Into the conformal window: multi-representation gauge theories

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(Dated: January 10, 2020)

Abstract

We investigate the conformal window of four-dimensional gauge theories containing fermionic matter fields in multiple representations. Of particularly relevant examples are the ultra-violet complete models with fermions in two distinct representations considered in the context of composite Higgs and top-partial compositeness. We first discuss various analytical approaches to unveil the lower edge of the conformal window and their extension to the multiple matter representations. In particular, we argue that scheme-independent series expansions for the anomalous dimension of a fermion bilinear $\gamma \bar{\psi} \psi$ at an infrared fixed point, combined with the conjectured critical condition $\gamma \bar{\psi} \psi = 1$ or equivalently $\gamma \bar{\psi} \psi (2 - \gamma \bar{\psi} \psi) = 1$, can be used to determine the boundary of conformal phase transition on fully physical grounds. In illustrative cases of $SU(2)$ and $SU(3)$ theories with $N_f$ fundamental flavors, we access our results by comparing to other analytical and lattice results.
I. INTRODUCTION

The existence of a non-zero infrared (IR) fixed point in the renormalization-group (RG) beta function of four-dimensional asymptotically free gauge theories with a sufficiently large number of massless fermions $N_f$ for a given number of colors $N_c$ has been of particular interests because of its potential application to phenomenological model buildings in the context of physics beyond the standard model (BSM), as well as its distinctive feature of conformal phase in contrast to nonconformal phase as in Quantum Chromodynamics (QCD). A perturbative calculation at two-loop order in the weak coupling regime finds an interacting IR fixed point \cite{1}, a Banks-Zaks (BZ) fixed point named after their work on the phase structure of vector-like gauge theories with massless fermions at zero temperature \cite{2}. As we vary the ratio of $N_f/N_c$, treated as a continuous variable, the IR fixed point approaches either zero, at which the theory loses the asymptotic freedom and becomes trivial, or runs away into the strong coupling regime where the theory loses the calculability of the perturbative expansion. For sufficiently small values of the ratio we expect the theory is in a chirally broken phase, that implies the presence of a zero-temperature quantum phase transition between the conformal and chirally broken phases at a critical value of the ratio. A finite range of the number of flavors for which the theory has a non-zero IR fixed point is called conformal window, and the chirally-broken theories near the phase transition are expected to have quite different IR dynamics, compared to QCD-like theories.

Near-conformal dynamics are ubiquitous in BSM model buildings for which the underlying ultraviolet (UV) theory is a novel strongly coupled gauge theory. One of the crucial features is a would-be large dimensional transmutation of composite operators. In walking technicolor theories a large anomalous dimension of a fermion bilinear, which results in a slowly evolving coupling and thus provides a large separation in the scale between the flavor symmetry breaking $\Lambda_F$ and the confinement $\Lambda_{TC}$, is required to realize the dynamical electroweak symmetry breaking while avoiding constraints in flavor physics \cite{3, 5, 6}. In partial composite Higgs models which include both the pseudoscalar Nambu-Goldstone bosons (pNGB) composite Higgs \cite{7, 9} and partial compositeness \cite{10}, a large anomalous dimension of baryonic operators linearly coupled to the standard model (SM) top quark provides a natural explanation of the relatively large mass of top quark. This idea was originally proposed in the framework of wrapped extra dimensions \cite{11, 12}, and the corresponding
minimal models have been extensively studied in various phenomenological aspects at the level of effective theories (see [13, 14] for reviews, and references therein). However, it is relatively recent to consider the realistic candidates for the four-dimensional UV complete models based on strongly coupled gauge theories, containing two different representations of fermionic matter fields [15–18]. The anomalous dimension of the baryonic operators, considered as top-partner, within the conformal window relevant to these UV models was calculated at one-loop order in the perturbative expansion [20, 21]. Furthermore, substantial efforts have been devoted to investigate the low-energy dynamics of such theories from the first-principle lattice calculations, where particularly interesting lattice models are based on \( SU(4) \) [22–27] and \( Sp(4) \) [28, 29] gauge theories.

The other common non-trivial features of near-conformal gauge theories are the emergence of light scalar resonances. Such a new degree of freedom at low energy may be identified as a dilaton arising from the spontaneous breaking of scale symmetry, and can be used to extend the Higgs sector of the standard model of particle physics [30–40]. Interestingly, recent lattice studies of \( SU(3) \) gauge theories with \( N_f = 8 \) fundamental Dirac fermions [41–44], as well as \( N_f = 2 \) two-index symmetric Dirac fermions [45–49], performed with moderate sizes of the fermion mass found a relatively light scalar particle in the spectrum. There have been several attempts to analyze these results by incorporating the scalar degrees of freedom within an effective field theory (EFT) [50–55]. The dilaton potential also inherently possesses the possibility of a strong finite-temperature first-order phase transition, which provides a natural set-up for the realization of electroweak baryon genesis [56–59] and the supercooled universe [60–64] within composite Higgs scenarios.

While phenomenological model buildings could be carried out with some working assumptions and qualitative features of near-conformal dynamics at low energy, in order to take the full advantage it is necessary to perform quantitative studies started from the underlying strongly-coupled gauge theories. As mentioned above, lattice Monte Carlo (MC) calculations are highly desired in this respect, where most of the modern technologies developed for lattice QCD can be applied without additional difficulties. However, lattice calculations are expensive and thus practically not suitable to explore all the possibilities in the theory space at arbitrary numbers of \( N_c \) and \( N_f \). Therefore, any analytical calculations for the

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1 Although it does not have to be relevant to near-conformal dynamics, the two-representation composite Higgs models can have additional light and non-anomalous scalars which may have a phenomenological impact on the collider physics, e.g. see [19].
determination of the conformal window are welcome to find the most promising UV models exhibiting near-conformal dynamics. While various analytical proposals are found in the literature \cite{65-67} besides the traditional Schwinger-Dyson analysis in the ladder approximation, we suggest to use the critical condition on the anomalous dimension of a fermion bilinear operator at an IR fixed point, $\gamma_{\text{IR}} = 1$ or equivalently $\gamma_{\text{IR}}(2 - \gamma_{\text{IR}}) = 1$, which is responsible for the conformal phase transition within the perturbation theory \cite{68}. We do not claim the originality of this idea, but we stress that it becomes an alternative method to unveil the conformal window in a scheme-independent way if we adopt the series expansion of $\gamma_{\text{IR}}$ recently developed by T. Ryttov and R. Shrock \cite{69-75}. We find that this method is particularly useful to discuss the sequential IR evolution for the theories containing multiple matter representations. Although we restrict our attention to the asymptotically free gauge theories containing both fermionic matter fields in the two different representations, relevant to the pNGB composite Higgs models as summarized in Ref. \cite{76}, the methodology discussed in this work can straightforwardly be extended to any number of distinct matter representations.

The paper is organized as follows. In Sec. II we provide some general remarks on the conformal window of a generic nonabelian gauge theory with fermion matter in multiple representations. We then describe several known analytical methods used for the determination of the lower bound of the conformal window, where we revisit the critical condition on the anomalous dimension of the fermion bilinear operators $\gamma_{\bar{\psi}\psi}$ in the onset of the conformality lost. In Sec. III we briefly review the scheme independent calculation of $\gamma_{\bar{\psi}\psi}$ in IR conformal gauge theories with fermions in up to two different representations, and determine the lower bound of the conformal window in the exemplified cases of $SU(2)$ and $SU(3)$ gauge theories with $N_f$ fermions in the fundamental representation. We access our results by comparing to several scheme dependent calculations as well as other analytical and lattice results. We also discuss the convergence of the scheme independent expansion for the critical condition. In Sec. IV we present our main results on the conformal window of the two-representation gauge theories relevant to the pNGB composite Higgs and top partial compositeness. We present some results of group invariants used to compute the coefficients of the scheme-independent series expansions in Appendix A and lower-order results to the conformal window in Appendix B. Finally, we conclude by summarizing our findings in Sec. V.
II. CONFORMAL WINDOW: ANALYTICAL APPROACHES

We start by providing a general remark on the conformal window of four-dimensional
gauge theories containing fermionic matter in the multiple representations $R_i$ with \{${N_{R_i}}$\},
$i = 1, 2, \cdots, k$, denoting a set of the numbers of flavors. In a small coupling regime, the
perturbative beta function in powers of the gauge coupling $\alpha = g^2/(4\pi)$ is given as

$$\beta(\alpha) \equiv \frac{\partial \alpha(\mu)}{\partial \ln \mu} = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \left( \frac{\alpha}{4\pi} \right)^\ell,$$

(1)

where $b_\ell$ is the $\ell$-loop coefficient and $\mu$ is the renormalization scale. The coefficients of the
lowest two terms, $b_1$ [17, 18] and $b_2$ [1], are renormalization scheme-independent and given
as

$$b_1 = \frac{11}{3}C_2(G) - \frac{4}{3} \sum_i N_{R_i} T(R_i),$$

(2)

and

$$b_2 = \frac{34}{3}C_2(G)^2 - \frac{4}{3} \sum_i (5C_2(G) + 3C_2(R_i)) N_{R_i} T(R_i).$$

(3)

The generators in the representation $R_i$ of an arbitrary gauge group $G$ are denoted by
$T^a_{R_i}$, $a = 1, \cdots, d(G)$, where $d(G)$ is the dimension of the adjoint representation. The
trace normalization factor $T(R_i)$ and the quadratic Casimir $C_2(R_i)$ are defined through
$\text{Tr}[T^a_{R_i} T^b_{R_i}] = T(R_i) \delta^{ab}$ and $T^a_{R_i} T^a_{R_i} = C_2(R_i) I$, respectively. These two group-theoretical
factors are related by $C_2(R_i) d(R_i) = T(R_i) d(G)$. Note that $b_\ell$ with $\ell \geq 3$ are renormalization scheme-dependent.
Throughout this work we denote $N_{R_i}$ for the number of Weyl
spinors if the representation is real or pseudoreal, and for the number of Dirac flavors if the
representation is complex.

As far as the UV completion is concerned, we require the theory is asymptotically free
or $b_1 > 0$. (We do not consider the scenarios of UV safety in this work.) This condition
leads to the maximum numbers of flavors above which we lose the asymptotic freedom. For
a single representation, it is given as

$$N_{R_i}^{AF} = \frac{11C_2(G)}{4T(R)},$$

(4)

while for multiple representations it spans the points on the $(k - 1)$-dimensional surface in
the space of the fermion representations, that satisfies $b_1 = 0$. Since the discussion below
is independent of whether the representations are multiple or not, we simply consider the case of a single representation \( R \). For a sufficiently small and positive value of \( b_1 \), the theory develops a non-zero IR fixed point (BZ fixed point), if the two-loop coefficient \( b_2 \) is negative,

\[
\alpha_{\text{BZ}} \simeq -4\pi \frac{b_1}{b_2}.
\]

Therefore, the perturbative analysis guarantees the existence of the conformal theory at a small coupling if the numbers of flavors are sufficiently large but bounded by \( N_{AF}^R \), i.e. \( b_1 \ll 1 \).

As we decrease \( N_R \), however, \( \alpha_{\text{BZ}} \) increases in general and at some point the two-loop result is no longer reliable. Higher order corrections should be then included to extend the perturbative two-loop results, but are largely limited due to its scheme dependence. If \( \alpha_{\text{BZ}} \sim \mathcal{O}(1) \), the perturbative expansion even breaks down and one has to take account for nonperturbative effects in order to describe the IR dynamics correctly. Nevertheless, if we keep decreasing \( N_R \), the (negative) slope of the beta function at UV becomes large enough that the theory becomes strongly coupled at low energy and eventually falls into the chirally broken phase. One of the extreme case is pure Yang-Mills at \( N_R = 0 \), which is confining therefore nonconformal. We therefore expect that there is a finite range of window in the number of flavors \( N_R \), namely the conformal window (CW), where the theory is IR conformal. While the upper bound of CW is identical to that for the loss of asymptotic freedom, \( N_{AF}^R \), its lower bound \( N_c^R \) is not easy to determine because of the difficulties mentioned above. In the following sections, we briefly discuss several analytical but approximate approaches being used to determine the lower bound of CW. Of our particular interest is the one obtained from the critical condition on the anomalous dimension of fermion bilinear operators, discussed in Sec. II D.

### A. 2-loop beta function

A naive estimation of the lower bound of CW might be obtained by losing the BZ fixed point to infinity. If we neglect the scheme-dependent higher order corrections, from the 2-loop beta function we find the condition \( b_2 = 0 \) for the lower bound. Analogous to the upper bound of CW the solution lives on the \((k-1)\)-dimensional surface for multiple representations.
\{R_1, R_2, \cdots, R_k\}. For a single representation, we obtain the simple expression

\[ N_{c, 2-\text{loop}}^R = \frac{17C_2(G)^2}{T(R) \left[ 10C_2(G) + 6C_2(R) \right]} \].

(6)

**B. (traditional) Schwinger-Dyson approach with the ladder approximation**

It is well known that the Schwinger-Dyson (SD) gap equation for the fermion propagator in the ladder (or rainbow) approximation yields the critical coupling, a minimal coupling strength required to trigger the chiral symmetry breaking, given as

\[ \alpha_c = \frac{\pi}{3C_2(R)}. \]

(7)

The traditional way to determine the number of flavors for the onset of the conformality lost is to equate the 2-loop IR fixed point, \( \alpha_{\text{BZ}} \) in Eq. 5, with \( \alpha_c \), where the result for the single representation \( R \) is

\[ N_{c, \text{SD}}^R = \frac{C_2(G)(17C_2(G) + 66C_2(R))}{T(R)(10C_2(G) + 30C_2(R))}. \]

(8)

Note that the critical coupling is proportional to the inverse of \( C_2(R) \). If more than one representation is present, fermions in different representations form chiral condensates sequentially if exist: one expects that fermions in the representation having the largest value of \( C_2(R) \), denoted by \( R_1 \) for convenience, would first be integrated out from the theory at the scale \( \Lambda_1 \) by developing a non-zero vacuum expectation value of the corresponding fermion bilinear. For \( \mu < \Lambda_1 \) the beta function will change into a new one which includes the effective degrees of freedom by excluding the fermions in \( R_1 \), and this procedure will sequentially occur as we decrease the scale \( \mu \). In this case, therefore, the theory leaves the conformal window when the fermions in \( R_1 \) form a non-zero fermion condensate.

**C. All-orders beta function**

The coefficients of the lowest two terms in the perturbative beta function do not depend on the renormalization scheme, so does the determination of \( N_c \) discussed in the previous two sections. While this is no longer true if one considers higher order terms, it is believed that there exists a certain scheme such that all higher order terms (\( \ell \geq 3 \)) vanish or at least the beta function is written in a closed form. Along the line of this idea all-orders
beta functions are suggested in Refs. [66, 67] inspired by the Novikov-Shifman-Vainshtein-Zakharov (NSVZ) beta function for supersymmetric theories [80]. The conjectured beta function for generic gauge theories with Dirac fermions in multiple representations, proposed in [66], is written in the following form

$$
\beta^{\text{all-orders}}(\alpha) = -\frac{\alpha^2}{2\pi} b_1 - \frac{2}{3} \sum_{i=1}^{k} T(R_i) N_{R_i} \gamma_{R_i}(\alpha) \left( 1 - \frac{\alpha}{2\pi C_2(G)} \frac{2b_1'}{b_1} \right) 
$$

(9)

where $$\gamma_{R_i}$$ is the anomalous dimension of a fermion bilinear for a given representation $$R_i$$, $$b_1' = C_2(R) - \sum_{i=1}^{k} T(R_i) N_{R_i}$$, and $$b_1$$ is in Eq. [3]. Using the leading-order expression for $$\gamma_{R}(\alpha)$$ this beta function reproduces the universal two-loop results. Note that the IR fixed point is determined by taking $$\beta^{\text{all-orders}}(\alpha) = 0$$, which is physical in the sense that it only involves scheme-independent quantities such as the anomalous dimension $$\gamma_{R_i}$$.

In the case of the single representation $$R$$, the anomalous dimension $$\gamma(R)$$ at the IR fixed point is given by

$$
\gamma_{\text{IR}} = \frac{11C_2(G)}{2T(R)N_R} - 2.
$$

(10)

The lower bounds are typically determined by taking $$\gamma_{\text{IR}} = 2$$, implied from the unitarity [81]. Unfortunately, this value of $$\gamma_{\text{IR}}$$ turns out to be inconsistent with the perturbative result at the IR fixed point. A modified version of the all-orders beta functions such that the inconsistency is resolved was later proposed in [67]. But now the unitarity condition leads to too small values of $$N_R$$ for the lower edge of the conformal window, e.g. smaller than the value obtained from Eq. [6] which shows the unitarity condition is too weak.

In contrast to the case of a single representation, the anomalous dimensions at the IR fixed point are indeterminable if multiple representations are present:

$$
\frac{2}{11} \sum_{i=1}^{k} T(R_i) N_{R_i} (2 + \gamma_{R_i}) = C_2(G).
$$

(11)

As for the single representation, the lower bound can be obtained by applying the unitarity condition to all the representations, $$\gamma(R_i) = 2$$ with $$i = 1, 2, \cdots, k$$. However, this approach does not give any informations on the sequencial chiral symmetry breaking. Note that in general the anomalous dimensions of the fermion bilinear in different representations are expected to be different at the IR fixed point.
D. Critical condition for the anomalous dimension of a fermion bilinear

The critical coupling in Eq. 7 being equal to $\alpha_{\text{BZ}}$ has been widely used to estimate the phase boundary of the conformal window. However, the essence of the critical condition is actually hidden in the anomalous dimension of the fermion bilinear $\gamma_{\bar{\psi}\psi}$ [68, 82, 83]. To see this, let us recall the Schwinger-Dyson equation for the massless fermion, where the full inverse propagator in the momentum space is

$$iS^{-1}(p) = Z(p)\left(\not{p} - \Sigma(p)\right), \quad (12)$$

with $Z(p)$ and $\Sigma(p)$ being the wave-function renormalization constant and the self-energy function, respectively. In the Euclidean space the ladder approximation to the SD equation leads to the integral gap equation

$$\Sigma(p) = 3C_2(R) \int \frac{d^4k}{(2\pi)^4} \frac{\alpha((k-p)^2)}{(k-p)^2} \frac{\Sigma(k^2)}{Z(k^2)k^2 + \Sigma^2(k^2)}. \quad (13)$$

In the Landau gauge, $Z(k^2) = 1$, this equation can be linearized by neglecting $\Sigma^2(k^2)$ in the regime of sufficiently large momenta. The slowly varying coupling $\alpha(\mu) \approx \alpha_{\text{IR}}$, which is the key assumption of near-conformal dynamics, further simplifies Eq. 13 and one obtains two scale-invariant solutions for $\Sigma(p^2)$ of the form, $(p^2)^{-\gamma/2}$, in the deep UV with [82]

$$\gamma_{\text{IR}}(2 - \gamma_{\text{IR}}) = \frac{\alpha_{\text{IR}}}{\alpha_c}, \quad (14)$$

where $\alpha_c$ is given in Eq. 7. For $\alpha_{\text{IR}} < \alpha_c$ the first and second solutions can be understood as the RG running of a renormalized mass $m(\mu)$ and a fermion bilinear operator $\langle \bar{\psi}\psi \rangle(\mu)$ within the operator product expansion (OPE) at large Euclidean momentum. In this case no solution is found for which the chiral symmetry is broken with a vanishing mass term, indicating that no spontaneous chiral symmetry breaking occurs. [83]

For $\alpha_{\text{IR}} \geq \alpha_c$ both solutions show the same $p$ dependence up to a phase difference (at $\alpha_{\text{IR}} = \alpha_c$, $\Sigma(p) \sim (1/p^2)^{-1/2}$), and the OPE identification is not possible. As discussed in details in Ref. [83], in fact, this situation can be described by the underdamping of an anharmonic oscillator and the corresponding interpretation is the existence of spontaneous symmetry breaking. In the same paper, the authors pointed out that the generic feature of the transition between conformal and chirally broken phases imposed in the critical condition, $\alpha_{\text{IR}} = \alpha_c$ or equivalently $\gamma_{\text{IR}} = 1$, persists beyond the ladder approximation, even though the details such as the value of $\alpha_c$ do not. Utilizing the critical condition on the
anomalous dimension instead of the gauge coupling makes more sense to determine the phase boundary between conformal and non-conformal phases since it is physical and thus free from the scheme dependence.

Interestingly, the critical condition derived from the truncated SD analysis is in agreement with the conjectured mechanism responsible for the zero-temperature conformal phase transition, featured by an annihilation of IR and UV fixed points [84]. As we approach the lower edge of the conformal window from above, in particular, the dimension of the operator \( \bar{\psi} \psi \) at IR fixed point \( \Delta_+ \) decreases, while that of the counterpart at UV fixed point \( \Delta_- \) increases, and becomes identical to each other in the onset of the transition, i.e. \( \Delta_+ = \Delta_- = 2 \). We assume that the space-time dimension is \( d = 4 \). In a simplified holographic model [85] the loss of conformality occurs when the mass squared of a bulk scalar in a higher dimensional theory violates the Breitenlohner-Freedman (BF) bound, and the AdS/CFT correspondence implies that the dimension of the fermion bilinear operator is equal to 2 at the conformal phase transition. As we discussed above if we cross the phase boundary from inside of the conformal window, the truncated SD equations no longer have the valid scale-invariant solutions. Analogously, the solutions to the beta function describing the fixed point merger become complex and give arise to a mass gap \( m \sim \Lambda_{\text{UV}} \exp \left( -c / \sqrt{\alpha_{\text{IR}} - \alpha_c} \right) \) with \( c > 0 \) [84]. Recently, it is proposed that such IR dynamics (walking dynamics), slightly below the conformal window, could approximately be analyzed by using conformal perturbation theory in the vicinity of a complex pair of fixed points [86].

In the onset of the conformality lost, \( \alpha_{\text{IR}} = \alpha_c \), the solution to Eq. 14 is equivalent to the condition \( \gamma_{\text{IR}} = 1 \), which results in the nonperturbative, gauge invariant and scheme-independent definition for the critical condition. In this work, we attempt to calculate \( \gamma_{\text{IR}} \) in a perturbative manner. Note that, although both conditions of \( \gamma(2 - \gamma) = 1 \) in [82] and \( \gamma = 1 \) are not distinguishable to all orders in perturbation theory, they provide two different definitions when the perturbative expansion is truncated at finite order. Practically the former condition has been adopted since the leading-order expression of \( \gamma \) leads to the critical coupling \( \alpha_c \) in Eq. 17, where higher order estimates in the modified minimal subtraction scheme (\( \overline{\text{MS}} \)) were considered in Ref. 68. In general, this perturbative approach suffers from scheme dependence when higher-order terms \( (\ell \geq 3) \) are concerned. As we will discuss in Sec. III, however, it turns out that we are able to circumvent this problem by adopting scheme-independent series expansions for \( \gamma \) at IR fixed points. We will also discuss the
convergence of the perturbative expansions for both definitions of the critical condition.

We have so far restricted our attention to gauge theories containing fermions in a single representation. In order to account for the two-representation theories relevant to composite Higgs models with partial compositeness, we should extend the critical condition discussed above to the case of fermions in multiple representations. Analogous to the critical couplings considered in the traditional SD approach in Sec. II B, in fact, the anomalous dimensions of fermions in the different representations give rise to different values at the IR fixed point \[79\]. The fermion representation for which the anomalous dimension first reaches the unity, say \(R_1\), first develops a non-vanishing fermion condensate and thus provides the critical condition of the whole theory, unless the effective theory after integrating out the fermions in the representation \(R_1\) does have an IR fixed point. While keeping an eye on the sequence of critical conditions in the case of multiple representations, we impose the following conditions to determine the lower edge of conformal window,

\[
\text{Max}\{\gamma_{IR}(R_i)\} \equiv 1, \quad \text{or} \quad \text{Max}\{\gamma_{IR}(R_i)(2 - \gamma_{IR}(R_i))\} \equiv 1. \tag{15}
\]

Again, we note that these two conditions are equivalent if all orders in the perturbative expansion are considered, but could in general result in two different sets of \(\{N^c_{R_i}\}\) if the expansion is truncated at finite order.

E. Comparison between various analytical approaches

We conclude this section by comparing the analytical approaches to determine the lower edge of the conformal window. For convenience let us use the abbreviations 2-loop, SD, BF and \(\gamma\text{CC}\) to denote the methods discussed in Sections II A, II B, II C and II D, respectively. Both of 2-loop and SD use the gauge coupling at the BZ fixed point, where we take \(\alpha_{\text{BZ}} = \infty\) and \(\alpha_{\text{BZ}} = \alpha_c\) for the former and the latter, respectively. One could extend these methods by including higher-loop order terms, but immediately encounters the complication of scheme dependence. On the other hands, BF and \(\gamma\text{CC}\) use the anomalous dimension of a fermion bilinear \(\gamma\), and thus in principle determine the conformal window on physical grounds. If we restrict to the case of a single representation, BF provides the exact value of \(\gamma\) at the IR fixed point in a scheme-independent way. In the onset of the conformality lost one typically chooses \(\gamma = 2\) inspired by NSVZ super-symmetric theories. To use \(\gamma\text{CC}\) one needs the value
of $\gamma$, which could be obtained from the perturbative calculation. As we will discuss in details in the following sections, one can still maintain the scheme independence of $\gamma_{CC}$ beyond the 2-loop orders by incorporating the scheme-independent series expansions [69].

We now turn our attention to the case of multiple representations. The 2-loop method can easily be extended by taking $b_2$ in Eq. [3] to be zero. In contrast to the case of a single representation, BF neither provides the values of $\gamma_{IR}(R_i)$ nor the sequence of chiral symmetry breaking. But, one can still estimate the lower bound of CW by taking $\gamma_{IR}(R_i) = 2$ for all the representations. In the cases of SD and $\gamma_{CC}$ one can use the dynamical results of $\alpha_c(R_i)$ and $\gamma_{IR}(R_i)$ as they have different values for different representations. Assuming the theory falls into the chirally broken phase below the conformal window, the representation having the maximum values of $\gamma_{IR}(R_i)$ and $\alpha_c^{-1}(R_i)$ determines the lower edge of the conformal window. Of course we note that all of the above discussions are limited by the fact that nonperturbative effects are not in the consideration.

III. SCHEME-INDEPENDENT DETERMINATION OF CONFORMAL WINDOW USING $\gamma_{CC}$

In this section we first briefly review the scheme-independent (SI) series expansion of physical quantities at an IR fixed point in asymptotically free gauge theories with fermions in a representation, and its extension to multiple representations. We only present the essential ingredients, focusing on the calculation of the anomalous dimension of fermion bilinear operators, needed to our discussions. The great details including calculations of other physical quantities can be found in a series of works done in [69–75, 87]. We then describe how to determine the conformal window using the critical condition for the anomalous dimension $\gamma_{CC}$ with this new technique. In the illustrative examples of $SU(2)$ and $SU(3)$ gauge theories with $N_f$ fundamental fermions, we discuss the consequence of two critical conditions in Eq. [15] at finite order in the SI expansion, and compare our results to various scheme-dependent expansions and other analytical (but approximate) approaches together with non-perturbative lattice results.
A. Scheme independent series expansion of $\gamma_{\text{IR}}$

A series expansion of the anomalous dimension of a fermion bilinear, composed of fermions in the representation $R$, at an IR fixed point using the scheme-independent variable $\Delta_{N_R} \equiv (N_{R}^{\text{AF}} - N_R)$ has been proposed by Ryttov [69] to write

$$\gamma_{\text{IR}}(\Delta_{N_R}) = \sum_{\ell=1}^{\infty} c_{\ell}(\Delta_{N_R})^{\ell}. \quad (16)$$

The coefficients of each term are clearly scheme-independent because the anomalous dimension in the left-handed side is physical, scheme-independent, and $N_{R}^{\text{AF}}$ is defined from the scheme-independent one-loop beta function Eq. (4). Furthermore it has been shown that the $\ell$-th order coefficient $c_{\ell}$ depends only on the coefficients of the beta function and the anomalous dimension to the $(\ell + 1)$- and $\ell$-th loops, evaluated at $\Delta_{N_R} = 0$, respectively. Namely, there are no higher-loop corrections to the coefficient $c_{\ell}$, though the coefficient is scheme-independent.

To determine the coefficients $c_{\ell}$ in Eq. (16) we first note that the coupling at the IR fixed point can be expanded as

$$\frac{\alpha_{\text{IR}}}{4\pi} = \sum_{j=1}^{\infty} a_j(\Delta_{N_R})^j. \quad (17)$$

Using this, we expand the anomalous dimension $\gamma_{\text{IR}}$ as

$$\gamma_{\text{IR}}(\Delta_{N_R}) = \sum_{i=1}^{\infty} k_i \left( \frac{\alpha_{\text{IR}}}{4\pi} \right)^i = \sum_{i=1}^{\infty} k_i \left( \sum_{j=1}^{\infty} a_j \Delta_{N_R}^j \right)^i. \quad (18)$$

Similarly, the beta function is expanded as

$$\beta_{\text{IR}}(\Delta_{N_R}) = -8\pi \sum_{i=1}^{\infty} b_i \left( \frac{\alpha_{\text{IR}}}{4\pi} \right)^{i+1} = -8\pi \sum_{i=1}^{\infty} b_i \left( \sum_{j=1}^{\infty} a_j \Delta_{N_R}^j \right)^{i+1}$$

$$= \sum_{\ell=2}^{\infty} d_{\ell} \Delta_{N_R}^{\ell} = 0. \quad (19)$$

Since the coefficients of the beta function $b_i$ depend on $N_R$, we need to expand them in powers of $\Delta_{N_R}$ to find $d_{i}$'s:

$$b_i(\Delta_{N_R}) = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n \frac{\partial^n b_i}{\partial N_R^n} \bigg|_{N_R = N_R^{AF}} \Delta_{N_R}^n. \quad (20)$$
As $\Delta_{N_R}$ is an arbitrary positive number less than $N_R^{AF}$, the coefficients $d_i$’s can be read off to find

\begin{align*}
d_2 &= 0, \\
d_3 &= -8\pi \left[ -a_1^2 \frac{\partial b_1}{\partial N_R} + a_1^3 b_2 \right] \bigg|_{N_R = N_R^{AF}}, \\
d_4 &= -8\pi \left[ -2a_1 a_2 \frac{\partial b_1}{\partial N_R} - a_1^3 \frac{\partial b_2}{\partial N_R} + 3a_1^2 a_2 b_2 + a_1^4 b_3 \right] \bigg|_{N_R = N_R^{AF}}, \\
d_5 &= -8\pi \left[ - (2a_1 a_3 + a_2^2) \frac{\partial b_1}{\partial N_R} + 3 \left( a_1 a_2^2 + a_1^2 a_3 \right) b_2 ight. \\
&\quad \left. - 3a_1^2 a_2^2 \frac{\partial b_2}{\partial N_R} - a_1^4 \frac{\partial b_3}{\partial N_R} + 4a_1^3 a_2 b_3 + a_1^5 b_4 \right] \bigg|_{N_R = N_R^{AF}}, \\
\cdots,
\end{align*}

where we use the properties that $b_1$ and $b_2$ have the terms proportional to a constant and $N_R$ only, and the one-loop coefficient $b_1$ is zero at $N_R = N_R^{AF}$. Now, since the beta function at the IR fixed point vanishes for any $\Delta_{N_R}$, all $d_i$’s are identically zero and we obtain the coefficients of $\alpha_{1\text{IR}}$, $a_i$, from Eq. (21)

\begin{align*}
a_1 &= \frac{1}{b_2} \frac{\partial b_1}{\partial N_R} \bigg|_{N_R = N_R^{AF}}, \\
a_2 &= \frac{1}{b_2^2} \left( \frac{\partial b_1}{\partial N_R} \right) \left[ b_2 \frac{\partial b_2}{\partial N_R} - \frac{\partial b_1}{\partial N_R} b_3 \right] \bigg|_{N_R = N_R^{AF}}, \\
a_3 &= \frac{1}{b_2^2} \frac{\partial b_1}{\partial N_R} \left[ b_2^2 \left( \frac{\partial b_2}{\partial N_R} \right)^2 + 2b_3^2 \left( \frac{\partial b_1}{\partial N_R} \right)^2 - 3b_2 b_3 \frac{\partial b_1}{\partial N_R} \frac{\partial b_2}{\partial N_R} ight. \\
&\quad \left. + b_2^2 \frac{\partial b_1}{\partial N_R} \frac{\partial b_3}{\partial N_R} - b_2 b_4 \left( \frac{\partial b_1}{\partial N_R} \right)^2 \right] \bigg|_{N_R = N_R^{AF}}, \\
\cdots.
\end{align*}

(22)

Once $a_i$’s are given, we can similarly determine the coefficients $c_i$’s from Eq. (18)

\begin{align*}
c_1 &= a_1 k_1 \bigg|_{N_R = N_R^{AF}}, \\
c_2 &= \left[ a_2 k_1 + a_1^2 k_2 \right] \bigg|_{N_R = N_R^{AF}}, \\
c_3 &= \left[ a_3 k_1 + 2a_1 a_2 k_2 - a_1^2 \frac{\partial k_2}{\partial N_R} + a_1^3 k_3 \right] \bigg|_{N_R = N_R^{AF}}, \\
\cdots,
\end{align*}

(23)

where we used the fact that $\frac{\partial k_2}{\partial N_R} = 0$ ($k_1$ is a constant).
B. Scheme (in)dependence of the critical condition $\gamma_{CC}$

As discussed in section II D, two different forms of critical condition, $\gamma_{IR} = 1$ and $\gamma_{IR}(2 - \gamma_{IR}) = 1$, are of course same in full theory. However, they can differ and give different conformal windows, if the critical conditions are evaluated at a finite order in perturbative expansion. Furthermore, the anomalous dimension expanded in the gauge coupling $\alpha$ depends on the renormalization scheme when higher-loop order terms ($\ell \geq 3$) are considered, and thus the critical condition also involves the scheme-dependence. However, using the scheme-independent series expansion of $\gamma_{IR}$ discussed in the previous section, we could develop scheme-independent critical conditions. Below, we discuss these issues in an examplified case of $SU(3)$ gauge theory with $N_f$ Dirac fermions in the fundamental representation.

To see this we first consider the scheme-dependent loop-expansion of the anomalous dimension up to the 4-loop,

$$\gamma_{IR}(\alpha_{IR})(= 1) \approx \sum_{i=1}^{n} k_i \left( \frac{\alpha_{IR}}{4\pi} \right)^i,$$

where $n = 1, 2, 3$ and 4. The values of the coefficient $k_i$, as well as $b_i$ in Eq. 19 have been computed in the modified minimal subtraction (MS) scheme [88, 89], in the modified regularization invariant (RI') scheme [90], and the minimal momentum subtraction (mMOM) scheme [91]. Note that one needs the coefficients $b_i$’s to obtain the values of $\alpha_{IR}$ to the same order. Accordingly, the critical condition $\gamma_{IR}(2 - \gamma_{IR}) = 1$ gives at each order

$$2k_1 \left( \frac{\alpha_{IR}}{4\pi} \right) + (2k_2 - k_1^2) \left( \frac{\alpha_{IR}}{4\pi} \right)^2 = 1,$$

for the 2-loop,

$$2k_1 \left( \frac{\alpha_{IR}}{4\pi} \right) + (2k_2 - k_1^2) \left( \frac{\alpha_{IR}}{4\pi} \right)^2 + (2k_3 - 2k_1k_2) \left( \frac{\alpha_{IR}}{4\pi} \right)^3 = 1,$$

for the 3-loop, and

$$2k_1 \left( \frac{\alpha_{IR}}{4\pi} \right) + (2k_2 - k_1^2) \left( \frac{\alpha_{IR}}{4\pi} \right)^2 + (2k_3 - 2k_1k_2) \left( \frac{\alpha_{IR}}{4\pi} \right)^3 + (2k_4 - 2k_1k_3 - k_2^2) \left( \frac{\alpha_{IR}}{4\pi} \right)^4 = 1,$$

for the 4-loop.

Using the these critical conditions, we obtain the lower boundaries of the conformal window in the above three different schemes, MS, RI', and mMOM, for the SU(3) gauge
FIG. 1. The lower-bound of the conformal window in SU(3) gauge theory with $N_f$ fermions in the fundamental representation using the critical condition $\gamma_{\text{CC}}$ evaluated at $\ell$-th loop order. The green dashed line in both panels denotes the lower-bound from the 2-loop beta function in Eq. [8]. In the left panel we present the lower bounds of the conformal window obtained from the critical condition $\gamma_{\text{IR}} = 1$, and the right panel we present the results from the condition $\gamma_{\text{IR}}(2 - \gamma_{\text{IR}}) = 1$. The blue circle is for $\overline{\text{MS}}$, the red triangle for $\text{RI}'$, and the black square for mMOM schemes.

theory with $N_f$ Dirac fermions in the fundamental representation at 2-loop, 3-loop and 4-loop orders, separately. The results are shown in Fig. 1. As seen in the figures, the two different conditions, as well as the different schemes, give different results on the conformal window. Note that in some cases at 3- and 4-loop orders we could not even find reasonable values of $N_c^f$. For comparison we also present the result obtained by the 2-loop method in Eq. [8] by green dashed line.

Next, applying the scheme-independent expression of the anomalous dimension at the $n$-th order in $\Delta N_f$, defined in Eq. (16), to the critical condition of $\gamma_{\text{IR}} = 1$, we obtain

$$\sum_{i=1}^{n} c_i (\Delta N_f)^i = 1.$$  \hspace{1cm} (28)

For the condition of $\gamma_{\text{IR}}(2 - \gamma_{\text{IR}}) = 1$, we find

$$2c_1 (\Delta N_f) + (2c_2 - c_1^2) (\Delta N_f)^2 = 1,$$  \hspace{1cm} (29)

at the second order,

$$2c_1 (\Delta N_f) + (2c_2 - c_1^2) (\Delta N_f)^2 + (2c_3 - 2c_1 c_2) (\Delta N_f)^3 = 1,$$  \hspace{1cm} (30)
2c_1 (\Delta N_f) + (2c_2 - c_1^2) (\Delta N_f)^2 + (2c_3 - 2c_1c_2) (\Delta N_f)^3 + (2c_4 - 2c_1c_3 - c_2^2) (\Delta N_f)^4 = 1, \quad (31)

at the third order, etc. These conditions are clearly scheme-independent at each order, since the coefficients c_i’s and \Delta N_f are invariant under the change of schemes. Therefore, we can determine the physical and scheme-independent lower edge of the conformal window in perturbation of the critical conditions, \gamma_{\text{IR}} and \gamma_{\text{IR}}(2 - \gamma_{\text{IR}}), expanded in powers of \Delta N_f. Using the values of the coefficient c_i to i = 4 computed in Ref. [73], we determine the lower boundaries of the conformal window for SU(3) gauge theory with N_f fundamental fermions at different orders in \Delta N_f. As seen in Fig. 2, the \gamma_{\text{IR}}(2 - \gamma_{\text{IR}}) = 1 condition shows much better convergence than the \gamma_{\text{IR}} = 1 condition, and the resulting values are largely consistent with those estimated from the scheme-dependent calculations at 3-rd and 4-th loop orders in Fig. 1. We find that such a behavior persists to all the other cases considered in this work. We therefore use the scheme-independent critical condition \gamma_{\text{IR}}(2 - \gamma_{\text{IR}}) = 1 for the definition of \gamma_{\text{CC}}.

Compared to other analytical approaches, our results of N_f^c computed at \( (\Delta N_f)^\ell \) with \( \ell = 2, 3, 4 \) are placed in between those from SD and 2-loop, BF methods. We present the resulting values in Table I. We also list the values for SU(2) gauge theories with N_f fundamental Dirac flavors, where similar trends are found. Nonperturbative lattice results on the conformal window for fundamental SU(2) and SU(3) theories are not settled down, yet. In the case of SU(2), the lattice results indicate that N_f = 6 is likely at the boundary of conformal window (e.g. see [92] and references therein). For SU(3), N_f = 12 is likely to be inside the conformal window, though still controversial, N_f = 8 is below the conformal window, but close enough so that it exhibits very different IR behaviors compared to QCD, and N_f = 10 is largely unknown. (See [44, 93] and references therein.) These lattice results are largely consistent with various analytical approaches except the SD method as shown in Table I. In particular, the \gamma_{\text{CC}} method predicts that N_f = 9 \sim 10 and 6 are likely at the boundary of the conformal window in SU(3) and SU(2) theories, respectively.
FIG. 2. Scheme-independent lower conformal-boundaries of SU(3) gauge theory with $N_f$ fundamental fermions calculated from the series expansion in $\Delta_{N_f}$ truncated at the order of $\ell = 2, 3, 4$. The red square is from the condition $\gamma_{\text{IR}}(2 - \gamma_{\text{IR}}) = 1$, while the green circle is from the $\gamma_{\text{IR}} = 1$ condition.

| Gauge group | 2-loop ($b_1 = 0$) | SD | BF($\gamma = 2$) | $\gamma_{\text{CC}} (\Delta_{N_f}^2)$ | $\gamma_{\text{CC}} (\Delta_{N_f}^3)$ | $\gamma_{\text{CC}} (\Delta_{N_f}^4)$ |
|-------------|-------------------|----|-----------------|---------------------------------|---------------------------------|---------------------------------|
| $SU(2)$     | 5.55              | 8  | 5.5             | 5.69                            | 5.82                            | 6.22                            |
| $SU(3)$     | 8.05              | 11.2 | 8.25           | 9.2                             | 9.4                             | 9.8                             |

TABLE I. Comparison of the lower bounds of conformal window on the number of fundamental flavors $N_f$, determined from various analytical approaches for $SU(2)$ and $SU(3)$ gauge groups: 2-loop denotes the two-loop beta function analysis, SD denotes the (traditional) Schwinger-Dyson analysis, BF denotes the unitarity bound on the all-orders beta function, and $\gamma_{\text{CC}} (\Delta_{N_f}^\ell)$ denotes the scheme-independent analysis of the critical anomalous dimension, expanded up to $\ell$-th order in perturbation.

C. Scheme-independent critical conditions for multiple representations

As explained in section II, the upper-bound of the conformal window in multiple representations does not uniquely determine the critical number of flavors, $N_{R}^{AF}$. Instead they span a hyper-surface of co-dimension one in the space of representations. So, we here consider the conformal window for a certain representation, while keeping the number of flavors in other representations fixed. For a two-representation case, widely used in the composite Higgs model, the upper bound of its conformal window is defined by pairs of flavor numbers
\((N_R, N_{R'})\) of fermions in two different representations. From the one-loop beta function Eq. 2 for a given number of fermions in the representation \(R'\), the maximum number of flavors in the representation \(R\) is given as

\[
N_{R}^\text{AF} = \frac{11C_2(G) - 4N_{R'}T(R')}{4T(R)}.
\] (32)

Analogous to the case of a single representation, we then calculate the lower-bound from the scheme-independent critical condition by expanding the anomalous dimension \(\gamma_{\text{IR}}\) in powers of \(\Delta_{N_R} \equiv N_{R}^\text{AF} - N_R\),

\[
\gamma_{\text{IR}}(\Delta_{N_R}) = \sum_{\ell=1}^{\infty} C_\ell(R, R')(\Delta_{N_R})^\ell.
\] (33)

The coefficients \(C_\ell(R, R')\) have been computed to the 3-rd order in Ref. [75] using the known pertubative results of the beta function and the anomalous dimension for the multiple representations calculated up to the four-loop order in \(\overline{\text{MS}}\) scheme [94].

IV. APPLICATIONS TO TWO-REPRESENTATION COMPOSITE HIGGS MODELS

We now turn our attention to the determination of conformal windows in 4-dimensional gauge theories with fermion matter fields in the two distinct representations relevant to composite Higgs and partial compositeness. The wish list of the underlying gauge models was first proposed in [16] and further refined in [17, 76], resulting in the most promising 12 models. Some of these models share the same gauge group and the same representations, but the details of the symmetry breaking patterns and/or the charge assignment under the non-anomalous \(U(1)\) symmetry are different. Since we are interested in the possible extension of these models towards the conformal window, we rather classify them according to the gauge group: \(SO(7), SO(9)\) with fermions in the real fundamental and spinorial representations, \(SO(11)\) with fermions in the real fundamental and pseudo-real spinorial representations, \(SO(10)\) with fermions in the real fundamental and complex (chiral) spinorial representations, \(Sp(4)\) with fermions in the pseudo-real fundamental and real two-index antisymmetric representations, \(SU(4)\) with fermions in the complex fundamental and real two-index antisymmetric representations, and \(SU(5)\) with fermions in the complex fundamental and two-index antisymmetric representations. In Ref. [95] another type of UV
complete composite Higgs models with fermion partial compositeness based on $Sp(4)$ gauge theories with 6 antisymmetric and 12 fundamental Weyl flavors was considered, which will be denoted by CVZ in this work.

In phenomenological two-representation composite Higgs models the global symmetries are spontaneously broken by fermion condensates at the scale of $\Lambda_{HC}$, where part of pNGBs are identified by SM-like complex Higgs doublets. However, a partial compositeness prefers the gauge theories to be either conformal or near-conformal such that the baryonic operators and the SM quarks are linearly coupled for a wide range of energy scale, between $\Lambda_{HC}$ and $\Lambda_{F}$. This situation can simply be realized by introducing additional fermions which decouple at $\Lambda_{HC}$, and the extended gauge models eventually fall into the chirally broken phase with the expected symmetry breakings of the original models. Although in principle the scaling dimension of the baryonic operators can take any value between the classical dimension of $9/2$ and the unitary bound of $3/2$, the most natural value considered for the top-partner is $\sim 5/2$ so that the size of the linear coupling is the order of unity, $O(1)$. Note that in this work we do not discuss the phenomenological aspects of near conformal dynamics for the composite Higgs and partial compositeness, but instead we estimate the phase boundary of the conformal window which would be useful to provide a guidance for more dedicated nonpertubative studies on the IR dynamics.

In Table II we summarize our findings of the minimal (integer) numbers of fermion flavors required for the aforementioned gauge groups to be in the conformal window. We report the results as pairs of integer numbers for the two distinct representations. In the first column we also present the corresponding model names introduced in Ref. [76]. As we discussed in the previous section, our choice for $\gamma_{CC}$ is $\gamma_{IR}(2 - \gamma_{IR}) = 1$, which provides better convergence than $\gamma_{IR} = 1$ even for the considered cases of two representations. Since we only consider an extension of the partial composite Higgs models, we exclude the cases of which either of $N_{\psi}$ and $N_{\chi}$ is smaller than any of the values considered in the original models. Following the discussion in Sec. IID we determine the phase boundary of the conformal transition when one of the representations first reaches the critical condition. Let us denote the corresponding representation by $R_{\chi}$ for the moment. We found that with the above restrictions on $(N_{\psi}, N_{\chi})$ the effective theories below which $N_{\chi}$ fermions are integrated out always develop non-zero fermion condensates of $\bar{\psi}$, i.e. $\gamma_{\bar{\psi}\psi}(2 - \gamma_{\bar{\psi}\psi}) > 1$, and thus no question arises to use the $\gamma_{CC}$ on $\gamma_{IR}(R_{\chi})$ for the determination of the conformal
TABLE II. Pairs of minimum integer values for the numbers of flavors in two distinct representations for a given gauge group $G$ considered in the pNGB composite Higgs models to be in the conformal window. Note that $F$, $A_2$, and $Sp$ denote for the fundamental, the two-index antisymmetric and the spinorial irreducible representations, respectively, where a bar notation stands for the complex conjugate. In the first column we present the relevant models found in Refs. [76, 95].

In gauge theories with two different representations scheme-independent calculations of $\gamma_{\bar{\chi}\chi}$ at a conformal IR fixed point are known to the cubic order in $\Delta N_\chi$ [75]. The coefficients in the scheme-independent series expansions are functions of group invariants as well as the numbers of flavors in both the representations. In Appendix A we present some relevant group theoretical quantities. Here, we show the results of the critical numbers of flavors obtained by applying $\gamma_{CC}$ to the highest order of $(\Delta N_\chi)^3$. The results obtained at lower orders of $\Delta N_\chi$ and $(\Delta N_\chi)^2$ are shown in Appendix B.

In Figs. 3–5 we present the estimation of the conformal window in the considered two-representation gauge theories. The upper and lower bounds of the shaded region are obtained by using Eq. 32 and the critical condition $\gamma_{CC}$, respectively. For comparison we also show the lower bounds estimated from other analytical approaches, where green, red and black dashed lines are for 2-loop, BF and SD methods. In the case of $Sp(4)$ gauge
FIG. 3. Estimation of the conformal window in $Sp(4)$ gauge theories containing $N_f$ fundamental and $n_f$ antisymmetric flavors. The upper bound the shaded region is associated with the lost of asymptotic freedom, while the lower bound is determined by the critical condition $\gamma_{CC}$. For comparison we also present the lower bounds of the conformal window estimated by other analytical methods: black, red and green dashed lines are for SD, BF, and 2-loop results, respectively. M5, M8 and CVZ models are denoted by blue circle, red diamond and black square, respectively.

In the left and right panels of Fig. 4 we present the results for $SU(4)$ and $SU(5)$ gauge theories containing $N_f$ fundamental ($\psi$) and $n_f$ two-index antisymmetric ($\chi$) fermions, respectively. Blue circles are for the models M6 and M12, while red diamond and black square are for the model M11 and the lattice $SU(4)$ model considered in Refs. \[22, 27\]. As seen in the figure, all the models are outside the conformal window. In particular, the lattice $SU(4)$ model which contains $N_f = 2$ fundamental and $n_f = 2$ antisymmetric Dirac fermions is deep inside the chirally broken phase, which is consistent with the fact that numerical results showed the nonperturbative features of confinement and (spontaneous) global symmetry breaking \[22, 24\].

In Fig. 5 from left-top to right-bottom panels, we show the estimated conformal window
FIG. 4. Estimation of the conformal window in $SU(4)$ (left) and $SU(5)$ (right) gauge theories containing $N_f$ fundamental and $n_f$ antisymmetric flavors. The upper bound of the shaded region is associated with the loss of asymptotic freedom, while the lower bound is determined by the critical condition $\gamma_{\text{CC}}$. For comparison we also present the lower bounds of the conformal window estimated by other analytical methods: black, red and green dashed lines are for SD, BF, and 2-loop results, respectively. M6 and M12 models are denoted by blue circles, M11 by red diamond, and the lattice $SU(4)$ model by black square.

for $SO(7)$, $SO(9)$, $SO(10)$ and $SO(11)$ gauge theories containing $N_f$ fundamental ($\psi$) and $n_f$ spinorial ($\chi$) representations. Note that for $SO(7)$ we found that the roles of $\psi$ and $\chi$ are interchanged. Although we only learn this fact a posteriori since full IR dynamics are encoded in $\gamma_{\bar{\psi}\psi}$ and $\gamma_{\bar{\chi}\chi}$ in a complicated way, we can obtain some clues from the ladder approximation for multiple representations. As discussed in Sec. II B, if we depart from the conformal window we first reach the critical condition associated with the representation having the largest value of the quadratic Casimir operator. In the case of $SO(7)$ we find that $C_2(\text{Sp}) < C_2(\mathbf{F})$, while in the other cases $C_2(\mathbf{F}) < C_2(\text{Sp})$. In Fig. 5 blue circles denote the models M1, M2, M7, M9, and red diamonds denote the models M3, M4 and M10. All the models are outside the conformal window.
FIG. 5. Estimation of the conformal window in $SO(7)$ (top-left), $SO(9)$ (top-right), $SO(10)$ (bottom-left), $SO(11)$ (bottom-right) gauge theories containing $N_f$ fundamental and $n_f$ spinorial flavors. The upper bound of the shaded region is associated with the lost of asymptotic freedom, while the lower bound is determined by the critical condition $\gamma_{CC}$. For comparison we also present the lower bounds of the conformal window estimated by other analytical methods: black, red and green dashed lines are for SD, BF, and 2-loop results, respectively. M1, M2, M7, M9 models are denoted by blue circles, while M3, M4, M10 are denoted by red diamonds.

V. CONCLUSION

We have proposed an analytical approach to determine the lower edge of the conformal window in a scheme-independent way by combining the conjectured critical condition on the anomalous dimension of a fermion bilinear, $\gamma_{IR} = 1$, which is responsible for the chiral
phase transition, and the scheme-independent series expansion of $\gamma_{\text{IR}}$ at a conformal IR fixed point with respect to $\Delta_{N_f} = (N^A_{f} - N_f)$. If all orders in the perturbative expansion are considered, this critical condition is identical to $\gamma_{\text{IR}}(2 - \gamma_{\text{IR}}) = 1$, which is obtained from the Schwinger-Dyson analysis in the ladder approximation along with some working assumptions. However, at finite order they yield different values for the critical number of flavors $N^c_f$ on the boundary of conformal and chirally broken phases. And it turns out that the latter shows much better convergence in the series expansion, while the resulting values obtained from both critical conditions approach to each other as we include higher order terms.

In the illustrative examples of $SU(2)$ and $SU(3)$ gauge theories with fundamental Dirac fermions we have determined $N^c_f$ using the scheme-independent critical condition on $\gamma_{\text{IR}}$ up to $\mathcal{O}(\Delta_{N_f}^4)$, and compared to other analytical calculations. We find that the resulting values are larger than those estimated from the vanishing 2-loop coefficients and the all-order beta function with $\gamma_{\text{IR}} = 2$, but smaller than those from the traditional Schwinger-Dyson analysis in the ladder approximation. We also find that our values are largely consistent with the lattice results in the literature.

We have extended the method of $\gamma_{\text{CC}}$ to the case of fermions in the $k$ different representations, where the critical $(k - 1)$-dimensional surface can be determined by the representation $R_1$ first delivering $\gamma_{\text{IR}}(R_1)$ to the unity. Here, we assume that fermions in the representations other than $R_1$ develop non-zero fermion condensates if do the ones in $R_1$. We have applied this method to the gauge theories containing fermionic matter fields in the two distinct representations relevant to the models of composite Higgs and partial compositeness, and estimated the critical numbers of flavors $(N^c_f, n^c_f)$ in the two dimensional space of $N_f$ and $n_f$ using $\gamma_{\text{IR}}(R_1)$ at the 3-rd order in $\Delta_{n_f}$. We find that all the considered partial composite Higgs models are in the chirally broken phase, while the CVZ model resides slightly inside the conformal window so that it is highly expected to have a large anomalous dimension of composite operators. While some of them are deep inside the broken phase (even below the 2-loop estimation), some are relatively close to the conformal window, such as M5, M8, M9, M10 and M12 models.

Recent nonperturbative lattice studies on $SU(3)$ gauge theory containing fermions of 8 fundamental flavors found very different features of IR dynamics, e.g. light scalar resonances, compared to QCD-like theories. Such findings may reflect the near-conformal dynamics.
Note that, according to the analytical results presented in Table I, the 8-flavor SU(3) model is slightly below all the estimates. Although we cannot simply generalize this specific case to generic multi-representation gauge theories, some of the partial composite Higgs models mentioned above can be good candidates for near-conformal theories. Hence, it would be encouraging to investigate such models in further details by means of nonperturbative lattice calculations.

ACKNOWLEDGMENTS

The authors would like to thank G. Cacciapaglia, G. Ferretti, V. Leino and M. Piai for useful discussions. The work of JWL and BSK is supported in part by the National Research Foundation of Korea grant funded by the Korea government (MSIT) (NRF-2018R1C1B3001379). The work of JWL and DKH is supported in part by Korea Research Fellowship program funded by the Ministry of Science, ICT and Future Planning through the National Research Foundation of Korea (2016H1D3A1909283). The work of DKH is also supported in part by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2017R1D1A1B06033701).

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| SO(N)                  | $d_R$  | $T(R)$ | $C_2(R)$    | $I_4(R)$ |
|-----------------------|--------|--------|-------------|----------|
| Fundamental           | $N$    | 1      | $\frac{N-1}{2}$ | 1        |
| Chiral spinor (even N)| $2^{\frac{N-2}{2}}$ | $2^{\frac{N-8}{2}}$ | $\frac{N(N-1)}{16}$ | $-2^{\frac{N-10}{2}}$ |
| Real spinor (odd N)   | $2^{\frac{N+1}{2}}$ | $2^{\frac{N-7}{2}}$ | $\frac{N(N-1)}{16}$ | $-2^{\frac{N-9}{2}}$ |
| Adjoint               | $\frac{N(N-1)}{2}$ | $N-2$  | $N-2$       | $N-8$    |
| Rank-2 symmetric      | $\frac{(N-1)(N+2)}{2}$ | $N+2$  | $N$         | $N+8$    |

TABLE III. Group invariants for various representations in $SO(N)$ gauge group.

| SU(N)                  | $d_R$  | $T(R)$ | $C_2(R)$    | $I_4(R)$ |
|-----------------------|--------|--------|-------------|----------|
| Fundamental           | $N$    | $\frac{1}{2}$ | $\frac{N^2-1}{2N}$ | 1        |
| Adjoint               | $N^2 - 1$ | $N$   | $N$         | $2N$     |
| Rank-2 symmetric      | $\frac{N(N+1)}{2}$ | $N+2$ | $\frac{(N-1)(N+2)}{N}$ | $N+8$    |
| Rank-2 antisymmetric  | $\frac{N(N-1)}{2}$ | $N-2$ | $\frac{(N+1)(N-2)}{N}$ | $N-8$    |

TABLE IV. Group invariants for various representations in $SU(N)$ gauge group.

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Appendix A: Group invariants

In this appendix, we summarize the group invariants required to calculate the coefficients $C_\ell(R, R')$ in Eq. [33] for the scheme-independent series expansions of $\gamma_{IR}(\Delta N_R)$ to $\ell = 3$ in the cases of two different representations. As the coefficients for $\ell \geq 3$ involve four-loop results of the RG beta function, we need the group-invariant products of four-index quantities such as $d_R^{abcd}d_{R'}^{abed}/d_A$ in addition to the trace normalization factor $T(R) (T(R'))$ and the eigenvalues of the quadratic Casimir operator $C_2(R) (C_2(R'))$. Here, we denote $A$
Sp(N)

$\frac{d}{dA} R^C (R) I_4(R)$

| Fundamental   | $N$ | $\frac{1}{2}$ | $\frac{N+1}{4}$ | 1 |
|---------------|-----|---------------|-----------------|----|
| Adjoint       | $\frac{N(N+1)}{2}$ | $\frac{N+2}{2}$ | $\frac{N+2}{2}$ | $N+8$ |
| Rank-2 antisymmetric | $\frac{(N+1)(N-2)}{2}$ | $\frac{N-2}{2}$ | $\frac{N}{2}$ | $N-8$ |

TABLE V. Group invariants for various representations in $Sp(N)$ gauge group.

| $d_{abcd} d_{abcd}' / d_A$ | SO(N) | SU(N) | Sp(N) |
|-----------------------------|-------|-------|-------|
| $(d_A-1)(d_A-3)$            | $\frac{(d_A-3)(d_A-8)}{96(d_A+2)}$ | $\frac{(d_A-1)(d_A-3)}{192(d_A+2)}$ |

TABLE VI. Values of $d_{abcd} d_{abcd}' / d_A$ for $SO(N)$, $SU(N)$, and $Sp(N)$ gauge groups.

for the adjoint representation of gauge group $G$ and $d_A$ for its dimension. For the details discussed below, we refer the reader to Refs. [74, 75] and references therein.

For a given representation $R$ the totally symmetric four-index quantity is defined as

$$d_{abcd}^R = \frac{1}{3!} \text{Tr} \left[ T^a \left( T^b T^c T^d + T^b T^d T^c + T^c T^b T^d + T^d T^b T^c + T^d T^c T^b \right) \right], \quad (A1)$$

where $T^a$ is the generators in $R$. In terms of group invariants this can be rewritten as

$$d_{abcd}^R = I_{4,R} d_{abcd} + \left( \frac{T(R)}{d_A+2} \right) \left( C_2(R) - \frac{1}{6} C_2(A) \right) \left( \delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} \right). \quad (A2)$$

Here, $d_{abcd}$ is a traceless tensor, satisfying $\delta_{ab} d_{abcd} = 0$, etc., which only depends on the group $G$. $I_{4,R}$ is a quartic group invariant. We list the values of the group invariants $d_A$, $T(R)$, $C_2(R)$, $I_{4,R}$ for the relevant representations in Tables III, IV and V for $SO(N)$, $SU(N)$ and $Sp(N)$, respectively.

Using the expression for $d_{abcd}^R$ in Eq. (A2), one can obtain

$$\frac{d_{abcd}^R d_{abcd}'^R}{d_A} = I_{4,R} I_{4,R'} d_{abcd} d_{abcd}' / d_A$$

$$+ \left( \frac{3}{d_A+2} \right) T(R) T(R') \left( C_2(R) - \frac{1}{6} C_2(A) \right) \left( C_2(R') - \frac{1}{6} C_2(A) \right). \quad (A3)$$

The gauge invariant products of $d_{abcd}$ in the first term are independent on the representation. In Table VI we present the results for $SO(N)$, $SU(N)$ and $Sp(N)$ gauge groups [77]. Finally, we present the resulting values of $d_{abcd}^R d_{abcd}'^R / d_A$ for $SO(N)$, $SU(N)$, and $Sp(N)$ gauge groups in Tables VII and VIII. Note that in the tables we only present the results of the two different
TABLE VIII. Values of $A_{ab}^{abcd}d_{F}^{abcd}/d_{A}$ in SO(N) gauge groups with $N \geq 3$. We denote $F$, $Sp$, and $A$ for fundamental, spinor, adjoint representations.

| $d_{F}^{abcd}d_{F}^{abcd}/d_{A}$ | SO(N) (even N) | SO(N) (odd N) |
|---------------------------------|----------------|----------------|
| $\frac{1}{24}(N^2 - N + 4)$    |                |                |
| $\frac{1}{3}2^{N-15}(13N^2 - 61N + 76)$ |                |                |
| $\frac{1}{2}(N - 2)$ (N$^3 - 15N^2 + 138N - 296$) |                |                |
| $\frac{1}{24}(N - 2)$ (N$^2 - 7N + 22$) |                |                |
| $\frac{1}{3}2^{N-15}(N^2 - 7N + 7)$ |                |                |
| $\frac{1}{2}N^2 - 24N^2 + 96N - 104$ |                |                |

TABLE VII. Values of $F_{abcd}d_{Sp}^{abcd}/d_{A}$ in SO(N) gauge groups with $N \geq 3$. We denote $F$, $Sp$, and $A$ for fundamental, spinor, adjoint representations.

| $d_{Sp}^{abcd}d_{Sp}^{abcd}/d_{A}$ | SU(N) | $Sp(N)$ |
|-------------------------------|-------|---------|
| $\frac{N^3 - 6N^2 + 18}{96N^2}$ | $\frac{1}{384}(N^2 + N + 4)$ |        |
| $\frac{(N - 2)(N^5 - 14N^4 + 72N^3 + 48N^2 - 288N - 576)}{96N^2}$ | $\frac{1}{384}(N - 2)(N^3 - 13N^2 + 110N - 104)$ |        |
| $\frac{N^3 - 8N^4 + 6N^3 + 48N^2 - 144}{96N^2}$ | $\frac{1}{384}(-20 + 20N - 7N^2 + N^3)$ |        |
| $\frac{1}{24}N^2 (N^2 + 36)$ | $\frac{1}{384}(N + 2)(N^3 + 15N^2 + 138N + 296)$ |        |
| $\frac{1}{48}N (N^2 + 6)$ | $\frac{1}{384}(N + 2)(N^2 + 7N + 22)$ |        |
| $\frac{1}{48}N(n - 2)(N^2 - 6N + 24)$ | $\frac{1}{384}(N + 2)(N - 2)(N^2 + N + 28)$ |        |

Appendix B: Results on $\gamma$CC from lower orders in the scheme-independent series expansions

In Figs. 6, 7, and 8, we present the critical values $(N_f^c, n_f^c)$ in the plane of $N_f$ and $n_f$, by treating them as continuous variables, corresponding to the lower edge of the conformal window estimated by applying the critical condition $\gamma$CC to two-representation gauge representations relevant to the models for composite Higgs and partial compositeness. It is straightforward to obtain the results of other possibilities by using Eq. A3 and the group invariants presented in Tables III, IV, V, and VI. Part of the results are found in Ref. [74].

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FIG. 6. Estimation of the lower edge of the conformal window in $Sp(4)$ gauge theories containing $N_f$ fundamental and $n_f$ antisymmetric flavors using the critical condition $\gamma CC$ and the scheme-independent series expansions truncated at $\Delta_{n_f}$ (green), $(\Delta_{n_f})^2$ (yellow) and $(\Delta_{n_f})^3$ (blue) orders.

FIG. 7. Estimation of the lower edge of the conformal window in $SU(4)$ (left) and $SU(5)$ (right) gauge theories containing $N_f$ fundamental and $n_f$ antisymmetric flavors using the critical condition $\gamma CC$ and the scheme-independent series expansions truncated at $\Delta_{n_f}$ (green), $(\Delta_{n_f})^2$ (yellow) and $(\Delta_{n_f})^3$ (blue) orders.

groups discussed in Sec. [IV]. In the figures, green, yellow and blue solid lines denote for the results obtained from the scheme-independent series expansions truncated at $\Delta_{n_f}$, $(\Delta_{n_f})^2$ and $(\Delta_{n_f})^3$ orders, respectively.
FIG. 8. Estimation of the lower edge of the conformal window in $SO(7)$, $SO(9)$, $SO(10)$ and $SO(11)$ gauge theories (from left-top to right-bottom) containing $N_f$ fundamental and $n_f$ spinorial flavors using the critical condition $\gamma_{CC}$ and the scheme-independent series expansions truncated at $\Delta_{n_f}$ (green), $(\Delta_{n_f})^2$ (yellow) and $(\Delta_{n_f})^3$ (blue) orders.