Transition probability of perturbative form for $\nu_\mu \to \nu_\tau$ oscillations in matter of constant density

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We give a convenient expression for the appearance probability $P(\nu_\mu \to \nu_\tau)$ describing neutrino oscillations in matter of constant density, derived using textbook quantum mechanics stratagems. Our formulation retains the clarity of an expansion in $\alpha = \Delta m_{21}^2/\Delta m_{31}^2$ exhibited by the popular Cervera et al. formula [Nucl. Phys. B 579, 17 (2000)] while enabling more accurate evaluation of oscillations over terrestrial baselines.

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I. INTRODUCTION

Analytic forms for flavor transition probabilities of neutrino oscillations in matter can facilitate studies of measurement sensitivity afforded by proposed new experimental facilities \[1\]-\[5\]. Of particular current interest is the transition probability for subdominant $\nu_\tau$ appearance, $\nu_\mu \to \nu_\tau$, using $\nu_\mu$ beams propagating over long baselines through terrestrial matter. Indeed, the size of the neutrino mixing angle $\theta_{13}$ has been dramatically clarified during the past year as the result of a number of recent experimental measurements \[6\]-\[10\].

A determination which is representative of the new level of precision is reported by the Daya Bay reactor experiment: $\sin^2 2\theta_{13} = 0.092 \pm 0.016$ (stat) $\pm 0.005$ (sys). With $\theta_{13} \sim 9^\circ$, the study of CP violation and mass hierarchy in the neutrino sector can proceed in earnest. Analytic investigations in this new era can benefit from the availability of convenient analytic forms for $P(\nu_\mu \to \nu_\tau)$ which are accurate to within a few percent for neutrino baselines through the Earth’s mantle. This work provides such a probability expression for $\nu_\mu \to \nu_\tau$ oscillations in a constant-density matter field.

A number of exact derivations for neutrino propagation with oscillations among three active flavors in constant-density matter have appeared in the literature over the past several decades \[11\]-\[15\]. In general these formulations are complicated and do not readily yield insights. A degree of clarity is achieved with the exact probability expressions of Kimura, Takamura, and Yokomaku in which matter effects are disentangled from CP violation effects \[16\]-\[17\]. Their formulation has been extended to include, for example, nonstandard interaction matter effects \[18\]-\[19\]. Nevertheless, a desire for more transparent formulations has led to the development of various approximation expansions; a review with comparisons can be found in Ref. \[20\]. A treatment which incorporates the magnitude of $\theta_{13}$ as recently measured into a perturbative framework is presented in \[21\].

For the $\nu_\tau$ appearance probability $P(\nu_\mu \to \nu_\tau)$ the three-term formula of Cervera et al. \[22\] (see also \[20\]) is frequently utilized in analytic studies. This formula has the form of a perturbative expansion in terms of the small mass hierarchy ratio $\alpha \equiv \Delta m_{21}^2/\Delta m_{31}^2 \simeq 1/32$. For propagation through the Earth’s crust such as occurs with the T2K (295 km), MINOS (735 km), and NO$\nu$A (810 km) baselines, the formula of Ref. \[22\] is adequate for most purposes. However for terrestrial baselines which exceed the proposed LBNE baseline of 1300 km, such as the “bimagic” 2540 km baseline \[3\] and the “magic” 7500 km baseline \[1\]-\[2\] which have significant pathlength through the Earth’s mantle, more accurate formulations are desirable. The Cervera et al. formula can be written as follows:

$$
P_{\text{approx}}(\nu_\mu \to \nu_\tau) \simeq \sin^2 2\theta_{13} \sin^2((1 - A)\Delta)/(1 - A)^2 + \alpha \sin 2\theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \cdot \sin(2\Delta)/A + \alpha^2 \sin^2 2\theta_{12} \cdot \sin^2(2\Delta)/A^2.
$$

In the above expression and throughout this paper, we use $\Delta \equiv \Delta m_{21}^2 \ell/(4E_\nu)$ for the atmospheric oscillation phase at baseline $\ell$. The symbol $A$ refers to the matter potential $A \equiv (2\sqrt{2}G_F n_e E_\nu)/\Delta m_{31}^2$ where $G_F$ is the Fermi coupling constant and $n_e$ is the electron density in matter. The sign of $A$ is determined by the sign of $\Delta m_{31}^2$ and choice of neutrino or antineutrino propagation. As a matter of convention, the formalism of this work refers to neutrino propagation (the $+$ sign) and assumes the normal mass hierarchy for the neutrino mass eigenstates ($\Delta m_{31}^2$ positive). We also use the compact notations $s_{ij} \equiv \sin \theta_{ij}$ and $c_{ij} \equiv \cos \theta_{ij}$ with $i,j = 1,2,3$.

We present a formulation of the $\nu_\tau$ appearance probability $P(\nu_\mu \to \nu_\tau)$ which retains the convenient perturbative form of Eq. \[1\] in the leading three terms of a six-term expansion. Our derivation uses conventional non-relativistic quantum mechanics methods to construct an approximate form for the time evolution operator for neutrino states in flavor basis. Amplitudes of useful precision are thereby implied for all oscillation transitions accessible to neutrinos of three active flavors. In this work however we focus upon $\nu_\mu \to \nu_\tau$ oscillations. Our approach is amenable to augmentations such as obtaining oscillation amplitudes with inclusion of selected non-standard inter-
action (NSI) matter potentials, but such developments are left for a future work.

II. OUTLINE

The paper proceeds as follows: We define some convenient notations and proceed straightaway in Sec. IIIA and the consequent six-term oscillation probability (Sec. IIIB) which comprises our formulation for $P(\nu_\mu \to \nu_e)$ in a constant density medium. We then show that our leading terms have resemblances to the probability terms of Eq. (1). The remaining three terms of our expression entail relatively small contributions to $P(\nu_\mu \to \nu_e)$. The extent of variations is illustrated using plots of our six-term probability versus Eq. (1) for $\nu_e$ appearing at baseline $\ell$ from an initial $\nu_\mu$ beam which propagates through matter of constant density, can be expressed as a sum of three terms:

$$A(\nu_\mu \to \nu_e) = T_1 + T_2 + T_3 .$$

The individual $T_i$ terms are the

$$T_1 = (-i) \sin 2\theta_{13} s_{23} \cdot \frac{\sin(N\ell)}{4\ell c_T N} \cdot e^{-i\Delta CP},$$

$$T_2 = (-i) c_{13} c_{23} \cdot \sin(\eta\ell) \cdot e^{iGL},$$

and

$$T_3 = \sin 2\theta_{13} s_{23} \cdot \sin^2 \left(\frac{\eta\ell}{2}\right) \cdot \left[\cos(N\ell) + iF_A \sin^2 \left(\frac{\eta\ell}{2}\right)\right] \cdot e^{-i\Delta CP} .$$

The factor $N$ which appears in oscillation phases in Eqs. (6) and (8) is defined by

$$N \equiv \frac{1}{4\ell v} \left[\sin(2\theta_{13})^2 + (\cos 2\theta_{13} - A)^2\right]^{1/2} .$$

In Eqs. (7) and (8) we use

$$\eta \equiv \frac{\alpha'}{4\ell v} \quad \text{and} \quad G \equiv \frac{1}{4\ell v} \left[1 + A + \alpha''\right] .$$

The quantity $F_A$ appearing in Eq. (8) is

$$F_A \equiv \left[c_{13}^2(1 - s_{12}^2\alpha) - (\cos 2\theta_{13} - A)\right] \simeq A .$$

The variables designated in (9), (10), and (11) are ones which arise naturally in the derivation of Sec. IV. To facilitate comparison with Eq. (1) we also define as convenient quantities $D$ and $\Delta'$:

$$D \equiv 4\ell v N \simeq |1 - A| ,$$

$$\Delta' \equiv GL = \Delta \left[(1 + A) + \alpha''\right] .$$

Thus the oscillation phases $N\ell$ and $\eta\ell$ can be written as $(D/4\ell v)\ell = D\Delta$ and as $(\alpha'/4\ell v)\ell = \alpha'\Delta$ respectively.

C. Oscillation probability of six terms

The $\nu_\mu \to \nu_e$ oscillation probability is constructed from $|A(\nu_\mu \to \nu_e)|^2$. Referring to Eq. (5), we express the result as a sum of six terms:

$$P(\nu_\mu \to \nu_e) = |T_1|^2 + (T_1 T_2^* + T_1 T_2) +$$

$$|T_2|^2 + (T_1 T_3^* + T_1 T_3) +$$

$$|T_3|^2 + (T_2^* T_3 + T_3 T_2^*).$$

The probability terms of Eq. (1) for $\nu_e$ appearing at baseline $\ell$ from an initial $\nu_\mu$ beam which propagates through matter of constant density, can be expressed as a sum of three terms:

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and

$T_3 = \sin 2\theta_{13} s_{23} \cdot \sin^2 \left(\frac{\eta\ell}{2}\right) \cdot \left[\cos(N\ell) + iF_A \sin^2 \left(\frac{\eta\ell}{2}\right)\right] \cdot e^{-i\Delta CP} .$

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$P(\nu_\mu \to \nu_e) = |T_1|^2 + (T_1 T_2^* + T_1 T_2) +$$

$|T_2|^2 + (T_1 T_3^* + T_1 T_3) +$$

$|T_3|^2 + (T_2^* T_3 + T_3 T_2^*).$
We proceed to construct the individual probability terms which appear in Eq. (14). Squaring the amplitude of Eq. (9) we obtain
\[ |T_1|^2 = (\sin 2\theta_{13})^2 s_{23}^2 \cdot \sin^2(D\Delta) . \] (15)

Similarly, we obtain from Eq. (7)
\[ |T_2|^2 = c_{13}^2 s_{23}^2 \cdot \sin^2(\eta\ell) = c_{13}^2 s_{23}^2 \cdot \sin^2(\alpha'\Delta) . \] (16)

Considering the \( T_1 T_2 \) cross terms, we write
\[ (T_1 T_2^* + T_1^* T_2) = \sin 2\theta_{13} c_{13} s_{23} \cdot \sin(\eta\ell) \cdot \frac{\sin(N\ell)}{4\ell_e N} \cdot e^{-i(G\ell + \delta_{CP})} + e^{i(G\ell + \delta_{CP})} . \] (17)

Upon introducing the notations of (12) and (13) and reducing the bracket expression, we obtain
\[ (T_1 T_3^* + T_1^* T_3) = \sin 2\theta_{13} c_{13} \sin 2\theta_{23} \cdot \sin(\alpha'\Delta) \cdot \frac{\sin(D\Delta)}{D}. \] (18)

\[ \left[ \cos \Delta' \cdot \cos \delta_{CP} - \sin \Delta' \cdot \sin \delta_{CP} \right]. \]

There are two more sets of cross terms to consider. We first determine \( (T_1 T_3^* + T_1^* T_3) \):
\[ (T_1 T_3^* + T_1^* T_3) = -2 \sin 2\theta_{13} \sin 2\theta_{13} s_{23}^2 F_A \cdot \sin^2 \frac{\alpha'\Delta}{2} \cdot \frac{\sin^2(D\Delta)}{D^2}. \] (19)

The remaining cross term is \( (T_2 T_3^* + T_2^* T_3) \). Now \( T_2 T_3^* \) is
\[ T_2 T_3^* = (-i) \sin 2\theta_{13} c_{13} s_{23} \cdot \sin(\eta\ell) \cdot \sin^2 \frac{\eta\ell}{2} \cdot \left[ \cos(N\ell) - i F_A \cdot \frac{\sin(N\ell)}{D} \right] \cdot e^{i(G\ell + \delta_{CP})}. \] (20)

Adding \( T_2 T_3 \) to \( T_2 T_3^* \) and extracting the common factors, we obtain
\[ (T_2 T_3^* + T_2^* T_3) = \]
\[ \left\{ \frac{1}{2} \sin 2\theta_{13} c_{13} \sin 2\theta_{23} \cdot \sin(\eta\ell) \cdot \sin^2 \frac{\eta\ell}{2} \right\} \cdot \left\{ (-i) e^{i(G\ell + \delta_{CP})} \left[ \cos(N\ell) - i F_A \cdot \frac{\sin(N\ell)}{D} \right] + \text{c.c.} \right\}. \] (21)

Equation (21) reduces to
\[ (T_2 T_3^* + T_2^* T_3) = \]
\[ \sin 2\theta_{13} c_{13} \sin 2\theta_{23} \cdot \sin(\alpha'\Delta) \cdot \sin^2 \frac{\alpha'\Delta}{2}. \]
\[ \left\{ \cos(D\Delta) \cdot \sin(\Delta' + \delta_{CP}) - F_A \cdot \frac{\sin(D\Delta)}{D} \cdot \cos(\Delta' + \delta_{CP}) \right\} \] (22)

The final term is \( |T_3|^2 \). Referring to Eq. (8) we obtain
\[ |T_3|^2 = \sin^2 2\theta_{13} s_{23}^2 \cdot \sin^4 \frac{\alpha'\Delta}{2} \cdot \left\{ \cos^2(D\Delta) + F_A^2 \cdot \frac{\sin^2(D\Delta)}{D^2} \right\}. \] (23)

The sum of the six probability terms of Eqs. (15), (16), (18), (19), (22), and (23) comprise our rendering of the probability for \( \nu_e \) appearance in an initial \( \nu_\mu \) beam. For convenience we state the total probability as a single expression in the conclusion (Sec. V) of this work.

D. Comparison to the Cervera et al. probability

In Eq. (14) for the \( \nu_\mu \to \nu_e \) oscillation probability, the three leading terms are \( |T_1|^2, (T_1 T_2^* + T_1^* T_2), \) and \( |T_3|^2 \), for which explicit expressions are given by Eqs. (15), (16), and (17) respectively. These terms have a clear resemblance to the corresponding three terms of the formula given in Eq. (1), however our extended perturbative form gives rise to certain modifications. In the two leading terms, our variables \( 2\theta_{13} \) and \( D \) replace \( \sin 2\theta_{13} \) and \( 1 - A \) respectively. For the second and third terms, the correspondence between our result versus the Cervera et al. expression can be seen by invoking small angle approximations and by recognizing that \( \Delta' \approx \Delta \) and that \( c_{13}^2 \approx 1.0 \).

Equation (14) contains three additional terms which arise from the presence of the very small \( T_3 \) amplitude. These terms go beyond the level of accuracy intended with Eq. (1). We find that these extra terms contribute amounts to the probability of less than one percent for terrestrial baselines accessible to accelerator-based oscillation experiments. Thus there is justification for neglecting these relatively complicated higher-order terms. The improved accuracy afforded by our formulation arises in the main with the refinements introduced into the three probability terms already present in the Cervera et al. formula [22], rather than in the extra terms.

To illustrate the level of improvement, we show the \( \nu_\mu \to \nu_e \) probability for neutrinos of energies between 1.0 and 10 GeV, propagating through the Earth for three baselines of interest to future experimentation. The nominal value reported by the Daya Bay experiment, \( \sin^2 2\theta_{13} = 0.092 \) [9], is used throughout, and the normal mass hierarchy is everywhere assumed. Figure 1 shows the \( \nu_e \) appearance probability for the Fermilab to Homestake baseline of 1300 km as envisaged for the Long Baseline Neutrino Experiment (LBNE). At this baseline neutrino propagation is entirely through the terrestrial crust, and a uniform density of 2.72 g/cm\(^3\) is assumed. Our six-term formula (solid curve) agrees with the more approximate probability (dashed curve) fairly well, however a small reduction in \( \nu_e \) appearance is indicated throughout the peak oscillation region. The difference between
the predictions becomes negligible at shorter terrestrial baselines.

This work
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FIG. 1. Probability for $\nu_e$ appearance from accelerator $\nu_\mu$ neutrino beams propagating in constant density matter, for the (a) LBNE and (b) bimagic baselines. In each plot our six-term transition probability (solid curve) is compared to the three-term probability of (1) (dashed curve). Our formula predicts the probability to be somewhat lower throughout the $E_\nu$ region of peak oscillation. This disparity is more pronounced at the longer baseline.

For baselines longer than LBNE the reduction in $\nu_e$ appearance probability becomes more pronounced. Figure 1b compares expectations at the bimagic baseline of 2540 km. At this baseline, there is propagation through the Earth’s mantle as well as the crust, giving rise to a mean density of 3.2 g/cm$^3$. In the vicinity of the oscillation peak at $\sim 4$ GeV our result falls below the Cervera et al. probability by several percent; this trend that persists at higher energies. The same general trend has been shown by other improved approximation forms – see for example, Fig. 3 of Ref. [21].

With even longer baselines, the MSW resonance in the mantle [24] greatly enhances the $\nu_\mu \to \nu_e$ oscillation. Figure 2 compares Eqs. (14) and (1) at the so-called magic baseline which occurs for propagation paths in the neighborhood of 7500 km. Here the propagation is predominantly mantle traversal and the mean density is 4.3 g/cm$^3$. The resonance-driven appearance probability is nearly 50% at its peak, consequently Fig. 2 is plotted with a distinctly larger abscissa range than is used in Figs. 1a,b. Our probability formula exhibits the same shape as predicted by Eq. (1) over most of the $E_\nu$ range. However it shows the appearance probability to be over-estimated by Cervera et al. throughout the region of the main oscillation peak. Figures 1 and 2 indicate the extent to which the six-term probability of this work may offer improved accuracy for long baseline oscillations.

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Figure 3 compares $|T_1|^2$ of Eq. (15), with the lead term of Eq. (1). For the Cervera et al. result, the leading term gives the entire probability contribution at 7500 km whereas the second and third terms, which carry the CP-violating phase and parameters from solar-scale mixing, have negligible probability at the magic baseline.

Figure 3 shows our $|T_1|^2$ term corresponds to a probability which is nearly 20% below the Cervera et al. prediction in the maximum oscillation region. This disagreement is partially alleviated by modest contributions, mostly of positive sign, which in our formula arise...
from the terms \((T_1 T_2^* + T_2 T_1^*)\) and \(|T_2|^2\). The three other terms in our formula, namely those which involve the \(T_3\) amplitude, contribute an amount which is only \(-0.002\) throughout the interval \(2.0 \leq E_\nu \leq 10.0\) GeV.

\[
|T_1|^2 = \begin{cases} 
\ell = 7500 \text{ km} 
\end{cases}
\]

\(\delta_{CP} = 0\)

**FIG. 3.** Comparison of the leading term in \(\mathcal{P}(\nu_\mu \rightarrow \nu_e)\) from the formula of this work (Eq. (15)) to the corresponding term in Eq. (1) for the 7500 km baseline. The leading terms exhibit similar shape dependence with respect to neutrino energy, however \(|T_1|^2\) of our formulation gives a lower probability through the region of the MSW resonance.

IV. DERIVATION OF \(\mathcal{A}(\nu_\mu \rightarrow \nu_e)\)

A. Hamiltonian in flavor basis

For neutrino propagation in vacuum, the Hamiltonian in the basis of three mass eigenstates \(\nu_i\) (\(i = 1, 2, 3\)) is

\[
\hat{H}_0^{(i)} = \text{diag} \left( \frac{\triangle m^2_{12}}{2E}, \frac{\triangle m^2_{23}}{2E}, \frac{\triangle m^2_{31}}{2E} \right) = \frac{1}{2E} \cdot \text{diag} (0, \alpha, 1).
\]

The transformation from mass basis \(\{\nu_1\}\) to neutrino flavor basis \(\{\nu_\phi\}\) (\(\phi = e, \mu, \tau\)) is provided by the unitary mixing matrix

\[
\hat{U}(\phi) = \hat{U}_{\text{mix}} \cdot \hat{U}^{(i)}.
\]

The Dirac-type CP-violating phase, \(\delta_{CP}\), can be conveniently incorporated into \(\hat{U}_{\text{mix}}\) by including auxiliary matrices \(\hat{\delta}_{CP}\), within the standard factored form:

\[
\hat{U}_{\text{mix}} = \hat{R}_1(\theta_{23}) \cdot \hat{R}_2(\theta_{13}) \cdot \hat{R}_3(\theta_{12}) \cdot \hat{\delta}_{CP} \cdot \hat{\delta}_{CP}^{-1}
\]

where \(\hat{\delta}_{CP} \equiv \text{diag}(1, 1, e^{i\delta_{CP}})\) and \(\hat{\delta}_{CP}^{-1} = \hat{\delta}_{CP} = \text{diag}(1, 1, e^{-i\delta_{CP}}).\) In Eq. (25) the atmospheric and solar mixings are accounted for via the rotation matrices \(\hat{R}_i(\theta_{23})\) and \(\hat{R}_3(\theta_{12})\), and the product \(\hat{\delta}_{CP} \cdot \hat{R}_2(\theta_{13}) \cdot \hat{\delta}_{CP}^{-1}\) carries the CP-violating phase:

\[
\hat{\delta}_{CP} \cdot \hat{R}_2(\theta_{13}) \cdot \hat{\delta}_{CP}^{-1} = \begin{pmatrix}
0 & s_{13} e^{-i\delta_{CP}} & 0 \\
1 & 0 & 0 \\
s_{13} e^{i\delta_{CP}} & 0 & 0
\end{pmatrix}.
\]

The effective wave equation for vacuum propagation of flavor states is

\[
i \frac{d}{dt} \hat{\rho}(\phi)(t) = \hat{H}_0^{(i)} \hat{\rho}(\phi)(t).
\]

Here, the vacuum Hamiltonian in flavor basis is given by the unitary transform of \(\hat{H}_0^{(i)}\),

\[
\hat{H}_0^{(i)} = \left( \hat{R}_1 \hat{\delta}_{CP} \hat{R}_2 \hat{\delta}_{CP}^{-1} \hat{R}_3 \right) \hat{H}_0^{(i)} \left( \hat{R}_3 \hat{\delta}_{CP} \hat{R}_2 \hat{\delta}_{CP}^{-1} \hat{R}_1 \right)
\]

(27)

Since \(\hat{\delta}_{CP} \cdot (\hat{\delta}_{CP})^\dagger\) commutes with \(\hat{R}_3 \hat{R}_2 \hat{R}_1\), and since \(\hat{\delta}_{CP} \hat{H}_0^{(i)} \hat{\delta}_{CP}^{-1} = \hat{H}_0^{(i)},\) Eq. (27) can be simplified:

\[
\hat{H}_0^{(i)} = \left( \hat{R}_1 \hat{\delta}_{CP} \right) \left( \hat{R}_2 \hat{R}_3 \hat{R}_0^{(i)} \hat{R}_2 \hat{R}_1 \right) \left( \hat{\delta}_{CP} \hat{R}_1 \right) \cdot \hat{H}_0^{(23)} \left( \hat{\delta}_{CP} \hat{R}_1 \right),
\]

(28)

where

\[
\hat{H}_0^{(23)} = \frac{1}{2E} \begin{pmatrix}
\frac{1}{2}c_{13}^2 \alpha + \frac{1}{2}c_{13} \alpha' & \frac{1}{2}c_{13} \alpha' & -\frac{1}{2}s_{13}^2 \alpha' \\
\frac{1}{2}c_{13} \alpha' & \frac{1}{2}c_{13}^2 \alpha + \frac{1}{2}c_{13} \alpha' & -\frac{1}{2}s_{13}^2 \alpha' \\
\frac{1}{2}s_{13}^2 \alpha' & -\frac{1}{2}s_{13}^2 \alpha' & s_{12}^2 s_{13}^2 \alpha + c_{13}^2
\end{pmatrix}
\]

(29)

Upon inclusion of the MSW matter interaction

\[
\hat{H}_{\text{matter}} = \text{diag}(V_e, 0, 0),
\]

the total Hamiltonian including matter effects as well as CP violation can be written in flavor basis as

\[
\hat{H}(\phi) = \left( \hat{R}_1 \hat{\delta}_{CP} \hat{R}_2 \hat{R}_3 \hat{R}_0^{(i)} \hat{R}_2 \hat{R}_1 \right) \left( \hat{\delta}_{CP} \hat{R}_1 \right) \cdot \hat{H}_0^{(23)} \left( \hat{\delta}_{CP} \hat{R}_1 \right),
\]

with \(\hat{H}(\phi)\) defined by Eq. (29) replacing \(\hat{H}_0^{(i)}\) on the right-hand side.

B. Hamiltonian in propagation basis

Neutrino propagation is usefully re-cast by transforming to the propagation basis. The latter basis is defined via

\[
\hat{\rho}(\phi) = \hat{\delta}_{CP} \hat{R}_1^{(23)} \hat{\rho}(\phi) \cdot \hat{\rho}(\phi) = \hat{R}_1 \hat{\delta}_{CP} \hat{\rho}(\phi).
\]

(30)

Multiplication of the wave equation in flavor basis from the left by \(\hat{\delta}_{CP} \hat{R}_1^{(23)}\) yields

\[
i \frac{d}{dt} \hat{\rho}(\phi)(t) = \hat{H}(\phi) \hat{\rho}(\phi),
\]

(31)
where $\hat{H}^{(p)} = \left( \hat{H}^{(23)}_0 + \hat{H}^{(\nu)}_{\text{matter}} \right)$ is the effective Hamiltonian in the propagation basis. The matrix $\hat{H}^{(p)}$ is nearly identical to $\hat{H}^{(23)}_0$ of Eq. (29) but includes the matter term $V_e = A/2v_e$ added to the element $(\hat{H}^{(23)}_0)_{11}$. Hamiltonian $\hat{H}^{(p)}$ is a real-valued, symmetric matrix devoid of the CP-violating phase $\delta_{CP}$. Of course the CP-phase reappears when one transforms from propagation basis back into flavor basis.

The matrix $\hat{H}^{(p)}$ can be “re-phased”. That is, we perform an algebraic manipulation leading to removal of a term proportional to $\hat{I}$, which merely contributes an overall phase to the oscillation amplitudes. Specifically we subtract (and also add, but then discard) the following term proportional to $\hat{\nu}$:

\[
\hat{H}^{(p)} = \frac{\hat{H}^{(23)}_0}{2v_e}, \quad \frac{1}{2} \left( A - \cos 2\theta_{13} \right) \hat{I}, \quad \text{and} \quad \frac{s_{2\theta_{13}}^2 \alpha^2}{2v_e} \left( \frac{c_{13}^2}{2} + \frac{s_{13}^2}{2} \right) = \frac{1}{4} s_{2\alpha}^2 \alpha.
\]

Upon extraction of a factor $\frac{1}{2}$ we obtain

\[
\hat{H}^{(p)} = \frac{1}{4v_e} \left( \right) \equiv \frac{1}{4v_e} (\cos 2\theta_{13} - A), \quad \frac{1}{4v_e} \sin 2\theta_{13},
\]

Using the variable $G$ defined in Eq. (10), we identify $\hat{H}^{(p)}_{22} = -G$. To represent the other elements of $\hat{H}^{(p)}$ in a compact form, we define:

\[
Q \equiv \frac{1}{4v_e} (\cos 2\theta_{13} - A), \quad f \equiv \frac{1}{4v_e} \sin 2\theta_{13},
\]

\[
a \equiv \frac{1}{4v_e} [c_{13} \alpha'], \quad b \equiv \frac{1}{4v_e} [-s_{13} \alpha'].
\]

The full Hamiltonian in propagation basis can then be written as

\[
\hat{H}^{(p)} = \begin{pmatrix} -Q & a & f \\ a & -G & b \\ f & b & +Q \end{pmatrix}.
\]

C. Formulation in an interaction picture

We separate $\hat{H}^{(p)}$ into an “unperturbed” part, $\hat{H}^{(p)}_0$, plus an interaction potential, $\hat{\nu}$ comprised of elements proportional to $\alpha'$:

\[
\hat{H}^{(p)} = \hat{H}^{(p)}_0 + \hat{\nu} = \begin{pmatrix} -Q & 0 & f \\ 0 & -G & 0 \\ f & 0 & +Q \end{pmatrix} + \begin{pmatrix} 0 & a & 0 \\ a & 0 & b \\ 0 & b & 0 \end{pmatrix}.
\]

Wave equation (31) can then be re-cast into an Interaction picture:

\[
\dot{\nu}^{(I)}(t) = e^{i\hat{H}^{(p)}_0 t} \dot{\hat{\nu}}(t), \quad \nu^{(I)}(t) = e^{-i\hat{H}^{(p)}_0 t} \nu^{(I)}(t),
\]

so that

\[
i \frac{d}{dt} \nu^{(I)}(t) = \dot{V}_i \cdot \nu^{(I)}(t)
\]

where

\[
\dot{V}_i(t) = e^{i\hat{H}^{(p)}_0 t} \cdot \nu \cdot e^{-i\hat{H}^{(p)}_0 t}.
\]

Our approach to Eq. (37) is to construct a heuristic functional form $\dot{U}^{(est)}_i(t, 0)$ which serves as an estimator of the time evolution operator in the Interaction picture:

\[
\nu^{(I)}(t) \simeq \dot{U}^{(est)}_i(t, 0) \cdot \nu^{(I)}(0).
\]

The wave equation which defines our $\dot{U}^{(est)}_i(t, 0)$ is

\[
i \frac{d}{dt} \dot{U}^{(est)}_i(t, 0) = \dot{V}_i(t) \cdot \dot{U}^{(est)}_i(t, 0),
\]

where $t$ is variable but $\dot{V}_i(t)$ is set to its value at the final baseline distance $t=L$.

To obtain $\dot{V}_i(t)$ we require the matrix representation (in propagation basis) of the unitary operator forms $\exp(\pm i\hat{H}^{(p)}_0 t)$, where $\hat{H}^{(p)}_0$ has the elements given in Eq. (35). Considering the series expansion

\[
\dot{W} \equiv e^{i\hat{H}^{(p)}_0 t} = \sum_{n=0}^{\infty} \frac{(i\hat{H}^{(p)}_0 t)^n}{n!},
\]

it is readily seen that neither the middle row nor middle column of matrix $\hat{H}^{(p)}_0$ mixes with other elements. Then the matrix $\dot{W}$ has the form

\[
e^{i\hat{H}^{(p)}_0 t} = \begin{pmatrix} W_{11} & 0 & W_{13} \\ 0 & e^{-iGt} & 0 \\ W_{31} & 0 & W_{33} \end{pmatrix}.
\]

Thus we may work with the reduced $2 \times 2$ matrix

\[
\dot{H}^{(p)}_R = \begin{pmatrix} -Q & f \\ f & +Q \end{pmatrix} = f \sigma_z - Q \sigma_x
\]

where $\sigma_x, \sigma_z$ are the Pauli spinor matrices. We write $\dot{H}^{(p)}_R = \hat{N} \cdot \hat{\sigma}$, where $\hat{N} = (f, 0, -Q)$, with $|\hat{N}| = \sqrt{f^2 + Q^2}$ (see Eq. (36)). The unit vector $\hat{n} \equiv N/N = (f^2 + Q^2)^{-1/2} (f, 0, -Q)$ serves as the axis-of-rotation in the reduced spinor space. With $\dot{H}^{(p)}_R = N \hat{n} \cdot \hat{\sigma}$, and recognizing that $t = \ell$ in natural units, we write

\[
e^{i\hat{H}^{(p)}_R (t=\ell)} = e^{i\hat{n} \cdot \hat{\sigma} (N) t} = e^{i\hat{n} \cdot \hat{\sigma} \phi},
\]
where $\phi \equiv N\ell$ designates the rotation angle. With $\hat{n} = (n_x, 0, n_z)$, the spinor identity is
\begin{equation}
\hat{e}^{in\cdot\hat{a}\phi} = \hat{I} \cos \hat{a} \hat{\phi} + i \hat{\sigma} \cdot \hat{n} \sin \hat{\phi} = \begin{pmatrix}
\cos \hat{\phi} + in_z \sin \hat{\phi} & -in_x \sin \hat{\phi} \\
in_x \sin \hat{\phi} & \cos \hat{\phi} - in_z \sin \hat{\phi}
\end{pmatrix}.
\end{equation}
(45)
We define $\gamma \equiv \cos \hat{\phi} + in_z \sin \hat{\phi}$ and $\beta \equiv n_x \sin \hat{\phi}$, and we write
\begin{equation}
\hat{e}^{-i\hat{H}(p)\ell} = \begin{pmatrix}
\gamma & 0 & i\beta \\
0 & e^{-iG\ell} & 0 \\
i\beta & 0 & \gamma^*
\end{pmatrix}.
\end{equation}
(46)
Note that $n_x$ and $n_z$ are real-valued, hence $\beta$ is real-valued, however $\gamma$ is complex. Evaluation of Eq. (38) yields
\begin{equation}
\hat{V}_I(\ell) = \begin{pmatrix}
0 & \gamma & i\beta \\
0 & e^{-iG\ell} & 0 \\
i\beta & 0 & \gamma^*
\end{pmatrix}.
\end{equation}
(47)
The complex matrix elements of $\hat{G}$ are usefully expressed as
\begin{equation}
u \equiv (\gamma a + i\beta b)e^{iG\ell},
\end{equation}
\begin{equation}v \equiv (\gamma b - i\beta a)e^{-iG\ell}.
\end{equation}
(48)
Then we have
\begin{equation}
\hat{V}_I(\ell) = \begin{pmatrix}
0 & u & 0 \\
u^* & 0 & v \\
v^* & 0 & 0
\end{pmatrix}.
\end{equation}
(49)
where $|u|^2 + |v|^2 = a^2 + b^2 = (\alpha'/4\ell_v)^2 = \eta^2$.

\section{Heuristic construction for $\hat{U}_I(t, 0)$}

Exact solution of wave equation requires a time evolution operator $\hat{U}_I(t, 0)$ which solves Eq. (40) for the case wherein $\hat{V}_I(t)$ is a function of “live” variable $t$. A formal solution is provided in principle by the Dyson series. In practice, the series is always truncated at low order; Sec. III of Ref. 21 gives a clear discussion. As an alternative approximation which retains the perturbative expansion structure of the Dyson series, we introduce the exponentiation of $\hat{V}_I(\ell)$ as a heuristic form:
\begin{equation}
\hat{U}^{(est)}_I(\ell, 0) = e^{-i\hat{V}_I(\ell)t}.
\end{equation}
To obtain the matrix representation of $\hat{U}^{(est)}_I(\ell, 0)$, we take a brute force approach, rather than e.g., harnessing the Cayley-Hamilton theorem. We observe that
\begin{equation}
(\hat{V}_I)^2 = \begin{pmatrix}
|u|^2 & 0 & uv \\
0 & |u|^2 + |v|^2 & 0 \\
(uv)^* & 0 & |v|^2
\end{pmatrix}.
\end{equation}
(51)
Furthermore, $(\hat{V}_I)^3 = \eta^2 (\hat{V}_I)$, $(\hat{V}_I)^4 = \eta^2 (\hat{V}_I)^2$, and by induction $(\hat{V}_I)^{n=odd} = \eta^{n-1}\hat{V}_I$, and $(\hat{V}_I)^{n=even} = \eta^{n-2}(\hat{V}_I)^2$. Thus
\begin{equation}
e^{-i\hat{V}_I t} = \sum_{n=0}^{\infty} \left(\frac{i\hat{V}_I t}{\eta}\right)^n/n! = \hat{I} - \left(\frac{\hat{V}_I}{\eta}\right)^2 (1 - \cos \eta t) - i \frac{\hat{V}_I}{\eta} \sin \eta t.
\end{equation}
(52)
The time evolution operator for neutrino propagation in our interaction picture obeys a matrix identity reminiscent of that for rotations generated by $\hat{J}^{(j=1)}[25]$. Equation (52) yields an explicit representation for $e^{-i\hat{V}_I t}$. We write $\theta \equiv \eta t$, $\pi \equiv u/\eta$, $\sigma \equiv v/\eta$, and use $(1 - \cos \theta) = 2 \cdot \sin^2 \theta/2$. Then the evolution operator of Eq. (50) is
\begin{equation}
\hat{U}^{(est)}_I(\ell, 0) = \begin{pmatrix}
1 - 2|\pi|^2 \sin^2 \theta/2 & -i\pi \sin \theta & -2i\pi \sin^2 \theta/2 \\
i\pi \sin \theta & \cos \theta & -i\pi \sin \theta \\
-2(|\pi|^2 + i\sigma)^* \sin^2 \theta/2 & -i\sigma \sin \theta & 1 - 2|\pi|^2 \sin^2 \theta/2
\end{pmatrix}.
\end{equation}
(53)
We proceed to work our way back, first to the three neutrino propagation basis and then to the neutrino flavor basis, wherein the matrix elements of $\hat{U}^{(est)}_I(\ell, 0)$ correspond to the possible neutrino oscillation amplitudes for 3-flavor mixing. The return of the time evolution operator to propagation basis requires that we calculate
\begin{equation}
\hat{U}^{(p)}(\ell, 0) = e^{-i\hat{H}_0(\eta)\ell} \cdot \hat{U}^{(est)}_I(\ell, 0).
\end{equation}
(54)
The required matrix forms are [53] and the adjoint of $\hat{U}_I$. As a preliminary to this multiplication, we define some compact forms. For the diagonal elements we define
\begin{equation}
D_u = 1 - 2|\pi|^2 \sin^2 \theta/2, \quad d \equiv \cos \theta,
\end{equation}
\begin{equation}
D_v = 1 - 2|\pi|^2 \sin^2 \theta/2.
\end{equation}
(55)
For the off diagonal elements, we define
\begin{equation}
e \equiv \pi \sin \theta, \quad p \equiv -2i\pi \sin^2 \theta/2, \quad k \equiv \pi \sin \theta.
\end{equation}
(56)
Then we write
\begin{equation}
\hat{U}^{(est)}_I(\ell, 0) = \begin{pmatrix}
D_u & -ie & p \\
-ie^* & d & -ik \\
p^* & -ik^* & D_v
\end{pmatrix}.
\end{equation}
(57)
Proceeding with the evaluation of $\hat{U}^{(p)}(\ell, 0)$:
\begin{equation}
\hat{U}^{(p)}(\ell, 0) = \begin{pmatrix}
(\gamma^* D_u - i\beta p^*) & (\gamma^*(-ie) - \beta k^*) & (\gamma^* p - i\beta D_u) \\
(-ie^*) e^{iG\ell} & d e^{iG\ell} & (-ik) e^{iG\ell} \\
(\gamma p^* - i\beta D_u) & (\gamma(-ik^*) - \beta e) & (\gamma D_v - i\beta p)
\end{pmatrix}.
\end{equation}
Finally, we return to neutrino flavor basis via
\[ \hat{U}^{(\nu)}(\ell, 0) = \hat{R}_1(\theta_{23}) \cdot \hat{I} \cdot \hat{U}_{\theta_{13}} \cdot \hat{R}^T_1(\theta_{23}) \cdot \hat{I} \cdot \hat{U}_{\phi} \cdot \hat{R}^T_1(\theta_{23}). \]

All of the physically relevant neutrino flavor oscillation amplitudes are contained in the matrix \( \hat{U}^{(\nu)}(\ell, 0) \); however our focus here is upon the element \( U_{12}^{(\nu)} \).

### E. \( \nu_\mu \to \nu_e \) oscillation amplitude

Element \( U_{12}^{(\nu)} \) provides the \( \nu_e \) appearance amplitude from an initial beam of \( \nu_\mu \) neutrinos:
\[ U_{12}^{(\nu)} = A(\nu_\mu \to \nu_e) = c_{23} U_{12}^{(p)} + s_{23} U_{13}^{(p)} e^{-i\delta_{CP}}. \tag{58} \]

Unwinding notations and rearranging, we obtain
\[ A(\nu_\mu \to \nu_e) = (-i) s_{23} \beta e^{-i\delta_{CP}} \]
\[ + (-i) c_{23} [\gamma^* \bar{u} - i \beta \bar{v}^*] \cdot \sin \theta \]
\[ + 2 s_{23} \left[ i \bar{u} \bar{v} - \gamma^* \bar{u} \bar{v} \right] \cdot \sin^2 \frac{\theta}{2} \cdot e^{-i\delta_{CP}}. \tag{59} \]

We identify the three terms of Eq. [59] using the expression \( A(\nu_\mu \to \nu_e) = T_1 + T_2 + T_3 \). For \( T_1 \) we have
\[ T_1 = (-i) s_{23} \beta e^{-i\delta_{CP}} = (-i) s_{23} n_e \sin \phi \cdot e^{-i\delta_{CP}} \]
\[ = (-i) \sin 2\theta_{13} \cdot s_{23} \cdot \sin \phi \cdot 4\ell c_N \cdot e^{-i\delta_{CP}}. \tag{60} \]

which coincides with Eq. [6].

For \( T_2 \) we have
\[ T_2 = -i c_{23} \left[ \gamma^* \bar{u} - i \beta \bar{v}^* \right] \sin \theta. \tag{61} \]

Assembling the various factors,
\[ T_2 = -i \frac{c_{23}}{\eta} \left[ \gamma^* (\gamma a + i \beta b) - i \beta \gamma^* b + \beta^2 a \right] \cdot \sin \theta \cdot e^{iG_\ell} \]
\[ = -i c_{23} \frac{\alpha}{\eta} \cdot \sin(\theta) \cdot e^{iG_\ell}, \]
which is the same as Eq. [7].

For the remaining term \( T_3 \) we have
\[ T_3 = 2 s_{23} \left[ i \bar{u} \bar{v} - \gamma^* \bar{u} \bar{v} \right] \cdot \sin^2 \frac{\theta}{2} \cdot e^{-i\delta_{CP}}. \tag{62} \]

Reduction within the bracket leads to
\[ T_3 = -2 s_{23} \left[ \gamma \left( \frac{ab}{\eta^2} \right) - i \beta \frac{a^2}{\eta^2} \right] \cdot \sin^2 \frac{\theta}{2} \cdot e^{-i\delta_{CP}}. \tag{63} \]

Now \( ab/\eta^2 = -c_{13}s_{13} \) and \( a^2/\eta^2 = c_{13}^2 \). Recall that \( \gamma = \cos \phi + i n_z \sin \phi \), and that \( \beta = n_x \sin \phi = \frac{\sin \phi}{\sin \phi_{\text{ti}}} \). Working within the bracket of Eq. [63], we separate the real and imaginary pieces:
\[ T_3 = 2 s_{23} \left\{ \frac{\sin 2\theta_{13}}{2} \cos \phi + i \left( \frac{\sin 2\theta_{13}}{2} n_z + c_{13}^2 n_x \right) \sin \phi \right\} \cdot \sin^2 \left( \frac{\eta_\ell}{2} \right) \cdot e^{-i\delta_{CP}}. \]

Then
\[ T_3 = \sin 2\theta_{13} \cdot s_{23} \cdot \left[ \cos(\eta_\ell) + i F_A \sin(\eta_\ell) \right] \cdot \sin^2 \left( \frac{\eta_\ell}{2} \right) \cdot e^{-i\delta_{CP}} \]

which is identical to Eq. [8].

### V. CONCLUSION

Having presented and subsequently derived each of the probability terms individually, we conclude by stating the entire six-term formula for \( P(\nu_\mu \to \nu_e) \):
\[ P(\nu_\mu \to \nu_e) = \]
\[ \left( \sin 2\theta_{13} \right)^2 \cdot \frac{\sin^2(D\Delta)}{D^2} \]
\[ + \sin 2\theta_{13} c_{13} \sin 2\theta_{23} \cdot \frac{\sin(\Delta') \cdot \sin(\Delta) \cdot \frac{\sin(D\Delta)}{D}}{D}. \]
\[ + c_{13}^2 c_{23}^2 \cdot \sin^2(\alpha' \Delta) \]
\[ - 2 \sin 2\theta_{13} \sin 2\theta_{13} c_{23}^2 \cdot F_A \cdot \sin \left( \frac{\alpha' \Delta}{2} \right) \cdot \frac{\sin^2(D\Delta)}{D^2} \]
\[ + \sin 2\theta_{13} c_{13} \sin 2\theta_{23} \cdot \sin(\alpha' \Delta) \cdot \sin^2 \left( \frac{\alpha' \Delta}{2} \right). \]
\[ \left\{ \cos(D\Delta) \cdot \sin(\Delta' + \delta_{CP}) - F_A \cdot \frac{\sin(D\Delta)}{D} \cdot \cos(\Delta' + \delta_{CP}) \right\} \]
\[ + \sin^2 2\theta_{13} s_{23}^2 \cdot \sin^4 \left( \frac{\alpha' \Delta}{2} \right) \cdot \left\{ \cos^2(D\Delta) + F_A^2 \cdot \frac{\sin^2(D\Delta)}{D^2} \right\}. \tag{64} \]

As discussed in Sec. [1111], the three leading terms of Eq. [64] are reminiscent of the perturbative expansion of Eq. [1]. These three terms account for essentially all of the \( \nu_\mu \to \nu_e \) appearance probability.

### VI. ACKNOWLEDGMENTS

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