Higher-degree Smoothness of Perturbations II

Gang Liu
Department of Mathematics
UCLA
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1 Introduction

In this paper we generalize part of the higher-degree smoothness results in perturbation theory in [3] from the case that the stable maps have the fixed domain $S^2$ to the general genus zero case. Note that genus zero case already captures all analytic difficulties related to the lack of differentiability of transition functions between local slices (see [6] for the discussion on this). The results in this paper and [4] together give one of the two methods for the infinite dimensional set-up used in [5] (compare the other infinite dimensional set-up in [1]).

The main result of this paper is the following theorem (see the relevant definitions in the later sections).

Theorem 1.1 Let $K \simeq K_t$ with $t \in \tilde{W}(\Sigma)$ be the fixed part of the local universal family of stable curves $S \to N(\Sigma)$, where $N(\Sigma)$ is a small open neighborhood of $[\Sigma]$ in $\bar{M}_{0,k}$ with the local coordinate chart $\tilde{W}(\Sigma)$. Consider the local uniformizer (slice) $W(f, H_f)$ centered at $f : \Sigma \to M$ of stable $L^p_k$-maps with domains $S_t, t \in \tilde{W}(\Sigma)$ and the corresponding space $\tilde{W}(f_K)$ of $L^p_k$ maps with domain $K$ with associated bundle $\mathcal{L}^K \to \tilde{W}(f_K)$. Let $\xi^K : \tilde{W}(f_K) \to \mathcal{L}^K$ be a smooth section satisfying the condition $C_1$ and $C_2$. Then $\xi^K$ gives rise a stratified $C^{m_0}$-smooth section $\xi : W(f, H_f) \to \mathcal{L}$ on the local slice $W(f, H_f)$, which, viewed in any other local slice, is still stratified $C^{m_0}$-smooth on their "common intersections" (=the fiber product over the space of unparametrized stable maps).
Here the conditions $C_1$ and $C_2$ are defined in [3] using the bi-grading there (see the definition in [3]) as follows.

$C_1$: The section $\eta: S_f \to L|S_f$ can be extended into a $C^{m_0}$-smooth section $\eta_{-m}: (S_f)^{-m} \to \mathcal{L}_0$ for some $m \geq m_0$.

$C_2$: The image of $\eta(h)$ is lying in $L^p_{k+m}(\Sigma, E_h) =: (\mathcal{L}_h)_m$ with $m \geq m_0$ so that $\eta_{-m,m}: (S_f)^{-m} \to \mathcal{L}_m$ is smooth.

Recall that $m_0 = [k - 2/p]$. We will assume that $p > 2$ and $m_0 > 1$ throughout this paper as in [3].

This theorem is proved in Sec.5. The two kinds of Banach neighborhoods on an end near a stable nodal map are defined in section 3. The corresponding (stratified) smooth structures on each of such neighborhoods are defined in section 4.

Only elementary facts on Sobolev spaces and standard calculus on Banach spaces are used in this paper, for which we refer to [2, 7].

2 Local universal family of stable curves

The starting point of this paper is the local deformation theory of stable maps. To this end, we need recall the local deformation of the stable curves first.

2.1 Stable curves

Given an ”initial” stable curve $\Sigma^0$, let $T_0$ be the tree associated to the domain of the stable curve $\Sigma^0 = (S^0, d^0, x^0)$ so that the underlying curve $S^0$ is a nodal surface with desingularization $\hat{S}^0 = \bigsqcup_{v \in T_0} S^0_v$ as the disjoint union of its components labeled by the vertices $v \in T_0$. Here the double points $d^0_v = \bigsqcup_{v \in T_0} d^0_v$ with each $d^0_v = \{d^0_{vw}, [vw] \in E(T_0)\}$, where each double point $d^0_{vw}$ on $S^0_v$ is labeled by an edge in the set of edges $E(T_0)$ of $T_0$; the marked points $x^0_v = \bigsqcup_{v \in T_0} x^0_v$ with $x^0_v$ to be the marked points on $S^0_v$. Clearly the nodal surface $S^0$ is obtained form $\hat{S}^0$ by identifying the double points.

In above, we have abused notations using $d$ to denote both double points and the set of their collections. Similarly for $x$, and we will continue do so for other similar notations.

The distinguished points (=the double points and marked points) of $\Sigma^0$ on $S^0_v/S^0$, will be denoted by $p^0_v/p^0$. Note that the stable curve $\Sigma^0$ determines and is determined by $p^0$ upto the actions of $G(= \Pi_{v \in T_0} SL(2, \mathbb{C})_v)$. 

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Let $N_{T_0}^{T_0}(\Sigma^0)$ be a small neighborhood of $[\Sigma^0]$ in the moduli space $\mathcal{M}_{0,k}^{T_0}$ with fixed topological type given by $T_0$, where $[\Sigma^0] \in \mathcal{M}_{0,k}^{T_0}$ is the "moduli point" of $\Sigma^0$. Then $N_{T_0}^{T_0}(\Sigma^0)$ parametrizes the stable curves near $\Sigma_0$, or equivalently the nearby distinguished points $p$ on the same fixed $S^0$. Hence we may introduce the parameters $b = \{b_{vu}; v \in T_0, p_{vu}^0 \in \mathbf{p}_v^0\}$ with $b_{vu}$ in a small disc $D_{vu}(p_{vu}^0)$ on $S_v^0$ centered at $p_{vu}^0$. In order to quotient out the (local) actions of $SL(2, \mathbb{C})_v$, for each $v \in T_0$, we fix the last three parameters in $b_{vu}$. Note this also selects the three corresponding distinguished pints on $S_v$ that makes it marked so that the identification $S_v \simeq S^2$ is specified.

Then the parameter $b$ is corresponding to the stable curve $\Sigma_b = (S_b, \mathbf{p}_b)$. The collection of such parameters $b$ will be denoted by $W_{T_0}^{T_0}(\Sigma^0)$, considered as one of the natural holomorphic coordinate charts of $N_{T_0}^{T_0}(\Sigma^0)$. Of course, different choices of fixing three elements in each set $b_v, v \in T_0$ above give other but same kind of coordinate charts of $N_{T_0}^{T_0}(\Sigma^0)$.

In this notation, the initial surface, $\Sigma^0 = \Sigma_0$ or $\Sigma_b$ with $b = 0$. In the following, the notations $\Sigma^0$ are used interchangeably with $\Sigma_0$ so that $N_{T_0}^{T_0}(\Sigma^0)$, $W_{T_0}^{T_0}(\Sigma^0)$ etc. will also be denoted by $N_{T_0}^{T_0}(\Sigma_0)$, $W_{T_0}^{T_0}(\Sigma_0)$ accordingly. Similarly $\mathbf{p}_v/\mathbf{p}_0 = \mathbf{p}_{0,v}/\mathbf{p}_0$.

Note that the desingularization $\hat{S}_b$ is same as $\hat{S}_0 = \hat{S}_0$ so that $S_b$ has the same components as $S = S_0$ has. As before, $\mathbf{p}_b = \cup_{v \in T_0} \mathbf{p}_{b,v}$, and $\mathbf{p}_{b,v} = \mathbf{x}_{b,v} \cup \mathbf{d}_{b,v}$ lying on $S_v^0$.

Let $\tilde{N}_{T_1}^{T_1}(\Sigma_0)$ be a small (full) neighborhood of $\Sigma_0$ in the moduli space $\overline{\mathcal{M}}_{0,k}^{T_1}$, and $N_{T_1}^{T_1}(\Sigma_0)$ be its top stratum as an open set in stratum $\mathcal{M}_{0,k}^{T_1}$ with fixed topological type given by $T_1$. Then $\tilde{N}_{T_1}^{T_1}(\Sigma_0)$ parametrizes the stable curves $\Sigma_t$ near $\Sigma_0$ whose topological types are "between $T_0$ and $T_1$". In particular, when $T_1$ is the top stratum of $\mathcal{M}_{0,k}$, $\tilde{N}_{T_1}^{T_1}(\Sigma_0)$ parametrizes the stable curves $\Sigma_t$ of all types near $\Sigma_0$. Here $t \in W_{T_1}^{T_1}(\Sigma_0)$ where $W_{T_1}^{T_1}(\Sigma_0)$ is one of the natural coordinate charts of $\tilde{N}_{T_1}^{T_1}(\Sigma_0)$ extending $W_{T_0}^{T_0}(\Sigma_0)$. Thus each parameter $t \in \tilde{N}_{T_1}^{T_1}(\Sigma_0)$ has the form $t = (b, a)$ with $b \in W_{T_0}^{T_0}(\Sigma_0)$. Here $a = \{a_{vw}; [vw] \in E(T_0)\}$ is the collection of the gluing parameters describes the gluing pattern from $S_b$ to the glued surface $S_t$ defined below. The non-zero entries of $a$, denoted by $a^c = \{a_{vw}; [vw] \in E(T_0), C(v, w) = u \in T_1\}$ are the effective parameters $a_{vw}$ gluing the components $S_v$ and $S_w$. Here the map $C : T_0 \times T_0 \rightarrow T_1$ is partially defined on $T_0 \times T_0$, and for each $(v, w)$ with $(v, w) \in E(T_0)$ it is defined by $C(v, w) = u \in T_1$ if the component $S_{t,vw}$ of $S_t$ is obtained from the components $S_{b,v}$ and $S_{b,w}$.

Recall the definition of $\Sigma_t = (S_t; \mathbf{p}_t)$ with $\mathbf{p}_t = \mathbf{d}_t \cup \mathbf{x}_t$ as follows.
(1) $S_t$ is obtained from $S_b$ by gluing at those double points $d_{vw}$ with $a_{vw} \neq 0$. Hence a component $S_{t,u} = \#\{u = \{a_{v_i,v_j}\}; C(v_i,v_j) = u \in T_1\} (S_{v_1},\ldots,S_{v_{k(u)})}.$ Here the right-hand side above is the gluing of the components $S_{v_1},\ldots,S_{v_{k(u)}}$ in $\Sigma_b$ with the gluing parameter $a_u$.

For each $a_{v_i,v_j} \neq 0$, denote $(v_i,v_j)$ by $(u_+,v_-)$ and $a_{v_i,v_j}$ by $a$ temporarily. Let $D_{\pm}$ be the small discs on $S_{\pm} \cong S^2$ with complex coordinate $w_{\pm}$. Then the gluing $\#_a(D_-,D_+) = D_- \coprod D_+$ quotient out the relation that $w_- \cdot w_+ = a$. Applying this to each nonzero $a_{v_i,v_j}$ above gives the desired gluing.

(2) The double points on $\Sigma_t = \Sigma_{b,a}$ are exactly the part of the double points $d_b = \{d_{b;u'u'}\}$ such that $a_{u'u'} = 0$. Since we only consider the local deformations, we may assume that $|a|$ is sufficient small so that the marked points $x_b$ become the corresponding ones, denoted by $x_t$ through the gluing.

• • • Fixed part $K_t \cong K_0$ in $S_t$.

The ”fixed part” $K_{\epsilon,t,u}$ of $S_{t,u}$ defined by

$$K_{\epsilon,t,u} = S_u \setminus \{\cap_{a_{v_i,v_j} \neq 0} N_{\epsilon}(b,a_{v_i,v_j}) \cup \cap_{a_{v_i,v_j} = 0} D_{\epsilon}(d_{b;v_i,v_j})\}.$$  

Here $N_{\epsilon}(b,a_{v_i,v_j})$ is the ”neck” part near the double point $d_{b;v_i,v_j}$ obtained by gluing the two corresponding small discs $D_{\epsilon}(d_{b;v_i,v_j})$ and $D_{\epsilon}(d_{b;v_j,v_i})$ with gluing parameter $a_{v_i,v_j} \neq 0$. Thus $K_{\epsilon,t,u} \subset S_{t,u}$ becomes a fixed subset of $S_{b,u}$ independent of $a$ with $t = (b,a)$.

To get rid of the $b$-dependency of the fixed part, go back to the desingularization $\hat{S}$ of initial underlying curve $S(= S^0)$ of $\Sigma^0(= \Sigma_0)$. For each double point $d_{vu}^0$ on the component $S_v$, choose another small disc $D_{\epsilon_2}(d_{vu})$ of radius $\epsilon_2 > \epsilon$ such that $D_{\epsilon}(d_{vu}(b)) \subset D_{\epsilon_2}(d_{vu})$ for all $b \in W^{T_0}(\Sigma_0)$. Then define the (”smallest”) fixed part $K$ to be the complement of the union of all the discs $D_{\epsilon_2}(d_{vu})$ in $S$. Then first of all, $K$ can be considered as a subset in $S_b$, denote by $K_b$, since both $S$ and $S_b$ have the same components; secondly by the construction of the gluing, it can also be considered as a subset of $S_t$, denoted by $K_t$. Note that for $|a|$ small enough, the marked points $x_t$ are lying on $K_t$.

Thus for $|a|$ small enough with the types between $T_0$ and $T_1$, $K_t \cong K_b \cong K_0 \subset S$ as the fixed part independent of $t$ while $\{S_t\}$ is a family of curves that are deforming.

### 2.2 Local universal family of the first kind

It is well-known that the total family obtained from the gluing construction above, $S =: S(\Sigma_0) \rightarrow \tilde{N}^{T_1}(\Sigma_0)$ with the fiber $S_t = S_t$ is a proper morphism
of complex manifolds/orbifolds.

Lemma 2.1 Given $T_1 \geq T_0$, let $t_0 = (b_0, a_0)$ be the center of $W^{T_1}(\Sigma_{t_0})$. Then there is a smooth but non-holomorphic family of identifications $\{\lambda^t_{t_0} : (S_t, x_t) \rightarrow (S_{t_0}, x_{t_0})\}$ for $t \in W^{T_1}(\Sigma_{t_0})$. Under this identification, the smallest fixed part $K_t$ is identified with $K_{t_0}$ so that $K_t \simeq K_{t_0} \simeq K_b \simeq K_{b_0} \simeq K_0$, the small disks or ”neck” areas at or near double points on $S_t$ are identifies with the corresponding ones on $S_{t_0}$. Away from the small annuli of the tubular neighborhoods of the boundaries of $K_t$, $\lambda^t_{t_0}$ is holomorphic and preserves any of the natural metrics.

Moreover, these maps together give rise a smooth map $\lambda^{t_0} : S|_{N^{T_1}(\Sigma_{t_0})} \rightarrow S_{t_0} = S_{t_0}$ and hence the induced smooth the map $\hat{\lambda}^{t_0} = (\lambda^{t_0}, \pi) : S|_{N^{T_1}(\Sigma_{t_0})} \rightarrow S_{t_0} \times N^{T_1}(\Sigma_{t_0})$. Here $\pi : S|_{N^{T_1}(\Sigma_{t_0})} \rightarrow N^{T_1}(\Sigma_{t_0})$ is the projection map.

The proof of this lemma is the immediate consequence of the construction of these diffeomorphisms below.

The map $\lambda^{t_0}$ above will be used to define the smooth structure of the first kind on the corresponding neighborhood of the first kind of a stable map in Sec. 4. Thus $W^{T_1}(\Sigma_{t_0})$ together with the map $\hat{\lambda}^{t_0}$ will be refer to as a local model of the first kind for the local universal family of stable curves.

It is the precise version of the intuitive notion that the parameter $t \in W^{T_1}(\Sigma_0)$ is considered as a point near the ends of the stratum $T_1$ representing the stable curve $\Sigma_t = (S_t; d_t, x_t)$ whose underlying surface $S_t$ is deforming and degenerating along the ends, while its fixed part $K_t$ remains fixed such that the relative locations of $x_t$ in $K_t$ are the same as the ones of $x$ in $K_0 = K$. Note that in the case used in [?], the initial curve $\Sigma_0$ is ”minimally” stabilized. In this case, $x_t$ is indeed fixed in the above model.

Now we define the required diffeomorphisms. In the case that $T_1 = T_0$, the lowest stratum with $t_0 = (b_0, 0) \in W^{T_0}(\Sigma_0)$, $\lambda^{t_0}$ is just $\lambda^{b_0} =: \{\lambda^{b_0}_{b,v}, v \in T_0, b \in W^{T_0}(\Sigma_0)\}$ defined as follows.

It is more convenient to define the inverse map of $\lambda^{b_0}_{b,v}$. For $v \in T_0, b \in W^{T_0}(\Sigma_0)$, $(\lambda^{b_0}_{b,v})^{-1} : (S_{b_0,v}, d_{b_0,v}) \rightarrow (S_{b,v}, d_{b,v})$ is defined by the following conditions:

(i) It is the ”identity” map on the complement of the union of all disks of radius $\epsilon_1$ centered at the double points of the component $S_{b_0,v}$, denoted by $K_{b_0,v,\epsilon_1}$ under the identifications of $\hat{S}_b \simeq \hat{S}_{b_0} \simeq \hat{S}_{b=0}$ for $|b|$ and $|b_0|$ sufficiently small. In fact under above identifications, we get the corresponding identification $K_{b,v,\epsilon_1} \simeq K_{b_0,v,\epsilon_1}$ of the fixed parts given by $\lambda^{b_0}_{b,v}$.
(ii) On the disks $D_\epsilon(d_{b_0;vu})$ centered at the double point $d_{b;vu}$, it is the translation that brings $d_{b_0;vu}$ to $d_{b;vu}$, and $D_\epsilon(d_{b_0;vu})$ to $D_\epsilon(d_{b;vu})$.

(iii) Under the above identification $\hat{S}_b \simeq \hat{S}_{b_0}$, the image $D_\epsilon(d_{b;vu})$ of $D_\epsilon(d_{b_0;vu})$ can be considered as a subset of $D_{\epsilon_0}(d_{b_0;vu}) \subset D_{\epsilon_1}(d_{b_0;vu})$ in $S_{b_0,v}$. Here of course, we assume that $\epsilon < \epsilon_0 < \epsilon_1$ and $|b|$ and $|b_0|$ sufficiently small. Hence by (i) and (ii) above, using the above identification again, the definition for the rest of $(\lambda_{b,v}^{b_0})^{-1}$ is reduced to find a self diffeomorphism of $D_{\epsilon_1}(d_{b_0;vu})$ that is the identity map near the boundary extending the map already defined on $D_{\epsilon_1}(d_{b_0;vu})$ in (ii). This can be done by extending the corresponding vector field. Then the desired diffeomorphism is the time-1 map of the flow of the extended vector field.

Note that the restriction map

$$\lambda_{b,v}^{b_0} : \cup_{vu \in E(T_0)} D_\epsilon(d_{b;vu}) \cup K_{b,v,\epsilon_1} \to \cup_{vu \in E(T_0)} D_\epsilon(d_{b_0;vu}) \cup K_{b_0,v,\epsilon_1}$$

is holomorphic and preserves the "natural" metrics (Spheric, cylindrical or 'flat' ones).

To move to the higher stratum $T_1$, recall that for given $t = (b, a) \in W^{T_1}(\Sigma_0)$ with $t$ close $t_0 = (b_0, a_0)$, at a double point $d_{b,vu} = d_{b,uv}$ where the gluing parameter $a_{vu} = a_{uv} \neq 0$, the gluing of the pair of disks of radius $\epsilon$ on the components $S_{b,v}$ and $S_{b,u}$ at the double points, $D_\epsilon(d_{b,vu}) \#_{a_{vu}=a_{uv}} D_\epsilon(d_{b,uv})$ was defined in this section.

The identification of the pair of disks $D_\epsilon(d_{b,vu})$ and $D_\epsilon(d_{b,uv})$ with $D_\epsilon(d_{b_0,vu})$ and $D_\epsilon(d_{b_0,uv})$ by $\lambda_{b,v}^{b_0}$ and $\lambda_{b,u}^{b_0}$ induces the corresponding identification

$$D_\epsilon(d_{b,vu}) \#_{a_{vu}=a_{uv}} D_\epsilon(d_{b,uv}) \simeq D_\epsilon(d_{b_0,vu}) \#_{a_{vu}=a_{uv}} D_\epsilon(d_{b_0,uv}).$$

Applying this to each double point with $a_{uv} \neq 0$, we get a family of identifications $S_{b_0,a} \simeq S_{b,a}$ smooth in $b$.

Thus the construction of $\lambda_t^{l_0}$ with $t_0 = (b_0, a_0)$, $t = (b, a)$ and $|t_0|$ and $|t|$ small can be obtained by using the following two families of identifications of finite cylinders.

(A) When $l$ is close to $l_0$, there is a family of identifications $[-l_0, l_0] \times S^1 \to [-l, l] \times S^1$ smooth in $l$ and induced from the corresponding identifications $[-l_0, l_0] \to [-l, l]$. We may assume that the restriction identifications to $[-l_0 = 1, l_0 - 1] \times S^1$ is the identity map.

Applying these identifications, we get a family of identifications $S_{b_0,a'} \simeq S_{b,a}$ with $|a'| = |a_0|$ and $\arg a' = \arg a$. 


(B) When $\theta \in S^1$ close to 1, there is a smooth $\theta$-dependent family of identifications $[-l_0, l_0] \times S^1 \to [-l_0, l_0] \times S^1$ that is the identity map on $[-l_0, l_0-1] \times S^1$ and is the rotation of angle $\theta$ on $\{l_0\} \times S^1$. The effect of these identifications is to untwist the angular twisting in the gluing construction. Applying this to the identifications obtained above so far, we finally get the desire family of identifications $\lambda_t^0 = \{\lambda_t^0 : S_t \to S_{t_0}\}$.

Let $\epsilon_2 > \epsilon_1$ and assume that $D_{\epsilon_1}(d_b) \subset D_{\epsilon_2}(d_{b_0})$. Then define the ("smallest") fixed part $K =: K_{\epsilon_2}$ to be the complement of the union of all disks centered at double points of radius $\epsilon_2$ on $S_0$. Then we have the fixed parts $K_t \simeq K_{t_0} \simeq K_b \simeq K_{b_0} \simeq K$ in the corresponding surfaces, on which the maps $\lambda_t^0$ above are the "identity" map. Note that the marked points $x_t$ are lying on $K_t$.

It follows from the construction above, the family of identification has the properties in the above lemma.

### 2.3 Local universal family of the second kind

An important property the restriction above local universal family $\mathcal{S}(\Sigma_0)$, a fixed stratum of type $T_1$ with $T_1 \geq T \geq T_0$, can be considered as a set of "moving" marked points $x_t$ on a fixed reference surface $\hat{S}_{t_0}$. Then the parameter $t \in W^{T_1}(\Sigma_{t_0})$, as part of the local coordinates of the moduli space $\mathcal{M}^{T_1}_{0,k}$ also describes the local moduli of the ("moving") distinguished points $p_t$ on the (fixed) reference curve $\hat{S}_{t_0}$ of type $T_1$.

The local universal family considered this way will be regarded as the second model.

Thus in the second model, we are in the exact same situation as we were at the beginning of this section: to parametrize stable curves (considered as moving distinguished points on a fixed reference surface) in a fixed stratum $T$, but with the initial curve $\Sigma_{t_0}$ instead of $\Sigma_0$. In particular in this model the metric on $\hat{S}_{t_0}$ is fixed by the marking $\hat{S}_{t_0} \simeq S^2$.

To distinguish the situation here with the one before, instead of using the corresponding coordinate charts of the form $W^T(\Sigma_{t_0})$ with the local parameter $b = b(t) \in W^T(\Sigma_{t_0})$ representing the moduli point of $\Sigma_t$, we will denote $b = b(t)$ by $u := u(t)$, and accordingly $p_{b(t)}$ by $p_u := p_{u(t)}$, $W^T(\Sigma_{t_0})$ by $U^T(\Sigma_{u_0})$ with $u_0 = u(t_0)$, etc.

We state this more formally as following: the two models of local universal family over the open set $N^T(\Sigma_{t_0})$ of $\mathcal{M}^{T}_{0,k}$ centered at $\Sigma_{t_0} = \Sigma_{u_0}$, denoted by $\mathcal{S}|_{W^T(\Sigma(t))}$ and $\mathcal{S}|_{U^T(\Sigma(u_0))}$ are identified by a fiber-wise analytic map $\phi^{-1} =$
\[\{\phi_t^{-1}\} \text{ with } \phi_{t_0}^{-1} : (S_{t_0}, p_{t_0}) \to (S_{u_0}, p_{u_0}) \text{ being the identification of the two (but the "same") central fibers.} \]

**Lemma 2.2** The identification \(\phi_{t_0}^{-1} : (S_{t_0}, p_{t_0}) \to (S_{u_0}, p_{u_0})\) induces a family of component-wise biholomorphic identifications, denoted by \(\phi_t^{-1} : (S_t, p_t) \to (S_u, p_u)\) with \(\hat{S}_u\) being the fixed \(\hat{S}_{u_0}\) such that the induced map \(\phi^{-1} : W^T(\Sigma_{t_0}) \to U^T(\Sigma_{u_0})\) on the coordinate charts given by \(t \to u\) is a holomorphic identification. Moreover these maps \(\{\phi_t^{-1}\}\) fit together to form an analytic identification between the corresponding universal families, denoted by \(\phi^{-1} : S|_{W^T(\Sigma_{t_0})} \to S|_{U^T(\Sigma_{u_0})}\) that are holomorphic on the desingularizations \(\hat{S}|_{W^T(\Sigma_{t_0})}\) and \(\hat{S}|_{U^T(\Sigma_{u_0})}\).

The above lemma is essentially a tautology in the situation above. However, it also follows from the local universal property of the family \(S\).

## 3 Two types of neighborhoods near ends

### 3.1 "Base" deformations of a stable map \(f = f_0 : \Sigma_0 \to M\)

- "Base" deformations of an initial stable map \(f = f_0 : \Sigma_0 \to M\) within the same stratum of type \(T_0\).

There are two kinds of such "base" deformations of \(f_0 : \Sigma_0 \to M\), denoted by \(\{f_b : \Sigma_b \to M, b \in W^{T_0}(\Sigma_0)\}\) and \(\{f_b : \Sigma_b \to M, b \in W^{T_0}(\Sigma_0)\}\) respectively.

The first one is simply defined to be \(\tilde{f}_b := \cup_{v \in T_0} f_{b:v}\) with \(\tilde{f}_{b:v} = f_v\) under the assumption that \(f\) is constant on all the small disks near double points.

The last identity makes sense as the domains of the two maps are the same \(S^0_v\) after forgetting the distinguishing points \(p_{b:v}\) and \(p^0_{v}\).

To define the second deformation, let \(\lambda^{b_0}_{b:v} : S_{b:v} \to S_v =: S^0_v\) in last section. Then we define \(f_{b:v} : S_{b:v} \to M\) to be \(f_{b:v} = f_v \circ \lambda^{b_0}_{b:v}\) and \(f_b = \cup_{v \in T_0} f_{b:v} : S_b \to M\).

Next we extend \(\{f_b, b \in W^{T_0}(\Sigma_0)\}\) to a higher stratum \(T_1 > T_0\).

- "Base" deformation \(\{f_t, t \in \tilde{N}(\Sigma_0)\}\).

Given \(f = f_- \bigvee f_+ ; (D_-, d_-) \bigvee_{d_=d_+} (D_+, d_+) \to M\) and a gluing parameter \(a_0 = exp\{- (s_0 + t_0 t)\} \neq 0\), to define the gluing \(#_{a_0}(f_-, f_+ : \#_{a_0}(D_-, D_+) \to M\) below, we introduce the cylindrical coordinate \((s_\pm, t_\pm) \in \mathbb{C}\).
\( \mathbb{R}^\pm \times S^1 \) on \( D_\pm \) by the identification of \( D_\pm \simeq \mathbb{R}^\pm \times S^1, D_\pm \). Then \( \#_{a_0}(D_-, D_+) \) is obtained by cutting of the part of \( D_\pm \) with \( |s_\pm| > -\log|a_0| \) and glue the rest along the boundaries twisted with an angle \( \arg a_0 \). Thus \( \#_{a_0}(D_-, D_+) \simeq [-\log|a_0|, \log|a_0|] \times S^1 \) with the induced cylindrical coordinate \((s, t)\) with \( s = 0 \) corresponding to the ”middle circle” and the other two cylindrical coordinates \((s_\pm, t_\pm)\) with \( s_\pm = 0 \) corresponding to the two boundary circles. Note that \( s_\pm = s = \log|a_0| \). Then \( \#_{a_0}(f_-, f_+) \) is defined to be \( \#_{a_0}(f_-, f_+)(s, t) = \exp_{f_\pm(d_\pm)}(\beta_-(s)\hat{f}_-(s, t) + (\beta_+(s)\hat{f}_+(s, t)) \). Here \( \hat{f}_\pm \) is a vector field over \( D_\pm \) such that \( f_\pm = \exp_{f_\pm(d_\pm)}\hat{f}_\pm \), and \( \beta_\pm \) are cut-off functions supported on \([-1, 1]\) with \( \beta_- + \beta_+ = 1 \).

The deformation \( f_t = (f_b)_a \) is then defined by implanting above construction to each double points \( d_{vw} \) of \( f_b \) with \( a_{vw} \neq 0 \).

### 3.2 Neighborhoods of the first kind

Let \( f_t, t \in W(\Sigma_0) \) be the ”base” deformation of \( f_0 \) constructed above over a small coordinate/uniformizer \( W(\Sigma_0) \) of \( \tilde{N}(\Sigma_0) \) centered at \( \Sigma_0 \).

A neighborhood \( W_{\nu'}(a)(f_0, H_{f_0}) \) of \([f_0]\) in the space of unparametrized stable maps, as a slice, is defined to be a family of Banach manifolds over the ”base” deformation \( \{f_t\} \),

\[
W_{\nu'}(a)(f_0, H_{f_0}) = \bigcup_{t \in W(\Sigma_0)} W_{\epsilon;\nu'}(f_t, H_{f_0}).
\]

Here for each fixed \( t \),

\[
W_{\epsilon;\nu'}(f_t, H_{f_0}) = \{h_t : (S_t, x_t) \to (M, H_{f_0}) \mid \|h_t - f_t\|_{k,p;\nu'} < \epsilon\}.
\]

Here ”a” is gluing parameter in \( t = (b, a) \).

- The \( \nu(a) \)-exponentially weighted norm

The norm \( \| - \|_{k,p;\nu(a)} \) used in the definition above is the \( \nu \)-exponentially weighted norm along the all ”neck” areas \( N(b, a_{vi}, a_{vj}) \) of \( S_t \) with \( a_{vi}, a_{vj} \neq 0 \) obtained by gluing from \( S_0 \) at the double points \( d_{vi}, d_{vj}(b) \); on the rest of \( S_t \), the norm is just the usual \( L_k^p \)-norm. More specifically, recall that ”neck” areas of \( S_t \) with \( a_{vi}, a_{vj} \neq 0 \) and \( t = (b, a) \), \( N(b, a_{vi}, a_{vj}) \simeq (-|\log|a_{vi}, a_{vj}||, |\log|a_{vi}, a_{vj}||) \times S^1 \). The weight function \( \nu(a_{ij}) \) is equal to \( \exp{\nu|s_{\pm}|} \) for points in \( N(b, a_{vi}, a_{vj}) \) with \( s_\pm \in (0, |\log|a_{vi}, a_{vj}|| - 2) \) and is a smooth function equal to 1 on two ends of the neck. Outside these necks, \( \nu(a) = 1 \) so that these weight \( \nu(a_{ij}) \) functions together defines a smooth weight function \( \nu(a) \). Here \( \nu \) is a fixed
positive constant with \( \nu < (p - 2)/p \). The \( L^p_{k,\nu(a)} \)-norm \( \|h_t\|_{k,p,\nu(a)} \) then is defined to be \( \|\nu(a) \cdot h_t\|_{k,p} \).

For each fixed \( t \), the norm so defined makes \( W^{\nu(a)}_\epsilon (f_t, H_{f_0}) \) became a Banach manifold. However, on \( W^{\nu(a)}_\epsilon (f_0, H_{f_0}) \), this \( t \)-dependent family of norms does not necessarily define a topology without further conditions as in [L6?]. The reason for this is that the mixed \( L^p_k/L^p_{k,\nu(a)} \)-norm used here is not continuous when \( h_t \) is moving from higher stratum to the lower ones. On the other hand, on each fixed stratum, near any given point the \( L^p_k \) and \( L^p_{k,\nu(a)} \) norms are "locally" equivalent so that the resulting space is at least a (topological) Banach manifold.

More specifically, consider the decomposition of \( W^{\nu(a)}_\epsilon (f_0, H_{f_0}) \) into its open strata,

\[
W^{\nu(a)}_\epsilon (f_0, H_{f_0}) =: \cup_{T \geq T_0} W^{\nu(a)}_\epsilon (f_0, H_{f_0})
\]

with each

\[
W^{\nu(a)}_\epsilon (f_0, H_{f_0}) =: \cup_{t \in W^{T}(\Sigma_0)} W^{\nu(a)}_\epsilon (f_t, H_{f_0}).
\]

Then the collection of all such neighborhoods \( W^{\nu(a)}_\epsilon (f_0, H_{f_0}) \) of a fixed stratum \( T \) generate a topology by the above mentioned local equivalence of the \( L^p_{k,\nu(a)} \)-norm with standard \( L^p_k \)-norm.

Since in order to prove the main results of this paper on the higher smoothness of the admissible perturbations, we need to localize further by using small neighborhoods of any given element \( g_{t_0} \) in \( W^{\nu(a)}_\epsilon (f_0, H_{f_0}) \). We need spell out more on the existence of such neighborhoods.

- Neighborhoods of the first kind on \( W^{\nu(a)}_\epsilon (f_0, H_{f_0}) \):

Fix a stratum of type \( T = T_1 \geq T_0 \). Recall that \( T_0 \) is the tree associated with the lowest stratum that \( f_0 \) lies on. Give \( \{g_{t_0} : (S_{t_0}, x_{t_0}) \to (M, H_{f_0})\} \in W^{\nu(a)}_\epsilon (f_0, H_{f_0}) \) with \( t_0 = (b_0, a_0) \in W^{T_1}(\Sigma_0) \) and \( \Sigma_{t_0} = (S_{t_0}, p_{t_0}) \), a neighborhood of \( g_{t_0} \) of the first kind in \( W^{\nu(a)}_\epsilon (f_0, H_{f_0}) \), denoted by

\[
W^{\nu(a)}_\epsilon (f_0, H_{f_0}) =: \cup_{t \in W^{T_1}(\Sigma_0)} W^{\nu(a)}_\epsilon (f_t, H_{f_0}; g_t) = \{h_t : S_t \to M| \|h_t-g_t\|_{k,p,\nu(a)} < \epsilon'\}
\]

with \( \epsilon' \ll \epsilon \), can be defined in a few equivalent ways that we describe now.

Here \( g_t : S_t \to M \) is the "base" deformation of \( g_{t_0} \) inside \( W^{\nu(a)}_\epsilon (f_0, H_{f_0}) \), similar to the initial deformation \( f_t \). Since topological type of \( S_t \) is fixed, we require that the deformation has a form \( g_t = g_{t_0} \circ T^{t_0}_t \), where

\[
T := T^{t_0} = \{T^{t_0}_t : S|_{W^{T_1}_t(\Sigma_{t_0})} = \cup_{t \in W^{T_1}_t(\Sigma_{t_0})} S_t \to S_{t_0} \}.
\]
is a smooth family of diffeomorphisms.

Thus we need establish the existence of the required deformation $g_t$. The key step is the following lemma.

**Lemma 3.1** Given the base deformation $\{f_t\}$ of $f_0$ defined earlier in this section, fix a member $f_{t_0} : S_{t_0} \to M$ of type $T_1$ in the deformation, there is a smooth family of diffeomorphisms $T =: T_{t_0}^{t_0} = \{T_t\} : S|_{W_{t_0}^{T_1}(S_{t_0})} \to S_{t_0}$ such that $\lim_{t \to t_0} \|f_t - f_{t_0} \circ T_{t_0}^t\|_{k,p,\nu(a)} = 0$ in an uniform manner in $t_0$ for $t_0$ varying in a compact set.

Above lemma implies the following two lemmas

**Lemma 3.2** Given $g_{t_0} \in W_{\nu(a_T), T_1}(f_0, H_{f_0})$. There exists a "base" deformation $g_t$ inside $W_{\nu(a_T), T_1}(f_0, H_{f_0})$ for $|t - t_0| << \epsilon$ with $g_t = g_{t_0} \circ T_{t_0}^t$.

**Lemma 3.3** These $W_{\nu(a_T), T_1}(g_{t_0}, H_{f_0})$ generate a topology on $W_{\nu(a_T), T_1}(f_0, H_{f_0})$.

We note that the required $T_{t_0}^t$ can be taken as the particular family of diffeomorphisms $\lambda_{t_0}^t$ defined before. Since our main concerns of this paper is the stratified smoothness of generic perturbations, we will not give the proofs of above lemmas. They will be given in [L?].

The neighborhoods $W_{\nu(a_T), T_1}(g_{t_0}, H_{f_0})$ here will be call the ones of the first kind.

Here are some variations or related constructions:

1. In the definition of $W_{\nu(a_T), T_1}(g_{t_0}, H_{f_0})$, the conditions that $|t - t_0| < \epsilon'$ and $\|h_t - g_t\|_{k,p,\nu(a)} < \epsilon'$ can be replaced by $|t - t_0| + \|h_t - g_t\|_{k,p,\nu(a)} < \epsilon'$.

2. In the above definition of the $L_{k,\nu(a)}^p$-norm, a $t$-dependent metric $m_t$ on the domain $S_t$. is used. Using the diffeomorphisms $T_{t_0}^t : S_t \to S_{t_0}$ to pull-back the fixed metric $m_{t_0}$, we get a family of metrics $(T_{t_0}^t)^*(m_{t_0})$ on $S_t$ and the corresponding $L_{k,\nu(a)}^p$-norms and neighborhoods $W_{\nu(a_T), T_1}(g_{t_0}, H_{f_0})$. Since these families of the metrics are uniformly equivalent for $|t - t_0| \leq \epsilon''$, the resulting neighborhoods defined this way are equivalent to the previous one. Thus, uptp the effect of $T_{t_0}^t$, we can use a fixed reference metric on $m_{t_0}$ to define the norm.

This implies that the second type of the neighborhoods defined below is (topologically) equivalent to the ones above.
Consider the deformations of \( h_t = h_{t_0} \circ T^{t_0}_t \) for all \( h_{t_0} : S_0 \to M \) with \( h_{t_0} \) in the central slice \( W^{\nu(a_{t_0}), T_1}_{\epsilon_1}(g_{t_0}, H_{f_0}) \). Denote the collection of such \( h_t \) with \( |t - t_0| < \epsilon'_1 \) by \( W^{\nu(a_{T_1}), T_1}_{\epsilon_1}(g_{t_0}, H_{f_0}) \).

**Lemma 3.4** The neighborhoods \( W^{\nu(a_{T_1}), T_1}_{\epsilon_1}(g_{t_0}, H_{f_0}) \) so defined are equivalent to the ones defined before.

For the proof of this lemma we refer to [L?] again.

### 3.3 Neighborhoods of the second kind

Recall that there is a family of biholomorphic identifications \( \phi_t^{-1} : (S_t, p_t) \to (S_u, p_u) \) which transforms the family of varying curves \( \{(S_t, p_t)\} \) with fixed \( (K_t, x_t) = (K_0, x_0) \) with the curve with **fixed** components but with a family of varying distinguished points, \( \{(\hat{S}_{u_0}, p_u)\} \).

It induces a bijection

\[
\Phi = \bigcup_t \Phi_t : W^{\nu(a_{T_1}), T_1}_{\epsilon_1}(g_{t_0}, H_{f_0}) =: \bigcup_{t \in W^{T_1}(\Sigma_{u_0})} W^{\nu(a), T}_t(g_{t_0}, H_{f_0}) \\
\to U^{T_1}_{\epsilon_1}(g_{u_0}, H_{f_0}) = \bigcup_{u \in U^{T_1}(\Sigma_{u_0})} U^{\nu}_t(g_{u_0}, H_{f_0})
\]

by pull-backs given by \( \Phi_t(h_t) = h_t \circ \phi_t \) denoted by \( h_u \).

The subspace \( U^{\nu}_{\epsilon_1}(g_{u_0}, H_{f_0}) \) here consists those stable maps \( h_u : (S_u, x_u) \to (M, H_{f_0}) \) with **fixed** marked points \( x_u \) on the fixed domain \( \hat{S}_u = \hat{S}_{u_0} \) such that \( \| h_u - g_u \|_{k,p} < \epsilon' \). Here we give each component of \( \hat{S}_{u_0} \) the spherical metric and use it to defined the above \( L^p_k \)-norm, and \( \{g_u = g_t \circ \phi_t\} \) is the transformed base family by \( \{\phi_t\} \).

Since for \( |t - t_0| < \epsilon' \), the norms of the above two spaces are equivalent, \( \Phi \) is a homeomorphism to its image.

The neighborhoods \( U^{T_1}_{\epsilon_1}(g_{u_0}, H_{f_0}) \) will be refereed as of the **second** kind. By the remark (2) above, the neighborhoods here are equivalent to the ones of the first kind above.

Note that the lowest stratum, \( W^{\nu(a_{T_0}), T_0}_{\epsilon_1}(f_0, H_{f_0}) \) is the same as \( U^{T_0}_{\epsilon_1}(f_{u=0}, H_{f_0}) \) since the norm used here is just the usual \( L^p_k \)-norm without exponential weight.

It is easy to see that \( \Phi_t \) is a diffeomorphism.
Lemma 3.5 For each fixed $t$ and hence $u$,
\[ \Phi_t : W^{v(a),t} \rightarrow U^{u}(g_{u0}, H_{f0}) \]
is a diffeomorphism.

Proof:
The only difference between $U^{T_1}(g_{u0}, H_{f0})$ and $W^{v(aT_1),T_1}(g_{t0}, H_{f0})$ is that
the domains of stable maps in $W^{v(aT_1),T_1}(g_{t0}, H_{f0})$ are varying depending on
the parameter $t$ while the domains of elements in $U^{T_1}(g(t_0), H_{f0})$ are the fixed
$\hat{S}_{u0}$ but with moving distinguished points parametrized by $u = u(t)$. When $t$
is fixed, the elements in $U^{u}(g_{u0}, H_{f0})$ or $W^{v(a),t}(g_{t0}, H_{f0})$ have the same domain
with fixed components and distinguished points under the identification map $\phi_t$. Moreover the Banach norm on these two spaces are equivalent. Hence
the induced map $\Phi_t$ is a diffeomorphism.

$\square$

From now on, if there is no confusion, we will drop the subscript $\epsilon$ that
describes the size of a neighborhood.

Next we define the neighborhoods that are still second type obtained from
$U^{T_1}(g_{u0}, H_{f0})$ by dropping some marked points.

Given $\{ h : S_u \rightarrow M \} \in U^{T_1}(g_{u0}, H_{f0})$ with fixed $\hat{S}_u = \hat{S}_{u0}$, among the $k$
marked points $x_u$ on $\hat{S}_{u0}$ we select $m$ points, denoted by $x^r_u$, such that $(S_u, x^r_u)$
is still stable. Denote the resulting stable curve with $m$ marked points by
$\Sigma^r_u := (S^r_u, x^r_u)$ with fixed $\hat{S}^r_u = \hat{S}^r_{u0}$, which is the same as $\hat{S}_{u0}$ as a surface.

Then the map $h^r : (S^r_u, x^r_u) \rightarrow (M, H^r_{f0})$ is defined to be the same map $h$
as before but forgetting the rest of the marked points, denoted by $x^c_u$ in $x_u$, where
$H^r_{f0}$ is the corresponding selection of local hypersurfaces. Note that
$h^r_u(x^c_u) \in H^r_{f0}.$

Then the collection of all such $h^r$ obtained from $h \in U^{T_1}(g_{u0}, H_{f0})$ by
dropping $k - m$ marked points will be denoted by $U^{T_1}(g^r_{u0}, H^r_{f0})$ with the centered $g^r_{u0}$.

The process above of course depends on the following choices: (i) the
selection of $x^r_u$, (ii) an order for $p_u \subset \hat{S}_u = \hat{S}_{u0}$ that induces an order for
$p^r_u \subset S^r_u = S^r_{u0}$. In the following, we fix one of such choices, labeled by
the superscript $r$ in the notations here. For each $v \in T_1$, by identifying the
first three points in $(p^r_u)_v$ and $(p_u)_v$, we get the holomorphic identifications
ψ_v : (\hat{S}_u^r)_v = (\hat{S}_{u_0}^r)_v \rightarrow (\hat{S}_u)_v = (\hat{S}_{u_0})_v \text{ and } \psi = \cup_{v \in T_1} \psi_v : \hat{S}_{u_0}^r \rightarrow \hat{S}_{u_0}, \text{ which induces the above drop-marking map, denoted by }

\Psi = \cup_{v \in T_1} \Psi_v : U^{T_1}(g_{u_0}, H_{f_0}) \rightarrow U^{T_1}(g_{u_0}, H_{f_0}^r)

given by pull-back by ψ, h \rightarrow h \circ ψ \text{ denoted by } h^r.

In next section we will show that this map is a diffeomorphism.

Note that the identifications ψ_v : (\hat{S}_u^r)_v = (\hat{S}_{u_0}^r)_v \rightarrow (\hat{S}_u)_v = (\hat{S}_{u_0})_v above also give canonical identifications of these surfaces with the fixed (S^2, ; 0, 1, \infty) so that they become ”marked” surfaces.

4 Stratified smooth structures on neighborhoods

The smooth structures on U^{T_1}(g_{u_0}, H_{f_0}^r) and U^{T_1}(g_{u_0}, H_{f_0}) are defined similarly, obtained as C^{m_0} submanifolds E^{-1}_{m_0}(H_{f_0}^r) of \tilde{U}^{T_1}(g_{u_0}) \times \hat{S}_{u_0}^m and E^{-1}_{k}(H_{f_0}) of \tilde{U}^{T_1}(g_{u_0}) \times \hat{S}_{u_0}^k, respectively. Here E_l : \tilde{U}^{T_1}(g_{u_0}) \times \hat{S}_{u_0}^l \rightarrow M^l is the l-fold total evaluation map at l selected marked points among the k marked points on the domain (\hat{S}_u, x_u) = (\hat{S}_{u_0}, x_u). Since E_l is of class C^{m_0} (see [? ] ), it is easy to see that it is a C^{m_0}-submersion so that above two subsets have C^{m_0}-smooth structures.

Proposition 4.1 \Psi : U^{T_1}(g_{u_0}, H_{f_0}) \rightarrow U^{T_1}(g_{u_0}, H_{f_0}^r) is a local diffeomorphism at g_{u_0}.

Proof:

Note that \Psi : U^{T_1}(g_{u_0}, H_{f_0}) \rightarrow U^{T_1}(g_{u_0}, H_{f_0}^r) is a bijection. Indeed since \tilde{g}_{u_0}(x_{u_0}^c) \in H_{f_0}^c and for any h \in U^{T_1}(g_{u_0}, H_{f_0}^r), we already have h(x_{u_0}^c) \in H_{f_0}^r, when U^{T_1}(g_{u_0}, H_{f_0}^r) is small enough and x_{u_0}^c as above is fixed, by implicit function theorem, the equation on x_{u_0}^c, h(x_{u_0}^c, x_{u_0}^c) \in H_{f_0}^c has an unique solution that is close to x_{u_0}^c. This proves that \Psi =: \Psi^r is a bijection with \Psi^{-1} sending

\{h^r : (\hat{S}_u^r = \hat{S}_{u_0}^r, x_{u_0}^r) \rightarrow M\} to \{h : (\hat{S}_u = \hat{S}_{u_0}, x_{u_0}, x_{u_0}^c) \rightarrow M\}.

Thus the map \Psi is the restriction to a C^{m_0} submanifold of the obvious smooth projection π : \tilde{U}^{T_1}(g_{u_0}) \times \hat{S}_{u_0}^k \rightarrow \tilde{U}^{T_1}(g_{u_0}) \times \hat{S}_{u_0}^m sending k marked points to the corresponding m marked points.

□
In this section we give a proof that the $\xi$ is of class $C^m$ on $\nu(a_0)_1(g_{t_0}, H_{f_0})$ defined below is of class $C^{m_0}$ viewed in any other local slices.

We assume that $\xi$ is obtained from $\xi^K = \oplus_{v \in T_0} \xi^K_v$ defined below.

Here $K = K_0 = \cup_{v \in T_0} K_v \subset S_0$ is the fixed part lying on the initial curve $S_0$ with subscript "0" corresponding to $t = (a, b) = (0, 0)$.

For each $v \in T_0$, let $\tilde{W}(f_{K_v})$ be the collection of $L^p_k$-maps $g = g_v : K_v \to M$ such that $\|g - f_{K_v}\|_{k,p} < \epsilon$. Here $f_{K_v} = f|_{K_v}$, the restriction of the initial map $f$ to $K_v$. We give $K_v$ the induced metric from $S_v$. The bundle $(\mathcal{L}^K_v, \tilde{W}(f_{K_v}))$ is defined as following: for any $h_v \in \tilde{W}(f_{K_v})$, the fiber $\mathcal{L}^K_v|_{h_v} = (L^p_{k-1})_0(K_v, h_v^*(E))$ consists of all $L^p_{k-1}$-sections with compact support in the interior of $K_v$.

Let $\tilde{W}(f_K) = \prod_{v \in T_0} \tilde{W}(f_{K_v})$ and $\mathcal{L}^K = \oplus_{v \in T_0} \mathcal{L}^K_v$.

Now for each $v \in T_0$, fix a section $\xi^K_v : \tilde{W}(f_{K_v}) \to \mathcal{L}^K_v$ of class $C^\infty$ satisfying the conditions $C_1$ and $C_2$. Then the section $\xi^K =: \oplus_{v \in T_0} \xi^K_v : \tilde{W}(f_K) \to \mathcal{L}^K$.

By the identification of $K_t \simeq K = K_0$, for any fixed $t \in W^{T_1}(\Sigma_{t_0})$, 

5 Higher-degree stratified smoothness of the perturbations

In this section we give a proof that the $\xi$ on $W^{\nu(a_0)_1}(g_{t_0}, H_{f_0})$ defined below is of class $C^{m_0}$ viewed in any other local slices.

Next we defined a smooth structure centered at $g_{t_0}$ for $W^{\nu(a_1)_1}(g_{t_0}, H_{f_0})$. This can be done by using the family of smooth identifications $\Lambda^0 = \{\lambda^0_t : \Sigma_t \to \Sigma_{t_0}, t \in W^{T_1}(\Sigma_{t_0})\}$ that is the identity map on the "small fixed part" $K_{t_1}(d(b_0))$ "centered" at $d(b_0)$ defined before in Sec. 2. These maps give rise a smooth trivialization of the local universal family $(S|_{W^{T_1}(\Sigma_{t_0})} \to W^{T_1}(\Sigma_{t_0})) \simeq \Sigma_{t_0} \times W^{T_1}(\Sigma_{t_0})$.

Now the smooth structure on $W^{\nu(a_1)_1}(g_{t_0}, H_{f_0})$ can be defined by the identification $\Lambda^0 : W^{\nu(a_0)_0}(g_{t_0}, H_{f_0}) \times N^{T_1}(\Sigma_{t_0}) \to W^{\nu(a)_1}(g_{t_0}, H_{f_0})$ defined by $\Lambda^0(h, t) = h \circ \lambda^0_t$. Note that the norms are equivalent under the map $\Lambda^0$ by the remark/note (3) in last section. This gives a smooth structure on each $W^{\nu(a)_1}(g_{0}, H_{f_0})$. Of course transition functions between two such neighborhoods of type $T_1$ are only continuous. The end $W^{\nu(a_1)_1}(f_0, H_{f_0})$ then is covered by such neighborhoods.
and \( g_t \in W^{\nu(a),t}(g_{t_0}, H_{f_0}) \subset W^{\nu(a_{T_1}),T_1}(g_{t_0}, H_{f_0}) \), we get the induced section \( \xi^t : W^{\nu(a),t}(g_{t_0}, H_{f_0}) \to \mathcal{L}^t \) defined by \( \xi^t(g_t) =: \xi^K(g_t)|_{K_t} \). It is easy to see that the standard local trivializations for the bundle \( (\mathcal{L}^K \to \tilde{W}(f_K)) \) and \( (\mathcal{L}^t \to \tilde{W}^{\nu(a),t}(g_{t_0}, H_{f_0})) \) are compatible with respect to the above identifications of \( K_1 \cong K_0 \) so that for each fixed \( t \), \( \xi^t \) is still smooth and satisfies the condition \( C_1 \) and \( C_2 \).

It follows that \( \xi^K \) becomes a section \( \xi \) on \( W^{\nu(a_{T_1}),T_1}(g_{t_0}, H_{f_0}) \), defined by \( \xi = \cup_{\xi \in \mathcal{W}(\Sigma_{t_0})} \xi^t \).

In fact it becomes a section \( \tilde{\xi} \) on the larger space \( \tilde{W}^{\nu(a_0),T_1}(g_{t_0}) \) without the constraints given by \( H_{f_0} \).

Recall that these identifications \( \lambda^{t_0} =: \{ \lambda_t^{t_0} : S_t \to S_{t_0} \} \), \( S|_{W^{T_1}(\Sigma_{t_0})} \to S_{t_0} \), induce a product structure

\[
\tilde{W}^{\nu(a_{T_1}),T_1}(g_{t_0}) \cong \tilde{W}^{\nu(a_0),t_0}(g_{t_0}) \times W^{T_1}(\Sigma_{t_0}),
\]

which in turn gives a smooth structure on \( \tilde{W}^{\nu(a_{T_1}),T_1}(g_{t_0}) \). Moreover, for \( t \) is sufficiently close to \( t_0 \) the map \( \lambda_t^{t_0} : S_t \to S_{t_0} \) induces an identification of \( \mathcal{L}_{g_{t_0}}^t \) with \( \mathcal{L}_{g_t}^{t_0} \) for \( \tilde{g}_t = \tilde{g}_{t_0} \circ (\lambda_t^{t_0}) \) essentially by the pull-back of \( \lambda_t^{t_0} \). Indeed, in the case, that \( E = TM \) and \( \mathcal{L}_{g_t}^t = L^p_{k-1}(S_t, g_t^*(E)) \) or \( \mathcal{L}_{g_{t_0}}^{t_0} = L^p_{k-1}(S_{t_0}, g_{t_0}^*(E) \otimes \Lambda^1) \), it is exactly given by the pull-back.

For \( \mathcal{L}_{g_{t_0}}^t = L^p_{k-1}(S_t, g_t^*(E) \otimes \Lambda^{0,1}) \) and \( \mathcal{L}_{g_{t_0}}^{t_0} = L^p_{k-1}(S_{t_0}, g_{t_0}^*(E) \otimes \Lambda^{0,1}) \), the identification is given by composition of the pull-back by \( \lambda_t^{t_0} \) with the map induced by the projection \( \Lambda^1 \to \Lambda^{0,1} \) since \( \lambda_t^{t_0} \) is not holomorphic away from the fixed part \( K \) as already observed in [?]. Combing this with the local trivializations of the bundles \( (\mathcal{L}^t \to \tilde{W}^{\nu(a_0),t}(g_t)) \) above, this gives a trivialization of the bundle \( \{ \mathcal{L}^t_{\Sigma_{t_0}} = \cup_{\xi \in \mathcal{W}(\Sigma_{t_0})} \mathcal{L}^t \to \tilde{W}^{\nu(a_{T_1}),T_1}(g_{t_0}) = \cup_{\xi \in \mathcal{W}(\Sigma_{t_0})} \tilde{W}^{\nu(a_0),t}(g_t) \} \) centered at \( g_{t_0} \).

Using the fact that \( \lambda_t^{t_0} \) is just the identity map on \( K = K_t = K_{t_0} \) (and hence holomorphic) it is easy to check that with respect to this product smooth structure and local trivialization, \( \xi/\tilde{\xi} \) so defined is ”constant” along \( W^{T_1}(\Sigma_{t_0}) \)-directions in the sense that for \( (h, t) \in \tilde{W}^{\nu(a_0),t_0}(g_{t_0}) \times W^{T_1}(\Sigma_{t_0}) \cong \tilde{W}^{\nu(a_{T_1}),T_1}(g_{t_0}), \) \( \xi(h, t) = \xi^{t_0}(h)(= \xi^K(h|_K)) \) (similarly for \( \tilde{\xi} \)). Hence it is a smooth section. Clearly the condition \( C_1 \) and \( C_2 \) still hold for \( \tilde{\xi} \).

The discussion here can be reformulated in the lemma.

**Lemma 5.1** On \( \tilde{W}^{\nu(a_{T_1}),T_1}(g_{t_0}), \) \( \tilde{\xi} = (\lambda^{t_0})^*(\tilde{\xi}^{t_0}) \). Here the section \( \tilde{\xi}|_{\tilde{W}^{\nu(a_0),t_0}(g_{t_0})} \) along the central slice \( \tilde{W}^{\nu(a_0),t_0}(g_{t_0}) \) is denoted by \( \tilde{\xi}^{t_0} \).
Proof:
This essentially is a tautology. \(\square\)

Thus \(\tilde{\xi}\) is the \(G^1_e\)-extension of \(\tilde{\xi}^t_0\). Here \(G^1_e = W^{T_1}(\Sigma_{t_0})\) considers as a family of diffeomorphisms \(\{\lambda^t_0 : S_t \to S_{t_0}\}\) parametrized by \(t \in W^{T_1}(\Sigma_{t_0})\). This interpretation proves the above lemma again. Similar interpretations using \(G^i_e\)-extensions with \(i > 1\) prove the two main theorems below.

The first main theorem is the following.

**Theorem 5.1** The section \(\tilde{\xi}^\Phi\) is of class \(C^{m_0}\). Consequently its restriction \(\xi^\Phi\) to the \(C^{m_0}\)-submanifold \(U^{T_1}(g_{u_0}, H_{f_0})\) with respect to the second smooth structure is of class \(C^{m_0}\). Here \(\xi^\Phi\) is obtained from the section \(\xi\) on \(W^{u(a_0), T_1}(g_{t_0}, H_{f_0})\) by the transformation \(\Phi\), similarly for \(\tilde{\xi}^\Phi\). The section \(\xi^\Phi, \Psi\) on \(U^{T_1}(g^r_{u_0}, H^r_{f_0})\) induced by the diffeomorphism \(\Psi : U^{T_1}(g_{u_0}, H_{f_0}) \to U^{T_1}(g^r_{u_0}, H^r_{f_0})\) is of class \(C^{m_0}\) as well.

**Proof:**

The first statement follows from the fact that \(\tilde{\xi}^\Phi\) is the \(G^2_e\)-extension of \(\xi^t_0\) by pull-backs of the elements in \(G^2_e\). Here \(G^2_e = W^{T_1}(\Sigma_{t_0}) = U^{T_1}(\Sigma_{u(t_0)})\) considered as the family of diffeomorphisms \(\{\lambda^{t_0}_0 \circ \phi_t : S_{u(t)} \to S_t \to S_{t_0}\}\) parametrized by \(t \in W^{T_1}(\Sigma_{t_0})\). Since \(\xi^t_0\) satisfies the conditions \(C_1\) and \(C_2\), a obvious generalization of the main theorem in the first paper of these sequel implies that the \(C^{m_0}\)-smoothness of the \(G^2_e\)-extension of \(\xi^{t_0}\) above so that \(\tilde{\xi}^\Phi\) is of class \(C^{m_0}\).

To prove the last statement, we note that in addition to diffeomorphism \(\Psi : U^{T_1}(g_{u_0}, H_{f_0}) \to U^{T_1}(g^r_{u_0}, H^r_{f_0})\) of that identified the bases, the standard local trivializations of the bundles \(\mathcal{L}\) centered at \(g_{u_0}\) and \(g^r_{u_0}\) using parallel transport along shortest connecting geodesics are also the ”same” in the sense that the trivialization for \(\mathcal{L}\) on \(U^{T_1}(g^r_{u_0}, H^r_{f_0})\) automatically give the one for \(\mathcal{L}\) on the other by our assumption that each local hypersurfaces in \(H_{f_0}\) is geodesic submanifold. \(\square\)

The idea above can be used to proof the smoothness of \(\xi\) viewed in any other chart \(W^{u(T_1)}(f'_0, H')\): that is to define the corresponding \(G^i_e\)-extension of the same section \(\xi^{t_0}\).

We start with the neighborhood \(U^{T_1}(g'_{u_0}, H')\) with the class \([g'_{u_0}] = [g_{u_0}]\). Here \(g_{u_0} = g_u(t_0) : (S_{u_0}, x(u)) \to (M, H_{f_0})\) is the center of \(U^{T_1}(g_{u_0}, H_{f_0})\), similarly for \(g'_{u_0} = g_u'(t_0) : (S'_{u_0}, x'(u')) \to (M, H'_{f_0})\). Since adding-dropping
making points does not affect the smoothness, we may assume that the number of marked points of $g_u$ and $g'_u$ are the same.

Now fix a dropping marking map $r = r^m_k$ that selects $m$ elements $x^r(u)$ from the $k$ marked points $x(u)$ satisfies the condition that each free component of an (hence any ) element $g_{u_0} \in U^{T_1}(g_{u_0})$ is minimally stabilized. This gives a “new” marking, an identification $(S_u)_v \simeq S^2_v \times S^2$ of a free component $(S_u)_v, v \in T_1$. Choose a ”compatible marking” $r'$ for an (hence any ) element $g'_u \in U^{T_1}(g'_u)$ accordingly.

Let $\Gamma = \Pi_{v \in T_1} \Gamma_v$ acting on the free components of $S_u$. Here each $\Gamma_v$ is a subgroup of $PSL(2, \mathbb{C})$ depending on the number of doubles points on the component $S^2_v$. In particular, if $S^2_v$ is stable $\Gamma_v = \{ e \}$. Now the assumption $[g'_u] = [g_{u_0}]$ of the centers above implies that there is a $\tilde{\gamma}_0 : S'_{u_0} \to S_{u_0}$ such that $g_{u_0} \circ \tilde{\gamma}_0 = g'_u$. Using the identifications (markings) of the free components with (a collection of ) $S^2(s)$ given by $r$ and $r'$, and denoted by $\psi^r$ and $\psi'^r$, the map $\tilde{\gamma}_0 = \psi^r \circ \gamma_0 \circ (\psi'^r)^{-1}$ for an element $\tilde{\gamma}_0 \in \Gamma$, unique upto the finite isotropies of $g_{u_0}$. Thus we can define the $\Gamma$-action on $U^{T_1}(g_{u_0})$ by a similar formula, $\gamma \cdot g_u =: g_u \circ (\psi^r \circ \gamma \circ (\psi'^r)^{-1})$. Here $\psi^r = \{ \psi^r_v, v \in T_1 \}$ with $\psi^r_v : (S_u)_v \to S^2$, similarly for $\psi'^r$. Note that for all elements in $g_u \in U^{T_1}(g_{u_0})$ the domains are all the ”same”; the parameter $u$ in te notation $\Sigma_u$ or $S_u$ only describes the locations of the distinguished points. In term of this ”action” of $\Gamma$, we have $g'_u = \gamma \cdot g_{u_0}$ between the two centers.

Denote $(\psi^r \circ \gamma \circ (\psi'^r)^{-1}) : S'_{u_0} \to S_u$ by $\tilde{\gamma}$. Let $x^{r_0}(u'_0) := \tilde{\gamma}_0^{-1}(x(u) \in S'_u$ be the ”new” marked points. Note that $g'_u(x^{r_0}(u'_0)) = g_{u_0} \circ \tilde{\gamma}_0^{-1}(x(u)) = g_{u_0}(x(u)) \in H_{f_0}$. Now consider the collection of the $L^p_k$ maps $g'_u$ of type $T_1$ near $g'_u$ with the constrains on the new marked points: $g'_u(x^{r_0}(u')) \in H_{f_0}$, denoted by $U^{T_1}(g'_{u_0}, H_{f_0}^{r_0})$.

**Theorem 5.2** Let $\xi^{r_0, r_0}$ be the section $\xi^r$ viewed in $U^{T_1}(g'_{u_0}, H_{f_0}^{r_0})$. Then $\xi^{r_0, r_0}$ is of class $C^{m_0}$.

**Proof:**

Let $G^3 = \Gamma \times W^{T_1}(f_{l_0}, H_{f_0})(= \Gamma \times G^1_e = \Gamma \times G^2_e$ with action of $G^2_e$ first then composing wit the action of $\Gamma$. Then the discussion above shows that $\xi^{r_0, r_0}$ is the restriction to the slice $U^{T_1}(g'_{u_0}, H_{f_0}^{r_0})$ of the $G^3$-extension of $\xi^{r_0}$. Hence it is of class $C^{m_0}$.

Applying implicit function theorem in a similar way to the proof that $\Psi$ induced by dropping makings is a differomorphism implies the next corollary.
Corollary 5.1 Let $\xi'^\Phi$ be the section $\xi^\Phi$ viewed in $U'^T_1(g'_u, H'_f)$. Then $\xi'^\Phi$ is of class $C^{m_0}$.

This proves the half of the main theorem below.

Theorem 5.3 Let $\xi^K : \tilde{W}(f_K) \to \mathcal{L}^K$ be a smooth section satisfying the condition $C_1$ and $C_2$. Then the smooth section $\xi$ on $W^{\nu(aT_1), T_1}(g_{t_0}, H_{f_0})$ is of class $C^{m_0}$ viewed in any other such local slices $W^{\nu(aT_1), T_1}(g'_{t_0}, H'_{f_0})$ or $U'T_1(g'_{u_0}, H'_{f_0})$ with respect their own smooth structures. Here $[g'_{t_0}] = [g_{t_0}]$ as unparametrized maps.

Proof:
The other half essentially follows from the obvious "extension" of the action of $G^3_{\epsilon}$ defined above by composing further the identifications given by inverse of $\phi'_t$ (with $t'$ and $t$ corresponding to each other) first, then the inverse of $\lambda'_t$. Indeed, let $\xi'$ be the section $\xi$ viewed in $W'^T_1(f_{t_0}, H_0)$. Then upto the effect of using different (but fixed ) markings, $\xi'$ is the restriction to $W'^T_1(f_{t_0}, H_0)$ of the new $G^3_{\epsilon}$-extension of $\xi^{t_0}$. A similar argument to the proof of the corollary above will eliminate the effect of different markings so that $\xi'$ is of class $C^{m_0}$.

□

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