Invariance-like Results for Nonautonomous Switched Systems

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Abstract

This paper generalizes the LaSalle-Yoshizawa Theorem to switched nonsmooth systems. Filippov and Krasovskii regularizations of a switched system are shown to be contained within the convex hull of the Filippov and Krasovskii regularizations of the subsystems, respectively. A candidate common Lyapunov function that has a negative semidefinite derivative along the trajectories of the subsystems is shown to be sufficient to establish LaSalle-Yoshizawa results for the switched system. Results for regular and non-regular candidate Lyapunov functions are developed via appropriate generalization of the notion of a time derivative. The developed generalization is motivated by adaptive control of switched systems where the derivative of the candidate Lyapunov function is typically negative semidefinite.

Index Terms

switched systems, differential inclusions, adaptive systems, nonlinear systems

I. INTRODUCTION

The focus of this paper is Lyapunov-based stability analysis of switched nonautonomous systems that admit non-strict candidate Lyapunov functions (cLfs) (i.e., cLfs with time derivatives bounded by a negative semidefinite function of the state). The theoretical development is motivated by the application of adaptive control methods to systems where either the control design or the dynamics dictate the need for a switched systems analysis. For example, neuromuscular electrical stimulation applications such as [1]–[4] involve switching between different muscle groups during different phases of operation to reduce fatigue [1], [4], to compensate for changing muscle geometry [3], or to perform functional tasks that require multi-limb coordination [2]. Such applications stand to

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This research is supported in part by NSF award numbers 1509516 and 1508757, ONR grant number N00014-13-1-0151, AFOSR Award Number FA9550-15-1-0155, and a contract with the AFRL, Munitions Directorate at Eglin AFB. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the sponsoring agency.
benefit from adaptive methods where the controller adapts to the uncertain dynamics without strictly relying on high gain or high frequency feedback often associated with robust control methods that can lead to overstimulation.

Switched dynamics are inherent in a variety of modern adaptation strategies. For example, in sparse neural networks [5], the use of different approximation architectures for different regions of the state-space introduce switching via the feedforward part of the controller. In adaptive gain scheduling methods [6], switching is introduced due to changing feedback gains. Switching is also utilized as a tool to improve transient response of adaptive controllers by selecting between multiple estimated models of stable plants (see, e.g., [7]–[16]). In addition, switched systems theory can be utilized to extend the scope of existing adaptive solutions to more complex circumstances that involve switched dynamics.

Lyapunov-based stability analysis of switched nonautonomous adaptive systems is challenging because adaptive update laws typically result in non-strict Lyapunov functions for the individual subsystems. For each subsystem, convergence of the error signal to the origin is typically established using Barbălat’s lemma (e.g., [17, Lemma 8.2]). In traditional methods that utilize multiple Lyapunov functions (e.g., [18, Theorem 3.2]) the class of admissible switching signals is restricted based on the rate of decay of the cLf (cf. [18, Eq. 3.10]). Since Barbălat’s lemma provides no information about the rate of decay of the cLf, it alone is insufficient to establish stability of a switched system using multiple Lyapunov functions. Approaches based on common cLfs have been developed for systems with negative definite Lyapunov derivatives; however, common cLf-based approaches do not trivially extend to systems with non-strict Lyapunov functions (cf. [19]–[21] and [18, Example 2.1]).

Because of complications resulting from a negative semidefinite Lyapunov derivative, few results are available in literature that examine adaptive control of uncertain nonlinear switched systems. An adaptive controller for switched nonlinear systems is developed in [22] using a generalization of Barbălat’s lemma from [23]. The controller is shown to asymptotically stabilize a switched system with parametric uncertainties in the subsystems. Multiple Lyapunov functions are utilized to analyze the stability of the switched system. However, the generalized Barbălat’s Lemma in [23] requires a minimum dwell time, and in general, minimum dwell time cannot be guaranteed when the switching is state-dependent.

Results such as [24]–[27] extend the Barbashin-Krasovskii-LaSalle invariance principle to discontinuous systems. However, these results are for autonomous systems, whereas the development in this paper is focused on nonautonomous systems. An extension of the LaSalle-Yoshizawa Theorem to nonsmooth nonautonomous systems is provided in [28, Theorem 2.5]; however, the result requires the cLf to be continuously differentiable, whereas the approach developed here uses a more general framework that utilizes locally Lipschitz-continuous cLfs.

This paper generalizes the LaSalle-Yoshizawa Theorem (see, e.g., [29] and [17, Theorem 8.4]) and its nonsmooth extensions in results such as [28, Theorem 2.5], and [30] to switched nonsmooth systems and nonregular Lyapunov functions. A non-strict common Lyapunov function (i.e., a common cLf with a negative semidefinite derivative) is used to establish boundedness of the system state and convergence of a positive semidefinite function of the system state to zero under arbitrary switching between nonsmooth nonlinear systems.

The paper is organized as follows. Notation is defined in Section II. Section III defines the class of systems considered along with the objectives. Sections IV and V are dedicated to the development of the main results of the
paper. Section VII provides a discussion on the merits of the generalized time derivatives defined in Section V. Section VIII presents illustrative examples, and Section IX provides concluding remarks. The appendix includes supplementary proofs.

II. NOTATION

The \( n \)-dimensional Euclidean space is denoted by \( \mathbb{R}^n \) and \( \mu \) denotes the Lebesgue measure on \( \mathbb{R}^n \). Elements of \( \mathbb{R}^n \) are interpreted as column vectors and \( (\cdot)^T \) denotes the vector transpose operator. The set of positive integers excluding 0 is denoted by \( \mathbb{N} \). For \( a \in \mathbb{R} \), the notation \( \mathbb{R}_{\geq a} \) denotes the interval \( [a, \infty) \) and the notation \( \mathbb{R}_{>a} \) denotes the interval \( (a, \infty) \). For a relation \( (\cdot) \), the notation \( (\cdot)^{\text{a.e.}} \) implies that the relation holds for almost all \( t \in I \), for some interval \( I \). Unless otherwise specified, an interval \( I \) is assumed to be right open, of nonzero length, and \( t_0 := \min I \).

The notation \( F : A \rightrightarrows B \) is used to denote a set-valued map from \( A \) to the subsets of \( B \). The notations \( \text{co} A \) and \( \overline{\text{co}} A \) are used to denote the convex hull and the closed convex hull of the set \( A \), respectively. If \( a \in \mathbb{R}^m \) and \( b \in \mathbb{R}^n \) then the notation \([a; b]\) denotes the concatenated vector \( \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^{m+n} \). For \( A \subseteq \mathbb{R}^m \), \( B \subseteq \mathbb{R}^n \) the notations \( \begin{bmatrix} A \\ B \end{bmatrix} \) and \( A \times B \) are interchangeably used to denote the set \( \{[a; b] \mid a \in A, b \in B\} \).

The notations \( \overline{B}(x, r) \) and \( \overline{B}(x, r) \) for \( x \in \mathbb{R}^n \) and \( r > 0 \) are used to denote the sets \( \{y \in \mathbb{R}^n \mid \|x - y\| \leq r\} \) and \( \{y \in \mathbb{R}^n \mid \|x - y\| < r\} \), respectively. The notation \( |(\cdot)| \) denotes the absolute value if \( (\cdot) \in \mathbb{R} \) and the cardinality if \( (\cdot) \) is a set. The notation \( \mathcal{L}_\infty(A, B) \) denotes essentially bounded functions from \( A \) to \( B \).

III. SWITCHED SYSTEMS AND DIFFERENTIAL INCLUSIONS

Consider a switched system of the form

\[
\dot{x} = f_\rho(x, t) \ (x, t),
\]

where \( \rho : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightarrow \mathcal{N}^{\sigma} \) denotes a state-dependent switching signal, \( \mathcal{N}^{\sigma} \subseteq \mathbb{N} \) is the set of all possible switching indices, and \( x \in \mathbb{R}^n \) denotes the system state. The collection \( \{f_\sigma : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightarrow \mathbb{R}^n\}_{\sigma \in \mathcal{N}^{\sigma}} \) is assumed to be locally bounded, uniformly in \( \sigma \) and \( t \), and the functions \( t \mapsto f_\sigma(x, t) \) and \( t \mapsto \rho(x, t) \) are assumed to be Lebesgue measurable \( \forall x \in \mathbb{R}^n \) and \( \forall \sigma \in \mathcal{N}^{\sigma} \).

Let \( f : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightarrow \mathbb{R}^n \) denote the function \( f(x, t) := f_\rho(x, t) \ (x, t) \). Since the collection \( \{f_\sigma\}_{\sigma \in \mathcal{N}^{\sigma}} \) locally bounded, uniformly in \( \sigma \) and \( t \), the function \( f \) is locally bounded, uniformly in \( t \). To establish measurability of \( f \), consider the representation \( f(x, t) = \sum_{\sigma \in \mathcal{N}^{\sigma}} I_\sigma(\rho(x, t)) f_\sigma(x, t) \), where

\[
I_\sigma(i) := \begin{cases} 
1, & i = \sigma, \\
0, & i \neq \sigma.
\end{cases}
\]

Since \( I_\sigma : \mathbb{N} \rightarrow \mathbb{R} \) is continuous \( \forall \sigma \in \mathcal{N}^{\sigma} \), \( t \mapsto I_\sigma(\rho(x, t)) \) is Lebesgue measurable \( \forall (\sigma, x) \in \mathcal{N}^{\sigma} \times \mathbb{R}^n \). Lebesgue measurability of \( t \mapsto f(x, t) \), \( \forall x \in \mathbb{R}^n \) then follows from that of \( t \mapsto f_\sigma(x, t) \), \( \forall (\sigma, x) \in \mathcal{N}^{\sigma} \times \mathbb{R}^n \).

\(^1\text{A collection of functions } \{f_\sigma : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightarrow \mathbb{R}^n \mid \sigma \in \mathcal{N}^{\sigma}\} \text{ is locally bounded, uniformly in } t \text{ and } \sigma, \text{ if for every compact } K \subseteq \mathbb{R}^n, \text{ there exists } M > 0 \text{ such that } \|f_\sigma(x, t)\|_2 \leq M, \forall (x, t) \in K \times \mathbb{R}_{\geq t_0} \text{ and } \forall \sigma \in \mathcal{N}^{\sigma}.\)
The main objective of this paper is to establish asymptotic properties of the generalized solutions to the system
\[ \dot{x} = f(x, t), \] (2)
using asymptotic properties of the generalized solutions to the individual subsystems
\[ \dot{x} = f_\sigma(x, t). \] (3)

The advantage of the strategy pursued in this paper, as opposed to directly analyzing (2), is that the analysis can be made independent of the switching function. That is, when established through the subsystems in (3), the stability properties of (2) are invariant with respective to the switching function over a wide range of switching functions. On the other hand, a direct analysis of (2) is valid only for the specific $\rho$ used to construct (2).

For a measurable function $g : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}$ the Filippov regularization is defined as [31, p. 85]
\[ K_F[g](x, t) := \bigcap_{\delta > 0, \mu(N) = 0} \overline{co} \{ g(y, t) \mid y \in B(x, \delta) \setminus N \}, \] (4)
and the Krasovskii regularization is defined as [32, p. 17]
\[ K_K[g](x, t) := \bigcap_{\delta > 0} \overline{co} \{ g(y, t) \mid y \in B(x, \delta) \}. \] (5)

In the following, generalized solutions of the systems in (2) and (3), defined using Filippov and Krasovskii regularization are analyzed. When a Filippov regularization is considered, the local boundedness requirement on the map $x \mapsto f_\sigma(x, t)$ is relaxed to essential local boundedness and a stronger measurability requirement is imposed so that $(x, t) \mapsto f_\sigma(x, t)$ and $(x, t) \mapsto \rho(x, t)$ are Lebesgue measurable $\forall \sigma \in N^\sigma$.

To achieve the stated objective, the differential inclusion that results from regularization of the overall switched system is proven to be contained within the convex combination of the differential inclusions that result from regularization of the subsystems under mild assumptions on the switching signal (Proposition 1, Section IV). To facilitate the discussion that follows, the existence of a non-strict Lyapunov function is shown to be sufficient to infer certain asymptotic properties of solutions to differential inclusions (Theorem 1, Section V). It is then established that a common non-strict Lyapunov function for the differential inclusions that result from regularization of the individual subsystems is also a non-strict Lyapunov function for the differential inclusion that results from regularization of the switched system (Proposition 2, Section VI). The main result of the paper then follows, i.e., conclusions about asymptotic properties of generalized solutions to (2) can be drawn from the asymptotic properties of the generalized solutions to (3) (Theorem 2).

The following section develops the aforementioned relationship between the differential inclusions resulting from regularization of the subsystems and the switched system.

IV. SWITCHING AND REGULARIZATION

Let $\dot{x} \in F(x, t) := K_F[f](x, t)$ and $\dot{x} \in F_\sigma(x, t) := K_F[f_\sigma](x, t)$ be Filippov regularizations and $\dot{x} \in K_\sigma(x, t) := K_K[f_\sigma](x, t)$ and $\dot{x} \in K_\sigma(x, t) := K_K[f_\sigma](x, t)$ be Krasovskii regularizations of (2) and (3), respectively. The following assumption imposes a mild restriction on the switching function $\rho$ to establish a relationship between $F$, $\{F_\sigma\}$, $K$, and $\{K_\sigma\}$.

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Assumption 1. For each \((x, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq t_0}\), there exists \(\delta^* > 0\) such that \(|\rho(B(x, \delta^*), t)| < \infty\). △

That is, \(\rho\) is locally bounded in \(x\) for each \(t\). Roughly speaking, Assumption 1 restricts infinitely many subsystems from being active in a small neighborhood of the state space. It does not restrict Zeno behavior and arbitrary time-dependent switching, and as such, is not restrictive. For further insight into why Assumption 1 is invoked, see Example 1. The following proposition states that under general conditions, the set-valued maps \(F\) and \(K\) are contained, pointwise, within the convex combination of the collections \(\{F_\sigma\}\) and \(\{K_\sigma\}\), respectively.

Proposition 1. Provided \(\rho : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathcal{N}^o\) satisfies Assumption 1, there exists a set \(E \subseteq \mathbb{R}_{\geq t_0}\) with \(\mu(\mathbb{R}_{\geq t_0} \setminus E) = 0\) such that the set-valued maps \(F, F_\sigma, K,\) and \(K_\sigma\) satisfy

\[
\begin{align*}
K(x, t) &\subseteq \bigcup_{\sigma \in \mathcal{N}^o} K_\sigma(x, t), \quad \forall (x, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq t_0}, \quad (6) \\
F(x, t) &\subseteq \bigcup_{\sigma \in \mathcal{N}^o} F_\sigma(x, t), \quad \forall (x, t) \in \mathbb{R}^n \times E. \quad (7)
\end{align*}
\]

Under the additional assumption that \(\forall \sigma \in \mathcal{N}^o\), there exist countable collections of measure-zero sets \(\{N_{\sigma i}\}_{i \in \mathbb{N}}\), and a constant \(\delta > 0\) such that \(\forall (x, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq t_0}\) and \(\forall \delta \in (0, \delta]\)

\[
\bigcap_{\mu(N) = 0} \left\{ f_\sigma(y, t) \mid y \in B(x, \delta) \setminus N \right\}
= \bigcap_{i \in \mathbb{N}} \left\{ f_\sigma(y, t) \mid y \in B(x, \delta) \setminus N_{\sigma i} \right\}, \quad (8)
\]

the inclusion in (7) holds with \(E = \mathbb{R}_{\geq t_0}\).

Proof for Krasovskii regularization: Fix \((x, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq t_0}\), select \(\delta^* > 0\) such that \(|\rho(B(x, \delta^*), t)| < \infty\) and let \(\mathcal{N} := \rho(B(x, \delta^*), t)\). Observe that the containment in (6) is straightforward if the union over \(\sigma\) is placed inside the convex closure operation. That is,

\[
\bigcap_{\delta > 0} \left\{ f_{\rho(y, t)}(y, t) \mid y \in B(x, \delta) \right\} \subseteq \bigcap_{\delta > 0} \bigcup_{\sigma \in \mathcal{N}} \left\{ f_\sigma(y, t) \mid y \in B(x, \delta) \right\}. \quad (9)
\]

The rest of the proof shows that the right hand side (RHS) of (9) is contained within the RHS of (6) in two steps. The first step is to show that

\[
\bigcap_{\delta > 0} \bigcup_{\sigma \in \mathcal{N}} \left\{ f_\sigma(y, t) \mid y \in B(x, \delta) \right\} \subseteq \bigcap_{\delta > 0} \bigcup_{\sigma \in \mathcal{N}} \left\{ f_\sigma(y, t) \mid y \in B(x, \delta) \right\}. \quad (10)
\]

The condition in (8) is satisfied by most discontinuous dynamical systems encountered in practice. For example, discontinuities resulting from sliding mode controllers, piecewise continuous reference signals, etc., satisfy (8). Hence, (8) is not restrictive in practice.

 Existence of such a \(\delta^*\) is guaranteed by Assumption 1.

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The second step is to show that
\[ \bigcap_{\delta > 0} \bigcup_{\sigma \in \mathcal{N}} \mathcal{C} \{ f_\sigma (y, t) \mid y \in B(x, \delta) \} \subseteq \bigcup_{\sigma \in \mathcal{N}} \mathcal{C} \{ f_\sigma (y, t) \mid y \in B(x, \delta) \}. \tag{11} \]

The result in (6) then follows from (9), (10), and (11).

To prove (10), fix \( \delta \in (0, \delta^*) \) and let \( z \in \mathcal{C} \bigcup_{\sigma \in \mathcal{N}} \{ f_\sigma (y, t) \mid y \in B(x, \delta) \} \). There exists a sequence \( \{ z_i \}_{i \in \mathbb{N}} \in \mathbb{R}^n \) such that \( z_i \in \bigcup_{\sigma \in \mathcal{N}} \{ f_\sigma (y, t) \mid y \in B(x, \delta) \}, \forall i \in \mathbb{N} \), and \( \lim_{i \to \infty} z_i = z \). Furthermore, by Carathéodory’s Theorem \([33, p. 103]\), there exists collection of \( m \leq n + 1 \) points \( \{z_{i_1}, \ldots, z_{i_m}\} \subset \mathbb{R}^n \), positive real numbers \( \{a_{i_1}, \ldots, a_{i_m}\} \) for which \( \sum_{j=1}^{m} a_{ij} = 1 \), and integers \( \{\sigma_{i_1}, \ldots, \sigma_{i_m}\} \in \mathcal{N} \), such that \( z_{ij} \in \{ f_{\sigma_j} (y, t) \mid y \in B(x, \delta) \} \) and \( z_i = \sum_{j=1}^{m} a_{ij} z_{ij} \). Hence, \( z = \lim_{i \to \infty} \sum_{j=1}^{m} a_{ij} z_{ij} \), that is, \( z = \lim_{i \to \infty} Z_i A_i \), where \( A_i = [a_{i_1} \; \cdots \; a_{i_m}] \) and \( Z_i = [z_{i_1}; \ldots; z_{i_m}]^T \).

Since the coefficients \( a_{ij} \geq 0 \) are bounded, the sequence \( \{A_i\}_{i \in \mathbb{N}} \) is a bounded sequence. Hence, there exists a subsequence \( \{A_{ik}\}_{k \in \mathbb{N}} \) such that \( \lim_{k \to \infty} A_{ik} = A \), for some \( A = [a_{1} \; \cdots \; a_{m}] \). Since the function \( A_i \mapsto \sum_{j=1}^{m} a_{ij} \) is continuous, \( \sum_{j=1}^{m} a_{j} = 1 \). Since the set \( \bigcup_{\sigma \in \mathcal{N}} \{ f_\sigma (y, t) \mid y \in B(x, \delta) \} \) is bounded, the sequence \( \{Z_{ik}\}_{k \in \mathbb{N}} \) is bounded. Hence, there exists a subsequence \( \{Z_{ik}\}_{k \in \mathbb{N}} \) such that \( \lim_{k \to \infty} Z_{ik} = Z \), element-wise, for some \( Z = [z_1; \ldots; z_m]^T \). Hence, \( z = \lim_{k \to \infty} Z_{ik} A_{ik} = Z A \), where the columns \( z_j \) of the matrix \( Z \) are the limits \( \lim_{k \to \infty} z_{i_k j} \). Hence, \( z \in \mathcal{C} \bigcup_{\sigma \in \mathcal{N}} \{ f_{\sigma_j} (y, t) \mid y \in B(x, \delta) \} \). Therefore, the point \( z \) is a convex combination of points from \( \mathcal{C} \bigcup_{\sigma \in \mathcal{N}} \{ f_{\sigma_j} (y, t) \mid y \in B(x, \delta) \} \). That is, \( z \in \bigcup_{\sigma \in \mathcal{N}} \mathcal{C} \{ f_\sigma (y, t) \mid y \in B(x, \delta) \} \). This proves (10).

To establish (11), let \( z \in \bigcap_{\delta > 0} \bigcup_{\sigma \in \mathcal{N}} \mathcal{C} \{ f_\sigma (y, t) \mid y \in B(x, \delta) \} \). Note that if \( 0 < \delta_1 \leq \delta_2 \), then
\[ \bigcup_{\sigma \in \mathcal{N}} \mathcal{C} \{ f_\sigma (y, t) \mid y \in B(x, \delta_1) \} \subseteq \bigcup_{\sigma \in \mathcal{N}} \mathcal{C} \{ f_\sigma (y, t) \mid y \in B(x, \delta_1) \}. \]

That is, if \( z \in \bigcup_{\sigma \in \mathcal{N}} \mathcal{C} \{ f_\sigma (y, t) \mid y \in B(x, \delta_1) \} \) for some \( 0 < \delta_1 \), then \( z \in \bigcap_{\delta > \delta_1} \bigcup_{\sigma \in \mathcal{N}} \mathcal{C} \{ f_\sigma (y, t) \mid y \in B(x, \delta) \} \). Hence, \( \forall k \in \mathbb{N} \), such that \( k \geq 1 \), there exist \( \{z_{k1}, \cdots, z_{k|\mathcal{N}|}\} \subset \mathbb{R}^n \), nonnegative real numbers \( \{a_{k1}, \cdots, a_{k|\mathcal{N}|}\} \) for which \( \sum_{j=1}^{|\mathcal{N}|} a_{kj} = 1 \), such that \( z_{kj} \in \bigcup_{\delta > \frac{1}{k}} \mathcal{C} \{ f_{\sigma_j} (y, t) \mid y \in B(x, \delta) \} \) and \( z = \sum_{j=1}^{|\mathcal{N}|} a_{kj} z_{kj} \). That is, \( z = Z_k A_k \), where \( A_k = [a_{k1} \; \cdots \; a_{k|\mathcal{N}|}] \) and \( Z_k = [z_{k1}; \cdots; z_{k|\mathcal{N}|}]^T \).

Since the sequences \( \{Z_k\}_{k \in \mathbb{N}} \) and \( \{A_k\}_{k \in \mathbb{N}} \) are bounded, there exist subsequences \( \{Z_{ik}\}_{i \in \mathbb{N}} \) and \( \{A_{ik}\}_{i \in \mathbb{N}} \) and vectors \( Z := [z_1; \cdots; z_{|\mathcal{N}|}]^T \) and \( A := [a_1; \cdots; a_{|\mathcal{N}|}] \) such that \( A = \lim_{i \to \infty} A_{ik}, \sum_{j=1}^{|\mathcal{N}|} a_{j} = 1 \), and \( Z = \lim_{i \to \infty} Z_{ik} \). Since \( z = Z_{ik} A_{ik}, \forall k \in \mathbb{N} \), it can be concluded that \( z = Z A \).

It is now claimed that \( \forall j \in \{1, \cdots, |\mathcal{N}|\}, z_j \in \bigcup_{\delta > 0} \mathcal{C} \{ f_{\sigma_j} (y, t) \mid y \in B(x, \delta) \} \). To prove the claim by contradiction, assume that \( \exists \delta^* > 0 \) such that \( z_j \notin \mathcal{C} \{ f_{\sigma_j} (y, t) \mid y \in B(x, \delta^*) \} \). Since
\[ \mathcal{C} \{ f_{\sigma_j} (y, t) \mid y \in B(x, \delta_1) \} \subseteq \mathcal{C} \{ f_{\sigma_j} (y, t) \mid y \in B(x, \delta_2) \}, \tag{12} \]

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\[ \forall \sigma_j \in \mathcal{N} \text{ and } \forall \delta_1 \leq \delta_2, \, z_j \notin \operatorname{co} \{ f_{\sigma_j}(y,t) \mid y \in B(x,\delta) \}, \forall \delta \geq \delta^* \text{. That is, } \exists k^*_j \in \mathbb{N} \text{ such that } z_j \notin \bigcap_{\delta \geq \frac{\delta^*}{k^*_j}} \operatorname{co} \{ f_{\sigma_j}(y,t) \mid y \in B(x,\delta) \}, \forall k_j \geq k^*_j \text{. From (12) and the fact that the sets } \bigcap_{\delta \geq \frac{\delta^*}{k^*_j}} \operatorname{co} \{ f_{\sigma_j}(y,t) \mid y \in B(x,\delta) \} \text{ are closed, it can be concluded that there exists } \epsilon > 0 \text{ such that } \forall k_l \geq k^*_l, \]

\[ B(z_j, \epsilon) \notin \bigcap_{\delta \geq \frac{\delta^*}{k^*_l}} \operatorname{co} \{ f_{\sigma_j}(y,t) \mid y \in B(x,\delta) \}. \tag{13} \]

Since \( z_{k_l} \in \bigcap_{\delta \geq \frac{\delta^*}{k^*_l}} \operatorname{co} \{ f_{\sigma_j}(y,t) \mid y \in B(x,\delta) \}, \forall k_l \in \mathbb{N} \), (13) contradicts \( z_j = \lim_{t \to \infty} z_{k_l} \).

Hence, \( z \) is a convex combination of points from \( \bigcap_{\delta \geq \frac{\delta^*}{k^*_l}} \operatorname{co} \{ f_{\sigma_j}(y,t) \mid y \in B(x,\delta) \} \). That is, \( z \in \operatorname{co} \bigcup_{\sigma \in \mathcal{N}} \bigcap_{\delta > 0} \operatorname{co} \{ f_{\sigma_j}(y,t) \mid y \in B(x,\delta) \} \), which proves (11), and hence, (6). The proof for Filippov regularization involves technical details related to exclusion of measure-zero sets that are provided in the appendix. \[ \blacksquare \]

The following example demonstrates that Assumption 1 is not vacuous.

**Example 1.** Let \( \mathcal{N}^\circ = \mathbb{N} \) and for \( \sigma \in \mathcal{N}^\circ \), let \( f_{\sigma} \) be defined as

\[ f_{\sigma}(x) := \begin{cases} 0 & |x| < \frac{1}{2^\sigma}, \\ 1 & |x| \geq \frac{1}{2^\sigma}, \end{cases} \]

so that \( K_\sigma(0) = F_\sigma(0) = \{0\}, \forall \sigma \in \mathcal{N}^\circ \). Let

\[ \rho(x) = \begin{cases} \sigma & x \in (-\frac{1}{2^{\sigma+1}}, -\frac{1}{2^\sigma}] \cup [\frac{1}{2^\sigma}, \frac{1}{2^{\sigma-1}}), \\ 1 & \text{otherwise} \end{cases} \]

Clearly, \( \rho \) violates Assumption 1 at \( x = 0 \). In this case, \( f(x) = \begin{cases} 1 & x \neq 0, \\ 0 & x = 0 \end{cases} \). Therefore, \( K(0) = [0,1] \) and \( F(0) = \{1\} \), and hence, Proposition 1 does not hold without the switching restriction in Assumption 1. \[ \blacksquare \]

To facilitate the analysis of \( F \) and \( K \) based on the analysis of the individual subsystems \( F_\sigma \) and \( K_\sigma \), respectively, a stability result for differential inclusions that relies on non-strict Lyapunov functions is developed in the following section. While the results developed in Section V are specific to differential inclusions that arise from Filippov and Krasovskii regularization of differential equations with discontinuous right-hand sides, the results developed in the following sections are more general in the sense that they apply to generic set-valued maps not necessarily resulting from Filippov or Krasovskii regularization.

**V. Non-strict Lyapunov functions for differential inclusions**

Let \( F : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^n \) be a set-valued map. Consider a differential inclusion of the form

\[ \dot{x} \in F(x,t). \tag{14} \]

A locally absolutely continuous function \( x : \mathcal{I} \to \mathbb{R}^n \) is called a solution of (14) over the closed interval \( \mathcal{I} \) provided

\[ \dot{x}(t) \in F(x(t),t), \tag{15} \]

\[ \text{The authors are grateful to the anonymous reviewer who suggested this example.} \]
for almost all $t \in I$ \cite[p. 50]{31}. To ensure existence of local solutions, the following restrictions are placed on the map $F$.

**Assumption 2.** For all $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, there exist $a, b \in \mathbb{R}_{>0}$ such that $F : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \mapsto \mathbb{R}^n$ satisfies the hypothesis of \cite[p. 83, Theorem 5]{31}.

Under Assumption 2 local solutions of \eqref{14} exists starting from any $(t_0, x_0) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ (cf. \cite[p. 83, Theorem 5]{31}).

In this paper, the solutions to \eqref{14} are analyzed using Lyapunov-like comparison functions with negative semidefinite derivatives. To this end, generalized time derivatives and non-strict Lyapunov functions are defined as follows.

**Definition 1.** Let $F : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \mapsto \mathbb{R}^n$ be nonempty and compact valued. The generalized time derivative of a locally Lipschitz-continuous function $V : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}$ with respect to $F$ is the function $\dot{\bar{V}}_F : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}$ defined as (cf. \cite{34})

$$
\dot{\bar{V}}_F (x, t) := \max_{p \in \partial V (x, t)} \max_{q \in F (x, t)} p^T [q; 1],
$$

where $\partial V$ denotes the Clarke gradient of $V$ \cite[p. 39]{35}.

See Section VII for a detailed comparison of Definition 1 with more popular set-valued notions of generalized time derivatives (i.e., \cite[eq. 13]{36} and \cite[p. 364]{37}).

**Definition 2.** Let $D \subseteq \mathbb{R}^n$ be an open and connected set and let $\Omega := D \times I$ for some interval $I$. Let $F : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \mapsto \mathbb{R}^n$ be nonempty and compact valued over $\Omega$. Let $V : \Omega \to \mathbb{R}$ be a locally Lipschitz-continuous positive definite function. Let $\underline{W}, \overline{W} : D \to \mathbb{R}$ be continuous positive definite functions and let $W : D \to \mathbb{R}$ be a continuous positive semidefinite function. If

$$
\underline{W} (x) \leq V (x, t) \leq \overline{W} (x), \quad \forall (x, t) \in \Omega,
$$

and

$$
\dot{\bar{V}}_F (x, t) \leq -W (x),
$$

$\forall x \in D$ and for almost all $t \in \mathbb{R}_{\geq t_0}$, then $V$ is called a non-strict Lyapunov function for $F$ over $\Omega$ with the bounds $\underline{W}, \overline{W},$ and $W$.

The following theorem establishes the fact that the existence of a non-strict Lyapunov function implies that $t \mapsto W (x (t))$ asymptotically decays to zero.

**Theorem 1.** Let $D \subseteq \mathbb{R}^n$ be an open and connected set, $r > 0$ be selected such that $\overline{B} (0, r) \subset D$ and $\Omega := D \times \mathbb{R}_{\geq t_0}$. Let $F : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \mapsto \mathbb{R}^n$ be a map that satisfies Assumption 2 and is locally bounded, uniformly in $t$, over $\Omega$. If $V : \Omega \to \mathbb{R}$ is a non-strict Lyapunov function for $F$ over $\Omega$ with the bounds $\underline{W} : D \to \mathbb{R}$,

\footnote{A set valued map $F : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^n$ is locally bounded, uniformly in $t$, over $\Omega$, if for every compact $K \subset D$, there exists $M > 0$ such that $\forall (x, t, y)$ such that $(x, t) \in K \times \mathbb{R}_{\geq t_0}$, and $y \in F (x, t)$, $\|y\|_2 \leq M$.}
∀ \text{if the sets } \{ x \in \mathbb{R}^n | W(x) \leq c \}, \text{ are complete, bounded, and satisfies } \lim_{t \to \infty} W(x(t)) = 0. \text{ In addition, if } D = \mathbb{R}^n \text{ and if the sets } \{ x \in \mathbb{R}^n | W(x) \leq c \} \text{ are compact, } \forall c \in \mathbb{R}_{>0}, \text{ then the result is global. Furthermore, if the non-strict Lyapunov function is regular } [35, \text{ Definition 2.3.4}], \text{ then (18) can be relaxed to } V_{\bar{F}}(x,t) \leq W(x), \text{ where}

\begin{equation}
V_{\bar{F}}(x,t) := \min_{p \in \partial V(x,t)} \max_{q \in \bar{F}(x,t)} p^T[q;1].
\end{equation}

\text{Proof: See the appendix.} \hfill \blacksquare

The following section utilizes the results of Sections IV and V to develop the main results of this paper.

VI. INVARIANCE-LIKE RESULTS FOR SWITCHED SYSTEMS

The following proposition states that a common non-strict Lyapunov function for a family of differential inclusions is also a non-strict Lyapunov function for the closure of their convex combination.\[^{6}\]

\textbf{Proposition 2.} \textit{Let } \Omega \subseteq \mathbb{R}^n \times \mathbb{R}_{\geq t_0}. \textit{Let } \{ F_{\sigma} : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \Rightarrow \mathbb{R}^n | \sigma \in \mathcal{N}^0 \} \textit{be a family of set-valued maps with compact and nonempty values that is locally bounded, uniformly in } \sigma, \textit{over } \Omega \times \mathcal{N}^0. \textit{If } V : \Omega \Rightarrow \mathbb{R} \textit{is a common non-strict Lyapunov function for the family } \{ F_{\sigma} \} \textit{over } \Omega \textit{with the bounds } W : D \rightarrow \mathbb{R}, \bar{W} : D \rightarrow \mathbb{R}, \textit{and } W : D \rightarrow \mathbb{R} \textit{ (i.e., } V \textit{is a non-strict Lyapunov function for } F_{\sigma} \textit{for each } \sigma \in \mathcal{N}^0 \textit{and the bounds } W, \bar{W}, \textit{and } W \textit{are independent of } \sigma). \textit{Then } V \textit{is also a non-strict Lyapunov function for } \overline{\cup}_{\sigma \in \mathcal{N}^0} F_{\sigma}(x,t) \textit{over } \Omega \textit{ with the bounds } W, \bar{W}, \textit{and } W. \textit{Proof: Since the maps } \{ F_{\sigma} \} \textit{are locally bounded, uniformly in } \sigma, \textit{over } \Omega \times \mathcal{N}^0, \overline{\cup}_{\sigma \in \mathcal{N}^0} F_{\sigma}(x,t) \textit{is nonempty and compact for all } (x,t) \in \Omega. \textit{Since } V \textit{is a common non-strict Lyapunov function, } \max_{p \in \partial V(x,t)} \max_{q \in F_{\sigma}(x,t)} p^T[q;1] \leq -W(x), \forall \sigma \in \mathcal{N}^0. \textit{Let } q^* \in F(x,t) := \overline{\cup}_{\sigma \in \mathcal{N}^0} F_{\sigma}(x,t). \textit{There exists a sequence } \{ q_j \}_{j \in \mathbb{N}} \textit{such that } \lim_{j \to \infty} q_j = q^* \textit{and } q_j \in \co \cup_{\sigma \in \mathcal{N}^0} F_{\sigma}(x,t). \textit{By Carathéodory’s theorem } [33, \text{ p. 103}], \textit{there exist } \sum_{i=1}^{m} a_i z_1^i, \textit{where } \sum_{i=1}^{m} a_i = 1, a_i \geq 0, \textit{and } z_i \in F_{\sigma_i}(x,t), \forall i \in \{1, \cdots, m\}. \textit{For any fixed } p \in \partial V(x,t), \textit{and } p^T[z_i;1] = \max_{q \in F_{\sigma_i}(x,t)} p^T[q;1], \forall i \in \{1, \cdots, m\} \textit{and } \forall j \in \mathbb{N}. \textit{Hence,}

\begin{equation}
\max_{p \in \partial V(x,t)} p^T[z_i;1] \leq \max_{p \in \partial V(x,t)} \max_{q \in F_{\sigma_i}(x,t)} p^T[q;1] \leq -W(x), \forall i \in \{1, \cdots, m\} \textit{ and } \forall j \in \mathbb{N}. \textit{Since } \sum_{i=1}^{m} a_i = 1, \textit{max}_{p \in \partial V(x,t)} p^T[q_j;1] \leq -W(x), \forall j \in \mathbb{N}. \textit{Now, since } (p,q) \mapsto p^T[q;1] \textit{is continuous, and } \partial V(x,t) \textit{and } \overline{\cup}_{\sigma \in \mathcal{N}^0} F_{\sigma}(x,t) \textit{are compact, the function } q \mapsto \max \{ p^T[q;1] | p \in \partial V(x,t) \} \textit{is continuous on } \overline{\cup}_{\sigma \in \mathcal{N}^0} F_{\sigma}(x,t). \textit{Hence, max}_{p \in \partial V(x,t)} p^T[q;1] \leq -W(x), \forall q \in \overline{\cup}_{\sigma \in \mathcal{N}^0} F_{\sigma}(x,t). \textit{\hfill \blacksquare}}

\[^{6}\]The observation that a common (strong) continuously differentiable Lyapunov function for a family of finitely many differential inclusions is also a Lyapunov function for the closure of their convex combination is stated in [19, Proposition 1]. In this paper, it is proved and extended to families of countably infinite differential inclusions and semidefinite locally Lipschitz-continuous Lyapunov functions.

\[^{7}\]A collection of set valued maps \{ F_{\sigma} : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \Rightarrow \mathbb{R}^n | \sigma \in \mathcal{N}^0 \} \text{ is locally bounded, uniformly in } \sigma, \text{ over } \Omega \times \mathcal{N}^0, \text{ if for every compact } K \subset \Omega, \text{ there exists } M > 0 \text{ such that } V(x,t,\sigma, y) \text{ such that } (x,t,\sigma) \in K \times \mathcal{N}^0 \text{ and } y \in F_{\sigma}(x,t), \|y\|_2 \leq M.
The following corollary demonstrates that the bound \( \frac{c}{10} \) in Proposition 2 can be relaxed to utilize \( \hat{V}_F \) instead of \( \hat{W} \) at the expense of a stricter continuity assumption on the set-valued maps \( \{ F_\sigma \} \).

**Corollary 1.** Let the family of set-valued maps \( \{ F_\sigma : \mathbb{R}^n \times \mathbb{R}^{\geq t_0} \to \mathbb{R}^n \mid \sigma \in \mathcal{N}_0 \} \) satisfy the hypothesis of Proposition 2. If \( V : \Omega \to \mathbb{R} \) is a common non-strict regular Lyapunov function for the family \( \{ F_\sigma \} \), over \( \Omega \), with the bounds \( W : \mathcal{D} \to \mathbb{R} \), \( \overline{W} : \mathcal{D} \to \mathbb{R} \), and \( W : \mathcal{D} \to \mathbb{R} \), and with \( \frac{c}{10} \) in Definition 2 relaxed to \( \frac{c}{10} \), \( \forall \sigma \), \( \forall (x, t) \in \mathbb{R}^n \times \mathcal{N}_0 \), for almost all \( t \in \mathbb{R}^{\geq t_0} \), then \( \hat{V}_{\mathcal{F}_\sigma}(x, t) \leq -W(x), \forall (x, \sigma) \in \mathbb{R}^n \times \mathcal{N}_0 \) and for almost all \( t \in \mathbb{R}^{\geq t_0} \), provided the set-valued maps \( \{ F_\sigma \} \) are continuous (in the sense of \([38] \) Definition 1.4.3) and convex valued.

**Proof:** See the appendix.

The main result of the paper can now be summarized in the following theorem.

**Theorem 2.** Let \( r > 0 \) be selected such that \( \overline{B}(0, r) \subset \mathcal{D} \) and let \( \Omega := \mathcal{D} \times \mathbb{R}^{\geq t_0} \). If Assumption 1 holds and the (Filippov) Krasovskii regularizations of the subsystems in (2) admit a common non-strict Lyapunov function \( V : \Omega \to \mathbb{R} \), over \( \Omega \), with the bounds \( W : \mathcal{D} \to \mathbb{R} \), \( \overline{W} : \mathcal{D} \to \mathbb{R} \), and \( W : \mathcal{D} \to \mathbb{R} \), then every solution of the (Filippov) Krasovskii regularization of the switched system in (2) such that \( x(t_0) \in \{ x \in B(0, r) \mid W(x) \leq c \} \), for some \( c \in (0, \min_{\|x\|_2} W(x)) \), is complete, bounded, and satisfies \( \lim_{t \to \infty} W(x(t)) = 0 \). In addition, if \( \mathcal{D} = \mathbb{R}^n \) and if the sets \( \{ x \in \mathbb{R}^n \mid W(x) \leq c \} \) are compact, \( \forall c \in \mathbb{R}^n \), then the result is global.

**Proof:** Since the collection \( \{ f_\sigma \mid \sigma \in \mathcal{N}_0 \} \) is locally bounded, uniformly in \( t \) and \( \sigma \), over \( \Omega \times \mathcal{N}_0 \), the collections \( \{ F_\sigma \mid \sigma \in \mathcal{N}_0 \} \) and \( \{ \mathcal{K}_\sigma \mid \sigma \in \mathcal{N}_0 \} \) are also locally bounded, uniformly in \( t \) and \( \sigma \), over \( \Omega \times \mathcal{N}_0 \). Hence, by Proposition 2, \( V \) is also a non-strict Lyapunov function for the set-valued maps \( (x, t) \to \bigcup_{\sigma \in \mathcal{N}_0} F_\sigma(x, t) \) and \( (x, t) \to \bigcup_{\sigma \in \mathcal{N}_0} \mathcal{K}_\sigma(x, t) \), over \( \Omega \), with the bounds \( W, \overline{W}, \) and \( W \). From Proposition 2, \( \mathcal{F}(x, t) \subseteq \bigcup_{\sigma \in \mathcal{N}_0} F_\sigma(x, t) \) and \( \mathcal{K}(x, t) \subseteq \bigcup_{\sigma \in \mathcal{N}_0} \mathcal{K}_\sigma(x, t) \). Hence, \( V \) is also a non-strict Lyapunov function for \( \mathcal{F} \) and \( \mathcal{K} \), over \( \Omega \), with the bounds \( W, \overline{W}, \) and \( W \). Since \( f \) is locally bounded, uniformly in \( t \) over \( \Omega \), \( \mathcal{F} \) and \( \mathcal{K} \) are also locally bounded, uniformly in \( t \) over \( \Omega \). The conclusion then follows by Theorem 1.

**Remark 1.** The geometric condition in \( \frac{c}{10} \) can be relaxed to the following trajectory-based condition. For all generalized solutions \( x_\sigma : \mathcal{I} \to \mathbb{R}^n \) to (3), let the subsystems in (3) satisfy

\[
\hat{V}_{F_\sigma}(x_\sigma(t), t) \leq -W(x_\sigma(t)),
\]

\( \forall \sigma \in \mathcal{N}_0 \) and for almost all \( t \in \mathcal{I} \). In addition, for a specific generalized solution \( x^* : \mathcal{I} \to \mathbb{R}^n \) to (2), if the set \( \{ t \subseteq \mathcal{I} \mid \rho(x^*(\cdot), \cdot) \text{ is discontinuous at } t \} \) is countable for every \( \mathcal{I} \subseteq \mathbb{R}^{\geq t_0} \), then weak versions of Theorem 1 and Proposition 2 that establish the convergence of \( W(x^*(t)) \) to the origin as \( t \to \infty \) can be proven using techniques similar to \([30]\) Corollary 1.

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8 Example 2 demonstrates that there are collections of upper semicontinuous set-valued maps for which Corollary 1 fails to hold, i.e., the continuity hypothesis in Corollary 1 is not vacuous.
Remark 2. If the subsystems are autonomous, and if they admit a common non-strict Lyapunov function that is regular, then by applying the invariance principle (e.g., [37, Theorem 3]) to the differential inclusions \( \dot{x} \in \overline{\mathcal{U}} \cup_{\sigma \in \mathcal{N}_x} \mathcal{F}_\sigma (x) \) and \( \dot{\bar{x}} \in \overline{\mathcal{U}} \cup_{\sigma \in \mathcal{N}_x} \mathcal{K}_\sigma (x) \), it can be shown that all solutions of (4) that start in the set \( L_1 \) converge to the largest weakly forward invariant set contained within \( L_1 \cap \{ x \in \mathcal{D} \mid V(x) = 0 \} \), where \( L_1 \) is a bounded connected component of the level set \( \{ x \in \mathcal{D} \mid V(x) \leq l \} \). Hence, Propositions 1 and 2 also generalize results such as [26] to switched nonsmooth systems. A similar result can also be obtained for the case where the subsystems are periodic.

VII. COMMENTS ON THE GENERALIZED TIME DERIVATIVE

If \( V \) is regular then the generalized time derivative obtained using Definition 1 is generally more conservative than (i.e., greater than or equal to) the maximal element of the more popular set-valued generalized derivatives defined in [36] and [37]. The motivation behind the use of the seemingly restrictive definition is twofold: (a) through a reduction of the admissible directions in \( F \) using locally Lipschitz-continuous regular functions, a generalized time derivative that is less conservative than the set-valued derivatives in [36] and [37] can be obtained (see Lemma 1 and Corollary 2) and (b) the invariance-like results in Section VI do not hold if the time derivative of the cLf is interpreted in the set-valued sense (see Example 2).

Lemma 1. Let \( \mathcal{D} \subseteq \mathbb{R}^n \) be open and connected and let \( \Omega := \mathcal{D} \times [0, T] \). Let \( \mathcal{F} : \mathcal{D} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a map that satisfies Assumption 2 and let \( G_i, \mathcal{G}_i, F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) be defined as

\[
G_i (x, t) := \{ q \in F(x, t) \mid \exists a_f \mid p^T[q; 1] = a_f, \forall p \in \partial V_i (x, t) \},
\]

\[
G (x, t) := \{ q \in F(x, t) \mid \exists a_f \mid p^T[q; 1] = a_f, \forall p \in \partial V (x, t) \},
\]

\[
\tilde{F} (x, t) := F (x, t) \cap (\bigcap_{i=1}^n G_i (x, t)),
\]

\( \forall (x, t) \in \Omega \). If

\[
\dot{V}_{\tilde{F}} (x, t) \leq -W (x), \quad \forall (x, t) \in \Omega,
\]

(21)

where \( \dot{V}_{\tilde{F}} \) is the \( V \)-generalized time derivative of \( V \) with respect to \( F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), defined as

\[
\dot{V}_{\tilde{F}} (x, t) := \min_{p \in \partial V (x, t)} \max_{q \in F (x, t)} p^T[q; 1],
\]

\( \forall (x, t) \in \Omega \), and \( \dot{V}_{\tilde{F}} (x, t) \) is understood to be \(-\infty\) when \( \tilde{F} (x, t) \) is empty, then each solution of (14), such that \( x(t_0) \in \mathcal{D} \), satisfies \( \dot{V} (x(t), t) \leq -W (x(t)), \) for almost all \( t \in [t_0, T) \), where \( T := \min (\sup \mathcal{I}, \inf \{ t \in \mathcal{I} \mid x(t) \notin \mathcal{D} \}) \).

Proof: See the appendix.
Lemma 1 implies that to establish Lyapunov stability and asymptotic behavior of all solutions of (14), examination of the set \( \hat{F} \), reduced from \( F \) using the functions in \( \mathcal{V} \), is sufficient. In [37, p. 364], the maximization is performed over the set \( G \) instead of \( \hat{F} \), i.e.,

\[
\max \dot{V}(F) (x, t) = \min_{p \in \partial V(x, t)} \max_{q \in \mathbb{G}(x, t)} p^T [q; 1].
\]

Note that if \( V \in \mathcal{V} \) then \( \hat{V}_F = \hat{V}_\tilde{F} \) and \( \tilde{F} \subseteq G \), and hence, \( \hat{V}_F (x, t) = \hat{V}_\tilde{F} (x, t) \leq \max \dot{V}(F) (x, t), \forall (x, t) \in \Omega \).

Thus, depending on the functions \( \mathcal{V} \) selected to reduce the inclusions, both \( \hat{V}_F \) and \( \hat{V}_\tilde{F} \) can provide a notion of the generalized time derivative of \( V \) that is less conservative than the set-valued derivative in [37] (and hence, the set-valued derivative in [36]). Naturally, if \( \mathcal{V} = \{V\} \) then the three are equivalent. The following definition is inspired by Lemma 1 and the corollary that follows is a straightforward consequence of Theorem 1 and Lemma 1.

**Definition 3.** Let \( D \subseteq \mathbb{R}^n \) be open and connected and let \( \Omega := D \times \mathbb{R}_{\geq t_0} \). Let \( V : \Omega \to \mathbb{R} \) be a locally Lipschitz-continuous regular function and let \( \overline{W}, \underline{W} : \Omega \to \mathbb{R} \) be continuous positive definite functions that satisfy (17). Let \( \mathcal{V} \) be a countable collection of locally Lipschitz-continuous regular functions. If there exists a continuous positive semidefinite function \( W : D \to \mathbb{R} \) such that \( \dot{V}_F (x, t) \leq - W(x), \forall x \in D \) and for almost all \( t \in \mathbb{R}_{\geq t_0}, \) then \( V \) is called a \( \mathcal{V} \)-non-strict Lyapunov function for \( F : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^n \) over \( \Omega \) with the bounds \( \overline{W}, \underline{W}, \) and \( W \).

**Corollary 2.** Let \( D \subseteq \mathbb{R}^n \) be an open and connected set containing the origin and let \( \Omega := D \times \mathbb{R}_{\geq t_0} \). Assume that the differential inclusion in (14) admits a \( \mathcal{V} \)-non-strict Lyapunov function over \( \Omega \) with the bounds \( \overline{W}, \underline{W}, D \to \mathbb{R}, \) and \( W : D \to \mathbb{R}. \) If \( F : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^n \) satisfies Assumption 2 and is locally bounded, uniformly in \( t \), over \( \Omega \), then every solution of (14) such that \( x(t_0) \in \{ x \in B(0, r) \mid \overline{W}(x) \leq c \}, \) for some \( c \in (0, \lim_{\|x\| \to r} \underline{W}(x)), \) is complete, bounded, and satisfies \( \lim_{t \to \infty} W(x(t)) = 0. \)

At this juncture, it would be natural to ask whether the result in Theorem 2 can be established using the set-valued derivatives in [36] and [37] or a common \( \mathcal{V} \)-non-strict Lyapunov function. The following example demonstrates that a common \( \mathcal{V} \)-non-strict Lyapunov function is not sufficient to establish the results in Section VI and neither are the set-valued derivatives in [36] or [37]. Furthermore, the example also demonstrates that the continuity assumption in Corollary 1 is not vacuous.

**Example 2.** Let \( g_1, g_2, g_3 : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined as \( g_1 (x) := [x_1; 0], g_2 (x) := [0; x_2], \) and \( g_3 (x) := [-x_1; -x_2]. \) Let the subsystems be defined by \( f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}^2 \) as

\[
f_1 (x) = \begin{cases} 
g_1 (x) & |x_1| < |x_2|, 
g_3 (x) & |x_1| \geq |x_2|,
\end{cases} \quad f_2 (x) = \begin{cases} 
g_2 (x) & |x_1| < |x_2|, 
g_3 (x) & |x_1| \geq |x_2|,
\end{cases}
\]

The subsystems have identical Krasovskii and Filippov regularizations, given by

\[
F_1 (x) = \begin{cases} 
g_1 (x), g_3 (x) & |x_1| = |x_2|, 
f_1 (x) & \text{otherwise},
\end{cases}
\]

The minimization here serves to maintain consistency of notation, but is in fact, redundant.
\[ F_2(x) = \begin{cases} \text{co} \{g_2(x), g_3(x)\} & |x_1| = |x_2| \\ f_2(x) & \text{otherwise}. \end{cases} \]

The function \( V : \mathbb{R}^2 \to \mathbb{R} \), defined as \( V(x) := \max(|x_1|, |x_2|) \), is a locally Lipschitz-continuous regular function\(^{[10]}\) that satisfies (17) and

\[ \partial V(x) = \begin{cases} v_1(x) & |x_1| < |x_2| \\ v_2(x) & |x_1| > |x_2| \\ \text{co} \{v_1(x), v_2(x)\} & |x_1| = |x_2|, \end{cases} \]

where \( v_1(x) = [\text{sgn}(x_1); 0] \) and \( v_2(x) = [0; \text{sgn}(x_2)] \). Hence, with \( V = \{V\} \), \( \hat{F}_i(x) = \begin{cases} 0 & |x_1| = |x_2| \\ F_i(x) & \text{otherwise} \end{cases} \), for \( i = 1, 2 \).

In this case, \( (v_1(x))^T f_2(x) = (v_2(x))^T f_1(x) = 0 \), \( (v_1(x))^T f_3(x) = -|x_1| \), and \( (v_2(x))^T f_3(x) = -|x_2| \). It follows that \( \hat{V}_{F_i}(x) \leq 0 \) and \( \hat{V}_{F_i}(x) = \hat{V}_{F_i}(x) \leq 0 \), \( \forall x \in \mathbb{R}^2 \) and \( i = 1, 2 \). It is also easy to see that \( \max \hat{V}(F_i)(x) \leq 0 \) and \( \max \hat{V}(F_i)(x) \leq 0 \), \( \forall x \in \mathbb{R}^2 \) and \( i = 1, 2 \), where \( \hat{V}(F_i) \) is defined in [36, eq. 13]. Thus, \( V \) is a common non-strict cLf for the subsystems in the sense of \( \hat{V}_{F_i} \), Definition [3, 37], and [36].

Let \( F := x \mapsto \text{co} F_1(x) \cup F_2(x) \). For any \( x \in \mathbb{R}^2 \) such that \( |x_1| = |x_2| \), \( q := \frac{1}{2} [x_1; x_2] \in \text{co} \{g_1(x), g_2(x), g_3(x)\} = F(x) \). Thus, whenever \( |x_1| = |x_2| = V(x) > 0 \), \( \min_{p \in \partial V(x)} p^T q = 0.5V(x) > 0 \), i.e., Proposition 2 does not hold. Furthermore, a solution of \( \dot{x} \in F(x) \), starting at \( x = [1; 1] \), is \( x(t) = e^{0.5t} [1; 1] \), i.e., Theorem 2 does not hold.

Thus, Proposition 2 and Theorem 2 may not hold if the generalized time derivative is understood in the sense of Definition 3, [37] or [36]. If \( \hat{V}_{F_i} \) is used as the generalized time derivative instead of \( \hat{V}_{F_i} \), then Proposition 2 may not hold if the set-valued maps \( \{F_\sigma\} \) are not continuous.

\[ \triangle \]

**VIII. DESIGN EXAMPLES**

Many of the applications discussed in the opening paragraphs of Section IV can be represented by the following example problems. The first example demonstrates the utility of the developed technique on a simple problem where only the regression dynamics are discontinuous. In the second example, an adaptive controller for a switched system that exhibits arbitrary switching between subsystems with different parameters and disturbances is analyzed.

**Example 3.** Consider the nonlinear dynamical system

\[ \dot{x} = Y_\rho(x,t) (x) \theta + u + d(t), \]  

(22)

where \( x \in \mathbb{R}^n \) denotes the state, \( u \in \mathbb{R}^n \) denotes the control input, \( d : \mathbb{R} \to \mathbb{R}^n \) denotes an unknown disturbance, \( \rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{N} \) denotes the switching signal, \( Y_\sigma : \mathbb{R}^n \to \mathbb{R}^n \times L \), for each \( \sigma \in \mathbb{N} \), is a known continuous function, and \( \theta \in \mathbb{R}^L \) is the vector of constant unknown parameters. The control objective is to regulate the system.

\(^{[10]}\) Pointwise maxima of locally Lipschitz-continuous regular functions is locally Lipschitz-continuous and regular.
state to the origin. The disturbance is assumed to be bounded, with a known bound \( \bar{d} \) such that \( \|d(t)\|_\infty \leq \bar{d} \), for almost all \( t \in \mathbb{R}_{\geq t_0} \). Furthermore, \( t \mapsto d(t) \) is assumed to be Lebesgue measurable.

The adaptive controller designed to satisfy the control objective is \( u = -k x - Y_{\rho(x,t)}(x) \hat{\theta} - \beta \text{sgn}(x) \), where \( \hat{\theta} : \mathbb{R}_{\geq t_0} \rightarrow \mathbb{R}^L \) denotes an estimate of the vector of unknown parameters, \( \theta, k, \beta \in \mathbb{R}_{>0} \) are positive constant control gains, and \( \text{sgn}(\cdot) \) is the signum function. The estimate, \( \hat{\theta} \), is obtained from the update law \( \dot{\hat{\theta}} = (Y_{\rho(x,t)}(x))^T x \).

For each \( \sigma \in \mathbb{N} \), the closed-loop error system can then be expressed as

\[
\begin{align*}
\dot{x} &= -k x + Y_\sigma(x) \hat{\theta} + d(t) - \beta \text{sgn}(x), \\
\dot{\hat{\theta}} &= -(Y_\sigma(x))^T x,
\end{align*}
\]

where \( \hat{\theta} := \theta - \hat{\theta} \) denotes the parameter estimation error. The closed-loop system in (23) and (24) is discontinuous, and hence, does not admit classical solutions. Thus, the analysis will focus on generalized solutions to (23) and (24). Since Filippov and Krasovskii regularizations of the closed-loop system in (23) and (24) are identical, the solutions to the corresponding differential inclusions are hereafter simply referred to as generalized solutions.

To analyze the developed controller, consider the cLf \( V : \mathbb{R}^{n+L} \rightarrow \mathbb{R}_{\geq t_0} \), defined as

\[
V(z) := \frac{1}{2} x^T x + \frac{1}{2} \hat{\theta}^T \hat{\theta},
\]

where \( z := [x; \hat{\theta}] \). Since the cLf is continuously differentiable, the Clarke gradient reduces to the standard gradient, i.e., \( \partial V(z,t) = \{z\} \). Using the calculus of K [\( \cdot \)] from [39], a bound on the regularization of the system in (23) and (24) can be computed as \( F_\sigma(z,t) \subseteq F'_\sigma(z,t) \), where

\[
F'_\sigma(z,t) = \left\{ \begin{array}{l}
-k x + Y_\sigma(x) \hat{\theta} + d(t) \\
-\beta K[\text{sgn}](x)
\end{array} \right\}.
\]

Using the Definition 1 and the fact that \( x^T K[\text{sgn}](x) = \{\|x\|_1\} \), a bound on the generalized time derivative of the cLf can be computed as

\[
\dot{V}_\sigma(z,t) = \max_{q \in F_\sigma(z,t)} z^T q,
\]

\[
\leq \max_{q \in F'_\sigma(z,t)} z^T q,
\]

\[
= -k \|x\|^2 + x^T d(t) - \beta \|x\|_1.
\]

Provided \( \beta > \bar{d} \),

\[
\dot{V}_\sigma(z,t) \leq -W(z),
\]

\( \forall (z, \sigma) \in \mathbb{R}^{n+L} \times \mathbb{N} \) and for almost all \( t \in \mathbb{R}_{\geq t_0} \), where \( W(z) = k \|x\|^2 \) is a positive semidefinite function. Using (25), (26), and Theorem 2 all the generalized solutions of the switched nonsmooth system in (23) and (24) are complete, bounded, and satisfy \( \|x(t)\|_2 \to 0 \) as \( t \to \infty \). \( \triangle \)

**Example 4.** Arbitrary switching between systems with different parameters and disturbances can be achieved in the case where the number of subsystems is finite. For example, consider the nonlinear dynamical system

\[
\dot{x} = Z_{\rho(x,t)}(x,t) \theta_{\rho(x,t)} + d_{\rho(x,t)}(x,t) + u,
\]

\( \forall (z, \sigma) \in \mathbb{R}^{n+L} \times \mathbb{N} \) and for almost all \( t \in \mathbb{R}_{\geq t_0} \), where \( W(z) = k \|x\|^2 \) is a positive semidefinite function. Using (25), (26), and Theorem 2 all the generalized solutions of the switched nonsmooth system in (23) and (24) are complete, bounded, and satisfy \( \|x(t)\|_2 \to 0 \) as \( t \to \infty \). \( \triangle \)
where \( \rho : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightarrow \mathcal{N}^o \) such that \( \mathcal{N}^o \) is a finite set, \( Z_\sigma : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightarrow \mathbb{R}^{n \times L} \), are known functions, \( \theta_\sigma \in \mathbb{R}^L \) are vectors of constant unknown parameters corresponding to each \( \sigma \in \mathcal{N}^o \), and \( d_\sigma : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightarrow \mathbb{R}^n \) are unknown disturbances such that for each \( \sigma \in \mathcal{N}^o \), \( \|d_\sigma (x, t)\|_\infty \leq \mathcal{D}_\sigma \), \( \forall (x, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \) and some \( \mathcal{D}_\sigma > 0 \). Furthermore, for each \( \sigma \in \mathbb{R}^n \), \((x, t) \mapsto d_\sigma (x, t)\) and \((x, t) \mapsto Z_\sigma (x, t)\) are continuous in \( x \), uniformly in \( t \) and Lebesgue measurable in \( t \), \( \forall x \in \mathbb{R}^n \). Let \( \theta := [\theta_1; \theta_2; \cdots ; \theta_{|\mathcal{N}^o|}] \in \mathbb{R}^{L|\mathcal{N}^o|} \) and let \( Y_\sigma := 1_\sigma \otimes Z_\sigma \), where \( 1_\sigma \in \mathbb{R}^{1 \times L} \) is a matrix defined by

\[
(1_\sigma)_{1,j} = \begin{cases} 
1, & j = \sigma, \\
0, & \text{otherwise}.
\end{cases}
\]

The adaptive controller designed to satisfy the control objective is

\[
u = -k_\rho(x, t)x - Y_{\rho(x, t)} (x, t) \hat{\theta} - \beta_\rho(x, t) \text{sgn} (x),
\]

where \( \beta_\sigma \in \mathbb{R}_{>0} \) and \( k_\sigma \in \mathbb{R}_{>0} \) are control gains corresponding to \( \sigma \in \mathcal{N}^o \) and \( \hat{\theta} : \mathbb{R}_{\geq t_0} \rightarrow \mathbb{R}^{L|\mathcal{N}^o|} \) is updated according to \( \hat{\theta} = (Y_{\rho(x, t)} (x, t))^T x \). A stability analysis similar to Example 3 can then be utilized to conclude asymptotic convergence of the state \( x \) to the origin provided \( \beta_\sigma > \mathcal{D}_\sigma, \forall \sigma \in \mathcal{N}^o \).

\[\Box\]

\section{Conclusion}

Motivated by applications in switched adaptive control, the generalized LaSalle-Yoshizawa corollary in [30] is extended to switched nonsmooth systems. The extension facilitates the analysis of the asymptotic characteristics of a switched system based on the asymptotic characteristics of the individual subsystems where a non-strict common Lyapunov function can be constructed for the subsystems. Application of the developed extension to a switched adaptive system is demonstrated through simple examples. Motivated by results such as [40], further research could potentially extend the developed method to utilize indefinite Lyapunov functions.

In Lemma 1 it is shown that arbitrary locally Lipschitz-continuous regular functions can be used to reduce the differential inclusion to a smaller set of admissible directions. This observation indicates that there may be a smallest set of admissible directions corresponding to each differential inclusion. Further research is needed to establish the existence of such a set and to find a representation of it that facilitates computation.

The developed method requires a strong convergence result for the subsystems, i.e., the existence of a common cLf that satisfies (18) implies that all the generalized solutions to the individual subsystems are bounded and asymptotically converge to the origin. Future research will focus on the development of results for switched nonsmooth systems where only weak convergence results (that is, only a subset of the generalized solutions to the individual subsystems are bounded and asymptotically converge to the origin) are available for the subsystems.

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APPENDIX

Proof of Theorem 1 Similar to the proof of [30, Corollary 1], it is established that the bounds on \( \dot{V}_F \) in (16) and (17) imply that the cLf is nonincreasing along all the solutions of (14). The nonincreasing property of the cLf is used to establish boundedness of \( x \), which is used to prove the existence and uniform continuity of complete solutions. Barbálat’s lemma [17, Lemma 8.2] is then used to conclude the proof.

To show that the cLf is nonincreasing, let \( x : \mathcal{I} \to \mathbb{R}^n \) be a maximal solution [24, Definition 2.1] of (14) such that \( x(t_0) \in \Omega_c := \{ x \in \overline{B}(0, r) \mid W(x) \leq c \} \). Define \( T > t_0 \) be the first exit time of \( x \) from \( D \), i.e.,

\[
T := \min \{ \sup \mathcal{I}, \inf \{ t \in \mathcal{I} \mid x(t) \notin D \} \},
\]

where \( \inf \emptyset \) is assumed to be \( \infty \). If \( V \) is locally Lipschitz-continuous but not regular, then [42, Proposition 4] (see also, [43, Theorem 2]) can be used to conclude that, for almost every \( t \in [t_0, T) \), the time derivative \( \dot{V}(x(t), t) \) exists, and \( \exists p_0 \in \partial V(x(t), t) \) such that

\[
\dot{V}(x(t), t) = p_0^T [\dot{x}(t) ; 1].
\]

Thus, (16) and (18) imply that \( \dot{V}(x(t), t) \leq -W(x(t)) \) for almost every \( t \in [t_0, T) \). If \( V \) is regular, then the relaxation in Footnote 7 and [36, Equation 22] can be used to conclude that for almost every \( t \in [t_0, T) \), the time derivative \( \dot{V}(x(t), t) \) exists and \( \dot{V}(x(t), t) \leq -W(x(t)) \). The conclusion that

\[
V(x(t_0), t_0) \geq V(x(t), t), \quad \forall t \in [t_0, T)
\]

then follows from [30, Lemma 2].

\[ \] Similar to [41, Proposition 1], it can be shown that any solution of (14) can be extended to a maximal solution; hence, if a solution exists, then it can be assumed to be maximal without loss of generality.

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Using \( (28) \), it can be shown that (see, e.g., \([17] \) Theorem 4.8) every solution of \( (14) \) that starts in \( \Omega_c \) stays in \( \mathbb{B} (0, r) \) on every interval of its existence. Therefore, all maximal solutions of \( (14) \) such that \( x(t_0) \in \Omega_c \) are precompact \( [24] \) Definition 2.3 and \( T = \sup T \). In the following, arguments similar to \([41] \) Proposition 2] are used to show that precompact solutions are complete.

For the sake of contradiction, assume that \( T < \infty \). Since \( F \) is locally bounded, uniformly in \( t \), over \( \Omega \), and \( x(t) \in \mathbb{B} (0, r) \) on \( [t_0, T) \), the map \( t \mapsto F (x(t), t) \) is uniformly bounded on \( [t_0, T) \). Hence, \( (15) \) implies that \( \dot{x} \in \mathcal{L}_\infty ([t_0, T), \mathbb{R}^n) \). Since \( t \mapsto x(t) \) is locally absolutely continuous, \( \forall t_1, t_2 \in [t_0, T), \|x(t_2) - x(t_1)\|_2 = \left\| \int_{t_1}^{t_2} \dot{x}(\tau) \, d\tau \right\|_2 \). Since \( \dot{x} \in \mathcal{L}_\infty ([t_0, T), \mathbb{R}^n) \), \( \left\| \int_{t_1}^{t_2} \dot{x}(\tau) \, d\tau \right\|_2 \leq \int_{t_1}^{t_2} M \, d\tau \), where \( M \) is a positive constant. Thus, \( \|x(t_2) - x(t_1)\|_2 \leq M |t_2 - t_1| \), and hence, \( t \mapsto x(t) \) is uniformly continuous on \( [t_0, T) \). Therefore, \( x \) can be extended into a continuous function \( x' : [t_0, T] \to \mathbb{R}^n \). Invoking \([31] \) p. 83, Theorem 5, \( x' \) can be extended into a solution of \( (14) \) on the interval \( [t_0, T'] \), for some \( T' > T \), which contradicts the maximality of \( x \). Hence, \( T = \infty \) and all precompact solutions of \( (14) \) are complete.

Since \( x \mapsto W(x) \) is continuous and \( \mathbb{B} (0, r) \) is compact, \( x \mapsto W(x) \) is uniformly continuous on \( \mathbb{B} (0, r) \). Since \( t \mapsto x(t) \) is uniformly continuous on \( \mathbb{R}_{\geq t_0} \), \( t \mapsto W(x(t)) \) is uniformly continuous on \( \mathbb{R}_{\geq t_0} \). Furthermore, \( t \mapsto \int_{t_0}^{t} W(x(\tau)) \, d\tau \) is monotonically increasing and from \( (18) \) and the fact that \( V \) is positive definite,

\[
\int_{t_0}^{t} W(x(\tau)) \, d\tau \leq V(x(t_0), t_0) - V(x(t), t) \leq V(x(t_0), t_0).
\]

Hence, \( \lim_{t \to \infty} \int_{t_0}^{t} W(x(\tau)) \, d\tau \) exists and is finite. By Barbălat’s Lemma \([17] \) Lemma 8.2, \( \lim_{t \to \infty} W(x(t)) = 0 \).

**Proof of Proposition 1 for Filippov regularization:** Fix \( (x, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \), select \( \delta^* > 0 \) such that \( |\rho(\mathbb{B}(x, \delta^*), t)| < \infty \), and let \( \mathcal{N} := \rho(\mathbb{B}(x, \delta^*), t) \). Similar to the proof for Krasovskii regularization, the proof proceeds in three steps. First, it is observed that

\[
\bigcap_{\delta > 0} \bigcap_{\mu(N)=0} \mathfrak{M} \{ f_{\rho(y, t)}(y, t) \mid y \in \mathbb{B}(x, \delta) \setminus N \} \subseteq \bigcap_{\delta > 0} \bigcap_{\mu(N)=0} A_N^x (x, t), \tag{29}
\]

where \( A_N^x := \mathfrak{M} \bigcup_{\sigma \in \mathcal{N}} \{ f_\sigma(y, t) \mid y \in \mathbb{B}(x, \delta) \setminus N \} \). Second, it is established that

\[
\bigcap_{\delta > 0} \bigcap_{\mu(N)=0} A_N^x (x, t) \subseteq \bigcap_{\delta > 0} \bigcap_{\mu(N)=0} B_N^x (x, t). \tag{30}
\]

where \( B_N^x (x, t) := \mathfrak{M} \bigcup_{\sigma \in \mathcal{N}} B_{N, \delta^*}(x, t) \) and \( B_{N, \delta^*}(x, t) := \mathfrak{M} \{ f_\sigma(y, t) \mid y \in \mathbb{B}(x, \delta) \setminus N \} \). Finally, it is shown that \( \forall x \in \mathbb{R}^n \) and almost all \( t \in \mathbb{R}_{\geq t_0} \),

\[
\bigcap_{\delta > 0} \bigcap_{\mu(N)=0} B_N^x (x, t) \subseteq \mathfrak{M} \bigcup_{\sigma \in \mathcal{N}} \bigcap_{\delta > 0} \bigcap_{\mu(N)=0} B_{N, \delta^*}(x, t). \tag{31}
\]

The conclusion of the proposition then follows. Apart from the technical detail required to handle the exclusion of measure-zero sets in the Filippov inclusion, the methods utilized to prove \( (30) \) and \( (31) \) are similar to those used...
in the proof for Krasovskii inclusions. Thus, in the following, only the techniques used to handle the exclusion of measure-zero sets are illustrated.

The containment in (29) is self-evident. To prove (30), define \( \mathcal{F}(\delta) := \{ N \subset B(x, \delta) \mid \mu(N) = 0 \} \), and let \( N^*(\delta) \subset 2^B(x, \delta) \) be a collection of sets of zero measure such that \( \sup \{ ||\theta|| \mid \theta \in A_N^\delta \} < \infty \), \( \forall N \in N^*(\delta) \). Since the functions \( f_\sigma(x, t) \) are locally essentially bounded, uniformly in \( t \) and \( \sigma \), the collection \( N^*(\delta) \) is nontrivial. Fix \( N \in N^*(\delta) \) and \( z \in A_N^\delta \). Using arguments similar to Part 1 of the proof it can be shown that the point \( z \) is a convex combination of points from \( B_{N\delta, t}(x, t) \). That is, \( z \in \text{co} B_N^\delta(x, t) \), and hence,

\[
\bigcap_{\sigma \in N^*(\delta)} A_N^\delta(x, t) \subseteq \bigcap_{\sigma \in N^*(\delta)} B_N^{\delta}(x, t).
\]

(32)

To establish (30) the intersection in (32) needs to include all of \( \mathcal{F}(\delta) \), not just the subset \( N^*(\delta) \). Since \( N^*(\delta) \in \mathcal{F}(\delta) \), the inclusion \( \bigcap_{\sigma \in N^*(\delta)} A_N^\delta(x, t) \subseteq \bigcap_{\sigma \in \mathcal{F}(\delta)} A_N^\delta(x, t) \) follows. Let \( M \in \mathcal{F}(\delta) \). There exist \( N^1 \in \mathcal{F}(\delta) \setminus N^*(\delta) \) and \( N^0 \in N^*(\delta) \) such that \( M = N^1 \cup N^0 \). Since \( N^0 \subseteq M \), \( A_{N^0}^\delta(x, t) \subseteq A_{N^0}^\delta(x, t) \). Therefore, \( \bigcap_{\sigma \in \mathcal{F}(\delta)} A_N^\delta(x, t) \subseteq \bigcap_{\sigma \in N^\delta} A_N^\delta(x, t) \), which implies \( \bigcap_{\sigma \in N^\delta} A_N^\delta(x, t) = \bigcap_{\sigma \in \mathcal{F}(\delta)} A_N^\delta(x, t) \). A similar reasoning for \( B_N^\delta(x, t) \) yields \( \bigcap_{\sigma \in N^\delta} B_N^\delta(x, t) = \bigcap_{\sigma \in \mathcal{F}(\delta)} B_N^\delta(x, t) \), \( \forall \delta \in (0, \delta^*) \), which proves (30).

As an intermediate step towards proving (31), the containment

\[
\bigcap_{\mu(N)=0} B_N^\delta(x, t) \subseteq \text{co} \bigcup_{\sigma \in N^\delta} B_{N, \delta, \sigma}(x, t), \quad \forall \delta \in (0, \delta^*),
\]

(33)
is established in the following. Let \( z \in \bigcap_{\mu(N)=0} B_N^\delta(x, t) \). The objective now is to show that \( z \in \text{co} \bigcup_{\sigma \in N^\delta} B_{N, \delta, \sigma}(x, t) \). The inclusions in (29) and (30) are valid \( \forall (x, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \). For the development hereafter, \( (x, t) \) is restricted to a set \( \mathbb{R}^n \times E \) for some \( E \subseteq \mathbb{R}_{\geq t_0} \) such that \( \forall \sigma \in N^* \), the Filippov inclusions \( F_\sigma(x, t) \) can be expressed as \( \bigcap_{\delta>0} \bigcap_{i \in \mathbb{N}} B_{N, \delta, \sigma} \) for some countable collection of measure zero sets \( \{ N_{\sigma_i}(t) \}_{i \in \mathbb{N}} \). Under the additional assumption in (8), the set \( E \) can be selected to equal to \( \mathbb{R}_{\geq t_0} \). Define \( N^* := \bigcup_{\sigma \in N} \bigcup_{i \in \mathbb{N}} N_{\sigma_i} \). Since \( N^*_\mu(N) = 0 \). Since \( z \in \bigcap_{\mu(N)=0} B_N^\delta(x, t) \), by Carathéodory’s Theorem [33, p. 103], there exist \( \{ z_1, \cdots, z_m \} \) such that each \( z_j \in B_{N, \delta, \sigma_j}(x, t) \) for some \( \sigma_j \in N^* \), and positive real numbers \( \{ a_1, \cdots, a_m \} \) with \( \sum_{j=1}^m a_j = 1 \), such that \( z = \sum_{j=1}^m a_j z_j \). Using (8) and De-Morgan’s laws, \( B_{N, \delta, \sigma}(x, t) \subseteq \bigcap_{\mu(N)=0} B_{N, \delta, \sigma}(x, t), \forall \sigma \in N^* \). Hence, for each \( j \in \{1, \cdots, m\} \), \( z_j \in \bigcap_{\mu(N)=0} B_{N, \delta, \sigma_j}(x, t) \) for some \( \sigma_j \in N^* \), which implies (33), \( \forall \delta \in (0, \delta^*) \). Using a nesting argument similar to the proof for Krasovskii inclusions, the containment in (31) follows \( \forall (x, t) \in \mathbb{R}_n \times E \).

To complete the proof of (31), it needs to be established that even without the additional assumption in (8), the set \( E \) can be selected such that \( \mu(\mathbb{R}_{\geq t_0} \setminus E) = 0 \). Since the functions \( (x, t) \rightarrow f_\sigma(x, t) \) are measurable, [31, Equation 27, p. 85] can be used to conclude that \( \forall \sigma \in N^* \) there exist sets \( \{ E_\sigma \subseteq \mathbb{R}_{\geq t_0} \}_{\sigma \in N^*} \) with \( \mu(\mathbb{R}_{\geq t_0} \setminus E_\sigma) = 0 \), such that for each \( \sigma \) and \( \forall (x, t) \in \mathbb{R}_n \times E_\sigma \), there exists a measure zero set \( N_\sigma(t) \subseteq \mathbb{R}_n \) such that \( F_\sigma(x, t) = \bigcap_{\delta>0} \text{co} \{ f_\sigma(y, t) \mid y \in B(x, \delta) \setminus N_\sigma(t) \} \). The selection \( E = \bigcap_{\sigma \in N} E_\sigma \) then satisfies \( \mu(\mathbb{R}_{\geq t_0} \setminus E) = 0 \), which, along with (29) and (30), proves (7).

**Proof of Lemma 7** The proof closely follows the proof of Lemma 1 in [37]. Let \( x : \mathcal{T} \rightarrow \mathbb{R}^n \) be a solution of (14) such that \( x(t_0) \in \mathcal{D} \). Consider the set of times \( \mathcal{T} \subseteq [t_0, T] \) where \( \dot{x}(t) \) is defined, \( \dot{x}(t) \in F(x(t), t) \), and...
\( \hat{V}_i(x(t), t) \) is defined \( \forall i \geq 0 \). Since \( x \) is a solution of (14) and the functions \( V_i \) are locally Lipschitz-continuous, \( \mu([t_0, T] \setminus \mathcal{T}) = 0 \), where \( \mu \) denotes the Lebesgue measure on \( \mathbb{R} \). The idea is to show that \( \hat{x}(t) \in \dot{F}(x(t), t) \), not just \( F(x(t), t) \). Indeed since \( V_i \) is locally Lipschitz-continuous, for \( t \in \mathcal{T} \) its time derivative can be expressed as

\[
\dot{V}_i(x(t), t) = \lim_{h \to 0} \frac{V_i(x(t) + h\hat{x}(t), t + h) - V_i(x(t), t))}{h}.
\]

Since each \( V_i \) is regular, for \( i \geq 1 \), \( \dot{V}_i(x(t), t) = V_{i+}([x(t):t],[\hat{x}(t):1]) = V_{i}^{o}([x(t):t],[\hat{x}(t):1]) = \max(p^{T}[\hat{x}(t):1], p \in \partial V_i(x(t), t)), \) and \( \dot{V}_i(x(t), t) = V_{i-}([x(t):t],[\hat{x}(t):1]) = V_{i}^{o}([x(t):t],[\hat{x}(t):1]) = \min(p^{T}[\hat{x}(t):1], p \in \partial V_i(x(t), t)), \) where \( V_{i}^{+} \) and \( V_{i}^{-} \) denote the right and left directional derivatives and \( V_{i}^{o} \) denotes the Clarke-generalized derivative [35, p. 39]. Hence, \( p^{T}[\hat{x}(t):1] = \dot{V}_i(x(t), t), \forall p \in \partial V_i(x(t), t), \) which implies \( \hat{x}(t) \in G_i(x(t), t), \) for each \( i \). Therefore, \( \hat{x}(t) \in \dot{F}(x(t), t). \) Hence, (21), along with the fact that \( \hat{V}(x(t), t) = p^{T}[\hat{x}(t);1], \forall p \in \partial V(x(t), t), \) implies that \( \forall t \in \mathcal{T}, \hat{V}(x(t), t) \leq -W(x(t)) \). Since \( \mu([t_0, T] \setminus \mathcal{T}) = 0, \hat{V}(x(t), t) \leq -W(x(t)) \) for almost all \( t \in [t_0, T]. \)

In the following, three technical Lemmas are stated to facilitate the proof of Corollary 1.

**Lemma 2.** If \( \{F_{\sigma} : \mathbb{R}^{n} \times \mathbb{R}_{\geq t_0} \supseteq \mathbb{R}^{n} \mid \sigma \in \mathbb{N} \} \) is a collection of locally bounded, continuous, compact-valued, and convex-valued maps, then the set-valued map \( F : (x, t) \mapsto \bigcup_{\sigma \in \mathbb{N}} F_{\sigma}(x, t) \) is continuous.

**Proof:** Let \( H : \mathbb{R}^{n} \times \mathbb{R}_{\geq t_0} \supseteq \mathbb{R}^{n} \) be defined as \( H(x, t) = \text{co}(F_1(x,t) \cup F_2(x,t)) \). If \( N \subset \mathbb{R}^{n} \) is an open set containing \( H(x, t), \) then \( \exists \varepsilon > 0 \) such that \( H(x, t) + B((x, t), \varepsilon) \subset N \). Since \( F_1 \) and \( F_2 \) are upper semicontinuous (USC), there exist open sets \( M_1, M_2 \subset \mathbb{R}^{n} \times \mathbb{R}_{\geq t_0} \) such that \( (x, t) \subset M_1 \cap M_2, F_1(M_1) \subset H(x, t) + B((x, t), \varepsilon), \) and \( F_2(M_2) \subset H(x, t) + B((x, t), \varepsilon). \) Therefore, \( F_1(x, t) \cup F_2(x, t) \subset H(x, t) + B((x, t), \varepsilon). \) Since \( H(x, t) + B((x, t), \varepsilon) \) is convex, \( \text{co}(F_1(x,t) \cup F_2(x,t)) \subset H(x, t) + B((x, t), \varepsilon). \) Thus, \( H \) is USC.

It is easy to see that \( (x, t) \mapsto F_1(x, t) \cup F_2(x, t) \) is lower semicontinuous (LSC). Using [44, Theorem 5.9 (c)], \( H \) is also LSC. Inductively, the map \( (x, t) \mapsto \text{co}\bigcup_{k=1}^{K} F_k(x, t) \) is continuous \( \forall K < \infty. \) Thus, the collection \( \{F_k\}_{k \in \mathbb{N}} \) defined as \( F_k(x, t) = \text{co}\bigcup_{\sigma=1}^{k} F_{\sigma}(x, t) \) is a collection of nondecreasing continuous set-valued maps. By [44, Exercise 4.3], the sequence \( \{F_k\}_{k \in \mathbb{N}} \) converges pointwise to the map \( (x, t) \mapsto \bigcup_{k \in \mathbb{N}} F_k(x, t). \) Since the sets \( \{F_k\} \) are nested, \( \bigcup_{k \in \mathbb{N}} F_k(x, t) = \text{co}\bigcup_{\sigma \in \mathbb{N}} F_{\sigma}(x, t). \) Hence, by [44, Theorem 5.48 (a)], the map \( (x, t) \mapsto \text{co}\bigcup_{\sigma \in \mathbb{N}} F_{\sigma}(x, t), \) is continuous [12].

**Lemma 3.** Let \( g : \mathbb{R}^{n} \to \mathbb{R} \) be continuous and let \( F : \mathbb{R}^{n} \times \mathbb{R}_{\geq t_0} \supseteq \mathbb{R}^{n} \) be a locally bounded, continuous, and compact-valued map. If \( \phi := (x, t) \mapsto \max_{q \in F(x, t)} g(q) \), then \( \phi \) is continuous at \( (x, t), \forall (x, t) \in \mathbb{R}^{n} \times \mathbb{R}_{\geq t_0}. \)

**Proof:** If not, then \( \exists \varepsilon > 0 \) such that \( \forall \delta > 0, \exists (y, \tau) \in B((x, t), \delta) \) such that \( |\phi(y, \tau) - \phi(x, t)| \geq \varepsilon. \) If \( \phi(y, \tau) - \phi(x, t) \geq \varepsilon \) then \( \arg \max_{q \in F(y, \tau) \cup F(x, t)} g(q) \subset F(y, \tau) \setminus F(x, t). \) If \( \phi(y, \tau) - \phi(x, t) \geq \varepsilon, \) then \( \arg \max_{q \in F(y, \tau) \cup F(x, t)} g(q) \subset F(x, t) \setminus F(y, \tau). \) That is, \( \arg \max_{q \in F(y, \tau) \cup F(x, t)} g(q) \subset F(x, t) \Delta F(y, \tau). \) Let \( \beta > 0. \) If \( \{(y_k, \tau_k)\}_{k \in \mathbb{N}} \subset B((x, t), \beta) \) is a sequence converging to \( (x, t) \) such that \( |\phi(y_k, \tau_k) - \phi(x, t)| \geq \varepsilon, \)

[12] By [44, Theorem 5.7 (c)], the notion of LSC in this paper is equivalent to the notion of inner semicontinuity in [44]. Since the all the maps under consideration are locally bounded and compact valued, by [44, Theorem 5.19], the notion of USC in this paper is equivalent to the notion of outer semicontinuity in [44].
then, ∀k ∈ N, max_{q ∈ F(y_k, τ_k), p ∈ F(x, t)} g(q) = max_{q ∈ F(x, t)} g(q). Since g and F are continuous and F is locally bounded, the sequence {max_{q ∈ F(y_k, τ_k), p ∈ F(x, t)} g(q)}_{k∈N} is a bounded sequence. On the other hand, since F is continuous, the sequence \{F(x, t) ∆ F(y_k, τ_k)\}_{k∈N} converges to the null set, and hence, the sequence \{max_{q ∈ F(y_k, τ_k), p ∈ F(x, t)} g(q)\}_{k∈N} converges to −∞, which is a contradiction.

**Lemma 4.** Let \( g : \mathbb{R}^n × \mathbb{R}^n → \mathbb{R} \) be a continuous function and let \( F : \mathbb{R}^n × \mathbb{R}_{≥ 0} → \mathbb{R}^n \) be a locally bounded, USC, and compact-valued map. Let \( h := (p, x, t) ↦ max_{q ∈ F(x, t)} g(p, q) \). If \( C_x ⊂ \mathbb{R}^n × \mathbb{R}_{≥ 0} \) and \( C_p ⊂ \mathbb{R}^n \) are compact, then \( h \) is continuous in \( p \), uniformly in \( (x, t) \) over \( C_p × C_x \).

**Proof:** Since \( g \) is continuous, and \( F(C_x) \) and \( C_p \) are compact\(^{13}\), it is uniformly continuous on \( C_p × F(C_x) \). Thus, given \( ε > 0, \exists \delta > 0 \), independent of \( (p, x, t) \), such that \( ∀p, p_0 ∈ C_p \) and \( ∀q, q_0 ∈ F(C_x) \), \( ∥p − p_0∥ < \delta ∧ ∥q − q_0∥ < \delta \implies g(p_0, p_0) < g(p, q) + ε \). In particular, \( ∥p − p_0∥ < \delta \implies g(p_0, p_0) < g(p_0, q_0) + ε \). For any fixed \( p_0 ∈ C_p \) and \( (x, t) ∈ C_x \), \( ∃q_0 ∈ F(x, t) \) such that \( h(p_0, x, t) = g(p_0, q_0) \), and hence, \( h(p_0, x, t) < g(p, q_0) + ε \). Since \( g(p, q_0) ≤ h(p, x, t) \) by definition, \( h(p_0, x, t) < h(p, x, t) + ε \). That is, \( ∀p, p_0 ∈ C_p \) and \( ∀(x, t) ∈ C_x \), \( ∥p − p_0∥ < \delta \implies h(p_0, x, t) < h(p, x, t) + ε \). By symmetry, \( |h(p_0, x, t) − h(p, x, t)| < ε \).

**Proof of Corollary**\(^7\) Rademacher’s theorem \(^{45}\) Theorem 3.2] and \(^{35}\) Proposition 2.3.6 (d)] imply that \( ∂V \) is single-valued for almost all \( (x, t) ∈ \mathbb{R}^n × \mathbb{R}_{≥ 0} \). As a result, for almost all \( (x, t) ∈ \mathbb{R}^n × \mathbb{R}_{≥ 0} \), \( \hat{V}_F(x, t) = \mathcal{V}_F(x, t) \).

By Proposition 2 for any \( (x, t) ∈ \mathbb{R}^n × \mathbb{R}_{≥ 0} \) and \( β > 0 \), there exists a sequence \( \{(y_k, τ_k)\}_{k∈N} ⊂ \mathcal{B}((x, t), β) \), converging to \( (x, t) \) such that \( ∂V(y_k, τ_k) = \{∇V(y_k, τ_k)\} = \{p_k\} \) and \( max_{q ∈ F(y_k, τ_k)} p_kq \leq −W(y_k) \).

Let \( q_k ∈ arg max_{q ∈ F(y_k, τ_k)} p_kq \). Since the set-valued map \( F \) is locally bounded and USC, the sequence \( \{q_k\}_{k∈N} \) is bounded, and hence, admits a convergent subsequence \( \{q_{k_l}\}_{l∈N} \) converging to some \( q^* ∈ \mathbb{R}^n × \mathbb{R}_{≥ 0} \).

Since \( ∂V \) is locally bounded and USC (cf. \(^{46}\) p. 4), the sequence \( \{p_{k_l}\}_{l∈N} \) is bounded. Hence, there exists a subsequence \( \{p_{k_{l_m}}\}_{m∈N} \) converging to some \( p^* ∈ \mathbb{R}^n \). Hence,

\[
(p^*)^T[q^*;1] ≤ \lim_{m→∞} −W(y_{k_{l_m}}) = −W(x).
\]

Using the characterization of the generalized gradient from \(^{35}\) p. 11, eq. (4)], \( p^* ∈ ∂V(x, t) \). From Lemma 2 \( F \) is continuous, and hence, \( q^* ∈ F(x, t) \).

Let \( h := (p, x, t) ↦ max_{q ∈ F(x, t)} p^T[q;1] \). To prove the corollary, it needs to be established that \( h(p^*, x, t) = (p^*)^T[q^*;1] \). The inequality \( h(p^*, x, t) ≥ (p^*)^T[q^*;1] \) is immediate from the definitions. Also,

\[
\begin{align*}
 & h(p^*, x, t) − (p^*)^T[q^*;1] = h(p^*, y_{k_{l_m}}, τ_{k_{l_m}}) − h(p^*, x, t) − h(p^*, y_{k_{l_m}}, τ_{k_{l_m}}) \\
 & h(p^*, y_{k_{l_m}}, τ_{k_{l_m}}) − h(p_{k_{l_m}}, y_{k_{l_m}}, τ_{k_{l_m}}) \\
 & + h(p_{k_{l_m}}, y_{k_{l_m}}, τ_{k_{l_m}}) − (p^*)^T[q^*;1].
\end{align*}
\]

Let \( ε > 0 \). By definition of \( p^* \) and \( q^* \), \( ∃M_1 ∈ N \) such that \( ∀m ≥ M_1, \ |h(p_{k_{l_m}}, y_{k_{l_m}}, τ_{k_{l_m}}) − (p^*)^T[q^*;1]| < \frac{ε}{3} \). Since \( ∂V \) and \( F \) are USC, \( ∂V(\mathcal{B}((x, t), β)) \) and \( F(\mathcal{B}((x, t), β)) \) are closed by \(^{44}\) Theorem 5.25. and

\(^{13}\) \( F(C_x) \) is bounded by \(^{31}\) Lemma 15, p. 66], and since \( F \) is USC and \( C_x \) is compact, \( F(C_x) \) is also closed by \(^{44}\) Theorem 5.25 (a).
hence, compact. Since \((p, q) \mapsto p^T [q; 1]\) is continuous, Lemma 4 implies that the function \(h\) is continuous in \(p\), uniformly in \((x, t)\), over \(\partial V(\mathbb{B}((x, t), \beta)) \times \mathbb{B}((x, t), \beta)\). Hence, \(\exists M_2 \in \mathbb{N}\) such that \(\forall m \geq M_2, |h(p^*, y_{k_{lm}}, \tau_{k_{lm}}) - h(p_{k_{lm}}, y_{k_{lm}}, \tau_{k_{lm}})| < \frac{\varepsilon}{3}\). Lemma 3 implies that the function \(\phi := (x, t) \mapsto h(p^*, x, t)\) is continuous. Hence, \(\exists M_3 > 0\) such that \(\forall m \geq M_3, |h(p^*, x, t) - h(p^*, y_{k_{lm}}, \tau_{k_{lm}})| \leq \frac{\varepsilon}{3}\).

Thus, for \(m \geq \max\{M_1, M_2, M_3\}\), \(h(p^*, x, t) \leq (p^*)^T [q^*; 1] + \varepsilon\). Since \(\varepsilon\) was arbitrary, \(h(p^*, x, t) = (p^*)^T [q^*; 1]\). Hence, from (34) and the definition of \(h\), \(\exists p^* \in \partial V(x, t)\) such that \(\max_{q \in F(x, t)} (p^*)^T [q; 1] \leq W(x)\), and hence, \(\min_{p \in \partial V(x, t)} \max_{q \in F(x, t)} p^T [q; 1] \leq -W(x)\).

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