Feshbach resonance [1] has opened up a new playground for the field of cold trapped atoms. Using this resonance, the effective interaction between the atoms can be varied over a wide range. In particular, for two fermion species with a Feshbach resonance between them, the ground state can be tuned from a weak-coupling Bardeen Cooper Schrieffer (BCS) superfluid to a strong coupling regime where the Fermions pair-up to form Bosons which in turn undergo Bose Einstein Condensation (BEC) [2,3].

Though this problem has been under intense theoretical [4] and experimental [5] investigations, almost all works thus far are restricted to the case where the concentrations of the two fermionic species are equal. We here generalize this study to the case of unequal populations of the two species, and investigate in detail the thermodynamic stability of this system, in particular the question when the uniform state can be stable.

Studies of fermions with unequal populations or mismatched Fermi surfaces and a pairing interaction have a long history. It was studied by Fulde and Ferrell, Larkin and Ovchinnikov (FFLO) [6] in the 1960’s with relation to superconductivity in materials with ferromagnetically coupled paramagnetic impurities. It was found that in this case the system is likely to have an inhomogeneous gapless superconducting phase. Advances in techniques of manipulating dilute ultracold atoms have revived interests in the related problems [7]. These studies, in our present language, are still restricted to the weak-coupling regime. We, however, would extend our analysis to all coupling strengths.

In the “canonical” problem of two species of fermions with equal mass (say, spin up and spin down electrons) and equal concentrations (thus a single Fermi surface), if the cross-species interaction is varied from weak to strong coupling, at low temperatures the system would undergo a smooth crossover from a superfluid with loosely bound Cooper pairs (the “BCS” limit) to one with condensation of tightly bound bosonic molecules (the “BEC” limit) [2]. The situation, however, can be very different if one considers two species of fermions with unequal concentrations (i.e. mismatched Fermi surfaces), even if they have identical mass. This can be anticipated because, on the one hand, far into the BCS side, the system is basically in the FFLO regime [7] and therefore must go into a spatially inhomogeneous phase. On the other hand, in the far end of the BEC side, the system is expected to behave like an ordinary (weakly interacting) Bose-Fermi mixture and thus has a stable homogeneous phase. Here the bosons are the fermion pairs and the fermions are the “leftover” unpaired atoms of the majority species. Deep into the BEC regime, the size of the Fermion pairs is small and the interaction between the bosons and the leftover unpaired fermions are expected to be weak. It is therefore a very interesting question as to what happens in between. This is the question we want to address in this paper.

For simplicity, we shall assume that the resonance is sufficiently wide that the physics reduces effectively to a single channel regime. This is probably valid [8] for many Feshbach resonances under current experimental investigations. Thus, in our calculations, we would not invoke explicitly the presence of the “closed channel” which leads to this Feshbach resonance. We simply model the fermions as interacting through a short-range, s-wave effective interaction (dependent on the external magnetic field) characterized by the corresponding scattering length $a$. $1/a$ varies from $\infty$ for large negative detuning (closed channel bound state energy much below continuum threshold) to $-\infty$ for large positive detuning.

Now we proceed to the details of our calculation and results. We consider two fermion species, denoted as “spin” $\uparrow$ and $\downarrow$, of equal mass $m$. Because of the unequal concentrations of the two species and the possible existence of pairing, it is useful to introduce three fields: the chemical potentials $\mu_\sigma (\sigma = \uparrow$ or $\downarrow$) and the pairing field $\Delta$. We shall confine ourselves to zero temperature and generalize the BCS mean field approach of [3]. The excitation spectrum for each spin is (see e.g. [11] for details)

$$E_\sigma(k) = \frac{\xi_\sigma(k) - \xi_{-\sigma}(k)}{2} + \sqrt{\left(\frac{\xi_\sigma(k) + \xi_{-\sigma}(k)}{2}\right)^2 + \Delta^2},$$

(1)

where $\xi_\sigma(k) = \hbar^2 k^2 / 2m - \mu_\sigma$ are the quasi-particle excitation energies for normal fermions, and $-\uparrow \equiv \downarrow$. The
density of each spin species is then
\[ n_\sigma = \int \frac{d^3k}{(2\pi)^3} \left[ u_k^2 f(E_\sigma) + v_k^2 f(-E_\sigma) \right], \quad (2) \]
with the coherence factors
\[ u_k^2 = 1 - v_k^2 = \frac{E_T(k) + \xi(k)}{E_T(k) + E_L(k)}. \]
Here \( f \) is the Fermi function. The equation for the order parameter \( \Delta \) reads:
\[ -\frac{m}{4\pi a} \Delta = \Delta \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1 - f(E_T) - f(E_L)}{E_T + E_L} - \frac{m}{\hbar^2 k^2} \right]. \quad (3) \]
We solve equations (2) and (3) self-consistently for fixed total density \( n \equiv n_\uparrow + n_\downarrow \) and density difference \( n_d \equiv n_\downarrow - n_\uparrow \). We shall always take \( \uparrow \) to be the majority species so that \( n_d \geq 0 \).

It is convenient to introduce the average chemical potential \( \mu \equiv (\mu_\uparrow + \mu_\downarrow)/2 \) and the difference \( h \equiv (\mu_\uparrow - \mu_\downarrow)/2 \geq 0 \). Then we have
\[ E_{T,L}(k) = \sqrt{\xi(k)^2 + \Delta^2} \mp h, \quad (4) \]
where \( \xi(k) \equiv \hbar^2 k^2/2m - \mu \). Hence \( E_L(k) > 0 \) always. From Eq. (2) we get
\[ n_d = \int \frac{d^3k}{(2\pi)^3} f(E_T(k)) \quad (5) \]
and so the integration is only over the region where \( E_T(k) < 0 \). In the following, it is useful to note that the smallest (or most negative) \( E_T(k) \) occurs at \( \xi(k) = 0 \) for \( \mu > 0 \), where it is \( \Delta - h \), and at \( k = 0 \) for \( \mu < 0 \), where it is \( \sqrt{\mu^2 + \Delta^2} - h \).

As in the case of equal concentrations, it is convenient to express our results in dimensionless variables. We shall define an inverse length scale \( k_F \) through the total density \( n \equiv k_F^d \equiv (3\pi^2 n)^{1/3} \), and an energy scale \( \varepsilon_F \equiv \hbar^2 k_F^2/2m \). We thus write \( \tilde{\mu} \equiv \mu/\varepsilon_F, \tilde{h} \equiv h/\varepsilon_F, \Delta \equiv \Delta/\varepsilon_F, \tilde{n}_d \equiv n_d/n \), and define the dimensionless coupling constant \( g \equiv 1/(\pi k_F a) \), which varies from \( \infty \) for large negative detuning to \( -\infty \) for large positive detuning.

We now describe the results of our calculations. We first make contact with the BCS-BEC cross-over for equal concentrations. The inset of Fig. 1 shows the typical behaviors of \( \tilde{\mu}, \Delta \) and \( \tilde{h} \) as a function of \( g \) for a given density difference \( \tilde{n}_d \). The behavior of \( \tilde{\mu} \) or \( \Delta \) is similar to that in the case of equal concentrations \( \tilde{n}_d = 0 \). For example, \( \tilde{\mu} \) is large and negative in the BEC limit whereas it is of order 1 in the BCS regime. Unlike that case however, \( g \) has to be larger than a minimum coupling \( g_c \) in order for a finite order parameter \( \Delta \) to exist. For \( g < g_c \), Eq. (3) requires that \( \Delta = 0 \) and the system is in the normal state. (For clarity of this inset, we plot only the \( \Delta \neq 0 \) solutions.) The main Fig. 1 shows \( \tilde{h} \) as function of \( g \) in the intermediate regime \( (|g| \lesssim 1) \) for three different \( \tilde{n}_d \) (0.2, 0.5, and 0.8). The horizontal dotted lines indicate the normal state in which the gap function \( \Delta \) is zero (described above).

The behavior for \( g < g_c \) is easy to understand. For sufficiently large and negative \( g \), the interaction is too weak to produce pairing since the concentrations are unequal. The system reduces to a Fermi gas. In this case, the chemical potentials are given by \( \mu_\sigma = (6\pi^2 n_\sigma)^{2/3}/(2m) \) which implies \( \mu = ((6\pi^2/3)^{2/3}/4m)(n_{\uparrow}^{2/3} + n_{\downarrow}^{2/3}) \) and \( h = ((6\pi^2/3)^{2/3}/4m)(n_{\uparrow}^{2/3} - n_{\downarrow}^{2/3}) \) (both independent of \( g \)). On the other hand, for large and positive \( g \) (the strong-coupling BEC limit), one can show \( \tilde{n}_d = 0, \tilde{\mu} = 0 = \tilde{n}_d \) and \( \tilde{h} \) with \( \mu_\uparrow = 0 = \mu_\downarrow \). These expressions simply reflect that the system becomes a Bose-Fermi mixture with boson concentration \( n_{\downarrow} \) and free fermion concentration \( n_{\uparrow} \).

Notice that the lines (for \( \Delta \neq 0 \) of \( \tilde{h} \) versus \( g \)) is plotted for three different \( \tilde{n}_d \) near \( g \sim 0.15 \). For \( g \approx 0.17, \tilde{h} \) increases with \( \tilde{n}_d \) for fixed \( g \). For \( g \lesssim 0.15, \tilde{h} \) decreases as \( \tilde{n}_d \) increases when the coupling strength is fixed. We shall return to these features again below.

Now we make contact with the superconductivity literature. It is helpful here to note that \( h \) plays the role of an effective external Zeeman field. We plot in Fig. 2 \( \Delta \) as a function of \( \tilde{h} \) for various coupling strengths \( g \). The horizontal portion of each curve corresponds to \( n_d = 0 \). In this region, \( \tilde{h} < \Delta \) and so that \( E_T(k) > 0 \) for all \( k \) (see Eq. (2)). The other part of the curve corresponds to \( n_d > 0 \), and exists only in the region \( h > \Delta \) (More precisely, for larger \( h \) where \( \mu \) becomes negative, this condition should read \( h > \sqrt{\mu^2 + \Delta^2} \)). For small \( g \) (\( \lesssim 0.1 \)), \( \Delta \) decreases with decreasing \( \tilde{h} \). This solution is the generalization of that first discovered by Sarma \( (10) \). We find that this superfluid state corresponds to one where \( n_d \) increases with decreasing \( \tilde{h} \), and hence unstable (to be discussed again below). For sufficiently large coupling (\( g \gtrsim 0.17 \)), \( \Delta \) decreases with increasing \( \tilde{h} \). This state has \( \tilde{n}_d \) decreases with \( \tilde{h} \), and satisfies one of the stability conditions to be discussed below.

In Fig. 3 the chemical potential difference \( \tilde{h} \) is plotted as a function of \( \tilde{n}_d \). These results correspond to those of Fig. 1 presented in a different manner. Let us explain how this graph should be read, with \( g = -0.1 \) as an example. For \( \tilde{n}_d = 0 \), \( \tilde{h} \) can take any value up to \( \tilde{h}_1 \approx 0.5 \) given by the intersection of the line labelled by \( g = -0.1 \) with the \( \tilde{n}_d = 0 \) axis. This portion corresponds to the line with \( \Delta \) being a constant in Fig. 2. For \( 0 < \tilde{n}_d < 0.46 \) the dependence of \( \tilde{h} \) on \( \tilde{n}_d \) is given by the solid line labelled by \( g = -0.1 \). This line corresponds to the state with \( \Delta \neq 0 \) but \( \tilde{h} < \Delta \) in Fig. 2 discussed above. For \( \tilde{n}_d > 0.46 \), the system enters into the normal state with the \( (\tilde{h}, \tilde{n}_d) \) relation represented by the dotted lines, given by \( \tilde{h} = 0.5((1 + \tilde{n}_d)^2/3 - (1 - \tilde{n}_d)^2/3) \) (see the discussion on Fig. 1 above). Lastly, for \( \tilde{n}_d = 1 \), \( \tilde{h} \) can take any value larger than \( \tilde{h}_2 \approx 0.6 \times 25/3 \approx 0.79 \). This is because
this line corresponds simply to a Fermi gas with only ↑ particles, and \( h \) can take any value larger than \( \mu \) so that \( \mu_\uparrow = (\mu - h)/2 < 0 \). For \( g \gtrsim 0.1 \), the graph can be read in a similar manner except that the dotted line representing the normal state is not involved.

For the uniform superfluid to be stable, two criterions must be fulfilled \([11, 12]\). First, the susceptibilities matrix \( \partial\tilde{n}_d/\partial\tilde{h}_1 \) can have only positive eigenvalues. One can show that this requires that \( \partial\tilde{n}_d/\partial\tilde{h}_1 \), evaluated at constant \( g \), must be positive \([13]\). That is, the plot of \( \tilde{h} \) versus \( \tilde{n}_d \) must have positive slope. From Fig. 3, we see that for small \( g \) (\( \lesssim 0.1 \)), the slope of each curve is always negative which indicates the superfluid state is unstable. For \( g \) greater than about 0.1, the slope of these curves change sign at some \( \tilde{n}_d \) between 0 and 1. In this case a stable superfluid state may occur for sufficiently large \( \tilde{n}_d \). After \( g \gtrsim 0.17 \), the slopes of these curves are positive for all \( n \geq 0 \) and the above stability criterion is satisfied for all \( n \).

The second stability criterion is that the superfluid density \( \rho_s \) must be positive \([11]\). \( \rho_s \) can be evaluated as discussed in \([11]\). The analytic result can be expressed as

\[
\rho_s = \left[ 1 - \frac{\theta(\sqrt{\mu^2 + \Delta^2} - h)\hat{k}_{1,2}}{2\sqrt{1-(\Delta/h)^2}} \right],
\]

with \( \hat{k}_{1,2} = [(\mu \mp \sqrt{h^2 - \Delta^2})]^{1/2} \) and \( \theta(x) \) is the step function. The line \( \rho_s = 0 \) is plotted as the dashed lines in Fig. 3, with \( \rho_s < 0 \) below and \( \rho_s > 0 \) above. Thus the states correspond to \( \Delta \neq 0 \) below this dashed line are all unstable. \( \rho_s < 0 \) indicates that the system is unstable towards a state with spatially varying \( \Delta \) and hence a state such as FFLO can be more preferable. From our results, it turns out that the condition for \( \rho_s > 0 \) is actually a slightly weaker requirement than the positive susceptibility discussed in the last paragraph (though we are not aware of any reason why it must be so).

We here note that, for \( \tilde{n}_d \to 0 \), the location where \( \rho_s \) changes sign is exactly at \( \mu = 0 \). Though this can be seen from Eq. \([5]\), it is more instructive to return to the more basic equation for the superfluid density: \( \rho_s = n + \rho_p \). Here \( \rho_p \), the paramagnetic response, is related to the backflow of quasiparticles and is given by \([11]\)

\[
\rho_p = -\frac{1}{\mu} \int_{-\infty}^{\infty} dk k^4 \delta(E_\uparrow(k)).
\]

For \( \mu < 0 \), the solution to \( E_\uparrow(k) = 0 \) exists only when \( h > \sqrt{\mu^2 + \Delta^2} \) and takes place at \( k = k_2 \) with \( k_2^2/2m = \sqrt{h^2 - \Delta^2 + \mu} \). For \( n_d \to 0^+ \), \( h \) is just slightly larger than \( \sqrt{\mu^2 + \Delta^2} \) [see Eq. \([14]\)]. \( k_2 \) is small and hence \( \rho_p \to 0^+ \) and \( \rho_s \approx n > 0 \) for \( \mu > 0 \). For \( \mu > 0 \), \( E_\uparrow(k) = 0 \) happens when \( h > \Delta \) and take place at two \( k \) values: \( k = k_2 \) as already given in the \( \mu < 0 \) case above and \( k = k_1 \) with \( k_1^2/2m = -\sqrt{\mu^2 + \Delta^2 + \mu} \). For \( n_d \to 0^+ \), \( h \) is just slightly larger than \( \Delta \) and the \( E_\uparrow(k) = 0 \) points occur near \( \xi(k) \approx 0 \) hence \( \partial E_\uparrow(k)/\partial k \to 0 \). Since \( k_1 \) and \( k_2 \) are finite, \( \rho_p \to -\infty \) and \( \rho_s < 0 \).

Finally we show our phase diagram in Fig. 4 covering the entire BEC to BCS regimes. On the BEC side, with \( g \gtrsim 0.17 \), the superfluid state is stable in which both the slope of \( \tilde{h}(\tilde{n}_d) \) and the superfluid density are positive (see Fig. 3). On the upper right of Fig. 4, the pairing order parameter \( \Delta \) is zero and the system is in the normal state. In constructing the boundary of this phase, we have simply taken the intersection of the full lines in Fig. 3 with the dotted lines. A more correct approach would involve the Maxwell construction. However, to do this we also need to know the equation of state for the non-uniform FFLO superfluid state beyond the weak-coupling regime. Since this information is not yet available, we shall leave this investigation to the future.

Lastly we discuss the “breached gap” state of Liu and Wilczek \([14]\). For this state to exist, one need \( E_\uparrow(k) < 0 \) for a region of \( k \) that lies between \( k_1 < k < k_2 \) where \( k_1 > 0 \). \( k_1, k_2 \) already given in the paragraph before last). This is possible only if \( E_\uparrow(k) \) is not monotonic with \( k \) and hence \( \mu > 0 \). Moreover, since \( E_\uparrow(0) > 0 \), we have \( \sqrt{\mu^2 + \Delta^2} > h \) yet \( h > \Delta \). Though there is a region of stability with \( \mu > 0 \) (the region near the upper right of Fig. 3), we find that it rather corresponds to \( \sqrt{\mu^2 + \Delta^2} < h \). (That is the case near \( \tilde{n}_d \approx 1 \) is obvious, since the line \( \tilde{n}_d = 1 \) corresponds to \( h > \mu \) and \( \Delta \to 0 \). Note also on the dash-dotted lines where \( \mu = 0 \), we also have \( \sqrt{\mu^2 + \Delta^2} < h \) since \( h > \Delta \). Therefore gapless excitations exist only near \( k = k_2 \). Moreover this region has \( n_\uparrow(k) = 1 \) for \( 0 \leq k \leq k_2 \) (and \( n_\uparrow(k) = v^2(k) < 1 \) for \( k > k_2 \) ), thus the leftover unpaired majority spin-up particles form a rather normal Fermi sphere with radius \( k_2 \), although the energy required to create a hole \( -E_\uparrow(k) \) can actually be non-monotonic as a function of \( k \), being maximum at an intermediate value \( k = \sqrt{2m\mu \tilde{s}} \) (where it is \( h - \Delta \)) but not \( k = 0 \) (where it is \( h - \sqrt{\mu^2 + \Delta^2} \)).

In conclusion, we have investigated the stability of a fermion mixture with unequal concentrations under a Feshbach resonance. We show that, in contrast to the case of equal concentrations, there is no smooth BCS-BEC crossover. The system is a uniform stable superfluid Bose-Fermi mixture only for sufficiently large coupling. For weak interactions the normal state is the only stable uniform state. The uniform state is unstable for intermediate coupling strengths. Phase transitions must occur when the Feshbach resonance is varied between large positive detuning and large negative detuning.

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FIG. 1: Main figure: scaled chemical potential difference $h/\varepsilon_F = \tilde{h}$ versus the coupling constant $g$ for three different values of $n_d/n = \tilde{n}_d$. Inset includes also the results for $\tilde{\Delta}$ and $\tilde{\mu}$ for $n_d/n = 0.5$.

FIG. 2: Scaled pair order parameter $\Delta/\varepsilon_F = \tilde{\Delta}$ versus $h/\varepsilon_F = \tilde{h}$ for given coupling constants $g$. 
FIG. 3: $h/\varepsilon_F = \tilde{h}$ versus $n_d/n = \tilde{n}_d$ for constant $g$'s. Full lines are for $\Delta \neq 0$ and the dotted lines are for $\Delta = 0$. The dashed line indicates where $\rho_s = 0$, with $\rho_s > 0$ above it. The dot-dash line indicates $\mu = 0$, with $\mu < 0$ above this line.

FIG. 4: Phase diagram for the two Fermion species with interspecies coupling $g$, with the stable uniform phases (white regions) shown. No uniform phases are stable in the shaded region (except when $n_d/n = 0$ or 1).
\[
\frac{h}{\epsilon_F} \quad \text{vs.} \quad g
\]

- \( n_{d}/n \) values: 0.2, 0.5, 0.8

- Graph shows \( n_{d}/n = 0.5 \) inset

\[\Delta \quad \mu \quad n_{d}/n = 0.5\]
