Weighted Super Poincaré Inequalities for
Infinite-Dimensional Extension of the
Dirichlet Distribution

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Abstract

For the infinite-dimensional extension of the Dirichlet distribution, the super Poincaré
inequality does not hold based on the result in [14], so we establish the weighted su-
per Poincaré inequalities for this measure with respect to two different Dirichlet forms
respectly.

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super Poincaré inequality, Fleming-Viot process.

1 Introduction

Let $N \geq 1$ be a natural number, and let $\alpha = (\alpha_1, \cdots, \alpha_{N+1}) \in (0, \infty)^{N+1}$. The Dirichlet
distribution $\mu_{\alpha}^{(N)}$ with parameter $\alpha$ is a probability measure on the set

$$\Delta^{(N)} := \left\{ x = (x_i)_{1 \leq i \leq N} \in [0, 1]^N : |x|_1 := \sum_{i=1}^{N} x_i \leq 1 \right\}$$

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with density function

\[ \rho(x) := \frac{\Gamma(|\alpha_1|)}{\prod_{1 \leq i \leq N+1} \Gamma(\alpha_i)} (1 - |x|_1)^{\alpha_{N+1} - 1} \prod_{1 \leq i \leq N} x_i^{\alpha_i - 1}, \quad x = (x_i)_{1 \leq i \leq N} \in \Delta^{(N)}, \]

where \( |\alpha|_1 := \sum_{i=1}^{N+1} \alpha_i \), denoted by \( D(\alpha_1, \alpha_2, \ldots, \alpha_N, \alpha_{N+1}) \). This distribution arises naturally in Bayesian inference as conjugate prior for categorical distribution, and it describes the distribution of allelic frequencies in population genetics, see for instance [7]. We introduce two generators of diffusion processes whose stationary distributions both are the Dirichlet distribution.

\[ L^{(N)}_{\alpha_1} f(x) = \sum_{n=1}^{N} x_n (1 - |x|_1) \left( \frac{\partial^2 f}{\partial x_n^2} \right)(x) + \left\{ \alpha_n (1 - |x|_1) - \alpha_\infty x_n \right\} \partial_n f(x), \quad f \in C^2(\mathbb{R}^N). \]

\[ E^{(N)}_{\alpha_1} (f, g) := \mu^{(N)}_{\alpha, \alpha} \left( 1 - |x|_1 \right) \sum_{n=1}^{N} x_n \partial_n f \partial_n g \quad f, g \in C^1(\mathbb{R}^N). \]

\[ L^{(N)}_{\alpha_2} f(x) = \sum_{i,j=1}^{N} x_i (\delta_{ij} - x_j) (\partial_i f \partial_j f)(x) + \sum_{i=1}^{N} (\alpha_i - |\alpha|_1 x_i) \partial_i f(x), \quad f \in C^2(\mathbb{R}^N). \]

\[ E^{(N)}_{\alpha_2} (f, g) := \mu^{(N)}_{\alpha, \alpha} \left( \sum_{i,j=1}^{N} x_i (\delta_{ij} - x_j) \partial_i f \partial_j g \right) \quad f, g \in C^1(\mathbb{R}^N). \]

[3] extend the measure \( \mu^{(N)}_{\alpha} \) and the operator (1.2) to the infinite-dimensional case. Consider the infinite-dimensional simplex

\[ \Delta^{(\infty)} := \left\{ x \in [0, 1]^N : \|x\|_1 := \sum_{i=1}^{\infty} x_i \leq 1 \right\}, \]

which is equipped with the \( L^1 \)-metric \( |x - y|_1 \). Let \( \alpha \in (0, \infty)^N \) with \( |\alpha|_1 = \Sigma_{i=1}^{\infty} \alpha_i < \infty \), and let \( \alpha_\infty > 0 \) which refers to \( \alpha_{N+1} \) in the infinite-dimensional case as \( N \to \infty \). Let

\[ \alpha^{(N)} = \left( \alpha_1, \ldots, \alpha_{N-1}, \Sigma_{i=1}^{N} \alpha_i, \alpha_\infty \right) \in (0, \infty)^{N+1}, \quad N \geq 1. \]
Then for any $N \geq 1$,

$$
\mu_{\alpha,\alpha}^{(N)} := \mu_{\alpha,\alpha}^{(N)}(dx_1, \ldots, dx_N) \prod_{i=N+1}^{\infty} \delta_0(dx_i)
$$

is a probability measure on $\Delta^{(\infty)}$. By Theorem 1.2 in [3], they proved the sequence $\{\mu_{\alpha,\alpha}^{(N)}\}_{N \geq 1}$ converges weakly to a probability measure $\mu_{\alpha,\alpha}^{(\infty)}$ on $\Delta^{(\infty)}$. $\mathcal{FC}^p$ is the $C^p$-cylindrical functions for $p \geq 1$:

$$
\mathcal{FC}^p := \{\Delta^{(\infty)} \ni x := (x_i)_{i \geq 1} \mapsto f(x_1, \ldots, x_m) : m \geq 1, f \in C^p(\Delta^{(m)})\}.
$$

In [3], S. Feng, L. Miclo and F.-Y. Wang established the Poincaré inequalities for the Dirichlet distribution with respect to the first type Dirichlet form for the finite-dimensional case and infinite-dimensional case. In [14], F.-Y. Wang and the author of this paper established the super Poincaré inequality for the finite-dimensional case. [3] also proved that the form

$$
E_{\alpha,1}^{(\infty)} := \mu_{\alpha,\alpha}^{(\infty)} \left( (1 - |x_1|) \sum_{n=1}^{\infty} n \partial_n f \partial_n g \right) \quad f, g \in \mathcal{FC}^1
$$

is closable in $L^2(\mu_{\alpha,\alpha}^{(\infty)})$ whose closure is a symmetric Dirichlet form. The generator $(L_{\alpha,2}^{(\infty)}, \mathcal{D}(L_{\alpha,2}^{(\infty)}))$ of the Dirichlet form $E_{\alpha,1}^{(\infty)}$ satisfies $\mathcal{FC}^2 \subset \mathcal{D}(L_{\alpha,2}^{(\infty)})$ and

$$
L_{\alpha,1}^{(\infty)} f(x) = \sum_{n=1}^{\infty} \left( x_n (1 - |x_1|) \partial^2_i f(x) + \{\alpha_n (1 - |x_1|) - \alpha_{\infty} x_n \} \partial_n f(x) \right), \quad f \in \mathcal{FC}^2.
$$

In [10], Stannat established the Poincaré inequality for the Dirichlet distribution with respect to the second type Dirichlet form and also established the Poincaré inequality for the Fleming-Viot process. In [14], F.-Y. Wang and the author of this paper established the super Poincaré inequality for the second type Dirichlet form about finite-dimensional case. Following the idea of [3], we can also prove that the form

$$
E_{\alpha,2}^{(\infty)} := \mu_{\alpha,\alpha}^{(\infty)} \left( \sum_{i,j=1}^{\infty} x_i (\delta_{ij} - x_j) \partial_i f \partial_j g \right) \quad f, g \in \mathcal{FC}^1
$$

is closable in $L^2(\mu_{\alpha,\alpha}^{(\infty)})$ whose closure is a symmetric Dirichlet form. The generator $(L_{\alpha,2}^{(\infty)}, \mathcal{D}(L_{\alpha,2}^{(\infty)}))$ of the Dirichlet form $E_{\alpha,2}^{(\infty)}$ satisfies $\mathcal{FC}^2 \subset \mathcal{D}(L_{\alpha,2}^{(\infty)})$ and

$$
L_{\alpha,2}^{(\infty)} f(x) = \sum_{i,j=1}^{\infty} x_i (\delta_{ij} - x_j) (\partial_i f \partial_j f)(x) + \sum_{i=1}^{\infty} (\alpha_i - |\alpha| x_i) \partial_i f(x), \quad f \in \mathcal{FC}^2.
$$
In [10], Stannat proved the invalidity of the log-Sobolev inequality when the state space contains infinite states. In fact, by the same method, we can prove the F-Sobolev inequality doesn’t hold, then neither does the super Poincaré inequality hold.

In this paper, we firstly introduce the property of the Dirichlet distribution in section one, then we establish the weighted super Poincaré inequality for the Dirichlet form $E^{\alpha,1}_{\infty}$ and the measure $\mu^{(\infty)}_{\alpha,\alpha_{\infty}}$ in section two. Then we try to construct a measure-valued process based on the Dirichlet distribution following the idea of [4] but fail. At last, we obtain the weighted super Poincaré inequality for the Dirichlet form $E^{\alpha,2}_{\infty}$ and the measure $\mu^{(\infty)}_{\alpha,\alpha_{\infty}}$ in section three. Then we try to establish the weighted super Poincaré inequality for the Fleming-Viot process but fail.

2 Property of the Dirichlet distribution

For any $m > n$, we define the map

$$T_m : \Delta^{(n)} \times \Delta^{(m-n)} \rightarrow \Delta^{(m)},$$

$$T_m : (x_1, x_2, \ldots, x_m) \mapsto (x_1(1 - \Sigma_{n<i\leq m} x_i), \ldots, x_n(1 - \Sigma_{n<i\leq m} x_i), x_{n+1}, \ldots, x_m).$$

Set $T$ is the limit of $T_m$, so

$$T : \Delta^{(n)} \times \Delta^{(\infty)} \rightarrow \Delta^{(\infty)},$$

$$T : (x_1, x_2, \ldots) \mapsto (x_1(1 - \Sigma_{n<i\leq \infty} x_i), \ldots, x_n(1 - \Sigma_{n<i\leq \infty} x_i), x_{n+1}, \ldots).$$

We denote

$$\mu^{(m)} = D(\alpha_1, \alpha_2, \ldots, \alpha_{m-1}, \Sigma_{i\geq m} \alpha_i, \alpha_{\infty}),$$

According to [3], we know $\{\mu^{(m)}\}_{m\geq 1}$ weakly converges to $\mu^{(\infty)}_{\alpha,\alpha_{\infty}}$.

**Proposition 2.1.** For fixed $n \geq 1, \forall f \in FC^1(\Delta^{(\infty)})$, such that $f$ has $m$ variations. We denote

$$\mu_1 = D(\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{\infty}),$$

$$\mu_2 = D(\alpha_{n+1}, \alpha_{n+2}, \ldots, \alpha_{m-1}, \Sigma_{i\geq m} \alpha_i, \Sigma_{i=1}^n \alpha_i + \alpha_{\infty}),$$

Then for this $f$, we have

$$\mu^{(\infty)}_{\alpha,\alpha_{\infty}}(f) = \mu_1(\mu_2^{(\infty)}(f \circ T)) := \int_{\Delta^{(n)}} \int_{\Delta^{(\infty)}} f \circ T(x,y) \mu_2^{(\infty)}(dy) \mu_1(dx).$$
Proof. \( \forall f \in FC^1(\Delta^{(\infty)}) \) and \( f \) has \( m \) variations.

\[
\mu_1(\mu_2(f \circ T_m)) = \frac{\Gamma(\sum_{i=1}^{n} \alpha_i + \alpha_{\infty})}{\prod_{i=1}^{m} \Gamma(\alpha_i) \prod_{i=n+1}^{m-1} \Gamma(\alpha_i) \Gamma(\sum_{i=1}^{i=_{m}} \alpha_i) \Gamma(\sum_{i=1}^{n} \alpha_i + \alpha_{\infty})} \cdot \int_{\Delta^{(m)}} f(T_m(x)) x_1^{\alpha_1-1} \cdots x_n^{\alpha_n-1} \cdot (1 - \sum_{i=1}^{n} x_i)^{\alpha_{\infty}-1} x_n+1^{\alpha_{n+1}-1} \cdots x_m^{\alpha_m-1} x_m^{\sum_{i\geq m} \alpha_i-1} \\
\cdot \left(1 - \sum_{i=n+1}^{m} x_i\right)^{\sum_{i=1}^{n} \alpha_i + \alpha_{\infty}-1} dx_1 \cdots dx_m
\]

\[
= \frac{\Gamma(\sum_{i=1}^{\infty} \alpha_i)}{\prod_{i=1}^{m-1} \Gamma(\alpha_i) \Gamma(\sum_{i=m}^{\infty} \alpha_i) \Gamma(\alpha_{\infty})} \cdot \int_{\Delta^{(m)}} f(y)^{\alpha_1-1} \cdots y_m^{\alpha_m-1} \left(1 - \sum_{i=1}^{m} y_i\right)^{\alpha_{\infty}-1} \left(1 - \sum_{i=n+1}^{m} x_i\right)^n dy_1 \cdots dy_m
\]

\[
\cdot \det(\nabla T^{-1}_m(y)) dy_1 \cdots dy_m
\]

\[
= \frac{\Gamma(\sum_{i=1}^{\infty} \alpha_i)}{\prod_{i=1}^{m-1} \Gamma(\alpha_i) \Gamma(\sum_{i=m}^{\infty} \alpha_i) \Gamma(\alpha_{\infty})} \cdot \int_{\Delta^{(m)}} f(y)^{\alpha_1-1} \cdots y_m^{\alpha_m-1} \sum_{i\geq m} \alpha_i-1 \\
\cdot \left(1 - \sum_{i=1}^{m} y_i\right)^{\alpha_{\infty}-1} dy_1 \cdots dy_m
\]

\[
= \mu^{(m)}(f).
\]

We have used

\[
\det(\nabla T^{-1}_m(y)) = \left(1 - \sum_{i=n+1}^{m} x_i\right)^{-n},
\]
\[
\left(1 - \sum_{i=n+1}^{m} x_i\right) \left(1 - \sum_{i=1}^{n} x_i\right) = \left(1 - \sum_{i=n+1}^{m} x_i\right) - \left(1 - \sum_{i=1}^{m} x_i\right) \sum_{i=1}^{n} x_i = 1 - \sum_{i=1}^{m} y_i.
\]

Let \(m \to \infty\), we get

\[
\mu_{\alpha, \alpha_\infty}^{(\infty)}(f) = \mu_1(\mu_2^{(\infty)}(f \circ T)) := \int_{\Delta^{(n)}} \int_{\Delta^{(\infty)}} f \circ T(x, y) \mu_2^{(\infty)}(dy) \mu_1(dx).
\]

\[\Box\]

3 The first type Dirichlet form

According to Theorem 1.1 in [14], from the Nash inequality, we can get the super Poincaré inequality for the Dirichlet distribution and Dirichlet form \(\mathcal{E}^{(N)}_{\alpha, 1}\). Combining the Poincaré inequality, we can obtain the weighted super Poincaré inequality for infinite-dimensional Dirichlet distribution and Dirichlet form \(\tilde{\mathcal{E}}^{(\infty)}_{\alpha, 1}\).

**Theorem 3.1.** Let \(\gamma_i \geq 1, i \geq 1\). Denote \(\tilde{\mathcal{E}}^{(\infty)}_{\alpha, 1}(f, f) = \mu_{\alpha, \alpha_\infty}^{(\infty)}\left(\frac{1}{1 - \sum_{i=n+1}^{m} y_i \sum_{i=1}^{\infty} y_i \gamma_i y_\infty (\partial_i f)^2}\right),\)

\(\forall f \in \mathcal{D}(\tilde{\mathcal{E}}^{(\infty)}_{\alpha, 1})\). When \(\alpha = (\alpha_1, \cdots, \alpha_\infty)\) satisfies \(1 \leq i \leq n, \alpha_i \geq 1,\) and \(0 < \alpha_\infty \leq 1\), then the weighted super Poincaré inequality

\[
\mu_{\alpha, \alpha_\infty}^{(\infty)}(f^2) \leq r \tilde{\mathcal{E}}^{(\infty)}_{\alpha, 1}(f, f) + \beta^{(1)}(r) \mu_{\alpha, \alpha_\infty}^{(\infty)}(|f|)^2, \forall f \in \mathcal{F}C^1(\Delta^{(\infty)})
\]

holds, where \(n\) comes from the smallest value satisfies

\[
\frac{1}{(\sum_{i=1}^{n} \alpha_i + \alpha_\infty) \inf_{i>n} \gamma_i} \leq r;
\]

\[
\beta_n(r) \leq c_n r^{-[\sum_{i=1}^{n} 1 + (\alpha_i + (\alpha_\infty - 1))].}
\]

\[
\beta^{(1)}(r) = \beta_n \left(\frac{r}{3}\right).
\]

**Proof.** \(\forall r > 0, n\) is the smallest value which satisfies

\[
\frac{1}{(\sum_{i=1}^{n} \alpha_i + \alpha_\infty) \inf_{i>n} \gamma_i} \leq r.
\]
∀ f ∈ \mathcal{F}\mathcal{C}^{1}(\Delta^{(\infty)})$, there exists \( m \geq 1 \), such that \( f \) has \( m \) variations. If \( m \leq n \), by the result of [14], we have the super Poincaré inequality

\[
\mu_{\alpha,\alpha_{\infty}}^{(\infty)}(f^2) 
\leq r \mu_{\alpha,1}^{(\infty)}\left(\sum_{i=1}^{\infty} x_{i} x_{\infty}(\partial_{i} f)^{2}\right) + \beta_{n}(r) \mu_{\alpha,\alpha_{\infty}}^{(\infty)}(|f|)^{2} 
\leq r \delta_{\alpha,1}^{(\infty)}(f, f) + \beta_{n}(r) \mu_{\alpha,\alpha_{\infty}}^{(\infty)}(|f|)^{2} 
\leq r \delta_{\alpha,1}^{(\infty)}(f, f) + \beta^{(1)}(r) \mu_{\alpha,\alpha_{\infty}}^{(\infty)}(|f|)^{2},
\]

where

\[
\beta_{n}(r) \leq c_{n} r^{-\left[\sum_{i=1}^{n} 1 + 2\alpha_{i} + (\alpha_{\infty} - 1)^{+}\right]}.
\]

\[
\beta^{(1)}(r) = \beta_{n}\left(\frac{r}{3}\right).
\]

If \( m > n \), according to [3], Theorem 1.1, we have the Poincaré inequality for \( \mu_{2}^{(\infty)} \), i.e.

\[
\mu_{2}^{(\infty)}(f^2) \leq \frac{1}{\sum_{i=1}^{n} \alpha_{i} + \alpha_{\infty}} \mu_{2}^{(\infty)}\left(\sum_{i=n+1}^{\infty} x_{i} x_{\infty}(\partial_{i} f)^{2}\right) + \mu_{2}^{(\infty)}(|f|)^{2}.
\]

According to [14], Theorem 1.1, we have the super Poincaré inequality for \( \mu_{1} \), i.e.

\[
\mu_{1}(f^2) \leq \mu_{1}\left(\sum_{i=1}^{n} x_{i} x_{\infty}(\partial_{i} f)^{2}\right) + \beta_{n}(r) \mu_{1}(|f|)^{2},
\]

where

\[
\beta_{n}(r) \leq c_{n} r^{-\left[\sum_{i=1}^{n} 1 + 2\alpha_{i} + (\alpha_{\infty} - 1)^{+}\right]}.
\]

\[
\mu^{(\infty)}_{\alpha,\alpha_{\infty}}(f^2) = \mu_{1}\left(\mu_{2}^{(\infty)}(f^2 \circ T)\right)
\leq \mu_{1}\left(\frac{1}{\sum_{i=1}^{n} \alpha_{i} + \alpha_{\infty}} \mu_{2}^{(\infty)}\left(\sum_{i=n+1}^{\infty} x_{i} x_{\infty}(\partial_{i} (f \circ T))^2\right)\right) + \mu_{1}(\mu_{2}^{(\infty)}(|f \circ T|)^{2})
\]

\[
= \mu_{1}\left(\frac{1}{\sum_{i=1}^{n} \alpha_{i} + \alpha_{\infty}} \mu_{2}^{(\infty)}\left(\sum_{i=n+1}^{\infty} x_{i} x_{\infty}\left((\partial_{i} f) \circ T - \sum_{j=1}^{n} x_{j}(\partial_{j} f) \circ T\right)^2\right)\right)
\]

\[
+ r \mu_{1}\left(\mu_{2}^{(\infty)}\left(\sum_{i=1}^{n} x_{i} x_{\infty}(\partial_{i} (f \circ T))^2\right)\right) + c \beta_{n}(r) \mu_{\alpha,\alpha_{\infty}}^{(\infty)}(|f|)^{2}
\]

7
\[
\begin{align*}
&\leq 2\mu_1 \left( \sum_{i=1}^{\infty} \frac{1}{\alpha_i + \alpha_\infty} \mu_2^{(\infty)} \left( \sum_{i=n+1}^{\infty} x_i x_\infty (\partial_i f)^2 \circ T \right) \right) \\
&+ 2\mu_1 \left( \sum_{i=1}^{\infty} \frac{1}{\alpha_i + \alpha_\infty} \mu_2^{(\infty)} \left( \sum_{i=n+1}^{\infty} x_i x_\infty \left( \sum_{j=1}^{\infty} x_j^2 (\partial_j f)^2 \circ T \right) \right) \right) \\
&+ r\mu_1 \left( \mu_2^{(\infty)} \left( \sum_{i=1}^{n} x_i x_\infty \left( \partial_i f \cdot \left( 1 - \sum_{i=n+1}^{\infty} x_i \right) \right)^2 \circ T^{-1} \circ T \right) \right) + c\beta_n(r)\mu_\alpha^{(\infty)} (| f |)^2 \\
&\leq 2\mu_1 \left( \sum_{i=1}^{\infty} \frac{1}{\alpha_i + \alpha_\infty} \mu_2^{(\infty)} \left( \sum_{i=n+1}^{\infty} y_i y_\infty (\partial_i f)^2 \right) \right) \\
&+ 2\mu_1 \left( \sum_{i=1}^{\infty} \frac{1}{\alpha_i + \alpha_\infty} \mu_2^{(\infty)} \left( \sum_{i=n+1}^{\infty} x_i x_\infty \left( \sum_{j=1}^{\infty} x_j^2 (\partial_j f)^2 \circ T \right) \right) \right) \\
&+ r\mu_1 \left( \mu_2^{(\infty)} \left( \sum_{i=1}^{n} x_i x_\infty \left( 1 - \sum_{i=n+1}^{\infty} x_i \right) (\partial_i f)^2 \circ T^{-1} \circ T \right) \right) \\
&+ c\beta_n(r)\mu_\alpha^{(\infty)} (| f |)^2 \\
&= \frac{2}{\sum_{i=1}^{\infty} \alpha_i + \alpha_\infty} \mu_\alpha^{(\infty)} \left( \sum_{i=n+1}^{\infty} y_i y_\infty (\partial_i f)^2 \right) \\
&+ \frac{2}{\sum_{i=1}^{\infty} \alpha_i + \alpha_\infty} \mu_\alpha^{(\infty)} \left( \frac{1}{\sum_{i=n+1}^{\infty} y_i \sum_{j=1}^{\infty} y_j y_\infty (\partial_j f)^2} \right) \\
&+ c\beta_n(r)\mu_\alpha^{(\infty)} (| f |)^2 + r\mu_\alpha^{(\infty)} \left( \sum_{i=1}^{n} x_i x_\infty (\partial_i f)^2 \circ T^{-1} \circ T \right) \\
&\leq \frac{2}{\sum_{i=1}^{\infty} \alpha_i + \alpha_\infty} \mu_\alpha^{(\infty)} \left( \sum_{i=n+1}^{\infty} y_i y_\infty (\partial_i f)^2 \right) \\
&+ \frac{2}{\sum_{i=1}^{\infty} \alpha_i + \alpha_\infty} \mu_\alpha^{(\infty)} \left( \frac{1}{\sum_{i=n+1}^{\infty} y_i \sum_{j=1}^{\infty} y_j y_\infty (\partial_j f)^2} \right) \\
&+ r\mu_\alpha^{(\infty)} \left( \sum_{i=1}^{n} x_i x_\infty (\partial_i f)^2 \circ T^{-1} \circ T \right) \\
&+ c\beta_n(r)\mu_\alpha^{(\infty)} (| f |)^2 \\
&\leq 2r\mu_\alpha^{(\infty)} \left( \frac{1}{\sum_{i=n+1}^{\infty} y_i \sum_{i=1}^{\infty} y_i y_\infty (\partial_i f)^2} \right) + r\mu_\alpha^{(\infty)} \left( \sum_{i=1}^{n} x_i x_\infty (\partial_i f)^2 \circ T^{-1} \circ T \right) \\
&+ c\beta_n(r)\mu_\alpha^{(\infty)} (| f |)^2
\end{align*}
\]
\[ + c\beta_n(r)\mu^{(\infty)}_{\alpha,\alpha\infty}(|f|^2) \]
\[ \leq 2r\mu^{(\infty)}_{\alpha,\alpha\infty}\left(\frac{1}{1 - \sum_{i=n+1}^{\infty} y_i} \sum_{i=1}^{\infty} y_i \gamma_i y_\infty (\partial_i f)^2\right) + r \int_{\Delta^\infty} \sum_{i=1}^{n} x_i \gamma_i x_\infty (\partial_i f)^2 d\mu^{(\infty)}_{\alpha,\alpha\infty}(Tx) \]
\[ + c\beta_n(r)\mu^{(\infty)}_{\alpha,\alpha\infty}(|f|^2) \]
\[ \leq 3r \int_{\Delta(\infty)} \frac{1}{1 - \sum_{i=n+1}^{\infty} y_i} \sum_{i=1}^{\infty} y_i \gamma_i y_\infty (\partial_i f)^2 d\mu^{(\infty)}_{\alpha,\alpha\infty}(x) + c\beta_n(r)\mu^{(\infty)}_{\alpha,\alpha\infty}(|f|^2). \]

We have used that when \(1 \leq i \leq n\), \(\alpha_i \geq 1\), and \(0 < \alpha_\infty \leq 1\), so we have
\[ \mu^{(\infty)}_{\alpha,\alpha\infty} \circ T \leq \mu^{(\infty)}_{\alpha,\alpha\infty}, \]
\[ \inf_{i>n} \left( \sum_{i=1}^{n} \alpha_i + \alpha_\infty \right) \gamma_i \leq r, \text{ and } \gamma_i \geq 1, \ 1 \leq i \leq n. \]

So we get
\[ \mu^{(\infty)}_{\alpha,\alpha\infty}(f^2) \leq r\mu^{(\infty)}_{\alpha,\alpha\infty}\left(\frac{1}{1 - \sum_{i=n+1}^{\infty} y_i} \sum_{i=1}^{\infty} y_i \gamma_i y_\infty (\partial_i f)^2\right) + c\beta_n\left(\frac{r}{3}\right)\mu^{(\infty)}_{\alpha,\alpha\infty}(|f|^2), \]

where \(n\) is the smallest value which satisfies
\[ \inf_{i>n} \left( \sum_{i=1}^{n} \alpha_i + \alpha_\infty \right) \gamma_i \leq r. \]

\[ \square \]

### 3.1 Construct measure-valued process

We firstly connect this Dirichlet distribution with Poisson-Dirichlet distribution according to the property of Dirichlet distribution. If \(\{U_i\}_{i \geq 1}\) is a series of independent random variations, such that

\[ U_i \sim \text{Beta}\left(\alpha_i, \sum_{j=i+1}^{\infty} \alpha_j + \alpha_\infty\right). \]

Let

\[ X_1 = U_1, \ X_2 = U_2(1 - U_1), \cdots, \ X_i = U_i \prod_{1 \leq j \leq i-1} (1 - U_j), \cdots, \]

then the law of \(\{X_i\}_{i \geq 1}\) is Dirichlet distribution.
According to the property of Poisson-Dirichlet distribution, if \( \{U_i\}_{i \geq 1} \) is a series of independent random variations, such that 
\[
U_i \sim \text{Beta}(1 - \alpha, \theta + i\alpha).
\]

Let 
\[
X_1 = U_1, \; X_2 = U_2(1 - U_1), \; \ldots, \; X_i = U_i \prod_{1 \leq j \leq i-1} (1 - U_j), \; \ldots,
\]
then the law of \( \{X_i\}_{i \geq 1} \) is Poisson-Dirichlet distribution.

To make them equivalent, let 
\[
\alpha_i = 1 - \alpha, \; \theta = \sum_{j \geq i+1} \alpha_j + \alpha_\infty - i\alpha.
\]

But 
\[
\sum_{j \geq i+1} \alpha_j + \alpha_\infty - i\alpha = \sum_{j \geq i+1} (1 - \alpha) + \alpha_\infty - i\alpha = \infty,
\]
so we can’t connect them together. Thus, we can’t construct a measure-valued process whose stationary distribution is same as the stationary distribution of the Fleming-Viot process based on the Poisson-Dirichlet distribution as in [4].

### 4 The second type Dirichlet form

According to Lemma 3.2 in [10], for the Fleming-Viot process, let \( (A_n)_{n \leq d+1} \) be a measurable partition of \( S \) such that \( \nu_0(A_i) > 0 \) and \( F(\mu) = \varphi((1_{A_1}, \mu), \ldots, (1_{A_d}, \mu)) \), \( \varphi \in C^\infty(\mathbb{R}^d) \). Then \( \forall F \in H^{1, 2}(m_{\theta, \nu_0}) \), we have 
\[
\mathcal{E}_{\theta, \nu_0}(F, F) = \mathcal{E}_{\theta}(\nu_0(A_1), \ldots, \nu_0(A_d))(\varphi, \varphi),
\]
where 
\[
\mathcal{E}_{\theta}(F, F) = \int \text{var}_{\mu}(\nabla F(\mu)) m_{\theta, \nu_0}(d\nu).
\]

Denote \( q = \theta(\nu_0(A_1), \ldots, \nu_0(A_d)) \),
\[
\mathcal{E}_{\theta(q)}(\varphi, \varphi) = \sum_{i,j=1}^d \int x_i(\delta_{ij} - x_j) \partial_i \varphi(x) \partial_j \varphi(x) \rho_q(x) dx,
\]
\( \rho_q(x) \) is the density function of Dirichlet distribution. As Theorem 3.1, we get the weighted super Poincaré inequality.
Theorem 4.1. Let $\gamma_i \geq 1$, $i \geq 1$. Denote
\[
\mathring{\tilde{E}}_{\alpha,2}(f, f) = \mu_{\alpha,\infty}^{(\infty)} \left( \sum_{i,j=1}^{\infty} x_i (\delta_{ij} - x_j) \gamma_i \gamma_j (\partial_i f \partial_j f)(x) \right), \quad \forall f \in \mathcal{D}(\mathring{\tilde{E}}_{\alpha,2}).
\]

When $\alpha = (\alpha_1, \cdots, \alpha_\infty)$ satisfies
\[
1 \leq i \leq n, \quad \alpha_i \geq 1, \quad 0 < \alpha_\infty \leq 1,
\]
then the weighted super Poincaré inequality
\[
\mu_{\alpha,\infty}^{(\infty)} (f^2) \leq r \mathring{\tilde{E}}_{\alpha,2}^{(\infty)} (f, f) + \beta_n^{(2)}(r) \mu_{\alpha,\infty}^{(\infty)} (|f|)^2, \quad \forall f \in \mathcal{F}C^1(\Delta^{(\infty)})
\]
holds, where $n$ comes from the smallest value satisfies
\[
\frac{1}{(\sum_{i=1}^{n} \alpha_i + \alpha_\infty) \inf_{i>n} \gamma_i} \leq r;
\]
\[
\beta_n(r) \leq c_n r^{-[\sum_{i=1}^{n} 1 \vee (2\alpha_i) + (\alpha_\infty - 1)^+]},
\]
\[
\beta^{(2)}(r) = \beta_n(r).
\]

Proof. \forall r > 0, $n$ is the smallest value which satisfies
\[
\frac{1}{(\sum_{i=1}^{n} \alpha_i + \alpha_\infty) \inf_{i>n} \gamma_i} \leq r.
\]
\forall f \in \mathcal{F}C^1(\Delta^{(\infty)})$, there exists $m \geq 1$, such that $f$ has $m$ variations. If $m \leq n$, by the result of [14], we have the super Poincaré inequality
\[
\mu_{\alpha,\infty}^{(\infty)} (f^2)
\]
\[
\leq r \mu_{\alpha,\infty}^{(\infty)} \left( \sum_{i,j=1}^{n} x_i (\delta_{ij} - x_j) (\partial_i f \partial_j f)(x) \right) + \beta_n(r) \mu_{\alpha,\infty}^{(\infty)} (|f|)^2
\]
\[
= r \mathring{\tilde{E}}_{\alpha,2}^{(\infty)} (f, f) + \beta_n(r) \mu_{\alpha,\infty}^{(\infty)} (|f|)^2
\]
\[
\leq r \mathring{\tilde{E}}_{\alpha,2}^{(\infty)} (f, f) + \beta^{(2)}(r) \mu_{\alpha,\infty}^{(\infty)} (|f|)^2,
\]
where
\[
\beta_n(r) \leq c_n r^{-[\sum_{i=1}^{n} 1 \vee (2\alpha_i) + (\alpha_\infty - 1)^+]},
\]
\[
\beta^{(2)}(r) = \beta_n(r).
\]
If $m > n$, according to [10], Proposition 3.3, we have the Poincaré inequality for $\mu_2^{(\infty)}$ i.e.

$$\mu_2^{(\infty)}(f^2) \leq \frac{1}{\sum_{i=n+1}^{\infty} \alpha_i + \alpha_{\infty}} \mu_2^{(\infty)} \left( \sum_{i,j=n+1}^{\infty} x_i(\delta_{ij} - x_j)(\partial_i f \partial_j f)(x) \right) + \mu_2^{(\infty)}(|f|)^2.\]

According to [14], Theorem 1.1, we have the super Poincaré inequality for $\mu_1$, i.e.

$$\mu_1(f^2) \leq r \mu_1 \left( \sum_{i,j=1}^{n} x_i(\delta_{ij} - x_j)(\partial_i f \partial_j f)(x) \right) + \beta_n(r) \mu_1(|f|)^2,$$

where

$$\beta_n(r) \leq c_n r^{-[\sum_{i=1}^{\infty} 1/(2a_i) + (\alpha_{\infty} - 1)^+]}.\]
\[
\frac{1}{\left(\sum_{i=1}^{n} \alpha_i + \alpha_\infty\right) \inf_{i>n} \gamma_i} \mu_{\alpha,\alpha_\infty} \left( \sum_{i,j=n+1}^{\infty} \gamma_i \gamma_j y_i (\delta_{ij} - y_j) \partial_i f \partial_j f \right) \\
- \frac{2}{\left(\sum_{i=1}^{n} \alpha_i + \alpha_\infty\right) \inf_{i>n} \gamma_i} \mu_{\alpha,\alpha_\infty} \left( \sum_{i=n+1}^{\infty} \sum_{k=1}^{n} \gamma_i \gamma_k y_i y_k \partial_i f \partial_k f \right) \\
+ r \mu_{\alpha,\alpha_\infty} \left( \sum_{i,j=1}^{n} y_i (\delta_{ij} - y_j) \partial_i f \partial_j f \right) + c \beta_n (r) \mu_{\alpha,\alpha_\infty} (|f|)^2 \\
\leq \mu_{\alpha,\alpha_\infty} \left( \sum_{i,j=1}^{\infty} \gamma_i \gamma_j y_i (\delta_{ij} - y_j) \partial_i f \partial_j f \right) + c \beta_n (r) \mu_{\alpha,\alpha_\infty} (|f|)^2.
\]

\[
\frac{1}{\left(\sum_{i=1}^{n} \alpha_i + \alpha_\infty\right) \inf_{i>n} \gamma_i} \leq r, \text{ and } \gamma_i \geq 1, \; 1 \leq i \leq n.
\]

So we get

\[
\mu_{\alpha,\alpha_\infty} (f^2) \leq r \mu_{\alpha,\alpha_\infty} \left( \sum_{i,j=1}^{\infty} \gamma_i \gamma_j y_i (\delta_{ij} - y_j) \partial_i f \partial_j f \right) + c \beta_n (r) \mu_{\alpha,\alpha_\infty} (|f|)^2,
\]

where \( n \) is the smallest value which satisfies

\[
\frac{1}{\inf_{i>n} \left(\sum_{i=1}^{n} \alpha_i + \alpha_\infty\right) \gamma_i} \leq r.
\]

\[\square\]

### 4.1 the Fleming-Viot process

Following the idea to prove the Poincaré inequality for the Fleming-Viot process in [10], we can’t get the weighted super Poincaré inequality for the Fleming-Viot process.

From Theorem 4.1, we know the second type Dirichlet form

\[
\tilde{\mathcal{E}}_{\theta(q),2}(\varphi, \varphi) = \sum_{i,j=1}^{d} \gamma_i \gamma_j \int x_i (\delta_{ij} - x_j) \partial_i \varphi (x) \partial_j \varphi (x) \rho_q (x) dx.
\]

To make

\[
\tilde{\mathcal{E}}_{\theta,\nu}(F, F) = \tilde{\mathcal{E}}_{\theta(q),2}(\varphi, \varphi)
\]

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holds for
\[ F(\mu) = \varphi((1_{A_1}, \mu), \cdots, (1_{A_d}, \mu)), \]
we define the weighted Dirichlet form
\[
\tilde{E}_{\theta, \nu_0}(F, F) := \int \text{var}_\mu(\tilde{\nabla}F(\mu))m_{\theta, \nu_0}(d\mu)
\]
\[ = \sum_{i,j=1}^{d} \gamma_i \gamma_j \int \mu(A_i)(\delta_{ij} - \mu(A_j))(\partial_i \varphi \partial_j \varphi)(\mu(A_1), \cdots, \mu(A_d))m_{\theta, \nu_0}(d\mu), \]
where \(\{\gamma_i\}_{i \geq 1}\) come from Theorem 4.1. Following the idea to establish the Poincaré inequality in [10], we want to establish the super Poincaré inequality for the Fleming-Viot process. In [10], we know \(\forall F \in \mathcal{FC}_{\infty}, \) there is \(d > 0,\) such that
\[ F(\mu) = \varphi((f_1, \mu), \cdots, (f_d, \mu)), \varphi \in C_{\infty}(\mathbb{R}^d). \]
We can construct
\[ f_i^m := \sum_{n=1}^{m+1} c_{i,n}^m 1_{A_n^m} \to f_i, \text{ as } m \to \infty, \ 1 \leq i \leq d, \]
\[ \varphi_m(x) := \varphi(\sum_{n=1}^{m+1} c_{1,n}^m x_n, \cdots, \sum_{n=1}^{m+1} c_{d,n}^m x_n), \ x \in \mathbb{R}^{m+1}. \]
Then for all \(\mu \in E\)
\[ F_m(\mu) := \varphi_m((1_{A_1^m}, \mu), \cdots, (1_{A_{m+1}^m}, \mu)) = \varphi((f_1^m, \mu), \cdots, (f_d^m, \mu)) \to F(\mu) \]
in \(L^p(\theta, \nu_0)\) for all \(p \geq 1.\) Similarly,
\[ \nabla F_m(\mu) = \sum_{i=1}^{m} \partial_i \varphi((f_1^m, \mu), \cdots, (f_d^m, \mu)) f_i^m \to \nabla F(\mu). \]
But the weighted Dirichlet form \(\{\tilde{E}_{\theta, \nu_0}(F_m, F_m)\}_{m \geq 1}\) does not converge.

**Proposition 4.2.** \(\forall \{\gamma_i\}_{i \geq 1}\) satisfies the condition in Theorem 4.1, when \(m \to \infty,\) the weighted Dirichlet form \(\{\tilde{E}_{\theta, \nu_0}(F_m, F_m)\}_{m \geq 1}\) does not converge.
Proof. As

\[ F_m(\mu) := \varphi_m(\langle 1_{A_1^m}, \mu \rangle, \ldots, \langle 1_{A_{m+1}^m}, \mu \rangle), \]

so by Lemma 3.1 in [10], we get

\[ \tilde{\nabla}_x F(\mu) := \frac{dF}{ds}(\mu + s\delta_x)|_{s=0} \]

\[ = \sum_{i=1}^{m+1} (\partial_x \varphi_m)(\langle 1_{A_i^m}, \mu \rangle, \ldots, \langle 1_{A_{m+1}^m}, \mu \rangle)1_{A_i^m} \]

\[ = \sum_{i=1}^{m+1} \sum_{j=1}^{d} \partial_j \varphi(\langle f_1^m, \mu \rangle, \ldots, \langle f_d^m, \mu \rangle) c_{ji}^m \gamma_i 1_{A_i^m}. \]

As

\[ \sum_{i=1}^{m+1} \sum_{j=1}^{d} c_{ji}^m 1_{A_i^m} \partial_j \varphi(\langle f_1^m, \mu \rangle, \ldots, \langle f_d^m, \mu \rangle) \to \sum_{j=1}^{d} \partial_j \varphi(\langle f_1, \mu \rangle, \ldots, \langle f_d, \mu \rangle) f_j \]

when \( m \to \infty \). We also know \( \gamma_i \to \infty \), when \( i \to \infty \). So \( \sum_{i=1}^{m+1} \sum_{j=1}^{d} c_{ji}^m \gamma_i 1_{A_i^m} \) does not converge when \( m \to \infty \). As

\[ \tilde{\mathbb{E}}_{\theta, \nu_0}(F_m, F_m) = \int \text{var}_\mu(\tilde{\nabla}_x F_m(\mu)) m_{\theta, \nu_0}(d\mu), \]

so \( \{\tilde{\mathbb{E}}_{\theta, \nu_0}(F_m, F_m)\}_{m \geq 1} \) does not converge as \( m \to \infty \). \( \square \)

According to Proposition 4.2, we can’t establish the weighted super Poincaré inequality for the Fleming-Viot process.

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