BLOW-UP CRITERIA OF SMOOTH SOLUTIONS TO THE THREE-DIMENSIONAL MICROPOLAR FLUID EQUATIONS IN BESOV SPACE

Baoquan Yuan* and Xiao Li

School of Mathematics and Information Science
Henan Polytechnic University, Henan 454000, China

ABSTRACT. In this paper, we investigate the blow-up criteria of smooth solutions and the regularity of weak solutions to the micropolar fluid equations in three dimensions. We obtain that if $\nabla h u, \nabla h \omega \in L^1(0, T; \dot{B}^0_{\infty, \infty})$ or $\nabla h u, \nabla h \omega \in L^8_3(0, T; \dot{B}^{-1}_{\infty, \infty})$ then the solution $(u, \omega)$ can be extended smoothly beyond $t = T$.

1. Introduction. In this paper we consider the following Cauchy problem for the incompressible micropolar fluid equations:

$$
\begin{aligned}
\frac{\partial u}{\partial t} - (\mu + \chi)\Delta u + u \cdot \nabla u + \nabla p - \chi \nabla \times \omega &= 0, \\
\frac{\partial \omega}{\partial t} - \gamma \Delta \omega - \kappa \text{div} \omega + 2\chi \omega + u \cdot \nabla \omega - \chi \nabla \times u &= 0, \\
\text{div} u &= 0, \\
u(x, 0) &= u_0(x), \quad \omega(x, 0) = \omega_0(x),
\end{aligned}
$$

(1)

where $u = (u_1(x, t), u_2(x, t), u_3(x, t))$ denotes the velocity of the fluid at a point $x \in \mathbb{R}^3$ and time $t \in [0, T]$; $\omega = (\omega_1(x, t), \omega_2(x, t), \omega_3(x, t))$ and $p = p(x, t)$ stand for the micro-rotational velocity and the hydrostatic pressure, respectively. $u_0$ and $\omega_0$ are the prescribed initial data for the velocity and angular velocity with property $\text{div} u_0 = 0$. $\mu$ is the kinematic viscosity, $\chi$ is the vortex viscosity, $\kappa$ and $\gamma$ are spin viscosities.

Micropolar fluid system was firstly developed by Eringen [9]. It is a type of fluids which exhibits micro-rotational effects and micro-rotational inertia, and can be viewed as a non-Newtonian fluid. Physically, micropolar fluids represent fluids that consist of rigid, randomly oriented (or spherical particles) suspended in a viscous medium, where the deformation of fluid particles is ignored. It can describe many phenomena that appear in a large number of complex fluids such as the suspensions, animal blood, liquid crystals which cannot be characterized appropriately by the Navier-Stokes system. For more background, we refer to [17] and references therein. When the micro-rotation effects are neglected or $\omega = 0$, the micropolar fluid flows reduces to the incompressible Navier-Stokes equations, which has been greatly analyzed, see, for example, the classical books by Ladyzhenskaya [14], Lions [16] or

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* Corresponding author: Baoquan Yuan.
Lemarié-Rieusset [15]. From the viewpoint of the model, therefore, Navier-Stokes flow is viewed as the flow of a simplified micropolar fluid.

Besides their physical applications, the micropolar fluid equations are also mathematically significant. Fundamental mathematical issues such as the global regularity of their solutions have generated extensive research, and many interesting results have been obtained (see, for example, [4, 6, 12, 17, 20, 23] and references therein). Galdi and Rionero considered the weak solutions in [11]. Using linearization and an almost fixed point theorem, Lukaszewicz [18] established the global existence of weak solutions with sufficiently regular initial data. And using the same technique, Lukaszewicz [19] proved the local and global existence and the uniqueness of the strong solutions under asymmetric condition. Yamaguchi [24] proved the existence theorem of global in time solution for small initial data. The regularity of weak solutions and blow-up criteria of smooth solutions to the micropolar fluid equations are important topic in the research of global well-posedness. Yuan [26] established classical Serrin-type regularity criteria in terms of the velocity or its gradient

\[ u \in L^q(0, T; L^p)(\mathbb{R}^3), \quad \frac{2}{q} + \frac{3}{p} = 1, 3 < p \leq \infty, \]

and

\[ \nabla u \in L^q(0, T; L^p)(\mathbb{R}^3), \quad \frac{2}{q} + \frac{3}{p} = 2, \frac{3}{2} < p \leq \infty. \]

Particularly, in the end-point case \( p = \infty \), the blow-up criteria can be extended to more general spaces \( \nabla u \in L^1(0, T; \dot{B}^0_{\infty, \infty}) \). Later Yuan [25] extended the Serrin’s regularity criteria to Lorentz spaces, and Gala [12] extended the Serrin’s regularity criteria to the Morrey-Campanato spaces.

Dong [7] further refined the velocity regularity in general Besov spaces

\[ u \in L^{\frac{2}{1-r}}(0, T; \dot{B}^r_{\infty, \infty})(\mathbb{R}^3), \quad -1 < r \leq 1. \]

Recently, two new logarithmically blow-up criteria of smooth solution to the equations (1) in the Morrey-Campanato space are established by Wang and Zhao [23], and Zhang [27] established an improved blow-up criteria in terms of vorticity of velocity in Besov space, i.e.

\[ \int_0^T \frac{\| \nabla \times u \|_{\dot{B}^0_{\infty, \infty}}}{\sqrt{1 + \log(1 + \| \nabla \times u \|_{\dot{B}^0_{\infty, \infty}})}} dt < \infty. \]

On the other hand, it is desirable to show the regularity of the weak solutions if some partial derivatives of the velocity satisfy certain growth conditions. For the 3D Navier-Stokes equations, there are many results to show such regularity of weak solutions in terms of partial derivative of the velocity \( u \) (see, for example, [1, 8, 10, 13, 22, 28, 29]). In particular, as to the endpoint case, Fang and Qian in [10] showed that if

\[ \nabla_h u \in L^{\frac{2}{1}}(0, T; \dot{B}^{-1}_{\infty, \infty})(\mathbb{R}^3), \]

then the weak solution \( u \) was regular. Here, \( \nabla_h = (\partial_1, \partial_2) \) is the horizontal gradient operator.

Motivated by the reference mentioned above, the purpose of the present paper is to extend the blow-up criteria of smooth solutions and the regularity of weak solutions to the micropolar fluid equations (1) in terms of partial derivatives of the velocity and the micro-rotational velocity.
Before stating our main results we introduce some function spaces and notations. Let $C^{∞}_{0,r}(R^3)$ denote the set of all $C^∞$ vector functions $f(x) = (f_1(x), f_2(x), f_3(x))$ with compact support such that $div f(x) = 0$. $L^r_{\sigma}(R^3)$ is the closure of $C^{∞}_{0,r}(R^3)$-function with respect to the $L^r$-norm $\| \cdot \|_r$, for $1 \leq r \leq ∞$. $H^r_{\sigma}(R^3)$ denotes the closure of $C^{∞}_{0,r}(R^3)$ with respect to the $H^r$-norm $\| f \|_{H^s} = \| (1 - Δ)^{\frac{s}{2}} f \|_2$, for $s \geq 0$.

Now, we state our results as follows.

Theorem 1.1. Let $u_0 \in H^1_{\sigma}(R^3)$ and $ω_0 \in H^1(R^3)$. Suppose that the pair $u \in C([0,T];H^1_{\sigma}(R^3)) \cap C((0,T);H^2_{\sigma}(R^3))$ and $ω \in C([0,T];H^1(R^3)) \cap C((0,T);H^2(R^3))$ is a smooth solution to the equations (1). If $(u,ω)$ satisfies the condition
\[
\int_0^T (\| \nabla u(t) \|_{H^0_{\sigma,∞}} + \| \nabla ω(t) \|_{H^2_{\sigma,∞}}) dt < ∞, \tag{2}
\]
then the solution $(u,ω)$ can be extended smoothly beyond $t = T$.

Theorem 1.2. Let $u_0 \in H^1_{\sigma}(R^3)$ and $ω_0 \in H^1(R^3)$. Suppose that the pair $u \in C([0,T];H^1_{\sigma}(R^3)) \cap C((0,T);H^2_{\sigma}(R^3))$ and $ω \in C([0,T];H^1(R^3)) \cap C((0,T);H^2(R^3))$ is a smooth solution to the equations (1). If $(u,ω)$ satisfies the condition
\[
\int_0^T (\| \nabla u(t) \|_{H^0_{\sigma,∞}} + \| \nabla ω(t) \|_{H^2_{\sigma,∞}}) dt < ∞, \tag{3}
\]
then the solution $(u,ω)$ can be extended smoothly beyond $t = T$.

Remark 1. If $ω = 0$, it is clear that theorem 1.2 reduces the results of Fang and Qian [10] for 3D Navier- Stokes equations.

Next, in order to derive the criteria on regularity of weak solutions to the micropolar fluid equations (1), we introduce the definition of a weak solution.

Definition 1.3. Let $u_0(x) \in L^2_{\sigma}(R^3)$ and $ω_0(x) \in L^2(R^3)$. A measurable function $(u(x,t),ω(x,t))$ is called a weak solution to the micropolar equations (1) on $[0,T]$ if
(a) $u(x,t) \in L^∞(0,T;L^2_{\sigma}(R^3)) \cap L^2(0,T;H^1_{\sigma}(R^3))$,
(b) and $ω \in L^∞(0,T;L^2(R^3)) \cap L^2(0,T;H^1(R^3))$;
(b) $\int_0^T \{ -(u,∂_r φ) + (μ + χ)(\nabla u,\nabla φ) + (u,\nabla φ) \} - χ(∇ × φ) dτ = -(u_0,φ(0)),$
\[
\int_0^T \{ -(ω,∂_r φ) + χ(∇ ω,∇ φ) + κ(div ω,div φ) + 2χ(ω,φ) + (u,∇ ω,φ) \} - χ(∇ × φ,φ) dτ = -(ω_0,φ(0)),
\]
for any $φ(x,t) \in H^1([0,T];H^1_{\sigma}(R^3))$ and $φ(x,t) \in H^1([0,T];H^1(R^3))$ with $φ(T) = 0$ and $φ(T) = 0$. 
In the reference [21], Rojas-Medar and Boldrini proved the global existence of weak solutions to the equations (1) of the magneto-micropolar fluid motion by the Galerkin method. The weak solutions also satisfy the strong energy inequality
\[
\|(u, \omega)\|_2^2 + 2\mu \int_0^t \|\nabla u\|_2^2 \, ds + 2\gamma \int_0^t \|\nabla \omega\|_2^2 \, ds + 2\kappa \int_0^t \|\text{div} \omega\|_2^2 \, ds \\
+ 2\chi \int_0^t \|\omega\|_2^2 \, ds \leq \|(u_0, \omega_0)\|_2^2,
\]
for \(0 \leq \varepsilon \leq t \leq T\).

**Corollary 1.** Let \(u_0 \in H^1_0(\mathbb{R}^3)\) and \(\omega_0 \in H^1(\mathbb{R}^3)\). Suppose that \((u(t, x), \omega(t, x))\) is a weak solution to the equations (1) and satisfies the strong energy inequality. If \((u, \omega)\) satisfies
\[
\int_0^T (\|\nabla_h u(t)\|_{B_{\infty, \infty}} + \|\nabla_h \omega(t)\|_{B_{\infty, \infty}}) \, d\tau < \infty,
\]
then the weak solution \((u, \omega)\) is regular on \((0, T]\).

**Corollary 2.** Let \(u_0 \in H^1_0(\mathbb{R}^3)\) and \(\omega_0 \in H^1(\mathbb{R}^3)\). Suppose that \((u(t, x), \omega(t, x))\) is a weak solution to the equations (1) and satisfies the strong energy inequality. If \((u, \omega)\) satisfies
\[
\int_0^T (\|\nabla_h u(t)\|_{B_{\infty, \infty}}^{\frac{3}{2}} + \|\nabla_h \omega(t)\|_{B_{\infty, \infty}}^{\frac{3}{2}}) \, d\tau < \infty,
\]
then the weak solution \((u, \omega)\) is regular on \((0, T]\).

The proofs of Corollary 1 and 2 are standard. For the sake of completeness, we give the sketch of the proof of Corollary 1 only. Since \(u_0 \in H^1_0(\mathbb{R}^3)\) and \(\omega_0 \in H^1(\mathbb{R}^3)\), by the classical local existence theorem of strong solutions to the equations (1) that there exist a time \(T' > 0\) and a unique solution \(u' \in C([0, T'); H^1_0(\mathbb{R}^3))\) and \(\omega' \in C([0, T'); H^1(\mathbb{R}^3))\). Since \((u, \omega)\) is a weak solution satisfying the energy inequality (4), we conclude that \((u', \omega') = (u, \omega)\) on \([0, T').\) Thus it is sufficient to show that \(T = T'\). If not, let \(T' < T\), then without loss of generality, we may assume that \(T'\) is the maximal existent time for \((u', \omega').\) By condition (5) in Corollary 1, we have
\[
\int_0^{T'} (\|\nabla_h u'(t)\|_{B_{\infty, \infty}} + \|\nabla_h \omega'(t)\|_{B_{\infty, \infty}}) \, d\tau < \infty,
\]
because \((u', \omega') = (u, \omega)\) on \([0, T]\), it follows that \((u'(t), \omega'(t))\) can be extended to interval \((0, T_1)\) for some \(T_1 > T'\), which is contradictory to the maximality of \(T'\). Therefore the weak solution \((u, \omega)\) is regular on \([0, T]\) and we thus complete the proof of Corollary 1.

The plan of the paper is arranged as follows. We first state some preliminary on functional settings and some important inequalities in Section 2 and then prove Theorem 1.1 in Section 3. Finally the proof of Theorem 1.2 is presented in Section 4.

2. **Preliminaries.** Let us choose a nonnegative radial function \(\chi(\xi) \in C_0^\infty(\mathbb{R}^n)\) such that \(0 \leq \chi(\xi) \leq 1\) and
\[
\chi(\xi) = \begin{cases} 
1, & \text{for } |\xi| \leq \frac{3}{4}, \\
0, & \text{for } |\xi| > \frac{4}{3},
\end{cases}
\]
and let \( \hat{\phi}(\xi) = \chi(\xi/2) - \chi(\xi) \), \( \chi_j(\xi) = \chi(\frac{\xi}{2^j}) \) and \( \hat{\phi}_j(\xi) = \hat{\phi}(\frac{\xi}{2^j}) \) for \( j \in \mathbb{Z} \). Write
\[
\begin{align*}
    h(x) &= \mathcal{F}^{-1}\chi(\xi), \quad h_j(x) = 2^{nj}h(2^j x); \\
    \varphi_j(x) &= 2^{nj}\varphi(2^j x),
\end{align*}
\]
where \( \hat{f}(\xi) \) and \( \mathcal{F}^{-1}f(\xi) \) denote the Fourier transform and inverse transform, respectively. Define the Littlewood-Paley projection operators \( S_j \) and \( \triangle_j \), respectively, as
\[
\begin{align*}
    S_j u(x) &= h_j * u(x), \quad \text{for } j \in \mathbb{Z}, \\
    \triangle_j u(x) &= \varphi_j * u(x) = S_{j+1} u(x) - S_j u(x), \quad \text{for } j \in \mathbb{Z}.
\end{align*}
\]
Formally \( \triangle_j \) is a frequency projection to the annulus \( |\xi| \sim 2^j \), while \( S_j \) is a frequency projection to the ball \( |\xi| \leq 2^j \) for \( j \in \mathbb{Z} \). For any \( u(x) \in L^2(\mathbb{R}^n) \) we have the Littlewood-Paley decomposition
\[
\begin{align*}
    u(x) &= h * u(x) + \sum_{j \geq 0} \varphi_j * u(x), \\
    u(x) &= \sum_{j = -\infty}^{\infty} \varphi * u(x),
\end{align*}
\]
where the series is convergent in the sense of \( L^2 \) norm. Clearly,
\[
\begin{align*}
    \text{supp} \chi(\xi) \cap \text{supp} \hat{\varphi}_j(\xi) &= \emptyset \quad \text{for } j \geq 1, \\
    \text{supp} \hat{\phi}_j(\xi) \cap \text{supp} \hat{\varphi}_{j'}(\xi) &= \emptyset, \quad \text{for } |j - j'| \geq 2.
\end{align*}
\]

Next, we recall the definition of Besov spaces. Let \( s \in \mathbb{R} \) and \( 1 \leq p \leq q \leq \infty \), the Besov space \( B^{s}_{p,q}(\mathbb{R}^n) \), abbreviated as \( B^{s}_{p,q} \), is defined by
\[
B^{s}_{p,q} = \{ f(x) \in \mathcal{S}(\mathbb{R}^n); \| f \|_{B^{s}_{p,q}} < \infty \},
\]
where
\[
\| f \|_{B^{s}_{p,q}} = (\| h * f \|_p^q + \sum_{j \geq 0} 2^{jsq}\| \varphi_j * f \|_p^q)^{1/q}
\]
is the Besov norm. The homogeneous Besov space \( \dot{B}^{s}_{p,q} \) is defined by the dyadic decomposition as
\[
\dot{B}^{s}_{p,q} = \{ f(x) \in \mathcal{S}'(\mathbb{R}^n); \| f \|_{\dot{B}^{s}_{p,q}} < \infty \},
\]
where
\[
\| f \|_{\dot{B}^{s}_{p,q}} = (\sum_{j = -\infty}^{\infty} 2^{jsq}\| \varphi_j * f \|_p^q)^{1/q}
\]
is the homogeneous Besov norm, and \( \mathcal{S}'(\mathbb{R}^n) \) denotes the dual space of \( \mathcal{S}(\mathbb{R}^n) = \{ f(x) \in \mathcal{S}(\mathbb{R}^n); D^\alpha f(0) = 0, \text{ for any } \alpha \in \mathbb{N}^n \text{ multi-index} \} \) and can be identified by the quotient space \( \mathcal{S}'/\mathcal{P} \) with the polynomial functional set \( \mathcal{P} \). In particular, if \( p = q = 2 \) and \( s = m \) is positive integer, \( \dot{B}^{s}_{2,2}(\mathbb{R}^n) \) and \( B^{s}_{2,2}(\mathbb{R}^n) \) are equivalent to the Sobolev spaces \( H^m(\mathbb{R}^n) \) and \( \dot{H}^m(\mathbb{R}^n) \), respectively. For details, refer to [3] and [2].

Below we recall the Bernstein’s lemma that will be used in the proofs of our results.
Lemma 2.1. (Bernstein’s inequality)

(a) Let $g(x) \in L^p(\mathbb{R}^n) \cap L^{p_1}(\mathbb{R}^n)$, and supp $g \subset \{ |\xi| \leq r \}$. Then there exists a constant $C$ such that

$$
\|g\|_{p_1} \leq C r^{n(\frac{1}{p} - \frac{1}{p_1})} \|g\|_p,
$$

for $1 \leq p \leq p_1 \leq \infty$.

(b) Assume that $f(x) \in L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$ and supp $\hat{f} \subset \{ 2^{j-1} \leq |\xi| \leq 2^{j+1} \}$ for $j \in \mathbb{Z}$, then there exists a constant $C_k$ so that the following inequality holds, i.e.

$$
C_k^{-1} 2^{jk} \|f\|_p \leq \|\partial_k f\|_p \leq C_k 2^{jk} \|f\|_p.
$$

The proof is an immediate consequence of Young’s inequality (see [2] for details).

Lemma 2.2. (page 82 in [2]) Let $1 \leq q < p < \infty$ and $\alpha$ be a position real number. A constant $C$ exists such that

$$
\|f\|_{L^p} \leq C \|f\|_{B_{\alpha,\infty}^\theta} \|f\|_{B_{q,q}^\beta}, \text{ with } \beta = \frac{p}{q} - 1 \text{ and } \theta = \frac{q}{p}.
$$

In particular, for $\beta = 1, q = 2$ and $p = 4$, we get $\alpha = 1$ and

$$
\|f\|_{L^4} \leq C \|f\|_{B_{\infty,\infty}^{\frac{1}{2}}} \|f\|_{H^{\frac{1}{2}}}.
$$

We will also use the following Sobolev inequality in $\mathbb{R}^3$, which is proved by Cao and Wu [5]

$$
\|u\|_r \leq C \|u\|_{H^r} \|\nabla u\|_2 \|\Delta u\|_2 + \|\partial_1 u\|_2 \|\partial_2 u\|_2 \|\partial_3 u\|_2
$$

$$
\leq C \|u\|_r \|\nabla u\|_2 \leq C \|u\|_r \|\nabla u\|_2 \leq C \|u\|_r \|\nabla u\|_2,
$$

for every $u \in H^1(\mathbb{R}^3)$ and every $r \in [2, 6]$, where $C$ is a constant depending only on $r$.

3. Proof of Theorem 1.1.

Proof. Taking the inner product of $-\Delta u$ with the first equation of (1), and taking the inner product of $-\Delta \omega$ with the second equation of (1). By integrating by parts and using the incompressibility condition, we have

$$
\frac{1}{2} \frac{d}{dt} (\|\nabla u\|_2^2 + \|\nabla \omega\|_2^2) + (\mu + \chi) \|\Delta u\|_2^2 + \gamma \|\Delta \omega\|_2^2 + \kappa \|\nabla \omega\|_2^2
$$

$$
\leq \chi \int_{\mathbb{R}^3} \nabla \times \nabla \omega \cdot \nabla u dx + \chi \int_{\mathbb{R}^3} \nabla \times \nabla u \cdot \nabla \omega dx + \int_{\mathbb{R}^3} \nabla u \cdot \nabla u dx + \int_{\mathbb{R}^3} \nabla \omega \cdot \nabla \omega dx.
$$

(10)

Employing the Hölder and Young inequalities and integration by parts, we derive the estimation of the first three terms on the right-hand side of (10) as

$$
\chi \int_{\mathbb{R}^3} \nabla \times \nabla \omega \cdot \nabla u dx + \chi \int_{\mathbb{R}^3} \nabla \times \nabla u \cdot \nabla \omega dx - 2\chi \|\nabla \omega\|_2^2
$$

$$
\leq 2\chi \|\nabla \omega\|_2^2 + \frac{\chi}{2} \|\Delta u\|_2^2 + 2\chi \|\nabla \omega\|_2^2 = \frac{\chi}{2} \|\Delta u\|_2^2,
$$

(11)
Arguing similarly as the above estimate, we have
\[
\int_{\mathbb{R}^3} \nabla u \cdot \nabla u \cdot \nabla u \, dx
= \sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j \, dx
= \sum_{j=1}^{3} \int_{\mathbb{R}^3} \partial_3 u_3 \partial_3 u_3 \, dx
+ \sum_{i=1}^{2} \sum_{j=1}^{3} \int_{\mathbb{R}^3} \partial_3 u_i \nabla h u_j \partial_3 u_j \, dx
+ \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{\mathbb{R}^3} \nabla h u_i \partial_i u_j \nabla h u_j \, dx
\leq C \int_{\mathbb{R}^3} |\nabla h u| |\nabla u|^2 \, dx + C \int_{\mathbb{R}^3} |\nabla h u| |\nabla u| |\nabla u| \, dx, \tag{12}
\]

Arguing similarly as the above estimate, we have
\[
\int_{\mathbb{R}^3} \nabla u \cdot \nabla \omega \cdot \nabla \omega \, dx
= \sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} \partial_k u_i \partial_i \omega_j \partial_k \omega_j \, dx
= \sum_{j=1}^{3} \int_{\mathbb{R}^3} \partial_3 u_3 \partial_3 \omega_j \partial_3 \omega_j \, dx
+ \sum_{i=1}^{2} \sum_{j=1}^{3} \int_{\mathbb{R}^3} \partial_3 u_i \nabla h \omega_j \partial_3 \omega_j \, dx
+ \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{\mathbb{R}^3} \nabla h u_i \partial_i \omega_j \nabla h \omega_j \, dx
\leq C \int_{\mathbb{R}^3} |\nabla h u| |\nabla \omega|^2 \, dx + C \int_{\mathbb{R}^3} |\nabla h \omega| |\nabla u| |\nabla \omega| \, dx, \tag{13}
\]

Inserting (11)-(13) into (10), we find
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2 + |\nabla \omega|_2^2 + (\mu + \frac{\chi}{2}) |\Delta u|_2^2 + \gamma |\Delta \omega|_2^2 + \kappa |\text{div} \nabla \omega|_2^2
\leq C \int_{\mathbb{R}^3} |\nabla h u| |\nabla u|^2 \, dx + C \int_{\mathbb{R}^3} |\nabla h u| |\nabla \omega|^2 \, dx + C \int_{\mathbb{R}^3} |\nabla h \omega| |\nabla u| |\nabla \omega| \, dx \tag{14}
\]
\[= I_1 + I_2 + I_3.\]
Next we estimate $I_1, I_2$ and $I_3$. First employing the Littlewood-Paley decomposition for $u$

$$
u = \sum_{j \in \mathbb{Z}} \Delta_j u = \sum_{j < -N} \Delta_j u + \sum_{|j| \leq N} \Delta_j u + \sum_{j > N} \Delta_j u,$$  

(15)

where $N$ is a positive integer that will be chosen later. And inserting (15) into $I_1$ one has

$$I_1 = C \sum_{j < -N} \int_{\mathbb{R}^3} |\Delta_j \nabla_h \nu| |\nabla \nu|^2 dx + C \sum_{|j| \leq N} \int_{\mathbb{R}^3} |\Delta_j \nabla_h \nu| |\nabla \nu|^2 dx$$

$$+ C \sum_{j > N} \int_{\mathbb{R}^3} |\Delta_j \nabla_h \nu| |\nabla \nu|^2 dx$$

(16)

$$= I_{11} + I_{12} + I_{13}.$$

For $I_{11}$, by applying the Hölder, Bernstein and Cauchy inequalities, we obtain that

$$I_{11} \leq C \|\nabla \nu\|^2 \sum_{j < -N} \|\Delta_j \nabla_h \nu\| \leq C \|\nabla \nu\|^2 \sum_{j < -N} 2^{j \frac{N}{2}} \|\Delta_j \nabla_h \nu\|$$

$$\leq \|\nabla \nu\| \left( \sum_{j < -N} 2^{j} \left( \sum_{j < -N} \|\Delta_j \nabla_h \nu\|^2 \right) \right)^{\frac{1}{2}} \leq C 2^{-\frac{3}{2}N} \|\nabla \nu\|^3.$$  

(17)

By virtue of the Hölder and Bernstein inequalities and the definition of Besov space, we have

$$I_{12} \leq C \|\nabla \nu\|^2 \sum_{|j| \leq N} \|\Delta_j \nabla_h \nu\| \leq C N \|\nabla \nu\|^2 \|\nabla_h \nu\|_{\dot{B}^{0}_{2,\infty}}.$$  

(18)

For $I_{13}$, by Gagliardo-Nirenberg inequality we have

$$I_{13} \leq C \|\nabla \nu\|^2 \sum_{j > N} 2^{j \frac{N}{2}} \|\Delta_j \nabla_h \nu\|$$

$$\leq C \|\nabla \nu\|^2 \|\Delta \nu\| \left( \sum_{j > N} 2^{j} \left( \sum_{j > N} \|\Delta_j \nabla_h \nu\|^2 \right) \right)^{\frac{1}{2}}$$

$$\leq C 2^{-\frac{3}{2}} \|\nabla \nu\|^2 \|\Delta \nu\|^2.$$  

(19)

Thus, collecting terms of (17)-(19) yields

$$I_1 \leq C 2^{-\frac{3}{2}} \|\nabla \nu\|^3 + C N \|\nabla \nu\| \|\nabla_h \nu\|_{\dot{B}^{0}_{2,\infty}} + C 2^{-\frac{3}{2}} \|\nabla \nu\| \|\Delta \nu\|^2.$$  

(20)

Similar to the estimate of $I_1$, the term $I_2$ can be bounded as

$$I_2 \leq C 2^{-\frac{3N}{2}} \left( \|\nabla \omega\|^3 + \|\nabla \nu\|^3 \right) + C N \|\nabla \omega\| \|\nabla_h \nu\|_{\dot{B}^{0}_{2,\infty}} + C 2^{-\frac{3}{2}} \|\nabla \omega\| \|\Delta \nu\|^2 + \|\Delta \omega\|^2.$$  

(21)

For $I_3$, write $I_3$ as

$$I_3 \leq C \int_{\mathbb{R}^3} \|\nabla h \omega\| \left( |\nabla \nu|^2 + |\nabla \omega|^2 \right) dx.$$  

(22)

Using an argument similar to that used in deriving $I_1$ and $I_2$ to obtain that

$$I_3 \leq C 2^{-\frac{3N}{2}} \left( \|\nabla \omega\|^3 + \|\nabla \nu\|^3 \right) + C N \|\nabla \omega\| \|\nabla_h \nu\|_{\dot{B}^{0}_{2,\infty}}$$

$$+ C 2^{-\frac{3}{2}} \left( |\nabla \nu|^2 + |\nabla \omega|^2 \right) \|\Delta \nu\|^2 + \|\Delta \omega\|^2.$$  

(23)
Substituting (20), (21) and (23) into (14) yields that
\[
\frac{1}{2} \frac{d}{dt} (\|\nabla u\|^2 + \|\nabla \omega\|^2) + (\mu + \chi)\|\Delta u\|^2 + \gamma \|\Delta \omega\|^2 + \kappa \|\text{div}{\nabla \omega}\|^2 \\
\leq 2^{-\frac{\lambda}{2\beta'}} (\|\nabla u\|^2 + \|\nabla \omega\|^2) + C N (\|\nabla u\|^2 + \|\nabla \omega\|^2) (\|\nabla h \omega\|_{B^3_{\infty, \infty}} + \|\nabla h \omega\|_{B^3_{\infty, \infty}}) \\
+ C 2^{-\frac{\lambda}{2} (\|\nabla u\|^2 + \|\nabla \omega\|^2)} (\|\Delta u\|^2 + \|\Delta \omega\|^2).
\] (24)

We choose
\[
N \geq \left[ \frac{2}{\log 2} \log C (\|\nabla u\|^2 + \|\nabla \omega\|^2 + e) \right] + 2.
\]

Therefore, (24) can be reduced to
\[
\frac{d}{dt} (\|\nabla u\|^2 + \|\nabla \omega\|^2) + (\mu + \chi)\|\Delta u\|^2 + \gamma \|\Delta \omega\|^2 + \kappa \|\text{div}{\nabla \omega}\|^2 \\
\leq C + C (\|\nabla u\|^2 + \|\nabla \omega\|^2) (\|\nabla h \omega\|_{B^3_{\infty, \infty}} + \|\nabla h \omega\|_{B^3_{\infty, \infty}}) (\log (\|\nabla u\|^2 + \|\nabla \omega\|^2 + e)).
\] (25)

Integrating in time and applying the Gronwall inequality, we infer that
\[
\|\nabla u\|^2 + \|\nabla \omega\|^2 \leq (\|u_0\|^2 + \|\omega_0\|^2 + CT) \\
\exp \left( \int_0^t C (\|\nabla h \omega\|_{B^3_{\infty, \infty}} + \|\nabla h \omega\|_{B^3_{\infty, \infty}}) \log (\|\nabla u\|^2 + \|\nabla \omega\|^2 + e) d\tau \right).
\] (26)

Taking logarithmic on both sides of (26) and applying the Gronwall inequality to it, we obtain that
\[
\log (\|\nabla u\|^2 + \|\nabla \omega\|^2 + e) \leq C (\|u_0\|^2 + \|\omega_0\|^2 + CT) \\
\exp \left( \int_0^t (\|\nabla h \omega\|_{B^3_{\infty, \infty}} + \|\nabla h \omega\|_{B^3_{\infty, \infty}}) d\tau \right),
\] (27)

for any \( t \in [0, T) \). Therefore, by the standard arguments of continuation of local solutions, we complete the proof of Theorem 1.1. \( \square \)

4. Proof of Theorem 1.2.

Proof. The proof of Theorem 1.2 is divided into two steps.

**Step I.** \((\|\nabla_h u\|^2 + \|\nabla_h \omega\|^2)\)-estimates.

Firstly, taking the inner product of \(\nabla_h u\) with \(\nabla_h\) of the first equation of (1), and taking the inner product of \(\nabla_h \omega\) with \(\nabla_h\) of the second equation of (1). It follows that
\[
\frac{1}{2} \frac{d}{dt} (\|\nabla_h u\|^2 + \|\nabla_h \omega\|^2) + (\mu + \chi)\|\nabla_h u\|^2 + \gamma \|\nabla_h \omega\|^2 + \kappa \|\text{div}{\nabla_h \omega}\|^2 \\
\leq \int_{\mathbb{R}^3} \nabla_h u \cdot \nabla u \cdot \nabla_h u dx + \int_{\mathbb{R}^3} \nabla_h u \cdot \nabla \omega \cdot \nabla_h \omega dx \\
+ 2\chi \int_{\mathbb{R}^3} \nabla \times \nabla_h u \cdot \nabla_h \omega dx - 2\chi \|\nabla_h \omega\|^2,
\] (28)
where we have used the facts that
\[
\int_{\mathbb{R}^3} u \cdot \nabla \nabla_h u \cdot \nabla_h u dx = \int_{\mathbb{R}^3} u \cdot \nabla \nabla_h \omega \cdot \nabla_h \omega dx = 0,
\] (29)
and
\[\int_{\mathbb{R}^3} \nabla \times \nabla h u \cdot \nabla h \omega dx = \int_{\mathbb{R}^3} \nabla \times \nabla h \omega \cdot \nabla h u dx. \tag{30}\]

By means of the Hölder and Young inequalities, as well as (8), we have
\[\int_{\mathbb{R}^3} \nabla h u \cdot \nabla u \cdot \nabla h \omega dx \leq \|\nabla h u\|_2\|\nabla u\|_2\|\nabla h \omega\|_2\leq C\|\nabla h u\|_2\|\nabla\nabla h u\|_2\|\nabla u\|_2\\tag{31}\]
Arguing similarly as the above estimate, one has
\[\int_{\mathbb{R}^3} \nabla h u \cdot \nabla \omega \cdot \nabla h \omega dx \leq \|\nabla h u\|_2\|\nabla \omega\|_2\|\nabla h \omega\|_2\leq C\|\nabla h u\|_2\|\nabla \omega\|_2\|\nabla h \omega\|_2\\tag{32}\]

Combining (31)-(32) and (11) into (28) and integrating by parts, we obtain
\[\|\nabla h u\|_2^2 + \|\nabla h \omega\|_2^2 + (\mu + \chi) \int_0^t \|\nabla h \nabla u\|_2^2\text{d}r + \gamma \int_0^t \|\nabla h \nabla \omega\|_2^2\text{d}r\]
\[= 2\kappa \int_0^t |\text{div}\nabla h \omega\|_2^2\text{d}r\]
\[\leq \|\nabla h u_0(x)\|_2^2 + \|\nabla h \omega_0(x)\|_2^2 + C \int_0^t \|\nabla h u\|_2^2\|\nabla u\|_2^2\text{d}r + C \int_0^t \|\nabla h \omega\|_2^2\|\nabla \omega\|_2^2\text{d}r.\tag{33}\]

**Step II.** \((\|\nabla u\|_2 + \|\nabla \omega\|_2)-estimates.\)

Similar to the proof of Theorem 1.1, we start this subsection from the term in equation (14). This time we estimate \(I_1, I_2\) and \(I_3\) in another way. By using of Hölder inequality and (9) that
\[I_1 \leq C \int_{\mathbb{R}^3} |\nabla h u| |\nabla u|^2\text{d}x \leq C\|\nabla h u\|_2\|\nabla u\|_2^2 \leq C\|\nabla h u\|_2\|\nabla u\|_2\|\nabla h u\|_2\|\Delta u\|_2^2.\tag{34}\]

Then the Hölder inequality and (9) yield that for \(I_2, I_3\)
\[I_2 \leq C \int_{\mathbb{R}^3} |\nabla h u| |\nabla \omega|^2\text{d}x \leq C\|\nabla h u\|_2\|\nabla \omega\|_2^2 \leq C\|\nabla h u\|_2\|\nabla \omega\|_2\|\nabla h \omega\|_2\|\Delta \omega\|_2^2.\tag{35}\]
Thus, inserting (34)-(36) into (14), we infer that
\[
\frac{1}{2} \frac{d}{dt} (\|\nabla u\|^2 + \|\nabla \omega\|^2) + (\mu + \chi) \|\Delta u\|^2 + \gamma \|\Delta \omega\|^2 + \kappa \|\text{div} \nabla \omega\|^2 \\
\leq C \|\nabla u\|_2 \|\nabla \nabla u\|_2 \|\Delta u\|^2 + C \|\nabla u\|_2 \|\nabla \nabla u\|_2 \|\Delta \omega\|^2 \\
+ C \|\Delta \omega\|_2 \|\nabla u\|^2 + C \|\nabla \omega\|^2 \|\nabla \omega\|^2 \|\Delta \omega\|^2. \tag{37}
\]
Integrating (37), applying Hölder inequality and combing the energy inequality (4) and (33), we obtain
\[
\|\nabla u\|^2 + \|\nabla \omega\|^2 + 2(\mu + \chi) \int_0^t \|\Delta u\|^2 \, dt + 2\gamma \int_0^t \|\Delta \omega\|^2 \, dt + 2\kappa \int_0^t \|\text{div} \nabla \omega\|^2 \, dt \\
\leq \|\nabla u_0(x)\|^2 + \|\nabla \omega_0(x)\|^2 \\
+ C \sup_{0 \leq r \leq t} \left[ \left( \int_0^t \|\nabla u\|^2 \, dt \right)^{\frac{1}{2}} \left( \int_0^t \|\nabla \nabla u\|^2 \, dt \right)^{\frac{1}{2}} \left( \int_0^t \|\Delta u\|^2 \, dt \right)^{\frac{1}{2}} \right] \\
+ C \sup_{0 \leq r \leq t} \left[ \left( \int_0^t \|\nabla \omega\|^2 \, dt \right)^{\frac{1}{2}} \left( \int_0^t \|\nabla \nabla \omega\|^2 \, dt \right)^{\frac{1}{2}} \left( \int_0^t \|\Delta \omega\|^2 \, dt \right)^{\frac{1}{2}} \right] \\
+ C \sup_{0 \leq r \leq t} \left[ \left( \int_0^t \|\Delta \omega\|^2 \, dt \right)^{\frac{1}{2}} \left( \int_0^t \|\nabla \nabla \omega\|^2 \, dt \right)^{\frac{1}{2}} \left( \int_0^t \|\Delta \omega\|^2 \, dt \right)^{\frac{1}{2}} \right] \\
\leq \|\nabla u_0(x)\|^2 + \|\nabla \omega_0(x)\|^2 + C \left( \|\nabla u_0(x)\|^2 + \|\nabla \omega_0(x)\|^2 \right) \int_0^t \|\nabla \omega\|^2 \, dt \\
+ \int_0^t \|\nabla \nabla u\|^2 \, dt + \int_0^t \|\nabla \nabla \omega\|^2 \, dt \right] \\
\times \left[ \left( \int_0^t \|\Delta u\|^2 \, dt \right)^{\frac{1}{2}} \left( \int_0^t \|\Delta \omega\|^2 \, dt \right)^{\frac{1}{2}} \right]. \tag{38}
\]
By virtue of the Hölder and Young inequalities and energy inequality (4), we get
\[
\|\nabla u\|^2 + \|\nabla \omega\|^2 + (\mu + \chi) \int_0^t \|\Delta u\|^2 \, dt + \int_0^t \|\Delta \omega\|^2 \, dt + 2\kappa \int_0^t \|\text{div} \nabla \omega\|^2 \, dt \\
\leq \|\nabla u_0(x)\|^2 + \|\nabla \omega_0(x)\|^2 + C \|\nabla u_0(x)\|^2 + C \|\nabla \omega_0(x)\|^2 \right] \\
+ C \left( \int_0^t \|\nabla \nabla u\|^2 \, dt \right)^{\frac{1}{2}} \\
+ C \left( \int_0^t \|\nabla \nabla \omega\|^2 \, dt \right)^{\frac{1}{2}} + C \left( \int_0^t \|\Delta u\|^2 \, dt \right)^{\frac{1}{2}} \\
\leq \|\nabla u_0(x)\|^2 + \|\nabla \omega_0(x)\|^2 + C \|\nabla u_0(x)\|^2 + C \|\nabla \omega_0(x)\|^2 \right] \\
+ C \left( \int_0^t \|\nabla \nabla u\|^2 \, dt \right)^{\frac{1}{2}} \left( \int_0^t \|\nabla \nabla \omega\|^2 \, dt \right)^{\frac{1}{2}} \right]. \tag{39}
\]
Then, by means of Gronwall inequality, we finally have
\[
\|\nabla u\|^2 + \|\nabla \omega\|^2 + (\mu + \chi) \int_0^t \|\Delta u\|^2 \, dt + \int_0^t \gamma \|\Delta \omega\|^2 \, dt + 2\kappa \int_0^t \|\text{div} \nabla \omega\|^2 \, dt \\
\leq \left( \|\nabla u_0(x)\|^2 + \|\nabla \omega_0(x)\|^2 \right) + C \|\nabla u_0(x)\|^2 + C \|\nabla \omega_0(x)\|^2 \right) \\
\exp \left( \int_0^t C \left( \|\nabla u\|^{\frac{8}{3}}_{B^{\infty,\infty}} + \|\nabla \omega\|^{\frac{8}{3}}_{B^{\infty,\infty}} \right) \, dt \right) \\
< \infty.
\]
(40)

Thus, we complete the proof of Theorem 1.2.

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E-mail address: bqyan@hpu.edu.cn

E-mail address: lixiaoyingchao@163.com