Recognition Principle for Generalized Eilenberg–Mac Lane Spaces

Bernard Badzioch

Abstract. We give a homotopy theoretical characterization of generalized Eilenberg–Mac Lane spaces which resembles the Γ-space structure used by Segal to describe infinite loop spaces.

1. Introduction

A generalized Eilenberg–Mac Lane space (GEM) is a space weakly equivalent to a product of Eilenberg–Mac Lane spaces \( \prod_{i=1}^{\infty} K(\pi_i, i) \) with \( \pi_i \) abelian. The goal of this note is to prove the following characterization of GEMs:

**Theorem 1.1.** Let \( R \) be a commutative ring with a unit and let \( F \text{Mod}_R \) be the category of finitely generated free \( R \)-modules. If \( H: F \text{Mod}_R \to \text{Spaces} \) is a functor such that

- \( H(0) \) is a contractible space,
- \( H(R) \) is connected and for every \( n \) the projections \( \text{pr}_k: R^n \to R \) induce a weak equivalence \( H(R^n) \cong H(R) \)

then the space \( H(R) \) is weakly equivalent to a product \( \prod_{i=1}^{\infty} K(M_i, i) \) where \( M_i \) is an \( R \)-module.

**Notation 1.2.** By \( \text{Spaces} \) above and in the rest of this paper we denote the category of simplicial sets. Consequently, by ‘space’ we always mean an object of this category.

The above description of GEMs is modeled after the Γ-space structure introduced by Segal in [Se] to characterize infinite loop spaces. Just as for infinite loop spaces one gets the following corollary which is implicitly present in the work of Bousfield [Bo] and Dror [D1] who apply it to localization functors.

**Corollary 1.3.** If \( F: \text{Spaces} \to \text{Spaces} \) is a functor preserving weak equivalences and preserving products up to weak equivalence then \( F \) preserves GEMs.

The rest of the paper is organized as follows. In section 2 the Grothendieck construction on a diagram of small categories is recalled and some of its properties are stated. In section 3 we give a description suitable for our purposes of \( \text{Sp}^{\infty} X \).
the infinite symmetric product on a space \( X \). In section 4 the proof of theorem 1.1 is presented. It essentially amounts to showing that for a functor \( H : \text{FMod}_R \to \text{Spaces} \) satisfying the assumptions of the theorem the space \( H(R) \) is a homotopy retract of \( \text{Sp}^{\infty} H(R) \). Since \( \text{Sp}^{\infty} H(R) \) is a GEM and the class of GEMs is closed under homotopy retraction the claim of the theorem will follow.

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2. The Grothendieck construction

**Definition 2.1.** Let \( \text{Cat} \) denote the category of small categories and let \( \mathbf{N} \) be the telescope category:

\[
\mathbf{N} := (0 \to 1 \to \cdots \to n \to \cdots)
\]

For a functor \( P : \mathbf{N} \to \text{Cat} \) the Grothendieck construction on \( P \) [Th] is the category \( \text{Gr}(P) \) whose objects are pairs \( (n, c) \) where \( n \in \mathbf{N} \) and \( c \in P(n) \). A morphism \( (n, c) \to (n', c') \) is a pair \( (i, \varphi) \) where \( i \) is the unique morphism \( n \to n' \) in \( \mathbf{N} \) and \( \varphi \in \text{Mor}_{P(n)}(P(i)c, c') \). The composition of \( (i, \varphi) : (n, c) \to (n', c') \) and \( (i', \varphi') : (n', c') \to (n'', c'') \) is defined to be the pair \( (i' \circ i, \varphi' \circ P(i') \varphi) : (n, c) \to (n'', c'') \).

For every \( n \in \mathbf{N} \) there is a functor

\[
I_n : P(n) \to \text{Gr}(P) \quad I_n(c) := (n, c)
\]

which lets us identify \( P(n) \) with a subcategory of \( \text{Gr}(P) \). It follows that any functor \( F : \text{Gr}(P) \to \mathbf{C} \) defines a sequence of functors \( F_n : P(n) \to \mathbf{C} \), \( n = 0, 1, \ldots \). Moreover, for \( n \in \mathbf{N} \) and \( c \in P(n) \) let \( \beta_{n,c} \) be the image under \( F \) of the morphism \( (i_n, \text{id}_{P(i_n)(c)}) \in \text{Mor}_{\text{Gr}(P)}((n, c), (n+1, P(i_n)(c))) \). It is easy to check that the morphisms \( \{\beta_{n,c} : c \in P(n)\} \) define a natural transformation of functors

\[
\beta_n : F_n \to F_{n+1} \circ P(i_n)
\]

The converse is also true [ChS, A.9]: any sequence of functors \( \{F_n : P(n) \to \mathbf{C}\}_{n \geq 0} \) and natural transformations \( \{\beta_n : F_n \to F_{n+1} \circ P(i_n)\}_{n \geq 0} \) can be used to define a functor \( F : \text{Gr}(P) \to \mathbf{C} \), such that \( F|_{P(n)} = F_n \).

**Proposition 2.2.** For any functor \( F : \text{Gr}(P) \to \mathbf{C} \) the natural morphism

\[
\text{colim}_{\mathbf{N}} \text{colim}_{P(n)} F_n \to \text{colim}_{\text{Gr}(P)} F
\]

is an isomorphism. Moreover, if \( \mathbf{C} = \text{Spaces} \) then the natural map

\[
\text{hocolim}_{\mathbf{N}} \text{hocolim}_{P(n)} F_n \to \text{hocolim}_{\text{Gr}(P)} F
\]

is a weak equivalence.
Proof. The first statement follows directly from the definition of $Gr(P)$. The proof of the second can be found in [Sl, Prop. 0.2] or [ChS, Cor. 24.6].

3. Infinite symmetric products

Let $\Sigma_n$ be the permutation group of the set $\{1, \ldots, n\}$. We will denote by $O_{\Sigma_n}$ the orbit category of $\Sigma_n$ whose objects are sets $\Sigma_n/G$ for $G \subseteq \Sigma_n$ and whose morphisms are $\Sigma_n$–equivariant maps $\Sigma_n/G \to \Sigma_n/H$. Let $O_{\Sigma_n}^{op}$ be the opposite category. We can identify $\Sigma_n$ with the subgroup of all elements of $\Sigma_{n+1}$ which leave the element $n+1$ fixed. The inclusion $\Sigma_n \subset \Sigma_{n+1}$ induces a functor

$$J_n : O_{\Sigma_n}^{op} \to O_{\Sigma_{n+1}}^{op}, \quad J_n(\Sigma_n/G) := \Sigma_{n+1}/G$$

This data in turn can be used to define a functor $O : \mathbb{N} \to \text{Cat}$:

$$O(n) := O_{\Sigma_n}^{op}, \quad O(i) := J_n$$

Let $\text{Spaces}_*$ denote the category of pointed spaces and let $X \in \text{Spaces}_*$. The group $\Sigma_n$ acts on $X^n$ by permuting the coordinates. As usual we have the fixed point functor

$$F_nX : O_{\Sigma_n}^{op} \to \text{Spaces}_*$$

defined by $F_nX(\Sigma_n/G) := (X^n)^G$ – the fixed point set of the action of $G$ on $X^n$.

**Remark 3.1.** For $G \subseteq \Sigma_n$ let $|\text{orb}G|$ denote the number of orbits of the action of $G$ on $\{1, \ldots, n\}$. Then there is a natural isomorphism $(X^n)^G \cong X^{\text{orb}G}$.

Using the embedding $\Sigma_n \subset \Sigma_{n+1}$ one can think of $G \subseteq \Sigma_n$ as a subgroup of $\Sigma_{n+1}$. There is an obvious isomorphism

$$(X^{n+1})^G \cong (X^n)^G \times X$$

and since $X$ is a pointed space, we have a map

$$\beta_{n,G} : (X^n)^G \to (X^{n+1})^G$$

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, \ast)$$

One can check that the morphisms $\{\beta_{n,G}\}_{G \subseteq \Sigma_n}$ give a natural transformation

$$\beta_n : F_nX \to F_{n+1}X \circ J_n$$

and from the remarks made in section 2 it follows that the functors $\{F_n\}_{n \geq 0}$ and natural transformations $\{\beta_n\}_{n \geq 0}$ can be assembled to define a functor

$$FX : Gr(O) \to \text{Spaces}_*$$

**Lemma 3.2.** $\text{hocolim}_{Gr(O)}FX \simeq Sp^\infty X$
Proof. By proposition 2.2 we have a weak equivalence
\[ \hocolim_{\Gr(O)} FX \simeq \hocolim_{\hocolim \O_n} F_n X \]
Moreover, by [D1, Ch.4, Lemma A.3]
\[ \hocolim_{\O_n} F_n X \simeq \colim_{\O_n} F_n X \]
(this follows from the fact that \( F_n X \) is a free diagram of spaces [D2, 2.7], and that for free diagrams homotopy colimits coincide with ordinary colimits). But
\[ \colim_{\O_n} F_n X \cong \Sp^n X \]
and so we have
\[ \hocolim_{\O_n} F_n X \simeq \hocolim_{\Sp^n} F_n X \simeq \colim_{\Sp^n} X \]
where the second equivalence is a consequence of [BK, Ch. XII, 3.5].
\[ \Box \]

4. The proof of theorem 1.1

Let \( H : \FMod_R \to \Spaces \) be a functor as in the theorem. One can assume that \( H \) takes its values in the category \( \Spaces_\ast \) of pointed spaces (if not, replace \( H \) with \( \tilde{H} \), \( \tilde{H}(M) = \cofib(H(0) \to H(M)) \) for \( M \in \FMod_R \)). Moreover, once we know that the theorem holds for \( R = \Z \), the ring of integers, the embedding \( \Z \to R \) will induce a functor \( \FMod_{\Z} \to \FMod_R \), and so the space \( H(R) \) will have the structure of a GEM. Furthermore the action of the ring \( R \) on its free module \( R \in \FMod_R \) via multiplications will induce an action of \( R \) on \( H(R) \) and so the homotopy groups \( \pi_i(H(R)) \) will be \( R \)-modules as claimed. Therefore for the rest of this paper we will assume that \( R = \Z \) and that \( H : \FMod_{\Z} \to \Spaces_\ast \).

For a free abelian group on \( n \) generators \( \Z^n \in \FMod_{\Z} \) the group \( \Sigma_n \) acts on \( \Z^n \) by permuting the set of generators. For \( G \subseteq \Sigma_n \) let \( (\Z^n)_G \) be the subgroup of all elements of \( \Z^n \) which are fixed by the action of \( G \).

Remark 4.1. It is not difficult to check that, using the notation of 3.1, there is a natural isomorphism of groups \( (\Z^n)_G \cong \Z_{|\orb(G)} \).

For any \( n \in \N \) we have a functor
\[ F_n Z : \O_{\Sigma_n} \to \FMod_{\Z} \]
\[ F_n Z(\Sigma_n/G) := (\Z^n)_G \]
Arguments similar to those in section 3 show that one can define a functor
\[ FZ : \Gr(O) \to \FMod_{\Z} \]
such that \( FZ|_{\O_n} = F_n Z \). Observe that \( \Sigma_1 \) is a trivial group and so \( FZ|_{\O_1} = F_1 Z \) is the constant functor with value \( \Z \).

Lemma 4.2. Let \( \tilde{Z} : \Gr(O) \to \FMod_{\Z} \) be the constant functor with value \( \Z \). There exists a natural transformation
\[ \theta : FZ \to \tilde{Z} \]
such that \( \theta|_{\O_1} \) is an isomorphism.
Proof. For $\Sigma_n/G \in O_{\Sigma_n} \subset Gr(O)$ define

$$\theta_{\Sigma_n/G} : (Z^n)_G \rightarrow \mathbb{Z}$$

$$(k_1, k_2, \ldots, k_n) \mapsto \sum_i k_i$$

It is easy to check that these maps give the required transformation of functors.

Let $H : \text{FMod}_Z \rightarrow \text{Spaces}$ be a functor satisfying the conditions of theorem 1.1; that is the projections $Z^n \rightarrow Z$ induce weak equivalences $H(Z^n) \rightarrow H(Z)^n$.

Lemma 4.3. $\text{hocolim}_{Gr(O)} H \circ FZ \simeq \text{Sp}^\infty H(Z)$

Proof. It follows from 3.1 and 4.1 that for $G \subseteq \Sigma_n$ we have isomorphisms

$$H((Z^n)_G) \cong H(Z^{\text{orb}G})$$

and

$$(H(Z^n)_G \cong H(Z)^{\text{orb}G})$$

Their composition with the map $H(Z^{\text{orb}G}) \rightarrow H(Z)^{\text{orb}G}$ induced by projections $Z^{\text{orb}G} \rightarrow Z$ gives a map $\varphi_{n,G} : H((Z^n)_G) \rightarrow (H(Z)^n)_G$ which, in view of the properties of $H$, is a weak equivalence. Moreover, the maps $\{\varphi_{n,G}\}_{n \geq 0, G \subseteq \Sigma_n}$ define a natural transformation of functors

$$\varphi : H \circ FZ \rightarrow FH(Z)$$

Therefore we have a weak equivalence

$$\text{hocolim}_{Gr(O)} H \circ FZ \simeq \text{hocolim}_{Gr(O)} FH(Z)$$

But by lemma 3.2 $\text{hocolim}_{Gr(O)} FH(Z) \simeq \text{Sp}^\infty H(Z)$.

To conclude the proof of the theorem observe that the natural transformation $\theta$ from lemma 4.2 gives a transformation

$$H(\theta) : H \circ FZ \rightarrow H \circ \tilde{Z}$$

and so induces a map

$$\text{hocolim}_{Gr(O)} H \circ FZ \rightarrow \text{hocolim}_{Gr(O)} H \circ \tilde{Z} \rightarrow \text{colim}_{Gr(O)} H \circ \tilde{Z} \cong H(Z)$$

On the other hand the inclusion $O_{\Sigma_1}^{op} \subseteq Gr(O)$ gives a map

$$H(Z) \simeq \text{hocolim}_{O_{\Sigma_1}^{op}} H \circ F_1 Z \rightarrow \text{hocolim}_{Gr(O)} H \circ FZ$$

and since $\theta \mid_{O_{\Sigma_1}^{op}}$ is an isomorphism it is easy to see that the composition

$$H(Z) \rightarrow \text{hocolim}_{Gr(O)} H \circ FZ \rightarrow H(Z)$$

has to be a weak equivalence. But by 4.3 $\text{hocolim}_{Gr(O)} H \circ FZ \simeq \text{Sp}^\infty H(Z)$, and so $H(Z)$ must be a GEM as a homotopy retract of a GEM (see [D1, Ch.4, Thm B.2]).

Remark 4.4. The above proof remains valid if we replace $\text{FMod}_Z$ by the category of free, finitely generated abelian monoids.
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Department of Mathematics,
University Notre Dame,
Notre Dame, IN 46556
USA

E-mail address: badzioch.1@nd.edu