A NEW EXTRAPOLATION METHOD FOR WEAK APPROXIMATION
SCHEMES WITH APPLICATIONS

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Abstract. We review Fujiwara’s scheme, a sixth order weak approximation scheme for the
numerical approximation of SDEs, and embed it into a general method to construct weak ap-
proximation schemes of order \(2^m\) for \(m \in \mathbb{N}\). Those schemes cannot be seen as cubature schemes,
but rather as universal ways how to extrapolate from a lower order weak approximation scheme,
namely the Ninomiya-Victoir scheme, for higher orders.

1. Introduction

The Ninomiya-Victoir scheme for the weak approximation of solutions of stochastic differential
equations can be described in the following framework: let \((\Omega, \mathcal{F}, P)\) be a probability space
and let \(\{B^1_t, \ldots, B^d_t\}_{t \in \mathbb{R}^+}\) be a \(d\)-dimensional standard Brownian motion. Define \(B^0_t := t\) and
\(B_t := (B^0_t, B^1_t, \ldots, B^d_t)\). We consider stochastic differential equations driven by the Brownian
motion \(\{B_t\}_{t \in \mathbb{R}^+}\)

\[
X(t, x) = x + \sum_{i=0}^{d} \int_0^t V_i(X(s, x)) \circ dB^i_s
\]

where \(x\) is in \(\mathbb{R}^N\), \(V_i \in C^\infty_b(\mathbb{R}^N; \mathbb{R}^N)\) and \(\circ\) stands for Stratonovich integral. We associate for
later use the following simple stochastic differential equations to equation (1)

\[
X^{(i)}(t, x) = x + \int_0^t V_i(X^{(i)}(s, x)) \circ dB^i_s.
\]

Let \(\{P_t\}_{t \in \mathbb{R}^+}\) and \(\{P^{(i)}_t\}_{t \in \mathbb{R}^+}\) be the associated heat semigroups on \(C^\infty_b(\mathbb{R}^d)\) such that \(P_tf(x) := E[f(X(t, x))]\) for \(t \geq 0\), and \(P^{(i)}_tf(x) := E[f(X^{(i)}(t, x))]\) for \(t \geq 0\). Notice here that the equation
associated to the index 0 is a pure drift equation, the semigroup a transport semigroup. Denote
furthermore by

\[
\mathcal{A} := V_0 + \frac{1}{2} \sum_{i=1}^{d} V^2_i,
\]

\[
\overline{Q}^0_t := (P^{(0)}_{t/\theta} \circ \cdots \circ P^{(d)}_{t/\theta})^\theta,
\]

\[
\overline{Q}^0_{\bar{t}} := (P^{(d)}_{t/\theta} \circ \cdots \circ P^{(0)}_{t/\theta})^\theta,
\]

\[
\overline{Q}^0_{t/\theta} := \frac{1}{2} (\overline{Q}^0_{\bar{t}} + \overline{Q}^0_{t/\theta}).
\]

the generator of the diffusion process (1), two ordered products of (semi-)flows with generators
\(V_0\) and \(V^2\) and the average of the two ordered products \(Q^0_t\). Then we have the well-known short
time asymptotics, formulated in the language of \(k\)-norms (see Definition 4)

\[
|P_tf(x) - Q^0_t g(x)| \leq Ct^3 \|g\|_{_6(d+1)}
\]
as \(t \to 0\), leading – by iteration – to the Ninomiya-Victoir scheme. Indeed, when we define \(n\)-fold
iteration of the operator \(Q^0_{t/\theta}\)

\[
Q^0_{T, n} = Q^0_{t/\theta} \circ \cdots \circ Q^0_{t/\theta},
\]

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we obtain a scheme of weak approximation order \( r = 2 \), i.e.,
\[
|P_T g(x) - Q_{T,n}^m g(x)| \leq \frac{C}{n^2} \|g\|_{69(d+1)}.
\]

Let us define formally weak approximations of \( P_T \) for some fixed, finite \( T \in \mathbb{R}_+ \) of weak approximation order \( r \).

**Definition 1** (scheme of weak approximation order \( r \)). A family of linear operators \( \{Q_{T,n}\}_{n \in \mathbb{N}} \) on \( C_b^\infty(\mathbb{R}^d) \), continuous with respect to the supremum norm topology, is called a scheme of weak approximation order \( r \) if there exists \( C > 0 \) and some number \( k \geq 0 \) such that
\[
(P_T f(x) - Q_{T,n} f(x)) \leq \frac{C}{n^r} \|f\|_k
\]
for all \( x \in \mathbb{R}^N \) and all \( f \in C_b^\infty(\mathbb{R}^d) \).

Notice that the operator \( Q_{T,n} \) is only supposed to be linear and continuous with respect to the supremum norm topology on the set of \( C_b^\infty \)-function, but not necessarily of sub-Markovian type. This means in particular that classical (Romberg-)extrapolations belong to this class.

In [5] T. Fujiwara constructs a sixth order scheme for smooth functions \( C_b^\infty(\mathbb{R}^N) \) which consists of a linear combination of the previously described Ninomiya-Victoir scheme. Through the linear combination \( T \). Fujiwara can “extrapolate” the weak approximation order to \( r = 6 \). In this paper, we define generalized Fujiwara schemes of order \( r = 2m \) including the scheme in [5] by refining Fujiwara’s technique to prove the convergence order and construct versions of weak approximation order \( r = 2m \) for \( m \in \mathbb{N} \). We finally obtain the following Theorem 4, whose proof can be found in Section 4. Notations can be found in the subsequent sections:

Let \( \{g_n\}_{n \in \mathbb{N}} \) be a generalized Fujiwara scheme of order \( 2m \), then
\[
Q_{T,n} := \sum_{i=1}^m f_{\theta_i} (Q_{T,\theta_i})^n
\]
for \( n \geq 0 \) is a scheme of weak approximation of order \( 2m \), where a choice of \( k \) is given by
\[
k = 2(2m + 1)(d+1) \sum_{i=1}^m \theta_i,
\]
that means
\[
|P_T g(x) - Q_{T,n} g(x)| \leq \frac{C}{n^{2m}} \|g\|_k
\]
for test functions \( g \in C_b^\infty(\mathbb{R}^N) \).

The remainder of the article is organized as follows: in Section 2 we introduce all algebraic prerequisites, in Section 3 we show the main algebraic result of this article, which is then applied in Section 4 to prove the existence of generalized Fujiwara schemes. In Section 5 we provide an implementation result, where the results can be compared to [10]. The appendix is devoted to an original proof of Fujiwara’s basic algebraic result.

2. Algebraic Prerequisites and Their Relation to Weak Approximation

Let \( A \) be a set whose elements are \( a_0, \ldots, a_d \). We call \( A \) an alphabet and \( a_0, \ldots, a_d \) letters. A word in alphabet \( A \) is a finite sequence of letters. Let 1 be an empty word and \( A^* \) a set of words including 1. If we impose a total ordering on \( A \), then \( A^* \) together with word concatenation and lexicographic ordering becomes an ordered unital semigroup. Let \( \mathbf{R}(A) \) be a set of noncommutative polynomials on \( A^* \) over \( \mathbf{R} \) i.e. a set of \( \mathbf{R} \)-linear combinations of elements of \( A^* \) and let \( \mathbf{R}(\langle A \rangle) \) be a set of noncommutative series of elements of \( A^* \) with coefficients in \( \mathbf{R} \), i.e. a set of functions \( f : A^* \rightarrow \mathbf{R} \) with well ordered support. Using componentwise addition and multiplication, which is induced by word concatenation, makes \( \mathbf{R}(\langle A^* \rangle) \) a \( \mathbf{R} \)-algebra (see [4] for more details). The degree of a monomial is a number of letters contained in the monomial and the degree of a noncommutative polynomial and a noncommutative series are the maximum degree of monomials contained in them. Let \( \mathbf{R}(\langle A \rangle)_m \) and \( \mathbf{R}(\langle A \rangle)_{\leq m} \) be the set of homogeneous polynomials of the degree \( m \) and the set of polynomials of the degree less or equal to \( m \) respectively. Define \( \mathbf{R}(\langle A \rangle)_m \) and \( \mathbf{R}(\langle A \rangle)_{\leq m} \) in the same manner. Since every \( u \in \mathbf{R}(\langle A \rangle) \) has a well ordered support, we can define \( \mathbf{R}(\langle A \rangle)_{>m} = \{ u \in \mathbf{R}(\langle A \rangle) | \deg(\inf(\text{supp}(u))) > m \} \) and \( \mathbf{R}(\langle A \rangle)_{\geq m} = \{ u \in \mathbf{R}(\langle A \rangle) | \deg(\inf(\text{supp}(u))) \geq m \} \) and it
is easy to see that $R\langle\langle A\rangle\rangle_{\geq m}$ and $R\langle\langle A\rangle\rangle_{= m}$ are double sided ideals in algebra $R\langle\langle A\rangle\rangle$. Let $j_m$ and $j_{\leq m}$ be the natural surjective maps from $R\langle\langle A\rangle\rangle$ onto $R\langle\langle A\rangle\rangle_{\geq m}$ and $R\langle\langle A\rangle\rangle_{\leq m}$ respectively.

Since every subset of $A^*$ has a least element regarding lexicographical ordering, we have $R\langle\langle A\rangle\rangle = R^{A^*}$. The set $A^*$ is countable, therefore taking metric topology in $R$ makes $R^{A^*}$ with induced product topology into a Polish space. Hence, we can consider its Borel $\sigma$-algebra $\mathcal{B}(R\langle\langle A\rangle\rangle)$, $R\langle\langle A\rangle\rangle$–valued random variables and expectations, and other notions as usual.

For $u \in R\langle\langle A\rangle\rangle$ we define the exponential map

$$\exp (u) := \sum_{n \geq 0} \frac{u^n}{n!},$$

and for $u \in R\langle\langle A\rangle\rangle$ with vanishing constant term, we define the logarithm,

$$\log (1 + u) := \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} u^n.$$

It is easy to check that

$$\log (\exp (u)) = u,$$

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on the respective domains. For $\theta \in \mathbb{N}$ define

$$p : = \exp (\sum_{i=0}^{d} a_i),$$

$$q^{\theta} : = \left( \exp \left( \frac{1}{\theta} a_0 \right) \cdots \exp \left( \frac{1}{\theta} a_d \right) \right)^\theta,$$

$$\overline{q}^{\theta} : = \left( \exp \left( \frac{1}{\theta} a_d \right) \cdots \exp \left( \frac{1}{\theta} a_0 \right) \right)^\theta,$$

$$q^{\theta} : = \frac{1}{2} (q^{\theta} + \overline{q}^{\theta}).$$

Let us make the substitution, which is the heart of the transfer from algebra to numerical schemes, $a_0 = V_0, a_1 = V^2_1/2, \ldots, a_d = V^2_d/2$ formally correct. Let $B$ be another alphabet including $v_0, v_1, \ldots, v_d$ and set $B^*, R\langle\langle B\rangle\rangle, \ldots$, in the same manner. For all $t \in R_+$ define an algebra homomorphism $\Psi_t : R\langle\langle A\rangle\rangle \rightarrow R\langle\langle B\rangle\rangle$ by setting

$$\Psi_t (a_0) := tv_0,$$

$$\Psi_t (a_i) := tv^2_i/2.$$

for all $i \in \{1, \ldots, d\}$.

Define next an algebra homomorphism $\Phi : R\langle\langle B\rangle\rangle \rightarrow C^\infty_b (R^N; R^N)$ by setting

$$\Phi (v_i) = V_i.$$  

Let $D = \{ \sum_{w \in B^*} a_w w | \sum_{w \in B^*} a_w \Phi (w) \text{ is well defined } \}$. Clearly, $R\langle\langle B\rangle\rangle \subset D$ and $D$ is a $R$–subalgebra of $R\langle\langle B\rangle\rangle$. The homomorphism $\Phi$ can then be uniquely extended to an $R$–algebra homomorphism $\Phi : D \rightarrow C^\infty_b (R^N; R^N)$.

The algebra of non-commutative words plays a major role in the analysis of weak approximation schemes due to the following well-known asymptotic expansion theorem, which allows to approximate the truncated exponential series in $A$ by other simpler expressions.

**Theorem 1.** For all function $f \in C^\infty_b (R^N)$, $x \in R^N$ and $n \in \mathbb{N}$, it holds that

$$P_n f(x) = \sum_{k=0}^{n} \frac{t^k}{k!} A^k f(x) + O(t^{n+1}) = \Phi(\Psi_t(j_{\leq n} p))f(x) + O(t^{n+1}).$$

as $t \rightarrow 0$.

**Proof.** See [6].
Hence we can, e.g., express the generator $\mathcal{A}$ of the diffusion process (1) by

$$\Phi(\Psi_1(a_0 + \ldots + a_d)) = \mathcal{A},$$

in particular we obtain the following crucial asymptotic formulas,

$$\Phi\Psi_t(j_{\leq n}(\exp(a_i))) = P^{(i)}_t + \mathcal{O}(t^{n+1})$$

as $t \to 0$ and $i = 0, \ldots, d$ again due to Theorem [1]

To be more precise on the goal of our paper, Theorem[1] also means that if we approximated $p$ by linear combinations of $(q^{[\theta]})_n$ up to a certain degree $2m - 1$ within the algebra $\mathbb{R}\langle A \rangle$ such that the remainder term is of order $\mathcal{O}(\frac{1}{m^n})$, then $P_t f(x)$ could be approximated by linear combinations of $\Phi(\Psi_1((q^{[\theta]}))_n) f(x)$ in a weak sense of order $2m$.

Notice that the letters $a_i$ correspond to squares of vector fields under $\Phi \circ \Psi_t$, hence one has to work out the correspondence to exponentials of first order terms, too. The next lemma shows how to relate those linear semi-flows of PDEs $P^{(i)}_t$ to non-linear flows of ODEs $\mathcal{F}_t(V)(x)$ up to a certain degree $m$, namely by replacing the normal random variable $Z$ by a random variable taking finitely many values and sharing moments up to order $2m$. This finally means that we can approximate $q^{[\theta]}$ by convex combinations of exponentials of first degree terms, i.e. $a_0, \ldots, a_d$ leading to weak approximation schemes.

**Lemma 1.** For all $i \in \{1, \ldots, d\}$ we have that

$$E[\exp(B_i v_i)] = \exp(i \frac{v_i^2}{2})$$

holds true. This formula also holds true under the homomorphism $\Phi \circ \Psi_1$, i.e.,

$$E[\exp(B_i v_i)] = \exp(i \frac{v_i^2}{2})$$

for test functions $f$ and $x \in \mathbb{R}^N$.

**Proof.** Proof by applying the Fourier transform of Brownian motion and classical subordination results. $\square$

3. How to approximate $p$ by $q$?

An alternative proof of this result can be found in the appendix:

**Lemma 2 (Lemma 2.1).** We have

$$\log q^{[1]} = \sum_{i=1}^{\infty} (-1)^{i+1} j_i (\log q^{[1]}).$$

**Proposition 1 (Proposition 2.2).** There exists $c_i \in \mathbb{R}\langle \langle A \rangle \rangle_{\geq 2i+1}$ such that for all $\theta \in \mathbb{N}$,

$$q^{[\theta]} = p + \sum_{i=1}^{\infty} c_i \frac{1}{\theta^{2i}}$$

holds.

**Corollary 1.** Let $q$ be a linear combination of $q^{[\theta]}$ for some $\theta \in \mathbb{N}$. If there exists $n \in \mathbb{N}$ such that $j_{\leq 2n-1}(q) = j_{\leq 2n-1}(p)$, then $j_{\leq 2n}(q) = j_{\leq 2n}(p)$.

**Proof.** For all $\theta \in \mathbb{N}$, $j_{\leq 2}(q^{[\theta]}) = j_{\leq 2}(p)$ holds. Hence, the case $n = 1$ is clear. Suppose $n \geq 2$ and $j_{\leq 2}(q^{[\theta]}) = j_{\leq 2}(p)$. Since $q = \sum_{i=1}^{k} \alpha_i q^{[\theta]}$ for some $\theta_i \in \mathbb{N}$ and since $j_{\leq 2}(q^{[\theta]}) = j_{\leq 2}(p)$ for all $\theta \in \mathbb{N}$, it follows $\sum_{i=1}^{k} \alpha_i = 1$. According to Proposition [1]

$$q = p + \sum_{i=1}^{\infty} c_i \left( \sum_{j=1}^{k} \alpha_j \frac{1}{\theta_{2j}} \right)$$

for some $c_i \in \mathbb{R}\langle \langle A \rangle \rangle_{\geq 2i+1}$. Since $j_{\leq 2n-1}(q) = j_{\leq 2n-1}(p)$, we have

$$\sum_{j=1}^{k} \alpha_j \frac{1}{\theta^{2j}} = 0$$
for all $i = 1, \ldots, n - 1$. Then
\[ q - p = \sum_{i=n}^{\infty} c_i \left( \sum_{j=1}^{k} \alpha_j \frac{1}{q_j^2} \right), \quad \text{where } c_n \in \mathbb{R}_{\langle\langle A \rangle\rangle} \geq 2n + 1, \]
which proves the corollary. \qed

Set
\[ A := \begin{bmatrix} 1 & \cdots & 1 \\ 1/\theta_1^2 & \cdots & 1/\theta_m^2 \\ \vdots & \ddots & \vdots \\ 1/\theta_1^{2(m-1)} & \cdots & 1/\theta_m^{2(m-1)} \end{bmatrix}. \]

**Corollary 2.**
\[ j \leq 2m \left( \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right)^T \begin{bmatrix} \frac{1}{\theta_1^n} - p \\ \vdots \\ \frac{1}{\theta_m^n} - p \end{bmatrix} = 0 \]
holds.

**Corollary 3.** For all $l \in \{1, \ldots, m - 1\}$,
\[ \left( \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right)^T \begin{bmatrix} \frac{1}{\theta_1^n} \\ \vdots \\ \frac{1}{\theta_l^n} \\ \vdots \\ \frac{1}{\theta_m^n} \end{bmatrix} = 0 \]

### 4. Generalized Fujiwara scheme and its property

**Definition 2** (Generalized Fujiwara scheme). A family of series,
\[ \{ q_n := \sum_{i=1}^{m} \hat{f}_i (q^{[\theta_i]})^n \}_{n \in \mathbb{N}} \]
is called a *generalized Fujiwara scheme* of order $2m$ if
\[ f = \begin{bmatrix} f_{\theta_1} & \cdots & f_{\theta_m} \end{bmatrix}^T = A^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \]
holds.

A straightforward calculation involving induction gives the following connection concerning the powers of series in $\mathbb{R}_{\langle\langle A \rangle\rangle}$. Notice that we split the product $q^n - p^n$ into telescoping summands, where one, two up to $m$ terms of the form $q - p$ appear.
Proposition 2. For \( p, q \in R\langle A \rangle \) and \( 2 \leq m \leq n \), we have
\[
q^n - p^n = \sum_{k=0}^{n-1} p^k (q - p) p^{n-k-1} + \sum_{l=2}^{m-1} \left( \sum_{k_1=l-1}^{n-1} \sum_{k_2=1}^{k_1-1} \ldots \sum_{k_m=1}^{k_{m-1}-1} p^{k_1} (q - p) p^{k_2-k_1-1} (q - p) \times \ldots \right.
\]
\[
\times p^{(k_m-k_{m-1}-1)} (q - p) p^{n-k_m-1} + \sum_{k_m=m}^{n-1} \sum_{k_{m-1}=m-2}^{k_m-1} \ldots \sum_{k_2=1}^{k_1-1} q^{k_1} (q - p) p^{k_2-k_1-1} (q - p) \times \ldots
\]
\[
\times p^{k_{m-k_{m-1}-1}} (q - p) p^{n-k_{m-1}-1}.
\]
In particular for \( m = 1 \),
\[
q^n - p^n = \sum_{k=0}^{n-1} q^k (q - p) p^{n-k-1}
\]
holds true.

Lemma 3. For \( z_1, z_2 \in R\langle A \rangle \), if \( j_{\leq l}(z_1) = 0 \) and \( j_{\leq m}(z_2) = 0 \), then \( j_{\leq l+m+1}(z_1 z_2) = 0 \).

Proof. By the assumption, monomials with the lowest degree contained in \( z_1 \) and \( z_2 \) are of the degree \( l+1 \) and \( m+1 \). Then, monomial with the lowest degree contained in \( z_1 z_2 \) has the degree \( l+m+2 \). Hence \( j_{\leq l+m+1}(z_1 z_2) = 0 \).

Corollary 4. For \( z_1, z_2, z_3 \in R\langle A \rangle \), if \( j_{\leq l}(z_1) = j_{\leq l}(z_2) \) and \( j_{\leq m}(z_3) = 0 \), then \( j_{\leq l+m+1}(z_1 z_3) = j_{\leq l+m+1}(z_2 z_3) \).

Corollary 5. For \( z \in R\langle A \rangle \), if \( j_{\leq l}(z) = 0 \), then \( j_{\leq l+m+1}(z^m) = 0 \).

Theorem 2. If a series,
\[
(15) \quad q_n := \sum_{i=1}^{m} f_{\theta_i} (q^{[i_1, \ldots, i]})^n,
\]
is a 2m-th order generalized Fujiwara scheme, then for all \( l \in \{2, \ldots, m-1\} \),
\[
(16) \quad j_{\leq 2m+l-1} \left( \sum_{i=1}^{m} f_{\theta_i} (q^{[i_1, \ldots, i]})^l \right) = 0
\]
holds true.

Proof. Fix \( l \in \{2, \ldots, m-1\} \). By Proposition \( \Box \)
\[
(17) \quad \sum_{i=1}^{m} f_{\theta_i} (q^{[i_1, \ldots, i]})^l = \sum_{i=1}^{m} f_{\theta_i} \sum_{i_1, \ldots, i_t}^{\infty} \frac{c_{i_1} \cdots c_{i_t}}{\theta^{2(i_1 + \cdots + i_t)}}
\]
holds. It is easy to see that
\[
c_{i_1} \cdots c_{i_t} \in R\langle A \rangle \geq 2(i_1 + \cdots + i_t) + l.
\]
Hence, we have
\[
j_{\leq 2m+l-1} \left( \sum_{i_1, \ldots, i_t=1}^{\infty} \frac{c_{i_1} \cdots c_{i_t}}{\theta^{2(i_1 + \cdots + i_t)}} \right)
\]
\[
= j_{\leq 2m+l-1} \left( \sum_{i_1, \ldots, i_t \geq 1}^{i_1 + \cdots + i_t \leq m-1} \frac{c_{i_1} \cdots c_{i_t}}{\theta^{2(i_1 + \cdots + i_t)}} \right)
\]
\[
= j_{\leq 2m+l-1} \left( \sum_{k=0}^{m-1} \sum_{i_1, \ldots, i_t \geq 1}^{i_1 + \cdots + i_t = k} \frac{c_{i_1} \cdots c_{i_t}}{\theta^{2k}} \right).
\]
Thus, we have
\[
\begin{align*}
&= j_{\leq 2m+l-1}(\sum_{i=1}^{m} f_{\theta_i}(q^{[\theta_i]} - p)^{l}) \\
&= \sum_{k=0}^{m-1} \sum_{i=1}^{m} f_{\theta_i} \frac{1}{\eta^{2k+1}} c_{i_1} \cdots c_{i_k} \\
&= \sum_{k=1}^{m-1} \sum_{i=1}^{m} f_{\theta_i} \left(\begin{array}{c}
\frac{1}{\eta^{2k+1}} \\
\eta^{2k+1}
\end{array}\right) c_{i_1} \cdots c_{i_k} \\
&= 0
\end{align*}
\]
by Corollary 3.

**Definition 3.** A generalized power series \( a \in \mathbb{R} \langle \{A\} \rangle \) is an element of \( O(s) \) if for every \( n, N \in \mathbb{N} \), \( n \leq N \) there exists a uniform bound for all the coefficients of the terms of \( \frac{1}{s} a \) with degree \( k \), which satisfies \( n \leq k \leq N \) as \( s \to 0 \).

**Theorem 3.** If \( \{q_n\}_{n \in \mathbb{N}} \) is an \( m \)-th order generalized Fujiwara scheme, then
\[
\Psi_{1/n} q_n - \Psi_{1/p} = \sum_{l=1}^{m} \sum_{k_1=0, k_2=1, k_3=1, \ldots, k_{m-1}=1, k_m=0}^{n-1} \sum_{k_1=0}^{k_2-1} \sum_{k_2=0}^{k_3-1} \cdots \sum_{k_{m-1}=0}^{k_m-1} \Psi_{1/n} a_{l,(k_1,\ldots,k_m)},
\]
where \( j_{\leq 2m+l-1}(a_{l,(k_1,\ldots,k_m)}) = 0 \) for \( l = 1, \ldots, m \), and \( \Psi_{1/n} q_n - \Psi_{1/p} \in O\left(\frac{1}{n^m}\right) \).

**Proof.** Let \( m \geq 2 \). The case \( m = 1 \) is trivial. Let \( \{q_n := \sum_{i=1}^{m} f_{\theta_i}(q^{[\theta_i]} \}_{n \in \mathbb{N}} \) be an \( m \)-th order generalized Fujiwara scheme. Note that \( \Psi_{1/p} = (\Psi_{1/n} q_n)^{n} \). Then by Proposition 2, we have,
\[
\Psi_{1/n} q_n - \Psi_{1/p} = \sum_{l=1}^{m} \sum_{k_1=0, k_2=1, k_3=1, \ldots, k_{m-1}=1, k_m=0}^{n-1} \sum_{k_1=0}^{k_2-1} \sum_{k_2=0}^{k_3-1} \cdots \sum_{k_{m-1}=0}^{k_m-1} (\Psi_{1/n} q^{[\theta_1]} - \Psi_{1/p})^{n-k_1-1} \\
\times (\Psi_{1/n} q^{[\theta_2]} - \Psi_{1/p})^{n-k_2-1} \cdots (\Psi_{1/n} q^{[\theta_m]} - \Psi_{1/p})^{n-k_m-1}.
\]

Set
\[
a_{1,(k_1)} = \sum_{i=1}^{m} f_{\theta_i} p^{k_1}(q^{[\theta_i]} - p)^{n-k_1-1}.
\]

For \( l \in \{2, \ldots, m-1\} \) set
\[
a_{l,(k_1,\ldots,k_l)} = \sum_{i=1}^{m} f_{\theta_i} p^{k_1}(q^{[\theta_i]} - p)^{k_2-k_1-1}(q^{[\theta_i]} - p) \cdots p^{k_l-k_{l-1}-1}(q^{[\theta_i]} - p)^{n-k_l-1},
\]
and for \( l = m \) define
\[
a_{m,(k_m,\ldots,k_m)} = \sum_{i=1}^{m} f_{\theta_i}(q^{[\theta_i]} p^{k_1-k_1-1}(q^{[\theta_i]} - p) \cdots p^{k_m-k_{m-1}-1}(q^{[\theta_i]} - p)^{n-k_m-1}.
\]

In particular the summand \( a_{1,(k_1)} \) can be written as
\[
(18) \quad a_{1,(k_1)} = p^{k_1} \sum_{i=1}^{m} f_{\theta_i}(q^{[\theta_i]} - p)^{n-k_1-1}.
\]
Let $c_i \in \mathbb{R} \langle \langle A \rangle \rangle_{\geq 2i+1}$, $i \in \mathbb{N}$ be as in Proposition 1. By Theorem 2

$$j_{\leq 2m} \left( \sum_{i=1}^{m} f_{\theta_i}(q^{[\theta_i]} - p) \right) = 0,$$

thus $j_{\leq 2m}(a_{1,(k_1)}) = 0$. Also it holds that

$$j_{2m+1} \left( \sum_{i=1}^{m} f_{\theta_i}(q^{[\theta_i]} - p) \right) = j_{2m+1}(c_m)[1/\theta^2_m, \ldots, 1/\theta^2_m].$$

By Corollary 2 for all $\theta \in \mathbb{N}$, $j_{\leq 2}(q^{[\theta]} - p) = 0$ holds. Thus, by Corollary 3, Corollary 1 and Proposition 1

$$(19) \quad j_{\leq 3m-1}((q^{[\theta]})^{k_1} - p)q^{k_2-k_1-1}(q^{[\theta]} - p)\ldots q^{k_m-k_{m-1}-1}(q^{[\theta]} - p)p^{n-k_m-1} = 0$$

holds, hence $j_{\leq 3m-1}(a_{m,(k_m,\ldots,k_1)}) = 0$. Moreover,

$$j_{3m}((q^{[\theta]})^{k_1} - p)q^{k_2-k_1-1}(q^{[\theta]} - p)\ldots q^{k_m-k_{m-1}-1}(q^{[\theta]} - p)p^{n-k_m-1} = (j_m(1))^{m} \frac{1}{\theta^2_m}.$$

Let $p_1, \ldots, p_{l+1} \in \mathbb{R} \langle \langle A \rangle \rangle$ with property $j_0(p_i) = 1$ for $i \in \{1, \ldots, l+1\}$. By using similar arguments as in the proof of Theorem 2 we get

$$j_{\leq 2m+l-1} \left( \sum_{i=1}^{m} f_{\theta_i}p_1(q^{[\theta_i]} - p)p_2(q^{[\theta_i]} - p)\ldots p_l(q^{[\theta_i]} - p)p_{l+1} \right)$$

$$= j_{\leq 2m+l-1} \left( \sum_{k=l}^{m-1} \sum_{i_1, \ldots, i_l \geq 1, i_1 + \ldots + i_l = k} f^{T} \begin{bmatrix} \frac{1}{\theta^2_1} \\ \vdots \\ \frac{1}{\theta^2_m} \end{bmatrix} p_1c_{i_1}p_2c_{i_2}\ldots p_lc_{i_l}p_{l+1} \right)$$

$$= 0,$$

and

$$j_{2m+l} \left( \sum_{i=1}^{m} f_{\theta_i}p_1(q^{[\theta_i]} - p)p_2(q^{[\theta_i]} - p)\ldots p_l(q^{[\theta_i]} - p)p_{l+1} \right)$$

$$= \sum_{i_1, \ldots, i_l \geq 1, i_1 + \ldots + i_l = m} f^{T} \begin{bmatrix} \frac{1}{\theta^2_1} \\ \vdots \\ \frac{1}{\theta^2_m} \end{bmatrix} j_{2m+l}(c_{i_1}, \ldots, c_{i_l})$$

for all $l \in \{2, \ldots, m-1\}$. We conclude, that $j_{\leq 2m+l-1}(a_{l,(k_1,\ldots,k_1)}) = 0$.

It remains to prove that $\Psi_{1/\theta_{n+1}} - \Psi_{1}p \in O\left(\frac{1}{n^m r^2}\right)$.

First, let us observe $a_{l,(k_1,\ldots,k_1)}$ for $l \in \{1, \ldots, m-1\}$. Choose $M \in \mathbb{N} \cup \{0\}$. As above we can write

$$j_{2m+l+M}(a_{l,(k_1,\ldots,k_1)})$$

$$= j_{2m+l+M} \left( \sum_{i=1}^{m} f_{\theta_i}p_1^{k_1}(q^{[\theta_i]} - p)p_2^{k_2-k_1-1}(q^{[\theta_i]} - p)\ldots p_l^{k_i-k_{i-1}-1}(q^{[\theta_i]} - p)p^{n-k_i-1} \right)$$

$$= \sum_{k=m}^{m+M/2} \sum_{i_1, \ldots, i_l \geq 1, i_1 + \ldots + i_l = k} f^{T} \begin{bmatrix} \frac{1}{\theta^2_1} \\ \vdots \\ \frac{1}{\theta^2_m} \end{bmatrix} j_{2m+l+M}(p^{k_1}c_{i_1}p^{k_2-k_1-1}\ldots p^{k_{i-1}-k_{i-2}-1}c_{i_l}p^{n-k_i-1})$$

The coefficient of the power $p^k$ of the term of degree $l$ is of the form $k^lc$, where $c$ is the coefficient of $p$ of the same degree. Hence, the coefficient of the term of degree $2m + l + M$ of

$$p^{k_1}c_{i_1}p^{k_2-k_1-1}\ldots p^{k_{i-1}-k_{i-2}-1}c_{i_l}p^{n-k_i-1}$$

is a finite sum, namely

$$\sum_{i \in I} k_1^{n_1} (k_2 - k_1 - 1)^{n_2} \ldots (n - k_l - 1)^{n_{l+1}+1} b_i.$$
where $b_i$ and the number of summands do not depend on $n$, and $n_{j,i} \in \mathbb{N}$, $\sum_{j=1}^{t+1} n_{j,i} \leq 2m + M - 2 \sum_{k=1}^{l} i_k$. Let us denote $b'_{t,k} = \left[ \frac{1}{x_i} \ldots \frac{1}{x_m} \right] f b_i$. Thus, the coefficient of a term of degree $2m + l + M$ of

$$\Psi_{1/n} \left( \sum_{k_l=1}^{n-1} \sum_{k_{l-1}=l-2}^{k_{l-1}=l-2} \cdots \sum_{k_2=1}^{k_2=1} \sum_{k_1=0}^{k_1=0} a_{i_1, \ldots, i_1} \right)$$

has the following upper bound

$$\frac{1}{n^{2m+M+l}} \left| \sum_{k_l=1}^{n-1} \sum_{k_{l-1}=l-2}^{k_{l-1}=l-2} \cdots \sum_{k_2=1}^{k_2=1} \sum_{k_1=0}^{k_1=0} b'_{t,k} \right|$$

Let us observe the number of all coefficients at the terms which are derived from $a_{i_1, \ldots, i_1}$ and the number of summands do not depend on $n$, there exists a uniform bound for all of the coefficients, which proves our assertion for $l < m$.

Now we are able – by means of our homomorphisms $\Psi$ and $\Phi$ to transfer the algebraic results into the realm of weak approximation schemes.
Definition 4. Let \( k \in \mathbb{N} \) and let \( g \in C^\infty_b(\mathbb{R}^d) \). Define
\[
\|g\|_k := \sup_{i \leq k} \| \nabla^i g \|_\infty.
\]

Remark 1. The function \( \| \cdot \|_k \) is a norm on \( C^\infty_b(\mathbb{R}^d) \).

Definition 5. Let \( B_k \) denote the space of bounded linear operators on \( (C^\infty_b(\mathbb{R}^d), \| \cdot \|_k) \). We can regard \( B_k \) as a normed space with the operator norm.

Proposition 3 ([1]). Fix a \( k \in \mathbb{N} \). The following assertions hold:

1. The family \( (P_t)_{t \geq 0} \) is a uniformly bounded subset of \( B_k \).
2. Let \( A \) be the generator of the diffusion process \( \Xi \) and let \( N \in \mathbb{N} \). Then, for \( g \in C^\infty_b(\mathbb{R}^d) \) we have
\[
(P_t g)(x) = \sum_{k=0}^{N} \frac{t^k}{k!} (A^k g)(x) + \frac{1}{N!} \int_0^t (t-s)^N (P_s A^{N+1} g)(x) q, ds.
\]

Theorem 4. Let \( \{q_n\}_{n \in \mathbb{N}} \) be a generalized Fujiwara scheme of order \( 2m \), then
\[
Q_{T,n} := \sum_{i=1}^{m} f_{\theta_i}(Q_{\frac{\theta_i}{m}})^n
\]
for \( n \geq 0 \) is a scheme of weak approximation of order \( 2m \), where a choice of \( k \) is given by
\[
k = 2(2m+1)(d+1) \sum_{i=1}^{m} \theta_i,
\]
that means
\[
|P_T g(x) - Q_{T,n} g(x)| \leq \frac{C}{n^{2m}} \|g\|_k
\]
for test functions \( g \in C^\infty_b(\mathbb{R}^N) \).

Proof. Due to asymptotic formulas
\[
\Phi \Psi_t(j \leq 2m)(p) = P_t + \mathcal{O}(t^{2m+1})
\]
and
\[
\Phi \Psi_t(j \leq 2m)(\exp(a_t)) = P_t + \mathcal{O}(t^{2m+1})
\]
where the constants in the Landau symbol depend on the derivatives of order at most \( 2(2m+1) \).

Remark: We can simply copy the proof of Theorem 4 by first replacing \( q_n \) with \( Q_{T,n} \) and \( p \) by \( P_T \). In the appearing sums we have to use the previous asymptotic formulas, namely
\[
(Q_{\frac{\theta_i}{m}})^n - P_{\frac{\theta_i}{m}} = (Q_{\frac{\theta_i}{m}})^n - \Phi \Psi_{\frac{\theta_i}{m}}(j \leq 2m)((Q_{\frac{\theta_i}{m}})^n) + \\
+ \Phi \Psi_{\frac{\theta_i}{m}}(j \leq 2m)((Q_{\frac{\theta_i}{m}})^n - p) + \\
+ \Phi \Psi_{\frac{\theta_i}{m}}(j \leq 2m(p) - P_{\frac{\theta_i}{m}})
\]
where the order behavior of the middle part has been shown in Theorem 4 and the order behavior of the other two summands follows from the previous asymptotic formulas. Apparently each term in \( Q_{\frac{\theta_i}{m}} \), which is approximated due to the asymptotic formulas, increases the number of derivatives necessary to do the estimation by \( 2(2m+1) \), which leads to the formula for \( k \).

Example 1. The case \( m = 1 \) apparently corresponds to a version of the original Ninomiya-Victoir scheme.
Example 2. The case \( m = 2 \) corresponds to a scheme already presented in [5]. One can choose \( \theta_1 = 1 \) and \( \theta_2 = 2 \) and \( f_{\theta_1} = -\frac{1}{3} \) and \( f_{\theta_2} = \frac{1}{3} \).

Example 3. The case \( m = 3 \) corresponds to Fujiwara’s originally presented scheme, which in our language reads like follows. Notice that we do not need the full strength of our previous proof, which is built on Theorem 2.

For all mutually different numbers \( \theta_1, \theta_2, \theta_3 \in \mathbb{N} \), we can construct 6-th order generalized Fujiwara scheme \( q \) with a form:

\[
q = f_{\theta_1}(q^{[\theta_1]}) + f_{\theta_2}(q^{[\theta_2]}) + f_{\theta_3}(q^{[\theta_3]}).
\]

For the proof, which is presented for convenience here, we assume without loss of generality that \( \theta_1 < \theta_2 < \theta_3 \). We have,

\[
f = \begin{bmatrix} f_{\theta_1} \\ f_{\theta_2} \\ f_{\theta_3} \end{bmatrix} = \begin{bmatrix} \frac{\theta_1^4}{\theta_2^3} & -\frac{\theta_1^4}{\theta_3^3} \\ -\frac{\theta_2^4}{\theta_1^3} & \frac{\theta_2^4}{\theta_3^3} \\ \frac{\theta_3^4}{\theta_1^3} & -\frac{\theta_3^4}{\theta_2^3} \end{bmatrix}.
\]

By Corollary [2] we have

\[
j \leq 4(q^{[\theta_2]} - p) = j \leq 4\left(\frac{\theta_2^4}{\theta_3^3}(q^{[\theta_1]} - p)\right),
\]

\[
j \leq 4(q^{[\theta_3]} - p) = j \leq 4\left(\frac{\theta_3^4}{\theta_2^3}(q^{[\theta_1]} - p)\right).
\]

Then, by Corollary [1] we have,

\[
j \leq 7(q^{[\theta_2]} - p)^2 = j \leq 7\left(\frac{\theta_2^4}{\theta_3^3}(q^{[\theta_1]} - p)^2\right),
\]

\[
j \leq 7(q^{[\theta_3]} - p)^2 = j \leq 7\left(\frac{\theta_3^4}{\theta_2^3}(q^{[\theta_1]} - p)^2\right).
\]

Thus,

\[
j \leq 7\left(\sum_{i=1}^{3} f_{\theta_i}(q^{[\theta_i]} - p)\right)^2
\]

\[
= j \leq 7((f_{\theta_1} + f_{\theta_2} \frac{\theta_1^4}{\theta_2^3} + f_{\theta_3} \frac{\theta_1^4}{\theta_3^3})(q^{[\theta_1]} - p)^2)
\]

\[
= (\frac{\theta_1^4}{\theta_2^3} - \frac{\theta_2^4}{\theta_1^3})(\theta_2^3 - \theta_1^3) + \frac{\theta_1^4}{\theta_3^3} - \frac{\theta_3^4}{\theta_2^3})(\theta_3^3 - \theta_2^3) + (\frac{\theta_2^4}{\theta_1^3} - \frac{\theta_1^4}{\theta_2^3})(\theta_2^3 - \theta_1^3)
\]

\[
= 0.
\]

5. IMPLEMENTATION OF A \( m \)-TH ORDER GENERALIZED FUJIWARA SCHEME

A scheme for approximation of expectation of order six was first introduced by Fujiwara [5]. In previous sections we theoretically constructed schemes for approximation of expectation of order \( 2m \) for arbitrary \( m \in \mathbb{N} \). In this section we show how to construct a practical scheme with approximating flow of vector fields \( V_i \), which drive the SDE [11], by some suitable integration schemes. The usual choice for the integration schemes are Runge-Kutta methods. In our concrete example from mathematical finance we will use a seventh-order nine-stage explicit Runge-Kutta method with a very good stability, given by M.Tanaka et al. (see [11], [12] and [13]). Higher order Runge-Kutta methods often lose stability with respect to rounding error, truncated error and piling error. In addition, these effect decrease order of approximating error. Since in a concrete application of the algorithm, e.g. in mathematical finance, some of the ODEs can be very close to being stiff, the stability of the Runge–Kutta algorithm is of high importance. We show a relation between convergence order of weak approximation scheme and \( m \)-th order Runge-Kutta method.

In addition we construct a concrete algorithm of a \( m \)-th order generalized Fujiwara scheme and analyze its computational cost and its approximating error. At the end we present a concrete numerical experiment. Tanaka’s result is presented in the Appendix since we could not find any of his papers written in English.

The results of this section can be compared to those from [10].
5.1. Runge-Kutta method. For $V \in C^\infty_b(\mathbb{R}^N, \mathbb{R}^N)$, the map $\exp : C^\infty_b(\mathbb{R}^N, \mathbb{R}^N) \times \mathbb{R}_+ \times \mathbb{R}^N$ represents the flow driven by the vector field $V$ starting at $x_0$, i.e., the solution of the ordinary differential equation:

$$\begin{align*}
\frac{d}{dt}x(t) &= V(x(t)), \\
x(0) &= x_0.
\end{align*}$$

**Definition 6** ($s$ stage explicit Runge–Kutta method of order $m$ for autonomous systems). A $s$ stage explicit Runge–Kutta method of order $m$ for autonomous systems is determined by a lower triangular matrix $A = [a_{ij}]_{i,j=1}^s$ and a row $b = [b_1 \cdots b_s]$ such that the following hold:

- Let $h \in \mathbb{R}$, $t_0 \in \mathbb{R}$ and let $t_n = t_{n-1} + h$ for all $n \in \mathbb{N}$. Given the vector $x_{n-1}$ as an approximation to $x(t_{n-1})$, where $x$ satisfies the equation (20), the approximation $x_n$ to $x(t_n)$ is computed by evaluating, for $i = 1, 2, \ldots, s$,

$$F_i = V(X_i),$$

where $X_1, X_2, \ldots, X_s$ are given by

$$X_i = x_{n-1} + h \sum_{j<i} a_{ij} F_j$$

and then evaluating

$$y_n = y_{n-1} + \sum_{j=1}^s b_j F_j.$$

- The Taylor expansion of $x_n$ as a function of $h$ around 0 should coincide with the Taylor expansion of $x(t_n) = x(t_{n-1} + h)$ up to (including) the term at the power $h^m$.

**Remark 2.** Usually Runge–Kutta methods are studied for general non-autonomous systems. In these cases the method is uniquely identified by a triplet $A$, $b$ and $c$, where $A$ and $b$ are as above and $c = [c_1 \ldots c_s]^T$ is a suitable column vector.

See Butcher [2] and [3] for more details about the theory of Runge–Kutta method.

The next theorem shows that we need at least 12-th order Runge-Kutta method for 3-rd order generalized Fujiwara scheme.

**Theorem 5.** For all $f \in C^\infty_b(\mathbb{R}^N \to \mathbb{R}^N)$, $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^N$, there exists $C_i > 0$ such that

$$|f(\exp(t V_0)) - f(R_m(t, V_0)(x))| \leq C_0 t^{m+1},$$

$$|E[f(\exp((\sqrt{i} Z V_i)) - f(R_{2m}(\sqrt{i} Z, V_i)(x))| \leq C_4 t^{m+1},$$

where $i \in \{1, \ldots, d\}$ and $Z \sim \mathcal{N}(0, 1)$.

**Proof.** The first inequality follows from the definition of $m$-th order Runge-Kutta method and Taylor’s theorem. Set $i \in \{1, \ldots, d\}$. By the definition of Runge-Kutta method and Taylor’s theorem again, we have,

$$f(\exp((\sqrt{i} Z V_i)) - f(R_{2m}(\sqrt{i} Z, V_i)(x))$$

$$= \frac{i^{m+1/2} 2^m m^{m+1}}{(m+1)!} V_i^{2m+1} f(x) + O(t^{m+1}).$$

Note that for all $k \in \mathbb{N}$, $E[Z^{2k+1}] = 0$ holds. Thus the conclusion is true.

The next theorem shows that if we do not urge to have $O(n)$ computational cost, 4th order Runge-Kutta method is enough for sixth order scheme.

**Theorem 6.** For $k, n \in \mathbb{N}$, for all $f \in C^\infty_b(\mathbb{R}^N)$, for all $i \in \{1, \ldots d\}$, and for all $x \in \mathbb{R}^N$, there exists $C_i > 0$ such that

$$|E[f\left(\exp\left(\frac{Z}{\sqrt{n}} V_i\right)\right) - f\left(R_m\left(\frac{Z}{n^k} \sqrt{V_i} \right)^n(x)\right)]| \leq \frac{C_i}{n^{km+k+m/2+1}}$$

holds where $Z \sim \mathcal{N}(0, 1)$. 

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Proof.

\[
E[f(\exp (\frac{Z}{\sqrt{n}} V_i)(x)) - f(R_m(\frac{Z}{n^{k/2}}, V_i)^{n_k}(x))] \\
= |E(\exp (\frac{Z}{n^{k/2}} V_i)^{n_k} f(x) - R_m(\frac{Z}{n^{k/2}}, V_i)^{n_k} f(x))] \\
= E(\sum_{l=0}^{n_k-1} (\exp (\frac{Z}{n^{k/2}} V_i)^l((\exp (\frac{Z}{n^{k/2}} V_i) - R_m(\frac{Z}{n^{k/2}}, V_i)) \\
R_m(\frac{Z}{n^{k/2}}, V_i)^{n_k-l-1} f(x)] \\
\leq \frac{C_i}{n^{2(k+1/2)(m/2+1)}} n^{k} \\
\leq \frac{C_i}{n^{(km+k+m/2+1)}}
\]

\[\square\]

5.2. Recipe for \textit{m–th order generalized Fujiwara scheme}. In the following subsection we will provide the pseudocode for implementation of the \textit{m–th order generalized Fujiwara scheme} with fixed coefficients \(\theta_1, \theta_2, \ldots, \theta_m\). Let \(f = [f_1 \cdots f_m]^T\) be as in the section 4 and let the function \(\text{solveDE}(V, x_0, t)\) return the solution of the ODE \((20)\) at time \(t\) with initial condition \(x(0) = x_0\).

\underline{Algorithm 1: Fujiwara}

\begin{verbatim}
Data: function \(g\), vector fields \(V_0, V_1, \ldots, V_d\), time \(T\), initial condition \(x_0\), number of partition points \(n\), number of samples \(M\)
Result: approximation \(E\) of the expectation \(E[f(X_T)]\), where \(X_t\) is a process defined by the SDE \((1)\)
\(Q \leftarrow 0 \in \mathbb{R}^{1 \times m};\)
for \(o \leftarrow 1\) to \(M\) do /* expectation (MonteCarlo or quasi Monte Carlo) */
    \(Q \leftarrow Q + \text{samplePath}(g, V_0, \ldots, V_d, T, x_0, n);\)
end
\(Q \leftarrow \frac{1}{M} Q;\)
/* approx. for \(E(g(X(T, x_0)))\) is the linear combination \(\sum_i f_i * Q_i\) */
\(E \leftarrow Q f;\)
return \(E\)
\end{verbatim}
it is to be expected that many of the ODEs of the type (20) have a nice enough explicit solution suitable for building a universal solver for SDEs of the type (1). In concrete practical applications it seems sensible to generate all needed random variables in advance. Namely, the random variables for various \( \theta \)’s do not have to be independent, therefore we can reduce its number by reusing them, and there exist efficient algorithms which speed up the process of their generation if we do it in one batch instead of step by step as it is written in Algorithm 2.

Remark 3. Usually in modern computers memory size is no longer an issue. From this perspective it seems sensible to generate all needed random variables in advance. Namely, the random variables for various \( \theta \)’s do not have to be independent, therefore we can reduce its number by reusing them, and there exist efficient algorithms which speed up the process of their generation if we do it in one batch instead of step by step as it is written in Algorithm 2.

5.3. Computational cost.

**Theorem 7.** Let \( d, n, M, m, T, \theta_1, \ldots, \theta_m \) be as above, such that \( T/n \) is sufficiently small. Furthermore, assume that each step of the method \( \text{solveDE} \) needs \( a \) operations, i.e., additions, multiplications and function evaluations, that \( B \) operations are needed to generate a (pseudo or quasi) Bernoulli random variable and that \( Z \) operations are needed to generate a standard \( d \)-dimensional normally distributed (pseudo or quasi) random variable. Then the computational cost of Algorithm 2 is

\[
M \left( 5m + n((d + 1)a + Z + 1) \sum_{k=1}^{m} \theta_k + nB + 1 \right) + 2m.
\]

**Proof.** Let us denote the computational cost of the Algorithm 2 by \( C \). A straightforward calculation shows that the computational cost of the Algorithm 2 is equal to \( M(C + 1) + 2m \).

For fixed \( j \in \{1, \ldots, n\} \) in Algorithm 2 we have \( \theta_k((d + 1)a + Z + 1) \) operations. Hence, for fixed \( k \in \{1, \ldots, m\} \) there are \( 5 + n \theta_k((d + 1)a + Z + 1) \) operations. It follows that \( C = 5m + n((d + 1)a + Z + 1) \sum_{k=1}^{m} \theta_k + nB + 1 \).

**Remark 4.** Rigorous use of Runge-Kutta algorithms for solving ODEs in the algorithm is only suitable for building a universal solver for SDEs of the type (1). In concrete practical applications it is to be expected that many of the ODEs of the type (20) have a nice enough explicit solution.
The error of the algorithm consists of discretization part, i.e. the error due to numerical solution of ODEs and the error which comes from the scheme, and of the convergence error which comes from the Monte Carlo or quasi Monte Carlo simulation.

Theorem 8. For \( n, M \in \mathbb{N} \) such that \( T/n \) is sufficiently small. The approximation error of Algorithm 1 is \( O(1/n^{2m}) + O(1/\sqrt{M}) \).

Remark 5. One should take great care when choosing a suitable subdivision of the interval, since the coefficient of the discretisation error directly depends on function \( f \) and vector fields \( \mathbf{V}_i \), thus, although bounded, the coefficient can get fairly large in some cases. Moreover, the convergence error of the Monte Carlo simulation is directly proportional to the square root of variance of \( f(X(T, x)) \). As in the case of discretisation error this should be taken into account, since, although constant, the variance can be large comparing to the size of error we would like to achieve.

5.4. Numerical example. For our numerical example we have chosen the generalised Fujiwara scheme of order 8 with the choice of parameters \( \theta_1 = 1 \), \( \theta_2 = 2 \), \( \theta_3 = 3 \) and \( \theta_4 = 4 \).

In order to compare the algorithm to the basic Ninomiya-Victoir scheme we consider an Asian call option written on an asset whose price process follows the Heston stochastic volatility model. Let \( X_1 \) be the price process of an asset following the Heston model:

\[
X_1(t, x) = x_1 + \int_0^t \mu X_1(s, x) \, ds + \int_0^t X_1(s, x) \sqrt{X_2(s, t)} \, dB^1(s)
\]

\[
X_2(t, x) = x_2 + \int_0^t \alpha(\theta - X_2(s, x)) \, ds + \int_0^t \beta \sqrt{X_2(s, t)}(\rho dB^1(s) + \sqrt{1 - \rho^2} dB^2(s)),
\]

where \( x = (x_1, x_2) \in (\mathbb{R}_{>0})^2 \), \( (B^1(t), B^2(t)) \) is a two-dimensional standard Brownian motion, \(-1 \leq \rho \leq 1\) and \( \alpha, \theta, \mu \) are some positive coefficients satisfying \( 2\alpha\theta - \beta^2 > 0 \) to ensure that the volatility does not reach zero. The payoff of the Asian call option on this asset with maturity \( T \) and strike \( K \) is \( \max(X_3(T, x)/T - K, 0) \), where

\[
X_3(t, s) = \int_0^t X_1(s, x) \, ds.
\]

Hence, the price of this option becomes \( D \times E[\max(X_3(T, x)/T - K, 0)] \) where \( D \) is an appropriate discount factor on which we do not focus in this experiment. As in [9], take \( T = 1 \), \( K = 1.05 \), \( \mu = 0.05 \), \( \alpha = 2.0 \), \( \beta = 0.1 \), \( \theta = 0.09 \), \( \rho = 0 \) and \( x = (1.0, 0.09) \).

Up to the error of the magnitude \( 10^{-6} \) we have

\[
E[\max(X_3(T, x)/T - K, 0)] = 6.0473534496 \times 10^{-2}
\]

obtained from [8]. Let \( X(t, x) = (X_1(t, x), X_2(t, x), X_3(t, x))^T \). SDEs (21) and (22) can be transformed in the Stratonovich form since \( X_2 \neq 0 \):

\[
X(t, x) = \sum_{i=0}^{2} \int_0^t V_i(X(s, x)) \circ dB^i(s),
\]

where

\[
V_0(y_1, y_2, y_3) = (y_3 - \frac{\rho \beta}{4}, \alpha(\theta - y_2) - \frac{\beta^2}{4}, y_1)^T
\]

\[
V_1(y_1, y_2, y_3) = (y_1 \sqrt{y_2}, \rho \beta \sqrt{y_2}, 0)^T
\]

\[
V_2(y_1, y_2, y_3) = (0, \beta \sqrt{1 - \rho^2} y_2, 0)^T.
\]

Taking our choice of \( \rho = 0 \) into consideration we get exact solutions of ODEs of the type (20) driven by vector fields \( V_1 \) and \( V_2 \) (see [9] for more details):

\[
\exp(tV_1)(x_1, x_2, x_3)^T = (x_1 e^{\sqrt{x_2}}, x_2, x_3),
\]

\[
\exp(tV_2)(x_1, x_2, x_3)^T = \left( x_1, \left( \frac{\beta t}{2} + \sqrt{x_2} \right)^2, x_3 \right).
\]
According to the proof of Theorem 5 we need a Runge–Kutta method of order at least 6 to approximate the solution $\exp(tV_0)(x_1, x_2, x_3)^T$ for generalized Fujiwara scheme of order 6 and a Runge–Kutta method of order at least 8 for a generalized Fujiwara scheme of order 8 if we want a linear algorithm. If we allow quadratic computational cost for the the generalized Fujiwara scheme of the weak order 8, it is sufficient to use a Runge–Kutta method of order 4. In our example we used 9 stage 7-th order Runge–Kutta method from [11], defined by the Butcher’s tableau presented in the Appendix.

The pseudorandom numbers in MC were generated by the Mersenne twister algorithm. The QMC was performed using Sobol sequence, generated by the library SobolSeq51.dll provided by Broda (see [1]). Both MC and QMC integration were performed using $10^8$ sample paths.

The use of exact solutions of ODEs driven by vector fields $V_1$ and $V_2$ reduces the computational cost of the algorithm by $2Mn\sum_{k=1}^{\omega} \theta_k$, where $\omega$ designates the order of weak generalized Fujiwara scheme divided by 2, $M$ denotes the number of MC/QMC sample paths, $n$ is the number of subdivision points and $\alpha$ is the number of operations required for solving ODE’s driven by $V_1$ or $V_2$, if we compare it to the results of Theorem 7.

| Method / n | 2 | 3 | 4 | 5 |
|------------|---|---|---|---|
| NV         | 0.00208536744970740 | 0.00095536839891733 | 0.00055694952885933 |
| GF (order 6) MC | 0.00006154245956983 | 0.00002578946759 | 0.00003522768790512 |
| GF (order 6) QMC | 0.00005526280089 | 0.0000105789197729 | 0.0000040357269938 |
| GF (order 8) MC | 0.00004536485526115 | 0.00003694928288030 | 0.000055051968504230 |
| GF (order 8) QMC | 0.0000178413262662 | 0.0000013695959963 | 0.0000010913411477 |

The graph in Fig. 1 clearly shows that the new extrapolation method reduces the order of the discretization error in comparison to the original Ninomiya-Victoir algorithm for several magnitudes. In the MC case the discretization error almost immediately converges to the integration error (see Fig 1 and Fig. 2). Also in the QMC case the discretization error is soon (for small $n$) overshadowed by the integration error caused by QMC integration (see Fig. 2), the weak order of the extrapolated algorithms can still be observed from the slope of curves in the graph in Fig. 1.

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6. APPENDIX

We give our original proof of Lemma

Definition 7. Let \( \mathfrak{g} \) be a Lie algebra. For \( X, Y \in \mathfrak{g} \) define \( c_1(X, Y) = X + Y \) and \( c_n(X, Y) \) by the following recursion formula

\[
(n + 1)c_{n+1}(X, Y) = \frac{1}{2}[X - Y, c_n(X, Y)] + \sum_{p \geq 1, 2p \leq n} K_{2p} \sum_{k_1, \ldots, k_{2p} \geq 0, k_1 + \cdots + k_{2p} = n} [c_{k_1}(X, Y), \ldots, [c_{k_{2p}}(X, Y), X + Y], \ldots],
\]

where \( K_{2p} \) are coefficients defined in [14 2.15.9]
Lemma 4.

Proof. For $n = 1$ the assertion is clear.

Suppose we have $c_m(X, Y) = (-1)^{m+1}c_m(Y, X)$ for all $m \leq n$. By recursion we obtain

$$(n + 1)c_{n+1}(Y, X) = \frac{1}{2}[Y - X, c_n(Y, X)] +$$

$$+ \sum_{p \geq 1, 2p \leq n} K_{2p} \sum_{k_1, \ldots, k_{2p} \geq 0, k_1 + \ldots + k_{2p} = n} [c_{k_1} (Y, X), [\ldots, [c_{k_{2p}} (Y, X), X + Y] \ldots]].$$

Using the induction hypothesis and bilinearity of Lie brackets, the above equation transforms into

$$(n + 1)c_{n+1}(Y, X) = \frac{1}{2} (-1)^{n+2}[X - Y, c_n(X, Y)] +$$

$$+ \sum_{p \geq 1, 2p \leq n} K_{2p} \sum_{k_1, \ldots, k_{2p} \geq 0, k_1 + \ldots + k_{2p} = n} (-1)^{k_1 + \ldots + 2p} [c_{k_1} (X, Y), [\ldots, [c_{k_{2p}} (X, Y), X + Y] \ldots]]$$

$$= (-1)^{n+2} \left( \frac{1}{2} [X - Y, c_n(X, Y)] + \right.$$

$$\left. + \sum_{p \geq 1, 2p \leq n} K_{2p} \sum_{k_1, \ldots, k_{2p} \geq 0, k_1 + \ldots + k_{2p} = n} [c_{k_1} (X, Y), [\ldots, [c_{k_{2p}} (X, Y), X + Y] \ldots]] \right)$$

$$= (-1)^{n+2} (n + 1)c_{n+1}(X, Y)$$

which proves the assertion. \qed

Let $\tau_{l, d}$ denote $j_l(\overline{\tau}^{[1]})$.

Proof of Lemma 3. The case $d = 0$ is trivial. Next we consider the case $d = 1$. Using Baker–Campbell–Hausdorff formula to expand $\tau_{l, 1}$ and $j_l(\log(\overline{\tau}^{[1]}(1)))$ and applying (25) proves the formula (11). By applying Baker–Campbell–Hausdorff formula to the definition of $\tau_{l, d}$ we get

$$\tau_{l, d} = j_l(\log(\overline{\tau}^{[1]}(d))) = j_l(\log(\exp(\log(\overline{\tau}^{[1]}(d-1))) \exp(a_d)))$$

$$= j_l \left( \sum_{k = 1}^l c_k \left( \sum_{j = 1}^l \tau_{j, d-1}, a_d \right) \right).$$

Suppose that for all $n \in \mathbb{N}$, $n < d$ we have

$$\log(\overline{\tau}^{[1]}(n)) = \sum_{i=1}^{n} (-1)^{i+1} \tau_{i, n}.$$

Using Lemma 4 the induction hypothesis and the BCH-formula on $j_l(\log(\overline{\tau}^{[1]}(d)))$ gives us

$$j_l(\log(\overline{\tau}^{[1]}(d))) = j_l \left( \log \left( \exp(a_d) \exp(\log(\overline{\tau}^{[1]}(d-1))) \right) \right)$$

$$= j_l \left( \sum_{k = 1}^l c_k (A_d, \log(\overline{\tau}^{[1]}(d-1))) \right)$$

$$= j_l \left( \sum_{k = 1}^l c_k (a_d, \sum_{j = 1}^l (-1)^{j+1} \tau_{j, d-1}) \right)$$

$$= j_l \left( \sum_{k = 1}^l (-1)^{k+1} c_k (\sum_{j = 1}^l (-1)^{j+1} \tau_{j, d-1}, a_d) \right).$$
Thus, it is sufficient to show that for all \( k \in \{1, \ldots, l\} \) and \( l \in \mathbb{N} \) we have

\[
ji\left(c_k\left(\sum_{j=1}^{l} (-1)^{j+1} \tau_{j,d-1}, a_d\right)\right) = (-1)^{k+l} ji\left(c_k\left(\sum_{j=1}^{l} \tau_{j,d-1}, a_d\right)\right).
\]

Note that the equality in (26) holds trivially for \( k > l \).

Since \( \tau_{j,d-1} \) is a homogeneous polynomial of degree \( j \), the assertion is clear for \( k = 1 \) and all \( l \in \mathbb{N} \). It is easy to see that for \( l' < l \) we have

\[
ji\left(c_m\left(\sum_{j=1}^{l} (-1)^{j+1} \tau_{j,d-1}, a_d\right)\right) = j_l'\left(c_m\left(\sum_{j=1}^{l'} (-1)^{j+1} \tau_{j,d-1}, a_d\right)\right).
\]

Let now \( ji\left(c_m\left(\sum_{j=1}^{l} (-1)^{j+1} \tau_{j,d-1}, a_d\right)\right) \) for all \( m \in \{1, \ldots, k\} \) and \( l \in \mathbb{N} \), then we have

\[
ji\left(c_{k+1}\left(\sum_{j=1}^{l} (-1)^{j+1} \tau_{j,d-1}, a_d\right)\right) = \frac{1}{k+1} ji\left(\frac{1}{2}\left(\sum_{j=1}^{l} (-1)^{j+1} \tau_{j,d-1} - a_d, c_k\left(\sum_{j=1}^{l} (-1)^{j+1} \tau_{j,d-1}, a_d\right)\right)\right) + \sum_{p \geq 1} K_{2p} \sum_{k_1, \ldots, k_{2p} > 0} \sum_{k_1 + \ldots + k_{2p} = k+1} [c_{k_1}\left(\sum_{j=1}^{l} (-1)^{j+1} \tau_{j,d-1}, a_d\right), \ldots, [c_{k_{2p}}\left(\sum_{j=1}^{l} (-1)^{j+1} \tau_{j,d-1}, a_d\right), \ldots, \sum_{j=1}^{l} (-1)^{j+1} \tau_{j,d-1} + a_d]\ldots] = \frac{1}{k+1}\left(\frac{1}{2}\sum_{j=1}^{l-1} \left[(-1)^{j+1} \tau_{j,d-1}, ji-\left(c_k\left(\sum_{j=1}^{l} (-1)^{j+1} \tau_{j,d-1}, a_d\right)\right)\right] + [\cdots - a_d, ji-\left(c_k\left(\sum_{j=1}^{l} (-1)^{j+1} \tau_{j,d-1}, a_d\right)\right)]\right) + \sum_{p \geq 1} K_{2p} \sum_{k_1, \ldots, k_{2p} > 0} \sum_{m_1, m_2, \ldots, m_{2p+1}} [j_{m_1}\left(c_{k_1}\left(\sum_{j=1}^{l} (-1)^{j+1} \tau_{j,d-1}, a_d\right)\right), \ldots, j_{m_{2p}}\left(c_{k_{2p}}\left(\sum_{j=1}^{l} (-1)^{j+1} \tau_{j,d-1}, a_d\right)\right), (-1)^{m_{2p+1}} \tau_{m_{2p+1},d-1} + j_{m_{2p+1}}(a_d)\ldots]}.\]
Using (27), the induction hypothesis and the bilinearity of Lie brackets the above expression transforms into

\[ j_l(c_{k+1} \left( \sum_{j=1}^l (-1)^{j+1} \tau_{j,d-1}, a_d \right)) = \frac{1}{k+1} \left( \frac{1}{2} \sum_{j=1}^{l-1} (-1)^{j+1+l-j+k} \tau_{j,d-1}, a_d \right) \]

\[ j_{l-j}(c_k \left( \sum_{j=1}^l \tau_{j,d-1}, a_d \right)) + (-1)^{l+k-1} \left[ -a_d, j_{l-1}(c_k \left( \sum_{j=1}^l \tau_{j,d-1}, a_d \right)) \right] \]

\[ + \sum_{p \geq 1} K_{2p} \sum_{k_1, \ldots, k_2p > 0} \sum_{m_1, \ldots, m_{2p+1} > 0} (-1)^{m_1 + \cdots + m_{2p+1} + 1 + k_1 + \cdots + k_{2p}} \]

\[ j_{m_1}(c_{k_1} \left( \sum_{j=1}^l \tau_{j,d-1}, a_d \right)), j_{m_2}(c_{k_2} \left( \sum_{j=1}^l \tau_{j,d-1}, a_d \right)), \ldots, \]

\[ j_{m_{2p}}(c_{k_{2p}} \left( \sum_{j=1}^l \tau_{j,d-1}, a_d \right)), \tau_{m_{2p+1}, d-1} + j_{m_{2p+1}}(a_d) \ldots)) \right) \].

Thus,

\[ j_l(c_{k+1} \left( \sum_{j=1}^l (-1)^{j+1} \tau_{j,d-1}, a_d \right)) \]

\[ = (-1)^{k+l+1} \frac{1}{k+1} \left( \frac{1}{2} \sum_{j=1}^{l} \tau_{j,d-1} - a_d, c_k \left( \sum_{j=1}^{l} \tau_{j,d-1}, a_d \right) \right) \]

\[ + \sum_{p \geq 1} K_{2p} \sum_{k_1, \ldots, k_{2p} > 0} \sum_{k_1 + \cdots + k_{2p} = k} \left[ c_{k_1} \left( \sum_{j=1}^{l} \tau_{j,d-1}, a_d \right), c_{k_2} \left( \sum_{j=1}^{l} \tau_{j,d-1}, a_d \right), \ldots, \right. \]

\[ \left. c_{k_{2p}} \left( \sum_{j=1}^{l} \tau_{j,d-1}, a_d \right), \tau_{m_{2p+1}, d-1} + a_d, \ldots) \right) \right) \]

\[ = (-1)^{k+l+1} j_l(c_{k+1} \left( \sum_{j=1}^l \tau_{j,d-1}, a_d \right)) \]

which is the desired result. \[ \square \]

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