Some Grüss-type results via Pompeiu's-like inequalities

This is the Published version of the following publication

Dragomir, Sever S (2015) Some Grüss-type results via Pompeiu's-like inequalities. Arabian Journal of Mathematics, 4 (3). 159 - 170. ISSN 2193-5343

The publisher’s official version can be found at http://link.springer.com/article/10.1007/s40065-015-0135-8
Note that access to this version may require subscription.

Downloaded from VU Research Repository https://vuir.vu.edu.au/31733/
S. S. Dragomir

Some Grüss-type results via Pompeiu’s-like inequalities

Received: 9 March 2015 / Accepted: 6 August 2015 / Published online: 27 August 2015 © The Author(s) 2015. This article is published with open access at Springerlink.com

Abstract In this paper, some Grüss-type results via Pompeiu’s-like inequalities are proved.

Mathematics Subject Classification 26D15 · 25D10

1 Introduction

In 1946, Pompeiu [18] derived a variant of Lagrange’s mean value theorem, now known as Pompeiu’s mean value theorem (see also [18, p.83]).

Theorem 1.1 (Pompeiu [18]) For every real valued function \( f \) differentiable on an interval \([a, b]\) not containing 0 and for all pairs \( x_1 \neq x_2 \) in \([a, b]\), there exists a point \( \xi \) between \( x_1 \) and \( x_2 \) such that

\[
\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi) .
\] (1.1)

The following inequality is useful to derive some Ostrowski-type inequalities; see [9].

Corollary 1.2 (Pompeiu’s inequality) With the assumptions of Theorem 1.1 and if \( \| f - \ell f' \|_\infty \) = \( \sup_{t \in (a,b)} |f(t) - tf'(t)| < \infty \) where \( \ell(t) = t, t \in [a, b] \), then

\[
|tf(x) - xf(t)| \leq \| f - \ell f' \|_\infty |x - t|
\] (1.2)

for any \( t, x \in [a, b] \).

S. S. Dragomir (✉)
Mathematics, College of Engineering and Science, Victoria University, PO Box 14428, Melbourne, MC 8001, Australia
E-mail: sever.dragomir@vu.edu.au
http://rgmia.org/dragomir

S. S. Dragomir
School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa
The inequality (1.2) was obtained by the author in [9].

For other Ostrowski-type inequalities concerning the $p$-norms $\|f-\ell f\|_p$, see [1,2,17,19].

For two Lebesgue integrable functions $f, g : [a, b] \to \mathbb{R}$, consider the Čebyšev functional:

$$C(f, g) := \frac{1}{b-a} \int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2} \int_a^b f(t)dt \int_a^b g(t)dt. \quad (1.3)$$

Grüss [10] showed that

$$|C(f, g)| \leq \frac{1}{4} (M - m)(N - n), \quad (1.4)$$

provided that there exists the real numbers $m, M, n, N$ such that

$$m \leq f(t) \leq M \quad \text{and} \quad n \leq g(t) \leq N \quad \text{for a.e. } t \in [a, b]. \quad (1.5)$$

The constant $\frac{1}{4}$ is best possible in (1.3) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known, result, though it was obtained by Čebyšev [7], states that

$$|C(f, g)| \leq \frac{1}{12} \left\| f' \right\|_\infty \left\| g' \right\|_\infty (b - a)^2, \quad (1.6)$$

provided that $f', g'$ exist and are continuous on $[a, b]$ and $\left\| f' \right\|_\infty = \sup_{t \in [a, b]} |f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (1.6) also holds if $f, g : [a, b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_\infty[a, b]$, while $\left\| f' \right\|_\infty = \text{ess sup}_{t \in [a, b]} |f'(t)|$.

A mixture between Grüss’ result (1.4) and Čebyšev’s one (1.6) is the following inequality obtained by Ostrowski [15]:

$$|C(f, g)| \leq \frac{1}{8} (b - a)(M - m) \left\| g' \right\|_\infty, \quad (1.7)$$

provided that $f$ is Lebesgue integrable and satisfies (1.5), while $g$ is absolutely continuous and $g' \in L_\infty[a, b]$. The constant $\frac{1}{8}$ is best possible in (1.7).

The case of Euclidean norms of the derivative was considered by Lupaş [12], in which he proved that

$$|C(f, g)| \leq \frac{1}{\pi^2} \left\| f' \right\|_2 \left\| g' \right\|_2 (b - a), \quad (1.8)$$

provided that $f, g$ are absolutely continuous and $f', g' \in L_2[a, b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Recently, Cerone and Dragomir [3] have proved the following results:

$$|C(f, g)| \leq \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_q \cdot \frac{1}{b-a} \left( \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)ds \right|^p dt \right)^{\frac{1}{p}}, \quad (1.9)$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ or $p = 1$ and $q = \infty$, and

$$|C(f, g)| \leq \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_1 \cdot \frac{1}{b-a} \text{ess sup}_{t \in [a, b]} \left| f(t) - \frac{1}{b-a} \int_a^b f(s)ds \right|, \quad (1.10)$$

provided that $f \in L_p[a, b]$ and $g \in L_q[a, b]$ ($p > 1, \frac{1}{p} + \frac{1}{q} = 1; p = 1, q = \infty$ or $p = \infty, q = 1$).

Notice that for $q = \infty, p = 1$ in (1.9), we obtain

$$|C(f, g)| \leq \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)ds \right| dt$$

$$\leq \|g\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)ds \right| dt \quad (1.11)$$
and, if \( g \) satisfies \((1.5)\), then

\[
|C(f, g)| \leq \inf_{y \in \mathbb{R}} \|g - y\|_\infty \cdot \frac{1}{b - a} \int_a^b \left| f(t) - \frac{1}{b - a} \int_a^b f(s) \, ds \right| \, dt
\]

\[
\leq \left\| g - \frac{n + N}{2} \right\|_\infty \cdot \frac{1}{b - a} \int_a^b \left| f(t) - \frac{1}{b - a} \int_a^b f(s) \, ds \right| \, dt
\]

\[
\leq \frac{1}{2} (N - n) \cdot \frac{1}{b - a} \int_a^b \left| f(t) - \frac{1}{b - a} \int_a^b f(s) \, ds \right| \, dt. \tag{1.12}
\]

The inequality between the first and the last term in \((1.12)\) has been obtained by Cheng and Sun \([8]\). However, the sharpness of the constant \( \frac{1}{2} \), a generalization for the abstract Lebesgue integral and the discrete version of it have been obtained in \([4]\).

For other recent results on the Grüss inequality, see \([5, 6, 11, 13, 14, 16, 20]\) and the references therein.

In this paper, some Grüss-type results via Pompeiu’s-like inequalities are proved.

\section{Some Pompeiu’s-type inequalities}

We can generalize the above inequality for the larger class of functions that are absolutely continuous and complex valued as well as for other norms of the difference \( f - \ell f' \).

\begin{theorem}
Let \( f : [a, b] \rightarrow \mathbb{C} \) be an absolutely continuous function on the interval \([a, b]\) with \( b > a > 0 \). Then for any \( t, x \in [a, b] \), we have

\[
|tf(x) - xf(t)| \leq \begin{cases}
\left\| f - \ell f' \right\|_\infty |x - t| & \text{if } f - \ell f' \in L_\infty [a, b], \\
\left( \frac{1}{2} \right)^{1/q} \left\| f - \ell f' \right\|_p \left| \frac{x^q}{p} - \frac{t^q}{p} \right|^{1/q} & \text{if } f - \ell f' \in L_p [a, b],
\end{cases}
\]

or equivalently

\[
\left| \frac{f(x)}{x} - \frac{f(t)}{t} \right| \leq \begin{cases}
\left\| f - \ell f' \right\|_\infty \left| \frac{1}{x} - \frac{1}{t} \right| & \text{if } f - \ell f' \in L_\infty [a, b], \\
\left( \frac{1}{2} \right)^{1/q} \left\| f - \ell f' \right\|_p \left| \frac{1}{x^{2q-1}} - \frac{1}{t^{2q-1}} \right|^{1/q} & \text{if } f - \ell f' \in L_p [a, b],
\end{cases}
\]

\tag{2.2}
\end{theorem}

\begin{proof}
If \( f \) is absolutely continuous, then \( f/\ell \) is absolutely continuous on the interval \([a, b]\) that does not contain \( 0 \) and

\[
\int_t^x \left( \frac{f(s)}{s} \right)' \, ds = \frac{f(x)}{x} - \frac{f(t)}{t}
\]

for any \( t, x \in [a, b] \) with \( x \neq t \).

Since

\[
\int_t^x \left( \frac{f(s)}{s} \right)' \, ds = \int_t^x \frac{f'(s) s - f(s)}{s^2} \, ds,
\]

we get the following identity:

\[
tf(x) - xf(t) = xt \int_t^x \frac{f'(s) s - f(s)}{s^2} \, ds. \tag{2.3}
\]
for any \( t, x \in [a, b] \).

We notice that the equality (2.3) was proved for the smaller class of differentiable function and in a different manner in [17].

Taking the modulus in (2.3), we have

\[
|tf(x) - xf(t)| = \left| xt \int_t^x \frac{f'(s) s - f(s)}{s^2} \, ds \right| 
\leq xt \left| \int_t^x \frac{f'(s) s - f(s)}{s^2} \, ds \right| := I,
\]

and utilizing Hölder’s integral inequality we deduce

\[
I \leq xt \left\{ \sup_{s \in [t,x]} |f'(s) s - f(s)| \right\} \left( \int_t^x \frac{1}{s^2} \, ds \right)^{1/2} \leq \left( \frac{1}{2q - 1} \right)^{1/2} \|f - \ell f'\|_{\infty} |x - t|, 
\]

\[
= \left( \frac{1}{2q - 1} \right)^{1/2} \|f - \ell f'\|_{\infty} \frac{|x^q - t^q|^{1/2}}{s^q - t^q} \left( \frac{1}{2q - 1} \right)^{1/2} < \frac{1}{p + \frac{1}{q} = 1},
\]

and the inequality (2.2) is proved. \( \square \)

**Remark 2.2** The first inequality in (2.1) also holds in the same form for \( 0 > b > a \).

### 3 Some Grüss-type inequalities

We have the following result of Grüss type.

**Theorem 3.1** Let \( f, g : [a, b] \to \mathbb{C} \) be absolutely continuous functions on the interval \( [a, b] \) with \( b > a > 0 \). If \( f', g' \in L_\infty[a, b] \), then

\[
\left| \frac{b^3 - a^3}{3} \int_a^b f(t) g(t) \, dt - \int_a^b tf(t) \, dt \int_a^b tg(t) \, dt \right| 
\leq \frac{1}{12} (b - a)^4 \left\| f - \ell f' \right\|_{\infty} \left\| g - \ell g' \right\|_{\infty}. 
\]

The constant \( \frac{1}{12} \) is best possible.

**Proof** From the first inequality in (2.1), we have

\[
\left| \int_a^b \int_a^b (tf(x) - xf(t)) (tg(x) - xg(t)) \, dr \, dx \right| 
\leq \int_a^b \int_a^b |(tf(x) - xf(t)) (tg(x) - xg(t))| \, dr \, dx 
\leq \left\| f - \ell f' \right\|_{\infty} \left\| g - \ell g' \right\|_{\infty} \int_a^b (x - t)^2 \, dr \, dx.
\]

\[ \square \]
Observe that
\[
\int_a^b \int_a^b (tf(x) - xf(t)) (tg(x) - xg(t)) \, dx \, dt
= \int_a^b \int_a^b \left[ t^2 f(x) g(x) + x^2 f(t) g(t) - tg(t) xf(x) - f(t) txg(x) \right] \, dx \, dt
= 2 \left[ \int_a^b t^2 \, dt \int_a^b f(t) g(t) \, dt - \int_a^b tf(t) \, dt \int_a^b g(t) \, dt \right]
\]
and
\[
\int_a^b \int_a^b (x-t)^2 \, dx \, dt = \frac{1}{3} \int_a^b [(b-x)^3 + (x-a)^3] \, dx = \frac{1}{6} (b-a)^4.
\]
Utilizing the inequality (3.2), we deduce the desired result (3.1).

Now, assume that the inequality (3.1) holds with a constant $B > 0$ instead of $\frac{1}{12}$, i.e.,
\[
\left| \frac{b^3 - a^3}{3} \int_a^b f(t) g(t) \, dt - \int_a^b tf(t) \, dt \int_a^b g(t) \, dt \right|
\leq B \left( b-a \right)^4 \| f - \ell f' \|_\infty \| g - \ell g' \|_\infty .
\] (3.3)

If we take $f(t) = g(t) = 1$, $t \in [a, b]$, then
\[
\frac{b^3 - a^3}{3} \int_a^b f(t) g(t) \, dt - \int_a^b tf(t) \, dt \int_a^b g(t) \, dt
= \frac{b^3 - a^3}{3} (b-a) - \left( \frac{b^2 - a^2}{2} \right)^2 = \frac{1}{12} (b-a)^4
\]
and
\[
\| f - \ell f' \|_\infty = \| g - \ell g' \|_\infty = 1
\]
and by (3.3) we get $B \geq \frac{1}{12}$, which proves the sharpness of the constant. \(\square\)

The following result for the complementary $(p, q)$-norms, with $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, holds.

**Theorem 3.2** Let $f, g : [a, b] \to \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$ with $b > a > 0$. If $f' \in L_p[a, b]$, $g' \in L_q[a, b]$ with $p, q > 1$, $p, q \neq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, then
\[
\left| \frac{b^3 - a^3}{3} \int_a^b f(t) g(t) \, dt - \int_a^b tf(t) \, dt \int_a^b g(t) \, dt \right|
\leq \frac{1}{2} \frac{1}{(2q-1)^{1/q} (2p-1)^{1/p}} \, \| f - \ell f' \|_p \| g - \ell g' \|_q \, M_{p}^{1/p} (a, b) \, M_{q}^{1/q} (a, b) .
\] (3.4)

where
\[
M_q(a, b) := \int_a^b \int_a^b \left| t^q x^q - \frac{x^q}{t^q} \right| \, dx \, dt .
\]
We have the bounds
\[
M_q(a, b) \leq (b-a) \, N_{q}^{1/2} (a, b)
\]
and
\[
M_p(a, b) \leq (b-a) \, N_{p}^{1/2} (a, b)
\]
where, for $r > 1$,

$$N_r (a, b) := \begin{cases} \frac{1}{2} \left( \frac{b^{2r+1}-a^{2r+1}}{2r+1}, \frac{b^{2r+3}-a^{2r+3}}{2r+3} - \left( \frac{b^2-a^2}{2} \right)^2 \right), r \neq \frac{3}{2} \\ (b^2-a^2) \left( \frac{b^2+a^2}{2}, \ln \frac{b}{a} - \left( \frac{b^2-a^2}{2} \right) \right), r = \frac{3}{2}. \end{cases}$$

**Proof** From the second inequality in (2.1), we have

$$|tf(x) - xf(t)| \leq \frac{1}{2q-1} \left\| f - \ell f' \right\|_p \left( \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right)^{1/q}$$

and

$$|rg(x) - xg(t)| \leq \frac{1}{2p-1} \left\| g - \ell g' \right\|_q \left( \frac{x^p}{t^{p-1}} - \frac{t^p}{x^{p-1}} \right)^{1/p}$$

for any $t, x \in [a, b]$.

If we multiply these inequalities and integrate, then we get

$$\left| \int_a^b \int_a^b (tf(x) - xf(t))(tg(x) - xg(t)) \, dr \, dx \right|$$

$$\leq \int_a^b \int_a^b |(tf(x) - xf(t))(tg(x) - xg(t))| \, dr \, dx$$

$$\leq \frac{1}{(2q-1)^{1/q} (2p-1)^{1/p}} \left\| f - \ell f' \right\|_p \left\| g - \ell g' \right\|_q 
\times \int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right|^{1/q} \left| \frac{x^p}{t^{p-1}} - \frac{t^p}{x^{p-1}} \right|^{1/p} \, dr \, dx. \quad (3.5)$$

Utilizing Hölder’s integral inequality for double integrals, we have

$$\int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right|^{1/q} \left| \frac{x^p}{t^{p-1}} - \frac{t^p}{x^{p-1}} \right|^{1/p} \, dr \, dx$$

$$\leq \left( \int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right| \, dr \, dx \right)^{1/q} \left( \int_a^b \int_a^b \left| \frac{x^p}{t^{p-1}} - \frac{t^p}{x^{p-1}} \right| \, dr \, dx \right)^{1/p}$$

$$= M_q^{1/q} (a, b) M_p^{1/p} (a, b) \quad (3.6)$$

for $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Utilizing Cauchy–Bunyakowsky–Schwarz integral inequality for double integrals, we have

$$M_q (a, b) = \int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right| \, dr \, dx$$

$$\leq \left( \int_a^b \int_a^b \, dr \, dx \right)^{1/2} \left( \int_a^b \int_a^b \left( \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right)^2 \, dr \, dx \right)^{1/2}$$

$$= (b-a) \left( \int_a^b \int_a^b \left( \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right)^2 \, dr \, dx \right)^{1/2}.$$
Theorem 3.4

Observe that

\[ N_q(a, b) := \int_a^b \int_a^b \left( \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right)^2 \, dx \, dt \]

\[ = \int_a^b \int_a^b \frac{x^{2q}}{t^{2(q-1)}} \, dx \, dt - 2 \int_a^b \int_a^b \frac{x^q}{t^{q-1}} \frac{t^q}{x^{q-1}} \, dx \, dt + \int_a^b \int_a^b \frac{t^{2q}}{x^{2(q-1)}} \, dx \, dt \]

\[ = 2 \int_a^b x^{2q} \, dx \int_a^b t^{-2(q-1)} \, dt - 2 \left( \int_a^b x \, dx \right)^2 \]

\[ = 2 \left( \frac{b^{2q+1} - a^{2q+1}}{2q+1} - \frac{b^{-2q+3} - a^{-2q+3}}{-2q+3} - \left( \frac{b^2 - a^2}{2} \right)^2 \right) \]

provided \( q \neq \frac{3}{2} \).

If \( q = \frac{3}{2} \), then

\[ N_q(a, b) = (b^2 - a^2) \left[ \frac{b^2 + a^2}{2} \cdot \ln \frac{b}{a} - \frac{b^2 - a^2}{2} \right]. \]

Therefore,

\[ M_q(a, b) \leq (b - a) N^{1/2}_q(a, b) \]

and, similarly,

\[ M_p(a, b) \leq (b - a) N^{1/2}_q(a, b). \]

\[ \square \]

Remark 3.3 The double integral

\[ M_q(a, b) := \int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right| \, dx \, dt \]

can be computed exactly by iterating the integrals. However, the final form is too complicated to be stated here.

The Euclidean norms case is as follows:

**Theorem 3.4** Let \( f, g : [a, b] \to \mathbb{C} \) be absolutely continuous functions on the interval \([a, b] \) with \( b > a > 0 \). If \( f', g' \in L_2[a, b] \), then

\[
\left| \frac{b^3 - a^3}{3} \int_a^b f(t) g(t) \, dt - \int_a^b t f(t) \, dt \int_a^b t g(t) \, dt \right|
\]

\[
\leq \frac{1}{9} \left\| f - \ell f' \right\|_2 \left\| g - \ell g' \right\|_2 \left[ (b^3 + a^3) \ln \frac{b}{a} - \frac{2}{3} (b^3 - a^3) \right]. \tag{3.7}
\]

**Proof** From the second inequality in (2.1), we have

\[
|tf(x) - xf(t)| \leq \frac{1}{\sqrt{3}} \left\| f - \ell f' \right\|_2 \left| \frac{x^2}{t} - \frac{t^2}{x} \right|^{1/2}
\]

and

\[
|tg(x) - xg(t)| \leq \frac{1}{\sqrt{3}} \left\| g - \ell g' \right\|_2 \left| \frac{x^2}{t} - \frac{t^2}{x} \right|^{1/2}
\]

for any \( t, x \in [a, b] \).
If we multiply these inequalities and integrate, then we get
\[
\left| \int_a^b \left( \int_a^b (tf(x) - xf(t)) (tg(x) - xg(t)) \, dt \right) \, dx \right|
\leq \int_a^b \left( \int_a^b |tf(x) - xf(t)| (tg(x) - xg(t)) \, dt \right) \, dx
\leq \frac{1}{3} \| f - \ell f' \|_2 \| g - \ell g' \|_2 \int_a^b \int_a^b \left| \frac{x^2}{t} - \frac{t^2}{x} \right| \, dx.
\tag{3.8}
\]
Since
\[
\int_a^b \int_a^b \left| \frac{x^2}{t} - \frac{t^2}{x} \right| \, dx
= \int_a^b \left( \int_a^x \left( \frac{x^2}{t} - \frac{t^2}{x} \right) \, dt + \int_x^b \left( \frac{t^2}{x} - \frac{x^2}{t} \right) \, dt \right) \, dx
= \int_a^b \left( x^2 (2 \ln x - \ln a - \ln b) + \frac{b^3 + a^3 - 2x^3}{3x} \right) \, dx
\]
and
\[
\int_a^b x^2 (2 \ln x - \ln a - \ln b) \, dx
= \int_a^b 2x^2 \ln x \, dx - \ln (ab) \int_a^b x^2 \, dx
= \frac{(b^3 + a^3) \ln \frac{b}{a}}{3} - \frac{2}{9} (b^3 - a^3),
\]
while
\[
\int_a^b \frac{b^3 + a^3 - 2x^3}{3x} \, dx = \frac{(b^3 + a^3) \ln \frac{b}{a}}{3} - \frac{2}{9} (b^3 - a^3),
\]
then we conclude that
\[
\int_a^b \int_a^b \left| \frac{x^2}{t} - \frac{t^2}{x} \right| \, dx \, dt
= \frac{2}{3} \left[ (b^3 + a^3) \ln \frac{b}{a} - \frac{2}{3} (b^3 - a^3) \right].
\]
Making use of the inequality (3.8), we deduce the desired result (3.7). \( \square \)

**Remark 3.5** It is an open question to the author if \( \frac{1}{9} \) is best possible in (3.7).

**Theorem 3.6** Let \( f, g : [a, b] \to \mathbb{C} \) be absolutely continuous functions on the interval \([a, b]\) with \( b > a > 0 \). Then,
\[
\left| \int_a^b f(t) g(t) \, dt - \int_a^b tf(t) \, dt \int_a^b tg(t) \, dt \right|
\leq \| f - \ell f' \|_1 \| g - \ell g' \|_1 \frac{2b^3 + a^3 - 3ab^2}{6a}.
\tag{3.9}
\]

**Proof** From the third inequality in (2.1), we have
\[
\left| \int_a^b \left( \int_a^b (tf(x) - xf(t)) (tg(x) - xg(t)) \, dt \right) \, dx \right|
\leq \int_a^b \left( \int_a^b |tf(x) - xf(t)| (tg(x) - xg(t)) \, dt \right) \, dx
\leq \| f - \ell f' \|_1 \| g - \ell g' \|_1 \int_a^b \int_a^b \left( \max \{t, x\} \right) \frac{2b^3 + a^3 - 3ab^2}{6a} \, dx.
\tag{3.10}
\]
Observe that

\[
\int_a^b \int_a^b \left( \frac{\max\{t, x\}}{\min\{t, x\}} \right)^2 \, dt \, dx
= \int_a^b \left[ \int_x^a \left( \frac{\max\{t, x\}}{\min\{t, x\}} \right)^2 \, dt \right] \, dx
= \int_a^b \left[ \int_a^x \left( \frac{t}{x} \right)^2 \, dt + \int_x^b \left( \frac{t}{x} \right)^2 \, dt \right] \, dx
= \frac{2b^3 + a^3 - 3ab^2}{6a},
\]

which together with (3.10) produces the desired inequality (3.9).

\[\Box\]

4 Some related results

The following result holds.

**Theorem 4.1** Let \( f, g : [a, b] \to \mathbb{C} \) be absolutely continuous functions on the interval \([a, b]\) with \( b > a > 0 \). If \( f', g' \in L_{\infty} [a, b] \), then

\[
\left| (b - a) \int_a^b \frac{f(t)g(t)}{t^2} \, dt - \int_a^b \frac{f(t)}{t} \, dt \int_a^b \frac{g(t)}{t} \, dt \right|
\leq (b - a)^2 \frac{L^2(a, b) - G^2(a, b)}{L^2(a, b) G^2(a, b)} \left\| f - \ell f' \right\|_\infty \left\| g - \ell g' \right\|_\infty,
\]

where \( G(a, b) := \sqrt{ab} \) is the geometric mean and

\[ L(a, b) := \frac{b - a}{\ln b - \ln a} \]

is the Logarithmic mean.

The inequality (4.1) is sharp.

**Proof** From the first inequality in (2.2), we have

\[
\left| \left( f(x) - \frac{f(t)}{t} \right) \left( g(x) - \frac{g(t)}{t} \right) \right|
\leq \left\| f - \ell f' \right\|_\infty \left\| g - \ell g' \right\|_\infty \left( \frac{1}{t} - \frac{1}{x} \right)^2
\]

for any \( t, x \in [a, b] \).

Integrating this inequality on \([a, b]^2\), we get

\[
\left| \int_a^b \int_a^b \left( f(x) - \frac{f(t)}{t} \right) \left( g(x) - \frac{g(t)}{t} \right) \, dt \, dx \right|
\leq \left\| f - \ell f' \right\|_\infty \left\| g - \ell g' \right\|_\infty \int_a^b \int_a^b \left( \frac{1}{t} - \frac{1}{x} \right)^2 \, dt \, dx.
\]

(4.3)
We have
\[
\int_a^b \int_a^b \left( \frac{f(x)}{x} - \frac{f(t)}{t} \right) \left( \frac{g(x)}{x} - \frac{g(t)}{t} \right) \, dx \, dt = 2 \left[ (b-a) \int_a^b \frac{f(t)g(t)}{t^2} \, dt - \int_a^b \frac{f(t)}{t} \, dt \int_a^b \frac{g(t)}{t} \, dt \right]
\]
and
\[
\int_a^b \int_a^b \left( \frac{1}{t} - \frac{1}{x} \right)^2 \, dx \, dt = 2 (b-a)^2 \frac{L^2(a,b) - G^2(a,b)}{L^2(a,b) G^2(a,b)}.
\]
Making use of (4.3), we get the desired result (4.1).

If we take \( f(t) = g(t) = 1 \), then we have
\[
(b-a) \int_a^b \frac{f(t)g(t)}{t^2} \, dt - \int_a^b \frac{f(t)}{t} \, dt \int_a^b \frac{g(t)}{t} \, dt = (b-a)^2 \frac{L^2(a,b) - G^2(a,b)}{L^2(a,b) G^2(a,b)}
\]
and
\[
\| f - \ell f' \|_\infty = \| g - \ell g' \|_\infty = 1,
\]
and we obtain in both sides of (4.1) the same quantity
\[
(b-a)^2 \frac{L^2(a,b) - G^2(a,b)}{L^2(a,b) G^2(a,b)}.
\]

The case of Euclidian norms is as follows:

**Theorem 4.2** Let \( f, g : [a, b] \to \mathbb{C} \) be absolutely continuous functions on the interval \([a, b]\) with \( b > a > 0 \). If \( f', g' \in L_2[a, b] \), then
\[
\left| (b-a) \int_a^b \frac{f(t)g(t)}{t^2} \, dt - \int_a^b \frac{f(t)}{t} \, dt \int_a^b \frac{g(t)}{t} \, dt \right| \leq \frac{1}{6} \| f - \ell f' \|_2 \| g - \ell g' \|_2 \frac{2(b-a)^3}{a^3 b^3}.
\]

**Proof** From the second inequality in (2.2) for \( p = q = 2 \), we have
\[
\left| \frac{f(x)}{x} - \frac{f(t)}{t} \right| \leq \frac{1}{\sqrt{3}} \| f - \ell f' \|_2 \left| \frac{1}{t^3} - \frac{1}{x^3} \right|^{1/2}
\]
and
\[
\left| \frac{g(x)}{x} - \frac{g(t)}{t} \right| \leq \frac{1}{\sqrt{3}} \| g - \ell g' \|_2 \left| \frac{1}{t^3} - \frac{1}{x^3} \right|^{1/2}
\]
for any \( t, x \in [a, b] \).

On multiplying (4.5) with (4.6), we derive
\[
\left| \left( \frac{f(x)}{x} - \frac{f(t)}{t} \right) \left( \frac{g(x)}{x} - \frac{g(t)}{t} \right) \right| \leq \frac{1}{3} \| f - \ell f' \|_2 \| g - \ell g' \|_2 \left| \frac{1}{t^3} - \frac{1}{x^3} \right|
\]
for any \( t, x \in [a, b] \).
Integrating this inequality on \([a, b]^2\), we get
\[
\int_a^b \int_a^b \left| \frac{f(x)}{x} - \frac{f(t)}{t} \right| \left( \frac{g(x)}{x} - \frac{g(t)}{t} \right) \, dx \, dt \\
\leq \int_a^b \int_a^b \left| \left( \frac{f(x)}{x} - \frac{f(t)}{t} \right) \left( \frac{g(x)}{x} - \frac{g(t)}{t} \right) \right| \, dx \, dt \\
\leq \frac{1}{3} \left\| f - \ell f' \right\|_2 \left\| g - \ell g' \right\|_2 \int_a^b \int_a^b \left| \frac{1}{x^3} - \frac{1}{x^3} \right| \, dx \, dt = \frac{(b - a)^3}{a^2 b^2}.
\]

From (4.8), we then obtain the desired result (4.4). \(\square\)

**Remark 4.3** It is an open question to the author if \(\frac{1}{6}\) is the best possible constant in (4.4).

The interested reader may obtain other similar results in terms of the norms \(\left\| f - \ell f' \right\|_p \left\| g - \ell g' \right\|_q\) with \(p, q > 1\), \(p, q \neq 2\) and \(\frac{1}{p} + \frac{1}{q} = 1\). However, the details are omitted.

**Acknowledgments** The author would like to thank the anonymous referees for their valuable comments that have been implemented in the final version of the paper.

**Open Access** This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

**References**

1. Acu, A.M.; Sofonea, F.D.: On an inequality of Ostrowski type. *J. Sci. Arts* 3(16), 281–287 (2011)
2. Acu, A.M.; Baboș, A.; Sofonea, F.D.: The mean value theorems and inequalities of Ostrowski type. Sci. Stud. Res. Math. Inform. 21(1), 5–16 (2011)
3. Cerone, P.; Dragomir, S.S.: New bounds for the Čebyšev functional. Appl. Math. Lett. 18, 603–611 (2005)
4. Cerone, P.; Dragomir, S.S.: A refinement of the Grüss inequality and applications. Tamkang J. Math. 38(1), 37–49 (2007)
5. Cerone, P.; Dragomir, S.S.: Some bounds in terms of \(\Delta\)-seminorms for Ostrowski-Grüss type inequalities. Soochow J. Math. 27(4), 423–434 (2001)
6. Cerone, P.; Dragomir, S.S.; Roumeliotis, J. Grüss inequality in terms of \(\Delta\)-seminorms and applications. Integr. Transforms Spec. Funct. 14(3), 205–216 (2003)
7. Chebyshev, P.L.: Sur les expressions approximatives des intégraux définis par les prises paires entre les même limites. Proc. Math. Soc. Charkow 2, 93–98 (1882)
8. Cheng, X-L.; Sun, J.: Note on the perturbed trapezoid inequality. J. Ineq. Pure Appl. Math. 3(2), (2002) (art. 29, 7 pp)
9. Dragomir, S.S.: An inequality of Ostrowski type via Pompeiu’s mean value theorem. J. Inequal. Pure Appl. Math. 6(3), (2005) (article 83, 9 pp)
10. Grüss, G.: Über das Maximum des absoluten Betrages von \(\int \frac{1}{b-a} \int_0^b f(x)g(x) \, dx - \int \frac{1}{b-a} \int_0^b f(x) \, dx \int_0^b g(x) \, dx\). Math. Z. 39, 215–226 (1935)
11. Li, X.; Mohapatra, R.N.; Rodriguez, R.S.: Grüss-type inequalities. J. Math. Anal. Appl. 267(2), 434–443 (2002)
12. Lupaș, A.: The best constant in an integral inequality. Mathematica (Cluj, Romania) 15(38(2)), 219–222 (1973)
13. Mercer, A.M.: An improvement of the Grüss inequality. J. Inequal. Pure Appl. Math. 6(4), (2005) (article 93, 4 pp)
14. Mitrović, D.S.; Pečarić, J.E.; Fink, A.M.: Classical and New Inequalities in Analysis. Kluwer Academic Publishers, Dordrecht/Boston/London (1993)
15. Ostrowski, A.M.: On an integral inequality. Aequat. Math. 4, 358–373 (1970)
16. Pachpatte, B.G.: On Grüss like integral inequalities via Pompeiu’s mean value theorem. J. Inequal. Pure Appl. Math. 6(3), (2005) (article 82, 5 pp)
17. Pečarić, J.; Ungar, Š.: On an inequality of Ostrowski type. J. Inequal. Pure Appl. Math. 7(4), (2006) (art. 151, 5 pp)
18. Pompeiu, D.: Sur une proposition analogue au théorème des accroissements fins. Mathematica (Cluj, Romania) 22, 143–146 (1946)
19. Popa, E.C.: An inequality of Ostrowski type via a mean value theorem. Gen. Math. 15(1), 93–100 (2007)
20. Sahoo, P.K.; Riedel, T.: Mean Value Theorems and Functional Equations. World Scientific, Singapore, New Jersey, London, Hong Kong (2000)