compute upper and lower bounds on \( \lambda^* \). In Section 3, we obtain
the existence of the maximal and minimal branches $u_\lambda$ of solutions to $(P_\lambda)$ and the monotonicity of $u_\lambda$ in $\lambda$. Furthermore, we obtain the existence and uniqueness of a touchdown solution to problem $(P_\lambda)$. In Section 4, we obtain the $C^2$-regularity of integral solutions, and, in particular, of the touchdown solution, to $(P_\lambda)$. Finally, Section 5 is dedicated to obtaining the asymptotic representation (3) of the touchdown solution to problem $(\hat{P}_\lambda)$.

2. Finite pull-in voltage for Problem $(P_\lambda)$

2.1 Preliminaries

In this subsection, we give definitions of integral solutions, sub- and supersolutions, maximal and minimal solutions which will be used throughout the paper.

**Definition 2.1 (Integral solution).** We say that $u \in C^1(0,1) \cap C[0,1]$ is an integral solution to problem $(P_\lambda)$ if $\int_0^1 \frac{f(s)}{g(u(s))} \, ds < +\infty$ and $u(\cdot)$ solves $(P_\lambda)$.

**Definition 2.2 (Sub- and supersolutions).** We say that a measurable function $\bar{u} : [0,1] \to [0,1]$ is a supersolution to problem $(P_\lambda)$ if $\bar{u}(1) \geq 0$, $\int_0^1 \frac{f(r)}{g(\bar{u}(r))} \, dr < +\infty$, and

$$
\bar{u}(r) \geq \int_r^1 \varphi^{-1} \left( t, \lambda \int_0^t \frac{f(s)}{g(\bar{u}(s))} \, ds \right) \, dt.
$$

Furthermore, we say that a measurable function $u : [0,1] \to [0,1]$ is a subsolution to problem $(P_\lambda)$ if for some $r_0 \in (0,1]$, $u(r) = 0$ on $[r_0,1]$, $\int_0^1 \frac{f(r)}{g(u(r))} \, dr < +\infty$, and

$$
u(r) \leq \int_r^1 \varphi^{-1} \left( t, \lambda \int_0^t \frac{f(s)}{g(u(s))} \, ds \right) \, dt.
$$

**Definition 2.3 (Minimal or maximal integral solution).** We call a function $u \in C^1(0,1) \cap C[0,1]$ a minimal (maximal) integral solution to problem $(P_\lambda)$ if it is an integral solution solution to $(P_\lambda)$ in the sense of Definition 2.1 and for any other integral solution $v$ to $(P_\lambda)$ it holds that $u(r) \leq v(r)$ (resp. $u(r) \geq v(r)$) for all $r \in [0,1]$.

**Definition 2.4.** The value

$$
\lambda^* = \sup \{ \lambda > 0 : (P_\lambda) \text{ possesses an integral solution} \}.
$$

is called a pull-in voltage for problem $(P_\lambda)$.

**Definition 2.5.** An integral solution $u$ to $(P_\lambda)$ is called a touchdown solution if $u(0) = 1$.

2.2 Verification of hypotheses $(H_1)$–$(H_4)$ for models $(E_1)$–$(E_4)$

Here, we will verify assumptions $(H_3)$ and $(H_4)$ for the functions $\varphi$ involved in the models $(E_1)$–$(E_4)$. First, we verify $(H_4)$. Let $(r,v) \in [0,1] \times [0, +\infty)$. In $(E_1)$, one defines $a(r) = r^\alpha$ and $P(v) = v^{\beta+1}$, so $\varphi(r,v) = a(r)P(v)$. In $(E_2)$, $\varphi(r,v) \leq r^{\tilde{\alpha}} \sum_{i=1}^n v^{\tilde{\beta}_i+1}$, where $\tilde{\alpha} = \min_{i=1,\ldots,n} \alpha_i$; therefore, $a(r) = r^{\tilde{\alpha}}$ and $P(v) = v^{\beta+1}$.
Further, in (E3), $\varphi(r, v) \leq r^{N-1}(v^{\tilde{p}-1} + v^{\tilde{p}-1})$, where $\tilde{p} = \min_{r \in [0,1]} p(r)$ and $\bar{p} = \max_{r \in [0,1]} p(r)$, so that we can set $a(r) = r^{N-1}$ and $\mathcal{P}(v) = v^{\tilde{p}-1} + v^{\tilde{p}-1}$.

Finally, in (E4), $\varphi(r, v) \leq \rho(\sin(t))^{N-1}v$, so one defines $a(r) = \rho(\sin(t))^{N-1}$ and $\mathcal{P}(v) = v$. Note that in all cases, a verification of the property $\sup_{r \in [0,1]} a(r) < +\infty$ is straightforward.

To verify (H3), we only have to check that $\int_0^1 \varphi^{-1}(r, F(r)) dr < +\infty$ since the rest of the conditions in (H3) is obviously fulfilled for all four models. To this end, we will show that for all $r \in (0,1]$, $\varphi^{-1}(r, F(r)) \leq G\left(\int_0^r h(s)ds\right)$ in each of the cases (E1)–(E4), where $G$ is an increasing function. As before, we let $(r, v) \in [0,1] \times [0, +\infty)$. In the example (E2), (we consider (E1) as a particular case of (E2)), we have that $\varphi(r, v) \geq r^{\alpha} \sum_{i=1}^n v^{\beta_i+1} \geq r^{\alpha} v^{\beta_i+1}$, where $\alpha = \max_{i=1,\ldots,n} \alpha_i$. We also note that $\alpha \leq \gamma$. Therefore,

$$\varphi^{-1}(r, F(r)) \leq \left(\int_0^r h(s)ds\right)\frac{1}{\beta_1+1} \leq \left(\int_0^r h(s)ds\right)\frac{1}{\beta_1+1} \quad \text{in (E1) and (E2)};$$

$$\varphi^{-1}(r, F(r)) \leq \left(\int_0^r h(s)ds\right)\frac{1}{\alpha(r)-1} \leq \left(\int_0^r h(s)ds\right)\frac{1}{\gamma} + \left(\int_0^r h(s)ds\right)\frac{1}{\gamma-1} \quad \text{in (E3)};$$

$$\varphi^{-1}(r, F(r)) \leq \text{const} \int_0^r h(s)ds \quad \text{in (E4)}.$$
In what follows, for simplicity of notations, we define

\[ \chi(r, u) = \lambda \frac{f(r)}{g(u)}. \]

**Step 1.** The sequence \( \{u_k\} \) is well-defined and \( u \leq u_k \leq \bar{u} \). We show the following two inequalities at the same time: \( \int_0^1 \chi(s, u_k(s)) < \infty \) and \( u \leq u_k \leq \bar{u} \) for all \( k \). When \( k = 0 \), it follows from the definition of a subsolution and supersolution and from the fact that \( u_0 = \bar{u} \). Suppose, as the induction hypothesis, that \( \int_0^1 \chi(s, u_k(s)) < \infty \) and \( u \leq u_k \leq \bar{u} \). Since \( \chi(s, u) \) is increasing in \( u \), we have that

\[ \chi(r, u)(r)) \leq \chi(r, u_k(r)) \leq \chi(r, \bar{u}(r)). \] (10)

Therefore, by (5) and (6),

\[ u \leq \int_r^1 \varphi^{-1} \left( t, \int_0^t \chi(r, u)(r))dt \right) \psi^{-1} \left( t, \int_0^t \chi(s, u_k(s))ds \right) dt = u_{k+1}(r) \leq \int_r^1 \varphi^{-1} \left( t, \int_0^t \chi(r, \bar{u}(r))ds \right) dt \leq \bar{u}. \]

This implies that \( \chi(r, u(r)) \leq \chi(r, u_{k+1}(r)) \leq \chi(r, \bar{u}(r)) \) and \( \int_0^1 \chi(s, u_{k+1}(s)) < \infty \).

**Step 2.** \( u_{k+1}(r) \leq u_k(r) \) for all \( k \) and \( r \in [0, 1] \). As before, we show this affirmation by induction on \( k \). When \( k = 0 \), by Step 1, we have \( u_0(r) = \bar{u}(r) \geq u_1(r) \). Suppose we have proved that \( u_k \leq u_{k-1} \). This implies that \( \chi(s, u_k(s)) \leq \chi(s, u_{k-1}(s)) \). Therefore,

\[ u_{k+1}(r) = \int_r^1 \varphi^{-1} \left( t, \int_0^t \chi(s, u_k(s))ds \right) \leq \int_r^1 \varphi^{-1} \left( t, \int_0^t \chi(s, u_{k-1}(s))ds \right) = u_k(r). \]

**Step 3.** The pointwise limit (7) solves (9). Since \( u_k(r) \) is non-increasing in \( k \) and \( u \leq u_k \leq \bar{u} \), there exists a finite pointwise limit \( u(r) \) given by (7). Let us show that \( u(r) \) solves (9). Indeed, by (10) and the continuity of \( \varphi^{-1} \), we can pass to the limit in (8) by the dominated convergence theorem.

Since \( u(s) \leq \bar{u} \), we have that \( 0 \leq \int_0^t \chi(s, u(s))ds < \infty \). Therefore, we can differentiate (9) with respect to \( r \). Applying \( \varphi(r, \cdot) \) to the both sides of the resulting equation, we obtain the equation in \( (P_{\lambda}) \). Thus, we conclude that \( u \) is an integral solution to \( (P_{\lambda}) \). \( \Box \)

**Remark 2.7.** In fact, we do not have to assume the existence of a subsolution to \( (P_{\lambda}) \), because \( u = 0 \) is a subsolution.

### 2.4 Existence of the finite pull-in voltage

**Proof of Theorem 1.3.** **Step 1.** Integral solutions to \( (P_{\lambda}) \) exist for small values of \( \lambda > 0 \). Define

\[ \bar{u}(r) = \int_r^1 \varphi^{-1} (t, \lambda_0 F(t)) dt, \]
where $F(\cdot)$ is defined in $(H_2)$. Note that the right-hand side is continuous in $\lambda_0 \in [0, 1]$ by the dominated convergence theorem and the continuity of $\varphi^{-1}$ in the second argument. By $(H_3)$ and since $\varphi^{-1}(t,0) = 0$, we can choose $\lambda_0 > 0$ small enough, such that $\rho_{\lambda_0} = \sup_{r \in [0,1]} \bar{u}(r) = \int_0^1 \varphi^{-1}(t, \lambda_0 F(t)) \, dt < 1$. This implies that $\int_0^1 \frac{f(s)}{g(u(s))} \, ds \leq g(\rho_{\lambda_0})^{-1} F(1) < \infty$. Furthermore, since $\frac{g(\rho_{\lambda_0})}{g(u(0))} \leq 1$, for $\lambda \in (0, \lambda_0 g(\rho_{\lambda_0})]$, we obtain

$$
\bar{u}(r) = \int_r^1 \varphi^{-1}(t, \lambda_0 F(t)) \, dt \geq \int_r^1 \varphi^{-1}\left(t, \lambda_0 g(\rho_{\lambda_0}) \int_0^t \frac{f(s)}{g(u(s))} \, ds\right) \, dt 
\geq \int_r^1 \varphi^{-1}\left(t, \lambda \int_0^t \frac{f(s)}{g(u(s))} \, ds\right) \, dt.
$$

Therefore, $\bar{u}$ is a supersolution for $(P_3)$. Since 0 is a subsolution for $(P_3)$, by Proposition 2.6, for each $\lambda \in (0, \lambda_0 g(\rho_{\lambda_0})]$, there exists an integral solution to $(P_3)$.

**Step 2. Upper bound $(1)$**. For $v \geq 0$, define $\tilde{P}_{d,e}(v) = \max\{v^d, v^e\}$ and note that $\tilde{P}_{d,e} = p_{d,e}^\ast$. The invertibility of $\varphi(r, \cdot)$, assumption $(H_4)$, and the fact that $\mathcal{P}(1) = \sum c_i$ imply that for all $r \in (0, 1)$,

$$
\varphi(r, v) \leq \mathcal{P}(1) a(r) \tilde{P}_{d,e}(v) \quad \text{and} \quad \varphi^{-1}(r, v) \geq p_{d,e} \left(\frac{v}{\mathcal{P}(1) a(r)}\right) \geq \frac{p_{d,e}(v)}{\tilde{P}_{d,e}(\mathcal{P}(1) a(r))}.
$$

Let $u(r)$ be an integral solution to $(P_\lambda)$. Then, for all $r \in (0, 1)$,

$$
u(r) = \int_r^1 \varphi^{-1}\left(t, \lambda \int_0^t \frac{f(s)}{g(u(s))} \, ds\right) \, dt \geq \int_r^1 \mathcal{A}^{-1} p_{d,e}(\lambda) p_{d,e} \left(\int_0^t \frac{f(s)}{g(u(s))} \, ds\right) \, dt
\geq \mathcal{A}^{-1} p_{d,e}(\lambda) p_{d,e} \left(\int_0^r \frac{f(s)}{g(u(s))} \, ds\right) (1 - r) \geq \mathcal{A}^{-1} p_{d,e}(\lambda) \frac{p_{d,e}(F(r))}{\tilde{P}_{d,e}(r)} (1 - r),
$$

where $\mathcal{A} = p_{d,e}(\mathcal{P}(1) \sup_{r \in [0,1]} a(r))$. By (11) and the monotonicity of $g$,

$$
\mathcal{A} \tilde{P}_{d,e}(g(0)) \geq \mathcal{A} \sup_{u \in [0,1]} \{uP_{d,e}(g(u))\} \geq \mathcal{A} u(r) \tilde{P}_{d,e}(g(u(r))) \geq p_{d,e}(\lambda) p_{d,e}(F(r)) (1 - r).
$$

Evaluating the right-hand side at $r = \frac{1}{2}$, we obtain that $p_{d,e}(\lambda)$ is bounded from above. Therefore, the pull-in voltage $\lambda^\ast$ is finite and

$$p_{d,e}(\lambda^\ast) \leq 2 \mathcal{A} P_{d,e}(g(0)) \left[ p_{d,e}(F(1/2)) \right]^{-1}.
$$

The above inequality implies (1).

**Step 3. Lower bound $(2)$**. To obtain the lower bound, consider the function $\Phi(\lambda) = \int_0^1 \varphi^{-1}(s, \lambda F(s)) \, ds$ for $\lambda \in [0, 1]$, and note that $\Phi(1) < +\infty$ by $(H_3)$.

Suppose first that $\Phi(1) < 1$. Then, in Step 1, we can take $\lambda_0 = 1$. Also, note that $\rho_{\lambda_0} := \Phi(\lambda_0) = \Phi(1) < 1$. By Step 1, the integral solution to $(P_\lambda)$ exists for all $\lambda \in [0, g(\Phi(1))]$. Therefore, $\lambda^\ast \geq g(\Phi(1))$.

Now suppose that $\Phi(1) \geq 1$. Since $\Phi(0) = 0$ and the function $\lambda \to \Phi(\lambda)$ is strictly increasing, find $\lambda_0$ such that $\Phi(\lambda_0) = \delta$, where $\delta \in (0, 1)$. Note that $\lambda_0 = \Phi^{-1}(\delta)$.
and \( \rho_{\lambda_0} = \Phi(\lambda_0) = \delta \). By Step 1, the integral solution to \((P_\lambda)\) exists for all \( \lambda \in (0, \Phi^{-1}(\delta)g(\delta)) \). This implies that \( \lambda^* \geq \sup_{\delta \in [0,1]} \Phi^{-1}(\delta)g(\delta) \). \( \square \)

3. Touchdown solution

3.1 Maximal and minimal branches of solutions

Proof of Theorem 1.4. We prove the existence of a maximal branch of solution. The existence of a minimal branch is obtained in the same way.

Step 1. The set of integral solutions to \((P_\lambda)\) for a given \( \lambda \) is partially ordered. We denote the above-mentioned set by \( \Lambda_\lambda \). By Theorem 1.3, for each \( \lambda \in (0, \lambda^*) \), there exists an integral solution to \((P_\lambda)\), so \( \Lambda_\lambda \) is non-empty.

Below, we assume that \( \lambda \in (0, \lambda^*) \). The set \( \Lambda_\lambda \) is partially ordered. Indeed, we can compare two solutions \( u_\lambda \) and \( v_\lambda \) from \( \Lambda_\lambda \) if \( u_\lambda(r) \leq v_\lambda(r) \) for all \( r \in [0,1] \) or vice versa. On the other hand, two solutions \( u_\lambda \) and \( v_\lambda \) from \( \Lambda_\lambda \) cannot be compared if there are subintervals of \((0,1)\) where \( u_\lambda > v_\lambda \) and where \( v_\lambda > u_\lambda \).

Step 2. Each chain of \( \Lambda_\lambda \) has a maximal element. If \( \Lambda_\lambda \) is finite, the statement is obvious. Below, we consider the case when \( \Lambda_\lambda \) is infinite. Note that the set of comparable solutions corresponding to the same \( \lambda \) forms a chain, which we denote by \( C_\lambda \). We show that \( C_\lambda \) has a maximal element. For any two elements \( u, v \in C_\lambda \), \( u \leq v \), for all \( r \in (0,1] \) we have

\[
0 \leq v(r) - u(r) = v(0) - u(0) - \int_0^r \left[ \varphi^{-1} \left( t, \lambda \int_0^t \frac{f(s)}{g(v(s))} ds \right) - \varphi^{-1} \left( t, \lambda \int_0^t \frac{f(s)}{g(u(s))} ds \right) \right] dt \leq v(0) - u(0). \tag{12}
\]

Define the number

\[
a_{\text{max}} = \sup_{u \in C_\lambda} \{u(0)\}.
\]

Let \( u_n \in C_\lambda \) be a non-decreasing sequence such that \( a_{\text{max}} = \lim_{n \to \infty} u_n(0) \). Note that \( \{u_n\} \) is a Cauchy sequence in \( C[0,1] \). Indeed, since \( \{u_n(0)\} \) is a Cauchy sequence, for \( m > n \), we have \( 0 \leq u_n(r) - u_m(r) \leq u_n(0) - u_m(0) \to 0 \) as \( m, n \to +\infty \). Therefore, there exists a function \( v \in C[0,1] \), \( v = \lim_{n \to \infty} u_n \). The dominated convergence theorem implies that \( v \) is a solution to equation \((9)\). Let us first show that \( v \in C_\lambda \), i.e., \( v \) can be compared with any other element from \( C_\lambda \). Let \( u \in C_\lambda \) be arbitrary.

Case 1. \( v(0) < u(0) \). Then, for all \( n \in \mathbb{N} \), \( u_n(0) < u(0) \). Since \( u_n \) and \( u \) are comparable, we have that \( u_n(r) \leq u(r) \) for all \( r \in [0,1] \). This implies that \( v(r) \leq u(r) \) for all \( r \in [0,1] \).

Case 2. \( v(0) > u(0) \). Then, there exists \( N > 0 \) such that \( u_n(0) > u(0) \) for all \( n \geq N \). Since \( u_n \) and \( u \) are comparable, we have that \( u_n(r) \geq u(r) \) for all \( r \in [0,1] \). This implies that \( v(r) \geq u(r) \) for all \( r \in [0,1] \).

Case 3. \( v(0) = u(0) \). Suppose \([0,r_0]\) be the maximal interval where \( v(r) = u(r) \) and \( r_0 < 1 \). In a small right neighborhood of \( r_0 \), denote it by \((r_0,r_0 + \varepsilon)\), we either have
$v(r) > u(r)$ or $v(r) < u(r)$. However, this contradicts to the following equation which has to be fulfilled for $r \in (r_0, r_0 + \varepsilon)$:

$$v(r) - u(r) = -\int_{r_0}^r \left[ \varphi^{-1} \left( t, \lambda \int_0^t \frac{f(s)}{g(v(s))} ds \right) - \varphi^{-1} \left( t, \lambda \int_0^t \frac{f(s)}{g(u(s))} ds \right) \right] dt$$

Hence, in the Case 3, we have that $r_0 = 1$ and $u = v$.

Therefore, $v$ can be compared with other elements from $C_\lambda$. Let us show that $v$ is a maximal element of $C_\lambda$. Suppose there is another element $w \in C_\lambda$ such that $w \geq v$. However, one must have $w(0) = v(0)$. By (12), $w(r) = v(r)$ for all $r \in [0, 1]$.

**Step 3.** $\Lambda_\lambda$ contains a unique maximal element. By the Zorn lemma, the set $\Lambda_\lambda$ contains a maximal element $u_{\max}(r)$. Suppose there is another maximal element $v \in \Lambda_\lambda$. Since $u_{\max} \in \Lambda_\lambda$, then $v \leq u_{\max}$. By the same reasoning, $u_{\max} \leq v$. This proves the statement of Step 3.

Maximal solutions, which we proved to exist for each $\lambda \in (0, \lambda^*)$, form a maximal branch. □

**Proposition 3.1.** Let $u_\lambda$, $\lambda \in (0, \lambda^*)$, be the maximal branch for $(P_\lambda)$. Then, for each pair $\lambda_1, \lambda_2 \in (0, \lambda^*)$, $\lambda_1 < \lambda_2$, it holds that $u_{\lambda_1} \leq u_{\lambda_2}$.

**Proof.** Below, for simplicity of notation, we introduce

$$\chi(s, u) = \frac{f(s)}{g(u)}.$$

Take $\lambda_3 \in (\lambda_2, \lambda^*)$. We have that $u_{\lambda_1}$ is a subsolution to $(P_{\lambda_2})$ and $u_{\lambda_3}$ is a supersolution for the same problem. Indeed,

$$u_{\lambda_1} = \int_0^1 \varphi^{-1} \left( t, \lambda_1 \int_0^t \chi(s, u_{\lambda_1}(s)) ds \right) dt < \int_0^1 \varphi^{-1} \left( t, \lambda_2 \int_0^t \chi(s, u_{\lambda_1}(s)) ds \right) dt.$$

In the same way,

$$u_{\lambda_3} > \int_0^1 \varphi^{-1} \left( t, \lambda_2 \int_0^t \chi(s, u_{\lambda_3}(s)) ds \right) dt.$$

By Proposition 2.6, there exists an integral solution $v_{\lambda_2}$ to $(P_{\lambda_2})$ such that $u_{\lambda_1} \leq v_{\lambda_2}$. This implies that $u_{\lambda_1} \leq u_{\lambda_2}$ by the maximality of $u_{\lambda_2}$. □

By the same argument, we have the following corollary.

**Corollary 3.2.** Let $u_\lambda$, $\lambda \in (0, \lambda^*)$ be the minimal branch for $(P_\lambda)$. Then, for each pair $\lambda_1, \lambda_2 \in (0, \lambda^*)$, $\lambda_1 < \lambda_2$, it holds that $u_{\lambda_1} \leq u_{\lambda_2}$.

**Corollary 3.3.** Let $u_\lambda$ be the maximal or minimal branch. Then, as $\lambda \to \lambda^*$, $u_\lambda$ converges pointwise to a finite function.

**Proof.** For each $r \in [0, 1]$, $u_\lambda(r)$ is bounded and increasing in $\lambda$. This implies the statement of the corollary. □
3.2 Existence, non-existence, and uniqueness of a touchdown solution

In what follows, we will need the following lemma.

**Lemma 3.4.** For any \( \sigma \in (0, 1) \), the family of integral solutions \( u_\lambda, \lambda \in [\lambda_0, \lambda^*], \lambda_0 \in (0, \lambda^*) \), to problem \( (P_\lambda) \) is uniformly bounded by a number \( 1 - \delta \) (where \( \delta \in (0, 1) \) depends on \( \sigma \)) on the subinterval \([\sigma, 1] \subset (0, 1)\).

**Proof.** For simplicity of notation, we write \( u \) instead of \( u_\lambda \) for the solution to \( (P_\lambda) \).

Let \( q_\delta = 1 - \delta \) and let \( \mu_\delta \in (0, 1) \) be such that \( u(\mu_\delta) = q_\delta \) (if exists). Note that if \( u(0) > 1 - \delta \), then \( \mu_\delta \) always exists and is uniquely defined (since \( u'(r) < 0 \)). Otherwise, if \( u(0) \leq 1 - \delta \), then \( u(r) \leq 1 - \delta \) for all \( r \in [0, 1] \). Below, we consider the case \( u(0) > 1 - \delta \).

Evaluating the both sides of the equation in \( (P_\lambda) \) at \( \mu_\delta t, t \in (0, 1) \) and integrating with respect to \( t \) from \( r \) to \( 1 \), where \( r \in [0, 1] \), we obtain

\[
u(\mu_\delta r) - q_\delta = \mu_\delta \int_r^1 \varphi^{-1}\left(\mu_\delta t, \lambda \int_0^{\mu_\delta t} \frac{f(s)}{g(u(s))} ds\right) dt.
\]

Now the same computation as \( (11) \), for all \( r \in (0, 1) \), implies

\[
u(\mu_\delta r) - q_\delta \geq A^{-1} p_{d,e}(\lambda) \frac{\mu_\delta p_{d,e}(F(\mu_\delta r))}{P_{d,e}(g(\mu_\delta r))}(1 - r),
\]

where \( A \) is the same as in \( (11) \). From here, we obtain

\[
K \sup_{u \in [q_\delta, 1]} \{(u - q_\delta)P_{d,e}(g(u))\} \geq K(u(\mu_\delta r) - q_\delta)P_{d,e}(g(\mu_\delta r)) \geq \mu_\delta p_{d,e}(F(\mu_\delta r))(1 - r),
\]

where \( K = \frac{A}{p_{d,e}(\lambda_0)} \). As in the case of \( (11) \), the above inequality holds for \( r \in (0, 1) \).

Evaluating the right-hand side at \( r = \frac{1}{2} \), we obtain

\[
Q(\mu_\delta) := \mu_\delta p_{d,e}(F\left(\frac{\mu_\delta}{2}\right)) \leq 2K \sup_{u \in [1 - \delta, 1]} (u - 1 + \delta)P_{d,e}(g(u)) \leq 2K \delta P_{d,e}(g(1 - \delta)).
\]

Note that \( p_{d,e}(\cdot) \) is strictly increasing. Next, since \( F(r) > 0 \) for all \( r \in (0, 1) \) by \( (H_2) \), \( f(r) \) cannot be an identical zero in a small right neighborhood \((0, \epsilon)\) of \( 0 \). Therefore, \( F \) is strictly increasing on \((0, \epsilon)\). Therefore, \( Q(\cdot) \) is strictly increasing on \((0, 2\epsilon)\). Hence, \( \mu_\delta \to 0 \) as \( \delta \to 0 \) uniformly in \( u \), an integral solution to \( (P_\lambda) \), and \( \lambda \in [\lambda_0, \lambda^*] \). Fix \( \sigma \in (0, 1) \) and choose \( \delta > 0 \) such that \( Q^{-1}(2K \delta P_{d,e}(g(1 - \delta))) < \sigma \). Then, on \([\sigma, 1] \), integral solutions to \( (P_\lambda) \) are bounded by \( 1 - \delta \) uniformly in \( \lambda \in [\lambda_0, \lambda^*] \). \( \square \)

**Proof of Theorem 1.5.** By Lemma 3.4, for any \( N \in \mathbb{N} \), the family \( u_\lambda, \lambda \in [\lambda_0, \lambda^*) \) \((\lambda_0 > 0)\), is bounded on \([\frac{1}{N}, 1]\) by a constant \( 1 - \delta \), where \( \delta \) depends on \( N \). Therefore, by the bounded convergence theorem, for all \( r \in [0, 1) \) and \( N \in \mathbb{N} \),

\[
\int_r^1 \varphi^{-1}\left(t, \lambda^* \int_0^t \frac{f(s)}{g(u_\lambda(s))} ds\right) dt = \lim_{\lambda \to \lambda^*} \int_r^1 \varphi^{-1}\left(t, \lambda \int_0^t \frac{f(s)}{g(u_\lambda(s))} ds\right) dt \leq \lim_{\lambda \to \lambda^*} u_\lambda \leq 1.
\]

TOUCHDOWN SOLUTIONS IN GENERAL MEMS MODELS 12
By the monotone convergence theorem, the sequence
\[
\left\{ \varphi^{-1}\left(t, \lambda^* \int_0^t \frac{f(s)}{g(u_{\lambda^*}(s))} \, ds \right) \right\}_{N=1}^{\infty}
\]
converges to an integrable function as \( N \to \infty \). Therefore, the limit function
\[
\varphi^{-1}\left(t, \lambda^* \int_0^t \frac{f(s)}{g(u_{\lambda^*}(s))} \, ds \right)
\]
is integrable on \([r, 1]\), for any \( r \in [0, 1] \), and one can pass to the limit as \( \lambda \to \lambda^* \) in equation (9) by the dominated convergence theorem. Therefore, \( u_{\lambda^*} \) is an integral touchdown solution to problem \((P_\lambda)\).

The proof of Theorem 1.6 on the uniqueness of a touchdown solution, whenever it exists, is a direct consequence of the following proposition.

**Proposition 3.5.** Assume \((H_1)-(H_4)\). Let \( u_{\lambda^*}, \lambda^* \in (0, \lambda^*] \), be an integral touchdown solution to \((P_{\lambda^*})\), i.e., \( u_{\lambda^*}(0) = 1 \). Then, \( \lambda^* \) is the pull-in voltage, i.e., \( \lambda^* = \lambda^* \).

Moreover, \( u_{\lambda^*} = u_{\lambda^*} \) and the solution \( u_{\lambda^*} \) is maximal.

**Proof.** Suppose \( \lambda^* < \lambda^* \). By Theorem 1.4, there exists a (unique) maximal branch of integral solutions \( v_\lambda \) to \((P_\lambda)\) when \( \lambda \) varies over \([\lambda^*, \lambda^*] \). By Corollary 3.2,

\[
u_{\lambda^*} \leq v_{\lambda^*} \leq v_{\lambda}, \quad \text{for all } \lambda \in [\lambda^*, \lambda^*].\]

However, since \( u_{\lambda}(0) = 1 \), then \( v_{\lambda}(0) = 1 \) for all \( \lambda \in [\lambda^*, \lambda^*] \). By Theorem 1.5, the pointwise limit \( \lim_{\lambda \to \lambda^*} v_\lambda(r) = u_{\lambda^*}(r) \) exists and is an integral touchdown solution to \((P_{\lambda^*})\). Clearly, \( u_{\lambda^*}(0) = u_{\lambda^*}(0) = 1 \) and \( u_{\lambda^*}(r) \leq u_{\lambda^*}(r) \) by (13). Let \([0, r_0] \subset [0, 1]\) be the maximal interval where \( u_{\lambda^*}(r) = u_{\lambda^*}(r) \). If \( r_0 < 1 \), then in a small right neighborhood of \( r_0 \), denote it by \((r_0, r_0 + \varepsilon)\), we have that \( u_{\lambda^*}(r) < u_{\lambda^*}(r) \). However, this contradicts to the following equation which has to be fulfilled for all \( r \in (r_0, r_0 + \varepsilon)\):

\[
u_{\lambda^*}(r) - u_{\lambda^*}(r) = -\int_{r_0}^{r} \left[ \varphi^{-1}\left(t, \lambda^* \int_0^t \frac{f(s)}{g(u_{\lambda^*}(s))} \, ds \right) - \varphi^{-1}\left(t, \lambda^* \int_0^t \frac{f(s)}{g(u_{\lambda^*}(s))} \, ds \right) \right] \, dt.
\]

Therefore, \( r_0 = 1 \), and hence, \( u_{\lambda^*} = u_{\lambda^*} \). This implies that \( \lambda^* = \lambda^* \) since the above equation should be fulfilled for all \( 0 \leq r_0 < r \leq 1 \). Furthermore, \( v_{\lambda^*} = v_{\lambda^*} \) by exactly the same argument. Hence, \( u_{\lambda^*} \) is maximal.

**Proposition 3.6.** Assume \((H_1)-(H_4)\) and let \( u_\lambda \) be as in Theorem 1.5. Further assume that \( \lim_{\lambda \to \lambda^*} u_\lambda(0) < 1 \). Then, the pointwise limit \( u_{\lambda^*}(r) = \lim_{\lambda \to \lambda^*} u_\lambda(r) \) is an integral solution to \((P_\lambda)\) which is not a touchdown solution.

**Proof.** Since in this case \( g(u_\lambda(r)) \) is bounded by \( g(u_{\lambda^*}(0)) \), where \( u_{\lambda^*}(0) < 1 \), we can pass to the limit as \( \lambda \to \lambda^* \) in equation (9) by the bounded convergence theorem.

**Corollary 3.7** (Corollary of Theorem 1.5 and Proposition 3.6). Assume \((H_1)-(H_4)\). Then, there exists a positive number \( \lambda^* > 0 \) such that for all \( \lambda \in (0, \lambda^*] \), there is an integral solution to \((P_\lambda)\) and for all \( \lambda > \lambda^* \), there is no integral solution to \((P_\lambda)\).
4. \(C^2\)-regularity of integral solutions

**Theorem 4.1.** Assume \((H_1)-(H_4)\). Furthermore, we assume that \(\varphi \in C^1([0,1] \times \mathbb{R})\). Let \(u(r)\) be an integral solution to \((P_\lambda)\). Then, \(u \in C[0,1] \cap C^2(0,1)\). In particular, the touchdown solution to \((P_\lambda)\) is \(C[0,1] \cap C^2(0,1)\)-regular.

**Proof.** The equation in \((P_\lambda)\) is equivalent to

\[-u'(r) = \varphi^{-1}\left(r, \lambda \int_0^r \frac{f(s)}{g(u(s))}\right).\]

By the inverse function theorem, the map \(\varphi^{-1}(r,v)\) is differentiable in the second argument. Thus, in order to show that \(u \in C^2(0,1)\), we have to show that \(\varphi^{-1}(r,v)\) is differentiable in \(r\). For each fixed \(v \in \mathbb{R}\), the equation \(\Psi(r,y) := \varphi(r,y) - v = 0\) defines \(y\) as an implicit function of \(r\). By the theorem on the differentiability of an implicit function, we have that \(\frac{dy}{dr} = -\frac{\partial_r \Psi(r,y)}{\partial_y \Psi(r,y)}\). However, the implicit function \(y(r) = \varphi^{-1}(r,v)\) is defined uniquely. Consequently, \(\varphi^{-1}(r,v)\) is differentiable in \(r\) and \(\frac{\partial_r \varphi^{-1}(r,v)}{y = \varphi^{-1}(r,v)}\). \(\square\)

5. Touchdown solution to \((\hat{P}_\lambda)\): existence and asymptotics

This section is dedicated to model \((E_1)\). Here we prove Theorem 1.7.

5.1 Equivalent problem in a neighborhood of infinity

As we mentioned before, the equation in \((P_\lambda)\) can be written in the form \((\hat{P}_\lambda)\) which, in turn, implies the following equation obtained by differentiation:

\[L(\alpha, \beta, \gamma)[u] = \lambda \frac{h(r)}{g(u(r))}.\] (14)

Note that its solution \(u\) is an integral solution if and only if

\[\lim_{r \to 0} r^\alpha |u'(r)|^\beta u'(r) = 0.\] (15)

It is straightforward to verify that for \(r \in (0,1)\), equation (14) can be rewritten as

\[u''(r)|u'(r)|^{\beta} + \frac{u'(r)|u'(r)|^\beta + \lambda r^{\gamma-\alpha} h(r)}{g(u(r))} = 0.\]

By the change of variable \(r = ce^{-t}\), where \(c = \lambda^{-\frac{1}{\beta}}\), we transform (14) to the problem

\[(\beta + 1)|v'|^\beta v'' - (\alpha - \beta - 1)|v'|^\beta v' + \frac{e^{-\theta t} h(t)}{g(v)} = 0, \quad t \in (\ln c, +\infty),\] (16)

where \(\theta = \gamma + 2 + \beta - \alpha\) and, with a slight abuse of notation, we re-used the symbol \(h(t)\) for the function \(h(ce^{-t})\). The boundary condition \(u(1) = 0\) becomes \(v(\ln c) = 0\). Note that (16) is equivalent to (14), but given in a neighborhood of infinity.
Since we are interested in integral solutions, condition (15) implies that
\[
\left| v'(t) \right|^\beta v'(t) = \int_t^{+\infty} e^{(\alpha-\beta-1)(t-s)} e^{-\theta s} h(s) g(v(s)) ds \tag{17}
\]
and it is equivalent to the convergence of the integral on the right-hand side of (17).

While we call a solution \( u(r) \) to \( \hat{P}_{\lambda} \) with \( u(0) = 1 \) a touchdown solution, the respective solution \( v(t) = u(ce^{-t}) \) to (17), such that \( \lim_{t \to +\infty} v(t) = 1 \), will be referred to below as a singular solution.

### 5.2 Construction of a singular solution to equation (17)

The following below Schauder’s fixed point theorem can be found in [21], Section 4.

**Theorem 5.1.** (Schauder’s fixed point theorem) Let \( X \) be a normed space and let \( T \) be a compact continuous map \( X \to X \). Then, \( T \) has a fixed point.

**Theorem 5.2.** Assume \((A_1)-(A_4)\). Then, there exists \( T > 0 \) such that on \( [T, +\infty) \), equation (17) possesses the singular solution
\[
v^*(t) = g^{-1}(\kappa e^{-\theta t} + x(t)), \quad x(t) = o(e^{-\theta t}), \tag{18}
\]
where the constants \( \kappa > 0 \) and \( \kappa \in (0,1) \) are given by (4).

**Proof.** The idea of the proof is to substitute \( v^* \), defined by (18), into equation (16) and to set up a fixed point argument, with respect to \( x \), in a certain normed space by means of Theorem 5.1.

**Step 1. The set of equations defining the fixed point map.** Thus, we search for a singular solution in the form (18), where the choice of \( \kappa \in (0,1) \) and \( \kappa > 0 \) will be explained later. The last term in (16) transforms via the substitution (18) as follows:
\[
\frac{e^{-\theta t} h(t)}{\kappa e^{-\theta t} + x(t)} = \kappa^{-1} e^{-\theta(1-\kappa) t} h(t) \left( 1 - \frac{x(t)}{1 + x(t)} \right), \tag{19}
\]
where \( x(t) = x(t)e^{\theta t} \kappa^{-1} \).

Note that if \( T > 0 \) is large enough, then \( |x(t)| < \frac{1}{4} \) for all \( t \geq T \). Hence, by \((A_4)\), if the solution in the form (18) exists for a large enough \( T > 0 \), it is an integral solution. In particular, \( v^*(t) > 0 \), so \( v^*(t) \) increases in \( t \).

For the fixed point argument, we will use the following set of equations. First, by using (17) and (19), we represent \( v^*(t)^{\beta+1} \) as
\[
v^*(t)^{\beta+1} = \psi(t) + y(t), \tag{21}
\]
where
\[
\psi(t) = \kappa^{-1} \int_t^{+\infty} e^{(\alpha-\beta-1)(t-s)} e^{-\theta(1-\kappa) s} h(s) ds,
\]
\[
y(t) = -\kappa^{-1} \int_t^{+\infty} e^{(\alpha-\beta-1)(t-s)} e^{-\theta(1-\kappa) s} h(s) \frac{x(s)}{1 + x(s)} ds. \tag{22}
\]
By boundedness of \( h \), the integrals (22) converge for sufficiently large \( T > 0 \). Derivating (18) with respect to \( t \), gives

\[
\left[ \frac{-\theta z \kappa e^{-\theta z t} + x'}{g'(g^{-1}(\kappa e^{-\theta z t} + x))} \right]^{\beta+1} = \psi(t) + y(t).
\] (23)

**Step 2. The fixed point map.** Introduce the normed space

\[
X_T = \left\{ x \in C[T, +\infty) \mid x(t) = o(e^{-\theta z t}) \text{ as } t \to +\infty \right\}
\]

with the norm \( \|x\|_T := \sup_{t \geq T} |x(t)| \).

Furthermore, introduce the subset of \( X_T \):

\[
\Xi_T = \left\{ x \in X_T : \|x\|_t \leq \frac{\kappa}{4} e^{-\theta z t} \quad \forall t \geq T \right\} = \left\{ x \in X_T : \|x\|_T \leq \frac{1}{4} \right\},
\]

\( \|x\|_t = \sup_{\kappa \geq t} |x(s)| \),

where \( x \) is defined via \( x \in X_T \) by (20). Note that for all \( x \in \Xi_T \), the argument of \( g^{-1} \) in (18) is positive, and hence, \( v^* \) is well-defined.

First, we will define the map \( \Gamma : \Xi_T \to X_T \) while fixing the parameters \( z \) and \( \kappa \) in such a way that \( \Gamma(x) \in X_T \) for \( x \in \Xi_T \). We then extend the map \( \Gamma \) from \( \Xi_T \) to \( X_T \), obtaining by this a map \( \hat{\Gamma} : X_T \to X_T \). We will then apply Schauder’s fixed point theorem to show that the map \( \hat{\Gamma} : X_T \to X_T \) has a fixed point \( x^0 \). By choosing \( T > 0 \) sufficiently large, we achieve that \( x^0 \) is a fixed point of \( \Gamma \).

More specifically, we define \( \Gamma(x) \) by expressing \( x' \) from the left-hand side of (23), and integrating the resulting equation from \( t \) to \( +\infty \). Namely, given \( x \in \Xi_T \),

\[
\Gamma(x)(t) = -\int_t^{+\infty} \left( (\psi(s) + y(s))^{\frac{1}{\beta+1}} g' \left( g^{-1}(\kappa e^{-\theta z s} + x(s)) \right) + \theta z \kappa e^{-\theta z s} \right) ds,
\]

where \( y \) is defined, also via \( x \in \Xi_T \), by the second equation in (22). The function \( \psi \), explicitly given by the first equation in (22), can be regarded as known. Moreover, since \( h(t) = C(1 + o(1)) \) as \( t \to +\infty \), the first expression in (22) implies that

\[
\psi(t) = \frac{C}{\kappa(\gamma + 1 - \theta z)} e^{-\theta(1-\kappa)t} (1 + o(1)),
\]

(25)

where we have taken into account that \( \theta(1-z) + (\alpha - \beta - 1) = \gamma + 1 - \theta z \). Since for \( x \in \Xi_T \), \( |x(t)| \leq \frac{1}{4} \) for all \( t \geq T \), it holds that

\[
|y(t)| \leq \frac{4}{3} \|x\|_t \psi(t) < \frac{\psi(t)}{3},
\]

(26)

Therefore, \((\psi(t) + y(t))^{\frac{1}{\beta+1}}\) in (24) is well-defined and

\[
(\psi(t) + y(t))^{\frac{1}{\beta+1}} = \psi(t)^{\frac{1}{\beta+1}} (1 + o(1)).
\]
Next, by Remark 1.2, which is equivalent to \((A_2)\), we have
\[
g' \left( g^{-1}(\kappa e^{-\theta xs} + x(s)) \right) = -A(\kappa e^{-\theta xs} + x)^\theta + o(e^{-\theta q s})
\]
\[
= -A \kappa^3 e^{-\theta q s x}(1 + o(1))^\theta + o(e^{-\theta q s}) = -A \kappa^3 e^{-\theta q s x}(1 + o(1)).
\]
By the above formula, (24) and (25), we obtain
\[
\Gamma(x) = \int_t^{x^+} \left( A\kappa^q \left[ \frac{C}{\kappa(\gamma + 1 - \theta x)} \right] \right) \exp \left\{ -\theta q s - \frac{\theta(1 - x)}{\beta + 1} \right\} (1 + o(1))
\]
\[
- \theta q s \exp\{-\theta q s\} ds. \tag{28}
\]
Thus, by setting
\[
\theta q s + \frac{\theta(1 - x)}{\beta + 1} = \theta x \quad \text{and} \quad A\kappa^q \left[ \frac{C}{\kappa(\gamma + 1 - \theta x)} \right] = \theta x k,
\]
we obtain that \(\Gamma(x)(t) = o(e^{-\theta q x})\). Therefore, \(\Gamma(x) \in X_T\) if \(x \in \Xi_T\). Furthermore, the first equality in (29) implies the expression for \(x\) in (4), while the second equality in (29) implies the expression for \(\kappa\).

Thus, we have constructed the map \(\Gamma : \Xi_T \to X_T\). Let for \(x \in X_T\), \(x_{(1/4)}(t) = \min\{x(t), 1/4\}\) if \(x(t) \geq 0\) and \(x_{(1/4)}(t) = \max\{x(t), -1/4\}\) if \(x(t) < 0\), where (we recall) \(x\) is defined via \(x \in X_T\) by (20). Further, if \(x \in X_T\), we define \(x_{(1/4)}(t) = \kappa e^{-\theta q t} x_{(1/4)}(t)\).

Clearly, \(x_{(1/4)} \in \Xi_T\) if \(x \in X_T\). Thus, we define
\[
\hat{\Gamma} : X_T \to X_T, \quad \hat{\Gamma}(x) = \Gamma(x_{(1/4)}).
\]

**Step 3. Continuity of \(\hat{\Gamma}\).** Take \(x^1, x^2 \in \Xi_T\). By (24) and (26),
\[
|\Gamma(x^1)(t) - \Gamma(x^2)(t)| \leq \int_t^{+\infty} \left| \left( g' \left( g^{-1}(\kappa e^{-\theta q s x} + x) \right) \right) \right| |(\psi + y^1)^{1/\beta} - (\psi + y^2)^{1/\beta}| + \left( \frac{4}{3} \right) \frac{1}{\psi(s)} \left| g' \left( g^{-1}(\kappa e^{-\theta q s x} + x) \right) - g' \left( g^{-1}(\kappa e^{-\theta q s x} + x^1) \right) \right| ds,
\]
where \(y^1\) and \(y^2\) are functions associated with \(x^1\) and \(x^2\), respectively, by formula (22). By Remark 1.2, which is equivalent to \((A_2)\), \(|g' \left( g^{-1}(\kappa e^{-\theta q t x} + x(t)) \right) | \leq K_1 e^{-\theta q t} \), where \(K_1 > 0\) is a constant. Further, by (22),
\[
|y^1(t) - y^2(t)| \leq \kappa^{-1} \int_t^{+\infty} e^{(\alpha - \beta - 1)(s - t) - \theta(1 - \kappa x)} \frac{h(s)}{1 + x^1}(1 + x^1) ds \leq \frac{16}{9} \kappa^{-1} \psi(t) e^{\theta q t} \|x^1 - x^2\|_t,
\]
and consequently, there exists a constant \(K_2 > 0\) such that
\[
|\psi + y^1)^{1/\beta} - (\psi + y^2)^{1/\beta}| \leq K_2 \psi(t)^{1/\beta} e^{\theta q t} \|x^1 - x^2\|_t.
\]
Furthermore,
\[
|g'(g^{-1}(ke^{-\theta t} + x^1)) - g'(g^{-1}(ke^{-\theta t} + x^2))| \leq \sup_{x \in \Xi_T} \left| \frac{g''(g^{-1}(ke^{-\theta t} + x))}{g'(g^{-1}(ke^{-\theta t} + x))} \right| |x^1(t) - x^2(t)|.
\]
(30)

The supremum on the right-hand side, can be estimated by \((A_2)\) and L’Hopital’s rule:

\[
-A = \lim_{u \to 1} \frac{g'(u)}{g(u)^q} = \lim_{u \to 1} \frac{g''(u)}{qg(u)^q-1g'(u)}.
\]

From here, we obtain that as \(u \to 1\),

\[
\frac{g''(u)}{g'(u)} = -A q g(u)^q-1(1 + o(1)).
\]

Therefore, the left-hand side of (30) can be estimated from above by the term

\[
K_3 e^{\theta(1-q)t} ||x^1 - x^2||_t,
\]

where \(K_3 > 0\) is a constant. The above estimates, the first equality in (29), and expression (25) imply that there exists a constant \(K_4 > 0\) such that for all \(t \geq T\),

\[
||\Gamma(x^1) - \Gamma(x^2)||_t \leq K_4 \int^t_t ||x^1 - x^2||_s ds.
\]

From here we obtain that for \(x^1, x^2 \in X_T\),

\[
||\hat{\Gamma}(x^1) - \hat{\Gamma}(x^2)||_T \leq K_4 \int^T_T ||x^1_{(1/4)} - x^2_{(1/4)}||_s ds.
\]

Suppose \(x^n, x \in X_T\) and \(||x^n - x||_T \to 0\). Then, for all \(s \geq T\), \(||x^n_{(1/4)} - x_{(1/4)}||_s \leq ||x^n - x||_T \to 0\). On the other hand, \(||x^n_{(1/4)}||_s\) and \(||x_{(1/4)}||_s\) are bounded by \(\frac{\eta}{4} e^{-\theta s}\) on \([T, \infty)\). This implies that \(||\hat{\Gamma}(x^n) - \hat{\Gamma}(x)||_T \to 0\) by the dominated convergence theorem. Hence, the map \(\hat{\Gamma} : X_T \to X_T\) is continuous.

**Step 4. Compactness of \(\hat{\Gamma}\).** Let us show that \(\hat{\Gamma} : X_T \to X_T\) is a compact map. Note that if \(x \in \Xi_T\), by (26), \(y(t) = \psi(t)O(||x||_t)\). A more careful consideration than (28) and the definition (24) of the map \(\Gamma\) show that

\[
\Gamma(x)(t) = \theta_2 \int^t_t e^{-\theta xs}(O(||x||_t)(1+\bar{x}(s))^q - 1 + o(1))ds.
\]

Note that \(o(1)\) in the latter formula expresses a function which tends to zero uniformly in \(x \in \Xi_T\). Indeed, it comes from the second expression in the first line in (27) which is uniform in \(x \in \Xi_T\). Since \(\hat{\Gamma}(x) = \Gamma(x_{(1/4)})\), there exists a constant \(K > 0\) such that

\[
|\hat{\Gamma}(x)(t)| \leq K e^{-\theta t} \quad \text{for all } x \in X_T \text{ and } t \geq T.
\]

It is sufficient to prove that for any \(\varepsilon > 0\), for the family of functions \(\{\hat{\Gamma}(x)\}_{x \in X_T}\), there is a finite \(\varepsilon\)-net. Note that the family \(\{\hat{\Gamma}(x)\}_{x \in X_T}\) is uniformly bounded and equicontinuous.
Let $\varepsilon > 0$ be arbitrary. Find $S > T$ such that $Ke^{-\varepsilon t} < \frac{\varepsilon}{2}$ for $t \geq S$. By the Arzelà-Ascoli theorem, find a finite $\frac{\varepsilon}{2}$-net for the family of functions $\{\hat{\Gamma}(x)\}_{x \in X_T}$ restricted to $[T, S]$; moreover, choose this $\frac{\varepsilon}{2}$-net from the functions of the aforementioned family. Next, we extend each function forming the $\frac{\varepsilon}{2}$-net on $[T, S]$ to $[T, +\infty)$ in such a way that its modulus is bounded by $Ke^{-\varepsilon t}$. The extended functions form an $\varepsilon$-net for the family $\{\Gamma(x)\}_{x \in X_T}$ considered on $[T, +\infty)$. Hence, $\Gamma : X_T \to X_T$ is a compact map.

**Step 5. Existence of a fixed point of $\Gamma$.** By Schauder’s fixed point therem (Theorem 5.1), the map $\hat{\Gamma} : X_T \to X_T$ has a fixed point $x^0 \in X_T$. Let $x^0$ be associated with $x^0$ by (20). Choose $S > T$ such that $\|x^0\|_S \leq \frac{1}{4}$. Then, $x^0 = x^0(t)$ for $t \geq S$. Hence, for all $t \geq S$, $\Gamma(x^0)(t) = x^0(t)$. Without loss of generality, we can set $S = T$ again, so $x^0$ is a fixed point of the map $\Gamma$. The proof is now complete. □

**Proposition 5.3.** Assume (A1)–(A4). Then, the solution $v^*$, constructed in Theorem 5.2, can be extended to a solution for (17) on $(-\infty, +\infty)$. Moreover, the extended solution $v^*$ is strictly increasing and there exists $T^* \in (-\infty, T)$ such that $v^*(T^*) = 0$.

**Proof.** The proof follows the scheme of the proof of Proposition 2.2 in [19]. However, it naturally differs from the latter due to a different nonlinearity, so we present the proof here for the reader’s convenience.

Equation (17) immediately implies that $v''(t) > 0$ for any possible extension of $v^*$. Let $w(t) = e^{(\beta+1-\alpha)t}v^*(t)^{\beta+1}$. Then, we obtain the following system with respect to $w$ and $v^*$:

$$
\begin{align*}
\begin{cases}
  w'(t) &= -e^{-(\gamma+1)t} \frac{h(t)}{g(v^*(t))}, \\
  v^*(t) &= e^{(\frac{\alpha}{\beta+1}-1)t}w^{\frac{1}{\beta+1}}.
\end{cases}
\end{align*}
$$

(31)

Pick $S > T$ and define $c_1 = w(S)$. Integrating the first equation in (31) gives

$$
 w(t) = w(S) + \int_t^S e^{-(\gamma+1)s} \frac{h(s)}{g(v^*(s))} ds, \quad t \leq S.
$$

(32)

By (A4), on $(-\infty, S]$, $c_1 \leq w(t) \leq c_1 + c_2 e^{-(\gamma+1)t} = c_t$ for some constant $c_2 > 0$. Define $c_0 = v^*(S)$, $\nu = 1_{(-\infty, c_0+c_2]} * \rho_\varepsilon$, and $\varsigma_t = 1_{[c_1, c_1+c_2]} * \rho_\varepsilon$, where $\rho_\varepsilon, \varepsilon \leq c_2$, is a standard mollifier supported on the ball of radius $\varepsilon$. Note that $\nu(\cdot) = 1$ on $(-\infty, c_0]$ and $\varsigma_t(\cdot) = 1$ on $[c_1, c_t]$. Instead of (31), consider the system of first-order ODEs on $(-\infty, S]$

$$
\begin{align*}
\begin{cases}
  w'(t) &= -e^{-(\gamma+1)t} \frac{h(t)v^*(t)}{g(v^*(t))}, \\
  v^*(t) &= e^{(\frac{\alpha}{\beta+1}-1)t}(\varsigma_t(w(t))^{\frac{1}{\beta+1}}.
\end{cases}
\end{align*}
$$

(33)

On $[T, S]$, $(v^*(t), w(t))$ is also a solution to (33), since over this interval $\nu(v^*(t)) = 1$ and $\varsigma_t(w(t)) = 1$. Further, since $\frac{h(t)v^*(t)}{g(v^*(t))} \leq \sup h(t)$ \(\frac{1}{g(c_0+2\varepsilon)}\) and $0 < (\varsigma_t(w(t))^{\frac{1}{\beta+1}} \leq (c_t + 2\varepsilon)^{\frac{1}{\beta+1}}$, one can extend $(v^*, w)$ to $(-\infty, S]$ (see, e.g., [7], Chapter 2, §6), obtaining by this a solution $(\tilde{v}, \tilde{w})$ to (33) which coincides with $(v^*, w)$ on $[T, S]$. Since on $(-\infty, S]$, $\nu(\tilde{v}) = 1$ and $\varsigma_t(\tilde{w}) = 1$, $(\tilde{v}, \tilde{w})$ is also a solution to (31). Hence, we write $(v^*, w)$ (instead of $(\tilde{v}, \tilde{w})$) for the extended solution. There are two possible situations: either
v* is strictly positive over (−∞, S], or there exists T* ∈ R such that v* > 0 on (T*, S] and v∗′(T*) = 0. Let us show that the first situation cannot be realized. If v* is strictly positive over (−∞, S], then there exists a finite limit L = limt→−∞ v*(t). We use equation (32) with t = −R and S = 0, where R > 0 is sufficiently large. Since \( \frac{h(t)}{g(v^*)} > \inf \frac{h(t)}{g(L)} \), we obtain that

\[
v'^*(-R)^{\beta+1} \geq e^{-(\alpha-\beta-1)R}v^*(0)^{\beta+1} + \frac{\inf h(t)}{(\gamma+1)g(L)}(e^{\theta R} - e^{-(\alpha-\beta-1)R}).
\]

Therefore, \( v'^*(-R) \to +\infty \) as \( R \to +\infty \). This implies that \( v^*(-R) = v^*(0) - \int_{-R}^{\infty} v'^*(t)dt \to -\infty \) as \( R \to +\infty \). This contradicts the fact that \( \lim_{t \to -\infty} v^*(t) = L < +\infty \). Thus, we conclude that the solution \( v* \) to (17) can be extended to \( \mathbb{R} \) in such a way that it crosses the x-axis at some point \( T* \in \mathbb{R} \).

**5.3 Proof of Theorem 1.7**

**Proof.** Step 1. Expressing \( u^*(r) \) via \( v^*(t) \). Let \( T^* \) be as in Proposition 5.3. Recall that equation (16) was obtained by the change of variable \( r = \lambda^* \frac{q}{\theta} e^{-t} \). Since \( v^*(T^*) = 0 \), to satisfy the boundary condition \( u^*(1) = 0 \), we choose \( \lambda^* \) in such a way that \( (\lambda^*)^{-1} \frac{q}{\theta} e^{-T^*} = 1 \), i.e., \( \lambda^* = e^{-\theta T^*} \). The corresponding change of variable then becomes \( r = e^{T^*-t} \). Therefore, \( u^*(r) = v^*(T^* - \ln r) \) is the touchdown solution, solving the equation in \( (\tilde{P}_\lambda) \) with \( \lambda = \lambda^* \) and satisfying the boundary condition \( u^*(1) = 0 \).

Step 2. Asymptotic behavior of \( v^*(t) \). By (18), Taylor’s formula, and Remark 1.2,

\[
v^*(t) = g^{-1}(ke^{-\theta t}) + x(t) \int_0^1 \frac{d\xi}{g'(g^{-1}(ke^{-\theta \xi} + \xi x(t)))} = g^{-1}(ke^{-\theta t}) - \frac{e^{-\theta t} o(1)}{Ak\theta e^{-\theta qt}}
\]

\[
1 + ke^{-\theta t} \int_0^1 \frac{d\xi}{g'(g^{-1}(\xi ke^{-\theta \xi}))} + o(e^{-\theta t(1-\gamma)t}) = 1 - \frac{k e^{-\theta t}}{Ak\theta e^{-\theta qt}} \int_0^1 \frac{d\xi}{\xi^q} + o(e^{-\theta t(1-\gamma)t})
\]

\[
= 1 - \frac{k^{1-q}}{A(1-q)} e^{-\theta t(1-q)t} + o(e^{-\theta t(1-\gamma)t}) (t \to +\infty).
\]

Step 3. Obtaining (3). Since \( e^{-t} = r e^{-T^*} \) and \( \lambda^* = e^{-\theta T^*} \), we obtain that \( e^{-\theta t(1-\gamma)t} = r^{\theta t(1-\gamma)(\lambda^*)^{x(1-\gamma)}} \). This implies (3).

Step 4. \( \lambda^* \) is the pull-in voltage. This follows immediately from Proposition 3.5. □

**Remark 5.4.** Note that by uniqueness of a touchdown solution (Proposition 3.5), the solution \( u^* \), constructed above, is the unique integral touchdown solution to \( (\tilde{P}_\lambda) \).

**Acknowledgements**

J.M. do Ó acknowledges partial support from CNPq through the grants 312340/2021-4 and 429285/2016-7 and Paraíba State Research Foundation (FAPESQ), grant no 3034/2021. E. Shamarova acknowledges partial support from Universidade Federal da Paraíba (PROPESQ/PRPG/UFPB) through the grant PIA13631-2020 (public calls no 03/2020 and 06/2021).
References

[1] D. Castorina, P. Esposito, and B. Sciuinzi. p-MEMS equation on a ball. Methods Appl. Anal., 15(3):277–283, 2008.
[2] D. Castorina, P. Esposito, and B. Sciuinzi. Degenerate elliptic equations with singular nonlinearities. Calc. Var. Partial Differential Equations, 34(3):279–306, 2009.
[3] P. Clément, D. G. de Figueiredo, and E. Mitidieri. Quasilinear elliptic equations with critical exponents. Topol. Methods Nonlinear Anal., 7(1):133–170, 1996.
[4] M. G. Crandall and P. H. Rabinowitz. Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems. Arch. Ration. Mech. Anal., 58(3):207–218, 1975.
[5] J. Dávila. Singular solutions of semi-linear elliptic problems. Handbook of differential equations: stationary partial differential equations. Vol. VI, Handb. Differ. Equ., 83–176. Elsevier/North-Holland, Amsterdam, 2008.
[6] P. Esposito, N. Ghoussoub, and Y. Guo. Mathematical analysis of partial differential equations modeling electrostatic MEMS, volume 20 of Courant Lecture Notes in Mathematics. Courant Inst. Math. Sci., New York; American Mathematical Society, Providence, RI, 2010.
[7] A.F. Filippov, Introduction to the theory of differential equations (Vvedenie v teoriyu differencialnih uravneniiy, in Russian), KomKniga, 2007.
[8] Y. Guo, Z. Pan, M.J. Ward, Touchdown and pull-in voltage behavior of a MEMS device with varying dielectric properties, SIAM J. Appl. Math. 66, 309–338, 2005.
[9] J.-S. Guo, P. Souplet, No touchdown at zero points of the permittivity profile for the MEMS problem, SIAM J. Math. Anal. 47, 614–625, 2015.
[10] Y. Guo, On the partial differential equations of electrostatic MEMS devices III: refined touchdown behavior, J. Differential Equations 244, 2277–2309, 2008.
[11] M. Ghergu and Y. Miyamoto, Radial single point rupture solutions for a general MEMS model. Calc. Var. Partial Differential Equations, 61, 47, 2022.
[12] J. Jacobsen and K. Schmitt. The Liouville-Bratu-Gelfand problem for radial operators. J. Differential Equations, 184(1):283–298, 2002.
[13] J. Jacobsen and K. Schmitt. Radial solutions of quasilinear elliptic differential equations. Handbook of differential equations, pages 359–435. Elsevier/North-Holland, Amsterdam, 2004.
[14] N. Kavallaris, T. Miyasita, and T. Suzuki, Touchdown and related problems in electrostatic MEMS device equation, NoDEA Nonlinear Differential Equations Appl. 15, 363–385, 2008.
[15] P. Korman, Infinitely many solutions for three classes of self-similar equations with p-Laplace operator: Gelfand, Joseph-Lundgren and MEMS problems, Proc. Roy. Soc. Edinburgh Sect. A 148, 341–356, 2018.
[16] J. M. do Ó and R. G. Clemente. Some elliptic problems with singular nonlinearity and advection for Riemannian manifolds. J. Math. Anal. Appl., 460(2):582–609, 2018.
[17] J. M. do Ó and E. da Silva, Quasilinear elliptic equations with singular nonlinearity. Adv. Nonlinear Stud., 16(2):363–379, 2016.
[18] J. M. do Ó and E. da Silva. Some results for a class of quasilinear elliptic equations with singular nonlinearity. Nonlinear Anal., 148:1–29, 2017.
[19] J.M. do Ó, E. Shamaraova, and E. da Silva, Singular solutions to k-Hessian equations with fast-growing nonlinearities, Nonlinear Anal. 222, Paper no 113000, 2022.
[20] J. A. Pelesko and D. H. Bernstein. Modeling MEMS and NEMS. Chapman & Hall/CRC Math., Boca Raton, FL, 2003.
[21] D.R. Smart. Fixed point theorems. Cambridge University Press, 1980.
(R. Clemente) Departamento de Matemática, Universidade Federal Rural de Pernambuco, 52171-900, Recife, Pernambuco, Brazil
Email address: rodrigo.clemente@ufrpe.br

(*Corresponding author: J.M. do Ó) Departamento de Matemática, Universidade Federal da Paraíba, 58051-900, João Pessoa, Brazil
Email address: jmbo@pq.cnpq.br

(E. da Silva) Departamento de Matemática, Universidade Federal do Rio Grande do Norte, 59078-970, Natal, Brazil
Email address: esteban.silva@ufrn.br

(E. Shamarova) Departamento de Matemática, Universidade Federal da Paraíba, 58051-900, João Pessoa, Brazil
Email address: evelina.shamarova@academico.ufpb.br