Quantum toboggans

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Abstract

Among all the $\mathcal{PT}$–symmetric potentials defined on complex coordinate contours $C(s)$, the name “quantum toboggan” is reserved for those whose $C(s)$ winds around a singularity and lives on at least two different Riemann sheets. An enriched menu of prospective phenomenological models is then obtainable via the mere changes of variables. We pay thorough attention to the harmonic oscillator example with a fractional screening and emphasize the role of the existence and invariance of its quasi-exact states for different tobogganic $C(s)$.

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1 Introduction

In spite of an impressive mathematical universality of Quantum Mechanics, many of its physical implementations refrain merely to a quantization of a classical point particle. In particular (i.e., say, in one dimension and for a single particle), the idea of a measurability of the coordinate $x$ leads immediately to the most popular differential Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \Psi(x) + V(x) \Psi(x) = E \Psi(x) \tag{1}$$

defined over the real axis of coordinates $x \in \mathbb{R}$. In addition, whenever we need that the energy remains observable, the potential $V(x)$ is being assumed real.

Bender and Boettcher [1, 2] initiated a small revolution by conjecturing that the similar differential Schrödinger equations might remain phenomenologically useful even if the coordinate-like variable $x$ itself ceases to be observable. In the other words, they endorsed a new model-building philosophy based on a tentative transition from the real line of $x \in \mathbb{R}$ to the suitable complex contours,

$$x \in C(s) \subset \mathbb{C}, \quad s \in (-\infty, \infty). \tag{2}$$

They claimed that such a complexification of $x$ need not necessarily be in conflict with the basic principles of quantum mechanics. Quantitatively, they illustrated this possibility by the (mainly, numerical and WKB) study of spectra of a family of the $\mathcal{PT}$-invariant potentials of the generic power-law multinomial form

$$V(x) = \sum_{\beta} g_{\beta} (ix)^{\beta} \quad g_{\beta} \in \mathbb{R} \tag{3}$$

where the linear involution operator $\mathcal{P}$ represents parity while the antilinear complex-conjugation involution $\mathcal{T}$ mimics time reversal [1].

The energy spectrum appears to remain real for an unexpectedly broad class of the $\mathcal{PT}$-symmetric potentials (3). In fact, the “weakening” of the usual Hermiticity of the Hamiltonian to the mere $\mathcal{PT}$-symmetry may prove useful in formal sense. Thus, in ref. [2] the pair of complexifications (2) and (3) has been shown to broaden the class of the partially solvable (usually called quasi-exact, QE) models [3] which, in the new setting, incorporates also the quartic-oscillator family

$$V_{BB}(x) = a (ix) + b (ix)^2 + c (ix)^3 + (ix)^4.$$

In ref. [4] we noticed that the latter QE construction may find an efficient extension to the singular interactions

$$V_Z(x) = a (ix) + b (ix)^2 + c (ix)^3 + (ix)^4 + e (ix)^{-1} + f (ix)^{-2}.$$
Now we intend to proceed one step further along this line.

In section 2 we start by a brief review of the current PT-symmetric way of dealing with the singular forces. For illustrations of the most relevant technical details we pick up the decadic-oscillator model of ref. [5].

In the next section 3 we come with the idea of equivalence between different singular potentials. This map is mediated by the suitable, PT-symmetry-preserving changes of the variables which leave the form of the Schrödinger equations unchanged. This idea leads directly to our present constructive approach to the toboggans in paragraph 3.3.

For the sake of clarity, we just give full details for a very specific class of potentials (“screened harmonic oscillators”). Only in discussion in section 4 we then return back to a broader context. We mention both some peculiarities of simpler models (not so well suited for illustrating the subtleties of tobogganic structures) and some complications related to the higher-degree polynomial potentials (requiring a more or less purely numerical treatment). We also briefly mention some recent progress concerning the possible physical interpretation and applicability of the general PT-symmetric models.

Section 5 finally summarizes our present results.

2 Boundary conditions, PT-symmetric way

2.1 Illustration: Asymptotics of decadic oscillators

One of the most user-friendly guides towards complex contours (2) is provided by the oscillators (3) with max $\beta = 10$,

$$V(x) = x^{10} + \text{asymptotically smaller terms}.$$  \hfill (4)

The standard asymptotic boundary conditions on the real axis may be employed in the first step,

$$\psi(x) = e^{-x^6/6 + \text{asymptotically smaller terms}}.$$  \hfill (5)

We may re-parametrize the large values of $x = \varrho \exp i\varphi \in C(s)$ by real $\varrho \gg 1$ and angle $\varphi \in (0, 2\pi)$ in order to see that

$$\psi(x) = \exp \left[-\frac{1}{6}\varrho^6 \cos 6\varphi + \text{asymptotically less relevant terms}\right], \quad \varrho \gg 1.$$

This implies that in the limit $|x| \to \infty$ our “physical prototype” wavefunction (5) vanishes whenever the asymptotic angle $\varphi$ falls inside one of the six intervals

$$\Omega_{(first\ right)} = \left(-\frac{\pi}{2} + \frac{\pi}{12}, -\frac{\pi}{2} + \frac{3\pi}{12}\right), \quad \Omega_{(third\ right)} = \left(-\frac{\pi}{2} + \frac{5\pi}{12}, -\frac{\pi}{2} + \frac{7\pi}{12}\right), \ldots$$

2
which represent the wedges visible on the first Riemann sheet in Figure 1 (where the upwards-running cut is assumed though not indicated). In the $\mathcal{PT}$-symmetric context this observation enables us, first of all, to choose the simplest curve $C(s) \neq \mathbb{R}$ in the “minimal” manner proposed by Buslaev and Grecchi [6],

$$C_{(BG)}(s) = s - i\varepsilon, \quad \varepsilon > 0, \quad s \in \mathbb{R}. \quad (6)$$

For small $\varepsilon$ it does not deviate too much from the real axis $\mathbb{R}$.

Currently we are going to assume the analyticity of our potential (4) in the whole complex plane of $x$ admitting, possibly, just a singularity at $x = 0$ from which a cut is to be oriented upwards. Then, an almost arbitrary smooth deformation $C_{(third-third)}(s)$ of $C_{(BG)}(s)$ is admissible reflecting just the uniqueness of the analytic continuation of the wavefunction. The ends of all these deformations $C_{(third-third)}(s)$ cannot leave the original pair of the subscript-indicated asymptotic wedges of course.

A nontrivial change of the energy spectrum may be expected when a new curve $C(s) \neq C_{(third-third)}(s)$ is picked up as connecting another pair of the wedges. Once we do so in the left-right-symmetric (i.e., $\mathcal{PT}$-symmetric) manner, two new possibilities emerge in Figure 1, viz., the downwards- or upwards-bent integration contours $C_{(first-first)}(s)$ and $C_{(fifth-fifth)}(s)$, respectively. The spectra of the respective energies $E_{(first-first)}$, $E_{(third-third)}$ and $E_{(fifth-fifth)}$ will be different in general [7, 8].

A less common possibility is encountered when our ansatz (5) is replaced by the alternative prescription

$$\psi(x) = e^{+x^6/6+\text{asymptotically smaller terms}}. \quad (7)$$

The corresponding change of the menu of the eligible decadic-oscillator boundary-condition wedges is displayed, in Figure 2, on the same Riemann sheet as used in Figure 1. The two topologically nonequivalent $\mathcal{PT}$-symmetric contours $C(s)$ will connect the second or the fourth left and right wedges.

### 2.2 Quasi-exact states in the decadic model

In extensive literature on QE bound states [9], the majority of their relevant properties has already been explored. In contrast, $\mathcal{PT}$-symmetric quantum mechanics still keeps some of its secrets so that the new QE constructions are quite common there [10, 11]. Once we restrict our considerations to the polynomial version of our previous example with $x \in C_{(third-third)}(s)$,

$$-\frac{d^2}{dx^2} \varphi(x) + \frac{L(L+1)}{x^2} \varphi(x) + \left[x^{10} + g_8 x^8 + g_6 x^6 + g_4 x^4 + g_2 x^2\right] \varphi(x) = E \varphi(x), \quad (8)$$

we discover the existence of the two nonequivalent threshold components of $\varphi(x)$ near the origin,

$$\varphi(x) = c_1 \varphi_1(x) + c_2 \varphi_2(x), \quad \varphi_1(x) \sim x^{-L}, \quad \varphi_2(x) \sim x^{L+1}. \quad (9)$$
In order to circumvent this technical subtlety, we shall restrict attention to the mere half-integer angular momenta \( L \). This will enable us to construct the QE bound-state solutions to eq. (8) by the method of ref. [5] incorporating both the components (9) in a single QE ansatz at any \( L + 1/2 = M = 1, 2, \ldots \),

\[
\varphi(x) = \exp \left( -\frac{x^6}{6} - \alpha \frac{x^4}{4} - \beta \frac{x^2}{2} \right) \sum_{n=0}^{N-1} h_n x^{2n-L}, \quad \alpha = \frac{g_8}{2}, \quad \beta = \frac{g_6 - \alpha^2}{2}. \quad (10)
\]

The assumption of the existence of such a type of an “exceptional” bound-state solution(s) represents the very core of the concept of the QE solvability [3]. Tacitly, it is assumed that the degree-of-polynomiality integer parameter \( N \geq 1 \) is arbitrary and that all the non-QE normalizable bound-state solutions, whenever needed, may be also constructed by some numerical technique.

From eq. (10) one may extract all the QE-termination conditions. The first one fixes \( g_4 = g_4(M, N) = 2\alpha \beta + 2M - 4N - 2 \) and enables us to transform our prototype differential equation (8) into the following finite set of recurrences,

\[
A_n h_{n+1} + B_n h_n + C_n h_{n-1} + D_n h_{n-2} = 0, \quad n = 0, 1, \ldots, N \quad (11)
\]

with coefficients

\[
A_n = (2n + 2)(2n + 2 - 2M), \quad B_n = E - \beta (4n + 2 - 2M),
C_n = \beta^2 - g_2 - \alpha (4n - 2M), \quad D_n = 4(N + 1 - n). \quad (12)
\]

Let us merely briefly recollect some basic features of their solution.

### 2.3 Elementary QE families at the smallest \( M \)

One of the most unpleasant features of the algebraic system of \( N + 1 \) equations (11) for \( N \) coefficients \( h_n \) is that it is overcomplete and, hence, nonlinear. Fortunately, a simplification occurs whenever \( 1 \leq M \leq N - 1 \) since one of the important coefficients vanishes in such a case, \( A_{M-1} = 0 \). This means that an upper subsystem of eq. (11) acquires an \( M \)-dimensional matrix form,

\[
\begin{pmatrix}
B_0 & A_0 \\
C_1 & B_1 & A_1 \\
D_2 & C_2 & \ddots & \ddots \\
& \ddots & \ddots & B_{M-2} & A_{M-2} \\
& & D_{M-1} & C_{M-1} & B_{M-1}
\end{pmatrix}
\begin{pmatrix}
h_0 \\
h_1 \\
h_2 \\
\vdots \\
h_{M-1}
\end{pmatrix} = 0. \quad (13)
\]
Once we satisfy its secular equation

\[
\det \begin{pmatrix}
B_0 & A_0 \\
C_1 & B_1 & A_1 \\
D_2 & C_2 & B_2 & \ddots \\
& \ddots & \ddots & \ddots \\
& & D_{M-1} & C_{M-1} & B_{M-1} \\
& & & D_M & C_M & B_M & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix} = 0
\] (14)

one of the first \( M \) lines of eq. (11) may be omitted as linearly dependent. This is precisely what we need. After any such omission, all our remaining recurrences become tractable as another linear and homogeneous matrix problem, solvable by the standard numerical techniques at any finite dimension \( N \) [12].

The practical solution of our two coupled matrix equations will be significantly facilitated whenever the “smaller” eq. (13) remains solvable in closed form. Of course, the constraint (14) remains trivial and gives the unique QE-compatible energy \( E = 0 \) at \( M = 1 \). An exhaustive analysis of the related \( N \)-dimensional QE conditions has been given in ref. [5]. It has been emphasized that the QE wavefunctions form the so called Sturmian \( N \)-plets at any positive integer \( N \). The corresponding QE-compatible couplings \( g_2 \) remain real (and, hence, compatible with the physical \( \mathcal{PT} \)-symmetry requirement) in a fairly large domain of the two freely variable couplings \( g_6 \) and \( g_8 \).

Whenever we decide to omit the first linearly dependent line from eq. (11), the resulting reduced secular equation contains \( g_2 \)'s just on the main diagonal. Of course, at any \( M \geq 2 \), the apparent linearity of such a problem is lost due to the nontrivial \( g_2 \)-dependence of the roots of the “auxiliary” constraint (14). Fortunately enough, once written in the form

\[ g_2 = \frac{E^2}{4}, \quad M = 2, \]

the solution of eq. (14) remains sufficiently compact and \( N \)-independent at \( M = 2 \).

We see that with the growth of \( M \), the determination of our Sturmian \( N \)-plets of the QE-compatible energies and couplings becomes more complicated, though not at a really prohibitive rate. Even in the next step with \( M = 3 \), the formula for \( g_2 \) remains fairly similar and linear though \( N \)-dependent,

\[ g_2 = \frac{8}{E} \left( 2N - 2 - \alpha \beta \right) + \frac{E^2}{16}, \quad M = 3. \]

### 3 Transformations changing \( C'(s) \)

The key technical ingredient with which we intend to present quantum toboggans will be certain equivalence transformation between different \( \mathcal{PT} \)-symmetric models.
Let us first assume that the Schrödinger eq. (1) + (3) has the following slightly more specific, perturbative, anharmonic-oscillator form

\[ -\frac{d^2}{dx^2} - (ix)^2 + \lambda W(ix) \] \( \psi(x) = E(\lambda) \psi(x) , \) \( W(ix) = \sum g_\beta(ix)^\beta . \) (15)

Many phenomenological considerations were based on such a type of models in the past [13]. People usually assume that \( \lambda \) is “sufficiently small” so that the reliable approximations of the energies pertaining to eq. (15) may be calculated using perturbation expansions in the powers of \( \lambda \). Our present approach will be different.

3.1 Liouvillean changes of variables

Let us try to change the variables in eq. (15) in the manifestly \( \mathcal{PT} \)–symmetric manner,

\( ix = (iy)^\alpha , \) \( \psi(x) = y^\varrho \varphi(y) . \) (16)

This means that at any real exponent \( \alpha > 0 \) we have

\[ i dx = i^\alpha \alpha y^{\alpha-1} dy , \quad \frac{(iy)^{1-\alpha}}{\alpha} \frac{d}{dy} = \frac{d}{dx} . \]

Our “old” Schrödinger equation (15) acquires the “new”, mathematically equivalent form

\[ y^{1-\alpha} \frac{d}{dy} y^{1-\alpha} \frac{d}{dy} y^\varrho \varphi(y) + i^{2\alpha} \alpha^2 \left[ -(iy)^{2\alpha} + \lambda W[(iy)^\alpha] - E(\lambda) \right] y^\varrho \varphi(y) = 0 . \]

Its first term is a sum

\[ y^{1-\alpha} \frac{d}{dy} y^{1-\alpha} \frac{d}{dy} y^{[(\alpha-1)/2]} \varphi(y) = y^{2+\varrho-2\alpha} \frac{d^2}{dy^2} \varphi(y) + \varrho(\varrho - \alpha) y^{\varrho - 2\alpha} \varphi(y) , \quad \varrho = \frac{\alpha - 1}{2} \]

where the first derivatives of \( \varphi(y) \) dropped out after our choice of the value of \( \varrho \) (this idea dates back to Liouville [14]). Thus, the new Schrödinger equation preserves the standard form,

\[ -\frac{d^2}{dy^2} \varphi(y) + \alpha^2 - \frac{1}{4y^2} \varphi(y) + (iy)^{2\alpha - 2} \alpha^2 \left[ -(iy)^{2\alpha} + \lambda W[(iy)^\alpha] - E(\lambda) \right] \varphi(y) = 0 . \) (17)

It naturally contains the \( \alpha \)–dependent singularity so that it does not make any sense to insist on its absence in our initial problem (15).
3.2 The screening choice of $W(ix)$

It is well known that the requirement of a feasibility of the evaluation of the individual perturbative corrections leads to the preference of integers $\beta = 3, 4, \ldots$ [15]. In parallel, the requirement of the convergence of the infinite perturbation series may only be met when operators or functions $W$ are relatively bounded [16]. In application to our particular ansatz (15) this means that we must consider only $\beta < 2$.

One of the ways out of such a trap lies in a transition to semi-numerical strong-coupling expansions [17], a remarkable technical simplification of which is easily found to occur at the rational exponents $\beta$ [18]. This motivates our present specific choice of

$$W(ix) = a(ix)^{4/3} + b(ix)^{2/3} + c(ix)^0 + d(ix)^{-2/3} + e(ix)^{-4/3} + f(ix)^{-2}$$  \hspace{1cm} (18)$$

parameterized by the six real coupling constants. Note that the exponents $\beta$ are both rational and smaller than two here. In a phenomenological perspective, such a function seems suited to screen the original oversimplified force $x^2$ in both the short- and long-distance regimes.

Once we pick up the specific exponent $\alpha = 3$ in (16) and scaling $\lambda = 1/9$ in eq. (15), our Schrödinger equation (17) + (18) precisely coincides with the decadic oscillator prototype problem (8). This is one of our present key observations. One is only required to abbreviate $L(L + 1) \equiv 2 - f$ (or, if you wish, $L = L(f) = \sqrt{9/4 - f - 1/2}$) and to perform a series of the following identifications: $g_8 \equiv a$, $g_6 \equiv -b$, $g_4 \equiv c - E(1/9)$, $g_2 \equiv -d$ while $E \equiv -e$. All these formulae express just a freedom of the mathematical identification of our two phenomenologically different decadic and screened harmonic oscillators.

It is quite amazing that their one-to-one equivalence mapping requires just an appropriate re-interpretation of the constants. We are now, at last, prepared to discuss the particular implications of this equivalence on the related, topologically nontrivial changes of the “coordinate” contours $C(s)$.

3.3 The emergence of tobogganic contours

The introduction of a complex deformation $\mathbb{R} \rightarrow C(s)$ of the domain of $x$ is a mere trivial analytic-continuation trick for mathematicians. In the context of physics, nevertheless, it requires an explanation [19]. One may, for example, re-classify the complex coordinate $x$ as a mere auxiliary variable in a representation of the Hilbert space of states [20]. In some physical systems, one may also try to construct a suitable, usually quite complicated operator of certain quantity interpreted as an “observable of position” [21].

In both of these two situations we are entirely free to adapt the choice of $C(s)$ to our phenomenological needs emphasizing, for example, an expected impact of
the choice of the wedges upon the spectrum of the observed energies. As we have emphasized in the Introduction, the “topologically trivial” contours (which may be depicted in a single Riemann sheet) look, paradoxically, intuitively less difficult to grasp and accept. Even then, we might re-read Figures 1 and 2 as a demonstration that there exists a wealth of the non-equivalent boundary conditions even in such a simpler case. An even stronger support of such a point of view might be found in the asymptotically exponential potentials of refs. [8, 22]. With their infinitely many (sic!) alternative wedges (which are all visible on the first Riemann sheet) they represent a certain quite difficult mathematical challenge, say, when treated in the so called Stokes-geometry language [23].

The main purpose of our present paper is to analyze the contours $C(s)$ which would become topologically nontrivial. Without any real loss of generality, we may start from the the most elementary choice of $\max \beta = 2$ in our particular model (18). In such a case with the harmonic-oscillator asymptotics $\psi(x) \sim \exp(-x^2/2)$, there only exist the two “standard” harmonic-oscillator wedges which are visible on the first Riemann sheet of Figure 3. In this picture with a branch singularity at $x = 0$, the complex plane is cut upwards.

Figure 3 also displays the first example of a quantum-toboggan contour $C(s)$ which leaves the first Riemann sheet. In fact, many different quantum toboggans will share the part which is visible in Figure 3. Once these contours $C(s)$ remain $\mathcal{PT}$-symmetric, they will always connect the same wedges. Thus, in our notation we shall have $C_{(\text{third-third})}(s)$ connecting the third left with the third right wedge, etc.

In Figure 4 we return to the contour $C_{(\text{second-second})}(s)$ which is derived from the alternative asymptotics $\psi(x) \sim \exp(+x^2/2)$ and which, in essence, remains non-tobogganic. It is still worth noting that its “tobogganic” version could be obtained by the mere analytic continuation since the picture is able to display just the halves of the relevant wedges. The second interesting observation is that the curve $C(s)$ cannot be deformed to any vicinity of the real line - similar shape is also exhibited by the Coulomb/Kepler contours [24].

The first nontrivial toboggan appears in Figure 5 where the “left” (!) branch of the toboggan $C_{(\text{third-third})}(s)$ is made visible by the clockwise rotation of the cut of Figure 3 by 90 degrees. The picture is made complete by Figure 6 where the remaining “right” half of the “third-third” toboggan is made visible by the anticlockwise rotation of the cut of Figure 3 by 90 degrees.

In our final Figure 7, a “left” branch of the fourth-fourth toboggan is displayed. It corresponds to the “anomalous” asymptotics $\psi(x) \sim \exp(+x^2/2)$ and is made visible by the clockwise rotation of the cut of Figure 3 by 180 degrees. The visualization of the rest of this curve $C(s)$ is trivial as it would just require an opposite, anticlockwise rotation of the cut.
4 Discussion

In spite of their entirely natural character, the studies of all the specific “far-from-the-real-axis” contours $C(s)$ are by far not frequent in the literature on $\mathcal{PT}$ symmetry even if we restrict our attention just to the QE context [4, 5, 11]. In such a setting, our present further move towards tobogganic $C(s)$ might look almost like a heresy. At present we see at least two reasons for an appeal of such a study. Firstly, we have already demonstrated that the Liouvillean changes of variables might make the explicit constructions of toboggans quite straightforward. Secondly, we believe that a particularly appealing aspect of all the modifications of boundary conditions is revealed by the QE solvable models. They represent very specific systems where just some of the levels remain unchanged due to the elementary, exceptionally easily analytically continued form of their QE wavefunctions.

4.1 Back to the simpler models

For the above-mentioned reasons we tried to keep our overall considerations in close contact with their specific screened-harmonic-oscillator illustration. Now, it is time to notice that the winding-independent coincidence of the energy levels must occur also in the simpler solvable potentials.

4.1.1 The unscreened harmonic oscillator

It is well known [25] that the harmonic-oscillator Schrödinger equation

\[
\left( -\frac{d^2}{dz^2} + \frac{\alpha^2 - 1/4}{z^2} + z^2 \right) h_n^{(\pm)}(z) = E_n^{(\pm)} h_n^{(\pm)}(z), \quad z \in \mathbb{R}
\]

is exactly solvable in terms of Laguerre polynomials $L_n^{(\pm\alpha)}(z)$ at $\alpha = 1/2$,

\[
E_n^{(\pm)} = 4n + 2 \pm 2\alpha, \quad h_n^{(\pm)}(z) = c_n^{(\pm)} \sqrt{z^{1+2\alpha} e^{-z^2/2}} L_n^{(\pm\alpha)}(z^2), \quad n = 0, 1, \ldots
\]

As long as these solutions are analytic in the whole complex plane of $z$, the model offers in fact one of the simplest possible illustrative examples of the existence of the real spectrum of energies generated by a manifestly non-Hermitian, $\mathcal{PT}$–symmetric Hamiltonian at $\alpha = 1/2$ [1].

At $\alpha \neq 1/2$, the model gets regularized on the straight line (6) at any $\varepsilon > 0$. One arrives at a merely inessentially modified ordinary differential equation

\[
\left( -\frac{d^2}{dx^2} + x^2 - 2i\varepsilon x + \frac{\alpha^2 - 1/4}{x^2 - 2i\varepsilon x - \varepsilon^2} \right) h_n^{(\pm)}(x-i\varepsilon) = \left( E_n^{(\pm)} + \varepsilon^2 \right) h_n^{(\pm)}(x-i\varepsilon) \quad (19)
\]

where $x \in \mathbb{R}$. In contrast to the original real-line problem where the parity has been conserved, $\mathcal{P} h_n^{(\pm)}(z) = h_n^{(\pm)}(-z) = \mp h_n^{(\mp)}(z)$, the new eigenfunctions $h_n^{(\pm)}(x - i\varepsilon)$
only remain eigenstates of the product of the two (mutually commuting) operators \( P \) and \( T \). Their \( PT \)-eigenvalue might give a new meaning to the superscripts of \( h_\pm^n(x - i\varepsilon) \) but one usually opts for complex normalization constants \( c_\pm^n \) chosen in such a way that all \( PT \)-eigenvalues of \( h_\pm^n(x - i\varepsilon) \) become the same and equal, say, to one [26].

Of course, the analytic continuation will work at all the accessible Riemann sheets so that all the tobogganic versions of the harmonic oscillator will have the same spectrum. In this sense, the example itself is degenerate and trivial.

4.1.2 The Magyari’s family of quasi-exact toboggans

In QE models, only the levels with QE structure are easy to study. Their formal resemblance to harmonic-oscillator states implies, in particular, that the different physical situations may be represented by the same formula since, as we have seen, the single QE bound state may fit several boundary conditions at once.

We have shown that once we fixed our “prototype” decadic polynomial problem (8), we were able to map some of its non-tobogganic versions upon the tobogganic variants of our illustrative harmonic oscillator screened by our specific six-term perturbation (18). For pedagogical reason we choose such a set of examples because, on one side, the Magyari’s [27] entirely general QE and \( PT \)-symmetric prototype problem

\[
-\frac{d^2}{dx^2} \varphi(x) + \frac{L(L + 1)}{x^2} \varphi(x) + \left[ x^{4q+2} + g_{4q} x^{4q} + \ldots + g_2 x^2 \right] \varphi(x) = E \varphi(x) \tag{20}
\]

is only exceptionally tractable non-numerically at \( q > 2 \) [28]. We were still able to recollect a few explicit formulae for \( q = 2 \) [5]. On the other side, there obviously exist also the simpler and more popular QE constructions at \( q = 1 \). We felt that they do not offer a sufficiently transparent and persuasive introduction of a sufficiently large variety of tobogganic paths \( C(s) \). Of course, having now gained the overall experience at \( q = 2 \), we might easily return to the multinomial \( q = 1 \) models and study, say, the set of simple QE examples

\[
\begin{align*}
V_f(x) &= x^6 + f_4 x^4 + f_2 x^2 + f_{-2} x^{-2}, \\
V_g(x) &= -(ix)^2 + i g_1 x + g_{-1} (ix)^{-1} + g_{-2} (ix)^{-2}, \\
V_h(x) &= -(ix)^{2/3} + h_{-2/3} (ix)^{-2/3} + h_{-4/3} (ix)^{-4/3} + h_{-2} (ix)^{-2},
\end{align*}
\]

by choosing \( \alpha = q + 1 = 2 \) for \( V_f \) or \( \alpha = 1 \) for \( V_g \) or \( \alpha = 2/3 \) for \( V_h \) in eq. (17). Expecting no really new observations, we leave this analysis as an exercise to the reader.

4.2 Physics with complex pseudo-coordinates \( x \in C(s) \)

Our study is based on the expected compatibility of a change of the integration path \( C(s) \) (and of the resulting loss of the Hermiticity of the Hamiltonian \( H \)) with the
observability (i.e., reality) of the energies. For the sake of completeness, let us now briefly summarize that a tenable physical background of such a construction requires that

- for all the operators $\mathcal{O}$ in our Hilbert space $\mathcal{L}$ we replace the usual conjugation $\mathcal{O} \rightarrow \mathcal{O}^\dagger$ by an alternative involutive operation

$$\mathcal{O} \rightarrow \mathcal{O}^{\sharp} = \eta^{-1} \mathcal{O}^\dagger \eta$$  \hspace{1cm} (21)

where the positive definite operator $\eta \neq I$ represents a certain non-standard metric in our Hilbert space. In this language, let the family of the observables be characterized by the so called quasi-Hermiticity property,

$$\mathcal{O}^{\dagger} = \eta \mathcal{O} \eta^{-1}$$  \hspace{1cm} (22)

(see an older general review [29] for more details);

- in the next step let us admit that eq. (22) (taken as an implicit definition of an unknown $\eta$ from a given Hamiltonian $\mathcal{O} = H$) has more solutions. One of them is assumed to coincide with the parity operator $\mathcal{P}$. In this manner we quite naturally arrive at the concept of the $\mathcal{PT}$-symmetric quantum Hamiltonians of ref. [1];

- at the same time we must be permitted to represent all the observables as operators compatible with the original quasi-Hermiticity condition (22). In this sense, the complex coordinates are in general not observable.

In terms of physics, we have to proceed in opposite direction. The standard probabilistic physical interpretation of the $\mathcal{PT}$-symmetric theory remains only achieved after our construction of the modified scalar product mediated by the (not necessarily unique [29]) metric operator $\eta = \eta^{\dagger} > 0$. This operator must satisfy all the subtle mathematical conditions of quasi-Hermiticity (cf. [29, 30]). For physical reasons, people often denote $\eta \equiv \mathcal{C}\mathcal{P}$ and speak about a charge-symmetry operator $\mathcal{C}$ and about the fundamental $\mathcal{CPT}$-symmetry in field theory [20].

In our present tobogganic context, we may conclude that one is allowed to introduce complex “coordinates” $x \notin (-\infty, \infty)$ provided only that they are not treated as “measurable” [31]. The spectrum of the related non-Hermitian Hamiltonians $H \neq H^{\dagger}$ (i.e., by assumption, the observable values of the bound-state energy levels) must remain real of course.

One of the important open questions is how one could guarantee this reality but, of course, this problem is shared by both the tobogganic and non-tobogganic models.
5 Summary

Among innovations accepted in the context of the \( PT \)-symmetric version of quantum mechanics, one of the most unusual ones concerns the admissibility of the various complex contours of the (by assumption, unmeasurable) “coordinates” \( x \in C(s) \). This possibility is comparatively rarely emphasized in the related literature, presumably, due to the expected purely technical complications. Only very recently, A. Mostafazadeh [32] reminded us that such an expectation would be over-pessimistic since all the \( PT \)-symmetric (i.e., left-right symmetric) contours \( C(s) \) in the complex plane of \( x \) may be perceived simply as their properly matched left and right sub-contours.

Whenever the potential in our Schrödinger equation contains a centrifugal-like singularity as contemplated, probably for the first time, by Buslaev and Grecchi in ref. [6], the situation remains very similar - in the complex plane of \( x \), it is only necessary to introduce a (say, upwards-running) branch cut from zero to infinity (cf., e.g., Figure 3 above). In our paper we tried to persuade the readers that in such a situation, the existence of the centrifugal-like singularity simplifies the picture since it allows us to perform a fairly general change of variables (cf. eq. (16) in our present context). This transformation mediates a one-to-one correspondence between the models defined over different contours of coordinates \( C(s) \). In our considerations we emphasized that many \( PT \)-symmetric contours may be visualized as maps of some other contours lying sufficiently close to the real axis.

Our main use of the latter correspondence between the \( PT \)-symmetric integration contours pertaining to the different potentials lied in the use of the mapping of the “elementary” contours \( C(e)(s) \) (i.e., the ones confined to a single Riemann sheet) onto the “tobogganic” contours \( C(t)(s) \) (i.e., the ones which encircle the branch points and become defined over several Riemann sheets in general).

In spite of a certain counter-intuitive character of the tobogganic case, one should keep in mind that both \( C(e)(s) \) and \( C(t)(s) \) are equally counter-intuitive since, as we already mentioned (cf. also [33]), our “coordinates” \( x \in C \) do not represent a physically measurable quantity. Hence, there is no reason for omitting the topologically nontrivial “toboggan-shaped” curves \( C(t)(s) \) from \( PT \)-symmetric quantum mechanics.

In our paper we showed that the use of the simplest versions of the toboggans \( C(t)(s) \) need not necessarily lead to any perceivable technical difficulties. As long as we paid attention just to the simplest possible class of \( V(x) \) (possessing just the single strong singularity at \( x = 0 \)), our “recipe” of an interpretation of the bound states degenerated simply to the inverse Liouvillian change of variables.

In the conclusion, let us express our belief that even the “next move” to the study of some multiply tobogganic \( PT \)-symmetric potentials (containing more than one strong singularity) need not still lead to unsurmountable difficulties: As an
encouragement, one might cite, say, the existence of the nice and solvable potentials of this type described in ref. [34].

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Figure captions

Figure 1. Asymptotic wedges for decadic oscillators.
Figure 2. The “unphysical” alternative to Figure 1.
Figure 3. Cut plane with wedges for harmonic oscillator.
Figure 4. The “left-second” – “right-second” contour $C(s)$
Figure 5. The “left” half of the “third-third” toboggan
Figure 6. The remaining part of the third-third toboggan.
Figure 7. The fourth-fourth toboggan.
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Figure 1. Asymptotic wedges for decadic oscillators
Figure 2. The "unphysical" alternative to Figure 1

(here: an upwards running branch cut)
Figure 3. Cut plane with wedges for harmonic oscillator

\[ C(x) \]

Im x 0

first left

first right

Re x 0
Figure 4. The "left-second" - "right-second" toboggan

\[ C(x) \]
Figure 5. The "left" half of the "third-third" toboggan
Figure 6. The remaining part of the third-third toboggan
Figure 7. The fourth-fourth toboggan

The second left wedge
rotated cut
a part of the fourth left wedge

$C(x)$