A CUBICAL MODEL FOR $(\infty, n)$-CATEGORIES

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ABSTRACT. We propose a new model for the theory of $(\infty, n)$-categories (including the case $n = \infty$) in the category of marked cubical sets with connections, similar in flavor to complicial sets of Verity. The model structure characterizing our model is shown to be monoidal with respect to suitably defined (lax and pseudo) Gray tensor products; in particular, these tensor products are both associative and biclosed. Furthermore, we show that the triangulation functor to pre-complicial sets is a left Quillen functor and is strong monoidal with respect to both Gray tensor products.

INTRODUCTION

The theory of $(\infty, n)$-categories is becoming an important tool in a number of areas of mathematics, including manifold topology, where it is used in the definition and classification of extended topological quantum field theories [Lur09], and in derived algebraic geometry, where it is used to capture certain properties of the “category” of correspondences [GR17b, GR17a]. There are several equivalent models for this theory, including: $n$-quasicategories [Ara14], $\Theta_n$-spaces [Rez10], and $n$-fold complete Segal spaces [Bar05]. There is also the model of complicial sets, which is perhaps the most well-developed one (cf. [Ver08b, Ver07] and [RV20, App. D]), but which has not yet been shown to be equivalent to others.

In this paper, we propose a new model for the theory of $(\infty, n)$-categories, using comical sets (composition + cubical sets). Comical sets are certain marked cubical sets (having marked $n$-cubes for all values $n \geq 1$), just like complicial sets are certain marked simplicial sets. Our model allows for a particularly elegant and simple treatment of the (lax and pseudo) Gray tensor products since they are inherently cubical in nature. One can find drawings of cubes in Gray’s book [Gra74], and the simplest ways of defining the lax Gray tensor product of strict $\omega$-categories are via cubical sets [Cra95, AABS02].

Because of the obvious similarities with complicial sets, there is a natural comparison functor to marked simplicial sets. To obtain it, we extend the usual triangulation functor $T: c\Set \to s\Set$ from cubical sets to simplicial sets to a marked version $T: c\Set^+ \to PreComp$. Here $T$ is valued not in the whole category $s\Set^+$ of marked simplicial sets but in the reflective subcategory $PreComp$ of pre-complicial sets so that our results hold up to isomorphism rather than homotopy. $PreComp$ supports a model structure that is Quillen equivalent to the complicial model structure on $s\Set^+$, and the lax Gray tensor product on $s\Set^+$ is more well-behaved when restricted to $PreComp$.

With that, our main results (cf. Theorems 2.16, 3.3, 6.5, 6.10 and 7.1) can be summarized as follows:

**Theorem.** The category $c\Set^+$ of marked cubical sets carries a model structure whose cofibrations are the monomorphisms and whose fibrant objects are the comical sets. This model structure is monoidal with respect to both the lax and pseudo Gray tensor products, which are simultaneously associative and biclosed.

Furthermore, the triangulation functor $T: c\Set^+ \to PreComp$ is left Quillen and strong monoidal with respect to both Gray tensor products.

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We expect that $T$ is in fact a Quillen equivalence, although we have not pursued this direction in the present paper. The “special cases” of this result are known for $(\infty,0)$-categories (i.e., $\infty$-groupoids) \cite{Cis06} and $(\infty,1)$-categories \cite{DKLS20}, although these papers consider slightly different versions of the cubical site from us.

In particular, our model validates the assertions \cite[Props. 3.2.6 and 3.2.9]{GR17b}, given there without a proof. We should note however that these assertions were previously proven in \cite{Ver08b} and \cite{Mae20} in the contexts of complicial sets and 2-quasicategories respectively.

Finally, our work owes a great deal to \cite{Ste06}, where the (semi-)cubical nerves of strict $\omega$-categories are analyzed. In particular, our definition of comical open boxes in Section \ref{section:comical-sets} follows \cite[Ex. 2.9]{Ste06}.

**Organization of the paper.** We begin in Section \ref{section:background} by reviewing the necessary background on model categories, cubical sets, and complicial sets. In Section \ref{section:marked-cubical-sets} we introduce marked cubical sets, study their basic properties, and construct both the lax and the pseudo Gray tensor products. In Section \ref{section:comical-sets} we define comical sets and construct the model structure for them. As a proof of concept, we define in Section \ref{section:1-category} the homotopy 1-category of a comical set and show that it has expected properties. We then turn our attention to the comparison between the cubical and the simplicial approaches. We extend the triangulation functor to marked cubical sets in Section \ref{section:triangulation} and show that it is strong monoidal with respect to both the lax and the pseudo Gray tensor products in Section \ref{section:monoidal} and that it is a left Quillen functor in Section \ref{section:quillen}.

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1. Background

In this section we introduce the notation and collect preliminary results to be used later in the paper.

1.1. Model categories. In this subsection, we review (a special case of) the theory of Olschok \cite{Ols09} for constructing combinatorial model structures with all objects cofibrant, which generalizes the theory of Cisinski \cite{Cis06} for constructing combinatorial model structures on presheaf categories with cofibrations the monomorphisms. This theory will be used to construct the model structures for comical sets.

**Definition 1.1** \cite{Sim12}. We say a set $\Lambda$ of trivial cofibrations in a model category $\mathcal{M}$ is *pseudo-generating* if any map $f$ that has a fibrant domain and the right lifting property with respect to $\Lambda$ is a fibration.

Now fix a locally presentable category $\mathcal{K}$.

Given a bifunctor $\otimes: \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ and maps $f: A \to A'$, $g: B \to B'$ in $\mathcal{K}$, we denote

$$f \otimes g: (A' \otimes B) \coprod_{A \otimes B} (A \otimes B') \to A' \otimes B'$$

the *Leibniz $\otimes$-product* of $f$ and $g$. Similarly, for any natural transformation $\phi: F \Rightarrow G$ between endofunctors $F,G: \mathcal{K} \to \mathcal{K}$ and for any $f: A \to A'$ in $\mathcal{K}$, we denote

$$\hat{\phi}: G(A) \coprod_{F(A)} F(A') \to G(A')$$

the *Leibniz product* of $\phi$ and $f$. 

By the cellular closure of a set $S$ of maps in $\mathcal{K}$, we mean the closure of $S$ under pushouts along arbitrary maps and transfinite composition. In the rest of this subsection, assume that we are given a small set $I$ of maps in $\mathcal{K}$ whose cellular closure is precisely the monomorphisms.

**Definition 1.2.** A functorial cylinder on $\mathcal{K}$ is a functor $C: \mathcal{K} \to \mathcal{K}$ equipped with natural transformations $\partial^0, \partial^1 : \text{Id} \Rightarrow C$ and $\sigma : C \to \text{Id}$ such that $\sigma \partial^0 = \sigma \partial^1 = \text{id}$. We also write $\partial_X = [\partial_X^0, \partial_X^1] : X + X \to CX$. We say that $C$ is a cartesian cylinder if the functor $C$ preserves colimits and moreover $\partial_X$ is a monomorphism for all $X$.

**Definition 1.3.** Suppose that $\mathcal{K}$ admits a cartesian functorial cylinder $C$. Let $S$ be a set of morphisms in $\mathcal{K}$. We define $\Lambda(\mathcal{K}, I, C, S) \subseteq \text{Mor} \mathcal{K}$ to be the smallest class of morphisms containing

$$S \cup \{ \partial^0_i \mid i \in I \} \cup \{ \hat{\partial}^1_i \mid i \in I \},$$

and closed under the operation $f \mapsto \hat{\partial} f$.

**Theorem 1.4** ([Ols09] Thm. 2.2.5). Let $\mathcal{K}$ and $I$ be as above. Suppose we are given a cartesian functorial cylinder $C$ on $\mathcal{K}$ and a set $S$ of morphisms in $\mathcal{K}$. Then there exists a model structure on $\mathcal{K}$ uniquely characterized by the following properties:

- The cofibrations are the monomorphisms.
- The set $\Lambda(\mathcal{K}, I, C, S)$ is a pseudo-generating set of trivial cofibrations.

This model structure is combinatorial and left proper.

**Proposition 1.5.** Let $\mathcal{K}$ and $I$ be as above. Suppose $\mathcal{K}$ admits a model structure whose cofibrations are the monomorphisms, and a pseudo-generating set $\Lambda$ of trivial cofibrations. Suppose further that $\mathcal{K}$ is equipped with a tensor product $\boxtimes : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ that forms part of a biclosed monoidal structure. Then these data form a monoidal model structure if and only if:

- $f \circ g$ is a cofibration whenever $f, g \in I$;
- $f \circ g$ is a trivial cofibration whenever $f \in \Lambda$ and $g \in I$; and
- $f \circ g$ is a trivial cofibration whenever $f \in I$ and $g \in \Lambda$.

**Proof.** This is an instance of [Mac20] Prop. A.4. See also [Hen18] Lem. C.11. \qed

1.2. Cubical sets. We will define cubical sets as presheaves on the box category, denoted $\square$. The category $\square$ is the (non-full) subcategory of the category of posets whose objects are the posets of the form $[1]^n := \{0 \leq 1\}^n$ and whose maps are generated by the cubical operators:

- **faces** $\partial^n_{i, \varepsilon} : [1]^{n-1} \to [1]^n$ for $i = 1, \ldots, n$ and $\varepsilon = 0, 1$ given by:
  $$\partial^n_{i, \varepsilon}(x_1, x_2, \ldots, x_{n-1}) = (x_1, x_2, \ldots, x_{i-1}, \varepsilon, x_i, \ldots, x_{n-1});$$

- **degeneracies** $\sigma^n_i : [1] \to [1]^{n-1}$ for $i = 1, 2, \ldots, n$ given by:
  $$\sigma^n_i(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n);$$

- **max-connections** $\gamma^n_{i,1} : [1] \to [1]^{n-1}$ for $i = 1, 2, \ldots, n - 1$ given by:
  $$\gamma^n_{i,1}(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{i-1}, \max(x_i, x_{i+1}), x_{i+2}, \ldots, x_n);$$

- **min-connections** $\gamma^n_{i,0} : [1] \to [1]^{n-1}$ for $i = 1, 2, \ldots, n - 1$ given by:
  $$\gamma^n_{i,0}(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{i-1}, \min(x_i, x_{i+1}), x_{i+2}, \ldots, x_n);$$

We will omit the superscript $n$ when no confusion is possible.

A straightforward computation shows that cubical operators satisfy the following cubical identities.
The monomorphisms of Corollary 1.9.

The Reedy part follows immediately from Theorem 1.6. For elegance, note that sections of \( k \) with \( \partial \) are determined by their sections and all are split epimorphisms.

Alternatively, one can describe \( \square \) as the category generated by the cubical operators, subject to the cubical identities.

We also have the following normal form of morphisms in \( \square \).

**Theorem 1.6** ([GM03 Thm. 5.1]). Every map in the category \( \square \) can be factored as a composite

\[
(\partial_{k_1,\epsilon_1} \ldots \partial_{k_r,\epsilon_r})(\gamma_{j_1,\delta_1} \ldots \gamma_{j_r,\delta_r})(\sigma_{i_1} \ldots \sigma_{i_r}),
\]

where \( 1 \leq i_1 < \ldots < i_r, 1 \leq j_1 \leq \ldots \leq j_r, \) and \( k_1 > \ldots > k_r \geq 1 \). Moreover, this factorization is unique if we further impose the additional condition: if \( j_p = j_{p+1} \), then \( \delta_p = \delta_{p+1} \).

**Remark 1.7.** In particular, any composite face map can be written uniquely as \( \alpha = \partial_{k_1,\epsilon_1} \ldots \partial_{k_r,\epsilon_r} \) with \( k_1 > \ldots > k_r \). Geometrically, such \( \alpha \) is the intersection of all \( \partial_{k_r,\epsilon_r} \)'s.

This theorem allows us to prove the following key property of \( \square \):

**Proposition 1.8.** The category \( \square \) is an EZ Reedy category with the structure defined as:

- \( \text{deg}[1]^n = n \);
- \( \square^- \) is generated under composition by degeneracies and (both kinds of) connections;
- \( \square^+ \) is generated under composition by face maps.

**Proof.** The Reedy part follows immediately from Theorem 1.6. For elegance, note that sections of \( \sigma_i \) are \( \partial_{0,i} \) and \( \partial_{1,i} \); sections of \( \gamma_{i,+} \) are \( \partial_{i,0} \) and \( \partial_{i+1,0} \); and sections of \( \gamma_{i,-} \) are \( \partial_{i,1} \) and \( \partial_{i+1,1} \). Thus all maps in \( \square^- \) are determined by their sections and all are split epimorphisms.

**Corollary 1.9.** The monomorphisms of \( \text{cSet} \) are the cellular closure of the set

\[
\{ \partial \square^n \rightarrow \square^n \mid n \geq 0 \}.
\]

The category \( \square \) carries a canonical strict monoidal product \( \otimes \) given by \([1]^n \otimes [1]^m = [1]^{m+n} \) with unit given by \([1]^0 \). Note that this product is not cartesian since, for instance, there is no ‘diagonal’ map \([1]^1 \rightarrow [1]^2 \) in \( \square \). This monoidal structure leads to another characterization of our box category, due to Grandis and Mauri [GM03 §5], as a certain kind of a free monoidal category.

A cubical monoid in a monoidal category \((\mathcal{C}, \otimes, I)\) is an object \( X \) equipped with maps:

\[
\partial_0, \partial_1 : I \rightarrow X, \quad \sigma : X \rightarrow I, \quad \gamma_0, \gamma_1 : X \otimes X \rightarrow X,
\]

subject to the axioms:

\[
\sigma \partial_\varepsilon = \text{id for } \varepsilon = 0, 1; \quad \gamma_\varepsilon (\gamma_\varepsilon \otimes \text{id}_X) = \gamma_\varepsilon (\text{id}_X \otimes \gamma_\varepsilon) \text{ for } \varepsilon = 0, 1;
\]

\[
\sigma \gamma_\varepsilon = \sigma (\sigma \otimes \text{id}_X) = \sigma (\text{id}_X \otimes \sigma) \text{ for } \varepsilon = 0, 1; \quad \gamma_\varepsilon (\partial_\delta \otimes \text{id}_X) = \gamma_\varepsilon (\text{id}_X \otimes \partial_\delta) \text{ for } \varepsilon = 0, 1;
\]

\[
\gamma_\varepsilon (\partial_\delta \otimes \text{id}_X) = \partial_\delta \sigma = \gamma_\varepsilon (\text{id}_X \otimes \partial_\delta) \text{ for } \delta \neq \varepsilon.
\]

**Theorem 1.10** ([GM03 Thm. 5.2.(d)]). The box category \( \square \) is the free strict monoidal category equipped with a cubical monoid.
Having established basic properties of the box category, we can now define cubical sets and fundamental constructions on them.

**Definition 1.11.** A cubical set is a presheaf $X : \square^{\text{op}} \to \text{Set}$. A cubical map is a natural transformation of such presheaves. The category of cubical sets and cubical maps will be denoted $\text{cSet}$.

We write $\square^n$ for the cubical set represented by $[1]^n$ and call it the (generic) $n$-cube. The boundary of the $n$-cube, denoted $\partial \square^n \to \square^n$, is the maximal proper subobject of the representable $\square^n$, i.e., the union of all of its faces. The subobject of $\square^n$ given by the union of all faces except the $(i, \varepsilon)$-th one is called the $(i, \varepsilon)$-open box and denoted $\square^n_{i, \varepsilon} \to \square^n$.

The monoidal product $\otimes$ can be extended via Day convolution from $\square$ to $\text{cSet}$, making $(\text{cSet}, \otimes, \square^0)$ a biclosed monoidal category. We refer to this monoidal product as the geometric product of cubical sets.

We adopt the convention of writing the action of cubical operators on the right, e.g., the $(1,0)$-face of an $n$-cube $x : \square^n \to X$ will be denoted $x_{\partial 1,0}$.

**Proposition 1.12.** The geometric product $X \otimes Y$ of cubical sets $X$ and $Y$ admits the following description.

- For $n \geq 0$, the $n$-cubes in $X \otimes Y$ are the formal products $x \otimes y$ of pairs $x \in X_k$ and $y \in Y_\ell$ such that $k + \ell = n$, subject to the identification $(x\sigma_{k+1}) \otimes y = x \otimes (y\sigma_1)$.
- For $x \in X_k$ and $y \in Y_\ell$, the faces, degeneracies, and connections of the $(k + \ell)$-cube $x \otimes y$ are computed as follows:

  \[
  (x \otimes y)\partial_{i,\varepsilon} = \begin{cases} 
  (x\partial_{i,\varepsilon}) \otimes y & 1 \leq i \leq k \\
  x \otimes (y\partial_{i-k,\varepsilon}) & k + 1 \leq i \leq k + \ell 
  \end{cases}
  \]

  \[
  (x \otimes y)\sigma_i = \begin{cases} 
  (x\sigma_i) \otimes y & 1 \leq i \leq k + \ell \\
  x \otimes (y\sigma_{i-k}) & k + 1 \leq i \leq k + \ell + 1 
  \end{cases}
  \]

  \[
  (x \otimes y)\gamma_{i,\varepsilon} = \begin{cases} 
  (x\gamma_{i,\varepsilon}) \otimes y & 1 \leq i \leq k \\
  x \otimes (y\gamma_{i-k,\varepsilon}) & k + 1 \leq i \leq k + \ell 
  \end{cases}
  \]

**Proof.** This is proven in [DKLS20] Prop. 1.20] in the case of cubical sets with one kind of connections. The proof given there works almost verbatim in our case.

Given cubes $x \in X_k$ and $y \in Y_\ell$, we may regard them as cubical maps $x : \square^k \to X$ and $y : \square^\ell \to Y$. Then applying the geometric product to these maps yields a map $x \otimes y : \square^{k+\ell} \to X \otimes Y$ which corresponds precisely to the $(k + \ell)$-cube with the same name. Moreover, every $n$-cube of $X \otimes Y$ arises via this construction for some, perhaps non-unique, pair of cubes $(x : \square^k \to X, y : \square^\ell \to Y)$ for $k + \ell = n$.

Since the identification in Proposition 1.12 only concerns degenerate cubes, we obtain the following corollary.

**Corollary 1.13.** A pair of non-degenerate cubes $x \in X_k$, $y \in Y_\ell$ yields a non-degenerate $(k + \ell)$-cube $x \otimes y$ in $X \otimes Y$. Conversely, every non-degenerate cube in $X \otimes Y$ arises this way from a unique pair of non-degenerate cubes.\[ \]

**Remark 1.14.** In particular, when $X = \square^{m}$ and $Y = \square^n$ are representable this pairing is given by the formula

$$
(\partial_{k_1, \varepsilon_1} \ldots \partial_{k_l, \varepsilon_l}) \otimes (\partial_{\ell_1, \eta_1} \ldots \partial_{\ell_s, \eta_s}) = \partial_{m+\ell_1, \eta_1} \ldots \partial_{m+\ell_s, \eta_s} \partial_{k_1, \varepsilon_1} \ldots \partial_{k_l, \varepsilon_l}
$$

where all strings of $\partial$’s are in the normal form specified by Theorem 1.6.
Proposition 1.15. For natural numbers \(k, m,\) and \(n,\) and \(\varepsilon = 0, 1,\) we have natural isomorphisms:
\[
\begin{align*}
(\partial \partial^m \rightarrow \square^m) \hat{\otimes} (\partial \square^n \rightarrow \square^n) & \cong (\partial \partial^{m+n} \rightarrow \square^{m+n}) \\
(\varepsilon_n^m \rightarrow \square^m) \hat{\otimes} (\partial \square^n \rightarrow \square^n) & \cong (\varepsilon_{n+k,\varepsilon}^m \rightarrow \square^{m+n}) \\
(\partial \varepsilon_n^m \rightarrow \square^m) \hat{\otimes} (\varepsilon_{n+k,\varepsilon}^n \rightarrow \square^n) & \cong (\varepsilon_{m+k,m+n} \rightarrow \square^{m+n})
\end{align*}
\]
\(\square\)

Using the above proposition and the fact that \(\square\) is an elegant Reedy category, we obtain:

Corollary 1.16. If \(f\) and \(g\) are monomorphisms in \(\text{cSet},\) then \(f \circ g\) is again a monomorphism. \(\square\)

The category \(\square\) admits two canonical identity-on-objects automorphisms \((-)^{\co}, (-)^{\co\text{-op}} : \square \rightarrow \square.\) The first one takes \(\partial_{i,\varepsilon}^n\) to \(\partial_{i+1,\varepsilon}^n\), \(\sigma_i^n\) to \(\sigma_{i+1}^n\), and \(\gamma_{i,\varepsilon}^n\) to \(\gamma_{i+1,\varepsilon}^n.\) The second one takes \(\partial_{i,\varepsilon}^n\) to \(\partial_{i-1,\varepsilon}^n\), \(\sigma_i^n\) to \(\sigma_{i-1}^n\), and \(\gamma_{i,\varepsilon}^n\) to \(\gamma_{i-1,\varepsilon}^n.\) (Their names are motivated by the fact that, according to the source/target distinction described in Section 3 below, \((-)^{\co}\) reverses the direction of even-dimensional cubes and \((-)^{\co\text{-op}}\) reverses the direction of all cubes.) Precomposition with these automorphisms induces functors also denoted \((-)^{\co}, (-)^{\co\text{-op}} : \text{cSet} \rightarrow \text{cSet},\) and their “contravariant” behavior with respect to the cubical structure can be seen via its interaction with the geometric product. We write \((-)^{\op}\) for the composite of those.

Proposition 1.17.

1. The functor \((-)^{\co} : \text{cSet} \rightarrow \text{cSet}\) is strong anti-monoidal, i.e., \((X \otimes Y)^{\co} \cong Y^{\co} \otimes X^{\co}\).
2. The functor \((-)^{\co\text{-op}} : \text{cSet} \rightarrow \text{cSet}\) is strong monoidal i.e., \((X \otimes Y)^{\co\text{-op}} \cong X^{\co\text{-op}} \otimes Y^{\co\text{-op}}\).
3. The functor \((-)^{\op} : \text{cSet} \rightarrow \text{cSet}\) is strong anti-monoidal, i.e., \((X \otimes Y)^{\op} \cong Y^{\op} \otimes X^{\op}\). \(\square\)

Finally, the composite \(\square \rightarrow \text{Cat} \rightarrow \text{sSet}\) given by \(\square^n \mapsto (\Delta^1)^n\) defines a co-cubical object in the category of simplicial sets. Taking the Yoneda extension, we obtain a adjoint pair
\[
T : \text{cSet} \rightleftarrows \text{sSet} : U.
\]
We will call \(T : \text{cSet} \rightarrow \text{sSet}\) the triangulation functor.

In view of Theorem 1.10 \(T\) can also be seen as a monoidal functor associated to the cubical monoid \(\Delta^1\) in \(\text{sSet}\). We conclude this section by recording some properties of the triangulation functor.

Proposition 1.18.

1. \(T\) is strong monoidal.
2. \(T\) preserves monomorphisms.

Proof. The first statement follows by the fact that \(T\) preserves colimits and \(\text{sSet}\) is cartesian closed.

The second statement follows from first, since \(T\) takes boundary inclusions, i.e., the elements of the cellular model, to monomorphisms. \(\square\)

1.3. Complicial sets. In this section, we introduce marked simplicial sets and model structures for \((n\text{-trivial})\) complicial sets. There are many similarities between the theory recalled here and that developed in Section 3.

Just as in the case of cubical sets, when working with simplicial sets, we will write the action of simplicial operators on the right.

Definition 1.19. A marked simplicial set is a simplicial set \(X\) equipped with a subset of its simplices \(eX \subseteq \bigcup_{n \geq 1} X_n\) called the marked simplices such that
- No 0-simplex is marked, and
- Every degenerate simplex is marked.
A map of marked simplicial sets $f : X \to Y$ is a map of simplicial sets which carries marked simplices to marked simplices. We denote $\mathbf{sSet}^+$ for the category of marked simplicial sets with maps for morphisms.

Marked simplicial sets used to be called stratified simplicial sets (cf. e.g., [Ver08a]), but the name ‘marked’ is more descriptive and has since become more popular.

There is a natural forgetful functor $\mathbf{sSet}^+ \to \mathbf{sSet}$, which has both left and right adjoints. The left adjoint $X \mapsto X^\flat$ endows a simplicial set $X$ with the minimal marking, marking only the degenerate simplices. The right adjoint $X \mapsto X^\sharp$ endows a simplicial set $X$ with the maximal marking, marking all simplices.

If $X$ is a simplicial set, we will by default consider it as a marked simplicial set with its minimal marking $X^\flat$.

**Definition 1.20.** We say that $X \in \mathbf{sSet}^+$ is $n$-trivial if every simplex of dimension $\geq n + 1$ is marked.

Given a marked simplicial set $X$, we will write $\text{core}_n X$ for its maximal $n$-trivial subset. In other words, the $k$-simplices of $\text{core}_n X$ are precisely those $k$-simplices $x$ in $X$ such that $x\alpha$ is marked in $X$ for any $\alpha : [m] \to [k]$ with $m > n$. This assignment extends to a functor $\text{core}_n : \mathbf{sSet}^+ \to \mathbf{sSet}^+$, which admits a left adjoint $\tau_n : \mathbf{sSet}^+ \to \mathbf{sSet}^+$, which we call $\tau_n X$. Explicitly, $\tau_n$ acts as the identity on the underlying simplicial set and a $k$-simplex is marked in $\tau_n X$ if either $k \leq n$ and $x$ is marked in $X$ or $k \geq n + 1$.

**Definition 1.21.** A map $X \to Y$ of marked simplicial sets is:

- **regular** if it creates markings, i.e., for an $n$-simplex $x$ of $X$ we have: $x \in eX_n$ if and only if $f(x) \in eY_n$;
- **entire** if the induced map between the underlying simplicial sets is invertible.

We now define several distinguished objects and maps in $\mathbf{sSet}^+$. These will be essential to the description of various model structures we will be considering.

We denote $\Delta^n = \tau_{n-1}(\Delta^n)$ the $n$-simplex with the non-degenerate $n$-simplex marked and no other non-degenerate simplices marked. We call the canonical map $\Delta^n \to \Delta^n$ the $n$-marker.

For $n \geq 1$ and $0 \leq k \leq n$, we denote $\Delta^k_n$ the $n$-simplex with the following marking: a non-degenerate simplex is marked if it contains all of the points $\{k - 1, k, k + 1\} \cap [n]$ among its vertices. We call $\Delta^k_n$ the $k$-compound $n$-simplex. We denote $\Lambda^k_n \subset \Delta^k_n$ the $k$-horn of dimension $n$ (i.e. the simplicial subset missing the non-degenerate $n$-simplex and the $k$th $(n - 1)$-face) endowed with the marking making it a regular subset of $\Delta^k_n$. We call $\Lambda^k_n$ the compound $k$-horn of dimension $n$. We call the inclusion $\Lambda^k_n \to \Delta^k_n$ the $k$-compound horn inclusion of dimension $n$. We denote $\Delta^{n''}_k = \tau_{n-2} \Delta^{n'}_k$, and we denote $\Delta^{n''}_k = \Lambda^k_n \cup \Lambda^k_n \tau_{n-2} \Delta^k_n$. The canonical inclusion $\Lambda^k_n \to \Delta^{n''}_k$ is called the elementary $k$-compound marking extension of dimension $n$.

There are two standard model structures on marked simplicial sets:

**Theorem 1.22.** The category $\mathbf{sSet}^+$ carries the complicial model structure characterized by the following properties:

- The cofibrations are the monomorphisms.
- The set of
  - complicial horn inclusions, and
  - elementary complicial marking extensions
forms a pseudo-generating set of trivial cofibrations.

This model structure is cartesian.

**Proof.** This is a combination of [Ver08b, Thm. 100 & Lem. 105].

Note that since the terminal object is always fibrant, this includes a characterization of the fibrant objects of the model structure, which are called complicial sets.
Definition 1.23. A map of marked simplicial sets $X \to Y$ is
- a complicial marking extension if it is in the cellular closure of the elementary complicial marking extensions;
- complicial if it is in the cellular closure of the complicial horn inclusions and the elementary complicial marking extensions.

There is also the $n$-trivial version of the complicial model structure.

Theorem 1.24. The category $\text{sSet}^+$ carries the $n$-trivial complicial model structure characterized by the following properties:
- The cofibrations are the monomorphisms.
- The set of
  - complicial horn inclusions,
  - elementary complicial marking extensions of dimension $\leq n + 1$, and
  - markers of dimension $> n$
forms a pseudo-generating set of trivial cofibrations.

Proof. This is a combination of [Ver08b, Ex. 104 & Lem. 105]. □

In $\text{sSet}^+$, the pseudo Gray tensor product is modelled by the cartesian product. We will adopt the following notation from [Ver08b, Ver08b] which emphasizes this view.

Definition 1.25. The cartesian product on $\text{sSet}^+$ (and its reflective subcategory $\text{PreComp}$ described below) is denoted by $\otimes$.

Thus Theorem 1.22 in particular says that Verity’s model structure is monoidal with respect to the pseudo Gray tensor product.

The following proposition will be useful later.

Proposition 1.26. Let $f:A \to X$ and $g:B \to Y$ be entire maps in $\text{sSet}^+$. Then their Leibniz pseudo Gray tensor $f \otimes g$ is a complicial marking extension.

Proof. Since the forgetful functor $\text{sSet}^+ \to \text{sSet}$ preserves colimits, $f \otimes g$ is entire. We assume for the sake of simplicity that $f \otimes g$ is an inclusion. Let $(x,y)$ be an $n$-simplex that is marked in $X \otimes Y$ but not in $\text{dom}(f \otimes g) = (A \otimes Y) \cup (X \otimes B)$. Equivalently, $x$ is marked in $X$ but not in $A$, and $y$ is marked in $Y$ but not in $B$. Then we must have $n \geq 1$, so the $(n+1)$-simplex $z = (x\sigma_0, y\sigma_1)$ is well-defined. We claim that this simplex $z$ extends as indicated below:

$$\Delta^{n+1} \xrightarrow{z} (A \otimes Y) \cup (X \otimes B) \xrightarrow{\partial_0} \Delta^{n+1}_1$$

To see that $z$ at least extends to $\Delta^{n+1}_1$, let $\alpha:[m] \to [n+1]$ be a simplicial operator with $0,1,2 \in \text{im} \alpha$. Then both $x(\sigma_0 \circ \alpha)$ and $y(\sigma_0 \circ \alpha)$ are degenerate, so $z\alpha$ is marked in $\text{dom}(f \otimes g)$. Since the face $z\partial_0 = (x,y(\partial_0 \circ \sigma_1))$ is marked in $X \otimes B$ and the face $z\partial_2 = (x(\partial_2 \circ \sigma_0), y)$ is marked in $A \otimes Y$, we indeed have an extension as indicated. Therefore we have a pushout square

$$\biguplus \Delta^{n+1}_1' \xrightarrow{} (A \otimes Y) \cup (X \otimes B) \xrightarrow{} X \otimes Y$$

$$\biguplus \Delta^{n+1}_1'' \xrightarrow{} (A \otimes Y) \cup (X \otimes B) \xrightarrow{} X \otimes Y$$
where the coproducts are taken over all \( n\)-simplices \((x,y)\) that are marked in \(X \otimes Y\) but not in \(\text{dom}(f \otimes g)\), and the upper horizontal map is induced by the simplices of the form \((x\sigma_0, y\sigma_1)\). This completes the proof.

**Definition 1.27.** Let \([n] \in \Delta\) and let \(0 \leq p,q \leq n\) be such that \(p + q = n\). Then we write \(\llbracket_1^{p,q} : [p] \to [n]\) for the simplicial operator \(i \mapsto i\), and \(\llbracket_2^{p,q} : [q] \to [n]\) for the operator \(i \mapsto p + i\).

**Definition 1.28.** Let \(X,Y \in \sSet^*\), let \((x,y) \in X_n \times Y_n\) be a simplex of \(X \times Y\), and let \(0 \leq i \leq n\). We say that \((x,y)\) is \(i\)-cloven if either \(x \llbracket_1^{i,n-i}\) is marked in \(X\) or \(y \llbracket_2^{i,n-i}\) is marked in \(Y\). We say that \((x,y)\) is fully cloven if it is \(i\)-cloven for all \(0 \leq i \leq n\).

The Gray tensor product of \(X\) and \(Y\), denoted \(X \otimes Y\), is defined to be the marked simplicial set with underlying simplicial set \(X \times Y\), where a simplex \((x,y) \in X_n \times Y_n\) is marked if and only if it is fully cloven.

**Theorem 1.29** ([Ver08a, Lem. 131]). The Gray tensor product endows \(\sSet^*\) with a (nonsymmetric) monoidal structure, such that the forgetful functor \((\sSet^*, \otimes) \to (\sSet, \times)\) is strict monoidal.

**Definition 1.30.** A pre-complicial set is a marked simplicial set \(X\) with the right lifting property with respect to the complicial marking extensions. These form a reflective subcategory of \(\sSet^*\) which we will denote \(\PreComp\). We will denote the localization functor \(X \mapsto X^{\text{pre}}\).

**Theorem 1.31.** The category \(\PreComp\) carries the following model structures:

1. the complicial model structure characterized by the following conditions:
   - The cofibrations are the monomorphisms.
   - The set of pre-complicial reflections of complicial horn inclusions forms a pseudo-generating set of trivial cofibrations.
2. the \(n\)-trivial complicial model structure characterized by the following conditions:
   - The cofibrations are the monomorphisms.
   - The set of pre-complicial reflections of complicial horn inclusions, and pre-complicial reflections of markers of dimension \(\geq n\) forms a pseudo-generating set of trivial cofibrations.

The localization \((\cdot)^{\text{pre}}\) is a left Quillen equivalence between the complicial model structures (resp. the \(n\)-trivial complicial model structures).

**Proof.** The proofs of both cases are analogous, so we only prove (1). We use [HKRS17, Prop. 2.1.4] to right induce the model structure along the inclusion \(\PreComp \to \sSet^*\). It suffices to show that the reflections of the complicial maps are weak equivalences. This is true by 2-out-of-3 because the unit of the localization is a natural weak equivalence (indeed, it is a complicial marking extension).

It follows that the adjunction \(\sSet^* \rightleftarrows \PreComp\) is a Quillen adjunction. Finally, since the unit is a natural weak equivalence this is in fact a Quillen equivalence. \(\square\)

Now we analyse the restriction of the Gray tensor product on \(\PreComp\).

**Theorem 1.32.** \(\PreComp\) is closed under the Gray tensor product \(\otimes\).

**Proof.** Let \(X,Y \in \PreComp\) and suppose we are given a map \((x,y) : \Delta^n_{\text{op}} \to X \otimes Y\).

Suppose for contradiction that \((x,y)\delta_k = (x\delta_k, y\delta_k)\) is unmarked. Then there exists a partition \(p + q = n - 1\) such that \((x\delta_k) \llbracket_1^{p,q}\) is unmarked in \(X\) and \((y\delta_k) \llbracket_2^{p,q}\) is unmarked in \(Y\).
Proposition 2.1. The category $\text{sSet}$ subject to the usual cubical identities and the following additional relations: every $\text{PreComp}$ on $\Delta^p_q$. Let $\alpha : [r] \to [p + 1]$ be a face operator such that $\{k - 1, k, k + 1\} \cap [p + 1] \subset \text{im}(\alpha)$. Then we have $\{k - 1, k, k + 1\} \cap [n + 1] \subset \text{im}(\alpha \ast \text{id}_{\{q-1\}})$ where $\ast$ denotes the join operation. So our assumption implies that $(x, y)(\alpha \ast \text{id}_{\{q-1\}})$ is marked in $X \otimes Y$. Since $(y(\alpha \ast \text{id}_{\{q-1\}})) \uparrow_{2,q} = y \uparrow_{2,q} \ast \alpha$ is marked in $X$. Similarly, $z \delta_k$ is marked for $\ell \in \{k - 1, k, k + 1\} \cap [p + 1]$. Thus we obtain $z : \Delta^{p+1}_k \to X$. But since $X$ is pre-complicial, this map must further extend to $\Delta^{p+1}_k$. Equivalently, $z \delta_k = (x \uparrow_{1,q} \ast \delta_k) \uparrow_{1,q}$ must be marked in $X$, which is our desired contradiction. \hfill \Box

Theorem 1.33. The complicial model structure (resp. $n$-trivial complicial model structure) on $\text{PreComp}$ is monoidal with respect to the Gray tensor product $\otimes$. In particular, this monoidal structure is biclosed.

Proof. The (reflection of the) Gray tensor product $\otimes$ on $\text{PreComp}$ is proven to be biclosed in [Ver08a, Thm. 148]. It is straightforward to check that the Leibniz Gray tensor product preserves monomorphisms. Since the complicial model structure on $\text{sSet}^+\ast$ is monoidal with respect to the Gray tensor product [Ver08b, Thm. 109] (although it is not biclosed on $\text{sSet}^+\ast$) and the unit of the pre-complicial reflection is a levelwise complicial marking extension, it follows that the complicial model structure on $\text{PreComp}$ is monoidal. To extend this fact to the $n$-trivial model structure, observe that the Leibniz Gray tensor product of the $m$-marker with any monomorphism is in the cellular closure of the $m'$-markers for $m' \geq m$. \hfill \Box

2. Marked cubical sets and Gray tensor products

In this section, we introduce marked cubical sets and define their Gray tensor product.

2.1. Marked cubical sets. In order to define marked cubical sets, we need to introduce certain enlargement $\square^+$ of the box category. The objects of $\square^+$ consist of: $[1]^n$ for every $n \geq 0$ and $[1]^n_\varepsilon$ for every $n \geq 1$. The morphisms of $\square^+$ are generated by the maps

- $\partial^+_i : [1]^{n-1} \to [1]^n$ for every $n \geq 1$, $i = 1, \ldots, n$, and $\varepsilon = 0, 1$,
- $\sigma^+_i : [1]^n \to [1]^{n+1}$ for every $n \geq 1$, $i = 1, \ldots, n$,
- $\gamma_i^n : [1]^n \to [1]^{n-1}$ for every $n \geq 2$, $i = 1, \ldots, n - 1$, and $\varepsilon = 0, 1$,
- $\phi^n : [1]^n \to [1]^n_\varepsilon$ for every $n \geq 1$,
- $\zeta_i^n : [1]^n_\varepsilon \to [1]^{n-1}$ for every $n \geq 1$, $i = 1, \ldots, n$,
- $\xi_i^n : [1]^n_\varepsilon \to [1]^{n+1}$ for every $n \geq 1$, $i = 1, \ldots, n$, and $\varepsilon = 0, 1$,

subject to the usual cubical identities and the following additional relations:

- $\zeta_i \varphi = \sigma_i$;
- $\xi_i \varepsilon \varphi = \gamma_i, \varepsilon$;
- $\sigma_i \zeta_j = \sigma_j \zeta_{i+1}$ for $i \leq j$;
- $\gamma_j, \varepsilon \zeta_i, \delta = \left\{ \begin{array}{ll} \gamma_i, \delta \zeta_j, \varepsilon & \text{for } j > i; \\
\gamma_i, \delta \zeta_{i+1}, \varepsilon & \text{for } j = i \text{ and } \delta = \varepsilon; \end{array} \right.$
- $\sigma_j \zeta_i, \delta = \left\{ \begin{array}{ll} \gamma_i, \delta \zeta_j & \text{for } j < i; \\
\sigma_i \zeta_j & \text{for } j = i; \\
\gamma_i, \delta \zeta_{j+1} & \text{for } j > i. \end{array} \right.$

Proposition 2.1. The category $\square^+$ is an EZ Reedy category with the following Reedy structure:

- $\deg[1]^0 = 0$; $\deg[1]^n = 2n - 1$ for $n \geq 1$; $\deg[1]^n_\varepsilon = 2n$ for $n \geq 1$;
- $\square^+$ is generated by the maps: $\sigma_i^n$, $\gamma_i, \varepsilon^n$, $\zeta_i^n$, and $\xi_i^n$;
- $\square^-$ is generated by the maps: $\partial^+_i \varepsilon$ and $\varphi^n$. 

The proof of this fact follows the same steps as established in [OR18, App. C]. We begin by noting the following simple lemma:

**Lemma 2.2.**

1. There are no non-identity maps in $\Box^+$ whose target is in $\Box^+ \setminus \Box$.
2. There are no non-identity maps in $\Box^+_i$ whose source is in $\Box^+ \setminus \Box$. □

**Proof of Proposition 2.7.** We first note that the sections of $\zeta_i$ are $\varphi \partial_{i,0}$ and $\varphi \partial_{i,1}$; the sections of $\xi_{i,1}$ are $\varphi \partial_{i,0}$ and $\varphi \partial_{i,1,0}$; and the sections of $\xi_{i,0}$ are $\varphi \partial_{i,1}$ and $\varphi \partial_{i+1,1}$. This proves the Eilenberg–Zilber property, and hence we only need to show the existence and uniqueness of factorizations. We consider four cases, depending on whether or not the domain and codomain of the map to be factored belong to $\Box \subseteq \Box^+$.

1. $[1]^m \to [1]^n$. The existence of a factorization follows from the existence of such a factorization in $\Box$. If such a factorization were to be non-unique, we would need to have $[1]^m \to [1]^k \to [1]^n$ with $[1]^k \to [1]^n \in \Box^+_i$, which is impossible by Lemma 2.2.2.

2. $[1]^m \to [1]^n$. We obtain the factorization by taking $[1]^m \to [1]^n \overset{\varphi}{\to} [1]^n$ and factoring the first map in $\Box$. Again, Lemma 2.2 implies uniqueness.

3. $[1]^m \to [1]^n$. Such a map must factor as $[1]^m \to [1]^n \to [1]^k \to [1]^n$, where the first map is either $\zeta_i$ or $\xi_{i,e}$, and the remaining two are obtained by factoring the composite $[1]^m \to [1]^n$ in $\Box$. Of course, the choice of $\zeta_i$ or $\xi_{i,e}$ may not be unique, but the composite $[1]^m \to [1]^n \to [1]^k$ must be — this is because of the additional relations relating $\zeta_i$’s to $\sigma_i$’s and $\xi_{i,e}$’s to $\gamma_{i,e}$’s. In light of Lemma 2.2.1, this proves the uniqueness.

4. $[1]^m \to [1]^n$. In this case, we obtain the factorization by combining the techniques from the previous two cases, namely

$$[1]^m \to [1]^m \to [1]^k \to [1]^n \overset{\varphi}{\to} [1]^n,$$

where again the first map is one of $\zeta_i$ or $\xi_{i,e}$ and the composite $[1]^m \to [1]^k \to [1]^n$ is obtained in $\Box$. Since there are no relations of the form $\varphi \alpha = \beta$, this reduces the proof of uniqueness to the previous case. □

**Definition 2.3.** A structurally marked cubical set is a presheaf $X : (\Box^+)^{op} \to \text{Set}$. A map of structurally marked cubical sets is a natural transformation of such presheaves.

Given a structurally marked cubical set $X$, we will write $X_n$ for $X([1]^n)$ and $eX_n$ for $X([1]^n)$. Just as in the case of cubical sets, we adopt the convention of writing cubical operators on the right, e.g., for $x \in eX_1$, we write $x\varphi$ for the resulting element of $X_1$.

**Definition 2.4.** A marked cubical set is a structurally marked cubical set $X : (\Box^+)^{op} \to \text{Set}$ for which the map $\varphi^* : eX_n \to X_n$ is a monomorphism for all $n \geq 1$. We write $\text{cSet}^+$ for the full subcategory of $\text{Set}((\Box^+)^{op})$ spanned by the marked cubical sets.

We think of a marked cubical set $X$ as a cubical set in which certain $n$-cubes have been designated as equivalences, i.e., those in $eX_n \subseteq X_n$. The maps $\zeta_i$ and $\xi_{i,e}$ ensure that every degenerate cube is marked.

We may apply the same intuition to structurally marked cubical sets. However, failure of $\varphi^n$’s to be monomorphisms means that being an equivalence is not a property of an $n$-cube, but a structure, as there can be multiple markings on it.

Every (structurally) marked cubical set has an underlying cubical set, which defines a functor $v : \text{cSet}^+ \to \text{cSet}$. Given a cubical set $X$, we can form a marked cubical set in two ways:

- the minimal marking functor takes a cubical set $X$ to a marked cubical set $X^!$, where only degenerate $n$-cubes are marked;
the maximal marking functor, assigns to \( X \) the marked cubical set \( X^1 \) in which all cubes marked (i.e., \( \varphi_n \)'s are identities).

This gives two functors \( (-)^\natural, (-)^\natural: cSet \to cSet^+ \). A straightforward verification shows:

**Proposition 2.5.** We have the following string of adjoint functors \( (-)^\natural \dashv \nu \dashv (-)^\natural \).

**Remark 2.6** (Limits and colimits of marked cubical sets). The proposition above gives a recipe for computing limits and colimits of diagrams \( F: \mathcal{I} \to cSet^+ \). In both cases, we first compute the underlying cubical set by taking the (co)limit of \( vF \) in \( cSet \), and then equipping it with the minimal marking making the colimit inclusions maps of marked cubical sets, or the maximal marking making the limit projects maps of marked cubical sets. It follows, e.g., that a cube in a colimit is marked if and only if it is in the image of a marked cube under one of the colimit inclusions.

Furthermore, the canonical embedding \( cSet^+ \to Set^{(G^+)^{op}} \) of marked cubical sets into structurally marked cubical sets admits a left adjoint, denoted \( \text{Im}: Set^{(G^+)^{op}} \to cSet^+ \). Explicitly, \( \text{Im} X \) is obtained by factoring all \( \varphi_n \)'s via their image \( eX_n \to (eX_n)\varphi_n \to X_n \) and taking the resulting object as a new set of marked \( n \)-cubes. We may summarize it with the following statement:

**Proposition 2.7.** Marked cubical sets form a reflective subcategory of the structurally marked cubical sets with the reflector given by \( \text{Im}: Set^{(G^+)^{op}} \to cSet^+ \).

**Corollary 2.8.** The category \( cSet^+ \) of marked cubical sets is locally presentable.

**Definition 2.9.** A map \( f: X \to Y \) of marked cubical sets is:

- **regular** if it creates markings, i.e., for an \( n \)-cube \( x \) of \( X \) we have: \( x \in eX_n \) if and only if \( f(x) \in eY_n \);
- **entire** if the induced map between the underlying cubical sets is invertible.

When a cubical set is considered as a marked cubical set, it will almost always be considered with its minimal marking. The only exception is the open boxes; see Section 3. We denote \( \square^n = (\square^n)^{\natural} \) the \( n \)-cube regarded as a marked cubical set and likewise \( \partial \square^n = (\partial \square^n)^{\natural} \). Just as in the case of cubical sets, we call the inclusion map \( \partial \square^n \to \square^n \) the boundary inclusion. We denote \( \tilde{\square}^n = \tau_{n-1}(\square^n) \) the \( n \)-cube with the nondegenerate \( n \)-cube marked and no other non-degenerate cubes marked. We call the canonical map \( \square^n \to \tilde{\square}^n \) the \( n \)-marker.

**Proposition 2.10.** The monomorphisms of \( cSet^+ \) (and \( Set^{(G^+)^{op}} \)) are the cellular closure of the set

\[ \{ \partial \tilde{\square}^n \to \square^n \mid n \geq 0 \} \cup \{ \tilde{\square}^n \to \tilde{\square}^n \mid n \geq 1 \}. \]

**Definition 2.11.** We say that \( X \in cSet^+ \) is \( n \)-trivial if every cube of dimension \( \geq n + 1 \) is marked.

Given a marked cubical set \( X \), we will write \( \text{core}_n X \) for its maximal \( n \)-trivial subset. In other words, the \( k \)-cubes of \( \text{core}_n X \) are precisely those \( k \)-cubes \( x \) such that \( x\alpha \) is marked for all \( \alpha: [1]^m \to [1]^k \) with \( m > n \). This assignment extends to a functor \( \text{core}_n: cSet^+ \to cSet^+ \), which admits a left adjoint \( \tau_n: cSet^+ \to cSet^+ \). Explicitly, \( \tau_n \) acts as the identity on the underlying cubical set and a \( k \)-cube is marked in \( \tau_n X \) if either \( k \leq n \) and \( x \) is marked in \( X \) or \( k \geq n + 1 \).

The functors \( (-)^{co}, (-)^{co-op}, (-)^{op}: cSet \to cSet \) generalize to the marked setting in the straightforward manner. For \( (-)^{co} \) we send \( \varphi^n \) to itself, \( \zeta^n_1 \) to \( \zeta^n_{1-i} \), and \( \xi^n_1 \) to \( \xi^n_{1-i} \). For \( (-)^{co-op} \), we send \( \varphi^n \) and \( \zeta^n_1 \) to themselves and \( \xi^n_1 \) to \( \xi^n_{1-i} \). These then induce functors by precomposition \( (-)^{co}, (-)^{co-op}, (-)^{op}: cSet^+ \to cSet^+ \).
2.2. Gray tensor products.

**Definition 2.12.** The (lax) Gray tensor product $X \otimes Y$ of two marked cubical sets $X, Y \in \text{cSet}^*$ is the geometric product $vX \otimes vY$ wherein a non-degenerate cube $x \otimes y$ is marked if and only if either $x$ is marked in $X$ or $y$ is marked in $Y$. This extends to a functor $\otimes : \text{cSet}^+ \times \text{cSet}^+ \to \text{cSet}^+$ in the obvious way.

**Definition 2.13.** The pseudo Gray tensor product $X \odot Y$ is the geometric product $vX \odot vY$ wherein a non-degenerate cube $x \odot y$ is unmarked if and only if:

- $x$ is a 0-cube and $y$ is unmarked in $Y$; or
- $x$ is unmarked in $X$ and $y$ is a 0-cube.

This extends to a functor $\odot : \text{cSet}^+ \times \text{cSet}^+ \to \text{cSet}^+$ in the obvious way.

**Remark 2.14.** Since no 0-cubes are marked, one can easily check that $X \odot Y$ may be obtained from $X \otimes Y$ by marking those $x \otimes y$ such that $x \in X_m$, $y \in Y_n$ with $m, n \geq 1$. Thus the identity at $vX \otimes vY$ lifts to an entire map $\mu_{X,Y} : X \otimes Y \to X \odot Y$. This map is clearly natural in $X$ and $Y$, and moreover $\mu_{X,Y}$ is invertible if either $X$ or $Y$ is 0-trivial.

**Remark 2.15.** The Gray tensor products $\otimes$ and $\odot$ share many properties and often a statement or a proof applies equally well to both tensor products. In such situations, we write $\otimes$ to mean either. Of course the interpretation of $\otimes$ is to be kept consistent within each statement and its proof.

**Theorem 2.16.** The Gray tensor product $\otimes$ forms part of a biclosed monoidal structure on $\text{cSet}^+$ such that the forgetful functor $v : (\text{cSet}^+, \otimes) \to (\text{cSet}, \odot)$ is strict monoidal. The entire inclusions $\mu_{X,Y} : X \otimes Y \to X \odot Y$ together with $\mu_0 = \text{id}_{\text{cSet}^+}$ equips the identity functor with a monoidal structure $(\text{id}_{\text{cSet}^+}, \mu) : (\text{cSet}^+, \otimes) \to (\text{cSet}^+, \odot)$.

**Proof.** Suppose we are given non-degenerate cubes $x \in X_m$, $y \in Y_n$, $z \in Z_k$ in $X, Y, Z \in \text{cSet}^+$. Then the $(m+n+k)$-cube $(x \odot y) \odot z$ in $(X \odot Y) \odot Z$ is unmarked if and only if:

1. none of $x, y, z$ is marked; or
2. (at least) two of $x, y, z$ are 0-cubes and the last is unmarked.

One can give a similar characterization of when $x \otimes (y \odot z)$ is unmarked, and it follows that the associativity isomorphism $(vX \otimes vY) \odot vZ \cong vX \otimes (vY \odot vZ)$ in $\text{cSet}$ lifts to an isomorphism $(X \otimes Y) \odot Z \cong X \otimes (Y \odot Z)$ in $\text{cSet}^+$. The unit isomorphisms can be lifted similarly, and moreover these lifted isomorphisms are suitably natural and coherent. Thus we indeed obtain a monoidal structure on $\text{cSet}^+$ such that $v$ is strict monoidal. The assertion regarding the monoidal structure on the identity functor is obvious.

It remains to show that this monoidal structure is biclosed. Let $F : \mathcal{F} \to \text{cSet}^+$ and $X \in \text{cSet}^+$. Since the geometric product is cocontinuous in each variable and $v$ is cocontinuous and strict monoidal, the canonical comparison map

$$\colim(X \odot F) \to X \odot \colim F$$

is $v$-invertible. Moreover one can check using Remark 2.7 that a non-degenerate cube in either side is marked if and only if it is the image of some marked cube under the canonical map from $X \odot Fi$ for some $i \in \mathcal{F}$. It follows that this comparison map is invertible. Dually, $(-) \odot X$ preserves colimits. Since $\text{cSet}^+$ is locally finitely presentable, the existence of the desired biclosed structure now follows.

**Lemma 2.17.** Let $f : A \to X$ and $g : B \to Y$ be monomorphisms in $\text{cSet}^+$.

1. If both $f$ and $g$ are regular then so is $f \odot g$.
2. If either $f$ or $g$ is entire then so is $f \odot g$.
3. If both $f$ and $g$ are entire then $f \odot g$ is invertible.
(4) If either \( f \) or \( g \) is entire then the square
\[
\begin{array}{ccc}
(A \otimes Y) \cup (X \otimes B) & \longrightarrow & (A \otimes Y) \cup (X \otimes B) \\
\downarrow f \circ g & & \downarrow f \circ g \\
X \otimes Y & \xrightarrow{\mu_{X,Y}} & X \otimes Y
\end{array}
\]
is a pushout in \( \mathbf{cSet}^+ \) where the upper horizontal map is induced by \( \mu \).

**Proof.** In each case, the underlying cubical map is a monomorphism by Corollary 1.16. We will assume for the sake of simplicity that \( f \circ g \) is an inclusion.

(18) Let \( x \otimes y \) be a non-degenerate cube in \( \text{dom}(f \circ g) \). By duality, we may assume that \( x \) is in \( A \). If \( x \otimes y \) is marked in \( X \otimes Y \), then either \( x \) is marked in \( X \) or \( y \) is marked in \( Y \). It follows (by the regularity of \( f \) in the former case) that \( x \otimes y \) is marked in \( A \otimes Y \). This shows that \( f \circ g \) is regular.

(18) Let \( x \otimes y \) be a marked non-degenerate cube in \( X \otimes Y \). Suppose that \( x \otimes y \) is in the image of \( f \circ g \). The case (18) combined with the commutativity of the square in (4) imply that if \( x \otimes y \) is marked in \( X \otimes Y \) then it is also marked in \( A \otimes Y \). Thus by Remark 2.14 it suffices to consider the case where \( x \in X_m \) and \( y \in Y_n \) for some \( m, n \geq 1 \). But in this case \( x \otimes y \) is marked in \( \text{dom}(f \circ g) \) by the definition of \( \circ \).

(2) Since \( \nu \) preserves colimits, we have \( \nu(f \circ g) \cong \nu f \circ \nu g \). Thus this assertion follows from the fact that the pushout of an isomorphism along any map is itself an isomorphism.

(38) We know from (2) that \( f \circ g \) is entire, so it suffices to show that this map is also regular. Let \( x \otimes y \) be a marked non-degenerate cube in \( X \otimes Y \). Then either \( x \) is marked in \( X \) or \( y \) is marked in \( Y \). The cube \( x \otimes y \) is then marked in \( X \otimes B \) in the first subcase and it is marked in \( A \otimes Y \) in the second subcase. Thus \( f \circ g \) is indeed regular.

(4) By (2), each map in this square is entire. Thus its image under \( \nu \) is trivially a pushout in \( \mathbf{cSet} \). Moreover, for each of the horizontal maps, Remark 2.14 implies that the codomain is obtained from the domain by marking those cubes \( x \otimes y \) such that \( x \in X_m \) and \( y \in Y_n \) with \( m, n \geq 1 \). Now the assertion follows by Remark 2.16.

(38) This case follows from (38) and (4). \( \square \)

3. Model structure for comical sets

In this section, we construct two families of model structures on the category \( \mathbf{cSet}^+ \) of marked cubical sets. The former of those has as its fibrant objects (saturated) comical sets, which we will define, and it is our tentative model for the theory of weak \( \omega \)-categories. The fibrant objects of the latter are the \( n \)-trivial comical sets, and it is our tentative model for the theory of \((\infty, n)\)-categories.

A comical set is to be thought of as a kind of weak \( \omega \)-category, and an \( n \)-cube therein represents an \( n \)-dimensional morphism. The \((n-1)\)-source of such an \( n \)-cube is the “composite” of the faces \( \partial_{k,\varepsilon} \) with \( k + \varepsilon \) odd, and similarly the \((n-1)\)-target is given by the even faces. (This idea of parity-based decomposition into source and target goes back to Street [Str87], where Street considers the free \( \omega \)-categories on simplices. In the case of cubes, see e.g., [Alt86, Str87, Ste93, AAB02]). For instance, a 2-cube can be seen as a morphism of the form:

```
\[
\begin{array}{ccc}
\partial_{1,0} & \longrightarrow & \partial_{1,1} \\
\partial_{1,0} \searrow & & \swarrow \partial_{1,1} \\
\partial_{2,1} & \downarrow & \\
\partial_{2,1} & & \\
\end{array}
\]
```
and a 3-cube represents a morphism between the following composites:

Marked \( n \)-cubes are to be thought of as being (weakly) invertible, although not every invertible cube is marked unless the comical set is saturated.

Before defining comical sets, we will need a few auxiliary definitions.

For \( n \geq 1, \ 0 \leq k \leq n \), and \( \varepsilon \in \{0,1\} \), we denote \( \square^n_{k,\varepsilon} \) the \( n \)-cube with the following marking: a non-degenerate cube \( \partial_{k,1-\varepsilon} \cdots \partial_{k,\varepsilon} \), written in the form specified by Theorem 1.6, is marked whenever this string does not contain \( \partial_{k-1,\varepsilon} \), \( \partial_{k,1-\varepsilon} \), or \( \partial_{k+1,\varepsilon} \). (This is exactly the marking described in [Ste06, Ex. 2.9].) We call this the \( (k,\varepsilon) \)-comical \( n \)-cube. We denote \( \square^n_{k,\varepsilon} \subset \square^n_{k,\varepsilon} \) the \( (k,\varepsilon) \)-open box of dimension \( n \) (i.e., the cubical subset missing the non-degenerate \( n \)-cube and the \( (k,\varepsilon) \)th \((n-1)\)-face) endowed with the marking making it a regular subset of \( \square^n_{k,\varepsilon} \). We call \( \square^n_{k,\varepsilon} \) the comical \( (k,\varepsilon) \)-open box of dimension \( n \). We call the inclusion \( \square^n_{k,\varepsilon} \rightarrow \square^n_{k,\varepsilon} \) the \( (k,\varepsilon) \)-comical open box inclusion of dimension \( n \).

The elementary \( (k,\varepsilon) \)-comical marking extension of dimension \( n \), denoted by \( \square^n_{k,\varepsilon} \rightarrow \square^n_{k,\varepsilon} \), is the Leibniz product of the unit \( \rightarrow \tau_{n-2} \) and the comical box inclusion \( \square^n_{k,\varepsilon} \rightarrow \square^n_{k,\varepsilon} \), i.e., the dashed map in:

For each \( x,y \in \{ \nabla, \rho \} \), we define the basic Rezk map \( L_{xy} \rightarrow L'_{xy} \) as the entire inclusion depicted below:
Here thick arrows indicate marked cubes. More precisely, \( L_{xy} \) is the pushout of the span

\[
\begin{array}{ccc}
X & \xleftarrow{\partial_{1,1}} & \square^1 & \xrightarrow{\partial_{1,0}} & Y \\
\end{array}
\]

where \( X \) is obtained from \( \square^2 \) by marking \( \partial_{1,0} \) and \( \partial_{2,1} \), and \( Y \) is obtained from \( \square^2 \) by marking \( \partial_{1,1} \) and \( \partial_{2,0} \). The codomain \( L'_{xy} \) is the 0-trivialization \( \tau_0(L_{xy}) \). The marked cubical sets \( L_{xy}, L'_{xy} \) are defined similarly for other choices of \( x, y \in \{ \uparrow, \downarrow \} \). By a Rezk map we mean any map of the form

\[
(\partial \square^n \to \square^n) \otimes (L_{xy} \to L'_{xy}) \otimes (\partial \square^n \to \square^n).
\]

**Definition 3.1.**

(1) A comical set is a marked cubical set with the right lifting property with respect to the comical open box inclusions and the elementary comical marking extensions.

(2) A saturated comical sets is a marked cubical set with the right lifting property with respect to the comical open box inclusions, the elementary comical marking extensions, and the Rezk maps.

**Remark 3.2.** We briefly explain how the definition of (saturated) comical set should be interpreted. In the comical \( n \)-cube \( \square^n_{k,\varepsilon} \), any sub-cube not contained in \( \partial_{k-1,\varepsilon}, \partial_{k,\varepsilon}, \partial_{k,1-\varepsilon}, \) or \( \partial_{k+1,\varepsilon} \) is marked. In particular the unique non-degenerate \( n \)-cube is marked, so it can be thought of as an equivalence between the composite of its odd faces and the composite of even faces. In other words, the comical \( n \)-cube \( \square^n_{k,\varepsilon} \) exhibits \( \partial_{k,\varepsilon} \) as a composite of \( \partial_{k-1,\varepsilon}, \partial_{k,1-\varepsilon}, \) and \( \partial_{k+1,\varepsilon} \). e.g., \( \square^2_{2,0} \) looks like:

One can thus interpret the right lifting property with respect to the comical box inclusions and the comical marking extensions respectively as the existence of composites and the closure of marked cubes under composition. In Section 4 we show how these conditions additionally encode such expected properties of composition as the unit and associative laws, at least for 1-cubes.

There are two standard model structures on marked cubical sets:

**Theorem 3.3** (Model structure for comical sets). The category \( \text{cSet}^+ \) carries two model structures:

(1) the comical model structure characterized by the following properties:
   - The cofibrations are the monomorphisms.
   - The set of
     - comical open box inclusions, and
     - elementary comical marking extensions
   forms a pseudo-generating set of trivial cofibrations.

(2) the saturated comical model structure characterized by the following properties:
   - The cofibrations are the monomorphisms.
   - The set of
     - comical open box inclusions,
     - elementary comical marking extensions, and
     - Rezk maps
   forms a pseudo-generating set of trivial cofibrations.
Both of these model structures are combinatorial, left proper, monoidal with respect to either of the Gray tensor products, and have all objects cofibrant.

The proof of this theorem is an application of the Cisinski–Olschok theory and verification of the closure of anodyne maps under pushout-product. The latter part is contained in Lemma 3.5 below.

**Definition 3.4.** We say that a map of marked cubical sets $X \to Y$ is

1. a **comical marking extension** if it is in the cellular closure of the elementary comical marking extensions.
2. **comical** if it is in the cellular closure of the comical open box inclusions and the elementary comical marking extensions.

**Lemma 3.5.** For any $1 \leq k \leq m$, $\varepsilon \in \{0, 1\}$ and $n \geq 0$, the Leibniz Gray tensor products

\[ f = (\square_m^{m+n} \to \square_k^{m+n}) \hat{\circ} (\partial \square^n \to \square^n) \]
\[ g = (\square_k^{m+n} \to \square_k^{m+n}) \hat{\circ} (\square^n \to \square^n) \]
\[ h = (\square_k^{m+n} \to \square_k^{m+1,n}) \hat{\circ} (\partial \square^n \to \square^n) \]

are all comical.

**Proof.** Consider a face of $\square^m+n$ whose normal form $\partial_{k_1,\varepsilon_1} \ldots \partial_{k_c,\varepsilon_c}$ does not involve $\partial_{k-1,\varepsilon}$, $\partial_{k,0}$, $\partial_{k,1}$ or $\partial_{k+1,\varepsilon}$. Then clearly any terminal segment of this normal form does not involve any of these four $\partial$’s. This observation implies that the second isomorphism of Proposition 1.15 may be lifted to the following commutative square:

\[
\begin{array}{ccc}
\cap_{k,\varepsilon}^{m+n} & \to & (\square_k^{m+n} \cap \partial \square^n) \cup (\cap_{k,\varepsilon}^{m+n} \cap \square^n) \\
\downarrow & & \downarrow f \\
\square_k^{m+n} & \to & \square_k^{m+n} \cap \square^n
\end{array}
\]

We claim this square to be a pushout. This is clear on the underlying cubical set level since the horizontal maps are invertible. Thus it suffices to check that the marking on $\square_k^{m+n} \cap \square^n$ agrees with that described in Remark 2.6. This is indeed the case since $f$ is regular by Lemma 2.17(1) and the only marked non-degenerate cube in $\text{cod}(f) \setminus \text{dom}(f)$ is the $(m+n)$-cube, which is the image of a marked cube under the lower horizontal map. Thus $f$ is indeed comical.

The map $g$ is entire by Lemma 2.17(2). Similarly to the above argument, one can deduce the existence of the following commutative square of entire monomorphisms:

\[
\begin{array}{ccc}
\square_k^{m+n} & \to & (\square_k^{m+n} \cap \square^n) \cup (\cap_{k,\varepsilon}^{m+n} \cap \square^n) \\
\downarrow & & \downarrow g \\
\square_k^{m+n} & \to & \square_k^{m+n} \cap \square^n
\end{array}
\]

One can moreover check that the only cube in $\text{cod}(g)$ that is not marked in $\text{dom}(g)$ is $\partial_{k,\varepsilon}$ and it is the image of a marked cube under the lower horizontal map. This shows that the above square is a pushout. Hence $g$ is a comical marking extension.
Similarly, one can check that the following square is a pushout:

\[
\begin{array}{c}
\square_{k,e}^{m+n'} \\
\downarrow \quad h \\
\square_{k,e}^{m+n''} \to \square_{k,e}^{m''} \odot \square^n
\end{array}
\]

Therefore \( h \) is a comical marking extension. \( \square \)

**Proof of Theorem 3.3.** We apply the Cisinski–Olschok theory, i.e., Theorem 1.4 with \( K = \mathbf{cSet}^+ \) and \( I \) the set of boundary inclusions and markers. The set \( S \) consists of the comical open box inclusions and the comical marking extensions in (1), and it additionally contains all Rezk maps in (2). For our cylinder functor \( C \), we can use either \( \square^1 \otimes (-) \) or \( \square^1 \odot (-) \) as they are equal by Remark 2.14. This produces a model structure on \( \mathbf{cSet}^+ \) in which the cofibrations are the monomorphisms and \( \Lambda(\mathbf{cSet}^+, I, C, S) \) is a pseudo-generating set of trivial cofibrations.

It remains to prove that the set \( S \) is in fact pseudo-generating, and moreover the model structure is monoidal with respect to either of the Gray tensor products. By duality and Proposition 1.5 it suffices to show that:

- \( f \hat{\otimes} g \) is in the cellular closure of \( I \) whenever \( f, g \in I \); and
- \( f \hat{\otimes} g \) is in the cellular closure of \( S \) whenever \( f \in S \) and \( g \in I \).

The first clause is straightforward to verify. We now treat the second clause.

There are three kinds of maps in \( S \), namely:

(A) comical box inclusions;
(B) elementary comical marking extensions; and
(C) Rezk maps,

and two kinds of maps in \( I \), namely:

(a) boundary inclusions; and
(b) markers.

The case (Ca \( \otimes \)) is obvious from our definition of Rezk maps. The case (Ca \( \odot \)) then follows by Lemma 2.17(4). In the cases (Bb) and (Cb), the map \( f \hat{\otimes} g \) is invertible by Lemma 2.17(3). The remaining cases are treated in Lemma 3.5. \( \square \)

There are also \( n \)-trivial versions of these model structures.

**Theorem 3.6 (Model structure for \( n \)-trivial comical sets).** The category \( \mathbf{cSet}^+ \) carries two families of model structures:

1. The \( n \)-trivial comical model structure characterized by the following properties:
   - The cofibrations are the monomorphisms.
   - The set of
     - comical open box inclusions,
     - elementary comical marking extensions of dimension \( \leq n + 1 \), and
     - markers of dimension \( > n \)
   forms a pseudo-generating set of trivial cofibrations.

2. The saturated \( n \)-trivial comical model structure characterized by the following properties:
   - The cofibrations are the monomorphisms.
   - The set of
     - comical open box inclusions,
     - elementary comical marking extensions of dimension \( \leq n + 1 \),
markers of dimension > n, and
- Rezk maps form a pseudo-generating set of trivial cofibrations.

Proof. Analogous to the proof of Theorem 3.3. Note that the Leibniz Gray tensor product of the m-marker with any monomorphism is in the cellular closure of the m'-markers with m' ≥ m. □

Proposition 3.7. The functor $\tau_n: \mathcal{C} \rightarrow \mathcal{C}^+$ is a left Quillen functor from the n-trivial comical model structure (resp. saturated n-trivial comical model structure) to the comical model structure (resp. saturated comical model structure).

Lemma 3.8. For any $n \geq 0$, $1 \leq k \leq n + 2$ and $\varepsilon \in \{0, 1\}$, the map $\tau_n \cap_{k,\varepsilon}^{n+2} \rightarrow \tau_n \square_{k,\varepsilon}^{n+2} = \square_{k,\varepsilon}^{n+2''}$ is comical.

Proof. Recall the defining pushout square of the comical marking extension:

$$
\begin{array}{c}
\cap_{k,\varepsilon}^{n+2} \\
p.o. \\
\square_{k,\varepsilon}^{n+2} \\
\end{array}
\xrightarrow{}
\begin{array}{c}
\tau_n \cap_{k,\varepsilon}^{n+2} \\
\tau_n \square_{k,\varepsilon}^{n+2} \\
\end{array}

$$

This diagram exhibits the desired result. □

Proof of Proposition 3.7. That $\tau_n$ preserves cofibrations is obvious. Moreover, observe that if a map $f: X \rightarrow Y$ is surjective on unmarked cubes of dimension > n then the naturality square for the unit

$$
\begin{array}{c}
X \\
f \\
Y \\
\end{array}
\xrightarrow{}
\begin{array}{c}
\tau_n X \\
\tau_n f \\
\tau_n Y \\
\end{array}
$$

is a pushout in $\mathcal{C}^+$ by Remark 3.4. Thus it remains to check that $\tau_n$ sends $\cap_{k,\varepsilon}^{n+2} \rightarrow \square_{k,\varepsilon}^{n+2}$ to a trivial cofibration, which follows from the previous lemma. □

As we mentioned earlier, our definition of comical box inclusion uses the marking described in [Ste06] where Steiner characterizes the nerves of strict $\omega$-categories. Phrased in the language of comical sets, his characterization implies the following result:

Theorem 3.9 (cf. [Ste06] Thm. 3.16). The cubical nerve of a globular $\omega$-category, or equivalently the underlying cubical set of a cubical $\omega$-category with connections, is a comical set. □

This is analogous to the statement that the simplicial nerve of a strict $\omega$-category is a complicial set [Ver08a].

We conclude this section with the following observation which will be useful in Section 7.

Proposition 3.10. For any $n \geq 2$, $1 \leq k \leq n$ and $\varepsilon \in \{0, 1\}$, the comical box inclusion $\cap_{k,\varepsilon}^{n} \rightarrow \square_{k,\varepsilon}^{n}$ may be written as:

$$
\begin{align*}
(\cap_{1,\varepsilon}^{2} \rightarrow \square_{1,\varepsilon}^{2}) \hat{\otimes} (\partial \square_{1,\varepsilon}^{n-2} \rightarrow \square_{1,\varepsilon}^{n-2}), & \quad k = 1, \\
(\partial \square_{k,\varepsilon}^{k-2} \rightarrow \square_{k,\varepsilon}^{k-2}) \hat{\otimes} (\cap_{2,\varepsilon}^{3} \rightarrow \square_{2,\varepsilon}^{3}) \hat{\otimes} (\partial \square_{2,\varepsilon}^{n-k-1} \rightarrow \square_{2,\varepsilon}^{n-k-1}), & \quad 1 < k < n, \\
(\partial \square_{n,\varepsilon}^{n-2} \rightarrow \square_{n,\varepsilon}^{n-2}) \hat{\otimes} (\cap_{2,\varepsilon}^{2} \rightarrow \square_{2,\varepsilon}^{2}), & \quad k = n.
\end{align*}
$$
Proof. It is easy to check that the underlying cubical maps match, and also the markings on the codomains match. Now observe that these Leibniz Gray tensor products are regular by Lemma 2.17(1).

4. Homotopy 1-categories of comical sets

Suppose we are given two 1-cubes \( f, g : x \rightarrow y \) in a comical set \( X \). Then a marked 2-cube satisfying any one of the following boundary conditions may be reasonably regarded as a homotopy \( f \sim g \):

Here equalities indicate degenerate (and hence marked) 1-cubes.

**Proposition 4.1.** If any one of the above boundary conditions admits a marked solution in the comical set \( X \) then so do the others.

Proof. Consider the following picture:

If we have a marked 2-cube \( \omega \) satisfying the boundary condition specified above, then this picture may be interpreted as a map \( \tau_1 \cap \triangle^3_{1,1} \rightarrow X \) which may be extended to \( \triangle^3_{1,1} \) by Lemma 3.8 yielding a marked 2-cube \( \omega' \). Conversely, if we are given \( \omega' \) then this picture specifies a map \( \tau_1 \cap \triangle^3_{1,1} \rightarrow X \) and extending it to \( \triangle^3_{1,1} \) yields a marked 2-cube \( \omega \).
Similarly, the following picture shows that $\chi$ exists if and only if $\omega$ does:

Dually, $\chi'$ exists if and only if $\omega'$ does.

The following picture shows that $\psi$ exists if and only if $\omega'$ does (and dually $\psi'$ exists if and only if $\omega$ does):

Finally the following picture shows that $\phi$ exists if and only if $\chi'$ does (and dually $\phi'$ exists if and only if $\chi$ does):

This completes the proof. □

**Definition 4.2.** We say two 1-cubes $f, g$ in a comical set $X$ are homotopic and write $f \sim g$ if any one of the above marked 2-cubes exists in $X$.

**Proposition 4.3.** For any pair of 0-cubes $x, y$ in a comical set $X$, the homotopy relation is an equivalence relation on the set of all 1-cubes $x \to y$.

**Proof.** The reflexivity and symmetry of $\sim$ are obvious. For transitivity, suppose we are given two homotopies:
Then the following picture specifies a map \( \tau_1 \cap \Delta_{2,0}^3 \to X \):

This map extends to \( \square_{2,0}^3 \), which in particular yields a homotopy \( f \sim h \).

Now consider a “composable” pair of 1-cubes \( f : x \to y \) and \( g : y \to z \) in a comical set \( X \). We may “compose” \( f \) and \( g \) by filling any one of the open boxes \( \cap \Delta_{2,0}^3, \cap \Delta_{1,1}^3, \cap \Delta_{2,0}^3, \cap \Delta_{2,1}^3 \) as follows:

We will temporarily call such \( a \) a \((1,0)\)-composite of \( f \) and \( g \), and similarly call \( b \), \( c \) and \( d \) \((1,1)\)-, \((2,0)\)- and \((2,1)\)-composites of \( f \) and \( g \) respectively.

**Proposition 4.4.** Any two composites of \( f \) and \( g \) are homotopic to each other.

**Proof.** Consider the four kinds of composites of \( f \) and \( g \) as above. The following picture specifies a map \( \tau_1 \cap \Delta_{2,0}^3 \to X \):

This map extends to \( \square_{2,0}^3 \), which yields a homotopy between the \((1,0)\)-composite \( a \) and the \((2,0)\)-composite \( c \). Similarly a homotopy \( b \sim c \) can be obtained via the following open box:
and a homotopy \( b \sim d \) can be obtained using:

Finally, if \( a' \) is another \((1,0)\)-composite of \( f \) and \( g \) then we have \( a \sim b \sim a' \). Similarly any two \((k,\epsilon)\)-composites of \( f \) and \( g \) (for fixed \( k \) and \( \epsilon \)) are homotopic to each other. This completes the proof. \( \square \)

**Proposition 4.5.** Let \( f, f': x \to y \) and \( g, g': y \to z \) be 1-cubes in a comical set \( X \) such that \( f \sim f' \) and \( g \sim g' \). Then any composite of \( f \) and \( g \) is homotopic to any composite of \( f' \) and \( g' \).

**Proof.** Choose witnesses of the following forms for compositions and a homotopy \( f \sim f' \):

Then extending the following map \( \tau_1 \cap_{2,1}^n \to X \) to \( \cap_{2,1}^n \) yields a homotopy \( a \sim b \):

Similarly, we may combine marked 2-cubes of the forms.
into a map $\gamma_{3,0}^3 \to X$ as follows:

Extending this map to $\Delta_{2,0}^3$ yields a homotopy $c \sim d$. The desired result now follows by Propositions 4.3 and 4.4.

**Definition 4.6.** We define the homotopy 1-category $\text{ho}_1 X$ of a comical set $X$ to be the category of 0-cubes and homotopy classes of 1-cubes in $X$.

**Proposition 4.7.** For a comical set $X$, $\text{ho}_1 X$ is indeed a 1-category.

*Proof.* Proposition 4.5 implies that we have a well-defined composition operation on $\text{ho}_1 X$. For any 0-cube $x$ in $X$, we claim that the homotopy class containing $x \sigma_1$ is the identity at $x$. Indeed for any $f : x \to y$, the degenerate 2-cube

exhibits $f$ as a $(1, 0)$-composite of $x \sigma_1$ and $f$, and also as a $(1, 1)$-composite of $f$ and $y \sigma_1$.

For associativity, suppose we are given 1-cubes $f : x \to y$, $g : y \to z$ and $h : z \to w$ in $X$. Compose these 1-cubes as follows:

Then we may combine them into a map $\gamma_{3,1}^3 \to X$:
Extending this map to $\Box_{3,1}^{3,1}$ then yields a marked 2-cube that witnesses the desired associativity.

The following proposition is straightforward to verify.

**Proposition 4.8.** The assignment $X \mapsto ho_{1}X$ extends to a functor from the category of comical sets to $\text{Cat}$. Moreover there is a natural isomorphism $ho_{1}(X^{\text{op}}) \cong (ho_{1}X)^{\text{op}}$. □

5. Triangulation

In this section, we upgrade the triangulation adjunction described in Section 1.2 to a marked version. We start by recalling the basic combinatorics of simplicial cubes, which can be found in [Ver07].

Given an $r$-simplex $\phi$ in the simplicial set $(\Delta^{1})^{n}$, we can define a function

$$
\{1, \ldots, n\} \to \{1, \ldots, r, \pm \infty\}
$$

by declaring

$$
i \mapsto \begin{cases} +\infty, & \pi_{i} \circ \phi(r) = 0, \\
p, & \pi_{i} \circ \phi(p-1) = 0 \text{ and } \pi_{i} \circ \phi(p) = 1, \\
-\infty, & \pi_{i} \circ \phi(0) = 1. \end{cases}
$$

In fact, any function $\{1, \ldots, n\} \to \{1, \ldots, r, \pm \infty\}$ determines a unique $r$-simplex in $(\Delta^{1})^{n}$, so we will identify these things.

**Remark 5.1.** In what follows, we sometimes write such expressions as $p \pm k$ for $p \in \{1, \ldots, r, \pm \infty\}$ and finite $k$. These expressions are to be interpreted as $p$ when $p \in \{\pm \infty\}$. We will never consider expressions involving more than one $\pm \infty$.

**Definition 5.2.** We will write $\iota = \iota_{n} : \{1, \ldots, n\} \to \{1, \ldots, n, \pm \infty\}$ for the inclusion regarded as an $n$-simplex in $(\Delta^{1})^{n}$.

We will think of any set of the form $\{1, \ldots, r, \pm \infty\}$ as a linearly ordered set

$$
-\infty < 1 < \cdots < r < +\infty.
$$

Note however that simplices $\phi : \{1, \ldots, n\} \to \{1, \ldots, r, \pm \infty\}$ are not necessarily order-preserving.

**Proposition 5.3.** Under this identification, a simplicial operator $\alpha : [q] \to [r]$ sends an $r$-simplex $\phi$ to the $q$-simplex $\phi \alpha$ given by

$$
(\phi \alpha)(i) = \begin{cases} +\infty, & \phi(i) > \alpha(q), \\
p, & \alpha(p-1) < \phi(i) \leq \alpha(p), \\
-\infty, & \phi(i) \leq \alpha(0). \end{cases}
$$

**Example 5.4.** For any $0 \leq m \leq r$,

$$
(\phi \downarrow_{1}^{m, r-m})(i) = \begin{cases} +\infty, & \phi(i) > m, \\
\phi(i), & \phi(i) \leq m. \end{cases}
$$

and

$$
(\phi \downarrow_{2}^{m, r-m})(i) = \begin{cases} \phi(i) - m, & \phi(i) > m, \\
-\infty, & \phi(i) \leq m. \end{cases}
$$

It is easy to verify the following proposition using Proposition 5.3.

**Proposition 5.5.** An $r$-simplex $\phi$ in $(\Delta^{1})^{n}$ is non-degenerate if and only if $\phi^{-1}(p) \neq \emptyset$ for each $1 \leq p \leq r$. 

Now we upgrade the codomain of the triangulation functor to a marked version. More precisely, we first consider the functor $\square \rightarrow \text{PreComp}$ associated (in the sense of Theorem 1.10) to the cubical monoid $\Delta^1$, where $\text{PreComp}$ is considered to be monoidal with respect to the Gray tensor product $\otimes$. Its object part is thus given by $[1]^n \mapsto (\Delta^1)^{\otimes n}$. This functor induces a strong monoidal left adjoint $T: \text{cSet} \rightarrow \text{PreComp}$ with right adjoint $U$. Now we regard functions $\phi: \{1, \ldots, n\} \rightarrow \{1, \ldots, r, \pm \infty\}$ as simplices in the marked simplicial set $(\Delta^1)^{\otimes n}$.

**Proposition 5.6.** An $r$-simplex $\phi$ in $(\Delta^1)^{\otimes n}$ is unmarked if and only if there exist

$$1 \leq i_1 < \cdots < i_r \leq n$$

such that $\phi(i_p) = p$ for all $1 \leq p \leq r$. In particular, the only unmarked $n$-simplex in $(\Delta^1)^{\otimes n}$ is $\iota_n$.

**Proof.** This is proved in [Ver07, Obs. 27]. Note however that the indexing in that paper is reversed. □

We would like to upgrade the domain of $T$ to a marked version too, by sending the marked $n$-cube to $\tau_{n,1}((\Delta^1)^{\otimes n})$. Proposition 5.6 implies that this marked simplicial set may be obtained from $(\Delta^1)^{\otimes n}$ by marking $\iota_n$. The following lemma shows that it is indeed an object in $\text{PreComp}$.

**Lemma 5.7.** The marked simplicial set $\tau_{n,1}((\Delta^1)^{\otimes n})$ is pre-complicial for any $n \geq 1$.

**Proof.** Suppose for contradiction that we are given a map $\phi: \Delta^m_k \rightarrow \tau_{n,1}((\Delta^1)^{\otimes n})$ that cannot be extended to $\Delta^{m''}_k$. Then $\phi$ must not factor through the pre-complicial set $(\Delta^1)^{\otimes n}$, so $\phi$ sends at least one of the marked, non-degenerate simplices in $\Delta^{m''}_k$ to $\iota_n$. Since all simplices in $\Delta^{m''}_k$ of dimension $> m$ are degenerate, it follows that $m \geq n$. On the other hand, we cannot have $m > n$ since $\phi\partial_k$ is unmarked in the $(n-1)$-trivial marked simplicial set $\tau_{n,1}((\Delta^1)^{\otimes n})$. Thus we must have $m = n$ and $\phi = \iota_n$. But at least one of $\partial_{k-1}$ and $\partial_{k+1}$ is a well-defined face of $\Delta^{m''}_k$, and it can be easily checked using Proposition 5.6 that $\phi$ sends this face to an unmarked simplex. This is the desired contradiction. □

Thus we have defined the object part of $T: \square^+ \rightarrow \text{PreComp}$, but we still need to define its value on the generating morphisms $\varphi^n$, $\zeta^n_i$, and $\xi^n_{i,c}$, and verify the co-cubical identities. The maps $T\varphi^n:T[1]^n \rightarrow T[1]_c^n$ are identity on the underlying cubical sets and add the additional marking on $\iota_n$. Note that they are epimorphisms. To define $T\zeta^n_i$ (resp. $T\xi^n_{i,c}$), recall that $\zeta^n_i \varphi^n = \sigma^n_i$ (resp. $\xi^n_{i,c} \varphi^n = \gamma^n_{i,c}$), and this yields a unique choice of $T\zeta^n_i$ (resp. $T\xi^n_{i,c}$). Finally, to see that this definition satisfies the additional identities, we note that these involving $\varphi$ are clear, whereas the remaining ones can be reduced to the usual cubical identities by precomposing with $\varphi$ and using the fact that it is an epimorphism.

Hence we obtain a left adjoint functor $T$ from structurally marked cubical sets to pre-complicial sets. Moreover, the right adjoint $U$ takes values in marked cubical sets, because the map $\square^n \rightarrow \square^n$ is carried by $T$ to an epimorphism. Thus by restricting the domain of $T$, we have constructed an adjunction $T \dashv U$ between marked cubical sets and pre-complicial sets. In the remainder of the paper, we show that $T$ is strong monoidal with respect to either version of the Gray tensor products and moreover left Quillen with respect to suitable model structures. We will make use of the following observation.

**Proposition 5.8.** There are isomorphisms $T(X^{op}) \cong (TX)^{op}$ natural in $X \in \text{cSet}^+$. 

**Proof.** Since both $X \mapsto T(X^{op})$ and $X \mapsto T(X)^{op}$ are cocontinuous, it suffices to verify the assertion for $X = \square^n$ for $n \geq 0$ and $X = \square^n$ for $n \geq 1$. It is straightforward to check that these isomorphisms exist and are natural for $X = \square^n$, $\square_1^+$ and $\square_2^+$. Since both $X \mapsto T(X^{op})$ and $X \mapsto (TX)^{op}$ are antimonoial, we obtain the isomorphisms for $X = \square^n$ with arbitrary $n$ by Theorem 1.10. The rest is straightforward. □
6. Triangulating Gray tensor product

We now prove that the triangulation functor is strong monoidal with respect to either version of the Gray tensor product. We begin by describing a proof strategy that will be used in both the lax and the pseudo cases.

6.1. Proof strategy. The proof typically reduces to showing an entire inclusion \( A \to B \) to be a complicial marking extension where \( A, B \) are certain entire supersets of \( (\Delta^1)^\oplus N \). (The integer \( N \) will be of the form \( N = m + n \) in the actual proofs, but this is irrelevant in this subsection.) There are three kinds of simplices of interest, namely those that are:

- (i) marked in \( (\Delta^1)^\oplus N \);
- (ii) marked in \( A \) but not in \( (\Delta^1)^\oplus N \); and
- (iii) marked in \( B \) but not in \( A \).

The simplices of type (i) are characterized by Proposition 5.6. The first step of the proof will be to (define suitable \( A, B \) and) characterize simplices of type (ii) and (iii).

Before proceeding, we need the following definitions.

**Definition 6.1.** For any \( r \)-simplex \( \phi : \{1, \ldots, N\} \to \{1, \ldots, r, \pm \infty\} \) in \( (\Delta^1)^N \), define:

\[
\mathcal{D}(\phi) = |\phi^{-1}(\{1, \ldots, r\})| - r,
\]

\[
\mathcal{O}(\phi) = \{(i, j) \in \{1, \ldots, N\}^2 : i < j, \phi(i) < \phi(j)\}.
\]

The integer \( \mathcal{D}(\phi) \) measures how “diagonal” \( \phi \) is, and the set \( \mathcal{O}(\phi) \) measures how “in order” \( \phi \) is.

We complicially extend the marking on \( A \) to those simplices \( \phi \) of type (iii) by nested induction on \( \mathcal{D}(\phi) \) and \( |\mathcal{O}(\phi)| \). More precisely, consider the lexicographical ordering on \( \mathbb{Z} \times \mathbb{N} \) so that \( (u_1, v_1) \leq (u_2, v_2) \) if and only if:

- \( u_1 < u_2 \); or
- \( u_1 = u_2 \) and \( v_1 \leq v_2 \).

For each \( (u, v) \in \mathbb{Z} \times \mathbb{N} \), let \( A(u, v) \) denote the marked simplicial set obtained from \( A \) by marking those simplices \( \phi \) such that \( \phi \) is marked in \( B \) and \( (\mathcal{D}(\phi), |\mathcal{O}(\phi)|) < (u, v) \). Then:

- \( A(u_1, v_1) \) is an entire subset of \( A(u_2, v_2) \) for any \( (u_1, v_1) \leq (u_2, v_2) \);
- \( \text{colim}_v A(u, v) = A(u + 1, 0) \) for any \( u \in \mathbb{Z} \);
- \( \text{colim}_u A(u, v) = B \); and
- \( A(0, 0) = A \) (by Proposition 5.5).

Now we assume the following.

**Assumption 1.** Any marked simplex \( \phi \) in \( B \) with \( \mathcal{D}(\phi) = 0 \) is marked in \( A \).

Then we may upgrade the last bulleted item to \( A(1, 0) = A \). Thus to prove that \( A \to B \) is a complicial marking extension, it suffices to exhibit the map \( A(u, v) \to A(u, v + 1) \) as a complicial marking extension for each \( (u, v) \geq (1, 0) \).

So fix \( (u, v) \geq (1, 0) \) and suppose that we are given an \( r \)-simplex \( \phi \) of type (iii) with \( (\mathcal{D}(\phi), |\mathcal{O}(\phi)|) = (u, v) \). Then in particular \( \mathcal{D}(\phi) \geq 1 \). So by the pigeon hole principle, we can choose \( 1 \leq p_\phi \leq r \) such that \( |\phi^{-1}(p_\phi)| \geq 2 \). Let \( i_\phi = \min \phi^{-1}(p_\phi) \). Let \( \tilde{\phi} \) be the \( (r + 1) \)-simplex given by

\[
\tilde{\phi}(i) = \begin{cases} 
\phi(i), & \phi(i) \leq p_\phi \text{ and } i \neq i_\phi, \\
\phi(i) + 1, & \phi(i) > p_\phi \text{ or } i = i_\phi.
\end{cases}
\]
Observe that we have $\tilde{\phi} \partial p \phi = \phi$. We wish to show that this simplex $\tilde{\phi}$ extends to $\Delta_{p_0}^{r+1'}$:

$\xymatrix{ \Delta^{r+1} \ar[r]^{\tilde{\phi}} & A(u,v) \\ \Delta_{p_0}^{r+1'} \ar[u] }$

Assuming this fact, we can deduce that we have a pushout square

$\xymatrix{ \coprod \Delta_{p_0}^{r+1'} \ar[r] & A(u,v) \\ \coprod \Delta_{p_0}^{r+1''} \ar[u] & A(u,v+1) \ar[u] }$

where the coproducts are taken over all $r$-simplices $\phi$ of type (iii) with $(D(\phi), |O(\phi)|) = (u,v)$ (for various $r$) and the horizontal maps are induced by $\tilde{\phi}$.

The following lemma implies that $\tilde{\phi}$ at least extends to $\Delta_{p_0}^{r+1}$.

**Lemma 6.2.** Let $\alpha : [q] \to [r+1]$ be a face operator with $\{p_0, p_0 \pm 1\} \subset \text{im} \alpha$. Then $\tilde{\phi} \alpha$ is marked in $(\Delta^1)^{\otimes N}$.

*Proof.* Let $p \in [q]$ be the necessarily unique element with $\alpha(p) = p_0$. Then we must have $\alpha(p-1) = p_0 - 1$ and $\alpha(p + 1) = p_0 + 1$. Now one can check using Proposition 5.3 and the minimality of $i_\phi$ that $(\tilde{\phi} \alpha)^{-1}(p + 1) = \{i_\phi\}$ and moreover any $1 \leq i \leq N$ satisfying $(\tilde{\phi} \alpha)(i) = p$ must also satisfy $i > i_\phi$. Thus $\tilde{\phi} \alpha$ is marked in $(\Delta^1)^{\otimes N}$ by Proposition 5.6.

Therefore it remains to prove that the faces $\chi = \tilde{\phi} \partial p_0 - 1$ and $\psi = \tilde{\phi} \partial p_0 + 1$ are marked in $A(u,v)$. First, we describe these simplices explicitly.

**Lemma 6.3.** The simplex $\chi$ is given by

$\chi(i) = \begin{cases} -\infty, & \phi(i) = p_0 \text{ and } i \neq i_\phi, \\ \phi(i), & \text{otherwise.} \end{cases}$

if $p_0 = 1$ and

$\chi(i) = \begin{cases} p_0 - 1, & \phi(i) = p_0 \text{ and } i \neq i_\phi, \\ \phi(i), & \text{otherwise} \end{cases}$

if $p_0 \geq 2$. The simplex $\psi$ is given by

$\psi(i) = \begin{cases} +\infty, & i = i_\phi \\ \phi(i), & \text{otherwise} \end{cases}$

if $p_0 = r$ and

$\psi(i) = \begin{cases} p_0 + 1, & i = i_\phi \\ \phi(i), & \text{otherwise} \end{cases}$

if $p_0 < r$.  

*Proof.* This is a routine application of Proposition 5.3.

These explicit descriptions allow us to prove the following.
Lemma 6.4. The simplices $\chi$ and $\psi$ satisfy

$$\left(\mathcal{D}(\chi), |\mathcal{O}(\chi)| \right) < \left(\mathcal{D}(\phi), |\mathcal{O}(\phi)| \right),$$

$$\left(\mathcal{D}(\psi), |\mathcal{O}(\psi)| \right) < \left(\mathcal{D}(\phi), |\mathcal{O}(\phi)| \right).$$

Proof. If $p_\phi = 1$ then clearly $\mathcal{D}(\chi) < \mathcal{D}(\phi)$.

Suppose $p_\phi \geq 2$. Then we have $\mathcal{D}(\chi) = \mathcal{D}(\phi)$. We claim that $\mathcal{O}(\chi)$ is a proper subset of $\mathcal{O}(\phi)$. Indeed, it can be seen from Lemma 6.3 that if a pair $(i, j)$ satisfies $\phi(i) \geq \phi(j)$ and $\chi(i) < \chi(j)$ then we must have $\phi(i) = \phi(j) = p_\phi$ and $i \neq i_\phi = j$. But then the minimality of $i_\phi$ implies $i > j$, and this shows that there is no pair $(i, j)$ in $\mathcal{O}(\chi) \setminus \mathcal{O}(\phi)$. On the other hand, our choice of $p_\phi$ guarantees the existence of some $j \neq i_\phi$ satisfying $\phi(j) = p_\phi$, and the pair $(i_\phi, j)$ is then in $\mathcal{O}(\phi) \setminus \mathcal{O}(\chi)$. Therefore $\mathcal{O}(\chi)$ is a proper subset of $\mathcal{O}(\phi)$, and this proves the lexicographical inequality concerning $\chi$.

The simplex $\psi$ can be treated dually. \qed

The last missing piece of the proof (that $A \rightarrow B$ is a complicial marking extension) is the following.

Assumption 2. The simplices $\chi$ and $\psi$ are marked in $B$.

This completes the proof strategy. (Whether Assumptions 1 and 2 hold depend on the exact definitions of $A$ and $B$, so there is no general strategy for verifying them.)

6.2. Triangulating the lax Gray tensor product. The goal of this subsection is to prove the following theorem.

Theorem 6.5. The adjunction $T : \mathcal{U} \rightarrow \mathcal{V}$ is monoidal with respect to the lax Gray tensor products. Equivalently, $T : (\mathcal{CSet}^+, \otimes) \rightarrow (\mathcal{PreComp}, \otimes)$ is strong monoidal.

Fix $m \geq 1$ and $n \geq 0$. Observe that

$$\begin{array}{ccc}
\coprod^{m+k} & \longrightarrow & \coprod^{m+n} \\
\varnothing & \downarrow & \downarrow \\
\coprod\Delta^{m+k} & \longrightarrow & \Delta^m \otimes \Delta^n \\
\end{array}$$

is a pushout square in $\mathcal{CSet}^+$ where the coproducts are taken over all face maps $[1]^k \rightarrow [1]^n$. This pushout is preserved by $T$, so the right square in

$$\begin{array}{ccc}
\coprod\Delta^{m+k} & \longrightarrow & \coprod(\Delta^1)^{\otimes (m+k)} \\
\varnothing & \downarrow & \downarrow \\
\coprod\Delta^{m+k} & \longrightarrow & T((\Delta^1)^{\otimes (m+k)}) \\
\end{array}$$

is a pushout square in $\mathcal{PreComp}$ where the upper horizontal map is induced by $\text{id}_{(\Delta^1)^{\otimes m}} \otimes T(\phi)$ for various face maps $\phi : [1]^k \rightarrow [1]^n$. The left square is also a pushout by Proposition 6.6 so the pasted square is a pushout too. In this subsection, we define $A$ to be the corresponding pushout in $\mathcal{Set}^+$ (and not in $\mathcal{PreComp}$):

$$\begin{array}{ccc}
\coprod\Delta^{m+k} & \longrightarrow & (\Delta^1)^{\otimes (m+n)} \\
\varnothing & \downarrow & \downarrow \\
\coprod\Delta^{m+k} & \longrightarrow & A \\
\end{array}$$

p.o.

so that its pre-complicial reflection $A^{\text{pre}}$ is precisely $T(\Delta^m \otimes \Delta^n)$. 

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Lemma 6.6. An \( r \)-simplex \( \phi \) is marked in \( A \) but not in \( (\Delta^1)^{\otimes (m+n)} \) if and only if \( r \geq m \), \( \phi(i) = i \) for all \( 1 \leq i \leq m \) and the restriction 
\[
\phi^{-1}(\{1, \ldots, r\}) \rightarrow \{1, \ldots, r\}
\]
of \( \phi \) is an isomorphism of linearly ordered sets.

Proof. Compute the colimit. \qed

Lemma 6.7. Let \( \phi \) be an unmarked \( r \)-simplex in \( (\Delta^1)^{\otimes (m+n)} \). Then \( \phi \) is marked in \( T(\bar{\square}_m) \otimes T(\bar{\square}_n) \) if and only if:

1. \( r \geq m \);
2. \( \phi(i) = i \) for all \( 1 \leq i \leq m \); and
3. there does NOT exist a sequence \( m < i_m < i_{m+1} < \cdots < i_r \leq m+n \) such that \( \phi(i_p) = p \) for all \( m \leq p \leq r \).

Proof. Write \( \phi = (\phi_1, \phi_2) \) for \( \phi \) regarded as a simplex in the product simplicial set \( (\Delta^1)^m \times (\Delta^1)^n \).

Lemma 6.8. An \( r \)-simplex \( \phi \) in \( T(\bar{\square}_m) \otimes T(\bar{\square}_n) \) with \( r \geq m \) is unmarked if and only if there exist 
\[
1 \leq i_1 < \cdots < i_r \leq m+n
\]
such that \( \phi(i_p) = p \) for all \( 1 \leq p \leq r \). Clearly the non-existence for \( q = m-1 \), which is precisely (3), implies the non-existence for all other values of \( q \). This completes the proof. \qed

Lemma 6.9. There is a complicial marking extension \( A \rightarrow T(\bar{\square}_m) \otimes T(\bar{\square}_n) \) that commutes with the evident inclusions of \( (\Delta^1)^{\otimes (m+n)} \).

Proof. We apply the proof strategy from Section 6.1 with \( B = T(\bar{\square}_m) \otimes T(\bar{\square}_n) \).

One can easily check using Lemmas 6.6 and 6.7 that any marked simplex \( \phi \) in \( T(\bar{\square}_m) \otimes T(\bar{\square}_n) \) with \( D(\phi) = 0 \) must also be marked in \( A \). This verifies Assumption \( \alpha \).

To verify Assumption \( \beta \) let \( \phi \) be an \( r \)-simplex that is marked in \( T(\bar{\square}_m) \otimes T(\bar{\square}_n) \) but not in \( A \). Then we necessarily have \( r \geq m \) by Lemma 6.7.
Consider the simplex $\chi = \tilde{\phi} \tilde{\partial}_{p_\phi - 1}$. Suppose for contradiction that $\chi$ is unmarked in $T(\square^m) \otimes T(\square^n)$. Then by Lemma 6.8 there exist $1 \leq i_1 < \cdots < i_r \leq m + n$ such that $\chi(i_p) = p$ for all $1 \leq p \leq r$ and $i_m > m$.

- If $p_\phi = 1$, then we also have $\phi(i_p) = p$ for all $p$ by Lemma 6.3; thus $\phi$ is unmarked in $T(\square^m) \otimes T(\square^n)$. This is the desired contradiction.

- Suppose $p_\phi \geq 2$. We claim that $\phi(i_p) = p$ holds for all $p$ in this case too. According to Lemma 6.3 the only thing we must check is that $\phi(i_{p_\phi - 1}) = p_\phi - 1$ holds (as opposed to $\phi(i_{p_\phi - 1}) = p_\phi$). To see that this is indeed the case, observe that $\chi^{-1}(p_\phi) = \{i_\phi\}$ by Lemma 6.3. Thus we must have $i_{p_\phi} = i_\phi$. Since $i_{p_\phi - 1} < i_{p_\phi}$, the minimality of $i_\phi$ implies that $i_{p_\phi - 1} \notin \phi^{-1}(p_\phi)$.

Therefore we have obtained the desired contradiction.

The simplex $\psi = \tilde{\phi} \tilde{\partial}_{p_\phi + 1}$ can be similarly checked to be marked in $T(\square^m) \otimes T(\square^n)$. This completes the proof. \)

**Proof of Theorem 6.10.** Since both $T(- \otimes -)$ and $T(-) \otimes T(-)$ preserve colimits in each variable, it suffices to check the existence of natural isomorphisms $T(X \otimes Y) \cong T(X) \otimes T(Y)$ for $X, Y$ generic (possibly marked) cubes.

By construction of $T$, we have $T(\square^m \otimes \square^n) \cong T(\square^m) \otimes T(\square^n)$ for any $m, n \geq 0$.

For any $m \geq 1$ and $n \geq 0$, we may obtain an isomorphism $T(\square^m \otimes \square^n) \cong T(\square^m) \otimes T(\square^n)$ by reflecting the complicial marking extension of Lemma 6.9 into $\text{PreComp}$. Dually, we have $T(\square^m \otimes \square^n) \cong T(\square^m) \otimes T(\square^n)$ for any $m \geq 0$ and $n \geq 1$.

Let $m, n \geq 1$. Observe that the left square below is a pushout in $\text{cSet}^+$ by Lemma 2.17(3):

\[
\begin{array}{ccc}
\square^m \otimes \square^n & \longrightarrow & \square^m \otimes \square^n \\
\downarrow & & \downarrow \\
\square^m \otimes \square^n & \longrightarrow & \square^m \otimes \square^n \\
\end{array}
\]

Since $T$ is cocontinuous, it follows that the right square is a pushout in $\text{PreComp}$. On the other hand, since both $T(\square^n) \rightarrow T(\square^m)$ and $T(\square^n) \rightarrow T(\square^n)$ are entire, the square below is a pushout in $\text{PreComp}$ by [Ver08a, Lem. 140]:

\[
\begin{array}{ccc}
T(\square^m) \otimes T(\square^n) & \longrightarrow & T(\square^m) \otimes T(\square^n) \\
\downarrow & & \downarrow \\
T(\square^m) \otimes T(\square^n) & \longrightarrow & T(\square^m) \otimes T(\square^n) \\
\end{array}
\]

Thus by comparing the two pushout squares in $\text{PreComp}$, we obtain $T(\square^m \otimes \square^n) \cong T(\square^m) \otimes T(\square^n)$. The naturality of these isomorphisms is evident, and this completes the proof.

**6.3. Triangulating the pseudo Gray tensor product.** The goal of this subsection is to prove the following theorem.

**Theorem 6.10.** The adjunction $T \dashv U$ is monoidal with respect to the pseudo Gray tensor products. Equivalently, $T: (\text{cSet}^+, \otimes) \rightarrow (\text{PreComp}, \otimes)$ is strong monoidal.

Fix $m, n \geq 1$. By Remark 2.14 the square

\[
\begin{array}{ccc}
\sqcup \square^{k+\ell} & \longrightarrow & \square^{m+n} \\
\downarrow & & \downarrow \\
\sqcup \square^{k+\ell} & \longrightarrow & \square^n \otimes \square^n \\
\end{array}
\]

is a pushout in $\text{PreComp}$.
is a pushout in $\mathbf{cSet}$ where the coproducts are taken over all pairs of face maps $\square^k \to \square^m$ and $\square^\ell \to \square^n$ such that $k, \ell \geq 1$. This pushout is preserved by $T$, so the right square in

$$
\begin{array}{ccc}
\bigcup \Delta^{k+\ell} & \xrightarrow{\bigcup i_{k+\ell}} & \bigcup (\Delta^1)^{\otimes (k+\ell)} \\
\downarrow & & \downarrow \\
\bigcup \Delta^{k+\ell} & \xrightarrow{\bigcup \tau_{m+n}} & \bigcup \tau_{m+n+1} (\bigcup (\Delta^1)^{\otimes (k+\ell)}) \\
\end{array}
$$

is a pushout in $\mathbf{Pre Comp}$. The left square is also a pushout by Proposition 5.6, so the pasted square is a pushout too. In this subsection, we define $A$ to be the corresponding pushout in $\mathbf{sSet}$ (and not in $\mathbf{Pre Comp}$):

$$
\begin{array}{ccc}
\bigcup \Delta^{k+\ell} & \xrightarrow{(\Delta^1)^{\otimes (m+n)}} & A \\
\downarrow & \text{p.o.} & \downarrow \\
\bigcup \Delta^{k+\ell} & \xrightarrow{A} & T(\square^m \otimes \square^n) \\
\end{array}
$$

so that its pre-complicial reflection $A^{\text{pre}}$ is precisely $T(\square^m \otimes \square^n)$.

**Lemma 6.11.** An $r$-simplex $\phi$ is marked in $A$ but not in $(\Delta^1)^{\otimes (m+n)}$ if and only if the restriction $\phi^{-1}(\{1, \ldots, r\}) \to \{1, \ldots, r\}$ of $\phi$ is an isomorphism of linearly ordered sets and moreover $\phi^{-1}(\{1, \ldots, r\})$ intersects both $\{1, \ldots, m\}$ and $\{m+1, \ldots, m+n\}$.

**Proof.** Compute the colimit. \(\square\)

**Lemma 6.12.** An $r$-simplex $\phi$ in $T(\square^m) \otimes T(\square^n)$ is unmarked if and only if there exist either

$$
1 \leq i_1 < \cdots < i_r \leq m
$$

or

$$
m + 1 \leq i_1 < \cdots < i_r \leq m + n
$$

such that $\phi(i_p) = p$ for all $1 \leq p \leq r$.

**Proof.** Since $\otimes$ is the categorical product on $\mathbf{Pre Comp}$, $\phi$ is marked if and only if both $\pi_1(\phi)$ and $\pi_2(\phi)$ are marked. Equivalently, $\phi$ is unmarked if and only if either $\pi_1(\phi)$ or $\pi_2(\phi)$ is unmarked. Thus the assertion follows from Proposition 5.6. \(\square\)

**Lemma 6.13.** There is a complicial marking extension $A \to T(\square^m) \otimes T(\square^n)$ that commutes with the evident inclusions of $(\Delta^1)^{\otimes (m+n)}$.

**Proof.** We apply the proof strategy from Section 6.1 with $B = T(\square^m) \otimes T(\square^n)$.

One can easily check using Proposition 5.6 and Lemmas 6.11 and 6.12 that any marked simplex $\phi$ in $T(\square^m) \otimes T(\square^n)$ with $D(\phi) = 0$ must also be marked in $A$. This verifies Assumption 1.

That Assumption 2 holds can be checked similarly to the corresponding part in the proof of Lemma 6.9. \(\square\)

Let $m \geq 1$ and $n \geq 0$. Observe that the square below is a pushout in $\mathbf{cSet}$:

$$
\begin{array}{ccc}
\bigcup \square^m & \xrightarrow{\square^m \otimes \square^n} & \square^m \otimes \square^n \\
\downarrow & & \downarrow \\
\bigcup \square^m & \xrightarrow{\square^m \otimes \square^n} & \square^m \otimes \square^n \\
\end{array}
$$
where the coproducts are taken over all \([1]^0 \to [1]^n\). This pushout is preserved by \(T\), so the right square in

\[
\begin{array}{c}
\coprod \Delta^m \xrightarrow{\coprod i_m} \coprod T(\square^m) \\
\downarrow \\
\coprod \Delta^m \xrightarrow{T(\square^m)} T(\square^m \oplus \square^n)
\end{array}
\]

is a pushout in \(\text{PreComp}\). The left square is also a pushout by Proposition 5.6, so the pasted square is a pushout too. Let \(A'\) denote the “corresponding” pushout in \(\text{sSet}^+\) (and not in \(\text{PreComp}\)):

\[
\begin{array}{c}
\coprod \Delta^m \xrightarrow{\coprod i_m} \coprod A \\
\downarrow \\
\coprod \Delta^m \xrightarrow{A'} A'
\end{array}
\]

so that its pre-complicial reflection \((A')^{\text{pre}}\) is precisely \(T(\square^m \oplus \square^n)\).

**Lemma 6.14.** The marked simplicial set \(A'\) is obtained from \(A\) by marking those \(m\)-simplices \(\phi\) such that \(\phi(i) = i\) for \(1 \leq i \leq m\) and \(\phi(i) \in \{\pm \infty\}\) for \(i > m\).

The unmarked simplices in \(B' = T(\square^m) \oplus T(\square^n)\) admit a characterization similar to Lemma 6.12.

**Lemma 6.15.** An \(r\)-simplex in \(T(\square^m) \oplus T(\square^n)\) with \(r \neq m\) is unmarked if and only if it is unmarked in \(T(\square^m) \oplus T(\square^n)\). An \(m\)-simplex \(\phi\) in \(T(\square^m) \oplus T(\square^n)\) is unmarked if and only if there exist

\[m + 1 \leq i_1 < \cdots < i_m \leq m + n\]

such that \(\phi(i_p) = p\) for all \(1 \leq p \leq m\).

**Lemma 6.16.** There is a complicial marking extension \(A' \to T(\square^m) \oplus T(\square^n)\) that commutes with the evident inclusions of \((\Delta^1)^{\oplus(m+n)}\).

**Proof.** The proof is similar to those of Lemmas 6.9 and 6.13.

**Proof of Theorem 6.17.** Since both \(T(- \oplus -)\) and \(T(-) \oplus T(-)\) preserve colimits in each variable, it suffices to check the existence of natural isomorphisms \(T(X \oplus Y) \cong T(X) \oplus T(Y)\) for \(X, Y\) generic (possibly marked) cubes.

For the appropriate values of \(m\) and \(n\), we may obtain isomorphisms

\[
T(\square^m \oplus \square^n) \cong T(\square^n) \oplus T(\square^n),
\]

\[
T(\square^m \oplus \square^n) \cong T(\square^m) \oplus T(\square^n),
\]

\[
T(\square^m \oplus \square^n) \cong T(\square^m) \oplus T(\square^n)
\]

by reflecting to \(\text{PreComp}\) the complicial marking extensions of Lemmas 6.13 and 6.16 and the dual of the latter respectively.

Let \(m, n \geq 1\). Observe that the left square below is a pushout in \(\text{cSet}^+\) by Lemma 2.17(3):

\[
\begin{array}{c}
\square^m \oplus \square^n \xrightarrow{\square^m \oplus \square^n} \square^m \oplus \square^n \\
\downarrow \\
\square^m \oplus \square^n \xrightarrow{\square^m \oplus \square^n} \square^m \oplus \square^n
\end{array}
\]

\[
\begin{array}{c}
T(\square^m \oplus \square^n) \xrightarrow{T(\square^m \oplus \square^n)} T(\square^m \oplus \square^n) \\
\downarrow \\
T(\square^m \oplus \square^n) \xrightarrow{T(\square^m \oplus \square^n)} T(\square^m \oplus \square^n)
\end{array}
\]
Since $T$ is cocontinuous, it follows that the right square is a pushout in $\text{PreComp}$. On the other hand, since both $T(\square^m) \to T(\tilde{\square}^m)$ and $T(\square^n) \to T(\tilde{\square}^n)$ are entire, the square below is a pushout in $\text{PreComp}$ by Proposition 1.26.

$$
\begin{array}{ccc}
T(\square^m) \oplus T(\square^n) & \longrightarrow & T(\tilde{\square}^m) \oplus T(\tilde{\square}^n) \\
\downarrow & & \downarrow \\
T(\square^m) \oplus T(\tilde{\square}^n) & \longrightarrow & T(\tilde{\square}^m) \oplus T(\tilde{\square}^n)
\end{array}
$$

Thus by comparing the two pushout squares in $\text{PreComp}$, we obtain $T(\tilde{\square}^m \oplus \tilde{\square}^n) \cong T(\tilde{\square}^m) \oplus T(\tilde{\square}^n)$. The naturality of these isomorphisms is evident, and this completes the proof. \qed

7. Triangulating model structures

The main theorem of our final section is the following.

**Theorem 7.1.** The adjunction $T \dashv U$ is a Quillen adjunction with respect to the comical model structure on $\text{eSet}$ and the complicial model structure on $\text{PreComp}$.

In the following proof, we denote a non-degenerate $r$-simplex $\phi: \{1, 2, 3\} \to \{1, \ldots, r, \pm \infty\}$ in the simplicial set $(\Delta^1)^{\oplus n}$ by the sequence $\phi(1)\phi(2)\phi(3)$ and omitting the letter $\infty$. For instance, $21-$ denotes the 2-simplex $\phi$ given by $\phi(1) = 2$, $\phi(2) = 1$ and $\phi(3) = -\infty$. Note that since $\phi$ is assumed to be non-degenerate, the dimension of $\phi$ can be recovered as the maximum integer appearing in the sequence.

**Proof.** Clearly $T$ sends the boundary inclusions $\partial \square^0 \to \square^0$ and $\partial \square^1 \to \square^1$ to (maps that are isomorphic to) the boundary inclusions $\partial \Delta^0 \to \Delta^0$ and $\partial \Delta^1 \to \Delta^1$ respectively. For any $n \geq 2$, we have

$$
T(\partial \square^n \to \square^n) \cong T\left(\left(\partial \square^1 \to \square^1\right)^{\oplus n}\right)
\cong (T(\partial \square^1 \to \square^1))^{\oplus n}
\cong (\partial \Delta^1 \to \Delta^1)^{\oplus n}
$$

by Theorem 6.5. Also, $T$ sends the marker $\square^n \to \tilde{\square}^n$ to the monomorphism $(\Delta^1)^{\oplus n} \to \tau_{n-1}(\Delta^1)^{\oplus n}$ by definition. This shows that $T$ preserves cofibrations.

Next we show that $T$ sends the open box inclusions to trivial cofibrations. We will check this “by hand” on the boxes of dimension $\leq 3$. This will imply the general case since the higher dimensional box inclusions are generated by these low dimensional ones in the sense of Proposition 3.10. $T$ is strong monoidal with respect to $\oplus$ (Theorem 6.10), and the Leibniz Gray tensor product of a complicial horn inclusion and a monomorphism (in $\text{PreComp}$) may be obtained as a composite of pushouts of complicial horn inclusions [Ver08b, Lem. 72].

Clearly $T$ sends $\cap_{1, \epsilon} \to \square_{1, \epsilon}$ to the trivial cofibration $\Lambda^1_{1-\epsilon} \to \Delta^1_{1-\epsilon}$. Consider the open box inclusion $\cap_{1,0} \to \square_{1,0}$. Its image under $T$ may be written as a pushout of the horn inclusion $\Lambda^2_2 \to \Delta^2_2$ followed by a pushout of $\Lambda^2_2 \to \Delta^2_1$. The following pictures (in which thick arrows indicate marked simplices) depict this factorization:

$$
\begin{array}{ccc}
\cap_{1, \epsilon} & \to & \square_{1, \epsilon} \\
\cap_{1,0} & \to & \square_{1,0}
\end{array}
$$

The box inclusions $\cap_{k, \epsilon} \to \square_{k, \epsilon}$ for other values of $k$ and/or $\epsilon$ can be treated similarly.
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Now consider the open box inclusion \( \bigtriangleup^3_{2,0} \to \square^3_{2,0} \). Observe that the only marked, non-degenerate cubes in \( \square^3_{2,0} \) are \( \partial_{1,1}, \partial_{3,1}, \) and \( \partial_{3,1}\partial_{1,1} \):

\[
\begin{array}{c}
\partial_{1,0} \\
downarrow \\
\partial_{2,1}
\end{array}
\quad
\begin{array}{c}
\partial_{1,0} \\
\partial_{2,1} \\
downarrow \\
\partial_{1,0}
\end{array}
\quad
\begin{array}{c}
\partial_{2,0} \\
\partial_{3,1} \\
downarrow \\
\partial_{2,0}
\end{array}
\quad
\begin{array}{c}
\partial_{3,1} \\
\partial_{1,0} \\
downarrow \\
\partial_{3,1}
\end{array}
\]

Let \( B \) denote the marked simplicial set obtained from \((\Delta^1)^\otimes 3\) by marking the 3-simplex \( \iota_3 = 123 \), the 2-simplices \( 12- \) and \( -12 \), and the 1-simplex \( -1- \). Then the pre-complicial reflection \( B^{\text{pre}} \) is precisely \( T(\square^3_{2,0}) \). We define \( A \) to be the regular subset of \( B \) consisting of those simplices \( \phi \) such that:

- \( \phi(1) \in \{\pm \infty\} \);
- \( \phi(2) = -\infty \); or
- \( \phi(3) \in \{\pm \infty\} \)

so that the pre-complicial reflection \( A^{\text{pre}} \) is \( T(\bigtriangleup^3_{2,0}) \). Then we have a sequence of inclusions

\[
A = A^0 \to A^1 \to \ldots \to A^8 = B
\]

where \( A^{s-1} \to A^s \) is the pushout of a suitable trivial cofibration as indicated in Table 1. This table is to be interpreted as saying, for example, that the inclusion \( A^0 \to A^1 \) fits into the pushout square

\[
\begin{array}{c}
\Lambda^1_2 \\
downarrow \\
\Delta^2_1 \longrightarrow \\
\mid \\
A^0 \\
\Delta^1_2 \\
\longrightarrow \\
A^1
\end{array}
\]

in \( \text{sSet}^+ \) where the composite \( \Delta^2_1 \to A^1 \to B \) corresponds to the simplex \( \phi = 211 \), and the face \( \phi\partial_1 \) corresponding to the missing face in the horn is 111. One can check that every non-degenerate face in \( B \setminus A \) appears exactly once in Table 1 and moreover it is marked if and only if it appears either in the “interior” column or in the sixth or seventh row. It is also straightforward to verify using Proposition 5.6 that, for each \( 1 \leq s \leq 8 \), the marked simplicial set \( A^{s-1} \) indeed contains enough (marked) simplices to support a map from the domain in the “pushout of” column. By reflecting everything to \( \text{PreComp} \), we can deduce that \( T \) sends the box inclusion \( \bigtriangleup^3_{2,0} \to \square^3_{2,0} \) to a trivial cofibration. The case \( \bigtriangleup^3_{2,1} \to \square^3_{2,1} \) is dual.
It remains to prove that \(T(\square^n_{k,\epsilon'} \to \square^n_{k,\epsilon''})\) is a trivial cofibration for any \(n, k, \epsilon\). We show that this map is in fact invertible. Note that, by unwinding the above argument, one can express \(T(\square^n_{k,\epsilon'} \to \square^n_{k,\epsilon''})\) as a composite

\[
T(\square^n_{k,\epsilon'}) = X^0 \to X^1 \to \cdots \to X^N = T(\square^n_{k,\epsilon''})
\]

where each map \(X^{s-1} \to X^s\) is a pushout (in \text{PreComp}\) of the pre-complicial reflection of a complicial horn inclusion. We show by induction on \(s\) that all \((n-1)\)-simplices contained in \(X^s\) are marked in \(T(\square^n_{k,\epsilon'})\). This indeed implies that \(T(\square^n_{k,\epsilon'} \to \square^n_{k,\epsilon''})\) is invertible since this map fits into the pushout square

\[
\begin{array}{ccc}
\Delta^{n-1} & \xrightarrow{\phi} & T(\square^n_{k,\epsilon'}) \\
\downarrow & & \downarrow \\
\Delta^{n-1} & \longrightarrow & T(\square^n_{k,\epsilon''})
\end{array}
\]

in \text{PreComp}\) where

\[
\phi(i) = \begin{cases} 
  i, & i < k, \\
  +\infty, & i = k, \quad \epsilon = 0, \\
  -\infty, & i = k, \quad \epsilon = 1, \\
  i-1, & i > k.
\end{cases}
\]

For the base case, write \(\square^n_{k,\epsilon'}\) as a pushout

\[
\bigsqcup \square^m \longrightarrow \square^n \\
\downarrow & \downarrow \\
\bigsqcup \widetilde{\square}^m \longrightarrow \square^n_{k,\epsilon'}
\]

where the coproducts are taken over all marked faces of \(\square^n_{k,\epsilon'}\), which in particular include all faces \(\partial_{k,\eta}\) of codimension 1 with \((\ell, \eta) \neq (k, \epsilon)\). By applying \(T\) to this pushout square, we can deduce that any \((n-1)\)-simplex of the form

\[
\Lambda^{n-1} \xrightarrow{\iota_{n-1}} (\Lambda^1)^\otimes(n-1) \xrightarrow{T(\partial_{k,\eta})} T(\square^n)
\]

is marked in \(T(\square^n_{k,\epsilon'})\). By combining this observation with Proposition \[5.6\] one can deduce that any \((n-1)\)-simplex contained in \(X^0\) is marked in \(T(\square^n_{k,\epsilon'})\).

For the inductive step, suppose that all \((n-1)\)-simplices contained in \(X^{s-1}\) are marked in \(T(\square^n_{k,\epsilon'})\). Suppose further that \(X^s\) contains a non-degenerate \((n-1)\)-simplex \(\phi\) that \(X^{s-1}\) does not contain (for otherwise we are done). Then \(X^{s-1} \to X^s\) fits into either a pushout square of the form

\[
\begin{array}{ccc}
(\Lambda^{n-1})^{pre} & \longrightarrow & X^{s-1} \\
\downarrow & & \downarrow \\
(\Lambda^{n-1})^{pre} & \phi \longrightarrow & X^s \\
\end{array}
\]

or one of the form

\[
\begin{array}{ccc}
(\Lambda^n)^{pre} & \longrightarrow & X^{s-1} \\
\downarrow & & \downarrow \\
(\Delta^n)^{pre} & \chi \longrightarrow & X^s \\
\end{array}
\]

or one of the form

\[
\begin{array}{ccc}
(\Lambda^n)^{pre} & \longrightarrow & X^{s-1} \\
\downarrow & & \downarrow \\
(\Delta^n)^{pre} & \chi \longrightarrow & X^s \\
\end{array}
\]
with \( \chi T \phi = \phi \). In the former case, \( \phi \) is marked in \( X^* \) and hence in \( T(\Delta^n, c^s) \) since the unique non-degenerate \((n-1)\)-simplex in \( \Delta^{n-1} \) is marked. In the latter case, the inductive hypothesis implies that \( \chi \) extends to the marked simplicial set \( \Delta^n \). Since \( X^* \) is a pre-complicial set, it follows that \( \phi = \chi T \phi \) is marked in \( X^* \) and hence in \( T(\Delta^n, c^s) \). This completes the proof.

The \( n \)-trivial version can be proved analogously.

**Theorem 7.2.** For each \( 0 \leq n < \infty \), the adjunction \( T \dashv U \) is a Quillen adjunction with respect to the \( n \)-trivial comical model structure on \( cSet^+ \) and the \( n \)-trivial complicial model structure on \( PreComp \).

**Proof.** Analogous to the proof of Theorem 7.1. Observe that \( T \) sends the (cubical) \( m \)-marker to a pushout of the (simplicial) \( m \)-marker.

**Remark 7.3.** We expect the saturated (\( n \)-trivial) complicial model structure on \( PreComp \) (obtained by transferring the model structure established in [OR18, Thm. 1.25]) to be monoidal with respect to the lax Gray tensor product \( \otimes \). If this is indeed the case, then the saturated version of Theorems 7.1 and 7.2 can be proved similarly.

We conclude this section with a conjecture.

**Conjecture 7.4.** The functor \( T: cSet^+ \to PreComp \) is a Quillen equivalence with respect to the (resp. saturated, \( n \)-trivial) comical model structure and the (resp. saturated, \( n \)-trivial) complicial model structure.

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