Constrained Mechanics and Noiseless Subsystems

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Many theories are formulated as constrained systems. We provide a mechanism that explains the origin of physical states of a constrained system by a process of selection of noiseless subsystems when the system is coupled to an external environment. Effectively, physical states that solve all the constraints are selected by a passive error correction scheme which has been developed in the context of quantum information processing. We apply this mechanism to several constrained theories including the relativistic particle, electromagnetism, and quantum gravity, and discuss some interesting (and speculative) implications on the problem of time and the status of symmetries in nature.

I. INTRODUCTION

Quantum information processing is a relatively young field of research compared to, for example, the study of constrained systems in classical and quantum mechanics. The amount of research in this new field is largely due to its possible exciting applications in computation and communication [1]. There are also indications, however, that some quantum information processing techniques may inspire new ways to look at some old problems such as quantum gravity [2]. In particular, the framework of noiseless subsystems has been argued to provide an intuitive picture for the emergence of particles and space-time from a fully quantum system [3]. Here, we show that the framework of noiseless subsystem can provide a basis for thinking about any (quantum) mechanical system that is subject to constraints.

The framework of noiseless subsystems has been developed as a tool to preserve fragile quantum information against decoherence [4]. In brief, when a quantum register (a Hilbert space) is subjected to decoherence due to an interaction with an external and uncontrollable environment, information stored in the register is, in general, degraded. It has been shown that when the source of decoherence exhibits some symmetries, certain subsystems of the quantum register are unaffected by the interactions with the environment and are thus noiseless. These noiseless subsystems are therefore very natural and robust tools that can be used for processing quantum information.

The outline of the paper is as follows. In the next section, we discuss noiseless subsystems
in more detail and argue that the algebraic structures used in their descriptions are very similar to the structures familiar from the study of constrained quantum systems. The technical result of this section is the construction of a map from a constrained system to the quantum theory of a new system coupled to an environment; physical states of the constrained system correspond to noiseless subsystems in the new theory. Section III applies this map to several well-known constrained systems, leading to new and sometimes speculative interpretations on the role and the origin of symmetries such as gauge invariance and time re-parametrization invariance. Each example brings out or emphasizes different aspects of the noiseless subsystem picture. We end with a discussion of the results in section IV.

II. EMERGENCE OF SYMMETRIES

Our goal in this section is to review the formalism of noiseless subsystems [4] and to compare it with Dirac’s method for dealing with constrained mechanics. A new picture will appear in which solutions to constrained dynamics problems can be thought of as noiseless subparts of some quantum information-theoretic system.

Noiseless Subsystems

We consider a quantum theory of a system \( S \) coupled to an environment (bath) \( B \). The full Hilbert space is given by the tensor product

\[
\mathcal{H}_{\text{full}} = \mathcal{H}_S \otimes \mathcal{H}_B.
\]

We assume that the system and bath Hilbert spaces are compact and discrete. The full Hamiltonian describing evolution is

\[
H_{\text{full}} = H_S \otimes I_B + I_S \otimes H_B + H_I,
\]

where \( H_S \) and \( H_B \) are operators acting on the system and bath, \( I_S \) and \( I_B \) are unit operators on \( S \) and \( B \), and \( H_I \) encodes the interactions between \( S \) and \( B \). This last term can generically be decomposed as

\[
H_I = \sum_\alpha N_\alpha \otimes B_\alpha, \tag{1}
\]

where \( N_\alpha \) and \( B_\alpha \) are a set of operators acting only on the system and bath respectively. It is worth pointing out that the terms \( H_S \otimes I_B \) and \( I_S \otimes H_B \) that are singled out in \( H_{\text{full}} \) are only special cases of interactions and could in principle be included in the expansion of \( H_I \).

Operators \( H_S, I_S \) and \( N_\alpha \), the parts of the Hamiltonian that act on the system, generate an algebra which is usually called \( \mathcal{A} \). The interpretation of \( \mathcal{A} \) is that it comprises all the possible operations (as part of the system or the interaction Hamiltonians) that change the
state of the system. It follows from the fact the Hamiltonian is hermitian that $\mathcal{A}$ is a $\dagger$-closed, unital algebra. It can be decomposed as

$$\mathcal{A} = \bigoplus_j I_{n_j} \otimes \mathcal{M}_{d_j},$$

where the tensor sum is over independent algebras $\mathcal{M}_{d_j}$ of $d_j \times d_j$ matrices, each occurring with multiplicity $n_j$. Following through, the Hilbert space of the system can similarly be decomposed as

$$\mathcal{H}_S = \bigoplus_j C_{n_j} \otimes C_{d_j}.$$ 

An operator $N_\alpha \in \mathcal{A}$ acts on $|a b\rangle$, where $a$ and $b$ denote states according to the above decomposition, to give

$$N_\alpha |a b\rangle = \sum_{b'} M_{b b'}^\alpha |b'\rangle |a b\rangle.$$  

(2)

The matrix $M$ is a rotation of the states labelled by $b$ and does not depend on the label $a$. One sees that the subspaces labelled by states $a$ in $C_{n_j}$ are acted upon with the unit operator by elements of $\mathcal{A}$ so that they are therefore left unchanged during evolution - these subspaces are said to be ‘noiseless’ or ‘decoherence free.’

From another point of view, consider the density matrix $\rho = |a b\rangle\langle a b|$. The action of the operators $N_\alpha$ on $\rho$ is

$$N_\alpha \rho N_\alpha^\dagger = \sum_{b'} \sum_{b''} M_{b'b}^\alpha |a b'\rangle \langle a b''| M_{b'b''}^\alpha \dagger.$$ 

Tracing out the subsystem spanned by the $|b\rangle$ states gives

$$\text{Tr}_B N_\alpha \rho N_\alpha^\dagger = |a\rangle \langle a|,$$

(3)

which is also equal to $\text{Tr}_B \rho$, the partial trace of the original density matrix. This further shows that the subspace spanned by states $|a\rangle$ (density matrices $|a\rangle \langle a|$) is left invariant by the noise operations. Note that if a density matrix $\rho$ is not in the special product form $|a b\rangle\langle a b|$, then $\text{Tr}_B N_\alpha \rho N_\alpha^\dagger \neq \text{Tr}_B \rho$, giving the appearance of non-unitary evolution.

There is an interesting specialization of noiseless subsystems in the event where the algebra $\mathcal{A}$ decomposes so that all the matrix algebras $\mathcal{M}_{d_j}$ are one-dimensional. In this case, the form of operators $N_\alpha$ is

$$N_\alpha |a, b\rangle = p_{a b} |a, b\rangle,$$

(4)

where the phases $p_{a b}$ replace the rotations $M_{b b'}^\alpha$ of (2). The phases can be avoided by using the density matrix formalism,

$$N_\alpha |a, b\rangle \langle a, b| N_\alpha^\dagger = p_{a b} |a, b\rangle \langle a, b| p_{a b}^\dagger = |a b\rangle \langle a b|.$$ 

In such special cases, the operators $N_\alpha$ are called ‘stabilizer’ elements and the invariance is apparent without having to trace out a particular subsystem.
Now, recall that the system Hilbert space is coupled to an environment. Thus a full state can be written as \( \rho = |\psi \phi \rangle \langle \psi \phi | \) where \( \psi \) is a state of the system Hilbert space and \( \phi \) is a state of the environment. Evolution is generated by the unitary operator \( U_{\text{full}} = \exp(i\tau H_{\text{full}}) \). In time \( \tau \), dropping the subscripts, the density matrix changes

\[
\rho \rightarrow U \rho U^\dagger \sim \rho + \frac{\tau^2}{2} \left( 2H \rho H - H^2 \rho - \rho H^2 \right),
\]

the approximation being made in the limit of short evolution times. For the case of the interaction Hamiltonian (1), consider the operators \( N \) to act as \( N |\psi \rangle = |\psi' \rangle \) and \( N |\psi' \rangle = |\psi'' \rangle \). Then, after a short evolution and after tracing out states of the environment, one has

\[
|\psi \rangle \langle \psi | \rightarrow |\psi \rangle \langle \psi | + \frac{\tau^2}{2} \left( 2|\psi' \rangle \langle \psi' | - |\psi'' \rangle \langle \psi | - |\psi \rangle \langle \psi'' | \right).
\]

The nature of the resulting state depends on the type of initial state that we start with. In the special case where the original state is noiseless and obeys (4) with unit phase, i.e. \( |\psi' \rangle = |\psi'' \rangle = |\psi \rangle \), the terms in the parenthesis cancel and the evolution is trivial. In other cases, the interpretation of the resulting state is less self-evident. For example, when the original state \( |\psi \rangle \) is noiseless and obeys (2), the evolution appears to be trivial only after tracing out a subsystem similarly as in (3). A different situation arises when the original state does not satisfy any of the noiselessness conditions. Then, the result of the evolution is a state that is, generically, mixed and whose ‘extent’ of mixture grows as the evolution progresses (the new terms in density matrix are proportional to \( \tau \)). Thus, from the point of view of the system, noisy states evolve according to a dissipative, non-unitary, dynamics.

Decoherence-free states are of importance for quantum information processing because they can be used to reliably store information for long periods of time. For information processing, however, it is also very important to be able to manipulate or change information in order to perform computations. To this end, it is interesting to define the possible operations that can be applied to a noiseless states without ruining its noiseless feature. These operations are elements of the algebra \( \mathcal{A}' \) of all elements that commute with the interaction algebra \( \mathcal{A} \). That is, an operator \( A' \in \mathcal{A}' \) can be used to manipulate a noiseless state if and only if \( [A, A'] = 0 \) for all \( A \in \mathcal{A} \). For more details on identifying noiseless subsystems via the commutant of the algebra of operators on the system see (4).

**Constrained Systems**

Shifting slightly, we now briefly review the standard method of dealing with quantum constrained systems (see, for example, (4)). The Hilbert space of an unconstrained system is called the kinematical Hilbert space \( \mathcal{H}_{\text{kin}} \). Constraints are represented by operators \( C_a \) that form a closed, first-class algebra \( [C_a, C_b] = f_{ab}^c C_c \) for some structure constants \( f_{ab}^c \). Physical states of the system are defined to be those that satisfy the constraint equations
\( C_a |\psi\rangle_{\text{phys}} = 0 \); the span of these states forms the physical Hilbert space, \( \mathcal{H}_{\text{phys}} \). An important aspect of understanding constrained systems is the construction of the algebra \( \mathcal{D} \) of Dirac observables. Operators in this algebra commute with the constraints and thus measure physical (invariant) properties of physical states. In other words, \( D \) is an observable, \( D \in \mathcal{D} \), if and only if \([D, C_a] = 0\).

There are significant similarities in the algebraic structures that are relevant to the constrained systems and that appear in the discussion of noiseless subsystems. Specifically, in each case one has two distinct algebras that commute. The aim of the present work is to probe this similarity and establish a connection between constrained systems and noiseless subsystems. This is accomplished by constructing a mapping from a constrained system to a noiseless subspace.

Consider a system subject to a set of first-class constraints \( C_a \). Consider also an identity operator \( I \) on the kinematical Hilbert space \( \mathcal{H}_{\text{kin}} \) and define new operators \( N_a \lambda = (I + \lambda C_a) \). Then if \( C_a |\psi\rangle_{\text{phys}} = 0 \), operators \( N_a \lambda \) stabilize physical states for all \( \lambda \), \( N_a \lambda |\psi\rangle_{\text{phys}} = |\psi\rangle_{\text{phys}} \). Thus, an alternative description of the constrained system starts to develop in which \( \mathcal{H}_{\text{kin}} \) can be identified with \( \mathcal{H}_S \) and the new stabilizer elements \( N_a \lambda \) generate the algebra \( \mathcal{A} \). Recall that elements of \( \mathcal{A} \) have the interpretation of being operations that couple the system to an environment. Thus, this approach suggests \( \mathcal{H}_{\text{kin}} \) should be coupled to a new Hilbert space \( \mathcal{H}_B \) representing an environment or bath.

The interaction Hamiltonian (1) for the constrained system and environment will have the form

\[
H_I = \sum_a N_a \otimes B_a = \sum_a (1 \otimes B_a + \lambda C_a \otimes B_a),
\]

for some operators \( B_a \) acting on the environment. Incidentally, the decomposition of the interaction terms on the right hand side make the first term appear as operators acting on the environment only, i.e. being part of \( 1 \otimes H_B \) of (1). Only the terms proportional to the constraints are therefore part of the ‘true’ interaction Hamiltonian,

\[
H_I \rightarrow \sum_a C_a \otimes B_a.
\]

In short, what we now have is a new quantum system with a full Hilbert space \( \mathcal{H}_{\text{full}} = \mathcal{H}_{\text{kin}} \otimes \mathcal{H}_B \) governed by a Hamiltonian of the form (1) with \( H_S \) given by the Hamiltonian of the constrained problem, \( H_B \) given by the operators \( B_a \), and \( H_I \) given by (6).

The noiseless states of this new theory are, by construction, solutions to the constraints \( C_a \) that we started with. They therefore exhibit all the physical properties that the solutions to the constrained problem do. Since the environment in the quantum information theoretic description is not really of interest from the point of view of the constrained dynamics problem, it should be traced out. As a result, the noiseless states evolve unitarily under the full Hamiltonian while the noisy states, which do not satisfy the constraint equations, decay non-unitarily and as such are not of physical interest.
The commutant $\mathcal{A}'$ in the noiseless subspace picture is the set of all operators that commute with the constraints $C_a$. Thus, there is also a close correspondence between $\mathcal{A}'$ and $\mathcal{D}$, up to the status of the unit operator. The unit is always a Dirac observable and is thus in $\mathcal{D}$. On the noiseless subsystem side, however, the unit operator is also included in the algebra $\mathcal{A}$ (recall that $\mathcal{A}$ is assumed unital). The interesting correspondence, therefore, should be made between the non-trivial elements of $\mathcal{A}'$ and the non-trivial Dirac observables.

III. EXAMPLES

The transition from constrained dynamics to noiseless subsystems is developed further in the following series of examples, starting from a straight-forward non-relativistic particle subject to a linear constraint, through gauge theory and the relativistic particle, and culminating in a discussion of quantum gravity.

Momentum Constraint

A very simple example of a constrained system is a classical non-relativistic particle moving in two spatial dimensions $x$ and $y$ under the restriction that its $y$ momentum be zero. The particle is described by a Hamiltonian

$$H = \left( \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \beta C \right)$$

that consists of kinetic terms and the constraint $C$ with its Lagrange multiplier $\beta$. For this example, $C = p_y$. The discussion below of the quantum version is only an outline but fully conveys the spirit of the mapping from constrained dynamics problems to noiseless subsystems.

The kinematical Hilbert space for the particle is taken to be the momentum space where states are labelled $|\psi\rangle = |p_x, p_y\rangle$. The constraint requires that $p_y|p_x, p_y\rangle = 0$. Intuitively, the states that make up the physical Hilbert space are $|p_x, 0\rangle$. Now consider the description of the particle motion from the noiseless subspace point of view. In this view, the full system is composed of the particle coupled to an environment. We define an identity operator $I$ such that $I|\psi\rangle = |\psi\rangle$ for all $|\psi\rangle$. Then, if a state satisfies $p_y|\psi\rangle = 0$ it also satisfies $(I + p_y\lambda)|\psi\rangle = |\psi\rangle$ for all $\lambda$. Thus, $N_\lambda = (I + p_y\lambda)$ is a stabilizer of physical states. (Of course, the operator $I$ by itself is also a stabilizer of physical states, but it is not interesting since it also stabilizes non-physical states.)

According to the noiseless subsystems approach, the particle is coupled to an environment. The full system of the particle and environment evolve under the Hamiltonian

$$H_{full} = \left( \frac{p_x^2}{2m} + \frac{p_y^2}{2m} \right) \otimes 1 + 1 \otimes B + p_y \otimes B,$$
where $B$ is some unspecified unitary operator that describes the evolution of the environment as well as the coupling of the particle to the environment. It is interesting to note that the states $|p_x, 0\rangle$ are not the only ones that can be considered as noiseless. Other momentum eigenstates, such as $|p_x, p_y\rangle$ with arbitrary $p_y$ can also be used. Operators $N_\lambda$ act on such states as

$$N_\lambda |p_x, p_y\rangle = |p_x\rangle \otimes (1 + P_y)|p_y\rangle = (1 + p_y)|p_x, p_y\rangle,$$

which is a (unnormalized) form of of (4) with non-trivial phase. However, superpositions of states with different $p_y$ do not satisfy the stabilizer condition and would thus appear to be noisy in evolution. Thus the system can be viewed as containing multiple (indeed, an infinite number of) noiseless subspaces, one for each value of $p_y$. The equal status of all the noiseless subspaces can be traced back to translational symmetry in the original constrained system picture, i.e. the freedom to rewrite the constraint from $p_y = 0$ in one reference frame to another value of $p_y$ in some other reference frame.

In both the constrained dynamics picture and the noiseless subspace picture, operators $D$ such that $[D, C] = 0$ are important. In the constrained dynamics picture, such operators are called Dirac observables. For example, $p_x$ and $x$ can be Dirac observables. In the noiseless subsystem picture, the operators $D$ can be used to manipulate the information stored in the particle state. For example, the operator $p_x$ can be used to read off the $x$ momentum in an eigenstate. As another example, an operator $x$ can be used to shift the momentum of the particle by a certain amount. It should be stressed that these operators could not be used reliably on noisy states.

An attractive feature of this simple example is that the role of the environment, which is fundamental to the description of the particle in the noiseless subsystem picture, can also be understood from the perspective of the constrained system. There, the Lagrange multiplier $\beta$ is interpreted as a force that determines the particle’s momentum. Since the source of the force is external to the particle, the constrained dynamics description of the particle also implicitly makes use of an environment. In some sense, then, the noiseless subsystem worldview can be seen to emphasize a feature that is present but that is often overlooked in the constrained mechanics formalism. These external forces, usually hidden in the constrained dynamics framework, are brought to the fore in the quantum information theoretic description of the system in terms of noiseless subsystems. In fact, the strategy for describing solutions to constrained problems in terms of noiseless subsystems will be to introduce/postulate a new environment system for every Lagrange multiplier and write down suitable interactions to generate the desired noiseless dynamics.
Gauge Theory

As another example, we consider electromagnetism with the action

\[ S = -\frac{1}{4} \int d^4 x \, F_{\mu \nu} F^{\mu \nu} , \]

where \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). The action is invariant under gauge transformations \( A_\mu \to A_\mu + \partial_\mu \alpha \) where \( \alpha \) is any space-time function. The momenta that are conjugate to \( A_\mu \) are \( \pi^0 = 0 \) and \( \pi^i = F^{0i} \). The Hamiltonian for the theory is

\[ H = \int d^3 x \left( \frac{1}{2} \pi_i \pi^i + \frac{1}{2} F_{ij} F^{ij} - A_0 \partial_i \pi^i \right) . \]

In the last term, \( A_0 \) appears as a Lagrange multiplier imposing the constraint

\[ C = \partial_0 \pi^i = \partial_i \partial^0 A_0 - \partial_i \partial^0 A^i . \]

The effect of the constraint is to reduce the number of physical, propagating degrees of freedom in the vector potential down to two, giving electromagnetism an interpretation as a theory of massless spin one particles (photons) propagating at the speed of light.

Since electromagnetism is a theory of a four-dimensional vector field, the Hamiltonian should be a function of all four components of \( A_\mu \) and their conjugate momenta, \( H = H(A_0, p_0, A_i, p_i) \). The Hamiltonian above, however, is of the special form

\[ H = H_S(\pi_i, A_i) - C(\pi^i, A^i) A_0 \]

whereby \( H_S \) does not depend on the scalar potential \( A_0 \) nor its momentum, and the second term is clearly split into two factors, one of which depends \( A_0 \) and another one that does not. This kind of splitting suggests writing the Hilbert space of electromagnetism as \( \mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B \), treating the the vector potential \( A_i \) as the ‘system’ and the scalar potential \( A_0 \) as the ‘bath.’ The Hamiltonian is therefore composed of a piece \( H_S \otimes 1 \) that evolves the system only and an interaction term \( H_I = C \otimes A_0 \).

Drawing on the earlier general discussion, we would like the system part of the interaction term to act as a stabilizer on physical states. At the moment, that part of \( H_I \) annihilates physical states. To make electromagnetism match the noiseless subsystems scheme, we replace the existing interaction term by \( H_I \to N \otimes A_0 \) where \( N = 1 + C \). The effect of this exchange is to introduce a new term into the Hamiltonian:

\[ H = H_S(\pi_i, A_i) - C(\pi^i, A^i) A_0 - A_0 . \]

Now the noiseless states in the system part, after tracing out the scalar potential degrees of freedom, should be exactly the ones that correspond to the transverse polarizations of photons.
The physical picture that emerges from the noiseless subsystem framework is as follows. The full set of fields \((A_0, A_i)\) evolves according to a well defined Hamiltonian which treats the vector potential as the ‘system’ and the scalar potential as the environment. Due to the interaction term, only certain states of vector potential are noiseless. So only those states can be expected to be preserved in the ‘long term’ and be seen/observed/detected in the laboratory - photons are interpreted as excitations in the noiseless sector of the vector potential. Having experience dealing only with the noiseless states, we are usually inclined to describe the experimental results using a theory that is re-parametrization invariant, i.e. a gauge theory. However, if experiments could give access to the full spectrum of states as opposed only to the noise-free ones, in effect allowing us to measure the scalar potential directly, then the gauge invariance would not be a true symmetry of the full system. This is reminiscent of the ‘random dynamics’ research program where gauge symmetry is emergent.

The extra term in the Hamiltonian has a similar effect to introducing a gauge fixing condition. This is not a problem in the present sense because, at the same time, we postulate that the scalar potential is not observable and focus the discussion on noiseless states in the vector potential Hilbert space; tracing out the scalar potential gets rid of the rigidity of the fixed gauge. At the end, the quantum information theoretic description ends up with the same physical solutions as the original gauge theory. In this sense we can say that gauge invariance is an emergent property in the noiseless states.

**Relativistic Particle**

Another familiar example of a constrained system is the relativistic particle moving in a flat background (metric \(\eta_{\mu\nu} = \text{diag}(-1,1,1,1)\)). Its Lagrangian is given by

\[
L = -m\sqrt{-\eta_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu}}.
\]

The overdot denotes time derivatives \(\partial/\partial \tau\). The conjugate momenta to \(q_{\mu}\) are \(p_{\mu} = (\partial L)/(\partial \dot{q}^{\mu}) = (m^2 \dot{q}_\mu)/(L)\), and the standard Hamiltonian \(H = p_{\mu} \dot{q}^{\mu} - L = 0\) vanishes due to time-reparametrization invariance of the Lagrangian. The system is instead characterized by the constraint

\[
C = -\dot{p}_0^2 + p_i^2 + m^2,
\]

which puts the relativistic particle on-shell. The total Hamiltonian therefore consists of only this constraint,

\[
H = \beta C,
\]

where \(\beta\) is a Lagrange multiplier. To obtain the physical states of the particle, consider first \(H_{\text{kin}}\) to be the space of wavefunction of four momentum variables, spanned by the states \(|p_0, p_i\rangle\). Due to the constraint, only three of the four momenta can be independent. It is
convenient to choose $p_0$ as the dependent variable. States

$$|\psi\rangle_{\text{phys}} = |p_0 \text{phys}, p_i \rangle \quad p_0 \text{phys} = \sqrt{p_i^2 + m^2}$$

satisfy the constraint, $C|\psi\rangle_{\text{phys}} = 0$.

To view the relativistic particle from a noiseless subsystem point of view, we consider the algebra $A$ generated by the constraint $C$ and the identity operator. We define operators $N_\lambda = 1 + \lambda C$ acting on the system Hilbert space spanned by states labelled by four-momenta. The full Hamiltonian describing the evolution of the particle coupled to the environment is defined as

$$H_{\text{full}} = 1 \otimes B + C \otimes B,$$

in analogy to (1) but with $H_S = 0$ due to the actual Hamiltonian of the relativistic particle being zero in the constrained dynamics picture. The particle and the environment evolve in the usual way via an evolution operator $U = \exp(i\tau H_{\text{full}})$; the evolution is naturally parametrized by a new external time variable $\tau$.

In the quantum information theoretic picture, a generic initial state evolves into a totally mixed background with particle-like excitations. This situation can be compared to the description of signals in liquid-state Nuclear Magnetic Resonance experiments. There, a liquid sample is viewed as consisting of many small randomly-oriented spins. When a low-frequency pulse is applied, the spins tilt slightly with the effect of generating a net magnetization in the sample. An effective density matrix can be used to describe the state of the sample which is composed of a sum of a total mixed state and a small particle contribution. Remarkably, the particle contribution actually behaves like a real particle and can even be successfully employed in experimental quantum information processing [6]. The proposal here is to view the relativistic particle in a similar manner - as an excitation over a noisy background.

Note that noiseless states evolve as if the Hamiltonian were zero exactly as in the original constrained system. Thus, these states exhibit an emergent time-reparametrization invariance property. However, in the noiseless subsystems picture, the ‘true’ Hamiltonian is actually $H_{\text{full}}$ and is nonzero. There is no ‘problem of time’ as the evolution of the environment provides a well defined clock to measure time flow by. This is another novel feature introduced by the noiseless subsystems viewpoint, and it may be helpful in reducing the conceptual difficulties that arise in the study of time re-parametrization invariant systems.

The proposed viewpoint, in a sense, is orthogonal to the much discussed relational approach (see for example [10, 11]) where the introduction of a background time is seen as something that should be avoided. Of particular interest is the work of Poulin [11] who uses noiseless subsystems and quantum information theoretic tools to write a relational formulation of quantum theory that is originally expressed in terms of a background time. In contrast, we argue in the reverse direction that the relational features usually ascribed to physical systems such as the relativistic particle can be understood as arising out of a non-relational theory of the system under consideration coupled to an environment.
Relationalism can be restored, however, by considering this non-relational theory as part of another, larger relational theory. Then, density matrices form a hierarchy

$$\rho_{rel} \leftrightarrow \rho_{non\,rel} \leftrightarrow \rho_{new\,rel}$$

where $\rho_{rel}$, $\rho_{non\,rel}$ and $\rho_{new\,rel}$ describe, respectively, the usual relativistic particle, the particle together with the environment, and the particle together with an environment as well as another auxiliary system (a clock). The transition between a density matrix in a large Hilbert space to a density matrix in a smaller space is performed by tracing out the redundant degrees of freedom. The bottom line is that the fixed background structure of the environment can be treated in a relational manner if relationalism is desired.

Quantum Gravity

As a final example, we consider the quantization of general relativity. As is well known, gravity, like the relativistic particle, is a totally constrained system [7]. The Hamiltonian is a sum of first-class constraints,

$$H = \int_{\Sigma} A^i_0 G^i + N_a D_a + ND.$$ 

The integral is taken over a three-dimensional manifold $\Sigma$. The Lagrange multiplier $A^i_0$ is of the kind appearing in the gauge theory example and implements the Gauss constraint $G^i$. The lapse function $N$ that implements the Hamiltonian constraint $D$ is akin to the Lagrange multiplier appearing in the relativistic particle example. The remaining multipliers $N_a$ that implement the three diffeomorphism constraints $D_a$ are characteristic to this example, but they can be treated using similar methods.

Multiple constraints in the gravity Hamiltonian can be treated in sequence and on an individual basis. By this we mean that, in general, each constraint (suppose there are $n$ of them) can have a separate environment Hilbert space $\mathcal{H}_{B_n}$ associated with it giving $\mathcal{H}_{full} = \mathcal{H}_S \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_n}$. Solutions to the $n$-th constraint can be found in the space $\mathcal{H}_S \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B(n-1)}$ (or its dual) as the noiseless states with respect to the appropriate coupling. After having characterized the solutions/noiseless states of this constraint, another constraint can be considered to further restrict the set of states that are of physical significance, and so on until all the constraints are taken care of. At each step, the size of the Hilbert space decreases until one finally determines the noiseless states in the original system Hilbert space $\mathcal{H}_S$.

If we are interested in the exact solutions of the constraints, it is of no significance whether the characterization of the solutions is done via standard methods or via quantum information theoretic tools. In particular, we can simplify our discussion of quantum gravity by using the well-known result that states invariant under gauge transformations and spatial diffeomorphisms can be labelled by spin networks [8]. To formally obtain solutions to full
quantum general relativity, then, we should couple the spin-networks to an environment and define the interaction Hamiltonian in terms of the Hamiltonian constraint. The noiseless spin networks in this scheme are the physical solutions of interest; it is likely that these noiseless states would have proper descriptions in terms of classical spacetimes. Unfortunately, the quantum information theoretic approach does not make the problem of actually writing down simple expressions for these states any easier than in the standard Dirac quantization program. A perhaps promising feature of the noiseless subsystem approach, however, is that these physical states should appear dynamically as invariant states out of a generic initial state of a system and environment.

Backtracking to the core picture of coupling the kinematical Hilbert space of gravity to an environment, observe that the symmetries such as gauge-invariance, diffeomorphism-invariance, and time re-parametrization invariance are not fundamental features of the full system comprising the various environments. In the quantum information theoretic picture, states in the full Hilbert space spanned by gravitational and bath degrees of freedom act as if they were coupled to a fixed space and time background. Thus an observer having access to the full Hilbert space can follow the evolution of a gravity state using a set of external variables using the methods of standard quantum mechanics. It is only the process of ignoring, or tracing out the background environment that reproduces the background-independent features of general relativity. Tracing out the environment and focusing attention on the noiseless states is of course motivated by observations of a four-dimensional universe obeying the Einstein’s equations to a high degree of accuracy.

Adding an environment to the universe is certainly a strange move with interpretational issues if the quantum theory of gravity is simply the quantization of the known gravity and matter. In that case the noisy states are unphysical. However, the situation is different in quantum gravity approaches, such as condensed matter approaches, in which general relativity is expected to be an effective theory describing the behavior of the low energy excitations of an underlying system. In that case, an environment and usually a true Hamiltonian is already present in the fundamental theory and the question is how a constrained theory can arise at the effective level. There are similar questions in Causal Dynamical Triangulations, in which the full theory has a time parameter. In this note we wish to suggest that general relativity may be the noiseless sector of the underlying quantum theory of gravity.

IV. DISCUSSION

The main result of this work is the connection of physical states of constrained systems to noiseless subsystems of quantum systems coupled to an environment. An explicit and simple construction is provided to map a system subject to first class constraints to another system that interacts with an external environment, where the interactions provide a mechanism for implementing the symmetries of the constrained system as noiseless states. The equivalence
of the two formalisms can be seen in the fact that the algebra of Dirac observables of the constrained system is isomorphic to the algebra of non-trivial unitary operations that can be performed on the new coupled system.

The relation of constrained systems to noiseless subsystems offers a fresh perspective on the status of several important symmetries such as gauge invariance in electromagnetism and quantum gravity, and time-reparametrization invariance in the relativistic particle and quantum gravity. Indeed, the quantum information theoretic point of view implies that symmetries of physical states may arise, or emerge, dynamically in noiseless states. These states are defined by interactions of a system with an environment and as a result of ignoring the evolution of that environment. In the case of the Gauss constraint (in electromagnetism as well as in quantum gravity), the environment can be economically thought of as being the scalar potential. In the case of other constraints, such as the Hamiltonian constraint (in the relativistic particle and quantum gravity), the environment should be thought of as a truly external object. In all cases, however, symmetries of physical states generated by the constraints loose their fundamental status and are in fact not present in generic states of the full system including the environment.

Emergence of symmetries is a concept that has been already studied from various perspectives and discussed in the context of quantum gravity. Notably, the notion of emergence is a natural one in the context of condensed matter and has been used to study many aspects of gravitation \[12\] including the role of the cosmological constant in quantum gravity \[13\]. A well known result supporting the emergence approaches is the equivalence of Euclidean quantum theory in the Feynman path integral formulation and statistical mechanics. Other indicative works reveal connections between general relativity and thermodynamics \[14\], and quantum features in Bohmian mechanics to the concept of equilibrium \[15\]. Discussions of emergent features can also be found in the literature on causal sets \[16\], causal dynamical triangulations \[17\], string-net condensation \[18\], and quantum information theory \[3, 19\]. Emergent gauge symmetry is also discussed in lattice theories \[9\].

Although the notion of an ‘environment of the universe’ that arises in our discussion of quantum gravity may sound slightly offbeat, it should be noted that the idea of coupling gravitational degrees of freedom to external systems is in fact in common use under different names. In quantum cosmology, for example, the Hilbert space of an FRW spacetime can be coupled to a scalar field to model inflation. Coupling gravity to a scalar field is also a useful technique for introducing clock variables to be employed in defining Dirac observables for quantum gravity \[20\]. The novelty in this work is the proposal for a concrete kind of coupling between the gravitational and external Hilbert spaces given in terms of the stabilizer operators, and the resultant dynamical emergence of physical states as noiseless subsystems during evolution.

The introduction of an environment can be criticized as bringing a background to theories such as the relativistic particle or gravity that are otherwise thought of as relational and background independent. We can defend the environment against this argument in two
ways. First, note that the presence of a constraint in classical mechanics implies that there is an external force acting on the system. Making a straight-forward generalization that all constraints arise as a result of an interaction with an external source, the presence of an environment actually appears quite natural. Second, it is known that a non-relational theory can be mapped into a relational one, for example by enlarging the configuration space by some clock variables. Thus, the fixed-background formulation of the environment can in principle be generalized to obtain a relational version of the noiseless subsystems-based theory. The above criticism, therefore, is not a fundamental obstacle and it should not deter us from considering theories in which symmetries arise dynamically out of interactions with a fixed structure. On the contrary, this formalism is suggestive of the existence of recently proposed dualities between background-independent and background-dependent theories \[21\]. An important benefit is the resulting true Hamiltonian instead of the constraint, a feature that is expected to bring the low-energy problem for quantum gravity theories to a level similar to that of ordinary condensed matter systems.

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