QUASI-HERMITIAN PAIR AND CO-AMENABILITY

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Abstract. We adapt the notion of quasi-Hermition group to the pairs \((G, H)\) of discrete group \(G\) and its subgroup \(H\). We show that a quasi-Hermitian pair is amenable in the sense of Eymard.

1. Introduction

Hermitian Banach \(*\)-algebras make an important class of \(*\)-algebras that extends the class of \(C^*\)-algebras. It has been an effectively elaborated in approximation theory, time-frequency analysis and signal process, and non-commutative geometry. In this paper, we are mainly focus on the (quasi-)Hermitian property of Banach \(*\)-algebras derived from a group.

Spectrum of a function in group algebra plays a crucial role in study of analytic group theory. For example, amenability is characterized by the fact that the spectrum of a normalized characteristic function has 1 as a spectral point in the reduced group \(C^*\)-algebra. The property of being Hermitian is a variant of such spectral study. A locally compact group \(G\) is called Hermitian (resp. quasi-Hermitian) if the spectrum \(\text{Sp}(f, L^1(G))\) is in the real line for all self-adjoint functions \(f = f^*\) in \(L^1(G)\) (resp. in \(C_c(G)\)). It was a long-standing conjecture that Hermitian groups are amenable until it was affirmatively proven in [SW20] even for quasi-Hermitian groups. In this paper, we extend the notion of quasi-Hermition group to pairs \((G, H)\) of discrete group \(G\) and its subgroup \(H\) and adapt the Samei-Wiersma’s results for quasi-Hermition pairs.

As amenability is characterized in many ways, it has many versions such as weak amenability, a-T-menability (also known as Haagerup property), Haagerup-Kraus’s approximation property, co-amenability, etc. We are more interested in co-amenability in the sense of Eymard. Recall the original definition of amenability given by John von Neumann: A discrete group \(G\) is amenable if there is a \(G\)-invariant state on \(\ell^\infty(G)\). Similarly, a subgroup \(H\) of \(G\) is co-amenable in \(G\) if there is a \(G\)-invariant state on \(\ell^\infty(G/H)\). Our main result is as follows.

Theorem 1. If \((G, H)\) is quasi-Hermitian, then \(H\) is co-amenable in \(G\).

The notion of quasi-Hermitian pair extends the usual notion of quasi-Hermitian group. In particular, if we take \(H\) to be the trivial subgroup, we recover Samei-Wiersma’s result for discrete groups. Indeed, \(G\) is quasi-Hermitian if and only if \((G, \{e\})\) is quasi-Hermitian, and \(G\) is amenable if and only if \(\{e\}\) is co-amenable in \(G\). This extension allows us to target a richer class.

About the proof of Theorem 1, we essentially follow the steps of [SW20]. The first step is that we construct Banach \(*\)-algebras \(PF^*_a(G : H)\) associated to the pair \((G, H)\) and
1 \leq p \leq 2. It generalizes the usual Banach \ast\text{-}algebras \(PF^\ast_p(G)\). When \(p = 2\), we have \(PF^\ast_p(G : H) = C^\ast_{\Lambda_{G/H}}(G)\). The second step is to see that the triple

\[(PF^\ast_{p_1}(G : H), PF^\ast_{p_2}(G : H), PF^\ast_{p_3}(G : H))\]

is a spectral interpolation of \ast\text{-}semisimple Banach \ast\text{-}algebras relative to \(PF^\ast_1(G : H)\) for any \(1 \leq p_1 < p_2 < p_3 \leq 2\). Then in Theorem 10, we will prove that \((G, H)\) is quasi-Hermitian if and only if the spectrum of \([f] \in [C G]\) in \(PF^\ast_p(G : H)\) does not depend on \(1 \leq p \leq 2\). This latter was proven in [Bar90] for amenable Hermitian groups, during which time it was unknown that Hermitian groups are amenable. The third step is to see that if the universal \(C^\ast\)-enveloping algebra of \(PF^\ast_1(G, H)\) is canonically isomorphic to \(C^\ast_{\Lambda_{G/H}}(G)\), then \(H\) is co-amenable in \(G\). Then Proposition 2.6 of [SW20] concludes our proof.

2. Preliminaries

We recall some definitions and useful results.

2.1. Banach \ast\text{-}algebras. Let \(A\) be a Banach \ast\text{-}algebra and let \(S\) be an \ast\text{-}subalgebra of \(A\). For \(a \in A\), we denote by \(Sp(a, A)\) the spectrum of \(a\) in \(A\), and by \(\tau(a, A)\) the spectral radius of \(a\) in \(A\). Denote by \(S_h\) the self-adjoint elements of \(S\). We say that \(S\) is quasi-Hermitian in \(A\) if \(Sp(a, A) \subseteq \mathbb{R}\) for all \(a \in S_h\). Similarly, we say that \(S\) is quasi-symmetric in \(A\) if \(Sp(a^\ast a, A) \subseteq \mathbb{R}_+\) for all \(a \in S\). Banach \ast\text{-}algebra \(A\) is called Hermitian (resp. symmetric) if it is quasi-Hermitian (resp. quasi-symmetric) in itself. Shirali-Ford’s theorem states that a Banach \ast\text{-}algebra is Hermitian if and only if it is symmetric (see for example [BD73, Theorem 41.5]).

By a \ast\text{-}representation of \(A\), we understand a pair \((\pi, \mathcal{H})\) where \(\mathcal{H}\) is a Hilbert space and \(\pi : A \to B(\mathcal{H})\) is a \ast\text{-}homomorphism into bounded operators on \(\mathcal{H}\). It is a well known fact that any \ast\text{-}representation is norm decreasing. We say that \(A\) is \ast\text{-}semisimple if for any non-zero element \(a \in A\), there is a \ast\text{-}representation that sends \(a\) into a non-zero element. In the sequel, \(A\) is always assumed to be \ast\text{-}semisimple. For a \ast\text{-}representation \((\pi, \mathcal{H})\) of \(A\), we denote by \(C^\ast_{\pi}(A)\) the completion of \(\pi(A)\) in \(B(\mathcal{H})\). Let \((\pi, \mathcal{H})\) and \((\rho, \mathcal{K})\) be \ast\text{-}representations of \(A\). We say that \(\pi\) weakly contains \(\rho\) and write \(\rho \prec \pi\) if \(\|\rho(a)\| \leq \|\pi(a)\|\) for all \(a \in A\). We say that \(\pi\) and \(\rho\) are weakly equivalent and write \(\pi \sim \rho\) if they weakly contains each other. We say that \(\pi\) and \(\rho\) are isomorphic and write \(\pi \cong \rho\) if there is a unitary operator \(U \in B(\mathcal{H}, \mathcal{K})\) such that \(\pi(a) = U^\ast \rho(a) U\) for all \(a \in A\). Note that for any family \(\{(\pi_i, \mathcal{H}_i)\}\) of \ast\text{-}representations, their \(\ell^2\)-direct-sum gives a \ast\text{-}representation \((\pi, \mathcal{H})\) such that \(\pi_i \prec \pi\) for all \(a \in A\). This yields the existence of the universal \ast\text{-}representation \((\pi_u, \mathcal{H}_u)\) of \(A\). This is the unique \ast\text{-}representation satisfying the following universal property: Any \ast\text{-}representation of \(A\) uniquely factors through \(C^\ast_{\pi_u}(G)\). The \(C^\ast\)-algebra \(C^\ast(A) = C^\ast_{\pi_u}(G)\) is called the universal \(C^\ast\text{-}enveloping algebra of \(A\). Since \(A\) is \ast\text{-}semisimple, we can see \(A\) as a dense \ast\text{-}subalgebra of \(C^\ast(G)\). If \(A\) is \ast\text{-}semisimple, then the inclusion \(A \subseteq C^\ast(A)\) is spectral, i.e.

\[Sp(a, A) = Sp(a, C^\ast(A)) \quad \text{for all} \quad a \in A.\]

The converse is rather trivial.

The main tool applied in [SW20] was the notion of spectral interpolation which was inspired by [Pty82]. Let us explain it more in detail. We say that \(A \subseteq B\) is a nested pair of \((\ast\text{-}semisimple)\) Banach \ast\text{-}algebras if \(A\) and \(B\) are \((\ast\text{-}semisimple)\) Banach \ast\text{-}algebras and \(A\) embeds continuously into \(B\) as a dense \ast\text{-}subalgebra. A nested triple of \((\ast\text{-}semisimple)\)
Banach algebras is defined similarly. Let \( S \) be a \( * \)-subalgebra of \( A \). We say that \( S_h \) has \textit{invariant spectral radius} in \( (A, B) \) if \( r(a, A) = r(a, B) \) for all \( a \in S \). We say that \( S \) is \textit{spectral subalgebra} of \( (A, B) \) if \( \text{Sp}(a, A) = \text{Sp}(a, B) \) for all \( a \in S \). Suppose that \( A \subseteq B \subseteq C \) is a nested triple of Banach \( * \)-algebras and \( S \) is a dense \( * \)-subalgebra of \( A \). We say that \( (A, B, C) \) is a \textit{spectral interpolation} of triple Banach \( * \)-algebras relative to \( S \) if there is a constant \( 0 < \theta < 1 \) such that
\[
 r(a, B) \leq r(a, A)^{1-\theta} r(a, C)^{\theta} \quad \text{for all} \quad a \in S_h.
\]
The following two results are crucial to our main result.

**Proposition 2** ([SW20, Proposition 2.6]). Let \( A \subseteq B \) be a nested pair of \(*\)-semisimple Banach \( * \)-algebras, and \( S \) be a dense \(*\)-subalgebra of \( A \). Suppose that \( S_h \) has an invariant spectral radius in \( (A, B) \). Then \( A \) and \( B \) have the same universal \( C^* \)-enveloping algebra. In particular, if \( B \) is a \( C^* \)-algebra, then \( B \) is the universal \( C^* \)-enveloping algebra of \( A \).

**Theorem 3** ([SW20, Theorem 3.4]). If \( (A, B, C) \) is a spectral interpolation of triple \(*\)-semisimple Banach \(*\)-algebras relative to a quasi-Hermitian dense \(*\)-subalgebra \( S \) of \( A \), then \( S_h \) has invariant spectral radius in \( (B, C) \). In particular, we have the canonical \(*\)-isomorphism \( C^*(B) \cong C^*(C) \).

### 2.2. Group representation

Let \( G \) be a discrete group. A \textit{unitary representation} of \( G \) is a pair \((\pi, \mathcal{H})\) where \( \mathcal{H} \) is a Hilbert space and \( \pi : G \to \mathcal{U}(\mathcal{H}) \) is a group homomorphism into the group of unitary operators on \( \mathcal{H} \). It can be extended linearly to the \(*\)-representation \( \pi : \ell^1(G) \to \mathcal{B}(\mathcal{H}) \). Conversely, any \(*\)-representation of \( \ell^1(G) \) restricts to a unitary representation of \( G \). Thus the representation theory of \( G \) and \( \ell^1(G) \) coincide.

Let \( H \leq G \) be a subgroup. Consider the regular representation
\[
\lambda_{G/H} : \ell^1(G) \to \mathcal{B}(\ell^2(G/H))
\]
defined by
\[
\lambda_{G/H}(f)(g)g(xH) = (f \ast g)(xH) = \sum_{y \in G} f(y)g(y^{-1}xH)
\]
for all \( f \in \ell^1(G), g \in \ell^2(G/H) \), and \( x \in G \). The kernel of \( \text{Ker} = \ker(\lambda_{G/H}) \) consists of the functions \( g \in \ell^1(G) \) such that \( \sum_{y \in G} g(xH) = 0 \) for all \( x, y \in G \). Let us denote
\[
\ell^1(G : H) = \ell^1(G)/\text{Ker}.
\]
Then the regular representation gives rise to the faithful \(*\)-representation
\[
\tilde{\lambda}_{G/H} : \ell^1(G : H) \to \mathcal{B}(\ell^2(G/H)).
\]
For \( f \in \ell^1(G) \), denote by \([f] \in \ell^1(G : H)\) the corresponding class, and by \( \tilde{f} \in \ell^1(G/H) \) the function defined by \( \tilde{f}(xH) = \sum_{y \in G} f(xH) \) for all \( x \in G \). Put
\[
[C^*G] = \{[f] \in \ell^1(G : H) : f \in C^*G\}.
\]

**Definition 4.** A \((G, H)\)-\textit{representation} is a unitary representation \((\pi, \mathcal{H})\) of \( G \) such that \( \pi(f) = 0 \) for all \( f \in \text{Ker} \).
Remark that \((\pi, \mathcal{H})\) is a \((G, H)\) representation if and only if \((\pi, \mathcal{H})\) is a \(*\)-representation of \(\ell^1(G)\) such that \(\pi(Ker) = \{0\}\) if and only if \((\pi, \mathcal{H})\) induces a \(*\)-representation of \(\ell^1(G : H)\) if and only if \((\pi, \mathcal{H})\) induces a \(*\)-representation of \([CG]\). We will denote by \(C^*(G : H)\) the universal \(C^*\)-enveloping algebra of \(\ell^1(G : H)\) (or equivalently \([CG]\)). If we denote by \((\pi_u, \mathcal{H}_u)\) the largest \((G : H)\) representation of \(G\) (in the sense of weak containment), then we have \(C^*_u(G) \cong C^*(G : H)\).

2.3. Pseudo-function \(*\)-algebras. We will now construct an analogue of pseudo-function \(*\)-algebras for the pair \((G, H)\). Let \(1 \leq p \leq q \leq \infty\) be conjugate numbers, i.e. \(p^{-1} + q^{-1} = 1\). Consider the regular representation

\[
\lambda^p_{G/H} : \ell^1(G) \to \mathcal{B}(\ell^p(G/H))
\]

defined by

\[
\lambda^p_{G/H}(f)g(xH) = (f \ast g)(xH) \sum_{y \in G} f(y)g(y^{-1}xH)
\]

for all \(f \in \ell^1(G)\), \(g \in \ell^p(G/H)\), and \(x \in G\). Since its kernel is independent of the choice of \(1 \leq p \leq \infty\), we get the faithful representation

\[
\tilde{\lambda}^p_{G/H} : \ell^1(G : H) \to \mathcal{B}(\ell^p(G/H)).
\]

We denote by

\[
[f] \in \ell^1(G : H) \mapsto \|f\|_{PF_p} := \|\lambda^p_{G/H}(f)\|
\]

the induced norm on \(\ell^1(G : H)\). For \([f] \in \ell^1(G : H)\), its \(PF^*_p\)-norm is given by

\[
\|f\|_{PF^*_p} = \max\{\|f\|_{PF_p}, |f|_{PF_q}\}.
\]

Denote by \(PF^*_p(G : H)\) the completion of \(\ell^1(G : H)\) with respect to \(PF^*_p\)-norm. Note that \(\|f\|_{PF_p} = \|f\|_{PF^*_p}\) for all \(f \in \ell^1(G)\), and consequently \(PF^*_p(G : H)\) becomes a unital Banach \(*\)-algebra. For instance, when \(p = 2\), we have \(PF^*_2(G : H) \cong C^*_{\lambda G/H}(G)\). The following lemma helps to understand the image of the algebras \(PF^*_p(G : H)\).

Lemma 5. Let \(1 \leq p_1 \leq p_2 \leq 2\). The identity map on \(\ell^1(G : H)\) extends to the norm decreasing injective \(*\)-homomorphisms

\[
\ell^1(G : H) \to PF^*_{p_1}(G : H) \to PF^*_{p_1}(G : H) \to PF^*_{p_2}(G : H) \to C^*_{\lambda G/H}(G).
\]

In particular, these Banach \(*\)-algebras are \(*\)-semisimple.

Proof. Let \(q_i\) be the conjugate of \(p_i\), i.e. \(p_i^{-1} + q_i^{-1} = 1\). For any \(f \in \ell^1(G)\), we have by complex interpolation

\[
\|f\|_{PF^*_p} \geq \|\lambda^p_{G/H}(f)\|^{1-\theta} \|\lambda^q_{G/H}(f)\|^\theta \geq \|\lambda^p_{G/H}(f)\|
\]

where \(\theta \in [0; 1]\) is such that \(p_2^{-1} = (1 - \theta)p_1^{-1} + \theta q_1^{-1}\). Using the same inequality for \(q_2\), we get \(\|f\|_{PF^*_p} \geq \|f\|_{PF^*_p}\). In other words, the identity on \(\ell^1(G : H)\) extends to a norm decreasing \(*\)-homomorphism \(i_{p_1, p_2} : PF^*_{p_1}(G : H) \to PF^*_{p_2}(G : H)\). Take any \(T \in \ker(i_{p_1, p_2})\).
Then there is a sequence of functions $f_n \in \mathbb{C}G$ such that $[f_n] \to T$ in $PF_{p_1}^*$-norm and $[f_n] \to 0$ in $PF_{p_2}^*$-norm. Seeing $T$ as an operator on $\ell^{p_1}(G/H)$, we get
\[
\langle T\delta_yh, \delta_xh \rangle_{\ell^{p_1}, \ell^{p_1}} = \lim_n \langle f_n * \delta_yh, \delta_xh \rangle_{\ell^{p_1}, \ell^{p_1}}
= \lim_n \sum_{h \in H} f_n(xhy)
= \lim_n \langle f_n * \delta_yh, \delta_xh \rangle_{\ell^{p_2}, \ell^{p_2}} = 0
\]
for all $x, y \in G$. Since $\{\delta_xh : x \in G\}$ is total in $\ell^p(G/H)$ for any $1 \leq p < \infty$, we get $T = 0$. This proves that $i_{p_1, p_2}$ is injective. 

3. Main section

3.1. Quasi-Hermitian pair. We are ready to give the definition of quasi-Hermitian pair.

**Definition 6.** Let $G$ be a discrete group, and let $H$ be its subgroup. The pair $(G, H)$ is called quasi-Hermitian (resp. quasi-symmetric) if $[\mathbb{C}G]$ is quasi-Hermitian (resp. quasi-symmetric) in $PF_1^*(G : H)$. The group $G$ is Hermitian (resp. symmetric) if the pair $(G, \{e\})$ is quasi-Hermitian (resp. quasi-symmetric).

**Example 7.** If $H$ has finite index in $G$, then the Banach $*$-algebras $PF_1^*(G : H)$ are finite dimensional and Hermitian. In particular, the pair $(G, H)$ is quasi-Hermitian.

**Example 8.** Suppose that $H$ is a non-amenable normal subgroup of $G$ such that $Q = G/H$ is quasi-Hermitian. Then $G$ is not quasi-Hermitian, and $(G, H)$ is quasi-Hermitian. Indeed, the quotient map $p : G \to Q$ induces an isomorphism $\ell^1(G : H) \cong \ell^1(Q)$ that is stable on $[\mathbb{C}G] \cong \mathbb{C}Q$. Since $\ell^1(Q)$ is isometrically embedded in $B(\ell^1(G/H))$ and $B(\ell^2(G/H))$, we have $\ell^1(G : H) \cong \ell^1(Q) \cong PF_1^*(G : H)$. Thus, $(G : H)$ is quasi-Hermitian if and only if $[\mathbb{C}G]$ is quasi-Hermitian in $PF_1^*(G : H)$ if and only if $\mathbb{C}Q$ is quasi-Hermitian in $\ell^1(Q)$ if and only if $Q$ is quasi-Hermitian.

**Proposition 9.** Let $1 \leq p_1 < p_2 < p_3 \leq 2$. Then
\[
(PF_{p_1}^*(G : H), PF_{p_2}^*(G : H), PF_{p_3}^*(G : H))
\]
is a spectral interpolation of triple Banach $*$-algebras relative to $PF_{p_1}^*(G : H)$.

**Proof.** The proof is a verbatim of [SW20, Proposition 4.5]. The spectral radius inequality part is a well known consequence of complex interpolation, and Lemma 5 shows that $PF_{p_1}^*(G : H) \subseteq PF_{p_2}^*(G : H) \subseteq PF_{p_3}^*(G : H)$ is a nested triple of $*$-subalgebras. 

**Theorem 10.** Let $G$ be a discrete group, and let $H \leq G$ be a subgroup. The following are equivalent.

(i) $(G, H)$ is quasi-symmetric.

(ii) $(G, H)$ is quasi-Hermitian.

(iii) For all $f = f^* \in \mathbb{C}G$, we have $r([f], PF_{p}^*(G : H)) = \|\lambda_{G/H}(f)\|$. 

(iv) For all $f = f^* \in \mathbb{C}G$, $r([f], PF_{p}^*(G : H))$ is independent of $p \in [1, 2]$.

(v) For all $f \in \mathbb{C}G$, we have $\text{Sp}([f], PF_{1}^*(G : H)) = \text{Sp}([f], C^*_G(H))$.

(vi) For all $f \in \mathbb{C}G$, $\text{Sp}([f], PF_{p}^*(G : H))$ is independent of $p \in [1, 2]$. 


Proposition 12. Let $C$ be a canonical equality of the full group is co-amenable. Since the converse inequality is always true, the above is equality.

Let $S$ be a symmetric finite subset of $G$ containing the support of $f(n) * g$. Then $\lambda_{G/H}^1(f(n))\tilde{g}$ is supported on $[S] = \{sH \in G/H : s \in S\}$ and

$$
(1 - \varepsilon)\|f^{(n)}\|_{PF^*_p(G : H)} \leq \|\lambda_{G/H}^1(f^{(n-1)})\tilde{g}\|_{\ell^1(G/H)} = \|\lambda_{G/H}^1(f^{(n-1)})f * g\|_{\ell^1(G/H)}.
$$

Let $S$ be a symmetric finite subset of $G$ containing the support of $f^{(n)} * g$. Then $\lambda_{G/H}^1(f^{(n)})\tilde{g}$ is supported on $[S] = \{sH \in G/H : s \in S\}$ and

$$
(1 - \varepsilon)\|f^{(n)}\|_{PF^*_p(G : H)} \leq \|\lambda_{G/H}^1(f^{(n-1)})\tilde{g}\|_{\ell^1(G/H)} = \|\lambda_{G/H}^1(f^{(n-1)})f * g\|_{\ell^1(G/H)}.
$$

Taking the $n$th root on both sides and lim sup over $p \to 1$, we get

$$
(1 - \varepsilon)^{1/n}\|f^{(n)}\|_{PF^*_p(G : H)} \leq \limsup_{p \to 1} \|f^{(n-1)}\|_{PF^*_p(G : H)}^{1/n} = \|f^{(n-1)}\|_{PF^*_p(G : H)}^{1/n} + \varepsilon,
$$

for some $1 < p_0$. Now letting $n \to \infty$ and then $\varepsilon \to 0$, we get

$$
r([f], PF^*_p(G : H)) \leq r([f], PF^*_p(G : H)) = \|\lambda_{G/H}(f)\|.
$$

Since the converse inequality is always true, the above is equality.

\[\square\]

3.2. **Co-amenable subgroup.** Co-amenableability was introduced in [Eym72] for a pair $(G, H)$ of locally compact group $G$ and its closed subgroup $H$. It extends the usual amenability. Basic examples are when the quotient $G/H$ is compact, and when $H$ is normal and the quotient group $G/H$ is amenable. We recall the definition.

**Definition 11.** Let $G$ be a discrete group, and let $H \leq G$ be a subgroup. We say that $H$ is co-amenable in $G$ if there exists a left $G$-invariant mean on $\ell^2(G/H)$.

The usual amenability is characterized in many different ways. One of them is the canonical equality of the full group $\text{C}^*$-algebra $\text{C}^*(G)$ and the reduced group $\text{C}^*$-algebra $\text{C}^*_\chi(G)$. A partial analogue of this result is observed for co-amenability.

**Proposition 12.** Let $G$ be a discrete group and let $H \leq G$ be a subgroup. Consider the following statements.

(i) The canonical map $\text{C}^*(G : H) \to \text{C}^*_{\chi_{G/H}}(G)$ is injective.
(iii) The canonical map $C^*(PF_1^*(G : H)) \to C^*_{\lambda_{G/H}}(G)$ is injective.

(iii) The subgroup $H$ is co-amenable in $G$.

We have (i) ⇒ (ii) ⇒ (iii). If $H$ is normal, we have (iii) ⇒ (i).

**Proof.** The implication (i) ⇒ (ii) follows from the surjective *-homomorphisms

$$C^*(G : H) \to C^*(PF_1^*(G : H)) \to C^*_{\lambda_{G/H}}(G).$$

Let us prove (ii) ⇒ (iii). Suppose that the map $C^*(PF_1^*(G : H)) \to C^*_{\lambda_{G/H}}(G)$ is injective, hence *-isomorphism. Consider the character $\sigma : [f] \in \ell^1(G : H) \to \sum_{x \in G} f(x)$. Since we have

$$|\sigma([f])| = \left| \sum_{x \in G} f(x) \right| \leq \sum_{x \in G} \left| \sum_{h \in H} f(xh) \right| = \|\lambda_{G/H}^1(f)\delta_H\|_1 \leq \|f\|_{PF_1^*(G,H)}.$$

$\sigma$ extends to a character on $C^*(PF_1^*(G : H)) = C^*_{\lambda_{G/H}}(G)$. It also extends to a state on $B(\ell^2(G/H))$ which we also denote by $\sigma$. Observe that $\sigma(\lambda_{G/H}(x)) = \sigma([\delta_x]) = 1$ for all $x \in G$. In particular, the left translation operators $\{\lambda_{G/H}^1(x) : x \in G\}$ are in the multiplicative domain of $\sigma$, that is $\sigma(\lambda_{G/H}(x)\lambda_{G/H}(x)) = \sigma(\lambda_{G/H}(x))\sigma(\lambda_{G/H}(x))$ and $\sigma(\lambda_{G/H}(x)\lambda_{G/H}(x)^*) = \sigma(\lambda_{G/H}(x))\sigma(\lambda_{G/H}(x)^*)$ for all $x \in G$. It follows from Bimodule Property (cf. [BO08, Proposition 1.5.7]) that

$$\sigma(\lambda_{G/H}(x)T\lambda_{G/H}(x^{-1})) = \sigma(\lambda_{G/H}(x))\sigma(M)\sigma(\lambda_{G/H}(x^{-1})) = \sigma(T)$$

for all $T \in B(\ell^2(G/H))$. Recall that the pointwise multiplication operators give faithful *-representation

$$f \in \ell^2(G/H) \mapsto M_f \in B(\ell^2(G/H))$$

and we have $M_{\lambda_{G/H}} = \lambda_{G/H}(x)M_f\lambda_{G/H}(x^{-1})$. Now, it is clear that $\sigma$ gives a $G$-invariant state on $\ell^2(G/H)$.

For the direction (iii) ⇒ (i), when $H$ is normal and co-amenable in $G$, the quotient group $Q = G/H$ is amenable. Moreover, we have the canonical isomorphisms $\ell^1(Q) \cong \ell^1(G : H)$, $C^*(Q) \cong C^*(G : H)$, and $C^*_{\lambda}(Q) \cong C^*_{\lambda_{G/H}}(G)$. Thus (iii) ⇒ (i) follows. \(\square\)

Now, we are ready to prove Theorem 1.

**Proof of Theorem 1.** Assume that $(G, H)$ is a quasi-Hermitian pair. Then $[CG]$ is spectral in $(PF_1^*(G : H), C^*_{\lambda_{G/H}}(G))$ by Theorem 10. Proposition 2 yields that we have canonical *-isomorphism $C^*(G : H) \to C^*_{\lambda_{G/H}}(G)$. Then $H$ is co-amenable in $G$ by Proposition 12. \(\square\)

**Corollary 13.** Quasi-Hermitian discrete groups are amenable.

**Proof.** Choose $H = \{e\}$ and apply Theorem 1. \(\square\)

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