Rational soliton on the plane wave background for the $(1+2) \, |\phi|^4$-model with negative coupling

Alla A. Yurova* and Artyom V. Yurov**

*Mathematical Department
Kaliningrad Technical University, Russia
yurov@freemail.ru

**Theoretical Physics Department,
I. Kant University of Russia
artyom_yurov@mail.ru

Abstract

We show that $(1+2)$ nonlinear Klein-Gordon equation with negative coupling admits an exact solution which appears to be the linear superposition of the plane wave and the nonsingular rational soliton.

1 Introduction

The nonlinear Klein-Gordon equation ($\phi^4$- or $|\phi|^4$-models) is one of those equations that frequently arise in seemingly different physical applications [1], [2], [3], [4], [5]. Unfortunately, this equation is nonintegrable. In the simplest case of $(1+1)$ it is relatively easy to find out the stationary solutions (kinks for the $\phi^4$-model, as a particular example). In case of general position $(1+D)$, however, there exist no general methods of construction of the exact solutions of Klein-Gordon equation - bar the case, when the solutions by definition possess enough symmetries. Such symmetries usually result in possibility of reduction of the problem into the one-dimensional case, allowing for complete integration of the initial equation. For example, in the massless theory with potential

$$V(\phi) = -\frac{\lambda \phi^4}{4},$$

the Euclidean solution with the $O(4)$ symmetry can be obtained upon the introduction of the proper boundary conditions. As the result one will get the so-called Fubini instanton: one-parameter nonsingular solution $\phi(r)$ ($r = \sqrt{\sum_{i=1}^{4} x_i^2}$, the $x_i$ are the Euclidean coordinates) with finite Euclidean action [6]. What is somewhat surprising is the fact that for any nonvanishing $m$ and the potential of the form

$$V(\phi) = \frac{m^2 \phi^2}{2} - \frac{\lambda \phi^4}{4},$$
the instanton solutions does not exist. This tiny example gives but a glimpse of what highly nontrivial properties does the solutions of the nonlinear multidimensional equations (like the $\phi^4$-model) have.

In this article we shall consider the $(1+D) |\phi|^4$-model in Minkowski space (summation is implied over the repeating contravariant and covariant indices):

$$\partial_\mu \partial^\mu \phi + m^2 \phi + \lambda |\phi|^2 \phi = 0,$$
with metric
$$g_{\mu\nu} = \text{diag}(+1, -1, -1, ..., -1).$$

As we shall see, in the case $D = 2$ for the negative coupling $\lambda$ the equation (1) admits the nonsingular solutions $\phi(x^\mu), \mu = 0, 1, 2$ such that
$$|\phi(x^\mu)| \to B = \text{const} \quad \text{at} \quad x^2 + y^2 \to \infty.$$

Of course, at first sight models with the negative coupling are quite suspicious from the physical point of view. In fact, since the potential is negative, the action will not be bounded from below and therefore the model may not be stable quantum-mechanically. On the other hand, we have learned recently that the models with negative potentials might be extremely important in cosmology and strings theory [7]. Thus, the problem to find out the interesting exact solutions of such models is alive and kicking.

We would also like to note, that it is still unclear, whether similar results can also be obtained for the case $D > 2$, or not. We believe in former, but for now it is just a hypothesis.

2 Equations

In [8] there has been obtained the novel exact solution of the Davey-Stewartson II (DS-II) equations describing the soliton on the plane wave background (see also [9]). This solution was constructed via Darboux transformations [10], [11] and has the form

$$\phi(x, y, t) = B e^{i s(x, y, t)} \left( -1 + \frac{P_1(x, y, t)}{P_2(x, y, t)} \right),$$

where $s(x, y, t)$ and $P_1(x, y, t)$ are linear functions whereas $P_2(x, y, t)$ is the polynomial of the second order such that $P_2(x, y, t) > 0$ for all values of $x, y, t$.

The aim of this work is to show that the nonintegrable $(1+2) |\phi|^4$ with negative coupling admits the similar solution (see (2)) with

$$P_1 = a_\mu x^\mu + a, \quad P_2 = \eta_{\mu\nu} x^\mu x^\nu + b_\mu x^\mu + A^2,$$

$$s = s_\mu x^\mu,$$
where
$$\eta_{\mu\nu} = \eta_{\nu\mu}, \quad \eta_{\mu\nu} = (\eta_{\mu\nu})^*,$$
$$\eta_{\mu\nu} = (\eta_{\mu\nu})^*, \quad (s_\mu)^* = s_\mu.$$

Substituting (3) into the (1) leads to

$$i P_2 [P_2 (P_1 - P_2) \partial_\mu \partial^\mu s + 2 J^\mu \partial_\mu s] + P_2 \partial_\mu J^\mu - 2 J^\mu \partial_\mu P_2 +$$
$$+ (P_1 - P_2) [\lambda B^2 (P_1 - P_2) (P_2 - P_1) + (m^2 - \partial_\mu s \partial^\mu s) P_2^2] = 0,$$
where

\[ J^\mu = P_2 \partial^\mu P_1 - P_1 \partial^\mu P_2. \]

Using (4) in the particular case \( b_\mu = 0 \) will result in the following system:

\[ s_\mu s^\mu - \lambda B^2 - m^2 = 0, \]

\[
(2is_\mu a^\mu + \lambda B^2(a + a^*))A^4 - a(\lambda B^2(a + 2a^*) + 2\eta^\mu_\nu)A^2 + \lambda B^2|a|^2a = 0,
\]

\[ a_\mu + a^*_\mu = 0, \]

\[
(2is_\mu a^\mu + \lambda B^2(a + a^*)) \eta_{\alpha\beta} - 4is^\mu \eta_{\mu\alpha} a_{\beta} + \lambda B^2 a_\alpha a_\beta = 0,
\]

\[
2\eta_{\alpha\beta} \left( \lambda B^2 a^\mu + \eta^\mu_\nu a_\rho + 2(a_\mu + ias_\mu) \eta^\mu_\rho \right) + a_\rho \left[ \lambda B^2 a_\alpha a_\beta - \lambda B^2(\lambda B^2 + a^* - 2\lambda a_\alpha a_\beta - 8\alpha_a \eta_{\mu\alpha} \eta_{\mu\beta} + 4iA^2 s^\mu \eta_{\mu\alpha} a_\beta = 0, \right.
\]

\[
\left[ (a + 2a^*) - 2A^2(a + a^*) \lambda B^2 + 8iA^2 \eta^\mu_\nu s^\nu a_\nu \right] \eta_{\alpha\beta} - \lambda B^2 A^2 + a^* - 2a) a_\alpha a_\beta - 8\alpha_a \eta_{\mu\alpha} \eta_{\mu\beta} + 4iA^2 s^\mu \eta_{\mu\alpha} a_\beta = 0,
\]

\[
\left[ \lambda B^2 (2a^2 - 2|a|^2 + a^2) + 2\eta^\mu_\nu A^2 \right] a_\alpha + 4A^2 (a^\mu + ias^\mu) \eta_{\mu\alpha} = 0.
\]

Thus in the case \((1 + D)\) one has to solve the system of

\[ N(D) = \frac{(D + 2)(2D^2 + 5D + 7)}{2} \]

algebraic equations, provided that:

(i) \( P_2 > 0 \) for any \( x^\mu \) (if this is the case then solution (2) will be nonsingular) and

(ii) the level lines of the function \( P_2 \) will be closed curves.

### 3 \( D = 2 \) Solutions

In the case \( D = 2 \): \( N(D) = 50 \). The inequalities (i) shall be valid if

\[
\eta_{11} > 0, \quad \begin{vmatrix} \eta_{11} & \eta_{12} \\ \eta_{12} & \eta_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} \eta_{00} & \eta_{01} & \eta_{02} \\ \eta_{01} & \eta_{11} & \eta_{12} \\ \eta_{02} & \eta_{12} & \eta_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} 0 & b_0 & b_1 & b_2 \\ b_0 & \eta_{00} & \eta_{01} & \eta_{02} \\ b_1 & \eta_{01} & \eta_{11} & \eta_{12} \\ b_2 & \eta_{02} & \eta_{12} & \eta_{22} \end{vmatrix} > 0.
\]

Let \( \eta_0, \alpha, \beta, \gamma, \rho \) and \( b \) be the arbitrary real parameters. Lets define three Lorentzian vectors

\[
\xi_\mu = (\xi_0, \xi_1, \xi_2), \quad \eta_\mu = (\eta_0, \eta_1, \eta_2), \quad \theta_\mu = (\theta_0, \theta_1, \theta_2),
\]

such that

\[
\eta_1 = \eta_0 \cos \alpha, \quad \eta_2 = \eta_0 \sin \alpha, \quad \sigma = \frac{\rho^2 \sin(2(\beta - \alpha))}{4\eta_0 \sin(\alpha - \gamma)},
\]

\[
\xi_1 = \rho \cos \beta, \quad \xi_2 = \rho \sin \beta, \quad \xi_0 = \rho \cos(\beta - \alpha),
\]

\[
\theta_1 = \sigma \cos \gamma, \quad \theta_2 = \sigma \sin \gamma, \quad \theta_0 = \frac{\sigma \cos(\beta - \gamma)}{\cos(\beta - \alpha)}.
\]
and, finally
\[ \lambda = -2\rho^2 \sin^2(\alpha - \beta), \quad m^2 = \frac{\sigma^2 \sin(2\beta - \gamma - \alpha) \sin(\gamma - \alpha)}{\cos^2(\beta - \alpha)}. \]  
(8)

Then the solution of the system (5), such that (6) together with the conditions (i) and (ii) holds has the form:
\[ \eta_{\mu\nu} = \xi_{\mu} \xi_{\nu} - 2b\xi_{\mu} \eta_{\nu} + 4 \left( b^2 + B^2 \right) \eta_{\mu} \eta_{\nu}, \quad s_{\mu} = (2B^2 - b^2) \eta_{\mu} + b\xi_{\mu} + \theta_{\mu}, \]
\[ a_{\mu} = 4i\eta_{\mu}, \quad a = \frac{1}{B^2}, \quad A = \pm \frac{1}{2B}. \]  
(9)

This is true when \( b_{\mu} = 0 \). If \( b_{\mu} \neq 0 \) then, instead of (5) one will have a somehow more difficult system. The particular solution, however, can still be obtained:
\[ \eta_{\mu\nu} = 4B^2 \left( \xi_{\mu} \xi_{\nu} - 2b(\xi_{\mu} \eta_{\nu} + \xi_{\nu} \eta_{\mu}) + 4 \left( b^2 + B^2 \right) \eta_{\mu} \eta_{\nu} \right), \]
\[ b_{\mu} = 8B^2 \left( 2(\kappa \psi B + \chi b) \eta_{\mu} - \chi \xi_{\mu} \right), \quad a_{\mu} = 16iB^2 \eta_{\mu}, \]
\[ a = 4 \left( 1 + \frac{\kappa B}{|c_1|^2} (c_2 c_1^* - c_2^* c_1) \right), \quad A^2 = 4B^2(\chi^2 + \psi^2), \]  
(10)

where \( c_{1,2}, \kappa = \pm 1 \) are arbitrary complex constants and \( s_{\mu} \) is the same as in the (9).

Finally, it is easy to show that
\[ \eta_{\mu\nu} x^\mu x^\nu + b_{\mu} x^\mu + A^2 = 1 + 4B^2 \left[ (\Lambda_{\mu} x^\mu - \chi)^2 + (L_{\mu} x^\mu + \psi)^2 \right], \]
where
\[ \Lambda_{\mu} = \xi_{\mu} - 2b\eta_{\mu}, \quad L_{\mu} = 2\kappa B \eta_{\mu}, \]
which means that (6) and two conditions (i) and (ii) hold for our solution, and that concludes our proof.

4 Conclusion

As we have seen, the equation (1) with negative coupling \( \lambda \) admits the nonsingular solution which looks as the linear superposition of the plane wave and rational soliton, and the level lines of this solution on the plane \( xy \) are ellipses.

There remains two open questions, both of them being an interesting problem for the future investigations.

The first question is the possible generalization of the method to the case \( D > 2 \). As for now, we can’t see any possible reasons why our approach will not work in multidimensions. The only price we have to pay working there is a rapidly growing amount of calculations. In fact, the number of algebraic equations \( N(D) \), having to be solved grows as \( D^3 \): \( N(3) = 100 \), \( N(4) = 177 \) and \( N(10) = 1542 \). But then again: the usage of computer programs (for example, MAPLE), that are powerful enough to handle such systems in a reasonable periods of time allows for this problem to be solved.
The second question is the negative value of $\lambda$. Is it possible to construct the exact solution with $\lambda > 0$? The answer is yes, but in this case the solutions will be singular. The singularity will be located along the hyperbola and the physical meaning of such solution is unclear for us.

At last, the next step is the calculation of one-loop quantum corrections to these new classical nontrivial solutions [12] but this is subject of another investigation.

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