CALABI-YAU STRUCTURE AND
SPECIAL LAGRANGIAN SUBMANIFOLD
OF COMPLEXIFIED SYMMETRIC SPACE

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Abstract. It is known that there exist Calabi-Yau structures on the complexifications of symmetric spaces of compact type. In this paper, we describe the Calabi-Yau structures of the complexified symmetric spaces in terms of the Schwarz’s theorem in detail. We consider the case where the Calabi-Yau structure arises from the Riemannian metric corresponding to the Stenzel metric. In the complexified symmetric spaces equipped with such a Calabi-Yau structure, we give constructions of special Lagrangian submanifolds of any given phase which are invariant under the actions of symmetric subgroups of the isometry group of the original symmetric space of compact type.

1. Introduction

An $2n$-dimensional Riemannian manifold is called a Calabi-Yau manifold if the holonomy group is a subgroup of $SU(n)$. A Kaehler manifold is Calabi-Yau if and only if it is Ricci-flat. Let $(M,J,\omega)$ be a complex $n$-dimensional Kaehler manifold, where $J$ is the complex structure and $\omega$ is the Kaehler form. Also, let $g$ be the Kaehler metric associated to $(J,\omega)$. If there exists a non-vanishing holomorphic $(n,0)$-form $\Omega$ on $M$ (i.e., the holomorphic complex line bundle $\bigwedge^{(n,0)}(M)$ is trivial), then $(M,J,\omega)$ is called a almost Calabi-Yau manifold. In particular, if $(\omega,\Omega)$ satisfies

\[ \omega^n = (-1)^{n(n-1)/2}(\sqrt{-1})^n c(\Omega \wedge \overline{\Omega}) \]

for some positive real constant $c$, then $(M,J,\omega)$ is Ricci-flat and hence it is Calabi-Yau. By replacing $\Omega$ to a suitable positive real constant-multiple of $\Omega$ if necessary, we may assume that $c = \frac{n!}{2^n}$. In the sequel, the Calabi-Yau manifold (resp. the Calabi-Yau structure) means a quadruple $(M,J,\omega,\Omega)$ (resp. a triple $(J,\omega,\Omega)$) such that $(J,\omega)$ is a Kaehler structure and that $(\omega,\Omega)$ satisfies

\[ \omega^n = (-1)^{n(n-1)/2} n! \left( \frac{\sqrt{-1}}{2} \right)^n \Omega \wedge \overline{\Omega}. \]

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Let \((J, \omega, \Omega)\) be a Calabi-Yau structure on \(M\) and \(g\) the Kaehler metric associated to \((J, \omega)\). Then, for any real constant \(\theta\), a \(n\)-form \(\text{Re}(e^{\sqrt{-1} \theta} \Omega)\) is a calibration on \((M, g)\). A submanifold calibrated by \(\text{Re}(e^{\sqrt{-1} \theta} \Omega)\) is called a special Lagrangian submanifold of phase \(\theta\). According to Strominger-Yau-Zaslov’s conjecture ([SYZ]) for the Mirror symmetry in the string theory, it is important to construct special Lagrangian submanifolds in a Calabi-Yau manifold.

Let \(M\) be \(C^\omega\)-Riemannian manifold and \(M^\mathbb{C}\) its complexification. In 1991, V. Gillemín and M. Stenzel ([GS]) gave a construction of Ricci-flat metrics on a sufficiently small tubular neighborhood of \(M\) in \(M^\mathbb{C}\). Let \(G/K\) be a (Riemannian) symmetric space of compact type. The complexification \((G/K)^\mathbb{C}\) of \(G/K\) is defined as the complexified symmetric space \(G^\mathbb{C}/K^\mathbb{C}\) equipped with the \(G^\mathbb{C}\)-invariant anti-Kaehler metric \(\beta_A\). The anti-Kahler manifold \((G^\mathbb{C}/K^\mathbb{C}, \beta_A)\) is called an anti-Kaehler symmetric space. This space \((G^\mathbb{C}/K^\mathbb{C}, \beta_A)\) is identified with the tangent bundle \(T(G/K)\) of \(G/K\) under the one-to-one correspondence \(\Psi : T(G/K) \longrightarrow G^\mathbb{C}/K^\mathbb{C}\) defined by

\[
\Psi(v) := \text{Exp}_p((J_0)_p(v)) \quad (p \in G/K, \ v \in T_p(G/K))
\]

(see Figure 1), where \(\text{Exp}_p\) denotes the exponential map of \((G^\mathbb{C}/K^\mathbb{C}, \beta_A)\) at \(p\), \(J_0\) denotes the natural complex structure of \(G^\mathbb{C}/K^\mathbb{C}\) and \(v\) is regarded as a tangent vector of the submanifold \(G \cdot o \approx G/K\) \((o = eK^\mathbb{C})\) in \(G^\mathbb{C}/K^\mathbb{C}\). For each \(p \in G/K \approx G \cdot o\) set \(\Psi_p := \Psi|_{T_p(G/K)}(= \text{Exp}_p \circ (J_0)_p)\) and \((G/K)^d_p := \Psi(T_p(G/K))\). Note that \((G/K)^d_p\)'s equipped with the (Riemannian) metric induced from \(\beta_A\) are isometric to the symmetric space \(G^d/K\) of non-compact type given as the dual of \(G/K\) and they are totally geodesic submanifolds in \((G^\mathbb{C}/K^\mathbb{C}, \beta_A)\).

\[
G^\mathbb{C}/K^\mathbb{C}
\]

**Figure 1.**
We consider the case where $G/K$ is the sphere $SO(n + 1)/SO(n) (= S^n(1))$. Then the complexification $SO(n + 1, \mathbb{C})/SO(n, \mathbb{C})$ of $SO(n + 1)/SO(n)$ is embedded into $\mathbb{C}^{n+1}$ as the complex sphere $S^n(1) := \{(z_1, \ldots, z_{n+1}) | \sum_{i=1}^{n+1} z_i^2 = 1\}$ of complex radius 1. The natural embedding $\iota$ of $SO(n + 1, \mathbb{C})/SO(n, \mathbb{C})$ into $\mathbb{C}^{n+1}$ is given by

$$\iota(q) := \cosh(\|\Psi^{-1}(q)\|) \cdot \vec{O}p + \sqrt{-1} \cdot \frac{\sinh(\|\Psi^{-1}(q)\|)}{\|\Psi^{-1}(q)\|} \cdot \Psi^{-1}(q)$$

($q \in SO(n + 1, \mathbb{C})/SO(\mathbb{C})$),

where $p$ is the base point of $\Psi^{-1}(q)$, $O$ is the origin of of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ including $S^n(1)(= SO(n + 1)/SO(n))$, and $\vec{O}p$ and $\Psi^{-1}(q)$ are regarded as vectors of $\mathbb{R}^{n+1}$ (see Figure 2). Hence we have

$$\iota(\Psi_p(v)) = \cosh(\|v\|) \cdot \vec{O}p + \sqrt{-1} \cdot \frac{\sinh(\|v\|)}{\|v\|} \cdot v \quad (v \in T_pS^n(1)).$$

![Figure 2.](image-url)
n-dimensional sphere $SO(n + 1)/SO(n)$ which are invariant under the natural action $SO(n) \lhd SO(n + 1, \mathbb{C})/SO(n, \mathbb{C})$. M. Ionel and M. Min-Oo ([IO]) constructed cohomogeneity one special Lagrangian submanifolds of some phase in $SO(4, \mathbb{C})/SO(3, \mathbb{C})$ which are invariant under the natural action $SO(2) \times SO(2) \lhd SO(4, \mathbb{C})/SO(3, \mathbb{C})$. K. Hashimoto and T. Sakai ([HS]) constructed cohomogeneity one special Lagrangian submanifolds of any phase and in the complexification $SO(4, \mathbb{C})/SO(3, \mathbb{C})$. Recently M. Arai and K. Baba ([AB]) constructed cohomogeneity one special Lagrangian submanifolds of any phase and in the complexification $SO(n + 1, \mathbb{C})/SO(n, \mathbb{C})$ which are invariant under the natural action $SO(p) \times SO(n + 1 - p) \lhd SO(n + 1, \mathbb{C})/SO(n, \mathbb{C}) (1 \leq p \leq [(n + 1)/2])$. Later, K. Hashimoto and K. Mashino ([HM]) constructed cohomogeneity one special Lagrangian submanifolds of any phase in $SO(n + 1, \mathbb{C})/SO(n, \mathbb{C})$ which are invariant under the natural action $K \lhd SO(n + 1, \mathbb{C})/SO(n, \mathbb{C})$ induced from the linear isotropy action $K \lhd SO(n + 1)/SO(n) (= S^n(1) \subset T_e K(G/K))$ of any irreducible rank two symmetric space $G/K$, where $n := \dim G/K - 1$.

In this paper, we first construct an almost Calabi-Yau structure $(J_0, \omega_{\psi_J}, \Omega_0)$ on the complexification $G^C/K^C$, which is invariant under the natural action $G \lhd G^C/K^C$, in terms of a $C^\infty$-function $f$ over $\mathbb{R}^l (l : \text{a natural number})$ and investigate in what case it is a Calabi-Yau structure, where $J_0$ and $\Omega_0$ are the natural complex structure and the natural non-vanishing closed holomorphic $(n, 0)$-form on $G^C/K^C$ (Section 2). In Section 3, we investigate the 0-level set of the moment map of a Hamiltonian action on the Calabi-Yau manifold $(G^C/K^C, J_0, \omega_{\psi_J}, \Omega_0)$. Let $H$ be a symmetric subgroup of $G$. The natural action $H \lhd G/K$ (which is called a Hermann action) is extended to the action on $G^C/K^C$ naturally. This extended action $H \lhd G^C/K^C$ is a Hamiltonian action. In section 4, we investigate the orbit structure of this Hamiltonian action $H \lhd G^C/K^C$. In Section 5, in the case where $\beta_{\psi_J} : = \omega_{\psi_J}(J_0(\cdot), \cdot)$ is the metric generalized the Stenzel metric, we first give a construction of an $H$-invariant special Lagrangian submanifold of cohomogeneity $r$ in $(G^C/K^C, J_0, \omega_{\psi_J}, \Omega_0)$, where $r$ denotes the cohomogeneity of $H \lhd G/K$ (see Theorem 5.4 and Corollary 5.5).

2. CALABI-YAU STRUCTURES ON COMPLEXIFIED SYMMETRIC SPACES

Let $G$ be a compact semi-simple Lie group and $\theta$ an involutive automorphism of $G$. Let $K$ be a closed subgroup of $G$ with $(\text{Fix } \theta)_0 \subset K \subset \text{Fix } \theta$, where $\text{Fix } \theta$ is the fixed point group of $\theta$ and $(\text{Fix } \theta)_0$ is the identity component of $\text{Fix } \theta$. Denote by $\mathfrak{g}$ (resp. $\mathfrak{k}$) the Lie algebra of $G$ (resp. $K$) and $B$ the Killing form of $\mathfrak{g}$. Denote by the same symbol $\theta$ the involution of $\mathfrak{g}$ induced from $\theta$. Set $\mathfrak{p} : = \text{Ker}(\theta + \text{id}_{\mathfrak{g}})$, which is identified with the tangent space $T_o(G/K)$
of $G/K$ at $o := eK$ ($e$ : the identity element of $G$), where $id_o$ is the identity transformation of $\mathfrak{g}$. Since $B|_{p \times p}$ is the $\text{Ad}_{G/K}|_p$-invariant, we obtain a $G$-invariant metric $\beta$ on $G/K$ with $\beta_{pK} = B$, where $\text{Ad}_G$ is adjoint representation of $G$. This Riemannian manifold $(G/K, \beta)$ is called a (Riemannian) symmetric space of compact type. The dimension of maximal flat totally geodesic submanifold in $G/K$ is called the rank of $G/K$. Denote by $\tilde{r}$ the rank of $G/K$. Also, assume that $G$ and $K$ admit faithful real representations. Hence the complexifications $G^\mathbb{C}$ and $K^\mathbb{C}$ of $G$ and $K$ are defined. For the complexification $B^\mathbb{C}$ ; $p^\mathbb{C} \times p^\mathbb{C} \to \mathbb{C}$ of $B$, its real part $\text{Re} B^\mathbb{C}$ is a $\text{Ad}_{G^\mathbb{C}}(K^\mathbb{C})|_{p^\mathbb{C}}$-invariant non-degenerate bilinear form (of half index) of $p^\mathbb{C}$ and hence we obtain a $G^\mathbb{C}$-invariant neutral metric $\beta_\mathbb{C}$ on $G^\mathbb{C}/K^\mathbb{C}$ with $(\beta_\mathbb{C})_{pK} = \text{Re} B^\mathbb{C}$, where $\text{Ad}_{G^\mathbb{C}}$ is adjoint representation of $G^\mathbb{C}$. This pseudo-Riemannian manifold $(G^\mathbb{C}/K^\mathbb{C}, \beta_\mathbb{C})$ is called an anti-Kaehler symmetric space, which is one of semi-simple pseudo-Riemannian symmetric spaces. Note that the terminology “anti-Kaehler” is used in [BFV] and [Koi3, Koi4] for example. Define $j : p^\mathbb{C} \to p^\mathbb{C}$ by $j(v) := \sqrt{-1}v \ (v \in p^\mathbb{C})$. Since $j$ is the $\text{Ad}_{G^\mathbb{C}}(K^\mathbb{C})|_{p^\mathbb{C}}$-invariant, we obtain a $G^\mathbb{C}$-invariant almost complex structure $J_0$ of $G^\mathbb{C}/K^\mathbb{C}$ with $(J_0)_{pK} = j$. Take an orthonormal base $(e_1, \ldots, e_n)$ of $p$ with respect to $B$ and let $(\theta^1, \ldots, \theta^n)$ be the dual basis of $(e_1, \ldots, e_n)$. Set $(\theta^1) \wedge \cdots \wedge (\theta^n)^\mathbb{C}$. Since $(\theta^1) \wedge \cdots \wedge (\theta^n)^\mathbb{C}$ is $\text{Ad}_{G^\mathbb{C}}(K^\mathbb{C})|_{p^\mathbb{C}}$-invariant, we obtain a $G^\mathbb{C}$-invariant holomorphic $(n, 0)$-form $\Omega_0$ on $G^\mathbb{C}/K^\mathbb{C}$ with $(\Omega_0)_{pK} = (\theta^1) \wedge \cdots \wedge (\theta^n)^\mathbb{C}$. Let $\psi$ be a strictly plurisubharmonic function over $G^\mathbb{C}/K^\mathbb{C}$, where we note that “strictly plurisubharmonicity” means that the Hermitian matrix

$$
\begin{pmatrix}
\frac{\partial^2 \psi}{\partial z_i \partial \overline{z}_j} \\
\end{pmatrix}
$$

is positive (or equivalently, $(\sqrt{-1} \partial \overline{\partial} \psi)(Z, \overline{Z}) > 0$ holds for any nonzero $(1, 0)$-vector $Z$). Then $\omega_\psi := \sqrt{-1} \partial \overline{\partial} \psi$ is a real non-degenerate closed 2-form on $G^\mathbb{C}/K^\mathbb{C}$ and the symmetric $(0, 2)$-tensor field $\beta_\psi$ associated with $J_0$ and $\omega_\psi$ is positive definite. Hence $(J_0, \omega_\psi, \Omega_0)$ is an almost Calabi-Yau structure on $G^\mathbb{C}/K^\mathbb{C}$. Thus, from each strictly plurisubharmonic function over $G^\mathbb{C}/K^\mathbb{C}$, we obtain an almost Calabi-Yau structure on $G^\mathbb{C}/K^\mathbb{C}$. Hence we suffice to construct a strictly plurisubharmonic function on $G^\mathbb{C}/K^\mathbb{C}$ to construct an almost Calabi-Yau structure on $G^\mathbb{C}/K^\mathbb{C}$. Denote by $\text{Exp}_p$ the exponential map of the anti-Kaehler manifold $(G^\mathbb{C}/K^\mathbb{C}, \beta_\mathbb{C})$ at $p(\in G^\mathbb{C}/K^\mathbb{C})$ and $\text{exp}$ the exponential map of the Lie group $G^\mathbb{C}$. Set $\tilde{g}^d := \mathfrak{t} \oplus \sqrt{-1}\mathfrak{p} \subset \mathfrak{g}^\mathbb{C}$ and $G^d = \text{exp}(\tilde{g}^d)$. Denote by $\beta_{G/K}$ the $G$-invariant (Riemannian) metric on $G/K$ induced from $B|_{p \times p}$ and $\beta_{G^d/K}$ the $G^d$-invariant (Riemannian) metric on $G^d/K$ induced from $-\text{Re} B^\mathbb{C}|_{\sqrt{-1}\mathfrak{p} \times \sqrt{-1}\mathfrak{p}}$. We may assume that the metric of $G/K$ is equal to $\beta_{G/K}$ by homothetically transforming the metric of $G/K$ if necessary. On the other hand, the Riemannian manifold $(G^d/K, \beta_{G^d/K})$ is a (Riemannian) symmetric space of non-compact type. The orbit $G \cdot o$ is isometric to $(G/K, \beta_{G/K})$ and the normal umbrella $\text{Exp}_o(T_o^\perp(G \cdot o)) = G^d \cdot o$ is isometric to $(G^d/K, \beta_{G^d/K})$. The complexification $\mathfrak{p}^\mathbb{C}$ of $\mathfrak{p}$ is identified with $T_o(G^\mathbb{C}/K^\mathbb{C})$.
and $\sqrt{-1}p$ is identified with $T_o(\Exp_o(T_o^+(G \cdot o)))$. Let $a$ be a maximal abelian subspace of $p$, where we note that $\dim a = \bar{r}$. Denote by $W$ the Weyl group of $G^d/K$ with respect to $\sqrt{-1}a$. This group acts on $\sqrt{-1}a$. Let $C(\subset \sqrt{-1}a)$ be a Weyl domain (i.e., a fundamental domain of the action $W \curvearrowright \sqrt{-1}a$). Then we have $G \cdot \Exp_o(C) = G^c/K^c$, where $C$ is the closure of $C$. For a connected open neighborhood $D$ of $0$ in $\sqrt{-1}a$, we define a neighborhood $U_1(D)$ of $o$ in $\Exp_o(\sqrt{-1}a)$ by $U_1(D) := \Exp_o(D)$, a neighborhood $U_2(D)$ of $o$ in $G^d/K$ by $U_2(D) := K \cdot U_1(D)$ and a tubular neighborhood $U_3(D)$ of $G \cdot o$ in $G^c/K^c$ by $U_3(D) := G \cdot U_1(D)$ and (see Figure 3). Denote by $\text{Conv}_W(D)$ the space of all $W$-invariant strictly convex $(C^\infty)$-functions over $D$, $\text{Conv}_K(U_2(D))$ the space of all $K$-invariant strictly convex $(C^\infty)$-functions over $U_2(D)$ and $\text{PH}_G(U_3(D))$ the space of all $G$-invariant strictly plurisubharmonic $(C^\infty)$-functions over $U_3(D)$. The restriction map from $U_3(D)$ to $U_2(D)$ gives an isomorphism of $\text{PH}_G(U_3(D))$ onto $\text{Conv}_K(U_2(D))$ and the composition of the restriction map from $U_3(D)$ to $U_1(D)$ with $\Exp_o$ gives an isomorphism of $\text{PH}_G(U_3(D))$ onto $\text{Conv}_W(D)$ (see [AL]). Hence we suffice to construct $W$-invariant strictly convex functions over $D$ or $K$-invariant strictly convex functions over $U_2(D)$ to construct strictly plurisubharmonic functions over $U_3(D)$. Let $\psi$ be a $G$-invariant strictly plurisubharmonic $(C^\infty)$-functions over $U_3(D)$. Denote by $\tilde{\psi}$ the restriction of $\psi$ to $U_2(D)$ and $\tilde{\psi}$ the composition of the restriction of $\psi$ to $U_1(D)$ with $\Exp_o$. Denote by $\text{Ric}_\psi$ the Ricci form of $\beta_\psi$. By a result of R. Bielawski (Theorem 3.3 in [B2]), we have

\[
\text{Ric}_\psi = -\sqrt{-1}\partial \bar{\partial} \log \det \left( \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j} \right) = -\sqrt{-1}\partial \bar{\partial} \log \left( \frac{\det \nabla d\tilde{\psi}}{\det \beta_{G^d/K}} \right)^h,
\]

over $U_2(D)$ and $\text{PH}_G(U_3(D))$ the space of all $G$-invariant strictly plurisubharmonic $(C^\infty)$-functions over $U_3(D)$. The restriction map from $U_3(D)$ to $U_2(D)$ gives an isomorphism of $\text{PH}_G(U_3(D))$ onto $\text{Conv}_K(U_2(D))$ and the composition of the restriction map from $U_3(D)$ to $U_1(D)$ with $\Exp_o$ gives an isomorphism of $\text{PH}_G(U_3(D))$ onto $\text{Conv}_W(D)$ (see [AL]). Hence we suffice to construct $W$-invariant strictly convex functions over $D$ or $K$-invariant strictly convex functions over $U_2(D)$ to construct strictly plurisubharmonic functions over $U_3(D)$. Let $\psi$ be a $G$-invariant strictly plurisubharmonic $(C^\infty)$-functions over $U_3(D)$. Denote by $\tilde{\psi}$ the restriction of $\psi$ to $U_2(D)$ and $\tilde{\psi}$ the composition of the restriction of $\psi$ to $U_1(D)$ with $\Exp_o$. Denote by $\text{Ric}_\psi$ the Ricci form of $\beta_\psi$. By a result of R. Bielawski (Theorem 3.3 in [B2]), we have

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\]
where $\nabla$ denotes the Riemannian connection of $\beta_{G^d/K}$, $(z_1, \cdots, z_n)$ is any complex coordinate of $G^C/K^C$ and $\left( \frac{\det \nabla d\bar{\psi}}{\det \beta_{G^d/K}} \right)^h$ is the $G$-invariant function over $G^C/K^C$ satisfying

$$
\left. \left( \frac{\det \nabla d\bar{\psi}}{\det \beta_{G^d/K}} \right)^h \right|_{G^d/K} = \frac{\det \nabla d\bar{\psi}}{\det \beta_{G^d/K}}.
$$

According to the result of [B1], for any given $K$-invariant positive $C^\infty$-function $\varphi$ on $G^d/K$, the Monge-Ampère equation

$$(2.2) \quad \frac{\det \nabla d\bar{\psi}}{\det \beta_{G^d/K}} = \varphi$$

has a global $K$-invariant strictly convex $C^\infty$-solution.

Furthermore, we can derive the following fact directly.

Lemma 2.1.  (i) For any $G$-invariant strictly plurisubharmonic ($C^\infty$-)function $\psi$ over $U_3(D)$, we have

$$(2.3) \quad \text{Ric}_\psi = -\sqrt{-1} \partial \bar{\partial} \log \left( \left( \frac{\det \nabla^0 d\bar{\psi}}{\det \beta_0} \right)^h \right),$$

where $\beta_0$ is the Euclidean metric of $\sqrt{-1}a$ associated to $-\text{Re} B^C_{\sqrt{-1}a \times \sqrt{-1}a}$ and $\nabla^0$ is the Euclidean connection of $\beta_0$ and $\left( \frac{\det \nabla^0 d\bar{\psi}}{\det \beta_0} \right)^h$ is the $G$-invariant function over $G^C/K^C$ satisfying

$$
\left. \left( \frac{\det \nabla^0 d\bar{\psi}}{\det \beta_0} \right)^h \right|_{\text{Exp}(\sqrt{-1}a)} \circ \text{Exp}_o = \frac{\det \nabla^0 d\bar{\psi}}{\det \beta_0}.
$$

(ii) For any given $W$-invariant positive $C^\infty$-function $\varphi$ on $\sqrt{-1}a$, the Monge-Ampère equation

$$(2.4) \quad \frac{\det \nabla^0 d\rho}{\det \beta_0} = \varphi$$

has a global $W$-invariant strictly convex $C^\infty$-solution.

Proof. Since $\bar{\psi}$ is $K$-invariant, we have

$$
\left( \frac{\det \nabla d\bar{\psi}}{\det \beta_{G^d/K}} \right)^h = \left( \frac{\det \nabla^0 d\bar{\psi}}{\det \beta_0} \right)^h.
$$
Therefore the statement (i) is directly derived from the above result by R. Bielawski. The statement (ii) is trivial. \[\Box\]

From a global $W$-invariant strictly convex $C^\infty$-solution $\rho$ of the Monge-Ampère equation

\begin{equation}
\frac{\det \nabla^0 \rho}{\det \beta_0} = c \quad (c: \text{a positive constant})
\end{equation}

we can construct a complete Ricci-flat metric $\beta_\psi$ on $G^C/K^C$, where $\psi$ is the $G$-invariant strictly plurisubharmonic $C^\infty$-function satisfying $\psi|_{\Exp_o(\sqrt{-1}a)} \circ \Exp_o = \rho$. Hence we obtain a Calabi-Yau structure $(J_0, \omega_\psi, \Omega_0)$ on $G^C/K^C$ by replacing $\rho$ to a suitable positive constant-multiple of $\rho$ if necessary.

We consider the case of $D = \sqrt{-1}a$. Then, according to the Schwarz’s theorem ([Sc]), the ring $C^\infty_W(\sqrt{-1}a)$ of all $W$-invariant $C^\infty$-functions over $\sqrt{-1}a$ is given by

\begin{equation}
C^\infty_W(\sqrt{-1}a) = \{ f \circ (\rho_1, \cdots, \rho_l) | f \in C^\infty(\R^l) \},
\end{equation}

where $\rho_1, \cdots, \rho_l$ are generators of $C^\infty_W(\sqrt{-1}a)$ of the ring $\Pol_W(\sqrt{-1}a)$ of all $W$-invariant polynomials over $\sqrt{-1}a$. In the sequel, set $\vec{\rho} := (\rho_1, \cdots, \rho_l)$ for simplicity. Let $\psi_i$ ($i = 1, \cdots, l$) be the elements of $\PH^+_G(G^C/K^C)$ with $\psi_i = \rho_i$. In the sequel, set $\vec{\psi} := (\psi_1, \cdots, \psi_l)$ for simplicity. Hence any element $\psi$ of $\PH^+_G(G^C/K^C)$ is described as $\psi = f \circ \vec{\psi}$ in terms of some $f \in C^\infty(\R^l)$. As the first generator $\rho_1$ of $C^\infty_W(\sqrt{-1}a)$, we take

\begin{equation}
\rho_1(\sqrt{-1}v) := \| v \|^2 + 1 \quad (v \in a).
\end{equation}

In the following, set $\psi_f := f \circ \vec{\psi}$. By using Lemma 2.1, we can derive the following fact.

**Theorem 2.2.** (i) The triple $(J_0, \omega_\psi, \Omega_0)$ is a Calabi-Yau structure of $G^C/K^C$ when

\begin{equation}
\det \left( \sum_{k=1}^l \sum_{k=1}^l \left( \left( \frac{\partial^2 f}{\partial y_i \partial y_k} \circ \vec{\rho} \right) \cdot \frac{\partial \rho_k}{\partial x_1} \cdot \frac{\partial \rho_k}{\partial x_2} + \left( \frac{\partial f}{\partial y_k} \circ \vec{\rho} \right) \cdot \frac{\partial^2 \rho_k}{\partial x_i \partial x_j} \right) \right) = c,
\end{equation}

where $c$ is a positive constant, and $(x_1, \cdots, x_r)$ and $(y_1, \cdots, y_l)$ are the natural coordinates of $\sqrt{-1}a$ and $\R^l$, respectively.

(ii) Assume that $\frac{\partial f}{\partial y_2} = \cdots = \frac{\partial f}{\partial y_l} = 0$. Then $(J_0, \omega_\psi, \Omega_0)$ is a Calabi-Yau structure of $G^C/K^C$ when

\begin{equation}
\det \left( 2x_i x_j \cdot \left( \frac{\partial^2 f}{\partial y_1 \partial y_1} \circ \vec{\rho} \right) + \left( \frac{\partial f}{\partial y_1} \circ \vec{\rho} \right) \cdot \delta_{ij} \right) = c,
\end{equation}

where $c$ is a positive constant, and $(x_1, \cdots, x_r)$ and $(y_1, \cdots, y_l)$ are as above.
Proof. By a simple calculation, we have
\[
(\nabla^0 d\bar{\psi}_f) \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \sum_{k=1}^{l} \sum_{\hat{k}=1}^{\hat{l}} \left( \frac{\partial^2 f}{\partial y_k \partial y_{\hat{k}}} \circ \overrightarrow{\rho} \right) \cdot \left( \frac{\partial \rho_{\hat{k}}}{\partial x_i} \frac{\partial \rho_k}{\partial x_j} \right) + \left( \frac{\partial f}{\partial y_{\hat{k}}} \circ \overrightarrow{\rho} \right) \cdot \left( \frac{\partial^2 \rho_k}{\partial x_i \partial x_j} \right).
\]

Hence, from (2.6), we obtain
\[
\det \left( (\nabla^0 d\bar{\psi}_f) \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \right) = c > 0,
\]
that is, $\bar{\psi}_f$ is convex. Also, we have
\[
\det \left( \beta_0 \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \right) = 1.
\]
Hence we have
\[
\frac{\det \nabla^0 d\bar{\psi}_f}{\det \beta_0} = c.
\]
Therefore, from Lemma 2.1, we obtain $\text{Ric}_{\psi_f} = 0$. Thus $(J_0, \omega_{\psi_f}, \Omega_0)$ is a Calabi-Yau structure of $G^C/K^C$. The statement (ii) follows from (i) directly.

Remark 2.1. (i) By using the result of [B1], we can show that the Monge-Ampère type equation (2.6) has global solution $f$.

(ii) The Monge-Ampère type equations (2.6) and (2.7) coincide in the case of rank $G/K = 1$.

From (ii) of Theorem 2.2, we can derive the following fact.

Corollary 2.3. Let $f$ be the $C^\infty$-function over $\mathbb{R}^l$ defined by
\[
f(y_1, \cdots, y_l) := \int_1^{y_1} (a \log s + b) ds + c,
\]
where $a, b$ and $c$ are positive constants. Then $(J_0, \omega_{\psi_f}, \Omega_0)$ is a Calabi-Yau structure of $G^C/K^C$.

Proof. By a simple calculation, we have
\[
\text{det} \left( 2x_i x_j \left( \frac{\partial^2 f}{\partial y_1^2} \circ \overrightarrow{\rho} \right) + \left( \frac{\partial f}{\partial y_1} \circ \overrightarrow{\rho} \right) \cdot \delta_{ij} \right)
= 2 \left( \frac{\partial^2 f}{\partial y_1^2} \circ \overrightarrow{\rho} \right) \cdot \left( \frac{\partial f}{\partial y_1} \circ \overrightarrow{\rho} \right)^{-1} \cdot \rho_1 = \frac{2a}{r}.
\]
Hence, it follows from (ii) of Theorem 2.2 that \((J_0, \omega_{\psi_f}, \Omega_0)\) is a Calabi-Yau structure of \(G^c/K^c\).

\(\square\)

**Remark 2.2.** For \(f\) as in (2.8), \(\beta_{\psi_f}\) coincides with the Stenzel metric in the case where \(G/K = SO(n + 1)/SO(n)(= S^n)\).

3. **Hamiltonian actions and the moment maps**

Let \((M, \omega)\) be a symplectic manifold and the action \(H \act M\) of a Lie group \(H\) on \((M, \omega)\). This action \(H \act M\) is called a Hamiltonian action if it satisfies the following conditions (i)~(iii):

(i) For any \(h \in H\), \(h^*\omega = \omega\) holds;

(ii) For any element \(X\) of the Lie algebra \(\mathfrak{h}\) of \(H\), \(i_X^*\omega\) is exact, where \(X^*\) denote the fundamental vector field on \(M\) associated to \(X\), that is,

\[ X^*_p := \frac{d}{dt} \bigg|_{t=0} (\exp tX) \cdot p \quad (p \in M) \]

and \(i_X^*\) denotes the inner product operator by \(X^*\);

(iii) There exists a family \(\{F_X\}_{X \in \mathfrak{h}}\) of \(C^\infty\)-functions over \(M\) such that \(dF_X = i_X^*\omega\ (X \in \mathfrak{h})\) and that the correspondence \(X \mapsto F_X\ (X \in \mathfrak{h})\) is a Lie algebra homomorphism of \(\mathfrak{h}\) into \(C^\infty(M)\).

Here we note that, by the condition (ii), it is assured that there exists a family \(\{F_X\}_{X \in \mathfrak{h}}\) of \(C^\infty\)-functions over \(M\) such that \(dF_X = i_X^*\omega\ (X \in \mathfrak{h})\) and that the correspondence \(X \mapsto F_X\ (X \in \mathfrak{h})\) is linear. For a function \(F\) over \((M, \omega)\), the \(s\)-gradient vector field \(sgrad\) is defined by \(dF(Y) = \omega(sgrad F, Y)\ (Y \in TM)\). Clearly we have \(sgrad F_X = X^*\). The moment map \(\mu : M \to \mathfrak{h}^*\) of this Hamiltonian action is defined by

\[(\mu(p))(X) := F_X(p) \quad (p \in M, \ X \in \mathfrak{h}).\]

Hence the level set \(\mu^{-1}(0)\) is given by

\[(3.1) \quad \mu^{-1}(0) = \bigcap_{X \in \mathfrak{h}} F_X^{-1}(0).\]

Let \((G^c/K^c, J_0, \omega_{\psi_f}, \Omega_0)\) be a Calabi-Yau manifold stated in the previous section and \(H\) be a closed subgroup of \(G\). Denote by \(\mathfrak{h}\) the Lie algebra of \(H\). Let \(n := \dim G/K\). For simplicity, set \(M := G \cdot o(\approx G/K)\), \(M^c := G^c/K^c\) and \(M^d := G^d \cdot o(\approx G^d/K)\). As stated in Introduction, set \(\Psi_p = \text{Exp}_p \circ (J_0)_p\) \((p \in M)\). Set \(M^d_\circ := \Psi_p(T_p(G \cdot o))\) \((p \in M)\), which is the normal umbrella of \(M\) in \((M^c, \beta_A)\). Note that \(M^d_\circ = M^d\).

**Lemma 3.1.** (i) The action \(H \act (M^c, J_0, \omega_{\psi_f}, \Omega_0)\) is a Hamiltonian action and its moment map \(\mu_{\psi_f}\) is given by

\[(3.2) \quad ((\mu_{\psi_f})(q))(X) = -(\text{Im} \overline{\mathcal{F}}_{\psi_f})_q(X^*_q) \quad (q \in M^c, \ X \in \mathfrak{h}),\]
where $\text{Im}(\cdot)$ denotes the imaginary part of $\cdot$.

(ii) The level set $\mu_{\psi_f}^{-1}(0)$ is given by

\begin{equation}
\mu_{\psi_f}^{-1}(0) = \{ q \in M^C | (\text{Im} \overline{\partial}\psi) q(X^*_q) = 0 \ (\forall X \in \mathfrak{h}) \}.
\end{equation}

**Proof.** Since $\omega_{\psi_f}$ is $G$-invariant and $H$ is a closed subgroup, it is $H$-invariant. Set $\alpha := -\text{Im} \overline{\partial}\psi_f$. For each $X \in \mathfrak{h}$, define a function $F_X$ over $M^C$ by $F_X(q) := \alpha_q(X^*_q) \ (q \in M^C)$. Then, for any tangent vector field $Y$ over $M^C$, we have

$$d\alpha(X^*, Y) = X^*(\alpha(Y)) - Y(\alpha(X^*)) - \alpha(\mathcal{L}_X \cdot Y) = \mathcal{L}_X \cdot \alpha(Y) - dF_X(Y),$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to $X^*$. Since $\alpha$ is $H$-invariant, we have $\mathcal{L}_X \cdot \alpha = 0$. Also, we have $d\alpha = -\omega$. Hence we obtain $dF_X = i_X \cdot \omega$. Also, it is clear that the correspondence $X \mapsto F_X \ (X \in \mathfrak{h})$ is a Lie algebra homomorphism of $\mathfrak{h}$ into $C^\infty(M)$. Therefore the action $H \curvearrowright (M^C, J_0, \omega_{\psi_f}, \Omega_0)$ is a Hamiltonian action and its moment map $\mu_{\psi_f}$ is given by

$$\mu_{\psi_f}(q)(X) = F_X(q) = -(\text{Im} \overline{\partial}\psi_f)_q(X^*_q) \ (q \in M^C, X \in \mathfrak{h}).$$

Thus the statement (i) has been shown. The statement (ii) follows from (i) directly. \qed

By using this lemma, we obtain the following fact.

**Lemma 3.2.** Let $f$ be as in (2.8). Then the level set $\mu_{\psi_f}^{-1}(0)$ is given by

\begin{equation}
\mu_{\psi_f}^{-1}(0) = \bigoplus_{p \in M} \Psi_p(T^\perp_p (H \cdot p)),
\end{equation}

where $T^\perp_p (H \cdot p)$ denotes the normal space of $H \cdot p$ in $M$ at $p$. Also, if $\text{cohom} (H \curvearrowright G/K) = r$, then we have $\dim \mu_{\psi_f}^{-1}(0) = n + r$.

**Proof.** Let $(U, (z_i = x_i + \sqrt{-1} y_i)_{i=1}^n)$ be a holomorphic coordinate of $M^C$ such that $\text{Span}\{(\frac{i}{\sqrt{-1}} \partial_{x_i})_p | i = 1, \cdots, n\} = T_p M$ holds for any $p \in U \cap M$. Note that, for $q \in U \cap M^d_{p}$, the following relation holds:

$$q = \Psi_p \left( \sum_{i=1}^n y_i(q) \left( \frac{\partial}{\partial x_i} \right)_p \right).$$
Fix \( p \in M \) and \( q \in U \cap M^d_p \). Take any \( X \in \mathfrak{h} \). Then, by a simple calculation, we have

\[
((\mu_{\psi_f}) (q)) (X) = -2 \left( \log \left( \sum_{i=1}^{n} y_i(q) y_j(q) g_{ij}(p) + 1 \right) + b \right) \sum_{i=1}^{j} X^*_i(q) y_j(q) g_{ij}(p),
\]

(3.5)

where \( g_{ij} := g \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \), \( X^* \) denotes the fundamental vector field on \( M^C \) associated to \( X \) and \( X^*_i \) is the function given by \( X^* = \sum_{i=1}^{n} \left( X_i^* \frac{\partial}{\partial x_i} + \hat{X}_i^* \frac{\partial}{\partial y_i} \right) \).

Hence \( q \in \mu^{-1}_{\psi_f}(0) \) if and only if

\[
g_p \left( \sum_{i=1}^{n} X^*_i(q) \left( \frac{\partial}{\partial x_i} \right)_p \right) \in T_p^+(H \cdot p)
\]

holds for any \( X \in \mathfrak{h} \). On the other hand, \( X \) moves over \( \mathfrak{h} \), \( \sum_{i=1}^{n} X^*_i(p) \left( \frac{\partial}{\partial x_i} \right)_p \) moves over the whole of \( T_p(H \cdot p) \). Therefore \( q \in \mu^{-1}_{\psi_f}(0) \) if and only if

\[
(J_0)_p \left( \left( \text{Exp}_p \left( (J_0)_p(T_pM) \right) \right)^{-1}(q) \right) \in T_p^+(H \cdot p)
\]

holds. From this fact, the relation (3.4) follows.

Let \( U \) be the open subset of \( G/K \) of all regular points of \( H \acts K \). Then \( \prod_{p \in U} \Psi_p \left( T_p^+(H \cdot p) \right) \) is an open subset of \( \mu^{-1}_{\psi_f}(0) \). It is clear that the dimension of this open subset is equal to \( n + r \). Hence we obtain \( \dim \mu^{-1}_{\psi_f}(0) = n + r \). \( \Box \)

4. The actions of symmetric subgroups on complexified symmetric spaces

Let \( (G^C/K^C, J_0, \omega \psi_f, \Omega_0) \) be a Calabi-Yau manifold stated in Section 2. As in the previous section, set \( M := G \cdot o(= G/K), M^C := G^C/K^C, M^d := G^d \cdot o(= G^d/K) \) and \( M^d_p := \Psi_p(T_p(G \cdot o)) \). Let \( H \) be a symmetric subgroup of \( G \) and \( \sigma \) the involutive automorphism of \( G \) satisfying \( (\text{Fix} \sigma)_0 \subset H \subset \text{Fix} \sigma \). The natural action \( H \) of on \( G/K (= M) \) is called a Hermann action. Assume that \( \sigma \circ \theta = \theta \circ \sigma \). Then the action is called a commutative Hermann action.

Set \( n := \dim M \) and denote by \( r \) the cohomogeneity of the action \( H \acts M \). The group \( H \) acts on \( M^C \) as a subaction of the natural action \( G \acts M^C \), where we note that \( G \acts M^C \) is a Hermann type action (this terminology was used
in [Koi1]). It is easy to show that the subaction $H \lhd M^C$ is a Hamiltonian action. Set $q := \text{Ker}(\sigma + \text{id}_b)$. From $\sigma \circ \theta = \theta \circ \sigma$, we have

$$p = p \cap b \oplus p \cap q \quad \text{and} \quad t = t \cap h \oplus t \cap q.$$ 

Take a maximal abelian subspace $b$ of $p \cap q$ and a maximal abelian subspace $a$ of $p$ including $b$. For $\beta \in b^*$, we define $p_\beta$ and $t_\beta$ by

$$t_\beta := \{ v \in t \mid \text{ad}(Z)^2(v) = -\beta(Z)^2v \forall Z \in b \}$$

and

$$p_\beta := \{ v \in p \mid \text{ad}(Z)^2(v) = -\beta(Z)^2v \forall Z \in b \}.$$

Also, we define $\Delta_b \subset b^*$ by

$$\Delta_b := \{ \beta \in b^* \mid p_\beta \neq \{0\} \},$$

which is the root system. Let $(\Delta_b)_+$ be the positive root subsystem of $\Delta_b$ with respect to some lexicographic ordering of $b^*$. Then we have

$$t = 3_t(b) \oplus \left( \bigoplus_{\beta \in (\Delta_b)_+} t_\beta \right),$$

$$p = 3_p(b) \oplus \left( \bigoplus_{\beta \in (\Delta_b)_+} p_\beta \right),$$

$$h = 3_h(b) \oplus \left( \bigoplus_{\beta \in (\Delta_b)_+} h_\beta \right),$$

$$q = 3_q(b) \oplus \left( \bigoplus_{\beta \in (\Delta_b)_+} q_\beta \right),$$

where $3_*(b)$ is the centralizer of $b$ in $t$. Set

$$\Sigma_b := \text{Exp}_o(b), \quad \Sigma_b^d := \text{Exp}_o(\sqrt{-1}b), \quad \Sigma_b^C := \text{Exp}_o(b^C),$$

$$\Sigma_a := \text{Exp}_o(a), \quad \Sigma_a^d := \text{Exp}_o(\sqrt{-1}a) \quad \text{and} \quad \Sigma_a^C := \text{Exp}_o(a^C).$$

Note that $\Sigma_a$ (resp. $\Sigma_a^d$) is included by $B$ (resp. $M^d$) because $\sqrt{-1}p$ is identified with $T_o(M^d)$. Set $H^d := \text{Exp}(h \cap t) \oplus \sqrt{-1}(h \cap p)$, $\theta^d := \theta^C|_{\sigma^d}$, $\sigma^d := \sigma^C|_{\sigma^d}$, $L := \text{Fix}(\sigma \circ \theta)$ and $L^d := \text{Fix}(\sigma^d \circ \theta^d)$. The normal umbrella $\text{Exp}_o(T_o^+(H^d \cdot o))$ of $H^d \cdot o$ in $M^d$ is isometric to the symmetric space $L^d/H \cap K$ and that the normal umbrella $\text{Exp}_o(T_o^+(H \cdot o) \cap T_oM)$ of $H \cdot o$ in $M$ is isometric to the symmetric space $L/H \cap K$ (see [Koi1, Koi3, Koi4]), where $T_o^+(H^d \cdot o)$ is the normal space of $H^d \cdot o$ in $M^d$ at $o$. It is shown that $T_o(L^d/H \cap K) = \sqrt{-1}(p \cap q)$, $T_o(H^d \cdot o) = \sqrt{-1}(p \cap h)$ and that all orbits of $G \lhd M^C$ meet $\Sigma_a$ orthogonally (see [Koi1, Koi3, Koi4]). Denote by $H_p$ the isotropy group of $H \lhd M$ at $p(\in M)$ and $h_p$ the Lie algebra of $H_p$. Also, let $h_p^\perp$ be the orthogonal complement of $h_p$ in $h$. Set $H_p^\perp := \{ \exp X \mid X \in h_p^\perp \}$. The group $H_p$ acts on the normal umbrella $M_p^d$. First we prove the following fact.
Lemma 4.1. For \( p \in M \) and \( q \in M^d_p \), we have
\[
H \cdot q = \bigcup_{X \in \mathfrak{h}_p^+} H_{(\text{Exp}_o X)_p} \cdot ((\text{Exp}_o X) \cdot q).
\]
Hence \( H \cdot q \) has the structure of the fiber bundle over \( H \cdot p \) with the standard fibre \( H_p \cdot q \) and the structure group \( H_p \).

Proof. Since \( (\text{Exp}_o X) \cdot M^d_p = M^d_{(\text{Exp}_o X)_p} \) holds for any \( X \in \mathfrak{h}_p^+ \), the first relation is derived. For any \( X \in (\text{Exp}_o, X) \), \( H_{(\text{Exp}_o X)_p} \) is conjugate to \( H_p \) and \( H_{(\text{Exp}_o X)_p} \cdot ((\text{Exp}_o X) \cdot q) \) is diffeomorphic to \( H_p \cdot q \). Hence the second-half part of the statement is derived. \( \square \)

Figure 4.

Lemma 4.2 Let \( q \in \Psi_p \left( T^\perp_p (H \cdot p) \right) \) and denote by \( \text{Hol}_{-\Psi_p^{-1}(q)}^\perp (H \cdot p) \) the normal holonomy bundle of the submanifold \( H \cdot p \) in \( M \) through \( -\Psi_p^{-1}(q) \). Then we have
\[
H \cdot q = \Psi \left( \text{Hol}_{-\Psi_p^{-1}(q)}^\perp (H \cdot p) \right).
\]

Proof. It is clear that \( \{(\text{Exp}_o X) \cdot p \mid X \in \mathfrak{h}_p^+ \} = H \cdot p \). Since \( H \lhd G/K \) is hyperpolar, \( \text{Exp}_{(\text{Exp}_o X)_p} \)
\( \left( T^\perp_{(\text{Exp}_o X)_p}(H \cdot p) \right) \) is totally geodesic in \( M \). From this fact, we can show that the orbit \( H_{(\text{Exp}_o X)_p} \cdot (\text{Exp}_o X) \cdot q \) is equal to the image of the fibre of the normal holonomy bundle \( \text{Hol}_{-\Psi_p^{-1}(q)}^\perp (H \cdot p) \) over \( (\text{Exp}_o X) \cdot p \) by \( \Psi_{(\text{Exp}_o X)_p} \). Hence it follows from Lemma 4.1 that \( H \cdot q \) is described as in the statement. \( \square \)
5. Special Lagrangian submanifolds in complexified symmetric spaces

Let \((G^C/K^C, J_0, \omega_{\psi_f}, \Omega_0)\) be the Calabi-Yau manifold stated in Section 2, where \(f\) is as in (i) of Theorem 2.2. As in the previous section, set \(M := G\cdot o(= G/K)\), \(M^C := G^C/K^C\), \(M^d := G^d/K(= G^d\cdot o)\) and \(M_p := \Psi_p(T_p(G\cdot o))\). Let \(H\) be a symmetric subgroup of \(G\) and \(r\) be the cohomogeneity of the Hermann action \(H \cdot \rho G/K\). The naturally extended action of \(H\) on \((M^C, J_0, \omega_{\psi_f}, \Omega_0)\) is a Hamiltonian action. Denote by \(\mu_{\psi_f}\), the moment map of this Hamiltonian action. Let \(Z(h^*)\) be the center of \(g^*\), that is,

\[
Z(h^*) := \{ X \in g^* \mid \text{Ad}^* (h)(X) = X \quad (\forall h \in H) \},
\]

where \(\text{Ad}^*\) denotes the coadjoint representation of \(H\). It is clear that \(\mu_{\psi_f}^{-1}(c)\) is \(H\)-invariant if and only if \(c\) belongs to \(Z(h^*)\). According to Proposition 2.5 of [HS], the following fact holds.

**Proposition 5.1 ([HS]).** Assume that \(L\) is a \(H\)-invariant connected isotropic submanifold in \((M^C, J_0, \omega_{\psi_f}, \Omega_0)\), where “isotropic” means that \(\omega_{\psi_f}(TL, TL) = 0\) holds. Then \(L \subset \mu_{\psi_f}^{-1}(c)\) holds for some \(c \in Z(h^*)\).

In the method of the proof of Proposition 2.6 of [HS], we can show the following fact.

**Proposition 5.2.** Let \(L\) be a \(H\)-invariant connected submanifold in \(M^C\) and \(r_0\) be the cohomogeneity of the action \(H \cdot \rho L\). Assume that \(L \subset \mu_{\psi_f}^{-1}(c)\) for some \(c \in Z(h^*)\) and that there exists a \(r_0\)-dimensional isotropic submanifold \(L_0\) in \((M^C, J_0, \omega_{\psi_f}, \Omega_0)\) satisfying the following conditions:

(i) \(L_0 \subset L\),
(ii) \(L_0\) is transversal to the principal orbits of the action \(H \cdot \rho L\),
(iii) \(H \cdot L_0 = L\).

Then \(L\) also is an isotropic submanifold in \((M^C, J_0, \omega_{\psi_f}, \Omega_0)\).

**Proof.** Take any \(X \in h\) and any \(Y \in T_pL\). From \(L \subset \mu_{\psi_f}^{-1}(c)\), we have \(d(\mu_{\psi_f})_p(Y) = 0\). On the other hand, we have \((d(\mu_{\psi_f})_p(Y))(X) = (\omega_{\psi_f})_p(Y, X^\ast_p)\), where \(X^\ast\) is the vector field on \(M^C\) associated to the one-parameter transformation group \(\{ \exp tX \}_{t \in \mathbb{R}}\) of \(M^C\) \((\exp : \text{the exponential map of } H)\). Hence we have \((\omega_{\psi_f})_p(Y, X^\ast_p) = 0\). Therefore, it follows from the arbitrariness of \(X\) and \(Y\) that \((\omega_{\psi_f})_p(T_pL, T_p(H \cdot p)) = 0\). Also, since \(L_0\) is isotropic, we have \((\omega_{\psi_f})_p(T_pL_0, T_pL_0) = 0\). Hence we obtain \((\omega_{\psi_f})_p(T_pL, T_pL) = 0\). Therefore, it follows from the arbitrariness of \(p\) that \(L\) is isotropic. \(\square\)
By Proposition 2.4 of [HS], we can show the following fact.

**Proposition 5.3.** Let $L$ be a $n$-dimensional connected submanifold in $(M^C, J_0, \omega_{\psi, f}, \Omega_0)$. Then $L$ is a special Lagrangian submanifold of phase $\theta$ if and only if
\[ \omega_{\psi, f}|_{TL \times TL} = 0 \text{ and } \operatorname{Im} \left( e^{\sqrt{-1} \theta} \Omega_0|_{(TL)^n} \right) = 0. \]

Let $f$ be as in (2.8). We give constructions of special Lagrangian submanifolds in the Calabi-Yau manifold $(M^C, J_0, \omega_{\psi, f}, \Omega_0)$. Let $U$ be the open subset of $M$ of all regular points of $H \curvearrowright M$. Then, as stated in the proof of Lemma 3.2, $\Sigma := \bigcup_{p \in U} \Psi_p \left( T^+_p(H \cdot p) \right)$ is an open subset of $\mu_{\psi, f}^{-1}(0)$. Since $H \curvearrowright M$ is a Hermann action, it is hyperpolar (see Subsections 3.1 in [HPTT]). Hence the principal orbit $H \cdot p_0$ ($p_0 \in U$) is an equifocal submanifold in $M$ and its section $\Sigma := \operatorname{Exp}_{p_0} \left( T^+_{p_0}(H \cdot p_0) \right)$ is an $r$-dimensional flat torus $T^r = S^1 \times \cdots \times S^1$ ($r$-times) embedded totally geodesically into $M$. Without loss of generality, we may assume that $\Sigma$ passes through $o$. Let $C$ be the component of $U \cap \Sigma$ containing $p_0$. Then we have $H \cdot C = U$. Set $\Sigma := \bigcup_{p \in U \cap \Sigma} \Psi_p \left( T^+_p(H \cdot p) \right)$. It is clear that $\Sigma$ is a dense open subset of $\Sigma^C := \bigcup_{p \in \Sigma} \Psi_p(T_p \Sigma)$ (\$T^C = S^1_b \times \cdots \times S^1_c\$, where $S^1_b \times \cdots \times S^1_c$ denotes the $r$-times of $S^1_b$’s. We identify $T^r$ and $T^r \times C$ with $\mathbb{C}^r$ and $\mathbb{C}^r$, respectively. Let $\tau_i : I_i \to \mathbb{C}$ ($i = 1, \ldots, r$) be regular curves, where $I_i$ is an open interval. Define an immersion $\tau : I_1 \times \cdots \times I_r \to \mathbb{C}^r$ by $\tau := \tau_1 \times \cdots \times \tau_r$. Set $T := \operatorname{Exp}_o \circ \tau : I_1 \times \cdots \times I_r \to S^1_b \times \cdots \times S^1_c(= \Sigma^C)$). Assume that $(L_r)_0 := T(I_1 \times \cdots \times I_r)$ is included by $\Sigma$. It is clear that $(L_r)_0$ is an isotropic submanifold in $\Sigma^C$ (hence in $(M^C, J_0, \omega_{\psi, f}, \Omega_0)$). Set $L_r := H \cdot (L_r)_0$. For any $p \in U$ and any $q \in \Psi_p \left( T^+_p(H \cdot p) \right)$ ($\subset \Sigma$), since $H \cdot p$ is an equifocal submanifold in $M$, the normal connection of the submanifold $H \cdot p$ in $M$ is flat and hence the (restricted) normal holonomy representation
\[ (H_{(\operatorname{Exp}_p X)_p})_0 \cap T^+_p(\operatorname{Exp}_p X)_p(H \cdot p) \]
is trivial, where $(H_{(\operatorname{Exp}_p X)_p})_0$ denotes the identity component of $H_{(\operatorname{Exp}_p X)_p}$. Hence the action
\[ (H_{(\operatorname{Exp}_p X)_p})_0 \cap \Psi_p(\operatorname{Exp}_p X)_p \left( T^+_p(\operatorname{Exp}_p X)_p(H \cdot p) \right) \]
also is trivial. Therefore it follows from Lemma 4.1 that each component of $H \cdot q$ is diffeomorphic to $\bigcup_{X \in h_p^r} (\operatorname{Exp}_p X) \cdot q$. From this fact, $\dim H \cdot q = n - r$ follows. Since $(L_r)_0$ is included by $\Sigma$, $L_r$ is an $n$-dimensional submanifold of cohomogeneity $r$ in $M^C$. By Proposition 5.2, $L_r$ is a Lagrangian submanifold.
Here we shall explain that the cohomogeneity of the Hamiltonian action $H \curvearrowright M^C$ is possible to be smaller than $(n + r)$. For $\tilde{q} \in \Psi_p(T_p(H \cdot p)) \subset \Sigma$, the (restricted) holonomy representation

$$(H_{(\text{Exp}_pX)_p})_0 \curvearrowright T_{(\text{Exp}_pX)_p}(H \cdot p)$$

of the Riemannian manifold $H \cdot p$ at $(\text{Exp}_pX) \cdot p$ is not necessarily trivial. Hence the action

$$(H_{(\text{Exp}_pX)_p})_0 \curvearrowright \Psi_{(\text{Exp}_pX)_p}(T_{(\text{Exp}_pX)_p}(H \cdot p))$$

also is not necessarily trivial. On the other hand, we can show that $H \cdot \tilde{q}$ is equal to the image of the holonomy bundle $\text{Hol}_{\Psi_p^{-1}(\tilde{q})}(H \cdot p)$ of $H \cdot p$ through $\Psi_p^{-1}(\tilde{q})$ by $\Psi$. From these facts, it follows that $\dim(H \cdot \tilde{q})$ is possible to be larger than $(n - r)$. That is, the cohomogeneity of the action $H \curvearrowright M^C$ is possible to be smaller than $(n + r)$. Set $b := T_p\Sigma$, which is a maximal abelian subspace of $p \cap q$. Note that $\tau$ is rearded as a regular curve in $b^C$ under the identification of $C\tau$ with $b^C$. Let $\Delta_b$, $(\Delta_b)_+$, $\mathfrak{t}_\beta$, $p_\beta$, $h_\beta$ and $q_\beta$ be as in Section 4. Define $(\Delta_b^V)_+$ and $(\Delta_b^H)_+$ by

$$(\Delta_b^V)_+ := \{ \beta \in (\Delta_b)_+ | p_\beta \cap q \neq \{0\} \}$$

and

$$(\Delta_b^H)_+ := \{ \beta \in (\Delta_b)_+ | p_\beta \cap h \neq \{0\} \},$$

respectively. Note that $\dim(p_\beta \cap q) = \dim(\mathfrak{t}_\beta \cap h)$ and $\dim(p_\beta \cap h) = \dim(\mathfrak{t}_\beta \cap q)$. Set $m_\beta^V := \dim(p_\beta \cap q) (\beta \in (\Delta_b^V)_+)$ and $m_\beta^H := \dim(p_\beta \cap h) (\beta \in (\Delta_b^H)_+)$. Let $\{X_{\beta,i}^V \mid i = 1, \cdots, m_\beta^V\}$ be a basis of $\mathfrak{t}_\beta \cap h (\beta \in (\Delta_b^V)_+)$ and $\{X_{\beta,i}^H \mid i = 1, \cdots, m_\beta^H\}$ be a basis of $p_\beta \cap h (\beta \in (\Delta_b^V)_+)$. Also, let $Y_{\beta,i}^V$ be the element of $\mathfrak{t}_\beta \cap h$ such that $\text{ad}(Z)(X_{\beta,i}^V) = \beta(Z)Y_{\beta,i}^V$ holds for any $Z \in \mathfrak{h}$. Define a Killing vector field $(Y_{\beta,i}^V)^*_{\mathfrak{h}}$ over $M^C$ by

$$((Y_{\beta,i}^V)^*_{\mathfrak{h}})_p := \left. \frac{d}{dt} \right|_{t=0} \exp(tY_{\beta,i}^V)(p) \quad (p \in M^C)$$

and a Killing vector field $(X_{\beta,i}^H)^*_{\mathfrak{h}}$ over $M^C$ by

$$((X_{\beta,i}^H)^*_{\mathfrak{h}})_p := \left. \frac{d}{dt} \right|_{t=0} \exp(tX_{\beta,i}^H)(p) \quad (p \in M^C).$$
For $Z \in b^C$, $(Y_{\beta,i})^*_{\text{Exp}_\beta(Z)}$ and $(X_{\beta,i})^*_{\text{Exp}_\beta(Z)}$ are described as

\begin{equation}
(5.1) \quad (Y_{\beta,i})^*_{\text{Exp}_\beta(Z)} = -\sin(\beta^C(Z))(\exp Z)_*(X_{\beta,i})^*
\end{equation}

and

\begin{equation}
(5.2) \quad (X_{\beta,i})^*_{\text{Exp}_\beta(Z)} = \cos(\beta^C(Z))(\exp Z)_*(X_{\beta,i})^*,
\end{equation}
respectively. A basis of $T_{\mathfrak{T}(s)}(H \cdot \mathfrak{T}(s)) = T_{\mathfrak{T}(s)}L_\tau \cap T_{\mathfrak{T}(s)}^1(L_\tau)_0$ is given by
\[
\left( \bigcup_{\beta \in (\Delta^Y)_+} \{ (Y^V_{\beta,i})^*_{\mathfrak{T}(s)} \mid i = 1, \ldots, m^V_\beta \} \right) \\
\bigcup \left( \bigcup_{\beta \in (\Delta^H)_+} \{ (X^H_{\beta,i})^*_{\mathfrak{T}(s)} \mid i = 1, \ldots, m^H_\beta \} \right).
\]
On the other hand, a basis of $T_{\mathfrak{T}(s)}(L_\tau)_0 \cap T_{\mathfrak{T}(s)}(\Sigma^C) (\approx \mathbb{C}^*)$ is given by
\[
\left\{ \frac{d\tau_1}{ds_i} e_1, \ldots, \frac{d\tau_r}{ds_r} e_r \right\},
\]
where $e_i := (0, \ldots, 0, 1, 0, \ldots, 0)$ ($1$ means that $i$-component is equal to 1). Let $(\Delta^Y)_+ = \{ \beta^V_i \mid i = 1, \ldots, k_V \}$ and $(\Delta^H)_+ = \{ \beta^H_i \mid i = 1, \ldots, k_H \}$. From (5.1) and (5.2), we have
\[
(\Omega_0)_{\mathfrak{T}(s)} \left( (Y^V_{\beta^V_1,i})^*_{\mathfrak{T}(s)}, \ldots, (Y^V_{\beta^V_{k_V},i})^*_{\mathfrak{T}(s)}, \ldots, (Y^V_{\beta^V_{m^V_\beta},i})^*_{\mathfrak{T}(s)}, \ldots, (X^H_{\beta^H_1,i})^*_{\mathfrak{T}(s)}, \ldots, (X^H_{\beta^H_{k_H},i})^*_{\mathfrak{T}(s)}, \ldots, (X^H_{\beta^H_{m^H_\beta},i})^*_{\mathfrak{T}(s)}, \ldots \right) \\
= \prod_{\beta \in (\Delta^Y)_+} \sin^{m^V_\beta} (-\beta^C(\mathfrak{T}(s))) \cdot \prod_{\beta \in (\Delta^H)_+} \cos^{m^H_\beta} (\beta^C(\mathfrak{T}(s))) \cdot \prod_{i=1}^r \frac{d\tau_i}{ds_i} \\
\times (\Omega_0)_a \left( X^V_{\beta^V_1,i}, \ldots, X^V_{\beta^V_{m^V_\beta},i}, \ldots, X^V_{\beta^V_{m^V_\beta},i}, \ldots, X^H_{\beta^H_1,i}, \ldots, X^H_{\beta^H_{m^H_\beta},i}, \ldots, X^H_{\beta^H_{m^H_\beta},i}, \ldots \right) \\
(\exp(\mathfrak{T}(s)))^{-1} (e_1), \ldots, (\exp(\mathfrak{T}(s)))^{-1} (e_r)\)
\[= - \prod_{\beta \in (\Delta_+^V)} \sin^{m_{\beta}} \left( \sum_{i=1}^{r} \tau_i(s_i) \beta(e_i) \right) \cdot \prod_{\beta \in (\Delta_+^H)} \cos^{m_{\beta}} \left( \sum_{i=1}^{r} \tau_i(s_i) \beta(e_i) \right) \times \frac{r}{d \delta} \cdot (\Omega_0) \left( X^V_{\beta_1,1}, \ldots, X^V_{\beta_r m_{\beta_1} V}, \ldots, X^V_{\beta_r m_{\beta_r} V}, \right) \]

In the last equality, we used the fact that \( \exp(\tau(s))^{-1}(e_i) = e_i \) \( (i = 1, \ldots, r) \) hold under the identification \( T_{\tau(s)} \Sigma = T_o \Sigma = \mathbb{C}^r \) because \( \exp(\tau(s))^{-1} \) is the parallel translation along the geodesic \( t \mapsto \text{Exp}_o(t \tau(s)) \) in \( \Sigma^C \). It is clear that

\[(\Omega_0) \left( X^V_{\beta_1,1}, \ldots, X^V_{\beta_r m_{\beta_1} V}, \ldots, X^V_{\beta_r m_{\beta_r} V}, \right) \]

is a nonzero real constant independent of \( s = (s_1, \ldots, s_r) \). From these facts and Proposition 5.3, we obtain the following fact for \( L_{\tau} \).

**Theorem 5.4.** The submanifold \( L_{\tau} \) is a special Lagrangian submanifold of phase \( \theta \) if and only if \( \tau_1, \ldots, \tau_r \) satisfy the following ordinary differential equation:

\[
(5.3) \quad \text{Im} \left( e^{-\sqrt{-1} \theta} \prod_{\beta \in (\Delta_+^V)} \sin^{m_{\beta}} \left( \sum_{i=1}^{r} \tau_i(s_i) \beta(e_i) \right) \times \prod_{\beta \in (\Delta_+^H)} \cos^{m_{\beta}} \left( \sum_{i=1}^{r} \tau_i(s_i) \beta(e_i) \right) \cdot \frac{r}{d \delta} \cdot \frac{d \tau_i}{d \delta} \right) = 0.
\]

Next we shall give solutions of the ordinary differential equation (5.3). Let \( \tau_i(s_i) = \varphi_i(s_i) + \sqrt{-1} \rho_i(s_i) \) \( (i = 1, \ldots, r) \), where \( \varphi_i \) and \( \rho_i \) are real-valued
functions. Set

\[ F(\tau_1(s_1), \cdots, \tau_r(s_r))(= F(\varphi_1(s_1), \rho_1(s_1), \cdots, \varphi_r(s_r), \rho_r(s_r))) \]

\[ := e^{\sqrt{-1} \theta} \cdot \prod_{\beta \in (\Delta^r_+) \setminus \{\rho\}} \sin^{m_\beta} \left( \sum_{i=1}^r \tau_i(s_i) \beta(e_i) \right) \times \prod_{\beta \in (\Delta^r_+) \setminus \{\rho\}} \cos^{m_\beta} \left( \sum_{i=1}^r \tau_i(s_i) \beta(e_i) \right), \]

Also, let

\[ F(\varphi_1, \rho_1, \cdots, \varphi_r, \rho_r) = u_0(\varphi_1, \rho_1, \cdots, \varphi_r, \rho_r) + \sqrt{-1} V_0(\varphi_1, \rho_1, \cdots, \varphi_r, \rho_r) \]

and

\[ \int F(\varphi_1, \rho_1, \cdots, \varphi_r, \rho_r) \, d\varphi_1 = U_0(\varphi_1, \rho_1, \cdots, \varphi_r, \rho_r) + \sqrt{-1} V_0(\varphi_1, \rho_1, \cdots, \varphi_r, \rho_r), \]

where \( u_0, v_0, U_0 \) and \( V_0 \) are real-valued functions. Define \( u_1 \) and \( v_1 \) by

\[ u_1(\varphi_1(s_1), \rho_1(s_1), \varphi'_1(s_1), \rho'_1(s_1), \varphi_2(s_2), \rho_2(s_2), \cdots, \varphi_r(s_r), \rho_r(s_r)) \]

\[ := \frac{\partial}{\partial s_1} (U_0(\varphi_1(s_1), \rho_1(s_1), \cdots, \varphi_r(s_r), \rho_r(s_r))) \]

and

\[ v_1(\varphi_1(s_1), \rho_1(s_1), \varphi'_1(s_1), \rho'_1(s_1), \varphi_2(s_2), \rho_2(s_2), \cdots, \varphi_r(s_r), \rho_r(s_r)) \]

\[ := \frac{\partial}{\partial s_1} (V_0(\varphi_1(s_1), \rho_1(s_1), \cdots, \varphi_r(s_r), \rho_r(s_r))). \]

It is clear that

\[ u_1(\varphi_1(s_1), \rho_1(s_1), \varphi'_1(s_1), \rho'_1(s_1), \varphi_2(s_2), \rho_2(s_2), \cdots, \varphi_r(s_r), \rho_r(s_r)) \]

\[ + \sqrt{-1} v_1(\varphi_1(s_1), \rho_1(s_1), \varphi'_1(s_1), \rho'_1(s_1), \varphi_2(s_2), \rho_2(s_2), \cdots, \varphi_r(s_r), \rho_r(s_r)) \]

\[ = \frac{\partial}{\partial s_1} \left( \left( \int F(\varphi_1, \rho_1, \cdots, \varphi_r, \rho_r) \, d\varphi_1 \right)(\varphi_1(s_1), \rho_1(s_1), \cdots, \varphi_r(s_r), \rho_r(s_r)) \right). \]

Let

\[ \int (u_1(\varphi_1, \varphi'_1, \rho_1, \rho'_1, \varphi_2, \rho_2, \cdots, \varphi_r, \rho_r) \]

\[ + \sqrt{-1} v_1(\varphi_1, \varphi'_1, \rho_1, \rho'_1, \varphi_2, \rho_2, \cdots, \varphi_r, \rho_r) ) \, d\varphi_2 \]

\[ = U_1(\varphi_1, \varphi'_1, \rho_1, \rho'_1, \varphi_2, \rho_2, \cdots, \varphi_r, \rho_r) \]

\[ + \sqrt{-1} V_1(\varphi_1, \varphi'_1, \rho_1, \rho'_1, \varphi_2, \rho_2, \cdots, \varphi_r, \rho_r), \]
where $U_1$ and $V_1$ are real-valued functions. In the sequel, we define $u_i, v_i, U_i$ and $V_i$ ($i = 2, \cdots, r$) by repeating the same process. Set
\[
\hat{F}(\varphi_1, \rho_1, \ldots, \varphi_r, \rho_r) := \int \cdots \int F(\varphi_1, \rho_1, \ldots, \varphi_r, \rho_r) \, d\varphi_1 \cdots d\varphi_r.
\]

It is easy to show that
\[
(u_r + \sqrt{-1}v_r)(\varphi_1(s_1), \rho_1(s_1), \varphi'_1(s_1), \rho'_1(s_1), \ldots, \\
\varphi_r(s_r), \rho_r(s_r), \varphi'_r(s_r), \rho'_r(s_r)) = \frac{\partial^r}{\partial s_1 \cdots \partial s_r} \left( \hat{F}(\varphi_1(s_1), \rho_1(s_1), \ldots, \varphi_r(s_r), \rho_r(s_r)) \right)
\]

**Corollary 5.5.** Let $F$ be the complex-valued function over $\mathbb{R}^{2r}$ defined by
\[
F(\varphi_1, \rho_1, \ldots, \varphi_r, \rho_r) := e^{\sqrt{-1}y} \cdot \prod_{\beta \in (\Delta^u_+)} \sin^{m_\beta} \left( \sum_{i=1}^r (\varphi_i + \sqrt{-1}\rho_i) : \beta(e_i) \right) \\
\cdot \prod_{\beta \in (\Delta^u_+)} \cos^{m_\beta} \left( \sum_{i=1}^r (\varphi_i + \sqrt{-1}\rho_i) : \beta(e_i) \right).
\]

If $\tau_i(s_i) = \varphi_i(s_i) + \sqrt{-1}\rho_i(s_i)$ ($i = 1, \ldots, r$) satisfy
\[
\operatorname{Im} \left( \hat{F}(\varphi_1(s_1), \rho_1(s_1), \ldots, \varphi_r(s_r), \rho_r(s_r)) \right) = 0,
\]
then they are a solution of (5.3) and hence $L_\tau$ ($\tau := \tau_1 \times \cdots \times \tau_r$) is a special Lagrangian submanifold of phase $\theta$.

**Proof.** Since $F$ is a holomorphic function, we have $\frac{\partial u_0}{\partial \varphi_1} = \frac{\partial v_0}{\partial \rho_1}$ and $\frac{\partial u_0}{\partial \rho_1} = -\frac{\partial v_0}{\partial \varphi_1}$. From these relations and the definitions of $U_0$ and $V_0$, we have
\[
\frac{\partial U_0}{\partial \varphi_1} = \frac{\partial V_0}{\partial \rho_1} = u_0, \quad \frac{\partial U_0}{\partial \rho_1} = -\frac{\partial V_0}{\partial \varphi_1} = v_0.
\]

Hence we obtain
\[
F(\tau_1(s_1), \ldots, \tau_r(s_r)) \cdot \tau'_1(s_1) \\
= \frac{\partial}{\partial s_1} \left( \int F(\varphi_1, \rho_1, \ldots, \varphi_r, \rho_r) \, d\varphi_1 \right) (\varphi_1(s_1), \rho_1(s_1), \ldots, \varphi_r(s_r), \rho_r(s_r))
\]

Since $u_1 + \sqrt{-1}v_1$ is holomorphic with respect to $\tau_2(= \varphi_2 + \sqrt{-1}\rho_2)$, we have $\frac{\partial u_1}{\partial \varphi_2} = \frac{\partial v_1}{\partial \rho_2}$ and $\frac{\partial u_1}{\partial \rho_2} = -\frac{\partial v_1}{\partial \varphi_2}$. From these relations and the definitions of $U_1$ and $V_1$, we have
\[
\frac{\partial U_1}{\partial \varphi_2} = \frac{\partial V_1}{\partial \rho_2} = u_1, \quad \frac{\partial U_1}{\partial \rho_2} = -\frac{\partial V_1}{\partial \varphi_2} = v_1.
\]
Hence we obtain

\[
F(\tau_1(s_1), \ldots, \tau_r(s_r)) \cdot \tau'_1(s_1) \cdot \tau'_2(s_2) = \\
\frac{\partial^2}{\partial s_1 \partial s_2} \left( \int F(\varphi_1, \rho_1, \ldots, \varphi_r, \rho_r) d\varphi_1 d\varphi_2 \right) (\varphi_1(s_1), \rho_1(s_1), \ldots, \varphi_r(s_r), \rho_r(s_r)).
\]

In the sequel, by repeating the same discussion, we obtain

\[
F(\tau_1(s_1), \ldots, \tau_r(s_r)) \cdot \tau'_1(s_1) \cdots \tau'_r(s_r) = \\
\frac{\partial^r}{\partial s_1 \cdots \partial s_r} \left( \hat{F} (\varphi_1(s_1), \rho_1(s_1), \ldots, \varphi_r(s_r), \rho_r(s_r)) \right).
\]

From this relation, we can derive the statement of this corollary directly. □

We consider the case where \( N = G/K \) is an \( dm \)-dimensional simply connected rank one symmetric space of compact type and constant maximal sectional curvature \( 4c \), that is, \( N = F \text{P}^m(4c) \) (\( F = \mathbb{C}, \mathbb{Q} \) or \( \mathbb{O} \)), where \( \mathbb{Q} \) (resp. \( \mathbb{O} \)) denotes the quaternion algebra (resp. the Octonian) and \( d \) is given by 

\( d = 2 \) (when \( F = \mathbb{C} \)), \( d = 4 \) (when \( F = \mathbb{Q} \)) or \( d = 8 \) (when \( F = \mathbb{O} \)). Note that

\[
\text{FP}^m(4c) = \begin{cases} 
SU(m + 1)/SU(1) \times U(m) & (F = \mathbb{C}) \\
Sp(m + 1)/(Sp(1) \times Sp(m)) & (F = \mathbb{Q}) \\
F_4/Spin(9) & (F = \mathbb{O}, m = 2).
\end{cases}
\]

In these cases, we have \( \Delta_+ = \{ \sqrt{c} e^*, 2\sqrt{c} e^* \} \), where \( e^* \) denotes the dual 1-form of the unit vector \( e \) of \( b \) (\( \dim b = 1 \) in these cases). Also, we have \( \dim p_{\sqrt{c} e^*} = d(m - 1) \) and \( \dim p_{2\sqrt{c} e^*} = d - 1 \). Hence, as a corollary of Theorem 5.4, we obtain the following fact.

**Corollary 5.6.** Let \( H \subset \text{FP}^m(4c) \) be a Hermann action. Then the submanifold \( L_r \) is a special Lagrangian submanifold of phase \( \theta \) if and only if \( \tau \) satisfies the following ordinary differential equation:

\[
(5.4) \quad \text{Im} \left( e^{-i\tau}\cdot \sin^{m V} (\sqrt{c} \tau(s)) \cdot \sin^{d-1} (2\sqrt{c} \tau(s)) \cdot \cos^{m H} (\sqrt{c} \tau(s)) \cdot \frac{d\tau}{ds} \right) = 0,
\]

where \( m^V \) (resp. \( m^H \)) denotes \( m^V_{\sqrt{c} e^*} \) (resp. \( m^H_{2\sqrt{c} e^*} \)).

**Proof.** Let \( \{ J_1, \ldots, J_{d-1} \} \) be the complex structure, the canonical local basis of the quaternionic structure or the Cayley structure of \( \text{FP}^m(4c) \). Then, since \( \text{FP}^m(4c) \) is of rank one, this action is of cohomogeneity one. It is shown that this action is commutative (i.e., \( \theta \circ \sigma = \sigma \circ \theta \)). In fact, Hermann actions on \( \text{FP}^m(4c) \) are classified as in Table 1 and all of Hermann actions in Table 1 are commutative. Since \( H \subset \text{FP}^m(4c) \) is commutative, it is shown that \( H \cdot o \) and the normal umbrella \( \text{Exp}_o(T_o^\perp (H \cdot o)) \) are reflective submanifolds, that is, they are \( J_i \)-invariant \((i = 1, \ldots, d - 1)\). This implies that \( T_o^\perp (H \cdot o) = p \cap q \) includes \( p_{2\sqrt{c} e^*} \). Hence we have \( m_{2\sqrt{c} e^*}^V = d - 1 \) and \( m_{2\sqrt{c} e^*}^H = 0 \). Therefore, the statement of this corollary follows from Theorem 5.4 directly. □
K. Hashimoto and K. Mashimo ([HM]) gave the ordinary differential equation corresponding to (5.3) for the Hamiltonian action $K \xrightarrow{H} TS^n(1)(= SO(n+1, \mathbb{C})/SO(n, \mathbb{C}))$ induced from the restricted action $K \xrightarrow{H} S^n(1)(= SO(n+1)/SO(n))$ of the linear isotropy action of any irreducible rank two symmetric space $G/K$ (see Theorem 5.2 of [HM]), where $n := \dim G/K - 1$ and $S^n(1)$ is the unit sphere of $T_o(G/K)$. Here we note that the action $K \xrightarrow{H} S^n(1)$ is a non-Hermann action of cohomogeneity one (stated in (ii) of Theorem A in [Kol]).

Recently, M. Arai and K. Baba ([AB]) gave the ordinary differential equation corresponding to (5.3) for the Hamiltonian action $K \xrightarrow{H} T\mathbb{C}P^n(4)(= SL(n+1, \mathbb{C})/(SL(1, \mathbb{C}) \times SL(n, \mathbb{C}))$ induced from the action $K \xrightarrow{H} \mathbb{C}P^n(4)$ (see Theorems 2.1-2.4 of [AB]).

According to the classification of cohomogeneity one actions on irreducible spaces $G/K$ of compact type such that $G$ is simple (i.e., $G/K$ is of type I in [H]) by A. Kollross (see Theorem B of [Kol]), any Hermann action on $\mathbb{C}P^n(4c)$ is orbit equivalent to one of Hermann actions in Table 1.

| $G/K(= \mathbb{F}P^n(4c))$ | isotropy action |
|---------------------------|----------------|
| $S(U(1) \times U(m)) \xrightarrow{H} SU(m+1)/S(U(1) \times U(m))$ | – |
| $SO(m+1) \xrightarrow{H} SU(m+1)/S(U(1) \times U(m))$ | isotropy action |
| $Sp(1) \times Sp(m) \xrightarrow{H} Sp(m+1)/(Sp(1) \times Sp(m))$ | isotropy action |
| $Sp(m) \times Sp(1) \xrightarrow{H} Sp(m+1)/(Sp(1) \times Sp(m))$ | – |
| $U(m+1) \xrightarrow{H} Sp(m+1)/(Sp(1) \times Sp(m))$ | – |
| $Spin(9) \xrightarrow{H} F_4/Spin(9)$ | isotropy action |
| $Sp(3) \cdot Sp(1) \xrightarrow{H} F_4/Spin(9)$ | – |

**Table 1 : Hermann actions on $\mathbb{F}P^n(4c)$**

For all Hermann actions of cohomogeneity two on irreducible rank two symmetric spaces of compact type, we shall give the following datas:

$$(\Delta_b)_+, (\Delta_b)^V, (\Delta_b)^H, m_{\beta}^V (\beta \in (\Delta_b)_+), m_{\beta}^H (\beta \in (\Delta_b)^H).$$

All of such Hermann actions and the above datas for the actions are given as in Table 2. By using Table 2, we can explicitly describe the ordinary differential equation (5.3) for the Hermann actions of cohomogeneity two on irreducible rank two symmetric spaces of compact type. In Table 2, in the case where $(\Delta_b)_+ = (\Delta_b)^V \cup (\Delta_b)^H$ is $\{\beta_1, \beta_2, \beta_1 + \beta_2\}$, it implies a positive root system.
of the root system of \((a_2)\)-type \(((\beta_1(e_1), \beta_1(e_2)) = (2, 0), (\beta_2(e_1), \beta_2(e_2)) = (-1, \sqrt{3})\), in the case where \((\Delta_0)_+\) is \(\{\beta_1, \beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2\}\), it implies a positive root system of the root system of \((a_2)\)-type \(((\beta_1(e_1), \beta_1(e_2)) = (1, 0), (\beta_2(e_1), \beta_2(e_2)) = (-1, 1)\) and in the case where \((\Delta_0)_+\) is \(\{\beta_1, \beta_2, \beta_1 + \beta_2, \beta_1 + 2\beta_2, 2\beta_1 + 3\beta_2\}\), it implies a positive root system of the root system of \((a_2)\)-type \(((\beta_1(e_1), \beta_1(e_2)) = (2, \sqrt{3}, 0), (\beta_2(e_1), \beta_2(e_2)) = (-\sqrt{3}, 1)\). Also, \(\rho_i\) \((i = 1, \ldots, 16)\) imply automorphisms of \(G\) whose dual actions are given as in Table 3 and \((\mathbb{m})^2\) implies the product Lie group \((\mathbb{m}) \times (\mathbb{m})\) of a Lie group \((\mathbb{m})\), \(\mathbb{m}\) in the column of \((\Delta_0)_+\), \((\Delta_0)_+^V\) and \((\Delta_0)_+^H\) imply \(m^V = \mathbb{m}^H = \mathbb{m}\), respectively. Note that Tables 2 and 3 are based on Tables 1 and 2 in [Koi2].

| \(H \cap G/K\) | \((\Delta_0)_+^V\) | \((\Delta_0)_+^H\) |
|-----------------|-----------------|-----------------|
| \(\rho_1(SO(4)) \cap SU(3)/SO(3)\) | \(\{\beta_1\}\) | \(\{\beta_2, \beta_1 + \beta_2\}\) |
| \(SO(6) \cap SU(6)/Sp(4)\) | \(\{\beta_1, \beta_2, \beta_1 + \beta_2\}\) | \(\{\beta_2, \beta_1 + \beta_2\}\) |
| \(\rho_2(Sp(3)) \cap SU(6)/Sp(3)\) | \(\{\beta_1\}\) | \(\{\beta_2, \beta_1 + \beta_2\}\) |
| \(SO(q + 2) \cap SU(q + 2)/SU(2) \times SU(q)\) | \(\{\beta_1, \beta_2, \beta_1 + \beta_2, \beta_1 + 2\beta_2, 2\beta_1 + 3\beta_2\}\) | \(\{\beta_1, \beta_2, \beta_1 + 2\beta_2, 2\beta_1 + 3\beta_2\}\) |
| \(S(U(j + 1) \cap U(q + j + 1))\) | \(\{\beta_1, \beta_2, \beta_1 + \beta_2, \beta_1 + 2\beta_2, 2\beta_1 + 3\beta_2\}\) | \(\{\beta_1, \beta_2, \beta_1 + 2\beta_2, 2\beta_1 + 3\beta_2\}\) |
| \(S(U(q + 2)/SU(2) \times SU(q)\) | \(\{\beta_1, \beta_2, \beta_1 + \beta_2\}\) | \(\{\beta_1, \beta_2, \beta_1 + 2\beta_2, 2\beta_1 + 3\beta_2\}\) |
| \(SO(8) \cap SO(4) \times SO(4)\) | \(\{2\beta_1 + \beta_2\}\) | \(\{2\beta_1 + \beta_2\}\) |
| \(\rho_3(SO(4) \times SO(4)) \cap SO(8)/SU(4)\) | \(\{\beta_1, \beta_2, \beta_1 + \beta_2\}\) | \(\{\beta_1, \beta_2, \beta_1 + \beta_2\}\) |
| \(\rho_4(U(4) \cap \mathbb{C} \times SU(8)/SU(4)\) | \(\{\beta_1, \beta_2, \beta_1 + \beta_2\}\) | \(\{\beta_1, \beta_2, \beta_1 + \beta_2\}\) |
| \(SO(4) \times SO(6) \cap SO(10)/SU(5)\) | \(\{2\beta_1 + \beta_2\}\) | \(\{2\beta_1 + \beta_2\}\) |
| \(SO(5) \times SU(5) \cap SO(10)/SU(5)\) | \(\{\beta_1, \beta_2, \beta_1 + \beta_2\}\) | \(\{\beta_1, \beta_2, \beta_1 + \beta_2\}\) |

Table 2: Hermann actions on rank two symmetric spaces
Table 2: Hermann actions on rank two symmetric spaces (continued)

| $H \curvearrowright G/K$ | $(\Delta)_h^U \cdot \mathcal{m}_V^U$ | $(\Delta)_h^H \cdot \mathcal{m}_V^H$ |
|--------------------------|-------------------------------------|-------------------------------------|
| $\rho_s(U(5)) \sim SO(10)/U(5)$ | $\{ \beta_1, 2\beta_1, 2\beta_1 + 2\beta_2 \} (1) (4)$ | $\{ 2\beta_1, 2\beta_2, 2\beta_1 + 2\beta_2 \} (4) (1)$ |
| $SO(2)^2 \times SO(3)^2 \sim$ | $\{ \beta_1, \beta_2, \beta_1 + \beta_2, 2\beta_1 \} (1) (1) (1)$ | $\{ \beta_1, \beta_2, \beta_1 + \beta_2, 2\beta_1 \} (1) (1) (1)$ |
| $(SO(5) \times SO(5))/SO(5)$ | $\rho_s(SO(5)) \sim$ | $\rho_s(U(3)) \sim Sp(2(2)/U(2)$ |
| $Sp(2)^2 \times Sp(2) \times Sp(2)$ | $\{ \beta_1, \beta_1 + \beta_2 \} (1) (1)$ | $\{ \beta_1, 2\beta_1 + \beta_2 \} (1) (1)$ |
| $SU(q + 2) \sim$ | $SU(4) \sim$ | $SU(4)/Sp(2) \times Sp(2)$ |
| $Sp(q + 2)/Sp(2) \times Sp(q)$ | $Sp(4)/Sp(2) \times Sp(2)$ | $Sp(j + 1) \times Sp(q - j + 1) \sim$ |
| $(q > 2)$ | $(2) (2) (1) (1)$ | $\{ \beta_1, \beta_1 + \beta_2 \} (2) (2) (1) (1)$ |
| $Sp(q + 2)/Sp(2) \times Sp(q)$ | $Sp(4)/Sp(2) \times Sp(2)$ | $(q > 2)$ |
| $(q > 2)$ | $2\beta_1 + \beta_2 \} (2) (2) (1) (1)$ | $2\beta_1 + \beta_2 \} (2) (2) (1) (1)$ |
| $Sp(2) \times Sp(2) \sim$ | $Sp(4)/Sp(2) \times Sp(2)$ | $Sp(2) \times Sp(2) \sim$ |
| $(q > 2)$ | $\{ \beta_1, \beta_1 + \beta_2 \} (3) (3) (3) (3)$ | $\{ \beta_1, \beta_1 + \beta_2 \} (3) (3) (3) (3)$ |
| $SU(2)^2 \cdot SO(2)^2 \sim$ | $SU(2)^2 \cdot SO(2)^2 \sim$ | $\rho_s(Sp(2)) \sim$ |
| $(Sp(2) \times Sp(2))/Sp(2)$ | $(Sp(2) \times Sp(2))/Sp(2)$ | $(Sp(2) \times Sp(2))/Sp(2)$ |
| $\{ \beta_1, \beta_1 + \beta_2 \} (2) (2) (2) (2)$ | $\{ \beta_1, \beta_1 + \beta_2 \} (2) (2) (2) (2)$ | $\{ \beta_1, \beta_1 + \beta_2 \} (2) (2) (2) (2)$ |
| $H \rtimes G/K$                              | $(\Delta_H)^{\upsilon}$, $m_{\upsilon}$ | $(\Delta_H)^{\mu}$, $m_{\mu}$ |
|---------------------------------------------|------------------------------------------|---------------------------------|
| $\rho_0(\text{Sp}(2)) \rtimes (\text{Sp}(2) \times \text{Sp}(2))/\text{Sp}(2)$ | {\beta_1, \beta_1 + \beta_2}           | {\beta_1, 2\beta_1 + \beta_2}  |
| $\rho_1(\text{Spin}(10) \cdot U(1))$      | {\beta_1, \beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2} | {\beta_1, \beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2} |
| $\rho_{10}(\text{SU}(6) \cdot \text{SU}(2)) \rtimes E_6/\text{Spin}(10) \cdot U(1)$ | {\beta_1, \beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2} | {\beta_1, \beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2} |
| $\rho_{111}(\text{Spin}(10) \cdot U(1)) \rtimes E_6/\text{Spin}(10) \cdot U(1)$ | {\beta_1, \beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2} | {\beta_1, \beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2} |
| $\rho_{12}(\text{Spin}(10) \cdot U(1)) \rtimes E_6/\text{Spin}(10) \cdot U(1)$ | {\beta_1, \beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2} | {\beta_1, \beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2} |
| $\rho_{13}(F_4) \rtimes E_6/F_4$ | {\beta_1} | {\beta_1, \beta_1 + \beta_2} |
| $\rho_{14}(\text{SO}(4)) \rtimes G_2/\text{SO}(4)$ | {\beta_1, 3\beta_1 + 2\beta_2} | {\beta_1, \beta_1 + \beta_2, 2\beta_1 + \beta_2, 3\beta_1 + \beta_2} |
| $\rho_{15}(\text{SO}(4)) \rtimes G_2/\text{SO}(4)$ | {\beta_1, 3\beta_1 + 2\beta_2} | {\beta_1, \beta_1 + \beta_2, 2\beta_1 + \beta_2, 3\beta_1 + \beta_2} |
| $\rho_6(G_2) \rtimes (G_2 \times G_2)/G_2$ | {\beta_1, 3\beta_1 + 2\beta_2} | {\beta_1, \beta_1 + \beta_2, 2\beta_1 + \beta_2, 3\beta_1 + \beta_2} |
| $\rho_7(\text{SU}(2)^3) \rtimes (G_2 \times G_2)/G_2$ | {\beta_1, \beta_2, \beta_1 + \beta_2} | {\beta_1, \beta_2, \beta_1 + \beta_2} |

Table 2: Hermann actions on rank two symmetric spaces (continued)
Table 3: The dual actions of $\rho_i$

| $H \sim G/K$ | $H^* \sim G^*/K$ |
|----------------|------------------|
| $\rho_1(SO(3)) \sim SU(3)/SO(3)$ | $SO_0(1,2) \sim SL(3,R)/SO(3)$ |
| $\rho_2(Sp(3)) \sim SU(6)/Sp(3)$ | $Sp(1,2) \sim SU^*(6)/Sp(3)$ |
| $\rho_3(SO(4) \times SO(4)) \sim SO(8)/U(4)$ | $SO(4,C) \sim SO^*(8)/U(4)$ |
| $\rho_4(U(4)) \sim SO(8)/U(4)$ | $U(2,2) \sim SO^*(8)/U(4)$ |
| $\rho_5(U(5)) \sim SO(10)/U(5)$ | $U(2,3) \sim SO^*(10)/U(5)$ |
| $\rho_6(SO(5)) \sim (SO(5) \times SO(5))/SO(5)$ | $SO_0(2,3) \sim SO(5,C)/SO(5)$ |
| $\rho_7(U(2)) \sim Sp(2)/U(2)$ | $U(1,1) \sim Sp(2,R)/U(2)$ |
| $\rho_8(Sp(2)) \sim (Sp(2) \times Sp(2))/Sp(2)$ | $Sp(2,R) \sim Sp(2,C)/Sp(2)$ |
| $\rho_9(Sp(2)) \sim (Sp(2) \times Sp(2))/Sp(2)$ | $Sp(1,1) \sim Sp(2,C)/Sp(2)$ |
| $\rho_{10}(SU(6) \cdot SU(2)) \sim E_6/Spin(10) \cdot U(1)$ | $SU(1,5) \cdot SL(2,R) \sim E_6^{14}/Spin(10) \cdot U(1)$ |
| $\rho_{11}(Spin(10) \cdot U(1)) \sim E_6/Spin(10) \cdot U(1)$ | $SO^{10}(10)/U(1) \sim E_6^{14}/Spin(10) \cdot U(1)$ |
| $\rho_{12}(Spin(10) \cdot U(1)) \sim E_6/Spin(10) \cdot U(1)$ | $SO_0(2,8)/U(1) \sim E_6^{14}/Spin(10) \cdot U(1)$ |
| $\rho_{13}(F_4) \sim E_6/F_4$ | $F_4^{20} \sim E_6^{20}/F_4$ |
| $\rho_{14}(SO(4)) \sim G_2/SO(4)$ | $SL(2,R) \times SL(2,R) \sim G_2^2/SO(4)$ |
| $\rho_{15}(SO(4)) \sim G_2/SO(4)$ | $\rho_{16}(SO(4)) \sim G_2^2/SO(4)$ |
| $\rho_{16}(G_2) \sim (G_2 \times G_2)/G_2$ | $G_2^2 \sim G_2^2/G_2$ |

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