Topologically Massive Gauge Theory:
A Lorentzian Solution

K. Saygili

Department of Mathematics, Yeditepe University,
Kayisdagi, 34755 Istanbul, Turkey

Abstract:

We obtain a lorentzian solution for the topologically massive non-abelian
gauge theory on AdS space $\tilde{H}^3$ by means of a $SU(1,1)$ gauge transforma-
tion of the previously found abelian solution. There exists a natural scale of
length which is determined by the inverse topological mass $\nu \sim ng^2$. In the
topologically massive electrodynamics the field strength locally determines
the gauge potential up to a closed 1-form via the (anti-)self-duality equation.
We introduce a transformation of the gauge potential using the dual
field strength which can be identified with an abelian gauge transforma-
tion. Then we present the map $\pi : \tilde{H}^3 \rightarrow \tilde{H}^2_+$ including the topological mass
which is the lorentzian analog of the Hopf map. This map yields a global
decomposition of $\tilde{H}^3$ as a trivial $\tilde{S}^1$ bundle over the upper portion of the
pseudo-sphere $\tilde{H}^2_+$ which is the Hyperboloid model for the Lobachevski ge-
ometry. This leads to a reduction of the abelian field equation onto $\tilde{H}^2_+$ using
a global section of the solution on $\tilde{H}^3$. Then we discuss the integration of
the field equation using the Archimedes map $A : \tilde{H}^2_+ - \{N\} \rightarrow \tilde{C}_2^\rho$. We
also present a brief discussion of the holonomy of the gauge potential and
the dual field strength on $\tilde{H}^2_+$.

1Electronic address: ksaygili@yeditepe.edu.tr
1 Introduction

Topologically massive gravity and gauge theory are dynamical theories which are specific to three dimensions [1, 2], [3]. They are qualitatively different from Einstein gravity and Yang-Mills gauge theory beside their mathematical elegance and consistency.

The most distinctive feature of the topologically massive gauge theories is the existence of a natural scale of length which is introduced by the topological mass: $[\nu] = [g]^2 = L^{-1}$ [4], [5], in the geometric units ($h = 1$, $c = 1$). The euclidean Dirac “monopole” type solution of the Maxwell-Chern-Simons (MCS) electrodynamics on the de Sitter (dS) space $\tilde{S}^3$ is an example of the essential new feature introduced by the topological mass [6, 7]. This is a physical system which intrinsically possesses the features of both gauge theory and gravity [6].

This solution can be embedded into the Yang-Mills-Chern-Simons (YMCS) gauge theory by means of a $SU(2)$ gauge transformation [7]. The Hopf map $\pi : \tilde{S}^3 \rightarrow \tilde{S}^2$ including the topological mass $\nu = ng^2$ yields a decomposition of $\tilde{S}^3$ as a non-trivial $\tilde{S}^1$ bundle over $\tilde{S}^2$. The stereographic projection $S : \tilde{S}^3 \rightarrow \mathbb{R}^3$ provides a geometric view of the $\tilde{S}^1$ fibres in the Hopf map. This leads to a reduction of the abelian field equation onto $\tilde{S}^2$ using local sections of $\tilde{S}^3$ [7]. In the topologically massive electrodynamics the field strength locally determines the gauge potential up to a closed 1-form [7] via the self-duality equation [8], [9]. A Wu-Yang type construction [10], [11] leads to a gauge function which can be expressed in terms of the magnetic or the electric charges [7]. In geometrical terms, the quantization of the topological mass reduces to the quantization of the inverse of the natural scale of length $L = 2\pi^{-\frac{1}{2}}$ in units of the inverse of the fundamental length scale $\lambda = 2\pi^{-\frac{1}{2}}g$. The fundamental scale $\lambda$ is the least common multiple of intervals over which the gauge function is single-valued and periodic for any integer $n$ [7]. The integral of the field equation reduces to the formula for the area of a rectangle [7] using the Archimedes map $A : \tilde{S}^2 - \{N, S\} \rightarrow \mathbb{C}^2$ [12, 13]. The geometric phase suffered by a vector upon a parallel transport on $\tilde{S}^2$ [14, 15], [16] can be expressed in terms of the holonomy of the gauge potential or the dual-field strength [7].

We present an analogous discussion of the lorentzian case by replacing the group $SU(2)$ with $SU(1, 1)$ [17], [18] and the dS space $\tilde{S}^3$ with anti-de Sitter (AdS) $\tilde{H}^3$ [19]. We embed the abelian solution on the AdS space $\tilde{H}^3$.
which is given in [6] into the YMCS theory by means of a $SU(1,1)$ gauge transformation.

Then we return to the abelian case. The discussion of (anti-)self-duality follows the same line of reasoning [7] for the self-duality [8], [9]. We introduce a transformation of the gauge potential using the dual field strength [7]. This transformation can be identified with an abelian gauge transformation.

We present the map $\pi : \tilde{H}^3 \longrightarrow \tilde{H}^2_+^{\ast}$ including the topological mass. This is the lorentzian analogue of the Hopf map in the euclidean case [7]. This yields a global decomposition of the AdS space $\tilde{H}^3$ as a trivial $\tilde{S}^1$ bundle over the upper portion of the pseudo-sphere $\tilde{H}^2_+^{\ast}$ [20]. This leads to a reduction of the field equation onto $\tilde{H}^2_+$ using a global section of the solution on $\tilde{H}^3$. The pseudo-sphere $\tilde{H}^2_+$ is the Hyperboloid (Minkowski) model of the Lobachevski geometry [21, 22]. This can be mapped to the Poincare disc $\tilde{D}^2_P$ of radius $r = \frac{1}{\nu}$ by a stereographic projection [21].

We discuss the integration of the field equation on $\tilde{H}^2_+$ using the lorentzian Archimedes map $A : \tilde{H}^2_+ - \{N\} \longrightarrow \tilde{C}^2_0$ from the pseudo-sphere $\tilde{H}^2_+ - \{N\}$ to the cylinder $\tilde{C}^2_0 = R \times \tilde{S}^1_\nu$, where $\tilde{S}^1_\nu$ is the ideal circle enclosing the Poincare disc $\tilde{D}^2_P$. We also present a brief discussion of the holonomy [14, 15], [16] of the gauge potential and the dual-field strength on $\tilde{H}^2_+$.

The topological mass $\nu \sim ng^2$ is not quantized in the present discussion.

\section{The Non-Abelian Gauge Theory}

\subsection{Yang-Mills-Chern-Simons Theory}

The topologically massive YMCS theory is given by the dimensionless action

\begin{equation}
S_{YMCS} = S_{YM} + S_{CS} = -\frac{1}{2} \left[ \int tr (F \wedge *F) + \nu \int tr \left( F \wedge A + \frac{1}{3} g A \wedge A \wedge A \right) \right] ,
\end{equation}

where $\nu$ is the topological mass. We include the factor, in the action (1), containing the gauge coupling constant $g$ in the expressions for the field. Because in our solution the underlying geometry forces the introduction of the gauge coupling constant into the formulas. This provides the strength
\[ g \] for the potentials upon assuming quantization of the topological mass: \[ \nu \sim ng^2. \] Note that the sign of the topological mass in the action (1) is opposite to the euclidean case [7] because of the conventions. The YMCS action (1) yields the field equation

\[ D \star F + \nu F = 0, \quad (2) \]

where \( D \) is the gauge covariant exterior derivative. The field 2-form also satisfies the Bianchi identity

\[ DF = 0. \quad (3) \]

The action (1) is not invariant under non-abelian gauge transformations

\[ A' = U^{-1}AU - \frac{1}{g}U^{-1}dU. \quad (4) \]

It changes by

\[ W = -8\pi^2 \frac{\nu}{g^2} w, \quad (5) \]

neglecting a surface term that vanishes under suitable asymptotic convergence conditions on \( U \) [1, 2], [3]. Here the expression \( w \) which is given as

\[ w = \frac{1}{48\pi^2} \int \text{tr} \left( U^{-1}dU \wedge U^{-1}dU \wedge U^{-1}dU \right), \quad (6) \]

corresponds to the winding number of the gauge transformation [1, 2], [3], if the gauge group is compact, for example \( SU(2) \). In this case the large gauge transformations, which are labeled by the winding number \( w \), have a non-trivial contribution to the action.

In the lorentzian case if we demand the expression \( \exp(iS) \) to be gauge invariant in order to have a well-defined quantum theory via path-integrals [1, 2], [3], then the change (5) can be tolerated if the topological mass is quantized as
\[ \nu = \frac{1}{4\pi}ng^2. \]  

We shall refer the relation: \( \nu \sim ng^2 \) as the quantization of the topological mass no matter if \( n \) is an integer or not. Because this relation will naturally arise in our solutions. Further, note that the group \( SU(1, 1) \) is not compact.

### 2.2 The Natural Scale of Length

We shall consider the YMCS theory over a spacetime with a co-frame consisting of the “modified” left-invariant basis 1-forms of Bianchi type \( VIII \) in Euler parameters \([6]\). The topological mass introduces a geometric scale of length. We shall use an intrinsic arclength parameterization where the arclength parameters are independent of the length scale determined by the inverse topological mass.

We scale the unmodified co-frame with the dimensionful factor \( \frac{1}{\nu} \). This yields

\[
\begin{align*}
\omega^1 &= -\cos(\nu\psi)d\theta - \sin(\nu\psi)\sinh(\nu\theta)d\phi, \\
\omega^2 &= -\sinh(\nu\psi)d\theta + \cos(\nu\psi)\sinh(\nu\theta)d\phi, \\
\omega^3 &= d\psi + \cosh(\nu\theta)d\phi,
\end{align*}
\]

in terms of the intrinsic (half) arclength parameters

\[
\begin{align*}
\theta &= \frac{1}{2}R\tilde{\theta}, & \phi &= \frac{1}{2}R\tilde{\phi}, & \psi &= \frac{1}{2}R\tilde{\psi}, \\
\tilde{\theta} &= \frac{1}{\nu}\tilde{\theta}, & \tilde{\phi} &= \frac{1}{\nu}\tilde{\phi}, & \tilde{\psi} &= \frac{1}{\nu}\tilde{\psi}
\end{align*}
\]

which have the dimension of length: \([\theta] = [\phi] = [\psi] = L\). The modification given in equation (8) amounts to scaling the Cartan-Killing metric by the factor \( \frac{1}{\nu^2} \)

\[
\begin{align*}
ds^2 &= \eta_{ab}\omega^a\omega^b, & \eta_{ab} = diag(-1, -1, 1) \\
&= -d\theta^2 + d\phi^2 + 2\cosh(\nu\theta)d\phi d\psi + d\psi^2 \\
&= -[d\theta^2 + \sinh^2(\nu\theta)d\phi^2] + [d\psi + \cosh(\nu\theta)d\phi]^2,
\end{align*}
\]
which yields the AdS space $\tilde{H}^3$. The \textit{radius} of the AdS space is also scaled by the same factor: $R = \frac{1}{\nu} \tilde{R} = \frac{1}{\nu^2}$. The parameters $\theta, \phi, \psi$ (9) respectively represent the half-length of arcs which correspond to the Eulerian parameters $\tilde{\theta} = \nu \theta, \tilde{\phi} = \nu \phi, \tilde{\psi} = \nu \psi$ on the AdS space $\tilde{H}^3$ of \textit{radius} $\tilde{R} = \frac{2}{\nu}$. Thus (10) is the metric on the AdS space $\tilde{H}^3$ of \textit{radius} $R = \frac{2}{\nu}$ which is parameterized in terms of the (half) Eulerian arclengths. The scalar curvature $\mathcal{R}$ of this space is determined by its \textit{radius} $R$

\[ \mathcal{R} = -\frac{6}{R^2} = -\frac{3}{2} \nu^2. \]  

(11)

This can also be verified from the metric (10). Note that the arclength parameters are independent of the length scale which is determined by the inverse topological mass. We shall only consider the degrees of freedom which are associated with the intrinsic arclengths.

The Maurer-Cartan equations $d\omega^i = \frac{1}{2} C_{jk}^i \omega^j \wedge \omega^k$ for the co-frame (8) yields

\[ d\omega^1 = \nu \omega^2 \wedge \omega^3, \quad d\omega^2 = \nu \omega^3 \wedge \omega^1, \quad d\omega^3 = -\nu \omega^1 \wedge \omega^2. \]  

(12)

The co-frame (8) determines a unique orientation on the AdS space $\tilde{H}^3$ and these satisfy the Hodge duality relations

\[ *\omega^1 = -\omega^2 \wedge \omega^3, \quad *\omega^2 = -\omega^3 \wedge \omega^1, \quad *\omega^3 = \omega^1 \wedge \omega^2. \]  

(13)

The Maurer-Cartan equations (12) and the Hodge-duality relations (13) for the basis (8) immediately lead to the result that the MCS field equation: $d(*F + \nu A) = 0$ will be identically satisfied for the gauge potential 1-form: $A = -\frac{\nu}{g} \omega^3$ [6]. The field equation is given by the derivative of the (anti-)self-duality condition: $*F + \nu A = 0$ for the topologically massive abelian gauge fields [8], [9]. The MCS action vanishes for this potential: $S_{\text{MCS}}[A] = 0$.

This AdS space $\tilde{H}^3$ can be embedded into the space $\mathbb{R}^4$ with signature $(+,+,−,−)$

\[ (y^1)^2 + (y^2)^2 - (y^3)^2 - (y^4)^2 = R^2, \quad R = \frac{2}{\nu}. \]  

(14)
by the correspondence
\begin{align}
y^1 &= R \cosh \left( \frac{\nu \theta}{2} \right) \cos \left( \frac{\nu \psi + \phi}{2} \right), \quad y^2 = R \cosh \left( \frac{\nu \theta}{2} \right) \sin \left( \frac{\nu \psi + \phi}{2} \right), \\
y^3 &= R \sinh \left( \frac{\nu \theta}{2} \right) \cos \left( \frac{\nu \psi - \phi}{2} \right), \quad y^4 = R \sinh \left( \frac{\nu \theta}{2} \right) \sin \left( \frac{\nu \psi - \phi}{2} \right),
\end{align}
(15)
where $R = \frac{2}{\nu}$. The flat metric
\begin{equation}
ds^2 = (dy^1)^2 + (dy^2)^2 - (dy^3)^2 - (dy^4)^2,
\end{equation}
(16)
on $\mathbb{R}^4$ reduces to (10) with this correspondence.

We shall define the map $\pi : \tilde{H}^3 \rightarrow \tilde{H}_2^+ \uparrow$ including the topological mass in section 3.2. We also scale the radius of the unit hyperboloid of 2-sheets $H^2$ (in euclidean $\mathbb{R}^3$) by the same factor $\frac{1}{\nu}$
\begin{equation}
(x^1)^2 + (x^2)^2 - (x^3)^2 = -r^2, \quad r = \frac{1}{\nu},
\end{equation}
(17)
The correspondence with the $\tilde{H}^2$ metric is given by
\begin{align}
x^1 &= r \sinh(\nu \theta) \cos(\nu \phi), \quad x^2 = r \sinh(\nu \theta) \sin(\nu \phi), \quad x^3 = r \cosh(\nu \theta),
\end{align}
(18)
where $r = \frac{1}{\nu}$. This provides an embedding of the upper portion of the hyperboloid $\tilde{H}_2^+$ with radius $r = \frac{1}{\nu}$ into the space $\mathbb{R}^3$ with signature $(+,+,−)$. The flat metric
\begin{equation}
ds^2 = (dx^1)^2 + (dx^2)^2 - (dx^3)^2,
\end{equation}
(19)
on $\mathbb{R}^3$ reduces to
\begin{equation}
ds^2 = d\theta^2 + \sinh^2(\nu \theta) \, d\phi^2,
\end{equation}
(20)
on $\tilde{H}_2^+$ with this correspondence. Thus (20) is the metric on the pseudo-sphere $\tilde{H}_2^+$ of radius $r = \frac{1}{\nu}$ which is parameterized in terms of the length of arcs corresponding to the parameters $\tilde{\theta} = \nu \theta, \tilde{\phi} = \nu \phi$. The $(-)$ factor for the metric (20) in (10) is due to our conventions. Note that the pseudo-sphere $\tilde{H}_2^+$ is a space-like surface [21, 22].

\begin{align*}
\mathcal{S}(x^1, x^2, x^3) &= \left( u^1 = \frac{r}{r + x^3} x^1, \ u^2 = -\frac{r}{r + x^3} x^2 \right) \\
&= \left( u^1 = r' \cos(\nu \phi), \ u^2 = -r' \sin(\nu \phi) \right),
\end{align*}

This is the Hyperboloid (Minkowski) model of the Lobachevski geometry [21]. The pseudo-sphere $\tilde{H}_2^+$ can be mapped to the Poincare disc with radius $r$ by a stereographic projection through the point $S(0,0,-r)$ as shown in Figure 1, [21]. The stereographic projection $\mathcal{S} : \tilde{H}_2^+ \rightarrow \tilde{D}_P^2$ is given as

Figure 1: The Stereographic projection $\mathcal{S} : \tilde{H}_2^+ \rightarrow \tilde{D}_P^2$ and The Archimedes map $\mathcal{A} : \tilde{H}_2^+ \rightarrow \tilde{C}^2$
\[ w = u^1 + iu^2 = r' \exp(-i\nu\phi), \quad r' = r \tanh \left( \frac{\nu\theta}{2} \right). \]

Note that \(|w| = r' < r\). The metric (20) becomes

\[ ds^2 = \frac{4r^4}{(r^2 - |w|^2)^2} |dw|^2 = \frac{4r^4}{[r^2 - (u^1)^2 - (u^2)^2]^2} \left[ (du^1)^2 + (du^2)^2 \right], \]

on \( \tilde{D}_p^2 \). More precisely, the parameter \( \theta \) is the length of an arc on the hyperbola which is determined by the intersection of the pseudo-sphere with a plane containing the \( z \)-axis. This becomes the hyperbolic radial arclength in \( \tilde{D}_p^2 \). The parameter \( \phi \) is the length of an arc on the ideal circle \( \tilde{S}_1^p \) of radius \( r \) which encloses the Poincaré disc.

The range of the parameters \( \theta, \phi, \psi \) are determined by the topological mass. The arclength \( \phi \) uniquely defines the angle \( \tilde{\phi} = \nu\phi \) for any parallel on \( \tilde{H}_+^2 \). The length of an arc, on the parallel determined by \( \theta = \frac{1}{\nu}\tilde{\theta} \), which corresponds to the angle \( \tilde{\phi} \) is

\[ \frac{\sinh(\nu\theta)}{\nu} \tilde{\phi} = \sinh(\nu\theta)\phi. \]

This reduces to the arclength \( \phi = \frac{1}{\nu}\tilde{\phi} \) on the equator: \( \cosh(\nu\theta) = \sqrt{2} \).

The coordinate \( \phi \) which is well-defined for any point except the north pole \( N(0,0,r) \) on \( \tilde{H}_+^2 \) changes from \( \phi = 0 \) to \( \phi = 2\pi \frac{1}{\nu} \) round about any parallel.

### 2.3 The Topologically Massive Non-abelian Solution

First consider the abelian solution [6] which is given by the gauge potential 1-form:

\[ A = -\frac{\nu}{g} \omega^3 \tau_3 \]

\[ = -\frac{\nu}{g} \left[ d\psi + \cos(\nu\theta)d\phi \right] \tau_3. \]
Here \( \tau_i = \frac{\theta_i}{2\pi} \) where \( \rho_i \)

\[
\rho_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(25)

are the generators of the group \( SU(1,1) \). Note that \( \rho_1 = i\sigma_2 \), \( \rho_2 = -i\sigma_1 \), \( \rho_3 = \sigma_3 \) where \( \sigma_1, \sigma_2, \sigma_3 \) are the Pauli spin matrices [7]. These satisfy the commutation relations

\[
[\rho_1, \rho_2] = -2i\rho_3, \quad [\rho_2, \rho_3] = 2i\rho_1, \quad [\rho_3, \rho_1] = 2i\rho_2.
\]

(26)

The factor \( \frac{\nu}{g} \) yields \( \frac{1}{4\pi} ng \) for the strength of the potential \( A \) (24) upon quantization of the topological mass (7). This is analogous to Dirac’s quantization condition \( eg = n \) which leads to the strength \( g \) for the non-abelian solution upon a gauge transformation of the Dirac’s monopole potential [10], [11].

A gauge transformation of the potential (24) necessarily contains the topological mass inherently in the gauge function. Because it also appears in the potential (24). This is basically due to the arclength parameterization. The gauge function which embeds the topologically massive abelian solution (24) into the YMCS theory is an element of the group \( SU(1,1), [17], [18] \). An element of this group is given by normalizing the radius \( R \) of \( H^3 \) which is parameterized with the Eulerian arclengths as

\[
U = \frac{1}{R} \begin{pmatrix} z^1 & z^2 \\ \bar{z}^2 & \bar{z}^1 \end{pmatrix}
\]

(27)

\[
= \exp(-\nu\gamma\tau_3) \exp(-\nu\beta\tau_2) \exp(-\nu\alpha\tau_3).
\]

Here \( z^1 = y^1 + iy^2, z^2 = y^3 + iy^4 \). We identify the Euler parameters as \( \alpha = \phi, \beta = \theta, \gamma = \psi \). Note that \( U \to 1 \) as \( \nu \to 0 \).

A gauge transformation (4) with the gauge function (27) of the potential (24) yields the potential 1-form

\[
A' = U^{-1}AU - \frac{1}{g}U^{-1}dU
\]

(28)

\[
= -\frac{\nu}{g} \left\{ \left[ \sin(\nu\phi)\tau_1 - \cos(\nu\phi)\tau_2 \right] d\theta \\
+ \left[ \cosh(\nu\theta)\cos(\nu\phi)\tau_1 + \cosh(\nu\theta)\sin(\nu\phi)\tau_2 - \sinh(\nu\theta)\tau_3 \right] \sinh(\nu\theta)\,d\phi \right\}.
\]
This gives rise to the field 2-form

\[ F' = dA' - gA' \wedge A' = U^{-1}FU \]  

\[ = -\frac{\nu^2}{g} \left[ \sinh(\nu\theta) \cos(\nu\phi)\tau_1 + \sinh(\nu\theta) \sin(\nu\phi)\tau_2 + \cosh(\nu\theta)\tau_3 \right] \]

\[ \sinh(\nu\theta)d\theta \wedge d\phi. \]

The dual-field 1-form covariantly transforms as: \( \ast F' = U^{-1} \ast FU \). The field equation (2) and the Bianchi identity (3) are identically satisfied since they also covariantly transform. This is the lorentzian analog of the topologically massive euclidean solution [7] where the parameter \( \tilde{\theta} \) becomes a pseudo-angle.

We have three observations which are based on simple dimensional arguments in the equations (24), (27), (28), (29). These are exactly the same as those of the euclidean case [7].

i) The gauge function necessarily contains the topological mass as in (27).

ii) The strength \( \frac{\nu}{g} \) of the abelian gauge potential (24) is crucial for finding the field 2-form (29) in the non-abelian case. One finds the correct expression for the field 2-form with this choice.

iii) This is associated with the quantization of the topological mass.

Thus if the strength of the potential is given as \( \frac{\nu}{g} \) then we inevitably arrive at the condition (7), because this yields \( \frac{\nu}{g} = \frac{1}{4\pi}ng \) as the strength of the potential. Conversely, if one starts with a potential with the strength \( \frac{1}{4\pi}ng \), then one needs to use (7) in finding the non-abelian gauge potential \( A' \) (28) and the field \( F' \) (29). Moreover the YMCS field equation reduces to the condition (7) which will be identically satisfied. Note that in this discussion the number \( n \) is a free parameter. The gauge coupling strength and the electric charge are denoted with \( g \). The factor of \( \frac{1}{4\pi} \) in the topological mass (7) is included in the strength of the potential for the sake of conciseness.

The action (1) for the potential (28) reduces to the Yang-Mills term because the Chern-Simons piece vanishes. This is also equal to the change \( W \) (5) in the action due to the gauge transformation (27). These are given as

\[ S_{YMCS}[A'] = S_{YM}[A'] = W[U] \]

\[ = -\infty. \]
The other term which arise in the action as a result of the gauge transformation given by the function (27) vanishes. The quantization of the topological mass also leads to

\[ R \sim n^2 g^4, \]  
for the scalar curvature (11).

### 3 The Abelian Gauge Theory

#### 3.1 Maxwell-Chern-Simons Theory

The action for the MCS theory is given as

\[ S_{MCS} = S_M + S_{CS} \]

\[ = -\frac{1}{2} \frac{1}{2\pi} \left( \int F \wedge \ast F + \nu \int F \wedge A \right). \]

Note that the sign of the topological mass is opposite to that in [7]. In order to preserve the other conventions of [7] we explicitly write \( \frac{1}{2\pi} \) in (7) as an overall factor in the action. We also adopt a slight change of convention in the abelian potential: \( A = -\frac{1}{2 g} \omega^3 \) and the topological mass is now given as: \( \nu = ng^2 \). We shall refer \( \nu = ng^2 \) as the quantization of the topological mass without addressing whether \( n \) is an integer. The MCS action (32) yields the field equation

\[ (\ast d + \nu) \ast F = 0. \]

The field 2-form also satisfies the Bianchi identity \( dF = 0 \).

The discussion of (anti-)self-duality follows the same line of reasoning for the self-duality [8], [9] which is given in [7]. The topologically massive field \( F \) locally determines the potential \( A' \) up to a gauge term via the (anti-)self-duality equation

\[ \ast F + \nu A' = 0 \quad , \quad F = dA', \]
where $A' = A - \alpha$, $d\alpha = 0$. Furthermore we can treat the dual-field 1-form $*F$ as a gauge potential in the field equation (33) because of the symmetry under the interchange $*F \leftrightarrow \nu A$ of the equations (34) and (33). Thus the field equation (33) reduces to finding the strength of the gauge potential which is determined by the field strength itself. The Maxwell and the Chern-Simons terms in the MCS action (32) interchange under this symmetry as $F' = F$ is kept fixed.

Furthermore we can define another potential $\tilde{A}$ by the transformation

$$\tilde{A} = A - \frac{1}{\nu} * F,$$  \hspace{1cm} (35)

which is motivated by this symmetry [7]. A variant of this transformation is also observed in [23]. In fact, with a similar reasoning, one can introduce higher order terms of type $(\frac{i}{\nu} * d)^i A$ where $i = 1, 2, 3, ...$. The topologically massive gauge theory is related to the CS theory through such an expansion of field redefinition in [24], [25, 26]. The new potential $\tilde{A}$ transforms as a connection under the abelian gauge transformations. We can interpret (35) as an abelian gauge transformation

$$A' = A - \frac{i}{g} U^{-1} dU,$$  \hspace{1cm} (36)

of the potential $A$. We find that the gauge function is given as

$$U = \exp \left( -\frac{g}{\nu} \oint *F \right)$$

$$= \exp \left[ ig \oint (A - \alpha) \right],$$  \hspace{1cm} (37)

upon the identification

$$-\frac{1}{\nu} * F = -\frac{i}{g} d \ln U,$$  \hspace{1cm} (38)

and using the (anti-)self-duality equation (34).
We have noted that the Wu-Yang construction on $S^2$ reduces to integration of the field equation in case one local potential is used on the appropriate local chart of $S^2$ except a pole [7]. Because the field strength locally determines the potential via the self-duality equation. In the present case we shall see that we have a trivial bundle $H^3 = S^1 \times H^2_\pm$. Thus there is a local gauge potential which is globally well-defined on $H^2_\pm$. This gauge potential is given by a gauge transformation from $A = 0$ with the gauge function $U$

$$U = \exp \left( -i \frac{g}{\nu} \Phi \right) = \exp (igQ) = \exp \left[ ig \oint A \right].$$

Here $\Phi$ and $Q$ are respectively the electric flux/circulation and the magnetic flux which will be defined below. These are related via the integral of the field equation on $H^2_\pm$.

### 3.2 The $U(1)$ Gauge Field on the Pseudo-sphere: $\tilde{H}^2_+$

We shall present the map $\pi : \tilde{H}^3 \rightarrow \tilde{H}^2_+$ including the topological mass in this section. For a brief discussion of this map in a different context without the topological mass see [27, 28]. The AdS space $\tilde{H}^3$ is globally given as a $S^1$ bundle over the base manifold $\tilde{H}^2_+$: $\tilde{H}^3 = S^1 \times \tilde{H}^2_+$, [20]. Thus the bundle is trivial and we have a global section over the whole pseudosphere: $\tilde{H}^2_+$. We remind that the effect of the topological mass is to introduce a natural scale of length. This map leads to a reduction of the field equation onto the pseudo-sphere $H^2_+$ using a global section of the (anti-)self-dual solution $A = -\frac{1}{2g} \omega^3$ on $\tilde{H}^3$ for which the action vanishes. This gives rise to a topologically massive potential which is well-defined on $H^2_+$.

The map $\pi : \tilde{H}^3 \rightarrow \tilde{H}^2_+$ is defined as

$$x^1 = 2 \frac{r}{R^2} \left( y^1 y^3 + y^2 y^4 \right),$$

$$x^2 = 2 \frac{r}{R^2} \left( y^2 y^3 - y^1 y^4 \right),$$

$$x^3 = \frac{r}{R^2} \left[ (y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2 \right].$$

14
The section of the 3-sphere (15) corresponding to \( N : \psi = -\phi, y^2 = 0 \) is given as

\[
\begin{align*}
    z^1 &= R \cosh \left( \frac{\nu \theta}{2} \right) = R \frac{r}{\sqrt{r^2 - |w|^2}}, \\
    z^2 &= R \sinh \left( \frac{\nu \theta}{2} \right) \exp(-i\nu \phi) = R \frac{w}{\sqrt{r^2 - |w|^2}}, \\
    w &= r \frac{x^1 - ix^2}{r + x^3} = r \frac{z^2}{z^1} = r \tanh \left( \frac{\nu \theta}{2} \right) \exp(-i\nu \phi).
\end{align*}
\]

Here \( w \) is the stereographic projection coordinate on \( \tilde{H}^2_+ \) projected from the south pole \( S(0,0,-r) \) to the \( xy \)-plane (21). The local section given in the equation (41) is well-defined on whole \( \tilde{H}^2_+ \). Thus this provides a global section since it can be extended for any value of \( \theta \). This leads to a global trivialization of the bundle over whole \( \tilde{H}^2_+ \). Therefore we have a trivial bundle. Thus we have the global equivalence: \( \tilde{H}^3 = \tilde{S}^1 \times \tilde{H}^2 \).

The inverse image of a point in \( \tilde{H}^2_+ \) with the stereographic coordinate \( w \) is given by the equation : \( z^2 = \frac{1}{r} wz^1 \) where \( |z^1|^2 - |z^2|^2 = R^2 \). This is the intersection of \( \tilde{H}^3 \) with a hyperplane passing through the origin which is defined by this equation. The inverse image in \( \tilde{H}^3 \) (15) of a point with the coordinate \( w \) (41) in \( \tilde{H}^2_+ \) is parameterized by \( \exp(-i\psi \tau_3) \) or simply \( \exp(i\nu \frac{\psi}{2}) \) (27) which generates a circle. The image of this circle is again the same point (41).

The 1-form \( \frac{1}{2} \omega^3 \) (8) defines a connection on the AdS space \( \tilde{H}^3 \) which is considered as a circle \( \tilde{S}^1 \) bundle over the pseudosphere \( \tilde{H}^2_+ \). This gives rise to the potential 1-form \( A = \frac{1}{2} \frac{\nu g}{2} \omega^3 \). The strength \( \frac{1}{2} \frac{\nu g}{2} \) of the potential reduces to \( -\frac{1}{2} ng \) in terms of the gauge coupling constant \( g \) upon adopting the quantization of the topological mass: \( \nu = ng^2 \).

The abelian potential \( A = \frac{1}{2} \frac{\nu g}{2} \omega^3 \) on \( \tilde{H}^3 \) which is given as

\[
A = -\frac{1}{2} \frac{\nu}{g} \left[ d\psi + \cosh(\nu \theta) d\phi \right] \quad (42)
\]

\[
= -\frac{1}{g} \frac{1}{R^2} \left( -y^2 dy^1 + y^1 dy^2 + y^4 dy^3 - y^3 dy^4 \right).
\]
yields the field 2-form

\[ F = dA = -\frac{1}{2} \frac{\nu^2}{g} \sinh(\nu \theta) d\theta \wedge d\phi \]
\[ = -\frac{1}{2} \frac{2}{g R^2} \left( dy^1 \wedge dy^2 - dy^3 \wedge dy^4 \right). \]

There exists a globally defined potential 1-form \( A^N \) on \( \tilde{H}^2_+ \) such that the field 2-form \( F \) is globally expressed as \( F = dA^N \). This potential 1-form \( A^N \) on \( \tilde{H}^2_+ \) is locally given by the projection of the specific section \( N : \psi = -\phi \) of \( A \) (42) on \( \tilde{H}^3 \) onto \( \tilde{H}^2_+ \) using the map \( \pi : \tilde{H}^3 \to \tilde{H}^2_+ \)

\[ A^N = + \frac{1}{2} \frac{\nu}{g} \left[ 1 - \cosh(\nu \theta) \right] d\phi = -\frac{1}{2} \frac{1}{g} \frac{-x^2 dx^1 + x^1 dx^2}{r(r + x^3)}. \]

The potential \( A^N \) is well-defined on whole \( \tilde{H}^2_+ \) since \( x^3 \geq r \). This potential yields the field 2-form

\[ F^N = F = -\frac{1}{2} \frac{\nu^2}{g} \sinh(\nu \theta) d\theta \wedge d\phi \]
\[ = -\frac{1}{2} \frac{1}{g} \frac{1}{r^3} \left( x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2 \right), \]

on \( \tilde{H}^2_+ \).

We could have considered the section \( N : \psi = -\phi \) for the potential (24) and the gauge function (27) in embedding the abelian solution into the YMCS theory. It is straightforward to check that the equations (28), (29) are satisfied for this section.

The potential (42) is determined by the field (43) on \( \tilde{S}^3 \) because of the Hodge-duality relations (13) for the basis (8). Hence it is self-dual (34). The extra \( \alpha \) term (34) for the potential \( A^N \) (44) on \( \tilde{H}^2_+ \) vanishes upon a consistent choice of the appropriate local section \( N : \phi = -\psi \) for the dual-field: \( *F^N = (F^N)^N \) while projecting \( \tilde{H}^3 \) onto \( \tilde{H}^2_+ \). Otherwise this gives rise to an extra term which can be made to vanish by an abelian gauge
transformation with $U = \exp[-i\nu(\phi + \psi)]$ on $\tilde{H}^3$. The MCS field equation (33)

$$d \ast F^N + \nu F^N = 0,$$

(46)

and the (anti-)self-duality equation (34)

$$*F^N + \nu A^N = 0,$$

(47)

are satisfied for the potential $A^N$ (44) on $\tilde{H}^2_\pm$.

We remark that the choice of the local section $\tilde{N} : \psi = -\phi$ endows us with a potential as a local expression for the connection which is determined by the field itself via the (anti-)self-duality condition [7].

### 3.3 Integration of the Field Equation

The integration of the field equation (46) follows the same line of reasoning as in the euclidean case [7]. We can use geometric quantities such as area and arclength on the pseudo-sphere $\tilde{H}^2_\pm$ or the Poincare disc $\tilde{D}^2_\pm$.

The integration of the field equation (46) over the local chart $U(P: \theta)$ of $\tilde{H}^2_\pm$ which is determined by $r \leq x^3 \leq r \cosh(\nu\theta)$ yields

$$\Phi^N + \nu Q^N = 0.$$  

(48)

Here $\Phi$ and $Q$ are respectively the electric flux/circulation and the magnetic flux through $U(P: \theta)$ which are defined as

$$\Phi^N \equiv \int_{U(P: \theta)} d \ast F^N , \quad Q^N \equiv \int_{U(P: \theta)} F^N.$$  

(49)

These reduce to loop integrals over the boundary of the chart $U(P: \theta)$ which is given by the parallel $P : \theta$ upon using the Stokes theorem

$$\Phi^N = -\oint_{P: \theta} *F^N , \quad Q^N = -\oint_{P: \theta} A^N.$$  

(50)
The factors of \((-\) sign are due to orientation on the parallel \(P : \theta\) since it encircles the chart \(U(P : \theta)\) in the left-handed sense. Thus we conclude, from the equations (48-50), that the magnetic flux \(Q^N\) through \(U(P : \theta)\) is determined by the electric circulation \(\Phi^N\) on the boundary \(P : \theta\). The electric flux/circulation and the magnetic flux associated with the potential \(A^N\) are given as

\[
\Phi^N = -\frac{\nu g}{1} \left[ 1 - \cosh(\nu \theta) \right], \quad Q^N = \frac{1}{g} \left[ 1 - \cosh(\nu \theta) \right]. \quad \text{(51)}
\]

These diverge as \(\theta \to \infty\).

The cylindrical coordinates leads to an interpretation of the electric flux/circulation and the magnetic flux in terms of area and arclength as in the euclidean case [7]. We can construct the Archimedes map \(A : \tilde{H}^2_+ - \{N\} \rightarrow \tilde{C}^2_P\) as a perpendicular projection from the \(z\)-axis as shown in Figure 1. The cylinder \(\tilde{C}^2_P = R \times \tilde{S}^1_P\) is given as the product of the real line \(R\) and the ideal Poincare circle \(\tilde{S}^1_P\) of radius \(r = \frac{1}{\nu}\). The image in \(\tilde{C}^2_P\) of a point \(P(x^1, x^2, x^3)\) in \(\tilde{H}^2_+ - \{N\}\) is given as

\[
A(x^1, x^2, x^3) = \left( \frac{r}{\sqrt{(x^3)^2 - r^2}} x^1, \frac{r}{\sqrt{(x^3)^2 - r^2}} x^2, x^3 \right). \quad \text{(52)}
\]

The Archimedes map \(A\) is locally area preserving. The gauge potential 1-form and the field 2-form are given as

\[
A^N = -\frac{1}{2} \frac{\nu^2}{g} a, \quad F^N = -\frac{1}{2} \frac{\nu^2}{g} \sigma. \quad \text{(53)}
\]

Here the 1-form \(a\) is the area of a thin strip on \(\tilde{C}^2_P\) of base length \(d\phi\) and height \(h = x^3(\theta) - x^3(0)\) and \(\sigma\) is the area 2-form on \(\tilde{C}^2_P\). The electric flux/circulation and the magnetic flux (51) are given as

\[
\Phi^N = \frac{1}{2} \frac{\nu^3}{g} \Sigma_U, \quad Q^N = -\frac{1}{2} \frac{\nu^2}{g} \Sigma_U, \quad \text{(54)}
\]

\[
= \frac{\nu^2}{g} h_U
\]
in terms of the area $\Sigma_U$ and height $h_U$ of the chart $U(P : \theta)$ or its image in $\tilde{C}_P^2$. Thus we can interpret the integrated field equation (48) simply as the area formula: $\text{height} = \frac{1}{\text{base-area}}$ for a rectangle of height $h = x^3(\theta) - x^3(0)$ and base $2\pi \nu$ up to an overall factor of $\frac{1}{2} g^2$.

### 3.4 The Holonomy

The Berry phase [14], which corresponds to holonomy in a line bundle [15], for the group $SU(1, 1)$ is investigated in various contexts [16]. The classical example of this is the geometric phase suffered by a vector upon parallel transport on the pseudosphere $H^2_\pm$. The geometric phase is given by a measure of the area suspended by a loop analogous to the euclidean case [16], [7]. It has become easy to express this in terms of the holonomy of the topologically massive gauge potential or the dual-field.

Consider a vector $X$ which is tangent to $\tilde{H}^2_\pm$ at longitude $\phi = 0$ and at latitude $\theta$. If we parallel transport this vector along the latitude $P : \theta$, it does not coincide with its initial direction after a complete revolution. But it suffers a phase $\tilde{\gamma}$ which is determined by a measure of the area $\Sigma_U$ suspended by the loop $P : \theta$ in $\tilde{H}^2_\pm$ [16]. This is given as

$$\tilde{\gamma} = \tilde{\Omega} = \frac{1}{r^2} \Sigma_U$$

$$= -2\pi [1 - \cosh(\nu \theta)],$$

where $\tilde{\Omega} = \frac{1}{r^2} \Sigma$ is the normalized area. It is straightforward to verify this by solving the equation for parallel transport: $\nabla_{(\phi)} X = 0$ with the metric (20) on $\tilde{H}^2_\pm$. This equation reduces to

$$\frac{dZ}{d\phi} + i \nu \cosh(\nu \theta) Z = 0.$$  \hspace{1cm} (56)

We find

$$Z(2\pi \nu) = Z(0) \exp\{-i \int_P \nu [1 - \cosh(\nu \theta)] d\phi\},$$

where $\nu$ is the topological number of the gauge potential. This is in agreement with the dual-field result [7].
for a complete revolution. Here $Z(0)$ and $Z(2\pi \nu)$ respectively correspond to the initial and the final vectors $X_i$, $X_f$. The phase in (57) reduces to the normalized area in (55) of the region which is subtented by the parallel $P : \theta$ via the Stokes theorem. The phase (55) can be written in terms of the holonomy $\tilde{\Gamma}$ of the topologically massive gauge potential or the dual-field over the parallel $P : \theta$ as

$$\tilde{\gamma} = -2\tilde{\Gamma},$$

(58)

where

$$\tilde{\Gamma} = Q^N = -\frac{1}{\nu} \Phi^N,$$

(59)

(51) and the factor of $\frac{1}{g}$ is ignored.

4 Conclusion

We have considered lorentzian solutions of the MCS and the $SU(1, 1)$ YMCS theories on the AdS space $\tilde{H}^3$. We have embedded the abelian solution into the YMCS theory by means of a $SU(1, 1)$ gauge transformation. The action for the abelian solution vanishes. Meanwhile the action for the non-abelian solution consists of only the Yang-Mills term and this is infinite.

In the abelian case the topologically massive field locally determines the potential up to a closed 1-form via the (anti-)self-duality equation. We have introduced a transformation of the gauge potential using the dual-field strength. This transformation can be identified with an abelian gauge transformation. The gauge function is given in terms of the magnetic flux or the electric flux/circulation.

Then we have introduced the map $\tilde{H}^3 \rightarrow \tilde{H}^2_\pm$ including the topological mass which is the lorentzian analogue of the Hopf map in the euclidean case. This map yields a global decomposition of $\tilde{H}^3$ as a trivial $S^1$ bundle over the pseudo-sphere $\tilde{H}^2_\pm$. This leads to a reduction of the abelian field equation onto $\tilde{H}^2_\pm$ using the global section $N : \psi = -\phi$ of the solution on $\tilde{H}^3$. This solution carry both magnetic flux and electric flux/circulation. The magnetic flux $Q^N$ through a finite chart $U(P : \theta)$ of $\tilde{H}^2_\pm$ is determined by the electric circulation.
$\Phi^N$ on the boundary of this chart. The Archimedes map $A : \tilde{H}_2^\pm - \{N\} \longrightarrow \tilde{C}_p$ has led to a simple interpretation of this composite structure in terms of the area of a rectangle. We have also expressed the geometric phase suffered by a vector upon parallel transport on the pseudo-sphere $\tilde{H}_2^\pm$ in terms of the holonomy of the topologically massive gauge potential or the dual-field.

These are analogous to the euclidean solutions [6, 7] of the MCS and the $SU(2)$ YMCS theories on the 3-sphere $\tilde{S}^3$. There exists a natural scale of length which is determined by the inverse topological mass $\nu \sim ng^2$. We have used an intrinsic arclength parameterization. The arclength parameters are taken to be independent of the length scale which is introduced by the topological mass. In geometrical terms, the quantization of the topological mass requires the quantization of the inverse natural scale of length in units of the inverse fundamental length scale $g^2$ as in the euclidean case [7]. However in the present discussion the parameter $n$ is free and it can assume an arbitrary value.

References

[1] S. Deser, R. Jackiw, S. Templeton, Phys. Rev. Lett., 48 (1982), 975.

[2] S. Deser, R. Jackiw, S. Templeton, Ann. Phys., 140 (1982), 372.

[3] J. F. Schonfeld, Nuc. Phys. B, 185 (1981), 157.

[4] S. Deser, Phys. Rev. Lett., 64 (1990), 611.

[5] R. Jackiw in “Relativity, Groups and Topology II”, (Les Houches Session XL, 1983, ed. B. DeWitt, R. Stora), North-Holland, 1984.

[6] A. N. Aliev, Y. Nutku, K. Saygili, Class. Quant. Grav., 17 (2000), 4111.

[7] K. Saygili, preprint, hep-th/0610307.

[8] P. K. Townsend, K. Pilch, P. Van Nieuwenhuizen, Phys. Lett. B, 136 (1984), 38.

[9] S. Deser, R. Jackiw, Phys. Lett. B, 139 (1984), 371.

[10] T. T. Wu, C. N. Yang, Phys. Rev. D, 12 (1975), 3845.
[11] L. H. Ryder, “Quantum Field Theory”, Cambridge University Press, 2006.

[12] D. McDuff, D. Salamon, “Introduction to Symplectic Topology”, Clarendon Press, Oxford, 1998.

[13] I. Agricola, T. Friedrich, “Global Analysis, Differential Forms in Analysis, Geometry and Physics”, Graduate Studies in Mathematics 52, American Mathematical Society, 2002.

[14] M. V. Berry, Proc. R. Soc. Lond., A, 392 (1984), 45.

[15] B. Simon, Phys. Rev. Lett., 51 (1983), 2167.

[16] A. Mostafazadeh, “Dynamical Invariants, Adiabatic Approximation and the Geometric Phase”, Nova Science Publishers, Inc., 2001.

[17] V. Bargmann, Annals of Math., 48 (1947), 568.

[18] R. Gilmore, “Lie Groups, Lie Algebras and Some of Their Applications”, John Wiley & Sons, 1974.

[19] R. W. Sharpe, “Differential Geometry: Cartan’s Generalization of Klein’s Erlangen Program”, Springer-Verlag, 1997.

[20] J. A. De Azcarraga, J. M. Izquierdo, “Lie groups, Lie algebras, cohomology and some applications in physics”, Cambridge University Press, 1998.

[21] B. A. Dubrovin, A. T. Fomenko, S. P. Novikov, “Modern Geometry-Methods and Applications”, Springer-Verlag, 1992.

[22] W. Kühnel, “Differential Geometry, Curves-Surfaces-Manifolds”, AMS, 2006.

[23] N. Itzhaki, Phys. Rev. D, 67 (2003), 065008.

[24] M. A. M Gomes, R. R. Landim J. Phys. A: Math. Gen. 38 (2005), 257.

[25] V. E. R. Lemes, C. Linhares de Jesus, C. A. G. Sasaki, S. P. Sorella, L. C. Q. Vilar, O. S. Ventura, Phys. Lett. B, 418 (1998), 324.
[26] V. E. R. Lemes, C. Linhares de Jesus, S. P. Sorella, L. C. Q. Vilar, O. S. Ventura, Phys. Rev. D, 58 (1998), 045010.

[27] V. Aldaya, J. A. de Azcarraga, J. Bisquert, J. M. Cervero, J. Phys. A: Math. Gen., 23 (1990), 707.

[28] J. G. Miller, Commun. Math. Phys., 52 (1977), 1.