The Eigenvalue Problem for Linear and Affine Iterated Function Systems

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Abstract
The eigenvalue problem for a linear function \( L \) centers on solving the eigen-equation \( Lx = \lambda x \). This paper generalizes the eigenvalue problem from a single linear function to an iterated function system \( F \) consisting of possibly an infinite number of linear or affine functions. The eigen-equation becomes \( F(X) = \lambda X \), where \( \lambda > 0 \) is real, \( X \) is a compact set, and \( F(X) = \bigcup_{f \in F} f(X) \). The main result is that an irreducible, linear iterated function system \( F \) has a unique eigenvalue \( \lambda \) equal to the joint spectral radius of the functions in \( F \) and a corresponding eigenset \( S \) that is centrally symmetric, star-shaped, and full dimensional. Results of Barabanov and of Dranishnikov-Konyagin-Protasov on the joint spectral radius follow as corollaries.

Keywords: eigenvalue problem, iterated function system, joint spectral radius
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1 Introduction
Let \( L : \mathbb{R}^2 \to \mathbb{R}^2 \) be a linear map with no nontrivial invariant subspace, equivalently no real eigenvalue. We use the notation \( L(X) := \{Lx : x \in X\} \). Although \( L \) has no real eigenvalue, \( L \) does have an eigen-ellipse. By eigen-ellipse we mean an ellipse \( E \), centered at the origin, such that \( L(E) = \lambda E \), for some real \( \lambda > 0 \). An example of an eigen-ellipse appears in Example 1 of Section 2 and in Figure 1. Although easy to prove, the existence of an eigen-ellipse appears not to be well known.
Theorem 1 If \( L : \mathbb{R}^2 \to \mathbb{R}^2 \) is a linear map with no real eigenvalue, then there is an ellipse \( E \) and a \( \lambda > 0 \) such that \( L(E) = \lambda E \).

Proof: Using the real Jordan canonical form for \( L \), there exists an invertible \( 2 \times 2 \) matrix \( S \) such that

\[
M := S^{-1}LS = \lambda \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},
\]

for some angle \( \theta \) and \( \lambda > 0 \). If \( D \) is the unit disk centered at the origin and if \( E = S(D) \), then

\[
L(E) = SMS^{-1}(E) = SM(D) = \lambda S(D) = \lambda E.
\]

The intent of this paper is to investigate the existence of eigenvalues and corresponding eigensets in a more general setting.

Definition 1 (iterated function system) Let \( \mathbb{X} \) be a complete metric space. If \( f_i : \mathbb{X} \to \mathbb{X}, i \in I \), are continuous mappings, then \( F = (\mathbb{X}; f_i, i \in I) \) is called an \textbf{iterated function system} (IFS). The set \( I \) is the index set. Call IFS \( F \) \textbf{linear} if \( \mathbb{X} = \mathbb{R}^n \) and each \( f \in F \) is a linear map and \textbf{affine} if \( \mathbb{X} = \mathbb{R}^n \) and each \( f \in F \) is an affine map.

In the literature the index set \( I \) is usually finite. This is because, in constructing deterministic fractals, it is not practical to use an infinite set of functions. We will, however, allow an infinite set of functions in order to obtain certain results on the joint spectral radius. In the case of an infinite linear IFS \( F \) we will always assume that the set of functions in \( F \) is compact. For linear maps, this just means, regarding each linear map as an \( n \times n \) matrix, that the set \( F \) of linear maps is a compact subset of \( \mathbb{R}^{n \times n} \).

Let \( \mathbb{H} = \mathbb{H}(\mathbb{X}) \) denote the collection of all nonempty compact subsets of \( \mathbb{X} \), and, by slightly abusing the notation, let \( F : \mathbb{H}(\mathbb{X}) \to \mathbb{H}(\mathbb{X}) \) also denote the function defined by

\[
F(B) = \bigcup_{f \in F} f(B).
\]

Note that, if \( B \) is compact and \( F \) is compact, then \( F(B) \) is also compact. Let \( F^k \) denote \( F \) iterated \( k \) times with \( F^0(B) = B \) for all \( B \). Our intention is to investigate solutions to the eigen-equation

\[
F(X) = \lambda X, \tag{1}
\]

where \( \lambda \in \mathbb{R}, \lambda > 0 \), and \( X \neq \{0\} \) is a compact set in Euclidean space.

Definition 2 (eigenvalue-eigenset) The value \( \lambda \) in Equation (1) above will be called an \textbf{eigenvalue} of \( F \), and \( X \) a corresponding \textbf{eigenset}. 

\[2\]
When $F$ consists of a single linear map on $\mathbb{R}^2$, the eigen-ellipse is an example of an eigenset. Section 2 contains other examples of eigenvalues and eigensets of linear IFSs. Section 3 contains background results on the joint spectral radius of a set of linear maps and on contractive IFSs. Both of these topics are germane to the investigation of the IFS eigenvalue problem. Section 4 contains the main result on the eigenvalue problem for a linear IFS.

**Theorem 2** A compact, irreducible, linear IFS $F$ has exactly one eigenvalue which is equal to the joint spectral radius $\rho(F)$ of $F$. There is a corresponding eigenset that is centrally symmetric, star-shaped, and full dimensional.

If $F = \{\mathbb{R}^n; f_i, i \in I\}$ is an IFS, let $F_\lambda := \frac{1}{\lambda} F = \{\mathbb{R}^n; \frac{1}{\lambda} f_i, i \in I\}$. Another way to view the above theorem is to consider the family $\{F_\lambda : \lambda > 0\}$ of IFSs. If $\lambda > \rho(F)$, then the attractor of $F_\lambda$ (defined formally in the next section) is the trivial set $\{0\}$. If $\lambda < \rho(F)$, then $F_\lambda$ has no attractor. So $\lambda = \rho(F)$ can be considered as a “phase transition”, at which point a somewhat surprising phenomenon occurs - the emergence of the centrally symmetric, star-shaped eigenset.

Theorems of Dranisnikov-Konyagin-Protasov and of Barabanov follow as corollaries of Theorem 2. These results are discussed in Section 5.

No such transition phenomenon occurs in the case of an affine, but not linear, IFS. A result for the affine case is the following, whose proof appears in Section 6.

**Theorem 3** For a compact, irreducible, affine, but not linear, IFS $F$, a real number $\lambda > 0$ is an eigenvalue if $\lambda > \rho(F)$ and is not an eigenvalue if $\lambda < \rho(F)$. There are examples where $\rho(F)$ is an eigenvalue and examples where it is not.

The transition phenomenon resurfaces in the context of projective IFSs, which will be the subject of a subsequent paper.

2 Examples

**Example 1** Figure 1 shows the eigen-ellipse for the the IFS $F = (\mathbb{R}^2; L)$, where

$$L = \begin{pmatrix} 65.264 & -86.116 \\ 156.98 & 62.224 \end{pmatrix}.$$  

The eigenvalue is approximately 97.23.

**Example 2** Figure 2 shows an eigenset for the IFS $F = (\mathbb{R}^2; L_1, L_2)$, where

$$L_1 = \begin{pmatrix} 10 & 10 \\ 8 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 8 & 0 \\ 10 & 10 \end{pmatrix}.$$
Figure 1: The eigen-ellipse for Example 1

Figure 2: The eigenset of Example 2
The eigenvalue is approximately 14.9. The picture on the right is the image of the picture of the left after applying both transformations, then shrinking the result about its center by a factor 14.9. The green and brown colors help to show how the image is acted on by the two transformations. The dots are an artifact of rounding errors, and serve to emphasize that the pictures are approximate.

Example 3 Figure 3 shows the eigenset for the the IFS $F = (\mathbb{R}^2; L_1, L_2)$, where

$$L_1 = \begin{pmatrix} 0.02 & 0 \\ 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0.0594 & -1.98 \\ 0.495 & 0.01547 \end{pmatrix}.$$  

The eigenvalue is approximately 1.

3 Background

This section concerns the following three basic notions: (1) the joint spectral radius of an IFS, (2) contractive properties of an IFS, and (3) the attractor of an IFS. Theorems 4 and 5 provides the relationship between these three notions for a linear and an affine IFS, respectively.

3.1 Norms and Metrics

Any vector norm $\| \cdot \|$ on $\mathbb{R}^n$ induces a matrix norm on the space of linear maps taking $\mathbb{R}^n$ to $\mathbb{R}^n$:

$$\|L\| = \max \left\{ \frac{\|Lx\|}{\|x\|} : x \in \mathbb{R}^n \right\}.$$  

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Since it is usually clear from the context, we use the same notation for the vector norm as for the matrix norm. This induced norm is sub-multiplicative, i.e., $\|L \circ L'\| \leq \|L\| \cdot \|L'\|$ for any linear maps $L, L'$.

Two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent if there are positive constants $a, b$ such that $a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$ for all $x \in \mathbb{R}^n$. Two metrics $d_1(\cdot, \cdot)$ and $d_2(\cdot, \cdot)$ are equivalent if there exist positive constants $a, b$ such that $a d_1(x, y) \leq d_2(x, y) \leq b d_1(x, y)$ for all $x, y \in \mathbb{R}^n$. It is well known that any two norms on $\mathbb{R}^n$ are equivalent [1]. This implies that any two $n \times n$ matrix norms are equivalent. Any norm $\| \cdot \|$ on $\mathbb{R}^n$ induces a metric $d(\cdot, \cdot) = \|x - y\|$. Therefore any two metrics induced from two norms are equivalent.

A set $B \subset \mathbb{R}^n$ is called centrally symmetric if $-x \in B$ whenever $x \in B$. A convex body in $\mathbb{R}^n$ is a convex set with nonempty interior. If $C$ is a centrally symmetric convex body, define the Minkowski functional with respect to $C$ by $\|x\|_C = \inf \{ \mu \geq 0 : x \in \mu C \}$.

The following result is well known.

**Lemma 1** The Minkowski functional is a norm on $\mathbb{R}^n$. Conversely, any norm $\| \cdot \|$ on $\mathbb{R}^n$ is the Minkowski functional with respect to the closed unit ball $\{x : \|x\| \leq 1\}$.

Given a metric $d(\cdot, \cdot)$, there is a corresponding metric $d_H$, called the Hausdorff metric, on the collection $\mathcal{H}(\mathbb{R}^n)$ of all non-empty compact subsets of $\mathbb{R}^n$: $d_H(B, C) = \max \left\{ \sup_{b \in B} \inf_{c \in C} d(b, c), \sup_{c \in C} \inf_{b \in B} d(b, c) \right\}$.

### 3.2 Joint Spectral Radius

The joint spectral radius of a set $\mathbb{L} = \{L_i, i \in I\}$ of linear maps was introduced by Rota and Strang [2] and the generalized spectral radius by Daubechies and Lagarias [3]. Berger and Wang [4] proved that the two concepts coincide for bounded sets of linear maps. The concept has received much attention in the recent research literature; see for example the bibliographies of [4] and [6]. What follows is the definition of the joint spectral radius of $\mathbb{L}$. Let $\Omega_k$ be the set of all words $i_1 i_2 \cdots i_k$, of length $k$, where $i_j \in I$, $1 \leq j \leq k$. For $\sigma = i_1 i_2 \cdots i_k \in \Omega_k$, define $L_\sigma := L_{i_1} \circ L_{i_2} \circ \cdots \circ L_{i_k}$.

A set of linear maps is bounded if there is an upper bound on their norms. Note that if $\mathbb{L}$ is compact, then $\mathbb{L}$ is bounded. For a linear map $L$, let $\rho(L)$ denote the ordinary spectral radius, i.e., the maximum of the moduli of the eigenvalues of $L$.

**Definition 3** For any set $\mathbb{L}$ of linear maps and any sub-multiplicative norm, the joint spectral radius of $\mathbb{L}$ is $\hat{\rho} = \hat{\rho}(\mathbb{L}) := \limsup_{k \to \infty} \hat{\rho}_k^{1/k}$ where $\hat{\rho}_k := \sup_{\sigma \in \Omega_k} \|L_\sigma\|$.
The **generalized spectral radius** of $\mathbb{L}$ is

$$\rho = \rho(\mathbb{L}) := \lim_{k \to \infty} \sup \rho_k^{1/k} \quad \text{where} \quad \rho_k := \sup_{\sigma \in \Omega_k} \rho(L_\sigma).$$

The following are well known properties of the joint and generalized spectral radius \[6\].

1. The joint spectral radius is independent of the particular sub-multiplicative norm.

2. For an IFS consisting of a single linear map $L$, the generalized spectral radius is the ordinary spectral radius of $L$.

3. For any real $\alpha > 0$ we have $\rho(\alpha \mathbb{L}) = \alpha \rho(\mathbb{L})$ and $\hat{\rho}(\alpha \mathbb{L}) = \alpha \hat{\rho}(\mathbb{L})$.

4. For all $k \geq 1$ we have

$$\rho_k^{1/k} \leq \rho \leq \hat{\rho} \leq \hat{\rho}_k^{1/k},$$

independent of the norm used to define $\hat{\rho}$.

5. If $\mathbb{L}$ is bounded, then the joint and generalized spectral radius are equal.

From here on we always assume that $\mathbb{L}$ is bounded. So, in view of property 5, we denote by $\rho(\mathbb{L})$ the common value of the joint and generalized spectral radius.

If $F$ is an affine IFS, then each $f \in F$ is of the form $f(x) = Lx + a$, where $L$ is the linear part and $a$ is the translational part. Let $\mathbb{L}_F$ denote the set of linear parts of $F$.

**Definition 4** The **joint spectral radius of an affine IFS $F$** is the joint spectral radius of the set $\mathbb{L}_F$ of linear parts of $F$ and is denoted $\rho(F)$.

**Definition 5** A set $\{L_i, i \in I\}$ of linear maps is called **reducible** if these linear maps have a common nontrivial invariant subspace. The set is **irreducible** if it is not reducible. An IFS is **reducible** (irreducible) if the set of linear parts is reducible (irreducible).

As shown in \[6\], a set of linear maps is reducible if and only if there exists an invertible matrix $T$ such that each $L_i$ can be put simultaneously in a block upper-triangular form:

$$T^{-1}L_iT = \begin{pmatrix} A_i & * \\ 0 & B_i \end{pmatrix},$$

with $A_i$ and $B_i$ square, and $*$ is any matrix with suitable dimensions. The joint spectral radius $\rho(F)$ is equal to $\max(\rho(\{A_i\}), \rho(\{B_i\}))$. 

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3.3 A Contractive IFS

Definition 6 (contractive IFS) A function \( f : X \to X \) is a contraction with respect to a metric \( d \) if there is a \( s, 0 \leq s < 1 \), such that \( d(f(x), f(y)) \leq sd(x, y) \) for all \( x, y \in \mathbb{R}^n \). An IFS \( F = (X; f_i, i \in I) \) is said to be contractive if there is a metric \( d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty) \), equivalent to the standard metric on \( \mathbb{R}^n \), such that each \( f \in F \) is a contraction with respect to \( d \).

Definition 7 (attractor) A nonempty compact set \( A \subset \mathbb{R}^n \) is said to be an attractor of the affine IFS \( F \) if

1. \( F(A) = A \) and
2. \( \lim_{k \to \infty} F^k(B) = A \), for all compact sets \( B \subset \mathbb{R}^n \), where the limit is with respect to the Hausdorff metric.

Basic to the IFS concept is the relationship between the existence of an attractor and the contractive properties of the functions of the IFS. The following result makes this relationship explicit in the case of a linear IFS. A proof of this result for an affine, but finite, IFS appears in [8]. For completeness we provide the proof for the infinite linear case. The notation \( \text{int}(X) \) will be used to denote the interior of a subset \( X \) of \( \mathbb{R}^n \). The notation \( \text{conv}(X) \) is used for the convex hull of the set \( X \). In \( \mathbb{R}^n \) the Minkowski sum and scalar product are defined by \( Y + Z = \{y + z : y \in Y, z \in Z\} \) and \( \alpha Y = \{\alpha y : y \in Y\} \), respectively.

Theorem 4 For a compact, linear IFS \( F = (\mathbb{R}^n; L_i, i \in I) \) the following statements are equivalent.

1. [contractive] There exists a norm \( \| \cdot \| \) on \( \mathbb{R}^n \) and an \( 0 \leq s < 1 \) such that \( \|Lx\| \leq s \|x\| \) for all \( L \in F \) and all \( x \in \mathbb{R}^n \).
2. [F-contraction] The map \( F : \mathbb{H}(\mathbb{R}^n) \to \mathbb{H}(\mathbb{R}^n) \) defined by \( F(B) = \bigcup_{L \in F} L(B) \) is a contraction with respect to a Hausdorff metric.
3. [topological contraction] There is a compact, centrally symmetric, convex body \( C \) such that \( F(C) \subset \text{int}(C) \).
4. [attractor] The origin is the unique attractor of \( F \).
5. [JSR] \( \rho(F) < 1 \).

Proof: (attractor \( \Rightarrow \) topological contraction) Let \( A \) be the attractor of \( F \). Let \( A_\rho = \{x \in \mathbb{R}^m : d_\mathcal{H}(\{x\}, A) \leq \rho \} \) denote the dilation of \( A \) by radius \( \rho > 0 \). By the definition of the attractor, \( \lim_{k \to \infty} d_\mathcal{H}(F^k(A_\rho), A) = 0 \), so there is an integer \( m \) so that \( d_\mathcal{H}(F^m(A_1), A) < 1 \). Thus,

\[
F^m(A_1) \subset \text{int}(A_1).
\]
If $C_1 := \text{conv}(A_1 - A_1)$, then it is straightforward to check that $C_1$ is a centrally symmetric convex body and that $F^m(C_1) \subset \text{int}(C_1)$, which implies
\[
\text{conv} F^m(C_1) \subset \text{int}(C_1).
\]
Consider the Minkowski sum
\[
C := \sum_{k=0}^{m-1} \text{conv} F^k(C_1).
\]
For any $L \in F$
\[
L(C) = \sum_{k=0}^{m-1} L \left( \text{conv} F^k(C_1) \right) = \sum_{k=0}^{m-1} \text{conv} \left( L \left( F^k(C_1) \right) \right)
\leq \sum_{k=0}^{m-1} \text{conv} F^{k+1}(C_1) = \text{conv} F^m(C_1) + \sum_{k=1}^{m-1} \text{conv} F^k(C_1)
\leq \text{int}(C_1) + \sum_{k=1}^{m-1} \text{conv} F^k(C_1)
= \text{int}(C).
\]
The last equality follows from the fact that if $K$ and $K'$ are convex bodies in $\mathbb{R}^n$, then $\text{int}(K) + K' = \text{int}(K + K')$.

(topological contraction $\Rightarrow$ contractive) Let $C$ be a centrally symmetric, convex body such that $F(C) \subset \text{int}(C)$. Let $\| \cdot \|_C$ be the Minkowski functional with respect to $C$ and $d_C$ the metric corresponding to the norm $\| \cdot \|_C$. Let $L \in F$. Since $C$ is compact, the containment $L(C) \subset \text{int}(C)$ implies that there is an $s \in [0,1)$ such that $\| Lx \|_C \leq s \| x \|_C$ for all $x \in \mathbb{R}^n$. Therefore $d_C(L(x), L(y)) = \| L(x) - L(y) \|_C = \| L(x-y) \|_C \leq s \| x - y \|_C = s d_C(x,y)$, and so $d_C$ is a metric for which each function in the IFS is a contraction. Since any convex body contains a ball of radius $r$ and is contained in a ball of radius $R$ for some $r, R > 0$, the metric $d_C$ is equivalent to the standard metric.

(contractive $\Rightarrow$ F-contraction) In the case of an IFS $F = (\mathbb{R}^n; f_i, i \in I)$, where $I$ is finite (and the $f_i$ are assumed only to be continuous), this is a basic result whose proof can be found is most texts on fractal geometry, for example [7]. Since $F$ is assumed contractive,
\[
\sup \left\{ \frac{d(f_i(x), f_i(y))}{d(x,y)} : x \neq y \right\} = s_i < 1,
\]
for each $i \in I$. The only sticking point in extending the proof for the finite IFS case to the infinite IFS case is to show that $\sup \{ s_i : i \in I \} < 1$. But if there is a sequence $\{ s_k \}$ such that $\lim_{k \to \infty} s_k = 1$, then, by the compactness of $F$, the limit $f := \lim_{k \to \infty} f_k \in F$. Moreover,
\[
\frac{d(f(x), f(y))}{d(x,y)} = \lim_{k \to \infty} \frac{d(f_k(x), f_k(y))}{d(x,y)} = \lim_{k \to \infty} s_k = 1,
\]
contradicting the assumption that each function in $F$ is a contraction.

(F-contraction $\Rightarrow$ attractor) The existence of a unique attractor follows directly from the Banach contraction mapping theorem. When $F$ is linear, uniqueness immediately implies that the attractor is $\{0\}$.

(contractive $\iff$ JSR) First assume that $F$ is contractive. Hence there is an $0 \leq s < 1$ such that $\|Lx\| \leq s \|x\|$ for all $x \in \mathbb{R}^n$ and all $L \in F$. By property (4) of the joint spectral radius

$$\rho(F) \leq \hat{\rho}_1 = \sup_{L \in F} \frac{\|Lx\|}{\|x\|} \leq s < 1.$$ 

The last inequality is a consequence of the compactness of $F$, the argument identical to the one used above in showing that (contractive $\Rightarrow$ attractor).

Conversely, assuming

$$\limsup_{k \to \infty} \hat{\rho}_k^{1/k} = \rho(F) < 1,$$

we will show that $F$ has attractor $A = \{0\}$. The inequality above implies that there is an $s$ such that $\rho_k^{1/k} \leq s < 1$ for all but finitely many $k$. In other words

$$\sup_{\sigma \in \Omega_k} \|L_{\sigma}\| = \hat{\rho}_k \leq s^k$$

for all but finitely many $k$. For $k$ sufficiently large, this in turn implies, for any $x \in \mathbb{R}^n$ and any $\sigma \in \Omega_k$, that $\|L_{\sigma}x\| \leq s^k \|x\|$. Therefore, for any compact set $B \subset \mathbb{R}^n$, with respect to the Hausdorff metric, $\lim_{k \to \infty} F^k(B) = \{0\}$. So $\{0\}$ is the attractor of $F$. □

**Corollary 1** If a compact, linear IFS $F$ is contractive and $F(A) = A$ for $A$ compact, then $A = \{0\}$.

**Proof:** According to Theorem 4 the IFS has the $F$-contractive property. According to the Banach fixed point theorem, $F$ has a unique invariant set, i.e., a unique compact $A$ such that $F(A) = A$. Since $F$ is linear, clearly $F(\{0\}) = \{0\}$. □

The following theorem is an extension of Theorem 4 to the case of an affine IFS. The proof of the equivalence of the first three statements, for a finite affine IFS, appears in [8]. The modifications in the proof (of the equivalence of the first three statements) needed to go from the finite to the compact case is omitted since it is exactly as in the proof of Theorem 4. The proof of the equivalence of statement (4) is given below. Note that this last equivalence implies that, if a linear IFS $F$ has an attractor and $F'$ is obtained from $F$ by adding any translational component to each function in $F$, then $F'$ also has an attractor.
Theorem 5 If \( F = (\mathbb{R}^n; f_i, i \in I) \) is a compact, affine IFS, then the following statements are equivalent.

1. [contractive] The IFS \( F \) is contractive on \( \mathbb{R}^n \).

2. [topological contraction] There exists a compact set \( C \) such that \( F(C) \subset \text{int}(C) \).

3. [attractor] \( F \) has a unique attractor, the basin of attraction being \( \mathbb{R}^n \).

4. [JSR] \( \rho(F) < 1 \).

Proof: As explained above, we prove only the equivalence of statement (4) to the other statements. Assuming \( \rho(F) < 1 \) we will show that \( F \) is contractive. Let \( F' \) be the linear IFS obtained from \( F \) by removing the translational component from each function in \( F \). By Theorem 4, the IFS \( F' \) is contractive. Hence there is a norm \( \| \cdot \| \) with respect to which each \( L \in F' \) is a contraction. Define a metric by \( d(x, y) = \|x - y\| \) for all \( x, y \in \mathbb{R}^n \). For any \( f(x) = Lx + a \in F \) we have \( d(f(x), f(y)) = \|f(x) - f(y)\| = \|(Lx + a) - (Ly + a)\| = \|L(x - y)\| \). Therefore each function \( f \in F \) is a contraction with respect to metric \( d \).

Conversely, assume that the affine IFS \( F \) is contractive. With linear IFS \( F' \) as defined above, it is shown in [8, Theorem 6.7] that there is a norm with respect to which each \( L \in F' \) is a contraction. It follows from Theorem 4 that \( \rho(F) < 1 \).

4 The Eigenvalue Problem for a Linear IFS

Just as for eigenvectors of a single linear map, an eigenset of an IFS is defined only up to scalar multiple, i.e., if \( X \) is an eigenset, then so is \( \alpha X \) for any \( \alpha > 0 \). Moreover, if \( X \) and \( X' \) are eigensets corresponding to the same eigenvalue, then \( X \cup X' \) is also a corresponding eigenset. For an eigenvalue of a linear IFS, call a corresponding eigenset \( X \) decomposable if \( X = X_1 \cup X_2 \), where \( X_1 \neq X \) and \( X_2 \neq X \) are also corresponding eigensets. Call eigenset \( X \) indecomposable if \( S \) is not decomposable.

Example. It is possible for a linear IFS to have infinitely many indecomposable eigensets corresponding to the same eigenvalue. Consider \( F = \{\mathbb{R}^2; L_1, L_2\} \) where

\[
L_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}.
\]

Let

\[ S(r_1, r_2) = \{ (\pm r_1, \pm r_2/2^k), (\pm r_1, \mp r_2/2^k), (\pm r_2/2^k, \pm r_1), (\pm r_2/2^k, \mp r_1) : k \geq 0 \} \]

It is easily verified that, for any \( r_1 \geq r_2 > 0 \), the set \( S(r_1, r_2) \) is an eigenset corresponding to eigenvalue 1. In addition, the unit square with vertices \((1, 1), (1, -1), (-1, 1), (-1, -1)\) is also an eigenset corresponding to eigenvalue 1.

The proof of the following lemma is straightforward. A set \( B \subset \mathbb{R}^n \) is called star shaped if \( \lambda x \in B \) for all for all \( x \in B \) and all \( 0 \leq \lambda \leq 1 \).
Lemma 2  1. If \( \{A_k\} \) is a sequence of centrally symmetric, convex, compact sets and \( A \) is a compact set such that \( \lim_{k \to \infty} A_k = A \), then \( A \) is also centrally symmetric and convex.

2. If \( F \) is a compact, linear IFS, \( B \) a centrally symmetric, convex, compact set and \( A = \lim_{k \to \infty} F^k(B) \), then \( A \) is a centrally symmetric, star-shaped, compact set.

Lemma 3 If \( F \) is a compact, irreducible, linear IFS with \( \rho(F) = 1 \), then there exists a compact, centrally symmetric, convex body \( A \) such that \( F(A) \subseteq A \).

Proof: Since, for each \( k \geq 2 \), we have \( \rho((1 - \frac{1}{k})F) = 1 - \frac{1}{k} < 1 \), Theorem \( \square \) implies that there is a compact, centrally symmetric, convex body \( A_k \) such that \( (1 - \frac{1}{k}) F(A_k) \subseteq \text{int}(A_k) \).

Since \( F \) is linear and the above inclusion is satisfied for \( A_k \), it is also satisfied for \( \alpha A_k \) for any \( \alpha > 0 \). So, without loss of generality, it can be assumed that \( \max\{|x| : x \in A_k\} = 1 \) for all \( k \geq 2 \). Since the sequence of sets \( \{A_k\} \) is bounded in \( \mathbb{H}(\mathbb{R}^n) \), this sequence has an accumulation point, a compact set \( A \). Therefore, there is a subsequence \( \{A_{k_i}\} \) such that \( \lim_{i \to \infty} A_{k_i} = A \) with respect to the Hausdorff metric. Since

\[
(1 - \frac{1}{k_i}) F(A_{k_i}) \subseteq \text{int}(A_{k_i}),
\]

it is the case that \( (1 - \frac{1}{k_i}) f(A_{k_i}) \subseteq \text{int}(A_{k_i}) \) for all \( f \in F \). From this it is straightforward to show that \( f(A) \subseteq A \) for all \( f \in F \) and hence that \( F(A) \subseteq A \). Moreover, by Lemma \( \square \) since the \( A_{k_i} \) are centrally symmetric and convex, so is \( A \). Notice also that \( A \) is a convex body, i.e., has nonempty interior; otherwise \( A \) spans a subspace \( E \subset \mathbb{R}^n \) with \( \dim E < n \) and \( F(A) \subseteq A \) implies \( F(E) \subseteq E \), contradicting that \( F \) is irreducible.

The affine span \( \text{aff}(B) \) of a set \( B \) is the smallest affine subspace of \( \mathbb{R}^n \) containing \( B \). Call a set \( B \subset \mathbb{R}^n \) full dimensional if \( \dim(\text{aff}(B)) = n \). Given an affine IFS \( F = (\mathbb{R}^n; f_i, i \in I) \) let

\[
F\lambda = \left\{ \mathbb{R}^n; \frac{1}{\lambda} f_i, i \in I \right\}.
\]

Lemma 4 If an irreducible, affine IFS \( F \) has an eigenset \( X \), then \( X \) must be full dimensional.

Proof: Suppose that \( F(X) = \lambda X \), i.e. \( F_\lambda(X) = X \). For \( x \in X \), let \( g \) be a translation by \( -x \). For the IFS \( F \), let \( F_g = \{\mathbb{R}^n; g f g^{-1}, f \in F\} \). If \( Y = g(X) \), then \( 0 \in Y \) and \( F_g(Y) = Y \). In particular, \( Y \) is full dimensional if and only if \( X \) is full dimensional, and the affine span of \( Y \) equals the ordinary (linear) span \( E = \text{span}(Y) \) of \( Y \). Moreover, the
linear parts of the affine maps in $F_g$ are just scalar multiples of the linear parts of the affine maps in $F$. Therefore $F_g$ is irreducible if and only if $F$ is irreducible.

Let $f(x) = Lx + a$ be an arbitrary affine map in $F_g$. From $F_g(Y) \subset Y \subset E$ it follows that $L(Y) + a = f(Y) \subset E$. Since $0 \in Y$, also $a = L(0) + a = f(0) \in Y \subset E$. Therefore $L(Y) \subset -a + E = E$. Since $E = \text{span}(Y)$, also $L(E) \subset E$. Because this is so for all $f \in F_g$, the subspace $E$ is invariant under all linear parts of maps in $F_g$. Because $F_g$ is irreducible, $\dim(E) = n$. Therefore $Y$, and hence $X$, must be full dimensional. 

\textbf{Lemma 5} If $F = \{\mathbb{R}^n; L_i, i \in I\}$ is a bounded linear IFS, then there is an \( \alpha > 0 \) such that \( \alpha F = \{\mathbb{R}^n; \alpha L_i, i \in I\} \) is contractive.

\textit{Proof:} By the boundedness of $F$ there is an $R$ such that, for any $L \in F$, \( \frac{\|Lx\|}{\|x\|} \leq \|L\| \leq R \) for all $x \in \mathbb{R}^n$. Therefore, if $D_r$ denotes a disk of radius $r$ centered at the origin, then $F(D_1) \subseteq D_R$. Hence \( \frac{1}{2R} F(D_1) \subset \text{int}(D_1) \). By Theorem \[4\] the IFS \( \frac{1}{2R} F \) is contractive. \( \square \)

\textbf{Proof of Theorem 2} Given $F = (\mathbb{R}^n; L_i, i \in I)$, consider the family $\{F_\lambda\}$ of IFS’s for $\lambda > 0$. Recall that $F_\lambda = \{\mathbb{R}^n; \frac{1}{\lambda} f_i, i \in I\}$.

It is first proved that $F$ has no eigenvalue $\lambda > \rho(F)$. By way of contradiction assume that $\lambda > \rho(F)$, which implies that $\rho(F_\lambda) < 1$. According to Theorem \[4\] the IFS $F_\lambda$ is contractive. By Corollary \[1\] the only invariant set of $F_\lambda$ is $\{0\}$, which means that the only solution to the eigen-equation $F(X) = \lambda X$ is $X = \{0\}$. But by definition, $\{0\}$ is not an eigenset.

The proof that $F$ has no eigenvalue $\lambda < \rho(F)$ is postponed because the more general affine version is provided in the proof of Theorem \[3\] in Section \[6\].

We now show that $\rho(F)$ is an eigenvalue of $F$. Again let $F_\lambda = \frac{1}{\lambda} F$, so that $\rho(F_\lambda) = 1$. With $A$ as in the statement of Lemma \[3\] consider the nested intersection

$$S = \bigcap_{k \geq 0} F_\lambda^k(A) = \lim_{k \to \infty} F_\lambda^k(A).$$

That $S$ is compact, centrally symmetric, and star-shaped follows from Lemma \[2\]. Also

$$F_\lambda(S) = F_\lambda \left( \bigcap_{k \geq 0} F_\lambda^k(A) \right) = \bigcap_{k \geq 1} F_\lambda^k(A) = S,$$

the last equality because $A \supseteq F_\lambda(A) \supseteq F_\lambda^{(2)}(A) \supseteq \cdots$. From $F_\lambda(S) = S$ it follows that $F(S) = \lambda S$.

It remains to show that $S$ contains a non-zero vector. Since $A$ is a convex body and determined only up to scalar multiple, there is no loss of generality in assuming that $A$ contains a ball $B$ of radius 1 centered at the origin. Then

$$\sup \{ \|L_{\sigma}(x)\| : \sigma \in \Omega_k, x \in B \} = \hat{\rho}_k(F_\lambda) \geq (\rho(F_\lambda))^k = 1.$$
So there is a point \( a_k \in F^k(A) \) such that \( \|a_k\| \geq 1 \). If \( a \) is an accumulation point of \( \{a_k\} \), then \( \|a\| \geq 1 \), and there is a subsequence \( \{a_{k_i}\} \) of \( \{a_k\} \) such that

\[
\lim_{i \to \infty} a_{k_i} = a.
\]

Since the sets \( F^{(k_i)}(A) \) are closed and nested, it must be the case that \( a \in F^{(k_i)}(A) \) for all \( i \). Therefore \( a \in S \).

That \( S \) is full dimensional follows from Lemma 4.

5 Theorems of Dranisnikov-Konyagin-Protasov and of Barabanov

Important results of Dranisnikov-Konyagin-Protasov and of Barabanov on the joint spectral radius turn out to be almost immediate corollaries of Theorem 2. The first result is attributed to Dranisnikov and Konyagin by Protasov, who provided a proof in \([10]\). Barabanov’s theorem appeared originally in \([11]\).

**Corollary 2 (Dranisnikov-Konyagin-Protasov)** If \( F = (\mathbb{R}^n; L_i, i \in I) \) is a compact, irreducible, linear IFS with joint spectral radius \( \rho := \rho(F) \), then there exists a centrally symmetric convex body \( K \) such that

\[
\text{conv} F(K) = \rho K.
\]

**Proof:** According to Theorem 2 there is a centrally symmetric, full dimensional eigenset \( S \) such that \( F(S) = \rho S \). If \( K = \text{conv}(S) \), then \( K \) is also centrally symmetric and

\[
\text{conv} F(K) = \text{conv} F(\text{conv} S) = \text{conv} F(S) = \text{conv} (\rho S) = \rho \text{conv} S = \rho K.
\]

The second equality is routine to check. Since \( S \) is full dimensional, \( K \) is a convex body, i.e., has nonempty interior.

The original form of the Barabanov theorem is as follows:

**Theorem 6 (Barabanov)** If a set \( F \) of linear maps on \( \mathbb{R}^n \) is compact and irreducible, then there exists a vector norm \( \| \cdot \|_B \) such that

\[
\text{for all } x \text{ and all } L \in F \quad \|Lx\|_B \leq \rho(F) \|x\|_B,
\]

\[
\text{for any } x \in \mathbb{R}^n \text{ there exists an } L \in F \text{ such that } \|Lx\|_B = \rho(F) \|x\|_B.
\]

Such a norm is called a Barabanov norm. The first property says that \( F \) is extremal, meaning that

\[
\|L\|_B \leq \rho(F) \tag{2}
\]
for all $L \in F$. It is extremal in the following sense. By property (4) of the joint spectral radius in Section 3,
\[
\sup_{L \in F} \|L\| \geq \rho(F)
\]
for any matrix norm. Therefore, the joint spectral radius $\rho(F)$ can be characterized as the infimum over all possible matrix norms of the largest norm of linear maps in $F$. Since $F$ is assumed compact, the inequality cannot be strict for all $L \in F$. Hence there exists an $L \in F$ whose Barabanov norm achieves the upper bound $\rho(F)$.

Furthermore, the second property in the statement of Barabanov’s Theorem says that, for any $x \in \mathbb{R}^n$, there is such an $L$ achieving a value equal to the joint spectral radius at the point $x$. See [12] for more on extremal norms.

In view of Lemma 1, Barabanov’s theorem can be restated in the following equivalent geometric form. Here $\partial$ denotes the boundary.

**Corollary 3** If $F$ is a compact, irreducible, linear IFS with joint spectral radius $\rho := \rho(F)$, then there exists a centrally symmetric convex body $K$ such that
\[
F(K) \subseteq \rho K,
\]
and, for any $x \in \partial K$, there is an $L \in F$ such that $Lx \in \partial(\rho K)$.

**Proof:** Let $F^t = (\mathbb{R}^n; L_i^t, i \in I)$, where $L^t$ denotes the adjoint (transpose matrix) of $L$. For a compact set $Y$, the dual of $Y$ (sometimes called the polar) is the set
\[
Y^* = \{z \in \mathbb{R}^n : \langle y, z \rangle \leq 1 \text{ for all } y \in Y\}.
\]
The first two of the following properties are easily proved for any compact set $B$.

1. $B^*$ is convex.

2. If $B$ is centrally symmetric, then so is $B^*$.

3. If $L$ is linear and $L^t(S) \subseteq S$, then $L(S^*) \subseteq S^*$.

To prove the third property above, assume that $L^t(S) \subseteq S$. and let $x \in S^*$. Then
\[
x \in S^* \implies \langle x, y \rangle \leq 1 \text{ for all } y \in S \\
\implies \langle x, L^t y \rangle \leq 1 \text{ for all } y \in S \\
\implies \langle Lx, y \rangle \leq 1 \text{ for all } y \in S \\
\implies Lx \in S^*
\]

Since $F$ is a compact, irreducible, linear IFS, so is $F^t$. Let $S$ be a centrally symmetric eigenset for $F^t$ as guaranteed by Theorem 2. By properties 1 and 2 above, $S^*$ is a centrally symmetric convex body. From the eigen-equation $F^t(S) = \rho S$, it follows that
\( \frac{1}{\rho} L^t(S) \subseteq S \) for all \( L \in F \). From property 3 above it follows that \( \frac{1}{\rho} F(S^n) \subseteq S^n \) or \( F(S^n) \subseteq \rho S^n \). Setting \( K = S^n \) yields

\[
F(K) \subseteq \rho K.
\]

Concerning the second statement of the corollary, assume that \( x \in \partial K = \partial S^n \). Then \( \langle x, y \rangle \leq 1 \) for all \( y \in S \) and \( \langle x, y \rangle = 1 \) for some \( y \in S \). Since \( F(S) = \rho S \), the last equality implies that there is an \( L \in F \) such that \( \langle \frac{1}{\rho} Lx, z \rangle = \langle x, \frac{1}{\rho} L'z \rangle = 1 \) for some \( z \in S \). Now we have \( \langle \frac{1}{\rho} Lx, y \rangle \leq 1 \) for all \( y \in S \) and \( \langle \frac{1}{\rho} Lx, z \rangle = 1 \) for some \( z \in S \). Therefore, \( \frac{1}{\rho} Lx \in \partial S^n = \partial K \) or \( Lx \in \rho(\partial K) = \partial (\rho K) \).

6 The Eigenvalue Problem for an Affine IFS

For an affine IFS \( F \), there is no theorem analogous to Theorem 2. More specifically, there are examples where \( \rho(F) \) is an eigenvalue of \( F \) and examples where \( \rho(F) \) is not an eigenvalue of \( F \). For an example where \( \rho(F) \) is an eigenvalue, let

\[
F_1 = \{ \mathbb{R}^2; f \}, \quad f(x) = Lx + (1, 0), \quad L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Note that \( L \), a 90° degree rotation about the origin, is irreducible and \( \rho(F_1) = 1 \). If \( S \) is the unit square with vertices \((0,0), (1,0), (0,1), (1,1)\), then \( F_1(S) = S \). Therefore \( \rho(F_1) = 1 \) is an eigenvalue of \( F_1 \). On the other hand let

\[
F_2 = \{ \mathbb{R}; f \}, \quad f(x) = x + 1.
\]

In this case \( \rho(F_2) = 1 \), but it is clear that there exists no compact set \( X \) such that \( F(X) = X \). For the affine case, Theorem 3 as stated in the introduction, does hold. The proof is as follows.

**Proof of Theorem 3.** If \( \lambda > \rho(F) \), then \( \rho(F_\lambda) < 1 \). According to Theorem 3 the IFS \( F_\lambda \) has an attractor \( A \) so that \( F_\lambda(A) = A \). Since at least one function in \( F_\lambda \) is not linear, \( A \neq \{0\} \). Since \( F_\lambda(A) = A \), also \( F(A) = \lambda A \). Therefore \( \lambda \) is an eigenvalue of \( F \).

Concerning the second statement in the theorem assume, by way of contradiction, that such an eigenvalue \( \lambda < \rho(F) \) exists, with corresponding eigenset \( S \). Then \( F_\lambda(S) = S \) and \( \rho(F_\lambda) > 1 \). According to Lemma 4 since \( F \) is assumed irreducible, the eigenset \( S \) is full dimensional. Exactly as in the proof of Lemma 4, using conjugation by a translation, there is an affine IFS \( F' \) and a nonempty compact set \( S' \) such that

1. \( F'(S') = S' \),
2. \( 0 \in \text{int}(\text{conv}(S')) \),
3. The set \( \mathbb{L}_{F'} \) of linear parts of the functions in \( F' \) is equal to the set \( \mathbb{L}_{F_\lambda} \) of linear parts of the functions in \( F_\lambda \),
4. $\rho(F') = \rho(F_\lambda) > 1$,

5. $F'$ is irreducible.

In item 2 above, $\text{int}(\text{conv}(S'))$ denotes the interior of the convex hull of $S'$. If $K = \text{conv}(S')$ and $f(x) = Lx + a$ is an arbitrary affine function such that $f(S') \subseteq S'$, then

$$f(K) \subseteq K.$$ 

This follows from the fact that $f(S') \subseteq S'$ as follows. If $z \in K$, then $z = \alpha x + (1 - \alpha) y$ where $0 \leq \alpha \leq 1$ and $x, y \in S'$. Therefore

$$f(z) = \alpha Lx + (1 - \alpha) Ly + a = \alpha(Lx + a) + (1 - \alpha)(Ly + a)$$

$$= \alpha f(x) + (1 - \alpha) f(y) \in \text{conv}(f(S')) \subseteq \text{conv}(S') = K.$$

Let $r > 0$ be the largest radius of a ball centered at the origin and contained in $K$ and $R$ the smallest radius of a ball centered at the origin and containing $K$. Let $x \in K$ such that $0 < \|x\| \leq r$. If $f(x) = Lx + a$ is any affine function such that $f(S') \subseteq S'$, then we claim that $\|Lx\| \leq R + r$. To prove this, first note that $-x \in K$. From $f(K) \subseteq K$ it follows that

$$\|Lx + a\| = \|f(x)\| \leq R$$

$$\|- Lx + a\| = \|L(-x) + a\| = \|f(-x)\| \leq R$$

$$\|2a\| = \|(Lx + a) + (-Lx + a)\| \leq \|Lx + a\| + \|L(-x) + a\| \leq 2R$$

$$\|Lx\| = \|f(x) - a\| \leq \|f(x)\| + \|a\| \leq R + r.$$

From the definition of the joint spectral radius, $\rho(F') > 1$ implies that there is an $\epsilon > 0$ such that $(\hat{\rho}_k(F_\lambda))^{1/k} > 1 + \epsilon$ for infinitely many values of $k$. This, in turn, implies that, for each such $k$, there is an affine map $f_k \in \{f_\sigma : \sigma \in \Omega_k\}$ and its linear part $L_k \in \{L_\sigma : \sigma \in \Omega_k\}$ such that $\|L_k\| \geq (1 + \epsilon)^k$. Choose $k = k_0$ sufficiently large that $\|L_{k_0}\| \geq (1 + \epsilon)^{k_0} > \frac{R + r}{r}$. Then there is a $y \in K'$ with $\|y\| = r$ such that $\|L_{k_0}y\| > r \frac{R + r}{r} = R + r$. Since $L_{k_0}$ is the linear part of an affine function $f$ with the property $f(S') \subseteq S'$ (property 1 above), this is a contradiction to what was proved in the previous paragraph.

\[\square\]

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