Spectral backtests of forecast distributions with application to risk management*

Michael B. Gordy
Federal Reserve Board, Washington DC

Alexander J. McNeil
The York Management School, University of York

July 26, 2019

Abstract

We study a class of backtests for forecast distributions in which the test statistic depends on a spectral transformation that weights exceedance events by a function of the modeled probability level. The weighting scheme is specified by a kernel measure which makes explicit the user’s priorities for model performance. The class of spectral backtests includes tests of unconditional coverage and tests of conditional coverage. We show how the class embeds a wide variety of backtests in the existing literature, and further propose novel variants which are easily implemented, well-sized and have good power. In an empirical application, we backtest forecast distributions for the overnight P&L of ten bank trading portfolios. For some portfolios, test results depend materially on the choice of kernel.

JEL Codes: C52; G21; G28; G32

Keywords: Backtesting; Volatility; Risk management

*We thank Harrison Katz and Sathya Ramesh for excellent research assistance. We have benefitted from discussion with Mike Giles, Marie Kratz, Hsiao Yen Lok, David Lynch, David McArthur, Michael Milgram, and Johanna Ziegel. The opinions expressed here are our own, and do not reflect the views of the Board of Governors or its staff. Address correspondence to Michael Gordy, Federal Reserve Board, Washington DC 20551, USA, +1-202-452-3705, michael.gordy@frb.gov.
1 Introduction

In many forecasting exercises, fitting some range of quantiles of the forecast distribution may be prioritized in model design and calibration. In risk management applications, which motivate this study, accuracy near the median of the distribution or in the “good tail” of high profits is generally much less important than accuracy in the “bad tail” of large losses. Even within the region of primary interest, preferences may be nonmonotonic in probabilities. For example, the modeller may care a great deal about assessing the magnitude of once-in-a-decade market disruptions, but care much less about quantiles in the extreme tail that are consequent to unsurvivable cataclysmic events. In this paper, we study a class of backtests for forecast distributions in which the test statistic weights exceedance events by a function of the modeled probability level. The weighting scheme is specified by a kernel measure which makes explicit the priorities for model performance. The backtest statistic and its asymptotic distribution are analytically tractable for a very large class of kernels.

Our approach unifies a wide variety of existing approaches to backtesting. In the area of risk management, the time-honored test statistic (dating back to Kupiec, 1995) is simply a count of “VaR exceedances,” i.e., indicator variables equal to one whenever the realized trading loss is in excess of the day-ahead value-at-risk (VaR) forecast. In our framework, this is the case where the kernel is Dirac measure concentrated at the target VaR level. At the other extreme, the tests applied in Diebold et al. (1998) represent a special case in which weights are uniform across all probability levels. The likelihood-ratio test of Berkowitz (2001) and the expected shortfall and spectral risk measure tests of Du and Escanciano (2017) and Costanzino and Curran (2015) represent intermediate cases of a kernel truncated to tail probabilities. While these works are related to our own, we make a distinct threefold contribution: (i) we offer an overarching testing framework that embeds many existing tests and many new ones, including discrete spectral tests and multivariate spectral tests; (ii) we emphasize the idea that choice of backtest should be guided by a user’s preferences for model performance as expressed in kernel choice, rather than by the blind pursuit of power; and
(iii) we propose a general form of conditional test which may be combined with any kernel and which nests the unconditional test as a special case.

The application of a weighting function in this paper bears some similarity to the approach of Amisano and Giacomini (2007) and Gneiting and Ranjan (2011) in the literature on comparisons of density forecasts. In both of those papers, weights are applied to a forecast scoring rule to obtain measures of forecast performance that accentuate the tails (or other regions) of the distribution. However, the measure for any one forecasting method has no absolute meaning and is designed to facilitate comparison with other methods using the general comparative testing approach proposed by Diebold and Mariano (1995). In contrast, our tests are absolute tests of forecast quality in the spirit of Diebold et al. (1998). While the comparative testing approach is useful for the internal refinement of the forecasting method by the forecaster, the absolute testing approach in this paper facilitates external evaluation of the forecaster’s results by another agent, such as a regulator. In this paper we adopt the perspective of such an agent who must make a judgement based on a predefined set of data supplied by the forecaster and who has very limited information about the model choices made by the forecaster.

Our investigation is motivated in part by a major expansion in the data available to regulators for the backtesting exercise. Prior to 2013, banks in the US reported to regulators VaR exceedances at the 99% level. The new Market Risk Rule mandates that banks report for each trading day the probability associated with the realized profit-and-loss (P&L) in the prior day’s forecast distribution, which is equivalent to providing the regulator with VaR exceedances at every level $\alpha \in [0,1]$. The expanded reporting regime allows us to assess the tradeoff between power and specificity in backtesting. If a regulator is concerned narrowly with the validation of reported VaR at level $\alpha$, then a count of VaR exceedances is a sufficient statistic for a test for unconditional coverage. However, if the regulator is willing to assign positive weight to probability levels in a neighborhood of $\alpha$, we can construct more powerful backtests. Furthermore, our approach is consistent with a broader view of the
risk manager’s mandate to forecast probabilities over a range of large losses. The formal
guidance of US regulators to banks on internal model validation explicitly requires “checking
the distribution of losses against other estimated percentiles” (Board of Governors of the
Federal Reserve System, 2011, p. 15).

Under the reforms mandated by the Fundamental Review of the Trading Book (Basel
Committee on Bank Supervision, 2013), 99%-VaR is replaced by 97.5%-Expected Shortfall
(ES) as the determinant of capital requirements. While there has been a lot of debate around
the question of whether or not ES is amenable to direct backtesting (Gneiting, 2011; Acerbi
and Szekely, 2014; Fissler et al., 2016), our contribution addresses a different issue. We
device tests of the forecast distribution from which risk measures are estimated and not tests
of the risk measure estimates. When ES is of primary interest it may be argued that a
satisfactory forecast of the tail of the loss distribution is of even greater importance, since
the risk measure depends on the whole tail.

In Section 2, we lay out the statistical setting for the risk manager’s forecasting problem
and the data to be collected for backtesting. The transformation that underpins the class of
spectral backtests is introduced in Section 3. Spectral backtests of unconditional coverage
are described in Section 4. In Section 5, we develop tests of conditional coverage based on
the martingale difference property. As an application to real data, in Section 6 we backtest
ten bank models for overnight P&L distributions for trading portfolios.

2 Theory and practice of risk measurement

We assume that a bank models P&L on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{N}_0}, \mathbb{P})\) where
\(\mathcal{F}_t\) represents the information available to the risk manager at time \(t\), \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\) and \(\mathbb{N}\)
denotes the non-zero natural numbers. For any time \(t \in \mathbb{N}\), \(L_t\) is an \(\mathcal{F}_t\)-measurable random
variable representing portfolio loss (i.e., negative P&L) in currency units. We denote the
conditional loss distribution given information to time $t - 1$ by

$$F_t(x) = \mathbb{P}(L_t \leq x \mid \mathcal{F}_{t-1}).$$

The loss distribution cannot be assumed to be time-invariant. The distribution of returns on the underlying risk factors (e.g., equity prices, exchange rates) is time-varying, most notably due to stochastic volatility. Furthermore, $F_t$ depends on the composition of the portfolio. Because the portfolio is rebalanced in each period, $F_t$ can evolve over time even when factor returns are iid.

For $t \in \mathbb{N}$ we can define the process $(U_t)$ by $U_t = F_t(L_t)$ using the probability integral transform (PIT). Under the assumption that the conditional loss distributions at each time point are continuous, the result of Rosenblatt (1952) implies that the process $(U_t)_{t \in \mathbb{N}}$ is a sequence of iid standard uniform variables, notwithstanding the fact that $(L_t)$ is typically non-stationary. The risk manager builds a model $\hat{F}_t$ of $F_t$ based on information up to time $t - 1$. Reported PIT-values are the corresponding rvs $(P_t)$ obtained by setting $P_t = \hat{F}_t(L_t)$ for $t \in \mathbb{N}$. The regulator is assumed to have no direct knowledge of $\hat{F}_t$, but can draw inferences based on a sample of the PIT-values. If the models $\hat{F}_t$ form a sequence of ideal probabilistic forecasts in the sense of Gneiting et al. (2007), i.e., coinciding with the conditional laws $F_t$ of $L_t$ for every $t$, then we expect the reported PIT-values to behave like an iid sample of standard uniform variates; tests of this property are tests that the sequence of models is calibrated in probability.

For any $\alpha$ in the unit interval, let $\hat{\text{VaR}}_{\alpha,t} := \hat{F}^{-1}_t(\alpha)$ be an estimate of the $\alpha$-VaR constructed at time $t - 1$ by calculating the generalized inverse of $\hat{F}_t$ at $\alpha$. Since the VaR exceedance event $\{L_t \geq \hat{\text{VaR}}_{\alpha,t}\}$ is equal to the event $\{P_t \geq \alpha\}$, the PIT-value provides a sufficient statistic for the VaR exceedances at all possible levels. Thus, we would expect well-designed tests that use reported PIT-values to be more powerful than VaR exceedance tests in detecting deficiencies in the models $\hat{F}_t$. 
Our tests make no assumptions about the procedures and models used by the bank in forecasting. In practice, there is considerable heterogeneity in methodology. For nearly two decades, most large banks have relied primarily on some variant of historical sampling (HS), which is a nonparametric method based on re-sampling of historical risk-factor changes or returns. As HS fails to account for serial dependencies in returns due to time-varying volatility, some banks adopt filtered historical simulation (FHS) as suggested by Hull and White (1998) and Barone-Adesi et al. (1998). In this approach, the historical risk-factor returns are normalized by their estimated volatilities, which are typically obtained by taking an exponentially-weighted moving-average of past squared returns. Banks that do not use HS or FHS typically adopt a parametric model for the joint distribution of risk-factor changes.

In our empirical application, testing for delayed response to changes in volatility is of special interest. Assuming a roughly symmetric loss distribution centered at zero, the frequent switching between positive and negative values will tend to cause PIT values to be serially uncorrelated, even when volatility is misspecified in the model. However, extreme PIT-values (i.e., near 0 or 1) will tend to beget extreme PIT-values in high volatility periods, and middling PIT-values (i.e., near $\frac{1}{2}$) will tend to beget middling PIT-values in low volatility periods. This pattern can be inferred by examining autocorrelation in the transformed values $|2P_t - 1|$. We will exploit this transformation in implementing tests of conditional coverage in Section 6.

There are relatively few empirical studies of bank VaR forecasting. Berkowitz and O’Brien (2002) show that VaR estimates by US banks are conservative (i.e., there are fewer exceedances than expected) and that the forecasts underperform simple time-series models applied to daily P&L. Conversative forecasts have been documented as well for Canadian banks (Pérignon et al., 2008) and in a larger international sample (Pérignon and Smith, 2010). The sensitivity of such results to sample period is revealed by O’Brien and Szerszen (2017). In their sample of five large US banks from 2001–2014, tests of unconditional coverage reject VaR forecasts as excessively conservative for all banks in the periods of relative
In the crisis period of 2007–2009, however, O’Brien and Szerszen reject VaR forecasts as insufficiently conservative for all five banks, and serial independence is rejected for four of the banks. This pattern is consistent with a failure to model stochastic volatility.

3 Spectral transformations of PIT exceedances

The tests in this paper are based on transformations of indicator variables for PIT exceedances. The transformations take the form

\[ W_t = \int_{[0,1]} \mathbb{1}_{\{P_t \geq u\}} d\nu(u) \]  

where the kernel measure \( \nu \) is a Lebesgue-Stieltjes measure defined on \([0,1]\). The kernel measure is designed to apply weight to the probability levels of greatest interest, typically (in practice) in the region of the standard VaR level \( \alpha = 0.99 \). With any Lebesgue-Stieltjes measure \( \nu \) on domain \([0,1]\), there is an associated increasing right-continuous function \( G_\nu \) such that \( \nu([0,u]) = G_\nu(u) \). It is easily seen that (1) is equivalent to the closed-form expression

\[ W_t = \nu([0,P_t]) = G_\nu(P_t) \]  

which shows that \( W_t \) is increasing in \( P_t \). The measure can be normalized such that \( G_\nu(1) = 1 \) without loss of generality, but we do not require it. To streamline the presentation, we will henceforth impose the following mild regularity condition on \( \nu \).

Assumption 1. \( \nu(\{0\}) = \nu(\{1\}) = 0 \) and \( G_\nu \) is differentiable except at a finite set of points.

The kernel measure can be discrete, continuous or mixed. In the discrete case, it takes the form

\[ \nu = \sum_{i=1}^{m} \gamma_i \delta_{\alpha_i} \]  

for \( m \geq 1 \) where \( \delta \) denotes Dirac measure. This places positive
mass \(\gamma_1, \ldots, \gamma_m\) at the ordered values \(0 < \alpha_1 < \cdots < \alpha_m < 1\) leading to

\[
W_t = \sum_{i=1}^{m} \gamma_i \mathbb{1}_{\{P_t \geq \alpha_i\}}.
\]

For the continuous case, the measure has density \(d\nu(u) = g_\nu(u)du\) for some nonnegative \(g_\nu(u)\) defined on \([0, 1]\) which we refer to as the kernel density. The univariate transformation extends naturally to the multivariate case in which a set of distinct kernel measures \(\nu_1, \ldots, \nu_m\) is applied to PIT-values to obtain the vector-valued variables \(W_1, \ldots, W_n\) where

\[
W_t = (W_{t,1}, \ldots, W_{t,m})', \quad W_{t,j} = \nu_j([0, P_t]) = G_j(P_t), \ j = 1, \ldots, m.
\]

We will refer to any backtest based on spectrally transformed PIT exceedances as a spectral backtest. For the purposes of this paper, we assume that the regulator can utilize only present and past values of \(P_t\) in the backtest statistic. This restriction could be relaxed considerably.\(^1\) What is essential to our contribution is that the regulator does not observe the entire distribution \(\hat{F}_t\), but does observe more than the VaR exception indicator \(\mathbb{1}_{\{L_t \geq \hat{\text{VaR}}_{\alpha,t}\}}\).

Let \((\mathcal{F}^*_t)\) be the regulator’s filtration generated by the PIT values, i.e., \(\mathcal{F}^*_t = \sigma(\{P_s : s \leq t\}) \subset \mathcal{F}_t\). Regardless of the form of the test, the null hypothesis is

\[
H_0: \ W_t \sim F^0_W \text{ and } W_t \perp \perp \mathcal{F}^*_{t-1}, \ \forall t,
\]

where \(F^0_W\) denotes the distribution function of \(W_t\) when \(P_t\) is uniform. The null hypothesis (5) implies that the \((W_t)\) are iid but is weaker than a null hypothesis that the \((P_t)\) are iid Uniform. This is by intent. Since the regulator is free to choose \(\nu\) in accordance with her priorities, she should not object to departures from uniformity and serial independence that arise outside the support of her chosen kernel.

\(^1\)Our approach could easily be generalized to incorporate information in \((L_t, \hat{\text{VaR}}_{\alpha,t})\) and in publicly observed market variables (such as VIX). However, frequent change in portfolio composition implies that lagged VaR values are less reliably informative than lagged PIT values.
Several recent papers propose to correct tests of forecasts for estimation error; see, e.g., Escanciano and Olmo (2010); Du and Escanciano (2017); Hurlin et al. (2017). Implementation of these corrections generally requires knowledge of the forecasting model and estimation scheme and is thus infeasible in the regulatory context we describe. Our null hypothesis imposes the high standard that the forecaster is an ideal forecaster working with a sequence of correctly specified, perfectly estimated models.

The spectral class encompasses a great variety of tests but we prioritize two general testing approaches: Z-tests and likelihood ratio (LR) tests. In the univariate case, the spectral Z-test is based on the asymptotic normality of $W_n = n^{-1} \sum_{t=1}^{n} W_t$ under the null hypothesis (5). Writing $\mu_W = \mathbb{E}(W_t)$ and $\sigma^2_W = \text{var}(W_t)$ for the moments in the null model $F_0^W$, it follows from the central limit theorem that

$$Z_n = \frac{\sqrt{n}(W_n - \mu_W)}{\sigma_W} \xrightarrow{d} N(0, 1).$$

(6)

In the multivariate case ($\dim W_t = m$) we have $\sqrt{n}(\bar{W}_n - \mu_W) \xrightarrow{d} N_m(0, \Sigma_W)$ where $\bar{W}_n = n^{-1} \sum_{t=1}^{n} W_t$ and $\mu_W$ and $\Sigma_W$ are the mean vector and covariance matrix of the null distribution $F_0^W$. Hence a test can be based on assuming for large enough $n$ that

$$T_n = n (\bar{W}_n - \mu_W)' \Sigma_W^{-1} (\bar{W}_n - \mu_W) \sim \chi^2_m,$$

(7)

where we refer to $T_n$ as an $m$-spectral Z-test statistic.

The first moment of the transformed PIT-values under the null hypothesis is easily obtained as

$$\mu_W = \int_{[0,1]} (1-u) d\nu(u)$$

(8)

The variance $\sigma^2_W$ and the cross-moments in $\Sigma_W$ are obtained using a simple product rule for spectrally transformed PIT values.

Theorem 3.1. The set of spectrally transformed PIT values defined by $W_{t,j} = \nu_j([0,P_t])$ is
closed under multiplication. The product $W^*_t = W_{t,1}W_{t,2}$ is given by $W^*_t = \nu^*([0, P_t])$ where $\nu^*$ is a Lebesgue-Stieltjes measure which satisfies

$$\nu^*([0, u]) = \int_{[0, u]} \left( \nu_2([0, s]) - \frac{1}{2} \nu_2(\{s\}) \right) d\nu_1(s) + \int_{[0, u]} \left( \nu_1([0, s]) - \frac{1}{2} \nu_1(\{s\}) \right) d\nu_2(s).$$

It follows that $\sigma^2_W = \mu_{W^*} - \mu^2_W$, where $\mu_{W^*}$ is found by applying (8) under the measure $\nu^*$ obtained when $\nu_1 = \nu_2 = \nu$. This yields

$$\mu_{W^*} = \int_{[0,1]} (1 - u) \left( 2G_\nu(u) - \nu(\{u\}) \right) d\nu(u). \quad (9)$$

The central limit theorem underpinning the Z-test requires finite second moments. For the univariate case, the following proposition provides a sufficient condition on the tail behavior of $G_\nu$.

**Proposition 3.2.** If $G_\nu(u) = \mathcal{O}((1 - u)^{-0.5 + \epsilon})$ as $u \to 1$ for some $\epsilon > 0$, then $\sigma^2_W$ is finite.

In the multivariate setting, the asymptotic distribution in (7) holds if the condition in Proposition 3.2 are satisfied for each $\nu_j, j = 1, \ldots, m$.

Likelihood ratio tests are based on continuous parametric models $F_P(\cdot | \theta)$ for the PIT values $P_t$ that nest uniformity as a special case corresponding to $\theta = \theta_0$. The implied model $F_W(\cdot | \theta)$ for the values $W_t = G_\nu(P_t)$ is used to test the null hypothesis (5) with $F^0_W = F_W(\cdot | \theta_0)$; the alternative is that $W_t \sim F_W(\cdot | \theta)$ with $\theta \neq \theta_0$. Writing $L_W(\theta | W)$ for the likelihood function, the test is based on the asymptotic chi-squared distribution of the statistic

$$LR_{W,n} = \frac{L_W(\theta_0 | W)}{L_W(\hat{\theta} | W)} \quad (10)$$

where $\hat{\theta}$ denotes the maximum likelihood estimate based on the transformed sample $(W_t)$.

An important difference between the two classes of test is that the Z-test is sensitive to the choice of kernel whereas the LR-test is sensitive only to the support of the kernel. Considering the univariate case for simplicity, we show
Theorem 3.3. Let $\nu_1$ and $\nu_2$ be Lebesgue-Stieltjes measures satisfying Assumption 1, and let $W_{t,j} = G_j(P_t)$ for $j = 1, 2$ and $t = 1, \ldots, n$ be the respective samples of transformed PIT values. If $\text{supp}(\nu_1) = \text{supp}(\nu_2)$ then $\text{LR}_{W_1,n} = \text{LR}_{W_2,n}$ almost surely.

This result can be viewed as a generalization of the invariance property of LR-tests under one-to-one transformations of the data.

4 Tests of unconditional coverage

It is common to divide backtesting methods into tests of unconditional coverage and tests of conditional coverage. In our setting, an unconditional test is a test for the distribution $F_0^W$ implied by the uniformity of the PIT-values while a conditional test is a test for both the correct distribution and the independence of $W_t$ and $F_{t-1}^*$ for all $t$.

In this section we present a number of unconditional tests based on the Z-test and LR-test ideas discussed in Section 3. It is important to note that the convergence results on which these tests are based, although mostly stated under iid assumptions, do hold in situations where the independence assumption is relaxed, for example for stationary and ergodic martingale-difference processes (according to the martingale CLT of Billingsley (1961)). In the case of the univariate Z-test, the test will have no power to detect serial dependence whenever $\lim_{n \to \infty} \text{var}(\sqrt{n}W_n) \approx \sigma_W^2$. If, however, there is persistent positive serial correlation in $(W_t)$ leading to $\lim_{n \to \infty} \text{var}(\sqrt{n}W_n) > \sigma_W^2$ then the Z-test will have some power to detect dependencies; however, more targeted tests of the independence property are available and are the subject of Section 5.

Our unconditional testing approach subsumes a number of important published tests or close relatives thereof. The discrete weighting framework in Section 4.1 includes the binomial LR-test of Kupiec (1995) and Christoffersen (1998) for the number of VaR exceedances. It also includes the multilevel Pearson chi-squared test recommended by Campbell (2006) and the multilevel LR-test proposed in Pérignon and Smith (2008) which also underlies the work
of Colletaz et al. (2013); see Kratz et al. (2018) for a comparative study of multilevel tests.

The continuous weighting framework in Section 4.2 builds on the seminal paper of Diebold et al. (1998). It is close in spirit to the approach of Crnkovic and Drachman (1996), who apply a statistic based on a weighted Kuiper distance between the distribution of PIT values and the uniform, and subsumes the likelihood ratio test of Berkowitz (2001) based on fitting a truncated normal distribution to probit-transformed PIT-values. Most closely related to our work, Du and Escanciano (2017) and Costanzino and Curran (2015) have proposed test statistics for spectral risk measures which can be viewed as special cases of our univariate spectral Z-test.

4.1 Discrete weighting

Discrete tests are based on the univariate transformation
\[ W_t = \sum_{i=1}^{m} \gamma_i 1\{P_t \geq \alpha_i\} \]
as defined in (3) and the multivariate transformation
\[ W_t = (1\{P_{t \geq \alpha_1}\}, \ldots, 1\{P_{t \geq \alpha_m}\})' \]in (4) for the same set of ordered levels \( \alpha_1 < \cdots < \alpha_m \). Obviously, when \( m = 1 \) (and \( \gamma_1 = 1 \)) both transformations yield \( W_t = 1\{P_t \geq \alpha\} \), so that we obtain iid Bernoulli \((1 - \alpha)\) variables under the null hypothesis (5). This is the basis for standard VaR exceedance testing based on the binomial distribution. The Z-test statistic (6) for \( W_t = 1\{P_{t \geq \alpha}\} \) coincides with the binomial score test statistic
\[ Z_n = \frac{\sqrt{n} (W_n - (1 - \alpha))}{\sqrt{\alpha(1 - \alpha)}}. \]The LR-test uses an implicit nesting model for \( P_t \) in which the \( W_t \) are iid Bernoulli \((p)\) and tests \( p = 1 - \alpha \) against \( p \neq 1 - \alpha \) by comparing the statistic (10) to a \( \chi_1^2 \) distribution; this is the approach taken in Kupiec (1995) and Christoffersen (1998).

When \( m > 1 \) the variables \( W_t = \sum_{i=1}^{m} \gamma_i 1\{P_{t \geq \alpha_i}\} \) take the ordered values \( \Gamma_0 < \cdots < \Gamma_m \) where \( \Gamma_0 = 0 \) and \( \Gamma_k = \sum_{i=1}^{k} \gamma_i \) for \( k = 1, \ldots, m \). Under the null hypothesis (5) the distributions of \( W_t \) and \( W_t \) satisfy
\[ \mathbb{P}(W_t = \Gamma_i) = \mathbb{P}(1'W_t = i) = \alpha_{i+1} - \alpha_i, \quad i \in \{0, 1, \ldots, m\}, \]
where $\alpha_0 = 0$ and $\alpha_{m+1} = 1$. In both cases this describes a multinomial distribution.

The univariate and multivariate tranformations result in different Z-tests which can be considered as alternative generalizations of the binomial score test (11). Application of Theorem 3.1 to the univariate case and use of (9) delivers moments under the null $\mu_W = \sum_{i=1}^{m} \gamma_i (1 - \alpha_i)$ and $\sigma^2_W = \sum_{i=1}^{m} \gamma_i^* (1 - \alpha_i) - \mu^2_W$ where $\gamma_i^* = \frac{1}{2} \Gamma_i - \gamma_i \gamma_i$. In constructing the test statistic $Z_n$ in (6), we can vary the weights $\gamma_i$ to emphasize different levels $\alpha_i$ and obtain a variety of new tests.

In the multivariate case, we construct an $m$-spectral Z-test as in (7) with $\mu_W = (1 - \alpha_1, \ldots, 1 - \alpha_m)'$ and second moment matrix $\Sigma_W$ with $(i, j)$ element given by $\alpha_i \land j (1 - \alpha_i \lor j)$. We then obtain the classical Pearson chi-squared statistic as proposed by Campbell (2006).

**Theorem 4.1.**

\[
n (\mathbf{W}_n - \mu_W)' \Sigma_W^{-1} (\mathbf{W}_n - \mu_W) = \sum_{i=0}^{m} \frac{(O_i - n \theta_i)^2}{n \theta_i}
\]

where $O_i = \sum_{t=1}^{n} \mathbb{1}\{\mathbf{W}_t = i\}$ and $\theta_i = \alpha_{i+1} - \alpha_i$ for $i = 0, \ldots, m$.

To implement a multinomial (or multi-level) LR-test of (12) we use a nesting model for $P_t$ in which $P(W_t = \Gamma_i) = P(\mathbf{1}' \mathbf{W}_t = i) = p_i$ and $\sum_{i=0}^{m} p_i = 1$. The likelihoods based on $(W_t)$ and $(\mathbf{W}_t)$ yield the same sufficient statistics $O_i = \sum_{t=1}^{n} \mathbb{1}\{W_t = \Gamma_i\} = \sum_{t=1}^{n} \mathbb{1}\{\mathbf{W}_t = i\}$ for the cell probabilities $p_i$. By the likelihood principle the univariate and multivariate LR-tests are identical and depend only on the levels $(\alpha_1, \ldots, \alpha_m)$ and not the weights $\gamma_i$ in the univariate transformation. The invariance of the univariate LR-test under different choices for the weights is also a consequence of Theorem 3.3.

### 4.2 Continuous weighting

Consider a kernel measure $\nu$ with associated absolutely continuous $G_\nu$. We assume that the kernel density satisfies $g_\nu(u) > 0$ for $\alpha_1 < u < \alpha_2$ and $g_\nu(u) = 0$ for $u < \alpha_1$ and $u > \alpha_2$. We refer to $\text{supp}(\nu) = [\alpha_1, \alpha_2]$ as the *kernel window*.

When $\nu_1$ and $\nu_2$ are both continuous kernels with the same kernel window, Theorem 3.1
simplifies. The kernel $\nu^*$ for the product $W^*_t = W_{t,1}W_{t,2}$ is continuous on the same kernel window with density $g^*(u) = G_1(u)g_2(u) + G_2(u)g_1(u)$. Moments and cross-moments can be obtained analytically for a wide variety of kernel densities, e.g., based on polynomials, exponential functions, or on beta-type densities of the form $(u-\alpha_1)^{a-1}(\alpha_2-u)^{b-1}$ for $a, b > 0$; see Section 4.3 for examples of new tests based on this idea. Thus, our compact presentation of the continuous spectral Z-test subsumes a very large class of possible tests.

For the LR-test, recall that we require a family of distributions $F_P(\cdot | \theta)$ for the PIT values that nests uniformity as a special case corresponding to $\theta = \theta_0$. Since $\text{supp}(\nu) = [\alpha_1, \alpha_2]$ for all the kernels of this section, Theorem 3.3 implies that they all give rise to identical LR-tests, depending only on the kernel window $[\alpha_1, \alpha_2]$ and the nesting model $F_P(\cdot | \theta)$. The form taken by $G_\nu$ on $[\alpha_1, \alpha_2]$ is immaterial.

Drawing upon the probitnormal model, we assume that the PIT values $P_1, \ldots, P_n$ have a distribution satisfying $\Phi^{-1}(P_t) \sim N(\mu, \sigma^2)$. Writing $\theta = (\mu, \sigma)'$, the distribution function and density of $P_t$ are respectively

$$
F_P(p | \theta) = \Phi\left(\frac{\Phi^{-1}(p) - \mu}{\sigma}\right), \quad f_P(p | \theta) = \frac{\phi\left(\frac{\Phi^{-1}(p) - \mu}{\sigma}\right)}{\phi(\Phi^{-1}(p))\sigma}, \quad p \in [0, 1],
$$

and the uniform distribution corresponds to $\theta = \theta_0 = (0, 1)'.$ The test of Berkowitz (2001) is a special case of the LR-test under this nesting model: choosing the kernel $\nu$ with $G_\nu(u) = \Phi^{-1}(\alpha_1 \vee u) - \Phi^{-1}(\alpha_1)$ and kernel window $[\alpha_1, 1]$, we observe that $W_t = G_\nu(P_t)$ has a $N(\mu, \sigma^2)$ distribution truncated to $[\Phi^{-1}(\alpha_1), \infty)$ and then translated to the left.

The probitnormal model truncated to the window $[\alpha_1, \alpha_2]$ also yields a further new bispectral Z-test based on a classical score test of $\theta = \theta_0$ against the alternative $\theta \neq \theta_0$. Full details of this test are found in our companion paper [SUPPLEMENTARY MATERIAL]. The resulting truncated probitnormal score test has a mixed weighting scheme (with discrete and
continuous parts) in which

\[ W_{t,i} = \gamma_{i,1} \mathbb{1}_{\{P_t \geq \alpha_1\}} + \gamma_{i,2} \mathbb{1}_{\{P_t \geq \alpha_2\}} + \int_{\alpha_1}^{\alpha_2} g_i(u) \mathbb{1}_{\{P_t \geq u\}} du, \quad i = 1, 2, \] (14)

for known constants \( \gamma_{i,1} \geq 0, \gamma_{i,2} \geq 0 \) and known functions \( g_i(u) \) which are positive and differentiable on \([\alpha_1, \alpha_2] \).

**NOTE:** For completeness and to preserve blinding, material drawn from the “companion paper” is appended to this paper as “SUPPLEMENTARY MATERIAL”.

### 4.3 Size and power

We have performed extensive Monte Carlo analyses of the size and power of unconditional spectral backtests. In this section, we offer representative examples which illustrate how the size and power of Z-tests depend on the kernel, and then briefly summarize other key findings. Full details of all simulation experiments may be found in the companion paper [SUPPLEMENTARY MATERIAL].

We consider kernels of discrete, continuous and mixed form. Parameters \( \alpha_1 \) and \( \alpha_2 \) control the kernel window. For the continuous tests, \( \alpha_1 \) and \( \alpha_2 \) are the infimum and supremum of the kernel support. For the discrete case, we consider 3-level kernels at the set of points \((\alpha_1, \alpha^*, \alpha_2)\), where \( \alpha^* = 0.99 \) is the conventional VaR level. We define a narrow window for which \( \alpha_1 = 0.985 \) and \( \alpha_2 = 0.995 \), and a wide window for which \( \alpha_1 = 0.95 \) and \( \alpha_2 = 0.995 \). Observe that the narrow window is symmetric around \( \alpha^* \), whereas the wide window is asymmetric.

For the continuous case, there is a wide variety of plausible candidates for the kernel. Table 1 lists the kernels that we discuss below; each may be thought of as describing a family of kernel densities for different windows \([\alpha_1, \alpha_2]\). For parsimony, all are special cases of the beta kernel. The uniform and hump-shaped Epanechnikov kernels are commonly used in the nonparametric statistics literature. In the SUPPLEMENTARY MATERIAL, we provide
analytical solutions for the moments of transformed PIT values for the general beta (a, b) case.

| Kernel family       | Mnemonic | Density g(u) | Beta representation |
|---------------------|----------|--------------|---------------------|
| Uniform             | ZU       | 1            | 1,1                 |
| Arcsin              | ZA       | 1/√u*(1−u*)  | ½,½                 |
| Epanechnikov        | ZE       | 1−(2u*−1)²   | 2,2                 |
| Linear increasing   | ZL⁺      | u*          | 2,1                 |
| Linear decreasing   | ZL⁻      | 1−u*        | 1,2                 |

Table 1: Kernel density functions on [α₁, α₂].

*u* denotes the rescaled value *u* = (u − α₁)/(α₂ − α₁). Density functions are not scaled to integrate to 1.

We next list the backtests to be implemented, assigning to each a unique mnemonic.

**Binomial score test:** the two-sided binomial score test at level α* (BIN);

**Multinomial tests:** the 3-point discrete uniform kernel (ZU3) and 3-point Pearson test (PE3);

**Continuous spectral tests:** univariate tests based on the uniform kernel (ZU); the arcsin kernel (ZA); Epanechnikov kernel (ZE); and increasing (ZL⁺) and decreasing (ZL⁻) linear kernels;

**Continuous/mixed bispectral tests:** we combine the increasing and decreasing linear kernels (ZLL) and we also apply the truncated probitnormal score test (PNS).

We consider three different choices for the cdf F of the true model of L_t: the standard normal, the scaled t₅ and scaled t₃. The Student t distributions are scaled to have variance one so differences stem from different tail shapes rather than different variances. We take the risk manager’s model F̂ to be the standard normal, i.e., we transform the sampled L_t to PIT-values as Pt = Φ(L_t). Therefore, when the samples of L_t are drawn from the standard normal, the PIT-values are uniformly distributed and are used to evaluate the size of the tests. The PIT samples arising from the Student t distributions show the kind of departures from uniformity that are observed when the risk manager’s model is too thin-tailed.
We fix a sample size $n = 750$ corresponding approximately to the three-year samples of bank data studied in Section 6. In Table 2, we report the percentage of rejections of the null hypothesis at the 5% confidence level based on $2^{16} = 65,536$ replications. All reported $p$-values are based on two-sided tests, though one-sided versions of some tests are of course available.

| window | $F$ | kernel | BIN | ZU3 | PE3 | ZU | ZA | ZE | ZL+ | ZL− | ZLL | PNS |
|--------|-----|--------|-----|-----|-----|----|----|----|-----|-----|-----|-----|
| narrow |     | Normal | 6.1 | 4.9 | 5.3 | 4.7 | 4.7 | 4.7 | 4.6 | 4.8 | 4.8 | 4.9 |
|        |     | Scaled t5 | 33.9 | 35.0 | 40.3 | 33.8 | 34.4 | 33.0 | 40.3 | 27.1 | 40.0 | 44.7 |
|        |     | Scaled t3 | 24.0 | 24.8 | 43.4 | 23.9 | 24.3 | 23.3 | 32.7 | 16.5 | 43.3 | 50.5 |
| wide   |     | Normal | 6.1 | 5.0 | 5.1 | 4.9 | 4.9 | 4.9 | 4.9 | 4.9 | 5.0 | 5.0 |
|        |     | Scaled t5 | 33.9 | 10.7 | 55.5 | 6.4 | 6.6 | 6.1 | 11.9 | 5.8 | 45.1 | 57.5 |
|        |     | Scaled t3 | 24.0 | 13.5 | 90.6 | 17.7 | 20.4 | 15.4 | 7.4 | 31.9 | 85.8 | 93.1 |

Table 2: Estimated size and power of unconditional Z-tests.
We report the percentage of rejections of the null hypothesis at the 5% confidence level based on $2^{16} = 65,536$ replications. The number of days in each backtest sample is $n = 750$. The narrow window is $[0.985, 0.995]$ and the wide window is $[0.95, 0.995]$.

In both narrow and wide windows, we observe that the size of the Z-tests is very close to the nominal size of 5%, except in the case of the binomial score test, which is slightly oversized. The power of the tests, in contrast, is sensitive to the choice of kernel. We summarize the results as follows:

1. Differences across tests in power are more pronounced on the wide window than on the narrow window. The monospectral tests (BIN, ZU, ZA, ZE, ZL+, ZL−) are broadly similar in power to the binomial score test on the narrow window.

2. The monospectral tests offer more power against the scaled $t_5$ model than the more fat-tailed scaled $t_3$ model on the narrow window, but the opposite is true in most cases on the wide window.

3. Increasing the window width reduces the power of most of the monospectral tests.

4. For the wide window, the increasing linear kernel ZL+ offers more power than the
decreasing linear kernel $ZL$ when the true model is the scaled $t_5$, but the opposite holds when the true model is the scaled $t_3$.

5. The multispectral (PE3, ZLL, PNS) tests offer more power than the monospectral tests, but the differences are relatively small in the case of the narrow window when the true model is the scaled $t_5$.

The first finding is easily understood. For the families of kernel densities in Table 1, the associated function $G_\nu$ converges to a step function as the window narrows. Put another way, all kernel families degenerate to the binomial score kernel as the window shrinks around $\alpha^*$. To illuminate the second finding, we plot in Figure 1 the distribution function for reported PIT-values under each of the true models, i.e., $\Pr(P_t \leq u) = \Pr(L_t \leq \Phi^{-1}(u)) = F(\Phi^{-1}(u))$. The cdf is simply the identity line ($y = x$) when the null hypothesis is true. Within the narrow window of $[0.985, 0.995]$, the cdf for the scaled $t_3$ lies closer to the identity line on average than does the cdf for the scaled $t_5$. The monospectral tests, being tests of the first moment of $W_t = G_\nu(P_t)$, are sensitive to this distance, so have greater power against the scaled $t_5$ than the scaled $t_3$. On the wide window, however, the cdf for the scaled $t_5$ lies closer to the identity line on average, so the tests have greater power against the scaled $t_3$.

The figure also illuminates the third and fourth findings. Both of the scaled Student $t$ cdfs cross the identity line outside the boundaries of the narrow window, but near the middle of the wide window. Crossings within the window reduce the average distance, so pose a particular challenge for the monospectral tests. On the wide window, the scaled $t_5$ cdf crosses the identity line near 0.971, which is slightly below the midpoint. This slight asymmetry favors the $ZL_+$ kernel, which puts heavier weight on the upper side of the window. The scaled $t_3$ cdf crosses the identity line above the midpoint (near 0.982), and furthermore the distance between the cdf and identity line is much larger at the lower end of the window. This asymmetry favors the $ZL_-$ kernel.

Finally, the greater power of the multispectral tests is most apparent when the kernel window contains a crossing of the type just described. While the crossing reduces the average
distance between the cdf of the reported PIT-values and the identity line, the cdf will be too steep or too shallow. Cross-moments of the kernels in a multispectral test can effectively detect such a slope violation. In contrast, when the cdf of the reported PIT-values lies roughly parallel to the identity line throughout the kernel window (as is the case for the scaled $t_5$ in the narrow window), the advantage of the multispectral test is expected to be less pronounced.

![Figure 1: Distribution functions for reported PIT-values.](image)

CDFs for the reported PIT-values when the risk manager assumes standard normal losses ($\hat{F} = \Phi$) but the true loss model $F$ is standard normal (red line), scaled $t_5$ (green line) or scaled $t_3$ (blue line).

In our companion paper [SUPPLEMENTARY MATERIAL], we study three additional dimensions of test design. First within the set of discrete kernel $Z$-tests, we find that the 3-level tests considered in Table 2 perform very similarly to their 5-level counterparts. Second, our summary conclusions on Table 2 are robust to the choice of backtest sample size $n$. Third, and most importantly, we find that the multinomial $Z$-tests and the probitnormal score test outperform their corresponding likelihood ratio tests. The $Z$-tests are similar in power to
the LR-tests, but the LR-tests are less well sized.\footnote{The superior performance of the Z-tests holds with a smaller backtest sample of \( n = 250 \) as well.} Therefore, we will henceforth limit our attention to the class of Z-tests.

As a general caveat, we do not advocate that power alone should dictate the choice of kernel. A general intuition from our simulation studies is that a test is most powerful in rejecting a false model when the kernel weights heavily on probability levels for which the quantiles of the risk manager’s model diverge from the true quantiles. As historical simulation in particular tends to understate the tails of the distribution, in practice we expect that the most powerful tests will weight heavily on extreme probability levels. However, this can come at the expense of the stability of the test, in the sense that the outcome can be determined by the presence or absence of one or two very large reported PIT-values. Furthermore, testing at very extreme tail values of \( \alpha \) runs counter to a primary regulatory motivation for the backtest, which is to verify the bank’s 99% VaR.

5 Tests of conditional coverage

While the unconditional tests of Section 4 have some limited power to detect the presence of serial dependencies, the aim in this section is to propose conditional extensions of our spectral tests that \textit{explicitly} address the independence of \( W_t \) and \( \mathcal{F}_{t-1}^* \) as well as the correctness of the distribution of \( W_t \). These tests should have more power to detect departures from the null hypothesis resulting from a bank’s failure to use all the information in \( \mathcal{F}_{t-1} \) when building the predictive model \( \hat{\mathcal{F}}_t \), such as a failure to address time-varying volatility in adequate fashion.

Our tests of conditional coverage extend the regression-based approach to testing conditional coverage of VaR estimates and offer many new possibilities. They include a variant on the widely-applied test of Christoffersen (1998) and subsume the dynamic quantile (DQ) test of Engle and Manganelli (2004). The Christofferson test is an LR-test of first-order Markov dependence in VaR exceedances and has been generalized to a multilevel test by Leccadito et al. (2014). In the DQ test, VaR exceedance indicators are regressed on lagged exceedance...
indicators and VaR estimates to assess whether exceedances occur independently at the desired rate.

5.1 Tests of the martingale difference property

A necessary condition for null hypothesis (5) to hold is the martingale difference (MD) property with respect to the regulator’s filtration ($\mathcal{F}_t^*$):

$$E(W_t - \mu_W \mid \mathcal{F}_{t-1}^*) = 0$$  \hspace{1cm} (15)

where we recall that $\mathcal{F}_t^* = \sigma\{P_s : s \leq t\}$. When the MD property (15) holds, we must have $E(h_{t-1}(W_t - \mu_W)) = 0$ for any $\mathcal{F}_{t-1}^*$-measurable random variable $h_{t-1}$. Using a function $h$, which we refer to as a conditioning variable transformation (CVT), we form the $k+1$-dimensional lagged vector $h_{t-1} = (1, h(P_{t-1}), \ldots, h(P_{t-k}))'$. To guarantee the existence of the second moment of $h_{t-1}$, we assume that $(P_t)$ is covariance-stationary and that $h$ is bounded. Particular examples that we will use in our empirical analysis are $h(p) = 1_{\{p \geq \alpha\}}$ for some $\alpha$ and $h(p) = |2p - 1|^c$ for $c > 0$.

For convenience, let $(\tilde{W}_t)$ denote the sequence of transformed reported PIT-values $\tilde{W}_t = W_t - \mu_W$ centered at their theoretical mean $\mu_W$. We base our test on the vector-valued process $Y_t = h_{t-1}\tilde{W}_t$ for $t = k+1, \ldots, n$. Under the null hypothesis (5), $(Y_t)$ is a MD sequence satisfying $E(Y_t \mid \mathcal{F}_{t-1}^*) = 0$. We want to test that $Y_{k+1}, \ldots, Y_n$ are close to the zero vector on average. We apply the conditional predictive test of Giacomini and White (2006) which was developed for comparing forecasting methods, and which has also been used by Nolde and Ziegel (2017) in the backtesting context. Let $\overline{Y}_{n,k} = (n-k)^{-1} \sum_{t=k+1}^{n} Y_t$ and let $\hat{\Sigma}_Y$ denote a consistent estimator of $\Sigma_Y := \text{cov}(Y_t)$. Giacomini and White show that under very weak assumptions, for large enough $n$ and fixed $k$,

$$(n-k) \overline{Y}_{n,k}' \hat{\Sigma}_Y^{-1} \overline{Y}_{n,k} \sim \chi^2_{k+1}. \hspace{1cm} (16)$$
Giacomini and White (2006) use the estimator $\hat{\Sigma}_Y^{GW} = (n - k)^{-1} \sum_{t=k+1}^{n} Y_t Y'_t$ but we can use the fact that $\mathbb{E}(\hat{W}_t^2 | F_{t-1}^*) = \sigma_W^2$ for all $t$ under the null hypothesis (5) to form an alternative estimator. We compute that

$$\Sigma_Y = \mathbb{E}(\text{cov}(Y_t | F_{t-1}^*)) = \mathbb{E}\left(\mathbb{E}\left(\text{E}(Y_t Y'_t | F_{t-1}^*)\right)\right) = \mathbb{E}\left(\mathbb{E}\left(\hat{W}_{t+1} | F_{t-1}^*\right)\right) = \sigma_W^2 H$$

(17)

where $H = \mathbb{E}\left(\hat{h}_{t-1} \hat{h}'_{t-1}\right)$, which suggests the estimator $\hat{\Sigma}_Y = \sigma_W^2 \hat{H}$ where

$$\hat{H} = (n - k)^{-1} \sum_{t=k+1}^{n} \hat{h}_{t-1} \hat{h}'_{t-1}.$$  

(18)

The decomposition in (17) has the advantage that it generalizes our unconditional spectral Z-test, which corresponds to the case $k = 0$. The case $k = 1$ may be viewed as a Z-test version of the first-order Markov chain test of Christoffersen (1998). To see that the conditional test also embeds the DQ test statistic proposed by Engle and Manganelli (2004), let $X$ be the $(n - k) \times (k + 1)$ matrix whose rows are given by $h_{t-1}$ for $t = k + 1, \ldots, n$ and let $\tilde{W} = (\tilde{W}_{k+1}, \ldots, \tilde{W}_n)'$. It follows that

$$\hat{\Sigma}_Y = \sigma_W^2 (n - k)^{-1} \sum_{t=k+1}^{n} h_{t-1} h'_t = \sigma_W^2 (n - k)^{-1} X'X$$

and $\tilde{Y}_{n,k} = (n - k)^{-1} X'\tilde{W}$ so that (16) may be rewritten as

$$\sigma_W^{-2} \tilde{W}'X(X'X)^{-1} X'\tilde{W} \sim \chi^2_{k+1}$$

(19)

The DQ test corresponds to the binomial score case, i.e., the case where $W_t = 1_{\{P_t \geq \alpha\}}$ and the CVT is $h(p) = 1_{\{p \geq \alpha\}}^4$.

---

3 It would also be possible to use a heteroscedasticity and autocorrelation-consistent (HAC) estimator of $\Sigma_Y$ (Newey and West, 1987; Andrews, 1991). Our preferred estimator has the advantage of higher sensitivity to deviations from the null hypothesis of zero serial correlation.

4 Engle and Manganelli (2004) allow as well for lagged VaR values to be included as regressors, an extension.
It is straightforward to generalize the conditional spectral Z-test to a conditional bispectral Z-test; see Appendix B.

### 5.2 Size and power

We build on the Monte Carlo exercises of Section 4.3 to study the size and power of the conditional tests of coverage. Here we present a representative extract of simulation studies documented in our companion paper [SUPPLEMENTARY MATERIAL]. The data are generated from three different “true” models: iid standard normal; a time series model with underlying ARMA(1,1) structure and standard normal marginals; and a model with underlying ARMA(1,1) structure and scaled t₅ marginal distribution. Our calibration of the ARMA parameters (AR = 0.95, MA = -0.85) is described in the companion paper [SUPPLEMENTARY MATERIAL] and is designed to mimic the serial dependence in PIT values when stochastic volatility is neglected. As in Section 4.3, we assume the risk manager reports PIT-values based on the standard normal model \( \hat{F} = \Phi \).

In addition to a choice of kernel, the MD test requires the choice of the number \( (k) \) of lagged PIT values and the conditioning variable transformation \( h(P) \). Define \( V(u) = |2u - 1| \); this V-shaped transformation of PIT values is well-suited to uncover dependence arising from stochastic volatility. As listed in Table 3, we consider four candidates for the CVT. Whereas the DQ requires only a time-series of traditional exceedance indicators, the three CVT based on the \( V(u) \) transformation require that the regulator observe PIT values.

| Mnemonic | \( h(P) \) | Description |
|----------|-------------|-------------|
| DQ       | 1\(_{\{P>0.99\}}\) | Flags upper-tail PIT values, as in Engle and Manganelli (2004). |
| V.BIN    | 1\(_{\{V(P)>0.98\}}\) | Two-tailed version of DQ, flags PIT values near zero or one. |
| V.4      | \( V(P)^4 \) | Places heavier weight on tail PIT values in the recent past. |
| V.\( \frac{1}{2} \) | \( \sqrt{V(P)} \) | Dampens sensitivity to tail PIT values relative to V.4. |

Table 3: Conditioning variable transformations. \( V(u) = |2u - 1| \)

possible in our framework, but change in portfolio composition implies that lagged VaR values are less informative than lagged PIT values.
Table 4 gives a flavor of the main findings for the example of the uniform kernel (ZU) and the narrow kernel window of \([0.985, 0.995]\). We report the percentage of rejections of the null hypothesis at the 5% confidence level based on \(2^{16} = 65,536\) replications. In the first column (CVT=“None”), we set \(k = 0\) to obtain the unconditional Z-test. Each of the remaining columns corresponds to a CVT with \(k = 4\). As seen in the first row (iid standard normal model), size is more difficult to control in the conditional tests. However, the CVT choices V.4 and V.\(\frac{1}{2}\) are only slightly oversized whereas V.BIN and, in particular, DQ are very oversized.

The model depicted in the second row gives uniformly distributed PIT values with a serial dependence structure that is typical when stochastic volatility is ignored. The power of the unconditional test in this situation is very limited (10.8%), while the MD tests show power ranging from 21.7% to 32.6%. There is a further increase in power when the simulated PIT data are both non-uniform and serially dependent (third row).

| F           | Serial Dependence | CVT | None | DQ  | V.BIN | V.4  | V.\(\frac{1}{2}\) |
|-------------|-------------------|-----|------|-----|-------|------|-----------------|
| Normal      | None              |     | 4.8  | 14.4| 9.0   | 6.7  | 6.7             |
| Normal      | ARMA(1, 1)        |     | 10.8 | 31.5| 30.9  | 32.6 | 21.7            |
| Scaled t5   | ARMA(1, 1)        |     | 36.2 | 54.9| 52.7  | 60.7 | 54.5            |

Table 4: Estimated size and power of conditional tests.
MD tests using the ZU kernel on the narrow window [0.985, 0.995]. We report the percentage of rejections of the null hypothesis at the 5% confidence level based on \(2^{16} = 65,536\) replications. The number of days in each backtest sample is \(n = 750\). ARMA parameters are AR = 0.95, MA = -0.85.

6 Application to bank-reported PIT values

6.1 Data

Our data consist of ten confidential backtesting samples provided by US banks to the Federal Reserve Board at the subportfolio level. Mandatory reporting to bank regulators pursuant to the Market Risk Rule took effect on January 1, 2013. For each significant subportfolio and each business day, the bank is required to report the overnight VaR at the 99% level, the
realized clean P&L, and the associated PIT-value (Federal Register, 2012, p. 53105). While the first two fields have been available to regulators for a long time (at least at an aggregate trading book level), access to PIT values is new. Each of our ten samples represents returns on an equity or foreign exchange subportfolio, which can include derivative as well as cash positions. Our samples are taken from the three-year period from 2014–2016.

Summary statistics for the unconditional distributions are found in Table 5. As is often the case with new regulatory reporting requirements, the data are not uniform in quality. Two of the samples (coded Pf104 and Pf110) have missing values (0.9% and 3.2% of trading days, respectively). Furthermore, close inspection reveals that most of the samples contain a small number of observations that are potentially spurious. In a few extreme cases, a PIT value of 1 is matched to a realized loss smaller than the forecast VaR. We apply a heuristic procedure to identify spurious values based on the distance between the reported PIT-value and an imputed value. The latter is constructed using a portfolio-specific model that fits PIT to the ratio of realized loss to VaR; details are provided in Appendix C. In test results reported below, we treat spurious values as missing to make the tests less sensitive to reporting error. Our conclusions are robust to taking all non-missing observations as valid.

Remaining columns of the table provide a histogram of PIT values. For some portfolios, tail PIT values are underrepresented (e.g., Pf107) or overrepresented (e.g., Pf105) in the sample. For some other portfolios, the histograms appear to be close to uniform, e.g., for Pf110, 85.9% of PIT values lie in [0.05, 0.95) and remaining mass is distributed roughly symmetrically.

6.2 Tests of unconditional coverage

Due to the generality of our framework, application of spectral backtests to data involves choices along several dimensions. As in Section 4.3, we fix $\alpha^* = 0.99$ as the conventional VaR level, define a narrow window as [0.985, 0.995] and a wide window as [0.95, 0.995]. Kernels are drawn from Table 1. Guided by our simulation results and the need for brevity, we
| ID | Trading days of which: | Frequencies |
|----|------------------------|-------------|
|    | Missing | Spurious | [0, .005) | [.005, .015) | [.015, .05) | [.05, .95) | [.95, .985) | [.985, .995) | [.995, 1] |
| 101| 756     | 0        | 0.0132    | 0.0172     | 0.0357   | 0.8796   | 0.0317   | 0.0106   | 0.0119   |
| 102| 751     | 0        | 0.0027    | 0.0108     | 0.0215   | 0.9113   | 0.0323   | 0.0121   | 0.0094   |
| 103| 750     | 0        | 0.0040    | 0.0040     | 0.0189   | 0.9474   | 0.0162   | 0.0081   | 0.0013   |
| 104| 774     | 7        | 0.0000    | 0.0000     | 0.0026   | 0.9804   | 0.0104   | 0.0013   | 0.0052   |
| 105| 750     | 0        | 0.0174    | 0.0214     | 0.0294   | 0.8596   | 0.0388   | 0.0187   | 0.0147   |
| 106| 629     | 0        | 0.0111    | 0.0191     | 0.0510   | 0.8328   | 0.0557   | 0.0175   | 0.0127   |
| 107| 750     | 0        | 0.0000    | 0.0000     | 0.0013   | 0.9960   | 0.0027   | 0.0000   | 0.0000   |
| 108| 756     | 0        | 0.0000    | 0.0053     | 0.0267   | 0.9278   | 0.0174   | 0.0120   | 0.0107   |
| 109| 734     | 0        | 0.0082    | 0.0151     | 0.0495   | 0.8654   | 0.0371   | 0.0206   | 0.0041   |
| 110| 774     | 25       | 0.0134    | 0.0200     | 0.0507   | 0.8585   | 0.0427   | 0.0107   | 0.0040   |

Table 5: Sample statistics.
Missing and spurious observations excluded from the reported frequencies. Sample period is 2014-01-01 to 2016-12-31.
exclusively employ two-sided Z-tests in our empirical analysis.

Table 6 presents $p$-values for the tests of unconditional coverage. We find that the forecast models for portfolios $Pf105$, $Pf106$ and $Pf107$ are rejected at the 1% level for all kernels and on both the narrow and wide kernel windows. In view of the histograms observed in Table 5, this is unsurprising. When an empirical distribution function (edf) lies above the uniform cdf within the kernel window (as observed for $Pf107$), large PIT values are underrepresented in the sample, which suggests that the forecast model overstates the upper quantiles of the loss distribution. When an edf lies below the uniform cdf (as observed for $Pf105$ and $Pf106$), large PIT values are overrepresented in the sample, which suggests that the forecast model understates the upper quantiles. By contrast, there are no rejections at all for $Pf110$, for which the edf is reasonably close to the theoretical cdf throughout the upper tail.

For the remaining six portfolios, test results are sensitive to the choice of kernel. This is to be expected and desirable, as the different tests prioritize different quantiles of the unconditional distribution. To shed light on the differences, in Figure 2 we plot the edf for three portfolios on the narrow window (upper panel) and wide window (lower panel). This plot is the empirical counterpart to Figure 1 in Section 4.3. For $Pf104$, we observe that the edf intersects with the theoretical cdf at the common upper window boundary $\alpha_2 = 0.995$, but lies above the theoretical cdf at lower PIT values. Over the wide window, the average distance between edf and theoretical cdf is large, so any test that assigns significant weight to PIT values near the center of the window will reject. When restricted to the narrow window, the average distance is reduced, so the tests fail to reject. The edf for portfolio $Pf103$ (not shown) is qualitatively close to that of $Pf104$, which explains the similarity in test results.

The edf for portfolio $Pf108$ lies somewhat below and roughly parallel to the theoretical cdf throughout the narrow window. Tests reject at the 5% level for some of the kernels and

---

5 All $p$-values in the tables below should be interpreted in the context of a single test of the null hypothesis. If multiple tests are conducted, inferences would have to be based on a standard correction method such as that of Bonferroni; see Shaffer (1995) for a review.
| ID  | window | BIN  | ZU3  | PE3  | ZU   | ZA   | ZE   | ZL+  | ZL−  | ZLL  | PNS |
|-----|--------|------|------|------|------|------|------|------|------|------|-----|
| 101 | narrow | 0.1046 | 0.0340 | 0.0502 | 0.0356 | 0.0294 | 0.0436 | 0.0230 | 0.0572 | 0.0595 | 0.0352 |
|     | wide   | 0.1046 | 0.1407 | 0.0593 | 0.2050 | 0.2300 | 0.1718 | 0.0610 | 0.4243 | 0.0128 | 0.0296 |
| 102 | narrow | 0.0016 | 0.0200 | 0.0027 | 0.0800 | 0.0795 | 0.0802 | 0.1346 | 0.0558 | 0.1085 | 0.2153 |
|     | wide   | 0.0016 | 0.0063 | 0.0121 | 0.2456 | 0.2484 | 0.2797 | 0.2114 | 0.2898 | 0.4477 | 0.2238 |
| 103 | narrow | 0.1029 | 0.1158 | 0.4212 | 0.0987 | 0.1134 | 0.0882 | 0.1094 | 0.0995 | 0.2545 | 0.2804 |
|     | wide   | 0.1029 | 0.0035 | 0.0229 | 0.0058 | 0.0041 | 0.0083 | 0.0156 | 0.0038 | 0.0126 | 0.0091 |
| 104 | narrow | 0.1829 | 0.1691 | 0.1119 | 0.1651 | 0.1678 | 0.1633 | 0.3063 | 0.0987 | 0.0767 | 0.0657 |
|     | wide   | 0.1829 | 0.0010 | 0.0002 | 0.0004 | 0.0003 | 0.0005 | 0.0031 | 0.0002 | 0.0003 | 0.0001 |
| 105 | narrow | 0.0000 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
|     | wide   | 0.0000 | 0.0000 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0000 | 0.0000 |
| 106 | narrow | 0.0005 | 0.0004 | 0.0055 | 0.0010 | 0.0008 | 0.0012 | 0.0019 | 0.0007 | 0.0033 | 0.0036 |
|     | wide   | 0.0005 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 107 | narrow | 0.0059 | 0.0018 | 0.0097 | 0.0026 | 0.0020 | 0.0032 | 0.0062 | 0.0016 | 0.0053 | 0.0033 |
|     | wide   | 0.0059 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 108 | narrow | 0.0166 | 0.0213 | 0.0933 | 0.0311 | 0.0290 | 0.0301 | 0.0191 | 0.0520 | 0.0482 | 0.0635 |
|     | wide   | 0.0166 | 0.6971 | 0.0052 | 0.8422 | 0.8168 | 0.8545 | 0.5358 | 0.4445 | 0.0034 | 0.0070 |
| 109 | narrow | 0.5217 | 0.2499 | 0.0222 | 0.1358 | 0.1237 | 0.1712 | 0.3171 | 0.0634 | 0.0165 | 0.0225 |
|     | wide   | 0.5217 | 0.2584 | 0.3373 | 0.0563 | 0.0715 | 0.0525 | 0.0503 | 0.0706 | 0.1471 | 0.2814 |
| 110 | narrow | 0.8514 | 0.9479 | 0.8692 | 0.6660 | 0.7719 | 0.6123 | 0.6321 | 0.7032 | 0.8737 | 0.9856 |
|     | wide   | 0.8514 | 0.5407 | 0.6916 | 0.3017 | 0.2932 | 0.3442 | 0.4527 | 0.2325 | 0.3407 | 0.6253 |

Table 6: Tests of unconditional coverage.
We report $p$-values by portfolio, kernel window, and kernel family. Narrow kernel window is [0.985,0.995] and wide kernel window is [0.95,0.995]. Sample period is 2014-01-01 to 2016-12-31.
fail to reject for others, but the $p$-values all lie between 1.5% and 10%. When we consider
the wide window, we find that the edf lies above the theoretical cdf in the lower half of the
window and below in the upper half, which implies that the forecast model underestimates
quantiles at one boundary of the kernel window and overestimates quantiles at the other
boundary, i.e., a slope deviation from the uniform cdf. As shown in Section 4.3, bispectral
tests generally outperform monospectral tests in this situation. We find that the tests based
on the bivariate ZLL and PNS kernels and the trivariate Pearson kernel reject at the 1%
level. The edf for portfolio $Pf101$ (not shown) also displays a slope violation on the wide
window, and again we find the bivariate ZLL and PNS kernels most effective.

In the case of portfolio $Pf109$, the edf displays a slope violation within the narrow
window. As before, we find that the tests based on the bivariate ZLL and PNS kernels
and the trivariate Pearson kernel reject at the 5% level, whereas the monospectral tests all
fail to reject. On the wide window, the edf lies uniformly below the theoretical cdf, so the
slope violation loses salience. The bispectral and trispectral tests now fail to reject, whereas
several of the monospectral tests reject at a 10% level.

### 6.3 Tests of conditional coverage

In this section, we emphasize the role of the conditioning variable transformation $h(P)$
in revealing serial dependence in PIT-values. For parsimony, we consider only a subset
of the kernels used in the previous section. We include the binomial score kernel (BIN)
as representative of the traditional test, the uniform kernel (ZU) as representative of the
continuous monospectral tests, and the (ZLL) as representative of the multispectral tests.
We fix $k = 4$ lags in the monospectral tests which corresponds to looking at dependencies
over a time horizon of one trading week. To facilitate comparison to the monospectral tests,
we fix $(k_1 = 4, k_2 = 0)$ for the bispectral ZLL test.

Missing or spurious values may be especially troublesome in a test of conditional coverage
because a PIT value missing at time $t$ introduces missing regressors at $t + 1, \ldots, t + k$. To
Figure 2: Empirical distribution functions for select portfolios. EDFs for narrow kernel window (upper panel) and wide kernel window (lower panel). The uniform cdf is plotted as a dashed black line.
avoid losing the subsequent $k$ observations, we replace missing or spurious $P_{t-\ell}$ with an
inputed value when computing the lagged vector $h_{t-1}$. (As in the tests of unconditional
coverage, we do not impute missing $P_t$ to backfill the dependent variables $W_t$, but simply
drop these observations.) Details of our imputation algorithm are found in Appendix C.

Table 7 presents $p$-values for the tests of conditional coverage. For portfolios $Pf105$,
$Pf106$ and $Pf108$, forecast models are strongly rejected (always at the 1% level, and nearly
always at the 0.01% level) regardless of the choice of CVT or kernel; for brevity we drop
these portfolios from the table. For only a single portfolio ($Pf109$), the forecast model is
never rejected. In the other six cases, choice of CVT and kernel matters.

For portfolios $Pf102$ and $Pf110$, the V.4 CVT generally leads to rejection at the 5% level,
whereas tests using the DQ CVT generally do not. The V.BIN and V.$1/2$ CVT are effective
in many cases, but appear less robust than V.4. This reflects the greater sensitivity of the
V.4 transformation to local spikes in market volatility. In the case of portfolio $Pf103$, the
tests reject on the wide window except when using the DQ CVT.

For portfolios $Pf101$ and $Pf104$, variation in $p$-value across tests is driven primarily by
kernel choice, and in a manner consistent with the tests of unconditional coverage in Table
6. Thus, for these two portfolios, serial dependence in the PIT-values does not appear to be
the salient shortcoming in the forecast model.

In the case of $Pf107$, the test statistic is undefined for the DQ CVT and its two-tailed
counterpart (V.BIN). As there were no observed violations in either tail ($P_t < .01$ or $P_t >
.99$), in both cases the matrix $\hat{H}$ of (18) is singular, so $\hat{\Sigma}_Y$ in the test statistic cannot be
inverted. This demonstrates a practical limitation of a binary-valued CVT, as short samples
may often contain no tail values. Observe also that the backtest fails to reject for the
remaining two CVT on the narrow window, even though the forecast model for this portfolio
is strongly rejected by the unconditional tests. Since $P_t < \alpha_1 = 0.985$ for all $t$, $W_t$ has a
degenerate distribution in the sample. In this situation, it may be shown that the conditional
test statistic is invariant to the CVT and to $k$ and is equal to the unconditional test statistic.
Table 7: Tests of conditional coverage.
We report test $p$-values by portfolio, conditioning variable transformation, kernel window and kernel family. The monospectral tests utilize $k = 4$ lags, and for the ZLL bispectral test we set $(k_1 = 4, k_2 = 0)$. Narrow kernel window is $[0.985, 0.995]$ and wide kernel window is $[0.95, 0.995]$. Sample period is 2014-01-01 to 2016-12-31. Forecast models for $Pf_{105}$, $Pf_{106}$ and $Pf_{108}$ (not tabulated) are rejected at the 1% level for all choices of CVT and kernel.

| ID | CVT | BIN | narrow window | wide window |
|----|-----|-----|--------------|-------------|
|    |     |     | ZU           | ZLL         | ZU           | ZLL         |
| 101| DQ  | 0.0017 | 0.0002 | 0.0020 | 0.0342 | 0.0092 |
|    | V.BIN | 0.0084 | 0.0002 | 0.0009 | 0.0513 | 0.0115 |
|    | V.4  | 0.0062 | 0.0003 | 0.0007 | 0.0921 | 0.0230 |
|    | V.$\frac{1}{2}$ | 0.0658 | 0.0063 | 0.0078 | 0.1376 | 0.0301 |
| 102| DQ  | 0.0088 | 0.0975 | 0.1448 | 0.1873 | 0.1443 |
|    | V.BIN | 0.0119 | 0.4716 | 0.3441 | 0.0201 | 0.0147 |
|    | V.4  | 0.0000 | 0.0086 | 0.0023 | 0.0003 | 0.0006 |
|    | V.$\frac{1}{2}$ | 0.0000 | 0.0239 | 0.0111 | 0.0004 | 0.0005 |
| 103| DQ  | 0.7540 | 0.7429 | 0.8408 | 0.1787 | 0.1879 |
|    | V.BIN | 0.0773 | 0.0426 | 0.1331 | 0.0068 | 0.0149 |
|    | V.4  | 0.3064 | 0.1372 | 0.2286 | 0.0074 | 0.0088 |
|    | V.$\frac{1}{2}$ | 0.5209 | 0.4194 | 0.5181 | 0.0318 | 0.0276 |
| 104| DQ  | 0.8732 | 0.8501 | 0.5202 | 0.0295 | 0.0138 |
|    | V.BIN | 0.8700 | 0.8456 | 0.5167 | 0.0289 | 0.0135 |
|    | V.4  | 0.3225 | 0.2013 | 0.1287 | 0.0034 | 0.0022 |
|    | V.$\frac{1}{2}$ | 0.2085 | 0.0976 | 0.0689 | 0.0016 | 0.0011 |
| 107| DQ  | NA | NA | NA | NA | NA |
|    | V.BIN | NA | NA | NA | NA | NA |
|    | V.4  | 0.1844 | 0.1071 | 0.1085 | 0.0001 | 0.0000 |
|    | V.$\frac{1}{2}$ | 0.1844 | 0.1071 | 0.1085 | 0.0001 | 0.0000 |
| 109| DQ  | 0.9917 | 0.6351 | 0.0548 | 0.1465 | 0.1383 |
|    | V.BIN | 0.8041 | 0.7510 | 0.1179 | 0.4522 | 0.5170 |
|    | V.4  | 0.8929 | 0.8603 | 0.2067 | 0.5578 | 0.6225 |
|    | V.$\frac{1}{2}$ | 0.9313 | 0.6389 | 0.1327 | 0.4766 | 0.5266 |
| 110| DQ  | 0.2658 | 0.3058 | 0.4121 | 0.1661 | 0.1395 |
|    | V.BIN | 0.0041 | 0.0006 | 0.0009 | 0.0044 | 0.0108 |
|    | V.4  | 0.0093 | 0.0008 | 0.0012 | 0.0100 | 0.0352 |
|    | V.$\frac{1}{2}$ | 0.1403 | 0.0513 | 0.0840 | 0.1508 | 0.2823 |
Recalling that the test statistic has distribution $\chi^2_{1+k}$ under the null hypothesis, we find that the $p$-value increases with $k$. This explains why unconditional backtests may have greater power than conditional backtests in situations where an overly conservative forecast model leads to degeneracy in $W_t$.

7 Conclusion

The class of spectral backtests embeds many of the most widely used tests of unconditional coverage and tests of conditional coverage, including the binomial likelihood ratio test of Kupiec (1995), the interval likelihood ratio test of Berkowitz (2001), and the dynamic quantile test of Engle and Manganelli (2004). As we demonstrate with many examples, viewing these tests in terms of the associated kernels facilitates the construction of new tests. From the perspective of the practice of risk management, making explicit the choice of kernel measure may help to discipline the backtesting process because the kernel directly expresses the user’s priorities for model performance.

Different kernels are sensitive to different deviations from the null hypothesis. A tester who only cares about systematic under- or overestimation of quantiles within a narrow range is well served by a number of single kernels, discrete and continuous. A tester who wants to ensure maximum fidelity of the forecast models to the true distributions across a wider range of quantiles may worry more about slope violations (overestimation of quantiles at one end of a window and underestimation at the other). Such a tester may favor a multispectral test. However, to promote a single “best” test from the spectral family would be contrary to the philosophy of our contribution, and we refrain from doing so. The tester should reflect on performance priorities and select her kernel accordingly.

Our results illustrate the value to regulators of access to bank-reported PIT-values. Until recently, regulators effectively observed only a sequence of VaR exceedance event indicators at a single level $\alpha$, and therefore backtests were designed to take such data as input. In some
jurisdictions, including the United States, PIT-values have been collected for some time. Besides enabling the formation of spectral test statistics, lagged PIT-values are especially effective as conditioning variables in regression-based tests of conditional coverage.

A  Proofs

A.1  Proof of Theorem 3.1

Let $G_1$ and $G_2$ be the increasing, right-continuous functions associated with the measures $\nu_1$ and $\nu_2$. It follows that $W_t^* = G_1(P_t)G_2(P_t)$. The function $G^*(u) = G_1(u)G_2(u)$ must also be increasing and right-continuous and can thus be used to define a Lebesgue-Stieltjes measure $\nu^*$ by setting $\nu^*(\{0\}) = G^*(0) = 0$ and $\nu^*((a,b]) = G^*(b) - G^*(a)$ for any $0 \leq a < b \leq 1$. It follows that $W_t^* = G^*(P_t) = \nu^*([0,P_t])$.

The formula for $\nu^*$ is obtained by applying the integration-by-parts formula for the Lebesgue-Stieltjes integral (Hewitt, 1960, Theorem A).

A.2  Proof of Proposition 3.2

Since $G_\nu(u) = O((1-u)^{-1/2+\epsilon})$ as $u \to 1$ for some small $\epsilon$, there exists a value $u_0$ and a positive constant $C$ such that $G_\nu(u) \leq C(1-u)^{-1/2+\epsilon}$ for $u \geq u_0$. Let $\bar{u}$ be the larger of $u_0$ and the last point at which $G_\nu$ is not differentiable (there are only finitely so many points by Assumption 1). We can decompose (9) as

$$
\mathbb{E}(W_t^2) = \int_{[0,\bar{u}]} (1-u) (2G_\nu(u) - \nu(\{u\})) \, d\nu(u) + \int_{(\bar{u},1]} (1-u) (2G_\nu(u) - \nu(\{u\})) \, d\nu(u) \tag{A.1}
$$

The integrand in the first term is bounded above by $2G_\nu(\bar{u})$ and so the integral is finite. We only need to prove the finiteness of the second term which can be written as

$$
\int_{\bar{u}}^1 (1-u)2G_\nu(u)g_\nu(u)\,du = \int_{\bar{u}}^1 (1-u) \frac{d}{du} (G_\nu(u)^2) \, du = [G_\nu(u)^2(1-u)]_{\bar{u}}^1 + \int_{\bar{u}}^1 G_\nu(u)^2 \, du
$$
using integration by parts.

Since $0 \leq G_\nu(u)^2(1-u) \leq C^2(1-u)^{2\epsilon}$ for $u \geq \bar{u}$ and $(1-u)^{2\epsilon} \to 0$ as $u \to 1$, it follows that $[G_\nu(u)^2(1-u)]_a^1 = -G_\nu(\bar{u})^2(1-\bar{u})$. Moreover, the second term is finite because

$$\int_{\bar{u}}^{1} G_\nu(u)^2 du \leq C^2 \int_{\bar{u}}^{1} (1-u)^{-1+2\epsilon} du = \frac{1}{2\epsilon} (1-\bar{u})^{2\epsilon}.$$ 

A.3 Proof of Theorem 3.3

Let $p_t$ denote the realized value of $P_t$ and $w_{t,j} = G_j(p_t)$ the corresponding realized value of $W_{t,j}$ for $t = 1, \ldots, n$ and $j = 1, 2$. There are two cases to consider. Either $p_t$ occurs in an interval where the right derivative of $G_j$ is 0 or in an interval where the right derivative is positive. Let $G_j$ denote the subset of $[0, 1]$ consisting of all points for which the right derivative of $G_j$ equals zero.

If $p_t \in G_j$ then, by the right-continuity of $G_j$, $p_t$ must occur in an interval of the form $[a_{t,j}, b_{t,j})$ (if there is a jump in $G_j$ at $b_{t,j}$) or $[a_{t,j}, b_{t,j}]$ (if $G_j$ is continuous at $b_{t,j}$). In either case the contribution of $w_{t,j}$ to the likelihood is

$$\mathbb{P}(W_{t,j} = w_{t,j}) = \mathbb{P}(G_j(P_t) = G_j(p_t)) = F_{P}(b_{t,j}) - F_{P}(a_{t,j}).$$

If $P_t \notin G_j$ then $w_{t,j}$ satisfies $\mathbb{P}(W_{t,j} \leq w_{t,j}) = \mathbb{P}(P_t \leq p_t)$ and $p_t = G_j^{-1}(w_{t,j})$, the unique inverse of $G_j$ at $w_{t,j}$. The contribution to the likelihood is a density contribution given by

$$f_{W_t}(w_{t,j} \mid \theta) = \frac{f_{P}(p_t \mid \theta)}{G_j'(p_t)}.$$ 

The general form of the realized likelihood given $w_j = (w_{1,j}, \ldots, w_{n,j})'$ is thus

$$\mathcal{L}_{W_j}(\theta \mid w_j) = \prod_{p_t \in G_j} \left( F_{P}(b_{t,j}) - F_{P}(a_{t,j}) \right) \prod_{p_t \notin G_j} \frac{f_{P}(p_t \mid \theta)}{G_j'(p_t)}.$$ 

For the measures $\nu_1$ and $\nu_2$ the sets $G_1$ and $G_2$ may differ at most by a null set. Let us assume
that each realized point $p_t$ is either in both of the sets $G_1$ and $G_2$ or in neither of the sets.

If $p_t \in G_1$ and $p_t \in G_2$ then the agreement of the supports on $(0,1)$ implies that $a_{t,1} = a_{t,2}$ and $b_{t,1} = b_{t,2}$. Thus the likelihood contributions are identical.

If $p_t \not\in G_1$ and $p_t \not\in G_2$ then the likelihoods differ only by the scaling factor $G_j'(p_t)$ which does not involve the parameters $\theta$. This factor will appear in the log-likelihood only as an unimportant additive term and cancel out of the LR test statistic.

It follows that the likelihoods $L_{w_j}(\theta \mid w_j)$ are maximized by the same values $\hat{\theta}$ and the LR-test statistics are identical.

A.4 Proof of Theorem 4.1

Let $X_t = (X_{t,0}, \ldots, X_{t,m})'$ be the $(m + 1)$-dimensional random vector with $X_{t,i} = 1_{\{V_{w_t} = i\}}$ for $i = 0, \ldots, m$. Under (5) $X_t$ has a multinomial distribution satisfying $\mathbb{E}(X_{t,i}) = \theta_i$, $\text{var}(X_{t,i}) = \theta_i(1 - \theta_i)$ and $\text{cov}(X_{t,i}, X_{t,j}) = -\theta_i \theta_j$ for $i \neq j$.

Now define $Y_t$ to be the $m$-dimensional random vector obtained from $X_t$ by omitting the first component. Then $\mathbb{E}(Y_t) = \theta = (\theta_1, \ldots, \theta_m)'$ and $\Sigma_Y$ is the $m \times m$ submatrix of $\text{cov}(X_t)$ resulting from deletion of the first row and column. Let $\overline{Y} = n^{-1} \sum_{t=1}^n Y_t$. A standard approach to the asymptotics of the Pearson test is to show that

$$S_m = \sum_{i=0}^m \frac{(O_i - n \theta_i)^2}{n \theta_i} = n \frac{(\sum_{i=1}^m X_{t,i} - n \theta_i)^2}{n \theta_i} = n (\overline{Y} - \theta)' \Sigma_Y^{-1} (\overline{Y} - \theta), \quad (A.2)$$

and hence to argue that $S_m \sim \chi_m^2$ in the limit as $n \to \infty$ by the central limit theorem. It remains to show that the right-hand side of (A.2) has the spectral test representation (7).

Let $A$ be the $m \times m$ matrix with rows given by $(e_1 - e_2, e_2 - e_3, \ldots, e_m)$ where $e_i$ denotes the $i$th unit vector. It may be easily verified that $Y_t = AW_t$, $\theta = A \mu_W$ and $\Sigma_Y = A \Sigma_W A'$. It follows that

$$n (\overline{W} - \mu_W)' \Sigma_W^{-1} (\overline{W} - \mu_W) = n (\overline{Y} - \theta)' \Sigma_Y^{-1} (\overline{Y} - \theta) = S_m.$$
B  Conditional bispectral Z-test

The conditional spectral Z-test generalizes to a conditional multispectral Z-test. In the bispectral case, we construct two sets of transformed reported PIT-values \((W_{t,1}, W_{t,2})\) for \(t = 1, \ldots, n\), and form the vector \(Y_t\) of length \(k_1 + k_2 + 2\) given by

\[
Y_t = \left( h'_{t-1,1} \widehat{W}_{t,1}, h'_{t-1,2} \widehat{W}_{t,2} \right)',
\]  

where \(\widehat{W}_{t,i} = W_{t,i} - \mu_{W,i}\) and \(h_{t-1,i} = (1, h_i(P_{t-1}), \ldots, h_i(P_{t-k}))'\). Parallel to the univariate case, let \(\sum_{k=\max(k_1,k_2)+1}^{n} Y_t\) for \(k = k_1 \lor k_2\), and let \(\hat{\Sigma}_Y\) denote a consistent estimator of \(\Sigma_Y := \text{cov}(Y_t)\). By the theory of Giacomini and White (2006), for \(n\) large and \((k_1, k_2)\) fixed,

\[
(n-k) \sum_{k=\max(k_1,k_2)+1}^{n} Y_{n,k} \hat{\Sigma}_{Y}^{-1} Y_{n,k} \sim \chi^2_{k_1+k_2+2}.
\]  

(B.2)

Working under the null hypothesis, we can generalize (17) to \(\Sigma_Y = A_W \circ H\), where \(\circ\) denotes element-by-element multiplication (Hadamard product). The matrices are

\[
H = \begin{pmatrix}
\mathbb{E}(h_{t-1,1}h'_{t-1,1}) & \mathbb{E}(h_{t-1,1}h'_{t-1,2}) \\
\mathbb{E}(h_{t-1,2}h'_{t-1,1}) & \mathbb{E}(h_{t-1,2}h'_{t-1,2})
\end{pmatrix}, \quad A_W = \begin{pmatrix}
\sigma^2_{W,1}J_{k_1+1,k_1+1} & \sigma_{W,12}J_{k_1+1,k_2+1} \\
\sigma_{W,12}J_{k_2+1,k_1+1} & \sigma^2_{W,2}J_{k_2+1,k_2+1}
\end{pmatrix}
\]  

(B.3)

where \(J_{m,n}\) denotes the \(m \times n\) matrix of ones and \(\sigma_{W,12} = \mathbb{E}(\widehat{W}_{t,1}\widehat{W}_{t,2})\). Our tests use the estimator \(\hat{\Sigma}_Y = A_W \circ \hat{H}\), where \(\hat{H}\) generalizes (18) as

\[
\hat{H} = (n - (k_1 \lor k_2))^{-1} \sum_{t=(k_1 \lor k_2)+1}^{n} (h'_{t-1,1}, h'_{t-1,2})'(h_{t-1,1}, h_{t-1,2}).
\]  

(B.4)

C  Identification of spurious PIT values

Consider a stylized Gaussian model in which loss is given by \(L_t = \sigma_{t-1}Z_t\), where \((Z_t)\) is an iid sequence of standard normal random variables and volatility \(\sigma_{t-1}\) is \(\mathcal{F}_{t-1}\)-measurable. Time
variation in $\sigma_t$ may arise from stochastic volatility or from changes over time in portfolio composition. Suppose that the risk-manager knows the true underlying distribution and the volatility. The risk-manager’s ideal value-at-risk forecast at $\alpha = 0.99$ is then $\widehat{\text{VaR}}_t = \Phi^{-1}(0.99)\sigma_{t-1}$, where $\Phi$ is the standard normal cdf. We do not observe $\sigma_{t-1}$, but from observing $L_t$ and $\widehat{\text{VaR}}_t$, we can back out the realized value of $Z_t$ as

$$Z_t = \Phi^{-1}(0.99) \times \frac{L_t}{\widehat{\text{VaR}}_t}.$$  \hfill (C.1)

Furthermore, the PIT values can be expressed as

$$P_t = \widehat{F}_{t-1}(L_t) = \Phi(L_t/\sigma_{t-1}) = \Phi(Z_t).$$  \hfill (C.2)

In general, we would not expect the $Z_t$ to be Gaussian, so (C.2) will not hold. However, so long as $(Z_t)$ is iid, there will still be a monotonic relationship between $Z_t$ (as defined by (C.1)) and $P_t$. We find that the predicted relationship holds qualitatively for all bank-reported portfolios, but with more noise in some portfolios than in others. This suggests that we can use violations of monotonicity to identify spurious PIT values, but the threshold for identification must vary across portfolios.

Let $H(z; \theta_i) : \mathbb{R} \rightarrow [0, 1]$ be a family of fitting functions with parameter $\theta_i$ for portfolio $i$, and replace (C.2) by

$$P_{i,t} = H(Z_{i,t}; \hat{\theta}_i) + \epsilon_{i,t} \hfill (C.3)$$

where the $\epsilon_{i,t}$ are white-noise residuals. Since the $H$ function should be increasing, it is convenient to take $H$ to be a cdf, even though it does not have a statistical interpretation in our context. For convenience, we take $H$ to be the normal cdf with unrestricted $(\mu_i, \sigma_i)$ as $\theta_i$.

For each portfolio $i$, we proceed as follows:

1. Fit $\theta_i$ by nonlinear least squares, and construct residuals $\epsilon_{it} = P_{it} - H(Z_{it}; \hat{\theta}_i)$.
2. The \((\epsilon_{it})\) are bounded in the open interval \((-1, 1)\), because \(H(Z_{it})\) does not produce boundary values. We model \(\epsilon_{it}\) as drawn from a rescaled beta distribution on \((-1, 1)\) with parameters \((a = \tau_i/2, b = \tau_i/2)\). This distribution has mean zero and variance \(1/(\tau_i + 1)\), so we simply fit \(\tau_i\) to the variance of the regression residuals.

3. Let \(B(\epsilon; \hat{\tau}_i)\) be the fitted beta distribution. We flag an observation \(P_{it}\) as spurious whenever \(B(\epsilon_{it}; \hat{\tau}_i) < q/2\) or \(B(\epsilon_{it}; \hat{\tau}_i) > 1 - q/2\), where \(q\) is a tolerance parameter.

4. We reestimate \(\tau_i\) as in step 3 on a sample that excludes the spurious observations. Repeat step 4 with the updated \(\hat{\tau}_i\). An observation is flagged as spurious if it is rejected in either round of estimation.

In our baseline procedure, we set the tolerance parameter to \(q = 10^{-5}\), which is intended to flag only the most egregious inconsistencies between \(P_{it}\) and the pair \((L_{it}, \hat{\text{VaR}}_{it})\). A typical case involves a PIT value very close to zero or one associated with a modest P&L such that \(|L_{it}| < \hat{\text{VaR}}_{it}\). Setting \(q = 0\) is equivalent to shutting down the identification of spurious values.

The procedure yields imputed PIT values as \(\hat{P}_{it} = H(Z_{it}; \hat{\theta}_i)\). As noted in Section 6.3, we use the imputed values to fill in for spurious values in forming regressors in the tests of conditional coverage.

References

Acerbi, C., and B. Szekely, 2014, Back-testing expected shortfall, Risk 1–6.

Amisano, G., and R. Giacomini, 2007, Comparing density forecasts via weighted likelihood ratio tests, Journal of Business & Economic Statistics 25, 177–190.

Andrews, D.W.K., 1991, Heteroskedasticity and autocorrelation consistent covariance matrix estimation, Econometrica 59, 817–858.

Barone-Adesi, G., F. Bourgoin, and K. Giannopoulos, 1998, Don’t look back, Risk 11, 100–103.

Basel Committee on Bank Supervision, 2013, Fundamental review of the trading book: A revised market risk framework, Publication No. 265, Bank for International Settlements.
Berkowitz, J., 2001, Testing the accuracy of density forecasts, applications to risk management, *Journal of Business & Economic Statistics* 19, 465–474.

Berkowitz, J., and J. O’Brien, 2002, How accurate are Value-at-Risk models at commercial banks?, *The Journal of Finance* 57, 1093–1112.

Billingsley, P., 1961, The Lindeberg–Lévy theorem for martingales, *Proceedings of the American Mathematical Society* 12, 788–792.

Board of Governors of the Federal Reserve System, 2011, Supervisory guidance on model risk management, SR Letter 11-7.

Campbell, S.D., 2006, A review of backtesting and backtesting procedures, *Journal of Risk* 9, 1–17.

Christoffersen, P., 1998, Evaluating interval forecasts, *International Economic Review* 39.

Colletaz, Gilbert, Christophe Hurlin, and Christophe Pérignon, 2013, The risk map: A new tool for validating risk models, *Journal of Banking and Finance* 37, 3843–3854.

Costanzino, N., and M. Curran, 2015, Backtesting general spectral risk measures with application to expected shortfall, *The Journal of Risk Model Validation* 9, 21–31.

Crnkovic, C., and J. Drachman, 1996, Quality control, *Risk* 9, 139–143.

Diebold, F.X., T.A. Gunther, and A.S. Tay, 1998, Evaluating density forecasts with applications to financial risk management, *International Economic Review* 39, 863–883.

Diebold, F.X., and R.S. Mariano, 1995, Comparing predictive accuracy, *Journal of Business & Economic Statistics* 13, 253–265.

Du, Z., and J.C. Escanciano, 2017, Backtesting expected shortfall: accounting for tail risk, *Management Science* 63, 940–958.

Engle, R.F., and S. Manganelli, 2004, CAViaR: conditional autoregressive value at risk by regression quantiles, *Journal of Business & Economic Statistics* 22, 367–381.

Escanciano, J.C., and J. Olmo, 2010, Backtesting parametric value-at-risk with estimation risk, *Journal of Business & Economic Statistics* 28, 36–51.

Federal Register, 2012, Risk-based capital guidelines: Market risk.

Fissler, T., J.F. Ziegel, and T. Gneiting, 2016, Expected shortfall is jointly elicitable with value-at-risk: implications for backtesting, *Risk* 58–61.

Giacomini, R., and H. White, 2006, Tests of conditional predictive ability, *Econometrica* 74, 1545–1578.

Gneiting, T., 2011, Making and evaluating point forecasts, *Journal of the American Statistical Association* 106, 746–762.
Gneiting, T., F. Balabdaoui, and A.E. Raftery, 2007, Probabilistic forecasts, calibration and sharpness, *Journal of the Royal Statistical Society, Series B* 69, 243–268.

Gneiting, T., and R. Ranjan, 2011, Comparing density forecasts using threshold- and quantile-weighted scoring rules, *Journal of Business & Economic Statistics* 29, 411–422.

Hewitt, E., 1960, Integration by parts for Stieltjes integrals, *The American Mathematical Monthly* 67, 419–423.

Hull, J. C., and A. White, 1998, Incorporating volatility updating into the historical simulation method for Value-at-Risk, *Journal of Risk* 1, 5–19.

Hurlin, C., S. Laurent, R. Quaedvlieg, and S. Smeekes, 2017, Risk measure inference, *Journal of Business & Economic Statistics* 35, 499–512.

Kratz, M., Y.H. Lok, and A.J. McNeil, 2018, Multinomial VaR backtests: A simple implicit approach to backtesting expected shortfall, *Journal of Banking and Finance* 88, 393–407.

Kupiec, P. H., 1995, Techniques for verifying the accuracy of risk measurement models, *Journal of Derivatives* 3, 73–84.

Leccadito, Arturo, Simona Boffelli, and Giovanni Urga, 2014, Evaluating the accuracy of Value-at-Risk forecasts: New multilevel tests, *International Journal of Forecasting* 30, 206–216.

Newey, W., and K. West, 1987, A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix, *Econometrica* 55, 703–08.

Nolde, N., and J.F. Ziegel, 2017, Elicitability and backtesting: Perspectives for banking regulation, *Annals of Applied Statistics* 11, 1833–1874.

O’Brien, J., and P.J. Szerszen, 2017, An evaluation of bank measures for market risk before, during and after the financial crisis, *Journal of Banking and Finance* 80, 215–234.

Pérignon, C., Z.Y. Deng, and Z.J. Wang, 2008, Diversification and Value-at-Risk, *Journal of Banking and Finance* 32, 783–794.

Pérignon, C., and D. R. Smith, 2010, The level and quality of Value-at-Risk disclosure by commercial banks, *Journal of Banking and Finance* 34, 362–377.

Pérignon, C., and D.R. Smith, 2008, A new approach to comparing VaR estimation methods, *Journal of Derivatives* 16, 54–66.

Rosenblatt, M., 1952, Remarks on a multivariate transformation, *Annals of Mathematical Statistics* 23, 470–472.

Shaffer, J. P., 1995, Multiple hypothesis testing, *Annual Review of Psychology* 46, 561–584.
SUPPLEMENTARY MATERIAL
Spectral backtests of forecast distributions
with application to risk management

July 26, 2019

Abstract

We extract material from a companion paper (in progress) which elaborates on certain results in our main paper. To avoid confusion with references to tables, figures and equations in the main paper, we prepend “S” when numbering tables, figures and equations contained in this supplement.
Supplement A: Truncated probitnormal score test

The probitnormal model yields a further new spectral test based on a classical score test of \( \theta = \theta_0 \) against the alternative \( \theta \neq \theta_0 \). Let \( \mathcal{L}_P(\theta \mid P^*_t) \) denote the likelihood contribution of a truncated observation \( P^*_t = \alpha_1 \lor (P_t \land \alpha_2) \) when \( P_t \) follows (13) and write

\[
S_t(\theta) = \left( \frac{\partial}{\partial \mu} \ln \mathcal{L}_P(\theta \mid P^*_t), \frac{\partial}{\partial \sigma} \ln \mathcal{L}_P(\theta \mid P^*_t) \right)'
\]

for the corresponding score vector. Let \( \overline{S}_n(\theta_0) = \frac{1}{n} \sum_{t=1}^n S_t(\theta_0) \) be the mean of the observed score vectors under the null.

Standard likelihood theory implies that \( \sqrt{n} \overline{S}_n(\theta_0) \) converges in distribution as \( n \to \infty \) to \( \mathcal{N}_2(0, I(\theta_0)) \), where \( I(\theta_0) \) denotes the covariance matrix of \( S_t(\theta) \), i.e., the Fisher information matrix. For large \( n \) we have approximately that

\[
n \overline{S}_n(\theta_0)' I(\theta_0)^{-1} \overline{S}_n(\theta_0) \sim \chi^2_2.
\]

An analytical expression for \( I(\theta_0) \) is provided later in this appendix.

Under a restriction on the kernel window we can show that the score test (S.2) is a bispectral Z-test with kernel measures \( \nu_1 \) and \( \nu_2 \) given by sums of discrete and continuous parts.

**Theorem S.1.** Let \( z_0 \) be the unique solution to the equation

\[
z^2 + (\phi(z)/\Phi(z)) z - 1 = 0.
\]

Provided \( \Phi(z_0) \leq \alpha_1 < \alpha_2 \leq 1 \), then \( S_t(\theta_0) = W_t - \mu_W \), almost surely, where

\[
W_{t,i} = \gamma_{i,1} \mathbb{1}_{\{P_t \geq \alpha_1\}} + \gamma_{i,2} \mathbb{1}_{\{P_t \geq \alpha_2\}} + \int_{\alpha_1}^{\alpha_2} g_i(u) \mathbb{1}_{\{P_t \geq u\}} du
\]

for \( \gamma_{i,1} \geq 0, \gamma_{i,2} \geq 0 \) and \( g_i(u) \) positive and differentiable on \( [\alpha_1, \alpha_2] \).

**Proof.**

\[
L_P(\theta \mid P^*) = \prod_{t : P_t^* = \alpha_1} F_P(\alpha_1 \mid \theta) \prod_{t : \alpha_1 < P_t^* < \alpha_2} f_P(\alpha_1 \mid \theta) \prod_{t : P_t^* = \alpha_2} \bar{F}_P(\alpha_2 \mid \theta)
\]

where \( \bar{F}(u) \) denotes the tail probability \( 1 - F(u) \). The likelihood contributions \( \mathcal{L}_P(\theta \mid P^*_t) \) are given by the individual terms in (S.5) according to whether \( P_t^* = \alpha_1, \alpha_1 < P_t^* < \alpha_2 \) or
\[ P_t^* = \alpha_2. \] Computing the score statistic and evaluating it at \( \theta_0 = (0, 1)' \) yields

\[
S_t(\theta_0) = \begin{cases} 
\psi_1(\alpha_1) & P_t^* = \alpha_1, \\
\psi_s(P_t^*) & \alpha_1 < P_t^* < \alpha_2, \\
\psi_2(\alpha_2) & P_t^* = \alpha_2.
\end{cases}
\]

where \( \psi_1(u) = \left( \frac{-\phi(\Phi^{-1}(u))/u}{-\phi(\Phi^{-1}(u))\Phi^{-1}(u)/u} \right) \),

\[
\psi_s(u) = \left( \frac{\Phi^{-1}(u)}{\Phi^{-1}(u)^2 - 1} \right) \quad \text{and} \quad \psi_2(u) = \left( \frac{\phi(\Phi^{-1}(u))/(1-u)}{\phi(\Phi^{-1}(u))\Phi^{-1}(u)/(1-u)} \right).
\]

The discontinuities at \( \alpha_1 \) and \( \alpha_2 \) are given by

\[
(\gamma_{1,1}, \gamma_{2,1})' = \psi_s(\alpha_1) - \psi_1(\alpha_1), \quad (\gamma_{1,2}, \gamma_{2,2})' = \psi_2(\alpha_2) - \psi_s(\alpha_2)
\]

and non-negativity of the \( \gamma_{i,j} \) in all cases is guaranteed provided \( \psi_s(\alpha_1) - \psi_1(\alpha_1) \geq 0 \). The second component of this vector inequality leads to condition (S.3). The weighting functions can be obtained by differentiating \( \psi_s(u) \) with respect to \( u \) on \([\alpha_1, \alpha_2]\) and are thus

\[
g_1(u) = \frac{1}{\phi(\Phi^{-1}(u))} \mathbb{I}_{(\alpha_1 \leq u \leq \alpha_2)}, \quad g_2(u) = \frac{2\Phi^{-1}(u)}{\phi(\Phi^{-1}(u))} \mathbb{I}_{(\alpha_1 \leq u \leq \alpha_2)}.
\]

Finally, since \( \mu_W = W_t - S_t(\theta_0) \), we must have that \( \mu_W = -\psi_1(\alpha_1) \).

We find \( \Phi(z_0) \approx 0.8 \), so the constraint on \( \alpha_1 \) is unlikely to bind in application to the range of tail probability levels of practical interest. For \( \alpha_2 < 1 \) the \( W_{t,i} \) variables are bounded, guaranteeing that the elements of \( I(\theta_0) \) are finite. For \( \alpha_2 = 1 \) the \( G_\nu \) functions for \( \nu_1 \) and \( \nu_2 \) grow like \( \Phi^{-1}(u) \) and \( \Phi^{-1}(u)^2 \) respectively. We can use the asymptotic approximation \( \Phi^{-1}(u) \sim \sqrt{-2 \ln(1-u)} \) as \( u \to 1 \) to verify that the condition of Proposition 3.2 is satisfied in both cases.

**Fisher information matrix**

The following identities are useful for dealing with the probitnormal distribution:

\[
\int_{\alpha_1}^{\alpha_2} \Phi^{-1}(u) du = \phi(\Phi^{-1}(\alpha_1)) - \phi(\Phi^{-1}(\alpha_2)) \tag{S.6}
\]

\[
\int_{\alpha_1}^{\alpha_2} (\Phi^{-1}(u)^2 - 1) du = \Phi^{-1}(\alpha_1)\phi(\Phi^{-1}(\alpha_1)) - \Phi^{-1}(\alpha_2)\phi(\Phi^{-1}(\alpha_2)). \tag{S.7}
\]
Let \( \xi(p \mid \theta) = (\Phi^{-1}(p) - \mu)/\sigma \). The first derivatives of the log-likelihood of the truncated probitnormal distribution are

\[
\frac{\partial}{\partial \mu} \ln L(\theta \mid P_t^*) = \begin{cases} 
- \frac{\phi(\xi(\alpha_1|\theta))}{\sigma\Phi(\xi(\alpha_1|\theta))} & P_t^* = \alpha_1, \\
- \frac{\xi(P_t^*|\theta)}{\sigma} & \alpha_1 < P_t^* < \alpha_2, \\
\frac{\phi(\xi(\alpha_2|\theta))}{\sigma\Phi(\xi(\alpha_2|\theta))} & P_t^* = \alpha_2,
\end{cases}
\]  
(S.8)

and

\[
\frac{\partial}{\partial \sigma} \ln L(\theta \mid P_t^*) = \begin{cases} 
- \frac{\phi(\xi(\alpha_1|\theta))\xi(\alpha_1|\theta)}{\sigma\Phi(\xi(\alpha_1|\theta))} & P_t^* = \alpha_1, \\
- \frac{\xi(P_t^*|\theta)^2 + 1}{\sigma} & \alpha_1 < P_t^* < \alpha_2, \\
\frac{\phi(\xi(\alpha_2|\theta))\xi(\alpha_2|\theta)}{\sigma\Phi(\xi(\alpha_2|\theta))} & P_t^* = \alpha_2.
\end{cases}
\]  
(S.9)

Recall that the expected Fisher information matrix is defined as

\[
I(\theta)_{ij} = -\mathbb{E} \left( \frac{\partial^2}{\partial \theta_i \partial \theta_j} \ln L(\theta \mid P_t^*) \right).
\]

The conditional second derivatives of the log-likelihood are

\[
- \frac{\partial^2}{\partial \mu^2} \ln L(\theta \mid P_t^*) = \begin{cases} 
\frac{\phi(\xi(\alpha_1|\theta))\left(\xi(\alpha_1|\theta)\phi(\xi(\alpha_1|\theta)) + \xi(\alpha_1|\theta)\Phi(\xi(\alpha_1|\theta))\right)}{\sigma^2\Phi(\xi(\alpha_1|\theta))^2} & P_t^* = \alpha_1, \\
\frac{1}{\sigma^2} & \alpha_1 < P_t^* < \alpha_2, \\
\frac{\phi(\xi(\alpha_2|\theta))\left(\phi(\xi(\alpha_2|\theta)) - \xi(\alpha_2|\theta)\Phi(\xi(\alpha_2|\theta))\right)}{\sigma^2\Phi(\xi(\alpha_2|\theta))^2} & P_t^* = \alpha_2,
\end{cases}
\]  
(S.10)

\[
- \frac{\partial^2}{\partial \sigma^2} \ln L(\theta \mid P_t^*) = \begin{cases} 
\frac{\phi(\xi(\alpha_1|\theta))\left(\xi(\alpha_1|\theta)^2\phi(\xi(\alpha_1|\theta)) + \xi(\alpha_1|\theta)^2\Phi(\xi(\alpha_1|\theta)) - 2\xi(\alpha_1|\theta)\Phi(\xi(\alpha_1|\theta))\right)}{\sigma^2\Phi(\xi(\alpha_1|\theta))^2} & P_t^* = \alpha_1, \\
\frac{3\xi(P_t^*|\theta)^2 - 1}{\sigma^4} & \alpha_1 < P_t^* < \alpha_2, \\
\frac{\phi(\xi(\alpha_2|\theta))\left(\xi(\alpha_2|\theta)^2\phi(\xi(\alpha_2|\theta)) - \xi(\alpha_2|\theta)^2\Phi(\xi(\alpha_2|\theta)) + 2\xi(\alpha_2|\theta)\Phi(\xi(\alpha_2|\theta))\right)}{\sigma^2\Phi(\xi(\alpha_2|\theta))^2} & P_t^* = \alpha_2,
\end{cases}
\]  
(S.11)
- \frac{\partial^2}{\partial \mu \partial \sigma} \ln \mathcal{L}(\theta \mid P^*) = \begin{cases} \\
\frac{\phi(\xi(\alpha_1|\theta)) (\phi(\xi(\alpha_1|\theta)) \xi(\alpha_1|\theta) - \Phi(\xi(\alpha_1|\theta)) + \xi(\alpha_1|\theta)^2 \Phi(\xi(\alpha_1|\theta)))}{\sigma^2 \Phi(\xi(\alpha_1|\theta))^2} \\
\frac{2\xi(P^*_t|\theta)}{\sigma^2} \\
\frac{\phi(\xi(\alpha_2|\theta)) (\phi(\xi(\alpha_2|\theta)) \xi(\alpha_2|\theta) - \Phi(\xi(\alpha_2|\theta)) - \xi(\alpha_2|\theta)^2 \Phi(\xi(\alpha_2|\theta)))}{\sigma^2 \Phi(\xi(\alpha_2|\theta))^2} \\
\end{cases}
\begin{align*}
P_t^* &= \alpha_1, \\
\alpha_1 &< P_t^* < \alpha_2, \\
P_t^* &= \alpha_2.
\end{align*}
(S.12)

By taking expectations using (S.6) and (S.7) and evaluating at \( \theta_0 = (0, 1)' \) we obtain the elements of \( I(\theta_0) \):

\begin{align*}
I(\theta_0)_{1,1} &= \phi(\Phi^{-1}(\alpha_1))^2/\alpha_1 + \phi(\Phi^{-1}(\alpha_2))^2/(1 - \alpha_2) \\
&\quad + \phi(\Phi^{-1}(\alpha_1))\Phi^{-1}(\alpha_1) - \phi(\Phi^{-1}(\alpha_2))\Phi^{-1}(\alpha_2) + (\alpha_2 - \alpha_1), \quad (S.13)
\end{align*}

\begin{align*}
I(\theta_0)_{2,2} &= \phi(\Phi^{-1}(\alpha_1))^2\Phi^{-1}(\alpha_1)^2/\alpha_1 + \phi(\Phi^{-1}(\alpha_1))\Phi^{-1}(\alpha_1)^3 \\
&\quad + \phi(\Phi^{-1}(\alpha_1))\Phi^{-1}(\alpha_1) + \phi(\Phi^{-1}(\alpha_2))^2\Phi^{-1}(\alpha_2)^2/(1 - \alpha_2) \\
&\quad - \phi(\Phi^{-1}(\alpha_2))\Phi^{-1}(\alpha_2)^3 - \phi(\Phi^{-1}(\alpha_2))\Phi^{-1}(\alpha_2) + 2(\alpha_2 - \alpha_1), \quad (S.14)
\end{align*}

\begin{align*}
I(\theta_0)_{1,2} &= \phi(\Phi^{-1}(\alpha_1))^2\Phi^{-1}(\alpha_1)/\alpha_1 + \phi(\Phi^{-1}(\alpha_1))(1 + \Phi^{-1}(\alpha_1)^2) \\
&\quad + \phi(\Phi^{-1}(\alpha_2))^2\Phi^{-1}(\alpha_2)/\alpha_2 - \phi(\Phi^{-1}(\alpha_2))(1 + \Phi^{-1}(\alpha_2)^2). \quad (S.15)
\end{align*}

### Supplement B: Moments for the beta kernel

We provide a general solution to the moments and cross-moments of the transformed PIT values when the kernel densities take the form

\[ g_\nu(u) = \frac{(u - \alpha_1)^{a-1}(\alpha_2 - u)^{b-1}}{(\alpha_2 - \alpha_1)^{a+b-1}B(a, b)} \]

for parameters \( a > 0, b > 0 \) and \( \alpha_1 \leq u \leq \alpha_2 \). The normalization guarantees that \( G_\nu(\alpha_2) = 1 \), and helps align the solution with standard beta distribution functions provided by statistical packages. In \( \mathbb{R} \) notation, the kernel function is simply

\[ G_\nu(u) = \text{pbeta} \left( \frac{\max\{\alpha_1, \min\{u, \alpha_2\}\} - \alpha_1}{\alpha_2 - \alpha_1}, a, b \right). \]

Solving for moments and cross-moments of kernels \((g_1(P), g_2(P))\) for uniform \( P \) involves
the following integral:

\[ M(a_1, b_1, a_2, b_2) = \int_{a_1}^{a_2} (1 - u)g_1(u)G_2(u)du = \frac{B(a_1 + a_2, 1 + b_1)}{a_2B(a_1, b_1)B(a_2, b_2)} {}_3F_2(a_2, a_1 + a_2, 1 - b_2; 1 + a_2, 1 + a_1 + a_2 + b_1; 1) \]

where \( {}_3F_2(c_1, c_2, c_3; d_1, d_2; 1) \) denotes a hypergeometric function of order \((3, 2)\) and argument unity. The final line follows from the Thomae transformation T7 in Milgram (2010, Appendix A). Due to the normalization of the kernels, \( M \) does not depend on the choice of kernel window.

When its parameters are all positive, as in the final form in (S.16), numerical solution to \( {}_3F_2(c_1, c_2, c_3; d_1, d_2; 1) \) is straightforward via the standard hypergeometric series expansion. In practice, we are most often interested in integer-valued cases for which \( M \) has a simple closed-form solution.

For given kernel window and PIT value, let \( W_{a,b} \) be the transformed PIT value under a beta kernel with parameters \((a, b)\). A recurrence rule for the incomplete beta function (Abramowitz and Stegun, 1965, eq. 6.6.7) leads to a linear relationship among “neighboring” transformations:

\[ (a + b)W_{a,b} = aW_{a+1,b} + bW_{a,b+1} \]  

(S.17)

An immediate implication is that the uniform, linear increasing and linear decreasing transformations (parameter sets \((1,1)\), \((2,1)\) and \((1,2)\), respectively) are linearly dependent. Any pair of these kernels would yield an equivalent bispectral test, and a trispectral test using all three kernels would be undefined due to a singular covariance matrix \( \Sigma_W \). By iterating the recurrence relationship, we can derive linear relationships among sets of kernels with integer-valued parameter differences \( a_i - a_\ell \) and \( b_i - b_\ell \), which would lead to redundancies among the corresponding \( j \)-spectral tests.

**Supplement C: Monte Carlo simulations**

To compare unconditional tests we generate pseudo PIT values by first sampling from standard normal and scaled Student \( t_5 \) and \( t_3 \) distributions, which represent the true model. We then transform the values to the interval \((0,1)\) using the standard normal cdf which is taken to be the risk manager’s model. The Student distributions are scaled to have variance one so differences stem from different tail shapes rather than different variances. The PIT samples
arising from normal are uniformly distributed and are used to evaluate the size of the tests. The PIT samples arising from the Student $t$ distributions show the kind of departures from uniformity that are observed when tails are poorly estimated.

The majority of the kernel and test abbreviations are explained in Section 4.3 of the main paper. Additional mnemonics used in this Supplement are

**5-level multinomial tests**: we apply the Pearson test (PE5) and the Z-test with discrete uniform kernel (ZU5) in addition to the 3-level tests used in the main paper.

**Discrete LR-tests**: the binomial LR-test (LR1) and the 3-level multinomial LR-test (LR3).

**Continuous LR-test**: the LR-test described in Section 4.2 of the main paper which may be viewed as an extension of the test of Berkowitz (LRB).

The baseline sample size is $n = 750$, which corresponds approximately to the three-year samples of bank data in the main paper. In all simulation experiments number of replications is $2^{16} = 65,536$.

| window  | $F$ | kernel | BIN | ZU3 | ZU5 | PE3 | PE5 |
|---------|-----|--------|-----|-----|-----|-----|-----|
| narrow  | Normal       | 6.1   | 4.9 | 4.6 | 5.3 | 5.9 |
|         | Scaled t5    | 33.9  | 35.0| 34.0| 40.3| 33.5|
|         | Scaled t3    | 24.0  | 24.8| 24.1| 43.4| 33.5|
| wide    | Normal       | 6.1   | 5.0 | 4.8 | 5.1 | 5.6 |
|         | Scaled t5    | 33.9  | 10.7| 11.5| 55.5| 46.3|
|         | Scaled t3    | 24.0  | 13.5| 11.0| 90.6| 81.1|

Table S.1: Estimated size and power of unconditional discrete Z-tests.

We report the percentage of rejections of the null hypothesis at the 5% confidence level based on 65,536 replications. The number of days in each backtest sample is $n = 750$. The narrow window is [0.985, 0.995] and the wide window is [0.95, 0.995].

Table S.1 provides a comparison of the 5-level discrete Z-tests (ZU5 and PE5) with their 3-level counterparts (ZU3 and PE3) and the binomial score test (BIN). For the discrete uniform kernel, the 5-level test is similar in size and power to the 3-level test. For the Pearson kernel, however, the 5-level test is notably less powerful than the 3-level test, and is slightly oversized. On this evidence, there appears to be no advantage in choosing 5-level tests over 3-level tests.

Table S.2 demonstrates that the conclusions drawn from Table 2 in the main paper are robust to the choice of backtest sample size $n$. As one would expect, power typically increases with $n$. For both scaled $t$ distributions and both kernel windows, we find that the ordering
| window | F | n | kernel | BIN | ZU3 | PE3 | ZU | ZA | ZE | ZL_+ | ZL_- | ZLL | PNS |
|--------|---|---|--------|-----|-----|-----|----|----|----|------|------|-----|-----|
| narrow | Normal | 250 | 4.1 | 4.2 | 5.0 | 3.9 | 3.9 | 4.1 | 3.7 | 5.3 | 5.1 |
|        |      | 500 | 3.9 | 4.6 | 5.4 | 4.6 | 4.5 | 4.6 | 4.6 | 4.7 | 4.7 |
|        |      | 750 | 6.1 | 4.9 | 5.3 | 4.7 | 4.7 | 4.6 | 4.8 | 4.8 | 4.9 |
|        | Scaled t5 | 250 | 17.4 | 19.6 | 18.0 | 18.5 | 18.9 | 18.0 | 22.0 | 14.6 | 20.9 | 22.5 |
|        |      | 500 | 22.1 | 27.1 | 30.9 | 26.5 | 26.9 | 25.7 | 31.5 | 21.6 | 30.2 | 33.6 |
|        |      | 750 | 33.9 | 35.0 | 40.3 | 33.8 | 34.4 | 33.0 | 40.3 | 27.1 | 40.0 | 44.7 |
|        | Scaled t3 | 250 | 13.4 | 15.3 | 17.5 | 14.3 | 14.7 | 13.8 | 19.2 | 9.7 | 20.8 | 22.9 |
|        |      | 500 | 15.9 | 20.2 | 31.8 | 19.6 | 20.1 | 18.7 | 26.4 | 14.0 | 31.0 | 36.7 |
|        |      | 750 | 24.0 | 24.8 | 43.4 | 23.9 | 24.3 | 23.3 | 32.7 | 16.5 | 43.3 | 50.5 |
| wide   | Normal | 250 | 4.1 | 4.4 | 5.2 | 4.8 | 4.8 | 4.7 | 4.8 | 4.8 | 5.1 |
|        |      | 500 | 3.9 | 4.7 | 5.1 | 4.9 | 4.9 | 4.7 | 4.9 | 4.8 | 5.0 |
|        |      | 750 | 6.1 | 5.0 | 5.1 | 4.9 | 4.9 | 4.9 | 4.9 | 5.0 | 5.0 |
|        | Scaled t5 | 250 | 17.4 | 8.1 | 23.0 | 5.9 | 6.3 | 5.7 | 8.9 | 4.9 | 17.2 | 24.4 |
|        |      | 500 | 22.1 | 9.7 | 40.3 | 6.3 | 6.5 | 6.0 | 10.6 | 5.4 | 31.3 | 41.6 |
|        |      | 750 | 33.9 | 10.7 | 55.5 | 6.4 | 6.6 | 6.1 | 11.9 | 5.8 | 45.1 | 57.5 |
|        | Scaled t3 | 250 | 13.4 | 9.1 | 36.1 | 7.7 | 9.1 | 6.8 | 6.3 | 10.9 | 30.2 | 42.7 |
|        |      | 500 | 15.9 | 11.3 | 70.9 | 12.8 | 14.8 | 11.1 | 6.8 | 21.5 | 64.9 | 77.4 |
|        |      | 750 | 24.0 | 13.5 | 90.6 | 17.7 | 20.4 | 15.4 | 7.4 | 31.9 | 85.8 | 93.1 |

Table S.2: Effect of backtest sample size on size and power of unconditional Z-tests. We report the percentage of rejections of the null hypothesis at the 5% confidence level based on 65,536 replications. The narrow window is [0.985, 0.995] and the wide window is [0.95, 0.995].
across kernels in power is little changed. Most importantly, the five properties summarized in Section 4.3 hold regardless of $n$.

Table S.3 compares LR-tests with Z-tests. The binomial LR-test (LR1) is less powerful than the binomial score test; the former is slightly oversized while the latter is a touch undersized. The 3-level LR-test (LR3) is compared against a 3-level monospectral test (ZU3) and a trispectral Pearson test (PE3). The LR-test is similar to the Pearson test in power (and mostly more powerful than the ZU3 test), but the LR-test is oversized. The generalized Berkowitz test (LRB) performs slightly less well than the truncated probitnormal score test (PNS) for the narrow window and very slightly better for the wider window. Results of a similar exercise with smaller backtest samples ($n = 250$) are qualitatively similar (untabulated). We conclude that the Z-tests are preferred to their LR-test counterparts, particularly when size is a paramount concern.

The next two experiments relate to conditional Z-tests; the CVT choices are defined in Table 3 in the main paper. Note that when CVT takes the value None, the test is an unconditional test. Table S.4 estimates the size of the tests when samples are uniformly distributed and serially independent. The same messages emerge as in the excerpt in Section 5.2: when the CVT is DQ or V.BIN the tests are quite badly oversized, particularly for the former; choosing V.4 or V.$\frac{1}{2}$ as CVT substantially mitigates (but does not eliminate) oversizing.

Table S.5 is an examination of power for the same kernels and CVT functions. The aim of the underlying simulation is to produce pseudo PIT values that are (i) serially dependent with a dependence structure that is typical when stochastic volatility in the data is ignored and (ii) possibly non-uniform with the same distributions used for the unconditional tests.

The data generating process is designed to mimic the behavior of volatile financial return series, such as daily log-returns on a stock index. Suppose we have empirical return data $X_1, \ldots, X_n$ and we form a version of the empirical cdf bounded away from 0 and 1 by taking

| window | $F$ | test BIN | LR1 | ZU3 | PE3 | LR3 | PNS | LRB |
|--------|-----|---------|-----|-----|-----|-----|-----|-----|
| narrow | Normal | 6.1 | 4.1 | 4.9 | 5.3 | 8.2 | 4.9 | 5.5 |
|        | Scaled t5 | 33.9 | 24.0 | 35.0 | 40.3 | 34.3 | 44.7 | 37.6 |
|        | Scaled t3 | 24.0 | 16.1 | 24.8 | 43.4 | 46.5 | 50.5 | 49.2 |
| wide   | Normal | 6.1 | 4.1 | 5.0 | 5.1 | 6.1 | 5.0 | 5.1 |
|        | Scaled t5 | 33.9 | 24.0 | 10.7 | 55.5 | 52.2 | 57.5 | 57.7 |
|        | Scaled t3 | 24.0 | 16.1 | 13.5 | 90.6 | 92.7 | 93.1 | 95.0 |

Table S.3: Comparison of LR-tests and Z-tests in size and power. We report the percentage of rejections of the null hypothesis at the 5% confidence level based on 65,536 replications. The number of days in each backtest sample is $n = 750$. The narrow window is $[0.985, 0.995]$ and the wide window is $[0.95, 0.995]$. 


**Table S.4**: Estimated size of tests of conditional coverage.

We report the percentage of rejections of the null hypothesis at the 5% confidence level based on 65,536 replications. The number of days in each backtest sample is \( n = 750 \). ARMA parameters are \( AR = 0.95 \), \( MA = -0.85 \). The narrow window is \( [0.985, 0.995] \) and the wide window is \( [0.95, 0.995] \).

\[
\hat{F}_n(x) = (n + 1)^{-1} \sum_{t=1}^{n} 1_{\{X_t \leq x\}} \] and then construct data \( \hat{U}_t = \hat{F}_n(X_t) \). The transformed returns \( (\hat{U}_t) \) are close to uniformly distributed and show negligible serial correlation. However, under the v-shaped transformation \( V(u) = |2u - 1| \), we obtain data \( (V(\hat{U}_t)) \) which remain approximately uniformly distributed but show strong serial correlation. Let \( T \) denote the transformation \( T(u) = \Phi^{-1}(V(u)) \). If we fit Gaussian ARMA models to the transformed data \( (T(\hat{U}_t)) \) we find that an ARMA(1,1) process often fits well and typical values for the AR and MA parameters are around 0.95 and -0.85.

We want to generate losses \( (L_t) \) with marginal distribution \( F \) in Table S.5 such that, if \( U_t = F(L_t) \), the process \( (T(U_t)) \) is exactly a Gaussian ARMA(1,1) process with AR parameter 0.95, MA parameter -0.85, mean zero and variance one. Let \( (Z_t) \) be such a Gaussian ARMA process and let \( (D_t) \) be a series of iid Bernoulli variables with mean 0.5. We apply the following series of transformations to construct \( (L_t) \):

\[
\tilde{U}_t = \Phi(Z_t), \quad U_t = \frac{1}{2}(1 + \tilde{U}_t)^{D_t}(1 - \tilde{U}_t)^{(1-D_t)}, \quad L_t = F^{-1}(U_t). \tag{S.18}
\]

The first transformation induces uniformity; the second can be thought of as a method of stochastically inverting \( V(u) = |2u - 1| \) while preserving uniformity; the third transformation gives losses with df \( F \). As for the unconditional tests, pseudo PIT values are obtained by the transformation \( P_t = \Phi(L_t) \).

In Table S.5 we observe that there is generally a very large increase in power when we move from the unconditional tests (CVT = None) to the conditional tests. This is evident even when the distribution of the simulated data is uniform (\( F = \text{Normal} \)). The most
| window | $F$ | CVT | kernel | BIN | ZU | $ZL_+$ | $ZL_-$ | ZLL | PNS |
|--------|-----|-----|--------|-----|----|--------|--------|-----|-----|
| narrow | Normal | None | 12.1 | 10.8 | 10.0 | 11.2 | 9.1 | 9.5 |
|        | DQ | 30.0 | 31.5 | 32.0 | 29.4 | 29.1 | 28.1 |
|        | V.BIN | 28.1 | 30.9 | 31.8 | 30.2 | 29.5 | 30.5 |
|        | V.4 | 30.3 | 32.6 | 30.7 | 33.6 | 32.3 | 33.4 |
|        | V.$\frac{1}{2}$ | 19.3 | 21.7 | 19.9 | 22.5 | 22.0 | 23.2 |
| Scaled t5 | None | 36.0 | 36.2 | 40.9 | 31.4 | 41.2 | 45.1 |
|        | DQ | 47.4 | 54.9 | 59.7 | 46.1 | 53.4 | 53.4 |
|        | V.BIN | 48.4 | 52.7 | 56.8 | 49.5 | 54.7 | 56.4 |
|        | V.4 | 56.4 | 60.7 | 63.1 | 57.2 | 61.1 | 62.0 |
|        | V.$\frac{1}{2}$ | 49.6 | 54.5 | 57.3 | 50.5 | 55.6 | 56.8 |
| Scaled t3 | None | 28.1 | 28.3 | 35.0 | 22.2 | 44.1 | 50.7 |
|        | DQ | 42.2 | 50.4 | 56.1 | 39.7 | 53.3 | 54.3 |
|        | V.BIN | 42.5 | 47.3 | 53.5 | 42.6 | 53.3 | 55.6 |
|        | V.4 | 50.1 | 54.8 | 58.9 | 49.7 | 58.8 | 60.4 |
|        | V.$\frac{1}{2}$ | 44.1 | 49.5 | 54.1 | 43.7 | 54.2 | 56.2 |
| wide  | Normal | None | 12.1 | 17.8 | 15.9 | 18.3 | 14.6 | 15.4 |
|        | DQ | 30.0 | 31.1 | 30.1 | 31.6 | 30.4 | 30.3 |
|        | V.BIN | 28.1 | 36.0 | 34.5 | 36.2 | 34.6 | 34.9 |
|        | V.4 | 30.3 | 52.1 | 46.4 | 54.1 | 51.2 | 53.8 |
|        | V.$\frac{1}{2}$ | 19.3 | 44.9 | 36.9 | 48.0 | 44.7 | 48.6 |
| Scaled t5 | None | 36.0 | 19.5 | 22.3 | 19.2 | 52.9 | 63.9 |
|        | DQ | 47.4 | 35.4 | 38.9 | 31.5 | 49.8 | 57.9 |
|        | V.BIN | 48.4 | 41.0 | 45.3 | 37.7 | 55.1 | 62.6 |
|        | V.4 | 56.4 | 55.2 | 56.7 | 52.8 | 67.2 | 74.1 |
|        | V.$\frac{1}{2}$ | 49.6 | 51.1 | 51.4 | 49.2 | 65.7 | 73.5 |
| Scaled t3 | None | 28.1 | 28.4 | 18.5 | 39.3 | 87.6 | 93.9 |
|        | DQ | 42.2 | 32.8 | 32.6 | 34.2 | 71.1 | 82.1 |
|        | V.BIN | 42.5 | 38.8 | 38.7 | 40.1 | 75.3 | 85.2 |
|        | V.4 | 50.1 | 50.7 | 48.4 | 52.7 | 82.5 | 90.2 |
|        | V.$\frac{1}{2}$ | 44.1 | 48.1 | 43.4 | 51.2 | 83.6 | 91.1 |

Table S.5: Estimated power of tests of conditional coverage when DGP is based on an ARMA(1,1) model. We report the percentage of rejections of the null hypothesis at the 5% confidence level based on 65,536 replications. The number of days in each backtest sample is $n = 750$. ARMA parameters are $AR = 0.95$, $MA = -0.85$. The narrow window is $[0.985, 0.995]$ and the wide window is $[0.95, 0.995]$. 
powerful tests are bispectral tests applied to the wider window using the CVT functions V.4 and V.½.

References

Abramowitz, M., and I. A. Stegun, eds., 1965, *Handbook of Mathematical Functions* (Dover Publications, New York).

Milgram, Michael S., 2010, On hypergeometric 3F2(1) - a review, Working Paper 1011.4546, arXiv.