COHOMOLOGY CLASSIFICATION OF SPACES WITH FREE $S^3$-ACTIONS

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Abstract. This paper gives the cohomology classification of finitistic spaces $X$ equipped with free actions of the group $G = S^3$ and the orbit space $X/G$ is the integral or mod 2 cohomology quaternion projective space $\mathbb{HP}^n$. We have proved that $X$ is the integral or mod 2 cohomology $S^{4n+3}$ or $S^3 \times \mathbb{HP}^n$. Similar results for $G = S^1$ actions are also discussed.

1. INTRODUCTION

Let $G$ be a compact Lie group acting on a finitistic space $X$. There are interesting problems related to transformation groups, for example, to classify the fixed point set $X^G$, the existence of free/semifree actions and the study of the orbit space $X/G$ for free actions of $G$ on $X$. A number of results has been proved in the literature in this direction [1, 3, 5, 6, 10, 11]. An another thread of research is to classify $X$ for a given orbit space $X/G$ when $G$ acts freely on $X$. Su [12] proved that if $G = S^d$, $d = 0, 1$, acts freely on a space $X$ and the orbit space $X/G$ is cohomology $\mathbb{FP}^n$, then space $X$ is the cohomology sphere $S^{(d+1)n+d}$, when $d = 0$, $\mathbb{F} = \mathbb{R}$ with $\mathbb{Z}_2$ coefficients, and when $d = 1$, $\mathbb{F} = \mathbb{C}$ with integer coefficients. He also proved that if $G = \mathbb{Z}_p$, $p$ an odd prime, acting freely on a space $X$ with the orbit space the mod $p$ cohomology Lens space $I_{\mathbb{F}}^{2n+1}$, then $X$ is the mod $p$ cohomology $(2n + 1)$-sphere $S^{2n+1}$. Kaur et al. [8] shown that if $G = S^3$ acts freely on the mod 2 cohomology $n$-sphere $S^n$, then $n \equiv 3(\text{mod } 4)$ and the orbit space is the mod 2 cohomology quaternion projective space $\mathbb{HP}^n$. In this paper, we have shown that if $G = S^3$ acts freely on a finitistic space $X$ with the orbit space the mod 2 cohomology quaternion projective space, then $X$ is the mod 2 cohomology $S^{4n+3}$ or $S^3 \times \mathbb{HP}^n$ depending upon the Euler class of the associated bundle is nontrivial or trivial. A similar result with the integer
coefficient is also discussed. We have also proved Kaur’s results [8] with integer coefficients.

For the actions of $G = \mathbb{S}^1$, Su [12] proved that if $G = \mathbb{S}^1$ acts freely on a space $X$ such that $X/G$ is a cohomology complex projective space with $\dim_\mathbb{Z} X/G < \infty$ and $\pi^* : H^2(X/G) \to H^2(X)$, where $\pi : X \to X/G$ is the orbit map, is trivial, then $X$ is an integral cohomology $(2n + 1)$-sphere. We have discussed the case when the induced map $\pi^*$ is nontrivial. In this case, we have proved that $X$ is the integral cohomology $\mathbb{S}^{2n+1}$ or $\mathbb{S}^1 \times \mathbb{C}\mathbb{P}^n$ or $L_{p}^{2n+1}$.

2. Preliminaries

Let $G$ be a compact Lie group and $G \to E_G \to B_G$ be the universal principal $G$-bundle, where $B_G$ is the classifying space. Suppose $G$ acts freely on a space $X$. The associated bundle $X \hookrightarrow (X \times E_G)/G \to B_G$ is a fibre bundle with fibre $X$. Put $X_G = (X \times E_G)/G$. Then the bundle $X \hookrightarrow X_G \to B_G$ is called the Borel fibration. We consider the Leray-Serre spectral sequence for the Borel fibration. If $B_G$ is simply connected, then the system of local coefficients on $B_G$ is simple and the $E_2$-term of the Leray-Serre spectral sequence corresponding to the Borel fibration becomes

$$E_2^{k,l} = H^k(B_G; R) \otimes H^l(X; R).$$

For details about spectral sequences, we refer [9]. Let $h : X_G \to X/G$ be the map induced by the $G$-equivariant projection $X \times E_G \to X$. Then, $h$ is a homotopy equivalence [4].

The following results are needed to prove our results:

**Proposition 2.1 ([7]).** Let $R$ denote a ring and $\mathbb{S}^{n-1} \to E \xrightarrow{\pi} B$ be an oriented sphere bundle. The following sequence is exact with coefficients in $R$

$$\cdots \to H^i(E) \xrightarrow{\rho} H^{i-n+1}(B) \xrightarrow{\cup} H^{i+1}(B) \xrightarrow{\pi^*} H^{i+1}(E) \xrightarrow{\rho} H^{i-n+2}(B) \to \cdots$$

which start with

$$0 \to H^{n-1}(B) \xrightarrow{\pi^*} H^{n-1}(E) \xrightarrow{\rho} H^0(B) \xrightarrow{\cup} H^n(B) \xrightarrow{\pi^*} H^n(E) \to \cdots$$

where $\cup : H^i(B) \to H^{i+n}(B)$ maps $x \to x \cup u$ and $u \in H^n(B)$ denotes the Euler class of the sphere bundle. The above exact sequence is called the Gysin sequence. It is easy to observe that $\pi^* : H^i(E) \to H^i(B)$ is an isomorphism for all $0 \leq i < n - 1$. 
Proposition 2.2. Let $A$ be an $R$-module, where $R$ is PID, and $G = S^3$ acts freely on a finitistic space $X$. Suppose that $H^j(X, A) = 0$ for all $j > n$, then $H^j(X/G, A) = 0$ for all $j > n$.

We have taken Čech cohomology and all spaces are assumed to be finitistic. Note that $X \sim_R Y$ means $H^*(X; R) \cong H^*(Y; R)$, where $R = \mathbb{Z}_2$ or $\mathbb{Z}$.

3. Main Theorems

Recall that the projective spaces $\mathbb{P}^n$ are the orbit spaces of standard free actions of $G = S_d$ on $S(d+1)n + d$, where $F = \mathbb{C}$ or $\mathbb{H}$ for $d = 1$ or $3$, respectively. If we take a free action of $S_d$ on itself and the trivial action on $\mathbb{P}^n$, then the orbit space of this diagonal action is $\mathbb{P}^n$. Now, the natural question: Is the converse true? If $G$ acts freely on a finitistic space $X$ with $X/G \sim_R \mathbb{P}^n$, then whether $X \sim_R S^{dn + d}$ or $X \sim_R S^3 \times \mathbb{P}^n$. In the following theorems, we have proved that the converse of these statements are true.

Theorem 3.1. Let $G = S^3$ acts freely on a finitistic space $X$ with $X/G \sim_R \mathbb{H}P^n$, where $R = \mathbb{Z}_2$ or $\mathbb{Z}$, and $u \in H^4(X/G)$ be the Euler class of the bundle $G \hookrightarrow X \xrightarrow{\pi} X/G$. Then, $u$ is either trivial or generator of $H^*(X/G)$. Moreover,

(i) If $u$ is a generator, then $X \sim_R S^{4n+3}$, and
(ii) If $u$ is trivial, then $X \sim_R S^3 \times \mathbb{H}P^n$.

Proof. As $G$ is a compact Lie group which acts freely on $X$, we have the Gysin sequence of the sphere bundle $G \hookrightarrow X \xrightarrow{\pi} X/G$:

$$\cdots \rightarrow H^i(X) \xrightarrow{\rho} H^{i-3}(X/G) \xrightarrow{\cup} H^{i+1}(X/G) \xrightarrow{\pi^*} H^{i+1}(X) \xrightarrow{\rho} H^{i-2}(X/G) \rightarrow \cdots$$

which begins with

$$0 \rightarrow H^3(X/G) \xrightarrow{\pi^*} H^3(X) \xrightarrow{\rho} H^0(X/G) \xrightarrow{\cup} H^4(X/G) \xrightarrow{\pi^*} H^4(X) \rightarrow \cdots$$

Since $X/G \sim_R \mathbb{H}P^n$, we have $H^*(X/G) = R[a]/(a^{n+1})$, where $\deg a = 4$. Note that $H^i(X) \cong H^i(X/G)$ for $i = 0, 1, 2$. By the exactness of the Gysin sequence, $H^{4i+1}(X) = H^{4i+2}(X) = 0$ for all $i \geq 0$ and $H^2(X) = 0$ for all $j > 4n + 3$. There are three possibilities: If the Euler class is (i) generator, (ii) nontrivial but not a generator, and (iii) trivial.

If the Euler class $u \in H^4(X/G)$ is a generator then $\cup : H^{4i}(X/G) \rightarrow H^{4i+4}(X/G)$ is an isomorphism for all $0 \leq i < n$ and thus, the Euler class of the bundle $G \rightarrow X \xrightarrow{\pi} \mathbb{H}P^n$. 

$X/G$ is nonzero. By the exactness of the Gysin sequence $\rho : H^{4i+3}(X) \to H^{4i}(X/G)$ and $\pi^* : H^{4i+4}(X/G) \to H^{4i+4}(X)$ becomes trivial for all $0 \leq i < n$. This gives that $H^{4i+3}(X) = H^{4i+4}(X) = 0$ for all $0 \leq i < n$. As $H^{4n+4}(X/G) = 0$, we have $H^{4n+3}(X) \cong H^{4n}(X/G) \cong R$. Consequently,

$$H^i(X) = \begin{cases} R & \text{if } i = 0, 4n + 3 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $X \sim_R \mathbb{S}^{4n+3}$.

If $u \in H^4(X/G)$ is a nontrivial but not a generator then this is possible only when $R = \mathbb{Z}$ and the Euler class $u \in H^4(X/G)$ is $m.a$, where $m$ is an integer different from 0 and 1. Then, the Euler class of the associated bundle is $m.a$ and $\cup : H^4(X/G) \to H^{4i+4}(X/G)$ maps generator $a^i$ to $m.a^i$ for all $0 \leq i < n$. By the exactness of the Gysin sequence, $H^{4i+3}(X) = 0$ and $H^{4i+4}(X) \cong H^{4i+4}(X/G) / \ker \pi^* \cong \mathbb{Z}_m$ for all $0 \leq i < n$. As $H^{4n+4}(X/G) = 0$, we have $H^{4n+3}(X) \cong H^{4n}(X/G) \cong \mathbb{Z}$. Let $a_4 \in H^4(X)$ and $b_{4n+3} \in H^{4n+3}(X)$ be such that $\pi^*(a_i) = a_4$ and $\rho(b_{4n+3}) = a^n$. Thus, we have

$$H^i(X) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ or } 4n + 3 \\ \mathbb{Z}_m & \text{if } 0 < i \equiv 0 \pmod{4} \leq 4n \\ 0 & \text{otherwise.} \end{cases}$$

As $G$ acts freely on $X$ and $B_G$ is simply connected, the $E_2$-term of the associated Leray-Serre spectral sequence for the Borel fibration $X \hookrightarrow X_G \twoheadrightarrow B_G$ is given by $E_2^{pq} = H^p(B_G) \otimes H^q(X)$ which converges to $H^*(X_G)$ as an algebra. Now, $H^*(B_G) = H^*(\mathbb{H}^{\infty}) = \mathbb{Z}[t]$, where $\deg t = 4$. Note that the only possible nontrivial differentials are $d_{4r} : E_{4r}^{*,*} \to E_{4r}^{*,*}, 1 \leq r \leq n + 1$. As $4n + 4 \geq 8$, $t \otimes 1$ and $1 \otimes a_4$ are permanent cocycles. So, $H^4(X_G) \cong \mathbb{Z} \oplus \mathbb{Z}_m$, a contradiction.

If the Euler class $u \in H^4(X/G)$ is trivial then the Euler class of the bundle $G \to X \to X/G$ is zero and $\cup : H^{4i}(X/G) \to H^{4i+4}(X/G)$ is trivial for all $i \geq 0$. By the exactness of the Gysin sequence, $\rho : H^{4i+3}(X) \to H^{4i}(X/G)$ and $\pi^* : H^{4i}(X/G) \to H^{4i}(X)$ becomes isomorphism for all $0 \leq i \leq n$. Let $a_4 \in H^4(X)$ and $b_{4i+3} \in H^{4i+3}(X)$ be such that $\pi^*(a_i) = a_4$ and $\rho(b_{4i+3}) = a^i$ for all $0 \leq i \leq n$. This implies that $H^{4i+3}(X) \cong R$ with basis $\{b_{4i+3}\}$ and $H^{4i}(X) \cong R$ with basis $\{a_{4i}\}$ for all $0 \leq i \leq n$. Thus, we have

$$H^i(X) = \begin{cases} R & \text{if } 0 \leq i \equiv 0 \text{ or } 3 \pmod{4} \leq 4n + 3 \\ 0 & \text{otherwise.} \end{cases}$$
Note that $b_i b_j = 0$ for all $i$ and $j$ and $a_i^{n+1} = 0$. Next, we observe that $a_i^t b_3 = b_{4i+3}$ for all $1 \leq i \leq n$. In the associated Leray-Serre spectral sequence, the only possible nontrivial differentials are $d_r : E_{r*}^r \to E_{r*}^{r+1}$, for $0 \leq r \leq n + 1$. So, the first nonzero possible differential is $d_4$. Clearly, $d_4(1 \otimes a_i^t) = 0$ for all $i \geq 0$. Now, we consider two subcases for coefficient groups $R = \mathbb{Z}_2$ or $R = \mathbb{Z}$:

Let $R = \mathbb{Z}_2$ and $a_i^t b_3 = 0$ for some $1 \leq k \leq n$. If $d_4(1 \otimes b_3) = t \otimes a_i^t$, then $t \otimes a_i^t = d_4((1 \otimes a_i^t)(1 \otimes b_3)) = 0$ which is not possible. Therefore, $d_4(1 \otimes b_3) = 0$. As $d_4 : E_4^{4i-4r, 4r+2} \to E_4^{4i, 3}$ is trivial, $t \otimes b_3$ are permanent cocycles for all $i \geq 0$, a contradiction to the fact that $H^2(X/G) = 0$ for all $j > 4n$. Therefore, $a_i^t b_3 \neq 0$ for all $1 \leq i \leq n$. This implies that $b_{4i+3} = a_i^t b_3$ for all $1 \leq i \leq n$. Thus, the cohomology ring of $X$ is $\mathbb{Z}_2[a_4, b_3]/(a_4^{n+1}, b_3^2)$, $\deg a_4 = 4$, $\deg b_3 = 3$. It is clear that $X \sim_{\mathbb{Z}_2} S^3 \times \mathbb{H}P^n$. This realizes case(ii) of the theorem.

Now, let $R = \mathbb{Z}$ and $a_i^t b_3 \neq \pm b_{4i+3}$ for some $1 \leq j \leq n$. Let $i_0 \in \mathbb{Z}$ be the largest integer such that $a_i^{i_0} b_3 \neq \pm b_{4i_0+3}$. If $d_4(1 \otimes b_3) = 0$, then $\{t \otimes b_3\}$ are permanent cocycles for all $i \geq 0$, which is not possible as in subcase(i). So, let $d_4(1 \otimes b_{4i+3}) = m_0(t \otimes a_i^t)$, where $m_0 \in \mathbb{Z}$ and $m_0 \neq 0$. Then, $H^4(X/G) \cong \mathbb{Z} \oplus \mathbb{Z}_{m_0}$. This gives that $m_0 = \pm 1$. Clearly, $d_4 : E_4^{0, 4j+3} \to E_4^{4, 4j}$ is an isomorphism for $i_0 + 1 \leq j \leq n$. So, we have $E_5^{i, 4j} = E_5^{0, 4j+3} = 0$ for all $i \geq 0$, $j = 0$ and $i_0 + 1 \leq j \leq n$. Note that $E_5^{i, 4j} = \mathbb{Z}_{m_j}$, where $1 \leq j \leq i_0$, and $E_5^{i, 4j+3}$ is $\mathbb{Z}$ if $m_j = 0$, and trivial, otherwise. If $d_4 : E_4^{0, 4i_0+3} \to E_4^{4, 4i_0}$ is trivial, then $\{t \otimes b_{4i_0+3}\}_{i \geq 0}$ are permanent cocycles, a contradiction. So, let $d_4 : E_4^{0, 4i_0+3} \to E_4^{4, 4i_0}$ is nontrivial. Now, $d_4(1 \otimes (a_i^{i_0} b_3 \pm b_{4i_0+3})) = (m_0 \pm m_{i_0})(t \otimes a_i^{i_0})$. Consequently, $m_{i_0} \neq \pm 1$. Thus, $H^j(X/G)$ is nonzero for infinitely many values of $j$, a contradiction. Therefore, $a_i^t b_3$ is $b_{4j+3}$ or $-b_{4j+3}$ for all $j$. Hence, $X \sim_{\mathbb{Z}} S^3 \times \mathbb{H}P^n$. □

Now, we compute the orbit space of free actions of $G = S^3$ on a paracompact space with integral cohomology $n$-sphere:

**Theorem 3.2.** Let $G = S^3$ acts freely on a paracompact space $X$ with $X \sim_{\mathbb{Z}} S^n$. Then, $n = 4k + 3$, for some $k \geq 0$ and $X/G \sim_{\mathbb{Z}} \mathbb{H}P^k$.

**Proof.** By the Gysin sequence sequence of the 3-sphere bundle, we get $H^0(X/G) \cong \mathbb{Z}$ and $H^i(X/G) = 0$, for all $1 \leq i \leq 3$ when $n \neq 1, 2$ or 3. Then, for $0 \leq i \leq n - 4$, $\cup : H^i(X/G) \to H^{i+4}(X/G)$ is an isomorphism. This gives that $H^i(X/G) = 0$ for $0 < i \equiv j \mod 4 < n$, where $1 \leq j \leq 3$ and $H^i(X/G) \cong \mathbb{Z}$ for $0 \leq i \equiv 0 \mod 4 < n$ with basis $\{a^4\}$, where $a \in H^4(X/G)$ denotes its generator. Suppose $n \equiv j \mod 4$, for some $0 \leq j \leq 2$ then $H^{n-3}(X/G) = 0$. If $(n = 1$ or 2) or $(0 \leq j \leq 2)$, then by
the exactness of the Gysin sequence, \( H^n(X/G) \neq 0 \), which contradicts Proposition 2.2. Therefore, \( n \equiv 3 \pmod{4} \). Let \( n = 4k + 3 \) for some \( k \geq 0 \). For \( n = 3 \), the result is trivially true. So let \( n > 3 \). Again, by Proposition 2.2 \( H^j(X/G) = 0 \) for all \( j > n \), and hence \( a^{k+1} = 0 \). This implies that \( \rho : H^n(X) \to H^{n-3}(X/G) \) is an isomorphism. Consequently, \( H^n(X/G) = 0 \). Thus, we have, \( H^*(X/G) = \mathbb{Z}[a]/\langle a^{k+1} \rangle, \deg a = 4 \).

In 1963, Su [12] has shown that if \( G = S^1 \) acts freely on a space \( X \) with orbit space \( X/G \sim \mathbb{C}P^n \) and \( \pi^* : H^2(X/G) \to H^2(X) \) is trivial, then \( X \sim \mathbb{S}^{2n+1} \), where \( \pi : X \to X/G \) is the orbit map. In the next theorem, we discuss the case when \( \pi^* \) is nontrivial.

**Theorem 3.3.** Let \( G = S^1 \) acts freely on a finitistic space \( X \) with \( X/G \sim \mathbb{C}P^n \), and \( u \in H^2(X/G) \) be the Euler class of the bundle \( G \to X \xrightarrow{\pi} X/G \). If the induced map \( \pi^* : H^2(X/G) \to H^2(X) \) is nontrivial, then \( u \) is trivial and \( X \sim \mathbb{S}^1 \times \mathbb{C}P^n \).

**Proof.** As \( X/G \sim \mathbb{C}P^n \), \( H^*(X/G) = \mathbb{Z}[a]/\langle a^{n+1} \rangle \), where \( \deg a = 2 \). As \( \pi_1(B_G) = 1 \), \( E_2 \)-term of the Leray-Serre spectral sequence is \( E_2^{p,q} = H^p(B_G) \otimes H^q(X) \) for the Borel fibration \( X \xrightarrow{\pi} X_G \to B_G \). Note that the possible nontrivial differentials are \( d_2, d_4, \ldots, d_{2n+2} \). Suppose \( \pi^* : H^2(X/G) \to H^2(X) \) is nontrivial. Then the Euler class \( u \in H^2(X/G) \) is not a generator. So, first suppose that the Euler class of the principal bundle \( X \xrightarrow{\pi} X/G \) is m.a, where \( m \neq 0 \) in \( \mathbb{Z} \). As \( \pi^* : H^2(X/G) \to H^2(X) \) is nontrivial, \( m \neq \pm 1 \). Then by the exactness of the Gysin sequence \( H^i(X) \cong \mathbb{Z} \) for \( i = 0, 2n + 1 \); \( H^i(X) \cong \mathbb{Z}_m \) with basis \( \{a_2^i\} \) for \( i = 0, 2, 4, \ldots, 2n \); and trivial otherwise. It gives that \( t^i \otimes a_2^j \) are permanent cocycles for all \( i, j \geq 0 \), a contradiction. Next, suppose that the Euler class \( u \) of the principal bundle is zero. Consequently, we have

\[
H^j(X) = \begin{cases} 
\mathbb{Z} & \text{if } 0 \leq j \leq 2n + 1 \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( a_2 \in H^2(X) \) and \( b_{2i+1} \in H^{2i+1}(X) \) be such that \( \pi^*(a) = a_2 \) and \( \rho(b_{2i+1}) = a^i \) for all \( 0 \leq i \leq n \). This implies that \( H^{2i+1}(X) \cong \mathbb{Z} \) with basis \( \{b_{2i+1}\} \) and \( H^{2i}(X) \cong \mathbb{Z} \) with basis \( \{a_2^i\} \) for all \( 0 \leq i \leq n \). Let if possible \( a_2^i b_1 \neq \pm b_{2j+1} \) for some \( 1 \leq j \leq n \) and suppose \( i_0 \) be such an largest integer. As \( H^1(X_G) = 0 \), \( d_2(1 \otimes b_1) \neq 0 \). So, let \( d_2(1 \otimes b_{2i+1}) = m_i(t \otimes a_2^i) \), where \( m_i \in \mathbb{Z} \) and \( m_0 \neq 0 \). Note that \( E_2^{2i,2j} = \mathbb{Z}_{m_j} \) and \( E_3^{2i,2j+1} = \mathbb{Z} \) if \( m_j = 0 \) and trivial otherwise for all \( i \geq 0 \) and \( 0 \leq j \leq n \). Since \( H^2(X_G) \cong \mathbb{Z} \), we have \( d_2 : E_2^{0,1} \to E_2^{2,0} \) is an isomorphism. Therefore, \( E_3^{i,2j} = E_3^{i,2j+1} = 0 \) for all \( i \geq 0 \) and \( i_0 + 1 \leq j \leq n \). If \( d_2 : E_2^{0,2m+1} \to E_2^{2,2m} \) is trivial, then
\{t^i \otimes b_{2i+1}\}_{i \geq 0}\) are permanent cocycles, a contradiction. So, let \(d_2 : E_2^{0,2i+1} \to E_2^{2,2i}\) is nontrivial. As \(d_2(1 \otimes a_2) = 0\), we get \(m_{i_0} \neq m_0\), and hence \(t^i \otimes a_{2i}^{\prime}\) are permanent cocycles for all \(i \geq 0\), a contradiction. Thus,

\[
H^\ast(X) = \mathbb{Z}[a_2, b_1]/\langle a_2, b_1^2 \rangle,
\]

where \(\deg b_1 = 1\) and \(\deg a_2 = 2\). Hence, our claim.

Now, we prove similar results with coefficients in \(\mathbb{Z}_p\), \(p\) a prime.

**Theorem 3.4.** Let \(G = S^1\) acts freely on a finitistic space \(X\) with the orbit space \(X/G \sim_{\mathbb{Z}_p} \mathbb{C}P^n\), \(p\) a prime. Let \(\pi^\ast : H^2(X/G) \to H^2(X)\) be the map induced by the orbit map \(\pi : X \to X/G\).

1. If \(\pi^\ast : H^2(X/G) \to H^2(X)\) is trivial, then \(X \sim_{\mathbb{Z}_p} S^{2n+1}\).
2. If \(\pi^\ast : H^2(X/G) \to H^2(X)\) is nontrivial, then either \(X \sim_{\mathbb{Z}_p} S^1 \times \mathbb{C}P^n\) or \(L_p^{2n+1}\).

**Proof.** The Euler class of the principal bundle \(X \to X/G\) is either trivial or a generator of \(H^4(X/G; \mathbb{Z}_p)\). If the Euler class of the associated bundle is trivial, then \(X \sim_{\mathbb{Z}_p} S^{2n+1}\). So, let the Euler class be a generator of \(H^4(X; \mathbb{Z}_p)\). It is easy to see that

\[
H^\ast(X; \mathbb{Z}_p) \cong \mathbb{Z}_p[b_1, b_2, \ldots, b_{2n+1}, a_2]/\langle a_2^{n+1} \rangle, \quad \deg a_2 = 2, \deg b_i = i.
\]

In the Leray-Serre spectral sequence, we must have \(d_2(1 \otimes b_i) \neq 0\) for suitable choice of generator \(b_i\) and \(d_2(1 \otimes a_{2i}) = 0\) for all \(0 \leq i \leq n\). This implies that \(b_{2i+1} = a_{2i+1}b_i\) for all \(0 \leq i \leq n\). If \(b_i^2 = 0\), then \(X \sim_{\mathbb{Z}_p} \mathbb{R}P^{2n+1}\). If \(b_i^2 \neq 0\) and \(p = 2\), then \(a_2 = b_i^2\).

This gives that \(X \sim_{\mathbb{Z}_2} \mathbb{R}P^{2n+1}\). If \(b_i^2 \neq 0\) and \(p\) is an odd prime, then \(\beta(b_1) = a_2\), where \(\beta : H^1(X; \mathbb{Z}_p) \to H^2(X; \mathbb{Z}_p)\) is the Bockstein homomorphism associated to the coefficient sequence \(0 \to \mathbb{Z}_p \to \mathbb{Z}_p^2 \to \mathbb{Z}_p \to 0\), then \(X \sim_{\mathbb{Z}_p} L_p^{2n+1}\). \(\square\)

The next example realises the above theorem.

**Example 3.5.** Recall that the map \((\lambda, (z_0, z_1, \ldots, z_n)) \to (\lambda z_0, \lambda z_1, \ldots, \lambda z_n)\), where \(\lambda \in S^1\) and \(z_i \in \mathbb{C}, 0 \leq i \leq n\), defines a standard free action of \(G = S^1\) on \(S^{2n+1}\). The orbit space \(X/G\) under this action is \(\mathbb{C}P^n\). For \(p\) a prime, \(H = \langle e^{2\pi i/p} \rangle\) induces a free action on \(S^{2n+1}\) with the orbit space \(S^{2n+1}/H = L_p^{2n+1}\). Consequently, \(S^1 = G/H\) acts freely on \(L_p^{2n+1}\) with the orbit space \(\mathbb{C}P^n\). Recall that for \(p = 2\), \(L_p^{2n+1} = \mathbb{R}P^{2n+1}\).
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