Research Article

Mathematical Model for Analysis of Uniaxial and Biaxial Reinforced Concrete Columns

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Received 7 August 2020; Revised 4 November 2020; Accepted 11 November 2020; Published 25 November 2020

Academic Editor: Faiz U. A. Shaikh

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This paper presents a mathematical model for the analysis of reinforced concrete (RC) uniaxial and biaxial columns. This proposed model is a quick and faster approach for the analysis and design of reinforced concrete rectangular columns without going through the interaction charts procedure as well as other iterative methods for the computation of required axial load capacity $(P_c)$ and moment capacity $(M_c)$. A simplified flow chart has also been developed to find the required column capacity using this mathematical model. Eight uniaxial columns (C-1 to C-8) and seven biaxial columns (CB-1 to CB-7) are analysed in this study. Each column is analysed having different steel reinforcement ratios $(\rho)$ with different loading conditions. In addition, the studied columns are subjected to both tension and compression failures. The detailed examples for both uniaxial and biaxial columns (one for each case) are also presented in this study. The studied columns are also analysed using computer software spColumn. The average variation of the mathematically computed values to the finite element software is not more than 10%, showing promising computational results.

1. Introduction

Columns are the vertical compression members, which transmit loads from the upper floors to the lower levels and to the soil through the foundations [1]. Based on the position of the load on the cross section, columns are classified as concentrically loaded (Figure 1) or eccentrically loaded columns (Figure 2). Eccentrically loaded columns are subjected to moments, in addition to axial force. The moments can be converted to a load $P$ and eccentricities $e_x$ and $e_y$. The moments can be uniaxial, as in the case when two adjacent panels are not similarly loaded, such as columns $A$ and $B$ in Figure 3 [2]. A column is considered as biaxially loaded when the bending occurs about the $x$- and $y$-axis, such as in the case of corner column $C$ in Figure 3. In a recent study [3], Al-Ansari and Afzal also presented an analytical model for generating interaction diagram charts for biaxial columns.

The strength of reinforced concrete columns is normally expressed using interaction diagrams to relate the design axial load $2\varnothing P_n$ to the design bending moment $\varnothing M_n$ [4, 5]. Each control point on the column interaction curve $(\varnothing P_n - \varnothing M_n)$ represents one combination of design axial load, $\varnothing P_n$ and design bending moment, $\varnothing M_n$, corresponding to a neutral-axis location (Figure 4) [6].

Extensive studies have been carried out on the interaction diagrams (uniaxial and biaxial columns) of reinforced concrete (RC) rectangular columns [6–12]. Several studies have also been performed on providing numerical approaches for the analysis and design of reinforced concrete columns. Furlong et al. [13] provided an overview of the analysis and design of reinforced concrete columns subjected to biaxial bending. They reviewed several methods of analysis that use traditional design methods and compared their results with the obtained data from physical tests of normal strength concrete columns subjected to short-term axial loads and biaxial bending’s. They concluded that the elliptic load contour equation [14] and the reciprocal equation [15] are the simplest to use, as they do not require complicated calculations.
Chen et al. [16] proposed an iterative numerical method for rapid section analysis and design of short concrete composite columns subjected to biaxial bending. Wang and Hsu [17] proposed the numerical method approach for the determination of load-moment curvature relationship for short and slender columns. This numerical method approach is also applicable for columns made of different materials, and shows good agreement with the different experimental results obtained in their study.

Whitney [18] and Hsu et al. [19] provided major research studies on numerical method approaches. Whitney suggested an approximate equation to estimate the nominal compressive strength of columns subjected to compression failure. Hsu in different research projects [10, 17, 20, 21] also presented the results of experimental and analytical studies on the strength and deformation of biaxially loaded short and tied columns with L–, channel, and T-shaped cross sections. In another study, Hsu [22] suggested a general
Numerical examples for the selected reinforced concrete columns (uniaxial and biaxial columns) are also illustrated to check the adequacy of this proposed model. Eight uniaxial columns (C-1 to C-8) and seven biaxial columns (CB-1 to CB-7) are analysed in this study. These columns are analysed having different steel reinforcement ratios ($\rho$), different values of steel yield strength ($f_y$), concrete compressive strength ($f'_c$), and different load capacity conditions. Moreover, the results obtained from this proposed model are compared with computer software spColumn 2016 [24].

2. Mathematical Model Formulation: ACI Code Design

The stress and strain distribution of a rectangular column section (uniaxial column) for the calculation of $P_n$ and $M_n$ is given in Figure 5.

The resultant force $P_N$ is equal to the summation of all internal forces:

$$ P_N = C_{Con} - T_s + C_S. \quad (1) $$
Similarly, the resultant moment \( M_N \) is equal to the summation of all internal moments:

\[
M_N = M_{\text{conc}} + M_T + M_{\text{Cs}}
\]  

(2)

The following steps revealed the calculation of the required internal forces and internal moments for a rectangular uniaxial RC column.

### 2.1. Plain Concrete Section

The internal concrete compressive force \( C_{\text{conc}} \) is computed as

\[
C_{\text{conc}} = (0.85 f'_c b a),
\]

\[
C_{\text{conc}} = 0.85 f'_c b \beta c,
\]

(3)

where \( C_{\text{conc}} \) = internal concrete compression force, \( f'_c \) = compressive concrete strength, \( b \) = column width, \( a \) = depth of the compression stress block, \( \beta = 0.85 - 0.008 (f'_c - 30) \geq 0.65 \), and \( c \) = distance from extreme compression fiber to the neutral axis.

Referring to Figure 5, the moment about the midpoint of the section \( (M_{\text{conc}}) \) can be computed as

\[
M_{\text{conc}} = C_c \left( d - \frac{a}{2} - d'' \right),
\]

\[
M_{\text{conc}} = 0.85 f'_c b a \left( d - \frac{a}{2} - d'' \right),
\]

(4)

where \( h \) = column total depth, \( d'' = ((h/2) - d') \), \( d \) = column effective depth \( (h-d') \), and \( d' \) = distance from extreme compression fiber to centroid of top reinforcing steel.

### 2.2. Tension Steel Section

The internal tensile force \( T_s \) is computed as

\[
T_s = A_s f_y,
\]

(5)

where \( A_s \) = area of tensile steel reinforcement and \( f_y \) = yield stress of reinforcing steel. The internal moment \( M_T \) is

\[
M_T = A_s f_y d''.
\]

(6)

### 2.3. Compression Steel Section

The internal compressive force \( C_s \) is computed as [25]

\[
C_s = A_s' \left( f'_s \right),
\]

\[
C_s = A_s' \left( f'_s - 0.85 f'_c \right),
\]

(7)

where \( A_s' \) = area of compression steel reinforcement and \( f'_s = f_y \) (if the compression steel yields).

The internal moment \( M_T \) is

\[
M_T = A_s' \left( f'_s - 0.85 f'_c \right) (d - d' - d'').
\]

(8)

### 3. Mathematical Model Analysis

The following steps should be revealed to calculate the design axial load and moment capacity of the required rectangular RC column section. Columns may be subjected to tension failure or compression failure depends on the balanced eccentricity value \( (\epsilon_c) \):
\[ e_b = \frac{M_b}{P_b}, \]  
\[ M_b = \left( C_b \left( d - \frac{a_b}{2} - d'' \right) + C_s (d - d' - d'') + (T_s + d'') \right) \times 10^{-3}, \]  
\[ P_b = C_s + C_{cb} - T_s, \]

where \( a_b = \beta \times c_b, \) \( C_{cb} = \left( 600 \times d/600 + f_y \right), \)
\( C_b = \left( 0.85 \times f'_c \times a_b \times b \right) \times 10^{-3}, \) and \( C_s = A'_f \left( f'_c - 0.85 f'_c \right), \) \( A'_f \) \( \times 10^{-3} \) (if the compression steel yields, then \( f'_c = f_y \)).

### 3.1. Tension Failure Analysis

Tension failure will occur when the balanced eccentricity value \( (e_b) \) is less than load eccentricity \( (e) \). Substituting the values of \( C_s, C_{cb}, \) and \( T_s \) in equation (1) and solving for \( (a) \) will be a second-degree equation [14]:

\[ P_N = C_{Con} - T_s + C_s (\text{Equation (1)}) , \]
\[ Aa^2 + Ba + C = 0, \]
where \( A = (0.85 \times f'_c \times b/2), \) \( B = 0.85 \times f'_c \times b \times (e' - d), \) \( C = A'_f \left( f'_c - 0.85 f'_c \right) \left( e' - d + d'' \right) \times 10^{-3}, \)
\( e' = e + d'' \) (when \( A'_f = A'_f \)).

Solve for \( (a) \) to get

\[ a = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \]

Substitute the value of \( a \) in equation (3) to calculate \( C_s \)
and from equations (5) and (7) to compute \( T_s \) and \( C_s \) values. These obtained values are substituted in equation (1) for
\[ P_N = C_{Con} - T_s + C_s, \]
\[ P_N = \frac{1}{e'} \left( C_s \left( d - \frac{a}{2} \right) + C_s (d - d') \right), \]

\[ M_N = P_N \times e. \]

The column axial load capacity and moment capacity can therefore be computed as

\[ P_c = \varnothing P_N, \]
\[ M_c = \varnothing M_N, \]

(where \( \varnothing \) is the column reduction factor having the value of 0.65).

### 3.2. Compression Failure Analysis

Compression failure will occur when the balanced eccentricity value \( (e_b) \) is bigger than the load eccentricity \( (e) \). Substituting the values of \( C_s, C_{cb}, \) and \( T_s \) in equation (1) and solving for \( (a) \) will be a cubic equation [14]:

\[ Aa^3 + Ba^2 + Ca + D = 0, \]

where \( A = (0.85 \times f'_c \times b/2), \) \( B = 0.85 \times f'_c \times b \times (e' - d), \) \( C = A'_f \left( f'_c - 0.85 f'_c \right) \left( e' - d + d'' \right) \times 10^{-3}, \)
\( D = -600 A_e \beta d, \) where \( e' = e + d'' \) (when \( A_e = A'_f \)).

Once the values of \( A, B, C, \) and \( D \) are calculated, the value of \( a \) can be determined by the trial method or directly by using MATLAB or any scientific calculator. Moreover, the cubic equation can also be solved using different numerical methods, for example, Newton Raphson Method. After getting the required value of \( (a) \), similar equations from (14) to (18) (as mentioned in the Tension Failure Analysis) should be used to get the required value of column axial load capacity \( (P_c) \) and moment capacity \( (M_c) \).

The following flow chart (Figure 6) can be followed to find the required capacity of the rectangular uniaxial column section.

### 4. Numerical Examples for Uniaxial Columns

Eight reinforced rectangular columns (C-1 to C-8) having different column sizes are analysed using the numerical method approach. These columns are having different reinforcement ratios \( (\rho) \) in addition to different failure types, both tension and compression failures. The design input data for these columns are illustrated in Table 1.

The above eight columns C1 to C8 are analysed using the mathematical model approach to find the required values of axial load capacity, \( P_c, \) and moment capacity, \( M_c. \) Moreover, these values are also compared with the computer software spColumn. The results obtained are depicted in Table 2.

These above columns are also analyzed with different available methods, Whitney’s 1st approximation method [18], Whitney’s second approximation method [18], and the method provided by HSU [19]. These available methods are only available for the columns having the compression failure. There are no examples available for the columns with the tension failure cases. The results comparison is mentioned in Table 3.

#### 4.1. Detailed Numerical Example for Column C-4 (400 × 400)

**Input Data:** Figure 7

- \( Pu = 400 \text{kN} \)
- \( Mu = 100 \text{kN·m} \)
- \( f'_c = 30 \text{MPa} \)
- \( f_y = 415 \text{MPa} \)
- \( As = 1000 \text{mm}^2 \)
- \( A'_f = 1000 \text{mm}^2 \)
- \( d'' = 80 \text{mm} \)
- \( \phi = 0.65 \)

**Solution:**

1. Finding the value of \( e = \frac{M_u}{P_u} = 100/400 = 250 \text{mm} \)
2. \( c_b = (600 \times d/600 + f_y) = (600 \times 320/600 + 415) = 189.16 \text{mm} \)
3. \( a_b = \beta \times c_b = 0.85 \times 189.16 = 160.788 \text{mm} \)
(4) \( C_{cb} = 0.85 f'_c a b \)
\( C_s = A'_s (f'_s - 0.85 f'_c) \)
(If the compression steel yields, then \( f'_c = f_y \))

\( T_s = A'_s f_y \)

\( P_b = C_{cb} + C_s - T_s \)
\( M_b = (C_{cb} (d - (a/2) - d'') + C_s (d - d' - d'') + (T_s \times d'')) \)
\( e_b = (M_b/P_b) \)

\( d'' = ((h/2) - d') \)

\( \epsilon'_c = \epsilon_c + d'' \)
\( C_c = 0.85 f'_c ab \)

\( P_N = (1/\epsilon'_c)(C_c (d - (a/2)) + C_s (d - d')) \)
\( P_i = 0P_N \)

\( M_N = P_N \times \epsilon \)
\( M_i = 0M_N \)

End

**Figure 6:** Flow chart of the mathematical model for the rectangular uniaxial rectangular column.

(4) \( C_{cb} = 0.85 f'_c a b = 0.85 \times 30 \times 160.788 \times 400 \)
\( = 1.64 \times 10^3 \text{kN} \)

(5) \( C_s = A'_s (f'_s - 0.85 f'_c) = 1000 \)
\( (415 - 0.85 \times 30) = 3.895 \times 10^2 \text{kN} \)

(6) \( T_s = A'_s f_y = 1000 \times 415 = 4.15 \times 10^2 \text{kN} \)

(7) \( P_b = C_s + C_{cb} - T_s = 1.615 \times 10^3 \text{kN} \)

(8) \( M_b = (C_c (d - (a/2) - d'') + C_s (d - d' - d'') + (T_s \times d'')) \)
\( d'' = ((h/2) - d') = 120 \text{mm} \)

\( M_b = 2.927 \times 10^2 \text{kN\cdot m} \)

(9) \( e_b = (M_b/P_b) = 181.3 \text{mm} < \epsilon (250 \text{mm}) \) (TENSION Failure)
Table 1: Uniaxial column input data.

| Column identifier | Pu (kN) | Mu (kN·m) | As (mm²) | As′ (mm²) | f′c (MPa) | fy (MPa) | ϕ | d’ (mm) | e (mm) |
|-------------------|---------|-----------|----------|-----------|-----------|----------|----|---------|--------|
| C1 (200 × 400)    | 300     | 60        | 400      | 400       | 30        | 300      | 0.7| 80      | 200    |
| C2 (200 × 400)    | 200     | 50        | 400      | 400       | 20        | 300      | 0.7| 60      | 250    |
| C3 (300 × 500)    | 800     | 200       | 1000     | 1000      | 30        | 415      | 0.7| 75      | 250    |
| C4 (400 × 400)    | 400     | 100       | 1000     | 1000      | 30        | 415      | 0.65| 80      | 250    |
| C5 (300 × 450)    | 1300    | 157.4     | 1530     | 1530      | 25        | 300      | 0.65| 75      | 121    |
| C6 (200 × 400)    | 400     | 20        | 402      | 402       | 30        | 300      | 0.7| 60      | 50     |
| C7 (300 × 500)    | 600     | 100       | 1500     | 1500      | 30        | 415      | 0.65| 90      | 167    |
| C8 (400 × 800)    | 1000    | 200       | 2000     | 2000      | 30        | 415      | 0.65| 80      | 200    |

Table 2: Uniaxial column design results.

| Column identifier | e_b (mm) | Failure condition | Mathematical model | spColumn |
|-------------------|----------|-------------------|--------------------|----------|
|                   |          |                   | Pc (kN)            | Mc (kN·m) |
|                   |          |                   | Pc (kN)            | Mc (kN·m) |
| C1                | 140.7    | Tension           | 377.6              | 75.5     |
|                   |          |                   | 378.3              | 75.6     |
| C2                | 141.5    | Tension           | 234.8              | 58.7     |
|                   |          |                   | 247.4              | 61.84    |
| C3                | 233      | Tension           | 1033               | 258      |
|                   |          |                   | 1036               | 259      |
| C4                | 181.8    | Tension           | 650.3              | 162.6    |
|                   |          |                   | 650                | 163      |
| C5                | 222      | Compression       | 1310               | 158      |
|                   |          |                   | 1425               | 172.5    |
| C6                | 138      | Compression       | 1191               | 59.5     |
|                   |          |                   | 1810               | 90.52    |
| C7                | 276      | Compression       | 1495               | 250      |
|                   |          |                   | 1677               | 279.5    |
| C8                | 364      | Compression       | 3606               | 722      |
|                   |          |                   | 3975               | 795      |

Table 3: Column design results comparison with different methods.

| Column identifier | Failure condition | Mathematical model | Whitney 1st approximation | Whitney 2nd approximation | HSU |
|-------------------|-------------------|--------------------|---------------------------|---------------------------|-----|
|                   |                   | Pc (kN)            | Mc (kN·m)                 | Pc (kN)                   | Mc (kN·m) |
| C1                | Tension           | 377.6              | 75.5                      | —                         | —   |
| C2                | Tension           | 234.8              | 58.7                      | —                         | —   |
| C3                | Tension           | 1033               | 258                       | —                         | —   |
| C4                | Tension           | 650                | 162                       | —                         | —   |
| C5                | Comp.             | 1310               | 158                       | 1580                      | 191 |
| C6                | Comp.             | 1191               | 59.5                      | 1877                      | 94  |
| C7                | Comp.             | 1495               | 250                       | 1677                      | 280 |
| C8                | Comp.             | 3606               | 722                       | 4300                      | 860 |

Figure 7: Column C-4.
5. Numerical Examples for Biaxial Columns

Seven reinforced biaxial rectangular columns (CB-1 to CB-7) having different column sizes are also analysed using the proposed model. These columns are also having different reinforcement ratios ($\rho$) in addition to different failure types, that is, tension-tension, compression-compression, and tension-compression failures. The design input load data for these columns are illustrated in Table 4. The column cross section subjected to biaxial bending is shown in Figure 8. A similar flow chart has to be adopted (as discussed in the uniaxial column sections), once for the case of eccentricity in the $x$-direction ($e_x$) and later for the eccentricity in the $y$-direction ($e_y$) to obtain the required values of load capacities in x- and y-direction ($\varnothing P_x$,$\varnothing P_y$). These values are later used in Bresler’s formula [15] (equation (19)) to find the value of $P_c$. Moreover, the Mcx and Mcy values can be found by using equations (21) and (22) accordingly:

$$P_c = \frac{1}{(1/\varnothing P_x) + (1/\varnothing P_y) - (1/\varnothing P_{N\_{\text{max}}})}$$  \hspace{1cm} (19)

$$\varnothing P_{N\_{\text{max}}} = 0.8\phi (0.85f'_c (Ag - Ast) + fy Ast),$$  \hspace{1cm} (20)

where $\varnothing P_{N\_{\text{max}}}$ = maximum permissible column load, $Ast$ = total area of steel, and $Ag$ = (Gross area of cross section) – (sectional area of concrete member).

The moments in the $x$- and $y$-direction can be found as

$$M_{cx} = P_c \times e_{ux},$$  \hspace{1cm} (21)

$$M_{cy} = P_c \times e_{uy}. $$  \hspace{1cm} (22)

The above seven columns (CB-1 to CB-7) are analysed with mathematical model approach to find the required values of axial load capacity $P_c$, using reciprocal formula. Moreover, the values of $P_c$ are also compared with the computer software spColumn. The results obtained are depicted in Table 5.

5.1 Detailed Numerical Example for Column CB-7 (400 × 1200)

**Input Data:** Figure 9

- $Pu = 1500\text{ kN}$
- $Mux = 300\text{kN.m}$
- $Muy = 300\text{kN.m}$
- $f'_c = 20\text{ MPa}$
- $fy = 300\text{ MPa}$
- $As = 3080\text{ mm}^2$
- $A'_s = 3080\text{ mm}^2$
- $d' = 60\text{ mm}$
- $\phi = 0.65$

**Solution:** (Solving for the X-direction)

(1) Finding the value of $e_x = (M_{ux}/P_u) = 200\text{ mm}$

(2) $c_b = (600 \times d/600 + f'_y) = (600 \times 1140/600 + 760) = 706.8\text{ mm}$

(3) $a_b = \beta \times c_b = 0.93 \times 760 = 706.8\text{ mm}$

(4) $C_{cb} = 0.85f'_y a_b b = 0.85 \times 20 \times 706.8 \times 400 = 4.81 \times 10^5\text{ kN}$

(5) $C_s = A'_s (f_y - 0.85f'_y) = 3080$\hspace{0.5cm}$(300 - 0.85(20)) = 8.71 \times 10^2\text{ kN}$

(6) $T_s = A_s f_y = 3080 \times 300 = 9.24 \times 10^5\text{ kN}$

(7) $P_{bx} = C_s + C_{cb} - T_s = 4.754\text{ kN}$

(8) $M_{bx} = (C_s (d - (a_b/2) - d''') + C_s (d - d' - d''') (T_s/d'''))$

$$d''' = ((h/2) - d') = 540\text{ mm}$$

$$M_{bx} = 2.155 \times 10^3\text{ kN.m}$$

(9) $e_{bx} = (M_{bx}/P_{bx}) = 453\text{ mm} > e_y (200\text{ mm})$

(COMPRESSION Failure)

(10) Finding the value of $a$ using the Cubic Equation

$$Aa^3 + Ba^2 + Ca + D = 0$$

where $a = e' + d'' + 740\text{ mm}$

(11) Check if the tension steel has yielded;

$c = (a/b) = 1016\text{ mm}$

$$\varepsilon_s = (d - c)/c \times 0.003 = 0.0028, \varepsilon_y = (f_y/Es) = 0.0015$$

(12) $P_{NX} = (1/e')(C_s (d - (a_b/2) + C_s (d - d' - d''') \times 10^{-3}$

$$M_{NX} = P_{NX} \times ex = 1414\text{ kN.m}$$

(13) $P_{CXX} = \varnothing P_{NX} = 4594\text{ kN}$
Table 4: Biaxial column input data.

| Column identifier | Pu (kN) | Mux (kN-m) | Muy (kN-m) | As (mm²) | A’s (mm²) | f’c | fy (MPa) | ϕ | d' (mm) | ex (mm) | ey (mm) |
|-------------------|---------|------------|------------|----------|----------|-----|----------|---|--------|--------|--------|
| CB-1 (300 × 600)  | 300     | 100        | 80         | 1232     | 1232     | 30  | 400      | 0.65| 80     | 267    | 333    |
| CB-2 (200 × 400)  | 200     | 40         | 20         | 628.4    | 628.4    | 20  | 300      | 0.7  | 40     | 100    | 200    |
| CB-3 (300 × 300)  | 2500    | 250        | 120        | 1225     | 1225     | 30  | 415      | 0.65| 70     | 48     | 100    |
| CB-4 (375 × 500)  | 1700    | 200        | 100        | 2100     | 2100     | 30  | 415      | 0.65| 60     | 59     | 118    |
| CB-5 (400 × 500)  | 800     | 200        | 50         | 1413.8   | 1413.8   | 30  | 415      | 0.65| 60     | 62.5   | 250    |
| CB-6 (350 × 700)  | 400     | 60         | 40         | 1638     | 1638     | 20  | 300      | 0.65| 45     | 100    | 150    |
| CB-7 (400 × 1200) | 1500    | 300        | 300        | 3080     | 3080     | 20  | 300      | 0.65| 60     | 200    | 200    |

Table 5: Biaxial column design results.

| Column identifier | Failure condition | Mathematical model Pc (kN) | spColumn Pc (kN) |
|-------------------|-------------------|-----------------------------|-----------------|
| CB-1              | Tension and tension | 280                         | 343             |
| CB-2              | Tension and tension | 262                         | 228             |
| CB-3              | Compression and compression | 480                        | 415             |
| CB-4              | Compression and compression | 2132                       | 2035            |
| CB-5              | Tension and compression | 1291                       | 1142            |
| CB-6              | Compression and compression | 2852                       | 2805            |
| CB-7              | Compression and tension | 1993                       | 2081            |

(14) $M_{CX} = \Phi M_{NX} = 918.79$ kN-m

(Solving for the Y-direction) (Figure 10)

(1) Finding the value of $e_y = (M_{uy}/P_u) = 200$ mm

(2) $c_y = (600 \times d/600 + f_y) = (600 \times 1140/600 + 300) = 226.67$ mm

(3) $a_y = b_y = d_y = 93 \times 226.67 = 210.8$ mm

(4) $C_{cb} = 0.85 \times f_y \times a_y \times b_y = 0.85 \times 20 \times 210.8 \times 1200 = 4.3 \times 103$ kN

(5) $C_y = A_y' (f_y - 0.85 f_y') = 3080$

(6) $T_y = A_y f_y = 3080 \times 300 = 9.24 \times 102$ kN

(7) $P_{by} = C_y + C_{cb} - T_y = 4.248 \times 103$ kN

(8) $M_{by} = (C_y (d - (a_y/2) - d_y') + C_s (d - d' - d'')) / (T_y \times d'')$

$d'' = ((h/2) - d') = 140$ mm

$M_{by} = 6.582 \times 10^2$ kN-m

(9) $e_{by} = (M_{by}/P_{by}) = 154.94$ mm $> ex$ (200 mm) (TENSION Failure)

(10) Finding the value of $(a)$ using the Quadratic Equation

$Aa^2 + Ba + C = 0$
A = 0.425 \times f'_c \times b \times 10^{-3} = 10200 \\
B = 0.85 \times f'_c \times b \times (e' - d) \times 10^{6}, \text{ where; } \\
(e' = e + d') = 340 \text{ mm} \\
B = 0 \\
C = A'_t (f'_c - 0.85 f'_c) (e' - d + d') - f_y A_t e' = \\
-2.62 \times 10^{8} \\
a = 160.227 \text{ mm} \\
(11) \text{ Check if the tension steel has yielded; } \\
c = (a/\beta) = 172.287 \text{ mm} \\
\varepsilon_s = (d - c/c) \times 0.003 = 0.00292, \ \varepsilon_y = (f_y/E_s) = 0.0015 \\
\varepsilon_s > \varepsilon_y, \text{ steel yields; } (f_s = f_y) \\
(12) \ P_{NY} = (1/e') (C_e (d - (a/2)) + C_z (d - d') \times 10^{-3} \\
C_z = 0.85 f'_c a b = 3.27 \times 10^6 \text{ kN} \\
P_{NY} = 3216 \text{ kN} \\
M_{NX} = P_{NY} \times e_y = 643.25 \text{ kN-m} \\
(13) \ P_{CY} = \phi P_{NY} = 2091 \text{ kN} \\
(14) \ M_{CY} = \phi M_{NY} = 418.12 \text{ kN-m} \\

Finding the value of P_c using Bresler’s equation (19): \\
P_c = \frac{1}{(1/\phi P_{x}) + (1/\phi P_{y}) - (1/\phi P_{N_{max}})} \quad (23) \\
where \ \phi P_{N_{max}} = 0.8 \phi (0.85 f'_c (A_g - A_{st}) + f_y A_{st}) = 5150 \text{ kN}, \quad P_c = (1/(1/4594) + (1/2091) - (1/5150)) = 1993 \text{ kN}. \\

6. Validation of the Mathematical Model 

In order to validate the proposed mathematical model approach, the model is validated with the existing experimental results of columns subjected to uniaxial and biaxial loadings. The experimental results data has been extracted from the test results provided by HSU [22]. Two uniaxial columns as provided by Bresler (B-1 and B-2) and one biaxial column as provided by Anderson and Lee (SC-4) are selected from the research article [22] to compare the results with the mathematical model. 

Table 6 illustrates the experimental testing data provided by HSU. The data and the results are provided in imperial units (Kips-ft) units. Therefore, they are converted to metric units accordingly to compare the values with our results. 

Table 7 provides the experimental test results as well as the validation of the test data with the proposed mathematical model. The column capacity (Pc) results obtained from the experimental data are quite close to the mathematical model results, showing satisfactory computational results. 

7. Results and Discussions 

The results obtained from the mathematical model approach for both uniaxial and biaxial columns showed a safe and conservative column design method. The results of eight uniaxial column sections (C-1 to C-8) using the proposed model are also compared with different available mathematical models, provided by Whitney’s 1st approximation method, Whitney’s second approximation method, and the method provided by HSU. Columns C1 to C-4 were subjected to tension failure, whereas columns C-5 to C-8 were the compression failure cases. The other three mathematical studies (Whitney’s 1st approximation, Whitney’s second
### Table 6: Experimental testing data [22].

| Experimental investigator | Column identifier with size (in x in) b (mm x mm) | As (in²) (mm²) | f'_s (Ksi) (MPa) | f_y (Ksi) (MPa) | d' (in) (mm) | e_x (in) (mm) | e_y (in) (mm) |
|---------------------------|-------------------------------------------------|----------------|------------------|----------------|-------------|-------------|-------------|
| Bresler (uniaxial column) | B-1 (6 x 8) (152 x 203)                          | 1.24 (800)     | 3.7 (25.6)       | 53.5 (369)     | 1.75 (44.5) | 6 (152.4)   | 0           |
| Bresler (uniaxial column) | B-2 (6 x 8) (152 x 203)                          | 1.24 (800)     | 3.9 (27)         | 53.5 (369)     | 1.75 (44.5) | 3 (76.2)    | 0           |
| Anderson and Lee          | SC-4 (4 x 4) (102 x 102)                         | 0.8 (516)      | 5.435 (37.5)     | 45.6 (314.6)   | 0.75 (19)   | 2.82 (71.63)| 2.82 (71.63)|

### Table 7: Validation of experimental data [22].

| Experimental investigator | Column identifier with size (in x in) (mm x mm) | φc  | eb (in) (mm) | Failure condition | Experimental results P_c (kips) (kN) | Mathematical model P_c (kips) (kN) | (P_{exp}/P_{Math}) |
|---------------------------|-------------------------------------------------|-----|-------------|--------------------|-------------------------------------|-----------------------------------|---------------------|
| Bresler (uniaxial column) | B-1 (6 x 8) (152 x 203)                          | 0.65| 4.67 (118.5)| eb < ex (tension)  | 24 (107)                            | 29 (132)                         | 0.83                |
| Bresler (uniaxial column) | B-2 (6 x 8) (152 x 203)                          | 0.65| 4.6 (117)   | eb > ex (compression) | 60 (267)                         | 59 (263)                         | 1.01                |
| Anderson and Lee          | SC-4 (4 x 4) (102 x 102)                         | 0.65| 2.7 (69)    | eb < ex (tension)  | 13.5 (60)                          | 11 (49)                          | 1.22                |

Figure 11: Axial load capacity comparison for uniaxial columns (C1–C8).

Figure 12: Moment capacity comparison for uniaxial columns (C1–C8).
approximation, and the method provided by HSU) are only limited to the case where the columns are subjected to the compression failure only.

These studied columns (C-1 to C-8) are also analysed using the computer software spColumn and the comparison results for axial load capacities (Pc) and moment capacities (Mc) are displayed in bar charts (Figures 11 and 12).

For the biaxial columns (CB-1 to CB-7), the axial load capacity results for mathematical model approach using Bresler’s formula and the computer software spColumn are displayed in the bar chart (Figure 13).

The values of Pc obtained using the mathematical model are quite close to the computer software results, showing relatively satisfactory computational results.

8. Conclusion

In this study, the mathematical model is presented to analyse and design the uniaxial and biaxial columns without going through the column interaction charts to find the required axial load capacities and moment capacities. A simplified flow chart has also been developed to solve the required column section following the mathematical model steps.

Eight (RC) uniaxial columns (C-1 to C-8) and seven (RC) biaxial columns (CB-1 to CB-7) are analysed in this study. These columns are analysed having different steel reinforcement ratios (ρ), different values of steel yield strength (fy), concrete compressive strength (fc’), and different load capacity conditions. Moreover, the studied columns are subjected to both tension and compression failures.

For the uniaxial columns, the proposed mathematical model results are also compared with the different available numerical approaches done by Whitney’s 1st approximation, Whitney’s 2nd approximation, and the method provided by HSU. All of these three methods were formulated based on the case of compression failure only. These uniaxial columns are also analysed using the computer software spColumn. The results obtained showed that this proposed mathematical approach showed good agreement with the computer software spColumn showing relatively satisfactory results.

The studied biaxial columns are subjected to different failure conditions, that is, tension-tension failure, compression-compression failure, and tension-compression failure. Bresler’s formula was used to find the required capacity (Pc) after finding the (Px) and (Py) from the mathematical model approach. The biaxial columns were also analysed with the computer software. The average variation of the mathematically computed values for biaxial columns to the finite element software was not more than 10%. Moreover, the results obtained for the columns subjected to tension failure are quite close with the computer software spColumn. Moreover, this mathematical model has also been validated with the existing experimental results conducted by HSU.

In short, this newly proposed mathematical model is a good and quick approach to analyse the reinforced concrete uniaxial and biaxial columns. This model can also help the students and the academic researchers to find the column capacities without going through the column interaction charts and other long iterative approaches.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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