Entropy of Kerr–Newman–AdS black holes with torsion

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Abstract

The canonical approach to black hole entropy in Poincaré gauge theory without matter is extended to include Maxwell field as a matter source. The new formalism is used to calculate asymptotic charges and entropy of Kerr–Newmann–AdS black holes with torsion. The result implies that the first law, with a nontrivial contribution of the Maxwell field, takes the same form as in general relativity.

1 Introduction

The analysis of black hole spacetimes in general relativity (GR) shows that astrophysically, the most significant among them are those produced by rotating massive bodies [1]. The simplest spacetime of this type is the rotating, asymptotically flat solution found by Kerr [2]. The Kerr metric has been further generalized by including first the electric charge, and then a nonvanishing cosmological constant [3, 4]. The final result of these generalizations is the Kerr-Newman-Anti de Sitter (KN-AdS) black hole, which is the most general stationary, asymptotically anti-de Sitter solution of Einstein-Maxwell field equations [5].

Starting from the early 1980s, many well-known exact solutions of GR have been generalized to solutions of the Poincaré gauge theory (PG), a modern gauge theory of gravity in which both the curvature and the torsion have their own dynamical roles [6]. Successful constructions of exact solutions with torsion [7, 8, 9] have been followed, inter alia, by an intensive investigation of the concept of conserved charge [10, 11]. In contrast to that, a systematic investigation of black hole entropy in PG has long been neglected, although some incomplete attempts could have been noticed in the literature, as noted in [12].

A few years ago, a general canonical approach to black hole entropy in PG was proposed in [12]. The approach is based on a canonical formulation of the idea developed in GR, according to which entropy is just the Noether charge on horizon [13]. Applying this approach to a number of black holes with or without torsion [14, 15, 16], we found somewhat unexpected result: in spite of many geometric and dynamic differences with respect to GR, entropy of black holes in PG without matter, as well as the associated first law, follow essentially the same pattern as in GR, up to a multiplicative constant. In the present paper, we extend our

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investigation of entropy by introducing Maxwell field as a matter source for gravity (PG-Maxwell system). The analysis is focussed on exploring thermodynamic properties of the generalized KN-AdS black hole, constructed by Baekler et al. in the late 1980s.

The paper is organized as follows. In section 2, we present a brief account of the general thermodynamic aspects of the PG-Maxwell system. In particular, a new definition of the black hole entropy in the presence of Maxwell field is introduced as a natural generalization of the earlier definition, valid in vacuum. In section 3, we describe geometric aspects of the KN-AdS black hole as a solution of the PG-Maxwell system. Next, in sections 4 and 5, we use these results to calculate energy, angular momentum, and entropy. The thermodynamic role of the Maxwell field and the resulting first law are clarified in section 6, section 7 is devoted to concluding remarks, and appendixes contain some important technical details.

Our conventions are the same as in Ref. The Latin indices \( i, j, \ldots \) are the local Lorentz indices, the Greek indices \( \mu, \nu, \ldots \) are the coordinate indices, and both run over \( 0, 1, 2, 3 \). The orthonormal coframe (tetrad) \( \vartheta^i \) and the metric compatible (Lorentz) connection \( \omega^i_{jk} = -\omega^j_{ik} \) are 1-forms, the dual basis (frame) is \( e^i = e^i_{\mu} \partial_{\mu} \), the interior product of \( e^i \) with \( \vartheta^j \) is \( e^i \vartheta^j = \delta^i_j \), and \( A \) is the electromagnetic potential 1-form. The metric components in the local Lorentz and coordinate basis are \( \eta_{ij} = (1, -1, -1, -1) \) and \( g_{\mu\nu} = \eta_{ij} \partial_{\mu}^i \partial_{\nu}^j \), respectively, and \( \varepsilon_{ijmn} \) is the totally antisymmetric symbol with \( \varepsilon_{0123} = 1 \). The Hodge dual is marked by a star \( ^* \), and the wedge product of forms is implicit.

## 2 PG-Maxwell system

We begin with an overview of the general Lagrangian and thermodynamic aspects of the PG dynamics in the presence of Maxwell field; for more details, see Refs. 12, 17.

### 2.1 Lagrangian formalism

In PG, the structure of spacetime is characterised by a Riemann–Cartan geometry, in which the torsion \( T^i = d\vartheta^i + \omega^i_k \vartheta^k \) and the curvature \( R^{ij} = d\omega^{ij} + \omega^i_k \omega^{kj} \) (2-forms) are the gravitational field strengths, associated to the Poincaré (translational and Lorentz) gauge potentials, the tetrad \( \vartheta^i \) and the Lorentz connection \( \omega^{ij} \), respectively. Moreover, our physical system contains also the Maxwell field characterised by the field strength \( F = dA \) (2-form), where \( A \) is the electromagnetic gauge potential (1-form).

Dynamical properties of the PG-Maxwell system are defined by the total Lagrangian

\[
L = L_G + L_M ,
\]

where \( L_G = L_G(\vartheta^i, T^i, R^{ij}) \) is a parity even PG Lagrangian, assumed to be at most quadratic in the field strengths, and \( L_M = L_M(\vartheta^i, F) \) describes the Maxwell field interacting with gravity. The gravitational field equations are obtained by varying \( L \) with respect to the gravitational potentials \( \vartheta^i \) and \( \omega^{ij} \). Introducing the gravitational covariant momenta, \( H_i := \partial L_G / \partial T^i \) and \( H_{ij} := \partial L_G / \partial R^{ij} \), and the associated energy-momentum and spin currents, \( E_i = \partial L_G / \partial b^i \) and \( E_{ij} := \partial L_G / \partial \omega^{ij} \), these equations can be written in a compact form as

\[
\begin{align*}
\delta b^i : & \quad \nabla H_i + E_i = -\tau_i , \quad (2.2a) \\
\delta \omega^{ij} : & \quad \nabla H_{ij} + E_{ij} = 0 . \quad (2.2b)
\end{align*}
\]
The source term on the right-hand side of (2.2a) is the Maxwell energy-momentum current
\[ \tau_i := \partial L_M / \partial \dot{\vartheta}^i, \]
where the related spin current vanishes, \( \sigma_{ij} := \partial L_M / \partial \omega^{ij} = 0 \). Similarly, the variation of \( L \) with respect to the electromagnetic potential \( A \) yields the Maxwell equation,
\[ \delta A : \quad dH = 0, \quad (2.2c) \]
where \( H := \partial L_M / \partial A \) is the electromagnetic covariant momentum.

The PG part of the total Lagrangian (2.1) has the form
\[ L_G := -\* (a_0 R + 2 \Lambda) + T^i \sum_{n=1}^3 \* (a_n^{(n)} T_i) + \frac{1}{2} R^{ij} \sum_{n=1}^6 \* (b_n^{(n)} R_{ij}), \quad (2.3a) \]
where \((a_0, \Lambda, a_n, b_n)\) are the gravitational coupling constants, and \((n)T_i, (n)R_{ij}\) are irreducible parts of the field strengths. The Maxwell part reads
\[ L_M := 4 a_1 \left( -\frac{1}{2} F \* F \right), \quad F := dA, \quad (2.3b) \]
where \(4a_1\) is a suitably normalized coupling constant.

In the analysis of black hole thermodynamics, we need the following explicit formulas:
\[ H_i = 2 \sum_{m=1}^2 \* (a_n^{(m)} T_i), \quad (2.4a) \]
\[ H_{ij} = -2a_0 \* (\partial_i \partial_j) + 2 \sum_{n=1}^6 \* (b_n^{(n)} R_{ij}), \quad (2.4b) \]
\[ H = -4a_1 \* F. \quad (2.4c) \]

### 2.2 Thermodynamics

The Hamiltonian approach to black hole entropy in PG [12] is based on the ideas developed originally in GR [18, 13], according to which the asymptotic charges (energy and angular momentum) as well as entropy, can be defined by certain boundary terms. Here, we introduce an extended version of that approach, suitable for analysing non-vacuum solutions of the PG-Maxwell system.

Consider a stationary black hole spacetime whose spatial section \( \Sigma \) has a two-component boundary, one component at infinity and the other at horizon, \( \partial \Sigma = S_\infty \cup S_H \). Then, asymptotic charges and entropy of a PG-Maxwell black hole are determined by the boundary integral \( \Gamma := \Gamma_\infty - \Gamma_H \), determined by the following variational equations:
\[ \delta \Gamma_\infty = \oint_{S_\infty} \delta B(\xi), \quad \delta \Gamma_H = \oint_{S_H} \delta B(\xi), \quad (2.5a) \]
\[ \delta B(\xi) := (\xi \* \partial^i) \delta H_i + \delta \partial^i (\xi \* H_i) + \frac{1}{2} (\xi \* \omega^{ij}) \delta H_{ij} + \frac{1}{2} \delta \omega^{ij} (\xi \* \delta H_{ij}) + (\xi \* A) \delta H + (\delta A) (\xi \* H). \quad (2.5b) \]

By construction, \( \delta B \) is obtained from the canonical generator of local translations. It contains not only the gravitational term (upper line), but also the Maxwell term (bottom line),
extending thereby the construction adopted in [12] to non-vacuum solutions. Specific forms of the Killing vector \( \xi = \partial_t, \partial_\phi \) or a linear combination thereof are chosen so that the boundary integrals \( (\Gamma_\infty, \Gamma_H) \) could be physically interpreted in terms of the asymptotic charges, black hole entropy, and an external, Maxwell term. To have a consistent interpretation, we require the operation \( \delta \) to satisfy the following rules:

(r1) On \( S_\infty \), the variation \( \delta \) acts on the parameters of a black hole solution, but not on the parameters of the background configuration.

(r2) On \( S_H \), the variation \( \delta \) must keep surface gravity constant.

Moreover, mathematical consistency strongly depends on the boundary conditions:

(r3) When the boundary terms \( (\delta \Gamma_\infty, \delta \Gamma_H) \) are \( \delta \)-integrable and finite, they can be given the usual thermodynamic interpretation.

Finally, note that \( \Gamma_\infty \) and \( \Gamma_H \) are introduced as a priori independent objects. However, the analysis of their construction from the canonical gauge generator reveals that the regularity of the generator can be expressed by the relation

\[
\delta \Gamma_\infty - \delta \Gamma_H = 0 ,
\]

which is equivalent to the first law of black hole thermodynamics. The Maxwell contribution to \( \delta B \) is an essential part of the first law.

3  Geometry and dynamics

In this section, we analyse basic properties of KN-AdS black holes as solutions of the PG-Maxwell system [8].

3.1  Metric and tetrad

The metric of a KN-AdS black hole in Boyer-Lindquist coordinates has the form [11]

\[
ds^2 = \frac{\Delta}{\rho^2} \left( dt + \frac{a}{\alpha} \sin^2 \theta d\varphi \right)^2 - \frac{\rho^2}{\Delta} dr^2 - \frac{\rho^2}{f} d\theta^2 - \frac{f}{\rho^2} \sin^2 \theta \left[ a dt + \frac{(r^2 + a^2)}{\alpha} d\varphi \right]^2 ,
\]

where

\[
\Delta(r) := (r^2 + a^2)(1 + \lambda r^2) - 2(mr - q^2) , \quad \alpha := 1 - \lambda a^2 ,
\]

\[
\rho^2(r, \theta) := r^2 + a^2 \cos^2 \theta , \quad f(\theta) := 1 - \lambda a^2 \cos^2 \theta .
\]

Here, \( m, a \) and \( q \) are the parameters characterising energy (mass), angular momentum and electric charge of the solution, and \( \lambda = -\Lambda/3a_0 \). The orthonormal tetrad associated to the metric is chosen in the form

\[
\vartheta^0 = N \left( dt + \frac{a}{\alpha} \sin^2 \theta d\varphi \right) , \quad \vartheta^1 = \frac{dr}{N} ,
\]

\[
\vartheta^2 = P d\theta , \quad \vartheta^3 = \frac{\sin \theta}{P} \left[ a dt + \frac{(r^2 + a^2)}{\alpha} d\varphi \right] ,
\]

The electric charge is not defined by the Maxwell term in (2.5b), it is, by definition, related to the electromagnetic \( U(1) \) boundary term; see section [8].
where
\[ N(r, \theta) = \sqrt{\Delta / \rho^2}, \quad P(r, \theta) = \sqrt{\rho^2 / f} . \quad (3.2b) \]

The larger root of \( \Delta(r) = 0 \) defines the outer horizon,
\[ (r_+^2 + a^2)(1 + \lambda r_+^2) - 2(m r_+ - q^2) = 0 , \quad (3.3) \]
and the angular velocity and surface gravity have the same form as in GR [5, 15],
\[ \omega_+ = \frac{a \alpha}{r_+^2 + a^2}, \quad \Omega_+ := \omega_+ + \lambda a = \frac{a(1 + \lambda r_+^2)}{r_+^2 + a^2}, \quad (3.4) \]
\[ \kappa = \frac{r_+^2 + 3 \lambda r_+^4 + \lambda a^2 r_+^2 - a^2 - 2q^2}{2r_+(r_+^2 + a^2)}, \quad (3.5) \]
and the area of the horizon is
\[ A_H = \int_{r_+} b^2 b^3 = \frac{4\pi (r_+^2 + a^2)}{\alpha} . \quad (3.6) \]

The Riemannian connection \( \tilde{\omega}^{ij} \) is calculated in Appendix A.

### 3.2 Torsion, connection and curvature

Riemann-Cartan geometry of PG is characterized by a nonvanishing torsion. For KN-AdS black holes, the ansatz for torsion is formally the same as for the Kerr-AdS case [8, 14],
\[ T^0 = T^1 = \frac{1}{N} \left[ -V_1 \vartheta^0 \vartheta^1 - 2V_4 \vartheta^2 \vartheta^3 \right] + \frac{1}{N^2} \left[ V_2 \vartheta^- \vartheta^2 + V_3 \vartheta^- \vartheta^3 \right], \]
\[ T^2 := \frac{1}{N} \left[ V_5 \vartheta^- \vartheta^2 + V_4 \vartheta^- \vartheta^3 \right], \]
\[ T^3 := \frac{1}{N} \left[ -V_4 \vartheta^- \vartheta^2 + V_5 \vartheta^- \vartheta^3 \right], \quad (3.7) \]

where \( \vartheta^- = \vartheta^0 - \vartheta^1 \), but the metric function \( N \) and the torsion functions \( V_n \) are modified by the nonvanishing electric charge parameter \( q^2 \),
\[ V_1 = \frac{1}{\rho^4} [(mr - 2q^2)r - ma^2 \cos^2 \theta], \quad V_2 = -\frac{1}{\rho^4 P}(mr - q^2)a \sin \theta \cos \theta, \]
\[ V_3 = \frac{1}{\rho^4 P}(mr - q^2) a \sin \theta, \quad V_4 = \frac{1}{\rho^4 P}(mr - q^2) a \cos \theta, \]
\[ V_5 = \frac{1}{\rho^4 P}(mr - q^2)r. \quad (3.8) \]

Having introduced torsion, the Riemann-Cartan connection can be expresses as
\[ \omega^{ij} = \tilde{\omega}^{ij} + K^{ij} , \quad (3.9a) \]
where $K^{ij}$ is the contortion 1-form, implicitly defined by the relation $T^i = K^i_k b^k$,

\[
K^{01} = \frac{1}{N} V_1 \vartheta^-, \\
K^{02} = K^{12} = -\frac{1}{N^2} V_2 \vartheta^+ + \frac{1}{N} (V_5 \vartheta^2 - V_4 \vartheta^3), \\
K^{03} = K^{13} = -\frac{1}{N^2} V_3 \vartheta^- + \frac{1}{N} (V_5 \vartheta^2 + V_4 \vartheta^3), \\
K^{23} = -\frac{2}{N} V_4 \vartheta^-.
\]  

(3.9b)

The curvature 2-form $R^{ij} = d\omega^{ij} + \omega^{ik} \omega^{kj}$ has only two nonvanishing irreducible parts:

\[
(6) R^{ij} = \lambda \vartheta^i \vartheta^j, \\
(4) R^{Ac} = \lambda \Delta (mr - q^2) \vartheta^c.
\]  

(3.10)

The quadratic invariants (Euler, Pontryagin and Nieh-Yan) are given by

\[
I_E := \frac{1}{2} \varepsilon_{ijmn} R^{ij} R^{mn} = 12 \lambda^2 \hat{\epsilon}, \\
I_P := R^{ij} R_{ij} = 0, \\
I_{NY} := T^i T_i - R^{ij} b_i b^j = 0.
\]  

(3.11)

### 3.3 PG-Maxwell field equations

Since the only nonvanishing parts of the gravitational field strengths are $(1)T^i$, $(2)T^i$ and $(4)R^{ij}$, $(6)R^{ij}$, the “effective” form of the gravitational Lagrangian reads

\[
L_G = -*(a_0 R + 2\Lambda) + T^i *((a_1 (1)T_i + a_2 (2)T_i) + \frac{1}{2} R^{ij} *(b_4 (4) R_{ij} + b_6 (6) R_{ij})).
\]  

(3.12)

The covariant momenta $H_i$ and $H_{ij}$, appearing in the field equations (2.2), are given by

\[
H_i = 2a_1 *(1)T_i - 2(2)T_i, \\
H_{ij} = -2A'_0 *(\vartheta_i \vartheta_j) + 2b_4 * (4) R_{ij}, \\
A'_0 := a_0 - \lambda b_6,
\]  

(3.13)

and the corresponding spin currents are

\[
E_i = e_i \mathcal{J} L_G - (e_i \mathcal{J} T^m) H_m - \frac{1}{2} (e_i \mathcal{J} R^{mn}) H_{mn}, \\
E_{ij} = -(* \vartheta_i E_j - * \vartheta_j E_i).
\]  

(3.14)

The contribution of the electromagnetic sector to Eqs. (2.2) is described by the Maxwell energy-momentum current [17]

\[
\tau_i = e_i \mathcal{J} L_M - (e_i \mathcal{J} F) H.
\]  

(3.15)

The form of $\tau_i$ depends on the Maxwell potential in a KN-AdS spacetime [19],

\[
A := -\frac{qe}{\rho \sqrt{\Delta}} \vartheta^0 \equiv -\frac{qe}{\rho^2} \left( dt + \frac{a}{\alpha} \sin^2 \theta d\varphi \right),
\]  

(3.16)
where $q_e$ is the electromagnetic charge parameter. This expression is a natural generalization of the spherically symmetric form $A = -(q_e/r)dt$. The related field strength and the covariant momentum are

$$F = -\frac{q_e}{\rho^4} \left[ (r^2 - a^2 \cos^2 \theta) \partial^0 \vartheta^1 + 2ar \cos \theta \partial^2 \vartheta^3 \right], \quad (3.17a)$$

$$H = -4a_1 q_e \rho^4 \left[ (r^2 - a^2 \cos^2 \theta) \partial^2 \vartheta^1 - 2ar \cos \theta \partial^0 \vartheta^1 \right]. \quad (3.17b)$$

When all the previous results taken into account, the explicit calculation shows that basic dynamical variables ($\vartheta^i$, $\omega_{ij}$, $A$) of a KN-AdS black hole, which are defined in Eqs. (3.2a), (3.9a) and (3.16), solve the PG-Maxwell field equations (2.2) if the Lagrangian parameters $(a_n, b_n, \Lambda)$ and the solution parameters $(\lambda, q, q_e)$ satisfy the relations

$$2a_1 + a_2 = 0, \quad a_0 - a_1 - \lambda(b_4 + b_6) = 0, \quad 3\lambda a_0 + \Lambda = 0, \quad q_e^2 = 2q^2. \quad (3.18)$$

Thus, according to our conventions, the electromagnetic charge parameter $q_e$ differs from the metric charge parameter $q$. However, none of them coincides with the asymptotic Maxwell charge, as will be shown in section 4.

4 Asymptotic boundary terms

The asymptotic values of energy and angular momentum are defined by the boundary term $\delta B(\xi)$ in (2.5). Two aspects of explicit calculations deserve a special attention.

First, Carter [20] and Henneaux and Teitelboim [21] demonstrated that the asymptotic metric of Kerr-AdS spacetimes cannot be properly described in Boyer-Lindquist coordinates. They found a new set of coordinates in which this deficiency is brought under control. However, our variational approach (2.5) allows a simpler procedure [14, 15], in which only the subset $(t, \varphi)$ of the Boyer-Lindquist coordinates is transformed to the “untwisted” form,

$$T = t, \quad \phi = \varphi - \lambda at. \quad (4.1a)$$

Under these transformations, the components $(v_t, v_\varphi)$ of a 4-vector $v_\mu$ transform as

$$v_T = v_t + \lambda a v_\varphi, \quad v_\phi = v_\varphi. \quad (4.1b)$$

In particular,

$$g_{T\varphi} = g_{\varphi\varphi} = g_{\varphi\phi}, \quad \Omega_+ := \left( \frac{g_{T\varphi}}{g_{\varphi\varphi}} \right)_{r_+} = \omega_+ + \lambda a = \frac{a(1 + \lambda r_+^2)}{r_+^2 + a^2}. \quad (4.1c)$$

And second, the background configuration, defined by $m = q = 0$, depends on the parameter $a$. To avoid the variation of those $a$’s that are associated to the background, we introduce a more precise formulation of the rule (r1) for the variation $\delta$, given below Eq. (2.5):

($r1'$) In calculating $\delta \Gamma_\infty(\xi)$, first apply $\delta$ to all the parameters $(m, a, q)$, then subtract those $\delta a$ terms that survive the limit $m = q = 0$, as they come from the background.
Before continuing, it is interesting to note that the lower line in the expression \( \delta B(\xi) \), Eq. (2.5), which refers to the contribution of the Maxwell field, yields vanishing boundary terms at infinity, but not at horizon. This follows from the asymptotic behavior of the variables \( A \) and \( H \), defined by Eqs. (3.16) and (3.17). Hence, nontrivial energy and angular momentum are generated only by the contributions stemming from the gravitational sector.

In the subsequent calculations, we use the following notation:

\[
d\Omega := \sin \theta d\theta d\varphi \rightarrow 4\pi, \quad d\Omega' := \sin^3 \theta d\theta d\varphi \rightarrow \frac{2}{3}4\pi.
\]

### 4.1 Angular momentum

The angular momentum is defined by \( \delta E_\varphi := \delta \Gamma_\infty(\partial_\varphi) \). The calculation is performed by ignoring \((m, q)\)-independent \( \delta a \) terms (background), even when they are divergent, and by omitting asymptotically vanishing terms. The nonvanishing contributions are

\[
\omega^{13}_\varphi \delta H_{13} + \delta \omega^{13} H_{13\varphi} = 2a_1 \delta \left( \frac{ma}{\alpha^2} \right) d\Omega',
\]

\[
b^0_\varphi \delta H_0 + \delta b^0 H_0 \varphi = 4a_1 \delta \left( \frac{ma}{\alpha^2} \right) d\Omega'.
\]

Summing up the two terms, one obtains

\[
\delta E_\varphi = 16\pi a_1 \delta \left( \frac{ma}{\alpha^2} \right).
\] (4.2)

### 4.2 Energy

Going over to energy, we calculate the nonvanishing contributions to \( \delta E_t = \delta \Gamma_\infty(\partial_t) \),

\[
\delta \omega^{12} H_{12t} + \delta \omega^{13} H_{13t} = 2a_1 m \delta \left( \frac{1}{\alpha} \right) d\Omega,
\]

\[
b^0_t \delta H_0 = 4a_1 \delta \left( \frac{m}{\alpha} \right) d\Omega.
\]

Hence,

\[
\delta E_t = 16\pi a_1 \left[ \frac{m}{2} \delta \left( \frac{1}{\alpha} \right) + \delta \left( \frac{m}{\alpha} \right) \right].
\]

The result is not \( \delta \)-integrable but, as we mentioned above, it can be corrected by moving to the untwisted coordinates \((T, \phi)\):

\[
\delta E_T = \delta E_t + \lambda a \delta E_\varphi = 16\pi a_1 \delta \left( \frac{m}{\alpha^2} \right).
\] (4.3)

The expressions (4.2) and (4.3) are proportional to the corresponding GR values.

### 5 Entropy

In this section, we analyse the PG part of the boundary term at horizon, \( \delta \Gamma_H \), where the Killing vector \( \xi \) is given by

\[
\xi := \partial_T - \Omega_+ \partial_\phi = \partial_t - \omega_+ \partial_\varphi.
\] (5.1)
As well be shown, this part defines the black hole entropy. The Maxwell contribution to $\delta \Gamma_H$ will be discussed in the next section.

In what follows, we use the notation $v_\xi := \xi \downarrow v$ and $A'_0 := a_0 - \lambda b_6$.

5.1 Nonvanishing terms

The calculation entropy is organised in two technical steps.

$$\delta \Gamma_1 = \frac{1}{2} \omega^{ij} \xi \delta H_{ij} + \frac{1}{2} \delta \omega^{ij} H_{ij} \xi$$

The only nonvanishing contributions stemming from the first element of $\delta \Gamma_1$ are

$$\omega_0^1 \delta H_{01} \equiv \omega_0^1 \delta H_{01\theta \phi} = 2 A'_0 \left( k - V \rho_+^2 \frac{\rho_+^2}{r_+^2 + a^2} \right) \delta \left( \frac{r_+^2 + a^2}{\alpha} \right) \sin \theta, \quad (5.2a)$$

$$\omega_0^3 \delta H_{03} + \omega_0^3 \delta H_{13} \equiv \delta K_{03}^i \delta (H_{03\theta \phi} + H_{13\phi \theta}) + \omega_0^3 \delta H_{13 \phi \theta} = 2 A'_0 \left( \frac{V_3 \rho_+^2}{N \rho_+^2} \alpha \right) \sin \theta + 2 \lambda b_4 \left( \frac{r_+}{P \rho_+^2} \right) \frac{\rho_+^2}{\alpha} \sin \theta. \quad (5.2b)$$

Here, the symbol $\equiv$ stands for an equality up to the factor $d\theta d\phi$.

In the second element of $\delta \Gamma_1$, there are two more nonvanishing contributions,

$$\delta \omega^{02} H_{02\xi} + \delta \omega^{12} H_{12 \xi} \equiv \delta K_{02}^i \delta (H_{02\xi} + H_{12\xi}) + \delta \omega^{12} \alpha H_{12 \xi}$$

$$= -2 A'_0 \delta \left( \frac{(r_+^2 + a^2) \rho_+^2}{\rho_+^2} \alpha \right) \frac{N \rho_+^2}{P \rho_+^2} \sin \theta - 2 \lambda b_4 \delta \left( \frac{r_+^2 + a^2}{\rho_+^2} \right) \frac{\rho_+^2}{\alpha} \sin \theta, \quad (5.3a)$$

and

$$\delta \omega^{03} H_{03\xi} + \delta \omega^{13} H_{13 \xi} \equiv - \delta K_{03}^i \delta (H_{03\theta \phi} + H_{13\phi \theta}) - \delta \omega^{13} \phi H_{13 \xi}$$

$$= -2 A'_0 \delta \left( \frac{(r_+^2 + a^2) \rho_+^2}{\rho_+^2} \alpha \right) \frac{N \rho_+^2}{P \rho_+^2} \sin \theta - 2 \lambda b_4 \delta \left( \frac{r_+^2 + a^2}{\rho_+^2} \right) \frac{\rho_+^2}{\alpha} \sin \theta. \quad (5.3b)$$

$$\delta \Gamma_2 = b^i \xi \delta H_i + \delta b^i H_{i\xi}$$

In $\delta \Gamma_2$, the nonvanishing contributions are

$$b^i_0 \delta H_0 \equiv b^0_0 \delta H_{00\phi} = 2 a_1 N \rho_+^2 \frac{\rho_+^2}{r_+^2 + a^2} \delta \left( \frac{(r_+^2 + a^2) \rho_+^2}{N \rho_+^2} \right) \sin \theta, \quad (5.4a)$$

$$\delta b^0 H_{0\xi} \equiv - \delta b^0 \phi H_{0\xi} = - 2 a_1 \delta \left( \frac{V_3 P \rho_+^2}{N \rho_+^2} \alpha \right) \sin \theta \frac{\rho_+^2}{P \rho_+^2} \sin \theta, \quad (5.4b)$$

$$\delta b^2 H_{2\xi} \equiv \delta b^2 H_{2\phi} = 2 a_1 (\delta P) (V_1 - V_2) \frac{\rho_+^2}{P \rho_+^2} \sin \theta, \quad (5.4c)$$

$$\delta b^3 H_{3\xi} \equiv - \delta b^3 \phi H_{3\xi} = 2 a_1 (\delta P) \left( \frac{r_+^2 + a^2}{P \rho_+^2} \right) (V_1 - V_2) \frac{\rho_+^2}{P \rho_+^2} \sin \theta. \quad (5.4d)$$
5.2 Simplifications

The above contributions can be simplified using the following properties (see Appendix B):

S1. The sum of the terms proportional to $\delta N/N$ in (5.2)–(5.4) vanishes.
S2. The sum of the terms proportional to $\delta P/P$ in (5.2)–(5.4) vanishes.

Hence, the original contributions (5.2)–(5.4) can be simplified as follows:

\[ (5.2a) : 2A_0' \left( \kappa - V_1 \frac{\rho^2_+}{r^2_+ + a^2} \right) \delta \left( \frac{r^2_+ + a^2}{\alpha} \right) \sin \theta , \]

\[ (5.2b) : 2A_0 \frac{a(mr_+ - q^2)r_+}{\rho^2_+(r^2_+ + a^2)} \delta \left( \frac{a}{\alpha} \right) \sin^3 \theta + 2\lambda b_4 \frac{ar_+}{r^2_+ + a^2} \delta \left( \frac{mr_+ - q^2 a}{\rho^2_+ \alpha} \right) \sin^3 \theta . \]

\[ (5.3a) : -2A_0' \delta \left( \frac{(mr_+ - q^2)r_+}{\rho^2_+} \right) \frac{\rho^2_+}{\alpha} \sin \theta - 2\lambda b_4 \delta \left( \frac{r_+}{\rho^2_+} \right) \frac{mr_+ - q^2}{\alpha} \sin \theta , \]

\[ (5.3b) : -2A_0' \delta \left( \frac{(mr_+ - q^2)r_+}{\rho^2_+ \alpha} \right) \frac{\rho^2_+}{r^2_+ + a^2} \sin \theta - 2\lambda b_4 \delta \left( \frac{r_+}{\alpha} \right) \frac{mr_+ - q^2}{r^2_+ + a^2} \sin \theta . \]

\[ (5.4a) : 2a_1 \frac{\rho^2_+}{r^2_+ + a^2} \delta \left( \frac{(mr_+ - q^2)r_+}{\alpha \rho^2_+} (r^2_+ + a^2 + \rho^2_+) \right) \sin \theta \]

\[ (5.4b) : -2a_1 \delta \left( \frac{a}{\alpha} \right) \frac{mr_+ - q^2}{\rho^2_+} \frac{\rho^2_+}{r^2_+ + a^2} r_+ a \sin^3 \theta , \]

\[ (5.4c) : = 0 , \]

\[ (5.4d) : 2a_1 \delta \left( \frac{r^2_+ + a^2}{\alpha} \right) (V_1 - V_3) \frac{\rho^2_+}{r^2_+ + a^2} \sin \theta . \]

Next, we use the relation $A_0' = \lambda b_4 + a_1$ to express these contributions in terms of only two independent constants, $\lambda b_4$ and $a_1$. The analysis of the $\lambda b_4$ part leads to an additional simplification (Appendix B).

S3. When the $\lambda b_4$ part is integrated over $d\theta \delta \varphi$, it vanishes.

The conclusions S1, S2 and S3 are the KN-AdS extensions of the results found for the Kerr-AdS black holes in [14].
5.3 The terms proportional to $a_1$

The property S3 allows us to simply replace $A'_0$ by $a_1$ in (5.2) and (5.3), ignoring the vanishing $\lambda b_4$ terms. Then:

\[
\begin{align*}
(5.2a) + (5.2b)_{1} + (5.3a)_{1} + (5.3b)_{1} : \\
2a_1 \sin \theta \left[ \left( \kappa - V_1 \frac{\rho_{+}^2}{r_{+}^2 + a^2} \right) \delta\left( \frac{r_{+}^2 + a^2}{\alpha} \right) + \left( \frac{mr_{+} - q^2}{\rho_{+}^4} \right)r_{+}^2 \delta\left( \frac{a}{\alpha} \right) \sin^2 \theta \\
- \frac{\rho_{+}^2}{r_{+}^2 + a^2} \delta\left( \frac{(mr_{+} - q^2)r_{+}}{\rho_{+}^4} \right) - \frac{\rho_{+}^2}{r_{+}^2 + a^2} \delta\left( \frac{(mr_{+} - q^2)r_{+}}{\rho_{+}^2} \right) \right], \\
(5.4a) + (5.4b) + (5.4c) : \\
2a_1 \frac{\rho_{+}^2}{r_{+}^2 + a^2} \sin \theta \left[ V_1 \delta\left( \frac{r_{+}^2 + a^2}{\alpha} \right) - \frac{mr_{+} - q^2}{\rho_{+}^4} \delta\left( \frac{a}{\alpha} \right) \sin^2 \theta \\
+ \frac{r_{+}^2 + a^2}{\alpha} \delta\left( \frac{(mr_{+} - q^2)r_{+}}{\rho_{+}^4} \right) + \delta\left( \frac{(mr_{+} - q^2)r_{+}}{\alpha \rho_{+}^2} \right) \right].
\end{align*}
\]

All terms except the first one (proportional to $\kappa$) cancel each other, so that the sum becomes

\[
2a_1 \kappa \sin \theta \delta\left( \frac{r_{+}^2 + a^2}{\alpha} \right).
\]

Then, the integration over $d\theta d\varphi$ yields

\[
(\delta \Gamma_H)^{PG} = 8\pi a_1 \kappa \delta\left( \frac{r_{+}^2 + a^2}{\alpha} \right) = T \delta S, \quad S := 16\pi a_1 \frac{\pi(r_{+}^2 + a^2)}{\alpha},
\]

where $T := \kappa/2\pi$ is the temperature. Thus, entropy is also proportional to the GR value.

6 Maxwell boundary term and the first law

The standard canonical analysis of the Maxwell sector implies that the asymptotic electric charge $Q$ can be defined by the boundary integral

\[
Q = - \int_{S_{\infty}} H = 4a_1 \int_{S_{\infty}} \frac{q_e (r^2 - a^2 \cos^2 \theta)b^2 b^3}{\rho^4} = 16\pi a_1 \frac{q_e}{\alpha}.
\]

The minus sign is just a matter of convention. Next, following Ref. [19], we define the electric potential $\Phi$ by

\[
\Phi := A_\xi \bigg|_{r_{+}^\infty} = -\frac{q_e r_{+}^b b^0}{\rho_{+}^N} \bigg|_{r_{+}^\infty} = \frac{q_e r_{+}}{r_{+}^2 + a^2}.
\]

Then, the Maxwell contribution on horizon has the form

\[
(\delta \Gamma_H)^M = A_\xi \delta H + (\delta A)H_\xi = A_\xi \delta H = \Phi \delta Q.
\]

Combining this relation with the result obtained in Eqs. (4.2), (4.3) and (5.7), one can immediately conclude that the first law $\delta \Gamma_H = \delta \Gamma_\infty$ takes the form

\[
T \delta S + \Phi \delta Q = \delta E_T - \Omega_+ \delta E_\varphi.
\]

The result is confirmed by the identity (C.2). After removing the common factor $16\pi a_1$, the first law (6.4) becomes identical to its GR form.
7 Concluding remarks

The canonical approach to black hole entropy proposed in [12] has been successfully applied to a number of vacuum solutions of PG [14, 15, 16]. In the present paper, we introduce its natural extension to non-vacuum solutions, by including Maxwell field as a matter source of gravity. Using this formalism, we study thermodynamic properties of KN-AdS black holes, encoded in the boundary terms at infinity and horizon, $\delta \Gamma_\infty$ and $\delta \Gamma_H$, respectively.

Analysing energy and angular momentum as the boundary terms at infinity, we found that their KN-AdS values are exactly the same as for the uncharged Kerr-AdS solution [4, 14]. This is in agreement with the fact that the asymptotic Maxwell contribution vanishes. Moreover, these asymptotic charges are proportional to the related GR expressions.

The boundary term at horizon produces entropy and an external, Maxwell term, such that both of them are also proportional to the corresponding GR expressions [4, 15]. Then, the first law is described by the general relation $\delta \Gamma_\infty = \delta \Gamma_H$, which follows from the way the boundary terms are constructed, see subsection 2.2. Apart from this general argument, we give an explicit proof of the first law based on the identity derived in Appendix B. After removing the overall multiplicative factor, the first law becomes identical to its GR form.

Thus, although PG has a rather different dynamical structure from GR, the present description of the KN-AdS thermodynamics is rather close to the GR results. A reason for this “accidental” similarity might be hidden in the identity found in Appendix B.

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A Technical formulas

The condition of vanishing torsion, $d\theta^i + \omega^i_{jk} \partial^k = 0$, defines the Riemannian connection:

\[
\begin{align*}
\tilde{\omega}^{01} &= -N b^0 - \frac{ar}{P \rho^2} \sin \theta b^3, \\
\tilde{\omega}^{02} &= \frac{a^2 \sin \theta \cos \theta}{P \rho^2} b^0 - \frac{aN}{\rho^2} \cos \theta b^3, \\
\tilde{\omega}^{03} &= -\frac{ar}{P \rho^2} \sin \theta b^1 + \frac{aN}{\rho^2} \cos \theta b^2, \\
\tilde{\omega}^{12} &= \frac{a^2 \sin \theta \cos \theta}{\rho^2 P} b^1 + \frac{rN}{\rho^2} b^2, \\
\tilde{\omega}^{13} &= -\frac{ar}{P \rho^2} \sin \theta b^0 + \frac{Nr}{\rho^2} b^3, \\
\tilde{\omega}^{23} &= -\frac{aN}{\rho^2} \cos \theta b^0 + \frac{P \cos \theta - \partial_\theta P \sin \theta}{P^2 \sin \theta} b^3.
\end{align*}
\]

(A.1)
Some general relations:

\[ N \frac{\partial_r N}{r_+} = \frac{\kappa (r_+^2 + a^2)}{\rho_+^2}, \]

\[ (\xi \partial^0) \bigg|_{r_+} = N \frac{\rho_+^2}{r_+^2 + a^2}, \quad (\xi \partial^a) \bigg|_{r_+} = 0. \]  \hspace{1cm} (A.2)

Interior products \( \xi \tilde{\omega}^{ij} \):

\[ \xi \tilde{\omega}^{01} = -N' (\xi \omega^0) = -\kappa, \quad \xi \tilde{\omega}^{02} = \frac{Na^2 \sin \theta \cos \theta}{P(r_+^2 + a^2)}, \]

\[ \xi \tilde{\omega}^{13} = -\frac{Nar_+}{P(r_+^2 + a^2)} \sin \theta, \quad \xi \tilde{\omega}^{03} = \xi \tilde{\omega}^{12} = 0, \quad \xi \tilde{\omega}^{23} \sim N^2. \]

Explicit form of the covariant momenta \( H_i \) and \( H_{ij} \) is given by

\[ H_0 = \frac{4a_1}{N} \left[ -V_4 b^0 b^1 + V_5 b^2 b^3 \right] + \frac{2a_1}{N^2} \left[ -V_2 b^- b^3 + V_3 b^- b^2 \right], \]

\[ H_1 = -H_0, \]

\[ H_2 = \frac{2a_1}{N} \left[ (V_1 - V_5) b^- b^3 - V_4 b^- b^2 \right], \]

\[ H_3 = \frac{2a_1}{N} \left[ -(V_1 - V_5) b^- b^2 - V_4 b^- b^3 \right]. \]  \hspace{1cm} (A.3)

and

\[ H_{01} = -2A_0' b^2 b^3, \]

\[ H_{02} = 2A_0' b^1 b^3 + 2b_4 \frac{\lambda}{\Delta} (mr - q^2) b^- b^3, \]

\[ H_{12} = -2A_0' b^0 b^3 - 2b_4 \frac{\lambda}{\Delta} (mr - q^2) b^- b^3, \]

\[ H_{03} = -2A_0' b^1 b^2 - 2b_4 \frac{\lambda}{\Delta} (mr - q^2) b^- b^2, \]

\[ H_{13} = 2A_0' b^0 b^2 + 2b_4 \frac{\lambda}{\Delta} (mr - q^2) b^- b^2, \]

\[ H_{23} = -2A_0' b^0 b^1. \]  \hspace{1cm} (A.4)

B On the evaluation of entropy

In this appendix, we discuss certain technical details of the derivation of entropy.

B.1 Elimination of \( \delta N/N \) and \( \delta P/P \) terms

Starting from the basic results on entropy obtained in Eqs. (5.2)-(5.4) in subsection 5.1, we are now going to show that both \( \delta N/N \) and \( \delta P/P \) terms cancel out.
Consider first the coefficients of the $\delta N/N$ terms. By a suitable rearrangement of these coefficients, shown in the following formulas

$$ (5.3a) + (5.3b) : 2(A_0' - \lambda b_4) \frac{r_+ (mr_+ - q^2)}{\alpha \rho_+^2} \left(1 + \frac{\rho_+^2}{r_+^2 + a^2}\right) \sin \theta, $$

$$ (5.4a) : -2a_1 \frac{r_+ (mr_+ - q^2)}{\alpha \rho_+^2} \left(1 + \frac{\rho_+^2}{r_+^2 + a^2}\right) \sin \theta, $$

one can directly conclude that their sum vanishes, as a consequence of $A_0' \equiv a_1 + \lambda b_4$. There are two more contributions of this type,

$$ (5.2b) : 2(A_0' - \lambda b_4) \frac{(mr_+ - q^2) r_+ a^2}{\alpha \rho_+^2 (r_+^2 + a^2)} \sin^3 \theta, $$

$$ (5.4b) : -2a_1 \frac{(mr_+ - q^2) r_+ a^2}{\alpha \rho_+^2 (r_+^2 + a^2)} \sin^3 \theta, $$

whose sum also vanishes. Hence, all $(\delta N)/N$ terms in entropy can be simply ignored.

After removing $\delta N/N$ terms, one finds that the sum of $\delta P/P$ terms also vanishes:

$$ (5.2b) + (5.3a) + (5.3b) : 0, $$

$$ (5.4a) + (5.4b) : 0. $$

### B.2 Elimination of $\lambda b_4$ terms

After eliminating all $\delta N/N$ and $\delta P/P$ terms, one can use the relation $A_0' = a_1 + \lambda b_4$ in Eqs. (5.2) and (5.3), subsection 5.2, to express them in terms of only two independent parameters, $a_1$ and $\lambda b_4$. Focussing on the $\lambda b_4$ terms and omitting the overall factor $2\lambda b_4$, the resulting contributions take the form

$$ (5.2a) : \left[ \kappa - V_1 \frac{\rho_+^2}{r_+^2 + a^2} \right] \delta \left( \frac{r_+^2 + a^2}{\alpha} \right) \sin \theta, $$

$$ (5.2b) : \left[ \frac{a (mr_+ - q^2) r_+}{\rho_+^2 (r_+^2 + a^2)} \delta \left( \frac{a}{\alpha} \right) + \frac{a r_+}{r_+^2 + a^2} \delta \left( \frac{mr_+ - q^2 a}{\rho_+^2} \right) \right] \sin^3 \theta, $$

$$ (5.3a) : -\left[ \frac{\rho_+^2}{\alpha} \delta \left( \frac{(mr_+ - q^2) r_+}{\rho_+^2} \right) + \frac{mr_+ - q^2}{r_+^2 + a^2} \delta \left( \frac{r_+}{\alpha} \right) \right] \sin \theta, $$

$$ (5.3b) : -\left[ \frac{\rho_+^2}{r_+^2 + a^2} \delta \left( \frac{(mr_+ - q^2) r_+}{\rho_+^2} \right) + \frac{mr_+ - q^2}{r_+^2 + a^2} \delta \left( \frac{r_+}{\alpha} \right) \right] \sin \theta. $$

**Step 1.** Let us first transform the first term in (5.2a) using the identity (C.2),

$$ (5.2a) : 2 \left[ \delta \left( \frac{m}{\alpha^2} \right) - \Omega_+ \delta \left( \frac{am}{\alpha^2} \right) - \frac{2r_+ q}{r_+^2 + a^2} \delta \left( \frac{q}{\alpha} \right) - V_1 \frac{\rho_+^2}{r_+^2 + a^2} \delta \left( \frac{r_+^2 + a^2}{\alpha} \right) \right] \sin \theta. $$

The result can be conveniently written as a sum of two parts, proportional to $\delta(mr_+ - q^2)$ and $(mr_+ - q^2)$,

$$ (5.2a)_1 : \frac{2r_+}{\alpha} \frac{\sin \theta}{r_+^2 + a^2} \delta(mr_+ - q^2), $$

$$ (5.2a)_2 : -\frac{2r_+ (mr_+ - q^2)}{(r_+^2 + a^2) \rho_+^2} \left[ (r_+^2 + a^2 - 2 \rho_+^2) \delta \left( \frac{1}{\alpha} \right) + \frac{\delta(r_+^2 + a^2)}{\alpha} \right] \sin \theta, $$
where we used the identities

\[ V_1 \rho_+^2 = \frac{2r_+(mr_+ - q^2)}{\rho_+^2} - m, \]
\[ \Omega_+ = \frac{a\alpha}{r_+^2 + a^2} + \lambda a, \]
\[ (1 - \lambda a^2)\delta \left( \frac{1}{\alpha^2} \right) - \frac{\lambda a}{\alpha^2} \delta a = \frac{3}{2} \delta \left( \frac{1}{\alpha} \right). \]

**Step 2.** Looking at the remaining contributions in (5.2b), (5.3a) and (5.3b), one again finds two types of terms. The part proportional to \( \delta (mr_+ - q^2) \) is given by

\[ \left[ (5.2b) + (5.3a) + (5.3b) \right]_1 : -\frac{2r_+}{\alpha} \frac{\sin \theta}{r_+^2 + a^2} \delta (mr_+ - q^2), \]

and it directly cancels the contribution (5.2a)\(_1\) given above, as expected.

As far as the part proportional to \( (mr_+ - q^2) \) is concerned, we find it convenient to separate the terms proportional to \( \delta r_+ \), \( \delta (1/\alpha) \) and the remaining \( a\delta a \) terms:\footnote{The remaining \( a\delta a \) terms are those that do not stem from \( \delta \alpha \).}

\[ \left[ (5.2b) + (5.3a) + (5.3b) \right]_2 : \]

\[ \delta r_+ : -\frac{2(mr_+ - q^2)(r_+^2 + a^2 + \rho_+^2)}{\alpha \rho_+^2(r_+^2 + a^2)} \left( 1 - \frac{2r_+}{\rho_+^2} \right) \sin \theta, \]
\[ \delta \left( \frac{1}{\alpha} \right) : \frac{2r_+(mr_+ - q^2)}{(r_+^2 + a^2)\rho_+^2} \left( a^2 \sin^2 \theta - \rho_+^2 \right) \sin \theta, \]
\[ a\delta a : \frac{2r_+(mr_+ - q^2)}{\alpha \rho_+^2(r_+^2 + a^2)} \left( \sin^2 \theta + \frac{2(r_+^2 + a^2 + \rho_+^2)}{\rho_+^2} \cos^2 \theta \right) \sin \theta. \]

Summing these terms with the corresponding expressions in (5.2a)\(_2\), one obtains

\[ \delta r_+ : -\frac{2(mr_+ - q^2)}{\alpha} \left( \frac{1}{\rho_+^2} + \frac{1}{r_+^2 + a^2} - \frac{2r_+}{\rho_+^2} \right) \sin \theta, \]
\[ \delta \left( \frac{1}{\alpha} \right) : 0, \]
\[ a\delta a : \frac{2r_+(mr_+ - q^2)}{\alpha (r_+^2 + a^2)} \left( \frac{-\sin^2 \theta}{\rho_+^2} + \frac{2(r_+^2 + a^2 - \rho_+^2)}{\rho_+^1} \cos^2 \theta \right) \sin \theta. \]

Since the integrals over \( \theta \) of the underlined terms vanish, it follows that the total contribution proportional to \( \lambda b_4 \) also vanishes.

**C Proof of the first law**

In this appendix, we derive an identity which is of essential importance for understanding the kinematic origin of the first law.
We start by introducing the notation
\[ M := \frac{m}{\alpha^2}, \quad J := Ma, \quad \Phi := \frac{q_e r_+}{r_+^2 + a^2}. \]

After using the horizon equation to express \( \delta r_+ \) in terms of \((\delta m, \delta q_e, \delta a)\), one finds
\[
L := \frac{\kappa}{2} \delta \left( \frac{r_+^2}{\alpha} + a^2 \right) = L_m \delta m + L_a \delta a - \Phi \delta q_e, \tag{C.1a}
\]
\[
L_m := \frac{r_+^2}{\alpha(r_+^2 + a^2)},
\]
\[
L_a = \frac{a(1 + \lambda r_+^2)(-1 + 3\lambda r_+^2)}{2\alpha^2 r_+} - \frac{a(1 + \lambda r_+^2)q_e^2}{2\alpha^2 r_+(r_+^2 + a^2)}.
\]

In an analogous manner, one obtains the relation
\[
R := \delta M - \Omega_+ \delta J - \Phi \delta \left( \frac{q_e}{\alpha} \right) = R_m \delta m + R_a \delta a - \Phi \delta \left( \frac{q_e}{\alpha} \right), \tag{C.1b}
\]
\[
R_m := L_m, \quad R_a = \frac{a(-1 + 3\lambda r_+^2)(1 + \lambda r_+^2)}{2\alpha^2 r_+} + \frac{a(-1 + 3\lambda r_+^2)q_e^2}{2\alpha^2 r_+(r_+^2 + a^2)}.
\]

Then, a direct comparison shows that the relation \( L = R \) is identically satisfied:
\[
\frac{\kappa}{2} \delta \left( \frac{r_+^2}{\alpha} + a^2 \right) = \delta M - \Omega_+ \delta J - \Phi \delta \left( \frac{q_e}{\alpha} \right). \tag{C.2}
\]

This identity coincides with the first law in GR.

References

[1] J. B. Griffiths and J. Podolský, *Exact Space-Times in Einstein’s General Relativity* (Cambridge University Press, Cambridge, 2009).

[2] R. P. Kerr, Gravitational field of a spinning mass as an example of algebraically special metrics, Phys. Rev. Lett. 11 (1963) 237–238.

[3] E. T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, Metric of a rotating, charged mass, J. Math. Phys. 6 (1965) 918–919;

[4] G. W. Gibbons and S. W. Hawking, Phys. Rev. D 15 (1977) 2738–2751.

[5] G. W. Gibbons, M. J. Perry, and C. N. Pope, The first law of thermodynamics for Kerr-anti-de Sitter black holea, Class. Quant. Grav. 22 (2005) 1503–1526 [arXiv:0408217].

[6] M. Blagojević and F. W. Hehl (eds.), *Gauge Theories of Gravitation, A Reader with Commentaries* (Imperial College Press, London, 2013). Comments on exact solution of PG are presented in section 16.

[7] P. Baekler, A spherically symmetric vacuum solution of the quadratic Poincaré gauge field theory of gravitation with Newtonian and confinement potentials, Phys. Lett. 99 B (1981) 329–332.

Ch. H. Lee, A spherically symmetric electro-vacuum solution of the Poincaré gauge field theory of gravitation, Phys. Lett. 130 B, 257–259 (1983).
J. D. McCrea, P. Baekler, and M. Gürses, A Kerr-like solution of the Poincaré gauge field equations, Nuovo Cim. B 99, 171–177 (1987).

[8] P. Baekler, M. Gürses, F. W. Hehl, and J. D. McCrea, The exterior gravitational field of a charges spinning source in the Poincaré gauge theory: a Kerr-Newmann metric with dynamic torsion, Phys. Lett. A 128 (1988) 245–250.

[9] Yu. N. Obukhov, Exact solutions in Poincaré gauge gravity theory, Universe 5 (2019) 127, https://doi.org/10.3390/universe5050127 (13 pages); arXiv:1905.11906.

[10] Ch.-M. Chen, J. M. Nester, and R.-S. Tung, Gravitational energy for GR and Poincaré gauge theory: A covariant Hamiltonian approach, Int. J. Mod. Phys. D 24 (2015) 1530026 arXiv:1507.07300.

[11] R. D. Hecht and J. M. Nester, A new evaluation of PGT mass and spin, Phys. Lett A 180 (1993) 324–331; Ch.-M. Chen and J. M. Nester, Quasi local quantities for GR and other gravity theories, Classical Quantum Gravity 16 (1999) 1279–1301 [arXiv:9809020];

[12] M. Blagojević and B. Cvetković, Entropy in Poincaré gauge theory: Hamiltonian approach, Phys. Rev. D 99 (2019) 104058 [arXiv:1903.02263].

[13] R. M. Wald, Black hole entropy is the Noether charge, Phys. Rev. D 48 (1993) R3427–R3431 [arXiv:9307038]; The thermodynamics of black holes, Living Rev. Relativity 4 (2001) 6 [arXiv:9912119].

[14] M. Blagojević and B. Cvetković, Entropy in Poincaré gauge theory: Kerr-AdS solution, Phys. Rev. D 102 (2020) 064034 [arXiv:2007.10721].

[15] M. Blagojević and B. Cvetković, Entropy in general relativity: Kerr-AdS black hole, Phys. Rev D 101 (2020) 084023 [arXiv:2002.05029]; Thermodynamics of Riemannian Kerr-AdS black holes in Poincaré gauge theory, Phys. Lett. B 816 (2021) 136242 (5 pages) arXiv:2103.00330.

[16] M. Blagojević and B. Cvetković, Entropy of Reissner-Nordström-like black holes, Phys. Lett. B 824 (2022) 136815 (5 pages) arXiv:2112.02099.

[17] F. W. Hehl and Y. N. Obukhov, Foundations of Classical Electrodynamics (Birkhäuser, Boston, 2003).

[18] T. Regge and C. Teitelboim, Role of surface integrals in the Hamiltonian formulation of general relativity, Ann. Phys. (N.Y.) 88 (1974) 286–318.

[19] M. M. Caldarelli, G. Cognola and D. Klemm, Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories, Class. Quant. Grav. 17 (2000) 399–420 [arXiv:9908022].

[20] B. Carter, Black hole equilibrium states, in Black holes, 1972 Les Houches Lectures, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1973) pp. 58–214.

[21] M. Henneaux and C. Teitelboim, Hamiltonian treatment of asymptotically anti-de Sitter spaces, Phys. Lett. 142 B (1984) 355–358; Asymptotically anti-de Sitter spaces, Comm. Math. Phys. 98 (1985) 391–424.