Model-independent resonance parameter extraction using the trace of K and T matrices

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Abstract

A model-independent method for the determination of Breit-Wigner resonance parameters is presented. The method is based on eliminating the dependence on the choice of channel basis by analyzing the trace of the $K$ and $T$ matrices in the coupled-channel formalism, rather than individual matrix elements of the multichannel scattering matrix.

Key words: multichannel scattering, baryon resonances, pi and eta mesons
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1. Introduction

A general problem in theoretical baryon physics is to make a connection between resonances that are predicted by various quark models and experiment. A reasonable way to proceed is by identifying the poles of analytic functions that are able to describe simultaneously all experimental data in a multiplicity of existing channels with theoretically predicted resonant states. Therefore, properly and uniquely extracting resonance parameters from experiment is a task of primary importance. We emphasize the problem of uniqueness. The work described here is motivated by the need to extract Breit-Wigner resonance parameters from multichannel partial-wave analyses (PWAs) in a model-independent way. Many PWAs of similar experimental data produce similar partial waves, while the extracted Breit-Wigner resonance parameters are often quite different. This fact can easily be seen in the Review of Particle Physics \cite{1} by the Particle Data Group (PDG). Each resonance in the Review has been parametrized in two ways. First, there are Breit-Wigner parameters, i.e. the resonance mass $M^R$, decay width $\Gamma^R$, and branching ratios $x_a$ (the ratio between the partial width into channel $a$ and the total width). Alternatively, $T$-matrix complex poles ($\text{Re} W_p$ and $-2\text{Im} W_p$), as well as complex residues (moduli $|r|$, and phases $\theta$) are given in \cite{1}. Breit-Wigner parameters obtained in various partial-wave analyses vary quite substantially, partly because details of those analyses are different; i.e. the number and character of the included channels, different parameterization schemes, analyticity constraints for scattering amplitudes, the choice of background models, and the method of unitarization (if at all) of the $S$ matrix. However, it is also the case that the methods for extracting resonance parameters are different: Argand-plot fits \cite{2}, Breit-Wigner fits with background \cite{3}, direct fits of analysis parameters \cite{4}, or model specific schemes which extract $T$ matrix poles \cite{5,6,7}.

In this Letter we present a model-independent...
method for extracting Breit-Wigner parameters from any unitary multichannel analysis able to provide the full $T$ matrix, using the trace of the corresponding $K$ matrix. Since all Breit-Wigner parameterizations are equivalent at the energy of the $K$-matrix pole, the parameters obtained using this method should be directly compared to those from quark models and lattice QCD. In order to connect the results of a model-independent $K$ matrix extraction with those of a model-dependent analysis, e.g. based on the matrix extraction with those of a model-dependent

All equations given here are considered to be matrix relations, unless matrix indices are explicitly stated. The transition probability $P_{a \rightarrow b}$ of a two-body system from initial channel $|a; q \rangle$ to final two-body (or quasi-two-body) channel $|k; q \rangle$ is given by the absolute square of the scattering $S^q$-matrix element $P_{a \rightarrow b} = |\langle k; q | S^q | a; q \rangle|^2$, where $q$ designates all quantum numbers conserved in the scattering reaction, and $a$ and $b$ are channels. In the case of $\pi N$ scattering we have conserved spin, parity, and almost conserved isospin (charge symmetry is only slightly violated). Conservation of probability is ensured if the $S$ matrix (for simplicity, we drop $q$ henceforth) is unitary. Therefore, the $S$ matrix can be written as $S = e^{2i\delta}$, where $\delta$ is some matrix Hermitian in the channel indices. Hermitian matrices have real eigenvalues and are diagonalized by unitary matrices. The $\delta$ matrix is related to a real, diagonal matrix $\delta_D$ by a unitary transformation $\delta = U^\dagger \delta_D U$, where $U$ is a unitary matrix. The $S$ matrix is evidently diagonalized by the same transformation, so $S = U^\dagger e^{2i\delta_D} U$.

The $K$ matrix \ref{eq:K} is defined as $K = i(I - S)/(I + S)$, where $I$ is the unit matrix. The $K$ matrix can, in the eigenstate basis, be written using the diagonal matrix $\delta_D$ as $K = U^\dagger \tan \delta_D U$. The $K$ matrix is Hermitian because $S$ is unitary, and symmetric because of time-reversal invariance, so $K$ is, in fact, a real matrix. Thus, $U$ is a real orthogonal matrix that we henceforth designate as $O$.

Every diagonal $N \times N$ matrix can be spanned in the orthonormal vector basis $\{E^1, \ldots, E^N\}$:

$$E^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & \vdots \\ 0 & \ddots & 0 \end{pmatrix}, \quad \ldots, \quad E^N = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix},$$

so, in our case, we have

$$\tan \delta_D = \sum_{i=1}^N E^i \tan \delta^i,$$  \hspace{1cm} (2)

where $\delta^i$ is the $i$th diagonal element of $\delta_D$, also known as the eigenphase shift, and $N$ is the number of channels. We define the coupling matrices $\chi^i$ to be

$$\chi^i = O^T E^i O,$$

and these matrices turn out to be orthonormal projectors:

$$\sum_{i=1}^N \chi^i = I, \quad \chi^i \chi^j = \chi^j \delta_{ij},$$

where $\delta_{ij}$ is the Kronecker $\delta$ symbol.

The trace of a matrix is, by definition, a sum of its diagonal elements. A trace has two particularly important properties: i) the trace of a product of matrices is invariant with respect to cyclic permutations, $\text{Tr}[ABC] = \text{Tr}[BCA]$; and ii) the trace is a distributive function with respect to scalars $\alpha$ and $\beta$, $\text{Tr}[\alpha A + \beta B] = \alpha \text{Tr}[A] + \beta \text{Tr}[B]$.

The orthogonal transformation in definition \ref{eq:K} conserves the trace of a matrix, so

$$\text{Tr} \chi^i = 1.$$  \hspace{1cm} (5)

It follows that the matrices $K$ and $T$ can be written as the sums

$$K = \sum_{i=1}^N \chi^i \delta_D \chi^i,$$

$$T = \sum_{i=1}^N \chi^i \delta_D \chi^i.$$
is given by the relation
\[ K = \sum_{j=1}^{N} \chi^j \tan \delta^j, \quad T = \sum_{j=1}^{N} \chi^j e^{i\delta^j} \sin \delta^j, \] (6)
where the connection between the $K$ and $T$ matrices is given by the relation
\[ K = T/(I + iT). \] (7)

3. Breit-Wigner parameterization

Elements of $\tan \delta_B$, as well as the $\chi^j$, are functions of energy or a corresponding kinematical variable, and their description requires modeling of the energy dependence of numerous functions. We see resonances in scattering reactions as real poles of the energy dependence of numerous functions. We are able, and their description requires modeling of the occupations of energy or a corresponding kinematical variable.

\[ r \] represents $\Gamma_B$ the mass of the resonance, $\Gamma'_r$ and $\Gamma_B = \Gamma_r(W)$ evaluated at an energy equal to the $r$th element of the diagonal matrix $\tan \delta_B$ can be written as [9]
\[ \tan \delta^r = \frac{\Gamma_r/2}{M_r - W} + \tan \delta_B^r, \] (8)
where the selected pole term is parametrized in Breit-Wigner form, and it is singled out from other contributions, designated collectively as the background term at resonance $\tan \delta_B$. The Breit-Wigner mass ($M_r$) and total width ($\Gamma_r$) parameters are allowed to be functions of the center-of-mass total energy $W$. The reported Breit-Wigner parameters $M_r$ and $\Gamma_r$ are given by the values of $M_r(W)$ and $\Gamma_r(W)$ evaluated at an energy equal to the corresponding resonance mass $M_r^R$:
\[ M_r^R = M_r(M_r^R), \quad \Gamma_r^R = \Gamma_r(M_r^R), \] (9)
where we have explicitly written $M_r$ and $\Gamma_r$ from Eq. (9) as functions of energy $W$.

The corresponding $K$ and $T$ matrices are given by the equations
\[ K = \chi^r \frac{\Gamma_r^r/2}{M_r - W} + \sum_{j \neq r}^N \chi^j \tan \delta^j, \] (10)
\[ T = \chi^r \frac{\Gamma_r^r/2}{M_r - W - i\Gamma_r^r/2} + \sum_{j \neq r}^N \chi^j e^{i\delta^j} \sin \delta^j, \] (11)
where the second term in each equation is the coupled-channel background contribution, and $\Gamma_r^r/2$ represents $\Gamma_r/2 + (M_r - W)$ tan $\delta_B$. When $W$ equals the mass of the resonance, $\Gamma_r^r$ is manifestly equal to $\Gamma_r$. Although these relations are in general a sum over several resonances $r$, here they are written for one resonance for simplicity.

If there is a pole in the $K$ matrix at some energy $M_r^R$, then the matrix element $\chi^r_{ab}$ at that energy gives the coupling strength of the resonance with mass $M_r^R$ and total decay width $\Gamma_r^R$ from channel $a$ to channel $b$. The diagonal element of the matrix $\chi^r$ is the branching ratio $x^r_a$ of a given resonance to the channel $a$
\[ x^r_a = \chi^r_{aa}. \] (12)

4. Extraction procedure

The channel dependence of resonance parameters can be reduced significantly by using only diagonal elements of the $K$ and $T$ matrices. In practice, these matrices can be obtained either by unitary coupled-channel partial-wave analyses, or by using partial-wave $T$ matrices obtained in diverse single-channel PWAs as input to a unitary coupled-channel formalism, and refitting them to obtain a unitary set of all coupled-channel $T$ matrix elements.

Channel dependence is completely removed from the sums
\[ \text{Tr}(K) = \sum_{j=1}^{N} \tan \delta^j, \quad \text{Tr}(T) = \sum_{j=1}^{N} e^{i\delta^j} \sin \delta^j, \] (13)

because the traces of the $K$ and $T$ matrices are the same as the traces of their similar diagonal partners $\tan \delta_B$ and $e^{i\delta_B} \sin \delta_B$, respectively. The same is also evident from Eq. (3). Consequently, Eqs. (10) and (11) are simplified by taking the traces
\[ \text{Tr}(K) = \frac{\Gamma_r^r/2}{M_r - W} + \sum_{j \neq r}^N \tan \delta^j, \] (14)
\[ \text{Tr}(T) = \frac{\Gamma_r^r/2}{M_r - W - i\Gamma_r^r/2} + \sum_{j \neq r}^N \sin \delta^j. \] (15)

The last relation, i.e. the $T$-matrix trace, would be a good starting point for model-dependent extraction methods. However, instead of putting considerable effort into modeling the background and energy- and channel-dependent resonance parameters, we use the following procedure:

(i) The parameter extraction procedure starts when a full $T$ matrix has been obtained from an energy-dependent partial-wave analysis of experimental data.

(ii) Contrary to the usual prescription, where Eq. (11) is used to obtain resonance parameters from the $T$ matrix in a model-dependent way, we use Eq. (14) to obtain the full $K$ matrix from the known $T$ matrix.
(iii) Poles of \( \text{Tr} \, K \) are found to obtain a set of resonance masses \( M^R_1, \ldots, M^R_{N_R} \), where \( N_R \) is the number of resonances.

(iv) Multiplying both sides of Eq. (14) by \((M^R_k - W)\) and setting the energy \( W \) to the value of the \( k \)th resonance mass \((M^R_k)\), the corresponding resonance width is isolated:

\[
\Gamma^R_k = 2 \lim_{W \to M^R_k} \left( M^R_k - W \right) \text{Tr}(K). \tag{16}
\]

All other contributions to the \( K \) matrix trace, i.e. background, other resonances, and channel-couplings, are removed in this limiting process (this relation turns out to be similar to Eq. (16) in Ref. [10] for the case of the various \( \pi N \) isospin channels).

(v) The branching ratio of a resonance to a given channel can be obtained in similar manner, but this time using the diagonal \( K \)-matrix element, \( K_{aa} \) from Eq. (10) and definition (12)

\[
x^k_a = 2 \lim_{W \to M^R_k} \left( M^R_k - W \right) K_{aa}, \tag{17}
\]

where, as before, all undesired contributions vanish.

(vi) Steps (iv) and (v) are then repeated for all resonances found in (iii).

### 5. Results and discussions

To illustrate the usefulness of our method, resonance parameters from a unitary, multi-resonance, coupled-channel analysis [7] have been extracted. The channels used in the analysis were \( \pi N, \eta N \), and an effective two-body channel designated as \( \pi^2 N \). Extracted parameters are given in Table 1. The proposed model gives resonance parameters very close to the values obtained by a complicated method of diagonalizing the matrix of the generalized Breit-Wigner function denominator, with minimal calculation.

We have also compared the \( K \)-matrix trace to that of the \( T \) matrix. It can be seen in Fig. 4 that the Breit-Wigner resonance positions obtained by looking for the poles in \( \text{Tr} \, K \) (indicated by gray vertical lines) directly correspond to the positions of peaks in \( \text{Im}(\text{Tr} \, T) \), and of zeros in \( \text{Re}(\text{Tr} \, T) \). The peaks of the \( T \)-matrix elements corresponding to individual channels, however, show a certain deviation from that behavior. This suggests that fitting individual channels in order to obtain resonance parameters introduces an uncontrolled error, which is avoided if the trace of the \( T \) matrix is used.

Unexpectedly, and contrary to previous findings, the resonance parameters produced by the \( K \)-matrix extraction method presented here, are in accordance with values obtained by the original analysis as well as with the \( T \)-matrix trace. The procedure involves no fitting, diagonalizing, nor modeling of the energy

### Table 1

| \( L_{212J}[^{X \pi N}/^{x \eta N}/^{x \pi^2 N}] \) | \( M^R \) | \( \Gamma^R \) | \( x_{\pi N} \) | \( x_{\eta N} \) | \( x_{\pi^2 N} \) |
| --- | --- | --- | --- | --- | --- |
| PDG [1] | [MeV] | [MeV] | [%] | [%] | [%] |
| \( S_{11} \) \( 35.55/30 \pm 10/10 \) | 1543 | 165 | 39 | 54 | 7 |
| \( S_{11} \) \( 1535 \pm 10/150 \pm 50 \) | 1553 | 182 | 46 | 50 | 4 |
| \( S_{11} \) \( 1550 \pm 10/10 \pm 20 \) | 1680 | 233 | 64 | 16 | 20 |
| \( S_{11} \) \( \text{UNKNOWN} \) \( \pm 2090/\text{NE} \) | 2054 | 1926 | 47 | 3 | 50 |
| \( P_{11} \) \( 60 \pm 10 \) \( 1440 \pm 10 \) \( 100 \pm 50 \) | 1482 | 541 | 61 | 0 | 39 |
| \( P_{11} \) \( 10 \) \( 1710 \pm 100 \) \( 150 \pm 50 \) | 1738 | 170 | 44 | 12 | 44 |
| \( P_{11} \) \( \text{UNKNOWN} \) \( \pm 2400/\text{NE} \) | 2123 | 379 | 3 | 83 | 14 |
| \( P_{11} \) \( \pm 10 \) \( 1720 \pm 10 \) \( 100 \pm 50 \) | 2157 | 355 | 16 | 83 | 1 |
| \( D_{13} \) \( 50 \) \( 1520 \) \( 10 \) \( 120 \pm 10 \) | 1515 | 121 | 56 | 0 | 44 |
| \( D_{13} \) \( \pm 50 \) \( 1700 \pm 50 \) \( 100 \pm 50 \) | 1818 | 126 | 15 | 15 | 70 |
| \( D_{13} \) \( \text{UNKNOWN} \) \( \pm 2000/\text{NE} \) | 2359 | 1216 | 26 | 6 | 68 |
| \( D_{15} \) \( 40 \) \( 1675 \) \( 10 \) \( 150 \) \( 15 \) \( 30 \) \( 10 \) | 1674 | 144 | 36 | 0 | 64 |
| \( F_{15} \) \( 60 \) \( 1680 \) \( 10 \) \( 130 \) \( 10 \) | 1682 | 144 | 67 | 1 | 32 |
| \( F_{17} \) \( \text{UNKNOWN} \) \( \pm 1900/\text{NE} \) | 2139 | 412 | 7 | 3 | 90 |
| \( G_{17} \) \( \text{UNKNOWN} \) \( \pm 2190 \) \( 150 \) \( 10 \) | 1806 | 286 | 6 | 0 | 94 |
| \( G_{17} \) \( \text{UNKNOWN} \) \( \pm 2190 \) | 2397 | 1217 | 16 | 0 | 84 |
| \( G_{17} \) \( \text{UNKNOWN} \) \( \pm 2190 \) | 2125 | 381 | 18 | 0 | 82 |

Extracted parameters are given in Table 1. The procedure involves no fitting, diagonalizing, nor modeling of the energy.
Fig. 1. The trace of the $T$ matrix and its contributions for partial waves from $S_{11}$ to $G_{17}$. The thick black line represents the imaginary part (upper graph) and real part (lower graph) of the trace of the $T$ matrix. The thin line is the $\pi N$ elastic contribution, the dashed line shows the contribution from $\eta N$, while the dotted line gives the effective-channel contribution (unitarity channel). Gray vertical lines are plotted at the trace of $K$-matrix pole positions.

dependence of the resonance parameters and background. Furthermore, a model-independent procedure cannot be given with the $T$-matrix formalism, because background makes a substantial contribution to the $T$ matrix, even at an energy equal to the resonance mass, $M^R$. The $T$-matrix background is removed at a complex energy equal to the $T$-matrix pole position. This might be the reason why extractions of $T$-matrix poles work much better than $T$-matrix extractions of Breit-Wigner parameters. By using the trace of the $K$ matrix, background has been completely removed from consideration at the resonance energies.

With regard to the differences between the two approaches listed in Table1, it is rather striking that all of them can be explained by arguments presented in the original analysis. Since an effective $\pi^2 N$ channel was introduced in [7] to parametrize the first inelasticity in each partial wave, the parameters of low-lying resonances should be much better determined than those of heavier ones (especially the third resonances in $S_{11}$ and $D_{13}$). A better quality of parameters is also expected for resonances that couple more strongly to the measured channels considered here. Therefore, $N(1720)\,P_{13}$ and the resonance(s) in $G_{17}$ have unrealistic parameters since they are completely driven by the effective channel, as can be clearly seen from Fig.1. These problems should be removed by the explicit inclusion of additional channels in the partial-wave analysis.

The parameters of the two lowest resonances in the $S_{11}$ and $D_{13}$ partial waves, as well as those of the $D_{15}$, second $P_{11}$, and $F_{15}$ resonances, are in rough accordance with quark-model expectations for their masses and partial widths [11], with the exception of the mass of the second $D_{13}$, which is predicted to be roughly degenerate with the second $S_{11}$ and the $D_{15}$ resonance. This disagreement could be explained by the large coupling of this state to the effective channel. The large width and the somewhat
larger mass of the first \( P_{11} \) (Roper) resonance extracted using the \( K \)-matrix procedure bring these parameters closer to those of the class of quark-model calculations based on one-gluon exchange potentials and pair creation for strong decays.

6. Conclusions

We have presented a model-independent method for resonance parameter extraction using the \( K \)-matrix formalism. It is shown that real poles of the \( K \) matrix are related to the resonant behavior of the trace of the \( T \) matrix. Our resonance parameter extraction procedure is simple and straightforward once the full \( T \) matrix is known. Unrealistic extracted parameters for some higher mass resonances point to the need to include additional channels in partial-wave analyses.

At the energies of the \( K \)-matrix poles, the influence of background and channel mixing is eliminated, so only parameter values obtained at this particular energy should be compared directly to the predictions of quark model and lattice QCD calculations.

This model-independent procedure cannot be extended to the \( T \)-matrix formalism because background makes a substantial contribution to the \( T \) matrix, even at the resonance energies \( M^R \). This might be the reason why methods that extract \( T \)-matrix poles work much better than those which extract Breit-Wigner parameters from the \( T \) matrix. By using the trace of the \( K \) matrix, the background has been completely removed from consideration at resonance energies.

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