SOME EXPLICIT RESULTS ON THE SUM OF A PRIME AND
AN ALMOST PRIME

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ABSTRACT. Inspired by a classical result of Rényi, we prove that every even integer \( N \geq 4 \) can be written as the sum of a prime and a number with at most 369 prime factors. We also show, under assumption of the generalised Riemann hypothesis, that this result can be improved to 33 prime factors.

1. INTRODUCTION

In 1948, Rényi [Rény48] proved the following theorem as an approximation to Goldbach’s conjecture.

**Theorem 1.1 (Rény48, Theorem 1).** There exists a natural number \( K \) such that every even integer \( N \geq 4 \) can be written as the sum of a prime and a number with at most \( K \) prime factors.

Namely, the case \( K = 1 \) is equivalent to Goldbach’s conjecture. If \( N \) is sufficiently large, then Chen [Che66; Che73] proved that one could take \( K = 2 \).

**Theorem 1.2 (Chen’s Theorem).** Every sufficiently large even integer can be written as the sum of a prime and a number with at most 2 prime factors.

There has been little work done however, on determining an explicit value of \( K \) that holds for all even \( N \geq 4 \). One of the reasons for this may be that Rényi and Chen’s original proofs are ineffective, in that a lower bound for \( N \) cannot be determined by following their methods.

Despite this, in [BJS22], Bordignon and the authors of this paper recently built upon unpublished work of Yamada [Yam13] to prove an effective and explicit variant of Chen’s Theorem. Namely, they showed [BJS22, Corollary 4] that Chen’s Theorem holds for all even \( N \geq \exp(\exp(32.6)) \). Using this result, a simple but wasteful argument gives that one can take \( K = e^{29.3} \approx 3.2 \cdot 10^{13} \) for all \( N \geq 4 \) [BJS22, Theorem 5].

In this paper, by using a more sophisticated procedure that essentially generalises the work in [BJS22], we improve on this result as follows.

**Theorem 1.3.** Every even integer \( N \geq 4 \) can be written as the sum of a prime and a number with at most \( K = 369 \) (not necessarily distinct) prime factors.

The main difficulty in lowering the value of \( K = 369 \) comes from our knowledge of potential Siegel zeros and the error term in the prime number theorem for arithmetic progressions. As these problems are mitigated under the assumption of the Generalised Riemann Hypothesis (GRH), we also provide a conditional result.

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Theorem 1.4. Assume GRH. Then every even integer $N \geq 4$ can be written as the sum of a prime and a number with at most $K = 33$ (not necessarily distinct) prime factors.

It should be noted that obtaining $K = 33$ does not require the full-strength of GRH. Rather, if our knowledge of the zeros of Dirichlet $L$-functions were to improve (say with significant computation), then the unconditional result would approach the conditional one. We also remark that our proofs require additional levels of optimisation compared to the explicit version of Chen’s theorem in [BJS22]. This is because we consider a wider range of $N$, causing more error terms to become non-negligible. In particular, we make use of recent work by Hathi and the first author [HJ21] for smaller values of $N$.

An outline of the paper is as follows. In Section 2 we provide the main notation and definitions used throughout. In Section 3 we state some preliminary lemmas. In Section 4 we outline the main method of approach, and prove the unconditional result (Theorem 1.3). In Section 5 we prove the conditional result (Theorem 1.4). Finally in Section 6 we detail possible avenues for future improvements.

2. Notation and setup

Here and throughout, $p$ denotes a prime number, $0 < \delta < 2$, $\alpha > 0$ and $X_2$ are parameters we choose later, $N \geq X_2$ is even,

$$\gamma = 0.57721\ldots \text{ (Euler’s constant)},$$

$$\prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) = \prod_{p>2} \frac{p(p-2)}{(p-1)^2} = 0.66016\ldots \text{ (Twin prime constant), and}$$

$$R_1 = 2.0452 \text{ (See [BJS22, Theorem 26]).}$$

As on page 18 of [BJS22], we let $k_0 := k_0(N)$ be the exceptional modulus up to $Q_1(x_2(N))$ (if it exists),

$$k_1 := k_1(N) = \begin{cases} k_0, & \text{if } k_0 \text{ exists and } (k_0, N) = 1 \\ 0, & \text{otherwise,} \end{cases}$$

and let $q_1 > \ldots > q_\ell$ be the prime factors of $k_1$ provided $k_1 \neq 0$. We also let $\alpha_1, \alpha_2, Y_0$ and $C(\alpha_1, \alpha_2, Y_0)$ be as in [Bor21, Theorem 1.2] which we state here for clarity.

Theorem 2.1 ([Bor21, Theorem 1.2]). Let $X_1 = \exp(\exp(Y_0))$, $\alpha_1, \alpha_2$ and $C$ be as in Table 6 of [Bor21]. Let $x > X_1$ and $k < \log^{\alpha_1} x_2$ be an integer. Let $E_0 = 1$ and $\beta_0$ denote the Siegel zero modulo $k$ if it exists, and $E_0 = 0$ otherwise. Then for $(k,l) = 1$ we have

$$\frac{\varphi(k)}{x} \left| \psi(x; k, l) - \frac{x}{\varphi(k)} \right| < \frac{C}{\log^{\alpha_2} x} + E_0 x^{\beta_0 - 1} \frac{\beta_0}{\beta_0}.$$ (2.1)

2.1. List of definitions. The following list of definitions is adapted directly from [BJS22]. However, we have made small modifications so that everything is expressed in terms of $\alpha_1, \alpha_2, Y_0$ and $C(\alpha_1, \alpha_2, Y_0)$ rather than the special case $\alpha_1 = 10, \alpha_2 = 8, Y_0 = 10.4, C = 3.2 \cdot 10^{-8}$ used in [BJS22]. As in [BJS22] we also take $\beta_0$ to be bounded by

$$\beta_0 \leq 1 - \nu(N), \quad \nu(N) = \min \left\{ \frac{100}{\sqrt{K_0(x_2)} \log^2 K_0(x_2)}, \frac{1}{2R_1 \log(Q_1(x_2))} \right\}$$ (2.1)
However, since it is known that there are no Siegel zeros for moduli less than $4 \cdot 10^5$ [Pla16], we are able to bound $\beta_0$ by

$$\beta_0 \leq 1 - \frac{1}{2R_1 \log(Q_1(x_2))} \quad (2.2)$$

whenever $K_0(x_2) \leq 4 \cdot 10^5$ (see the proof of [BJS22, Lemma 33] for more details). Moreover, the function $p^*(X_2)$ (not written below) is equal to $p(X_2)$ but with the sharper bound (2.2) used for $\beta_0$ for all $N$. Now, without further ado, we define $A = \{N - p : p \leq N, \ p \nmid N\}$, $A_d = \{a \in A : d \mid a\}$, $S(A, n) = \left| A - \bigcup_{p \mid n} A_p \right|$, $P(z) = \prod_{p \leq z, p \nmid N} p$, $V(z) = \prod_{p \mid P(z)} \left(1 - \frac{1}{p - 1}\right)$, $U_N = 2e^\gamma \prod_{p > 2} \left(1 - \frac{1}{(p - 1)^2}\right) \prod_{p > 2} \frac{p - 1}{p - 2}$, $m_j = q_1 \ldots q_j$, $p^{(j)}(z) = \prod_{p < z, p \nmid N, p \nmid q_1, \ldots, q_j} p$, $V^{(j)}(z) = V(p^{(j)}(z))$, $U_N^{(j)} = 2e^{-\gamma} \prod_{p > 2} \left(1 - \frac{1}{(p - 1)^2}\right) \prod_{p > 2} \frac{p - 1}{p - 2}$, $x_2 = x_2(N) := \frac{N}{\log^\alpha N}$, $c_{\alpha, N} = K \left(\frac{1}{2} - \alpha - \frac{2\alpha_1 \log \log N}{\log N}\right)$, $K_0(N) = \log^k N$, $Q_1(N) = \log^{\alpha_1} N$, $r(d) = |A_d| - \frac{|A|}{\varphi(d)}$, $h(s) = \begin{cases} e^{-2}, & 1 \leq s \leq 2, \\ e^{-s}, & 2 \leq s \leq 3, \\ 3s^{-1}e^{-s}, & s \geq 3, \end{cases}$ $\mathcal{E}(y) = \frac{4y \log^\frac{3}{2} y}{\log^{\alpha_1} x_2(y)} + \frac{4y}{\log^{\alpha_1 - \frac{1}{2}} y} + \frac{18y \log^{1/2} y}{\log^{\alpha_2} y} + \frac{5}{2} y^{\frac{1}{3}} \log^{\frac{11}{2}} y$.
\[ p_2(x_2) = \max_{y \geq x_2(X_2)} \left[ \frac{\log^2 y}{y} \left( 1.1 \log(Q_1(x_2)) \left( \frac{C(\alpha_1, \alpha_2, Y_0)}{\log^{\alpha_2} y} + \frac{y^{\beta_0}}{\beta_0} \right) + 27 \cdot 2 \log(2) \log^{\alpha_1-2} y + 0.4 \log^3 y \right) \right] \]

\[ p_1(x_2) = p_2(x_2) + \frac{1}{\log^{\alpha_1-2} x_2(x_2)} \left( \frac{0.67}{x_2(x_2)^{\frac{1}{2}}} \right), \]

\[ p(x_2) = p_1(x_2) \left( 1 + \frac{1}{\log^2 X_2 \log^3 x_2(x_2)} + \frac{1}{\left( 1 - \frac{4}{\log x_2(x_2)} \right) \log x_2} \right) + \frac{2.2}{\log^2 X_2}, \]

\[ c(x_2) = c_1(x_2) \left( 1 + \frac{1}{\log^2(X_2) \log^3 x_2(x_2)} + \frac{1}{\left( 1 - \frac{4}{\log x_2(x_2)} \right) \log x_2} \right) + \frac{1}{\log^2 X_2}, \]

\[ c_1(x_2) = \max_{y \geq x_2(x_2)} \left[ \frac{C(\alpha_1, \alpha_2, Y_0)}{\log^{\alpha_2-2} y} + \log^2 y \left( 1 - \frac{1}{2N \log Q_1(y)} \right)^{-1} y^{1-2^{\alpha_1} \log \log x_2(X_2)} \right. \]

\[ + \left. Q_1(y) \left( \frac{1.09}{\sqrt{y}} + \frac{3}{y^{2/3}} \right) + 34(\log y)^{1.52} \exp(-0.8 \sqrt{\log y}) \right]. \]

\[ c_2(x_2) = c(x_2) + \frac{1.3841 \log^4 X_2}{X_2 \log \log x_2}, \]

\[ c_3(x_2) = \max_{N \geq x_2} \left[ \frac{1}{\log \log \log N} \cdot \left\{ \frac{3}{2 \log N} + \frac{\log(N \log^\alpha x_2(N))}{\log(N \log^\alpha x_2(N))} \right\} \frac{\log^\delta N}{\log^\delta x_2(N)} \right. \]

\[ \cdot \left( e^\gamma \log \log \log^\delta x_2(N) + \frac{5}{2 \log \log^\delta x_2(N)} \right) \]

\[ + \frac{1.3841 \log^{2+\delta} N}{N \log \log N \log \log \log N} \right], \]

\[ c_4(x_2) = p(x_2) + \frac{0.9 \sqrt{x_2(x_2)} \log^4 X_2}{X_2 \log^\alpha \log \log x_2}, \]

\[ c_4^*(x_2) = p^*(x_2) + \frac{0.9 \sqrt{x_2(x_2)} \log^4 X_2}{X_2 \log^\alpha \log \log x_2}. \text{(See discussion above)} \]

\[ a_1(x_2) = \max_{N \geq x_2} \left[ \frac{c_2(x_2)}{\log^{2-\delta} N \log \log N} \cdot \frac{1.3841 \log(\log^\alpha x_2(N))}{\log(\log^\alpha x_2(N))} \right] + c_3(x_2), \]

\[ a(x_2) = a_1(x_2) \max_{N \geq x_2} \left[ \log \log N \frac{\prod_{p \geq 2} (p-1)^2}{\log^3 N} \cdot \frac{2.5}{p(p-2)} \right. \]

\[ \cdot \left( e^\gamma \log \log(\log^\alpha x_2(N)) + \frac{2.5}{\log(\log^\alpha x_2(N))} \right) \]

For our application of the explicit linear sieve in [BJS22, §2] we also need to work

with the functions \( f(s) \) and \( F(s) \) defined by the differential difference equation

\[ F(s) = \frac{2e^\gamma}{s}, \quad f(s) = 0, \quad 0 < s \leq 2, \]

\[ (sF(s))' = f(s-1), \quad (sf(s))' = F(s-1), \quad s \geq 2. \]
From this definition, explicit expressions for $F(s)$ and $f(s)$ can be produced, getting more complicated as $s$ gets larger. In [Cai08, p. 1340–1341] some of these expressions are listed. For example, for $5 \leq s \leq 7$,

$$F(s) = \frac{2e^\gamma}{s} \left(1 + \int_2^{s-1} \frac{\log(t-1)}{t} \, dt + \int_2^{s-3} \frac{\log(t-1)}{t} \left(\int_{t+2}^{s-1} \frac{1}{u} \log \frac{u-1}{t+1} \, du \right) \, dt \right)$$

and for $4 \leq s \leq 6$,

$$f(s) = \frac{2e^\gamma}{s} \left(\log(s-1) + \int_3^{s-1} \frac{1}{t} \left(\int_2^{t-1} \frac{\log(u-1)}{u} \, du \right) \, dt \right).$$

Note that $F(s)$ monotonically decreases towards 1 and $f(s)$ monotonically increases towards 1 [HR74, p. 227]. Thus, for $s \geq 6$, we can bound $F(s)$ and $f(s)$ as

$$F(s) - 1 \leq F(6) - 1 = 1.049 \ldots \cdot 10^{-4},$$

$$1 - f(s) \leq 1 - f(6) = 1.056 \ldots \cdot 10^{-4}.$$ 

We also set

$$m_{\alpha,X_2} = \max\{(1 - f(c_{\alpha,X_2}), F(c_{\alpha,X_2}) - 1)\}.$$ 

Finally, for $K \geq 1$ we write $\pi_K(N)$ for the number of ways to write $N$ as the sum of a prime and a number with at most $K$ prime factors.

### 3. Some preliminary results

Here we provide some preliminary lemmas required for sieving, most of which are variants of lemmas from [BJS22]. All notation is as in Section 2.

We begin with two lemmas which are modifications on Lemmas 17 and 18 of [BJS22].

**Lemma 3.1.** We have

$$\sum_{p < x} \frac{1}{p} \geq \log \log x + M - \frac{2.964 \cdot 10^{-6}}{\log x}, \quad x \geq 2 \quad (3.1)$$

$$\sum_{p < x} \frac{1}{p} \leq \log \log x + M + \frac{1.445 \cdot 10^{-2}}{\log x}, \quad x > \exp(8.9) \quad (3.2)$$

$$\sum_{p < x} \frac{1}{p} \leq \log \log x + M + \frac{2.588 \cdot 10^{-6}}{\log x}, \quad x > 10^{12.} \quad (3.3)$$

**Proof.** To begin with, we note that (3.1) is the same as the lower bound in [BJS22, Lemma 17]. For (3.2) we use a direct computation for $\exp(8.9) \leq x \leq \exp(10)$ and then the fact that [BJS22, Lemma 16]

$$\sum_{p \leq x} \frac{1}{p} \leq \log \log x + M + \frac{2}{\sqrt{x} \log x}$$

for $\exp(10) < x \leq 10^{12}$. For $x > 10^{12}$ it suffices to prove (3.3).

The inequality (3.3) follows by the same method as the proof of [BJS22, Lemma 17]. Namely, most of the error terms in the proof of the lower bound in [BJS22, Lemma 17] are bounded in absolute value, so that the same reasoning also applies for an upper bound. The only exception occurs when bounding

$$\int_x^{10^{10}} \frac{(y - \theta(y))(1 + \log y)}{y^2 \log^2 y} \, dy$$
for $10^{12} < x \leq 10^{19}$. In this case, we use the bound $x - \theta(x) \leq 1.95 \sqrt{x}$ from Theorem 2] to obtain
\[
\int_x^{10^{19}} \frac{(y - \theta(y))(1 + \log y)}{y^2 \log^2 y} \, dy \leq \int_x^{10^{19}} \frac{1.95(1 + \log y)}{y^{3/2} \log^2 y} \, dy
\]
\[
= 1.95 \left[ \frac{1}{2} \log \left( \frac{1}{\sqrt{y}} \right) - \frac{1}{\sqrt{y}} \log y \right]_x^{10^{19}}
\]
\[
\leq \frac{1.95}{\sqrt{x} \log x} - 0.975 \cdot \log \left( \frac{1}{\sqrt{x}} \right) - 2.76 \cdot 10^{-11}. \quad \square
\]

**Lemma 3.2.** Suppose $z > \exp(8.9)$. Then for all $30 < u < z$, we have
\[
\prod_{u \leq p < z} \left( 1 - \frac{1}{p - 1} \right)^{-1} < \left( 1 + 1.31287 \cdot 10^{-2} + \frac{3.01 \cdot 10^{-6}}{\log u} \right) \frac{\log z}{\log u}. \quad (3.4)
\]

Now suppose $z > 10^{12}$. Then for all $400 < u < z$, we have
\[
\prod_{u \leq p < z} \left( 1 - \frac{1}{p - 1} \right)^{-1} < \left( 1 + 5.52843 \cdot 10^{-4} + \frac{2.97 \cdot 10^{-6}}{\log u} \right) \frac{\log z}{\log u}. \quad (3.5)
\]

**Proof.** We modify the proof of [BJS22, Lemma 18] but bound some of the terms with more care since we are dealing with much lower values of $u$. We will only prove (3.4) as the method of proof for (3.5) is identical. So to begin with,
\[
\prod_{u \leq p < z} \left( 1 - \frac{1}{p - 1} \right)^{-1} = \prod_{u \leq p < z} \left( 1 - \frac{1}{p} \right)^{-1} \prod_{u \leq p < z} \left( 1 - \frac{1}{p} \right)^{-1}.
\]

Now,
\[
\prod_{u \leq p < z} \left( 1 - \frac{1}{p - 1} \right)^{-1} \leq \prod_{p \geq u} \left( 1 - \frac{1}{p} \right)^{-1} = \prod_{p > 2} \left( 1 - \frac{1}{p} \right)^{-1} \prod_{2 \leq p < u} \left( 1 - \frac{1}{p} \right)^{-1} \prod_{2 \leq p < z} \left( 1 - \frac{1}{p} \right)^{-1} \prod_{2 \leq p < u} \left( 1 - \frac{1}{p} \right)^{-1}
\]
noting that $u > 30$ and $\prod_{p > 2} \left( 1 - \frac{1}{p} \right)^{-1} = 1.5147 \ldots$ is the reciprocal of the twin prime constant. Thus,
\[
\prod_{u \leq p < z} \left( 1 - \frac{1}{p - 1} \right)^{-1} < 1.00754 \prod_{u \leq p < z} \left( 1 - \frac{1}{p} \right)^{-1}. \quad (3.6)
\]

Next, we note that
\[
\prod_{u \leq p < z} \left( 1 - \frac{1}{p} \right)^{-1} = \exp \left( - \sum_{u \leq p < z} \log \left( 1 - \frac{1}{p} \right) \right). \quad (3.7)
\]

Now, by Lemma 3.1,
\[
\sum_{u \leq p < z} \frac{1}{p} = \sum_{p < z} \frac{1}{p} - \sum_{p < u} \frac{1}{p}
\]
Lemma 3.3. Let \( \imath \) improves on an earlier result of Dudek [Dud17] that was used in [BJS22].

\[
\text{Lemma 3.4.} \quad \text{Let } \imath \text{ be an integer, } \delta \geq \frac{1}{2}, \text{ and } \eta > 10. \text{ Using (3.6), (3.9), (3.10), (3.11) and } \eta > 10, \text{ we have,}
\]

\[\prod_{u \leq p < z} \left( 1 - \frac{1}{p} \right)^{-1} \leq \frac{\log z - \log u}{\log u} \exp \left( 1.6236 \cdot 10^{-3} + \frac{2.964 \cdot 10^{-6}}{\log u} \right) \exp \left( 0.54 \sum_{p \geq u} \frac{1}{p^2} \right) \]

with

\[
\exp \left( 1.6236 \cdot 10^{-3} + \frac{2.964 \cdot 10^{-6}}{\log u} \right) \leq \left( 1 + 1.6236 \cdot 10^{-3} + \frac{2.964 \cdot 10^{-6}}{\log u} + 0.52 \left( 1.6236 \cdot 10^{-3} + \frac{2.964 \cdot 10^{-6}}{\log u} \right)^2 \right)^{\frac{1}{2}}
\]

and

\[
\exp \left( 0.54 \sum_{p \geq u} \frac{1}{p^2} \right) \leq \exp \left( 0.54 \sum_{p \geq 2} \frac{1}{p^2} - 0.54 \sum_{p < u} \frac{1}{p^2} \right) \leq 0.00390788
\]

\[
(3.11)
\]

since \( u \geq 30 \) and \( \sum_{p} \frac{1}{p^2} = 0.452247 \ldots \) is known to a high degree of accuracy (see e.g. [Mer82]). Using (3.7), (3.8), and (3.9) and \( u \geq 30 \) then gives the desired result. \hfill \Box

We now recall [BJS22, Lemma 37], giving additional examples that we require.

**Lemma 3.3.** (BJS22, Lemma 37). For \( z \geq 285 \) and \( j = 0, \ldots, \ell \), we have

\[
V^{(j)}(z) = \frac{U^{(j)}(N)}{\log z} \left( 1 + \frac{\theta}{2 \log^2 z} \right) \left( 1 + \frac{2 \theta}{z} \right) \left( 1 + \frac{8 \theta \log N}{z} \right) \left( 1 + \frac{\theta}{z - 1} \right),
\]

where \( |\theta| \leq 1 \). In particular, for a choice of positive integer \( M \) we set \( z = N^{1/M} \) and \( z \geq z_0 \), allowing us to write

\[
\frac{U^{(j)}(N)}{\log z} \left( 1 - \frac{\xi(z_0, M)}{\log^2 N} \right) < V^{(j)}(z) < \frac{U^{(j)}(N)}{\log z} \left( 1 + \frac{\xi(z_0, M)}{\log^2 N} \right)
\]

for some constant \( \xi(z_0, M) > 0 \). For our purposes, we compute that \( \xi(10^{12}, 40) \leq 801 \) and \( \xi(\exp(8.9), 18) \leq 4685 \).

Finally we give a result that follows directly from [HJ21, Theorem 1.5]. This improves on an earlier result of Dudek [Dud17] that was used in [BJS22].

**Lemma 3.4.** Let \( p_i \) denote the \( i \)th prime and suppose \( X_2 \geq 4 \cdot 10^{18} \). Then every even integer \( 2 < N < X_2 \) can be written as the sum of a prime and a square-free number \( \eta > 1 \) with at most \( K \) prime factors, where \( K \geq 1 \) is the largest integer such that

\[
\theta(p_{K+6}) - \theta(13) < \log(X_2).
\]
Proof. For $2 < N \leq 4 \cdot 10^{18}$, the result is true since Goldbach’s conjecture holds in this range [OHP14]. For $4 \cdot 10^{18} < N < X_2$ we then have by [HJ21, Theorem 1.5] that $N = p + \eta$ where $p$ is a prime and $\eta$ is a square-free number coprime to the first 6 primes 2, 3, 5, 7, 11 and 13. Since $\eta < N \leq X_2$, the number of prime factors of $\eta$ is at most

$K = \max_{m} \left\{ \prod_{i=1}^{m} p_{i+6} < X_2 \right\}$

and if $\prod_{i=1}^{K} p_{i+6} < X_2$ then $\theta(p_{K+6}) - \theta(13) = \sum_{i=1}^{K} \log(p_{i+6}) < \log(X_2)$.

\[ \square \]

Remark. The condition $X_2 \geq 4 \cdot 10^{18}$ can be weakened to $X_2 \geq 40$. However, here we wish to highlight the usefulness of the Goldbach verification [OHP14].

4. The unconditional result

In this section, we prove Theorem 4.3. Namely, that every even integer $N \geq 4$ can be expressed as the sum of a prime and a number with at most $K = 369$ prime factors. The general idea will be to set $z = N^{1/M}$ for some positive integer $M$ satisfying $5 \leq M \leq K + 1$. We then have $\pi_{M-1}(N) \geq S(A, P(z))$ so that, if one can prove $S(A, P(z)) > 0$ for all $N \geq X_2$, then $\pi_K(N) \geq \pi_{M-1}(N) > 0$ for all $N \geq X_2$. Since we will be taking $X_2$ to be quite large, the case when $4 \leq N < X_2$ must be treated separately. This will be done using Lemma 3.4.

To bound $S(A, P(z))$ from below, we generalise Theorem 43 of [BJS22]. This is done by parameterising $\alpha_1, \alpha_2, Y_0$ and $M$ and making some other small changes.

**Theorem 4.1.** Let $M \geq 5$, and $\alpha, \delta, X_2, \alpha_1, \alpha_2$ and $Y_0$ be parameters as in Section 3. Also, let $\log \log x_2(X_2) \geq Y_0$, $\alpha > 0$, $N \geq X_2$ be even, $z = N^{1/M}$, $z_0 = X_2^{1/M}$ such that

\[
\frac{\sqrt{x_2}}{\log^{\alpha_1} x_2} \geq \log^{\alpha_1} N \geq 10^9, \quad 1 - \frac{\xi(z_0, M)}{\log^2 N} \geq 0, \quad X_2 \geq 4 \cdot 10^{18} \tag{4.1}
\]

where $\xi(z_0, M)$ is as in Lemma 3.3 and

\[
\frac{N^{\alpha}}{\log^{\alpha_1} x_2(N) \log^{2.5} N} \geq \exp \left( u \left( 1 + \frac{9 \cdot 10^{-7}}{\log u} \right) \right), \quad \frac{N^{\frac{1}{2} - \alpha}}{\log^{2\alpha_1} N} \geq z^2, \quad K_0(x_2) \geq 3022
\]

with $u > 400$ so that

\[
\epsilon = 5.52843 \cdot 10^{-4} + \frac{2.97 \cdot 10^{-6}}{\log u} < \frac{1}{1807.21138}.
\]

If $k_1 < K_0(x_2)$, we have

\[
S(A, P(z)) > M \frac{U_N}{\log^2 N} \left( 1 - \frac{\xi(z_0, M)}{\log^2 N} \right)
\]

\[
\cdot \left( f \left( M \left( \frac{1}{2} - \alpha \right) \right) - C_1(\epsilon) \epsilon^2 h \left( M \left( \frac{1}{2} - \alpha \right) \right) \right)
\]

\[
- \frac{1}{M} \left( 1 - \frac{\xi(z_0, M)}{\log^2 N} \right)^{-1} \left( 2\epsilon^2 \prod_{p > 2} \left( 1 - \frac{1}{(p-1)^2} \right) \right)^{-1} \frac{c_4(X_2)}{\log N}.
\]

\[1\text{To avoid confusion with notation, we remark that $\alpha_1$ means something different in [BJS22]. Namely, it corresponds to our variable $\alpha$.} \]
On the other hand, if $k_1 \geq K_0(x_2)$, we have

$$S(A, P(z)) >$$

$$M \frac{U_N N}{\log^2 N} \left( 1 + \frac{\xi(z_0, M)}{\log^2 N} \right) \left\{ f(c_0, x_2) - \epsilon_0(X_2, \delta)(1 - f(c_0, x_2)) - (1 + \epsilon_0(X_2, \delta))c_2(\epsilon) e^2 h(c_0, x_2)ight.$$ 

$$\left. - (3 \epsilon_0(X_2, \delta) + o(X_2)) \cdot (\overline{m}_{\alpha, X_2} + e C_1(\epsilon) e^2 h(c_0, x_2)) - a(X_2) - \frac{2 \xi(z_0, M)}{\log^2 N} \right.$$

$$\left. - \frac{1}{M} \left( 1 + \frac{\xi(z_0, M)}{\log^2 N} \right)^{-1} \left( 2 e^\gamma \prod_{p > 2} \left( 1 - \frac{1}{(p-1)^2} \right) \right)^{-1} c_1^2(X_2) 1.3841 \log(\log^{\alpha_2} x_2(N)) \right\},$$

where $C_1(\epsilon)$ and $C_2(\epsilon)$ are the values in \[\text{BJS22, Table 1}\], $\epsilon_0(X_2, \delta) = \frac{1}{\overline{p} - 2}$ with $\overline{p}$ the largest prime such that

$$\log^\delta x_2(X_2) \geq \prod_{2 < p \leq \overline{p}} p$$

and all other notation is as in Section 4.

We omit a full proof of Theorem 4.1 as it follows by essentially the same reasoning as the proof of \[\text{BJS22, Theorem 43}\]. However, we do make a few remarks.

Firstly, we note that the restriction $M \geq 5$ is so that the condition $\frac{\tilde{N}^{1/2 - \alpha}}{\log^{\alpha_1} N} \geq z^2$ is satisfied. This also means that our approach works for at best $K = 4$ prime factors. Next, we note that the definition of $\epsilon$ comes from using the second part of Lemma 3.2 in place of \[\text{BJS22, Lemma 17}\]. We also have some new conditions in \[\text{4.4}\]. These conditions were always satisfied in \[\text{BJS22, Theorem 43}\] where $\log \log x_2 \geq 10.4$ (which is not necessarily true here). Primarily, the condition $\sqrt{x_2}/\log^{\alpha_1} x_2 \geq \log^{\alpha_2} x_2 \geq 10^9$ is required so that one can use \[\text{BJS22, Lemma 22}\] in the proof of \[\text{BJS22, Lemma 28}\]. The condition $1 - \xi(z_0, M)/\log^2 N \geq 0$ is required to prevent any sign problems when applying Lemma 5.3. Finally, the condition $X_2 \geq 4 \cdot 10^{18}$ is chosen as for $N \leq 4 \cdot 10^{18}$ we always have $\pi_{M-1}(N) > 0$ by [OHP14]. Certainly, these conditions can be weakened if desired, but they are easily satisfied in all the scenarios we consider.

The condition $8 \alpha_1 + \frac{160 \log \log N}{\log N} < 1$ was also removed as this was only required in \[\text{BJS22, Theorem 43}\] to give an exact expression for $f(s)$ and ensure that the lower bound for $S(A, P(z))$ was asymptotically large enough to prove Chen’s theorem.

Proof of Theorem 4.3 From [Bor21, Table 6] we have that

\begin{align*}
(\psi_0, \alpha_1, \alpha_2, C) &= (7.8, 7, 1, 0.16), \quad \text{(4.2)} \\
(\psi_0, \alpha_1, \alpha_2, C) &= (7.9, 7, 2, 3.98), \quad \text{(4.3)}
\end{align*}

are valid choices of parameters. We cannot use (4.2) directly, as this would cause $p_2(X_2)$ to diverge. However, with the notation of Theorem 2.1 (4.2) means that

$$\frac{\varphi(k)}{x} \left| \psi(x; k, l) - \frac{x}{\varphi(k)} \right| < \frac{0.16}{\log x} + E_0 \frac{x^\beta_0 - 1}{\beta_0},$$

\[\text{2The first part will be used in the conditional case in Section 4.}\]
for all $x \geq \exp(\exp(7.8))$ and $k \leq \log^7 x$. Thus, for $\exp(\exp(7.8)) \leq x \leq \exp(\exp(7.9))$ and $k \leq \log^7 x$, we have
\[
\frac{\varphi(k)}{x} \left| \psi(x; k, l) - \frac{x}{\varphi(k)} \right| < \frac{0.16 \cdot \exp(7.9)}{\log^2 x} + E_0 \frac{x^{\beta_0 - 1}}{\beta_0}.
\]
Combining this with \(\text{[157]}\) means that \(Y_0, \alpha_1, \alpha_2, C = (7.8, 7.2, 431.57)\) is also a valid choice of parameters, and these are the choices we use for our computation.

We set $X_2 = \exp(\exp(7.816))$, $\delta = 1.3$, $\alpha = 0.25$, $M = 40$ and $\epsilon = 1/(1807.2114)$. Then for $N \geq X_2$, we have by Theorem 3.4 that
\[
S(A, P(z)) > \frac{37U_NN}{\log^2 N} \quad k_1 < K_0(x_2),
\]
\[
S(A, P(z)) > \frac{0.23U_NN}{\log^2 N} \quad k_1 \geq K_0(x_2).
\]
This tells us that every even integer $N \geq X_2$ can be written as the sum of a prime and a number with at most $M - 1 = 39$ prime factors. For the range $2 < N < X_2$ we then apply Lemma 3.4 and obtain the final value $K = 369$.\[\]
Remark. Although it may seem that taking $M$ larger than 40 would lead to a better result, this is not necessarily the case. In particular, as $M$ gets larger, so does $\xi(\alpha_0, M)$ to the point where it negatively affects the second condition in \(\text{[157]}\) and the bounds on $S(A, P(z))$. Moreover, as $M$ gets large, $z = N^{1/M}$ decreases and worse bounds must be used in Lemma 3.2.

5. The conditional result

In this section we prove Theorem \(\text{[14]}\). As assuming GRH allows for many improvements to the unconditional result, this section is quite large and has been split into three parts. To begin with, we will use some recent results of Ernvall-Hytönen and Palojärvi \(\text{[EHP22]}\) to obtain conditional bounds for the error terms \(\left| \pi(x; q, a) - \frac{\mu(x)}{\varphi(q)} \right|\) and \(\left| \theta(x; q, a) - \frac{x}{\varphi(q)} \right|\) appearing in the prime number theorem for arithmetic progressions. Next, we will extend Lemma 3.4 under assumption of GRH. Finally, we will prove a conditional lower bound on $S(A, P(z))$ and use this to prove Theorem 1.4.

We note that in Sections 5.1 and 5.3 there are some similarities with upcoming work due to Bordignon and the second author \(\text{[BS22]}\). However, we have still included all the details here to make this paper self-contained.

5.1. Conditional bounds on $\pi(x; q, a)$ and $\theta(x; q, a)$. First we give a bound on \(\left| \pi(x; q, a) - \frac{\mu(x)}{\varphi(q)} \right|\) which will later be used in Section 5.3 as part of the lower bound on $S(A, P(z))$.

Lemma 5.1. Let $x \geq X_2 \geq 4 \cdot 10^{18}$, and $q$ and $a$ be integers such that $3 \leq q \leq \sqrt{x}$ and $(a, q) = 1$. Then, assuming GRH,
\[
\left| \pi(x; q, a) - \frac{\mu(x)}{\varphi(q)} \right| \leq c_\pi(X_2) \sqrt{x} \log x,
\]
for all $x \geq \exp(\exp(7.8))$ and $k \leq \log^7 x$. Thus, for $\exp(\exp(7.8)) \leq x \leq \exp(\exp(7.9))$ and $k \leq \log^7 x$, we have
\[
\frac{\varphi(k)}{x} \left| \psi(x; k, l) - \frac{x}{\varphi(k)} \right| < \frac{0.16 \cdot \exp(7.9)}{\log^2 x} + E_0 \frac{x^{\beta_0 - 1}}{\beta_0}.
\]
Combining this with \(\text{[157]}\) means that \(Y_0, \alpha_1, \alpha_2, C = (7.8, 7.2, 431.57)\) is also a valid choice of parameters, and these are the choices we use for our computation.

We set $X_2 = \exp(\exp(7.816))$, $\delta = 1.3$, $\alpha = 0.25$, $M = 40$ and $\epsilon = 1/(1807.2114)$. Then for $N \geq X_2$, we have by Theorem 3.4 that
\[
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\]
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\]
This tells us that every even integer $N \geq X_2$ can be written as the sum of a prime and a number with at most $M - 1 = 39$ prime factors. For the range $2 < N < X_2$ we then apply Lemma 3.4 and obtain the final value $K = 369$.\[\]
Remark. Although it may seem that taking $M$ larger than 40 would lead to a better result, this is not necessarily the case. In particular, as $M$ gets larger, so does $\xi(\alpha_0, M)$ to the point where it negatively affects the second condition in \(\text{[157]}\) and the bounds on $S(A, P(z))$. Moreover, as $M$ gets large, $z = N^{1/M}$ decreases and worse bounds must be used in Lemma 3.2.

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We note that in Sections 5.1 and 5.3 there are some similarities with upcoming work due to Bordignon and the second author \(\text{[BS22]}\). However, we have still included all the details here to make this paper self-contained.

5.1. Conditional bounds on $\pi(x; q, a)$ and $\theta(x; q, a)$. First we give a bound on \(\left| \pi(x; q, a) - \frac{\mu(x)}{\varphi(q)} \right|\) which will later be used in Section 5.3 as part of the lower bound on $S(A, P(z))$.

Lemma 5.1. Let $x \geq X_2 \geq 4 \cdot 10^{18}$, and $q$ and $a$ be integers such that $3 \leq q \leq \sqrt{x}$ and $(a, q) = 1$. Then, assuming GRH,
\[
\left| \pi(x; q, a) - \frac{\mu(x)}{\varphi(q)} \right| \leq c_\pi(X_2) \sqrt{x} \log x,
\]
\footnote{Note $0.16 \cdot \exp(7.9) = 431.5651 \ldots$.}
Proof. Follows directly from [EHP22, Theorem 1] upon using the bounds on $x$ and $q$. Note that we have also used $\varphi(q) \geq 2$ since $q \geq 3$. 

We now provide a similar style result for $|\theta(x; q, a) - \frac{x}{\varphi(q)}|$ which will be useful in Section 5.2.

**Lemma 5.2.** Let $x \geq X_3 \geq 4 \cdot 10^{18}$, and $q$ and $a$ be integers such that $1 \leq q \leq x$ and $(a, q) = 1$. Then, assuming GRH,

$$|\theta(x; q, a) - \frac{x}{\varphi(q)}| < c_\theta(X_3) \sqrt{x} \log^2 x,$$

where

$$c_\theta(X_3) = \frac{1}{16\pi} + \frac{1}{6\pi} + 0.092 + \frac{12.683}{\log X_3} + \frac{254.9795}{\log^2 X_3} + \frac{2607.854}{\log^3 X_3}$$

$$+ \frac{11605.056}{\log X_3} + \frac{(0.092 \log X_3 + 8.250) \log \log X_2}{X_3^{1/4} \log X_2} + \frac{1.3135 \log^2 X_2 + 60.8825 \log X_2 + 939.260}{X_3^{1/4} \log X_2} \frac{273.934}{\sqrt{X_2 \log X_2}} \leq 0.640.$$ 

Proof. The result for $q = 1$ follows immediately from [Sch76, Theorem 10]. Then, since all primes are odd (except 2), the case $q = 2$ follows similarly. We thus assume $q \geq 3$ from here onwards.

We first obtain bounds for $|\psi(x; q, a) - x/\varphi(q)|$. So, using $3 \leq q \leq x$ and [EHP22, Theorem 3],

$$|\psi(x; q, a) - \frac{x}{\varphi(q)}| < c_\psi(X_3) \sqrt{x} \log^2 x,$$

where

$$c_\psi(X_3) = \frac{1}{16\pi} + \frac{1}{6\pi} + 0.092 + \frac{12.683}{\log X_3} + \frac{254.9795}{\log^2 X_3} + \frac{2607.854}{\log^3 X_3}$$

$$+ \frac{2.015}{X_3} + \frac{4.179}{\sqrt{X_3} \log X_3} + \frac{263.886}{\sqrt{X_3} \log^2 X_3} + \frac{(1 + 1.93378 \cdot 10^{-8})}{X_3^{1/6} \log^2 X_3} + \frac{1.04320}{X_3^{1/6} \log^2 X_3} \leq 0.83.$$ 

Next, by [Bro+21, Corollary 5.1], we have for all $x \geq X_3 \geq 4 \cdot 10^{18}$

$$\psi(x; q, a) - \theta(x; q, a) \leq \psi(x) - \theta(x) \leq (1 + 1.93378 \cdot 10^{-8}) \sqrt{x} + 1.04320 x^{1/3}.$$ 

Hence,

$$|\theta(x; q, a) - \frac{x}{\varphi(q)}| \leq c_\theta(X_3) \sqrt{x} \log^2 x,$$
with \( c_6(X_3) \) as in (5.1). \( \square \)

5.2. An extension of Lemma 5.1. Lemma 5.1 is based off a result of Hath and the first author [HJ21, Theorem 1.5] which gives that any even integer \( N \geq 40 \) can be expressed as the sum of a prime and a square-free number that is coprime to the primorial \( 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 30030 \). As this result is quite useful in obtaining our final value for \( K \), here we provide an extension which is conditional under GRH and involves larger primorials and values of \( N \).

We begin with a variation of [HJ21, Lemma 5.1] and then bound some terms for ease of computation.

**Lemma 5.3** (cf. [HJ21, Lemma 5.1]). Define \( \overline{R}_k(N) \) to be the logarithmically-weighted number of representations of \( N \) as \( N = p \cdot \eta \) where \( p \) is a prime, \( \eta \) is a square-free number coprime to \( k \) and \( \eta \neq 1 \). Now, assume GRH and let \( N \geq X_3 \geq 4 \cdot 10^{18} \) be even. Then for any \( C \in (0,1/2) \) and \( B < \sqrt{N} \),

\[
\frac{\overline{R}_k(N)}{N} > 2c \prod_{q|k/2} \left( 1 - \frac{q-1}{q^2-q-1} \right) - \frac{c_6(X_3) \log^2(N)}{\sqrt{N}} \sum_{d|k/2} \sum_{e|d} \sum_{a \leq B \sqrt{e/d}} \mu^2(a) \\
- \left( \frac{1 + 2C}{1 - 2C} \right) \sum_{d|k/2} \sum_{e|d} \frac{1}{\varphi(d/e)} \sum_{a>B \sqrt{e/d}} \frac{\mu^2(a)}{\varphi(a^2)} \phi(a) \\
- \log(N) \left( \sum_{d|k/2} \sum_{e|d} \left( N^{-\frac{1}{2}} \left( \sqrt{d} - \frac{1}{d} \right) + \frac{1}{\sqrt{d} e} N^{-C} + N^{-2C} \right) \right) \\
- \frac{\log(k)}{N} - \frac{\log(N)}{N}.
\]

(5.2)

Here, \( c = 0.37395 \ldots \) is Artin’s constant, \( \mu \) is the Möbius function, and \( c_6(X_3) \) is as in Lemma 5.2.

**Proof.** This is essentially identical to that of [HJ21, Lemma 5.1] with two main differences. First, we have introduced a parameter \( B \) which replaces the choice of \( 10^3 \) used in [HJ21]. Secondly, we have replaced “\( c_6(da^2/e) \log \log n \)” with \( c_6(X_3) \log^2(N) \) as a result of the stronger bounds we have under GRH. Note that there is also a slight notation clash with [HJ21, Lemma 5.1], namely, \( N \) and \( c_6 \) mean something different in [HJ21] and we have accounted for this accordingly. \( \square \)

**Theorem 5.4.** Keep the notation and conditions of Lemma 5.3 and let \( k \) be the product of the first \( L+1 \) primes. We then have, for \( B \geq \max\{45, 8 \sqrt{k/2}\} \),

\[
\frac{\overline{R}_k(N)}{N} > 2c \prod_{q|k/2} \left( 1 - \frac{q-1}{q^2-q-1} \right) - \frac{(4 + \sqrt{3})^L \cdot B \cdot c_6(X_3)}{3^L \sqrt{N}} \log^2(N) \\
- \left( \frac{1 + 2C}{1 - 2C} \right) \cdot 2^L \cdot G \left( \frac{B}{\sqrt{k/2}} \right) \\
- \log(N) \left( \frac{7^L}{3^L \sqrt{N}} + \frac{(4 + \sqrt{3})^L}{3^L N^C} + \frac{3^L}{N^{2C}} \right)
\]
where

\[ G(x) = e^\gamma \left( \frac{\log \log x}{x^2} - \text{li} \left( \frac{1}{x} \right) \right) + \frac{3}{x}. \]

Proof. We write \( k' = k/2 \) and bound each of the sums from Lemma 5.3. First,

\[
\sum_{d \mid k'} \sum_{e \mid d} \sum_{a \leq B \epsilon/e} \mu^2(a) \leq \sum_{d \mid k'} \sum_{e \mid d} \sum_{a \leq B \epsilon/e} \sum_{(a,d)=e} 1 \leq \sum_{d \mid k'} \sum_{e \mid d} B \sqrt{de}.
\]

To bound this expression further we note that \( k' \) (and each \( d \mid k' \)) is square-free and odd. Thus, for any \( x \mid k' \) with \( m \) prime divisors, we have \( x \geq 3^m \). So, writing \( \omega(d) \) for the number of unique prime factors of \( d \),

\[
\sum_{d \mid k'} \sum_{e \mid d} \frac{1}{\sqrt{de}} \leq \sum_{d \mid k'} \frac{1}{\sqrt{d}} \sum_{m=0}^{\omega(d)} \frac{1}{(\sqrt{3})^m} \binom{\omega(d)}{m} \leq \sum_{d \mid k'} \frac{1}{\sqrt{d}} \left( 1 + \frac{1}{\sqrt{3}} \right)^{\omega(d)} \leq \sum_{d \mid k'} \frac{L}{m} \sum_{m=0}^{(\sqrt{3})^m} \left( 1 + \frac{1}{\sqrt{3}} \right)^{m} \left( \frac{L}{m} \right) = \frac{4 + \sqrt{3}}{3} L \quad . \tag{5.3}
\]

Next,

\[
\sum_{d \mid k/2} \sum_{e \mid d} \frac{1}{\varphi(d/e)} \sum_{a \leq B \epsilon/e} \frac{\mu^2(a)}{\varphi(a^2)} \leq \sum_{a > B / \sqrt{k'}} \frac{\mu^2(a)}{\varphi(a^2)} \sum_{e \mid (a,k')} \sum_{(a,d)=e} 1 \leq 2L \sum_{a > B / \sqrt{k'}} \frac{\mu^2(a)}{\varphi(a^2)},
\]

and, by [RS62, Theorem 15],

\[
\sum_{a > B / \sqrt{k'}} \frac{\mu^2(a)}{\varphi(a^2)} \leq \sum_{a > B / \sqrt{k'}} \left( \frac{e^\gamma \log \log a^2}{a^2} + \frac{2.5}{a^2 \log \log a^2} \right) \leq \sum_{a > B / \sqrt{k'}} \left( \frac{e^\gamma \log \log a^2}{a^2} + \frac{1.76}{a^2} \right) \quad (\text{Since } B / \sqrt{k'} \geq 8)
\]

\[
\leq \int_{\lfloor B / \sqrt{k'} \rfloor}^{\infty} \left( \frac{e^\gamma \log \log x}{x^2} + \frac{3}{x^2} \right) dx = G \left( \left\lfloor \frac{B}{\sqrt{k/2}} \right\rfloor \right).
\]
Finally, we want to bound
\[
\sum_{d\mid k/2} \sum_{e\mid d} \left( N^{-\frac{k}{2}} \left( \frac{1}{e} - \frac{1}{d} \right) + \frac{1}{\sqrt{de}} N^{-C} + N^{-2C} \right). 
\] (5.4)

Each term is this double sum is bounded analogously to the double sum in (5.3). Namely,
\[
\sum_{d\mid k'} \sum_{e\mid d} \left( \frac{1}{e} - \frac{1}{d} \right) \leq \sum_{d\mid k'} \sum_{e\mid d} \frac{1}{e} = \left( \frac{7}{3} \right)^L,
\]
\[
\sum_{d\mid k'} \sum_{e\mid d} \frac{1}{\sqrt{de}} \leq \left( \frac{4 + \sqrt{3}}{3} \right)^L
\]
and
\[
\sum_{d\mid k'} \sum_{e\mid d} 1 = 3^L.
\]

As a result, (5.4) is bounded above by
\[
\frac{7^L}{3^L \sqrt{N}} + \frac{(4 + \sqrt{3})^L}{3^L N^C} + \frac{3^L}{N^{2C}}
\]
as desired. \(\square\)

**Corollary 5.5.** Assume GRH. Then every even integer \( N \geq \exp(109) \) can be written as the sum of a prime and a square-free number coprime to the product of the first 15 primes. In addition, every even integer \( N \geq \exp(158) \) can be written as the sum of a prime and a square-free number coprime to the product of the first 22 primes.

**Proof.** In Theorem 5.4, we set \( k \) to be the product of the first 15 primes, \( N \geq \exp(109), C = 0.13, \) and \( B = 10^{14.7} \) to get \( R_k(N)/N > 0.03 \). Next we set \( k \) to be the product of the first 22 primes, \( N \geq \exp(158), C = 0.13, \) and \( B = 10^{21.1} \) to get \( R_k(N)/N > 0.004 \). All computations are provided by Sage 9.3. \(\square\)

We finish this section with a generalised version of Lemma 3.4 for which Corollary 5.5 can be directly applied to.

**Proposition 5.6.** Let \( p_i \) denote the \( i^{th} \) prime and \( X_2 \geq 4 \cdot 10^{18} \). Suppose every even integer \( N \geq X_3 \) can be written as the sum of a prime and a square-free number coprime to the product of the first \( L + 1 \) primes. Then every even integer \( X_3 \leq N < X_2 \) can be written as the sum of a prime and a square-free number \( \eta \) with at most \( K \) prime factors, where \( K \geq 1 \) is the largest integer such that
\[
\theta(p_{K+L+1}) - \theta(p_{L+1}) < \log(X_2).
\]

**Proof.** Direct generalisation of the proof of Lemma 3.4. \(\square\)
5.3. A conditional lower bound for $S(A, P(z))$. We now prove an analogue of Theorem 4.3 assuming GRH. For this we will first need a variant of the Bombieri-Vinogradov theorem (cf. [BJS22, Lemmas 29 and 30]).

**Lemma 5.7.** Assume GRH and suppose $N \geq X_2 \geq 4 \cdot 10^{18}$ is even and $H := \sqrt[N]{\frac{\log^A N}{N}} \geq 45$. Then

$$\sum_{d \leq H \atop (d, N) = 1} \mu^2(d) \left| \pi(N; d, N) - \frac{\pi(N)}{\varphi(d)} \right| \leq \frac{p_G(X_2)N}{\log^A N},$$

where

$$p_G(X_2) = 0.65 \left( c_\pi(X_2) + \frac{1}{16\pi} \right) \leq 0.429$$

with $c_\pi(X_2)$ as defined in Lemma 5.1.

**Proof.** First note that we may assume $d \geq 3$ since for $d = 1$, we have $|E_\pi(N; d, N)| = 0$, and $d \neq 2$ since $N$ is even. Now, by the triangle inequality

$$\left| \pi(N; d, N) - \frac{\pi(N)}{\varphi(d)} \right| \leq \left| \pi(N; d, N) - \frac{\text{li}(N)}{\varphi(d)} \right| + \frac{1}{\varphi(d)} |\text{li}(N) - \pi(N)|.$$

We bound the first term using Lemma 5.1 and by [Sch76, Corollary 1] the second term is bounded above by

$$\frac{1}{8\pi \varphi(d)} \sqrt{N \log N} \leq \frac{1}{16\pi} \sqrt{N \log N}.$$

Therefore,

$$\left| \pi(N; d, N) - \frac{\pi(N)}{\varphi(d)} \right| \leq \left( c_\pi(X_2) + \frac{1}{16\pi} \right) \sqrt{N \log N}$$

so that, by [BJS22, Lemma 21]

$$\sum_{d \leq H \atop (d, N) = 1} \mu^2(d) \left| \pi(N; d, N) - \frac{\pi(N)}{\varphi(d)} \right| \leq 0.65H \left( c_\pi(X_2) + \frac{1}{16\pi} \right) \sqrt{N \log N} \leq \frac{p_G(X_2)N}{\log^A N},$$

as required. □

**Lemma 5.8** (cf. [BJS22, Lemma 42]). Keeping the notation and conditions from Lemma 5.7 we have

$$\sum_{d \leq H \atop d \mid P(z)} |r(d)| < \frac{c_{4,G}(X_2)N}{\log^A N}, \quad (5.5)$$

where

$$c_{4,G}(X_2) = p_G(X_2) + \frac{0.9}{\sqrt{X_2 \log \log X_2}} \leq 0.429.$$

**Proof.** Follows by the same reasoning as the proof of [BJS22, Lemma 41]. □

We now give a lower bound for $S(A, P(z))$ assuming GRH.
Theorem 5.9. Assume GRH. Let $\log\log X_2 \geq Y_0$, $M \geq 5$, $\alpha > 0$, $N \geq X_2$ be even, $z = N^{1/M}$, $z_0 = X_2^{1/M}$ such that

$$\frac{\sqrt{X_2}}{\log^{A+1} X_2} \geq 45, \quad 1 - \frac{\xi(z_0, M)}{\log^2 N} \geq 0, \quad X_2 \geq 4 \cdot 10^{18} \tag{5.6}$$

where $\xi(z_0, M)$ is as in Lemma 3.3 and

$$\frac{N^{\alpha}}{\log^{A+1} N} \geq \exp \left( u \left( 1 + \frac{9 \cdot 10^{-7}}{\log u} \right) \right), \quad N^{\frac{1}{2} - \alpha} \geq z^2,$$

with $u > 150$ so that

$$\epsilon = 1.31287 \cdot 10^{-2} + \frac{3.01 \cdot 10^{-6}}{\log u} < \frac{1}{76.16387}.$$

Then

$$S(A, P(z)) > M \frac{U N}{\log^2 N} \left( 1 - \frac{\xi(z_0, M)}{\log^2 N} \right)$$

$$\cdot \left( f \left( M \left( \frac{1}{2} - \alpha \right) \right) - C_1(\epsilon)e^2h \left( M \left( \frac{1}{2} - \alpha \right) \right) \right)$$

$$- \frac{1}{M} \left( 1 - \frac{\xi(z_0, M)}{\log^2 N} \right)^{-1} \left( 2e^7 \prod_{p > 2} \left( 1 - \frac{1}{(p - 1)^2} \right)^{-1} \right) c_{4, G}(X_2) \log^{A-2} N),$$

where $C_1(\epsilon)$ is from [BJS22, Table 1], $c_{4, G}(X_2)$ is defined in Lemma 5.8, and all other notation is as in Section 2.

Proof. We argue similarly to the case $k_1 < K_0(x_2)$ in the proof of [BJS22, Theorem 43]. So, set

$$D = N^{\frac{1}{2} - \alpha}, \quad s = \frac{\log D}{\log z} = M \left( \frac{1}{2} - \alpha \right) \quad \text{and} \quad Q(u) = \prod_{p \leq u \atop p \nmid N} p.$$

Since $D \geq z^2$, we have by [BJS22, Theorem 6]

$$S(A, P(z)) > M \frac{|A| U N}{\log N} \left( 1 - \frac{\xi(z_0, M)}{\log^2 N} \right) \left( f(s) - C_1(\epsilon)e^2h(s) \right) - \sum_{d \mid P(z) \atop d < QD} |r(d)|. \tag{5.7}$$

We now remark that the condition

$$\frac{N^{\alpha}}{\log^{A+1} N} \geq \exp \left( u \left( 1 + \frac{9 \cdot 10^{-7}}{\log u} \right) \right)$$

implies that

$$Q(u) \leq \frac{N^{\alpha}}{\log^{A+1} N} \tag{5.8}$$

by [BJS22, Lemma 23]. As a result, $QD \leq H := \frac{\sqrt{X_2}}{\log^{A+1} N}$ so that we may apply Lemma 5.8 to (5.7). This gives the desired result upon noting that $|A| > N/ \log N$ ([BJS22, Lemma 41]).

Equipped with this conditional lower bound on $S(A, P(z))$, we now finally prove Theorem 1.4.
Proof of 1.4. We set \( X_2 = \exp(\exp(5.087)) \), \( \alpha = 0.28365 \), \( A = 2.01 \), \( \epsilon = 1/(76.1639) \) and \( M = 18 \). Note that in this case, 
\[
f \left( M \left( \frac{1}{2} - \alpha \right) \right) = f(3.8943) = 0.97209 \ldots \quad \text{(See Cai08, p. 1340.)}
\]
Applying Theorem 5.9 we then obtain 
\[
S(A, P(z)) > \frac{0.005U_N N}{\log^2 N} > 0.
\]
This means that, assuming GRH, every even \( N \geq \exp(\exp(5.087)) \) can be written as the sum of a prime and a number with at most \( M - 1 = 17 \) prime factors.

For \( \exp(158) \leq N < \exp(\exp(5.087)) \), we then apply Corollary 5.5 and Proposition 5.6 with \( L + 1 = 22 \) to prove that \( K = 33 \) works in this range. Then, for the range \( \exp(109) \leq N < \exp(158) \) we again use Corollary 5.5 and 5.6 with \( L + 1 = 15 \). Finally, for \( 2 < N \leq \exp(109) \) we use Lemma 3.4. \( \square \)

6. Possible improvements

With more work, it should be possible to improve our main results (Theorems 1.3 and 1.4). There are many avenues to do this, so in what follows we detail what we believe are some of the most impactful approaches. If the reader is interested in pursuing any of these avenues, the authors are very open to correspondence on the matter.

Before we begin a general point is that we expect many of the explicit results that go into our proof to improve naturally in line with increased computational power. So in this regard, we remark that extending the computations of Platt [Pla16] regarding zeros of Dirichlet \( L \)-functions, would be a sure-fire way to improve the ingredients used for the unconditional result (Theorem 1.3).

6.1. Bounds on primes in arithmetic progressions. The main bottleneck to improving the unconditional result is our existing bounds on the error term in the prime number theorem for arithmetic progressions. In our approach, we used the recent bounds obtained by Bordignon in [Bor21]. Certainly, one could get a small improvement in our results by extending Table 6 in [Bor21] to give more optimal parameters. However, on inspection, it appears that there are several other aspects of Bordignon’s work that can be improved.

To begin with, in [Bor21] the error term in the explicit formula [Bor21, (1)] is obtained using a method due to Goldston [Gol83]. However, an asymptotically better error term can be obtained from the work of Wolke [Wol83] and Ramaré [Ram10]. An explicit form of such an error term was obtained recently by Cully-Hugill and the first author [CHJ21].

Moreover, the zero-free regions for Dirichlet \( L \)-functions could be improved. Namely, there is recent work of Kadiri [Kad18] which could be built upon to give better bounds on Siegel/exceptional zeros compared to [Bor21, Theorem 1.1]. This would also lead to a better (i.e. lower) value of \( R_1 \) that could be used in this work. The methods used in the recent work of Morrill and Trudgian [MT20] could also be useful in this regard.
6.2. **Explicit bounds on Siegel zeros.** In addition to the bounds one can obtain on Siegel zeros described in Section 6.1, we also seek to improve bounds of the form

\[ \beta \leq 1 - \frac{\lambda}{\sqrt{q \log^2 q}} \]  

(6.1)

where \( \lambda \) is a positive constant, and \( \beta \) is a (potential) Siegel zero mod \( q \). This bound is that which appears in (2.1) and is an important component in the proof of Theorem 1.3. For \( q > 4 \cdot 10^6 \), Bordignon [Bor19; Bor20] shows that one can take \( \lambda = 100 \) and this is what we use. Here, we note that for \( q \leq 4 \cdot 10^5 \) there are no Siegel zeros by a computation due to Platt [Pla16]. In fact, the relevant computation in [Pla16] was only a side result of the main computation, meaning a more targeted approach could pay dividends.

It also appears that the factor of \( \log^2 q \) can be removed from (6.1) by using an approach due to Goldfeld and Schinzel [GS75]. This has already been done for odd characters in [RR20] but a version that also works for even characters would be required in our setting.

6.3. **Bounds on sums and products of primes.** Another key component which goes into our results are bounds on

\[ \sum_{p < x} \frac{1}{p} \quad \text{and} \quad \prod_{u \leq p < z} \left( 1 - \frac{1}{p - 1} \right)^{-1} \]

(see Lemmas 3.1 and 3.2). Some of these bounds could be greatly improved by computation. For instance, if one were to extend the computation used for [BJS22, Lemma 16] to all \( 2 \leq x \leq 10^{13} \), then the constant \( (2.964 \cdot 10^{-6}) \) appearing in the bound

\[ \sum_{p < x} \frac{1}{p} \geq \log \log x + M - \frac{2.964 \cdot 10^{-6}}{\log x} \]

would be reduced to \( 1.483 \cdot 10^{-6} \).

6.4. **Further exploration of sieve methods.** Throughout recent history there have been numerous sieve-theoretic approaches to the problem of expressing even numbers as the sum of a prime and an almost prime. The overarching sieve used in this paper is the linear sieve, and we use the explicit version from [BJS22, §2]. One could further explore the existing literature on linear sieves (e.g. [HR74; FI10]) and likely find an approach that is superior to the one here.

In this direction we also remark, as discussed in Section 4, that our approach works for at best \( K = 4 \) prime factors. It would be interesting to explore simpler methods, such as those using Brun’s sieve (e.g. [HR74, §2.4]), which fail asymptotically for such low values of \( K \), but might give better explicit results than those in Theorems 1.3 and 1.4.
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