A Generalized Forced Oscillation Method for Tuning Proportional-Resonant Controllers

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Abstract—This brief proposes a tuning methodology for proportional-resonant (PR) controllers by using the design philosophy of the Ziegler–Nichols forced oscillation method. Unlike such related methods that are usual for proportional–integral–differential design, and those that have been recently proposed for PR controllers, the method in this brief is not restricted to plants whose Nyquist diagram crosses the negative real axis. It involves a feedback experiment with an adjustable phase relay, which allows the identification of the most appropriate point of the frequency response for each class of plants. The proposed method, which also includes tuning formulas that provide appropriate stability margins for each class of plants, is analyzed in detail for three examples, showing its wide applicability.

Index Terms—Frequency-domain controller design, process control, proportional-resonant (PR) controller, relay with adjustable phase (RAP) experiment, sinusoid tracking and rejection, Ziegler–Nichols (ZN) methods.

I. INTRODUCTION

The internal model principle [1] establishes that to asymptotically track a given reference, or to asymptotically reject a given disturbance, assuming a stable closed loop, the controller must have the nonvanishing modes of the reference and disturbance inputs. When these inputs are sinusoidal signals with frequency \( \omega_r \), the internal mode is formed by a pair of poles at \( \pm j\omega_r \). This characteristic leads to a controller with resonance frequency at \( \omega_r \), hence the denomination of the resonant controller. This kind of controller is widely applied in dc–ac inverters [2]–[5], high-precision positioning systems [6], and vibration control in flexible structures [7]. In some of these applications, the so-called high-Q resonant controllers are used, in which the controller poles are shifted to the left half of the complex plane, improving robustness at the cost of losing perfect tracking and rejection.

In order to contribute to the applicability and enlarge the dissemination of resonant controllers, whose parameters are typically tuned considering the knowledge of the plant model, a tuning method of a resonant structure for plants that have an ultimate frequency (that is, whose Nyquist plot crosses the negative real axis) has been proposed in [8]. This method can be implemented experimentally in a straightforward manner through the same standard relay feedback experiment [9] that is commonly applied to proportional–integral–differential (PID) controllers. Then, the parameters of the resonant structure can be calculated from simple tuning formulas, similar to the Ziegler–Nichols (ZN)-like methods.

On the other hand, a PID tuning method based on a modified relay feedback experiment has been proposed in [10] for plants with relative degree larger than one. Since it can be applied to plants that are not amenable to the application of the traditional ZN-like tuning methods—plants that have neither an ultimate frequency nor an S-like reaction curve—the class of plants for which ZN-like methods can be applied has been enlarged.

In this brief, a tuning method for proportional-resonant (PR) controllers is proposed that can be applied to quite general linear time-invariant plants, regardless of the existence of an ultimate frequency, of order or of relative degree. This method is based on the modified relay feedback experiment presented in [10], which enables the identification of the most appropriate point of the plant’s frequency response in a single experiment. Then, tuning formulas for the PR controller are developed to place this point of the loop frequency response at a specified location in the complex plane, chosen to provide appropriate stability margins and closed-loop performance. A detailed analysis of the proposed method is performed in different plants, which demonstrate its applicability to a large amount of plants with different characteristics. Thus, this method provides an alternative to tune PR controllers without the need of the plant model and with very little design effort.

This brief is organized as follows. In Section II, preliminary concepts are presented. The tuning method of the PR controller is proposed in Section III, where the PR tuning formulas are developed based on the knowledge of the controller resonance frequency and a particular point of the plant’s frequency response. In Section IV, the modified relay feedback experiment that allows obtaining this information in a single experiment is presented. The applicability of the proposed tuning methodology is analyzed with three different plants in Section V.

II. PRELIMINARIES

A. Plants

In this brief, linear time-invariant causal (LTIC) plants are considered, which are represented by \( Y(s) = G(s)U(s) \), where \( U(s) \) and \( Y(s) \) are the Laplace transforms of the input and the plant’s output (the controlled variable), respectively. \( G(s) \) is the plant transfer function, which is assumed to be...
strictly proper and bounded-input, bounded-output (BIBO)-
stable. The plant is controlled with unitary feedback by an
LTIC controller, that is, \( E(s) = R(s) - Y(s) \) with \( U(s) =
C(s)E(s) \), where \( R(s) \) is the reference, \( E(s) \) is the tracking
error, and \( C(s) \) is the controller transfer function, which is
assumed to be proper.

B. PR Controller

The PR controllers considered in this brief have the form
\[
C_{pr}(s) = K_p + \frac{K_{r1}s + K_{r2}}{s^2 + 2\xi\omega_n s + \omega_n^2}
\]
where \( \omega_n \) is the frequency that must be tracked and/or rejected,
\( \xi \) is the damping coefficient of the poles, and \( K_p, K_{r1}, K_{r2} \in \mathbb{R} \)
are the gains to be tuned.

A stable closed-loop system asymptotically tracks/rejects a
given sinusoidal signal with frequency \( \omega_n \) if the internal mode
formed by a pair of poles at \( \pm j\omega_n \) is present in the loop,
providing infinite gain at this frequency; this is achieved by
the PR controller (1) with damping coefficient \( \xi = 0 \). In many
situations, positive values of \( \xi \) are preferred [5]–[7], which
improves robustness and makes the tuning task easier at the
cost of losing perfect tracking. In this brief, we develop a
tuning method for the general case, i.e., arbitrary \( \xi \geq 0 \).

To assess the control systems performance criteria are evaluated
in terms of the system’s response to a sinusoidal reference
signal considering the settling time \( t_s \) and the maximum
overshoot \( M_o \), as defined in [8]. The settling time is also
represented in the number of periods of the reference signal,
that is, \( n_s = \omega_n t_s/(2\pi) \). The maximum overshoot is obtained through
\[
M_o = \max \left\{ \frac{y_{max} - y_r}{y_r}, 0 \right\} \times 100
\]
where \( y_{max} = \max_{t < t_s} |y(t)| \) and \( y_r = \max_{t > t_s} |y(t)| \).

C. Tuning Methods Based on Forced Oscillation

There are many research studies dedicated to PID tuning
rules, and many of the methods that have been proposed and
successfully applied (an overview is given in [11]) constitute variations to the method proposed in [12], which in this brief
is referred to as the classical forced oscillation (CFO) method.
The CFO method is based on the knowledge of the ultimate
point of the plant’s frequency response—the point at which
its Nyquist plot crosses the negative real axis, corresponding
to the lowest frequency where its phase is \( -\pi \). This point is
characterized by the ultimate frequency \( \omega_u \) and the ultimate
gain \( K_u \), defined as
\[
\omega_u = \min_{\omega > 0} \omega : \mathcal{L}G(j\omega) = -\pi, \quad K_u = \frac{1}{|G(j\omega_u)|}
\]
The CFO method can then be summarized as follows.

1) Identify the ultimate point of the plant’s frequency
response, that is, determine \( \omega_u \) and \( K_u \).
2) Design the parameters of the controller such that
\( C(j\omega_u)G(j\omega_u) = p \), or equivalently
\[
C(j\omega_u) = -K_up
\]
where \( p \) is a prespecified location in the complex plane.

The first step of the CFO method is usually performed by
means of the relay feedback experiment, which consists in a
closed-loop experiment with the nonlinear control action:
\( u(t) = d \operatorname{sign}(e(t)) + b \), where \( \operatorname{sign}(\cdot) \) is the sign function
[ \( \operatorname{sign}(x) = 1 \) for positive \( x \) and \( \operatorname{sign}(x) = -1 \) for negative \( x \) ],
\( d \in \mathbb{R}^+ \) is a parameter to be chosen, and \( b \in \mathbb{R} \) is the bias.
Once a symmetric oscillation is obtained, its amplitude \( A_u \)
and period \( T_u \) are measured and the ultimate quantities are calculated from [9]
\[
K_u = \frac{4d}{\pi A_u} \quad \text{and} \quad \omega_u = \frac{2\pi}{T_u}
\]
The second step of the CFO method is fulfilled by solving (2) for the controller gains with a chosen location \( p \). Over
the years, different locations \( p \) have been proposed; the tuning
formulas presented in [12] correspond to \( p = -0.4 + j0.08 \)
for PI controllers and \( p = -0.6 - j0.28 \) for PID controllers.

Plants that have no ultimate point are not amenable to
the application of the CFO method. This is the case of all
minimum-phase stable first- and second-order plants and most
plants with relative degree smaller than three. To overcome
this limitation, a PID tuning method based on a modified
relay feedback experiment was presented in [10] that is
applicable to a large class of plants that have no ultimate
point.

On the other hand, these methods were classically limited to
the tuning of PID controllers. Regarding the tuning of resonant
controllers, a methodology based on the CFO method has been
proposed in [8], where tuning formulas were developed by
solving (2) for a given resonant structure. Thus, the controller
parameters can be obtained experimentally in a straightforward
manner through the relay feedback experiment and simple
tuning formulas. Still, the tuning method and resulting tuning
formulas were restricted to plants whose frequency response
possesses an ultimate point.

In Section III, we present a tuning method for PR con-
trollers, which is developed in the same spirit of the methods
mentioned previously and that can be applied to quite generic
plants, whether or not with an ultimate point.

III. Generalized Forced Oscillation Method

In the same spirit of the CFO and related methods, the first
step of the generalized forced oscillation (GFO) method consists in identifying the point of the plant’s frequency response
at which the phase reaches a previously specified value \( \nu \), that is,
determine the quantities \( \omega_v \) and \( M_v \)
\[
\omega_v = \min_{\omega \geq 0} \omega : \mathcal{L}G(j\omega) = \nu \quad \text{and} \quad M_v = |G(j\omega_v)|
\]
which can also be written as
\[
G(j\omega_v) = M_v \mathbb{L}\nu = M_v (\cos(\nu) + j \sin(\nu))\,.
\]
In Section IV, the modified relay feedback experiment that
yields \( \omega_v \) and \( M_v \) will be presented. For the moment assume
that these quantities have somehow become available. Then,
design the controller parameters such that
\[
C(j\omega_v)G(j\omega_v) = p = M_v(\cos(\rho) + j \sin(\rho))
\]
or equivalently
\[ C(j\omega_v) = \frac{M_p}{M_v} (\cos(\rho - \nu) + j \sin(\rho - \nu)) \] (5)
where \( p \) is a previously specified location in the complex plane. In what follows, PR tuning formulas will be proposed through the solution of (5) for the transfer function (1) with \( s = j\omega_v \). We restrict our study to the case where \( \omega_r < \omega_u \).

A. Defining \( \nu \) and \( p \)
To proceed with the design, it is necessary to define what point of the plant’s frequency response will be identified, which corresponds to defining which value of \( \nu \) to use. This must be a value such that the identified point is relevant and representative of the stability margins that will be obtained. Then, it remains to specify what would be a reasonable stability margin to achieve, which corresponds to defining the value of \( p \). Control textbooks suggest values of phase margin of at least 45° to provide appropriate robustness and dynamic performance for typical practical situations [13].

To make these definitions, PR controllers have been designed for a wide array of plants (we have taken into account 123 plants that have an ultimate point and 140 plants that do not, whose transfer functions are presented in the test batch of [12] and [10]), considering different identified points of the plant’s frequency response and also with different specifications of stability margins. Several values of \( \nu \) and \( p \), which result in different closed-loop performance and stability margins, have been tested. Then, the resulting closed-loop performance from the response to a sinusoidal reference signal has been evaluated. The best values of \( \nu \) and \( p \) for each class of plants are presented next.

For plants with an ultimate point, this is clearly the point that must be used, so the choice \( \nu = -180^\circ \) is self-evident for this class of plants—which for future convenience, we will call Class A. In this particular case, (5) can be rewritten as (2). Thus, for the plants in Class A, the ultimate point of the plant’s frequency response is identified, that is,
\[ v = -180^\circ, \ \omega_v = \omega_u = \min_{\omega \geq 0} \angle G(j\omega) = -180^\circ \]
\[ M_v = M_u = |G(j\omega_u)| = 1/K_u. \] (6)
As for the choice of the location to which the ultimate point should be shifted, parameter \( p \), we took the classical ZN point for PI tuning (\( p = -0.4 + j0.08 \)) as a first approximation. After numerous tests, we have found the best results in terms of stability margins and closed-loop performance with
\[ p = 0.4 \left( \cos(-183^\circ) + j \sin(-183^\circ) \right) \] (7)
for \( 0 < \omega_r/\omega_u < 0.5 \), and
\[ p = 0.4 \left( \cos(-181^\circ) + j \sin(-181^\circ) \right) \] (8)
for \( 0.5 \leq \omega_r/\omega_u < 1 \).

Consider now the plants that do not possess an ultimate point. In a previous work [10], it was found that \( v = -120^\circ \) was the best choice for the tuning of PID controllers for plants without an ultimate point. As for \( p \), it has been proposed in [10] to pick it such that a phase margin of 50° was achieved, which corresponds to \( p = 1^\circ - 130^\circ \). We have successfully tested these same choices here for the tuning of PR controllers, so this is what we propose, provided that the plant’s frequency response achieves this phase for some frequency. This set of plants—that is, those that do not possess an ultimate point but whose frequency response reaches \(-120^\circ\) for some frequency—will be called Class B in this brief. Thus, for plants in Class B, we propose to use
\[ v = -120^\circ, \ \omega_v = \omega_{120} = \min_{\omega \geq 0} \angle G(j\omega) = -120^\circ \]
\[ M_v = M_{120} = |G(j\omega_{120})| \] (9)
\[ p = 1^\circ - 130^\circ = \cos(-130^\circ) + j \sin(-130^\circ). \] (10)

With these definitions, the phase margin will be 50°, provided that the magnitude of the loop transfer function monotonically decreases for frequencies higher than \( \omega_{120} \).

Finally, consider those plants whose frequency response never reaches \(-120^\circ\), which will be called Class C in this brief. This class of plants is the least problematic regarding stability margins, since the loop frequency response of such a plant with a PR controller is very unlikely to ever cross the negative real axis. Therefore, the choice of the design parameters \( \nu \) and \( p \) is not expected to be critical in this case; still, the best choices must be made. After a frequency response analysis of the problem and a large number of tests, the best results in terms of stability margins and closed-loop performance were obtained with the following values, which are, thus, the ones recommended for Class C:
\[ v = -60^\circ, \ \omega_v = \omega_{60} = \min_{\omega \geq 0} \angle G(j\omega) = -60^\circ \]
\[ M_v = M_{60} = |G(j\omega_{60})| \] (11)
\[ p = 1^\circ - 90^\circ = \cos(-90^\circ) + j \sin(-90^\circ) \] (12)
which provide a phase margin of 90°, assuming that the loop transfer function is monotonically decreasing for frequencies higher than \( \omega_{60} \).

In the following, a set of generic tuning formulas is developed from the solution of (5) with the proposed PR structure. Then, particular sets of PR tuning formulas are proposed for each of the classes of plants previously defined.

B. PR Tuning
The PR tuning formulas are obtained by substituting (1) with \( s = j\omega_v \) into the tuning equation (5), which results in
\[ K_p + \frac{j\omega_v K_r + K_r}{\omega_r^2 - \omega_v^2 + j2\xi\omega_v \omega_r} = \frac{M_p}{M_v} (\cos(\rho - \nu) + j \sin(\rho - \nu)). \]
After simplifications, the last expression can be represented by its real and imaginary parts as the following equations:
\[ K_p (\omega_r^2 - \omega_v^2) + K_r \]
\[ = \frac{M_p}{M_v} (\omega_r^2 - \omega_v^2) \cos(\rho - \nu) - 2\omega_v \xi \omega_r \sin(\rho - \nu)) \] (13)
\[ \omega_v (K_r + 2 K_p \xi \omega_r) \]
\[ = \frac{M_p}{M_v} (\omega_r^2 - \omega_v^2) \sin(\rho - \nu) + 2\omega_v \xi \omega_r \cos(\rho - \nu)). \] (14)
The generic tuning formula for \( K_r \) becomes the following expression:

\[
K_r = \frac{K_{r2}}{\omega_z^2 - \omega_r^2} + \frac{M_p (\omega_r^2 - \omega_z^2) \cos(\rho - \nu)}{M_\nu (\omega_r^2 - \omega_z^2)} - \frac{2M_p \omega_z \omega_r \sin(\rho - \nu)}{M_\nu (\omega_r^2 - \omega_z^2)} \tag{15}
\]

where a degree of freedom in the tuning of the controller parameters is verified, since this is a single equation with two unknowns: \( K_p \) and \( K_{r2} \). A second equation involving these unknowns is obtained by imposing the product of the controller zeros to be equal to \( \eta \omega_r^2 \), where \( \eta > 0 \) is a parameter to be determined next. This additional constraint allows to achieve controller zeros with absolute value \( \eta \omega_r \) when they are complex conjugate, as detailed in the following.

The transfer function (1) can be represented as

\[
C_{pr}(s) = \frac{K_p}{s^2 + (2\zeta \omega_z + K_{r1})s + (K_{r2} + 1)\omega_z^2} \tag{16}
\]

The numerator of \( C_{pr}(s) \) expressed as a function of \( K_p \) and its roots, say \( z_1 \) and \( z_2 \), is given by \( Z(s) = K_p(s^2 - (z_1 + z_2)s + z_1z_2) \). From the additional constraint \( z_1z_2 = \eta \omega_r^2 \), if the controller has a pair of complex zeros, i.e., \( z_{1,2} = -a \pm jb \), where \( a, b \in \mathbb{R} \), then \( z_1z_2 = a^2 + b^2 = \eta \omega_r^2 \) and \( |z_{1,2}| = \eta \omega_r \). Thus, we can write

\[
Z(s) = K_p(s^2 + 2as + \eta^2 \omega_r^2) \tag{17}
\]

By comparing the expressions in (16) and (17), a second equation involving the parameters \( K_p \) and \( K_{r2} \) is achieved

\[
K_p = \frac{K_{r2}}{(\eta^2 - 1)\omega_r^2} \tag{18}
\]

From (15) and (18), we obtain the generic tuning formula for \( K_{r2} \)

\[
K_{r2} = \frac{M_p (\omega_r^2 - \omega_z^2) \cos(\rho - \nu)(\eta^2 - 1)\omega_z^2}{M_\nu (\eta^2 \omega_r^2 - \omega_z^2)} - \frac{2M_p \omega_z \omega_r \sin(\rho - \nu)(\eta^2 - 1)}{M_\nu (\eta^2 \omega_r^2 - \omega_z^2)} \tag{19}
\]

Substitution of the previous equation into (18) gives the generic tuning formula for \( K_p \)

\[
K_p = \frac{M_p (\omega_r^2 - \omega_z^2) \cos(\rho - \nu) - 2\omega_z \omega_r \sin(\rho - \nu)}{M_\nu (\eta^2 \omega_r^2 - \omega_z^2)} \tag{20}
\]

The generic tuning formula for \( K_{r1} \) is obtained by substituting the last expression into (13). Then, after simplifications, it becomes the following expression:

\[
K_{r1} = \frac{M_p 2\omega_z \omega_r \omega_r^3 (\eta^2 - 1) \cos(\rho - \nu)}{M_\nu (\eta^2 \omega_r^2 - \omega_z^2)} + \frac{M_p (\omega_r^2 - \omega_z^2)(\eta^2 \omega_r^2 - \omega_z^2) + 4\omega_z \omega_r^2 \omega_r^2 \sin(\rho - \nu)}{M_\nu (\eta^2 \omega_r^2 - \omega_z^2)} \tag{21}
\]

After developing the set of generic tuning formulas presented in (19) to (21), sets of particular tuning formulas are obtained for each of the three classes of plants previously defined. As mentioned before, this task was performed by considering a wide array of plants and different identified points of the plant’s frequency response (for those without an ultimate point), for several choices of locations \( p \), damping coefficient \( \zeta \), and parameter \( \eta \). The parameter \( \eta \) was set to 0.1 in order to achieve controller zeros (when they are complex) a decade below than the controller poles. The damping coefficient \( \zeta \) is a parameter to be chosen by the controller designer, considering that a stable closed-loop system asymptotically tracks (rejects) a given sinusoidal reference (disturbance) with frequency \( \omega_r \) if the PR controller (1) with \( \zeta = 0 \) is inserted in the loop.

For Class A, the identified point of the plant’s frequency response is given in (6) and there are two proposed locations \( p \) depending on the relationship between \( \omega_r \) and \( \omega_u \). Consequently, two sets of tuning formulas of the PR controller (1) are proposed considering (7) for \( 0 < \omega_r/\omega_u < 0.5 \) and (8) for \( 0.5 \leq \omega_r/\omega_u < 1 \). Thus, for Class A plant, the set of tuning formulas of the PR controller is

\[
K_p = \frac{0.399(\omega_u^2 - \omega_r^2) - 0.0419 \omega_u \omega_r \zeta}{M_u(\omega_u^2 - 0.01 \omega_r^2)} \tag{22}
\]

when \( 0 < \omega_r/\omega_u < 0.5 \), whereas

\[
K_p = \frac{0.4(\omega_u^2 - \omega_r^2) - 0.0140 \omega_u \omega_r \zeta}{M_u(\omega_u^2 - 0.01 \omega_r^2)} \tag{23}
\]

must be applied when \( 0.5 \leq \omega_r/\omega_u < 1 \).

For Class B, the identified point of the plant’s frequency response is given in (9) and the location \( p \) is presented in (10). Hence, the set of tuning formulas of the PR controller for a Class B plant is

\[
K_p = \frac{0.985(\omega_u^2 - \omega_r^2) - 0.347 \omega_{1200} \omega_r \zeta}{M_{1200}(\omega_u^2 - 0.01 \omega_r^2)} \tag{24}
\]

For Class C, the identified point of the plant’s frequency response is given in (11) and the location \( p \) is defined in (12). Thus, the set of tuning formulas of the PR controller for a
Black dashed lines are at $\omega_0$ and at $-180^\circ$.

Class C plant is

$$
K_p = \frac{0.866(\omega_0^2 - \omega_r^2) - \omega_0 \omega_r \xi}{M_0(\omega_0^2 - 0.01\omega_r^2)} \\
K_{r_1} = \frac{0.5(\omega_0^2 - \omega_r^2) + 1.71\omega_0^2\xi + 2\omega_0\omega_r\xi^2}{M_60(\omega_0^2 - 0.01\omega_r^2)} \\
K_{r_2} = \frac{0.857\omega_0^2(\omega_0^2 - \omega_r^2) + 0.99\omega_0\omega_r\xi^2}{M_60(\omega_0^2 - 0.01\omega_r^2)}.
$$

(25)

The PR controller applied to a Class A plant is tuned using two sets of tuning formulas because there is a compromise between the controller contribution to the phase at the plant’s ultimate frequency, stability margins, and closed-loop performance. Moreover, a small change in the controller contribution to the phase at this specific frequency significantly changes the stability margins and the closed-loop performance. Unlike Classes B and C for which the PR controller guarantees the defined phase margins of 50° and 90°, respectively.

A frequency response analysis helps to evaluate the proposed PR tuning. Fig. 1 presents the frequency response of the Class A plant $G_a(j\omega) = e^{-\gamma}/(s + 1)^2$ and the loop $G_a(j\omega)C_{pr}(j\omega)$ with reference frequencies $\omega_r = 0.1\omega_0$ and $\omega_r = 0.9\omega_0$, for which the controllers are tuned through (22) and (23), respectively. For $\omega_r = 0.1\omega_0$, the controller resonance frequency is much lower than the plant’s ultimate frequency, then appropriate stability margins are achieved, and good closed-loop performance is expected. For $\omega_r = 0.9\omega_0$, the controller resonance frequency is close to the plant’s ultimate frequency, and consequently, stability margins are much smaller than the previous case. Thus, poor performance and even unstable closed-loop systems can be expected for $\omega_r \approx \omega_0$, since the controller structure presents very large gains in a range around the plant’s ultimate frequency. It is important to note that this is not a limitation of the tuning rules just proposed; it is rather a limitation of the controller structure.

Fig. 2 shows the frequency response of the Class B plant $G_b(j\omega) = 1/(s + 1)^2$ and the loop $G_b(j\omega)C_{pr}(j\omega)$ with the PR controller tuned through (24) having reference frequencies $\omega_r = 0.1\omega_{20}$ and $\omega_r = 0.9\omega_{20}$. Likewise, Fig. 3 presents the frequency response of the Class C plant $G_c(j\omega) = 1/(s + 1)$ and the loop $G_c(j\omega)C_{pr}(j\omega)$ with the PR controller tuned through (25) having reference frequencies $\omega_r = 0.1\omega_0$ and $\omega_r = 0.9\omega_0$, respectively.

IV. RELAY WITH ADJUSTABLE PHASE EXPERIMENT

Consider now the experiment setup shown in Fig. 4, where $F(s)$ is a known transfer function. When the self-oscillation condition is achieved, one obtains the ultimate frequency $\omega_1$ of the transfer function $F(s)G(s)$, i.e., $\omega_1 = \mathcal{L}F(j\omega_1)G(j\omega_1) = -180^\circ$. Thus, the plant’s magnitude and phase at this frequency can be calculated as

$$
|G(j\omega_1)| = \frac{\pi A}{4 d |F(j\omega_1)|}, \quad \angle G(j\omega_1) = -180^\circ - \angle F(j\omega_1)
$$

since $F(j\omega_1)$ is known (see [10] for details).

If the identification is performed through the ultimate point of $F(s)G(s)$, we must choose a $F(s)$ whose phase is:

$$
\gamma \triangleq -180^\circ - \nu, \quad \text{at the frequency } \omega_0.
$$

To overcome this difficulty, the use of $F(s)$ with (almost) constant phase, that is, $\mathcal{L}F(j\omega) = \gamma \forall \omega$, has been proposed in [10]. A transfer function with flat phase frequency response that is not necessarily an entire multiple of $-90^\circ$ can be obtained by a fractional order integrator (FOI): $FOI(s) = 1/(s^\mu)$, where it can be verified that $\mathcal{L}FOI(j\omega) = -\pi/2 \times m$ and defining $m = -\gamma/90^\circ$ yields $\mathcal{L}FOI(j\omega) = \gamma \forall \omega$, as desired. These fractional-order systems are usually approximately implemented by integer order systems and used in fractional-order PI$^d$D$^\mu$ controllers [14].
To obtain transfer functions that approximate the desired FOI, the MATLAB package FOMCON [15] was used, where the parametric transfer function is given by

$$F(s) = \frac{\sum_{k=1}^{11} b_k s^k}{\sum_{k=1}^{11} a_k s^k}$$

and the coefficients \(\{a_k, b_k\}\) are such that the FOI approximation has magnitude slope of \(-m \times 20 \text{ dB/decade} and a constant phase value of \(-m \times 90^\circ\) considering the range of frequencies from \(10^{-3}\) to \(10^3\) rad/s. Table I presents two sets of coefficients \(\{a_k, b_k\}\), one representing a FOI approximation with \(m = 1/3\) and the other with \(m = 2/3\). These transfer functions are actually used in the GFO method, that is, taking \(F(s) = \hat{F}(s)\). The FOI approximation with \(m = 2/3\) is used to identify the point of the plant’s frequency response whose phase is \(-120^\circ\), whereas the FOI approximation with \(m = 1/3\) is used to identify the point of the plant’s frequency response whose phase is \(-60^\circ\) by adding an integrator to the transfer function of this FOI approximation, that is, \(\hat{F}(s) = \hat{F}(s)/s\).

To identify the most appropriate point of the plant’s frequency response, we propose the relay with adjustable phase (RAP) experiment, whose block diagram is shown in Fig. 4. We propose to start the RAP experiment with a \(0^\circ\) phase relay, i.e., to perform the traditional relay experiment. If a self-oscillation condition is not obtained, that is, the plant’s output is not a well-defined signal with oscillation’s amplitude and period constants over time, the relay phase must be decreased from \(0^\circ\) to \(-60^\circ\) (or from \(-60^\circ\) to \(-120^\circ\)), until the experiment reaches a self-oscillatory behavior with the largest phase relay possible. Thus, the RAP experiment enables an automatic procedure for the identification of a defined point of the plant’s frequency response—whether having or not an ultimate point—in a single experiment without designer intervention.

![Fig. 4. Block diagram of the RAP experiment.](image)

| TABLE I |
| --- |
| Table of Coefficients of \(\hat{F}(s)\) |

\[
\begin{array}{c|c|c|c|c|c}
\text{m} & \text{m = 1/3 for } \gamma = -30^\circ & \text{m = 2/3 for } \gamma = -60^\circ \\
\hline
k & a_k & b_k & a_k & b_k \\
\hline
0 & 0 & 0.3452 & 0 & 0.7152 \\
1 & 111.1 & 1309 & 11.11 & 1446 \\
2 & 8.49 \times 10^4 & 5.4 \times 10^5 & 1.097 \times 10^4 & 4.387 \times 10^5 \\
3 & 1.15 \times 10^5 & 4.302 \times 10^5 & 1.918 \times 10^5 & 2.678 \times 10^6 \\
4 & 3.232 \times 10^5 & 7.22 \times 10^5 & 6.963 \times 10^5 & 3.473 \times 10^6 \\
5 & 1.942 \times 10^5 & 2.598 \times 10^6 & 5.403 \times 10^5 & 9.672 \times 10^5 \\
6 & 2.509 \times 10^6 & 2.013 \times 10^7 & 9.016 \times 10^5 & 5.799 \times 10^6 \\
7 & 6.866 \times 10^5 & 3.36 \times 10^7 & 3.24 \times 10^5 & 7.487 \times 10^6 \\
8 & 4.195 \times 10^5 & 1.211 \times 10^7 & 2.506 \times 10^7 & 2.08 \times 10^7 \\
9 & 5.462 \times 10^5 & 9.508 \times 10^5 & 4.164 \times 10^5 & 1.238 \times 10^5 \\
10 & 1569 & 167.8 & 1466 & 15.45 \\
11 & 1 & 0.06905 & 1 & 0.003576 \\
\end{array}
\]

V. BENCHMARK CLASS OF PLANTS

To evaluate the proposed tuning methodology, three different plants that represent the classes defined in Section III will be considered. For each of them, initially the RAP experiment is performed and the PR controllers are designed, then the closed-loop response to a sinusoidal reference is evaluated considering both the nominal plant and a perturbed plant to evaluate the system robustness with the same PR controller. In these examples, we consider \(\zeta = 0\) to achieve asymptotically tracking of the reference.

A. Class A Plant

The Class A plant is described by the transfer function

$$G_a(s) = \frac{e^{-s}}{(s + 1)^2}.$$  \hspace{1cm} (27)

The starting point of the tuning methodology is to identify a particular point of the plant’s frequency response. Since the plant \(G_a(s)\) has an ultimate point, it belongs to Class A, then a self-oscillatory behavior is achieved in the RAP experiment with \(0^\circ\) and the plant’s ultimate point is identified. The parameters obtained in this experiment are summarized in Table II. The second step is to design the PR controller (1) for a given reference frequency \(\omega_r\) using this identified point of the plant’s frequency response. For a Class A plant, the controller gains are calculated depending on the relationship between \(\omega_r\) and \(\omega_0\): if \(0 < \omega_r/\omega_0 < 0.5\), the PR controllers must be tuned through (22); whereas, if \(0.5 \leq \omega_r/\omega_0 < 1\), the PR controllers must be tuned through (23).

In this example, the reference is a sinusoidal signal with frequencies: \(\omega_r = 0.1\omega_0\) and \(\omega_r = 0.7\omega_0\). The set of controller parameters and performance measures are summarized in Table III. The reference and the output signals for each set of controller parameters are shown in Fig. 5, where we consider the nominal plant \(G_a(s)\) and also a perturbed plant \(G_{2a}(s) = e^{-s}/(s + 2)^2\) to evaluate the system robustness.

A frequency response analysis provides a clear picture of system stability. The Nyquist diagrams of \(G_a(s)\) and of the loop transfer function \(C_{pr(s)}G_a(s)\) for \(\omega_r = 0.1\omega_0\) and \(\omega_r = 0.7\omega_0\) are presented in Fig. 6, where it can be seen that for both \(\omega_r\), the Nyquist diagrams of \(C_{pr(s)}G_a(s)\) do not encircle the point \(-1 + j0\). For \(\omega_r = 0.1\omega_0\), the frequency response is smooth enough around the negative real axis so that shifting the ultimate point away from \(-1 + j0\) guarantees good stability margins. Unlike for \(\omega_r = 0.7\omega_0\), where proximity of \(\omega_r\) and \(\omega_0\) implies that the distance between the loop transfer function and \(-1 + j0\) is decreased. In this case, shifting the plant’s ultimate point away from \(-1 + j0\) does not guarantee good stability margins, since the nearby points are not shifted along because the frequency response is not sufficiently smooth in this range of frequencies. Consequently, stability
TABLE III
TUNING PARAMETERS AND PERFORMANCE MEASURES

| Plant  | $\omega_r$ (rad/s) | $\zeta$ | $K_p$ | $K_t_1$ | $K_t_2$ | $t_s$ (s) | $n_s$ | $M_s$ (%) |
|--------|-------------------|---------|-------|---------|---------|-----------|-------|-----------|
| $G_a(s)$ | 0.1 $\omega_u$ = 0.132 | 0.101 | 0.0699 | -0.0174 | 125.7 | 7.6 | 9.9 |
| $G_{2a}(s)$ | 0.7 $\omega_u$ = 0.924 | 0.524 | 0.0120 | -0.443 | 58.0 | 8.5 | 23 |
| $G_b(s)$ | 0.1 $\omega_{120} = 0.169$ | 3.82 | 1.14 | -10.8 | 76.9 | 2.1 | 7.9 |
| $G_{2b}(s)$ | 0.9 $\omega_{120} = 1.52$ | 0.740 | 0.220 | -16.9 | 26.3 | 6.4 | 3.0 |
| $G_c(s)$ | 0.1 $\omega_{60} = 0.168$ | 1.71 | 1.66 | -0.0479 | 93.9 | 2.5 | 6.3 |
| $G_{2c}(s)$ | 0.9 $\omega_{60} = 1.51$ | 0.332 | 0.319 | -0.751 | 24.1 | 5.8 | 0 |

Fig. 5. Closed-loop response considering the same PR controller with $G_a(s)$, signal $y(t)$, and $G_{2a}(s) = e^{-s}/(s + 2)^2$, signal $y_2(t)$. (a) $\omega_r = 0.1\omega_u$, (b) $\omega_r = 0.7\omega_u$.

Margins are much smaller than the case $\omega_r = 0.1\omega_u$ and, therefore, poorer transient response is expected for reference frequencies nearest to the plant’s ultimate point. Recall that this is a limitation imposed by the controller structure and not by the proposed tuning method.

B. Class B Plant

The Class B plant has the following transfer function:

$$G_b(s) = \frac{1}{(s + 1)^2}. \tag{28}$$

The plant described by $G_b(s)$ belongs to Class B since it has no ultimate point and its frequency response does cross the $-120^\circ$ phase line. An oscillatory condition is obtained for the RAP experiment with $-60^\circ$, then the point of the plant’s frequency response whose phase is $\nu = -120^\circ$ is identified. Fig. 7(a) shows the plant’s output in this experiment considering as reference input a step with amplitude 1, where from 0 to 10s, the RAP experiment with 0° is performed, and self-oscillatory behavior is not obtained. Thus, at 10s, the relay phase is changed to $-60^\circ$ and the self-oscillation condition is achieved, as expected. The parameters obtained from this experiment are summarized in Table II.

The controller gains are calculated from (24) for two frequencies: $\omega_r = 0.1\omega_{120}$ and $\omega_r = 0.9\omega_{120}$. The sets of controller parameters and performance measures are summarized in Table III. The reference and output signals for each set of the controller parameters are presented in Fig. 8, where we used the nominal plant $G_b(s)$ and also a perturbed plant $G_{2b}(s) = 1/((s + 1)(s + 2))$ to evaluate the system robustness.

C. Class C Plant

The Class C plant is described by the transfer function

$$G_c(s) = \frac{1}{s + 1}. \tag{29}$$

The plant $G_c(s)$ has no ultimate point and its frequency response does not cross the $-120^\circ$ phase line but it does cross the $-60^\circ$ phase line, thus it belongs to Class C. A self-oscillatory behavior is obtained for the RAP experiment with $-120^\circ$, and the point of the plant’s frequency response whose
phase is $\nu = -60^\circ$ is identified. Fig. 7(b) presents the plant’s output considering as reference input a step with amplitude one, where the RAP experiment with $0^\circ$ is implemented for $t$ ranging from 0 to 5s; at 5s, the relay phase is changed to $-120^\circ$, and self-oscillation condition is not observed. Therefore, at 10s, the RAP experiment with $-120^\circ$ is performed and a self-oscillatory behavior is achieved. The parameters obtained from this experiment are summarized in Table II.

The controller gains are calculated from (25) using two frequencies: $\omega_r = 0.1\omega_{60}$, and $\omega_r = 0.9\omega_{60}$. The sets of controller parameters and performance measures are summarized in Table III. The reference and the plant’s output signals for each set of controller parameters are shown in Fig. 9, considering the nominal plant $G_r(s)$ and also a perturbed plant $G_{2c}(s) = 1/(s + 2)$ to evaluate the system robustness.

VI. CONCLUSION

This brief has proposed the GFO method for tuning PR controllers including both plants that have an ultimate point and plants that do not. This method is based on the identification of the most appropriate point of the frequency response for each class of plants through the RAP experiment. Four sets of tuning formulas have been developed to obtain appropriate stability margins and closed-loop performance for each class of plants. The proposed methodology was validated considering a wide variety of plants and also different reference frequencies below the plant identified frequency. For all such cases, good closed-loop performance (in terms of settling time and maximum overshoot) and robustness (which is obtained through appropriate stability margins) have been obtained. Thus, the GFO method is an alternative to experimentally tune a PR controller without the plant model and also without the use of advanced control system techniques, which should contribute to the applicability and enlarge the dissemination of resonant controllers. The development of the GFO method to considerer proportional multiresonant controllers is an interesting subject for future research.

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