Randomly evolving trees III

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Abstract

The properties of randomly evolving special trees having defined and analyzed already in two earlier papers (arXiv:cond-mat/0205650 and arXiv:cond-mat/0211092) have been investigated in the case when the continuous time parameter converges to infinity. Equations for generating functions of the number of nodes and end-nodes in a stationary (i.e. infinitely old) tree have been derived. In order to solve exactly these equations we have chosen three different distributions for the number of new nodes $\nu$ produced by one dying node. By using appropriate method we have calculated step-by-step the probabilities of finding $n = 1, 2, \ldots$ nodes as well as end-nodes in a stationary random tree. Analyzing the results of numerical calculations we have observed that the qualitative properties of stationary random trees depend hardly on the character of distribution of $\nu$. The conclusion to be correct that in the evolution process the formation of a rod-like stationary tree is much more probable than the formation of a tree with many branches. We have established that the probability of finding $n$ nodes in a stationary tree depends sensitively on the average value of $\nu$ and has a maximum the location of which is increasing with $n$ but remains always smaller than unity. This is also true for the end-nodes.

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1 Introduction

In previous two papers [1], [2] we defined and analyzed random processes with continuous time parameter describing the evolution of special trees consisting of living and dead nodes connected by lines. It seems to be appropriate to
repeat briefly the characteristic features of the evolution process. The initial state $S_0$ of the tree corresponds to a single living node called root which at the end of its life is capable of producing $\nu = 0, 1, \ldots$ new living nodes, and after that it becomes immediately dead. If $\nu > 0$ then the new nodes are promptly connected to the dead node and each of them independently of the others can evolve further like a root. The random evolution of trees with continuous time parameter has not been investigated intensively recently. The main interest since the late 1990s has been focussed on the study of non-equilibrium networks occurring in common real world. The evolution mechanism of trees with living and dead nodes may be useful in some of biological systems where the branching processes are dominant.

In what follows, we will use notations applied in [1] and [2]. It seems to be useful to cite the basic definitions. The probability to find the number $\nu$ of living nodes produced by one dying precursor equal to $j$ was denoted by $f_j$ where $j \in \mathbb{Z}$.

For the generating function as well as the expectation value and the variance of $\nu$ we used the following notations:

$$E\{z^\nu\} = q(z), \quad E\{\nu\} = q_1 \quad \text{and} \quad D^2\{\nu\} = q_2 + q_1 - q_1^2,$$

where

$$q_j = \left[\frac{d^j q(z)}{dz^j}\right]_{z=1}, \quad j = 1, 2, \ldots$$

are factorial moments of $\nu$. It was shown in [1] that the time dependence of the random evolution is determined almost completely by the expectation value $q_1$. In accordance to this the evolution was called subcritical if $q_1 < 1$, critical if $q_1 = 1$ and supercritical if $q_1 > 1$.

In the further considerations we are going to use four distributions for the random variable $\nu$. As shown in [1] the equations derived for the first and the second moments of the number of nodes are independent of the detailed structure of the distribution of $\nu$ provided that the moments $q_1$ and $q_2$ are finite. We called distributions of this type for $\nu$ arbitrary and used the symbol $\nu \in a$ for its notation. In many cases it seems to be enough to apply the truncated distribution of $\nu$. If the possible values of the random variable $\nu$ are 0, 1 and 2 with probabilities $f_0$, $f_1$ and $f_2$, respectively, then in the previous paper [2] the distributions of this type were denoted by $\nu \in t$. Many times it is expedient to assume distributions to be completely determined by one parameter. As known the geometric and Poisson distributions are such

\footnote{$\mathbb{Z}$ is the set of non-negative integers.}
distributions. In paper [2] we used the symbols $\nu \in g$ and $\nu \in p$ to identify these distributions.

The distribution function of the lifetime $\tau$ of a living node will be supposed to be exponential, i.e. $\mathcal{P}\{\tau \leq t\} = 1 - e^{-Qt}$. In order to characterize the tree evolution two non-negative integer valued random functions $\mu_\ell(t)$ and $\mu_d(t)$ are introduced: $\mu_\ell(t)$ is the number of living nodes, while $\mu_d(t)$ is that of dead nodes at $t \geq 0$. The total number of nodes at $t \geq 0$ is denoted by $\mu(t)$.

Clearly, the nodes can be sorted into groups according to the number of outgoing lines. Following the notation in [2] the number of nodes with $k \geq 0$ outgoing lines at time instant $t \geq 0$ is denoted by $\mu(t, k)$. A node not having outgoing line is called end-node. It is obvious that an end-node is either live or dead. Therefore, the number of end-nodes $\mu(t, 0)$ can be written as a sum of numbers of living and dead end-nodes, i.e. $\mu(t, 0) = \mu_\ell(t, 0) + \mu_d(t, 0) = \mu_0(t)$.

Since all living nodes are end-nodes $\mu_\ell(t, 0)$ can be replaced by $\mu_\ell(t)$. The total number of dead nodes $\mu_d(t)$ is given by $\mu_d(t) = \sum_{k=0}^{\infty} \mu_d(t, k)$.

In this paper we are dealing with properties of $\mu(t)$, $\mu_\ell(t)$ and $\mu_0(t)$ when $t \to \infty$. We will call the random trees arising from a single root after elapsing infinite time stationary.

In Section 2 the basic properties of probability distributions of the number of nodes, living and end-nodes are investigated when $t \to \infty$. Special attention is paid in Section 3 to the effect of distribution law of the number of outgoing lines. Three different distributions of $\nu$ are investigated. In order to simplify the notation, indices referring to different distributions of $\nu$ are usually omitted in formulas. Finally, the characteristic properties of stationary random trees are summarized in Section 4.

2 General considerations

Let us introduce the notion of tree size which is nothing else but the total number of nodes $\mu(t)$ at time moment $t \geq 0$. We want to analyze the asymptotic behavior of the tree size, i.e. the behavior of the random function $\mu(t)$ when $t \to \infty$. We say the limit random variable

$$\tilde{\mu} \overset{d}{=} \lim_{t \to \infty} \mu(t),$$


exists in the sense that the relation:

$$\lim_{t \to \infty} \mathcal{P}\{\mu(t) = n|\mathcal{S}_0\} = \mathcal{P}\{\tilde{\mu} = n\}$$

(1)
is true for all positive integers $n$, where $\mathcal{S}_0$ denotes the initial state of the tree. A randomly evolving tree is called "very old" when $t \to \infty$, and a very old tree, as mentioned already, will be named stationary random tree.

It is elementary to prove that if the limit probability $\mathcal{P}\{\tilde{\mu} = n\} = p_n$ exists, then the generating function

$$g(z) = \sum_{n=1}^{\infty} p_n z^n$$

(2)
is determined by one of the fixed points of the equation

$$g(z) = z q[g(z)].$$

(3)

It can be shown that if $q_1 \leq 1$, then the fixed point to be chosen has to satisfy the limit relation

$$\lim_{z \uparrow 1} g(z) = 1,$$

(4)

while if $q_1 > 1$, then it should have the property

$$\lim_{z \uparrow 1} g(z) < 1$$

(5)

and independently of $q_1$ the equation $g(0) = 0$ must hold. The relation $q_1 \leq 1$ means that the probability to find stationary tree of finite size is evidently 1, if $q_1 \leq 1$, but if $q_1 > 1$, then

$$\lim_{z \uparrow 1} g(z) = \sum_{n=1}^{\infty} p_n < 1,$$

i.e. the probability to find a stationary tree of infinite size is equal to

$$w_\infty = 1 - \sum_{n=1}^{\infty} p_n.$$

The proof of the proposition is simple. Let us assume that

$$q(z) = \sum_{n=0}^{\infty} f_n z^n$$

is a probability generating function, i.e. $q(1) = \sum_{n=0}^{\infty} f_n = 1$ and $f_1 \neq 1$. We need the following theorem:
Theorem 1 If \( q'(1) = q_1 \leq 1 \), then \( q(z) > z \), \( \forall \ 0 \leq z < 1 \); while if \( q'(1) = q_1 > 1 \), then there is a point \( 0 < z_0 < 1 \) such that \( q(z_0) = z_0 \), i.e. \( q(z) > z \), if \( 0 < z < z_0 \) and \( q(z) < z \), if \( z_0 < z < 1 \).

Let us introduce the function \( \varphi(z) = q(z) - z \). Since \( q(z) \) is convex, i.e. all derivatives are positive in the interval \([0, 1]\) it is evident that

\[
\frac{d\varphi(z)}{dz} = \varphi'(z)
\]

is a nondecreasing function of \( z \), \( \forall \ z \in [0, 1] \). If \( q'(1) < 1 \), then \( \varphi'(z) < 0 \), i.e. \( \varphi'(z) \) is nondecreasing, negative valued function of \( z \) in \( 0 \leq z < 1 \). Since \( \varphi(1) = 0 \) it is obvious that \( \varphi(z) < 0 \), if \( 0 < z < 1 \), i.e. \( q(z) < z \), and this is the first statement of Theorem 1. If \( q'(1) > 1 \), then \( \varphi'(z) > 0 \), and since \( \varphi(1) = 0 \) the inequality \( \varphi(z) < 0 \) has to be true for all \( z < 1 \) lying near 1. On the other side \( \varphi(0) = f_0 > 1 \), hence there should exist one \( z_0 \) in \((0, 1)\) which satisfies the equation \( \varphi(z_0) = 0 \) and that implies the second statement of Theorem 1.

It seems to be important to investigate how the living nodes behave in very old, i.e. stationary trees. Intuitively one can say that stationary trees arising in subcritical evolution do not contain living nodes and they have finite average size. At the same time, it seems to be quite obvious that stationary trees originating in supercritical evolution could have living nodes with non-zero probability, i.e. they are entities of "eternal life".

In order to give more precise answer the generating function of the random variable

\[
\tilde{\mu}_t \overset{d}{=} \lim_{t \to \infty} \mu_t(t)
\]

should be derived. It is easy to show that

\[
E\{z^{\tilde{\mu}_t}\} = g^{(t)}(z) = \sum_{n=0}^{\infty} p^{(t)}_n z^n
\]

is one of the fixed points of the equation

\[
q\left[g^{(t)}(z)\right] = g^{(t)}(z). \tag{6}
\]

\^2Existence of more than one \( z_0 \) is excluded because \( q(z) \) is convex.
By Theorem 1, equation (6) has one fixed point if $q_1 < 1$, namely $g_1^{(t)}(z) = 1$, $\forall z \in [0, 1]$, and from this it follows that
$$p_n^{(t)} = \begin{cases} 
1, & \text{if } n = 0, \\
0, & \text{if } n > 0,
\end{cases}$$
i.e. the probability that a subcritical very old tree does not have living node, is exactly 1. If $q_1 > 1$, then besides $g_1^{(t)}(z) = 1$ equation (6) has another fixed point given by $g_2^{(t)}(z) < 1$, $\forall z \in [0, 1]$, hence the probability $P\{\tilde{\mu}_t > 0\} = 1 - g_2^{(t)}(0)$ should be larger than zero. In other words, a supercritical stationary tree may evolve infinitely with certain non-zero probability. \(^3\)

The expectation value and the standard deviation of the total number of nodes can be used to characterize the size of a stationary random tree. From (3), one obtains
$$E\{\tilde{\mu}\} = \left[ \frac{dg}{dz} \right]_{z=1} = \frac{1}{1-q_1}, \quad \text{if } q_1 < 1.$$In a supercritical evolution, i.e. when $q_1 > 1$, the expectation value $E\{\tilde{\mu}\}$ does not exist, but the limit relation
$$\lim_{t \to \infty} E\{\mu(t) e^{-(q_1-1)Qt}\} = \frac{q_1}{q_1 - 1}$$
can be simply proved. We note here that in the critical case, i.e. when $q_1 = 1$, the average tree size becomes infinite linearly, i.e. we have the relation $\lim_{t \to \infty} E\{\mu(t)/Qt\} = 1$.

The standard deviation of the total number of nodes in stationary random trees is given by
$$D\{\tilde{\mu}\} = \frac{D\{\nu\}}{(1-q_1)^{3/2}}, \quad \text{if } q_1 < 1.$$In a supercritical state the standard deviation does not exist, but it can be readily shown that
$$\lim_{t \to \infty} D\{\mu(t) e^{-2(q_1-1)Qt}\} = \frac{q_1}{\sqrt{q_1-1}} \left( 1 + \frac{D\{\nu\}}{q_1-1} \right), \quad \text{if } q_1 > 1.$$\(^3\)More precise formulation would be the following: in countable set $T$ of subcritical stationary trees the measure of subset containing trees with living nodes is zero, while in that of supercritical trees the measure of subset consisting of trees with living nodes is larger than zero.
Finally, we would like to deal briefly with some of the properties of end-nodes in stationary random trees. By using equation (14) of [2], it can be shown that the generating function
\[ g_0(z) = \mathbb{E}\{z^{\bar{\mu}_0}\} = \sum_{n=0}^{\infty} p_n^{(0)} z^n, \]
where \(\bar{\mu}_0 \xrightarrow{d} \lim_{t \to \infty} \mu(t, 0)\)
is nothing else than one of fixed points of the following simple equation:
\[ q[g_0(z)] = g_0(z) + (1 - z) f_0. \] (7)
Substituting \(z = 0\) we obtain
\[ g_0(0) \left\{ 1 - \sum_{n=1}^{\infty} f_n [g_0(0)]^n \right\} = 0, \]
i.e. \(g_0(0) = 0\), hence \(\mathcal{P}\{\bar{\mu}_0 = 0\} = 0\).

In the sequel we will investigate the properties of stationary trees when \(q(z)\) is known. In this case we can obtain exact expressions for probabilities \(\mathcal{P}\{\bar{\mu} = n\}, \ \mathcal{P}\{\bar{\mu}_\ell = n\}, \ \mathcal{P}\{\bar{\mu}_0 = n\}, \ \forall n = 1, 2, \ldots\) from the corresponding generating function.

3 Known distribution of \(\nu\)

3.1 Truncated arbitrary distribution of \(\nu\)
The generating function of \(\nu\) is given by
\[ q(z) = f_0 + f_1 z + f_2 z^2 = 1 + q_1 (z - 1) + \frac{1}{2} q_2 (z - 1)^2 \] (8)
with the restriction for \(q_1\) and \(q_2\) determined by the equality \(f_0 + f_1 + f_2 = 1\). (See Fig. 1 in [2].) By using Eq. (3) and applying (8) we have
\[ z \left[ 1 + q_1 (g - 1) + \frac{1}{2} q_2 (g - 1)^2 \right] = g. \]
The fixed point of this equation is
\[ g(z) = 1 + \frac{1}{q_2 z} \left[ 1 - q_1 z - \sqrt{(1 - q_1 z)^2 + 2 q_2 z (1 - z)} \right], \] (9)
is also the generating function of \( \tilde{\mu} \). It is an elementary task to show that

\[
g(z) = \begin{cases} 
0, & \text{if } z = 0, \\
1, & \text{if } z = 1, \text{ and } q_1 \leq 1 \\
1 - 2(q_1 - 1)/q_2, & \text{if } z = 1, \text{ and } q_1 > 1.
\end{cases}
\]  

(10)

Figure 1: Probabilities to find \( n = 1, 2, \ldots \) nodes in a "very old" \((t = \infty)\) tree when the distribution of \( \nu \) is truncated arbitrary. The dark and light bars correspond to subcritical and slightly supercritical tree evolutions, respectively.

Since

\[
g(z) = \sum_{n=1}^{\infty} p_n z^n,
\]

performing expansion of generating function \((9)\) into power series of \( z \), we can determine the probabilities \( p_n, \ \forall \ n \in \mathbb{Z} \) easily. After elementary calculations we obtain

\[
p_n = \frac{1}{2\sqrt{\pi}q_2} \left[ \frac{\Gamma(n+1/2)}{\Gamma(n+2)} \right] \left( U^{n+1} + V^{n+1} \right) - W_{n+1},
\]  

(11)

where

\[
U = q_1 - q_2 + \sqrt{2q_2} \sqrt{1 - q_1 + \frac{1}{2}q_2},
\]
\[ V = q_1 - q_2 - \sqrt{2q_2} \sqrt{1 - q_1 + \frac{1}{2}q_2}, \]

and

\[ W_{n+1} = \frac{1}{4\pi} \sum_{j=1}^{n} \frac{\Gamma(j-1/2)}{\Gamma(j+1)} \frac{\Gamma(n-j+1/2)}{\Gamma(n+j+2)} \left(\sum_{k=1}^{j} \frac{U_k V_n + 1}{V_{n+1-j}}\right). \]

Figure 2: Probabilities to find \( n = 1, 2, \ldots \) nodes in a "very old" \( (t = \infty) \) tree when the distribution of \( \nu \) is truncated arbitrary and \( f_1 = 0.1 \). The dark and light bars correspond to subcritical and slightly supercritical tree evolutions, respectively.

The probabilities \( p_n, \quad n = 1, 2, \ldots \) can be seen in Fig. 1. These are the probabilities to find \( n = 1, 2, \ldots \) nodes in a "very old" tree developed according to the distribution of \( \nu \in a \). The dark and light bars correspond to subcritical \( (q_1 = 0.95, q_2 = 0.5) \) and slightly supercritical \( (q_1 = 1.05, q_2 = 0.5) \) tree evolutions, respectively. In the last case \( w_\infty = 0.2 \) and it is not surprising that the probabilities \( p_n \) for finite \( n \) are larger in sub- than in supercritical trees.

Fig. 2 shows the dependence of \( p_n \) on \( n \) in sub- and supercritical evolutions, respectively. If the difference \( q_1 - q_2 = f_1 \) is small enough, let us say \( 0.1 \), then one observes a special phenomenon, namely, an "oscillation" in the dependence of \( p_n \) versus \( n \), what is clearly seen in Fig. 2.
In order to explain the origin of "oscillation" we calculated the dependence of probabilities $p_2$ and $p_3$ on $q_1$ at $q_2 = 0.5$. In Fig. 3 we can see that $p_3 > p_2$ in the interval $0 \leq q_1 < q_1^{(c)}(2)$. For the sake of simplicity the upper limit $q_1^{(c)}(2)$ will be called "critical $q_1$".

The reason of oscillation is trivial. Clearly, if $q_1 - q_2 = f_1 = 0$, then there are no random trees having nodes of even number, i.e. $p_{2j} = 0$, $j = 1, 2, \ldots$, therefore, it should be an interval $0 \leq f_1 < f_1^{(c)}(2j)$ in which

$$p_{2j} < p_{2j+1} \quad \forall \ j = 1, 2, \ldots$$

We calculated the critical values of $q_1$ corresponding to $f_1^{(c)}(2j)$ at $q_2 = 0.4(0.1)1.0$. The results are shown in Table 1. As expected $q_1^{(c)}(2j)$ is slightly decreasing with $j$ when $q_2$ is fixed.

At this point it is worthwhile to underline one of the most characteristic properties of stationary random trees. As known, all moments of the total number of nodes are converging to infinity when the evolution is supercritical and $t \to \infty$. However, the probability to find a "very old" (stationary) supercritical tree of finite size is always larger than zero and, therefore, the probability to find a supercritical tree of infinite size is $w_\infty = 1 - \sum_{n=1}^{\infty}$, and is always smaller than one.

In the case of truncated arbitrary distribution of $\nu$ one can write Eq. (10).
Table 1: Critical values of $q_1^{(c)}(2j)$ at $q_2 = 0.4(0.1)1.0$.

| $q_2$ | $p_2 = p_3$ | $p_4 = p_5$ | $p_6 = p_7$ |
|-------|-------------|-------------|-------------|
| 0.4   | 0.553       | 0.514       | 0.491       |
| 0.5   | 0.674       | 0.634       | 0.609       |
| 0.6   | 0.789       | 0.749       | 0.723       |
| 0.7   | 0.897       | 0.858       | 0.833       |
| 0.8   | 0.100       | 0.963       | 0.938       |
| 0.9   | 1.098       | 1.062       | 1.039       |
| 1.0   | 1.191       | 1.157       | 1.135       |

in the form:

$$\sum_{n=1}^{\infty} p_n = \begin{cases} 1, & \text{if } q_1 \leq 1, \\ 1 - 2(q_1 - 1)/q_2, & \text{if } q_1 > 1, \end{cases}$$

and so, one can formulate the statement as follows: a supercritical "very old" tree may be finite with probability $1 - 2(q_1 - 1)/q_2$, and consequently, infinite with probability $w_\infty = 2(q_1 - 1)/q_2$. It is elementary to prove that $2(q_1 - 1)/q_2 \leq 1$, if $q_1 > 1$.

Let us get some insight into the behavior of living nodes in stationary random trees. From Eq. (6) we obtain for all $|z| \geq 0$ that

$$g^{(\ell)}_{1,2} = \begin{cases} 1, & \text{if } q_1 < 1, \\ f_0/f_2 = 1 - 2(1 - q_1)/q_2, & \text{if } q_1 > 1, \end{cases}$$

and from this we can conclude that

$$p^{(\ell)}_0 = \begin{cases} 1, & \text{if } q_1 < 1, \\ f_0/f_2 = 1 - 2(1 - q_1)/q_2, & \text{if } q_1 > 1. \end{cases}$$

It seems to be appropriate to underline the notion of the second half of Eq. (13). It is expressing that the probability of finding living nodes in supercritical stationary random tree is $1 - f_0/f_2$, i.e.

$$\mathcal{P}\{\tilde{\mu}_\ell > 0\} = 2 \frac{q_1 - 1}{q_2}.$$
As to the probabilistic properties of end-nodes in stationary trees we have to go back to Eqs. (7) and (8). It is elementary to show that the generating function of \( \tilde{\mu}_0 \) is nothing else than

\[
\tilde{g}_0(z) = 1 + \frac{1}{2f_2} \left[ 1 - f_1 - 2f_2 - (1 - f_1) \sqrt{1 - \frac{4f_2 f_0}{(1 - f_1)^2} z} \right].
\] (14)

It is seen immediately that

\[
\lim_{z \uparrow 1} \tilde{g}_0(z) = \tilde{g}_0(1) = \begin{cases} 1, & \text{if } q_1 \leq 1, \\ 1 - 2(q_1 - 1)/q_2, & \text{if } q_1 > 1, \end{cases}
\]

i.e. the probability that the number of end-nodes in a supercritical random tree is infinite and is given by \( 2(q_1 - 1)/q_2 \). At this point one has to remember that in Eq. (12) it was shown that \( 2(q_1 - 1)/q_2 \), at the same time, is equal to the probability of finding infinite number of nodes in a supercritical random tree. At the first sight the result of this comparison is surprising, but considering that the statement expresses only the equality of measures characterizing sets of nodes and end-nodes in an infinite tree, it is far not unexpected. By expanding \( g_0(z) \) in power series at \( z = 0 \) one obtains

\[
p_n^{(0)} = \frac{1}{\sqrt{\pi}} f_0 \frac{\Gamma(n - 1/2)}{\Gamma(n + 1)} \frac{(4 f_0 f_2)^{n-1}}{(1 - f_1)^{2n-1}}, \quad n > 0.
\] (15)

Figure 4: Dependence of probabilities \( p_n^{(0)} \) on \( n \) at \( q_2 = 0.5 \).
In Fig. 4 we can see how the probability of finding \( n \) end-nodes in a stationary random tree depends on \( n \) when \( q_1 = 0.95 \), \( q_2 = 0.5 \) and \( q_1 = 1.05 \), \( q_2 = 0.5 \), respectively. It is remarkable that the probabilities are decreasing very rapidly with increasing \( n \), i.e. there are very few among the stationary trees containing a large number of end-nodes. The dependence of \( p_2^{(0)} \) and \( p_3^{(0)} \) on \( q_1 \) is shown in Fig. 5 at \( q_2 = 0.5 \). One can observe the formation of a maximum in both curves. The sites of maxima are lying in the interval \( 0 << q_1 < 1 \).

Figure 5: Dependence of probabilities \( p_n^{(0)} \) on \( q_1 \) at \( n = 2, 3 \) and \( q_2 = 0.5 \)

In order to compare the probabilities \( p_n \) and \( p_n^{(0)} \) the difference \( d_n = p_n - p_n^{(0)} \) has been calculated. For small values of \( n \) one can obtain \( d_n < 0 \). Fig. 6 shows the dependence of \( d_n \) on \( n \) in sub- and supercritical stationary random trees. For the sake of simple comparison of probabilities \( p_n \) and \( p_n^{(0)} \) for small \( n \) values Table 2 has been compiled.

### Table 2: Probabilities \( p_n \) and \( p_n^{(0)} \) for small \( n \)

\[
\begin{array}{|c|c|c|c|c|}
\hline
n & 1 & 2 & 3 & 4 \\
\hline
p_n & f_0 & f_0 f_1 & f_0 (f_1^2 + f_0 f_2) & f_0 f_1 (f_1^2 + 3f_0 f_2) \\
p_n^{(0)} & f_0/(1 - f_1) & f_0 f_2/(1 - f_1)^3 & 2f_0 f_2^2/(1 - f_1)^5 & 5f_0^2 f_2^3/(1 - f_1)^7 \\
\hline
\end{array}
\]
3.2 Geometric distribution of $\nu$

Let us investigate now the size distribution of stationary random trees when $\nu$ is of geometric distribution, i.e.

$$q(z) = \frac{1}{1 + q_1(1 - z)}.$$  \hspace{1cm} (16)

It can be immediately seen that the equation

$$g(z) = \frac{z}{1 + q_1[1 - g(z)]}$$

has to be solved and

$$g_{1,2}(z) = \frac{1}{2q_1} \left[ 1 + q_1 \pm \sqrt{(1 + q_1)^2 - 4q_1 z} \right]$$

are the two solutions. Since $g_1(0) = 1 + 1/q_1 > 1$ the root

$$g_2(z) = g(z) = \frac{1}{2q_1} \left[ 1 + q_1 - \sqrt{(1 + q_1)^2 - 4q_1 z} \right]$$  \hspace{1cm} (17)
has to be chosen as probability generating function for $\tilde{\mu}$. Clearly,

$$\lim_{z \uparrow 1} g(z) = \begin{cases} 
1, & \text{if } q_1 \leq 1, \\
1/q_1, & \text{if } q_1 > 1,
\end{cases}$$

(18)
i.e. the probability to find infinite number of nodes in a supercritical stationary random tree is $w_\infty = 1 - 1/q_1$.

By using the power series of $g(z)$ it can be shown that the probability of finding $n$ nodes in a stationary tree is nothing else, but

$$p_n = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n - 1/2)}{\Gamma(n + 1)} \frac{(4q_1)^{n-1}}{(1 + q_1)^{2n-1}}.$$

(19)

Figure 7: Probabilities to find $n = 1, 2, \ldots$ nodes in a "very old" ($t = \infty$) tree when the distribution of $\nu$ is geometric. The dark and light bars correspond to subcritical and slightly supercritical tree evolutions, respectively.

Fig. 7 shows the dependence of the probability $p_n$ on $n$. The dark bars denote probabilities to find $n = 1, 2, \ldots$ nodes in a stationary tree produced in a subcritical ($q_1 = 0.9$) evolution. At the same time, the light bars are related to probabilities that $n = 1, 2, \ldots$ nodes can be found in a stationary tree which was produced by a strongly supercritical ($q_1 = 1.5$) evolution.
We underline again that $\mathbb{E}\{\tilde{\mu}\}$ exists and can be calculated from Eq. (17) only if $q_1 < 1$.

The probability distribution of the number of end-nodes in stationary random trees has some new qualitative features in comparison with those obtained in the case $\nu \in \mathfrak{t}$. By using Eq. (7) and taking into account the generating function (16) we obtain

$$g_0(z) = \frac{1}{1 + q_1[1 - g_0(z)]} = \frac{1 - z}{1 + q_1},$$

and it is elementary to show that the solution

$$g_0(z) = \frac{1}{2q_1(1 + q_1)} \left[ 1 + q_1 + q_1^2 + q_1 z - r(q_1, z) \right],$$

where

$$r(q_1, z) = \sqrt{(1 + q_1 + q_1^2 + q_1 z)^2 - 4q_1(1 + q_1)^2 z},$$

is such that the requirement $g_0(1) = 1$, if $q_1 < 1$ is met. Expanding $g_0(z)$ into power series around $z = 0$ one finds \(^5\) that

$$p^{(0)}_n = \begin{cases} \frac{1}{2q_1(1 + q_1)} \left\{ q_1 + (1 + q_1 + q_1^2) C_1 [X(q_1) + Y(q_1)] \right\}, & \text{if } n = 1, \\ \frac{1 + q_1 + q_1^2}{2q_1(1 + q_1)} \left\{ C_n [X(q_1)^n + Y(q_1)^n] - Z_n(q_1) \right\}, & \text{if } n > 1, \end{cases} \quad (20)$$

\(^5\) The aim of the following elementary consideration is to give help to understand how the probability $p^{(0)}_n$ has been calculated. $r(q_1, z)$ can be rewritten into the form:

$$r(q_1, z) = q_1 \sqrt{(z - z_1)(z - z_2)},$$

where

$$z_1 = 1 + \frac{(1 + q_1)^2}{q_1} + 2 \frac{1 + q_1}{\sqrt{q_1}},$$

and

$$z_2 = 1 + \frac{(1 + q_1)^2}{q_1} - 2 \frac{1 + q_1}{\sqrt{q_1}}.$$

Introducing the notations $X(q_1) = 1/z_1$ and $Y(q_1) = 1/z_2$ one obtains

$$g_0(z) = \frac{1}{2q_1(1 + q_1)} \left\{ 1 + q_1 + q_1^2 + q_1 z - q_1 \sqrt{z_1 z_2} \sqrt{(1 - X(q_1)z)[1 - Y(q_1)z]} \right\},$$

and this formula is used to get the power series of $g_0(z)$. 

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where

\[ C_n = \frac{1}{2\sqrt{\pi}} \frac{\Gamma(n - 1/2)}{\Gamma(n + 1)} , \]

while

\[ X(q_1) = \frac{q_1}{q_1 + (1 + q_1)(1 + \sqrt{q_1})^2}, \quad Y(q_1) = \frac{q_1}{q_1 + (1 + q_1)(1 - \sqrt{q_1})^2} \]

and

\[ Z_n(q_1) = \sum_{k=1}^{n-1} C_k C_{n-k} X(q_1)^k Y(q_1)^{n-k}. \]

It is relevant to note that

\[ \sum_{n=1}^{\infty} p_n^{(0)} = \begin{cases} 1, & \text{if } q_1 \leq 1, \\ 1/q_1, & \text{if } q_1 > 1. \end{cases} \]

In other words, the probability to find infinite number of end-nodes in a supercritical stationary tree is nothing else than \( w_\infty = 1 - 1/q_1 \).

\[ \text{Figure 8: Probabilities to find } n = 1, 2, \ldots \text{ end-nodes in a stationary tree when the distribution of } \nu \text{ is geometric. The dark and light bars correspond to sub- and supercritical evolutions, respectively.} \]

Fig. 8 shows the probabilities of finding \( n = 1, 2, \ldots \) end-nodes in a stationary random tree when \( \nu \in g \). We have seen in Fig. 4 a similar bar chart but by making a comparison between the two bar charts we can conclude
that the ratio of probability $p_1^{(0)}$ to $p_2^{(0)}$ is much larger in the case of $\nu \in g$ than in that of $\nu \in t$. In other words, if $\nu$ is of geometric distribution then the probability of formation of rod like stationary random trees is significantly greater than in the case of $\nu \in t$.

The structure of stationary random trees can be better visualized by giving the formulas $p_n$ and $p_n^{(0)}$ for $n = 1, 2, 3, 4$ explicitly. Table 3 contains these formulas.

Table 3: Probabilities $p_n$ and $p_n^{(0)}$ for $n = 1, 2, 3, 4$

| $n$ | $1$ | $2$ | $3$ | $4$ |
|-----|-----|-----|-----|-----|
| $p_n$ | $1/(1 + q_1)$ | $q_1/(1 + q_1)^3$ | $2q_1/(1 + q_1)^5$ | $5q_1/(1 + q_1)^7$ |
| $p_n^{(0)}$ | $(1 + q_1)/(1 + q_1 + q_1^2)$ | $q_1^2(1 + q_1)/(1 + q_1 + q_1^2)^3$ | $q_1^3(1 + q_1)h_3(q_1)/(1 + q_1 + q_1^2)^5$ | $q_1^4(1 + q_1)h_4(q_1)/(1 + q_1 + q_1^2)^7$ |

where

$$h_3(q_1) = 1 + 3q_1 + q_1^2 \quad \text{and} \quad h_4(q_1) = 1 + 7q_1 + 13q_1^2 + 7q_1^3 + q_1^4$$

Figure 9: Dependence of probabilities $p_n$, $n = 2, 3$ on $q_1$ in the case of $\nu \in g$. 

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In the case of geometric distribution of $\nu$ the curves $p_2$ and $p_3$ versus $q_1$ are similar to those we have seen in Fig. 4 though the maxima are appearing at smaller values of $q_1$ than in the case of $\nu \in t$. The dependence of probabilities $p_n$, $n = 2, 3$ on $q_1$ is presented in Fig. 9.

![Figure 10: Variation of the difference $d_n = p_n - p_n^{(0)}$ with $n$ in the case of sub- and supercritical stationary random trees when $\nu \in g$.](image)

It seems to be instructive to show the variation of the probability difference $d_n = p_n - p_n^{(0)}$ with $n$ in the case of sub- and supercritical stationary random trees. In Fig. 10 one can see that $d_n > 0$ when $n > 1$.

### 3.3 Poisson distribution

In the sequel we would like to discuss briefly the case when $\nu$ has Poisson distribution with parameter $q_1 > 0$. As known, the generating function of $\nu$ is given by

$$q(z) = E\{z^{\nu}\} = e^{-(1-z)q_1},$$

and $E\{\nu\} = D^2\{\nu\} = q_1$.

It can be easily shown that the generating function $g(z) = E\{z^{\tilde{\mu}}\}$ of the random variable $\lim_{t \to \infty} \mu(t) \overset{d}{=} \tilde{\mu}$ should satisfy the following equation:

$$g(z) = z \cdot q[g(z)] = z \cdot e^{-q_1} \cdot e^{q_1 g(z)}.$$  \hspace{1cm} (21)

Applying Theorem 1 to Eq. (21) one can formulate the following statement:
If \( z \uparrow 1 \) and \( 0 < q_1 \leq 1 \), then the equation \( \tilde{g} = e^{-q_1} e^{q_1 \tilde{g}} \), where \( \tilde{g} = \lim_{z \uparrow 1} g(z) = \sum_{n=0}^{\infty} p_n \), has only one root in the interval \([0, 1]\), and that is \( \tilde{g} = g_1 = 1 \), while if \( q_1 > 1 \), then besides \( g_1 \) there is another root in \([0, 1]\), namely \( \tilde{g} = g_2 < 1 \).

The consequence of this statement means that if \( q_1 > 1 \), then the probability to find infinite number of nodes in a stationary random tree in the case of \( \nu \in \mathbf{p} \) is nothing else than

\[
w_{\infty} = 1 - g_2 = 1 - \sum_{n=0}^{\infty} p_n.
\]

Figure 11: Illustration of the appearance of roots \( g_2 < 1 \) when \( q_1 = 1.1 \) and 1.2.

In Fig. 11 we would like to illustrate the appearance of roots smaller than 1 of the equation \( R = e^{-q_1} e^{q_1 \tilde{g}} - \tilde{g} = 0 \). We can see, that the equation \( R = 0 \) has only one trivial root in \([0, 1]\), namely the \( g_1 = 1 \), when \( 0 < q_1 \leq 1 \). The black points are referring to roots due to four different values of \( q_1 \).

The probability to find infinite number of nodes in a stationary random tree has been calculated, and the \( w_{\infty} \) versus \( q_1 \) curve is plotted in Fig. 12. It is interesting to note that \( w_{\infty} > 0.5 \) when \( q_1 = 1.4 \).

Now, we want to calculate the probabilities \( p_n \) of finding \( n = 1, 2, \ldots \) nodes in a stationary random tree. Expanding the expression

\[
z \exp(-q_1) \exp \left[ -q_1 \sum_{n=1}^{\infty} p_n z^n \right]
\]
into power series of $z$ at $z = 0$ we can step-by-step compute the probabilities $p_n, \ n = 1, 2, \ldots$, in accordance with Eq. (21). We obtain that

$$p_n = C_n q_1 \left[e^{-q_1} q_1\right]^n,$$

(22)

where $C_n, \ n = 1, 2, \ldots$ are positive rational numbers. The first seven of them are given in Table 4. It seems to be hardly possible to obtain explicit an formula for $C_n$, but it is an easy task to compose an algorithm for its computation.

Table 4: Coefficients $C_n$ in $p_n$ for $n = 1, 2, 3, 4, 5$

| $n$ | 1 | 2 | 3/2 | 8/3 | 125/24 | 54/5 | 16807/720 |
|-----|---|---|-----|-----|--------|------|----------|
| $C_n$ | 1 | 1 | 3/2 | 8/3 | 125/24 | 54/5 | 16807/720 |

It follows from Eq. (22) immediately that the probability $p_n$ vs. $q_1$ has a maximum at

$$q_1 = 1 - \frac{1}{n}.$$

The appearance of maxima is well seen in Fig. 13 in the cases of $n = 2, 3, 4, 5$.

Finally, we would like to discuss the properties of end-nodes in stationary random trees evolved according to Poisson distribution of $\nu$. In order to
obtain the probabilities $p_n^{(0)}$, $n = 1, 2, \ldots$ we have to solve the equation
\[
\exp(-q_1) \exp[q_1 g_0(z)] = g_0(z) + (1 - z) \exp(-q_1),
\]
where
\[
g_0(z) = \sum_{n=1}^{\infty} p_n^{(0)} z^n.
\]
By using a special expansion procedure we can get the probabilities $p_1^{(0)}$, $p_2^{(0)}$, \ldots step-by-step. Introducing the notations
\[
u(q_1) = (e^{q_1} - q_1)^{-1},
\]
\[
\ell_3(q_1) = e^{q_1} + 2q_1,
\]
\[
\ell_4(q_1) = e^{2q_1} + 8e^{q_1}q_1 + 6q_1^2,
\]
\[
\ell_5(q_1) = e^{3q_1} + 22e^{2q_1}q_1 + 58e^{q_1}q_1^2 + 24q_1^3,
\]
\[
\ell_6(q_1) = e^{4q_1} + 52e^{3q_1}q_1 + 328e^{2q_1}q_1^2 + 444e^{q_1}q_1^3 + 120q_1^4
\]
the first six probabilities are given by the following Eqs.:
\[
p_1^{(0)} = \nu(q_1), \quad p_2^{(0)} = \frac{1}{2}q_1^2[\nu(q_1)]^3, \quad p_3^{(0)} = \frac{1}{6}q_1^3\ell_3(q_1)[\nu(q_1)]^5,
\]
\[
p_4^{(0)} = \frac{1}{24}q_1^4\ell_4(q_1)[\nu(q_1)]^7, \quad p_5^{(0)} = \frac{1}{120}q_1^5\ell_5(q_1)[\nu(q_1)]^9,
\]
\[
p_6^{(0)} = \frac{1}{720}q_1^6\ell_6(q_1)[\nu(q_1)]^{11}.
\]
Figure 14: Probabilities to find $n = 1, \ldots, 6$ end-nodes in a stationary random tree when $\nu \in p$.

Figure 15: Dependence of probabilities $p_n^{(0)}$, $n = 2, 3, 4, 5$ on $q_1$ when $\nu \in p$.

By using these formulas we are able to show the dependence of $p_n^{(0)}$ on $n$ and $q_1$, respectively. We see in Fig. 14 that the evolution process prefers the one end-node structures, i.e. stationary random trees in which the probability of finding $n > 1$ end-nodes is very small. The curves $p_n^{(0)}$, $n = 2, 3, 4, 5$ versus $q_1$ plotted in Fig. 15 are clearly demonstrating that the sites of maxima of probabilities are shifted to higher values of $q_1$ as the number of nodes $n$ is increasing.
4 Conclusions

We have investigated the main properties of evolution of special trees when the continuous time parameter tends to infinity. A tree of infinite age is called stationary or simply “very old”. We have stated that if the limit relations
\[
\lim_{t \to \infty} P\{\mu(t) = n|S_0\} = p_n \leq 1, \quad \forall \ n = 1, 2, \ldots
\]
are true, then the random function \(\mu(t)\) giving the number of nodes in a tree at \(t \geq 0\) converges in distribution to a random variable \(\tilde{\mu}\) which counts the number of nodes in a stationary tree.

For the generating function \(E\{z^{\tilde{\mu}}\} = g(z)\) a simple equation has been derived, namely
\[
g(z) = zq[g(z)], \quad (a)
\]
where \(q(z)\) is nothing else than the generating function of the number of new nodes \(\nu\) produced by one dying node. It has been proved that if \(\nu\) has finite first and second factorial moments, i.e. if \(E\{\nu\} = q_1\) and \(E\{\nu(\nu - 1)\} = q_2\) are finite, then \(\sum_{n=1}^{\infty} p_n = 1\), if \(q_1 \leq 1\), and \(\sum_{n=1}^{\infty} p_n \leq 1\), if \(q_1 > 1\). It means, if \(q_1 \leq 1\), then the probability \(w_\infty\) of finding infinite number of nodes in a stationary random tree is zero, while if \(q_1 > 1\), then that is larger than zero, namely, \(w_\infty = 1 - \sum_{n=1}^{\infty} p_n \leq 1\).

Here, one has to note that the expectation value \(E\{\tilde{\mu}\}\) exists only if the tree evolution is subcritical, i.e. if \(q_1 < 1\). In the case of \(q_1 > 1\), i.e. in supercritical evolution it has been shown that
\[
\lim_{t \to \infty} E\{\mu(t) \exp[-(q_1 - 1)Qt]\} = q_1/(q_1 - 1),
\]
while if \(q_1 = 1\), then \(\lim_{t \to \infty} E\{\mu(t)/Qt\} = 1\).

It has been shown also that the generating function \(g_0(z)\) of the number of end-nodes \(\tilde{\mu}_0\) in a stationary random tree has to satisfy the equation
\[
g_0(z) = q[g_0(z)] - (1 - z)f_0, \quad (b)
\]
which can be used to study end-node properties. When \(q(z) = E\{z^{\nu}\}\) is known then by using appropriate method for comparison of coefficients of \(z^n\) in both sides of equations \((a)\) and \((b)\) we could calculate step-by-step the probabilities \(p_1, p_1^{(0)}, p_2, p_2^{(0)}, \ldots\) to find \(n = 1, 2, \ldots\) nodes and end-nodes, respectively, in a stationary random tree.
For the calculations, we have chosen three different distributions of $\nu$. The generating functions of these distributions are given by the following formulas:

$$q(z) = \begin{cases} 
1 + q_1(z - 1) + \frac{1}{2}q_2(z - 1)^2, & \text{if } \nu \in t, \\
1/[1 + q_1(z - 1)], & \text{if } \nu \in g, \\
e^{q_1(z-1)}, & \text{if } \nu \in p.
\end{cases}$$

Analyzing the results of numerical calculations the first impression is that the qualitative properties of stationary random trees depend hardly on the character of distribution of $\nu$. We have seen that in all cases the probability to find $n = 1$ node (or end-node) in a stationary tree is significantly larger then to find $n > 1$ nodes. One can conclude that in the evolution process the formation of a rod-like stationary random tree is much more probable than that with many branches.

We have found special behavior in the dependence of $p_n$ on $n$ only in the case of $\nu \in t$ when $\mathcal{P}\{\nu = 1\} = f_1$ is smaller than a critical value. The appearance of the oscillation of $p_n$ versus $n$ is consequence of the following trivial statement: when $f_1 = 0$ then $p_{2j} = 0$, $j = 1, 2, \ldots$.

It has been demonstrated that the probabilities $p_n$ and $p_n^{(0)}$ versus $q_1$ show a maximum the location of which is increasing with $n$ but remains always smaller than 1. This property is best seen in the case of Poisson distribution of $\nu$.

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