Answer to Question 55.

Are there pictorial examples that distinguish covariant and contravariant vectors?

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The pair of terms: contravariant vectors and covariant vectors can be replaced by another pair: vectors and linear forms. The vectors constitute the linear space, that is a set with two operations: addition and multiplication by scalars. The linear form is a mapping of vectors into scalars which is additive and homogeneous under multiplication by scalars. The set of linear forms is itself a linear space – it is called dual space. Neuenschwander [1] claims that "the distinction between covariant and contravariant vectors is necessary when dealing with curved spaces." I would add that the distinction between vectors and linear forms is necessary also in a vector space devoid of scalar product and metric determined by it. As was shown in [2], the concept of flat space without metric is useful in describing electrostatics in anisotropic dielectric medium and magnetostatics in anisotropic magnetic medium, where various metrics can be introduced, determined by the permittivity or permeability tensors.

One should be aware of the fact that not all familiar geometric notions exist in a vector space devoid of the scalar product. Valid notions include: linearly dependent vectors, linearly independent vectors, parallel vectors and parallel planes. Undefined notions include: angles, perpendicular vectors, comparable length of non-parallel vectors, circles and spheres.

We depict vector as a directed segment. Its relevant features are direction and magnitude. The direction consists of a straight line (on which the vector lies) which, after Lounesto [3], can be called an attitude of the vector, and an arrow on that line which we call an orientation. Two vectors of the same attitude are parallel.

In the case of a vector space without a scalar product, there is a need to introduce dual objects known as linear forms which, in a sense, replace the scalar product. The linear form (or one-form or outer form of first order) is a linear mapping of vectors into scalars: \( f : x \rightarrow f(x) \in \mathbb{R} \). It is known from algebra that its kernel, i.e. the set \( M = \{ x : f(x) = 0 \} \) is a linear subspace of \( \mathbb{R}^3 \) with dimension two, so \( M \)

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1Strictly speaking, we should say "in an affine space". This remark is for reader with more rigorous mathematical knowledge.
is a plane. Thus, for each linear form, one can associate a plane \( M \) passing through the origin. One can also find other planes parallel to \( M \), on which the form \( f \) takes values 1, 2, 3 and so on. In this way, we get the geometric image of a form as a family of equidistant parallel planes with an arrow joining the neighbouring planes and showing the direction of increase of the form; see Figure 1. The angle between the arrow and the planes is not important.

Conversely, once we have such a family of planes, a number can be ascribed to any vector \( x \) as follows. If the origin of \( x \) lies on one of the planes, we count the number of planes pierced by \( x \) and this is equal to \( f(x) \); see Figure 1. If \( x \) ends on some plane, the number \( f(x) \) is integer; otherwise \( f(x) \) is not integer. If \( x \) intersects the planes in the direction of decreasing labels of planes, then \( f(x) \) is negative. This prescription, along with the corresponding pictures, can be found in Ref. [4]. Similar way of visualisation of the linear forms is in the books of Burke [5,6].

\[\begin{array}{c}
\includegraphics[width=0.4\textwidth]{figure1.png}
\end{array}\]

Figure 1. Geometric image of a linear form.

As a matter of fact, in order to describe a linear form one does not need to draw infinitely many parallel equidistant planes. Two neighbouring ones are sufficient; see Figure 2. Thus we claim that the geometric image of a linear form is a slab (a plane-parallel layer) with an arrow penetrating it from one boundary to the other. Pictures of this kind were shown already in Ref. [7].

\[\begin{array}{c}
\includegraphics[width=0.4\textwidth]{figure2.png}
\end{array}\]

Figure 2. Simplified geometric image of a linear form.

In this manner, we arrive to conclusion that direction of the linear form consists of attitude – a plane and orientation – an arrow piercing the plane. There are also pictorial prescriptions of adding linear forms and multiplying them by scalars, we refer the reader to Refs. [2] and [6].

Strictly speaking, in mathematics, the duality notion is reversible: when linear space \( V' \) is dual to a linear space \( V \), then \( V \) is also dual to \( V' \). Hence one could ask the question: Why do we visualise elements of \( V \) as directed segments, and elements of \( V' \) as families of parallel planes? Why not vice versa? My answer is as follows: We all live in the three-dimensional where the position vector \( \mathbf{r} \), called also radius vector, is naturally represented by the directed segment. Therefore, if we put \( \mathbf{r} \) in
V, all elements of V should be represented also by segments and, consequently, all elements of V’ by families of planes. When one puts r in V’, then everything is other way around.

It is useful to give some examples of physical quantities which can be regarded undoubtedly as vectors and others as linear forms. The most natural vectorial quantity is the \textit{radius vector} \( r \) of a point in space relative to the reference point, called the \textit{origin of a frame}. The \textit{displacement vector} \( \mathbf{l} \) is of the same nature. Hence the \textit{velocity} \( \mathbf{v} = \frac{d\mathbf{r}}{dt} \), the derivative of \( \mathbf{r} \) with respect to a scalar variable \( t \), is also a vector. The same is true of the \textit{acceleration} \( \mathbf{a} = \frac{d\mathbf{v}}{dt} \) and the \textit{electric dipole moment} \( \mathbf{d} = q\mathbf{l} \).

If one considers the potential energy \( U \) as a scalar, its relation to force in the traditional language is \( dU = -\mathbf{F} \cdot d\mathbf{r} \) where dot denotes the scalar product. This means that force is a linear map of the infinitesimal vector \( d\mathbf{r} \) into the infinitesimal scalar \( -dU = \mathbf{F}(d\mathbf{r}) \). Thus, after this observation, the force should be treated as a linear form. This in turn, by the Newton equation \( \mathbf{F} = \frac{d\mathbf{p}}{dt} \), implies that the momentum \( \mathbf{p} \) has to be a linear form too. This view is adopted in modern mathematical formulations of mechanics, see e.g. [8].

A one-form quantity occurs also in the description of the plane waves. The locus of points in space with the same phase of a wave is just a plane. The family of planes with phases differing by \( 2\pi n \) for natural \( n \) can be viewed as the geometric image of a one-form depicted in Figure 1. This one-form describes the physical quantity known as the \textit{wave vector} \( \mathbf{k} \) with magnitude \( 2\pi/\lambda \) (\( \lambda \) is the wavelength). Thus, in my opinion, the physical quantity \( \mathbf{k} \) should have a different name since, in its directed nature, it is not a vector. If I may create an English word I would propose the name \textit{wavity} for \( \mathbf{k} \).

Another one-form is the \textit{electric field strength} \( \mathbf{E} \), since we consider it to be a linear map of the infinitesimal vector \( d\mathbf{r} \) into the infinitesimal potential difference: \( -dV = \mathbf{E}(d\mathbf{r}) \). The attitude of \( \mathbf{E} \) at each point of space is plane tangent to the equipotential surface passing through the point.

It is worth mentioning that there is a plenty of other geometric objects which can be called \textit{directed quantities} in the sense that the direction and magnitude can be ascribed to them. Among them primary directed quantities are \textit{multivectors}: vectors, bivectors, and trivectors. Just as a vector, in the process of abstraction, arises from a straight line segment with an orientation, a \textit{bivector} originates from a plane segment with an orientation, and a \textit{trivector} from a solid body with an orientation. The connection of multivectors to straight lines, planes and bodies gives them the advantage of being easily depicted in illustrations. For their application in classical mechanics and in electromagnetism, see Ref. [9] and [10] respectively.

\textit{Outer forms} are quantities dual to multivectors. They are called \textit{differential forms} when they depend on their position in space. Differential forms are very popular in theoretical physics, see e.g. Refs. [11-13], but authors writing about them rarely use pictures to illustrate the concept to the reader; nice exceptions are already mentioned Refs. [4–6] in which outer forms are presented as specific “slicers”. No care is put there on directions of outer forms. Careful description of directions of the whole variety of directed quantities can be found in Refs. [2] and [14].
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