THE MILNOR INVARIANTS OF CLOVER LINKS

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Abstract. J.P. Levine introduced a clover link to investigate the indeterminacy of the Milnor invariants of a link. It is shown that for a clover link, the Milnor numbers of length at most $2k + 1$ are well-defined if those of length at most $k$ vanish, and that the Milnor numbers of length at least $2k + 2$ are not well-defined if those of length $k + 1$ survive. For a clover link $c$ with the Milnor numbers of length at most $k$ vanishing, we show that the Milnor number $\mu_c(I)$ for a sequence $I$ is well-defined up to the greatest common divisor of $\mu_c(J)$'s, where $J$ is a subsequence of $I$ obtained by removing at least $k + 1$ indices. Moreover, if $I$ is a non-repeated sequence with length $2k + 2$, the possible range of $\mu_c(I)$ is given explicitly. As an application, we give an edge-homotopy classification of 4-clover links.

1. Introduction

The Milnor invariant introduced by J. Milnor [7], [8]. For an oriented ordered $n$-component link $L$ in the 3-sphere $S^3$ with peripheral information, the Milnor number $\mu_L(I)$, which is an integer, is specified by a finite sequence $I$ in $\{1, 2, \ldots, n\}$. The Milnor $\overline{\pi}$-invariant $\overline{\pi}_L(I)$ is the residue class of $\mu_L(I)$ modulo the greatest common divisor of $\mu_L(J)$'s, where $J$ is obtained from proper subsequence of $I$ by permuting cyclicly. The length of the sequence $I$ is called the length of $\overline{\pi}_L(I)$ and denoted by $|I|$. His original definition of the Milnor invariant eliminates the indeterminacy of the possible variations of the Milnor numbers caused by different choices of peripheral elements.

In [6], J. P. Levine examined the Milnor invariants from the point of view of based links, in order to understand the indeterminacy. A based link is a link for which some peripheral information is specified, i.e., meridians (the weakest) or both meridians and longitudes (the strongest). It is known that these invariants are completely well-defined for the strongest form of basing (disk links [5] or string links [2]). As basing only slightly stronger than the specification of meridians, he introduced an $n$-clover link which is an embedded graph consisting of $n$ loops, each loop connected to a vertex by an edge in $S^1$. The Milnor number of a clover link is defined up to flatly isotopy and it is shown that those of length $\leq k$ vanish implies those of length $\leq 2k + 1$ are completely non-indeterminate [6].

The first author [10] redefined the Milnor number of a clover link up to ambient isotopy as follows. Given an $n$-clover link $c$, we construct an $n$-component bottom tangle $\gamma(F_c)$ by using a disk/band surface $F_c$ of $c$. In [6], Levine defined the Milnor number of a bottom tangle. Therefore we define the Milnor number $\mu_c$ of an $n$-clover link $c$ to be the Milnor number $\mu_{\gamma(F_c)}$. (In [6], a bottom tangle is called a string link. The name ‘bottom tangle’ follows K. Habiro [3].) In [10], it is shown that the same result as Levine [6] holds while there are infinitely many choices of $\mu_{\gamma(F_c)}$ for $c$. 

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Unfortunately, the Milnor numbers of length at least \(2k + 2\) are not well-defined if those of length \(k + 1\) survive. In this paper, we show that the Milnor number \(\mu_c(I)\) of a sequence \(I\) modulo \(\delta^k_c(I)\) is well-defined if the Milnor numbers of length \(\leq k\) vanish, where \(\delta^k_c(I)\) is the greatest common divisor of \(\mu_c(J)\)'s, where \(J\) is range to over all subsequences of \(I\) obtained by removing at least \(k + 1\) indices. In fact, we have the following theorem.

**Theorem 1.1.** Let \(c\) be an \(n\)-clover link and \(l_c\) a link which is the disjoint union of loops of \(c\). If the Milnor numbers of \(l_c\) for sequences with length \(\leq k\) vanish, then the residue class of \(\mu_c(I)\) modulo \(\delta^k_c(I)\) is an invariant for any sequence \(I\).

**Remark 1.2.** In contrast to the \(\overline{\mu}\)-invariant for a link in \(S^3\), we do not need to take cyclic permutation for getting \(\delta^k_c(I)\).

It is an important property that for non-repeated sequences, the Milnor invariants of links are link-homotopy invariants [4]. The first author showed that the Milnor numbers for any non-repeated sequence with length \(\leq 3\) give an edge-homotopy classification of 3-clover links [10], where edge-homotopy [9] is an equivalence relation, which is a generalization of link-homotopy, generated by crossing changes on the same spatial edge. We also discuss giving an edge-homotopy classification of 4-clover links. The Milnor numbers for non-repeated sequences with length 4 could be useful to have an edge-homotopy classification of 4-clover links. But they are not well-defined in general. Hence we consider the set of all Milnor numbers of length 4 for all disk/band surfaces of \(c\). More generally, we define the following set;

\[
H_c(2k + 2, j) = \left\{ \sum_{S \in S^2_{j+1}} \mu_\gamma(F_j)(S_j)X_S \mid F_c : \text{a disk/band surface of } c \right\}
\]

for each integer \(j\) \((1 \leq j \leq n)\), where \(S^2_{j+1}\) is the set of length-(2\(k + 1\)) non-repeated sequences without containing \(j\) and for a sequence \(S = i_1i_2\ldots i_{2k+1}\), \(X_S = X_{i_1}X_{i_2}\ldots X_{i_{2k+1}}\) is a monomial in non-commutative variables \(X_1, \ldots, X_n\). Since \(H_c(2k + 2, j)\) consists of the Milnor numbers \(\mu_\gamma(F_j)(S_j)\) \((S \in S^2_{j+1})\) for all disk/band surfaces of \(c\), it is an invariant of \(c\). While it seems too big to handle \(H_c(2k + 2, j)\), we have the following theorem.

**Theorem 1.3.** Let \(c\) be an \(n\)-clover link and \(F_c\) a disk/band surface of \(c\). If the Milnor numbers of \(l_c\) for non-repeated sequences with length \(\leq k\) vanish, then we have the following:

\[
H_c(2k + 2, j) =
\left\{ \sum_{\|J\| = |I| = k} \mu_\gamma(F_J)(H_I) \left( \sum_{t \in J} m_{t_J}X_{J_J}\gamma(I_{I_J} - I_{I_{J_J}} - I_{I_{J_J + 1}})X_{J_J} \right) \right. + m_{iJ}(X_{IJ_J} - X_{IJ_I} + X_{IJ_J}) + \sum_{S \in S^2_{j+1}} \mu_\gamma(F_j)(S_j)X_S \mid m_{pq} = m_{qp} \in \mathbb{Z} \right\},
\]

where for a sequence \(J = i_1\ldots i_m\), \(\{J\} = \{i_1, \ldots, i_m\}\) and \(J_{\leq s}\) (resp. \(J_{>s}\)) is a subsequence \(i_1\ldots i_{s-1}\) (resp. \(i_{s+1}\ldots i_m\)) of \(J\) for \(1 \leq s \leq m\), and \(X_{J_{\leq s}}\) and \(X_{J_{>s}}\) are defined to be 1.

This theorem implies that the set \(H_c(2k + 2, j)\) is obtained from the Milnor numbers of \(\gamma(F_c)\) for any single disk/band surface \(F_c\) of \(c\), that is, \(H_c(2k + 2, j)\) is specified explicitly.
By the following corollary we have that $H_c(2k + 2,j)$ is not only an invariant of $c$ but also an edge-homotopy invariant.

**Corollary 1.4.** For a clover link $c$, if the Milnor numbers of $I_c$ for non-repeated sequences with length $\leq k$ vanish, then $H_c(2k + 2,j)$ is an edge-homotopy invariant of $c$.

It is the definition that the Milnor numbers of length 1 are zero. If $k = 1$, then the theorem above holds without the condition. The following example follows directly from Theorem 1.6.

**Example 1.5.** For a 4-clover link $c$ and a disk/band surface $F_c$ of $c$, we have

\[
H_c(4,4) = \left\{ \left( \mu_{(F_c)}(1)\mu_{(F_c)}(23)(m_{14} - m_{34} - m_{12} + m_{24}) + \mu_{(F_c)}(12)\mu_{(F_c)}(34)(m_{24} - m_{23} - m_{14} + m_{13}) \right) \bigg| X_{123} \right. \\
+ \left( \mu_{(F_c)}(1)\mu_{(F_c)}(23)(m_{34} - m_{13} - m_{24} + m_{12}) + \mu_{(F_c)}(13)\mu_{(F_c)}(24)(m_{34} - m_{23} - m_{14} + m_{12}) \right) \bigg| X_{132} \right. \\
+ \left( \mu_{(F_c)}(13)\mu_{(F_c)}(24)(m_{23} - m_{34} - m_{12} + m_{14}) + \mu_{(F_c)}(12)\mu_{(F_c)}(34)(m_{24} - m_{23} - m_{13} + m_{14}) \right) \bigg| X_{213} \right. \\
+ \left( \mu_{(F_c)}(1)\mu_{(F_c)}(23)(m_{34} - m_{13} - m_{24} + m_{12}) + \mu_{(F_c)}(13)\mu_{(F_c)}(24)(m_{34} - m_{23} - m_{14} + m_{12}) \right) \bigg| X_{231} \right. \\
+ \left( \mu_{(F_c)}(13)\mu_{(F_c)}(24)(m_{23} - m_{34} - m_{12} + m_{14}) + \mu_{(F_c)}(12)\mu_{(F_c)}(34)(m_{24} - m_{23} - m_{13} + m_{14}) \right) \bigg| X_{312} \right. \\
+ \left( \mu_{(F_c)}(1)\mu_{(F_c)}(23)(m_{13} - m_{34} - m_{12} + m_{24}) + \mu_{(F_c)}(12)\mu_{(F_c)}(34)(m_{24} - m_{23} - m_{14} + m_{13}) \right) \bigg| X_{321} \right. \\
+ \sum_{S \in \mathcal{S}_3} \mu_{(F_c)}(S)X_S \bigg| m_{pq} \in \mathbb{Z} \right\}.
\]

This together with the theorem below gives us an edge-homotopy classification of 4-clover links, see Remark 1.7.

**Theorem 1.6.** Let $c$ and $c'$ be 4-clover links. They are edge-homotopic if and only if $H_c(4,4) \cap H_{c'}(4,4) = \emptyset$ and $\mu_c(I) = \mu_{c'}(I)$ for any non-repeated sequence $I$ with $|I| \leq 3$.

**Remark 1.7.** By Example 1.5 we are able to determine whether $H_c(4,4) \cap H_{c'}(4,4)$ is empty or not. Hence by combining Example 1.5 and Theorem 1.6 we obtain an edge-homotopy classification of 4-clover links.

2. The Milnor numbers of clover links

In this section we define the Milnor numbers for clover links.

2.1. A construction of bottom tangles. An $n$-component tangle is a properly embedded disjoint union of $n$ arcs in the 3-cube $[0,1]^3$. An $n$-component bottom tangle $\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$ defined by Levine [8] is a tangle with $\partial \gamma_i = \{(\frac{2i-1}{2n+1}, \frac{1}{2}, 0), (\frac{2i+1}{2n+1}, \frac{1}{2}, 0)\} \subset \partial [0,1]^3$ for each $i = 1, 2, \ldots, n$.

A spatial graph is an embedded graph in $S^3$. Let $C_n$ be a graph consisting of $n$ oriented loops $e_1, e_2, \ldots, e_n$, each loop $e_i$ connected to a vertex $v$ by an edge $f_i$ $(i = 1, 2, \ldots, n)$, see Figure 2.1. An $n$-clover link in $S^3$ is a spatial graph of $C_n$ [8]. The each part of a clover link corresponding to $e_i$, $f_i$ and $v$ of $C_n$ are called the leaf, stem and root, denoted by the same notations respectively.
L. Kauffman, J. Simon, K. Wolcott and P. Zhao \[4\] defined disk/band surfaces of spatial graphs. For a spatial graph $\Gamma$, a disk/band surface $F^\Gamma$ of $\Gamma$ is a compact, oriented surface in $S^3$ such that $\Gamma$ is a deformation retract of $F^\Gamma$ contained in the interior of $F^\Gamma$. Note that any disk/band surface of a spatial graph is ambient isotopic to a surface constructed by putting a disk at each vertex of the spatial graph, connecting the disks with bands along the spatial edges. We remark that for a spatial graph, there are infinitely many disk/band surfaces up to ambient isotopy.

Given an $n$-clover link, we construct an $n$-component bottom tangle using a disk/band surface of the clover link as follows:

1. For an $n$-clover link $c$, let $F_c$ be a disk/band surface of $c$ and let $D$ be a disk which contains the root. From now on, we may assume that the intersection $D \cap \bigcup_{i=1}^n f_i$ and orientations of the disks are as illustrated in Figure 2.2.

2. Let $N(D)$ be the regular neighborhood of $D$ and $\mathring{N}(D)$ the interior of $N(D)$. Since $S^3 \setminus N(D)$ is homeomorphic to the 3-ball, $F_c \setminus N(D)$ can be seen as a disjoint union of surfaces in the 3-ball. Hence $\partial F_c \setminus N(D)$ is a disjoint union of $n$-arcs and $n$-circles $\bigcup_{i=1}^n S^1_i$ in the 3-ball.

3. Since the 3-ball is homeomorphic to $[0, 1]^3$, we obtain an oriented ordered $n$-component bottom tangle $\gamma(F_c)$ from $(\partial F_c \setminus \mathring{N}(D)) \setminus \bigcup_{i=1}^n S^1_i$ as illustrated in (3) and (4) of Figure 2.3. We call $\gamma(F_c)$ an $n$-component bottom tangle obtained from $F_c$. 

Figure 2.3. A method for obtaining a bottom tangle from a disk-band surface of a clover link

While there are infinitely many disk/band surfaces of a clover link which satisfy the condition (1) above, they are related by certain local moves as follows.

Lemma 2.1. [10 Proposition 2.5] For an \( n \)-clover link \( c \), any two disk/band surfaces \( F_c \) and \( F'_c \) are transformed into each other by adding full-twists to bands (Figure 2.4 (a)) and a single move illustrated in Figure 2.4 (b).

Figure 2.4. Two local moves of disk/band surfaces
2.2. Milnor invariants. Let us briefly recall from [6] the definition of the Milnor number of a bottom tangle. Let \( \gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n \) be an oriented ordered \( n \)-component bottom tangle in \([0,1)^3\) with \( \partial \gamma_i = \{(\frac{2i-1}{2n+1}, \frac{1}{2}, 0), (\frac{2i}{2n+1}, \frac{1}{2}, 0)\} \subset \partial[0,1)^3 \) for each \( i = 1, 2, \ldots, n \). Let \( G \) be the fundamental group of \([0,1)^3\setminus \gamma \) with a base point \( p = (\frac{1}{2}, 0, 0) \) and \( G_q \) the \( q \)th lower central subgroup of \( G \). Let \( \lambda_i \) and \( \lambda_i \) be the \( i \)th meridian and \( i \)th longitude of \( \lambda \) respectively as illustrated in Figures 2.5.

We assume that \( \lambda_i \) is trivial in \( G/G_q \). Since the quotient group \( \mathcal{G}/G_q \) is generated by \( \alpha_1, \alpha_2, \ldots, \alpha_n \). \( \gamma \) is represented by \( \alpha_1, \alpha_2, \ldots, \alpha_n \) modulo \( G_q \).

We consider the Magnus expansion of \( \lambda_j \). The Magnus expansion is a homomorphism (denoted \( E \)) from a free group \( \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \) to the formal power series ring in non-commutative variables \( X_1, X_2, \ldots, X_n \) with integer coefficients defined as follows. \( E(\alpha_i) = 1 + X_i, \ E(\alpha_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \cdots (i = 1, 2, \ldots, n) \).

For a sequence \( I = i_1 i_2 \cdots i_{k-1} j \) \((i_m \in \{1, 2, \ldots, n\}, k \leq q)\), we define the Milnor number \( \mu_I(I) \) to be the coefficient of \( X_{i_1 i_2 \cdots i_{k-1}} \) in \( E(\lambda_j) \) (we define \( \mu_I(j) = 0 \), which is an invariant [6]. (In [6], the set of \( \lambda_i \)'s, without taking the Magnus expansion, is called the Milnor's \( \Pi \)-invariant.) For a bottom tangle \( \gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n \), an oriented link \( L(\gamma) = L_1 \cup L_2 \cup \cdots \cup L_n \) in \( S^3 \) can be defined by \( L_i = \gamma_i \cup a_i \), where \( a_i \) is a line segment connecting \((\frac{2i-1}{2n+1}, \frac{1}{2}, 0)\) and \((\frac{2i}{2n+1}, \frac{1}{2}, 0)\), see Figure 2.6. We call \( L(\gamma) \) the closure of \( \gamma \). On the other hand, for any oriented link \( L \) in \( S^3 \), there is a bottom tangle \( \gamma_L \) such that the closure of \( \gamma_L \) is equal to \( L \). So we define the Milnor number of \( L \) to be the Milnor number of \( \gamma_L \). Let \( \Delta_L(I) \) be the greatest common divisor of \( \mu_L(J) \)'s, where \( J \) is obtained from proper subsequence of \( I \) by permuting cyclicly. The Milnor invariant \( \Pi_L(I) \) is the residue class of \( \mu_L(I) \) modulo \( \Delta_L(I) \). We note that for a sequence \( I \), if we have \( \Delta_L(I) = 0 \), then the Milnor invariant \( \Pi_L(I) \) is equal to the Milnor number \( \mu(\gamma_L) \). Now we define the Milnor number of a clever link.
Definition 2.2. Let \( c \) be an \( n \)-clover link and \( F_c \) a disk/band surface of \( c \). Let \( \gamma(F_c) \) be the \( n \)-component bottom tangle obtained from \( F_c \). For a sequence \( I \), the Milnor number \( \mu_c(I) \) of \( c \) is defined to be the Milnor number \( \mu_{\gamma(F_c)}(I) \).

Remark 2.3. While \( \mu_c(I) \) depends on a choice of \( F_c \), the first author [10] proved the following result: Let \( l_c \) be a link which is the disjoint union of leaves of \( c \). If the Milnor numbers of \( l_c \) for sequences with length \( \leq k \) vanish, then \( \mu_c(I) \) is well-defined for any sequence \( I \) with \( |I| \leq 2k + 1 \).

3. Proof of Theorem [10]

In this section we will give a proof of Theorem [10].

An \( n \)-component tangle \( u = u_1 \cup u_2 \cup \cdots \cup u_n \) is an \( n \)-component string link if for each \( i (= 1, 2, \ldots, n) \), the boundary \( \partial u_i = \{(\frac{2i-1}{2n+1}, \frac{1}{2}, 0), (\frac{2i-1}{2n+1}, \frac{1}{2}, 1)\} \subset \partial[0, 1]^3 \) In particular, \( u \) is trivial if for each \( i (= 1, 2, \ldots, n) \), \( u_i = \{(\frac{2i-1}{2n+1}, \frac{1}{2}, 0), (\frac{2i-1}{2n+1}, \frac{1}{2}, 1)\} \subset \partial[0, 1]^3 \).

Here we introduce a SL-move [10] given by a string link \( u \) which is a transformation of an \( n \)-component bottom tangle \( \gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n \) with \( \partial \gamma_i = \{(\frac{2i-1}{2n+1}, \frac{1}{2}, 0), (\frac{2i-1}{2n+1}, \frac{1}{2}, 1)\} \subset \partial[0, 1]^3 \).

(1) Let \( u = u_1 \cup u_2 \cup \cdots \cup u_n \) be an oriented ordered \( n \)-component string link in \([0, 1]^3\). For each \( i (= 1, 2, \ldots, n) \), we consider an arc \( u'_i \) which is parallel to the \( i \)th component \( u_i \) of \( u \) with opposite orientation and \( \partial u'_i = \{(\frac{2i-1}{2n+1}, \frac{1}{2}, 0), (\frac{2i-1}{2n+1}, \frac{1}{2}, 1)\} \subset \partial[0, 1]^3 \), see Figure 3.7.

(2) Let \( \gamma' = \gamma'_1 \cup \gamma'_2 \cup \cdots \cup \gamma'_n \) be an \( n \)-component bottom tangle in \([0, 1]^3\) defined by

\[ \gamma'_i = h_0(u_i \cup u'_i) \cup h_1(\gamma_i) \]

for \( i = 1, 2, \ldots, n \), where \( h_0, h_1 : ([0, 1] \times [0, 1]) \times [0, 1] \to ([0, 1] \times [0, 1]) \times [0, 1] \) are embeddings defined by

\[ h_0(x, t) = (x, \frac{1}{2}t) \text{ and } h_1(x, t) = (x, \frac{1}{2} + \frac{1}{2}t) \]

for \( x \in ([0, 1] \times [0, 1]) \) and \( t \in [0, 1] \).

We say that \( \gamma' \) is obtained from \( \gamma \) by a SL-move. For example, see Figure 3.8.

We note that if \( u \) is trivial, a SL-move is just adding full-twists or nothing. A SL-move is determined by a string link and a number of full-twists, that is, ‘SL’ stands for String Link.

Proposition 3.1. Let \( \gamma \) be an \( n \)-component bottom tangle and \( \gamma' \) a bottom tangle obtained from \( \gamma \) by a SL-move. If the Milnor numbers of \( \gamma \) and \( \gamma' \) for sequences with length \( \leq k \) vanish, then for any sequence \( I \),

\[ \mu_{\gamma'}(I) \equiv \mu_{\gamma}(I) \mod \delta^k_n(I), \]

where \( \delta^k_n(I) \) is the greatest common divisor of \( \mu_{\gamma}(J)'s \) for a proper subsequence \( J \) of \( I \) which is obtained by removing at least \( k + 1 \) indices.
Proof. The proof is by induction on the length $|I| = q$. If $q \leq k$, the proposition clearly holds. We note that, by the induction hypothesis, $\delta^k_Y(I) = \delta^k_Y(I)$. Denote respectively by $\alpha_i, \lambda_i$ (resp. $\alpha_i', \lambda_i'$) the $i$th meridian and $i$th longitude of $\gamma$ (resp. $\gamma'$) for $1 \leq i \leq n$. Let $E_X$ (resp. $E_Y$) be the Magnus expansion in non-commutative variables $X_1, \ldots, X_n$ (resp. $Y_1, \ldots, Y_n$) obtained by replacing $\alpha_i$ by $1 + X_i$ (resp. $\alpha_i'$ by $1 + Y_i$) for $1 \leq i \leq n$. Fix $j$, by the assumption, the Milnor numbers for $\gamma$ and $\gamma'$ of length $\leq k$ vanish, so $E_X(\lambda_j)$ and $E_Y(\lambda_j')$ can be written respectively in the form

$$E_X(\lambda_j) = 1 + F_j(X) \quad \text{and} \quad E_Y(\lambda_j') = 1 + F_j'(Y),$$

where $F_j(X) = F_j(X_1, \ldots, X_n)$ and $F_j'(Y) = F_j'(Y_1, \ldots, Y_n)$ are terms of degree $\geq k$.

Here we define a set of polynomials:

$$D_j^k = \left\{ \sum \nu(i_1 \ldots i_m)Y_{i_1} \cdots Y_{i_m} \left| \begin{array}{c} \nu(i_1 \ldots i_m) \equiv 0 \mod \delta^k_Y(i_1 \ldots i_m), \\ \nu(i_1 \ldots i_m) \in \mathbb{Z}, \\ m < q \end{array} \right. \right\} \cup \left\{ \sum \nu(i_1 \ldots i_m)Y_{i_1} \cdots Y_{i_m} \left| \begin{array}{c} \nu(i_1 \ldots i_m) \equiv 0 \mod \delta^k_Y(i_1 \ldots i_m), \\ \nu(i_1 \ldots i_m) \in \mathbb{Z}, \\ m \geq q \end{array} \right. \right\}.$$

Then it is enough to show

$$F_j'(Y) - F_j(Y) \in D_j^k.$$

The following claims are shown by similar to the assertions (16) and (18) in [8].

**Claim 1.** $D_j^k$ is a two-sided ideal of the formal power series ring in non-commutative variables $Y_1, \ldots, Y_n$ with integer coefficients.

**Claim 2.** If at least $k$ variables are inserted anywhere in a term $\mu(i_1 i_2 \ldots i_m)Y_{i_1 i_2 \ldots i_m}$, then the resulting term belongs to $D_j^k$. 

Let $u_i$ be the $i$th longitude of a string link which gives the SL-move, see Figure 3.8. In the proof of [10] Lemma 2.6, it is shown that the degree of each term in $E_Y(u_i^{-1}) - 1$ is at least $k + 1$. Set $E_Y(u_i) = 1 + G_i(Y)$ and $E_Y(u_i^{-1}) = 1 + \overline{G_i}(Y)$, where $G_i(Y)$ and $\overline{G_i}(Y)$ mean the terms of degree $\geq k + 1$. Since $\alpha_i = u_i^{-1} \alpha_i' u_i$ (where $\alpha_i$ is assumed to be an element of $\pi_1([0, 1]^{3} \setminus \gamma)$), we have

$$E_Y(\alpha_i) = E_Y(u_i^{-1} \alpha_i' u_i) = (1 + \overline{G_i}(Y))(1 + Y_i)(1 + G_i(Y)) = (1 + \overline{G_i}(Y))(1 + G_i(Y)) + (1 + \overline{G_i}(Y))Y_i(1 + G_i(Y)) = 1 + Y_i + Y_i G_i(Y) + \overline{G_i}(Y) Y_i + \overline{G_i}(Y) Y_i G_i(Y).$$

Hence $E_Y(\lambda_j)$ is obtained from $E_X(\lambda_j)$ by substituting $X_i$ for $Y_i + Y_i G_i(Y) + \overline{G_i}(Y) Y_i + \overline{G_i}(Y) Y_i G_i(Y)$. Set $E_Y(\lambda_j) = 1 + H_j(Y)$, where $H_j(Y)$ is the terms of degree $\geq k$. Note that terms of degree $\leq 2k$ of $H_j(Y) - F_j(Y)$ vanish, and that any term of $H_j(Y) - F_j(Y)$ of
degree $\geq 2k + 1$ is obtained from $F_j(Y)$ by inserting at least $k + 1$ variables. By Claim 2, $H_j(Y) - F_j(Y) \in D^k_j$. Since $\lambda'_j = u_j \lambda_j u_j^{-1}$ (where $\lambda_j$ is assumed to be an element of $\pi_1([0,1] \setminus \gamma')$),

$E_Y(\lambda'_j) = E_Y(u_j \lambda_j u_j^{-1})$

$= (1 + G_j(Y))(1 + H_j(Y))(1 + G_j(Y))$

$= 1 + H_j(Y) + H_j(Y)G_j(Y) + G_j(Y)H_j(Y) + G_j(Y)H_j(Y)G_j(Y)$.

It follows from Claims 1 and 2 that we have

$F'_j(Y) - H_j(Y) = H_j(Y)G_j(Y) + G_j(Y)H_j(Y) + G_j(Y)H_j(Y)G_j(Y) \in D^k_j$. Since $H_j(Y) - F_j(Y) \in D^k_j$, by Claim 1 we have

$F'_j(Y) - F_j(Y) \in D^k_j$.

This completes the proof. □

Proof of Theorem 1.1. By Lemma 2.1, any two disk/band surfaces $F_c$ and $F'_c$ of an $n$-component clover link $c$ are transformed into each other by the moves (a) and (b) illustrated in Figure 2.4. So two bottom tangles $\gamma(F_c)$ and $\gamma(F'_c)$ are transformed into each other by a SL-move. Since the both closures $L(\gamma(F_c))$ and $L(\gamma(F'_c))$ are ambient isotopic to $l_c$, by the hypothesis of Theorem 1.1,

$0 = \mu_{l_c}(J) = \mu_{\gamma(F_c)}(J) = \mu_{\gamma(F'_c)}(J)$

for any sequence $J$ with $|J| \leq k$. Hence by Proposition 3.3, $\mu_{\gamma(F_c)}(I) \equiv \mu_{\gamma(F'_c)}(I) \mod \delta^k_{\gamma(F_c)}(I)$ for any sequence $I$. This completes the proof. □

4. Proof of Theorems 1.3 and 1.6

In this section we will give proofs of Theorems 1.3 and 1.6 and Corollary 1.4.

Proposition 4.1. Let $\gamma$ be an $n$-component bottom tangle and $\gamma'$ a bottom tangle obtained from $\gamma$ by a SL-move which is given by a string link $u$. If the Milnor numbers of $\gamma$ and $\gamma'$ for non-repeated sequences with length $\leq k$ vanish, then we have the following:
\[
\sum_{S \in S_{k+1}^{2k+1}} (\mu_{\gamma}(S) - \mu_{\gamma}(S_j))Y_S
\]
\[
= \sum_{|I|=|J|=k \atop J \in S_{k+1}^{2k+1}} \mu_{\gamma}(J)u_{\gamma}(I) \sum_{i \in (J)} \mu_{\gamma}(l_i)Y_{J_{<i}}(Y_{i,H} - Y_{i,H} - Y_{i,I} + Y_{i,I})Y_{J_{<i}} + \sum_{|I|=|J|=k \atop J \in S_{k+1}^{2k+1}} \mu_{\gamma}(J)\mu_{\gamma}(I)u_{\gamma}(l)(Y_{H,J} - Y_{H,J} - Y_{H,J} + Y_{H,J}).
\]

Proof. We compare with the Magnus expansions of the \(j\)th longitudes of \(\gamma\) and \(\gamma'\). Denote respectively by \(\alpha_i, \lambda_i\) (resp. \(\alpha'_i, \lambda'_i\)) the \(i\)th meridian and \(i\)th longitude of \(\gamma\) (resp. \(\gamma'\)) for \(1 \leq i \leq n\). Let \(E_X\) (resp. \(E_Y\)) be the Magnus expansion in non-commutative variables \(X_1, \ldots, X_n\) (resp. \(Y_1, \ldots, Y_n\)) obtained by replacing \(\alpha_i\) by \(1 + X_i\) (resp. \(\alpha'_i\) by \(1 + Y_i\)) for \(1 \leq i \leq n\). By the assumption, the Minor numbers for \(\gamma\) and \(\gamma'\) of degree \(k\) coincide, hence denote by
\[
E_X(\lambda_j) = 1 + \sum_{I \in S_k^j} \mu_{\gamma}(I)X_I + r_j(X) + O_X(2)
\]
and
\[
E_Y(\lambda'_j) = 1 + \sum_{I \in S_k^j} \mu_{\gamma}(I)Y_I + r'_j(Y) + O_Y(2),
\]
where \(r_j(X)\) and \(r'_j(Y)\) mean the terms of degree \(\geq k + 1\) and \(O_X(2)\) (resp. \(O_Y(2)\)) denotes the terms which contain \(X_i\) (resp. \(Y_i\)) at least 2 times for some \(i(=1, 2, \ldots, n)\). Let \(f_j(X) = \sum_{I \in S_k^j} \mu_{\gamma}(I)X_I\) and \(f_j(Y) = \sum_{I \in S_k^j} \mu_{\gamma}(I)Y_I\).

Let \(u_i\) be the \(i\)th longitude of \(u\) and let \(\beta_i = [\lambda'_i, \alpha'_i] = \lambda'_i^{-1}\alpha'_i^{-1}\lambda'_i\alpha'_i\), see Figure 3.9. Let \(E_Z\) be the Magnus expansion in non-commutative variables \(Z_1, \ldots, Z_n\) obtained by replacing \(\beta_i\) by \(1 + Z_i\) for \(1 \leq i \leq n\). Then we have
\[
E_Z(u_i) = 1 + \sum_{I \in S_k^j} \mu_{u}(I)Z_I + O(2).
\]

First, we observe \(E_Y(\beta_i)\) and \(E_Y(\alpha_i)\), where \(\alpha_i\) is assumed to be an element of \(\pi_1([0, 1]^{\lambda'}/\gamma')\). Since \(E_Y(\lambda'_i)E_Y(\lambda'_i^{-1}) = 1\), set \(E_Y(\lambda'_i^{-1}) = 1 - f_i(Y) + r'_i(Y) + O_Y(2)\), where \(r'_i(Y)\) is the terms of degree \(\geq k + 1\). Observe that
\[
E_Y(\beta_i) = E_Y(\lambda'_i^{-1}\alpha_i^{-1}\lambda'_i\alpha_i) = (1 - f_i(Y) + r'_i(Y) + O_Y(2))(1 - Y_i + O_Y(2))(1 + f_i(Y) + r'_i(Y) + O_Y(2))(1 + Y_i)
\]
\[
= 1 + f_i(Y)Y_i - Y_if_i(Y) + O(k + 2) + O_Y(2)
\]
\[
= 1 + \sum_{I \in S_k^j} \mu_{\gamma}(I)Y_I - Y_i \sum_{I \in S_k^j} \mu_{\gamma}(I)Y_I + O(k + 2) + O_Y(2)
\]
\[
= 1 + \sum_{I \in S_k^j} \mu_{\gamma}(I)(Y_{II} - Y_{II}) + O(k + 2) + O_Y(2).
\]
This implies that \(E_Y(u_i)\) is obtained from \(E_Z(u_i)\) by substituting \(Z_i\) for
\[
\sum_{I \in S_k^j} \mu_{\gamma}(I)(Y_{II} - Y_{II}) + O(k + 2) + O_Y(2).
\]
So we have
\[
E_Y(u_i) = 1 + \sum_{I \in S_k^j} \mu_{u}(I) \sum_{I \in S_k^j} \mu_{\gamma}(I)(Y_{II} - Y_{II}) + O(k + 2) + O_Y(2).
\]
Let \( g_i(Y) = \sum_{l \neq i} \mu_u(li) \sum_{l \in S^k_i} \mu_\gamma(II)(Y_{lI} - Y_{II}) \). Then we have
\[
E_Y(u_i^{-1}) = 1 - g_i(Y) + O(k + 2) + O_Y(2).
\]
Since \( \alpha_i = u_i^{-1} \alpha'_i u_i \), we have
\[
E_Y(\alpha_i) = E_Y(u_i^{-1} \alpha'_i u_i)
\]
\[
= (1 - g_i(Y) + O(k + 2) + O_Y(2))(1 + Y_i)(1 + g_i(Y) + O(k + 2) + O_Y(2))
\]
\[
= 1 + Y_i + \sum_{l \neq i} \mu_u(li) \sum_{l \in S^k_i} \mu_\gamma(II)(Y_{lI} - Y_{II} - Y_{III} + Y_{II}) + O(k + 3) + O_Y(2).
\]

Now we consider the difference \( d_j(Y) = E_Y(\lambda_j) - (1 + f_j(Y) + r_j(Y) + O_Y(2)). \) Since \( E_Y(\lambda_j) \) is obtained from \( E_X(\lambda_j) = 1 + f_j(X) + r_j(X) + O_X(2) \) by substituting \( X_i \) for
\[
Y_i = \sum_{l \neq i} \mu_u(li) \sum_{l \in S^k_i} \mu_\gamma(II)(Y_{lI} - Y_{II} - Y_{III} + Y_{II}) + O(k + 3) + O_Y(2),
\]
all terms of degree \( \leq 2k \) of \( d_j(Y) - O_Y(2) \) vanish. The terms of degree \( 2k + 1 \) in \( d_j(Y) - O_Y(2) \) is obtained from \( f_j(Y) \) by substituting \( Y_i \) for
\[
\mu_\gamma(II)(Y_{lI} - Y_{II} - Y_{III} + Y_{II})
\]
for some \( i \in \{1, 2, \ldots, n\} \). It follows that
\[
d_j(Y) - O_Y(2) = 0
\]
\[
= \sum_{j \in S^k_i} \mu_\gamma(jj) Y_{J_{J_{J_{<}}}} \sum_{i \in \{j\}} \mu_u(li) \sum_{l \neq i} \mu_\gamma(II)(Y_{iI} - Y_{II} - Y_{III} + Y_{II}) + O(k + 3) + O_Y(2)
\]
\[
= \sum_{|J|=|I|=k} \mu_\gamma(jj) \mu_\gamma(II) \mu_u(li) Y_{J_{J_{J_{<}}}} + O(k + 3) + O_Y(2)
\]
where \( O_Y(2) \) means the terms which contain \( Y_j \) at least one time.

Finally, we observe the difference \( E_Y(\lambda'_j) - (1 + f_j(Y) + r_j(Y) + O_Y(2)). \) Since \( \lambda'_j = u_j \lambda_j u_j^{-1} \) (where \( \lambda_j \) is assumed to be an element of \( \pi_1([0, 1]^3 \setminus \gamma') \)), we have
\[
E_Y(\lambda'_j) = E_Y(u_j \lambda_j u_j^{-1})
\]
\[
= 1 + (1 + g_j(Y))(f_j(Y) + r_j(Y) + d_j(Y))(1 - g_j(Y)) + O(2k + 2) + O_Y(2)
\]
\[
= 1 + f_j(Y) + r_j(Y) + d_j(Y) + g_j(Y) f_j(Y) - f_j(Y) g_j(Y) + O(2k + 2) + O_Y(2)
\]
So we have
\[
E_Y(\lambda'_j) - (1 + f_j(Y) + r_j(Y) + O_Y(2))
\]
\[
= d_j(Y) + \sum_{|J|=|I|=k} \mu_\gamma(jj) \mu_\gamma(II) \mu_u(li) Y_{J_{J_{J_{<}}}} + Y_{II} + Y_{III} + Y_{II} + Y_{III} + Y_{II} + Y_{III} + Y_{II} + Y_{III}
\]
\[
+ O(2k + 2) + O_Y(2)
\]
\[
= \sum_{|J|=|I|=k} \mu_\gamma(jj) \mu_u(li) Y_{J_{J_{J_{<}}}} + Y_{II} + Y_{III} + Y_{II} + Y_{III} + Y_{II} + Y_{III} + Y_{II} + Y_{III}
\]
\[
+ O(2k + 2) + O_Y(2)
\]
This completes the proof. \hfill \square

Proof of Theorem 1.3. By Lemma 2.1 any two disk/band surfaces \( F_c \) and \( F_{c'} \) of an \( n \)-component clover link \( c \) are transformed into each other by the moves (a) and (b) in Figure 2.4. So two bottom tangles \( \gamma(F_c) \) and \( \gamma(F_{c'}) \) are transformed into each other by a SL-move. Since the both closures \( L(\gamma(F_c)) \) and \( L(\gamma(F_{c'})) \) are ambient isotopic to \( l_c \) and the hypothesis of Theorem 1.3

\[ 0 = n_l(J) = \mu(\gamma(F_c))(J) = \mu(\gamma(F_{c'}))(J) \]

for any sequence \( J \) with \( |J| \leq k \). Since \( \mu(pq) \) is the ‘linking number’ of the \( p \)th component and the \( q \)th component of \( u \), \( \mu(pq) = \mu(qp) \) and the set

\[ \{ \mu(pq) \mid u : \text{a string link} \} = \mathbb{Z} \]

for any \( p \) and \( q \). This and Proposition 4.1 give us the Theorem 1.3 \hfill \square

In order to prove Corollary 1.4 and Theorem 1.6, we need the following lemma given in [10].

Lemma 4.2. [10] Lemma 4.1] Two \( n \)-clover links \( c \) and \( c' \) are edge-homotopic if and only if there exist disk/band surfaces \( F_c \) and \( F_{c'} \) of \( c \) and \( c' \) respectively such that the two bottom tangles \( \gamma(F_c) \) and \( \gamma(F_{c'}) \) are link-homotopic.

Proof of Corollary 1.4. Let \( c \) and \( c' \) be \( n \)-clover links. We assume that they are edge-homotopic. By Lemma 1.2 there exist disk/band surfaces \( F_c \) and \( F_{c'} \) of \( c \) and \( c' \) respectively such that \( \gamma(F_c) \) and \( \gamma(F_{c'}) \) are link-homotopic. This implies that \( \mu(\gamma(F_c))(I) = \mu(\gamma(F_{c'}))(I) \) for any non-repeated sequence \( I \). On the other hand, by Theorem 1.3 the set \( H_c(2k + 2, j) \) (resp. \( H_{c'}(2k + 2, j) \)) is obtained from the Milnor numbers of \( \gamma(F_c) \) (resp. \( \gamma(F_{c'}) \)) for \( F_c \) (resp. \( F_{c'} \)). Hence we have \( H_c(2k + 2, j) = H_{c'}(2k + 2, j) \). \hfill \square

Proof of Theorem 1.6. Suppose that two \( 4 \)-clover links \( c \) and \( c' \) are edge-homotopic. By Corollary 1.4 \( H_c(4, 4) = H_{c'}(4, 4) (\neq \emptyset) \). By Lemma 1.2 there exist disk/band surfaces \( F_c \) and \( F_{c'} \) such that \( \gamma(F_c) \) and \( \gamma(F_{c'}) \) are link-homotopic. This implies that the Milnor numbers of \( \gamma(F_c) \) and \( \gamma(F_{c'}) \) are equal for any non-repeated sequence \( I \). Since the Milnor numbers of length \( \leq 3 \) are always well-defined for clover links (see Remark 2.3), we have \( \mu(I) = \mu(I) \) for any non-repeated sequence \( I \) with \( |I| \leq 3 \).

Conversely if \( H_c(4, 4) \cap H_{c'}(4, 4) \neq \emptyset \), then there exist disk/band surfaces \( F_c \) and \( F_{c'} \) of \( c \) and \( c' \) respectively such that

\[ \sum_{S \in \mathcal{S}_2} \mu(\gamma(F_c))(S4)X_S = \sum_{S \in \mathcal{S}_2} \mu(\gamma(F_{c'}))(S4)X_S. \]

In particular,

\[ \mu(\gamma(F_c))(1234) = \mu(\gamma(F_{c'}))(1234) \]
\[ \mu(\gamma(F_c))(2134) = \mu(\gamma(F_{c'}))(2134). \]

According to the link-homotopy classification theorem for string links by N. Habegger and X. S. Lin [2], for two \( 4 \)-component string links (bottom tangles) that have common values of the Milnor numbers for non-repeated sequences with length \( \leq 3 \), they are link-homotopic if and only if their Milnor numbers for sequences 1234 and 2134 coincide, see also [11] Theorem 4.3]. This together with the hypothesis implies that \( \gamma(F_c) \) and \( \gamma(F_{c'}) \) are link-homotopic. Therefore \( c \) and \( c' \) are edge-homotopic by Lemma 4.1. This completes the proof. \hfill \square
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