ENERGY PRINCIPLES FOR SELF-GRAVITATING BAROTROPIC FLOWS:

I. GENERAL THEORY

Joseph Katz*

National Astronomical Observatory, Mitaka, Tokyo 181

Shogo Inagaki

Department of Astronomy, Faculty of Science, Kyoto University,

Sakyo-ku, Kyoto 606-01

and

Asher Yahalom

The Racah Institute of Physics, Jerusalem 91904, Israel

* Permanent address: The Racah Institute of Physics, Jerusalem 91904, Israel.
Abstract

The following principle of minimum energy may be a powerful substitute to the dynamical perturbation method, when the latter is hard to apply. Fluid elements of self-gravitating barotropic flows, whose vortex lines extend to the boundary of the fluid, are labelled in such a way that any change of trial configurations automatically preserves mass and circulation. The velocity field is given by a mass conserving Clebsch representation. With three independent Lagrangian functions, the total energy is stationary for all small variations about a flow with fixed linear and angular momenta provided Euler’s equations for steady motion are satisfied. Thus, steady flows are stable if their energy is minimum. Since energy is here minimized subject to having local and global constants of the motion fixed, stability limits obtained that way are expected to be close to limits given by dynamical perturbation methods. Moreover, the stability limits are with respect to arbitrary, not necessary small, perturbations. A weaker form of the energy principle is also given which may be easier to apply.

The Lagrangian functional, with the same three Lagrange variables is stationary for the fully time dependent Euler equations. It follows that the principle of minimum energy gives stability conditions that are both necessary and sufficient if terms linear in time derivatives (gyroscopic terms) are absent from the Lagrangian. The gyroscopic term for small deviations around steady flows is given explicitly.

Key words: Energy variational principle; Self-gravitating systems; Stability of fluids.
1. Introduction

The method of small perturbations and analyses of eigenmodes are difficult to apply to astrophysical models that are not at least axisymmetric (Binney & Tremaine 1987). Moreover, it gives good indications on how instabilities set in, but once a configuration has become unstable, the method is at a loss to say whether it tends towards a new steady figure or will oscillate like a pendulum or become chaotic. In this era of supercomputers, one might think that a dynamical follow-up is routine work, but this is not the case, yet. Moreover, as a rule, analytical methods give more insight than purely numerical ones. Therefore, there are still reasons for perfecting other methods like the principle of minimum energy which played important roles in stability theory [see in particular Antonov (1962), Lebovitz (1965) and Chandrasekhar’s (1969) works].

We shall be concerned with ideal fluids and, for definiteness, we limit ourselves to barotropic fluids. In our Lagrange variables, non-barotropic fluids are more or less a particular case than a different one and may be treated along the same lines. We shall not do this here.

From the pioneering work of Arnold (1966), see also Lynden-Bell & Katz (1981) on compressible flows (the paper will be referred to below as LBK81) we know that among all steady flows that are mass preserving and isocirculational, those whose energy is stationary for small variations satisfy Euler’s equations. By mass preserving we mean that in an arbitrary virtual displacement of a small element of fluid there is no loss of mass and by isocirculational we mean that the flux of the vortex lines, in any direction through a small area, is the same for the displaced element of fluid.
It follows that if a steady configuration has minimum energy, it is stable with respect to perturbations that are not necessarily small. In a perturbation analysis, perturbations are always assumed to be small. Thus, while there may be stable configurations with respect to small perturbations for which the energy is not a minimum, the stability limits given by the energy principle are stronger than those of a perturbation analysis.

This principle of minimum energy can be made even stronger by imposing further, as constraints, global invariants of the motions. In that way, virtual displacements may eventually be restricted as much as real motions. If, by varying some parameter, a series of stable configurations ceases to have minimum energy with such severe constraints, the limit may be quite close to the limit one would find in a perturbation analysis.

Another way of using the “maximally constrained” energy consists in using a computer to find a new minimum when the energy principle fails to indicate stability. Since time is not involved, calculations have one dimension less than the dynamical equations; this may be of great computational help. The new configuration, if any, will be one in which the unstable model is likely to settle. The absence of a new steady configuration may indicate that no new real stable configuration exists.

A key point in applying this energy principle is to make an appropriate parametrization of the fluid elements; this includes the topology of the flows that must be conserved as well. The parametrization used here has been found in LBK81 where labels are attached to vortex lines rather than to fluid elements. Such a labelling leads naturally to a particular Clebsch representation for the velocity of the flow. A Lagrangian and Energy Principle in terms of three independent functions was developed in Katz and Lynden-Bell(1985).
(For a formal generalization of that approach, see Simo, Lewis and Marsden 1991). The method contains no Lagrange multiplier as is usually done [Serrin 1959, Lin 1963, Seliger and Whitham 1968] as such multipliers are not very useful in second variations.

The new elements in this work, as compared to LBK 81 and KLB 85, are as follows. We specify completely the labelling in a class of flows. We use a new set of Lagrangian functions to prove energy principles for steady motions. We show the uniqueness of our choice and fix whatever freedom remains in positioning the coordinates. The principle of minimum energy is completed with the proof that our variables are true Lagrangian variables. That is, Euler’s dynamical equations are obtained by varying the Action. If the Lagrangian’s kinetic energy is purely quadratic in time derivatives (no gyroscopic term), the principle of energy minimum is a necessary and sufficient condition for stability. The form of the terms linear in time derivatives gives thus important information and have therefore been worked out explicitly. The formulation follows closely classical mechanics. Applications to two dimensional flows and in particular to MacLaurin disks, whose dynamical stability limits are well known, give a perfect illustration of the power of our method. To maintain this paper within reasonable limits, we delay the application to an accompanying paper II.

2. Stationary Barotropic Self-Gravitating Flows

It will be useful to write the equations of motion first to show our essentially standard notations: \( \vec{r} = (x^k) = (x, y, z) \) \( [k, l, m, n = 1, 2, 3] \) for the positions, \( \vec{W} \) for the steady velocity field in inertial coordinates, \( \rho \) the density, \( P(\rho) \) the pressure, \( h(\rho) = \int dP/\rho \) the
specific enthalpy and $\Phi(<0)$ the gravitational potential,

$$\Phi = -G \int \frac{\rho(r')}{|r-r'|} d^3 x'.$$

(2.1)

Euler’s equations for flows that are steady in coordinates with uniform velocity $\vec{b}$ and angular velocity* velocity $\vec{U}$ may be written

$$\vec{O} \equiv (\vec{U} \cdot \vec{\nabla}) \vec{W} + \vec{\Omega}_c \times \vec{W} + \vec{\nabla}(h + \Phi) = 0,$$

(2.2)

where, by definition,

$$\vec{W} = \vec{U} + \vec{b} + \vec{\Omega}_c \times \vec{r} \equiv \vec{U} + \vec{\eta}_c.$$

(2.3)

Another useful form of Euler’s equation is

$$\vec{O} = \vec{\omega} \times \vec{U} + \vec{\nabla} \Lambda = 0,$$

(2.4)

where

$$\vec{\omega} = \text{rot} \ \vec{W}$$

(2.5)

and

$$\Lambda = \frac{1}{2}(\vec{U}^2 - \vec{\eta}_c^2) + h + \Phi.$$

(2.6)

With (2.4) we also have the equation of mass conservation

$$\mathcal{U} \equiv \vec{\nabla} \cdot (\rho \vec{U}) = 0.$$

(2.7)

The equation of circulation conservation follows from (2.4):

$$\text{rot} \ (\vec{\omega} \times \vec{U}) = 0.$$

(2.8)

* The index $c$ of $\vec{\Omega}_c$ refers to rotating coordinates and is just here to distinguish it from $\vec{\Omega}$ with no index, commonly used for angular velocities in galactic disks or rotating ste
The boundary conditions are those of a self-gravitating flow in free space, the density and the pressure go to zero and the velocity of sound goes to zero as well:

\[
\rho|_s = 0 \quad P|_s = 0 \quad \frac{dP}{d\rho}|_{\rho=0} = 0.
\]  

(2.9)

Notice that (2.7) with (2.9) implies

\[
\vec{U} \cdot \vec{\nabla} \rho|_s = 0,
\]  

(2.10)

i.e., \(\vec{U}\) is in the tangent plane of the surface of the fluid. Apart from satisfying these boundary conditions, physical quantities are assumed to be bounded everywhere.

In time dependent flows, in addition to mass conservation and conservation of circulation, the total mass, the linear and the angular momentum are conserved; the center of mass moves with a uniform velocity or is at rest. The conserved global quantities are

\[
M = \int \rho d^3x, \quad \vec{P} = \int \vec{W} \rho d^3x, \quad \vec{J} = \int \vec{r} \times \vec{W} \rho d^3x.
\]  

(2.11)

It is always possible and worthwhile to take

\[
\vec{P} = 0.
\]  

(2.12)

What this has to do with steady flows will soon become clear.
3. Mass Preserving and Isocirculational Labelling

3.1. Labelling of Fluid Elements When the Vortex Lines Extend to the Boundary

Conservation of vorticity implies conservation of the topology of the vortex lines. Flows with given vorticity have thus also given topologies of $\vec{\omega}$-lines. For definiteness we shall consider flows with the same topology as in LBK81 (see figure 1). There, vortex lines were labelled by three parameters: the load $\lambda$, the metage $\mu$ and an “angle” $\beta$.

Surfaces of constant load $\lambda$ are surfaces of constant mass per unit vortex strength in a narrow tube of vortex lines: if $dM$ is the mass in a tube with vorticity flux $dC$,

$$\lambda = \frac{dM}{dC} = \int_{\text{Bottom}}^{\text{Top}} \frac{\rho}{\omega} dl.$$  
(3.1)

The integration is along $\vec{\omega}$-lines. The surfaces $\lambda =$ constant are embedded “cylinders” as shown in figure 1. The cylinders may be parametrized in any way we want, say $\alpha = \alpha(\lambda) = \text{const.}$, but the particular parametrization in which $\alpha(\lambda)$ is proportional to the circulation on a closed contour around $\lambda = \text{const.}$ has very special property (see below).

The metage $\mu$, is defined as another family of surfaces with the same integral as in (3.1) taken up to some “red mark” on the $\vec{\omega}$-line:

$$\mu = \int_{\text{Bottom}}^{\text{Red mark}} \frac{\rho}{\omega} dl.$$  
(3.2)

Surfaces of constant $\mu$ cut across $\lambda$-surfaces. If the red mark is on the bottom, $\mu = 0$. If it is at the top, $\mu = \lambda$.

The angular variable $\beta$ is defined by a family of surfaces of $\vec{\omega}$-lines as well; these are “cuts” hanging on a “central line” (see figure 1). Thus, by definition,

$$\vec{\omega} \cdot \vec{\nabla} \alpha = 0, \quad \vec{\omega} \cdot \vec{\nabla} \beta = 0 \quad \text{and} \quad \vec{\omega} \cdot \vec{\nabla} \mu = \rho.$$  
(3.3)
The parametrization of $\beta$ may be so chosen that

$$\vec{\omega} = \vec{\nabla} \alpha \times \vec{\nabla} \beta.$$  \hspace{1cm} (3.4)

This defines $\beta$ for given $\vec{\omega}$ and $\alpha$ up to an additive single valued function of $\alpha$, $B(\alpha)$. This $B(\alpha)$ is associated with the freedom to take any cut as $\beta = 0$. Having chosen $\alpha, \beta, \mu$, It also follows from (3.3) that we must have

$$\rho = \vec{\nabla} \alpha \times \vec{\nabla} \beta \cdot \vec{\nabla} \mu$$  \hspace{1cm} (3.5)

or that

$$\rho = \frac{\partial (\alpha, \beta, \mu)}{\partial (x, y, z)}.$$  \hspace{1cm} (3.6)

and thus

$$\frac{\partial (\alpha, \beta, \mu)}{\partial (x, y, z)} \neq 0$$ \hspace{1cm} (3.7)

everywhere except on the surface of the fluid since $\rho|_s = 0$,

$$\rho|_s = \vec{\omega} \cdot \vec{\nabla} \mu|_s = 0.$$  \hspace{1cm} (3.8)

With (3.4) we see that the velocity field has now a Clebsch form

$$\vec{W} = \alpha \vec{\nabla} \beta + \vec{\nabla} \nu.$$  \hspace{1cm} (3.9)

The function $\nu$ we define by the condition that mass be preserved i.e. by equation (2.7).

The function $\nu$ is thus defined by an elliptic equation since (2.7) can be written

$$\vec{\nabla} \cdot (\rho \vec{\nabla} \nu) = \vec{\nabla} \cdot [\rho(-\alpha \vec{\nabla} \beta + \vec{\eta}_c)].$$  \hspace{1cm} (3.10)

Since $\rho|_s = 0$, a regular single valued solution $\nu(\vec{r})$ (up to a constant) must be unique when it exists.
The main tool in practical calculations will thereby be the Green function $G(\vec{r}, \vec{r}')$, solution of

$$\vec{\nabla} \cdot (\rho \vec{\nabla} G) = \delta^3(\vec{r} - \vec{r}'). \quad (3.11)$$

### 3.2. Uniqueness of the Labelling

Consider now the effect of a reparametrization of the fluid elements in a flow with given $\rho(\vec{r})$, $\vec{W}(\vec{r})$ and thus $\vec{\omega}(\vec{r})$ and given $\lambda(\alpha)$. Let $\tilde{\alpha}, \tilde{\beta}, \tilde{\mu}$ be another labelling. Surfaces of constant load may be reparametrized in any way: $\alpha \rightarrow \tilde{\alpha}(\alpha)$. Since

$$\vec{\omega} = \vec{\nabla} \alpha \times \vec{\nabla} \beta = \vec{\nabla} \tilde{\alpha} \times \vec{\nabla} \tilde{\beta} = \frac{d\tilde{\alpha}}{d\alpha} \vec{\nabla} \alpha \times \vec{\nabla} \tilde{\beta} \quad (3.12)$$

We must change $\beta$ to

$$\tilde{\beta} = \frac{d\alpha}{d\tilde{\alpha}} \beta + B(\alpha). \quad (3.13)$$

With both $\mu = \tilde{\mu} = 0$ on the bottom of the fluid, the parametrization of $\mu$ for given $\rho$ and $\vec{\omega}$ is uniquely defined

$$\tilde{\mu} = \mu. \quad (3.14)$$

The corresponding change in $\nu$ is accordingly defined by

$$\vec{W} = \alpha \vec{\nabla} \beta + \vec{\nabla} \nu = \tilde{\alpha} \vec{\nabla} \tilde{\beta} + \vec{\nabla} \tilde{\nu} \quad (3.15)$$

or

$$\vec{\nabla}(\tilde{\nu} - \nu) = \alpha \vec{\nabla} \beta - \tilde{\alpha} \vec{\nabla} \tilde{\beta}. \quad (3.16)$$

From (3.13) it follows that

$$\tilde{\nu} = \nu - (\tilde{\alpha} \frac{d\alpha}{d\tilde{\alpha}} - \alpha) \beta - \int \tilde{\alpha} dB(\alpha) + \text{Const.} \quad (3.17)$$
α is single valued but β is not. If, however, we define and keep ν single valued, any reparametrization that preserves the single valuedness of ν must satisfy the condition

\[ \tilde{\alpha} \frac{d\alpha}{d\tilde{\alpha}} - \alpha = 0 \quad \text{or} \quad \tilde{\alpha} = l\alpha, \quad (3.18) \]

where \( l \) is a constant. The parametrization for which ν is single valued is easily found. On the one hand \( C(\lambda) \) is given by the flux of \( \vec{\omega} \) through a surface with boundary \( \alpha = \text{constant} \).

\[ C(\lambda) = \int \int \vec{\omega} \cdot d\vec{S} = \int \int d\alpha d\beta = \int_{\alpha_c}^{\alpha} [\beta] d\alpha, \quad (3.19) \]

where \([\beta]\) is the value of the discontinuity of β and \( \alpha_c \) is the value of α on the central line. On the other hand \( C(\lambda) \) is also equal to the circulation of \( \tilde{W} \) along a contour on a \( \alpha = \text{constant} \) surface:

\[ C(\lambda) = \oint_{\alpha} \vec{W} \cdot d\vec{r} = \oint_{\alpha} \alpha d\beta = \alpha [\beta]. \quad (3.20) \]

So, (3.19) and (3.20) give

\[ \frac{dC}{d\alpha} = [\beta] = [\beta] + \alpha \frac{d[\beta]}{d\alpha} \quad (3.21) \]

which means that

\[ [\beta] = q \quad \text{and} \quad \alpha = \frac{1}{q} C(\lambda), \quad q = \text{const.} \quad (3.22) \]

The parametrization of α by the circulation along λ-tubes is unique except again, as in (3.18), for the multiplication constant \( q \); with \( q = 2\pi \), the domains of the Lagrange variables \( \alpha, \beta, \mu \) is thus

\[ 0 \leq \alpha \leq \alpha_M \equiv C_{\text{MAX}}/2\pi, \quad B(\alpha) \leq \beta \leq B(\alpha) + 2\pi, \quad 0 \leq \mu \leq \lambda(\alpha), \quad (3.23) \]

where \( \alpha = \frac{C(\lambda)}{2\pi} \) must be a given function of the flow. \( C(\lambda) \) is similar to the amplitude \( J \) of the angular momentum. The only remaining arbitrariness of the parametrization is
$B(\alpha)$ associated with the cut where $\beta = 0$. We have thus found that the parametrization $\alpha = C/2\pi$ of LBK81 is the only one (up to $q$) that insure single valued $\nu$’s in a Clebsch representation.

3.3. The Fixation of Coordinates and of the Cut $\beta = 0$

Consider the trial configuration of figure 1. We are free to position the axis of coordinates in the simplest way; we shall make the following choice. The central line has two points $\mu = 0, \mu = \lambda_c$; we use this as the $z$ axis. The bottom of the central line is the origin of the coordinates. Unless the configuration is axially symmetrical, we have several ways, for orienting the $xy$ coordinates. For instance if the $yz$ plane cuts the $\lambda = 0$ line at $P$, the orientation may be chosen so that $y_P$ be maximum. There may be several extrema of $y_P$ and it does not matter which one we take. We shall only compare configurations with small differences. The cut $\beta = 0$ may be the ruled surface generated by lines parallel to the $x$ axis and sliding on the central string.

In this way, we fix uniquely the relative positions of trial configurations with respect to the coordinates and the parametrization of their points is:

$$0 \leq \alpha \leq \alpha_M, \quad 0 \leq \beta \leq 2\pi, \quad 0 \leq \mu \leq \lambda(\alpha)$$  \hspace{1cm} (3.24)

3.4. Proof That the Labelling Is Mass and Circulation Preserving

Consider a fluid element labeled $(\alpha, \beta, \mu) \equiv (\alpha^k)$ with the coordinates $x, y, z$ or $\vec{r}$. Any displacement from $\vec{r}$ to $\vec{r} + \Delta\vec{r}$ of that fluid element ($\Delta\alpha = \Delta\beta = \Delta\mu = 0$) is obviously mass preserving since according to (3.5)

$$\Delta(\rho d^3 x) = \Delta d^3 \alpha = 0$$  \hspace{1cm} (3.25)
This shows incidentally that if we set
\[ \Delta \vec{r} = \vec{\xi}(\vec{r}), \quad \text{then} \quad \Delta \rho = -\rho \vec{\nabla} \cdot \vec{\xi} \] (3.26)

Similarly from (3.4), the flux of the vorticity through any surface element \( d\vec{S} \) parametrized by \( \alpha, \beta \) is
\[ \vec{\omega} \cdot d\vec{S} = d\alpha d\beta \] (3.27)

which shows again that
\[ \Delta (\vec{\omega} \cdot d\vec{S}) = 0 \] (3.28)

or (see Arnold 1966, LBK1981)
\[ \Delta \vec{\omega} + \text{rot}(\vec{\omega} \times \vec{\xi}) = (\vec{\xi} \cdot \vec{\nabla})\vec{\omega} \] (3.29)

4. Energy Principles

We shall now show that the energy of steady flows is stationary compared to the energy of any nearby trial configuration with the same total mass, linear and angular momentum and the same mass and vortex flux in displaced fluid elements.

4.1. Some Differential Identities

Let \( F \) be a function of \((\alpha, \beta, \mu) = \alpha^k\). Since
\[ \frac{\partial \alpha^l}{\partial x^m} \frac{\partial x^m}{\partial \alpha^k} = \partial_m \alpha^l \partial_k x^m = \vec{\nabla} \alpha^l \cdot \vec{\nabla} \vec{r} = \delta^l_k, \] (4.1)

it follows that
\[ \Delta(\vec{\nabla} \alpha^l) = -\vec{\nabla} \vec{\xi}^k \partial_k \alpha^l = -\vec{\nabla} \vec{\xi} \cdot \vec{\nabla} \alpha^l \] (4.2)
and therefore
\[ \Delta \tilde{\nabla} F = -\tilde{\nabla} \xi^k \partial_k F + \tilde{\nabla} \Delta F = -\tilde{\nabla} \xi \cdot \tilde{\nabla} F + \tilde{\nabla} \Delta F \] (4.3)

In particular, see (3.9),
\[ \Delta \tilde{W} = \alpha \Delta \tilde{\nabla} \beta + \Delta \tilde{\nabla} \nu = -\tilde{\nabla} \xi \cdot \tilde{W} + \tilde{\nabla} \Delta \nu \]
\[ = \tilde{\nabla} \tilde{W} \cdot \xi + \tilde{\nabla}(\Delta \nu - \tilde{W} \cdot \bar{\xi}) \] (4.4)

Where \( \Delta \nu \) is defined by varying (3.10)
\[ \Delta [\tilde{\nabla} \cdot (\rho \tilde{\nabla} \nu)] = \Delta \tilde{\nabla} \cdot [\rho(-\alpha \tilde{\nabla} \beta + \bar{\eta} c)] \] (4.5)

This defines \( \Delta \nu \) by an elliptic equation of the same form as \( \nu \) itself
\[ \tilde{\nabla} \cdot [\rho \tilde{\nabla}(\Delta \nu)] = etc... \] (4.6)

and therefore \( \Delta \nu \) is defined with the same type of Green function as \( \nu \). Another needed variation is that of \( \Phi \) which, according to (2.1) and (3.5) may be written
\[ \Phi = -G \int \frac{d^3 \alpha'}{|\bar{r}(\alpha) - \bar{r}'(\alpha')}| = -G \int \frac{d^3 \alpha'}{R} \] (4.7)

Thus
\[ \Delta \Phi = -G \int \left( \xi \cdot \tilde{\nabla} \frac{1}{R} + \bar{\xi} \cdot \tilde{\nabla}' \frac{1}{R} \right) d^3 \alpha' \] (4.8)

and therefore the variation of the gravitational potential energy
\[ \Delta \int \frac{1}{2} \Phi \rho d^3 x = \Delta \int \frac{1}{2} \Phi d^3 \alpha = \int \frac{1}{2} \Delta \Phi d^3 \alpha = \int \xi \cdot \tilde{\nabla} \Phi \rho d^3 r. \] (4.9)
4.2. The Variation of the Energy

The energy of the flow

\[ E = \int \left( \frac{1}{2} \vec{W}^2 + \varepsilon(\rho) + \frac{1}{2} \Phi \right) \rho d^3x \]  
(4.10)

in which \( \varepsilon(\rho) \) is the specific internal energy of the barotropic fluid, related to the pressure and the specific enthalpy:

\[ \varepsilon(\rho) = h - \frac{P}{\rho} \quad P = -\rho^2 \frac{\partial \varepsilon}{\partial \rho} \]  
(4.11)

With \( \Delta \vec{W} \) given in (4.4) and with \( \Delta \int \frac{1}{2} \Phi \rho d^3x \) in (4.9), one readily finds the following identity for \( \Delta E \),

\[ \Delta E = \int \{ [\vec{W} \cdot \vec{\nabla}) \vec{W} + \vec{\nabla}(h + \Phi)] \cdot \vec{\xi} \rho - \vec{\nabla} \cdot (\rho \vec{W})(\Delta \nu - \vec{W} \cdot \vec{\xi}) \} d^3x \]

\[ + \int \vec{\nabla} \cdot [(\Delta \nu - \vec{W} \cdot \vec{\xi}) \rho \vec{W} - P \vec{\xi}] d^3x. \]  
(4.12)

4.3. The Constrained Variational Identity and the Principle of Stationary Energy

We can now also calculate \( \Delta \vec{P} \) and \( \Delta \vec{J} \) which according to (2.11) are given by

\[ \Delta \vec{P} = \int \Delta \vec{W} d^3\alpha \quad \text{and} \quad \Delta \vec{J} = \int (\vec{\xi} \times \vec{W} + \vec{r} \times \Delta \vec{W}) d^3\alpha \]  
(4.13)

Using (2.3) that defines \( \vec{U} \), we then obtain a new identity:

\[ \Delta E - \vec{b} \cdot \Delta \vec{P} - \vec{\Omega}_c \cdot \Delta \vec{J} = \int [\vec{O} \cdot \vec{\xi} \rho + \vec{U}(\vec{W} \cdot \vec{\xi} - \Delta \nu)] d^3x \]

\[ + \int_S [(\Delta \nu - \vec{W} \cdot \vec{\xi}) \rho \vec{W} - P \vec{\xi}] \cdot d\vec{S}, \]  
(4.14)

where \( \vec{O} \) and \( \vec{U} \) have been defined in (2.2) and (2.7), respectively.
We see from (4.14) that if $\rho|_s = P|_s = 0$, mass is preserved ($U = 0$) and Euler’s equations for steady flows in uniformly moving coordinates hold ($\vec{\mathcal{O}} = 0$) then the energy is stationary ($\Delta E = 0$) when linear and angular momenta are kept fixed ($\Delta \vec{P} = \Delta \vec{J} = 0$).

4.4. Weak and Strong Principles of Minimum Energy

Reciprocally if $\Delta E = 0$ with $\Delta \vec{P} = \Delta \vec{J} = 0$ (which defines $\Delta \vec{b}$ and $\Delta \vec{\Omega}_c$), then the following must hold:

$(\alpha)$ Either $\nu$ is defined by mass conservation and the principle of stationary energy provides Euler’s equations $\vec{\mathcal{O}} = 0$,

$(\beta)$ Or $\nu$ is an additional variable and $\Delta E = 0$ if, in addition to $\vec{\mathcal{O}} = 0$, we have mass conservation, $U = 0$.

The principle of energy with four instead of three independent functions, $\vec{r}$ and $\nu$, we shall call the weak energy principle, as opposed to that with only three independent functions which we call the strong energy principle.

As far as stationarity of energy is concerned, the distinction is of no consequence. But it becomes important when we consider second order variations and the principle of minimum energy. Second order variations of (4.14) near the extremum $\Delta E = 0$ are given by

$$\Delta^2 E - \vec{b} \cdot \Delta^2 \vec{P} - \vec{\Omega}_c \cdot \Delta^2 \vec{J} = \int [\Delta \vec{\mathcal{O}} \cdot \vec{\xi} \rho - \Delta U (\Delta \nu - \vec{W} \cdot \vec{\xi})] d^3 x$$  \hspace{1cm} (4.15)

Notice that with (2.9) and (2.10), the boundary term in (4.14) does not contribute to $\Delta^2 E$. The relations $\Delta^2 \vec{P} = \Delta^2 \vec{J} = 0$ define $\Delta^2 \vec{b}$ and $\Delta^2 \vec{\Omega}_c$. Stationary configurations are stable if

$$\Delta^2 E > 0.$$  \hspace{1cm} (4.16)
The strong energy principle reads then

\[ \Delta^2 E_{\text{strong}} = \int \Delta \vec{O} \cdot \vec{\xi} \rho d^3 x > 0 \]  

(4.17)

which contains \( \Delta \nu \) that must be obtained from (4.6). The weak energy principle is

\[ \Delta^2 E_{\text{weak}} = \int [\Delta \vec{O} \cdot \vec{\xi} \rho - \Delta U (\Delta \nu - \vec{W} \cdot \vec{\xi})] d^3 x > 0 \]  

(4.18)

and is manifestly simpler to apply since \( \Delta \nu \) is here independent. The weakness of this principle compared to (4.17) can be understood by the fact that variations of the trial functions are mass preserving and isocirculational but that the velocity trial field \( \vec{W} \) does not conserve mass. Only extremal \( \vec{W} \)'s do. Thus fluctuations in \( \Delta^2 E \) would include \( \Delta \vec{W} \)'s that do not necessarily satisfy the mass conservation equation. \( \vec{W} + \Delta \vec{W} \) being less restricted than a real dynamical \( \vec{W} + \Delta \vec{W} \), some instabilities might show up that cannot exist and would not show up in a perturbation analysis. For this reason this energy principle is weaker. But the weak principle may be helpful when the strong principle involves too hard calculations. Moreover, in numerical calculations, in which the computer ”searches” the minimum of \( E \), one has to find the Green functions \( G(\vec{r}, \vec{r}') \) for \( \nu \) at every step and that may be time consuming. Therefore the energy principle with four independent functions may indeed be useful.

5. Principles of Stationary Action

5.1. Introduction

It may appear to make little sense to set up a Lagrangian formulation in terms of variables that take a priori account of all conservation laws of motion. First the dynamical equations insure automatically that the constants of the motion are indeed constant.
Second, in hydrodynamics, fixation of the values of constants of motion does not reduce very much the number of independent variables but complicate considerably the equations of motion.

However, it is important to show that our labelling defines proper Lagrange variables, even if we are never going to use them, for the following reason. An ordinary Lagrangian has the form $L = T - V$ while the energy $E = T + V$ ($T$ the kinetic energy, $V$ the potential energy). If $T$ is purely quadratic in the time derivatives of the Lagrange variables, $\Delta^2 E > 0$ is a necessary and sufficient condition of stability. It is however well known that Lagrangians subject to non-holonomic constraints, like circulation conservation or fixed linear momentum, contain additional terms, so called gyroscopic terms $G$, that are linear in time derivatives. The form of $L$ is then $T + G - V$, the energy, however, is still $T + V$. In those circumstances, $\Delta^2 E > 0$ is only a sufficient condition of stability, not a necessary one. In our representation in which not only fixed $\vec{J}$ and $\vec{P}$ but also fixed circulation and mass conservation have been incorporated, we may have lots of gyroscopic terms in the Lagrangian. It is therefore useful to consult the form of $L$ and to see if there are gyroscopic terms so as to know when our energy principles give necessary and sufficient conditions of stability or only sufficient ones. For this reason we have first to proof that the Lagrangians of our flows provide indeed Euler’s equations.

5.2. *Time Dependent Lagrange Variables and Dynamical Equations*

We use the same Lagrange variables $\alpha^k$ introduced in section 3. Now, however, they depend also on the time: $\alpha^k = \alpha^k(\vec{r}, t)$. Surfaces of constant $\alpha^k$ define a velocity $\vec{w}$ which
satisfy the equations:

\[ \alpha^k(\vec{r} + \vec{w}dt, t + dt) = \alpha^k(\vec{r}, t); \quad (5.1) \]

in the limit \( dt \to 0 \):

\[ \dot{\alpha}^k + \vec{\omega} \cdot \vec{\nabla} \alpha^k = 0. \quad (5.2) \]

Since arbitrary displacements of a fluid element with labels \( \alpha^k \) (see section 3.4) are both mass preserving and isocirculational it follows from (5.2), (3.4) and (3.6) that mass and vortex strength are both preserved along a motion with velocity \( \vec{w} \):

\[ \dot{\rho} + \vec{\nabla} \cdot (\rho \vec{w}) = 0 \quad (5.3) \]

and

\[ \dot{\vec{\omega}} + \text{rot}(\vec{\omega} \times \vec{w}) = 0 \quad (5.4) \]

There exists a useful explicit expression for \( \vec{w} \) that simplifies various formulas; consider the following identity obtained from \( \vec{r}(t, \alpha^k) \):

\[ \vec{r} \equiv \vec{r}[t, \alpha^k(\vec{r}, t)] \quad (5.5) \]

Since the right hand side must be independent of \( t \), we have:

\[ \left( \frac{\partial \vec{r}}{\partial t} \right)_t = \left( \frac{\partial \vec{r}}{\partial t} \right)_\alpha + \dot{\alpha}^k \frac{\partial \vec{r}}{\partial \alpha^k} \quad (5.6) \]

extracting \( \dot{\alpha}^k \) from (5.2) and replacing it in (5.6) gives then:

\[ \left( \frac{\partial \vec{r}}{\partial t} \right)_\alpha = \vec{w} \quad (5.7) \]

and, quite generally,

\[ \left( \frac{\partial F(t, \vec{r})}{\partial t} \right)_\alpha = \dot{F} + \vec{w} \cdot \vec{\nabla} F \quad (5.8) \]
The "absolute" velocity of the fluid is, say, $\vec{W}$ plus the velocity of the vortex lines $\vec{w}$:

$$\vec{v} = \vec{w} + \vec{W} \quad (5.9)$$

The relative velocity is similarly:

$$\vec{u} = \vec{w} + \vec{U} \quad (5.10)$$

$\vec{W}$ and $\vec{U}$ are related by eq. (2.3). The equation of mass conservation along the motion of the fluid must be

$$\dot{\rho} + \vec{V} \cdot (\rho \vec{u}) = 0 \quad (5.11)$$

and (5.3), with (5.10) and (5.11), imply that $\vec{U}$ satisfies equation (2.7) again:

$$\mathcal{U} \equiv \vec{V} \cdot (\rho \vec{U}) = 0. \quad (5.12)$$

The dynamical equations of motion are slightly different from (2.2) and are well known:

$$\vec{O}_D \equiv \dot{\vec{v}} + (\vec{u} \cdot \vec{V})\vec{v} + \vec{\Omega}_c \times \vec{v} + \vec{V}(h + \Phi) = 0, \quad (5.13)$$

This familiar equation may also be written in a slightly less familiar form which will be useful; with (5.8) and (5.10), we have

$$\vec{O}_D \equiv \left(\frac{\partial \vec{v}}{\partial t}\right)_\alpha + (\vec{U} \cdot \vec{V})\vec{v} + \vec{\Omega}_c \times \vec{v} + \vec{V}(h + \Phi) = 0, \quad (5.14)$$

In stationary flows $\vec{w} = 0$, $\vec{v} = \vec{W}$ and $(\frac{\partial \vec{W}}{\partial t})_\alpha = 0$; $\vec{O}_D$ becomes then the $\vec{O}$ of (2.2).

5.3. The Action

The Action of the system is given (see KLB 85) by:

$$A = \int_{t_0}^{t_1} Ldt \equiv \int \int [\frac{1}{2} \vec{w}^2 - (\frac{1}{2} \vec{W}^2 + \varepsilon + \frac{1}{2} \Phi)]d^3xdt = \int \int [\frac{1}{2} \vec{w}^2 \rho d^3x - E(t)]dt \quad (5.15)$$
Notice the minus sign and the fact that $E$ appears here effectively as the potential. The Lagrangian of isocirculational flows is actually a Routhian that incorporates mass and circulation conservation. With (5.9) we may write $L$ as this:

$$L = \int [\vec{w} \cdot \vec{v} - \left( \frac{1}{2} \vec{v}^2 + \varepsilon + \frac{1}{2} \Phi \right)] \rho d^3 x$$

which is of great help in calculations. The Lagrange variables are thus $(\alpha, \beta, \mu) = \alpha^k$ as in stationary flows; $\vec{v}$, instead of $\vec{W}$, has now a Clebsch form:

$$\vec{v} = \alpha \vec{\nabla} \beta + \vec{\nabla} \nu.$$  

where $\nu$ is a single valued functional of $\alpha^k$ defined by (5.12) and is linear in $(\dot{\alpha}^k, \vec{b}, \vec{\Omega}_c)$ because $\vec{w}$ is linear and homogeneuos in $\dot{\alpha}^k$. Quantities like $\varepsilon$ and $\Phi$ depend on $\rho$ only which is a functional of $\alpha^k$. The Action is thus a functional of $\alpha^k$, quadratic in $\dot{\alpha}^k$ but not quadratic homogeneous. Notice that $L$ is apparently linear only in $\dot{\alpha}^k$ through $\vec{w}$. However, $L$ is quadratic in $\nu$ which is also linear in $\vec{w}, \vec{b}$ and $\vec{\Omega}_c$. The constants $\vec{b}$ and $\vec{\Omega}_c$ are defined by the values of linear and angular momentum which are now given by:

$$\vec{P} = \int \vec{v} \rho d^3 x = \int (\alpha \vec{\nabla} \beta + \vec{\nabla} \nu) \rho d^3 x = 0;$$  

and

$$\vec{J} = \int \vec{r} \times \vec{v} \rho d^3 x = \int \vec{r} \times (\alpha \vec{\nabla} \beta + \vec{\nabla} \nu) \rho d^3 x = \vec{J}_0;$$

$\vec{b}$ and $\vec{\Omega}_c$ are thus equally linear in $\dot{\alpha}^k$ through $\nu$.

### 5.4. Variational Identities for Calculating $\Delta A$

Let us now make small displacements $\xi$ of fluid elements and calculate the changes $\Delta A$, $\Delta \vec{P}$ and $\Delta \vec{J}$ in $A$, $\vec{P}$ and $\vec{J}$. Much of the calculation has actually been done in
Indeed notice first that $E(t)$, $\vec{P}$ and $\vec{J}$ are similar to $E$ in (4.10), $\vec{P}$ in (2.11b) and $\vec{J}$ in (2.11c) with $\vec{v}$ replacing $\vec{W}$. Since $\vec{v}$, in time dependent flows, and $\vec{W}$, in steady flows, are both represented by $\alpha \vec{\nabla} \beta + \vec{\nabla} \nu$, we obtain, straightaway $\Delta E(t) - \vec{b} \cdot \Delta \vec{P} - \vec{\Omega}_c \cdot \Delta \vec{J}$ 

$$\bar{\Omega}$$ by $\vec{v}$:

$$\Delta E(t) - \vec{b} \cdot \Delta \vec{P} - \vec{\Omega}_c \cdot \Delta \vec{J} = \int \{[(\vec{v} \cdot \vec{\nabla}) \vec{v} + \vec{\Omega}_c \times \vec{v} + \vec{\nabla} (h + \Phi)] \cdot \vec{\xi}_\rho + \vec{\nabla} \cdot (\rho \vec{v}) (\vec{v} \cdot \vec{\xi} - \Delta \nu)\} d^3 x$$

$$+ \int_S [(\Delta \nu - \vec{v} \cdot \vec{\xi}) \rho \vec{v} - P \vec{\xi}] \cdot d\vec{S}, \quad (5.19)$$

Notice that (5.19) is a variational identity; $\nu$ has been treated as an independant function.

To calculate $\Delta A$ there remains, however, some work to do on the $\vec{v} \cdot \vec{w}$ part of $L$. Following (5.7) and (5.17)

$$\Delta (\vec{w} \cdot \vec{v}) = \Delta (\frac{\partial \vec{r}}{\partial t})_\alpha \cdot \vec{v} + \vec{w} \cdot \Delta (\alpha \vec{\nabla} \beta + \vec{\nabla} \nu) \quad (5.20)$$

We can use (4.4) to rewrite $\Delta (\alpha \vec{\nabla} \beta + \vec{\nabla} \nu)$ and obtain

$$\Delta (\vec{w} \cdot \vec{v}) = (\frac{\partial \vec{\xi}}{\partial t})_\alpha \cdot \vec{v} + (\vec{w} \cdot \vec{\nabla}) \vec{v} \cdot \vec{\xi} + \vec{w} \cdot \vec{\nabla} (\Delta \nu - \vec{v} \cdot \vec{\xi}) \quad (5.21)$$

so that, with some integration by part and, with the help of (5.8), we can write

$$\Delta \int \vec{w} \cdot \vec{v} \rho d^3 x = \frac{\partial (\int \vec{v} \cdot \vec{\xi} \rho d^3 x)}{\partial t} + \int [-\vec{v} \cdot \vec{\xi} \rho + \vec{\nabla} \cdot (\rho \vec{w}) (\vec{v} \cdot \vec{\xi} - \Delta \nu)] d^3 x + \int_S (\Delta \nu - \vec{v} \cdot \vec{\xi}) \rho \vec{v} \cdot d\vec{S}, \quad (5.22)$$

Thus, $\int [\Delta L + \vec{b} \cdot \Delta \vec{P} + \vec{\Omega}_c \cdot \Delta \vec{J}] dt$ is given by (5.22) minus (5.19); in writing the result we shall use $\vec{O}_D$ defined in (5.13) and $U$ in (5.12):

$$\Delta A + \int (\vec{b} \cdot \Delta \vec{P} + \vec{\Omega}_c \cdot \Delta \vec{J}) dt = \int [\vec{v} \cdot \vec{\xi} \rho d^3 x]_{t_0}^{t_1} - \int \int [\vec{O}_D \cdot \vec{\xi} \rho + U (\vec{v} \cdot \vec{\xi} - \Delta \nu)] d^3 x dt$$

$$- \int \int_S [(\Delta \nu - \vec{v} \cdot \vec{\xi}) \rho \vec{W} - P \vec{\xi}] \cdot d\vec{S} dt, \quad (5.23)$$
5.5 Weak and Strong Principles of Stationary Action

Identity (5.23) leads straightaway to the following results. If \( \rho|_s = P|_s = 0 \), mass is preserved (\( \mathcal{U} = 0 \)) and Euler’s equations hold (\( \mathbf{\bar{O}}_D = 0 \)) then the Action is stationary (\( \Delta A = 0 \)) when linear and angular moment. Reciprocally, if \( \Delta A = 0 \) with \( \Delta \mathbf{\bar{P}} = \Delta \mathbf{\bar{J}} = 0 \) (which define \( \Delta \mathbf{\bar{b}} \) and \( \Delta \mathbf{\bar{\Omega}}_c \)) then the following must hold:

(\( \alpha \)) Either \( \nu \) is defined by \( \mathcal{U} = 0 \) and the principle of stationary Action provides Euler’s equation \( \mathbf{\bar{O}}_D = 0 \).

(\( \beta \)) Or \( \nu \) is an additional variable and \( A \) is stationary if in addition to \( \mathbf{\bar{O}}_D = 0 \), we have mass conservation \( \mathcal{U} = 0 \). The principle of stationary Action, with four independent functions, \( \alpha^k \) and \( \nu \), instead of three, we call the weak principle of stationary Action. It is the dynamical counterpart to the weak energy principle.

6. Gyroscopic Terms for Small Deviations from Steady Flows

6.1 The Action for Small Deviations

Small time dependent deviations from a stable flow are given by Euler’s linearized equations whose Action is the second order variation \( \Delta^2 A \) of \( A \) calculated at the stationary "point". Let \( \bar{r}_0(t, \alpha^k) \) be the coordinates of the fluid element of a particular stationary flow and \( \bar{r}_0 + \xi(t, \alpha^k) \) the coordinates in the perturbed flow. The gyroscopic terms in \( \Delta^2 A \) are the bilinear functionals of \( \xi \) and \( \dot{\xi} \). The gyroscopic terms of the fully constrained Action are given by the \( \xi \), \( \dot{\xi} \) terms in \( \int [\Delta^2 L + \mathbf{\bar{b}} \cdot \Delta^2 \mathbf{\bar{P}} + \mathbf{\bar{\Omega}}_c \cdot \Delta^2 \mathbf{\bar{J}}] dt \) as evaluated at the stationary point. Let the

\[
\begin{align*}
\bar{w}_0 &= 0, & \bar{v}_0 &= \bar{W}_0, & \bar{u}_0 &= \bar{U}_0, & \nabla \cdot (\rho_0 \bar{U}_0) &= 0 & \text{etc...} \quad (6.1)
\end{align*}
\]
Notice that:

\[
(\frac{\partial \vec{\xi}}{\partial t})_0 = \dot{\vec{\xi}} + (\vec{w}_0 \cdot \vec{\nabla})\vec{\xi} = \dot{\vec{\xi}}
\]

(6.2)

With (6.1) and (6.2) we readily obtain \(\Delta^2 L + \text{etc... from (5.23) at the stationary "point".}

Notice that because of (2.9) and (2.10), the second variations have no surface terms. Since the terms at \(t = t_0\) or \(t_1\) do not contribute either, we have:

\[
[\Delta^2 A + \int_{t_0}^{t_1} (\vec{b} \cdot \Delta^2 \vec{P} + \vec{\Omega}_c \cdot \Delta^2 \vec{J}) dt]_0 = \int \int [-\Delta \vec{O}_D \cdot \dot{\vec{\xi}} + \Delta U(\Delta \nu - \vec{v} \cdot \vec{\xi})]_0 dt d^3 x
\]

(6.3)

in which

\[
\Delta U|_0 = \Delta [\vec{\nabla} \cdot (\rho \vec{U})]|_0 = \vec{\nabla} \cdot [\rho_0(\vec{\nabla} \Delta \nu - \dot{\vec{\xi}} - (\vec{\nabla} \vec{\xi} \cdot \vec{U}_0 + \vec{U}_0 \cdot \vec{\nabla} \vec{\xi}) - \vec{\nabla} \vec{\xi} \cdot \vec{\eta})]
\]

where \(\vec{\eta}_c\) is the vector defined in (2.3); of \(\Delta \vec{O}_D\) we write only those parts susceptible to contribute to the gyroscopic term:

\[
(\int \int -\Delta \vec{O}_D \cdot \dot{\vec{\xi}} \rho_0 dt d^3 x)_{\text{Gyro}} = \int \int \{\dot{\vec{\xi}} \cdot (\vec{\nabla} \Delta \nu - \vec{\nabla} \vec{\xi} \cdot \vec{W}_0) + [(\vec{\xi} - \vec{\nabla} \Delta \nu + \Delta XX X X
\]

6.2 Gyroscopic Term for the Strong Principle.

In the strong energy and Action principle, \(\nu\) is defined by \(U = 0\) and \(\Delta \nu\) is defined by \(\Delta U|_0 = 0\). \(\Delta \nu\) may be decomposed into a "dynamical contribution" \(\Delta \nu_D\) which is zero when \(\dot{\vec{\xi}} = 0\) and a "steady" part \(\Delta \nu_S\):

\[
\Delta \nu = \Delta \nu_D + \Delta \nu_S
\]

(6.6)

Similarly, \(\vec{\eta}_c\) has a dynamical and steady contribution (see our remarks at the end of section 5.3) and we write
\[ \Delta \vec{n}_c = \Delta \vec{n}_{cD} + \Delta \vec{n}_{cS} \quad (6.7) \]

\[ \Delta U|_0 = 0 \] gives the following two equations for \( \Delta \nu_D \) and \( \Delta \nu_S \) deduced from (6.4)

\[ \vec{\nabla} \cdot (\rho_0 \vec{\nabla} \Delta \nu_D) = \vec{\nabla} \cdot [\rho_0 (\dot{\vec{\xi}} + \Delta \vec{n}_{cD}) ] \quad (6.8a) \]

\[ \vec{\nabla} \cdot (\rho_0 \vec{\nabla} \Delta \nu_S) = \vec{\nabla} \cdot [\rho_0 (\tilde{U}_0 \cdot \vec{\nabla} \vec{\xi} + \vec{\nabla} \vec{\xi} \cdot \tilde{U}_0) + \rho_0 \vec{\nabla} \vec{\xi} \cdot \vec{n}_c + \Delta \vec{n}_{cS} )] \quad (6.8b) \]

Gyroscopic terms in (6.3) will only come from \( \Delta \vec{O}_D \) since \( \Delta U|_0 = 0 \). Using eq. (6.8) the bilinear terms of \( [\Delta^2 A + \int_{t_0}^{t_1} (\vec{b} \cdot \Delta^2 \vec{F} + \vec{\Omega}_c \cdot \Delta^2 \vec{J}) dt]_0 \), which we denote by \( \Delta^2 G_{strong} \) are

\[ \Delta^2 G_{strong} = 2 \int \int [\vec{\nabla} \Delta \nu_D \cdot \vec{\nabla} \Delta \nu_S - \dot{\vec{\xi}} \cdot \vec{\nabla} \vec{\xi} \cdot \tilde{W}_0] \rho_0 d^3x dt \]

\[ - \int \int [\Delta \vec{n}_{cD} \cdot \vec{\nabla} \Delta \nu_S + \Delta \vec{n}_{cS} \cdot \vec{\nabla} \Delta \nu_D + \Delta \vec{n}_{cD} \cdot \vec{\nabla} \vec{\xi} \cdot \tilde{W}_0 \]

\[ + (\Delta \vec{\Omega}_{cD} \times \tilde{W}_0 + \vec{\Omega}_c \times \vec{\nabla} \Delta \nu_D) \cdot \vec{\xi}] \rho_0 d^3x dt, \quad (6.9) \]

This expression becomes somewhat simpler for motions that are steady in inertial coordinates \( (\vec{n}_c = 0) \), when, for some reason, \( \Delta \vec{n}_c \) does not contribute either:

\[ \Delta^2 G_{strong} = 2 \int \int [\vec{\nabla} \Delta \nu_D \cdot \vec{\nabla} \Delta \nu_S - \dot{\vec{\xi}} \cdot \vec{\nabla} \vec{\xi} \cdot \tilde{W}_0] \rho_0 d^3x dt \quad (6.10) \]

We conclude that the strong energy principle provides a necessary and sufficient condition of stability if:

\[ \Delta^2 E_{strong} > 0 \quad \text{and} \quad \Delta^2 G_{strong} = 0 \quad (6.11) \]
6.3 Gyroscopic Term and the Weak Energy Principle.

The weak Lagrangian is linear in time derivatives; $L$ is of the form $G - V$ since $T = 0$. The perturbed Action has no quadratic term in $\dot{\xi}$ which appears only in $\Delta^2 G_{weak}$. As a result, if $\Delta^2 G_{weak} \neq 0$, $\Delta^2 E_{weak} > 0$ is certainly a sufficient condition of stability (this can easily be shown). However, if $\Delta^2 G_{weak} = 0$ and $\Delta^2 E_{weak} > 0$, the linearized equation of motion have no solution at all! Usual arguments about stabil

It is than still interesting to obtain $\Delta^2 G_{weak}$ for, however, very different reasons than we wanted $\Delta^2 G_{strong}$. In the weak Action principle, $\bar{P}$ and $\bar{J}$ do not define $\bar{b}$ and $\bar{\Omega}_c$ [see (5.18)] because $\nu$ is independent of them. Instead of describing the motion in moving coordinates, we rather stay in inertial coordinates and fix $\bar{P} = \bar{J} - \bar{J}_0 = 0$ with Lagrange multipliers, say, $\bar{b}(t)$ and $\bar{\Omega}_c(t)$. The Action is then

$$A^\dagger = A + \int [\bar{b}(t) \cdot \bar{P} + \bar{\Omega}_c(t) \cdot (\bar{J} - \bar{J}_0)] dt \quad (6.12)$$

Among all the dynamical perturbations, we consider only those for which $\bar{P} = \bar{J} - \bar{J}_0 = 0$. The formal expression for $\Delta^2 A^\dagger$ is than exactly the same as the right hand side of (6.3).

The gyroscope term $\Delta^2 G_{weak}$

$$<\Delta^2 G>_{weak} = \int \int [(2\ddot{\xi} + \Delta \ddot{\eta}_{cD}) \cdot (\bar{\nabla} \Delta \nu - \bar{\nabla} \dot{\xi} \cdot \bar{W}_0) - \Delta \ddot{\Omega}_{cD} \times \bar{W}_0 \cdot \dot{\xi}] \rho_0 d^3 x dt \quad (6.13)$$

We conclude that a sufficient condition for stability is

$$\Delta^2 E_{weak} > 0 \quad and \quad \Delta^2 G_{weak} \neq 0 \quad (6.14)$$

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FIGURE CAPTIONS

Fig. 1. Representation of a trial configuration indicating a surface of constant load $\lambda = \lambda_0$, a cut of constant $\beta = \beta_0$ hanging on the central “string” and the line $\lambda = 0$. The point $P$, at a shortest distance from the $z$-axis helps to define the orientation of the $x, y$ plane.