Compressive sampling with chaotic dynamical systems

Venceslav Kafedziski, Member, IEEE and Toni Stojanovski, Member, IEEE

Abstract—We investigate the possibility of using different chaotic sequences to construct measurement matrices in compressive sampling. In particular, we consider sequences generated by Chua, Lorenz and Rössler dynamical systems and investigate the accuracy of reconstruction when using each of them to construct measurement matrices. Chua and Lorenz sequences appear to be suitable to construct measurement matrices. We compare the recovery rate of the original sequence with that obtained by using Gaussian, Bernoulli and uniformly distributed random measurement matrices. We also investigate the impact of correlation on the recovery rate. It appears that correlation does not influence the probability of exact reconstruction significantly.

Index Terms—Compressive sampling, Chaos, Correlation.

I. INTRODUCTION

According to the Nyquist-Shannon sampling theorem, if a signal is band limited to a bandwidth $B$, then it is completely determined by sampling it at discrete times, provided that the sampling rate is at least equal to $2B$. The original continuous-time signal can be reconstructed from the discrete-time samples via an interpolation process achieved with a low-pass filter.

Recently, work by Candès [1], Donoho [2], and others demonstrated that it is possible to exactly reconstruct some signals from undersampled data. If a signal is sparse in the original domain, or some transform domain (sparsifying domain), meaning that it does not have many features, we can take far fewer measurements and utilize knowledge of the structure of the signal to infer the rest. Thus, we can exactly reconstruct sparse signals with far fewer measurements than needed by Nyquist - Shannon theory.

This approach to sub-Nyquist sampling has been called compressive sampling (CS). Nice overviews of compressive sampling can be found in [3], [4].

The idea of CS is to combine the two stages, sampling and compression. Measurements of the signal are taken using the measurement matrix, which is supposed to be incoherent with the matrix describing the sparsifying transform. More formally, the so called Restricted Isometry Property (RIP) should be met. Since random signals are incoherent with almost anything, taking iid samples from a random distribution (for example, Gaussian or uniform) to create the measurement matrix, violates the RIP property with exponentially small probability.

Since chaotic signals exhibit similar properties to random signals, some authors have attempted using chaotic signals to construct the measurement matrix.

In [5] the authors continue on the work in [6] on random filters in CS, and examine the use of chaos filters in CS with filter taps calculated from the Logistic map. The authors claim that their numerical simulations indicate that chaos filters generated by the Logistic map outperform random filters.

In [7] the authors construct the measurement matrix with chaotic sequence from Logistic map and prove that, with overwhelming probability, the RIP of this kind of matrix is satisfied, which guarantees exact recovery. The authors experimentally show that chaotic matrix has similar performance to the Gaussian random matrix and sparse random matrix. In the subsequent work [8], the authors show that Toeplitz-structured measurement matrix constructed using a chaotic sequence is sufficient to satisfy RIP with high probability. This measurement matrix can be easily built as a filter with a small number of taps.

To the best of our knowledge in the literature on using chaotic sequences in CS, chaotic maps other than the Logistic map have not been used. Also, in the literature it is usually assumed that the chaotic sequence should be uncorrelated, and, therefore, every $d$-th sample is used to create the measurement matrix, where $d$ is such to ensure that the samples are uncorrelated. Although the performance of Logistic map has been shown to be very satisfactory and sometimes to outperform the performance using random matrices, such as Gaussian, Bernoulli or uniform, it is still desirable to examine and compare the performance of other chaotic maps.

Therefore, in this paper we address the following questions:

- Which chaotic signals can be used to construct measuring matrices?
- Does the correlation influence the probability of exact reconstruction significantly?
- How does the performance of chaos-based measurement matrices compare to the performance of random measurement matrices?

In Section II we give a brief overview of compressive sampling. Section III gives a brief overview of nonlinear dynamical systems that exhibit chaotic behaviour, and depicts the properties of three chaotic systems whose applicability in CS is examined. The main results of this work are presented in Section IV. Section V concludes the paper.

II. COMPRESSIVE SAMPLING

Compressive sampling answers the question if we can compress the signal $x \in \mathbb{R}^N$ into some compressed basis $\Psi$ where it can be represented sparsely as a signal $s$, then can we
recover the original signal if the number of measurements $M$ is approximately equal to the number of significant components of $s$? If our sparse signal $s$ is $k$-sparse, meaning it has $k$ significant components, we can fix our solution for $s$ in $k$ dimensions, and do some kind of optimization for the remaining $N - k$ elements. Mathematically, the measured samples are given by

$$y = \Phi x = \Phi \Psi s = \Theta s \tag{1}$$

where $\Phi$ is the measurement (sensing) basis, $\Psi$ is the compression basis, and $\Theta = \Phi \Psi$ is the compressive sensing matrix and is the product of the compression and measurement bases. So, we can take some small number of samples $y$, compute the sparse representation $s$ of our exact signal $x$, and then apply the inverse compression approximation to recover $x$.

Important property that $\Phi$ and $\Psi$ should meet is incoherence. The coherence measures the largest correlation between any two elements of $\Phi$ and $\Psi$. If $\Phi$ and $\Psi$ contain correlated elements, the coherence is large. Otherwise, it is small. Mathematically, coherence is defined as,

$$\mu(\Psi, \Phi) = \sqrt{N} \max_{1 \leq k \leq j \leq N} |\langle \psi_k, \phi_j \rangle| \tag{2}$$

This function takes on values between 1 and $\sqrt{N}$.

To guarantee that the compressive sensing matrix $\Theta$ is stable, it must meet the Restricted Isometry Property (RIP):

$$(1 - \delta_k)\|s\|_2^2 \leq \|\Theta s\|_2^2 \leq (1 + \delta_k)\|s\|_2^2 \tag{3}$$

In other words, $\Theta$ must be a distance preserving transformation for all $k$-sparse vectors $x$, bounded by some constant $\delta_k$, known as the restricted isometry constant. When this property holds, all $k$-subsets of the columns of $\Theta$ are nearly orthogonal. If this property does not hold then it is possible for a $k$-sparse signal to be in the null space of $\Theta$ and in this case it may be impossible to reconstruct these vectors.

It has been shown that bases $\Phi$ and $\Psi$ which are incoherent will satisfy RIP.

Regarding the reconstruction algorithms, different norms can be used. The $l_2$ norm does not favor a sparse solution and has been shown that cannot be used. The $l_0$ norm counts the number of zeros in the vector, and that’s exactly what we want to minimize. But, it turns out that we would have to try every combination of zeros to find the solution, which is NP-hard, and thus intractable. The researchers that worked in the CS area discovered that we can solve the problem using the $l_1$ norm and obtain exact results.

### III. Chaotic Dynamical Systems

Nonlinear dynamical systems are capable of exhibiting chaotic behavior for certain parameter values. Chaotic behavior exhibits exponential sensitivity to small changes in initial conditions: two chaotic trajectories starting from arbitrarily close initial conditions will eventually diverge from each other. Thus, even the smallest error in the measurement of initial conditions of a chaotic dynamical system precludes us from predicting its long term behavior. Despite their deterministic definition e.g. via ordinary differential equations, chaotic dynamical systems exhibit unpredictable behavior.

This duality in the nature of chaotic dynamical systems have sparked an immense interest in their potential applicability in a wide range of areas: cryptography, telecommunications, traffic modelling, medicine etc. Using chaotic dynamical systems for random number generation is a well researched area [9]. In this section we present the properties of three well-known chaotic dynamical systems which are relevant for generation of elements of measurement matrices in CS.

#### A. Chua

Properties of chaotic behaviour will be illustrated via an example based on Chua’s circuit which is the simplest electronic circuit capable of exhibiting chaos. Besides its simplicity, it is particularly useful and interesting because its chaotic behaviour has been proven analytically, numerically and experimentally which has not been accomplished for many other circuits.

In numerical simulations, one often exploits the following dimensionless form of Chua’s circuit

$$\dot{x} = \begin{bmatrix} \alpha x_2 - x_1 - g(x_1) \\ x_1 - x_2 + x_3 - \beta x_2 \\ -\gamma x_1 \end{bmatrix} \tag{4}$$

where $g(x_1) = bx_1 + \frac{1}{2}(a-b)(|x_1 + 1| - |x_1 - 1|)$. For the following parameter values $\alpha = -1.27$, $b = -0.68$, $\alpha = 10.0$, $\beta = 14.87$ Chua’s circuit exhibits chaotic behaviour. The chaotic attractor of Chua’s circuit for the previous parameter values is widely known as the double-scroll chaotic attractor. As an indication of the randomness of chaotic trajectories we show in Fig. 1 the normalised autocorrelation

$$R_{x_1, \text{norm}}(\tau) = \frac{R_{x_1}(\tau)}{R_{x_1}(0)}$$

of the signal $x_1(t)$ generated by Chua’s circuit [4], where

$$R_{x_1}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x_1(t + \tau)x_1(t)dt.$$  

Figure 1 reveals rapid decorrelation between close samples, thus making this sequence akin to a random trajectory.

**Fig. 1.** Autocorrelation of the chaotic signal generated by Chua’s circuit.

Figure 2 shows the probability density function of the chaotic signal $x_1(t)$ generated by Chua’s circuit [4].

#### B. Lorenz

We have numerically examined the Lorenz system

$$\dot{x} = \begin{bmatrix} 16.0(x_2 - x_1) \\ 45.6x_1 - x_1x_3 - x_2 \\ x_1x_2 - 4.0x_3 \end{bmatrix}. \tag{5}$$

Correlation between samples decreases rapidly with the sampling distance, as shown in Figure 3.

**Fig. 4** shows the probability density function of the chaotic signal $x_1(t)$ generated by Lorenz system [5].
IV. MAIN RESULTS

In this section we explain how we use chaotic signals to sample sparse signals. We compare their performance to the performance when sampling is done by random measurement matrices. We also investigate the impact of correlation in the measurement sequence, chaotic or random.

As an input sequence we use time-sparse signals: \( k \) spikes \( \in \{-1, +1\} \) randomly set in a sequence of \( N \) samples, with equal probability of \(-1\)’s and \(+1\)’s. A Bernoulli sequence of total of \( k \) \(-1\)’s and \(+1\)’s with probability 0.5 is generated by using Monte Carlo simulation. Then \( N-k \) zeros are added at the end of this sequence. Finally, the obtained sequence of length \( N \) is randomly permuted.

Next we explain the process of construction of the measurement matrix \( \Phi \in \mathbb{R}^{M \times N} \). The procedure is the same for both random signals and chaotic signals. We first generate a sequence \( \mathbf{c} = [c_0, c_1, \ldots, c_{MN-1}] \) of length \( M \times N \) and then we construct a matrix of size \( M \times N \) columnwise (taking \( M \) contiguous samples and putting them in a column of \( \Phi \)).

All elements of the matrix are scaled with the factor \( 1/(\sigma \sqrt{M}) \) where \( \sigma^2 \) is the variance of the sampled signal (or sequence) used to construct the measurement matrix. Thus

\[
\Phi = \frac{1}{\sigma \sqrt{M}} \begin{bmatrix}
    c_0 & c_M & \cdots & c_{M(N-1)} \\
    c_1 & c_{M+1} & \cdots & c_{M(N-1)+1} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{M-1} & c_{2M-1} & \cdots & c_{MN-1}
\end{bmatrix}
\]  

The vector of measurements \( \mathbf{y} \in \mathbb{R}^M \) is obtained as

\[
\mathbf{y} = \Phi \mathbf{x}
\]  

The problem of reconstruction can be stated formally as:

\[
\min_{\mathbf{s} \in \mathbb{R}^N} \|\mathbf{s}\|_1 \quad \text{subject to} \quad \mathbf{y} = \Theta \mathbf{s}
\]

This is a well-established problem known as basis pursuit. Basis pursuit problems can be easily transformed into a linear programming problem.

In our simulations we used the primal-dual interior point algorithm, whose Matlab implementation can be found here: [http://www-stat.stanford.edu/~candes/software.html](http://www-stat.stanford.edu/~candes/software.html) under L1 MAGIC. It requires as an input the initial guess \( \mathbf{x}_0 \) for the solution, the measurement matrix \( \Phi \), the measurements \( \mathbf{y} \), and the precision to which we want the problem solved.

We performed extensive simulations with various measurement matrices. We used two continuous time chaotic dynamical systems for creating the elements of the measurement matrix: Chua’s circuit and Lorenz system. For Chua’s circuit we sampled \( x_1 \) at sampling distance \( \tau = 1 \), while for Lorenz system we sampled \( x_1 \) at sampling distance \( \tau = 0.5 \).

We also used several random measurement matrices, obtained from sequences such as iid Gaussian, Gaussian with autocorrelation \( R(i) = \rho^{|i|} \), \( i \in \mathbb{Z} \) (we used \( \rho = 0.99 \)), Bernoulli, iid uniform in the range \([0,1]\) and iid uniform in the range \([-0.5,0.5]\).

Figure 5 depicts the dependence of the probability of incorrect reconstruction on the signal sparsity \( k \) when \( N = 100 \) and \( M = 50 \). As a criterion for exact reconstruction we used that the relative error \( e \) is smaller than \( \varepsilon = 0.01 \)

\[
e = \|\mathbf{x} - \mathbf{x}_r\|/\|\mathbf{x}\| < \varepsilon = 0.01
\]
where $x_r$ is the reconstructed vector and $\| \cdot \|$ is the $l_2$ norm.

As depicted in Fig. 5 there are no significant differences in the performance of different measurement matrices. This is despite the significant normalised autocorrelation $R_{x_1, \text{norm}}(\tau = 1) = 0.47$ for Chua’s circuit (4).

Gaussian sequence with correlation 0.99 also did not exhibit any noticeable performance loss. We should keep in mind that since the elements of the measurement sequence are written columnwise in matrix $\Phi$, the correlation is significantly decreased (the adjacent elements in each row come from elements in the sequence that are $M$ samples apart).

Maximum sparsity $k_{\text{max}}$ which allows for correct recovery of $x$ does not depend on the measurement matrix, and is solely determined by $N$ and $M$. Figure 6 depicts the dependence of the maximum sparsity $k_{\text{max}}$ on the ratio $\tau = N/M$ for $\varepsilon = 0.01$. $k_{\text{max}}$ is determined as the maximum sparsity for which error rate is smaller than 0.1. Linear interpolation is used to determine $k_{\text{max}}$. If $M$ is doubled, then $k_{\text{max}}$ increases roughly by factor 2.1 for each $N$.

Figure 7 depicts histograms of the logarithm of the relative error $e = \| |x - x_r|/\|x\| \|$ for fixed $N = 100$ and $M = 50$, and for various $k$, obtained from 5,000 simulation runs. When the original signal is correctly recovered, then a small but finite error $e$ occurs due to the finite precision to which the optimization problem is solved by the L1 MAGIC algorithm. If the recovery is incorrect, then a large relative error occurs. Consequently, wide range of values of $\varepsilon \in [10^{-2.5}, 10^{-0.5}]$ can distinguish between the correct and incorrect recovery.

V. CONCLUSION

We studied the use of different chaotic signals to construct measurement matrices in compressive sampling. We showed that Chua and Lorenz chaotic signals show performance comparable to that of random Gaussian, Bernoulli and uniformly distributed sequences. We determined that the correlation does not increase the probability of incorrect reconstruction. The performance is relatively insensitive on the value of the parameter $\varepsilon$ used to test the exact reconstruction.

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