Heisenberg Double and Pentagon Relation

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Abstract

It is shown that the Heisenberg double has a canonical element, satisfying the pentagon relation. From a given invertible constant solution to the pentagon relation one can restore the structure of the underlying algebras. Drinfeld double can be realized as a subalgebra in the tensor square of the Heisenberg double. This enables one to write down solutions to the Yang-Baxter relation in terms of solutions to the pentagon relation.
1 Introduction

The theory of quantum groups \[3\] appeared as an algebraic setting for the construction of solutions to the Yang-Baxter equation \[12, 2\]:

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.
\]

(1.1)

According to \[3\], for any Hopf algebra \(A\) one can construct a quasi-triangular Hopf algebra \(D(A)\), called Drinfeld double, where the universal \(R\)-matrix, satisfying relation (1.1), exists. In \[5\] it was shown how to reconstruct the underlying Hopf algebra from a given solution to equation (1.1). In \[9, 1, 10\] another, Heisenberg double \(H(A)\), has been introduced, which, unlike the Drinfeld double, is not a Hopf algebra.

The purpose of the present letter is to show that the constant “pentagon” relation

\[
S_{12}S_{13}S_{23} = S_{23}S_{12},
\]

(1.2)

plays the same role in Heisenberg double as Yang-Baxter relation (1.1) does in Drinfeld’s one. This is the content of Sect. \[3\]. In Sect. \[3\] we show that the Drinfeld double can be realized as a subalgebra in the tensor square of the Heisenberg double. This gives an explicit formula for solutions to the Yang-Baxter equation in terms of solutions to the pentagon equation. As an example, in Sect. \[3\] the Borel subalgebra of \(U_q(sl(2))\) at \(|q| < 1\) is considered and a slightly generalized form of the quantum dilogarithm identity of \[4\] is obtained.

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2 Heisenberg Double and Pentagon Relation

Let \(\{e_\alpha\}\) be a linear basis of associative and co-associative bialgebra \(A\), with the following multiplication and co-multiplication rules:

\[
e_\alpha e_\beta = m_{\alpha\beta}^\gamma e_\gamma, \quad \Delta(e_\alpha) = \mu_{\alpha\beta}^{\gamma\gamma} e_\beta \otimes e_\gamma,
\]

(2.1)

where summations over the repeated indices are implied. The Heisenberg double \(H(A)\) can be defined as an associative algebra with generating elements \(\{e_\beta, e_\alpha\}\) and the following defining relations:

\[
e_\alpha e_\beta = m_{\alpha\beta}^\gamma e_\gamma, \quad e_\alpha^\alpha e_\beta^\beta = \mu_{\alpha\beta}^{\gamma\gamma} e_\gamma, \quad e_\alpha e_\beta = m_{\rho\sigma}^{\alpha\beta} \mu_{\alpha\gamma}^{\sigma\sigma} e_\rho e_\sigma.
\]

(2.2)

There are two subalgebras of \(H(A)\) with the linear bases \(\{e_\alpha\}\) and \(\{e_\alpha^\alpha\}\), which are equivalent to the algebra \(A\) and its dual \(A^*\). The analog of the adjoint representation is given by a realization in terms of the structure constants:

\[
\langle \alpha | e_\beta | \gamma \rangle = m_{\alpha\beta}^\gamma, \quad \langle \alpha | e_\beta^\beta | \gamma \rangle = \mu_{\alpha\beta}^{\gamma\gamma}.
\]

(2.3)

The remarkable property of the Heisenberg double is described by the following theorem.
Theorem 1 The canonical element $S = e_\alpha \otimes e^\alpha$ in the Heisenberg double satisfies the pentagon relation (1.2).

Now, following [5], for a given invertible solution $S$ of the pentagon relation (1.2), define two bialgebras $B$ and $B^*$, generated with entries of matrices $F$ and $G$, respectively, and the following multiplication and co-multiplication rules:

$$F_1F_2S_{12} = S_{12}F_1, \quad S_{12}G_1G_2 = G_2S_{12}, \quad (2.4)$$

$$\Delta(F_1) = F_1 \otimes F_1, \quad \Delta(G_1) = G_1 \otimes G_1. \quad (2.5)$$

Note, that the associativity conditions are ensured by the pentagon relation (1.2). In fact, these algebras are dual as bialgebras with the following pairing:

$$\langle G_1, F_2 \rangle = S_{12}. \quad (2.6)$$

To extract the structure constants for given linear bases $\{e_\alpha\}$ in $B$ and $\{e^\alpha\}$ in $B^*$ with the canonical pairing:

$$\langle e^\beta, e_\alpha \rangle = \delta^\beta_\alpha, \quad (2.7)$$

represent matrices $F$ and $G$ as linear combinations of the base elements with some coefficients:

$$F_1 = F_1^\alpha e_\alpha, \quad G_1 = G_1^\alpha e^\alpha. \quad (2.8)$$

Substituting these expressions into (2.6), and using (2.7), we obtain:

$$S_{12} = G_1^\alpha F_2^\alpha. \quad (2.9)$$

Expanding the known matrix $S_{12}$ in the form of (2.9), one can calculate the matrices $G_1^\alpha$ and $F_1^\alpha$ up to an invertible trasformation over the index $\alpha$. From (2.9) it follows also, that for a finite dimensional matrix $S$ one associates in this way a finite dimensional algebra with the dimension, given by the formula:

$$\dim(B) = \text{rank}(P_{12}S_{12})^{t_1}, \quad (2.10)$$

where $P_{12}$ is the permutation matrix, and the upper index $t_1$ means the transposition in the first subspace. Introduce now the dual matrices $F_{1,\alpha}$ and $G_1^\alpha$ through the equations:

$$\text{tr}_1(F_{1,\alpha}F_1^\beta) = \delta^\beta_\alpha, \quad \text{tr}_1(G_1^\alpha G_1^\beta) = \delta^\beta_\alpha. \quad (2.11)$$

Using these, one can derive the following formulas for the structure constants:

$$m_{\alpha,\beta}^\gamma = \text{tr}_1(G_1^\alpha G_{1,\beta}G_1^\gamma), \quad \mu_{\gamma}^{\alpha,\beta} = \text{tr}_1(F_1^\alpha F_1^\beta F_{1,\gamma}). \quad (2.12)$$

There is an analog of the adjoint representation for these algebras, given by the $S$-matrix itself:

$$F_1 = S_{01}, \quad G_1 = S_{10}, \quad (2.13)$$

where the 0-th subspace corresponds to the representation space. In fact, formulas (2.13) realize the Heisenberg double, defined by (2.4) and the mixed permutation relation of the form:

$$G_1S_{12}F_2 = F_2G_1. \quad (2.14)$$
Formulas (2.10)–(2.12) make sense for finite dimensional case, while the infinite dimensional case needs a further qualification. Our result can be stated as the following theorem.

**Theorem 2** For any invertible solution to the constant pentagon relation (1.2) one can associate a pair of mutually dual associative and co-associative bialgebras.

### 3 Yang-Baxter and Pentagon Relations

Let $H(A)$ be the Heisenberg double, defined in Section 2. Co-multiplications of subalgebras $A$ and $A^*$ can not be extended to any co-multiplication of the whole algebra. This is the main difference between Heisenberg and Drinfeld doubles. It is possible, however, to realize Drinfeld double as a subalgebra in the tensor product of two Heisenberg’s, $H(A) \otimes \tilde{H}(A)$, where the second “tilded” double is defined as follows:

$$e_\alpha e_\beta = m_{\alpha \beta}^\gamma \tilde{e}_\gamma, \quad \tilde{e}_\alpha e_\beta = \mu_{\alpha \beta}^\gamma e_\gamma, \quad \tilde{e}_\beta \tilde{e}_\alpha = \mu_{\alpha \gamma}^{\alpha \beta} m_{\alpha \beta}^\gamma e_\sigma e_\rho. \quad (3.1)$$

The canonical element $\tilde{S} = \tilde{e}_\alpha \otimes \tilde{e}_\alpha$, satisfies the “reversed” pentagon relation:

$$\tilde{S}_{12} \tilde{S}_{23} = \tilde{S}_{23} \tilde{S}_{13} \tilde{S}_{12}. \quad (3.2)$$

Using relations (2.2) and (3.1) it is easy to show that the elements

$$E_\alpha = \mu_{\alpha \beta}^\gamma e_\beta \otimes \tilde{e}_\gamma, \quad E'^\alpha = m_{\alpha \beta}^\gamma e_\beta \otimes \tilde{e}_\gamma. \quad (3.3)$$

satisfy the defining relations of the Drinfeld double:

$$E_\alpha E_\beta = m_{\alpha \beta}^\gamma E_\gamma, \quad E'^\alpha E'^\beta = \mu_{\alpha \beta}^\gamma E'^\gamma, \quad \mu_{\alpha \gamma}^{\alpha \beta} m_{\alpha \beta}^\gamma E_\sigma E_\rho = m_{\rho \sigma}^{\rho \gamma} \mu_{\alpha}^{\alpha \gamma} E_\sigma E_\rho. \quad (3.4)$$

In particular, for the canonical element $R = E_\alpha \otimes E'^\alpha$ one gets a factorized formula:

$$R_{12,34} = S'_{14} S_{13} S_{24} S'_{23}. \quad (3.5)$$

where

$$S' = \tilde{e}_\alpha \otimes e_\alpha, \quad S'^\alpha = e_\alpha \otimes \tilde{e}_\alpha. \quad (3.6)$$

By construction, $R$-matrix (3.5) satisfies the Yang-Baxter relation:

$$R_{11,22} R_{11,33} R_{22,33} = R_{22,33} R_{11,33} R_{11,22}. \quad (3.7)$$

In fact, it is a consequence of eight different pentagon relations, two homogeneous ones, (1.2) and (1.2), and six mixed ones:

$$S'_{12} S'_{13} S'_{23} = S_{23} S'_{12}, \quad \tilde{S}_{12} S'_{23} = S'_{23} S'_{13} \tilde{S}_{12},$$

$$S_{12} S''_{12} S''_{23} = S''_{23} S'_{12}, \quad S''_{12} \tilde{S}_{23} = \tilde{S}_{23} S''_{13} S''_{12},$$

$$S'_{12} \tilde{S}_{13} S''_{23} = S_{23} S'_{12}, \quad S''_{12} S_{23} = S'_{23} S_{13} S'_{12}. \quad (3.8)$$
Consider now the case, where algebra $A$ is a Hopf algebra. The unit element
and co-unity map have the form:

$$1 = \varepsilon^\alpha e_\alpha, \quad \varepsilon(e_\alpha) = \varepsilon_\alpha, \quad (3.9)$$

and there are also the antipode and it’s inverse maps:

$$\gamma(e_\alpha) = \gamma^\beta_\alpha e_\beta, \quad \gamma(\gamma_\alpha e_\alpha) = \delta^\beta_\alpha. \quad (3.10)$$

In this case the second double $\tilde{H}(A)$ can be realized through the first one $
H(A)$:

$$\tilde{e}_\alpha = \gamma^\beta_\alpha e_\beta, \quad \tilde{e}_\alpha = \gamma^\alpha_\beta e_\beta, \quad \tilde{\gamma}_\alpha \gamma_\gamma = \delta^\beta_\alpha. \quad (3.11)$$

where the overline means the opposite multiplication, which can be realized
by the transposition operation. The four different $S$-matrices can be expressed
now only in terms of the original one:

$$\tilde{S} = S^t, \quad S' = (S^{-1})^{t_1}, \quad S'' = (S^{t_2})^{-1}, \quad (3.12)$$

where $t$ and $t_i$ mean the full and partial transpositions respectively. Clearly, all
the pentagon relations (3.2) and (3.8) are reduced to (3.2). Thus, formula (3.3)
enables to construct solutions to the Yang-Baxter relation from invertible and
cross-invertible solutions to the pentagon relation (1.2).

The above construction can be generalized also to the non-constant case. For
this consider some invertible and cross-invertible solution to the non-constant
pentagon relation (see [8] for the definition, and [7], for the example) which can
be written in the following form:

$$\begin{align*}
S_{12}(z_1, z_2, z_3, z_4) &= S_{13}(z_1, z_2, z_3, z_4) S_{23}(z_1, z_2, z_3, z_4) \\
&= \tilde{S}_{23}(z_1, z_2, z_3, z_4) S_{12}(z_1, z_2, z_3, z_4), \quad (3.13)
\end{align*}$$

where $z_0, \ldots, z_4$ are some parameters of any nature. Substituting non-constant
$S$’s into (3.3), we obtain an $R$-matrix, depending of sixteen parameters:

$$\begin{align*}
R_{12,34}(\tilde{z}) &= (S_{13}(z_{11}, \ldots, z_{14}))^{-1} S_{13}(z_{21}, \ldots, z_{24}) \\
&\times S_{23}(z_{31}, \ldots, z_{34}) (S_{23}^{-1}(z_{41}, \ldots, z_{44}))^{t_2}, \quad (3.14)
\end{align*}$$

where $\tilde{z}$ is a 4-by-4 matrix with entries $z_{ij}$ for $i, j = 1, \ldots, 4$. Substituting
six such $R$-matrices with different arguments into (3.7), we get the following
equation:

$$\begin{align*}
R_{11',22'}(\hat{x}) R_{11',33'}(\hat{y}) R_{22',33'}(\hat{z}) &= R_{22',33'}(\hat{z}') R_{11',33'}(\hat{y}') R_{11',22'}(\hat{x'}), \quad (3.15)
\end{align*}$$

which is satisfied provided the matrices $\hat{x}, \ldots, \hat{x'}$ are constrained in the special
way:

$$\begin{align*}
x_{23} &= x_{43} = z_{12} = z_{22} = x'_{23} = x'_{43} = z'_{12} = z'_{22}, \\
x_{13} &= x_{33} = z_{32} = z_{42} = x'_{13} = x'_{33} = z'_{32} = z'_{42},
\end{align*}$$

1 By cross-invertibility we mean the invertibility of the partially transposed matrix $S^{t_2}$.
\[x_{22} = x_{42} = x'_{22} = x'_{42} = y_{12}, \quad x_{12} = x_{32} = x'_{12} = x'_{32} = y'_{12},\]
\[z_{13} = z_{23} = z'_{13} = z'_{23} = y_{i3}, \quad z_{33} = z_{43} = z'_{33} = z'_{43} = y'_{i3}, \quad i = 1, \ldots, 4,\]

\[x_{11} = x'_{11} = y_{11} = y'_{21} = z'_{13} = z'_{14}, \quad z_{14} = z'_{14} = x'_{24} = x'_{44} = y_{24} = y_{44},\]
\[x_{21} = x'_{21} = y_{21} = y'_{11} = z'_{11} = z'_{21}, \quad z_{24} = z'_{24} = x_{24} = x_{44} = y_{14} = y_{34},\]
\[x_{31} = x'_{31} = y'_{31} = y_{11} = z_{31} = z_{21}, \quad z_{34} = z'_{34} = x_{14} = x_{24} = y_{14} = y_{34},\]
\[x_{41} = x'_{41} = y_{31} = y_{41} = z_{11} = z_{21}, \quad z_{44} = z'_{44} = x_{14} = x_{24} = y_{14} = y_{34}.\]

(3.16)

Apparently, there should exist a reparametrization, simplifying these constraints, and admitting some “rapidity” picture.

To conclude the section note, that formula (3.5) has a typical “box” structure, used for the construction of solutions to the Yang-Baxter equation in terms of those to the “twisted” Yang-Baxter equations [6].

4 Examples

Consider several examples, realizing the general constructions of Section 2.

For a given finite group \(G\) consider it’s group algebra as a Hopf algebra \(A\). The multiplication relations of the corresponding Heisenberg double are as follows:
\[e_g e_h = e_{gh}, \quad e^g e^h = \delta_{g,h} e^g, \quad e^g e_h = e_h e^g, \quad g, h \in G.\]

(4.1)

This example generates the infinite sequence of finite dimensional solutions to the pentagon relation (1.2). Particularly, in the adjoint representation (2.3), the matrix elements of the canonical element read:
\[\langle g, h | S | g', h' \rangle = \delta_{g' h} \delta_{h'}^{g h}.\]

(4.2)

Let Hopf algebra \(A\) be now the algebra \(C[x]\) with the co-multiplication
\[\Delta(x) = x \otimes 1 + 1 \otimes x.\]

(4.3)

For a linear base take the normalized monomials:
\[e_m = x^m / m!, \quad m = 0, 1, \ldots\]

(4.4)

The dual algebra \(A^*\) is also the algebra \(C[\bar{x}]\) with the co-multiplication of the form (4.3), the dual base being
\[e^m = \bar{x}^m, \quad m = 0, 1, \ldots\]

(4.5)

The Heisenberg double \(H(A)\) is defined by the Heisenberg permutation relation between \(x\) and \(\bar{x}\):
\[x \bar{x} - \bar{x} x = 1,\]

(4.6)

and the canonical element is just the usual exponent:
\[S = \sum_{m=0}^{\infty} e_m \otimes e^m = \exp(x \otimes \bar{x}).\]

(4.7)
The pentagon relation (1.2) is an evident consequence of the permutation relation (4.6).

The next example is less trivial. Let algebra $A$ be a deformed universal enveloping algebra of the Lie algebra:

$$HE - EH = E,$$

(4.8)

with the co-multiplications [3]:

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(E) = E \otimes \exp(hH) + 1 \otimes E,$$

(4.9)

where the complex parameter $h$ is chosen to have a positive real part. This algebra coincides with Borel subalgebra of $U_q(sl(2))$ where $q = \exp(-h)$. For a linear base take again normalized monomials:

$$e_{m,n} = H^mE^n / m!(q)_n, \quad m, n = 0, 1, \ldots,$$

(4.10)

where

$$(q)_n = \begin{cases} 1 & n = 0; \\ (1 - q) \cdots (1 - q^n) & n > 0. \\ \end{cases}$$

(4.11)

The dual algebra is generated also by two elements $\overline{H}$ and $\overline{F}$ with the permutation relation:

$$\overline{H}F - F \overline{H} = -hF,$$

(4.12)

and co-multiplications:

$$\Delta(\overline{H}) = \overline{H} \otimes 1 + 1 \otimes \overline{H}, \quad \Delta(\overline{F}) = \overline{F} \otimes \exp(-\overline{H}) + 1 \otimes \overline{F},$$

(4.13)

the dual base being

$$e^{m,n} = \overline{H}^m\overline{F}^n, \quad m, n = 0, 1, \ldots$$

(4.14)

The permutation relations of the Heisenberg double $H(A)$ read

$$H \overline{H} - \overline{H}H = 1, \quad E \overline{H} = \overline{H}E,$$

(4.15)

$$HF - FH = -F, \quad EF - FE = (1 - q)q^{-H}.$$ (4.16)

The canonical element is given by the formula:

$$S = \sum_{m,n=0}^\infty e_{m,n} \otimes e^{m,n} = \exp(H \otimes \overline{H})(E \otimes F; q)_\infty^{-1},$$

(4.17)

where

$$(x; q)_\infty = (1 - x)(1 - xq)\cdots$$

(4.18)

The pentagon relation (1.3) for the element (4.17) can be rewritten by the use of above permutation relations in the following form:

$$(U; q)_\infty([U, V]/(1 - q); q)_\infty(V; q)_\infty = (V; q)_\infty(U; q)_\infty,$$

(4.19)
where the square brackets denote the commutator, and

\[ U = E_2 F_3, \quad V = E_1 F_2. \]  

(4.20)

Operators \( U \) and \( V \) satisfy the following algebraic relations:

\[ W \equiv UV - qVU, \quad [U, W] = [V, W] = 0, \]  

(4.21)

which mean that the element \( W \) lies in the center of the algebra, generated by operators \( U \) and \( V \). In particular case, where \( W = 0 \), formula (4.19) coincides with the quantum dilogarithm identity of [4]. The generalized form (4.19) of the latter has been found earlier by Volkov [11].

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