Head-on collisions of boson stars

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We study head-on collisions of boson stars in three dimensions. We consider evolutions of two boson stars which may differ in their phase or have opposite frequencies but are otherwise identical. Our studies show that these phase differences result in different late time behavior and gravitational wave output.

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I. INTRODUCTION

Models of compact “stars” described by non-fluid sources serve both as probes of strong field gravity as well as exotic alternatives to the standard “regular” fluid stars. Among these non-standard star models, a geon—a self gravitating star consisting of electromagnetic fields—was first considered by Wheeler \cite{1}. Despite its appeal, no stable geon could be constructed and so interest diminished. However, borrowing from this idea, Kaup was able to construct the first stable configuration—which referred to as a Klein-Gordon geon—by considering a $U(1)$ classical scalar field \cite{2}. Soon after Kaup’s construction, Ruffini and Bonazzola revisited the solution, adopting a quantum point of view for the matter fields though maintaining the geometry as a classical entity \cite{3}. By employing a Hartree-Fock (multi-particle) approximation, they reobtain Kaup’s equations (and hence solutions) providing an alternative view to the resulting stars. These solutions are currently known as boson stars and describe a family of self-gravitating scalar field configurations within general relativity.

Boson stars have been studied in many different contexts (see \cite{4},\cite{5} for reviews). They have been proposed as models for dark matter (see \cite{6} for a recent review) and more recently have been used as convenient models for compact objects including black holes \cite{7}. One can consider strongly gravitating compact-star spacetimes without the worries associated with regular matter—such as shock fronts and discontinuities in the fluid variables—making boson stars very useful probes of strong-field general relativity.

Given this utility, a number of efforts have modeled the dynamical evolution of boson stars. For instance, they have been considered in one and two dimensions for studies of critical phenomena \cite{8,9}. In three dimensions, studies of the dynamics of perturbed boson stars have been presented in \cite{10,11} and preliminary collisions in \cite{12}. Works examining boson star collisions in two dimensions were carried out in \cite{8} while in three dimensions, in addition to our own, another effort is under way \cite{13}. Another work which studies the collisions of Q-balls also has relevance here \cite{14}. These Q-balls are solitonic solutions of the Klein-Gordon equations in the absence of gravity with an attractive potential.

In this paper we investigate several boson star configurations paying particular attention to head-on collisions in 3D. Our aim is to survey some of the phenomenology that can be found in these scenarios. Additionally, since one can employ this particular physical system to describe strongly gravitating compact stars, one can use them to model compact systems in general relativity in their early stages where tidal effects are not too severe. On a more speculative level, one can also examine the gravitational wave signature produced so that searches by gravitational wave detectors might place constraints on their existence.

The paper is organized as follows. In Section \textsection II we detail the formalism we use for the Einstein-Klein-Gordon system. Section \textsection III describes the numerical implementation of the governing equations. In Section \textsection IV we present our results. We conclude in Section \textsection V adding some final comments. The computation of the initial data for a single boson star is described in the Appendix \textsection A while the Appendix \textsection B is devoted to some (qualitative) physical considerations in order to explain the results of the simulations.

II. THE EINSTEIN-KLEIN-GORDON SYSTEM

The dynamics of a complex scalar field in a curved spacetime is described by the following Lagrangian density (adopting geometrical units, i.e. $G = c = 1$) \cite{15}

$$\mathcal{L} = -\frac{1}{16\pi}R + \frac{1}{2} \left[ g^{ab} \partial_a \phi \partial_b \phi + V(|\phi|^2) \right].$$

(1)

Here $R$ is the Ricci scalar, $g_{ab}$ is the spacetime metric, $\phi$ is the scalar field, $\bar{\phi}$ its complex conjugate, and $V(|\phi|^2)$ a potential depending only on $|\phi|^2$. Throughout this paper Roman letters at the beginning of the alphabet $a, b, c, \ldots$ denote spacetime indices ranging from 0 to 3, while letters near the middle $i, j, k, \ldots$ range from 1 to 3, denoting spatial indices. This Lagrangian gives rise to the equations determining the evolution of the metric (Einstein equations) and those governing the scalar field behavior (Klein-Gordon Equations).
A. Einstein Equations

The variation of the action associated with the Lagrangian (1) with respect to the metric $g_{ab}$ leads to the Einstein equations

$$R_{ab} - \frac{R}{2} g_{ab} = 8\pi T_{ab}, \quad (2)$$

where $R_{ab}$ is the Ricci tensor and $T_{ab}$ is the stress-energy tensor given by

$$T_{ab} = \frac{1}{2} \left[ \partial_a \phi \partial_b \phi + \partial_a \phi \partial_b \phi \right] - \frac{1}{2} g_{ab} \left( g^{cd} \partial_c \phi \partial_d \phi + V (|\phi|^2) \right). \quad (3)$$

The Einstein equations form a system of 10 non-linear partial differential equations for the spacetime metric components $g_{ab}$. A convenient way to express the equations can be obtained by the identity

$$R_{ab} = -\frac{1}{2} g^{cd} \left( \partial_c \partial_d g_{ab} + \partial_a \Gamma_{bc} - \Gamma_{cab} \Gamma^{c} \right) + g^{cd} g^{ef} \left( \partial_c g_{da} \partial_f g_{eb} - \Gamma^{abc} \Gamma_{def} \right). \quad (4)$$

where we have introduced the quantities

$$\Gamma_a \equiv g^{bc} \Gamma_{abc} \quad (5)$$

$$\Gamma_{abc} = \frac{1}{2} \left( \partial_b g_{ac} + \partial_c g_{ab} - \partial_a g_{bc} \right). \quad (6)$$

To this point we have considered arbitrary coordinates $x^a$ that label spacetime points. In order to define an initial value problem for the Einstein equations, a foliation defined by the hypersurfaces with $x^0 \equiv t = \text{const}$ is introduced with normalized normal vector $n_a = -\nabla_a t / ||\nabla_a t||$, where $\nabla_a$ is the covariant derivative associated to $\nabla_{ab}$. The harmonic formulation of Einstein equations exploits the fact that the coordinates $x^a$ can be chosen satisfying the harmonic $\nabla^a \nabla_c x^a = 0$ or their generalized version

$$\nabla^a \nabla_c x^a = H^a (t, x^i), \quad (7)$$

where the functions $H^a$ amount to prescribing coordinate conditions. These expressions can be combined to re-express Einstein equations (2) in their generalized harmonic form, as

$$g^{cd} \partial_c \partial_d g_{ab} + \partial_a H_b + \partial_b H_a = -16 \pi \left( T_{ab} - \frac{T}{2} g_{ab} \right)$$

$$+2 \Gamma_{cab} H^c + 2 g^{cd} g^{ef} \left[ \partial_c g_{da} \partial_f g_{eb} - \Gamma^{abc} \Gamma_{def} \right] \quad (8)$$

Notice that the partial derivatives of $H^a$ do not belong to the principal part of the system since they are prescribed spacetime functions. The principal part of (8) is a well posed system of decoupled wave equations for the 10 metric components. It is important to stress that equations (8) constitute an overdetermined system since $\Gamma_a$ can be obtained either from the choice of coordinates (7) or from the evolution of the metric itself (5). In the free-evolution approach, equation (8) is adopted to update the metric, and equation (7) is regarded as a constraint on the full system. Possible constraint violations can be measured by introducing a new four-vector (47)

$$2 Z^a \equiv -\Gamma^a - H^a (t, x^i). \quad (9)$$

Clearly, the physical solutions will be those satisfying the algebraic condition $Z_a = 0$ throughout the spacetime; we will refer to the $Z_a$ quantities as the Z-constraints from now on. Note that added flexibility, useful for numerical purposes, is attained by adding these constraints to the equations. In particular, since the addition of these Z-constraints to the evolution system can change the stability of the solutions against perturbations off the constraint surface one can exploit this fact to one’s advantage. Indeed, suitable terms can be added to the equations in order to construct an attractor for the physical solutions in certain spacetimes, in such a way that small Z-constraint violations will be damped during the evolution (20). With these damping terms, equation (8) can be written as

$$g^{cd} \partial_c \partial_d g_{ab} + \partial_a H_b + \partial_b H_a =$$

$$+2 \Gamma_{cab} H^c + 2 g^{cd} g^{ef} \left[ \partial_c g_{da} \partial_f g_{eb} - \Gamma^{abc} \Gamma_{def} \right] - 2 \sigma_0 \left[ n_a Z_h + n_b Z_a - g_{ab} n^c Z_c \right] - 16 \pi \left( T_{ab} - \frac{T}{2} g_{ab} \right)$$

where we have introduced $\sigma_0$, which is a free parameter that controls the damping of the Z-constraints (9).

B. Klein-Gordon Equations

As mentioned, the variation of the Lagrangian (11) with respect to the scalar field $\phi$ leads to the Klein-Gordon (KG) equations

$$\square \phi = \frac{dV}{d|\phi|^2} \phi, \quad (11)$$

where the box $\square = g^{ab} \nabla_a \nabla_b$ stands here for the wave operator on a curved background. For concreteness, from now on we will consider the free field case where the potential takes the form

$$V(|\phi|^2) = m^2 |\phi|^2, \quad (12)$$

with $m$ a parameter that can be identified with the bare mass of the field theory, although it has units of inverse length. For our simulations we have fixed its value to unity. The potential (12) leads to the so-called minihalon stars, because achievable stable configurations have small masses. More general terms can be included, such as the $\lambda |\phi|^4$ self-interaction term introduced in (21), leading to heavier boson stars which have masses and sizes.
more relevant to astrophysical applications when considering their compactification ratio. In the present work we restrict to the \( \lambda = 0 \) case, deferring to a future work the study of the effect of this self-interacting term.

C. Reduction of the complete system to first order

The second order Einstein-Klein-Gordon (EKG) equations (10)-(11), can be employed to obtain the future evolution of the spacetime if \( \{g_{ab}, \phi, g_{at}, \phi, t\} \) are provided at an initial hypersurface. For a numerical implementation, it is also useful to further reduce the system to fully first order to take advantage of several numerical techniques devised to exploit stability theorems for these kinds of systems [22, 23, 24, 25]. Consequently, these techniques provide a clean path to discretize the system in a robust manner [26, 27, 28]. Thus we begin by reducing these equations to first order form.

The reduction in time is achieved by introducing new independent variables related to the time derivatives of the fields. Following a similar route as in [19], we define

\[
Q_{ab} \equiv -n^c \partial_c g_{ab} , \quad \Pi \equiv -n^c \partial_c \phi .
\]

(13)

The evolution equations for \(Q_{ab}\) and \(\Pi\) are now given by the EKG equations (10)-(11), while the evolution equations for \(g_{ab}\) and \(\phi\) are simply given by their definition (13).

The reduction to first order in space is made by introducing new independent variables encoding the first space derivatives as

\[
D_{ab} \equiv \partial_t g_{ab} , \quad \Phi_i \equiv \partial_i \phi .
\]

(14)

The equations of motion for these quantities are obtained by applying the time derivative to their definition (14) and imposing that time and spatial derivatives commute. Notice that in this step, just as with the Z-constraints, one encounters an overdetermined system since spatial derivatives can be obtained either by deriving the fields or from the evolution equation of the new quantities (13). This fact can be re-interpreted as adding new constraints to the system

\[
C_{ab} = \partial_t g_{ab} - D_{ab} = 0 , \quad C_i = \partial_i \phi - \Phi_i = 0 ,
\]

(15)

which must be satisfied for a consistent solution. An analogous situation is encountered with the conditions

\[
C_{ijab} = \partial_t D_{jab} - \partial_t D_{jab} = 0 , \quad C_{ij} = \partial_t \Phi_j - \partial_j \Phi_i = 0
\]

(16)

resulting from the commutativity of the second spatial derivatives. Henceforth we will refer to the \(\{C_c, C_{ijab}, C_{ijab}\}\) quantities as the C-constraints. At this point the resulting first order system is described by the evolution equations for the array of fields \(\{g_{ab}, Q_{ab}, D_{ab}, \phi, \Pi, \Phi_i\}\) together with the Z-constraints [9] and C-constraints [15,16].

Additionally, as with the Z-constraints, the mathematical properties of the evolution system can be changed—when the constraints are not exactly satisfied—by judiciously adding the C-constraints to the right-hand-side of the equations. For instance, in [19] these constraints are incorporated in the equations in such a way that the physical solutions (i.e., those satisfying \(C_i = C_{ij} = C_{ijab} = C_{ijab} = 0\)) are an attractor in certain spacetimes. This means that small C-constraint violations will also be damped during the evolution. We follow the same approach here for the full first order reduction of the Einstein Klein-Gordon equations by writing them as:

\[
\partial_t g_{ab} = \beta^k D_{kab} - \alpha Q_{ab}
\]

(17)

\[
\partial_t Q_{ab} = \beta^k \partial_k Q_{ab} - \alpha \gamma^{ij} \partial_i D_{jab} - \sigma_1 \beta^k \partial_k g_{ab}
\]

(18)

\[
- \alpha \partial_a H_b - \alpha \partial_b H_a + 2 \alpha \Gamma_{abc} H_c
+ 2 \alpha g^{cd} (\gamma^{ij} D_{ca} D_{db} - Q_{ca} Q_{db} - g^{ij} \Gamma_{ace} \Gamma_{bdf})
- \frac{\alpha}{2} n^c n^d Q_{ca} Q_{db} - \alpha \gamma^{ij} D_{ab} Q_{je} n^c + \sigma_1 \beta^k D_{kab}
- 16 \pi \gamma (T_{ab} - \frac{g_{ab}}{2})
- 2 \alpha \sigma_0 [n_a Z_b + n_b Z_a - g_{ab} n^c Z_c]
\]

\[
\partial_t D_{ab} = \beta^k \partial_k D_{ab} - \alpha \partial_t Q_{ab} + \alpha \sigma_1 \partial_t g_{ab}
+ \frac{\alpha}{2} n^c n^d D_{ca} D_{db} + \alpha \gamma^{ij} n^c D_{ij} c D_{kab} - \alpha \sigma_1 D_{kab}
\]

\[
\partial_t \phi = \beta^k \Phi_k - \alpha \Pi
\]

\[
\partial_t \Pi = \beta^k \partial_k \Pi - \alpha \gamma^{ij} \partial_i \Phi_j - \sigma_1 \beta^k \partial_k \phi
- \frac{\alpha}{2} \Pi n^c n^d Q_{ca} - \alpha \gamma^{ij} \Phi_i n^c Q_{jc}
+ \alpha \Gamma_{ig} \Phi^g + \alpha m^2 \phi + \sigma_1 \beta^k \Phi_k
\]

\[
\partial_t \Phi_i = \beta^k \partial_k \Phi_i - \alpha \partial_i \Pi + \alpha \sigma_1 \partial_i \phi
+ \frac{\alpha}{2} \Pi n^c n^d D_{ca} + \alpha \gamma^{ij} \Phi_k n^c D_{ij} c - \alpha \sigma_1 \Phi_i
\]

where \(\sigma_1\) is another free parameter that controls the damping of the first order constraints [15,16]. Additionally, we have introduced the familiar lapse \(\alpha\) and shift \(\beta^i\) functions for convenience. These are defined through the relations \(\beta^i = \gamma^{ij} \beta_j\) and \(\gamma^{ij}\) being the spatial projection of \(g_{ab}\) satisfying \(\gamma^{ij} \gamma_{ij} = \delta_{ij}, g_{ii} = \beta_i\) and \(g^{ii} = \alpha^{-2}\). Equations (17,22) constitute the final set of equations being integrated in our implementation.

D. Analysis quantities

One way to identify the position of the boson star, which in the fully dynamical might prove difficult, is to use the energy density \(\rho = n^a n^b T_{ab}\). We have used the maximum of \(\rho\) at each temporal slice to represent the center of each star-like configuration.

Another useful quantity, which is of special interest in the case of the boson-antiboson collision, is the Noether charge of the theory. Due to the U(1) symmetry of the stress-energy tensor, there is a conserved quantity which
can be computed by
\[ N = \int J^0 \, dz^3 \quad J^0 = \frac{i}{2} \sqrt{-g} g^{0\nu} \left[ \vec{\phi} \, \partial_{\nu} \phi - \phi \partial_{\nu} \vec{\phi} \right]. \]
We employ the Noether density \( J^0 \) to give additional evidence as to the location of the boson star. As discussed in [3], this quantity can be associated with the number of bosonic particles.

The mass of the stars is computed by means of the ADM mass, which is defined as:
\[ M = \frac{1}{16\pi} \lim_{r \to \infty} \int g^{ij} \left[ \partial_j g_{ik} - \partial_k g_{ij} \right] N^k \, dS \]
where \( N^k \) stands here for the unit outward normal to the sphere.

The gravitational radiation is described asymptotically—when the correct tetrad and coordinates are adopted—by the Newman-Penrose (spin-weighted) \( \Psi_4 \) scalar. This quantity is defined as
\[ \Psi_4 = C_{abcd} \, n^a \, m^b \, n^c \, m^d, \]
with \( C_{abcd} \) being the Weyl tensor and where \( n^a \) denotes the incoming null normal to the extraction worldtube \( \Gamma \) while \( \bar{n}^a \) is the complex conjugate of the null vector tangent to \( \Gamma \) at a constant time slice.

To analyze the structure of the radiated waves it is convenient to decompose the signal in \( -2 \) spin weighted spherical harmonics as,
\[ r \, \Psi_4 = \sum_{l,m} C_{l,m} \, Y_{l,m} \]
where the factor \( r \) is included to better capture the \( 1/r \) leading order behavior of \( \Psi_4 \). For head-on collisions of non-spinning objects, one could take advantage of the natural axis of symmetry defined by the line joining the centers of the objects. In such a case, the (spin weighted) spherical harmonics \( S_{l,m} \) can be defined such that the radiation produced in the axisymmetric configuration would display essentially only \( m = 0 \) modes. However, as we are interested in more general cases also, we adopt spherical harmonics defined with respect to an axis that can be regarded as orthogonal with respect to the orbital plane, which in our present case is given by the \( z \) axis. In this case, the radiation extracted would have non-trivial components for \( m \neq 0 \).

We also focus in integral quantities that are independent of the specific basis of the spherical harmonics, such as the radiated energy. In terms of \( \Psi_4 \), we can write
\[ \frac{dE}{dt} = \frac{r_{ext}^2}{4\pi} \int_{\Omega} \left[ \int_{-\infty}^{t'} \Psi_4(t') \, dt' \right] \left[ \int_{-\infty}^{t'} \bar{\Psi}_4(t') \, dt' \right] \, d\Omega, \]
where \( \bar{\Psi}_4 \) denotes the complex conjugate of \( \Psi_4 \), and \( r_{ext} \) is the extraction radius. Using both the decomposition \[ \text{[26]} \] and the orthonormalization of the spherical harmonics, this expression can also be written as
\[ \frac{dE}{dt} = \frac{1}{4\pi} \sum_{l,m} |D_{l,m}|^2, \quad D_{l,m} = \int_{-\infty}^{t'} C_{l,m}(t') \, dt'. \]

Finally, we notice that we will assume the resulting coordinates in the neighborhood of the extraction surfaces satisfy, to a reasonable level, conditions ensuring \( \Psi_4 \) is essentially free of spurious effects (aside from those arising from the extraction at finite distances) [29, 30]. Detailed studies of the possible influences of these in more general settings will be presented elsewhere [31].

III. IMPLEMENTATION ISSUES

The first order symmetric hyperbolic system [17, 22] is implemented following techniques devised to take advantage of several useful theorems which guarantee the stable implementation of linear hyperbolic systems. In general these techniques involve: (i) discrete derivative operators satisfying summation by parts (SBP), which allow defining a semidiscrete norm and an energy estimate for the solution [23, 25], (ii) a method of lines employing a third order Runge-Kutta operator for the time integration that preserves the energy estimate in the fully discrete case [24], and (iii) maximally dissipative boundary conditions that keep this energy bounded in time [22] in presence of boundaries. The combination of techniques (i-iii) guarantee that linear problems are implemented stably and thus provides a direct route to a robust implementation of generic first order hyperbolic systems. Additionally, we introduce a Kreiss-Oliger type dissipative operator (suitably modified at/near boundary points so as not to spoil the energy estimate [24, 27]). The introduction of dissipation allows one to control the high frequency modes of the solution which are always poorly described by a finite difference approach. In what follows we provide further details of the particular application of these techniques in the present work.

A. Discrete operators and time evolution

We have adopted second order derivative operators satisfying SBP and defined a discrete energy essentially as the \( L_2 \) norm of the solution, that is
\[ E = \sum_A ||U_A|| = \sum_A \sum_{I,J,K} U_A(I, J, K)^2 \]
where \( U_A = \{ g_{ab}, Q_{ab}, D_{ab}, \phi, \Pi, \Phi \} \) is an array with all the evolved variables and \( U_A(I, J, K) \) is the value of \( U_A \) at the grid point \( (I, J, K) \). The energy estimate is obtained when the discrete operators reduce to the standard three-point centered derivative operator at interior points while the standard (first order) two-point sideways stencil at boundary points. The particular form
for the Kreiss-Oliger fourth order dissipation operator is described in [26, 27] as is the third order Runge-Kutta scheme employed to advance the solution in time.

We additionally employ the norm of the Z-constraints,

\[
\|Z\| = \sum_{a=0}^{3} \|Z_a\| \quad (30)
\]

to monitor the deviation of the numerical solution from the physical one.

### B. Characteristic structure

As mentioned, we employ maximally dissipative boundary conditions. These conditions rely on knowing the characteristic mode structure of the principal part of the system. In essence, these conditions bound the amount of “energy” the incoming modes introduce at the boundaries to remain below what the outgoing modes carry away from the computational domain. Schematically, this reduces to adopting conditions satisfying \( w^- = Lw^+ \) with \( w^-, w^+ \) denoting incoming/outgoing modes at a given boundary, and \( L \) a bounded matrix [23]. As a result, the norm of the solution remains bounded in the linear regime, which is key to ensure the well posedness of the IBVP (for details see [22, 23, 24]). In our particular case, we choose \( L = 0 \) thus setting the incoming modes to zero. The characteristic decomposition (eigenmodes \( w \) with velocity \( v \)) for the EKG system [17, 22] is given by

\[
\begin{align*}
  w_{ab}^0 &\equiv g_{ab}, & v^0 &= 0 \\
  w_{ab}^\beta &\equiv D_{ab} - N^k D_{kab} N_i, & v^\beta &= -N_k \beta^k \\
  w_{ab}^{(\pm)} &\equiv Q_{ab} - \sigma_2 g_{ab} \pm N^k D_{kab}, & v^{\pm} &= -N_k \beta^k \pm \alpha
\end{align*}
\]

with \( N^k \) the outgoing normal vector to the boundary surface and the supra-indices \( \{0, \beta, \pm\} \) denote modes propagating with the different speeds \( \{v^0, v^\beta, v^{\pm}\} \). Thus the incoming modes are given by \( w_{ab}^0 \) and \( w_{ab}^\beta \) as long as \( N_k \beta^k < 0 \). To apply the maximally dissipative conditions at the numerical level, we follow the prescription described in [23, 26] and simply enforce

\[
\begin{align*}
  \partial_t w_{ab}^0 &= 0, \\
  \partial_t w_{ab}^\beta &= 0, \text{ if } N_k \beta^k < 0. \quad (33)
\end{align*}
\]

### C. Mesh Refinement

Adaptive mesh refinement (AMR) provides increased resolution where and when needed, and therefore becomes an essential tool in resolving a wide range of dynamical scales, both spatial and temporal. As described by Berger and Oliger [32], a given numerical grid can be further refined based on dynamical measures of the numerical error by the creation of refined grids which are evolved in sync with the coarse grid. We have built our code taking advantage of the computational infrastructure called HAD (an outgrowth of the code presented in [33, 34]). HAD implements a (slight) modification of the standard Berger-Oliger strategy that guarantees preserving the stability properties of the unigrid implementation and significantly reduces spurious reflections at interface boundaries [28]. In HAD creation or destruction of finer grids is automated so that points which display an error above some threshold are “covered” by a finer grid. The error associated with each point is computed via a self-shadow hierarchy [30]. In this technique, the error is computed on any given grid by subtracting the solution on that grid with that on the next coarser level. Because these errors evolve with different resolutions, their difference represents a measure of the local truncation error without recourse to more complicated shadow schemes. In particular, at any given point, the \( L_2 \) norm of the difference between the solutions over all evolved fields is taken as the truncation error estimate. Although we have performed binary boson star runs employing AMR, in this work we take only partial advantage of the mesh refinement capabilities of HAD. One reason for this is that we want to compare significantly different scenarios and we find it useful to ensure similar discretizations are employed across the cases considered. Our procedure is to let the infrastructure define the grid structures at the initial time and keep it fixed throughout the simulations. The truncation error value employed by the self-shadow hierarchy to define the grids is chosen such that the original stars are contained within the finest grid and that this grid covers the whole region in between. Fixing the grid structure at the initial time reduces some of the overhead and memory required by dynamic regridding.

### IV. SIMULATIONS AND RESULTS

In this section we describe the simulations we have performed both to check the correctness of the implementation and investigate interesting aspects of boson star head-on collisions. We begin in [IVA] by considering a single isolated boson star configuration and verify the code is able to reproduce known features of the solution. The evolution of the star is performed employing two different coordinate systems, a static one and the harmonic one. The former coordinate system is the natural one employed in constructing a single static boson star solution, and so the evolution with it should be trivial. The latter one does not conform to the ansatz employed to obtain the solution and some non-trivial dynamics can be expected.

In Section [IVB] we consider different binary systems described by boson stars with zero initial momentum. Initial data for these systems is obtained by a simple superposition of a single boson star with another which is identical to it up to a possible phase or reflection of its natural frequency. The stars are placed sufficiently far from each other to ensure the constraints are satisfied to
a degree consistent with the value obtained in the single boson star case.

We have performed three different types of head-on collisions: collisions of equal boson stars (IV B 1), collisions of boson stars whose phase are in opposition (IV B 2) and finally collisions of boson stars with antiboson stars (IV B 3) (these differences are explained in detail below).

For all of the simulations presented here we have adopted damping parameters $\sigma_0 = \sigma_1 = 1$, a Courant factor $\Delta t / \Delta x = 0.25$ and Kreiss-Oliger dissipation parameterized by 0.03 (as described in [26, 27]). The simulations have been performed on 64 to 128 processors employing 3 refinement levels, placed as defined by the shadow hierarchy.

### A. The single boson star

As described in the introduction, a boson star is a self gravitating configuration of scalar field with Lagrangian given by $H^4$. In a particular coordinate system, one assumes a particular form for the spherically symmetric scalar field

$$\phi = \phi_0(r) e^{-i\omega t}.$$ (34)

Therefore the field values at any given point oscillate with a constant frequency $\omega$. Details of the calculation of $\phi_0(r)$ are presented in Appendix A. The geometry, on the other hand, remains static in these same coordinates. The resulting spacetime provides a reasonable model for that resulting from a compact object in equilibrium.

We have carried out simulations of single boson stars under two coordinate choices. In IV A 1 we summarize the results in the coordinate system on which the solution is static and the results in the harmonic gauge is presented in IV A 2. The simulations presented for both cases employ a grid structure consisting of three levels of refinement. The finest level covers the radius $R_{95}$ of the star (defined as region of space containing approximately 95% of the star’s mass) and the other boxes surrounding it with half the resolution each respectively. For all these runs the star is located at $x^i = 0$ ($i = 1..3$) and the computational domain is given by $x^i \in [-144, 144]$.

To obtain a measure of the convergence rate of the solution, we have performed simulations with different resolutions for the boxes. The finest grid of each resolution considered has $\Delta x \in \{0.375, 0.25, 0.156\}$ for the static coordinates and $\Delta x \in \{0.5, 0.375, 0.25\}$ for the harmonic ones. The others grids, as mentioned, have a gradual 2:1 ratio in their discretization length.

Below we detail certain convergence tests of the code about which we would like to make a general comment. Residuals of constraints are expected to approach zero with increases in resolution while field variables should approach a unique solution, and this behavior is indeed what we find. The rate of convergence can also be examined and compared to the expected approximation order of the difference equations solved, and we found the expected second order convergence in the unigrid case. However when dealing with a complicated grid hierarchy, precise measurements of this rate become somewhat problematic, especially when the hierarchies produced vary with the truncation error threshold chosen. Also, we should mention that more detailed convergence tests with this code infrastructure applied to other problems demonstrate the expected order of convergence.

#### 1. Single boson star in static coordinates

In order to maintain the coordinates such that the spacetime remains static the $H^a(t,x^i)$ functions in need to be chosen in a suitable way. Since we are considering a static situation, the condition on $H^a$ can be read directly from the equation by imposing the constraints (i.e., $Z^a = 0$) and the staticity of the spacetime (i.e., $\partial_t g_{ab} = 0$). In an explicitly static scenario, this initial value of the $H^a(t = 0, x^i)$ is preserved for all times.

We begin with an initial data describing a star where the value of the scalar field defining it at the origin is $|\phi_0(r = 0)| = 0.01$. This star, which is obtained by solving the initial data problem with a high resolution one-dimensional code as described in the Appendix A has an ADM mass of $M_{1D} = 0.361$ and radius $R_{95} = 19.6$, and lies well inside the stable branch of solutions.

The evolution of an isolated star displays the expected
behavior. In particular, the geometry is static and the behavior of the scalar field agrees with Eq. (24) as demonstrated in Fig. 1 where the central value of the real part of the scalar field versus time describes an oscillatory behavior with a frequency $\omega = 0.96 \pm 0.03$. This is in excellent agreement with the frequency obtained with the one-dimensional initial-data problem code $\omega_{1D} = 0.976$.

2. Single boson star in harmonic coordinates

In this section we revisit the initial data of the previous one, but in this case we do not adopt coordinate conditions that explicitly demonstrate the staticity of the spacetime. One goal of this exercise is to test the ability of the harmonic-coordinate condition to adapt to the physical problem under consideration. This will be important for the binary cases considered later, as that problem is certainly not static.

The implementation evolves the system for as long as the code was run without any signs of instabilities. For this particular case we have evolved the system for more than $t = 1500$. During this evolution, we have monitored the ADM mass which we extract at $r_{\text{ext}} = 100$ and obtain $M = 0.364$ which decreases by less than 3% / 1% for the coarsest/finest resolution considered ($\Delta x = 0.5 / 0.25$ for the innermost grid) by the time we stop the code. This illustrates the ability of the code to preserve this conserved quantity for long times.

We have also checked that for this particular solution, both the harmonic and the static coordinates agree at the center of symmetry. This implies that the scalar field should have the same local oscillatory behavior (54) at $r = 0$ in both coordinate systems. This is indeed the case and is illustrated in Fig. 1. In particular, the measured oscillation frequency is $\omega = 0.97 \pm 0.06$. Additionally, we have checked that the absolute value of scalar field at the center of the star varies by at most $12\% / 2\%$ for the coarsest/highest resolution considered.

Coordinate effects do arise however. As mentioned above the geometry should have some non-trivial coordinate-induced dynamics since we are not adopting a coordinate system on which the solution is explicitly static. This effect can be seen in Fig. 2 where the evolution of the maximum of $g_{xx}$ as a function of coordinate time $t$ for 3 different resolutions is displayed. As evident in the figure, there is an initial transient variation of the metric value which later approaches a constant value.

Finally, we monitor the constraints in Fig. 4. The figure shows the $L_2$ norm (30) of the physical constraints (i.e., the Z-constraints (9)) and its behavior when the resolution is improved, converging clearly to zero. The measured convergence rate is 2.9 between the low-and-medium and 2.8 between the medium-and-high resolutions.

B. Binary boson-stars head-on collisions

Initial data for the collision of two boson stars is obtained by a simple superposition of the single boson solution in a way described below. Additionally, in order to consider different configurations, we take advantage of the following observations:

(i) The solution of the initial data problem is unaffected by a phase difference in the ansatz assumed for the scalar field. Namely, $\phi = \phi_0(r)e^{i(\omega t + \theta)}$ gives rise to the same initial data for $\{g_{ab}, \partial_t g_{ab} = 0\}$.

(ii) if $\{g_{ab}, \phi_0, \partial_t g_{ab} = 0, \partial_t \phi = i\omega \phi_0\}$ provides consistent initial data, so does $\{g_{ab}, \phi_0, \partial_t g_{ab} = 0, \partial_t \phi = -i\omega \phi_0\}$. The difference is solely in a frequency-reflection of the boson star and is known as an anti-boson star.

We will exploit these observations to consider what we will refer to as a boson in phase opposition by taking $\theta = \pi$ in (i) and/or an anti-boson in (ii).

The initial data we consider is schematically represented by considering the following construction

\[
\phi = \phi^{(1)}(x - x_1, y, z) + \phi^{(2)}(x - x_2, y, z)
\]
\[
\psi = \psi^{(1)}(x - x_1, y, z) + \psi^{(2)}(x - x_2, y, z) - 1
\]
\[
\alpha = \alpha^{(1)}(x - x_1, y, z) + \alpha^{(2)}(x - x_2, y, z) - 1
\]
where $u^{(i)}$ denotes the corresponding field of the boson $i$, centered at $(x_i, 0, 0)$ and the value of $\phi^{(i)}$ will be dictated by the type of boson star considered, that is: a boson star, a boson star in phase opposition or an antiboson star. Notice that three fields (the scalar field $\phi$, the conformal factor $\psi$ and the lapse $\alpha$) are enough to set an initial data consistent with the EKG equations, as it is described in the appendix. Under this approach we are making the assumption that the boson stars are described with a single global scalar field $\phi$ instead of considering two sets of complex scalar fields to represent each star. This choice fixes in a straightforward way the interaction of the stars.

In order to consistently choose the initial position of the boson stars we have measured, at the initial time, the $L_\infty$ norm of the Hamiltonian constraint for different coordinate separations of the centers $D$. Since at sufficiently large distances the adopted data satisfies the constraints by construction, the constraint’s behavior versus $D$ provides a measure of the distance that must be adopted. As illustrated in Fig. 5, the error decreases very rapidly with separation and we have chosen the value $D = 50$ (which corresponds to $x_1 = -x_2 = 25$) that is nearing the asymptotic behavior. In addition, we will use for all the head-on collisions the same form of the scalar function $\phi_0(r)$ in order to compare easily the different cases.

In this case, two identical boson star configurations are adopted to define the initial data. Since the stars adopted have no initial velocity, their initial behavior is marked by a slow approach towards each other as they feel their gravitational attraction. As the evolution progresses however, the resulting behavior depends on the mass of the single initial boson stars. Previous studies which concentrated on a particular set of initial masses, always displayed a collision of stars giving rise to the formation of a black hole [12]. More recently, the work of [8, 38], which considered boosted stars with large kinetic energy, presented cases where the stars appear to pass through each other in a sort of “solitonic” behavior.

Here we want to study the situation in which there is a final regular object (that is, containing no singularities). To do so we begin with a broad parameter search along the masses of the boson stars searching for cases where the final compact object does not collapse to a black hole. As shown in Fig. 6 for stars with masses around $M = 0.26$ (corresponding to an amplitude of $\phi_0(r = 0) = 0.005$) the final object appears to avoid black hole formation. This avoidance is indicated by the geometric variables tending to a smooth slow oscillatory behavior for $M < 0.26$ while a marked violent behavior is displayed for larger values, together with a collapse of the lapse function.

We thus focus on the particular case where each star has masses $M = 0.26$, which is approximately 40% of
the maximum allowed mass on the stable branch for the potential \( V_0 \). The radius of the single star is around \( R_0 = 27 \), and in this case we extend the computational domain to \( x^i \in [-320, 320] \). As mentioned above, the refinement regions are rectangular boxes covering the centers of the stars and the span between them with a grid spacing at the region of the star of \( \Delta x = 0.50 \) for the lowest resolution and \( \Delta x = 0.375 \) for the highest one. We compute the values of the ADM mass and \( \Psi_4 \) at extraction surfaces located at \( r_{\text{ext}} = 140, 170 \) and \( 200 \), where the grid spacing is given by \( \Delta x = 4 \) for the lowest resolution and \( \Delta x = 3 \) for the highest one (i.e., there are 3 levels of refinement). Since the wavelength of the observed radiation is of the order \( \lambda \simeq 100 \), the traveling wave is well represented by the grid-structure at the extraction location.

In an effort to ensure fidelity to the continuum problem, we monitor constraint residuals and convergence of the metric variables for these evolutions. Notice however that, again, the grid structure for different resolutions differs and so our evolutions do not lend themselves to a traditional convergence study. However we do see the expected behavior, in that the constraint residuals decrease in regions where the resolutions differ and the fields indicate convergence to a common value. This is illustrated in Fig. 7 where the maximum value of \( g_{xx} \) versus time is displayed. As evident in the plots, the values for different resolutions agree quite well until a time of about \( t \simeq 800 \) where accumulation of errors and boundary effects become apparent.

To illustrate the dynamics displayed by the solution, we present in Fig. 8 and 9 a sequence of plots illustrating the behavior of the Noether density \( J^0 \). The extremes of this function correspond, at least initially, to the centers of the stars and as the evolution proceeds, the extremes give an indication as to the movement of the stars. Fig. 8 presents two-dimensional slices (at \( z = 0 \)) of \( J^0 \). Notice that the maximum value of \( J^0 \) is reached after the collision. This suggests that the boson stars have merged and oscillations of the merged object are clearly distinguishable. Fig. 9 shows the nature of these oscillations in more detail by presenting contour plots of \( J^0 = 2 \times 10^{-5} \) at more frequent times.

The gravitational radiation produced in the collision, as encoded in \( \Psi_4 \), is presented in Fig. 10 which illustrates the (real) coefficients of its \( l = 2 \) spin-weighted spherical harmonic modes. Due to the symmetry of this particular problem and with the spherical harmonics defined with respect to the \( z \) axis, there are the following simple
FIG. 7: Boson/Boson pair. $L_\infty(g_{xx})$ as function of time for three different resolutions and their differences. We can observe clearly that the merger occurs at approximately $t \simeq 550$. The solution is qualitatively convergent during the first part of the merger until $t \simeq 800$ when boundary effects negatively affect the convergence.

The gravitational radiation emitted during the collision reaches the first extraction surface (at $r_{ext} = 140$) at $t = 600$. It takes around $t = 200 - 300$ to reach the boundary, be reflected and pass again through the same extraction surface, as can be seen in Fig. 8. Notice that, despite having the surface extraction far from the sources, they are still not well within the “wave-zone” since $r\Psi_4$ displays still a slight dependence on $r_{ext}$. Nevertheless, the structure of the radiated wave is evident in the plot.

In Fig. 8 we have plotted the emitted energy (or power) $dE/dt$ (eqn. 28) as a function of time for different resolutions. We find that practically all energy is radiated in the $l = 2$ mode which would simplify the parameterization of the obtained waveforms for their use in data analysis searches [30].

These evolutions therefore suggest that the merger of the two boson stars produces an oscillating single boson star. However, we note that the resulting star does not correspond to the boson star with mass equal to the sum of the individual masses. One might suspect the collision dispersed energy in the form of scalar and gravitational radiation, but that does not appear to be the case. The ADM mass hardly decreases and the gravitational wave output is minimal. Instead, it would appear that the resultant object has yet to settle into a stationary star.

2. Collision of boson/boson in opposition of phase pair

The previous case considered, in its early stages, can be regarded as generic for equal mass objects in a

relations between the non-trivial coefficients $C_{2,m}$:

$$
Re\{C_{2,2}\} = Re\{C_{2,-2}\}
$$

$$
Re\{C_{2,2}\} = -\sqrt{3}/2 Re\{C_{2,0}\} .
$$

Numerically we have found that $Re\{C_{2,2}\} = Re\{C_{2,-2}\}$ and $Re\{C_{2,2}\} = -1.22 Re\{C_{2,0}\}$, which is in excellent agreement with the expected analytical relations.
FIG. 9: Boson/Boson pair. Contours corresponding to the value $J^0 = 2 \times 10^{-5}$ at the $z = 0$ plane. From top to bottom the contours are shown for times $\{0, 160, 240, 300\}$, $\{340, 380, 420, 460\}$, $\{500, 540, 580, 640\}$, $\{680, 760, 840, 900\}$ and $\{980, 1060, 1140, 1200\}$ indicated by solid, solid-with-crosses, solid-with-squares and solid-with-triangles respectively in each plot. As the stars get closer, the initially spherical contours deform until merging as illustrated by a cusp on the top-most figure. Afterwards, $J^0$ exhibits essentially quadrupolar-type oscillations.

head-on configuration. The inner-structure details only become relevant when the objects become sufficiently close to each other as indicated by the effacement theorem \[33, 40\].

However, as the stars approach each other, the particular details of the star under consideration can have strong consequences. Let us consider the more general possibility of having a relative phase difference between

the single boson stars that we will use to construct the global solution \[33\], as for instance,

$$
\phi^{(1)} = \phi_0(r) e^{-i\omega t} \tag{37}
$$

$$
\phi^{(2)} = \phi_0(r) e^{-i(\omega t + \theta)} \tag{38}
$$

with $\theta$ the relative phase. We will concentrate here on the extreme case, $\theta = \pi$ and so the stars are in phase-opposition. Recall that while Einstein equations are insensitive to this phase difference, the Klein-Gordon equations are. Notice that in our head-on configuration the surface $x = 0$ is an anti-symmetry plane for the scalar field, so it must remain zero in that plane. In the present case, the observed behavior can be regarded, to a certain extent, as being influenced by a “repulsive” interaction of two objects trapped in a gravitational potential well. This leads to the objects oscillating around the coordinate position $(x = \pm 15, y = 0, z = 0)$. Elucidating the final fate of this system will require much longer evolutions (making sure there is no causal interaction with the boundaries). Additional comments about this case in light of result obtained in the other ones will be discussed in the conclusion section.

The gravitational radiation produced in this scenario is considerably weaker than the previous case, as the involved objects never acquire significant velocities to induce a strong-time dependent quadrupole. In fact, our extracted radiation is of the same order as the spurious signal produced by the initial data. From these results

FIG. 10: Boson/Boson pair. Coefficients corresponding to the $l = 2$ modes of $r\Psi_4$ as a function of time, extracted at $r_{\text{ext}} = 200$. After some initial transient due to spurious radiation in the initial data, the signal is clearly visible corresponding to the merger of the stars and later decaying to smaller values. At late times, $t > 800$, contamination with boundary effects obscures the extracted signals.
coupled with the analysis presented in [13] we conjecture that the maximum of the radiation happens at $\theta = 0$ while the minimum corresponds to $\theta = \pi$ being the radiation a smooth function of the phase difference. However this must be substantiated by studying several representative cases.

3. **Boson against antiboson**

Another interesting case is the collision of a boson star with an otherwise identical star except with the opposite charge density. Such a star is called an antiboson star and rotates in the opposite direction as its partner in the complex plane. Recall that the initial value problem solution was degenerate upon the reflection $\omega \rightarrow -\omega$. Additionally, while this change leaves the geometry unchanged, the associated Noether change is affected by an overall sign change. In this section, we study the dynamical behavior of a binary system composed of a boson and an antiboson star. The initial conditions for such a scenario are simply obtained with the following choice,

$$
\phi^{(1)} = \phi_0(r) e^{-i\omega t},
$$

$$
\phi^{(2)} = \phi_0(r) e^{+i\omega t}.
$$

We obtain the evolution of such a system, and as with the phase-opposition case, the early behavior agrees with that of the boson-boson case. As time progresses however, notable differences arise which are illustrated in Fig. 16. The dynamical behavior when the stars are close is less strong than in the boson-boson case but more so than in the boson-phase opposition case. As a result, there is non-trivial generation of gravitational waves though their time-scale is longer than and their strength smaller than the boson-boson case. The total radiated energy however, is similar as illustrated in Fig. 15.

**V. CONCLUSIONS**

We have studied the behavior of boson stars both in isolation and head-on collisions with a three-dimensional implementation of the Einstein equations. Studies of the isolated star in Section IV A provided a non-trivial test of the implementation and showed it capable of accurately extracting delicate features of the solution such as the oscillation frequency. Additionally by considering a coordinate condition not adapted to the static spacetime under study, the solution obtained further illustrated the ability of the harmonic coordinates to deal with compact objects in a smooth manner.

We then considered head-on collisions of different configurations of boson stars. Our studies revealed the interesting and varied phenomenology interacting boson stars.
can give rise to. Additionally, these scenarios provided an excellent test of the implementation under strongly dynamical conditions. This implementation is being employed to deal with black hole spacetimes, and the results will be communicated elsewhere.

In addition to the notable differences obtained, it is interesting to pay a closer look at the position of the boson stars as a function of time. The positions are determined by the location where the energy density reaches a maximum in a given neighborhood. Fig. 18 shows the maximums of the energy density $\rho$ while Fig. 19 shows the proper distance between the origin and these maximums (i.e., that can be identified with the center of the boson stars) located on the $+x$ direction. Due to the symmetry of the problem this is enough to draw conclusion on both stars. Included in the figure is the position a star would have as dictated by a simple Newtonian behavior of two
FIG. 16: Boson/antiBoson pair. 2D $z = 0$ cuts of the Noether density $J^0$ at different times. Although the stars come closer and merge, the bosonic part is distinguishable from the antibosonic one by means of the different sign of the Noether density.

A further remark about the particular cases that we have evolved is that the difference between the coordinate and the proper distances, during most part of the evolution (the early part), is given in all cases by a slowly varying function of time, compared to the time scale of the solution.

Additionally, these plots also explain, qualitatively, why the radiated energy is stronger for the boson-boson case since it is the one where the source's quadrupole has the most dynamic behavior. It would be tempting to regard this as further evidence that in rather simple scenarios, the coordinate behavior provides a reasonable description of the objects dynamics as seen in the context of equal-mass binary black hole cases [41, 42, 43, 44].

A priori, since the initial separation of the stars is of the same order of their radii, non-Newtonian effects would be expected to become relevant early on. This is clearly visible for the boson-boson and boson-antiboson in phase opposition cases, however, the Newtonian result is a reasonably good approximation for the boson-antiboson collision. This results from the significantly different interaction between their scalar fields in these cases, while in the boson-boson case a merger takes place at relatively early times, in the boson-antiboson in phase opposition the stars approach and essentially go through each-other.

A further remark about the particular cases that we have evolved is that the difference between the coordinate and the proper distances, during most part of the evolution (the early part), is given in all cases by a slowly varying function of time, compared to the time scale of the solution.
FIG. 18: The maximum of the energy density \( \rho \) is plotted as a function of time for the three different cases.

However one must be cautious in the present case as we are considering a particular model to define the scalar fields through a single complex field.

Also, interesting differences are observed when comparing our results with those of the study of Q-balls [14]. Here, the inclusion of gravitational effects modify the dynamics strongly. Besides not needing to boost the stars for a collision to take place, the gravitational attractions of the stars in our case is evidenced by the “trapping” of the stars instead of “bouncing apart” (for the boson-boson in phase opposition) or going through each other (for the “boson-antiboson”) case as in [14].

Finally, the results presented here illustrate the rich behavior displayed by binary boson star dynamics even in the rather simple scenario of a head-on collision. In particular, the dependence on possible phase differences revealed these phases strongly influence the outcome of the problem. In appendix B we present a simple treatment that sheds light in the influence different phases or frequency-reflection in one of the stars’ description have in the dynamics of the problem. In particular, this treatment suggests that energy arguments explain why the boson-boson case gives rise to a merge while in the boson-opposition in phase boson case they seem to repel each other.

Having examined the landscape of head-on boson star collisions behavior, near future studies will concentrate on the more complex situations of orbiting cases as well as boson-star black hole binaries.

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APPENDIX A: INITIAL DATA FOR AN ISOLATED BOSON STAR

The initial data for the boson star configuration is computed first in spherical symmetry with a one-dimensional code. The resulting solution is then extended to three dimensions. The one-dimensional solution is obtained in the following way. First, we adopt a specific ansatz for the scalar field,

\[ \phi(t, r) = \phi_o(r) e^{-i\omega t}. \]  

(A1)

With this assumption, our goal is then to to find \( \phi_o(r) \), \( \omega \) and the metric coefficients such that the spacetime gen-
erated by this matter configuration is static. We begin by considering the problem in polar-areal coordinates as is done in, for instance, \textsuperscript{2, 8, 9}. The line element in these coordinates takes the form,

\[ ds^2 = -\alpha (r)^2 \, dt^2 + a (r)^2 + r^2 d\Omega^2. \quad (A2) \]

The equilibrium equations in this coordinate system are then given by,

\[ a' = \frac{a}{2} \left\{ -\frac{a^2 - 1}{r} + 4\pi r \left[ \left( \frac{\omega^2}{\alpha^2} + m^2 \right) a^2 \phi_\alpha^2 + \Phi_\alpha^2 \right] \right\}, \quad (A3) \]

\[ \alpha' = \frac{\alpha}{2} \left\{ -\frac{a^2 - 1}{r} + 4\pi r \left[ \left( \frac{\omega^2}{\alpha^2} - m^2 \right) a^2 \phi_\alpha^2 + \Phi_\alpha^2 \right] \right\}, \quad (A4) \]

\[ \phi_\alpha' = \Phi_\alpha, \quad (A5) \]

\[ \Phi_\alpha' = -\left( 1 + 2 a^2 - 4\pi r^2 m^2 \phi_\alpha \right) \frac{\Phi_\alpha}{r} - \left( \frac{\omega^2}{\alpha^2} - m^2 \right) \phi_\alpha^2, \quad (A6) \]

where a prime denotes differentiation with respect to \( r \).

In order to obtain a solution of this system adapted to the desired physical situation we provide the following boundary conditions:

\[ a (0) = 1, \quad (A7) \]

\[ \phi_\alpha (0) = \phi_c, \quad (A8) \]

\[ \Phi_\alpha (0) = 0, \quad (A9) \]

\[ \partial_r a (0) = 0, \quad (A10) \]

\[ \lim_{r \to \infty} a (r) = 1, \quad (A11) \]

\[ \lim_{r \to \infty} \phi_\alpha (r) = 0, \quad (A12) \]

\[ \lim_{r \to \infty} \alpha (r) = 1; \quad (A13) \]

which guarantee regularity at the origin and asymptotic flatness. For a given value of central value of the field \( \phi_c \) these equations and boundary conditions only admit solutions for a discrete set of values of \( \omega \). In our particular case, we are interested on the fundamental (lowest frequency) solution. The problem is solved by integrating from \( r = 0 \) outwards using a second order shooting method implemented using the numerical package LSODA\textsuperscript{12}. The equations are integrated setting \( a(r = 0) = 1 \) initially, after all the equations are solved, the lapse is rescaled in order to obtain a function which asymptotes to 1 at infinity\textsuperscript{13}. The same rescaling is then performed to the frequency \( \omega \).

Once the solution is computed in this coordinate system a change of coordinates is performed to maximal isotropic ones:

\[ ds^2 = a^2 (\tilde{r}) \, d\tilde{t}^2 + \psi^4 (\tilde{r}) \left( d\tilde{r}^2 + \tilde{r}^2 d\Omega^2 \right). \quad (A14) \]

In these coordinates the extension to three dimensions is direct since the solution is explicitly spatially conformally flat. We perform the transformation of coordinates as in \textsuperscript{8}. Since \( \alpha (\tilde{r}) \) and \( \psi (\tilde{r}) \) (and the scalar field \( \phi_i (\tilde{r}) \)) are scalar functions of the spacelike hypersurface, and the time-slicing is not changed, the extension to three-dimensions involves only writing these functions as functions of \( x, y \) and \( z \) such that \( \tilde{r}^2 = x^2 + y^2 + z^2 \). This task is performed using a 5 point Lagrange interpolation.

This way we obtain initial data for \( g_{ab} \) and \( \phi \), the rest of the fields for the three dimensional code are chosen as follows: \( Q_{ab} = 0, \) \( \Pi \) is computed from ansatz (A11) and \( D_{ab} \) and \( \Phi_i \) are calculated using constraint equations (A14). That completely defines the initial data for a boson star.

**APPENDIX B: ENERGY CONSIDERATIONS**

To gain some physical insight into these results, we consider here how the energy density behaves for the three different cases studied. To simplify the discussion, we consider the energy density in flat spacetime, i.e. we assume a Minkowski metric. Under this assumption, the energy density for a complex scalar field \( \phi \) can be written as

\[ \rho = \frac{1}{2} \left[ |\Pi|^2 + |\Phi|^2 + m^2 |\phi|^2 \right] \quad (B1) \]

where \( \Pi \) and \( \Phi \) are the derivatives of \( \phi \) defined as before in Eqs. (B3) and (B4). Now, we treat the scalar field as given by the superposition \( \phi = \phi_1 + \phi_2 \) with \( \phi_1, \phi_2 \) describing the different stars we study. The energy density associated with this superposition can be expressed as

\[ \rho = \rho_1 + \rho_2 + \Delta \quad (B2) \]

where \( \rho_i \) the energy density that corresponds to the field \( \phi_i \) in isolation and \( \Delta \) is the interaction potential which vanishes when the stars are well separated. The interaction potential can then be written as

\[ \Delta = \frac{1}{2} \left[ \Pi_1 \Pi_2 + \Pi_1 \bar{\Pi}_2 + \Phi_1 \Phi_2 + \Phi_1 \bar{\Phi}_2 \right] + \frac{m^2}{2} \left( \bar{\phi}_1 \phi_2 + \phi_1 \bar{\phi}_2 \right) \quad (B3) \]

We next assume the scalar field assumes the form

\[ \phi_1 = \phi_1^0 (t, x^i) e^{i\omega t} \quad (B4) \]

\[ \phi_2 = \phi_2^0 (t, x^i) e^{i(\omega t + \theta)} \quad \epsilon = \pm 1 \quad (B5) \]

where \( \phi_1 \) represents the complex field configuration associated with a dynamic boson star and \( \phi_2 \) represents the field configuration of the other star. Both stars are characterized by the same frequency \( \omega \) but the second star can be a regular boson (\( \epsilon = 1, \theta = 0 \)), a boson in opposition of phase (\( \epsilon = 1, \theta = \pi \)), or an antiboson (\( \epsilon = -1, \theta = 0 \)). The profile of each is described by the real field \( \phi_3^0 (t, x^i) \) which is assumed to be a slowly varying function of...
time so the most dynamical part of the time dependence is represented by the harmonic factor $e^{i\omega t}$.

Under these assumptions, a straightforward computation reveals that the effective interaction potential can be expressed as

$$\Delta = \Delta_0 \cos[(1 - \epsilon) \omega t - \theta]. \quad \text{(B6)}$$

Notice that when the two stars are centered in the same position the function $\Delta_0$ is strictly non-negative, since it is a sum of quadratic terms,

$$\Delta_0 = \left[ \Pi^2_1 \Pi^2_2 + \Pi^2_1 \Pi^2_2 + \Phi^2_1 \Phi^2_2 + m^2 \left( \phi^2_1 \phi^2_2 + \phi^2_1 \phi^2_2 \right) \right] , \quad \text{(B7)}$$

and the behavior of the interaction term for each case considered is governed by that of $\cos[(1 - \epsilon) \omega t - \theta]$. Thus, for the problems analyzed in the present work, the interaction potential for each case, is,

$$\Delta_{B-B} = \Delta_0 , \quad \text{(B8)}$$

$$\Delta_{B-\text{OPB}} = -\Delta_0 ,$$

$$\Delta_{B-\text{AB}} = \Delta_0 \cos(2\omega t) .$$

Since the contribution of $\Delta_0$ is positive for the boson-boson case, the resulting gravitational potential associated with this case would be deeper than if not present. As a result, a merge of the two stars would seem a natural consequence. On the other hand the $\Delta_0$ contribution has the opposite sign for the boson-opposition phase boson case, translating into a gravitational potential exhibiting a bump at the center with two minima around it. Such a bump would suggest it energetically preferable for the stars not to merge. Finally, the boson-antiboson interaction potential is governed by a varying function of time. If the interaction time-scale is much longer than $\approx 1/\omega$, the interaction term essentially integrates away to 0. Whether this is the case depends on the collision under study and, in particular, the stars’ momentum as they come close to each other.

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[47] Notice the factor of 2 difference between the formulation presented here and [19]. This is for convenience as unnecessary factors of 2 are hence removed

[48] this can be done because the linearity of the slicing equation