COMPLETION OF THE MODULI SPACE FOR
POLARIZED CALABI-YAU MANIFOLDS

YUGUANG ZHANG

Abstract. In this paper, we construct a completion of the moduli space for polarized Calabi-Yau manifolds by using Ricci-flat Kähler-Einstein metrics and the Gromov-Hausdorff topology, which parameterizes certain Calabi-Yau varieties. We then study the algebro-geometric properties and the Weil-Petersson geometry of such completion. We show that the completion can be exhausted by sequences of quasi-projective varieties, and new points added have finite Weil-Petersson distance to the interior.

1. Introduction

A Calabi-Yau manifold $X$ is a simply connected complex projective manifold with trivial canonical bundle $\omega_X \cong \mathcal{O}_X$, and a polarized Calabi-Yau manifold $(X,L)$ is a Calabi-Yau manifold $X$ with an ample line bundle $L$. Let $\mathcal{M}^P$ be the moduli space of polarized Calabi-Yau manifolds $(X,L)$ of dimension $n$ with a fixed Hilbert polynomial $P = P(\mu) = \chi(X, L^\mu)$, i.e.

$$
\mathcal{M}^P = \{(X,L)|P(\mu) = \chi(X, L^\mu)\}/\sim,
$$

where $(X_1, L_1) \sim (X_2, L_2)$ if and only if there is an isomorphism $\psi : X_1 \to X_2$ such that $L_1 = \psi^*L_2$. We denote the equivalent class $[X,L] \in \mathcal{M}^P$ represented by $(X,L)$.

The compactifications of moduli spaces were studied in various cases, for example, the Mumford’s compactification of moduli spaces for curves (cf. [27]), the Satake compactification of moduli spaces for Abelian varieties (cf. [38]), and more recently the compact moduli spaces for general type stable varieties of higher dimension (cf. [21]). Because of the importance of Calabi-Yau manifolds in mathematics and physics (cf. [49]), it is also desirable to have compactifications of $\mathcal{M}^P$. The purpose of this paper is to construct a completion of $\mathcal{M}^P$ in a certain sense, which can be viewed as a partial compactification.

There are several perspectives towards this moduli space $\mathcal{M}^P$. First of all, the Bogomolov-Tian-Todorov’s unobstructedness theorem of Calabi-Yau manifolds implies that $\mathcal{M}^P$ is a complex orbifold (cf. [41, 43]). The variation of Hodge structures gives a natural orbifold Kähler metric on $\mathcal{M}^P$, called the Weil-Petersson metric, which is the curvature of the first Hodge bundle with a natural Hermitian metric (cf. [41]). Other natural metrics

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were also studied in [17] and [47] etc. From the algebro-geometric point of view, Viehweg proved in [45] that $\mathcal{M}^P$ is a quasi-projective variety, and coarsely represents the moduli functor $\mathfrak{M}^P$ for polarized Calabi-Yau manifolds with Hilbert polynomial $P$. The third perspective is to understand $\mathcal{M}^P$ by considering Ricci-flat Kähler-Einstein metrics.

For a polarized Calabi-Yau manifold $(X, L)$, Yau’s theorem on the Calabi conjecture, so called Calabi-Yau theorem, asserts that there exists a unique Ricci-flat Kähler-Einstein metric $\omega$ with $\omega \in c_1(L)$, i.e. the Ricci curvature $\text{Ric}(\omega) \equiv 0$ (cf. [48]). This theorem is obtained by showing the existence and the uniqueness of the solution of the Monge-Ampère equation

\begin{equation}
(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n = (-1)^{n^2} \Omega \wedge \Omega, \quad \sup_{X} \varphi = 0,
\end{equation}

for any background Kähler metric $\omega_0 \in c_1(L)$, and letting $\omega = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi$, where $\Omega$ is a holomorphic volume form, i.e. a nowhere vanishing section of $\mathcal{O}_X$. We can regard $\mathcal{M}^P$ as a parameter space of Ricci-flat Calabi-Yau manifolds.

In [10], Gromov introduced the notion of Gromov-Hausdorff distance between metric spaces, which provides a frame to study families of Riemannian manifolds. For any two compact metric spaces $A$ and $B$, the Gromov-Hausdorff distance of $A$ and $B$ is

$$d_{GH}(A, B) = \inf \{d^Z_H(A, B) | \exists \text{ isometric embeddings } A, B \hookrightarrow Z \},$$

where $Z$ is a metric space, $d^Z_H(A, B)$ is the standard Hausdorff distance between $A$ and $B$ regarded as subsets by the isometric embeddings, and the infimum is taken for all possible $Z$ and isometric embeddings. We denote $\text{Met}$ the space of the isometric equivalent classes of all compact metric spaces equipped with the topology, called the Gromov-Hausdorff topology, induced by the Gromov-Hausdorff distance $d_{GH}$. Then $\text{Met}$ is a complete metric space (cf. [10, 34]). The Gromov-Hausdorff topology was used to study moduli spaces for Einstein metrics by various authors (See for instance [1, 6, 7] etc.).

The Calabi-Yau theorem gives a continuous map

\begin{equation}
\mathcal{CY}: \mathcal{M}^P \to \text{Met}, \quad \text{by } [X, L] \mapsto (X, \omega),
\end{equation}

where $\omega$ is the unique Ricci-flat Kähler-Einstein metric representing $c_1(L)$. However, $\mathcal{CY}$ is not injective in general since $\mathcal{M}^P$ contains the information of complex structures.

For constructing compactifications of $\mathcal{M}^P$, Yau suggested that one uses the Weil-Petersson metric to obtain a metric completion of $\mathcal{M}^P$ first, and then tries to compactify this completion (cf. [17]). In [47], an alternative approach is proposed by using the Gromov-Hausdorff distance, instead of the Weil-Petersson metric, to construct a completion. Let $\overline{\mathcal{CY}(\mathcal{M}^P)}$ be the closure of the image $\mathcal{CY}(\mathcal{M}^P)$ in $\text{Met}$. There is a natural metric space structure on $\overline{\mathcal{CY}(\mathcal{M}^P)}$ by restricting the Gromov-Hausdorff distance. A
question is to understand $\mathcal{CY}(\mathcal{MP})$ from the algebraic geometry and the Weil-Petersson geometry viewpoints.

A normal projective variety $X$ is called 1-Gorenstein if the dualizing sheaf $\omega_X$ is an invertible sheaf, i.e. a line bundle, and is called Gorenstein if furthermore $X$ is Cohen-Macaulay. A variety $X$ has only canonical singularities if $X$ is 1-Gorenstein, and for any resolution $\tilde{\pi} : \tilde{X} \to X$, $\tilde{\pi}_* \omega_{\tilde{X}} = \omega_X$, which is equivalent to that the canonical divisor $K_X$ is Cartier, and

$$K_{\tilde{X}} = \tilde{\pi}^* K_X + \sum_E a_E E,$$

where $E$ are exceptional prime divisors. If $X$ has only canonical singularities, then the singularities are rational, and $X$ is Cohen-Macaulay (cf. (C) of [33, Section 3]), which implies that $X$ is Gorenstein. A Calabi-Yau variety $X$ is a normal projective Gorenstein variety with trivial dualizing sheaf $\omega_X \cong \mathcal{O}_X$, and having at most canonical singularities. A polarized Calabi-Yau variety $(X,L)$ is a normal projective Gorenstein variety with an ample line bundle $L$.

If $(X,d_Y) \in \mathcal{CY}(\mathcal{MP})$, then there is a sequence $\{[X_k,L_k]\} \subset \mathcal{MP}$ such that $\mathcal{CY}([X_k,L_k]) = (X_k,\omega_k)$ converge to $(Y,d_Y)$ in the Gromov-Hausdorff sense. Note that the diameters and the volumes satisfy that $\operatorname{diam}_{\omega_k}(X_k) \to \operatorname{diam}_{d_Y}(Y)$, $\operatorname{Vol}_{\omega_k}(X_k) = \frac{1}{n!} c_1(L_k)^n \equiv \operatorname{cont}$, independent of $k$, which imply that $Y$ is a non-collapsed limit. In [8], Donaldson and Sun studied the algebro-geometric structure of $Y$, and proved that $Y$ is homeomorphic to a Calabi-Yau variety $X_0$ of dimension $n$. Hence loosely speaking, $\mathcal{CY}(\mathcal{MP})$ can be regarded as a parameter space of certain Calabi-Yau varieties.

A degeneration of polarized Calabi-Yau manifolds $(\pi_{\Delta} : X \to \Delta, L)$ is a flat morphism from a variety $X$ of dimension $n + 1$ to a disc $\Delta \subset \mathbb{C}$ such that for any $t \in \Delta^* = \Delta \setminus \{0\}$, $X_t = \pi_{\Delta}^{-1}(t)$ is a Calabi-Yau manifold, the central fiber $X_0 = \pi_{\Delta}^{-1}(0)$ is singular, and $L$ is a relative ample line bundle on $X$. If we further assume that $X_0$ is a Calabi-Yau variety, then the total space $X$ is normal as any fiber $X_t$ is reduced and normal. Thus the relative dualizing sheaf $\omega_{X/\Delta}$ is defined, i.e. $\omega_{X/\Delta} \cong \omega_X \otimes \pi_{\Delta}^* \omega_{\Delta}^{-1}$, and is trivial, i.e. $\omega_{X/\Delta} \cong \mathcal{O}_X$, since every fiber is normal, Cohen-Macaulay and Gorenstein.

Let $(\pi_{\Delta} : X \to \Delta, L)$ be a degeneration of polarized Calabi-Yau manifolds with a Calabi-Yau variety $X_0$ as the central fiber, and $\omega_t$ be the unique Ricci-flat Kähler-Einstein metric on $X_t$ representing $c_1(L|_{X_t})$, $t \in \Delta^*$. The asymptotic behaviour of $\omega_t$ when $t \to 0$ is studied in [36, 35], and it is shown that $(X_t,\omega_t)$ converges to a compact metric space of the same dimension in the Gromov-Hausdorff sense. This result, together with Donaldson-Sun’s theorem, shows the equivalence between the algebro-geometric degenerating Calabi-Yau manifolds to a Calabi-Yau variety and the non-collapsing Gromov-Hausdorff convergence of Ricci-flat Kähler-Einstein metrics.
The first goal of the present paper is to investigate the algebro-geometric structure of $\mathcal{CY}(\mathcal{M}^P)$.

**Theorem 1.1.** There is a Hausdorff topological space $\overline{\mathcal{M}}^P$, and a surjection $\overline{\mathcal{CY}} : \overline{\mathcal{M}}^P \to \overline{\mathcal{CY}(\mathcal{M}^P)}$ satisfying the follows.

i) $\mathcal{M}^P$ is an open dense subset of $\overline{\mathcal{M}}^P$, and $\overline{\mathcal{CY}}|_{\mathcal{M}^P} = \mathcal{CY}$.

ii) For any $p \in \mathcal{M}^P$, $\overline{\mathcal{CY}(p)}$ is homeomorphic to a Calabi-Yau variety.

iii) There is an exhaustion $\mathcal{M}^P \subset \mathcal{M}_{m(1)} \subset \mathcal{M}_{m(2)} \subset \cdots \subset \mathcal{M}_{m(l)} \subset \cdots \subset \overline{\mathcal{M}}^P = \bigcup_{l \in \mathbb{N}} \mathcal{M}_{m(l)}$, where $m(l) \in \mathbb{N}$ for any $l \in \mathbb{N}$, such that $\mathcal{M}_{m(l)}$ is a quasi-projective variety, and there is an ample line bundle $\lambda_{m(l)}$ on $\mathcal{M}_{m(l)}$.

iv) Let $(\pi_\Delta : \mathcal{X} \to \Delta, \mathcal{L})$ be a degeneration of polarized Calabi-Yau manifolds with a Calabi-Yau variety $X_0$ as the central fiber. Assume that for any $t \in \Delta^*$, there is an ample line bundle $L_t$ on $X_t$ such that $L_t^k \cong \mathcal{L}|_{X_t}$ for a $k \in \mathbb{N}$, and $[X_t, L_t] \in \mathcal{M}^P$. Then there is a unique morphism $\rho : \Delta \to \mathcal{M}_{m(l)}$, for $l \gg 1$, such that $\overline{\mathcal{CY}}(\rho(t))$ is homeomorphic to $X_t$ for any $t \in \Delta$, and

$$\overline{\mathcal{CY}}(\rho(t)) \to \overline{\mathcal{CY}}(\rho(0)),$$

when $t \to 0$, in the Gromov-Hausdorff sense. Furthermore, $\rho^* \lambda_{m(l)} = \pi_{\Delta,*} \omega_{\mathcal{X}/\Delta}^{\nu(l)}$ for a $\nu(l) \in \mathbb{N}$.

**Remark 1.2.** In general, we do not expect $\mathcal{M}_{m(l)} = \overline{\mathcal{M}}^P$ for some $m(l)$ because of the lack of the boundedness condition for singular Calabi-Yau varieties (cf. Section 3 in [11]).

When $n = 2$, a Calabi-Yau variety is a K3 orbifold, and a degeneration of K3 surfaces to a K3 orbifold is called a degeneration of type I. It is well-known that one can fill the holes in the moduli space of Kähler polarized K3 surfaces by some Kähler K3 orbifolds, and obtain a complete moduli space (cf. [19] 20). The relationship between such moduli space and the degeneration of Ricci-flat Kähler-Einstein metrics is also established in [19] 20. Theorem 1.1 is a generalization of [19] 20 to higher dimensional polarized Calabi-Yau manifolds.

In [25], Kähler-Einstein metrics are used to construct compactifications of moduli spaces for Kähler-Einstein orbifolds, and it is proved that such compactification coincides with the standard Mumford’s compactification in the case of curves. The moduli space of Fano manifolds admitting Kähler-Einstein metrics is constructed in a recent preprint [30], which generalizes the earlier work [42] for del Pezzo surfaces. The Gromov-Hausdorff compactification of such moduli space for del Pezzo surfaces of each degree is
studied in [31], and it is proven to agree with certain algebro-geometric compactification. We can regard Theorem 1.1 as an analog result of [31] for the Calabi-Yau case.

Now we study the Weil-Petersson geometry of $M_P$. Note that for any flat family $(\pi_\Delta : X \to \Delta, L)$ of polarized Calabi-Yau manifolds with Hilbert polynomial $P$, there is a unique morphism $f : \Delta \to M_P$, since $M_P$ coarsely represents the moduli functor $\mathcal{M}_P$. The Weil-Petersson metric $\omega_{WP}$ is an orbifold Kähler metric on $M_P$ (cf. [41]) characterized by $f^* \omega_{WP} = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \int_{X_t} (-1)^n \Omega_t \wedge \bar{\Omega}_t$, where $\Omega_t$ is a relative holomorphic volume form, i.e. a nowhere vanishing section of $\pi_{X/\Delta}$. The metric $\omega_{WP}$ is the curvature of the first Hodge bundle with a natural Hermitian metric.

In [4], Candelas, Green and Hübisch found some nodal degenerations of Calabi-Yau 3-folds with finite Weil-Petersson distance. In general, [46] shows that if $(\pi_\Delta : X \to \Delta, L)$ is a degeneration of polarized Calabi-Yau manifolds, and if the central fiber $X_0$ is a Calabi-Yau variety, then the Weil-Petersson distance between $x \in M_P$ and the interior $\Delta^*$ is finite, i.e. $\omega_{WP}$ is not complete on $\Delta^*$. Conversely, if we assume that the Weil-Petersson distance of $x \in M_P$ is finite, then $\pi_\Delta : X \to \Delta$ is birational to a degeneration $\pi_\Delta' : X' \to \Delta$ such that $X' \setminus X_0 \cong X' \setminus X_0'$, and $X_0'$ is a Calabi-Yau variety by recent papers [41] and [40]. As a consequence, the algebro-geometric degenerating Calabi-Yau manifolds to a Calabi-Yau variety is equivalent to the finiteness of the Weil-Petersson distance.

Our next result shows that the points in $\overline{M}_P \setminus M_P$ have finite Weil-Petersson distance.

**Theorem 1.3.** Let $\overline{M}_P$ and $\overline{CY}$ be the same as in Theorem 1.1.

i) For any point $x \in \overline{M}_P \setminus M_P$, there is a curve $\gamma$ such that $\gamma(0) = x$, $\gamma([0,1]) \subset M_P$ and the length of $\gamma$ under the Weil-Petersson metric $\omega_{WP}$ is finite, i.e.

$$\text{length}_{\omega_{WP}}(\gamma) < \infty.$$

ii) Let $(\pi_\Delta : X \to \Delta, L)$ be a degeneration of polarized Calabi-Yau manifolds such that for any $t \in \Delta^*$, $L^k_t \cong L|_{X_t}$, for a $k \in \mathbb{N}$, and $[X_t, L_t] \in M_P$, where $L_t$ is an ample line bundle. If the Weil-Petersson distance between $0 \in \Delta$ and the interior $\Delta^*$ is finite, then there is a unique morphism $\varphi : \Delta \to M_{m(l)}$, for $l \gg 1$, such that $\overline{CY}(\varphi(t))$ is homeomorphic to $X_t$, $t \in \Delta^*$.

This paper is organized as follows. Section 2 studies Ricci-flat Kähler-Einstein metrics. In Section 2.1, we recall the generalized Calabi-Yau theorem in [9], and then in Section 2.2, we use [8] to improve the earlier work in [35, 36], i.e. we show that along a degeneration of polarized Calabi-Yau manifolds with a Calabi-Yau variety as the central fiber, the Gromov-Hausdorff
limit of Ricci-flat Kähler-Einstein metrics on general fibers is homeomorphic to the central fiber. All of properties about the Gromov-Hausdorff topology in Theorem 1.1 are from this section. The technique developed in this section can also be used to the unique filling-in problem for degenerations of Calabi-Yau manifolds, i.e. Corollary 2.3 which has independent interests. In Section 3, we study the algebraic geometry of the moduli space. Firstly, we recall the Viehweg’s construction of quasi-projective moduli space for polarized Calabi-Yau manifolds (cf. [45]) in Section 3.1. Secondly, in Section 3.2, we construct an enlarged moduli space of $\mathcal{M}^P$ by using the construction of moduli spaces for varieties with at worst canonical singularities (cf. Section 8 of [45]). More precisely, for any $m > 0$, we construct a moduli functor $\mathcal{M}_m$ for polarized Calabi-Yau varieties that can be embedded in $\mathbb{CP}^N$, $N = N(m)$. Then we use the results in Section 8 of [45] to prove that $\mathcal{M}_m$ can be coarsely represented by a quasi-projective variety. Any $\mathcal{M}_{m(l)}$ in Theorem 1.1 comes from this construction. We prove Theorem 1.1 and Theorem 1.3 in Section 4, and finally, we give a remark for compactifications in Section 5.

In this paper, the notion scheme stands for separated schemes of finite type over $\mathbb{C}$, and the notion variety stands for either a reduced irreducible scheme or the set of its closed points with the natural analytic topology depending on the content. A point in a scheme means a closed point. For a flat family of schemes $\pi_T : \mathcal{X} \rightarrow T$ over $T$, we denote $X_t = \pi_T^{-1}(t)$ the fiber $\mathcal{X} \times_T \{t\}$ over a point $t \in T$. Since a Calabi-Yau manifold $X$ is defined to be simply connected, the natural map $\mathcal{M}^P \rightarrow \mathcal{M}^{P^\mu}$ by $(X, L) \mapsto (X, L^\mu)$ for any $\mu \in \mathbb{N}$ is injective, where $P_{\mu}(k) = P(\mu k)$, and thus is an isomorphism. Thus, we identify $\mathcal{M}^P$ and $\mathcal{M}^{P^\mu}$ in this paper.

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2. Ricci-flat Kähler-Einstein metrics

In this section, we study the Gromov-Hausdorff convergence of Ricci-flat Kähler-Einstein metrics along degenerations of polarized Calabi-Yau manifolds.

2.1. Singular Kähler-Einstein metric. There is a notion of Kähler metric for normal varieties (cf. [4, Section 5.2]). A smooth Kähler metric $\omega$ on a normal variety $X$ is an usual Kähler metric on the regular locus $X_{reg}$ such that for any singular point $p \in X$, there is a neighborhood $U_p$ with an
embedding $U_p \hookrightarrow \mathbb{C}^N_p$, and a smooth strongly pluri-subharmonic function $v_p$ on $\mathbb{C}^N_p$ satisfying $\omega|_{U_p \cap X_{\text{reg}}} = \sqrt{-1} \partial \bar{\partial} v_p|_{U_p \cap X_{\text{reg}}}$. If these functions $v_p$ are not smooth, we call $\omega$ a singular Kähler metric. A Kähler metric $\omega$, possibly singular, defines a class $[\omega]$ in $H^1(X, \mathcal{P}\mathcal{H}_X)$, where $\mathcal{P}\mathcal{H}_X$ denotes the sheaf of pluri-harmonic functions on $X$.

If $L$ is an ample line bundle on $X$, there is an $m > 0$ such that $L^m$ is very ample, and $H^i(X, L^m) = \{0\}$ for any $i > 0$ and $\mu \geq m$. A basis $\Sigma = \{s_0, \cdots, s_N\}$ of $H^0(X, L^m)$ gives an embedding $\Phi_{\Sigma} : X \hookrightarrow \mathbb{CP}^N$ by $x \mapsto [s_0(x), \cdots, s_N(x)]$, which satisfies $L^m = \Phi_{\Sigma}^* \mathcal{O}_{\mathbb{CP}^N}(1)$, where $N = \dim_{\mathbb{C}} H^0(X, L^m) - 1$. The pullback $\omega_{\Sigma} = \Phi_{\Sigma}^* \omega_{FS}$ of the Fubini-Study metric is a smooth Kähler metric in the above sense such that $[\omega_{\Sigma}] = m c_1(L) \in NS_{\mathbb{R}}(X)$. The Hermitian metric $h_{FS}$ of $\mathcal{O}_{\mathbb{CP}^N}(1)$, whose curvature is the Fubini-Study metric, restricts to an Hermitian metric $h_{\Sigma} = \Phi_{\Sigma}^* h_{FS}$ on $L^m$, which satisfies that $\omega_{\Sigma} = -\frac{1}{2} \sqrt{-1} \partial \bar{\partial} \log |\vartheta|_{h_{\Sigma}}^2$ on $X_{\text{reg}}$ for any local section $\vartheta$ of $L^m$. We regard $\Phi_{\Sigma}(X)$ as a point in $\mathcal{H}ilb^P_N$, denoted still by $\Phi_{\Sigma}(X)$, where $\mathcal{H}ilb^P_N$ is the Hilbert scheme parametrizing subschemes of $\mathbb{C}^P$ with the Hilbert polynomial $P = P(k) = \chi(X, L^m k)$. If $\Sigma' = \{s'_0, \cdots, s'_N\}$ is another basis of $H^0(X, L^m)$, we have a matrix $u = (u_{ij}) \in SL(N + 1)$ such that $[s'_0, \cdots, s'_N] = \sum_{i=0}^{N} s_i u_{i0}, \cdots, \sum_{i=0}^{N} s_i u_{iN}$, denoted by $[\Sigma'] = [\Sigma] \cdot u$, and thus, $\Phi_{\Sigma'}(x) = \sigma(u, \Phi_{\Sigma}(x))$ for any $x \in X$, where $\sigma : SL(N + 1) \times \mathbb{C}^P \rightarrow \mathbb{C}^P$ is the natural $SL(N + 1)$-action on $\mathbb{C}^P$. Note that $\sigma$ induces an $SL(N + 1)$-action on the Hilbert scheme $\mathcal{H}ilb^P_N$, denoted still by $\sigma : SL(N + 1) \times \mathcal{H}ilb^P_N \rightarrow \mathcal{H}ilb^P_N$. We have $\Phi_{\Sigma'}(X) = \sigma(u, \Phi_{\Sigma}(X))$, and we denote the orbit

$$O(X, L^m) = \{\sigma(u, \Phi_{\Sigma}(X)) \mid u \in SL(N + 1)\} \subset \mathcal{H}ilb^P_N.$$ 

In [9], a generalized Calabi-Yau theorem is obtained for polarized Calabi-Yau varieties, i.e. the existence and the uniqueness of singular Ricci-flat Kähler-Einstein metrics with bounded potentials. More precisely, for a polarized Calabi-Yau variety $(X, L)$, Theorem 7.5 of [9] says that there is a unique bounded function $\varphi$ satisfying the following Monge-Ampère equation

$$(\omega_{\Sigma} + \sqrt{-1} \partial \bar{\partial} \varphi)^n = (-1)^\frac{n^2}{2} \Omega \wedge \Omega, \quad \sup_X \varphi = 0, \text{ and } \varphi \geq -C,$$

where $\Omega$ is a holomorphic volume form, i.e. a nowhere vanishing section of the dualizing sheaf $\varpi_X$. The restriction of the singular Kähler metric $\omega = \omega_{\Sigma} + \sqrt{-1} \partial \bar{\partial} \varphi$ on the regular locus $X_{\text{reg}}$ is a smooth Ricci-flat Kähler-Einstein metric, and $\omega \in [\omega_{\Sigma}] = m c_1(L)$. Furthermore, $\omega$ is unique in $m c_1(L)$, and particularly is independent of the choice of $\Phi_{\Sigma}$. By the boundedness of $\varphi$, we have that $h = \exp(-\varphi) h_{\Sigma}$ is an Hermitian metric on $L^m$ whose curvature is $\omega$. 


We define an $L^2$-norm $\| \cdot \|_{L^2(h)}$ on $H^0(X, L^m)$ by

$$\|s\|^2_{L^2(h)} = \int_X |s|^2 \omega^n = \int_X e^{-\varphi} |s|^2 \omega^n.$$  

If $h'$ is another Hermitian metric with the same curvature $\omega$, then $\partial \bar{\partial} \log \frac{h'}{h} \equiv 0$, i.e. $\log \frac{h'}{h}$ is a pluriharmonic function on a closed normal variety $X$, and thus $h = e^\varphi h'$ for a constant $\varphi$. If $\Sigma_h = \{s_0, \cdots, s_N\}$ is an orthonormal basis of $H^0(X, L^m)$ respecting to $\| \cdot \|_{L^2(h)}$, then $\Sigma_{h'} = \{e^{-\varphi}s_0, \cdots, e^{-\varphi}s_N\}$ is orthonormal respecting to $\| \cdot \|_{L^2(h')}$, and furthermore $\Sigma_h$ and $\Sigma_{h'}$ induce the same embedding $\Phi_{\Sigma_h} = \Phi_{\Sigma_{h'}}$.

If $\Sigma_h$ and $\Sigma_{h'}$ are two orthonormal bases of $H^0(X, L)$ respecting to $\| \cdot \|_{L^2(h)}$, there is an $u \in SU(N + 1) \subset SL(N + 1)$ such that $[\Sigma_h] = [\Sigma_{h'}] \cdot u$, $\Phi_{\Sigma_{h'}}(x) = \sigma(u, \Phi_{\Sigma_h}(x))$ for any $x \in X$, and thus $\Phi_{\Sigma_{h'}}(X) = \sigma(u, \Phi_{\Sigma_h}(X))$ in $\mathcal{H}ilb^N$. The action $\sigma$ and $h$ induce an $SU(N + 1)$-orbit

$$RO(X, L^m) = \{\sigma(u, \Phi_{\Sigma_h}(X))|u \in SU(N + 1)\} \subset O(X, L^m).$$

Note that $RO(X, L^m)$ is compact, and depends only on the singular Kähler metric $\omega$, but not on the choice of $h$, even the norm $\| \cdot \|_{L^2(h)}$ does.

2.2. Gromov-Hausdorff convergence of Ricci-flat Kähler-Einstein metrics. Let $(\pi_\Delta : \mathcal{X} \to \Delta, \mathcal{L})$ be a degeneration of polarized Calabi-Yau manifolds with a Calabi-Yau variety $X_0$ as the central fiber. By taking certain power of $\mathcal{L}$, we assume that $\mathcal{L}$ is relative very ample. There is a morphism $\hat{\Phi} : \mathcal{X} \to \mathbb{C}P^N \times \Delta \to \mathbb{C}P^N$ by composing an embedding and the projection such that $\mathcal{L} \cong \hat{\Phi}^* \mathcal{O}_{\mathbb{C}P^N}(1)$. We always assume that any $\hat{\Phi}(X_t)$ does not belong to a proper subspace of $\mathbb{C}P^N$ by shrinking $\Delta$ if necessary. We denote $\omega_{0,t} = \hat{\Phi}^* \omega_{FS}|_{X_t}$, and denote $\omega_t$ the unique Ricci-flat Kähler-Einstein metric in $[\omega_{0,t}]$ for any $t \in \Delta$. Note that $\omega_t = \omega_{0,t} + \sqrt{-1} \partial \bar{\partial} \varphi_t$ for a unique bounded potential function $\varphi_t$ with $\sup \varphi_t = 0$, which satisfies the Monge-Ampère equation (1.1) and (2.2) respectively.

The limiting behaviour of $\omega_t$, when $t \to 0$, is studied intensively in [37], [35] and [36]. Theorem 1.4 in [35] asserts that the diameter has a uniform upper bound $D > 0$, i.e.

$$\text{diam}_{\omega_t}(X_t) \leq D,$$

for any $t \in \Delta^*$. Furthermore, for any smooth family of embeddings $F_t : X_{0, \text{reg}} \to X_t$ with $F_0 = \text{Id}$, we have

$$F_t^* \omega_t \to \omega_0, \quad \varphi_t \circ F_t \to \varphi_0, \quad \text{and} \quad \varphi_t > -C$$

for a constant $C > 0$, when $t \to 0$ in the $C^\infty_{\text{loc}}$-sense, where $\varphi_0$ is the solution of (2.2), and $\omega_0$ is the unique singular Ricci-flat Kähler-Einstein metric in $c_1(\mathcal{L}|_{X_0})$. In [36], it is proved that, when $t \to 0$, $(X_t, \omega_t)$ converges to a compact metric space $X_\infty$ in the Gromov-Hausdorff topology, and $X_\infty$ is the metric completion of $(X_{0, \text{reg}}, \omega_0)$. Actually, $X_\infty$ is a Calabi-Yau variety by the following theorem due to Donaldson and Sun (cf. [8, 3]).
Theorem 2.1 (Theorem 1.2 of [8]). Let \((X_k, L_k)\) be a sequence of polarized Calabi-Yau manifolds of dimension \(n\) with the same Hilbert polynomial \(P\), and \(\omega_k \in c_1(L_k)\) be the unique Ricci-flat Kähler-Einstein metric. We assume that

\[
\operatorname{Vol}_{\omega_k}(X_k) = \frac{1}{n!} c_1(L_k)^n \equiv v, \quad \operatorname{diam}_{\omega_k}(X_k) \leq D
\]

for constants \(D > 0\) and \(v > 0\), and furthermore, \((X_k, \omega_k)\) converges to a compact metric space \(X_\infty\) in the Gromov-Hausdorff sense. Then we have the follows.

i) \(X_\infty\) is homeomorphic to a Calabi-Yau variety, denoted still by \(X_\infty\).

ii) There are constants \(m > 0\) and \(N > 0\) satisfying the following. For any \(k\), there is an orthonormal basis \(\Sigma_k\) of \(H^0(X_k, L_k^m)\) respecting to the \(L^2\)-norm induced by \(\omega_k\), which induces an embedding \(\Phi_{\Sigma_k} : X_k \hookrightarrow \mathbb{C}P^N\) with \(L_k^m = \Phi_{\Sigma_k}^* \mathcal{O}_{\mathbb{C}P^N}(1)\). And \(\Phi_{\Sigma_k}(X_k)\) converges to \(X_\infty\) in the Hilbert scheme \(\mathcal{H}il^P_N\).

iii) The metric space structure on \(X_\infty\) is induced by the unique singular Ricci-flat Kähler-Einstein metric \(\omega \in \frac{1}{m!} c_1(\mathcal{O}_{\mathbb{C}P^N}(1)|_{X_\infty})\).

By Proposition 4.15 of [8], \(X_\infty\) is a projective normal variety with only log-terminal singularities. Note that the holomorphic volume forms \(\Omega_k\) are parallel respecting to \(\omega_k\), and converge to a holomorphic volume form \(\Omega_\infty\) on the regular locus \(X_{\infty, \text{reg}}\) along the Gromov-Hausdorff convergence by normalizing \(\Omega_k\) if necessary. Thus the dualizing sheaf \(\omega_{X_\infty} \cong \mathcal{O}_{X_\infty}\), and \(X_\infty\) is 1–Gorenstein. Furthermore, the canonical divisor \(K_{X_\infty}\) is Cartier and trivial, which implies that \(X_\infty\) has at worst canonical singularities. Then \(X_\infty\) has only rational singularities, \(X_\infty\) is Cohen-Macaulay and is Gorenstein. Consequently, \(X_\infty\) is a Calabi-Yau variety.

A natural question is what’s the relationship between these two Calabi-Yau varieties \(X_0\) and \(X_\infty\) in our setting.

Lemma 2.2. Let \((\pi_\Delta : \mathcal{X} \to \Delta, \mathcal{L})\) be a degeneration of polarized Calabi-Yau manifolds with a Calabi-Yau variety \(X_0\) as the central fiber. If \(\omega_t\), \(t \in \Delta^*\), is the unique Ricci-flat Kähler metric on \(X_t\) with \(\omega_t \in c_1(\mathcal{L}|_{X_t})\), then \((X_t, \omega_t)\) converges to a compact metric space \(X_\infty\) homeomorphic to \(X_0\) in the Gromov-Hausdorff sense. As a consequence, the singular Ricci-flat Kähler metric \(\omega_0 \in c_1(\mathcal{L}|_{X_0})\) induces a compact metric space structure on \(X_0\).

Proof. We denote \(L_t = \mathcal{L}|_{X_t}\), and denote \(h_t\) the Hermitian metric on \(L_t\), whose curvature is the Ricci-flat Kähler-Einstein metric \(\omega_t\). We apply Theorem 2.1 to a sequence \(t_k \to 0\), and then \(X_\infty\) is a Calabi-Yau variety. Furthermore, there are constants \(m > 0\) and \(N > 0\) such that, for any \(k\), there is an orthonormal basis \(\Sigma_{t_k}\) of \(H^0(X_{t_k}, L_{t_k}^m)\) respecting to the \(L^2\)-norm \(\| \cdot \|_{L^2(\Lambda_{t_k}^m)}\) inducing an embedding \(\Phi_{\Sigma_{t_k}} : X_{t_k} \hookrightarrow \mathbb{C}P^N\) with \(L_{t_k}^m = \Phi_{\Sigma_{t_k}}^* \mathcal{O}_{\mathbb{C}P^N}(1)\). And \(\Phi_{\Sigma_{t_k}}(X_{t_k})\) converges to \(X_\infty\) in the Hilbert scheme \(\mathcal{H}il^P_N\) under the natural analytic topology.
Corollary 2.3. Let $(\pi_\Delta : \mathcal{X} \to \Delta, \mathcal{L})$ and $(\pi'_\Delta : \mathcal{X}' \to \Delta, \mathcal{L}')$ be two degenerations of polarized Calabi-Yau manifolds with Calabi-Yau varieties $X_0$ and $X'_0$ as the central fibers respectively. If there is a sequence of points...
$t_k \to 0$ in $\Delta$, and there is a sequence of isomorphism $\psi_k : X_{t_k} \to X'_{t_k}$ such that $\psi_k^* \mathcal{L}|_{X_{t_k}} \cong \mathcal{L}'|_{X'_{t_k}}$, then $X_0$ is isomorphic to $X'_{0}$.

Proof. If $\omega_{t_k}$ and $\omega'_{t_k}$ are Ricci-flat Kähler-Einstein metrics representing $c_1(\mathcal{L}|_{X_{t_k}})$ and $c_1(\mathcal{L}'|_{X'_{t_k}})$ respectively, then $(X_{t_k}, \omega_{t_k})$ is isometric to $(X'_{t_k}, \omega'_{t_k})$ as compact metric spaces. By Lemma 2.2, $(X_{t_k}, \omega_{t_k})$ (also $(X'_{t_k}, \omega'_{t_k})$) converges to a compact metric space $X_\infty$ homeomorphic to both $X_0$ and $X'_0$. We further claim that $X_0$ is isomorphic to $X'_0$.

By the proof of Lemma 2.2 and taking some powers of $\mathcal{L}$ and $\mathcal{L}'$, we can assume that there are morphisms $\Psi : \mathcal{X} \to \mathbb{CP}^N$ and $\Psi' : \mathcal{X}' \to \mathbb{CP}^N$ with $\mathcal{L} = \Psi^* \mathcal{O}_{\mathbb{CP}^N}(1)$ and $\mathcal{L}' = \Psi'^* \mathcal{O}_{\mathbb{CP}^N}(1)$ respectively such that $\Psi|_{\mathcal{X}_t}$ and $\Psi'|_{\mathcal{X}'_t}$ are embeddings for any $t \in \Delta$. Furthermore, there are embeddings $\Phi_{\Sigma_k} : X_{t_k} \to \mathbb{CP}^N$ induced by an orthonormal basis $\Sigma_k$ of $H^0(X_{t_k}, \mathcal{L}|_{X_{t_k}})$ for each $k$ such that $\Phi_{\Sigma_k}(X_{t_k})$ converges to a Calabi-Yau variety homeomorphic to $X_\infty$ in the Hilbert scheme $\mathcal{H}il^P_N$, denoted still by $X_\infty$. The same arguments as in the proof of Lemma 2.2 show that there are $w$ and $w' \in SL(N+1)$ such that $X_\infty = \sigma(w, \Psi(X_0))$ and $X'_\infty = \sigma(w', \Psi'(X'_0))$ where $\sigma : SL(N+1) \times \mathcal{H}il^P_N \to \mathcal{H}il^P_N$ is a natural $SL(N+1)$-action on $\mathcal{H}il^P_N$. Hence $X_0$ is isomorphic to $X'_0$. \hfill $\square$

**Remark 2.4.** Note that $\mathcal{X}$ may not be birational to $\mathcal{X}'$ in this corollary. If we have a stronger assumption that $\mathcal{X}/X_0$ is isomorphic to $\mathcal{X}'/X'_0$, then the conclusion is a direct consequence of [2] Theorem 2.1 and [30] Corollary 4.3.

## 3. Quasi-projective moduli space

In this section, we construct an enlarged moduli space parameterizing certain polarized Calabi-Yau varieties.

### 3.1. Moduli space for Calabi-Yau manifolds

In [45], Viehweg constructed the coarse moduli space of polarized Calabi-Yau manifolds with a fixed Hilbert polynomial $P$ by using the Geometric Invariant Theory (GIT), and it was shown to be a quasi-projective variety. Let’s recall the relevant notions and the basic steps of the construction.

The moduli functor $\mathcal{M}^P$ for polarized Calabi-Yau manifolds with Hilbert polynomial $P$ is a functor from the category of schemes to the category of sets such that $\mathcal{M}^P(Spec(\mathbb{C})) = \mathcal{M}^P$ as a set, and for any scheme $T$, $\mathcal{M}^P(T) = \{(\pi_T : \mathcal{X} \to T, \mathcal{L})/ \sim \}$. Here $\pi_T : \mathcal{X} \to T$ is a flat family of schemes, and $\mathcal{L}$ is a relative ample line bundle on $\mathcal{X}$ such that for any point $t \in T$, $(X_t = \pi_T^{-1}(t), \mathcal{L}|_{X_t}) \in \mathcal{M}^P$. And we say $(\pi_T : \mathcal{X} \to T, \mathcal{L}) \sim (\pi'_T : \mathcal{X}' \to T, \mathcal{L}')$ if there is a $T$-isomorphism $\tau : \mathcal{X} \to \mathcal{X}'$ and an invertible sheaf $\mathcal{B}$ on $T$ such that $\tau^* \mathcal{L}' \cong \mathcal{L} \otimes \pi_1^* \mathcal{B}$. Theorem 1.13 and Corollary 7.22 of [45] assert that there is a quasi-projective scheme $\tilde{\mathcal{M}}^P$ coarsely representing the functor $\mathcal{M}^P$, i.e. the followings hold. There is a natural transformation $\Theta : \mathcal{M}^P \to \hom(\cdot, \tilde{\mathcal{M}}^P)$ such that $\Theta(Spec(\mathbb{C})) : \mathcal{M}^P(Spec(\mathbb{C})) \to \hom(Spec(\mathbb{C}), \tilde{\mathcal{M}}^P)$ is...
bijective, and, for any scheme $W$ and a natural transformation $\Xi : M^P \to \text{hom}(. , W)$, there is a unique natural transformation $\Pi : \text{hom}(. , \tilde{M}^P) \to \text{hom}(. , W)$ such that $\Xi = \Pi \circ \Theta$. This property implies that for any $(\mathcal{X} \to T, \mathcal{L}) \in M^P(T)$, there is a unique morphism $T \to \tilde{M}^P$. We identify $\tilde{M}^P$ with the set of closed points of $\tilde{M}^P$ by $\Theta(\text{Spec}(\mathbb{C}))$.

Now we recall the construction of $\tilde{M}^P$ in [45]. Firstly, the functor $M^P$ is bounded. More precisely, by Matsusaka's Big Theorem (cf. [26]), for any polarized Calabi-Yau manifold $(X, L)$, there is an $m_0 > 0$ depending only on $P$ such that for any $m \geq m_0$, $L^m$ is very ample, and $H^i(X, L^m) = \{0\}$, $i > 0$. By choosing a basis $\Sigma$ of $H^0(X, L^{m_0})$, we have an embedding $\Phi_\Sigma : X \to \mathbb{CP}^N$ such that $L^{m_0} = \Phi_\Sigma^* \mathcal{O}_{\mathbb{CP}^N}(1)$. We regard $\Phi_\Sigma(X)$ as a point in the Hilbert scheme $\text{Hil}^{P_{m_0}}$ parametrizing the subshemes of $\mathbb{CP}^N$ with Hilbert polynomial $P_{m_0}$, where $N = h^0(X, L^{m_0}) - 1$. For any other choice $\Sigma'$, $\Phi_{\Sigma'}(X) = \sigma(u, \Phi_\Sigma(X))$ for a $u \in SL(N + 1)$ where $\sigma : SL(N + 1) \times \text{Hil}^{P_{m_0}} \to \text{Hil}^{P_{m_0}}$ is the $SL(N + 1)$-action on $\text{Hil}^{P_{m_0}}$ induced by the natural $SL(N + 1)$-action on $\mathbb{CP}^N$.

Secondly, $M^P$ is open (see [45, Lemma 1.18]), i.e. for any flat family of polarized varieties $(\pi_Y : \mathcal{X} \to Y, \mathcal{L})$, there is an open subscheme $Y' \subset Y$ such that a morphism $T \to Y$ factors through $T \to Y'$ if and only if $(\mathcal{X} \times_Y T \to T, p^* \mathcal{L}) \in M^P(T)$ where $p : \mathcal{X} \times_Y T \to \mathcal{X}$ denotes the projection. This is equivalent to that there is an open subscheme $\mathcal{H}'_N$ of $\text{Hil}^{P_{m_0}}$ (cf. [45, Notions 7.2]) such that a point $p \in \mathcal{H}'_N$ if and only if $(X_p = \pi_{\mathcal{H}}^{-1}(p), L_p) \in M^P$ and $L_p^{m_0} \cong \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p}$ where $\pi_{\mathcal{H}} : \mathcal{U}_N \to \text{Hil}^{P_{m_0}}$ is the universal family over the Hilbert scheme $\text{Hil}^{P_{m_0}}$. The $SL(N + 1)$-action $\sigma$ on $\text{Hil}^{P_{m_0}}$ induces an $SL(N + 1)$-action on $\mathcal{H}'_N$, denoted still by $\sigma : SL(N + 1) \times \mathcal{H}'_N \to \mathcal{H}'_N$. The moduli scheme $\tilde{M}^P$ is constructed by showing that a certain quotient of the $SL(N + 1)$-action on $\mathcal{H}'_N$ exists.

Thirdly, $M^P$ is separated (cf. [45, Lemma 1.18]), i.e. for any two $(\mathcal{X}_i \to S, \mathcal{L}_i) \in M^P(S)$, $i = 1, 2$, any isomorphism of $(\mathcal{X}_1, \mathcal{L}_1)$ onto $(\mathcal{X}_2, \mathcal{L}_2)$ over $S \setminus \{0\}$ extends to a $S$-isomorphism from $(\mathcal{X}_1, \mathcal{L}_1)$ to $(\mathcal{X}_2, \mathcal{L}_2)$, where $(S, 0)$ is a germ of smooth curve. Thus the $SL(N + 1)$-action on $\mathcal{H}'_N$ is proper and the stabilizers are finite by [45, Lemma 7.3]. The separateness condition implies that if the moduli space exists, i.e. the quotient of the $SL(N + 1)$-action $\sigma$ exists, then it is Hausdorff under the analytic topology.

Finally, $M^P$ satisfies further properties, so called the weak positivity and the weak stability, i.e. Assumption 7.19 of [45] holds by the proof of Theorem 1.13 in [45]. Then the geometric quotient $Q' : \mathcal{H}'_N \rightarrow \tilde{M}^P$ of the $SL(N + 1)$-action $\sigma$ exists (cf. Corollary 3.33 and the proof of Theorem 7.20 in [45]). Here the geometric quotient means that $Q'$ is a morphism from $\mathcal{H}'_N$ to a scheme $\tilde{M}^P$ satisfying the following (cf. [45, Definition 3.6]). For any $p \in \mathcal{H}'_N$ and any $u \in SL(N + 1)$, $Q'(\sigma(u, p)) = Q'(p)$, $\mathcal{O}_{\tilde{M}^P} = (Q'_* \mathcal{O}_{\mathcal{H}'_N})^{SL(N + 1)}$, and, for any two disjoint $SL(N + 1)$-invariant closed subscheme $W_1$ and $W_2$,
we have that $Q^o(W_1) \cap Q^o(W_2) = \emptyset$, and both of $Q^o(W_1)$ and $Q^o(W_2)$ are closed. Furthermore, for any $p \in \mathcal{M}^P$, the fiber $Q^{o,-1}(p)$ consists of exactly one $SL(N+1)$-orbit. The existence of $Q^o$ implies that the $SL(N + 1)$-action $\sigma$ is closed, and the dimension of the stabilizers is constant on connected components. Moreover, $\mathcal{M}^P$ is a quasi-projective variety, and there is an ample sheaf $\lambda$ on $\mathcal{M}^P$ such that $Q^{o,*}\lambda = \pi_{H,*}\mathcal{O}_{\mathcal{U}_N/H_N}^{\nu}$ for a $\nu \geq 1$ by [45 Corollary 7.22].

Viehweg also showed in Section 8 of [45] that the above construction works for more general moduli functor of certain polarized varieties with semi-ample dualizing sheaf and at worst canonical singularities as long as the boundedness condition, the openness (or the local closeness) condition and the separateness condition hold. In Section 3.2, we will use the construction in [45 Section 8] to obtain an enlarged moduli space of $\mathcal{M}^P$, which also parameterizes certain Calabi-Yau varieties.

We remark that there is an analogue construction of the symplectic reduction (cf. [28 Section 8]) to obtain $\mathcal{M}^P$ by using Ricci-flat Kähler-Einstein metrics. If we define a real slice

\[(3.1) \quad R^o_N = \bigcup_{p \in H^o_N} RO(X_p, \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p}) \subset H^o_N, \text{red,}\]

where $RO(X_p, \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p})$ is defined by (2.24), then there is a natural $SU(N + 1)$-action on $R^o_N$, and the set theory quotient is $\mathcal{M}^P$. The real slice $R^o_N$ is an analog of the zero level set of a momentum map in the symplectic reduction.

### 3.2. Enlarged moduli space.

Now we construct the enlarged moduli space. For any polarized Calabi-Yau manifold $(X, L)$ of dimension $n$ with Hilbert polynomial $P = P(\mu) = \chi(X, L^\mu)$, we assume that $L$ is very ample, and $H^i(X, L^\mu) = \{0\}$ for any $i > 0$ and $\mu \geq 1$ without loss of generality.

For any $m \geq 1$, there is an $N = N(m) > 0$ such that a basis $\Sigma$ of $H^0(X, L^m)$ induces an embedding $\Phi_\Sigma : X \hookrightarrow \mathbb{CP}^N$. Let $\pi_H : U_N \to \mathcal{H}b^F_N$ be the universal family over the Hilbert scheme $\mathcal{H}b^F_N$ of the Hilbert polynomial $P_m(\mu) = P(m\mu)$, and $H^o_N \subset \mathcal{H}b^F_N$ be the open subscheme whose set of closed points parameterizes smooth varieties. The moduli space $\mathcal{M}^P$ is constructed in [45] as the geometric quotient $Q^o : H^o_N \to \mathcal{M}^P$ under the natural $SL(N + 1)$-action $\sigma$ on $H^o_N$ as explained above.

**Lemma 3.1.** There is an open subscheme $\mathcal{H}_N$ of $\mathcal{H}b^F_N$ such that $H^o_N \subset \mathcal{H}_N \subset \overline{H^o_N}$ where $\overline{H^o_N}$ denotes the Zariski closure of $H^o_N$ in $\mathcal{H}b^F_N$, and a point $p \in \mathcal{H}_N$ if and only if $X_p = \pi^{-1}_H(p)$ is a Calabi-Yau variety.

**Proof.** This result is undoubtedly well-known to experts, and the proof was explained to the author by Chenyang Xu. We use the closed subscheme $\overline{H^o_N}$ to replace $\mathcal{H}b^F_N$, and try to prove that $\mathcal{H}_N$ is an open subscheme of $\overline{H^o_N}$. 
Note that Calabi-Yau varieties are normal projective varieties, the
ormality is an open condition for flat families, and any subset of \( \mathcal{H}_N^0 \) containing
the open subset \( \mathcal{H}_N^0 \) is constructible (cf. [16] Proposition 10.14). The
main theorem in [18] shows that if \( \pi_S : X \to S \) is a flat morphism from
a germ of a variety to a germ of smooth curve \((S, 0)\) whose special fiber
\( X_0 = \pi_S^{-1}(0) \) has only canonical singularities, then both \( X \) and fibers \( X_t \)
have only canonical singularities. Thus for any point \( p \in \mathcal{H}_N \), if \( X_p = \pi_H^{-1}(p) \)
has at worst canonical singularities, then by taking curves passing \( p \) and
normalizations of curves, there is a neighborhood of \( p \) over which fibers of
\( \pi_H \) have at worst canonical singularities, i.e. having only canonical singularities
is an open condition. We denote \( W \) the open subscheme of \( \mathcal{H}_N^0 \) whose
set of closed points parameterizes normal varieties with at worst canonical singularities.
By [45] Lemma 1.19], there is a locally closed subscheme \( \mathcal{H}_N \) of \( W \) such that a morphism \( T \to W \) factors through \( T \to \mathcal{H}_N \), if
and only if \( (U_N \times \mathcal{T} \to T, p^*\mathcal{O}_{U_N/W}) \sim (U_N \times \mathcal{T} \to T, p^*\mathcal{O}_{U_N}) \), where
\( p : U_N \times \mathcal{T} \to T \) is the projection. Hence a point \( p \in \mathcal{H}_N \) if and only if
\( \mathcal{O}_{X_p} = p^*\mathcal{O}_{X_p} \), which implies that \( X_p \) is a Calabi-Yau variety. Furthermore,
\( \mathcal{H}_N^0 \subset \mathcal{H}_N \), and \( \mathcal{H}_N \) is open.

We define a moduli subfunctor \( \mathcal{M}_m \) of the moduli functor of polarized
Gorenstein varieties, i.e. 3) of Examples 1.4 in [45], such that
\[
\mathcal{M}_m(\text{Spec}(\mathbb{C})) = \{(X_p = \pi_H^{-1}(p), \mathcal{O}_{CP^N(1)}|_{X_p}) | \ p \ \text{a point of} \ \mathcal{H}_N \}/\sim,
\]
where \( (X_{p_1}, \mathcal{O}_{CP^N(1)}|_{X_{p_1}}) \sim (X_{p_2}, \mathcal{O}_{CP^N(1)}|_{X_{p_2}}) \) if and only if there is an
isomorphism \( \psi : X_{p_1} \to X_{p_2} \) such that \( \mathcal{O}_{CP^N(1)}|_{X_{p_1}} \cong \psi^*\mathcal{O}_{CP^N(1)}|_{X_{p_2}} \),
which is equivalent to \( p_1 = \sigma(u, p_2) \) for an \( u \in SL(N + 1) \). The functor \( \mathcal{M}_m \)
is bounded by the definition, and is open by Lemma 3.1. By [2] Theorem 2.1] and [30] Corollary 4.3], if \( (X_1 \to S, \mathcal{L}_1) \) and \( (X_2 \to S, \mathcal{L}_2) \) are two
flat families of polarized Calabi-Yau varieties over a germ of smooth curve \((S, 0)\), then any isomorphism of these two families over \( S \setminus \{0\} \) extends to an
isomorphism over \( S \), i.e. the moduli functor \( \mathcal{M}_m \) is also separated.

Now we use the construction in [45] Section 8] to prove that \( \mathcal{M}_m \) can be
coarsely represented by a quasi-projective variety.

**Lemma 3.2.** The coarse moduli space of \( \mathcal{M}_m \) is a quasi-projective variety
\( \mathcal{M}_m \), which is constructed as a geometric quotient \( Q : \mathcal{H}_N \to \mathcal{M}_m \). There is
a positive integer \( \nu = \nu(m) \), and an ample line bundle \( \lambda_m \) on \( \mathcal{M}_m \) such that
\( Q^*\lambda_m = \pi_{H,*}\mathcal{O}_{U_N/H,N} \). Furthermore, \( \mathcal{M}_m^P \) is an open subscheme of \( \mathcal{M}_m \).

**Proof.** Note that \( \mathcal{M}_m \) satisfies [45] Assumptions 8.22], i.e. \( \mathcal{M}_m \) is bounded,
open, separated, and moreover is a moduli functor of varieties with semiample dualizing sheaf. Then [45] Theorem 8.23] shows that \( \mathcal{M}_m \) can be
coarsely represented by a quasi-projective scheme \( \mathcal{M}_m \). More precisely, the
basis changing, local freeness condition, the weak positivity and the weak
stability are verified in [45] Section 8.6], and then [45] Theorem 7.20] shows
the existence of the geometric quotient \( Q : \mathcal{H}_N \to \mathcal{M}_m \). Furthermore,
there is a positive integer \( \nu = \nu(m) \), and an ample line bundle \( \lambda_m \) on \( \tilde{M}_m \) such that \( \mathcal{O}^*\lambda_m = \pi_{H^*\mathcal{W}_{UN}/\mathcal{H}_N} \) by [45, Theorem 8.23, Theorem 7.20 and Corollary 7.22]. Finally, since \( \mathcal{H}_N^o \subset \mathcal{H}_N \) is \( SL(N + 1) \)-invariant Zariski open, and \( \mathcal{Q}|_{\mathcal{H}_N^o} = \mathcal{Q}^o \), we obtain that \( \tilde{M}^P \) is open in \( M_m \).

\[ (3.3) \]

\[ \mathcal{M}_m = \mathcal{R}_N / SU(N + 1) = \mathcal{H}_{N,\text{red}} / SL(N + 1) \]

with the quotient topology induced by the analytic topology of \( \mathcal{H}_{N,\text{red}} \), is homeomorphic to the underlying variety of \( \mathcal{M}_m \). Note that the reduced Hilbert scheme \( \overline{\mathcal{H}_{N,\text{red}}}^{P_m} \) is Hausdorff, and so is the subset \( \mathcal{R}_N \). Thus the quotient by a compact Lie group \( \mathcal{M}_m = \mathcal{R}_N / SU(N + 1) \) is also Hausdorff, which has already been implied by the separateness of \( \mathcal{M}_m \). For a point \( p \in \mathcal{H}_N \), we denote \( [X_p] \in \mathcal{M}_m \) the image of \( p \) under the quotient map, i.e. \( [X_p] = \mathcal{Q}(p) \).

4. Proof of Theorem 1.1 and Theorem 1.3

In this section, we prove Theorem 1.1 and Theorem 1.3. By Matsusaka’s Big Theorem, for any polarized Calabi-Yau manifold \( (X, L) \in \mathcal{M}^P \), we assume that for any \( \mu \geq 1 \), \( L^\mu \) is very ample, and \( H^i(X, L^\mu) = \{0\} \), \( i > 0 \) without loss generality.

For any \( D > 0 \), we define a subset \( \mathcal{M}^P(D) \) of \( \mathcal{M}^P \) by

\[ \mathcal{M}^P(D) = \{ [X, L] \in \mathcal{M}^P | \text{Ricci-flat metric } \omega \in c_1(L) \text{ with diam}_\omega(X) \leq D \} \]

We have that if \( D_1 \leq D_2 \), then \( \mathcal{M}^P(D_1) \subset \mathcal{M}^P(D_2) \), and \( \mathcal{M}^P = \bigcup_{D>0} \mathcal{M}^P(D) \).

Let’s consider an exhaustion

\[ \mathcal{M}^P(1) \subset \cdots \subset \mathcal{M}^P(j) \subset \cdots \subset \mathcal{M}^P = \bigcup_{j \in \mathbb{N}} \mathcal{M}^P(j). \]
Note that for a sequence \([X_k, L_k] \in \mathcal{M}^P(j)\), if \((X_k, \omega_k)\) converges to a compact metric space \(X_\infty\) in the Gromov-Hausdorff sense, then by Theorem 2.1, there are embeddings \(\Phi_k : X_k \hookrightarrow \mathbb{C}P^{N_j}\) for an \(N_j > 0\) independent of \(k\) such that \(L_k^{m_j} \cong \Phi_k^*\mathcal{O}_{\mathbb{C}P^{N_j}}(1)\) for an \(m_j > 0\), and \(\Phi_k(X_k)\) converges to a Calabi-Yau variety in the Hilbert scheme \(\text{Hilb}^P_{N_j}\), which is homeomorphic to \(X_\infty\), denoted still by \(X_\infty\).

**Lemma 4.1.** If we denote \(m(l) = \prod_{j=1}^l m_j\), and the sequence \([X_k, L_k] \in \mathcal{M}^P(l_0)\), for an \(l_0 \leq l\), i.e. \(\text{diam}_{\omega_k}(X_k) \leq l_0\), then \([X_\infty] \in \mathcal{M}_{m(l)}\), where \(\mathcal{M}_{m(l)}\) is the underlying quasi-projective variety of \(\tilde{\mathcal{M}}_{m(l)}\) constructed in Lemma 3.2.

**Proof.** If \(\mathcal{V}_{m(l)/m_{l_0}} : \mathbb{C}P^{N_{l_0}} \to \mathbb{C}P^{N_{m(l)}}\) is the Veronese map, then \(\mathcal{V}_{m(l)/m_{l_0}}(X_\infty)\) has the Hilbert polynomial \(P_{m(l)}\), and thus \(\mathcal{V}_{m(l)/m_{l_0}}(X_\infty) \in H_{m_{l_0}(l)} \subset \text{Hilb}_N^{P_{m(l)}}\). Note that \([\mathcal{V}_{m(l)/m_{l_0}}(X_\infty)] = Q_l(\mathcal{V}_{m(l)/m_{l_0}}(X_\infty))\), where \(Q_l : H_{m_{l_0}(l)} \to \tilde{\mathcal{M}}_{m(l)}\) is the quotient map in Lemma 3.2. We obtain the conclusion by identifying \(\mathcal{V}_{m(l)/m_{l_0}}(X_\infty)\) with \(X_\infty\). \(\square\)

**Lemma 4.2.** There is a continuous inclusion \(i_l : \mathcal{M}_{m(l)} \hookrightarrow \mathcal{M}_{m(l+1)}\).

**Proof.** Let \(\mathcal{V}_{m_l} : \mathbb{C}P^{N_{m(l)}} \to \mathbb{C}P^{N_{m(l+1)}}\) be the Veronese map. For a point \(p \in H_{N_{m(l)}}\), let \(X_p = \pi^{-1}\mathcal{H}^{-1}(p)\) where \(\pi_\mathcal{H} : \mathcal{U}_{N_{m(l)}} \to H_{N_{m(l)}}\) is the universal family. If \(\Phi_{m_l} : X_p \hookrightarrow \mathbb{C}P^{N_{m(l)}}\) is the embedding induced by the Ricci-flat Kähler-Einstein metric \(\omega_p\) in \(c_1(\mathcal{O}_{\mathbb{C}P^{N_{m(l)}}}(1)|_{X_p})\), then the embedding \(\mathcal{V}_{m_l} \circ \Phi_{m_l} : X_p \hookrightarrow \mathbb{C}P^{N_{m(l+1)}}\) is induced by \(m_{l+1}\omega_p\). Thus \(\mathcal{V}_{m_l}\) gives a continuous map \(\tilde{\mathcal{V}}_{m_l} : \mathcal{R}_{N_{m(l)}} \to \mathcal{R}_{N_{m(l+1)}}\) which is equivariant under the \(SU(N_{m(l)} + 1)\) and \(SU(N_{m(l+1)} + 1)\) actions. We obtain a continuous \(i_l : \mathcal{M}_{m(l)} \to \mathcal{M}_{m(l+1)}\) by taking the quotients.

If \(i_l([X_1]) = i_l([X_2])\), for two \([X_1]\) and \([X_2]\) \(\in \mathcal{M}_{m(l)}\), then there is an isomorphism \(\tilde{\psi} : \mathcal{V}_{m_l}(X_1) \to \mathcal{V}_{m_l}(X_2)\) with \(\tilde{\psi}(\mathcal{O}_{\mathbb{C}P^{N_{m(l)}+1}}(1)|_{\mathcal{V}_{m_l}(X_1)}) \cong (\mathcal{O}_{\mathbb{C}P^{N_{m(l)}+1}}(1)|_{\mathcal{V}_{m_l}(X_2)})\). Thus we obtain an isomorphism \(\psi : X_1 \to X_2\) with \(\psi(\mathcal{O}_{\mathbb{C}P^{N_{m(l)}+1}}(1)|_{X_1})^{m_{l+1}} \cong (\mathcal{O}_{\mathbb{C}P^{N_{m(l)}+1}}(1)|_{X_2})^{m_{l+1}}\), since the Veronese map \(\mathcal{V}_{m_{m(l)}}(X_i) : X_i \to \mathcal{V}_{m_{m(l)}}(X_i), i = 1, 2\), are isomorphisms. Hence \(\psi(\mathcal{O}_{\mathbb{C}P^{N_{m(l)}+1}}(1)|_{X_1}) \cong \mathcal{O}_{\mathbb{C}P^{N_{m(l)}+1}}(1)|_{X_1}\), and \([X_1] = [X_2]\), i.e, \(i_l\) is injective. \(\square\)

**Proof of Theorem 7.1.** We define

\[
\overline{\mathcal{M}}^P = \bigcup_{l \in \mathbb{N}} \mathcal{M}_{m(l)}
\]

by using the inclusions \(i_l\). Note that \(\overline{\mathcal{M}}^P\) is an open dense subset of each \(\mathcal{M}_{m(l)}\) by Lemma 3.2 and thus of \(\overline{\mathcal{M}}^P\).
We extend the map $\overline{CY} : \mathcal{M}^P \to \text{Met}$ to a map $\overline{CY} : \overline{\mathcal{M}}^P \to \overline{\mathcal{CY}}(\overline{\mathcal{M}}^P)$ by the following. For any $x \in \mathcal{M}_{m(l)} \subset \overline{\mathcal{M}}^P$, let $\omega_l$ be the Ricci-flat Kähler-Einstein metric on $X_p$ representing $c_1(O_{\mathbb{P}^{N_m(l)}}|X_p)$, where $p \in \mathcal{H}_{N_m(l)} \subset \text{Hilb}_{N_m(l)}^{m(l)}$, $Q_{m(l)}(p) = x$, and $X_p = \pi_{-1}(p)$ from the construction in Section 3.2. For a normalized curve $f : \Delta \to \mathcal{H}_{N_m(l)}$ with $f(\Delta^*) \subset \mathcal{H}_{N_m(l)}^m$, we have a degeneration of polarized Calabi-Yau manifolds $(\mathcal{U}_{N_m(l)} \times \mathcal{H}_{N_m(l)}) \Delta \to \Delta, \rho^*O_{\mathbb{P}^{N_m(l)}}(1)|\mathcal{U}_{N_m(l)}$ with central fiber $X_p$, where $p$ is the projection to the first factor. By Lemma 2.2, $\omega_l$ induces a compact metric space structure on $X_p$.

We define $\overline{CY}(x) = (X_p, \frac{1}{m(l)}\omega_l)$. If we consider $i_l(x) \in \mathcal{M}_{m(l+1)}$, then $X_p$ is isomorphic to $\mathcal{V}_{m_l}(X_p)$, $(X_p, \frac{1}{m(l)}\omega_l)$ is isometric to $(\mathcal{V}_{m_l}(X_p), \frac{1}{m(l+1)}\omega_{l+1})$ as metric spaces by $i_l(x) = Q_{m(l+1)}(\mathcal{V}_{m_l}(X_p))$, and $\mathcal{V}_{m_l}O_{\mathbb{P}^{N_m(l+1)}}(1)|\mathcal{V}_{m_l}(X_p) \cong O_{\mathbb{P}^{N_m(l)}}(1)|X_p$ where $\mathcal{V}_{m_l}$ is the Veronese map. Thus $\overline{CY}$ is well-defined. If $X_p$ is smooth, there is an ample line bundle $L_p$ such that $L_p^{m(l)} \cong O_{\mathbb{P}^{N_m(l)}}|X_p$, and $[X_p, L_p] \in \mathcal{M}^P$. Hence $rac{1}{m(l)}\omega_l \in c_1(L_p)$ and $\overline{CY}|_{\mathcal{M}^P} = CY$.

For any compact metric space $(Y, d_Y) \in \overline{CY}(\overline{\mathcal{M}}^P)$, we have a sequence $[X_k, L_k] \in \mathcal{M}^P(l_k)$, for an $l_k > \text{diam}_{d_Y}(Y)$, such that $(X_k, \omega_k)$ converges to $(Y, d_Y)$ in the Gromov-Hausdorff sense, where $\omega_k \in c_1(L_k)$ is the Ricci-flat Kähler-Einstein metric. By Lemma 3.1 there is a Calabi-Yau variety $X_\infty$ homeomorphic to $Y$ and satisfying that $X_\infty$ can be embedded in $\mathbb{P}^{N_m(l_k)}$ and $[X_\infty] \in \mathcal{M}_{m(l_k)}$. Furthermore, the metric structure $d_Y$ is induced by the singular Ricci-flat Kähler-Einstein metric $\omega \in \frac{1}{m(l_k)}c_1(O_{\mathbb{P}^{N_m(l_k)}}(1)|X_\infty)$ by Theorem 2.1 which implies that $\overline{CY}([X_\infty]) = (X_\infty, \omega)$, i.e. $\overline{CY}$ is surjective. We obtain i), ii) and iii).

Let $(\pi_\Delta : \mathcal{X} \to \Delta, \mathcal{L})$ be a degeneration of polarized Calabi-Yau manifolds satisfying the condition in iv). We assume that $\mathcal{L}$ is relative very ample, and $[X_t, \mathcal{L}|_{X_t}] \in \mathcal{M}^P$, $t \in \Delta^*$. By (2.5), there is an $l_2 > 0$ such that $\text{diam}_{\omega_t}(X_t) \leq l_2$ for $t \in \Delta^*$, where $\omega_t$ is the unique Ricci-flat Kähler-Einstein metric representing $c_1(\mathcal{L}|_{X_t})$. Then we have $m_{l_2} > 0$ and $m(l_2) > 0$ such that we have a morphism $\overline{\Psi} : \mathcal{X} \hookrightarrow \Delta \times \mathbb{P}^{N_{m_{l_2}}} \hookrightarrow \Delta \times \mathbb{P}^{N_{m_{l_2}}} \to \mathbb{P}^{N_{m_{l_2}}}$ with $\mathcal{L}^{m_{l_2}}|_{X_t} \cong \overline{\Psi}^*O_{\mathbb{P}^{N_{m_{l_2}}}}(1)|X_t$ by composing an embedding, the Veronese map $\mathcal{V}_{m_{l_2}}$, and the projection. There is a unique morphism $\rho : \Delta \to \mathcal{M}_{m_{l_2}}$ such that $\rho(t) = [\overline{\Psi}(X_t)]$ by the conclusion of $\mathcal{M}_{m_{l_2}}$ coarsely representing the functor $\mathfrak{M}_{m_{l_2}}$ in Lemma 3.2. The Gromov-Hausdorff convergence in iv) is a consequence of Lemma 2.2. \(\square\)
Proof of Theorem 1.3. For any point \( x \in \overline{\mathcal{M}}^P \setminus \mathcal{M}^P \), we assume that \( x \in \mathcal{M}_{m(l)} \subset \overline{\mathcal{M}}^P \) for an \( m(l) > 0 \). Let \( p \in \mathcal{H}_{N_{m(l)}} \subset \mathcal{H}_{\text{ilb}_{N_{m(l)}}} \), \( Q_{m(l)}(p) = x \), and \( X_p = \pi^{-1}_H(p) \), where \( \pi_H : \mathcal{U}_{N_{m(l)}} \rightarrow \mathcal{H}_{N_{m(l)}} \) is the universal family. Let \( \tau : \Delta \rightarrow \mathcal{H}_{N_{m(l)}} \) be a morphism such that \( \tau(0) = p \) and \( \tau(\Delta^*) \subset \mathcal{H}_{N_{m(l)}} \), and \( X = \mathcal{U}_{N_{m(l)}} \times_{\mathcal{H}_{N_{m(l)}}} \Delta \rightarrow \Delta \) be the degeneration of Calabi-Yau manifolds.

Since the central fiber \( X_p \) is a Calabi-Yau variety, [44 Proposition 2.3] shows that the Weil-Petersson distance between the interior \( \Delta^* \) and \( p \) is finite. Hence we obtain i) by composing the quotient map \( Q_{m(l)} \).

Let \( (\pi_\Delta : \mathcal{X} \rightarrow \Delta, \mathcal{L}) \) be a degeneration of polarized Calabi-Yau manifolds. If we assume that the Weil-Petersson distance between \( \Delta^* \) and 0 is finite, then \( (\pi_\Delta : \mathcal{X} \rightarrow \Delta, \mathcal{L}) \) is birational to a new family \( (\pi'_\Delta : \mathcal{X}' \rightarrow \Delta, \mathcal{L}') \) such that \( (\mathcal{X}' \setminus X_0, \mathcal{L}') \cong (\mathcal{X}'' \setminus X_0', \mathcal{L}'') \), and \( X_0' \) is a Calabi-Yau variety by [44 Theorem 1.2]. We obtain ii) by the same argument as in the proof of iv) in Theorem 1.1.

□

Remark 4.3. Note that \( \overline{\mathcal{M}}^P \) parameterizes certain Calabi-Yau varieties, which are proven to be K-stable by [29]. Hence Theorem 1.1 gives an evidence to the conjecture of the existence of the K-moduli spaces (cf. [30 Conjecture 3.1]).

5. A REMARK FOR COMPACTIFICATIONS

Finally, we remark that there actually is a natural Gromov-Hausdorff compactification of \( \overline{\mathcal{M}}^P \). If we define the normalized Calabi-Yau map \( \overline{\mathcal{NCY}} : \mathcal{M}^P \rightarrow \text{Met} \), by \([X, L] \mapsto (X, \text{diam}_{\omega}^{-2}(X_\omega))\),

where \( \omega \in c_1(L) \) is the unique Ricci-flat Kähler-Einstein metric, then the Gromov’s precompactness theorem (cf. [10] [44]) asserts that the closure \( \overline{\mathcal{NCY}(\mathcal{M}^P)} \) of \( \mathcal{NCY}(\mathcal{M}^P) \) in \( \text{Met} \) is compact. Moreover, the map \( \overline{\mathcal{CY}(\mathcal{M}^P)} \rightarrow \overline{\mathcal{NCY}(\mathcal{M}^P)}, \ (Y, d_Y) \mapsto (Y, \text{diam}_{d_Y}^{-1}(Y)d_Y) \)

is injective and continuous. However, because of the collapsing phenomenon, the algebro-geometric structure of \( \overline{\mathcal{NCY}(\mathcal{M}^P)} \) is unclear, and it is not a compactification in the usual algebraic geometry sense. The Gromov-Hausdorff compactification is studied for the moduli spaces of compact Riemann surfaces and Abelian varieties in a recent preprint [32].

Let \( (\mathcal{X} \rightarrow \Delta, \mathcal{L}) \) be a degeneration of polarized Calabi-Yau manifolds of dimension \( n \) such that the diameter of the Ricci-flat Kähler metric \( \omega_t \in c_1(\mathcal{L}|_{X_t}) \) tends to infinite when \( t \rightarrow 0 \), i.e. \( \text{diam}_{\omega_t}(X_t) \rightarrow \infty \). Since \( \text{Vol}_{\omega_t}(X_t) = \frac{1}{m}c_1^{\mathbb{P}}(\mathcal{L}|_{X_t}) \equiv \text{const.} \), \( (X_t, \omega_t) \) must collapse (cf. [1]), i.e. for metric 1-balls \( B_{\omega_t}(1) \), \( \text{Vol}_{\omega_t}(B_{\omega_t}(1)) \rightarrow 0 \) when \( t \rightarrow 0 \). If \( 0 \in \Delta \) is a large complex limit point (cf. [12]), a refined version of the Strominger-Yau-Zaslow (SYZ) conjecture (cf. [39]) due to Gross, Wilson, Kontsevich and Soibelman (cf. [15] [23] [24]) says that \( \text{diam}_{\omega_t}(X_t) \sim \sqrt{-\log |t|} \), and \( (X_t, \text{diam}_{\omega_t}^{-2}(X_t)|_{\omega_t}) \) converges to a compact metric space \( (B, d_B) \) in the
Gromov-Hausdorff sense. If $h^{i,0}(X_t) = 0$, $1 \leq i < n$, then $B$ is homeomorphic to $S^n$. Furthermore, there is an open subset $B_0 \subset B$ with $\text{codim}_{\mathbb{R}} B \setminus B_0 \geq 2$, $B_0$ admits a real affine structure, and the metric $d_B$ is induced by a Monge-Ampère metric $g_B$ on $B_0$, i.e. under affine coordinates $x_1, \cdots, x_n$, there is a potential function $\phi$ such that

$$g_B = \sum_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx_i dx_j, \quad \text{and} \quad \det \left( \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) = 1.$$ 

This conjecture was verified by Gross and Wilson for fibred K3 surfaces with only type I$_1$ singular fibers in [15], and was studied for higher dimensional HyperKähler manifolds in [13, 14]. In [24], it was further conjectured that the Gromov-Hausdorff limit $B$ is homeomorphic to the Calabi-Yau skeleton of the Berkovich analytic space associated to $X \times \Delta$ by taking some base change if necessary, which gives an algebro-geometric description of $B$. If we grant this version of SYZ conjecture, we will have a nice algebro-geometric structure for the compactification at least for some one dimensional moduli space $\mathcal{M}^P$.

**Example 5.1.** A simple concrete example for Theorem 1.1 and Theorem 1.3 is the mirror Calabi-Yau 3-fold of the quintic 3-fold constructed in [5] (cf. Section 18 in [12]), i.e. $X_t$ is the crepant resolution of the quotient

$$Y_s = \{ [z_0, \cdots, z_4] \in \mathbb{CP}^4 | z_0^5 + \cdots + z_4^5 + sz_0 \cdots z_4 = 0 \}/(\mathbb{Z}_5^5/\mathbb{Z}_5)$$

of the quintic by $\mathbb{Z}_5^5/\mathbb{Z}_5$, where $s^5 = t \in \mathbb{C}$. By choosing a polarization, $\mathcal{M}^P = \mathbb{C} \setminus \{1\}$, and 0 is an orbifold point of $\mathcal{M}^P$. When $t = 1$, $X_1$ is a Calabi-Yau variety with finite ordinary double points, and however, $t = \infty$ is a large complex limit point, which implies that $t = \infty$ is the cusp end of $\mathcal{M}^P$ and has infinite Weil-Petersson distance. Thus $\overline{\mathcal{M}^P} = \mathbb{C}$. Again, if we grant the refined version of SYZ conjecture, the point $t = \infty$ corresponds to the $S^3$ with a Monge-Ampère metric on an open dense subset, and consequently, $\overline{\mathcal{M}^P}$ has a natural Gromov-Hausdorff compactification $\mathbb{CP}^1$ with a continuous surjection $\overline{\mathcal{CY}} : \mathbb{CP}^1 \to \overline{\mathcal{CY}}(\mathcal{M}^P)$ extending $\overline{\mathcal{CY}}$.

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Mathematical Sciences Center, Tsinghua University, Beijing 100084, P.R.China.  
E-mail address: yuguangzhang76@yahoo.com