Constructing Two Edge-Disjoint Hamiltonian Cycles in Locally Twisted Cubes

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Abstract

The $n$-dimensional hypercube network $Q_n$ is one of the most popular interconnection networks since it has simple structure and is easy to implement. The $n$-dimensional locally twisted cube, denoted by $LTQ_n$, an important variation of the hypercube, has the same number of nodes and the same number of connections per node as $Q_n$. One advantage of $LTQ_n$ is that the diameter is only about half of the diameter of $Q_n$. Recently, some interesting properties of $LTQ_n$ were investigated. In this paper, we construct two edge-disjoint Hamiltonian cycles in the locally twisted cube $LTQ_n$, for any integer $n \geq 4$. The presence of two edge-disjoint Hamiltonian cycles provides an advantage when implementing algorithms that require a ring structure by allowing message traffic to be spread evenly across the locally twisted cube.

Keywords: edge-disjoint Hamiltonian cycles; locally twisted cubes; inductive construction; parallel computing system

1 Introduction

Parallel computing is important for speeding up computation. The design of an interconnection network is the first thing to be considered. Many topologies have been proposed in the literature [3, 4, 5, 6], and the desirable properties of an interconnection network include symmetry, relatively small degree, small diameter, embedding capabilities, scalability, robustness, and efficient routing. Among those proposed interconnection networks, the hypercube is a popular interconnection network with many attractive properties such as regularity, symmetry, small diameter, strong connectivity, recursive construction, partition ability, and relatively low link complexity [12]. The architecture of an interconnection network is usually modeled by a graph, where the nodes represent the processing elements and the edges represent the communication links. In this paper, we will use graphs and networks interchangeably.

The $n$-dimensional locally twisted cube, denoted by $LTQ_n$, was first proposed by Yang et al. [15, 16] and is a better hypercube variant which is conceptually closer to the comparable hypercube $Q_n$ than existing variants. The $n$-dimensional locally twisted cube $LTQ_n$ is similar to $n$-dimensional hypercube $Q_n$ in the sense that the nodes can be one-to-one labeled with 0-1 binary strings of length $n$, so that the labels of any two adjacent nodes differ in at most two successive bits. One advantage is that the diameter of locally twisted cubes is only about half the diameter of hypercubes [16]. Recently, some interesting properties, such as conditional link faults, of the locally twisted cube $LTQ_n$...
were investigated. Yang et al. proved that $LTQ_n$ has a connectivity of $n$ \cite{10}. They also showed that locally twisted cubes are 4-pancyclic and that a locally twisted cube is superior to a hypercube in terms of ring embedding capability \cite{15}. Ma and Xu \cite{10} showed that for any two different nodes $u$ and $v$ in $LTQ_n (n \geq 3)$, there exists a $uv$-path of length $l$ with $d(u,v) + 2 \leq l \leq 2^n - 1$ except for a shortest $uv$-path, where $d(u,v)$ is the length of a shortest path between $u$ and $v$. In \cite{14}, Yang et al. addressed the fault diagnosis of locally twisted cubes under the $MM^*$ comparison model. Hsieh et al. constructed $n$ edge-disjoint spanning trees in an $n$-dimensional locally twisted cube \cite{7}. Recently, Hsieh et al. showed that for any $LTQ_n (n \geq 3)$ with at most $2n - 5$ faulty edges in which each node is incident to at least two fault-free edges, there exists a fault-free Hamiltonian cycle \cite{8}.

Two Hamiltonian cycles in a graph are said to be edge-disjoint if they do not share any common edge. The edge-disjoint Hamiltonian cycles can provide advantage for algorithms that make use of a ring structure \cite{13}. The following application about edge-disjoint Hamiltonian cycles can be found in \cite{14}. Consider the problem of all-to-all broadcasting in which each node sends an identical message to all other nodes in the network. There is a simple solution for the problem using an $n$-node ring that requires $n - 1$ steps, i.e., at each step, every node receives a new message from its ring predecessor and passes the previous message to its ring successor. If the network admits edge-disjoint rings, then messages can be divided and the parts broadcast along different rings without any edge contention. If the network can be decomposed into edge-disjoint Hamiltonian cycles, then the message traffic will be evenly distributed across all communication links. Edge-disjoint Hamiltonian cycles also form the basis of an efficient all-to-all broadcasting algorithm for networks that employ warmhole or cut-through routing \cite{9}.

The edge-disjoint Hamiltonian cycles in $k$-ary $n$-cubes and hypercubes has been constructed in \cite{1}. Barden et al. constructed the maximum number of edge-disjoint spanning trees in a hypercube \cite{2}. Petrovic et al. characterized the number of edge-disjoint Hamiltonian cycles in hyper-tournaments \cite{11}. Hsieh et al. constructed edge-disjoint spanning trees in locally twisted cubes \cite{7}. Hsieh et al. investigated the edge-fault tolerant Hamiltonicity of an $n$-dimensional locally twisted cube \cite{8}. The existence of a Hamiltonian cycle in locally twisted cubes has been verified \cite{15}. However, there has been little work reported so far on edge-disjoint properties in the locally twisted cubes. In this paper, we show that, for any integer $n \geq 4$, there are two edge-disjoint Hamiltonian cycles in the $n$-dimensional locally twisted cube $LTQ_n$.

The rest of the paper is organized as follows. In Section 2 the structure of the locally twisted cube is introduced, and some definitions and notations used in this paper are given. Section 3 shows the construction of two edge-disjoint Hamiltonian cycles in the locally twisted cube. Finally, we conclude this paper in Section 4.

## 2 Preliminaries

We usually use a graph to represent the topology of an interconnection network. A graph $G = (V, E)$ is a pair of the node set $V$ and the edge set $E$, where $V$ is a finite set and $E$ is a subset of $\{\langle u, v \rangle | \langle u, v \rangle$ is an unordered pair of $V\}$. We will use $V(G)$ and $E(G)$ to denote the node set and the edge set of $G$, respectively. If $\langle u, v \rangle$ is an edge in a graph $G$, we say that $u$ is adjacent to $v$. A neighbor of a node $v$ in a graph $G$ is any node that is adjacent to $v$. Moreover, we use $N_G(v)$ to denote the neighbors of $v$ in $G$. The subscript $'G'$ of $N_G(v)$ can be removed from the notation if it has no ambiguity.

A path $P$, represented by $\langle v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{l-1} \rangle$, is a sequence of distinct nodes such that two consecutive nodes are adjacent. The first node $v_0$ and the last node $v_{l-1}$ visited by $P$ are called the path-start and path-end of $P$, denoted by $\text{start}(P)$ and $\text{end}(P)$,
Let $\mathbf{n} \geq 3$, the recursive definition of the $(2)$ for connected by four edges $(00, 01), (00, 10), (01, 11), \text{and} (10, 11)$. Let $\mathbf{n}$ denote by $\mathbf{\oplus}$. For each $\mathbf{n}$ in each $\mathbf{i}$ of graph $\mathbf{Q} \mathbf{\oplus}$ edges. Note that $0 \rightarrow \mathbf{n}$ start to each $\mathbf{Q}$ can be decomposed into two sub-locally twisted cubes $\mathbf{Q}$. A path (or cycle) in $\mathbf{v}$ is called a Hamiltonian path (or Hamiltonian cycle) if it contains every node of $\mathbf{G}$ exactly once. Two paths (or cycles) $\mathbf{P}_1$ and $\mathbf{P}_2$ connecting a node $u$ to a node $v$ are said to be edge-disjoint iff $E(\mathbf{P}_1) \cap E(\mathbf{P}_2) = \emptyset$. Two paths (or cycles) $\mathbf{Q}_1$ and $\mathbf{Q}_2$ of graph $\mathbf{G}$ are called node-disjoint if $V(\mathbf{Q}_1) \cap V(\mathbf{Q}_2) = \emptyset$. Two node-disjoint paths $\mathbf{Q}_1$ and $\mathbf{Q}_2$ can be concatenated into a path, denoted by $\mathbf{Q}_1 \Rightarrow \mathbf{Q}_2$, if $end(\mathbf{Q}_1)$ is adjacent to $\mathbf{Q}_2$.

Now, we introduce locally twisted cubes. A node of the $\mathbf{n}$-dimensional locally twisted cube $LTQ_n$ is represented by a 0-1 binary string of length $\mathbf{n}$. A binary string $\mathbf{b}$ of length $\mathbf{n}$ is denoted by $\mathbf{b}_0 \mathbf{b}_1 \mathbf{b}_2 \cdots \mathbf{b}_{\mathbf{n}-1}$, where $\mathbf{b}_{\mathbf{n}-1}$ is the most significant bit. We then give the recursive definition of the $\mathbf{n}$-dimensional locally twisted cube $LTQ_n$, for any integer $\mathbf{n} \geq 2$, as follows.

**Definition 2.1.** [15] [16] Let $\mathbf{n} \geq 2$. The $\mathbf{n}$-dimensional locally twisted cube, denoted by $LTQ_n$, is defined recursively as follows.

1. $LTQ_2$ is a graph consisting of four nodes labeled with 00, 01, 10, and 11, respectively, connected by four edges (00, 01), (00, 10), (01, 11), and (10, 11).
2. For $\mathbf{n} \geq 3$, $LTQ_n$ is built from two disjoint copies $LTQ_{\mathbf{n}-1}$ according to the following steps. Let $LTQ_{\mathbf{n}-1}^0$ denote the graph obtained by prefixing the label of each node of one copy of $LTQ_{\mathbf{n}-1}$ with 0, let $LTQ_{\mathbf{n}-1}^1$ denote the graph obtained by prefixing the label of each node of the other copy of $LTQ_{\mathbf{n}-1}$ with 1, and connect each node $\mathbf{b} = 0\mathbf{b}_{\mathbf{n}-2} \mathbf{b}_{\mathbf{n}-3} \cdots \mathbf{b}_{\mathbf{1}} \mathbf{b}_0$ of $LTQ_{\mathbf{n}-1}^0$ with the node $1(\mathbf{b}_{\mathbf{n}-2} \oplus \mathbf{b}_0)\mathbf{b}_{\mathbf{n}-3} \cdots \mathbf{b}_{\mathbf{1}} \mathbf{b}_0$ of $LTQ_{\mathbf{n}-1}^1$ by an edge, where ‘$\oplus$’ represents the modulo 2 addition.

According to Definition 2.1, $LTQ_n$ is an $n$-regular graph with $2^n$ nodes and $n2^{n-1}$ edges. Note that $0 \oplus 0 = 1 \oplus 1 = 0$ and $0 \oplus 1 = 1 \oplus 0 = 1$. The $\mathbf{n}$-dimensional locally twisted cube $LTQ_n$ is closed to an $\mathbf{n}$-dimensional hypercube $Q_n$ except that the labels of any two adjacent nodes in $LTQ_n$ differ in at most two successive bits. In addition, $LTQ_n$ can be decomposed into two sub-locally twisted cubes $LTQ_{\mathbf{n}-1}^0$ and $LTQ_{\mathbf{n}-1}^1$, where for each $\mathbf{i} \in \{0, 1\}$, $LTQ_{\mathbf{n}-1}^i$ consists of those nodes $b = b_{\mathbf{n}-1}b_{\mathbf{n}-2} \cdots b_1 b_0$ with $b_{\mathbf{n}-1} = i$. For each $\mathbf{i} \in \{0, 1\}$, $LTQ_{\mathbf{n}-1}^i$ is isomorphic to $LTQ_{\mathbf{n}-1}^i$. For example, Fig. (a) shows $LTQ_3$ and Fig. (b) depicts $LTQ_4$ containing two sub-locally twisted cubes $LTQ_3^0$ and $LTQ_3^1$.

Let $\mathbf{b}$ be a binary string $b_{\mathbf{t}-1}b_{\mathbf{t}-2} \cdots b_1 b_0$ of length $\mathbf{t}$. We denote $\mathbf{b}^t$ the new binary string...
obtained by repeating $b$ string $i$ times. For instance, $(10)^2 = 1010$ and $0^3 = 000$.

3 Two Edge-Disjoint Hamiltonian Cycles

Obviously, $LTQ_3$ has no two edge-disjoint Hamiltonian cycles since each node is incident to three edges. Our method for constructing two edge-disjoint Hamiltonian cycles of $LTQ_n$, with integer $n \geq 4$, is based on an inductive construction. Initially, we construct two edge-disjoint Hamiltonian paths, $P$ and $Q$, of $LTQ_4$ so that $\text{start}(P) = 0010$, $\text{end}(P) = 0000$, $\text{start}(Q) = 0110$, and $\text{end}(Q) = 0100$. Clearly, these two paths are two edge-disjoint Hamiltonian cycles of $LTQ_4$. For $n \geq 5$, we will construct two edge-disjoint Hamiltonian paths $P$ and $Q$ in $LTQ_n$ such that $\text{start}(P) = 00(0)^n-5010$, $\text{end}(P) = 10(0)^n-5010$, $\text{start}(Q) = 00(0)^n-5110$, and $\text{end}(Q) = 10(0)^n-5110$. By Definition 2.1, $\text{start}(P) \in N(\text{end}(P))$ and $\text{start}(Q) \in N(\text{end}(Q))$. Thus, $P$ and $Q$ are two edge-disjoint Hamiltonian cycles.

We first show that $LTQ_4$ contains two edge-disjoint Hamiltonian paths in the following lemma.

**Lemma 3.1.** There are two edge-disjoint Hamiltonian paths $P$ and $Q$ in $LTQ_4$ such that $\text{start}(P) = 0010$, $\text{end}(P) = 0000$, $\text{start}(Q) = 0110$, and $\text{end}(Q) = 0100$.

**Proof.** We prove this lemma by constructing such two paths. Let $P = (0010 \rightarrow 0110 \rightarrow 0111 \rightarrow 0101 \rightarrow 0100 \rightarrow 1100 \rightarrow 1110 \rightarrow 1010 \rightarrow 1000 \rightarrow 1001 \rightarrow 1011 \rightarrow 1101 \rightarrow 1111 \rightarrow 0011 \rightarrow 0001 \rightarrow 0000)$, and let $Q = (0110 \rightarrow 1110 \rightarrow 1111 \rightarrow 1001 \rightarrow 0101 \rightarrow 0011 \rightarrow 0010 \rightarrow 1010 \rightarrow 1011 \rightarrow 1111 \rightarrow 0011 \rightarrow 1101 \rightarrow 1100 \rightarrow 1000 \rightarrow 0000 \rightarrow 0100)$. Fig. 2 depicts the constructions of $P$ and $Q$. Clearly, $P$ and $Q$ are edge-disjoint Hamiltonian paths in $LTQ_4$.

According to Definition 2.1, nodes 0010 and 0000 are adjacent, and nodes 0110 and 0100 are adjacent. Thus, the following corollary immediately holds true from Lemma 3.1.

**Corollary 3.2.** There are two edge-disjoint Hamiltonian cycles in $LTQ_4$.

Using Lemma 3.1 we show that $LTQ_5$ has two edge-disjoint Hamiltonian paths in the following lemma.

**Lemma 3.3.** There are two edge-disjoint Hamiltonian paths $P$ and $Q$ in $LTQ_5$ such that $\text{start}(P) = 00010$, $\text{end}(P) = 10010$, $\text{start}(Q) = 00110$, and $\text{end}(Q) = 10110$.
Fig. 3: Two edge-disjoint Hamiltonian paths in $LTQ_5$, where solid arrow lines indicate a Hamiltonian path $P$ and dotted arrow lines indicate the other edge-disjoint Hamiltonian path $Q$.

**Proof.** We first partition $LTQ_5$ into two sub-locally twisted cubes $LTQ^0_4$ and $LTQ^1_4$. By Lemma 3.1 there are two edge-disjoint Hamiltonian paths $P^i$ and $Q^i$ in $LTQ^i_4$, for $i \in \{0, 1\}$, such that $\text{start}(P^i) = 00010$, $\text{end}(P^i) = 0000$, $\text{start}(Q^i) = 0010$, and $\text{end}(Q^i) = 0100$. By Definition 2.1 we have that $\text{end}(P^0) \in N(\text{end}(P^1))$ and $\text{end}(Q^0) \in N(\text{end}(Q^1))$.

Let $P = P^0 \Rightarrow P^1_{\text{rev}}$ and let $Q = Q^0 \Rightarrow Q^1_{\text{rev}}$, where $P^1_{\text{rev}}$ and $Q^1_{\text{rev}}$ are the reversed paths of $P^1$ and $Q^1$, respectively. Then, $P$ and $Q$ are two edge-disjoint Hamiltonian paths in $LTQ_5$ such that $\text{start}(P) = 00010$, $\text{end}(P) = 0010$, $\text{start}(Q) = 00110$, and $\text{end}(Q) = 10110$. Fig. 3 shows the constructions of such two edge-disjoint Hamiltonian paths in $LTQ_5$.

Thus, the lemma holds true. □

According to Definition 2.1 nodes 00010 and 10010 are adjacent, and nodes 00110 and 10110 are adjacent. Thus, the following corollary immediately holds true from Lemma 3.3.

**Corollary 3.4.** There are two edge-disjoint Hamiltonian cycles in $LTQ_5$.

Based on Lemma 3.3 we prove the following lemma.

**Lemma 3.5.** For any integer $n \geq 5$, there are two edge-disjoint Hamiltonian paths $P$ and $Q$ in $LTQ_n$ such that $\text{start}(P) = 00(0)^{n-5}010$, $\text{end}(P) = 10(0)^{n-5}010$, $\text{start}(Q) = 00(0)^{n-5}110$, and $\text{end}(Q) = 10(0)^{n-5}110$.

**Proof.** We prove this lemma by induction on $n$, the dimension of the locally twisted cube. It follows from Lemma 3.3 that the lemma holds for $n = 5$. Suppose that the lemma is true for the case $n = k$ ($k \geq 5$). Assume that $n = k + 1$. We first partition $LTQ_{k+1}$ into two sub-locally twisted cubes $LTQ^0_k$ and $LTQ^1_k$. By the induction hypothesis, there are two edge-disjoint Hamiltonian paths $P^i$ and $Q^i$ in $LTQ^i_k$, for $i \in \{0, 1\}$, such that $\text{start}(P^i) = 00(0)^{k-5}010$, $\text{end}(P^i) = 00(0)^{k-5}010$, $\text{start}(Q^i) = 10(0)^{k-5}110$, and $\text{end}(Q^i) = 10(0)^{k-5}110$. By Definition 2.1 we have that $\text{end}(P^0) \in N(\text{end}(P^1))$ and $\text{end}(Q^0) \in N(\text{end}(Q^1))$.

Let $P = P^0 \Rightarrow P^1_{\text{rev}}$ and let $Q = Q^0 \Rightarrow Q^1_{\text{rev}}$, where $P^1_{\text{rev}}$ and $Q^1_{\text{rev}}$ are the reversed paths of $P^1$ and $Q^1$, respectively. Then, $P$ and $Q$ are two edge-disjoint Hamiltonian paths in $LTQ_{k+1}$ such that $\text{start}(P) = 00(0)^{k-4}010$, $\text{end}(P) = 10(0)^{k-4}010$, $\text{start}(Q) = 00(0)^{k-4}110$, and $\text{end}(Q) = 10(0)^{k-4}110$. Fig. 4 depicts the constructions of such two edge-disjoint Hamiltonian paths in $LTQ_{k+1}$. Thus, the lemma holds true when $n = k + 1$. By induction, the lemma holds true. □
Fig. 4: The constructions of two edge-disjoint Hamiltonian paths in $LTQ_{k+1}$, with $k \geq 5$, where dotted arrow lines indicate the paths and solid arrow lines indicate concatenated edges.

By Definition 2.1, nodes $\text{start}(P) = 00(0)^{n-5}010$ and $\text{end}(P) = 10(0)^{n-5}010$ are adjacent, and nodes $\text{start}(Q) = 00(0)^{n-5}110$ and $\text{end}(Q) = 10(0)^{n-5}110$ are adjacent. It immediately follows from Lemma 3.5 that the following corollary holds true.

**Corollary 3.6.** For any integer $n \geq 5$, there are two edge-disjoint Hamiltonian cycles in $LTQ_n$.

It immediately follows from Lemmas 3.1 and 3.5, and Corollaries 3.2 and 3.6, that the following two theorems hold true.

**Theorem 3.7.** For any integer $n \geq 4$, there are two edge-disjoint Hamiltonian paths in $LTQ_n$.

**Theorem 3.8.** For any integer $n \geq 4$, there are two edge-disjoint Hamiltonian cycles in $LTQ_n$.

4 Concluding Remarks

In this paper, we construct two edge-disjoint Hamiltonian cycles (paths) in a $n$-dimensional locally twisted cubes $LTQ_n$, for any integer $n \geq 4$. In the construction of two edge-disjoint Hamiltonian cycles (paths) of $LTQ_n$, some edges are not used. It is interesting to see if there are more edge-disjoint Hamiltonian cycles of $LTQ_n$ for $n \geq 6$. We would like to post it as an open problem to interested readers.

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