A DIRECT SOLUTION TO THE GENERIC POINT PROBLEM

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Abstract. We provide a new proof of a recent theorem of Ben Yaacov, Melleray, and Tsankov. If $G$ is a Polish group and $X$ is a minimal, metrizable $G$-flow with all orbits meager, then the universal minimal flow $M(G)$ is non-metrizable. In particular, we show that given $X$ as above, the universal highly proximal extension of $X$ is nonmetrizable.

1. Introduction

In this paper, we are concerned with actions of a topological group $G$ on a compact space $X$. All groups and spaces are assumed Hausdorff. A compact space $X$ equipped with a continuous $G$-action $a : G \times X \to X$ is called a $G$-flow. The action $a$ is often suppressed in the notation, i.e., $gx$ is written for $a(g, x)$. A $G$-flow $X$ is called minimal if every orbit is dense. It is a fact that every topological group $G$ admits a universal minimal flow $M(G)$, a minimal flow which admits a $G$-map onto any other minimal flow. A $G$-map is a continuous map respecting the $G$-action. The flow $M(G)$ is unique up to $G$-flow isomorphism.

We can now recall the following theorem of Ben Yaacov, Melleray, and Tsankov [4].

Theorem 1.1. Let $G$ be a Polish group, and let $M(G)$ be the universal minimal flow of $G$. If $M(G)$ is metrizable, then $M(G)$ has a comeager orbit.

The question of whether or not metrizability of $M(G)$ was enough to guarantee a comeager orbit was first asked by Angel, Kechris, and Lyons [5]. In [6], the current author proved Theorem 1.1 in the case when $G$ is the automorphism group of a first-order structure. The proof given there used topological properties of the largest $G$-ambit $S(G)$ along with combinatorial reasoning about the structures. In [4], the authors also use topological properties of $S(G)$, but the combinatorics are replaced by the following theorem due to Rosendal; see [4] for a proof.

Theorem 1.2. Let $G$ be a Polish group acting continuously on a compact metric space $X$. Assume the action is topologically transitive. Then the following are equivalent:

(1) $G$ has a comeager orbit.

(2) For any open $1 \in V \subseteq G$ and any open $B \subseteq X$, there is open $C \subseteq B$ so that for any open $D \subseteq C$, the set $C \setminus VD$ is nowhere dense.
It is proven in [3] that comeager orbits push forward; namely, if $X$ is a minimal $G$-flow, $x \in X$ is a point whose orbit is generic, and if $\pi : X \to Y$ is a surjective $G$-map, then $\pi(x)$ has generic orbit in $Y$. Theorem 1.1 then becomes equivalent to the following: whenever $G$ is a Polish group and $X$ is a minimal metrizable flow with all orbits meager, then $G$ must admit some minimal, nonmetrizable flow. Remarkably, neither [4] nor [6] prove Theorem 1.1 in this direct fashion.

We provide a direct proof of Theorem 1.1. For any topological group $G$ and any $G$-flow $X$, we construct a new $G$-flow denoted $S_G(X)$. We then show that if $X$ is minimal, then so is $S_G(X)$. Lastly, if $G$ is Polish and $X$ is metrizable and has all orbits meager, we use Theorem 1.2 to show that $S_G(X)$ is nonmetrizable.

After providing our new proof of Theorem 1.1, we investigate the flow $S_G(X)$ in more detail. For any $G$-flow $X$, there is a natural map $\pi_X : S_G(X) \to X$. When $X$ is minimal, we show that $\pi_X$ is the universal highly proximal extension of $X$. The notion of a highly proximal extension was introduced by Auslander and Glasner in [2]. If $X$ and $Y$ are minimal $G$-flows, a $G$-map $\varphi : Y \to X$ is highly proximal if for any $x \in X$ and nonempty open $U \subseteq Y$, there is $g \in G$ with $g\pi^{-1}(\{x\}) \subseteq U$. Auslander and Glasner prove in [2] that for every minimal $G$-flow $X$, there is a universal highly proximal extension $\pi : \tilde{X} \to X$. This means that $\pi$ is highly proximal, and for every other highly proximal $\varphi : Y \to X$, there is a $G$-map $\psi : \tilde{X} \to Y$ so that $\pi = \varphi \circ \psi$. The map $\pi$ is unique up to $G$-flow isomorphism over $X$. Our construction of the flow $S_G(X)$ provides a new construction of the universal highly proximal extension of $X$ and hints at a generalization of this notion even when $X$ is not minimal.

2. The flow $S_G(X)$ and proof of Theorem 1.1

All groups and spaces will be assumed Hausdorff. In this section, fix a topological group $G$ and a $G$-flow $X$. Write $\mathcal{N}_G$ for the collection of symmetric open neighborhoods of the identity in $G$ and write $\text{op}(X)$ for the collection of nonempty open subsets of $X$.

**Definition 2.1.** A near filter is any $\mathcal{F} \subseteq \text{op}(X)$ so that for any $A_1, \ldots, A_k \in \mathcal{F}$ and any $U \in \mathcal{N}_G$, we have $U A_1 \cap \cdots \cap U A_k \neq \emptyset$. A near ultrafilter is a maximal near filter.

Near ultrafilters exist by an application of Zorn’s lemma. Near ultrafilters on a uniform space have been considered in [1] and [3]. Two aspects of our approach are slightly different. First, the notion of nearness is not given by the natural uniform structure on the compact Hausdorff space $X$. Second, instead of working with a notion of nearness on $\mathcal{P}(X)$, we are more or less working with the regular open algebra on $X$ (see item (2) in Lemma 2.2).

Let $S_G(X)$ denote the space of near ultrafilters on $\text{op}(X)$.

**Lemma 2.2.**

1. Let $p \in S_G(X)$, and let $A \subseteq X$ be open. If $A \notin p$, then there is some $V \in \mathcal{N}_G$ with $VA \notin p$.

2. Let $A \subseteq X$ be open, and let $B_1, \ldots, B_k \subseteq A$ be open with $B_1 \cup \cdots \cup B_k$ dense in $A$. If $p \in S_G(X)$ and $A \in p$, then $B_i \in p$ for some $i \leq k$. 

Proof.

(1) As $A \notin p$, find $B_1, \ldots, B_n \in p$ and $U \in N_G$ with $UA \cap UB_1 \cap \cdots \cap UB_n = \emptyset$. Let $V \in N_G$ with $VV \subseteq U$. Then $V(VA) \cap VB_1 \cap \cdots \cap VB_n = \emptyset$.

(2) Towards a contradiction, assume $B_i \notin p$ for each $i \leq k$. For each $i \leq k$, find $B_i^1, \ldots, B_i^{n_i} \in p$ and a $U \in N_G$ so that $UB_i \cap UB_i^1 \cap \cdots \cap UB_i^{n_i} = \emptyset$. We can take the same $U \in N_G$ for each $i \leq k$ by intersecting. Let $C = \bigcap_{i \leq k} \bigcap_{j \leq n_i} UB_i^j$. Then since $A \in p$, we have $UA \cap C \neq \emptyset$. Let $ga \in UA \cap C$, where $g \in U$ and $a \in A$. Since $UA \cap C$ is open, there is open $A' \subseteq A$ with $gA' \subseteq UA \cap C$. As $B_1 \cup \cdots \cup B_k$ is dense in $A$, there is some $i \leq k$ and some $b \in B_i$ with $gb \in UA \cap C$. Since $gb \in UB_i$, this is a contradiction. \qed

Definition 2.3. If $A \in \text{op}(X)$, set $N_A := \{p \in S_G(X) : A \notin p\}$. We endow $S_G(X)$ with the topology whose typical basic open neighborhood is $N_A$ for $A \in \text{op}(X)$.

Proposition 2.4. The topology from Definition 2.3 is compact Hausdorff.

Proof. To show that $S_G(X)$ is Hausdorff, let $p \neq q \in S_G(X)$. Find some $A \in p \setminus q$. As $A \notin q$, find some $V \in N_G$ so that $VA \notin q$. Set $B = \text{int}(X \setminus VA)$. Then $B \notin p$. So $p \in N_B$, $q \in N_VA$, and $N_VA \cap N_B = \emptyset$.

To show that $S_G(X)$ is compact, suppose $C := \{N_{A_i} : i \in I\}$ is a collection of basic open sets without a finite subcover. Then for any $i_1, \ldots, i_k \in I$, we can find $p \in \bigcap_{j \leq k} S_G(X) \setminus N_{A_{i_j}}$, equivalently, with $A_{i_1}, \ldots, A_{i_k} \in p$. But this implies that $\{A_i : i \in I\}$ is a near filter and can be extended to a near ultrafilter $q$. Therefore $C$ is not an open cover. \qed

Definition 2.5. If $p \in S_G(X)$ and $g \in G$, we let $gp \in S_G(X)$ be defined by declaring $A \in gp$ iff $g^{-1}A \in p$ for each $A \in \text{op}(X)$.

Proposition 2.6. The action in Definition 2.5 is continuous.

Proof. First note that for a fixed $g \in G$, the map $p \to gp$ is continuous. So let $p, p_i \in S_G(X)$ and $g_i \in G$ with $p_i \to p$ and $g_i \to 1$. Suppose $A \notin p$. Find $V \in N_G$ with $VA \notin p_i$. So eventually $VA \notin p_i$. Also, as $g_i \to 1$, eventually we have $g_i^{-1} \in V$. Whenever $g_i^{-1}A \subseteq VA$, we must have $g_i^{-1}A \notin p_i$. So eventually $A \notin g_i p_i$. \qed

Up until now, no assumptions on $G$ and $X$ have been needed. In fact, we did not even need $X$ to be compact to construct $S_G(X)$. We now begin adding extra assumptions to $G$ and $X$ to obtain stronger conclusions about $S_G(X)$.

Proposition 2.7. Suppose $X$ is a minimal $G$-flow. Then so is $S_G(X)$.

Proof. Let $p \in S_G(X)$, and let $A \in \text{op}(X)$ with $N_A \neq \emptyset$. Find some $V \in N_G$ with $N_VA \neq \emptyset$. Then $B := \text{int}(X \setminus VA) \neq \emptyset$. As $X$ is minimal, find $g_1, \ldots, g_k$ with $X = \bigcup_{i \leq k} g_i B$. For some $i \leq k$, we must have $g_i B \in p$. Then $B \in g_i^{-1} p$, so we must have $A \notin g_i^{-1} p$, and the orbit of $p$ is dense as desired. \qed

Before proving Theorem 1.1, we need a sufficient criterion for when $S_G(X)$ is nonmetrizable.

Proposition 2.8. Suppose there are $\{A_n : n < \omega\} \subseteq \text{op}(X)$ and $V \in N_G$ so that the collection $\{VA_n : n < \omega\}$ is pairwise disjoint. Then $S_G(X)$ is nonmetrizable.
Proof. If $S \subseteq \omega$, let $A_S = \bigcup_{n \in S} A_n$ and let $Y = \{p \in S_G(X) : A_\omega \in p\}$. Then $Y \subseteq S_G(X)$ is a closed subspace. To show that $S_G(X)$ is nonmetrizable, we will exhibit a continuous surjection $\pi : Y \rightarrow \beta\omega$. First note that if $S \subseteq \omega$, then $VA_S \cap VA_\omega \setminus S = \emptyset$. Therefore, if $p \in Y$, $p$ contains exactly one of $A_S$ or $A_\omega \setminus S$ for each $S \subseteq \omega$. We let $\pi : Y \rightarrow \beta\omega$ be defined so that for $S \subseteq \omega$, $S \in \pi(p)$ iff $A_S \in p$. It is immediate that $\pi$ is continuous. To see that $\pi$ is surjective, let $q \in \beta\omega$. Then $\{A_S : S \in q\}$ is a near filter; any near ultrafilter $p$ extending it is a member of $Y$ with $\pi(p) = q$. □

Proof of Theorem 1.1. We now fix a Polish group $G$ and a minimal $G$-flow $X$ whose orbits are all meager. Then by Theorem 1.2, there is so that $\pi_0 : X \rightarrow \beta\omega$. Therefore, if $p \in Y$, $p$ contains exactly one of $A_S$ or $A_\omega \setminus S$ for each $S \subseteq \omega$. We let $\pi : X \rightarrow \beta\omega$ be defined so that for $S \subseteq \omega$, $S \in \pi(p)$ iff $A_S \in p$. It is immediate that $\pi$ is continuous. To see that $\pi$ is surjective, let $q \in \beta\omega$. Then $\{A_S : S \in q\}$ is a near filter; any near ultrafilter $p$ extending it is a member of $Y$ with $\pi(p) = q$. □

3. Universal highly proximal extensions

Let $\varphi : Y \rightarrow X$ be a $G$-map between minimal flows. There are several equivalent definitions which all say that $\varphi$ is highly proximal. The definition we will use here is that $\varphi$ is highly proximal iff every nonempty open $B \subseteq Y$ contains a fiber $\varphi^{-1}(\{x\})$ for some $x \in X$. Define the fiber image of $B$ to be the set $\varphi_{fib}(B) := \{x \in X : \varphi^{-1}(\{x\}) \subseteq B\}$. Notice that $\varphi_{fib}(B)$ is open, and $\varphi$ is highly proximal iff $\varphi_{fib}(B) \neq \emptyset$ for every nonempty open $B \subseteq Y$. It follows that this definition is the same as the one given in the introduction.

Now let $X$ be a $G$-flow, and form $S_G(X)$. We define the map $\pi_X : S_G(X) \rightarrow X$ as follows. For each $p \in S_G(X)$, there is a unique $x_p \in X$ so that every neighborhood of $x_p$ is in $p$. The existence of such a point is an easy consequence of the compactness of $X$ and the second item of Lemma 2.2. For uniqueness, notice that if $x \neq y \in X$, we can find open $A \ni x$, $B \ni y$, and $U \in N_G$ with $UA \cap UB = \emptyset$. We set $\pi_X(p) = x_p$. This map clearly respects the $G$-action. To check continuity, one can check that if $K \subseteq X$ is closed, then $\pi_X^{-1}(K) = \{p \in S_G(X) : A \in p \text{ for every open } A \supseteq K\}$, and this is a closed condition.

Proposition 3.1. Let $X$ be minimal. Then the map $\pi_X : S_G(X) \rightarrow X$ is highly proximal.

Proof. By Proposition 2.7, $S_G(X)$ is a minimal flow. So let $N_A \subseteq S_G(X)$ be a nonempty basic open neighborhood. This implies that $\text{int}(X \setminus A) \neq \emptyset$. Let $x \in \text{int}(X \setminus A)$. Then there are open $B \ni x$ and $U \in N_G$ with $UB \cap A = \emptyset$. It follows that any $p \in S_G(X)$ containing $B$ cannot contain $A$. In particular, we have $\pi_X^{-1}(\{x\}) \subseteq N_A$. □
Theorem 3.2. Let $X$ be minimal. Then the map $\pi_X : S_G(X) \to X$ is the universal highly proximal extension of $X$.

Proof. Fix a highly proximal extension $\varphi : Y \to X$. For each $y \in Y$, let $F_y := \{\varphi_{fib}(B) : B \ni y \text{ open}\}$. Then $F_y \subseteq \text{op}(X)$ is a filter of open sets, so in particular it is a near filter. We will show that for each $p \in S_G(X)$, there is a unique $y \in Y$ with $F_y \subseteq p$. This will define the map $\psi : S_G(X) \to Y$.

We first show that for each $p \in S_G(X)$, there is at least one such $y \in Y$. To the contrary, suppose for each $y \in Y$, there were $B_y \ni y$ open so that $\varphi_{fib}(B_y) \not\subseteq p$. Find $y_1, \ldots, y_k$ so that $\{B_{y_1}, \ldots, B_{y_k}\}$ is a finite subcover. Let $A_i = \varphi_{fib}(B_{y_i})$. Each $A_i$ is open, so we will reach a contradiction once we show that $\bigcup_{i \leq k} A_i$ is dense. Let $A \subseteq X$ be open. Then $C := B_{y_i} \cap \varphi^{-1}(A) \neq \emptyset$ for some $i \leq k$. As $C$ is open, $\varphi_{fib}(C) \neq \emptyset$, and $\varphi_{fib}(C) \subseteq A \cap A_i$.

Now we consider uniqueness. Let $p \in S_G(X)$ and consider $y \neq z \in Y$. Find open $B \ni z$ and $C \ni z$ and some $V \in \mathcal{N}_G$ so that $VB \cap VC = \emptyset$. It follows that $\varphi_{fib}(V) \cap \varphi_{fib}(VC) = \emptyset$. Now notice that $V \varphi_{fib}(B) \subseteq \varphi_{fib}(VB)$, and likewise for $C$. Hence $p$ cannot contain both $F_y$ and $F_z$.

The map $\psi$ clearly respects the $G$-action and satisfies $\pi_X = \varphi \circ \psi$. To show continuity, let $K \subseteq Y$ be closed. Let $F_K := \{\varphi_{fib}(B) : B \ni K \text{ open}\}$. We will show that $\psi(p) \in K$ iff $F_K \subseteq p$. From this it follows that $\psi^{-1}(K)$ is closed. One direction is clear. For the other, suppose $\psi(p) = y \not\in K$. Find open sets $B \ni y$, $C \ni K$, and $V \in \mathcal{N}_G$ with $VB \cap VC = \emptyset$. As in the proof of uniqueness, $p$ cannot contain both $F_y$ and $F_K$. \hfill \Box

By combining the main results of the previous two sections, we obtain the following.

Corollary 3.3. Let $G$ be a Polish group, and let $X$ be a minimal, metrizable $G$-flow with all orbits meager. Then the universal highly proximal extension of $X$ is nonmetrizable.

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