Perturbative BF–Yang–Mills theory on noncommutative $\mathbb{R}^4$

H. B. Benaoum
Institut für Physik, Theoretische Elementarteilchenphysik, Johannes Gutenberg–Universität, 55099 Mainz, Germany
email : benaoum@thep.physik.uni-mainz.de

Abstract

A $U(1)$ BF–Yang–Mills theory on noncommutative $\mathbb{R}^4$ is presented and in this formulation the $U(1)$ Yang–Mills theory on noncommutative $\mathbb{R}^4$ is seen as a deformation of the pure BF theory. Quantization using BRST symmetry formalism is discussed and Feynman rules are given. Computations at one–loop order have been performed and their renormalization studied. It is shown that the $U(1)$ BFYM on noncommutative $\mathbb{R}^4$ is asymptotically free and its UV–behaviour in the computation of the $\beta$–function is like the usual $SU(N)$ commutative BFYM and Yang Mills theories.

PACS : 11.10. Gh , 10.10. -z , 11.15. q , 11.15. Bt

Keywords : Renormalization, Noncommutative Geometry, Quantum Field Theory, Yang–Mills Theory, BF Theory.

MZ–TH/99–48

To the memory of my father
1 Introduction:

It is generally believed that our picture of the space–time as a manifold locally viewed as a flat Minkowski space should be modified drastically at the Planck scale [1]. One possibility is to consider that the space, at this scale, is a noncommutative space and field theories on these spaces have to be formulated in the framework of noncommutative geometry [2]. Constructions of the fundamental interactions along Connes’s approach are described in the literature [3].

Moreover, it has been shown by Connes, Douglas and Schwarz [4] that a supersymmetric Yang–Mills theory on noncommutative torus is naturally related to compactification of Matrix theory which means that noncommutative geometry can be useful in string theory [5].

Yet, the most urgent problem waiting to be solved, is whether or not quantum field theory on noncommutative space is well–defined. Specifically, it has been established in the case of noncommutative space–time, that two dimensional theories on the Fuzzy sphere [6] and on the quantum cylinder possess no UV–divergences at all. In contrast, field theories on noncommutative $\mathbb{R}^4$ [7], non–commutative 3–tori [8] and quantum plane have UV–divergences [9]. Thus, ultraviolet behaviour of a field theory on noncommutative space is sensitive to the topology.

Recently, one–loop renormalizability of Yang–Mills on noncommutative $\mathbb{R}^4$ [10] and on noncommutative torus [11] has been demonstrated. Particularly, it has also been shown that these fields theories have an asymptotical free behaviour. A general discussion about renormalizability of a massive scalar quantum field theory on non–commutative $\mathbb{R}^d$ has been carried out in [12] and unfamiliar mixing effects of IR and UV due to non–commutativity has been found in [13]. Such a mixing has no analog in conventional quantum field theory.

Here, we will be concerned with the perturbative quantization of the so–called BF–Yang–Mills theory ( BFYM ) on noncommutative $\mathbb{R}^4$. We recall that BFYM formulation on commutative $\mathbb{R}^4$ has been used in [14] to introduce an explicit representation of the ‘t Hooft algebra, making closer connection
between Yang–Mills theory and topological field theories of BF type \[15\].

The euclidean BFYM on commutative $\mathbb{R}^4$ is described by the action:

$$S_{\text{com BFYM}}^\text{com} = \int \text{Tr} \left( iB \wedge F + g^2 B \wedge \ast B \right)$$

$$= \int d^4 x \left( \frac{i}{2} \epsilon^{\mu\nu\alpha\beta} B_{\mu\nu}^a F_{\alpha\beta}^a + g^2 B_{\mu\nu}^a B_{\mu\nu}^a \right). \quad (1)$$

where $F = F_{\mu\nu}^a dx^\mu \wedge dx^\nu \; T^a$ is the usual field strength, $B$ is a Lie valued 2–form and $\ast$ is the Hodge product for a $p$–form with $T^a$ as generators of $SU(N)$ Lie algebra in the fundamental representation normalized as $\text{Tr}(T^a T^b) = \frac{1}{2} \delta_{ab}$.

In this framework Yang–Mills theory is seen in the first order formalism as a deformation of a topological theory of BF type. A proof of the full equivalence of BFYM with the standard second order formalism has been achieved through path integral and algebraic methods \[16\].

Its UV–behaviour and computation of the $\beta$–function have been carried out showing that it is equal to the Yang–Mills case \[17\].

The paper is organized as follows. In section 2 we introduce the $U(1)$ BFYM on noncommutative $\mathbb{R}^4$ and study its quantization with the BRST formalism. In section 3 we derive the Feynman rules and compute the one–loop divergent diagrams. Section 4 is concerned with the one–loop renormalizability and explicit one–loop computation of the $\beta$–function. Finally in section 5, we summarize our results.

### 2 BF Yang Mills on noncommutative $\mathbb{R}^4$:

The noncommutative $\mathbb{R}^4$ is defined as the algebra $A_\theta$ generated by $x_\mu, \mu = 1, 2, 3, 4$ satisfying the commutation relations:

$$[x_\mu, x_\nu] = i \theta_{\mu\nu}, \quad [x_\mu, \theta_{\rho\nu}] = 0. \quad (2)$$

where $\theta_{\mu\nu}$ is a real constant antisymmetric matrix with rank 4.

In the non–commutative geometry framework, the geometrical features of the noncommutative manifold are described by a $C^*$–algebra. The algebra of functions on noncommutative $\mathbb{R}^4$ can be considered as the deformation of the $C^*$–algebra of continuous complex functions over $\mathbb{R}^4$ vanishing at infinity, using the Weyl product,

$$(f \ast g)(x) = e^{i \frac{\theta_{\mu\nu}}{2 \hbar} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \eta}} f(x + \xi) g(x + \xi) \mid_{\xi = \eta = 0}$$

$$= \int \int \frac{d^4 p}{(2\pi)^2} \frac{d^4 q}{(2\pi)^2} e^{i \omega(p, q)} e^{i(p+q) \cdot x} \check{f}(p) \check{g}(q). \quad (3)$$
where $\omega(p,q) = \frac{1}{2} \theta_{\mu\nu} p^\mu q^\nu$ and $\tilde{f}$ (resp. $\tilde{g}$) is the Fourier transform of $f$ (resp. $g$).

The $\star$ product satisfies the following identities:

$$\int d^4x \ (f \ast g)(x) = \int d^4x (g \ast f)(x) = \int d^4x f(x) g(x)$$

$$\int d^4x \ (f \ast g \ast h)(x) = \int d^4x \ (h \ast f \ast g)(x) = \int d^4x (g \ast h \ast f)(x) \quad (4)$$

The last identity follows from the associativity of the $\ast$ product, i.e. $(f \ast g) \ast h = f \ast (g \ast h)$.

An alternative reformulation of the $U(1)$ Yang–Mills theory on noncommutative $\mathbb{R}^4$ is through the first order formalism. This model, named ”gaussian” $U(1)$ BF–Yang–Mills on noncommutative $\mathbb{R}^4$ is described by the action:

$$S_{BFYM} = \int d^4x \ (\frac{i}{2} \epsilon^{\mu\nu\alpha\beta} B_{\mu\nu} \ast F_{\alpha\beta} + g^2 B_{\mu\nu} \ast B^{\mu\nu})(x). \quad (5)$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \{A_\mu, A_\nu\}$ is the field strength for the antihermitian gauge field $A_\mu$ where $\{A_\mu, A_\nu\}(x) = (A_\mu \ast A_\nu)(x) - (A_\nu \ast A_\mu)(x)$ is the Moyal bracket and $B_{\mu\nu}$ is an antisymmetric tensor.

It is easy to see that this action is on–shell equivalent to the classical $U(1)$ Yang–Mills action on noncommutative $\mathbb{R}^4$. This latter can also be recovered by performing a path integration over $B_{\mu\nu}$. Indeed, after a field redefinition $B \to B/g$ we get:

$$\int DB_{\mu\nu} \ e^{-S_{BFYM}} \propto e^{-S_{YM}}. \quad (6)$$

where $S_{YM}$ is the Yang–Mills action on noncommutative $\mathbb{R}^4$ given by:

$$S_{YM} = \frac{1}{4g^2} \int d^4x \ (F^{\mu\nu} \ast F_{\mu\nu})(x). \quad (7)$$

The BFYM action is invariant under the usual gauge symmetry:

$$\delta A_\mu = D_\mu \epsilon, \quad \delta F_{\mu\nu} = \{F_{\mu\nu}, \epsilon\}, \quad \delta B_{\mu\nu} = \{B_{\mu\nu}, \epsilon\}. \quad (8)$$

with $D_\mu \epsilon = \partial_\mu \epsilon + \{A_\mu, \epsilon\}$.

In the following we will use the BRST formalism [19], which requires the introduction of Faddeev–Popov ghost $c$ and anti–ghost $\bar{c}$, auxiliary field $b$ and a $s$–BRST operator defined as:

$$sA_\mu = D_\mu c, \quad sF_{\mu\nu} = \{F_{\mu\nu}, c\}, \quad sc = -c \ast c, \quad s\bar{c} = b, \quad sb = 0, \quad sB_{\mu\nu} = \{B_{\mu\nu}, c\}. \quad (9)$$
which is off–shell nilpotent $s^2 = 0$.

Correspondingly, one introduces a gauge–fixing term action:

$$S_{gf} = \int d^4x \, s \left( \bar{c} \left( \frac{\alpha}{2} b - \partial_\mu A^\mu \right) \right)(x). \quad (10)$$

and an external field contribution:

$$S_{ext} = \int d^4x \left( \Omega_A^\mu \ast sA_\mu + \Omega_B^{\mu\nu} \ast sB_{\mu\nu} + \Omega_c \ast sc \right)(x). \quad (11)$$

Then the complete tree–level action is:

$$\Sigma [A_\mu, B_{\mu\nu}, c, \bar{c}, b, \Omega_A, \Omega_B, \Omega_c] = S_{BFYM} + S_{gf} + S_{ext} = \int d^4x \left( \frac{i}{2} \epsilon^{\mu\nu\alpha\beta} B_{\mu\nu} \ast F_{\alpha\beta} + g^2 B_{\mu\nu} \ast B^{\mu\nu} \\
+ b \ast \left( \frac{\alpha}{2} b - \partial_\mu A^\mu \right) + \bar{c} \ast \partial_\mu D^\mu c + \Omega_A^{\mu} \ast D_\mu c + \Omega_B^{\mu\nu} \ast \left( B_{\mu\nu} \ast c \right) - \Omega_c \ast (c \ast c) \right). \quad (12)$$

which satisfies the Slavnov–Taylor identity:

$$S(\Sigma) = 0. \quad (13)$$

where

$$S(\Sigma) = \int d^4x \left( \frac{\delta \Sigma}{\delta \Omega_A^\mu} \frac{\delta \Sigma}{\delta A_\mu} + \frac{\delta \Sigma}{\delta \Omega_B^{\mu\nu}} \frac{\delta \Sigma}{\delta B_{\mu\nu}} + b \frac{\delta \Sigma}{\delta \bar{c}} + \frac{\delta \Sigma}{\delta \Omega_c} \frac{\delta \Sigma}{\delta c} \right). \quad (14)$$

### 3 One–loop calculations:

To check the one–loop UV–behaviour of the BFYM on noncommutative $\mathbb{R}^4$, we derive the Feynman rules by expanding $S_{BFYM} + S_{gf}$ and separating it into quadratic and higher order pieces. After field rescaling, $B \rightarrow B/g$ and $A \rightarrow gA$, the quadratic pieces give the propagators in momentum space for the fields $A, B$ and $c$, and the higher order pieces give the vertices $BAA$ and $\bar{c}Ac$.

After expansion, the quadratic action is found to be:

$$S_0 = \int d^4x \left( i \epsilon^{\mu\nu\alpha\beta} B_{\mu\nu} \ast \partial_\alpha A_\beta + B_{\mu\nu} \ast B^{\mu\nu} + \frac{\alpha}{2} b \ast b - b \ast \partial_\mu A^\mu + \bar{c} \ast \Box c \right). \quad (15)$$

While the interaction action is given by:

$$S_{int} = \int d^4x \left( \frac{ig}{2} \epsilon^{\mu\nu\alpha\beta} B_{\mu\nu} \ast \left\{ A_\alpha, A_\beta \right\} + g \bar{c} \ast \partial_\mu \left\{ A^\mu, c \right\} \right). \quad (16)$$

Green’s functions are derived by using the generating functional $Z_0 [J_A, J_B, J_b, J_c, J_{\bar{c}}]$ with sources terms for each of the fields:

$$Z_0 [J_A, J_B, J_b, J_c, J_{\bar{c}}] = N \int DADBD\bar{D}cD\bar{c} \, e^{−S_0 − S_\ast}. \quad (17)$$
where

\[
S_s = \int d^4x \left( A^\mu J_{A \mu} + B^{\mu\nu} J_{B \mu\nu} + b J_b + c J_c + \bar{c} J_{\bar{c}} \right).
\]  

(18)

is the source term action.

Since \( S_0 \) is not diagonal in fields, and cannot be diagonalized, the resulting propagators of the fields are not diagonal either. This means that there are cross–propagators that do not vanish at tree level. To derive explicitly Green’s functions for various fields, we work in momentum space and shift all the fields by field independent functions \( C \):

\[
A_\mu(p) \rightarrow A_\mu(p) + C_A \mu(p)
\]

\[
B_{\mu\nu}(p) \rightarrow B_{\mu\nu}(p) + C_B \mu\nu(p)
\]

\[
b(p) \rightarrow b(p) + C_b(p)
\]

\[
c(p) \rightarrow c(p) + C_c(p)
\]

\[
\bar{c}(p) \rightarrow \bar{c}(p) + C_{\bar{c}}(p).
\]

(19)

By making the linear terms in the fields vanish and solving for \( C \)'s, the following Feynman rules for propagators \( A, B, c \) and \( \bar{c} \) are then obtained:

\[
\mu, p \rightarrow \underbrace{\nu, q}_{\epsilon} \quad D_{AA}^{\mu\nu}(p, q) = \frac{1}{p^2} \left( \delta_{\mu\nu} + (\alpha - 1) \frac{p^\mu p^\nu}{p^2} \right) \delta(p + q)
\]

\[
\alpha, p \rightarrow \underbrace{\nu, q}_{\epsilon} \quad D_{BA}^{\mu\nu\alpha}(p, q) = -\frac{1}{2} \epsilon_{\mu\nu\rho\alpha} \frac{p^\rho}{p^2} \delta(p + q)
\]

\[
\mu\nu, p \rightarrow \underbrace{\alpha, q}_{\epsilon} \quad D_{BB}^{\mu\nu\alpha\beta}(p, q) = -\frac{1}{4} \epsilon_{\mu\nu\lambda\gamma} \epsilon_{\alpha\beta\rho\gamma} \frac{p^\rho}{p^2} \delta(p + q)
\]

\[
p \rightarrow \underbrace{q}_{\epsilon} \quad D_{c\bar{c}}(p, q) = -\frac{i}{p^2} \delta(p + q).
\]

\[
\text{figure 1}
\]
Interaction vertices arise from $S_{\text{int}}$ as usual. Indeed, in momentum space $BAA$ and $cA\bar{c}$ vertices are:

$$V^{BAA}_{\mu\nu\alpha\beta}(p,q,r) = -2g \epsilon_{\mu\nu\alpha\beta} \sin \omega(p,q) \delta(p + q + r)$$

$$V^{cA\bar{c}}_{\mu}(p,q,r) = -2ig q_{\mu} \sin \omega(p,q) \delta(p + q + r) .$$

The above Feynman rules for $U(1)$ BFYM on noncommutative $\mathbb{R}^4$ are the same as the usual euclidean BFYM on commutative $\mathbb{R}^4$ for $SU(N)$ Lie algebra in which group indices $a$ are identified with the momentum $p_a$ and the structure constants $f_{abc}$ with $-2i \sin \omega(p_b,p_c) \delta(p_a + p_b + p_c) , \ (\text{see } [20] \text{ and } [10, 11] \text{ for Yang–Mills theory on noncommutative torus and noncommutative } \mathbb{R}^4 )$. In figure 3 and 4 (see Appendix), all relevant one–loop diagrams for self–energies and vertices are presented. We have explicitly calculated the one–loop diagrams of BFYM on noncommutative $\mathbb{R}^4$ where only planar contributions are considered. Every diagram gets then multiplied by a phase factor that depends only on the momenta of the external lines. We point out that non–planar contributions may create some trouble at higher order.

These calculations are done by using dimensional regularization in $D = 4 - 2\epsilon$ dimension $[21, 22]$ where the 4–dimensional measure $\frac{d^4k}{(2\pi)^4}$ is replaced by the $D$–dimensional one $\frac{d^Dk}{(2\pi)^D}$, and before performing the $D$–dimensional integration, some care has to be taken according to the rules in $[22]$. Moreover, whenever a product of sines appears, we express each $\sin \omega(p,q)$ as $\frac{1}{2i} (e^{i\omega(p,q)} - e^{-i\omega(p,q)})$ and transform these sines into a first term depending on the internal momentum which provides the Feynman integrals by oscillatory factors making them finite and a second term independent of the internal momentum so that the integrals have poles. Then we extract the one–loop UV divergent contribution to all the divergent $1PI$ Green functions.
The obtained results for self–energies are then given as follows:

a) AA self energy,
\[
\tilde{D}^{AA}_{\mu\nu}(p,q) = \left(\frac{1}{3} - \alpha\right) \frac{g^2}{(4\pi)^2} \Gamma(\epsilon) \left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right) \delta(p + q).
\]

b) AB self energy,
\[
\tilde{D}^{AB}_{\alpha\beta\mu}(p,q) = \frac{1}{2} (3 - \alpha) \frac{g^2}{(4\pi)^2} \Gamma(\epsilon) \epsilon_{\alpha\beta\mu\rho} p^\rho \delta(p + q).
\]

c) BB self energy,
\[
\tilde{D}^{BB}_{\mu\nu\alpha\beta}(p,q) = -(1 + \alpha) \frac{g^2}{(4\pi)^2} \Gamma(\epsilon) \left(\delta_{\mu\alpha} \delta_{\nu\beta} - \delta_{\mu\beta} \delta_{\nu\alpha}\right) \delta(p + q).
\]

d) ghosts c\overline{c} self energy,
\[
\tilde{D}^{c\overline{c}}(p,q) = \frac{3}{2} (1 - \alpha) \frac{g^2}{(4\pi)^2} \Gamma(\epsilon) p^2 \delta(p + q).
\]

Similarly the divergent parts for the two vertices BAA and \(\overline{c}Ac\) read,
\[
\tilde{V}^{BAA}_{\mu\nu\alpha\beta}(p,q,r) = 2 \alpha \frac{g^3}{(4\pi)^2} \Gamma(\epsilon) \epsilon_{\mu\nu\alpha\beta} \sin\omega(p,q) \delta(p + q + r).
\]
\[
\tilde{V}^{c\overline{c}A}_{\mu}(p,q,r) = 4 i \alpha \frac{g^3}{(4\pi)^2} \Gamma(\epsilon) q_\mu \sin\omega(p,q) \delta(p + q + r).
\]

We see from the power counting argument, that the superficial degree of divergence \(\omega_s\) for these diagrams is given by:
\[
\omega_s = 4 - (E_A + E_c) - 2E_B.
\]

where \(E_A, E_B\) and \(E_c\) represent the number of A, B and ghost c external lines respectively.

Then we remark that the one–loop divergent contributions for self–energies and vertices are like the usual BFYM theory on commutative \(\mathbb{R}^4\) for the Lie algebra \(SU(N)\) with structure constants \(f_{abc}\) replaced by \(-2i\sin\omega(p_b,p_c)\) and the quadratic Casimir \(C_2(G)\) equal to 2.

Up on these replacements, our results agree with the usual \(SU(N)\) BFYM on commutative \(\mathbb{R}^4\) \([7]\) where their calculations have been done in the Landau gauge (\(\alpha = 0\)).

We see also that the divergent part of the ghost vertex \(\tilde{V}^{c\overline{c}A}(p,q,r)\) vanishes in the
Landau gauge due to the transversality of the propagators in this gauge and the vertex $\tilde{V}^{BA}_{\mu\alpha}(p,q,r)$ is finite.

At the one–loop level, two other trilinear and quadrilinear vertices $\tilde{V}^{AAA}_{\mu\nu\alpha}(p_1,p_2,p_3)$ and $\tilde{V}^{AAAA}_{\mu\nu\alpha\beta}(p_1,p_2,p_3,p_4)$ respectively appear and do not belong to the tree–level $U(1)$ BFYM action on noncommutative $\mathbb{R}^4$. They correspond to the nonlinear self–interactions of $U(1)$ Yang–Mills action on noncommutative $\mathbb{R}^4$ and arise from term $(F^{\mu\nu} * F^{\mu\nu})(x)$ which is allowed to enter at the quantum level due to the symmetries of the theory. We will see in the next section how such a counter–term can arise.

4 Renormalization and $\beta$– function :

Before we perform the renormalization, we consider the gauge fixing action $S_{gf}$ and integrate out the auxiliary fields $b$. After this manipulation, the $S_{BFYM} + S_{gf} = S$ becomes:

$$S = \int d^4x \left( \frac{i}{2} \epsilon^{\mu\nu\alpha\beta} B_{\mu\nu} * F_{\alpha\beta} + B_{\mu\nu} * B^{\mu\nu} - \frac{1}{2\alpha} (\partial_{\mu} A^{\mu}) * (\partial_{\nu} A^{\nu}) + \bar{c} * \partial_{\mu} D^{\mu} c \right). \tag{20}$$

To perform the renormalization, we follow standard techniques [23, 24] and substitute the bare quantities for the renormalized ones, where in general an operational mixing is allowed by the symmetry and parity properties of the fields. We then have for the fields:

$$\begin{pmatrix} B_0 \mu\nu \\ F_0 \mu\nu \end{pmatrix} = \begin{pmatrix} Z_{BB} \delta_{\mu\alpha} \delta_{\nu\beta} & \frac{i}{2} Z_{BA} \epsilon_{\mu\nu\alpha\beta} \\ 0 & Z_{AA} \delta_{\mu\alpha} \delta_{\nu\beta} \end{pmatrix} \begin{pmatrix} B_R \alpha\beta \\ F_R \alpha\beta \end{pmatrix}$$

\begin{align*}
\bar{c}_0 &= Z_c c_R \\
\bar{c}_0 &= Z_c \bar{c}_R. \tag{21}
\end{align*}

and the parameters $g$ and $\alpha$:

$$g_0 = Z_g g_R, \quad \alpha_0 = Z_\alpha \alpha_R. \tag{22}$$

where the constants $Z_{AA}, Z_{BA}$ and $Z_{BB}$ and $Z_c$ are the $A$ field, $BA$ fields, $BB$ fields and ghost–field renormalization constants, respectively, while the constants $Z_g$ and $Z_\alpha = Z_{AA}^2$ are the coupling constant and gauge parameter renormalization constants. Here $F_R \mu\nu = \partial_{\mu} A_R \nu - \partial_{\nu} A_R \mu + Z_{AA} Z_g g_R \{ A_R \mu, A_R \nu \}$ is the renormalized field strength.

Notice that the presence of the factor $i$ is necessary in (21) since $B_{\mu\nu}$ and $F_{\mu\nu}$ have
opposite parity and a mixing of $B_R^{\mu\nu}$ is not allowed since $F^{\mu\nu}$ must be a curvature tensor. With the above field redefinitions, a counter–term $(F^{\mu\nu} * F^{\mu\nu})(x)$, absent at tree level in the theory, appears at the quantum level which is required to renormalize the trilinear $AAA$ and quadrilinear $AAAA$ vertices arising at one–loop level.

Now let us analyze the structure of the counter–terms that make finite the self–energies and the vertices of the $U(1)$ BF–Yang–Mills theory on noncommutative $\mathbb{R}^4$. It is necessary first to begin by writing the action in terms of bare quantities:

$$S = \int d^4x \left( i \frac{\epsilon^{\mu\nu\alpha\beta}}{2} B_{0\mu\nu} * F_{0\alpha\beta} + B_{0\mu\nu} * B_{0\mu\nu} - \frac{1}{2\alpha_0} (\partial_\mu A_0^\mu) * (\partial_\nu A_0^\nu) + \bar{c}_0 * \partial_\mu D^\mu c_0 \right). \quad (23)$$

and then in terms of renormalized fields and renormalization constants:

$$S = \int d^4x \left( i \frac{1}{2} Z_{BB} (Z_{AA} + 2Z_{BA}) \epsilon^{\mu\nu\alpha\beta} B_R^{\mu\nu} * F_R^{\alpha\beta} + Z_{BB}^2 B_R^{\mu\nu} * B_R^{\mu\nu} 
- Z_{BA} (Z_{AA} + Z_{BA}) F_R^{\mu\nu} * F_R^{\mu\nu} - \frac{1}{2\alpha_R} (\partial_\mu A_R^\mu) * (\partial_\nu A_R^\nu) + Z_c \bar{c}_R * \partial_\mu D^\mu c_R \right). \quad (24)$$

where $D_\mu c_R = \partial_\mu c_R + Z_{AA} Z_g g_R \{ A_R^\mu, c_R \}$ is the renormalized covariant derivative. Apart from the constraints on the renormalization constants by the gauge Ward identities, the $Z$’s are in principle completely arbitrary. In practice, because the $U(1)$ BFYM theory on noncommutative $\mathbb{R}^4$ needs regularizing, the arbitrariness of the $Z$’s is only in the finite parts. Since Feynman rules of the $U(1)$ BFYM theory on noncommutative $\mathbb{R}^4$ at tree level should not be modified we expect:

$$Z_{AA} \simeq 1 + a(\alpha_R) \frac{g_R^2}{(4\pi)^2} \Gamma(\epsilon) + (\cdots) + O(g_R^2)$$

$$Z_{BB} \simeq 1 + b(\alpha_R) \frac{g_R^2}{(4\pi)^2} \Gamma(\epsilon) + (\cdots) + O(g_R^2)$$

$$Z_{BA} \simeq c(\alpha_R) \frac{g_R^2}{(4\pi)^2} \Gamma(\epsilon) + (\cdots) + O(g_R^2)$$

$$Z_c \simeq 1 + d(\alpha_R) \frac{g_R^2}{(4\pi)^2} \Gamma(\epsilon) + (\cdots) + O(g_R^2). \quad (25)$$

where $(\cdots)$ represents finite terms at order $g_R^2$. Note that $Z_{BA} \sim g_R^2 \Gamma(\epsilon)$ in order not to modify the Feynman rules at the tree level. Moreover, the $Z$’s renormalization constants should be determined by adjusting the counter–terms so as to cancel overall divergences appearing in one–loop Feynman amplitudes. In our case, they are obtained by straightforward comparison between the Feynman rules for the quadratic (24) and the divergent parts of self–energies.
Consequently, the following system appears:

\[ Z_{BB} (Z_{AA} + 2Z_{BA}) = 1 + \frac{1}{2} (3 - \alpha_R) \frac{gR^2}{(4\pi)^2} \Gamma(\epsilon) + (\cdots) + O(gR^2) \]

\[ Z_{BB}^2 = 1 - (1 - \alpha_R) \frac{gR^2}{(4\pi)^2} \Gamma(\epsilon) + (\cdots) + O(gR^2) \]

\[ 4Z_{BA} (Z_{AA} + Z_{BA}) = \left( \frac{1}{3} - \alpha_R \right) \frac{gR^2}{(4\pi)^2} \Gamma(\epsilon) + (\cdots) + O(gR^2) \]

\[ Z_c^2 = 1 + \frac{3}{2} (1 - \alpha_R) \frac{gR^2}{(4\pi)^2} \Gamma(\epsilon) + (\cdots) + O(gR^2). \quad (26) \]

Solving (26) by a use of (25) gives:

\[ a(\alpha_R) = \frac{1}{2} \left( \frac{13}{3} - \alpha_R \right), \quad b(\alpha_R) = -\frac{1}{2} (1 + \alpha_R), \quad c(\alpha_R) = -\frac{1}{4} \left( \frac{1}{3} - \alpha_R \right), \]

\[ d(\alpha_R) = \frac{3}{4} (1 - \alpha_R). \quad (27) \]

The structure of renormalization constants that remove the divergent quantities are:

\[ Z_{AA} \simeq 1 + \frac{1}{2} \left( \frac{13}{3} - \alpha_R \right) \frac{gR^2}{(4\pi)^2} \Gamma(\epsilon) \]

\[ Z_{BB} \simeq 1 - \frac{1}{2} (1 + \alpha_R) \frac{gR^2}{(4\pi)^2} \Gamma(\epsilon) \]

\[ Z_{BA} \simeq -\frac{1}{2} \left( \frac{1}{3} - \alpha_R \right) \frac{gR^2}{(4\pi)^2} \Gamma(\epsilon) \]

\[ Z_c \simeq 1 + \frac{3}{4} (1 - \alpha_R) \frac{gR^2}{(4\pi)^2} \Gamma(\epsilon). \quad (28) \]

Our remaining task is now to determine the renormalization constant \( Z_g \). Indeed from the \( cA\bar{c} \) vertex we get:

\[ Z_g Z_{AA} Z_c^2 = 1 - 2 \alpha_R \frac{gR^2}{(4\pi)^2} \Gamma(\epsilon). \quad (29) \]

It is easy to extract from (29) the renormalization of the coupling constant which turns out to be:

\[ Z_g = 1 - \frac{11}{3} \frac{gR^2}{(4\pi)^2} \Gamma(\epsilon). \quad (30) \]

The \( \beta \)-function can now be easily read from (30):

\[ \beta_1 = -\frac{11}{3}. \quad (31) \]

which ensures that the theory is asymptotically free [24]. Moreover, as expected, the UV–behaviour of the \( U(1) \) BFYM on noncommutative \( \mathbb{R}^4 \) is the same as the usual \( SU(N) \) BFYM and Yang–Mills theories on commutative \( \mathbb{R}^4 \).

We also notice that the Weyl–Moyal matrix \( \theta_{\mu\nu} \) expressing the non–local character of the interaction is not renormalized at the one–loop order.
5 Conclusions:

In summary we have introduced the $U(1)$ BFYM theory on noncommutative $\mathbb{R}^4$ and shown its equivalence to $U(1)$ Yang–Mills on noncommutative $\mathbb{R}^4$ after integrating out the antisymmetric $B_{\mu\nu}$ field. Quantization of this theory in the BRST symmetry formalism is studied where the full quantum action with the Slavnov–Taylor identity is obtained.

After extracting the Feynman rules, one–loop calculations have been performed and particularly the one–loop UV–divergent contribution to the divergent 1PI Green functions. Its renormalization at one–loop level has been described and its asymptotical free behaviour checked. Moreover we have shown that the UV–behaviour of the $U(1)$ BFYM theory on noncommutative $\mathbb{R}^4$ is similar to the usual $SU(N)$ BFYM and Yang–Mills theories on commutative $\mathbb{R}^4$. We have also seen that – at least – at the one–loop order no renormalization of the Weyl–Moyal matrix $\theta_{\mu\nu}$ is needed. Within the one–loop level, apart from some complications in the computations due to the non–local terms in the action that result in Feynman integrals with oscillatory factors, no new phenomena appear. It becomes clear that the $U(1)$ BFYM on noncommutative $\mathbb{R}^4$ behaves like the commutative case.

For $g = 0$, the action (5) is just the $U(1)$ BF theory on noncommutative $\mathbb{R}^4$ 1:

$$S_{BF} = \int d^4x \frac{i}{2} \epsilon^{\mu\nu\alpha\beta} B_{\mu\nu} \ast F_{\alpha\beta}. \quad (32)$$

As for the commutative BF, this $U(1)$ BF on noncommutative $\mathbb{R}^4$ action is invariant under the gauge symmetry:

$$\delta_\epsilon A_\mu = D_\mu \epsilon, \quad \delta_\epsilon F_{\mu\nu} = \{ F_{\mu\nu}, \epsilon \} , \quad \delta_\epsilon B_{\mu\nu} = \{ B_{\mu\nu}, \epsilon \}. \quad (33)$$

and an extra "topological " symmetry:

$$\delta_\psi A_\mu = 0 , \quad \delta_\psi F_{\mu\nu} = 0 , \quad \delta_\psi B_{\mu\nu} = D_\mu \psi_\nu - D_\nu \psi_\mu. \quad (34)$$

Among the problems – that could be studied– are the quantization and the perturbative renormalization of the BF theory on noncommutative spaces. These will be

---

1 For commutative spaces, the BF is a topological theory of Schwarz type. Indeed for this theory on $S^4$ or $\mathbb{R}^4$, the partition function is just one whereas for $T^4$, it is proportional to $\text{vol}(T^*)$ where $T^*$ is the dual torus. A question – that can be adressed - is the value of the partition function for different topologies on noncommutative spaces. Thanks to Sheikh–Jabbari for giving rise to this point.
adressed elsewhere.

We should mention that the \( U(1) \) BFYM on noncommutative \( \mathbb{R}^4 \) can also be interpreted as a deformation (perturbation) of the pure \( U(1) \) BF theory on noncommutative \( \mathbb{R}^4 \) having besides the gauge symmetry, the topological one.

Prototype of topological gauge field theory of Schwarz type is the Chern–Simons theory. Renormalization and finiteness of this system on noncommutative 3d spaces can also be carried out and will be reported in a future work.

In ref [5], Seiberg and Witten established a relation between noncommutative Yang–Mills and commutative one. Indeed, they obtained a transformation from commutative gauge field \( A \) with gauge parameter \( \lambda \) to noncommutative gauge field \( \hat{A} \) with gauge parameter \( \hat{\lambda} \) by requiring the equivalence of the gauge transformation of \( A \) and \( \hat{A} \). We have checked and found that this mapping exists also for BF–Yang Mills theory [25].

Acknowledgment

I would like to thank the DAAD for its financial support and A. Davydychev for the axodraw file. I’m grateful to R. Coquereaux, T. Krajewski, C.P. Martin, F. Scheck and M.M. Sheikh–Jabbari for reading the manuscript and their comments. I thank also I. Chepelev for his email concerning the new version his work [12], J.M. Gracia–Bondia for discussions and the referee for his remarks.
Appendix:

Here are the different one loop Feynman diagrams relevant for BFYM theory:

\[ A \quad A = \quad \text{diagram 1} + \quad \text{diagram 2} + \quad \text{diagram 3} \]

\[ A \quad B = \quad \text{diagram 4} \]

\[ B \quad B = \quad \text{diagram 5} \]

\[ c \quad \bar{c} = \quad \text{diagram 6} \]

**Figure 3:** Self-energies

\[ c \quad \bar{c} = \quad \text{diagram 7} + \quad \text{diagram 8} + \quad \text{permutations} \]

**Figure 4:** Vertex

\[ A \quad A = \quad \text{diagram 9} + \quad \text{diagram 10} + \quad \text{permutations} \]

\[ B \quad A = \quad \text{diagram 11} + \quad \text{diagram 12} + \quad \text{permutations} \]
References

[1] S. Doplicher, K. Fredenhagen and J.E. Roberts, Phys. Lett. B331, 39 (1994); Commun. Math. Phys. 172, 187 (1995).

[2] A. Connes, Noncommutative Geometry, Academic Press (1994).

[3] A. Connes and J. Lott, Nucl. Phys. Suppl. B11, 19 (1990); H.–H. Chamseddine, G. Felder and J. Fröhlich, Nucl. Phys. B395, 672 (1993).

[4] A. Connes, M.R. Douglas and A. Schwarz, JHEP 02, 003 (1998).

[5] N. Seiberg and E. Witten, JHEP 09, 032 (1999).

[6] H. Grosse, C. Klimešik and P. Prešnajder, Int. J. Theor. Phys. 35, 231 (1996); Commun. Math. Phys. 178, 507 (1996); 180, 429 (1996).

[7] M. Chaichian, A. Demichev and P. Prešnajder, Quantum Field Theory on Noncommutative Space–Times and the Persistence of Ultraviolet Divergences, hep–th/9812180; see also, Quantum Field Theory on the Noncommutative Plane with $E_8(2)$ symmetry, hep–th/9904132.

[8] J. C. Várilly and J.M. García–Bondía, Int. J. Mod. Phys. A14, 1305 (1999).

[9] T. Filk, Phys. Lett. B376, 53 (1996).

[10] C.P. Martin and D. Sanchez–Ruiz, Phys. Rev. Lett. 83, 476 (1999).

[11] M.M. Sheikh–Jabbari, JHEP 06, 015 (1999); T. Krajewski and R. Wulkenhaar, Perturbative quantum gauge fields on the noncommutative torus, hep–th/9903187.

[12] I. Chepelev and R. Roiban, Renormalization of Quantum Field Theories on Noncommutative $\mathbb{R}^d$, I. Scalars, hep–th/9911098.

[13] S. Minwalla, M. Van Raamsdonk and N. Seiberg, Noncommutative perturbative dynamics, hep–th/9912172.

[14] F. Fucito, M. Martellini and M. Zeni, Nucl. Phys. B496, 259 (1997); M. Martellini and M. Zeni, The BF Formalism for Yang–Mills Theory and the ‘t Hooft Algebra, hep–th/9610090.

[15] G. Horowitz, Commun. Math. Phys. 125, 417 (1989); M. Blau and G. Thompson, Ann. Phys. 205, 130 (1991); D. Birmingham, M. Blau, M. Rakowski and G. Thompson, Phys. Rep. 209, 129 (1991).
[16] F. Fucito, M. Martellini, S.P. Sorella, A. Tanzini, L.C.Q. Vilar and M. Zeni, Phys. Lett. B404, 94 (1997); A.S. Cattaneo, P. Cotta–Ramusino, F. Fucito, M. Martellini, M. Rinaldi, A. Tanzini and M. Zeni, Commun. Math. Phys. 197, 571 (1998).

[17] M. Martellini and M. Zeni, Phys. Lett. B401, 62 (1997).

[18] N. Nekrasov and A. Schwarz, Commun. Math. Phys. 198, 689 (1998).

[19] O. Piguet and S.P. Sorella, *Algebraic Renormalization*, Springer–Verlag, Berlin (1995).

[20] M. Douglas, *Two lectures on D–geometry and noncommutative geometry*, hep–th/9901146.

[21] G. ’t Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972).

[22] P. Breitenlohner and D. Maison, Commun. Math. Phys. 52, 11 (1977).

[23] C. Itzykson and J.–B. Zuber, *Quantum Field Theory*, McGraw–Hill, New York (1980).

[24] T. Muta, *Foundations in Quantum Chromodynamics*, World Scientific, Singapore (1987).

[25] H.B. Benaoum, *On noncommutative and commutative equivalence for BFYM theory : Seiberg–Witten map*, hep–th/0004002.