Phase-locking of driven vortex lattices with transverse ac-force and periodic pinning

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For a vortex lattice moving in a periodic array we show analytically and numerically that a new type of phase locking occurs in the presence of a longitudinal dc driving force and a transverse ac driving force. This phase locking is distinct from the Shapiro step phase locking found with longitudinal ac drives. We show that an increase in critical current and a fundamental phase locked step width scale with the square of the driving ac amplitude. Our results should carry over to other systems such as vortex motion in Josephson-junction arrays.

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I. INTRODUCTION

When an external ac drive is applied to a dc driven system interacting with a periodic potential, resonance between the external ac frequency and the frequency of motion in the periodic potential can give rise to phase locking. This phenomenon is found in a wide variety of nonlinear systems in condensed matter physics. A particularly well known example is the ac/dc-driven single small Josephson-junction$^{[1,2]}$ and Josephson-junction arrays$^{[3]}$, where Shapiro steps are observed in the current-voltage characteristics. Shapiro type phase locking has recently also been studied in experiments$^{[4]}$ and theory$^{[3]}$ for vortex motion in two-dimensional superconductors with periodic arrays of pinning sites, as well as in superconductors where the vortices are driven over a periodic potential generated by thickness modulations$^{[5]}$. Moreover, driven systems with many degrees of freedom in the presence of quenched disorder can exhibit phase-locking when there is a dynamically induced periodicity. Examples of this are charge density waves$^{[6]}$ and vortex lattices in superconductors with random pinning$^{[3,7]}$. In the case of vortices in superconductors with periodic pinning$^{[6,8]}$, a vortex can be viewed as an overdamped particle moving along a tilted washboard potential where the tilt is produced by the dc force from an applied current as well as by the superimposed ac currents in the same direction.

An important difference of vortex arrays from charge density wave systems and single degree of freedom systems (like a small Josephson junction) is that the displacement field acting on vortices is a two-dimensional vector. This means that displacements can be induced in two different directions, and therefore a new kind of phase locking is possible when an ac force is applied transverse to the direction of the longitudinally applied dc force. The type of phase-locking observed in this case is qualitatively different from the Shapiro type phase-locking observed when dc and ac forces are in parallel. We first show analytically the possibility of the phase locked states for certain commensurate vortex configurations and predict that the critical current increases quadratically with the ac amplitude and that the width of some of the phase-locked steps scale as the square of the ac amplitude. We also find scaling of the critical current as well as the first phase locked region as a function of the ac frequency and the pinning geometry. The predictions of the perturbation analysis are confirmed with numerical simulations. We analyze the validity of the perturbation approach and demonstrate how deviations occur. Our results suggest that more pronounced transverse phase-locking may be observed at lower commensurate magnetic fields, such as $B = 1.5B_o$ or $B = 1.25B_o$. The general model and perturbation results are easily generalizable to other systems which exhibit Shapiro step-like phase locking such as Josephson-junction arrays and Frenkel-Kontorova type models.

Our studies are directly relevant for several contemporary efforts since vortex matter interacting with nanostructured pinning arrays of holes$^{[10,12]}$ and dots$^{[13,14]}$ has been attracting increasing attention due to the easily tunable pinning properties. Pronounced commensuration effects are observed in these systems when the density of vortices matches to integer or fractional multiples of the density of pinning sites. In addition to square pinning arrays, recent experiments have been conducted on rectangular pinning arrays$^{[4]}$. Simulations have shown several interesting dynamical phases of dc driven vortices in periodic pinning systems$^{[5,11]}$. Imaging$^{[12]}$ and transport experiments$^{[13]}$ along with simulations$^{[14]}$ have found that the vortex motion above the first matching field can occur by the flow of interstitial vortices between the pinning sites. These vortices still experience a periodic potential created by the vortices located at the pinning sites. Recently, phase locking was observed for dc and ac driven vortices interacting with periodic...
pinning at $B = 2B_\phi$, where $B_\phi$ is the field at which the number of vortices equals the number of pinning sites [3], where the dc and ac drive where in the same direction. The values of the voltage response in Ref. [4] strongly suggest that is is the interstitial vortices that are mobile while the vortices on pinning sites remain immobile.

II. THE MODEL

We will adopt a simple overdamped particle model for describing vortices. The overdamped normalized equation of motion for vortex $i$ is

$$\dot{r}_i = -\nabla i U(\{r_j\}) + f_d ,$$

(1)

where $r_i = (x_i, y_i)$ is the coordinate of vortex $i$ normalized to the effective 2D penetration depth, $\Lambda = 2\lambda^2/d$, $\lambda$ being the penetration depth and $d \ll \lambda$ is the sample thickness. The energy surface, $U$, is of the order of the sample size.

We have previously demonstrated that phase-locking of the vortex motion can exist if a certain periodicity of the global pinning potential is present. This was shown for driving forces of the form, $f_d = (\eta + \varepsilon \sin \Omega t) \hat{x}$, where $\hat{x}$ is a unit vector in the x-direction. Analogous to the well-known Shapiro steps [4] in the current-voltage characteristics of an ac/dc-driven Josephson junction, it was demonstrated that nonzero intervals of $\eta$ will result in phase-locking of vortex motion with vortex velocities resonating with the external ac-field. We will in this paper demonstrate that one can also expect phase-locking in the transversely ac-driven case, i.e., for

$$f_d = \left\{ \begin{array}{ll} \eta \\ \varepsilon \sin \Omega t \end{array} \right. ,$$

(5)

where $\eta$ and $\varepsilon$ are the magnitudes of dc and ac forces and $\Omega$ is the frequency of the ac drive. The relevant equations of motion are therefore,

$$\dot{x}_i + \frac{\partial}{\partial x_i} U(\{r_j\}) = \eta$$

(6)

$$\dot{y}_i + \frac{\partial}{\partial y_i} U(\{r_j\}) = \varepsilon \sin \Omega t$$

(7)

III. ANALYTICAL RESULTS

We will consider a system with periodic pinning in a rectangular periodic lattice with plaquette area $A = l_x l_y$, where $l_x$ and $l_y$ are the dimensions of the rectangular plaquettes. We will assume that the applied magnetic field is larger than the first matching field, such that all pinning centers are occupied by trapped vortices. We will in the following assume that the trapped vortices remain trapped at the pinning sites and immobile at all times. Thus the interstitial vortices move in a strictly periodic lattice of immobile vortices with periodicity $(l_x, l_y)$. We can then perform a perturbational analysis of the interstitial vortex behavior through the following assumption: any interstitial vortices moves such that the forces from other interstitial vortices cancel by symmetry [5]. Thus, analyzing the behavior of the interstitial (mobile) vortices, we will adopt the potential function for a single interstitial vortex moving in a lattice of pinned vortices (at positions $(nl_x, (n + 1/2)l_y)$), as given in Eq. (4). Thus, the interstitial vortex position is subject to a potential as in Eq. (4) where the coordinate is the vortex position, $(x_i, y_i)$, and with the lattice constants here being the distances between pinning centers, $(l_x, l_y)$, instead of the computational system dimensions, $(L_x, L_y)$.

A. Critical dc-force without ac-drive

The first result to evaluate for the purely dc-driven case is the “critical dc current”, $\eta_{c}^{0}$, below which the
interstitial vortex is trapped by the pinned vortex lattice. As \( \eta \) is increased from \( \eta = 0 \), the interstitial vortex will find an equilibrium point for which the pinned vortex lattice provides a cancelling force to the dc bias. However, for a given (critical) value of \( \eta \), namely \( \eta_c(0) \), the pinned vortex lattice can no longer provide enough force to resist motion of the interstitial, which therefore begins to propagate in the \( x \)-direction. Assuming that the pinned vortices do not move and that an interstitial vortex only interacts with pinned vortices, this critical force can be found as the maximum gradient value of Eq. (4) for \( y_{ij} = mL_\phi \). The corresponding equation of motion is given by Eq. (8):

\[
\dot{x} + \frac{2\pi}{l_x} \sum_{n=-\infty}^{\infty} \frac{\sin \left( \frac{2\pi x}{l_x} \right)}{\cosh \left( \frac{2\pi l_y}{l_x} \left( \frac{1}{2} + n \right) \right) - \cos \left( \frac{2\pi l_x}{l_y} \right)} = \eta.
\]

(8)

This equation can produce a critical current by requiring \( \dot{x} = 0 \) and optimizing the left hand side through the varying the position, \( x_i \). In general, this must be done numerically. The result is shown in Fig. (1), where the critical dc force (thick solid) is shown together with the optimized value of the equilibrium value of \( x_i = x_* \). In the limit of sech(\( \pi l_y/l_x \)) \( \ll 1 \), we can derive the approximate expression of the critical current by noticing that only two terms (\( n = -1, 0 \)) are required to describe the critical current, which is then always given by \( x_i = x_* = \frac{1}{4} x_c \):

\[
\eta_c(0) = k \text{sech} \left( \frac{\pi l_y}{l_x} \right), \quad \text{for } \frac{\pi}{2} \gg \frac{l_x}{l_y}
\]

(9)

\[
k = \frac{2\pi}{l_x}
\]

(10)

The approximate expression, Eq. (9), is shown in figure 1 as a thin solid curve, validating that the interaction potential between interstitial and pinned vortices can be simplified to the terms \( n = -1, 0 \) for \( l_x \gg l_y \). One can also see that in the extreme opposite limit, \( l_x \gg l_y \), the critical dc force, \( \eta_c(0) \), can be approximated by \( \pi/l_y \). This is shown as a dashed line in figure 1.

It is here important to remember that the analysis, which is based on a magnetic field of “matching field plus one flux quantum”, is actually valid for magnetic fields, \( B_\phi < B \leq 2B_\phi \), for which the vortex configuration results on forces between interstitial vortices cancelling due to symmetry; such magnetic fields can be, e.g., \( B = 2B_\phi, \frac{3}{2}B_\phi, \frac{5}{4}B_\phi, \cdots \).

B. Transverse ac-drive included

Let us assume that \( \text{sech} \left( \frac{\pi l_y}{l_x} \right) \ll 1 \), so that we can write the approximate equations of motion for the interstitial vortices as,

\[
\dot{x} + \eta_c(0) \left[ 1 + \frac{1}{4} k^2 \varepsilon \right] \sin (x, k) = \eta
\]

(11)

\[
\dot{y} + \frac{2\pi}{l_x l_y} y_i = \varepsilon \sin \Omega t,
\]

(12)

where we have retained terms in \( y_i \) only to lowest order. We have required \( \pi |y_i| \ll l_x \) in order to obtain Eqs. (11) and (12).

The solution to the second of those equations is easily found,

\[
y_i = \varepsilon \sin (\Omega t + \theta)
\]

(13)

with

\[
\varepsilon = \frac{\varepsilon}{\sqrt{1 + \left( \frac{2\pi}{\Omega l_x l_y} \right)^2}}
\]

(14)

\[
\theta = \tan^{-1} \frac{2\pi}{\Omega l_x l_y}
\]

(15)

Inserting Eq. (13) into Eq. (11) yields,

\[
\dot{x} + \eta_c(0) \left[ 1 + \frac{1}{4} k^2 \varepsilon \right] \sin kx_i
\]

\[
\left. - \eta_c(0) \frac{1}{4} k^2 \varepsilon^2 \cos (2\Omega t + \theta) \sin kx_i \right) = \eta
\]

This equation is the effective equation of motion for the vortex behavior in the longitudinal (\( x \)) direction given an ac-force in the transverse (\( y \)) direction, derived within the approximations listed above. The equation is equivalent to that of an overdamped pendulum (\( kx_i \) being the pendulum phase) with dc-torque (\( \eta \)) and a pivot vertically oscillating with frequency \( 2\Omega \) and amplitude proportional to \( \varepsilon^2 \) — i.e., an overdamped equivalent of the classic Kapitza problem [13] for underdamped and parametrically driven pendula.

We will now consider a few separate cases of vortex responses to the transverse ac-force: \( \langle x_i \rangle = 0 \Rightarrow x_i(t) = x_0 \). We will omit the last term on the left hand side of Eq. (11). The remaining (dc) terms are:

\[
\eta_c \sin kx_0 = \eta
\]

(17)

\[
\Leftrightarrow |\eta| \leq \eta_c = \eta_c(0) \left[ 1 + \frac{1}{4} k^2 \varepsilon^2 \right] = \eta_c(0) + \delta \eta_c
\]

(18)

\[
\Leftrightarrow \eta_c = \eta_c(0) \left[ 1 + \frac{1}{4} k^2 \varepsilon^2 \right] = \eta_c(0) + \delta \eta_c
\]

(19)

where \( \eta_c \) is the critical dc force. It is here worthwhile to notice that the critical current, \( \eta_c \), increases \( (\delta \eta_c \) positive) quadratically with the transverse ac-amplitude, \( \varepsilon \).
This is in direct contrast to the case when the ac-drive is longitudinal, in which case the critical dc force decreases quadratically \[\varepsilon^2\]. We remember that validity of the expressions requires \(\varepsilon \ll \Omega \). \[\langle \dot{x}_i \rangle > 0 \Leftrightarrow |\eta| > \eta_c\]: Without the parametric term in Eq. (16), the solution, for \(\eta^2 \geq \eta_c^2\), is given by \[\theta = \eta_0 = \eta_0^2\], is neglected. Let us assume the following simple ansatz for the vortex motion,

\[
\dot{x}_i = \eta_0 \frac{1 - \eta_0^2}{1 - k_0 \eta_0} v_0 = \eta_0 \langle x_0 \rangle = \frac{1}{k_0} \eta_0^2 v_0, \tag{21}
\]

where \(\eta_0\) is the dc force when the parametric term in Eq. (16) is neglected. Let us assume the following simple ansatz for the vortex motion,

\[
x_i(t) = v_0 t + x_0 \tag{22}
\]

where \(x_0\) is a constant. Inserting this ansatz into the effective equation of motion, Eq. (16), while only maintaining dc terms, yields,

\[
v_0 = \langle \dot{x}_i \rangle = \sqrt{\eta_0^2 - \eta_c^2}, \tag{20}
\]

\[
\dot{x}_i = \eta_0 \frac{1 - \eta_0^2}{1 - k_0 \eta_0} \frac{c}{\pi \varepsilon} \equiv \langle \varepsilon \rangle \langle \dot{x}_i \rangle = \eta_0 + \delta \eta.
\]

With the ansatz of Eq. (22) we can therefore expect the term \(\langle \dot{x}_i \rangle\) to contribute to \(\delta \eta\) if \(kv_0 = 2\Omega\) (see figure 2), and the resulting relationship between the internal phase, \(kx_0 - 2\theta\), and the dc current, \(\eta_0 + \delta \eta\), is:

\[
2\Omega/k - \frac{1}{2} \delta \eta \sin(kx_0 - 2\theta) = \eta_0 + \delta \eta. \tag{24}
\]

As the phase, \(kx_0 - 2\theta\), can be adjusted to balance this equation for different choices of \(\delta \eta\), we can argue for a nonzero range, \(\Delta \eta\), in dc-force for which the average speed of the interstitial vortices is unchanged. This phase-locking range has the magnitude:

\[
\Delta \eta = \delta \eta_c. \tag{25}
\]

Thus, we can predict phase-locking in the transversely ac-driven vortex system and the predicted total range in phase-locking is equal to the increase in the critical dc-force due to the ac-drive. This prediction is correct up to and including terms \(\propto \varepsilon^2\).

This range in bias current for which the average speed (voltage) of the interstitial vortex is constant will manifest itself as a step in the dc current-voltage characteristics of the system. It is important to emphasize that this step is very different in origin from the Shapiro steps recently demonstrated for longitudinal ac-drive. The essential difference lies in the fact that the Shapiro steps arise from a “direct” driving term in an effective pendulum equation, whereas the present phase-locking for the transverse ac-drive arises from an effective parametric ac-driving term in the longitudinal equation of motion.

If one chooses a better (and more complicated) ansatz for the vortex motion (e.g., Eq. (21) instead of Eq. (22)) it becomes evident that an \(\varepsilon^2\) phase-locked step also exists for \(kv_0 = \Omega\) and that higher order phase-locked steps may exist at any sub and super harmonic of the driving frequency. However, we will not go into detail with other phase-locked modes in this presentation.

**IV. NUMERICAL SIMULATIONS**

In order to validate the analysis of Sec.III predicting some basic effects of a transverse ac-force on vortices, we have conducted numerical simulations of driven interstitial vortices in a rectangular lattice of pinned vortices.

Figure 3 shows a series of simulated (normalized) current (dc-force) voltage (average vortex velocity, \(v_0\)) for different values of the transverse ac-drive. The system is a square array of \(4 \times 4\) pinned vortices with periodicity \((l_x, l_y) = (2, 2)\) in a computational simulation box of size \((L_x, L_y) = (4l_x, 4l_y)\). The simulated magnetic field corresponds to \(B/B_0 = 1.5\) such that each pinning center is occupied by a vortex and every other plaquette has an interstitial vortex. The applied normalized frequency is \(\Omega = 4\). The four curves (shifted vertically for clarity) in figure 3 are for (from top) \(\varepsilon = 0, 1, 2, 3\). According to the analysis above, the critical dc-force, \(\eta_c\), resulting in onset of voltage (vortex transport) should increase quadratically for increasing \(\varepsilon\). This is clearly visible from figure 3. A phase-locked step in dc bias current should develop for increasing \(\varepsilon\) around \(kv_0 = 2\Omega\). This is also clearly visible for \(\eta \approx 2.5\). As the ac amplitude is increased we observe more steps in the IV characteristics. However, while the simplest of these, \(kv_0 = \Omega\), can be analyzed in some detail, we will be concerned with only the steps at \(kv_0 = 0\) and \(kv_0 = 2\Omega\) here.

Two characteristically different vortex evolutions are shown as insets to figure 3. At bottom right we show a snapshot (●) of a vortex configuration within the parameter range of the phase-locked step for \(\varepsilon = 2\) together with a long time trace (thin line) of the vortex trajectories. The vortex paths are obviously periodic and the interstitial vortices seem to move in a geographically simple configuration where all interactions between the interstitials cancel due to symmetry. The top left inset shows the traces outside of the phase-locked region. Here we observe the pinned vortices as static dots, while the interstitials move in a chaotic or quasiperiodic band between the pinning sites.

A detailed investigation of the validity of the above perturbation results was performed for several different sets of system parameters. For a system of a single interstitial vortex, we show in figure 4 the detailed comparisons between the analytical predictions (lines), Eq. (19), and numerical simulations (markers) of the increase, \(\delta \eta_c\), in critical dc force as a function of ac-amplitude. Figure 4a shows data for square pinning arrays of different plaquette areas \((l_xl_y = 4, 16)\) and with different driving
we can correlate the magnitude of lattice deformation for relatively large values of \( \epsilon \). The interstitial vortex-vortex interaction is non-zero. The lattice deforms into a configuration where the effective interaction for \( L_x \) interstitial vortices will break down for densely populated systems with \( l = 2l_y \). Once again we observe a measurable phase-locked step for very small ac-amplitudes. The measures of lattice deformation are defined as,

\[
\xi_i^{(x)} = (x_i \mod l_x) \\
\xi_i^{(y)} = (y_i \mod l_y) \\
\xi_{i,j}^{(x)} = \begin{cases} 
|\xi_j^{(x)} - \xi_i^{(x)}|, & |\xi_j^{(x)} - \xi_i^{(x)}| \leq \frac{L_x}{2} \\
|\xi_j^{(x)} - \xi_i^{(x)}|, & |\xi_j^{(x)} - \xi_i^{(x)}| > \frac{L_x}{2} 
\end{cases} \\
\xi_{i,j}^{(y)} = \begin{cases} 
|\xi_j^{(y)} - \xi_i^{(y)}|, & |\xi_j^{(y)} - \xi_i^{(y)}| \leq \frac{L_y}{2} \\
|\xi_j^{(y)} - \xi_i^{(y)}|, & |\xi_j^{(y)} - \xi_i^{(y)}| > \frac{L_y}{2} 
\end{cases} \\
\langle |\xi^{(x)}| \rangle_{i,j} = \langle |\xi_{i,j}^{(x)}| \rangle_{i,j} \\
\langle |\xi^{(y)}| \rangle_{i,j} = \langle |\xi_{i,j}^{(y)}| \rangle_{i,j} \\
\sigma_x = \sqrt{\langle |\xi_i^{(x)}|^2 \rangle_t - \langle |\xi_{i,j}^{(x)}|^2 \rangle_{i,j}} \\
\sigma_y = \sqrt{\langle |\xi_i^{(y)}|^2 \rangle_t - \langle |\xi_{i,j}^{(y)}|^2 \rangle_{i,j}}
\]

Finally, these averages are averaged over the phase-locked step to provide a single measure for a given \( \epsilon \) (\(|\langle |\xi^{(x,y)}| \rangle \rangle \) and \( \sigma_{x,y} \) are typically smallest near the center of the locking range). Thus, \( 0 \leq \langle |\xi^{(x,y)}| \rangle \leq l_{x,y}/2 \) is a measure of the average spatial lattice deformation and \( \sigma_{x,y} \) is a measure of the average temporal fluctuations in the lattice deformation, \( \langle |\xi^{(x,y)}| \rangle \).

The minimum averaged lattice deformation parameters for a given phase-locked step are shown in figure 8 as solid markers (\( \bullet \): \( \langle |\xi^{(x)}| \rangle \) and \( \langle |\xi^{(y)}| \rangle \)). We observe large lattice deformations at small ac-amplitudes, where the agreement between the observed range of phase-locking is poor, while the deformations “collapse” for larger values of \( \epsilon \), thereby giving rise to better agreement between simulations and analysis. We have also shown the average of the temporal fluctuations, \( \sigma_x \) and \( \sigma_y \), in order to illustrate that the lattice deformations depend on time when \( \langle |\xi^{(x)}| \rangle \) and \( \langle |\xi^{(y)}| \rangle \) are non-zero. Thus, it indicates that internal modes of the lattice deformation are excited and may contribute significantly to the phase-locking range in the low ac-amplitude cases. For large ac-amplitudes we observe a saturation of the range in phase-locking accompanied by reappearance of the deformation lattice. This is consistent with chaotic behavior which are expected for strong ac-driven nonlinear systems.


V. CONCLUSION

We have developed a simple analysis of the behavior of interstitial vortices in systems with periodic pinning and transverse ac-drive. The result indicates that the dc-driven longitudinal motion can phase-lock to the transverse ac-signal and that the range of phase-locking (in dc-current) is quadratic in the ac-amplitude and we have developed a quantitative expression providing detailed dependencies of also other relevant system parameters, such as pinning geometry and driving frequency. We have further demonstrated that the critical current increases quadratically with the ac-amplitude. The perturbation results have been validated by numerical simulations which show good agreement with the analytical predictions in the expected range of system parameters. The mechanism of phase-locking discussed in this paper is distinct from phase-locking to a longitudinally applied ac-signal. The latter case was studied in [3] and exhibits qualitatively different responses to ac-perturbations, such as decreasing critical dc-current with increasing ac-amplitude and phase-locked steps that grow linearly with ac-amplitude. The results shown in this paper are for commensurate fields. Incommensurate fields, where no simple geometrical relationship can exist between the pinned and interstitial lattices, are characterized by non-cancelling interactions between interstitial vortices. Thus, phase-locking ranges for the non-commensurate fields usually have magnitudes less than predicted by the single interstitial analysis.

Our results and analyses indicate that the most important phase-locking appears at relatively small magnetic fields (with respect to the first matching field) where the inter-vortex repulsion is not deforming the interstitial vortex lattice.

The predictions in this paper should be directly applicable for experimental verification in superconductors (or Josephson junction arrays) where dc and ac fields are orthogonal.

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FIG. 1. Static critical force, $\eta_0^{(0)} x$, for $\varepsilon = 0$ as a function of rectangular periodic pinning aspect ratio. Thick lines represent the maximum force derived from Eq. (3) for the optimal position, $x_\circ$, along the $x$-axis. Thin line represent Eq. (4), which is valid for $\text{sech}(\pi l_x/l_x) \ll 1$, dashed line represent $\pi/l_x$, which is asymptotically correct for $l_x \gg l_y$.

FIG. 2. Sketch of interstitial vortex (●) trajectory in rectangular lattice of pinned vortices (○). Upper part of sketch shows a case where $\langle \dot{x}_i \rangle = 0$ while the lower part shows a case where $\langle \dot{x}_i \rangle = l_x \Omega/\pi$. Ac-force is transverse, dc-force is longitudinal.

FIG. 3. Simulated IV characteristics for different values of transverse ac-amplitude, $\varepsilon = 0, 1, 2, 3$ (top down). IV curves are vertically offset for clarity. System parameters are: $L_x = L_y = 4l_x = 4l_y = 8$, $\Omega = 4$, and $B = 1.5B_\phi$. Lower inset shows vortex trajectories in a phase-locked state while upper inset shows trajectories outside a phase locked regime.

FIG. 4. Increase of critical dc current, $\delta \eta_0$, as a function of ac-amplitude, $\varepsilon$. Markers represent numerical simulations of equations (3) and (4). Solid lines are the corresponding predictions from Eq. (14). Simulations are conducted for a single interstitial vortex. (a) $\Delta$: $l_x = l_y = 2$ and $\Omega = 4$; $\bigcirc$: $l_x = l_y = 2$ and $\Omega = 8$; $\bigodot$: $l_x = l_y = 4$ and $\Omega = 8$. (b) $\Delta$: $l_x = 2l_y = 4$ and $\Omega = 4$; $\bigodot$: $2l_x = l_y = 4$ and $\Omega = 8$.

FIG. 5. Magnitude, $\Delta \eta$, of the phase-locked step at $kv_0 = 2\Omega$ as a function of ac-amplitude, $\varepsilon$. Markers represent numerical simulations of equations (3) and (4). Solid lines are the corresponding predictions from Eq. (25). Simulations are conducted for a single interstitial vortex. (a) $\Delta$: $l_x = l_y = 4$ and $\Omega = 4$; $\bigodot$: $l_x = l_y = 4$ and $\Omega = 8$; $\bigcirc$: $l_x = l_y = 2$ and $\Omega = 4$; $\bigodot$: $2l_x = l_y = 4$ and $\Omega = 8$. (b) $\Delta$: $l_x = 2l_y = 4$ and $\Omega = 4$; $\bigodot$: $2l_x = l_y = 4$ and $\Omega = 8$; $\bigcirc$: $2l_x = l_y = 4$ and $\Omega = 8$.
