GALOIS CORRESPONDENCE FOR GALOIS COVERINGS ON TROPICAL CURVES

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Abstract. We define Galois coverings on tropical curves for which a Galois correspondence and a universal mapping property hold.

1. Introduction

Let $\varphi : \Gamma \to \Gamma'$ be a map between tropical curves and $G$ a finite group. For an isometric action of $G$ on $\Gamma$, $\varphi$ is Galois if (1) $\varphi$ is a finite harmonic morphism, (2) the order of $G$ coincides with the degree of $\varphi$, (3) the action of $G$ on $\Gamma$ induces a transitive action on every fiber, and (4) every stabilizer subgroup of $G$ with respect to all but a finite number of points is trivial. We also say $\varphi$ to be $G$-Galois. Here, a tropical curve is a metric graph that may have edges of length $\infty$, and a finite harmonic morphism is defined as a morphism of our category of tropical curves (see Section 2 for more details). When $\varphi$ is $G$-Galois, for a finite harmonic morphism $\psi : \Gamma \to \Gamma''$, we define $G(\psi)$ as the subset of $G$ consisting of all elements $g$ of $G$ satisfying $\psi \circ g = \psi$. For a fixed $G$-Galois covering $\varphi : \Gamma \to \Gamma'$, let $A'$ be the set of all finite harmonic morphisms $\tilde{\psi} : \Gamma \to \tilde{\Gamma''}$ each of which is $G(\tilde{\psi})$-Galois. For two elements $\psi_1$ and $\psi_2$ of $A'$, we write $\psi_1 \sim \psi_2$ if there exists a finite harmonic morphism of degree one $\psi_{12}$ satisfying $\psi_1 = \psi_{12} \circ \psi_2$. This $\sim$ becomes an equivalence relation. Let $A := A'/\sim$. For $[\psi_i] \in A$, where $[\psi_i]$ denotes the equivalence class of $\psi_i$, we write $[\psi_1] \leq_A [\psi_2]$ if for any $\psi_1' \in [\psi_1]$ and $\psi_2' \in [\psi_2]$, there exists a finite harmonic morphism $\theta$ satisfying $\psi_1' = \theta \circ \psi_2'$. This $\leq_A$ is well-defined and becomes a partial order. Let $B$ be the set of all subgroups of $G$. For $G_i \in B$, we define a partial order $\leq_B$ such that $G_1 \leq_B G_2$ if $G_1 \subset G_2$. Then, we have the following:

Theorem 1.1 (Galois correspondence for Galois coverings). In the above setting, there exists a one-to-one correspondence between $(A, \leq_A)$ and $(B, \leq_B)$ reversing (partial) orders.

For Galois coverings, the following two theorems hold. Here, for a map between tropical curves $\varphi : \Gamma \to \Gamma'$ and an isometric action of a 2020 Mathematics Subject Classification. Primary 14T15; Secondary 14T20.

Key words and phrases. Galois correspondence for Galois coverings on tropical curves, universal mapping property.
finite group $H$ on $\Gamma$, $\varphi$ is $H$-normal if $\varphi$ satisfies the conditions (1), (3), and (4) above.

**Theorem 1.2.** Let $\varphi : \Gamma \to \Gamma'$ be a $G$-Galois covering, $\psi : \Gamma \to \Gamma''$ in $A'$, and $\theta : \Gamma'' \to \Gamma'$ the unique continuous map satisfying $\varphi = \theta \circ \psi$. Then, the following are equivalent:

1. $G(\psi)$ is a normal subgroup of $G$,
2. there exists a finite group $H$ isometrically acting on $\Gamma''$ such that $\theta$ is $H$-Galois, and
3. there exists a finite group $H'$ isometrically acting on $\Gamma''$ such that $\theta$ is $H'$-normal.

Moreover, suppose in addition to (2) that for any $g \in G$, there exists $h \in H$ satisfying $\psi \circ g = h \circ \psi$, then $H$ is isomorphic to the quotient group $G/G(\psi)$.

**Theorem 1.3** (Universal mapping property). Let $\varphi : \Gamma \to \Gamma'$ be $G$-Galois. Then, for any finite harmonic morphism $\psi : \Gamma \to \Gamma''$ satisfying for any $g \in G$, $\psi \circ g = \psi$, there exists a unique finite harmonic morphism $\theta$ satisfying $\psi = \theta \circ \varphi$.

Our definition of Galois coverings on tropical curves is an enhanced version of the definition in [2] (the condition (4) is added). By adding the condition (4), Theorems 1.1 and 1.3 above and Theorem 1.4 below, which will be shown in [3], hold; see Remark 3.7 and Examples 3.8, 3.9, 3.10.

**Theorem 1.4** ([3]). Let $\varphi : \Gamma \to \Gamma'$ be a finite harmonic morphism between tropical curves and $G$ a finite group isometrically acting on $\Gamma$. Then, $\varphi$ is $G$-Galois if and only if the action of $G$ on $\operatorname{Rat}(\Gamma)$ naturally induced by the action of $G$ on $\Gamma$ is Galois for the semifield extension $\operatorname{Rat}(\Gamma)/\varphi^*(\operatorname{Rat}(\Gamma'))$.

Here, $\operatorname{Rat}(\Gamma)$ denotes the semifield of all rational functions on $\Gamma$ and $\varphi^*(\operatorname{Rat}(\Gamma'))$ the pull-back. In the setting of Theorem 1.4, “the action of $G$ on $\operatorname{Rat}(\Gamma)$ is Galois for $\operatorname{Rat}(\Gamma)/\varphi^*(\operatorname{Rat}(\Gamma'))$” means that the $G$-invariant subsemifield of $\operatorname{Rat}(\Gamma)$ is $\varphi^*(\operatorname{Rat}(\Gamma'))$ and that subgroups of $G$ whose invariant semifields coincide are equal.

This paper is organized as follows. In Section 2, we give the definitions of tropical curves and finite harmonic morphisms between tropical curves. Section 3 gives proofs of Theorems 1.1, 1.2, and 1.3.

**Acknowledgements**

The author thanks my supervisor Masanori Kobayashi, Yuki Kageyama, and Yasuhito Nakajima for helpful comments. This work was supported by JSPS KAKENHI Grant Number 20J11910.
2. Preliminaries

In this section, we prepare basic definitions related to tropical curves which we need later. See, for example, [1] for details.

In this paper, a graph is an unweighted, undirected, finite, connected nonempty multigraph that may have loops. For a graph $G$, the set of vertices is denoted by $V(G)$ and the set of edges by $E(G)$. The degree of a vertex is the number of edges incident to it. Here, a loop is counted twice. A leaf end is a vertex of degree one. A leaf edge is an edge incident to a leaf end. A tropical curve is the underlying topological space of the pair $(G,l)$ of a graph $G$ and a function $l: E(G) \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ called a length function, where $l$ can take the value $\infty$ on only leaf edges, together with an identification of each edge $e$ of $G$ with the closed interval $[0,l(e)]$. When $l(e) = \infty$, the interval $[0,\infty]$ is the one point compactification of the interval $[0,\infty)$ and the leaf end of $e$ must be identified with $\infty$. We regard $[0,\infty]$ not just as a topological space but as almost a metric space. The distance between $\infty$ and any other point is infinite. If $E(G) = \{e\}$ and $l(e) = \infty$, then we can identify either leaf ends of $e$ with $\infty$. When a tropical curve $\Gamma$ is obtained from $(G,l)$, the pair $(G,l)$ is called a model for $\Gamma$. There are many possible models for $\Gamma$. A model $(G,l)$ is loopless if $G$ is loopless. Let $\Gamma_\infty$ denote the set of all points of $\Gamma$ identified with $\infty$. An element of $\Gamma_\infty$ is called a point at infinity. We frequently identify a vertex (resp. an edge) of $G$ with the corresponding point (resp. the corresponding closed subset) of $\Gamma$. The relative interior $e^\circ$ of an edge $e$ is $e \setminus \{v,w\}$ with the endpoint(s) $v, w$ of $e$.

Let $\varphi: \Gamma \rightarrow \Gamma'$ be a continuous map between tropical curves. $\varphi$ is a finite morphism if there exist loopless models $(G,l)$ and $(G',l')$ for $\Gamma$ and $\Gamma'$, respectively, such that (1) $\varphi(V(G)) \subset V(G')$ holds, (2) $\varphi(E(G)) \subset E(G')$ holds, and (3) for any edge $e$ of $G$, there exists a positive integer $\deg_e(\varphi)$ such that for any points $x, y$ of $e$, $\text{dist}(\varphi(x), \varphi(y)) = \deg_e(\varphi) \cdot \text{dist}(x, y)$ holds. Here, $\text{dist}(x, y)$ denotes the distance between $x$ and $y$. Assume that $\varphi$ is a finite morphism. $\varphi$ is harmonic if for every vertex $v$ of $G$, the sum $\sum_{e \in E(G): v \rightarrow e', e \in e} \deg_e(\varphi)$ is independent of the choice of $e' \in E(G')$ incident to $\varphi(v)$. This sum is denoted by $\deg_v(\varphi)$. Then, the sum $\sum_{e \in V(G): v \rightarrow v'} \deg_e(\varphi)$ is independent of the choice of vertex $v'$ of $G'$. It is said the degree of $\varphi$ and written by $\deg(\varphi)$. If both $\Gamma$ and $\Gamma'$ are singletons, we regard $\varphi$ as a finite harmonic morphism that can have any number as its degree. Note that if $\varphi \circ \psi$ is a composition of finite harmonic morphisms, then it is also a finite harmonic morphism of degree $\deg(\varphi) \cdot \deg(\psi)$, and thus tropical curves and finite harmonic morphisms between them make a category.

Remark 2.1. Let $\varphi: \Gamma \rightarrow \Gamma'$ be a map between tropical curves. Then $\varphi$ is a continuous map whose restriction on $\Gamma \setminus \Gamma_\infty$ is an isometry if and only if it is a finite harmonic morphism of degree one. In this
paper, we will use the word “a finite group $G$ isometrically acts on a tropical curve $Γ$” as the meaning that $G$ continuously acts on $Γ$ and it is isometric on $Γ \setminus Γ_∞$.

**Remark 2.2.** Let $Γ$ be a tropical curve and $G$ a finite group isometrically acting on $Γ$. Let $Γ/G$ be the quotient space (as a topological space) and $π_G : Γ \to Γ/G$ be the natural surjection. Fix a loopless model $(V, E, l)$ ($V$ is a set of vertices and $E$ is a set of edges) for $Γ$ such that for any $g \in G$ and $e \in E$, $g(V) = V$ holds and if $g(e) = e$, then the restriction of $g$ on $e$ is the identity map of $e$. Note that by the proof of [1, Lemma 3.4], we can choose such a loopless model. Let $V′ := π_G(V)$, $E′ := π_G(E)$ and for any $e \in E$, $l′(π_G(e)) := |G_e| \cdot l(e)$, where $G_e$ denotes the stabilizer subgroup of $G$ with respect to $e$ and $|G_e|$ the order of $G_e$. Then, $(V′, E′, l′)$ gives $Γ/G$ a tropical curve structure and is a loopless model for the quotient tropical curve $Γ/G$. By loopless models $(V, E, l)$ and $(V′, E′, l′)$ for $Γ$ and $Γ/G$, respectively, $π_G$ is a finite harmonic morphism of degree $|G|$.

3. Main results

In this section, we give our definition of Galois coverings on tropical curves and proofs of Theorems [1][1][1] 1.2 and 1.3.

**Definition 3.1.** Let $Γ$ be a tropical curve and $G$ a finite group. An isometric action of $G$ on $Γ$ is Galois if there exists a finite subset $U′$ of $Γ/G$ such that for any $x′ ∈ (Γ/G) \setminus U′$, $|π_G^{-1}(x′)| = |G|$ holds.

**Definition 3.2.** Let $φ : Γ \to Γ′$ be a map between tropical curves. $φ$ is Galois if there exists a Galois action of a finite group $G$ on $Γ$ such that there exists a finite harmonic morphism of degree one $θ : Γ/G → Γ′$ satisfying $φ \circ g = θ \circ π_G$ for any $g ∈ G$. Then, we say that $φ$ is a G-Galois covering on $Γ′$ or just $G$-Galois.

**Proof of Theorem [1][1] 1.1** Let

$$Φ : A → B; \quad [ψ] ↦ G(ψ)$$

and

$$Ψ : B → A; \quad G′ ↦ [π_{G′}].$$

We shall show that if $ψ_1 ∼ ψ_2$ holds, then $G(ψ_1) = G(ψ_2)$ holds. For any $f ∈ G(ψ_2)$, we have

$$ψ_1 ∘ f = (ψ_1 ∘ ψ_2) ∘ f = ψ_1 ∘ (ψ_2 ∘ f) = ψ_1 ∘ ψ_2 = ψ_1.$$

Thus $f ∈ G(ψ_1)$ holds. The inverse inclusion can be shown similarly. Therefore, $Φ$ is well-defined.

We shall show that for any $G′ ∈ B$, the action of $G′$ on $Γ$ induced by that of $G$ on $Γ$ is Galois. Let $U′$ be the finite subset of $Γ/G$ in Definition [1][1] 3.1. For any $x ∈ Γ \setminus π_G^{-1}(U′)$, we have $|G_x| = \frac{|G|}{|G_x|} = 1$, where we use
the orbit-stabilizer theorem (cf. [1] Chapter 6) and \(Gx\) stands for the orbit of \(x\) by \(G\), and thus \(|G'_x| = 1\). Therefore, \(\pi_{G'}(\pi_{G'}^{-1}(U'))\) is finite and for any \(x' \in (\Gamma/G') \setminus \pi_{G'}(\pi_{G'}^{-1}(U'))\), we have \(|\pi_{G'}^{-1}(x')| = |G'|\).

We shall show that for any \(G' \in G\), \(G' = G(\pi_{G'})\) holds. By the definition of \(G(\pi_{G'})\), \(G' \subset G(\pi_{G'})\) holds. Let \(g \in G(\pi_{G'})\). For any \(x \in \Gamma\), since \((\pi_{G'} \circ g)(x) = \pi_{G'}(x)\), there exists \(g' \in G'\) such that \(g(x) = g'(x)\). Let us assume that \(x\) is in \(\Gamma \setminus \pi_{G'}^{-1}(U')\). Then \(g = g'\) must hold. In fact, if \(g \neq g'\), then \(g^{-1}g \neq 1\) and \(g^{-1}g(x) = x\). Thus, \(g^{-1}g' \in G_x \setminus \{1\}\) and it contradicts that \(|\pi_{G'}^{-1}(x)| = |G|\). By these arguments, we have \([\pi_{G'}] \in A\) and \(\Phi \circ \Psi = id_B\), where \(id_B\) denotes the identity map of \(B\).

We shall show that \(\Psi \circ \Phi = id_A\) holds. By the definitions of \(A\) and Galois coverings, for any \(\psi \in A', \psi \sim \pi_{G(\psi)}\) holds, and hence we have

\[
(\Psi \circ \Phi)(\psi) = [\pi_{G(\psi)}] = [\psi].
\]

We shall show that if \([\psi_1] \leq_A [\psi_2]\) holds, then \(G(\psi_1) \supseteq_B G(\psi_2)\) holds. For \(g \in G(\psi_2)\), we have \(\psi_2 \circ g = \psi_2\). Since \([\psi_1] \leq_A [\psi_2]\), there exists a finite harmonic morphism \(\theta\) satisfying \(\psi_1 = \theta \circ \psi_2\). Therefore

\[
\psi_1 \circ g = (\theta \circ \psi_2) \circ g = \theta \circ (\psi_2 \circ g) = \theta \circ \psi_2 = \psi_1
\]

hold. Hence \(g \in G(\psi_1)\).

Finally, we shall show that if \(G_1 \leq_B G_2\) holds, then \([\pi_{G_1}] \geq_A [\pi_{G_2}]\) holds. Since \(G_1 \subset G_2\), there exists a unique continuous map \(\theta\) satisfying \(\theta \circ \pi_{G_1} = \pi_{G_2}\). We shall show \(\theta\) is a finite harmonic morphism. Let \((V, E, l_1, (V_1, E_1, l_1))\) and \((V_2, E_2, l_2)\) be loopless models for \(\Gamma\), \(\Gamma/G_1\) and \(\Gamma/G_2\), respectively, compatible with \(\pi_{G_1}\), i.e., \(\pi_{G_1}(V) = V_i\) for \(i = 1, 2\). Let \(e \in E\). Since each \(\pi_{G_i}\) is Galois, for any points \(x, y, e\) of \(e\), we have

\[
dist(\pi_{G_2}(x), \pi_{G_2}(y)) = dist(x, y) = dist(\pi_{G_1}(x), \pi_{G_1}(y)).
\]

Hence \(\theta\) is a finite morphism and \(deg_{\pi_{G_1}(e)}(\theta) = 1\). Next, we shall show that \(\theta\) is harmonic. Fix a vertex \(v' \in V_1\) and an edge \(e'' \in E_2\) incident to \(\theta(v')\). Since the action of \(G\) on \(\Gamma\) is isometric and \(G_1 \subset G_2\), with any \(v \in \pi_{G_2}^{-1}(\theta(v'))\), we have

\[
\sum_{\theta(e')} \deg_{\pi_{G_1}(e')}(\theta) = \frac{1}{|\theta^{-1}(\theta(v'))|} \cdot \sum_{\theta(e')} \deg_{\pi_{G_1}(e')} = \frac{1}{|\theta^{-1}(\theta(v'))|} \cdot |\{e' \in E_1 | \theta(e') = e''\}|
\]

and

\[
|\theta^{-1}(\theta(v'))| = \frac{|\pi_{G_2}^{-1}(\theta(v'))|}{|\pi_{G_1}^{-1}(\theta(v'))|} = \frac{|G_2 v|}{|G_1 v|}.
\]

With any edge \(e \in E\) such that \(\pi_{G_2}(e) = e''\), the ratio \(\frac{|G_2 e|}{|G_1 e|} = \frac{|G_2|}{|G_1|}\) coincides with the number of elements of the inverse image of \(e''\) by \(\theta\).
Therefore we have
\[
\sum_{e' \in E_1: e' \to e''} \deg_{e'}(\theta) = \frac{|G_1v|}{|G_2v|} \cdot \frac{|G_2v|}{|G_1v|} = |G_2v| \in \mathbb{Z}_{\geq 0},
\]
where we use the orbit-stabilizer theorem at the second equality. Hence the sum is independent of the choice of \(e''\) and thus \(\theta\) is harmonic. □

Next, we will prove Theorem 12. For this purpose, we define normal coverings on tropical curves:

**Definition 3.3.** Let \(\varphi: \Gamma \to \Gamma'\) be a map between tropical curves. \(\varphi\) is normal if there exists a Galois action of a finite group \(H\) on \(\Gamma\) and a finite harmonic morphism \(\theta: \Gamma/H \to \Gamma'\) such that for any \(h \in H\), \(\varphi \circ h = \theta \circ \pi_H\) holds. Then, we say that \(\varphi\) is a \(H\)-normal covering on \(\Gamma'\) or just \(H\)-normal.

By definition, if \(\varphi\) is \(G\)-Galois, then it is \(G\)-normal. Normal coverings have the following property:

**Proposition 3.4.** Let \(\varphi: \Gamma \to \Gamma'\) be a \(H\)-normal covering on \(\Gamma'\). Then, for any finite harmonic morphism \(\psi: \Gamma'' \to \Gamma\), there exists a loopless model \((\Gamma'', l'')\) for \(\Gamma''\) such that for any \(e'' \in E(\Gamma'')\) and any automorphism \(f\) of \(\Gamma''\), i.e., a finite harmonic morphism of degree one \(\Gamma'' \to \Gamma''\), satisfying \(\varphi \circ \psi \circ f = \varphi \circ \psi\), there exists \(h \in H\) such that \((\psi \circ f)|_{e''} = (h \circ \psi)|_{e''}\), where \(|_{e''}\) denotes the restriction of the map on \(e''\).

**Proof.** By the proof of [1, Lemma 3.4], we can choose \((\Gamma'', l'')\) such that \(\psi(V(\Gamma''))\) (resp. \((\varphi \circ \psi)(V(\Gamma''))\)) induces a loopless model for \(\Gamma\) (resp. \(\Gamma'\)) compatible with \(\psi\) (resp. \(\varphi \circ \psi\)) and for any \(e'' \in E(\Gamma'')\) and \(h \in H\), if \(h(\psi(e'')) = \psi(e'')\), then the restriction \(h|_{\psi(e'')}\) is the identity map on \(\psi(e'')\). For any edge \(e'' \in E(\Gamma'')\) and any point \(x''\) of \((e'')^\circ\), since \((\varphi \circ \psi \circ f)(x'') = (\varphi \circ \psi)(x'')\), there exists \(h \in H\) satisfying \((\psi \circ f)(x'') = (h \circ \psi)(x'').\) Since the action of \(H\) on \(\Gamma\) is isometric and \(h\) does not invert any edge, \((\psi \circ f)|_{(e'')^\circ} = (h \circ \psi)|_{(e'')^\circ}\) holds. Since both \(\psi \circ f\) and \(h \circ \psi\) are continuous, \((\psi \circ f)|_{e''} = (h \circ \psi)|_{e''}\) holds. □

This proposition is an analogue of the fact in the field theory that for an algebraic extension \(L/K\) of fields, it is normal if and only if all embeddings \(\sigma\) of \(L\) to the algebraic closure of \(K\) containing \(L\) which is the identity on \(K\) satisfy \(\sigma(L) = L\). In the tropical curve case, however, we must consider only an edge instead of the whole tropical curve.

Let \(\varphi: \Gamma \to \Gamma'\) be a \(G\)-Galois covering on \(\Gamma'\). For any \(\psi: \Gamma \to \Gamma''\) in \(A'\), since \(G(\psi)\) is a subgroup of \(G\) and \(\Gamma'\) and \(\Gamma''\) have quotient topologies, there exists a unique continuous map \(\theta: \Gamma'' \to \Gamma'\) such that \(\varphi = \theta \circ \psi\).

**Proposition 3.5.** In the above setting, the following are equivalent:

1. \(G(\psi)\) is a normal subgroup of \(G\),

   \[2.\]
(2) there exists a finite group $H$ isometrically acting on $\Gamma''$ such that $\theta$ is $H$-Galois and for any $g \in G$, there exists $h \in H$ satisfying $\psi \circ g = h \circ \psi$, and

(3) there exists a finite group $H'$ isometrically acting on $\Gamma''$ such that $\theta$ is $H'$-normal and for any $g \in G$, there exists $h' \in H'$ satisfying $\psi \circ g = h' \circ \psi$.

Moreover, if the above conditions hold, then $H$ in (2) is isomorphic to the quotient group $G/G(\psi)$.

For a map between tropical curves, we will use the word “it is a local isometry” as the meaning that it is continuous on the whole domain tropical curve and that it is a local isometry on the domain tropical curve except all points at infinity.

**Proof of Proposition 3.5.** Clearly (2) implies (3).

Assume that (3) holds. For $g \in G$ and $h' \in H'$, if $\psi \circ g = h' \circ \psi$, then $\psi \circ g^{-1} = h'^{-1} \circ \psi$, and thus for any $g' \in G(\psi)$, we have

$$\psi \circ (g^{-1}g'g) = (\psi \circ g^{-1}) \circ g'g = (h'^{-1} \circ \psi) \circ g'g$$

$$= h'^{-1} \circ (\psi \circ g') \circ g = h'^{-1} \circ \psi \circ g$$

$$= h'^{-1} \circ (\psi \circ g) = h'^{-1}h' \circ \psi = \psi.$$ 

This means that $g^{-1}g'g \in G(\psi)$. Thus (1) holds.

Assume that (1) holds. By the same argument of the last part of the proof of Theorem 1.1, $\theta$ is a finite harmonic morphism of degree $[G]$. Here, $G(\psi)$ corresponds to $G_1$ and $G$ to $G_2$ in this case. We define an isometric action of $G/G(\psi)$ on $\Gamma''$ as $[g](x'') := \psi(g(x))$ for $[g] \in G/G(\psi)$, $x'' \in \Gamma''$ and any $x \in \psi^{-1}(x'')$. It is well-defined. In fact, if $x_1 \in \psi^{-1}(x'')$, then there exists $g_1 \in G(\psi)$ such that $g_1(x_1) = x$. Since $G(\psi)$ is a normal subgroup of $G$, there exists $g_2 \in G(\psi)$ such that $g(x) = gg_1(x_1) = g_2g(x_1)$. Thus we have $\psi(g(x)) = \psi(g_2g(x_1)) = \psi(g(x_1))$. If $[g] = [g_1]$, then there exists $g_1 \in G(\psi)$ such that $g = g_1g_2$. Hence $\psi(g(x)) = \psi(g_1g_2(x)) = \psi(g(x))$. Since the action of $G$ on $\Gamma$ is Galois, there exists a finite subset $U'$ of $\Gamma/G$ such that for any $x' \in (\Gamma/G) \setminus U'$, $|\pi_{\Gamma}^{-1}(x')| = |G|$ holds. Let $O$ be the finite subset $(\pi_{G/G(\psi)} \circ \psi)(\pi_{\Gamma}^{-1}(U'))$ of $(\Gamma''/(G/G(\psi)))$, where $\pi_{G/G(\psi)} : \Gamma'' \to (\Gamma''/(G/G(\psi)))$ is the natural surjection. Let $[x''] \in (\Gamma''/(G/G(\psi))) \setminus O$. We have $\pi_{G/G(\psi)}^{-1}([x'']) = (G/G(\psi))x''$. Since $|\varphi^{-1}(\theta(x''))| = |G|$ and $\psi$ is $G(\psi)$-Galois, we have

$$|(G/G(\psi))x''| = |\psi(\varphi^{-1}(\theta(x'')))| = \frac{|G|}{|G(\psi)|},$$

and thus the action of $G/G(\psi)$ on $\Gamma''$ is Galois. Let

$$\phi : \Gamma''/(G/G(\psi)) \to \Gamma'; \quad \pi_{G/G(\psi)}(x'') \mapsto \theta(x''),$$

then $\phi$ is an isomorphism.
where $x'' \in \Gamma''$. With any fixed element $x \in \psi^{-1}((G/G(\psi))x'')$, since
\[
\theta((G/G(\psi))x'') = \theta(\psi(Gx)) = \theta(\psi(x)) = \varphi(x),
\]
we have $\theta(x'') = \varphi(x)$, and thus $\phi$ is well-defined. By definition, for any $[g] \in G/G(\psi)$, $\phi \circ \pi_{G/G(\psi)} = \theta \circ [g]$ holds. Since $\pi_{G/G(\psi)}$ and $\theta$ are local isometries and $[g]$ is an automorphism, $\phi$ is a local isometry. When $\theta(x'') = \theta(y'')$, with any $x \in \psi^{-1}((G/G(\psi))y'')$ and $y \in \psi^{-1}((G/G(\psi))y'')$, $(G/G(\psi))x'' = \psi(Gx) = \psi(Gy) = (G/G(\psi))y''$ hold, and thus $\pi_{G/G(\psi)}(x'') = \pi_{G/G(\psi)}(y'')$ holds. This means that $\phi$ is injective and hence a finite harmonic morphism of degree one. In conclusion, $\theta$ is $G/G(\psi)$-Galois.

We shall show the last assertion. By (1), the quotient group $G/G(\psi)$ naturally acts on $\Gamma''$ as above. Since each stabilizer subgroup of $G$ with respect to each point of $\Gamma$ except a finite number of points is trivial, the natural action is faithful, i.e., for any equivalence class $[g] \in G/G(\psi)$ other than $[1]$, there exists a point $x'' \in \Gamma''$ such that $[g](x'') \neq x''$. For any $g \in G$, $\psi \circ g = [g] \circ \psi$ and there exists $h \in H$ such that $\psi \circ g = h \circ \psi$. Since the natural action above is faithful, if $[g] \neq [\bar{g}]$, then $[g] \circ \psi \neq [\bar{g}] \circ \psi$. Thus, the map
\[
G/G(\psi) \to H; \quad [g] \mapsto h
\]
is injective. Since
\[
|H| = \text{deg}(\theta) = \frac{|G|}{|G(\psi)|} = |G/G(\psi)|
\]
holds, this map is surjective. By definition, it is a group homomorphism. In conclusion, we have the desired group isomorphism. □

To remove the last conditions in (2) and (3) of Proposition 3.5, we prove the following lemma by using Proposition 3.4.

Lemma 3.6. In the same setting in Proposition 3.5, the following are equivalent:

1. there exists a finite group $H$ isometrically acting on $\Gamma''$ such that $\theta$ is $H$-normal, and
2. there exists a finite group $H'$ isometrically acting on $\Gamma''$ such that $\theta$ is $H'$-normal and for any $g \in G$, there exists $h' \in H'$ satisfying $\psi \circ g = h' \circ \psi$.

Proof. (2) clearly implies (1).

Assume that (1) holds. For any $g \in G$, we define $h'_g : \Gamma'' \to \Gamma''; \psi(x) \mapsto (\psi \circ g)(x)$, where $x \in \Gamma$. Then, by Proposition 3.4, $h'_g$ is a local isometry. Since $\psi \circ g$ is surjective and continuous, $h'_g$ is an automorphism of $\Gamma''$. Set $H'$ as the group generated by these $h'_g$. Then $H'$ induces an isometric and faithful action on $\Gamma''$. For any $x \in \Gamma$, we check that $H'\psi(x) = H\psi(x)$. Since for any $g \in G$, there exists $h \in H$
such that 
\[(h'_g \circ \psi)(x) = (\psi \circ g)(x) = (h \circ \psi)(x),\]
we have \(H'\psi(x) \subset H\psi(x).\) Conversely, for any \(h \in H^\prime,\) since \(\psi\) holds, we have 
\[\varphi(\psi^{-1}((h \circ \psi)(x))) = \theta((h \circ \psi)(x)) = (\theta \circ \psi)(x) = \varphi(x),\]
\[\psi^{-1}((h \circ \psi)(x)) \text{ is contained in } Gx.\] Therefore, for any \(\tilde{x} \in \psi^{-1}((h \circ \psi)(x)),\) there exists \(g \in G\) such that \(\tilde{x} = g(x).\) Thus, we have 
\[(h \circ \psi)(x) = \psi(\tilde{x}) = \psi(g(x)) = (\psi \circ g)(x) = (h'_g \circ \psi)(x),\]
and hence \(H'\psi(x) \supset H\psi(x).\) From this, the action of \(H'\) on \(\Gamma^\prime\) induces a transitive action on each fiber by the transitivity of the action of \(H\) on each fiber. In conclusion, \(\theta\) is \(H'\)-normal and satisfies that \(\theta \circ \psi = \varphi\) and for any \(g \in G,\) there exists \(h'_g \in H'\) such that \(\psi \circ g = h'_g \circ \psi.\) \(\square\)

By Proposition \ref{prop:harmonic-morphisms} and Lemma \ref{lem:harmonic-morphisms} we prove theorem \ref{thm:Galois-coverings}.

Finally, we shall prove Theorem \ref{thm:Galois-coverings}. This theorem means that all Galois coverings are categorical quotients.

**Proof of Theorem \ref{thm:Galois-coverings}** Since \(\Gamma'\) has the quotient topology, we have a unique continuous map \(\theta\) satisfying \(\psi = \theta \circ \varphi,\) and thus it is enough to check that \(\theta\) is a finite harmonic morphism. Fix loopless models \((V, E, l), (V', E', l')\) and \((V'', E'', l'')\) for \(\Gamma, \Gamma'\) and \(\Gamma'',\) respectively, compatible with \(\varphi\) and \(\psi.\) We can choose such loopless models by the assumptions. Let \(e \in E.\) As for any \(x, y \in e,\) \(\text{dist}(x, y) = \text{dist}(\varphi(x), \varphi(y))\) holds, we have \(\text{deg}_{e} (\psi) = \text{deg}_{\varphi(e)} (\theta).\) Therefore, \(\theta\) is a finite morphism. Let \(v\) be in \(V\) and fix an edge \(e'' \in E''\) incident to \(\psi(v) = \theta(\varphi(v)).\) Then, since \(\psi\) is a finite harmonic morphism, we have 
\[
\text{deg}_{e'} (\psi) = \sum_{e \in E; e \rightarrow e', v \in e} \text{deg}_{e'} (\psi) \\
= \sum_{e \in E; \varphi(e) \rightarrow e', \varphi(v) \in \varphi(e)} \text{deg}_{\varphi(e)} (\theta) \cdot \frac{|Gv|}{|G|} \\
= \sum_{e' \in E'; e' \rightarrow e'', \varphi(v) \in \varphi(e')} \text{deg}_{e'} (\theta) \cdot \frac{|Gv|}{|G|} \\
= \frac{|G|}{|Gv|} \sum_{e' \in E'; e' \rightarrow e'', \varphi(v) \in \varphi(e')} \text{deg}_{e'} (\theta).
\]
Thus we have 
\[
\sum_{e' \in E'; e' \rightarrow e'', \varphi(v) \in \varphi(e')} \text{deg}_{e'} (\theta) = \text{deg}_{e} (\psi) \cdot \frac{|Gv|}{|G|} = \text{deg}_{v} (\psi) \cdot \frac{|Gv|}{|G|}.
\]
It is independent of the choice of \(e''\) and since \(\text{deg}_{e'} (\theta) = \text{deg}_{e} (\psi)\) with any \(e \in E\) such that \(\varphi(e) = e',\) it is a positive integer. These mean that \(\theta\) is harmonic as \(v\) is an element of \(V.\) The uniqueness comes from the universal mapping property of quotient topology. \(\square\)
Remark 3.7. In [2], the author gave another definition of Galois coverings on tropical curves. With that definition, for a map between tropical curves $\varphi : \Gamma \to \Gamma'$, if there exists an isometric action of a finite group $G$ on $\Gamma$ such that there exists a finite harmonic morphism of degree one $\theta : \Gamma/G \to \Gamma'$ satisfying $\varphi \circ g = \theta \circ \pi_G$ for any $g \in G$, we call $\varphi$ a $G$-Galois covering on $\Gamma'$. In this paper, let us call it a $G$-preGalois covering on $\Gamma'$ or a preGalois covering simply. There exists a preGalois covering on a tropical curve for which the Galois correspondence and the universal mapping property do not hold. See Examples 3.8, 3.9, and 3.10. By Examples 3.9 and 3.10, we know that for a preGalois covering $\varphi : \Gamma \to \Gamma'$, even there exists a model $(G, l)$ for $\Gamma$ such that the greatest common divisor of $\{\deg_e(\varphi) | e \in E(G)\}$ is one, $\varphi$ may not be a Galois covering.

\[\text{Figure 1. On each figure, black dots (resp. lines) stand for vertices (resp. edges).}\]

Example 3.8. Let $G$ be the graph whose set of vertices consists of two vertices and whose set of edges consists of three multiple edges $\{e_1, e_2, e_3\}$ between them (the left figure of Figure 1). Let $l$ be the length function such that $l(E(G)) = \{1\}$ and $\Gamma$ the tropical curve obtained from $(G, l)$. Then, the symmetric group of degree three $\Sigma_3$ isometrically acts on $\Gamma$ in a natural way. The quotient graph $G/\Sigma_3$ and the length function $E(G/\Sigma_3) \to \mathbb{R} \cup \{\infty\}$; $[e_i] \mapsto 2$ give the quotient space $\Gamma/\Sigma_3$ a tropical curve structure. Let $\Gamma'$ stand for this quotient tropical curve. Then, the natural surjection $\pi_{\Sigma_3} : \Gamma \to \Gamma'$ is a $\Sigma_3$-preGalois covering on $\Gamma'$. Note that $\Gamma'$ is isometric to the closed interval $[0, 2]$.

Let $\sigma$ be the cyclic permutation of multiple edges $(e_1e_2e_3)$. For a finite harmonic morphism $\psi : \Gamma \to \Gamma''$, if $\psi \circ \sigma = \psi$, then for any $\beta \in \Sigma_3$, $\psi \circ \beta = \psi$ holds. Hence, Theorem 1.1 does not hold for $\pi_{\Sigma_3}$.

$\pi_{\Sigma_3}$ is not $\Sigma_3$-Galois and does not have a universal mapping property. In fact, since $\deg_{e_1}(\pi_{\Sigma_3}) = 2$, $\pi_{\Sigma_3}$ is not $\Sigma_3$-Galois. Since for any $\beta \in \Sigma_3$, $\pi_{\langle\sigma\rangle} \circ \beta = \pi_{\langle\sigma\rangle}$ holds, there exists a unique continuous map $\theta$ satisfying $\pi_{\langle\sigma\rangle} = \theta \circ \pi_{\Sigma_3}$. Here, $\langle\sigma\rangle$ stands for the subgroup of $G$ generated by $\sigma$. On the other hand, since $\deg(\pi_{\Sigma_3}) = 6$ and $\deg(\pi_{\langle\sigma\rangle}) = 3$, $\theta$ is not a finite harmonic morphism. Hence, $\pi_{\Sigma_3}$ does not have a universal mapping property.
Example 3.9. Let $e_1, \ldots, e_6$ be the edges of the star of six edges $S_6$, i.e., the complete bipartite graph $K_{1,6}$ (the center figure of Figure 1). Let $l$ be the length function such that $l(E(S_6)) = \{1\}$ and $\Gamma$ be the tropical curve obtained from $(S_6, l)$. Let $\sigma$ (resp. $\beta$) be the cyclic permutation $(e_1e_2)$ (resp. $(e_3e_4e_5e_6)$) and $G$ the finite group generated by $\sigma$ and $\beta$. Then, the quotient graph $S_6/G$ and the length function $E(S_6/G) \to \mathbb{R} \cup \{\infty\}$; $[e_1] \mapsto 4; [e_3] \mapsto 2$ give the quotient space $\Gamma/G$ a tropical curve structure. Let $\Gamma'$ stand for this quotient tropical curve. Then, the natural surjection $\pi_G : \Gamma \to \Gamma'$ is a $G$-preGalois covering.

For a finite harmonic morphism $\psi : \Gamma \to \Gamma''$, if $\psi \circ (\sigma\beta^2) = \psi$, then $\psi \circ \beta^2 = \psi$ holds. Hence, Theorem 1.1 does not hold for $\pi_G$.

Let $H$ be the subgroup of $G$ generated by $\beta$. Then, the natural surjection $\pi_H : \Gamma \to \Gamma'/H$ is $H$-preGalois that is not $H$-Galois and does not have a universal mapping property. In fact, since $\deg_{e_5}(\pi_H) = 4$, $\pi_H$ is not $H$-Galois. Note that the quotient graph $S_6/H$ is the tree consisting of three vertices and two edges $[e_1], [e_3]$, and $S_6/H$ and the length function $E(S_6/H) \to \mathbb{R} \cup \{\infty\}$; $[e_1] \mapsto 4; [e_3] \mapsto 1$ give the quotient space $\Gamma/H$ a tropical curve structure. Let $\gamma$ be the cyclic permutation $(e_1e_2e_3e_4e_5e_6)$. Then, the quotient tropical curve $\Gamma/(\gamma)$ is isometric to the closed interval $[0, 1]$. Since $\pi_{\gamma} \circ \beta = \pi_{\gamma}$ holds, there exists a unique continuous map $\theta$ satisfying $\pi_{\gamma} = \theta \circ \pi_H$. On the other hand, since $\deg(\pi_H) = 4$ and $\deg(\pi_{\gamma}) = 6$, $\theta$ is not a finite harmonic morphism. Hence, $\pi_H$ does not have a universal mapping property.

Example 3.10. Let $e_1, \ldots, e_5$ be the edges of the star of five edges $S_5$, i.e., the complete bipartite graph $K_{1,5}$ (the right figure of Figure 1). Let $l$ be the length function such that $l(E(S_5)) = \{1\}$ and $\Gamma$ be the tropical curve obtained from $(S_5, l)$. Let $\sigma$ (resp. $\beta$) be the cyclic permutation $(e_1e_2)$ (resp. $(e_3e_4e_5)$) and $G$ the finite group generated by $\sigma$ and $\beta$. Then, the quotient graph $S_5/G$ and the length function $E(S_5/G) \to \mathbb{R} \cup \{\infty\}$; $[e_1] \mapsto 3; [e_3] \mapsto 2$ give the quotient space $\Gamma/G$ a tropical curve structure. Let $\Gamma''$ stand for this quotient tropical curve. Then, the natural surjection $\pi_G : \Gamma \to \Gamma''$ is a $G$-preGalois covering.

$\pi_G$ is not $G$-Galois and does not have a universal mapping property. In fact, since $\deg_{e_5}(\pi_G) \geq 2$, $\pi_G$ is not $G$-Galois. Let $\Gamma''$ be the tropical curve isometric to the closed interval $[0, 1]$. Note that $\Gamma''$ is the quotient tropical curve of $\Gamma$ by the natural action of the group generated by the cyclic permutation $\gamma := (e_1e_2e_3e_4e_5)$ on $\Gamma$. Let $\pi_{\gamma} : \Gamma \to \Gamma''$ be the natural surjection. Since for any $\delta \in G$, $\pi_{\gamma} \circ \delta = \pi_{\gamma}$, there exists a unique continuous map $\theta$ satisfying $\pi_{\gamma} = \theta \circ \pi_G$. On the other hand, since $\deg(\pi_{\gamma}) = 6$ and $\deg(\pi_G) = 5$, $\theta$ is not a finite harmonic morphism. Hence, $\pi_G$ does not have a universal mapping property.

Note that since Galois coverings are preGalois coverings, for our definition of Galois coverings, all corresponding assertions in [2] hold and in addition, we need not consider tropical curves with edge-multlicities.
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