Identities of nonterminating series by Zeilberger’s algorithm

Tom H. Koornwinder

Abstract: This paper argues that automated proofs of identities for non-terminating hypergeometric series are feasible by a combination of Zeilberger’s algorithm and asymptotic estimates. For two analogues of Saalschütz’ summation formula in the non-terminating case this is illustrated.

Last modified: February 13, 1998

1. Introduction

The Gosper algorithm and in particular the subsequent Zeilberger algorithm and related WZ method have been extremely successful for an approach by computer algebra to identities involving \((q)\)-hypergeometric functions (see the book by Petkovšek, Wilf & Zeilberger [7] and the references given there). Most of the current theory and all implementations of Zeilberger’s algorithm remain restricted to the case of identities for terminating hypergeometric series, while many of the known identities in the literature involve non-terminating hypergeometric identities. The book [7] discusses in Chapter 7 some nonterminating cases obtained from terminating cases by the WZ method. A very convincing demonstration that Zeilberger’s algorithmic is suitable for non-terminating cases is given by Gessel [3, p.547]. He demonstrates Gauss’ summation formula for the Gaussian hypergeometric series of argument 1 by means of a combination of Zeilberger’s algorithm and the asymptotic estimate

\[
\frac{\Gamma(a+k)}{\Gamma(b+k)} \sim k^{a-b} \quad \text{as } k \to \infty
\]  

(1.1)

(see for instance Olver [6, Ch.4, §5.1] for a proof of (1.1)).

I will demonstrate in this paper that the method extends to the non-terminating generalization of Saalschütz’ summation formula for a terminating Saalschützian \(3F_2\) hypergeometric series of argument 1. This nonterminating case has the form \(A + B = C\), where \(A\) is a non-terminating Saalschützian \(3F_2(1)\), \(B\) is a quotient of Gamma functions times another \(3F_2(1)\) and \(C\) is a quotient of Gamma functions. Without knowing the formula explicitly, one can start with \(A\), apply Zeilberger’s algorithm to the terms of \(A\), and finally arrive by some limit transition using (1.1) at the desired formula \(A + B = C\).

It is important to notice that all steps in the proof, both in the Gauss case and in the non-terminating Saalschütz case, may be automated, including the application of the asymptotic formula (1.1). I strongly believe that this idea may lead to an algorithmic approach to most identities for non-terminating hypergeometric series in literature. For the \(q\)-analogues of such identities an algorithmic approach should be feasible as well.

For an introduction to Zeilberger’s and related algorithms I refer to the book [7]. For hypergeometric series see Erdélyi [1] and Gasper & Rahman [2]. Although the emphasis
in [2] is on the $q$-case, it also contains some information on the $q = 1$ case. Anyhow, many formulas for $q = 1$ can be looked up from the corresponding $q$-case in [2], by silently taking the formal limit for $q \uparrow 1$, and by using that

$$\Gamma_q(z) := \frac{(1-q)^{1-z}(q; q)_\infty}{(q^z; q)_\infty} \to \Gamma(z) \quad \text{as} \quad q \uparrow 1$$

(see [5, Appendix B] and references given there).

The contents of this paper are as follows. Section 2 discusses Gauss’ summation formula following Gessel [3, p.547]. Section 3 discusses the case of a non-terminating Saalschützian $3F_2(1)$. Section 4 gives three other identities related to the one derived in section 3, and it is shown how these identities follow analytically from each other. Finally, in section 5, we consider one of the other identities in section 4 ($A + B = C$ with $C$ a Gamma quotient and $A$ and $B$ a Gamma quotient times a non-terminating $3F_2(1)$ with one of the upper parameters equal to 1). As is demonstrated, it can be derived by computer algebra and asymptotics, similarly as for the case of section 3, but the asymptotics is quite tricky (maybe interesting in its own right) and not yet suitable to be automated.

2. Gauss’ summation formula

Gauss’ summation formula (Gauss, 1813; see for instance [1, 2.8(46) with proof in §2.1.3]) is the non-terminating analogue of the Chu-Vandermonde summation formula. It reads as follows:

$$2F_1 \left[\begin{array}{c} a, b \\ c \end{array}; 1 \right] := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)}, \quad (2.1)$$

where we assume that $\text{Re} (c - a - b) > 0$ and $c \notin \{0, -1, -2, \ldots\}$ in order to ensure that the hypergeometric series on the left-hand side is well-defined and absolutely convergent. Gessel [3, §7] shows how to prove this identity by the WZ method, as I will recapitulate now.

Formula (2.1), with $c$ replaced by $c + n$ ($n \in \{0, 1, 2, \ldots\}$), can be written as

$$\sum_{k=0}^{\infty} f(n, k) = s(n), \quad (2.2)$$

where

$$f(n, k) := \frac{(a)_k (b)_k}{(c + n)_k k!}, \quad s(n) := \frac{\Gamma(c + n) \Gamma(c + n - a - b)}{\Gamma(c + n - a) \Gamma(c + n - b)}. \quad (2.3)$$

The sum on the left-hand side of (2.2) still absolutely converges (since $n \geq 0$). Formula (2.2) can be rewritten as

$$\sum_{k=0}^{\infty} F(n, k) = 1, \quad (2.4)$$
where
\[ F(n, k) := \frac{f(n, k)}{s(n)} = \frac{\Gamma(a + k) \Gamma(b + k) \Gamma(c + n - a) \Gamma(c + n - b)}{\Gamma(1 + k) \Gamma(c + n + k) \Gamma(a) \Gamma(b) \Gamma(c + n - a - b)}. \] (2.5)

By (1.1) we have
\[ F(n, k) \sim k^{a+b-c-1-n} \quad \text{as} \quad k \to \infty. \]

Thus the sum on the left-hand side of (2.4) converges absolutely (as we already knew). We want to prove (2.4) by the WZ method. Gosper’s algorithm applied to \( F(n+1, k) - F(n, k) \) or, equivalently, the WZ method applied to \( F(n, k) \) succeeds. In Maple V4, for instance, call
\[
\text{read `hsum.mpl`: gosper(F(n+1,k)-F(n,k),k);
or
}\]
\[
\text{read `hsum.mpl`: WZcertificate(F(n,k),k,n);}
\]
from Koepf’s package hsum.mpl [4], or call
\[
\text{read ekhad; ct(F(n,k),1,k,n,N);
}
\]
from Zeilberger’s package EKHAD [8]. The resulting formula is:
\[ F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k), \] (2.6)
where
\[ G(n, k) := \frac{k}{c + n - a - b} F(n, k). \]

Note that
\[ G(n, 0) = 0, \quad G(n, k) \sim \frac{k^{a+b-c-n}}{c + n - a - b} \quad \text{as} \quad k \to \infty. \] (2.7)

It follows from (2.6) and (2.7) that
\[ \sum_{k=0}^{K} F(n+1, k) - \sum_{k=0}^{K} F(n, k) = G(n, K+1) - G(n, 0) = G(n, K+1) \to 0 \quad \text{as} \quad K \to \infty. \]

Hence \( \sum_{k=0}^{\infty} F(n, k) \) is independent of \( n \). Thus
\[ \sum_{k=0}^{\infty} F(n, k) = \lim_{n \to \infty} \sum_{k=0}^{\infty} F(n, k) = \sum_{k=0}^{\infty} \left( \lim_{n \to \infty} F(n, k) \right) = \sum_{k=0}^{\infty} \delta_{k,0} = 1, \] (2.8)
where the second equality is justified by dominated convergence. Indeed, for some \( C > 0 \) we have \( 1/|s(n)| \leq C \) if \( n \in \{0, 1, 2, \ldots\} \) (use (2.3) and (1.1)). Thus, by (2.5) and (2.3) we obtain
\[ |F(n, k)| \leq C |f(n, k)| \leq \left| \frac{(a)_k (b)_k}{(\text{Re} \, c + n_0)_k k!} \right| \] (2.9)
for \( n \in \{n_0, n_0 + 1, \ldots\} \), where \( n_0 \) is such that \( \text{Re} \, c + n_0 > 0 \). Again by (1.1), the right-hand side of (2.9) summed over \( k = 0, 1, 2, \ldots \) yields a convergent sum, thus justifying the application of the dominated convergence theorem in (2.8).
3. Non-terminating analogue of Saalschütz’ summation formula

Saalschütz’ summation formula (which actually goes back to Pfaff, 1797) for a terminating Saalschützian $3F_2$ series with argument 1 reads as follows:

$$3F_2\left[\binom{-m, b, c}{e, -m + b + c - e + 1}; 1\right] := \sum_{k=0}^{m} \frac{(-m)_k (b)_k (c)_k}{(e)_k (-m + b + c - e + 1)_k k!} \tag{3.1}$$

$$= \frac{(e - b)_m (e - c)_m}{(e)_m (e - b - c)_m} \quad (m \in \{0, 1, 2, \ldots\}, \quad e, e - b - c \notin \{0, -1, -2, \ldots\}).$$

See for instance [1, 4.4(3) with proof in §2.1.5]. Now replace $-m$ by $a$ on the left-hand side of (3.1), where $a$ is arbitrarily complex:

$$3F_2\left[\binom{a, b, c}{e, a + b + c - e + 1}; 1\right] := \sum_{k=0}^{m} \frac{(a)_k (b)_k (c)_k}{(e)_k (a + b + c - e + 1)_k k!}, \tag{3.2}$$

and try to generalize identity (3.1) for this case. Such generalizations, with one additional $3F_2(1)$ term, are known in literature (see for instance ...), but we want to find a formula of this type from scratch, just starting from (3.2) and working with the Zeilberger algorithm.

In (3.2) replace $c$ by $c + n$, where $n \in \{0, 1, 2, \ldots\}$. Then

$$3F_2\left[\binom{a, b, c + n}{e, a + b + c - e + n + 1}; 1\right] = \sum_{k=0}^{\infty} f(n, k), \tag{3.3}$$

where

$$f(n, k) := \frac{(a)_k (b)_k (c + n)_k}{(e)_k (a + b + c - e + 1 + n)_k k!};$$

so

$$f(n, k) \sim \frac{\Gamma(e) \Gamma(a + b + c - e + 1 + n)}{\Gamma(a) \Gamma(b) \Gamma(c + n)} k^{-2} \quad \text{as } k \to \infty$$

by (1.1). Thus we have absolute convergence in the sum in (3.3). Zeilberger’s algorithm applied to $f(n, k)$ succeeds. In Maple V4, for instance, call

```plaintext
read ekhad: ct(f(n,k),1,k,n,N);
```

from Zeilberger’s package EKHAD [8]. The resulting recurrence is:

$$(c - e + a + n + 1) (b + c - e + n + 1) f(n + 1, k) - \frac{(c - e + n + 1) (b + c - e + a + n + 1)}{c + n} f(n, k) = g(n, k + 1) - g(n, k), \tag{3.4}$$

where

$$g(n, k) := \frac{(b + c - e + a + n + 1) (e + k - 1)}{c + n} f(n, k).$$
We can rewrite (3.4) in the form

\[ F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k), \tag{3.5} \]

where

\[ F(n,k) := \frac{\Gamma(a + c - e + 1 + n) \Gamma(b + c - e + 1 + n)}{\Gamma(c - e + 1 + n) \Gamma(a + b + c - e + 1 + n)} f(n,k), \]
\[ G(n,k) := \frac{k(e+k-1)}{(c+n)(c-e+n+1)} F(n,k). \]

Observe that

\[ F(n,k) \sim \frac{\Gamma(e) \Gamma(a + c - e + 1 + n) \Gamma(b + c - e + 1 + n)}{\Gamma(a) \Gamma(b) \Gamma(c + n) \Gamma(c - e + 1 + n)} k^{-2} \quad \text{as } k \to \infty, \]

and

\[ G(n,0) = 0, \quad \lim_{k \to \infty} G(n,k) = \frac{\Gamma(e) \Gamma(a + c - e + 1 + n) \Gamma(b + c - e + 1 + n)}{\Gamma(a) \Gamma(b) \Gamma(c + 1 + n) \Gamma(c - e + 2 + n)}. \tag{3.6} \]

Put

\[ S(n) := \sum_{k=0}^{\infty} F(n,k) = \frac{\Gamma(a + c - e + 1 + n) \Gamma(b + c - e + 1 + n)}{\Gamma(c - e + 1 + n) \Gamma(a + b + c - e + 1 + n)} \, {}_{2}F_{1}\left[ a, b, c + n \mid e, a + b + c - e + 1 + n \right]. \tag{3.7} \]

Then by (3.5) and (3.6) we obtain

\[ S(n+1) - S(n) = \lim_{k \to \infty} G(n,k+1) - G(n,0) = \frac{\Gamma(e) \Gamma(a + c - e + 1 + n) \Gamma(b + c - e + 1 + n)}{\Gamma(a) \Gamma(b) \Gamma(c + 1 + n) \Gamma(c - e + 2 + n)}. \tag{3.8} \]

Hence

\[ S(n) = S(0) + \frac{\Gamma(e) \Gamma(a + c - e + 1) \Gamma(b + c - e + 1)}{\Gamma(a) \Gamma(b) \Gamma(c + 1) \Gamma(c - e + 2)} \sum_{k=0}^{n-1} \frac{(a + c - e + 1)k(b + c - e + 1)k}{(c + 1)k(c - e + 2)k}. \tag{3.9} \]

Note that

\[ \frac{(a + c - e + 1)k(b + c - e + 1)k}{(c + 1)k(c - e + 2)k} \sim \frac{\Gamma(c + 1) \Gamma(c - e + 2)}{\Gamma(a + c - e + 1) \Gamma(b + c - e + 1)} k^{a+b-e-1} \quad \text{as } k \to \infty. \]
Therefore assume that \( \text{Re} (e - a - b) > 0 \). Then, for \( n \to \infty \), the right-hand side of (3.9) tends to

\[
\frac{\Gamma(a + c - e + 1) \Gamma(b + c - e + 1)}{\Gamma(c - e + 1) \Gamma(a + b + c - e + 1)} \, _3F_2 \left[ \begin{array}{c} a, b, c \\ e, a + b + c - e + 1 \end{array} ; 1 \right] + \frac{\Gamma(e) \Gamma(a + c - e + 1) \Gamma(b + c - e + 1)}{\Gamma(a) \Gamma(b) \Gamma(c + 1) \Gamma(c - e + 2)} \, _3F_2 \left[ \begin{array}{c} a + c - e + 1, b + c - e + 1, 1 \\ c + 1, c - e + 2 \end{array} ; 1 \right].
\]

Here we substituted the right-hand side of (3.7) with \( n = 0 \) for \( S(0) \). Next let \( n \to \infty \) on the left-hand side of (3.9). From (3.7) we obtain

\[
\lim_{n \to \infty} S(n) = \, _2F_1 \left[ \begin{array}{c} a, b \\ e \end{array} ; 1 \right] = \frac{\Gamma(e) \Gamma(e - a - b)}{\Gamma(e - a) \Gamma(e - b)} \text{ (Re} (e - a - b) > 0) \text{).} \quad (3.10)
\]

The second identity is Gauss’ summation formula (2.1). The first identity follows by applying (1.1) to the limit of the quotient of Gamma functions in (3.7) combined with a formal limit (to be justified in the Lemma below) within the \(_3F_2(1)\) in (3.7). Thus the limit case of (3.9) for \( n \to \infty \) is the identity we looked for:

\[
\frac{\Gamma(a + c - e + 1) \Gamma(b + c - e + 1)}{\Gamma(c - e + 1) \Gamma(a + b + c - e + 1)} \, _3F_2 \left[ \begin{array}{c} a, b, c \\ e, a + b + c - e + 1 \end{array} ; 1 \right] + \frac{\Gamma(e) \Gamma(a + c - e + 1) \Gamma(b + c - e + 1)}{\Gamma(a) \Gamma(b) \Gamma(c + 1) \Gamma(c - e + 2)} \, _3F_2 \left[ \begin{array}{c} a + c - e + 1, b + c - e + 1, 1 \\ c + 1, c - e + 2 \end{array} ; 1 \right]
\]

\[
= \frac{\Gamma(e) \Gamma(e - a - b)}{\Gamma(e - a) \Gamma(e - b)}, \quad (3.11)
\]

valid for \( \text{Re} (e - a - b) > 0) \text{.}\)

Let us now give the promised Lemma which will justify the first identity in (3.10).

**Lemma 3.1** Let \( \text{Re} (e - a - b) > 0 \). Then

\[
\lim_{n \to \infty} \, _3F_2 \left[ \begin{array}{c} a, b, c + n \\ e, a + b + c - e + 1 + n \end{array} ; 1 \right] = \, _2F_1 \left[ \begin{array}{c} a, b \\ e \end{array} ; 1 \right].
\]

**Proof** The limit formally holds because

\[
\lim_{n \to \infty} \frac{(a)_k (b)_k (c + n)_k}{(e)_k (a + b + c - e + 1 + n)_k k!} = \frac{(a)_k (b)_k}{(e)_k k!}.
\]

We will justify this limit by dominated convergence. Take \( \varepsilon \in (0, 1) \) such that also \( \varepsilon < \text{Re} (e - a - b) \). It follows from (1.1) that
Then the dominated convergence follows because
\[
\sum_{k=1}^{\infty} k^{Re(a+b-c)-1} n^{Re(a+b-c)+1} (n+k)^{Re(e-a-b)-1}
\]
\[
= \text{const.} \left( \frac{n}{n+k} \right)^{1-\varepsilon} k^{-1-\varepsilon} (n-1+k)^{Re(e-a-b)-\varepsilon}
\]
\[
\leq \begin{cases} \text{const.} k^{-Re(e-a-b)-1} & \text{if } k \leq n, \\ \text{const.} k^{-\varepsilon-1} & \text{if } k \geq n. \end{cases}
\]
Then the dominated convergence follows because \( \sum_{k=1}^{\infty} k^{-\varepsilon-1} < \infty. \)

4. Other non-terminating identities related to Saalschütz’ summation formula

Formula (3.11), which I derived in the previous section by a mixture of computer algebra and asymptotic techniques, is possibly not in the literature, but it can be derived from some formulas of similar nature which are in the literature.

The best known non-terminating generalization of Saalschütz’ formula (3.1) is:

\[
\begin{align*}
\sum_{n=1}^{\infty} & \left[ \frac{(a)_k (b)_k (c+n)_k}{(e)_k (a+b+c-e+1+n)_k k!} \right] = \text{const.} \left[ \frac{\Gamma(a+k) \Gamma(b+k)}{\Gamma(e+k) \Gamma(1+k)} \right] \\
& \times \left[ \frac{\Gamma(c+a+b-e+1+n)}{\Gamma(c+n)} \right] \left[ \frac{\Gamma(c+n+k)}{\Gamma(c+a+b-e+1+n+k)} \right] \\
& \leq \text{const.} k^{Re(a+b-c)-1} n^{Re(a+b-c)+1} (n+k)^{Re(e-a-b)-1}
\end{align*}
\]

\( 3F_2 \left[ a, b, c \middle| e, a+b+c-e+1 ; 1 \right] 
\]

\[
\begin{align*}
&+ \frac{\Gamma(e-1) \Gamma(a-e+1) \Gamma(b-e+1) \Gamma(c-e+1) \Gamma(a+b+c-e+1)}{\Gamma(1-e) \Gamma(a) \Gamma(b) \Gamma(c) \Gamma(a+b+c-2e+2)} \\
& \times \sum_{n=1}^{\infty} \left[ \frac{a-e+1, b-e+1, c-e+1}{2-e, a+b+c-2e+2} ; 1 \right]
\end{align*}
\]

\[
= \frac{\Gamma(a-e+1) \Gamma(b-e+1) \Gamma(c-e+1) \Gamma(a+b+c-e+1)}{\Gamma(1-e) \Gamma(b+c-e+1) \Gamma(a+c-e+1) \Gamma(a+b+e+1)},
\quad (4.1)
\]

see [2, (II.24)]. Since both \( 3F_2(1) \) series are Saalschützian, there are no conditions on the parameters necessary for convergence.

The second formula we will use involves two non-terminating \( 3F_2(1) \) series with one upper parameter equal to 1, like the second \( 3F_2(1) \) in formula (3.11). The formula reads as follows.

\[
\begin{align*}
&\sum_{n=1}^{\infty} \left[ \frac{a, b, 1}{d, e} ; 1 \right] + \frac{1-e}{e-b-1} \sum_{n=1}^{\infty} \left[ \frac{d-a, b, 1}{d, b-e+2} ; 1 \right] \\
&= \frac{\Gamma(d) \Gamma(e) \Gamma(a-e+1) \Gamma(b-e+1) \Gamma(d+e-a-b-1)}{\Gamma(a) \Gamma(b) \Gamma(d-a) \Gamma(d-b)},
\quad (4.2)
\end{align*}
\]

\( \text{Re} (d+e-a-b-1) > 0, \text{Re} (a-e+1) > 0. \)
It can be derived by specialization of [2, (3.3.1)] combined with Gauss’ summation formula (2.1).

The third formula transforms a nonterminating Saalschützian \( _3F_2(1) \) into a nonterminating \( _3F_2(1) \) with upper parameter 1:

\[
_3F_2 \left[ \begin{array}{c}
a, b, c \\
e, a + b + c - e + 1
\end{array} ; 1 \right] = \frac{\Gamma(e) \Gamma(a + b + c - e + 1)}{\Gamma(a) \Gamma(b + 1) \Gamma(c + 1) \times _3F_2 \left[ \begin{array}{c}
e - a, b + c - e + 1, 1 \\
b + 1, c + 1
\end{array} ; 1 \right] (\text{Re } a > 0). \tag{4.3}
\]

It follows by specialization of [2, (3.1.2)].

Now replace in (4.3) \( a, b, c, e \) by \( b - e + 1, a - e + 1, c - e + 1, 2 - e \). Next substitute in the right-hand side of (4.3) (with parameters replaced as above) formula (4.2) with \( a, b, d, e \) replaced by \( 1 - b, a + c - e + 1, a - e + 2, c - e + 2 \). This yields

\[
_3F_2 \left[ \begin{array}{c}
b - e + 1, a - e + 1, c - e + 1 \\
2 - e, a + b + c - 2e + 2
\end{array} ; 1 \right] = -\frac{\Gamma(2 - e) \Gamma(a + b + c - 2e + 2)}{e \Gamma(b - e + 1) \Gamma(a - e + 1) \Gamma(c - e + 2)} \times _3F_2 \left[ \begin{array}{c}
1 + b + c - e, 1 + a + c - e, 1 \\
2 + c - e, c + 1
\end{array} ; 1 \right] + \frac{\Gamma(e - a - b) \Gamma(c) \Gamma(2 - e) \Gamma(a + b + c - 2e + 2)}{\Gamma(1 - b) \Gamma(1 + b + c - e) \Gamma(1 - a) \Gamma(a + c - e + 1)} \tag{4.4}
\]

(Re (1 + b - e) > 0, Re (e - a - b) > 0).

Now formula (3.11) follows from (4.1) by substituting (4.4) for the second \( _3F_2(1) \) in (4.1), and by using that

\[
\Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}.
\]

So we have seen that formulas (4.1), (4.2) and (4.3) imply formula (3.11) via formula (4.4). Conversely, formulas (3.11), (4.2) and (4.3) imply formula (4.1) via formula (4.4). In the next section we will derive formulas (4.2) and (4.3) by methods of computer algebra and asymptotics.

5. Further proofs of identities of non-terminating series by computer algebra and asymptotics

Let us try to prove formula (4.2) in a similar way as formula (3.11). Put

\[
f(n, k) := \frac{(a)_k (b + n)_k}{(d + n)_k (e)_k}.
\]
By Zeilberger’s algorithm we find that

\[(a - n - d) (n + b) f(n + 1, k) - (n + b - e + 1) (n + d) f(n, k) = g(n, k + 1) - g(n, k),\]

where \( g(n, k) := -(n + d) (e + k - 1) f(n, k). \)

Hence

\[F(n + 1, k) - F(n, k) = G(n, k + 1) - G(n, k),\]

where

\[F(n, k) := \frac{(b)_n (d - a)_n}{(d)_n (b - e + 1)_n} \frac{(a)_k (b + n)_k}{(d)_k (e)_k},\]

\[G(n, k) := \frac{(e + k - 1) (a)_k (b + n)_k (b)_n (d - a)_n}{(e - n - b - 1)_n (d + n)_k (e)_k (d)_n (b - e + 1)_n}.\]

Assume

\[\text{Re} (d + e - a - b - 1) > 0, \quad \text{Re} (a - e + 1) > 0. \quad (5.1)\]

Then

\[S(n) := \sum_{k=0}^{\infty} F(n, k) = \frac{(b)_n (d - a)_n}{(d)_n (b - e + 1)_n} 3F_2 \left[\frac{a, b + n, 1}{d + n, e}; 1\right] \quad (5.2)\]

has absolutely convergent series and

\[S(n + 1) - S(n) = \lim_{k \to \infty} G(n, k + 1) - G(n, 0) = -G(n, 0)\]

\[= -\frac{(e - 1) (d - a)_n (b)_n}{(e - b - 1)_n (d)_n (b - e + 2)_n}. \quad (5.3)\]

Hence

\[S(0) = S(\infty) + \frac{e - 1}{e - b - 1} \sum_{n=0}^{\infty} \frac{(d - a)_n (b)_n}{(d)_n (b - e + 2)_n},\]

where \( S(\infty) := \lim_{n \to \infty} S(n). \)

Thus

\[3F_2 \left[\frac{a, b, 1}{d, e}; 1\right] = \frac{e - 1}{e - b - 1} 3F_2 \left[\frac{d - a, b, 1}{d, b - e + 2}; 1\right] + S(\infty).\]

So formula (4.2) will be proved if we can show that

\[\lim_{n \to \infty} \frac{(b)_n (d - a)_n}{(d)_n (b - e + 1)_n} 3F_2 \left[\frac{a, b + n, 1}{d + n, e}; 1\right] \]

\[= \frac{\Gamma(d) \Gamma(e) \Gamma(a - e + 1) \Gamma(b - e + 1) \Gamma(d + e - a - b - 1)}{\Gamma(a) \Gamma(b) \Gamma(d - a) \Gamma(d - b)}. \quad (5.4)\]
This limit result is not evident. A formal limit would yield

\[
\left( \lim_{n \to \infty} n^{e-a-1} \right) {}_2F_1 \left[ \begin{array}{c} a, 1 \\ e \end{array} ; 1 \right] = 0 \cdot \infty \quad \text{since } \Re (a-e+1) > 0.
\]

We will give a proof of (5.4) at the end of this section.

Let us first turn to formula (4.3), and rewrite it in the following equivalent form.

\[
\frac{(b)_n (d-a)_n}{(d)_n (b-e+1)_n} {}_3F_2 \left[ \begin{array}{c} a, b+n, 1 \\ d+n, e \end{array} ; 1 \right] = \frac{\Gamma(d) \Gamma(e) \Gamma(b-e+1) \Gamma(d+e-a-b-1)}{\Gamma(b) \Gamma(d-a) \Gamma(d+e-b-1)} \frac{\Gamma(b+n) \Gamma(d-a+n)}{\Gamma(b-e+1+n) \Gamma(d+e-a-1+n)} \times {}_3F_2 \left[ \begin{array}{c} d+n-1, d+e-a-b-1, e-1 \\ d+e-a+n-1, d+e-b-1 \end{array} ; 1 \right] \quad \text{(Re } \quad (d+e-a-b-1) > 0) \quad . \quad (5.5)
\]

Observe that we can take limits on the right-hand side of (5.5) for \( n \to \infty \) (see (3.10)), and thus arrive at (5.4). However, I only want to use (5.5) if I can derive it by methods of computer algebra and asymptotics. Let us try.

Write (5.5) more compactly as

\[
S(n) = C \tilde{S}(n), \quad \text{(5.6)}
\]

where \( S(n) \) is given by (5.2),

\[
C := \frac{\Gamma(d) \Gamma(e) \Gamma(b-e+1) \Gamma(d+e-a-b-1)}{\Gamma(b) \Gamma(d-a) \Gamma(d+e-b-1)},
\]

and

\[
\tilde{S}(n) := \frac{\Gamma(b+n) \Gamma(d-a+n)}{\Gamma(b-e+1+n) \Gamma(d+e-a-1+n)} {}_3F_2 \left[ \begin{array}{c} d+n-1, d+e-a-b-1, e-1 \\ d+e-a+n-1, d+e-b-1 \end{array} ; 1 \right].
\]

Note that \( \tilde{S}(n) \) is just (3.7) with \( a, b, c, e \) replaced by \( e-1, d+e-a-b-1, d-1, d+e-b-1 \).

Thus (3.8) yields that

\[
\tilde{S}(n+1) - \tilde{S}(n) = \frac{1}{C} \frac{(e-1)(d-a)_n(b)_n}{(b-e+1)(d)_n(b-e+2)_n}.
\]

Comparison with (5.3) yields that \( S(n) - C \tilde{S}(n) \) is independent of \( n \). Hence

\[
S(n) - C \tilde{S}(n) = S(\infty) - C \tilde{S}(\infty).
\]

Thus (5.6) holds iff (5.4) holds. So we have not made real progress. Formula (5.4) will follow from (5.6), but for the proof of (5.6) we need (5.4). Let us therefore give an independent proof of (5.4).
Proof of (5.4) Observe that
\[
\frac{(b)_n (d - a)_n}{(d)_n (b - e + 1)_n} {}_{3}F_{2} \left[ \begin{array}{c} a, b + n, 1 \\ d + n, e \end{array} ; 1 \right]
= \frac{\Gamma(d) \Gamma(e) \Gamma(b - e + 1)}{\Gamma(a) \Gamma(b) \Gamma(d - a)} \sum_{k=0}^{\infty} \frac{\Gamma(a + k) \Gamma(b + n + k)}{\Gamma(e + k) \Gamma(d + n + k)} \frac{\Gamma(d - a + n)}{\Gamma(b - e + 1 + n)}. \quad (5.7)
\]

We will first work formally, and consider the sum obtained by replacing in the above infinite sum the gamma quotients by their asymptotic estimates (for big \(n\) and \(k\)) as given by (1.1). This yields
\[
\sum_{k=0}^{\infty} \frac{\Gamma(a + k) \Gamma(b + n + k) \Gamma(d - a + n)}{\Gamma(e + k) \Gamma(d + n + k) \Gamma(b - e + 1 + n)} \quad (5.8)
\]
\[
\sim \sum_{k=0}^{\infty} (k + 1)^{a - e} (n + k + 2)^{b - d} (n + 1)^{d + e - a - b - 1} \quad (5.9)
\]
\[
= \frac{1}{n + 1} \sum_{k=0}^{\infty} \left( 1 + \frac{k + 1}{n + 1} \right)^{b - d} \left( \frac{k + 1}{n + 1} \right)^{a - e} \quad (5.10)
\]
\[
\sim \int_{t=0}^{\infty} (1 + t)^{b - d} t^{a - e} dt \quad \text{as } n \to \infty \quad (5.11)
\]
\[
= \frac{\Gamma(a - e + 1) \Gamma(d + e - a - b - 1)}{\Gamma(d - b)}. \quad (5.12)
\]

Here we used that the integral in (5.11), which converges because of assumptions (5.1), is approximated by its Riemann sum (5.10) as \(n \to \infty\). Thus formally the right-hand side of (5.7) indeed equals the right-hand side of (5.4).

We will justify this formal proof by Lebesgue’s dominated convergence theorem. Rewrite (5.8) as
\[
\int_{0}^{\infty} \gamma_n(t) h_n(t) dt, \quad (5.12)
\]
where, for \(\frac{k}{n + 1} < t \leq \frac{k + 1}{n + 1}\),
\[
h_n(t) := \left( 1 + \frac{k + 1}{n + 1} \right)^{b - d} \left( \frac{k + 1}{n + 1} \right)^{a - e}
\]
and
\[
\gamma_n(t) := \frac{\Gamma(a + k)}{\Gamma(e + k) (k + 1)^{a - e}} \frac{\Gamma(b + n + k)}{\Gamma(d + n + k) (n + k + 2)^{b - d}} \frac{\Gamma(d - a + n)}{\Gamma(b - e + 1 + n) (n + 1)^{d + e - a - b - 1}}. \quad (5.13)
\]
Then
\[
|\gamma_n(t)| \leq \text{const.} \quad \text{and} \quad |h_n(t)| \leq \text{const.} (1 + t)^{\Re(b - d) t^{\Re(a - e)}}. \quad (5.14)
\]
Furthermore,
\[
\lim_{n \to \infty} \gamma_n(t) = 1 \quad \text{and} \quad \lim_{n \to \infty} h_n(t) = (1 + t)^{b - d} t^{a - e}. \quad (5.15)
\]
It follows by dominated convergence that the integral (5.12) tends to the integral (5.11) as \(n \to \infty\).
References

[1] A. Erdélyi e.a., *Higher transcendental functions, Vol. I*, McGraw-Hill, 1953. Reprinted in 1981 by R.E. Krieger.

[2] G. Gasper & M. Rahman, *Basic hypergeometric series*, Cambridge University Press, 1990.

[3] I. M. Gessel, *Finding identities with the WZ method*, J. Symb. Comp. 20 (1995), 537–566.

[4] W. Koepf, *Maple package hsum.mpl*, Version 1.0, February 1, 1998; obtainable by email from koepf@imn.htwk-leipzig.de.

[5] T. H. Koornwinder, *Jacobi functions as limit cases of q-ultraspherical polynomials*, J. Math. Anal. Appl. 148 (1990), 44–54.

[6] F. W. J. Olver, *Asymptotics and special functions*, Academic Press, New York, 1974. Reprinted in 1997 by A. K. Peters.

[7] M. Petkovšek, H. S. Wilf and D. Zeilberger, *A = B*, A. K. Peters, 1996.

[8] D. Zeilberger, *Maple package EKHAD*, Version of March 27, 1997; obtainable from URL http://www.math.temple.edu/~zeilberg.

University of Amsterdam, Korteweg-de Vries Institute for Mathematics
Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands
email thk@wins.uva.nl