Research Article

Bayesian Analysis of Inverted Kumaraswamy Mixture Model with Application to Burning Velocity of Chemicals

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Burning velocity of different chemicals is estimated using a model from mixed population considering inverted Kumaraswamy (IKum) distribution for component parts. Two estimation techniques maximum likelihood estimation (MLE) and Bayesian analysis are applied for estimation purposes. BEs of a mixture model are obtained using gamma, inverse beta prior, and uniform prior distribution with two loss functions. Hyperparameters are determined through the empirical Bayesian method. An extensive simulation study is also a part of the study which is used to foresee the characteristics of the presented model. Application of the IKum mixture model is presented through a real dataset. We observed from the results that Linex loss performed better than squared error loss as it resulted in lower risks. And similarly gamma prior is preferred over other priors.

1. Introduction

Mixture models appear as obvious candidates whenever datasets that consist of two or more heterogeneous populations are mixed together. Due to its modeling versatility, the finite mixture model has attracted a great deal of attention in the history of statistics. To analyze the heterogeneous nature of processes, the mixture models are comparatively more suitable than the simple models. A mixture model with finite components is suitable to use when data are overdispersed, to fit a zero-expansion model, to measure heavy-tailed density, and to test for heterogeneity in cluster analysis. Mixture models have been effectively used in many areas such as industrial engineering (Ali et al. [1]), biology (Bhattacharya [2]), social sciences (Harris [3]), economics (Jedidi et al. [4]), and reliability (Sultan et al. [5]). For more detail about the finite mixture models, see Everitt [6], Ali [7], Feroze and Aslam [8], Zhang and Huang [9], Fundi et al. [10], Tripathi et al. [11], Noor et al. [12], and Feroze and Aslam [13].

Many researchers have provided valuable literature on inverted distribution, for example, Aljuaid [14] studied inverse Weibull, Noor and Aslam [15] analyzed inverse Weibull mixture distribution, Abd EL-Kader et al. [16] analyzed inverted Pareto type I distribution, Basheer [17] proposed generalized alpha power inverse Weibull distribution, and Hassan and Zaky [18] present study on estimation of entropy for inverse Weibull distribution under multiple censored data. Kumaraswamy [19] proposed a distribution which has widespread applications, particularly in situations that are bounded from below and above, such as individual’s height, test scores, atmospheric temperature, and hydrological data. AL-Fattah et al. [20] obtained the IKum distribution from the Kumaraswamy distribution.
using the transformation \( X = (1/T) - 1 \) when random variable \( T \) has Kumaraswamy distribution with \( \alpha \) and \( \beta \) as shape parameters. They discussed important properties of inverted Kumaraswamy distribution and obtained parameters of the proposed model by using MLE and Bayesian technique. The IKum distribution has a long tail to the right; as a result, it can effectively be used for long-term reliability predictions and producing optimistic predictions as compared to other distributions.

Censoring is an important factor of experiments measuring life/failure times. Censored samples are encountered in life test whenever the experimenter has some obligations on the cost or available time for the experiment. Different censoring schemes are used for different experiments, but type I censoring is the most commonly used censoring scheme.

Our aim is to analyze inverted Kumaraswamy distribution in a different way as no other work after AL-Fattah et al. [20] is found on the IKum distribution. We propose a mixture model whose component densities are formed by IKum density and estimate the parameters and reliability function of the mixture model under study using Bayesian as well as frequentist method.

2. Methodology

2.1. Two-Component Mixture Model of IKum Distribution

A random variable \( X \) supposed to have a \( k \) component mixture model is defined as follows:

\[
f(x) = \sum_{i=1}^{k} \delta_i f_i(X|\Theta),
\]

where \( \Theta = (\delta_1, \alpha_1, \beta_1), i = 1, 2, \ldots, k \).

The probability density function (pdf) and reliability function of the mixture model whose component densities are characterized by IKum distribution are given by

\[
f(x; \Theta) = \delta_1 f_1(x; \Theta_1) + (1 - \delta_1) f_2(x; \Theta_2), \quad 0 < \delta_1 < 1,
\]

\[
R(x; \Theta) = \delta_1 R_1(x; \Theta_1) + (1 - \delta_1) R_2(x; \Theta_2), \quad 0 < \delta_1 < 1,
\]

where pdf and reliability function of \( i^{th} \) IKum density, respectively, are

\[
\begin{align*}
f_i(x; \Theta_i) &= \alpha_i \beta_i (1 + x_i)^{-(\alpha_i + 1)} \left[ 1 - (1 + x_i)^{-(\alpha_i + 1)} \right]^{\beta_i - 1}, \\
R_i(x; \Theta_i) &= \left[ 1 - \delta_i \left( 1 - (1 + x_i)^{-(\alpha_i + 1)} \right)^{\beta_i} \right], \\
x > 0, \alpha_i, \beta_i > 0, i = 1, 2.
\end{align*}
\]

2.2. Sampling and Likelihood Function under Type 1 Censoring

Suppose \( m \) items are taken from a population which is a mixture of two-component IKum model with prespecified termination time \( T_0 \). Let the test be conducted, \( s \) items are failed from \( m \) items, and \( (m-s) \) items are still in working position. As per the work of Mendenhall and Hader [21], in many problems, only the futile (useless) items are easily marked as a family of first population and second population. For example, an engineer may divide failed items of electronic as a first population and second population. \( s_1 \) units belong to the first population, \( s_2 \) to the second population, and \( m-s \) items do not give us any information about the population to which they belong to. It is obvious that \( s = s_1 + s_2 \) are the number of uncensored items. Suppose that \( x_{ij} \) denote the failure times of \( j^{th} \) item which are belonging to the \( i^{th} \) subpopulation and that \( x_{ij} \leq t_0, i = 1, 2 \) and \( j = 1, \ldots, s_i \).

The likelihood function for the IKum mixture model using the above-discussed sampling scheme given by Mendenhall and Hader [21] is given by

\[
L(\Theta|x) \propto \prod_{j=1}^{s_1} \delta_1 f_1(x_{ij}; \Theta_1) \prod_{j=1}^{s_2} (1 - \delta_1) f_2(x_{ij}; \Theta_2) \prod_{j=1}^{s_2} \delta_1 R_1(x; \Theta_1) + (1 - \delta_1) R_2(x; \Theta_2) \]^{m-s}

\[
= \prod_{j=1}^{s_1} \delta_1 \alpha_1 \beta_1 \left( 1 + x_{ij} \right)^{-(\alpha_1 + 1)} \left[ 1 - \left( 1 + x_{ij} \right)^{-(\alpha_1 + 1)} \right]^{\beta_1 - 1} \prod_{j=1}^{s_2} \left( 1 - \delta_1 \right) \alpha_2 \beta_2 \left( 1 + x_{ij} \right)^{-(\alpha_2 + 1)}
\]

\[
\left[ 1 - \left( 1 + x_{ij} \right)^{-(\alpha_2 + 1)} \right]^{\beta_2 - 1} \left[ 1 - \delta_1 \left( 1 - \left( 1 + x_{ij} \right)^{-(\alpha_1 + 1)} \right)^{\beta_1} - \delta_2 \left( 1 - \left( 1 + x_{ij} \right)^{-(\alpha_2 + 1)} \right)^{\beta_2} \right]^{m-s}
\]

where \( x = (x_{i1}, x_{i2}, \ldots, x_{i_{s_1}}, x_{s_1+1}, \ldots, x_{s_2}) \) are uncensored observations for failure time.

2.3. Maximum Likelihood Estimation. Taking log of the likelihood function (4) and differentiating w.r.t. parameters
result in five nonlinear equations. A solution of these nonlinear equations gives MLEs for the vector of parameters. We use the SAS package to compute ML estimates of the parameter and their MSEs.

\[
\frac{\partial Q(\Theta|x)}{\partial \delta_1} = \frac{s_1}{\delta_1} - \frac{s_2}{1 - \delta_1} \frac{(m - s)(F_2(t) - F_1(t))}{R(t)},
\]

\[
\frac{\partial Q(\Theta|x)}{\partial \alpha_1} = \frac{s_1}{\alpha_1} - \sum_{j=1}^{s_1} \ln(1 + x_{ij}) - (\beta_1 - 1)w_1(x) - \frac{(m - s)\delta_1 \beta_1 \phi_1(t)F_1(t)}{R(t)(1 - (1 + t)^{-\alpha_1})},
\]

\[
\frac{\partial Q(\Theta|x)}{\partial \alpha_2} = \frac{s_2}{\alpha_2} - \sum_{j=1}^{s_2} \ln(1 + x_{2j}) - (\beta_2 - 1)w_2(x) - \frac{(m - s)(1 - \delta_1)\beta_2 \phi_2(t)F_2(t)}{R(t)(1 - (1 + t)^{-\alpha_2})},
\]

\[
\frac{\partial Q(\Theta|x)}{\partial \beta_1} = \frac{s_1}{\beta_1} - \sum_{j=1}^{s_1} \ln(1 - (1 + x_{ij})^{-\beta_1}) - \frac{(m - s)\delta_1 \beta_1 \phi_1(t)\ln(1 - (1 + t)^{-\alpha_1})}{R(t)},
\]

\[
\frac{\partial Q(\Theta|x)}{\partial \beta_2} = \frac{s_2}{\beta_2} - \sum_{j=1}^{s_2} \sum_{i=1}^{s_1} \ln(1 - (1 + x_{ij})^{-\beta_2}) - \frac{(m - s)(1 - \delta_1)\beta_2 \phi_2(t)\ln(1 - (1 + t)^{-\alpha_2})}{R(t)},
\]

where

\[
F_i(t) = [1 - (1 + t)^{-\alpha_i}]^\beta_i,
\]

\[
w_i(x) = \sum_{j=1}^{s_i} \frac{(1 + x_{ij})^{-\alpha_i} \ln(1 + x_{ij})}{(1 - (1 + x_{ij})^{-\alpha_i})},
\]

\[
\phi_i = (1 + t)^{-\alpha_i} \ln(1 + t),
\]

\[
R(t) = \left[1 -\delta_i [1 - (1 + t_o)^{-\alpha_i}]^\beta_i - \left(1 -\delta_i [1 - (1 + t_o)^{-\alpha_i}]^\beta_i \right) \right], \quad i = 1, 2.
\]

2.4. Bayes Estimation. The Bayesian approach is a powerful statistical tool used to reduce uncertainty in complex problems. Bayesian theory basically relies upon prior distribution and the use of loss functions. Loss function represents the loss incurred when the real parameter is derived from the estimated value. Square error loss function (SELF) used in the study is a symmetric loss function. In many situations, overestimation is more serious than underestimation, or vice versa. Asymmetric loss functions are those loss functions in which negative and positive errors of the same or different dimensions cause different losses. To compensate the situation, an asymmetric loss function is also used.

2.4.1. Posterior Density Assuming Informative (Gamma) Prior. It is assumed that \(\alpha_i\) and \(\beta_i\) each have gamma prior distribution with \((a_i, b_i)\) and \((c_i, d_i)\) hyperparameters, respectively, and \(\delta_i\) assumes a uniform prior so joint prior density for \(\alpha_i, \beta_i, \delta_i\) and \(\delta_i\) is

\[
g(\Theta) \propto \alpha_i^{-\alpha_i-1}e^{-\alpha_i} \beta_i^{-\beta_i-1}e^{-\beta_i} \phi_i^{-\phi_i-1}e^{-\phi_i}\beta_i^{-\beta_i-1}e^{-\beta_i}, \quad i = 1, 2.
\]

Thus, posterior density using the likelihood function and joint prior in proportional form is as follows:
Integration of the posterior density does not produce estimators in compact and simple form; therefore, we use Lindley’s approximation to obtain Bayes estimators, posterior risks, and reliability estimates for the shape parameters of the IKum mixture model.

### 2.4.2. Lindley’s Procedure for Estimation of Parameters

Lindley [22] proposed an approximation known as Lindley’s approximation to obtain Bayes estimators, posterior risks, and reliability estimates for the shape parameters of the IKum mixture model. This approximation, Bayes estimator expands as a function that involves a posterior mode of $\Theta$. Lindley’s approximation has been utilized by many authors for the estimation of the parameters for the simple as well as mixture models; see Jaheen [23], Ahmad et al. [24], Sultan et al. [25], etc.

Consider the following integral

$$
\int U(\Theta) e^{Q(\Theta)} d(\Theta),
$$

where $\Theta = (\alpha_1, \alpha_2, \beta_1, \beta_2, \delta_1)$ is a vector of parameters, $U(\Theta)$ is an arbitrary function of $\Theta$, and $Q(\Theta)$ is the logarithm of a posterior function for $n$ observation. Lindley [22] suggested the following approximate Bayes estimator under SELF:

$$
\widehat{U}_{BL}(\Theta) = E[U(\Theta)|x] = U(\widehat{\Theta}) + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \left\{ U_{ij}(\Theta) \tau_{ij} + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{l=1}^{m} Q_{ijl}(\Theta) U_{ij}(\Theta) \tau_{ij} \tau_{il} \right\},
$$

where $i, j, s = 1, 2, \ldots, m$

$$
U_i(\widehat{\Theta}) = \frac{\partial U(\Theta)}{\partial \theta_i} \bigg|_{\Theta=\widehat{\Theta}},
$$

$$
U_{ij}(\widehat{\Theta}) = \frac{\partial^2 U(\Theta)}{\partial \theta_i \partial \theta_j} \bigg|_{\Theta=\widehat{\Theta}},
$$

$$
\tau_{ij} = (-Q)^{-1}_{m \times m} = \sum_{m \times m},
$$

$$
Q_{ij}(\widehat{\Theta}) = \frac{\partial^3 Q(\Theta)}{\partial \theta_i \partial \theta_j \partial \theta_j} \bigg|_{\Theta=\widehat{\Theta}},
$$

$$
Q_{ijl}(\Theta) = \frac{\partial^3 Q(\Theta)}{\partial \theta_i \partial \theta_j \partial \theta_j} \bigg|_{\Theta=\widehat{\Theta}}.
$$

All the functions on the right-hand side are to be obtained as the posterior mode. $Q_{ijl}$ is given in Appendix B. Parameters of the proposed IKum mixture model using Lindley’s approximation may be obtained as

$$
\widehat{U}(\Theta) = U(\widehat{\Theta}) + \frac{1}{2} \left( D + \sum_{i=1}^{5} B_i A_k \right), \quad i = 1, 2, \ldots, 5,
$$

where
\[ \tilde{\beta}_{2,\text{SELF}} = \tilde{\beta}_2 + \frac{1}{2} \sum_{i=1}^{5} B_i \tau_{i5}, \] (18)

where \( B_i, i = 1, 2, \ldots, 5 \) are given earlier and \( \tau_{ik} \) are the elements of the inverse of the matrix \( Q_{ik} \). Posterior risk under SELF is the variance which can be evaluated as \( \text{var}(\Theta) = \mathbb{E}[U(\Theta)|x]^2 - (\mathbb{E}[U(\Theta)|x])^2 \) by using Mathematica 12.

\[ \tilde{U}_{\text{LLF}}(\Theta) = -\frac{1}{\eta} \ln \left\{ e^{(-\eta \theta_i)} + \frac{1}{2} \left( D_{\text{LLF}} + \sum_{i=1}^{5} B_i A_i \right) \right\}, \] (20)

\[ \tilde{\delta}_{1,\text{LLF}} = -\frac{1}{\eta} \ln \left\{ e^{(-\eta \theta_1)} + \frac{1}{2} \left( U_{11} + \sum_{i=1}^{5} B_i \tau_{i1} \right) \right\}, \] (21)

\[ \tilde{\alpha}_{1,\text{LLF}} = -\frac{1}{\eta} \ln \left\{ e^{(-\eta \theta_5)} + \frac{1}{2} \left( U_{22} + \sum_{i=1}^{5} B_i \tau_{i2} \right) \right\}, \] (22)

\[ \tilde{\alpha}_{2,\text{LLF}} = -\frac{1}{\eta} \ln \left\{ e^{(-\eta \theta_5)} + \frac{1}{2} \left( U_{33} + \sum_{i=1}^{5} B_i \tau_{i3} \right) \right\}, \] (23)

\[ \tilde{\beta}_{1,\text{LLF}} = -\frac{1}{\eta} \ln \left\{ e^{(-\eta \theta_5)} + \frac{1}{2} \left( U_{44} + \sum_{i=1}^{5} B_i \tau_{i4} \right) \right\}, \] (24)

\[ \tilde{\beta}_{2,\text{LLF}} = -\frac{1}{\eta} \ln \left\{ e^{(-\eta \theta_5)} + \frac{1}{2} \left( U_{55} + \sum_{i=1}^{5} B_i \tau_{i5} \right) \right\}, \] (25)

and posterior risks under LLF are

\[ \rho(\tilde{\delta}_{1,\text{LLF}}) = \ln \left\{ e^{(-\eta \theta_1)} + \frac{1}{2} \left( D_{\text{LLF}} + \sum_{i=1}^{5} B_i A_i \right) \right\} + \eta \left( \delta_{1,\text{MLE}} + \frac{1}{2} \sum_{i=1}^{5} B_i \tau_{i1} \right), \] (26)

\[ \rho(\tilde{\alpha}_{1,\text{LLF}}) = \ln \left\{ e^{(-\eta \theta_5)} + \frac{1}{2} \left( D_{\text{LLF}} + \sum_{i=1}^{5} B_i A_i \right) \right\} + \eta \left( \alpha_{1,\text{MLE}} + \frac{1}{2} \sum_{i=1}^{5} B_i \tau_{i2} \right), \] (27)

\[ \rho(\tilde{\alpha}_{2,\text{LLF}}) = \ln \left\{ e^{(-\eta \theta_5)} + \frac{1}{2} \left( D_{\text{LLF}} + \sum_{i=1}^{5} B_i A_i \right) \right\} + \eta \left( \alpha_{2,\text{MLE}} + \frac{1}{2} \sum_{i=1}^{5} B_i \tau_{i3} \right), \] (28)

\[ \rho(\tilde{\beta}_{1,\text{LLF}}) = \ln \left\{ e^{(-\eta \theta_5)} + \frac{1}{2} \left( D_{\text{LLF}} + \sum_{i=1}^{5} B_i A_i \right) \right\} + \eta \left( \beta_{1,\text{MLE}} + \frac{1}{2} \sum_{i=1}^{5} B_i \tau_{i4} \right), \] (29)

\[ \rho(\tilde{\beta}_{2,\text{LLF}}) = \ln \left\{ e^{(-\eta \theta_5)} + \frac{1}{2} \left( D_{\text{LLF}} + \sum_{i=1}^{5} B_i A_i \right) \right\} + \eta \left( \beta_{2,\text{MLE}} + \frac{1}{2} \sum_{i=1}^{5} B_i \tau_{i5} \right). \] (30)
2.4.3. Posterior Density Assuming Informative (Inverse Beta) Prior. It is assumed that the shape parameters $\alpha_i$ and $\beta_i$ have inverse beta prior with hyperparameters $a_i$, $b_i$, and $c_i d_i$, respectively,

$$
g(\alpha_i) \propto \alpha_i^{(a_i-1)} (1 - \alpha_i)^{-(a_i + b_i)} \alpha_i > 0, \quad i = 1, 2, \tag{31}
$$

$$
g(\beta_i) \propto \beta_i^{(c_i-1)} (1 - \beta_i)^{-(c_i + d_i)} \beta_i > 0, \quad i = 1, 2.
$$

Joint prior density for $\alpha_i, \beta_i$, and $\delta_i$ is given by

$$
g(\Theta) \propto \alpha_i^{(a_i-1)} (1 - \alpha_i)^{-(a_i + b_i)} \beta_i^{(c_i-1)} (1 - \beta_i)^{-(c_i + d_i)} \delta_i^{(-c_i d_i)}.
$$

Combining the likelihood function (4) and joint prior (32), the following joint posterior density of the IKum mixture model is obtained:

$$
p(\Theta|x) \propto \sum_{i=0}^{m-1} \sum_{n=0}^{l-1} (m - s) l \left( \delta_1^{\alpha_1} (1 - \delta_1)^{\beta_1} \right)^{(a_i-1)} (1 - \alpha_i)^{-(a_i + b_i)} \beta_i^{(c_i-1)} (1 - \beta_i)^{-(c_i + d_i)} \alpha_i^{(a_i-1)} \beta_i^{(c_i-1)} (1 - \beta_i)^{-(c_i + d_i)} \delta_i^{(-c_i d_i)} \left[ 1 - \left( 1 - \delta_i \right)^{\alpha_1} \right]^{\beta_1} \left[ 1 - \left( 1 - \delta_i \right)^{\beta_1} \right]^{\delta_1} \left( 1 - \delta_i \right)^{n \ln(1 - (1 - x_{1j})^{-\alpha_1}) + n \ln(1 - (1 - t_{ij})^{-\delta_1})} \delta_i^{(-c_i d_i)}.
$$

Bayesian tool and are considered for Bayesian analysis when there is little or no prior information available. Let $\alpha_i, \beta_i \sim U(0, \infty)$, and $\delta_i \sim U(0, 1)$, $i = 1, 2$. So the joint prior is

$$
g(\Theta) \propto 1. \tag{34}
$$

Assuming independence combining the prior with likelihood function (4). The joint posterior density of $\alpha_i, \beta_i$, and $\delta_i$ is obtained as

$$
p(\Theta|x) \propto \prod_{i=1}^{s} \delta_i^{\alpha_i} \beta_i^{(c_i-1)} (1 + x_{1j})^{-(a_i + 1)} (1 - x_{1j})^{-a_i} \beta_i^{(c_i-1)} \left[ 1 - \left( 1 - \delta_i \right)^{\alpha_1} \right] \left[ 1 - \left( 1 - \delta_i \right)^{\beta_1} \right] \left[ 1 - \left( 1 - \delta_i \right)^{\alpha_1} \right] \left[ 1 - \left( 1 - \delta_i \right)^{\beta_1} \right] \delta_i^{(-c_i d_i)} \left( 1 - \delta_i \right)^{n \ln(1 - (1 - x_{1j})^{-\alpha_1}) + n \ln(1 - (1 - t_{ij})^{-\delta_1})} \delta_i^{(-c_i d_i)}.
$$

2.5. Posterior Density Assuming Noninformative (Uniform) Prior. Noninformative priors are an important part of the

$$
p(\Theta|x) \propto \prod_{i=1}^{s} \delta_i^{\alpha_i} \beta_i^{(c_i-1)} \left[ 1 - \left( 1 - \delta_i \right)^{\alpha_1} \right] \left[ 1 - \left( 1 - \delta_i \right)^{\beta_1} \right] \left[ 1 - \left( 1 - \delta_i \right)^{\alpha_1} \right] \left[ 1 - \left( 1 - \delta_i \right)^{\beta_1} \right] \delta_i^{(-c_i d_i)} \left( 1 - \delta_i \right)^{n \ln(1 - (1 - x_{1j})^{-\alpha_1}) + n \ln(1 - (1 - t_{ij})^{-\delta_1})} \delta_i^{(-c_i d_i)} \left( 1 - \delta_i \right)^{n \ln(1 - (1 - x_{1j})^{-\alpha_1}) + n \ln(1 - (1 - t_{ij})^{-\delta_1})} \delta_i^{(-c_i d_i)}.
$$

Mathematical expressions of the Bayes estimators and posterior risk can be obtained by assigning hyperparameters a zero value in the expressions of Bayes estimators and posterior risks in Sections 2.4.1 and 2.4.3 under SELF and LLF.

2.6. Reliability Estimation. The objective of assessing the reliability of estimates is to determine how much of the variability in the data is due to errors in measurement. And how much is in the true parameters. Approximate Bayes estimator of reliability function of IKum at some value $t$ can be obtained as

$$
\tilde{R}(t) = R(x; \Theta) + \frac{1}{2} \left( D_R + \sum_{i=1}^{5} B_i A_i \right). \tag{36}
$$
Here,

\[ D_R = \left| l_1 (r_{12} + r_{21}) + l_2 (r_{13} + r_{31}) + k_1 (r_{14} + r_{41}) + k_2 (r_{15} + r_{51}) + \delta_1 m_1 (r_{24} + r_{42}) + (1 - \delta_1) m_2 (r_{35} + r_{53}) + \delta_1 \theta_1 r_{22} + (1 - \delta_1) \theta_2 r_{33} + \delta_1 z_i r_{44} + (1 - \delta_1) z_i r_{55}, \right| \]

\[ A_i = \left| (F_2 (t) - F_1 (t)) r_{11} + \delta_1 l_1 r_{12} + (1 - \delta_1) l_2 r_{13} + \delta_1 k_1 r_{14} + (1 - \delta_1) k_2 r_{15}, \right|, \quad i = 1, 2, 3, 4, 5, \]

where \( B_i \) is defined earlier, \( r_{ik} \) are the elements of the inverses of the matrix \( Q_{ik} \), and \( l_i, k_i, m_i, z_i, \theta_i \) \((i = 1, 2)\) are given in Appendix B.

3. Results and Discussion

3.1. Monte Carlo Simulation. Simulation is performed to get insight into properties/trends of the obtained Bayes estimators. For this purpose, a Monte Carlo simulation is performed for 1000 samples of size 30, 50, and 100 for each selection of the vector of parameters \( \Theta = (\delta_1, \alpha_1, \alpha_2, \beta_1, \beta_2) \) \((0.4, 1, 1.5, 0.8, 0.5), (0.4, 2, 3, 1.3, 1.5)\) using the inverse transformation method as follows:

- Generate a random sample of different selected sample sizes from the proposed mixture model using the inverse transformation method
- If \( u_i < \delta_1 \), then use \( u_i \) to generate random variate \( x \) from the mixture of two-component IKum as \( x = ((1 - u_i)^{1/\beta_1} - \tau_1) - 1 \).
- If \( u_i \geq \delta_1 \), then use \( u_i \) to generate random variate \( x \) from the mixture of two-component IKum as \( x = ((1 - u_i)^{1/\beta_2} - \tau_2) - 1 \).

Select a sample censored at a fixed test termination time \( t \) and only take censored observations.

For the different choice of parameters, hyperparameter for the informative priors (gamma) are selected for \( i = 1, 2 \) to satisfy \( E(\beta_i) = c_i/d_i \) and inverse beta \( E(\beta_i) = c_i/(d_i - 1) \),

\[
E(\beta_1) = \frac{c_1}{d_1} = \frac{5}{2}, \\
E(\beta_2) = \frac{c_2}{d_2} = \frac{2}{2}, \\
E(\beta_3) = \frac{c_3}{d_3} = \frac{10}{2}, \\
E(\beta_4) = \frac{c_4}{d_4} = \frac{6}{2}.
\]

The above steps are repeated 1000 times. The Bayes estimates are computed over 1000 repetitions by averaging the estimate and the squared deviation, respectively. Estimates are computed using two informative (gamma and inverse beta) priors and uniform noninformative prior.

Results presented in Tables 1 and 2 (Appendix A) are obtained through simulation procedure which narrates the properties of the derived Bayes estimators and posterior risks of parameters and reliability estimate of the IKum mixture model. Different sample sizes, i.e., \( n = 30, 50, \) and \( 100 \) are taken to perform a simulation study. It is observed that as we increase the sample size, the estimate of parameters converges to a true parametric value. It is also observed that the use of LLF assuming gamma prior produces less posterior risk, hence can be thought of as a best loss function. An experimenter always tries to choose such a loss function for which he has to bear the minimum loss for estimation. In the same context, gamma prior resulted in smaller posterior risks as compared to other priors.

Bayes estimates are found overestimated for few cases and underestimated for few cases. It is all because the complex mixture model of IKum densities is considered and these Bayes estimates are obtained by using the approximate method. MLEs obtained in the simulation study are found to be somewhat inconsistent, and when compared with Bayes estimates, we found that PRs are lesser than MSEs of MLEs.

Results of simulation are also exhibited graphically which are given in Figure 1. From graphs of simulated Bayes estimates, we can see that we almost get the same results as obtained under simulation study numerically. It means we can obtain the same Bayes estimates through graphs.

3.2. Real Dataset Example. This dataset consists of 56 observations related to the burning velocity of different chemical materials. The burning speed/velocity is the laminar flame speed under the specified composition, temperature, and pressure conditions. It decreases as the inhibitor concentration increases and can be checked by analyzing the pressure distribution in the spherical vessel and by observing the flame propagation directly. Data related to burning velocity (cm/sec) of different chemical materials are given in Table 3 and are available at http://www.cheresources.com/mists.pdf.

The following information is extracted from the above real data to analyze the mixture of the IKum model when \( T = 85 \):
Similarly considering \( T = 65 \), we obtain
Table 2: BEs, PRs, and reliability estimates using UP, IBeta, and gamma prior under LLF and SELF parameters \((\delta_1, \alpha_1, \alpha_2, \beta_1, \beta_2)\) (0.4, 2, 3, 1.3, 1.5) and \(T = 5\).

| \(n\) | Parameters | MLEs | SELF | Bayes estimates |
|-------|------------|------|------|-----------------|
|       | \(\hat{\alpha}_1\) | \(\hat{\alpha}_2\) | \(\hat{\beta}_1\) | \(\hat{\beta}_2\) | \(\hat{\delta}_1\) | \(\hat{R}(t)\) | \(\hat{\alpha}_1\) | \(\hat{\alpha}_2\) | \(\hat{\beta}_1\) | \(\hat{\beta}_2\) | \(\hat{\delta}_1\) | \(\hat{R}(t)\) |
| 30    | 2.320675   | 0.610604 | 1.764759 | 0.60725  | 0.866185 | 0.068320 | 1.36015   | 0.098695 | 0.370755 | 0.007937 | 0.029892 | 0.001955 | 1.63092 |
|      | 2.00356    | 0.00001  | 2.99899  | 0.00087 | 1.23822  | 0.00026  | 1.51891   | 0.00026  | 0.42826  | 0.00084  | 0.037247 | 0.001369 | 2.00311 |
|      | 2.00598    | 0.00000  | 2.99979  | 0.00077 | 1.236987 | 0.00020  | 1.51633   | 0.00001  | 0.428555 | 0.00012  | 0.037346 | 0.001367 | 2.00300 |
|      | 2.00398    | 0.00001  | 2.99999  | 0.00073 | 1.23589  | 0.00019  | 1.51478   | 0.00001  | 0.42877  | 0.00087  | 0.037267 | 0.001364 | 2.00301 |
| 50    | 1.63092    | 0.196138 | 1.986596 | 0.191953 | 1.55010  | 0.124060 | 1.51610   | 0.00026  | 0.40795  | 0.00101  | 0.037346 | 0.001367 | 2.00300 |
|      | 2.00311    | 0.00000  | 2.99599  | 0.00077 | 1.23674  | 0.00020  | 1.51610   | 0.00001  | 0.40755  | 0.00057  | 0.037267 | 0.001364 | 2.00301 |
|      | 2.00315    | 0.00000  | 2.99909  | 0.00073 | 1.23504  | 0.00019  | 1.51401   | 0.00001  | 0.40755  | 0.00057  | 0.037267 | 0.001364 | 2.00301 |
|      | 2.00300    | 0.00000  | 2.99590  | 0.00073 | 1.235103 | 0.00026  | 1.51698   | 0.00001  | 0.40755  | 0.00057  | 0.037267 | 0.001364 | 2.00301 |
| 100   | 1.63092    | 0.196138 | 1.986596 | 0.191953 | 1.55010  | 0.124060 | 1.51610   | 0.00026  | 0.40795  | 0.00101  | 0.037346 | 0.001367 | 2.00300 |
|      | 2.00311    | 0.00000  | 2.99599  | 0.00077 | 1.23674  | 0.00020  | 1.51610   | 0.00001  | 0.40755  | 0.00057  | 0.037267 | 0.001364 | 2.00301 |
|      | 2.00315    | 0.00000  | 2.99909  | 0.00073 | 1.23504  | 0.00019  | 1.51401   | 0.00001  | 0.40755  | 0.00057  | 0.037267 | 0.001364 | 2.00301 |
|      | 2.00300    | 0.00000  | 2.99590  | 0.00073 | 1.235103 | 0.00026  | 1.51698   | 0.00001  | 0.40755  | 0.00057  | 0.037267 | 0.001364 | 2.00301 |

\(\delta_1 = 0.40,\) 
\(m_1 = 22,\) 
\(m_2 = 34,\) 
\(s_1 = 21,\) 
\(s_2 = 22,\) 
\(\sum_{j=1}^{s_1} \ln(1 - (1 + x_{1j})^{-\alpha_1}) = -37.657,\) 
\(\sum_{j=1}^{s_2} \ln(1 - (1 + x_{2j})^{-\alpha_2}) = -34.689.\) 

To analyze the data of the burning velocity of chemical materials, we consider three different censoring times. The data thus acquired are used to get MLEs and BEs of the parameter of the IKum mixture model that are given in Table 4. Bayes estimates are obtained assuming uniform, gamma, and inverse beta priors. However, two loss functions SELF and LLF are used for Bayesian estimation. It is found that MLEs are a bit lower than Bayes estimates. The mixture model comprises five parameters whose all four parameters are shape parameters except mixing weight. Mixing weight which is considered 0.40 in the mixture data is almost ideally estimated. Shape parameter \(D\) is estimated to be about on average 1.5 to 1.8 cm/sec for the first component, and for the second component density, it ranges from almost 0.90 to 1.00 cm/s.

3.3. Simulation Study of Real Data. In this section, using the estimates of the real dataset, we determine the estimates through a simulation study. It is observed that the Bayes estimators obtained through simulation are very close to the true values of estimators. One can easily observe that the suitable prior for these data is gamma prior and the loss function is LLF because they provide less posterior risk. Results are given in Table 5.
Figure 1: Continued.
Figure 1: Graphical representation of the Bayes estimators, posterior risk, and reliability function of the IKum mixture model based on simulation studies.

Table 3: Real dataset of the burning velocity of chemical materials.

|        |        |        |        |        |        |        |        |        |        |        |        |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 68     | 61     | 64     | 55     | 51     | 68     | 44     | 50     | 82     | 60     | 89     | 61     |
| 66     | 50     | 87     | 48     | 42     | 58     | 46     | 67     | 46     | 46     | 44     | 48     |
| 54     | 47     | 89     | 38     | 108    | 46     | 40     | 44     | 312    | 41     | 31     | 40     |
| 56     | 45     | 43     | 46     | 46     | 46     | 46     | 52     | 58     | 82     | 71     | 48     |
|        |        |        |        |        |        |        |        |        |        |        | 41     |

Table 4: BEs and their PRs and reliability estimate using UP and IP (inverse beta and gamma) under LLF and SELF.

| Parameters | MLEs | SELF | Bayes estimates | LLF |
|------------|------|------|-----------------|-----|
|            |      |      | UP              |     |
|            |      |      | IBeta           |     |
|            |      |      | Gamma           |     |
|            |      |      | UP              |     |
|            |      |      | IBeta           |     |
|            |      |      | Gamma           |     |
| $T = 85$, $n = 56$ |      |      |                 |     |
| $\hat{a}_1$ | 1.37051 | 1.83988 | 1.83980 | 1.85330 | 1.78270 | 1.78143 | 1.54997 |
| $\hat{a}_2$ | 0.08924 | 0.11546 | 0.11540 | 0.11538 | 0.01123 | 0.01120 | 0.00301 |
| $\hat{\beta}_1$ | 0.59800 | 0.92089 | 0.92087 | 0.92880 | 0.92446 | 0.92450 | 0.99108 |
| $\hat{\beta}_2$ | 0.01446 | 0.00627 | 0.00626 | 0.00624 | 0.00301 | 0.00309 | 0.00025 |
| $\hat{\delta}_1$ | 0.57709 | 0.85566 | 0.85560 | 0.86400 | 0.91727 | 0.91769 | 0.81050 |
| $\hat{\delta}_2$ | 0.013332 | 0.00301 | 0.00309 | 0.00025 | 0.00021 | 0.00024 | 0.00009 |
| $\hat{\gamma}_1$ | 0.70302 | 0.51546 | 0.51544 | 0.57839 | 0.64110 | 0.64105 | 0.52460 |
| $\hat{\gamma}_2$ | 0.01606 | 0.00025 | 0.00024 | 0.00023 | 0.00055 | 0.00055 | 0.00035 |
| $\hat{R}(t)$ | 0.44828 | 0.42232 | 0.42202 | 0.42709 | 0.42662 | 0.42618 | 0.40190 |
| $\hat{R}(t)$ | 0.00446 | 0.00048 | 0.00048 | 0.00041 | 0.00010 | 0.00011 | 0.00001 |
| $\hat{R}(t)$ | 0.94381 | 0.99511 | 0.99562 | 0.99403 | 0.99675 | 0.99820 | 0.99862 |
| $\hat{R}(t)$ | 0.00196 | 0.00001 | 0.00001 | 0.00006 | 0.00012 | 0.00013 | 0.00012 |
In this paper, we conduct a Bayesian estimation of the unknown parameters and reliability function of the inverted Kumaraswamy mixture model under type 1 right censoring. For the choice of different sample sizes, $n = 30, 50$, and 100 are taken to perform a simulation study. It is observed as the sample size increases the parameters converge to their true parametric value. And it is also noted that LLF is found to be the best loss function assuming gamma prior because it has a less posterior risk.

### Table 4: Continued.

| Parameters | MLEs UP | SELF Ib | Gamma UP | LLF Ib | Gamma |
|------------|---------|---------|----------|--------|--------|
| $T = 70, n = 56$ |         |         |          |        |        |
| $a_1$      | 1.06862 | 1.83988 | 1.80750  | 1.81851 | 1.78271 | 1.78120 | 1.54971 |
| $a_2$      | 0.06010 | 0.11546 | 0.09458  | 0.10148 | 0.01123 | 0.01120 | 0.00305 |
| $\alpha_1$ | 1.30658 | 0.92089 | 0.92659  | 0.93250 | 0.92446 | 0.92449 | 0.99104 |
| $\alpha_2$ | 0.06097 | 0.00627 | 0.00539  | 0.00455 | 0.00034 | 0.00023 | 0.00006 |
| $\beta_1$  | 0.57246 | 0.85566 | 0.84716  | 0.85708 | 0.91727 | 0.91766 | 0.81049 |
| $\beta_2$  | 0.01725 | 0.00309 | 0.00220  | 0.00325 | 0.00260 | 0.00261 | 0.00009 |
| $\delta_1$ | 0.59864 | 0.51546 | 0.51452  | 0.57248 | 0.64110 | 0.64101 | 0.52459 |
| $\delta_2$ | 0.01280 | 0.00024 | 0.00021  | 0.00025 | 0.00055 | 0.00055 | 0.00035 |
| $\hat{R}(t)$ | 0.04025 | 0.42232 | 0.42029  | 0.42469 | 0.42662 | 0.42612 | 0.40197 |
| $\hat{R}(t)$ | 0.00520 | 0.00048 | 0.00041  | 0.00060 | 0.00011 | 0.00011 | 0.00001 |
| $T = 65, n = 56$ |         |         |          |        |        |
| $a_1$      | 1.18110 | 1.6711  | 1.67010  | 1.67012 | 1.52543 | 1.52441 | 1.52420 |
| $a_2$      | 0.06643 | 0.02896 | 0.02899  | 0.02894 | 0.00115 | 0.00116 | 0.00100 |
| $\alpha_1$ | 1.02660 | 0.95929 | 0.95929  | 0.95892 | 0.99544 | 0.99546 | 0.99540 |
| $\alpha_2$ | 0.04790 | 0.00162 | 0.00626  | 0.00160 | 0.00002 | 0.00002 | 0.00002 |
| $\beta_1$  | 0.56466 | 0.82492 | 0.82492  | 0.82482 | 0.80514 | 0.80543 | 0.80501 |
| $\beta_2$  | 0.01518 | 0.00062 | 0.00309  | 0.00062 | 0.00005 | 0.00006 | 0.00005 |
| $\delta_1$ | 0.63268 | 0.50818 | 0.50819  | 0.50812 | 0.51218 | 0.51267 | 0.51215 |
| $\delta_2$ | 0.01819 | 0.00006 | 0.00023  | 0.00006 | 0.00017 | 0.00020 | 0.00010 |
| $\hat{R}(t)$ | 0.48837 | 0.41121 | 0.41189  | 0.41120 | 0.40130 | 0.40124 | 0.40103 |
| $\hat{R}(t)$ | 0.00581 | 0.00012 | 0.00012  | 0.00012 | 0.00002 | 0.00002 | 0.00001 |
| $\hat{R}(t)$ | 0.93409 | 0.99653 | 0.99654  | 0.99489 | 0.99663 | 0.99693 | 0.99889 |

| Parameters | MLEs UP | SELF Ib | Gamma UP | LLF Ib | Gamma |
|------------|---------|---------|----------|--------|--------|
| $\alpha_1$ | 2.32066 | 1.51740 | 1.48950  | 1.85330 | 1.78270 | 1.78120 | 1.54970 |
| $\alpha_2$ | 0.60160 | 0.00834 | 0.01369  | 0.11538 | 0.01123 | 0.01120 | 0.00304 |
| $\alpha_1$ | 1.76476 | 1.01512 | 1.01593  | 0.92883 | 0.92446 | 0.92449 | 0.99104 |
| $\alpha_2$ | 0.60725 | 0.00071 | 0.00075  | 0.00624 | 0.00034 | 0.00023 | 0.00009 |
| $\beta_1$  | 0.86619 | 0.93457 | 0.93516  | 0.86400 | 0.91727 | 0.91766 | 0.81049 |
| $\beta_2$  | 0.09869 | 0.00127 | 0.00212  | 0.00023 | 0.00055 | 0.00055 | 0.00035 |
| $\delta_1$ | 0.37076 | 0.45179 | 0.45830  | 0.42709 | 0.42662 | 0.42612 | 0.40197 |
| $\delta_2$ | 0.00794 | 0.00420 | 0.00932  | 0.00484 | 0.00012 | 0.00011 | 0.00002 |
| $\hat{R}(t)$ | 0.94176 | 0.95196 | 0.98027  | 0.98909 | 0.95199 | 0.98027 | 0.99693 |
| $\hat{R}(t)$ | 0.00196 | 0.00004 | 0.00000  | 0.00000 | 0.00000 | 0.00000 | 0.00000 |

### 4. Conclusion

In this paper, we conduct a Bayesian estimation of the unknown parameters and reliability function of the inverted Kumaraswamy mixture model under type 1 right censoring. For the choice of different sample sizes, $n = 30, 50$, and 100 are taken to perform a simulation study. It is observed as the sample size increases the parameters converge to their true parametric value. And it is also noted that LLF is found to be the best loss function assuming gamma prior because it has a less posterior risk.

Bayes estimates are found overestimated for some values and underestimated for few values. From the real dataset, it is observed that as the censoring times $T = 85, 70$, and 65 decrease, the posterior risk also decreases respectively. The BEs $\hat{a}_1, \hat{a}_2, \hat{\beta}_1$, and $\hat{\beta}_2$ represent the mean value of the burning velocity of the chemical material. And $\hat{R}(t)$ represents the reliability of the estimates. Simulation of real data is carried out to compare the parametric values. Graphical representation of parameters is also presented by taking the number of iterations on the $x$-axis and different parametric values on the $y$-axis. From these graphs, we
come up with almost the same results of Bayes estimates. Bayes estimates are found better than MLEs as PRs are smaller as compared to MSEs.

Appendix

A. Numerical Results of Simulation Study

BEs, PRs, and reliability estimates using UP, IBeta, and gamma prior under LLF and SELF for parameters \((\delta_1, \alpha_1, \alpha_2, \beta_1, \beta_2)\) 0.4, 1.5, 1, 0.8, 0.5 and \(T = 5\) and (0.4, 2, 3, 1.3, 1.5) and \(T = 5\) are shown.

B. Derivation for the Elements of Lindley's Approximation

Taking logarithm of posterior density given in equation (8)

\[
Q(\Theta|x) = \log p(\Theta|x) \propto s_1 \log \delta_1 + s_2 \log (1 - \delta_1) + (a_1 + s_1 - 1) \log \alpha_1 + (a_2 + s_2 - 1) \log \alpha_2 + (c_1 + s_1 - 1) \log \beta_1 + (c_2 + s_2 - 1) \log \beta_2 - b_i \alpha_1 - b_2 \alpha_2 - c_i \beta_1 - c_2 \beta_2 - (\alpha_1 - 1) \sum_{j=1}^{s_1} \log (1 + x_{ij}) \]

\[
= -(\alpha_2 - 1) \sum_{j=1}^{s_2} \log (1 + x_{ij}) + (\beta_1 - 1) \sum_{j=1}^{s_1} \log (1 - (1 + x_{ij})^{-\alpha_1}) + (\beta_2 - 1) \sum_{j=1}^{s_2} \log (1 - (1 + x_{ij})^{-\alpha_2}) + (m - s) \log (1 - \delta_1 (1 - (1 + t_a)^{-\alpha_1})) - (1 - \delta_1) (1 - (1 + t_a)^{-\alpha_1}) \]

From equations (B.1) and (10), the elements \(Q_{ij}, Q_{ij} = Q_{ji}, t_a = T\) for \(i, j, s = 1, 2, \ldots, 5\) are derived as follows:

\[
\begin{align*}
\varphi_i &= (1 + T)^{-\alpha_i} \ln (1 + T), \\
\psi_i &= \sum_{j=1}^{s_i} \frac{(1 + x_{ij})^{-\alpha_i} \ln (1 + x_{ij})}{(1 - (1 + x_{ij})^{-\alpha_i})}, \\
k_i &= -F_i \ln (1 - (1 + T)^{-\alpha_i}), \\
l_i &= \frac{\beta_i F_i \varphi_i}{(1 - (1 + T)^{-\alpha_i})}, \\
m_i &= \frac{\delta_i F_i \varphi_i}{(1 - (1 + T)^{-\alpha_i}) (1 + \beta_i \ln (1 - (1 + T)^{-\alpha_i}))}, \\
\theta_i &= \frac{\delta_i F_i \varphi_i \beta_i \ln (1 + T) - (\beta_i - 1) \varphi_i}{(1 - (1 + T)^{-\alpha_i})^2}, \\
z_i &= -\delta_i (1 - (1 + T)^{-\alpha_i})^2 F_i, \\
Q_{11} &= \left[ \frac{s_1}{\delta_1^2} + \frac{s_2}{(1 - \delta_1)^2} + \frac{(m - s)(F_2 - F_1)^2}{R^2} \right], \\
Q_{12} &= \frac{(m - s)F_1 \varphi_i \beta_i (p_1 (F_2 - F_1) - R)}{(1 - (1 + T)^{-\alpha_2}) R^2}, \\
Q_{13} &= \frac{(m - s)F_2 \varphi_i \beta_2 ((1 - p_1) (F_2 - F_1) + R)}{(1 - (1 + T)^{-\alpha_2}) R^2}, \\
Q_{14} &= \frac{(m - s)F_1 \ln (1 - (1 + T)^{-\alpha_1}) ((F_2 - F_1) p_1 - R)}{R^2}, \\
Q_{15} &= \frac{(m - s)F_2 \ln (1 - (1 + T)^{-\alpha_2}) ((F_2 - F_1) (1 - p_1) + R)}{R^2}.
\end{align*}
\]
\[
Q_{22} = \frac{\delta_1}{\alpha_1^2} - (\beta_1 - 1) \psi_{11} + \psi_{22}^2 + \frac{(m-s)\delta_1 \beta_1 \varphi_{11} F_1}{(1 - (1 + T)^{-\alpha_1})^2 R} \left( \ln(1 + T) - \frac{(\beta_1 - 1) \varphi_{11}}{(1 - (1 + T)^{-\alpha_1})} - \frac{\delta_1 \beta_1 \varphi_{11} F_1}{(1 - (1 + T)^{-\alpha_1})^2 R} \right),
\]

\[
Q_{23} = Q_{32} = \frac{(m-s) \delta_1 (1 - \delta_1) \beta_1 \varphi_{11} F_1}{(1 - (1 + T)^{-\alpha_1}) (1 - (1 + T)^{-\alpha_2}) R^2},
\]

\[
Q_{24} = Q_{42} = \frac{\psi_1 - \frac{(m-s) \delta_1 \varphi_{11} F_1}{(1 - (1 + T)^{-\alpha_1}) R^2} \left( \frac{1}{(1 - (1 + T)^{-\alpha_1})} + \ln(1 - (1 + T)^{-\alpha_1}) \delta_1 \beta_1 F_1 + \ln(1 - (1 + T)^{-\alpha_1}) \beta_1 \right)}{R},
\]

\[
Q_{25} = Q_{52} = \frac{- (m-s) \beta_1 (1 - \delta_1) \beta_1 \varphi_{11} F_1}{(1 - (1 + T)^{-\alpha_1}) R^2} \ln(1 - (1 + T)^{-\alpha_1}).
\]

\[
Q_{33} = \frac{- \delta_2}{\alpha_2^2} - (\beta_2 - 1) \left( \psi_2^2 + \psi_{22} \right) - \frac{(m-s) (1 - \delta_1) \beta_2 \varphi_{22} F_2}{(1 - (1 + T)^{-\alpha_1}) R} \left( \ln(1 + T) + \frac{(\beta_2 - 1) \varphi_{22}}{(1 - (1 + T)^{-\alpha_1})} + \frac{(1 - \delta_1) \beta_2 \nu_{22} F_2}{(1 - (1 + T)^{-\alpha_1}) R} \right),
\]

\[
Q_{34} = Q_{43} = \frac{(m-s) \delta_1 (1 - \delta_1) \beta_2 \varphi_{22} F_2}{(1 - (1 + T)^{-\alpha_1}) (1 - (1 + T)^{-\alpha_2}) R^2} \ln(1 - (1 + T)^{-\alpha_1}),
\]

\[
Q_{35} = Q_{53} = \frac{-(m-s) (1 - \delta_1) \varphi_{22} F_2}{(1 - (1 + T)^{-\alpha_1}) R^2} \left( 1 + \beta_2 \ln(1 - (1 + T)^{-\alpha_1}) + \frac{(1 - \delta_1) \beta_2 \varphi_{22} F_2 \ln(1 - (1 + T)^{-\alpha_1})}{R} \right),
\]

\[
Q_{44} = \frac{- \delta_1}{\beta_1} \frac{s_{12}}{F_2} \ln(1 - (1 + T)^{-\alpha_1}) \left( 1 + \frac{\delta_{12} F_2}{R} \right),
\]

\[
Q_{45} = Q_{54} = \frac{- (m-s) \delta_1 (1 - \delta_1) F_1 F_2 \ln(1 - (1 + T)^{-\alpha_1})}{R^2} \ln(1 - (1 + T)^{-\alpha_1}),
\]

\[
Q_{55} = \frac{- \delta_2}{\beta_2} \frac{s_{12}}{F_2} \ln(1 - (1 + T)^{-\alpha_1}) \left( 1 + \frac{\delta_{12} F_2}{R} \right) \ln(1 - (1 + T)^{-\alpha_1}) + (1 - \delta_1) F_2 + 1.
\]

\[
Q_{111} = \frac{2s_{11}}{\delta_{11}} - \frac{2s_{11}}{(1 - \delta_{11})^2} + \frac{2(m-s) (F_2 - F_1)^3}{R^3}.
\]

\[
Q_{112} = Q_{112} = \frac{2(m-s) (F_2 - F_1) \beta_1 \varphi_{11}}{(1 - (1 + T)^{-\alpha_1}) R^2} \left( 1 - \delta_1 \frac{(F_2 - F_1)}{R} \right),
\]

\[
Q_{113} = Q_{113} = \frac{2(m-s) (F_2 - F_1) \beta_2 \varphi_{22}}{(1 - (1 + T)^{-\alpha_2}) R^2} \left( 1 + \delta_1 \frac{(F_2 - F_1)}{R} \right).
\]

\[
Q_{114} = Q_{114} = \frac{2(m-s) (F_2 - F_1) \ln(1 - (1 + T)^{-\alpha_1})}{R^2} \left( 1 - \delta_1 \frac{(F_2 - F_1)}{R} \right),
\]

\[
Q_{115} = Q_{115} = \frac{2(m-s) (F_2 - F_1) \ln(1 - (1 + T)^{-\alpha_1})}{R^2} \left( 1 - \delta_1 \frac{(F_2 - F_1)}{R} \right).
\]

\[
Q_{222} = Q_{221} = \frac{(m-s) F_1 \beta_1 \varphi_{11}}{(1 - (1 + T)^{-\alpha_1}) R} \left( \frac{- \delta_1 (F_2 - F_1) \ln(1 + T)}{R} + \ln(1 + T) + \beta_1 \varphi_{11} \delta_1 (F_2 - F_1) \ln(1 + T) - (\beta_1 - 1) \varphi_{11} \right) \left( 1 - (1 + T)^{-\alpha_1} \right) R + \delta_1 (F_2 - F_1) \beta_1 \ln(1 + T) - \frac{(\beta_1 - 1) \varphi_{11}}{(1 - (1 + T)^{-\alpha_1}) R} \left( 1 - (1 + T)^{-\alpha_1} \right),
\]

\[
Q_{232} = Q_{321} = \frac{(m-s) F_1 \beta_1 \varphi_{11}}{(1 - (1 + T)^{-\alpha_1}) (1 - (1 + T)^{-\alpha_2}) R^2} \left( \frac{2 \delta_1 (F_2 - F_1)}{R} \right) + \delta_1 (F_2 - F_1) \beta_1 \ln(1 - (1 + T)^{-\alpha_1}) - \beta_1 \ln(1 - (1 + T)^{-\alpha_1}) R^2 \right) + 2 \delta_1 - 1,
\]

\[
Q_{122} = Q_{212} = \frac{(m-s) F_2 \beta_1 \varphi_{22}}{(1 - (1 + T)^{-\alpha_2}) R} \left( \frac{\delta_1 (F_2 - F_1)}{R} - 1 + \frac{2 \delta_1^2 F_2 \beta_1 \varphi_{22} (1 - (1 + T)^{-\alpha_2})}{R^2} + \frac{2 \delta_1^2 F_1 \beta_1 \varphi_{22} (1 - (1 + T)^{-\alpha_2})}{R^2} \right) + 2 \delta_1 - 1.
\]
\[ Q_{133} = Q_{331} = \frac{(m-s)F_2\beta_2\varphi_2}{R(1-(1+T)^{-\alpha_2})} \left( -\ln(1+T) - \frac{(1-\delta_1)\ln(1+T)(F_2 - F_1)}{R} + \frac{(1-\delta_1)(\beta_2-1)\varphi_2(F_2 - F_1)}{R(1-(1+T)^{-\alpha_2})} + \frac{(\beta_2-1)\varphi_2}{(1-(1+T)^{-\alpha_2})} \right) + \frac{2(1-\delta_1)^2F_2\beta_2\varphi_2(F_2 - F_1)}{R^2(1-(1+T)^{-\alpha_2})} + \frac{2(1-\delta_1)^2\beta_2\varphi_2}{R(1-(1+T)^{-\alpha_2})}, \]
\[ Q_{144} = Q_{341} = \frac{(m-s)F_2F_1\beta_2\varphi_2}{R^2(1-(1+T)^{-\alpha_2})} \left( \frac{2\delta_1(1-\delta_1)(F_2 - F_1)}{R} + 2\delta_1 - 1 \right), \]
\[ Q_{135} = Q_{351} = \frac{(m-s)F_2\varphi_2}{R(1-(1+T)^{-\alpha_2})} \left( 1 + \beta_2 \ln(1-(1+T)^{-\alpha_2}) \right) \left( \frac{1 - \delta_1}{R} + \frac{2(1-\delta_1)^2F_2F_1(1-\delta_1)(F_2 - F_1)\ln(1-(1+T)^{-\alpha_2})}{R^2} \right) + \frac{2\beta_2(1-\delta_1)n(1-(1+T)^{-\alpha_2})F_2}{R} + \frac{2(1-\delta_1)\beta_2(F_2 - F_1)\ln(1-(1+T)^{-\alpha_2})}{R}, \]
\[ Q_{144} = Q_{414} = \frac{(m-s)F_2n(1-(1+T)^{-\alpha_2})}{R^2} \left( \frac{2\delta_1F_1(1-\delta_1)(F_2 - F_1)}{R} - 2\delta_1F_1 + \delta_1(F_2 - F_1) - 1 \right), \]
\[ Q_{145} = Q_{451} = \frac{(m-s)F_2F_1}{R^2} \ln(1-(1+T)^{-\alpha_2}) \left( \frac{2\delta_1(1-\delta_1)(F_2 - F_1)}{R} + 2\delta_1 - 1 \right). \]

(B.8)

\[ Q_{155} = Q_{551} = \frac{(m-s)F_2}{R} \ln(1-(1+T)^{-\alpha_2}) \left( 1 + \frac{2(1-\delta_1)F_2}{R} + \frac{2(1-\delta_1)F_2}{R^2} + \frac{2(1-\delta_1)^2F_2F_1(1-\delta_1)(F_2 - F_1)}{R} + \frac{(1-\delta_1)(F_2 - F_1)}{R} \right). \]

(B.9)
\[ Q_{332} = Q_{233} = Q_{323} = -\frac{(n - r)(1 - \delta_1)\delta_1 F_1 F_2 \beta_2 \varphi_1 \varphi_2}{(1 - (1 + T)^{-a_2})(1 - (1 + T)^{-a_3}) R^2} \left( \ln(1 + T) - \frac{2(1 - \delta_1)\beta_2 \varphi_2 F_2}{R(1 - (1 + T)^{-a_2})} - \frac{(1 - \beta_2) \varphi_2}{(1 - (1 + T)^{-a_3})} \right), \]

\[ Q_{234} = Q_{442} = Q_{424} = -\frac{(m - s)\delta_1 F_1 \varphi_1 \ln(1 - (1 + T)^{-a_1})}{(1 - (1 + T)^{-a_2}) R} \left( 2 + \beta_2 \ln(1 - (1 + T)^{-a_2}) + \frac{2 \delta_1 \beta_2 F_1}{R} + \frac{2 \delta_1^2 \beta_1 \ln(1 - (1 + T)^{-a_1}) F_1^2}{R^2} + \frac{3 \delta_1 \beta_1 \ln(1 - (1 + T)^{-a_1}) F_1}{R} \right), \]

\[ Q_{245} = Q_{254} = Q_{452} = -\frac{(m - s)(1 - \delta_1)\delta_1 F_1 \varphi_1 \ln(1 - (1 + T)^{-a_1})}{(1 - (1 + T)^{-a_2}) R^2} \left( 1 + \frac{\beta_1 \ln(1 - (1 + T)^{-a_2})}{R} + \frac{2 \delta_1 \beta_1 \ln(1 - (1 + T)^{-a_1}) F_1}{R} \right), \]

\[ Q_{255} = Q_{552} = Q_{525} = -\frac{(m - s)(1 - \delta_1)\delta_1 F_1 F_2 \beta_2 \varphi_1 \varphi_1 (\ln(1 - (1 + T)^{-a_2}))}{(1 - (1 + T)^{-a_1}) R^2} \left( 1 + \frac{2(1 - \delta_1) F_2}{R} \right), \]

\[ Q_{222} = -\left( \beta_2 - 1 \right) \left( 3 \psi_{222} + 2 \psi_2^3 + \psi_{222} \right) + \frac{2s_2}{\alpha_2^2} + \frac{(m - s)\beta_2 (1 - \delta_1) \varphi_2 F_2}{(1 - (1 + T)^{-a_2}) R} \left( -\ln(1 + T)^2 - \frac{2F_2^2 \beta_2^2 \varphi_1 \varphi_1^2}{(1 - (1 + T)^{-a_2}) R^2} \left( F_1 \beta_1 (2\beta_2 - 2) \delta_1 \varphi_1^2 - \frac{2F_2^2 \beta_1^2 \delta_1^2 \varphi_1^2}{(1 - (1 + T)^{-a_2}) R^2} \left( F_1 \beta_1 (2\beta_2 - 2) \delta_1 \varphi_1^2 \right) (1 - (1 + T)^{-a_2}) R \right), \right) \]

\[ Q_{334} = Q_{343} = Q_{433} = -\frac{(m - s)(1 - \delta_1)\delta_1 F_1 F_2 \beta_2 \varphi_2 (\ln(1 - (1 + T)^{-a_1}))}{(1 - (1 + T)^{-a_2}) R^2} \left( \left( \ln(1 + T) \right) - \frac{2(1 - \delta_1)\beta_2 \varphi_2 F_2}{R(1 - (1 + T)^{-a_2})} - \frac{(\beta_2 - 1) \varphi_2}{(1 - (1 + T)^{-a_2})} \right), \]

\[ Q_{344} = Q_{443} = Q_{434} = -\frac{(m - s)(1 - \delta_1)\delta_1 F_1 F_2 \beta_2 \varphi_2 (\ln(1 - (1 + T)^{-a_1}))}{(1 - (1 + T)^{-a_2}) R^2} \left( 2 + \beta_2 \ln(1 - (1 + T)^{-a_2}) + \frac{2 \delta_1 \beta_2 F_1}{R} + 1 \right), \]

\[ Q_{345} = Q_{543} = Q_{533} = -\frac{(m - s)(1 - \delta_1)\delta_1 F_2 \varphi_2 (\ln(1 - (1 + T)^{-a_1}))}{(1 - (1 + T)^{-a_2}) R^2} \left( 1 + \beta_2 \ln(1 - (1 + T)^{-a_2}) \right) + \frac{2(1 - \delta_1) \beta_2 \ln(1 - (1 + T)^{-a_2}) F_2}{R}, \]

\[ Q_{355} = Q_{553} = Q_{535} = -\frac{(m - s)(1 - \delta_1)\delta_1 F_2 \varphi_2 (\ln(1 - (1 + T)^{-a_1}))}{(1 - (1 + T)^{-a_2}) R^2} \left( 2 + \beta_2 \ln(1 - (1 + T)^{-a_2}) + \frac{2(1 - \delta_1) F_2}{R} + \frac{3(1 - \delta_1 \beta_2 \ln(1 - (1 + T)^{-a_1}) F_2^2}{R^2} \right), \]

\[ Q_{444} = Q_{445} = Q_{454} = -\frac{2s_1}{\beta_1^2} - \frac{(m - s)\delta_1 F_1 \ln(1 - (1 + T)^{-a_1})^2}{R^2} \left( \ln(1 - (1 + T)^{-a_1}) + \frac{3 \delta_1 F_1 (\ln(1 - (1 + T)^{-a_1}) + \frac{2 \delta_1^2 \beta_2 \ln(1 - (1 + T)^{-a_1}) F_1^2}{R^2} \right), \]

\[ Q_{445} = Q_{454} = Q_{545} = -\frac{(m - s)(1 - \delta_1) F_1 F_2 \ln(1 - (1 + T)^{-a_1}) \ln(1 - (1 + T)^{-a_1})^2}{R^2} \left( 2 + \beta_2 \ln(1 - (1 + T)^{-a_2}) + \frac{2(1 - \delta_1) F_2}{R} + 1 \right), \]

\[ Q_{455} = Q_{554} = Q_{545} = -\frac{(m - s)(1 - \delta_1) F_1 F_2 \ln(1 - (1 + T)^{-a_1})^2}{R^2} \left( \ln(1 - (1 + T)^{-a_1}) + \frac{3 \delta_1 F_2}{R} + \frac{2(1 - \delta_1)^2 F_2^2}{R^2} \right), \]

(B.10)
Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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