GL(n, R)-Equivalence of a Pair of Curves in Terms of Invariants

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Abstract: In this paper, the generator set of $G = GL(n, R)$ is obtained in accordance to the group $G = GL(n, R)$. The conditions of $G = GL(n, R)$-equivalence of a pair of curves are found in terms of $G = GL(n, R)$-invariants. And the independence of $GL(n, R)$-invariants is shown.

Keywords: $GL(n, R)$-invariants, differential invariants of curves, equivalence of curves.

1. Introduction

The theory of differential invariants consists of three fundamental theorems. The first of these is finding the generators for invariant functions. The second is finding the conditions of equivalence for curves and the third one is finding the relations (if it exists) between these generators. We give the generator set of differential invariants for two curves and investigate the relations among them.

Let $R$ be the field of real numbers and $R^n$ be $n$-dimensional Euclidean space. The set

$$GL(n, R) = \left\{ A = \begin{bmatrix} a_{ij} \end{bmatrix} : i, j = 1, \ldots, n \right\}$$

is a group in accordance to multiplication of matrix. The action of the group $GL(n, R)$ on $R^n$ is given by

$$g x = \begin{bmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} g_{11}x_1 + \cdots + g_{1n}x_n \\ \vdots \\ g_{n1}x_1 + \cdots + g_{nn}x_n \end{bmatrix}$$

for $g \in GL(n, R)$ and $x \in R^n$.

Invariant theory is studied since earlier times [5, 6, 13, 14]. There are a lot of paper and books about the invariant theory of curves and surfaces [2, 3, 7, 8, 9, 10, 11, 12]. The generator set of differential invariants and the relations of them is obtained in [1] for special groups. For two curves, it is investigated the differential invariants and its applications to ruled surfaces for the group $SL(n, R)$ in [4].

In this paper, we investigate the differential invariants of a pair of curves for the group $GL(n, R)$. In section 1, we give some introductory definitions. In section 2, the generator system of differential invariants is found for the rational functions of a pair of curves. Then the conditions of equivalence for two pairs of curves is given by the differential invariants. Also it is shown that the set of generator invariants is minimal.

Definition 1.1. A $C^\infty$-function $x : I \to R^n$ will be called a parametric curve or briefly a curve in $R^n$.

Definition 1.2. Let $\{x_1, x_2\}$ and $\{y_1, y_2\}$ be two pairs of curves. If $y_i = g x_i, i = 1, 2$ for some $g \in GL(n, R)$, then these curve families will be called $GL(n, R)$-equivalent and denoted by $\{x_1, x_2\} \overset{G}{\approx} \{y_1, y_2\}$ for the group $G = GL(n, R)$.

Definition 1.3. Let $x_1$ and $x_2$ be two curve in $R^n$. The polynomial
GL(n, R) -Equivalence of a Pair of Curves in Terms of Invariants

\[ P \{x_1, x_2\} = P(x_1, x_2, x'_1, x'_2, \ldots, x^{(m)}_1, x^{(m)}_2) \]

for some natural number \( m \) will be called the differential polynomial of \( x_1 \) and \( x_2 \).

The derivation of \( P \{x_1, x_2\} \) will be denoted by \( P' \) and this derivation is obtained as follows:

\[ x_i^{(0)} = x_i, \quad x_i^{(m)} = x_i^{(m)} \]

**Definition 1.4.** Let \( P_1 \) and \( P_2 \) be two differential polynomials. Then the function

\[ f < x_1, x_2 > \equiv \frac{P_1 \{x_1, x_2\}}{P_2 \{x_1, x_2\}}, \quad P_2 \{x_1, x_2\} \neq 0 \]

will be called a differential rational function.

If \( f < g, y > \equiv f < x, y > \) for all \( g \in GL(n, R) \), the differential rational function \( f \) all called centro-affine invariant differential rational function. Centro-affine differential polynomial is defined by the same way. There no exists centro-affine invariant differential polynomial except constant. But there exists the centro-affine invariant differential rational function different from constant.

The set of all differential rational functions will be denoted by \( R \{x_1, x_2\} \). It is a field and \( R \)-algebra. Let \( G \) be the group \( GL(n, R) \). The set of all centro-affine invariant differential rational functions will be denoted by \( R \{x_1, x_2\}^G \). \( R \{x_1, x_2\}^G \) is a differential subfield and subalgebra of \( R \{x_1, x_2\} \).

**Definition 1.5.** Let \( f_1, f_2, \ldots, f_k \in R \{x_1, x_2\}^G \). If the differential field and algebra generated by these functions is equal to \( R \{x_1, x_2\}^G \), then these functions will be called the generator set of \( R \{x_1, x_2\}^G \).

2. Centro-Affine Invariants of a Pair of Curves

Let \( x_1, x_2, \ldots, x_n \in R^n \). We will be denoted the determinant

\[
\begin{vmatrix}
x_{11} & \ldots & x_{1n} \\ \\
\vdots & \ddots & \vdots \\ \\
x_{n1} & \ldots & x_{nn}
\end{vmatrix}
\]

by \( [x_1 \ldots x_n] \). In here, \( k \).

The column of this determinant is consist of the components of \( x_k \), which are \( x_{k1}, x_{k2}, \ldots, x_{kn} \).

**Lemma 2.1.** Let \( x_0, x_1, \ldots, x_n, y_1, \ldots, y_n \) be vectors in \( R^n \). Then the following equality holds:

\[
\begin{align*}
[x_0, x_1, \ldots, x_n] [x_0 y_1, \ldots, y_n] - [x_0 y_1, \ldots, y_n] [x_0, x_1, \ldots, x_n] \\
&- \cdots - [x_0, x_1, \ldots, x_n] [x_0, y_1, \ldots, y_n] = 0.
\end{align*}
\]

**Proof.** Page 53 in [1].

**Definition 2.1.** A curve \( x \) in \( R^n \) will be called \( GL(n, R) - \text{regular} \) (briefly regular) if \( [x_1 x'_1 \ldots x^{(n-1)}_1] \neq 0 \). Hence for all \( t \),

\[ [x_i(t) x'_i(t) \ldots x^{(n-1)}_i(t)] \neq 0. \]

Let \( G \) be the group \( GL(n, R) \).

**Theorem 2.1.** Let \( x_1 \) and \( x_2 \) be two curve in \( R^n \) such that \( x_i \) is regular. Then the generator set of \( R \{x_1, x_2\}^G \) is

\[
\begin{align*}
&\begin{bmatrix}
x_1 \ldots x^{(i-1)}_1 & x^{(n)}_1 & x^{(i+1)}_1 & \ldots & x^{(n-1)}_1 \\
x_1 x'_1 \ldots x^{(n-1)}_1 \\
x_1 \ldots x^{(i-1)} & x_1 x^{(i+1)}_1 & \ldots & x^{(n-1)}_1 \\
x_1 x'_1 \ldots x^{(n-1)}_1
\end{bmatrix},
\end{align*}
\]

(2.2)

for \( i = 0, \ldots, n-1 \).

**Proof.** For the group \( G = GL(n, R) \), the generator set of \( R(x, \tau) \in \Delta \) is

\[
\begin{bmatrix}
x_1 \ldots x_n, x_1 x_{i+1} \ldots x_n \\
x_1 \ldots x_n
\end{bmatrix}, i = 1, \ldots, n, \tau \in \Delta / \{1, \ldots, n\}
\]

[1]. Let us take \( x_1, x_2, x'_1, x'_2, \ldots, x^{(K)}_1, x^{(K)}_2, \ldots \) instead of the vectors \( x \). Then the generator set of \( R(x_1, x_2, x'_1, x'_2, \ldots, x^{(K)}_1, x^{(K)}_2, \ldots)^G \) is

\[
\begin{align*}
&\begin{bmatrix}
x_1 \ldots x^{(i-1)}_1 & x^{(i)}_1 x^{(i+1)}_1 & \ldots & x^{(n-1)}_1 \\
x_1 x'_1 \ldots x^{(n-1)}_1 \\
x_1 \ldots x^{(i-1)}_1 & x_1 x^{(i+1)}_1 & \ldots & x^{(n-1)}_1 \\
x_1 x'_1 \ldots x^{(n-1)}_1
\end{bmatrix}, \\
i = 0, \ldots, n-1, \tau \in \Delta / \{0, \ldots, n-1\}
\]

\[
\begin{bmatrix}
x_1 \ldots x^{(i-1)}_1 & x^{(i)}_1 x^{(i+1)}_1 & \ldots & x^{(n-1)}_1 \\
x_1 x'_1 \ldots x^{(n-1)}_1 \\
x_1 \ldots x^{(i-1)}_1 & x_1 x^{(i+1)}_1 & \ldots & x^{(n-1)}_1 \\
x_1 x'_1 \ldots x^{(n-1)}_1 
\end{bmatrix}, \tau \geq 0
\]
GL(n, R)-Equivalence of a Pair of Curves in Terms of Invariants

Firstly, we want to show that
\[
\frac{[x_1 \ldots x_i \ldots (t) \ldots (i) \ldots x_{i+1}]}{[x_{i+1} \ldots x_n]} \quad \tau \geq n \text{ is generated by}
\]
\[
\frac{[x_1 \ldots x_i \ldots (n)] \ldots (n) \ldots x_{i+1}]}{[x_{i+1} \ldots x_n]} \quad i = 0, \ldots, n-1.
\]
Let \( \tau = n \).

Then the generator set of
\[
R(x_1, x_2, x_3, \ldots, x^{(K)}_1, x^{(K)}_2 \ldots) \overset{G}{\to}
\]
\[
\left[\frac{x^{(n)}_1 x^{(n)}_2 \ldots x^{(n)}_i \ldots x^{(n)}_n}{x^{(n-1)}_1 x^{(n-1)}_2 \ldots x^{(n-1)}_i \ldots x^{(n-1)}_n}ight] \ldots \left[\frac{x^{(n)}_1 x^{(n)}_2 \ldots x^{(n)}_i \ldots x^{(n)}_n}{x^{(n-1)}_1 x^{(n-1)}_2 \ldots x^{(n-1)}_i \ldots x^{(n-1)}_n}ight]
\]

So these are generated by the set (2.2).

Let \( \tau > n \). By induction, for \( \tau - 1 \) let the set (2.2) be the generator set. Therefore
\[
\frac{[x_1 \ldots x_i \ldots (t) \ldots (i) \ldots x_{i+1}]}{[x_{i+1} \ldots x_n]} \quad \tau > n
\]

get
\[
\frac{[x_1 \ldots x_i \ldots (t) \ldots (i) \ldots x_{i+1}]}{[x_{i+1} \ldots x_n]} = \left[\frac{x_1 \ldots x_i \ldots (t) \ldots (i) \ldots x_{i+1}}{x_{i+1} \ldots x_n}ight] - \left[\frac{x_1 \ldots x_i \ldots (t) \ldots (i) \ldots x_{i+1}}{x_{i+1} \ldots x_n}ight]
\]

If we divide this equation by \( [x_1 x_1 \ldots x^{(n)}_n] \), it is obtained that
\[
\frac{[x_1 \ldots x_i \ldots (t) \ldots (i) \ldots x_{i+1}]}{[x_{i+1} \ldots x_n]} = \left[\frac{x_1 \ldots x_i \ldots (t) \ldots (i) \ldots x_{i+1}}{x_{i+1} \ldots x_n}ight] - \left[\frac{x_1 \ldots x_i \ldots (t) \ldots (i) \ldots x_{i+1}}{x_{i+1} \ldots x_n}ight]
\]

The first term in the right of equality (2.3) is obtained by the derivation of
\[
\frac{[x_1 \ldots x_i \ldots (t) \ldots (i) \ldots x_{i+1}]}{[x_{i+1} \ldots x_n]}
\]

The second term in the equality (2.3) is generated by the set (2.2) in according to induction hypothesis. In Lemma 1, if we take
\[
x_1 = x_1, \quad x_2 = x_2', \ldots, \quad x_n = x_n^{(i-1)}
\]

\[
x_0 = x_0^{(i)}, \quad y_2 = y_2', \ldots, \quad y_{i+1} = x_{i+1}^{(i-1)}
\]

\[
y_{i+2} = x_{i+2}^{(i-1)}, \quad y_{i+3} = x_{i+3}^{(i-1)}, \ldots, \quad y_n = x_n^{(i-2)}
\]

eliminate the zero terms and divide \( [x_1 x_1 \ldots x^{(n)}_n] \) it is obtained that
\[
\left[\frac{x_1 \ldots x_i \ldots (t) \ldots (i) \ldots x_{i+1}}{x_{i+1} \ldots x_n}ight] - \left[\frac{x_1 \ldots x_i \ldots (t) \ldots (i) \ldots x_{i+1}}{x_{i+1} \ldots x_n}ight] = 0
\]

So the term \( [x_1 x_1 \ldots x^{(n)}_n] \) generated by the set (2.2). Therefore the third term in the equality (2.3) is generated by the set (2.2).

Similarly,
\[
\frac{[x_1 \ldots x_i \ldots (t) \ldots (i) \ldots x_{i+1}]}{[x_{i+1} \ldots x_n]} \quad \tau \geq 0
\]

obtained by induction on \( \tau \). For \( \tau = 0 \),
\[
\left[\frac{x_1 \ldots x_i \ldots (t) \ldots (i) \ldots x_{i+1}}{x_{i+1} \ldots x_n}ight] \quad \text{is the generator. Let for } \tau = n,
\]

\[
\left[\frac{x_1 \ldots x_i \ldots (t) \ldots (i) \ldots x_{i+1}}{x_{i+1} \ldots x_n}ight] \quad \text{generated by the set (2.2). Let us show that this is true for } \tau = n+1.
\]
Let \( \mathbf{A} \) be a matrix, \( \mathbf{A}' \) a generator, and \( \mathbf{A}_1, \mathbf{A}_2 \) be two curve families such that \( \mathbf{A}_1 = \mathbf{A}' \mathbf{A}_2 \mathbf{A}_1 \). By the induction hypothesis, The set (2.2) is generator set. \( \square \)

**Theorem 2.2.** Let \( G = GL(n, R) \) and \( \{x_i, x_2\} \), \( \{y_1, y_2\} \) be two curve families such that \( x_i \) and \( y_i \) are regular. If for \( i = 0, \ldots, n-1 \)

\[
\begin{bmatrix}
    x_1 \cdots x_i (n-1) (n) x_{i+1} \cdots x_{n-1} \\
    x_1' \cdots x_i' (n-1) \\
\end{bmatrix} =
\begin{bmatrix}
    y_1 \cdots y_i (n-1) (n) y_{i+1} \cdots y_{n-1} \\
    y_1' \cdots y_i' (n-1) \\
\end{bmatrix}
\]

then \( \{x_i, x_2\} \sim \{y_1, y_2\} \).

**Proof.** Since \( x_i \) and \( y_i \) are regular, we get \( \{x_i x_1' \cdots x_i (n-1) \} \neq 0 \) and \( \{y_i y_1' \cdots y_i' (n-1) \} \neq 0 \). Let us take the matrixes

\[
\mathbf{A}_i = \begin{bmatrix}
    x_{11}(t) & \cdots & x_{11}(n-1)(t) \\
    \vdots & \ddots & \vdots \\
    x_{in}(t) & \cdots & x_{in}(n-1)(t) \\
\end{bmatrix}
\]

and

\[
\mathbf{A}_i' = \begin{bmatrix}
    x_{11}'(t) & \cdots & x_{11}'(n)(t) \\
    \vdots & \ddots & \vdots \\
    x_{in}'(t) & \cdots & x_{in}'(n)(t) \\
\end{bmatrix}
\]

Since \( \{x_i x_1' \cdots x_i (n-1) \} \neq 0 \), there exists the inverse of \( \mathbf{A}_i \). Take the matrix \( \mathbf{A}_i^{-1} \mathbf{A}_i' = \mathbf{C} \). Then

\[
\mathbf{A}_i' = \mathbf{A}_i \mathbf{C} \]. So the matrix \( \mathbf{C} \) has the form

\[
\begin{bmatrix}
    0 & \cdots & 0 & c_{1n} \\
    1 & \cdots & 0 & c_{2n} \\
    \vdots & \ddots & \vdots & \vdots \\
    0 & \cdots & 1 & c_{nn} \\
\end{bmatrix}
\]
where
\[
\begin{pmatrix}
\frac{x_1^{(n)}y_1^{(n)} \cdots x_1^{(n-1)}y_1^{(n-1)}}{x_1y_1 \cdots x_1^{(n-1)}y_1^{(n-1)}} & \cdots & \frac{x_1^{(n)}y_1^{(n)} \cdots x_2^{(n-1)}y_2^{(n-1)}}{x_1y_1 \cdots x_2^{(n-1)}y_2^{(n-1)}} \\
\vdots & & \vdots \\
\frac{x_n^{(n)}y_n^{(n)} \cdots x_n^{(n-1)}y_n^{(n-1)}}{x_ny_n \cdots x_n^{(n-1)}y_n^{(n-1)}} & \cdots & \frac{x_n^{(n)}y_n^{(n)} \cdots x_n^{(n-1)}y_n^{(n-1)}}{x_ny_n \cdots x_n^{(n-1)}y_n^{(n-1)}}
\end{pmatrix} = \begin{pmatrix}
c_1 \\
c_2 \\
\vdots \\
c_n
\end{pmatrix}
\]
From the equalities (2.5), it is obtained that
\[
A_{n_i}^{-1}A_{n_i}' = A_{n_j}^{-1}A_{n_j}'.
\]
So we have that
\[
\begin{aligned}
(A_{n_i}^{-1}A_{n_i})' &= A_{n_i}^{-1}A_{n_i} + A_{n_i}'A_{n_i}^{-1}'
\end{aligned}
\]
\[
= A_{n_i}'A_{n_i}^{-1} + A_{n_i}(-A_{n_i}^{-1}A_{n_i}'A_{n_i}^{-1})
\]
\[
= A_{n_i}'(A_{n_i}^{-1}A_{n_i}' - A_{n_i}^{-1}A_{n_i}')A_{n_i}^{-1} = 0
\]
Therefore
\[
A_{n_i}^{-1}A_{n_i}' = g, \quad g \text{ is constant and }
\]
\[
det(A_{n_i}^{-1}A_{n_i}') = detA_{n_i}^{-1}detA_{n_i} = detg \neq 0.
\]
So
\[
g \in GL(n, R)
\]And we get
\[
A_{n_i} = gA_{n_i}.
\]If we write this equality obviously, we have that
\[
\begin{pmatrix}
y_{11} & \cdots & y_{11}^{(n-1)} \\
y_{1n} & \cdots & y_{1n}^{(n-1)} \\
g_{11} & \cdots & g_{1n} \\
g_{n1} & \cdots & g_{nn}
\end{pmatrix}
\begin{pmatrix}
x_{11} & \cdots & x_{11}^{(n-1)} \\
x_{1n} & \cdots & x_{1n}^{(n-1)} \\
x_{11} & \cdots & x_{11}^{(n-1)} \\
x_{1n} & \cdots & x_{1n}^{(n-1)}
\end{pmatrix}
\]
and then
\[
y_i(t) = gx_i(t), \quad \forall t \in I.
\]
Let us take the matrix
\[
D_{x_2} = \begin{pmatrix}
x_{21} \\
x_{22} \\
\vdots \\
x_{2n}
\end{pmatrix}
\]
and take
\[
\begin{pmatrix}
x_1 & x_1' & \cdots & x_1^{(n-1)} \\
x_2 & x_2' & \cdots & x_2^{(n-1)} \\
\vdots & \vdots & \ddots & \vdots \\
x_n & x_n' & \cdots & x_n^{(n-1)}
\end{pmatrix}
\begin{pmatrix}
h_{11} \\
h_{22} \\
\vdots \\
h_{nn}
\end{pmatrix}
= \begin{pmatrix}
x_{21} \\
x_{22} \\
\vdots \\
x_{2n}
\end{pmatrix}
\]
Therefore
\[
A_{n_i}^{-1}D_{x_2} = H = \begin{pmatrix} h_{in} \
\end{pmatrix}, i = 1, \ldots, n.
\]
Let us find the element of this matrix. We have that
\[
x_1h_{1n} + x_1'h_{2n} + \cdots + x_1^{(n-1)}h_{nn} = x_{21}
\]
\[
x_2h_{1n} + x_2'h_{2n} + \cdots + x_2^{(n-1)}h_{nn} = x_{22}
\]
\[
\vdots
\]
\[
x_nh_{1n} + x_n'h_{2n} + \cdots + x_n^{(n-1)}h_{nn} = x_{2n}
\]
The solution of this equation system in according to Cramer’s rule;
\[
\begin{pmatrix} h_{1n} \\
h_{2n} \\
\vdots \\
h_{nn}
\end{pmatrix} = \begin{pmatrix}
x_1x_1' & \cdots & x_1^{(n-1)}x_1^{(n-1)} \\
x_1x_1' & \cdots & x_1^{(n-1)}x_1^{(n-1)} \\
\vdots & \vdots & \ddots & \vdots \\
x_1x_1' & \cdots & x_1^{(n-1)}x_1^{(n-1)}
\end{pmatrix}^{-1}
\]
\[
\begin{pmatrix} x_2x_2' & \cdots & x_2^{(n-1)}x_2^{(n-1)} \\
x_2x_2' & \cdots & x_2^{(n-1)}x_2^{(n-1)} \\
\vdots & \vdots & \ddots & \vdots \\
x_2x_2' & \cdots & x_2^{(n-1)}x_2^{(n-1)}
\end{pmatrix}
\]
Similarly, we can find the matrix
\[
A_{n_1}^{-1}D_{x_2} = (gA_{n_1})^{-1}D_{x_2} = A_{n_1}^{-1}g^{-1}D_{x_2}
\]so, we will get
\[
D_{x_2} = g^{-1}D_{x_2} \quad \text{and} \quad D_{x_2} = gD_{x_2}.
\]
If we write this equality as matrixes
\[
\begin{pmatrix}
y_1(t) \\
y_2(t)
\end{pmatrix} = \begin{pmatrix}
g_{11} & \cdots & g_{1n} \\
g_{n1} & \cdots & g_{nn}
\end{pmatrix}
\begin{pmatrix}
x_1(t) \\
x_2(t)
\end{pmatrix}
\]
\[
\begin{pmatrix}
y_1(t) \\
y_2(t)
\end{pmatrix} = \begin{pmatrix}
g_{11} & \cdots & g_{1n} \\
g_{n1} & \cdots & g_{nn}
\end{pmatrix}
\begin{pmatrix}
x_1(t) \\
x_2(t)
\end{pmatrix}
\]
\[
\begin{pmatrix}
y_1(t) \\
y_2(t)
\end{pmatrix} = \begin{pmatrix}
g_{11} & \cdots & g_{1n} \\
g_{n1} & \cdots & g_{nn}
\end{pmatrix}
\begin{pmatrix}
x_1(t) \\
x_2(t)
\end{pmatrix}
\]
Then we get
\[
y_2(t) = gx_2(t), \quad \forall t \in I.
\]
So for the same
\[
g \in GL(n, R),
\]
it is obtained that
\[
y_1(t) = gx_1(t) \quad \text{and} \quad y_2(t) = gx_2(t).
\]
Hence
\[
\begin{pmatrix}
x_1, x_2
\end{pmatrix} \sim \begin{pmatrix}
y_1, y_2
\end{pmatrix}.
\]

Theorem 2.3. Let
\[
G = GL(n, R)
\]and
$f_i(t), f_2(t), \ldots, f_n(t), f_{2i}(t), i = 0, \ldots, n - 1$, \hspace{1em} $(t \in I)$

be $C^\infty$-functions. Then there exists curves $x_1, x_2$ which $x_i$ is regular such that

$$\begin{bmatrix}
    x_1, x_1^{(i-1)}, x_1^{(i+1)}, \ldots, x_1^{(n-1)} \\
    x_2, x_2^{(i-1)}, x_2^{(i+1)}, \ldots, x_2^{(n-1)} \\
    \vdots \quad \vdots \quad \vdots \\
    x_n, x_n^{(i-1)}, x_n^{(i+1)}, \ldots, x_n^{(n-1)}
\end{bmatrix}
= f_{i1}(t), i = 0, \ldots, n - 1$$

and

$$\begin{bmatrix}
    x_1, x_1^{(i-1)}, x_1^{(i+1)}, \ldots, x_1^{(n-1)} \\
    x_2, x_2^{(i-1)}, x_2^{(i+1)}, \ldots, x_2^{(n-1)} \\
    \vdots \quad \vdots \quad \vdots \\
    x_n, x_n^{(i-1)}, x_n^{(i+1)}, \ldots, x_n^{(n-1)}
\end{bmatrix}
= f_{2i}(t), i = 0, \ldots, n - 1$$

**Proof.** From the previous proof, we take the matrix multiplication $A_{x_i}^{-1}A_{x_i}' = B$ such that $A_{x_i}' = A_{x_i}B$.

In here, matrix $B$ has the form

$$B = \begin{pmatrix}
    0 & \ldots & 0 & f_1(t) \\
    1 & \ldots & 0 & f_2(t) \\
    \vdots & \ddots & \vdots & \vdots \\
    0 & \ldots & 1 & f_n(t)
\end{pmatrix}$$

Then we have the following differential equation system from this multiplication:

$$x_{11}f_1(t) + x_{11}'f_2(t) + \ldots + x_{11}^{(n-1)}f_n(t) = x_{11}^{(n)}$$

$$x_{12}f_1(t) + x_{12}'f_2(t) + \ldots + x_{12}^{(n-1)}f_n(t) = x_{12}^{(n)}$$

$$\vdots$$

$$x_{1n}f_1(t) + x_{1n}'f_2(t) + \ldots + x_{1n}^{(n-1)}f_n(t) = x_{1n}^{(n)}$$

Let us take $x_{ii} = y_i, i = 1, \ldots, n$. So we can write the above differential equation system as

$$f_1(t)y + f_2(t)y' + \ldots + f_n(t)y^{(n-1)} - y^{(n)} = 0$$

It is known that the theory of differential equations, there exist one solution of this differential equation. Let $x_i(t) = (y_1, y_2, \ldots, y_n)$ be the solution. Then the curve $x_i(t)$ satisfies the conditions of the theorem.

Take the matrixes

$$A_2 = \begin{pmatrix}
    x_{11} & \ldots & x_{11}^{(n-2)} & x_{21} \\
    x_{12} & \ldots & x_{12}^{(n-2)} & x_{22} \\
    \vdots & \ddots & \vdots & \vdots \\
    x_{1n} & \ldots & x_{1n}^{(n-2)} & x_{2n}
\end{pmatrix}$$

and let $A_{x_i}^{-1}A_{x_2} = H$. So $A_{x_2} = A_{x_i}H$. Then we get the matrix $H$ as:

$$H = \begin{pmatrix}
    1 & 0 & \ldots & 0 & f_{20}(t) \\
    0 & 1 & \ldots & 0 & f_{21}(t) \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \ldots & 1 & f_{2n-2}(t) \\
    0 & 0 & \ldots & 0 & f_{2n-1}(t)
\end{pmatrix}$$

Since $A_2 = A_{x_i}H$, we have the following differential equation system:

$$x_{21} = x_{11}f_{20}(t) + x_{11}'f_{21}(t) + \ldots + x_{11}^{(n-1)}f_{2n-1}(t)$$

$$x_{22} = x_{12}f_{20}(t) + x_{12}'f_{21}(t) + \ldots + x_{12}^{(n-1)}f_{2n-1}(t)$$

$$\vdots$$

$$x_{2n} = x_{1n}f_{20}(t) + x_{1n}'f_{21}(t) + \ldots + x_{1n}^{(n-1)}f_{2n-1}(t)$$

So we get the curve $x_2 = \begin{pmatrix}
    x_{21} \\
    \vdots \\
    x_{2n}
\end{pmatrix}$, hence curves $x_1$ and $x_2$ satisfies the theorems rules.

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