The Construction of Field Operators

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Abstract. We draw attention to some tune problems in constructions of the quantum-field operators for spins 1/2 and 1. They are related to the existence of negative-energy solutions and acausal solutions of relativistic wave equations. Particular attention is paid to neutral particles, chiral theories, and to the method of the Lorentz boosts.

1. Introduction.
The Dirac equation can be deduced either by the Dirac method [1] (the Hamiltonian should be linear in $\partial/\partial x^i$, and be compatible with $E^2 - p^2 c^2 = m^2 c^4$); or by the Sakurai method [2] (based on the equation $(E - \sigma \cdot p)(E + \sigma \cdot p) \phi = m^2 \phi$); or by the Ryder one [3] (the relation between 2-spinors at rest is $\phi_R(0) = \pm \phi_L(0)$). The $\gamma^\mu$ obey the Clifford algebra anticommutation relations.

Usually, everybody uses the following definition of the field operator [4]:

$$\Psi(x) = \frac{1}{(2\pi)^3} \sum_\sigma \int \frac{d^3p}{2E_p} [u_\sigma(p) a_\sigma(p)e^{-ip \cdot x} + v_\sigma(p) b_\sigma(p)^*e^{ip \cdot x}],$$  \hspace{1cm} (1)

as given ab initio. I studied in the previous works [13, 14, 17]: $\sigma \rightarrow h$ (the helicity basis); the modified Sakurai derivation (the additional $m_2 \gamma^5$ term in the Dirac equation); the derivation of the Barut equation [18] from the first principles, namely based on the generalized Ryder relation, $(\phi_L^h(0) = \hat{A} \phi_L^h(0) + \hat{B} \phi_R^h(0))$. In fact, we have the second mass state ($\mu$-meson) from that equation:

$$[i\gamma^\mu \partial_\mu - \alpha \partial_\mu \partial^\mu/m - \beta]\psi = 0;$$  \hspace{1cm} (2)

the self/anti-self charge-conjugate Majorana 4-spinors [7, 8] in the momentum representation.

We begin with the method of constructions of the field operators given in [5]. The Wigner rules [6] of the Lorentz transformations for the (0, S) left- $\phi_L(p)$ and the (S, 0) right- $\phi_R(p)$ spinors are used. We can recover the Dirac equation in the matrix form:

$$\begin{pmatrix} \mp \frac{1}{m} & p_0 + \sigma \cdot p \\ p_0 - \sigma \cdot p & \mp \frac{1}{m} \end{pmatrix} \psi(p^\mu) = 0,$$  \hspace{1cm} (3)

or $(\gamma \cdot p - m)u(p) = 0$ and $(\gamma \cdot p + m)v(p) = 0$. The solutions of the Dirac equation can be denoted as $u(p) = column(\phi_R(p), \phi_L(p))$ and $v(p) = \gamma^5 u(p)$, ref. [3, p.53] with $E = \sqrt{p^2 + m^2} > 0$, $p_0 = \pm E$, $p^\pm = E \pm p_z$, $p_{r,l} = p_x \pm ip_y$. They are the parity eigenstates with the eigenvalues of $\pm 1$. They also describe eigenstates of the charge operator. Thus, in this Section we have used...
the basis for charged particles in the \((S,0) \oplus (0,S)\) representation (in general)

\[
\begin{align*}
\begin{pmatrix}
1 \\
0 \\
. \\
. \\
0
\end{pmatrix}, &
\begin{pmatrix}
0 \\
1 \\
. \\
. \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
. \\
. \\
1
\end{pmatrix},
\end{align*}
\]

Sometimes, the normalization factor is convenient to choose \(N(\sigma) = m^\sigma\) in order the rest spinors to vanish in the massless limit. However, other constructs are possible in the \((1/2,0) \oplus (0,1/2)\) representation.

2. Majorana Spinors in the Momentum Representation.

During the 20th century various authors introduced self/anti-self charge-conjugate 4-spinors (including in the momentum representation), see [7, 8, 9, 10]. Later, they [11, 13, 14, 12] etc studied these spinors, they found corresponding dynamical equations, gauge transformations and other specific features of them. The definitions are:

\[
\begin{align*}
C &= e^{i\theta} \begin{pmatrix}
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
0 & i & 0 & 0 \\
-i & 0 & 0 & 0
\end{pmatrix},
K &= -e^{i\theta} \gamma^2 K
\end{align*}
\]

is the anti-linear operator of charge conjugation. \(K\) is the complex conjugation operator. We define the self/anti-self charge-conjugate 4-spinors in the momentum space:\(^1\)

\[
C\lambda^{S,A}(p) = \pm \lambda^{S,A}(p),
C\rho^{S,A}(p) = \pm \rho^{S,A}(p).
\]

The rest \(\lambda\) and \(\rho\) spinors are:

\[
\begin{align*}
\lambda^S_{\uparrow}(0) &= \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\
i \\
1 \\
0 \end{pmatrix},
\lambda^S_{\downarrow}(0) &= \sqrt{\frac{m}{2}} \begin{pmatrix} -i \\
0 \\
0 \\
i \end{pmatrix},
\lambda^A_{\uparrow}(0) &= \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\
-i \\
1 \\
0 \end{pmatrix},
\lambda^A_{\downarrow}(0) &= \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\
i \\
0 \\
1 \end{pmatrix};
\rho^S_{\uparrow}(0) &= \sqrt{\frac{m}{2}} \begin{pmatrix} 1 \\
0 \\
0 \\
i \end{pmatrix},
\rho^S_{\downarrow}(0) &= \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\
i \\
1 \\
0 \end{pmatrix},
\rho^A_{\uparrow}(0) &= \sqrt{\frac{m}{2}} \begin{pmatrix} 1 \\
0 \\
0 \\
-i \end{pmatrix},
\rho^A_{\downarrow}(0) &= \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\
i \\
-i \\
0 \end{pmatrix}.
\end{align*}
\]

\(^1\) Such definitions of 4-spinors differ, of course, from the original Majorana definition in x-representation:

\[
\nu(x) = \frac{1}{\sqrt{2}} (\Psi_D(x) + \Psi^\tau_D(x))
\]

\(C\nu(x) = \nu(x)\) that represents the positive real \(C-\) parity field operator. However, the momentum-space Majorana-like spinors open various possibilities for description of neutral particles (with experimental consequences, see [12]). For instance, "for imaginary \(C\) parities, the neutrino mass can drop out from the single \(\beta\) decay trace and reappear in \(0\nu\beta\beta\), a curious and in principle experimentally testable signature for a non-trivial impact of Majorana framework in experiments with polarized sources."
It is easy to derive the explicit forms of the 4-spinors of the second kind $\lambda_{\uparrow \downarrow}^{S,A}(p)$ and $\rho_{\uparrow \downarrow}^{S,A}(p)$ in this basis. Here they are:

$$\lambda_{\uparrow}^S(p) = \frac{1}{2\sqrt{E + m}} \begin{pmatrix} i p_t \\ i(p^- + m) \\ p^- + m \\ -p_r \end{pmatrix}, \quad \lambda_{\downarrow}^S(p) = \frac{1}{2\sqrt{E + m}} \begin{pmatrix} -i(p^+ + m) \\ -i p_r \\ -p_t \\ (p^+ + m) \end{pmatrix},$$

(10)

$$\lambda_{\uparrow}^A(p) = \frac{1}{2\sqrt{E + m}} \begin{pmatrix} -i p_i \\ -i(p^- + m) \\ (p^- + m) \\ p_r \end{pmatrix}, \quad \lambda_{\downarrow}^A(p) = \frac{1}{2\sqrt{E + m}} \begin{pmatrix} i(p^+ + m) \\ i p_r \\ p_t \\ (p^+ + m) \end{pmatrix},$$

(11)

$$\rho_{\uparrow}^S(p) = \frac{1}{2\sqrt{E + m}} \begin{pmatrix} p^+ + m \\ p_r \\ i p_t \\ -i(p^+ + m) \end{pmatrix}, \quad \rho_{\downarrow}^S(p) = \frac{1}{2\sqrt{E + m}} \begin{pmatrix} p_t \\ (p^- + m) \\ -i p_r \\ -i(p^- + m) \end{pmatrix},$$

(12)

$$\rho_{\uparrow}^A(p) = \frac{1}{2\sqrt{E + m}} \begin{pmatrix} p^+ + m \\ p_r \\ i p_t \\ i p_t \end{pmatrix}, \quad \rho_{\downarrow}^A(p) = \frac{1}{2\sqrt{E + m}} \begin{pmatrix} p_t \\ (p^- + m) \\ -i p_r \\ -i(p^- + m) \end{pmatrix}.$$  (13)

As we showed, the $\lambda$ and $\rho$ 4-spinors are NOT the eigenspinors of the helicity. Moreover, the $\lambda$ and $\rho$ are NOT the eigenspinors of the parity (in this representation $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$), as opposed to the Dirac case. The indices $\uparrow \downarrow$ should be referred to the chiral helicity quantum number introduced in the 60s, $\eta = -\gamma^5 i$.

While $Pu_\sigma(p) = +u_\sigma(p)$, $Pv_\sigma(p) = -v_\sigma(p)$, we have $P\lambda_{\uparrow \downarrow}^{S,A}(p) = \rho_{\uparrow \downarrow}^{A,S}(p)$, $P\rho_{\uparrow \downarrow}^{S,A}(p) = \lambda_{\uparrow \downarrow}^{A,S}(p)$ for the Majorana-like momentum-space 4-spinors on the first quantization level.

The dynamical coordinate-space equations are:

$$i\gamma^\mu \partial_\mu \lambda^S(x) - m\rho^A(x) = 0, \quad i\gamma^\mu \partial_\mu \rho^A(x) - m\lambda^S(x) = 0,$$  (14)

$$i\gamma^\mu \partial_\mu \lambda^A(x) + m\rho^S(x) = 0, \quad i\gamma^\mu \partial_\mu \rho^S(x) + m\lambda^A(x) = 0.$$  (15)

These are NOT the Dirac equation. However, they can be written in the 8-component form as follows:

$$[i\Gamma^\mu \partial_\mu - m] \Psi_+(x) = 0, \quad [i\Gamma^\mu \partial_\mu + m] \Psi_-(x) = 0.$$  (16)

One can also re-write the equations into the two-component form. Similar formulations have been presented by M. Markov [15], and A. Barut and G. Ziino [9]. The group-theoretical basis for such doubling has been given in the papers by Gelfand, Tsetlin and Sokolik [16].

The Lagrangian is

$$\mathcal{L} = \frac{i}{2} \left[ \bar{\lambda}^S \gamma^\mu \partial_\mu \lambda^S - (\partial_\mu \bar{\lambda}^S) \gamma^\mu \lambda^S + \bar{\rho}^A \gamma^\mu \partial_\mu \rho^A - (\partial_\mu \bar{\rho}^A) \gamma^\mu \rho^A + \bar{\lambda}^A \gamma^\mu \partial_\mu \lambda^A - (\partial_\mu \bar{\lambda}^A) \gamma^\mu \lambda^A + \bar{\rho}^S \gamma^\mu \partial_\mu \rho^S - (\partial_\mu \bar{\rho}^S) \gamma^\mu \rho^S - m(\bar{\lambda}^S \rho^A + \bar{\lambda}^A \rho^S - \bar{\lambda}^S \rho^S + \bar{\lambda}^A \rho^A) \right].$$  (17)

The connection with the Dirac spinors has been found. For instance,

$$\begin{pmatrix} \lambda_{\uparrow}^S(p) \\ \lambda_{\downarrow}^S(p) \\ \lambda_{\uparrow}^A(p) \\ \lambda_{\downarrow}^A(p) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & i & -1 & i \\ -i & 1 & 1 & -1 \\ 1 & -i & -1 & -i \\ i & 1 & i & -1 \end{pmatrix} \begin{pmatrix} u_{+1/2}(p) \\ u_{-1/2}(p) \\ v_{+1/2}(p) \\ v_{-1/2}(p) \end{pmatrix}. $$  (18)
See also ref. [16, 9]. A possibility was noted for generalizations of the concept of the Fock space, which lead to the “doubling” Fock space [16, 9].

It was shown [13] that the covariant derivative (and, hence, the interaction) can be introduced in this construct in the following way: \( \partial_{\mu} \rightarrow \nabla_{\mu} = \partial_{\mu} - igL^{5}A_{\mu} \), where \( L^{5} = \text{diag}(\gamma^{5} - \gamma^{5}) \), the \( 8 \times 8 \) matrix. With respect to the transformations

\[
\lambda'(x) \rightarrow (\cos \alpha - i\gamma^{5} \sin \alpha)\lambda(x), \quad \overline{\lambda}'(x) \rightarrow \overline{\lambda}(x)(\cos \alpha - i\gamma^{5} \sin \alpha),
\]

the spinors retain their properties to be self/anti-self charge conjugate spinors and the proposed Lagrangian [13, p.1472] remains to be invariant. This tells us that while self/anti-self charge conjugate states have zero eigenvalues of the ordinary (scalar) charge operator but they can possess the axial charge.\(^2\)

Next, because the transformations

\[
\lambda_{S}^{\prime}(p) = \begin{pmatrix} \Xi & 0 \\ 0 & \Xi \end{pmatrix} \lambda_{S}(p) \equiv \lambda_{S}^{*}(p), \\
\lambda_{S}^{\prime\prime}(p) = \begin{pmatrix} i\Xi & 0 \\ 0 & -i\Xi \end{pmatrix} \lambda_{S}(p) \equiv -i\lambda_{S}^{*}(p), \\
\lambda_{S}^{\prime\prime\prime}(p) = \begin{pmatrix} 0 & i\Xi \\ i\Xi & 0 \end{pmatrix} \lambda_{S}(p) \equiv i\gamma^{0}\lambda_{S}^{*}(p), \\
\lambda_{S}^{IV}(p) = \begin{pmatrix} 0 & \Xi \\ -\Xi & 0 \end{pmatrix} \lambda_{S}(p) \equiv \gamma^{0}\lambda_{S}^{*}(p)
\]

with the \( 2 \times 2 \) matrix \( \Xi \) defined as (\( \phi \) is the azimuthal angle related with \( p \))

\[
\Xi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}, \quad \Xi \Lambda_{R,L}(p \leftarrow 0)\Xi^{-1} = \Lambda^{*}_{R,L}(p \leftarrow 0),
\]

and because the corresponding transformations for \( \lambda^{A} \) do not change the properties of bispinors to be in the self/anti-self charge conjugate spaces, the Majorana-like field operator (\( b^{\dagger} \equiv a^{\dagger} \)) admits additional phase (and, in general, normalization) transformations:

\[
\nu^{ML}(x^{\mu}) = [c_{0} + i(\tau \cdot c)]\nu^{ML}(x^{\mu}),
\]

where \( c_{\alpha} \) are arbitrary parameters. The \( \tau \) matrices are defined over the field of \( 2 \times 2 \) matrices and the Hermitian conjugation operation is assumed to act on the \( c \)-numbers as the complex conjugation. One can parametrize \( c_{0} = \cos \phi \) and \( c = \mathbf{n}\sin \phi \) and, thus, define the \( SU(2) \) group of phase transformations. One can select the Lagrangian which is composed from the both field operators (with \( \lambda \) spinors and \( \rho \) spinors) and which remains to be invariant with respect to this kind of transformations. The conclusion is: it is permitted a non-Abelian construct which is based on the spinors of the Lorentz group only. This is not surprising because both the \( SU(2) \) group and \( U(1) \) group are the sub-groups of the extended Poincaré group (cf. [3]).

\(^2\) In fact, from this consideration one can also recover the Feynman-Gell-Mann equation (and its charge-conjugate equation). It is re-written in the two-component form

\[
\left\{ \begin{array}{l}
\bar{\sigma}_{\mu}^{+}\pi_{\mu}^{-} - m^{2} - \frac{g}{2}\sigma_{\mu}^{\rho}\pi_{\mu}^{\rho} = 0, \\
\overline{\sigma}_{\mu}^{-}\pi_{\mu}^{+} - m^{2} + \frac{g}{2}\sigma_{\mu}^{\rho}\pi_{\mu}^{\rho} = 0,
\end{array} \right.
\]

where already one has \( \pi_{\mu}^{\pm} = \pm i\partial_{\mu} + gA_{\mu} \).
The Dirac-like and Majorana-like field operators can be built from both $\lambda^{S,A}(p)$ and $\rho^{S,A}(p)$, or their combinations. The anticommutation relations are the following ones (due to the bi-orthonormality):

\[ [a_{\eta'}(p'), a_{\eta}^\dagger(p)]_{\pm} = (2\pi)^{3}2E_{p}\delta(p - p')\delta_{\eta,-\eta'}, \]

and

\[ [b_{\eta'}(p'), b_{\eta}^\dagger(p)]_{\pm} = (2\pi)^{3}2E_{p}\delta(p - p')\delta_{\eta,-\eta'} \]

Other (anti)commutators are equal to zero: \([a_{\eta'}(p'), b_{\eta}^\dagger(p)] = 0\).

In the Fock space operations of the charge conjugation and space inversions can be defined through unitary operators such that:

\[ U_{[1/2]}^{c}(x^{\mu})(U_{[1/2]}^{c})^{-1} = \mathcal{C}_{[1/2]}\Psi_{[1/2]}^\dagger(x^{\mu}), \quad U_{[1/2]}^{s}(x^{\mu})(U_{[1/2]}^{s})^{-1} = \gamma_{0}\Psi(x^{\mu}), \]

the time reversal operation, through an antunitary operator

\[ \left[V_{[1/2]}^{T}\Psi(x^{\mu})(V_{[1/2]}^{T})^{-1}\right]^\dagger = S(T)\Psi^\dagger(x^{\mu''}) \]

with $x^{\mu''} \equiv (x^{0}, -\mathbf{x})$ and $x^{\mu''} = (-x^{0}, \mathbf{x})$. We further assume the vacuum state to be assigned an even $P$- and $C$-eigenvalue and, then, proceed as in ref. [4]. As a result we have the following properties of creation (annihilation) operators in the Fock space:

\[ U_{[1/2]}^{s}a_{\downarrow}(p)(U_{[1/2]}^{s})^{-1} = -ia_{\downarrow}(-p), \quad U_{[1/2]}^{s}a_{\downarrow}(p)(U_{[1/2]}^{s})^{-1} = +ia_{\downarrow}(-p), \]

\[ U_{[1/2]}^{s}b_{\downarrow}(p)(U_{[1/2]}^{s})^{-1} = +ib_{\downarrow}(-p), \quad U_{[1/2]}^{s}b_{\downarrow}(p)(U_{[1/2]}^{s})^{-1} = -ib_{\downarrow}(-p), \]

what signifies that the states created by the operators $a_{\downarrow}^\dagger(p)$ and $b_{\downarrow}^\dagger(p)$ have very different properties with respect to the space inversion operation, comparing with Dirac states (the case also regarded in [9]):

\[ U_{[1/2]}^{s}|p, \uparrow\rangle^{\pm} = +|p, \downarrow\rangle, \quad U_{[1/2]}^{s}|p, \downarrow\rangle^{\pm} = +|p, \uparrow\rangle, \]

\[ U_{[1/2]}^{s}|p, \uparrow\rangle^{\pm} = -|p, \downarrow\rangle, \quad U_{[1/2]}^{s}|p, \downarrow\rangle^{\pm} = -|p, \uparrow\rangle. \]

For the charge conjugation operation in the Fock space we have two physically different possibilities. The first one, e.g.,

\[ U_{[1/2]}^{c}a_{\uparrow}(p)(U_{[1/2]}^{c})^{-1} = +b_{\downarrow}(p), \quad U_{[1/2]}^{c}a_{\uparrow}(p)(U_{[1/2]}^{c})^{-1} = +b_{\downarrow}(p), \]

\[ U_{[1/2]}^{c}b_{\uparrow}(p)(U_{[1/2]}^{c})^{-1} = -a_{\downarrow}^\dagger(p), \quad U_{[1/2]}^{c}b_{\uparrow}(p)(U_{[1/2]}^{c})^{-1} = -a_{\downarrow}^\dagger(p). \]

The action of this operator on the physical states are

\[ U_{[1/2]}^{c}|p, \uparrow\rangle^{\pm} = +|p, \uparrow\rangle^{\pm}, \quad U_{[1/2]}^{c}|p, \uparrow\rangle^{\pm} = +|p, \uparrow\rangle^{\pm}, \]

\[ U_{[1/2]}^{c}|p, \downarrow\rangle^{\pm} = -|p, \downarrow\rangle^{\pm}, \quad U_{[1/2]}^{c}|p, \downarrow\rangle^{\pm} = -|p, \downarrow\rangle^{\pm}. \]

The second one acts in the following manner:

\[ \tilde{U}_{[1/2]}^{c}a_{\uparrow}(p)(\tilde{U}_{[1/2]}^{c})^{-1} = -b_{\downarrow}(p), \quad \tilde{U}_{[1/2]}^{c}a_{\uparrow}(p)(\tilde{U}_{[1/2]}^{c})^{-1} = -b_{\downarrow}(p), \]

\[ \tilde{U}_{[1/2]}^{c}b_{\uparrow}(p)(\tilde{U}_{[1/2]}^{c})^{-1} = +a_{\downarrow}^\dagger(p), \quad \tilde{U}_{[1/2]}^{c}b_{\uparrow}(p)(\tilde{U}_{[1/2]}^{c})^{-1} = +a_{\downarrow}^\dagger(p), \]
and, therefore,
\[ \tilde{U}^*_{[1/2]}|p, \uparrow >^+ = -|p, \downarrow >, \quad \tilde{U}^*_{[1/2]}|p, \downarrow >^+= -|p, \uparrow >. \]  
\[ \tilde{U}^*_{[1/2]}|p, \uparrow >^- = +|p, \downarrow >^+, \quad \tilde{U}^*_{[1/2]}|p, \downarrow >^- = +|p, \uparrow >^+. \]  

It is possible a situation when the operators of the space inversion and charge conjugation commute each other in the Fock space [19]. Finally, the time reversal anti-unitary operator in the Fock space should be defined in such a way that the formalism to be compatible with the CPT theorem. We obtain for the \( \Psi(x^\mu) \) field:
\[ V^T a^\dag_1(p)(V^T)^{-1} = a^\dag_1(-p), \quad V^T a^\dag_1(p)(V^T)^{-1} = -a^\dag_1(-p), \]  
\[ V^T b_1(p)(V^T)^{-1} = b_1(-p), \quad V^T b_1(p)(V^T)^{-1} = -b_1(-p). \]  

Thus, this construct has very different properties with respect to \( C, P \) and \( T \) comparing with the Dirac construct.

The dependence of the physical results on the choice of the basis is a bit strange thing. Somewhat similar things have been presented in [17] when compared the Dirac-like constructs in the parity basis and the helicity basis. It was shown that the helicity eigenstates of the operator \( (\sigma \cdot n) \otimes I \) are NOT the parity eigenstates, and vice versa, in the helicity basis (cf. with [20], while they obey the same Dirac equation. This is not surprising because the \( P \) operator does not commute with the \( h \) operator. The bases are connected by the unitary transformation. The both sets of 4-spinors form the complete system in a mathematical sense.

3. The Spin 1.
3.1. Maxwell Equations as Quantum Equations.
In refs. [21, 22] the Maxwell-like equations have been derived from the D’Alembert equation. Here they are:
\[ \nabla \times E = \frac{1}{c} \frac{\partial B}{\partial t} + \nabla Im \chi, \quad \nabla \cdot E = -\frac{1}{c} \frac{\partial}{\partial t} Re \chi, \]  
\[ \nabla \times B = \frac{1}{c} \frac{\partial E}{\partial t} + \nabla Re \chi, \quad \nabla \cdot B = \frac{1}{c} \frac{\partial}{\partial t} Im \chi. \]  

Of course, similar equations can be obtained in the massive case \( m \neq 0 \), i.e., within the Proca-like theory. Similar to the spin-1/2 case we have in the spin-1 case:
\[ (EI^{(3)} - S \cdot p)(EI^{(3)} + S \cdot p)\Psi^{(3)} - p(p \cdot \Psi^{(3)}) = m^2 \Psi^{(3)}. \]  

These lead to (45-46), when \( m = 0 \) provided that the \( \Psi^{(3)} \) is chosen as a superposition of a vector (the electric field) and an axial vector (the magnetic field). When \( \chi = 0 \) we recover the common-used Maxwell equations.

Otherwise, we can start with (\( c = \hbar = 1 \))\(^3\)
\[ \frac{\partial E}{\partial t} = curl B, \quad \frac{\partial B}{\partial t} = -curl E. \]  

\(^3\) The question of both explicite and implicite dependences of the fields on the time (and, hence, the "whole-partial derivative") has been studied in [23, 24].
It has solutions with relativistic dispersion relations $E_{\nu} = \nu c$. The new equation is written (equation. One can add the Klein-Gordon equation with arbitrary multiple factor to the Weinberg (Euclidean metric is now used). In fact, they added the Klein-Gordon equation to the Weinberg

In the spin-1 case it is:

$$\Xi = \gamma^{\mu\nu\alpha\beta} \partial_\mu \partial_\nu \partial_\alpha \partial_\beta - 2m^2 \Psi(x) = 0,$$

with the Barut-Muzinich-Williams covariantly-defined matrices [26, 27]. For the spin-1 they are:

$$\gamma_{44} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_{ii} = \gamma_{44} = \begin{pmatrix} 0 & iS_i \\ -iS_i & 0 \end{pmatrix},$$

$$\gamma^{ij} = \begin{pmatrix} 0 & \delta_{ij} - S_iS_j - S_jS_i \\ \delta_{ij} - S_iS_j - S_jS_i & 0 \end{pmatrix}. $$

Later Sankaranarayanan and Good considered another version of this theory [27] (see also [29]). For the $S = 1$ case they introduced the Weaver-Hammer-Good sign operator, ref. [28], $m^2 \to m^2 ((\partial^2/\partial t^2) + E) $, which led to the different parity properties of an antiparticle with respect to a boson particle. Next, Tucker and Hammer et al [30] introduced another higher-spin equations. In the spin-1 case it is:

$$[\gamma_{\mu\nu} \partial_\mu \partial_\nu + \partial_\mu \partial_\mu - 2m^2] \Psi^{(s=1)} = 0$$

(Euclidean metric is now used). In fact, they added the Klein-Gordon equation to the Weinberg equation. One can add the Klein-Gordon equation with arbitrary multiple factor to the Weinberg equation. The new equation is written ($p_\mu = -i\partial/\partial x^\mu$):

$$[\gamma_{\alpha\beta} p_\alpha p_\beta + A p_\alpha p_\alpha + B m^2] \Psi = 0.$$ 

It has solutions with relativistic dispersion relations $E^2 - p^2 = m^2$, ($c = \hbar = 1$) provided that $B/A = 1$, or $B/A = 1$. This can be proven by considering the algebraic equation $Det[\gamma_{\alpha\beta} p_\alpha p_\beta + A p_\alpha p_\alpha + B m^2] = 0$. It is of the 12th order in $p_\mu$. Solving it with respect to energy one obtains the above conditions. Unlike the Maxwell equations there are NO any $E = 0$ solutions. The solutions in the momentum representation have been explicitly presented in [29] in the standard representation of $\gamma_{\mu\nu}$ matrices. If the 6-component $v(p)$ are defined in such way, we inevitably would get the additional energy-sign operator, refs. [28, 27], $\epsilon = i\hbar / E = \pm 1$ in the dynamical equation, and the different parities of the corresponding boson and antiboson, $\hat{P} u_\sigma (p) = + u_\sigma (p)$ and $\hat{P} v_\sigma (p) = - v_\sigma (p)$.
4. The Construction of Quantum Field Operators.

The method for constructions of field operators has been given in [5]:

\[ \phi(x) = \frac{1}{(2\pi)^{3/2}} \int dk e^{ikx} \tilde{\phi}(k). \] (57)

From the Klein-Gordon equation we know \((k^2 - m^2)\tilde{\phi}(k) = 0\). Finally, we come to

\[ \phi(x) = \frac{1}{(2\pi)^{3/2}} \int dk e^{ikx} \delta(k^2 - m^2)(\theta(k_0) + \theta(-k_0))\phi(k) = \]

\[ = \frac{1}{(2\pi)^{3/2}} \int dk \left[ e^{ikx} \delta(k^2 - m^2)\phi^+(k) + e^{-ikx} \delta(k^2 - m^2)\phi^-(k) \right], \] (58)

where

\[ \phi^+(k) = \theta(k_0)\phi(k), \text{ and } \phi^-(k) = \theta(k_0)\phi(-k). \] (59)

In the spinor case (the \((1/2, 0) \oplus (0, 1/2)\) representation space) we have more components. We have \((\hat{k} + m)\psi(k)|_{k^2 = m^2} = 0\). However, we use again

\[ \psi(x) = \frac{1}{(2\pi)^{3/2}} \int dk e^{ikx} \delta(k^2 - m^2)(\theta(k_0) + \theta(-k_0))\psi(k) = \]

\[ = \frac{1}{(2\pi)^{3}} \int \frac{d^3k}{2E} \left[ e^{ikx} \theta(k_0)\psi(k) + e^{-ikx} \theta(k_0)\psi(-k) \right], \] (60)

where \(k_0 = E = \sqrt{k^2 + m^2}\) is positive in this case, \((\hat{k} + m)\psi^+(k) = 0\), \((-\hat{k} + m)\psi^-(k) = 0\).

Please note that the momentum-space Dirac equations have solutions \(k_0 = \pm \sqrt{k^2 + m^2}\), both for \(u^-\) and \(v^-\) spinors. This can be checked by calculating the determinants. Usually, one chooses \(k_0 = E\) in the \(u^-\) and in the \(v^-\). This is because on the classical level (better to say, on the first quantization level) the negative-energy \(u^-\) can be transformed in the positive-energy \(v^-\), and vice versa. This is not precisely so, if we go to the secondary quantization level. The introduction of creation/annihilation noncommuting operators gives us more possibilities in constructing generalized theory even on the basis of the Dirac equation.

Various-type field operators are possible in the \((1/2, 1/2)\) representation. During the calculations below we have to present \(1 = \theta(k_0) + \theta(-k_0)\) (as previously) in order to get positive- and negative-frequency parts.

\[ A_\mu(x) = \frac{1}{(2\pi)^{3}} \int d^4k \delta(k^2 - m^2)e^{+ikx}A_\mu(k) = \]

\[ = \frac{1}{(2\pi)^{3}} \int \frac{d^4k}{2E} \left[ \delta(k_0 - E_k) + \delta(k_0 + E_k) \right]\theta(k_0) + \theta(-k_0)\epsilon^{+ikx}A_\mu(k) \]

\[ = \frac{1}{(2\pi)^{3}} \int \frac{d^4k}{2E_k} \theta(k_0)\left[A_\mu(k)e^{+ikx} + A_\mu(-k)e^{-ikx}\right] = \]

\[ = \frac{1}{(2\pi)^{3}} \sum_\lambda \int \frac{d^4k}{2E_k} \left[ \epsilon_\mu(k, \lambda)a_\lambda(k)e^{+ikx} + \epsilon_\mu(-k, \lambda)a_\lambda(-k)e^{-ikx}\right]. \] (62)

\[ ^4 \text{In this book a bit different notation for positive- (negative-) energy solutions has been used comparing with the general accepted one [4].} \]
In general, due to theorems for integrals and for distributions the presentation \(1 = \theta(k_0) + \theta(-k_0)\) is possible because we use this in the integrand. However, remember, that we have the \(k_0 = E = 0\) solution of the Maxwell equations. Moreover, it has the experimental confirmation (for instance, the stationary magnetic field \(\text{curl}B = 0\)). Meanwhile the \(\theta\) function is NOT defined in \(k_0 = 0\). Do we not lose this solution in the above construction of a quantum field operator? Of course, the same arguments can be applied in the construction of the quantum field operator for \(F_{\mu\nu}\).

Next, we should transform the second part to \(\epsilon^*_\mu(k, \lambda)b^\dagger_\lambda(k)\) as usual. In such a way we obtain the charge-conjugate states. In the Dirac case we should assume the following relation in the field operator:

\[
\sum_\lambda v_\lambda(k)b^\dagger_\lambda(k) = \sum_\lambda u_\lambda(-k)a_\lambda(-k).
\]

Hence, \(\Lambda_{\mu\lambda} = -im(\sigma \cdot n)_{\mu\lambda}\) and \(b^\dagger_\lambda(k) = i(\sigma \cdot n)_{\mu\lambda}a_\lambda(-k)\). In the \((1, 0) \oplus (0, 1)\) representation we have somewhat different situation. Namely, \(a_\mu(k) = [1 - 2(S \cdot n)^2]_{\mu\lambda}a_\lambda(-k)\). This signifies that in order to construct the Sankaranarayanan-Good field operator (which was used by Ahluwalia, Johnson and Goldman [29], it satisfies \([\gamma_{\mu\nu}\partial_\mu\partial_\nu - (\partial_\mu \partial^\mu)E^2] \Psi = 0\), we need additional postulates. We can set for the 4-vector field operator:

\[
\sum_\lambda \epsilon_\mu(-k, \lambda)a_\lambda(-k) = \sum_\lambda \epsilon^*_\mu(k, \lambda)b^\dagger_\lambda(k).
\]

However, in the \((\frac{1}{2}, \frac{1}{2})\) representation we can also expand (apart the equation (64)) in the different way:

\[
\sum_\lambda \epsilon_\mu(-k, \lambda)a_\lambda(-k) = \sum_\lambda \epsilon^*_\mu(k, \lambda)a_\lambda(k).
\]

From the first definition we obtain

\[
b^\dagger_\sigma(k) = \frac{E^2_k}{m^2} \begin{pmatrix}
1 + \frac{k^2}{E_k^2} & \sqrt{\frac{k^2}{E_k^2}} & -\sqrt{\frac{k^2}{E_k^2}} & -\frac{2k_0}{E_k} \\
-\sqrt{\frac{k^2}{E_k^2}} & -\frac{k^2}{E_k^2} & \frac{m^2k^2}{E_k^2} + \frac{k_0^2}{E_k^2} & \frac{\sqrt{3k_3k_0}}{k^2} \\
\sqrt{\frac{k^2}{E_k^2}} & \frac{\sqrt{3k_3k_0}}{k^2} & \frac{\sqrt{3k_3k_0}}{k^2} & -\frac{2k_3}{k^2} \\
\frac{2k_3}{E_k} & -\frac{\sqrt{3k_3k_0}}{k^2} & -\frac{\sqrt{3k_3k_0}}{k^2} & \frac{m^2}{E_k^2} - \frac{2k_3}{k^2}
\end{pmatrix} \begin{pmatrix}
a_{00}(-k) \\
a_{11}(-k) \\
a_{10}(-k) \\
a_{10}(-k)
\end{pmatrix}.
\]

From the second definition \(a^2_{\lambda\lambda} = \sum_{\nu\mu} \epsilon^*_\mu(k, \sigma)[\gamma_{44}]_{\nu\mu} \epsilon_\nu(-k, \lambda)\) we have:

\[
a_{\sigma}(k) = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & k^2 & k^2 & \sqrt{3k_3k_0} \\
0 & k^2 & k^2 & -\sqrt{3k_3k_0} \\
0 & \sqrt{3k_3k_0} & \sqrt{3k_3k_0} & 1 - \frac{2k_3}{k^2}
\end{pmatrix} \begin{pmatrix}
a_{00}(-k) \\
a_{11}(-k) \\
a_{10}(-k) \\
a_{10}(-k)
\end{pmatrix}.
\]

It is the strange case: the field operator will only destroy particles (like in the \((1, 0) \oplus (0, 1)\) case). Possibly, we should think about modifications of the Fock space in this case, or introduce several field operators for the \((\frac{1}{2}, \frac{1}{2})\) representation. However, other way is possible: to construct the left- and right- parts of the \((1, 0) \oplus (0, 1)\) field operator separately each other. In this case the commutation relations may be more complicated.

Finally, going back to the rest \((S, 0) \oplus (0, S)\) objects. Bogoliubov et al. constructs them introducing the products with \(\delta\) functions like \(\delta(k_0 - m)\). Then, he makes the boost of the ”spinors” only, and changes by hand the above \(\delta\) to \(\delta(k^2 - m^2)\) (where we
already have \( k_0 = E = \sqrt{k^2 + m^2} \). Is it possible to make the boost of the \( \delta \) functions consistently in such a way?

The conclusion is: we still have few questions unsolved in the bases of the quantum field theory, which open a room for generalized theories.

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