Nearest Neighbor Two-Point Correlation Function of the \(Z\)-Invariant Eight-Vertex Model

Michael Lashkevich\(^1\) and Yaroslav Pugai\(^1,2\)
\(^1\)L. D. Landau Institute for Theoretical Physics, 142432 Chernogolovka, Russia
\(^2\)Department of Mathematics, University of Melbourne, Parkville, Victoria 3052, Australia

The nearest neighbor two-point correlation function of the \(Z\)-invariant inhomogeneous eight-vertex model in the thermodynamic limit is computed using the free field representation.

May 1998

Recently, a free field construction for correlation functions of the (\(Z\)-invariant) eight-vertex model\(^{1,2}\) has been proposed\(^3\) within the algebraic approach to integrable models of statistical mechanics\(^1,4-7\). The free field representation provides explicit formulas for multipoint correlation functions on the infinite lattice. However, the resulting expressions given in terms of a certain series of multiple integrals turn out to be rather cumbersome. In this letter we give an explicit expression for the nearest neighbor two-point correlation function in terms of a single two-fold integral, and perform some checks. We also discuss the independence of the integral representations of the free parameter \(u_0\) of the vertex-face correspondence entering into the free fields construction.\(^3\)

Let us briefly recall the notations used in Ref. 3 (see Ref. 1 for a complete definition of the eight-vertex model). The fluctuating variables \(\varepsilon = \pm 1\) are situated at edges of the square lattice. To each configuration of variables \(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\) ordered around a vertex a local Boltzmann weight \(R_{\varepsilon_1\varepsilon_2}\) is associated, as it is shown in Fig. 1a. The nonzero Boltzmann weights can be parametrized as follows\(^1\):

\[
\begin{align*}
R(u)_{++} &= R(u)_{--} = a(u) = -i\rho(u) \theta_4 \left( \left( \frac{i\pi}{2} ; i\frac{2\pi}{\tau} \right) \theta_4 \left( \left( \frac{i\pi}{2} ; i\frac{2\pi}{\tau} \right) \theta_1 \left( \left( \frac{i\pi}{2} (1-u) ; i\frac{2\pi}{\tau} \right) \right) \right), \\
R(u)_{+-} &= R(u)_{-+} = b(u) = -i\rho(u) \theta_4 \left( \left( \frac{i\pi}{2} ; i\frac{2\pi}{\tau} \right) \theta_1 \left( \left( \frac{i\pi}{2} u ; i\frac{2\pi}{\tau} \right) \theta_4 \left( \left( \frac{i\pi}{2} (1-u) ; i\frac{2\pi}{\tau} \right) \right) \right), \\
R(u)_{++} &= R(u)_{--} = c(u) = -i\rho(u) \theta_1 \left( \left( \frac{i\pi}{2} ; i\frac{2\pi}{\tau} \right) \theta_4 \left( \left( \frac{i\pi}{2} u ; i\frac{2\pi}{\tau} \right) \theta_4 \left( \left( \frac{i\pi}{2} (1-u) ; i\frac{2\pi}{\tau} \right) \right) \right), \\
R(u)_{++} &= R(u)_{--} = d(u) = -i\rho(u) \theta_1 \left( \left( \frac{i\pi}{2} ; i\frac{2\pi}{\tau} \right) \theta_1 \left( \left( \frac{i\pi}{2} u ; i\frac{2\pi}{\tau} \right) \theta_1 \left( \left( \frac{i\pi}{2} (1-u) ; i\frac{2\pi}{\tau} \right) \right) \right),
\end{align*}
\]

where \(\theta_j(u;\tau)\) is the standard \(j\)th theta function with the basic periods 1 and \(\tau\) (\(\text{Im} \tau > 0\)). The normalization factor \(\rho(u)\) is irrelevant for correlation functions.

For definiteness, let us consider the model in the antiferroelectric phase restricting values of the parameters \(\varepsilon, r, u\) to be real numbers in the region \(\varepsilon > 0, r > 1, -1 < u < 1\). For fixed \(r\) the parameter \(\varepsilon\) measures deviation from the criticality. In the limit \(\varepsilon \to 0\) the model has a second order phase transition. In the ‘low temperature’ limit \(\varepsilon \to \infty\) the system falls into one of two ground states (Fig. 1b) indexed by \(i = 0, 1\).

Let \(P_{\varepsilon}^{(i)}\) be the probability in the thermodynamic limit that the spin in the ‘central’ edge is fixed to be \(\varepsilon\). The label \((i)\) indicates that spins at the edges situated ‘far away’ from the origin are the same as in the \(i\)th ground state so that in the low-temperature limit \(\varepsilon \to \infty\) the probability is nonvanishing for \(\varepsilon = (-)^i\). It has been shown in Ref. 3 that the one-point correlation function

\[
\theta_1^{(i)} = \sum_{\varepsilon} \varepsilon P_{\varepsilon}^{(i)}
\]
is recovered from the bosonization procedure. The resulting integral representation

\[
R(u - v)^{\varepsilon_1 \varepsilon_2}_{\varepsilon_3 \varepsilon_4} = v^{\varepsilon_2}_{\varepsilon_1} \frac{\varepsilon_3}{u} \frac{\varepsilon_4}{\varepsilon_1} - \frac{+}{-} \frac{+}{-} \frac{+}{-} \frac{+}{-} \frac{+}{-} \frac{+}{-} \frac{+}{-} \frac{+}{-}
\]

The integration contours can be reduced to the Baxter–Kelland formula for the spontaneous staggered polarization. Here we used the notations \( h_j(u) = \vartheta_j(u/r; i\pi/\epsilon) \) and \( \vartheta_j(u) = \vartheta_j(u; i\pi/\epsilon) \). The integration contour \( C_0 \) goes over the imaginary period of the theta functions (from some complex two adjacent rows are \( 0 < \varepsilon_1 < \varepsilon_2 < 1 \), while \( \varepsilon_1 = -\varepsilon_2 = (-)^i \) as \( \epsilon \to \infty \). The main statement of this letter is that the free field construction allows one to express the nearest neighbor two-point correlation function

\[
g_2^{(i)}(u_1 - u_2) = \sum_{\varepsilon_1 \varepsilon_2 = \pm} \varepsilon_1 \varepsilon_2 P^{(i)}_{\varepsilon_1 \varepsilon_2}(u_1, u_2)
\]

in terms of a two-fold contour integral as follows:

\[
g_2^{(i)}(u) = -2 \left( \frac{\vartheta_4'(0)}{\vartheta_4(0)} \right)^2 \vartheta_1(u) \vartheta_1(1) \int_{C_1} \int_{C_2} \frac{dv_1}{2\pi i} \frac{dv_2}{2\pi i} \frac{\vartheta_4(v_1 + v_2 - u)}{\vartheta_1(v_1 - u) \vartheta_1(v_2 - u)} \times
\]

\[
\times \vartheta_1(v_1 - v_2) \frac{\vartheta_4(v_1 - v_2 + 1)}{\vartheta_1(v_1 - v_2 + 1)} \prod_{j=1}^{2} \frac{1}{\vartheta_1(v_j) \vartheta_1(v_j)}
\]

The integration contours \( C_{1,2} \) go over the same imaginary period as \( C_0 \) so that \(-1 < \text{Re } v_1 < u < \text{Re } v_2 < 1\), \( \text{Re } v_2 - \text{Re } v_1 < 1 \).

Briefly, to get this formula we applied the standard procedure of computing traces of bosonic operators over Fock spaces and then proceeded as in the one-point function case (see Appendix D of Ref. 3) by
applying various identities for theta functions to provide summation over the SOS variables including infinite summation over Fock spaces. We will not go into technical details of this procedure since they are more cumbersome than instructive.

Let us only make a remark on the free parameter $u_0$ in the free field representation. Although the explicit formulas for correlation functions in terms of traces of bosonic fields do contain $u$ and checked numerically that it does not depend on $u_0$ to make sure, we also obtained the formula for $g_2^{(i)}$ with general $u_0$ (which turns out to be more cumbersome) and checked numerically that it does not depend on $u_0$.

To support the validity of the integral representation (3) we compared it with the known results.

- The partition function differentiation method.\(^{2a}\) The probabilities $P_{\epsilon, r}^{(i)}(u_1, u_2)$ are equal to those shown in Fig. 2b because of the Z-invariance. So the correlation function $g_2^{(i)}(u)$ can be calculated as

$$g_2^{(i)}(u) = \left. \left( a \frac{\partial}{\partial a} - b \frac{\partial}{\partial b} - c \frac{\partial}{\partial c} + d \frac{\partial}{\partial d} \right) \log \kappa(a, b, c, d) \right|_{u, c, r}.$$  \(4\)

Here $\kappa$ is the partition function per site as a function of the Boltzmann weights,\(^1\)

$$\log \kappa(a, b, c, d) = \log(c + d) + \sum_{m=1}^{\infty} \frac{(x^{3m} - p^{m/2})(1 - p^{m/2}z^{-m})(z^m + x^{-m} - z^{-m})}{m(1 - p^m)(1 + z^{2m})}$$

with $x = e^{-\epsilon}$, $p = x^{2r}$, $z = e^{\epsilon(1-2u)}$, and the derivatives are taken at the point characterized by the parameters $u$, $\epsilon$, $r$ according to Eq. (1).

We checked numerically that the results of (3) and (4) coincide at least up to the fifth decimal digit in a wide region of values of $\epsilon$ and $r$.

- The limiting case $r \rightarrow \infty$. In this limit the Boltzmann weights (1) become those of the six-vertex model in the antiferroelectric regime. The integral representation for correlation functions of the antiferroelectric six-vertex model is known.\(^{9,5}\) Analytically the formula (3) gives another integral representation in this limit, but numerically the results of integration according to both formulas coincide up to the sixth decimal digit.

- The limiting case $\epsilon \rightarrow 0$. This is the critical region of the eight-vertex model. Using the Baxter duality transformation for the Boltzmann weights\(^1\) one can map the model in this region onto the six-vertex model in the gapless regime whose correlation functions were found in Ref. 11 (see also Ref. 12). Performing the duality transform at the level of correlation functions\(^{10}\) one has to compare our answer with the following correlation function

$$g_2^{JM}(\beta_1 - \beta_2) \equiv -2 < E_{-+}^{(1)} E_{+-}^{(2)} > (\beta_1, \beta_2)$$

in the notations of Ref. 11 with identification $\nu = 1/r$, $\beta_j = i\pi u_j$. The integral representation for this quantity can be written as follows\(^{11}\)

$$g_2^{JM}(\beta) = -2 \frac{\text{sh} \beta}{\text{sh} \nu \beta} \int_{C_-} \frac{dv_1}{2\pi i} \int_{C_+} \frac{dv_2}{2\pi i} \frac{1}{\text{sh}(v_1 - \beta) \text{sh}(v_2 - \beta)} \times$$

$$\times \frac{\text{sh}(v_1 - v_2)}{\text{sh} \nu (v_1 - v_2 + i\pi)} \prod_{j=1}^{2} \frac{\text{sh} \nu v_j}{\text{sh} v_j}.$$  \(5\)

Here the contours $C_{\pm}$ are from $-\infty$ to $+\infty$. They are chosen in such a way that $\beta + i\pi$ (resp. $\beta$) is above (resp. below) $C_-$ and $\beta$ (resp. $\beta - i\pi$) is above (resp. below) $C_-$. By checking it directly one shows that the limit $\epsilon \rightarrow 0$ of Eq. (3) coincides with Eq. (5).

\(^a\) We would like to thank Prof. J. H. H. Perk for pointing to us the possibility of this check.
• The \( r = 2 \) case. Under such specification the eight-vertex model is equivalent to a two of non-interacting Ising models. In this case \(-g_2^{(r)}\) coincides with the nearest neighbor diagonal correlator of the inhomogeneous (\(Z\)-invariant) Ising model in the ferromagnetic regime\(^{13}\) (see also Refs. 2, 14, 15),

\[
-g_2^{(r)}(u) = \langle \sigma_{m,n}\sigma_{m+1,n+1} \rangle = \frac{1}{\pi} \frac{\theta_4(0; i\frac{2\pi}{r}) \theta'_2(i\frac{2\pi}{r}; i\frac{2\pi}{r})}{\theta_3(0; i\frac{2\pi}{r}) \theta_1(i\frac{2\pi}{r}; i\frac{2\pi}{r})}.
\]

where \( \sigma_{m,n} \) is the spin variable at the site \((m, n)\) of the square lattice. As we show in the Appendix, Eq. (3) reduces to this formula at \( r = 2 \).

We hope that a similar integral representation can be obtained starting from the free field representation for other multipoint correlation functions, in particular, for two-point functions with separation of 2 lattice sites or more. It would be also very interesting to understand whether the elliptic algebra approach proposed in Refs. 16 would lead to another bosonization prescription and give a direct procedure of obtaining the integral representations of the correlation functions of the eight-vertex model.

Acknowledgments

We are very grateful to M. Jimbo and T. Miwa for their kind hospitality and support in Kyoto University and RIMS where a part of this work has been done. We would like to thank M. Jimbo, K. Hasegawa, H. Konno, T. Miwa, S. Odake, J. Shiraishi and all participants of the seminar “Elliptic algebras and bosonization” in Kyoto for useful discussions. We are indebted to R. J. Baxter, B. Davies, O. Foda, S. Lukyanov, and J. H. H. Perk for their interest to the work and valuable remarks improving the manuscript. Ya. P. is very grateful to R.J. Baxter for his kind hospitality in ANU and to O. Foda for constant attention and support. The work was supported, in part, by CRDF under the grant RP1–277 and by INTAS and RFBR under the grant INTAS–RFBR–95–0690, and by RFBR under the grants 96–15–96821 and 96–02–16507. Ya. P. was also supported by the Australian Research Council.

Appendix

Let us obtain (6) from (3) in the Ising model case \( r = 2 \). The integrand of (3) is antisymmetric with respect to the permutation of \( v_1 \) and \( v_2 \). This allows one to perform the second integration simply by taking the residues at the pole \( v_2 = u \). Using the identity

\[
\frac{\vartheta_4(u) \vartheta_1(u)}{\vartheta_4(0) \vartheta_1(0)} = \frac{\vartheta_4(0) \vartheta'_1(0 \vartheta_1(0) \vartheta_4(0) \vartheta_4(1) \vartheta_1(u + 1))}{\vartheta'_1(0) \vartheta_4(1) \vartheta_1(u + 1))}
\]

which is valid for \( r = 2 \), the resulting expression can be represented in the following form

\[
g_2^{(r)}(u) = 2 \frac{\vartheta_4(0) \vartheta_1(1)}{\vartheta_4(0) \vartheta_4(1)} \frac{\vartheta_1(0)}{\vartheta_1(0 + 1)} \frac{\vartheta_1(u)}{\vartheta_1(u + 1)}
\]

The function \( J(u) \) satisfies the following determining properties

\[
J(u + \frac{1}{2} \frac{\pi}{\epsilon}) = J(u), \quad J(-u) = J(u), \quad J\left(\frac{i\pi}{2\epsilon}\right) = \frac{1}{2\epsilon},
\]

\[
J(u + 2) = -J(u) + \frac{h_4(u)h_4(0)}{h_1(u)h_1(0)},
\]

\( J(u) \) is regular on the strip \(-2 < \text{Re} \, u < 2\),

which fix it completely to be

\[
J(u) = \frac{1}{2\epsilon} \frac{\vartheta_2(0; i\frac{2\pi}{r}) \vartheta'_2(i\frac{2\pi}{r}; i\frac{2\pi}{r})}{\vartheta'_1(0; i\frac{2\pi}{r}) \vartheta_1(i\frac{2\pi}{r}; i\frac{2\pi}{r})}
\]

Passing to the conjugate module in the coefficient at \( J(u) \) in Eq. (A.1) one gets (6).
References

1. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, 1982
2. R. J. Baxter, *Phil. Trans. Royal Soc. London* **289**, 315 (1978)
3. M. Lashkevich and Ya. Pugai, Free field construction for correlation functions of the eight vertex model, *Nucl. Phys.* **B516**, 623 (1998) [hep-th/9710099]
4. B. Davies, O. Foda, M. Jimbo, T. Miwa, and A. Nakayashiki, *Commun. Math. Phys.* **151**, 89 (1993)
5. M. Jimbo and T. Miwa, *Algebraic Analysis of Solvable Lattice Models*, CBMS Regional Conference Series in Mathematics, **85**, AMS, 1994
6. M. Jimbo, T. Miwa, and A. Nakayashiki, *J. Phys.* **A26** 2199 (1993)
7. S. Lukyanov and Ya. Pugai, *Nucl. Phys.* **B[FS]473**, 631 (1996) [hep-th/9602074]
8. R. J. Baxter and S. B. Kelland, *J. Phys.* **C7**, L403 (1974)
9. M. Jimbo, K. Miki, T. Miwa, and A. Nakayashiki, *Phys. Lett.* **A168**, 256 (1992)
10. M. Yu. Lashkevich, *Mod. Phys. Lett.* **B10**, 101 (1996) [hep-th/9408131]
11. M. Jimbo, T. Miwa, *J. Phys.* **A29** 2923 (1996) [hep-th/9601113] [hep-th/9311189]
12. S. Lukyanov, *Commun. Math. Phys.* **167**, 183 (1995) [hep-th/9307196]; *Phys. Lett.* **B325**, 409 (1994) [hep-th/9307196]
13. R. J. Baxter, I. G. Enting *J. Phys.* **A11**, 2463 (1978)
14. O. Foda, M. Jimbo, T. Miwa, K. Miki, and A. Nakayashiki, *J. Math. Phys.* **35**, 13 (1994)
15. J. R. Reyes Martinez, *Phys. Lett.* **A227** 203 (1997) [hep-th/9609135]
16. M. Jimbo, H. Konno, S. Odake, J. Shiraishi, Quasi-Hopf twistors for elliptic quantum groups, [q-alg/9712029](December 1997); M. Jimbo, H. Konno, S. Odake, J. Shiraishi, Elliptic algebra \( U_{q,p}(\hat{sl}_2)\): Drinfeld currents and vertex operators, [q-alg/9802002](February 1998)