Damaging 2D Quantum Gravity

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Abstract

We investigate numerically the behaviour of damage spreading in a Kauffman cellular automaton with quenched rules on a dynamical $\phi^3$ graph, which is equivalent to coupling the model to discretized 2D gravity. The model is interesting from the cellular automaton point of view as it lies midway between a fully quenched automaton with fixed rules and fixed connectivity and a (soluble) fully annealed automaton with varying rules and varying connectivity. In addition, we simulate the automaton on a fixed $\phi^3$ graph coming from a 2D gravity simulation as a means of exploring the graph geometry.
Cellular automata have found applications in many areas of physics and other sciences. One particularly intriguing idea was put forward by Kauffman in the 1960s
, who suggested that a random Boolean network might serve as a model for cell differentiation. This predated the more recent explosion of papers on the subject that was largely inspired by the work of Wolfram
. In the Kauffman model each of \( i = 1, 2, \ldots, N \) lattice sites contains a spin which can take the values \( \pm 1 \) (or equivalently a Boolean variable that can take the values 0, 1) and the spins evolve according to

\[
\sigma_i(t+1) = f_i(\sigma_{j_1}(t), \sigma_{j_2}(t), \ldots, \sigma_{j_k}(t))
\]

where the functions \( f_i \) are chosen independently and each site \( i \) has \( K \) specified inputs sites \( j_1(i), \ldots, j_k(i) \).

The original biologically motivated model of Kauffman chose the \( f_i \) randomly at \( t = 0 \) with a probability \( p = 1/2 \) for \( f_i = +1 \) and a probability \( 1 - p = 1/2 \) for \( f_i = -1 \). For each site and also chose the \( j_1(i), j_2(i), \ldots, j_k(i) \) inputs randomly from the \( N \) lattice sites at \( t = 0 \). This version of the model thus displays quenched randomness as both the functions \( f_i \) and the input sites are fixed and has been called “infinite dimensional” because of the non-locality of the inputs. The model must display periodic behaviour for finite \( N \) as there are only \( 2^N \) possible spin configurations, so one can enquire about the length and number of limit cycles as well as quantities such as the Hamming distance between different configurations 1 and 2, \( D_{12}(t) \), defined as

\[
D_{12}(t) = \frac{1}{N} \sum_{i=1}^{N} (\sigma_1^i(t) - \sigma_2^i(t))^2
\]

where \( \sigma^1 \) is a spin in configuration 1 and \( \sigma^2 \) is a spin in configuration 2. The application to genetics arises from identifying the stable limit cycles with different stable genotypes.

It was found (numerically)
 that for \( K \leq 2 \) the length of cycles was small, of order \( \beta(K)N^{1/2} \), whereas for \( K \geq 3 \) it increased as \( \exp(\alpha(N)N) \), which implied a critical \( K_c \) between 2 and 3, with corresponding singular behaviour in \( \alpha(K) \) and \( \beta(K) \). The \( K \leq 2 \) phase was called a frozen phase and had

\[
\lim_{t \to \infty} D_{12}(t) = 0 \tag{3}
\]

for any starting configurations 1 and 2, whereas the \( K > 2 \) phase was chaotic, having

\[
\lim_{t \to \infty} D_{12}(t) = d_{final} \tag{4}
\]

with the final distance \( d_{final} \) independent of the initial distance. A simple annealed approximation to the model was solved by Derrida and Pomeau
 and captured these features giving a critical value of \( K = 2 \). One can think of implementing the approximation in a simulation by changing both the functions and the inputs and each time step, which amounts to neglecting all the correlations.

In \( \square \) Derrida and Stauffer investigated a finite dimensional variant of the model in which the \( K \) inputs were chosen from the neighbours of a site on various lattices. They varied the probability \( p, (1-p) \) for choosing \( f_i = +1, (-1) \), which can be shown to give the condition

\[
2Kp_c(1-p_c) = 1 \tag{5}
\]

for the transition between the frozen and chaotic phases, where \( p \) is now used as a control parameter rather than \( K \). Derrida and Stauffer pointed out that the Kauffman model with varying \( f_i \) on a \( d \) dimensional lattice was equivalent (as far as distances and overlaps between configurations were concerned) to \( d + 1 \) dimensional directed percolation \( \square \) on the appropriate lattices. The qualitative properties of this annealed model were similar to the infinite dimensional case, although the numerical values of thresholds and exponents were different.

They also investigated numerically a version with fixed \( f_i \) and found rather different behaviour from the infinite dimensional case, with the final distance in the chaotic phase depending on the initial distance (cf equ.4)). The behaviour is summarized as:

\[
\lim_{D_{12}(0) \to 0} D_{12}(\infty) = Q(p) > 0, \text{ if } p > p_c.
\]
\[ \lim_{D_{12}(0) \to 0} \frac{D_{12}(\infty)}{D_{12}(0)} = \begin{cases} 0, & \text{if } p < p_c \\ \chi(p), & \text{if } p < p_c \end{cases} \]

where the susceptibility \( \chi(p) \) diverges at \( p = p_c \). The strategy adopted in \[3\] was to look at the time variation of the distance between two configurations, where one of the initial configurations contained some “damaged” spins that were different to the other reference configuration. It is possible to think of the transition from the frozen phase to the chaotic phase as damage percolating through the lattice, which allows another estimate of \( p_c \) if one neglects spin-spin correlations, namely

\[ 2p_c(1 - p_c) = x_c \]

where \( x_c \) is the bond percolation threshold for the lattice. Numerical simulations of the quenched model on various lattices \[4, 5\] showed that frozen and chaotic phases existed on square, triangular and various cubic lattices, whereas the honeycomb (hexagonal) lattice possessed only a frozen phase. It was found that estimates of \( p_c \) were subject to strong finite size effects, with only the square lattice value being pinned down with precision. In addition, various interesting quantities such as the fractal dimension of the cluster of damaged spins at threshold were measured. The analogy with percolation for the quenched model seems to be rather less solid than the annealed version, as various thresholds and fractal dimensions differ from the percolative case \[3\].

The work of \[3, 4, 5\] is much closer in spirit to standard statistical simulations of spin models on various lattices than the earlier “infinite dimensional” models. It is now well known both from analytical work \[6, 7\] and numerical work \[8\] that the critical behaviour of spin models such as the Ising model can be modified by placing them on appropriate dynamical (connectivity) lattices, where the spins interact with the lattice and vice-versa. The methods of \[8\] can also be adapted to solve bond percolation on dynamical lattices by treating it as the limit of a \( q \) state Potts model as \( q \to 1 \) \[12\], again leading to different critical behaviour from fixed lattice counterparts. In the light of the different behaviour of spin models on dynamical lattices and the relation between percolation and damage spreading in the finite range Kauffman models a natural question to ask is whether the behaviour of the quenched rule automaton is also changed by placing it on a dynamical lattice, which would then correspond to annealing the inputs. Such a model lies halfway between the (soluble) model of Derrida and Pomeau with annealed rules and inputs and the models with quenched rules and inputs simulated by Derrida and Stauffer, and might \textit{a priori} lie in either universality class. There has been some scepticism of the relevance of quenched rule Kauffman automata with nearest neighbour interactions on regular lattices for biological applications \[3\], on the grounds that gene interactions are known to have considerable temporal and spatial variability. In view of this, our model might even be considered to be as appropriate as the original Kauffman infinite range model since we have included both spatial and \( \textit{temporal} \) variation in using a dynamical lattice, whilst preserving a characteristic fixed rule for each lattice point or “gene”.

A second motivation for performing a simulation on a dynamical lattice comes from 2D gravity: in \[13\] percolation on triangulations arising from simulations of 2D gravity coupled to matter with various central charges was used to investigate the geometry of the triangulations. In view of the links between damage spreading in the Kauffman model and percolation one might also hope to learn something about discretized 2D gravity, at least qualitatively, from simulating the Kauffman model on suitable graphs. The geometry of the graphs is certain to be reflected in some of the features of the damage clusters and their manner of spreading. There is no back reaction from the automaton spins on the lattice because we change the connectivity without reference to the spins so we will, in effect, be investigating the lattices of pure 2D gravity with no matter. For the case of pure 2D gravity the appropriate choice of graphs would be either \( \phi^3 \) graphs or their dual triangulations as in \[14\]. As there is no transition for a quenched rule Kauffman automaton on a regular fixed connectivity \( \phi^3 \) lattice - the honeycomb lattice, we choose to simulate \( \phi^3 \) graphs to see if the introduction of dynamical connectivity gives a transition.

In the current work we use the methods employed \[13\] in our simulations of Potts models coupled to 2D gravity where \( \phi^3 \) graphs of spherical topology with \( N = 10000 \) points were generated. To avoid unnecessarily singular graphs, tadpoles and self energy bubbles were not allowed (ie no rings of length
one or two in the graphs). The dynamical connectivity was implemented by performing a series of local “flip” moves, which are the dual of those employed on triangulations, and can be shown to be ergodic. We found that it was a good rule of thumb to carry out $\text{NFLIP} = N$ local “flip” moves on the lattice in the spin model simulations between spin updates in order to ensure sufficient coupling to 2D gravity so we do the same between the automaton time steps. The rules $f_i$ are chosen at random for each graph point at the start of each run, using the probabilities $p$ and $1 - p$ for “up” and “down” rules. As the region $p > 0.5$ is equivalent to $p \leq 0.5$ on inversion of $p$ and $1 - p$ we only simulate up to $p = 0.5$. The $f_i$ are then held fixed during the run, whereas the flips ensure that the inputs are shuffled between each automaton update, which is performed according to equ.(1) simultaneously for each spin across the entire graph. We look at the evolution of the Hamming distance in equ.(2) for configurations with 4, 100, 400 and 1000 differing initial spins over 500 automaton timesteps and repeat this process 50 times for each $p$ to obtain averaged values and errors. This proved sufficient to obtain reasonable error bars. The initial spin configurations were taken as cold (ordered) starts and the initial sites to be damaged were chosen at random on each run. We also simulate the automaton on fixed 2D gravity graphs, where we switch off the flips in between the automaton updates. In this case we expect to learn something about the geometry of a given $\phi^3$ graph by comparing the results with other fixed graphs, such as the honeycomb graph.

In Fig.1 we have plotted the time evolution of the Hamming distance for $p = 0.10$ and $p = 0.40$ on both fixed and dynamical $\phi^3$ graphs for a starting damage of 100 to give some idea of the general behaviour. One qualitative fact is immediately clear from the graph: there is no transition as $p$ is varied for the fixed $\phi^3$ graph, whereas the damage clearly spreads to reach an asymptotic value for $p = 0.4$ when the flips are switched on. Thus the fixed 2D gravity graph, a highly disordered $\phi^3$ graph with a complicated internal structure, behaves like its smooth $\phi^3$ honeycomb counterpart in so far as it displays no transition to a chaotic phase at larger $p$ values. The most interesting observation for the dynamical graph simulations, on the other hand, is how close they come to the analytical results for the totally annealed automaton of Derrida and Pomeau. For a totally annealed Kauffman model with $K = 3$, which is also the value for the dynamical $\phi^3$ graphs here, one expects a critical value of

$$p_c = \frac{1}{2} - 1/(2\sqrt{3}) \simeq 0.21$$

and an asymptotic value of the damage $d_{\text{final}} \simeq 0.38$ at $p = 0.5$ that is independent of the starting damage. In Fig.2 we plot the asymptotic value of the damage after 500 time steps against $p$ for the various starting damages, from which it is clear that the automaton is almost certainly behaving identically to the fully annealed model. There is a transition from a frozen to a chaotic phase in the region of $0.21 - 0.22$ and the final damage is independent of the starting damage, for all but the smallest starting damage, which is probably too small to avoid an atypical choice of starting sites. We find $d_{\text{final}} \simeq 0.366(1)$, which is a little lower than the analytical value. The changes in connectivity in the simulation thus appear to be sufficient to destroy the correlations between the spin values and allow the application of the annealed approximation.

The results for the asymptotic values of the damage for a fixed 2D gravity graph are shown in Fig.3, clearly demonstrating the absence of a transition in this case. The bond percolation threshold on 2D gravity graphs was calculated to be $x_c \simeq 0.78$ in [13], so the formula $2p_c(1 - p_c) = x_c$ of [10] correctly predicts no transition for the quenched Kauffman automaton in this case (as it does for the quenched automaton on a honeycomb lattice). It is instructive to compare the results from a similarly sized honeycomb lattice in Fig.4 with those in Fig.3. The curves for the final damage vs $p$ in Fig.4 for a lattice of 8192 points look, in fact, as if there is a transition. Much larger simulations however, show convincingly that a transition is absent on a honeycomb lattice, in particular by regarding the behaviour of the final damage as the initial damage is varied. From comparing Fig.3 and Fig.4 we can therefore deduce that it is a lot easier to see the absence of damage spreading on the 2D gravity $\phi^3$ lattice than on the honeycomb (ie regular $\phi^3$) lattice. This suggests that the spreading of a damage cluster encounters more obstacles on the 2D gravity graph than the honeycomb.

Such a conclusion is consistent with the behaviour observed for the critical slowing down of the magnetization in cluster algorithms for spin models on dynamical surfaces, which turned out to be anomalously large [14]. It was suggested in [14] that this was due to the inhibition of cluster growth from bottlenecks and “baby-universes” in the graph geometry that only changed slowly due to the local “flip”
graph update. Such bottle-necks and baby universes would also trap the damage clusters in the quenched Kauffman model and account for both the much clearer absence of a transition than on the honeycomb lattice and the much smaller final damage on the fixed 2D gravity graphs compared with the honeycomb lattice. The relative smallness of the clusters is also an indication of the larger fractal dimension of the 2D gravity graphs, which appears to be close or equal to 3 [17].

In summary, we simulate a quenched rule Kauffman automaton on dynamical $\phi^4$ graphs (as used in 2D gravity simulations) and find that the annealed results of Derrida and Kauffman are applicable. A simulation on a fixed $\phi^4$ graph on the other hand, although there is no transition to a chaotic phase, gives some indication of the complicated internal geometry that is present in such graphs. It would be interesting to look at the crossover between the effectively annealed behaviour on dynamical graphs and that on fixed graphs as the number of flip updates per automaton update was reduced, as well as examining the relation between damage spreading and thermodynamic quantities [18] for more standard spin models on dynamical lattices.

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Figure Captions

Fig. 1. The damage vs time for $p = 0.1, 0.4$ on both fixed and dynamical $\phi^3$ graphs with fixed rules.

Fig. 2. The final damage vs $p$ for various initial damages on a dynamical $\phi^3$ graph with fixed rules. The various initial damages are indicated.

Fig. 3. The final damage vs $p$ for various initial damages on a fixed $\phi^3$ graph with fixed rules. The various initial damages are indicated.

Fig. 4 The final damage on a similarly sized honeycomb lattice, again with fixed rules. The various initial damages are indicated.
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