UNIQUENESS AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A BIHARMONIC EQUATION WITH SUPERCRITICAL EXPONENT

ZONGMING GUO, XIAOHONG GUAN AND YONGGANG ZHAO*

Department of Mathematics, Henan Normal University
Xinxiang 453007, China

(Communicated by Bernhard Ruf)

Abstract. Existence and uniqueness of positive radial solution $u_p$ of the Navier boundary value problem:

\[
\begin{align*}
\Delta^2 u &= u^p & \text{in } \mathbb{R}^N \setminus \overline{B}, \\
u > 0 & \quad \text{in } \mathbb{R}^N \setminus \overline{B}, \\
u &= \Delta u = 0 & \text{on } \partial B,
\end{align*}
\]

where $B \subset \mathbb{R}^N$ ($N \geq 5$) is the unit ball and $p > \frac{N+4}{N-4}$, are obtained. Meanwhile, the asymptotic behavior as $p \to \infty$ of $u_p$ is studied. We also find the conditions such that $u_p$ is non-degenerate.

1. Introduction. We consider existence and uniqueness of positive radial solutions of the Navier boundary value problem:

\[
\begin{align*}
\Delta^2 u &= u^p & \text{in } \mathbb{R}^N \setminus \overline{B}, \\
u > 0 & \quad \text{in } \mathbb{R}^N \setminus \overline{B}, \\
u &= \Delta u = 0 & \text{on } \partial B,
\end{align*}
\]

where $B \subset \mathbb{R}^N$ ($N \geq 5$) is the unit ball, i.e., $B = \{x \in \mathbb{R}^N : |x| < 1\}$ and $p > \frac{N+4}{N-4}$.

The structure of positive solutions of the equation

\[
\Delta^2 u = u^p \quad \text{in } \mathbb{R}^N \quad (N \geq 5), \quad p > 1
\]

is considered by many authors recently, see [1, 3, 6, 9, 10, 12, 13, 15, 16, 17, 19, 23, 25]. The classification of positive entire solutions of (1) via Morse index has also been obtained, see [5, 21, 23, 24, 30].

In the supercritical case, i.e., when $p > \frac{N+4}{N-4}$, there are no positive solutions of the Navier boundary value problem:

\[
\begin{align*}
\Delta^2 u &= u^p & \text{in } \Omega, \\
u &= \Delta u = 0 & \text{on } \partial\Omega,
\end{align*}
\]

2010 Mathematics Subject Classification. Primary: 35B45; Secondary: 35J40.

Key words and phrases. Biharmonic equations, supercritical exponent, uniqueness, exterior domains, non-degenerate.

The first author is supported by NSF grants 11171092 and 11571093. The research of the third author is supported by Key Scientific Research Project for Colleges and Universities of Henan Province grant 18A110024, Ph.D. Research Foundation of Henan Normal University (QD16149) and Natural Science Foundation of Henan Normal University (2016QK01).

* Corresponding author: ygzhao@aliyun.com.
when $\Omega$ is star-shaped or a starlike domain (see [27, 29]). On the other hand, the existence results of positive solutions have been established when $\Omega$ is topologically nontrivial in the spirit of Bahri-Coron (see [2, 8]) or when it is contractible with some special geometry (see [11]). We are able to say that both topology and geometry of the domain $\Omega$ play important roles in the existence of solutions for (2).

In this paper, we establish the existence and uniqueness of positive radial solution $u_p$ of (P) and obtain the asymptotic behavior as $p \to \infty$. To obtain the existence and uniqueness of $u_p$, by using the Kelvin transformation, we need to consider existence and uniqueness of positive radial solution $v_{p,\alpha}^*$ of the problem:

$$\begin{cases}
\Delta^2 v = |x|^\alpha v^p & \text{in } B, \\
v = 0, \quad \Delta v - 4 \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial B,
\end{cases}$$  

(3)

where $\alpha^*_\alpha = (N - 4)p - (N + 4) > 0$.

The existence and uniqueness of the positive least energy radial solution $w_p$ of the problem

$$\begin{cases}
\Delta^2 w = w^p & \text{in } \mathbb{R}^N \setminus B, \\
w > 0 & \text{in } \mathbb{R}^N \setminus B, \\
w = |\nabla w| = 0 & \text{on } \partial B
\end{cases}$$  

(4)

with $p > \frac{N+4}{N-4}$ are obtained in [20], which was used to construct nontrivial solutions to the problem

$$\begin{cases}
\Delta^2 u = u^p & \text{in } \Omega \setminus B_r(x_0), \\
u = |\nabla u| = 0 & \text{on } \partial \Omega \cup \partial B_r(x_0),
\end{cases}$$  

(5)

where $B_r(x_0) \subset \subset \Omega$ and $p > \frac{N+4}{N-4}$. A variant of the arguments in [7, 20] implies that we can also construct nontrivial solutions to the problem

$$\begin{cases}
\Delta^2 u = u^p & \text{in } \Omega \setminus B_r(x_0), \\
u = \Delta u = 0 & \text{on } \partial \Omega \cup \partial B_r(x_0),
\end{cases}$$  

(6)

where $B_r(x_0) \subset \subset \Omega$ and $p > \frac{N+4}{N-4}$ if we understand the properties $u_p$ of (P), in particular, the non-degeneracy of $u_p$. We can show that the unique positive least energy radial solution obtained in [20] is actually the unique positive radial solution to (4). Indeed, we can show that the equation in (3) with the boundary conditions: $v = \frac{\partial v}{\partial \nu} = 0$ on $\partial B$ admits a unique positive radial solution by using a variant of the arguments in [4]. The main purpose of this paper is the uniqueness of $u_p$, the asymptotic behavior of $u_p$ as $p \to \infty$ and the conditions such that $u_p$ is non-degenerate. Meanwhile, we also provide the asymptotic behavior (as $p \to \infty$) of the unique positive radial solution $w_p$ of (4).

Our main results of this paper are the following propositions and theorems.

**Theorem 1.1.** Let $N \geq 5$, $p > \frac{N+4}{N-4}$. The problem (P) admits a unique positive radial solution $u_p \in C^4(\mathbb{R}^N \setminus \overline{B})$ satisfying

$$\lim_{|x| \to \infty} \sup |x|^{N-4} u(x) < \infty.$$  

**Theorem 1.2.** Let $u_p$ be the unique positive radial solution of (P) obtained in Theorem 1.1. Then, as $p \to \infty$,

$$u_p(|x|) \to g(|x|) \quad \text{in } C^2_{loc}(\mathbb{R}^N \setminus B),$$  

(7)

with

$$g(|x|) = \begin{cases}
\omega_1(|x|) & \text{for } 1 \leq |x| \leq r_0, \\
\omega_2(|x|) & \text{for } r_0 \leq |x| < \infty,
\end{cases}$$  

(8)
where
\[
\omega_1(|x|) = -\frac{(N-4)}{(r_0^{-N-2}-1)}r_0^{N-4}\left(\frac{1}{2}r_0^2 - \frac{1}{N}r_0^N\right)(1-|x|^{2-N})
+ \frac{(N-2)(N-4)}{2(r_0^{-N-2}-1)}r_0^{N-4}\left[\frac{1}{(N-4)}(1-|x|^{4-N}) - \frac{1}{N}(|x|^2 - 1)\right],
\]
\[
\omega_2(|x|) = \frac{(N-2)(N-4)}{2}r_0^{-N-4}\left[\frac{1}{(N-4)}|x|^{4-N} - \frac{1}{(N-2)}r_0^2|x|^{2-N}\right],
\]
where \(r_0 > 1\) is the only root of the equation
\[
(N-2)^2 - N(N-3)r^2 + (N-4)r^N = 0
\]
in \((1, \infty)\).

**Proposition 1.3.** Let \(N \geq 5\) and \(p > \frac{N+4}{N-4}\). Then, the linearized problem
\[
\begin{cases}
\Delta^2 h = pu_{p}^{p-1}h & \text{in } \mathbb{R}^N \setminus \overline{B}, \\
h = \Delta h = 0 & \text{on } \partial B, \ \lim_{|x| \to \infty} h(x) = 0
\end{cases}
\]
does not admit any nontrivial radial solution.

**Theorem 1.4.** The following estimate holds
\[
\frac{\|u_{p}\|_{\infty}^{p+1}}{p+1} \to 0 \text{ as } p \to \infty. \quad (10)
\]

**Remark 1.5.** The consequence of Theorem 1.4 provides an important difference between the fourth order case and the second order case. In [14], by using \(\|w_{p}\|_{\infty}^{p+1} \to D > 0\) as \(p \to \infty\), where \(w_{p}\) is the unique positive radial solution of the Dirichlet problem:
\[
-\Delta w = w^p \text{ in } \Omega, \ w = 0 \text{ on } \partial \Omega
\]
with \(\Omega\) being an annulus, the author obtains the relation between \(w_{p}\) and a solution of an initial value problem of a second order ODE with an exponential nonlinearity via a blow-up argument. The authors in [7] mentioned that the similar behavior also holds for \(\Omega = \mathbb{R}^N \setminus \overline{B}\). We can not use the similar arguments to our \(u_{p}\) here since the limit as \(p \to \infty\) of \(\frac{\|u_{p}\|_{\infty}^{p+1}}{p+1}\) is 0.

The organization of this paper is as follows. In section 2 we establish the existence and uniqueness of positive radial solution \(u_{p}\) of (P). In section 3, we obtain the asymptotic behavior of \(u_{p}\) as \(p \to \infty\) and provide the proof of Theorems 1.2 and 1.4. In the final section, we find the conditions such that \(u_{p}\) is non-degenerate. In this paper, we use \(C\) to denote a universal positive constant, which may change from one line to another line.

2. Existence and uniqueness of positive radial solution \(u_{p}\) of (P). In this section we mainly demonstrate the existence and uniqueness of positive radial solution \(u_{p}\) of (P). To do this, we first obtain the existence and uniqueness of positive radial solution of the problem (3) with \(\alpha_*\) being replaced by a nonnegative number \(\alpha\).

**Proposition 2.1.** Let \(N \geq 5, \ \alpha \geq 0\) and \(1 < p < \frac{N+4+2\alpha}{N-4}\). The problem
\[
\begin{cases}
\Delta^2 v = |x|^\alpha v^p & \text{in } B, \\
v > 0 & \text{in } B, \\
v = 0, \ \Delta v - 4\frac{\partial v}{\partial v} = 0 & \text{on } \partial B
\end{cases}
\]
admits a unique positive radial solution \( v_{p,\alpha} \in C^4(B) \cap C^2(\overline{B}) \).

The existence and uniqueness of positive radial solution \( u_p \) of (P) can be obtained from Proposition 2.1 and the Kelvin transformation.

Proof of Proposition 2.1. To obtain the existence of \( v_{p,\alpha} \), we consider
\[
A_{p,\alpha} = \inf_{v \in H^2_{rad}(B), \|v\|_{L^{p+1}(B)} = 1} \left[ \int_0^1 r^{N-1} (\Delta v(r))^2 dr - 4(v'(1))^2 \right]
\]
where
\[
H^2_{rad}(B) = \{ v \in H^2(B) \cap H^1_0(B) : v(x) = v(|x|) \}
\]
with the norm \( \|v\|_{H^2_{rad}(B)} = \left( \int_0^1 r^{N-1} (\Delta v(r))^2 dr \right)^{\frac{1}{2}} \) and \( L^p_0(B) \) is the weighted Sobolev space with the norm \( \|w\|_{L^p_0(B)} = \left( \int_0^1 r^{N-1+\alpha} |w(r)|^p dr \right)^{\frac{1}{p}} \).

Note that if \( \psi \in H^2_{rad}(B) \), we see that
\[
|r^{N-1} \psi'(r)| \leq \int_0^r s^{N-1} |\Delta \psi(s)| ds \leq N^{-\frac{1}{2}} r^N \|\psi\|_{H^2_{rad}(B)}.
\]
Therefore,
\[
|\psi'(r)|^2 \leq N^{-1} r^{2-N} \|\psi\|^2_{H^2_{rad}(B)},
\]
\[
4|\psi'(1)|^2 \leq \frac{4}{N} \|\psi\|^2_{H^2_{rad}(B)}.
\]
This implies that the norm of \( H^2_{rad}(B) \) is equivalent to \( \left( \int_0^1 r^{N-1} (\Delta v(r))^2 dr - 4(v'(1))^2 \right)^{\frac{1}{2}} \).

It is known from [18] that the embedding:
\[
H^2_{rad}(B) \hookrightarrow L^{p+1}_\alpha(B)
\]
is compact for \( 1 \leq p < \frac{N+\delta+2}{N-\delta} \). Meanwhile, choosing \( V(x) \equiv 1 \) and \( W(x) \equiv 0 \), it follows from (3) of Theorem 1.1 in [28] that
\[
\int_0^1 r^{N-1} |\Delta \psi|^2 dr \geq (N-1) \int_0^1 r^{N-3} |\psi'(r)|^2 dr + (N-1)(\psi'(1))^2, \quad \forall \psi \in H^2_{rad}(B).
\]
(Note that the number \( \theta \) in [28] is 0.) This implies that, for \( N \geq 5 \),
\[
\int_0^1 r^{N-1} (\Delta \psi)^2 dr - 4(\psi'(1))^2 \geq (N-1) \int_0^1 r^{N-3} |\psi'(r)|^2 dr, \quad \forall \psi \in H^2_{rad}(B).
\]
Both (12) and (13) imply that \( A_{p,\alpha} \) is attained at some \( \overline{\psi}_{p,\alpha} \in H^2_{rad}(B) \) and (12) implies that \( A_{p,\alpha} > 0 \). Note that we see from (14) that \( \psi \in H^2(1-\delta, 1) \). The embedding \( H^2(1-\delta, 1) \hookrightarrow C^{1,\tau}([1-\delta, 1]) \) for some \( 0 < \tau < \frac{1}{2} \) implies that the embedding \( H^2(1-\delta, 1) \hookrightarrow C^1([1-\delta, 1]) \) is compact.

We claim that \( \overline{\psi}_{p,\alpha} \) is nonnegative. On the contrary, we consider the solution of
\[-\Delta w = |\Delta \overline{\psi}_{p,\alpha}| \text{ in } B, \quad w = 0 \text{ on } \partial B.
\]
In particular \( w \in H^2_{rad}(B) \) and observe that \( -\Delta (w \pm \overline{\psi}_{p,\alpha}) \geq 0 \) in \( B \). Then, the strong maximum principle implies \( w > |\overline{\psi}_{p,\alpha}| \) in \( B \) and \( |w'(1)| > |\overline{\psi}_{p,\alpha}'(1)| \). Using \( \|w\|_{L^{p+1}(B)} \) in the definition of \( A_{p,\alpha} \), we derive a contradiction. It is easily seen that \( v_{p,\alpha} = \frac{1}{A_{p,\alpha}} \overline{\psi}_{p,\alpha} \) is a nontrivial nonnegative radial solution of (11) in \( H^2_{rad}(B) \).
regularity results in [18] imply that \( v_{p,\alpha} \in C^4(B \setminus \{0\}) \cap C^2(B) \). It is easily known from the regularity of \( \Delta^2 \) that \( v_{p,\alpha} \in C^4(B) \cap C^2(B) \) (note \( \alpha \geq 0 \)).

We now show \( v_{p,\alpha} > 0 \) and \( \Delta v_{p,\alpha} < 0 \) in \( B \). It easily follows from (11) that \( (\Delta v_{p,\alpha})'(r) > 0 \) for \( r \in (0,1) \). Since \( \Delta v_{p,\alpha}(1) = 4v_{p,\alpha}'(1) \leq 0 \) (note that \( v_{p,\alpha}(r) \geq 0 \) for \( r \in [0,1) \) and \( v_{p,\alpha}(1) = 0 \), we see that \( \Delta v_{p,\alpha}(r) < 0 \) for \( r \in [0,1) \). The strong maximum principle again implies that \( v_{p,\alpha} > 0 \) in \( [0,1) \).

We now show the uniqueness of \( v_{p,\alpha} \). We will see that we can not even obtain our uniqueness of the least energy radial solution by using arguments of [20] here. In [20], by the Pohozaev’s identity, the authors notice that if \( z_p(\rho) := r^{N-4}w_p(r) \) and \( \rho = r^{-1} \) with \( w_p \) being a positive least energy radial solution of (4), then \( z_p'(1) \) depends only on \( p \). Since the boundary conditions of (11) is a little more complicated than that of [20] (note that the boundary conditions in [20] are \( z_p(1) = z_p'(1) = 0 \)), if we use the similar arguments in \( v_{p,\alpha}(\rho) := r^{N-4}u_p(r) \) with \( \rho = r^{-1} \), where \( u_p \) is a positive least energy radial solution of (P), we can obtain the related Pohozaev’s identity

\[
\left( \frac{N + \alpha}{p + 1} - \frac{N - 4}{2} \right) \int_B |x|^{\alpha + p + 1} v_{p,\alpha}^p = |\partial B| \left[ \frac{(\Delta v_{p,\alpha}(1))^2}{2} - \frac{v_p'(1)(\Delta v_p)'(1)}{2} - \frac{N}{2}(\Delta v_p)(1)v_p'(1) \right]
\]

and that \( 2(4 - N)(v_p'(1))^2 - v_p'(1)(\Delta v_p)'(1) \) depends only on \( p \), which can not be used to obtain our uniqueness of by arguments as in [20]. So, we will use different arguments to obtain our uniqueness here.

Suppose by contradiction that the problem (11) admits two different positive radial solutions \( \bar{\tau} \) and \( \tau \) (we omit the subscripts \( p \) and \( \alpha \)). Let \( \lambda^{\frac{N-4}{p-4}} = \frac{\bar{\tau}(0)}{\tau(0)} \). We define the function:

\[
w(r) = \lambda^{\frac{N-4}{p-4}} \bar{\tau}(\lambda r) \quad \text{for} \quad r \in [0, \frac{1}{\lambda}].
\]

Clearly we have

\[
\left\{
\begin{array}{l}
\Delta^2 w = r^\alpha w^p \quad \text{in} \quad (0, \frac{1}{\lambda}), \\
w(\frac{1}{\lambda}) = 0, \quad \Delta w(\frac{1}{\lambda}) - 4\lambda w'(\frac{1}{\lambda}) = 0
\end{array}
\right.
\]

and

\[
w(0) = \bar{\tau}(0)
\]

Moreover, we can also see that

\[
(\Delta w)(r) < 0, \quad w'(r) < 0 \quad \text{for} \quad r \in (0, \frac{1}{\lambda}].
\]

We now show that

\[
(\Delta \bar{\tau})(0) = (\Delta w)(0).
\]

Suppose that \((\Delta \bar{\tau})(0) < (\Delta w)(0)\). If there exists \( e \in (0, R(\lambda)) \) with \( R(\lambda) = \min\{1, \frac{1}{\lambda} \} \), such that \( \Delta(\bar{\tau} - w) < 0 \) on \( [0,e) \) and \( \Delta(\bar{\tau} - w)(e) = 0 \), we easily see that \( \bar{\tau}(r) < w(r) \) for \( r \in (0, e] \). Therefore,

\[
\Delta^2(\bar{\tau} - w) = r^\alpha[\bar{\tau}^p - w^p] < 0 \quad \text{on} \quad (0,e]
\]

and the maximum principle implies that \( \Delta(\bar{\tau} - w) > 0 \) in \( [0,e] \), a contradiction. Thus,

\[
\Delta(\bar{\tau} - w) < 0 \quad \text{on} \quad [0,R(\lambda)].
\]

This also implies that

\[
(\bar{\tau} - w)'(r) < 0 \quad \forall r \in (0,R(\lambda)].
\]

Therefore,
This and (16) imply that
\[
\forall r \in (0, R(\lambda)), v(r) < w(r).
\] (20)

We now consider three cases: (i) \( \lambda < 1 \), (ii) \( \lambda = 1 \), (iii) \( \lambda > 1 \).

For the case (i), we see that \( \frac{1}{\lambda} > 1 \) and \( R(\lambda) = 1 \). Let
\[
\hat{v} = \begin{cases} v & \text{in } [0, 1], \\ \hat{v} & \text{in } (1, \frac{1}{\lambda}], \end{cases}
\]
where \( \hat{v} \) satisfies the problem
\[
\begin{cases} \Delta^2 \hat{v} = 0 & \text{in } (1, \frac{1}{\lambda}), \\ \hat{v}(1) = \hat{v}(\frac{1}{\lambda}) = 0, \\ (\Delta \hat{v})(1) = 4v'(1), \\ (\Delta \hat{v})(\frac{1}{\lambda}) = 0. \end{cases}
\] (21)

We easily see that \( (\Delta \hat{v})(r) \equiv g(r) \) for \( r \in [1, \frac{1}{\lambda}] \) and \( g \) satisfies the problem
\[
\begin{cases} \Delta g = 0 & \text{in } (1, \frac{1}{\lambda}), \\ g(1) = 4v'(1), \\ g(\frac{1}{\lambda}) = 0. \end{cases}
\] (22)

Therefore, \( \Delta \hat{v} \in C^0([0, \frac{1}{\lambda}]) \). Note that \( \Delta v(1) = 4v'(1) \). We also see that
\[
g(r) = 4v'(1) + \frac{c}{N-2} \left(1 - r^2 - N\right), \text{ for } r \in [1, \frac{1}{\lambda}]
\]
and
\[
c = \frac{(N-2)[-4v'(1)]}{1 - \lambda N^{-2}}.
\]

It follows from (19) that \( c > 0 \). Moreover,
\[
\Delta^2(\hat{v} - w) = m(r) := \begin{cases} r^\alpha [v^p - w^p] < 0 & \text{in } [0, 1], \\ -r^\alpha w^p < 0 & \text{in } (1, \frac{1}{\lambda}). \end{cases}
\] (23)

Since we can fix the value of \( m \) at \( r = 1 \) such that \( m \in C^0([0, \frac{1}{\lambda}]) \), we easily see that \( \Delta(\hat{v} - w) \in C^1([0, \frac{1}{\lambda}]) \). It follows from (23) that \( \Delta(\hat{v} - w)(r) \) is decreasing in \( (0, \frac{1}{\lambda}) \) and \( \Delta(\hat{v} - w)(\frac{1}{\lambda}) < 0 \). This is a contradiction since
\[
\Delta(\hat{v} - w)\left(\frac{1}{\lambda}\right) = g\left(\frac{1}{\lambda}\right) - \Delta w\left(\frac{1}{\lambda}\right) = -4\lambda w'(\frac{1}{\lambda}) > 0.
\]

For the second case, we have \( R(\lambda) = 1 \). Arguments similar to those in the proof of the case (i) imply that
\[
\Delta(\bar{v} - w) < 0 \text{ on } [0, 1], \quad \bar{v}(1) = w(1) = 0.
\]

It follows from the maximum principle that
\[
\bar{v} > w \quad \text{in } (0, 1).
\]

This contradicts (16).

For the third case, we have \( R(\lambda) = \frac{1}{\lambda} < 1 \). It follows from (19) that \( \Delta(\bar{v} - w) < 0 \) in \([0, R(\lambda)]\). This and (16) imply that \( (\bar{v} - w)(R(\lambda)) < 0 \). But this is a contradiction since \( \bar{v}(R(\lambda)) > 0 \) and \( w(R(\lambda)) = 0 \).

These contradictions imply that \( (\Delta \bar{v})'(0) \geq (\Delta w)'(0) \). Using the similar arguments, we can also show \( (\Delta \bar{v})'(0) \leq (\Delta w)'(0) \). Therefore, (18) holds. The standard ODE theory implies that \( \bar{v} \equiv w \) in \([0, 1]\) and hence \( \lambda = 1 \), \( \bar{v} \equiv \bar{v} \) in \([0, 1]\). This contradicts the assumption: \( \bar{v} \neq \bar{v} \) in \([0, 1]\) and the uniqueness is obtained. \( \Box \)
Corollary 2.2. We have that
\[
\|v_{p,\alpha}\|_{L^\infty(B)}^{p-1} \geq C, \tag{24}
\]
\[
\|\Delta v_{p,\alpha}\|_{L^\infty(B)} \geq C, \tag{25}
\]
where \(C > 0\) is independent of \(p\) and \(\alpha\). Moreover, if \(\alpha^*_s = (N-4)p - (N+4) > 0\), there is \(\hat{\alpha} \in (0,1)\) independent of \(p\) such that
\[
v_{p,\alpha^*_s}(r) \geq C \quad \forall r \in (0,\hat{\alpha}), \tag{26}
\]
where \(C\) is independent of \(p\).

**Proof.** Arguments similar to those in the proof of Proposition 2.1 imply that the eigenvalue problem:
\[
\begin{align*}
\Delta^2 \varphi &= \sigma \varphi & \text{in } B, \\
\varphi &= 0, \quad \Delta \varphi - 4 \frac{\partial \varphi}{\partial \nu} &= 0 & \text{on } \partial B
\end{align*}
\tag{27}
\]
admits an eigenvalue \(\sigma_1 > 0\) and an eigenfunction \(\varphi_1 > 0\) corresponding to \(\sigma_1\). Multiplying \(\varphi_1\) on both the sides of the equation of \(v_{p,\alpha}\) and integrating it on \(B\), we see that
\[
\sigma_1 \int_B v_{p,\alpha} \varphi_1 dx = \int_B |x|^\alpha v_{p,\alpha}^p \varphi_1 dx \leq \|v_{p,\alpha}\|_{L^\infty(B)}^{p-1} \int_B v_{p,\alpha} \varphi_1 dx. \tag{28}
\]
This implies that
\[
\|v_{p,\alpha}\|_{L^\infty(B)}^{p-1} \geq \sigma_1. \tag{29}
\]
Since \(\alpha \geq 0\), we see from (29) that
\[
\|v_{p,\alpha}\|_{L^\infty(B)}^{p-1} \geq \sigma_1. \tag{30}
\]
This implies that (24) holds.

We can easily obtain (25) from (24) by a simple contradiction argument.

We now show (26). Note that \(\alpha^*_s > 0\) implies that \(p > \frac{N+4}{N-4}\) and \(1 < p < \frac{N+4+2\alpha^*_s}{N-4}\).

Suppose that there are sequences \(\{p_i\}\) with \(p_i \rightarrow \infty\) and \(\{r_i\}\) with \(r_i \rightarrow 0\) as \(i \rightarrow \infty\) such that \(v_{p_i,\alpha^*_i}(r_i) \rightarrow 0\) (note that \(\alpha^*_i = (N-4)p_i - (N+4) > 0\)), we have that
\[
v_{p_i,\alpha^*_i}(r) \rightarrow 0 \quad \text{for } r \in (r_i,1) \text{ as } i \rightarrow \infty, \tag{31}
\]
since \(v_{p_i,\alpha^*_i}'(r) < 0\) for \(r \in (0,1]\). On the other hand, it follows from (41) of Proposition 3.4 in next section and Arzela-Ascoli’s theorem that there is \(\hat{v}\) such that (up to a subsequence)
\[
v_{p_i,\alpha^*_i} \rightarrow \hat{v} \quad \text{uniformly on } [0,1] \text{ as } i \rightarrow \infty.
\]
This and (31) imply that \(\hat{v} \equiv 0\) in \([0,1]\). This contradicts (24). This contradiction implies that (26) holds. \(\square\)

**Proof of Theorem 1.1.** Let \(\alpha^*_s = (N-4)p - (N+4)\). We see that \(\alpha^*_s > 0\) provided \(p > \frac{N+4}{N-4}\) and
\[
\frac{N+4}{N-4} - p = \frac{N+4+2\alpha^*_s}{N-4}. \tag{32}
\]
This implies that \(1 < p < \frac{N+4+2\alpha^*_s}{N-4}\) provided \(p > \frac{N+4}{N-4}\).

Let \(u(x)\) be a solution to (P). Making the Kelvin transformation:
\[
v(y) = |x|^{N-4} u(x), \quad y = \frac{x}{|x|^2}.
\]
we know from Lemma 3.1 of [19] that \( v(y) \) satisfies the problem
\[
\begin{cases}
\Delta^2 v = |y|^\alpha v^p & \text{in } B,

v = 0, \quad \Delta v - 4 \frac{\partial v}{\partial v} = 0 & \text{on } \partial B.
\end{cases}
\]
(33)
Since \( 1 < p < \frac{N+4+2\alpha}{N-4} \), it follows from Proposition 2.1 that (33) admits a positive radial solution \( v_{p,\alpha} \in C^4(B) \cap C^2(\overline{B}) \). This implies that \( u_p(x) := |x|^{4-N} v_{p,\alpha} \left( \frac{x}{|x|^2} \right) \) is a positive radial solution to (P) in \( C^4(R^N \setminus \overline{B}) \) satisfying
\[
\lim_{|x| \to \infty} \sup |x|^N u_p(x) < \infty.
\]
The uniqueness of \( v_{p,\alpha} \) implies the uniqueness of \( u_p \). This completes the proof of this theorem.

3. Asymptotic behavior of \( u_p \) as \( p \to \infty \): Proof of Theorems 1.2 and 1.4.
In this section, we study the asymptotic behavior of \( u_p \) obtained in Theorem 1.1 as \( p \to \infty \). The asymptotic behaviour of the solution as \( p \to \infty \) of the second semilinear elliptic problem
\[
\begin{cases}
-\Delta u = u^p & \text{in } \Omega,

u > 0 & \text{in } \Omega,

u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where \( \Omega \) is an annulus of \( \mathbb{R}^N, N \geq 2 \) is studied in [14].
Let
\[
\mathcal{G} = \left\{ \psi \in C^2[1, \infty) : \psi(1) = 0, \lim_{r \to \infty} \psi(r) = 0, \int_1^\infty r^{-N+1}[(\Delta \psi)(r)]^2 dr < \infty \right\}
\]
with \( \|\psi\|_{\mathcal{G}} = \left( \int_1^\infty r^{-N+1}[(\Delta \psi)(r)]^2 dr \right)^{1/2} \). We have the following proposition.

**Proposition 3.1.** For \( p > \frac{N+4}{N-4} \) let us denote by
\[
I_p = \inf_{\psi \in \mathcal{G}} \left[ \int_1^\infty r^{N-1}[(\Delta \psi)(r)]^2 dr \right].
\]
Then we have that for \( p > \frac{N+4}{N-4} \) sufficiently large,
\[
I_p \leq 2 \left[ \frac{(N-2)(N-4)r_0^{-4}}{r_0^{N-2} - 1} \right]^2 \left[ \frac{(N-2)^2}{N(N-4)} + \frac{r_0^N}{N} - \frac{r_0^4 - N^4}{N - 4} \right]
+ 2(N-2)^2(N-4)r_0^{N-4},
\]
(35)
where \( r_0 \) is given in Theorem 1.2.

**Proof.** Both the results of Proposition 2.1 and the Kelvin transformation imply that \( I_p \) attains at some nonnegative \( \tilde{u}_p \in \mathcal{G} \). Note that a simple calculation implies that, for any \( \psi \in \mathcal{G} \),
\[
\int_1^\infty r^{N-1}[(\Delta \psi)^2] dr = \int_0^1 s^{-N-1} \left( \Delta_s \phi - \frac{4}{s} \phi_s - 2(N-4) \frac{\phi}{s^2} \right)^2 ds,
\]
where \( \psi(r) = s^{N-4} \phi(s), r = \frac{1}{s} \) and for any \( \phi \in H^2_{rad}(B) \),
\[
\int_0^1 s^{-N-1}((\Delta_s \phi)^2) ds = \int_1^\infty r^{N-1} \left( \Delta_r \psi - \frac{4}{r} \psi_r - 2(N-4) \frac{\psi}{r^2} \right)^2 dr,
\]
where \( \phi(s) = r^{N-4}\psi(r) \) and \( s = \frac{1}{r} \). The inequalities
\[
\int_0^1 s^{N-3}(\phi_s)^2 ds \leq C \int_0^1 s^{N-1}(\Delta_s\phi)^2 ds, \ \forall \phi \in H^2_{\text{rad}}(B) \quad \text{(see [13, 28])},
\]
\[
\int_0^1 s^{N-5}\phi^2 ds \leq C \int_0^1 s^{N-1}(\Delta_s\phi)^2 ds, \ \forall \phi \in H^2_{\text{rad}}(B),
\]
\[
\int_1^\infty r^{N-3}(\psi_r)^2 dr \leq C \int_1^\infty r^{N-1}(\Delta_r\psi)^2 dr, \ \forall \psi \in \mathcal{G},
\]
and
\[
\int_1^\infty r^{N-5}\psi^2 dr \leq C \int_1^\infty r^{N-1}(\Delta_r\psi)^2 dr, \ \forall \psi \in \mathcal{G}
\]
imply that \( \|\psi\|_\mathcal{G} \) is equivalent to \( \|\phi\|_{H^2_{\text{rad}}(B)} \). The embedding \( H^2_{\text{rad}}(B) \hookrightarrow L^p_{\text{rad}}(B) \) and the Kelvin transformation imply that there is \( C > 0 \) such that
\[
\left( \int_1^{\infty} r^{N-1}|\psi|^{p+1} dr \right)^{\frac{1}{p+1}} \leq C \int_1^{\infty} r^{N-1}(\Delta\phi)^2 dr, \ \forall \psi \in \mathcal{G}. \quad (36)
\]
Considering the function \( \varrho(|x|) \) given in (8), we see that \( \varrho \in \mathcal{G} \) and \( \|\varrho\|_\infty := \max_{0 \leq r \leq 1} \varrho(r) = 1 \) (this also applies in the following). It follows from the definition of \( I_p \) that
\[
I_p \leq \frac{\int_1^{\infty} r^{N-1}|\Delta\varrho(r)|^2 dr}{\left( \int_1^{\infty} r^{N-1}|\varrho(r)|^{p+1} dr \right)^{\frac{1}{p+1}}}
\]
Recalling
\[
\lim_{p \to \infty} \left[ \int_1^{\infty} r^{N-1}|\varrho(r)|^{p+1} dr \right]^{\frac{1}{p+1}} = \|\varrho\|_\infty = 1,
\]
we see that
\[
\frac{\int_1^{\infty} r^{N-1}|\Delta\varrho(r)|^2 dr}{\left( \int_1^{\infty} r^{N-1}|\varrho(r)|^{p+1} dr \right)^{\frac{1}{p+1}}} \to \int_1^{\infty} r^{N-1}|(\Delta\varrho(r)|^2 dr \quad \text{(as } p \to \infty) \]
\[
= \left[ \frac{(N-2)(N-4)r_0^{N-2}}{r_0^{N-2} - 1} \right]^{\frac{1}{2}} \left[ \frac{(N-2)^2}{N(N-4)} + \frac{r_0^N}{N(N-4)} - r_0^2 - \frac{r_0^{N-4}}{N-4} \right] + (N-2)(N-4)r_0^{N-4}.
\]
This completes the proof of this proposition. \( \square \)

**Corollary 3.2.** Let \( u_p \) be the unique positive radial solution of (P). Then we have that for \( p \) sufficiently large,
\[
\int_1^{\infty} r^{N-1}(\Delta u_p)^2 dr \leq C, \quad \int_1^{\infty} r^{N-1}u_p^{p+1} dr \leq C,
\]
where \( C \) is a positive constant independent of \( p \).

**Proof.** Let us consider a minimizer \( \bar{u}_p \) to \( I_p \). We have that \( \bar{u}_p \) solves the problem
\[
\begin{aligned}
\Delta^2 \bar{u}_p &= I_p \bar{u}_p^p \quad \text{in } \mathbb{R}^N \setminus \overline{B} \\
\bar{u}_p &= \Delta \bar{u}_p = 0 \quad \text{on } \partial B.
\end{aligned}
\]
Since the radial solution \( u_p \) to (P) is unique, we derive that \( u_p = I_p^{\frac{1}{p+1}} \bar{u}_p \). Then
\[
\int_1^{\infty} r^{N-1}(\Delta u_p)^2(r) dr = I_p^{\frac{2}{p+1}} \int_1^{\infty} r^{N-1}(\Delta \bar{u}_p)^2(r) dr = I_p^{\frac{2}{p+1}} \leq C,
\]
\[
\int_1^\infty r^{N-1}u_p^{p+1}(r)dr = I_p^{\frac{p+1}{p-1}} \int_1^\infty r^{N-1}\hat{u}_p^{p+1}(r)dr = I_p^{\frac{p+1}{p-1}} \leq C.
\]

**Proposition 3.3.** We have
\[
|\nabla(\Delta u_p)| \leq C,
\]
where \( C \) is a constant independent of \( p \).

**Proof.** We have that \( u_p \) satisfies
\[
(r^{N-1}(\Delta u_p)'(r))' = r^{N-1}u_p^p \quad \text{in} \quad (1, \infty).
\]
From the equation of \( u_p \), we easily see that there is exactly one \( r_p \in (1, \infty) \) such that \( u_p'(r_p) = 0 \) and \( r_p \) is the maximum point of \( u_p \). Indeed, suppose that \( u_p \) admits a local maximum point and a local minimum point in \((1, \infty)\), then \( u_p \) has another maximum point and there are \( r_p^1, r_p^2 \in (1, \infty) \) such that \((\Delta u_p)'(r_p^1) = (\Delta u_p)'(r_p^2) = 0\). Integrating the equation of \( u_p \) in \((r_p^1, r_p^2)\), we derive a contradiction. Similar arguments imply that there is exactly one \( r_p^* \in (1, \infty) \) such that \((\Delta u_p)'(r_p^*) = 0\).

Note that \( r_p \) and \( r_p^* \) are the unique maximum and minimum point of \( u_p \) and \( \Delta u_p \) in \((1, \infty)\) respectively. Integrating (39) in \((r, r_p^*) \subset (1, r_p^*) \) and \((r_p^*, r) \subset (r_p^*, \infty)\), we obtain
\[
|\Delta u_p|(r) \leq \frac{1}{r^{N-1}} \int_1^r s^{N-1}u_p^p(s)ds.
\]
By Corollary 3.2 the claim follows. \( \square \)

The following proposition mainly present the uniform boundedness of \( \nabla(\Delta v_{p,\alpha}) \) for any \( p > \frac{N+4}{N-4} \). We would like to point out that this result can not be obtained directly by arguments similar to those in the proof of Proposition 3.3.

**Proposition 3.4.** Let \( v_{p,\alpha} \) be given in Proposition 2.1. Then
\[
|\nabla(\Delta v_{p,\alpha})| \leq C,
\]
where \( C \) is a constant independent of \( p \). Moreover,
\[
|\Delta v_{p,\alpha}(r)| \leq C, \quad |v_{p,\alpha}'(r)| \leq C, \quad |v_{p,\alpha}(r)| \leq C \quad \forall r \in [0, 1],
\]
where \( C > 0 \) is independent of \( p \). Furthermore, for \( |x| \) sufficiently large,
\[
u_p(|x|) \leq C|x|^{4-N}, \quad (\Delta u_p)(|x|) \leq C|x|^{2-N},
\]
where \( C \) is independent of \( p \).

**Proof.** Let
\[
\tilde{\varrho}(|y|) = |x|^{N-4}\varrho(|x|), \quad |y| = \frac{1}{|x|}.
\]
We see that \( \tilde{\varrho} \in H^2_{rad}(B) \). Using \( \tilde{\varrho} \) as the test function, arguments similar to those in the proof of Proposition 3.1 imply that
\[
A_{p,\alpha} \leq C
\]
and
\[
\int_0^1 \xi^{N-1+\alpha}v_{p,\alpha}^{p+1}(\xi)d\xi = \int_0^\infty r^{N-1}u_p^{p+1}dr = I_p^{\frac{p+1}{p-1}} \leq C,
\]
where \( C > 0 \) is independent of \( p \). It follows from (14) and the embeddings that
\[
\int_0^1 r^{N-1}(\Delta v_{p,\alpha})^2dr \leq C,
\]
where $C > 0$ is independent of $p$. To see this, we notice that $v_{p,\alpha} = \frac{1}{N\alpha - 2}A_{p,\alpha}^{\frac{2}{N\alpha - 2}}v_{p,\alpha}$ and
\[
\int_0^1 r^{N-1}(\Delta v_{p,\alpha})^2 dr = A_{p,\alpha}^{\frac{2}{N\alpha - 2}} \int_0^1 r^{N-1}(\Delta \overline{v}_{p,\alpha})^2 dr. \tag{46}
\]
We also know that
\[
\int_0^1 r^{N-1}(\Delta \overline{v}_{p,\alpha})^2 dr = A_{p,\alpha} + 4(\overline{v}_{p,\alpha}(1))^2. \tag{47}
\]

It is known from (14) that for any sufficiently small $\delta > 0$, $\overline{v}_{p,\alpha} \in H^2(1-\delta, 1)$. Moreover, the embedding $H^2(1-\delta, 1) \hookrightarrow C^{1,\tau}([1-\delta, 1])$ for some $0 < \tau < \frac{1}{2}$ implies that
\[
\left( \frac{\max_{r \in [1-\delta, 1]} |\overline{v}_{p,\alpha}(r)|}{r} \right)^2 \leq C \int_{1-\delta}^1 r^{N-1}(\Delta \overline{v}_{p,\alpha})^2 dr, \tag{48}
\]
where $C > 0$ is independent of $\delta$ and $p$. Since we can choose $\delta$ such that
\[
\frac{\int_{1-\delta}^1 r^{N-1}(\Delta \overline{v}_{p,\alpha})^2 dr}{\int_0^1 r^{N-1}(\Delta \overline{v}_{p,\alpha})^2 dr} \leq \frac{1}{16C}, \tag{49}
\]
we see from (47)-(49) that
\[
\int_0^1 r^{N-1}(\Delta \overline{v}_{p,\alpha})^2 dr \leq \frac{4}{3}A_{p,\alpha} \leq C, \tag{50}
\]
where $C > 0$ is independent of $p$. We obtain (45) from (46) since $A_{p,\alpha}^{\frac{2}{N\alpha - 2}} \leq C$ and $C > 0$ is independent of $p$.

Define $z_p(r) = -\Delta v_{p,\alpha}(r)$. We see that $z_p(r) > 0$ for $r \in [0, 1)$ and $(v_{p,\alpha}, z_p)$ satisfies the system of equations
\[
\begin{cases}
-(r^{N-1}v_{p,\alpha}'(r))' = r^{N-1}z_p(r) & \text{in } (0, 1), \\
-(r^{N-1}z_p(r))' = r^{N-1}v_{p,\alpha}'(r) & \text{in } (0, 1), \\
v_{p,\alpha}(1) = 0, \quad z_p(1) = -4v_{p,\alpha}'(1). \tag{51}
\end{cases}
\]
Moreover, we easily see from the equations in (50) that $r^{N-1}v_{p,\alpha}'(r)$ and $r^{N-1}z_p(r)$ are decreasing functions for $r \in (0, 1)$ and thus both $\lim_{r \to 0} r^{N-1}v_{p,\alpha}'(r)$ and $\lim_{r \to 0} r^{N-1}z_p(r)$ exist (maybe $\infty$). The facts $v_{p,\alpha} \in H^2_{rad}(B)$ and the embedding $H^2_{rad}(B) \hookrightarrow L^q_{rad}(B)$ for $1 \leq q \leq \frac{N+4+2\alpha}{N-2} + 1$ imply that
\[
\int_B |x|^\alpha v_{p,\alpha} p dx \leq C p, \quad \int_B z_p^2 dx \leq C, \tag{51}
\]
and
\[
\int_B |z_p| dx \leq \left( \int_B z_p^2 dx \right)^{\frac{1}{2}} |B|^\frac{1}{2} \leq C, \tag{52}
\]
where $C > 0$ is independent of $p$. Therefore, for any $\epsilon > 0$,
\[
- \int_{B_r} |x|^\alpha v_{p,\alpha} p dx = \int_{B_r} \Delta z_p dx = C \epsilon^{N-1} z_p' (\epsilon),
\]
\[
- \int_{B_r} z_p dx = \int_{B_r} \Delta v_{p,\alpha} p dx = C \epsilon^{N-1} v_{p,\alpha}' (\epsilon).
\]
These, (51) and (52) imply that, for any $p > \frac{N+4}{N-2}$,
\[
v_{p,\alpha}'(r) \to 0, \quad z_p(r) \to 0 \text{ as } r \to 0.
\]
Therefore, for any $p > \frac{N+4}{N-4}$,

$$v_{p, \alpha_*}(r) \leq C, \quad z_p(r) \leq C \quad \forall r \in [0, \epsilon]. \tag{53}$$

To see this, we notice from the embedding and (45) that there is $C > 0$ independent of $p$ such that

$$\int_0^1 r^{N+\alpha_*-1} \frac{2^{(N+\alpha_*)}}{v_{p, \alpha_*}}(r)dr \leq C \frac{2^{(N+\alpha_*)}}{N+\alpha_*}. \tag{54}$$

Using the fact that $v_{p, \alpha_*}(r)$ is decreasing, we have that for any $\epsilon > 0$ and $r \in (0, \epsilon]$,

$$\int_0^r 2^{(N+\alpha_*)} s^{N+\alpha_*-1} \frac{1}{v_{p, \alpha_*}}(s)ds \geq r^{N+\alpha_*} C \frac{2^{(N+\alpha_*)}}{N+\alpha_*}. \tag{55}$$

This and (54) imply that for $r \in (0, \epsilon]$,

$$r^{N+\alpha_*} v_{p, \alpha_*}(r) \leq (N + \alpha_*) C \frac{2^{(N+\alpha_*)}}{N+\alpha_*},$$

and

$$r^{\frac{N-4}{2}} v_{p, \alpha_*}(r) \leq (N + \alpha_*) C \frac{2^{(N+\alpha_*)}}{N+\alpha_*}. \tag{55}$$

where $C > 0$ is independent of $p$. Similar arguments imply that for $r \in (0, \epsilon]$,

$$r^{\frac{N-4}{2}} z_p(r) \leq C, \tag{56}$$

where $C > 0$ is independent of $p$.

Let

$$w(t) = r^{\frac{N-4}{2}} v_{p, \alpha_*}(r), \quad y(t) = r^{\frac{N}{2}} z_p(r), \quad t = -\ln r.$$

(We omit the subscript $p$ here.) We see that $(w(t), y(t))$ satisfies the problem

$$w_{tt} - 2w_t - \frac{N(N-4)}{4} w + y = 0, \quad t \in (0, \infty),$$

$$y_{tt} + 2y_t - \frac{N(N-4)}{4} y + e^{-p \cdot t} w^p = 0, \quad t \in (0, \infty) \tag{57}$$

where and in the following

$$p_* = \frac{[N + 4 + 2\alpha_*] - [N - 4]p}{2} = \frac{\alpha_*}{2}.$$

Note that $p_* > 0$ provided $1 < p < \frac{N+4+2\alpha_*}{N-4}$ or $p > \frac{N+4}{N-4}$. We know from (55), (56) and $1 < p < \frac{N+4+2\alpha_*}{N-4}$ (or $p > \frac{N+4}{N-4}$) that $w(t) \leq C, y(t) \leq C$ for $t \in [-\ln \epsilon, \infty)$, where $C > 0$ is independent of $p$. We also know that

$$e^{-p \cdot t} w^p(t) = (e^{-\frac{\alpha_*}{2} t} w(t))^p$$

and

$$\frac{p_*}{p} = \frac{(N-4)p - (N + 4)}{2p} \rightarrow \frac{N-4}{2} \text{ as } p \rightarrow \infty.$$

Therefore, for any $p > \frac{N+4}{N-4}$,

$$e^{-\frac{\alpha_*}{2} t} w(t) \leq e^{-\frac{\alpha_*}{2} t} C \quad \text{for } t \in [-\ln \epsilon, \infty)$$

and

$$e^{-p \cdot t} w^p(t) = O(e^{-\frac{\alpha_*}{2} t}) \quad \text{for } t \in [-\ln \epsilon, \infty).$$

It follows from the ODE theory on perturbation of linear systems (see [22]) that

$$e^{\frac{N-4}{4} t} w(t) \leq C, \quad e^{\frac{N}{4} t} y(t) \leq C \quad \text{for } t \text{ near } \infty, \tag{58}$$
where $C > 0$ is independent of $p$. Note that the system (57) can be written to the following system

\[
\begin{aligned}
\dot{w}_1 &= w_2, \\
\dot{w}_2 &= \frac{N(N-4)}{4} w_1 + 2w_2 - w_3, \\
\dot{w}_3 &= w_4, \\
\dot{w}_4 &= \frac{N(N-4)}{4} w_3 - 2w_4 - e^{-p_1 t} w_1^p,
\end{aligned}
\]

where $(w_1, w_2, w_3, w_4) = (w, \dot{w}, y, \dot{y})$. The matrix of (59) is

\[
A = \begin{pmatrix}
0 & \frac{N(N-4)}{4} & 1 & 0 \\
0 & 2 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & \frac{N(N-4)}{4} & -2
\end{pmatrix}.
\]

By simple calculations, we see that the four eigenvalues of $A$ are: $\lambda_{1,2} = \pm \frac{N-4}{2}$, $\lambda_{3,4} = \mp \frac{N}{2}$. We only choose $\lambda_1 = -\frac{N-4}{2}$ and $\lambda_3 = -\frac{N}{2}$, since $\lambda_2 = \frac{N-4}{2} > 0$ and $\lambda_4 = \frac{N}{2} > 0$ which do not meet our requirement. Where we use the facts that $w(t)$ and $y(t)$ are bounded for $t$ near $\infty$. We also see that there are eigenvectors for $\lambda_1$ and $\lambda_3$ respectively: $(1, -\frac{N-4}{2}, 0, 0)$ and $(0, 0, 1, -\frac{N}{2})$. These imply that (58) holds and our claim (53) holds. Since, for $r \in (0, \epsilon)$,

\[
|\langle \Delta v_{p, \alpha}, \rangle(r) | = \left| \frac{1}{r^{N-1}} \int_0^r \xi^{N-1+\alpha} v_{p, \alpha}^p \, d\xi \right| \leq \frac{Cp}{N+\alpha^*} r^{1+\alpha^*},
\]

we can choose $\epsilon > 0$ such that, for sufficiently large $p$ and $r \in (0, \epsilon]$, $C r^{N-4-\frac{N+3}{p}} \leq 1$, then

\[
|\nabla (\Delta v_{p, \alpha})| \leq C,
\]

where $C > 0$ is independent of $p$. It is known from (44) that

\[
|\langle \Delta v_{p, \alpha}, \rangle(r) | = \left| \frac{1}{r^{N-1}} \int_0^r \xi^{N-1+\alpha} v_{p, \alpha}^p \, d\xi \right| \leq \epsilon^{1-N} C \forall r \in (\epsilon, 1]
\]

and hence

\[
|\nabla (\Delta v_{p, \alpha})| \leq C \forall r \in (\epsilon, 1],
\]

where $C > 0$ is independent of $p$. Both (61) and (62) imply that

\[
|\nabla (\Delta v_{p, \alpha})| \leq C \forall r \in [0, 1],
\]

where $C > 0$ is independent of $p$. It follows from (63) and a simple calculation that (41) and (42) hold. The proof of Proposition 3.4 is completed.

\[\square\]

Lemma 3.5. We have that

\[
\|u_p\|_\infty \geq C,
\]

where $C > 0$ is independent of $p$.

Proof. It follows from (26) and $u_p(s) = s^{4-N} v_{p, \alpha}(1/s)$ that

\[
u_p(s) \geq C s^{4-N} \forall s \in \left( \frac{1}{r}, \infty \right),
\]

where $C > 0$ is independent of $p$. Therefore,

\[
\max_{s \in [1, \infty)} u_p(s) \geq C s_0^{4-N}
\]
for some \( s_0 \in (\frac{1}{2}, \infty) \). Note that \( C_{s_0}^{4-N} \) is independent of \( p \). This implies that (64) holds. Both (42) and (64) imply that there is \( M > 0 \) independent of \( p \) such that

\[
1 < r_p \leq M, \quad 1 < r_p^* \leq M,
\]

where \( r_p \) and \( r_p^* \), defined in the proof of Proposition 3.3, are the unique maximum and minimum point of \( u_p \) and \( \Delta u_p \) in \((1, \infty)\), respectively.

To obtain (65), we also use contradiction arguments. Suppose that there is a sequence \( \{p_i\} \) with \( p_i \to \infty \) as \( i \to \infty \) such that

\[
\|\Delta u_{p_i}\|_\infty \leq \epsilon_i \to 0 \quad \text{as} \quad i \to \infty.
\]

Since (67) implies that \( 1 < r_{p_i} \leq M \), a simple calculation gives that

\[
\|u_{p_i}\|_\infty \leq O(\epsilon_i) \to 0 \quad \text{as} \quad i \to \infty.
\]

This contradicts (64) and completes the proof of this lemma.

**Corollary 3.6.** We have

\[
u_p \to \bar{u} \not\equiv 0 \text{ in } C^2_{\text{loc}}[1, \infty) \text{ as } p \to \infty.
\]

**Proof.** Let \( r_p^* \) be defined as in the proof of Proposition 3.3. We integrate \((\Delta u_p)'(r)\) in \((1, r_p^*)\) and obtain from (38) that

\[
|((\Delta u_p)(r_p^*)) = \left| \int_1^{r_p^*} (\Delta u_p)'(r)dr \right| \leq C,
\]

where \( C > 0 \) is independent of \( p \). Thus

\[
|\Delta u_p| \leq C.
\]

Similar arguments imply

\[
|u_p'| \leq C, \quad |u_p| \leq C.
\]

From (38), (70), (71) and Ascoli-Arzela’s theorem we obtain that \( u_p \to \bar{u} \) in \( C^2_{\text{loc}}[1, \infty) \) as \( p \to \infty \). By Lemma 3.5, we derive that \( \bar{u} \not\equiv 0 \) and \( \Delta \bar{u} \not\equiv 0 \).

Let

\[
r_0 = \lim_{p \to \infty} r_p, \quad r_0^* = \lim_{p \to \infty} r_p^*.
\]

The boundary conditions of \( u_p \) and \( \Delta u_p \) and the conclusions in Lemma 3.5 imply

\[
1 < r_0 \leq M, \quad 1 < r_0^* \leq M.
\]

**Lemma 3.7.** For any \( r \neq r_0 \) there exists \( p_0 > \frac{N+4}{N-4} \) such that for any \( p \geq p_0 \), we have

\[
u_p(r) < 1.
\]

**Proof.** Let us consider the case \( r < r_0 \) (the proof of the case \( r > r_0 \) is the same). By contradiction, let us suppose that there exist \( \tilde{r} < r_0 \) and a sequence \( \{p_i\} \) with \( p_i \to \infty \) as \( i \to \infty \) such that

\[
u_{p_i}(\tilde{r}) \geq 1.
\]

Since \( u_{p_i} \) is strictly increasing in \([1, r_{p_i}]\), we derive that

\[
u_{p_i}(r) > 1 \quad \text{for any } r \in (\tilde{r}, r_{p_i}].
\]

Let us denote by \( r_{p_i}^* \) the unique minimum point of \( \Delta u_{p_i} \) and \( r_0^* = \lim_{i \to \infty} r_{p_i}^* \). We consider three cases here: (i) \( r_0^* < r_0 \), (ii) \( r_0^* = r_0 \), (iii) \( r_0^* > r_0 \).
We only consider the first case, other two cases can be studied similarly. We integrate the equation of $u_{p_i}$ in $(r^*, r_{p_i})$ and obtain that
\[
|((\Delta u_{p_i})'(r_{p_i}))| = \frac{1}{r_{p_i}^{N-1}} \int_{r_{p_i}}^{r^*} s^{N-1} u_{p_i}(s) ds \to \infty \quad \text{as } i \to \infty \quad (74)
\]
no matter $r^*_0 \leq \tilde{r}$ or $r^*_0 > \tilde{r}$. This contradicts to (65). For the other two cases, we integrate the equation of $u_{p_i}$ in $(\tilde{r}, r^*_0)$ respectively, we can also derive contradictions. \hfill \Box

**Corollary 3.8.** Let $r_p$ and $r^*_p$ be the maximum and minimum points of $u_p$ and $\Delta u_p$ respectively. We have
\[
r^*_p = \lim_{p \to \infty} r^*_p = \lim_{p \to \infty} r_p = r_0.
\]
Therefore,
\[
\lim_{p \to \infty} (\Delta u_p)(r^*_p) = (\Delta \varphi)(r_0) = -(N-2)(N-4)r_0^{-2}.
\] (76)

Suppose that $r_0 \neq r^*_0$. There are two cases: (a) $r_0 > r^*_0$, (b) $r_0 < r^*_0$.

For the first case, it follows from Lemma 3.7 that $u_p(r) < 1$ for $r \in (r^*_p, r_0)$. Therefore, for any sufficiently small $\epsilon > 0$, there is $\tilde{p}_0 > 1$ such that for $p > \tilde{p}_0$,
\[
(r^{N-1}(\Delta u_p)'(r))' < \epsilon \quad \forall r \in (r^*_p, 1).
\]
Integrations imply that
\[
|((\Delta u_p)(r^*_p))| \leq \frac{(r^*_p - 1)(1 - (r^*_p)^{2-N})}{N-2} \epsilon.
\]
This contradicts to (65).

For the second case, we first notice that we can choose $R_* > r^*_0$ such that $|((\Delta u_p)(R_*))| < \frac{C}{10}$, where $C$ is given in (65). It follows from Lemma 3.7 that $u_p(r) < 1$ for $r \in (r^*_p, R_*)$. Therefore, for any sufficiently small $\epsilon > 0$, there is $\tilde{p}_1 > \tilde{p}_0 > 1$ such that for $p > \tilde{p}_1$,
\[
(r^{N-1}(\Delta u_p)'(r))' < \epsilon \quad \forall r \in (r^*_p, R_*).
\]
Integrations imply that, if we choose $\epsilon > 0$ sufficiently small,
\[
|((\Delta u_p)(r^*_p))| \leq |((\Delta u_p)(R_*))| + \left[ \frac{(r^*_p)^{3-N} - R_*^{3-N}}{N-3} + \frac{r^*_p[R_*^{2-N} - (r^*_p)^{2-N}]}{N-2} \right] \epsilon
\]
\[
< \frac{C}{10} + \frac{3C}{10} = \frac{4C}{10}.
\]
This contradicts to (65). \hfill \Box

**Lemma 3.9.** Let $\varpi$ be the function defined in (69). Then we have
\[
\varpi(r) < 1 \quad \text{for any } r \neq r_0 \quad (77)
\]
and
\[
\varpi(r_0) = 1. \quad (78)
\]

**Proof.** By Lemma 3.7 we have that $\varpi(r) \leq 1$. Suppose that there exists $r' \neq r_0$ such that $\varpi(r') = 1$. Without loss of generality, we assume $r' < r_0$. Since $u_p(r)$ is increasing in $[1, r_0]$, we have that $\varpi(r) \equiv 1$ for any $r \in [r', r_0]$ and then $\Delta \varpi \equiv 0$ in $[r', r_0)$. Since $\Delta u_p$ is decreasing in $(1, r^*_p)$ and $(\Delta u_p)(1) = 0$, we have that $(\Delta \varpi)(r) \equiv 0$ for $r \in (1, r_0)$ (note that $r_0 = r^*_0$). This implies that $\varpi'(r) = cr^{1-N}$ and $\varpi(r) = \frac{c}{r_N}(r^{2-N} - 1)$ for $r \in (1, r_0)$. No matter $c = 0$ or not, we can not
obtain \( \varpi \equiv 1 \) in \((r', r_0) \subset (1, r_0)\). This is a contradiction and completes the proof of this lemma. \( \square \)

**Proof of Theorem 1.2.** It follows from Corollary 3.6 and Lemma 3.9 that \( \varpi \) satisfies

\[
\begin{align*}
\Delta^2 \varpi &= 0 \quad \text{in } (1, \infty) \setminus \{r_0\}, \\
\varpi(1) &= (\Delta \varpi)(1) = 0, \quad \varpi(\infty) = (\Delta \varpi)(\infty) = 0, \\
\varpi(r_0) &= 1.
\end{align*}
\tag{79}
\]

A straightforward computation shows that \( \varpi \equiv \varphi \). This completes the proof of this theorem. \( \square \)

**Remark 3.10.** Arguments similar to those in the proof of Theorem 1.2 and simple calculations imply that the following conclusion holds (note that \( (\Delta u_p)(1) > 0 \) and \( (\Delta u_p)'(1) < 0 \)):

Let \( w_p \) be the unique positive radial solution of (4). Then, as \( p \to \infty \),

\[
w_p(r) \to \zeta(r) \quad \text{in } C^2_{loc}(\mathbb{R}^N \setminus B),
\]

with

\[
\zeta(r) = \begin{cases}
\zeta_1(r) & \text{for } 1 \leq r \leq \hat{r}_0, \\
\zeta_2(r) & \text{for } \hat{r}_0 < r < \infty,
\end{cases}
\]

where

\[
\zeta_1(r) = \frac{m_1}{N} \left[ \frac{1}{2} (r^2 - 1) - \frac{1}{N-2} (1 - r^{2-N}) \right] + \frac{m_2}{2N} \left[ \frac{1}{2} (r^2 - 1) + \frac{1}{2N} (1 - r^{2-N}) - \frac{1}{2(N-4)} (1 - r^{4-N}) \right],
\]

\[
\zeta_2(r) = \frac{(N-2)(N-4)}{2\hat{r}_0^{N-4}} \left[ \frac{1}{N-4} \hat{r}_0^{4-N} - \frac{1}{N-2} \hat{r}_0^{2-N} \right].
\]

The numbers \( m_1 > 0, m_2 < 0 \) and \( \hat{r}_0 > 1 \) can be determined by the following system of equations:

\[
\begin{align*}
\frac{m_1}{N} (1 - \hat{r}_0^{-N}) + \frac{m_2}{N-2} \left[ \frac{1}{N} (1 - \hat{r}_0^{-N}) - \frac{1}{2} (\hat{r}_0^{2-N} - \hat{r}_0^{-N}) \right] &= 0, \\
(N-2)m_1 + (1 - \hat{r}_0^{-2-N})m_2 + (N-2)(N-4)\hat{r}_0^{-2} &= 0,
\end{align*}
\]

\[
\begin{align*}
&\frac{m_1}{N} \left[ \frac{1}{2} (\hat{r}_0^2 - 1) - \frac{1}{N-2} (1 - \hat{r}_0^{-2}) \right] + \frac{m_2}{2N} \left[ \frac{1}{2N} (1 - \hat{r}_0^{-2-N}) - \frac{1}{2(N-4)} (1 - \hat{r}_0^{-4-N}) \right] = -1 = 0.
\end{align*}
\]

**Proof of Theorem 1.4.** Multiplying \( u_p' \) on both the sides of the equation

\[
(\Delta u_p)' + \frac{N-1}{r} (\Delta u_p)' = u_p^p
\]

and integrating it on \((r_p, \infty)\), we see that

\[
\frac{\|u_p\|_{p+1}^p}{p+1} = \int_{r_p}^{\infty} (\Delta u_p)(\Delta u_p)' dr - 2(N-1) \int_{r_p}^{\infty} \frac{1}{r} u_p'(\Delta u_p)' dr
\]

\[
= -\frac{1}{2} (\Delta u_p(r_p))^2 - 2(N-1) \int_{r_p}^{\infty} \frac{1}{r} u_p'(\Delta u_p)' dr.
\]

Therefore,

\[
\lim_{p \to \infty} \frac{\|u_p\|_{p+1}^p}{p+1} = -\frac{1}{2} ((\Delta \varphi)(r_0))^2 - 2(N-1) \int_{r_0}^{\infty} \frac{1}{r} \varphi'(\Delta \varphi)' dr = 0
\]

via a simple calculation. \( \square \)
Remark 3.11. We can obtain the following conclusion by using Remark 3.10 and arguments similar to those in the proof of Theorem 1.4.

Let \( w_p \) be the unique positive radial solution of (4). Then, the following estimate holds

\[
\frac{\|w_p\|_{p}^{p+1}}{p+1} \to 0 \text{ as } p \to \infty.
\]  
(82)

4. Non-degeneracy of \( u_p \). The purpose of this section is to see that under what conditions the problem

\[
\begin{cases}
\Delta^2 \phi - p u_p^{p-1} \phi = 0 & \text{in } \mathbb{R}^N \setminus \overline{B}, \\
\phi(1) = (\Delta \phi)(1) = 0 & \lim_{|x| \to \infty} \phi(x) = 0
\end{cases}
\]  
(83)

admits only the trivial solution \( \phi \equiv 0 \).

We first study the problem

\[
\begin{cases}
\Delta^2 \phi - p u_p^{p-1} \phi - \nu \frac{\varphi}{r} = 0 & \text{in } (1, \infty), \\
\phi(1) = (\Delta \phi)(1) = 0 & \lim_{r \to \infty} \phi(r) = 0.
\end{cases}
\]  
(84)

This eigenvalue is characterized as

\[
\nu(p) = \inf_{\psi \in \phi} \frac{\int_1^\infty r^{N-1}(\Delta \psi)^2 dr - \frac{4}{p-1} \int_1^\infty r^{N-1} u_p^{p-1} \psi^2 dr}{\int_1^\infty r^{N-5} \psi^2 dr}
\]  
(85)

The number \( \nu(p) \) is negative, since this Rayleigh quotient gets negative when evaluated at \( \psi = u_p \). This fact, the Hardy’s inequality

\[
\frac{N^2(N-4)^2}{16} \int_1^\infty r^{N-5} \psi^2 dr \leq \int_1^\infty r^{N-1}(\Delta \psi)^2 dr
\]

and a simple compactness argument involving the fast decay \( u_p^{p-1} = o(r^{-4}) \) at \( r = \infty \), yield the existence of an extremal for \( \nu(p) \) which represents a positive solution to (84) for \( \nu = \nu(p) \).

We first present the proof of Proposition 1.3.

Proof of Proposition 1.3. To obtain the conclusion of Proposition 1.3, we first consider the function \( \rho(p) := ru_p'(r) + \frac{4}{p-1} u_p(r) \). For convenience, we omit the subscript \( p \) of \( \rho(p) \) in the following. It is easily seen via a simple calculation that \( \rho \) satisfies the equation

\[
\Delta^2 \rho - pu_p^{p-1} \rho = 0 \text{ in } (1, \infty).
\]  
(86)

We now claim that there is \( \hat{r}_p \in (1, \infty) \) such that

\[
(\Delta \rho)(r) < 0 \forall r \in [1, \hat{r}_p).
\]  
(87)

We see that

\[
(\Delta \rho)(r) = \Delta (ru_p'(r)) + \frac{4}{p-1} (\Delta u_p)(r) = r(\Delta u_p)'(r) + \frac{2(p+1)}{p-1} (\Delta u_p)(r).
\]

Then \( (\Delta \rho)(1) < 0 \) and \( \hat{r}_p \) exists. We can easily see that \( \hat{r}_p \geq r_p^* \).

Suppose that (83) has a nontrivial radial solution \( h(r) \) (we omit the subscript \( p \) of \( h_p \)). We claim that

\[
h'(1) \neq 0.
\]

Otherwise, we have \( h(1) = h'(1) = (\Delta h)(1) = 0 \) and \( (\Delta h)'(1) \neq 0 \) (if \( (\Delta h)'(1) = 0 \), we see that \( h \equiv 0 \) in \( (1, \infty) \)). Without loss of generality, we assume \( (\Delta h)'(1) > 0 \), then it follows from the equation of \( h \) that \( h(r) > 0 \) for all \( r \in (1, \infty) \) and \( h \) is an increasing function of \( r \). This contradicts to the fact that \( h(r) \to 0 \) as \( r \to \infty \).
Without loss of generality, we assume \( h'(1) > 0 \). The similar arguments imply that \((\Delta h)'(1) < 0\). Therefore, there is \( 0 < r_1 \leq \hat{r}_p \) such that
\[
h(r) > 0, \quad (\Delta h)(r) < 0 \quad \forall r \in (1, r_1).
\]
Multiplying the equation of \( h \) by \( \rho \) and the equation of \( \rho \) by \( h \) and integrating on \((r, \infty)\), we see that, for \( r \in [1, \infty)\),
\[
[(\Delta h)(r)\rho'(r) - (\Delta h)'(r)\rho(r)] + [(\Delta \rho)''(r)h(r) - (\Delta \rho)(r)h'(r)] = 0.
\]
Let
\[
I_1(r) = [(\Delta h)(r)\rho'(r) - (\Delta h)'(r)\rho(r)], \quad I_2(r) = [(\Delta \rho)''(r)h(r) - (\Delta \rho)(r)h'(r)].
\]
Since \( h(1) = (\Delta h)(1) = 0 \), \( \rho(1) > 0 \) and the claim \( (87) \) implies that \( (\Delta \rho)(1) < 0 \), we see that
\[
I_1(1) = -(\Delta h)'(1)\rho(1) > 0, \quad I_2(1) = -(\Delta \rho)(1)h'(1) > 0.
\]
This contradicts \( (89) \) and the proof of this proposition is completed. \( \square \)

**Remark 4.1.** It follows from Proposition 1.3 and the Kelvin transformation that the problem
\[
\left\{
\begin{array}{ll}
\quad h^{(4)}(r) + \frac{2(N-1)}{r} h''(r) + \frac{(N-1)(N-3)}{r^2} h''(r) - \frac{(N-1)(N-3)}{r^3} h'(r) = 0, & \forall r \in [0, 1),

h(1) = 0, & (\Delta h)(1) = 4h'(1),
\end{array}
\right.
\]
does not admit any nontrivial solution \( h(r) \) satisfying \( \lim_{r \to 0^+} r^{N-4} h(r) = 0 \). Making the transformations:
\[
k(t) = h(r), \quad t = \log r,
\]
we see that \( k(t) \) satisfies the problem:
\[
k^{(4)}(t) + 2(N-4)k'''(t) + (N^2 - 10N + 20)k''(t) - 2(N^2 - 6N + 8)k'(t) = 0, & \forall t \in (-\infty, 0).
\]
Moreover, \( k(0) = 0 \). The characteristic equation of \( (91) \) is
\[
\lambda^4 + 2(N-4)\lambda^3 + (N^2 - 10N + 20)\lambda^2 - 2(N^2 - 6N + 8)\lambda = 0.
\]
A simple calculation shows that the 4 roots of \( (92) \) are \( 2, 0, -(N-2), -(N-4) \). The standard ODE theory implies that for \( t \to -\infty \),
\[
k(t) = M_1(1+o(1))e^{2t} + M_2(1+o(1)) + M_3(1+o(1))e^{-(N-2)t} + M_4(1+o(1))e^{-(N-4)t}.
\]
This implies that the equation in \( (90) \) has four solutions \( h_i(r) \) \( i = 1, 2, 3, 4 \) such that \( h_1(0) = h_1'(0) = 0 \), \( |h_2(0)| < \infty \), \( \lim_{r \to 0^+} r^{(N-2)}|h_3(r)| < \infty \), and
\[
\lim_{r \to 0^+} r^{(N-4)}|h_4(r)| < \infty.
\]
Proposition 1.3 implies that \( h_1(r), h_2(r) \) and \( h_3(r) \) can not satisfy the boundary conditions of \( (90) \).

As in the previous section let \( u_p \) be the unique positive radial solution of \( (P) \). We want to analyze the possible degeneracy of the linearized operator \( L_{u_p} = \Delta^2 - pu_p^{-1}I \). To this aim let us denote by \( \beta \) a generic eigenvalue of the problem
\[
\left\{
\begin{array}{ll}
\Delta^2 z - pu_p^{-1}z = \beta z & \text{in } \mathbb{R}^N \setminus \overline{B},

z = \Delta z = 0 & \text{on } \partial B, \quad \lim_{|x| \to \infty} z(x) = 0.
\end{array}
\right.
\]
We will find the conditions such that \( \beta \neq 0 \).
Therefore, we choose \( k \) with \( w \) and an eigenfunction \( \psi \) of \( -\Delta_{S^{N-1}} \) associated to \( \lambda_k \). Then the function

\[
\hat{w}(r) := \int_{S^{N-1}} \psi(r, \theta) \phi(\theta) d\theta
\]

satisfies

\[
\hat{w}^{(4)} + \frac{2(N-1)}{r} \hat{w}''' + \frac{(N-1)(N-3)}{r^2} \hat{w}'' - \frac{(N-1)(N-3)}{r^3} \hat{w}'
\]

\[
= \int_{S^{N-1}} \left( \psi^{(4)} + \frac{2(N-1)}{r} \psi_{rrr} + \frac{(N-1)(N-3)}{r^2} \psi_{rr} \right) \phi d\theta
\]

\[
- \int_{S^{N-1}} \frac{(N-1)(N-3)}{r^3} \psi_r \phi d\theta
\]

\[
= \int_{S^{N-1}} \Delta^2 \psi - \frac{(8-2N)}{r^4} \Delta_{S^{N-1}} \psi - \frac{(2N-6)}{r^3} \Delta_{S^{N-1}} \psi_r \phi d\theta
\]

\[
- \int_{S^{N-1}} \left( \frac{2}{r^2} \Delta_{S^{N-1}} \psi_r + \frac{\Delta_{S^{N-1}}^2 \psi}{r^4} \right) \phi d\theta
\]

\[
= \int_{S^{N-1}} \mu w^{p-1} w \phi d\theta + \frac{\mu}{r^4} w - \frac{2(N-6)}{r^4} \Lambda_{S^{N-1}} \psi \phi d\theta
\]

\[
- \int_{S^{N-1}} \left( \frac{(2N-6)}{r^3} \Delta_{S^{N-1}} \psi_r + \frac{2}{r^2} \Delta_{S^{N-1}} \psi_{rr} + \frac{1}{r^4} \Delta_{S^{N-1}}^2 \psi \right) \phi d\theta
\]

\[
= \mu w^{p-1} w + \frac{\mu}{r^4} w - \frac{(8-2N)}{r^4} (-\lambda_k) w
\]

\[
- \frac{(2N-6)}{r^3} (-\lambda_k) w' - \frac{2}{r^2} (-\lambda_k) w'' - \frac{1}{r^4} \lambda_k^2 w
\]

\[
= \mu w^{p-1} w + \frac{\mu - \lambda_k^2}{r^4} w + 2 \lambda_k \left[ \frac{1}{r^2} w'' + \frac{(N-3)}{r^3} w' + \frac{(4-N)}{r^4} w \right].
\]

Therefore,

\[
\hat{L}_{p,k} w = \mu w.
\]
This implies that \( \mu \) is an eigenvalue of the operator \( \hat{L}_{p,k} \). Note that we can show \( w(1) = (\Delta w)(1) = 0 \) by the expansion of \( \psi(r,\theta) \) with respect to the eigenfunctions of \(-\Delta_{S^{n-1}}\) and a simple calculation.

In order to see the converse, we consider \( \lambda_k \in \sigma(-\Delta_{S^{n-1}}) \) and \( \beta_k \in \sigma(\hat{L}_{p,k}) \), and choose corresponding eigenfunctions \( \phi \) and \( w \). Setting

\[
v(x) = w(|x|)\phi\left(\frac{x}{|x|}\right),
\]

there holds

\[
\Delta^2 v = \left(\Delta^2 + \frac{(8-2N)}{r^4}(-\lambda_k)w + \frac{(2N-6)}{r^3}(-\lambda_k)w' + \frac{2}{r^2}(-\lambda_k)w'' + \frac{\lambda_k^2}{r^4}w \right)\phi
\]

\[
= \left[\Delta^2 + \frac{\lambda_k^2}{r^4} - 2\lambda_k\left(\frac{w''(r)}{r^2} + (N-3)\frac{w'}{r^3} + \frac{4-N}{r^4}w\right)\right] \phi
\]

\[
= \left(\Delta - \frac{\lambda_k}{r^2}\right)^2 (w)\phi.
\]

This implies that

\[
\hat{L}_p v = \hat{L}_{p,k}(w)\phi = \beta_k v.
\]

Hence, \( \beta_k \in \sigma(\hat{L}_p) \). Therefore, \( \mu = \beta_k \in \sigma(\hat{L}_p) \). Note that we can easily see \( v = \Delta v = 0 \) on \( \partial B \) since \( w(1) = (\Delta w)(1) = 0 \).

**Corollary 4.3.** The problem (83) has a nontrivial solution if and only if there exists \( k \geq 1 \) such that

\[
\beta_k = 0,
\]

where \( \beta_k \in \sigma(\hat{L}_{p,k}) \). Moreover the solutions \( v \) to (83) can be written as

\[
v(x) = w(|x|)\phi_k\left(\frac{x}{|x|}\right)
\]

where \( w(r) \) is a solution of the problem

\[
\begin{align*}
\left\{ \begin{array}{l}
\left(\Delta - \frac{\lambda_k}{r^2}\right)^2 w - pu^{p-1}_w = \beta_k \frac{w}{r^2} & \text{in } (1, \infty), \\
w(1) = (\Delta w)(1) = 0, \lim_{r \to \infty} w(r) = 0. 
\end{array} \right.
\end{align*}
\]

**Proof.** First of all we point out that any nontrivial solution \( v \) to (83) satisfies \( \hat{L}_p v = 0 \) with \( v = \Delta v = 0 \) on \( \partial B \) and \( \lim_{|x| \to \infty} v(x) = 0 \). Hence any solution to (83) corresponds to an eigenfunction of \( \hat{L}_p \) relative to the zero eigenvalue.

So, it follows from Lemma 4.2 that, if zero is an eigenvalue of \( \hat{L}_p \) then \( \beta_k^{(i)} = 0 \) for some \( i \geq 1 \) and \( k \geq 0 \) (note that \( \beta_k^{(i)} = \beta_k^{(i)}(p) \) and the corresponding eigenfunction is \( v(x) = w_i(|x|)\phi_k\left(\frac{x}{|x|}\right) \). Hence we have to show that \( \beta_k^{(i)} = 0 \) only if \( k \neq 0 \). On the contrary, since \( \lambda_0 = 0 \), we see that \( \beta_k^{(i)} = 0 \in \sigma(\hat{L}_{p,0}) \). This contradicts Proposition 1.3. \( \square \)

**Lemma 4.4.** Let \( \beta_k^{(i)} \in \sigma(\hat{L}_{p,k}) \). Then, if \( l \geq m \geq 1 \),

\[
\beta_l^{(i)} \geq \beta_m^{(i)}.
\]

Moreover,

\[
\beta_k^{(i)} > 0
\]

for \( k \geq 1 \) and \( i \geq 2 \).
Proof. By the Kelvin transformation, we can change the eigenvalue problem (96) to a related eigenvalue problem on \( B \). We see that
\[
\beta_k^{(i)} = \tilde{\beta}_k^{(i)} + \lambda_k^2,
\]
where
\[
\tilde{\beta}_k^{(i)} = \inf_{W \subset \mathbb{R}^N, \dim W = i} \max_{u \in W, u \neq 0} \frac{\int_1^\infty r^{N-1} \left[ (\Delta u)^2 - pu^{p-1}u'^2 + G(r, w) \right] dr}{\int_1^\infty r^{N-5} u'^2 dr}
\]
with \( G(r, w) = 2 \lambda_k \left( r^{-2} (w')^2 + (N - 4) r^{-4} w^2 \right) \). Then (97) can be easily obtained from (99) and the expression of \( \tilde{\beta}_k^{(i)} \).

To prove (98), we only need to show
\[
\beta_0^{(2)} > 0.
\]
We see that \( \beta_0^{(1)} = \tilde{\beta}_0^{(1)} = \nu(p) < 0 \). Since \( u_p \) is a mountain pass solution, the radial Morse index of \( u_p \) is at most 1, we have that \( \beta_0^{(2)} \geq 0 \). To see this, let us suppose that the operator \( \tilde{L}_{p,r} := r^4 (\Delta^2 - pu^{p-1}I) \) admits at least 2 negative (radial) eigenvalues, say, \( \beta_0^{(1)} < \beta_0^{(2)} < 0 \), since \( \beta_0^{(1)} \) is simple. We assume that \( \varphi_k \) are the associated eigenfunctions of \( \tilde{L}_{p,r} \). Since \( \varphi_1 \) is orthogonal to \( \varphi_2 \) in \( L^2_{rad}(\mathbb{R}^N \setminus \overline{B}, |x|^{-4}) \), we have that
\[
\int_{\mathbb{R}^N \setminus \overline{B}} (\Delta \varphi_1 \Delta \varphi_2 dx - pu_p^{p-1} \varphi_1 \varphi_2) = 0.
\]
From this we see that
\[
I(\psi) := \int_{\mathbb{R}^N \setminus \overline{B}} [(\Delta \psi)^2 - pu_p^{p-1} \psi^2] dx,
\]
is negative on \( X := \{ s \varphi_1 + t \varphi_2 : s, t \in \mathbb{R} \} \) except at the origin and hence the radial Morse index of \( u_p \) is at least two. This contradicts the fact that the radial Morse index of \( u_p \) is at most 1. Therefore, \( \beta_1^{(2)} > \beta_0^{(2)} \geq 0 \). Then, (98) can be obtained from (100) and (97).

We now have the following theorem.

**Theorem 4.5.** Assume that \( p \) such that
\[
\beta_k^{(1)} \neq 0 \quad \text{for} \quad k = 1, 2, 3, \ldots
\]
Then \( u_p \) is non-degenerate.

Proof. It follows from (98) that \( \beta_k^{(i)} \neq 0 \) for \( k \geq 1 \) and \( i \geq 2 \). The only possibility for \( \beta_k = 0 \) is \( \beta_k^{(1)} = 0 \). Assume that \( p \) is such that (101) holds, then problem (83) admits only the solution \( \phi \equiv 0 \) and \( u_p \) is non-degenerate.

It is remarkable to note that \( \beta_k^{(1)} = 0 \) is equivalent to \( \sigma_k^{(1)} = p \), where \( \sigma_k^{(1)} \) is the first eigenvalue of the eigenvalue problem:
\[
\left\{ \begin{array}{l}
\left( \Delta - \frac{\lambda_k}{r^2} \right)^2 \psi = \sigma u_p^{p-1} \psi \quad \text{in} \quad (1, \infty), \\
\psi(1) = (\Delta \psi)(1) = 0, \quad \lim_{r \to \infty} \psi(r) = 0.
\end{array} \right.
\]
Since
\[
\Delta - \frac{\lambda_k}{r^2} = r^k \left[ \frac{\partial^2}{\partial r^2} + \frac{N + 2k - 1}{r} \frac{\partial}{\partial r} \right] r^{-k},
\]

(103)
the problem (102) becomes

\[
\begin{cases}
\left( \frac{\partial^2}{\partial r^2} + \frac{N+2k-1}{r} \frac{\partial}{\partial r} \right) \tilde{\psi} = \sigma u_p^{p-1} \tilde{\psi} \text{ in } (1, \infty), \\
\tilde{\psi}(1) = (\Delta \tilde{\psi})(1) + 2k \tilde{\psi}'(1) = 0, \quad \lim_{r \to \infty} \tilde{\psi}(r) = 0,
\end{cases}
\]

(104)

where \( \psi = r^k \tilde{\psi} \).

By Kelvin’s transformation in dimension \( N + 2k \), (104) is equivalent to

\[
\begin{cases}
\Delta_k^2 \tilde{\psi} = \sigma r^{4-\alpha} \psi_p^{p-1} \tilde{\psi} \text{ in } [0, 1), \\
\tilde{\psi}(1) = (\Delta_k \tilde{\psi})(1) - 4\tilde{\psi}'(1) = 0,
\end{cases}
\]

(105)

where \( \tilde{\psi}(r) = r^{4-(N+2k)} \tilde{\psi} \left( \frac{1}{r} \right) \) and \( \Delta_k = \frac{\partial^2}{\partial r^2} + \frac{N+2k-1}{r} \frac{\partial}{\partial r} \) is the Laplace operator in \( \mathbb{R}^{N+2k} \). As in the proof of Proposition 2.1, the new eigenvalue problem (105) admits a variational structure: in fact it can be rewritten as the eigenvalue problem in \( B_1^k \) with \( B_1^k \) being the unit ball in \( \mathbb{R}^{N+2k} \). The standard spectrum theory implies that (105) admits an infinite sequence of eigenvalues

\[ \sigma_k^{(1)} < \sigma_k^{(2)} < \ldots \]

Note that \( \sigma_k^{(i)} = \sigma_k^{(i)}(p) \). Therefore, we have arrived from Theorem 4.5 the following corollary.

**Corollary 4.6.** Assume that \( p \) is such that

\[ \sigma_k^{(1)}(p) \neq p \text{ for all } k = 1, 2, \ldots, \]

(106)

where \( \sigma_k^{(1)}(p) \) is the first eigenvalue of (105). Then \( u_p \) is nondegenerate.

Using arguments similar to those in the proof of Proposition 1.3 to (104), we can obtain that \( \sigma_k^{(1)}(p) \neq p \). Note that the function \( \rho_p(r) \) in the proof of Proposition 1.3 satisfies \( \rho_p(1) \leq 0 \). It remains to show that the set \( \{ p : \sigma_k^{(1)}(p) = p \} \) is finite. We use arguments similar to those in the proof of Proposition 3.5 of [20]. We first claim that under the assumption \( \frac{N+4}{N} < p \leq \frac{N+4}{N-4} + M \), if \( \sigma_k^{(1)}(p) = p \), then

\[ k \leq K_M, \]

(107)

where \( K_M \) depends only on \( M \). It is known from (41) that, for \( p > \frac{N+4}{N-4} \), \( v_{p,\alpha} \leq C \) in \([0, 1] \), where \( C > 0 \) is independent of \( p \). Therefore,

\[ \sigma_k^{(1)} \geq \frac{1}{C_M} \sigma_k^{(1)} \]

where \( \sigma_k^{(1)} \) is the first eigenvalue of

\[
\begin{cases}
\Delta_k^2 \tilde{\phi} = \sigma \tilde{\phi} \text{ in } [0, 1), \\
\tilde{\phi}(1) = (\Delta_k \tilde{\phi})(1) - 4\tilde{\phi}'(1) = 0.
\end{cases}
\]

(108)

Since \( \sigma_k^{(1)} \to +\infty \) as \( k \to +\infty \), we deduce that \( k \leq K_M \) if \( \sigma_k^{(1)}(p) = p \). Our claim (107) holds.

We then claim that the eigenvalues \( \sigma_k^{(1)} \) are simple and analytic in \( p \). We can show that \( \sigma_k^{(1)} \) is simple for each \( k \) by the comparison principle of \( \Delta^2 \) in the radial form (see [26]). We can show that \( \sigma_k^{(1)}(p) \) is analytic in \( p \) by arguments similar to those in the proof of Lemma 3.3 of [20]. Therefore, a variant of the proof of Lemma 3.4 of [20] and arguments similar to those in the proof of Proposition 3.5 of [20] imply that the following proposition holds.
Proposition 4.7. For each $k$ the set of numbers $p$ for which $\sigma_k^{(1)}(p) = p$ is finite (maybe empty). In particular, there exist countably many supercritical exponents $\{p_1, \ldots, p_j, \ldots\}$ with $p_j > \frac{N+4}{N-4}$ such that (106) holds if and only if $p \neq p_j$ for all $j = 1, 2, \ldots$.

Remark 4.8. Since we do not know the exact profile of $u_p$ near the maximum point $r_p$, we do not claim that $p_j \to +\infty$ as $j \to +\infty$. Neither do we claim the set $\{p_j\}$ is nonempty.

Combining Corollary 4.6 and Proposition 4.7, we obtain the following theorem

Theorem 4.9. Assume that $p$ is such that $p \neq p_j$ for all $j = 1, 2, \ldots$. Then $u_p$ is nondegenerate.

REFERENCES

[1] G. Arioli, F. Gazzola, H. C. Grunau and E. Mitidieri, A semilinear fourth order elliptic problem with exponential nonlinearity, SIAM J. Math. Anal., 36 (2005), 1226–1258.
[2] A. Bahri and J. M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of the topology of the domain, Comm. Pure Applied Math., 41 (1988), 253–294.
[3] E. Berchio, A. Farina, A. Ferrero and F. Gazzola, Existence and stability of entire solutions to a semilinear fourth order elliptic problem, J. Differential Equations, 252 (2012), 2596–2612.
[4] R. Dalmasso, Uniqueness theorems for some fourth-order elliptic equations, Proc. Amer. Math. Soc., 123 (1995), 1177–1183.
[5] J. Davila, L. Dupaigne, K. L. Wang and J. C. Wei, A monotonicity formula and a Liouville-type theorem for a fourth order supercritical problem, Adv. Math., 258 (2014), 240–285.
[6] J. Davila, I, Flores and I. Guerra, Multiplicity of solutions for a fourth order problem with power-type nonlinearity, Math. Ann., 348 (2010), 143–193.
[7] M. del Pino and J. C. Wei, Supercritical elliptic problems in domains with small holes, Ann. I. H. Poincaré-AN, 24 (2007), 507–520.
[8] F. Ebobisse and M. O. Ahmedou, On a nonlinear fourth order elliptic equation involving the critical Sobolev exponent, Nonlinear Anal., 52 (2003), 1535–1552.
[9] A. Ferrero, H. C. Grunau and P. Karageorgis, Supercritical biharmonic equations with power-type nonlinearity, Annali di Matematica, 188 (2009), 171–185.
[10] F. Gazzola and H. C. Grunau, Radial entire solutions for supercritical biharmonic equations, Math. Ann., 334 (2006), 905–936.
[11] F. Gazzola, H. C. Grunau and M. Squassina, Existence and nonexistence results for critical growth biharmonic elliptic equations, Calc. Var. PDEs, 18 (2003), 117–143.
[12] N. Ghoussoub and A. Moradifam, Bessel pairs and optimal Hardy and Hardy-Rellich inequalities, Math. Ann., 349 (2011), 1–57.
[13] N. Ghoussoub and A. Moradifam, Functional Inequalities: New Perspectives and New Applications, Mathematical Surveys and Monographs, vol. 187, American Mathematical Society, 2013.
[14] M. Grossi, Asymptotic behaviour of the Kazdan-Warner solution in the annulus, J. Differential Equations, 233 (2006), 96–111.
[15] Z. M. Guo, Further study of entire radial solutions of a biharmonic equation with exponential nonlinearity, Ann. di Matematica, 193 (2014), 187–201.
[16] Z. M. Guo, X. Huang and F. Zhou, Radial symmetry of entire solutions of a bi-harmonic equation with exponential nonlinearity, J. Funct. Anal., 268 (2015), 1972–2004.
[17] Z. M. Guo and J. C. Wei, Qualitative properties of entire radial solutions for a biharmonic equation with supercritical nonlinearity, Proc. Amer. Math. Soc., 138 (2010), 3957–3964.
[18] Z. M. Guo, F. S. Wan and L. P. Wang, Embeddings of weighted Sobolev spaces and a weighted fourth order elliptic equation, arXiv:1803.11298 (2018), in press.
[19] Z. M. Guo, J. C. Wei and F. Zhou, Singular radial entire solutions and weak solutions with prescribed singular set for a biharmonic equation, J. Differential Equations, 263 (2017), 1188–1224.
[20] Y. X. Guo and J. C. Wei, Supercritical biharmonic elliptic problems in domains with small holes, Math. Nachr., 282 (2009), 1724–1739.
[21] H. Hajlaoui, A. Harrabi and D. Ye, On stable solutions of biharmonic problem with polynomial growth, *Pacific J. Math.*, 270 (2014), 79–93.

[22] P. Hartman, *Ordinary Differential Equations*, 2nd edition. Birkhäuser, Boston, 1982.

[23] P. Karageorgis, Stability and intersection properties of solutions to the nonlinear biharmonic equation, *Nonlinearity*, 22 (2009), 1653–1661.

[24] S. Khenissy, Nonexistence and uniqueness for biharmonic problems with supercritical growth and domain geometry, *Diff. and Integr. Equations*, 24 (2011), 1093–1106.

[25] C. S. Lin, A classification of solutions of a conformally invariant fourth order equation in $\mathbb{R}^n$, *Comment. Math. Helv.*, 73 (1998), 206–231.

[26] P. J. McKenna and W. Reichel, Radial solutions of singular nonlinear biharmonic equations and applications to conformal geometry, *Electron. J. Differential Equations*, 37 (2003), 1–13.

[27] E. Mitidieri, A Rellich type identity and applications, *Comm. PDEs*, 18 (1993), 125–151.

[28] A. Moradifam, Optimal weighted Hardy-Rellich inequalities on $H^2 \cap H^1_0$, *J. Lond. Math. Soc.*, 85 (2012), 22–40.

[29] R. C. A. M. Van Der Vorst, Fourth order elliptic equations with critical growth, *C. R. Acad. Sci. Paris*, 320 (1995), 295–299.

[30] J. C. Wei and D. Ye, Liouville theorems for stable solutions of biharmonic problem, *Math. Ann.*, 356 (2013), 1599–1612.

Received April 2018; revised August 2018.

*E-mail address*: gzm@htu.cn

*E-mail address*: guanxiaohong2011@126.com

*E-mail address*: ygzhao@aliyun.com