Study of defects of elastic thermal protection by inverse methods of nonlinear acoustics

A V Nenarokomov1,3, O M Alifanov1, K A Nenarokomov1, D M Titov1 and V S Finchenko2

1 Moscow Aviation Institute, Russia
2 Lavochkin Association, Khimki, Russia
3 Author to whom any correspondence should be addressed: aleksey.nenarokomov@mai.ru

Abstract. The purpose of this research was to develop an approved experimental technique for remote (non-contact) diagnostics of structural defects in elastic materials of thermal protection of inflatable re-entry vehicles [1]. The performance of equipment is based on nonlinear interaction of acoustic beam of finite amplitude in analyzed materials with structural defects. Such problems are of great practical importance in the study of properties of composite materials used as non-destructive elastic surface coating in objects of space technology, power engineering etc. The most promising direction in non-destructive diagnostics (research methods) for the elastic composite materials is to use the procedure of inverse problems [2-8]. Corresponded inverse problem is considered as minimization of least-squares residual functional for calculated and measured acoustic pressure at external surface of specimen. The minimization of the residual functional is executed by conjugate gradient method. Determination of spatial distribution of structural acoustic nonlinearity in the analysed specimen allows spotting the defects. In the paper the implementation of experimental technique for remote diagnostics of subsurface flaws in elastic materials is considered. Results of analysis of computational effectiveness of inverse problems algorithms are included in this paper. Experimental validation of the method is based on experimental testing of specimens with a-priori installed artificial defects.

Nomenclature
Latin symbols
\( d \) Thickness of specimen
\( c \) Velocity of sound
\( f(\tau) \) Acoustic pressure measurement
\( g^s \) Vector of gradient minimization method at current iteration
\( J \) Least-square minimized functional
\( J^{(s)}_p \) Gradient of the functional \( J \) at current iteration
\( M \) Number of pressure sensors
\( N_c \) Dimension of approximated function
1. Introduction
Remote (non-contact) and non-destructive diagnostics of specimens of elastic composite thermal protective materials of inflatable re-entry vehicles (figure 1) were executed to estimate nonlinearity coefficient in domain of possible defects of materials based on inverse method of nonlinear acoustics [8]. The value of the nonlinearity coefficient can be used to estimate geometric characteristics of defects [9-10]. The experimental specimens of elastic material were submitted for acoustic tests as a slab about 60x60x6 mm (figure 2) with thickness-to-length ratio not less than 1:10, such ratio of the geometrical characteristics, provide in the course of testing a pressure distribution in the specimen close to one-dimensional. Based on these speculations, the mathematical model of wave propagation in the specimen can be covered the following equations [8]:

\[
\frac{\partial^2 p}{\partial \tau^2} - c^2 \frac{\partial^2 p}{\partial x^2} = \frac{1}{\rho^2} \frac{\partial^2 p}{\partial \tau^2}, \quad x \in (0,d), \tau \in (0,\tau_{\text{max}}] \quad (1)
\]

\[
p(x,0) = p_0, \quad x \in [0,d] \quad (2)
\]

\[
\frac{\partial p}{\partial \tau}(x,0) = 0, \quad x \in [0,d] \quad (3)
\]

\[
\frac{\partial p}{\partial x}(0,\tau) = \rho_0 \frac{\partial^2 \xi}{\partial \tau^2}, \quad \tau \in (0,\tau_{\text{max}}] \quad (4)
\]

\[
\frac{\partial^2 p}{\partial x^2}(d,\tau) = 0, \quad \tau \in (0,\tau_{\text{max}}] \quad (5)
\]
Figure 1. Re-entry vehicle with inflatable structure: a - initial position, b - 1st stage, c – 2nd stage

Figure 2. Specimen of material

In model (1)-(5) the nonlinearity coefficient \( \varepsilon(x) \) is unknown. The physical meaning of this coefficient is spatial distribution of uniformity or non-uniformity of the material under analysis. \( \varepsilon(x) \) is equal 0 for perfect structure of materials.

The results of pressure measurements at the left side (boundary) of the material’s specimen are assigned as necessary additional information to solve an inverse problem

\[
p^{\exp}(0, \tau) = f(\tau), \tau \in (0, \tau_{\text{max}})
\]

Writing down a least-square discrepancy of the calculated and experimental pressure values in points of measuring we receive the residual functional

\[
J(\varepsilon(x)) = \int_{0}^{\tau_{\text{max}}} (p(0, \tau) - f(\tau))^2 d\tau
\]

To construct an iterative algorithm of the inverse problem solving a conjugate gradient method has been used [9]. This inverse problems is conditionally well-posed, therefore special regularization technique is not necessary. The uniform mesh with number of points \( N_\varepsilon \) is formed in
domain \( x \in [0, d] \)
\[ \omega_{\varepsilon} = \{x_k = (k - 1)\Delta x, \ k = 1, N_{\varepsilon} - 1\} , \] and the search function can be approximated as
\[ \varepsilon(x) = \sum_{k=1}^{N_{\varepsilon}} \varepsilon_k \varphi_k(x) \] (8)
where \( \varphi_k(x) \), \( k=1,\ldots, N_{\varepsilon} \) are the basic approximating functions (B-splines, piecewise functions, etc.), the coefficients of approximation \( \varepsilon_k \), \( k=1,\ldots, N_{\varepsilon} \) are unknown. In this case the functional (7) is reduced to the function of \( N_{\varepsilon} \) variables.

\[ J(\varepsilon) = \int_0^{\tau_{\text{max}}} (p(0, \tau) - f(\tau))^2 d\tau \] (9)
Following [9] the gradient of the residual functional can be presented as
\[
\frac{\partial J}{\partial \varepsilon_k} = -\int_0^{\tau_{\text{max}} \int_0^d \psi(x, \tau) \varepsilon \frac{\partial^2 p}{\partial t^2} dx d\tau}
\]
\[ k = 1,\ldots, N_{\varepsilon} \] (10)
where \( \psi(x, \tau) \) is the solution of adjoint problem to linear form of (1) – (5):
\[ \frac{\partial^2 \psi}{\partial \tau^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2 \rho} \varepsilon(x) \frac{\partial^2 \psi}{\partial \tau^2}, \ x \in (0, d), \ t \in [0, \tau_{\text{max}}) \] (11)
\[ \psi(x, \tau_{\text{max}}) = 0, \ x \in [0, d] \] (12)
\[ \frac{\partial \psi}{\partial t}(x, \tau_{\text{max}}) = 0, \ x \in [0, d] \] (13)
\[ c^2 \frac{\partial \psi}{\partial x}(0, \tau) + 2(p(0, \tau) - f(\tau)) = 0, \ \tau \in [0, \tau_{\text{max}}) \] (14)
\[ \frac{\partial^2 \psi}{\partial x^2}(d, \tau) = 0, \ \tau \in [0, \tau_{\text{max}}) \] (15)
The boundary conditions in inverse problems of non-linear acoustics are based on results of measurements and can be Dirichlet or Von Neumann conditions. Type of boundary conditions should satisfy the uniqueness of considered inverse problem. The uniqueness usually demands the minimum number of measurement in one experiment. And so to estimate the coefficient \( \varepsilon(x) \) in (1)-(7) it is necessary to measure the gradient of pressure at the left surface and to provide arbitrary conditions at the right surface, and to measure pressure at the left boundary as additional information to solve inverse problem.

The corresponded scheme of testing is presented in figure 3. In such case we can to estimate the spatial distribution of defect according estimations of \( \varepsilon(x) \).
2. Analysis of influence of experiment conditions to the accuracy of inverse problems solution

The successive application of remote (non-contact) diagnostics of structural defects in elastic materials demands correct development of computational algorithm of experimental data processing. At this stage of research computational experiments become the most effective. In this case all coefficients of mathematical model (1)-(5) are supposed to be known as well as size and position of defect and the direct problem of acoustics (1)-(5) was solving. After that the additional “experimental measured” pressure (6) in the point of sensor’s installation are formed based on the pressure distribution from the solution of (1)-(5) and corresponded inverse problem of estimation of $\varepsilon(x)$ (which at this stage is supposed to be unknown). This inverse problem is formulated as minimization of least-squares residual functional for calculated and measured acoustic pressure at external surface of specimen (7). The minimization of the residual functional is executed by conjugate gradient method [9]. Such approach provides analysis of influence of errors in input data to the accuracy of inverse problem’s solution, as well as search of optimal experimental conditions.

The stochastic errors in “experimental measurements” for simulation of inverse problem’s solving are formed by

$$ f_m(r) = \bar{f}_m(r)(1 + \alpha \delta_f(r)), \quad m = 1, M, $$

where $\bar{f}_m(r)$ is “exact” pressure measurement from the direct problem solution, $\alpha \delta_f(r)$ is produced by pseudorandom number generator with Gaussian distribution, expected value 1 and dispersion 0; $\delta(r)$ is maximum relative error.

The “exact” value of parameter $\varepsilon(x)$ (which was used for direct problem solving) is presented in figure 4. The initial value $\varepsilon_i$ at the iterative procedure of $\varepsilon(x)$ estimating was used 0.5. The thickness of specimens was supposed as 0.005 m. The relative error of inverse problem solution was calculated as
\[ \delta \varepsilon = \| e(x) - \tilde{e}(x) \|_{L^2} / \| \tilde{e}(x) \|_{L^2} \]  

(17)

where \( \varepsilon \) is solution of simulated inverse problem, \( \tilde{e} \) is “exact” value of parameter.

The result of “reconstruction” of unknown parameter \( \varepsilon(x) \) and comparing calculated pressure \( p(0, \tau) \) with “measured” \( f(\tau) \) in the case without errors of measurements \( (\delta_f = 0) \) are presented in figure 4. If \( \varepsilon(x) \) is not equal to 0, therefore some defect is positioned at these points. There are the results of inverse problems solution for three different initial approximations. The results are practically identical and are equal the “exact” value. This fact confirms the uniqueness of this nonlinear inverse problem solution.

Let us consider the influence of the number of parameters of approximation \( N_\varepsilon (8) \) to the accuracy of inverse problem solution for various levels of errors in input data \( \delta_f \), as well as compare the different types of approximations. The results of using various basic functions of approximation (B-splines, piecewise functions (8)) are presented in figures 5 – 6. These results confirm the sufficient computational stability of developed algorithm to the random errors of input data.
Figure 5. Estimated values of $\varepsilon(x), p(0, \tau)$ and $f(\tau)$ for input data with relative error $\delta_f = 3\%$: 1, 2, 3 – approximation by cubic B-splines with numbers of approximating parameters $N_\varepsilon = 5, 9$ and 13 respectively, 4 - approximation by piecewise functions ($N_\varepsilon = 20$), 5 – “exact” value $\tilde{\varepsilon}(x)$ and “measured” pressure $f(\tau)$ respectively.

Than the influence of errors in values of function $\xi_\tau \partial \xi_\tau \partial (4)$, which determines the value of the gradient of acoustics pressure at the external surface of specimen is analyzed. The bias errors with relative level $\delta = 5\%$ were simulated by

$$
\frac{\partial^2 \xi}{\partial \tau^2} (\tau) = \frac{\partial^2 \xi}{\partial \tau^2} (\tau)(1 + \delta) \tag{18}
$$

The result of inverse problems solving in the case of errors in the left boundary condition (18) are presented in figure 7. The presented results demonstrated very small influence of these errors to the shape of the desired function, as well as the influence to the value of it is not small. Taking into account that the developed method is the remote diagnostics of defects, just shape of function $\varepsilon(x)$ is important to estimate the coordinates and sizes of defects.
Figure 6. Estimated values of $\varepsilon(x)$, $P(0, \tau)$ and $f(\tau)$ for input data with relative error $\delta_f = 10\%$: 1, 2, 3 – approximation by cubic B-splines with numbers of approximating parameters $N_\varepsilon$ 5, 9 and 13 respectively, 4 - approximation by piecewise functions ($N_\varepsilon = 20$), 5 – “exact” value $\varepsilon(x)$ and “measured” pressure $f(\tau)$ respectively.

The next very important question for iterative methods of inverse problems solving of conditionally well-posed problems [11] as considered problem (1)-(7) is the criterion of the stopping of iteration. From the pure mathematical point of view the minimization procedure can be executed till zero value of functional (7), but for ill-posed problems the discrepancy criterion [8] based on iterative regularization approach is usually used. In spite of the problem (1-7) is conditionally well-posed, the input data with experimental errors are used for its solving. This is the question of reasonability of using the discrepancy criterion for stopping the iterative procedure for the conditionally well-posed problem. In the case of implementation of this criterion the functional (7) is minimized till condition

$$J(\tilde{\varepsilon}) \leq \delta^2$$

(19)

where $\delta^2 = \int_0^{T_{\text{step}}} \sigma^2(\tau) d\tau$ is integral error of measurement $f(\tau)$, and $\sigma$ is known dispersion of measurements. This approach is not regularization, it just improve the accuracy of calculation.

The values of minimized functional and correspondent values of estimated function for using of various approximations are presented in figures 8-10. These results also confirm the reasonability of...
choice the optimal number of approximation parameters based on (19): the number \( N_\varepsilon \) should be minimal, which provide the achievement of level \( \delta^2 \).

Figure 7. Estimated values of \( \varepsilon(x), \rho(0,\tau) \) and \( f(\tau) \) for errors in the left boundary condition \( \frac{\partial^2 \varepsilon}{\partial \tau^2} \): 1, 2 – with errors \( \delta_\varepsilon = +10\% \) and \( \delta_\varepsilon = 0-10\% \) respectively, 3 – “exact” value \( \varepsilon(x) \) and “measured” pressure \( f(\tau) \) respectively.

Figure 8. Influence of the number of approximation parameters \( N_\varepsilon \) to the value of minimized functional \( J \) as the function of the number of iteration \( s \): 1, 2, 3 – approximation by cubic B-splines with numbers of approximating parameters \( N_\varepsilon = 5, 9 \) and 13 respectively, 4 - approximation by piecewise functions \( (N_\varepsilon = 20) \), 5 – value of integral error of measurement \( \delta^2 \).
Figure 9. Estimated values of $\varepsilon(x)$: 1 – B-splines with $N_\varepsilon = 5$ (minimization without criterion (19)), 2 – B-splines with $N_\varepsilon = 5$ (stopping criterion (19)), 3 – B-splines with $N_\varepsilon = 9$ (minimization without criterion (19)), 4 – B-splines with $N_\varepsilon = 5$ (stopping criterion (19)), 5 – “exact” value $\tilde{\varepsilon}(x)$.

Figure 10. Estimated values of $\varepsilon(x)$: 1 – B-splines with $N_\varepsilon = 13$ (minimization without criterion (19)), 2 – B-splines with $N_\varepsilon = 13$ (stopping criterion (19)), 3 – approximation by piecewise functions $N_\varepsilon = 20$ (minimization without criterion (19)), 4 – approximation by piecewise functions $N_\varepsilon = 20$ (stopping criterion (19)), 5 – “exact” value $\tilde{\varepsilon}(x)$.

The executed study confirm sufficient stability of the developed algorithm to the various disturbance, but of course it cannot be considered as completed study and just confirm the reasonability of computational experiments technique for analysis of effectiveness of inverse problems algorithm.

3. Experimental testing
The purpose of acoustics testing is measurements of:
- transient acoustics pressure \( p^{\exp}(0, \tau), \tau \in (0, \tau_{\max}) \) at the surface of specimen;
- gradient of pressure \( \frac{\partial p}{\partial x}(0, \tau), \tau \in (0, \tau_{\max}) \).

The scheme of experimental facility is presented at figure 11. It includes:
- generators of high frequency signals of special shape 4Mhz Synthesized Function Generator SFG-2004;
- US receiver;
- US transmitter;
- digital oscillograph LECROY WaveAce 204;
- experimental module EMA-1.

Using technology of specimen’s manufacturing provides installation of artificial defects at specimens during manufacturing [10-13]. To approve developed technique it is necessary to provide maximum accuracy of a-priori measurement of coordinates of defects by X-ray facilities. A few specimens were manufactured; parameters and defects of one of them are presented in table 1. The results of pressure measurements \( p^{\exp}(0, \tau), \tau \in (0, \tau_{\max}) \) are presented in figure 12.

![Figure 11. The scheme of experimental facility.](image)

**Table 1.** Parameters and characteristics of defects of specimens

| # of specimen | length \times width \times thickness [mm] | Mass [g] | Density [kg/m³] | Description of defects |
|---------------|------------------------------------------|---------|-----------------|-----------------------|
| 2             | 60.0 \times 60.0 \times 5.65             | 19.72   | 972.87          | Silicon sphere with diameter 1.8 mm is installed at the center of the slab and at depth 1.7 mm from back surface. |
Figure 12. Experimental and calculated pressure at the front side of the specimen as function of time (test #2): 1 - calculated, 2 – experimental.

The result of inverse problems solving are presented in figure 13, where $\varepsilon(x)$ is

$$
\overline{\varepsilon}(x) = \frac{\varepsilon(x) - \varepsilon_0}{\varepsilon_{\text{max}} - \varepsilon_0}
$$

(20)

Figure 13. Value of undimensional coefficient $\overline{\varepsilon}(x)$ (test # 2)

The comparing of experimental and calculated data are presented in figures 12. Maximum discrepancy of calculated pressure from experimental measured are calculated as

$$
\delta_{\text{max}} = \max_j \left| P(0, \tau_j) - P^{\text{exp}}(0, \tau_j) \right|
$$

(21)
where $p^{\text{exp}}(0, \tau_j)$ - acoustic pressure measured at $j$-th time, $p(0, j)$ - acoustic pressure calculated by (1)-(5) based on inverse problem solving. Calculated values are presented in table 2. The integral discrepancy was calculated as

$$\delta^2 = \sum_{j=1}^{N} \left( p(0, \tau_j) - p^{\text{exp}}(0, \tau_j) \right)^2 / N_j$$

(22)

Calculated values are presented in table 3.

**Table 2.** Maximum errors of inverse problems results

| Test # | $\delta_{\text{max}}$ |
|--------|---------------------|
| 2      | 1,7                 |

**Table 3.** Integral errors of inverse problems results

| Test # | $\delta^2$ |
|--------|------------|
| 2      | 0,12       |
| 3      | 0,04       |
| 5      | 0,05       |

**Conclusions**

The executed tests have shown the corresponded increasing of pressure according defects of structure. The experimental facility and corresponded software were developed, allowing with the high accuracy to define the defects in elastic TPS materials. The executed tests have shown the corresponded increasing of pressure according defects of structure. The exception is test #4, where the power is not enough to provide resonance oscillation of high density inclusion. The deviations of the calculated pressure (using $\varepsilon(x)$ estimations) from the pressure measured in the experiments are insignificant showing sufficient accuracy in the estimations of $\varepsilon(x)$ of the analyzed materials.

**Acknowledgments**

This work was supported by the Russian Ministry of Science and Education in the frame of the Project Part of the financial support (grant # 9.3917.2017).

**References**

[1] Bogdanov V, Rodimov R, Pichkhadze K, Tertarashvili A, Finchenco V, Kouznetsovcov V, Inflatable Ballutes to Provide Aerodynamic Shape to the Payload Bus Enabling its Atmosphere Entry. 48-th International Aeronautical Congress, 6-10 Oct., 1997, Turin, Italy, IAF-97-I.6.03

[2] Norton S J 1999 Iterative Inverse Scattering Algorithms: Methods of Computing Frechet Derivatives Journal of Acoustical Society of America 106, 2653

[3] Valdivia N P, Williams E G and Herdic P C 2008 Approximations of Inverse Boundary Element Methods with Partial Measurements of the Pressure Field Journal of Acoustical Society of America 123 109

[4] Astral D E and Gustafsson M G 2003 Optimal Detection of Crack Echo Families in Elastic Solides Journal of Acoustical Society of America 113 2732
[5] Simonetti F and Cawley P 2003 A Guided Wave Technique for the Characterization of Highly Attenuative Viscoelastic Materials Journal of Acoustical Society of America 114 158

[6] Bucaro J and Romano A J 2004 Detection and Localization of inclusions in Plates Using Inversion of Point Actuated Surface Displacement Journal of Acoustical Society of America 115 201

[7] Romano A J, Bucaro J A, Vignola J F and Abraham P B 2007 Detection and Localization of Rib Detachment in Thin Metal and Composite Plates by Inversion of Laser Doppler Vibrometry Scans Journal of Acoustical Society of America 121 2667

[8] Rudenko O V 1993 Non-linear methods in acoustic diagnostics De Haller K. Nonlinear acoustics applied to nondestructive testing Defectoscopia N8 24

[9] Alifanov O M, Artyukhin E A and Rumyantsev S V 1995 Extreme Methods for Solving Ill-Posed Problems with Applications to Inverse Problems (New York: Begell House)

[10] Alifanov O M, Nenarokomov A V, Nenarokomov K A, Terentieva A V, Titov D M and Finchenko V S 2014 Experimental-computational system for noncontact diagnostics of elastic materials Proceedings of 8th International Conference on Inverse Problems in Engineering (May 12–15, 2014, Poland).- Silesian TU Publ., Gliwice-Wroclaw, Poland 13

[11] Alifanov O M, Nenarokomov A V and Nenarokomov K A 2011 Non-Contact Parameter Estimation of Elastic Materials by Inverse Methods of Nonlinear Acoustics Proceedings of 2nd African Conference on Computational Mechanics (January 2011), UoK Publ.

[12] Alifanov O M, Nenarokomov A V and Nenarokomov K A 2011 Non-Contact Testing of Elastic Materials by Inverse Methods of Nonlinear Acoustics Proceedings of 7th International Conference on Inverse Problems in Engineering: Theory and Practice (May 4 – 6, 2011, Orlando, Florida, USA).- University of Central Florida Publ.