A Dirichlet-type integral on spheres, applied to the fluid/gravity correspondence

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Abstract

We evaluate an analogue of an integral of Dirichlet over the sphere $S^D$, but with an integrand that is independent of $[(D + 1)/2]$ Killing coordinates. As an application, we evaluate an integral that arises when comparing a conformal fluid on $S^D$ and black holes in $(D + 2)$-dimensional anti-de Sitter spacetime.
1 Introduction

There is a class of functions that are particularly easy to integrate over the \( n \)-dimensional unit sphere \( S^n \), namely monomials in the cartesian coordinates for \( \mathbb{R}^{n+1} \supset S^n \). Let \( x_i, i = 1, \ldots, n+1 \) be such coordinates, so \( S^n \) is the hypersurface \( \sum_{i=1}^{n+1} x_i^2 = 1 \). A well-known result of Dirichlet \footnote{\cite{1}} is that, for non-negative integers \( \alpha_j \),

\[
\int_{S^n} \prod_{j=1}^{n+1} x_j^{\alpha_j} = \begin{cases} 
0, & \text{some } \alpha_j \text{ is odd,} \\
\frac{2 \prod_{j=1}^{n+1} \Gamma\left(\frac{1}{2} + \frac{1}{2} \alpha_j\right)}{\Gamma\left(\frac{n+1}{2} + \frac{1}{2} \sum_{i=1}^{n+1} \alpha_i\right)}, & \text{all } \alpha_j \text{ are even.} 
\end{cases}
\]  

(1.1)

More generally, we have, for any real and non-negative \( \alpha_j \),

\[
\int_{S^n} \prod_{j=1}^{n+1} |x_j|^{\alpha_j} = \frac{2 \prod_{j=1}^{n+1} \Gamma\left(\frac{1}{2} + \frac{1}{2} \alpha_j\right)}{\Gamma\left(\frac{n+1}{2} + \frac{1}{2} \sum_{i=1}^{n+1} \alpha_i\right)}. 
\]  

(1.2)

A simple direct proof is given in, for example, \cite{2}. For a historical review of a wider class of integrals, see \cite{3}. Taking linear combinations of these results allows one to integrate polynomials and more general power series in \( x_i \) over spheres.

In applications, it may be necessary to use some angular coordinates intrinsic to the sphere. Consider the \( D \)-dimensional unit sphere \( S^D \), and let \( D = 2n + \epsilon \), with \( \epsilon = 0, 1 \) according to whether \( D \) is even or odd. By introducing plane polar coordinates \((\mu_i, \phi_i)\) for orthogonal 2-planes in \( \mathbb{R}^{D+1} \), we have \([\lfloor (D+1)/2 \rfloor]\) angular coordinates \( \phi_i, i = 1, \ldots, n+\epsilon \), with independent periods \( 2\pi \). The flat metric on \( \mathbb{R}^{D+1} \) induces the round metric on \( S^D \) given by

\[
ds_D^2 = \sum_{i=1}^{n+\epsilon} d\mu_i^2 + \sum_{i=1}^{n+\epsilon} \mu_i^2 d\phi_i^2, 
\]  

(1.3)

where \( \mu_i \) satisfy the constraint

\[
\sum_{i=1}^{n+\epsilon} \mu_i^2 = 1. 
\]  

(1.4)

The metric coefficients are independent of \( \phi_i \), i.e. \( \partial/\partial \phi_i \) are commuting Killing vectors; they represent rotational symmetries. One can imagine situations in which one has to consider functions that are independent of \( \phi_i \), and so are expressible in terms of \( \mu_i \) only. These are a generalization to higher dimensions of axisymmetric functions on \( S^2 \), which in 3-dimensional spherical polar coordinates depend on \( \mu = \cos \theta \) but not the azimuthal coordinate \( \phi \). This motivates us to consider integrals that are analogous to (1.2), but over \( S^D \) and involving powers of \( \mu_i \). The main result that we shall prove is that, for \( \alpha_j \geq -1 \),

\[
\int_{S^D} \prod_{j=1}^{n+\epsilon} \mu_j^{\alpha_j} = \frac{2\pi^{(D+1)/2} \prod_{j=1}^{n+\epsilon} \Gamma\left(1 + \frac{1}{2} \alpha_j\right)}{\Gamma\left(\frac{D+1}{2} + \frac{1}{2} \sum_{i=1}^{n+\epsilon} \alpha_i\right)}. 
\]  

(1.5)

As an application, we shall evaluate an integral arising in \cite{4}, which concerns a correspondence between fluid mechanics on spheres and black holes in AdS (anti-de Sitter) spacetime.

2 Proof of general result

Let \( X_I, I = 1, \ldots, D+1 \) be cartesian coordinates for \( \mathbb{R}^{D+1} \). We introduce sets of plane polar coordinates \((\mu_i, \phi_i)\) for the \((X_{2i-1}, X_{2i})\)-planes by

\[
(X_{2i-1}, X_{2i}) = (\mu_i \cos \phi_i, \mu_i \sin \phi_i), 
\]  

(2.1)
for \( i = 1, \ldots, n + \epsilon \). If \( D \) is even, then we instead define \( \mu_{n+1} \) by

\[
X_{2n+1} = \mu_{n+1}. \tag{2.2}
\]

The coordinates \((\mu_1, \ldots, \mu_{n+1}, \phi_1, \ldots, \phi_{n+\epsilon})\) cover \( \mathbb{R}^{D+1} \), with ranges \( \mu_i \geq 0 \) for \( i = 1, \ldots, n+\epsilon \), \( \mu_{n+1} \) unrestricted if \( D \) is even, and \( 0 \leq \phi_i < 2\pi \) for all \( i \).

The \( D \)-dimensional unit sphere \( S^D \subset \mathbb{R}^{D+1} \) is the hypersurface \( \sum_{i=1}^{D+1} X_i^2 = 1 \), on which the round metric is \((1.3)\). Bearing in mind the constraint \((1.4)\), it can be expressed as

\[
ds_D^2 = ds_n^2 + \sum_{i=1}^{n+\epsilon} \mu_i^2 d\phi_i^2, \tag{2.3}
\]

where

\[
ds_n^2 = \sum_{i=1}^{n+1} d\mu_i^2. \tag{2.4}
\]

If we regard \( \mu_i \) as cartesian coordinates for \( \mathbb{R}^{n+1} \), then \((2.4)\) can be interpreted as the round metric on \( S^n \subset \mathbb{R}^{n+1} \). A difference is that there no constraints on the signs of \( \mu_i \) as coordinates for \( \mathbb{R}^{n+1} \). On the sphere \( S^n \), we again have the constraint \((1.4)\).

The interpretation of \( \mu_i \) as either coordinates for \( \mathbb{R}^{D+1} \) or for \( \mathbb{R}^{n+1} \) enables us to reduce an integral over \( S^D \) that is independent of the \( \phi_i \) coordinates to an integral over \( S^n \): we have a “sphere within a sphere”. Note that

\[
\prod_{l=0}^{n+\epsilon} \int_0^{2\pi} d\phi_l \int_{\sum_{i=1}^{n+\epsilon} \mu_i^2 = 1, \mu_1, \ldots, \mu_{n+\epsilon} \geq 0} d^n \mu \prod_{j=1}^{n+\epsilon} \mu_j^{\alpha_j+1} = \pi^{n+\epsilon} \int_{\sum_{i=1}^{n+\epsilon} \mu_i^2 = 1} d^n \mu \prod_{j=1}^{n+\epsilon} |\mu_j|^{\alpha_j+1}, \tag{2.5}
\]

because the \( \phi_l \) integrals give a factor of \((2\pi)^{n+\epsilon}\), and removing the sign constraints on \( \mu_1, \ldots, \mu_{n+\epsilon} \) gives a factor of \( 2^{-(n+\epsilon)} \). The meaning of \( d^n \mu \) should be clear. Explicitly, one can, for example, eliminate \( \mu_{n+1} \) from the integrand in favour of \( \mu_1, \ldots, \mu_n \) using the constraint \((1.4)\). Then \( d^n \mu \) means \( \prod_{k=1}^n d\mu_k \), bearing in mind that for each choice of \((\mu_1, \ldots, \mu_n)\) we must account for both signs of \( \mu_{n+1} \) on the right. Expressing this in terms of integrals over \( S^D \) and \( S^n \), with respective metrics \((1.3)\) and \((2.4)\), we have

\[
\int_{S^D} \prod_{j=1}^{n+\epsilon} \mu_j^{\alpha_j} = \pi^{n+\epsilon} \int_{S^n} \prod_{j=1}^{n+\epsilon} |\mu_j|^{\alpha_j+1}. \tag{2.6}
\]

Using the Dirichlet integral \((1.2)\) for integration over \( S^n \), remembering for even \( D \) that it includes a factor of \( \Gamma(\frac{1}{2}) = \sqrt{\pi} \), we hence obtain our main result \((1.3)\).

### 3 Application: fluid/gravity correspondence

An explicit application of our main result is to a missing step in [4], which studies the fluid/gravity correspondence. It is argued that there is a duality between a conformal fluid on \( S^D \) that solves the relativistic Navier–Stokes equations and a large black hole in \( \text{AdS}_{D+2} \) that solves the Einstein equations. For one specific example, in arbitrary dimensions, the fluid is uncharged and rigidly rotating, and the black hole is the Kerr–AdS solution, with a horizon radius much larger than the AdS radius. One can compare the thermodynamics of both sides of the correspondence. From the correspondence for non-rotating solutions, one can make predictions for rotating solutions.
On the fluid side of the correspondence, one considers the spacetime
\[ ds^2 = -dt^2 + ds_D^2, \] (3.1)
where \( ds_D^2 \) is the round metric on \( S^D \) (1.3). The spacetime is filled with a fluid with velocity
\[ u^a \partial_a = \gamma \left( \frac{\partial}{\partial t} + \sum_{i=1}^{n+\epsilon} \omega_i \frac{\partial}{\partial \phi_i} \right), \] (3.2)
where
\[ \gamma = \frac{1}{\sqrt{1 - v^2}}, \quad v^2 = \sum_{i=1}^{n+\epsilon} \mu_i^2 \omega_i^2, \] (3.3)
and \( \omega_i^2 < 1 \). One computes the energy-momentum tensor and currents. Integration gives conserved charges, which can be compared with the gravity side of the correspondence. A missing step in [4] is a proof for all \( D \) of a certain integral, namely
\[ \int_{S^D} \left( 1 - \sum_{j=1}^{n+\epsilon} \mu_j^2 \omega_j^2 \right)^{-(D+1)/2} = \frac{2\pi^{(D+1)/2}}{\Gamma\left(\frac{D+1}{2}\right) \prod_{j=1}^{n+\epsilon} (1 - \omega_j^2)}, \] (3.4)
Equivalently, we have
\[ \int_{S^D} \gamma^{D+1} = \frac{V_D}{\prod_{j=1}^{n+\epsilon} (1 - \omega_j^2)}, \quad V_D = \frac{2\pi^{(D+1)/2}}{\Gamma\left(\frac{D+1}{2}\right)}, \] (3.5)
where \( V_D \) is the volume of \( S^D \). Using this result, one then finds agreement between the two sides of the correspondence.

To prove the required integral, we first use binomial expansions to obtain
\[ \left( 1 - \sum_{j=1}^{n+\epsilon} \mu_j^2 \omega_j^2 \right)^{-(D+1)/2} = \sum_{k_1, \ldots, k_{n+\epsilon} \geq 0} (D+1)(D+1) \ldots (D+1 + k - 1) \prod_{j=1}^{n+\epsilon} \left( \frac{\mu_j \omega_j}{k_j} \right)^{2k_j}, \] (3.6)
where
\[ k = \sum_{j=1}^{n+\epsilon} k_j. \] (3.7)
From the main result (1.5), if \( k_j \) are non-negative integers, then
\[ \int_{S^D} \prod_{j=1}^{n+\epsilon} \mu_j^{2k_j} = \frac{2\pi^{(D+1)/2} \prod_{j=1}^{n+\epsilon} (k_j)!}{\Gamma\left(\frac{D+1}{2} + k\right)}. \] (3.8)
Using this and the expansion of \( \prod_{j=1}^{n+\epsilon} (1 - \omega_j^2)^{-1} \), we hence obtain the integral (3.4).

References

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