Elliptic Integrals for Calculation of the Propagation Time of a Signal, Emitted and Received by Satellites on One Orbit

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Abstract. The propagation time of a signal, emitted by a moving along an elliptical orbit satellite from the GPS (or GLONASS) satellite configurations is a very important ingredient of the theory, based on the formalism of the null cone and accounting for the effects of the General Relativity Theory. For the case of satellites, orbiting along a plane elliptic orbit, it has been proved that the propagation time for the signal between the satellites is given by a combination of elliptic integrals of the first, second and third kind. For the more general case of satellites on a space-distributed elliptic orbit, the propagation time is expressed by higher (fourth) order elliptic integrals, which according to the standard theory can be expressed recurrently by means of lower-order elliptic integrals. In the concrete case, the elliptic integrals of the second and the fourth order are expressed by means of a combination of irrational functions and the zero-order elliptic integral in the Legendre form. It has been proved that for the investigated case, second-order elliptic integrals can be expressed by elementary functions.

INTRODUCTION

In the past 15 – 20 years the problem about GPS satellite-ground station communications has been replaced by the problem about autonomous navigation and intersatellite communications (ISC) (links), which has been mentioned yet in 2005 in the monograph [1]. Autonomous navigation means that satellites on one orbit or on different orbits around the Earth should have the capability to transmit data between them via intersatellite cross-link ranging [2] and thus, to ensure navigation control and data processing without commands from Earth stations in the course of six months. However, since the transmitted signals are propagating in the space around the Earth and are thus experiencing the influence of the gravitational field of the Earth, their propagation should take into account the General Relativity effects.

The theory of intersatellite communications (ISC) is developed in the series of papers by S. Turyshev, V. Toth, M. Sazhin [3], [4] and S. Turyshev, N. Yu, V. Toth [5]. This theory concerns the space missions GRAIL (Gravity Recovery and Interior Laboratory), GRACE-FOLLOW-ON (GRACE-FO - Gravity Recovery and Climate Experiment - Follow On) mission and the Atomic Clock Ensemble in Space (ACES) experiment on the International Space Station (ISS). However, it should be stressed that all these missions are realized by means of low-orbit satellites (about 450 km above the Earth), while the satellites from the GPS, GLONASS and the Galileo constellations are on much higher orbits (from 19140 km for GLONASS and ranging to 26560 km for GPS, even higher). Consequently, since the main effect of the gravitational field is to curve the trajectory of the signal and thus to increase the propagation time of the signal and also the distance, travelled by the signal (compared to the case of a flat spacetime), this effect will be considerable for such large distances.

In the above-mentioned papers [3], [4] and [5], the basic theoretical instrument for calculating the propagation time of the signal is the Shapiro delay formulae. If the coordinates of the emitting and of the receiving satellite are

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correspondingly $|x_A(t_A)| = r_A$ and $|x_B(t_B)| = r_B$, and $R_{AB} = |x_A(t_A) - x_B(t_B)|$ is the Euclidean distance between the signal - emitting satellite and the signal - receiving satellite, then from the null cone equation, the signal propagation time $T_{AB} = T_B - T_A$ between two space points can be expressed by the known formulae [6] (see also the review article [7] by Sovers, Fanselow and Jacobs on VLBI radio interferometry)

$$T_{AB} = \frac{R_{AB}}{c} + \frac{2GM_⊕}{c^3} \ln \left( \frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}} \right),$$

where $G_⊕M_⊕$ is the geocentric gravitational constant, $M_⊕$ is the Earth mass and $G_⊕$ is the gravitational constant. We shall denote also by $t=TCG$ the Geocentric Coordinate Time (TCG). The second term in formulae Eq. (1) is the Shapiro time delay term, accounting for the signal delay due to the curved space-time. However, in this general form the Shapiro delay formulae cannot account for the effect of the ellipticity of the orbit on the propagation of the signal and also, the initial and the final radius-vectors $r_A$ and $r_B$ (related to the initial point of emission and the final point of reception of the signal) should be dependent on some parameter, uniquely determining the position of the satellite on the orbit. For the case of a plane elliptical orbit this parameter will be the eccentric anomaly angle $E$ and for space distributed orbits (parametrized by the full number of 6 Keplerian elements $(f, a, e, Ω, I, ω)$, the parameter is the true anomaly angle $f$. In other words, in this paper the investigated problem will be about propagation of a signal between satellites on one and the same orbit.

The main goal of this paper is to find the explicite formulae for the propagation time of the signal for the two relevant cases - signal emitted by a satellite, orbiting a plane orbit (and percepted by a satellite on the same orbit), and signal, emitted by a satellite along a space-distributed orbit (again, the signal-receiving satellite is on the same orbit). Although in both cases the dynamical parameter will be just one (however, the parameters will be different), the second case will be characterized by more complicated formulae, related to elliptical integrals of higher order. So the general calculation should not be based on any direct application of the Shapiro delay formulae Eq. (1), but rather than that, on a calculation, starting from the light null cone formalism.

There is a second motivation for finding the propagation time for the various cases, related to the formalism of the two intersecting null cones with origins at the signal-emitting and signal-receiving satellites, developed in the paper [8] (see also a shorter version in [9]). This formalism enables to find the propagation time of the signal, when both satellites are moving, meaning that the propagation time of the signal takes into account the motion of the second (signal-receiving) satellite during the time for propagation of the signal between the two satellites. However, the first ingredient of such a theory is to find the s.c. "first propagation time", which is the propagation time for the signal, emitted by the first satellite. In these papers, the theory is developed only for the plane elliptical orbit case. In this paper, the calculation for the first time is extended to the case of signal-emitting and signal-receiving satellites on a space-oriented orbit. This might be considered as the second step towards constructing a theory for propagation of signals between moving satellites on different, space-oriented orbits. The main motivation for such a complicated research (both from a physical point of view and especially from a mathematical one) comes from the requirement that the Global Navigation Satellite System (GNSS), consisting of 30 satellites on space-distributed orbits and orbiting the Earth at a height of 23616 km, should be interoperable with the other two navigational systems GPS and GLONASS [10]. This is important from an operational point of view, since a combined GNSS of 75 satellites from the GPS, GLONASS and the Galileo constellations may increase greatly the visibility of the satellites, especially in critical areas such as urban canyons [11].

**PROPOSITION TIME FOR THE CASE OF A SIGNAL-EMITTING SATELLITE, MOVING ALONG A PLANE ELLIPTICAL ORBIT**

Propagation time without any approximations

Let us take the null-cone metric in the standard form

$$ds^2 = -c^2 \left( 1 + \frac{2V}{c^2} \right) (dT)^2 + \left( 1 - \frac{2V}{c^2} \right) \left( (dx)^2 + (dy)^2 + (dz)^2 \right) = 0,$$

where $V = \frac{G_⊕M_⊕}{r}$ is the standard gravitational potential of the Earth. No account is taken of any any harmonics due to the spherical form of the Earth since the GPS orbits are situated at a distance more than 20000 km (considered from
the centre of the Earth). Further, the Kepler parametrization of the space coordinates \( x = x(a,e,E) \) and \( y = y(a,e,E) \) for the plane elliptical satellite orbit

\[
x = a(\cos E - e) \quad , \quad y = a\sqrt{1 - e^2}\sin E
\]

is used, where \( a \) is the semi-major axis of the orbit, \( e \) is the eccentricity and \( E \) is the eccentric anomaly angle, related to the motion of the satellite along the plane elliptical orbit.

If the right-hand side of Eq. (1) is divided and multiplied by \((dt)^2\), where \( t = t_{cel} \) is the celestial time from the Kepler equation \( E - e\sin E = n(t_{cel} - t_p) \) (\( t_p \) is the time of perigee passage), one can obtain

\[
(c^2 + 2V)(dT)^2 = (1 - \frac{2V}{c^2})v^2 ,
\]

where \( v \) is the square of the satellite velocity along the orbit

\[
v^2 = v_x^2 + v_y^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 .
\]

Taking into account the Kepler parametrization Eq. (2), the above expression for the velocity for the case of plane motion can be rewritten as

\[v = \sqrt{v_x^2 + v_y^2} = \frac{na}{1 - e\cos E}\sqrt{1 - e^2\cos^2 E} \]

and \( n = \frac{\sqrt{G\,M_\oplus}}{a^3} \) is the mean motion. Expressing \( dT \) from Eq. (3), finding the celestial time \( t_{cel} \) in terms of \( E \) from the Kepler equation and performing the integration over the eccentric anomaly angle \( E \), one can obtain the expression for the propagation time (for the case of no restrictions imposed)

\[
T = \int \frac{\sqrt{(c^2 - 2V)}}{c\sqrt{(c^2 + 2V)}} \, dE + C = \frac{a}{c} \int \sqrt{\frac{a_1 y^3 + a_2 y^2 + a_3 y}{b_1 y^3 + b_2 y^2 + b_3 y + b_4}} \, dy ,
\]

where the numerical constants \( a_1, a_2, a_3, b_1, b_2, b_3, b_4 \) can easily be calculated, \( y \) is the variable \( y = 1 - e\cos E \) and \( C \) is some integration constant, which can be assumed to be zero. Integrals of the type Eq. (6) are not abelian ones (see the monograph by Prasolov and Solovyev [12]), because abelian integrals are related to algebraic curves \( F(x,y) := x^2 - P(y) = 0 \), where \( P(y) \) is an algebraic polynomial. In the case, the integrand expression is a rational function of \( y \). Further the expression Eq. (6) will not be used, because a justifiable and reasonable physical approximation shall be applied. Nevertheless, in view of the constantly improving accuracy for measuring the propagation time, this might represent an interesting problem for future mathematical research.

The useful approximation for a small gravitational potential, compared to the square of the velocity of light

Further we shall be interested in the case

\[
\beta = \frac{2V}{c^2} = \frac{2G_\oplus M_\oplus}{c^2 a} \ll 1 ,
\]

which can be assumed to be fulfilled. For the parameters of the GPS orbit - \( a = 26561 \, [km] \), in the review paper [13] the constant \( \beta \) can exactly be calculated to be \( 0.334 \times 10^{-9} \), with the velocity of light taken to be \( c = 299792458 \, [m/sec] \). The geocentric gravitational constant \( G_\oplus M_\oplus \) (obtained from the analysis of laser distance measurements of artificial Earth satellites) can be taken to be equal to \( G_\oplus M_\oplus = (3986004.405 \pm 1) \times 10^8 \, [m^3/sec^2] \), but the value of \( G_\oplus M_\oplus \) can vary also in another range from \( G_\oplus M_\oplus = 3986056.75236 \times 10^8 \, [m^3/sec^2] \) to the value \( G_\oplus M_\oplus = 3987999.07898 \times 10^8 \, [m^3/sec^2] \) due to the uncertainties in measuring the Newton gravitational constant \( G_\oplus \). The mass of the Earth can be taken approximately to be \( M_\oplus \approx 5.97 \times 10^{24} \, [kg] \). One of the latest values for \( G_\oplus \) from deep space experiments was reported in the paper [14] to be \((6.674 + 0.0003) \times 10^{-11} \, [m^3/kg\cdot sec^2] \).
Derivation of the propagation time under the approximation $\beta = \frac{2V}{c} \ll 1$ and physical justification of the obtained result

After decomposing the under-integral expression with the square root in Eq. (6) and leaving only the first-order term in $\frac{2V}{c}$, one can obtain

$$T = \int \frac{\nu}{c} \sqrt{\frac{(1 - \frac{2V}{c})}{(1 + \frac{2V}{c})}} \, dt \approx \int \frac{\nu}{c} (1 - \frac{2V}{c^2}) dt = I_1 + I_2 =$$

$$= \frac{a}{c} \int \sqrt{1 - e^2 \cos^2 E} \, dE - \frac{2GM_{\odot}}{c^3} \int \frac{\sqrt{1 + e \cos E}}{1 - e \cos E} \, dE . \quad (9)$$

This expression is consistent from a physical point of view due to the following reasons:

1. The coefficient $\frac{a}{c}$ as a ratio of the large semi-major axis of the orbit and the velocity of light $c = 299792458 \, \text{[m/s]}$ will have a dimension $[m/\text{sec}] = \text{[sec]}$, as it should be. The second coefficient $\frac{2GM_{\odot}}{c^3}$ has a corresponding dimension $\frac{m^3}{\text{sec}^3} : \frac{m^3}{\text{sec}} = \text{[sec]}$, which clearly proves that formulae Eq. (9) has the proper dimensions.

2. The formulae in fact gives the propagation time $T$ of the signal, emitted by the satellite at some initial position (given by the eccentric anomaly angle $E_{\text{in}}$), and the final point (given by $E_{\text{fin}}$) of reception of the signal by another satellite. So both emission- and reception- signal points should remain on one and the same satellite orbit. In this sense, the integrals Eq. (8) and Eq. (9) represent curvilinear integrals (depending on the initial and the final points of integration), derived after the application of the null-cone equation (in a general form written as $F = g_{\alpha\beta} dx^\alpha dx^\beta = 0$) and consequently, they are related to the propagation time of the signal and not to the motion of the satellite. Moreover, the condition $F = 0$ is proved to be compatible with the geodesic equation [15] for light-like geodesics

$$\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \rightleftharpoons$$

where $\tau$ is the proper time along the light ray and $\nu, \alpha, \beta = 0, 1, 2, 3$. In other words, the zero-length (light-like) geodesics (representing the trajectory of the signal) are determined also by the null cone equation $g_{\alpha\beta} dx^\alpha dx^\beta = 0$, because the null equation is a first integral of the geodesic equation Eq. (10).

3. Let now $x, y, z$ be the space coordinates, reached by the signal after it has been emitted. From the null cone equation Eq. (11), after using the approximation $\beta = \frac{2V}{c} \ll 1$, it can be obtained

$$((dx)^2 + (dy)^2 + (dz)^2) \approx c^2 \left[1 + \frac{4V}{c^2}\right] (dT)^2 \ll 3c^2 (dT)^2 . \quad (11)$$

Since $cdT$ is the infinitesimal distance, travelled by the light signal, the above inequality means that the distance travelled by light is much greater than the Euclidean distance. In other words, the light trajectory signal is a curved one, in accord with the meaning of the Shapiro delay formulae Eq. (1) about the delay of the signal under the action of the gravitational field.

4. The trigonometric function $\cos^2 E$ has the same values for $E = \alpha$ and $E = 180 - \alpha$, so the correspondence eccentric anomaly angle $E$ and the propagation time $T$ is not reliable for angles in the second quadrant.

5. The propagation time $T$ Eq. (9) should be a real-valued expression, since time cannot be complex. This fact is not evident from the beginning of the calculations, but will be proved in the next subsections. Remarkably, this important property will turn out to be valid also for the next case of a signal, emitted and percepted by satellites on a space-distributed orbit.
Mathematical structure of the expression for the propagation time, related to zero-order elliptic integrals of the first, second and the third rank

The first term in expression Eq. (9) after performing the simple transformation \( \frac{x}{2} - E = \mathcal{E} \) can be brought to the following form

\[
T_1 = \int_0^E \frac{E}{\sqrt{1 - e^2 \cos^2 E}} dE = \int_0^E \frac{E}{\sqrt{1 - e^2 \sin^2 \left( \frac{\pi}{2} - E \right)}} dE = - \int_0^E \frac{\mathcal{E}}{\sqrt{1 - e^2 \sin^2 \mathcal{E}}} d\mathcal{E} . \tag{12}
\]

This represents an elliptic integral of the first kind.

The second term in Eq. (9) after performing the substitution

\[
\sqrt{1 + e \cos \mathcal{E}} \quad \frac{1}{1 - e \cos \mathcal{E}} = \mathcal{y}
\]

and introducing the notations

\[
\tilde{k}^2 = \frac{1 - e}{1 + e} = q , \quad \mathcal{y} = y , \quad \left( \frac{\mathcal{y}}{\tilde{k}} \right)^2 = \tilde{y} , \quad \tilde{y} = -\tilde{y}
\]

can be written as a sum of two integrals, i.e.

\[
T_2 = \frac{2GM_0}{c^3} \int \frac{1 + e \cos \mathcal{E}}{1 - e \cos \mathcal{E}} dE = T_{2A} + T_{2B} . \tag{15}
\]

It should be noted that \( \tilde{k} \) in eq. (14) is merely a notation, representing the real expression \( \tilde{k} = \sqrt{\frac{1 - e}{1 + e}} = \sqrt{q} \), since \( 0 < \frac{1 - e}{1 + e} < 1 \) (the eccentricity of the orbit \( e \) is always less than one, \( 0 < e < 1 \)). Using this notation, the first term \( T_{2A} \) in eq. (15) can be represented as

\[
T_{2A} = \frac{4GM}{c^3} \frac{1}{\tilde{k} \sqrt{e^2 - 1}} \int \frac{d\tilde{y}}{\sqrt{\tilde{y} \left( \tilde{y} - \frac{1}{\tilde{k}} \right) \left( \tilde{y} - 1 \right)}} . \tag{16}
\]

Due to the presence of the \( \sqrt{e^2 - 1} \) term in the denominator, it might seem that expression Eq. (16) is imaginary. However, this is not true, because it can be rewritten as

\[
T_{2A} = \frac{4GM}{c^3} \frac{1}{\tilde{k} \sqrt{1 - e^2}} \left( -i \right) \int \frac{d\tilde{y}}{\sqrt{\tilde{y} \left( \tilde{y} - \frac{1}{\tilde{k}} \right) \left( \tilde{y} - 1 \right)}} , \tag{17}
\]

representing an elliptic integral of zero order and of the first kind. Since \( \tilde{k} \) is real-valued expression, the two imaginary units \( i \) in the denominator appear from \( \sqrt{e^2 - 1} = i \sqrt{1 - e^2} \) and also from the square root, after performing the variable transformation from \( \tilde{y} \) to \( \tilde{y} \) in \( \tilde{y} \left( \tilde{y} - \frac{1}{\tilde{k}} \right) \left( \tilde{y} - 1 \right) \) in eq. (16). As a result, a factor \( \sqrt{(-1)^3} = \sqrt{(-1)} = i^3 = -i \) will appear in the denominator of eq. (17). The whole expression eq. (17) turns out to be a real-valued one with a negative sign, because \( d\tilde{y} = -d\tilde{y} \). Since this expression is a part of the expression for the propagation time, this is physically reasonable.

The second term \( T_{2B} \) in Eq. (15) can also be written in the form of a real-valued expression, as it should be

\[
T_{2B} = \frac{4GM}{c^3 q^2} \frac{1}{\sqrt{1 - e^2}} \int \frac{d\tilde{y}}{(\tilde{y} - \frac{1}{q}) \sqrt{\tilde{y} \left( \tilde{y} - \frac{1}{q} \right) \left( \tilde{y} + 1 \right)}} . \tag{18}
\]

This is an elliptic integral of the third kind. Consequently, the whole expression for the propagation time can be represented as a sum of elliptic integrals of the second, the first and the third kind. [12]
PROPAGATION TIME FOR A SIGNAL, EMITTED AND PERCEPTED BY SATELLITES ON A SPACE-DISTRIBUTED ORBIT

Three-dimensional orbit parametrization and the general formulae for the orbit parametrization

Instead of the simple two-dimensional parametrization Eq. (2), the three-dimensional parametrization of the space-distributed orbit to be used is

\[ x = \frac{a(1-e^2)}{1+e\cos f} \left[ \cos \Omega \cos (\omega + f) - \sin \Omega \sin (\omega + f) \cos \hat{i} \right], \quad (19) \]

\[ y = \frac{a(1-e^2)}{1+e\cos f} \left[ \sin \Omega \cos (\omega + f) + \cos \Omega \sin (\omega + f) \cos \hat{i} \right], \quad (20) \]

\[ z = \frac{a(1-e^2)}{1+e\cos f} \sin (\omega + f) \sin \hat{i}, \quad (21) \]

where \( r = \frac{a(1-e^2)}{1+e\cos f} \) is the radius-vector in the orbital plane, the angle \( \Omega \) of the longitude of the right ascension of the ascending node is the angle between the line of nodes and the direction to the vernal equinox, the argument of perigee (periapsis) \( \omega \) is the angle within the orbital plane from the ascending node to perigee in the direction of the satellite motion \((0 \leq \omega \leq 360^\circ)\), the angle \( i \) is the inclination of the orbit with respect to the equatorial plane and the true anomaly \( f \) geometrically represents the angle between the line of nodes and the position vector \( r \) on the orbital plane has an initial point at the centre of the ellipse. Since the angle \( f \) is related to the motion of the satellite and all the other parameters of the orbit do not change during the motion of the satellite, it can easily be found that

\[ \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{(v_x^f)^2 + (v_y^f)^2 + (v_z^f)^2} df = v_f df, \quad (22) \]

where the velocity \( v_f \) (with velocity components \((v_x^f, v_y^f, v_z^f)\), associated to the true anomaly angle \( f \) is given by \n
\[ v_f = \frac{na}{\sqrt{1-e^2}} \sqrt{1+e^2+2e\cos f}. \quad (23) \]

Further, again making use of the null cone equation Eq. (3) and also of the approximation Eq. (7) \( \beta = \frac{2V}{c^2} \ll 1 \), one can obtain the general formulae for the propagation time

\[ T = \int \frac{v}{c}(1 - \frac{2V}{c^2}) dt = \tilde{T}_1 + \tilde{T}_2 = \frac{1}{c} \int v dt - \frac{2}{c^3} \int vV dt. \quad (24) \]

Note the important fact that in the above formulae one can replace \( vdt \) by \( v_f df \), because

\[ vdt = \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2} dt = \sqrt{\left( \frac{dx}{df} \right)^2 + \left( \frac{dy}{df} \right)^2 + \left( \frac{dz}{df} \right)^2} df = v_f df. \quad (25) \]

It is worth mentioning that without the approximation Eq. (7), a similar formulae to Eq. (6) can be obtained. Here we shall not write this formulae.

**Analytical calculation of the first integral (first \( O(\frac{1}{c}) \)) correction without the use of elliptic integrals**

We shall present a calculation, which shows that some integrals can be calculated both analytically, and also by the use of elliptic integrals.
Let us take the first $O(\frac{1}{c^2})$ propagation time correction
\[\tilde{T}_1 = \frac{1}{c} \int v_j df = \frac{n a}{c \sqrt{1 - e^2}} \int \sqrt{1 + e^2 + 2e \cos f} df = \frac{n a}{c \sqrt{1 - e^2}} \tilde{T}^{(1)}_1\] (26)
and let us perform the series of six subsequent variable transformations
\[\sqrt{1 + e^2 + 2e \cos f} = y , \quad \eta = (e + 1)^2 - y^2 , \quad \sqrt{1 - \frac{\eta q^2}{y + B_1}} = z , \quad z^2 = z_1 , \quad 1 - z_1 = m , \quad \tilde{m} = m - \frac{q^2}{2} . \] (27)
(28)
Then for $\tilde{T}_1$ the following sum of two integrals will be obtained
\[\tilde{T}_1 = \frac{n a i}{c \sqrt{(1 - e^2)q}} \left[ \int \frac{d\tilde{m}}{\sqrt{\tilde{m} - \frac{q^2}{2}}} + \int \frac{d\tilde{m}}{\sqrt{\tilde{m} + \frac{q^2}{2}}} \right] , \] (29)
where again the notation $q = \frac{1+e}{1-e}$ has been used and each of the integrals inside the square bracket will be denoted correspondingly as $\tilde{T}^{(1)}_1$ and $\tilde{T}^{(2)}_1$. Making use of the analytically calculated integral from the book [17]
\[\int \frac{dx}{(x \pm p)\sqrt{a + 2bx + cx^2}} = -\frac{1}{\sqrt{a + 2bp + cp^2}} \ln \frac{\sqrt{a + 2bx + cx^2} + \sqrt{a + 2bp + cp^2}}{x \pm p} , \] (30)
one can derive for the second integral $\tilde{T}^{(2)}_1$ in Eq. (29) the following expression
\[\tilde{T}^{(2)}_1 = -\frac{n a (1 + e)\sqrt{2}}{c \sqrt{1 - e^2} \sqrt{3e^2 + 2e + 3}} \ln \left( \frac{\sqrt{2}m_1(f_b; r_b) + \frac{1}{2}m_2(f_b; r_b)}{\sqrt{2}m_1(f_a; r_a) + \frac{1}{2}m_2(f_a; r_a)} \left( \frac{m_3(f_a; r_a)}{m_3(f_b; r_b)} \right) \right) \] (31)
and $m_1(f; r)$, $m_2(f; r)$, $m_3(f; r)$ are expressions, written in terms either of the initial and final true anomaly angles $f_a$ and $f_b$ or, of the initial distance $r_a$ (at which the emission of the signal takes place) and the final point $r_b$ of reception on the same orbit, corresponding to the propagation time $\tilde{T}^{(2)}_1$. The first integral $\tilde{T}^{(1)}_1$ can be calculated analogously. It should be noted that both $\tilde{T}^{(1)}_1$ and $\tilde{T}^{(2)}_1$ are real-valued expressions and moreover, the logarithmic term is again present, as in the original Shapiro delay formulae Eq. (1).

**Analytical calculation of the first $O(\frac{1}{c^2})$ correction by means of elliptic integrals**

Let us calculate the integral $\tilde{T}_1 = \frac{n a}{c \sqrt{1 - e^2}} \int \sqrt{1 + e^2 + 2e \cos f} df$ by making use of the substitution
\[y = \sqrt{\frac{1}{q} \frac{(1 + e \cos E)}{(1 - e \cos E)}} , \quad q = \frac{1 - e}{1 + e} \] (32)
and also of the well-known relation from celestial mechanics between the eccentric anomaly angle $E$ and the true anomaly angle $f$ [16]
\[\tan \frac{f}{2} = \sqrt{\frac{1 - \cos f}{1 + \cos f}} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2} . \] (33)
Then we can write the integral $\tilde{T}_1$ in the form of an elliptic integral of the second order and of the first kind in the
Legendre form

\[ \tilde{T}_1 = -2i \frac{na}{c} q^2 \int \frac{y^2 dy}{\sqrt{(1-y^2)(1-q^2y^2)}} = -2i \frac{na}{c} q^2 I . \]  

(34)

The integral \( I \) can also be represented as

\[ I = \tilde{I}_2^{(4)}(y; q) = \int \frac{y^2 dy}{\sqrt{(1-y^2)(1-q^2y^2)}} = -\frac{1}{q^2} \int \frac{(1-q^2y^2)dy}{\sqrt{(1-y^2)(1-q^2y^2)}} + \frac{1}{q^2} \int \frac{dy}{\sqrt{(1-y^2)(1-q^2y^2)}} = \]  

(35)

\[ = -\frac{1}{q^2} \int \sqrt{1-q^2 \sin^2 \varphi} d\varphi + \frac{1}{q^2} \int \frac{dy}{\sqrt{(1-y^2)(1-q^2y^2)}} . \]  

(36)

The first integral in Eq. (36) is an elliptic integral of the second kind (denoted usually by \( E(\varphi) = \int \sqrt{1-q^2 \sin^2 \varphi} d\varphi \), where \( x = \sin \varphi \)). So we obtain a relation between the second-order elliptic integral \( \tilde{I}_2^{(4)}(y; q) \) of the first kind in the Legendre form, the zero-order elliptic integral of the first kind in the Legendre form \( \tilde{I}_0^{(4)}(y; q) \) (the second integral in Eq. (36)) and the elliptic integral \( E(\varphi) \)

\[ \tilde{J}_2^{(4)}(y; q) = \int \frac{y^2 dy}{\sqrt{(1-y^2)(1-q^2y^2)}} = -\frac{1}{q^2} E(\varphi) + \frac{1}{q^2} \tilde{I}_0^{(4)}(y; q) . \]  

(37)

Since in the preceding section an analytical expression was obtained for \( \tilde{T}_1 \) without the use of any elliptic functions, the obtained result in Eq. (36) means that for the investigated case, elliptic integrals of second order can be expressed through elementary functions, contrary to the claims of some authors that this is not possible for all kinds of elliptic integrals.

Using the relation between the eccentric anomaly angle \( E \) and the true anomaly angle \( f \) (derived from Eq. (33)), it is interesting to express the first \( (O(\frac{1}{e^2})) \) propagation time correction Eq. (34) also in terms of the variable \( E \)

\[ \tilde{T}_1 = \frac{na}{c} \sqrt{1-e^2} \int \frac{1}{(1-e \cos E)} \sqrt{1+e \cos E} dE . \]  

(38)

This integral resembles the second integral \( -\frac{2G M}{c e} \int \sqrt{1+e \cos E} dE \) in the \( O(\frac{1}{e^2}) \) time correction Eq. (9) for the case of plane elliptical orbit, but in the case the integral is with another coefficient and is modified with the term \( \frac{1}{(1-e \cos E)} \), multiplying the square root.

**Second analytical calculation of the first time \( O(\frac{1}{e}) \) correction in terms of second-order elliptic integrals**

This second calculation will not make any use of the eccentric anomaly angle variable \( E \). Let us apply the variable transformation

\[ \tilde{y} = \frac{\sqrt{1+2e \cos f + e^2}}{1+e} = \frac{\tilde{y}}{1+e} , \]  

(39)

after which the integral \( \tilde{T}_1 \) acquires the form

\[ \tilde{T}_1 = \frac{na}{c q \sqrt{1-e^2}} \int \sqrt{1+e^2+2e \cos f} df = \]  

(40)

\[ = i 2na(1+e) \int \frac{\tilde{y}^2 d\tilde{y}}{(1-\tilde{y}^2)\left(1-\frac{\tilde{y}^2}{q^2}\right)} = 2na(1+e) \int \frac{\tilde{y}^2 d\tilde{y}}{c q \sqrt{1-e^2}} \int \frac{\tilde{y}^2 d\tilde{y}}{\sqrt{(1-\tilde{y}^2)\left(\frac{\tilde{y}^2}{q^2} - 1\right)}} . \]  

(41)
Note that the coefficient in front of the integral is modified in comparison with the one in Eq. (43), and more importantly, the resulting expression is again a real-valued one due to the property of the elliptic integral

$$J_2^{(4)}(\tilde{y}, \frac{1}{q}) = \frac{1}{t} \int \frac{\tilde{y}^2 d\tilde{y}}{\sqrt{(1 - \tilde{y}^2) \left( \frac{x}{q^2} - 1 \right)}} .$$

(42)

This formulae is valid, because due to the inequality \( \cos f \leq 1 \) and the choice of the variable \( \tilde{y} \) in Eq. (39), it can easily be proved that

$$\tilde{y} \leq 1 \ , \ \triangleright 1 - \tilde{y}^2 \geq 0 \ , \ 1 - \frac{\tilde{y}^2}{q^2} \geq - \frac{(1 - q^2)}{q^2} \ .$$

(43)

Consequently, since \( 1 - \frac{\tilde{y}^2}{q^2} \) can take negative values (note that \( q^2 = (\frac{1 - \epsilon}{1 + \epsilon})^2 < 1 \)), the representation Eq. (42) is correct.

DEFINITIONS OF ELLIPTIC INTEGRALS OF HIGHER ORDER AND COMPARISON WITH SOME STATEMENTS FROM STANDARD TEXTBOOKS

Higher-order elliptic integrals

This definition is necessary to be given because further, when calculating the second part of the propagation time, we shall encounter elliptic integrals of the fourth-order.

According to the general definition, elliptic integrals are of the type

$$\int R(y, \sqrt{G(y)})dy \quad \text{or} \quad \int R(x, \sqrt{P(x)})dx \ ,$$

(44)

where \( R(\tilde{y}, y) \) and \( R(\tilde{x}, x) \) are rational functions of the variables \( \tilde{y}, y \) or \( \tilde{x}, x \) and \( \tilde{y} = \sqrt{G(y)}, \tilde{x} = \sqrt{P(x)} \) are arbitrary polynomials of the fourth or of the third degree respectively. In accord with the notations in the monograph [12], the fourth-order and the third-order polynomials will be of the form

$$\tilde{y}^2 = G(y) = a_0 y^4 + 4 a_1 y^3 + 6 a_2 y^2 + 4 a_3 y + a_4 \quad \text{and} \quad \tilde{x}^2 = P(x) = a x^3 + b x^2 + c x + d .$$

(45)

So in the general case, elliptic integrals of the \( n \)-th order and of the first kind are defined as

$$J_n^{(4)} = \int \frac{y^n dy}{\sqrt{a_0 y^4 + 4 a_1 y^3 + 6 a_2 y^2 + 4 a_3 y + a_4}} \quad \text{and} \quad J_n^{(3)} = \int \frac{x^n dx}{\sqrt{a x^3 + b x^2 + c x + d}} \ .$$

(46)

Elliptic integrals of the \( n \)-th order and of the third kind are defined as

$$H_n^{(4)} = \int \frac{dy}{(y - c_1)^n \sqrt{a_0 y^4 + 4 a_1 y^3 + 6 a_2 y^2 + 4 a_3 y + a_4}} \quad \text{and} \quad H_n^{(3)} = \int \frac{dx}{(x - c_2)^n \sqrt{a x^3 + b x^2 + c x + d}} .$$

(47)

In the above definitions, the upper indices "(3)" or "(4)" denote the order of the algebraic polynomial under the square root in the denominator, while the lower indice "\( n \)" denotes the order of the elliptic integral, related to the "\( y^n \)" or "\( x^n \)" terms in the nominator of Eq. (46) and in the denominator of Eq. (47) (for \( n \)-th order elliptic integrals of the third kind).

Comparison with some definitions from standard textbooks

In the textbook [18] a statement is expressed (although without any proof) that "higher-order elliptic integrals in the Legendre form can be expressed by means of the three standard integrals

$$\int \frac{dy}{\sqrt{(1 - y^2)(1 - q^2 y^2)}} \ , \ \int \frac{y^2 dy}{\sqrt{(1 - y^2)(1 - q^2 y^2)}} \ , \ \int \frac{dy}{(1 + hy^2)\sqrt{(1 - y^2)(1 - q^2 y^2)}} ,$$

(48)
where the parameter $h$ can be also a complex number", and these integrals, as stated, "cannot be expressed through elementary functions". This assertion is not precise, because in the preceding sections an example was found, when the second integral in Eq. (45) in fact can be expressed through elementary function - for this case, by the logarithmic term in Eq. (31). A similar statement to the one in [18] is given also in the monograph [19]: "Integrals of the type Eq. (44) $\int R(y, \sqrt{G(y)})dy$ and $\int R(x, \sqrt{P(x)})dx$ cannot be expressed by elementary functions in their final form, even if an extended understanding of this notion is considered."

In the monograph [12] a statement is expressed in a correct and precise manner, namely: "Every elliptic integral can be represented in the form of a linear combination of a rational function of the variables $y$ and $\tilde{y}$, integrals of a rational function of the variable $y$ and also the integrals

$$\int \frac{dy}{\sqrt{G(y)}} , \int \frac{ydy}{\sqrt{G(y)}} , \int \frac{y^2dy}{\sqrt{G(y)}} , \int \frac{dy}{(y-c)\sqrt{G(y)}} .$$  \hspace{1cm} (49)

In this statement it is not mentioned that elliptic integrals cannot be expressed by elementary functions.

**FOURTH-ORDER ELLIPTIC INTEGRALS AND THE SECOND $O(\frac{1}{c^4})$ CORRECTION FOR THE PROPAGATION TIME FOR THE CASE OF A SPACE-DISTRIBUTED ORBIT**

Taking into account the general equality Eq. (24), the second $O(\frac{1}{c^4})$ correction can be written as

$$T_2 = -\int \frac{\nu}{c} \frac{2V}{c^2} dt = -\frac{2G\mu M}{c^3} \int \frac{v_f}{r} df , \hspace{1cm} (50)$$

where $v_f$, the radius-vector $r$ in the orbital plane and the potential $V$ of the gravitational field around the Earth are defined in the usual way

$$v_f = \frac{na}{\sqrt{1-e^2}} \sqrt{1+2e\cos f + e^2} , \hspace{1cm} r = a(1-e^2) , \hspace{1cm} V = \frac{GM}{r} = \frac{GM}{a(1-e^2)}(1+e\cos f) . \hspace{1cm} (51)$$

The final integral for the second correction is of the form

$$T_2 = -\frac{2G\mu M}{c^3} \cdot \frac{na}{a^2(1-e^2)} \int (1+e\cos f)\sqrt{1+2e\cos f + e^2} df = T_2^{(1)} + T_2^{(2)} , \hspace{1cm} (52)$$

where

$$T_2^{(1)} = -\frac{2G\mu M}{c^3} \cdot \frac{n}{(1-e^2)\frac{1}{2}} \int \sqrt{1+2e\cos f + e^2} df = -\frac{2G\mu M}{c^3} \cdot \frac{n}{(1-e^2)\frac{1}{2}} \tilde{T}_1 , \hspace{1cm} (53)$$

and $\tilde{T}_1$ is the previously calculated integral $\tilde{T}_1 = \int \sqrt{1+2e\cos f + e^2} df$. The second term $T_2^{(2)}$ in the second correction $T_2$ is

$$T_2^{(2)} = -\frac{2G\mu M}{c^3} \cdot \frac{ne}{(1-e^2)\frac{1}{2}} \int \cos f\sqrt{1+2e\cos f + e^2} df = -\frac{2G\mu M}{c^3} \cdot \frac{ne}{(1-e^2)\frac{1}{2}} \tilde{T}_2^{(2)} , \hspace{1cm} (54)$$

where $\tilde{T}_2^{(2)}$ is the notation for the more complicated integral

$$\tilde{T}_2^{(2)} = \int \cos f\sqrt{1+2e\cos f + e^2} df . \hspace{1cm} (55)$$

The second part of the second $O(\frac{1}{c^4})$ correction expressed in terms of elliptic integrals of the second and fourth order

It can be proved that

$$T_2^{(2)} = -\frac{2G\mu M}{c^3} \cdot \frac{ne}{(1-e^2)\frac{1}{2}} \tilde{T}_2^{(2)} = \hspace{1cm} (56)$$
where

\[ J_2^{(4)}(\bar{y},q) = \int \frac{\bar{y}^2 d\bar{y}}{\sqrt{(1-\bar{y}^2)(1-q^2\bar{y}^2)}} = \frac{1}{3} \int \frac{q^3 \bar{y}^2 d\bar{y}}{\sqrt{(\bar{y}^2 - 1)(1 - q^2\bar{y}^2)}} = \frac{q^3}{3} J_2^{(4)}(\bar{y},q) \]  

(58)

and \( \bar{y} \) is the notation for the variable \( \bar{y} = \frac{q}{y} \). The last formulae can also be compared to Eq. (42). Analogously, the formulae for the fourth-order elliptic integral \( J_4^{(4)}(\bar{y},q) \) can be written as

\[ J_4^{(4)}(\bar{y},q) = \frac{q^4}{3} \int \frac{\bar{y}^4 d\bar{y}}{\sqrt{(\bar{y}^2 - 1)(1 - q^2\bar{y}^2)}} \]  

(59)

Relation between the fourth-order and the second-order elliptic integrals in the expression for \( J_2^{(2)} \)

It can very easily be proved that

\[ \int \sqrt{(1-\bar{y}^2)(1-q^2\bar{y}^2)} d\bar{y} = \frac{2}{3} J_0^{(4)} - \frac{1}{3} (1+q^2) J_2^{(4)} + \frac{1}{3} \left[ \bar{y} \sqrt{(1-\bar{y}^2)(1-q^2\bar{y}^2)} \right]_{\bar{y}=\bar{y}_1}^{\bar{y}=\bar{y}_0} \]  

(60)

We have already proved that \( J_2^{(4)} \) can be expressed in elementary functions, but yet we do not know whether \( J_0^{(4)} \) or the second-rank integral \( \int \sqrt{(1-\bar{y}^2)(1-q^2\bar{y}^2)} d\bar{y} \) can also be expressed through elementary functions. However, this will be proved in a forthcoming publication.

Further, calculating the derivatives \( \frac{d}{d\bar{y}} \left( \sqrt{(1-\bar{y}^2)(1-q^2\bar{y}^2)} \right) \) and \( \frac{d}{d\bar{y}} \left( \bar{y} \sqrt{(1-\bar{y}^2)(1-q^2\bar{y}^2)} \right) \), integrating the resulting equations from \( \bar{y}_0 \) to \( \bar{y}_1 \), and combining all the three equations, the following two equations can be derived for \( J_4^{(4)} \) and \( J_3^{(4)} \):

\[ J_4^{(4)} = \frac{1}{3q^2} \left[ \bar{y} \sqrt{(1-\bar{y}^2)(1-q^2\bar{y}^2)} \right]_{\bar{y}=\bar{y}_1}^{\bar{y}=\bar{y}_0} + \frac{2(1+q^2)}{3q^2} J_2^{(4)} - \frac{1}{3q^2} J_0^{(4)} \]  

(61)

In expressions Eq. (59) - Eq. (61) the symbol \( \left. \frac{d}{d\bar{y}} \right|_{\bar{y}=\bar{y}_0}^{\bar{y}=\bar{y}_1} \) means that the corresponding expression is taken at \( \bar{y} = \bar{y}_1 \) and from it the value of the expression at \( \bar{y} = \bar{y}_0 \) is substracted.

We should also add to this system of equations the earlier derived equation for \( J_2^{(4)} \)

\[ J_2^{(4)}(\bar{y},q) = \int \frac{\bar{y}^2 d\bar{y}}{\sqrt{(1-\bar{y}^2)(1-q^2\bar{y}^2)}} = \frac{1}{q^2} E(\varphi) + \frac{1}{q^2} J_0^{(4)}(\bar{y}^2;q) \]  

(62)

The second equation Eq. (61) for \( J_3^{(4)} \) clearly suggests that \( J_3^{(4)} \) is expressed in elementary functions because the first-order elliptic integral \( J_1^{(4)} = \int \frac{\bar{y}^2 d\bar{y}}{\sqrt{(1-\bar{y}^2)(1-q^2\bar{y}^2)}} \) can be analytically calculated after performing the Euler substitution

\[ \sqrt{\bar{y}^2 - \frac{(1+q^2)}{q^2}} \bar{y}^2 + \frac{1}{q^2} = u + \bar{y}^2 \]  

(63)

The result is

\[ J_1^{(4)} = -\frac{1}{2q^3} \ln \left| \sqrt{(1-\bar{y}^2)(1-q^2\bar{y}^2)} - \bar{y}^2 + \frac{(1+q^2)}{2q^2} \right|_{\bar{y}=\bar{y}_0}^{\bar{y}=\bar{y}_1} \]  

(64)
Note that the variable $\tilde{y}$ is not related to any elliptic function! Concerning the fourth-order elliptic integral $\tilde{J}_4^{(4)}$, taking into account the already proved fact that $\tilde{J}_4^{(4)}$ can be expressed in elementary functions, it would follow that $\tilde{J}_4^{(4)}$ can also be expressed by elementary functions, but it is necessary to prove that the zero-order elliptic integral $\tilde{J}_0^{(4)}$ can be expressed in elementary functions. In another publication, it will be proved that the integral can be expressed by an analytical formulae, but also it can be numerically calculated, since the modulus of the elliptic integral is equal to the eccentricity $e = 0.01323881349526$ of the GPS orbit, which is exactly known [20]. The general mathematical theory for the recurrent relations between $n$-th order elliptic integrals of any kind and the lower-order elliptic integrals has been developed in the monographs [12], [21] and [19].

**CONCLUSION**

In this paper we considered two cases of propagation of a signal in the gravitational field around the Earth, when General Relativity effects have to be taken into account: 1 case. Signals, exchanged between satellites on one and the same plane elliptical orbit; 2 case. Signals, exchanged between satellites on a space-distributed orbit. It has been shown that in the first case the propagation time is given by a sum of elliptic integrals of first, second and the third kind, all of which however are of zero-order. The integrals are represented in the Legendre form. In the second case, the propagation time is expressed by a more complicated combination of elliptic integrals of the second- and of the fourth-order. It is remarkable also that the fourth-order elliptic integral can be decomposed into elliptic integrals of the second kind and of the first kind, which are characteristic for the first case. Yet, an elliptic integral of the third kind appears in the first case and does not appear in the second case. All the relevant integrals in the expressions for the propagation time have coefficients, which have definite numerical values, so from the found analytical expressions the numerical value for the propagation time can also be calculated. In principle, elliptic integrals of zero order and of the first-, second- and third-kind can be evaluated also numerically.

The method of calculation in both cases is based on the null cone formalism in General Relativity, which previously has been applied in many other papers. This method is a consequence of the light-like geodesics formalism in GR, so it is a reliable one. However, in this aspect some theoretical problems yet remain not investigated- for example, if light-like geodesics are applied, will they give some new solution about the propagation time. Such an investigation is very complicated from a mathematical point of view, because the system of equations for the null geodesics is a nonlinear one and probably will require methods from the theory of dynamical systems and not the iterative approximation methods, applied for the moment. An inconvenience of the proposed in this paper method is that one should know the position of the signal-receiving satellite at the moment of reception of the signal. This position depends on the time of motion of the satellite, but since it is equal to the propagation time, it is also an unknown variable. This problem can be solved if the formalism of two null cones, developed in [8] and [9] is applied. In fact, this inconvenience is typical also for the Shapiro delay formulae.

Another problem which for the moment is not yet solved is about finding the propagation time for signals, exchanged between satellites on different space-distributed orbits, which is important from an experimental point of view in reference to the NASA and ESA concept about autonomous navigation. Since the Keplerian parameters of the two orbits are different, one might propose again to use the null cone formalism, but in some way it might be thought that the orbit is "gradually deformed" from the first set of Keplerian parameters $(f_1, a_1, e_1, \Omega_1, i_1, \omega_1)$ to the second set of parameters $(f_2, a_2, e_2, \Omega_2, i_2, \omega_2)$. This means that in the null-cone equation not only the true anomaly angle $f$ will change, but also all the other parameters of the space-distributed orbit, and integration has to be performed over all these parameters. This will represent a considerable mathematical difficulty.

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