Fast and Reliable Dispersal of Crash-Prone Agents on Graphs

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Abstract

We study the ability of mobile agents performing simple local computations to completely cover an unknown graph environment while implicitly constructing a distributed spanning tree. Whenever an agent moves, it may crash and disappear from the environment. The agents activate autonomously at exponential waiting times of mean 1 and enter the graph over time at a source vertex \( s \). They are able to settle at vertices of the graph and mark a neighbour. The agents are identical and make decisions driven by the same local rule of behaviour. The local rule is only based on the presence of neighbouring agents, and whether a neighbour marks the agent’s current location.

An implicit spanning tree is gradually constructed by having certain agents settle and act as its vertices, marking their parent. Each vertex in the environment has limited physical space and may contain at most a settled and an unsettled agent. Our goal is to show that even under conditions of asynchronicity, frequent crashing, and limited physical space, the simple mobile agents successfully cover the environment and construct a distributed spanning tree in linear time. Specifically, we show that, assuming at most \( ct/4 \) crashes happen before time \( t \) for all times \( t \), the agents can complete the tree asymptotically almost surely in \( 8((1-c)^{-1} + o(1))n \) time where \( n \) is the number of vertices. The analysis relates our algorithm to the “totally asymmetric simple exclusion process” in statistical mechanics.

1 Introduction

In swarm robotics, a vast number of autonomous mobile robots cooperate to achieve complex goals [19]. Individual members of the swarm are usually assumed to be simple, expendable, and computationally limited, and to move and act according to local rules of behaviour.

Swarms are often claimed to be highly fault-tolerant, as redundancy can enable the swarm to go on with its mission even if some robots malfunction [28]. However, as the size of a robotic fleet grows, so too does the opportunity for error. We cannot expect to release a huge swarm of simple robots to an unknown environment without the frequent occurrence of failures such as dead battery, mechanical faults, destructive crashes, and traffic jams. If the number of errors scales with the number of robots, are swarms really worth the trouble?

In this work, we ask this question in the context of an algorithmic coverage problem for swarms. “Coverage” algorithms that enable a single- or multi-robot system to cover or explore unknown environments are an important topic in mobile robotics. There has been great interest in practical applications, for example, to search and rescue operations [13], and a rich body of theoretical work exists (we refer the reader to the surveys [4], [16]). A natural coverage problem for robotic swarms is sometimes called the uniform dispersal problem, introduced in [19]. In uniform dispersal, the swarm’s robots enter an unknown graph environment over time through one or several source vertices and are tasked to eventually occupy every vertex with a robot while avoiding collisions.
1.1 Contribution

We show that a swarm whose members frequently crash and block each other off can achieve uniform dispersal quickly and reliably in an arbitrary graph environment. Hence, we claim that in this setting many robots can win against many errors. The swarm consists of anonymous, identical, and autonomous mobile agents that enter the graph over time through a source vertex $s$. They act based on a simple rule of local behaviour (Algorithm 1). We assume they have the ability to mark neighbouring agents and to sense when an agent in their location is marked, and that they activate independently at exponential waiting times of mean 1. They do not communicate directly and rely only on sensing marking agents and the number of agents in neighbouring vertices.

We assume there can be at most two agents per vertex. Hence, due to asynchronicity, agents frequently block each other off. Agents can crash whenever they try to move, resulting in their deletion. We assume the number of crashes before time $t$ is bounded by $ct/4$, for fixed $0 \leq c < 1$. Larger environments require more agents and more time to cover. Since agents enter the environment over time, the more agents that are required for covering, the more crashes we must expect. Agents enacting our local rule arrive in the graph environment at a rate slightly faster than $t/4$ (the precise rate depends on the environment), so when $c$ is close to 1 the vast majority of agents to enter the environment will crash before achieving anything.

We show that dispersal completes in time $8 \cdot (1 - c)^{-1} + o(1)^n$ asymptotically almost surely, where $n$ is the number of vertices in the environment. No dispersal algorithm can complete in less than $O(n)$ expected time, since at least $n$ agents must first enter the graph, so when there are no crashes our algorithm’s performance is tight up to perhaps the factor 8. When crashes are allowed, we conjecture a more precise analysis will improve the $(1 - c)^{-1}$ factor (see Section 3.1).

In many mobile agent systems one wishes to construct a spanning tree of the environment for purposes of mapping, routing or broadcasting [1, 2, 10, 14, 15]. We do so implicitly by having agents act as nodes of the tree, and making them aware of their immediate descendants.

Our algorithm works also in a synchronous time setting, and when agents enter from multiple source vertices. Multiple source vertices result instead in the agents constructing an implicit spanning forest. In both these settings performance is faster.

By postulating that the agents activate at random rather than arbitrary waiting times, we depart from the way asynchronicity is often modelled in distributed algorithms (e.g., in [22]). This departure is necessary, as our results rely on randomness (or complete synchronicity). Our model of asynchronicity is inspired by the totally asymmetric simple exclusion process (TASEP) in statistical mechanics. TASEPs and their generalizations have been studied extensively as models of a great variety of transport phenomena, such as biological transport [11] and traffic flow [12], and many exact and asymptotic results are known [20, 27]. Our analysis relates our algorithm to TASEPs.

1.2 Previous work

Uniform dispersal was introduced by Hsiang, Arkin et al. in [19] for discrete grid environments. They considered a synchronous time setting where robots are allowed to send short messages to nearby robots, and showed time-optimal algorithms for this setting. Many variations have since been studied been studied. Barrameda et al. extends the problem to the asynchronous setting with no explicit communication [8, 7] (without definite performance guarantees). Recent works include dispersal with weakened sensing [17], dispersal in arbitrary graph environments [22], and dispersal while attempting to minimize individual robot movements [5]. Our model differs from previous work on several central points, including the presence of crashes, and the ability to mark neighbours (which is weaker than the communication available to robots in [19]). To our knowledge,
we are also the first to give an effective performance guarantee in a non-synchronous setting.

Robotic coverage, patrolling, and exploration with adversarial interference, as well as crashes, have been studied in different problem settings from our own. Handling crash faults in asychronous settings is discussed as a central direction of research for swarm robotics in [24]. Agmon and Peleg studied a gathering problem for robots where a single robot may crash [3], and gathering with multiple crashes was later discussed by Zohir et al. in a similar setting [9]. Robotic exploration in an environment containing threats has been studied in [29, 30]. Moreover, adversarial crashes of processes are often studied in general distributed algorithms (e.g., [13]). Our problem and model are different from these works. We model crashes differently, and assume crashed robots are deleted from the environment. Such an assumption is applicable when crashed robots can be ignored or pushed aside, or the crash causes the robot to disappear, such as crashed air-based robots falling to the ground. We differ also in the scope of the crashes we allow: we account for situations where the number of crashes scales with the mission’s complexity (the time it takes to cover the environment), and where even the vast majority of robots may crash. However, to enable this, we assume access to a huge reservoir of robots waiting to replace crashed robots.

A fascinating introduction to TASEP-like processes and their connection to other fields is [21].

2 Model

We consider a swarm of mobile robotic agents performing world-embedded calculations on an environment represented by a connected graph $G$. The vertices of $G$ represent spatial locations, and the edges represent connections between these locations, such that the existence of an edge $(u, v)$ indicates that an agent may move from $u$ to $v$.

We assume an infinite collection of agents attempt to enter $G$ over time through a source vertex $s \in G$. The agents are identical and execute the same algorithm. They begin in the mobile state, and eventually enter the settled state. Settled agents are stationary, and are capable of marking another settled agent located at a neighbouring vertex. Mobile agents move between the vertices of $G$ and sometimes crash while in motion. They are oblivious, and decide where to move based only on local information provided by their sensors: the number of agents at neighbouring vertices, and whether any of the neighbouring settled agents mark their current location. Each vertex has limited capacity: it can contain at most one settled and one mobile agent. We find the image of a UAV that explores the environment before finally landing and becoming a beacon to be helpful.

Mobile agents are only allowed to move to a neighbouring vertex when they are activated. Each agent, including agents outside $G$, reactivates infinitely often and independent of other agents, at random exponential waiting times of mean 1.

When the source vertex contains less than two agents, an agent from outside $G$ that was activated attempts to enter $s$. It is convenient to give the agents arbitrary labels $A_1, A_2, \ldots$ and assume that $A_i$ cannot enter $s$ before all agents with lower indices entered or crashed. This assumption makes the analysis simpler, but the performance bound we prove in this work holds also for the entrance model where agent entrance depends only on which agent is activated first. Hence, whenever the current lowest-index agent outside of $G$ activates and there is no mobile agent at $s$, it moves to $s$. If $s$ is completely empty, the agent settles upon arrival and becomes the root of the spanning tree. Otherwise it remains a mobile agent.

We denote by $G(t)$ the graph whose vertices are vertices of $G$ containing settled agents at time $t$. There is a directed edge $(u, v) \in G(t)$ if a settled agent at $u$ is marked by a settled agent at $v$. The goal of the agents is to reach a time $T$ wherein $G(T)$ is a spanning tree of the entire environment $G$. The makespan of an algorithm is the first time $T_0$ when this occurs.
Crashes are modelled as follows: when an agent $A_i$ is activated and attempts to enter $s$ or move from $u$ to $v$ via the edge $(u, v)$, occasionally an adversarial event occurs, causing the deletion of $A_i$ from $G$. Agents are safe while they reside at vertices—they do not crash unless attempting to move. We assume the number of adversarial events before time $t$ is bounded by a fraction of $t$. Adversarial events may otherwise be as “inconvenient” as possible: we may assume there is an adversary choosing crashes to maximize the makespan of our algorithm. We show an algorithm whose performance degrades gracefully with the density of adversarial events (Theorem 3.1).

Unless stated otherwise, when discussing the configuration of agents “at time $t$”, we always refer to the configuration before any activation at time $t$ has occurred.

3 Dispersal and Spanning Trees

We study a simple local behaviour that disperses agents and incrementally constructs a distributed spanning tree of $G$. We prove this algorithm’s makespan is linear in the number of vertices of $G$ asymptotically almost surely, and that performance degrades gracefully with the number of crashes.

Algorithm 1 Local rule for a mobile agent $A$.

Let $v$ be the current location of $A$ in $G$ (or undefined, if $A$ is outside $G$).

if a neighbour $u$ of $v$ contains exactly one agent, and this agent marks an agent in $v$ then

   Attempt to move to $u$.

else if a neighbour $u$ of $v$ contains no agents then

   Attempt to move to $u$ and become settled if no crash.

   Mark the vertex $v$ (unless just entered from outside $G$).

else

   Stay put.

end if

This rule of behaviour causes the growth of $G(t)$ as a partial spanning tree of $G$. It acts as a kind of depth first search that splits into parallel processes whenever a mobile agent is blocked by another mobile agent. Every vertex of the tree $G(t)$ is marked by settled agents at its descendants. Mobile agents follow these marks to discover the leaves of the current tree $G(t)$ and expand it. Agents grow the tree by settling at unexplored vertices that then become new leaves.

Our main result is the following theorem:

**Theorem 3.1.** If for all $t$ the number of adversarial events before time $t$ is allowed to be at most $ct/4$, $0 \leq c < 1$, then the makespan of Algorithm 1 over graph environments with $n$ vertices is at most $8((1 - c)^{-1} + o(1))n$ asymptotically almost surely as $n \to \infty$.

Figure 1 shows an execution of our algorithm on a grid environment with $n = 62$ square vertices (represented by the white region). We allowed an adversary to delete at most $ct/4$ agents before time $t$, with $c = 0.8$. This corresponded to a deletion of 56% of agents that entered the environment before the makespan. In a more constrained topology (such as a path graph, see Section 3.1.3), the agents would progress more slowly, and a greater percentage would be deleted under the same limitations on the adversary. The makespan (bottom right figure) was 613, consistent with the upper bound of Theorem 3.1. After the spanning tree completes, agents keep entering the region until there are two agents at every vertex (i.e., until every arrow in Figure 1 is red). This is related to the “slow makespan”, which we will later define. The slow makespan was 831.
Figure 1: An execution of Algorithm 1 on a grid environment. The source is denoted by a square box in the center. The arrows denote settled agents, and their direction points to the neighbouring settled agent they mark. Red arrows indicate a mobile agent is on top of the settled agent (note that by the algorithm, a mobile agent will never occupy a vertex that does not have a settled agent).

3.1 Analysis

We study the makespan of Algorithm 1. Due to space constraints, some of our claims in this section have their proofs placed instead in the Appendix.

For the analysis, we will assume that agents from $A_1, A_2, \ldots$ that settle or crash keep being activated. This is a purely “virtual” activation: such agents of course do and affect nothing upon being activated. We start with a structural Lemma:

Lemma 3.2. $G(t)$ is a tree at all times $t$ with probability 1.

Proof. When the first agent enters and successfully settles, $G(t)$ contains only $s$. No settled agents are ever deleted, so $G(t)$ can only gain new vertices. Whenever a mobile agent settles, it extends the tree $G(t)$ by one vertex, connecting its current location $v$ to $G(t)$ via a single directed edge. By definition, the edge is directed from the vertex the settled agent marks—which is its previous location—to $v$. This turns $v$ into a leaf of $G(t)$. With probability 1 no two agents on $G$ activate at the exact same time, so no two agents settle the same vertex. Hence $G(t)$ remains a tree. \qed

3.1.1 Event orders

We explain how we intend to bound the makespan. Our strategy shall be to use coupling to compare the performance of Algorithm 1 with that of agents moving on different structures. Coupling is a technique in probability theory for comparing different random processes (see [23]).

The basic idea is this: whenever we run Algorithm 1 on $G$, we can log the exact times at which the agents activate, as well as the times adversarial events happen and which agents they affect. This gives us an order of events $S$ sampled from some random distribution. Note that agents
keep activating forever (but these activations do nothing once the graph is full), so $S$ is infinitely
long. We then “re-enact” or “simulate” $S$ on a new environment (or several new environments)
involving the agents $A_1, A_2, \ldots$ by activating and deleting the agents according to $S$.

To make things more precise, by “simulating” $S$ on different environments we mean that we
consider the coupled process $(G, G_2, \ldots, G_m)$ wherein different environments $G, G_2, \ldots, G_m$ have
agents that are paired such that whenever $A_i$ in $G$ is scheduled for an activation or a deletion
according to the event order $S$ (which is simply an infinite list of scheduled activation and deletion
times), the copy of $A_i$ in all the environments $G_2, \ldots G_m$ also activates or is deleted. When the
copies of $A_i$ are activated they do exactly what Algorithm 1 tells them to according to their local
neighborhood. Agents entrances are modelled as usual (Section 2), but note that even if $A_i$ manages
to enter $G$ following an activation, its copy might not enter one of the other environments because
in that environment the entrance is blocked, or there is a lower-index agent waiting to enter. During
Algorithm 1's analysis, we will often be talking about a deterministic event order $S$ being simulated
over different environments. The end-goal, however, is to say something about the event order $S$
when it is randomly sampled from the execution of Algorithm 1 on $G$.

The event order $S$ must be a possible set of events that occurred during an execution of our
algorithm on the base graph environment $G$. This means, due to our model, that an agent $A_i$ in
$G$ will never be scheduled for deletion except at times when it is activated and attempts to move.
However, while simulating $S$ on the environments $G_2, \ldots G_m$, we must be allowed to break the rules
of the model: we might delete agents even when they don’t attempt to move, or while they are
outside of the new graph environment. Whenever we say “for any event order $S$”, we mean event
orders $S$ that could have happened over $G$.

In $S$, define $t_0$ to be the first time $A_1$ activates, $t_1$ to be the first time after $t_0$ that either $A_1$ or
$A_2$ activate, and $t_i$ to be the first time $t > t_{i-1}$ that any agent in the set $\{A_1, \ldots, A_{i+1}\}$ is activated.

**Definition 3.3.** The times $t_0 < t_1 < t_2 < \ldots$ in $S$ are called the meaningful event times of $S$.

For meaningful event times to be well-defined there must be a minimal time $t > t_{i-1}$ where
one of the agents $A_1, \ldots, A_i$ activates. Because the activation times of the agents are independent
exponential waiting times of mean 1, this is true with probability 1 for a randomly sampled $S$.
Moreover, with probability 1, at any time $t_i$ there is precisely one agent $A$ of $A_1, A_2, \ldots, A_{i+1}$
scheduled for activation by $S$. Because both these things are true with probability 1, we assume
they are true for any event order $S$ referred to at any point in this analysis. This does affect our
main result (Theorem 3.1), which is probabilistic.

Our end-goal is randomly sample $S$ from $G$ and simulate it on 4 increasingly “slower”
environments: $\mathcal{P}(n), \mathcal{P}(\infty), \mathcal{P}^*(\infty), B$, so that the 5 environments ($G$ and these four) are coupled.
Meaningful event times are so called because, prior to the first activation of $A_i$, any of the agents
$A_{i+1}, A_{i+2}, \ldots$ cannot enter or move in any of these environments, and activating them causes noth-
ing. Hence, at any time $t$ which is not a meaningful event time, the configuration of agents cannot
change (no agents move and no agents are deleted in any of the environments $S$ is simulated on).

The possibility to create an event order $S$ is the only reason we labelled the agents and made
the assumption about entrance orders in Section 2.

### 3.1.2 $\mathcal{P}(n)$ versus $G$

Let $n$ be the number of vertices of $G$. The path graph $\mathcal{P}(n)$ over $n$ vertices is a graph over the
vertices $v_1 v_2 \ldots v_n$ such that there is an edge $(v_i, v_{i+1})$ for all $1 \leq i \leq n - 1$. We simulate $S$ on the
graph environment $\mathcal{P}(n)$ where the source vertex $s$ is $v_1$. Simulating $S$ on $\mathcal{P}(n)$ results in what is
Figure 2: The processes $\mathcal{P}(n), \mathcal{P}(\infty), \mathcal{P}^*(\infty), \mathcal{B}$ that we will be interested in. White vertices are empty, black vertices contain a settled agent, and red vertices contain both a mobile and a settled agent. Edge directions indicate edge directions in $G(t)$. Note that $\mathcal{B}$ does not have a source vertex.

mostly a normal-looking execution of Algorithm 1 on $\mathcal{P}(n)$, but as discussed, it might lead to some oddities such as agents being deleted while they are still outside the graph environment.

Let us introduce some notation. $A_G^i$ refers to the copy of $A_i$ being simulated by $S$ on $G$, and $A_{\mathcal{P}(n)}^i$ is similarly defined.

Definition 3.4. The depth of $A_G^i$ at time $t$, written $d(A_G^i, t)$, is the number of times $A_G^i$ has successfully moved before time $t$. Depth is initially 0. Entering at $s$ is considered a movement, so agents entering $s$ have depth 1.

$d(A_{\mathcal{P}(n)}^i, t)$ is similarly defined with respect to $\mathcal{P}(n)$.

Definition 3.5. Let $T$ be a tree graph environment (such as $\mathcal{P}(n)$) with source vertex $s$. A vertex $v$ of $T$ becomes slow at time $t$ if a mobile agent on $v$ was activated and found no vertex it could move to, and also, either $v$ is a leaf of $T$ or all of its descendants in $T$ are slow at time $t$.

An agent $A_i$ is slow at time $t$ if it is located at a slow vertex at time $t$.

Definition 3.6. The slow makespan of $S$ on $T$, $M_{\text{slow}}^T$, is the first time all vertices of $T$ are slow when simulating the event order $S$.

$G$ is not always a tree, but given a fixed event order $S$, we can associate to $S$ a spanning tree of $G$, $T_S$, containing $G(t)$ as a subtree for all times $t$. Lemma 3.2 says agents only use edges of $T_S$, so we may define the slow makespan of $S$ on the $G$-simulation as the slow makespan on $T_S$. Slow makespan is clearly also defined for the $\mathcal{P}(n)$-simulation. Furthermore, $M_{\text{slow}}^G$ is an upper bound on the (regular) makespan of the $G$-simulation, since every vertex must have a settled agent before it becomes slow and, as the settled agents of $G$ never move, they cannot be deleted by $S$.

Lemma 3.7. A slow agent $A_G^i$ is forever unable to move and never deleted in the event order $S$.

Proof. Only agents attempting to move can be deleted. If $A_G^i$ is at a leaf of $T_S$, it can never move, since its parent vertex in $T_S$ contains a settled agent marking an agent in a different location, and settled agents are never deleted. Hence, $A_G^i$ is never deleted. Slow vertices propagate upwards from the leaves of $T_S$, so the statement of the lemma follows by induction.

Proposition 3.8. For any event order $S$, $M_{\text{slow}}^G \leq M_{\text{slow}}^{\mathcal{P}(n)}$.

We prove this proposition by induction on the meaningful event times $t_0, t_1, \ldots$ in the event order $S$. We show the following statements to be true for non-deleted agents at all times $t_m$:

(a) If $A_G^i$ is not slow or settled, then $d(A_G^i, t_m) \geq d(A_{\mathcal{P}(n)}^i, t_m)$.
(b) If \( A_i^{P(n)} \) is slow or settled, then \( A_i^{G}(t_m) \) is slow or settled, and \( d(A_i^{G}, t_m) \leq d(A_i^{P(n)}, t_m) \).

We note that both statements are (trivially) true at time \( t_0 \), as no event has occurred yet.

**Lemma 3.9.** If statement (b) is true up to time \( t_m \), settled and slow agents of \( P(n) \) neither move nor get deleted as a result of an event of \( S \) scheduled for time \( t_m \) (i.e., the agents still exist and are in the same place at time \( t_{m+1} \)).

Assuming (a) and (b) hold at all times, let us see how to infer Proposition 3.8. If statements (a) and (b) are true up to time \( t \), we do this by comparing simulations of \( S \) on different environments. To start, let \( P(\infty) \) be the path graph with infinite vertices, and where \( s = v_1 \). We may simulate \( S \) on \( P(\infty) \) as we did on \( P(n) \).

**Lemma 3.10.** If statements (a) and (b) are true up to time \( t_m \), statement (a) is true at time \( t_{m+1} \).

**Lemma 3.11.** If statements (a) and (b) are true up to time \( t_m \), statement (b) is true at time \( t_{m+1} \).

### 3.1.3 \( P(n) \) versus \( P(\infty) \)

We wish to bound the slow makespan \( M_{\text{slow}}^{P(n)} \) (which is determined by \( S \)). We do this by comparing simulations of \( S \) on different environments. To start, let \( P(\infty) \) be the path graph with infinite vertices, and where \( s = v_1 \). We may simulate \( S \) on \( P(\infty) \) as we did on \( P(n) \).

**Lemma 3.12.** For any event order \( S \) simulated on \( P(n) \) and \( P(\infty) \) and any time \( t < M_{\text{slow}}^{P(n)} \), \( P(n) \) and \( P(\infty) \) contain the exact same number of agents.

**Proof.** The configuration of agents in the first \( n \) vertices of \( P(n) \) and \( P(\infty) \) is identical until \( v_n \) becomes slow in \( P(n) \). After \( v_n \) becomes slow, the configuration of agents in the first \( n - 1 \) vertices is still the same in both graphs until an agent in \( v_{n-1} \) is prevented from moving by an agent in \( v_n \), meaning \( v_{n-1} \) becomes slow. By induction, the configuration of agents in the first \( k \) vertices of both graphs is identical until \( v_k \) in \( P(n) \) becomes slow (we use Lemma 3.9 to infer that the slow agents at \( v_{k+1} \) are never deleted). Hence, until \( v_1 \) becomes slow, agents enter at the same times in \( P(n) \) and \( P(\infty) \). \( v_1 \) becomes slow precisely at time \( M_{\text{slow}}^{P(n)} \).

### 3.1.4 \( P(\infty) \) versus \( P^*(\infty) \)

We simulate \( S \) on the environment \( P^*(\infty) \). \( P^*(\infty) \) is \( P(\infty) \) with the modification that there is at time \( t = 0 \) a settled agent at every vertex \( v_i \). The settled agent at \( v_i \) marks the settled agent at \( v_{i-1} \). These “dummy” agents are never activated, and are not of the indexed agents \( A_1, A_2, \ldots \). Because there is already a settled agent at every vertex, the agents \( A_1, A_2, \ldots \) never become settled. Call this environment \( P^*(\infty) \). Lemma 3.13 shows \( P^*(\infty) \) is strictly slower than \( P(\infty) \):

**Lemma 3.13.** For any event order \( S \) and at any time \( t \), the amount of mobile agents in \( P^*(\infty) \) at time \( t \) is at most the total amount of agents in \( P(\infty) \).
3.1.5 $\mathcal{P}^*(\infty)$ versus totally asymmetric simple exclusion

We bound the arrival rate of agents at $\mathcal{P}^*(\infty)$ by another, even slower process. This process, $B$, takes place on the path graph $\mathcal{P}(\infty)$ where we also have non-positive vertices $v_0, v_{-1}, v_{-2}, \ldots$, and such that there is an edge $(v_i, v_{i+1})$ for every $i$. Like $\mathcal{P}^*(\infty)$ there is initially a settled agent at every vertex, marking the vertex before it. Unlike the other processes, agents do not enter at $s$: the agent $A_i$ begins inside the graph environment as a mobile agent located at $v_{-i+1}$. To compare $B$ with $\mathcal{P}^*(\infty)$, we count the agents that cross the edge $(v_0, v_1)$. There is one more crucial feature of $B$: agents are never deleted from $B$. Scheduled agent deletions at $S$ are treated as a regular activation of the agent. Besides these differences, $S$ can be simulated on $B$ as before.

**Lemma 3.14.** For any event order $S$ and at any time $t$, the number of mobile agents that crossed the $(v_0, v_1)$ edge of $B$ is at most the number of agents that entered or were deleted before entering $\mathcal{P}^*(\infty)$.

Recall that $S$ is an event order of some execution of Algorithm 1 on the graph environment of interest, $G$. We may randomly sample $S$ by running Algorithm 1 on $G$ and logging the events.

The stochastic process resulting from simulating a randomly sampled event order $S$ on $B$ is called a totally asymmetric simple exclusion process (TASEP) with step initial condition, first introduced in [26]. In this process, agents (called also “particles”) are activated at exponential rate 1 and attempt to move rightward whenever no other agent blocks their path. This is precisely the outcome of simulating $S$ on $B$ (since agent activations that lead to a deletion in the other processes are treated as a regular activation in $B$).

In TASEP with step initial condition, let us write $B_t$ to denote the number of agents that have crossed $(v_0, v_1)$ at time $t$. It is shown in [25] that $B_t$ converges to $\frac{t}{4}$ asymptotically almost surely (i.e., with probability 1 as $t \to \infty$). [20] shows that the deviations are of order $t^{1/3}$. Specifically we have in the limit:

$$\lim_{t \to \infty} \mathbb{P}(B_t - \frac{t}{4} \leq 2^{-4/3}st^{1/3}) = 1 - F_2(-s)$$

Valid for all $s \in \mathbb{R}$, where $F_2$ is the Tracy-Widom distribution and obeys the asymptotics $F_2(-s) = O(e^{-c_1s^3})$ and $1 - F_2(s) = O(e^{-c_2s^{3/2}})$ as $s \to \infty$. We employ Equation 1 and the prior analysis to prove Theorem 3.1.

**Proof.** Let $G$ be a graph environment with $n$ vertices. Let $S$ be the randomly sampled event order of an execution of Algorithm 1 on $G$. We will bound the slow makespan, $M^G_{slow}$.

We simulate $S$ over the environments $\mathcal{P}(n), \mathcal{P}(\infty), \mathcal{P}^*(\infty)$, and $B$. From Lemma 3.14 we know that at all times the number of agents that crossed the $(v_0, v_1)$ edge of $B$, meaning $B_t$, is less than the number of agents that entered $\mathcal{P}^*(\infty)$ or were deleted before entering. At most $ct/4$ agents are deleted by time $t$, so the number of mobile agents at $\mathcal{P}^*(\infty)$ at time $t$ is at least $B_t - ct/4$. Lemmas 3.12 and 3.13 imply this is at least the number of agents at $\mathcal{P}(n)$ at any time $t < M^G_{slow}$.

At any time $t$, there cannot be more than $2n$ agents at $\mathcal{P}(n)$. Hence, if $B_t - ct/4 > 2n$, then $t \geq M^P(n)$. By Proposition 3.8, we shall then also have $t \geq M^G_{slow}$.

Write $t_n = 8((1 - c)^{-1} + n^{-1/3})n$. We are interested in $\mathbb{P}(B_{t_n} - 2n \leq 0)$, which we rewrite as

$$\mathbb{P}(B_{t_n} - \frac{t_n}{4} \leq \frac{1}{t_n^{1/3}}(2n - \frac{t_n}{4})t_n^{1/3})$$

As $n \to \infty$, $\frac{1}{t_n^{1/3}}(2n - \frac{t_n}{4})$ approaches $-\infty$, so Equation 1 implies (2) is 0 in the limit. By the above, this implies $t_n$ is an upper bound on $M^G_{slow}$ asymptotically almost surely.
In empirical evaluation, the slow makespan is often faster than estimated in the proof of Theorem 3.1, because agent deletions actually speed up the pace of the other agents by freeing space. We do not expect \( 8 \cdot (1 - c)^{-1} \) to be tight, and it would be interesting to know more exact bounds.

We describe some extensions of Algorithm 1 to different settings.

**Synchronous time.** We may consider a time setting that is discretized to steps \( t = 1, 2, \ldots \) such that at every step, all the agents activate at once. It can be shown for Algorithm 1 that there will be no collisions, i.e., no two agents will attempt to enter the same vertex. Analysis similar to the asynchronous case shows that agents then enter at rate \( t/2 \) (instead of approximately \( t/4 \)) on \( \mathcal{P}(n) \), and analogous reasoning to Lemma 3.8 and Theorem 3.1 gives an upper bound of \( 4(1 - c)^{-1} n \) on the makespan of a graph with \( n \) vertices, assuming \( c t/2 \) adversarial events. Consider the path graph \( \mathcal{P}(n) \) with \( s = v_2 \) (not the usual \( s = v_1 \)), and where the agents first fill the vertices \( v_3, v_4, \ldots \) with a double layer before reaching \( v_1 \). The synchronous makespan of this environment is asymptotically \( 4(1 - c)^{-1} n \). Hence, the bound on the makespan in the synchronous case is precise.

**Multiple source vertices.** Instead of just having a single source vertex \( s \), we may consider a setting with multiple source vertices such that each of them corresponds to its own set of agents \( A_1, A_2, \ldots \) entering over time. In the asynchronous setting, Lemma 3.2 can be generalized to show that \( G(t) \) is then a forest, and the agents attempt to create a spanning forest of \( G \). The technique in this paper can be generalized to show that the makespan bound of Theorem 3.1 holds. In general graph environments multiple sources may not improve the makespan by much. For example, consider the path graph \( \mathcal{P}(n) \) with \( k \) sources on \( v_1, v_2, \ldots, v_k \). The makespan of this graph is bounded below by the makespan of the path graph \( \mathcal{P}(n - k - 1) \) with a single source vertex \( v_1 \).

### 4 Discussion

It is often difficult to handle faulty agents. In swarm robotics, where one has access to an enormous robotic fleet, we must anticipate many faults, such as crashing. Because robots in the swarm are usually assumed to be autonomous and have limited computational power, it is important to ask whether there are simple rules of local behaviour that mitigate such failures. To this end, we investigated the problem of covering an unknown graph environment, and constructing an implicit spanning tree, with a swarm of frequently crashing agents. We showed a simple and local rule of behaviour that enables the swarm to quickly and reliably finish this task in the presence of crashes. The swarm’s performance degrades gracefully as crash density increases.

We outline here several directions for future research. First, our model interprets the “swarm” part of swarm robotics as a vast and redundant number of robots that we can disperse to the environment over time. We used this model for uniform dispersal, but it would be interesting to adapt it to other kinds of missions, and design algorithms for those missions that can handle crashes or other forms of interference. For example, in [3], mobile agents entering at a source vertex \( s \) over time sequentially pursue each other to discover shortest paths between \( s \) and some target node. The algorithm succeeds even if some of the agents are interrupted and have their location changed (but crashes are not handled).

Next, in this work, we made the simplifying assumption that the environment of the agents is represented as a discrete graph \( G \). If the agents instead attempted to cover a continuous planar domain by an algorithm similar to ours, the agents would need to construct a shared graph representation of the environment through the settled agents in \( G(t) \) and their markings. It would be interesting to see our algorithm extended to such settings.
Lastly, can we handle more than just crashes of robots in motion? There are many other situations and modes of failure that can be discussed, such as Byzantine agents, or dynamic changes to the graph environment.

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Appendix

Reminder:

(a) If $A_i^G$ is not slow or settled at time $t_m$, then $d(A_i^G, t_m) \geq d(A_i^{P(n)}, t_m)$.

(b) If $A_i^{P(n)}$ is slow or settled at time $t_m$, then $A_i^G(t_m)$ is slow or settled, and $d(A_i^G, t_m) \leq d(A_i^{P(n)}, t_m)$.

5.1 Proof of Lemma 3.9

Proof. Referring to the Lemma’s statement, we remind that here “time $t_m$” refers to the configuration of agents at time $t_m$ before any scheduled events. Hence, even if something is true at time $t_m$, we still need to show that it remains true after the events that happen at time $t_m$.

Let $A_i^{P(n)}$ be slow or settled at time $t_m$. To show $A_i^{P(n)}$ will not be deleted, it suffices to show the event order $S$ will not delete $A_i^G$. (b) implies $A_i^G$ is settled or slow at time $t_m$. Lemma 3.7 says $S$ never deletes slow agents. $S$ never deletes settled agents of $G$ as, in our model, agents are only deleted when they move, and $S$ obeys the rules of the model when simulated on $G$. Hence, $S$ will not delete $A_i^G$.

Next we show that $A_i^{P(n)}$ will not move as a result of an event scheduled for time $t_m$. If $A_i^{P(n)}$ is settled, this is true by definition. Otherwise, $A_i^{P(n)}$ is slow. By assumption, (b) is true at all times up to $t_m$. Hence, by the same reasoning as the above paragraph, agents of $P(n)$ that became slow or settled at or prior to time $t_m$ have not been deleted. Consequently, the argument of Lemma 3.7 applies also here, allowing us to conclude that agents cannot move after they become slow. In particular this applies to $A_i^{P(n)}$.

5.2 Proof of Lemma 3.10

Proof. Only one event occurs at time $t_m$. This event is either an uninterrupted activation of an agent (meaning the agent is not deleted), or an activation that leads to a deletion. If the event is a deletion, (a) holds at time $t_{m+1}$ trivially, so we assume that it is an uninterrupted activation.

Let $A_i^G$ and $A_i^{P(n)}$ be the agents that are activated at time $t_m$. The depth of any other agent is unchanged, so we need only verify (a) for these two agents. Assuming (a) it true at time $t_m$, it is only possible for (a) to become false at time $t_{m+1}$ if $A_i^G$ did not move, but $A_i^{P(n)}$ did. We assume this is the case.

If $A_i^G$ does not move as a result of its activation at time $t_m$, then either it is settled, in which case (a) is true and we are done, or there is a mobile agent at every neighbouring vertex in $G(t_m)$. If $A_i^G$ is mobile and all of its neighbours are slow at time $t_m$, then $A_i^G$ becomes slow at time $t_{m+1}$.
and (a) is true. Otherwise there is a mobile agent, \( A_j^G \), that is preventing \( A_i^G \) from moving and is not slow. We must have that

\[
d(A_i^G, t_{m+1}) + 1 = d(A_j^G, t_{m+1})
\]  (3)

Because \( A_i^G \) and \( A_j^G \) are always moving down a spanning tree \( T_S \) of \( G \), hence the depth of \( A_j^G \) must be precisely one greater than \( A_i^G \)'s in order to prevent movement.

Because \( A_j^G \) is not activated at time \( t_m \), (a) and (b) are still true for it at time \( t_{m+1} \). Because \( A_j^G \) is not slow or settled, (a) implies that

\[
d(A_j^G, t_{m+1}) \geq d(A_j^{P(n)}, t_{m+1})
\]  (4)

And the contrapositive of (b) implies that \( A_j^{P(n)} \) is not settled. However, consider the structure of the graph \( P(n) \): if \( A_j^{P(n)} \) is mobile, then since it entered before \( A_i^{P(n)} \), it must be further ahead. In particular, we must have

\[
d(A_j^{P(n)}, t_{m+1}) \geq d(A_j^{P(n)}, t_{m+1}) + 1
\]  (5)

As otherwise \( A_j^{P(n)} \) would have prevented \( A_j^{P(n)} \) from moving when activated at time \( t_m \).

(In)equalities 3, 4 and 5 imply \( d(A_i^G, t_{m+1}) \geq d(A_i^{P(n)}, t_{m+1}) \). This shows (a) is true at time \( t_{m+1} \). \( \square \)

5.3 Proof of Lemma 3.11

Proof. As in Lemma 3.10 we can assume that the event at time \( t_m \) is the uninterrupted activation of a pair of agents \( A_i^G \) and \( A_i^{P(n)} \), and we need only verify that (b) is still true for this pair of agents. We separate our proof into cases.

Case 1: Assume \( A_i^{P(n)} \) is settled at time \( t_{m+1} \). Because \( P(n) \) is a path graph and using Lemma 3.9, \( A_i^{P(n)} \) can only be settled if every non-deleted agent that entered before it is settled behind it. At time \( t_{m+1} \) (b) is still true for all agents other than \( A_i^G \) and \( A_i^{P(n)} \). Hence, it follows from (b) that for any non-deleted agent \( A_j^G \) where \( j < i \) we have:

\[
d(A_j^G, t_{m+1}) \leq d(A_j^{P(n)}, t_{m+1})
\]  (6)

Algorithm 1 guarantees that any agent in \( G \) always neighbours a settled agent or is at the same location as a settled agent. Thus, we know that \( d(A_i^G, t_{m+1}) \leq d(A_j^G, t_{m+1}) + 1 \) for some settled agent \( A_j \). Furthermore, this inequality must hold for some \( A_j^G \) that entered before \( A_i^G \) (i.e., \( j < i \)), because any settled agent that entered after \( A_i^G \) must have gone down a different branch of \( T_S \), otherwise it would be blocked by \( A_i^G \) and unable to settle. Let \( j_{\text{max}} = \max_{j < i} d(A_j^G, t_{m+1}) \). Then \( d(A_i^G, t_{m+1}) \leq j_{\text{max}} + 1 \). If this is an equality, \( A_i^G \) is necessarily settled.

From Inequality 3 we infer

\[
d(A_i^{P(n)}, t_{m+1}) \geq \max_{j < i} d(A_j^{P(n)}, t_{m+1}) + 1 \geq j_{\text{max}} + 1 \geq d(A_i^G, t_{m+1})
\]  (7)

Where \( d(A_i^{P(n)}, t_{m+1}) \geq \max_{j < i} d(A_j^{P(n)}, t_{m+1}) + 1 \) follows from the fact that \( A_i^{P(n)} \) is ahead of all non-deleted agents that came before it. In the case of equality, \( A_i^G \) must be settled. If \( A_i^G \) isn’t settled, then the inequality is strict. Consequently, it follows from the fact that (a) holds at
time \( t_{m+1} \) (Lemma 3.10) that \( A_i^G \) must be slow. Otherwise, (a) implies \( A_i^G \)'s depth is greater than \( A_i^{P(n)} \)'s, contradicting the inequality. Either way, (b) is true.

**Case 2:** Assume \( A_i^{P(n)} \) is slow and not settled at time \( t_{m+1} \). If \( A_i^{P(n)} \) is slow at \( t_m \), then it follows from (b) that \( A_i^G \) is slow or settled at \( t_m \), and so activation cannot affect either of these agents, meaning (b) remains true at \( t_{m+1} \) and we are done. Thus, we may assume \( A_i^{P(n)} \) is not slow at time \( t_m \).

Using Lemma 3.9, \( A_i^{P(n)} \) can only become slow at time \( t_{m+1} \) if all vertices behind it contain settled agents, and all vertices ahead of it contain two slow agents (one settled and one mobile). If \( d(A_i^{P(n)}, t_m) \) is \( k \) there are \( n + (n - k) = 2n - k \) slow or settled agents in \( P(n) \) at time \( t_m \). These \( 2n - k \) agents must have entered \( P(n) \) before \( A_i^{P(n)} \), because any agent that enters after \( A_i^{P(n)} \) must pass it to become slow or settled, and this is impossible because \( A_i^{P(n)} \) is not settled.

Using (b) we learn from the above that in \( G \), at time \( t_m \) there are at least \( 2n - k \) slow agents that entered before \( A_i^G \). Of these, at least \( n - k \) agents are slow and mobile, and have greater depth than \( A_i^G \) or are in a different branch of \( T_S \) (because they arrived before \( A_i^G \) and \( A_i^G \) could not have passed them). There are thus at most \( n - (n - k) = k \) vertices \( A_i^G \) could have visited since entering \( G \), meaning its depth is at most \( k \), and we have \( d(A_i^G, t_m) \leq d(A_i^{P(n)}, t_m) \).

If this inequality is strict, then from statement (a) we learn that \( A_i^G \) is settled or slow, so (b) is true and we are done. Otherwise, \( d(A_i^G, t_m) = k \). We saw there are (at least) \( n - k \) slow mobile agents in \( G \) that have greater depth than \( A_i^G \) or are in a different branch of \( T_S \). From this, we infer that any descendant of \( A_i^G \) must contain a slow mobile agent, or that \( A_i^G \) is at a leaf of \( T_S \) and has no descendants. Thus, if \( A_i^G \) is not already settled or slow, it will become slow after the activation at time \( t_m \), since its slow descendants will prevent it from moving. This completes the proof.

\[ \square \]

### 5.4 Proof of Lemma 3.13

**Proof.** Let \( A_i^* \) be the copy of \( A_i \) simulated over \( P^*(\infty) \). Let \( t_0, t_1, t_2, \ldots \) be the meaningful event times of \( S \). We show by induction that at any time \( t_m \), for all non-deleted agents:

\[
\text{either } A_i^{P(\infty)} \text{ is settled or } d(A_i^*, t_m) \leq d(A_i^{P(\infty)}, t_m) \quad (8)
\]

This implies any agent that enters \( P^*(\infty) \) must have already or concurrently entered \( P(\infty) \), completing the proof.

The induction statement is trivially true at time \( t_0 \), as no event has occurred yet. We assume it is true up to time \( t_m \), and show it remains true at \( t_{m+1} \).

If the event scheduled for time \( t_m \) was a deletion of some agent, the statement remains trivially true (as both simulated versions of the agent are deleted). Otherwise, the scheduled event is the uninterrupted activation of some pair of agents \( A_i^* \) and \( A_i^{P(n)} \).

Any agent \( A_j \) where \( j \neq i \) does not move, so we need only verify the inductive statement remains true for \( A_i^* \) and \( A_i^{P(n)} \). The only situation in which Inequality (8) is falsified at time \( t_{m+1} \) if it is true at time \( t_m \) is if \( d(A_i^*, t_m) = d(A_i^{P(\infty)}, t_m) \) and \( A_i^{P(\infty)} \) is mobile at time \( t_{m+1} \), but \( A_i^* \) manages to move whereas \( A_i^{P(\infty)} \) is blocked by a mobile agent \( A_j^{P(\infty)} \). By the inductive hypothesis, \( d(A_j^*, t_m) \leq d(A_j^{P(\infty)}, t_m) \). Because \( P^*(\infty) \) is a path graph and \( j < i \), we know that \( d(A_j^*, t) > d(A_i^*, t) \) at all times \( t \) after \( A_j^* \) entered the environment. Hence, if \( d(A_i^*, t_m) = d(A_i^{P(\infty)}, t_m) \) and \( A_j^{P(\infty)} \) blocks \( A_i^{P(\infty)} \), then \( A_i^* \) must also block \( A_j^* \) when it attempts to move. This shows that the inductive hypothesis is correct at time \( t_{m+1} \).
5.5 Proof of Lemma 3.14

Proof. Unlike Lemma 3.13, here we count the number of agents that enter $\mathcal{P}^*(\infty)$, and not the number of currently existing agents that entered it. This means we count also agents that entered at $\mathcal{P}^*(\infty)$ but were deleted. This difference is necessary for the comparison, because agents cannot be deleted from $B$.

Despite this difference, the proof is very similar to Lemma 3.13. One shows by induction on the meaningful event times $t_0, t_1, t_2, \ldots$ that at any time $t_m$, for any $i$ such that $A^B_i$ and $A^*(\infty)_i$ were not deleted we have:

\[d(A^B_i, t_m) - i + 1 \leq d(A^*(\infty)_i, t_m)\]  \hfill (9)

Note that $d(A^B_i, t_m) - i + 1$ is the index of the vertex of $A^B_i$ at time $t_m$. If $A^B_i$ crossed $(v_0, v_1)$ we must have $d(A^B_i, t_m) - i + 1 \geq 1$. Recalling that if $A^*(\infty)_i$ is outside of $G$ at time $t$ then $d(A^*(\infty)_i, t) = 0$, we see by (9) that crossing $(v_0, v_1)$ can only happen if $A^*(\infty)_i$ entered $\mathcal{P}^*(\infty)$, or if $A^*(\infty)_i$ was deleted before entering. Hence, the Lemma follows from (9).

Let us show (9) holds by induction. It holds trivially for all $i$ at $t_0$. Now, assume (9) holds at time $t_m$, and we will show it holds at time $t_{m+1}$.

Suppose the pair of agents activated at $t_m$ is $A^B_i$ and $A^*(\infty)_i$. Then these are the only agents for which (9) might be false at $t_{m+1}$. Assuming (9) is true at $t_m$, it can only become false at $t_{m+1}$ if $d(A^B_i, t_m) - i + 1 = d(A^*(\infty)_i, t_m)$, but $A^B_i$ successfully moves as a result of activation at time $t_m$ whereas $A^*(\infty)_i$ does not and also is not deleted. If $A^*(\infty)_i$ does not move this means some $A^*(\infty)_j, j < i$ is blocking it. Hence, we must have $d(A^*(\infty)_j, t_{m+1}) = d(A^*(\infty)_i, t_{m+1}) + 1$. By the inductive hypothesis we have $d(A^*(\infty)_j, t_{m+1}) - j + 1 \leq d(A^*(\infty)_i, t_{m+1})$. Since $j < i$, $A^B_j$ is always ahead of $A^B_i$, meaning $d(A^B_j, t_{m+1}) - j + 1 < d(A^B_i, t_{m+1}) - j + 1$. Combining these (in)equalities we get $d(A^B_i, t_{m+1}) - i + 1 < d(A^*(\infty)_i, t_{m+1}) + 1$, hence $d(A^B_i, t_{m+1}) - i + 1 \leq d(A^*(\infty)_i, t_{m+1})$. This completes the proof by induction of (9).