REPRESENTATIONS OF THE CYCLICALLY SYMMETRIC
$q$-DEFORMED ALGEBRA $so_q(3)$

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Abstract

An algebra homomorphism $\psi$ from the nonstandard $q$-deformed (cyclically symmetric) algebra $U_q(so_3)$ to the extension $\hat{U}_q(sl_2)$ of the Hopf algebra $U_q(sl_2)$ is constructed. Not all irreducible representations of $U_q(sl_2)$ can be extended to representations of $\hat{U}_q(sl_2)$. Composing the homomorphism $\psi$ with irreducible representations of $\hat{U}_q(sl_2)$ we obtain representations of $U_q(so_3)$. Not all of these representations of $U_q(so_3)$ are irreducible. Reducible representations of $U_q(so_3)$ are decomposed into irreducible components. In this way we obtain all irreducible representations of $U_q(so_3)$ when $q$ is not a root of unity. A part of these representations turns into irreducible representations of the Lie algebra $so_3$ when $q \to 1$. Representations of the other part have no classical analogue. Using the homomorphism $\psi$ it is shown how to construct tensor products of finite dimensional representations of $U_q(so_3)$. Irreducible representations of $U_q(so_3)$ when $q$ is a root of unity are constructed. Part of them are obtained from irreducible representations of $\hat{U}_q(sl_2)$ by means of the homomorphism $\psi$. 
I. INTRODUCTION

It is well-known that the Lie algebras sl₂ and so₃ of the Lie groups SL(2, C) and SO(3), respectively, are isomorphic. But these algebras differ from each other if we consider their embedding to the wider Lie algebra sl₃. There is no automorphism of sl₃ which transfers the embedding sl₂ ⊂ sl₃ to the embedding so₃ ⊂ sl₃. Note that the embedding so₃ ⊂ sl₃ is of great importance for nuclear physics: it is used in spectroscopy.

The definition of the q-analogue of the universal enveloping algebra $U(sl_2)$ is well-known. It is the quantum algebra $U_q(sl_2)$ which is a Hopf algebra. If we wish to have a q-analogue of the universal enveloping algebra $so_q$ such that at $q \to 1$ we obtain the classical embedding $so_3 \subset sl_3$, then the algebra $sl_2$ is not appropriate for this role. By other words, an algebra $U_q(so_3)$ must differ from $U_q(sl_2)$. This algebra $U_q(so_3)$ is well. It is the associative algebra generated by three elements $I_1$, $I_2$ and $I_3$ satisfying the relations

\begin{align}
q^{1/2}I_1I_2 - q^{-1/2}I_2I_1 &= I_3, \\ q^{1/2}I_2I_3 - q^{-1/2}I_3I_2 &= I_1, \\ q^{1/2}I_3I_1 - q^{-1/2}I_1I_3 &= I_2.
\end{align}

Such (and more general) deformation of the commutator $[I_i, I_j] = I_iI_j - I_jI_i$ was defined at 1967 by R. Santilli in the paper [1] (see also [2] and [3]) under studying a generalization of the Lie theory. Afterwards (in 1990), the algebra $U_q(so_3)$ with commutation relations (1)–(3) was determined by D. Fairlie [4]. An algebra which can be reduced to $U_q(so_3)$ was defined in 1986 by M. Odesski [5].

Fairlie [4] gave finite dimensional irreducible representations of the algebra $U_q(so_3)$ which at $q \to 1$ give the well-known finite dimensional irreducible representations of the Lie algebra so₃. These representations are given by integral or half-integral non-negative numbers. Odesski [5] also gave some classes of irreducible representations.

It was shown (see [5–7]) that the algebra $U_q(so_3)$ has irreducible finite dimensional representations which have no classical analogue (that is, which do not admit the limit $q \to 1$). It was not clear why such strange representations of the algebra $U_q(so_3)$ appear. What is their nature? The answer to this question is one of the aims of this paper.

We construct a homomorphism from $U_q(so_3)$ to the algebra $\hat{U}_q(sl_2)$ which is an extension of the well-known quantum algebra $U_q(sl_2)$ (note that there is no homomorphism from $U_q(so_3)$ to $U_q(sl_2)$). Irreducible finite dimensional representations of $U_q(sl_2)$ (but not all) can be extended to finite dimensional representations of the algebra $\hat{U}_q(sl_2)$. Composing a homomorphism $U_q(so_3) \to \hat{U}_q(sl_2)$ with these representations of $\hat{U}_q(sl_2)$, we obtain representations of the algebra $U_q(so_3)$. But some of irreducible representations of $\hat{U}_q(sl_2)$ lead to reducible representations of the algebra $U_q(so_3)$. Decomposing these reducible representations of $U_q(so_3)$ we obtain irreducible representations of this algebra which have no analogue for the Lie algebra so₃. If $q$ is not a root of unity, then in this way we obtain all finite dimensional irreducible representations of $U_q(so_3)$. But there are infinite dimensional irreducible representations of $U_q(so_3)$ which cannot be obtained in this way.

Existence of the homomorphism $U_q(so_3) \to \hat{U}_q(sl_2)$ allows us to define tensor products of representations of the algebra $U_q(so_3)$ which is not a Hopf algebra.

Using the homomorphism $U_q(so_3) \to \hat{U}_q(sl_2)$ and irreducible representations of $\hat{U}_q(sl_2)$ we obtain representations of $U_q(so_3)$ when $q$ is a root of unity. Taking irreducible representations of $U_q(so_3)$ obtained in this way and decomposing reducible representations, we obtain several series
of irreducible representations of $U_q(\mathfrak{so}_3)$. In addition, we construct irreducible representations of $U_q(\mathfrak{so}_3)$ which cannot be derived from $\hat{U}_q(\mathfrak{sl}_2)$.

When $q$ is not a root of unity, then each irreducible (finite or infinite dimensional) representation of $U_q(\mathfrak{so}_3)$ is equivalent to one of the representations constructed below. (We do not give a proof of this assertion in this paper because it would take much place; this proof will be given in a separate paper.) We think that in this paper we constructed also all irreducible representations of $U_q(\mathfrak{so}_3)$ when $q$ is a root of unity. But in this case we have no proof of this assertion. The reason of this is that in this case there are many classes of irreducible representations and a proof of completeness of irreducible representations becomes very tedious.

Let us remark that in [5] there were constructed irreducible finite dimensional representations of $U_q(\mathfrak{so}_3)$ when $q$ is not a root of unity and a part of irreducible infinite dimensional representations. In [6] and [7], there were constructed irreducible representations of $U_q(\mathfrak{so}_3)$ which satisfy the conditions of $\ast$-representations (that is, such that $T(I_j^\ast) = -T(I_j)$, $j = 1, 2$). These $\ast$-representations are a part of irreducible representations of $U_q(\mathfrak{so}_3)$ constructed in this paper. We started to study irreducible representations of $U_q(\mathfrak{so}_3)$ for $q$ a root of unity in [8], where a part of irreducible representations for this case were constructed. Note that in [5–8] there are no relations of representations of $U_q(\mathfrak{so}_3)$ to representations of $\hat{U}_q(\mathfrak{sl}_2)$. This relation makes representations of $U_q(\mathfrak{so}_3)$ clear and understandable.

We suppose that in Sections II and III $q$ is any complex number different from $-1$. In Sections IV–VII, $q$ is not a root of unity. In Sections VIII–X, $q$ is a root of unity.

II. THE ALGEBRAS $U_q(\mathfrak{so}_3)$ AND $\hat{U}_q(\mathfrak{sl}_2)$

The algebra $U_q(\mathfrak{so}_3)$ is obtained by a $q$-deformation of the standard commutation relations

$$[I_1, I_2] = I_3, \quad [I_2, I_3] = I_1, \quad [I_3, I_1] = I_2$$

of the Lie algebra $\mathfrak{so}_3$. So, $U_q(\mathfrak{so}_3)$ is defined as the complex associative algebra with unit element generated by the elements $I_1$, $I_2$, $I_3$ satisfying the defining relations

$$[I_1, I_2]_q := q^{1/2}I_1I_2 - q^{-1/2}I_2I_1 = I_3,$$  

(4)

$$[I_2, I_3]_q := q^{1/2}I_2I_3 - q^{-1/2}I_3I_2 = I_1,$$  

(5)

$$[I_3, I_1]_q := q^{1/2}I_3I_1 - q^{-1/2}I_1I_3 = I_2.$$  

(6)

Unfortunately, a Hopf algebra structure is not known on $U_q(\mathfrak{so}_3)$. However, it can be embedded into the Hopf algebra $U_q(\mathfrak{sl}_3)$ as a Hopf coideal (see [9]). This embedding is very important for the possible application in spectroscopy.

It follows from the relations (4)–(6) that for the algebra $U_q(\mathfrak{so}_3)$ the Poincaré–Birkhoff–Witt theorem is true and this theorem can be formulated as: The elements $I_k^\ast I_m^\ast I_n^\ast$, $k, m, n = 0, 1, 2, \cdots$, form a basis of the linear space $U_q(\mathfrak{so}_3)$. Indeed, by using the relations (4)–(6) any product $I_{j_1}I_{j_2}\cdots I_{j_s}$, $j_1, j_2, \cdots, j_s = 1, 2, 3$, can be reduced to a sum of the elements $I_k^\ast I_m^\ast I_n^\ast$ with complex coefficients.

Note that by (4) the element $I_3$ is not independent: it is determined by the elements $I_1$ and $I_2$. Thus, the algebra $U_q(\mathfrak{so}_3)$ is generated by $I_1$ and $I_2$, but now instead of quadratic relations (4)–(6) we must take the relations

$$I_1I_2^2 - (q + q^{-1})I_2I_1I_2 + I_2^2I_1^2 = -I_1,$$  

(7)
\[ I_2I_1^2 - (q + q^{-1})I_1I_2I_1 + I_1^2I_2 = -I_2, \]  

which are obtained if we substitute the expression (4) for \( I_3 \) into (5) and (6). The equation \( I_3 = q^{1/2}I_1I_2 - q^{-1/2}I_2I_1 \) and the relations (7) and (8) restore the relations (4)–(6).

Remark that the definition of \( U_q(\mathfrak{so}_3) \) by means of relations (7) and (8) was used in [9] for the embedding of \( U_q(\mathfrak{so}_3) \) to \( U_q(\mathfrak{sl}_3) \). The relations (7) and (8) differ from Serre’s relations in the definition of quantum algebras by V. Drinfeld and M. Jimbo by appearance of non-vanishing right hand sides.

The algebra \( U_q(\mathfrak{so}_3) \) is closely related to (but not coincides with) the quantum algebra \( U_q(\mathfrak{sl}_2) \). The last algebra is generated by the elements \( q^H, q^{-H}, E, F \) satisfying the relations

\[ q^Hq^{-H} = q^{-H}q^H = 1, \quad q^H E q^{-H} = qE, \quad q^H F q^{-H} = q^{-1}F, \]  

\[ [E, F] := EF - FE = \frac{q^{2H} - q^{-2H}}{q - q^{-1}}, \]  

Note that \( U_q(\mathfrak{sl}_2) \) is the associative algebra equipped with a Hopf algebra structure (a comultiplication, a counit and an antipode). In particular, the comultiplication \( \Delta \) is determined by the formulas

\[ \Delta(q^\pm H) = q^\pm H \otimes q^\pm H, \quad \Delta(E) = E \otimes q^H + q^{-H} \otimes E, \]  

\[ \Delta(F) = F \otimes q^H + q^{-H} \otimes F. \]  

In order to relate the algebras \( U_q(\mathfrak{so}_3) \) and \( U_q(\mathfrak{sl}_2) \) we need to extend \( U_q(\mathfrak{sl}_2) \) by the elements \((q^kq^H + q^{-k}q^{-H})^{-1}\) in the sense of [10]. We denote by \( \hat{U}_q(\mathfrak{sl}_2) \) the associative algebra with unit element generated by the elements

\[ q^H, \quad q^{-H}, \quad E, \quad F, \quad (q^kq^H + q^{-k}q^{-H})^{-1}, \quad k \in \mathbb{Z}, \]  

satisfying the defining relations (9) and (10) of the algebra \( U_q(\mathfrak{sl}_2) \) and the following natural relations:

\[ (q^kq^H + q^{-k}q^{-H})^{-1}(q^kq^H + q^{-k}q^{-H}) = (q^kq^H + q^{-k}q^{-H})(q^kq^H + q^{-k}q^{-H})^{-1} = 1, \]  

\[ q^{\pm H}(q^kq^H + q^{-k}q^{-H})^{-1} = (q^kq^H + q^{-k}q^{-H})^{-1}q^{\pm H}, \]  

\[ (q^kq^H + q^{-k}q^{-H})^{-1}E = E(q^{k+1}q^H + q^{-k-1}q^{-H})^{-1}, \]  

\[ (q^kq^H + q^{-k}q^{-H})^{-1}F = F(q^{k-1}q^H + q^{k+1}q^{-H})^{-1}. \]  

Note that the algebra \( U_q(\mathfrak{sl}_2) \) has finite dimensional irreducible representations \( T_l \equiv T_l^{(1)}, T_l^{(-1)}, T_l^{(1)}, T_l^{(-1)}, l = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots, \) acting on the vector spaces \( \mathcal{H}_l \) with bases \( |m\rangle, m = -l, -l + 1, \cdots, l \). These representations are given by the formulas

\[ T_l^{(1)}(q^H)|m\rangle = q^m|m\rangle, \quad T_l^{(1)}(E)|m\rangle = [l - m]|m + 1\rangle, \]  

\[ T_l^{(1)}(F)|m\rangle = [l + m]|m - 1\rangle, \]

where a number in square brackets means a \( q \)-number, defined by the formula

\[ [a] = \frac{q^a - q^{-a}}{q - q^{-1}}, \]
and by the formulas
\[ T_i^{(i)}(q^H) | m \rangle = -q^m | m \rangle, \quad T_i^{(-1)}(E) = T_i^{(1)}(E), \quad T_i^{(-1)}(F) = T_i^{(1)}(F), \]
\[ T_i^{(i)}(q^H) | m \rangle = i q^m | m \rangle, \quad T_i^{(i)}(E) = T_i^{(i)}(E), \quad T_i^{(i)}(F) = -T_i^{(i)}(F), \]
\[ T_i^{(-1)}(q^H) | m \rangle = -i q^m | m \rangle, \quad T_i^{(-1)}(E) = T_i^{(1)}(E), \quad T_i^{(-1)}(F) = -T_i^{(1)}(F). \]
The representations \( T_i^{(i)}, T_i^{(-1)}, T_i^{(i)}, T_i^{(-i)}, l = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots \), are pairwise non-equivalent, and any finite dimensional irreducible representation of \( U_q(\mathfrak{sl}_2) \) is equivalent to one of these representations (see, for example, [11], Chapter 3).

Now we wish to extend these representations of \( U_q(\mathfrak{sl}_2) \) to the representations of \( \hat{U}_q(\mathfrak{sl}_2) \) by using the relation
\[ T((q^k q^H + q^{-k} q^{-H})^{-1}) := (q^k T(q^H) + q^{-k} T(q^{-H}))^{-1}. \]
Clearly, only those irreducible representations \( T \) of \( U_q(\mathfrak{sl}_2) \) can be extended to \( \hat{U}_q(\mathfrak{sl}_2) \) for which the operators \( q^k T(q^H) + q^{-k} T(q^{-H}) \) are invertible. From formulas (15)–(19) it is clear that these operators are always invertible for the representations \( T_i^{(i)}, T_i^{(-1)}, l = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots \), and for the representations \( T_i^{(i)}, T_i^{(-i)}, l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots \). For the representations \( T_i^{(i)}, T_i^{(-i)}, l = 0, 1, 2, \cdots \), some of these operators are not invertible since they have zero eigenvalue. Denoting the extended representations by the same symbols, we can formulate the following statement:

**Proposition 1.** The algebra \( \hat{U}_q(\mathfrak{sl}_2) \) has the irreducible finite dimensional representations \( T_i^{(i)}, T_i^{(-1)}, l = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots \), and \( T_i^{(i)}, T_i^{(-i)}, l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots \). Any irreducible finite dimensional representation of \( \hat{U}_q(\mathfrak{sl}_2) \) is equivalent to one of these representations.

**III. THE ALGERBRA HOMOMORPHISM** \( U_q(\mathfrak{so}_3) \to \hat{U}_q(\mathfrak{sl}_2) \)

The aim of this section is to give (in an explicit form) the homomorphism of the algebra \( U_q(\mathfrak{so}_3) \) to \( \hat{U}_q(\mathfrak{sl}_2) \). This homomorphism is described by the following proposition:

**Proposition 2.** There exists a unique algebra homomorphism \( \psi : U_q(\mathfrak{so}_3) \to \hat{U}_q(\mathfrak{sl}_2) \) such that
\[ \psi(I_1) = \frac{i}{q - q^{-1}}(q^H - q^{-H}), \]
\[ \psi(I_2) = (E - F)(q^H + q^{-H})^{-1}, \]
\[ \psi(I_3) = (i q^{H-1/2} E + i q^{-H-1/2} F)(q^H + q^{-H})^{-1}, \]
where \( q^{H+a} := q^H q^a \) for \( a \in \mathbb{C} \).

**Proof.** In order to prove this proposition we have to show that
\[ q^{1/2} \psi(I_1) \psi(I_2) - q^{-1/2} \psi(I_2) \psi(I_1) = \psi(I_3), \]
\[ q^{1/2} \psi(I_2) \psi(I_3) - q^{-1/2} \psi(I_3) \psi(I_2) = \psi(I_1), \]
\[ q^{1/2} \psi(I_3) \psi(I_1) - q^{-1/2} \psi(I_1) \psi(I_3) = \psi(I_2). \]
Let us prove the relation (23). (Other relations are proved similarly.) Substituting the expressions (20)–(22) for \( \psi(I_i), i = 1, 2, 3, \) into (23) we have (after multiplying both sides of equality by \( (q^H + q^{-H}) \) on the right) the relation
\[ q(E - F) E q^H (q q^H + q^{-1} q^{-H})^{-1} + q(E - F) F q^{-H} (q^{-1} q^H + q q^{-H})^{-1} - \]
\[ -qE^2q^H(qq^H + q^{-1}q^{-H})^{-1} - q^{-1}Fq^{-H}(qq^H + q^{-1}q^{-H})^{-1} + 
+ q^{-1}Fq^H(q^{-1}q^H + qq^{-H})^{-1} + qF^2q^{-H}(q^{-1}q^H + qq^{-H})^{-1} = i\frac{q^{2H} - q^{-2H}}{q - q^{-1}}. \]

The formula (23) is true if and only if this relation is correct. We multiply both its sides by 
\((qq^H + q^{-1}q^{-H})(q^{-1}q^H + qq^{-H})\) on the right and obtain the relation in the algebra \(U_q(sl_2)\) (that is, 
without the expressions \((q^kq^H + q^{-k}q^{-H})^{-1}\)). This relation is easily verified by using the defining 
relations (9) and (10) of the algebra \(U_q(sl_2)\). Proposition is proved.

**IV. finite dimensional representations of \(U_q(so_3)\): \(q\) is not a root of unity**

We assume in Sections IV–VII that \(q\) is not a root of unity.

If \(T\) is a representation of the algebra \(\hat{U}_q(sl_2)\) on a linear space \(V\), then the mapping \(R : \ U_q(so_3) \rightarrow V\) defined as the composition \(R = T \circ \psi\), where \(\psi\) is the homomorphism from Proposition 2, is a representation of \(U_q(so_3)\). Let us consider the representations

\[ \begin{align*}
R^{(1)}_i &= T^{(1)}_i \circ \psi, \\
R^{(-1)}_i &= T^{(-1)}_i \circ \psi, \\
R^{(i)}_i &= T^{(i)}_i \circ \psi, \\
R^{(-i)}_i &= T^{(-i)}_i \circ \psi
\end{align*} \]

of \(U_q(so_3)\), where \(T^{(1)}_i, T^{(-1)}_i, T^{(i)}_i, T^{(-i)}_i\) are the irreducible representations of \(\hat{U}_q(sl_2)\) from Proposition 1.

Using formulas for the representations \(T^{(\pm1)}_i\) of \(U_q(sl_2)\) and the expressions (20)–(22) for \(\psi(I_j)\), \(j = 1, 2, 3\), we find that

\[ \begin{align*}
R^{(1)}_i(I_1)|m\rangle &= i|m\rangle|m\rangle, \\
R^{(1)}_i(I_2)|m\rangle &= \frac{1}{q^m + q^{-m}}\{[l - m]|m + 1\rangle - [l + m]|m - 1\rangle\}, \\
R^{(1)}_i(I_3)|m\rangle &= \frac{iq^{1/2}}{q^m + q^{-m}}\{q^m[l - m]|m + 1\rangle + q^{-m}[l + m]|m - 1\rangle\}
\]

for the representation \(R^{(1)}_i\) and

\[ \begin{align*}
R^{(-1)}_i(I_1)|m\rangle &= -i|m\rangle|m\rangle, \\
R^{(-1)}_i(I_2) &= -R^{(1)}_i(I_2), \\
R^{(-1)}_i(I_3) &= R^{(1)}_i(I_3).
\]

Denoting the vectors \(|m\rangle\) by \(|-m\rangle\) for the representations \(R^{(-1)}_i\) we easily find that the matrices of

the representation \(R^{(-1)}_i\) in the basis \(|-m\rangle\), \(m = -l, -l+1, \cdots, l\), coincide with the corresponding

matrices of the representation \(R^{(1)}_i\). Thus, the non-equivalent representations \(T^{(1)}_i\) and \(T^{(-1)}_i\) of

the algebra \(\hat{U}_q(sl_2)\) lead to equivalent representations of \(U_q(so_3)\).

For the representations \(R^{(i)}_i\) and \(R^{(-i)}_i\) we have

\[ \begin{align*}
R^{(i)}_i(I_1)|m\rangle &= -\frac{q^m + q^{-m}}{q - q^{-1}}|m\rangle, \\
R^{(i)}_i(I_2)|m\rangle &= \frac{i[l - m]}{q^m - q^{-m}}|m + 1\rangle + \frac{i[l + m]}{q^m - q^{-m}}|m - 1\rangle, \\
R^{(i)}_i(I_3)|m\rangle &= -\frac{iq^{m+1/2}[l - m]}{q^m - q^{-m}}|m + 1\rangle - \frac{iq^{-m+1/2}[l + m]}{q^m - q^{-m}}|m - 1\rangle
\]

and

\[ \begin{align*}
R^{(-i)}_i(I_1)|m\rangle &= \frac{q^m + q^{-m}}{q - q^{-1}}|m\rangle, \\
R^{(-i)}_i(I_2)|m\rangle &= \frac{i[l - m]}{q^m - q^{-m}}|m + 1\rangle - \frac{i[l + m]}{q^m - q^{-m}}|m - 1\rangle, \\
R^{(-i)}_i(I_3)|m\rangle &= -\frac{iq^{m+1/2}[l - m]}{q^m - q^{-m}}|m + 1\rangle + \frac{iq^{-m+1/2}[l + m]}{q^m - q^{-m}}|m - 1\rangle
\]
\begin{align*}
R_i^{(-i)}(I_2)|m\rangle &= -i\frac{[l-m]}{q^m - q^{-m}}|m+1\rangle - i\frac{[l+m]}{q^m - q^{-m}}|m-1\rangle, \\
R_i^{(-i)}(I_3)|m\rangle &= -i\frac{q^{m+1/2}[l-m]}{q^m - q^{-m}}|m+1\rangle - i\frac{q^{-m+1/2}[l+m]}{q^m - q^{-m}}|m-1\rangle.
\end{align*}

**Proposition 3.** The representations $R_i^{(1)}$ of $U_q(\mathfrak{so}_3)$ are irreducible. The representations $R_i^{(-1)}$ and $R_i^{(-i)}$ are reducible.

**Proof.** To prove the first part of the proposition we first note that since $q$ is not a root of unity, the eigenvalues $i[m]$, $m = -l, -l+1, \cdots, l$, of the operator $R_i^{(1)}(I_1)$ are pairwise different.

Let $V$ be an invariant subspace of the space $\mathcal{H}_i$ of the representation $R_i^{(1)}$, and let $\mathbf{v} \equiv \sum_m \alpha_i|m_i\rangle \in V$, where $|m_i\rangle$ are eigenvectors of $R_i^{(1)}(I_1)$. Then $|m_i\rangle \in V$. We prove this for the case when $\mathbf{v} = \alpha_1|m_1\rangle + \alpha_2|m_2\rangle$. (The case of more number of summands is proved similarly.) We have $R_i^{(1)}(I_1)\mathbf{v} = \alpha_1|m_1\rangle + \alpha_2|m_2\rangle |m_2\rangle$. Since

\[
\mathbf{v} = \alpha_1|m_1\rangle + \alpha_2|m_2\rangle \in V, \quad \mathbf{v}' \equiv i\alpha_1|m_1\rangle|m_1\rangle + i\alpha_2|m_2\rangle|m_2\rangle \in V
\]

one derives that

\[
i[m_1]\mathbf{v} - \mathbf{v}' = i\alpha_2([m_1] - [m_2])|m_2\rangle \in V.
\]

Since $[m_1] \neq [m_2]$, then $|m_2\rangle \in V$ and hence $|m_1\rangle \in V$.

In order to prove that $V = \mathcal{H}_i$ we obtain from the above formulas for $R_i^{(1)}(I_2)|m\rangle$ and $R_i^{(1)}(I_3)|m\rangle$ that

\[
\begin{align*}
\{R_i^{(1)}(I_3) - iq^{m+1/2}R_i^{(1)}(I_2)\}|m\rangle &= iq^{1/2}|m-1\rangle, \\
\{R_i^{(1)}(I_3) + iq^{-m+1/2}R_i^{(1)}(I_2)\}|m\rangle &= iq^{1/2}|m+1\rangle.
\end{align*}
\]

Since $V$ contains at least one basis vector $|m\rangle$, it follows from these relations that $V$ contains the vectors $|m-1\rangle, |m-2\rangle, \cdots, |-l\rangle$ and the vectors $|m+1\rangle, |m+2\rangle, \cdots, |l\rangle$. This means that $V = \mathcal{H}_i$ and the representation $R_i^{(1)}$ is irreducible.

Let us show that the representations $R_i^{(-1)}$ are reducible. The eigenvalues of the operator $R_i^{(-1)}(I_1)$ are

\[
-\frac{q^m + q^{-m}}{q - q^{-1}}, \quad m = -l, -l+1, \cdots, l,
\]

that is, every spectral point has multiplicity 2. Namely, the pairs of vectors $|m\rangle$ and $|-m\rangle$ are of the same eigenvalue. Let $V_1$ be the subspace of the representation space $\mathcal{H}_i$ spanned by the vectors

\[
|\frac{1}{2}\rangle + i|-\frac{1}{2}\rangle, \quad |\frac{3}{2}\rangle - i|-\frac{3}{2}\rangle, \quad |\frac{5}{2}\rangle + i|-\frac{5}{2}\rangle, \quad |\frac{7}{2}\rangle - i|-\frac{7}{2}\rangle, \quad \cdots,
\]

and let $V_2$ be the subspace spanned by the vectors

\[
|\frac{1}{2}\rangle - i|-\frac{1}{2}\rangle, \quad |\frac{3}{2}\rangle + i|-\frac{3}{2}\rangle, \quad |\frac{5}{2}\rangle - i|-\frac{5}{2}\rangle, \quad |\frac{7}{2}\rangle + i|-\frac{7}{2}\rangle, \quad \cdots.
\]

We denote the vectors (24) by

\[
|\frac{1}{2}\rangle', \quad |\frac{3}{2}\rangle', \quad |\frac{5}{2}\rangle', \quad |\frac{7}{2}\rangle', \quad \cdots
\]

and the vectors (25) by

\[
|\frac{1}{2}\rangle'', \quad |\frac{3}{2}\rangle'', \quad |\frac{5}{2}\rangle'', \quad |\frac{7}{2}\rangle'', \quad \cdots.
\]
Then
\[ R^{(i)}_t(I_1)|m\rangle' = -\frac{q^m + q^{-m}}{q - q^{-1}}|m\rangle', \quad R^{(i)}_t(I_1)|m\rangle'' = -\frac{q^m + q^{-m}}{q - q^{-1}}|m\rangle''. \]
We also have
\[
R^{(i)}_t(I_2)|\frac{1}{2}\rangle' = i\frac{[l - \frac{1}{2}]}{q^{1/2} - q^{-1/2}}|\frac{1}{2}\rangle' + i\frac{[l + \frac{1}{2}]}{q^{1/2} - q^{-1/2}}|\frac{3}{2}\rangle' + \frac{[l + \frac{1}{2}]}{q^{1/2} - q^{-1/2}}|\frac{1}{2}\rangle' + \frac{[l - \frac{1}{2}]}{q^{1/2} - q^{-1/2}}|\frac{3}{2}\rangle'.
\]
We derive similarly that
\[
R^{(i)}_t(I_2)|\frac{1}{2}\rangle'' = -\frac{[l + \frac{1}{2}]}{q^{1/2} - q^{-1/2}}|\frac{1}{2}\rangle'' + i\frac{[l - \frac{1}{2}]}{q^{1/2} - q^{-1/2}}|\frac{3}{2}\rangle''.
\]
and that
\[
R^{(i)}_t(I_2)|m\rangle' = i\frac{[l - m]}{q^m - q^{-m}}|m + 1\rangle' + i\frac{[l + m]}{q^m - q^{-m}}|m - 1\rangle', \quad m > \frac{1}{2};
\]
\[
R^{(i)}_t(I_2)|m\rangle'' = i\frac{[l - m]}{q^m - q^{-m}}|m + 1\rangle'' + i\frac{[l + m]}{q^m - q^{-m}}|m - 1\rangle'', \quad m > \frac{1}{2}.
\]
Thus, the subspaces \(V_1\) and \(V_2\) are invariant with respect to the operators \(R^{(i)}_t(I_1)\) and \(R^{(i)}_t(I_2)\). This means that they are invariant with respect to the representation \(R^{(i)}_t\).

It is proved similarly that the subspace \(V_1\) of the space \(\mathcal{H}_t\) of the representation \(R^{(-i)}_t\) spanned by the vectors (24) and the subspace \(V_2\) of \(\mathcal{H}_t\) spanned by the vectors (25) are invariant with respect to the operators \(R^{(-i)}_t(I_1)\) and \(R^{(-i)}_t(I_2)\). That is, the representation \(R^{(-i)}_t\) is also reducible. Proposition is proved.

Let \(R^{(i,+)}_n\) and \(R^{(i,-)}_n\), \(n = l + \frac{1}{2} = \dim V_1 = \dim V_2\), be the representations of \(U_q(sl_2)\) which are restrictions of \(R^{(i)}_t\) to the subspaces \(V_1\) and \(V_2\), respectively. Denoting the vectors (26) of the subspace \(V_1\) by
\[
|1\rangle, \ |2\rangle, \ |3]\rangle, \ |4]\rangle, \cdots, \ |n\rangle \equiv |l + \frac{1}{2}\rangle, \tag{28}
\]
respectively, we have
\[
R^{(i,+)}_n(I_1)|k\rangle = -\frac{q^{k-1/2} + q^{-k+1/2}}{q - q^{-1}}|k\rangle,
\]
\[
R^{(i,+)}_n(I_2)|1\rangle = \frac{n}{q^{1/2} - q^{-1/2}}|1\rangle + i\frac{n - 1}{q^{1/2} - q^{-1/2}}|2\rangle,
\]
\[
R^{(i,+)}_n(I_2)|k\rangle = i\frac{n - k}{q^{k-1/2} - q^{-k+1/2}}|k + 1\rangle + i\frac{n + k - 1}{q^{k-1/2} - q^{-k+1/2}}|k - 1\rangle, \quad k \neq 1.
\]
For the operator \(R^{(i,+)}_n(I_3)\) we have
\[
R^{(i,+)}_n(I_3)|1\rangle = -\frac{n}{q^{1/2} - q^{-1/2}}|1\rangle - i\frac{q(n - 1)}{q^{1/2} - q^{-1/2}}|2\rangle,
\]
\[ R^{(i,+)}_{n}(I_{3})|k\rangle = -i \frac{q^{k[2k]}n - k}{q^{k-1/2} - q^{-k+1/2}}|k + 1\rangle - i \frac{q^{-k+1}[2k]n + k - 1}{q^{k-1/2} - q^{-k+1/2}}|k - 1\rangle, \quad k \neq 1. \]

Denoting the vectors (27) of the subspace \( V_{2} \) by the symbols (28), respectively, we obtain

\[ R^{(i,-)}_{n}(I_{1})|k\rangle = -\frac{q^{k-1/2} + q^{-k+1/2}}{q - q^{-1}}|k\rangle, \]

\[ R^{(i,-)}_{n}(I_{2})|1\rangle = -\frac{[n]}{q^{1/2} - q^{-1/2}}|1\rangle + i \frac{[n - 1]}{q^{1/2} - q^{-1/2}}|2\rangle, \]

\[ R^{(i,-)}_{n}(I_{2})|k\rangle = R^{(i,+)}_{n}(I_{2})|k\rangle, \quad k \neq 1. \]

For the operator \( R^{(i,-)}_{i}(I_{3}) \) we find that

\[ R^{(i,-)}_{n}(I_{3})|1\rangle = \frac{[n]}{q^{1/2} - q^{-1/2}}|1\rangle - i \frac{q[n - 1]}{q^{1/2} - q^{-1/2}}|2\rangle, \]

\[ R^{(i,-)}_{n}(I_{3})|k\rangle = R^{(i,+)}_{n}(I_{3})|k\rangle, \quad k \neq 1. \]

Let now \( R^{(i,+)}_{n} \) and \( R^{(i,-)}_{n} \), \( n = l + \frac{1}{2} \), be the representations of \( U_{q}(so_{3}) \) which are restrictions of the representation \( R^{(i)}_{n} \) to the subspaces \( V_{1} \) and \( V_{2} \), respectively. Introducing the vectors similar to the vectors (28), for the representation \( R^{(i,+)}_{n} \) we have

\[ R^{(i,+)}_{n}(I_{1})|k\rangle = \frac{q^{k-1/2} + q^{-k+1/2}}{q - q^{-1}}|k\rangle, \quad R^{(i,+)}_{n}(I_{2}) = -R^{(i,+)}_{n}(I_{2}), \]

\[ R^{(i,+)}_{n}(I_{3})|1\rangle = \frac{[n]}{q^{1/2} - q^{-1/2}}|1\rangle + i \frac{q[n - 1]}{q^{1/2} - q^{-1/2}}|2\rangle, \]

\[ R^{(i,+)}_{n}(I_{3})|k\rangle = i \frac{q^{k[2k]}n - k}{q^{k-1/2} - q^{-k+1/2}}|k + 1\rangle + i \frac{q^{-k+1}[2k]n + k - 1}{q^{k-1/2} - q^{-k+1/2}}|k - 1\rangle, \quad k \neq 1. \]

For the representation \( R^{(i,-)}_{i}(I_{3}) \) we obtain

\[ R^{(i,-)}_{n}(I_{1})|k\rangle = \frac{q^{k-1/2} + q^{-k+1/2}}{q - q^{-1}}|k\rangle, \quad R^{(i,-)}_{n}(I_{2}) = -R^{(i,-)}_{n}(I_{2}), \]

\[ R^{(i,-)}_{n}(I_{3})|1\rangle = -\frac{[n]}{q^{1/2} - q^{-1/2}}|1\rangle + i \frac{q[n - 1]}{q^{1/2} - q^{-1/2}}|2\rangle, \]

\[ R^{(i,-)}_{n}(I_{3})|k\rangle = R^{(i,+)}_{n}(I_{3})|k\rangle. \]

Thus, we constructed the representations \( R^{(i,+)}_{n} \), \( R^{(i,-)}_{n} \), \( R^{(i,+)}_{n} \) and \( R^{(i,-)}_{n} \) of the algebra \( U_{q}(so_{3}) \). The following theorem characterizes them.

**Theorem 1.** The representations \( R^{(i,+)}_{n} \), \( R^{(i,-)}_{n} \), \( R^{(i,+)}_{n} \) and \( R^{(i,-)}_{n} \) are irreducible and pairwise nonequivalent. For any \( l \) the representation \( R^{(i)}_{l} \) is not equivalent to any of these representations.

**Proof.** The irreducibility is proved exactly in the same way as in Proposition 3. Equivalence relations may exist only for irreducible representations of the same dimension. That is, we have to show that under fixed \( n \) no pair of the representations \( R^{(i,+)}_{n} \), \( R^{(i,-)}_{n} \), \( R^{(i,+)}_{n} \) and \( R^{(i,-)}_{n} \) is
equivalent. It follows from the above formulas that the operators $R_n^{(i,+)}(I_1)$ and $R_n^{(i,-)}(I_1)$ as well as the operators $R_n^{(-i,+)}(I_1)$ and $R_n^{(-i,-)}(I_1)$, have the same set of eigenvalues. Moreover, the spectrum of the first pair of operators differs from that of the second pair. Hence, no of representations $R_n^{(i,+)}$ and $R_n^{(i,-)}$ is equivalent to $R_n^{(-i,+)}$ or $R_n^{(-i,-)}$. The representations $R_n^{(i,+)}$ and $R_n^{(i,-)}$ are not equivalent since the operators $R_n^{(i,+)}(I_2)$ and $R_n^{(i,-)}(I_2)$ have different traces (for equivalent representations these operators must have the same trace). For the same reason, the representations $R_n^{(-i,+)}$ and $R_n^{(-i,-)}$ are not equivalent. The last assertion of the theorem follows from the fact that the spectrum of the operator $R_t^{(1)}(I_1)$ differs from the spectra of the operators $R_n^{(i,+)}(I_1)$, $R_n^{(i,-)}(I_1)$, $R_n^{(-i,+)}(I_1)$ and $R_n^{(-i,-)}(I_1)$. Theorem is proved.

Clearly, the reducible representations $R_n^{(i)}$ and $R_n^{(-i)}$ decomposes into irreducible components as

$$R_n^{(i)} = R_n^{(i,+)} \oplus R_n^{(i,-)}, \quad R_n^{(-i)} = R_n^{(-i,+)} \oplus R_n^{(-i,-)}.$$  \hfill (29)

It can be proved that every irreducible finite dimensional representation of $U_q(so_3)$ is equivalent to one of the representations $R_t^{(30)}, R_t^{(31)}, R_t^{(+31)}, R_t^{(-31)}, R_t^{(+30)}, R_t^{(-30)}$. That is, these representations exhaust, up to equivalence, all irreducible finite dimensional representations of $U_q(so_3)$. A proof of this statement will be given in a separate paper.

V. TENSOR PRODUCTS OF REPRESENTATIONS OF $U_q(so_3)$

As mentioned above, no Hopf algebra structure is known for the algebra $U_q(so_3)$. Therefore, we cannot construct tensor product of finite dimensional representations of $U_q(so_3)$ by using a comultiplication as we do in the case of the quantum algebra $U_q(sl_2)$. However, we may construct some tensor product representations by using the algebra homomorphism of Proposition 2.

First we determine which tensor products of irreducible representations of $U_q(sl_2)$ can be extended to representations of the algebra $U_q(sl_2)$. Verifying for which tensor products $T = T' \otimes T''$ of irreducible representations of $U_q(sl_2)$ the operators

$$q^kT(q^H) + q^{-k}T(q^{-H}), \quad k \in \mathbb{Z},$$

are invertible, we conclude that only the tensor products

$$T_t^{(\pm l)} \otimes T_{l'}^{(\pm 1)}, \quad T_t^{(\pm l)} \otimes T_{l'}^{(\pm 1)}, \quad l, l' = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots,$$

$$T_t^{(\pm l)} \otimes T_{l'}^{(\pm 1)}, \quad l = 0, 1, 2, \ldots, \quad l' = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots,$$

$$T_t^{(\pm l)} \otimes T_{l'}^{(\pm 1)}, \quad l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \quad l' = 0, 1, 2, \ldots,$$

$$T_t^{(\pm l)} \otimes T_{l'}^{(\pm 1)}, \quad l, l' = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots,$$

can be extended to the algebra $U_q(sl_2)$. Taking into account the decompositions of tensor products of irreducible representations of $U_q(sl_2)$ (see, for example, the end of Subsection 3.2.1 and Proposition 3.22 in [11]) we find that

$$T_t^{(\omega)} \otimes T_{l'}^{(\omega')} \simeq T_{t+l'}^{(\omega \omega')} + T_{t+l'-1}^{(\omega \omega')} + \cdots + T_{|t+l'|}^{(\omega \omega')}, \quad (30)$$

$$T_t^{(\omega)} \otimes T_{l'}^{(\pm \omega)} \simeq T_{t+l'}^{(\pm \omega)} + T_{t+l'-1}^{(\pm \omega)} + \cdots + T_{|t+l'|}^{(\pm \omega)}, \quad (31)$$

$$T_t^{(\pm l)} \otimes T_{l'}^{(\omega)} \simeq T_{t+l'}^{(\pm \omega)} + T_{t+l'-1}^{(\pm \omega)} + \cdots + T_{|t+l'|}^{(\pm \omega)}.$$ \hfill (32)
\[ T_{i}^{(\omega i)} \otimes T_{i'}^{(\omega' i')} \simeq T_{i+i'}^{(-\omega \omega')} \oplus T_{i+i'-1}^{(-\omega \omega')} \oplus \cdots \oplus T_{|i+i'|-1}^{(-\omega \omega')}, \]

where \(\omega, \omega' = \pm 1\).

Now we define tensor products of representations of \(U_q(so_3)\) corresponding to the above tensor product representations of \(\hat{U}_q(sl_2)\) as

\[ R \otimes R' = (T \otimes T') \circ \psi, \]

where \(R = T \circ \psi\) and \(R' = T' \circ \psi\). Taking into account the definitions of tensor products of representations of \(U_q(sl_2)\) by means of the comultiplication and the definition of the mapping \(\psi\) we have

\[ (R \otimes R')(I_1) = (T \otimes T' \circ \psi(I_1) = \frac{i}{q - q^{-1}} \left( T(q^H) \otimes T'(q^H) - T(q^{-H}) \otimes T'(q^{-H}) \right). \]

Similarly,

\[ (R \otimes R')(I_2) = \left( T(E) \otimes T'(q^H) + T(q^{-H}) \otimes T'(E) - T(F) \otimes T'(q^H) - -(T(q^{-H}) \otimes T'(F)) \right) \left( T(q^H) \otimes T'(q^H) + T(q^{-H}) \otimes T'(q^{-H}) \right)^{-1}. \]

Composing both sides of the relations (30)–(33) with the mapping \(\psi\) of Proposition 2, we find the decomposition into representations of \(U_q(so_3)\) for the tensor products

\[ R_{i}^{(1)} \otimes R_{i'}^{(1)}, \quad R_{i}^{(1)} \otimes R_{i'}^{(\pm i)}, \quad R_{i'}^{(\pm i)} \otimes R_{i}^{(1)}, \quad R_{i}^{(\pm i)} \otimes R_{i'}^{(\pm i)}, \quad R_{i'}^{(\pm i)} \otimes R_{i}^{(\pm i)}, \]

where the second and the third tensor products are defined only for \(l = 0, 1, 2, \ldots\) (Note that the representations \(R_{i}^{(\pm i)}\) are defined only for \(l = 1, 2, 2, \ldots\)). We have

\[ R_{i}^{(1)} \otimes R_{i'}^{(1)} \simeq R_{i+i'}^{(1)} \oplus R_{i+i'-1}^{(1)} \oplus \cdots \oplus R_{|i+i'|}^{(1)}, \]

\[ R_{i}^{(1)} \otimes R_{i'}^{(\pm i)} \simeq R_{i+i'}^{(\pm i)} \oplus R_{i+i'-1}^{(\pm i)} \oplus \cdots \oplus R_{|i+i'|}^{(\pm i)}, \]

\[ R_{i}^{(\pm i)} \otimes R_{i'}^{(1)} \simeq R_{i+i'}^{(\pm i)} \oplus R_{i+i'-1}^{(\pm i)} \oplus \cdots \oplus R_{|i+i'|}^{(\pm i)}, \]

\[ R_{i}^{(\omega i)} \otimes R_{i'}^{(\omega' i')} \simeq R_{i+i'}^{(\omega i)} \oplus R_{i+i'-1}^{(\omega i)} \oplus \cdots \oplus R_{|i+i'|}^{(\omega i)}. \]

In these formulas the representations \(R_{i}^{(\pm i)}\) are reducible. Unfortunately, our definition of tensor products of representations of \(U_q(so_3)\) does not allow to determine the tensor products containing the irreducible representations \(R_{n}^{(\pm i, \pm)}\) and \(R_{n}^{(\pm i, \mp)}\).

VI. INFINITE DIMENSIONAL REPRESENTATIONS OF \(U_q(so_3)\) OBTAINED FROM REPRESENTATIONS OF \(U_q(sl_2)\)

By using the homomorphism \(\psi : U_q(so_3) \rightarrow \hat{U}_q(sl_2)\) from Proposition 2 and infinite dimensional irreducible representations of the algebra \(\hat{U}_q(sl_2)\) we can construct infinite dimensional irreducible representations of the algebra \(U_q(so_3)\).

Let us first describe irreducible infinite dimensional representations of the algebra \(U_q(sl_2)\). Note that by an infinite dimensional representation \(T\) of \(U_q(sl_2)\) we mean a homomorphism of \(U_q(sl_2)\) into the algebra of linear operators (bounded or unbounded) on a Hilbert space, defined on an
everywhere dense invariant subspace $D$, such that the operator $T(q^H)$ can be diagonalized, has a discrete spectrum and its eigenvectors belong to $D$. Infinite dimensional representations $T$ of $U_q(so_3)$ are described in the same way replacing the operator $T(q^H)$ by $T(I_1)$.

Two representations $T$ and $T'$ of $U_q(\mathfrak{sl}_2)$ on spaces $\mathcal{H}$ and $\mathcal{H}'$, respectively, are called (algebraically) equivalent if there exist everywhere dense invariant subspaces $V \subset \mathcal{H}$ and $V' \subset \mathcal{H}'$ and a one-to-one linear operator $A : V \to V'$ such that $AT(a)v = T'(a)Av$ for all $a \in U_q(\mathfrak{sl}_2)$ and $v \in V$. Equivalence of infinite dimensional representations of $U_q(so_3)$ is defined in the same way.

Let $\epsilon$ be a fixed complex number such that $0 \leq \Re \epsilon < 1$, and let $\mathcal{H}_\epsilon$ be a complex Hilbert space with the orthonormal basis

$$|m\rangle, \quad m = n + \epsilon, \quad n = 0, \pm 1, \pm 2, \cdots. \quad (34)$$

For every complex number $a$ we construct the representation $T_{ae}$ on the Hilbert space $\mathcal{H}_\epsilon$ defined by

$$T_{ae}(q^H)|m\rangle = q^m|m\rangle, \quad T_{ae}(E)|m\rangle = [a - m]|m + 1\rangle - [a + m]|m - 1\rangle, \quad T_{ae}(F)|m\rangle = [a + m]|m + 1\rangle - [a - m]|m - 1\rangle,$$

where $[a \pm m]$ is the $q$-number (see, for example, [12]). The equivalence relations in the set of the representations $T_{ae}$ can be extracted from the paper [12].

Note that the representation $T_{ae}$ is irreducible if and only if $a \neq \pm \epsilon \ (\text{mod} \ Z)$.

All the representations $T_{ae}$ can be extended to representations of the algebra $\hat{U}_q(\mathfrak{sl}_2)$ except for the case when $\epsilon = \pm i\pi/2\tau$, where $q = e^\epsilon$. (We suppose below that $\epsilon \neq \pm i\pi/2\tau$. We denote these extended representations by the same symbols $T_{ae}$.

The formula $R_{ae} = T_{ae} \circ \psi$ associates with every irreducible representation $T_{ae}$, $\epsilon \neq \pm i\pi/2\tau$, of $\hat{U}_q(\mathfrak{sl}_2)$ a representation of the algebra $U_q(so_3)$.

Let $\epsilon \neq \pm i\pi/2\tau$ and $\epsilon \neq \pm i\pi/2\tau + \frac{1}{2}$. Then for the representations $R_{ae}$ of $U_q(so_3)$ we have

$$R_{ae}(I_1)|m\rangle = i|m\rangle, \quad (35a)$$

$$R_{ae}(I_2)|m\rangle = \frac{1}{q^m + q^{-m}} \left\{ [a - m]|m + 1\rangle - [a + m]|m - 1\rangle \right\}, \quad (35b)$$

$$R_{ae}(I_3)|m\rangle = \frac{iq^{1/2}}{q^m + q^{-m}} \left\{ q^m[a - m]|m + 1\rangle + q^{-m}[a + m]|m - 1\rangle \right\}. \quad (35c)$$

If $\epsilon = i\pi/2\tau + \frac{1}{2}$, then denoting the basis elements $|m\rangle$, $m = n + \epsilon$, $n \in \mathbb{Z}$, by $|n + \frac{1}{2}\rangle$, $n \in \mathbb{Z}$, respectively, we obtain

$$R_{ae}(I_1)|k\rangle = -\frac{q^k + q^{-k}}{q - q^{-1}}|k\rangle,$$

$$R_{ae}(I_2)|k\rangle = i\left[ \frac{a' - k}{q^k - q^{-k}} \right]|k + 1\rangle + i\left[ \frac{a' + k}{q^k - q^{-k}} \right]|k - 1\rangle,$$

$$R_{ae}(I_3)|k\rangle = -\frac{iq^{k+1/2}[a' - k]}{q^k - q^{-k}}|k + 1\rangle - \frac{iq^{k+1/2}[a' + k]}{q^k - q^{-k}}|k - 1\rangle,$$

where $a' = a + i\pi/2\tau$ and $k = n + \frac{1}{2}$. If $\epsilon = -i\pi/2\tau + \frac{1}{2}$, then using the same notations for basis elements we obtain

$$R'_{ae}(I_1)|k\rangle = \frac{q^k + q^{-k}}{q - q^{-1}}|k\rangle,$$

$$R'_{ae}(I_2)|k\rangle = -i\left[ \frac{a' - k}{q^k - q^{-k}} \right]|k + 1\rangle - i\left[ \frac{a' + k}{q^k - q^{-k}} \right]|k - 1\rangle.$$
\[ R_{\alpha'}(I_3)|k\rangle = -i q^{k+1/2}[a' - k] \frac{k + 1}{q^k - q^{-k}} |k - 1\rangle \]

(to distinguish these representations from the previous ones we supplied \( R_{\alpha'} \) by prime).

**Proposition 4.** The representations \( R_{\alpha'} \) of \( U_q(\mathfrak{so}_3) \) are irreducible for irreducible representations \( T_{\alpha'} \), \( \epsilon \neq \pm i\pi/2\tau + \frac{1}{2} \), of \( \hat{U}_q(\mathfrak{sl}_2) \). The representations \( R_{\alpha'}, \epsilon = i\pi/2\tau + \frac{1}{2} \), and \( R_{\alpha'}', \epsilon = -i\pi/2\tau + \frac{1}{2} \), are reducible.

Proof is given in the same way as in the case of Proposition 3.

As in the case of finite dimensional representations in Section IV, decomposing the representations \( R_{\alpha'}, \epsilon = \pm i\pi/2\tau + \frac{1}{2} \), we obtain irreducible infinite dimensional representations of \( U_q(\mathfrak{so}_3) \) which will be denoted by \( R_{\alpha'}^{(i,\pm)} \) and \( R_{\alpha'}^{(-i,\pm)} \), \( a' = a + i\pi/2\tau \). In the basis

\[ |n\rangle, \quad n = 1, 2, 3, \ldots, \]

they are given by the formulas

\[ R_{\alpha'}^{(i,\pm)}(I_1)|k\rangle = -\frac{q^{k-1/2} + q^{-k+1/2}}{q - q^{-1}} |k\rangle, \]

\[ R_{\alpha'}^{(i,\pm)}(I_2)|1\rangle = \pm \frac{[a']}{q^{1/2} - q^{-1/2}} |1\rangle + \frac{i[a' - 1]}{q^{1/2} - q^{-1/2}} |2\rangle, \]

\[ R_{\alpha'}^{(i,\pm)}(I_2)|k\rangle = i \frac{[a' - k]}{q^{k-1/2} - q^{-k+1/2}} |k + 1\rangle + \frac{i[a' + k - 1]}{q^{k-1/2} - q^{-k+1/2}} |k - 1\rangle, \quad k \neq 1. \]

\[ R_{\alpha'}^{(i,\pm)}(I_3)|1\rangle = \mp \frac{[a']}{q^{1/2} - q^{-1/2}} |1\rangle - \frac{i q[a' - 1]}{q^{1/2} - q^{-1/2}} |2\rangle, \]

\[ R_{\alpha'}^{(i,\pm)}(I_3)|k\rangle = -i \frac{q[k'+k]}{q^{k-1/2} - q^{-k+1/2}} |k + 1\rangle - \frac{q^{-k+1}[a' + k - 1]}{q^{k-1/2} - q^{-k+1/2}} |k - 1\rangle, \quad k \neq 1. \]

and by the formulas

\[ R_{\alpha'}^{(-i,\pm)}(I_1)|k\rangle = \frac{q^{k-1/2} + q^{-k+1/2}}{q - q^{-1}} |k\rangle, \quad R_{\alpha'}^{(-i,\pm)}(I_2) = -R_{\alpha'}^{(i,\pm)}(I_2), \]

\[ R_{\alpha'}^{(-i,\pm)}(I_3)|1\rangle = \pm \frac{[a']}{q^{1/2} - q^{-1/2}} |1\rangle + \frac{i q[a' - 1]}{q^{1/2} - q^{-1/2}} |2\rangle, \]

\[ R_{\alpha'}^{(-i,\pm)}(I_3)|k\rangle = i \frac{q[k'+k]}{q^{k-1/2} - q^{-k+1/2}} |k + 1\rangle + \frac{q^{-k+1}[a' + k - 1]}{q^{k-1/2} - q^{-k+1/2}} |k - 1\rangle, \quad k \neq 1. \]

**Theorem 2.** The representations \( R_{\alpha'}^{(i,\pm)} \) \( R_{\alpha'}^{(-i,\pm)} \) are irreducible and pairwise nonequivalent. For any \( a \) the irreducible representation \( R_{\alpha'} \) is not equivalent to some of these representations.

Proof is given in the same way as in the finite dimensional case (see the proof of Theorem 1).

The algebra \( U_q(\mathfrak{sl}_2) \) has also irreducible infinite dimensional representations with highest weights or with lowest weights. They are classified in the paper [12]. All of these representations can be extended to the algebra \( \hat{U}_q(\mathfrak{sl}_2) \). Using the composition \( R = T \circ \psi \) we obtain the corresponding
representations $R$ of $U_q(sl_2)$. As above, it can be easily proved that to nonequivalent representations $T$ of $\hat{U}_q(sl_2)$ with highest or lowest weight there correspond nonequivalent irreducible representations of $U_q(so_3)$. We give a list of these representations.

Let $l = \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$. We denote by $R^+_l$ the representation of $U_q(so_3)$ acting on the Hilbert space $H_l$ with the orthonormal basis $|m\rangle$, $m = -l, -l+1, \cdots$, and given by formulas (35) with $a = -l$. By $R^-_l$ we denote the representation of $U_q(so_3)$ acting on the Hilbert space $H_l$ with the orthonormal basis $|m\rangle$, $m = -l, -l+1, -l-2, \cdots$, and given by formulas (35) with $a = l$.

Now let $a \neq 0 \pmod{Z}$ and $a \neq \frac{1}{2} \pmod{Z}$. We denote by $H_a$ the Hilbert space with the orthonormal basis $|m\rangle$, $m = -a, -a+1, -a+2, \cdots$. On this space the representation $R^+_a$ acts which is given by formulas (35). On the Hilbert space $H_a$ with the orthonormal basis $|m\rangle$, $m = a, a-1, a-2, \cdots$, the representation $R_a^-$ acts which is given by formulas (35).

Proposition 5. The above representations $R^+_l$ and $R^-_a$ are irreducible and pairwise nonequivalent.

Proof of this proposition is contained in [13].

VII. OTHER INFINITE DIMENSIONAL REPRESENTATIONS OF $U_q(so_3)$

The algebra $U_q(so_3)$ has also irreducible infinite dimensional representations which cannot be obtained from representations of $\hat{U}_q(sl_2)$. We describe these representations in this section.

Let $H$ be the infinite dimensional vector space with the basis $|m\rangle$, $m = 0, \pm 1, \pm 2, \cdots$, and let $\lambda = q^\tau$ be a nonzero complex number such that $0 \leq \text{Re} \tau < 1$. Then a direct calculation shows that the operators $Q^+_\lambda(I_1)$ and $Q^+_\lambda(I_2)$ given by the formulas

$$Q^+_\lambda(I_1)|m\rangle = \frac{\lambda q^m + \lambda^{-1}q^{-m}}{q - q^{-1}} |m\rangle,$$

$$Q^+_\lambda(I_2)|m\rangle = \frac{1}{q - q^{-1}} |m + 1\rangle + \frac{1}{q - q^{-1}} |m - 1\rangle$$

satisfy the relations (7) and (8) and hence determine a representation of $U_q(so_3)$ which will be denoted by $Q^+\lambda$. Similarly, the operators $Q^-\lambda(I_1)$ and $Q^-\lambda(I_2)$ given on the space $H$ by

$$Q^-\lambda(I_1)|m\rangle = -\frac{\lambda q^m + \lambda^{-1}q^{-m}}{q - q^{-1}} |m\rangle, \quad Q^-\lambda(I_2) := Q^+_\lambda(I_2)$$

determine a representation of $U_q(so_3)$ which is denoted by $Q^-\lambda$. The operators $Q^\lambda(I_3)$ can be calculated by means of formula (4).

Proposition 6. If $\lambda \neq 1$ and $\lambda \neq q^{1/2}$, then the representations $Q^\lambda$ and $Q^-\lambda$ are irreducible. The representations $Q^\lambda_1$ and $Q^\lambda_{\sqrt{q}}$ are reducible.

Proof. The first part is proved in the same way as that of Proposition 3. Let us prove the second part. The representations $Q^\lambda_1$ and $Q^\lambda_{\sqrt{q}}$ are the only representations in the set $\{Q^\lambda\}$ for which the operator $Q^\pm\lambda(I_1)$ has not a simple spectrum. The operators $Q^\pm\lambda(I_1)$ has the spectrum

$$\cdots, q^{-2} + q^2, q^{-1} + q, 2, q + q^{-1}, q^2 + q^{-2}, \cdots$$

Thus, only the spectral point 2 has multiplicity 1. All other points have multiplicity 2. Let $V_1$ and $V_2$ be the vector subspaces of $H$ with the bases

$$|0\rangle, \quad |m\rangle' = |m\rangle - |-m\rangle, \quad m = 1, 2, \cdots,$$
and

\[ |m\rangle'' = |m\rangle + |-m\rangle, \quad m = 1, 2, \ldots, \]

respectively. These basis vectors are eigenvectors of the operator \( Q_1^\pm(I_1) \):

\[ Q_1^+(I_1)|m\rangle' = \pm \frac{q^m + q^{-m}}{q - q^{-1}} |m\rangle', \quad Q_1^+(I_1)|m\rangle'' = \pm \frac{q^m + q^{-m}}{q - q^{-1}} |m\rangle'', \]

and

\[ Q_1^+(I_2)|0\rangle = \frac{1}{q - q^{-1}} |1\rangle', \quad Q_1^+(I_2)|1\rangle'' = \frac{1}{q - q^{-1}} |2\rangle'', \]

\[ Q_1^+(I_2)|m\rangle' = \frac{1}{q - q^{-1}} |m + 1\rangle' + \frac{1}{q - q^{-1}} |m - 1\rangle', \quad m > 0, \]

\[ Q_1^+(I_2)|m\rangle'' = \frac{1}{q - q^{-1}} |m + 1\rangle'' + \frac{1}{q - q^{-1}} |m - 1\rangle'', \quad m > 1. \]

Thus, the subspaces \( V_1 \) and \( V_2 \) are invariant with respect to the representation \( Q_1^+ \) (and the representation \( Q_1^- \)). We denote the subrepresentations of \( Q_1^\pm \) realized on \( V_1 \) and \( V_2 \) by \( Q_1^{1,\pm} \) and \( Q_1^{2,\pm} \), respectively.

The eigenvalues of the operators \( Q_{\sqrt{q}}^\pm(I_1) \) are

\[ \cdots, q^{-3/2} + q^{3/2}, q^{-1/2} + q^{1/2}, q^{1/2} + q^{-1/2}, q^{3/2} + q^{-3/2}, \cdots. \]

Thus, every spectral point has multiplicity 2. We denote by \( W_1 \) and \( W_2 \) the vector subspaces of \( \mathcal{H} \) spanned by the basis vectors

\[ |1/2\rangle' = |0\rangle - |1\rangle, \quad |3/2\rangle' = |1\rangle - |2\rangle, \cdots, |m + \frac{1}{2}\rangle' = |m\rangle - |m - 1\rangle, \cdots \]

and

\[ |1/2\rangle'' = |0\rangle + |1\rangle, \quad |3/2\rangle'' = |1\rangle + |2\rangle, \cdots, |m + \frac{1}{2}\rangle'' = |m\rangle + |m - 1\rangle, \cdots, \]

respectively. These basis vectors are eigenvectors of the operator \( Q_{\sqrt{q}}^\pm(I_1) \):

\[ Q_{\sqrt{q}}^+(I_1)|m + \frac{1}{2}\rangle' = \pm \frac{q^{m+1/2} + q^{-m-1/2}}{q - q^{-1}} |m + \frac{1}{2}\rangle', \]

\[ Q_{\sqrt{q}}^+(I_1)|m + \frac{1}{2}\rangle'' = \pm \frac{q^{m+1/2} + q^{-m-1/2}}{q - q^{-1}} |m + \frac{1}{2}\rangle'', \]

and

\[ Q_{\sqrt{q}}^+(I_2)|\frac{1}{2}\rangle' = -\frac{1}{q - q^{-1}} |\frac{1}{2}\rangle' + \frac{1}{q - q^{-1}} |\frac{3}{2}\rangle', \]

\[ Q_{\sqrt{q}}^+(I_2)|m + \frac{1}{2}\rangle' = \frac{1}{q - q^{-1}} |m + \frac{3}{2}\rangle' + \frac{1}{q - q^{-1}} |m - \frac{1}{2}\rangle', \quad m > 0, \]

\[ Q_{\sqrt{q}}^+(I_2)|\frac{1}{2}\rangle'' = \frac{1}{q - q^{-1}} |\frac{1}{2}\rangle'' + \frac{1}{q - q^{-1}} |\frac{3}{2}\rangle'', \]

\[ Q_{\sqrt{q}}^+(I_2)|m + \frac{1}{2}\rangle'' = \frac{1}{q - q^{-1}} |m + \frac{3}{2}\rangle'' + \frac{1}{q - q^{-1}} |m - \frac{1}{2}\rangle'', \quad m > 0. \]
Thus, the subspaces $W_1$ and $W_2$ are invariant with respect to the representations $Q_{\sqrt{q}}^\pm$. We denote the subrepresentations of $Q_{\sqrt{q}}^\pm$ realized on $W_1$ and $W_2$ by $Q_{\sqrt{q}}^{1,\pm}$ and $Q_{\sqrt{q}}^{2,\pm}$, respectively. Proposition is proved.

**Theorem 3.** The representations $Q_{1,\pm}^{1,\pm}$, $Q_{1,\pm}^{2,\pm}$, $Q_{\sqrt{q}}^{1,\pm}$ and $Q_{\sqrt{q}}^{2,\pm}$ are irreducible and pairwise nonequivalent. For any admissible value of $\lambda$ the representation $Q_{\sqrt{q}}^{\lambda}$ (as well as the representation $Q_{\sqrt{q}}^{-\lambda}$) is not equivalent to some of these representations.

**Proof.** Proof is similar to that of Theorem 1 if to take into account spectra of the operators $Q_{1,\pm}^{1,\pm}(I_1)$, $Q_{1,\pm}^{2,\pm}(I_1)$, $Q_{\sqrt{q}}^{1,\pm}(I_1)$, $Q_{\sqrt{q}}^{2,\pm}(I_1)$ and traces of the operators $Q_{1,\pm}^{1,\pm}(I_2)$, $Q_{1,\pm}^{2,\pm}(I_2)$, $Q_{\sqrt{q}}^{1,\pm}(I_2)$, $Q_{\sqrt{q}}^{2,\pm}(I_2)$.

It will be proved in a separate paper that every irreducible infinite dimensional representation of $U_q(\mathfrak{so}_3)$ is equivalent to one of the representations described in this and previous sections.

**VIII. FINITE DIMENSIONAL REPRESENTATIONS OF $\hat{U}_q(\mathfrak{sl}_2)$: $q$ IS A ROOT OF UNITY**

Everywhere below $q$ is a root of unity, that is, there is a smallest positive integer $p$ such that $q^p = 1$. We suppose that $p \neq 1, 2$. We introduce the number $p'$ setting $p' = p$ if $p$ is odd and $p' = p/2$ if $p$ is even.

As in the case of the algebra $U_q(\mathfrak{sl}_2)$ (see [11], Chapter 3), if $q$ is a root of unity, then $U_q(\mathfrak{so}_3)$ is a finite dimensional vector space over the center of $U_q(\mathfrak{so}_3)$. If $q$ is a primitive root of unity, then this assertion is stated in [5]. If $q$ is any root of unity, then this assertion may be proved in the following way. If $q^p = 1$, then the center $C$ of $U_q(\mathfrak{so}_3)$ contains the elements

$$P_p = I_j^p + aI_j^{p-2} + bI_j^{p-4} + \cdots + dI_j^r, \quad j = 1, 2, 3,$$

where $r = 0$ if $p$ is even and $r = 1$ if $p$ is odd and $a, b, \cdots, d$ are certain fixed complex numbers expressed in terms of $q$. (They are the polynomials $P$ defined in [5] if $q$ is a primitive root of unity. Unfortunately, we could not find the explicit expressions for the coefficients $a, b, \cdots, d$. But note that $P_3 = I_j^3 + I_j$, $P_4 = I_j^4 + I_j^2$ and $P_5 = I_j^5 + (1 + (q + q^{-1})^{-1}I_j^3 + (q + q^{-1})^{-1}I_j)$. Therefore, $I_j^s$, $s > n$, can be reduced to the linear combination of $I_j^i$, $i < n$, with coefficients from the center $C$. Now our assertion follows from this and from Poincaré–Birkhoff–Witt theorem for $U_q(\mathfrak{so}_3)$.

**Theorem 4.** If $q$ is a root of unity, then any irreducible representation of $U_q(\mathfrak{so}_3)$ is finite dimensional.

**Proof.** Let $T$ be an irreducible representation of $U_q(\mathfrak{so}_3)$. Then $T$ maps central elements into scalar operators. Since the linear space $U_q(\mathfrak{so}_3)$ is finite dimensional over the center $C$ with the basis $I_1^kI_2^mI_3^n$, $k, m, n < p$, then for any $a \in U_q(\mathfrak{so}_3)$ we have $T(a) = \sum_{k, m, n < p} T(I_1^kI_2^mI_3^n)$. Hence, if $v$ is a nonzero vector of the representation space $\mathcal{V}$, then $T(U_q(\mathfrak{so}_3))v = \mathcal{V}$ and $\mathcal{V}$ is finite dimensional. Theorem is proved.

Taking into account Theorem 4, below we consider only finite dimensional representations of $U_q(\mathfrak{so}_3)$.

In order to find irreducible representations of $U_q(\mathfrak{so}_3)$ for $q$ a root of unity, we use the same method as before, that is, we apply the homomorphism $\psi$ from Proposition 2 and irreducible representations of the algebra $\hat{U}_q(\mathfrak{sl}_2)$ for $q$ a root of unity.

Let us find irreducible representations of $\hat{U}_q(\mathfrak{sl}_2)$ for $q$ a root of unity. The quantum algebra $U_q(\mathfrak{sl}_2)$ for $q$ a root of unity has the following irreducible representations (see [11], Subsection 3.3.2):
(a) The representations $T_i^{(1)}$, $T_i^{(-1)}$, $T_i^{(0)}$, $T_i^{(-i)}$, $2l < p'$, given by the formulas (15)–(19).
(b) The representations $T_{a,b}$, $a,b, \lambda \in \mathbb{C}$, $\lambda \neq 0$, acting on a $p'$-dimensional vector space $\mathcal{H}$ with the basis $|j\rangle$, $j = 0, 1, 2, \cdots, p' - 1$, and given by the formulas

$$T_{a,b}(q^H)|i\rangle = q^{-i}b|b\rangle, \quad T_{a,b}(F)|p' - 1\rangle = b|0\rangle,$$  

$$T_{a,b}(F)|i\rangle = |i + 1\rangle, \quad i < p' - 1, \quad T_{a,b}(E)|0\rangle = a|p' - 1\rangle,$$  

$$T_{a,b}(E)|i\rangle = \left(ab + [i] \frac{\lambda^2 q^{-1-i} - \lambda^{-2} q^{i-1}}{q - q^{-1}}\right)|i - 1\rangle, \quad i > 0.$$  

The representations $T_{a,b}$ with $(a,b) = (0,0)$ and $\lambda = \pm q^n$, $n = 0, 1, 2, \cdots, p' - 2$, are reducible and must be taken out from this set.

(c) The representations $T_{b,0}$, $b, \lambda \in \mathbb{C}$, $\lambda \neq 0$, acting on a $p'$-dimensional vector space $\mathcal{H}$ with the basis $|j\rangle$, $j = 0, 1, 2, \cdots, p' - 1$, and given by the formulas

$$T_{b,0}(q^H)|i\rangle = q^{i}b^{-1}|i\rangle, \quad T_{b,0}(E)|p' - 1\rangle = b|0\rangle,$$  

$$T_{b,0}(E)|i\rangle = |i + 1\rangle, \quad i < p' - 1, \quad T_{b,0}(F)|0\rangle = 0,$$  

$$T_{b,0}(F)|i\rangle = [i] \frac{\lambda^2 q^{-1-i} - \lambda^{-2} q^{i-1}}{q - q^{-1}}|i - 1\rangle, \quad i > 0.$$  

The representations $T_{b,0}$ with $\lambda = \pm q^n$, $n = 0, 1, 2, \cdots, p' - 2$, are reducible and must be taken out from this set.

**Remark 1.** In the set of representations (a)–(c) there exist equivalent representations (see, for example, Propositions 3.17 and 3.18 in [11]).

**Remark 2.** In [11], Subsection 3.3.2, irreducible representations of the algebra generated by the elements $E, F, K := q^{2H}, K^{-1} := q^{-2H} \in U_q(\mathfrak{sl}_2)$ are given. Clearly, this algebra is a subalgebra in $U_q(\mathfrak{sl}_2)$. It is easy to generalize the results of Subsection 3.3.2 in [11] for $U_q(\mathfrak{sl}_2)$. Let us note that the algebra $U_q(\mathfrak{sl}_2)$ has a unique automorphism $\varphi$ such that $\varphi(q^H) = iq^H, \varphi(E) = -E$ and $\varphi(F) = F$. (If $q$ is not a root of unity, then this automorphism transforms the representations $T_i^{(1)}$ to the representations $T_i^{(ii)}$, respectively.) Therefore, the mapping $\tilde{T}_{a,b,\lambda} = T_{a,b,\lambda} \circ \varphi$ is also a representation of $U_q(\mathfrak{sl}_2)$. We have

$$\tilde{T}_{a,b}(q^H)|i\rangle = iq^{-i}b|b\rangle, \quad \tilde{T}_{a,b}(F)|p' - 1\rangle = b|0\rangle,$$  

$$\tilde{T}_{a,b}(F)|i\rangle = |i + 1\rangle, \quad i < p' - 1, \quad \tilde{T}_{a,b}(E)|0\rangle = a|p' - 1\rangle,$$  

$$\tilde{T}_{a,b}(E)|i\rangle = \left(ab - [i] \frac{\lambda^2 q^{1-i} - \lambda^{-2} q^{-i+1}}{q - q^{-1}}\right)|i - 1\rangle, \quad i > 0.$$  

However, it is easy to see by comparing (36)–(38) with (42)–(44) that the representation $\tilde{T}_{a,b,\lambda}$ is equivalent to $T_{a,b,\lambda}$. This means that for $q$ a root of unity we do not obtain new representations of $U_q(\mathfrak{sl}_2)$ from $T_{a,b,\lambda}$ applying the automorphism $\varphi$ as in the case of the representations $T_i^{(1)}$.

We have described irreducible representations of the algebra $U_q(\mathfrak{sl}_2)$. Now we wish to extend these representations to obtain representations of the algebra $\hat{U}_q(\mathfrak{sl}_2)$ by using the relation

$$T((q^k q^H + q^{-k} q^{-H})^{-1}) := (q^k T(q^H) + q^{-k} T(q^{-H}))^{-1}.$$
Clearly, only those irreducible representations \( T \) of \( U_q(\mathfrak{sl}_2) \) can be extended to \( \hat{U}_q(\mathfrak{sl}_2) \) for which the operators \( q^k T(q^H) + q^{-k} T(q^{-H}) \) are invertible. From formulas (15)–(19) it is clear that these operators are always invertible for the irreducible representations \( T^{(1)}_l, T^{(-1)}_l, l = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{p-1}{2} \), and for the irreducible representations \( T^{(i)}_l, T^{(-i)}_l, l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots, \frac{p-1}{2} \) (or \( \frac{p'-2}{2} \)). (For the representations \( T^{(i)}_l, T^{(-i)}_l, l = 0, 1, 2, \cdots \), some of these operators are not invertible since they have zero eigenvalue.) We denote the extended representations by the same symbols \( T^{(1)}_l, T^{(-1)}_l, T^{(i)}_l, T^{(-i)}_l \), respectively.

Similarly, the representation \( T_{ab\lambda} \) (and the representation \( T_{0b\lambda}' \)) can be extended to a representation of the algebra \( \hat{U}_q(\mathfrak{sl}_2) \) if and only if \( \lambda \neq \pm iq^k, k \in \mathbb{Z} \).

**Proposition 7.** The algebra \( \hat{U}_q(\mathfrak{sl}_2) \) for a root of unity has the irreducible representations \( T^{(1)}_l, T^{(-1)}_l, l = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{p-1}{2} \), the irreducible representations \( T^{(i)}_l, T^{(-i)}_l, l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots, \frac{p-1}{2} \) (or \( \frac{p'-2}{2} \)), and the irreducible representations \( T_{ab\lambda}, T_{0b\lambda}' \), \( \lambda \neq \pm iq^k, k \in \mathbb{Z} \). Any irreducible representation of \( \hat{U}_q(\mathfrak{sl}_2) \) for a root of unity is equivalent to one of these representations.

**IX. REPRESENTATIONS OF \( U_q(\mathfrak{so}_3) \) FOR A ROOT OF UNITY OBTAINED FROM THOSE OF \( \hat{U}_q(\mathfrak{sl}_2) \)**

As in Section IV, we shall obtain representations of \( U_q(\mathfrak{so}_3) \) for a root of unity by applying the homomorphism \( \psi \) from Proposition 2. Namely, if \( T \) is a representation of \( \hat{U}_q(\mathfrak{sl}_2) \), then

\[
R = T \circ \psi
\]

is a representation of \( U_q(\mathfrak{so}_3) \). As in Section IV, application of this method to the pair of the irreducible representations \( T^{(1)}_l \) and \( T^{(-1)}_l \) of \( \hat{U}_q(\mathfrak{sl}_2) \) leads to the same representation of \( U_q(\mathfrak{so}_3) \) which will be denoted by \( R^{(1)}_l \). Applying the formula (45) to the irreducible representations \( T^{(i)}_l \) and \( T^{(-i)}_l \) of \( \hat{U}_q(\mathfrak{sl}_2) \) give the representations of \( U_q(\mathfrak{so}_3) \) which will be denoted by \( R^{(i)}_l \) and \( R^{(-i)}_l \), respectively.

**Proposition 8.** The representations \( R^{(1)}_l \) of \( U_q(\mathfrak{so}_3) \) are irreducible. The representations \( R^{(i)}_l \) and \( R^{(-i)}_l \) are reducible.

**Proof** of this proposition is the same as that of Proposition 3.

Repeating word-by-word the reasoning of Section IV, we decompose the representations \( R^{(i)}_l \) and \( R^{(-i)}_l \) into the direct sums of representations of \( U_q(\mathfrak{so}_3) \) which are denoted by \( R^{(+i,+)}_n \) and \( R^{(+i,-)}_n \):

\[
R^{(i)}_l = R^{(+i,+)}_n \oplus R^{(+i,-)}_n, \quad R^{(-i)}_l = R^{(-i,+)}_n \oplus R^{(-i,-)}_n, \quad n = l + \frac{1}{2}.
\]

Moreover, the representations \( R^{(+i,+)}_n \) and \( R^{(+i,-)}_n \) are given in the appropriate bases \( |1\rangle, |2\rangle, \cdots, |n\rangle \) by the corresponding formulas of Section IV.

**Theorem 5.** The representations \( R^{(+i,+)}_n, R^{(+i,-)}_n, R^{(-i,+)}_n, R^{(-i,-)}_n, n = 1, 2, 3, \cdots, \frac{p}{2} \) (or \( \frac{p'-1}{2} \)) are irreducible and pairwise nonequivalent. For any \( l, \ l = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{p-1}{2} \), the representation \( R^{(1)}_l \) is not equivalent to some of these representations.

**Proof** is the same as that of Theorem 1.
Now we apply formula (45) to the representations $T_{ab\lambda}$ and $T'_{0b\lambda}$. As a result, we obtain the representations

$$R_{ab\lambda} = T_{-a,b,-\lambda} \circ \psi, \quad R'_{0b\lambda} = T'_{0,b,-\lambda},$$

given in the bases $|j\rangle$, $j = 0, 1, 2, \ldots, p'-1$, by the formulas

$$R_{ab\lambda}(I_1)|i\rangle = \frac{-1}{q - q^{-1}}(q^{-i}\lambda + q^{i}\lambda^{-1})|i\rangle,$$

$$R_{ab\lambda}(I_2)|0\rangle = \frac{i}{\lambda - \lambda^{-1}}(a|p' - 1\rangle + |1\rangle),$$

$$R_{ab\lambda}(I_2)|p' - 1\rangle = \frac{i}{q^{-p' + 1}\lambda - q^{p'}\lambda^{-1}}\left\{b|0\rangle + \left(ab + [p' - 1]\frac{q^{-p' + 2}\lambda^2 - q^{p'}\lambda^{-2}}{q - q^{-1}}\right)|p' - 2\rangle\right\},$$

$$R_{ab\lambda}(I_2)|i\rangle = \frac{i}{q^{-i}\lambda - q^{i}\lambda^{-1}}\left\{(ab + [i]\frac{q^{-i + 1}\lambda^2 - q^{i - 1}\lambda^{-2}}{q - q^{-1}})|i - 1\rangle + \right.$$ \left.+ |i + 1\rangle\right\}, \quad 0 < i < p' - 1.$$

and by the formulas

$$R'_{0b\lambda}(I_1)|i\rangle = \frac{1}{q - q^{-1}}(q^{-i}\lambda + q^{i}\lambda^{-1})|i\rangle, \quad R'_{0b\lambda}(I_2)|0\rangle = \frac{-i}{\lambda - \lambda^{-1}}|1\rangle,$$

$$R'_{0b\lambda}(I_2)|p' - 1\rangle = \frac{-i}{q^{-p' + 1}\lambda - q^{p'}\lambda^{-1}}\left(b|0\rangle + [p' - 1]\frac{q^{-p' + 2}\lambda^2 - q^{p'}\lambda^{-2}}{q - q^{-1}}|p' - 2\rangle\right),$$

$$R'_{0b\lambda}(I_2)|i\rangle = \frac{-i}{q^{-i}\lambda - q^{i}\lambda^{-1}}\left(|i + 1\rangle + \right.$$ \left.+ [i]\frac{q^{-i + 1}\lambda^2 - q^{i - 1}\lambda^{-2}}{q - q^{-1}}|i - 1\rangle\right), \quad 0 < i < p' - 1.$$

The operators $R_{ab\lambda}(I_3)$ and $R'_{0b\lambda}(I_3)$ can be calculated by means of the relation

$$R(I_3) = q^{1/2}R(I_1)R(I_2) - q^{-1/2}R(I_2)R(I_1).$$

Recall that the representations $R_{ab\lambda}$ and $R'_{0b\lambda}$ are determined for $\lambda \neq 0$ and $\lambda \neq \pm q^k$, $k \in \mathbb{Z}$.

It is seen from the above formulas that

$$R'_{0b\lambda}(I_1) = R_{0,0,-\lambda}(I_1), \quad R'_{0b\lambda}(I_2) = R_{0,b,-\lambda}(I_2),$$

that is, the representations $R_{0,0,-\lambda}$ and $R'_{0b\lambda}$ are equivalent. For this reason, we consider below only the representations $R_{ab\lambda}$.

In order to study the representations $R_{ab\lambda}$ of $U_q(so_3)$ we consider the spectrum of the operator $R_{ab\lambda}(I_1)$. It coincides with the set of points

$$-\frac{\lambda + \lambda^{-1}}{q - q^{-1}}, -\frac{q^{-1}\lambda + q\lambda^{-1}}{q - q^{-1}}, -\frac{q^{-2}\lambda + q^2\lambda^{-1}}{q - q^{-1}}, \ldots, -\frac{q^{1-p'}\lambda + q^{p'}\lambda^{-1}}{q - q^{-1}}.$$
It is easy to see that there exist coinciding points in this set if and only if \( \lambda \) is equal to one of the numbers

\[
\pm q^{1/2}, \pm q^{3/2}, \pm q^{5/2}, \ldots, \pm q^{(p'-1)/2} \text{ (or } \pm q^{(p'-2)/2} \text{).}
\]

(Here we have to take \( \pm q^{(p'-1)/2} \) if \( p' \) is even and \( \pm q^{(p'-2)/2} \) if \( p' \) is odd.) Moreover, the set (50) splits into pairs of coinciding points if and only if \( \lambda = \pm q^{(p'-1)/2} \). In all other cases there exists at least one spectral point which coincides with no other point. In particular, if \( \lambda = \pm q^{(p'-2)/2} \), then in this set there exists only one eigenvalue with multiplicity 1. In all other cases there are more than one eigenvalues with multiplicity 1.

**Proposition 9.** If \( \lambda \neq \pm q^{(p'-1)/2} \) for even \( p' \) and \( \lambda \neq \pm q^{(p'-2)/2} \) for odd \( p' \), then the representation \( R_{ab\lambda} \) is irreducible.

**Proof.** Let \( \lambda \neq \pm q^{(p'-1)/2} \) for even \( p' \) and \( \lambda \neq \pm q^{(p'-2)/2} \) for odd \( p' \). We distinguish two cases: when the spectrum of the operator \( R_{ab\lambda}(I_1) \) is simple and when there exists at list one spectral point of this operator having multiplicity 2. In the first case the proof is the same as the first part of the proof of Proposition 3. For the second case, we give a proof only for \( \lambda = q^{1/2} \). (Proofs for other values of \( q \) are similar.) Then in the set (50) there are only two coinciding points \( -\frac{\lambda + \lambda^{-1}}{q - q^{-1}} \) and \( -\frac{q^{-1} \lambda + q\lambda^{-1}}{q - q^{-1}} \) corresponding to the eigenvectors \(|0\rangle \) and \(|1\rangle \). Let \( V \) be an invariant subspace of the representation space \( \mathcal{H} \). As in the proof of Proposition 3, it is shown that \( V \) is a linear span of eigenvectors of the operator \( R_{ab\lambda}(I_1) \), that is, a certain part of the vectors \(|i\rangle, i \neq 0, 1, \alpha_0|0\rangle + \alpha_1|1\rangle, \beta_0|0\rangle + \beta_1|1\rangle \) constitutes a basis of \( V \). Let \( V \) contain some basis vector \(|j\rangle \). Then as in the proof of Proposition 3, acting successively upon \(|j\rangle \) by certain linear combinations of the operators \( R_{ab\lambda}(I_2) \) and \( R_{ab\lambda}(I_3) \) we generate all the vectors \(|i\rangle, i = 0, 1, \ldots, \frac{1}{2}(p' - 1) \). This means that \( V = \mathcal{H} \) and the representation \( R_{ab\lambda} \) is irreducible. If \( V \) contains no vector \(|j\rangle, j \neq 0, 1 \), then some linear combination \( \alpha_0|0\rangle + \alpha_1|1\rangle \) belongs to \( V \). Then the vector \( v = R_{ab\lambda}(I_2)(\alpha_0|0\rangle + \alpha_1|1\rangle) \) belongs to \( V \). Since \( v \) contains the summand \( \alpha|2\rangle \) with nonzero coefficient \( \alpha \), then \(|2\rangle \in V \). This is a contradiction. Hence, the representation \( R_{ab\lambda} \) is irreducible. Proposition is proved.

Let \( p' \) be even. Let us study the representations \( R_{ab\lambda} \) for \( \lambda = \pm q^{(p'-1)/2} \). For \( \lambda = q^{(p'-1)/2} \) we have

\[
R_{ab\lambda}(I_1)|i\rangle = \frac{-1}{q - q^{-1}}(q^{-i+(p'-1)/2} + q^{i-(p'-1)/2})|i\rangle,
\]

(51)

\[
R_{ab\lambda}(I_2)|0\rangle = c_{(p'-1)/2}(a|p' - 1\rangle + |1\rangle),
\]

(52)

\[
R_{ab\lambda}(I_2)|p' - 1\rangle = -c_{(p'-1)/2}((ab + |p' - 1\rangle|p' - 2\rangle + b|0\rangle),
\]

(53)

\[
R_{ab\lambda}(I_2)|i\rangle = c_{-i+(p'-1)/2}((ab + |i\rangle|i - 1\rangle + |i + 1\rangle),
\]

(54)

where

\[
c_j = \frac{i}{q^i - q^{-j}}.
\]

The operator \( R_{a,b,(p'-1)/2}(I_1) \) has the spectrum

\[
\frac{-1}{q - q^{-1}}(q^{-i+(p'-1)/2} + q^{i-(p'-1)/2}), \quad i = 0, 1, 2, \ldots, p' - 1,
\]

that is, if \( p' \) is even, then all spectral points are of multiplicity 2.
We assume that $ab \neq j^2$, $j = 0, 1, \cdots, p' - 1$, and go over from the basis \{$|i\rangle\$ to the basis \{$|i\rangle^o\$}, where

\[
|i\rangle^o = \prod_{j=0}^i (ab + [j]^2)^{-1/2} |i\rangle, \quad i = 0, 1, 2, \cdots, p' - 1.
\]

Then the formula (51) does not change and the formulas (52)–(54) turn into

\[
R_{ab\lambda}(I_2)|0\rangle^o = c_{(p'-1)/2} \left( a \prod_{j=1}^{p'-1} (ab + [j]^2)^{1/2} |p' - 1\rangle + (ab + 1)^{1/2} |1\rangle^o \right),
\]

\[
R_{ab\lambda}(I_2)|p' - 1\rangle^o = -c_{(p'-1)/2} \left( (ab + 1)^{1/2} |p' - 2\rangle + \frac{b}{\prod_{j=1}^{p'-1} (ab + [j]^2)^{1/2}} |0\rangle^o \right),
\]

\[
R_{ab\lambda}(I_2)|i\rangle^o = c_{-i+(p'-1)/2} \left( (ab + [i]^2)^{1/2} |i - 1\rangle + (ab + [i + 1]^2)^{1/2} |i + 1\rangle \right).
\]

We split the representation space $\mathcal{H}$ into the direct sum of two linear subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ spanned by the basis vectors $|j\rangle'$, $j = 0, 1, 2, \cdots, \frac{p'}{2}(p' - 2)$, and $|j\rangle''$, $j = 0, 1, 2, \cdots, \frac{p'}{2}(p' - 2)$, where

\[
|j\rangle' = |j\rangle^o + i(-1)^{-j+1} j^{p'}/2 |p' - j - 1\rangle, \quad |j\rangle'' = |j\rangle^o + i(-1)^{-j+1} j^{p'}/2 |p' - j - 1\rangle^o.
\]

Then as in Section IV, we derive

\[
R_{a,b,(p'-1)/2}(I_1)|j\rangle' = \frac{-1}{q - q^{-1}} (q^{-j+(p'-1)/2} + q^{j-(p'-1)/2}) |j\rangle',
\]

\[
R_{a,b,(p'-1)/2}(I_1)|j\rangle'' = \frac{-1}{q - q^{-1}} (q^{-j+(p'-1)/2} + q^{j-(p'-1)/2}) |j\rangle''
\]

for the operator $R_{a,b,(p'-1)/2}(I_1)$ and

\[
R_{a,b,(p'-1)/2}(I_2)|j\rangle' = c_{-j+(p'-1)/2} \left( (ab + [j + 1]^2)^{1/2} |j + 1\rangle' + (ab + [j]^2)^{1/2} |j - 1\rangle' \right),
\]

\[
R_{a,b,(p'-1)/2}(I_2)|j\rangle'' = c_{-j+(p'-1)/2} \left( (ab + [j + 1]^2)^{1/2} |j + 1\rangle'' + (ab + [j]^2)^{1/2} |j - 1\rangle'' \right),
\]

where $j \neq 0, \frac{p'}{2} - 1$,

\[
R_{a,b,(p'-1)/2}(I_2)|\frac{p'}{2} - 1\rangle' = \frac{1}{q^{1/2} - q^{-1/2}} \left( ab + [\frac{p'}{2}]^2)^{1/2} |\frac{p'}{2} - 1\rangle' + \frac{i}{q^{1/2} - q^{-1/2}} (ab + [\frac{p'}{2} - 1]^2)^{1/2} |\frac{p'}{2} - 2\rangle',
\]

\[
R_{a,b,(p'-1)/2}(I_2)|\frac{p'}{2} - 1\rangle'' = -\frac{1}{q^{1/2} - q^{-1/2}} \left( ab + [\frac{p'}{2}]^2)^{1/2} |\frac{p'}{2} - 1\rangle'' + \frac{i}{q^{1/2} - q^{-1/2}} (ab + [\frac{p'}{2} - 1]^2)^{1/2} |\frac{p'}{2} - 2\rangle'',
\]

\[
R_{a,b,(p'-1)/2}(I_2)|0\rangle' = c_{(p'-1)/2} \left( a \prod_{j=1}^{p'-1} (ab + [j]^2)^{1/2} |p' - 1\rangle + (ab + 1)^{1/2} |1\rangle \right)
\]
When
\[ a \prod_{j=1}^{p'-1} (ab + [j]2)^{1/2} = \frac{b}{\prod_{j=1}^{p'-1} (ab + [j]2)^{1/2}}, \] (55)
then the last relation reduces to
\[ R_{ab,(p'-1)/2}(I_2)|0\rangle' = \frac{(-1)^{p'-2}}{q^{(p'-1)/2} - q^{-(p'-1)/2}} a \prod_{j=1}^{p'-1} (ab + [j]2)^{1/2} |0\rangle' + c^{(p'-1)/2} (ab + 1)^{1/2} |1\rangle'. \]

Similarly, if the condition (55) is fulfilled, then
\[ R_{ab,(p'-1)/2}(I_2)|0\rangle'' = \frac{(-1)^{p'/2}}{q^{(p'-1)/2} - q^{-(p'-1)/2}} a \prod_{j=1}^{p'-1} (ab + [j]2)^{1/2} |0\rangle'' + c^{(p'-1)/2} (ab + 1)^{1/2} |1\rangle''. \]

Thus, the subspaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are invariant with respect to the representation \( R_{ab,(p'-1)/2} \) if the condition (55) is fulfilled. We denote the corresponding subrepresentations by \( R_{ab,(p'-1)/2}^{1,+} \) and \( R_{ab,(p'-1)/2}^{2,+} \), respectively.

Similarly, if \( \lambda = -q^{(p'-1)/2} \), then
\[ R_{ab,-(p'-1)/2}(I_1) = -R_{ab,(p'-1)/2}(I_1), \quad R_{ab,-(p'-1)/2}(I_2) = -R_{ab,(p'-1)/2}(I_2) \]
and the subspaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are invariant with respect to the representation \( R_{ab,-(p'-1)/2} \) if the condition (55) is fulfilled. We denote the corresponding subrepresentations by \( R_{ab,-(p'-1)/2}^{1,-} \) and \( R_{ab,-(p'-1)/2}^{2,-} \), respectively.

**Proposition 10.** Let the condition (55) is satisfied. Then the representations \( R_{ab,(p'-1)/2}^{i,+} \) and \( R_{ab,(p'-1)/2}^{i,-} \), \( i = 1, 2 \), of the algebra \( U_q(so_3) \) are irreducible and pairwise nonequivalent. If the condition (55) is not satisfied, then the representations \( R_{ab,(p'-1)/2} \) and \( R_{ab,-(p'-1)/2} \) are irreducible.

**Proof** is similar to that of the previous propositions and we omit it.

Remark that the representations \( R_{ab,(p'-1)/2}^{i,+} \) and \( R_{ab,-(p'-1)/2}^{i,-} \), \( i = 1, 2 \), have two nonzero diagonal matrix elements \( \langle \frac{p'}{2} - 1 | R_{ab}^{i} \rangle \) and \( \langle 0 | R_{ab}^{i} \rangle \).

Let now \( p' \) be odd and \( \lambda = q^{(p'-2)/2} \). For this value of \( \lambda \) we have
\[ R_{ab\lambda}(I_1)|i\rangle = \frac{-1}{q - q^{-1}} (q^{-i+(p'-2)/2} + q^{i-(p'-2)/2}) |i\rangle, \]
\[ R_{ab\lambda}(I_2)|0\rangle = c^{(p'-2)/2} (a|p' - 1\rangle + |1\rangle), \]
\[ R_{ab\lambda}(I_2)|p' - 1\rangle = -c^{p'/2} ( (ab + e[p' - 1][p'])(p' - 2) + b|0\rangle), \]
\[ R_{ab\lambda}(I_2)|i\rangle = c^{-i+(p'-2)/2} ( (ab + e[i][i + 1])(i - 1) + |i + 1\rangle). \]
where \( \epsilon = 1 \) for \( p' = p/2 \), \( \epsilon = -1 \) for \( p' = p \) and \( c_j \) is such as in (51)-(54). The operator \( R_{a,b,(p'-2)/2}(I_1) \) has the spectrum

\[
-\frac{1}{q-q^{-1}}(q^{-i+(p'-2)/2} + q^{-(p'-2)/2}), \quad i = 0, 1, 2, \ldots, p' - 1,
\]

that is, all spectral points are of multiplicity 2 except for the point \( -(q^{p'/2} + q^{-p'/2})/(q - q^{-1}) \) which is of multiplicity 1.

We assume that \( ab \neq -\epsilon[j][j+1], \ j = 0, 1, \ldots, p' - 1 \), and go over from the basis \( \{|i\} \) to the basis \( \{|i\}^{o} \), where

\[
|i\rangle^{o} = \prod_{j=0}^{i}(ab + \epsilon[j][j+1])^{-1/2}|i\rangle, \quad i = 0, 1, 2, \ldots, p' - 1.
\]

Then

\[
R_{ab\lambda}(I_1)|i\rangle^{o} = -\frac{1}{q-q^{-1}}(q^{-i+(p'-2)/2} + q^{-(p'-2)/2}) |i\rangle^{o},
\]

\[
R_{ab\lambda}(I_2)|0\rangle^{o} = c_{(p'-2)/2} \left( a \prod_{j=1}^{p'-1} (ab + \epsilon[j][j+1])^{1/2} |p' - 1\rangle^{o} + (ab + \epsilon[2])^{1/2} |1\rangle^{o} \right),
\]

\[
R_{ab\lambda}(I_2)|p' - 1\rangle^{o} = -c_{p'/2}((ab + \epsilon[p' - 1][p'])^{1/2} |p' - 2\rangle^{o} +
\]

\[
+ b \prod_{j=1}^{p'-1} (ab + \epsilon[j][j+1])^{-1/2} |0\rangle^{o},
\]

\[
R_{ab\lambda}(I_2)|i\rangle^{o} = c_{i+(p'-2)/2}((ab + \epsilon[i][i+1])^{1/2}|i - 1\rangle^{o} +
\]

\[
+ (ab + \epsilon[i + 1][i + 2])^{1/2}|i + 1\rangle^{o},
\]

where \( \lambda = q^{(p'-2)/2} \). Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be two linear subspaces of the representation space \( \mathcal{H} \) spanned by the basis vectors

\[
|j\rangle' = |j\rangle^{o} + i(-1)^j |p' - j - 2\rangle^{o}, \quad j = 0, 1, 2, \ldots, \frac{p' - 3}{2},
\]

and the basis vectors

\[
|j\rangle'' = |j\rangle^{o} + i(-1)^{j+1} |p' - j - 2\rangle^{o}, \quad j = 0, 1, 2, \ldots, \frac{p' - 3}{2},
\]

respectively. Then the operator \( R_{a,b,(p'-2)/2}(I_1) \) acts on the basis elements \( |j\rangle' \) and \( |j\rangle'' \) as on the vectors \( |j\rangle \) and

\[
R_{a,b,(p'-1)/2}(I_2)|j\rangle' = c_{-j+(p'-2)/2} \left( (ab + \epsilon[j + 1][j + 2])^{1/2}|j + 1\rangle'
\]

\[
+ (ab + \epsilon[j][j + 1])^{1/2}|j - 1\rangle' \right),
\]

\[
R_{a,b,(p'-2)/2}(I_2)|j\rangle'' = c_{-j+(p'-2)/2} \left( (ab + \epsilon[j + 1][j + 2])^{1/2}|j + 1\rangle''
\]

\[
+ (ab + \epsilon[j][j + 1])^{1/2}|j - 1\rangle'' \right),
\]

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where \( j \neq 0, \frac{p' - 3}{2} \),

\[
R_{a,b,(p'-2)/2}(I_2)|\frac{p'-3}{2}\rangle' = \frac{(-1)^{(p'-3)/2}}{q^{1/2} - q^{-1/2}}(ab + \epsilon[\frac{p'-1}{2}]|\frac{p'+1}{2}\rangle|\frac{p'-3}{2}\rangle' +
\]
\[
\quad + \frac{i}{q^{1/2} - q^{-1/2}}(ab + \epsilon[\frac{p'-1}{2}]|\frac{p'-3}{2}\rangle'|\frac{p'-5}{2}\rangle'),
\]

\[
R_{a,b,(p'-2)/2}(I_2)|\frac{p'-3}{2}\rangle'' = -\frac{(-1)^{(p'-3)/2}}{q^{1/2} - q^{-1/2}}(ab + \epsilon[\frac{p'-1}{2}]|\frac{p'+1}{2}\rangle|\frac{p'-3}{2}\rangle'' +
\]
\[
\quad + \frac{i}{q^{1/2} - q^{-1/2}}(ab + \epsilon[\frac{p'-1}{2}]|\frac{p'-3}{2}\rangle'|\frac{p'-5}{2}\rangle''),
\]

\[
R_{a,b,(p'-2)/2}(I_2)|0\rangle' = c_{(p'-2)/2}\left(a \prod_{j=1}^{p'-1} (ab + \epsilon[j][j + 1])^{1/2} |p' - 1\rangle + (ab + \epsilon[2])^{1/2}|1\rangle\right.
\]
\[
\quad - i(ab + \epsilon[2])^{1/2}|p' - 1\rangle - i(ab + \epsilon[p' - 2][p' - 1])^{1/2} |p' - 3\rangle),
\]

\[
R_{a,b,(p'-2)/2}(I_2)|0\rangle'' = c_{(p'-2)/2}\left(a \prod_{j=1}^{p'-1} (ab + \epsilon[j][j + 1])^{1/2} |p' - 1\rangle + (ab + \epsilon[2])^{1/2}|1\rangle
\]
\[
\quad + i(ab + \epsilon[2])^{1/2}|p' - 1\rangle + i(ab + \epsilon[p' - 2][p' - 1])^{1/2} |p' - 3\rangle\right).
\]

If

\[
a \prod_{j=1}^{p'-1} (ab + \epsilon[j][j + 1])^{1/2} + i(ab + \epsilon[2])^{1/2} = 0,
\]

\[
(ab + \epsilon[2])^{1/2} \prod_{j=1}^{p'-1} (ab + \epsilon[j][j + 1])^{1/2} = ib,
\]

then

\[
R_{a,b,(p'-2)/2}(I_2)|p' - 1\rangle = \frac{-bc_{p'/2}}{\prod_{j=1}^{p'-1} (ab + \epsilon[j][j + 1])^{1/2}}|0\rangle',
\]

\[
R_{a,b,(p'-2)/2}(I_2)|0\rangle' = \frac{i(ab + \epsilon[2])^{1/2}}{q^{(p'-2)/2} - q^{-(p'-2)/2}}|1\rangle' + c|p' - 1\rangle',
\]

\[
R_{a,b,(p'-2)/2}(I_2)|0\rangle'' = \frac{i(ab + \epsilon[2])^{1/2}}{q^{(p'-2)/2} - q^{-(p'-2)/2}}|1\rangle'',
\]

where \( c \) is a nonzero coefficient easily determined from the above formulas. Hence, the subspaces \( \mathcal{H}_1 + \mathbb{C}|p' - 1\rangle \) and \( \mathcal{H}_2 \) of the representation space are invariant with respect to the representation \( R_{a,b,(p'-2)/2} \) (we denote these subrepresentations by \( R_{a,b,(p'-2)/2}^1 \) and \( R_{a,b,(p'-2)/2}^2 \), respectively). Remark that

\[
\dim \mathcal{H}_1 + \mathbb{C}|p' - 1\rangle = \frac{1}{2}(p' + 1), \quad \dim \mathcal{H}_2 = \frac{1}{2}(p' - 1).
\]

If

\[
a \prod_{j=1}^{p'-1} (ab + \epsilon[j][j + 1])^{1/2} - i(ab + \epsilon[2])^{1/2} = 0,
\]

\[24\]
\[(ab + \epsilon[2])^{1/2} \prod_{j=1}^{p'-1} (ab + \epsilon[j][j+1])^{1/2} = -ib, \quad (59)\]

then
\[R_{a,b,(p'-2)/2}(I_2)|p' - 1\rangle \frac{\Pi_{j=1}^{p'} (ab + \epsilon[j][j+1])^{1/2}}{i(ab + \epsilon[2])^{1/2}} |0\rangle'',
\[R_{a,b,(p'-2)/2}(I_2)|0\rangle' = \frac{i(ab + \epsilon[2])^{1/2}}{q^{(p'-2)/2} - q^{-(p'-2)/2}} |1\rangle',
\[R_{a,b,(p'-2)/2}(I_2)|0\rangle'' = \frac{i(ab + \epsilon[2])^{1/2}}{q^{(p'-2)/2} - q^{-(p'-2)/2}} |1\rangle'' + c|p' - 1\rangle,

where \(c\) is a nonzero coefficient. Hence, now the subspaces \(H_1\) and \(H_2 + C|p' - 1\rangle\) of the representation space are invariant. We denote the subrepresentations on these subspaces by \(\hat{R}_{a,b,(p'-2)/2}\) and \(\hat{R}_{a,b,(p'-2)/2}\), respectively). Note that the representation \(\hat{R}_{a,b,(p'-2)/2}\) is not equivalent to \(R_{a,b,(p'-2)/2}\) (and the representation \(\hat{R}_{a,b,(p'-2)/2}\) is not equivalent to \(R_{a,b,(p'-2)/2}\)) since the parameters \(a\) and \(b\) determining these representations satisfy different equations.

If \(a\) and \(b\) do not satisfy the relations (56) and (57) or the relations (58) and (59), then the representation \(R_{a,b,(p'-2)/2}\) is irreducible.

Let now \(p'\) be odd and \(\lambda = -q^{(p'-2)/2}\). In this case, the representation \(R_{a,b,-(p'-2)/2}\) is irreducible if \(a\) and \(b\) do not satisfy the relations (56) and (57) or the relations (58) and (59). If \(a\) and \(b\) satisfy the relations (56) and (57), then \(R_{a,b,-(p'-2)/2}\) is a reducible representation and decomposes into the direct sum of two subrepresentations acting on the subspaces \(H_1 + C|p' - 1\rangle\) and \(H_2\). These subrepresentations are denoted by \(R_{a,b,-(p'-2)/2}\) and \(R_{a,b,-(p'-2)/2}\), respectively, and are determined as
\[R_{a,b,-(p'-2)/2}(I_1) = -R_{a,b,-(p'-2)/2}(I_1), \quad R_{a,b,-(p'-2)/2}(I_2) = -R_{a,b,-(p'-2)/2}(I_2), \quad i = 1, 2.

Similarly, if \(a\) and \(b\) satisfy the relations (58) and (59), then \(R_{a,b,-(p'-2)/2}\) is a reducible representation and decomposes into the direct sum of two subrepresentations acting on the subspaces \(H_1\) and \(H_2 + C|p' - 1\rangle\). These subrepresentations are denoted by \(\hat{R}_{a,b,-(p'-2)/2}\) and \(\hat{R}_{a,b,-(p'-2)/2}\), respectively, and are determined as
\[\hat{R}_{a,b,-(p'-2)/2}(I_1) = -\hat{R}_{a,b,-(p'-2)/2}(I_1), \quad \hat{R}_{a,b,-(p'-2)/2}(I_2) = -\hat{R}_{a,b,-(p'-2)/2}(I_2), \quad i = 1, 2.

**Proposition 11.** Let the conditions (56) and (57) are satisfied. Then the representations \(R_{a,b,(p'-2)/2}, R_{a,b,(p'-2)/2}, R_{a,b,-(p'-2)/2}\) and \(R_{a,b,-(p'-2)/2}\) are irreducible and pairwise nonequivalent. If the conditions (58) and (59) are satisfied, then the representations \(\hat{R}_{a,b,(p'-2)/2}, \hat{R}_{a,b,(p'-2)/2}, \hat{R}_{a,b,-(p'-2)/2}\) and \(\hat{R}_{a,b,-(p'-2)/2}\) are irreducible and pairwise nonequivalent.

Proof is similar to that of the previous propositions and we omit it.

**X. OTHER REPRESENTATIONS OF** \(U_q(\text{so}_3)\) **FOR** \(q\) **A ROOT OF UNITY**

In the previous section we described irreducible representations of \(U_q(\text{so}_3)\) obtained from irreducible representations of the algebra \(\hat{U}_q(\text{sl}_2)\) for \(q\) a root of unity. However, at \(q\) a root of unity the algebra \(U_q(\text{so}_3)\) has irreducible representations which cannot be derived from those of \(\hat{U}_q(\text{sl}_2)\). They are obtained as irreducible components of the representations \(Q_\lambda\) from Section VII when one put \(q\) equal to a root of unity. We describe these representations of \(U_q(\text{so}_3)\) in this section.
Let \( \lambda = q^\tau \) be a nonzero complex number such that \( 0 \leq \text{Re} \tau < 1 \) and let \( \mathcal{H} \) be the \( p' \)-dimensional complex vector space with basis

\[
|m\rangle, \quad m = 0, 1, 2, \ldots, p' - 1.
\]

We define on this space the operators \( Q'_\lambda(I_1) \) and \( Q'_\lambda(I_2) \) determined by the formulas

\[
Q'_\lambda(I_1)|m\rangle = \frac{\lambda q^m + \lambda^{-1} q^{-m}}{q - q^{-1}} |m\rangle,
\]

\[
Q'_\lambda(I_2)|0\rangle = \frac{1}{q - q^{-1}} |1\rangle + \frac{1}{q - q^{-1}} |p' - 1\rangle,
\]

\[
Q'_\lambda(I_2)|p' - 1\rangle = \frac{1}{q - q^{-1}} |p' - 2\rangle + \frac{1}{q - q^{-1}} |0\rangle,
\]

\[
Q'_\lambda(I_2)|m\rangle = \frac{1}{q - q^{-1}} |m - 1\rangle + \frac{1}{q - q^{-1}} |m + 1\rangle, \quad m \neq 0, p' - 1.
\]

A direct computation shows that these operators satisfy the relations (7) and (8) and hence determine a representation of \( U_q(\text{so}_3) \) which will be denoted by \( Q'_\lambda \).

**Theorem 6.** If \( \lambda \neq 1 \) and \( \lambda \neq q^{1/2} \), then the representation \( Q'_\lambda \) is irreducible.

Proof of this proposition is the same as that of the first part of Proposition 3.

The representations \( Q'_1 \) and \( Q'_{\sqrt{q}} \) are studied in the same way as the representations \( Q_1 \) and \( Q_{\sqrt{q}} \) in Section VII. This study leads to the irreducible representations of \( U_q(\text{so}_3) \) which are described below. (Note that the description of these representations for \( p' \) even and for \( p' \) odd is deferent.)

Let \( p' \) be odd. We denote by \( \mathcal{H}_r \) and \( \mathcal{H}_s, r = \frac{1}{2}(p' + 1), s = \frac{1}{2}(p' - 1) \), the complex vector spaces with the bases

\[
|0\rangle, \quad |1\rangle, \quad |2\rangle, \ldots, |\frac{1}{2}(p' - 1)\rangle
\]

and

\[
|1\rangle, \quad |2\rangle, \ldots, |\frac{1}{2}(p' - 1)\rangle,
\]

respectively. Four representations \( Q_{1}^{\pm, \pm} \) act on the space \( \mathcal{H}_r \) and are given by the formulas

\[
Q_{1}^{\pm, \pm}(I_1)|m\rangle = \frac{q^m + q^{-m}}{q - q^{-1}} |m\rangle, \quad m = 0, 1, 2, \ldots, \frac{1}{2}(p' - 1),
\]

\[
Q_{1}^{\pm, \pm}(I_2)|\frac{1}{2}(p' - 1)\rangle = \pm \frac{1}{q - q^{-1}} |\frac{1}{2}(p' - 1)\rangle + \frac{1}{q - q^{-1}} |\frac{1}{2}(p' - 3)\rangle,
\]

\[
Q_{1}^{\pm, \pm}(I_2)|m\rangle = \frac{1}{q - q^{-1}} |m + 1\rangle + \frac{1}{q - q^{-1}} |m - 1\rangle, \quad m < \frac{1}{2}(p' - 1),
\]

and by the formulas

\[
Q_{1}^{\pm, \pm}(I_1)|m\rangle = -\frac{q^m + q^{-m}}{q - q^{-1}} |m\rangle, \quad m = 0, 1, 2, \ldots, \frac{1}{2}(p' - 1),
\]

\[
Q_{1}^{\pm, \pm}(I_2) := Q_{1}^{\pm, \pm}(I_2).
\]

Note that the upper sign corresponds to the representations \( Q_{1}^{+, +} \) and \( Q_{1}^{-, +} \) and the lower sign to the representations \( Q_{1}^{+, -} \) and \( Q_{1}^{-, -} \).
On the space $\mathcal{H}_s$, four representations $\hat{Q}^{\pm,\pm}_s$ act by the corresponding formulas (60)–(64), but now $m$ runs over the values $1, 2, 3, \ldots, \frac{1}{2}(p’ - 1)$.

Let now $\mathcal{H}_r$ and $\mathcal{H}_s$, $r = \frac{1}{2}(p’ + 1)$, $s = \frac{1}{2}(p’ - 1)$, be the complex vector spaces with the bases

$$|m + \frac{1}{2}\rangle, \quad m = 0, 1, 2, \ldots, \frac{1}{2}(p’ - 1),$$

and

$$|m + \frac{1}{2}\rangle, \quad m = 0, 1, 2, \ldots, \frac{1}{2}(p’ - 3),$$

respectively. The four representations $Q^{\pm,\pm}_s$ act on the space $\mathcal{H}_r$ and are given by the formulas

$$Q^{\pm,\pm}_s(I_1)|m + \frac{1}{2}\rangle = \frac{q^{m+1/2} + q^{-m-1/2}}{q - q^{-1}}|m + \frac{1}{2}\rangle,\quad m = 0, 1, 2, \ldots, \frac{1}{2}(p’ - 1),$$

$$Q^{\pm,\pm}_s(I_2)|\frac{1}{2}\rangle = \pm \frac{1}{q - q^{-1}}|\frac{1}{2}\rangle + \frac{1}{q - q^{-1}}|\frac{3}{2}\rangle,$$

$$Q^{\pm,\pm}_s(I_2)|m + \frac{1}{2}\rangle = \frac{1}{q - q^{-1}}|m + \frac{3}{2}\rangle + \frac{1}{q - q^{-1}}|m - \frac{1}{2}\rangle,\quad m \neq 0,$$

where $|m + \frac{3}{2}\rangle \equiv 0$ if $m = \frac{1}{2}(p’ - 1)$, and by the formulas

$$Q^{\pm,\pm}_s(I_1)|m + \frac{1}{2}\rangle = -\frac{q^{-m+1/2} + q^{m-1/2}}{q - q^{-1}}|m + \frac{1}{2}\rangle,\quad m = 0, 1, 2, \ldots, \frac{1}{2}(p’ - 1),$$

$$Q^{\pm,\pm}_s(I_2) := Q^{\pm,\pm}_s(I_2).$$

On the space $\mathcal{H}_s$, four representations $\hat{Q}^{\pm,\pm}_s$ act by the corresponding formulas (65)–(69), but now $m$ runs through the values $0, 1, 2, \ldots, \frac{1}{2}(p’ - 3)$.

Let now $p’$ be even. We denote by $\mathcal{H}_r$ and $\mathcal{H}_s$, $r = \frac{1}{2}(p’ + 2)$, $s = \frac{1}{2}(p’ - 2)$, the complex vector spaces with the bases

$$|0\rangle,\quad |1\rangle,\quad |2\rangle,\quad \ldots,\quad |\frac{1}{2}(p’ - 2)\rangle,$$

and

$$|1\rangle,\quad |2\rangle,\quad \ldots,\quad |\frac{1}{2}(p’ - 2)\rangle,$$

respectively. The representations $Q^{\pm,\pm}_1$ act on $\mathcal{H}_r$ and $\mathcal{H}_s$, respectively, which are given by the formulas

$$Q^{\pm,\pm}_1(I_1)|m\rangle = \frac{q^m + q^{-m}}{q - q^{-1}}|m\rangle,\quad i = 1, 2,$$

$$Q^{\pm,\pm}_1(I_2)|m\rangle = \frac{1}{q - q^{-1}}|m + 1\rangle + \frac{1}{q - q^{-1}}|m - 1\rangle,\quad i = 1, 2,$$

where $|m + 1\rangle$ or $|m - 1\rangle$ must be put equal to 0 if the corresponding vector does not exist.

Let $\mathcal{H}_{p’/2}$ be the complex vector space with the basis

$$|m + \frac{1}{2}\rangle,\quad m = 0, 1, 2, \ldots, \frac{1}{2}(p’ - 2).$$

Four representations $\hat{Q}^{\pm,\pm}_s$ act on this space which are given by the formulas

$$\hat{Q}^{\pm,\pm}_s(I_1)|m + \frac{1}{2}\rangle = \frac{q^{m+1/2} + q^{-m-1/2}}{q - q^{-1}}|m + \frac{1}{2}\rangle.$$
\begin{align*}
\hat{Q}^{\pm,\pm}(I_2)|\frac{1}{2}\rangle &= \pm \left( \frac{1}{q - q^{-1}} |\frac{1}{2}\rangle + \frac{1}{q - q^{-1}} |\frac{3}{2}\rangle \right), \\
\hat{Q}^{\pm,\pm}(I_2)|\frac{1}{2}(p' - 2)\rangle &= \pm \left( \frac{1}{q - q^{-1}} |\frac{1}{2}(p' - 2)\rangle + \frac{1}{q - q^{-1}} |\frac{1}{2}(p' - 4)\rangle \right), \\
\hat{Q}^{\pm,\pm}(I_2)|m + \frac{1}{2}\rangle &= \frac{1}{q - q^{-1}} |m - \frac{1}{2}\rangle + \frac{1}{q - q^{-1}} |m + \frac{3}{2}\rangle, \quad m \neq \frac{1}{2}, \frac{1}{2}(p' - 2),
\end{align*}

and by the formulas
\begin{align*}
\hat{Q}^{-,\pm}(I_1)|m + \frac{1}{2}\rangle &= - \frac{q^{m+1/2} + q^{-m-1/2}}{q - q^{-1}} |m + \frac{1}{2}\rangle, \\
\hat{Q}^{-,\pm}(I_2) &= \hat{Q}^{+,\pm}(I_2).
\end{align*}

Let us mention peculiarities of the representations described above. The operators \(Q^{\pm,\pm}(I_2), Q^{\pm,\pm}(I_2), \hat{Q}^{\pm,\pm}(I_2)\) have nonzero diagonal matrix elements and nonzero traces. Moreover, the operators \(\hat{Q}^{\pm,\pm}(I_2)\) have two such diagonal elements. Spectra of the operators \(Q^{\pm,\pm}(I_1), Q^{\pm,\pm}(I_1), Q^{\pm,\pm}(I_1)\) and \(\hat{Q}^{\pm,\pm}(I_1)\) are not symmetric with respect to the zero point.

**Proposition 12.** The representations \(Q^{\pm,\pm}, Q^{\pm,\pm}, \hat{Q}^{\pm,\pm}, Q^{\pm,\pm}, Q^{\pm,\pm}, \hat{Q}^{\pm,\pm}\) are irreducible and pairwise nonequivalent. No representation \(Q^{\lambda}_\lambda\) is equivalent to any of these representations.

**Proof** is the same as that of Proposition 3.

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