STRATIFYING $q$-SCHUR ALGEBRAS OF TYPE $D$

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Abstract. Two families of $q$-Schur algebras associated to Hecke algebras of type $D$ are introduced, and related to a family used by Geck, Gruber and Hiss [10], [11]. We prove that the algebras in one family, called the $q$-Schur$^{1.5}$ algebras, are integrally free, stable under base change, and are standardly stratified if the base field has odd characteristic. In the so-called linear prime case of [10],[11], all three families give rise to Morita equivalent algebras. A final section discusses a different example, and speculates on the direction of a general theory.

Following up our recent work [9] for type $B$ (and $C$), we introduce here some endomorphism algebras of type $D$. These are of possible use in determining irreducible representations of finite groups of Lie type $D$, especially in non-defining characteristics, as in the work of Geck-Hiss [11] and Gruber-Hiss [1] following the spirit of Dipper-James' work in type $A$. (See Remark 2.11 below for further discussion.)

We organize the paper as follows: Section 1 sets up preparation on Weyl groups of classical types and distinguished coset representatives, especially those with the trivial intersection property. The various $q$-Schur algebras are introduced in Section 2 and connections between them are also discussed. The linear prime Morita equivalence theorems are proved in Section 3. In Sections 4, 5 and 6, we establish further results on $q$-permutation and twisted $q$-permutation modules of type $B$. These results are supplementary to those given in [9]. In particular, we prove, in the type $B$ setting, the homological property required (see [9]) in stratifying an endomorphism algebra for type $D$. The main results are given in Section 7, where the quasi-heredity in the odd degree case is proved for one of our algebras, called the $q$-Schur$^{1.5}$ algebra. A standard stratification in the even degree case is constructed for the same algebra when 2 is invertible in the base ring. Section 8 gives an effective approach to the bad prime case $p = 2$, and concludes with speculations regarding a general theory.

Date: 6 December, 1999.

1991 Mathematics Subject Classification. 20G05, 16S80.

The authors would like to thank ARC for support as well as NSF, and the Universities of Virginia and New South Wales for their cooperation.
A preliminary version of the work was posted at the Newton Institute workshop, June 1997, for the program on the representation theory of algebraic and related finite groups.

1. Weyl groups of classical types

**Notation.** Consider the following Coxeter systems:

- \((W_r, S)\) the Weyl group of type \(B_r\) with \(S = \{s_0, s_1, \ldots, s_{r-1}\}\)
- \((\tilde{W}_r, \tilde{S})\) the Weyl group of type \(D_r\) with \(\tilde{S} = \{u, s_1, \ldots, s_{r-1}\}\)
- \((\tilde{W}_r, \tilde{S})\) the Weyl group of type \(A_{r-1}\) with \(\tilde{S} = \{s_1, \ldots, s_{r-1}\}\)

We often drop the subscript \(r\) and will choose \(u = s_0 s_1 s_0 \in W\) in the sequel. Then \(\tilde{W} \subseteq \tilde{W} \subseteq W\). Let \(t_1 = s_0\), \(t_i = s_{i-1} t_{i-1} s_{i-1}\), \(u_i = t_1 t_i\) for \(2 \leq i \leq r\), and let \(C = \langle t_1, \ldots, t_r \rangle\) and \(\tilde{C} = \langle u_2, \ldots, u_r \rangle\). Then, we have \(W = C \times \tilde{W}\) and \(\tilde{W} = \tilde{C} \times W\). Moreover, we may identify \(\tilde{W}\) with the symmetric group \(S_r\) on \(r\) letters.

Let \(\ell\) (resp. \(\tilde{\ell}\)) be the length function on \(W\) (resp. \(\tilde{W}\)) with respect to \(S\) (resp. \(\tilde{S}\)) and \(n_0\) the function giving the number of \(s_0\) in a reduced expression of an element of \(W\) (see [3, \$2.1]). It is well-known that \(W\) identifies with a subgroup of the symmetric group \(S_{2r}\) (see, e.g., [3, §2.1W3]). In this identification, the restriction to \(W\) of the signature function on \(S_{2r}\) induces the group homomorphism \(\rho_0 : W \rightarrow \{1, -1\}\) defined by \(\rho_0(w) = (-1)^{n_0(w)}\) for all \(w \in W\). The kernel of \(\rho_0\), \(\ker(\rho_0)\), is the intersection of \(W\) with the alternating group \(A_{2r}\). Clearly, the subgroup \(\ker(\rho_0)\) is generated by the set \(\{s_0 s_1 s_0, s_1, \ldots, s_{r-1}\}\), and is isomorphic to the Weyl group \(\tilde{W}\) of type \(D_r\) by the map sending \(u\) to \(s_0 s_1 s_0\) and \(s_i\) to \(s_i\). This agrees with our identification above of \(\tilde{W}\) with a subgroup of \(W\). For convenience, we call \(\tilde{C} = \langle u_2, \ldots, u_r \rangle\) the bottom part of \(\tilde{W}\), where \(u_i = t_1 t_i\) for \(2 \leq i \leq r\). Note that the subgroup generated by \(s_2, \ldots, s_{r-1}\) and \(\tilde{C}\) is the Weyl group of type \(B_{r-1}\). Let \(f\) be the automorphism of \(\tilde{W}\) induced by flipping the Coxeter graph. Then, \(f\) on \(\tilde{W}\) is now the restriction to \(\tilde{W}\) of the inner automorphism \(g \mapsto s_0 g s_0\) of \(W\). Note that \(f\) fixes each element of \(\tilde{C}\), and interchanges the two parabolic copies of \(S_r\).

Fix positive integers \(n, r\), and let \(\Lambda(n, r)\) (resp. \(\Lambda^+(n, r)\)) be the set of compositions (resp. partitions) of \(r\) with \(n\) parts (counting zeros in both cases). Thus, \(\Lambda^+(r) = \Lambda^+(r, r)\) is the set of partitions of \(r\). Let \(\Pi_2 = \Pi_2(n, r) = \bigcup_{r_1 + r_2 = r} \Lambda(n, r_1) \times \Lambda(n, r_2)\), where \(n_r\) is the maximum of \(n\) and \(r\). The elements \(\lambda\) in \(\Pi_2\) will be written in the sequel as \(\lambda = (\lambda^{(1)}, \lambda^{(2)})\), and will be called bicompositions of \(r\). Let \(\Pi_2^+\) be the subset consisting of all \((\lambda^{(1)}, \lambda^{(2)}) \in \Pi_2\) such
that both $\lambda^{(1)}$ and $\lambda^{(2)}$ are partitions. The elements in $\Pi_2^+$ are called bipartitions. Put $\Pi^+(r) = \Pi^+(r, r)$. We define
\[
\Pi_1 = \Pi_1(n, r) = \{ \lambda \in \Pi_2 \mid |\lambda^{(2)}| = 0 \}
\]
\[
\Pi_{1b} = \Pi_{1b}(n, r) = \{ \lambda \in \Pi_2 \mid |\lambda^{(1)}| = 0 \},
\]
where $|\lambda^{(i)}|$ denotes the sum of the parts of $\lambda^{(i)}$. Each $\lambda \in \Pi_2$ gives naturally a composition $\hat{\lambda} \in \Lambda(n_r + n, r)$ by juxtaposition. For a bi-composition $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Pi_2$ with $|\lambda^{(1)}| = a$, let $C_\lambda = \langle t_1, \cdots, t_a \rangle$, let $\mathfrak{S}_\hat{\lambda}$ the Young subgroup defined by $\hat{\lambda}$, and define
\[
W_\lambda = C_\lambda \mathfrak{S}_\hat{\lambda}, \quad \hat{W}_\lambda = W_\lambda \cap \hat{W} \quad \text{and} \quad \mathring{W}_\lambda = \mathfrak{S}_\hat{\lambda}.
\]
In the spirit of [9], we shall call these subgroups quasi-parabolic subgroups. Note that $\mathring{W}_\lambda$ is not parabolic. See [13] or [9] for more details.

For $W_\lambda$, the notion of ‘distinguished’ coset representative was introduced in [9, §2]. Following [9], $D_\lambda$ denotes the set of distinguished right $W_\lambda$-coset representatives, and $D_{\lambda, \mu} = D_\lambda \cap D_\mu^{-1}$ the set of distinguished double $W_\lambda$-$W_\mu$-coset representatives. Let $W_\lambda$ be the minimal parabolic subgroup containing $W_\lambda$. (Thus, $\hat{\lambda} = (|\lambda^{(1)}|, \lambda^{(2)})$.) Then, for $d \in D_{\lambda, \mu}$, we may write $d = udv$, where $d \in W_\lambda dW_\mu$ is distinguished (for parabolic subgroups), $u \in W_\lambda$ and $v \in D_{W_\lambda \cap W_\mu} \cap W_\mu$. This is called the Howlett right distinguished decomposition of $d$ (see [9, 2.3]). Putting $D_\lambda = D_\lambda \cap \hat{W}$ and $D_{\lambda, \mu} = D_{\lambda, \mu} \cap \hat{W}$, we obtain distinguished $\hat{W}_\lambda$-coset and double $\hat{W}_\lambda$-$\mathring{W}_\mu$-coset representatives. Let
\[
D_{\lambda, \mu}^0 = \{ d \in D_{\lambda, \mu} \mid W_\lambda^d \cap W_\mu = \{1\} \}
\]
be the set of double coset representatives in $D_{\lambda, \mu}$ with the trivial intersection property. For non-negative integer $a$, let $\omega_a$ denote the bipartition $(1^a, 1^{r-a})$. Then $\hat{\omega}_a = (a, 1^{r-a})$ and $W_a = W_{\hat{\omega}_a}$.

**Lemma 1.1.** Let $a, b$ be non-negative integers.

(a) The set $D_{\omega_a, \omega_b}^0$ is empty unless $a + b \leq r$.

(b) For $x \in W_{(a, r-a)}$, $y \in W_{(b, r-b)}$, we have $d \in D_{\omega_a, \omega_b}^0$ if and only if $xyd \in D_{\omega_a, \omega_b}$.

(c) For $d \in W$, $d \in D_{\omega_a}$ if and only if $n_0(xd) = n_0(x) + n_0(d)$ for all $x \in C_{\omega_a}$, and $d \in D_{\omega_a, \omega_b}^0$ if and only if $n_0(xyd) = n_0(x) + n_0(d) + n_0(y)$ for all $x \in C_{\omega_a}$ and $y \in C_{\omega_b}$.

1A distinguished coset representative $d$ is the unique element of minimal length in $dW_\lambda$ or $W_\lambda d$. It is not necessarily minimal with respect to the Bruhat order if $W_\lambda$ is not parabolic. See [13] or [9] for more details.
For every choice $\varepsilon_j \in \{0, 1\}$ with $1 \leq j \leq i$, we have
\[ t_{a+1}^{\varepsilon_1} \cdots t_{a+i}^{\varepsilon_i} D_{\omega_{a+1}, \omega_{b}}^0 \subseteq D_{\omega_{a}, \omega_{b}}^0. \]

Proof. The statement (a) is obvious, and (b) follows from [11 (2.2.7)]
and the fact that $C_{\omega_{a}}^d \cap C_{\omega_{b}} = \{1\}$ if and only if $C_{\omega_{a}}^{w_0} \cap C_{\omega_{b}} = \{1\}$
for all $x \in W_{a,r-a}$, $y \in W_{b,r-b}$. We now prove (c). Write $d = w d'$
with $w \in W_a$ and $d' \in D_{\omega_a}$. Then $d \in D_{\omega_b}$ if and only if $w \in W_a$, which is
equivalent to $n_0(xd) = n_0(x) + n_0(d)$ for all $x \in C_{\omega_a}$, proving the first
part. Now, let $d = w d'$ be the Howlett decomposition of $d$, where $d$
is the distinguished double coset representative in $W_a d W_b$ (for parabolic
subgroups), $u \in W_a$ and $v \in W_b$. Then, by (b), $d \in D_{\omega_{a}, \omega_{b}}^0$ iff $u \in W_a$
and $v \in W_b$ (or $d \in D_{\omega_{a}, \omega_{b}}$) and $d' \in D_{\omega_{a}, \omega_{b}}$.
The latter is equivalent
to $s_0 \not\in W_a \cap W_b$, which means no cancellation for $s_0$ when writing the
products $xdy$ ($x \in W_a$ and $y \in W_b$) in reduced form. (When $s_0$ is in the
intersection, we have $s_0 d = d s_0$, and cancellation could occur.)

From this together with the first part of (c), we obtain that $d \in D_{\omega_{a}, \omega_{b}}^0$ iff
$n_0(xdy) = n_0(x) + n_0(d) + n_0(y)$ for all $x \in C_{\omega_a}$ and $y \in C_{\omega_b}$, proving
(c). To prove (d), let $w \in D_{\omega_{a+1}, \omega_{b}}$. Putting $t = t_{a+1}^{\varepsilon_1} \cdots t_{a+i}^{\varepsilon_i}$, we have,
for $x \in C_{\omega_a}$ and $y \in C_{\omega_b}$, $n_0(x t y) = n_0(x t) + n_0(w) + n_0(y)$ by (c) since
$w \in D_{\omega_{a+1}, \omega_{b}}$. But $n_0(x t) = n_0(x) + n_0(t)$, and $n_0(t) + n_0(w) = n_0(t w)$.
Therefore, $n_0(x t y) = n_0(x) + n_0(t w) + n_0(y)$ and therefore, $t w \in D_{\omega_{a}, \omega_{b}}$
by (c) again. \hfill \square

Corollary 1.2. Let $a \geq 1$, and assume that $d \in D_{\omega_{a}, \omega_{b}}^0$. Then
\[ \{d, t_a d, s_a d, s_a t_a d\} \subseteq D_{\omega_{a-1}, \omega_{b}}^0. \]

If, in addition, $d \not\in D_{\omega_{a+1}, \omega_{b}} \cup t_{a+1} D_{\omega_{a+1}, \omega_{b}}$, then every element in
\[ \{d, t_a d, s_a d, s_a t_a d\} \] is not in $\cup_{\varepsilon_1 = 0, 1} t_{a+1}^{\varepsilon_1} t_{a+2}^{\varepsilon_2} D_{\omega_{a+1}, \omega_{b}}^0$. In other words, if
we put $U = D_{\omega_{a}, \omega_{b}} \cup t_{a+1} D_{\omega_{a+1}, \omega_{b}}$, we have
\[ \{1, t_a, s_a, s_a t_a\} (D_{\omega_{a}, \omega_{b}}^0 \setminus U) \cap (\cup_{\varepsilon_1 = 0, 1} t_{a+1}^{\varepsilon_1} t_{a+2}^{\varepsilon_2} D_{\omega_{a+1}, \omega_{b}}^0) = \emptyset. \]

Proof. The inclusion can be proved by using Lemma [11c] and the disjunctness follows easily from the facts $C_{\omega_{a+1}}^d \cap C_{\omega_{b}} \neq \{1\}$ and
\[ U = \{x \in D_{\omega_{a}, \omega_{b}}^0 \mid C_{\omega_{a+1}}^x \cap C_{\omega_{b}} = \{1\} \}. \]

(If $x$ is in the right hand side, then we may write $x = tx'$ where $t \in C_{\omega_{a+1}}$ and $x' \in D_{\omega_{a+1}}$. This is just a right coset decomposition. Next,
an argument using Lemma [11c] shows that $x' \in D_{\omega_{a+1}}^{-1}$. That is, $x' \in D_{\omega_{a+1}}$, and even $x' \in D_{\omega_{a+1}, \omega_{b}}^0$. Finally, $x \in D_{\omega_{a}}$ forces $t = t_{a+1}$ or $1$.) \hfill \square
2. \textit{q-Schur algebras of classical types}

We start with the definition of the \textit{q-Schur} algebra of type $B$, that is, the \textit{q-Schur}$_2$ algebra.

Let $\mathbb{Z}_0 = \mathbb{Z}[q, q^{-1}, q_0, q_0^{-1}]$ be the ring of Laurent polynomials in two variables, and let $\mathcal{H} = \mathcal{H}_{q_0,q}$ be the (two-parameter generic) Hecke algebra over $\mathbb{Z}_0$ associated to $W$ with defining basis $\{T_w\}_{w \in W}$. The subalgebra $\overline{\mathcal{H}}$ generated by $T_{s_i}$, $1 \leq i \leq r - 1$, is the Hecke algebra associated to $\overline{W}$. We denote by $\mathcal{H}' = \mathcal{H}'_{q_0,q} = \mathcal{H}_{Z'}$ etc. the Hecke algebras obtained by changing base to a commutative $\mathbb{Z}_0$-algebra $Z'$. Also, for any subset $X$ of $W$, we denote by $\mathcal{H}'(X)$ the $Z'$-submodule generated by all $T_w$, $w \in X$.

Define $\pi_0 = \pi_0^- = 1$, and for $a \geq 1$, let

$$\pi_a = \prod_{i=1}^{a} (q^{i-1} + T_{t_i}), \quad \pi_a^- = \prod_{i=1}^{a} (q_0 q^{i-1} - T_{t_i}).$$

Note that there is a $\mathbb{Z}[q, q^{-1}]$-algebra automorphism on $\mathcal{H}'$:

$$\eta : \mathcal{H}' \rightarrow \mathcal{H}'; \quad q_0 \mapsto q_0^{-1}, T_{s_0} \mapsto -q_0^{-1} T_{s_0} \text{ and } T_{s_i} \mapsto T_{s_i}, \forall i \geq 1.$$

Then, $\eta(\pi_a) = q_0^{-a} \pi_a^-$ for each $a \geq 0$.

The $q$-permutation $\mathcal{H}'$-modules associated to $W_\lambda$, $\lambda \in \Pi_2$, are defined as cyclic $\mathcal{H}'$-modules $x_\lambda \mathcal{H}'$ and $\mathcal{H}' x_\lambda$ with generators

$$x_\lambda = \pi_a \bar{x}_\lambda, \text{ where } a = |\lambda^{(1)}| \text{ and } \bar{x}_\lambda = \sum_{w \in W_\lambda} T_w$$

\textbf{Definition 2.1.} ([9]) The endomorphism algebra

$$\mathcal{S}^q_2(n,r;Z') = \text{End}_{\mathcal{H}'}(\oplus_{\lambda \in \Pi_2} x_\lambda \mathcal{H}')$$

is called the \textit{q-Schur}$_2$ algebra of degree $(n,r)$.

\textbf{Remark 2.2.} (a) A Morita equivalent version of the \textit{q-Schur}$_2$ algebra, called the $(Q, q)$-\textit{Schur} algebra, was also introduced by Dipper-James-Mathas in [4].

(b) Replacing $\Pi_2$ by $\Pi_1$ in the definition above, we see from [4, (6.3.1)] that the centralizer subalgebra $\mathcal{S}^q_1(n,r;Z')$ of the \textit{q-Schur}$_2$ algebra defined by using $\Pi_1$ is isomorphic to (and hence, will be identified with) the usual \textit{q-Schur} algebra $\mathcal{S}_q(n,r;Z')$. (Recall $n_r = \max(n, r)$.) This algebra was used by Dipper and James to parametrize the irreducible modular characters of finite $\text{GL}_n$ in the non-defining characteristic case, while the one defined by using $\Pi_{1b}$ was considered in the work of Geck-Hiss and Gruber-Hiss on finite orthogonal and symplectic groups. We call this latter algebra here the \textit{q-Schur}$_{1b}$ algebra. Note
that its identity is an idempotent in the \( q \)-Schur\(^2 \) algebra. We agree to not insist on the same identity element when using the terminology ‘centralizer subalgebra’. Also, we shall write \( B \leq A \) if \( B \) is a centralizer subalgebra of \( A \). That is, \( B = eAe \) where \( e \) is an idempotent in \( A \). Recall from [9, (6.3.2)] that, if \( B \leq A \) where \( A \) is an \( \mathcal{O} \)-free algebra for a regular ring \( \mathcal{O} \), then the decomposition matrix of \( B \) is part of the decomposition matrix of \( A \).

(c) Using the twisted permutation module \( y_\lambda \mathcal{H}' \), where

\[
y_\lambda = \pi_{|\lambda(1)|}^{-1} \bar{y}_\lambda, \quad \text{with} \quad \sum_{w \in \bar{W}_\lambda} (-q)^{-\ell(w)} T_w,
\]

we may define another endomorphism algebra \( \tilde{S}_q^2(n, r; \mathcal{Z}') \). It is known [1] that \( \tilde{S}_q^2(n, r; \mathcal{Z}') \) is isomorphic to \( S_q^2(n, r; \mathcal{Z}') \) as \( \mathcal{Z}' \)-algebras.

**Theorem 2.3.** ([9]) The \( q \)-Schur\(^2 \) algebra \( S_q^2(n, r; \mathcal{Z}') \) is (integrally) quasi-hereditary.

In the so-called linear prime case (a fairly strong restriction), the following result was first obtained by Gruber and Hiss [11] in the case \( n = r \). It follows from (2.3) and [9, (6.3.2)] in general.

**Corollary 2.4.** The decomposition matrix for the \( q \)-Schur\(^{1b} \) algebra \( S_q^2(n, r; \mathcal{Z}') \) contains an \( m \times m \) upper unitriangular block, where \( m \) is the number of non-isomorphic irreducible modular representations.

In the linear prime case studied in [10] and [11], the decomposition matrix is square. That is, \( m \) is the same as the number of ordinary irreducible representations (the number of bipartitions). The validity of the conjecture [10, 3.4] would imply this is true for all odd primes [4].

We now turn to defining the \( q \)-Schur\(^{1a} \), \( q \)-Schur\(^{2.5} \) and \( q \)-Schur\(^{1b} \) algebras associated to the Hecke algebra of type \( D \).

Let \( \mathcal{H} \) be the (generic) Hecke algebra over \( \mathcal{Z} = \mathbb{Z}[q, q^{-1}] \) associated to the Weyl group \( \bar{W} \), and let \( \mathcal{H}' = \mathcal{H} \otimes_{\mathcal{Z}} \mathcal{Z}' \) for any commutative \( \mathcal{Z} \)-algebra \( \mathcal{Z}' \). We define (compare [13])

\[
\bar{\pi}_0 = \bar{\pi}_1 = 1, \quad \bar{\pi}_a = \prod_{i=2}^{a} (q^{i-1} + T_{u_i}), \quad 2 \leq a \leq r,
\]

and put \( \bar{x}_\lambda = \bar{\pi}_a \bar{x}_\lambda \), where \( a = |\lambda^{(1)}| \). Then the \( \mathcal{H}' \)-modules \( \bar{x}_\lambda \mathcal{H}' \) and \( \mathcal{H}' \bar{x}_\lambda \) are called the \( q \)-permutation modules associated to the subgroup \( \bar{W}_\lambda \).

\(^2\)In the context of [10, 11], \( q \) is a power of a rational integer prime, which is reduced modulo a different prime when discussing decomposition numbers.
For \( \lambda \in \Pi_{\mathfrak{h}} \) (i.e. \(|\lambda^{(1)}| = 0\)), \( f(W_\lambda) \) is a parabolic subgroup, and is different from \( W_\lambda \) (not even conjugate to it) if \( s_1 \in W_\lambda \). We define the notation \( f(\lambda) \) by \( W_{f(\lambda)} = W_{f(J(\lambda))} = f(W_\lambda) \) and put

\[
\Pi_{2.5} = \Pi_{2.5}(n, r) = \Pi_2(n, r) \cup f\Pi_{\mathfrak{h}}(n, r)
\]

\[
\Pi_{1.5} = \Pi_{1.5}(n, r) = \{ \lambda \in \Pi_2(n, r) \mid |\lambda^{(1)}| \geq |\lambda^{(2)}| \}.
\]

**Definition 2.5.** For any positive integers \( n, r \) with \( r \geq 4 \), let \( \mathcal{T}_{2,1}^g = \oplus_{\lambda \in \Pi_{\mathfrak{h}}} \tilde{\lambda} \mathcal{H}' \) for \( k = 1.5, 2.5 \). The endomorphism algebra

\[
\mathcal{S}_q^g(n, r; \mathcal{Z}') = \text{End}_{\mathcal{H}'}(\mathcal{T}_{2,1}^g)
\]

is called the \( q\)-Schur algebra of degree \((n, r)\). By using the disjoint union \( \Pi_{\mathfrak{h}} = \Pi_{\mathfrak{h}} \sqcup f(\Pi_{\mathfrak{h}}) \), we similarly define \( \mathcal{T}_{2,1}^B \). The corresponding centralizer subalgebra of the \( q\)-Schur\(^{2.5} \) algebra, an analog for type \( D \) to the \( q\)-Schur\(^{\mathfrak{h}} \) algebra for type \( B \), is called a \( q\)-Schur\(^{\mathfrak{h}} \) algebra. When \( \mathcal{Z}' = \mathcal{Z} \), we will denote \( \mathcal{S}_q^g(n, r; \mathcal{Z}) \) simply by \( \mathcal{S}_q^g(n, r) \).

**Remarks 2.6.** (1) Using (2.7b) below, we see that the \( q\)-Schur\(^{\mathfrak{h}} \) algebra is isomorphic to the endomorphism algebra

\[
\text{End}_{\mathcal{H}'}(\oplus_{\lambda \in \Pi_{\mathfrak{h}}} \tilde{\lambda} \mathcal{H}'|_{\mathcal{H}'}).
\]

(2) The \( q\)-Schur algebra of type \( D \) defined in [11, 7.2] uses only one parabolic copy of \( \mathcal{G}_r \). In terms of our notation, it is (Morita equivalent to) the endomorphism algebra of the \( \mathcal{H}'\)-module \( \oplus_{\lambda \in \Pi_{\mathfrak{h}}} \tilde{\lambda} \mathcal{H}' \) (or \( \oplus_{\lambda \in f(\Pi_{\mathfrak{h}})} \tilde{\lambda} \mathcal{H}' \)). We shall denote this algebra by \( \mathcal{S}_q^{\mathfrak{h}'}(n, r; \mathcal{Z}') \). Clearly, \( \mathcal{S}_q^{\mathfrak{h}'}(n, r; \mathcal{Z}') \leq \mathcal{S}_q^{\mathfrak{h}'}(n, r; \mathcal{Z}) \) and therefore, its decomposition matrix is determined by that of the \( q\)-Schur\(^{\mathfrak{h}} \) algebra. Actually, Gruber and Hiss, later in their paper, appear to be implicitly using \( \mathcal{S}_q^{\mathfrak{h}'}(r, r) \) rather than \( \mathcal{S}_q^{\mathfrak{h}'}(r, r) \). See Remark 3.3(2) below.

When the \( \mathcal{Z}_0\)-algebra \( \mathcal{Z}' \) has the property that \( q_0 \) is specialized to 1, that is, \( \mathcal{H}' = \mathcal{H}'_{1,q} = \mathcal{H} \otimes_{\mathcal{Z}_0} \mathcal{Z}' \), we have in \( \mathcal{H}'_{1,q} \)

\[
T_{s_0}^2 = 1 \text{ and } T_{s_0}T_w = T_{s_0}T_w, \quad \forall w \in \tilde{W}.
\]

**In the rest of the section, we assume \( \mathcal{H}' = \mathcal{H}'_{1,q} \).** The first part of the following has been observed by Gruber-Hiss.

**Lemma 2.7.** (a) The algebra \( \mathcal{H}' \) is isomorphic to the subalgebra of \( \mathcal{H}' = \mathcal{H}'_{1,q} \) generated by \( T_s, s \in \tilde{S} \). So we identify \( \mathcal{H}' \) with this subalgebra and obtain a graded Clifford system for \( \mathcal{H}' \) over \( \mathcal{H}' \).

(b) We have \( \mathcal{H}'\)-module decompositions

\[
x_\lambda \mathcal{H}' = \begin{cases} 
  x_\lambda \mathcal{H}', & \text{if } |\lambda^{(1)}| \neq 0 \\
  x_\lambda \mathcal{H}' + T_{s_0} \tilde{f}(x_\lambda) \tilde{H}', & \text{if } |\lambda^{(1)}| = 0,
\end{cases}
\]
where \( \hat{f} \) is the induced flipping automorphism on \( \mathcal{H}' \) defined by \( \hat{f}(T_w) = T_{s_0}T_wT_{s_0} = T_{\hat{f}(w)} = T_{f(w)} \).

(c) For any \( 1 \leq a \leq r \), we have
\[
\pi_a = (1 + T_{s_0})\tilde{\pi}_a = \tilde{\pi}_a(1 + T_{s_0}), \quad \pi_a^- = (1 - T_{s_0})\tilde{\pi}_a = \tilde{\pi}_a(1 - T_{s_0}).
\]

Thus, \( \tilde{x}_\lambda \mathcal{H}' \cong x_\lambda \mathcal{H}' = (1 + T_{s_0})\tilde{x}_\lambda \mathcal{H}' \), for any \( \lambda \in \Pi_2 \) with \( |\lambda^{(1)}| \geq 1 \). In particular, the permutation modules \( \tilde{x}_\lambda \mathcal{H}' \) are \( \mathcal{Z}' \)-free and pure in \( \mathcal{H}' \).

Proof. The statements (a) and (b) are clear, noting \( \mathcal{H}' = \mathcal{H}' \oplus T_{s_0} \mathcal{H}' \).

The first assertion in (c) follows from induction and the relation
\[
(1 + T_{s_0})(q^{i-1} + T_{t_i}) = (q^{i-1} + T_{t_i})(1 + T_{s_0}) = (1 + T_{s_0})(q^{i-1} + T_{u_i}) \quad \forall i,
\]
noting \( u_i = s_0t_i = t_is_0 \) and \( T_{t_i} = T_{s_0}T_{u_i} \) for all \( i \geq 2 \). For the second displayed equation in (c), use
\[
(1 - T_{s_0})(q^{i-1} - T_{t_i}) = (1 - T_{s_0})(q^{i-1} + T_{u_i}), \, i \geq 2.
\]

The freeness follows from (b) and \([4 \text{ (3.2.2a)})\], and the purity follows from the fact that \( x_\lambda \mathcal{H}' = x_\lambda \mathcal{H}' \) is pure in \( \mathcal{H}' = (1 + T_{s_0})\mathcal{H}' \oplus T_{s_0} \mathcal{H}' \), and hence pure in \( (1 + T_{s_0})\mathcal{H}' \). \( \square \)

Using the decomposition for \( x_\lambda \mathcal{H}' \) above and noting \( S_q^1(n, r; \mathcal{Z}') \leq S_q^1(n, r; \mathcal{Z}') \), we obtain immediately the following.

**Corollary 2.8.** Let \( T_{\mathcal{Z}'}^2 = \oplus_{\lambda \in \Pi_2} x_\lambda \mathcal{H}' \). Then \( T_{\mathcal{Z}'}^2|_{\tilde{\mathcal{H}'}} \cong T_{\mathcal{Z}'}^{2.5} \). Hence, the \( q \)-Schur\(^{2.5} \) algebra is isomorphic to \( \text{End}_{\tilde{\mathcal{H}'}}(T_{\mathcal{Z}'}^2) \), and we have
\[
\begin{align*}
(1) & \quad S_q^1(n, r; \mathcal{Z}') \leq S_q^{1.5}(n, r; \mathcal{Z}') \leq S_q^{2.5}(n, r; \mathcal{Z}') \\
(2) & \quad S_q^1(n, r; \mathcal{Z}') \leq S_q^2(n, r; \mathcal{Z}') \leq S_q^{2.5}(n, r; \mathcal{Z}') \\
(3) & \quad S_q^n(n, r; \mathcal{Z}') \leq S_q^{2.5}(n, r; \mathcal{Z}').
\end{align*}
\]

In this paper, we will mainly investigate the structure of the \( q \)-Schur\(^{1.5} \) algebras, though we keep the \( q \)-Schur\(^{10} \) algebras in mind. See \([2.11), \,(3.7), \,(7.5b) \) and \((8.1)\).

Specht modules \( S_{\lambda K} \) over the quotient field \( K \) of \( \mathcal{Z} \) is a complete set of irreducible \( \mathcal{H}_K \)-modules. Thus, their restrictions to \( \mathcal{H}_K \) are \( \mathcal{H}_K \)-modules. The following result is an easy consequence of Tits’ deformation theory and standard facts about ordinary characters of Weyl groups of type \( D \).

**Lemma 2.9.** Let \( r \) be odd. Then the set \( \{ S_{\lambda K} \mid \lambda \in \Pi_{1,5}^+(r, r) \} \) is a complete set of distinct irreducible \( \mathcal{H}_K \)-modules. If \( r \) is even, then \( S_{\lambda K} \), for a bipartition \( \lambda = (\alpha, \beta) \) of \( r \), is irreducible iff \( \alpha \neq \beta \), and \( S_{(\alpha, \alpha)K} \) has two (distinct) composition factors. Moreover, we have in general \( S_{\lambda K} \cong S_{\lambda^* K} \) on \( \mathcal{H}' \) for \( \lambda^* = (\beta, \alpha) \).
Recall the automorphism \( \eta \) defined at the beginning of this section, and note that \( \eta(T_{s_0}) = -T_{s_0} \). (Also, \( \eta(T_{u_i}) = T_{u_i} \) for all \( i \geq 2 \), and the automorphism \( \eta \) fixes \( \mathcal{H}' \).) Let \( S^n_{\lambda} \) be the module \( S_\lambda \) with \( \mathcal{H} \)-action twisted by \( \eta \). We abbreviate below the induction functor \( \text{Ind} \mathcal{H}' \) to \( \uparrow \mathcal{H}' \).

Lemma 2.10. (a) For any \( \lambda \in \Pi^+(r) \), let \( \lambda^* = (\lambda^{(2)}, \lambda^{(1)}) \). Then we have \( S^n_{\lambda} \approx S_{\lambda^*} \mathcal{H}_K \) as \( \mathcal{H}_K \)-modules. Moreover, if \( |\lambda^{(1)}| \geq |\lambda^{(2)}| \) and \( |\mu^{(1)}| \geq |\mu^{(2)}| \), but not both equalities, then \( \text{Hom}_{\mathcal{H}_K}(x^n_{\lambda} \mathcal{H}_K, S_{\mu} \mathcal{H}_K) = 0 \); in particular, when \( r \) is odd, we always have \( \text{Hom}_{\mathcal{H}_K}(x^n_{\lambda} \mathcal{H}_K, x_{\lambda} \mathcal{H}_K) = 0 \).

(b) If \( |\lambda^{(1)}| = 0 \), then \( x_{\lambda} \mathcal{H}' \uparrow \mathcal{H}' \cong x_{\lambda} \mathcal{H}' \).

(c) If \( 2 \) is invertible in \( \mathcal{Z}' \), then we have \( x_{\lambda} \mathcal{H}' \uparrow \mathcal{H}' \cong x_{\lambda} \mathcal{H}' \oplus x^n_{\lambda} \mathcal{H}' \) for any \( \lambda \in \Pi_2 \) with \( |\lambda^{(1)}| \geq 1 \).

Proof. By the discussion in [9, (5.1.2)], it suffices to look at the specializations at \( q_0 = q = 1 \). In this case, the construction given at the beginning of [9, §5.1] gives the required isomorphism. Using this, the composition factors \( S_{\lambda} \) appearing in \( x^n_{\lambda} \mathcal{H}_K \approx (x_{\lambda} \mathcal{H}_K)^n \) (the module \( x_{\lambda} \mathcal{H}_K \) with action twisted by \( \eta \)) have the property \( |\nu^{(1)}| \leq |\nu^{(2)}| \). So the last assertion in (a) follows. The statement (b) follows from the fact \( x_{\lambda} = \tilde{x}_{\lambda} = x_{\lambda} \) in this case. Finally, since \( 2 \) is invertible, we have \( \mathcal{H}' = (1 + T_{s_0}) \mathcal{H}' \oplus (1 - T_{s_0}) \mathcal{H}' \), and therefore,

\[
\tilde{x}_{\lambda} \mathcal{H}' \uparrow \mathcal{H}' = \tilde{x}_{\lambda} \mathcal{H}' (1 + T_{s_0}) \mathcal{H}' \oplus \tilde{x}_{\lambda} \mathcal{H}' (1 - T_{s_0}) \mathcal{H}' \cong x_{\lambda} \mathcal{H}' \oplus x^n_{\lambda} \mathcal{H}'.
\]

\[\blacksquare\]

Remark 2.11. (1) Similar to [6, 2c), we define for a bicomposition \( \lambda \slash \eta \lambda = \tilde{\eta}_{|\lambda^{(1)}} \eta \). Then, we may establish a parallel theory as above with all \( \tilde{x}_{\lambda} \) replaced by \( \tilde{y}_{\lambda} \). In particular, the resulting \( q \)-Schur algebras are isomorphic copies of the original ones and all \( \tilde{y}_{\lambda} \mathcal{H}' \) are pure in \( \mathcal{H}' \).

(2) Let \( G \) denote the finite orthogonal group \( \text{SO}_{2n}(q) \) of even degree and fix a splitting \( p \)-modular system \( \{ K, \mathcal{O}, k \} \). Then \( \mathcal{H}_{\mathcal{Z}'} \) (\( \mathcal{Z}' \in \{ K, \mathcal{O}, k \} \)) is isomorphic to the endomorphism algebra of the induced module \( M_{\mathcal{Z}'} \) of the trivial \( B \)-module \( \mathcal{Z}' \), where \( B \) is a fixed Borel subgroup. Then the purity of the submodules \( \tilde{y}_{\lambda} \mathcal{H}_{\mathcal{O}} \) of the symmetric algebra \( \mathcal{H}_{\mathcal{O}} \) over \( \mathcal{O} \) gives rise to isomorphisms by [11, Thm1]

\[
S^\kappa_q(n, r; \mathcal{O}) \cong \text{End}_{\mathcal{O}G}(\oplus_{\lambda \in \Pi_k, \sqrt{\mathcal{O}M_{\mathcal{O}}}} \tilde{y}_{\lambda} \mathcal{M}_{\mathcal{O}}), \text{ for } \kappa = 1.5, 2.5, 10.
\]

The arguments in [11, §4] shows that, for \( \kappa = 10 \), the decomposition matrix of the \( \mathcal{O}G \)-module \( \oplus_{\lambda \in \Pi_9, \sqrt{y_{\lambda}M_{\mathcal{O}}}} \) is part of the decomposition matrix of \( \mathcal{O}G \), and is even decisive in determining the unipotent part of the latter decomposition matrix in good cases. Probably the determination of this decomposition matrix is about as hard as determining that of the larger algebra \( S^{2.5}_q(n, r; \mathcal{Z}') \), though not all irreducible modular
representations of the latter may be required in every case. At the same
time, it may be hoped that the decomposition matrix for $S_{q,5}^2(n, r; \mathbb{Z}')$
will be helpful in determining decomposition numbers and modular ir-
reducible representations of $S_{q}^2(n, r; \mathbb{Z}')$, since the two algebras have the same
ordinary irreducible representations. This philosophy works well in at least two quite different cases, presented in section 3 below
and in section 8.

3. The linear prime case: Morita equivalence theorems

In this section, we shall prove that, in the linear prime case (where
much is already known, cf. [10]), the $q$-Schur$^2$ and $q$-Schur$^{1,5}$ algebras are Morita equivalent. In the type $D$ linear prime case, both the $q$-
Schur$^{2,5}$ and $q$-Schur$^{1,5}$ algebras are Morita equivalent to the $q$-Schur$^{1,5}$
algebra.

We need some preparation at the group algebra level. For $0 \leq a \leq r,$ let
\[
\varepsilon_{a,r-a} = 2^{-r} (1 + t_1) \cdots (1 + t_a)(1 - t_{a+1}) \cdots (1 - t_r)
= 2^{-r} (1 + t_1) \cdots (1 + t_a) w_{a,r-a}(1 - t_1) \cdots (1 - t_{r-a}) w_{a,r-a}^{-1},
\]
where $w_{a,b} = s_a \cdots s_1 s_{a+1} \cdots s_{a+b-1} \cdots s_b = (\frac{1}{b+1} \cdots \frac{a}{b+a} \cdots \frac{a+b}{1} \cdots 1)$. Note that $\bar{W}_{a,r-a} w_{a,r-a} = w_{a,r-a} \bar{W}_{r-a,a}$. For a bicomposition $\lambda \in \Pi_2(n, r)$, we shall consider a decomposition for the second part of $\lambda$:
\[
\lambda^{(2)} = \beta + \gamma = (\beta_1 + \gamma_1, \cdots, \beta_n + \gamma_n)
\]
where $\beta = (\beta_1, \cdots, \beta_n) \in \Lambda(n, |\beta|)$ and $\gamma = (\gamma_1, \cdots, \gamma_n) \in \Lambda(n, |\gamma|)$. Associated to such a decomposition $\lambda^{(2)} = \beta + \gamma$, we have two compositions:
\[
(\beta|\gamma) = (\beta_1, \gamma_1, \cdots, \beta_n, \gamma_n), \quad (\beta, \gamma) = (\beta_1, \cdots, \beta_n, \gamma_1, \cdots, \gamma_n).
\]
Write $a = |\lambda^{(1)}|$, $b = |\beta|$ and $c = |\gamma|$, and let $w_{\beta,\gamma}^{(2)} \in \bar{W}_{(1^a, r-a)} = S_{(r-r_2+1, \cdots, r)}$ be the (unique in $\bar{W}_{(1^a, r-a)}$) distinguished double coset representative\footnote{See, for example, [3, 3.4] for a construction of such a representative.} for the subgroups $\bar{W}_{(1^a, \lambda^{(2)})}$ and $\bar{W}_{(1^a, b,c)}$ of $\bar{W}_{(1^a, r-a)}$ such that
\[
\bar{W}_{(1^a, \lambda^{(2)})} \cap w_{\beta,\gamma}^{(2)} \bar{W}_{(1^a, b,c)} w_{\beta,\gamma}^{(2)-1} = \bar{W}_{(1^a, (\beta|\gamma))},
\]
and
\[
(w_{\beta,\gamma}^{(2)})^{-1} \bar{W}_{(1^a, \lambda^{(2)})} w_{\beta,\gamma}^{(2)} \cap \bar{W}_{(1^a, b,c)} = \bar{W}_{(1^a, \beta,\gamma)}.
\]
Lemma 3.1. Maintain the notation introduced above. Let $R$ be a commutative ring containing $2^{-1}$. Then, for any bicomposition $\lambda$ of $r$ with $|\lambda^{(1)}| = a$,
\[
\mathbf{x}_\lambda RW = \bigoplus_{\beta \in \Lambda(a,b), \gamma \in \Lambda(n,c)} \mathbf{x}_\lambda w_{\beta,\gamma}^{(2)} \varepsilon_{a+b,c} RW
\]
where $\mathbf{x}_\lambda = \sum_{w \in W_\lambda} w$ and $\mathbf{x}_\lambda = \sum_{w \in W_\lambda} w$

Proof. Since $2$ is invertible in $R$, we have all $\varepsilon_{a,r-a} \in RC$ and $RC = \bigoplus_{a=0}^r \bigoplus_{w \in W_\lambda} RW\varepsilon_{a,r-a} w^{-1}$. Thus,
\[
\mathbf{x}_\lambda RW \cong \text{Ind}_{RW_{\Lambda(C)}}^{RW_{\Lambda(C)}}(\text{Ind}_{RW_{\Lambda(C)}}^{RW_{\Lambda(C)}}(Rx_\lambda))
\]
\[
\cong \bigoplus_{\beta + \gamma = \lambda^{(2)}} R\mathbf{x}_\lambda w_{\beta,\gamma}^{(2)} \varepsilon_{a+b,c} RW_{(\lambda^{(1)},(\beta,\gamma))} RW.
\]
We certainly have
\[
R\mathbf{x}_\lambda w_{\beta,\gamma}^{(2)} \varepsilon_{a+b,c} RW_{(\lambda^{(1)},(\beta,\gamma))} RW \cong \mathbf{x}_\lambda w_{\beta,\gamma}^{(2)} \varepsilon_{a+b,c} RW;
\]
proving the lemma. \hfill \Box

The type $B$ case. Let $\mathcal{H} = \mathcal{H}_{q_0,q}$ be the Hecke algebra of type $B$ over $\mathbb{Z}_0$ defined in §2. Recall from [3, (3.8),(4.4)] the elements $v_{a,b} = \pi_a T_{\omega_a,b} \pi_b^-$ and the polynomial
\[
g_r = g_r(q_0,q) = \prod_{i=-(r-1)}^{r-1} (q_0 + q^i).
\]
Let $\mathbb{Z}_{0,g_r}$ be the ring obtained by localizing $\mathbb{Z}_0$ at $g_r$. Thus, $g_r$ is invertible in $\mathbb{Z}_{0,g_r}$. Let $\mathcal{H}_{g_r} = \mathcal{H} \otimes_{\mathbb{Z}_0} \mathbb{Z}_{0,g_r}$. Then, by [3, (3.27-8)], there are orthogonal idempotents $\{e_{a,r-a} \mid 0 \leq i \leq r\}$ in $\mathcal{H}_{g_r}$, such that
\[
e_{a,r-a} \mathcal{H}_{g_r} = e_{a,r-a} \mathcal{H}_{g_r}, \quad \text{and} \quad \mathcal{H}(\mathfrak{S}_{(a,b)}) e_{a,b} = e_{a,b} \mathcal{H}(\mathfrak{S}_{(a,b)}).
\]
Also, $e_{a,r-a}$ specializes to $\varepsilon_{a,r-a}$ in $\mathbb{Q}W$. The following lemma gives a new basis for $\mathcal{H}_{g_r}$.

Lemma 3.2. The set
\[
\mathcal{B} = \{T_x e_{a,r-a} T_y \mid 0 \leq a \leq r, x \in \mathcal{D}_{(a,r-a)}^{-1}, y \in \hat{W}\}
\]
forms a basis for $\mathcal{H}_{g_r}$.

Proof. Let $H$ be the $\mathbb{Z}_{0,g_r}$-submodule generated by the set $\mathcal{B}$. Specializing both $q_0$ and $q$ to $1$, we obtain an algebra homomorphism $\mathbb{Z}_{0,g_r} \to \mathbb{Q}$. Since $\mathcal{H}_{g_r} \otimes_{\mathbb{Z}_{0,g_r}} \mathbb{Q} \cong \mathbb{Q}W$ and the image of the set $\mathcal{B}$ becomes the basis
\[
\{((w \varepsilon_{a,r-a} w^{-1})wy \mid 0 \leq a \leq r, x \in \hat{W}/\hat{W}_{(a,r-a)}, y \in \hat{W}\}.
\]
So $B$ is linearly independent. To check that it spans $H_{gr}$, it suffices to prove the corresponding spanning property over any field $k$ which is a $\mathbb{Z}_{0,gr}$-algebra. Let $e_{a,r-a,k}$ be the image of $e_{a,r-a}$ in $H_{gr,k}$. Note that $H_{gr,k}e_{a,r-a,k}H_{gr,k}$ is spanned by $T_x e_{a,r-a,k} T_y$ ($x \in D_{(a,r-a)}, y \in W$). So, if $H_{gr,k} = \sum_{a=0} e_{a,r-a,k} H_{gr,k} H_{gr,k}$, where $e_k = \sum_{a=0} e_{a,r-a,k}$, then $B_k$ must span $H_{gr,k}$.

It remains to prove the equality $H_{gr,k} = H_{gr,k} e_k H_{gr,k}$. Suppose this is not the case. Then there is a maximal left ideal in $H_{gr,k}$ such that

$$H_{gr,k} e_k H_{gr,k} \subset I.$$

Since the functor $M \mapsto e_k M$ defines an equivalence from the category $H_{gr,k} - \text{mod}$ to the category $e_k H_{gr,k} - \text{mod}$, it must be that $e_k (H_{gr,k} / I) \neq 0$. However, on the other hand,

$$e_k H_{gr,k} = e_k H_{gr,k} e_k H_{gr,k} \subset e_k I \subset e_k H_{gr,k}$$

which implies that $e_k (H_{gr,k} / I) = 0$, a contradiction. \hfill $\square$.

We now have the following.

**Lemma 3.3.** Let $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Pi_2(n, r)$ be a bicomposition with $|\lambda^{(1)}| = a$. Then the set, denoted by $B_\lambda$,

$$\{ \bar{x}_\lambda T_w(2) e_{a+|\beta|,|\gamma|} T_w \mid \beta \in \Lambda(n, b), \gamma \in \Lambda(n, c), \beta + \gamma = \lambda^{(2)}, w \in D_{(\lambda^{(1)}, \beta, \gamma)} \},$$

where $D_{(\lambda^{(1)}, \beta, \gamma)}$ is the set of distinguished right $\bar{W}(\lambda^{(1)}, \beta, \gamma)$-coset representatives in $\bar{W}$, forms a basis for $x_\lambda H_{gr}$.

**Proof.** Write $\bar{x}_\lambda = \bar{x}_{\lambda^{(1)} h_{\beta, \gamma}} x'_{\beta, \gamma}$ with $x'_{\beta, \gamma} = \sum_{w \in W(1, a, \lambda^{(2)} \cap D_{(\lambda^{(1)}, \beta, \gamma)})} T_w$ and $h_{\beta, \gamma} = \sum_{d \in W(1, a, \lambda^{(2)}) \cap D_{(\lambda^{(1)}, \beta, \gamma)}} T_d$. Since $w_{a, \beta}$ is distinguished, we have $\bar{x}_\lambda T_w(2) = h_{\beta, \gamma} T_w(2) x'_{(\lambda^{(1)}, \beta, \gamma)}$, and, for each $d \in W(1, a, \lambda^{(2)}) \cap D_{(\lambda^{(1)}, \beta, \gamma)}$, $T_d T_w(2) = T_{dw(2)}$ and $dw(2) \in D_{(a+h, c)}$. Note also that the union of all $(W(1, a, \lambda^{(2)}) \cap D_{(\lambda^{(1)}, \beta, \gamma)}) w(2)$ with $\lambda^{(2)} = \beta + \gamma$ is disjoint. Thus, for any field which is a $\mathbb{Z}_{0,gr}$-algebra, the image of $B_\lambda$ in $x_\lambda H_{gr,k}$ is linearly independent by (3.2). Therefore, if $M$ denotes the $\mathbb{Z}_{0,gr}$-submodule generated by the set $B_\lambda$, then the image $M_k$ of $M$ in $x_\lambda H_{gr,k}$ has dimension $|W / W| \lambda$ by (3.2a), which is equal to $\dim x_\lambda H_{gr,k}$ by (3.2.2a). Therefore, $M_k = H_{gr,k}$ for any field $k$ which is a $\mathbb{Z}_{0,gr}$-algebra. Consequently $M = x_\lambda H_{gr}$, and hence $B_\lambda$ forms a basis for $x_\lambda H_{gr}$. \hfill $\square$. 


Corollary 3.4. For any \( \lambda \in \Pi_2(n, r) \) with \( |\lambda|^{(1)} = a \), we have the following decomposition

\[
x_\lambda \mathcal{H}'_{gr} = \bigoplus_{b,c} \bigoplus_{\beta \in \Lambda(n,b), \gamma \in \Lambda(n,c)} x_\lambda T_{w_{\beta, \gamma}}^{(2)} e_{a+b,c} \mathcal{H}'_{gr}.
\]

Moreover, we have a module isomorphism

\[
x_\lambda \mathcal{H}'_{gr} \cong \bigoplus_{b,c} \bigoplus_{\beta \in \Lambda(n,b), \gamma \in \Lambda(n,c)} x_{(\lambda(1), \beta, \gamma)} e_{a+b,c} \mathcal{H}'_{gr}.
\]

Proof. The equality (3.4.1) follows from Lemma 3.3 and base change. To prove the isomorphism, we use the notation introduced above to note that

\[
\bar{x}_\lambda T_{w_{\beta, \gamma}}^{(2)} e_{a+b,c} \mathcal{H}'_{gr} = h_{\beta, \gamma} T_{w_{\beta, \gamma}}^{(2)} \bar{x}_{(\lambda(1), \beta, \gamma)} e_{a+b,c} \mathcal{H}'_{gr}.
\]

So we have an epimorphism from \( \bar{x}_{(\lambda(1), \beta, \gamma)} e_{a+b,c} \mathcal{H}'_{gr} \) to \( \bar{x}_\lambda T_{w_{\beta, \gamma}}^{(2)} e_{a+b,c} \mathcal{H}'_{gr} \), via left multiplication by \( h_{\beta, \gamma} T_{w_{\beta, \gamma}}^{(2)} \). By looking at the image of the “standard” basis \( \{ \bar{x}_{(\lambda(1), \beta, \gamma)} e_{a+b,c} T_w \mid w \in \bar{D}_{(\lambda(1), \beta, \gamma)} \} \), we see from Lemma 3.3 that this map is in fact an isomorphism. That is, we have

\[
(3.4.2) \quad \bar{x}_{(\lambda(1), \beta, \gamma)} e_{a+b,c} \mathcal{H}'_{gr} \sim \bar{x}_\lambda T_{w_{\beta, \gamma}}^{(2)} e_{a+b,c} \mathcal{H}'_{gr}.
\]
Lemma 3.6. Let \(a, b\) be non-negative integers. Then:

\[
\begin{aligned}
(a) \quad \check{v}_{a,b} &= \begin{cases} 
(1 + T_{s_0})\check{v}_{a,b}, & \text{if } a \geq 1, \\
\check{v}_{a,b}(1 - T_{s_0}), & \text{if } b \geq 1.
\end{cases}
\end{aligned}
\]

(b) \(\check{v}_{a,b} \mathcal{H}(\mathfrak{S}_{(a,b)}) = \mathcal{H}(\mathfrak{S}_{(b,a)})\check{v}_{a,b}\).

(c) If \(a \geq 1\) and \(b \geq 1\), then \(\check{v}_{a,b}(T_{w_{b,a}} - T_{f_{w_{b,a}}})\check{v}_{a,b} = \check{v}_{a,b}z\) for some central element \(z\) of \(\mathcal{H}(\mathfrak{S}_{(b,a)})\). Moreover, \(z\) is invertible in \(\mathcal{H}_{g_r}\).

(d) \(\check{v}_{a,r-a} \mathcal{H}_{g_r} \cong \check{v}_{r-a} \mathcal{H}_{g_r}\) as \(\mathcal{H}_{g_r}\)-module.

Proof. The statement (a) follows from a direct computation (see [12, 1.8]). (b) is [13, (1.14)]. To see (c), we use (a) to have

\[
(1 + T_{s_0})\check{v}_{a,b}(T_{w_{b,a}} - T_{f_{w_{b,a}}})\check{v}_{a,b}
= \check{v}_{a,b}(1 - T_{s_0})T_{b,2}(T_{s_1} - s_{0,s_{1}s_{0}})T_{b+1,2} \cdots T_{a+b-1,a}\check{v}_{a,b}
= \check{v}_{a,b}T_{b,2}(1 - T_{s_0})T_{s_1}(1 + T_{s_0})T_{b+1,2} \cdots T_{a+b-1,a}\check{v}_{a,b}
= v_{a,b}T_{w_{b,a}}v_{a,b} = v_{a,b}z = (1 + T_{s_0})\check{v}_{a,b}z
\]

by [3, (3.23)], where \(T_{b,i} = T_{s_i} \cdots T_{s_j}\) for \(j > i\). Cancellation of \((1 + T_{s_0})\), we obtain the required relation.

We now prove (d). The result is clear if \(a = 0\) or \(r\). Assume now \(0 < a < r\) and let \(b = r - a\). Then

\[
\check{\pi}_a(T_{w_{a,b}} - T_{f_{w_{a,b}}})\check{v}_{b,a} \mathcal{H}_{g_r} = \check{v}_{a,b}(T_{w_{b,a}} - T_{f_{w_{b,a}}})\check{\pi}_a \mathcal{H}_{g_r}
\]

\[\supseteq \check{v}_{a,b}(T_{w_{b,a}} - T_{f_{w_{b,a}}})\check{v}_{a,b} \mathcal{H}_{g_r}
= \check{v}_{a,b}z \mathcal{H}_{g_r} = \check{v}_{a,b} \mathcal{H}_{g_r}\]

by (c).

So the inclusion has to be an equality, and the required isomorphism follows. \(\square\)

Theorem 3.7. For any \(\mathcal{Z}_{g_r}\)-algebra \(\mathcal{Z}'\), the algebras \(S_q^{1.5}(r, r; \mathcal{Z}')\) and \(S_q^{1.0}(r, r; \mathcal{Z}')\) are Morita equivalent to the algebra \(S_q^{1.5}(n, r; \mathcal{Z}')\). In particular, this Morita equivalence holds in the linear prime case.

Proof. We first look at the Morita equivalence between \(S_q^{2.5}\) and \(S_q^{1.5}\). Since \(\check{T}_{2.5}^{1.5} \cong T_{2.5}^{1.5} |_{\mathcal{H}'} \) (2.8) and (3.4.1) holds when restricted to \(\mathcal{H}'\), it suffices to prove that any direct summand appeared in (3.4.1) is isomorphic to a direct summand of \(\check{T}_{2.5}^{1.5}\). Equivalently, by (3.4.2), we only need to prove that, if \(a + b = r\) and \(\alpha, \beta\) are compositions of \(a, b\) respectively, then we have an \(\mathcal{H}'_{g_r}\)-module isomorphism

\[
(3.7.1) \quad \check{e}_{(\alpha,\beta)} e_{a,b} \mathcal{H}'_{g_r} \cong \check{e}_{(\beta,\alpha)} e_{b,a} \mathcal{H}'_{g_r}.
\]
This is certainly true if \( a = 0 \) since
\[
\bar{x}_{(0,\beta)}e_{0,r}\mathcal{H}'_{\bar{g}_r} = (1 - T_{s_0})\bar{x}_{\beta}\bar{\pi},\mathcal{H}'_{\bar{g}_r} \cong \bar{x}_{\beta}\bar{\pi}_{0,0}\mathcal{H}'_{\bar{g}_r} \cong (1 + T_{s_0})\bar{x}_{\beta}\bar{\pi}_{r,0}\mathcal{H}'_{\bar{g}_r}.
\]

Assume now \( a \geq 1 \). Recall from [3, 3.27] that \( e_{a,b} = v_{a,b}z_{a,b}T_{w_{b,a}} \), where \( z_{a,b} \) is central in \( \mathcal{H}(\mathcal{S}_{a,b}) \), and invertible in \( \mathcal{H}'_{\bar{g}_r} \). Write \( \tilde{e}_{a,b} = \tilde{v}_{a,b}z_{a,b}T_{w_{b,a}} \). Then \( \tilde{e}_{a,b} \) commutes with the elements of \( \mathcal{H}(\mathcal{S}_{a,b}) \) (3.6), and \( e_{a,b} = (1 + T_{s_0})\tilde{e}_{a,b} \). Thus,
\[
\bar{x}_{(a,\beta)}e_{a,b}\mathcal{H}'_{\bar{g}_r} = (1 + T_{s_0})\bar{x}_{(a,\beta)}\tilde{e}_{a,b}\mathcal{H}'_{\bar{g}_r} \cong \bar{x}_{(a,\beta)}\tilde{e}_{a,b}\mathcal{H}'_{\bar{g}_r} = \bar{x}_{(a,\beta)}\tilde{v}_{a,b}\mathcal{H}'_{\bar{g}_r}.
\]

We need to deal with two cases: \( a = 1 \) and \( a > 1 \). If \( a = 1 \), then there is no \( T_{s_1} \) involved in \( \bar{x}_{(1,\beta)} \) and so \( T_{f(w_{r-1,1})}\bar{x}_{(1,\beta)} = \tilde{f}(\bar{x}_{(1,\beta)})T_{f(w_{r-1,1})} \). Since \( \tilde{\pi}_aT_{s_1} = \tilde{\pi}_aT_u \) for all \( a \geq 2 \), we have
\[
\tilde{\pi}_{r-1}(T_{w_{r-1,1}} - T_{f(w_{r-1,1})})\bar{x}_{(1,\beta)}\tilde{v}_{1,r-1}\mathcal{H}'_{\bar{g}_r} = \bar{x}_{(1,\beta)}\tilde{\pi}_{r-1}(T_{w_{r-1,1}} - T_{f(w_{r-1,1})})\tilde{v}_{1,r-1}\mathcal{H}'_{\bar{g}_r} = \bar{x}_{(1,\beta)}\tilde{v}_{r-1,1}\mathcal{H}'_{\bar{g}_r},
\]
which implies (3.7.1) in this case.

Finally, if \( a \geq 2 \), then, by symmetry, we may assume that \( b \geq 2 \).

Noting \( \tilde{\pi}_aT_{s_1} = \tilde{\pi}_aT_u \) again, we have
\[
\tilde{\pi}_b(T_{w_{b,a}} - T_{f(w_{b,a})})\bar{x}_{(a,\beta)}\tilde{v}_{a,b}\mathcal{H}'_{\bar{g}_r} = \bar{x}_{(\beta,a)}T_{w_{b,a}} - T_{f(w_{b,a})}\tilde{f}(\bar{x}_{(a,\beta)})\tilde{v}_{a,b}\mathcal{H}'_{\bar{g}_r} = \bar{x}_{(\beta,a)}\tilde{\pi}_b(T_{w_{b,a}} - T_{f(w_{b,a})})\tilde{v}_{a,b}\mathcal{H}'_{\bar{g}_r} = \bar{x}_{(\beta,a)}\tilde{v}_{b,a}\mathcal{H}'_{\bar{g}_r},
\]
which implies (3.7.1).

We now prove the Morita equivalence between \( S^b_q \) and \( S^1_q \). Since \( \tilde{T}^b_{\mathcal{Z}} \cong T^b_{\mathcal{Z}}|_{\mathcal{H}_r} \), the proof above together with an argument similar to the proof of (3.3) shows that, when \( \tilde{T}^b_{\mathcal{Z}} \) and \( T^b_{\mathcal{Z}} \) decompose via (3.4.1), they have the same direct summands (up to multiplicity). Therefore, the required Morita equivalence follows.

**Remarks 3.8.** (1) We point out that, in the type B case, results [3, (4.17)] and [11, (7.6)] give an interpretation of the q-Schur^B algebra in terms of q-Schur algebras, effectively making these algebras quasi-hereditary. However, such a result as [3, (4.17)] is not available in the type D case, even with the linear prime assumption, so that (3.7) is new.
in this case. Together with (7.4) below and a result [11, 7.15] of Gruber-Hiss, it implies that the \( q \)-Schur\(^{\text{b}} \) algebra is also quasi-hereditary under the linear prime assumption. See Remark 7.5b below.

(2) Theorem 3.7 does not give a Morita equivalence between \( S_q^{\text{b}} \) and \( S_q^{\text{b}'} \) (see (2.9(2))). It can be checked that such a Morita equivalence exists for group algebras, but with a proof involving, when \( r \) is even, non-parabolic subgroups. So there is no obvious "\( q \)-analogue", and we do not know if it exists, when \( r \) is even. We point out that, without such a Morita equivalence, result [11, 7.17] holds only for the \( q \)-Schur\(^{\text{b}} \) algebra in the type \( D \) case.

4. Hom spaces between \( x_{\lambda} \mathcal{H}' \) and \( y_{\mu} \mathcal{H}' \)

We shall assume \( \mathcal{H}' = \mathcal{H}'_{q_0,q} \) until further notice. (We do not assume \( q_0 = 1 \) in the next three sections.) From Lemma 2.10(a), we see that the study of the Hom space \( \text{Hom}_{\mathcal{H}'}(x_{\lambda}^{\pi} \mathcal{H}', y_{\mu} \mathcal{H}') \) is important. In particular we need the base change property

\[
\text{Hom}_{\mathcal{H}}(x_{\lambda}^{\pi} \mathcal{H}, y_{\mu} \mathcal{H}) \otimes \mathcal{Z}' \cong \text{Hom}_{\mathcal{H}'}(x_{\lambda}^{\pi} \mathcal{H}', y_{\mu} \mathcal{H}') \cong x_{\mu} \mathcal{H}' \cap \mathcal{H}' x_{\lambda}^{\pi},
\]

which eventually gives the homological property Theorem 5.4 below as described in [7] (1.2.9-10)]. We need some preparation in this section and the next. We first look at the intersection \( \pi_a \mathcal{H}' \cap \mathcal{H}' \pi_b \).

Lemma 4.1. For any non-negative integers \( a,b \) with \( a+b \leq r \), the set \( \mathcal{B}_{a,b^-} := \{ \pi_a T_x \pi_b^- \mid x \in \mathcal{D}_0^{\omega_a,\omega_b} \} \) forms a basis of the \( \mathcal{Z}' \)-module \( \pi_a \mathcal{H}' \pi_b^- \). Moreover, we have \( \mathcal{B}_{a,b^-} = \{ \pi_a T_x \pi_b^- \mid x \in \mathcal{D}_{\omega_a,\omega_b} \} \setminus \{ 0 \} \). (That is, every \( \pi_a T_x \pi_b^- \) is either zero or a basis element.) A similar basis \( \mathcal{B}_{a,b^-} \) may be constructed for \( \pi_a \mathcal{H}' \pi_b \).

Proof. We first prove that the set generates \( \pi_a \mathcal{H}' \pi_b^- \). Clearly, \( \pi_a \mathcal{H}' \pi_b^- \) is generated by \( \pi_a T_w \pi_b^- \) (\( w \in \mathcal{W} \)). If \( w = x \hat{d} y \) with \( x \in W_a, y \in W_b \) is the Howlett decomposition, then \( \pi_a T_w \pi_b^- = \pi_a T_x T_y \pi_b^- = \pi_a h_1 T_{\hat{d}} h_2 \pi_b^- \) for some \( h_1 \in \mathcal{H}'(\hat{W}_a) \) and \( h_2 \in \mathcal{H}'(\hat{W}_b) \). By [4, 2.2.7], we have \( h_1 T_{\hat{d}} h_2 = \sum z \in \mathcal{D}_{\omega_a,\omega_b} a_z T_z \). It follows that \( \pi_a \mathcal{H}' \pi_b^- \) is generated by \( \{ \pi_a T_{\hat{d}} \pi_b^- \} \) \( \forall \hat{d} \in \mathcal{D}_{\omega_a,\omega_b} \). Suppose now that \( d \in \mathcal{D}_{\omega_a,\omega_b} \setminus \mathcal{D}_0^{\omega_a,\omega_b} \). Let \( d = u \hat{d} v \) be the right distinguished decomposition ([4, §2.3]), where \( \hat{d} \in \mathcal{D}_{\omega_a,\omega_b} \). Then \( s_0 \in C_{\omega_a} \cap C_{\omega_b} \), since \( C_{\omega_a} \cap C_{\omega_b} = C_{\omega_a} \cap C_{\omega_b} \). Thus, \( (1 + T_{s_0}) T_{\hat{d}} = T_{\hat{d}} (1 + T_{s_0}) \). Therefore, \( \pi_a T_{\hat{d}} \pi_b^- = \pi_a T_{\hat{d}} T_{\hat{d}} \pi_b^- T_{\hat{d}} = 0 \). This proves that \( \mathcal{B}_{a,b^-} \) generates \( \pi_a \mathcal{H}' \pi_b^- \), and the last assertion follows.
We now prove that the elements of \( \mathcal{B}_{a,b} \) are linearly independent. Assume \( h = \sum_{x \in \mathcal{D}^0_{\omega_a, \omega_b}} \alpha_d \pi_a T_d \bar{\pi}_b = 0 \) for some \( \alpha_d \in \mathcal{Z} \). Since
\[
T_{t_{1}^{1} \cdots t_{a}^{a}} T_{t_{1}^{a+1} \cdots t_{a+b}^{b}} \in \mathcal{H}'(W_a d W_b) := \sum_{w \in W_a d W_b} \mathcal{Z}' T_w \ (\varepsilon_i \in \{0, 1\}),
\]
we have \( 0 = pr_d(h) = \sum_{x \in \mathcal{D}^0_{\omega_a, \omega_b} \cap W_a d W_b} \alpha_d \pi_a T_x \bar{\pi}_b = 0 \), where \( pr_d \) is the projection onto \( \mathcal{H}'(W_a d W_b) \). Now, by a further consideration of the projection onto \( \mathcal{H}'(W_a d W_b) \), we see that all \( \alpha_x = 0 \). This proves the linear independence, and hence, the lemma.

\[ \square \]

**Corollary 4.2.** For any non-negative integers \( a, b \) with \( a \geq 1 \), the set
\[
\{ \pi_{a-1} T_{t_{a}^{a}} T_{t_{a+1}^{a+1}} T_{y^{-} n_{b}} | y \in \mathcal{D}^0_{\omega_{a+1}, \omega_{b}}, \varepsilon_j \in \{0, 1\} \forall j, d \in \mathcal{D}^0_{\omega_a, \omega_b} \setminus \{(\mathcal{D}^0_{\omega_{a+1}, \omega_{b}} \cup t_{a+1} \mathcal{D}^0_{\omega_{a+1}, \omega_{b}})\}
\]
is part of a basis for \( \pi_{a-1} \mathcal{H}' \bar{\pi}_b \).

**Proof.** Let \( t = t_{a}^{1} t_{a+1}^{2} \). We see by writing \( t = gz \), where \( z \in W_{a+1} \) and \( g \in tW_{a+1} \) is a distinguished left coset representative, that, for any \( u \in W_{a+1} \), if \( T_x \) appears as a term in \( T_u \) (i.e. with non-zero coefficient), then \( x = tx' \) for some \( x' \in W_{a+1} \). So \( n_0(cx) = n_0(c) + n_0(x) \) for all \( c \in C_{\omega_{a-1}} \). By Lemma \[\mathcal{L}\], we have \( x \in \mathcal{D}_{\omega_{a-1}} \cap W_{a+1} \). For \( w \in \mathcal{D}^0_{\omega_{a+1}, \omega_{b}} \), let \( w = uvw \) be the right distinguished decomposition with \( \bar{w} \in W_{a+1} w W_b \) distinguished (for parabolic subgroups). Recall from \[\mathcal{B}\] (2.3.1)] that \( v \in W_b \). Also, since \( C^w_{\omega_{a+1}} \cap C_{\omega_{b}} = 1 \), the distinguished left decomposition \( u' \bar{w} v' \) of \( x \bar{w} \) has \( v' \in W_b \). Consequently, \[\mathcal{L}\] (2.2.5)], \( x \bar{w} \in \mathcal{D}^{-1}_{\omega_{b}} \). On the other hand, since \( x \in \mathcal{D}_{\omega_{a-1}} \cap W_{a+1} \), we find \( x \bar{w} \in \mathcal{D}_{\omega_{a-1}} \). (Write \( x = x_1 x_2 \) with \( x_2 \) distinguished in \( W_{a-1} x \), and apply \[\mathcal{B}\] (2.2.7)).) Then
\[
T_{t} T_{w} = (T_{t} T_{u})T_{\bar{w}} T_{v} = \sum_{x \in \mathcal{D}_{\omega_{a-1}} \cap W_{a+1}} \alpha_x T_{x \bar{w}} T_{v} = q^m T_{tw} + \sum_{tw<z} \beta_z T_z,
\]
where all \( z \in \mathcal{D}_{\omega_{a-1}, \omega_{b}} \) by \[\mathcal{B}\] (2.2.7)] again. Since \( \pi_{a-1} T_{z} \pi_{b} = 0 \) for \( z \in \mathcal{D}_{\omega_{a-1}, \omega_{b}} \), we have
\[
\pi_{a-1} T_{t} T_{w} \pi_{b} = q^m \pi_{a-1} T_{tw} \pi_{b} + \sum_{tw<z} \beta_z \pi_{a-1} T_{z} \pi_{b}.
\]
A similar argument shows that this is also true for \( t = s_{a}^{1} t_{a}^{2} \) and \( w \in \mathcal{D}^0_{\omega_a, \omega_b} \setminus \{(\mathcal{D}^0_{\omega_{a+1}, \omega_{b}} \cup t_{a+1} \mathcal{D}^0_{\omega_{a+1}, \omega_{b}})\} \). Therefore, the given set is linearly independent and can be extended to a basis for \( \pi_{a-1} \mathcal{H}' \pi_{b} \) by Cor. \[\mathcal{L}\] 2 and Lemma \[\mathcal{L}\].
We now can show that the intersection \( \pi_a \mathcal{H}' \cap \mathcal{H}' \pi_b^- \) is free.

**Theorem 4.3.** Assume that \( q_0 + 1 \) is not a zero divisor in \( \mathbb{Z}' \). Then, for any positive integers \( a, b \leq r \), we have

\[
\pi_a \mathcal{H}' \cap \mathcal{H}' \pi_b^- = \pi_a \mathcal{H}' \pi_b^-.
\]

In particular, if \( a + b > r \), then the intersection is 0.

**Proof.** We apply induction on \( a \). Note first the elements, \( T_w \pi_w^- \), \( w \in \mathcal{D}_{\mathcal{w}}^{-1} \), form a basis of \( \mathcal{H}' \pi_b^- \) as in (the proof of) [3] (3.2.2a). Now let \( a = 1 \) and consider \( h = \sum_{w \in \mathcal{D}_{\mathcal{w}}^{-1}} \alpha_w T_w \pi_w^- \in \pi_1 \mathcal{H}' \). We have \( T_w h = q_0 h \).

For \( w \in \mathcal{D}_{\mathcal{w}}^{-1} \) write \( w = uwv \) with \( \hat{w} \in \langle s_0 \rangle \mathcal{W}_b \) distinguished and \( u \in \langle s_0 \rangle \), \( v \in \hat{W}_b \), and put \( \mathcal{V}(w) := \langle s_0 \rangle \hat{w} \cap \mathcal{W}_b \), a parabolic subgroup. Then \( T_w T_{\hat{w}} \pi_{\hat{w}}^- = -T_{u} \pi_{u}^- \) if \( s_0 \in \mathcal{V}(w) \). Thus,

\[
T_{s_0} h = \sum_{s_0 w > w} \alpha_w T_{s_0 w} \pi_w^- + \sum_{s_0 w < w} \alpha_w T_{s_0 w} \pi_w^- \\
= \sum_{s_0 w > w} -\alpha_w T_{w} \pi_w^- + \sum_{s_0 w > w} \alpha_w T_{s_0 w} \pi_w^- \\
+ \sum_{s_0 w < w} (q_0 - 1) \alpha_w T_{w} \pi_w^- + \sum_{s_0 w < w} q_0 \alpha_w T_{s_0 w} \pi_w^- 
\]

Note that the last equation gives \( T_{s_0} h \) as a linear combination of basis elements \( T_y \pi_y^- \), \( y \in \mathcal{D}_{\mathcal{w}}^{-1} \). Thus, equating the coefficients of \( T_w \pi_w^- \) in \( T_{s_0} h = q_0 h \), we obtain that

\[
\begin{cases}
(1) & \text{if } s_0 w < w, s_0 \not\in \mathcal{V}(w), \quad \text{then } \alpha_{s_0 w} + (q_0 - 1) \alpha_w = q_0 \alpha_w; \\
(2) & \text{if } s_0 w < w, s_0 \in \mathcal{V}(w), \quad \text{then } (q_0 - 1) \alpha_w = q_0 \alpha_w; \\
(3) & \text{if } s_0 w > w, s_0 \in \mathcal{V}(w), \quad \text{then } -\alpha_w + q_0 \alpha_{s_0 w} = q_0 \alpha_w.
\end{cases}
\]

There is a fourth case, but we do not require it. From (2), we have \( \alpha_w = 0 \), for \( s_0 w < w, s_0 \in \mathcal{V}(w) \). By (3) and noting that \( q_0 + 1 \) is not a zero-divisor, we have \( \alpha_w = 0 \), for \( s_0 w > w, s_0 \in \mathcal{V}(w) \). Thus, \( \alpha_{s_0 w} = \alpha_w = 0 \) for all \( w \) with \( s_0 \in \mathcal{V}(w) \). From (1) we see that \( \alpha_{s_0 w} = \alpha_w \) for all \( w \) with \( s_0 \not\in \mathcal{V}(w) \). Therefore, \( h \) can be written as \( (1 + T_{s_0}) h' \pi_b^- = \pi_1 h' \pi_b^- \) for some \( h' \in \mathcal{H}' \), proving the case for \( a = 1 \).

Assume now \( a \geq 1 \) and \( \pi_a \mathcal{H}' \cap \mathcal{H}' \pi_b^- = \pi_a \mathcal{H}' \pi_b^- \). Then, \( T_{s_0} \pi_a \mathcal{H}' \cap \mathcal{H}' \pi_b^- = T_{s_0} \pi_a \mathcal{H}' \pi_b^- \). We first prove that the result holds for \( a + 1 \) with \( a + 1 + b \leq r \). By [4] (4.1.2), we have \( \pi_{a+1} \mathcal{H}' = \pi_a \mathcal{H}' \cap T_{s_0} \pi_a \mathcal{H}' \). Thus, by induction, we obtain

\[
\pi_{a+1} \mathcal{H}' \cap \mathcal{H}' \pi_b^- = \pi_a \mathcal{H}' \cap T_{s_0} \pi_a \mathcal{H}' \pi_b^- = \pi_a \mathcal{H}' \pi_b^- \cong T_{s_0} \pi_a \mathcal{H}' \pi_b^-.
\]
We need to prove that
\[ \pi_a \mathcal{H}' \pi_b^- \cap T_{s_a} \pi_a \mathcal{H}' \pi_b^- = \pi_{a+1} \mathcal{H}' \pi_b^- . \]
The inclusion \( \supseteq \) is clear. Suppose \( h \in \pi_a \mathcal{H}' \pi_b^- \cap T_{s_a} \pi_a \mathcal{H}' \pi_b^- \). Then, by Lemma [4.4], \( h = \pi_ah_1 \pi_b^- = T_{s_a} \pi_ah_2 \pi_b^- \) where \( h_1 = \sum_{x \in \mathcal{D}_0} \alpha_x T_x \) and \( h_2 = \sum_{y \in \mathcal{D}_0} \beta_y T_y \). So we have \((\pi_ah_1 - T_{s_a} \pi_ah_2)\pi_b^- = 0\). Since \( \pi_{a+1} \mathcal{H}' = \pi_a \mathcal{H}' \cap T_{s_a} \pi_a \mathcal{H}' \), if we could prove that \( h_0 = \pi_ah_1 - T_{s_a} \pi_ah_2 = 0 \), i.e., \( h' = \pi_ah_1 = T_{s_a} \pi_ah_2 \), then we would have \( h' \in \pi_{a+1} \mathcal{H}' \), and therefore, \( h = h' \pi_b^- \in \pi_{a+1} \mathcal{H}' \pi_b^- \), as desired. So it remains to prove \( h_0 = 0 \). For simplicity of notation, we put \( D = \mathcal{D}_{\omega_a, \omega_b}^0 \setminus (\mathcal{D}_{\omega_{a+1}, \omega_b}^0 \cup t_{a+1} \mathcal{D}_{\omega_{a+1}, \omega_b}^0) \) and \( B = B_1 \cup B_2 \) where
\[
B_1 = \{ T_x, T_{ta} T_x, T_{ta+1} T_x, T_a T_{ta+1} T_x \mid x \in \mathcal{D}_{\omega_{a+1}, \omega_b}^0 \}, \quad \text{and} \quad
B_2 = \{ T_d, T_{td} T_d, T_{tad} T_d, T_{ta} T_d T_d \mid d \in D \}.
\]
Since \( a > 0 \), Cor. [4.2] gives the linear independent set \( B = \pi_{a-1} B \pi_b^- \) in \( \pi_{a-1} \mathcal{H}' \pi_b^- \). Write
\[ T_{s_a} \pi_ah_2 = \pi_{a-1}T_a (q^{a-1} - T_{s_a})h_2 = \pi_{a-1}(q^{a-1}T_{s_a}h_2 + T_{s_a}T_{ta}h_2), \]
and
\[ h_2 = \sum_{y \in \mathcal{D}_{\omega_{a+1}, \omega_b}^0} \beta_y T_y \]
and
\[ h_2 = \sum_{y \in \mathcal{D}_{\omega_{a+1}, \omega_b}^0} \beta_y T_y + \sum_{y \in \mathcal{D}_{\omega_{a+1}, \omega_b}^0} \beta_{ta+1} y T_{ta+1} T_y + \sum_{d \in D} \beta_d T_d . \]
Since \( y \in \mathcal{D}_{\omega_{a+1}, \omega_b}^0 \) implies \( s_a y, s_at_{a+1} y, t_{a+1}y \in \{ 1, t_a, t_{a+1} \} \mathcal{D}_{\omega_{a+1}, \omega_b}^0 \) (see Lemma [4.4(b)], and \( d \in D \) implies \( s_a d \in s_a D \), we see that \( T_{s_a}h_2 \) is a linear combination of the elements in \( B \). Likewise, \( T_{s_a} T_{ta} T_y \) (resp. \( T_{t_a} T_{ta} T_{ta+1} T_y \)) is a linear combination of \( T_{ta+1} T_w \) (resp. \( T_{ta} T_{ta+1} T_w \)) with \( w \in \mathcal{D}_{\omega_{a+1}, \omega_b}^0 \) and \( T_{s_a} T_{ta} T_d \) is a linear combination of the elements in \( B_2 \). Hence \( T_{s_a} T_{ta} h_2 \) is a linear combination of the elements of \( B \). On the other hand, \( \pi_ah_1 = \pi_{a-1}(q^{a-1} + T_{s_a})h_1 \) and \( (q^{a-1} + T_{s_a})h_1 \) is a linear combination of the elements of \( B \). Therefore, we see that \( h_0 \) is a linear combination of \( \pi_{a-1} T_{ta} w \) with \( w \in B \). Since \( B = \pi_{a-1} B \pi_b^- \) is a linear independent set, it follows from \( h_0 \pi_b^- = 0 \) that \( h_0 = 0 \).

Assume now that \( a + b \geq r \). Then \( \pi_{a+1} \mathcal{H}' \pi_b^- = 0 \) by [3.16]. We need to prove that \( \pi_{a+1} \mathcal{H}' \cap \mathcal{H}' \pi_b^- = 0 \). By \( (1.3.1) \), it suffices to prove that \( \pi_a \mathcal{H}' \pi_b^- \cap T_{s_a} \pi_a \mathcal{H}' \pi_b^- = 0 \) if \( a + b = r \). Let \( w_{a,b} \) be the (unique) distinguished \( \hat{W}_a \circ \hat{W}_b \) double coset representative with the trivial intersection property: \( \hat{W}_a \cap \hat{W}_b = \{ 1 \} \), as given at the beginning of §3, and let \( \pi(a, b^-) = \pi_a T_{w_{a,b}} \pi_b^- \) (cf. footnote 5 below). Then, by [3.11], \( \pi_a \mathcal{H}' \pi_b^- = \mathcal{H}'(\hat{W}_{(a,b)}) \pi(a, b^-) \). Now the linear independence (see [3.3]) between bases \( B = \{ T_w \pi(a, b^-) \mid w \in \hat{W}_{(a,b)} \} \) for \( \pi_a \mathcal{H}' \pi_b^- \).
and $T_{x_0}\mathcal{B}$ for $T_{x_0}\pi_a\mathcal{H}'\pi_b^{-1}$ implies that the intersection is trivial. This completes the proof of the theorem.

We have the following generalization which has been known for parabolic subgroups. Recall the definition of $y_\lambda$ in Remark 2.2c.

**Theorem 4.4.** Assume that both $q_0+1$ and $q+1$ are not zero divisors in $\mathcal{Z}'$. For bicompositions $\lambda, \mu$ of $r$, we have $x_\lambda\mathcal{H}' \cap \mathcal{H}'y_\mu = x_\lambda\mathcal{H}'y_\mu$. Moreover, this intersection is $\mathcal{Z}'$-free with basis $\{x_\lambda T_w y_\mu \mid w \in \mathcal{D}_{\lambda,\mu}^0\}$. A similar result holds for $y_\lambda\mathcal{H}' \cap \mathcal{H}'x_\mu$.

**Proof.** Clearly, we have $x_\lambda\mathcal{H}' \cap \mathcal{H}'y_\mu \supseteq x_\lambda\mathcal{H}'y_\mu$. Applying (1, (4.1.3)) and Theorem 1.1, we obtain $x_\lambda\mathcal{H}' \cap \mathcal{H}'y_\mu = \bar{x}_\lambda\mathcal{H}' \cap \mathcal{H}'\bar{y}_\mu \cap \pi_a\mathcal{H}'\pi_b^{-1}$. By (8, 1.1h), we have $\bar{x}_\lambda\mathcal{H}' \cap \mathcal{H}'\bar{y}_\mu = \bar{x}_\lambda\mathcal{H}'\bar{y}_\mu$ since both $\bar{W}_\lambda$ and $\bar{W}_\mu$ are parabolic. (This requires that $q+1$ is not a zero divisor.) Suppose $h \in \bar{x}_\lambda\mathcal{H}'\bar{y}_\mu \cap \pi_a\mathcal{H}'\pi_b^{-1}$ and write $h = \sum_{w \in \mathcal{D}_{\omega_a,\omega_b}} \alpha_w \pi_a T_w \pi_b^{-1}$. Then, $T_y h = qh$ and $hT_t = -h$ for all $s = s_t \in \bar{W}_\lambda$ and $t = s_j \in \bar{W}_\mu$. Since we have $sw, wt \in \mathcal{D}_{\omega_a,\omega_b}^0$ whenever $w \in \mathcal{D}_{\omega_a,\omega_b}^0$ (see Lemma 1.1b), it follows from Lemma 1.1, for $h' = \sum_{w \in \mathcal{D}_{\omega_a,\omega_b}^0} \alpha_w \pi_a T_w$ with $h = \pi_a h' \pi_b^{-1}$, that $T_y h' = qh'$ and $h'T_t = -h'$ for all $s = s_t \in \bar{W}_\lambda$ and $t = s_j \in \bar{W}_\mu$. Therefore, $h' = \sum_{w \in \mathcal{D}_{\omega_a,\omega_b}^0 \cap \mathcal{D}_{\lambda,\mu}^0} \alpha_w \bar{x}_\lambda T_w \bar{y}_\mu$ and $h = \sum_{w \in \mathcal{D}_{\lambda,\mu}^0 \cap \mathcal{D}_{\omega_a,\omega_b}^0} \alpha_w x_\lambda T_w y_\mu$, proving the first assertion. Since $\mathcal{D}_{\lambda,\mu}^0 \cap \mathcal{D}_{\omega_a,\omega_b}^0 = \mathcal{D}_{\lambda,\mu}^0$, the basis assertion follows immediately. The final claim may be established by applying the standard anti-automorphism $T_w \mapsto T_{w^{-1}}$. □

**Corollary 4.5.** Maintain the notation above and assume that both $q_0+1$ and $q+1$ are not zero divisors in $\mathcal{Z}'$. Then $\text{Hom}_{\mathcal{H}'}(y_\mu\mathcal{H}', x_\lambda\mathcal{H}')$ is free and

$$\text{Hom}_{\mathcal{H}'}(y_\mu\mathcal{H}', x_\lambda\mathcal{H}') \cong x_\lambda\mathcal{H}'y_\mu.$$ A similar statement holds with the roles of $x_\lambda$ and $y_\mu$ interchanged.

5. **Induced bistandard bases**

Throughout the section, we assume $a+b=r$ and $q_0+1$ is not a zero-divisor in $\mathcal{Z}'$. Recall from the proof of Theorem 4.4 that $w_{a,b}$ is the (unique) distinguished $\tilde{W}_a \tilde{W}_b$ double coset representative with the trivial intersection property.

**Lemma 5.1.** If $a+b=r$, then $\mathcal{D}_{\omega_a,\omega_b}^0 = \tilde{W}_a w_{a,b} \tilde{W}_b$.

---

4If we put $q_{s_0} = q_0$, $q_{s_i} = q$ for $1 \leq i \leq r-1$, and define for $w \in W$, $q_w = q_{s_i} \cdots q_{s_m}$, where $w = s_i \cdots s_m$ is a reduced expression, then the ring homomorphism on $\mathcal{H}'$ sending $q_s$ to $q_s^{-1}$ and $T_w$ to $(-1)^{\ell(w)} q_w T_w$ will interchange $x_\lambda$ with $y_\lambda$. Thus, (8, (4.1.3)) holds for $y_\lambda\mathcal{H}'$, and also for $\mathcal{H}'y_\lambda$. 


Proof. Let \( w \in D_{\omega_1, \omega_2}^d \), and let \( w = udv \) be the right distinguished decomposition of \( w \) with \( d \in W_a w W_b \) distinguished. Since \( C_{\omega_1}^d \cap C_{\omega_2}^d = \{1\} \), it follows that \( n_0(t_1 \cdots t_a d t_1 \cdots t_b) = a + n_0(d) + b \geq r \), and hence \( n_0(d) = 0 \) and \( d \in \bar{W} \). Since \( d \) is distinguished and
\[
\{(1)d, \cdots (a)d\} = \{b+1, \cdots r\},
\]
we must have \( d = w_{a,b} \). The rest of the proof follows immediately from the trivial intersection property of \( w_{a,b} \).

\[\square\]

Put \( \pi(a, b^-) = \pi_a T_{w_{a,b}} \pi_b^- \) and \( \pi(a^-, b) = \pi_a^- T_{w_{a,b}} \pi_b \) and write \( \mathcal{H}'_{\pi_m} = \mathcal{H}'(W_n) \). The latter may be regarded as the image of \( \mathcal{H}' \) under the obvious \((Z'-module)\) epimorphism \( h \mapsto \bar{h} \) from \( \mathcal{H}' \) to \( \mathcal{H}' \).

Corollary 5.2. If \( a + b = r \), then \( \pi_a \mathcal{H}' \pi_b^- \) has basis \( \{T_u \pi(a, b^-) T_v \mid u \in W_a, v \in W_b\} \). A similar result holds for \( \pi_a^- \mathcal{H}' \pi_b \). In particular, we have \( \pi_a \mathcal{H}' \pi_b^- = \mathcal{H}_a \pi(a, b^-) \mathcal{H}_b' \) and \( \pi_a^- \mathcal{H}' \pi_b = \mathcal{H}_a' \pi(a^-, b) \mathcal{H}_b \).

Proof. Immediately follows from Lemma [3] and the lemma above. \[\square\]

For partitions \( \alpha, \beta \) of \( a \), let \( \mathcal{H}_{a, \alpha} = \bar{x}_a \mathcal{H}_a' \cap \bar{x}_a \mathcal{H}_a \mathcal{H}_a \beta \). This is a free \( Z' \)-module.

Theorem 5.3. Let \( \lambda, \mu \) be bicompositions of \( r \) such that \( |\lambda(1)| + |\mu(1)| = r \). Then
\[
x_{\lambda}^\mu \mathcal{H}' \cap \mathcal{H}' x_{\mu} = \mathcal{H}_{a, \alpha}^{(1)} \pi_{\lambda(1)} \mu(2) \mathcal{H}_b' \mathcal{H}_b \pi_{\lambda(2)} \mu(1),
\]
where \( a = |\lambda(1)| \) and \( b = |\mu(1)| \).

Proof. We first note that \( w_{a,b} \in \bar{W}_{(a,b)} \bar{W}_{(b,a)} \) is distinguished and \( \bar{W}_{(a,b)} \bar{W}_{(b,a)} = \bar{W}_{(b,a)} \). Thus, we have
\[
\bar{x}_\alpha \mathcal{H}'(\bar{W}_{(a,b)}) \pi(a^-, b) = \bar{x}_\lambda(1) \mathcal{H}_a \pi(a^-, b) \bar{x}_\lambda(2) \mathcal{H}_b = \pi(a^-, b) \bar{x}_\lambda \mathcal{H}'(\bar{W}_{(b,a)})
\]
for all \( \lambda \) with \( |\lambda(1)| = a \) and \( |\lambda(2)| = b = r - a \). Thus, by [3] (4.1.3) and Theorem 4.4 (applying \( \eta \) if necessary), we have
\[
(5.3.1)\quad x_{\lambda}^\mu \mathcal{H}' \cap \mathcal{H}' x_{\mu} = (x_{\lambda}^\mu \mathcal{H}' \pi_{\lambda}) \cap (\mathcal{H}' x_{\mu} \pi_{\mu} - \mathcal{H}' \pi_{\mu})
\]
\[
= (x_{\lambda}^\mu \mathcal{H}' \pi_{\lambda} \mathcal{H}' \pi_{\mu}) \cap (\pi_{\lambda} \mathcal{H}' \pi_{\mu} - \mathcal{H}' x_{\mu}) = x_{\lambda}^\mu \mathcal{H}' x_{\mu} \mathcal{H}' \pi_{\mu} \pi_{\lambda} \mathcal{H}' x_{\mu}
\]
\[
= (\bar{x}_\lambda(1) \mathcal{H}_a \pi(a^-, b) \bar{x}_\lambda(2) \mathcal{H}_b) \cap (\mathcal{H}_a = \mathcal{H}_b \mathcal{H}_a \mathcal{H}_b ' = \mathcal{H}_a \mathcal{H}_b ' = \mathcal{H}_a' \mathcal{H}_b' \mathcal{H}_b \pi_{\mu} \pi_{\lambda} \mathcal{H}_b ')
\]
By Corollary 5.2, we have clearly that, if \( h_1 \pi(a^-, b) h_2 = h_1' \pi(a^-, b) h_2' \), where \( h_1 = \bar{x}_\lambda(1) \mathcal{H}_b ' \), \( h_1 ' = \bar{x}_\lambda(2) \mathcal{H}_b ' \), \( h_2 = \bar{x}_\lambda(2) \mathcal{H}_b ' \) and \( h_2 ' = \bar{x}_\lambda(2) \mathcal{H}_b ' \), then \( h_1 = h_1 ' \) and \( h_2 = h_2 ' \). Thus, \( h_1 \in \mathcal{H}_a^{(2)} \mathcal{H}_b ' \), \( h_2 \in \mathcal{H}_a^{(2)} \mathcal{H}_b ' \), and
\[^5\text{We use superscript } - \text{ to indicate the "minus" part. The notation } \pi(a, b^-) \text{ is denoted by } v_{a,b} \text{ in [3].}\]
consequently, we have \( x_\lambda^\eta \mathcal{H}' \cap \mathcal{H}' x_\mu = \mathcal{H}^{\lambda(1)\mu(2)}_a \pi(a^-, b) \mathcal{H}^{\lambda(2)\mu(1)}_b \), proving the theorem.

The theorem above guarantees the existence of a Murphy-type basis for the intersection \( x_\lambda^\eta \mathcal{H}' \cap \mathcal{H}' x_\mu \). We are now going to describe this basis.

Recall from [14] the Murphy basis \( \{ x_{st}^a := T_{\delta(u^{-1})} \} \) of \( \mathcal{H}'_a \), where \( s, t \) are standard \( \beta \)-tableaux for all partitions \( \beta \) of \( a \) and \( \delta(u) \) is a distinguished right \( \mathfrak{S}_\beta \)-coset representative defined by \( u \). Let \( T^s(\beta) \) denote the set of standard \( \beta \)-tableaux, and let, for \( \alpha \in \Lambda(n, r) \), \( \mathfrak{S}^{ss}(\beta, \alpha) \) be the set of semi-standard \( \beta \)-tableaux of type \( \alpha \). These have all been generalized to bipartitions and bicompositions in [9, §1]. Thus, for \( \lambda \in \Pi_2 \) and \( \mu \in \Pi_2 \), the corresponding sets are denoted by \( T^s(\mu) \) and \( \mathfrak{S}^{ss}(\mu, \lambda) \), respectively. We refer the reader to [9, §1] for the notion of standard and semi-standard bitableaux. Recall also from [9, §1] the \( \delta \)-function which takes a standard (bi)tableau or a semi-standard (bi)tableau to a certain distinguished coset representative in \( \mathfrak{H} \). In particular, the Murphy basis induces a basis (see [14] and [9]) \( m^s_{st} \) for the module \( \mathcal{H}^{\alpha\beta}_a \), for any given compositions \( \alpha, \beta \) of \( a \), where \( s, t \in \mathfrak{S}^{ss}(\alpha, \beta) \), and \( \gamma \) is a partition of \( a \). Note that the basis element is simply a sum of certain Murphy basis elements:

\[
m^s_{st} = \sum_{s \in T_s, t \in T_t} x^a_{st}.
\]

We have immediately the following.

**Theorem 5.4.** Let \( \lambda, \mu \) be bicompositions of \( r \) such that \( |\lambda(1)| + |\mu(1)| = r \). Then the set \( \mathcal{Y}_\lambda^\mu = \{ m^a_{s_1 t_1} \pi(a^-, b) m^b_{s_2 t_2} \} \), where \( a = |\lambda(1)| \), \( b = |\mu(1)| \) and \( \{ m^a_{s_1 t_1} \} \) and \( \{ m^b_{s_2 t_2} \} \) are bases for \( \mathcal{H}^{\lambda(1)\mu(2)}_a \) and \( \mathcal{H}^{\lambda(2)\mu(1)}_b \), respectively, forms a basis for \( x_\lambda^\eta \mathcal{H}' \cap \mathcal{H}' x_\mu \). In particular, the Murphy bases for \( \mathcal{H}'_a \) and \( \mathcal{H}'_b \) induce a Murphy type basis \( \mathcal{Y}_{a^{-}, b} = \mathcal{Y}_{\omega_a^{-}, \omega_b} \) for \( \mathcal{H}' \cap \mathcal{H}' \cap \mathcal{H}' \).

The bases \( \mathcal{Y}_{\lambda^{-}, \mu} \) for \( x_\lambda^\eta \mathcal{H}' \cap \mathcal{H}' x_\mu \) are constructed by using the Murphy type bases for the type \( A \) intersections \( \mathcal{H}^{\alpha\beta}_a \). However, we are also able to construct them directly by using Murphy type bases \( \mathcal{X}_{\lambda, \mu} \), defined in [9] and called the bistandard bases, for the type \( B \) intersections \( x_\lambda \mathcal{H}' \cap \mathcal{H}' x_\mu \). We will see that such a construction gives us certain homological properties required in stratifying an endomorphism algebra.
Let $\mathcal{Y}_{a,b^*} = \eta(\mathcal{Y}_{a^*,b})$ and $\Pi_{2,a} = \{ \lambda \in \Pi_2 \mid |\lambda(1)| = a \}$. Then

$$\mathcal{Y}_{a,b^*} = \{ x_{s,t}^a, \pi(a, b^*)x_{s,t}^b \mid s, t \in T^s(\nu(i)), \nu \in \Pi_{2,a}^+ \},$$

replacing $\pi(a, b^*)$ by $\pi(a^*, b)$, we obtain $\mathcal{Y}_{a^*,b}$. We shall call the bases $\mathcal{Y}_{a,b^*}$ and $\mathcal{Y}_{a^*,b}$ induced bistandard bases.

Recall from [9, §1.2.2] the bistandard basis of $x_\lambda \mathcal{H}' \cap \mathcal{H}'x_\mu$:

$$X_{\lambda,\mu} = \{ X_{st}^\nu \mid \nu \in \Pi_2^+, s \in \mathcal{T}_{ss}(\nu, \lambda), t \in \mathcal{T}_{ss}(\nu, \mu) \},$$

where $X_{st}^\nu = \sum_{t \in T_1} T_{\delta(s)-1}^t x_\nu T_{\delta(t)}$. Since $\mathcal{T}_{ss}(\nu, \omega_0) = \mathcal{T}(\eta)$ where $\omega_0 = (-, 1^r)$, the set

$$X_{\lambda,\omega_0} = \{ X_{st}^\nu \mid \nu \in \Pi_2^+, s \in \mathcal{T}_{ss}(\nu, \lambda), t \in \mathcal{T}(\eta) \},$$

respectively,

$$X_{\lambda,\omega_0} = \{ X_{st}^\nu \mid \nu \in \Pi_2^+, t \in \mathcal{T}_{ss}(\nu, \mu), s \in \mathcal{T}(\eta) \}$$

is a basis for $x_\lambda \mathcal{H}'$, respectively, for $\mathcal{H}'x_\mu$.

**Theorem 5.5.** Assume $a + b = r$. We have that $\mathcal{Y}_{a^*,b} = \pi_a^- \mathcal{X}_{\omega_0,\omega_b} \setminus \{0\}$ and $\mathcal{Y}_{a,b} = \mathcal{X}_{\omega_0,\omega_b} \pi_b^- \setminus \{0\}$ form bases for $\pi_a^- \mathcal{H}' \pi_b$ and $\pi_a \mathcal{H}' \pi_b^-$, respectively.

**Proof.** We prove the first case. The second case can be obtained by applying the involution $\iota : T_w \mapsto T_{w^{-1}}$ to the first. Since $X_{st}^\nu = \sum_{t \in T_1} T_{\delta(s)-1}^t x_\nu T_{\delta(t)}$, we have by Lemma [4.1] $\pi_a^- X_{st}^\nu = 0$ whenever $|\nu(1)| > b = r - a$, or $|\nu(1)| = b$ and $\delta(s)^{-1} \notin \mathcal{D}_{\omega_0,\omega_b}$. Assume now $\pi_a^- X_{st}^\nu \neq 0$. Then $|\nu(1)| = b$ and $\delta(s)^{-1} \notin \mathcal{D}_{\omega_0,\omega_b}$.

We now note that, if $t = (t_1, t_2) \in \mathcal{T}_{ss}(\nu, \omega_b)$, then, by the definition [3, (2.2)], $t_1$ is semi-standard and contains a $\nu(1)$-tableau of type $(1^r)$. Thus, $|\nu(1)| = b$ forces that $t_1$ is a standard $\nu(1)$-tableau, i.e., $t_1 \in \mathcal{T}(\nu(1))$. Therefore, the set $\mathcal{T}_{ss}(\nu, \omega_b)$ can be identified with $\mathcal{T}(\nu(1)) \times \mathcal{T}(\nu(2))$. So $|T_1| = 1$ and, for $t \in \mathcal{T}_{ss}(\nu, \omega_b), s \in \mathcal{T}(\nu)$, $X_{st}^\nu = \sum_{t \in T_1} X_{st}^\nu = X_{st}^\nu$ where $t = \{ t \} = \{ (t_1, t_2), w_{a,b}, \}$. On the other hand, since $\mathcal{D}_{\omega_0,\omega_b} = \mathcal{W}_{a,b} \mathcal{W}_b$ by Lemma [5.1], it follows that $\delta(s) = \delta(s_1) w_{a,b} \delta(s_2)$ for some $(s_1, s_2) \in \mathcal{T}(\nu(1)) \times \mathcal{T}(\nu(2))$. Therefore, we have

$$\begin{align*}
\pi_a^- X_{st}^\nu &= \pi_a^- T_{\delta(s)^{-1}} \pi_b \bar{x}_\nu T_{\delta(t)} \\
&= \pi_a^- T_{\delta(s_1)^{-1}} T_{w_{a,b}} T_{\delta(s_2)^{-1}} \pi_b \bar{x}_\nu T_{\delta(t)} \\
&= (T_{\delta(s_1)^{-1}} \pi_b \bar{x}_\nu T_{\delta(t_2)}) \pi(a, b^*)(T_{\delta(s_1)^{-1}} \bar{x}_\nu T_{\delta(t_1)}) \\
&= x_{s,t}^a \pi(a, b^*) x_{s,t}^b,
\end{align*}$$

proving the inclusion $\pi_a^- \mathcal{X}_{\omega_0,\omega_b} \setminus \{0\} \subseteq \mathcal{Y}_{a^*,b}$. The above relation also proves that $\mathcal{Y}_{a^*,b}$ is a subset of $\pi_a^- \mathcal{X}_{\omega_0,\omega_b}$, hence they are equal. \(\square\)
For $\nu \in \Pi_{2,a}^+$, let $T_s^*(\nu) = T_s^*(\nu^{(1)}) \times T_s^*(\nu^{(2)})w_{b,a}$. Then $T_s^*(\nu)$ is the subset of $T_s^*(\nu)$ consisting of all standard $\nu$-bitableau $t = (t_1, t_2)$ such that $t_1$ has entries $1, 2, \ldots, a$.

We put, for $\lambda \in \Pi_{2,a}$,
\[
\Sigma^s_\lambda(\nu, \lambda) = \{s \in \Sigma^s_\lambda(\nu, \lambda) | T_s \subseteq T^s_\lambda(\nu)\} = \delta(T^s_\lambda(\nu)).
\]

**Corollary 5.6.** We have for $\lambda \in \Pi_{2,a}$ and $\mu \in \Pi_{2,b}$

\[
\mathcal{Y}_{\omega_\lambda, \mu} = \pi^-_\lambda X_{\omega_\lambda, \mu} \setminus \{0\}
\]

\[
= \{\pi^-_\lambda X^\nu_{st} | \nu \in \Pi_{2,b}^+, t \in \Sigma^s_\lambda(\nu, \mu), s \in T^s_\lambda(\nu), \delta(s) \in \mathcal{D}_{\omega_\mu, \omega_\lambda}^0\}, \text{ and}
\]

\[
\mathcal{Y}_{\lambda^- \omega_\mu} = \eta(X_{\lambda \omega_\mu}) \pi^-_\lambda \setminus \{0\}
\]

\[
= \{\eta(X^\nu_{st}) \pi^-_\lambda | \nu \in \Pi_{2,a}^+, s \in \Sigma^s_\lambda(\nu, \lambda), t \in T^s_\lambda(\nu), \delta(t) \in \mathcal{D}_{\omega_\mu, \omega_\lambda}^0\}.
\]

**Proof.** First, using an argument similar to (5.3.1), we see that $\mathcal{Y}_{\omega_\lambda, \mu} \subset \pi^-_\lambda X_{\omega_\lambda, \mu}$. Suppose $X^\nu_{st} \in X_{\omega_\lambda, \mu}$. Then, by a similar argument as above, $\pi^-_\lambda X^\nu_{st} \neq 0$ implies that $\delta(s)^{-1} \in \mathcal{D}_{\omega_\lambda, \omega_\mu}^0$ and $\nu \in \Pi_{2,b}^+$. Thus, $\delta(s)^{-1} = \delta(s_2)^{-1}w_{b,a}\delta(s_1)^{-1}$ for some $s_i \in T(\nu^{(i)})$. On the other hand, since $\nu \in \Pi_{2,b}^+$, we have $T_1 \subseteq T_b(\nu)$ and so $t \in \Sigma^s_{a, b}(\nu, \mu)$. Therefore, by (5.3.1), one sees easily that $\pi^-_\lambda X^\nu_{st} \in \mathcal{Y}_{\omega_\lambda, \mu}$. $\square$

For any bitableau $u = (u_1, u_2)$, define $u^* = (u_2, u_1)$. Then, $*$ sends a standard $\nu$-tableau to a standard $\nu^*$-tableau.

**Lemma 5.7.** If $\nu \in \Pi_{2,b}^+$ and $t \in T_b^s(\nu)$, then $\delta(t^*) = w_{b,a}\delta(t) \in \mathcal{D}_{\omega_\mu, \omega_\lambda}^0$, and

\[
\{w_{b,a}\delta(t) | t \in T_b^s(\nu^*)\} = \{\delta(s) | s \in T^s(\nu), \delta(s) \in \mathcal{D}_{\omega_\lambda, \omega_\mu}^0\}.
\]

In particular, we have

\[
T_b^s(\nu^*) = \{s | s \in T^s(\nu), \delta(s) \in \mathcal{D}_{\omega_\lambda, \omega_\mu}^0\}.
\]

**Proof.** Write $t = (t_1, t_2w_{a,b})$ for some $t_i \in T^s(\nu^{(i)})$. Then $\delta(t) = \delta(t_1)(w_{b,a}\delta(t_2)w_{a,b})$ and $t^* = (t_2w_{a,b}, t_1)$. If $t^\nu$ denotes the standard $\nu$-bitableau in which the numbers $1, 2, \ldots, r$ appear in the same order down successive rows in the first diagram of $\lambda$ and then in the second diagram, then

\[
t^\nu\delta(t^*) = (t_2w_{a,b}, t_1) = (t_2, t_1w_{b,a})w_{a,b}
\]

\[
= (t_1^*, t_2^\nu)(\delta(t_2)w_{a,b}\delta(t_1))w_{a,b} = t^\nu\delta(t_2)w_{a,b}\delta(t_1).
\]

So $\delta(t^*) = w_{a,b}\delta(t)$.

Now the inclusion $\subseteq$ follows immediately with $\nu$ replaced by $\nu^*$. Conversely, suppose $s \in T^s(\nu)$ and $\delta(s) \in \mathcal{D}_{\omega_\mu, \omega_\lambda}^0$. Then $\delta(s) = \delta(s_1)w_{b,a}\delta(s_2)$ for some $s_i \in T(\nu^{(i)})$, and $s = t^\nu\delta(s) = (s_1w_{b,a}, s_2)$. 

So \( s^* = (s_2, s_1 w_{b, a}) \in T_b^s(\nu^*) \) and \( \delta(s) = w_{b, a} \delta(s^*) \), proving the equality. \( \square \)

We now have an alternate description of the basis \( \mathcal{Y}_{\lambda, \mu} \) defined in \([4]\). For \( \nu \in \Pi^*_{2, b} \), \( s^* \in \mathcal{Y}^s_b(\nu, \lambda^*) \) and \( t \in \mathcal{Y}^s_b(\nu, \mu) \), let

\[
Z^\nu_{st} = \sum_{s \in T_s} \pi_a X^\nu_{st}.
\]

**Theorem 5.8.** We have, for \( \lambda \in \Pi_{2, a}, \mu \in \Pi_{2, b} \), the set

\[
\{ Z^\nu_{st} \mid \nu \in \Pi^*_{2, b}, s^* \in \mathcal{Y}^s_b(\nu, \lambda^*), t \in \mathcal{Y}^s_b(\nu, \mu) \}
\]

forms a basis for \( x^0_\lambda \mathcal{H}' \cap \mathcal{H}' x_\mu \). Moreover, it coincides with the basis \( \mathcal{Y}_{\lambda, \mu} \) defined in \([4]\).

**Proof.** Note first that \( s \in \mathcal{Y}_b(\nu^*, \lambda) \) is a \( \nu^* \)-tableau of type \( \lambda \) (not necessarily semi-standard) and that, if \( t \in T_t \) with \( t \in \mathcal{Y}^s_b(\nu, \mu) \), then \( t = (t_1, t_2 w_{a, b}) \) and \( \delta(t) = \delta(t_1)(w_{b, a} \delta(t_2) w_{a, b}) \) for some \( t_1 \in T(\nu^0) \), where \( \delta(t_1) \in S_{(1, \ldots, b)} \) and \( w_{a, b} \delta(t_2) w_{a, b} \in S_{(b+1, \ldots, r)} \). Thus, \( t^* = (t_2 w_{a, b}, t_1) \in T_{t^*} \) and \( t^* \in \mathcal{Y}_a(\nu^*, \mu^*) \). So we have

\[
Z^\nu_{st} = \sum_{s \in T_s, t \in T_t} \pi_a T_{\delta(s)}^{-1} \pi_b \pi_t T_{\delta(t)}
\]

\[
= \sum_{s \in T_s, t \in T_t} \pi_a T_{\delta(s_2)}^{-1} T_{w_{a, b} \delta(s_1)}^{-1} \pi_b \pi_t T_{w_{b, a} \delta(t_2) w_{a, b} \delta(t_1)}
\]

\[
= \sum_{s^* \in T_{s^*}, t \in T_t} T_{\delta(s_2)}^{-1} T_{w_{a, b} \delta(s_1)}^{-1} \pi_a \pi_t T_{w_{b, a} \delta(t_2) w_{a, b} \delta(t_1)}
\]

\[
= \sum_{s^* \in T_{s^*}, t^* \in T_{t^*}} T_{\delta(s^*)}^{-1} \pi_a \pi_t T_{\delta(t^*) \pi_b}
\]

\[
= \sum_{t^* \in T_{t^*}} \eta(X^*_{s^*, t^*}) \pi_b
\]

So \( Z^\nu_{st} \in x^0_\lambda \mathcal{H}' \cap \mathcal{H}' x_\mu \) since \( x^0_\lambda \mathcal{H}' \cap \mathcal{H}' x_\mu = x^0_\lambda \mathcal{H}' \pi_b \cap \pi_a \mathcal{H}' x_\mu \). Clearly, the elements \( Z^\nu_{st} \) are linearly independent.

We now prove the spanning condition. Suppose \( h \in x^0_\lambda \mathcal{H}' \cap \mathcal{H}' x_\mu \). Then \( h \in x^0_\lambda \mathcal{H}' \pi_b \cap \pi_a \mathcal{H}' x_\mu \), and, by Cor. \([5, 6]\),

\[
h = \sum_{\nu, s, t} \alpha_{s, t} \pi_a X^\nu_{st} = \sum_{\nu', s, t} \beta_{s, t} \eta(X^\nu_{s, t}) \pi_b.
\]
Equating the coefficients with respect to the basis $\mathcal{Y}_{\alpha^{-}, b}$, we see that $\alpha_{s, t}^{\nu} = \beta_{s, t}^{\nu}$ whenever $s \in T_{{\nu}}^*$ and $t \in T_{{\nu}}^*$. Consequently, $h$ is a linear combination of the elements $Z_{st}^\nu$.

To see the last assertion, we observe from the third equality above that

$$Z_{st}^\nu = \sum_{s' \in T_{{\nu}}, t \in T_1} T_{\delta(s_2)}^{-1} T_{\delta(s_1)}^{-1} \bar{\alpha}_{\nu}(2) \bar{\pi}(a^{-}, b) \bar{\alpha}_{\nu}(1) T_{\delta(t_1)} T_{\delta(t_2)} T_{\delta(t_2)} T_{\delta(t_1)}$$

which is an element in $\mathcal{Y}_{\lambda^{-}, \mu}$. \qed

6. The homological property

In this section, we will prove the homological property required in stratifying the $q$-Schur algebra. The validity of such property is closely related to the existence of nice bases as is seen in [7] where a Kazhdan-Lusztig basis was used for parabolic subgroups. We will see below that the bistandard basis and those they induce for quasi-parabolic groups of type $B$ Hecke algebras are sufficient to guarantee the homological property. We need some preparation. Recall from [3] (1.2.1a)] the definition of the dominance order $\succeq$ on bicompositions. That is, $\lambda \succeq \mu$ iff $\sum_{i=1}^{j} \lambda(i)_i \leq \sum_{i=1}^{j'} \mu(i)_i$ for all $j$ and $|\lambda^{(1)}| + \sum_{i=1}^{j'} \lambda(2)_i \leq |\mu^{(1)}| + \sum_{i=1}^{j'} \mu(2)_i$ for all $j'$.

For $\mu \in \Pi_{2,b}^+$, let $\Pi_{\mu}^+ = \{ \nu \in \Pi_{2,b}^+ | \nu \succeq \mu \}$ and $\Pi_{\mu, b}^+ = \{ \nu \in \Pi_{2,b}^+ | \nu \succeq \mu \}$. Note that $\Pi_{\mu}^+$ indexes the composition factors of $\mathcal{H}_K x_\mu$, where $K$ is the quotient field of $\mathcal{Z}$. We linearly order $\Pi_{\mu}^+$ by $\nu(1) \leq \nu(2) \leq \cdots$ such that $\nu(i) \succeq \nu(j)$ implies $i \leq j$. Fix an ordering on each $\mathcal{S}^{ss}(\nu(i), \mu)$ and concatenate them via the linear ordering on the $\nu(i)$ to obtain a linear ordering $t_1 \preceq t_2 \preceq \cdots \preceq t_n$ on $\cup_{i=1}^{n_\mu} \mathcal{S}^{ss}(\nu(i), \mu)$. Thus, using this ordering, the basis $X_{\omega_{0}, \mu}$ allows us to define an integral left (twisted) “Specht” filtration for $\mathcal{H}_t x_\mu$ by setting:

$$\mathcal{E}_{\nu}^\mu = \text{span}\{ X_{\nu}^{s(t_j)} | 1 \leq j \leq i, s \in T_{{\nu}}(\nu(t_j)) \}, \ 1 \leq i \leq n_\mu,$$

$$\mathcal{E}_{0, \nu}^\lambda = 0.$$

Here $\nu(t_j)$ is the shape of $t_j$, that is, the partition whose Young diagram is the underlying diagram of $t_j$. We shall write $\mathcal{E}_{st}^\nu$ for $\mathcal{E}_{st}^\nu$. 

Note that the restriction of the ordering on $\Pi_{\mu}^+$ induces an ordering $\nu(i_1) \preceq \nu(i_2) \preceq \cdots$ on $\Pi_{\mu, b}^+$, and hence an ordering $t_{j_1} \preceq t_{j_2} \preceq \cdots \preceq t_{j_n}$ on $\cup_{i=1}^{n_\mu} \mathcal{S}^{ss}(\nu(i), \mu)$. The following lemma is an easy consequence of Theorem 5.8.
Lemma 6.1. Let $\lambda, \mu \in \Pi^+(r)$ such that $|\lambda(1)| + |\mu(1)| = r$, and let 

$$0 = \mathcal{E}^\mu_{0 Z'} \subseteq \mathcal{E}^\mu_{1 Z'} \subseteq \cdots \subseteq \mathcal{E}^\mu_{n_\mu Z'} = H' x_\mu$$

be the filtration defined as above. For each $i$, let $j_{m(i)}$ be the maximal index in the sequence $j_1, \ldots, j_{m(i)}$ such that $j_{m(i)} \leq i$. Then $x_\lambda^n H' \cap \mathcal{E}^\mu_i$ is $Z'$-free for every $i$, with basis 

$$\mathcal{Y}_{\lambda^-, \mu, i} = \{ Z^\nu_{st} | t \in \{t_{j_1}, \ldots, t_{j_{m(i)}}\}, s^* \in \Xi^{ss}_b(\nu(t), \lambda^*) \}.$$ 

Proof. Let $|\lambda(1)| = a$ and $|\mu(1)| = b$. By the definition above, $\mathcal{E}^\mu_i$ has basis 

$$\{ X_{st_j}^{\nu(t)} | 1 \leq j \leq i, s \in T^s(\nu(t_j)) \}.$$ 

Thus, $\pi_a^{-1} \mathcal{E}^\mu_{i Z'}$ has basis 

$$\{ \pi_a^{-1} X_{st_j}^{\nu(t)} | t \in \{t_{j_1}, \ldots, t_{j_{m(i)}}\}, s^* \in T^s_b(\nu(t)) \}.$$ 

Since $x_\lambda^n H' \cap \mathcal{E}^\mu_i = x_\lambda^n H' \pi_b \cap \pi_a^{-1} \mathcal{E}^\mu_{i Z'}$, it follows from a similar argument for Theorem 5.8 that $x_\lambda^n H' \cap \mathcal{E}^\mu_{i Z'}$ is free with the required basis $\mathcal{Y}_{\lambda^-, \mu, i}$. \hfill \square

We will also need the following result which can be obtained by applying $\iota : T_w \mapsto T_{w^{-1}}$ to [8, (6.1.6)].

Lemma 6.2. Keep the notation introduced above and let $\lambda, \mu \in \Pi^+_2$. Then $x_\lambda H' \cap \mathcal{E}^\mu_{i Z'}$ is free with basis 

$$X_{\lambda, \mu, i} = \{ X_{st_j}^{\nu(t)} | 1 \leq j \leq i, s \in \Xi^{ss}(\nu(t_j), \lambda) \}.$$ 

By taking duals, we turn the twisted Specht filtration $\mathcal{E}^\mu$ above to a Specht filtration $\mathcal{F}^\mu$: 

$$0 = \mathcal{F}^{0 \mu} \subseteq \mathcal{F}^{1 \mu} \subseteq \cdots \subseteq \mathcal{F}^{n_\mu \mu} = x_\mu H,$$ 

where 

$$\mathcal{F}^{i \mu} = (H x_\mu / \mathcal{E}^\mu_{n_\mu-i})^*, \quad 0 \leq i \leq n_\mu,$$ 

(compare [8, (5.2.3)]). We define the Specht module 

$$S_\lambda = \mathcal{F}^1_\lambda.$$ 

Theorem 6.3. For $\lambda, \mu \in \Pi^+_2(r)$ and any commutative $Z$-algebra $Z'$, let $a = |\lambda(1)|$ and $b = |\mu(1)|$. Then base change defines isomorphisms 

$$\begin{cases} 
(1) \text{Hom}_H(x_\mu H / \mathcal{F}^1_\mu, x_\lambda H)_{Z'} \xrightarrow{\sim} \text{Hom}_{H Z'}((x_\mu H / \mathcal{F}^1_\mu)_{Z'}, x_\lambda H_{Z'}), \\
(2) \text{Hom}_H(x_\mu H / \mathcal{F}^1_\mu, x_\lambda^n H)_{Z'} \xrightarrow{\sim} \text{Hom}_{H Z'}((x_\mu H / \mathcal{F}^1_\mu)_{Z'}, x_\lambda^n H_{Z'}), 
\end{cases}$$

for all $i$, assuming $a + b = r$ in the latter case. Also, the $\mathcal{Z}'$-modules in (6.3(1) (resp. 6.3(2)) is free of rank $r_{\lambda,\mu,i} = \#\lambda_{\lambda,\mu,i}$ (resp. $r_{\lambda,\mu,i} = \#\lambda_{\lambda,\mu,i}$). Furthermore, for $i < j$, the natural maps

\[
\begin{aligned}
(1') & \quad \text{Hom}_\mathcal{H}(x_\mu \mathcal{H}/\mathcal{F}_\mu^j, x_\lambda \mathcal{H}) \to \text{Hom}_\mathcal{H}(x_\mu \mathcal{H}/\mathcal{F}_\mu^i, x_\lambda \mathcal{H}), \\
(2') & \quad \text{Hom}_\mathcal{H}(x_\mu \mathcal{H}/\mathcal{F}_\mu^j, x_\lambda \mathcal{H}) \to \text{Hom}_\mathcal{H}(x_\mu \mathcal{H}/\mathcal{F}_\mu^i, x_\lambda \mathcal{H}), \quad \text{if } a + b = r,
\end{aligned}
\]

have $\mathcal{Z}'$-free cokernels of rank $r_{\lambda,\mu,i} - r_{\lambda,\mu,j}$ (resp. $r_{\lambda,\mu,i} - r_{\lambda,\mu,j}$).

**Proof.** Since $(x_\mu \mathcal{H}/\mathcal{F}_\mu^i)^* \cong E_{n_{\mu-i}}^\mu$, $(x_\lambda \mathcal{H})^* \cong \mathcal{H}x_\lambda^\lambda (\text{[4, 4.3]})$ and $(x_\lambda \mathcal{H})^* \cong \mathcal{H}x_\lambda^\lambda$, (1) and (2) are equivalent to showing that base change defines isomorphisms

\[
\text{Hom}_\mathcal{H}(x_\lambda \mathcal{H}, E_{i_{\mathcal{Z}'}}^\mu) \cong \text{Hom}_{\mathcal{H}'}(x_\lambda \mathcal{H}', E_{i_{\mathcal{Z}'}}^\mu),
\]

and

\[
\text{Hom}_\mathcal{H}(x_\lambda \mathcal{H}, E_{i_{\mathcal{Z}'}}^\mu) \cong \text{Hom}_{\mathcal{H}'}(x_\lambda \mathcal{H}', E_{i_{\mathcal{Z}'}}^\mu) \quad \text{if } a + b = r,
\]

for all $i$. Since $\mathcal{H}'$ is a Frobenius algebra over $\mathcal{Z}'$ (and thus “injective” for $\mathcal{H}'$-modules relative to $\mathcal{Z}'$-split exact sequences), we have

\[
\text{Hom}_{\mathcal{H}'}(x_\lambda \mathcal{H}', E_{i_{\mathcal{Z}'}}^\mu) \cong x_\lambda \mathcal{H}' \cap E_{i_{\mathcal{Z}'}}^\mu,
\]

e tc., reducing the isomorphisms to the isomorphisms

\[
(x_\lambda \mathcal{H} \cap E_{i_{\mathcal{Z}'}}^\mu) \cong x_\lambda \mathcal{H}' \cap E_{i_{\mathcal{Z}'}}^\mu, \quad \text{and } (x_\lambda \mathcal{H} \cap E_{i_{\mathcal{Z}'}}^\mu) \cong x_\lambda \mathcal{H}' \cap E_{i_{\mathcal{Z}'}}^\mu \quad \text{if } a + b = r \text{ in the latter case. Now, Theorems 5.1 and 6.2 have established the above isomorphisms, and the first assertion is proved.}
\]

The discussion in the previous paragraph shows that the maps in (1') and (2') are determined by the inclusions

\[
x_\lambda \mathcal{H}' \cap E_{n_{\mu-j},\mathcal{Z}'} \subseteq x_\lambda \mathcal{H}' \cap E_{n_{\mu-i},\mathcal{Z}'}, \quad x_\lambda \mathcal{H}' \cap E_{n_{\mu-j},\mathcal{Z}'} \subseteq x_\lambda \mathcal{H}' \cap E_{n_{\mu-i},\mathcal{Z}'}.
\]

Since the basis described in 6.1 or 6.2 for $n_{\mu-j}$ is a subset of the basis for $n_{\mu-i}$, the last assertion now follows easily. \[\square\]

We observe that the isomorphism (2) in (5.3) holds for $a + b > r$ as both sides would be zero in this case. Also, one would expect that the base change property in (6.3(2)) holds in general, though we do not need the case $a + b < r$.

We now can apply [4, 1.2.13] to obtain the following homological vanishing property. Note that $\mathcal{H}_K$, where $K$ is the quotient filed of $\mathcal{Z}$, is semisimple.

**Theorem 6.4.** For $\lambda, \mu \in \Pi_2^+(r)$, we have for all $i$,

\[
\begin{aligned}
(1) & \quad \text{Ext}_\mathcal{H}^1(x_\mu \mathcal{H}, \mathcal{F}_\mu^i, x_\lambda \mathcal{H}) = 0 \\
(2) & \quad \text{Ext}_\mathcal{H}^1(x_\mu \mathcal{H}, \mathcal{F}_\mu^i, x_\lambda \mathcal{H}) = 0 \quad \text{if } |\lambda^{(1)}| + |\mu^{(1)}| = r.
\end{aligned}
\]

Moreover, both $\text{Hom}_\mathcal{H}(\mathcal{F}_\mu^i, x_\lambda \mathcal{H})$ and $\text{Hom}_\mathcal{H}(\mathcal{F}_\mu^i, x_\lambda \mathcal{H})$ (assuming $|\lambda^{(1)}| + |\mu^{(1)}| = r$).
Theorem 7.1. Let $H$ that $r$ and $\mu$ be an isomorphism $H$ (resp. Hom$_H$) and $r \in \mathcal{D}_H$. Then the algebra $S$ is free and base change induces an isomorphism $S_q(1,5)(n, r; \mathcal{Z}) \simeq S_q(1,5)(n, r; \mathcal{Z})$.

Proof. The first assertion follows from [4, (1.2.13)] and the previous theorem.

Next, observe that for any $j$, Theorem 6.3 implies that Hom$_H(F_{q,j}, x_\lambda \mathcal{H})$ (resp. Hom$_H(F_{q,j}, x_\mu \mathcal{H})$) identifies with the cokernel of the natural map

$$\text{Hom}_H(x_\mu \mathcal{H}/F_{q,j}, x_\lambda \mathcal{H}) \rightarrow \text{Hom}_H(x_\mu \mathcal{H}, x_\lambda \mathcal{H})$$

(resp. Hom$_H(x_\mu \mathcal{H}/F_{q,j}, x_\mu \mathcal{H}) \rightarrow \text{Hom}_H(x_\mu \mathcal{H}, x_\mu \mathcal{H})$.) Thus, the last assertion follows from the last assertion of the previous theorem, together with elementary properties of distinguished double coset representatives.

7. Stratifying the $q$-Schur$^{1,5}$ Algebra

We now turn back to the type $D$ case. So we assume in this section that $H' = H'_{1,q}$. We have first the following base change property.

Theorem 7.1. Let $\mathcal{Z}'$ be a commutative $\mathcal{Z}$-algebra in which $2$ is invertible. Then the algebra $S_q(1,5)(n, r; \mathcal{Z}')$ is free and base change induces an isomorphism

$$S_q(1,5)(n, r; \mathcal{Z}) \simeq S_q(1,5)(n, r; \mathcal{Z}).$$

Proof. For $\lambda, \mu \in \Pi_{1,5}$, we have, using Lemma 2.10(c),

$$\text{Hom}_{H'}(x_\mu \mathcal{H'}, x_\lambda \mathcal{H'}) \simeq \text{Hom}_{H'}(x_\mu \mathcal{H'}, x_\mu \mathcal{H'} \uparrow \mathcal{H'})$$

$$\simeq \text{Hom}_{H'}(x_\mu \mathcal{H'}, x_\lambda \mathcal{H'}) \oplus \text{Hom}_{H'}(x_\mu \mathcal{H'}, x_\mu \mathcal{H'}).$$

Now, the Hom space $\text{Hom}_{H'}(x_\mu \mathcal{H'}, x_\lambda \mathcal{H'})$ has always the base change property by Theorem 6.3(1), while the Hom space $\text{Hom}_{H'}(x_\mu \mathcal{H'}, x_\mu \mathcal{H'}) \simeq x_\lambda \mathcal{H'} \cap \mathcal{H'}x_\mu$ is 0 if $|\lambda(1)| + |\mu(1)| > r$ (Theorem 4.3), or free of rank independent of $\mathcal{Z}'$ if $|\lambda(1)| + |\mu(1)| = r$. Hence the result follows.

With this Theorem, we can determine the structure of the $q$-Schur$^{1,5}$ algebra immediately when $r$ is odd, at least relative to the $q$-Schur$^2$ algebra.

Corollary 7.2. Assume again that $2$ is invertible in $\mathcal{Z}'$ and $r$ is odd. Then $S_q(1,5)(n, r; \mathcal{Z}')$ is a centralizer subalgebra of $S_q(2)(n, r; \mathcal{Z})$ associated to the coideal $\Pi_{1,5}^+(n, r)$ of $\Pi_{1,5}^+(n, r)$. Hence, it is quasi-hereditary, and

$$(7.2.1) \quad S_q(1,5)(n, r; \mathcal{Z}) \leq S_q(1,5)(n, r; \mathcal{Z}') \leq S_q(2)(n, r; \mathcal{Z}).$$
Proof. When $r$ is odd, we have always $|\lambda^{(1)}| + |\mu^{(1)}| > r$. It follows that $\text{Hom}_{H'}(\bar{x}_\mu H', x_\mu H') = 0$. Therefore,

$$\text{Hom}_{H'}(x_\lambda H', x_\mu H') \cong \text{Hom}_{\tilde{H}'}(\bar{x}_\lambda \tilde{H}', \bar{x}_\mu \tilde{H}')$$

for all $\lambda, \mu \in \Pi_2$, and consequently, $S_q^{1,5}(n, r; Z')$ is a centralizer subalgebra of $S_q^2(n, r; Z)$ defined by the coideal $\Pi_1^+$. The result follows. Note that the left hand inequality has been proved in (2.8) above. \hfill \Box

When $r$ is even, it is very unlikely that $S_q^{1,5}(n, r; Z')$ is quasi-hereditary in general.\footnote{A bad case is $r = 4$, $q = 1$, and $Z'$ the localization $Z_{(2)}$. The five ordinary irreducible representations associated to the four bipartitions $(2;2), (2;1,1), (1,1;2), (1,1;1,1)$ may be shown to have only four distinct 2-modular constituents for $S_q^{1,5}$. This precludes any integral quasi-hereditary structure compatible with our partial order.} However, it can be integrally and standardly stratified naturally, using the type $B$ Specht modules filtrations $F^\lambda$. It suffices now, by the base change property, to look at the integral case. Recall that $S_\lambda = F^\lambda_1$.

**Theorem 7.3.** Let $Z'$ be the ring obtained by localizing $Z = Z[q, q^{-1}]$ at 2. For any $\lambda \in \Pi_{1,5}^+$, let $\Delta^{1,5}(\lambda) = \text{Hom}_{\tilde{H}'}(S_{\lambda Z'}, \tilde{T}^{1,5}_{Z'})$. Then $\Delta^{1,5}(\lambda)$ is $Z'$-free and $\{\Delta^{1,5}(\lambda)\}_{\lambda \in \Pi_{1,5}^+}$ is a standard stratifying system for the category of $S_q^{1,5}(n, r; Z')$-modules.

Proof. The freeness follows from the definition of $S_\lambda$ and Theorem 6.4. It remains to check the Hypothesis given in (1.2.9).

Let $\Lambda = \Pi_{1,5}(n, r)$ and define a new order $\leq$ on $\Lambda$ by setting $\lambda \leq \mu$ if $\lambda \leq \mu$ or $\lambda \leq \mu^*$ with $\lambda, \mu, \mu^* \in \Lambda$. Clearly, $(\Lambda, \leq)$ is a quasi-poset. Let $\tilde{\Lambda}$ denote the associated poset. By (3.2.2c), we have $\tilde{T}^{1,5}_{Z'} = \oplus_{\mu \in \Lambda} \tilde{T}^\mu_{\mu}$, where $\tilde{T}^\mu_{\mu} \cong x_\mu H'$.

For $\mu \in \Lambda$, $\tilde{T}^\mu_{\mu}$ has a filtration $F^\mu_\mu$ (see (3.3)) for which $G^r_i F^\mu_\mu \cong S_{\nu_{\mu,i} Z'}$, for $\nu_{\mu,i} > \mu$ if $i < n_{\mu}$. Note that this can be replaced by $\tilde{\nu}_{\mu,i} > \tilde{\mu}$.

For $\lambda, \mu \in \Lambda$, if $\text{Hom}_{\tilde{H}'}(S_{\mu Z'}, \tilde{T}_\lambda) \neq 0$, then $\text{Hom}_{\tilde{H}_K}(\tilde{T}_\lambda, S_{\mu K}) \neq 0$. Thus, we have either $\text{Hom}_{\tilde{H}_K}(\tilde{T}_\lambda, S_{\mu K}) \neq 0$ or $\text{Hom}_{\tilde{H}_K}(\tilde{T}_\lambda, S_{\mu K}) \neq 0$ by Lemma 2.4(b). The former implies that $\lambda \leq \mu$, while the latter implies that $|\lambda^{(1)}| = |\lambda^{(2)}| = |\mu^{(1)}| = |\mu^{(2)}|$ (thus, $\mu^* \in \Lambda$), and $\lambda \leq \mu^*$. Therefore, $\lambda \leq \mu$.

It remains to check the homological condition

$$\text{Ext}^1_{\bar{H}'}(\bar{x}_\mu \tilde{H}' / F^\mu_{\mu Z'}, \bar{x}_\lambda \tilde{H}') = 0$$
for all $i$ and $\lambda \in \Pi_{1,5}^+$. However, one sees easily that
\[
\text{Ext}^1_{\mathcal{H}}(\hat{x}_\mu \mathcal{H}' / \mathcal{F}_\mu, \hat{x}_\lambda \mathcal{H}')
\cong \text{Ext}^1_{\mathcal{H}'}(x_\mu \mathcal{H}' / \mathcal{F}_\mu, x_\lambda \mathcal{H}' \uparrow \mathcal{H}')
\cong \text{Ext}^1_{\mathcal{H}'}(x_\mu \mathcal{H}' / \mathcal{F}_\mu, x_\lambda \mathcal{H}') \oplus \text{Ext}^1_{\mathcal{H}'}(x_\mu \mathcal{H}' / \mathcal{F}_\mu, x_\lambda \mathcal{H}') = 0,
\]
by Theorem 6.4. Now, the theorem follows from [7, (1.2.10)(1.2.12)].

Let $\mathcal{O}$ be regular local ring of Krull dimension $\leq 2$ with the field $K$ of fractions and residue field $k$. Assume further that $2$ is invertible in $\mathcal{O}$.

**Corollary 7.4.** The decomposition matrix of $S_{q}^{1,5}(n, r)_\mathcal{O}$ contains an upper unitriangular block of size equal to the number of irreducible modular representations of $S_{q}^{1,5}(n, r)_k$. Moreover, if all $\Delta(\lambda)_k$ with $\lambda^{(1)} = \lambda^{(2)}$ are decomposable, then $S_{q}^{1,5}(n, r)_k$ is quasi-hereditary.

**Proof.** Let $\tilde{A} = S_q^{1,5}(n, r)$. Since $\tilde{A}_K$ is split semisimple, we may assume that $\mathcal{O}$ is complete. By Theorem 7.3 and [7, (1.2.5),(1.2.8)], $A := \tilde{A}_k$ has a standard stratification
\[
0 = J_0 \subset J_1 \subset \cdots \subset J_m = A
\]
such that (1) $m = \#\tilde{A}$; (2) $J_i/J_{i-1} = (A/J_{i-1})e_i(A/J_{i-1})$ for some idempotent $e_i$ with $(A/J_{i-1})e_i \cong \Delta(\lambda(i))_k$. Moreover, there are $\tilde{A}_\mathcal{O}$-projective modules $P(\lambda(i))$ such that $P(\lambda(i))$ has a $\Delta$-filtration (see [7, (1.2.4)]) with top section isomorphic to $\Delta(\lambda(i))_\mathcal{O}$, other sections $\Delta(\mu)_\mathcal{O}$ satisfying $\mu > \lambda(i)$, and such that $P(\lambda(i))_k$ is the projective cover of $\Delta(\lambda(i))_k$.

Now, if $\lambda = \lambda(i)$ is not of the form $(\alpha, \alpha)$, then $\Delta(\lambda(i))_k$ is indecomposable and $\text{End}_A(\Delta(\lambda(i))_k)$ is a division ring, and hence $J_i/J_{i-1}$ is a heredity ideal of $A/J_{i-1}$. If $\lambda = (\alpha, \alpha)$ and $S_{\lambda_0}|_{\tilde{H}_\mathcal{O}}$ is decomposable, then $S_{\lambda_0}|_{\tilde{H}_\mathcal{O}}$ a direct sum of two non-isomorphic indecomposable modules, and hence, $\Delta(\lambda(i))_k$ is a direct sum of two distinct PIMs (of $A/J_{i-1}$). Thus, in this case, $J_i/J_{i-1}$ is also a heredity ideal of $A/J_{i-1}$, and the second assertion follows.

Finally, if $\lambda = (\alpha, \alpha)$ and $S_{\lambda_0}|_{\tilde{H}_\mathcal{O}}$ is indecomposable, then $\Delta(\lambda(i))_k$ is indecomposable and $\text{End}_A(\Delta(\lambda(i))_k)$ is a local ring (not a division ring!). Note that $\Delta(\lambda(i))_K$ is a sum of two non-isomorphic simple modules. In all cases, we see that the matrix $(d_{ij})$, where $d_{ij} = \dim \text{Hom}_A(P(\lambda(i))_K, \Delta(\lambda(j))_K)$, is upper unitriangular, and hence the result follows from [7, (1.1.3)].
Remarks 7.5. (a) It is very likely that the odd characteristic condition in the main results 7.1-4 can be removed. To do this we need a direct argument for type $D$ parallel to the work for type $B$ in [9]. For example, when $r$ is odd, results (7.1) and (7.2) can be proved as follows: Put
\[
\bar{D}_\lambda = \begin{cases} 
(D_\lambda \cap \bar{W}) \cup (s_0D_\lambda \cap \bar{W}), & \text{if } |\lambda^{(1)}| \geq 1 \\
D_\lambda \cap \bar{W}, & \text{if } |\lambda^{(1)}| = 0.
\end{cases}
\]
Then one may prove that $\#D_{\lambda,\mu} = \#\bar{D}_{\lambda\mu}$, where $\lambda, \mu \in \Pi_{1,5}$ and $\bar{D}_{\lambda\mu} = \bar{D}_\lambda \cap \bar{D}_{\mu}^{-1}$. By directly constructing a basis for $\text{Hom}_{\bar{H}}(\bar{T}_{\mu}', \bar{T}_\lambda')$ (compare [9, (4.2.6)]), one could prove that $\text{Hom}_{\bar{H}}(\bar{T}_{\mu}', \bar{T}_\lambda') = \text{Hom}_{\bar{H}}(T_{\mu}', T_\lambda')$.

Thus, the quasi-heredity of the $q$-Schur$^{1,5}$ algebra when $r$ is odd follows immediately.

(b) Note that (7.4) applies as well to the $q$-Schur$^{2,5}$ and $q$-Schur$^{1b}$ algebras in the linear prime case, because of (3.7) above. Also, together with [11, 7.15], we see that, in the linear prime case, all $\Delta(\lambda)_k$ with $\lambda^{(1)} = \lambda^{(2)}$ are decomposable. Therefore, by (7.4), the $q$-Schur$^{1,5}$ algebra is quasi-hereditary in this case. It is likely that this result always holds when the polynomial $\bar{g}_r$ is invertible.

8. The bad prime case

A case at “the other extreme” compared with the linear prime case (§3) is $p = 2$, and hence, for finite groups of Lie type, $q = 1$ taking $q$ to be prime power with $p \nmid q$. (See the footnote below.) In this case, the group algebra of $\bar{C} = \langle t_1t_2, \ldots, t_1t_r \rangle$ has only one irreducible modular representation, and is a local ring. (In the linear prime case the Hecke algebra behaves as if it had a semisimple subalgebra associated to $\bar{C}$.) Nevertheless, it is quite easy to find the modular irreducible representation of $S_q^{1b}(r, r)$ (or $S_q^{1b}(r, r)$) in terms of those for $S_q^{1}(r, r)$ (which maybe regarded as part of $S_q^{1,5}(r, r)$). For simplicity, we will stick to the $S_q^{1b}$ case.

To fix notation, we get $q_0 = q = 1$ and let $Z'$ be a commutative local ring with residue field $k = Z'/\mathfrak{m}$ of characteristic 2. Let $S_q^{1b}$ denote $S_q^{1b}(r, r, Z')$ for some fixed $r > 0$, and let $S_q^{1}(r, r, Z')$. The latter is just the classical Schur algebra.

It is interesting to note that “decomposition numbers” for $S_q^{1b}$, with $Z' = Z_{(q)}$ (or any char. 0 DVR with 2 in its maximal ideal), are the same for $q = 1$ and for $q$ equal to any odd prime power. This follows using, say, [3, (1.1.2)] and the fact that $S_q^{1b}$ is split semisimple for each of these specializations of $q$. 
Theorem 8.1. There is a natural surjective homomorphism
\[ \theta : S^k_{Z'} \to S^1_{Z}, \]
whose kernel is contained in the radical \( \text{rad}(S^k_{Z'}) \). In particular, the irreducible representations of \( S^k_{Z'} \) over \( k \) are all obtained from those \( S^1_k \) by factorization through \( \theta \).

Also, if \( Z' \to Z'' \) is a commutative local rings, we have the base change properties
\[ Z'' \otimes S^k_{Z'} \cong S^k_{Z''}, \text{ and } Z'' \otimes S^1_{Z'} \cong S^1_{Z''}. \]

Proof. Write \( \bar{W} = \check{C} \rtimes \bar{W} \) where \( \check{C} \) is the normal elementary abelian 2-group \( \langle t_1, t_2, \ldots, t_r \rangle \). If \( \bar{W}_\mu \) is any subgroup of \( \bar{W} \), and \( M \) is any right \( \bar{Z}'\bar{W}_\mu \)-module, then the semidirect product decomposition gives
\[ \text{Ind}^{\bar{Z}'\bar{W}_\mu}_{\bar{Z}'\bar{W}_\mu} (M) \cong \text{Ind}^{\bar{Z}'M}_{\bar{Z}'\bar{W}_\mu} (M) \otimes \bar{Z}'\check{C} \]
where the right hand side of the tensor product is regarded as a \( \bar{Z}'\bar{W} \)-module by inflation, and \( \bar{Z}'\check{C} \) is regarded as a \( \bar{W} \)-module with \( \check{C} \) acting by right multiplication and \( \bar{W} \) acting by conjugation. Thus,
\[ \check{T}^\bar{Z}' \cong \bar{T}_Z \otimes \bar{Z}'\check{C} \]
with \( \check{T}^\bar{Z}' \) as in \((2.3)\), and \( \bar{T}_Z \) defined similarly for the classical symmetric group \( \bar{W} \). Passing to the endomorphism algebras, we have
\[ S^k_{Z'} = \text{End}_{\bar{Z}'\check{C}}(\check{T}^\bar{Z}') \cong (\text{End}_{\bar{Z}'\check{C}}(\bar{T}_Z) \otimes \bar{Z}'\check{C})^{\bar{W}} \]
where the right hand side is the fixed points for an action of \( \bar{W} \) on the tensor product. Note that the augmentation \( \bar{Z}'\check{C} \to \bar{Z}' \) is a \( \bar{Z}'\bar{W} \)-homomorphism, split by the natural inclusion \( \bar{Z}' \cong \bar{Z} \cdot 1 \subseteq \bar{Z}'\check{C} \). If \( I(\bar{Z}'\check{C}) \) denotes the augmentation ideal of \( \bar{Z}'\check{C} \), then a power of \( I(\bar{Z}'\check{C}) \) is contained in \( 2\bar{Z}'\check{C} \). The same is true for the ideal
\[ (\text{End}_{\bar{Z}'\check{C}}(\bar{T}_Z) \otimes I(\bar{Z}'\check{C}))^{\bar{W}} \]
of \( (\text{End}_{\bar{Z}'\check{C}}(\bar{T}_Z) \otimes \bar{Z}'\check{C})^{\bar{W}} \cong S^1_{Z'} \). The quotient of \( S^k_{Z'} \) by the ideal is
\[ (\text{End}_{\bar{Z}'\check{C}}(\bar{T}_Z) \otimes \bar{Z}')^{\bar{W}} \cong S^1_{Z''}. \]
This gives the surjective homomorphism \( \theta \) described in the theorem.

The base change properties follows easily from the fact that both \( \check{T}^\bar{Z}' \) and \( \bar{T}_Z \) are permutation modules for \( \bar{Z}'\bar{W} \). The proof is complete. \( \square \)

Remarks 8.2. (a) The \( q \)-analogue of the homomorphism \( \theta \) always exists. This is because \( \check{T}^\bar{Z}' = \pi_q^n \bar{T}^\bar{Z}' \) (or, in the type \( D \) case, \( \bar{T}^\bar{Z}' = \pi_q^n \bar{T}^\bar{Z}' \)) and \( \theta \) is simply the restriction map. In the type \( B \) case, we even know that it is onto and what the kernel of \( \theta \) is. Using the bistandard (or
cellular) basis for $S^B_q(n, r; \mathbb{Z}')$ (see [3, (6.1.1)]), we see easily that the basis elements $\Phi^\lambda_{st}$ with $|\lambda^{(1)}| = 0$ are sent to a basis for $S^B_q(n, r; \mathbb{Z}')$, while those with $|\lambda^{(1)}| > 0$ are sent to 0 under $\theta$. The surjectivity in the type $D$ case should follow from a direct construction of a standard basis for $S^D_q$, cf. (7.5).

(b) The bad prime case for finite groups of Lie type generalizes, at the Hecke algebra level, to the case where the factor $q_0 + 1$ of $g_r$ (or the factor 2 of $\tilde{g}_r$ for type $D$) is zero in the base ring. However, if $q \neq 1$, the kernel of $\theta$ defined in (b) is not nilpotent in general. For example, in the type $B_2$ case, consider the cellular basis elements at the $\lambda = (1; 1)$ level. They are $1 + T_{s_0}$, $(1 + T_{s_0})T_{s_1}$, $T_{s_1}(1 + T_{s_0})$ and $T_{s_1}(1 + T_{s_0})T_{s_1}$. Since

$$
(1 + T_{s_0})T_{s_1}(1 + T_{s_0})T_{s_1} = (1 + T_{s_0})T_{s_1}^2 + (1 + T_{s_0})T_{t_2} \\
= (1 + T_{s_0})T_{s_1}^2 - q(1 + T_{s_0}) + \pi_2 \\
= (q - 1)(1 + T_{s_0})T_{s_1} + \pi_2.
$$

We see that the Hecke algebra $\mathcal{H}'$ has three distinct irreducible modular representations, while $\mathcal{Y}'$ has only two. It is easy to check that the same is true for $S^B_q(2, 2; \mathbb{Z}')$ and $S^B_q(2, 2; \mathbb{Z}')$.

(c) It is reasonable to speculate, based on the section 3 and 8, that the irreducible modular representations of $S^B_q$, in any characteristic $p$, are determined by some of the irreducible modular representations of $S_q^{1.5}$, at least when $q$ itself is a power of a different prime. In the linear prime case, all the irreducible modular representations are required, while only those associated to $S_q^1 \leq S_q^{1.5}$ are needed for $p = 2$ and $q = 1$. It would be very satisfactory if one could predict just from the order of $q$ modulo $p$ just what part of $S_q^{1.5}$ were required. We hope to explore possible general theories along these lines in the future.

References

[1] M. Cabanes, Algèbres de Hecke comme algèbres symétriques et théorème de Dipper., C.R. Acad. Sci. Paris, 327 (1998), 531-536.
[2] E. Cline, B. Parshall and L. Scott, Stratifying endomorphism algebras, Memoirs Amer. Math. Soc. 591, 1996.
[3] R. Dipper and G. James, Representations of Hecke algebras of type $B_n$, J. Algebra, 146 (1992), 454–481.
[4] R. Dipper, G. James and A. Mathas, The $(Q, q)$-Schur algebra, Proc. London Math. Soc. 77 (1998), 327–361.
[5] J. Du, Cells in certain sets of matrices, Tôhoku Math. J. 48 (1996), 417–427.
[6] J. Du, Generalized $q$-Schur algebras and the ways to approach them, Virginia Conference Proceedings (to appear).
[7] J. Du, B. Parshall and L. Scott, Stratifying endomorphism algebras associated to Hecke algebras, J. Algebra, 203 (1998), 169–210.
[8] J. Du, B. Parshall and L. Scott, *Quantum Weyl reciprocity and tilting modules*, Commun. Math. Phys. **195** (1998), 321–352.
[9] J. Du and L. Scott, *The q-Schur algebra*, Trans. Amer. Math. Soc. (to appear).
[10] M. Geck and G. Hiss, *Modular representations of finite groups of Lie type in non-defining characteristics*, in *Finite reductive groups*, M. Cabanes, ed. (1996) 173–227.
[11] J. Gruber and G. Hiss, *Decomposition numbers of finite classical groups for linear primes*, J. reine angew. Math. **485** (1997), 55-91.
[12] J. Hu and J.-p. Wang, *Hecke algebras of type $D_n$ at roots of unity*, J. Algebra **212** (1999), 132-160.
[13] C. Mak, *Quasi-parabolic subgroups of $(\mathbb{Z}/m\mathbb{Z}) \wr S_r$*, preprint, UNSW.
[14] G. Murphy, *The representations of Hecke algebras of type $A_n$*, J. Algebra **173** (1995), 97-121.
[15] C. Pallikaros, *Representations of Hecke algebras of type $D_n$*, J. Algebra **169** (1994), 20-48.

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