Inversion formula and Parseval theorem for complex continuous wavelet transforms studied by entangled state representation

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In a preceding Letter (Opt. Lett. 32, 554 (2007)) we have proposed complex continuous wavelet transforms (CCWTs) and found Laguerre–Gaussian mother wavelets family. In this work we present the inversion formula and Parseval theorem for CCWT by virtue of the entangled state representation, which makes the CCWT theory complete. A new orthogonal property of mother wavelet in parameter space is revealed.

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I. INTRODUCTION

Wavelet transforms (WTs) are very useful in signal analysis and detection [1, 2, 3] since it can overcome the shortcomings of nonlocality behavior of classical Fourier analysis and thus enriches the theory of Fourier optics [4]. The continuous WT of a signal function \(f(x) \in L^2(\mathbb{R})\) by a mother wavelet \(\psi(x)\) (restricted by the admissibility condition \(\int_{-\infty}^{\infty} \psi(x) \, dx = 0\)) is defined by

\[
W_f(\mu, s) = \frac{1}{\sqrt{C_0}} \int_{-\infty}^{\infty} f(x) \, \psi^*(\frac{x-s}{\mu}) \, dx,
\]

where \(\mu > 0\) is a scaling parameter and \(s \in \mathbb{R}\) is a translation parameter. The inversion of (1) is

\[
f(x) = \frac{1}{C_0} \int_{0}^{\infty} \frac{d\mu}{\mu^2} \int_{-\infty}^{\infty} W_f(\mu, s) \, \psi(x-s/\mu) \, ds \sqrt{\mu},
\]

where \(C_0 = \int_{0}^{\infty} |\psi(p)|^2 \, dp < \infty\) and \(\psi(p)\) is the Fourier transform of \(\psi(x)\), for proving (2) we have employed the Dirac’s representation theory [5], which has the merit of rigour and simplicity.

In Ref. [6, 7], Fan and Lu have linked the one-dimensional (1D) WT with the unitary transform (squeezing and displacement) in quantum mechanics, i.e., expressing the WT as a matrix element of the single-mode squeezing-displacing operator between the mother wavelet state vector \(\langle \psi \rangle\) and the state vector to be transformed, such that the admissibility condition for mother wavelets is examined in the context of quantum mechanics, in so doing a family of the Hermite–Gaussian mother wavelets are found. Further, by introducing the bipartite entangled state representation \(|\eta\rangle\) [8]

\[
|\eta\rangle = \exp\left(\frac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right)|00\rangle,
\]

Fan and Lu then proposed the continuous complex wavelet transforms (CCWT) for \(g(\eta) \equiv \langle \eta | g \rangle\),

\[
W_\psi g(\mu, \kappa) = \frac{1}{\mu} \int \frac{d^2\eta}{\pi} \, g(\eta) \, \psi^*(\frac{\eta-\kappa}{\mu}),
\]

where \(\kappa \in \mathbb{C}\). Correspondingly, the admissibility condition for mother wavelets, \(\int_{-\infty}^{\infty} \psi(\eta) = 0\), is examined in the entangled state representations and a family of new mother wavelets (named the Laguerre–Gaussian wavelets) are found to match the CCWT [9], i.e., the shortcomings of nonlocality behavior of classical Fourier analysis and detection [1, 2, 3] since it can overcome the contrast to the direct-product of two single-mode squeezing (displacement) operators, and the two-mode squeezed state is simultaneously an entangled state.

In order to complete the CCWT theory, we must ask if the corresponding Parseval theorem exists. This is important since the inversion formula of CCWT may appear as a lemma of this theorem. We shall solve this issue by virtue of the merits of entangled state in quantum mechanics, to be more specific, we shall use the property that the two-mode squeezing operator has its natural representation in the entangled state basis (see [10] below).

Noting that CCWT involves two-mode squeezing transform, so the corresponding Parseval theorem differs from that of the direct-product of two 1D wavelet transforms, too.
II. THE QUANTUM MECHANICAL VERSION OF CCWT

Let us begin with putting the CCWT into the context of quantum mechanics. Based on the idea of quantum entanglement initiated by Einstein-Podolsky-Rosen (EPR) [13], Fan and Klauder constructed the entangled state representation in two-mode Fock space $|\eta\rangle$ in (3) as the common eigenvector of two particles' relative position $X_1 - X_2$ and their momentum $P_1 + P_2$,

$$\langle X_1 - X_2 | \eta \rangle = \sqrt{2\eta_1} |\eta\rangle, \quad \langle P_1 + P_2 | \eta \rangle = \sqrt{2\eta_2} |\eta\rangle,$$

where $X_j = (a_j + a_j^\dagger)/\sqrt{2}$, $P_j = (a_j - a_j^\dagger)/\sqrt{2i}$, ($j = 1, 2$). $|\eta\rangle$ is complete $\int d^2\eta |\eta\rangle \langle \eta | = 1$ ($d^2\eta \equiv d\eta_1 d\eta_2$, $\eta = \eta_1 + i\eta_2$), and orthonormal $|\eta\rangle |\eta\rangle' = \pi \delta(\eta - \eta') \delta(\eta' - \eta''') \equiv \pi \delta(\eta - \eta')$.

Using $|\eta\rangle$ and $\psi(\eta) = \langle \eta | \psi \rangle$ we can recast the CCWT in (4) as

$$W_\psi g(\mu, \kappa) = \langle \psi | U_2(\mu, \kappa) | \eta \rangle,$$

and $U_2(\mu, \kappa)$ is a two-mode squeezing-translation operator, which has its natural expression in EPR entangled state representation,

$$U_2(\mu, \kappa) \equiv \frac{1}{\mu} \int \frac{d^2\eta}{\pi} \left| \frac{\eta - \kappa}{\mu} \right\rangle \langle \eta |,$$

when $\kappa = 0$, $U_2(\mu, 0) = S_2$.

III. PARSEVAL THEOREM IN THE CCWT

Now let us prove the Parseval theorem for CCWT,

$$\int_0^\infty \frac{d\mu}{\mu^2} \int \frac{d^2\kappa}{\pi} W_\psi g_1(\mu, \kappa) W_\psi g_2(\mu, \kappa) = C'_\psi \int \frac{d^2\eta}{\pi} g_1(\eta) g_2(\eta) U_2(\mu, \kappa) |\xi\rangle$$

where $\kappa = \kappa_1 + i\kappa_2$,

$$C'_\psi = 4 \int_0^\infty \frac{d|\xi|}{|\xi|} |\psi(\xi)|^2.$$

$\psi(\xi)$ is the Fourier transform of $\psi(\eta)$, a mother wavelet.

According to (10) and (9) the quantum mechanical version of Parseval theorem should be

$$\int_0^\infty \frac{d\mu}{\mu^2} \int \frac{d^2\kappa}{\pi} \langle \psi | U_2(\mu, \kappa) | g_1 \rangle \langle g_2 | U_2(\mu, \kappa) | \psi \rangle = C'_\psi \langle g_2 | g_1 \rangle,$$

where $\psi(\eta) = \langle \eta | \psi \rangle$, so $\psi(\xi) = \langle \xi | \psi \rangle$, $|\xi\rangle$ is the conjugate state to $|\eta\rangle$,

$$|\xi\rangle = \exp\left\{-\frac{1}{2}|\eta|^2 + \xi a_1^\dagger + \xi^* a_2^\dagger - a_1^\dagger a_2^\dagger \right\} |00\rangle$$

$$= (-1)^{a_1 a_2} |\eta\rangle_{\eta = \xi}, \quad \xi = \xi_1 + i\xi_2,$$

which is the common eigenstate of center-of-mass coordinate and the relative momenta operators, i.e.,

$$\langle X_1 + X_2 | \xi \rangle = \sqrt{2\xi_1} |\xi\rangle, \quad \langle P_1 - P_2 | \xi \rangle = \sqrt{2\xi_2} |\xi\rangle,$$

and is complete

$$\int \frac{d^2\xi}{\pi} |\xi\rangle \langle \xi | = 1.$$  

The overlap between $|\xi\rangle$ and $|\eta\rangle$ is [14]

$$\langle \xi | |\eta\rangle = \frac{1}{2} \exp\left\{\frac{1}{2}(\xi^* \eta - \eta^* \xi) \right\} = \frac{1}{2} \exp\left\{i(\xi_1 \eta_2 - \xi_2 \eta_1) \right\}.$$  

Using (11) and (17), we have

$$\psi(\xi) = 4 \pi \int \frac{d^2\eta}{\pi} |\xi\rangle \langle \eta | \psi \rangle$$

$$= \int \frac{d^2\eta}{2\pi} \exp\left\{\xi^* \eta - \xi \eta^* \right\} 2 |\psi(\eta)|^2.$$  

Eq. (11) indicates that once the state vector $|\psi\rangle$ corresponding to mother wavelet is known, for any two states $|\alpha\rangle$ and $|\beta\rangle$, their overlap up to the factor $C_\psi$ (determined by (12)) is just their corresponding overlap of CCWTs in the $(\mu, \kappa)$ parametric space.

Proof of Eq. (11) or (13)

We start with calculating $U_2^\dagger(\mu, \kappa) |\xi\rangle$. Using (10) and (17), we have

$$U_2^\dagger(\mu, \kappa) |\xi\rangle = \frac{1}{\mu} \int \frac{d^2\eta}{\pi} |\xi\rangle \langle \eta | \psi \rangle$$

$$= \frac{1}{\mu} |\xi\rangle e^{\frac{\pi}{4}(\xi_1 \eta_2 - \xi_2 \eta_1)}.$$  

it follows

$$\int_0^\infty \frac{d\mu}{\mu^2} \int \frac{d^2\kappa}{\pi} \langle \psi | U_2(\mu, \kappa) | g_1 \rangle \langle g_2 | U_2(\mu, \kappa) | \psi \rangle = C'_\psi \langle g_2 | g_1 \rangle.$$  

Using (10) and (20) the left-hand side (LHS) of (13) can
be reformulated as

\[ LHS \text{ of Eq.} (13) = \int_0^\infty \frac{d\mu}{\mu^3} \int \frac{d^2\xi d^2\xi'}{\pi^2} \langle \psi | \xi \rangle \times (\xi | U_2 (\mu, \kappa) | g_1) (g_2 \xi', \kappa) | \xi' | \psi \rangle = \int_0^\infty \frac{d\mu}{\mu^3} \int \frac{d^2\xi d^2\xi'}{\pi^2} (g_2 \xi', \kappa) | \xi' | g_1 \psi (\xi') \delta (\xi'_1 - \xi_1) \delta (\xi'_2 - \xi_2) \]

where the integration value in \{..\} is actually \( \xi \)-independent. Noting that the mother wavelet \( \psi (\eta) \) in Eq. (5) is just the function of \( |\eta| \), so \( \psi (\xi) \) is also the function of \( |\xi| \). In fact, using Eqs. (5), (6), and (13), we have

\[ \psi (\xi) = e^{-1/2|\xi|^2} \sum_{n=0}^{\infty} K_{n,n} H_{n,n} (|\xi|, |\xi|), \tag{22} \]

where we have used the integral formula

\[ \int \frac{d^2\xi}{2\pi} e^{\xi^2 \xi + z^2} = \frac{1}{\xi} e^{-\xi^2} \exp \left( \frac{1}{2} \xi \right), \quad \text{Re} (\zeta) < 0. \tag{23} \]

So we can rewrite (21) as

\[ LHS \text{ of Eq.} (13) = C'_{\psi} \int \frac{d^2\xi}{\pi} \langle g_2 | \xi \rangle \langle \xi | g_1 \rangle = C'_{\psi} \langle g_2 | g_1 \rangle, \tag{24} \]

where

\[ C'_{\psi} = 4 \int_0^\infty \frac{d\mu}{\mu^3} |\psi (\mu \xi)|^2 = 4 \int_0^\infty \frac{d\xi}{|\xi|} |\psi (\xi)|^2. \tag{25} \]

Then we can complete the proof of the Parseval theorem for CCWT in (13). Here, we should emphasize that (13) is not only different from the product of two 1D WTs, but also different from the usual wavelet transform in 2D.

When \( |g_2| = |\eta| \), by using (10) we see

\[ \langle \eta | U_2 (\mu, \kappa) | \psi \rangle = \frac{1}{\mu} \psi \left( \frac{\eta - \kappa}{\mu} \right), \tag{26} \]

then substituting it into (13) yields

\[ g_1 (\eta) = \frac{1}{C'_{\psi}} \int_0^\infty \frac{d\mu}{\mu^3} \int \frac{d^2\kappa}{\pi \mu} W_\psi g_1 (\mu, \kappa) \psi \left( \frac{\eta - \kappa}{\mu} \right) \]

which is just the inverse transform of the CCWT.

Especially, when \( |g_1| = |g_2| \), (13) reduces to

\[ \int_0^\infty \frac{d\mu}{\mu^3} \int \frac{d^2\kappa}{\pi \mu} |W_\psi g_1 (\mu, \kappa)|^2 = C'_{\psi} \int \frac{d^2\eta}{\pi} |g_1 (\eta)|^2, \tag{27} \]

or

\[ \int_0^\infty \frac{d\mu}{\mu^3} \int \frac{d^2\kappa}{\pi \mu} |\langle \psi | U_2 (\mu, \kappa) | g_1 \rangle|^2 = C'_{\psi} \langle g_1 | g_1 \rangle, \tag{28} \]

which is named isometry of energy.

### IV. NEW ORTHOGONAL PROPERTY OF MOTHER WAVELET IN PARAMETER SPACE

On the other hand, when \( |g_1| = |\eta| \), \( |g_2| = |\eta'| \), (13) becomes

\[ \frac{1}{C'_{\psi}} \int_0^\infty \frac{d\mu}{\mu^3} \int \frac{d^2\kappa}{\pi} \psi \left( \frac{\eta' - \kappa}{\mu} \right) \psi^* \left( \frac{\eta - \kappa}{\mu} \right) = \pi \delta (\eta - \eta'), \tag{29} \]

which is a new orthogonal property of mother wavelet in parameter space spanned by \( (\mu, \kappa) \). In a similar way, we take \( |g_1| = |g_2| = \eta, n \), a two-mode number state, since \( \langle m, n | m, n \rangle = 1 \), then we have

\[ \int_0^\infty \frac{d\mu}{\mu^3} \int \frac{d^2\kappa}{\pi} |\langle \psi | U_2 (\mu, \kappa) | m, n \rangle|^2 = C'_{\psi}, \tag{30} \]

or take \( |g_1| = |g_2| = |z_1, z_2 \rangle \), \( |z\rangle = \exp \left( - |z|^2 / 2 + za^\dagger \right) |0\rangle \) is the coherent state, then

\[ \int_0^\infty \frac{d\mu}{\mu^3} \int \frac{d^2\kappa}{\pi} |\langle \psi | U_2 (\mu, \kappa) | z_1, z_2 \rangle|^2 = C'_{\psi}. \tag{31} \]

This indicates that \( C'_{\psi} \) is \( |g_1| \)-independent, which coincides with the expression in (12).

Next we examine a special example. When the mother wavelet \( \psi (\eta) \) is taken as the following form

\[ \psi_M (\eta) = \langle \eta | \psi \rangle = e^{-1/2|\eta|^2} \left( 1 - \frac{1}{2} |\eta|^2 \right), \tag{32} \]

which is different from \( e^{-(x^2 + y^2)/2} (1 - x^2) (1 - y^2) \), the direct-product of two 1D Mexican hat wavelets (we name entangled mexican hat wavelets (EMHWs)), using (18) we have

\[ \psi (\xi) = \frac{1}{2} |\xi| e^{-\xi^2}, \tag{33} \]

which leads to

\[ C'_{\psi} = \int_0^\infty |\xi|^3 e^{-|\xi|^2} d\xi = \frac{1}{2} \tag{34} \]

Thus for the EMHWs (32), we see

\[ 2 \int_0^\infty \frac{d\mu}{\mu^3} \int \frac{d^2\kappa}{\pi} \psi_M \left( \frac{\eta' - \kappa}{\mu} \right) \psi_M^* \left( \frac{\eta - \kappa}{\mu} \right) = \pi \delta (\eta - \eta'). \tag{35} \]

Eq. (35) can be checked as follows. Using (32) and the integral formula

\[ \int_0^\infty \frac{1-u^2}{2} \left( 1 - \frac{uy^2}{2} \right) e^{-u^2/2} du = -\frac{4(4x^2+4y^2+y^2)}{(x^2+y^2)^4}, \quad \text{Re} (x^2 + y^2) > 0, \tag{36} \]

The end.
we can put the left-hand side (LHS) of (35) into

\[
\text{LHS of (35)} = -\int_{0}^{\infty} \frac{d^2\kappa}{\mu^2} \int \frac{d^2\kappa}{\pi} e^{-\frac{x^2+y^2}{2\mu^2}} \left( 1 - \frac{x^2}{2\mu^2} \right) \left( 1 - \frac{y^2}{2\mu^2} \right) e^{-\frac{x^2+y^2}{2}} \]

\[
= \int_{0}^{\infty} u du \int \frac{d^2\kappa}{\pi} \left( 1 - \frac{ux^2}{2} \right) \left( 1 - \frac{uy^2}{2} \right) e^{-\frac{x^2+y^2}{2}} \]

\[
= -\int \frac{d^2\kappa}{\pi} 4(x^2 - 4xy^2 + y^4)/(x^2 + y^2)^4, \tag{37}
\]

where \(x^2 = |\eta' - \kappa|^2, y^2 = |\eta - \kappa|^2\).

When \(\eta' = \eta, x^2 = y^2\),

\[
\text{LHS of (35)} = \int \frac{d^2\kappa}{2\pi |\kappa - \eta|^4} = \int_{0}^{\infty} \int_{0}^{2\pi} \frac{dr d\theta}{2\pi r^3} \to \infty. \tag{38}
\]

On the other hand, when \(\eta \neq \eta'\) and noticing that

\[
x^2 = (\eta_1' - \kappa_1)^2 + (\eta_2' - \kappa_2)^2, \]
\[
y^2 = (\eta_1 - \kappa_1)^2 + (\eta_2 - \kappa_2)^2, \tag{39}
\]

which leads to

\[
dx^2 dy^2 = 4 |J| d\kappa_1 d\kappa_2, \tag{40}
\]

where \(J(x, y) = \left| \begin{array}{cc} \kappa_1 - \eta_1' & \kappa_2 - \eta_2' \\ \kappa_1 - \eta_1 & \kappa_2 - \eta_2 \end{array} \right|\).

As a result of (39), (37) reduces to

\[
\text{LHS of (35)} = -4 \int_{-\infty}^{\infty} \frac{dx dy}{\pi} \frac{x^4 - 4xy^2 + y^4}{|J|(x^2 + y^2)^4} = 0, \tag{41}
\]

where we have noticed that \(J(x, y)\) is the function of \((x^2, y^2)\). Thus we have

\[
\text{LHS of (35)} = \left\{ \begin{array}{ll} \infty, & \eta = \eta', \\
0, & \eta \neq \eta'. \end{array} \right. = \text{RHS of (35)}. \tag{42}
\]

In sum, we have proposed the Parseval theorem corresponding to the CCWT in the context of quantum mechanics. Our calculations are simplified greatly by using the quantum state representations of two-mode squeezing operators. Finally, we should emphasize that since the CCWT corresponds to two-mode squeezing transform which differs from two single-mode squeezing operators’ direct product, the Parseval theorem of CCWT defined in this paper differs from that of the direct product of two 1-dimensional wavelet transforms.

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