RESTRICTED DYCK PATHS ON VALLEYS SEQUENCE

RIGOBERTO FLÓREZ, TOUFIK MANSOUR, JOSÉ L. RAMÍREZ, FABIO A. VELANDIA, AND DIEGO VILLAMIZAR

Abstract. In this paper we study a subfamily of a classic lattice path, the Dyck paths, called restricted \(d\)-Dyck paths, in short \(d\)-Dyck. A valley of a Dyck path \(P\) is a local minimum of \(P\); if the difference between the heights of two consecutive valleys (from left to right) is at least \(d\), we say that \(P\) is a restricted \(d\)-Dyck path. The area of a Dyck path is the sum of the absolute values of \(y\)-components of all points in the path. We find the number of peaks and the area of all paths of a given length in the set of \(d\)-Dyck paths. We give a bivariate generating function to count the number of the \(d\)-Dyck paths with respect to the semi-length and number of peaks. After that, we analyze in detail the case \(d = -1\). Among other things, we give both, the generating function and a recursive relation for the total area.

1. Introduction

A classic concept, the Dyck paths, has been widely studied. Recently, a subfamily of these paths, non-decreasing Dyck paths, has received certain level of interest due to the good behavior of its recursive relations and generating functions. In this paper we keep studying a generalization of the non-decreasing Dyck paths. Other generalizations of non-decreasing Dyck paths have been given for Motzkin paths and for Lukasiewicz paths \([14, 16]\).

We now recall, to avoid ambiguities, some important definitions that we need in this paper. A Dyck path is a lattice path in the first quadrant of the \(xy\)-plane that starts at the origin, ends on the \(x\)-axis, and consists of (the same number of) North-East steps \(U := (1, 1)\) and South-East steps \(D := (1, -1)\). The semi-length of a path is the total number of \(U\)'s that the path has. A valley (peak) is a subpath of the form \(DU\) (UD) and the valley vertex of \(DU\) is the lowest point (a local minimum) of \(DU\). Following \([17, 18]\) we define the valley vertices of a Dyck path \(P\) as the vector \(\nu = (\nu_1, \nu_2, \ldots, \nu_k)\) formed by all \(y\)-coordinates (listed from left to right) of all valley vertices of \(P\). For further recent work about different combinatorial aspects of Dyck paths, see for instance \([2, 3, 5, 10, 15, 20]\).

For a fixed \(d \in \mathbb{Z}\), a Dyck path is a lattice path in the first quadrant of the \(xy\)-plane that starts at the origin, ends on the \(x\)-axis, and consists of (the same number of) North-East steps \(U := (1, 1)\) and South-East steps \(D := (1, -1)\). The semi-length of a path is the total number of \(U\)'s that the path has. A valley (peak) is a subpath of the form \(DU\) (UD) and the valley vertex of \(DU\) is the lowest point (a local minimum) of \(DU\). Following \([17, 18]\) we define the valley vertices of a Dyck path \(P\) as the vector \(\nu = (\nu_1, \nu_2, \ldots, \nu_k)\) formed by all \(y\)-coordinates (listed from left to right) of all valley vertices of \(P\). For further recent work about different combinatorial aspects of Dyck paths, see for instance \([2, 3, 5, 10, 15, 20]\).

For a fixed \(d \in \mathbb{Z}\), a Dyck path \(P\) is called restricted \(d\)-Dyck or \(d\)-Dyck (for simplicity), if either \(P\) has at most one valley, or if its valley vertex vector \(\nu\) satisfies that \(\nu_{i+1} - \nu_i \geq d\), where \(1 \leq i < k\). The set of all \(d\)-Dyck paths is denoted by \(\mathcal{D}_d\), the set of all \(d\)-Dyck paths of semi-length \(n\) is denoted \(\mathcal{D}_d(n)\), and the cardinality of \(\mathcal{D}_d(n)\) is denoted by \(r_d(n)\).

The first well-known example of these paths is the set of 0-Dyck paths; in the literature, \([1, 6, 7, 9, 11, 13]\), this family is known as non-decreasing Dyck paths. The whole family of
Dyck paths can be seen as a limit of $d$-Dyck and it occurs when $d \to -\infty$. Another example is from Figure 1; we observe that $\nu = (0,1,0,3,4,3,2)$ and that $\nu_{i+1} - \nu_i \geq -1$, for $i = 1, \ldots, 6$, so the figure depicts a $(-1)$-Dyck path of length 28 (or semi-length 14).

![Figure 1. A $(-1)$-Dyck path of length 28.](image)

The recurrence relations and/or the generating functions for $d$-Dyck when $d \geq 0$ have different behavior than the case $d < 0$. For example, the generating functions for known aspects, in $d$-Dyck when $d \geq 0$, are all rational (see [1, 6, 7, 11, 13, 17, 18]). However, the aspects that we analyze in this paper, when $d < 0$, give that the generating functions are all algebraic (non-rational). In this paper we give a bivariate generating function to count the number of paths in $D_d(n)$, for $d \leq 0$, with respect to the number of peaks and semi-length. We also give a relationship between the total number of $d$-Dyck paths and the Catalan numbers. Additionally, we give an explicit symbolic expression for the generating function with respect to the semi-length. For the particular case $d = -1$ we give a combinatorial expression and a recursive relation for the total number of paths. We also analyze the asymptotic behavior for the sequence $r_{-1}(n)$. It would be very interesting if we can understand better the behavior of $d$-Dyck paths for $d < -1$.

The area of a Dyck path is the sum of the absolute values of $y$-components of all points in the path. That is, the area of a Dyck path corresponds to the surface area under the paths and above of the $x$-axis. For example, the path $P$ in Figure 1 satisfies that $\text{area}(P) = 70$. We use generating functions and recursive relations to analyze the distribution of the area of all paths in $D_{-1}(n)$.

## 2. Number of $d$-Dyck Paths and Peaks Statistic

Given a family of lattice paths, a classic question is how many lattice paths are there of certain length, and a second classic question is how many peaks are there depending on the length of the path. These questions have been completely answered, for instance, for Dyck paths [8], $d$-Dyck paths for $d \geq 0$ [1, 18], and Motzkin paths [21] among others. In this section we give a bivariate generating function to enumerate the peaks and semi-length of the $d$-Dyck paths for $d < 0$.

We now give some notation needed for this paper, including the parameters needed for the generating function in this section. The level of a valley is the $y$-component of its valley vertex. We recall that the set of all $d$-Dyck paths is denoted by $D_d$, the set of all $d$-Dyck paths of semi-length $n$ is denoted $D_d(n)$, and the cardinality of $D_d(n)$ is denoted
by \( r_d(n) \). Given a \( d \)-Dyck path \( P \), we denote the semi-length of \( P \) by \( \ell(P) \) and denote the number of peaks of \( P \) by \( \rho(P) \). So, the bivariate generating function to count the number of paths and peaks of \( d \)-Dyck paths is defined by

\[
L_d(x, y) := \sum_{P \in \mathcal{D}_d} x^{\ell(P)} y^{\rho(P)}.
\]

2.1. Some facts known when \( d \geq 0 \). These results can be found in [18].

- If \( d \geq 0 \), then the generating function \( F_d(x, y) \) is given by

\[
L_d(x, y) = 1 + xy(1 - 2x + x^2 + xy - x^{d+1}y) \over (1 - x)(1 - 2x + x^2 - x^{d+1}y).
\]

- If \( d \geq 1 \),

\[
r_d(n) = \sum_{k=0}^{\left\lfloor \frac{n+d-2}{d} \right\rfloor} \left( n - (d-1)(k-1) \right) \over 2k.
\]

- If \( n > d \), then we have the recursive relation

\[
r_d(n) = 2r_d(n - 1) - r_d(n - 2) + r_d(n - d - 1),
\]

with the initial values \( r_d(n) = \binom{n}{2} + 1 \), for \( 0 \leq n \leq d \).

- Let \( p_d(n, k) \) be the number of \( d \)-Dyck paths of semi-length \( n \), having exactly \( k \) peaks. If \( d \geq 0 \), then

\[
p_d(n, k) = \binom{n + k - d(k - 2) - 2}{2(k - 1)}.
\]

For the whole set of Dyck paths, the number \( p_{-\infty}(n, k) \), is given by the Narayana numbers \( N(n, k) = \binom{n}{k} \binom{n}{k-1} \).

2.2. Peaks statistic for \( d \) a negative integer. For the remaining part of the paper we consider only the case \( d < 0 \) and use \( e \) to denote \( |d| \).

Theorem 2.1. If \( d \) is a negative integer and \( e := |d| \), then the generating function \( L_e(x, y) \) satisfies the functional equation

\[
L_e(x, y) = xy + xL_e(x, y) + xS_e(x, y)L_e(x, y),
\]

where \( S_e(x) \) satisfies the algebraic equation

\[
(1 - xS_e(x, y))^e(y + (1 - y)xS_e(x, y)) - S_e(x, y)(1 - xS_e(x, y))^{e+1} - \frac{x^{e+2}y}{1 - x}S_e(x, y) = 0.
\]

Proof. We start this proof introducing some needed notation. The set \( Q_{d,i} \subseteq \mathcal{D}_d \) denotes the family of nonempty paths where the last valley is at level \( i \). We consider the generating function

\[
Q_i^{(e)}(x, y) := \sum_{P \in Q_{d,i}} x^{\ell(P)} y^{\rho(P)}.
\]
It is convenient to consider the sum over the $Q_{e}^{(e)}(x, y)$. We also consider the generating function, with respect to the lengths and peaks, that counts the $d$-Dyck paths that have either no valleys or the last valley is at level less than $e$. That is,

$$S_{e}(x, y) = \frac{y}{1 - x} + \sum_{j=0}^{e-1} Q_{j}^{(e)}(x, y).$$

A path $P$ can be uniquely decomposed as either $UD, UTD$, or $UQDT$ (by considering the first return decomposition), where $T \in \mathcal{D}_d$ and $Q$ is either a path without valleys or is a path in $\cup_{i=0}^{e-1} Q_{d,i}$ (see Figure 2, for a graphical representation of this decomposition). Notice that the decomposition $UQDT$ ensures that the condition $\nu_{i+1} - \nu_i \geq d$ holds for all $i \geq 1$.

![Figure 2. Decomposition of a $d$-Dyck path.](image)

From the symbolic method we obtain the functional equation

$$L_{e}(x, y) = xy + xL_{e}(x, y) + xS_{e}(x, y)L_{e}(x, y).$$

Now we are going to obtain a system of equations for the generating functions $Q_{i}(x, y)$. Let $Q$ be a path in the set $Q_{d,i}$. If $i = 0$, then the path $Q$ can be decomposed uniquely as either $UQ'D\Delta$ or $UQ'DR$, where $\Delta$ is a pyramid, $R$ is a path in $Q_{d,0}$, and $Q'$ is either a path without valleys or $Q' \in \cup_{i=0}^{e-1} Q_{d,i}$. Therefore, we have the functional equation

$$Q_{0}^{(e)}(x, y) = xS_{e}(x, y) \frac{xy}{1 - x} + xS_{e}(x, y)Q_{0}^{(e)}(x, y).$$

For $i > 0$, any path $Q$ can be decomposed uniquely in one of these two forms $UR_{1}D$ or $UQDR_{2}$, where $R_1 \in Q_{d,i-1}$, $R_2 \in Q_{d,i}$, and $Q$ is either a path without valleys or $Q \in \cup_{i=0}^{e-1} Q_{d,i}$. So, we have the functional equation

$$Q_{i}^{(e)}(x, y) = xQ_{i-1}^{(e)}(x, y) + xS_{e}(x)Q_{i}^{(e)}(x, y).$$
Summarizing the above discussion, we obtain the system of equations:

\[
\begin{align*}
Q_0^{(e)}(x, y) &= xS_e(x, y)\frac{xy}{1-x} + xS_e(x, y)Q_0^{(e)}(x, y) \\
Q_1^{(e)}(x, y) &= xQ_0^{(e)}(x, y) + xS_e(x, y)Q_1^{(e)}(x, y) \\
&\vdots \\
Q_i^{(e)}(x, y) &= xQ_{i-1}^{(e)}(x, y) + xS_e(x, y)Q_i^{(e)}(x, y) \\
&\vdots \\
Q_{e-1}^{(e)}(x, y) &= xQ_{e-2}^{(e)}(x, y) + xS_e(x, y)Q_{e-1}^{(e)}(x, y).
\end{align*}
\]

(3)

Summing the equations in (3), we obtain that

\[
\sum_{j=0}^{e-1} Q_j^{(e)}(x, y) = xS_e(x, y) \left( \sum_{j=0}^{e-1} Q_j^{(e)}(x, y) + \frac{xy}{1-x} \right) + x \sum_{j=0}^{e-2} Q_j^{(e)}(x, y).
\]

From this and (2) we have

\[
S_e(x, y) - \frac{y}{1-x} = x \left( S_e(x, y) - \frac{y}{1-x} - Q_{e-1}^{(e)}(x, y) \right) + xS_e(x, y) \left( S_e(x, y) - \frac{y}{1-x} \right) + \frac{x^2y}{1-x}S_e(x, y).
\]

Now solving this in previous equation for $S_e(x, y)$ we have

\[
S_e(x, y) = \frac{1 - x + xy - \sqrt{1 - 2x + x^2 - 2xy - 2x^2y + x^2y^2 + 4x^2Q_{e-1}^{(e)}(x, y)}}{2x}.
\]

(5)

Notice that all of the $Q_i^{(e)}(x, y)$, with $i \geq 0$, can be expressed as

\[
Q_i^{(e)}(x, y) = \frac{x^{i+2}yS_e(x, y)}{(1-x)(1-xS_e(x, y))^{i+1}}.
\]

(6)

Substituting (6) into (4) we obtain the desired functional equation.

We observe that substituting (5) into (1), we have

\[
L_e(x, y) = \frac{xy}{1 - x - xS_e(x, y)} = \frac{xy}{1 - x - \frac{1 - x + xy - \sqrt{1 - 2x + x^2 - 2xy - 2x^2y + x^2y^2 + 4x^2Q_{e-1}^{(e)}(x, y)}}{2x}}.
\]

From the combinatorial description of $Q_{e-1}^{(e)}(x, y)$, we obtain $Q_{e-1}^{(e)}(x, y) \to 0$, as $e \to \infty$. Therefore,

\[
\lim_{e \to \infty} L_e(x, y) = \lim_{e \to \infty} \frac{xy}{1 - x - xS_e(x, y)} = \frac{1 - x - xy - \sqrt{1 - 2x + x^2 - 2xy - 2x^2y + x^2y^2}}{2x}.
\]
This last generating function is the distribution of the Narayan sequence. This corroborates that the restricted \((-\infty)\)-Dyck paths coincides with the non-empty Dyck paths.

**Theorem 2.2.** If \(1 \leq k \leq |d| + 3\), then the \(k\)-th coefficient of the generating function \(L_e(x, 1)\) coincides with the Catalan number \(C_k\).

**Proof.** We first observe that the shortest Dyck path that contains a forbidden sequence of valleys is \(P = U^{e+2}DUD^{e+2}UD\) (clearly, \(\ell(P) = e + 4\)) with \(e = |d|\). Therefore, if \(d < 0\), then \(r_d(n) = C_n\) for \(n = 1, 2, \ldots, |d| + 3\). □

The first few values for the sequence \(r_d(n)\), for \(d \in \{-1, -2, -3, -4\}\) are

- \(\{r_{-1}(n)\}_{n \geq 1} = \{1, 2, 5, 14, 41, 123, 375, 1157, 3603, \ldots\}\),
- \(\{r_{-2}(n)\}_{n \geq 1} = \{1, 2, 5, 14, 42, 131, 419, 1365, 4511, \ldots\}\),
- \(\{r_{-3}(n)\}_{n \geq 1} = \{1, 2, 5, 14, 42, 132, 428, 1419, 4785, \ldots\}\),
- \(\{r_{-4}(n)\}_{n \geq 1} = \{1, 2, 5, 14, 42, 132, 429, 1429, 4850, \ldots\}\).

For example, there are 41 \((-1)\)-Dyck paths out of the 42 Dyck paths of length 10. The Figure 3, depicts the only Dyck path of length 10 that is not a \((-1)\)-Dyck path.

![Figure 3](image-url)

**Figure 3.** The only Dyck path of length 10 that is not a \((-1)\)-Dyck path.

Recall that \(d\) is a negative integer and that \(e := |d|\). Then by Theorem 2.1, we have

\[
(L_e(x, y) + y)^e \left(x L_e^2(x, y) + (xy + x - 1)L_e(x, y) + xy\right)
- \frac{x}{1-x}((1-x)L_e(x, y) - xy)(L_e(x, y))^{e+1} = 0.
\]

This implies that

\[
\sum_{j=2}^{e+1} x \left(\begin{array}{c} e \\ j - 2 \end{array}\right)y^{e+2-j}(L_e(x, y))^j + \sum_{j=1}^{e+1} (xy + x - 1) \left(\begin{array}{c} e \\ j - 1 \end{array}\right)y^{e+1-j}(L_e(x, y))^j
- \sum_{j=0}^{e} x \left(\begin{array}{c} e \\ j \end{array}\right)y^{e+1-j}(L_e(x, y))^j + \frac{x^2y}{1-x}(L_e(x, y))^{e+1} = 0.
\]

Hence, by taking \(y = 1\), we have

\[
L_e(x, 1) = Z \left(a_0 + \sum_{j=2}^{e+1} a_j(x)(L_e(x, 1))^j\right),
\]
where \( Z = 1 \), and
\[
a_0 = \frac{x}{1 - (e + 2)x},
\]
\[
a_j = \frac{1}{1 - (e + 2)x} \left( x \binom{e + 2}{j} - \binom{e}{j - 1} \right), \quad j = 2, 3, \ldots, e,
\]
\[
a_{e+1} = \frac{(e + 2)x(1 - x) - 1 + x(1 + x)}{(1 - x)(1 - (e + 2)x)}.
\]

Hence, by the Lagrange inversion formula, we expand the generating function \( L_e(x, 1) \) as a power series in \( Z \) to obtain
\[
L_e(x, 1) = \sum_{n \geq 1} \frac{[Z^{n-1}]}{n} \sum_{i_0 + i_2 + i_3 + \cdots + i_{e+1} = n} \frac{n!}{i_0! i_2! \cdots i_{e+1}!} a_0^{i_0} Z^{i_2 + \cdots + (e+1)i_{e+1}} \prod_{j=2}^{e+1} a_j^{i_j},
\]
that leads to the following result.

**Theorem 2.3.** We have
\[
L_e(x, 1) = \sum_{n \geq 1} \frac{\sum_{i_2 + \cdots + (e+1)i_{e+1} = n} n! \prod_{j=2}^{e} x^{\binom{e+2}{j} - \binom{e}{j-1}}}{(1 - (e + 2)x)^n} \cdot \frac{(e + 2)x(1 - x) - 1 + x(1 + x)}{1 - x},
\]
where
\[
\binom{n}{i_2, \ldots, i_{e+1}} = \frac{n!}{i_2! \cdots i_{e+1}!(n - i_2 - \cdots - i_{e+1})!}
\]
and \( t = \frac{(e + 2)x(1 - x) - 1 + x(1 + x)}{1 - x} \).

For example, Theorem 2.3 with \( e = 2 \) gives
\[
L_2(x, 1) = \sum_{n \geq 1} \frac{\sum_{i_3 = n-1}^{n} \binom{n}{i_3}(6x - 2)^{i_3}(1 - x)^{3i_3}}{n(1 - 4x)^n}.
\]

Thus,
\[
L_2(x, 1) = \frac{x}{1 - 4x} + \frac{x^2(6x - 2)}{(1 - 4x)^3} + \frac{x^3t}{(1 - 4x)^4} + \frac{2x^3(6x - 2)^2}{(1 - 4x)^5} + \frac{5x^4t(6x - 2)}{(1 - 4x)^6} + \frac{5x^4(6x - 2)^3 + 3x^5t^2}{(1 - 4x)^7} + \frac{21x^5(6x - 2)^2t}{(1 - 4x)^8} + \frac{28x^6(-2 + 6x)t^2 + 14x^5(-2 + 6x)^4}{(1 - 4x)^9} + \cdots,
\]
where \( t = (-3x^2 + 5x - 1)/(1 - x) \).

### 3. Some results for the case \( d = -1 \)

In this section we keep analyzing the bivariate generating function given in previous section for the particular case \( d = -1 \). For this case, we provide more detailed results. We denote by \( Q \) the set of all nonempty paths in \( D_{-1} \) having at least one valley, where the last valley is at ground level. We denote by \( Q_n \) the subset of \( Q \) formed by all paths of semi-length \( n \) and denote by \( q_n \) the cardinality of \( Q_n \). For simplicity, when \( d = -1 \) (or \( e = 1 \)) we use \( L(x, y) \) instead of \( L_1(x, y) \).
Theorem 3.1. The bivariate generating function \( L(x, y) \) is given by
\[
L(x, y) = \frac{(x - 1)y \left(1 - x(2 + y) - \sqrt{(1 - x - 2xy - 2x^2y + x^2y^2 - x^3y^2)/(1 - x)}\right)}{2(1 - 2x + x^2 - 2xy + x^2y)}.
\]

Proof. A path \( P \in \mathcal{Q} \) can be uniquely decomposed as either \( UD, UTD, U\Delta DT \), or \( UQDT \), where \( \Delta \) is a pyramid, \( T \in \mathcal{D}_{-1} \), and \( Q \in \mathcal{Q} \). Therefore, we obtain the following functional relation
\[
L(x, y) = xy + xL(x, y) + x \left(\frac{y}{1 - x}\right) L(x, y) + xL(x, y)Q(x, y),
\]
where
\[
Q(x, y) := \sum_{Q \in \mathcal{Q}} x^{\ell(Q)} y^{\rho(P)}.
\]
We are going to obtain an explicit expression for the generating function \( Q(x, y) \). Additionally, a path \( Q \in \mathcal{Q} \) can be uniquely decomposed as either \( U\Delta DU\Delta 'D \), \( U\Delta DR \), \( UR_1DR_2 \), or \( URDU\Delta D \), where \( \Delta, \Delta ' \) are pyramids, and \( R, R_1, R_2 \in \mathcal{Q} \) (see Figure 4 for a graphical representation of this decomposition).

![Figure 4. Decomposition of a (-1)-Dyck path in \( \mathcal{Q} \).](image)

Using the symbolic method, we obtain the functional equation
\[
Q(x, y) = x^2 \left(\frac{y}{1 - x}\right)^2 + x \left(\frac{y}{1 - x}\right) Q(x, y) + x(Q(x, y))^2 + x^2 \left(\frac{y}{1 - x}\right) Q(x, y).
\]
Solving the equation above for \( Q(x, y) \), we find that
\[
Q(x, y) = \frac{1 - x - xy - x^2y - \sqrt{(1 - x)(1 - x - 2xy - 2x^2y + x^2y^2 - x^3y^2)}}{2(1 - x)x}.
\]
This and solving (7) for \( L(x, y) \) imply the desired result. \( \square \)
Expressing \( L(x, y) \) as a series expansion we obtain these first few terms.

\[
L(x, y) = xy + x^2 (y^2 + y) + x^3 (y^3 + 3y^2 + y) + x^4 (y^4 + 6y^3 + 6y^2 + y) + \cdots
\]

\[
+ x^5 (y^5 + 10y^4 + 19y^3 + 10y^2 + y) + x^6 (y^6 + 15y^5 + 46y^4 + 45y^3 + 15y^2 + y) + \cdots
\]

Figure 5 depicts all six paths in \( D_{-1}(4) \) with exactly 3 peaks. Notice that this is the bold coefficient of \( x^4 y^3 \) in the above series.

\[
\text{Figure 5. All six paths in } D_{-1}(4) \text{ with exactly 3 peaks.}
\]

The generating function for the \((-1)\)-Dyck paths is given by

\[
L(x) := L(x, 1) = \frac{-1 + 4x - 3x^2 + \sqrt{1 - 4x + 2x^2 + x^4}}{2(1 - 4x + 2x^2)}.
\]

Thus,

\[
L(x) := x + 2x^2 + 5x^3 + 14x^4 + 41x^5 + 123x^6 + 375x^7 + 1157x^8 + \cdots.
\]

For simplicity for the remaining part of the paper, if there is not ambiguity, we use \( r(n) \) instead of \( r_{-1}(n) \). Our interest here is to give a combinatorial expression for this sequence. First of all, we give some preliminary results. Let \( b(n) \) be the number of \((-1)\)-Dyck paths of semi-length \( n \) that either have no valleys or the last valley is at ground level. Note that \( b(n) - 1 \) is the \( n \)-th coefficient of the generating function \( Q(x, 1) \), see (8), or equivalently

\[
\sum_{n \geq 0} b(n)x^n = Q(x, 1) + \frac{1}{1 - x} = \frac{1 - x^2 - \sqrt{1 - 4x + 2x^2 + x^4}}{2(1 - x)x} = 1 + x + 2x^2 + 4x^3 + 9x^4 + 22x^5 + 57x^6 + 154x^7 + 429x^8 + \cdots.
\]

This generating function coincides with the generating function of the number of Dyck paths of semi-length \( n \) that avoid the subpath \( UUDU \). From Proposition 5 of [22] and [4, pp. 10] we conclude the following proposition.

**Proposition 3.2.** For all \( n \geq 0 \) we have

\[
b(n) = 1 + \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j} \binom{2n-3j}{n-j+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-k} \binom{n-k}{j} N(j, k),
\]

where \( N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1} \) are the Narayana numbers, with \( N(0, 0) = 1 \).
Remark 3.3. Let $\mathcal{B}(n) = Q_n \cup \{ U^n D^n \}$ denote the set of $(-1)$-Dyck paths having either no valleys or the last path is at height zero. Denote by $\mathcal{B}_{j,k}(n)$ the subset of $\mathcal{B}(n)$ of paths that contain exactly $j$ valleys where $k$ of those valleys are $(-1)$-valleys, i.e., there is a valley to the left of it, no valleys in between, and the heights differ by $-1$. Take a path $P \in \mathcal{B}_{j,k}(n)$, then $\rho(P) = j + 1$. Decompose the path as

$$P = U^n \Delta_i D^n U^s \Delta_{i_2} \cdots \Delta_{i_k} U^h D^b,$$

where there are $k$ occurrences of $D \Delta_p U$. So, there are $k$ red down steps indicating a $(-1)$-valley. Notice then that there are $n - k$ down steps that are not labeled red and they belong to pyramids, there is no restriction on those, so they can be represented as compositions of $n - k$ down steps on $j + 1$ parts, one per peak. They are counted by $\binom{n - k - 1}{j + 1 - 1} = \binom{n - k - 1}{j}$. This means that, numerically, $N(j, k + 1)$ corresponds to the sequence of $(-1)$-Dyck paths of semilength $j + k + 1$ containing $j$ valleys with $k$ of them being $(-1)$-valleys.

Theorem 3.4. The total number of paths in $\mathcal{D}_{-1}(n)$ is given by

$$r(n) = \sum_{\ell=0}^{n} \sum_{i=0}^{n-\ell-1} \binom{n-\ell-1}{i} q^{(i)}(\ell),$$

where

$$q^{(i)}(n) = \sum_{n_1 + n_2 + \cdots + n_i = n} b(n_1)b(n_2) \cdots b(n_i).$$

Proof. Let $Q^{(i)}(n)$ denote the set of $i$–tuples $(P_1, \ldots, P_i)$ of paths $P_j \in \mathcal{B} = \bigcup_{n \geq 0} \mathcal{B}(n)$, using the notation in Remark 3.3 above, such that $\ell(P_1) + \cdots + \ell(P_i) = n$. It is clear that $|Q^{(i)}(n)| = q^{(i)}(n)$. Notice that by definition of $\mathcal{B}$ we allow the empty path, $\lambda$, to be counted. Recall, also, that the number of compositions of $n - \ell$ on $i + 1$ positive integer parts, denoted as $C_{i+1}(n - \ell)$, is given by the binomial coefficient $\binom{(n-\ell)-1}{i+1-1}$.

Consider, then, the function

$$\varphi : \bigcup_{i,\ell} (Q^{(i)}(\ell) \times C_{i+1}(n - \ell)) \longrightarrow \mathcal{D}_{-1}(n),$$

defined by $\varphi((P_1, \ldots, P_i), (C_1, \ldots, C_{i+1})) = P$, where $P$ is the path described as $P = U^{C_1} M U^{C_2} \cdots$, where

$$M = \begin{cases} 
D^{C_1}, & \text{if } P_1 = \lambda; \\
P_1, & \text{if } P_1 = \Delta; \\
P_1 D, & \text{otherwise}. 
\end{cases}$$

Figure 6 shows two examples of how the function $\varphi$ works.
This function is a bijection, where the inverse function is given by decomposing a path by using the following algorithm:

**Algorithm 1** Inverse function $\phi$ (or reverse)

1. If there are $(-1)$-valleys, go to step 2. If there are no consecutive valleys with difference equal to $-1$, then the path is increasing and can be decomposed by using only pyramids $\Delta$ and $\lambda$ in the following way:
   - If there are no valleys, then the path is $\Delta_a = U^a D^a$ for some $a \geq 1$, return $(a)$.
   - If there is just one valley, the path is $U^a D^b U^c D^d$ for $a \geq b$ and $a + c = b + d$, if $a > b$ return $(a - b, U^b D^b, c)$ and if $a = b$ return $(a, \lambda, c)$.
   - For two consecutive valleys at the same height (not in the ground), place $(a, \Delta, b, \lambda)$ and start at the second valley. If they are in the ground, return $(a, \lambda, b, \lambda)$.
   - For two consecutive valleys that are not at the same height return $(a, \Delta_1, b, \Delta_2)$.
2. Find the rightmost $(-1)$-valley, i.e., locate the rightmost occurrence of $DU^k D^k D^U$. The right part of this string is increasing: go to step 1. Extract the maximal subword that is a Dyck path and call this path $P_i$, increase the value of $i$ and go to step 1 with the left part of the path.

For example, consider the path given in Figure 7. First one locates the rightmost $(-1)$-valley (all denoted by a red circle around them) and to the right then the first step says that it is (1). We take out the path $P_1$ and we locate the next $(-1)$-valley, the right part corresponds to $(1, \lambda, 1, \lambda, 1)$ and the left part of $P_2$ corresponds to $(1, UD, 1)$, and so the whole path is encoded by $(1, UD, 1, P_2, 1, \lambda, 1, \lambda, 1, P_1, 1)$. \[\Box\]
The Corollary 3.5 is direct consequence of the decomposition given in the proof of Theorem 3.1. The first result follows from Figure 4 and the second result uses the first part and the decomposition $UTD, U\Delta DT$, or $UQDT$ as given in the proof of Theorem 3.1.

**Corollary 3.5.** If $n > 1$, then these hold

1. If $q_n = |Q_n|$, then
   
   $$ q_n = 2q_{n-1} + q_{n-2} + q_{n-3} + \sum_{i=2}^{n-4} q_i(q_{n-i-1} - q_{n-i-2}) + 1, $$
   
   for $n > 3$, with the initial values $q_1 = 0$, $q_2 = 1$, and $q_3 = 3$.

2. If $r(n) = |D_d(n)|$, then
   
   $$ r(n) = 3r(n-1) - r(n-2) + q_{n-2} + \sum_{i=2}^{n-3} q_i(r(n-i-1) - r(n-i-2)), $$
   
   for $n > 3$, with the initial values $r(1) = 1$, $r(2) = 2$, and $r(3) = 5$.

The generating function of the sequence $r(n)$ is algebraic of order two, then $r(n)$ satisfies a recurrence relation with polynomial coefficients. This can be automatically solved with Kauers’s algorithm [19]. In particular we obtain that $r(n)$ satisfies the recurrence relation:

$$ 2nr(n) - 4nr(n+1) + (12 + 5n)r(n+2) - 4(15 + 4n)r(n+3) $$

$$ + 10(9 + 2n)r(n+4) - 2(21 + 4n)r(n+5) + (6 + n)r(n+6) = 0, \quad n \geq 6 $$

with the initial values $r(0) = 0, r(1) = 1, r(2) = 2, r(3) = 5, r(4) = 14$, and $r(5) = 41$.

In Theorem 3.6 we give an asymptotic approximation for the sequence $r(n)$. To accomplish this goal we use the singularity analysis method to find the asymptotes of the coefficients of a generating function (see, for example, [12] for the details).

**Theorem 3.6.** The number of $(−1)$-Dyck paths has the asymptotic approximation

$$ r(n) \sim \frac{\rho^{-n}}{\sqrt{n^3\pi}} \cdot \frac{\sqrt[4]{\rho(4 - 4\rho - 4\rho^3)}}{4(-1 + 4\rho - 2\rho^2)}. $$
Proof. The dominant singularity $\rho$ of the generating function $L(x)$ is the smallest real positive root of $1 - 4x + 2x^2 + x^4$. From a symbolic computation we find that

$$\rho = \frac{1}{3} \left( -1 - \frac{42^{2/3}}{\sqrt{13 + 3\sqrt{33}}} + \sqrt{2(13 + 3\sqrt{33})} \right) \approx 0.295598.$$

From the expression given in (9) for $L(x)$ we have

$$L(x) = \frac{-1 + 4x - 3x^2}{2(1 - 4x + 2x^2)} + \frac{\sqrt{1 - 4x + 2x^2 + x^4}}{2(1 - 4x + 2x^2)} \sim (x - \rho)^{1/2} \frac{\sqrt{\rho(4 - 4\rho - 4\rho^3)}}{2(1 - 4\rho + 2\rho^2)}$$

as $x \to \rho$.

Therefore,

$$r(n) \sim \frac{n^{-1/2-1} \sqrt{\rho(4 - 4\rho - 4\rho^3)}}{\rho^n(2\sqrt{\pi})} \frac{2(1 - 4\rho + 2\rho^2)}{\rho^n \sqrt{4(-1 + 4\rho - 2\rho^2)}}.$$ 

\[ \square \]

4. **The Area of the ($-1$)-Dyck paths**

In this section we use generating functions and recursive relations to analyze the distribution of the area of the paths in the set of restricted ($-1$)-Dyck paths. We recall that the area of a Dyck path is the sum of the absolute values of $y$-components of all points in the path. We use $\text{area}(P)$ to denote the area of a path $P$. From Figure 1 on Page 2, we can see that $\text{area}(P) = 70$. We use $a(n)$ to denote the total area of all paths in $\mathcal{D}_{-1}(n)$. In Theorem 4.1 we give a generating function for the sequence $a(n)$. We now introduce a bivariate generating function depending on this previous parameter and $\ell(P)$ (the semi-length of $P$). So,

$$A(x, q) := \sum_{P \in \mathcal{D}_{-1}} x^{\ell(P)} q^{\text{area}(P)}.$$

We now give again some terminology needed for the following theorems. Let $\mathcal{Q} \subset \mathcal{D}_{-1}(n)$ be the set formed by all paths having at least one valley, were the last valley is at ground level; let $\mathcal{Q}_n \subset \mathcal{Q}$ be the set formed by all paths of semi-length $n$, and let $q_n = |\mathcal{Q}_n|$.

**Theorem 4.1.** The generating function for the sequence $a(n)$ is given by

$$V(x) = \sum_{n \geq 0} a(n) x^n = \frac{b(x) - c(x) \sqrt{1 - 4x + 2x^2 + x^4}}{(1 - x)^2(1 - 4x + 2x^2)^3(1 - 3x - x^2 - x^3)},$$

where

$$b(x) = 2x - 23x^2 + 107x^3 - 262x^4 + 359x^5 - 256x^6 + 82x^7 - 5x^8 - 10x^9 + 6x^{10},$$

$$c(x) = x - 10x^2 + 41x^3 - 89x^4 + 108x^5 - 73x^6 + 18x^7 + 2x^8.$$

Proof. From the decomposition $UD$, $UTD$, $U\Delta DT$, or $UQDT$ given in the proof of Theorem 3.1 we obtain the functional equation

$$A(x, q) = xq + xqA(xq^2, q) + E(x, q)A(x, q) + xqB(xq^2, q)A(x, q),$$

(10)
where \( E(x, q) := \sum_{j \geq 1} x^j q^j \) and \( B(x, q) := \sum_{P \in \mathcal{Q}} x^{\ell(P)} q^{area(P)} \). Note that \( E(x, q) \) corresponds to the generating function that counts the total number of non-empty pyramids in the given decomposition.

From the decomposition given in Figure 4, we obtain the functional equation

\[
B(x, q) = E(x, q)^2 + E(x, q)B(x, q) + xqB(q^2x, q)B(x, q) + xqB(q^2x, q)E(x, q).
\]

Let \( M(x) \) be the generating function of the total area of the \((-1)\)-Dyck paths in \( \mathcal{Q} \). From the definition of \( A(x, q) \) and \( B(x, q) \) we have

\[
V(x) = \left. \frac{\partial A(x, q)}{\partial q} \right|_{q=1} \quad \text{and} \quad M(x) = \left. \frac{\partial B(x, q)}{\partial q} \right|_{q=1}.
\]

Therefore, differentiating (11) with respect to \( q \) we obtain,

\[
M(x) = \frac{2x^2(1 + x)}{(1 - x)^4} + \frac{x(x + 1)}{(1 - x)^3} Q(x) + \frac{x}{1 - x} M(x) + xQ(x)^2 + x \left( M(x) + 2x \frac{\partial Q(x)}{\partial x} \right) \left( Q(x) + \frac{x}{1 - x} \right) + xQ(x) \left( M(x) + \frac{x}{1 - x} \right) + xQ(x) \frac{x(x + 1)}{(1 - x)^2},
\]

where \( Q(x) := Q(x, 1) \) and \( Q(x, y) \) is the generating function given in (8) on Page 8.

Now, differentiating (10) with respect to \( q \) we obtain,

\[
V(x) = x + xL(x) + x \left( V(x) + 2x \frac{\partial L(x)}{\partial x} \right) + \frac{x(x + 1)}{(1 - x)^3} L(x) + \frac{x}{1 - x} V(x) + xQ(x)L(x) + x \left. \frac{\partial B(xq^2, q)}{\partial q} \right|_{q=1} L(x) + xQ(x)V(x).
\]

Solving (11) for \( B(x, q) \) and substituting into (12) and then solving the resulting expression for \( V(x) \) we obtain the desired result. \( \square \)

The first few values of the series of \( V(x) \) are

\[
V(x) = \sum_{n \geq 1} a(n)x^n = x + 6x^2 + 29x^3 + 130x^4 + 547x^5 + 2198x^6 + 8551x^7 + 32508x^8 + \cdots.
\]

We recall that for simplicity we use \( r(n) \) instead of \( r_{-1}(n) \).

**Theorem 4.2.** If \( n \geq 1 \), then these hold

1. If \( A_n \) is the total area of all paths in \( \mathcal{Q}_n \), then
   \[
   A_n = 2A_{n-1} + A_{n-2} + 2A_{n-3} + q_n - q_{n-1} + 2nq_{n-2} + 2(n - 5)q_{n-3} + 4n^2 - 14n + 13 + \sum_{i=2}^{n-4} 2(A_i + iq_i + i(i + 1))(q_{n-i-1} - q_{n-i-2}), \quad n > 4,
   \]
   with the initial values \( A_1 = 0, A_2 = 2, A_3 = 13, \) and \( A_4 = 58 \).
(2) The sequence \(a(n)\) satisfies the recursive relation

\[
a(n) = 3a(n - 1) - a(n - 2) + A_{n-2} + 2(n - 1)q_{n-2} + 2nr(n - 1) + 2(3 - n)r(n - 2) - 4r(n - 3) + (n - 1)^2 + \sum_{i=3}^{n-2}q_{i-1}(a(n - i) - a(n - i - 1)) + \sum_{i=3}^{n-2}(A_{i-1} + (2i - 1)q_{i-1} + i^2)(r(n - i) - r(n - i - 1)).
\]

Proof. We prove Part (1), constructing a recursive relation for the total area of \(Q_n\). This part of the proof is divided into four cases, and we use \(P \setminus T\) to denote the subpath resulting after removing the subpath \(T\) from the path \(P\).

Case 1. We observe that for a fixed \(i \in \{1, 2, \ldots, n - 1\}\) there is exactly one path in \(Q_n\) of the form \((XY)^i(XY)^{n-i}\), where its area is equal to \(i^2 + (n - i)^2\). So, the total area of these types of paths is \(\sum_{i=1}^{n-1}(i^2 + (n - i)^2) = n(n - 1)(2n - 1)/3\).

Case 2. In this case, we find the area of all paths of the form \(P_i := X'Y'Q\). Note that in \(P_i\) the first pyramid is of height one and \(Q \in Q_{n-1}\) and in \(P_{n-2}\) the first pyramid is of height \(n - 2\) and \(Q \in Q_2\). These give that all first pyramids \((XY)^i\) run for \(i \in \{1, 2, \ldots, n - 2\}\) and \(Q_j\) runs for \(j \in \{2, \ldots, n - 1\}\).

Now from the definition of \(P_i\), we have that for a fixed \(i\) there are \(q_{n-i}\) paths of form \(P_i\) (that is, having a pyramid \((XY)^i\) in the beginning of the path). So, the contribution to the area given by all first pyramids of the form \((XY)^i\), overall paths of the form \(P_i\), is equal to \(i^2 \times q_{n-i}\). This and the fact that \(A_{n-j}\) is the area of \(Q_{n-j}\), imply that the total area of all paths of the form \(P_i\) is given by \(i^2 q_{n-i} + A_i\). Therefore, the total area of these types of paths is \(\sum_{i=1}^{n-2}i^2 q_{n-i} + \sum_{j=2}^{n-1}A_j\).

Case 3. In this case we find the area of all paths of the form \(H_i := XQ_iY(XY)^i\) where \(Q_{\ell} \in Q_{n_i-1}\) Note that similar to the Case 2, the last pyramids \((XY)^i\) run for \(i \in \{1, 2, \ldots, n - 3\}\) and \(Q_j\) runs for \(j \in \{2, \ldots, n - 2\}\). We now observe that for a fixed \(i\) there are \(q_{n-i-1}\) paths of form \(H_i\) (that is, having a pyramid \((XY)^i\) in the end of the path). The contribution to the area given by all last pyramids of the form \((XY)^i\), overall paths of the form \(H_i\), is equal to \(i^2 \times q_{n-i-1}\).

We analyze the contribution to the desired area given by \(XQ_{\ell}Y = H_{n-i-1} \setminus (XY)^{n-i-1}\) with \(Q_{\ell} \in Q_i\). For a fixed \(i \in \{2, 3, \ldots, n - 2\}\) there are \(q_i\) paths of form \(H_{n-i-1}\) having a first subpath of the form \(XQ_{\ell}Y\). Note that \(X\) and \(Y\) give rise to a trapezoid, where the two parallel sides have lengths 2\(i\) and 2\(i + 2\), giving rise to an area of 2\(i + 1\). So, for a fixed \(i\), the contribution to the area given by all first subpaths of the form \(XQ_{\ell}Y\) is equal to the area of the trapezoids plus the area of all paths of the form \(Q_{\ell}\) (these are on top of the trapezoids). That is, the area of a trapezoid multiplied by the total number of the paths of the form \(Q_{\ell}\), plus the area of all paths of the form \(Q_{\ell}\). Thus, the contribution to the area given by first subpaths of the form \(XQ_{\ell}Y\) (overall paths of the form \(H_i\), for a fixed \(i\)), is \(((2i + 1) \times q_i + A_i)\).
We conclude that the total area of these types of paths is
\[
\sum_{i=1}^{n-3} i^2 \times q_{n-i-1} + \sum_{i=2}^{n-2} ((2i + 1) \times q_i + A_i).
\]

**Case 4.** Finally, we find the area of all paths of the form \( T_i := XQ'YQ'' \) where \( Q' \in Q_i \) and \( Q'' \in Q_{n-i-1} \) for \( i \in \{2, 3, \ldots, n-3\} \). First of all, we analyze the contribution to the desired area given by all paths of the form \( Q'' \in Q_{n-i-1} \) (overall paths of the form \( T_i \) for a fixed \( i \)). Since \( Q' \in Q_i \), we know that for a given path \( Q \in Q_{n-i-1} \) there are as many paths of the form \( XQ'YQ \) as paths in \( Q_i \). Thus, for a fixed \( i \in \{2, 3, \ldots, n-3\} \) we find

\[
\text{the area given by all subpaths } T_i \setminus XQ'Y \text{ for every } Q' \in Q_i.
\]

Thus, the area of all subpaths of the form \( Q'' \in Q_{n-i-1} \) is, that is, clearly, equal to \( A_{n-i-1}q_i \).

We now analyze the contribution to the desired area given by all subpaths of the form \( XQ'Y \). That is, the area of all subpaths \( T_i \setminus Q'' \) (overall paths of the form \( T_i \) for a fixed \( i \)). It is easy to see that for a fixed \( i \in \{2, 3, \ldots, n-3\} \) there are \( q_{n-i-1} \) subpaths of the form \( XQ'Y \). Note that \( X \) and \( Y \) give rise to a trapezoid, where the two parallel sides have lengths \( 2i \) and \( 2i + 2 \), giving rise to an area of \( 2i + 1 \). So, the contribution to the area given by the first subpaths of the form \( XQ'Y \) is equal to the area of the trapezoids plus the area of all paths of the form \( Q' \) (these are on top of the trapezoids). Thus, the area of a trapezoid multiplied by the total number of the paths of the form \( Q' \) plus the area of all paths of the form \( Q' \) and then all of these multiplied by the total number of paths of the form \( Q'' \). Thus, the contribution to the area given by the first subpaths of the form \( XQ'Y \) (overall paths of the form \( T_i \) for a fixed \( i \)), is

\[
((2i + 1) \times q_{i}q_{n-i-1} + A_{i}q_{n-i-1}).
\]

We conclude that the total area of these types of paths is

\[
\sum_{i=2}^{n-3} A_{n-i-1}q_i + \sum_{i=2}^{n-3} ((2i + 1) \times q_{i}q_{n-i-1} + A_{i}q_{n-i-1}).
\]

Adding the results from Cases 1-4, we obtain that the recursive relation for the area \( A_n \) is given by

\[
A_n = \sum_{i=1}^{n-1} (i^2 + (n - i)^2) + \sum_{i=1}^{n-2} i^2 q_{n-i} + \sum_{i=2}^{n-1} A_i + \sum_{i=2}^{n-3} (2i + 1)q_{i}q_{n-(i+1)} + \sum_{i=2}^{n-3} A_{i}q_{n-(i+1)} + \sum_{i=2}^{n-3} A_{i}q_{n-(i+1)} + \sum_{i=1}^{n-2} i^2 q_{n-(i+1)} + \sum_{i=2}^{n-2} (2i + 1)q_i.
\]

Subtracting \( A_n \) from \( A_{n+1} \) and simplifying we have

\[
A_n = 2A_{n-1} + A_{n-2} + 2A_{n-3} + (2n - 5)q_{n-3} + (2n - 4)q_{n-2} + q_{n-1} + 4n^2 - 14n + 15 + \sum_{i=2}^{n-4} (2A_i + (2i + 1)q_i)(q_{n-i-1} - q_{n-i-2}) + \sum_{i=2}^{n-3} (2i^2 - 2i + 1)(q_{n-i} - q_{n-i-1}).
\]

We now rearrange this expression to obtain \( q_n \) (see the expression within brackets) given in Corollary 3.5.
\[ A_n = 2A_{n-1} + A_{n-2} + 2A_{n-3} + (2n - 6)q_{n-3} + (2n - 4)q_{n-2} - q_{n-1} + 4n^2 - 14n + 13 + \]
\[ \sum_{i=2}^{n-4} 2(A_i + iq_i)(q_{n-i-1} - q_{n-i-2}) + \sum_{i=2}^{n-3} 2 \left( i^2 - i \right)(q_{n-i} - q_{n-i-1}) \]
\[ + [2q_{n-1} + q_{n-2} + q_{n-3} + \sum_{i=2}^{n-4} q_i(q_{-i+n-1} - q_{-i+n-2}) + 1]. \]

After some simplifications we obtain the desired recursive relation.

Proof of Part (2). This part is similar to Part 1. However, in this proof we need to use: \( D_{-1}(j) \), \( r(i) = |D_{-1}(i)| \), \( Q_j, q_j = |Q_j| \), and \( A_i \).

**Case 1.** We find the area of all paths of the form \( XQY \), where \( Q \in D_{-1}(n - 1) \). Note that \( X \) and \( Y \) give rise to a trapezoid of area equal to \( 2n - 1 \); this area multiplied by \( r(n - 1) = |D_{-1}(n - 1)| \) gives that the total area of the trapezoids is \( (2n - 1)r(n - 1) \). The total area of all paths of the form \( XQY \) is given by the area of all trapezoids and the area of all paths that are on top of the trapezoids. That is, the area of these types of paths is \( (2n - 1)r(n - 1) + a(n - 1) \).

**Case 2.** In this case, we find the area of all paths of the form \( K_i := X^iY^iQ_{\ell} \), where \( Q_{\ell} \in D_{-1}(n - i) \) and \( i \in \{1, 2, \ldots, n - 1\} \). Since \( r(n - i) = |D_{-1}(n - i)| \), we conclude that for a fixed \( i \) there are \( r(n - i) \) paths of form \( K_i \). So, the contribution to the area given by all first pyramids of the form \( (XY)^i \), overall paths of the form \( K_i \), is equal to \( i^2 \times r(n - i) \). This and the fact that \( a(n - j) \) is the area of \( Q_{n-j} \), imply that the total area of all paths of the form \( K_i \) is given by \( i^2 \times r(n - i) + a(n - i) \). Therefore, the total area of these type of paths is \( \sum_{i=1}^{n-1} i^2 \times r(n - i) + a(n - i) \).

**Case 3.** Finally, we find the area of all paths of the form \( M_i := XQ'YD \) where \( Q' \in Q_i \) and \( D \in D_{-1}(n - i - 1) \) for \( i \in \{2, 3, \ldots, n - 2\} \). First of all, we analyze the contribution to the desired area given by all paths \( D \in D_{-1}(n - i - 1) \) (overall paths of the form \( M_i \) for a fixed \( i \)). Since \( Q' \in Q_i \), we know that for a given path \( D' \in D_{-1}(n - i - 1) \) there are as many paths of the form \( XQ'YD' \) as paths in \( Q_{i} \). Thus, for a fixed \( i \in \{2, 3, \ldots, n - 2\} \) we find the area given by all subpaths \( M_i \setminus XQ'Y \) for every \( Q' \in Q_i \). That is, the area of all subpaths of \( M_i \) of the form \( D \in Q_{n-i-1} \) is equal to \( a(n - i - 1)q_i \).

We now analyze the contribution to the desired area given by all subpaths of the form \( XQ'Y \) for every \( Q' \in Q_i \). That is, the area of all subpaths \( M_i \setminus D \) (overall paths of the form \( M_i \) for a fixed \( i \)). It is easy to see that for a fixed \( i \in \{2, 3, \ldots, n - 2\} \) there are \( r(n - i - 1) \) subpaths of the form \( XQ'Y \). Note that \( X \) and \( Y \) give rise to a trapezoid, where the two parallel sides have lengths \( 2i \) and \( 2i + 2 \), giving rise to an area of \( 2i + 1 \). So, the contribution to the area given by the first subpaths of the form \( XQ'Y \) is equal to the area of the trapezoids plus the area of all paths of the form \( Q' \) (these are on top of the trapezoids). Thus, the area of a trapezoid multiplied by the total number of the paths of the form \( Q' \) plus the area of all paths of the form \( Q' \) and then all of these multiplied by the total number of paths of the form \( D \). Thus, the contribution to the area given
by the first subpaths of the form $XQ'Y$ (overall paths of the form $M_i$ for a fixed $i$), is

\((2i + 1) \times q_ir(n - i - 1) + A_i r(n - i - 1)).\)

We conclude that the total area of these types of paths is

\[
\sum_{i=2}^{n-2} A_i r(n - i - 1) + \sum_{i=2}^{n-2} (2i + 1) \times q_ir(n - i - 1).
\]

Adding the results from Cases 1-3, we obtain that the recursive relation for the area $a(n)$ is given by

\[
a(n) = a(n-1) + (2n-1)r(n-1) + \sum_{i=1}^{n-1} i^2 r(n - i) + \sum_{i=1}^{n-1} a(n - i)

+ \sum_{i=2}^{n-2} q_i a(n - i - 1) + \sum_{i=2}^{n-2} A_i r(n - i - 1) + \sum_{i=2}^{n-2} (2i + 1)q_ir(n - i - 1).
\]

Subtracting $a(n)$ from $a(n + 1)$ and simplifying we have

\[
a(n) = 3a(n-1) - a(n-2) + A_{n-2} + 2(n-1)q_{n-2} + (2n-1)r(n-1) + (3-2n)r(n-2) + (n-1)^2

+ \sum_{i=3}^{n-2} q_{i-1}(a(n - i) - a(n - i - 1)) + \sum_{i=3}^{n-2} A_{i-1}(r(n - i) - r(n - i - 1))

+ \sum_{i=3}^{n-2} (2i - 1)q_{i-1}(r(n - i) - r(n - i - 1)) + \sum_{i=1}^{n-2} i^2(r(n - i) - r(n - i - 1)).
\]

After some other simplifications we have that

\[
a(n) = 3a(n-1) - a(n-2) + A_{n-2} + 2(n-1)q_{n-2} + 2nr(n-1)

+ 2(3-n)r(n-2) - 4r(n-3) + (n-1)^2 + \sum_{i=3}^{n-2} q_{i-1}(a(n - i) - a(n - i - 1))

+ \sum_{i=3}^{n-2} (A_{i-1} + (2i - 1)q_{i-1} + i^2) (r(n - i) - r(n - i - 1)).
\]

This completes the proof. \(\square\)

Notice that the total area of the Dyck paths (cf. [23]) is given by $4^n - \binom{2n+1}{n}$.

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