Calculation of the Width of the $\tau \rightarrow K^- K^0 \nu_\tau$ Decay in the Extended Nambu–Jona-Lasinio Model with Estimation of the Contribution from the Final-State Interaction

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The partial width of the $\tau \rightarrow K^- K^0 \nu_\tau$ decay has been calculated within the extended Nambu–Jona-Lasinio model including the contact and vector channels. The contributions from the $\rho$ meson in the ground and first radially excited states have been taken into account in the vector channel. The results are in satisfactory agreement with experimental data. The inclusion of the final-state interaction of kaons results only in insignificant corrections within the accuracy of the model.

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1. INTRODUCTION

It was shown in recent works [1–3], where decays of the tau lepton into two pseudoscalar mesons were described, that the final-state interaction between produced particles plays an important role in the description of these processes within the Nambu–Jona-Lasinio model [4, 5], particularly when the produced particles include a pion. In the last case, the correction to the decay width can reach 30%. This correction decreases with an increase in the total mass of produced mesons. In particular, the correction in the case of the production of a kaon and an $\eta$ meson is about 15%. In the case of the production of two kaons, it is about 6%, which is negligibly small compared to the accuracy of our model, which is about 15%. Thus, there is a tendency toward a decrease in the contribution of the final-state interaction with an increase in the total mass of the produced mesons, as well as when excited intermediate meson states should be taken into account.

The decay considered in this work is still of interest for experimental collaborations and theoretical groups. For example, it was studied recently by the Belle and BaBar Collaborations [6, 7]. It was also analyzed in many theoretical works [8–11] with the resonance chiral perturbation theory, vector dominance model, algebra of angular momenta, etc.

2. LAGRANGIAN OF THE EXTENDED NAMBU–JONA-LASINIO MODEL

The fragment of the quark–meson Lagrangian of the extended Nambu–Jona-Lasinio model that contains the required vertices has the form [5, 12–14]

$$\Delta L_{\text{int}} = \bar{q} \frac{1}{2} \gamma^\mu \sum_{j=\pm} \lambda_j^0 \left( A_{\rho} \phi_\rho^0 + B_{\rho} \phi_\rho^j \right) + i \gamma^5 \sum_{j=\pm, 0} \lambda_j^K \left( A_{K} K^j + B_{K} K^{j'} \right) q,$$

where $q$ and $\bar{q}$ are the fields of the $u$, $d$, and $s$ quarks with the constituent masses $m_u = m_d = 280$ MeV and $m_s = 420$ MeV, and excited meson states are marked with the prime symbol. The factors $A$ and $B$ have the form

$$A_M = \frac{1}{\sin(2\theta_M^0)} \times \left[ g_M \sin(\theta_M + \theta_M^0) + g_M' k_M f_M(k_M^2) \sin(\theta_M - \theta_M^0) \right]$$

and

$$B_M = \frac{-1}{\sin(2\theta_M^0)} \times \left[ g_M \cos(\theta_M + \theta_M^0) + g_M' k_M f_M(k_M^2) \cos(\theta_M - \theta_M^0) \right].$$

Here, the subscript $M$ specifies the corresponding meson;

$$f(k_M^2) = \left( 1 + d k_M^2 \right) \Theta(\Lambda^2 - k_M^2)$$
is the form factor that describes the first radially excited meson states [14], where \( d \) is the slope parameter depending on the quark composition of the meson [14] and \( k \) is the relative momentum of quarks in the meson; \( \theta_M \) are the mixing angles appearing when diagonalizing the free Lagrangian that contains mesons in the ground and first radially excited states [14]:

\[
\begin{align*}
\theta_\rho &= 81.8^\circ, \quad \theta_\rho^0 = 61.5^\circ, \\
\theta_K &= 58.11^\circ, \quad \theta_K^0 = 55.52^\circ;
\end{align*}
\]

\( \lambda \) are linear combinations of the Gell–Mann matrices; and the coupling constants are given by the expression

\[
\begin{align*}
g_\rho &= \left( \frac{3}{2f_{20}} \right)^{1/2}, \\
g_\rho^0 &= \left( \frac{3}{2f_{20}} \right)^{1/2}, \\
g_K &= \left( \frac{Z_K}{4f_{11}} \right)^{1/2}, \\
g_K^0 &= \left( \frac{1}{4f_{11}} \right)^{1/2},
\end{align*}
\]

where \( Z_K \) is the additional renormalization constant appearing in the \( K - K^\prime \) transitions:

\[
Z_K = \left( 1 - \frac{3(m_u + m_d)^2}{2M_{K_{1,4}}^2} \right)^{-1/2},
\]

\[
M_{K_{1,4}} = \left( \frac{\sin^2 \alpha}{M_{K_{(1270)}}^2} + \frac{\cos^2 \alpha}{M_{K_{(1400)}}^2} \right)^{1/2}.
\]

Here, the splitting of the \( K_{1,4} \) state into two physical \( K_{(1270)} \) and \( K_{(1400)} \) mesons with the mixing angle \( \alpha = 57^\circ \) was taken into account [15]. We used the masses of mesons \( M_{K_{(1270)}} = (1253 \pm 7) \) MeV and \( M_{K_{(1400)}} = (1403 \pm 7) \) MeV [16].

The renormalization of the Lagrangian results in the appearance of the following integrals in quark loops:

\[
I_{nq_0}^{(\prime\prime)} = -i \frac{N_c}{(2\pi)^4} \int \frac{f^\prime(m(k)^2)}{(m_u^2 - k^2)^{n/2}(m_d^2 - k^2)^{n/2}} \Theta(\Lambda^2 - k^2) d^4k,
\]

where \( \Lambda = 1.03 \) GeV is the three-dimensional cutoff parameter in quark loops [14].

3. \( \tau \to K^- K^0 \nu \), PROCESS IN THE EXTENDED NAMBU–JONA–LASINIO MODEL

This process is described by the diagrams shown in Figs. 1 and 2. In the Nambu–Jona-Lasinio model, the amplitude corresponding to these diagrams has the form

\[
M_{\text{tree}} = -2 \sqrt{2} G_F V_{ud} T_{11}^{KK} \left[ T_{k}^{(c)} + \frac{C_\rho}{g_\rho} \right]
\]

\[
\times \frac{I_{11}^{KK}}{I_{11}^{KK}} \frac{T_{k}^{(p)}}{M_{\rho}^2 - q^2 - i\sqrt{q^2} \Gamma_\rho} \frac{q^2}{g_\rho}
\]

\[
\times \frac{I_{11}^{KK}}{I_{11}^{KK}} \frac{T_{k}^{(p')}}{M_{\rho'}^2 - q^2 - i\sqrt{q^2} \Gamma_{\rho'}} \frac{q^2}{g_{\rho'}}
\]

\[
\left[ L_{11}(p_{K^0} - p_{K^-}^\prime) \right],
\]

where \( G_F \) is the Fermi constant; \( V_{ud} \) is the element of the Cabibbo–Kobayashi–Maskawa matrix; \( L_{11} \) is the lepton current; \( M_{\rho} = (775.11 \pm 0.34) \) MeV and \( \Gamma_{\rho} = (149.1 \pm 0.8) \) MeV are the mass and width of the \( \rho \) (770) meson, respectively; \( M_{\rho'} = (1465 \pm 25) \) MeV and \( \Gamma_{\rho'} = (400 \pm 60) \) MeV are the mass and width of the \( \rho (1450) \) meson, respectively [16];

\[
T_{k}^{(c)} = 1 - \frac{\{ I_{11}^{KK}\}^2}{I_{11}^{KK}} \frac{M_{K_{1,4}}^2}{M_{K_{1,4}}^2},
\]

\[
T_{k}^{(p)} = 1 - \frac{I_{11}^{KK} I_{11}^{KK}}{I_{11}^{KK}} \frac{M_{K_{1,4}}^2}{M_{K_{1,4}}^2},
\]

\[
T_{k}^{(p')} = 1 - \frac{I_{11}^{KK} I_{11}^{KK}}{I_{11}^{KK}} \frac{M_{K_{1,4}}^2}{M_{K_{1,4}}^2}.
\]
are constants describing the $K_1^0 - K$ transitions;
\[
I_{m_0 \ldots m_n}^{M_1 \ldots M_r} = \frac{i N_\pi}{(2\pi)^4} \times \int \frac{A_M \ldots B_M \ldots}{(m^2_\pi - k^2)^6(m^2_\pi - k^2)^6} \Theta(\Lambda^2 - k^2) d^4k,
\]
where $A_M$ and $B_M$ are the constants given by Eqs. (2), are integrals with the Lagrangian vertices in the numerator, and
\[
C_\rho = \frac{1}{\sin^2(2\theta^\rho)} \sin(\theta^\rho + \theta^\rho) + R_{\rho} \sin(\theta^\rho - \theta^\rho),
\]
\[
C_{\rho} = -\frac{1}{\sin(2\theta^\rho)} \cos(\theta^\rho + \theta^\rho) + R_{\rho} \cos(\theta^\rho - \theta^\rho)
\]
are the constants appearing in transitions between the $W$ boson and the intermediate vector meson. Here, $\theta$ and $\theta^\rho$ are the mixing angles of the ground and excited states defined in Eq. (4) and
\[
R_{\rho} = \frac{I_{\rho}^{(\omega)}}{\sqrt{I_{20}^{(\omega)} I_{20}^{(\omega)}}}.
\]

As a result, the branching ratio of this process is obtained:
\[
\text{Br}(\tau \to K^0 \nu) = 13.95 \times 10^{-4}.
\]
The experimental value is [16]
\[
\text{Br}(\tau \to K^0 \nu)_{\text{exp}} = (14.86 \pm 0.34) \times 10^{-4}.
\]

### 4. INCLUSION OF THE FINAL-STATE INTERACTION

To take into account the final-state interaction, it is necessary to take into account the triangular meson diagrams with exchange by the $\rho$, $\omega$, and $\phi$ neutral mesons shown in Fig. 3. The corresponding vertices can be found in [4]. They lead to the integrals for meson triangles:

\[
F^{(\omega)}_{\mu} = \int \left[ \frac{(k - 2p^\omega)}{k^2 - M^2_\omega} \right] \frac{(2k + p^\omega - p^\omega)}{(k - p^\omega)^2 - M^2_\omega} \left( g^{\nu\lambda} - \frac{\epsilon^{\nu\lambda\omega}}{M_\omega} \right) d^4k
\]
\[
F^{(\omega)}_{\mu} = \int \left[ \frac{(k - 2p^\omega)}{k^2 - M^2_\omega} \right] \frac{(2k + p^\omega - p^\omega)}{(k - p^\omega)^2 - M^2_\omega} \left( g^{\nu\lambda} - \frac{\epsilon^{\nu\lambda\omega}}{M_\omega} \right) d^4k
\]
\[
F^{(\omega)}_{\mu} = \int \left[ \frac{(k - 2p^\omega)}{k^2 - M^2_\omega} \right] \frac{(2k + p^\omega - p^\omega)}{(k - p^\omega)^2 - M^2_\omega} \left( g^{\nu\lambda} - \frac{\epsilon^{\nu\lambda\omega}}{M_\omega} \right) d^4k
\]

As seen, these integrals coincide in structure with the integral obtained in [1] and are
\[
F^{(\omega)}_{\mu} = i \left[ \frac{I_{1(\omega)}^{(\omega)}}{M_\rho} + I_{2M}^{(\omega)} \right] \left[ p^\omega - p^\omega \right]_{\mu
\nu},
\]
\[
F^{(\omega)}_{\mu} = i \left[ \frac{I_{1M}^{(\omega)}}{M_\omega} + I_{2M}^{(\omega)} \right] \left[ p^\omega - p^\omega \right]_{\mu
\nu},
\]
\[
F^{(\omega)}_{\mu} = i \left[ \frac{I_{1(\omega)}^{(\omega)}}{M_\rho} + I_{2M}^{(\omega)} \right] \left[ p^\omega - p^\omega \right]_{\mu
\nu}.
\]

As a result, the total amplitude including the final-state interaction has the form
\[
M_{\text{tot}} = -2\sqrt{2} G_f V_{ud} J^{KK}_{11}
\]
The expression in the braces contains contributions from meson triangles. The contributions from triangles with exchange by ρ and ω mesons approximately cancel each other and the main contribution comes from the triangle with exchange by the φ meson. The inclusion of the final-state interaction leads to the appearance of a new parameter, the meson-loop cutoff $\Lambda_M$.

The partial width of the $\tau \to K K \nu_\tau$ decay calculated in the preceding section within the extended Nambu–Jona-Lasinio model is in satisfactory agreement with experimental data within errors allowed in our model. Taking into account the final-state interaction of mesons and accepting $\Lambda_M = 610$ MeV, one can reach complete agreement with experimental data. We note that this cutoff is close to a value of 740 MeV that was used to describe the $\tau \to K \pi \nu_\tau$ decay and was obtained from the related $e^+e^- \to \pi^+\pi^- \nu\bar{\nu}$ process [1]. However, the final-state interaction played a very important role in that process and provided a very significant correction. In our case, this correction is much smaller and lies within the error of the Nambu–Jona-Lasinio model. Corrections of the same order of magnitude can be obtained by varying the width of the intermediate radially excited meson within the allowed experimental errors. It is also noteworthy that the description of the related $e^+e^- \to K^+K^-$ process within the extended Nambu–Jona-Lasinio model disregarding the final-state interaction of mesons also gives satisfactory agreement with experimental data [17].

5. CONCLUSIONS

In our last works, we described all possible decays of the tau lepton into two pseudoscalar mesons. The calculations have shown that the final-state interaction plays an important role in the $\tau \to \pi\pi\nu_\tau$ and...
\(\tau \to K\pi\nu\) processes and makes a contribution of about 30%. It is noteworthy that the contribution from the ground state has been taken into account in the indicated decays in the vector channel, whereas intermediate radially excited vector mesons make an insignificant contribution. The role of the final-state interaction in the \(\tau \to K\eta\nu\) and \(\tau \to KK\nu\) decays decreases noticeably with an increase in the total mass of produced mesons. The contribution from the final-state interaction in the \(\tau \to K\eta\nu\) decays decreases to 15 and 6%, respectively, which is within the accuracy of the model. It is worth noting that the contribution of excited intermediate mesons in the vector channel in these processes increases sharply possibly because the role of the final-state interaction decreases. As shown in Section 3, the effect of a change in the width of the excited intermediate meson on the result is larger than that of the final-state interaction. The origin of the mutual effect of the radially excited intermediate states and the final-state interaction on the decay width requires a more comprehensive analysis.

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