Absolute moments in terms of characteristic functions

Gwo Dong Lin and Chin-Yuan Hu
Academia Sinica and National Changhua University of Education

Abstract. The absolute moments of probability distributions are more complicated than conventional ones. By using a direct and simpler approach, we retreat P. L. Hsu’s (1951, J. Chinese Math. Soc., Vol. 1, pp. 257–280) formulas in terms of the characteristic function and provide some new results as well. The case of nonnegative random variables is also investigated through both characteristic function and Laplace–Stieltjes transform. Besides, we prove that the distribution of a nonnegative random variable with a finite fractional moment can be completely determined by a proper subset of the translated fractional moments. This improves significantly Hall’s (1983) result for distributions on the right-half line.

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Postal addresses: Gwo Dong Lin, Institute of Statistical Science, Academia Sinica, Taipei 11529, Taiwan. (E-mail: gdlin@stat.sinica.edu.tw)
Chin-Yuan Hu, Department of Business Education, National Changhua University of Education, Changhua 50058, Taiwan. (E-mail: buhuua@gmail.com)
1. Introduction

The absolute moments of probability distributions play important roles in both theoretical and applied fields (see, e.g., von Bahr 1965, Hall 1983, and Matsui and Pawlas 2016). Our main purpose in this paper is to investigate the presentation of (fractional) absolute moments of distributions in terms of their Fourier–Stieltjes or Laplace–Stieltjes transforms.

We review first some important properties of the characteristic function (ch.f.). Consider a random variable \( X \) with distribution \( F(x) = \Pr(X \leq x) \) on \( \mathbb{R} \equiv (-\infty, \infty) \), denoted \( X \sim F \), and let \( \phi \) be the ch.f. of \( F \), namely, the Fourier–Stieltjes transform
\[
\phi(t) = \int_{-\infty}^{\infty} e^{itx} dF(x) = \int_{-\infty}^{\infty} \cos(tx) dF(x) + i \int_{-\infty}^{\infty} \sin(tx) dF(x), \quad t \in \mathbb{R},
\]
(1)
or, \( \phi(t) = \mathbb{E}[\exp(itX)] = \mathbb{E}[\cos(tX)] + i \mathbb{E}[\sin(tX)] \equiv \text{Re}(\phi(t)) + i \text{Im}(\phi(t)), \quad t \in \mathbb{R} \). It is known that for any distribution \( F \), its ch.f. \( \phi \) in (1) always exists and uniquely determines \( F \) by the famous Weierstrass Approximation Theorem. Therefore, we can recover \( F \) and derive its properties via the ch.f. \( \phi \) theoretically. In fact, we have the classical Lévy inverse formula:
\[
F(x + h) - F(x) = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{1 - e^{-ith}}{it} e^{-itx} \phi(t) \, dt,
\]
provided that both \( x \) and \( x + h \) (with \( h > 0 \)) are continuity points of \( F \) (see, e.g., Lukacs 1970, Chapter 3). Another explicit expression is the following (improper) integral: for any continuity point \( x \) of \( F \) on \( \mathbb{R} \),
\[
F(x) = \frac{1}{2} - \frac{1}{\pi} \int_{0}^{\infty} \text{Im}(\phi(t)e^{-itx}) \frac{dt}{t} = \frac{1}{2} - \frac{1}{\pi} \lim_{T \to \infty} \int_{0}^{T} \text{Im}(\phi(t)e^{-itx}) \frac{dt}{t}
\]
(2)
(see Zolotarev 1957, or Kawata 1972, p. 130, or Rossberg et al. 1985, p. 45).

On the other hand, if \( X \geq 0 \), then for any continuity point \( x \) of \( F \) on \( \mathbb{R}_+ \equiv [0, \infty) \), we have the inverse formulas via (2):
\[
F(x) = \frac{2}{\pi} \int_{0}^{\infty} (\sin xt) \text{Re}(\phi(t)) \frac{dt}{t}, \quad x \in \mathbb{R}_+,
\]
(3)
\[
F(x) = 1 - \frac{2}{\pi} \int_{0}^{\infty} (\cos xt) \text{Im}(\phi(t)) \frac{dt}{t}, \quad x \in \mathbb{R}_+
\]
(4)
(Laue 1983, 1986). This means that for any nonnegative random variable, one of the real and imaginary parts of its ch.f. is enough to recover the corresponding distribution.
It is also well known that if the random variable $X \sim F$ on $\mathbb{R}$ has finite $n$th moment $m_n = \mathbb{E}[X^n]$ for some positive integer $n$, then its ch.f. $\phi$ is $n$-time differentiable and $\mathbb{E}[X^n] = (-i)^n \phi^{(n)}(0)$ (in terms of the ch.f.). Unlike the conventional moment, the absolute moment $\mu_s = \mathbb{E}[|X|^s]$ is, however, much complicated. For example, if $\mu_s$ is finite for an odd positive integer $s$, then we can derive

$$\mu_s = \frac{1}{2\pi i^{s+1}} \int_{-\infty}^{\infty} \left[ \phi^{(s)}(t) - \phi^{(s)}(-t) \right] \frac{dt}{t}$$

(which corrects a result in Lukacs 1970, p. 26). In particular,

$$\mu_1 = -\frac{1}{\pi} \int_{-\infty}^{\infty} \text{Re}(\phi'(t)) \frac{dt}{t}.$$  \hspace{1cm} (5)

Moreover, $\mu_1$ has another formula

$$\mu_1 = \frac{2}{\pi} \int_{0}^{\infty} \frac{1 - \text{Re}(\phi(t))}{t^2} dt,$$

which is a special case of Theorem 1 or Theorem 1’ below (see also Remark 1 for discussions on (5) and the absolute moment $\mu_{-1}$ of negative order). It is worth mentioning that the distribution of $X$ with finite $\mu_s$ is completely determined by the set

$$M_s = \{ \mathbb{E}[|X + a|^s] : a \in \mathbb{R} \}$$

of translated absolute moments, where $s > 0$ is not an even integer (Hall 1983).

We will focus on the formulas of (fractional) absolute moments of positive real orders. The main results are stated in the next section, and all the proofs are given in Section 4. The case of nonnegative random variables is also studied through Laplace–Stieltjes transform. Besides, we improve significantly Hall’s (1983) characterization result (6) for distributions on the right-half line (Theorems 7 and 8). In Section 3, we give the necessary lemmas to prove the main results. Finally, in Section 5 some illustrative remarks are provided.

2. Main results

Let $X \sim F$ on $\mathbb{R}$ with ch.f. $\phi$ and define the polynomial

$$P_n(t) = \sum_{k=0}^{n} \frac{1}{k!} \phi^{(k)}(0) t^k, \quad t \in \mathbb{R},$$
provided \(\phi\) is \(n\) time differentiable. Hsu (1951) might be the first one to derive formulas of absolute moments. For example, he obtained the following results in terms of the ch.f. (by using the orthogonality property of the Hermite polynomials), in which \([s]\) denotes the largest integer less than or equal to \(s\).

**Theorem 1** (Hsu 1951, Theorems 2.1 and 4.1). Let \(X \sim F\) on \(\mathbb{R}\).

(a) If \(X\) has finite absolute moment \(\mu_{2n+1}\) of positive odd order \(2n+1\), then

\[
\mu_{2n+1} = (-1)^{n+1} \frac{2}{\pi} (2n+1)! \int_0^\infty \frac{\text{Re}(\phi(t) - P_{2n}(t))}{t^{2n+2}} dt.
\]

(b) If the fractional absolute moment \(\mu_{n+r} < \infty\) for some integer \(n \geq 0\) and real \(r \in (0,1)\),

\[
\mu_{n+r} = (-1)^{n/2+1} \frac{2}{\pi} \Gamma(n+r+1) \sin \left( \left(\frac{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + \frac{r}{2}\right) \pi \right) \int_0^\infty \frac{\text{Re}(\phi(t) - P_n(t))}{t^{n+r+1}} dt.
\]

Note that (7) can be rewritten as

\[
\mu_{2n+1} = (-1)^{n+1} \frac{2}{\pi} (2n+1)! \int_0^\infty t^{-2n-2} \left[ \text{Re}(\phi(t)) - \sum_{k=0}^n \frac{(-1)^k}{(2k)!} m_{2k} t^{2k} \right] dt.
\]

In order to get a better insight into this result, let us recall the following:

(i) for \(n \geq 0\), the even function

\[
g_n(t) = (-1)^{n+1} \left( \cos t - \sum_{k=0}^n \frac{(-1)^k}{(2k)!} t^{2k} \right) \geq 0, \quad t \in \mathbb{R},
\]

and if \(n \geq 1\), \(g_n\) is increasing and convex on \([0, \infty)\);

(ii) for \(n \geq 1\), the odd function

\[
h_n(t) = (-1)^n \left( \sin t - \sum_{k=1}^n \frac{(-1)^{k+1}}{(2k-1)!} t^{2k-1} \right) \geq 0, \quad t \geq 0,
\]

increases on \([0, \infty)\). (See, e.g., Hu and Lin 2005, Lemma 4.)

Moreover, we define the auxiliary functions:

\[
G_n(t) = (-1)^{n+1} \left( \text{Re}(\phi(t)) - \sum_{k=0}^n \frac{(-1)^k}{(2k)!} m_{2k} t^{2k} \right), \quad t \in \mathbb{R}, \quad n \geq 0,
\]

\[
H_n(t) = (-1)^n \left( \text{Im}(\phi(t)) - \sum_{k=1}^n \frac{(-1)^{k+1}}{(2k-1)!} m_{2k-1} t^{2k-1} \right), \quad t \in \mathbb{R}, \quad n \geq 1,
\]

provided the moments are finite. Also, define \(h_0(t) = \sin t\) and \(H_0(t) = \mathbb{E}[\sin(tX)]\), \(t \in \mathbb{R}\).
Combining (7) and (8), we have the following slight extension of Theorem 1. Note that we don’t assume the finiteness of expectations in (9) because of the fact \( g_n \geq 0 \), and hence both sides could be infinite simultaneously.

**Theorem 1’**. Let \( X \sim F \) on \( \mathbb{R} \), \( n \geq 0 \) be an integer and \( 2n < s < 2n + 2 \). Then

\[
\mu_s = \mathbb{E}[|X|^s] = \frac{2}{\pi} \Gamma(s + 1) \sin((s - 2n)\pi/2) \int_0^{\infty} \frac{\mathbb{E}[g_n(tX)]}{t^{s+1}} \, dt \quad \text{(finite or infinite).} \tag{9}
\]

If, in addition, \( \mu_s < \infty \) for some \( s \in (2n, 2n + 2) \) with integer \( n \geq 0 \), then

\[
\mu_s = \frac{2}{\pi} \Gamma(s + 1) \sin((s - 2n)\pi/2) \int_0^{\infty} \frac{G_n(t)}{t^{s+1}} \, dt. \tag{10}
\]

Theorems 1 and 1’ and some of their special cases have been rediscovered in the literature (see, e.g., von Bahr 1965, von Bahr and Esseen 1965, Brown 1970, 1972, Kawata 1972, Section 11.4, Chung 1974, Section 6.2, and Chow and Teicher 1997, Section 8.4). Surprisingly, few of the above authors mentioned the original article Hsu (1951). A direct and probably simpler proof of Theorems 1’ will be given later.

Similarly, we have the following result for distributions on the right-half line, which modifies a result of Brown (1970, Lemma 1) (noting that the function \( h_0(t) = \sin t, \ t \geq 0 \), is not of constant sign) and retreats those of Laue (1986) in a simpler way. The latter applied the calculus of fractional derivatives and integrals.

**Theorem 2**. Let \( 0 \leq X \sim F \). Then the following statements are true.

(a) If \( s \in (2n - 1, 2n + 1) \) for some integer \( n \geq 1 \), the moment

\[
m_s = \mathbb{E}[X^s] = \frac{2}{\pi} \Gamma(s + 1) \cos((s - 2n)\pi/2) \int_0^{\infty} \frac{\mathbb{E}[h_n(tX)]}{t^{s+1}} \, dt \quad \text{(finite or infinite).} \tag{11}
\]

(b) If, in addition, \( m_s < \infty \) for some \( s \in (2n - 1, 2n + 1) \) with integer \( n \geq 1 \),

\[
m_s = \frac{2}{\pi} \Gamma(s + 1) \cos((s - 2n)\pi/2) \int_0^{\infty} \frac{H_n(t)}{t^{s+1}} \, dt. \tag{12}
\]

(c) If \( m_s < \infty \) for some \( s \in (0, 1) \),

\[
m_s = \frac{2}{\pi} \Gamma(s + 1) \cos(s\pi/2) \int_0^{\infty} \frac{\mathbb{E}[\sin(tX)]}{t^{s+1}} \, dt = \frac{2}{\pi} \Gamma(s + 1) \cos(s\pi/2) \int_0^{\infty} \frac{H_0(t)}{t^{s+1}} \, dt. \tag{13}
\]

(d) If \( 0 < X \sim F \), the improper integral

\[
\int_0^{\infty} \frac{\mathbb{E}[\sin(tX)]}{t} \, dt = \int_0^{\infty} \frac{\text{Im}(\phi(t))}{t} \, dt = \frac{\pi}{2}. \tag{14}
\]
regardless of the distribution $F$.

In view of (14), we can define a density function
\begin{equation}
    g(t) = \frac{2}{\pi} \frac{\mathbb{E}[\sin(tX)]}{t}, \quad t > 0,
\end{equation}
provided $\mathbb{E}[\sin(tX)] \geq 0$ on $(0, \infty)$. This leads to the following reciprocal moment relation. Two illustrative examples are given in Remark 3 below. See also Remark 4 and Laue (1986) for other moment relations by using the mixture of exponential distribution.

**Theorem 3.** Let $0 < X \sim F$ with $\mathbb{E}[\sin(tX)] \geq 0$ for all $t \in [0, \infty)$, and let $0 < Y \sim G$ with density function $g$ defined in (15).

(a) If $\mathbb{E}[X^s] < \infty$ for some $s \in (0, 1)$, then
\begin{equation}
    \mathbb{E}[Y^{-s}] = \frac{1}{\Gamma(s + 1) \cos(s\pi/2)} \mathbb{E}[X^s].
\end{equation}

(b) If, instead, $\mathbb{E}[X^s] < \infty$ for all $s \in (-1, 1)$, then (16) holds true for all $s \in (-1, 1)$.

It is surprising that the condition $\mathbb{E}[\sin(tX)] \geq 0$ in Theorem 3 is fulfilled by many distributions as shown in Lemma 3 and Corollary 1 below.

**Corollary 1.** Let $Z$ have a Pólya-type ch.f. $\phi$ and let $X > 0$ obey the distribution $F = 1 - \phi$. Assume further that $\mathbb{E}[X^s] < \infty$ for all $s \in (-1, 1)$. Then
\begin{equation}
    \mathbb{E}[|Z|^{-s}] = \frac{1}{\Gamma(s + 1) \cos(s\pi/2)} \mathbb{E}[X^s], \quad |s| < 1.
\end{equation}

Similarly, for length-biased distributions, we have the following reciprocal moment relation by Theorem 1’.

**Theorem 4.** Let $0 \leq X \sim F$ with mean $m_1 \in (0, \infty)$. Define the random variable $Z \sim H$ with the length-biased distribution $H(z) = \frac{1}{m_1} \int_0^z t dF(t)$, $z \geq 0$, and the random variable $Y \sim G$ with the density
\begin{equation}
    g(y) = \frac{2}{\pi} \frac{1 - \mathbb{E}[\cos(yX)]}{m_1 y^2}, \quad y > 0.
\end{equation}
If $\mathbb{E}[X^s] < \infty$ for some $s \in (0, 2)$, then
\begin{equation}
    \mathbb{E}[Y^{1-s}] = \frac{1}{\Gamma(s + 1) \sin(s\pi/2)} \mathbb{E}[X^s] = \frac{1}{\Gamma(s + 1) \sin(s\pi/2)} \mathbb{E}[Z^{s-1}].
\end{equation}
For nonnegative random variables $X \sim F$, we define the Laplace–Stieltjes transform
$L(\lambda) = \mathbb{E}[\exp(-\lambda X)], \lambda \geq 0$. It is known that the distribution $F$ of $X$ is uniquely determined
by $L$; more precisely, for each continuity point $x \geq 0$ of $F$,

$$F(x) = \lim_{n \to \infty} \sum_{k \leq nx} (-1)^k \frac{n^k}{k!} L^{(k)}(n)$$

(see also Remark 2 below). We can apply $L$ (instead of Fourier–Stieltjes transform) to get
the formulas for fractional moments. To do this, let us define the auxiliary functions

$$q_n(\lambda) = (-1)^{n+1} \left( e^{-\lambda} - \sum_{k=0}^{n} \frac{(-1)^k}{k!} \lambda^k \right) \geq 0, \lambda \geq 0, \ n \geq 0, \ (17)$$

and

$$Q_n(\lambda) = (-1)^{n+1} \left( L(\lambda) - \sum_{k=0}^{n} \frac{(-1)^k}{k!} m_k \lambda^k \right) \geq 0, \lambda \geq 0, \ n \geq 0, \ (18)$$

provided the moments are finite (see, e.g., Hu and Lin 2008). Note that the derivative
$q'_n = q_{n-1}$ for $n \geq 1$. Then we have the following result. See also Urbanik (1993, p. 326) and
Klar (2003) for the formula (20) under the assumption of finite fractional moments.

**Theorem 5.** Let $0 \leq X \sim F$ and $n < s < n + 1$, where $n \geq 0$ is an integer. Then the
fractional moment (finite or infinite)

$$m_s = \frac{1}{\pi} \Gamma(s+1) \sin((s-n)\pi) \int_0^\infty \frac{\mathbb{E}[q_n(\lambda X)]}{\lambda^{s+1}} d\lambda = \frac{(-1)^{n+1}}{\Gamma(-s)} \int_0^\infty \frac{\mathbb{E}[q_n(\lambda X)]}{\lambda^{s+1}} d\lambda. \ (19)$$

If, in addition, $m_s < \infty$ for some $s \in (n, n + 1)$ with integer $n \geq 0$, then

$$m_s = \frac{1}{\pi} \Gamma(s+1) \sin((s-n)\pi) \int_0^\infty \frac{Q_n(\lambda)}{\lambda^{s+1}} d\lambda = \frac{(-1)^{n+1}}{\Gamma(-s)} \int_0^\infty \frac{Q_n(\lambda)}{\lambda^{s+1}} d\lambda. \ (20)$$

As shown in Lemma 5, the function $Q_n$ in (18) is closely related to the high-order equi-
librium distribution $F_{(n+1)}$ with respect to (w.r.t.) $F$ defined below. For $0 \leq X \sim F$
with mean $m_1 \in (0, \infty)$, we define the first-order equilibrium distribution by $F_{(1)}(x) =
m_1^{-1} \int_0^x \overline{F}(y) dy, \ x \geq 0$, where $\overline{F} = 1 - F$. The high-order equilibrium distributions are
defined iteratively. Namely, the $n$th-order equilibrium distribution (w.r.t. $F$) is $F_{(n)}(x) =
m_{(n-1)}^{-1} \int_0^x \overline{F}_{(n-1)}(y) dy, \ x \geq 0$, provided the mean $m_{(n-1)}$ of $F_{(n-1)}$ is finite. It is known that
$m_{(n-1)} = m_n / (n m_{n-1})$ (see, e.g., Lin 1998b, p. 265, or Harkness and Shantaram 1969). An
application of Theorem 5 leads to the next result similar to Theorems 3 and 4.
Theorem 6. Let $0 \leq X \sim F$ with $n$th moment $m_n \in (0, \infty)$ for some integer $n \geq 1$. Let $X_{(n)}$ have the $n$th-order equilibrium distribution $F_{(n)}$ and Laplace–Stieltjes transform $L_{(n)}(\lambda) \equiv \mathbb{E}[\exp(-\lambda X_{(n)})]$, $\lambda \geq 0$. Assume further that $Y > 0$ has distribution $G = 1 - L_{(n)}$.

(a) If $\mathbb{E}[X^s] < \infty$ for some $s \in (n-1, n)$, then

$$E[Y^{n-s}] = \frac{n!(n-s)\pi}{m_n(s+1)\sin((n-s)\pi)} E[X^s]. \quad (21)$$

(b) If, instead, $\mathbb{E}[X^s] < \infty$ for all $s \in (n-1, n+1)$, then (21) holds true for all $s \in (n-1, n+1)$.

The constant is defined for $s = n$ by continuity to be equal to $1/m_n$.

Finally, we present the counterpart result of Hall (1983) for distributions on the right-half line, which claims that a proper subset of the previous translated moments in (6) is enough to characterize such a distribution. Theorem 7 treats the case of finite $s$th fractional moment with $s \in (0, 1)$, while Theorem 8 deals with the remaining cases which are more involved.

Theorem 7. Let $0 \leq X \sim F$ and let the fractional moment $m_s \in (0, \infty)$ for some $s \in (0, 1)$. Then the distribution $F$ is completely determined by the sequence

$$M_s^+ = \{E[X^s]\} \cup \{E[(X + a_k)^s]\}_{k=1}^\infty$$

of translated fractional moments, where $\{a_k\}_{k=1}^\infty$ is a sequence of positive and distinct real numbers satisfying one of the following conditions:

(a) $\lim_{k \to \infty} a_k = \infty$ and $\sum_{k=1}^\infty 1/a_k = \infty$;
(b) $\lim_{k \to \infty} a_k = a_0 \in (0, \infty)$;
(c) $\lim_{k \to \infty} a_k = 0$ and $\sum_{k=1}^\infty a_k = \infty$.

Theorem 8. Let $0 \leq X \sim F$ and let the fractional moment $m_s \in (0, \infty)$ for some $s \in (n, n+1)$, where $n \geq 1$ is an integer. Further, suppose that $\{a_k\}_{k=1}^\infty$ is a sequence of positive and distinct real numbers satisfying: $\lim_{k \to \infty} a_k = \infty$ and $\sum_{k=1}^\infty 1/a_k = \infty$. Then the distribution $F$ is completely determined by the sequence

$$M_s^{++} = \{E[X^s]\} \cup \{E[(X + a_k)^s]\}_{k=1}^\infty \cup \{E[(X + 2a_k)^s]\}_{k=1}^\infty$$

of translated fractional moments.

An immediate consequence of the last two theorems is the following.
Corollary 2. Let $0 \leq X \sim F$.

(a) If the fractional moment $m_s \in (0, \infty)$ for some $s \in (0, 1)$, then the distribution $F$ is characterized by any one of the two sequences: (i) $\{E[X^s]\} \cup \{E[(X+p_j)^s]\}_{j=1}^{\infty}$ and (ii) $\{E[X^s]\} \cup \{E[(X+1/p_j)^s]\}_{j=1}^{\infty}$, where $p_j$ is the $j$th prime number.

(b) If, instead, the fractional moment $m_s \in (0, \infty)$ for some non-integer $s > 1$, then the distribution $F$ is characterized by the sequence $\{E[X^s]\} \cup \{E[(X+p_j)^s]\}_{j=1}^{\infty} \cup \{E[(X+2p_j)^s]\}_{j=1}^{\infty}$.

3. Lemmas

To prove the main results, we need some preliminary lemmas in the sequel.

Lemma 1. Let $n \geq 0$ be an integer and $2n < s < 2n + 2$. Then the integral

$$\int_0^{\infty} \frac{g_n(t)}{t^{s+1}} dt = \pi \frac{\Gamma(s+1)}{2 \Gamma(r+1)} \sin((s-2n)\pi/2) \cdot (s-2n+1).$$

Proof. Note that $g_0(t) = 1 - \cos t$, $t \in \mathbb{R}$, and the derivatives $g_k' = h_k$, $h_k' = g_{k-1}$ for $k \geq 1$. Moreover, by L’Hospital rule, we have, for $2 \leq 2k < r < 2(k+1)$, the limits

$$\lim_{t \to 0^+} \frac{g_k(t)}{t^r} = 0 = \lim_{t \to \infty} \frac{g_k(t)}{t^r}, \quad \lim_{t \to 0^+} \frac{h_k(t)}{t^{r-1}} = 0 = \lim_{t \to \infty} \frac{h_k(t)}{t^{r-1}}.$$

First, we recall the identity:

$$\int_{-\infty}^{\infty} \frac{1 - \cos t}{|t|^{r+1}} dt = \pi \frac{1}{\Gamma(r+1)} \sin((s-2n)\pi/2) \cdot (s-2n+1), \quad r \in (0, 2)$$

(see, e.g., Chung 1974, p. 159), from which the case $n = 0$ in the lemma follows. Then, by integration by parts and induction, we have, for $n \geq 1$, the integral

$$\int_0^{\infty} \frac{g_n(t)}{t^{s+1}} dt = \frac{1}{s} \int_0^{\infty} \frac{h_n(t)}{t^s} dt = \frac{1}{s(s-1)} \int_0^{\infty} \frac{g_{n-1}(t)}{t^{s-1}} dt = \ldots$$

$$= \frac{1}{s(s-1) \cdots (s-2n+1)} \int_0^{\infty} \frac{g_0(t)}{t^{s-2n+1}} dt = \frac{1}{s(s-1) \cdots (s-2n+1)} \int_0^{\infty} \frac{1 - \cos t}{t^{s-2n+1}} dt = \frac{\pi}{2 \Gamma(s+1)} \sin((s-2n)\pi/2).$$

The proof is complete.

Lemma 2. Let $n \geq 0$ be an integer and $2n - 1 < s < 2n + 1$. Then the integral

$$\int_0^{\infty} \frac{h_n(t)}{t^{s+1}} dt = \frac{\pi}{2} \frac{\Gamma(s+1)}{\sin((s-2n)\pi/2)}.$$
Note that the above identity holds true in the sense of improper integral (not the Lebesgue integral on \((0, \infty)\)) when \(n = 0\) and \(s \in (-1, 0]\).

**Proof of Lemma 2.** First, we consider the case \(n \geq 1\) and \(2n - 1 < s < 2n + 1\), or, equivalently, \(2 \leq 2n < s + 1 < 2n + 2\). Then from the proof of Lemma 1 (replacing \(s\) by \(s + 1\)), we have the integral

\[
\int_0^\infty \frac{h_n(t)}{t^{s+1}} dt = \frac{(s + 1)\pi}{2\Gamma(s + 2) \sin((s + 1 - 2n)\pi/2)} = \frac{\pi}{2\Gamma(s + 1) \cos((s - 2n)\pi/2)}.
\]

Therefore, it remains to prove the case \(n = 0\). Namely, for \(-1 < s < 1\), we want to prove

\[
\int_0^\infty \frac{h_0(t)}{t^{s+1}} dt = \int_0^\infty \frac{\sin t}{t^{s+1}} dt = \frac{\pi}{2\Gamma(s + 1) \cos(s\pi/2)}.
\]

The case \(s = 0\) is trivial. For \(s \neq 0\), we recall the identity:

\[
\int_0^\infty \frac{\sin t}{t^{s+1}} dt = \Gamma(-s) \sin(-s\pi/2), \quad 0 < |s| < 1
\]

(see, e.g., Gradshteyn and Ryzhik 2014, Formulas 3.721(1) and 3.761(4)). Finally, Euler’s reflection formula, \(\Gamma(1 - z)\Gamma(z) = \pi/\sin(z\pi)\) for non-integer \(z\), completes the proof.

**Lemma 3.** Let \(0 < X \sim F\) with density function \(f(x)\) decreasing to zero as \(x \to \infty\). Then \(\mathbb{E}[\sin(tX)] \geq 0\) for all \(t \in [0, \infty)\).

**Proof.** See, e.g., Luckas (1970), p. 84.

**Lemma 4.** Let \(n \geq 0\) be an integer and \(n < s < n + 1\). Then the integral

\[
\int_0^\infty \frac{q_n(\lambda)}{\lambda^{s+1}} d\lambda = \frac{\Gamma(n + 1 - s)}{s(s - 1)(s - 2) \cdots (s - n)} = (-1)^{n+1}\Gamma(-s).
\]

**Proof.** Note that \(q_0(\lambda) = 1 - \exp(-\lambda), \ \lambda \geq 0\), and that the derivative \(q'_k = q_{k-1}\) for \(k \geq 1\).

Moreover, by L’Hospital rule, we have, for \(0 \leq k < r < k + 1\), the limits

\[
\lim_{\lambda \to 0^+} \frac{q_k(\lambda)}{\lambda^r} = 0 = \lim_{\lambda \to \infty} \frac{q_k(\lambda)}{\lambda^r}.
\]

First, it can be shown that for \(0 < r < 1\), we have the integral

\[
\int_0^\infty \frac{q_0(\lambda)}{\lambda^{r+1}} d\lambda = \int_0^\infty \frac{1 - e^{-\lambda}}{\lambda^{r+1}} d\lambda = \frac{\Gamma(1 - r)}{r}.
\]
This is exactly the case $n = 0$. Then, by integration by parts and induction, we have, for $n \geq 1$, the integral

$$\int_0^\infty \frac{q_n(\lambda)}{\lambda^{s+1}} d\lambda = \frac{1}{s} \int_0^\infty \frac{q_{n-1}(\lambda)}{\lambda^s} d\lambda = \cdots = \frac{1}{s(s-1) \cdots (s-n+1)} \int_0^\infty \frac{q_0(\lambda)}{\lambda^{s-n+1}} d\lambda = \frac{\Gamma(n+1-s)}{s(s-1) \cdots (s-n)} = (-1)^{n+1} \Gamma(-s).$$

The proof is complete.

The next lemma is an application of Lin’s (1994) theorem to the specific function $g(x) = \exp(-\lambda x)$, $x \geq 0$, where $\lambda > 0$ is a constant.

**Lemma 5.** Let $0 \leq X \sim F$ with $n$th moment $m_n \in (0, \infty)$ for some integer $n \geq 1$, and let $X_{(n)}$ have the $n$th-order equilibrium distribution $F_{(n)}$. Then the Laplace–Stieltjes transform of $X_{(n)}$ has the form

$$L_{(n)}(\lambda) = \mathbb{E}[\exp(-\lambda X_{(n)})] = n! Q_{n-1}(\lambda)/(m_n \lambda^n), \ \lambda \geq 0,$$

where the function $Q_{n-1}$ is defined in (18), and the RHS of the above equality is defined for $\lambda = 0$ by continuity to be equal to 1.

For a proof of the following variant of the Müntz–Szász Theorem, see Lin (1993), where ‘$\overset{d}{=} \cdot$’ means ‘equal in distribution’.

**Lemma 6.** Let $X_1$ and $X_2$ be two nonnegative random variables with Laplace–Stieltjes transforms $L_1$ and $L_2$, respectively. Let $\{\lambda_k\}_{k=1}^\infty$ be a sequence of positive and distinct real numbers satisfying one of the following conditions:

(a) $\lim_{k \to \infty} \lambda_k = \infty$ and $\sum_{k=1}^\infty 1/\lambda_k = \infty$;
(b) $\lim_{k \to \infty} \lambda_k = \lambda_0 \in (0, \infty)$;
(c) $\lim_{k \to \infty} \lambda_k = 0$ and $\sum_{k=1}^\infty \lambda_k = \infty$.

Further, assume that $L_1(\lambda_k) = L_2(\lambda_k)$ for all $k \geq 1$. Then $X_1 \overset{d}{=} X_2$.

4. Proofs of main results

Proof of Theorem 1’. It suffices to prove formula (9). Since the even function $g_n \geq 0$ for all $n \geq 0$, Tonelli’s Theorem applies (see, e.g., Royden 1988, p. 309) and we have, by
\[ g_n(0) = 0 \text{ and Lemma 1}, \]
\[
\int_{0}^{\infty} \mathbb{E}[g_n(tX)] \frac{dt}{t^{s+1}} = \frac{1}{2} \int_{-\infty}^{\infty} \mathbb{E}[g_n(tX)] \frac{dt}{t^{s+1}} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{g_n(tx)}{t^{s+1}} dF(x) dt
\]
\[
= \frac{1}{2} \int_{-\infty}^{\infty} \frac{g_n(tx)}{t^{s+1}} dt dF(x)
\]
\[
= \frac{1}{2} \int_{-\infty}^{\infty} |x|^s \frac{g_n(t)}{|t|^{s+1}} dt dF(x)
\]
\[
= \int_{0}^{\infty} \frac{g_n(t)}{t^{s+1}} dt \int_{-\infty}^{\infty} |x|^s dF(x)
\]
\[
= \frac{\pi}{2} [\Gamma(s+1) \sin((s-2n)\pi/2)]^{-1} \mu_s.
\]

This proves (9), and hence (10) follows immediately.

**Proof of Theorem 2.** We first prove formula (11). Since the odd function \( h_n(t) \geq 0 \) on \([0, \infty)\) for \( n \geq 1 \), Tonelli’s Theorem applies and we have, by \( h_n(0) = 0 \) and Lemma 2,
\[
\int_{0}^{\infty} \frac{\mathbb{E}[h_n(tX)]}{t^{s+1}} dt = \int_{0}^{\infty} \frac{h_n(tx)}{t^{s+1}} dF(x) dt = \int_{0}^{\infty} \frac{h_n(tx)}{t^{s+1}} dt dF(x)
\]
\[
= \int_{0}^{\infty} x^s \int_{0}^{\infty} \frac{h_n(t)}{t^{s+1}} dt dF(x)
\]
\[
= \int_{0}^{\infty} \frac{h_n(t)}{t^{s+1}} dt \int_{0}^{\infty} x^s dF(x)
\]
\[
= \frac{\pi}{2} [\Gamma(s+1) \cos((s-2n)\pi/2)]^{-1} m_s.
\]

This proves part (a), and hence part (b) with \( n \geq 1 \) follows immediately.

As for part (c), namely, the case \( n = 0 \) and \( s \in (0, 1) \), we first prove that
\[
\int_{0}^{\infty} \frac{|h_0(t)|}{t^{s+1}} dt = \int_{0}^{\infty} \frac{|\sin t|}{t^{s+1}} dt = \int_{0}^{1} \frac{|\sin t|}{t^{s+1}} dt + \int_{1}^{\infty} \frac{|\sin t|}{t^{s+1}} dt \equiv I_1 + I_2 < \infty. \tag{22}
\]

Since \( \lim_{t \to 0} (\sin t)/t = 1 \), \( \sup_{t \in [0,1]} |\sin t|/t \equiv M < \infty \), and hence the first integral \( I_1 \leq M \int_{0}^{1} t^{-s} dt = M/(1-s) < \infty \). On the other hand, the second integral
\[
I_2 \leq \int_{1}^{\infty} \frac{1}{t^{s+1}} dt = 1/s < \infty,
\]
which completes the proof of (22). This in turn implies that
\[
\int_{0}^{\infty} \int_{0}^{\infty} \frac{|\sin(tx)|}{t^{s+1}} dt dF(x) = \int_{0}^{\infty} x^s \int_{0}^{\infty} \frac{|\sin t|}{t^{s+1}} dt dF(x) = \mathbb{E}[X^s] \int_{0}^{\infty} \frac{|\sin t|}{t^{s+1}} dt < \infty,
\]

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provided $m_s < \infty$. Hence, the Fubini–Tonelli Theorem applies (see, e.g., Rudin 1987, p. 164) and (13) holds true by Lemma 2. Part (d) is an immediate consequence of (2) or (4) by letting $x = 0$. The proof is complete.

**Proof of Theorem 3.** To prove part (a), note that, by Theorem 2(c), the moment

$$
\mathbb{E}[Y^{-s}] = \int_0^\infty t^{-s}g(t) \, dt = \int_0^\infty \frac{2}{\pi} \frac{\mathbb{E}[\sin(tX)]}{t^{s+1}} \, dt = \frac{1}{\Gamma(s+1)\cos(s\pi/2)} \mathbb{E}[X^s].
$$

Part (b) follows from part (a) with the help of analytic continuation of Mellin transform.

**Proof of Corollary 1.** Since $\phi$ is a Pólya-type ch.f., $Z$ has a Pólya-type density function $p$, which is symmetric on $\mathbb{R}$ and of the form

$$
p(z) = \frac{1}{\pi} \int_0^\infty \cos(zt)\phi(t) \, dt = \frac{1}{\pi} \int_0^\infty \cos(zt)[1 - F(t)] \, dt, \quad z > 0
$$

(Lukacs 1970, pp. 83–84). By integration by parts, we rewrite the density

$$
p(z) = \frac{1}{\pi z} \int_0^\infty \sin(zt)dF(t) = \frac{1}{\pi z} \mathbb{E}(\sin(zX)) \geq 0, \quad z > 0.
$$

Note that $Y = |Z|$ has the density function $g(y) = 2p(y)$, $y > 0$. Therefore, the required result follows from Theorem 3 immediately.

**Proof of Theorem 4.** Note that, by Theorem 1' with $n = 0$, the moment

$$
\mathbb{E}[Y^{1-s}] = \int_0^\infty y^{1-s}g(y) \, dy = \int_0^\infty \frac{2}{\pi} \frac{1 - \mathbb{E}[\cos(yX)]}{m_1y^{s+1}} \, dy = \frac{1}{\Gamma(s+1)\sin(s\pi/2)} \frac{\mathbb{E}[X^s]}{m_1}.
$$

**Proof of Theorem 5.** It suffices to prove formula (19). Since the function $q_n \geq 0$, Tonelli’s Theorem applies and we have, by $q_n(0) = 0$ and Lemma 4,

$$
\int_0^\infty \frac{\mathbb{E}[g_n(\lambda X)]}{\lambda^{s+1}} \, d\lambda = \int_0^\infty \int_0^\infty \frac{q_n(\lambda x)}{\lambda^{s+1}} dF(x) \, d\lambda = \int_0^\infty \int_0^\infty q_n(\lambda x) \frac{d\lambda dF(x)}{\lambda^{s+1}} = \int_0^\infty q_n(\lambda) \frac{d\lambda}{\lambda^{s+1}} \int_0^\infty x^s dF(x) = \int_0^\infty q_n(\lambda) \frac{d\lambda}{\lambda^{s+1}} \int_0^\infty x^s dF(x) = (-1)^{n+1} \Gamma(-s) m_s.
$$

Therefore, (19) holds true, and (20) follows immediately.
**Proof of Theorem 6.** For $s \in (n - 1, n)$, let us consider the moment
\[
\mathbb{E}[Y^{n-s}] = \int_0^\infty \Pr(Y^{n-s} > y) dy = \int_0^\infty \Pr(Y > y^{1/(n-s)}) dy
\]
\[
= \int_0^\infty \mathcal{G}(y^{1/(n-s)}) dy = (n - s) \int_0^\infty \frac{\mathcal{G}(\lambda)}{\lambda^{s-n+1}} d\lambda
\]
\[
= (n - s) \int_0^\infty \frac{L_{(n)}(\lambda)}{\lambda^{s-n+1}} d\lambda = \frac{n!(n-s)}{m_n} \int_0^\infty \frac{Q_{n-1}(\lambda)}{\lambda^{s+1}} d\lambda.
\]
The last equality is due to Lemma 5. Therefore, part (a) follows from Theorem 5, while part (b) follows from part (a) with the help of analytic continuation of Mellin transform.

**Proof of Theorem 7.** Suppose that the random variables $0 \leq X \sim F$ and $0 \leq Y \sim G$ satisfy (i) $\mathbb{E}[X^s] = \mathbb{E}[Y^s] = m_s \in (0, \infty)$ for some $s \in (0, 1)$ and (ii)
\[
\mathbb{E}[(X + a_k)^s] = \mathbb{E}[(Y + a_k)^s], \quad k = 1, 2, \ldots, \tag{23}
\]
where the sequence $\{a_k\}_{k=1}^\infty$ is defined in the theorem. Then we want to prove that $F = G$.

Denote the constant $C_s = \frac{1}{\pi} \Gamma(s + 1) \sin(s\pi)$. Then, by Theorem 5, we have the moment
\[
\mathbb{E}[X^s] = C_s \int_0^\infty \frac{1 - L_X(\lambda)}{\lambda^{s+1}} d\lambda, \tag{24}
\]
where $L_X$ is the Laplace–Stieltjes transform of $X \sim F$. In other words, the function
\[
h_X(\lambda) = \frac{C_s}{m_s} \frac{1 - L_X(\lambda)}{\lambda^{s+1}} \geq 0, \quad \lambda > 0,
\]
is a bona fide density. Let $Z_X \sim H_X$ have the density $h_X$. Similarly, let $Z_Y \sim H_Y$ have the corresponding density (derived from $Y \sim G$)
\[
h_Y(\lambda) = \frac{C_s}{m_s} \frac{1 - L_Y(\lambda)}{\lambda^{s+1}} \geq 0, \quad \lambda > 0.
\]
Since $L_{X+a_k}(\lambda) = \mathbb{E}[\exp(-\lambda(X + a_k))] = \exp(-a_k\lambda)L_X(\lambda), \quad \lambda \geq 0$, it follows from (23) and (24) that
\[
\int_0^\infty e^{-a_k\lambda} \frac{L_Y(\lambda) - L_X(\lambda)}{\lambda^{s+1}} d\lambda = 0, \quad k = 1, 2, \ldots,
\]
or, equivalently,
\[
\int_0^\infty e^{-a_k\lambda} dH_X(\lambda) = \int_0^\infty e^{-a_k\lambda} h_X(\lambda) d\lambda = \int_0^\infty e^{-a_k\lambda} h_Y(\lambda) d\lambda
\]
\[
= \int_0^\infty e^{-a_k\lambda} dH_Y(\lambda), \quad k = 1, 2, \ldots.
\]
This implies that $H_X = H_Y$ by Lemma 6, and hence $h_X = h_Y$, which in turn implies that $L_X = L_Y$. Therefore, $X \overset{d}{=} Y$ by the uniqueness theorem for Laplace–Stieltjes transforms. The proof is complete.

**Proof of Theorem 8.** Suppose that the random variables $0 \leq X \sim F$ and $0 \leq Y \sim G$ satisfy (i) $\mathbb{E}[X^s] = \mathbb{E}[Y^s] = m_s \in (0, \infty)$ for some $s \in (n, n+1)$, where $n \geq 1$ is a fixed integer, and (ii)

$$\mathbb{E}[(X + a)^s] = \mathbb{E}[(Y + a)^s], \quad a = a_k, \ 2a_k, \ k = 1, 2, \ldots. \quad (25)$$

Then we want to prove that $F = G$.

Firstly, we show that

$$\mathbb{E}[X^k] = \mathbb{E}[Y^k] = m_k, \ 1 \leq k \leq n. \quad (26)$$

Mimicking the proof of Hall (1983), we have the relation: for all $0 \leq \ell < n$,

$$a^s \{2a^s \mathbb{E}[(X + a)^s] - 2^s(2a)^s \mathbb{E}[(X + 2a)^s]\} = \sum_{j=0}^{\ell} \binom{s}{j} a^{s+1-j} (2 - 2^{s+1-j}) \mathbb{E}[X^j] + \binom{s}{\ell+1} \mathbb{E}[X^{s+1}] + o(1) \text{ as } 0 < a \to \infty.$$ 

Therefore, the sequence $M_s^{++}$ determines the moments $\mathbb{E}[X^k], \ 1 \leq k \leq n$, and (26) holds true due to the assumption (25).

Secondly, let $L_X$ and $L_Y$ be the Laplace–Stieltjes transforms of $X$ and $Y$, respectively. Denote the constant $C_s = \frac{1}{\pi} \Gamma(s+1) \sin((s-n)\pi)$, and define the function

$$h_X(\lambda) = \frac{C_s}{m_s} \frac{Q_n(\lambda)}{\lambda^{s+1}} \geq 0, \ \lambda > 0,$$

in which $L = L_X$ in Theorem 5. Then it follows from (20) that $\int_0^\infty h_X(\lambda) d\lambda = 1$, namely, $h_X$ is a bona fide density function. Similarly, define the density function

$$h_Y(\lambda) = \frac{C_s}{m_s} \frac{Q_n^*(\lambda)}{\lambda^{s+1}} \geq 0, \ \lambda > 0,$$

where

$$Q_n^*(\lambda) = (-1)^{n+1} \left(L_Y(\lambda) - \sum_{k=0}^{n} \frac{(-1)^k}{k!} m_k \lambda^k\right) \geq 0, \ \lambda \geq 0.$$
Thirdly, recall from (19) that for \( a > 0 \), the translated moment

\[
\mathbb{E}[(X + a)^s] = C_s \int_0^\infty \frac{\mathbb{E}[g_\lambda((X + a))]\lambda^{s+1}}{\lambda^{s+1}} d\lambda
\]

\[
= C_s (-1)^{n+1} \int_0^\infty \frac{1}{\lambda^{s+1}} \left( e^{-a\lambda}L_X(\lambda) - \sum_{k=0}^{n} \frac{(-1)^k}{k!} \mathbb{E}[(X + a)^k] \right) d\lambda. \quad (27)
\]

Combining (25) through (27) yields

\[
\int_0^\infty e^{-a_k\lambda} \frac{L_X(\lambda) - L_Y(\lambda)}{\lambda^{s+1}} d\lambda = 0, \quad k = 1, 2, \ldots,
\]

or, equivalently,

\[
\int_0^\infty e^{-a_k\lambda} h_X(\lambda) d\lambda = \int_0^\infty e^{-a_k\lambda} h_Y(\lambda) d\lambda, \quad k = 1, 2, \ldots.
\]

By Lemma 6, we finally conclude that \( h_X = h_Y \), and hence \( L_X = L_Y \), \( X \overset{d}{=} Y \). The proof is complete.

5. Remarks

Remark 1. Note that the expression (5) improves Rényi’s (1970, p. 369) result by deleting the redundant condition \( \mu_2 < \infty \). Hsu (1951) also proved the curious result that the absolute moment (of negative order) \( \mu_{-1} = \infty \), if the ch.f. \( \phi \) has a nonnegative real part. The latter condition is clearly satisfied by all the Pólya-type ch.f.s (Lukacs 1970, p. 83).

Remark 2. Suppose that \( 0 < X \sim F \) has a continuous density \( f \) and that the power function \( f^r \) is integrable for some \( r > 1 \). Then \( f \) has the following representation in terms of the Laplace–Stieltjes transform \( L \):

\[
f(x) = \lim_{n \to \infty} \frac{(-1)^{n-1}}{(n-1)!} \frac{(n/x)^n L^{(n-1)}(n/x)}{L^{(n)}(n)}, \quad x > 0
\]

(see, e.g., Chung 1974, p. 140).

Remark 3. To illustrate the use of Theorem 3, let us take \( F \) to be the standard exponential distribution \( F(x) = 1 - \exp(-x) \), \( x > 0 \). Then \( Y \sim G \) has the half-Cauchy density \( g(t) = (2/\pi)(1 + t^2)^{-1}, \ t > 0 \), and hence, by (16), the moment relation

\[
\mathbb{E}[Y^{-s}] = \frac{1}{\Gamma(s + 1) \cos(s\pi/2)}, \quad \mathbb{E}[X^s] = \frac{1}{\cos(s\pi/2)}, \quad |s| < 1.
\]
More generally, let \( X \sim F \) have the Gamma density

\[
f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp(-x/\beta), \quad x > 0,
\]

where \( \alpha \in (0, 2] \) and \( \beta > 0 \). Then \( \mathbb{E}[\sin(tX)] \geq 0, \ t \geq 0 \), and \( Y \sim G \) has the density

\[
g(t) = \frac{2}{\pi} \frac{\mathbb{E}[\sin(tX)]}{t} = \frac{2}{\pi} \frac{\sin(\alpha \theta)}{t(1 + \beta^2 t^2)^{\alpha/2}}, \ t > 0,
\]

where \( \theta = \arctan(\beta t) \in (0, \pi/2) \). Thus, we have the moment relation

\[
\mathbb{E}[Y^{-s}] = \frac{1}{\Gamma(s + 1) \cos(s \pi/2)} \mathbb{E}[X^s] = \frac{1}{\Gamma(s + 1) \cos(s \pi/2)} \frac{\Gamma(s + \alpha)}{\Gamma(\alpha)\beta^s}, \ |s| < 1.
\]

**Remark 4.** Let \( 0 < X \sim F \) and define the random variable \( Z \sim H \) with the mixture of exponential distribution \( H(\lambda) = \int_0^\infty (1 - \exp(-\lambda x))dF(x) = 1 - L(\lambda), \ \lambda \geq 0 \). Then we can improve Laue’s (1986, Lemma 3.2) moment relation as follows:

\[
\mathbb{E}[Z^{-s}] = \mathbb{E}[\mathcal{E}^{-s} \mathbb{E}[X^s]] \quad \text{(finite or infinite)}, \quad s \in \mathbb{R},
\]

where the random variable \( \mathcal{E} \), independent of \( X \), obeys the standard exponential distribution and the moment \( \mathbb{E}[\mathcal{E}^{-s}] \) equals \( \Gamma(1 - s) \) or \( \infty \) according as \( s < 1 \) or \( s \geq 1 \). To see this, recall that \( Z \overset{d}{=} \mathcal{E}/X \) by the definition of \( H \) and hence

\[
\mathbb{E}[Z^{-s}] = \mathbb{E}[(\mathcal{E}/X)^{-s}] = \mathbb{E}[\mathcal{E}^{-s}X^s] = \mathbb{E}[\mathcal{E}^{-s}] \mathbb{E}[X^s], \quad s \in \mathbb{R},
\]

due to the Monotone Convergence Theorem (see Lin 1998a, Lemma 4).

**Remark 5.** It seems hard to present the absolute moments of **negative** order in terms of ch.f. (see, e.g., Hsu 1951, (15)). However, for positive random variable \( X \sim F \) with Laplace–Stieltjes transform \( L \), we have the elegant formula

\[
\mathbb{E}[X^{-s}] = (\Gamma(s))^{-1} \int_0^\infty \lambda^{s-1} L(\lambda) d\lambda, \ s > 0,
\]

by the Tonelli Theorem (see, e.g., Pitman and Yor 2003, p. 298, or Lin 1998b, p. 264).

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