Fractional topological insulators in three dimensions

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Topological insulators can be generally defined by a topological field theory with an axion angle $\theta$ of 0 or $\pi$. In this work, we introduce the concept of fractional topological insulator defined by a fractional axion angle and show that it can be consistent with time reversal ($T$) invariance if ground state degeneracies are present. The fractional axion angle can be measured experimentally by the quantized fractional bulk magnetoelastic polarization $P_3$, and a ‘halved’ fractional quantum Hall effect on the surface with Hall conductance of the form $\sigma_H = \frac{e^2}{2}\frac{q}{2\pi}$ with $p,q$ odd. In the simplest of these states the electron behaves as a bound state of three fractionally charged ‘quarks’ coupled to a deconfined non-Abelian SU(3) ‘color’ gauge field, where the fractional charge of the quarks changes the quantization condition of $P_3$ and allows fractional values consistent with $T$-invariance.

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Most states of quantum matter are classified by the symmetries they break. However, topological states of quantum matter \cite{1} evade traditional symmetry-breaking classification schemes, and are rather described by topological field theories (TFT) in the low-energy limit. For the quantum Hall effect, the TFT is the 2+1 dimensional Chern-Simons (CS) theory \cite{2} with coefficient given by the quantized Hall conductance. In the noninteracting limit, the integer quantized Hall (IQH) conductance in units of $\frac{e^2}{2}\pi$ is given by the TKNN invariant \cite{3} or first Chern number. In the presence of strong correlations, one can also observe the fractional quantum Hall effect (FQHE), where the Hall conductance is quantized in rational multiples of $\frac{e^2}{2}\pi$. In both cases however, these topological states can exist only in a strong magnetic field which breaks time reversal ($T$) symmetry.

More recently, $T$-invariant topological insulators (TI) have been studied extensively \cite{4,5,6}. The TI state was first predicted theoretically in HgTe quantum wells, and observed experimentally in Bi\textsubscript{2}Se\textsubscript{3} soon after. The theory of TI has been developed along two independent routes. Topological band theory identified $\mathbb{Z}_2$ topological invariants for noninteracting band insulators \cite{8,11,11}. The TFT of $T$-invariant insulators was first developed in 4+1 dimensions, where the CS term is naturally $T$-invariant \cite{12,13}. Dimensional reduction then gives the TFT for TI in 3+1 and 2+1 dimensions \cite{14}. The TFT is generally valid for interacting systems, and describes the experimentally measurable quantized magnetoelastic response. The coefficient of the topological term, the axion angle $\theta$, is constrained to be either 0 or $\pi$ by $T$-invariance. The TFT has been further developed in Ref. \cite{15,16}. More recently, it has been shown that it reduces to the topological band theory in the noninteracting limit \cite{17}.

By analogy with the relation between the IQHE and FQHE, one is naturally led to the question whether there can exist a ‘fractional TI’. In 2+1 dimensions, an explicit wavefunction for the fractional quantum spin Hall state was first proposed in Ref. \cite{9}, and the edge theory was investigated in Ref. \cite{18}. The $T$-invariant fractional topological state has also been constructed explicitly in 4+1 dimensions \cite{12}. Since $T$-invariant TI form a dimensional ladder in 4, 3 and 2 dimensions \cite{14,14,20}, it is natural to investigate the $T$-invariant TI in 3+1 dimensions. Fractional states generally arise from strong interactions. Since topological band theory cannot describe such interactions, we formulate the general theory in terms of the TFT. The TI is generally described by the effective action $S_\theta = \frac{\theta^2}{2\pi} \int d^3 x \, dt \, \mathbf{E} \cdot \mathbf{B}$ where $\mathbf{E}$ and $\mathbf{B}$ are the electromagnetic fields \cite{14}. Under periodic boundary conditions, the partition function and all physical quantities are invariant under shifts of $\theta$ by multiples of $2\pi$. Since $\mathbf{E} \cdot \mathbf{B}$ is odd under $T$, it appears that the only values of $\theta$ allowed by $T$ are 0 or $\pi$ mod $2\pi$.

In this paper, we show that there exist $T$-invariant insulating states in 3+1 dimensions with $P_3 \equiv \frac{\theta}{q}$ quantized in non-integer, rational multiples of $\frac{1}{2}$ of the form $P_3 = \frac{\theta}{q}$ with $p,q$ odd integers. The magnetoelastic polarization $P_3$ is defined by the response equation $\mathbf{P} = -\frac{\partial}{\partial \theta} (P_3 + \text{const.})$, where $\mathbf{B}$ is an applied magnetic field and $\mathbf{P}$ is the induced electric polarization. Such a fractionalized bulk topological quantum number leads to a fractional quantum Hall conductance of $\frac{e^2}{2}\frac{q}{2\pi}$ on the surface of the fractional TI. In contrast to the usual QHE in 2+1 dimensions, the surface QHE does not necessarily exhibit edge states and thus cannot be directly probed by transport measurements. Alternatively, it can only be experimentally observed through probes which couple to each surface separately, such as magneto-optical Kerr and Faraday rotation \cite{14,16}. Generically, a slab of fractional TI can have different fractional Hall conductance on the top and bottom surfaces, which can be determined separately by combined Kerr and Faraday
measurements, independent of non-universal properties of the material [21]. Our approach is inspired by the composite particle, or projective construction of FQH states [2, 22, 23]. The idea is to decompose the electron with charge $e$ into $N$ fractionally charged, fermionic ‘partons’, which have a dynamics of their own. One considers the case that the partons form a known topological state, say a topological band insulator. When the partons are recombined to form the physical electrons, a new topological state of electrons emerges. In the FQH case for example, the $\nu = \frac{1}{3}$ Laughlin state can be obtained by splitting each electron into $N = 3$ partons of charge $\frac{e}{3}$. Each parton fills the lowest Landau level and forms a noninteracting $\nu = 1$ IQH state. Ignoring the exponential factors, the parton wavefunction is the Slater determinant IQH wavefunction $\Psi(\{z_i\}) \propto \prod_{i<j} (z_i - z_j)$, and the electron wavefunction is obtained by gluing three partons together, which leads to the Laughlin wavefunction $\Psi_{1/3}(\{z_i\}) \propto \prod_{i<j} (z_i - z_j)^3$. Similarly, in 3+1 dimensions one can construct an interacting many-body wavefunction by gluing partons which are in a $\mathbb{Z}_2$ topological band insulator state. The parton ground state wavefunction $\Psi_{1}(\{r_n s_n\})$ is a Slater determinant describing the ground state of a noninteracting TI Hamiltonian such as the lattice Dirac model [24], with $\{r_n s_n\}$, $n = 1, \ldots, N$ the position and spin coordinates of the partons. The electron wavefunction is obtained by requiring the coordinates of all $N_e$ partons forming the same electron to be the same [22],

$$\Psi_{N_e}(\{r_n s_n\}) = [\Psi_{1}(\{r_n s_n\})]^{N_e}.$$  

Equation (1) is the $(3 + 1)$-dimensional generalization of the Laughlin wavefunction, and serves as a trial wavefunction for the simplest fractional TI phases we propose.

More generally, we can consider $N_f$ different ‘flavors’ of partons, with $N_{c\nu}^{(f)}$ partons of each flavor $f = 1, \ldots, N_f$. This decomposition has to satisfy two basic rules. First, to preserve the fermionic nature of the electron, the total number of partons per electron must be odd (Fig. 1b),

$$N_e^{(1)} + N_e^{(2)} + \cdots + N_e^{(N_f)} = \text{odd}.$$  

Second, if $q_f < e$ is the (fractional) charge of partons of flavor $f$, the total charge of the partons must add up to the electron charge $e$,

$$N_{c\nu}^{(1)} q_1 + N_{c\nu}^{(2)} q_2 + \cdots + N_{c\nu}^{(N_f)} q_{N_f} = e.$$  

For instance, the $\nu = \frac{1}{3}$ Laughlin state described above corresponds to $N_f = 1$, $N_{c\nu}^{(1)} = 3$, and $q_1 = \frac{e}{3}$, which satisfies both conditions. Here we consider that partons of each flavor $f$ condense in a (generally different) noninteracting $T$-invariant TI state with axion angle $\theta_f = \pi \mod 2\pi$. This is the analog of having partons condense in various IQH states in the FQH construction.

Finally, the partons have to be bound together to yield physical electrons. As we will see, this can be done by coupling partons of flavor $f$ to a $SU(N_{c\nu}^{(f)})$ gauge field, which can be interpreted as a ‘color field’ where partons of flavor $f$ come in $N_{c\nu}^{(f)}$ colors. Since the TI analog of the $\nu = \frac{1}{3}$ Laughlin state will involve three partons coupled to a $SU(3)$ gauge field in 3+1 dimensions, we dub our partons ‘quarks’ by analogy with quantum chromodynamics (QCD).

To obtain a more systematic understanding of the fractional TI, we now deduce its effective gauge theory by way of a gedanken experiment. We consider subjecting a noninteracting TI to strong electron-electron interactions, and start with the simplest case of $N_f = 1$ with $N_e^{(1)}$ odd. The electron being split into $N_e$ quarks of charge $\frac{e}{N_e^{(1)}}$, the electron operator will be written as a product of $N_e$ quark operators $\psi_i$, $i = 1, \ldots, N_e$. However, the quark operators act in a Hilbert space which is larger than the physical electron Hilbert space. We need to remove those states of the quark Hilbert space which are not invariant under unitary transformations which leave the electron operator unchanged, i.e. $SU(N_e)$ transformations with quarks in the $N_e$ representation. The projection onto the electron Hilbert space can therefore be implemented by coupling the quarks to a $SU(N_e)$ gauge field $a_\mu$ with a coupling constant $g$. Outside the fractional TI, we expect the system to be in the confined phase, in analogy to quark confinement in QCD, which has only $SU(N_e)$ singlet excitations in its low-energy spectrum, i.e. gauge-invariant ‘baryons’. Quarks of a given flavor within the baryon are antisymmetric in their $N_{c\nu}^{(f)}$ color indices; Fermi statistics then implies that their spins are aligned. In a relativistic theory this would imply that in the $N_f = 1$ theory the baryon has spin $\frac{1}{2}$. In nonrelativistic lattice models this is not a concern, but

![FIG. 1: a) Quark picture of fractional TI with flavor and color degrees of freedom; b) surface FQHE vs transport measurements [Eq. (7)]; c) nontrivial vs trivial fractional TI; d) Witten effect as a probe of bulk topology.](image)
even within the context of relativistic continuum field theories one can obtain composite spin-$\frac{1}{2}$ electrons for $N_f > 1$.

Inside the fractional TI, electron-electron interactions translate into complicated interactions among quarks. We consider the case that these interactions lead the quarks to condense at low energies into a noninteracting $T$-invariant TI state with axion angle $\theta$, and that the non-Abelian gauge field $a_\mu$ enters a deconfined phase [20]. We now show that such a phase is a fractional TI. A low-energy effective Lagrangian for $N_f = 1$ can be conjectured in the form

$$\mathcal{L} = \psi^\dagger \left(i D_0 - H_0[\pm i\mathbf{D}]\right) \psi + \mathcal{L}_{\text{int}}(\psi^\dagger, \psi),$$

where $D_0 = (D_0, -\mathbf{D}) = \partial_\mu + i \frac{N_c}{2} A_\mu + i g a_\mu$ is the $U(1)_{\text{em}} \times SU(N_c)$ gauge covariant derivative, and $H_0 = H_0(\mathbf{p})$ is the single-particle Hamiltonian for a $T$-invariant TI with axion angle $\theta$. $\mathcal{L}_{\text{int}}$ represents weak $T$-invariant residual interactions which do not destabilize the gapped phase of the quark mass [14]. In principle, the effective Yang-Mills Lagrangian for $\theta$ is adiabatically connected to a trivial vacuum with the phase of the quark mass [14].

We are now faced with our initial question of whether the effective theory [14, 15] breaks $T$-invariance. According to the first term in Eq. (5), $T$-invariance would require $\theta_{\text{eff}}$ to be quantized in integer multiples of $\pi$ if the minimal electric charge was $e$ [27]. However, the minimal charge in our theory is $\frac{e}{N_c}$, i.e., that of the quarks. Therefore, $\theta_{\text{eff}}$ has to be quantized in integer multiples of $\frac{\pi}{N_c}$. On the other hand, the second term in Eq. (5) requires $\theta$ to be quantized in integer multiples of $\pi$ [28], which means by Eq. (6) that $\theta_{\text{eff}}$ has to be quantized in units of $\frac{\pi}{N_c}$. This latter constraint is consistent with, but stronger than, the former [29], and the values of $\theta_{\text{eff}}$ allowed by $T$-invariance are thus correctly given by Eq. (6).

Equations (4) and (5) constitute a TFT which, precisely because it is topological, is insensitive to small $T$-invariant perturbations and defines a new stable phase of matter, the $T$-invariant fractional TI in $3 + 1$ dimensions. The effective theory can also be derived in the multimeter case $N_f \geq 1$, with $N_c(\theta)$ satisfying rules [22]. Considering that quarks of flavor $f$ form a noninteracting TI with axion angle $\theta_f = \pi$ mod $2\pi$ and integrating them out yields an effective Lagrangian in the form of (5), but with gauge group $U(1)_{\text{em}} \times \prod_{f} U(N_c(\theta_f))/U(1)_{\text{diag}}$. Here $U(1)_{\text{diag}}$ is the overall $U(1)$ gauge transformation of the electron operator. The electromagnetic axion angle $\theta_{\text{eff}}$ is given by $\theta_{\text{eff}} = \left(\sum_{f=1}^{N_f} N_c(\theta_f)\right)^{-1} \cdot \theta_f$ when $\theta_f$ is odd for each flavor, one can show that $\theta_{\text{eff}} = \pi p/q$ with $p, q$ odd integers.

Important physical properties of the fractional TI can be read off from Eq. (5). The surface of the fractional TI is an axion domain wall with the $U(1)_{\text{em}}$ axion angle jumping from $\theta_{\text{eff}}$ in the fractional TI to 0 in the vacuum. Such a domain wall has a surface QHE with surface Hall conductance $\sigma_{1,1} = \frac{\theta_{\text{eff}}}{2\pi} \frac{e^2}{h}$ [14]. Therefore, the surface Hall conductance of the fractional TI has the general form

$$\sigma_{H,1} = \frac{p}{q} \frac{e^2}{h}, \quad p, q \text{ odd.}$$

(7)

For example, in the simplest single-flavor case with $\theta = \pi$ in Eq. (6), we have $\sigma_{H,1} = \frac{1}{N_c} \frac{e^2}{h}$ with $N_c$ an odd integer, corresponding to half of a $\frac{1}{N_c}$ FQH Laughlin state. The more general result (7) corresponds to half of a generic Abelian FQH state [22, 30].

The fractional axion angle and the associated surface Hall conductance (7) are properties of the bulk topology. It is important to distinguish them from a TI with $\theta_{\text{eff}} = \pm \pi$ and where the surface Dirac fermions form a FQH state [31]. In a noninteracting TI with $\theta_{\text{eff}} = -\pi$ for example, both the axion domain wall and the surface FQH state contribute to $\sigma_{H,1}$.

$$\sigma_{H,1} = \left(\frac{1}{2} + \frac{n}{q}\right) \frac{e^2}{h} = \frac{2n - q}{q} \frac{2e^2}{h},$$

with $\frac{n}{q}$ an allowed filling fraction for a FQH state in $2 + 1$ dimensions. For Abelian FQH states $q$ is odd, hence the
surface Hall conductance $\sigma_{H,s} = \frac{2n-m}{q}$ has the same general form as for the fractional TI [Eq. (4)]. As the simplest example, the Laughlin state with $\frac{n}{q} = \frac{1}{3}$ leads to $\sigma_{H,s} = -\frac{1}{3} \frac{e^2}{h}$ (Fig. 1c, right) which is the same as for a genuine fractional TI with bulk $P_3 = -\frac{1}{3}$ (Fig. 1c, left). However, the bulk topology is very different in both cases. Therefore, surface measurements are not sufficient to determine the bulk topology and bulk measurements of $P_3$ are needed. One such measurement would consist in embedding a monopole with magnetic charge $q_m$ inside the fractional TI (Fig. 1b) and measuring its electric charge $q_e$ induced by the Witten effect [32, 33].

Another possible ‘experiment’ is to measure the ground state degeneracy (GSD) on topologically nontrivial spatial 3-manifolds. Consider for example a fractional TI on a manifold $\Sigma_g \times \mathcal{I}$ with $\Sigma_g$ a Riemann surface of genus $g$ and $\mathcal{I} = [0, 1]$ a bounded interval, where $L$ is the sample ‘thickness’ and the two copies of $\Sigma_g$ (at each end of $\mathcal{I}$) are the two bounding surfaces. We first discuss contributions to the GSD arising solely from the boundary, and comment on bulk contributions later on. A noninteracting TI with a $\nu = \frac{1}{2}$ Laughlin state deposited on both surfaces is described by two independent CS theories [2, 32] and has a GSD of $3^g$ ($m^g$ for $\nu = \frac{1}{m}$) on each surface for a total GSD of $(3^g)^2 = 3^{2g}$. The situation is different for a genuine fractional TI with $P_3 = \pm \frac{1}{3}$ (Fig. 1c, left). To study the GSD we set the external electromagnetic fields to zero in Eq. (4) and consider the internal $SU(3)$ $\theta$-term. Assuming that the system stays gapped as we take the limit of zero thickness $L \to 0$ where the gauge fields $a_\mu$ on both surfaces become identified, the system is described by a single $SU(3)_k$ CS theory on $\Sigma_g$ where the level $k$ is the sum of the contributions from both surfaces. If on both surfaces $\theta$ goes to the same value outside the TI, then $k = 0$ and there is no GSD. If $\theta = 0$ on one side and $\theta = 2\pi$ on the other, we have a $SU(3)_l$ CS theory with GSD $3^g \neq 3^{2g}$ [33]. Another example is the solid torus $S^1 \times T^2$. The unique boundary is a 2-torus $T^2$, hence a $\nu = \frac{1}{3}$ Laughlin state deposited on it has GSD 3 from the quantum mechanics of Wilson lines along the two non-contractible loops on $T^2$ [33]. However, since one of these loops extends into the bulk of the solid torus and is thus contractible, the $\theta$-term contributes no GSD to a genuine fractional TI. However, in addition to the boundary contributions of the $\theta$-term to the GSD, the gauge theory in the bulk can have a nontrivial GSD even in the absence of boundaries. For instance, the deconfined phase of $SU(N_c)$ gauge theory has a GSD of $N_c^3$ on $T^3$ [32]. The total GSD has in general both bulk and boundary contributions, and depends on the details of the gauge group.

It should be noted that several distinct fractional TI states can correspond to the same $\sigma_{H,s}$. For instance, $\sigma_{H,s} = \frac{3}{5} \frac{e^2}{2h}$ can correspond to a $\mathcal{N}_f = 1$, $U(1)_{em} \times SU(3)$ theory with $\theta = 3\pi$, or to a $\mathcal{N}_f = 2$, $U(1)_{em} \times U(3) \times U(4)/U(1)$ theory with $\theta_1 = \pi$ and $\theta_2 = -3\pi$. This is analogous to the well-known fact that topological orders in FQH states cannot be characterized by the Hall conductance alone [32]. The property of GSD provides a finer classification of fractional TI. Moreover, all the states discussed so far can be considered as $(3+1)$-dimensional generalization of Abelian FQH states. In principle, more generic non-Abelian fractional TI states with exotic even-denominator ‘halved’ non-Abelian FQH states on their surface can also be constructed using more generic parton decompositions, which correspond to effective theories with gauge groups other than $SU(m)$ [24].

In conclusion, we have shown that fractional TI states in $3 + 1$ dimensions with a quantized fractional bulk magnetoelectric polarization $P_3$ and a ‘halved’ odd-denominator surface FQHE are fully consistent with $T$-invariance, and can in principle be realized in strongly correlated systems with strong spin-orbit coupling.

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This can be achieved either by adding additional colored but electrically neutral matter or considering the special case with only Abelian groups, $N^{(f)} = 1$ for all $f$.

For a recent discussion, see A. Kapustin and E. Witten, hep-th/0604151; S. Gukov and E. Witten, hep-th/0612073.

Color neutral monopoles carry $U(1)_{em}$ magnetic charge $\frac{1}{N_c}$, and lead to $\theta_{eff} = \frac{n\pi}{N_c}$, $n \in \mathbb{Z}$. However, monopoles with smaller $U(1)_{em}$ magnetic charge $\frac{1}{2}$ are allowed but also carry color magnetic charge $\frac{1}{N_c}$, and lead to $\theta_{eff} = \frac{n\pi}{N_c}$, $n \in \mathbb{Z}$.

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