SCATTERING FOR THE QUADRATIC KLEIN-GORDON EQUATIONS

ZIHUA GUO AND JIA SHEN

Abstract. We study the scattering problems for the quadratic Klein-Gordon equations with radial initial data in the energy space. For 3D, we prove small data scattering, and for 4D, we prove large data scattering with mass below the ground state.

CONTENTS

1. Introduction 1
2. Small energy scattering and perturbed Strichartz estimates 5
2.1. 3D case 7
2.2. 4D case 10
3. Variational analysis and Virial/Morawetz estimate 13
3.1. Variation in $L^2$-critical case 14
3.2. Virial/Morawetz estimate 16
4. Large data scattering in 4D case 19
4.1. $L^3$ decay after large time 19
4.2. Proof of Theorem 1.3 21
Acknowledgements 23
References 23

1. Introduction

In this paper we study the Cauchy problems to the following quadratic Klein-Gordon equation

$$\begin{align*}
\partial_t^2 u - \Delta u + u &= u^2, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d \\
u(0, x) &= u_0, \\
\partial_t u(0, x) &= u_1,
\end{align*}$$

where $u(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $d = 3, 4$. The Klein-Gordon equation with various types of nonlinear terms ($u^2$ replaced by $f(u)$) has been extensively studied in a large amount of literatures, for example, see [19] and references therein for the detailed introduction. In particular, the existence of global solutions and study of their asymptotic behaviour are two important topics.

We first review the cases with the power type nonlinearity $f(u) = \lambda |u|^p u$. There are two special indices for $p$: mass-critical index $p = 4/d$ and energy-critical index $p = 4/(d-2)$. In view of the current studies, when $4/d < p \leq 4/(d-2)$ ($p > 4/d$ for $d = 1, 2$), the scattering problems were better understood. For the defocusing case $\lambda > 0$, see [1, 9, 25–27] and for the focusing case $\lambda < 0$, see [19, 20]. For small data,
one can have scattering in critical space $H^s$ (see [35, 36]). When $p \leq 4/d$, there are less results on the scattering problems in energy space. For the mass-critical case $p = 4/d$, it was observed in [28] that the scattering results for Klein-Gordon equation can imply the same results for mass critical nonlinear Schrödinger equation (NLS). On the other hand, scattering for the 2D cubic Klein-Gordon was established in [21] using the result for NLS in [5, 6]. When $p < 4/d$, the scattering results were usually obtained for small data in some weighted Sobolev space, for example, in [34] for $p_S(d) < p \leq 4/d$, where $p_S(d)$ is the Strauss exponent satisfying $dp(p+1) = 2(p+2)$, and in [18] for $p > 2/d$. When $0 < p \leq 2/d$ if $d \geq 2$, or $p = 3$ if $d = 1$, scattering operator does not exist, see [3, 8, 10, 24].

The quadratic term $u^2$ may be compared with $|u|^2$ (namely $p = 1$). It is mass-subcritical for 3D and mass-critical for 4D. However, due to the better regularity and algebraic structure of $u^2$, some new methods were developed to study the asymptotic behaviour. Let us mention Klainerman’s vector field method [22] and Shatah’s normal form method [33]. Both methods showed scattering of global small solutions for (1.1) with $d = 3$. For 2D, the global existence of small solutions and asymptotic behaviour were studied in [4, 30]. Note that the above two models are below the Strauss exponent, i.e. $p = 2 = p_S(3)$, and $p = 2 < p_S(2)$. The above results are for small data with sufficient regularity and decay (in weighted Sobolev space). Using the space-time resonance structure and $U^p, V^p$ space, Schottdorf [32] showed small data scattering in energy space for 3D quadratic Klein-Gordon equation. Recently, in [13], the first author and Nakanishi used a new approach to show the scattering for the 3D Zakharov system with small radial energy data. The idea is to combine the radially improved Strichartz estimates in [17] and (partial) normal form method in [33]. It turns out that this approach can deal with the scattering problems for a class of 3D quadratic dispersive equations and has been further extended. For example, see [11, 12, 16] for non-radial version generalization and applications to other equations.

The purpose of this paper is to study the asymptotic behaviour for the quadratic Klein-Gordon equation (1.1) using this approach. Comparing to the $U^p, V^p$ space methods used in [32], we used only Strichartz space that allows perturbation. This gives us the possibility to study the large data problem as [14, 15]. Our first result is the small data scattering in energy space for the quadratic Klein-Gordon equation (1.1) in 3D and 4D.

**Theorem 1.1.** Let $d = 3$ or $d = 4$, and $\kappa > 0$ be a sufficiently small constant. Suppose that $(u_0, u_1)$ is radial, and satisfies

$$\|(u_0, u_1)\|_{H^1 \times L^2} \ll 1,$$

then there exists a unique solution $u(t, x)$ to (1.1) in

$$C(\mathbb{R} : H^1) \cap \left( \frac{1}{2}, \frac{3}{10} - \kappa; \frac{2}{5} - 3\kappa\frac{7}{10} + \kappa \right) \mathbb{R}, \text{ when } d = 3,$$

and in

$$C(\mathbb{R} : H^1) \cap \left( \frac{1}{2}, \frac{5}{14} - \kappa; \frac{3}{7} - 4\kappa\frac{11}{14} + \kappa \right) \mathbb{R}, \text{ when } d = 4.$$

Moreover, scattering holds, namely, $\exists u_\pm(x) \in H^1$ such that

$$\|(u - i(D)^{-1}\partial_t u - e^{it(D)}u_\pm\|_{H^1} \to 0, \text{ as } t \to \pm \infty.$$
Remark 1.2.
(a) The notation \((1/q, 1/r; s_0, s_1)\) is the space given in \((1.4)\).
(b) The scattering part in the above Theorem is not new, but we can obtain stronger results that the solutions belong to a set of perturbed Strichartz spaces, see Proposition 2.5 and Proposition 2.10 below. This enable us to study large data scattering.
(c) The radial assumption could be replaced by additional angular regularity by the similar arguments in \([12]\).

Now we turn to the large data problem. On one hand, the quadratic Klein-Gordon equation \((1.1)\) has a conservation of energy

\[
E(u(t), u_t(t)) = \int_{\mathbb{R}^d} \left( \frac{1}{2} |\partial_t u(t, x)|^2 + \frac{1}{2} |\nabla u(t, x)|^2 + \frac{1}{2} |u(t, x)|^2 - \frac{1}{3} u(t, x)^3 \right) dx.
\]

On the other hand, the ground state \(Q\), that is the unique radial positive solution to the elliptic equation

\[-\Delta Q + Q = Q^2,
\]

is a stationary solution to \((1.1)\), which is non-scattering. It is well known that \(Q\) attains the best constant of Gagliardo-Nirenberg inequality

\[
\int |f(x)|^{2(d+2)/d} \leq \frac{d+2}{d} \left( \frac{\|f\|_2}{\|Q\|_2} \right)^{4d} \|\nabla f\|_2^2.
\]

We want to clarify the dichotomy behaviour into blowup and scattering with \(Q\) as a threshold. However, for the 3D case, we do not know how to prove scattering at the moment since it is \(L^2\)-subcritical and we do not have the variational analysis of the Virial estimate. So we only have the result in 4D.

**Theorem 1.3.** Let \(d = 4\) and \(\kappa > 0\) be a sufficiently small constant. Suppose that \((u_0, u_1)\) is radial, and satisfies

\[
E(u_0, u_1) < E(Q, 0).
\]

(a) If \(\|u_0\|_2 > \|Q\|_2\), the solution to \((1.1)\) must blow up in finite time.

(b) If \(\|u_0\|_2 < \|Q\|_2\), there exists a unique solution \(u(t, x)\) to \((1.1)\) in

\[
C \left( \mathbb{R} : H^1 \right) \cap \left( \frac{1}{2} \cdot \frac{5}{14} - \kappa, \frac{3}{7} - 4\kappa \cdot \frac{11}{14} + \kappa \right)_{\mathbb{R}},
\]

and \(u(t, x)\) satisfies

\[
\left\| u - i\langle D\rangle^{-1} \partial_t u - e^{it\langle D\rangle} u_\pm \right\|_{H^1} \to 0,
\]

when \(t \to \pm\infty\), for some \(u_\pm(x) \in H^1\).

**Remark 1.4.**
(a) Recently, Dodson and Murphy gave a new proof of the scattering for the focusing \(H^{1/2}\)-critical NLS in \([7]\), in which they used the Virial/Morawetz estimate (used in \([29]\) by Ogawa and Tsutsumi) to avoid the concentration compactness argument. We follow their idea to prove the large data scattering.

(b) There are some difficulties for the Klein-Gordon equation. Virial/Morawetz estimate yields small \(L^3\) norm at one large time, while we need small \(L^3\) norm on a suitable large time interval. However, on one hand, Klein-Gordon equation does not have almost finite propagation of localised \(L^2\) norm as NLS. On the other hand,
Virial/Morawetz estimate cannot give the decay of the localization of \( \| \partial_t u \|_2 \), so local energy estimate inside the light cone cannot be applied. To overcome this difficulty, we use Cazenave’s approach [2] to give a pointwise decay of \( \| u - i \langle D \rangle^{-1} \partial_t u \|_3 \) after large time.

An important similar equation is the following \( \phi^4 \) model, which arises in quantum field theory

\[
\partial_t^2 \phi - \Delta \phi = \phi - \phi^3,
\]

with the non-vanishing boundary condition \( \lim_{|x| \to \infty} |\phi(x)| = 1 \). It has the conservation of energy

\[
\dot{E}(\phi, \phi_t) = \int_{\mathbb{R}^d} \frac{1}{2} |\partial_t \phi(t, x)|^2 + \frac{1}{2} |\nabla \phi(t, x)|^2 + \frac{1}{4} (1 - |\phi|^2)^2 \, dx.
\]

In [23], Kowalczyk, Martel and Muñoz studied the asymptotic stability of some kink solutions in dimension one. Note that under the simpler boundary condition

\[
\lim_{|x| \to \infty} \phi(t, x) = 1,
\]

and let \( w(t, x) = \phi(t, x) - 1 \), then the equation (1.2) can be transformed into Klein-Gordon equation

\[
\partial_t^2 w - \Delta w + 2w = -3w^2 - w^3.
\]

Then the small data scattering results in Theorem 1.1 also hold for (1.3). It seems interesting to study the large data problem.

**Notations**

- \( \hat{f} \) or \( \mathcal{F} f \) denotes the Fourier transform of \( f \).
- \( C > 0 \) denotes some constant, and \( C(a) > 0 \) denotes some constant depending on coefficient \( a \).
- If \( f \leq C g \), we write \( f \lesssim g \). If \( f \leq C g \) and \( g \leq C f \), we write \( f \sim g \). Suppose further that \( C = C(a) \) depends on \( a \), then we write \( f \lesssim_a g \) and \( f \sim_a g \), respectively.
- For \( x \in \mathbb{R}^d \), \( \langle x \rangle := (1 + |x|^2)^{1/2} \).
- \( D := \mathcal{F}^{-1} \langle \xi \rangle \mathcal{F} \) and \( \langle D \rangle^s := \mathcal{F}^{-1} (1 + |\xi|^2)^{s/2} \mathcal{F} \).
- Take a cut-off function \( \chi \in C_0^{\infty}(0, \infty) \) such that \( \chi(r) = 1 \) if \( r \leq 1 \) and \( \chi(r) = 0 \) if \( r > 2 \). For \( k \in \mathbb{Z} \), let \( \chi_k(r) = \chi(2^{-r} k) \) and \( \phi_k(r) = \chi_k(r) - \chi_{k-1}(r) \). We define the Littlewood-Paley dyadic operator \( P_{\leq k} f := \mathcal{F}^{-1} \left( \chi_k(|\xi|) \hat{f}(\xi) \right) \) and \( P_k f := \mathcal{F}^{-1} \left( \phi_k(|\xi|) \hat{f}(\xi) \right) \).
- \( L^p(\mathbb{R}^d) \) and \( H^s(\mathbb{R}^d) \) denote the usual Lebesgue and Sobolev space. \( L^p_{\text{rad}}(\mathbb{R}^d) \) and \( H^s_{\text{rad}}(\mathbb{R}^d) \) denote the space of radial functions in \( L^p(\mathbb{R}^d) \) (or \( H^s(\mathbb{R}^d) \)).
- \( B^s_{p,q} \) and \( B^s_{p,q} \) denote the standard inhomogeneous and homogeneous Besov space, that is \( \| f \|_{B^s_{p,q}} := \| P_{\leq 0} f \|_p + \left( \sum_{k \geq 0} 2^{sk} \| P_k f(x) \|_p^q \right)^{1/q} \), and \( \| f \|_{B^s_{p,q}} := \left( \sum_{k \in \mathbb{Z}} 2^{sk} \| P_k f(x) \|_p^q \right)^{1/q} \) and we write \( B^s_{p} = B^s_{p,2} \).
- We define the following Besov type space

\[
\| f \|_{(B^s_{p,0})^{1/2}} := \left( \sum_{k \in \mathbb{Z}, k \leq 0} 2^{sk} \| P_k f(x) \|_p^2 \right)^{1/2} + \left( \sum_{k \in \mathbb{Z}, k \geq 0} 2^{sk} \| P_k f(x) \|_p^2 \right)^{1/2}.
\]
We define the norms of space-time function space
\[
\|F\|_{L^q(t; B^s_x)}(t, x) := \|F(t, x)\|_{L^q_t(B^s_x)}(t, x),
\]
\[
\|F\|_{C^{s_0}[t, s]} := \|F(t, x)\|_{C^{s_0}[t, s]}.
\] (1.4)

Sometimes we omit the interval \(I\) for abbreviation.

2. SMALL ENERGY SCATTERING AND PERTURBED STRICHARTZ ESTIMATES

Before starting our proof, we make some preliminaries. First, we need the radially improved Strichartz estimates.

**Lemma 2.1** ([11]). Suppose that \(\varphi \in L^2\) is radial, \(d \geq 2\), \(2 \leq q, r \leq +\infty\), and \((q, r, d) \neq (2, \infty, 2)\). If \((q, r)\) satisfies \(1/q + (d - 1)/r < (d - 1)/2\), we have
\[
\|e^{it\langle \xi \rangle} \hat{P}_k \varphi\|_{L^q_t(B^s_x)}(t, x) \lesssim \|P_k \varphi\|_2,
\] (2.1)
and if \((q, r)\) satisfies \(1/q + (d - 1)/r > (d - 1)/2\) and \(2/q + 2d - 1)/r < (2d - 1)/2\),
\[
\|e^{it\langle \xi \rangle} \hat{P}_k \varphi\|_{L^q_t(B^s_x)}(t, x) \lesssim \|P_k \varphi\|_2.
\] (2.2)

Another important tool used in this paper is the normal form method. In fact, we are going to use different normal forms for 3D and 4D Strichartz estimates, so we first introduce a general definition of normal form. By the change of variable
\[
U(t, x) = u(t, x) - i\langle D\rangle^{-1} u_0(t, x),
\]
we can transform the original equation into a first order one
\[
i\partial_t U + \langle D\rangle U = \langle D\rangle^{-1} u^2 = \frac{1}{4} \langle D\rangle^{-1} \left( U^2 + 2\hat{U} \hat{U} + \hat{U}^2 \right),
\] (2.3)
then the integral equation is
\[
U(t, x) = e^{it\langle \xi \rangle} U_0 - \frac{i}{4} \int_0^t e^{i(t-s)\langle \xi \rangle} \langle D\rangle^{-1} \left( U^2 + 2\hat{U} \hat{U} + \hat{U}^2 \right) ds.
\]
Let \(m(\xi_1, \xi_2)\) be some Coifman-Meyer bilinear multiplier. We write \(U^+(t, x) = U(t, x)\) and \(U^-(t, x) = \hat{U}(t, x)\). For \((\tau_1, \tau_2) \in \{(+, +), (+, -), (-, +), (-, -)\}\), we define the normal form for different nonlinear terms as
\[
\bar{\Omega}_{\tau_1, \tau_2}(U^{\tau_1}, U^{\tau_2})(t, \xi) = \int_{\mathbb{R}^d} \frac{1}{i\Phi(\xi, \eta)} \overline{m(\xi - \eta, \eta)} \hat{U}^{\tau_1}(t, \xi - \eta) \hat{U}^{\tau_2}(t, \eta) d\eta,
\]
where the modulation \(\Phi(\xi, \eta) := \Phi_{\tau_1, \tau_2}(\xi, \eta)\) is defined by
\[
\Phi_{\tau_1, \tau_2}(\xi, \eta) := -\langle \xi \rangle + \tau_1 \langle \xi - \eta \rangle + \tau_2 \langle \eta \rangle.
\]
Thus, the normal form transform adapted to the equation (2.3) is defined by
\[
(U + i\langle D\rangle^{-1} \Omega(U, U))(t, x) := U(t, x) + \frac{i}{4} \langle D\rangle^{-1} \sum \Omega_{\tau_1, \tau_2}(U^{\tau_1}, U^{\tau_2})(t, x),
\]
where the summation is over \((\tau_1, \tau_2) \in \{(+, +), (+, -), (-, +), (-, -)\}\). Note that the normal form is well-defined, if \(|\Phi| \neq 0\) for all \((\xi, \eta) \in \mathbb{R}^d \times \mathbb{R}^d\) satisfying \(m(\xi - \eta, \eta) \neq 0\). In this paper, we are going to use the normal form with \(m(\xi - \eta, \eta)\) satisfying \(m(\xi - \eta, \eta) = 0\) unless \(\min\{\|\xi - \eta\|, |\eta|\} \lesssim 2^\beta\) for some large constant \(\beta > 0\), so for any choice of \((\tau_1, \tau_2)\),
\[
|\Phi(\xi, \eta)| \geq |\langle \xi \rangle \pm \langle \xi - \eta \rangle \pm \langle \eta \rangle| \gtrsim \frac{1}{\min\{\|\xi - \eta\|, |\eta|\}} \gtrsim 1.
\]
Therefore, we can only consider nonlinear term $U^2$ for simplicity, and the proof of
the Strichartz estimates for other kinds of nonlinear term is essentially the same.

In this section, we focus on the simplified equation $i\partial_t U + \langle D \rangle U = \langle D \rangle^{-1}(U^2)$. For any functions $f(t, x)$ and $g(t, x)$, we define the normal form as

\[
\hat{\Omega}(f, g)(t, \xi) = \int_{\mathbb{R}^d} \frac{1}{i(-\langle \xi \rangle + \langle \xi - \eta \rangle + \langle \eta \rangle)} m(\xi - \eta) \hat{f}(t, \xi - \eta) \hat{g}(t, \eta) d\eta. \tag{2.4}
\]

Now we insert the normal form transform into the equation, and get

\[
(i\partial_t + \langle D \rangle) \left( U + i\langle D \rangle^{-1}\Omega(U, U) \right) \nonumber
\]
\[
= \langle D \rangle^{-1}(U^2) + i\Omega(U, U) \nonumber
\]
\[
+ i\Omega(i\partial_t U, U) + i\Omega(U, i\partial_t U) \nonumber
\]
\[
= \langle D \rangle^{-1}U^2 + i\Omega(U, U) + i\langle D \rangle^{-1}\Omega(-\langle D \rangle U, U) + i\langle D \rangle^{-1}\Omega(U, -\langle D \rangle U) \nonumber
\]
\[
+ i\Omega \langle \langle D \rangle^{-1}U^2, U \rangle + i\Omega \langle U, \langle D \rangle^{-1}U^2 \rangle. \nonumber
\]

The quadratic term is

\[
\langle D \rangle^{-1}U^2 + i\Omega(U, U) + i\langle D \rangle^{-1}\Omega(-\langle D \rangle U, U) + i\langle D \rangle^{-1}\Omega(U, -\langle D \rangle U) \nonumber
\]
\[
= \mathcal{F}^{-1} \left( \langle \xi \rangle^{-1} \int \hat{U}(t, \xi - \eta) \hat{U}(t, \eta) d\eta \right) \nonumber
\]
\[
+ \mathcal{F}^{-1} \left( i \int \frac{1}{i} m(\xi - \eta, \eta) \left( 1 - \frac{\langle \xi \rangle}{\langle \xi \rangle} - \frac{\langle \eta \rangle}{\langle \xi \rangle} \right) \hat{U}(t, \xi - \eta) \hat{U}(t, \eta) d\eta \right) \nonumber
\]
\[
= \mathcal{F}^{-1} \left( \langle \xi \rangle^{-1} \int \hat{U}(t, \xi - \eta) \hat{U}(t, \eta) d\eta - \langle \xi \rangle^{-1} \int m(\xi - \eta, \eta) \hat{U}(t, \xi - \eta) \hat{U}(t, \eta) d\eta \right). \nonumber
\]

In fact, the Coifman-Meyer bilinear operator with multiplier $1 - m(\xi - \eta, \eta)$ is the resonance term, namely

\[
\mathcal{F}(T_{\text{Res}}(f, g)) \langle \xi \rangle := \int (1 - m(\xi - \eta, \eta)) \hat{f}(t, \xi - \eta) \hat{g}(t, \eta) d\eta.
\]

After normal form reduction, we have

\[
U(t, x) = K(t) \left( U_0 + i\langle D \rangle^{-1}\Omega(U, U) \langle 0 \rangle \right) - i\langle D \rangle^{-1}\Omega(U, U) \nonumber
\]
\[
- i \int_0^t K(t - s) \langle D \rangle^{-1}T_{\text{Res}}(U, U) \, ds \nonumber
\]
\[
+ \int_0^t K(t - s) \langle D \rangle^{-1} \left( \langle D \rangle^{-1}U^2 \right) \, ds \nonumber
\]
\[
+ \int_0^t K(t - s) \langle D \rangle^{-1} \left( \Omega \left( U, \langle D \rangle^{-1}U^2 \right) \right) \, ds. \tag{2.5}
\]

Finally, our normal form transform is based on frequency decomposition. Fixed a
large parameter $\beta > 0$, for any two functions $U, U' \in H^1$, we split the decomposition
as
\[ U(x)U'(x) = \sum_{(j,k) \in \mathbb{Z}^2} P_j U P_k U' \]
\[ = \sum_{(j,k) \in HH} P_j U P_k U' + \sum_{(j,k) \in HL} P_j U P_k U' + \sum_{(j,k) \in LH} P_j U P_k U' \]
\[ + \sum_{(j,k) \in LL} P_j U P_k U', \]
where
\[ HH := \{(j,k) \in \mathbb{Z}^2 : j, k \geq -\beta - 10\}, \]
\[ HL := \{(j,k) \in \mathbb{Z}^2 : j \geq k + 5, j \geq -\beta - 10, -\beta + 10 \geq k\}, \]
\[ LL := \{(j,k) \in \mathbb{Z}^2 : j, k \leq -\beta + 10\}. \]
and \( LH := \{(j,k) \in \mathbb{Z}^2 : (k,j) \in HL\}. \) For any subset \( S \subset \mathbb{Z}^2 \), let
\[ m_S(\xi - \eta, \eta) := \sum_{(j,k) \in S} \phi_j(\xi - \eta) \phi_k(\eta), \]
and define the bilinear frequency cut-off to \( S \) by
\[ (UU')_S := \mathcal{F}^{-1} \int m_S(\xi - \eta, \eta) \hat{U}(\xi - \eta) \hat{U}'(\eta) d\eta = \sum_{(j,k) \in S} P_j U P_k U'. \]

2.1. 3D case. Let \( \kappa > 0 \) and \( \varepsilon > 0 \) be some small coefficients. In this section, we take the normal form (2.4) with \( m = m_{LL} + m_{LH} + m_{HL} \), then the resonance term is
\[ T_{Res}(U, U) = (UU')_{HH}. \]
We also have roughly
\[ \Omega(U, U') \sim (UU')_{LL+LH+HL}. \]
Let \( S(I) \) be the strong Strichartz norm
\[ S(I) = \left(0, \frac{1}{2}, 0|1\right) \cap \left(\frac{1}{2}, \frac{3}{10} - \kappa, \frac{2}{5} - 3\kappa, \frac{7}{10} + \kappa\right). \]
The interpolation space between \( L^\infty_t L^3_x \) and \( L^\infty_t H^1_x \) is defined as follows
\[ Z(I) = \left(0, 0, -\frac{5}{4}, -\frac{3}{4}\right). \]
Define weak Strichartz norm
\[ \tilde{S}(I) = \left(\frac{1}{2} - \varepsilon, \frac{1}{4} + \varepsilon, 5\varepsilon\right) \cap \left(\frac{1}{36}, \frac{1}{6}, -\varepsilon|\varepsilon\right). \]
Note that \( \tilde{S}(I) \) can be interpolated by \( S(I) \) and \( Z(I) \).

**Lemma 2.2** (Resonance term). For radial \( U \) and \( U' \), we have
\[ \left\| \int_0^t K(t - s) (D)^{-1} (UU')_{HH} ds \right\|_{S(I)} \lesssim_\beta \| U \|_{\tilde{S}(I)} \| U' \|_{\tilde{S}(I)}. \] (2.6)
Proof. By interpolation, for \( j \geq -\beta - 10 \), we have
\[
\|P_j U\|_{\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon, 5\varepsilon} \lesssim_\beta \|P_j U\|_{\frac{1}{2}-\varepsilon, (1-2\varepsilon)(1-\varepsilon)(\varepsilon, -\beta - 10)} \lesssim_\beta \|P_j U\|_{\frac{1}{2}-\varepsilon, (1-2\varepsilon)(1-\varepsilon)(\varepsilon, -\beta - 10)} \lesssim_\beta \|P_j U\|_{S(I)} \|P_j U\|_{Z(I)}^{2\varepsilon}.
\]
For \((j, k) \in HH\),
\[
\|P_j U P_k U'\|_{(1-2\varepsilon, 1+2\varepsilon, 2\varepsilon, 4\varepsilon)} \lesssim_\beta \|P_j U\|_{\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon, \varepsilon, 4\varepsilon} \|P_k U'\|_{\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon, \varepsilon, 4\varepsilon} \lesssim_\beta 2^{-\varepsilon(j+k)} \|P_j U\|_{\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon, \varepsilon, 4\varepsilon} \|P_k U'\|_{\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon, \varepsilon, 4\varepsilon}.
\]
Next, we sum over \( j \) and \( k \):
\[
\left\| \int_0^t K(t-s)(D)^{-1}(UU')_{HH} \right\|_{S(I)} \lesssim_\beta \sum_{(j,k) \in HH} \|P_j U P_k U'\|_{(1-2\varepsilon, 1+2\varepsilon, 2\varepsilon, 4\varepsilon)} \lesssim_\beta \sum_{j > -\beta - 10} 2^{-\varepsilon j} \|P_j U\|_{\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon, \varepsilon, 4\varepsilon} \times \sum_{j > -\beta - 10} 2^{-\varepsilon k} \|P_k U'\|_{\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon, \varepsilon, 4\varepsilon} \lesssim_\beta \|U\|_{\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon, \varepsilon, 4\varepsilon} \|U'\|_{\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon, \varepsilon, 4\varepsilon},
\]
then the lemma follows. Later, we will omit the details on the summation over \( j \) and \( k \).

Lemma 2.3 (Boundary term). For radial \( U \) and \( U' \), there exists \( \theta > 0 \), such that
\[
\| (D)^{-1} \Omega(U, U') \|_{S(I)} \lesssim 2^{-\theta \beta} \|U\|_{S(I)}^{1-2\varepsilon} \|U\|_{Z(I)}^{2\varepsilon} \|U'\|_{S(I)}^{1-2\varepsilon} \|U'\|_{Z(I)}^{2\varepsilon}. \tag{2.7}
\]
Proof. First, we estimate \( \|\Omega(U, U')\|_{(0,1/2,0)} \). For \((j, k)\) such that \( j \leq -\beta + 10 \) or \( k \leq -\beta + 10 \), we have
\[
\|P_j U P_k U'\|_{(0,1/2,0)} \lesssim \|P_j U\|_{(0,1/2,0)} \|P_k U'\|_{(0,1/2,0)} \lesssim \|P_j U\|_{(0,1/2,\varepsilon, 3\varepsilon)} \|P_k U'\|_{(0,1/2,\varepsilon, 3\varepsilon)} \lesssim 2^{2(j+k)-\frac{j}{2}+\frac{k}{2}} \frac{1}{2} \|P_j U\|_{(0,1/2,\varepsilon, -1/2, 3\varepsilon)} \|P_k U'\|_{(0,1/2,\varepsilon, -1/2, 3\varepsilon)}.
\]
As for the other norm in \( S(I) \), we have interpolation
\[
\|P_j U\|_{\frac{1}{2}, \frac{1}{2}+\varepsilon, \varepsilon} \lesssim \|P_j U\|_{\frac{1}{2}, \frac{1}{2}+\varepsilon, \varepsilon} + \frac{1}{2} \|P_j U\|_{\frac{1}{2}, \frac{1}{2}+\varepsilon, \varepsilon} \|P_j U\|_{\frac{1}{2}, \frac{1}{2}+\varepsilon, \varepsilon} \lesssim \|P_j U\|_{\frac{1}{2}, \frac{1}{2}+\varepsilon, \varepsilon} \|P_j U\|_{\frac{1}{2}, \frac{1}{2}+\varepsilon, \varepsilon} \|P_j U\|_{\frac{1}{2}, \frac{1}{2}+\varepsilon, \varepsilon}.
\]
Therefore, for \((j, k) \in LL\),
\[
\|P_j U P_k U'\|_{\frac{1}{2}, \frac{1}{2}+\varepsilon, \varepsilon} \lesssim \|P_j U\|_{\frac{1}{2}, \frac{1}{2}+\varepsilon, \varepsilon} \|P_k U'\|_{\frac{1}{2}, \frac{1}{2}+\varepsilon, \varepsilon} \lesssim 2^{2(j+k)} \|P_j U\|_{\frac{1}{2}, \frac{1}{2}+\varepsilon, \varepsilon} \|P_k U'\|_{\frac{1}{2}, \frac{1}{2}+\varepsilon, \varepsilon}.
\]
and for \((j, k) \in \text{LH}\),
\[
\|P_j U P_k U''\|_{\left(\frac{1}{4}+\frac{\kappa}{10}, \frac{1}{4}+\frac{\kappa}{10}+\kappa\right)} \lesssim \|P_j U\|_{\left(\frac{1}{4}+\frac{\kappa}{10}, 0\right)} \|P_k U''\|_{\left(\frac{1}{4}+\frac{\kappa}{10}, -\frac{1}{20}+\kappa\right)} \\
\lesssim 2^{j1+2^{-\frac{1}{2}}} \|P_j U\|_{\left(\frac{1}{4}+\frac{\kappa}{10}, -\frac{1}{2}\right)} 2^{-\frac{1}{50}k} \|P_k U''\|_{\left(\frac{1}{4}+\frac{\kappa}{10}, -\frac{1}{30}\right)}.
\]

Note that HL case is similar, then the lemma follows. \(\square\)

**Lemma 2.4 (Trilinear term).** For radial \(U, U'\) and \(U''\), we have
\[
\left\| \int_0^t K(t-s)\langle D\rangle^{-1} \Omega \left(\langle D\rangle^{-1} (UU'') \right), U' \right\|_{S(I)} \lesssim_{\beta} \|U\|_{\bar{S}(I)} \|U''\|_{\bar{S}(I)} \|U'\|_{\bar{S}(I)}.
\]  
(2.8)

**Proof.** By interpolation, for \(j \in \mathbb{Z}\), we have
\[
\|P_j U\|_{\left(\frac{1}{4}+\frac{\kappa}{10}, -\varepsilon\right)} \lesssim \|P_j U\|_{\left(\frac{1}{4}+\frac{\kappa}{10}, -\frac{1}{20}+\frac{\kappa}{10}\right)} \\
\lesssim \|P_j U\|_{\left(\frac{1}{4}+\frac{\kappa}{10}, -2\varepsilon\right) + \frac{1}{4} + \frac{\kappa}{10} + \frac{1}{10}} \\
\lesssim \|P_j U\|_{\left(\frac{1}{4}+\frac{\kappa}{10}, -2\varepsilon\right) + \frac{1}{4} + \frac{\kappa}{10} + \frac{1}{10}} \|P_j U\|_{\left(\frac{1}{4}+\frac{\kappa}{10}, 0\right)} \|P_j U\|_{\left(\frac{1}{4}+\frac{\kappa}{10}, -\varepsilon\right)}.
\]

By Strichartz estimate, we have
\[
\|\Omega \left(\langle D\rangle^{-1} (UU'') \right), U'\|_{L^1_t L^2_x} \lesssim \|UU''\|_{\left(\frac{1}{2}, 0\right)} \|P_k U''\|_{\left(\frac{1}{4}+\frac{\kappa}{10}, 0\right)} \|P_k U''\|_{\left(\frac{1}{4}+\frac{\kappa}{10}, -\varepsilon\right)} \\
\lesssim \|UU''\|_{\left(\frac{1}{2}, 0\right)} \|P_k U''\|_{\left(\frac{1}{4}+\frac{\kappa}{10}, -\varepsilon\right)}.
\]

For all \(j_1, j_2 \in \mathbb{Z}\), we have
\[
\|P_{j_1} U P_{j_2} U''\|_{\left(\frac{1}{2}, 0\right)} \lesssim 2^{j_1+2^{j_2}+j_2} \|P_{j_1} U\|_{\left(\frac{1}{2}, -\varepsilon\right)} \\
\times 2^{j_2+2\varepsilon} \|P_{j_2} U''\|_{\left(\frac{1}{2}, -\varepsilon\right)}.
\]

We also have
\[
\|P_{j_1} U P_{j_2} U'' P_k U'\|_{L^1_t L^2_x} \lesssim \|P_{j_1} U\|_{\left(\frac{1}{2}, 0\right)} \|P_{j_2} U''\|_{\left(\frac{1}{2}, 0\right)} \|P_{j_1} U\|_{\left(\frac{1}{2}, -\varepsilon\right)} \|P_{j_2} U''\|_{\left(\frac{1}{2}, -\varepsilon\right)} \|P_k U'\|_{\left(\frac{1}{2}, -\varepsilon\right)}.
\]

Thus, the lemma follows, noting that \(\bar{S}(I) \subset \left(\frac{1}{3}, \frac{1}{6}, -\varepsilon\right)\). \(\square\)

Combining all the above estimates, we obtain a perturbed Strichartz estimate in 3D case:

**Proposition 2.5.** Let \(d = 3, \varepsilon > 0\) and \(\kappa > 0\) are small constants. Assume that \(U\) is a solution of (2.5) with initial data \(U_0 \in H^1_{\text{rad}}\), then there exists \(\theta > 0\) such that
\[
\|U\|_{S(I)} \lesssim \|U_0\|_{H^1} + 2^{-\theta \beta} \|U\|_{\bar{S}(I)^2} \|U\|_{\bar{S}(I)^2} + \|U\|_{\bar{S}(I)^2} + \|U\|_{\bar{S}(I)^3}.
\]
(2.9)

Furthermore, we have small data scattering for (1.1) in 3D case.
2.4 **4D case.** Let $0 < \kappa \ll \varepsilon \ll 1$ and $\delta > 0$ be some small coefficients. In this section, we take the normal form (2.4) with $m = m_{LL}$, then the resonance term is

$$T_{\text{Res}}(U, U) = (UU')_{HH+HL+HL}.$$  

We also have roughly

$$\Omega(U, U') \sim (UU')_{LL}.$$

Let $S(I)$ be the strong Strichartz norm

$$S(I) = \left( 0, \frac{1}{2}, 0 | 1 \right) \cap \left( \frac{2}{5}, \frac{5}{14} - \kappa, \frac{3}{7} - 4\kappa, \frac{11}{14} + \kappa \right).$$

The interpolation space between $L_\infty^\infty L_2^3$ and $L_\infty^\infty H_\varepsilon^4$ is defined as follows

$$Z(I) = \left( 0, 0, -\frac{1}{2}, -\frac{4}{3} + \delta \right).$$

The weak norm $\tilde{S}(I)$ is

$$\|U\|_{\tilde{S}(I)} = \|P_{\geq 0} U\|_{\left( \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon, 7\varepsilon \right)} \cap \left( \frac{1}{2} - \varepsilon, \frac{1}{2} + 3\varepsilon \varepsilon, \frac{1}{2} \right)$$

$$+ \|P_{< 0} U\|_{\left( \frac{1}{2} - \varepsilon, \frac{1}{2} - \varepsilon, -\varepsilon \right)} \cap \left( \frac{1}{2}, -\frac{3}{4} \right).$$

**Lemma 2.6 (Resonance term).** Assume that $U$ and $U'$ are radial. For $0 < \kappa \ll \varepsilon \ll 1$, we have

$$\left\| \int_0^1 K(t-s)\langle D\rangle^{-1}(UU')_{HH+HL+HL} \ ds \right\|_{\tilde{S}(I)} \lesssim \beta \|U\|_{\tilde{S}(I)} \|U'\|_{\tilde{S}(I)}. \hspace{1cm} (2.10)$$

**Proof.** By interpolation, for $j \geq -\beta - 10$, we have

$$\|P_j U\|_{\left( \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon, 7\varepsilon \right)} \lesssim \beta \|P_j U\|_{\left( \frac{1}{2} - \varepsilon, (1-2\varepsilon)\left( \frac{1}{2} - \kappa \right), 7\varepsilon + 4\left(1-2\varepsilon\right)\left( \frac{1}{2} - \kappa \right) - \frac{1}{4} - \varepsilon \right)}$$

$$\lesssim \beta \|P_j U\|_{\left( \frac{1}{2} - \varepsilon, (1-2\varepsilon)\left( \frac{1}{2} - \kappa \right), (1-2\varepsilon)\left( \frac{1}{2} + \kappa \right) - 2\varepsilon - \frac{3}{4} + \delta \right)}$$

$$\lesssim \beta \|P_j U\|_{\tilde{S}(I)} \|P_j U\|_{Z(I)}^2.$$  

For $(j, k) \in HH$, we have

$$\|P_j U P_k U'\|_{\left( 1 - 2\varepsilon, \frac{1}{2} + 2\varepsilon, 4\varepsilon \right)} \lesssim \beta \|P_j U\|_{\left( \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon, 6\varepsilon \right)} \|P_k U'\|_{\left( \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon, 6\varepsilon \right)}$$

$$\lesssim \beta \|P_j U\|_{\tilde{S}(I)} \|P_j U\|_{Z(I)}^2 \|P_k U'\|_{\left( \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon, 7\varepsilon \right)}.$$  

Next, consider $(j, k) \in HL$, and $LH$ case follows easily. For $j \geq -\beta - 10$, we have interpolation

$$\|P_j U\|_{\left( \frac{1}{2} - \varepsilon, \frac{1}{2} + 3\varepsilon \varepsilon, \frac{1}{2} \right)} \lesssim \|P_j U\|_{\left( \frac{1}{2} - \varepsilon, (1-2\varepsilon)\left( \frac{1}{2} - \kappa \right), \frac{1}{2} + 4(1-2\varepsilon)\left( \frac{1}{2} - \kappa \right) - \frac{1}{4} - 3\varepsilon \right)}$$

$$\lesssim \|P_j U\|_{\left( \frac{1}{2} - \varepsilon, (1-2\varepsilon)\left( \frac{1}{2} - \kappa \right), \frac{1}{2} + \kappa \right)} \|P_j U\|_{\left( \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon, 7\varepsilon \right)}$$

and for $k \leq -\beta + 10$,

$$\|P_k U'\|_{\left( \frac{1}{2} - \varepsilon, \frac{1}{2} - \varepsilon \varepsilon, \frac{1}{2} \right)} \lesssim \|P_k U\|_{\left( \frac{1}{2} - \varepsilon, (1-2\varepsilon)\left( \frac{1}{2} - \kappa \right), \frac{1}{2} + 4(1-2\varepsilon)\left( \frac{1}{2} - \kappa \right) - \frac{1}{4} - \varepsilon \right)}$$

$$\lesssim \|P_k U\|_{\left( \frac{1}{2} - \varepsilon, (1-2\varepsilon)\left( \frac{1}{2} - \kappa \right), \frac{1}{2} \right)}$$

$$\lesssim \|P_j U\|_{\left( \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon, 7\varepsilon \right)} \|P_j U\|_{\left( \frac{1}{2} - \varepsilon, \frac{1}{2} - \varepsilon \varepsilon, \frac{1}{2} \right)}.$$
For \((j, k) \in HL\), we have
\[
\|P_jUP_kU'\|_{(1-2\varepsilon, \frac{1}{2}+2\varepsilon, 4\varepsilon, 6\varepsilon)} \lesssim \|P_jU\|_{\left(\frac{1}{2}-\varepsilon, \frac{1}{2}+3\varepsilon, 6\varepsilon\right)} \|P_kU'\|_{\left(\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon, 0\right)} \\
\lesssim 2^{-\frac{j+k}{2}} \|P_jU\|_{\left(\frac{1}{2}-\varepsilon, \frac{1}{2}+3\varepsilon, \frac{7}{2}\varepsilon\right)} \|P_kU'\|_{\left(\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon, -\varepsilon\right)}.
\]

**Lemma 2.7** (Boundary term). Assume that \(U\) and \(U'\) are radial. For \(0 < \kappa \ll \varepsilon \ll 1\), there exists \(\theta > 0\), such that
\[
\|\langle D \rangle^{-1}\Omega(U, U')\|_{S(I)} \lesssim 2^{-\theta\beta} \|U\|_{S(I)}^{1-2\varepsilon} \|U\|_{S(I)}^{2\varepsilon} \|U'\|_{S(I)}^{1-2\varepsilon} \|U'\|_{S(I)}^{2\varepsilon}.
\]

**Proof.** First, we estimate \(\|\Omega(U, U')\|_{(0, \theta_0)}\). For \((j, k) \in LL\), we have
\[
\|P_jUP_kU'\|_{(0, \theta_0)} \lesssim \|P_jU\|_{(0, \theta_0)} \|P_kU'\|_{(0, \theta_0)} \\
\lesssim \|P_jU\|_{(0, \frac{1}{2}-\varepsilon, 1-4\varepsilon)} \|P_kU'\|_{(0, \frac{1}{2}-\varepsilon, 1-4\varepsilon)} \\
\lesssim 2^{j+k}2^{-\frac{j+k}{2}} \|P_jU\|_{(0, \frac{1}{2}-\varepsilon, 2\varepsilon, -\frac{1}{2})} \|P_kU'\|_{(0, \frac{1}{2}-\varepsilon, 2\varepsilon, -\frac{1}{2})}.
\]

As for the other norm in \(S(I)\), we have interpolation
\[
\|P_jU\|_{\left(\frac{1}{2}-\varepsilon, (1-2\varepsilon)(\frac{7}{14}-\kappa), \frac{7}{2}\varepsilon\right)} \lesssim \|P_jU\|_{\left(\frac{1}{2}-\varepsilon, (1-2\varepsilon)(\frac{7}{14}-\kappa), (1-2\varepsilon)(\frac{7}{14}-\kappa), 2\varepsilon, -\frac{1}{2}\right)} \\
\lesssim \|P_jU\|_{\left(\frac{1}{2}-\varepsilon, (\frac{7}{14}-\kappa), \frac{7}{2}\varepsilon\right)} \|P_jU\|_{\left(0, 0, -\frac{1}{2}\right)}.
\]

and
\[
\|P_kU'\|_{\left(\varepsilon, (1-4\varepsilon)(\frac{1}{2}+2\varepsilon, (\frac{7}{14}-\kappa), \frac{7}{2}\varepsilon\right)} \lesssim \|P_kU'\|_{\left(\varepsilon, (1-4\varepsilon)(\frac{1}{2}+2\varepsilon, (\frac{7}{14}-\kappa), 2\varepsilon, -\frac{1}{2}\right)} \\
\lesssim \|P_kU'\|_{\left(0, 0, -\frac{1}{2}\right)} \|P_jU\|_{\left(\frac{1}{2}-\varepsilon, (1-2\varepsilon)(\frac{7}{14}-\kappa), \frac{7}{2}\varepsilon\right)}.
\]

Therefore, for \((j, k) \in LL\) and \(j \geq k\), we have
\[
\|P_jUP_kU'\|_{\left(\frac{1}{2}-\varepsilon, (\frac{1}{14}-\kappa), \frac{7}{2}\varepsilon, -\frac{1}{2}\varepsilon\right)} \lesssim \|P_jU\|_{\left(\frac{1}{2}-\varepsilon, (1-4\varepsilon)(\frac{1}{7}+\frac{1}{2}2\varepsilon, 4(1-4\varepsilon)(\frac{1}{7})\right)} \\
\lesssim \|P_jUP_kU'\|_{\left(\frac{1}{2}, (1-2\varepsilon)(\frac{7}{14}-\kappa), (1-4\varepsilon)(\frac{1}{7}+\frac{1}{2}2\varepsilon, (\frac{7}{14}-\kappa), 2\varepsilon\right)} \\
\lesssim 2^{j+k}2^{\frac{j+k}{2}} \|P_jU\|_{\left(\frac{1}{2}-\varepsilon, (1-2\varepsilon)(\frac{7}{14}-\kappa), \frac{7}{2}\varepsilon\right)} \|P_kU'\|_{\left(\varepsilon, (1-4\varepsilon)(\frac{1}{2}+2\varepsilon, (\frac{7}{14}-\kappa), -\frac{1}{2}\varepsilon\right)}.
\]

**Lemma 2.8** (Refined estimate for boundary term). Assume that \(U\) and \(U'\) are radial. For \(0 < \kappa \ll \varepsilon \ll 1\),
\[
\|\langle D \rangle^{-1}\Omega(U, U')\|_{S(I)} \lesssim 2^{-\beta} \|U\|_{(0, \beta_0)} \|U'\|_{S(I)}.
\]

**Proof.** This lemma is easy to obtain, since for \(j \leq -\beta + 10\),
\[
\|P_jU\|_{(0, 0, 0)} \lesssim \|P_jU\|_{(0, \beta_0)} \\
\lesssim 2^{j+\beta} \|P_jU\|_{(0, \beta_0)}.
\]
Lemma 2.9 (Trilinear term). Assume that $U$, $U''$ and $U'$ are radial. For $0 < \kappa \ll \varepsilon \ll 1$, we have
\[
\left\| \int_0^t K(t-s) \langle D \rangle^{-1} \Omega \left( \langle D \rangle^{-1} (UU''), U' \right) ds \right\|_{S(I)} \lesssim_{\beta} \|U\|_{S(I)} \|U''\|_{\tilde{S}(I)} \|U'\|_{\tilde{S}(I)}.
\]
(2.13)

Proof. Divide the normal form into two parts
\[
\Omega (UU'', U') = \Omega ((UU'')_{LL}, U') + \Omega ((UU'')_{HH}, U').
\]
By interpolation, for $j \leq -\beta + 10$, we have
\[
\|P_j U\| \left( \frac{1}{3}, \frac{1}{3} - \frac{1}{4} \right) \lesssim \|P_j U\| \left( \frac{1}{3}, \frac{1}{3} \right) + \left( \frac{1}{4} - 2\varepsilon \right), \left( \frac{1}{4} - 2\varepsilon \right) + 4 \left( \frac{1}{4} - \kappa \right) + \left( \frac{1}{4} - \kappa \right) + \left( \frac{1}{4} - \frac{1}{4} \right)
\]
\[
\lesssim \|P_j U\| \left( \frac{1}{3}, \frac{1}{3} - \frac{1}{4} \right) \|P_j U\| \left( \frac{1}{3}, \frac{1}{3} \right) \|P_j U\| \left( 0, 0, -\frac{1}{2} \right).
\]
By Strichartz estimate,
\[
\|\Omega ((UU'')_{LL}, U')\|_{L^2_t L^2_x} \lesssim \|UU''\| \left( \frac{1}{3}, \frac{1}{3} \right) \left\| P_k U'' \right\| \left( \frac{1}{3}, \frac{1}{3} \right) \|P_k U''\| \left( 0, 0, -\frac{1}{2} \right) \lesssim \|UU''\| \left( \frac{1}{3}, \frac{1}{3} \right) \left\| P_k U'' \right\| \left( \frac{1}{3}, \frac{1}{3} - \frac{1}{4} \right) \lesssim \|UU''\| \left( \frac{1}{3}, \frac{1}{3} \right) \left\| P_k U'' \right\| \left( \frac{1}{3}, \frac{1}{3} - \frac{1}{4} \right) \left\| P_k U'' \right\| \left( 0, 0, -\frac{1}{2} \right).
\]
For all $(j_1, j_2) \in LL$, we have
\[
\|P_{j_1} U P_{j_2} U'' \| \left( \frac{1}{3}, \frac{1}{3} \right) \lesssim 2^{\frac{1}{2} j_1 + j_2} \|P_{j_1} U\| \left( \frac{1}{3}, \frac{1}{3} - \frac{1}{4} \right) \|P_{j_2} U''\| \left( \frac{1}{3}, \frac{1}{3} - \frac{1}{4} \right).
\]
From the estimate of resonance term, we have
\[
\|P_{\leq -\beta + 10} (UU'')_{HH}\| \left( 1 - 2\varepsilon \frac{1}{3} + 2\varepsilon, 0 \right) \lesssim \|U\|_{S(I)} \|U''\|_{\tilde{S}(I)}.
\]
Therefore,
\[
\|\Omega ((UU'')_{HH}, P_k U')\| \left( 1 - \varepsilon \frac{1}{3} + 3\varepsilon, 10\varepsilon \right) \lesssim \|P_{\leq -\beta + 10} (UU'')_{HH}\| \left( 1 - 2\varepsilon \frac{1}{3} + 2\varepsilon, 0 \right) \|P_k U'\| \left( \varepsilon, \varepsilon, 0 \right) \lesssim \|U\|_{S(I)} \|U''\|_{\tilde{S}(I)} 2^{-\frac{1}{2} k} \|P_k U''\| \left( \varepsilon, \varepsilon, \frac{1}{2} \varepsilon - \kappa \right) \|P_k U''\| \left( \frac{1}{2}, \frac{1}{2} - \frac{1}{4} \right) \times \|P_k U''\| \left( 0, 0, -\frac{1}{2} \right) \|P_k U''\| \left( 0, 0, -\frac{1}{2} \right).
\]
\]
Combining all the above estimates, we obtain a perturbed Strichartz estimate in 4D case:

Proposition 2.10. Let $d = 4$, $0 < \kappa \ll \varepsilon \ll 1$ are small constants. Assume that $U$ is a solution of (2.5) with initial data $U_0 \in H^1_{rad}$, then there exists $\theta > 0$ such that
\[
\|U\|_{S(I)} \lesssim \|U_0\|_{H^1} + 2^{\theta \beta} \|U\|_{S(I)}^{2(1 - 2\varepsilon)} \|U''\|_{S(I)}^{2\varepsilon} + \|U\|_{S(I)}^{\frac{2}{3}} + \|U\|_{S(I)}^{\frac{3}{2}}.
\]
(2.14)
Furthermore, we have small data scattering for (1.1) in 4D case.
3. Variational analysis and Virial/Morawetz estimate

We first review a classical result on the global well-posedness and blow-up dichotomy for Klein-Gordon equations with general nonlinearity $u^{p+1}$, which is due to Payne and Sattinger (see [31]). Assume that $u(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a solution of Klein-Gordon equation

$$\partial_t^2 u - \Delta u + u = f(u),$$

where $f(u) = u^{p+1}$ with $p > 0$. Define that $F(u) := \int f(u) du$ and $G(u) := uf(u) - 2F(u)$. The energy is

$$E(u(t), u_t(t)) = \int_{\mathbb{R}^d} \frac{1}{2} |\partial_t u(t, x)|^2 + \frac{1}{2} |\nabla u(t, x)|^2 + \frac{1}{2} |u(t, x)|^2 - \frac{1}{p+2} u(t)^{p+2} \, dx.$$

Let $Q$ be the ground state, i.e. the unique radial positive solution to the elliptic equation

$$-\Delta Q + Q = Q^{p+1}.$$

Define the stationary energy

$$J(\varphi) = \frac{1}{2} \|\nabla \varphi\|^2_2 + \frac{1}{2} \|\varphi\|^2_2 - \frac{1}{p+2} \int_{\mathbb{R}} \varphi^{p+2} \, dx,$$

where $\varphi \in H^1$. The potential well is

$$j(\lambda) = \mathcal{L}_{\alpha, \beta} J(\varphi) := J(e^{\alpha \lambda} \varphi(e^{-\beta \lambda} x)) = \frac{1}{2} e^{(2\alpha + (d-2)\beta) \lambda} \|\nabla \varphi\|^2_2 + \frac{1}{2} e^{(2\alpha + d\beta) \lambda} \|\varphi\|^2_2 - \frac{1}{p+2} e^{((p+2)\alpha + d\beta) \lambda} \int_{\mathbb{R}} \varphi^{p+2} \, dx.$$

Define the sign functional $K_{\alpha, \beta}(\varphi) := \partial_{\lambda} |_{\lambda=0} \mathcal{L}_{\alpha, \beta} J(\varphi)$, then

$$K_{\alpha, \beta}(\varphi) = \frac{1}{2} (2\alpha + (d-2) \beta) \|\nabla \varphi\|^2_2 + \frac{1}{2} (2\alpha + d\beta) \|\varphi\|^2_2 - \frac{1}{p+2} ((p+2)\alpha + d\beta) \int_{\mathbb{R}} \varphi^{p+2} \, dx.$$

Minimal energy with respect to $K_{\alpha, \beta}$ is

$$m_{\alpha, \beta} := \inf \{ J(\varphi) : \varphi \in H^1 \setminus \{0\}, K_{\alpha, \beta}(\varphi) = 0 \}.$$

We take two subsets in energy space:

$$K^+_{\alpha, \beta} := \{ (u_0, u_1) \in H^1 \times L^2 : E(u_0, u_1) < m_{\alpha, \beta}, K_{\alpha, \beta} (u_0) \geq 0 \},$$

$$K^-_{\alpha, \beta} := \{ (u_0, u_1) \in H^1 \times L^2 : E(u_0, u_1) < m_{\alpha, \beta}, K_{\alpha, \beta} (u_0) < 0 \}.$$

Now, with the above notations, we are prepared to state the dichotomy result:

**Theorem 3.1 ([31]).** Let $f(u) = u^{p+1}$, and $p \in \mathbb{N}_+$, such that $0 < p$ for $d = 1, 2$, and $0 < p < 4/(d-2)$ for $d \geq 3$. Assume that $u(t) \in C(I : H^1)$ is the solution to (3.1) with initial data $u(0, x) = u_0$ and $u_t(0, x) = u_1$, where $I$ is the maximal lifespan interval. If $(u_0, u_1) \in K^+_{\alpha, \beta}$ or $K^-_{\alpha, \beta}$, $u(t)$ is global or blows up in finite time, respectively. Furthermore, we have that $m_{1,0} = E(Q, 0)$.

Particularly, the Theorem holds for quadratic equation (1.1) when $d \leq 5$. We remark that $K_{1,0}$ is used to prove global well-posedness and blow-up dichotomy, but is not sufficient for the scattering.
3.1. Variation in $L^2$-critical case. We consider another dichotomy below the ground state as follows:

$$
\mathcal{K}^+ := \{(u_0, u_1) \in H^1 \times L^2 : E(u_0, u_1) < E(Q, 0), \|u_0\|_2 < \|Q\|_2\},
$$

$$
\mathcal{K}^- := \{(u_0, u_1) \in H^1 \times L^2 : E(u_0, u_1) < E(Q, 0), \|u_0\|_2 > \|Q\|_2\}.
$$

In this subsection, we are going to review the result in [21] that under the $L^2$-critical assumption, the solution starting from $\mathcal{K}^+$ exists globally, and the Virial functional has a positive lower bound. Then, we will prove that the solution to 4D quadratic Klein-Gordon equation (1.1) with initial data in $\mathcal{K}^-$ blows up in finite time. This blow-up result seems to be new, but the proof is essentially the same as that in [20].

Now, suppose that $p = 4/d$. We first recall the classical sharp Gagliardo-Nirenberg inequality:

**Proposition 3.2.** For any $g \in H^1$, we have

$$
\int |g(x)|^{\frac{2(d+2)}{d}} \leq \frac{d+2}{d} \left( \|g\|_2 \right)^{\frac{4}{d}} \|\nabla g\|_2^2,
$$

where the equality holds if and only if $g(x) = \alpha Q(\lambda (x-x_0))$ for some $\alpha \in \mathbb{C}$, $\lambda \in (0, \infty)$, and $x_0 \in \mathbb{R}^d$. Furthermore, suppose that a function $g(x) \in H^1$ satisfies $\|g\|_2 < \|Q\|_2$, then we have

$$
\|\nabla g\|_2^2 - \frac{d}{d+2} \int |g(x)|^{\frac{2(d+2)}{d}} \, dx \geq \left( 1 - \left( \frac{\|g\|_2}{\|Q\|_2} \right)^{\frac{4}{d}} \right) \|\nabla g\|_2^2. \quad (3.2)
$$

As a corollary, the inequality gives us an equivalent characterization for Virial functional

$$
K(g) := \int |\nabla g|^2 - \frac{dp}{2(p+2)} \int |u|^{p+2} = \int |\nabla g|^2 - \frac{d}{d+2} \int |g|^{\frac{2(d+2)}{d}}.
$$

In general, $\|g\|_2 \leq \|Q\|_2$ implies that $K(g) \geq 0$. Recall the energy identity and the Pohozaev identity for the ground state $Q$, i.e.

$$
\|Q\|_{H^1}^2 = \|Q\|_{p+2}^{p+2},
$$

and

$$
\frac{d-2}{2} \|\nabla Q\|_2^2 + \frac{d}{2} \|Q\|_2^2 = \frac{d}{p+2} \|Q\|_{p+2}^{p+2},
$$

which imply

$$
E(Q, 0) = \frac{1}{2} \|Q\|_2^2.
$$

Note that

$$
E(u_0, u_1) = \frac{1}{2} K(u_0) + \frac{1}{2} \|u_0\|_2^2 + \frac{1}{2} \|u_1\|_2^2,
$$

so we have

**Corollary 3.3.** Suppose that $E(u_0, u_1) < E(Q, 0)$, then $\|u_0\|_2 < \|Q\|_2$ if and only if $K(u_0) > 0$ or $u_0 = 0$. 
Proposition 3.4. Let $p = 4/d$. Assume that $\|u_0\|_2 < \|Q\|_2$ and $E(u_0, u_1) < E(Q, 0)$. If $u(t, x) \in C(I : H^1)$ is a solution of (3.1) with initial data $u(0, x) = u_0$ and $u_t(0, x) = u_1$, for all $t \in I$, we have

$$\|u(t)\|_2 < A \|Q\|_2,$$

for some $A = A(E(u_0, u_1)) = A(E(u(t), u_t(t))) < 1$. Moreover,

$$E(u(t), u_t(t)) \sim \|u\|^2_{H^1} + \|u_t\|^2_2,$$

for all $t \in I$.

Proof. First, note that by Gagliardo-Nirenberg inequality, if for any $t \in I$, such that $\|u(t)\|_2 = \|Q\|_2$, we must have $K(u(t)) \geq 0$. Thus,

$$E(u(t), u_t(t)) > \frac{1}{2} \|u(t)\|^2_2 = \frac{1}{2} \|Q\|^2_2,$$

which contradicts to our assumption. Therefore,

$$\|u(t, \cdot)\|_2 < \|Q\|_2,$$

or equivalently, $K(u(t)) \geq 0$ for all $t \in I$.

Next, we are going to derive a gap between $\|u(t)\|_2$ and $\|Q\|_2$. From the assumption, there exists a constant $A < 1$ such that

$$E(u(t), u_t(t)) < \frac{A^2}{2} \|Q\|^2_2.$$

Therefore, $\|u(t)\|_2 < A \|Q\|_2$, for all $t \in I$. Using Gagliardo-Nirenberg inequality again,

$$A^{\frac{2}{d}} \|\nabla u\|^2_2 \geq \frac{d}{d+2} \|u(t, x)\|_{H^{\frac{2(d+2)}{d}}}. $$

Therefore, we have

$$K(u(t)) \geq \left(1 - A^{\frac{2}{d}}\right) \|\nabla u(t)\|^2_2$$

and

$$E(u(t), u_t(t)) > \frac{1}{2} \left(1 - A^{\frac{2}{d}}\right) \|\nabla u(t)\|^2_2 + \frac{1}{2} \|\partial_t u(t)\|^2_2.$$

From the local theory and the uniform bound of $\|u\|_{H^1} + \|\partial_t u\|_{L^2}$, we obtain the global well-posedness for $L^2$-critical equation in $\mathcal{K}^+$. It follows from Theorem 3.1 that for 4D quadratic Klein-Gordon equation (1.1), $\mathcal{K}^+ \subset \mathcal{K}^+_{1,0}$.

Now, we can prove blow-up result in $\mathcal{K}^-$ for 4D quadratic Klein-Gordon equation (1.1).

Proposition 3.5. Suppose that $d = 4$. If $\|u_0\|_2 > \|Q\|_2$ and $E(u_0, u_1) < E(Q, 0)$, the solution to (1.1) blows up in finite time.

Proof. The idea is to prove that $\mathcal{K}_{1,0}^+ = \mathcal{K}^\pm$. It follows from $m_{1,0} = E(Q, 0)$ that

$$\mathcal{K}_{1,0}^+ \cup \mathcal{K}_{1,0}^- = \mathcal{K}^+ \cup \mathcal{K}^-.$$ 

Note that $\mathcal{K}_{1,0}^\pm$ are two disjoint sets, and so do $\mathcal{K}^\pm$. From the definition, we have that $\mathcal{K}_{1,0}^-$ and $\mathcal{K}^\pm$ are open sets. Since $\mathcal{K}^+ \subset \mathcal{K}_{1,0}^+$, it suffices to prove that $\mathcal{K}_{1,0}^+$ is connected.
For any \( \varphi \neq 0 \), such that \((\varphi, u_1)\) in \( K_{1,0}^+ \), we define
\[
\begin{align*}
J_1(\lambda) := J(\lambda \varphi) &= \frac{1}{2} \lambda^2 (\| \nabla \varphi \|^2_2 + \| \varphi \|^2_2) - \frac{1}{3} \lambda^3 \int \varphi^3 \, dx,
\end{align*}
\]
then \( J_1'(\lambda) = \lambda^2 (\| \nabla \varphi \|^2_2 + \| \varphi \|^2_2) - \lambda^3 \int \varphi^3 \, dx \). We have that \( J_1'(1) \geq 0 \). If \( 0 \leq \lambda \leq 1 \), \( J_1'(\lambda) \geq \lambda^2 (1 - \lambda) (\| \nabla \varphi \|^2_2 + \| \varphi \|^2_2) \geq 0 \), so \( J_1(\lambda) \leq J_1(1) < J(Q) - \| u_1 \|^2_2 / 2 \). Note that \( K(\lambda \varphi) = J_1(\lambda) \geq 0 \), so \( \{ \lambda \varphi : 0 \leq \lambda \leq 1 \} \subset K_{1,0}^+ \). \{\lambda \varphi\} is a continuous orbit connecting \( \varphi \) and 0, which implies that \( K_{1,0}^+ \) is connected. Therefore the Proposition follows. 

\[\square\]

3.2. Virial/Morawetz estimate. Take two functions \( h(x) : \mathbb{R}^d \to \mathbb{R} \) and \( q(x) : \mathbb{R}^d \to \mathbb{R} \). Let \( h_j(x) \) be the \( j \)-th coordinate of the vector-valued function \( h(x) \). After integrating by parts, we obtain the Morawetz identity for the general equation (3.1):
\[
- \partial_t \left( \int u_t (h \cdot \nabla u + qu) \, dx \right) = \int \partial_k u \partial_k (h_j) \partial_j u \, dx + \frac{1}{2} \int |u|^2 (-\Delta q(x)) \, dx - \int q(x) G(u) \, dx + \int \left( q(x) - \frac{1}{2} \text{div} h(x) \right) \left( - |u_t|^2 + |\nabla u|^2 + |u|^2 + F(u) \right) \, dx.
\]

See [27] for a more general version of Morawetz identity for complex-valued solution. Using suitable cut-off function, we are able to obtain a decay estimate for focusing equation in radial case, in the spirit of recent work [7]:

**Proposition 3.6** (Virial/Morawetz estimate). Let \( d \geq 3 \) and \( u(t, x) \in C(I : H^4) \) be a solution of (3.1) with initial data \((u_0, u_1) \in H^4_{rad} \times L^2_{rad} \). Suppose that \( E := E(u_0, u_1) > 0 \) and \( \| u(t) \|_{H^3} + \| u_t(t) \|_2 \sim E \). If there exists \( A < 1 \) such that \( \| u(t) \|_2 \leq A \| Q \|_2 \) for all \( t \in I \), we have
\[
\int_{[T, 2T] \cap I} \int_{|x| \leq R} |u|^{p+2} \, dx \, dt \leq C(E, d, p) \left( R + TR^{-\min\{\frac{d+1}{2}, p, 2\}} \right),
\]
for any \( T > 0 \), \( R > 0 \).

**Proof.** Take a cut-off function \( \chi(r) \in C_0^\infty(0, \infty) \) such that \( \chi(r) = 1 \) if \( r \leq 1 \) and \( \chi(r) = 0 \) if \( r > 2 \). \( \chi(R^{-1}r) \) is denoted by \( \chi_R(r) \). Let
\[
\varphi(r) = \int_0^r \chi_R^2(s) \, ds
\]
and
\[
\Psi(x) = \frac{x}{|x|} \varphi(|x|).
\]
By simple computations, we have
\[
\partial_t \Psi_j = \delta_{jk} \frac{\varphi(|x|)}{|x|} + \frac{x_j x_k}{|x|^2} \left( \varphi'(|x|) - \frac{\varphi(|x|)}{|x|} \right),
\]
\[
\text{div}\Psi = \frac{d-1}{|x|} \varphi(|x|) + \varphi'(|x|),
\]
\[
\Delta \text{div}\Psi = \varphi''(|x|) + (d-1) \left( \frac{2}{|x|} \varphi''(|x|) + \frac{d-2}{|x|^2} \right) \varphi'(|x|) - \frac{d-3}{|x|^3} \varphi(|x|),
\]
\[
\varphi'(|x|) = \frac{2}{|x|} \varphi - \frac{1}{|x|^2} \varphi x x = \psi (t x),
\]
and it is easy to see that
\[
\text{div} \Psi \leq \frac{1}{2} \varphi + \frac{2}{|x|} \varphi' + \frac{d-2}{|x|^2} \varphi' + \frac{d-3}{|x|^3} \varphi.
\]

Let \( h(x) = \Psi(x) \) and \( q = \frac{1}{2} \text{div} h(x) \) in the Morawetz identity, then
\[
-\partial_t \left( \int u_t \left( \Psi \cdot \nabla u + \frac{1}{2} \text{div}(\Psi) u \right) \right) \, dx
\]
\[
= \int u_k \partial_k \Psi_j u_j \, dx - \frac{1}{4} \int \Delta \text{div}\Psi |u|^2 \, dx + \int \frac{1}{2} \text{div}(\Psi) G(u) \, dx
\]
\[
= \int \left( \frac{\varphi}{|x|} \nabla u^2 + \left( \varphi' - \frac{\varphi}{|x|} \right) \frac{x_j x_k}{|x|^2} \nabla u \right) \, dx + \int \left( \frac{d-1}{|x|} \varphi + \frac{1}{2} \varphi' \right) G(u) \, dx
\]
\[
- \frac{1}{4} \int \left( \varphi'' + (d-1) \left( \frac{2}{|x|} \varphi'' + \frac{d-2}{|x|^2} \right) \varphi' \right) |u|^2 \, dx + \frac{d-3}{4} \int \varphi |u|^2 \, dx
\]
\[
= \int \varphi' |\nabla u|^2 \, dx + \frac{d}{2} \int \varphi' G(u) \, dx + \frac{d-1}{2} \int \left( \frac{\varphi}{|x|} - \varphi' \right) G(u) \, dx
\]
\[
- \frac{1}{4} \int \left( \varphi'' + (d-1) \left( \frac{2}{|x|} \varphi'' + \frac{d-2}{|x|^2} \right) \varphi' \right) |u|^2 \, dx + \frac{d-3}{4} \int \varphi |u|^2 \, dx
\]
From the definition of \( \varphi \), we can estimate easily
\[
\left| \int \left( \varphi'' + (d-1) \left( \frac{2}{|x|} \varphi'' + \frac{d-2}{|x|^2} \right) \varphi' \right) |u|^2 \, dx \right| \leq \frac{1}{R^2} C(E).
\]

The Virial/Morawetz quantity is denoted by
\[
M(t) := - \left( \int u_t \left( \Psi \cdot \nabla u + \frac{1}{2} \text{div}(\Psi) u \right) \right) \, dx,
\]
and it is easy to see that \( |M(t)| \leq R \). Therefore, noting that \( G(u) = pu^{p+2}/(p+2) \),
\[
\partial_t M(t) \geq \int \chi_R^2 \left( |\nabla u|^2 + \frac{d}{2} \right) G(u) \, dx - C \int \left| \frac{\varphi}{|x|} - \varphi' \right| |u|^{p+2} \, dx - C(E) \frac{1}{R^2}.
\]
In order to deal with the main term, since \( \|\chi_R u\|_2 \leq \|u\|_2 \leq \|Q\|_2 \), by Gagliardo-Nirenberg inequality,
\[
\int \left( |\nabla (\chi_R u)|^2 - \frac{dp}{2 (p+2)} |\chi_R u|^{p+2} \right) \, dx \geq C \int |\chi_R u|^{p+2} \, dx.
\]
As for the remainder terms, we have that the cut-off function \( \varphi(r)/r - \varphi'(r) = 0 \) if \( r \leq R \), and \( 0 \leq \varphi/r - \varphi' \leq R/r \) if \( r \geq R \). We also have radial Sobolev inequality
\[
\| |x|^{-\frac{(d-1)p}{2}} |u|^p \|_{L^p} \leq C(E).
\]
Therefore,
\[
\partial_t M(t) \geq \int \chi_R^2 \left( |\nabla u|^2 - \frac{dp}{2(p+2)} |u|^{p+2} \right) \, dx \\
-C \int_{|x| \geq R} \frac{R}{r} |u|^{p+2} \, dx - \frac{1}{R^2} C(E) \\
\geq \int (|\nabla (\chi_R u)|^2 - \frac{dp}{2(p+2)} |\chi_R u|^{p+2}) \, dx \\
-C \int_{|x| \geq R} |u|^{p+2} \, dx - \frac{1}{R^2} C(E) \\
\geq A \int |\chi_R u|^{p+2} \, dx - \frac{C(E)}{R^{d+1}} \int_{|x| \geq R} |u|^2 \, dx - \frac{1}{R^2} C(E),
\]
where we use the identity
\[
\int \chi_R |\nabla u|^2 \, dx = \int |\nabla (\chi_R u)|^2 \, dx + \int \chi_R \Delta (\chi_R)|u|^2 \, dx
\]
for the second inequality. Integrate in \( t \) on \([T, 2T]\), then the Proposition follows. \( \square \)

**Corollary 3.7.** Let \( d = 4 \). Suppose that \( u \) is a radial solution of quadratic Klein-Gordon equation (1.1), whose initial data \((u_0, u_1)\) satisfies \( \|u_0\|_2 < \|Q\|_2, E(u_0, u_1) < E(Q, 0) \). Define that \( E := E(u_0, u_1) > 0 \), then for any \( \varepsilon_0 > 0 \), \( T > 1 \) and \( \tau > 0 \), there exists \( T_0 = T_0(\varepsilon_0, T, E) \geq T \), such that
\[
\int_{T_0}^{T_0 + \tau} \int_{|x| \leq |t|^{5/2}} |u(t, x)|^3 \, dx \, dt \leq \varepsilon_0. \quad (3.4)
\]

**Proof.** In this case, \( G(u) = 2u^2/3 \). First, by variation, it follows from \( (u_0, u_1) \in K^+ \) that the assumptions in Proposition 3.6 hold for 4D quadratic equation. Taking \( R = T^{2/3} \), we have
\[
\int_{T}^{2T} \int_{|x| \leq T^{2/3}} |u(t, x)|^{p+2} \, dx \, dt \leq C(E)T^{\frac{8}{5}},
\]
and then
\[
\int_{T}^{2T} t^{-1} \int_{|x| \leq t^{2/5}} |u(t, x)|^{p+2} \, dx \, dt \leq C(E)T^{-\frac{2}{5}}.
\]
Summation the above integral over \([2^kT, 2^{k+1}T]\) for \( k = 0, 1, 2, \ldots \) yeilds
\[
\int_{T}^{\infty} t^{-1} \int_{|x| \leq t^{2/5}} |u(t, x)|^{p+2} \, dx \, dt \leq C(E)T^{-\frac{3}{5}}.
\]
For any fixed \( \tau > 0 \), divide the above integral into \([T + k\tau, T + (k + 1)\tau]\) for non-negative integer \( k \), i.e.
\[
\sum_{k \in \mathbb{N}} \frac{1}{T + (k + 1)\tau} \int_{T + k\tau}^{T + (k+1)\tau} \int_{|x| \leq |t|^{2/5}} |u(t, x)|^{p+2} \, dx \, dt \leq C(E).
\]
Since the series
\[
\sum_{k=0}^{+\infty} \frac{1}{T + (k + 1)\tau}
\]
diverges, there exists a $T_0 = T + k_0 \tau$ such that
\[ \int_0^{T_0+\tau} \int_{|x| \leq |t|^{\frac{2}{3}}} |u(t, x)|^3 \, dx \, dt \leq \varepsilon_0, \]
and the Corollary follows. \(\square\)

4. LARGE DATA SCATTERING IN 4D CASE

4.1. $L^3$ decay after large time. Corollary 3.7 yields that localised $L^3_x$ norm of $u$ decays on arbitrarily large time interval, which is not sufficient for large data scattering. After normal form reduction, we need $L^3$ decay of $U = u - i(D)^{-1}u_t$ to establish the space-time bound. Now, we go back to the first order equation (2.3):
\[ U(t, x) = K(t)U_0(x) - i \int_0^t K(t - s)(D)^{-1}(u(s, x)^2) \, ds. \]

**Proposition 4.1.** Let $d = 4$. Suppose that $u$ is a radial solution of quadratic Klein-Gordon equation (1.1). For any $\varepsilon_1 > 0$ and $T > 0$, there exists $\tau_1 = \tau_1(E, \varepsilon_1) \geq C(E)\varepsilon_1^{-5}$ and $T_1 = T_1(E, \varepsilon_1, T)$, such that $T < T_1 - \tau_1$, and
\[ \sup_{t \in [T_1 - \tau_1, T_1]} \|U(t, x)\|_{L^2} \leq \varepsilon_1. \] (4.1)

**Proof.** Take a large constant $\tau_1 > 0$ and $R > 0$ that will be defined later. We estimate $L^3_x$ norm of $u(t, x)$, and divide it into four parts
\[ \|U(t, x)\|_{L^2} \leq \|K(t)U_0(x)\|_{L^2} \] (4.2)
\[ + \left\| \int_0^{t-\tau_1} K(t - s)(D)^{-1}(u(s, x)^2) \, ds \right\|_{L^3_x} \] (4.3)
\[ + \int_{t-\tau_1}^t |t - s|^{-\frac{2}{3}} \|u(s, x)\|_{L^3_x}^2 \, ds \] (4.4)
\[ + \int_{t-\tau_1}^t |t - s|^{-\frac{2}{3}} \|u(s, x)\|_{L^3_x}^2 \, ds. \] (4.5)

First, we bound (4.2). Let $v(t) := K(t)U_0$. From radially improved Strichartz estimates, for any $2 < q < 3$, we have
\[ \|v(t)\|_{L^q_x L^2_t} \leq C(E). \]

Note that $\|\partial_t v(t, x)\|_{L^2} \lesssim \|v(t, x)\|_{H^1}$, then $v(t)$ is Lipschitz continuous from $\mathbb{R} \to L^3_x$. Thus, we must have
\[ \|K(t)U_0(x)\|_{L^2} \to 0, \] (4.6)
when $t \to \pm \infty$.

Note that
\[ \int_0^{t-\tau_1} K(t - s)(D)^{-1}(u(s, x)^2) \, ds = K(t - t_1)U(t - \tau_1) - K(t)U_0, \]
and
\[ \left\| \int_0^{t-\tau_1} K(t - s)(D)^{-1}(u(s, x)^2) \, ds \right\|_{L^\infty_x} \leq C \int_0^{t-\tau_1} |t - s|^{-2} \|u\|_{L^2}^2 \, ds \leq C(E)\tau_1^{-1}. \]
Therefore, (4.3) can be bounded by
\[
\left\| \int_0^{t-\tau_1} K(t-s)\langle D \rangle^{-1} (u(s,x)^2) \, ds \right\|_{L^3_x} \\
\leq C \left\| \int_0^{t-\tau_1} K(t-s)\langle D \rangle^{-1} (u(s,x)^2) \, ds \right\|_{L^\infty_x} \left\| K(\tau_1) U(t-\tau_1) - K(t)U_0 \right\|_{L^2_x} \\
\leq C(E)\tau_1^{-\frac{1}{4}}.
\]

Let \( R = \|s\|^{2/5} \). From radial Sobolev inequality, we have
\[
\int_{|x| \geq R} |u(t,x)|^3 \, dx \leq CR^{-\frac{3}{2}} \int_{|x| \geq R} |u(t,x)|^2 |x|^{\frac{3}{2}} u \, dx \\
\leq C(E)\|s\|^{-\frac{3}{8}}.
\]

An elementary inequality yields
\[
\int_{t-\tau_1}^t |t-s|^{-\frac{3}{4}} \|u(s,x)\|_{L^3_x}^2 \, ds \leq C(E) \int_{t-\tau_1}^t |t-s|^{-\frac{3}{4}} |s|^{-\frac{3}{8}} \, ds \\
\leq C(E) \left( \int_{t-\tau_1}^{t-1} |s|^{-\frac{3}{8}} \, ds + t^{-\frac{3}{8}} \int_{t-1}^t |t-s|^{-\frac{3}{8}} \, ds \right) \\
\leq C(E) \left( \tau_1^{-\frac{1}{4}} + t^{-\frac{3}{8}} \right).
\]

For any \( \varepsilon_1 > 0 \) and \( T > 0 \), there exists \( \tilde{T} = \tilde{T}(\varepsilon_1,T) > T \) and \( \tau_1 = \tau_1(E,\varepsilon_1) \geq C(E)\varepsilon_1^{-5} \), such that for any \( t > \tilde{T} \),
\[
(4.2) + (4.3) + (4.4) \leq \frac{1}{2} \varepsilon_1.
\]

Finally,
\[
\int_{t-\tau_1}^t |t-s|^{-\frac{3}{4}} \|u(s,x)\|_{L^3_x}^2 \, ds \leq C(E) \int_{t-\tau_1}^t |t-s|^{-\frac{3}{4}} \|u(s,x)\|_{L^3_x}^{\frac{3}{2}} \, ds \\
\leq C(E)\tau_1^{\frac{3}{8}} \left( \int_{t-\tau_1}^t \int_{|x| \leq R} |u(s,x)|^{\frac{3}{2}} \, dx \, ds \right)^{\frac{1}{3}}.
\]

By Corollary 3.7, for the above \( \tilde{T} \), take \( \tau = 2\tau_1 \) and \( \varepsilon_0 \leq C(E)\tau_1^{-2}\varepsilon_1^{4} \). Therefore, there exists \( \tilde{T}_0 = \tilde{T}_0(E,\varepsilon_1,T) \geq \tilde{T} \) such that for all \( t \in [\tilde{T}_0 + \tau_1, \tilde{T}_0 + 2\tau_1] \),
\[
(4.5) \leq \frac{1}{2} \varepsilon_1.
\]

Now we take \( T_1 = \tilde{T}_0 + 2\tau_1 \), and the Proposition follows. \( \square \)
4.2. Proof of Theorem 1.3. Let $T_2 > 1$ and $\tau_2 > 0$ will be defined later. The equation after normal Form reduction can be rewritten as

$$U(t, x) = K(t) \left( U_0 + i \langle D \rangle^{-1} \Omega(U, U)(0) \right)$$

$$- i \langle D \rangle^{-1} \Omega(U, U)$$

$$- i \int_0^{T_2 - \tau_2} K(t - s) \langle D \rangle^{-1} \left( (UU)_{LH+HL+HH} + 2\Omega \left( -i \langle D \rangle^{-1} U^2, U \right) \right) \, ds$$

$$- i \int_{T_2 - \tau_2}^{T_2} K(t - s) \langle D \rangle^{-1} \left( (UU)_{LH+HL+HH} + 2\Omega \left( -i \langle D \rangle^{-1} U^2, U \right) \right) \, ds$$

$$- i \int_{T_2}^{t} K(t - s) \langle D \rangle^{-1} \left( (UU)_{LH+HL+HH} + 2\Omega \left( -i \langle D \rangle^{-1} U^2, U \right) \right) \, ds.$$

(4.7)

(4.8)

(4.9)

(4.10)

(4.11)

First, we have Strichartz bound

$$\| K(t) \left( U_0 + i \langle D \rangle^{-1} \Omega(U, U)(0) \right) \|_{\tilde{S}(I)} \lesssim \| U_0 \|_{H^1} + \| U_0 \|_{H^1}^2,$$

then for any $\varepsilon_1 > 0$, there exists $\tilde{T} = \tilde{T}(\varepsilon_1) > 0$, such that

$$\| K(t) \left( U_0 + i \langle D \rangle^{-1} \Omega(U, U)(0) \right) \|_{\tilde{S}(T_2, \infty)} \lesssim \varepsilon_1,$$

for all $T > \tilde{T}$. Thus, we take some $T_2 > \tilde{T}$. We also have

$$\| \Omega(U, U) \|_{\tilde{S}(T_2, \infty)} \lesssim 2^{-\beta} C(E) \| U \|_{\tilde{S}(T_2, \infty)}.$$

By interpolation,

$$\| f(x) \|_{Z(I)} \lesssim \| f \|_{L_{t,x}^1} \| f \|_{L_{t,x}^2}^{2\beta}.$$

Note we also have

(4.9) $$= -i[K(t - T_2 + \tau_2) \left( U(T_2 - \tau_2) + i \langle D \rangle^{-1} \Omega(U, U)(T_2 - \tau_2) \right)$$

$$- K(t) \left( U_0 + i \langle D \rangle^{-1} \Omega(U, U)(0) \right)].$$

Therefore,

$$\| (4.9) \|_{\tilde{S}(T_2, \infty)} \lesssim \| (4.9) \|_{\tilde{S}(T_2, \infty)}^{1 - 2\varepsilon} \| (4.9) \|_{\tilde{S}(0, \infty)}^{2\varepsilon} \| (4.9) \|_{\tilde{S}(0, \infty)}^{2\varepsilon} \| (4.9) \|_{\tilde{S}(0, \infty)}^{2\varepsilon} \| (4.9) \|_{\tilde{S}(0, \infty)}^{2\varepsilon}.$$

Take $p$ such that $\frac{1}{p} = \frac{1}{8} + \frac{2}{q}$, then by Sobolev embedding, we have

$$\| K(t - s) \langle D \rangle^{-1} \left( (UU)_{LH+HL+HH} + 2\Omega \left( -i \langle D \rangle^{-1} U^2, U \right) \right) \|_{\tilde{S}(0, \infty)}^{1 - 2\varepsilon} \| (4.9) \|_{\tilde{S}(0, \infty)}^{2\varepsilon} \| (4.9) \|_{\tilde{S}(0, \infty)}^{2\varepsilon} \| (4.9) \|_{\tilde{S}(0, \infty)}^{2\varepsilon} \| (4.9) \|_{\tilde{S}(0, \infty)}^{2\varepsilon}.$$

$$|t - s|^{-2(1 - \frac{3}{2})} \| 2^k \left( \| U \|_{L_{t,x}^1}^{2} + \| U \|_{L_{t,x}^3}^{3} \right) \|_{L_{t,x}^2}^{2\varepsilon} \| (4.9) \|_{\tilde{S}(0, \infty)}^{2\varepsilon} \| (4.9) \|_{\tilde{S}(0, \infty)}^{2\varepsilon} \| (4.9) \|_{\tilde{S}(0, \infty)}^{2\varepsilon} \| (4.9) \|_{\tilde{S}(0, \infty)}^{2\varepsilon}.$$

$$|t - s|^{-2(1 - \frac{3}{2})} + |t - s|^{-2}.$$
Thus,
\[
\| (4.9) \|_{\tilde{S}(T_2, +\infty)} \leq \| (4.9) \|_{\tilde{S}(T_2, +\infty)}^{1-2\varepsilon} \| (4.9) \|_{(0, -\frac{1}{4})}^{2\varepsilon} \\
\leq C(E) \int_{0}^{T_2-T_2} \left( |t-s|^{2(1-\frac{2}{q})} + |t-s|^2 \right) ds \|_{L^\infty_{\tilde{T}T_2}}^{2\varepsilon} \\
\leq C(E) T_2^{-\varepsilon-2\varepsilon}.
\]

Next, we estimate (4.10). From radially improved Strichartz estimate, we can take
\[
\| U \|_{\text{Strz}} := \bigcap_{2 \leq q \leq +\infty, 2 \leq r \leq \frac{3}{2} + \frac{1}{2}} \| U \|^{\frac{2}{3}}_{(\frac{1}{q}, \frac{2}{q}, \frac{3}{2} - 2\varepsilon, \frac{1}{2})}.
\]

Note that
\[
\| U \|_{(\frac{1}{q}, \frac{2}{q}, \frac{3}{2} - 2\varepsilon, \frac{1}{2})} \leq \| U \|_{\text{Strz}}.
\]

Therefore, we use a standard local theory to obtain
\[
\| U \|_{S(I)} \leq C(E) \| I \|^{\frac{1}{2}}.
\]

We also have interpolation
\[
\| f \|_{L^\infty(\mathcal{B}^{-\frac{1}{2}+\frac{3}{4}+\delta})} \leq \| f \|_{L^\infty}^{1-3\delta} \| f \|_{H^\frac{1}{2}}^{3\delta}.
\]

By Proposition 4.1, for \( \varepsilon_1 > 0 \) and the above \( \tilde{T} \), there exists \( \tilde{T}_1 = C(E)\varepsilon_1^{-5} \) and \( T_2 \), such that
\[
\| U \|_{L^\infty_1(T_2 - \tilde{T}_1; T_2; L^2_1)} \leq \varepsilon_1.
\]

Take \( T_2 = \varepsilon_1^{-3\varepsilon} \). Note that \( [T_2 - \tau_2, T_2] \subset [T_2 - \tilde{T}_1, T_2] \) for sufficiently small \( \varepsilon_1 \), then we have
\[
\| (4.10) \|_{\tilde{S}(T_2, +\infty)} \leq C(E) \left( \| U \|_{S(T_2, T_2)}^{2-4\varepsilon} \| U \|_{Z(T_2 - \tau_2; T_2)}^{4\varepsilon} \| U \|_{S(T_2, T_2)}^{6\varepsilon} \| Z(T_2 - \tau_2; T_2) \right) \\
\leq C(E) \left( \left\langle \tau_2 \right\rangle^{1-2\varepsilon} \| U \|_{L^{6(1-3\delta)}_3(T_2 - \tau_2; T_2: L^2_1)}^{4\varepsilon(1-3\delta)} \left\langle \tau_2 \right\rangle^{\frac{1}{2} - 3\varepsilon} \| U \|_{L^{6(1-3\delta)}_3(T_2 - \tau_2; T_2: L^2_1)}^{6\varepsilon(1-3\delta)} \right) \\
\leq C(E) \left( \left\langle \tau_2 \right\rangle^{1-2\varepsilon} \varepsilon_1^{4\varepsilon(1-3\delta)} + \left\langle \tau_2 \right\rangle^{\frac{1}{2} - 3\varepsilon} \varepsilon_1^{6\varepsilon(1-3\delta)} \right) \\
\leq C(E) \varepsilon_1^{\frac{1}{2} - \varepsilon}.
\]

Above all, we have
\[
\| U \|_{\tilde{S}(T_2, +\infty)} \leq C(E) \left( \varepsilon_1 + 2^{-\beta} \| U \|_{\tilde{S}(T_2, +\infty)}^{3\varepsilon^2} + \| U \|_{\tilde{S}(T_2, +\infty)}^{2} + \| U \|_{\tilde{S}(T_2, +\infty)}^{3} \right) \\
\leq C(E) \varepsilon_1^{\frac{1}{2} - \varepsilon}.
\]

Take a large \( \beta = \beta(E) > 0 \), such that
\[
C(E) 2^{-\beta} < \frac{1}{2}.
\]

A standard bootstrap argument yields that for some \( T_2 = T_2(\varepsilon_1) \), \( \| U \|_{\tilde{S}(T_2, +\infty)} \leq C(E)\varepsilon_1^{3\varepsilon^2} \). Thus, we have \( \| U \|_{\tilde{S}(0, +\infty)} \leq C(E) \).
Finally, we can prove the large data scattering. We need to show that when 
\( t \to \pm \infty \), \( e^{-it\langle D \rangle}U(t) \) has limit in \( H^1 \). Note that 
\[
K(-t)U(t) = U_0 + i(D)^{-1}\Omega(U,U)(0) - iK(-t)(D)^{-1}\Omega(U,U) \\
- \int_0^t K(-s)(D)^{-1}T_{\text{Res}}(U,U) \, ds \\
- 2i \int_0^t K(-s)(D)^{-1} \left( \Omega\left(-i(D)^{-1}U^2, U \right) \right) \, ds.
\]
By Strichartz estimates, we have 
\[
\left\| \int_{t_1}^{t_2} K(-s)(D)^{-1}T_{\text{Res}}(U,U) \, ds \right\|_{H^1} \lesssim \| U \|_{\tilde{S}(t_1,t_2)}^2
\]
and 
\[
\left\| \int_{t_1}^{t_2} K(-s)(D)^{-1} \left( \Omega\left(-i(D)^{-1}U^2, U \right) \right) \, ds \right\|_{H^1} \lesssim \| U \|_{\tilde{S}(t_1,t_2)}^3.
\]
By the boundedness of \( \tilde{S} \) norm, the integral term in \( K(-t)U \) has limit in \( H^1 \). For the scattering, it suffices to prove that \( \| K(-t)(D)^{-1}\Omega(U,U) \|_{H^1} \) tends to 0, when \( t \to \pm \infty \). We know that \( P_{<0}U \in (1/3, 1/6, -13/21)_R \) from the boundedness of \( \tilde{S} \), then \( \| P_{<0}U \|_{L^2([R, L^2])} \lesssim 1 \). Since \( u(t) \) is bounded in \( H^1 \), we also have that 
\[
\| P_{<0}U(t_1) - P_{<0}U(t_2) \|_{L^2} \lesssim \| P_{<0}U(t_1) - P_{<0}U(t_2) \|_{L^2} \\
\lesssim |t_1 - t_2| \sup_{[t_1, t_2]} \| P_{<0}\partial_t u(t) \|_{L^2} \\
\lesssim |t_1 - t_2| \sup_{[t_1, t_2]} \| P_{<0} \left( \langle D \rangle U + \langle D \rangle^{-1} (U^2) \right) \|_{L^2} \lesssim |t_1 - t_2|,
\]
which implies that \( \| P_{<0}U(t) \|_{L^2} \) is Lipschitz continuous in \( t \). Thus, we obtain that 
\[
\lim_{t \to \pm \infty} \| P_{<0}U(t) \|_{L^2} = 0.
\]
By the boundedness of Coifman-Meyer bilinear operator, 
\[
\| \Omega(U, U) \|_{H^1} \lesssim \| P_{<0}U \|_{L^2} \| P_{<0}U \|_{L^6}.
\]
Therefore, we have that \( \lim_{t \to \pm \infty} \| \Omega(U, U) \|_{H^1} = 0 \), which completes the proof of Theorem 1.3.

Acknowledgements
The authors are grateful to Professor Baoxiang Wang and Professor Kenji Nakanishi for helpful discussions. Z. Guo is partially supported by ARC DP170101060. J. Shen was supported by China Scholarship Council.

References
[1] P. Brenner. On space-time means and everywhere defined scattering operators for nonlinear Klein-Gordon equations. Mathematische Zeitschrift, 186(3):383–391, 1984.
[2] T. Cazenave. Semilinear schrödinger equations, volume 10. American Mathematical Soc., 2003.
[3] J.-M. Delort. Existence globale et comportement asymptotique pour l’équation de Klein–Gordon quasi linéaire à données petites en dimension 1. In Annales scientifiques de l’École normale supérieure, volume 34, pages 1–61. No longer published by Elsevier, 2001.

[4] J.-M. Delort, D. Fang, and R. Xue. Global existence of small solutions for quadratic quasilinear Klein–Gordon systems in two space dimensions. Journal of Functional Analysis, 211(2):288–323, 2004.

[5] B. Dodson. Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state. Advances in Mathematics, 285:1589–1618, 2015.

[6] B. Dodson. Global well-posedness and scattering for the defocusing, $L^2$-critical, nonlinear Schrödinger equation when $d = 2$. Duke Mathematical Journal, 165(18):3435–3516, 2016.

[7] B. Dodson and J. Murphy. A new proof of scattering below the ground state for the 3d radial focusing cubic NLS. Proceedings of the American Mathematical Society, 145(11):4859–4867, 2017.

[8] V. Georgiev and B. Yordanov. Asymptotic behaviour of the one-dimensional Klein–Gordon equation with a cubic nonlinearity. 1996.

[9] J. Ginibre and G. Velo. The global Cauchy problem for the non linear Klein-Gordon equation. Mathematische Zeitschrift, 189(4):487–505, 1985.

[10] R. T. Glassey. On the asymptotic behavior of nonlinear wave equations. Transactions of the American Mathematical Society, 182:187–200, 1973.

[11] Z. Guo, Z. Hani, and K. Nakanishi. Scattering for the 3D Gross–Pitaevskii Equation. Communications in Mathematical Physics, 359(1):265–295, 2018.

[12] Z. Guo, S. Lee, K. Nakanishi, and C. Wang. Generalized Strichartz estimates and scattering for 3D Zakharov system. Communications in Mathematical Physics, 331(1):239–259, 2014.

[13] Z. Guo and K. Nakanishi. Small Energy Scattering for the Zakharov System with Radial Symmetry. International Mathematics Research Notices, 2014(9):2327–2342, 2013.

[14] Z. Guo, K. Nakanishi, and S. Wang. Global dynamics below the ground state energy for the Zakharov system in the 3D radial case. Advances in Mathematics, 238:412–441, 2013.

[15] Z. Guo, K. Nakanishi, and S. Wang. Global dynamics below the ground state energy for the Klein-Gordon-Zakharov system in the 3D radial case. Communications in Partial Differential Equations, 39(6):1158–1184, 2014.

[16] Z. Guo, K. Nakanishi, and S. Wang. Small energy scattering for the Klein-Gordon-Zakharov system with radial symmetry. Mathematical Research Letters, 21(4):733–755, 2014.

[17] Z. Guo and Y. Wang. Improved Strichartz estimates for a class of dispersive equations in the radial case and their applications to nonlinear Schrödinger and wave equations. Journal d’Analyse Mathématique, 124(1):1–38, 2014.

[18] N. Hayashi and P. I. Naumkin. Scattering operator for nonlinear Klein–Gordon equations. Communications in Contemporary Mathematics, 11(05):771–781, 2009.

[19] S. Ibrahim, N. Masmoudi, and K. Nakanishi. Scattering threshold for the focusing nonlinear Klein-Gordon equation. Anal. PDE, 4(3):405–460, 2011.
[20] S. Ibrahim, N. Masmoudi, and K. Nakanishi. Threshold solutions in the case of mass-shift for the critical Klein-Gordon equation. *Transactions of the American Mathematical Society*, 366(11):5653–5669, 2014.

[21] R. Killip, B. Stovall, and M. Visan. Scattering for the cubic Klein–Gordon equation in two space dimensions. *Transactions of the American Mathematical Society*, 364(3):1571–1631, 2012.

[22] S. Klainerman. Global existence of small amplitude solutions to nonlinear klein-gordon equations in four space-time dimensions. *Communications on Pure and Applied Mathematics*, 38(5):631–641, 1985.

[23] M. Kowalczyk, Y. Martel, and C. Muñoz. Kink dynamics in the $\varphi^4$ model: Asymptotic stability for odd perturbations in the energy space. *Journal of the American Mathematical Society*, 30(3):769–798, 2017.

[24] A. Matsumura. On the asymptotic behavior of solutions of semi-linear wave equations. *Publications of the Research Institute for Mathematical Sciences*, 12(1):169–189, 1976.

[25] K. Nakanishi. Energy scattering for nonlinear Klein–Gordon and Schrödinger equations in spatial dimensions 1 and 2. *Journal of Functional Analysis*, 169(1):201–225, 1999.

[26] K. Nakanishi. Scattering theory for the nonlinear Klein-Gordon equation with Sobolev critical power. *International Mathematics Research Notices*, 1999(1):31–60, 1999.

[27] K. Nakanishi. Remarks on the energy scattering for nonlinear Klein-Gordon and Schrödinger equations. *Tohoku Mathematical Journal, Second Series*, 53(2):285–303, 2001.

[28] K. Nakanishi. Transfer of global wellposedness from nonlinear Klein-Gordon equation to nonlinear Schrödinger equation. *Hokkaido Mathematical Journal*, 37(4):749–771, 2008.

[29] T. Ogawa and Y. Tsutsumi. Blow-up of $H^1$ solution for the nonlinear Schrödinger equation. *Journal of Differential Equations*, 92(2):317–330, 1991.

[30] T. Ozawa, K. Tsutaya, and Y. Tsutsumi. Global existence and asymptotic behavior of solutions for the Klein-Gordon equations with quadratic nonlinearity in two space dimensions. *Mathematische Zeitschrift*, 222(3):341–362, 1996.

[31] L. E. Payne and D. H. Sattinger. Saddle points and instability of nonlinear hyperbolic equations. *Israel Journal of Mathematics*, 22(3-4):273–303, 1975.

[32] T. Schottdorf. Global existence without decay for quadratic Klein-Gordon equations. *arXiv preprint arXiv:1209.1518*, 2012.

[33] J. Shatah. Normal forms and quadratic nonlinear Klein-Gordon equations. *Communications on Pure and Applied Mathematics*, 38(5):685–696, 1985.

[34] W. A. Strauss. Nonlinear scattering theory at low energy. *Journal of functional analysis*, 41(1):110–133, 1981.

[35] B. Wang. On existence and scattering for critical and subcritical nonlinear Klein-Gordon equations in $H^s$. *Nonlinear Analysis: Theory, Methods & Applications*, 31(5-6):573–587, 1998.

[36] B. Wang. Scattering of solutions for critical and subcritical nonlinear Klein-Gordon equations in $H^s$. *Discrete & Continuous Dynamical Systems-A*, 5(4):753–763, 1999.
