Path Ramsey Number for Random Graphs

SHOHAM LETZTER
Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, UK
(e-mail: s.letzter@dpmms.cam.ac.uk)

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Answering a question raised by Dudek and Prałat, we show that if \( pn \to \infty \), w.h.p., whenever \( G = G(n, p) \) is 2-edge-coloured there is a monochromatic path of length \( (2/3 + o(1))n \). This result is optimal in the sense that \( 2/3 \) cannot be replaced by a larger constant.

As part of the proof we obtain the following result. Given a graph \( G \) on \( n \) vertices with at least \( (1 - \varepsilon)(n^2) \) edges, whenever \( G \) is 2-edge-coloured, there is a monochromatic path of length at least \( (2/3 - 110\sqrt{\varepsilon})n \). This is an extension of the classical result by Gerencsér and Gyárás which says that whenever \( K_n \) is 2-coloured there is a monochromatic path of length at least \( 2n/3 \).

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1. Introduction

Considering the richness of Ramsey theory and the great interest in random graphs, it is natural to consider Ramsey properties of random graphs. The study of random Ramsey theory has proved particularly useful in the establishment of upper bounds on the size Ramsey number. For graphs \( G, F, H \), we write \( G \to (F, H) \) if for every red–blue colouring of the edges of \( G \), there is either a red \( F \) or a blue \( H \). If \( F, H \) are isomorphic, we instead use the notation \( G \to H \). The size Ramsey number, denoted by \( \hat{r}(H) \), is defined to be

\[
\hat{r}(H) = \min \{|E(G)| : G \to H \}.
\]

Let \( P_n \) denote the path on \( n \) vertices. In [1], disproving a conjecture of Erdős [8], Beck showed that \( \hat{r}(P_n) \leq 900n \). In [5] Bollobás noted a slightly better bound, and recently Dudek and Prałat [7] gave an elementary proof of the bound \( \hat{r}(P_n) \leq 137n \). In fact, they proved that w.h.p. \( G(n, \alpha/n) \to P_{\beta n} \) for suitable constants \( \alpha, \beta \).

Dudek and Prałat [7] also showed that w.h.p. \( G(n, p) \to P_{(1/3+o(1))n} \) when \( pn \to \infty \) and raised the question of determining the maximum \( \ell \) such that \( G(n, p) \to P_{\ell} \) for \( pn \to \infty \). Inspired by the result of Gerencsér and Gyárás [9] which says that \( K_n \to P_{2n/3} \), they
Figure 1. A black and grey graph on $n$ vertices (the shaded part may be coloured arbitrarily) with no monochromatic path on more than $\lceil 2n/3 + 1 \rceil$ vertices.

asked if $G(n, p) \to P_l$ for $l = (2/3 + o(1))n$. Our main result answers this question in the affirmative.

**Theorem 1.1.** Let $0 < p = p(n) < 1$ satisfy $pn \to \infty$. Then w.h.p. $G(n, p) \to P_{(2/3 + o(1))n}$.

This result is essentially best possible since there is a 2-colouring of the edges of $K_n$ with no monochromatic path on more than $\lceil 2n/3 + 1 \rceil$ vertices. To see this, divide the vertex set of $K_n$ into two sets $A, B$ such that $|A| = \lfloor n/3 \rfloor$, let the edges spanned by $B$ be coloured red, and colour the other edges blue (see Figure 1). In fact, Gerencsér and Gyárfás [9] proved the following more general result.

**Theorem 1.2.** Let $n \geq k + \lceil (l + 1)/2 \rceil$. Then $K_n \to (P_{k+1}, P_{l+1})$.

In order to prove Theorem 1.1, we extend Theorem 1.2 to graphs with a large number of edges.

**Theorem 1.3.** Let $0 \leq \varepsilon \leq 1/64$, let $k \geq l$ and let $G$ be a graph on $n \geq k + \lceil (l + 1)/2 \rceil + 240\sqrt{\varepsilon}k$ vertices with at least $(1 - \varepsilon)\binom{n}{2}$ edges. Then $G \to (P_{k+1}, P_{l+1})$.

In particular, given $0 \leq \varepsilon \leq 1/64$, for every graph $G$ on $n$ vertices and at least $(1 - \varepsilon)\binom{n}{2}$ edges, $G \to P_k$, where $k = \lceil 2n/3 \rceil - 110\sqrt{\varepsilon}n$.

Theorem 1.3 is a consequence of the following similar result, in which we consider graphs with large minimum degree rather than a large number of edges.

**Theorem 1.4.** Let $0 < \varepsilon \leq 1/4$, let $k \geq l$ and let $G$ be a graph on $n \geq k + \lceil (l + 1)/2 \rceil + 100\sqrt{\varepsilon}k$ vertices with minimum degree at least $(1 - \varepsilon)n$. Then $G \to (P_{k+1}, P_{l+1})$.

It is easy to deduce Theorem 1.3 from Theorem 1.4. By an averaging argument, it suffices to prove the assertion for $n = k + \lceil (l + 1)/2 \rceil + 240\sqrt{\varepsilon}k$. By removing at most $\sqrt{\varepsilon}n$ vertices, we obtain a graph on $n' \geq (1 - \sqrt{\varepsilon})n$ vertices and minimum degree at least
$(1 - 2\sqrt{\varepsilon})n \geq (1 - 2\sqrt{\varepsilon})n'$ vertices. One can check that
\[(1 - \sqrt{\varepsilon})n \geq k + \lfloor (l + 1)/2 \rfloor + 200\sqrt{\varepsilon}k,
\]
so the assertion of Theorem 1.3 follows from Theorem 1.4, with $2\sqrt{\varepsilon}$ in place of $\varepsilon$. For the second part, it is easy to check that when $k = l = \lfloor 2n/3 \rfloor - 110\sqrt{\varepsilon}n$, it follows that $n \geq k + \lfloor (l + 1)/2 \rfloor + 240\sqrt{\varepsilon}k$.

In our proofs we shall use the following result, which was proved independently by Dudek and Prałat [7] and Pokrovskiy [12].

**Lemma 1.5.** For every graph $G$ there exist two disjoint subsets $U, W \subseteq V(G)$ of equal size such that there are no edges between them and $G \setminus (U \cup W)$ has a Hamilton path.

This result is the main tool used by Dudek and Prałat in [7] to prove the bound $\hat{r}(P_n) \leq 137n$. It turns out that their proof may be modified to give a better upper bound, as stated in the following theorem.

**Theorem 1.6.** For $n$ sufficiently large, $\hat{r}(P_n) \leq 91n$.

The rest of the paper is organized as follows. In Section 2 we give the proof of Lemma 1.5, as well as an easy but useful corollary. In Section 3 we give a short proof of a weaker version of Theorem 1.1 as well as the proof of the improved upper bound in Theorem 1.6.

We prove Theorem 1.4 in Section 4. In order to prove Theorem 1.1, we use the so-called sparse Regularity Lemma, due to Kohayakawa [11] and Rödl (see [6]). In Section 5 we state this result as well as some necessary notation. We prove Theorem 1.1 in Section 6 and finish with some concluding remarks in Section 7. Throughout the paper we omit floor and ceiling signs whenever they do not affect the arguments.

## 2. A useful lemma

In the proof of Theorems 1.1 and 1.4 we use the following lemma, which was obtained independently by Dudek and Prałat [7] and Pokrovskiy [12]. For the sake of completeness, we prove it here.

**Lemma 1.5.** For every graph $G$ there exist two disjoint subsets $U, W \subseteq V(G)$ of equal size such that there are no edges between them and $G \setminus (U \cup W)$ has a Hamilton path.

**Proof.** In order to find sets with the desired properties, we apply the following algorithm, maintaining a partition of $V(G)$ into subsets $U, W$ and a path $P$. Start with $U = V(G), W = \emptyset$ and $P$ an empty path. At each stage of the algorithm, do the following. If $|U| \leq |W|$, stop. Otherwise, if $P$ is empty, move a vertex from $U$ to $P$ (note that $U \neq \emptyset$). If $P$ is non-empty, let $v$ be its endpoint. If $v$ has a neighbour $u$ in $U$, put $u$ in $P$, otherwise move $v$ to $W$. 

Note that at any given point in the algorithm there are no edges between \( U \) and \( W \). Furthermore, the value \(|U| - |W|\) is positive at the beginning of the algorithm and decreases by one at every stage, thus at some point the algorithm will stop and will produce sets \( U, W \) with the required properties.

Occasionally it is easier to use the following immediate consequence of Lemma 1.5.

**Corollary 2.1.** Let \( G \) be a balanced bipartite graph on \( n \) vertices with bipartition \( \{V_1, V_2\} \) which has no path of length \( k \). Then there exist disjoint subsets \( X_i \subseteq V_i \) such that \(|X_1| = |X_2| \geq (n-k)/4\) and \( e(G[X_1, X_2]) = 0 \).

**Proof.** By Lemma 1.5, there exist disjoint subsets \( U, W \subseteq V(G) \) of equal size such that \( e(G[U, W]) = 0 \) and \( V(G) \setminus (U \cup W) \) has a Hamilton path \( P \). Note that \( P \) must alternate between \( V_1 \) and \( V_2 \) and has an even number of vertices, implying that \(|V_1 \cap V(P)| = |V_2 \cap V(P)|\). It follows that \(|U_1| + |W_1| = |U_2| + |W_2|\), where \( U_i = U \cap V_i \) and \( W_i = W \cap V_i \). Since \(|U| = |W|\), we conclude that \(|U_1| = |W_2|\) and \(|U_2| = |W_1|\). Without loss of generality, suppose that \(|U_1| \geq |U_2|\). Then \(|U_1| = |W_2| \geq (n - |V(P)|)/4 \geq (n-k)/4\). Take \( X_1 = U_1 \) and \( X_2 = W_2 \).

**3. An improved upper bound on the size Ramsey number for paths**

Before we turn to the proofs of Theorems 1.4 and 1.1, we demonstrate how Lemma 1.5 alone can be used to obtain results about the path Ramsey number of random graphs. We start by proving the following weaker version of Theorem 1.1, using only elementary tools.

**Lemma 3.1.** Let \( 0 < p = p(n) < 1 \) and assume that \( pn \to \infty \). Then w.h.p. \( G(n, p) \to P_l \) for some \( l = (1/2 + o(1))n \).

In the proof of Lemma 3.1 we use the following easy consequence of Corollary 2.1.

**Corollary 3.2.** Let \( G \) be a graph on \( n \) vertices such that \( G \not\to P_{k+1} \). Then there exist disjoint subsets \( X, Y \subseteq V(G) \) of size at least \((n-2k)/4\) such that \( e(G[X, Y]) = 0 \).

**Proof.** Consider a red–blue colouring of \( G \) with no monochromatic \( P_{k+1} \). By Lemma 1.5, there exist disjoint sets \( U, W \), both of size at least \((n-k)/2\), with no red edges between them. Considering the graph \( G[U, W] \), it follows from Corollary 2.1 that there exist sets \( X \subseteq U, Y \subseteq W \) of size at least \((n-2k)/4\), with no blue edges between them. We conclude that there are no edges of \( G \) between \( X \) and \( Y \).

We are now ready for the proof of Lemma 3.1, which is a relaxed version of Theorem 1.1.
Proof of Lemma 3.1. Let $G = G(n, p)$, where $np \to \infty$. Given $\alpha > 0$, suppose $G \not\xrightarrow{\text{a.s.}} P_{(1/2-\alpha)n}$. By Corollary 3.2, there exist disjoint subsets $X, Y \subseteq V(G)$ of size at least $\alpha n/2$ with no edges of $G$ between them. But this is a contradiction, since w.h.p. every two disjoint sets of at least $\alpha n/2$ vertices in $G$ have an edge between them. It follows that for every $\alpha > 0$ w.h.p. $G(n, p) \to P_{(1/2-\alpha)n}$.

Corollary 3.2 can be used to obtain an improvement over the upper bound $\hat{r}(P_n) \leq 137n$, which was obtained by Dudek and Praš [7].

**Theorem 1.6.** For $n$ sufficiently large, $\hat{r}(P_n) \leq 91n$.

We note that our proof is very similar to the proof in [7], the main difference being our use of Corollary 3.2. We shall use the following lemma.

**Lemma 3.3.** Let $c = 4.86$, $d = 7.7$ and $G = G(cn, d/n)$. Then w.h.p. the following two conditions hold.

(i) $|E(G)| \leq (1 + o(1)) c^2d n/2$.

(ii) For every two disjoint sets $U, W \subseteq V(G)$ of size at least $(c-2)n/4$, we have $e(G[U, W]) > 0$.

**Proof.** The number of edges in $G$ is a binomial random variable with mean $\binom{cn}{2} \cdot \frac{d}{n} = (1 + o(1)) \frac{c^2d}{2} n$.

Condition (i) follows from the concentration of binomial random variables around their mean.

We prove condition (ii) by the first moment method. Let $Z$ denote the number of pairs $(U, W)$ of disjoint subsets of $V(G)$ of size $(c-2)n/4$ with $e(G[U, W]) = 0$. The expectation of $Z$ satisfies the following, where $x = (c-2)/4$:

$$E[Z] = \binom{cn}{zn} \binom{(c-2)n}{zn} \left(1 - \frac{d}{n}\right)^{(zn)^2} \leq \frac{(zn)!}{((zn))!((c-2)n)!} \exp(-dx^2n) \leq \exp(\beta n).$$

By Stirling’s formula (which states that $n! = (1 + o(1))\sqrt{2\pi n}(n/e)^n$), we can take

$$\beta = (c \log c - 2x \log x - (c-2x) \log(c-2x) - dx^2) \leq -0.0005.$$ 

It follows that $E[Z] \to 0$, implying that w.h.p. $Z = 0$, and hence condition (ii) holds.

**Remark.** The constants $c, d$ in Lemma 3.3 were chosen to minimize the number of edges in $G$ under condition (ii).

The proof of Theorem 1.6 follows easily from Lemma 3.3.

**Proof.** Pick $c = 4.86$ and $d = 7.7$ as in Lemma 3.3 and denote $G = G(cn, d/n)$. If $G \not\xrightarrow{\text{a.s.}} P_n$ then by Corollary 3.2 there exist disjoint subsets $X, Y \subseteq V(G)$ of size at least $(c-2)n/4$
such that $e(G[X, Y]) = 0$, contradicting condition (ii) from Lemma 3.3. We conclude that w.h.p., $G \to P_n$. By condition (i), we have that w.h.p., $|E(G)| \leq 91n$. It follows that $\hat{r}(P_n) \leq 91n$ for large enough $n$.

\section{Path Ramsey number for dense graphs}

Before turning to the proof of Theorem 1.4, we remind the reader of its statement.

\textbf{Theorem 1.4.} Let $0 < \varepsilon < 1/4$, let $k \geq l$ and let $G$ be a graph on $n \geq k + \lfloor (l + 1)/2 \rfloor + 100\varepsilon k$ vertices with minimum degree at least $(1 - \varepsilon)n$. Then $G \to (P_{k+1}, P_{l+1})$.

\textbf{Proof.} We start by proving Theorem 1.4 under the assumption that $k < (1/2 - \varepsilon)n$. If there is no red $P_{k+1}$, by Lemma 1.5 there exist disjoint sets $W_1, W_2$ of size at least $(n - k)/2$ with no red edges between them. Since $G$ has minimum degree at least $n(1 - \varepsilon)$, we can greedily find a blue path on at least the following number of vertices, implying the existence of a blue $P_{k+1}$:

$$|W_1| + |W_2| - 2\varepsilon n \geq n - k - 2\varepsilon n = 2(1/2 - \varepsilon)n - k > k.$$  

Hence, we can assume from now on that $k \geq (1/2 - \varepsilon)n \geq n/4$, so every vertex in $G$ has at most $4\varepsilon k$ non-neighbours. Putting $\delta = 4\varepsilon$, we have $n \geq k + \lfloor (l + 1)/2 \rfloor + 25\delta k$. Furthermore, we can assume that $\delta k \geq 2$, otherwise $G$ is a complete graph and Theorem 1.4 follows directly from Theorem 1.2.

The proof proceeds by induction. Clearly, the assertion of Theorem 1.4 holds if $k = 1$, so we may assume that $k \geq 2$. If $k > l$, then by induction there is either a red $P_k$ or a blue $P_{l+1}$; in the latter case we are done. If $k = l$, then by induction there is either a red or a blue $P_k$. Thus, without loss of generality, there is a red path on $k$ vertices which we denote by $P = (v_1 \ldots v_k)$. Let $U = V(G) \setminus V(P)$. We consider three cases.

\textbf{Case 1:} $G[U]$ contains a blue path $Q$ on $13\delta k$ vertices. Let $Q_1$ be a maximal path extending $Q$ by alternating between vertices of $P$ and $U$ and which has both ends in $U$. Let $U' = U \setminus V(Q_1)$ and $V' = V(P) \setminus V(Q_1)$. Let $Q_2$ be a maximal path alternating between $U'$ and $V'$ which has both ends in $U'$. Denote the ends of $Q_1$ by $x_i, y_i$, for $i = 1, 2$. We show that $|V(Q_1)| + |V(Q_2)| \geq l + 3\delta k$.

Suppose this is not the case. In particular, since $|U| \geq l/2 + 25\delta k$, the paths $Q_1$ and $Q_2$ do not cover $U$. Pick a vertex $z \in U \setminus (V(Q_1) \cup V(Q_2))$. Note that all but at most $3\delta k$ vertices of $P$ are adjacent to all of $x_1, x_2, z$. By our assumption on the lengths of $Q_1$ and $Q_2$, the number of vertices of $P$ which are in $Q_1$ or $Q_2$ is at most $(1/2 - 5\delta)k$, hence there exist vertices $v_i, v_{i+1}$ which are adjacent to all of $x_1, x_2, z$. We assume that $v_i$ and $v_{i+1}$ have no common red neighbours in $x_1, x_2, z$ because otherwise we obtain a red $P_{k+1}$. It follows that, without loss of generality, $v_i$ is joined in blue to two of $x_1, x_2, z$, contradicting the maximality of $Q_1$ and $Q_2$.

Let $Q'_2$ be a subpath of $Q_2$ with ends $x'_2, y'_2 \in U$ satisfying

$$|V(Q'_2)| + |V(Q_1)| = l + 3\delta k.$$
Similarly to the above, there exist vertices $v_i, v_{i+1}$ which are both neighbours of all of $x_1, y_1, x_2', y_2'$. By the maximality of $Q_1$, none of $v_i, v_{i+1}$ is adjacent in blue to one of $x_1, y_1$ and to one of $x_2', y_2'$. Furthermore, we may assume that $v_i$ and $v_{i+1}$ have no common red neighbour in $\{x_1, y_1, x_2', y_2'\}$. Thus, without loss of generality, $x_1, y_1$ are blue neighbours of $v_i$ and $x_2', y_2'$ are blue neighbours of $v_{i+1}$. Let $C_1$ and $C_2$ denote the blue cycles obtained by adding $v_i$ to $Q_1$ and $v_{i+1}$ to $Q_2$, and let $U_1 = V(C_1) \cap U$.

We assume that there is no blue $P_{i+1}$. It follows that $|V(C_1)| \leq l$, implying that $|U_2| \geq \frac{3}{2} \delta k$, and that there are no blue edges between $C_1$ and $C_2$.

The number of vertices in $V(P) \setminus (V(C_1) \cup V(C_2))$ is at least $(1/2 + 3\delta)k$, hence there exists $j$ such that $v_j, v_{j+1} \notin V(C_1) \cup V(C_2)$. Note that none of $v_j$ and $v_{j+1}$ can have blue neighbours in both $U_1$ and $U_2$, because otherwise we obtain a blue $P_{i+1}$. Also, we can assume that $v_j, v_{j+1}$ have no red common neighbour in either $U_1$ or $U_2$, because otherwise there is a red $P_{i+1}$. Thus, recalling that $v_j, v_{j+1}$ have at most $\delta k$ non-neighbours in $G$, without loss of generality, $v_j$ is joined in red to all but at most $\delta k$ vertices of $U_1$ and $v_{j+1}$ is joined in red to all but at most $\delta k$ vertices in $U_2$.

Let $w_2 \in U_2$ be any red neighbour of $v_{j+1}$ (such a vertex exists because $|U_2| \geq \frac{3}{2} \delta k$). Since $w_2$ is connected to all but at most $\delta k$ vertices of $U_1$ and these edges must all be red, $U_1$ contains a vertex $w_1$ which is a red neighbour of both $w_2$ and $v_j$ (such a vertex exists because $|U_1| \geq |V(Q_1)| \geq 13\delta k$). We obtain a red path $(v_1 \ldots v_j w_1 w_2 v_{j+1} \ldots v_k)$ on $k + 2$ vertices. This finishes the proof of Theorem 1.3 in the first case.

**Case 2:** $l \leq (1 - 16\delta)k$. Let $Q_1$ be a maximal blue path alternating between $U$ and $P$ with both ends in $U$. Similarly, let $Q_2$ be a maximal blue path with both ends in $U$, alternating between $U \setminus V(Q_1)$ and $V(P) \setminus V(Q_1)$. As in the previous case, it can be shown that $|V(Q_1)| + |V(Q_2)| \geq l + 6\delta k$.

Let $Q'_2$ be a subpath of $Q_2$ such that $|V(Q_1)| + |V(Q'_2)| = l + 6\delta k$. As before, there exists $j$ such that $v_j, v_{j+1} \notin V(Q_1) \cup V(Q'_2)$ and $v_1, v_2$ are joined in $G$ to all ends of the two paths. The vertices $v_j, v_{j+1}$ can be used to extend $Q_1, Q'_2$ into blue vertex-disjoint cycles $C_1, C_2$, whose sum of lengths is $l + 6\delta k$ and each of which has length at least $6\delta k$. The proof of Theorem 1.4 can now be finished as in the first case.

**Case 3:** $l \geq (1 - 16\delta)k$ and $G[U]$ contains no blue path of length at least $13\delta k$. We conclude from Lemma 1.5 that there exist two disjoint sets $W_1, W_2 \subseteq U$ with no blue edges between them of equal size satisfying the following inequality:

$$|W_1| = |W_2| \geq \frac{1}{2}(|U| - 13\delta k) = \frac{1}{2}(l/2 + 12\delta k) \geq \frac{1}{2}(l/2 + 4\delta k).$$

Since every vertex in $G$ is adjacent to all but at most $\delta k$ vertices, we can greedily find a red path $Q$ in $U$ such that the following holds:

$$|V(Q)| \geq |W_1| + |W_2| - 2\delta k = (1/2 + 2\delta)k.$$ 

Let $X$ be the set of the first and last $(1/4 + \delta)k$ vertices of $P$. We assume that there is no red edge between $X$ and $Q$, because otherwise there is a red $P_{k+1}$. Note that $|V(Q)| \geq |X| \geq (1/2 + 2\delta)k$; hence we may greedily construct a blue path alternating between $X$ and $V(Q)$ on at least the following number of vertices:

$$2|X| - 2\delta k \geq (1 + 2\delta)k \geq l + 1.$$
Hence there exists a blue $P_{t+1}$, completing the proof of Theorem 1.4.

5. Sparse Regularity Lemma

We shall make use of a variant of Szemerédi’s Regularity Lemma [13] for sparse graphs, often referred to as the sparse Regularity Lemma, which was proved independently by Kohayakawa [11] and Rödl (see [6]). Before stating the theorem, we introduce some notation.

Given two disjoint sets of vertices $U, V$ in a graph, we define the density $d_p(U, V)$ of edges between $U$ and $V$ with respect to $p$ to be

$$d_p(U, V) = \frac{e(G[U, V])}{p|U||V|},$$

where $e(G[U, V])$ is the number of edges between $U$ and $V$. We say that a bipartite graph with bipartition $U, V$ is $(\varepsilon, p)$-regular if, for every $U' \subseteq U, V' \subseteq V$ with $|U'| \geq \varepsilon|U|, |V'| \geq \varepsilon|V|$, the density $d_p(U', V')$ satisfies $|d_p(U', V') - d_p(U, V)| \leq \varepsilon$.

A graph $G$, a partition $\{V_1, \ldots, V_t\}$ of $V(G)$ is called an $(\varepsilon, p)$-regular partition if it is an equipartition (i.e., the sizes of the sets differ by at most one), and if all but at most $\varepsilon$ of the pairs induce an $(\varepsilon, p)$-regular graph.

Given $0 < \eta, p < 1, D \geq 1$, a graph $G$ is called $(\eta, p, D)$-upper-uniform if, for all disjoint sets of vertices $U_1, U_2$ of size at least $\eta|V(G)|$, the density $d_p(U_1, U_2)$ is at most $D$.

We are now ready to state the sparse Regularity Lemma of Kohayakawa and Rödl.

**Theorem 5.1.** For every $\varepsilon > 0$, $t$ and $D > 1$ there exist $\eta > 0$ and $T$ such that for every $0 \leq p \leq 1$, every $(\eta, p, D)$-upper-uniform graph admits an $(\varepsilon, p)$-regular partition into $s$ parts where $t \leq s \leq T$.

We shall use a variant of Theorem 5.1, namely the coloured version of the sparse Regularity Lemma.

**Theorem 5.2.** For every $\varepsilon > 0$, $t, l$ and $D > 1$ there exist $\eta > 0$ and $T$ such that for every $0 \leq p \leq 1$, if $G_1, \ldots, G_l$ are $(\eta, p, D)$-upper-uniform graphs on vertex set $V$, there is an equipartition of $V$ into $s$ parts, where $t \leq s \leq T$, for which all but at most $\varepsilon$ of the pairs induce a regular pair in each $G_i$.

6. Path Ramsey number for random graphs

We are now ready to prove Theorem 1.1, whose statement is as follows.

**Theorem 1.1.** Let $0 < p = p(n) < 1$ satisfy $pn \to \infty$. Then w.h.p. $G(n, p) \to P_{(2/3+\alpha(1))n}$.

**Proof.** Let $0 < p < 1$ be such that $pn \to \infty$ and let $\alpha > 0$. We show that w.h.p. for every 2-edge-colouring of $G = G(n, p)$ there is a monochromatic path of length at least $(2/3 - \alpha)n$. 
Pick $\varepsilon > 0$ small and $t$ large (taking $t = 1/\varepsilon$ and $\varepsilon \leq 1/64$ small enough such that $110\sqrt{\varepsilon} + 7\varepsilon \leq \alpha$ would do). Let $\eta, T$ be the constants arising from the application of Theorem 5.2 with $\varepsilon, t, l = 2, D = 2$. Note that w.h.p. for every two disjoint subsets $U, W \subseteq V(G)$ of size at least $\eta n$, we have

$$1/2 \leq d_p(U, W) \leq 2. \quad (6.1)$$

In particular, $G$ is w.h.p. $(\eta, p, 2)$-upper-uniform. Thus, by Theorem 5.2, given a 2-edge-colouring of $G$, there exists an $(\varepsilon, p)$-regular partition $V_1, \ldots, V_s$ with $t \leq s \leq T$. By (6.1), we may assume that $d_p(V_i, V_j) \geq 1/2$ for every $1 \leq i < j \leq s$.

Let $H$ be the auxiliary graph with vertex set $[s]$, where $ij$ is an edge if and only if $V_i, V_j$ induce a regular bipartite graph in both red and blue. We colour an edge $ij$ in $H$ red if the red density $d_p(V_i, V_j)$ is at least $1/4$ and blue otherwise (so if $ij$ is blue, the blue density is at least $1/4$).

Since the partition $V_1, \ldots, V_s$ is $(\varepsilon, p)$-regular, the number of edges in $H$ is at least $(1 - \varepsilon)(\delta/2)$. It follows from Theorem 1.3 that $H$ contains a monochromatic path $P$ on at least $l = (2/3 - \delta)s$ vertices, where $\delta = 110\sqrt{\varepsilon}$ (assuming $\varepsilon > 0$ is small enough). Let $i_1, \ldots, i_l$ denote the vertices of $P$.

Assuming without loss of generality that $P$ is red, we show that $G$ contains a red path of length at least $(2/3 - \alpha)n$. We divide each set $V_{ij}$ into two sets $U_{ij}, W_{ij}$ of equal sizes, so $|U_{ij}| = n/2s$. Let $P_j$ be a longest red path in the bipartite graph $G[U_{ij}, W_{ij+1}]$. The following claim shows that $P_j$ covers most vertices in $U_j \cup W_{j+1}$.

**Claim 6.1.** For every $1 \leq j \leq l$, $P_j$ covers at least $1 - 4\varepsilon$ of the vertices of $U_j \cup W_{j+1}$.

**Proof.** Suppose that for some $j$, $P_j$ covers at most $1 - 4\varepsilon$ of the vertices of $U_j \cup W_{j+1}$. Set $U = U_j$ and $W = W_{j+1}$. By Corollary 2.1, there exist sets $X \subseteq U, Y \subseteq W$ with $|X| = |Y| \geq \varepsilon|U|$, such that there are no red edges between $X$ and $Y$. But by the regularity of the partition $V_1, \ldots, V_s$, the density $d_p(U, V)$ is within $\varepsilon$ of the density of red edges between $U$ and $W$, which is at least $1/4$. In particular, $G$ has a red edge between $X$ and $Y$, contradicting our assumption, thus proving Claim 6.1.

We now show that the paths $P_1, \ldots, P_{l-1}$ can be joined to a path $Q$ while losing only a few of the vertices. Let $X_j$ be the set of first $2\varepsilon|V_1|$ vertices of $P_j$ and similarly let $Y_j$ be the set of last $2\varepsilon|V_1|$ vertices of $P_j$. Since the paths $P_j$ alternate between the sets $U_{ij}, W_{j+1}$, we have that $|Y_j \cap V_{ij}|, |X_{j+1} \cap V_{ij+1}| \geq \varepsilon|V_1|$. It follows from the fact that $ij_{j+1}$ is a red edge in $H$ that there is a red edge between $Y_j$ and $X_{j+1}$. Hence $G$ has a red path $Q$ which contains all vertices of $V(P_1) \cup \ldots \cup V(P_{l-1})$ but at most $4\varepsilon|V_1|(l - 1)$.

By Claim 6.1, we have that $|P_j| \geq (1 - 4\varepsilon)|V_1|$, so the following holds:

$$|Q| \geq (1 - 8\varepsilon)(l - 1)|V_1| = (1 - 8\varepsilon)(s(2/3 - \delta) - 1) \cdot \frac{n}{s} \geq (2/3 - (\delta + 1/t + 6\varepsilon))n \geq (2/3 - \alpha)n.$$  

This completes the proof of Theorem 1.1. \qed
7. Concluding remarks

We remark that stronger versions of Theorem 1.4 for the symmetric case $k = l$ were proved by Benevides, Łuczak, Scott, Skokan and White [3] and by Gyárfás and Sárközy [10]. The results in [3] imply in particular that, for every $\varepsilon > 0$, there exists $n_0$ such that for every graph $G$ on $n \geq n_0$ vertices with minimum degree $\delta(G) \geq 3n/4$ satisfies $G \to P_{(2/3 - \varepsilon)n}$. The condition on the minimum degree is best possible. The proofs in [3] and [10] rely heavily on the Regularity Lemma, whereas our proof of Theorem 1.4 is elementary.

It may be interesting to strengthen Theorem 1.4 so as to prove a result similar to the aforementioned result of [3] in the non-diagonal case, namely when $k$ and $l$ are not necessarily equal. Furthermore, it may be interesting to obtain the result of [3] using elementary methods.

Finally, we note that the gap between the best known lower and upper bounds on the size Ramsey number is still very wide. Theorem 1.6 gives an upper bound of $\hat{r}(P_n) \leq 91n$, which is to our knowledge the best known upper bounds. Bollobás [4] proved the best known lower bound of $\hat{r}(P_n) \geq (1 + \sqrt{2})n - 2$, improving Beck’s result [2] who showed that $\hat{r}(P_n) \geq \frac{9}{4}n$. It would be very interesting to try to close this gap.

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