Abstract. Let \((Q, n, k)\) be a commutative local Noetherian ring, \(f_1, \ldots, f_c\) a \(Q\)-regular sequence in \(n\), and \(R = Q/(f_1, \ldots, f_c)\). Given a complex of finitely generated free \(R\)-modules, we give a construction of a complex of finitely generated free \(Q\)-modules having the same homology. A key application is when the original complex is an \(R\)-free resolution of a finitely generated \(R\)-module. In this case our construction is a sort of converse to a construction of Eisenbud-Shamash yielding a free resolution of an \(R\)-module \(M\) over \(R\) given one over \(Q\).

1. Introduction

Let \((Q, n, k)\) be a commutative local Noetherian ring and \(R = Q/I\) for some ideal \(I\) of \(Q\). Given a free resolution of an \(R\)-module \(M\), can one then describe a free resolution of \(M\) over \(Q\)? When \(I\) is generated by a \(Q\)-regular sequence \(f_1, \ldots, f_c\) contained in \(n\), we give a construction that provides a positive answer to this question. Our construction is a sort of converse of those of Nagata \[Na\] and Eisenbud-Shamash \[Ei\] in this same context.

We have recently learned that similar constructions are given by Eisenbud, Peeva and Schreyer in \[EiPeSc, Section 7\] and Burke in \[Bu\].

We present the construction in Section 2. Sections 3-5 consist of some applications.

2. The Construction

Let \((\mathcal{F}, \partial^\mathcal{F})\) be a complex of finitely generated free \(R\)-modules. Denote by \((F, \partial^F)\) a lifting to \(Q\) of \((\mathcal{F}, \partial^\mathcal{F})\), that is \(F\) is a graded module consisting of free \(Q\)-modules, \(\partial^F\) is an endomorphism of \(F\) of degree \(-1\), and \(F \otimes_Q R = \mathcal{F}\) with \(\partial^F \otimes R = \partial^\mathcal{F}\).

2.1. We denote by \(\text{Hom}_{gr}(F, F) = \bigsqcup_{n \leq 0} \text{Hom}_{gr}(F, F)_n\) the nonpositively graded \(Q\)-module of endomorphisms of \(F\), where \(\text{Hom}_{gr}(F, F)_n = \prod_{i \in \mathbb{Z}} \text{Hom}_Q(F_i, F_{i+n})\). Since \(\text{Hom}_{gr}(F, F)_n\) is a direct product of flat modules (and \(Q\) is Noetherian), a result of Chase \[Ch\] tells us that \(\text{Hom}_{gr}(F, F)_n\) is a flat \(Q\)-module. It follows that \(\text{Hom}_{gr}(F, F)\) is a flat \(Q\)-module.
Let $K$ denote the Koszul complex on $f_1, \ldots, f_c$, and $\mathcal{B}$ denote a basis of $K$ together with $0$, that is,
\[ \mathcal{B} = \{ e_{i_1} \wedge \cdots \wedge e_{i_j} \mid i_1 < \cdots < i_j, 1 \leq j \leq c \} \cup \{0, 1\} \]
We adhere to the convention that $1 \wedge \alpha = \alpha \wedge 1 = \alpha$, and assign degrees in the usual way: $|e_{i_1} \wedge \cdots \wedge e_{i_j}| = j, |1| = 0$; we set the degree of $0$ to be $-1$.

**Lemma 2.2.** There exist (graded) endomorphisms $\{t^\alpha\}_{\alpha \in \mathcal{B}}$ of $F$ of degree $-|\alpha| - 1$ such that for all $0 \neq \gamma \in \mathcal{B}$
\[
\sum_{\alpha \wedge \beta = \pm \gamma} (-1)^{|\beta|+(\alpha \beta)} t^\beta t^\alpha + \sum_{e_i|\gamma} (-1)^{|\gamma|+(e_i\gamma)} f_i t^{e_i\gamma} = 0
\]
where $(-1)^{(\alpha \beta)}$ is the appropriate sign such that $(-1)^{(\alpha \beta)} \alpha \wedge \beta = \gamma$, and $[e_i\gamma]$ is the element in $\mathcal{B}$ equal to $e_i \wedge \gamma$ up to sign.

Note that the first sum in (1) involves the terms $t^1 t^\gamma$ and $t^1 t^1$.

**Proof.** We begin by setting $t^0 = \text{Id}_F$ and $t^1 = \partial F$. Then $t^\alpha$ for $|\alpha| \geq 1$ will be defined inductively by degree.

Let $K^\sharp$ denote the augmented Koszul complex $0 \to K_c \to \cdots \to K_0 \to R \to 0$, which is an exact complex. Since the $Q$-module $\text{Hom}_{K^\sharp}(F, F)$ of graded $Q$-linear endomorphisms of $F$ of nonpositive degree is flat, the complex $\text{Hom}_{K^\sharp}(F, F) \otimes Q K^\sharp$ is exact.

For degree $1$, we have $\partial^1 \text{Hom}_{K^\sharp}(F, F) \otimes K^\sharp(1t^1 \otimes 1) = t^1 t^1 \otimes \partial F$. Since $t^1 = \partial F$, $t^1 \otimes \partial F = \partial F^\sharp$, and therefore $t^1 t^1 \otimes \partial F = (\partial F)^2 = 0$. By exactness of $\text{Hom}_{K^\sharp}(F, F) \otimes K^\sharp$, there exist endomorphisms $t^e_i$ of $F$ of degree $-2$ such that
\[
\partial \text{Hom}_{K^\sharp}(F, F) \otimes K^\sharp \left( - \sum_{i=1}^c (t^e_i \otimes e_i) \right) = t^1 t^1 \otimes 1
\]
That is, $- \sum f_i t^{e_i} \otimes 1 = t^1 t^1 \otimes 1$, which gives the desired equation $t^1 t^1 + \sum f_i t^{e_i} = 0$.

Now assume that the $t^\gamma$ have been defined, and satisfy (1), for all $\gamma \in \mathcal{B}$ with $|\gamma| \leq d$. We have
\[
\partial \text{Hom}_{K^\sharp}(F, F) \otimes K^\sharp \left( \sum_{|\gamma|=d} \left( \sum_{\alpha \wedge \beta = \pm \gamma} (-1)^{|\beta|+(\alpha \beta)} t^\beta t^\alpha \right) \otimes \gamma \right)
\]
\[
= \sum_{|\gamma|=d} \left( \sum_{\alpha \wedge \beta = \pm \gamma} (-1)^{|\beta|+(\alpha \beta)} t^\beta t^\alpha \right) \otimes \partial K(\gamma)
\]
\[
= \sum_{|\gamma|=d-1} (-1)^{(e_i\gamma)} f_i \left( \sum_{\alpha \wedge \beta = \pm [e_i \gamma]} (-1)^{|\beta|+(\alpha \beta)} t^\beta t^\alpha \right) \otimes \gamma
\]
We want to show the above quantity is zero. For this it suffices to show that for $\gamma \in \mathcal{B}$ with $|\gamma| = d - 1$,
\[
\sum_{e_i|\gamma} (-1)^{(e_i \gamma)} f_i \left( \sum_{\alpha \wedge \beta = \pm [e_i \gamma]} (-1)^{|\beta|+(\alpha \beta)} t^\beta t^\alpha \right) = 0
\]
We have

\[
\sum_{e_i \gamma} (-1)^{(e_i \gamma)} f_i \left( \sum_{\alpha, \beta = \pm \gamma} (-1)^{\beta} t^\beta f_i [e_i \beta] \right)
\]

\[
= \sum_{\alpha, \beta = \pm \gamma} \left[ t^\alpha \left( \sum_{e_i \gamma} (-1)^{\alpha + (e_i \beta) + (e_i \gamma)} f_i [e_i \beta] \right) + \left( \sum_{e_i \gamma} (-1)^{\beta + 1 + (e_i \beta) + (e_i \gamma)} f_i [e_i \beta] \right) t^\alpha \right]
\]

(2)

From the equalities

\[
[e_i \gamma] = (-1)^{(e_i \gamma)} e_i \land \gamma
\]

\[
= (-1)^{(e_i \gamma) + (\beta \alpha)} e_i \land \beta \land \alpha
\]

\[
= (-1)^{(e_i \gamma) + (\beta \alpha) + (e_i \beta) [e_i \beta] \land \alpha}
\]

\[
= (-1)^{(e_i \gamma) + (\beta \alpha) + (e_i \beta) + [e_i \beta] [e_i \gamma]}
\]

we see that \((-1)^{(e_i \beta) \alpha + (e_i \gamma)} = (-1)^{(\beta \alpha) + (e_i \beta)}\). Quantity (2) above thus becomes

\[
\sum_{\alpha, \beta = \pm \gamma} \left[ t^\alpha \left( \sum_{e_i \gamma} (-1)^{\alpha + (\beta \alpha) + (e_i \beta) f_i [e_i \beta] + (e_i \gamma) f_i [e_i \beta]} \right) \right]
\]

\[
+ \left( \sum_{e_i \gamma} (-1)^{\beta + 1 + (\beta \alpha) + (e_i \beta) + (e_i \gamma) f_i [e_i \beta]} \right) t^\alpha \right]
\]

(3)

Since \(|e_i \beta| \leq d\), we have by induction the relations

\[
\sum_{\delta \land \epsilon = \pm \beta} (-1)^{|\epsilon| + (\delta \epsilon) t^\delta + \sum_{e_i \beta} (-1)^{|\beta + (e_i \beta) f_i [e_i \beta] | 0}
\]

which we may rearrange to

\[
\sum_{e_i \gamma} (-1)^{(e_i \beta) f_i [e_i \beta]} = \sum_{e_i \gamma} (-1)^{(e_i \beta) + 1 f_i [e_i \beta]} + \sum_{e_i \beta} (-1)^{|\beta + (\delta \epsilon) + 1 t^\delta}
\]
Substituting this expression into (3) gives

\( \sum_{\alpha \land \beta = \pm \gamma} t^\alpha \left( \sum_{\epsilon_i \mid \gamma} (-1)^{|\alpha|+(\beta \alpha)+(e_i \beta)+1} f_i e_i[\epsilon_i, \beta] \right) \)

\( + \sum_{\alpha \land \beta = \pm \gamma} \left( \sum_{\epsilon_i \mid \beta} (-1)^{|\alpha|+(\beta \alpha)+(e_i \beta)+1} f_i e_i[\epsilon_i, \beta] \right) t^\alpha \)

We will show that the two outer sums in (4) are zero. The first outer sum may be broken up as

\( \sum_{\alpha \land \beta = \pm \gamma} t^\alpha \left( \sum_{\epsilon_i \mid \gamma} (-1)^{|\alpha|+(\beta \alpha)+(e_i \beta)+1} f_i e_i[\epsilon_i, \beta] \right) \)

Consider a term from the first sum:

\( (-1)^{|\alpha|+(\beta \alpha)+(e_i \beta)+1} f_i e_i[\epsilon_i, \beta] \)

Since \( e_i \mid \beta \) and \( e_i \mid \gamma \), we must have \( e_i \mid \alpha \). Therefore there is a corresponding term in the second sum:

\( (-1)^{|\alpha'|+1+(\beta' \alpha')+(e_i \beta')+1} f_i e_i[\epsilon_i, \beta'] t^\alpha' \)

where \( \alpha = [e_i \beta] \) and \( \alpha' = [e_i \beta] \). An easy calculation shows that \((-1)^{(\beta \alpha)+(e_i \beta)} = (-1)^{(\beta' \alpha')+(e_i \beta')+(e_i \beta)+(\beta \alpha)}\). Substituting this into (4), we see that the two terms in (6) and (4) are of opposite sign, and therefore cancel. It follows that the entire quantity (5) is zero.

The second outer sum in (5) may be broken up as

\( \sum_{\alpha \land \beta = \pm \gamma} \left( \sum_{\delta \land \epsilon = \pm \beta} (-1)^{|\alpha|+(\beta \alpha)+(\delta \epsilon)+1} t^\delta \right) \)

\( + \sum_{\alpha \land \beta = \pm \gamma} \left( \sum_{\delta \land \epsilon = \pm \beta} (-1)^{|\alpha|+(\beta \alpha)+(\delta \epsilon)+1} t^\delta \right) t^\alpha \)
Consider a term of the first sum:

\[(9) \quad (-1)^{\lvert \alpha \rvert + (\beta \alpha) + \lvert \delta \rvert + (\delta \epsilon) + 1} t^\alpha t^\epsilon t^\delta \]

There is a corresponding term in the second sum:

\[(10) \quad (-1)^{(\lvert \alpha' \rvert + 1)(\lvert \beta' \rvert + 1) + (\beta' \alpha') + (\delta' \epsilon') + 1} t^\alpha t^\epsilon t^\delta\]

where $\alpha = \epsilon'$, $\epsilon = \delta'$ and $\delta = \alpha'$. An easy calculation shows that $(-1)^{(\beta' \alpha') + (\delta' \epsilon')} = (-1)^{(\alpha || \delta| + || \delta \epsilon| + (\beta \alpha) + (\delta \epsilon))}$. Substituting this into (10) shows that the two terms (9) and (10) are of opposite sign, and therefore cancel. It follows that the entire quantity (8) is zero.

We have now shown that

\[
\partial \text{Hom}_{gr}(F, F) \otimes K^2 \left( \sum_{\lvert \gamma \rvert = d} \left( -1 ^{\lvert \beta \rvert + (\alpha \beta) t^\beta t^\alpha} \right) \otimes \gamma \right) = 0
\]

By exactness of $\text{Hom}_{gr}(F, F) \otimes K^2$, there exists for each $\mu \in B$ with $|\mu| = d + 1$, endomorphisms $t^\mu \in \text{Hom}_{gr}(F, F)$ of degree $-d - 2$ such that

\[
\partial \text{Hom}_{gr}(F, F) \otimes K^2 \left( (-1)^{d+1} \sum_{|\mu|=d+1} t^\mu \otimes \mu \right) = \sum_{|\gamma|=d} \left( \sum_{\alpha \land \beta \equiv \pm \gamma} (-1)^{|\beta| + (\alpha \beta) t^\beta t^\alpha} \right) \otimes \gamma
\]

It is easy to see that (11) holds for each $\gamma \in B$, $|\gamma| = d$, and thus induction is complete. \hfill \square

Consider the complex $F \otimes_Q K$. We perturb its differential $\partial^F \otimes K + F \otimes \partial^K$ to

\[
\partial = \sum_{\alpha \in B} t^\alpha \otimes s_\alpha
\]

where $t^0 = \text{Id}_F$, $t^1 = \partial^F$, $s_0 = \partial^K$, $t^\alpha$ are defined by Lemma 2.2 for $\alpha \neq 0$, and $s_\alpha$ is multiplication by $\alpha$ for $\alpha \neq 0$.

**Lemma 2.3.**

\[
\partial^2 = 0
\]
Now using the Leibniz rule \( \partial^2(x \otimes y) = \left( \sum_{\beta \in \mathcal{B}} t^\beta \otimes s_\beta \right) \left( \sum_{\alpha \in \mathcal{B}} t^\alpha \otimes s_\alpha \right) (x \otimes y) \)

\[
= \left( \sum_{\beta \in \mathcal{B}} t^\beta \otimes s_\beta \right) \left( \sum_{\alpha \neq 0} (-1)^{|\alpha|} t^\alpha (x) \otimes \alpha \wedge y + (-1)^{|x|} x \otimes \partial^K(y) \right) \\
= \left( \sum_{\beta \neq 0} t^\beta \otimes s_\beta + t^0 \otimes s_0 \right) \left( \sum_{\alpha \neq 0} (-1)^{|\alpha|} t^\alpha (x) \otimes \alpha \wedge y + (-1)^{|x|} x \otimes \partial^K(y) \right) \\
= \sum_{\alpha, \beta \neq 0} (-1)^{|x|(|\alpha|+|\beta|)+(|\alpha|+1)|\beta|} t^\alpha (x) \otimes \beta \wedge \alpha \wedge y \\
+ \sum_{\beta \neq 0} (-1)^{|x||\beta|+|x||\beta|} t^\beta (x) \otimes \beta \wedge \partial^K(y) \\
+ \sum_{\alpha \neq 0} (-1)^{|x||\alpha|+|x|(|\alpha|-1)} t^\alpha (x) \otimes \partial^K(\alpha \wedge y)
\]

Now using the Leibniz rule \( \partial^K(\alpha \wedge y) = \partial^K(\alpha) \wedge y + (-1)^{|\alpha|} \alpha \wedge \partial^K(y) \) in the third sum, we achieve cancellation in the second and third sums to obtain

\[
\partial^2(x \otimes y) = \sum_{\alpha, \beta \neq 0} (-1)^{|x|(|\alpha|+|\beta|)+(|\alpha|+1)|\beta|} t^\alpha (x) \otimes \beta \wedge \alpha \wedge y \\
+ \sum_{|\alpha| > 0} (-1)^{|x||\alpha|+|x|-|\alpha|-1} t^\alpha (x) \otimes \partial^K(\alpha) \wedge y \\
= \left( \sum_{\alpha, \beta \neq 0} (-1)^{|\alpha|(|\alpha|+1)|\beta|} t^\alpha \wedge \alpha + \sum_{|\alpha| > 0} (-1)^{-|\alpha|-1} t^\alpha \otimes \partial^K(\alpha) \right) (x \otimes y)
\]

We see then that for \( \partial^2(x \otimes y) = 0 \) for arbitrary \( x \otimes y \) we must have for all \( \gamma \in \mathcal{B} \)

\[
\sum_{\alpha, \beta = \pm \gamma} (-1)^{|\beta|+(\alpha \beta)|\beta|} t^\alpha + \sum_{e_i \gamma} (-1)^{|\gamma|+(e_i \gamma)} f_{e_i} = 0
\]

where \( (-1)^{(\alpha \beta) \gamma} \alpha \wedge \beta = \gamma \) and \( [e_i \gamma] \) is the element in \( \mathcal{B} \) equal to \( e_i \gamma \) up to sign. But this is exactly what we get from Lemma \( \ref{lem:lemma} \). \( \square \)

**Theorem 2.4.** The complex \((F \otimes_Q \mathcal{K}, \partial)\) of finitely generated free \( Q \)-modules has the same homology as \((\mathcal{F}, \partial^F)\).

**Proof.** We filter the complex \( F \otimes_Q \mathcal{K} \) as

\[
\cdots \subset \mathcal{F}^{p-1} \subset \mathcal{F}^p \subset \cdots
\]
for $p \in \mathbb{Z}$, where $\mathcal{F}^p$ is the subcomplex of $F \otimes Q K$ given by $\mathcal{F}^p = \bigsqcup_{i \leq p} F_i \otimes Q K$. We note that since $F \otimes Q K$ is a horizontal band in the upper half plane, this filtration is bounded. Therefore we have a convergent spectral sequence $E^2_{p,q} \Rightarrow H_n(F \otimes Q K)$.

In fact, for each $n \in \mathbb{Z}$, $\mathcal{F}^{n-1} = 0$ and $\mathcal{F}^n = (F \otimes Q K)_n$. Thus for the induced filtration $(\mathcal{F}^p_n(F \otimes Q K))$ of $H_n(F \otimes Q K)$ we have

$$0 = \mathcal{F}^{n-1} \subseteq \mathcal{F}^{n-2} \subseteq \cdots \subseteq \mathcal{F}^n H_n = H_n$$

Since the quotients $\mathcal{F}^p/\mathcal{F}^{p-1}$ are isomorphic to $F_p \otimes K$, we see that the $E^1_{p,q}$ terms of the spectral sequence are

$$E^1_{p,q} \cong \begin{cases} 0 & p > 0 \\ F_q \otimes Q R & p = 0 \end{cases}$$

subsequently, the spectral sequence collapses on the $p$-axis, and the nonzero $E^2$ terms are given by the homology of $\mathcal{F} = F \otimes Q R$. Specifically,

$$E^2_{p,q} \cong \begin{cases} 0 & q \neq 0 \\ H_p(\mathcal{F}) & q = 0 \end{cases}$$

It follows that

$$H_n(F \otimes Q K) \cong H_n(\mathcal{F})$$

and so $F \otimes Q K$ and $\mathcal{F}$ have the same homology. \hfill \Box

**Corollary 2.5.** Suppose that $(\mathcal{F}, \partial')$ is an $R$-free resolution of a finitely generated $R$-module $M$. Then $(F \otimes Q K, \partial)$ is a $Q$-free resolution of $M$.

**Remark.** We have from Lemma 2.2 that the endomorphisms $t^{e_1}, \ldots, t^{e_c}$ satisfy $t^1 t^1 + \sum f_i t^{e_i} = 0$, that is,

$$(\partial')^2 = -\sum_{i=1}^c f_i t^{e_i}$$

Thus $t^{e_i} \otimes R$ is nothing more than the negative of the Eisenbud operator $t_i(Q, \{f_i\}, \mathcal{F})$ defined in Section 1 of [3]. One therefore may think of the $t^{e_i} \otimes R$ for $\alpha \in \mathcal{B}$ with $|\alpha| > 1$ as higher-order Eisenbud operators.

As is shown in loc. cit., each $t^{e_i} \otimes R : \mathcal{F} \to \mathcal{F}$ is a chain map of degree $-2$. We remark that this fact follows directly from (1) when $\gamma = e_i$. Also from (1), for $\gamma = e_1 \wedge e_2$ for example, we have

$$t^{e_1 \wedge e_2} t^1 + t^1 t^{e_1 \wedge e_2} + t^{e_1} t^{e_2} - t^{e_2} t^{e_1} + \sum_{i \neq 1,2} (-1)^{|e_i(e_1 \wedge e_2)|} f_i t^{e_1 \wedge e_2 \wedge e_i} = 0$$

From which it follows that $t^{e_1} \otimes R$ and $t^{e_2} \otimes R$ commute with each other up to homotopy, with $t^{e_1 \wedge e_2}$ being the homotopy. the relations (4) show the same holds for all $t^{e_i}$ and $t^{e_j}$; this is [3] Corollary 1.5).

3. **CODIMENSION ONE**

When $c = 1$, in other words, when $R = Q/(f)$ for a single nonzerodivisor $f$, the complex over $Q$ we construct has a particularly simple form, and isomorphic incarnations of it have appeared in recent articles [St], [BeJo], [AtVr]. It is described as follows.

In each degree, say $n$, the complex $F \otimes Q K$ has just two summands, $F_{n-1} \otimes Q K_1$ and $F_n \otimes Q K_0$. Since $K_1 = Re$ and $K_0 = R$ are rank one free modules, we may
identify these two summands with \( F_{n-1} \) and \( F_n \), respectively. The differential of \( F \otimes_Q K \) is just \( t^0 \otimes s_0 + t^1 \otimes s_1 + t^e \otimes s_e \), and tracking through the identifications above we see that \( F \otimes_Q K \) is isomorphic to

(11)

\[
\cdots \to F_n \oplus F_{n+1} \begin{bmatrix}
\partial^e_n & (-1)^{n+1} t^e \\
(-1)^n f & \partial^e_{n+1}
\end{bmatrix} \to F_{n-1} \oplus F_n \begin{bmatrix}
\partial^e_{n-1} & (-1)^n t^e \\
(-1)^{n-1} f & \partial^e_n
\end{bmatrix} \to F_{n-2} \oplus F_{n-1} \to \\
\cdots \to F_1 \oplus F_2 \begin{bmatrix}
\partial^e_1 & 0 \\
-f & \partial^e_2
\end{bmatrix} \to F_0 \oplus F_1 \begin{bmatrix}
\partial^e_0 & 0 \\
-f & \partial^e_1
\end{bmatrix} \to F_0 \to 0
\]

Minimality. We say that a complex of finitely generated free \( Q \)-modules \((C, \partial^C)\) is minimal if \( \text{Im} \partial^C \subseteq n \cdot C_{n-1} \) for all \( n \). Assuming that \( F \) is taken minimal, the question of whether \( F \otimes_Q K \) is minimal then boils down to whether \( \text{Im} t^e \subseteq n F \).

One simple way this can happen is if \( t_n^e \) is actually zero for all \( n \):

3.1. Let \( R = Q/(f) \) where \( f \) is a nonzero divisor of \( Q \). Let \( \tilde{M} \) be a maximal Cohen-Macaulay \( Q \)-module, and \( F \) a minimal free resolution of \( \tilde{M} \) over \( Q \). Then \( \tilde{F} = F \otimes_Q R \) is a minimal free resolution of \( M = \tilde{M} \otimes_Q R \) over \( R \).

Choosing the lifting \( F \) of \( \tilde{F} \) to \( Q \), we see that \((\partial^F)^2 = 0\), so that \( t_n^e = 0 \) for all \( n \). It follows that \( F \otimes_Q K \) is minimal, in fact, it has the form

\[
\cdots \to F_2 \oplus F_3 \begin{bmatrix}
\partial^e_2 & 0 \\
f & \partial^e_3
\end{bmatrix} \to F_1 \oplus F_2 \begin{bmatrix}
\partial^e_1 & 0 \\
-f & \partial^e_2
\end{bmatrix} \to F_0 \oplus F_1 \begin{bmatrix}
\partial^e_0 & 0 \\
-f & \partial^e_1
\end{bmatrix} \to F_0 \to 0
\]

(In this case we say that \( M \) lifts to \( Q \).)

On the other hand, quite often \( \text{Im} t^e \nsubseteq n F \). For example, assume that \( Q \) is a regular local ring, and \( f \in n^2 \), so that \( R \) is a singular hypersurface ring. Let \( M \) be a maximal Cohen-Macaulay \( R \)-module. Then as described in [Ei], a free resolution \( \tilde{F} \) of \( M \) over \( R \) can be chosen periodic of period \( \leq 2 \), with lifting \( F \) to \( Q \) such that the maps \( \partial^F \) comprise a matrix factorization of \( f \). It follows that the \( t^e \) are identity maps, and so \( F \otimes_Q K \) is not minimal. In fact, it is an infinite resolution over a regular local ring, and so cannot be minimal by Hilbert’s Syzygy Theorem.

4. Rank

In this section we assume that \( R = Q/(f_1, \ldots, f_c) \) for a regular sequence \( f_1, \ldots, f_c \) of length \( c \). Let \( d \) denote the dimension of \( Q \).

4.1. We see from the construction of \( F \otimes_Q K \) the following comparison between the ranks of the free modules of \( \tilde{F} \) versus those of \( F \otimes_Q K \):

(12)

\[
\text{rank}_Q(F \otimes_Q K)_n = \sum_{i=0}^c \binom{c}{i} \text{rank}_R \tilde{F}_{n-i}
\]

Summing up we get

(13)

\[
\sum_{n \geq 0} \text{rank}_Q(F \otimes_Q K)_n = \sum_{n \geq 0} \left( \sum_{i=0}^c \binom{c}{i} \text{rank}_R \tilde{F}_{n-i} \right) = 2^c \sum_{n \geq 0} \text{rank}_R \tilde{F}_n
\]

Of course this equality is only interesting when \( \tilde{F} \) is a finite complex. In this case we have the following application. First we recall a well-known conjecture.
Buchsbaum and Eisenbud \cite[Proposition 1.4]{BucEi}, and Horrocks \cite{Ha} Problem 24] conjectured that a free resolution $F$ of a nonzero module of finite length over a local ring $R$ of dimension $d$ satisfies $\text{rank}_R F_i \geq \binom{d}{i}$ for all $i$. The weaker statement that $\sum_{i \geq 0} \text{rank}_R F_i \geq 2^d$ was later conjectured by Avramov \cite[pp. 63]{EvGr}. We will refer to this latter conjecture as the total rank conjecture; it was recently proved by Walker \cite[Theorem 1]{Wa} when $R$ is a complete intersection whose residual characteristic is not two, and also when $R$ is any local ring containing a field of positive characteristic not equal to two. In our application below, we show that the conjecture holds modulo nonzerodivisors.

**Application to rank conjectures.**

**Theorem 4.2.** Let $R = Q/(f)$, where $Q$ be a commutative local ring of dimension $d$, and $f$ is a nonzerodivisor contained in the maximal ideal of $Q$. The total rank conjecture holds for $R$ if it does so for $Q$.

**Proof.** Suppose that $M$ is an $R$-module with finite length having a finite free resolution $\mathcal{F}$ over $R$ with $\sum_{n \geq 0} \text{rank}_R \mathcal{F}_n < 2^{d-1}$. The from \cite{13} we see that $M$ has the finite free resolution $F \otimes Q K$ over $Q$ with $\sum_{n \geq 0} \text{rank}(F \otimes Q K)_n < 2^d$. \hfill $\Box$

Looking at (12) we have the following refinement.

**Proposition 4.3.** Assume that $Q$ is a local ring of dimension $d$, and $f_1, \ldots, f_c$ a $Q$-regular sequence contained in the maximal ideal of $Q$. Let $\mathcal{F}$ be a minimal $R$-free resolution of a finite length $R$-module $M$, and $F \otimes Q K$ the resolution from Theorem 2.4. If $\text{rank}_R \mathcal{F}_{n-i} \geq \binom{d-c}{n-i}$ for $i = 0, \ldots, c$, then $\text{rank}_Q (F \otimes Q K)_n \geq \binom{d}{n}$.

**Proof.** From (12) we have

$$\text{rank}_Q (F \otimes Q K)_n = \sum_{i=0}^c \binom{c}{i} \text{rank}_R \mathcal{F}_{n-i} \geq \sum_{i=0}^c \binom{c}{i} \frac{(d-c)(d-c-1)\ldots(d-c-i+1)}{i!(d-n-i)!} = \binom{d}{n},$$

where the last equality is a standard identity. \hfill $\Box$

5. Totally acyclic complexes

Our interest in this project is due to the adjoint pair of functors

$$K_{\text{tac}}(Q) \xrightarrow{S} K_{\text{tac}}(R)$$

defined in \cite{BeJoMo}, where $K_{\text{tac}}(Q)$ is the homotopy category of totally acyclic complexes over $Q$, and $K_{\text{tac}}(R)$ is the homotopy category of those over $R$. The main point of loc. cit. is that the adjoint pair of functors provides approximations of elements in $K_{\text{tac}}(R)$ by those in the image of the functor $S$. Specifically, for $C \in K_{\text{tac}}(R)$ we have a morphism $\epsilon_C : STC \to C$, and this morphism is the approximation. The functor $T$ is defined on objects as follows: for $C \in K_{\text{tac}}(R)$, $TC$ is a complete resolution of $\text{Im} \delta^n_C$. The functor $S$ is simply the base change $\underline{\mathcal{C}} \otimes_Q R$. The connection between \cite{BeJoMo} and the current project is expressed in the following theorem.

**Theorem 5.1.** Assume that $R = Q/(f_1, \ldots, f_c)$, where $f_1, \ldots, f_c$ is a $Q$-regular sequence, and let $C \in K_{\text{tac}}(R)$. Letting $\mathcal{F} = C$, the complex $F \otimes Q K$ defined above is $TC$. That is, $F \otimes Q K$ is a complete resolution of $\text{Im} \delta^n_C$. 

It follows that \( STC \) is \( (F \otimes_Q K) \otimes_Q R \), and the morphism
\[
\epsilon_C : STC \to C
\]
is the map that projects the copy \( \overline{F} \otimes Q K_0 \) of \( \overline{F} \) in \( (F \otimes_Q K) \otimes_Q R \) onto \( \overline{F} = C \).

We illustrate the whole affair with an example.

**Example.** Let \( R = k[x, y]/(x^2, y^2) \), and \( C \) be the totally acyclic \( R \)-complex with \( \text{Im} \, \partial_0^C = Rxy \cong k \):
\[
C = \overline{F} : \cdots \to R^3 \xrightarrow{[x \ 0 \ -y \ 0 \ y \ x]} R^2 \xrightarrow{[xy]} R \xrightarrow{[y]} R^2 \to \cdots
\]
A lifting of \( C \) to a sequence of homomorphisms over \( Q = k[x, y]/(x^2) \) is given by
\[
\widetilde{C} = F : \cdots \to Q^3 \xrightarrow{[x \ 0 \ -y \ 0 \ y \ x]} Q^2 \xrightarrow{[xy]} Q \xrightarrow{[y]} Q^2 \to \cdots
\]
We have
\[
\partial_1^F \partial_2^F = [x \ y] [x \ 0 \ -y \ 0 \ y \ x] = [0 \ y^2 \ 0] = y^2 \cdot [0 \ 1 \ 0]
\]
\[
\partial_0^F \partial_1^F = [xy] [x \ y] = [0 \ xy^2] = y^2 \cdot [0 \ x]
\]
\[
\partial_1^F \partial_0^F = [x \ y] [xy] = [0 \ y^2 \ x]
\]
We see that \( t_2 = [0 \ -1 \ 0] \), \( t_1 = [0 \ -x] \), and \( t_0 = [-x] \), where we have written \( t \) to represent \( t^e \). Therefore \( TC \) is
\[
\cdots \to Q^2 \oplus Q^3 \xrightarrow{[x \ y \ 0 \ -1 \ 0 \ -y]} Q^1 \oplus Q^2 \xrightarrow{[xy \ 0 \ x]} Q^1 \oplus Q^1 \xrightarrow{[x \ y \ 0 \ -x]} Q^2 \oplus Q^1 \to \cdots
\]
This section of the right approximation \( \epsilon_C : STC \to C \) thus takes the form
\[
\begin{array}{c}
R^2 \oplus R^3 \xrightarrow{[x \ y \ 0 \ -1 \ 0 \ -y]} R \oplus R^2 \xrightarrow{[xy \ 0 \ x]} \ R \oplus R \xrightarrow{[x \ y \ 0 \ -x]} R^2 \oplus R \\
R^3 \xrightarrow{[x \ y \ 0 \ -1 \ 0 \ -y]} R^2 \xrightarrow{[x \ y]} R \xrightarrow{[xy]} R
\end{array}
\]

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