Spontaneously Broken Translational Invariance of Compactified Space

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Abstract

We propose a mechanism to break the translational invariance of compactified space spontaneously. As a simple model, we study a real $\phi^4$ model compactified on $M^{D-1} \otimes S^1$ in detail, where we impose a nontrivial boundary condition on $\phi$ for the $S^1$-direction. It is shown that the translational invariance for the $S^1$-direction is spontaneously broken when the radius $R$ of $S^1$ becomes larger than a critical radius $R^*$ and also that the model behaves like a $\phi^4$ model on a single kink background for $R \to \infty$. It is pointed out that spontaneous breakdown of translational invariance is accompanied by that of some global symmetries, in general, in our mechanism.

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Spacetime translational invariance can be broken spontaneously if any local operators acquire spacetime-dependent vacuum expectation values. This situation, however, seems implausible because a configuration which minimizes a potential is, in general, independent of spacetime coordinates and further because the spacetime-dependent vacuum configuration would produce nonzero kinetic energy, so that energetically such the configuration would be unfavorable. One might expect that even if some of space dimensions are compactified, the translational invariance of the compactified space (if exists) could not be broken spontaneously.

The purpose of this paper is to propose a mechanism to break the translational invariance of compactified spaces spontaneously. To illustrate our mechanism, let us consider a real $\phi^4$ model in $D$ dimensions

$$S = \int d^Dx \left\{ -\frac{1}{2} \partial_M \phi \partial^M \phi - V(\phi) \right\} ,$$

where the index $M$ runs from 0 to $D - 1$ and

$$V(\phi) = \frac{\lambda}{8} \left( \phi^2 - \frac{2\mu^2}{\lambda} \right)^2 .$$

We should note that the action has a $Z_2$ symmetry

$$\phi \to -\phi .$$

It turns out that the existence of global symmetries is crucial to our mechanism and that the above $Z_2$ symmetry plays an important role in this model. One may conclude that the ground state would be given by $\phi = \pm \sqrt{2} \mu / \lambda$, at which the scalar potential $V(\phi)$ takes the minimum value and the $Z_2$ symmetry is broken. This is, however, a hasty conclusion, as we will see below.

Let us suppose that one of the space coordinates, say, $y \equiv x^{D-1}$ is compactified on a circle $S^1$ whose radius is $R$. Since $S^1$ is multiply-connected and the action has the $Z_2$ symmetry, we can impose the following nontrivial boundary condition associated with the $Z_2$ symmetry for the field $\phi$:

$$\phi(x^\mu, y + 2\pi R) = -\phi(x^\mu, y) ,$$

where $x^\mu$ denote the coordinates of the uncompactified spacetime. Thanks to the $Z_2$ symmetry, the action (1) is still single-valued even with the boundary condition (4). An
important consequence of the nontrivial boundary condition (4) is that any vacuum expectation value of $\phi(x^\mu, y)$ cannot be a ($y$-independent) nonzero constant, which is inconsistent with the boundary condition (4). In other words, any nonzero vacuum expectation value of $\phi(x^\mu, y)$ should have the $y$ dependence in order to satisfy the boundary condition (4), i.e.

$$\langle \phi(x^\mu, y) \rangle \neq 0 \quad \rightarrow \quad \frac{\partial}{\partial y} \langle \phi(x^\mu, y) \rangle \neq 0.$$  

(5)

It immediately follows that if the vacuum is translationally invariant the vacuum expectation value of $\phi(x^\mu, y)$ has to vanish, or conversely that if $\phi(x^\mu, y)$ acquires a nonzero vacuum expectation value, which implies the $y$ dependence of $\langle \phi(x^\mu, y) \rangle$, then the translational invariance for the $S^1$-direction is spontaneously broken.

In order to find a vacuum configuration, one might try to minimize the potential $V(\phi)$. This is, however, wrong in the present model. To find a vacuum configuration, we should take account of kinetic terms in addition to potential terms since the translational invariance could be broken and then the vacuum configuration might be coordinate-dependent. Since the translational invariance of the (uncompactified) $(D - 1)$-dimensional Minkowski spacetime is expected to be unbroken, finding a vacuum configuration of the model may be equivalent to solving a minimization problem of the following functional

$$\mathcal{E}[\phi, R] \equiv \int_0^{2\pi R} dy \left\{ \frac{1}{2} (\partial_y \phi)^2 + V(\phi) \right\}.$$  

(6)

In this paper, our analysis will be restricted to the tree level.

In the following, we shall ignore the $x^\mu$ dependence in $\phi$ since we are interested in a vacuum configuration, for which the translational invariance of the $(D - 1)$-dimensional Minkowski spacetime is assumed to be unbroken. It should be emphasized that $\phi(y)$ cannot be an arbitrary function but has to obey the (antiperiodic) boundary condition

$$\phi(y + 2\pi R) = -\phi(y).$$  

(7)

If the translational invariance for the $S^1$-direction is unbroken, the vacuum expectation value of $\phi$ has to vanish and then the functional $\mathcal{E}[\phi, R]$ becomes

$$\mathcal{E}[\phi = 0, R] = \frac{\pi R \mu^4}{\lambda}.$$  

(8)

$^\S$The $\mathcal{E}[\phi]$ may be regarded as a potential from a viewpoint of the $(D - 1)$-dimensional Minkowski spacetime.
If there exists any configuration of $\phi(y)$ such that
\[ E[\phi, R] < E[\phi = 0, R] = \frac{\pi R \mu^4}{\lambda}, \tag{9} \]
them the configuration $\phi = 0$ is no longer a vacuum configuration and hence the translational invariance for the $S^1$-direction has to be broken, as discussed previously. Before minimizing the functional $E[\phi, R]$, we would like to make a comment about solutions to the field equation for $\phi(y)$
\[ 0 = \frac{\delta E[\phi, R]}{\delta \phi(y)} = -\frac{d^2 \phi(y)}{dy^2} + \frac{\lambda}{2} \phi(y) \left( (\phi(y))^2 - 2 \frac{\mu^2}{\lambda} \right). \tag{10} \]
We first note that $\phi = 0$ is a trivial solution to eq.(10), while another trivial (would-be) solutions $\phi = \pm \sqrt{\frac{2}{\pi} \mu}$ have to be excluded due to the boundary condition (7). Using the above field equation to eliminate the term $\frac{1}{2} (\partial_y \phi(y))^2$ in eq.(8), we find
\[ E[\phi, R] \bigg|_{\phi = 0} = \frac{\pi R \mu^4}{\lambda} - \int_0^{2\pi R} dy \frac{\lambda}{8} (\phi(y))^4 \leq E[\phi = 0, R]. \tag{11} \]
Since the equality on the last line holds only when $\phi = 0$, we have thus arrived at an important conclusion that if there appear nontrivial solutions $\phi$ to the field equation (10), then $E[\phi, R]$ is lower than $E[0, R]$ so that the translational invariance for the $S^1$-direction is broken spontaneously with the $Z_2$ symmetry breaking.

Let us now proceed to find a vacuum configuration, which minimizes the functional $E[\phi, R]$. To this end, we shall first construct whole solutions to the field equation (10) with the boundary condition (7), which are candidates of a vacuum configuration. We shall then determine which configuration gives the lowest value of $E[\phi, R]$ (if there exist several solutions). The field equation (10) has been studied before in a quite different context [1, 2], though the boundary condition has been imposed to be periodic but not antiperiodic. It turns out that most of the results given in ref.[1, 2] are useful for our purposes and that the nontrivial solutions to our problem will be given by
\[ \phi(y) = \frac{2k \omega}{\sqrt{\lambda}} \text{sn}(\omega(y - y_0), k), \tag{12} \]
where
\[ \omega \equiv \frac{\mu}{\sqrt{1 + k^2}}. \tag{13} \]
Here, \( \text{sn}(u, k) \) is the Jacobi elliptic function whose period is \( 4K(k) \), where \( K(k) \) denotes the complete elliptic function of the first kind. Since the integration constant \( y_0 \) in eq. (12), which in fact reflects the translational invariance of the equation of motion, is irrelevant, we shall set \( y_0 \) to be equal to zero in the following analysis. The antiperiodic boundary condition (7) requires that the parameter \( k (0 \leq k < 1) \) and the radius \( R \) should be related mutually through

\[
R = (2n - 1) \frac{K(k)}{\pi \omega} \quad (14)
\]

for some positive integer \( n \). (For the periodic boundary condition, \( 2n - 1 \) in eq. (14) should be replaced by \( 2n \).) We may denote a solution specified by eq. (14) with an integer \( n \) by \( \phi_n(y) \). We note that as \( k \) runs from zero to one the right hand side of eq. (14) increases monotonically from \( R_n^* \equiv (n - \frac{1}{2})/\mu \) to infinity. Thus, \( \phi_n(y) \) is a solution only when \( R \geq R_n^* \).

For \( 0 < R \leq R_n^* \), there exists only one solution to the field equation (10), i.e. the trivial solution \( \phi = 0 \). Thus, the vacuum configuration is given by the trivial solution, and hence the translational invariance is unbroken for \( 0 < R \leq R_n^* \). For \( R_1^* < R \leq R_2^* \), there exist two solutions to eq. (10), i.e. the trivial one and \( \phi_1(y) \). It follows from eq. (11) that the trivial solution \( \phi = 0 \) is no longer the vacuum configuration. Since \( \phi_1(y) \) depends on \( y \), the translational invariance for the \( S^1 \)-direction is spontaneously broken.

For \( R_2^* < R \leq R_{n+1}^* \), there exist \( n + 1 \) solutions to eq. (10), i.e. the trivial one and \( \phi_m(y) \) for \( m = 1, 2, \cdots, n \). Since \( \mathcal{E}[\phi_m, R] < \mathcal{E}[0, R] \) for every \( m \), the trivial solution is no longer the vacuum configuration, and hence the translational invariance for the \( S^1 \)-direction is spontaneously broken for \( R_n^* < R \leq R_{n+1}^* \). Therefore, we have found that for \( 0 < R \leq R_1^* \) the translational invariance for the \( S^1 \)-direction is unbroken, while for \( R > R_1^* \) it is broken spontaneously with the \( Z_2 \) symmetry breaking.

Let us next discuss a problem which solution \( \phi_n(y) \) is the true vacuum configuration, i.e. which solution minimizes the functional \( \mathcal{E}[\phi, R] \). For \( R > R_n^* \), \( \phi_n(y) \) becomes a solution for which we obtain a rather complicated expression

\[
\mathcal{E}[\phi_n, R] = \frac{(2n - 1)\mu^3}{3\lambda(1 + k^2)^{3/2}} \left\{ -(1 - k^2)(5 + 3k^2)K(k) + 8(1 + k^2)E(k) \right\}, \quad (15)
\]

where \( E(k) \) is the complete elliptic function of the second kind. Although we could directly compare \( \mathcal{E}[\phi_n, R] \) for \( n = 1, 2, 3, \cdots \), we shall here take another approach to solve
the problem. It is not difficult to show
\[
\frac{dE[\phi_n, R]}{dR} = \frac{\pi \mu^4}{\lambda} \left( \frac{1 - k^2}{1 + k^2} \right)^2 \geq 0,
\]
which implies that \( E[\phi_n, R] \) is a monotonically increasing function of \( R \). At \( R = R^*_n \) \((k = 0)\) and \( \infty \) \((k = 1)\), \( E[\phi_n, R] \) takes the values
\[
E[\phi_n, R^*_n] = (2n - 1) \frac{\pi \mu^3}{2 \lambda},
\]
\[
E[\phi_n, \infty] = (2n - 1) \frac{4 \sqrt{2} \mu^3}{3 \lambda},
\]
respectively. It follows from eqs.\((16)\) and \((17)\) that
\[
(2n - 1) \frac{\pi \mu^3}{2 \lambda} \leq E[\phi_n, R] < (2n - 1) \frac{4 \sqrt{2} \mu^3}{3 \lambda}
\]
and especially
\[
E[\phi_1, R] < \frac{4 \sqrt{2} \mu^3}{3 \lambda}.
\]
The above observations will be enough to show that for \( R \geq R^*_n \) \((n \geq 2)\)
\[
E[\phi_1, R] < E[\phi_n, R] < E[0, R].
\]
Therefore, we have found the vacuum expectation value of \( \phi \) to be
\[
\langle \phi(x^n, y) \rangle = \begin{cases} 0 & \text{for } R \leq R^*_1 \\ \phi_1(y) & \text{for } R > R^*_1. \end{cases}
\]

It may be instructive to reanalyze the model from a viewpoint of the Fourier expansion.
It follows from the boundary condition \((7)\) that \( \phi(y) \) may be expanded in the Fourier-series for the \( S^1 \)-direction as
\[
\phi(y) = \frac{1}{\sqrt{\pi R}} \sum_{l=1}^\infty \left\{ a^{(2l-1)} \cos \left( (2l - 1) \frac{y}{2R} \right) + b^{(2l-1)} \sin \left( (2l - 1) \frac{y}{2R} \right) \right\},
\]
or equivalently,
\[
\phi(y) = \frac{1}{\sqrt{2\pi R}} \sum_{l \in \mathbb{Z}} \varphi^{(2l-1)} e^{i(2l-1)\frac{y}{2R}}
\]
with \( \varphi^{(2l-1)} = \frac{1}{\sqrt{2}}(a^{(2l-1)} - ib^{(2l-1)}) = \varphi^{(-2l+1)*} \). A key observation is that a constant zero mode is excluded in the above expansion due to the nontrivial boundary condition.
Inserting eq.\((22)\) into eq.\((8)\), we have, up to the quadratic terms with respect to \( \varphi^{(2l-1)} \),
\[
E_0[\varphi, R] = \sum_{l=1}^\infty m_{(2l-1)}^2 |\varphi^{(2l-1)}|^2,
\]
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\]
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\]
where
\[ m_{(2l-1)}^2 \equiv -\mu^2 + \left( \frac{2l-1}{2R} \right)^2. \] (24)

The second term in eq. (24) is the Kaluza-Klein mass, which comes from the “kinetic” term \( \frac{1}{2} (\partial_y \phi(y))^2 \), and which gives a positive contribution to the squared mass term. We can easily see that for \( R \leq R_1^* \) all the squared masses \( m_{(2l-1)}^2 \) are positive semi-definite because of the induced Kaluza-Klein masses \( \left( \frac{2l-1}{2R} \right)^2 \). On the other hand, for \( R > R_1^* \) it seems that negative squared masses appear. This is a signal of a phase transition and is consistent with the results obtained before.

It may be interesting to point out that the translational invariance for the \( S^1 \)-direction can be reinterpreted as a global \( U(1) \) symmetry, which is in fact possessed by the theory after the compactification. To see this, we note that an infinitesimal translation \( y \to y + a \) in eq. (22) can equivalently be realized, in terms of the Fourier modes, by the following transformation:
\[ \phi^{(2l-1)} \rightarrow e^{i \frac{2l-1}{2R} a} \phi^{(2l-1)}, \] (25)
from which we may assign a \( U(1) \) charge \( \frac{2l-1}{2R} \) to \( \phi^{(2l-1)} \). Thus, spontaneous breakdown of the translational invariance for the \( S^1 \)-direction may be interpreted as that of the \( U(1) \) symmetry. One might then ask about a Nambu-Goldstone mode associated with spontaneous breakdown of the translational invariance or the \( U(1) \) symmetry. It turns out that, to answer the question, the following mode expansion is more suitable than the Fourier mode expansion for \( R > R_1^* \) [1, 2]:
\[ \phi(y) = \sum_{l=1}^{\infty} \left\{ a^{(2l-1)} Ec^{2l-1}(\omega y, k) + b^{(2l-1)} Es^{2l-1}(\omega y, k) \right\}, \] (26)
where \( Ec^{2l-1}(u, k) \) and \( Es^{2l-1}(u, k) \) are eigenfunctions of the so-called Lamé equation with \( N = 2 \) [3]
\[ \left[ -\frac{d^2}{du^2} + N(N + 1)k^2 \text{sn}^2(u, k) \right] \Psi(u, k) = \Omega(k) \Psi(u, k), \] (27)
with the boundary condition
\[ \Psi(u + 2K(k), k) = -\Psi(u, k). \] (28)

The \( Ec^{2l-1}(u, k) \) and \( Es^{2l-1}(u, k) \) may further be supplemented by the following conditions [4]:
\[ Ec^{2l-1}(-u, k) = +Ec^{2l-1}(u, k), \] (29)
\[ Es^{2l-1}(-u, k) = -Es^{2l-1}(u, k). \] (30)

\*The eigenfunctions \( Ec \) and \( Es \) are differently defined from those in ref. [3].
\[ E_{s}^{2l-1}(-u, k) = -E_{s}^{2l-1}(u, k), \]  \hspace{1cm} (29)

and

\[ E_{c}^{2l-1}(u, k) \rightarrow \frac{1}{\sqrt{\pi R}} \cos ((2l-1)u) \quad \text{as} \quad k \rightarrow 0, \]

\[ E_{s}^{2l-1}(u, k) \rightarrow \frac{1}{\sqrt{\pi R}} \sin ((2l-1)u) \quad \text{as} \quad k \rightarrow 0. \]  \hspace{1cm} (30)

In the expansion (26), \( a'(2l-1) \) and \( b'(2l-1) \) correspond to normal modes around the background \( \phi = \phi_1(y) \). If we set \( \Omega(k) = (1 + k^2)(1 + \frac{m^2}{\mu^2}) \), \( m^2 \) may correspond to a squared mass in \((D - 1)\)-dimensional Minkowski spacetime. The lowest five eigenvalues and eigenfunctions for the Lamé equation with \( N = 2 \) are exactly known, and the eigenfunctions are given by so-called Lamé polynomials \([3]\). Only two of them satisfy the desired boundary condition (28), and are given by \( E_{c}^{1}(u, k) \propto \cn(u, k)dn(u, k) \) and \( E_{s}^{1}(u, k) \propto \sn(u, k)dn(u, k) \) with \( m^2 = 0 \) and \( \left( \frac{3k^2}{1+k^2} \right) \mu^2 \), respectively. Noting that \( E_{c}^{1}(\omega y, k) \propto \frac{d\phi_1(y)}{dy} \), we know that the mode \( a'(1) \) is really the massless Nambu-Goldstone mode associated with spontaneous breakdown of the translational invariance or the \( U(1) \) symmetry.

We have shown that the translational invariance for the \( S^1 \)-direction is spontaneously broken in the model \([1]\) with the boundary condition \([7]\) when the radius \( R \) becomes larger than a critical radius \( R_1^* \). Our mechanism to break the translational invariance is not specific to this model at all. Let us briefly discuss a general strategy to construct models in which the translational invariance of compactified spaces can be broken spontaneously. Suppose that some of space dimensions are compactified on a manifold with the translational invariance. Let \( V(\phi_i) \) be a scalar potential. Our mechanism may require \( V(\phi_i) \) to satisfy the following two conditions:

(i) The origin \( \phi_j = 0 \) is not the minimum of the potential \( V(\phi_i) \).

(ii) Let \( \bar{\phi}_j \) be a configuration which minimizes \( V(\phi_i) \). Then, some of \( \phi_j \) with \( \bar{\phi}_j \neq 0 \) have to be non-singlets for some global symmetries of the theory.

A key ingredient of our mechanism is to impose nontrivial boundary conditions on non-singlet fields \( \phi_j \) with \( \bar{\phi}_j \neq 0 \), which have to be consistent with global symmetries of the theory. We would have a variety of models since we have a wide choice of potentials, compactified spaces and boundary conditions. A general feature of our models will be
that the translational invariance of compactified spaces is expected to be unbroken when scales of the compactified spaces are sufficiently small and to be broken spontaneously with some global symmetries when the scales become large enough.

Finally, we would like to make some comments on vacuum structures of our models and on an application to supersymmetric field theories. In the limit of \( R \to \infty (k \to 1) \), \( \phi(y) \) in eq. (12) will reduce to \( \sqrt{2} \mu \tanh \left( \sqrt{\frac{\mu}{2}} (y - y_0) \right) \). This is nothing but a (static) single kink solution in \( D = 2 \) dimensions \[4\]. So, the model considered here may be regarded as a real \( \phi^4 \) model on a single kink background sitting on a line in the limit of \( R \to \infty \). This observation suggests that models based on our breaking mechanism might be regarded as quantum field theories on (topologically) nontrivial backgrounds in a broken phase of the translational invariance. The second comment is that vacuum structures of our models are expected to be quite nontrivial, in general. To see this, let us consider a simple extension of the model (1) by replacing the real field \( \phi \) by a complex one \( \Phi \) with a \( U(1) \) symmetry and also the boundary condition (7) by \( \Phi(y + 2\pi R) = e^{i2\pi \alpha} \Phi(y) \). One might expect that the vacuum structure is similar to the original real \( \phi^4 \) model. This is not, however, the case. In fact, as studied in ref. \[5\], this model admits a rich set of solutions to the field equation for \( \Phi(y) \), and the vacuum configuration is quite different from that of the real \( \phi^4 \) model \[4\]. The final comment is that it would be worth applying our mechanism to supersymmetric field theories. In ref. \[6\], it has been shown that our mechanism can be used to break supersymmetry spontaneously and that this SUSY breaking mechanism is new, that is, it is different from the O’Raifeartaigh \[7\] and Fayet-Iliopoulos \[8\] mechanisms. It would be of great importance to use this new SUSY breaking mechanism to construct phenomenologically interesting supersymmetric models.

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\[\text{\footnotesize Since in ref. [3] the field equation has been solved with the periodic boundary condition, we should reanalyze the field equation with a proper boundary condition.}\]
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