On the Mazur–Ulam theorem in fuzzy n–normed strictly convex spaces

M. Eshaghi Gordji
Department of Mathematics, Semnan University,
P. O. Box 35195-363, Semnan, Iran
Tel:0098-231-4459905
Fax:0098-231-3354082
e-mail: madjid.eshaghi@gmail.com

S. Abbaszadeh
Department of Mathematics, Semnan University,
P. O. Box 35195-363, Semnan, Iran
e-mail: s.abbaszadeh.math@gmail.com

Th. M. Rassias
Department of Mathematics, National Technical University of Athens,
Zografou, Campus 15780 Athens, Greece
e-mail: trassias@math.ntua.gr

Abstract. In this paper, we generalize the Mazur–Ulam theorem in the fuzzy real n-normed strictly convex spaces.

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1. Introduction

The theory of isometric began in the classical paper [16] by S. Mazur and S. Ulam who proved that every isometry of a real normed vector space onto another real normed vector space is a linear mapping up to translation. The property is not true for normed complex vector space(for instance consider the conjugation on C). The hypothesis of surjectivity is essential. Without this assumption, Baker [2] proved that every isometry from a normed real space into a strictly convex normed real space is linear up to translation. A number of the mathematicians have had dealt with the Mazur–Ulam theorem.

The main theme of this paper is the proof of the Mazur–Ulam theorem in a fuzzy n-normed strictly convex space.

In 1984, Katsaras [12] defined a fuzzy norm on a linear space and at the same year Wu and Fang [24] also introduced a notion of fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for fuzzy topological linear space. In [4], Biswas defined and studied fuzzy inner product spaces in linear space. In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [14]. In 2003, Bag and Samanta [1] modified the definition of Cheng and Mordeson [5] by removing a regular condition. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy norms (see [1]).

In [8, 9], Gähler introduced a new approach for a theory of 2-norm and n-norm on a linear space. In [10], Hendra Gunawan and Mashadi gave a simple way to derive an (n-1)-norm from the n-norm and realized that any n-normed space is an (n-1)-normed space. A. Narayanan and S. Vijayabalaji have introduced the notion of fuzzy n-normed linear space in [17]. Also, S. Vijayabalaji, N. Thillaigovindan and Y. B. Jun, extended n-normed linear
spaces to fuzzy n-normed linear spaces in [23]. We mention here the papers and monographs [3][5][7][11][13][15][18][19][20][21][22] and [25] concerning the isometries on metric spaces.

2. Preliminaries

In this section, we state some essential definitions and results which will be needed in the sequel.

Definition 2.1. Let $X$ be a real linear space. A function $N : X \times \mathbb{R} \rightarrow [0, 1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on $X$, if for all $x, y \in X$ and all $s, t \in \mathbb{R}$:

(N1) $N(x, t) = 0$ for $t \leq 0$;
(N2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
(N3) $N(tx, s) = N(x, \frac{s}{t})$ if $t \neq 0$;
(N4) $N(x + y, t + s) \geq \min\{N(x, t), N(y, s)\}$;
(N5) $N(x, \cdot)$ is non-decreasing function on $\mathbb{R}$ and $\lim_{t \to -\infty} N(x, t) = 1$;
(N6) For $x \neq 0$, $N(x, \cdot)$ is (upper semi) continuous on $\mathbb{R}$.

The pair $(X, N)$ is called a fuzzy normed linear space. One may regard $N(x, t)$ as the truth value of the statement "the norm of $x$ is less than or equal to the real number $t$".

Definition 2.2. Let $n \in \mathbb{N}$ (natural numbers) and let $X$ be a real vector space of dimension $d \geq n$. A real valued function $\|\cdot, \ldots, \cdot\|$ on $X \times \ldots \times X$ satisfying the following four properties:

(1) $\|x_1, \ldots, x_n\| = 0$, if and only if $x_1, \ldots, x_n$ are linearly dependent;
(2) $\|x_1, \ldots, x_n\|$ is invariant under any permutation;
(3) $\|x_1, \ldots, \alpha x_n\| = |\alpha|\|x_1, \ldots, x_n\|$, for any $\alpha \in \mathbb{R}$;
(4) $\|x_1, \ldots, x_{n-1}, y + z\| \leq \|x_1, \ldots, x_{n-1}, y\| + \|x_1, \ldots, x_{n-1}, z\|;

is called an n-norm on $X$ and the pair $(X, \|\cdot, \ldots, \cdot\|)$, is called an n-normed space.

Definition 2.3. Let $X$ be a real linear space over a real field $F$. A fuzzy subset $N$ of $X^n \times \mathbb{R}$ (is the set of real numbers) is called the fuzzy n-normed on $X$, if and only if for every $x_1, \ldots, x_n, x_n' \in X$:

(nN1) For all $t \in \mathbb{R}$ with $t \leq 0$, $N(x_1, \ldots, x_n, t) = 0$;
(nN2) For all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, \ldots, x_n, t) = 1$, if and only if $x_1, \ldots, x_n$ are linearly dependent;
(nN3) $N(x_1, \ldots, x_n, t)$ is invariant under any permutation of $x_1, \ldots, x_n$;
(nN4) For all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, \ldots, cx_n, t) = N(x_1, \ldots, x_n, \frac{t}{|c|})$, if $c \neq 0$, $c \in F$ (field);
(nN5) For all $s, t \in \mathbb{R}$, $N(x_1, \ldots, x_n + x_n', s + t) \geq \min\{N(x_1, \ldots, x_n, t), N(x_1, \ldots, x_n', s)\}$;
(nN6) $N(x_1, \ldots, x_n, t)$ is left continuous and non-decreasing function of $t \in \mathbb{R}$ and

$$\lim_{t \to -\infty} N(x_1, \ldots, x_n, t) = 1;$$

In this case, the pair $(X, N)$ is called a fuzzy n-normed linear space.

Example 2.4. Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space. We define

$$N(x_1, \ldots, x_n, t) := \begin{cases} \frac{t}{1 + \|x_1, \ldots, x_n\|}, & \text{when } t \in \mathbb{R} \text{ with } t > 0, (x_1, \ldots, x_n) \in X \times \ldots \times X, \\ 0, & \text{when } t \leq 0, \end{cases}$$

Then it is easy to show that $(X, N)$ is a fuzzy n-normed linear space.

Definition 2.5. A fuzzy n-normed space is called strictly convex, if and only if for every $x_1, \ldots, x_n, x_n' \in X$ and $s, t \in \mathbb{R}$, $N(x_1, \ldots, x_n + x_n', s + t) = \min\{N(x_1, \ldots, x_n, t), N(x_1, \ldots, x_n', s)\}$ and for any $z_1, \ldots, z_n \in X$, $N(z_1, \ldots, z_n, t) = N(z_1, \ldots, z_n, s)$ implies that $x_1 = z_1, \ldots, x_n = z_n$ and $s = t$. 

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Let \((X, N)\) and \((Y, N)\) be two fuzzy \(n\)-normed spaces. We call \(f : (X, N) \to (Y, N)\) a fuzzy \(n\)-isometry, if and only if
\[
N(x_1 - x_0, \ldots, x_n - x_0, t) = N(f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0), t),
\]
for all \(x_0, x_1, \ldots, x_n \in X\) and all \(t > 0\).

Let \(X\) be a real linear space and \(x, y, z\) mutually disjoint elements of \(X\). Then \(x, y\) and \(z\) are said to be \(2\)-collinear if \(y - z = t(x - z)\), for some real number \(t\).

3. MAZUR–ULAM PROBLEM

In this section we prove the Mazur–Ulam theorem in the fuzzy real \(n\)-normed strictly convex spaces. From now on, let \((X, N)\) and \((Y, N)\) be two fuzzy \(n\)-normed strictly convex spaces and \(f : (X, N) \to (Y, N)\) be a function.

Lemma 3.1. For each \(x_1, \ldots, x_n, x_n' \in X\) and \(t \in \mathbb{R}\),
(i) \(N(x_1, \ldots, x_n - x_n', t) = N(x_1, \ldots, x_n - x_n', t)\);
(ii) \(N(x_1, \ldots, x_i, x_j, \ldots, x_n, t) = N(x_1, \ldots, x_i + \alpha x_j, \ldots, x_j, \ldots, x_n, t)\), for all \(\alpha \in \mathbb{R}\).

Proof.
\[
N(x_1, \ldots, x_n - x_n', t) = N(x_1, \ldots, (-1)(x_n - x_n'), t) = N(x_1, \ldots, x_n - x_n, \frac{t}{-1}) = N(x_1, \ldots, x_n - x_n, t).
\]
To prove (ii), assume that \(s, t \in \mathbb{R}\) and \(s, t > 0\) and \(z = \frac{1}{t}x_i + x_j\). By using (i) and \((nN_2)\) and \((nN_0)\), we have
\[
N(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n, t) \leq N(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n, t + s)
= N(x_1, \ldots, \alpha(z - x_j), \ldots, x_j, \ldots, x_n, t + s)
= N(x_1, \ldots, z - x_j, \ldots, x_j, \ldots, x_n, \frac{t + s}{\alpha})
= \min\{N(x_1, \ldots, z, \ldots, x_j, \ldots, x_n, \frac{s}{\alpha}), N(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n, \frac{s}{\alpha})\}
= N(x_1, \ldots, z, \ldots, x_j, \ldots, x_n, \frac{t}{\alpha})
= N(x_1, \ldots, \alpha z, \ldots, x_j, \ldots, x_n, t)
= N(x_1, \ldots, x_i + \alpha x_j, \ldots, x_j, \ldots, x_n, t)
\leq N(x_1, \ldots, x_i + \alpha x_j, \ldots, x_j, \ldots, x_n, t + s)
= \min\{N(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n, t), N(x_1, \ldots, \alpha x_j, \ldots, x_j, \ldots, x_n, s)\}
= N(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n, t).
\]
Hence, \(N(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n, t) = N(x_1, \ldots, x_i + \alpha x_j, \ldots, x_j, \ldots, x_n, t)\), for all \(\alpha \in \mathbb{R}\). 

Lemma 3.2. Let \(x_0, x_1 \in X\) be arbitrary and \(t > 0\). Then \(u = \frac{x_0 + x_1}{2}\) is the unique element of \(X\) satisfying
\[
N(x_1 - u, x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0, t)
= N(x_0 - x_n, x_0 - u, x_2 - x_n, \ldots, x_n - x_0, t)
= N(x_0 - x_n, x_1 - x_n, \ldots, x_{n-1} - x_n, 2t)
\]
for every \(x_2, \ldots, x_n \in X\) and \(u, x_0\) and \(x_1\) are \(2\)-collinear.
exists a real number $s$.

Assume that $2 = \frac{x_0 + x_1}{2}$. We can write

$$x_0 - u = x_0 - \frac{x_0 + x_1}{2} = -\frac{x_0}{2} + \frac{x_1}{2} = -\frac{x_0 - x_1}{2} = -(x_1 - u).$$

Thus we conclude by the Definition 2.7 that $u$, $x_0$ and $x_1$ are 2–colinear.

By using Lemma 3.1, we can see that

$$N(x_1 - u, x_1 - x_n, x_{n-1} - x_n, t)$$

and similarly

$$N(x_0 - x_n, x_0 - u, x_2 - x_n, x_{n-1} - x_n, t) = N(x_0 - x_n, x_1 - x_n, x_{n-1} - x_n, 2t).$$

Now, we prove the uniqueness of $u$.

Assume that $v \in X$, satisfies the above properties. Since $v$, $x_0$ and $x_1$ are 2–colinear, there exists a real number $s$ such that $v := sx_0 + (1 - s)x_1$. In view of Lemma 3.1 and Definition 2.5, we obtain

$$N(x_0 - x_n, x_1 - x_n, x_{n-1} - x_n, 2t)$$

and similarly

$$N(x_0 - x_n, x_0 - u, x_2 - x_n, x_{n-1} - x_n, t) = N(x_0 - x_n, x_1 - x_n, x_{n-1} - x_n, 2t).$$

So, $2t = \frac{s}{|s|}$. Since $t > 0$, $|s| = \frac{t}{2}$. Also

$$N(x_0 - x_n, x_1 - x_n, x_{n-1} - x_n, 2t)$$

and similarly

$$N(x_0 - x_n, x_0 - u, x_2 - x_n, x_{n-1} - x_n, t) = N(x_0 - x_n, x_1 - x_n, x_{n-1} - x_n, 2t).$$

Hence $\frac{1}{2} = |s| = |1 - s|$ and so $s = \frac{1}{2}$. Thus we obtain that $u = v$ and this complete the proof.

\[\Box\]

**Lemma 3.3.** Let $f : (X, N) \to (Y, N)$ be a fuzzy $n$-isometry;

(i) For every $x_0, x_1, x_2 \in X$, if $x_0$, $x_1$ and $x_2$ are 2–colinear, then $f(x_0)$, $f(x_1)$ and $f(x_2)$ are 2–colinear.

(ii) If $f(0) = 0$, then for every $z_1, ..., z_n \in X$ and $t > 0$

$$N(z_1, ..., z_n, t) = N(f(z_1), ..., f(z_n), t)$$
Proof. Since $x_0$, $x_1$ and $x_2$ are 2–colinear, there exists a real number $s$ such that $x_1 - x_0 = s(x_2 - x_0)$. So, for each $x_3, \ldots, x_{n+1} \in X$ we have

$$N(f(x_1) - f(x_0), f(x_3) - f(x_0), \ldots, f(x_{n+1}) - f(x_0), t)$$

$$= N(x_1 - x_0, x_3 - x_0, \ldots, x_{n+1} - x_0, t)$$

$$= N(x_2 - x_0, x_3 - x_0, \ldots, x_{n+1} - x_0, \frac{t}{|s|})$$

$$= N(f(x_2) - f(x_0), f(x_3) - f(x_0), \ldots, f(x_{n+1}) - f(x_0), \frac{t}{|s|})$$

$$= N(s(f(x_2) - f(x_0)), f(x_3) - f(x_0), \ldots, f(x_{n+1}) - f(x_0), t),$$

and by definition 2.5, we conclude that $f(x_1) - f(x_0) = s(f(x_2) - f(x_0))$. To prove the property (ii), we can write

$$N(z_1, \ldots, z_n, t) = N(z_1 - 0, \ldots, z_n - 0, t)$$

$$= N(f(z_1) - f(0), \ldots, f(z_n) - f(0), t)$$

$$= N(f(z_1), \ldots, f(z_n), t).$$

\[ \square \]

**Theorem 3.4.** Every fuzzy $n$-isometry $f : (X, N) \to (Y, N)$ is affine.

**Proof.** $f : (X, N) \to (Y, N)$ is affine, if the function $g : (X, N) \to (Y, N)$ defined by $g(x) = f(x) - f(0)$, is linear. Its obvious that $g$ is an $n$-isometry and $g(0) = 0$. Thus, we may assume that $f(0) = 0$. Hence, it is enough to show that $f$ is linear. Let $x_0, x_1 \in X$. By Lemma 3.1, for every $x_2, \ldots, x_n \in X$ we have

$$N(f(x_0) - f(x_n), f(x_0) - f(x_0 + x_1), f(x_2) - f(x_n), \ldots, f(x_{n-1}) - f(x_n), t)$$

$$= N(f(x_0) - f(x_n), f(x_0) - f(x_0 + x_1), f(x_2) - f(x_n), \ldots, f(x_{n-1}) - f(x_n), t)$$

$$= N(x_0 - x_0, \frac{x_0 + x_1}{2} - x_0, x_2 - x_0, \ldots, x_{n-1} - x_0, t)$$

$$= N(x_0 - x_0, x_1 - x_0, x_2 - x_0, \ldots, x_{n-1} - x_0, 2t)$$

$$= N(f(x_0) - f(x_0), f(x_1) - f(x_0), f(x_2) - f(x_0), f(x_3) - f(x_0), \ldots, f(x_{n-1}) - f(x_0), 2t)$$

$$= N(f(x_0) - f(x_0), f(x_1) - f(x_0), f(x_2) - f(x_0), f(x_3) - f(x_0), \ldots, f(x_{n-1}) - f(x_0), 2t).$$

And we can obtain

$$N(f(x_1) - f(x_0), f(x_1) - f(x_n), f(x_2) - f(x_n), \ldots, f(x_{n-1}) - f(x_n), t)$$

$$= N(f(x_1) - f(x_0), f(x_1) - f(x_n), f(x_2) - f(x_n), \ldots, f(x_{n-1}) - f(x_n), t)$$

$$= N(x_0 - x_1, x_1 - x_1, x_2 - x_1, \ldots, x_{n-1} - x_1, t)$$

$$= N(x_0 - x_1, x_1 - x_1, x_2 - x_1, \ldots, x_{n-1} - x_1, 2t)$$

$$= N(f(x_0) - f(x_1), f(x_0) - f(x_1), f(x_2) - f(x_1), \ldots, f(x_{n-1}) - f(x_1), 2t)$$

$$= N(f(x_0) - f(x_0), f(x_1) - f(x_1), f(x_2) - f(x_1), f(x_3) - f(x_1), \ldots, f(x_{n-1}) - f(x_1), 2t).$$

By (i) of Lemma (3.3), we obtain that $f(\frac{x_0 + x_1}{2}), f(x_0)$ and $f(x_1)$ are 2–colinear. Now, from Lemma 3.2, we have

$$f(\frac{x_0 + x_1}{2}) = \frac{f(x_0)}{2} + \frac{f(x_1)}{2}$$

for all $x_0, x_1 \in X$. It follows that $f$ is $Q$-linear ($Q$ is the set of rational numbers). We have to show that $f$ is $R$-linear.

Let $r \in R^+$ and $x \in X$. By (i) of Lemma (3.3), $f(0), f(x)$ and $f(rx)$ are 2–colinear. Since
f(0) = 0, there exists s ∈ ℝ such that f(rx) = sf(x). From (ii) of Lemma (3.3), for every x₁, ..., xₙ₋₁ and t > 0, we have

\[ N(x, x₁, x₂, ..., xₙ₋₁; \frac{t}{r}) = N(rx, x₁, ..., xₙ₋₁, t) \]
\[ = N(f(rx), f(x₁), f(x₂), ..., f(xₙ₋₁), t) \]
\[ = N(s, f(x₁), f(x₂), ..., f(xₙ₋₁), t) \]
\[ = N(f(x), f(x₁), f(x₂), ..., f(xₙ₋₁); \frac{t}{|s|}) \]
\[ = N(x, x₁, x₂, ..., xₙ₋₁; \frac{t}{|s|}). \]

Hence s = ±r. The proof is completed if s = r. If s = −r, that is, f(rx) = −rf(x). Then there exists q₁, q₂ ∈ ℚ such that 0 < q₁ < r < q₂. For each x₁, ..., xₙ ∈ X, we have

\[ N(f(x), f(z₁) − f(q₂x), ..., f(zₙ₋₁) − f(q₂x); \frac{t}{q₂ + r}) \]
\[ = N(q₂f(x) − (−rf(x)), f(z₁) − f(q₂x), ..., f(zₙ₋₁) − f(q₂x), t) \]
\[ = N(f(rx) − f(q₂x), f(z₁) − f(q₂x), ..., f(zₙ₋₁) − f(q₂x), t) \]
\[ = N(rx − q₂x, z₁ − q₂x, ..., zₙ₋₁ − q₂x, t) \]
\[ = N(x, z₁ − q₂x, ..., zₙ₋₁ − q₂x; \frac{t}{q₂ − r}) \]
\[ ≥ N(x, z₁ − q₂x, ..., zₙ₋₁ − q₂x; \frac{t}{q₂ − q₁}) \]
\[ = N(q₁x − q₂x, z₁ − q₂x, ..., zₙ₋₁ − q₂x, t) \]
\[ = N(f(q₁x) − f(q₂x), f(z₁) − f(q₂x), ..., f(zₙ₋₁) − f(q₂x), t) \]
\[ = N(f(x), f(z₁) − f(q₂x), ..., f(zₙ₋₁) − f(q₂x); \frac{t}{q₂ − q₁}). \]

By \((nNₙ)\), we have q₂ + r ≤ q₂ − q₁, which is a contradiction. Hence s = r, that is, f(rx) = rf(x) for all positive real numbers r. Therefore f is ℝ-linear, as desired. □

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