On The Solution And Stability Analysis Of 6th order Bvp By Special Multistep Methods

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Abstract. An effective method of solving a 6th order nonlinear BVP based on the numerical differentiation is presented in this article. The Uniqueness and existence properties of the solution are established. A more accurate and reliable process is derived to know the solution of 6th order BVPs. The procedure is verified on nonlinear problem. The solutions are matched with exact solutions and absolute errors obtained in this method are compared with that of Galerkin method.

1. Introduction

Many problems in natural science, medicine, management, engineering, particle dynamics, fluid mechanics, elasticity and heat transfer can be turned into ordinary or partial differential equations using mathematical modelling. To obtain an approximate solution that is similar to an effective solution, we may use numerical methods.

Several researchers have carried out a comprehensive analysis of multi-step approaches for BVPs. Several authors such as Gear, Gragg and Statter, P. Henrici and M. K Jain [1 – 4] have provided a thorough treatment of the topic. In different fields of technology and science, 6th order BVPs occur which include astrophysics, structural engineering, etc. Applications of 6th order BVPs and existence and uniqueness properties can be had from the references [5–10]. For solving a specific case of 6th order BVPs, Siddiqi et al. [7, 8] discussed the quintic and septic spline techniques. Analytical results for these situations are available in occasional cases. In order to explain a special case of 6th order BVP, a sextic B-spline Galerkin technique is developed [9, 10]. Multistep methods using numerical integration and differentiation were discussed in [11, 12] to solve differential equations. Differential Transform Techniques were used to solve differential equations [13, 14]. These methods are also considered in solving some problems related to mathematical modelling and modelling of complex fluid dynamical problems [15 – 25]. For solving Bratu, Lane-Emden–type and other singular boundary value problems, B-spline collocation method was discussed in [26 – 28].

By polynomial replacement, the methods using Numerical Integration were obtained by Henrici [3]. Special multistep methods were developed by substituting y(x) on LHS of $F(x, y, y', y'', y''', y^{(4)}, y^{(5)}, y^{(6)}) = 0$ by a polynomial and differentiating it six times. We have examined
implicit methods and found that they are of order \((k - 5)\). The performance of the procedures are proven by solving 6th order BVPs and matched to exact solution.

2. **Uniqueness and Existence of two-point BVPs**
   Since the theory of existence and uniqueness is important in studying numerical methods for solving BVPs, some of its characteristics are presented here.

Consider the 2nd order linear differential equation

\[
f_0(u)u'' + f_1(x)u' + f_2(x)u = r(x), \quad a \leq x \leq b
\]

Here \(f_0, f_1, f_2\) and \(r\) are continuous on \([a, b]\).

Let the boundary conditions be of the form:

\[
\begin{align*}
l_1(u) &= \alpha_0 u(a) + \alpha_1 u'(a) + \beta_0 u(b) + \beta_1 u'(b) = \mu_0, \\
l_2(u) &= \gamma_0 u(a) + \gamma_1 u'(a) + \delta_0 u(b) + \delta_1 u'(b) = \mu_1,
\end{align*}
\]

Here \(\alpha_i, \beta_i, \gamma_i, \delta_i\) and \(\mu_i\) for \(i = 0, 1\) are constants.

The specific boundary constraints (2) are

(i) Dirichlet conditions \(u(a) = \mu_0, \ u(b) = \mu_1\);

(ii) Neumann’s boundary conditions \(u(a) = \mu_0, \ u'(b) = \mu_1, \ u(a) = \mu_0, \ u'(b) = \mu_1\);

(iii) Mixed boundary conditions \(\alpha_0 u(a) + \alpha_1 u'(a) = \mu_0, \ \alpha_0^2 + \alpha_1^2 \neq 0, \ \delta_0 u(b) + \delta_1 u'(b) = \mu_1, \ \delta_0^2 + \delta_1^2 \neq 0;\)

Periodic boundary conditions \(u(a) = u(b), \ u'(a) = u'(b)\). The homogenous BVP related with (1)-(2) takes the form

\[
\begin{align*}
f_0(u)u'' + f_1(x)u' + f_2(x)u &= 0, \quad a \leq x \leq b, \\
l_1(u) &= 0, l_2(u) = 0.
\end{align*}
\]

**Theorem 2.1** (Agarwal and O’Regan [6])

Let \(u_1(x)\) and \(u_2(x)\) be any two linear and independent solutions of (7). Then, the homogeneous BVP (7) - (8) has only the trivial solution iff

\[
\Delta = \begin{vmatrix} l_1(u_1) & l_1(u_2) \\ l_2(u_1) & l_2(u_2) \end{vmatrix} \neq 0.
\]

**Theorem 2.2** (Agarwal and O’Regan [6])

The non-homogeneous BVP (1) - (2) has a unique solution iff the homogeneous BVP (7)-(8) has trivial solution only.

Consider the 2nd order nonlinear DE

\[
u'' = f(x, u, u'), \quad x \in [a, b],
\]

With respect to the conditions \(u(a) - \mu_0 = 0, \ u(b) - \mu_1 = 0,\)

Where constants \(\mu_0\) and \(\mu_1\) are real finite and arbitrary.

The below given theorem assures the presence of a unique solution of the BVP (10) - (11).

**Theorem 2.3**

Suppose the function \(f\) in BVP (10)-(11) satisfies the below given conditions

(i) \(f\) and all of its partial derivatives are continuous on \(R = \{(x, u, u') : a \leq x \leq b, -\infty < u < \infty, \text{ and } -\infty < u' < \infty \};\)

(ii) \(f_u(x, u, u') \geq \delta > 0\) on some \(R\) for some \(\delta;\)

(iii) Constant \(K\) exists with

\[
\frac{\partial f(x, u, u')}{\partial u'} \leq K \text{ for all } (x, u, u') \in R,
\]

Then the solution of BVP (10)-(11) exists and is unique.
In ordinary differential equations, Agarwal[8] gives a detailed explanation of uniqueness and existence theorems for finding the solution of higher order BVPs.

3. Multistep methods for 6th order BVPs
The 6th order BVP
\[ F(x, y, y', y'', y''', y''''', y'''''v, y''''''v) = 0 \] (13)
Subject to constraints
\[ y(a) = a_0, y'(a) = a_1, y''(a) = a_2, \quad y(b) = b_0, y'(b) = b_1, y''(b) = b_2, \]
which regularly occurs in the applications of science and engineering.
For the solution of equation (13), a k-step multistep method is given by
\[ y_{n+1} = \sum_{j=1}^{k} a_j y_{n+1-j} + \square^6 \sum_{j=0}^{k} b_j y_{n+1-j} \] (14)
Here \( a_j, b_j \) are numeric constants.

Presenting the polynomials
\[ \rho(\xi) = \xi^k - \sum_{j=1}^{k} a_j \xi^{k-j} \quad \text{and} \quad \sigma(\xi) = \sum_{j=1}^{k} b_j \xi^{k-j} \] (15)
Equation (2) can be expressed as
\[ \rho(E)y_{n-k+1} - \square^6 \sigma(E)y_{n-k+1}^{vi} = 0, \text{ and here } E(y_n) = y_{n+1} \] (16)
Applying (4) to \( y^{vi} = \lambda y \), we get
\[ \rho(\xi) - \square^6 \sigma(\xi) = 0 \quad \text{and} \quad \square = \lambda \quad \text{complex} \] (17)
The zeros of (17) and \( \square \) are complex, they lie within the circle of unit radius whenever \( \square \) lies inside the region. Stability region \( R \), its boundary \( \partial R \) gives the locus of \( \partial R \) as
\[ \square(\theta) = \rho(e^{i\theta})/\sigma(e^{i\theta}), 0 \leq \theta \leq 2\pi \] (18)

4. Derivation of approaches
Let the interpolating polynomial of \( y(x) \) be \( p(x) \) which is written as
\[ p(x) = \sum_{m=0}^{k} (-1)^m \frac{(x-x_{n+1})^m}{m!} y_{n+1} \quad \text{s} = \frac{(x-x_{n+1})}{h} \] (19)
Taking derivative of (19) six times in respect of \( x \), we get
\[ p^{vi}(x) = \left( \frac{1}{h^6} \right) \sum_{m=0}^{k} \frac{d^6}{ds^6} \left[ (-1)^m \binom{-s}{m} \right] \frac{\partial}{\partial s} y_{n+1} \] (20)
Substituting \( y^{vi} \) with \( p^{vi}(x) \) in equation (1) and placing \( s = -r \) in \( x_{n+1-r} \), we have
\[ \sum_{m=0}^{k} \delta_{r,m} (\frac{\partial}{\partial s} y_{n+1}) = h^6 f_{n+1-r} \] (21)
Where \( \delta_{r,m} = \frac{d^6}{ds^6} \left[ (-1)^m \binom{-s}{m} \right] \) (22)

5. Determination of Generating function for \( \delta_{r,m} \)
Let us define \( D_{r,t} = \sum_{m=0}^{k} \delta_{r,m} t^m \) (23)
Substituting \( \delta_{r,m} \) from (21) in (22), we get
\[ D_{r,t} = \sum_{m=0}^{\infty} \delta_{r,m} t^m = \frac{[-(1-t)^{-s}] [\log(1-t)]^5}{s} \] (24)
\[ \sum_{m=0}^{\infty} \delta_{r,m} t^m = \frac{[-(1-t)^{-s}] [\log(1-t)]^5}{s} \] at \( s = -r \) (25)

6. Implicit methods
Choosing \( r = 0 \) in (20), the methods can be obtained as
\[ \sum_{m=0}^{k} \delta_{0,m} \frac{\partial}{\partial s} y_{n+1} = h^6 f_{n+1} \] (26)
From (24) it is evident that \( \delta_{0,m} \) is coefficient term of \( t^m \) in \( [\log(1-t)]^5 \) as powers of \( t \) which are presented in the table 1.
After the simplification, equation (24) can be written as
\sum_{j=0}^{k} a_j y_{n+1-j} = \square^6 f_{n+1} \tag{25}

Table 2 gives the coefficients \(a_j\).

The truncation error is \(LTE = \delta_{0,k+1} \square^{k+1} y^{k+1}(\eta)\) \tag{26}

\begin{table}
\[\begin{array}{cccccccccc}
\text{Table 1 Coefficients of } \delta_{0,m} ; \ m = 0 (1) 10 \\
\hline
m & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
\delta_{0,m} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{23}{4} & 9 & \frac{3013}{240} \\
\end{array}\]
\end{table}

\begin{table}
\[\begin{array}{cccccccccccc}
\text{Table 2 Coefficients of } a_j ; j = 0 (1) k, k = 6 (1) 11 \\
\hline
K & J & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
6 & 1 & 4 & -6 & 15 & -20 & 15 & -6 & 1 \\
7 & 8 & \frac{39}{4} & -73 & 239 & -447 & \frac{1045}{2} & -391 & 183 & -49 & \frac{23}{4} \\
9 & 9 & \frac{75}{4} & -154 & 563 & -1203 & \frac{3313}{2} & -1525 & 939 & -373 & \frac{347}{4} & -9 \\
10 & 7513 & \frac{7513}{240} & -6709 & 18047 & -5419 & 34343 & -93773 & 28603 & -3759 & 10427 & -3229 & 3013 \\
\end{array}\]
\end{table}

The method (25) is of order \(k-5\) and is happen to be absolute stable and we have
\[\rho(\xi) = \sum_{j=0}^{k} a_j \xi^{k-j} \text{ and } \sigma(\xi) = \xi^k.\] \tag{27}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The absolute stability region (ASR) of the method (25) for \(k = 6\) and 7.}
\end{figure}
Figure 2: The absolute stability region (ASR) of the method (25) for $k = 8, 9$ and 10.

The 5th order ND method (25) with $k = 6$ is

$$y_{n+1} = 6y_n - 15y_{n-1} + 20y_{n-2} - 15y_{n-3} + 6y_{n-4} - y_{n-5} + \frac{4}{5}f_{n+1}$$

(28)

The 6th order ND method (25) with $k = 7$ is

$$y_{n+1} = \frac{13}{4}y_n - 9y_{n-1} + \frac{55}{4}y_{n-2} - 12y_{n-3} + \frac{27}{4}y_{n-4} - 2y_{n-5} + \frac{1}{4}y_{n-6} + \frac{1}{4}f_{n+1}$$

(29)

The 7th order ND method (25) with $k = 8$ is

$$y_{n+1} = \frac{2922}{39}y_n - \frac{956}{39}y_{n-1} + \frac{1788}{39}y_{n-2} - \frac{4180}{39}y_{n-3} + \frac{1564}{39}y_{n-4} - \frac{732}{39}y_{n-5} + \frac{196}{39}y_{n-6} - \frac{23}{39}y_{n-7} + \frac{4}{39}f_{n+1}$$

(30)

The 8th order ND method (25) with $k = 9$ is

$$y_{n+1} = \frac{616}{75}y_n - \frac{2252}{75}y_{n-1} + \frac{4812}{75}y_{n-2} - \frac{6626}{75}y_{n-3} + \frac{6100}{75}y_{n-4} - \frac{3756}{75}y_{n-5} + \frac{1492}{75}y_{n-6} - \frac{347}{75}y_{n-7} + \frac{36}{75}y_{n-8} + \frac{4}{75}f_{n+1}$$

(31)

The 9th order ND method (25) with $k = 10$ is

$$y_{n+1} = \frac{67090}{7513}y_n - \frac{270705}{7513}y_{n-1} + \frac{650280}{7513}y_{n-2} - \frac{1030290}{7513}y_{n-3} + \frac{1125276}{7513}y_{n-4} - \frac{858090}{7513}y_{n-5} + \frac{451080}{7513}y_{n-6} - \frac{156405}{7513}y_{n-7} + \frac{32290}{7513}y_{n-8} - \frac{240}{7513}f_{n+1}$$

(32)

7. Numerical Illustrations and Results

The ND method derived here in this paper is used to solve the following 6th order nonlinear BVP.

Example: Consider the sixth order nonlinear BVP [10]

$$y^{(6)} - 20e^{-30y} = -40(1 + x)^{-6} \in [0, 1]$$

(33)

Subject to

$$y(0) = 0, \quad y(1) = \frac{\ln 2}{6}, \quad y'(0) = \frac{1}{6}, \quad y'(1) = \frac{1}{12}, \quad y''(0) = \frac{1}{6}, \quad y''(1) = \frac{1}{24}$$
in the interval \([0, 1]\) with \(h = 0.01\).
By partitioning \([0, 1]\) into 10 equal subintervals and applying to the problem, we got the exact solution
\[\text{asy}(x) = \frac{\ln(1 + x)}{6}\] and \(2.0495827258 \times 10^{-12}\) as the highest absolute error.
The derived ND method is applied to solve the above BVP and the results, i.e. exact solution, the numerical solution and absolute errors are presented in the table 3. Also the comparison of exact and approximate solution has been shown graphically.

| Table 3: Solution by 5th order ND method with \(k = 6\) and \(h = 0.01\) |
|-----------------|-----------------|-----------------|-----------------|
| \(x\) | Exact Solution | Solution by 5th order ND | Absolute Error |
| 0 | 0.0000000000 | 4.0372322741E-12 | 4.0372322741E-12 |
| 0.1 | 0.0158850300 | 1.5885029969E-02 | 2.0495827258E-12 |
| 0.2 | 0.0303869261 | 3.0386926133E-02 | 1.1047864013E-12 |
| 0.3 | 0.0437273774 | 4.3727377412E-02 | 6.2617272478E-13 |
| 0.4 | 0.0560787061 | 5.6078706104E-02 | 3.7125857943E-13 |
| 0.5 | 0.0675775180 | 6.7577518018E-02 | 2.2720714199E-13 |
| 0.6 | 0.0783339382 | 7.8333938208E-02 | 1.4327428133E-13 |
| 0.7 | 0.0884380418 | 8.8438041844E-02 | 9.306445039E-14 |
| 0.8 | 0.0979644442 | 9.796444150E-02 | 6.2283511681E-14 |
| 0.9 | 0.1069756477 | 1.0697564770E-01 | 4.3687276019E-14 |
| 1 | 0.1155245301 | 1.1552453009E-01 | 3.0225821845E-14 |

| Table 4: Comparison of Absolute Errors of fifth order ND method and Galerkin method [10] |
|-----------------|-----------------|-----------------|
| \(x\) | Absolute Error by ND method | Absolute Error by Galerkin method |
| 0.1 | 2.0495827258E-12 | 1.117587E-08 |
| 0.2 | 1.1047864013E-12 | 1.583248E-07 |
| 0.3 | 6.2617272478E-13 | 3.539026E-07 |
| 0.4 | 3.7125857943E-13 | 5.960464E-07 |
| 0.5 | 2.2720714199E-13 | 6.631017E-07 |
| 0.6 | 1.4327428133E-13 | 5.811453E-07 |
| 0.7 | 9.3064445039E-14 | 3.352761E-07 |
| 0.8 | 6.2283511681E-14 | 2.011657E-07 |
| 0.9 | 4.3687276019E-14 | 1.117587E-07 |
Figure 3: Comparison of ND Method solution with Exact Solution

8. Conclusion
The solution of sixth order nonlinear BVP by Numerical Differentiation method is presented. The solution obtained by ND methods in this article is more precise. The efficacy and positive side of the derived method is visible in its application, accuracy and efficiency. By analysing the results and values recorded in tables 3 and 4, it is apparent that the results by ND method have surpassed the results achieved by Galerkin method.

9. References
[1] Gear C W 1971 Numerical Initial Value Problems in Ordinary Differential Equations (Prentice Hall)
[2] Gragg W B and H J Statter 1964 Generalized multistep predictor–corrector methods Journal of the ACM 11 188–209
[3] Henrici P 1962 Discrete Variable Methods in Ordinary Differential Equations (Wiley Eastern Limited)
[4] Jain M. K 1984 Numerical Solution of Differential Equations (Wiley Eastern Limited)
[5] Agarwal R P 1986 Boundary Value Problems for Higher Order Differential Equations (Singapore : World Scientific)
[6] Agarwal R P and O’Regan D 2009 Ordinary and partial differential equations with Special functions Fourier series and boundary value problems (New York : Springer)
[7] Siddiqi S S Akram G and Nazeer S 2007 Quintic spline solutions of linear sixth order boundary value problems. Appl. Math. Comput. 189 887–92
[8] Siddiqi S S and Akram G 2008 Septic spline solutions of sixth order boundary value problems. J. Comput. Appl. Math. 215 288–301
[9] Kasi Viswanadham K N S and Murali Krishna P 2010 Sextic B-spline Galerkin method for sixth boundary value problems. Int. J. Math. Sci. Comput. Appl. 4 377–87
[10] Sreenivasulu Ballem and Kasi Viswanadham K N S Numerical solution of sixth order boundary value problems by Galerkin method with quartic B-splines, Numerical Heat Transfer and Fluid Flow, Lecture Notes in Mechanical Engineering. doi.org/10.1007/978-981-13-1903-7_58
[11] Eskandari Z and M Sh Dahaghin 2012 A special linear multistep method for special second order differential equations International Journal of Pure and Applied Mathematics 78(1) 1 – 8
[12] Rama Chandra Rao P S 2006 Special multistep methods based on numerical differentiation for solving the initial value problem Applied Mathematics and Computation 181 500–10 https://doi.org/10.1016/j.amc.2005.12.063
[13] Vishwa Prasad Rao S and Ram Chandra Rao P S 2008 Transform technique for the solution of a class of differential equation, International Journal of Applied Mathematical Analysis and Applications 3(2) 161 – 66
[14] Vishwa Prasad Rao S, Ram Chandra Rao P S and Prabhakara Rao C 2011 Solution of differential equation from the transform techniques, International Journal of Computational Science and Mathematics 3(1) 121 – 25
[15] Anusha G, Vishwa Prasad Rao S and Balarama Krishna C 2019 Designing of modelling and applications in typical engineering process, International Journal of Recent Technology and Engineering 8(2) 2289 – 91 DOI: 10.35940/ijrte.B2665.078219
[16] Anusha G, Balarama Krishna C and Vishwa Prasad Rao S 2019 Mathematical model of a competitive species pair of semi monod type and its stability analysis Journal of Advanced Research in Dynamical & Control Systems 11 (Special Issue-05) 1693 – 98
[17] Balarama Krishna C, Vishwa Prasad Rao S and Anusha G 2019 Solution analysis of a 3rd order initial value problem International Journal of Recent Technology and Engineering 8 (2) 3784 – 89 DOI: 10.35940/ijrte.B3484.078219
[18] Shamshuddin Md and Krishna C B 2019 Heat absorption and joule heating effects on transient free convective reactive micro polar fluid flow past a vertical porous plate Fluid Dynamics & Material Processing 15(3) 207 – 31 DOI:10.32604/fdmp.2019.00449
[19] Balarama Krishna C and Ramachandra Rao P S 2019 Numerical solution and stability analysis of special multistep methods for fifth order differential equations International journal of nonlinear science 27(3) 131 – 47
[20] Anusha G and Balarama Krishna C 2020 Auto ecology logistic model with population dependent carrying capacity International Journal of Mechanical and Production Engineering Research and Development 10(3) 8677-82
[21] Swamy Reddy G, Ravi Kiran G and Archana Reddy R 2019 Radiation impacts on free convection circulation of a power-law fluid past vertical plate filled along with Darcy Porous medium Int. J. Engg. Adv. Tech. 8(6) 4582-85 https://dx.doi.org/10.35940/ijeat.F8886.088619
[22] Ravi Kiran G, Radhakrishna Murthy V and Radhakrishnamacharya G 2019 Pulsatile flow of a dusty fluid thorough a constricted channel in the presence of magnetic field Mat. Today: Proc. 19(6) 2645–49 https://doi.org/10.1016/j.matpr.2019.10.116
[23] Ravi Kiran G, Swamy Reddy G, Devika B and Archana Reddy R 2019 Effect of magnetic field and constriction on pulsatile flow of a dusty fluid J. Mech. Cont. Math. Sci. 14(6) 67 – 82. https://doi.org/10.26782/jmcsms.2019.12.00006
[24] Ravi Kiran G, Radhakrishnamacharya G and Beg OA 2017 Peristaltic flow and hydrodynamic dispersion of a reactive micropolar fluid-simulation of chemical effects in the digestive process J. Mech. Med. Bio.17(1) 1750013. https://doi.org/10.1142/S0219519417500130
[25] Swamy Reddy G and Srinivasacharya D 2019 Heat and mass transfer by Natural convection in a doubly stratified porous medium saturated with Power-law fluid International Journal of Advanced Trends in Computer Applications, 1(1) 66 – 69
[26] Pradip Roul Kiran Thula and Ravi Agarwal 2019 Non-optimal fourth-order and optimal sixth-order B-spline collocation methods for Lane-Emden boundary value problems Applied Numerical Mathematics 145 342–60 https://doi.org/10.1016/j.apnum.2019.05.004
[27] Pradip Roul, Kiran Thula and Prasad Goura V M K 2019 An optimal sixth-order quartic B-spline collocation method for solving Bratu-type and Lane-Emden–type problems
[28] Kiran Thula 2020 A sixth-order numerical method based on shishkin mesh for singularly perturbed boundary value problems *Iranian Journal of Science and Technology, Transactions A: Science* https://doi.org/10.1007/s40995-020-00952-x