A nonparametric copula approach to conditional Value-at-Risk

GERY GEENENS∗ RICHARD DUNN
School of Mathematics and Statistics, School of Mathematics and Statistics,
UNSW Sydney, Australia UNSW Sydney, Australia

December 18, 2017

Abstract

Value-at-Risk and its conditional allegory, which takes into account the available information about
the economic environment, form the centrepiece of the Basel framework for the evaluation of market risk
in the banking sector. In this paper, a new nonparametric framework for estimating this conditional
Value-at-Risk is presented. A nonparametric approach is particularly pertinent as the traditionally used
parametric distributions have been shown to be insufficiently robust and flexible in most of the equity-
return data sets observed in practice. The method extracts the quantile of the conditional distribution
of interest, whose estimation is based on a novel estimator of the density of the copula describing the
dynamic dependence observed in the series of returns. Real-world back-testing analyses demonstrate
the potential of the approach, whose performance may be superior to its industry counterparts.

1 Introduction

In the past two decades the quantification of risk has become an area of high interest due to its paramount
role in modern financial sectors, as core business operations are predicated on mutually beneficial trades
of this risk [Segal [2011]]. While there are numerous methodologies for quantifying risk, few are as pop-
ular and widespread as Value-at-Risk (hereafter: VaR). VaR became a crucial means for financial risk
management after the stock market crash of 1987, and it is now globally accepted as benchmark for risk
management through its inclusion in the mandatory Basel II Banking standard in 2004. Practically, VaR
is the amount of capital that a firm has to secure aside to resist unlikely but not impossible adverse events
when engaging in risky trading activities. See [Duffie and Pan (1997), Jorion (2001) and Scaillet (2003)]
for the financial background and applications. Statistically speaking, VaR is nothing but an upper quantile

∗Corresponding author: ggeenens@unsw.edu.au, School of Mathematics and Statistics, UNSW Sydney, Australia, tel +61
2 938 57032, fax +61 2 9385 7123
of the distribution $F_X$ of some random loss $X$, potentially faced by the firm over a given period when engaging in those activities. For a certain fixed level $\alpha \in (0, 1)$, with $\alpha$ generally close to 1, VaR is thus defined as:

$$\text{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} : P(X > x) \leq 1 - \alpha\} = F_X^{-1}(\alpha),$$

where $F_X^{-1}(\alpha)$ is the (generalised) inverse of $F_X$, i.e., its quantile function.

It is, however, clear that the risk that some trading activities represents varies with the market conditions. Hence, it is generally paramount to assess that risk conditionally on some additional variables, say $Z \in \mathbb{R}^d$, reflecting the latest available information about the economic environment (Chernozhukov and Umanstev 2001, McNeil et alii 2005, Kuester et alii 2006, Escanciano and Olmo 2010). In a general framework, the conditional Value-at-Risk (cVaR) at level $\alpha \in (0, 1)$ for a random loss $X$ given some vector of covariates $Z$, is defined as

$$c\text{VaR}_\alpha(X|Z = z) = \inf\{x \in \mathbb{R} : P(X > x|Z = z) \leq 1 - \alpha\} = F_{X|Z}^{-1}(\alpha|z),$$

that is, the $\alpha$-quantile of the conditional distribution $F_{X|Z}$ of $X$ given $Z = z$. Among others, the variables $Z$ may be past observed losses, in a time series setting (see Section 2), and/or other exogenous economic and market covariates.

Remark 1.1. The idiom ‘conditional Value-at-Risk’ has sometimes been used differently in the previous literature, e.g. in Jorion (2001) or Rockafellar and Uryasev (2002), for what is now more commonly called the Expected Shortfall (ES). That should not bring any confusion here. Some comments about the related problem of estimating the conditional ES are provided in Section 6.

Attempts to estimate cVaR have historically centred on parametric models due to their familiar and convenient nature. The most well-known of these approaches is probably the industry-benchmark RiskMetrics (JP Morgan 1996), which attempts to characterise future returns through a normal distribution that is scaled by an Exponentially Weighted Moving Average (EWMA) estimate of the market volatility. This is despite empirical research revealing that the normal assumption usually falls short for fitting financial data, which typically show substantial skewness and kurtosis. This has motivated the development of EWMA-models replacing the normal innovations with ones from the $t$-distribution (So and Yu 2006), the Laplace distribution (Guermat and Harris 2001), the asymmetric-Laplace distribution (Gerlach et alii 2013), or even more sophisticated parametric distributions. In spite of this, RiskMetrics has arguably remained the industry standard since 1996.

RiskMetrics is based on modelling the volatility embedded in the series of returns via an integrated GARCH
model with Normal innovations. This belongs to a much wider class of parametric time series models, which can be specified as follows. Let $X_t$ be the return (or loss) of interest at time $t$. Then, for $t = 0, \ldots, T$, it is assumed that

$$X_{t+1}|F_t \sim D(\xi_t, \theta),$$  
(1.2)

$$\xi_{t+1} = h(\xi_t, \ldots, \xi_{t-p+1}, X_t, \ldots, X_{t-q+1}; \beta),$$  
(1.3)

where $F_t$ is the information gathered from time 0 up to time $t$, $D$ is a specified parametric innovation distribution which is parametrised by $\xi_t$, a vector of time-varying parameters, and $\theta$, a vector of static parameters, and $h$ is the function, parametrised by a static vector of parameters $\beta$, which sets the dynamics of the considered model. For instance, a popular approach for modelling the behaviour of volatility (i.e. the conditional variance of $X_t$), is through the GARCH(1,1) model (Bollerslev 1986), which satisfies the above specification with $\xi_t = \sigma^2_t$, the volatility parameter at time $t$, and $h$ given by $\sigma^2_{t+1} = \beta_0 + \beta_1 \epsilon^2_t + \beta_2 \sigma^2_t$. The conditional distribution $D$ is usually assumed to be Normal, Student or skew-Student. Further specifications of GARCH-like models satisfying dynamics like (1.2)-(1.3) include the above-mentioned integrated GARCH (iGARCH) model (Engle and Bollerslev 1986), Exponential GARCH model (eGARCH) (Nelson 1991) and the Asymmetric Power ARCH model (APARCH) (Ding et alii 1993), with the gjr-GARCH specification of Glosten et alii (1993) as special case. For a recent and extensive survey of GARCH models, see Francq and Zakoian (2011). Another class of models satisfying (1.2)-(1.3) are the Generalised Autoregressive Score models (GAS) recently suggested in Creal et alii (2013) and Harvey (2013), which seem to be serious alternatives to GARCH models when modelling highly non-linear volatility dynamics. Alternative parametric models for forecasting cVaR finally include approaches based on Extreme-Value-Theory, such as Block Maxima Model (BMM, McNeil 1998, 1999) or Peak over Threshold models (POT, Embrechts et alii 1997, 1999, Chavez-Dumoulin et alii 2014); or on quantile regression such as CaViaR (Engle and Manganelli 2004). Nieto and Ruiz (2016) offers a recent comprehensive review.

While those approaches have their merits, they typically suffer from the usual rigidity of the parametric setting. In particular, different parametric model specifications usually lead to disparate results which are hard to reconcile, see empirical illustration of this in Li and Racine (2008, Section 4.2). In addition, model misspecification can have serious consequences; e.g., it has been argued that the 2009 global financial crisis was mainly due to an unwarranted usage of the parametric Gaussian copula model for asset pricing (Salmon 2009). In answer to this lack of flexibility, nonparametric models, which discard any rigid structure for the data, have been suggested as well. The nonparametric methodology really lets the data ‘speak for itself’, hence its attractiveness. This is particularly important when estimating high quantiles of distributions,
such as VaR, as usually the tail behaviour of a probability density is entirely locked by its parametric specification. On the other hand, estimation of tails of distributions without any parametric guidelines is usually a challenging task, owing to the typical sparseness of data in those areas.

First nonparametric attempts centred on empirical methodologies in which past returns were assumed to represent the full distribution of future returns, see e.g. the historical simulation technique in [Hendricks (1996), Linsmeier and Pearson (2000), Dowd (2001) and Chen and Tang (2007)]. This basic approach ignores the changing nature of equity markets, hence Barone-Adesi et alii (2002), following Hull and White (1998), suggested a scaling of the historical returns by the implied market volatility in order to compensate for these changes. This method, known as Filtered Historical Simulation (FHS) has grown to be a forerunner in the market due to its simplicity and computational efficiency. More recently, Zikovic and Aktan (2011) and Dupuis et alii (2015) proposed ‘weighted’ versions of historical simulations along similar lines. Fan and Gu (2003) suggested what can be regarded as a semiparametric version of RiskMetrics, later extended in Martins-Filho and Yao (2006), Martins-Filho et alii (2016) and Wang and Zhao (2016).

Cai (2002) used a Nadaraya-Watson-type estimator of the conditional distribution of $X$ given $Z$ and obtained the conditional VaR by inverting it, an approach refined in Scaillet (2005), Cai and Wang (2008), Li and Racine (2008), Taylor (2008), Wu et alii (2008), Xu (2013) and Franke et alii (2015). Some of these estimators are discussed in more detail in Section 3 as this paper actually gives a continuation to them in that it studies a novel estimator of $c\text{VaR}_\alpha(X|Z = z)$ based on the direct inversion of a nonparametric estimator of the conditional distribution $F_{X|Z}$. What mostly differs from the previous contributions is that here $F_{X|Z}$ will be estimated by making use of new developments in the field of nonparametric copula modelling. This has various advantages, those being detailed throughout the paper. Of course, copulas have been around for a long time in finance and related fields, see e.g. Embrechts et alii (2002), Embrechts (2009), Chernobini et alii (2004) for comprehensive reviews, but the literature in the field is again overwhelmingly dominated by parametric methods. Through the particular application of estimating $c\text{VaR}$, this paper aims to demonstrate the capability of flexible nonparametric copula modelling methods for financial applications.

It is organised as follows: Section 2 sets the framework that will be considered. Section 3 provides a short review of existing nonparametric, kernel-based, methods for estimating conditional distributions and densities which this work will complement. Section 4 outlines, in detail, how some recent elements of nonparametric copula modelling can be amalgamated to provide an estimate of the conditional Value-at-Risk. Section 5 focuses on empirically illustrating and validating the idea through two real data applications. Section 6 concludes with some paths for future research.
2 Framework

Consider a sample \( \{X_t; t = 0, 1, \ldots, T-1\} \) of losses for a given portfolio, and assume that it is a realisation of a strictly stationary process \( X \) in discrete time, with marginal distribution \( F_X \) admitting a density \( f_X \).

The theoretical considerations exposed in Section 4.2 hold assuming that \( \{X_t\} \) forms an \( \alpha \)-mixing (i.e., strongly mixing) sequence, a dependence structure obeyed by most time series models (Doukhan 1994), hence are rather general.

For simplicity, this paper will only focus on the problem of estimating the one-step-ahead cVaR, that is, estimating at time \( T-1 \) the VaR at time \( T \); when taking as ‘influencing economic factors’, i.e. \( Z \) in (1.1), the just observed loss \( X_{T-1} \) only. Specifically, the parameter of interest is here

\[
c\text{VaR}_\alpha(X_T|X_{T-1} = x) = \inf\{y \in \mathbb{R} : P(X_T > y|X_{T-1} = x) \leq 1 - \alpha\} = F_{X_T|X_{T-1}}^{-1}(\alpha|x),
\]

the quantile of level \( \alpha \) of the conditional distribution of \( X_T \) given that \( X_{T-1} = x \), viz.

\[
F_{X_T|X_{T-1}}(y|x) = P(X_T \leq y|X_{T-1} = x).
\]

Although very basic, this Markov-type framework provides scope for the volatility clustering and serial correlation of returns seen in real datasets to be taken into account (McNeil et alii 2005). In addition, it covers the effect of return momentum (Carhart 1997), which has been demonstrated to have the strongest effect on future equity returns compared to any of the other popular risk factors (Bender et alii 2013). Finally, it is consistent with Fama (1965)’s Efficient Market Hypothesis. Hence it seems a valid basis for illustrating the idea.

Remark 2.1. It is stressed, though, that the suggested methodology can readily be extended to more general frameworks. In particular, conditioning on more than one variable would be conceptually straightforward, bearing in my mind that, the developed procedure being purely nonparametric, it could suffer from dimensionality issues (Geenens 2011). If needs be, dimension reduction can be achieved by introducing some structural assumptions, such as a Single-Index structure, see Fan et alii (2017).

Under the stationary assumption, the distribution (2.1), hence its quantiles, are independent of \( T \) and only depend on the observed value \( X_{T-1} = x \). Interestingly, \( F_{X_T|X_{T-1}}(y|x) \) can be regarded as a regression function, arguing that

\[
F_{X_T|X_{T-1}}(y|x) = \mathbb{E}(\mathbb{I}_{\{X_T \leq y\}}|X_{T-1} = x),
\]

where \( \mathbb{I}_{\{\cdot\}} \) is the indicator function, equal to 1 if the statement between brackets is true and 0 otherwise.

\(^1\)Here, the term ‘loss’ for \( X \) is quite generic. It may mean negative log-returns, for instance, or any other quantity that could be appropriate.
Naturally, it can also be written

\[ F_{X_T|X_{T-1}}(y|x) = \int_{-\infty}^{y} f_{X_T|X_{T-1}}(\xi|x) d\xi, \quad (2.3) \]

where \( f_{X_T|X_{T-1}} \) is the conditional density of \( X_T \) given \( X_{T-1} \) – provided it exists. One can, therefore, estimate \( F_{X_T|X_{T-1}} \) using either regression ideas, or by plugging an estimate of the conditional density, say \( \hat{f}_{X_T|X_{T-1}} \), in (2.3). Regression-based estimation has been quite popular (see next section), however, an estimator of type

\[ \hat{F}_{X_T|X_{T-1}}(y|x) = \int_{-\infty}^{y} \hat{f}_{X_T|X_{T-1}}(\xi|x) d\xi \quad (2.4) \]

offers substantial advantages. Indeed, provided that \( \hat{f}_{X_T|X_{T-1}}(\cdot|x) \) is a bona fide density, in the sense that \( \hat{f}_{X_T|X_{T-1}}(\cdot|x) \geq 0 \) for all \( (x,y) \) and \( \int_{-\infty}^{\infty} \hat{f}_{X_T|X_{T-1}}(\xi|x) d\xi = 1 \) for all \( x \), the so-obtained \( \hat{F}_{X_T|X_{T-1}} \) is always a bona fide distribution function as well, non-decreasing in \( y \) and with \( \hat{F}_{X_T|X_{T-1}}(-\infty|x) = 0, \hat{F}_{X_T|X_{T-1}}(\infty|x) = 1 \). This is essential when inverting it for obtaining its quantile. Regression-based approaches may lead to estimates not constrained to lie in \([0, 1]\) or to be monotonic in \( y \), which causes obvious issues and inconsistencies.

3 Kernel estimators of conditional distributions and densities

The most basic nonparametric regression estimator is arguably the Nadaraya-Watson (NW) estimator (Nadaraya 1964, Watson 1964), which for (2.2) writes

\[ \hat{F}_{X_T|X_{T-1}}(y|x) = \frac{\sum_{t=1}^{T-1} K_h(x - X_{t-1}) \mathbb{I}\{X_t \leq y\}}{\sum_{t=1}^{T-1} K_h(x - X_{t-1})}, \quad (3.1) \]

where \( K \) is a symmetric probability distribution (‘kernel’), \( h > 0 \) is a smoothing parameter (‘bandwidth’) and \( K_h(\cdot) = K(\cdot/h)/h \), see Härdle et alii (2004, Chapter 4) for details. Clearly, the so-defined \( \hat{F}_{X_T|X_{T-1}}(y|x) \) is a bona fide distribution function, always lying in \([0, 1]\) and non-decreasing in \( y \). This makes the inversion of (3.1) very easy, and Franke et alii (2015) studied in detail the estimator of the conditional VaR obtained by doing so.

It is usually accepted, though, that local polynomial regression estimators (Fan and Gijbels 1996) enjoy better theoretical properties than the Nadaraya-Watson estimator. This motivated Yu and Jones (1998) to suggest a local linear (LL) estimator of a conditional distribution function. Yet, the LL estimator is neither constrained to between 0 and 1, nor to be monotonic in \( y \), which violates common sense. Hence Hall et alii (1999) and Cai (2002) proposed the weighted Nadaraya-Watson estimator (WNW), which satisfies those constraints while sharing the same theoretical properties as the LL estimator. However, this estimator is not continuous in its first argument, which may not be ideal. In addition, for each \( x \), it requires numerically determining a set of weights through a constrained optimisation problem, which may
be computationally demanding. Note that Cai and Wang (2008) fixed the non-continuity issue through yet another layer of smoothing, suggesting their ‘weighted double kernel local linear estimator’ (WDKLL) of $F_{X_T|X_{T-1}}$, which nonetheless still necessitates numerical optimisation for determining the right weights, making the procedure rather cumbersome. An approach based on the conditional density through (2.3) may be simpler.

By definition, the conditional density $f_{X_T|X_{T-1}}$ is

$$f_{X_T|X_{T-1}}(y|x) = \frac{f_{X_{T-1},X_T}(x,y)}{f_{X_{T-1}}(x)}, \quad (3.2)$$

where $f_{X_{T-1},X_T}$ is the joint density of the vector $(X_{T-1}, X_T)$ and $f_{X_{T-1}}$ its marginal. Again, these quantities are independent of $T$, by the assumed stationarity of the process $\mathcal{X}$, hence $f_{X_{T-1},X_T} = f_1$, say, and $f_{X_T} = f_X$. This motivates a nonparametric estimator of type

$$\hat{f}_{X_T|X_{T-1}}(y|x) = \frac{\hat{f}_1(x,y)}{\hat{f}_X(x)}, \quad (3.3)$$

where both the numerator and denominator in (3.2) are estimated from the observed sample by usual kernel-type estimators $\hat{f}_1$ and $\hat{f}_X$ for bivariate and univariate densities (Härdle et alii 2004, Chapter 3).

This kind of ‘plug-in’ kernel conditional density estimator, initially proposed in Rosenblatt (1968), was studied in Hyndman et alii (1996), Bashtannyk and Hyndman (2001), Fan and Yim (2004) and Hall et alii (2004), among others. Plugging (3.3) into (2.3) yields the ‘double kernel’ Nadaraya-Watson estimator (DKNW):

$$\hat{F}_{X_T|X_{T-1}}(y|x) = \sum_{t=1}^{T-1} \frac{K_h(x-X_{t-1})K_0((y-X_t)/h_0)}{\sum_{t=1}^{T-1} K_h(x-X_{t-1})}, \quad (3.4)$$

where $K_0(u) = \int_{-\infty}^{u} K_0(u') du'$ is the ‘integrated’ version of a kernel $K_0$ and $h_0$ is a bandwidth. Essentially, (3.4) is (3.1) but with the indicator $\mathbf{1}_{\{X_t \leq y\}}$ replaced by a smoothed version of it, which makes $\hat{F}_{X_T|X_{T-1}}(y|x)$ continuous in $y$. This is usually beneficial to the estimator, as it has often been stressed in the classical literature on quantile estimation (Azzalini 1981, Falk 1985, Yang 1985). It is precisely this estimator (3.4) that Scaillet (2005) and Li and Racine (2008) inverted to produce their estimator of cVaR – see also Ferraty and Quintela-Del-Río (2016) for an extension of this to a functional context. In the study below, it will therefore be taken as the benchmark for the previously proposed kernel-type methods.

Yet, it can be understood that the ratio form of (3.3) creates issues (Faugeras 2009), both in theory and in practice: denominator close to 0, numerical instability, delicate choice of smoothing parameters, etc. What is suggested in this paper is to use a different type of kernel estimator of the conditional density in (2.4), based on the copula of the vector $(X_{T-1}, X_T)$. Given that $X_{T-1}$ and $X_T$ have the same distribution by stationarity, the only extra information contained in the joint distribution of $(X_{T-1}, X_T)$ relates to
their dependence. This evidences the appropriateness of an approach based on copulas, a.k.a. dependence functions, in this framework.

4 Nonparametric copula-based estimation of \( \tilde{c} \text{VaR} \)

4.1 Copulas, copula densities, conditional densities and conditional distributions

The central result of copula theory, known as Sklar’s theorem (Sklar, 1959), asserts that for any continuous bivariate random vector \((X, Y)\) with distribution function \(F_{XY}\) (and marginals \(F_X\) and \(F_Y\)), there exists a unique ‘copula’ function \(C\) such that

\[
F_{XY}(x, y) = C(F_X(x), F_Y(y)) \quad \forall (x, y) \in \mathbb{R}^2. \tag{4.1}
\]

Clearly, \(C\) describes how \(X\) and \(Y\) ‘interact’ to produce the joint behaviour of \((X, Y)\), hence it fully characterises the dependence structure between \(X\) and \(Y\) and isolates it from their marginal behaviours.

See Joe (1997) and Nelsen (2006) for general textbook treatment of these ideas, and Cherubini et alii (2004, 2012) for their implications in finance. Importantly, given that \(F_X(X), F_Y(Y) \sim U_{[0,1]}\) (probability integral transform), \(C\) is actually a bivariate distribution on the unit square \(I = [0, 1]^2\) with uniform marginals.

Under mild conditions, that distribution admits a density, known as the copula density:

\[
c(u, v) = \frac{\partial^2 C}{\partial u \partial v}(u, v), \quad (u, v) \in I.
\]

Writing (4.1) for the vector \((X_{T-1}, X_T)\) yields

\[
F_{X_{T-1},X_T}(x, y) = C_1(F_X(x), F_X(y)),
\]

for some copula \(C_1\) independent of \(T\) and \(F_X = F_{X_{T-1}} = F_{X_T}\) by stationarity. Now, differentiating both sides, the joint density \(f_1 = f_{X_{T-1},X_T}\) is seen to be

\[
f_1(x, y) = \frac{\partial^2 F_{X_{T-1},X_T}}{\partial x \partial y}(x, y) = c_1(F_X(x), F_X(y))f_X(x)f_X(y),
\]

by the chain rule, with \(c_1\) the copula density of \(C_1\). Hence, from (3.2), the conditional density of \(X_T\) given \(X_{T-1}\) can be written directly in terms of the copula density:

\[
f_{X_T|X_{T-1}}(y|x) = c_1(F_X(x), F_X(y))f_X(y). \tag{4.2}
\]

Replacing the unknown \(c_1, F_X\) and \(f_X\) by some estimators \(\hat{c}_1, \hat{F}_X\) and \(\hat{f}_X\) leads to a different, copula-based estimator of \(f_{X_T|X_{T-1}}\), viz.

\[
\hat{f}_{X_T|X_{T-1}}(y|x) = \hat{c}_1(\hat{F}_X(x), \hat{F}_X(y))\hat{f}_X(y). \tag{4.3}
\]
Under a product shape, this estimator is, unlike (3.3), free from any issues a random denominator creates, hence its attractiveness. Expression (4.2) is also very intuitive: the conditional density of $X_T$ given $X_{T-1}$ is the marginal density of $X_T$, corrected for the influence that $X_{T-1}$ may have on $X_T$ through the copula density $c_1$ of the vector $(X_{T-1}, X_T)$. Now, integrating (4.2) in (2.3) yields

$$F_{X_T | X_{T-1}}(y | x) = \int_{-\infty}^{y} c_1(F_X(x), F_X(\xi)) f_X(\xi) d\xi = \int_{0}^{F_X(y)} c_1(F_X(x), v) dv,$$

through the change-of-variable $F_X(\xi) = v$. Such an expression is known in the copula literature as a $h$-function.

### 4.2 Nonparametric copula density estimation and $c$VaR

Usually, $h$-functions are estimated by numerically integrating an estimator of $c_1$ (Nagler and Czado, 2016). However, seeing that

$$F_{X_T | X_{T-1}}(y | x) = \mathbb{E}_X \left( c_1(F_X(x), F_X(X)) \mathbb{I}_{\{X \leq y\}} \right),$$

the integral can easily be approximated by Monte-Carlo, viz.

$$\hat{F}_{X_T | X_{T-1}}(y | x) = \frac{1}{T} \sum_{t=0}^{T-1} \hat{c}_1(\hat{F}_X(x), \hat{F}_X(X_t)) \mathbb{I}_{\{X_t \leq y\}},$$

where $\hat{c}_1$ and $\hat{F}_X$ are appropriate estimators of the unknown $c_1$ and $F_X$ as in (4.3) (but here the estimation of the marginal density $f_X$ is not required). Interestingly, estimator (4.4) can be thought of as a weighted empirical distribution function, where each indicator $\mathbb{I}_{\{X_t \leq y\}}$ is weighted according to the chance of seeing $X_t$ if it is known that $X_{t-1} = x$, as measured by the (estimated) copula density $\hat{c}_1$. Note that, for some reasons briefly mentioned earlier, one might prefer the continuous version

$$\hat{F}_{X_T | X_{T-1}}(y | x) = \frac{1}{T} \sum_{t=0}^{T-1} \hat{c}_1(\hat{F}_X(x), \hat{F}_X(X_t)) \mathbb{K}_0((y - X_t)/h_0), \quad (4.4)$$

where $\mathbb{K}_0$ and $h_0$ are as in (3.4).

In the copula framework, it is customary to estimate $F_X$ by the rescaled empirical distribution function

$$\hat{F}_X(x) = \frac{1}{T + 1} \sum_{t=0}^{T-1} \mathbb{I}_{\{X_t \leq x\}},$$

known to be a simple and uniformly consistent estimator of $F_X$. It is, therefore, clear that the viability of estimator (4.4) is mostly conditional on a reliable estimator for $c_1$. However, good nonparametric estimation of a copula density has long proved elusive. This mainly for three reasons which make the case non-standard: 1) a copula density has bounded support $\mathcal{I}$, and kernel estimators are known to heavily suffer from boundary bias issues; 2) a copula density is often unbounded in some corners of $\mathcal{I}$, and kernel
estimators are known not to be consistent in that case; and 3) there is no access to genuine observations from the density to estimate, as \( c_1 \) is essentially the density of \((F_X(X_{T-1}), F_X(X_T))\) where \( F_X \) is unknown, and one must resort to ‘pseudo-observations’ \( \{\hat{F}_X(X_t)\} \).

Recently, though, Geenens et alii (2017), after an original idea of Geenens (2014), proposed a novel kernel-type estimator of the copula density along the following lines. Define

\[
S_{T-1} = \Phi^{-1}(F_X(X_{T-1})) \quad \text{and} \quad S_T = \Phi^{-1}(F_X(X_T)),
\]

where \( \Phi^{-1} \) is the ‘probit’ function, i.e. the inverse of the standard normal distribution function \( \Phi \). Through standard distributional arguments, one gets

\[
c_1(u,v) = g_1(\Phi^{-1}(u), \Phi^{-1}(v)) \frac{\phi(\Phi^{-1}(u))}{\phi(\Phi^{-1}(v))}, \quad \forall (u,v) \in \mathcal{I},
\]

where \( g_1 \) is the joint density of \((S_{T-1}, S_T)\) and \( \phi \) is the standard normal density. Hence an estimator \( \hat{c}_1 \) of \( c_1 \) can be directly obtained from any estimator \( \hat{g}_1 \) of \( g_1 \):

\[
\hat{c}_1(u,v) = \hat{g}_1(\Phi^{-1}(u), \Phi^{-1}(v)) \frac{\phi(\Phi^{-1}(u))}{\phi(\Phi^{-1}(v))}.
\] (4.5)

Estimating \( g_1 \), though, is much easier. For \( U \sim \mathcal{U}(0,1) \), \( \Phi^{-1}(U) \sim \mathcal{N}(0,1) \), hence \( g_1 \) has unconstrained support with standard normal marginals. Local likelihood methods, in particular Loader (1996)’s local log-quadratic estimator, are particularly good at estimating normal densities, hence the appropriateness of using this type of methodology for estimating \( g_1 \) in this context.

Geenens et alii (2017)’s estimator, called ‘LLTKDE2’, is actually (4.5) with \( \hat{g}_1 \) being the local log-quadratic estimator of the bivariate density \( g_1 \) based on pseudo-observations \( \{\Phi^{-1}(\hat{F}_X(X_t))\} \). Its theoretical properties were obtained. In particular, under mild assumptions, it was shown to be uniformly consistent on any compact proper subset of \( \mathcal{I} \), and asymptotically normal with known expressions of (asymptotic) bias and variance. In addition, a practical criterion for selecting the always crucial smoothing parameters was studied and tested. Combining transformation and local likelihood estimation, the procedure actually takes advantage of the known uniform margins of \( C_1 \), which results in remarkably accurate estimation (Geenens et alii 2017, Nagler and Czado 2016, De Backer et alii 2017). Besides, the LLTKDE2 estimates typically enjoy a visually pleasant appearance usually peculiar to parametric fits.

What is suggested in this paper is to use that LLTKDE2 estimator \( \hat{c}_1 \) in (4.4) and proceed with the extraction of cVaR. The nonparametric copula-based cVaR estimator is thus defined as

\[
c_{\text{VaR}}(x) = c_{\text{VaR}}(X_{T}|X_{T-1} = x) = \hat{F}_{X_T|X_{T-1}}^{-1}(\alpha|x),
\]

where \( \hat{F}_{X_T|X_{T-1}}^{-1} \) is the generalised inverse of (4.4). Given that \( \hat{F}_X \) is uniformly consistent for \( F_X \) on \( \mathbb{R} \) and...
\( c_1 \) is uniformly consistent for the integrable \( c_1 \) on any proper compact subset of \( I \), (4.4) is also uniformly consistent over any compact subset of \( \mathbb{R}^2 \) by the ergodic theorem. It classically follows that, provided that \( \inf_{x \in G} \int_{X_{T-1}} (c \text{VaR}_{\alpha,T}(x) | x) > 0 \) where \( G \) is any compact subset of \( \mathbb{R} \), \( c \text{VaR}_{\alpha,T}(x) \) is a uniformly consistent estimator of \( c \text{VaR}_{\alpha,T}(x) \):

\[
\sup_{x \in G} |c \hat{\text{VaR}}_{\alpha,T}(x) - c \text{VaR}_{\alpha,T}(x)| \xrightarrow{P} 0 \quad \text{as} \quad T \to \infty,
\]

for any fixed \( \alpha \in (0, 1) \). Further theoretical properties (such as asymptotic normality and rate of convergence) of this estimator would easily follow from the usual asymptotic representation of sample quantiles, viz.

\[
c \hat{\text{VaR}}_{\alpha,T}(x) - c \text{VaR}_{\alpha,T}(x) \approx \frac{\alpha - \int_{X_{T-1}} (c \text{VaR}_{\alpha,T}(x) | x)}{\int_{X_{T-1}} (c \text{VaR}_{\alpha,T}(x) | x)},
\]

but details are left aside owing to the rather unwieldy expressions in Geenens et alii (2017). Rather, its practical performance is assessed in Section 5 via a real-data analysis and back-testing.

5 Real data

5.1 Illustration - S&P500 data

Firstly the procedure described in the previous sections is illustrated on the intra-day returns from the S&P500 US index between November 2011 and October 2015. The S&P500 index is one of the most actively traded indices available to investors. Figure 5.1 (left) shows the evolution over time of the S&P500 index from the 9th of November 2011 until the 31st of October 2015 (which corresponds to \( T = 1,000 \) trading days). Figure 5.1 (right) shows the corresponding negative log-returns series \( \{X_t\} \), i.e., approximately the percentage changes in the value of the index. It is reasonable to posit that this series is (at least approximately) stationary.
The copula density \( c_1 \) of \((X_{T-1}, X_T)\) is estimated from these data by the LLTKDE2 estimator of Geenens et alii (2017), see Figure 5.2. The shape of the estimated copula density indicates that \((X_{T-1}, X_T)\) is not bivariate Gaussian (Gaussian copulas can only show peaks in opposite corners of \(I\), not adjacent corners).

The ‘heat’ map clearly shows two effects. First, some sort of negative effect, which shows that log-returns on two successive days may be negatively associated. If today’s return is low (large value of \( u \)), one can expect a return tomorrow in the middle range \((v \approx 0.5)\), while if today’s return is very high \((u \approx 0)\), chances are that tomorrow’s return will be very low (i.e., negative) \((v \approx 1)\). This is, however, largely balanced by the second effect, materialised by the peak in the lower-left corner of the unit square: there is also a substantial probability that, given a high return today \((u \approx 0)\), tomorrow’s return is very high as well \((v \approx 0)\). In other words, when \(X_{T-1}\) is seen to be small on a day, one expects a more extreme (in either direction) realisation of \(X_T\) the day after than in other situations. This, obviously, impacts the corresponding values of cVaR in a direct way.
This is illustrated by Figure 5.3. The conditional densities $f_{X_T|X_{T-1}}(\cdot|x)$ for $x = -0.02$ (high return today), $x = 0$ (medium return today) and $x = 0.02$ (low return today) have been estimated by (4.3). The stationary marginal (i.e., unconditional) density $f_X$ was estimated by the local log-quadratic estimator [Loader 1996], and is shown in Figure 5.3 (blue line). For $x = -0.02$, which here corresponds to $u \simeq 0.01$, $\hat{f}_X$ is multiplied on its domain by the ‘slice’ of $\hat{c}_1$ at $u \equiv 0.01$ which is a U-shaped function. Hence the (estimated) conditional density of $X_T$ given $X_{T-1} = -0.02$ has substantially fatter tails. This translates into a higher $c\text{VaR}$ at both levels $\alpha = 0.95$ and $\alpha = 0.99$ (i.e., higher risk). When $x_T = 0$, the risk is actually lower than average ($c\text{VaR}$ lower than the unconditional VaR), because for $u \simeq 0.5$ the ‘slice’ of $\hat{c}_1$ shows a mode around the middle-range and down-weighs the tails of the resulting conditional density. Along similar lines, when $x_T = 0.02$, one finds that $c\text{VaR}$ is slightly higher then the unconditional one.

Note that extracting $c\text{VaR}$ as a quantile of (4.4) does not require the estimation of the density $f_{X_T|X_{T-1}}$; here, those densities are shown in Figure 5.3 for illustration purpose only. The obtained values of $\text{VaR}$ and $c\text{VaR}$, at $x = -0.02$, 0 and 0.02, for $\alpha = 0.95$ and 0.99, are given in Table 5.1.

| $\alpha$ | $\text{VaR}_{\alpha,T}$ | $c\text{VaR}_{\alpha,T}(x)$ |
|----------|--------------------------|-----------------------------|
| 0.95     | 0.01278                  | 0.01543                     |
|          |                          | 0.01192                     |
|          |                          | 0.01379                     |
| 0.99     | 0.01968                  | 0.02263                     |
|          |                          | 0.01867                     |
|          |                          | 0.02095                     |

Table 5.1: Estimated $\text{VaR}$ and $c\text{VaR}$ for the S&P500 data at level $\alpha = 0.95$ and $\alpha = 0.99$. 

Figure 5.2: Estimated copula density of $(X_{T-1}, X_T)$ for the negative log-returns of the S&P500 index.
Figure 5.3: Estimated unconditional density \( \hat{f}_X \) (plain blue line) of negative log-returns, conditional densities \( \hat{f}_{X|X_{T-1}}(\cdot|x) \) for \( x = -0.02 \) (dashed), \( x = 0 \) (dotted) and \( x = 0.02 \) (dashed-dotted), and corresponding values of VaR and \( \hat{c}\text{VaR} \) at level 95\% (green) and 99\% (red).

5.2 Forecasting and Backtesting - IBM data

The previous analysis is, of course, purely descriptive. Here the proposed estimator of \( c\text{VaR} \) is evaluated by contrasting its forecasting performance against a range of reasonable alternatives. The IBM Corporation stock index from the 3rd of January 2011 to the 22nd of November 2017 (1,736 trading days) is considered. The raw series, as well as its negative log-returns, are shown in Figure 5.4.

The copula density \( c_1 \) of \((X_{T-1}, X_T)\) was first estimated on the first \( T = 252 \) observations, which corresponds to roughly one year of trading days, a common time horizon in rolling window analyses. The conditional VaR at time \( T+1 = 253 \) given \( X_T = x_T \) was then estimated via the procedure described in the
previous subsection. Then a rolling window of width $N = 252$ was used: each day, the oldest observation was discarded and the newly observed one included in the ‘learning sample’, thereby updating on a daily basis the estimation of the copula density for forecasting the next c VaR. The series of daily one-step-ahead c VaR forecasts at level 95% (yellow) and 99% (orange) are shown in Figure 5.5. Over the 1,484 c VaR forecasts, the proportion of VaR violations (i.e., when the realised value of the loss exceeds the forecast VaR) is 0.0498 for c VaR at level 95% and 0.013 for c VaR at level 1%. Both are very close to their target.

![IBM data: daily one-step-ahead c VaR forecasts at level 95% (yellow) and 99% (orange), rolling window of width $N = 252$ (roughly one year).](image)

Many procedures, essentially akin to statistical hypothesis tests, have been proposed for assessing and comparing such c VaR forecasts. The Basel accords appraise their accuracy through ‘back-testing’, hence the same methodology will be followed here. [Campbell (2007)](#) gives a comprehensive review of such back-testing procedures. The simplest version is simply to contrast the empirical proportions of violations against their targeted theoretical probability $1 - \alpha$. This is [Kupiec (1995)](#)’s ‘unconditional coverage’ test (UC). This, however, ignores the likely serial correlation of the violation events. [Christoffersen (1998)](#) suggested a ‘conditional coverage’ test (CC) taking this into account. Although by far the most popular among practitioners, these two tests have been criticised on different grounds, see [Escanciano and Olmo (2010)](#). The favoured option seems to be the ‘dynamic quantile’ test (DQ) of [Engle and Manganelli (2004)](#), which jointly tests for both UC and CC and has proved more powerful against some forms of malfunctions in the forecasting.

Table 5.2 shows the $p$-values associated to these 3 tests (UC, CC, DQ) for several c VaR forecasting procedures, at levels 95% and 99%. The forecasting performance of the procedure suggested in this paper
Table 5.2: \( p \)-values associated to the three backtesting procedures (UC, CC, DQ) for the NP-Cop procedure and its competitors. Underlined values highlight statistical significance (at 5% level).

| Procedure                      | \( \alpha = 0.95 \) |          |          | \( \alpha = 0.99 \) |          |          |
|--------------------------------|----------------------|----------|----------|----------------------|----------|----------|
| NP-Cop                         | 0.986                | 0.793    | 0.219    | 0.297                | 0.454    | 0.106    |
| DKNW                           | 0.649                | 0.814    | 0.007    | 0.580                | 0.360    | 0.011    |
| GARCH-\( \mathcal{N} \)       | 0.173                | 0.162    | 0.075    | 0.016                | 0.010    | 0.001    |
| GARCH-\( S \)                 | 0.206                | 0.171    | 0.003    | 0.049                | 0.021    | 0.000    |
| GARCH-\( S \)                 | 0.826                | 0.314    | 0.013    | 0.580                | 0.704    | 0.234    |
| iGARCH-\( \mathcal{N} \)      | 0.033                | 0.090    | 0.019    | 0.049                | 0.021    | 0.002    |
| iGARCH-\( S \)                | 0.797                | 0.422    | 0.067    | 0.763                | 0.802    | 0.923    |
| eGARCH-\( \mathcal{N} \)      | 0.323                | 0.155    | 0.016    | 0.827                | 0.854    | 0.912    |
| eGARCH-\( S \)                | 0.030                | 0.001    | 0.000    | 0.000                | 0.000    | 0.000    |
| eGARCH-\( S \)                | 0.030                | 0.063    | 0.056    | 0.028                | 0.060    | 0.104    |
| ejGARCH-\( S \)               | 0.649                | 0.900    | 0.001    | 0.200                | 0.335    | 0.015    |
| gjr-GARCH-\( \mathcal{N} \)   | 0.618                | 0.819    | 0.076    | 0.008                | 0.024    | 0.090    |
| gjr-GARCH-\( S \)             | 0.300                | 0.325    | 0.020    | 0.297                | 0.292    | 0.043    |
| gjr-GARCH-\( S \)             | 0.826                | 0.314    | 0.007    | 0.423                | 0.335    | 0.021    |
| GAS-\( \mathcal{N} \)         | 0.017                | 0.047    | 0.115    | 0.049                | 0.095    | 0.062    |
| GAS-\( S \)                   | 0.706                | 0.644    | 0.160    | 0.004                | 0.014    | 0.005    |
| GAS-\( S \)                   | 0.458                | 0.591    | 0.052    | 0.049                | 0.095    | 0.021    |

(NP-Cop, for nonparametric copula) is compared to its main nonparametric competitor, based on inverting the ‘double kernel’ Nadaraya-Watson estimator (DKNW), as well as to a battery of parametric alternatives: GARCH, iGARCH, eGARCH and gjr-GARCH, all with three different innovation distributions (Normal \( \mathcal{N} \), Student \( S \) and skewed-Student \( S \)S). These models have been fitted using the R package \texttt{rugarch} (Ghalanos, 2017). Also included in the study is the Generalised Autoregressive Score (GAS) model, again with the same three innovation distributions, which has been fitted via the R package \texttt{GAS} (Ardia et alii, 2016). Note that the DKNW estimator was fit from the R package \texttt{np} (Hayfield and Racine, 2008).

It appears that the adequacy of the forecasts is rejected (at significance 5%) by at least one test for at least one level for all but two of the tested procedures: the proposed nonparametric copula-based estimator, and the integrated GARCH model with Student innovations. For all other procedures there is some evidence that the \( c \)VaR forecasts are not adequate in some sense. In order to compare the proposed NP-Cop to the parametric iGARCH-\( S \) specification, one resorts to the ‘Quantile Loss’ function which, for the forecast at time \( t \), is

\[
\ell_{\alpha,t} = \ell \left( X_t, c \text{VaR}_{\alpha,t}(x_{t-1}) \right) = (\alpha - \mathbb{I}_{\{X_t \leq c \text{VaR}_{\alpha,t}(x_{t-1})\}})(X_t - c \text{VaR}_{\alpha,t}(x_{t-1})).
\]

This is obviously an asymmetric loss function which, inspired by similar ideas in quantile regression (Koenker and Bassett, 1978), penalises more heavily Value-at-Risk exceedance than otherwise (González-Rivera et alii, 2004). Averaging over the 1,484 predictions, one obtains for the nonparametric copula-based
procedure losses of
\[ \frac{1}{1484} \sum_{t=253}^{1736} \ell_{0.95,t} = 0.00139 \quad \text{and} \quad \frac{1}{1484} \sum_{t=253}^{1736} \ell_{0.99,t} = 0.00059, \]
while for the parametric iGARCH-S model, one gets
\[ \frac{1}{1484} \sum_{t=253}^{1736} \ell_{0.95,t} = 0.00139 \quad \text{and} \quad \frac{1}{1484} \sum_{t=253}^{1736} \ell_{0.99,t} = 0.00055. \]

On this criterion, one can say that both procedures are doing equally well. It is thus fair to conclude that the proposed nonparametric copula-based procedure is on par with the best parametric procedure when it comes to forecast the one-step ahead \( c \text{VaR} \) at level 95% and 99% for the IBM data. This is impressive, given that this was achieved without any parametric guidelines, and rather extracting the relevant information straight from the data.

Instead of using a rolling window as previously, one can also use an expanding window, that is, one keeps all the previous observations from time 0 when updating the estimate of the copula density from the newly observed returns. This is, obviously, meaningful if one believes in the stationarity of the series over the whole period of observation. If one does so on the IBM series, one obtains the daily one-step ahead \( c \text{VaR} \) forecasts shown in Figure 5.6.

![Figure 5.6: IBM data: daily one-step-ahead \( c \text{VaR} \) forecasts at level 95% (yellow) and 99% (orange), expanding window.](image)

The series of \( c \text{VaR} \) are naturally smoother than in the ‘rolling’ window case (especially at the 99%-level), as one can understand. What is remarkable, though, is that the procedure seems able to guess each time there is going to be an extreme loss: each time the realised series shows a peak higher than, say, 0.04, the
estimated $c \text{VaR}$ at level 99\% shows a peak of similar amplitude as well. This seems very promising and motivates further study of the proposed procedure.

6 Concluding remarks and future work

This paper investigates a novel nonparametric conditional Value-at-Risk forecast procedure. Like previously suggested nonparametric methods, the VaR is here obtained by direct inversion of a nonparametric estimator of the conditional cumulative distribution of interest. What differs from them is that estimation of that conditional distribution is based on the density of the copula describing the dynamic dependence observed in the series of returns. This has several advantages, as expounded in the paper. In particular, the so-produced estimated conditional distribution function is always constrained to between 0 and 1 and increasing, which makes it easy to invert for extracting the desired quantiles. The copula framework leads to intuitive interpretation of the results, see application on the S&P500 index in Section 5.1. In addition, the analysis of the value of the IBM Corporation stock from January 2011 to November 2017 (Section 5.2) has revealed that the suggested procedure performs as well as the best parametric model one can use on those data for one-step ahead Value-at-Risk forecasting.

It is important to note that VaR has recently been the object of discussion and criticism, mainly because it is not sub-additive which makes it a non-coherent risk measure. Hence other risk measures have been proposed as well, making the choice of the right risk measure a problem of theoretical interest of its own (Cherubini et alii [2012]). In any case, the Basel III accords recommend complemented the VaR by the Expected Shortfall (ES), owing to the coherence of the latter. Hence estimation methods for the conditional Expected Shortfall have been suggested and investigated (Scaillet 2005, Cai and Wang 2008, Chen 2008, Kato 2012, Linton and Xiao 2013, Xu 2016); see Nadarajah et alii (2014) for a review. For some level $\alpha$, the (conditional) ES is defined as

$$c\text{ES}_{\alpha,T}(x) = \mathbb{E}(X_T|X_T > c \text{VaR}_{\alpha,T}(x), X_{T-1} = x),$$

i.e., the expected loss given that the loss exceeds the corresponding (conditional) Value-at-Risk, that is,

$$c\text{ES}_{\alpha,T}(x) = \frac{1}{1 - \alpha} \int_{c\text{VaR}_{\alpha,T}(x)}^{\infty} y f_{X_T|X_{T-1}}(y|x) \, dy.$$ 

Define $W_T = X_T - c \text{VaR}_{\alpha,T}$, and see that

$$c\text{ES}_{\alpha,T}(x) = c \text{VaR}_{T,\alpha}(x) + \frac{1}{1 - \alpha} \int_{0}^{\infty} w f_{W_T|X_{T-1}}(w|x) \, dw.$$ 

18
From (4.2), one has

\[ f_{W|X_{T-1}}(w|x) = f_W(w) \times c_1^{(W)}(F_X(x), F_W(w)), \]

where \( c_1^{(W)} \) is the copula density of the vector \((X_{T-1}, W_T)\). Now, because \( W_T \) is just \( X_T \) plus a constant, the dependence within \((X_{T-1}, W_T)\) is the exact same as within \((X_{T-1}, X_T)\), and their copula is the same. Hence, the conditional Expected Shortfall can be written

\[ c_{ES\alpha,T}(x) = c_{VaR\alpha,T}(x) + \frac{1}{1 - \alpha} \int_0^\infty w c_1(F_X(x), F_X(w + c_{VaR\alpha,T}(x))) f_W(w) \, dw. \]

This, in turn, suggests a nonparametric estimator of type

\[ \hat{c}_{ES\alpha,T}(x) = \hat{c}_{VaR\alpha,T}(x) + \frac{1}{1 - \alpha} \frac{1}{T} \sum_{t=0}^{T-1} (X_t - \hat{c}_{VaR\alpha,T}(x)) \hat{c}_1(\hat{F}_X(x), \hat{F}_X(X_t)) \mathbb{I}_{\{X_t > \hat{c}_{VaR\alpha,T}(x)\}}, \]

with the same estimators as in (4.4). This estimator will be studied in detail in a forthcoming paper.

**Acknowledgements**

This research was supported by a Faculty Research Grant from the Faculty of Science, UNSW Sydney (Australia).

**References**

Ardia, D., Boudt, K. and Catania, L. (2016), Value-at-Risk prediction in R with the GAS Package, *Manuscript*, [http://arxiv.org/pdf/1611.06010](http://arxiv.org/pdf/1611.06010).

Azzalini, A. (1981), A note on the estimation of a distribution function and quantiles by a kernel method, *Biometrika*, 68, 326-328.

Bashtannyk, D.M. and Hyndman, R.J. (2001), Bandwidth selection for kernel conditional density estimation, *Comput. Statist. Data Anal.*, 36, 279-298.

Barone-Adesi, G., Giannopoulos, K. and Vosper, L. (2002), Backtesting Derivative Portfolios With Filtered Historical Simulation (FHS), *Europ. Finan. Manage.*, 8, 31-58.

Bender, J., Briand, R., Melas, D. and Subramanian, R. (2013), Foundations of Factor Investing, MSCI Research Insight.

Bollerslev, T. (1986), Generalized autoregressive conditional heteroskedasticity, *J. Econometrics*, 31, 307-327.
Carhart, M.M. (1997), On persistence in mutual fund performance, J. Finance, 52, 57-82.

Cai, Z. (2002), Regression quantiles for time series, Econometric theory, 18, 169-192.

Cai, Z. and Wang, X. (2008), Nonparametric estimation of conditional VaR and expected shortfall, J. Econometrics, 147, 1130.

Campbell, S.D. (2007), A review of backtesting and backtesting procedures, J. Risk, 9, 1-17.

Chavez-Dumoulin, V., Embrechts, P. and Sardy, S. (2014), Extreme-quantile tracking for financial time series, J. Econometrics, 181, 44-52.

Chen, S.X. and Tang, C.Y. (2007), Nonparametric inference of value-at-risk for dependent financial returns, J. Financial Econometrics, 3, 227-255.

Chen, S.X. (2008), Nonparametric estimation of expected shortfall, J. Financ. Econom., 6, 87-107.

Chernozhukov, V. and Umanstev, L. (2001), Conditional value-at-risk: aspect of modeling and estimation, Empir. Econom., 26, 271-292.

Cherubini, U., Luciano, E. and Vecchiato, W., Copula methods in finance, Wiley, New York, 2004.

Cherubini, U., Gobbi, F. Mulinacci, S. and Romagnoli S., Dynamic Copula Methods in Finance, Wiley, New York, 2012.

Christoffersen, P. (1998), Evaluating interval forecasting, Int. Econ. Rev., 39, 841-862.

Creal, D., Koopman, S.J. and Lucas, A. (2013), Generalized Autoregressive Score models with applications, J. Appl. Econometrics, 28, 777-795.

De Backer, M., El Ghouch, A. and Van Keilegom, I. (2017), Semiparametric copula quantile regression for complete or censored data, Electron. J. Statist., 11, 1660-1698.

Ding, Z., Granger, C. and Engle, R. (1993), A long memory property of stock market returns and a new model, J. Empirical Finance, 1, 83-106.

Doukhan, P., Mixing: properties and examples, Lecture Notes in Statistics, Vol. 85, Springer, Berlin, 1994.

Dowd, K. (2001), Estimating VaR with order Statistics, J. Derivatives, 8, 23-20.

Duffie, D. and Pan, J. (1997), An overview of value at risk, J. Derivatives, 4, 7-49.
Dupuis, D.J., Papageorgiou, N. and Rémillard, B. (2015), Robust conditional variance and value-at-risk, J. Financial Econometrics, 13, 896-921.

Embrechts, P., Kluppelberg, C. and Mikosh, T., Modelling Extremal Events for Insurance and Finance, Springer, Berlin, 1997.

Embrechts, P., Resnick, S. and Samorodnitsky, G. (1999), Extreme value theory as a risk management tool, N. Amer. Actuarial J., 26, 30-41.

Embrechts, P., McNeil, A. and Straumann, D. (2002), Correlation and dependence in risk management: properties and pitfalls, Risk management: value at risk and beyond, 176-223.

Embrechts, P. (2009), Copulas: a personal view, J. Risk Ins., 76, 639-650.

Engle, R.F. and Bollerslev, T. (1986), Modelling the persistence of conditional variances, Econometric Rev., 5, 1-50.

Engle, R.F. and Manganelli, S. (2004). CAViaR: Conditional Autoregressive Value at Risk by Regression Quantiles, J. Bus. Econ. Statist., 22, 367-381.

Escanciano, J.C. and Olmo, J. (2010), Backtesting parametric value-at-risk with estimation risk, J. Bus. Econ. Statist., 28, 36-51.

Falk, M. (1985), Asymptotic normality of the kernel quantile estimator, Ann. Statist., 13, 428-433.

Fama, E. (1965), The behavior of stock market prices, J. Bus., 38, 34-105.

Fan, J. and Gijbels, I., Local Polynomial Modelling and Its Applications, Chapman and Hall/CRC, 1996

Fan, J. and Gu, J. (2003), Semiparametric estimation of Value at Risk, Econometrics J., 6, 261-290.

Fan, J. and Yim, T.H. (2004), A crossvalidation method for estimating conditional densities, Biometrika, 91, 819-834

Fan, Y., Härdle, W.K., Wang, W. and Zhu, L. (2017), Single index based CoVar with very high dimensional covariates, J. Bus. Econ. Statist., In press.

Faugeras, O. (2009), A quantile-copula approach to conditional density estimation, J. Multivariate Anal., 100, 2083-2099.

Ferraty, F. and Quintela-Del-Rio, A. (2016), Conditional VaR and Expected Shortfall: a new functional approach, Econometric Rev., 35, 263-292.
Francq, C. and Zakoian, J.M., GARCH models: structure, statistical inference and financial applications, John Wiley and Sons, 2011.

Franke, J., Mwita, P. and Wang, W. (2015), Nonparametric Estimates For Conditional Quantiles Of Time Series, AStA Adv. Stat. Anal., 99, 107-130.

Geenens, G. (2011), Curse of dimensionality and related issues in nonparametric functional regression, Statist. Surv., 5, 30-43.

Geenens, G. (2014), Probit transformation for kernel density estimation on the unit interval, J. Amer. Statist. Assoc., 109, 346-358.

Geenens, G., Charpentier, A. and Paindaveine, D. (2017), Probit transformation for nonparametric kernel estimation of the copula density, Bernoulli, 23, 1848-1873.

Gerlach, R., Lu, Z. and Huang, H. (2013), Exponentially smoothing the skewed Laplace distribution for Value-at-Risk forecasting, J. Forecast., 32, 534-550.

Ghalanos, A. (2017), Introduction to the rugarch package (version 1.3-1), Manuscript, http://cran.r-project.org/web/packages/rugarch

Glosten, L.R., Jagannathan, R. and Runkle, D.E. (1993), On the relation between the expected value and the volatility of the nominal excess return on stocks, J. Finance, 48, 1779-1801.

González-Rivera, G., Lee, T.H. and Mishra, S. (2004), Forecasting volatility: a reality check based on option pricing, utility function, value-at-risk and predictive likelihood, Int. J. Forecast., 20, 629-645.

Guermat, C. and Harris, R.D.D. (2001), Robust conditional variance estimation and Value-at-Risk, J. Risk, 4, 25-41.

Hall, P., Wolff, R.C.L. and Yao, Q. (1999), Methods for estimating a conditional distribution function, J. Amer. Statist. Assoc., 94, 154-163.

Hall, P., Racine, J. and Li, Q. (2004), Cross-validation and the estimation of conditional probability densities, J. Amer. Statist. Assoc., 99, 1015-1026.

Härdle, W., Müller, M., Sperlich, S. and Werwatz, A., Nonparametric and semiparametric models, Springer-Verlag, Berlin, 2004. 2004.

Harvey, A.C, Dynamic models for volatility and heavy-tails, with applications to financial and economic time series, Cambridge University Press, 2013.
Hayfield, T. and Racine, J.S. (2008), Nonparametric econometrics: the np package, J. Stat. Softw., 27, 1-32.

Hendricks, D. (1996), Evaluation of value-at-risk models using historical data, Federal Reserve Bank of New York Economic Policy Review, 39-69.

Hull, J. ans White, A. (1998), Incorporating volatility updating into the historical simulation method for Value-at-Risk, J. Risk, 1, 5-19.

Hyndman, R.J., Bashtannyk, D.M. and Grunwald, G.K. (1996), Estimating and visualizing conditional densities, J. Comput. Graph. Statist., 5, 315-336.

Joe, H., Multivariate Models and Dependence Concepts, Chapman & Hall, London, 1997.

Jorion, P., Value at Risk, 2nd Edition, McGraw-Hill, New York, 2001.

Kato, K. (2012), Weighted Nadaraya-Watson estimation of conditional expected shortfall, J. Financial Econometrics, 10, 265-291.

Koenker, R. and Basset, G. (1978), Regression quantiles, Econometrica, 46, 33-50.

Kuester, K., Mittnik, S. and Paolella, M.S. (2006), Value-at-Risk predictions: a comparison of alternative strategies, J. Financial Econometrics, 4, 53-89.

Kupiec, P. (1995), Techniques for Verifying the Accuracy of Risk Management Models, J. Derivatives, 3, 73-84.

Li, Q. and Racine, J.S. (2008), Nonparametric estimation of conditional CDF and quantile functions with mixed categorical and continuous data, J. Bus. Econ. Stat., 26, 423-434.

Linsmeier, T.J. and Pearson, N.D. (2000), Value At Risk, Financial Analysts Journal, 56, 47-67.

Linton, O.B. and Xiao, Z. (2013), Estimation of and inference about the expected shortfall for time series with infinite variance, Econometric Theory, 29, 771-807.

Loader, C.R. (1996), Local likelihood density estimation, Ann. Statist., 24, 1602-1618.

Martins-Filho, C. and Yao, F. (2006), Estimation of value-at-risk and expected shortfall based on nonlinear models of returns dynamics and extreme value theory, Stud. Nonlinear Dynam. Econometrics, 10, 1-43.

Martins-Filho, C., Yao, F. and Torero, M. (2016), Nonparametric estimation of conditional value-at-risk and expected shortfall based on extreme value theory, Econometric Theory, 1-45.
McNeil, A. (1998), Calculating quantile risk measures for financial time series using extreme value theory, Swiss Federal Technical University E-collection.

McNeil, A.J. (1999), Extreme Value Theory for Risk Managers, Internal Modelling and CAD II, Risk books, 93-113.

McNeil, A.J., Frey, R. and Embrechts, P., Quantitative Risk Management, Princeton University Press, 2005.

JP Morgan, Risk Metrics - Technical document, 4th Edition, New York, 1996.

Nadarajah, S., Zhang, B. and Chan, S. (2014), Estimation methods for expected shortfall, Quant. Finance, 14, 271-291.

Nadaraya, E.A. (1964). On estimating regression, Theory Probab. Applic., 9, 141-142.

Nagler, T. and Czado, C. (2016), Evading the curse of dimensionality in nonparametric density estimation with simplified vine copulas, J. Multivariate Anal., 151, 69-89.

Nelsen, R.B., An Introduction to Copulas, Springer Verlag, 2006.

Nelson, D. (1991), Conditional heteroskedasticity in asset returns: a new approach, Econometrica, 59, 347-370.

Nieto, M.R. and Ruiz, E. (2016), Frontiers in VaR forecasting and backtesting, Int. J. Forecast., 32, 475-501.

Rockafellar, R.T. and Uryasev, S. (2002), Conditional value-at-risk for general loss distributions, J. Banking Finance, 26, 1443-1471.

Rosenblatt, M. (1968), Conditional probability density and regression estimators, In: Multivariate Analysis II, Academic Press, New York, 25-31.

Salmon, F. (2009), Recipe for disaster: the formula that killed Wall Street, Wired Magazine, February 23 2009.

Scaillet, O. (2003), The origin and development of VaR. In: Modern Risk Management: A History, 15th Anniversary of Risk Magazine. Risk Publications, London, 115-129.

Scaillet, O. (2005), Nonparametric estimation of conditional expected shortfall, Insurance and Risk Management Journal, 74, 639-660.
Segal, S., Corporate Value of Enterprise Risk Management: The Next Step in Business Management, Wiley, 2011.

Sklar, A. (1959), Fonctions de répartition à n dimensions et leurs marges, Publications de l’Institut de Statistique de l’Université de Paris, 8, 299-331.

So, M.K.P. and Yu, P.L.H. (2006), Empirical analysis of GARCH models in Value at Risk estimation, J. Int. Finan. Markets, Inst. Money, 16, 180-197.

Taylor, J.W. (2008), Using exponentially weighted quantile regression to estimate Value-at-Risk and Expected Shortfall, J. Financial Econometrics, 6, 382-406.

Wang, C.S. and Zhao, Z. (2016), Conditional Value-at-Risk: semiparametric estimation and inference, J. Econometrics, 195, 86-103.

Watson, G.S. (1964). Smooth regression analysis, Sankhya A, 26, 359-372.

Wu, W.B., Yu, K. and Mitra, G. (2008), Kernel conditional quantile estimation for stationary processes with application to conditional Value-at-Risk, J. Financial Econometrics, 6, 253-270.

Xu, K.L. (2013), Nonparametric inference for conditional quantiles of time series, Econometric Theory, 29, 673-698.

Xu, K.L. (2016), Model-Free Inference for Tail Risk Measures, Econometric Theory, 32, 122-153.

Yang, S.S. (1985), A Smooth nonparametric estimator of a quantile function, J. Amer. Stat. Assoc., 80, 1004-1011.

Yu, K. and Jones, M.C. (1998), Local linear quantile regression, J. Amer. Stat. Assoc., 93, 228-237.

Zikovic, S. and Aktan, B. (2011), Decay factor optimization in time weighted simulation - evaluating VaR performance, Int. J. Forecast., 27, 1147-1159.