“Supersolid” self-bound Bose condensates via laser-induced interatomic forces

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We show that the dipole-dipole interatomic forces induced by a single off-resonant running laser beam can lead to a self-bound pencil-shaped Bose condensate, even if the laser beam is a plane-wave. For an appropriate laser intensity the ground state has a quasi-one dimensional density modulation — a Bose “supersolid”.

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Electrostriction is the tendency of matter to become compressed in the presence of an electric field [4]. In an optically trapped atomic Bose-Einstein condensate (BEC) it is provided by the gradient of the incident electric field (one-body dipole forces). However, it can occur even if the external fields are homogeneous: the electrostriction effects of dipole-dipole interatomic forces induced by certain configurations of far-off-resonant laser beams are capable, for sufficiently high laser intensity, of self-binding a BEC within a wavelength of the laser (in the near-zone) [5]. Thereby, a physical situation can be realized which has analogies to self-gravity [3,4].

Here we show that even in the simplest case of a single homogeneous, circularly-polarized, far-off-resonant laser beam a BEC can be self-bound by electrostriction, due to the retarded part of the dipole-dipole interaction. The self-bound condensate will be pencil shaped, aligned along the direction of propagation of light, in principle even for arbitrarily small intensity. As the laser intensity increases, the electrostriction can give rise to a remarkable density modulation of the quasi-one dimensional condensate ground state. Such a condensate bears similarity to a “supersolid”, i.e., a long-range crystalline-like density modulation imposed upon a superfluid by inter-particle forces [3,4]. The formation of such structures is associated with a strong enhancement of the elastically scattered field (and, consequently, a large radiation pressure) and with suppression of the heating due to spontaneous Rayleigh scattering. The relation of our effects to (stimulated) Brillouin scattering, collective atomic recoil and superradiant Rayleigh scattering [3,4] is briefly discussed.

We consider the case of a far-off-resonant circularly polarized laser beam propagating along the positive \( \hat{z} \) direction with wavevector \( q \) (Fig. 1—insert). The dipole-dipole induced interatomic potential [4] then becomes

\[
V_{dd}(\vec{r}) = \frac{3}{4} \hbar \Gamma_{ray} \left[ \frac{2z^2 - x^2 - y^2}{q^4 r^5} (\cos qr + qr \sin qr) \right].
\]

Here \( x, y, z \) are the components of the interatomic separation \( \vec{r} \) and \( \hbar \Gamma_{ray} = \alpha^2 q^3 / 16 \pi \epsilon_0 z^2 \) is the single-atom (spontaneous) Rayleigh scattering rate, which is proportional to the laser intensity \( I \) and to \( \alpha^2 \), i.e. to the saturation parameter that scales as the inverse square of the detuning from the nearest atomic resonance [3]. We assume in the following that the condensate contains many atoms per cubic wavelength to ensure the validity of the mean-field approach. The condensate is taken to be at zero temperature and the rate of heating due to incoherent Rayleigh scattering is shown to be negligible.

The Gross-Pitaevskii equation [10] for the order parameter \( \Psi(\vec{r}, t) \) can be obtained starting from

\[
\frac{i \hbar}{\partial t} \frac{\partial \Psi}{\partial t} = \frac{\delta}{\delta \Psi} H_{\text{tot}}.
\]

Here the total mean-field energy functional \( H_{\text{tot}} = H_{\text{kin}} + H_{\text{ho}} + H_{\text{dd}} + H_{\text{scat}} \) consists of: (a) the kinetic energy, \( H_{\text{kin}} = \int d\vec{r} (\hbar^2 / 2m) |\nabla \Psi|^2 \); (b) the harmonic-trap energy, that can be due to the focus size of the beam, \( H_{\text{ho}} = \int d\vec{r} V_{\text{ho}} n(\vec{r}) \), where \( n = |\Psi(\vec{r}, t)|^2 \) is the atomic density and \( V_{\text{ho}} = m \omega^2 r^2 / 2 + m \omega_z^2 z^2 / 2 \) is the harmonic cylindrically symmetric potential, where \( \omega^2 = x^2 + y^2 \); (c) the mean-field energy due to the short-range (\( r^{-6} \)) van der Waals interaction, which will be treated within the delta-function pseudo-potential approximation, \( H_{\text{scat}} = (2 \pi a^2 / m) \int d\vec{r} n(\vec{r})^2 \), where \( a \) is the s-wave scattering length, and (d) the electromagnetically-induced mean-field energy, \( H_{\text{dd}} = (1/2) \int d\vec{r} d\vec{r}' V_{\text{dd}}(\vec{r} - \vec{r}') n(\vec{r}) n(\vec{r}') \) which corresponds to the electrostriction energy of the medium [3,4] to lowest order in the \( a n / \epsilon_0 \). The mean radii of the condensate can be estimated using the Gaussian ansatz \( \Psi(\rho, z) = N^{1/2} \exp \left( -\rho^2 / 2w_r^2 - z^2 / 2w_z^2 \right) / \pi^{3/4} w_z^{1/2} w_r \). The variational parameters \( w_z \) and \( w_r \), which are related to the mean quadratic radii through \( \langle \Delta x \rangle^2 \equiv \langle x^2 \rangle = w_r^2 / 2 \) and \( \langle \Delta z \rangle^2 \equiv \langle z^2 \rangle = w_z^2 / 2 \), are obtained by minimizing \( H_{\text{tot}} \).

A self-bound condensate is then found to exist in the Thomas-Fermi (TF) limit (negligible kinetic energy). External confinement is not necessary and the values of \( w_r \) and \( w_z \) are finite due only to the presence of a plane-wave laser. The laser intensities are below the threshold of the instability caused by the 1/r^3 part of the dipole-dipole potential [4] (static field limit) [4]. \( I \leq 12 \pi \hbar^2 c \epsilon_0 \alpha / m a^2 \), and can be, in principle, arbitrarily small. Here and henceforth we normalize the intensity to \( 8 \pi \hbar^2 c \epsilon_0 \alpha / m a^2 \) and the spatial scales are normalized to the laser wavelength \( \lambda_L = 2 \pi / q \).
Figure 1 shows the variational values of the mean condensate radii in the TF-limit for a plane wave laser and in the absence of external confinement. For $I \ll 1$ the radii of the condensate are given by $\Delta x = 0.1125 I^{-1/2}$ and $\Delta z = 0.7847 I^{-1}$. We note that for laser intensity $I > 0.1$ the condensate is strongly confined in the radial direction with $\Delta x \sim 0.2$, and less confined in the longitudinal direction, with typical size larger or of the order of the wavelength, $\Delta z > 0.6$.

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\[ \Delta x \sim 0.2, \quad \Delta z > 0.6 \]

The large extension of the condensate along the longitudinal \( z \) axis means that the oscillatory long-range behavior of the potential \( V_{\text{dd}} \) can become manifest in this direction. We study the effect of these oscillations using the ansatz

\[ \Psi(\rho, z) = \psi(z) \exp\left(-\rho^2/2w_r^2\right)/\pi^{1/2}w_r \]  

in order to reduce the problem to a quasi-one dimensional (1D) one. We then obtain $\psi(z)$, as well as $w_r$, by minimizing numerically the mean-field energy $H_{\text{tot}}$. The numerical solutions for the ground state, based on the Gaussian ansatz \( \Psi \), yield radial and longitudinal sizes of the self-confined condensate \( H_{\text{ho}} = 0 \) that are in good agreement with those based on the Gaussian ansatz, as displayed in Fig. 1.

The electromagnetically-induced mean-field energy, expressed through the linear density $n_{1D} = |\psi(z)|^2$, takes the 1D-reduced form

\[ H_{\text{dd}}^{1D} = (1/2) \int dz dz' V_{\text{dd}}^{1D}(z-z') n_{1D}(z)n_{1D}(z') \]

where the induced dipole-dipole potential, reduced to 1D, is given by

\[ V_{\text{dd}}^{1D}(z) = \frac{1}{2\pi w_r^2} \int d^2\rho \exp\left(-\rho^2/2w_r^2\right) V_{\text{dd}}(\rho, z). \]

The 1D-reduced interaction contains an attractive singular part $V_{\text{dd}}^{1D}(z)|_{\text{sing}} = -(h\Gamma_{\text{ray}}/q^2w_r^2)\delta(z)$. For $I < 1.5$ the singular part of the reduced interaction is balanced by a similar repulsive term $V_{\text{scat}}^{1D}(z) = (2a_0^2/mw_r^2)\delta(z)$ proportional to the (positive) $s$-wave scattering length. Figure 2 shows the non singular part of $V_{\text{dd}}^{1D}$ for three different values of the variational parameter $w_r$. For large values of $z$ the 1D-reduced potential oscillates as $V_{\text{dd}}^{1D}(z) \approx -(3h\Gamma_{\text{ray}}/2q)\cos(qz)^2/z$, giving rise to characteristic attractive logarithmic singularities in the Fourier transform at $k_z = 0$ and $k_z = \pm 2q$ (Fig. 2b).

In the TF-limit, the Bogoliubov dispersion relation for the condensate \( \vec{\mathbf{E}} \) is proportional to the square root of the Fourier transform of the interparticle potential. In a sense then, the $k_z = \pm 2q$ singularities give rise to a roton-like minimum (cf. He-II) in the dispersion relation. When the roton minimum touches the zero frequency axis there is no energy cost to the system adopting a density modulation in its ground state, and a 'supersolid' is created \( \vec{\mathbf{E}} \) (see below).

A new feature appearing in the TF-limit is the formation of single condensate droplets of size less than \( \approx 0.25\lambda_L \) located at an approximately regular distance of $\lambda_L/2$. The variational values for ground state energies based on \( \vec{\mathbf{E}} \) are substantially below the ones obtained by the Gaussian ansatz, as displayed in Fig. 3a, and indicate that such density modulation is quite stable as its energy reduction is of order of the Gaussian variational energy. The effect of density modulation arises as the field back-scattered by the condensate interferes with the incident field, thus modulating the total laser intensity and conse-
sequently creating a series of attractive dipole traps along the longitudinal axis that are separated by half a wavelength (see thick curve in Fig. 4). In this limit, the single condensate droplets become more and more confined and their number is reduced, when approaching the threshold of the \(1/r^3\) instability (\(I = 1.5\)).

\[ \eta = N a / \lambda L, \]  \hspace{1cm} (5)

which is approximately the ratio between the TF-ground state energy per particle for \(I = 1\) and the recoil energy \(E_R = \hbar^2 q^2 / 2m\) (see Fig. 3a). When the product \(\eta I\) decreases and becomes comparable to one, the radii of the condensate both increase with respect to those obtained in the TF-limit. This can be understood comparing the TF-energy, \(E_{tot} \sim N \eta E_R I^2\), with the kinetic energy, \(E_{kin} \sim N E_R I\), estimated using the TF-Gaussian solutions.

We can compensate the kinetic energy pressure in the radial direction, when necessary, by adding external radial confinement, e.g. by an appropriate choice of the focus of the laser beam. Fig. 3b shows the results of fixing the radial width \(w_r\) and decreasing the parameter \(\eta\). As soon as the kinetic energy associated with the density modulation, that is larger as \(N E_R\), exceeds the energy reduction caused by density modulation, e.g. \(\eta I^2 \sim 1\), we find the oscillatory pattern to be washed out. For \(\eta\) smaller than 2.8 at \(I = 1\), the condensate is still self-confined along the longitudinal direction, with a similar value of the longitudinal size \(\Delta z\), but the density oscillations are almost washed out. Only for much smaller values of \(\eta\), for instance \(\sim 0.02\) at \(I = 1\), does the kinetic energy also affect \(\Delta z\) (long dashed line in Fig. 3b).

Remarkably, when \(\eta I^2 \sim 1\), the single condensate droplets overlap, self-organizing into a spatially-coherent long-range density modulation — a Bose “supersolid” is formed. In this novel regime the atoms are coherently distributed among the wells allowing the establishment of a phase between the overlapping condensates. An example is given by the dot-dashed curve in Fig. 3b.

Using the Gaussian ansatz in its time-dependent form we then obtain analytically in the small intensity limit the values for the radial and longitudinal oscillation frequencies for a self-bound condensate in absence of the density modulation (\(I \ll \eta I^2 \ll 1\)): \(\Omega^2 = 2.88 \langle E_R / h \rangle^2 (I^3 \eta)\) and \(\Omega^2 = 0.0148 \langle E_R / h \rangle^2 (I^3 \eta)\). It is noteworthy that, although the condensate radii are much larger than the laser wavelength (far-zone), the frequency \(\Omega^2\) of the compressional mode along the radial direction is close to the “plasma” frequency \(\omega_p^2 = 4\pi n_0 u / m = 9.86 \langle E_R / h \rangle^2 (I^3 \eta)\) introduced for the condensate in the near-zone “self-gravity” regimes \(\Omega, n(0)\) being the peak density and \(u = (11/10) \Gamma_{ray}/q\) the “gravitational” coupling of the near zone laser-induced interatomic potential \(-u/r\). The longitudinal frequency \(\Omega_z\) is expected to be substantially modified in the presence of the oscillatory pattern.

The spontaneous Rayleigh scattering at a rate \(\Gamma_{ray}\) from the interaction-inducing laser beam leads to heating and depletes the condensate. Treating the scattering of photons as an incoherent process, we expect the total energy of the condensate to increase as

\[ \frac{d}{dt} (E_{tot}) = 2 N E_R \Gamma_{ray}. \]  \hspace{1cm} (6)

The characteristic time in which the condensate may heat up is \(\tau_{heat} = ((d E_{tot}/dt)/|E_{tot}|)^{-1}\). We estimate \(\tau_{heat} = 0.02145 N (h/E_R) I\), using the relations for the small intensity limit obtained using the Gaussian ansatz \(E_{tot}/N = -0.7189 \eta E_R I^2\), and compare it with the above expression for \(\omega_p\). The product \(\tau_{heat} \omega_p\), that can be expressed in terms of the number of atoms per cubic wavelength as \(\tau_{heat} \omega_p = 0.11 (a/\lambda L)^{1/2} (\lambda_e R_0)^{1/4} I^{-1/2}\), proves that heating due to spontaneous Rayleigh scattering events does not provide a fundamental limitation on the observability of the discussed effects for sufficient density.

For the density modulations discussed above, most of the photons are nearly elastically and coherently scattered. Their total cross section is given by \(N^2 f_{cm}\) times the single-atom cross section, where the fraction \(f_{cm}\) is defined as an appropriate average over all the possible directions of scattering (in the Born approximation).
Here $n(k)$ is the Fourier component of atomic density corresponding to $k = 2q(\sin(\theta)\hat{x}, 0, \cos(\theta)\hat{z})$, the momentum transferred by a single photon in the $x$-$z$ plane and $\theta$ is the angle between the incident beam and the scattered direction. Only the (complementary) fraction $(1 - f_{cm})$ of the rate of energy change (7), that is related to the incoherent part of the scattering cross-section, is absorbed by the interatomic degree of freedom in the center-of-mass frame, thereby providing a partial suppression of heating.

The center of mass is therefore subject to a constant radiation force that is enhanced by a factor $N f_{cm}$ by the density modulation. Correspondingly the electromagnetic field is strongly coherently back-scattered (diffracted). The radiation force can either shift the equilibrium position of the condensate, if it is located in a longitudinal trap, or else accelerate it uniformly. In the latter case the scattered light will be Doppler shifted. The effects described here are due to the same matter field interactions as those responsible for stimulated Rayleigh scattering [13] and collective atomic recoil (CARL) [1]. In fact, $f_{cm}$ will lead to similar effects. However, the change of atomic energy, unaccounted for by CARL, is the essence of our effects.

A central prediction of this paper is that induced dipole-dipole forces result in a stationary density modulation for the condensate ground-state. Density modulations in a condensate can also occur in the presence of phonon excitations, as observed in the experiment by Inouye et al [1] that has demonstrated the superradiant Rayleigh scattering of a pulse of light by an atomic BEC. However, traveling phonons represent excitations of the system and so are quite distinct from the ground-state density modulation proposed here. The two situations can, for instance, be distinguished by diffracting a nearly perpendicular probe laser with $z$-component of the wavevector, $k_z \approx 2q$. Sound waves at frequency $\Omega = v_s 2q$, where $v_s$ is the sound velocity, lead to a density modulation of the form $\cos(2qz - \Omega t)$ so that the first diffraction orders (i.e. Brillouin peaks) of the probe beam (wavenumber $k$), at angles $\theta \approx \pm 2q/k$, will be frequency shifted to $ck \pm \Omega$ [13]. By contrast, diffraction from the stationary density modulation discussed above will be elastic, with no frequency shift.

An example of the experimental conditions required for the predicted effects involves the following parameters for a cloud of $N \sim 10^3$ sodium atoms: a circularly polarized laser beam, red-detuned by 1.7 GHz from the $3S_{1/2}$ (F=1) → $3P_{3/2}$ (F = 0,1,2) transition, gives the threshold for $1/r^3$ instability at $\approx 525$ mW/cm$^2$ [1]. For such intensity, the acceleration of the center of mass is approximately $\approx 500Nf_{cm}$ [m/s$^2$], where $f_{cm}$ is a non-negligible fraction (for instance $f_{cm} \sim 0.1$ for $I \sim 1$ and $f_{cm} \sim 1$ for $I \ll 1$). It is possible to balance this force by combining longitudinal harmonic confinement with a magnetic field gradient, so that the condensate will not accelerate.

To conclude, we have demonstrated a new regime of self-confined and self-organized ground-state density modulations in a BEC illuminated by a single circular polarized laser beam. We have shown that non-linear scattering of light may arise, even in the small-saturation limit, and have estimated the enhancement of the total “elastic” cross section. This regime is inherently possible even for a plane-wave laser, although it is facilitated by the radial focusing of the beam.

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