We construct an infinite dimensional real analytic manifold structure for the space of real analytic mappings from a compact manifold to a locally convex manifold. Here a map is real analytic if it extends to a holomorphic map on some neighbourhood of the complexification of its domain. As is well known the construction turns the group of real analytic diffeomorphisms into a smooth locally convex Lie group. In the inequivalent “convenient setting of calculus” the real analytic diffeomorphisms even form a real analytic Lie group. However, we prove that the Lie group structure on the group of real analytic diffeomorphisms is in general not real analytic in our sense.

Keywords: real analytic, manifold of mappings, infinite-dimensional Lie group, diffeomorphism group, Silva space

MSC2010: 58D15 (primary); 58D05, 58B10, 26E05 (secondary)

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Introduction and statement of results

A classical result by Leslie states that the group of analytic diffeomorphisms of a compact analytic manifold is a smooth (infinite dimensional) Lie group (see [Les82]). Unfortunately, the proof given in the paper [Les82] contains a gap, as is pointed out in [KM90]. However, in the “convenient setting of analysis” it is possible to achieve an even stronger result (cf. [KM90,KM97]): The group of real analytic diffeomorphisms of a compact real analytic manifold is a regular real analytic (in the sense of convenient analysis) infinite dimensional Lie group.

In both approaches, the Lie group of real analytic diffeomorphisms has been modelled on Silva spaces. In the context of Silva spaces the concept of \(C^r\)-maps between locally convex spaces known as Keller’s \(C^r\)-theory (see [Glö02b] for a streamlined exposition) which is adopted in [Les82] and the “convenient setting of analysis” are equivalent. Hence, the result in [KM90] subsumes the earlier (but flawed) result in [Les82].

In the present paper we consider again the Lie group of real analytic diffeomorphisms on a compact real analytic manifold. The concept of real analyticity adopted in this work is based on an idea of Milnor. A map \(f: E \supseteq U \to F\) between real locally convex spaces will be called real analytic if it extends to a holomorphic map (i.e. a Keller \(C^\infty\)-map) \(\tilde{U} \to F_C\) on an open neighbourhood \(\tilde{U}\) of \(U\) in the complexification \(E_C\) of \(E\).

This notion of real analytic maps was introduced in [Glö02b] and diverges from the real analytic maps in the convenient setting (see [KM90,KM97]). It is known that both concepts coincide in the context of Fréchet spaces. However, for Silva spaces real analyticity in our sense is stronger than “convenient” real analyticity.

In the first part of the present paper, we construct a real analytic manifold structure on the set of real analytic mappings \(C^\omega_R(M,N)\). The line of thought follows the well known construction for smooth structures on spaces of (real) analytic mappings (cf. [KM90]). We obtain the following result.

**Theorem A** Let \(M, N\) be real analytic manifolds. Assume that \(M\) is compact and that \(N\) admits a local addition. Then \(C^\omega_R(M,N)\) is a real analytic manifold. The manifold structure does not depend on the choice of local addition.

Here the real analytic structure means real analyticity in the stronger sense explained above. To spell it out explicitly, this results is not surprising in any respect. In the inequivalent “convenient setting of real analytic maps” a version of Theorem A using the notion of “convenient real analytic” was already known. However, it is not possible to adapt the arguments establishing real analyticity to our setting, as the manifold is no longer modelled on a Silva space. Moreover, the stronger notion of real analyticity we adopt in this paper requires special care if one wants to deal with spaces \(C^\omega_C(M,N)\) where \(N\) is an infinite dimensional locally convex manifold. To obtain Theorem A for infinite dimensional manifolds, we need to consider the complexification for certain types of infinite-dimensional vector bundles. For finite dimensional vector bundles the
concrete constructions seem to be part of the folklore (see for example [KM90 7.1], an infinite dimensional analogue is now recorded in [DCSTJ]).

The manifold structure on $C^\infty(M, N)$ allows us to adapt the well known construction in [KM97 43.] to construct the Lie group $\text{Diff}^\infty(M)$.

**Theorem (Kriegl/Michor 1990)** For a compact real analytic manifold $M$ the smooth structure of $C^\infty(M, M)$ turns the group $\text{Diff}^\infty(M)$ of real analytic diffeomorphisms into a Lie group. It is even a real analytic Lie group in the convenient sense.

To say it once more, real analyticity in the sense adopted in this paper differs from “convenient real analyticity”, thus the construction does not provide a real analytic Lie group in our sense. As the ambient manifold $C^\infty(M, M)$ is a real analytic manifold in our sense the same is true for $\text{Diff}^\infty(M)$. Hence one would suspect that this structure turns $\text{Diff}^\infty(M)$ into a real analytic Lie group in our sense. In the second part of this paper we investigate this question for the group of analytic diffeomorphisms on the circle $S^1$. Contrary to the treatment in the “convenient setting of analysis“, we obtain the following surprising result.

**Theorem B** Let $S^1$ be the unit circle in $\mathbb{R}^2$ with the canonical real analytic manifold structure. Then the group multiplication of the Lie group $\text{Diff}^\infty(S^1)$ is not real analytic.

Thus $\text{Diff}^\infty(S^1)$ is a convenient real analytic Lie group (see [KM90]) but not a real analytic Lie group in our sense. The reason for this surprising behaviour is buried in the construction of the model space $X^\omega(S^1)$ of $\text{Diff}^\infty(S^1)$ and its complexification. Both are Silva spaces, i.e. well behaved inductive limits of Banach spaces. Note that with some care, one can extend the group multiplication of $\text{Diff}^\infty(M)$ to open zero-neighbourhoods of every individual steps of the inductive limit. However, we cannot choose an ascending sequence of zero-neighbourhoods leading to a zero-neighbourhood in the limit on which the multiplication is defined and complex analytic.

The counterexample in the second part of the paper is tailored to the manifold $S^1$. Hence it just indicates that $\text{Diff}^\infty(M)$ will in general not be a real analytic Lie group in our sense. Nevertheless, the construction of the counterexample should carry over to the general setting. The authors believe that a similar analysis will show that for an arbitrary compact real analytic manifold $M$ (except the zero-dimensional ones) the group $\text{Diff}^\infty(M)$ is not a real analytic Lie group.

1. The locally convex manifold structure for spaces of analytic maps

In this section we recall the construction of the manifold structure on spaces of analytic functions. The basic idea is not new and follows the exposition of the construction in the “convenient setting” (see [KM90]). Beyond the Fréchet setting our notion of real analytic maps is inequivalent to the notion in the convenient setting of global
analysis. Thus the arguments establishing analyticity in our sense are new and require the complexification of several (infinite-dimensional) vector bundles.

1.1 Notation We write \( \mathbb{N} := \{1, 2, \ldots\} \), respectively \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). As usual \( \mathbb{K} \) will denote either the field of real numbers \( \mathbb{R} \) or the field of complex numbers \( \mathbb{C} \), respectively. For a normed space \((E, \|\cdot\|)\), \( x \in E \) and \( R > 0 \), we let \( B_E^R(x) \) be the open ball of radius \( R \) around \( x \).

The setting of analytic mappings used in this paper was developed in \cite{Glö02b} building on a notion of complex analytic maps outlined in \cite{BS71b}. For the concrete definition and more information on the differential calculus, locally convex manifolds and Lie groups we refer to Appendix A.

1.2 Definition Let \( N \) be a real analytic manifold modelled on a locally convex space over \( \mathbb{R} \). We call a real analytic map \( \Sigma: TN \supseteq \Omega \to N \) defined on an open neighbourhood \( \Omega \) of the zero-section in \( TN \) a (real analytic) local addition if

(a) \((\pi_{TN}, \Sigma): TN \supseteq \Omega \to N \times N \) induces a \( C_\omega^\mathbb{R} \)-diffeomorphism onto an open neighbourhood of the diagonal in \( N \times N \),

(b) \( \Sigma(0_x) = x, \ \forall x \in N \), where \( 0_x \) is zero-element in the fibre over \( x \).

1.3 Remark (a) For finite dimensional paracompact manifolds \( N \) there always exists a real analytic local addition. It is given by a real analytic Riemannian exponential map \( \exp: TN \supseteq \Omega \to N \) (see \cite{Gra58} and \cite KM90, 7.5]).

(b) Note that also every (possibly infinite dimensional) (real analytic) Lie group admits a real analytic local addition due to the real analytic group structure and the fact that the tangent bundle is trivial (cf. \cite KM97, 42.4]).

Our approach uses complexifications of several (possibly infinite-dimensional) vector bundles on \( M \). The reader is referred to \cite{DGST14} Section 3] for remarks on the notation and details concerning these constructions.

1.4 Let \((F, \pi, M)\) be a real analytic vector bundle whose typical fibre \( E \) is a locally convex vector space, \( M \) is finite dimensional paracompact and \( F \) a locally convex manifold. By \cite{DGST14} Proposition 3.5] (also cf. \cite KM90, 7.1] for the finite dimensional case) the bundle \((F, \pi, M)\) admits a unique bundle complexification \((F_\mathbb{C}, \pi_\mathbb{C}, M_\mathbb{C})\).

For a compact subset \( K \subseteq M \) consider the real vector space of germs along \( K \) of real analytic sections \( \Gamma_\mathbb{R}(F|K) \). The complexification of \( \Gamma_\mathbb{R}(F|K) \) is given as

\[ \Gamma_\mathbb{C}(F|K)_C = \Gamma_\mathbb{C}(F_\mathbb{C}|K) \]

(cf. \cite KM90, 7.2]). As the bundle complexification is unique this construction does not depend on the choice of complexifications.
In the following we identify $\Gamma^\omega_\mathbb{R}(F|K)$ with the corresponding complemented subspace of $\Gamma^\omega_\mathbb{R}(F|K)$ and topologise $\Gamma^\omega_\mathbb{R}(F|K)$ with the subspace topology. If $F$ is a finite dimensional manifold, Lemma [A.16] shows that $\Gamma^\omega_\mathbb{R}(F|K)$ is a Silva space. Since closed subspaces of Silva spaces are Silva spaces by [BPS7] Corollary 8.6.9] $\Gamma^\omega_\mathbb{R}(F|K)$ is a Silva space if $\dim F < \infty$.

1.5 (Canonical charts for $C^\omega_\mathbb{R}(M,N)$) Fix a compact real analytic manifold $M$ and a (possibly infinite dimensional) real analytic manifold $N$. We require that $N$ admits a real analytic local addition $\Sigma: TN \supseteq \Omega \to N$.

Consider $f \in C^\omega_\mathbb{R}(M,N)$ and define the subset

$$U_f := \{ g \in C^\omega_\mathbb{R}(M,N) | (f(x), g(x)) \in (\pi_{TN}, \Sigma)(\Omega), \text{ for all } x \in M \}.$$ 

of $C^\omega_\mathbb{R}(M,N)$ together with a map $\Phi_f: U_f \to \Gamma^\omega_\mathbb{R}(f^*TN)$ given by

$$\Phi_f(\gamma) := (\text{id}_M, (\pi_{TN}, \Sigma)^{-1} \circ (f, \gamma)).$$

In the following, we identify $f^*\Omega = \{(x, X) \in M \times TN \mid X \in T_{f(x)}N \cap \Omega\}$ with an open submanifold of $f^*TN$. The topology on $\Gamma^\omega_\mathbb{R}(f^*TN)$ is the subspace topology of $\Gamma^\omega_\mathbb{R}((f^*TN)_C|M)$. Now $\Gamma^\omega_\mathbb{R}((f^*TN)_C|M)$ is the locally convex inductive limit of steps whose topology is the compact open topology (cf. Lemma [A.14]). We deduce that the topology of $\Gamma^\omega_\mathbb{R}((f^*TN)_C|M)$ is finer than the compact open topology and thus the same holds for $\Gamma^\omega_\mathbb{R}(f^*TN)$. As $M$ is compact, this shows that $\Phi_f(U_f) = \{ \sigma \in \Gamma^\omega_\mathbb{R}(f^*TN)| \sigma(M) \subseteq f^*\Omega \}$ is an open subset of $\Gamma^\omega_\mathbb{R}(f^*TN)$. Define the real analytic map

$$\tau_f: f^*\Omega \to (f \times \text{id}_N)^{-1}(\pi_{TN}, \alpha)(\Omega) \subseteq M \times N, \tau_f(x, X) := (x, \Sigma(X)).$$

Clearly $\tau_f$ is bijective and respect the fibres over $M$. Its inverse is the real analytic map

$$\tau_f^{-1}(y, z) := (y, (\pi_{TN}, \Sigma)^{-1}(f(y), z)).$$

By construction this map takes its image in $f^*\Omega \subseteq f^*TN$ and is continuous with respect to the subspace topology on this space induced by $M \times TN$ on the fibre product. We conclude that $\tau_f$ is a homeomorphism onto its (open) image and thus

$$\Omega_{f,g} := \tau^{-1}_g(\tau_f(f^*\Omega)) \subseteq g^*\Omega$$

is open. Let us now compute for $f, g \in C^\omega_\mathbb{R}(M,N)$ and $\sigma$ in $\Phi_g(U_f \cap U_g)$ a formula for $\Phi_f \circ \Phi_g^{-1}$. Denote by $\pi^*_{TN}: g^*TN \to TN$ the bundle map covering $g$. Then we obtain

$$\begin{align*}
(\Phi_f \circ \Phi_g^{-1})(\sigma) &= (\text{id}_M, (\pi_{TN}, \Sigma)^{-1} \circ (f, \Sigma \circ (\pi^*_{TN} g) \circ \sigma) \\
&= \tau_f^{-1} \circ \tau_g \circ \sigma =: (\tau_f^{-1} \circ \tau_g)_*(\sigma). 
\end{align*}$$

Here $(\tau_f^{-1} \circ \tau_g)_*$ is defined on $[M, \Omega_{f,g}] := \{ \sigma \in \Gamma^\omega_\mathbb{R}(g^*TN)| \sigma(M) \subseteq \Omega_{f,g} \}$ which is an open subset of $\Gamma^\omega_\mathbb{R}(g^*TN)$.

Thus $\Phi_g(U_f \cap U_g) = [M, \Omega_{f,g}]$ is open in the compact open topology and also in the finer topology on $\Gamma^\omega_\mathbb{R}(g^*TN)$.

$^1$As is customary, we denote for a smooth or analytic map $h$ by $h_*$ the map defined by $h_*(\gamma) := h \circ \gamma$ on a suitable open subset of a space of mappings.
We will now prove Theorem A. The manifold structure constructed on \( C^*_(M,N) \) is a manifold under the extended Definition \( A.5 \) of manifolds (i.e. the model space can depend on the chart).

1.6 Theorem Let \( M, N \) be real analytic manifolds such that \( M \) is compact and \( N \) admits a real analytic local addition.

(a) The family \((U_f, \Phi_f)_{f \in C^*_{M,N}}\) is a real analytic atlas for \( C^*_{M,N} \).

(b) The identification topology with respect to the atlas in (a) turns \( C^*_{M,N} \) into a (Hausdorff) real analytic manifold modelled on the spaces \( \Gamma^*_{R}(f^*TN) \) where \( f \) runs through \( C^*_R(M,N) \). If \( N \) is finite dimensional, \( C^*_{M,N} \) is a manifold modelled on Silva spaces.

(c) The manifold structure does not depend on the choice of local addition.

Proof. (a) Clearly \((U_f, \Phi_f)_{f \in C^*_{M,N}}\) is an atlas for \( C^*_{M,N} \). We have to show that the changes of charts are real analytic. To this end consider \( f, g \in C^*_R(M,N) \) with \( U_f \cap U_g \neq \emptyset \) and fix \( \sigma \in \Phi_g(U_f \cap U_g) \). We will now construct a complex analytic map on an open \( \sigma \)-neighbourhood in the complexification \( \Gamma^*_{C}(g^*TN)_C \) which extends \( \Phi_f \circ \Phi_g^{-1} \). To achieve this, proceed in several steps.

Step 1: Complexifications of bundles and local data.

We form the pullback bundles \((f^*TN, f^*\pi_{TN}, M)\) and \((g^*TN, g^*\pi_{TN}, M)\). These bundles are locally convex bundles over a finite dimensional paracompact base. Hence both pullback bundles admit unique bundle complexifications (cf. \([4]\)) which we denote by \((f^*TN)_C, (f^*\pi_{TN})_C, M^*\) and \((g^*TN)_C, (g^*\pi_{TN})_C, M^*\), respectively. Note that by passing to open subsets in the complexification, we may choose the same complexification \( M^* \) of \( M \) as base for the complex bundles.

Following \([4]\) we identify the complexifications \( \Gamma^*_{C}(f^*TN) \) and \( \Gamma^*_{C}(g^*TN)_C \) with \( \Gamma^*_{C}(f^*TN)_{\mathbb{C}}(M) \) and \( \Gamma^*_{C}(g^*TN)_{\mathbb{C}}(M) \). Hence \( \sigma \in \Phi_g(U_f \cap U_g) \subseteq \Gamma^*_{C}(g^*TN)_{\mathbb{C}}(M) \) is associated to a germ \( \tilde{\sigma} \in \Gamma^*_{C}(g^*TN)_{\mathbb{C}}(M) \).

Finally, we fix some local data. As \( M \) is compact, we can choose a finite set \( A \) with the following properties:

(i) For \( \alpha \in A \) there is a bundle trivialisations \( \kappa^f_{\alpha} : (f^*\pi_{TN})^{-1}(M_\alpha) \rightarrow M_\alpha \times E_C \) of \( (f^*TN)_{\mathbb{C}} \) which restricts on the real analytic submanifold \( f^*TN \) to a bundle trivialisations of \( f^*TN \).

(ii) For \( \alpha \in A \) there is a bundle trivialisations \( \kappa^g_{\alpha} : (g^*\pi_{TN})^{-1}(M_\alpha) \rightarrow M_\alpha \times E_C \) of \( (g^*TN)_{\mathbb{C}} \) which restricts on the real analytic submanifold \( g^*TN \) to a bundle trivialisations of \( g^*TN \).

(iii) There are compact sets \( K_\alpha \subseteq M_\alpha \cap M \) with \( M = \bigcup_{\alpha \in A} K_\alpha \).
Step 2: A suitable $\tilde{\sigma}$-neighbourhood $O_\sigma$ in $\Gamma_\mathbb{C}^e((g^*TN)_C|M)$. As $M$ is compact and thus finite dimensional, the complexification $M^*$ in Step 1 is paracompact and thus regular as a topological space. From Lemma [A.12] we deduce that thus $(f^*TN)_C$ and $(g^*TN)_C$ are regular as topological spaces. Observe that the image $\sigma(M)$ is a compact subset of $\Omega_{f,g} \subseteq g^*TN$. We thus deduce from [DGS14] Lemma 2.2 (a) that there is an open complex neighborhood $O_1 \subseteq (g^*TN)_C$ of $\sigma(M)$ on which $\tau_{f,g} \circ \tau_\sigma : \Omega_{f,g} \to f^*TN$ extends to a complex analytic map $\phi : (g^*TN)_C \supseteq O_1 \to (f^*TN)_C$.

Without loss of generality we can assume $O_1 \cap f^*TN \subseteq \Omega_{f,g}$. Shrinking $O_1$ further, we may assume that $\phi$ preserves the fibres of the complex bundle. To see this note that $\tau_{g}^{-1} \circ \tau_\sigma$ is fibre preserving. Then we compute in pairs of trivialisations which satisfy (i) and (ii) of Step 1. It is easy to construct a complex analytic extension on a complex neighborhood which preserves the fibres. Finally the identity theorem for real analytic functions shows that we can shrink $O_1$ such that $\phi$ is a fibre preserving map of the complex bundle.

As $\sigma(M)$ is contained in $O_1$, we deduce for each $\alpha \in \Lambda$ from (iii) in Step 1 that $\sigma(K_\alpha)$ is contained in the open set

$$U_\alpha := \phi^{-1}(\Gamma_\mathbb{C}^e((f^*\pi TN)_C^{-1}(M_\alpha)) \cap (g^*\pi TN)_C^{-1}(M_\alpha)) \cap O_1 \subseteq (g^*TN)_C.$$

Now fix $\alpha \in \Lambda$ and consider the family $(\kappa_\alpha^2(x))_{x \in K_\alpha}$. Recall that $K_\alpha$ is a compact subset of a finite dimensional manifold locally compact manifold. Apply now Wallace theorem [Eng89, 3.2.10] to obtain a finite family of compact sets $(K_{\alpha,k})_{1 \leq k \leq n_\alpha}$ which satisfy the following properties.

- For $1 \leq k \leq n_\alpha$ there are open subsets $O_{\alpha,k,1} \subseteq M_\alpha$ and $O_{\alpha,k,2} \subseteq E_C$ such that $\kappa_\alpha^2(\sigma(K_{\alpha,k})) \subseteq O_{\alpha,k,1} \times O_{\alpha,k,2}$. This entails $K_{\alpha,k} \subseteq O_{\alpha,k,1} \subseteq M_\alpha$.
- The open set $O_{\alpha,k} := (\kappa_\alpha^2)^{-1}(O_{\alpha,k,1} \times O_{\alpha,k,2})$ is contained in $U_\alpha$.
- $K_\alpha \subseteq \bigcup_{1 \leq k \leq n_\alpha} K_{\alpha,k}$.

Repeat this construction for each $\alpha \in \Lambda$. Then we can replace $(K_\alpha)_{\alpha \in \Lambda}$ with a finite family of compact subsets such that $n_\alpha = 1$ for all $\alpha \in \Lambda$ and the above conditions are satisfied. To shorten the notation denote this refinement again by $(K_\alpha)_{\alpha \in \Lambda}$ and write $O_{\alpha,i} := O_{\alpha,1,i}$ for $i \in \{1,2\}$ and $O_{\alpha} := O_{\alpha,1}$.

Observe that by construction $\tilde{\sigma}$ is contained in each of the sets

$$[K_\alpha, O_\alpha] := \{ s \in \Gamma_\mathbb{C}^e((g^*TN)_C|M) | s(K_\alpha) \subseteq O_\alpha \}.$$

The topology on $\Gamma_\mathbb{C}^e((g^*TN)_C|M)$ is finer than the compact open topology (see Definition [A.13] and Lemma [A.14]). Thus the sets $[K_\alpha, O_\alpha]$ are open. Since $A$ is finite, the set $O_\sigma = \bigcap_{\alpha \in \Lambda} [K_\alpha, O_\alpha]$ is an open $\tilde{\sigma}$-neighbourhood. Moreover, each $O_\alpha$ is contained in $O_1$ and thus property (iii) in Step 1 yields $O_\sigma \subseteq [M, O_1] = \{ s \in \Gamma_\mathbb{C}^e((g^*TN)_C|M) \mid s(M) \subseteq O_1 \}$. The topology on $\Gamma_\mathbb{C}^e((g^*TN)_C|M)$ is finer than the compact open topology, whence the set $[M, O_1]$ is open in $\Gamma_\mathbb{C}^e((g^*TN)_C|M)$. 

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Step 3: Embed the spaces of complex sections. With the notation of Lemma \[\text{(A.10)}\] we obtain topological embeddings

\[
\Theta_f : \Gamma_C^\infty((f^*TN)_C|\mathcal{M}) \to \bigoplus_{\alpha \in A} \text{Hol}(K_\alpha \subseteq M_\alpha^C, E_\mathbb{C}), s \mapsto (I_{\kappa^\alpha} \circ \text{res}_{K_\alpha}^M(s))_{\alpha \in A},
\]

\[
\Theta_g : \Gamma_C^\infty((g^*TN)_C|\mathcal{M}) \to \bigoplus_{\alpha \in A} \text{Hol}(K_\alpha \subseteq M_\alpha^C, E_\mathbb{C}), s \mapsto (I_{\kappa^\alpha} \circ \text{res}_{K_\alpha}^M(s))_{\alpha \in A}.
\]

Observe that the topology on \(\text{Hol}(K_\alpha \subseteq M_\alpha, E_\mathbb{C})\) is finer than the compact open topology (Lemma \[\text{(A.15)}\] (a) asserts \(\text{Hol}(K_\alpha \subseteq M_\alpha, E_\mathbb{C}) \cong \Gamma_C^\infty(((f^*TN)_C|\mathcal{M})\) and the topology on the latter space satisfies this property). Hence,

\[
I_{\kappa^\alpha} \circ \text{res}_{K_\alpha}^M([K_\alpha, O_\alpha]) = [K_\alpha, O_\alpha^2] \subseteq \text{Hol}(K_\alpha \subseteq M_\alpha, E_\mathbb{C})
\]

is open. We deduce that \(\Theta_g(O_g)\) is contained in the open box-neighbourhood \(\bigoplus_{\alpha \in A} [K_\alpha, O_\alpha^2] \subseteq \bigoplus_{\alpha \in A} \text{Hol}(K_\alpha \subseteq M_\alpha, E_\mathbb{C})\).

Step 4: A complex analytic extension on \(\mathcal{O}_\sigma\). We define the map

\[
\phi_* : \Gamma_C^\infty((g^*TN)_C|\mathcal{M}) \supseteq [M, O_1] \to \Gamma_C^\infty((f^*TN)_C|\mathcal{M}), s \mapsto \phi \circ s.
\]

Note that the map \(\phi_*\) makes sense, as \(\phi\) is fibre-preserving. Moreover, as \(\mathcal{O}_\sigma \subseteq [M, O_1]\) is a complex open neighborhood of \(\hat{\sigma}\), \(\phi_*\) extends \((\tau_f^{-1} \circ \tau_g)_*\), in an open neighborhood of \(\sigma\) in the complexification.

To see this define maps \(f_\alpha : [K_\alpha, O_\alpha^2] \to \text{Hol}(K_\alpha \subseteq M_\alpha^C, E_\mathbb{C})\) via

\[
f_\alpha(\gamma) := (\text{pr}_2 \circ \kappa^\alpha_\gamma \circ \phi \circ (\kappa^\alpha_\eta)^{-1})_* (\text{id}_{K_\alpha}, \gamma)
\]

By \[\text{Glö04b}\] Proposition 3.3., \(f_\alpha\) is a \(C^\infty\)-map and we obtain a map

\[
\oplus_{\alpha \in A} f_\alpha : \oplus_{\alpha \in A} \text{Hol}(K_\alpha \subseteq M_\alpha^C, E_\mathbb{C}) \supseteq \oplus_{\alpha \in A} [K_\alpha, O_\alpha^2] \to \oplus_{\alpha \in A} \text{Hol}(K_\alpha M_\alpha^C, E_\mathbb{C})
\]

This map satisfies \(\oplus_{\alpha \in A} f_\alpha \circ \Theta_g|_{\mathcal{O}_\sigma} = \Theta_f \circ \phi_*|_{\mathcal{O}_\sigma}\). Since every \(f_\alpha\) is \(C^\infty\), the map \(\oplus_{\alpha \in A} f_\alpha\) is \(C^\infty\) by \[\text{Glö11}\] Proposition 4.7. Recall from Lemma \[\text{(A.10)}\] that \(\Theta_f\) is a linear topological embedding with closed image. Hence \(\phi_*|_{\mathcal{O}_\sigma} = (\tau_f^{-1} \circ \tau_g)_*\) implies that \(\phi_*|_{\mathcal{O}_\sigma}\) is \(C^\infty\).

We summarise now the results from the Steps 1-4. We have seen that the map \((\tau_f^{-1} \circ \tau_g)_*\) extends to a complex analytic map on a neighborhood of each element in its domain. As real analyticity is a local property, this shows that \((\tau_f^{-1} \circ \tau_g)_*\) is real analytic. Hence \(\Phi_f \circ \Phi_g^{-1}\) is a real analytic map.

(b) Endow \(C^\infty_\mathbb{R}(M, N)\) with the identification topology of the atlas constructed in (a). The topological space \(C^\infty_\mathbb{R}(M, N)\) will be a real analytic locally convex manifold if we can show that the identification topology on \(C^\infty_\mathbb{R}(M, N)\) is Hausdorff. To see this, it suffices to prove that for all \(x \in M\) the point evaluations
\( ev_x : C^\infty_R(M, N) \to N, f \mapsto f(x) \) are continuous with respect to the identification topology. By definition we thus have to prove that for all \( f \in C^\infty_R(M, N) \) and \( x \in M \) the composition \( ev_x \circ \Phi_f^{-1} : \Phi_f(U_f) \to N \) is continuous. One easily computes for a section \( \sigma \in \Phi_f(U_f) \) the identity \( ev_x \circ \Phi_f^{-1}(\sigma) = \Sigma \circ \pi_{TN}^{-1}(\sigma(x)) \). Hence, it suffices to observe that the point evaluations on \( \Gamma^\infty_C((f^*TN)_{|M}) \) are continuous (as \( \Gamma^\infty_C(f^*TN) \) is topologised as a subspace). By definition \( \Gamma^\infty_C((f^*TN)_{|M}) \) is the inductive limit of spaces \( \Gamma^\infty_C((f^*TN)_{|W}) \) on which the point evaluations are continuous (see Definition A.13). As point evaluations are linear, an inductive limit argument shows that \( ev_x \) is continuous on the limit. In conclusion \( C^\infty_R(M, N) \) is a Hausdorff topological space.

(c) Let \( \Sigma^\# \) be another real analytic local addition on \( N \). Construct new charts \( \Phi^\#_f \) and maps \( \tau^\#_f \) as in \( \ref{1.5} \) with respect to \( \Sigma \). We have only used that \( (\pi_{TN}, \Sigma) \) restricts to a diffeomorphism on an open neighbourhood \( \Omega \) of the zero section in \( TN \). By definition of a local addition the same holds for \( \Sigma^\# \). Thus we can define \( \tau^\#_f \circ \tau^{-1} \) on an open subset \( \Omega_{f,g}^\# \subseteq g^*(\Omega \cap \Omega^\#) \) (depending both on \( \Sigma \) and \( \Sigma^\# \)). Furthermore, we obtain an identity analogous to \( \ref{1} \) (including now \( \Phi^\#_f \) and \( \tau^\#_f \)). Note that the arguments and constructions in Step 1, 3 and 4 of (a) do not depend on \( \Sigma \) and can thus be copied verbatim. Finally it is easy to see that also the construction in Step 2 of (a) can easily be adapted (as only the choices of compact and open sets have to be changed). Hence analogous arguments as in (a) show that the charts constructed with respect to different local additions are compatible, i.e. the resulting change of charts are real analytic. We conclude that the construction does not depend on the choice of local addition.

Recall that in \( \ref{10} \) a smooth manifold structure for \( C^\infty_R(M, N) \) has been constructed. Moreover, the construction in \( \ref{10} \) Theorem 10.4\] carries over to an infinite-dimensional manifold \( N \) which admits a local addition. The manifold \( C^\infty_R(M, N) \) is modelled on spaces of smooth sections \( \Gamma^\infty_C(f^*TN) \) for \( f \in C^\infty_R(M, N) \) with canonical charts defined analogously to the charts constructed in \( \ref{15} \). We remark that the notion of differentiability adopted in \( \ref{10} \) coincides with the one adopted in this paper. Hence, the set \( C^\infty_R(M, N) \) is a smooth manifold and we can copy the arguments in \( \ref{90} \) p.48f.\] verbatim to obtain the following results:

1.7. (\( \ref{90} \) Theorem 8.3\)] Let \( M, N \) be real analytic finite dimensional manifolds, with \( M \) compact. Then the smooth manifold \( C^\infty_R(M, N) \) with the structure from \( \ref{10} \) is a real analytic manifold. In fact a real analytic atlas is given by \( \Phi^\infty_f : C^\infty_R(M, N) \supseteq U_f \to \Gamma^\infty_C(f^*TN), g \mapsto \text{id}_M, (\pi_{TN}, \Sigma)^{-1}(f, g) \) where \( f \) runs through \( C^\infty_R(M, N) \) and \( U_f \) is defined as in \( \ref{15} \) with respect to the (real analytic) local addition \( \Sigma \). As \( M \) is compact, the model spaces \( \Gamma^\infty_C(f^*TN) \), are endowed with the compact open \( C^\infty_R \)-topology.

\footnote{In fact, one has only to replace the \( \Omega \)-Lemma in the proof of \( \ref{10} \) Theorem 10.4\] by Glöckner’s \( \Omega \)-Lemma \( \ref{04a} \) Theorem F.23].}
1.8 Proposition Let $M$ and $N$ be real analytic manifolds and assume that $M$ is compact. Consider the canonical inclusion $\iota: C^\infty_\mathbb{R}(M, N) \to \mathbb{C}^\infty_\mathbb{R}(M, N)$.

(a) If $N$ is finite dimensional, then $\iota$ is a real analytic map with respect to the real analytic manifold structures of Theorem 1.6 and \[\mathbb{C}^\infty_\mathbb{R}\].

(b) If $N$ is infinite-dimensional and admits a local addition, then $\iota$ is of class $\mathbb{C}^\infty_\mathbb{R}$ with respect to the smooth structures of Theorem 1.6 and \[\mathbb{M}ic80\], §10 on $\mathbb{C}^\infty(M, N)$.

Proof. Computing in canonical charts, we see that for $f \in \mathbb{C}^\infty_\mathbb{R}(M, N)$ the canonical inclusion maps $\text{dom} \Phi_j$ into $\text{dom} \Phi_j^\infty$. Thus it suffices to consider the local representative $\Phi_j^\infty = \iota \circ \Phi_j^{-1}$ which coincides with the restriction of the canonical inclusion $\Lambda: \Gamma_\mathbb{R}^\infty(f^*TN) \to \Gamma_\mathbb{R}^\infty(f^*TN)$. As $\Lambda$ is linear it is thus sufficient to prove that $\Lambda$ is continuous. Denote the typical fibre of the vector bundle $f^*TN$ by $E$. By definition of the compact open $\mathbb{C}^\infty_\mathbb{R}$-topology, the topology on $\Gamma_\mathbb{R}^\infty(f^*TN)$ is initial with respect to the linear mappings

$$
\theta_\psi: \Gamma_\mathbb{R}^\infty(f^*TN) \to C_\mathbb{R}^\infty(M_\psi, E), X \mapsto \text{pr}_2 \circ \psi \circ X|_{M_\psi}
$$

where $\psi$ runs through all (real analytic) bundle trivializations.

Fixing a trivialization $\psi$ we will show that $\theta_\psi^\infty \circ \Lambda$ is continuous. In the following we use standard multiindex notation to denote partial derivatives.

Recall from Definition \ref{0029} that a typical zero-neighbourhood $C_\mathbb{R}^\infty(M_\psi, E)$ is of the form

$$
\Omega_{\kappa, K, n, p} := \left\{ g \in C_\mathbb{R}^\infty(M_\psi, E) \mid \sup_{|\alpha| \leq n} P_{\alpha, \kappa, K, p}(g) = \sup_{|\alpha| \leq n} \sup_{x \in K} (\partial^p (g \circ \kappa^{-1})(x)) < 1 \right\},
$$

where $\kappa: M_\psi \supseteq U_\kappa \to V_\kappa \subseteq \mathbb{R}^k$ is a real analytic manifold chart, $K \subseteq V_\kappa$ is compact, $n \in \mathbb{N}_0$, and $p$ is a continuous seminorm on $E$. Fix $\kappa, K \subseteq V_\kappa$ compact, $n \in \mathbb{N}_0$ and a continuous seminorm $p$. We construct a zero-neighbourhood in $\Gamma_\mathbb{R}^\infty(f^*TN)$ which is mapped by $\theta_\psi^\infty \circ \Lambda$ to $\Omega_{\kappa, K, n, p}$.

The topology on $\Gamma_\mathbb{R}^\infty(f^*TN)$ is the subspace topology induced by $\Gamma_\mathbb{R}^\infty((f^*TN)_{|\cdot}|M)$. Since $\psi$ is a real analytic bundle trivialization, the complex analytic extension $\psi_C$ of $\psi$ yields a bundle trivialization for $(f^*TN)_{|\cdot}$ (cf. \[\mathbb{D}GS14\], Proposition 3.5). Analogously, we can extend $\kappa$ to a complex analytic chart $\kappa_C$ of the complexification $M_\psi^C$. Now by a combination of Lemma \ref{0029} \[(\mathbf{b})\) and \[(\mathbf{a})\) we obtain a continuous linear map

$$
\text{H}: \Gamma_\mathbb{R}^\infty((f^*TN)_{|\cdot}|M) \to \text{Hol}(K \subseteq M_{\psi_C}, E_C), X \mapsto \text{pr}_2 \circ \psi_C \circ \text{res}^M_{K}(X).
$$

Recall that $\text{Hol}(K \subseteq M_{\psi_C}, E_C)$ is the inductive limit of the spaces $\text{Hol}(U_k, E_C)$ (where $(U_k)_{k \in \mathbb{N}}$ is a fundamental neighbourhood of $K$ in $M_{\psi_C}$). Each of the steps carries the compact open topology, which coincides with the compact open $\mathbb{C}^\infty_\mathbb{R}$-topology for complex analytic maps by Lemma \ref{0029} \[(\mathbf{b})\). Hence by definition of the locally convex inductive limit, the set

$$
O_{\kappa, K, n, p} := \left\{ g \in \text{Hol}(K \subseteq M_{\psi_C}, E_C) \mid \sup_{|\alpha| \leq n} P_{\alpha, \kappa, K, p}(g) < 1 \right\}
$$

10
is an open zero-neighbourhood where we chose a seminorm \( p \) on \( E \) which induces the given seminorm \( \rho \) on the real subspace \( E \). (This is possible by [BS71a, Section 2].) Note that the partial derivatives in the definition of \( O_{K,n,p} \) are taken with respect to complex variables. Since \( \psi|_{\text{dom}\psi} = \psi \) and \( \kappa|_{M_0 \cap U_c} = \kappa \), it is easy to see that the open zero-neighbourhood \( H^{-1}(O_{K,n,p}) \cap \Gamma_R(f^*TN) \) is mapped by \( \theta^\psi_\psi \circ \Lambda \) into \( \Omega_{\kappa,K,n,1} \). This concludes the proof.

Keller \( C^\infty \)-maps coincide on Silva spaces with smooth mappings in the convenient sense (cf. [KM90, 1.3]). Thus we can almost copy the proof for the following result.

1.9 Proposition ([KM97, Theorem 43.3]) For a compact real analytic manifold \( M \) the group \( \text{Diff}^\omega(M) \) of all real analytic diffeomorphisms of \( M \) is an open submanifold of \( C^\infty_R(M,M) \).

Composition and inversion in this group are smooth, whence \( \text{Diff}^\omega(M) \) is a smooth Lie group modelled on the Silva space \( \mathfrak{X}^\omega(M) \). Its Lie algebra is the space \( \mathfrak{X}^\omega(M) \) of all real analytic vector fields on \( M \), equipped with the negative of the Lie bracket of vector fields. The associated exponential mapping \( \exp: \mathfrak{X}^\omega(M) \to \text{Diff}^\omega(M) \) is the flow mapping to time 1, and it is smooth.

Proof. Since \( M \) is compact and thus finite dimensional, Proposition 1.8 shows that the canonical inclusion \( \iota: C^\infty_R(M,M) \to C^\omega_R(M,M) \) is real analytic. Hence \( \iota \) is continuous and we have \( \text{Diff}^\omega(M) = \iota^{-1}(\text{Diff}^\infty(M)) \). Now by [Mic80, Theorem 11.11] \( \text{Diff}^\infty(M) \) is an open submanifold of \( C^\infty_R(M,M) \), whence \( \text{Diff}^\omega(M) \) is open in \( C^\infty_R(M,M) \). The rest of the proof can be copied verbatim from [KM97, Theorem 43.4].

1.10 Remark (a) As shown in [KM97, Theorem 43.3] the Lie group \( \text{Diff}^\omega(M) \) is even a real analytic Lie group in the convenient sense. Note that the notion of real analyticity we adopt in this paper is stronger than the notion of convenient real analyticity. Beyond the Fréchet-setting (e.g. for Silva spaces) both notions are inequivalent. We will show in the next section, that the Lie group \( \text{Diff}^\omega(M) \) is not a real analytic Lie group in our sense.

(b) The topology constructed on \( \mathfrak{X}^\omega(M) = \Gamma^\omega_R(\text{id}_M^*TM) \) coincides with the “Van Howe”-topology constructed in [Les82, §4] (this follows from [Les82, Lemma 4.1]). Thus the Lie group \( \text{Diff}^\omega(M) \) is modelled on the same topological vector space as the Lie group constructed in [Les82]. Although the construction in [Les82] of the Lie group structure is flawed, the Lie group structure obtained in Proposition 1.9 is precisely the one described in [Les82].

---

3 The authors were not able to follow the argument given in [KM97, Theorem 43.3] assuring that the identification topology on \( C^\omega_R(M,M) \) is finer than the compact open \( C^\infty_R \)-topology. Hence we chose to replace the argument showing that \( \text{Diff}^\omega(M) \) is an open subset in \( C^\infty_R(M,N) \).
2. The group of real analytic diffeomorphism on the circle is not real analytic

In this section, we will prove Theorem B. In particular this implies that $\text{Diff}^\omega(M)$ is not in general a real analytic Lie group. To this end consider the unit circle $S^1$ in $\mathbb{C} \cong \mathbb{R}^2$ with its canonical real analytic manifold structure. We begin with preparatory considerations concerning $C^\omega_\mathbb{R}(S^1, S^1)$.

2.1 (Real analytic local addition on $S^1$) The manifold $S^1$ carries the structure of a real analytic (one-dimensional) Lie group and hence its tangent bundle is trivial via the following canonical isomorphism of real analytic vector bundles:

$$S^1 \times L(S^1) \to T\mathbb{S}^1 : (z, v) \mapsto (z, z \cdot v).$$

Since the Lie algebra $L(S^1) = T_1S^1 = i\mathbb{R}$ is isomorphic to $\mathbb{R}$, we obtain the following isomorphism:

$$\psi: S^1 \times \mathbb{R} \to T\mathbb{S}^1, (z, r) \mapsto (z, z \cdot ie^r)$$

The space $C^\omega_\mathbb{R}(S^1)$ consists of all analytic sections in the tangent bundle $T\mathbb{S}^1$ and $C^\omega_\mathbb{R}(S^1, \mathbb{R})$ can be viewed as analytic sections in the trivial bundle $S^1 \times \mathbb{R}$. Hence, the spaces of sections are isomorphic as locally convex vector spaces. Note: The same argument also works if $S^1$ is replaced by any other analytic Lie group. For the rest of this section, we identify $T\mathbb{S}^1$ with $S^1 \times \mathbb{R}$ via the given isomorphism.

The set $\Omega := S^1 \times ]-\pi, \pi[$ is an open neighbourhood of the zero-section in $T\mathbb{S}^1 \cong S^1 \times \mathbb{R}$. The manifold $S^1$ admits a canonical real analytic local addition:

$$\Sigma: \Omega \to S^1, (z, r) \mapsto z \cdot e^{ir}.$$ 

In fact, the map

$$(\pi_{\mathbb{S}^1}, \Sigma): \Omega \to \{ (z, w) \in S^1 \times S^1 | z \neq -w \}, (z, r) \mapsto (z, z \cdot e^{ir})$$

is an analytic diffeomorphism with inverse:

$$(\pi_{\mathbb{S}^1}, \Sigma)^{-1}: \{ (z, w) \in S^1 \times S^1 | z \neq -w \} \to \Omega, (z, w) \mapsto (z, \text{arg}\left(\frac{w}{z}\right)),$$

where arg denotes the principal argument in the interval $]-\pi, \pi[$.

As a next step, we consider the analytic manifold $C^\omega_\mathbb{R}(S^1, S^1)$, constructed in Theorem 1.6.

2.2 We want to consider a chart of the manifold $C^\omega_\mathbb{R}(S^1, S^1)$ around the identity. First, we observe that $\text{id}|_{\mathbb{S}^1}^T T\mathbb{S}^1 = T\mathbb{S}^1 \cong S^1 \times \mathbb{R}$. Thus, the canonical chart in $1.5$ around
id$_{\Omega}$ is given by:
\[
\Phi_{id_{\Omega}} : U_{id_{\Omega}} \to V_{id_{\Omega}} \subset C^\omega_{\R}(S^1, \R), \gamma \mapsto \left( z \mapsto \arg \left( \frac{\gamma(z)}{z} \right) \right),
\]
\[
\Phi_{id_{\Omega}}^{-1} : V_{id_{\Omega}} \to U_{id_{\Omega}}, \eta \mapsto \left( z \mapsto z \cdot e^{i\eta(z)} \right).
\]
Using this local chart around id$_{\Omega}$, the composition map looks like
\[
\mu : C^\omega_{\R}(S^1, \R) \times C^\omega_{\R}(S^1, \R) \to C^\omega_{\R}(S^1, \R), (\eta_1, \eta_2) \mapsto \eta_1 \circ E(\eta_2),
\]
where $E(\eta) : S^1 \to S^1, z \mapsto z \cdot e^{i\eta(z)}$. We will now show that the map $\mu$ is not real analytic in any open neighbourhood of $(0, 0)$.

2.3 Viewing $S^1$ as a subset of $\C^\times := \C \setminus \{0\}$, we may consider $\C^\times$ to be a complexification of $S^1$. This allows us to fix a fundamental sequence of neighbourhoods
\[
U_n := \left\{ z \in \C \mid e^{-\frac{1}{n}} < |z| < e^\frac{1}{n} \right\}
\]
of $S^1$ in its complexification. By [Glö04b, 4.2], the complexification of the locally convex space $C^\omega_{\R}(S^1, \R)$ is the Silva space $\text{Hol}(S^1 \subseteq \C^\times, \C)$. Now $\text{Hol}(S^1 \subseteq \C^\times, \C)$ is the locally convex direct limit of the following sequence of complex Banach spaces:
\[
\text{Hol}(S^1 \subseteq \C^\times, \C) = \bigcup_{n \in \N} \text{Hol}_b(U_n, \C)
\]
To shorten the notation we set $E_n^b := \text{Hol}_b(U_n, \C)$ using the fundamental sequence $(U_n)_{n \in \N}$ defined in 2.1 (b).

2.4 Proposition The map $\mu : C^\omega_{\R}(S^1, \R) \times C^\omega_{\R}(S^1, \R) \to C^\omega_{\R}(S^1, \R)$ introduced in 2.2 is not real analytic on any neighbourhood of $(0, 0) \in C^\omega_{\R}(S^1, \R) \times C^\omega_{\R}(S^1, \R)$.

Proof. We argue by contradiction and assume that $\mu$ is real analytic in a neighbourhood of $(0, 0)$. Then by definition of real analyticity, there exists an open 0-neighbourhood $\Omega \subseteq \text{Hol}(S^1 \subseteq \C^\times, \C)$ and a complex analytic mapping $\mu_C : \Omega \times \Omega \to \text{Hol}(S^1 \subseteq \C^\times, \C)$ such that $\mu_C(\eta_1, \eta_2) := \mu(\eta_1, \eta_2)$ whenever $\eta_1, \eta_2 \in \Omega \cap C^\omega_{\R}(S^1, \R)$.

Since the linear map $E_n^b \to \text{Hol}(S^1 \subseteq \C^\times, \C) : f \mapsto f|_{\partial B}$ is continuous, there is a number $R > 0$ such that the closed ball $\overline{B}_R(0)$ is mapped into the open neighbourhood $\Omega \subseteq \text{Hol}(S^1 \subseteq \C^\times, \C)$.

Using this number $R$, we define the following meromorphic function:
\[
f : \C^\times \setminus \{e^R, e^{-R}\} \to \C, z \mapsto \frac{1}{z - e^R} + \frac{1}{z - e^{-R}}
\]
By construction, this function has the properties
\[
f(\overline{z}) = \overline{f(z)} \quad \text{and} \quad f(1/z) = f(z) \quad \text{for all } z.
\]
Combining these two properties, we may conclude that whenever \( z \in S^1 \), we have \( f(z) \in \mathbb{R} \), since
\[
\overline{f(z)} = f(\overline{z}) = f(1/z) = f(z).
\]
The function \( f \) has poles of order 1 at point \( e^R \) and \( e^{-R} \) (and a removable singularity at 0 which is not important for our discussion) and is holomorphic elsewhere.

As a next step, we fix a positive integer \( n \in \mathbb{N} \) such that \( \frac{1}{n} < R \) and find a number \( \delta > 0 \) such that the closed ball \( B^{R\delta}_R(0) \) is mapped into the open neighbourhood \( \Omega \subseteq \text{Hol}(S^1 \subseteq \mathbb{C}^\times, \mathbb{C}) \).

Since the (relatively compact) open set \( U_n \) has a positive distance from all the singularities of the function \( f \), the function \( f \) is bounded on the set \( U_n \). Hence, \( f|_{U_n} \in E^{R\delta}_R(0) \).

As a composition of complex analytic mappings, this function is itself complex analytic, i.e. holomorphic on the open disc \( B^R_R(0) \subseteq \mathbb{C} \).

Now, let \( z \in B^R_R(0) \cap \mathbb{R} = ]-R,R[ \). Then, we can evaluate \( h(z) \) explicitly:
\[
\begin{align*}
 h(z) &= \mu_C(r \cdot f|_{U_n}, z \cdot 1_{U_1})(1) = \mu_C(r \cdot f|_{U_n}, z \cdot 1_{U_1})(1) \\
 &= r \cdot f \circ E(z \cdot 1_{U_1})(1) = r \cdot f \left( e^{iz} \cdot 1_{U_1}(1) \right) \\
 &= r \cdot f \left( e^{iz} \right). 
\end{align*}
\]

Note that for \( |z| < R \), we have that
\[
|e^{iz}| = e^{\text{Re}(iz)} = e^{-i\text{Im}(z)}
\]
and since \( \text{Im}(z) \in ]-R,R[ \), we can conclude that \( e^{iz} \) is not one of the singularities of the function \( f \). This shows that the holomorphic function
\[
g: B^R_R(0) \to \mathbb{C}, z \mapsto r \cdot f \left( e^{iz} \right)
\]
makes sense and coincides with \( h \) for all real arguments \( z \) by (2).

By the Identity Theorem for holomorphic functions, we obtain \( h \equiv g \) and thus the formula (2) holds for all \( z \in B^R_R(0) \).
In particular, this allows us to conclude that $h(it) = r \cdot f(e^{it})$ for all real $t \in ]0, R[$ and hence
\[ \mu_C(r \cdot f|_{U_n}, it \cdot 1_{U_1})(1) = r \cdot f(e^{-t}). \]
Now, we take the limit $t \to R$ on both sides and obtain a contradiction, since the left hand side converges to the perfectly well-defined number
\[ \mu_C(r \cdot f|_{U_n}, iR \cdot 1_{U_1})(1) \]
while the right hand diverges since $e^{Rt}$ is a pole of the function $f$.

We can now deduce the content of Theorem B.

2.5 Theorem The group multiplication $\text{Diff}^\omega(S^1)$ is not real analytic whence the Lie group $(\text{Diff}^\omega(S^1), \circ)$ is not a real analytic Lie group in our sense.

Proof. By 1.9, the group $\text{Diff}^\omega(S^1)$ is an open neighbourhood of $\text{id}_{S^1}$ in the manifold $C^\omega_R(S^1, S^1)$. Pulling back the group multiplication by the canonical chart (2.3), we obtain the mapping $\mu$ from 2.2 on some open neighbourhood of $(0, 0)$ in $\text{Hol}(S^1 \subseteq \mathbb{C}^\times, \mathbb{C}) \times \text{Hol}(S^1 \subseteq \mathbb{C}^\times, \mathbb{C})$. Hence the assertion follows from Proposition 2.4.

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A. Locally convex calculus and spaces of germs of analytic mappings

In this appendix we recall several well known facts concerning calculus in locally convex spaces. Moreover, we discuss topologies on spaces of (germs) of analytic mappings. These results are well known, but it is sometimes difficult to extract the results and their proofs from the literature. Hence we repeat the results needed together with their proofs for the readers convenience.

A.1 Definition Let $r \in \mathbb{N}_0 \cup \{\infty\}$ and $E$, $F$ locally convex $\mathbb{K}$-vector spaces and $U \subseteq E$ open. We say a map $f: U \to F$ is a $C^r_\mathbb{K}$-map if it is continuous and the iterated directional derivatives

$$d^k f(x, y_1, \ldots, y_k) := (D_{y_k} \cdots D_{y_1} f)(x)$$

exist for all $k \in \mathbb{N}_0$ with $k \leq r$ and $y_1, \ldots, y_k \in E$ and $x \in U$, and the mappings $d^k f: U \times E^k \to F$ so obtained are continuous. If $f$ is $C^\infty_\mathbb{R}$, we say that $f$ is smooth. If $f$ is $C^\infty_C$, we say that $f$ is complex analytic or holomorphic and that $f$ is of class $C^\omega_\mathbb{C}$.

A.2 (Complexification of a locally convex space) Let $E$ be a real locally convex topological vector space. We endow the locally convex product $E_\mathbb{C} := E \times E$ with the following operation

$$(x + iy). (u, v) := (xu - yv, xv + yu) \quad \text{for } x, y \in \mathbb{R}, u, v \in E$$

The complex vector space $E_\mathbb{C}$ is called the complexification of $E$. We identify $E$ with the closed real subspace $E \times \{0\}$ of $E_\mathbb{C}$.

A.3 Definition Let $E, F$ be real locally convex spaces and $f: U \to F$ defined on an open subset $U$. We call $f$ real analytic (or $C^\omega_\mathbb{R}$) if $f$ extends to a $C^\infty_\mathbb{C}$-map $\tilde{f}: \tilde{U} \to F_\mathbb{C}$ on an open neighbourhood $\tilde{U}$ of $U$ in the complexification $E_\mathbb{C}$.

For $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$, being of class $C^r_\mathbb{K}$ is a local condition, i.e. if $f|_{U_\alpha}$ is $C^r_\mathbb{K}$ for every member of an open cover $(U_\alpha)_\alpha$ of its domain, then $f$ is $C^r_\mathbb{K}$. (See [Glö02b, pp. 51-52] for the case of $C^\infty_\mathbb{C}$, the other cases are clear by definition.) In addition, the composition of $C^r_\mathbb{K}$-maps (if possible) is again a $C^r_\mathbb{K}$-map (cf. [Glö02b] Propositions 2.7 and 2.9).

A.4 ($C^r_\mathbb{K}$-Manifolds and $C^r_\mathbb{K}$-mappings between them) For $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$, manifolds modelled on a fixed locally convex space can be defined as usual. The model space of a locally convex manifold and the manifold as a topological space will always be

---

Recall from [Dah11] Proposition 1.1.16] that $C^\omega_\mathbb{R}$ functions are locally given by series of continuous homogeneous polynomials (cf. [BS71a, BS71b]). This justifies our abuse of notation.
assumed to be Hausdorff spaces. However, we will not assume that manifolds are second countable or paracompact. Direct products of locally convex manifolds, tangent spaces and tangent bundles as well as $C^r_K$-maps between manifolds may be defined as in the finite dimensional setting.

For $C^r_K$-manifolds $M, N$ we use the notation $C^r_K(M, N)$ for the set of all $C^r_K$-maps from $M$ to $N$. Moreover, we let $\text{Diff}^r_K(M)$ denote the subset of all $C^r_K$-diffeomorphisms in $C^r_K(M, M)$. For $C^\infty_K$-manifolds, we will also write $\text{Hol}(M, N) := C^\infty_K(M, N) := C^\infty_K(M, N)$ for the set of all complex analytic maps from $M$ to $N$.

Furthermore, for $s \in \{\infty, \omega\}$, we define locally convex $C^s_K$-Lie groups as groups with a $C^s_K$-manifold structure turning the group operations into $C^s_K$-maps.

To deal with manifolds of analytic mappings we need to slightly extend the notion of locally convex manifold. This is needed only in Section I where we relax the definition of a manifold as follows.

**A.5 Definition** (Generalized manifolds) Let $M$ be a Hausdorff topological space.

(a) A pair $(U_\kappa, \kappa)$ with $U_\kappa \subseteq M$ open and $\kappa: U_\kappa \to V_\kappa \subseteq E_\kappa$ a homeomorphism onto an open subset of a locally convex space $E_\kappa$ over $K$ is called generalized manifold chart. Note that the model space may change depending on the chart.

(b) For $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ define $C^r_K$-compatibility and $C^r_K$-atlases for generalized manifold charts exactly as in the finite dimensional case. A generalized $C^r_K$-manifold is a Hausdorff topological space with a $C^r_K$-manifold structure, i.e. an equivalence class of $C^r_K$-atlases induced by an atlas of generalized charts.

Now we discuss the standard topologies on function spaces.

**A.6 (The Compact Open $C^\infty_K$-Topology)** Let $M$ be a finite dimensional $C^\infty_K$-manifold of dimension $d \in \mathbb{N}_0$ and let $E$ be any locally convex $K$-vector space. The compact open $C^\infty_K$-topology on the vector space $C^\infty_K(M, E)$ is the locally convex vector spaces topology, given by the family of seminorms

$$P_{\alpha, \phi, K, p}: C^\infty_K(M, E) \to [0, +\infty[, \gamma \mapsto \sup_{x \in K} p(\partial^\alpha(\gamma \circ \phi^{-1})(x)),$$

where $p$ is a continuous seminorm on $E$, $\alpha \in \mathbb{N}_0^d$ is a multi-index, $\phi: U_\phi \to V_\phi$ is a $C^\infty_K$-diffeomorphism of an open subset $U_\phi \subseteq M$ to an open subset $V_\phi \subseteq K^d$, and $K \subseteq V_\phi$ is a compact set.

In the case $K = \mathbb{C}$, the space $C^\infty_K(M, E)$ is endowed just with the compact open topology. However, it is a well-known fact that the compact open topology coincides in this case with the topology from Definition [A.6]. For the reader’s convenience we give a sketch of the proof.

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A.7 Lemma Let $M$ be a finite dimensional complex manifold and let $E$ be a complex locally convex vector space. Then the compact open $C_\infty^\infty$-topology on the space

$$\text{Hol}(M, E) = C_\infty^\infty(M, E),$$

defined in A.6 agrees with the usual compact open topology, which is the topology of uniform convergence on compact subsets.

Proof. Since $M$ is locally compact, we may work in local charts and hence, assume that $M = \Omega \subseteq \mathbb{C}^d$ is an open subset of a finite dimensional vector space $\mathbb{C}^d$. Let $a \in \Omega$ be a point. Then there is a number $R > 0$ such that $K := \overline{B_{R/2}}(a) \subseteq \Omega$. We fix the number $r := R/2$. Let $x \in \overline{B_{r/2}}(a)$ and let $v \in \mathbb{C}^d$ be any vector of norm 1. Then by Cauchy’s integral formula, we may write the derivative of $\gamma \in \text{Hol}(\Omega, E)$ at point $a$ in direction $v$ as

$$d^1\gamma(x, v) = \frac{1}{2\pi i} \int_{|z|=r} \frac{\gamma(x + zv)}{z^2} \, dz.$$

Applying a continuous seminorm $p$ on both sides, we obtain

$$p\left(d^1\gamma(x, v)\right) \leq \frac{1}{2\pi} \cdot 2\pi r \sup_{|z|=r} \frac{p(\gamma(x + zv))}{r^2} \leq \frac{1}{r} \sup_{y \in K} p(\gamma(y)).$$

In particular, if chose $v = e_j$ as a standard basis vector of $\mathbb{C}^d$, we obtain

$$p\left(\frac{\partial \gamma}{\partial x_j}(x)\right) \leq \frac{1}{r} \sup_{y \in K} p(\gamma(y))$$

and since $x \in \overline{B_{r/2}}(a)$ was arbitrary, this implies that uniform convergence on $K$ implies uniform convergence on the open ball $\overline{B_{r/2}}(a)$. Since every compact subset $K \subseteq \Omega$ can be covered by finitely many open balls, we have shown that taking the partial derivative $\frac{\partial}{\partial x_j}$ is continuous with respect to the compact open topology. The case of a derivative with respect to a multi-index follows by induction. \(\square\)

A.8 (The space of bounded holomorphic functions) For a finite dimensional complex manifold $M$ and complex Banach space $E$, the space of bounded holomorphic functions

$$\text{Hol}_b(M, E) := \{ \gamma \in \text{Hol}(M, E) : \gamma \text{ is bounded on } M \},$$

is a Banach space with respect to the supremum norm (cf. [BS71b, Proposition 6.5]).

A.9 (Fundamental sequence) Let $K$ be a compact subset of a finite dimensional $C_\infty^\infty$ manifold $M$. Then there always exists a sequence $U_1 \supseteq U_2 \cdots$ of metrisable open neighbourhoods of $K$ in $M$ such that

(a) For each $n \in \mathbb{N}$, the set $\overline{U_{n+1}}$ is compact in $U_n$. 

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(b) Each open neighbourhood $U$ of $K$ contains one of the sets $U_n, n \in \mathbb{N}$.

(c) Each connected component of each set $U_n$ intersects the compact set $K$ non-trivially.

Such a sequence will be called a fundamental sequence of open neighbourhoods of $K$ in $M$.

Proof. Note that in general, $M$ need not be metrisable. However, $M$ is locally compact. Thus, the compact set $K$ has a relatively compact neighbourhood $U$. The closure $\overline{U}$ is compact and locally metrisable, hence metrisable. Therefore, the compact set $K$ is contained in an open relatively compact and metrisable set $U$.

Using a metric on $U$, we construct a descending sequence $U_1 \supseteq U_2 \supseteq \ldots$ of open neighbourhoods of $K$ in $U$ such that every neighbourhood of $K$ in $M$ contains some $U_n$. We can and will always choose a fundamental sequence such that every connected component of each $U_n$ meets $K$ and $U_{n+1} \subseteq U_n$ for all $n$. Furthermore, the closure $\overline{U_{n+1}}$ is contained in $U$ for all $n \in \mathbb{N}$ whence it is compact (in $M$ and also in $U_n$).

A.10 (Germs of analytic mappings) Let $K$ be a compact subset of a finite dimensional $C^\omega_K$ manifold $M$ and let $E$ be a locally convex vector space over $\mathbb{K}$.

(a) Let $C^\omega_K(K \subseteq M, E)$ be the space of germs of $\mathbb{K}$-analytic maps along $K$ of $C^\omega_K$-functions from open $K$-neighbourhoods in $M$ to $E$.

Again we set $\text{Hol}(K \subseteq M, E) := C^\omega_K(K \subseteq M, E)$. By abuse of notation the germ of $f$ around $K$ will be denoted as $f$.

(b) Let $\mathbb{K} = \mathbb{C}$. Consider the directed set $(\mathcal{N}, \subseteq)$ of open neighbourhoods of $K$ in $M$ (partially) ordered by inclusion. Then for $U, W \in \mathcal{N}$ with $W \subseteq U$ we obtain continuous linear maps

$$\text{res}_W^U : \text{Hol}(U, E) \to \text{Hol}(W, E), f \mapsto f|_W,$$

which form bonding maps for an inductive system in the category of complex locally convex spaces. Passing to the limit of the system, we obtain limit maps

$$\text{res}_W^K : \text{Hol}(W, E) \to \text{Hol}(K \subseteq M, E), \quad W \in \mathcal{N},$$

assigning to each function $f \in \text{Hol}(W, E)$ the associated germ around $K$. We give $\text{Hol}(K \subseteq M, E)$ the (a priori not necessarily Hausdorff) inductive limit topology of the above inductive system. In Lemma A.11 we will see that this topology is indeed Hausdorff.

(c) Let again, $\mathbb{K} = \mathbb{C}$ and fix a fundamental sequence $U_1 \supseteq U_2 \supseteq \ldots$ of $K$ in $M$.

By A.9(c) and the Identity Theorem for analytic mappings the bonding maps $\text{res}_{U_m}^{U_n}$ are injective for $m \geq n$ and so are the limit maps. Now A.9(b) implies

For locally metrisable spaces, metrisability and paracompactness are equivalent.
that the direct limit topology on $\text{Hol}(K \subseteq M, E)$ discussed in part (b) equals the direct limit topology of the sequence $(\text{Hol}(U_n, E), \text{res}_{U_{n+1}}^{U_n})_{n \in \mathbb{N}}$.

For $E$ finite dimensional, the topology on $\text{Hol}(K \subseteq M, E)$ is nicer. By \cite{9}

(a) The bonding maps of the inductive limit factor in the obvious way through

$$\text{Hol}(U_n, E) \to \text{Hol}_b(U_{n+1}, E) \to \text{Hol}(U_{n+1}, E).$$

We conclude that $\text{Hol}(K \subseteq M, E) = \lim_{\to} \text{Hol}_b(U_n, E)$.

The topology on $\text{Hol}(K \subseteq M, \mathbb{C})$ coincides with the inductive topology induced by the system

$$\text{Hol}_b(U_n, \mathbb{C}) \to \text{Hol}(K \subseteq M, \mathbb{C}), g \mapsto \text{res}_K(U_n)(g)$$

for $n \in \mathbb{N}$.

In \cite{9} Theorem 3.4 it was proved that the bonding maps of this system are compact, whence $\text{Hol}(K \subseteq M, \mathbb{C})$ becomes a Silva space by \cite{9} Theorem 3.4. For $k \in \mathbb{N}$ recall from \cite{a} Lemma 3.4 that $\text{Hol}(K \subseteq M, \mathbb{C}^k)$ is a Silva space for all $k \in \mathbb{N}$ as a finite locally convex sums of Silva spaces.

\textbf{A.11 Lemma} (\text{Hol}(K \subseteq M, E) is Hausdorff) Let $K$ be a compact subset of a finite dimensional $C^\infty$ manifold $M$ and let $E$ be a complex locally convex vector space. Then $\text{Hol}(K \subseteq M, E)$ with the topology of \textbf{A.10(b)} is Hausdorff.

\textbf{Proof.} For each $a \in M$, let $V_a$ denote the set of all charts around $a$, i.e. the set $V_a$ consists of all $C^\infty$-diffeomorphisms $\phi: U_\phi \to V_\phi$ with $U_\phi$ open $a$-neighbourhood in $M$ and $V_\phi$ open in $\mathbb{C}^d$ with $d = \dim M$. Denote by $\mathcal{N}$ the family of all open $K$-neighborhoods in $M$. For each $W \in \mathcal{N}$, Lemma A.7 yields a continuous linear map

$$\Psi_W: \text{Hol}(W, E) \to \prod_{a \in K} \prod_{\phi \in V_a} \prod_{\alpha \in \mathbb{N}_0^d} E, \gamma \mapsto (\partial^{\alpha}(\gamma \circ \phi^{-1})(\phi(a))).$$

By construction, for $W, U \in \mathcal{N}$ with $W \subseteq U$ we have $\Psi_W \circ \text{res}_W^U = \Psi_U$. Hence on the locally convex limit $\text{Hol}(K \subseteq M, E)$ these maps induces a continuous linear map

$$\Psi: \text{Hol}(K \subseteq M, E) \to \prod_{a \in K} \prod_{\phi \in V_a} \prod_{\alpha \in \mathbb{N}_0^d} E.$$

This map is injective by the Identity Theorem for complex analytic maps. Hence, we obtain an injective continuous map from $\text{Hol}(K \subseteq M, E)$ into a Hausdorff space, implying the Hausdorff property of the space of germs.

We will now study sections of locally convex vector bundles. Our goal is now to topologise the spaces of germs of analytic sections around compact subsets.

We start with a small lemma about infinite dimensional vector bundles:

\footnote{Recall that a Silva space is defined as the inductive limit of a sequence of Banach spaces such that the bonding maps are compact.}
(Regularity of total spaces of bundles) Let \((F, \pi, M)\) be a topological vector bundle, i.e. a vector bundle whose typical fibre \(E\) is a topological convex space. The total space \(F\) is regular as a topological space if and only if the topological space \(M\) is regular.

Proof. The space \(M\) can be embedded in the total space \(F\) via the zero-section. Hence, \(M\) is regular if \(F\) is regular. To show the converse implication, we assume that \(M\) is regular.

Recall from general topology that \(F\) is regular if every open neighbourhood of every point contains a closed neighbourhood of that point. To check this criterion fix \(a \in F\) together with an open \(a\)-neighbourhood \(\Omega \subseteq F\).

Choose a trivialisation \(\kappa: \pi^{-1}(M_a) \to M_a \times E\) of the bundle with \(a \in \pi^{-1}(M_a)\). The set \(\kappa(\Omega \cap \pi^{-1}(M_a))\) is an open neighbourhood of \(\kappa(a)\) in the product \(M \times E\). Hence, we can find open subsets \(W \subseteq M_a\) and \(V \subseteq E\), respectively, such that \(\kappa(a) \in W \times V \subseteq \kappa(\Omega \cap \pi^{-1}(M_a)).\)

Now \(M_a\) is regular as a subspace of the regular space \(M\) and \(E\) is regular as a topological vector space. Therefore, we may choose \(W\) and \(V\) so small that \(\overline{W} \times \overline{V} \subseteq \kappa(\Omega \cap \pi^{-1}(M_a))\). We obtain an \(a\)-neighbourhood \(A := \kappa^{-1}(\overline{W} \times \overline{V}) \subseteq F\) contained in \(\Omega\). It remains to show that \(A\) is closed in \(F\).

Consider an element \(\overline{a} \in \overline{A}\) of the closure of \(A\). We apply the bundle projection \(\pi\) to this element and obtain \(\pi(\overline{a}) \in \pi(\overline{A}) \subseteq \overline{\pi(A)} = \overline{W} \times \overline{V} \subseteq M_a\). As \(\overline{a} \in \pi^{-1}(M_a)\) we apply \(\kappa\) to achieve \(\kappa(\overline{a}) \in \kappa(\overline{A}) \subseteq \kappa(A) = \overline{W} \times \overline{V} = \overline{W \times V}\). This shows that \(\overline{a} \in \kappa^{-1}(\overline{W} \times \overline{V}) = A\) which shows that \(A\) is closed and concludes the proof. \(\square\)

Let \((F, \pi, M)\) be a \(C^\infty\)-bundle whose typical fibre \(E\) is a complex locally convex vector space and \(M\) is finite dimensional.

(a) Let \(\Gamma^\subset_c(F)\) be the space of holomorphic sections of \((F, \pi, M)\). We topologise \(\Gamma^\subset_c(F)\) with the initial topology with respect to the maps

\[ \theta_\psi: \Gamma^\subset_c(F) \to \text{Hol}(M_\phi, E), X \mapsto pr_2 \circ \psi \circ X|_{M_\phi}. \]

Here \(\psi\) ranges through all bundle trivialisations of \(F\). Note that \(\Gamma^\subset_c(F)\) is Hausdorff as the point evaluations are continuous.

Consider a compact subset \(K \subseteq M\).

(b) Let \(\mathcal{N}_K\) be the set of all open \(K\)-neighbourhoods. For \(U \in \mathcal{N}_K\) we define the restricted bundle \((F|U) := \pi^{-1}(U), \pi|_{F|U}\).

(c) We denote by \(\Gamma^\subset_c(F|K)\) the space of germs of sections along \(K\), i.e. germs of sections in \(\Gamma^\subset_c(F|U)\) where \(U\) ranges \(\mathcal{N}_K\).

Topologise \(\Gamma^\subset_c(F|K)\) as the the locally convex inductive limit of the cone \((\Gamma^\subset_c(F|W) \to \Gamma^\subset_c(F|K))_{W \in \mathcal{N}_K}\) (where the limit maps send a section to its germ).
At this point it is not clear whether $\Gamma^c_\mathcal{C}(F|K)$ is Hausdorff. We will see in Lemma A.16 that the spaces $\Gamma^c_\mathcal{C}(F|K)$ are also Hausdorff.

A.14 Lemma Let $(F, \pi, M)$ be a $C^\infty_\mathcal{C}$-bundle whose typical fibre $E$ is a complex locally convex space and $M$ is finite dimensional. The topology of $\Gamma^c_\mathcal{C}(F)$ (cf. Definition A.13) coincides with the compact open topology. Hence a typical subsbasis for the topology is

$$[L, O] := \{ X \in \Gamma^c_\mathcal{C}(F)|X(L) \subseteq O \}$$

where $L$ is a compact subset of $M$ and $O$ is an open subset of $F$.

Proof. We show first that for a compact set $L \subseteq M$ and open subset $O \subseteq F$ the set $[L, O]$ is open in $\Gamma^c_\mathcal{C}(F)$. To this end, we will prove that $[L, O]$ is a neighbourhood for each fixed element $\sigma \in [L, O]$. Since $L \subseteq M$ is compact and $M$ is finite dimensional, there is a finite family of compact sets $K_\alpha \subseteq M$, $1 \leq \alpha \leq m$ such that:

(a) $L = \bigcup_\alpha K_\alpha$,

(b) for each $\alpha$ there is a bundle trivialisation $\psi_\alpha$ with $K_\alpha \subseteq M_{\psi_\alpha}$ and

(c) the compact set $K_\alpha \times \text{pr}_2 \circ \psi_\alpha \circ \sigma(K_\alpha)$ is contained in $M_{\psi_\alpha} \times \text{pr}_2 \circ \psi_\alpha (O \cap \text{dom} \psi_\alpha)$.

Recall that the topology on $\Gamma^c_\mathcal{C}(F)$ is initial with respect to maps $\theta_\psi$ where $\psi$ runs through all bundle trivialisations. Moreover, the space $\theta_\psi$ maps into carries the compact open topology. As we are dealing with sections the following identity holds

$$\Omega_\alpha := [K_\alpha, (\text{pr}_2 \circ \psi_\alpha)^{-1}(\text{pr}_2 \circ \psi_\alpha (O \cap \text{dom} \psi_\alpha))] = \theta_{\psi_\alpha}^{-1}([K_\alpha, \text{pr}_2 \circ \psi_\alpha (O \cap \text{dom} \psi_\alpha)].$$

In particular we observe that $\sigma$ is contained in each open set $\Omega_\alpha$ for $1 \leq \alpha \leq m$. Note that by construction the set $\bigcap_{1 \leq \alpha \leq m} \Omega_\alpha$ is contained in $[L, O]$ which proves that $[L, O]$ is a neighbourhood of $\sigma$.

Conversely, fix a section $\tau$ together with an arbitrary $\tau$-neighbourhood $\Omega$ in $\Gamma^c_\mathcal{C}(F)$. By definition of the initial topology, $\Omega$ contains an open $\tau$-neighbourhood of the form

$$\bigcap_{1 \leq k \leq n} \theta_{\psi_k}^{-1}([L_k, U_k]),$$

where $L_k \subseteq M_{\psi_k}$ is compact and $U_k \subseteq E$ is open. As above $\theta_{\psi_k}^{-1}([L_k, U_k]) = [L_k, (\text{pr}_2 \circ \psi_k)^{-1}(U_k)]$. Thus the assertion follows. \qed

A.15 Lemma Let $(F, \pi, M)$ be a $C^\infty_\mathcal{C}$-bundle whose typical fibre $E$ is a complex locally convex space and $M$ is finite dimensional. Consider a compact subset $K$ of $M$.

(a) If there is a trivialisation $\psi: F \supseteq \Omega \to M_\psi \times E$ such that $K \subseteq M_\psi$, then the map

$L_\psi: \Gamma^c_\mathcal{C}(F|K) \to C^\infty_\mathcal{C}(K \subseteq M_\psi, E), \gamma \mapsto \text{pr}_2 \circ \psi \circ \gamma$ is an isomorphism of locally convex spaces.

(b) If $L \subseteq K$ is another compact subset, then the canonical restriction map

$\text{res}^K_L: \Gamma^c_\mathcal{C}(F|K) \to \Gamma^c_\mathcal{C}(F|L)$ is continuous linear.
(c) Let \( K_1, K_2 \subseteq M \) be compact subsets with \( K = K_1 \cup K_2 \). Then the map

\[
R := (\text{res}_{K_1}^K, \text{res}_{K_2}^K) : \Gamma_c^\infty(F|K) \to \Gamma_c^\infty(F|K_1) \oplus \Gamma_c^\infty(F|K_2)
\]

is a topological embedding. In particular if \( K_1 \cap K_2 = \emptyset \) the mapping \( R \) is an isomorphism of locally convex spaces.

**Proof.** (a) Let \( W \subseteq M \) be an open neighbourhood of \( K \). From Lemma [A.14] we deduce that the linear bijective map \((\text{pr}_2 \circ \psi)_\ast : \Gamma_c^\infty(F|W) \to \text{Hol}(W, E), \sigma \mapsto \text{pr}_2 \circ \psi \circ \sigma\) is an isomorphism of topological vector spaces. Now the assertion follows from an easy inductive limit argument.

(b) Since the restriction map \( \text{res}_U^W : \text{Hol}(W, E) \to \text{Hol}(U, E) \) is continuous linear for all open \( U \subseteq W \), the assertion follows from an inductive limit argument.

(c) By (b) \( R \) is continuous linear. Clearly \( R \) is also injective. If \( K_1 \) and \( K_2 \) are disjoint, the assertion is trivial.

Thus we will now assume that \( K_1 \cap K_2 \neq \emptyset \). Fix fundamental sequences of neighbourhoods \((U_n^i)_{n \in \mathbb{N}}\) of \( K_i \) for \( i \in \{1, 2\} \). By construction the sets \( U_n := U_n^1 \cup U_n^2 \) form a fundamental sequence for \( K \).

It remains to show that the map \( R \) is a topological embedding. To this end let \( \Omega \) be an open zero-neighbourhood in \( \Gamma_c^\infty(F|K) \). It remains to show that \( R(\Omega) \) is an open zero-neighbourhood in \( R(\Gamma_c^\infty(F|K)) \). By [Wen03, p.109] we may assume that \( \Omega \) is a zero-neighbourhood of the form

\[
\Omega = \bigcup_{n \in \mathbb{N}} \sum_{1 \leq j \leq n} \Omega_j \subseteq \Gamma_c^\infty(F|U_j) \text{ open zero-neighbourhood}
\]

By Lemma [A.14] we may assume that \( \Omega_j = [L_j, O_j] \) with \( L_j \subseteq U_j \) compact and \( O_j \subseteq F \) open. Hence every \( \Omega_j \) gives rise to two open sets \([L_j \cap U_{j+1}^i, O_j] \subseteq \Gamma_c^\infty(F|U_j^i)\) for \( i \in \{1, 2\} \). As we deal with sections one easily obtains the equality

\[
R(\Omega) = \bigcup_{n \in \mathbb{N}} \sum_{1 \leq j \leq n} \left( [L_j \cap U_{j+1}^1, O_j] \times [L_j \cap U_{j+1}^2, O_j] \right) \cap R(\Gamma_c^\infty(F|K))
\]

As \( \Gamma_c^\infty(F|K_1) \oplus \Gamma_c^\infty(F|K_1) \) is the inductive limit of the system \( \Gamma_c^\infty(F|U_n^1) \oplus \Gamma_c^\infty(F|U_n^2) \), we see that \( R(\Omega) \) is open in the subspace topology. Summing up \( R \) is a topological embedding.

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**A.16 Lemma** Let \((F, \pi, M)\) be a \(C^\infty_c\)-bundle whose typical fibre \( E \) is a complex locally convex space and \( M \) is finite dimensional. Fix a compact subset \( K \subseteq M \) and a finite family of compact subsets \((K_\alpha)_{\alpha \in A}\) of \( M \) such that

(a) \( K = \bigcup_\alpha K_\alpha \)

(b) For each \( \alpha \) there is a bundle trivialisation \( \psi_\alpha \) with \( K_\alpha \subseteq M_{\psi_\alpha} \).

---
With the notation of Lemma A.15 we obtain a mapping
\[ \Theta := (I_\psi \circ \text{res}^K_{K_\alpha})_{\alpha \in A}: \Gamma^\omega_{\mathcal{C}}(F|K) \to \bigoplus_{\alpha \in A} \text{Hol}(K_\alpha \subseteq M_{\phi_\alpha}, E) \]
is a linear topological embedding, whose image is a closed vector subspace. Thus \( \Gamma^\omega_{\mathcal{C}}(F|K) \) is Hausdorff. If \( E \) is finite dimensional, the space \( \Gamma^\omega_{\mathcal{C}}(F|K) \) is a Silva space.

**Proof.** Iteratively applying Lemma A.15 (c) we obtain a linear topological embedding
\[ R_A = (\text{res}^K_{K_\alpha})_{\alpha \in A}: \Gamma^\omega_{\mathcal{C}}(F|K) \to \bigoplus_{\alpha \in A} \Gamma^\omega_{\mathcal{C}}(F|K_\alpha). \]

Apply Lemma A.15 (a) to each summand to see that \( \Theta \) is a topological embedding.

Now \( \Gamma^\omega_{\mathcal{C}}(F|K) \) embeds into a product of Hausdorff spaces (cf. Lemma A.11) and thus \( \Gamma^\omega_{\mathcal{C}}(F|K) \) is Hausdorff. Moreover, the image \( \text{im}\Theta \) is homeomorphic to \( \text{im}R_A \subseteq \bigoplus_{\alpha \in A} \Gamma^\omega_{\mathcal{C}}(F|K_\alpha) \).

It is easy to see that the image of \( R_A \) is the space
\[ \left\{ (\gamma_\alpha)_{\alpha \in A} \in \bigoplus_{\alpha \in A} \Gamma^\omega_{\mathcal{C}}(F|K_\alpha) \left| \text{res}^K_{K_\alpha \cap K_\beta}(\gamma_\alpha) = \text{res}^K_{K_\alpha \cap K_\beta}(\gamma_\beta), \text{ if } K_\alpha \cap K_\beta \neq \emptyset \right. \right\}. \]

As each \( \Gamma^\omega_{\mathcal{C}}(F|K_\alpha \cap K_\beta) \) is Hausdorff the image of \( R_A \) is obviously closed.

Let \( E \) now be finite dimensional. Then each \( \text{Hol}(K_\alpha \subseteq M_{\phi_\alpha}, E) \) is a Silva space by A.10 (c). Since any finite locally convex sum of Silva spaces and closed subspaces of Silva spaces are Silva spaces by [BP87 Corollary 8.6.9], the assertion follows.

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