Defining the integers in large rings of a number field using one universal quantifier

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dedicated to Yuri Matiyasevich on his 60th birthday

... the undecidable poem “В Петербурге мы сойдемся снова” ...(18)

Abstract

Julia Robinson has given a first-order definition of the rational integers \( \mathbb{Z} \) in the rational numbers \( \mathbb{Q} \) by a formula \((\forall\exists\forall\exists)(F = 0)\) where the \( \forall \)-quantifiers run over a total of 8 variables, and where \( F \) is a polynomial.

We show that for a large class of number fields, not including \( \mathbb{Q} \), for every \( \varepsilon > 0 \), there exists a set of primes \( \mathcal{S} \) of natural density exceeding \( 1 - \varepsilon \), such that \( \mathbb{Z} \) can be defined as a subset of the “large” subring

\[ \{ x \in K : \text{ord}_p x \geq 0, \forall p \notin \mathcal{S} \} \]

of \( K \) by a formula where there is only one \( \forall \)-quantifier. In the case of \( \mathbb{Q} \) we will need two quantifiers. We also show that in some cases one can define a subfield of a number field using just one universal quantifier.

Introduction

Julia Robinson’s work In 1949 (19) Julia Robinson showed that the set of integers \( \mathbb{Z} \) is definable in the language of rings in the field of rational numbers \( \mathbb{Q} \) by a first-order formula. This implies that the first order theory of fields in undecidable.
The quantifier complexity of this formula was analysed in [1]: it is equivalent to a formula of the form
\[(\forall x_1^{(1)} \ldots x_5^{(1)})(\exists y_1^{(1)} \ldots y_4^{(1)})(\forall x_1^{(2)} \ldots x_3^{(2)})(\exists y_1^{(2)}): F(x, y) = 0,\]
where \(F\) is a polynomial over \(\mathbb{Z}\) in multi-variables
\[x = (x_1^{(1)}, \ldots, x_5^{(2)})\text{ and } y = (y_1^{(1)}, \ldots, y_4^{(1)}, y_1^{(2)}).\]

Given the results of Julia Robinson one may ask by just how complex a formula can \(\mathbb{Z}\) be defined in \(\mathbb{Q}\)? Here, “complex” refers to how many quantifiers of what kind need to be used, and how many quantifier alterations are necessary. In view of Hilbert’s Tenth Problem, it is particularly relevant to reduce the number of universal quantifiers — since a definition of \(\mathbb{Z}\) in \(\mathbb{Q}\) without any universal quantifiers would imply that Hilbert’s Tenth Problem for \(\mathbb{Q}\) has a negative answer, and disprove a conjecture of Mazur on the topology of rational points, cf. [3]. The above formula has a total of eight universal quantifiers and three quantifier alterations.

**Recent developments**

(a) In [1], it was shown that a (heuristically probable) conjecture about elliptic curves allows one to give a model of \(\mathbb{Z}\) over \(\mathbb{Q}\) involving only one universal quantifier. Here, a model is essentially a countable definable set over \(\mathbb{Q}\) with a bijection to \(\mathbb{Z}\) such that via this bijection, the graphs of addition and multiplication on \(\mathbb{Z}\) are definable subsets over \(\mathbb{Q}\).

(b) Recall that any ring in between \(\mathbb{Z}\) and \(\mathbb{Q}\) is of the form \(\mathbb{Z}[\frac{1}{S}]\) for some set of primes \(S\). In [6], Poonen showed that there is a set of primes \(S\) of full natural density such that \(\mathbb{Z}\) has a model in \(\mathbb{Z}[\frac{1}{S}]\) involving no universal quantifiers whatsoever. Note however that this does not settle the question of defining \(\mathbb{Z}\) as a diophantine subset of \(\mathbb{Z}[\frac{1}{S}]\). The result was extended to number fields in [8].

(c) Let \(K\) be a number field and let \(\mathcal{S}_K\) be a set of primes of \(K\). Define \(O_{K,\mathcal{S}_K}\) to be the following ring:
\[O_{K,\mathcal{S}_K} := \{x \in K : \text{ord}_p x \geq 0, \forall p \notin \mathcal{S}_K\}.\]
If \(\mathcal{S}_K\) is infinite we will call these rings “big” or “large”. The second named author has given an existential definition of \(\mathbb{Z}\) in some large subrings of the following fields: totally real fields, their extensions of degree 2 and fields such that there exists an elliptic curve defined over \(\mathbb{Q}\), of positive rank over \(\mathbb{Q}\) and of the same rank over the field in question (see [11, 12, 13, 14, 15]).
The density of the set of inverted primes in these subrings is, however, always bounded away from 1, essentially by $1 - \frac{1}{[K:\mathbb{Q}]}$.

(d) In [4] Poonen showed that $\mathbb{Z}$ can be defined in any number field using two universal quantifiers and there are exist big subrings of $\mathbb{Q}$ with the set of inverted primes of density arbitrarily close to one and where $\mathbb{Z}$ is definable using just one universal quantifier.

Main results The aim of this work is to consider the problem of improving some of the above results in the following sense: (a) unconditional on any conjectures; (b) for subrings $\mathbb{Z}[\frac{1}{S}]$, where $S$ is “large” in the sense of arbitrary high density $< 1$; (c) such that it defines the actual subset $\mathbb{Z}$ of this large subring; (d) for other number fields instead of $\mathbb{Q}$.

The main results are as follows:

**Theorem 1.** Let $K \neq \mathbb{Q}$ be a number field of one of the following types:

1. $K$ is totally real;
2. $K$ is an extension of degree two of a totally real number field;
3. There exists an elliptic curve defined over $\mathbb{Q}$ and of positive rank over $\mathbb{Q}$ such that this curve preserves its rank over $K$;

Then for every $\varepsilon > 0$, there exists a set of primes $\mathfrak{W}_K$ of $K$ of natural density exceeding $1 - \varepsilon$, such that $\mathbb{Z}$ can be defined as a subset of $O_K,\mathfrak{W}_K$ by a formula with only one $\forall$-quantifier.

**Theorem 2.** Let $K$ be a number field, including $\mathbb{Q}$. Assume there exists an elliptic curve defined over $K$ of rank 1 over $K$. Then for every $\varepsilon > 0$, there exists a set of primes $\mathfrak{W}_K$ of $K$ of natural density exceeding $1 - \varepsilon$, such that $\mathbb{Z}$ can be defined as a subset of $O_K,\mathfrak{W}_K$ by a formula with only two $\forall$-quantifiers.

Observe that the fact that $\mathbb{Z}$ can be defined over $\mathbb{Q}$ using two quantifiers does not imply directly that $\mathbb{Z}$ can be defined over a ring of integers using two quantifiers: in translating a definition over $\mathbb{Q}$ to a ring of integers, one has to represent a rational number as a ratio of two elements of the ring. Thus a “mechanical” translation of Poonen’s result over $\mathbb{Q}$ would produce a definition with four universal quantifiers.

As we have mentioned above, in [4], Poonen also proved a that integers can be defined using just one quantifier over a big subring of $\mathbb{Q}$. His result was obtained by different techniques and the sets of inverted primes are different from ours: in [4], the inverted primes are inert in a finite union of
quadratic extensions, whereas in the above theorem, primes without relative degree one factors in a fixed extension are inverted, together with a density zero set related to the elliptic curve used in the construction. These results raise the question of characterization of large subrings of \( \mathbb{Q} \) in which \( \mathbb{Z} \) admits a diophantine definition, or a diophantine model, or a definition or model using \( n \geq 1 \) universal quantifiers; and in particular whether there is any difference between these rings. Results like those above and in [4] should be seen as a first contribution to this type of questions.

We also prove the following theorems concerning definability with only one quantifier.

**Theorem 3.** Let \( K \neq \mathbb{Q} \) be a number field. Assume there exists an elliptic curve defined over \( K \) of rank 1 over \( K \). Then for every \( \varepsilon > 0 \), there exists a set of primes \( \mathfrak{W}_K \) of \( K \) of natural density exceeding \( 1 - \varepsilon \), such that \( \mathbb{Q} \cap \mathcal{O}_{K,\mathfrak{W}_K} \) can be defined over \( \mathcal{O}_{K,\mathfrak{W}_K} \) by a formula with only one \( \forall \)-quantifier.

**Theorem 4.** Let \( M/K \) be a number field extension. Assume there exists an elliptic curve \( E \) defined over \( K \) such that \( \text{rank} E > 0 \) and \( [E(M) : E(K)] \neq \infty \). Let \( \mathfrak{W}_M \) be any set of \( M \) primes (including the set of all \( M \)-primes and the empty set). Then \( \mathcal{O}_{M,\mathfrak{W}_M} \cap K \) is definable over \( \mathcal{O}_{M,\mathfrak{W}_M} \) using just one universal quantifier.

1. **Elliptic Curves and Existential Models of \( (\mathbb{Z},+,|) \) over \( \mathcal{O}_{K,\mathfrak{W}} \)**

1.1. In this section we will use elliptic curves to define divisibility in large rings. Most of the technical details are taken from [5] and [8].

1.2 **Notation.** The following notation will be used for the rest of this section.

- \( K \) is a number field.
- \( E \) is an elliptic curve of rank 1 defined over \( K \) (in particular, we assume such an \( E \) exists).
- We fix a Weierstrass equation \( W : y^2 = x^3 + ax + b \) for \( E \) with coefficients in the ring of integers of \( K \).
- \( E(K)_{\text{tors}} \) is the torsion subgroup of \( E(K) \).
- \( t \) is an even multiple of \( \#E(K)_{\text{tors}} \).
• $Q \in E(K)$ is such that $Q$ generates $E(K)/E(K)_{\text{tors}}$.

• $P := tQ$.

• $\mathcal{P}_Q = \{2, 3, 5, \ldots\}$ is the set of rational primes.

• $\mathcal{P}_K$ is the set of all finite primes of $K$.

• Let $\mathcal{S}_{\text{bad}} = \mathcal{S}_{\text{bad}}(W, P, K) \subseteq \mathcal{P}_K$ consist of the primes that ramify in $K/Q$, the primes for which the reduction of the chosen Weierstrass model is singular (this includes all primes above 2), and the primes at which the coordinates of $P$ are not integral.

• $h_K$ is the class number of $K$.

• Write $nP = (x_n, y_n) = (x_n(P), y_n(P))$ where $x_n, y_n \in K$.

• Let the divisor of $x_n(P)$ be of the form
  \[
  \frac{a_n}{b_n} b_n = \frac{a_n(P)}{b_n(P)} b_n(P)
  \]
  where
  \begin{align*}
  - \quad & a_n = \prod q^{-a_q}, \text{ where the product is taken over all primes } q \text{ of } K \text{ not in } \mathcal{S}_{\text{bad}} \text{ such that } a_q = \text{ord}_q x_n < 0. \\
  - \quad & b_n = \prod q^{a_q}, \text{ where the product is taken over all primes } q \text{ of } K \text{ not in } \mathcal{S}_{\text{bad}} \text{ such that } a_q = \text{ord}_q x_n > 0. \\
  - \quad & b_n = \prod q^{a_q}, \text{ where the product is taken over all primes } q \in \mathcal{S}_{\text{bad}} \text{ and } a_q = \text{ord}_q x_n.
  \end{align*}

• For $n$ as above, let $\mathcal{I}_n = \mathcal{I}_n(P) = \{p \in \mathcal{P}_K : p | a_n\}$. By definition of $\mathcal{S}_{\text{bad}}$ and $\mathfrak{a}_n$, we have $\mathcal{I}_1 = \emptyset$.

• For $\ell \in \mathcal{P}_Q$, define $a_\ell$ to be the smallest positive number such that $\ell^{a_\ell} > C$, where $C$ is defined in Proposition 1.3 below. For all but finitely many primes $\ell$ we have that $a_\ell = 1$.

• For $j \in \mathbb{Z}_{\geq 1}$, $\ell \in \mathcal{P}_Q$, let $p_{\ell j}(P) = p_{\ell j}$ be a prime of largest norm in $\mathcal{S}_{\ell j} \setminus \mathcal{S}_{\ell j-1}$, if such a prime exists.

• Let $m_0 = \prod_{\ell > 1} \ell^{a_\ell}$.

• Let $\mathcal{Y}_K = \mathcal{Y}_K(P) = \{p_{\ell j} : \ell \in \mathcal{P}_Q, j \in \mathbb{Z}_{>0}\}$. 
Let $W_K \subset P_K$ satisfy the following requirements: $V_K \subseteq P_K \setminus W_K$ and $\mathcal{I}_{\text{bad}} \subset W_K$.

For $n$ as above, let $d_n = N_{K/Q} \mathfrak{d}_n \in \mathbb{Z}_{\geq 1}$.

The following results can be found in [8].

1.3 Lemma. Let $n \in \mathbb{Z}_{\geq 1}$. Suppose that $t \in P_K$ divides $\mathfrak{d}_n$, and $p$ is a rational prime.

1. If $t \mid p$, then $\text{ord}_t \mathfrak{d}_pn = \text{ord}_t \mathfrak{d}_n + 2$.

2. If $t \nmid p$, then $\text{ord}_t \mathfrak{d}_pn = \text{ord}_t \mathfrak{d}_n$.

Consequently if $j \mid k$ then $\mathfrak{d}_j \mid \mathfrak{d}_k$.

In [8] it is assumed that $p \neq 2$ but the proof is unchanged in that case also.

1.4 Proposition (divisibility properties). Let $\mathfrak{A}$ be an integral divisor of $K$. Then

$\{n \in \mathbb{Z} \setminus \{0\} : \mathfrak{A} \mid \mathfrak{d}_n(P)\} \cup \{0\}$

is a subgroup of $\mathbb{Z}$.

1.5 Proposition (growth rate). There exists $a \in \mathbb{R}_{>0}$ such that $\log d_n = (a - o(1))n^2$ as $n \to \infty$.

1.6 Proposition (existence of primitive divisors). There exists $C > 0$ such that for all $\ell, m \in P_Q$ with $\max(\ell, m) > C$ we have that $\mathcal{I}_m \setminus (\mathcal{I}_\ell \cup \mathcal{I}_m) \neq \emptyset$.

The following corollaries are easy consequences of the propositions above.

1.7 Corollary (strong divisibility). Let $m, n \in \mathbb{Z} \setminus \{0\}$, and let $(m, n)$ be their GCD. Then $\mathcal{I}_m \cap \mathcal{I}_n = \mathcal{I}_{(m, n)}$. In particular, if $(m, n) = 1$ then $\mathcal{I}_m \cap \mathcal{I}_n = \emptyset$.

1.8 Corollary. For any $z < k \in \mathbb{Z}_{>0}$ the following statements are true:

1. $\mathcal{I}_{km_0} \setminus \mathcal{I}_{zm_0} \neq \emptyset$.

2. $\mathcal{I}_{zm_0} \subset \mathcal{I}_{km_0}$ if and only if $z$ divides $k$.

3. $p^{\ell + \text{ord}_z m_0}$ exists for all $j > 0$. 
4. $p_{\ell^j + \operatorname{ord}_\ell m_0} \in S_{km_0}$ if and only if $\ell^j$ divides $k$. \hfill \Box

**Proof.** 1. Since $S_{km_0} \cap S_{zm_0} = S_{(k,z)m_0}$ by Corollary 1.7, without loss of generality we can assume that $z \mid k$. By construction, $m_0 \geq C$, where $C$ is the constant from Proposition 1.6. Thus, this part of the corollary holds.

2. This assertion follows directly from Corollary 1.7.

3. To insure existence of $p_{\ell^j + \operatorname{ord}_\ell m_0}$, we need to show that

$$S_{\ell^j + \operatorname{ord}_\ell m_0} \setminus S_{\ell^j + \operatorname{ord}_\ell m_0} \neq \emptyset.$$  

By construction either $\ell > C$ or $\ell^{\operatorname{ord}_\ell m_0} > C$. Thus this assertion follows from Proposition 1.6 also.

4. By Corollary 1.7 we have that $p_{\ell^j + \operatorname{ord}_\ell m_0} \in S_{km_0}$ only if $\ell^j + \operatorname{ord}_\ell m_0 \mid km_0$ if and only if $j \mid k$. Conversely, if $j \mid k$, then $\ell^j + \operatorname{ord}_\ell m_0 \mid km_0$ and $S_{\ell^j + \operatorname{ord}_\ell m_0} \subseteq S_{km_0}$ by Part (2) of the corollary. Thus we also have

$$p_{\ell^j + \operatorname{ord}_\ell m_0} \in S_{\ell^j + \operatorname{ord}_\ell m_0} \subseteq S_{km_0}.$$  

\hfill \Box

**1.9 Corollary.** For all $n \in \mathbb{Z}_{>0}$, for some positive constant $\kappa$, independent of $n$, we have that $n^2 < \kappa d_n$. \hfill \Box

We now proceed to define divisibility in a large ring.

**1.10 Lemma.** The equations

\begin{align*}
x_{km_0}^{hK} &= \frac{a_k}{b_k}; \quad A_k a_k + B_k b_k = 1, \quad (1) \\
x_{jm_0}^{hK} &= \frac{a_j}{b_j}; \quad A_j a_j + B_j b_j = 1, \quad (2) \\
b_k &= b_j z \quad (3)
\end{align*}

have a solution $A_j, B_j, A_k, B_k, a_j, b_j, a_k, b_k \in \mathcal{O}_K, \mathcal{O}_K'$ if and only if $j$ divides $k$ in $\mathbb{Z}$.
Proof. Observe that the set
\[
\{(a, b) \in O_K, W_K : (\exists n \in \mathbb{Z}_{>0})(x_{m+n} = \frac{a}{b})\}
\]
is certainly diophantine over $O_K, W_K$ given that we know how to define the set of non-zero elements of the ring.

Now suppose first that the equations are satisfied in $O_K, W_K$. Let
\[
j = \prod \ell^{|n_\ell|}, n_\ell > 0.
\]
Then by Corollary 1.8 we have that $p_{\ell^{n_\ell}m_\ell} \notin W_K$. Further, since $(a_j, b_j) = 1$ in $O_K, W_K$, and since for each $i$ we have that
\[
p_{\ell^{n_\ell}m_\ell} \notin W_K
\]
and
\[
\text{ord}_{p_{\ell^{n_\ell}m_\ell}} x_{j^m} < 0
\]
it is the case that
\[
\text{ord}_{p_{\ell^{n_\ell}m_\ell}} b_j > 0
\]
and
\[
\text{ord}_{p_{\ell^{n_\ell}m_\ell}} b_k > 0.
\]
Since $(a_k, b_k) = 1$ in $O_K, W_K$, we conclude that
\[
\text{ord}_{p_{\ell^{n_\ell}m_\ell}} x_{k^m} < 0
\]
or
\[
p_{\ell^{n_\ell}m_\ell} \notin W_K.
\]
But by Corollary 1.8 this is possible only if $\ell^{n_\ell}$ divides $k$. Thus, if the equations hold we have that $j$ divides $k$.

Conversely, suppose $j$ divides $k$. By the definition of the class number, we can let $a_k, b_k$ and $a_j, b_j$ be pairs of algebraic integers relatively prime in $O_K$. Observe that $\delta_{j^m}$ and $\delta_{k^m}$ are precisely the non-invertible-in-$O_K, W_K$-parts of the divisors of $b_j$ and $b_k$ respectively. Thus by Corollary 1.8 and Lemma 1.3 we have that $b_j$ divides $b_k$. 

Summarizing the results of this section we state the following theorem:
1.11 Theorem. \((\mathbb{Z}, +, |)\) has a Diophantine model over \(O_K, \mathbb{W}_K\).

We finish this section with a “vertical” definability result which exploits our ability to define divisibility existentially. First we need several technical propositions.

1.12 Proposition. Let \(M/K\) be a number field extension of degree \(n\). Let \(Q\) be a prime of \(K\) and let \(q_1, \ldots, q_m\) be all the primes of \(M\) lying above \(Q\). Let \(\alpha \in M\) be a generator of \(M\) over \(K\) such that \(\alpha\) is integral with respect to \(Q\). Let \(u \in M\) be integral at \(Q\). Assume further there exists a sequence \(\{k_i, y_i\}\) where \(k_{i+1} > k_i\) and \(y_i \in K\) and \(\text{ord}_{q_j} y_i \geq 0\). Finally assume that for all \(i, j\) we have that \(\text{ord}_{q_j}(u - y_i) \geq k_i\). Then \(u \in K\).

Proof. Let \(D\) be the discriminant of the power basis of \(\alpha\). Using this power basis we can write
\[
u = \sum_{r=0}^{n-1} a_r \alpha^r
\]
with \(D a_r \in K\) and integral at \(Q\). Then
\[
u - y_i = (a_0 - y_i) + \sum_{r=1}^{n-1} a_r
\]
and
\[
\text{ord}_{q_j}(u - y_i) > k_i, j = 1, \ldots, m.
\]
This implies that \(\text{ord}_Q a_r > \frac{k_i}{n} - \text{ord}_Q D\) for all \(i \in \mathbb{Z}_{\geq 0}\). Thus, \(a_r = 0, r = 1, \ldots, n - 1\) and \(u \in K\).

The following two proposition are taken from [5].

1.13 Lemma. There exists a positive integer \(m_1\) such that for any positive integers \(k, l\),
\[
d(x_{lm_1}) \bigg| n \left( \frac{x_{lm_1}}{x_{klm_1}} - k^2 \right)^2
\]
in the integral divisor semigroup of \(K\).

1.14 Lemma. Let \(J\) be an integral divisor of \(K\). Then for some \(m\) we have that \(J\) divides \(d(x_m)\) in the integral divisor semigroup of \(K\).

We are now ready to prove the definability result.

1.15 Theorem. Let \(K\) be a number field. Then \(O_K, \mathbb{W}_K \cap \mathbb{Q}\) is definable over \(O_K, \mathbb{W}_K\) using one universal quantifier.
Proof. If $K = \mathbb{Q}$, then the statement of the theorem is trivial. So without loss of generality we can assume that $K \neq \mathbb{Q}$. Let $q$ be a rational prime and let $q_1, \ldots, q_r$ be all the factors of $q$ in $K$. Let $u \in K$ be such that

1. $j | \ell$,

2. for all $i = 1, \ldots, r$ we have that $\text{ord}_{q_i} x_{jm_1 m_0} < \text{ord}_{q_i} v$, 

3. for all $i = 1, \ldots, r$ we have that

$$-\text{ord}_{q_i} x_{jm_1 m_0} < 2 \text{ord}_{q_i} \left( \frac{x_{jm_1 m_0}}{x_{\ell m_1 m_0}} - u \right). \quad (4)$$

We claim that $u \in \mathbb{Q}$.

Indeed, fix $v$, consider the corresponding $x_{jm_1 m_0}, x_{\ell m_1 m_0}$ and let $k = \frac{j}{f}$. Then by Lemma [1.13]

$$-\text{ord}_{q_i} v < -\text{ord}_{q_i} x_{jm_1 m_0} < 2 \text{ord}_{q_i} \left( k^2 - u \right).$$

Keeping in mind that $v$ is arbitrary, we can apply Proposition [1.12] to reach the desired conclusion.

Suppose now that $u$ is a square of a rational integer $k$. Then using Lemma [1.14] and Lemma [1.3] for any $v$ we can find $j > 0$ such that

$$\text{ord}_{q_i} x_{jm_1 m_0} < \text{ord}_{q_i} v$$

for all $i = 1, \ldots, m$. Set $\ell = k j$ and Lemma [1.13] assures us that (4) will hold. Finally we remind the reader that every positive integer can be written as a sum of squares and a every rational number is a ratio of integers.

\[ \square \]

2. From Divisibility to Multiplication

In this section we will address the issue of converting our existential model of $(\mathbb{Z}, +, |)$ to a model of $(\mathbb{Z}, +, \times)$. We will use the same notation as above. Our starting point is the following lemma

2.1 Lemma ([1], section 4). There exists a formula $\mathcal{F}(l, m, n)$ in $(\mathbb{Z}, +, \times, \neq)$ of the form $(\exists \forall \exists)\mathcal{G}$ with one universal quantifier and $\mathcal{G}$ a formula which is a conjunction of divisibility conditions and additions, such that for integers $m, n$, we have $l = m \cdot n \iff \mathcal{F}(l, m, n)$.

\[ \square \]
In our model of addition and divisibility we send a non-zero integer to triples \( \{(x, y, z) \in O_{K, W}^3\} \) where \((\frac{x}{z}, \frac{y}{z})\) are affine coordinates of points on \(E(K)\) with respect to our fixed affine Weierstrass equation \(W\). Thus, a direct translation of “\(\forall n \in \mathbb{Z}\)” becomes “for all \((x, y, z)\) such that \((\frac{x}{z}, \frac{y}{z})\) satisfy \(W\)”, which uses three universal quantifiers. However, a result of Poonen from [7] can be used to reduce the number of \(\forall\)-quantifiers by one. Indeed, in [7] it is shown that the set of non-squares of a number field is Diophantine. Thus, we get

**2.2 Lemma.** A sentence of the form “for all \((x, y, z) \in O_{K, W}\) such that \((\frac{x}{z}, \frac{y}{z})\) satisfy \(W\)” is equivalent to a sentence of the form

\[
\forall x, z \in O_{K, W} : ((z \neq 0) \land (\exists y \in O_{K, W} : \frac{x^3}{z^3} + \frac{ax}{z} + b = \frac{y^2}{z^2}) \lor \\
((x^3 z^3 + ax z^6 + b z^5 \text{ is not a square in } K) \lor z = 0),
\]

involving only two universal quantifiers.

**2.3 Remark.** The proof of Theorems 4.2 and 5.3 in [1] contains a gap that can be fixed by the same technique: expressions of the form “for all \((x, y) \in \mathbb{Q}^2\) satisfying \(W\)” can be replaced by

\[
\forall x \in \mathbb{Q}((\exists y \in \mathbb{Q} : W(x, y) = 0) \lor (x^3 + ax + b \text{ is not a square in } \mathbb{Q})).
\]

Combining the discussion above with Theorem 2.1 we obtain the following.

**2.4 Theorem.** The set

\[
\Pi = \{(A, B, C, D, Y, F) \in O_{K, W} \mid \exists j, k, z \in \mathbb{Z}_{>0} : \begin{cases}
z = jk \\
x_{jm_0} = A/D \\
x_{km_0} = B/Y \\
x_{zm_0} = C/F
\end{cases}\}
\]

is definable in \(O_{K, W}\) by a \(\exists\forall\exists\)-formula using two universal quantifiers. \(\square\)

This theorem says that there is a model of the integers over the big ring involving two universal quantifiers. We now have to work a bit more to define the actual subset of integers by a similar formula.
3. From Models to Subset-definitions

In this section we will use Theorem 2.4 to define the actual set of integers in large rings, using two universal quantifier. First we extend our list of notation and assumptions.

3.1 Notation and Assumptions. The notation and assumptions below will be used in the remainder of the paper.

- \( n = [K : \mathbb{Q}] \).
- \( L \) is any extension of \( K \) of degree \( r > 0 \).
- \( \gamma \in O_L \) generates \( L \) over \( K \).
- \( d \) is an integer greater than \( |\gamma - \sigma(\gamma)| \) for any embedding \( \sigma \) of \( L \) into its algebraic closure.
- \( G(T) = G_0(T) \) is the monic irreducible polynomial of \( \gamma \) over \( K \).
- \( G_i(T) = G_0(T - di), i = 1, \ldots, n \).
- Assume \( \mathcal{W}_K \) contains only the primes of \( K \) without relative degree one factors in \( L \) and not dividing the discriminant of \( G_0 \) (and consequently of any \( G_i, i = 1, \ldots, n \)).
- Assume \( \mathcal{S}_{\text{bad}} \subset \mathcal{W}_K \).
- Let \( l_0 = 0, \ldots, l_{rn} \) be distinct natural numbers.
- For \( x \in K \) let \( n(x) = \prod_{p \in \mathcal{P}_K, \text{ord}_p x > 0} p^{\text{ord}_p x} \) and let \( d(x) = \prod_{p \in \mathcal{P}_K, \text{ord}_p x < 0} p^{-\text{ord}_p x} \).

3.2 Lemma. With \( m_1 \) as in Lemma 1.13 \( (d(x_{im_1}), n(x_{klm_1}))) = 1 \) in the integral divisor semigroup of \( K \).}

From Lemma 1.13 and Lemma 3.2 we also deduce the following corollary.
3.3 Corollary.
\[ d(x_{lm_1}) \mid n \left( \frac{x_{lm_1}^{h_{K_{lm_1}}}}{x_{klm_1}^{h_{K_{klm_1}}}} - k^{2h_{K_{klm_1}}} \right)^2 \]

3.4 Lemma. For any \( k \in \mathbb{Z}_{>0} \) we have that \( d(x_k), d_k \) are squares of some integral divisors of \( K \).

Finally we state a lemma which will give us a handle on the bounds. The proof can be found in Chapter 5 of [16].

3.5 Lemma. Let \( v \in O_{K,\mathcal{W}_K}, \) let \( \alpha, \beta \in O_{K,\mathcal{W}_K} \) be relatively prime in \( O_{K,\mathcal{W}_K} \), \( v \) not a unit of \( O_{K,\mathcal{W}_K} \). Assume
\[ \frac{v^{h_{K}}}{\prod_{i=0}^{n} G_i(\alpha/\beta - l_i)} \in O_{K,\mathcal{W}_K}. \]
Then \( v^{h_{K}} = yw \), where \( y \in O_{K,\mathcal{W}_K}, w \in O_{K,\mathcal{W}_K} \) and all the primes occurring in the divisor of \( y \) are not in \( \mathcal{W}_K \), all the primes occurring in the divisor of \( w \) are in \( \mathcal{W}_K \). Further, there exists a positive constant \( c \) depending only on \( G_0, l_1, \ldots, l_{rn} \), such that for all embeddings \( \sigma \) of \( K \) into its algebraic closure, \( |\sigma(\alpha/\beta)| < |\mathcal{N}_{K/Q}(y)|^c \), \( |\mathcal{N}_{K/Q}(\beta)| < |\mathcal{N}_{K/Q}(y)|^c \) and all the coefficients of the characteristic polynomial of \( \mathcal{N}_{K(\gamma)/Q}(\beta)\alpha/\beta \) over \( \mathbb{Q} \) with respect to \( K(\gamma) \) are also less than \( |\mathcal{N}_{K/Q}(y)|^c \). (Here, given an element \( \beta \in K(\gamma) \), the characteristic polynomial of \( \beta \) is \( f(X) = \prod_{j=1}^{rn} (X - \sigma_j(\beta)) \), where \( \sigma_1, \ldots, \sigma_{rn} \) are all the embeddings of \( K(\gamma) \) into the algebraic closure of \( \mathbb{Q} \).)

3.6 Notation. We use the following notation in the sequel:

- Let \( m = m_0m_1 \) with \( m_0 \) as defined in Notation 1.2 and \( m_1 \) as defined in Lemma 1.13.

- Let \( Z \) be a positive integer not divisible by any primes of \( \mathcal{W}_K \) and greater than \( r_{nk^{nh_{K}}} \), where \( \kappa \) is the constant from Corollary 1.9.

- Let \( c \) be as in Lemma 3.5.

3.7 Proposition. Consider the following system of equations and conditions where all the variables besides \( x_{jm}, x_{km}, x_{zm} \) take their values in \( O_{K,\mathcal{W}_K} \).
\[ (A, B, C, D, Y, F) \in \Pi \] \[ \exists j, k, z \in \mathbb{Z}_{>0} : \frac{A}{D} = x_{jm}; \frac{B}{Y} = x_{km}; \frac{C}{F} = x_{zm} \]
\[
\begin{align*}
(A/D)^{h_K} &= A_1/D_1, & (B/Y)^{h_K} &= B_1/Y_1, & (C/F)^{h_K} &= C_1/F_1 \\
X_1A_1 + U_1D_1 &= 1; & X_2B_1 + U_2Y_1 &= 1; & X_3C_1 + U_3F_1 &= 1
\end{align*}
\] (7)

\[\nu^{h_K}Z \prod_{i=0}^{n} G_i(A_1/D_1 - l_i)G_i(x^{2h_K} - l_i) \in O_{K,K} \]

\[Y_1 = (Z^2(v_{5crn}T)^{2h_K})
\] (9)

\[(F_1B_1 - x^{2h_K}Y_1C_1)^{2h_K} = Y_1^{2h_K+1}w
\] (10)

We claim that these equations can be satisfied with variables as indicated above only if \(x^{2h_K}\) is an integer. At the same time, if \(x\) is a positive integer the equations above can be satisfied.

**Proof.** Assume that the equations above are satisfied with all the variables except for \(x_{jm}, x_{km}, x_{zm}\) taking values in \(O_{K,K}\). Then from equation (5) we conclude that \(z = jk\). Let \(\nu^{h_K} = yu\), where \(y \in O_{K}\) and does not have any primes from \(K_{K}\) in its divisor and is not a unit of \(O_{K}\), while all the primes occurring in the divisor of \(u\) are from \(K_{K}\). We can assume \(y\) is not a unit because from equation (9) we know that \(\nu^{h_K}\) is divisible by \(Z\) which is not a unit of \(O_{K,K}\). We now combine three inequalities described below. Throughout the proof, we will set

\[N := |N_{K/Q}(y^e)|.
\]

First, from (9), by Lemma 3.5 we have that

\[|N_{K/Q}\Phi(A_1/D_1)| \leq N.
\]

Further, from equations (7), we also know that

\[d_{jm}^{h_K} | N_{K/Q}(\Phi(A_1/D_1))
\]

with \(d_{jm}\) as in Notation 1.2 Thirdly, from Corollary 1.9 we find the bound

\[(jm)^2 \leq \kappa d_{jm},
\]

where \(\kappa\) is a fixed positive constant independent of \(j\) and \(y\), defined in Corollary 1.8 From these three inequalities, we conclude that

\[j^{2h_K} < \kappa^{h_K}N^2,
\] (12)
We now turn our attention to equation (11). Write \( n(Y_1) = (e_0)^{h_K}(e_1)^{h_K} \), where \( e_0 \) is an integral divisor not divisible by any prime of \( W_K \) and \( e_1 \) is a divisor consisting of \( W_K \)-primes only. We rewrite (11) as

\[
\frac{Y_1}{C_1^{2h_K}} \cdot w = \left( \frac{F_1B_1}{Y_1C_1} - x^{2h_K} \right)^{2h_K}.
\]  

(13)

Since by Lemma 3.2 and equation (8) we have that \( Y_1 \) and \( C_1 \) are coprime in \( O_K,W_K \), if we consider the non-\( W_K \)-part of the numerators of the divisors of the left and right sides of (13), we see that

\[ e_0 | n(Y_1C_1 - x^{2h_K})^2. \]

By Corollary 3.3,

\[ e_0 | n(Y_1C_1 - j^{2h_K})^2 \]

and by Lemma 3.4 we know that \( e_0 \) is a square of an integral divisor. Therefore we conclude

\[ \sqrt{e_0} | n(j^{2h_K} - x^{2h_K}). \]  

(14)

Next we write \( x^{h_K} = \frac{x_1}{x_2} \), where \( x_1, x_2 \neq 0 \) are relatively prime integers of \( K \). If we clear denominators in (14) using \( \mathcal{N}_{K/Q}(x_2^{2h_K}) \) we get

\[ \sqrt{e_0} | (\mathcal{N}_{K/Q}(x_2^{2h_K})j^{2h_K} - \mathcal{N}_{K/Q}(x_2^{2h_K})x^{2h_K}). \]  

(15)

We let \( H(T) \) be the characteristic polynomial of \( \mathcal{N}_{K(\gamma)/Q}(x_2^{2h_K})x^{2h_K} \) over \( Q \) with respect to \( K(\gamma) \). Then by (15)

\[ H(\mathcal{N}_{K(\gamma)/Q}(x_2^{2h_K})j^{2h_K})^2 \equiv 0 \mod \mathcal{N}_{K(\gamma)/Q}(e_0) \]

and therefore either

\[ H(\mathcal{N}_{K(\gamma)/Q}(x_2^{2h_K})j^{2h_K}) = 0 \]  

(16)

or

\[ |H(\mathcal{N}_{K(\gamma)/Q}(x_2^{2h_K})j^{2h_K})|^2 \geq |\mathcal{N}_{K(\gamma)/Q}(e_0)|. \]

However, we can estimate an expression such as \( |H(X)| \) by its degree (here, \( rn \)) times its leading monomial (here, \( X^{rn} \)) times any bound on its coefficients. Now from Lemma 3.5, we have that the coefficients of the characteristic polynomial of \( \mathcal{N}_{K(\gamma)/Q}(x_2^{2h_K})x^{2h_K} \) over \( Q \) with respect to \( K(\gamma) \) are bounded by \( N \). Therefore, we get

\[ |H(\mathcal{N}_{K(\gamma)/Q}(x_2^{2h_K})j^{2h_K})|^2 \leq |rnN\mathcal{N}_{K(\gamma)/Q}(x_2^{2h_K})j^{2h_K}rn|^2. \]
But now, we use equation (9) and Lemma 3.5 again to conclude that
\[ |N_{K/Q}(x_{2hK}^{2})| < N. \]

From equation (12), we find \( j^{2hK} < h^{K}N^{2} \), so that if we plug this into the previous inequality, we get
\[ |H(N_{K(\gamma)/Q}(x_{2hK}^{2}j^{2hK}))|^{2} \leq (rn^{h_{K}}N^{3rn+1})^{2}. \] (17)

With our definitions, equation (10) implies
\[ r_{0}^{h_{K}} n(\epsilon_{1}^{h_{K}}) = n(Z^{2}y^{5crn}w^{5crn-1})^{2h_{K}}. \]

Recall that \( Z \) is an integer such that \( Z \geq r_{0}^{h_{K}}h_{K} \). If we now only consider the non-\( W_{K} \)-part of the equality and take norms and then \( h_{K} \)-th roots, we find
\[ |N_{K(\gamma)/Q}(x_{2hK}^{2}j^{2hK})| \geq |N_{K(\gamma)/Q}(e_{0})|^{2} = (rn^{h_{K}}N^{3rn+1})^{2}. \] (18)

From (17) and (18) we conclude that
\[ |H(N_{K(\gamma)/Q}(x_{2hK}^{2}j^{2hK}))|^{2} < |N_{K(\gamma)/Q}(e_{0})|, \]

In the end, we find that the alternative (16) holds, so \( H \) has a rational root, and thus all its roots are rational (and equal). Hence \( x^{2hK} = j^{2hK} \) is a rational integer.

In the other direction, suppose that \( x = j \in \mathbb{Z}_{>0} \). Set \( x_{jm} = \frac{A}{D}, A, D \in O_{K} \) and set \( x_{jm}^{h_{K}} = \frac{A_{1}}{D_{1}}, \) where \( A_{1}, D_{1} \) are relatively prime elements of \( O_{K} \). Then the \( A_{1}, D_{1} \)-part of (8) will be satisfied. Set
\[ v = Z \prod_{i=0}^{rn} D_{i}(A_{i}/D_{i} - l_{i})G_{i}(x^{2h_{K}} - l_{i}). \]

By Lemma there exists \( k \in \mathbb{Z}_{>0} \) such that \( Z^{4}v^{10crn} \) divides \( \vartheta(x_{kn}) \). Let \( z = jk \) and define \( B, C, Y, F, B_{1}, C_{1}, Y_{1}, F_{1} \) so that (5), (6), (7) and (8) are satisfied. Observe that by choice of \( k \) we satisfy (10) also. Further by Lemma 1.13 we have that \( n(Y_{1}) \) divides \( n\left( \frac{F_{1}B_{1}}{Y_{1}C_{1}} - x^{2h_{K}} \right)^{2h_{K}} \) and therefore \( n(Y_{1}^{2h_{K}+1}) \) divides \( (F_{1}B_{1} - x^{2h_{K}}Y_{1}C_{1})^{2h_{K}} \). Thus the ratio
\[ \frac{(F_{1}B_{1} - x^{2h_{K}}Y_{1}C_{1})^{2h_{K}}}{Y_{1}^{2h_{K}+1}} \in O_{K} \subset O_{K,\vartheta_{K}} \]
and therefore (11) is also satisfied.
We summarize the previous discussion in the following result.

3.8 Theorem. Let $K$ be any number field such that there exists an elliptic curve $E$ defined over $K$ of rank 1 over $K$. Let $W$ be a Weierstrass equation of $E$ over $K$ and let $t$ be the size of the torsion group of $E(K)$. Let $L$ be any non-trivial extension of $K$ and let $\mathcal{W}_K \subset \mathcal{P}_K$ be any set of primes of $K$ satisfying the following conditions.

1. The complement of $\mathcal{W}_K$ in $\mathcal{P}_K$ contains all but finitely many elements of the set $\mathcal{V}_K(P) = \{p^j : \ell \in \mathcal{P}(\mathbb{Z}), j \in \mathbb{Z}_{>0}\}$ for some point $P \in tE(K)$ of infinite order.

2. $S_{\text{bad}}(P, W, K) \subset \mathcal{W}_K$.

3. All but finitely many primes of $\mathcal{W}_K$ do not have a relative degree one factor in the extension $L/K$.

Then $\mathbb{Z}$ can be defined in $O_K, \mathcal{W}_K$ using two universal quantifiers.

Proof. This follows from combining the above results with with Corollary B.10.10 from [16]. The only point which needs to be made is that we can existentially define integrality at finitely many primes (see Chapter 4 of [16]) and therefore the relaxation of assumptions on $\mathcal{W}_K$ or $P$ will not alter our conclusion. $\square$

3.9 Remark. For the construction of diophantine models of $\mathbb{Z}$ in [6] and [8] to go through, infinitely many elements of the set $\mathcal{V}_K(P)$ have to be inverted. This is very different from the situation in the above theorem.

4. Density computation

We first compute the density of the set $\mathcal{V}_K(P)$. For that, we need the following lemma:

4.1 Lemma. Let $l \in \mathcal{P}(\mathbb{Q})$ and suppose $p \in \mathcal{I}_{l^{n+1}} \setminus \mathcal{I}_l$ (if such $p$ exists, $n > a_l$). Then $l^{n+1} < 3Np$.

Proof. If $p \in \mathcal{I}_{l^{n+1}} \setminus \mathcal{I}_l$, then $p$ does not divide the discriminant of our Weierstrass equation and $\bar{E}$, the reduction of $E \pmod{p}$ is non-singular. Further, $x_{l^n}, y_{l^n}$ are integral at $p$, while $\text{ord}_p x_{l^{n+1}} < 0$, $\text{ord}_p y_{l^{n+1}} < 0$. Therefore, under reduction $\pmod{p}$, the image of $l^n P$ is not $\bar{O}$ – the image of $O \pmod{p}$, while $l^{n+1} \bar{P} = \bar{O}$. Thus we must conclude that $E(F_p)$ has an
element of order \( l^{n+1} \) and therefore \( l^{n+1} | E(\mathbb{F}_p) \). Let \( \# \mathbb{F}_p = N_p = q \). From a theorem of Hasse we know that \( \# E(\mathbb{F}_p) \leq q + 1 + 2\sqrt{q} \leq 3q \) (see [17], Chapter V, Section 1, Theorem 1.1).

4.2 Proposition. The set \( \mathcal{V}_K(P) \) has natural density zero.

Proof. Recall that \( p_{\ell k} \) is a primitive prime divisor of largest norm for \( \ell^k P \). For the proof, we first remark that it is proven in [6] and [8] (using properties of Galois representations) that the set of primitive largest norm divisors of \( \ell P \)

\[ \mathcal{B} = \{ p_\ell : \ell \in \mathcal{P}_Q \land a_\ell = 1 \} \]

has a natural density that is zero. To prove the theorem, it therefore suffices to consider the complement of \( \mathcal{B} \) in \( \mathcal{V}_K(P) \), as in the next proposition. It turns out this is much easier:

4.3 Lemma. The natural density of the set \( \mathcal{A} = \{ p_{\ell k} : \ell \in \mathcal{P}_Q, k \in \mathbb{Z}_{>1} \land k > a_\ell \} \) is zero.

Proof. For \( p = p_{\ell k} \in \mathcal{A} \), the previous Lemma says \( 3Np_{\ell k} > \ell^k \). Thus,

\[ \# \{ p \in \mathcal{A} : Np \leq X \} \leq \# \{(\ell, k) \in \mathcal{P}_Q \times \mathbb{Z}_{\geq 2} : \ell \leq \sqrt[3]{3X} \} \]

Clearly if \( \sqrt[3]{3X} < 2 \), there will be no prime \( \ell \) with \( \ell \leq \sqrt[3]{3X} \). Thus, we can limit ourselves to positive integers \( k \) such that \( k < \log 3X \).

By the Prime Number Theorem (see [22], Theorem 4, Section 5, Chapter XV), for some positive constant \( C \) we have that \( \# \{ \ell \in \mathcal{P}_Q : \ell \leq X \} \leq CX/\log X \) for all \( X \in \mathbb{Z}_{>0} \). From the discussion above we now have the following sequence of inequalities:

\[
\{ p \in \mathcal{A} : Np \leq X \} \leq \sum_{k=2}^{\lceil \log 3X \rceil} \# \{ \ell \in \mathcal{P}_Q : \ell \leq \sqrt[3]{3X} \} \\
\leq \sum_{k=2}^{\lceil \log 3X \rceil} \# \{ \ell \in \mathcal{P}_Q : \ell \leq \sqrt{3X} \} \\
\leq \log(3X)[C \frac{\sqrt{3X}}{\log \sqrt{3X}}] = \tilde{C} \sqrt[3]{X}
\]

for some positive constant \( \tilde{C} \). At the same time by the Prime Number Theorem again we also know that for some positive constant \( \tilde{C} \) we have \( \# \{ p \in \mathcal{P}_K : Np \leq X \} \geq \tilde{C}X/\log X \). Thus the upper density of \( \mathcal{A} \) is

\[
\limsup_{X \to \infty} \frac{\# \{ p \in \mathcal{A} : Np \leq X \}}{\# \{ p \in \mathcal{P}_K : Np \leq X \}} \leq \limsup_{X \to \infty} \frac{\tilde{C} \sqrt[3]{X} \log X}{CX} = 0.
\]
Hence $\mathcal{S}$ has a natural density, and it is zero.

We can prove Theorem 2 from the introduction:

**4.4 Theorem.** Let $K$ be a number field such that there exists an elliptic curve $E$ defined over $K$ of rank 1 over $K$. Then for any $\varepsilon > 0$ there exists a set of primes $\mathcal{W}_K$ of density greater than $1 - \varepsilon$ such that $\mathbb{Z}$ can be defined in $O_K, \mathcal{W}_K$ using two universal quantifiers.

**Proof.** First of all we observe that for any point $P \in E(K)$ of infinite order, the set $\mathcal{V}_K(P)$ is of natural density 0 by the previous proposition. Next let $L$ be an extension of $K$ of prime degree $p > \frac{1}{\varepsilon}$. Then, by the natural version of the Tchebotarev Density Theorem, the set of primes of $K$ having a degree one factor in the extension $L/K$ has natural density $\frac{1}{p}$. Adding primes of $\mathcal{V}_K(P)$ to this set does not change its density. We apply Theorem 3.8 with this $\mathcal{W}_K$.

5. **Proof of the First Main Theorem**

We will now use the following definability results, proofs of which can be found in [11], [12] and Section 7.8 of [16].

**5.1 Proposition.** Let $K \neq \mathbb{Q}$ be a number field of one of the following types:

1. $K$ is totally real;

2. $K$ is an extension of degree two of a totally real number field;

3. There exists an elliptic curve defined over $\mathbb{Q}$ and of positive rank over $\mathbb{Q}$ such that this curve preserves its rank over $K$;

Let $L$ be a totally real cyclic extension of $\mathbb{Q}$ of degree $p$ such that $p$ does not divide $[K_G : \mathbb{Q}]$, where $K_G$ is the Galois closure of $K$ over $\mathbb{Q}$. Let $\mathcal{W}_K$ be a set of $K$ such that all but finitely many primes in the set do not split in the extension $KL/K$. Then there exists a set of $K$-primes $\mathcal{W}_K$ containing $\mathcal{W}_K$ and such that $\mathcal{W}_K \setminus \mathcal{W}_K$ has at most finitely many elements, while the set $O_K, \mathcal{W}_K \cap \mathbb{Q}$ has a Diophantine definition in $\mathbb{Q}$.

We will also need the following property of natural density of sets of primes.
5.2 Lemma. Let $K$ be any number field, let $\mathcal{U}_Q$ be a set of rational primes of natural density 0 and let $\mathcal{U}_K$ be the set of all the primes of $K$ lying above the primes of $\mathcal{U}_Q$. Then the natural density of $\mathcal{U}_K$ is also 0.

Proof. This follows from the fact that $\#\{p \in \mathcal{P}_K : Np \leq X\} = O(X/\log X)$ for any number field.

We are now ready to prove Theorem 1 from the introduction. First we need a couple of technical propositions which will allow us to reduce the number of quantifiers.

5.3 Lemma. Let $M/K$ be a number field extension of degree $n$. Let $\mathcal{W}_K \subset \mathcal{P}_K$ contain all the $M$-factors of primes in $\mathcal{W}_K$ so that $\mathcal{O}_{M,\mathcal{W}_M}$ is the integral closure of $\mathcal{O}_{K,\mathcal{W}_K}$ in $K$. Let $\alpha \in \mathcal{O}_{M,\mathcal{W}_M}$ generate $M$ over $K$. Assume also that the discriminant of the power basis of $\alpha$ is $D$. Then for every $w \in \mathcal{O}_{M,\mathcal{W}_M}$ we have that either

$$w = \sum_{i=0}^{n-1} a_i \alpha^i, a_i \in \mathcal{O}_{K,\mathcal{W}_K}$$

or

$$Dw = \sum_{i=0}^{n-1} b_i \alpha^i, b_i \in \mathcal{O}_{K,\mathcal{W}_K} \land \exists a_i \text{ such that } a_i \not\equiv 0 \mod D$$

in $\mathcal{O}_{K,\mathcal{W}_K}$. Furthermore, both options cannot hold at the same time and every element of $\mathcal{O}_{K,\mathcal{W}_K}$ occurs as a coefficient in the first sum.

Proof. By a well-known number-theoretic fact (see for example Lemma B.4.12 of [16]), for any $w \in \mathcal{O}_{M,\mathcal{W}_M}$ we have that $Dw = \sum_{i=0}^{n-1} b_i \alpha^i, b_i \in \mathcal{O}_{K,\mathcal{W}_K}$. At the same time, if

$$w = \sum_{i=0}^{n-1} a_i \alpha^i, a_i \in \mathcal{O}_{K,\mathcal{W}_K}$$

(19)

and $D$ is not a unit in $\mathcal{O}_{K,\mathcal{W}_K}$, then the second option cannot hold. (If $D$ is a unit, then the second option cannot hold in any case.) Thus, for each $w$ one of the options holds and both cannot hold at the same time. Next it is clear that for any choice $(a_0, \ldots, a_{n-1}) \in \mathcal{O}_{K,\mathcal{W}_K}^n$ we have that $w = \sum_{i=0}^{n-1} a_i \alpha^i \in \mathcal{O}_{M,\mathcal{W}_M}$ and for each $w$ the choice of the $n$-tuple $(a_0, \ldots, a_{n-1})$ satisfying (19) is unique. □

5.4 Remark. Note that the condition $b_i \not\equiv 0 \mod D$ is actually Diophantine, since it is equivalent to a sentence $\bigvee_{j=1}^{n}(b_i \equiv A_j \mod D)$, where the $\{A_j\}$ contains a representative of every non-zero equivalence class modulo the principal ideal generated by $D$ in $\mathcal{O}_{K,\mathcal{W}_K}$. 

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5.5 Proposition. Let $M, K, \mathcal{W}_K, \mathcal{W}_M, \alpha$ be as in Lemma 5.3. Assume further that $O_K, \mathcal{W}_K$ is existentially definable over $O_M, \mathcal{W}_M$. Let $Z \subseteq O_K, \mathcal{W}_K$ be definable over $O_K, \mathcal{W}_K$ by a formula of the form $\forall X \exists Y P(T, \bar{X}, \bar{Y})$, or $\exists \bar{U} \forall \bar{X} \exists \bar{Y} P(T, \bar{U}, \bar{X}, \bar{Y})$ where $\forall X$ and $\exists Y$ represent a sequence of universal or existential quantifiers respectively. Assume that the sequence of universal quantifiers in the formula is of length less or equal to $n = [K : \mathbb{Q}]$. Then $Z$ is definable over $O_M, \mathcal{W}_M$ with a formula using just one universal quantifier.

Proof. The idea is to encode the variables over which there is universal quantification into a single universal quantifier over the larger ring, by using them as coefficients in the power basis of $\alpha$. It is enough to consider the “translation” of

$$\exists \bar{U} \in O_K, \mathcal{W}_K \forall \bar{X} \in O_K, \mathcal{W}_K \exists \bar{Y} \in O_K, \mathcal{W}_K P(T, \bar{U}, \bar{X}, \bar{Y})$$

into variables ranging over $O_M, \mathcal{W}_M$. Let $\Gamma(V, \bar{Z})$ be a Diophantine definition of $O_K, \mathcal{W}_K$ over $O_M, \mathcal{W}_M$. Let $\bar{U} = (u_1, \ldots, u_r), \bar{X} = (x_0, \ldots, x_\ell), \bar{R} = (x_{\ell+1}, \ldots, x_{n-1}), \bar{Y} = (y_1, \ldots, y_m)$. Let

$$F_1 := (\Gamma(T, \bar{Z}_0) = 0)$$
$$F_2 := (\bigwedge_{i=1}^r \Gamma(u_i, \bar{Z}_i) = 0)$$
$$F_3 := (\bigwedge_{i=0}^{n-1} \Gamma(x_i, \bar{Z}_{r+i+1}) = 0)$$
$$F_4 := (\bigwedge_{i=1}^m \Gamma(y_i, \bar{Z}_{r+n+i}) = 0)$$
$$F := (F_1 \land F_2 \land F_3 \land F_4)$$

(21)

Let

$$H := \left( \bigvee_{i=0}^{n-1} (x_i \not\equiv 0 \text{ mod } D) \right)$$

(22)

Then (20) becomes

$$\exists \bar{U} \exists \bar{Z}_0 \exists \bar{Z}_1, \ldots, \bar{Z}_r \forall w \exists \bar{X} \exists \bar{Z}_{r+1}, \ldots, \exists \bar{Z}_{r+n}, \exists \bar{Y} \exists \bar{Z}_{r+n+1} \ldots \bar{Z}_{r+n+m} \left( F \land (w = \sum_{i=0}^{n-1} x_i \alpha^i) \land P(T, \bar{U}, \bar{X}, \bar{Y}) = 0 \right)$$

(23)
5.6 **Theorem.** Let $K$ be a number field of one of the following types:

1. $K \neq \mathbb{Q}$ is totally real;
2. $K$ is an extension of degree two of a totally real number field;
3. There exists an elliptic curve defined over $\mathbb{Q}$ and of positive rank over $\mathbb{Q}$ such that this curve preserves its rank over $K$.

Let $L$ be a totally real cyclic extension of $\mathbb{Q}$ of degree $p$ such that $p$ does not divide $[K_G : \mathbb{Q}]$, where $K_G$ is the Galois closure of $K$ over $\mathbb{Q}$. Let $\mathcal{H}_K$ be a set of primes of $K$ such that all but finitely many primes in the set do not split in the extension $KL/K$. Let $E$ be an elliptic curve defined over $\mathbb{Q}$ and of rank one over $\mathbb{Q}$. Let $P \in E(\mathbb{Q})$ be a point of infinite order. Let $\mathcal{V}_Q(P)$ be defined as in Theorem 3.8 and let $\mathcal{V}_K(P)$ be a set of primes of $K$ containing at least one factor for every prime of $\mathcal{V}_Q(P)$. Let $\mathcal{U}_K = \mathcal{H}_K \setminus \mathcal{V}_K(P)$. Then for some set of $K$-primes $\mathcal{U}$ containing $\mathcal{U}$ and such that $\mathcal{U} \setminus \mathcal{U}_K$ is a finite set we have that

1. $Z$ is definable in $O_K, \mathcal{U}_K$ using one universal quantifier;
2. for any $\varepsilon > 0$, it can be arranged that the natural density of $\mathcal{U}_K$ is greater than $1 - \varepsilon$.

**Proof.** By Theorem 5.1, for some set of primes $\mathcal{U}$ as described above we have that $O_K, \mathcal{U}_K \cap \mathbb{Q}$ is existentially definable in $O_K, \mathcal{U}_K$. Let $O_Q, \mathcal{X}_Q = O_K, \mathcal{U}_K \cap \mathbb{Q}$. Then given our assumption on $L$, we have that all but finitely primes of $\mathcal{X}_Q$ do not split in the extension $L/\mathbb{Q}$. Further, by construction, $\mathcal{V}_Q(P) \cap \mathcal{X}_Q$ is at most finite set. Thus, $Z$ is definable using two universal quantifiers in $O_Q, \mathcal{X}_Q$ and therefore by Proposition 5.5 we can define $Z$ in $O_K, \mathcal{U}_K$ using just one universal quantifier.

Next let $\varepsilon > 0$ be given. Then choose $L$ to be of prime degree $p > \frac{1}{2}$ and let $\mathcal{U}_K$ be the set of all $K$-primes not splitting completely in the extension $KL/K$. Then $\mathcal{U}_K$ will be of natural density $\frac{1}{p-1}$. Next observe that by Proposition 5.2 we have that the density of $\mathcal{V}_K(P)$ is zero and therefore removing primes of $\mathcal{V}_K(P)$ from $\mathcal{U}_K$ to form $\mathcal{U}_K$ will not change the density. \[\square\]
6. Defining Subfields over Number Fields Using One Universal Quantifier

In this section we will produce another vertical definability result exploiting properties of elliptic curves and requiring just one universal quantifier.

6.1 Proposition. Let \( M, K, q_1, \ldots, q_k, \mathcal{O} \) be as in Proposition 1.12. Let \( E \) be an elliptic curve defined over \( K \) such that \( \text{rank } E(K) > 0 \). Then \( K \) is definable over \( M \) using just one universal quantifier.

Proof. Set \( r := [E(M) : E(K)] \). Fix an affine Weierstrass equation \( W \) for \( E \). Let \( u \in M, \text{ord}_{q_i} u > 0 \) for all \( i = 1, \ldots, n \) and consider the following formula:

\[
\forall z \in M \exists (a_1, b_1), (a_2, b_2) \in rE(M) : \bigwedge_{i=1}^{k} \text{ord}_{q_i} a_1 < \text{ord}_{q_i} z \land \text{ord}_{q_i} a_2 < z \land 2 \text{ord}_{q_i} (u - \frac{a_1}{a_2}) \geq -\text{ord}_{q_i} a_2.
\]

Here, as above, we identify non-zero points of \( E(M) \) with pairs of solutions to the chosen Weierstrass equation and \( rE(M) \) is the set of \( r \)-multiples of non-zero points of \( E(M) \). Suppose the formula is true for some value of \( u \in M \). Then by assumption \( \frac{a_1}{a_2} \in K \) and by Proposition 1.12 we have that \( u \in K \).

Now assume that \( u \in \mathbb{Z}, u \neq 0 \) and \( u \) is a square. Let \( (x_1, y_1) \in E(M) \) be the affine coordinates with respect to \( W \) of a point \( P \in E(M) \) of infinite order. Then by Lemma 1.13 there exists a positive integer \( m_1 \) such that for any positive integers \( l, k \),

\[
\mathfrak{d}(xl_{rm_1}) \mid n \left( \frac{x_{lrm_1}}{x_{rklm_1}} - l^2 \right)^2
\]

in the integral divisor semigroup of \( M \). Further, by Corollary 1.8 and Lemma 1.3 we have that for any positive \( N \) for some sufficiently large \( m \) it is the case that \( \text{ord}_{q_i} x_{rm_1} < -N \) for all \( i \). Finally we note that any positive integer can be written as a sum of four squares, and any element of \( K \) can be expressed as a linear combination of some basis elements with rational coefficients.

We can use the same method of proof over certain subrings of \( M \). Then only change we would have to make is to possibly represent coordinates of points on \( E \) as ratios of elements in the ring. Everything else remains the same since order at a prime is existentially definable in any \( O_{M,\mathcal{O}_M} \). In this way we arrive at the following.
6.2 Proposition. Let $M/K$ be a number field extension. Assume there exists an elliptic curve $E$ defined over $K$ such that $\text{rank } E > 0$. Let $\mathcal{W}_M$ be any set of $M$ primes (including the set of all $M$-primes and the empty set). Then $O_{M,\mathcal{W}_M} \cap K$ is definable over $O_{M,\mathcal{W}_M}$ using just one universal quantifier.

6.3 Remark. In connection with the results above we should note that the first-order definability of any subfield of a number field follows from the work of Julia Robinson also. (See [10].) However, her definition uses several quantifiers since it proceeds by defining the algebraic integers over the field first, and then defining $\mathbb{Z}$ over the ring of integers.

Of course for $\mathcal{W}_M = \emptyset, \mathcal{W}_M$ of finite size and many infinite sets $\mathcal{W}_M$ we actually have existential definability. However, we do not have a proof of existential definability for $\mathcal{W}_M = \mathcal{P}_M$. See [12] for more details.

References

[1] Gunther Cornelissen and Karim Zahidi. Elliptic divisibility sequences and undecidable problems about rational points. J. reine angew. Math., 613:1–33, 2007.

[2] Serge Lang. Algebraic Number Theory. Addison Wesley, Reading, MA, 1970.

[3] Barry Mazur. The topology of rational points. Experiment. Math., 1(1):35–45, 1992.

[4] Bjorn Poonen. Characterizing integers among rational numbers with a universal-existential formula. Preprint (2007) arXiv:math/0703907v1 [math.NT], to appear in Am. J. Math.

[5] Bjorn Poonen. Using elliptic curves of rank one towards the undecidability of Hilbert’s Tenth Problem over rings of algebraic integers. In C. Fieker and D. Kohel, editors, Algorithmic Number Theory, volume 2369 of Lecture Notes in Computer Science, pages 33–42. Springer Verlag, 2002.

[6] Bjorn Poonen. Hilbert’s tenth problem and Mazur’s conjecture for large subrings of $\mathbb{Q}$. J. Amer. Math. Soc., 16(4):981–990, 2003.

[7] Bjorn Poonen. The Set of Non-squares in a Number Field is Diophantine, Preprint (2007) arXiv:0712.1785v2 [math.NT]

[8] Bjorn Poonen and Alexandra Shlapentokh. Diophantine definability of infinite discrete non-archimedean sets and diophantine models for large subrings of number fields. J. reine angew. Math., 2005:27–48, 2005.

[9] Julia Robinson. Definability and decision problems in arithmetic. J. Symb. Logic, 14:98–114, 1949.
[10] Julia Robinson. The Undecidability of Algebraic Rings and Fields. *Proc. Am. Math. Soc.*, 10: 950–957, 1959.

[11] Alexandra Shlapentokh. Diophantine definability and decidability in the extensions of degree 2 of totally real fields. *J. Alg.*, 313:846–896, 2007.

[12] Alexandra Shlapentokh. Elliptic curves retaining their rank in finite extensions and Hilbert’s Tenth Problem, to appear in *Trans. Am. Math. Soc.*, DOI: 10.1090/S0002-9947-08-04302-X, 2008.

[13] Alexandra Shlapentokh. Diophantine definability over some rings of algebraic numbers with infinite number of primes allowed in the denominator. *Inv. Math.*, 129:489–507, 1997.

[14] Alexandra Shlapentokh. Defining integrality at prime sets of high density in number fields. *Duke Math. J.*, 101(1):117–134, 2000.

[15] Alexandra Shlapentokh. On diophantine definability and decidability in large subrings of totally real number fields and their totally complex extensions of degree 2. *J. Numb. Th.*, 95:227–252, 2002.

[16] Alexandra Shlapentokh. *Hilbert’s Tenth Problem: Diophantine Classes and Extensions to Global Fields*. Cambridge University Press, 2006.

[17] Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1992. Corrected reprint of the 1986 original.

[18] Donald Wesling. The Speaking Subject in Russian Poetry and Poetics since 1917. *New Literary History* 23(1):93–112, 1992. (“Versions of Otherness”)

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