PROOF OF A CONJECTURAL SUPERCONGRUENCE MODULO $p^5$

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Abstract. In this paper we prove the supercongruence
\[
(p-1)/2 \sum_{n=0}^{(p-1)/2} \frac{6n+1}{256n} \binom{2n}{n}^3 \equiv p(-1)^{(p-1)/2} + (-1)^{(p-1)/2} \frac{7}{24} p^4 B_{p-3} \pmod{p^5}
\]
for any prime $p > 3$, which was conjectured by Sun in 2019.

1. Introduction

In 1997, L. van Hamme [21] proposed many conjectural $p$-adic supercongruences motivated by corresponding Ramanujan-type series for $1/\pi$. For example, he conjectured the supercongruence
\[
\sum_{n=0}^{(p-1)/2} \frac{6n+1}{256n} \binom{2n}{n}^3 \equiv (-1)^{(p-1)/2} p \pmod{p^4}
\]
for any prime $p > 3$, inspired by the Ramanujan series (cf. [14])
\[
\sum_{n=0}^{\infty} \frac{6n+1}{256n} \binom{2n}{n}^3 = \frac{4}{\pi}.
\]
The congruence (1.1) was confirmed by L. Long [8] in 2011.

In 2011 Z.-W. Sun [16] formulated many conjectural supercongruences involving Bernoulli numbers or Euler numbers. Recall that the Bernoulli numbers $B_0, B_1, \ldots$ and the Euler numbers $E_0, E_1, \ldots$ are defined by
\[
\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} (|x| < 2\pi) \quad \text{and} \quad \frac{2}{e^x + e^{-x}} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!} (|x| < \frac{\pi}{2})
\]
respectively. For example, he conjectured the congruence
\[
\sum_{n=0}^{p-1} \frac{6n+1}{256n} \binom{2n}{n}^3 \equiv (-1)^{(p-1)/2} p - p^3 E_{p-3} \pmod{p^4}
\]
for any prime $p > 3$. This was later confirmed by G.-S. Mao and C.-W. Wen [11, Th. 1.2].

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In 2019 Z.-W. Sun [18, Conj. 22] conjectured that for any prime $p > 3$ and positive odd integer $m$ we have

$$16^{m-1} (pm)^4 (m-1)^3 \left( \sum_{n=0}^{m-1} \frac{6n+1}{256^n} \binom{2n}{n}^3 - (-1)^{(p-1)/2} p \sum_{r=0}^{m-1} \frac{6r+1}{256^r} \binom{2r}{r}^3 \right)$$

$$\equiv (-1)^{(p-1)/2} \frac{7}{24} B_{p-3} \pmod{p}.$$ 

In this paper we confirm this in the case $m = 1$. Namely, we establish the following result.

**Theorem 1.1.** Let $p > 3$ be a prime. Then

$$\sum_{n=0}^{(p-1)/2} \frac{6n+1}{256^n} \binom{2n}{n}^3 \equiv p(-1)^{(p-1)/2} + (-1)^{(p-1)/2} \frac{7}{24} p^4 B_{p-3} \pmod{p^5}. \quad (1.3)$$

Another similar congruence modulo $p^5$ states that

$$\sum_{n=0}^{p-1} \frac{3n+1}{16^n} \binom{2n}{n}^3 \equiv p + \frac{7}{6} p^4 \pmod{p^5}$$

for any prime $p > 3$, which was conjectured by Sun [16] in 2011 and confirmed by C. Wang and D.-W. Hu [22] in 2020.

In the next section, we provide some known lemmas. We will use the WZ method to prove Theorem 1.1 in Section 3.

2. Some known lemmas

In 1862 J. Wolstenholme [23] proved the classical congruence

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$$

for any prime $p > 3$. This was refined by J.W.L. Glaisher [2] in 1900.

**Lemma 2.1** (Glaisher [2]). For any prime $p > 3$, we have

$$\binom{2p-1}{p-1} \equiv 1 - \frac{2}{3} p^3 B_{p-3} \pmod{p^4}. \quad (2.1)$$

**Remark 2.2.** For modern references about (2.1), the reader may consult [12] and [3].

In 1895, F. Morley [13] got the following fundamental congruence:

$$\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} 4^{p-1} \pmod{p^3}$$

for any prime $p > 3$. This was refined by L. Carlitz [1] in 1953.
Lemma 2.3 (Carlitz[1]). For each odd prime \( p \), we have
\[
(-1)^{(p-1)/2} \binom{p-1}{(p-1)/2} \equiv 4^{p-1} + \frac{p^3}{12} B_{p-3} \pmod{p^4}.
\]

We also need the following result of E. Lehmer established in 1938.

Lemma 2.4 (E. Lehmer [7]). For any prime \( p > 3 \), we have
\[
\sum_{k=1}^{(p-1)/2} \frac{1}{k} \equiv -2 \sum_{k=1}^{(p-1)/2} \frac{1}{2k-1} \equiv -2q_p(2) + p q_p(2)^2 \pmod{p^2},
\]
where \( q_p(2) \) denotes the Fermat quotient \((2^{p-1} - 1)/p\).

Let \( a_1, a_2, \ldots, a_m \) be integers. For any integer \( n \geq m \), we define the alternating multiple harmonic sum
\[
H(a_1, a_2, \ldots, a_m; n) := \sum_{1 \leq k_1 < k_2 < \cdots < k_m \leq n} \prod_{i=1}^{m} \frac{\text{sign}(a_i)^{k_i}}{k_i^{a_i}},
\]
and call \( m \) and \( \sum_{i=1}^{m} |a_i| \) its depth and weight respectively. For convenience, we simply write \( H_n \) to stand for \( H(1; n) \).

We need the following known results as lemmas.

Lemma 2.5 ([6]). Let \( a, r \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\} \), For any prime \( p > ar + 2 \), we have
\[
H\big\{a^r; p - 1\big\} \equiv \begin{cases} 
(-1)^r \frac{a}{2(a+1)} p^2 B_{p-ar-2} \pmod{p^3} & \text{if } ar \text{ is odd,} \\
(-1)^{r-1} \frac{a}{ar+1} p B_{p-ar-1} \pmod{p^2} & \text{if } ar \text{ is even.}
\end{cases}
\]

Lemma 2.6 ([15]). For any \( a \in \mathbb{Z}^+ \) and prime \( p > a + 2 \), we have
\[
H\left(\frac{a}{2}; \frac{p-1}{2}\right)
\equiv \begin{cases} 
-2q_p(2) + pq_p(2)^2 - \frac{2}{7} p^2 q_p(2)^3 - \frac{7}{12} p^2 B_{p-3} \pmod{p^3} & \text{if } a = 1, \\
\frac{2a-2}{a} B_{p-a} \pmod{p} & \text{if } a > 1 \text{ is odd,} \\
\frac{a(2a+1)}{2(a+1)} B_{p-a-1} \pmod{p^2} & \text{if } a \text{ is even.}
\end{cases}
\]

Lemma 2.7 ([6]). For any \( a, b \in \mathbb{Z}^+ \) and any prime \( p > a + b + 1 \), we have
\[
H(a, b; p - 1) \equiv \frac{(-1)^b}{a+b} \left(\frac{a+b}{a}\right) B_{p-a-b} \pmod{p}.
\]

Lemma 2.8 ([19]). For any prime \( p > 3 \), we have
\[
H(1, -2; p - 1) \equiv H(-1, 2; p - 1) \equiv H(2, -1; p - 1) \equiv \frac{1}{4} B_{p-3} \pmod{p}
\]
and
\[
H(1, 1, -1; p - 1) \equiv -\frac{1}{3} q_p(2)^2 - \frac{7}{24} B_{p-3} \pmod{p}.
\]
Lemma 2.9 ([5]). Let $a, b \in \mathbb{Z}^+$ with $a + b$ odd. For any prime $p > a + b$, we have

$$H \left( a, b; \frac{p-1}{2} \right) \equiv \frac{B_{p-a-b}}{2(a+b)} \left( (-1)^b \left( \frac{a+b}{a} \right) + 2^{a+b} - 2 \right) \pmod{p}. $$

Lemma 2.10 (R. Tauraso and J. Q. Zhao [20]). For any prime $p > 3$, we have

$$H(1, -1; p - 1) \equiv q_p(2)^2 - pq_p(2)^3 \equiv \frac{13}{24}pB_{p-3} \pmod{p^2}. \quad (2.3)$$

3. PROOF OF THEOREM 1.1

We will use the following WZ pair appeared in [3] to prove Theorem 1.1. For $n, k \in \mathbb{N} = \{0, 1, 2, \ldots \}$, we define

$$F(n, k) = \frac{(6n - 2k + 1)}{2^{8n-2k}} \left( \binom{2n}{n} \binom{2n+2k}{n+k} \binom{2n-2k}{n-k} \binom{n+k}{n} \right)$$

and

$$G(n, k) = \frac{n^2 \binom{2n}{n} \binom{2n+2k}{n+k} \binom{2n-2k}{n-k} \binom{n+k}{n}}{2^{8n-2k-4}(2n + 2k - 1) \binom{2k}{k}}.$$ 

Clearly $F(n, k) = G(n, k) = 0$ if $n < k$. It is easy to check that

$$F(n, k - 1) - F(n, k) = G(n + 1, k) - G(n, k) \quad (3.1)$$

for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}^+$.

Summing (3.1) over $n \in \{0, \ldots, (p - 1)/2\}$ we get

$$\sum_{n=0}^{(p-1)/2} F(n, k - 1) - \sum_{n=0}^{(p-1)/2} F(n, k) = G \left( \frac{p + 1}{2}, k \right) - G(0, k) = G \left( \frac{p + 1}{2}, k \right).$$

Furthermore, summing both side of the above identity over $k \in \{1, \ldots, (p - 1)/2\}$, we obtain

$$\sum_{n=0}^{(p-1)/2} F(n, 0) = F \left( \frac{p - 1}{2}, \frac{p - 1}{2} \right) + \sum_{k=1}^{(p-1)/2} G \left( \frac{p + 1}{2}, k \right). \quad (3.2)$$

Lemma 3.1. Let $p > 3$ be a prime. Then

$$F \left( \frac{p - 1}{2}, \frac{p - 1}{2} \right) \equiv (-1)^{p-1}p \left[ 1 - qp_p(2) + p^2q_p(2)^2 - p^3q_p(2)^3 - \frac{7}{12}p^3B_{p-3} \right] \pmod{p^5}. $$

Proof. By the definition of $F(n, k)$, we have

$$F \left( \frac{p - 1}{2}, \frac{p - 1}{2} \right) = \frac{2p - 1}{2^{2p-3}} \left( \frac{2p - 2}{p - 1} \left( \frac{p - 1}{(p-1)/2} \right) \right) = p \frac{2^{p-1}}{(p-1)/2} \frac{p-1}{2^{p-3}}.$$
This, together with Lemma 2.1, Lemma 2.3 and the equality $2^{p-1} = 1 + p q(p(2))$, yields that

$$F \left( \frac{p-1}{2}, \frac{p-1}{2} \right) \equiv \frac{p \left( 1 - \frac{2}{3} p^3 B_{p-3} \right) (-1)^{(p-1)/2} (4^{p-1} + \frac{1}{12} p^3 B_{p-3})}{(1 + p q(p(2))^3} \equiv (-1)^{p-1} \frac{1}{2} p \left( 1 - p q(p(2)) + p^2 q(p(2))^2 - p^3 q(p(2))^3 - \frac{7}{12} p^3 B_{p-3} \right) \pmod{p^5}.$$}

This concludes the proof.

Lemma 3.2. For any prime $p > 3$, we have

$$\sum_{k=1}^{(p-1)/2} \frac{(p/2 - k)}{(p + 1 - 2k)(p + 2k)} \equiv \frac{1}{2} q(p(2)) - \frac{p}{4} q(p(2))^2 - 2 p q(p(2)) + \frac{1}{6} p^2 q(p(2))^3 + 4 p^2 q(p(2))^2 + p^2 q(p(2))^2 + \frac{7}{48} p^2 B_{p-3} \pmod{p^3}.$$}

Proof. In view of Lemma 2.4

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k(2k - 1)} \equiv 2 \sum_{k=1}^{(p-1)/2} \frac{1}{2k - 1} - H(p-1)/2 \equiv 4 q(p(2) - 2 p q(p(2))^2 \pmod{p^2} \quad (3.3)$$

and

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k(2k - 1)^2} = H(p-1)/2 - 2 \sum_{k=1}^{(p-1)/2} \frac{1}{2k - 1} + 2 \sum_{k=1}^{(p-1)/2} \frac{1}{(2k - 1)^2} \equiv -4 q(p(2) + \frac{1}{2} H(2; (p-1)/2) \pmod{p} \quad (3.4)$$

It is easy to see that

$$\sum_{k=1}^{(p-1)/2} \frac{(p/2 - k)}{(p + 1 - 2k)(p + 2k)} \equiv \frac{1}{4} H(p-1)/2 - \frac{1}{2} p \sum_{k=1}^{(p-1)/2} \frac{1}{k(2k - 1)} - p^2 \sum_{k=1}^{(p-1)/2} \frac{1}{k(2k - 1)^2}.$$}

Then we immediately obtain the desired result by Lemma 2.4 (3.3) and (3.4).
Lemma 3.3. For any prime \( p > 3 \), we have

\[
\frac{(p-1)/2}{(p/2 - k)H_k}{(p + 1 - 2k)(p + 2k)}
\equiv 2q_p(2) - q_p(2)^2 - 6pq_p(2) + 2pq_p(2)^2 + pq_p(2)^3 + \frac{7}{12}pB_{p-3} \pmod{p^2}.
\]

Proof. By Lemmas [2.5 and 2.6] and (2.3), we have

\[
\sum_{k=1}^{(p-1)/2} \frac{H_{2k}}{k} = \sum_{k=1}^{p-1} \frac{(1 + (-1)^k)H_k}{k}
\equiv H(1, 1; p - 1) + H(1, -1; p - 1) + \frac{1}{2}H(2; (p - 1)/2)
\equiv q_p(2)^2 - pq_p(2)^3 + \frac{7}{24}pB_{p-3} \pmod{p^2}. \quad (3.5)
\]

Noting \( 2H(1, 1; n) = H_n^2 - H(2; n) \), we get

\[
\sum_{k=1}^{(p-1)/2} \frac{H_k}{k} = H(1, 1; (p - 1)/2) + H(2; (p - 1)/2)
\equiv \frac{1}{2}H_{(p-1)/2}^2 + \frac{1}{2}H(2; (p - 1)/2)
\equiv 2q_p(2)^2 - 2pq_p(2)^3 + \frac{7}{6}pB_{p-3} \pmod{p^2}. \quad (3.6)
\]

It is easy to see that

\[H_{(p+1)/2-k} \equiv \frac{2}{p + 1 - 2k} + 2pH(2; 2k) - \frac{p}{2}H(2; k) + H_{(p-1)/2} + 2H_{2k} - H_k \pmod{p^2}.
\]

This, together with (3.5), (3.6) and [10] (2.2), (2.3), yields that

\[
\sum_{k=1}^{(p-1)/2} \frac{H_{(p+1)/2-k}}{k} \equiv -8q_p(2) + 4q_p(2)^2 + 4pq_p(2)^2 + 8pq_p(2)
\quad - 4pq_p(2)^3 - \frac{7}{3}pB_{p-3} \pmod{p^2} \quad (3.7)
\]

and

\[
\sum_{k=1}^{(p-1)/2} \frac{H_{(p+1)/2-k}}{k(2k - 1)} \equiv \sum_{k=1}^{(p-1)/2} \frac{H_k}{k(2k - 1)}
\equiv - \sum_{k=1}^{(p-1)/2} \frac{H_{(p+1)/2-k}}{k} - \sum_{k=1}^{(p-1)/2} \frac{H_k}{k}
\equiv 8q_p(2) - 6q_p(2)^2 \pmod p. \quad (3.8)
\]
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Since
\[
\sum_{k=1}^{(p-1)/2} \frac{(p/2 - k)H_k}{(p + 1 - 2k)(p + 2k)} = -\frac{1}{4} \sum_{k=1}^{(p-1)/2} \frac{H_{(p+1)/2 - k}}{k} - \frac{p}{2} \sum_{k=1}^{(p-1)/2} \frac{H_{(p+1)/2 - k}}{k(2k - 1)} \pmod{p^2},
\]
we immediately get the desired result with the aids of (3.7) and (3.8). □

Lemma 3.4. Let $p > 3$ be a prime. Then
\[
\sum_{k=1}^{(p-1)/2} \frac{(p/2 - k)H_{2k}}{(p + 1 - 2k)(p + 2k)} \equiv q_p(2) - \frac{1}{4} q_p(2)^2 - 3pq_p(2) + \frac{p}{2} q_p(2)^2 + \frac{p}{4} q_p(2)^3 + \frac{13}{32} pB_{p-3} \pmod{p^2}.
\]

Proof. It is easy to check that
\[
H_{p+1 - 2k} \equiv pH(2; 2k - 2) + H_{2k - 2} \pmod{p^2}.
\]
So
\[
\sum_{k=1}^{(p-1)/2} \frac{H_{p+1 - 2k}}{k} \equiv p \sum_{k=1}^{(p-1)/2} \frac{H(2; 2k - 2)}{k} + \sum_{k=1}^{(p-1)/2} \frac{H_{2k - 2}}{k}
\]
\[
= p \left( \sum_{k=1}^{(p-1)/2} \frac{H(2; 2k)}{k} - \sum_{k=1}^{(p-1)/2} \frac{1}{k(2k - 1)^2} - \frac{1}{4} H(3; (p - 1)/2) \right)
\]
\[
+ \left( \sum_{k=1}^{(p-1)/2} \frac{H_{2k}}{k} - \sum_{k=1}^{(p-1)/2} \frac{1}{k(2k - 1)} - \frac{1}{2} H(2; (p - 1)/2) \right).
\]

Combing this with Lemma [2.6, (3.5), (3.3), (3.4) and 10 (2.3)], we get
\[
\sum_{k=1}^{(p-1)/2} \frac{H_{p+1 - 2k}}{k} \equiv q_p(2)^2 - 4q_p(2) + 4pq_p(2) + 2pq_p(2)^2
\]
\[
- pq_p(2)^3 - \frac{13}{8} pB_{p-3} \pmod{p^2} \quad (3.9)
\]
and
\[
\sum_{k=1}^{(p-1)/2} \frac{H_{p+1 - 2k}}{k(2k - 1)} \equiv \sum_{k=1}^{(p-1)/2} \frac{H_{2k - 2}}{k(2k - 1)} \equiv 4q_p(2) - 2q_p(2)^2 \pmod{p}. \quad (3.10)
\]
Since
\[
\sum_{k=1}^{(p-1)/2} \frac{(p/2 - k)H_{2k}}{(p + 1 - 2k)(p + 2k)} = -\frac{1}{4} \sum_{k=1}^{(p-1)/2} \frac{H_{p+1 - 2k}}{k} - \frac{p}{2} \sum_{k=1}^{(p-1)/2} \frac{H_{p+1 - 2k}}{k(2k - 1)}.
\]
Lemma 3.5. For any prime \( p > 3 \), we have

\[
\frac{(p-1)/2}{(p/2 - k)H_k^2} \equiv 4q_p(2) - 6q_p(2)^2 + 2q_p(2)^3 + \frac{1}{8}B_{p-3} \pmod{p}.
\]

Proof. It is easy to verify that

\[
\frac{(p-1)/2}{(p/2 - k)H_k^2} \equiv \frac{1}{2} \sum_{k=1}^{(p-1)/2} H_k^2 \frac{H_{2k-1}}{2k-1} = \frac{1}{2} \sum_{k=0}^{(p-3)/2} \frac{H_{2k+1}}{2k+1}
\]

Observe that

\[
\frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{1}{(2k-1)k^2} = 2 \sum_{k=1}^{(p-1)/2} \frac{1}{2k-1} - H_{(p-1)/2} - \frac{1}{2}H(2; (p - 1)/2) \quad (3.11)
\]

and

\[
\sum_{k=1}^{(p-1)/2} \frac{H_{k-1}}{k(2k-1)} = 2 \sum_{k=1}^{(p-1)/2} \frac{H_{k-1}}{2k-1} - \sum_{k=1}^{(p-1)/2} \frac{H_{k-1}}{k}
\]

This, together with \([2.2]\), Lemma \([2.6]\), \([3.3]\), \([3.7]\) and \([10, (1.1)]\), yields the desired result. \(\square\)

Lemma 3.6. Let \( p > 3 \) be a prime. Then

\[
\sum_{k=1}^{(p-1)/2} \frac{(p/2 - k)H_kH_{2k}}{-(p + 1 - 2k)(p + 2k)} \equiv 2q_p(2) - \frac{5}{2}q_p(2)^2 + \frac{1}{2}q_p(2)^3 + \frac{5}{16}B_{p-3} \pmod{p}.
\]
**Proof.** By [10, Lemma 2.4, (3.12)], [3:10], and Lemmas 2.5, 2.6 and 2.8, we have

\[
\frac{(p-1)}{2} \sum_{k=1} \frac{H_{2k}H_{2k-2}}{k} = \frac{(p-1)}{2} \sum_{k=1} \frac{H_{2k}^2}{k} - \frac{(p-1)}{2} \sum_{k=1} \frac{H_{2k}}{k(2k-1)} - \frac{1}{2} \frac{(p-1)}{2} \sum_{k=1} \frac{H_{2k}}{k^2}
\]

\[
\equiv -4q_p(2) + 2q_p(2)^2 - \frac{2}{3}q_p(2)^3 - \frac{1}{12}B_{p-3} \pmod{p}.
\] (3.12)

In view of [9, Lemma 3.2], [10, Theorem 1.3] and (3.7), we have

\[
\frac{(p-1)}{2} \sum_{k=1} \frac{H_kH_{2k-2}}{k} = \frac{(p-1)}{2} \sum_{k=1} \frac{H_{2k}H_k}{k} - \frac{(p-1)}{2} \sum_{k=1} \frac{H_k}{k(2k-1)} - \frac{1}{2} \frac{(p-1)}{2} \sum_{k=1} \frac{H_k}{k^2}
\]

\[
\equiv -8q_p(2) + 6q_p(2)^2 - \frac{4}{3}q_p(2)^3 + \frac{13}{12}B_{p-3} \pmod{p}.
\] (3.13)

It is easy to see that

\[
\frac{(p-1)}{2} \sum_{k=1} \frac{(p/2 - k)H_kH_{2k}}{(p + 1 - 2k)(p + 2k)}
\]

\[
\equiv \frac{1}{2} \frac{(p-1)}{2} \sum_{k=1} \frac{H_{2k-2}}{k(2k-1)} - \frac{1}{4}H_{(p-1)/2} \sum_{k=1} \frac{H_{2k-2}}{k} - \frac{1}{2} \frac{(p-1)}{2} \sum_{k=1} \frac{H_{2k-2}}{k} - \frac{1}{4} \sum_{k=1} \frac{H_kH_{2k-2}}{k} - \frac{1}{4} \sum_{k=1} \frac{H_kH_{2k-2}}{k}.
\]

Combining this with (3.12), (3.13), (3.10), (3.5), (3.3) and Lemma 2.6, we immediately get the desired result.

\[\square\]

**Lemma 3.7.** For any prime \( p > 3 \), we have

\[
\frac{(p-1)}{2} \sum_{k=1} \frac{(p/2 - k)H_k^2}{(p + 1 - 2k)(p + 2k)} \equiv q_p(2) - q_p(2)^2 + \frac{1}{6}q_p(2)^3 + \frac{1}{3}B_{p-3} \pmod{p}.
\]

**Proof.** Replacing \( k \) by \( (p + 1)/2 - j \) in (3.3), we have

\[
\frac{(p-1)}{2} \sum_{j=1} \frac{1}{(2j-1)j^2} \equiv 8q_p(2) \pmod{p},
\]

and in view of [10, Lemma 2.4, (3.12)] and (ii), we can deduce that

\[
\frac{(p-1)}{2} \sum_{k=1} \frac{H_{2k-2}}{k^2} \equiv -8q_p(2) + \frac{5}{2}B_{p-3} \pmod{p}.
\]
This, together with \((3.10)\) and \((3.12)\), yields that
\[
\sum_{k=1}^{(p-1)/2} \frac{(p/2 - k)H_{2k}^2}{(p + 1 - 2k)(p + 2k)} \equiv -\frac{1}{4} \left( \sum_{k=1}^{(p-1)/2} \frac{H_{2k}H_{2k-2}}{k} - \sum_{k=1}^{(p-1)/2} \frac{H_{2k-2}}{k(2k-1)} - \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{H_{2k-2}}{k^2} \right)
\equiv q_p(2) - \frac{1}{6}q_p(2)^3 + \frac{1}{3}B_{p-3} \pmod{p}.
\]

This ends the proof. \(\square\)

**Lemma 3.8.** Let \(p > 3\) be a prime. Then
\[
\sum_{k=1}^{(p-1)/2} \frac{(p/2 - k)H(2; 2k)}{(p + 1 - 2k)(p + 2k)} \equiv q_p(2) - \frac{3}{16}B_{p-3} \pmod{p},
\]
\[
\sum_{k=1}^{(p-1)/2} \frac{(p/2 - k)H(2; k)}{(p + 1 - 2k)(p + 2k)} \equiv 4q_p(2) - \frac{7}{8}B_{p-3} \pmod{p}.
\]

**Proof.** In view of [11, (2.3)], \((3.4)\) and Lemma 2.6 we have
\[
\sum_{k=1}^{(p-1)/2} \frac{(p/2 - k)H(2; 2k)}{(p + 1 - 2k)(p + 2k)} \equiv \frac{1}{4} \left( \sum_{k=1}^{(p-1)/2} \frac{H(2; 2k)}{k} - \sum_{k=1}^{(p-1)/2} \frac{1}{k(2k-1)^2} - \frac{1}{4} \sum_{k=1}^{(p-1)/2} \frac{1}{k^3} \right)
\equiv q_p(2) - \frac{3}{16}B_{p-3} \pmod{p}.
\]

It is easy to see that
\[
\sum_{k=1}^{(p-1)/2} \frac{(p/2 - k)H(2; k)}{(p + 1 - 2k)(p + 2k)} \equiv \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{H(2; 2k; p + 1/2 - k)}{p - 2k}
\equiv \sum_{k=1}^{(p-1)/2} \frac{H(2; 2k - 2)}{k} - \frac{1}{4} \sum_{k=1}^{(p-1)/2} \frac{H(2; 2k - 2)}{k}.
\]

Observe that
\[
\sum_{k=1}^{(p-1)/2} \frac{H(2; 2k - 2)}{k} = \sum_{k=1}^{(p-1)/2} \frac{H(2; 2k)}{k} - \sum_{k=1}^{(p-1)/2} \frac{1}{k(2k-1)^2} - \frac{1}{4} \sum_{k=1}^{(p-1)/2} \frac{1}{k^3}
\equiv 4q_p(2) - \frac{3}{4}B_{p-3} \pmod{p}.
and
\[ \sum_{k=1}^{(p-1)/2} \frac{H(2; k - 1)}{k} = \sum_{k=1}^{(p-1)/2} \frac{H(2; k)}{k} - \sum_{k=1}^{(p-1)/2} \frac{1}{k^3} \equiv \frac{1}{2} B_{p-3} \pmod{p}. \]

So
\[ \sum_{k=1}^{(p-1)/2} \frac{(p/2 - k)H(2; k)}{(p + 1 - 2k)(p + 2k)} \equiv 4q_p(2) - \frac{7}{8} B_{p-3} \pmod{p}. \]

Therefore the proof of Lemma 3.8 is complete.

Lemma 3.9. For any primes \( p > 3 \), we have
\[ \sum_{k=1}^{(p-1)/2} G\left( \frac{p+1}{2}, k \right) \equiv (-1)^{\frac{p-1}{2}} p^2 \left( q_p(2) - pq_p(2)^2 + p^2 q_p(2)^3 + \frac{7}{8} p^3 B_{p-3} \right) \pmod{p^5}. \]

Proof. For any complex number \( a \), let \( (a)_0 = 1 \) and \( (a)_n = a(a + 1) \ldots (a + n - 1) \) for \( n \in \mathbb{Z}^+ \). By the definition of \( G(n, k) \), we have
\[ G(n, k) = \frac{n^2 \binom{2n}{n} \binom{2n+2k}{n+k}}{2^{8n-4-2k}(2n + 2k - 1)\binom{2k}{k}} = \frac{n^2 \binom{2n}{n} \left( \frac{1}{2} \right)_{n+k} \left( \frac{1}{2} \right)_{n-k} \binom{n}{k}}{2^{4n-4-2k}n!2(2n + 2k - 1)\binom{2k}{k}} \]
\[ = \frac{n^2 \binom{2n}{n} \left( \frac{1}{2} \right)_{n+k} \left( \frac{1}{2} \right)_{n-k} \binom{n}{k}}{2^{4n-4-2k}n!2(2n + 2k - 1)\binom{2k}{k}} = \frac{n^2 \binom{2n}{n} \left( \frac{1}{2} \right)_{n+k} \left( \frac{1}{2} \right)_{n-k} \binom{n}{k}}{2^{8n-6-2k}n!2(2n + 2k - 1)\binom{2k}{k}}, \] (3.14)

where we have used the equalities
\[ \frac{\left( \frac{1}{2} \right)_{n+k}}{(n+k)!} = \frac{\binom{2n+2k}{n+k}}{4^n k}, \]
\[ \left( \frac{1}{2} \right)_{n-k} = \frac{\left( \frac{1}{2} \right)_{n} \left( \frac{1}{2} + n \right)_{k}}{2^k} \]
and
\[ \left( \frac{1}{2} \right)_{n-k} \left( \frac{1}{2} + n - k \right)_{k-1} = \frac{\left( \frac{1}{2} \right)_{n-1}}{2^{k-1}} (1 \leq k \leq n). \]

It is easy to check that
\[ \frac{(p+1)}{(p-k)} \equiv \frac{k! \left( 1 + \frac{p}{2} H_k + \frac{p^2}{4} \sum_{1 \leq i < j \leq k} \frac{1}{ij} \right)}{(-1)^{k}! \left( 1 - \frac{p}{2} H_k + \frac{p^2}{4} \sum_{1 \leq i < j \leq k} \frac{1}{ij} \right)} \equiv (-1)^{k} \left( 1 + pH_k + \frac{p^2}{2} H^2_k \right) \pmod{p^3}. \]
In view of $[17, (4.4)]$, we have the following congruence modulo $p^3$

$$
\frac{(p-1)^2}{2k} (4)^k \equiv 1 - p \sum_{j=1}^{k} \frac{1}{2j-1} + \frac{p^2}{2} \left( \sum_{j=1}^{k} \frac{1}{2j-1} \right)^2 - \sum_{j=1}^{k} \frac{1}{(2j-1)^2} \mod p^3
$$

$$
= 1 - p \left( H_{2k} - \frac{1}{2} H_k \right) + \frac{p^2}{2} \left( H_{2k} - \frac{1}{2} H_k \right)^2 - H(2; 2k) + \frac{1}{4} H(2; k).
$$

By $[3,14]$, we have the following congruence modulo $p^5$

$$
\frac{(p-1)^2}{2k} \frac{(p+1)^2 (p+1)^2 q}{2^4 p - 1} \sum_{k=1}^{\frac{(p-1)/2}{2}} \frac{(p/2 - k)}{(p+1 - 2k)(p+2k)} 
\cdot \left( 1 + \frac{3p}{2} H_k - p H_{2k} + \frac{9p^2}{8} H_k^2 - \frac{3p^2}{2} H_k H_{2k} + \frac{p^2}{2} H_{2k}^2 - \frac{p^2}{2} \left( H(2; 2k) - \frac{H(2; k)}{4} \right) \right).
$$

In view of Lemmas $3.2, 3.8$ and Lemma $2.3$, we have the following congruence modulo $p^5$

$$
\frac{(p-1)^2}{2k} \frac{(p+1)^2 (p+1)^2 q}{2^4 p - 1} \sum_{k=1}^{\frac{(p-1)/2}{2}} \frac{(p/2 - k)}{(p+1 - 2k)(p+2k)} 
\cdot \left( 1 + \frac{3p}{2} H_k - p H_{2k} + \frac{9p^2}{8} H_k^2 - \frac{3p^2}{2} H_k H_{2k} + \frac{p^2}{2} H_{2k}^2 - \frac{p^2}{2} \left( H(2; 2k) - \frac{H(2; k)}{4} \right) \right).
$$

Then we obtain the desired result by noting that $4^{p-1} = 1 + 2 p q_p(2) + p^2 q_p(2)^2$.

**Proof of Theorem 1.1.** Substituting Lemmas $3.1$ and $3.9$ into $3.2$, we immediately get that

$$
\sum_{n=0}^{(p-1)/2} F(n, 0) \equiv p(-1)^{(p-1)/2} + (-1)^{(p-1)/2} \frac{7}{24} p^4 B_{p-3} \mod p^5,
$$

which is equivalent to our desired result.

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