The well known experiment with three linear polarizers clearly demonstrates interesting properties of light. The beam intensity of photon beam transmitted through one, two or three polarizers can be measured in dependence on the rotations of the polarizers. The experiment has attracted attention of many researchers because the measured intensities cannot be explained by photons being only absorbed by the polarizers; the polarizers must also change a property (called polarization) of the transmitted photons. However, contemporary theories of light have not allowed to determine the probability (density) functions characterizing the two effects. It has strongly limited understanding of the physical phenomenon. In this paper example data of beam intensities corresponding to 3 ideal identical linear polarizers are analyzed with the help of probabilistic model based on stochastic state-transition-change (STC) process. Using STC process, it has been possible to determine numerical values of the probability (density) functions for the first time. Measured beam intensities corresponding to sequences of different polarization sensitive devices and other optical elements may be analyzed in similar way, too. STC process extends applicability of the theory of stochastic processes in various fields of research. The results open up new possibilities of description of polarization of light, optical phenomena and particle-matter interaction in general.

Keywords: three polarizers experiment, photon, polarization of light, linear polarizer, light-matter interaction, stochastic state-transition-change process, optics, laser systems

CONTENTS

I. Introduction 1
II. Measurement of transmission of light through sequence of linear polarizers 2
   A. Experimental setup 2
   B. Beam intensity and number of particles 3
   C. Ideal linear polarizers and Malus’s law 3
   D. Real linear polarizers 3

III. (Artificial) input data - 3 ideal identical linear polarizers 3
   A. Relative number of transmitted photons 3
   B. Symmetries 4

IV. Probabilistic model 4
   A. STC process - definition 4
   B. STC process - main formulae 6
   C. Definition of linear polarizer 7
   D. Definition of unpolarized and polarized light 7
   E. Parameterization of unknown functions 7
      1. Parameterization of $\rho_{S,0}$ 7
      2. Parameterization of $P_T$ 8
      3. Parameterization of $p_C$ 8

V. Analysis of data using the probabilistic model 8
   A. Check of applicability of the probabilistic model to input data (consistency) 8
   B. Fitting of the probabilistic model to data 8
   C. Numerical results 8

VI. Open questions (problems) and how to address them 12
   A. List of open questions 12
   B. Possibilities of answering the open questions 12

VII. Conclusion 13

References 14

I. INTRODUCTION

The study of the interaction of light with matter has a very long tradition. Several distinct theoretical approaches have been developed to better understand huge amount of observed phenomena [1–11]. The measurement and description of effects related only to the polarization of light represent a very broad topic, see [12–17] and Part 3 in [9] devoted to optical polarization. A wide range of polarization phenomena, their applications and the history of the polarization of light are discussed in [18].

I. Newton tried to explain a number of phenomena with light by using the idea that light consists of particles (corpuscles). However, determination of properties of these particles varied greatly with time; useful historical references and many comments to different concept of light quanta are in [19]. The interaction of light with matter is still the subject of extensive scientific research. One of many important discoveries related to particle of light concerned its energy. M. Planck in 1900 [20, 21] introduced quantization of energy of electromagnetic radiation to describe spectrum of black-body radiation. A. Einstein in 1905 [22] developed further this quantum hypothesis. He basically introduced modern concept of particle of light - photon - and used it to explain the photoelectric effect. This phenomenon occurs when beam of light irradiates...
certain materials and electrons from the surface of the materials are ejected from it. Experimental results show that electrons will only be emitted when light of a certain frequency or higher is incident on materials - regardless of the intensity of the light beam or duration of exposure. Photon has been used for explanation of many other observed phenomena related to light since the beginning of the 20th century. It has led to numerous technological advances. There is now whole field called photonics [10, 23–25] which began with the invention of the laser in 1960 and which further demonstrate in various ways the usefulness of the concept of particles of light (photons).

Interesting properties of light related to polarization may be easily demonstrated with the help of the widely known and discussed three polarizers experiment. When incident unpolarized light passes through two crossed linear polarizers then the intensity of light is strongly reduced (nearly no light passes through). However, when another polarizer is placed between the two polarizers then some light can pass through the sequence of the three polarizing filters. The intensity of light behind the third polarizer in the sequence is maximal when the second polarizer has polarizing axis oriented at 45° relative to the polarizing axis of the first polarizer. This easy to observe experimental result has attracted attention of many researches as it is clear that the effect cannot be explained by only absorption of some photons from the beam by the polarizers; the polarizers must change also a property of the transmitted photons (called polarization). Moreover, both the effects must depend on the orientation of the axes of the polarizers.

The passage of a photon through a linear polarizer can be considered as a probabilistic process. The probability of transmission of the photon through the polarizer and the probability that the photon polarization state will change during the transmission depend on the orientation of the axis of the polarizer and the initial photon polarization state. However, the values of the corresponding probability (density) functions characterizing the transmission have never been determined which has limited understanding of the physical phenomenon.

Various matrix calculi have been developed to describe the effect of polarizing elements on the state of polarization of a light beam, see sect. 12.8 in [9] and, e.g., a historical revision of the development of the differential Stokes–Mueller matrix formalism [26]. The most widely known matrix calculi are the Mueller calculus and the Jones calculus. The various distinct matrix formalisms have been developed with different aims and purposes in order to better understand observed polarization phenomena. However, none of the matrix calculi has been designed to describe probabilistic character of transmission of light beam (photons) through a sequence of polarizing elements such as the sequence of 3 linear polarizers. Even the Maxwell’s equations [15] do not make it possible to determine the probability (density) functions mentioned above. The description of transmission of light through sequence of three polarizers by Jones [27] do not mention probability at all.

Statistical optics [2] deals mainly with a scalar theory of light waves. The scalar quantities are regarded as representing one polarization component of the electric or magnetic field, with the approximation that all such components can be treated independently. Its application to data concerning propagation of light phenomena is, therefore, based on several strong assumptions (such as introduction of some complex amplitudes, various approximations concerning the amplitudes etc.). This theoretical approach, therefore, does not fully explore the concept of quantum of light. This is also the case of other contemporary theoretical attempts [15] trying to describe various polarization (optical) phenomena with the help of statistics.

It will be shown in this paper that it is possible to determine the probability (density) functions characterizing transmission of light through sequence of 3 polarizers on the basis of experimental data with the help of state-transition-change (STC) stochastic process introduced in [28]. Several ideas of M. V. Lokajíček [29] will be developed further. The advantage of STC stochastic process is that it allows to unify description of many particle phenomena with the help of the theory of probability and theory of stochastic processes, see [28].

This paper is structured as follows. Sect. II briefly summarizes basic characteristics of measured particle numbers (beam intensities) corresponding to transmitted light through one, two or three (“ideal”) linear polarizers in dependence on the rotations of their axes. Example input data of particle numbers corresponding to transmission of light through 3 ideal identical linear polarizers are discussed in sect. III. Probabilistic model describing transmission of light through $M$ identical polarizers is formulated in sect. IV. The example (reference) input data are analyzed with the help of the probabilistic model in sect. V where numerical results are shown, too. The results of the analysis are discussed further in sect. VI where one can find a list of newly opened questions and proposed ways how to solve them. Concluding remarks are in sect. VII.

II. MEASUREMENT OF TRANSMISSION OF LIGHT THROUGH SEQUENCE OF LINEAR POLARIZERS

A. Experimental setup

Let us consider a photon beam passing through a sequence of $M$ linear polarizers. Let us further consider flat control surfaces $\Sigma_i$ ($i = 0, ..., M$) parallel to each other and orthogonal to the direction of the beam. Each polarizer is placed between two neighbouring control surfaces, each transport segment contains one linear polarizer, see fig. 1 in the case of $M = 3$. The control surfaces $\Sigma_i$ do not correspond to physical surfaces of the polarizers but to places where characteristics of the beam can be measured (they will be called surfaces for short).

Let $N_i$ be the number of particles which passed through surface $\Sigma_i$. The number of particles passed through $i$-th polarizer, i.e., $N_{1+i}$ ($i = 0, ..., M - 1$ in this case), in general depends on the rotation of the polarizer, the rotations of all the other polarizers placed in front of it, and the number of initial particles $N_0$. We may introduce vector $\vec{\alpha}$ having components $\alpha_i$ ($i = 0, ..., M - 1$) and representing rotations of all the individual polarizers. It is useful to introduce a convention that if a
function depends on \( \alpha \) then it may depend only on some of its components.

**B. Beam intensity and number of particles**

Sometimes *intensity* of a particle beam defined as energy incident on a surface per unit of time and per unit of area (i.e., having units of \( J s^{-1} m^{-2} = W m^{-2} \)) is measured in an experiment. The intensity of light beam passing through the surface \( \Sigma_i \) may be denoted as \( I_i(\alpha) \) (i.e., \( i = 0, \ldots, M \)), see fig. 1. The intensity \( I_0 \) is the initial intensity of the beam. If the intensity \( I_i \) and the number of photons \( N_i \) correspond to the same surface area and time interval *and the photons have the same energy* then it holds (if \( N_i(\alpha) \neq 0, i = 0, \ldots, M - 1 \))

\[
\frac{N_{i+1}(\alpha)}{N_i(\alpha)} = \frac{I_{i+1}(\alpha)}{I_i(\alpha)}
\]  

(1) and (if \( N_0 \neq 0 \), \( i = 0, \ldots, M \))

\[
\frac{N_i(\alpha)}{N_0} = \frac{I_i(\alpha)}{I_0} .
\]  

(2)

Transmittance \( T_i(\alpha) \) of \( i \)-th polarizer (\( i = 0, \ldots, M - 1 \)) is standardly defined as

\[
T_i(\alpha) = \begin{cases} 
\frac{N_{i+1}(\alpha)}{N_i(\alpha)} & \text{if } N_i(\alpha) \neq 0 \\
0 & \text{otherwise}
\end{cases}
\]  

(3)

It characterizes how much \( i \)-th polarizer transmit light in dependence on its rotation and rotations of all the preceding polarizers.

**C. Ideal linear polarizers and Malus’s law**

In textbooks on optics transmission of light through “ideal” linear polarizers is often discussed (see, e.g., sect. 12.4 in [9]). If initially unpolarized light beam is sent into a sequence of \( M \) ideal linear polarizers then the transmittance \( T_i(\alpha) \) of \( i \)-th ideal polarizer (\( i = 0, \ldots, M - 1 \)) is

\[
T_i(\alpha) = \begin{cases} 
\frac{1}{2} & \text{if } i = 0 \\
\cos^2(\alpha_i - \alpha_{i-1}) & \text{if } i > 0
\end{cases}
\]  

(4)

I.e., the initial number of particles \( N_0 \) after transmission through the first ideal polarizer drops to one half independently of the orientation of the polarization axis \( \alpha_0 \). The cosine-squared function expresses the known Malus’s law. It represents the first quantitative relationship for treating polarized light intensities describing measurements performed by É.-L. Malus [31] (see also summary of the life of Malus, the historical context, and further comments concerning his law in [32]). The transmittances \( T_i \) given by eq. (4) do not depend on the value of the initial number of particles \( N_0 \).

Eqs. (3) and (4) imply that the relative beam intensity corresponding to the surface \( \Sigma_i \) is

\[
\frac{N_i(\alpha)}{N_0} = \begin{cases} 
1 & \text{if } i = 0 \\
\frac{1}{2} & \text{if } i = 1 \\
\frac{1}{2} \prod_{j=1}^{i-1} \cos^2(\alpha_j - \alpha_{j-1}) & \text{if } i > 1
\end{cases}
\]  

(5)

**D. Real linear polarizers**

Number of photons transmitted through a sequence consisting of “real” linear polarizers may be very similar to the number of particles transmitted through sequence of ideal polarizers, see sect. II C. However, it is known that some other real polarizers differ significantly from the ideal polarizers. Every two real linear polarizers transmit some light even if they are crossed and absorb some light if their axes are parallel. This is often expressed by modifying the Malus’s law, as discussed in sect. 12.4 in [9]. In [33] it has been measured that two crossed real linear polarizers produce unpolarized light (value of \( N_2(\alpha)/N_0 \) behind the third, testing, linear polarizer is independent of its rotation angle). It has been measured in [34] that transmittance of a pair of real linear polarizers may have local maximum if the polarizer axes are mutually crossed, and around this point the transmittance is not fully symmetrical (the effect can be measured with sensitive measurement devices). Some smaller or bigger deviations of measured beam intensity from Malus’s law are visible in experiments conducted in [35–37], too. The deviations depend on polarizer properties. Some other differences of real linear polarizers from ideal ones are summarized in sect. 15.27 in [9]. It has not been possible to observe many of these differences by Malus at the beginning of 19th century, i.e., before the advent of photomultiplier tubes (PMTs), charge-coupled devices (CCDs) and other devices now commonly used for measurement of particle numbers (beam intensities).

**III. (ARTIFICIAL) INPUT DATA - 3 IDEAL IDENTICAL LINEAR POLARIZERS**

**A. Relative number of transmitted photons**

The relative particle numbers \( N_i(\alpha)/N_0 \) corresponding to transmission of light through one and two linear polarizers in dependence on their rotation angles are commonly measured (e.g., to compare the measured intensities with the Malus’s law). However, it seems that there are no publicly available experimental data of the relative beam intensities (number of transmitted photons) measured behind one, two and three linear polarizers in dependence on the rotations of the polarizers. In [33] one can find interesting experimental results con-

---

1 In radiometry this definition of intensity corresponds to irradiance (see chapter 34 in [30]).
Experimental setup for measurement of relative particle numbers $N_i/N_0$ (beam intensities $I_i(\vec{\alpha})/I_0$) for $i = 1, 2, 3$ in dependence on the rotation angles $\vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2)$ of the linear polarizers when light beam is transmitted through one, two or three polarizers. The measured relative particle numbers correspond to conveniently chosen control surfaces $\Sigma_i$ (they may not correspond to physical surfaces of the polarizers). Values of polarization $\theta_i$ of photons which passed through surface $\Sigma_i$ form state space $S_i$. The values of photon polarization angles must be determined with the help of a probabilistic model on the basis of experimental data.

Therefore, let us take (for the sake of simplicity) the dependences given by eq. (5) (corresponding to a sequence of ideal linear polarizers). Let us consider sequence of 3 ideal identical linear polarizers ($M = 3$). It will be assumed that photons in the beam have the same energy, i.e., that eq. (2) holds. These dependences will be used in our further considerations as an example of measured data. In the following we will not focus on possible differences existing between the relative particle numbers corresponding to real and ideal polarizers (see sect. II D). This well-defined possible example of measured data will be analyzed with the aim to determine probabilistic (statistical) characteristics of a photon transmitted through a sequence of linear polarizers. It has never been done until now. We will be, therefore, interested mainly in the concept and possibilities of this new kind of analysis.

### B. Symmetries

Experimental data often reveal symmetries. This is also the case of the (artificial) relative particle numbers $N_i(\vec{\alpha})/N_0$ ($i = 1, \ldots, M$) in dependence on the orientation of the axes of the polarizers given by eq. (5). The ratio $N_1(\vec{\alpha})/N_0$ does not depend on the orientation of the axis of the first polarizer; and the ratios $N_2(\vec{\alpha})/N_0$ and $N_3(\vec{\alpha})/N_0$ are periodic due to the cosine-squared function in eq. (5c). One may define the following function

$$
\text{sym}(x) = \begin{cases} 
\pi - y & \text{if } \frac{\pi}{2} < y \leq \pi \\
y & \text{otherwise}
\end{cases}
$$

(6a) (6b)

where $y = (|x| \mod \pi)$. This function will help to reflect the symmetries. The function is even ($\text{sym}(x) = \text{sym}(-x)$) and is plotted in fig. 2.

### IV. PROBABILISTIC MODEL

Measured (relative) beam intensities characterize a transmission of light through a medium, but they do not explain the phenomenon. If light beam consists of photons then one may ask how a single photon interacts with a polarizer (matter), what the probability (density) functions characterizing the transmission and mentioned in the abstract and sect. I are. In the following we will use the results obtained in [28] to define stochastic STC process suitable for analysis of transmission of light through sequence of polarizers.

#### A. STC process - definition

**Definition IV.1 (Photon polarization angle).** Transmission of a photon through $i$-th linear polarizer depends on a property of the photon (called polarization) having meaning of an angle $\theta_i$ which characterize the state of the photon when it passed through the surface $\Sigma_i$.

**Remark IV.1.** As to the definition IV.1, the photon polarization angle can specify the direction of a vector quantity characterizing the photon (spin, orientation vector, ...) projected onto the plane perpendicular to the direction of the photon velocity; or projection of direction of some photon oscillations onto the plane. However, the particular physical meaning of this variable is not important in the presented paper. In the following it is necessary to only know that it has meaning of an angle.
Assumption IV.1 (Variables). Let the random variables $X_i$ characterizing transmission of photon through $\Sigma_i$ for given index $i$ in sequence $(0, \ldots, M)$ be

$$X_i = (\theta_i),$$  

(7)

i.e., the photon polarization angles are random variables. Let the non-random variables be

$$X_{iNR} = (\alpha_0, \ldots, \alpha_{i-1}),$$

(8)

i.e., the rotation angles of the polarizers are non-random variables (parameters). $M + 1$ variables and $M$ non-random variables are used in total for description of the whole system.

Remark IV.2. The number of transmitted photons through polarizers may in general be measured as function of time. In some cases this needs to be taken into account. However, input data discussed in sect. III A do not depend on time (numbers of transmitted photons corresponding to the initial number of photons $N_0$ are counted behind each polarizer in a sequence independently on time when the photons were transmitted). Time variable is, therefore, not introduced for description of our system (see assumption IV.1), it will not be of our interest.

Remark IV.3. Transmission of a photon through polarizers can in general depend on, e.g., spatial coordinates $x_i$ and $y_i$ specifying where it passed through surface $\Sigma_i$. The transmission can strongly depend on microscopic structure of the polarizers which needs to be taken into account in some cases. However, input data discussed in sect. III A do not depend on the spatial coordinates (or the information is not known). Moreover, polarizers have typically a periodic structure on microscopic level (e.g., microscopic needle-like crystals are aligned in the same direction). It is mainly the position of transmitted particle with respect to these microscopic periodic structures on which the transmission in general depends. This “structural” spatial distribution may or may not be the same for two different distributions of positions of particles in dependence on $x_i$ and $y_i$. There exists a class of distributions of positions of incoming particles in dependence on $x_i$ and $y_i$ which leads to the same structural spatial distribution. No spatial variables are introduced by assumption IV.1, it means that the results of our analysis will be valid for an unknown fixed structural distribution without the necessity to know its dependence on spatial coordinates.

Assumption IV.2 (State spaces). Let state space $S_i$ be set of polarization states of photons, represented by variables $X_i$, when they pass through surface $\Sigma_i$.

Remark IV.4. Assumption IV.2 implies that number of states of system which were in any state in state space $S_i$ is the same as number of photons which passed through $\Sigma_i$.

Assumption IV.3 (Energy of photons). Photons have the same energy when they pass through a surface $\Sigma_i$ for all $i = 0, \ldots, M$.

Remark IV.5. The assumption IV.3 could be, for the purposes presented in this paper, replaced by an assumption that eq. (2) holds. It is irrelevant in the following whether the energy of photons is quantized or not.

Definition IV.2. Functions introduced in definition 3.18 in [28] can be written also as (see assumption IV.1)

$$\text{dos}_i(\theta_i, \vec{\alpha}) = \text{dos}_i(X_i) \quad i = 0, \ldots, M$$

(9)

$$N_i(\vec{\alpha}) = N_i \quad i = 0, \ldots, M$$

(10)

$$\rho_{S,i}(\theta_i, \vec{\alpha}) = \rho_{S,i}(X_i) \quad i = 0, \ldots, M$$

(11)

where the notation introduced in [28] is on the right-hand sides of the equations. Dependence of the functions on left-hand side on non-random variables (i.e., on rotation angles of polarizers) may or may not be written explicitly. $\rho_{S,i}(\theta_i, \vec{\alpha})$ has meaning of probability density function of polarization of photons which passed through surface $\Sigma_i$. By multiplying it by number of photons $N_i(\vec{\alpha})$ which passed through surface $\Sigma_i$ one obtains density of states $\text{dos}_i(\theta_i, \vec{\alpha})$. 

FIG. 2: The function $\text{sym}(x)$ given by eq. (6) in the interval from $-\pi$ rad to $\pi$ rad and having period $\pi$ rad.
**Definition IV.3** (Probability of transmission $P_T(\theta, \alpha)$). If the probabilities $P_{T,i}(X_i)$ are the same for all $i = 0, 1, ..., M - 1$, then the probability of transmission of photon through a polarizer can be denoted as $P_T(\theta, \alpha)$. It depends on the polarization of the incoming photon $\theta_{in}$ and the rotation angle of the polarizer $\alpha$.

**Assumption IV.4** (Identical probabilities $P_{T,i}(X_i)$). The probability functions $P_{T,i}(X_i)$ are the same for all $i = 0, 1, ..., M - 1$, i.e., for all polarizers. I.e., it holds

$$P_T(\theta, \alpha) = P_{T,i}(X_i) \quad i = 0, ..., M - 1$$

where the arguments of the function $P_T$ depend on $i$.

**Definition IV.4.** If the probability density functions $\rho_{C,i}(X_i, X_{i+1})$ are the same for all $i = 0, 1, ..., M - 1$, then the function can be denoted as $\rho_{C}(\theta_{in}, \theta_{out}, \alpha)$. It depends on the polarization of the outgoing photon $\theta_{out}$ and the rotation of the polarizer $\alpha$. This probability density function expresses possibility of change of polarization of transmitted photon through one polarizer given that the photon passed through the polarizer (i.e., it is conditional probability density function).

**Assumption IV.5** (Identical functions $\rho_{C,i}(X_i, X_{i+1})$). Let the probability density functions $\rho_{C,i}(X_i, X_{i+1})$ be the same for all transport segments, i.e., polarizers. It holds

$$\rho_C(\theta, \theta_{i+1}, \alpha_{i}) = \rho_{C,i}(X_i, X_{i+1}) \quad i = 0, ..., M - 1.$$  

**Assumption IV.6.** Let all the polarizers in the sequence be identical.

**Remark IV.6.** Assumption IV.6 implies assumptions IV.4 and IV.5.

**Definition IV.5.** If the probability density function $\rho_C(\theta_{in}, \theta_{out}, \alpha)$ does not depend on the polarization angle of incoming photons then the function may be denoted as $\rho_C(\theta_{out}, \alpha)$.

**Assumption IV.7** (Independence of $\rho_{C,i}(X_i, X_{i+1})$ on $X_i$). Let the probability density functions $\rho_{C,i}(X_i, X_{i+1})$ do not depend on $X_i$, i.e., it holds

$$\rho_{C,i}(\theta_{i+1}, \alpha_{i}) = \rho_{C,i}(X_i, X_{i+1}) \quad i = 0, ..., M - 1.$$  

**Remark IV.7.** The assumptions IV.5 and IV.7 are equivalent to the assumptions 3.8 and 3.9 in [28].

**Definition IV.6.** Let the probability density function corresponding to both the assumptions IV.5 and IV.7 be denoted as $\rho_C(\theta_{out}, \alpha)$, i.e.,

$$\rho_C(\theta_{out} = \theta_{i+1}, \alpha = \alpha_{i}) = \rho_{C,i}(X_i, X_{i+1}) \quad i = 0, ..., M - 1.$$  

**Definition IV.7.** Let the probability density functions $\rho_{S,i}(X_i)$ corresponding to the surface $\Sigma_i$ be denoted as $\rho_{S,i}(\theta, \alpha)$. It depends on photon polarization $\theta$ when the photon passed through the surface $\Sigma_i$, i.e., it holds

$$\rho_{S,i}(\theta, \alpha) = \rho_{S,i}(X_i) \quad i = 0, ..., M.$$  

**Remark IV.8.** The probability functions $P_{T,i}$ and the probability density functions $\rho_{C,i}$ characterize properties of the identical polarizers (how they transmit photon of given polarization). The functions $\rho_{S,i}$ characterize polarization of photons passing through surfaces $\Sigma_i$, i.e., property of the beam.

It will be explained later that the functions $\rho_{S,0}(\theta, \alpha)$, $P_{T,i}(\theta_{in}, \alpha)$ and $\rho_C(\theta_{in}, \theta_{out}, \alpha)$ will be parameterized and determined on the basis of experimental data. However, parameterizations of some a priori unknown functions in any model are typically accompanied by several additional assumptions. Let us, therefore, formulate for completeness the following assumption:

**Assumption IV.8.** Parameterization of unknown functions

Let functions $\rho_{S,0}(\theta, \alpha)$, $P_{T,i}(\theta_{in}, \alpha)$ and $\rho_C(\theta_{in}, \theta_{out}, \alpha)$ be parameterized.

**Definition IV.8** (Stochastic process). Let $\{X_i : i \in I\}$ be a stochastic process given by definition 3.25 in [28] which satisfies assumptions 3.3, 3.5 to 3.7, 3.8 and 3.9 in [28]. Let it satisfy also assumptions IV.1 to IV.3, IV.6 and IV.8. It implies that assumptions IV.4, IV.5 and IV.7 hold (see remarks IV.6 and IV.7).

### B. STC process - main formulae

Several formulae derived in [28] will be needed for data analysis using stochastic process given by definition IV.8. They can be rewritten using the notation introduced in sect. IV.A. The usage of eq. (2) is conditional on the validity of the assumption IV.3.

Eq. (75) in [28] implies

$$1 \geq \frac{N_1(\bar{\alpha})}{N_0} \geq \frac{N_2(\bar{\alpha})}{N_0} \geq \ldots \geq \frac{N_M(\bar{\alpha})}{N_0} \geq 0.$$  

Eq. (46) in [28] and eq. (11) imply the following normalization condition of $\rho_{S,i}(\theta, \bar{\alpha})$

$$\int \rho_{S,i}(\theta, \bar{\alpha})d\theta = 1 \quad i = 0, ..., M.$$  

Eq. (76) in [28] and eq. (13) imply

$$\int_{\theta_{out}} \rho_{C}(\theta_{in}, \theta_{out}, \alpha)d\theta_{out} = 1.$$  

The normalization condition (19) must hold for any value of $\theta_{in}$. Eq. (47) in [28] and eq. (9) imply

$$\text{dos}(\theta, \bar{\alpha}) = N_i(\bar{\alpha})\rho_{S,i}(\theta, \bar{\alpha}) \quad i = 0, ..., M.$$  

Eq. (40) in [28] and eq. (10) imply

$$N_i(\bar{\alpha}) = \int_{\theta} \text{dos}(\theta, \bar{\alpha})d\theta \quad i = 0, ..., M.$$  

The density of states $\text{dos}(\theta, \bar{\alpha})$ defined by eq. (20) represents spectrum of values of polarization angles of photons.
which passed through the surface $\Sigma_i$; the spectrum is normalized to the number of particles $N_i$ (see eq. (21)) which in total passed through the surface $\Sigma_i$. Dividing eqs. (20) and (21) by $N_0$ one obtains

\[
\frac{N_{i+1}(\tilde{\alpha})}{N_0} = \frac{N_i(\tilde{\alpha})}{N_0} \rho_{S,i}(\theta, \tilde{\alpha}) \quad i = 0, \ldots, M \tag{22}
\]

\[
\frac{N_i(\tilde{\alpha})}{N_0} = \int_\theta \frac{\text{d}S_i(\theta, \tilde{\alpha})}{N_0} \quad i = 0, \ldots, M . \tag{23}
\]

One of the key formulae given by eqs. (62) and (74) in [28] and describing transmission of a particle beam through a sequence of transport segments can be rewritten using eqs. (11) to (13) and (20)

\[
\frac{N_{i+1}(\tilde{\alpha})}{N_0} = \int_\theta \frac{N_i(\tilde{\alpha})}{N_0} \rho_{S,i}(\theta, \tilde{\alpha}) P_T(\theta, \alpha_i) \text{d}\theta, \tag{24}
\]

where

\[
\frac{N_{i+1}(\tilde{\alpha})}{N_0} \rho_{S,i+1}(\theta_{i+1}, \tilde{\alpha}) = \int_\theta \frac{N_i(\tilde{\alpha})}{N_0} \rho_{S,i}(\theta, \tilde{\alpha}) P_T(\theta, \alpha_i) P_C(\theta, \alpha_{i+1}, \alpha_i) \text{d}\theta, \tag{25}
\]

for $i \in (0, \ldots, M - 1)$. Eq. (25) is of key importance. If the initial probability density function $\rho_{S,0}$ and the functions $P_T$ and $P_C$ are given then the formula (25) may be used iteratively to calculate $\frac{N_{i+1}(\tilde{\alpha})}{N_0}\rho_{S,i}(\theta, \tilde{\alpha})$ for any $i = 1, \ldots, M$ (i.e., behind any polarizer). Let us emphasize that an outgoing photon after being transmitted through a polarizer can have different value of polarization, and it becomes incoming photon interacting with the next polarizer in the sequence. The photon polarization angles corresponding to the surfaces $\Sigma_i$ are, therefore, distinguished by index $i$ in eq. (25). The two functions $P_T$ and $P_C$ are the same for identical polarizers, but it must be correctly integrated over their arguments, see eq. (25). It allows to calculate the relative number of particles $N_i(\tilde{\alpha})/N_0$ for any $i = 1, \ldots, M$ according to eq. (24) if 3 functions are known: $\rho_{S,0}(\theta_{0}, \tilde{\alpha})$, $P_T(\theta_{0}, \alpha_{0})$ and $P_C(\theta_{0}, \theta_{0}, \alpha_{0}, \tilde{\alpha})$. These unknown functions may be parameterized and determined on the basis of measured (input) values of $N_i(\tilde{\alpha})/N_0$.

Adding assumption IV.7 then eq. (97) in [28] can be written using eq. (14) as

\[
\frac{N_{i+1}(\tilde{\alpha})}{N_0} \rho_{S,i+1}(\theta_{i+1}, \tilde{\alpha}) = \rho_C(\theta_{i+1}, \alpha_i) \left[ \int_{\theta_0} \rho_{S,0}(\theta_{0}, \tilde{\alpha}) P_T(\theta_{0}, \alpha_{0}) \text{d}\theta_0 \right] \prod_{j=1}^i \left[ \int_{\theta_j} \rho_C(\theta_j, \alpha_{j-1}) P_T(\theta_j, \alpha_j) \text{d}\theta_j \right]. \tag{26}
\]

where $\prod_{j=1}^i[...] = 1$ if $i = 0$. The functions $\frac{N_{i+1}(\tilde{\alpha})}{N_0} \rho_{S,i+1}(\theta_{i+1}, \tilde{\alpha})$ of $\theta_{i+1}$ given by eq. (26) have the same shapes for all $i = 0, \ldots, M - 1$; the functions differ only in normalizations.

### C. Definition of linear polarizer

Until now, we have not defined what a linear polarizer is. A linear polarizer may be defined as an optical element which may change polarization angle $\theta_0$ of an incoming photon (only one variable characterizing photon polarization state) such that the possible directions of polarization of the outgoing photon (specified by the polarization angle $\theta_{\text{out}}$) are predominantly parallel to an axis (called axis of the linear polarizer).

The functions $P_T$ and $P_C$ characterizing given optical element are, however, not a priori known and must be determined on the basis of experimental data represented by measured relative particle numbers transmitted through a sequence of optical elements (as it will be done in the next section). In general, the determined functions $P_T$ and $P_C$ of given optical element may or may not correspond to a linear polarizer.

### D. Definition of unpolarized and polarized light

The probability density function $\rho_{S,i}(\theta, \tilde{\alpha})$ for given $i = 0, \ldots, M$ in dependence on $\theta_i$ characterizes polarization of photons when they pass through the surface $\Sigma_i$. If polarization angles $\theta_i$ of the photons are distributed uniformly (i.e., the distribution does not depend on $\theta_i$) and normalized to 1 (see eq. (18)) then

\[
\rho_{S,i}(\theta, \tilde{\alpha}) = \frac{1}{2\pi}, \tag{27}
\]

and the photon beam (light) passing through the surface $\Sigma_i$ may be denoted as unpolarized (full angle in radian is $2\pi$). On the other side, if $\rho_{S,i}(\theta, \tilde{\alpha})$ depends on $\theta_i$ then the photon beam may be denoted as polarized.

### E. Parameterization of unknown functions

#### 1. Parameterization of $\rho_{S,0}$

If decrease of measured particle numbers behind the first polarizer $N_i(\tilde{\alpha})/N_0$ does not depend on the orientation of its polarization axis, see eq. (5b), then one may assume that the probability density function $\rho_{S,0}(\theta_{0}, \tilde{\alpha})$ does not depend on the photon polarization. I.e., the polarization is distributed uniformly and the initial photon beam is unpolarized (see eq. (27))

\[
\rho_{S,0}(\theta_{0}, \tilde{\alpha}) = \frac{1}{2\pi}. \tag{28}
\]

In more general case it would be necessary to introduce a parameterization depending on $\theta_{0}$ and some free parameters; it would correspond to polarized light.
2. Parameterization of $P_T$

We may assume that the probability of photon transmission through one polarizer depends only on the difference of the polarizer rotation and the photon polarization ($P_T(\theta_{in}, \alpha) = P_T(\theta_{in} - \alpha)$). The following parameterization may be chosen

$$P_T(\theta_{in}, \alpha) = 1 - \frac{1 - g(\theta_{in}, \alpha)}{1 + a_2 g(\theta_{in}, \alpha)}$$

(29)

where

$$g(\theta_{in}, \alpha) = e^{-a_0 \left(\frac{\text{sym}(\theta_{in} - \alpha)}{\sigma_0}\right)^{a_1}}$$

(30)

and $a_0$, $a_1$ and $a_2$ are free parameters ($a_0 = 1\text{rad}$). The function $P_T$ has a meaning of probability; its values should be, therefore, in the interval from 0 to 1 (not necessary in the full interval) for given values of the free parameters.

3. Parameterization of $\rho_C$

Parameterization of the function $\rho_C(\theta_{in}, \theta_{out}, \alpha)$ may be chosen in the form of Gaussian function

$$\rho_C(\theta_{in}, \theta_{out}, \alpha) = \frac{1}{2\sigma \sqrt{\pi} \text{erf}\left(\frac{\pi}{2\sigma}\right)} \exp\left(-\frac{(\text{sym}(\theta_{out} - \alpha))^2}{\sigma^2}\right)$$

(31)

where $\sigma$ is a free parameter and erf(x) is the error function defined as

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) \, dt.$$  

(32)

The probability density function $\rho_C$ given by eq. (31) is normalized to 1 when integrated over $\theta_{out}$ (see eq. (19)) and does not depend on the value of $\theta_{in}$. It is assumed that it depends only on the difference $\theta_{out} - \alpha$ (similarly as in the case of the parameterization of the $P_T$ function, see eq. (29)).

The parameterization of $\rho_C$ given by eq. (31) corresponds to continuous spectrum of polarization angle values $\theta_{out}$ centered around the value given by the rotation of the polarizer $\alpha$. If only a single discrete value $\theta_{out}$ (equal to $\alpha$) were admitted then the probability density function would be represented by corresponding delta function. The smaller the value of the free parameter $\sigma$ (i.e., the width of the corresponding peak), the closer the continuous spectrum is to the delta function.

The parameterizations of $P_T$ and $\rho_C$ given by eqs. (29) and (31) correspond to the definition of linear polarizer in sect. IV C. The parameterizations of the 3 functions $\rho_{S,0}$, $P_T$ and $\rho_C$ a priori restrict set of possible solutions which could describe measured data. The parameterizations represent additional assumptions in the probabilistic model, see the assumption IV.8, which will be used for analysis of date in the next section.

V. ANALYSIS OF DATA USING THE PROBABILISTIC MODEL

In the following the (artificial) input data of relative particle numbers $N_i(\bar{\alpha})/N_0$ ($i = 1, 2, 3$) corresponding to transmission of photon beam through the sequence of 3 ideal identical polarizers (see sect. III) will be analyzed with the help of the probabilistic model presented in sect. IV. The relative particle numbers may be calculated with the help of eqs. (24) and (25). We will not focus too much on numerical details but rather on conceptually important points and questions.

A. Check of applicability of the probabilistic model to input data (consistency)

Following the guidelines in sect. 4.2 in [28] one can check that the input data of particle numbers discussed in sect. III satisfy the basic inequalities given by eq. (17), and the relative particle numbers $N_i(\bar{\alpha})/N_0$ do not depend on $N_0$. The assumption IV.3 can be also tested experimentally for a photon beam passing through the surface $\Sigma$ ($i = 0, \ldots, 3$) and is satisfied in our case (see sect. III). The other assumptions (see sect. IV A) and consequences of the probabilistic model must be tested indirectly.

B. Fitting of the probabilistic model to data

Calculation of the relative particle numbers $N_i(\bar{\alpha})/N_0$ behind the third (i.e., the last) polarizer in the sequence according to eq. (24), and needed for comparison to input data, is the most computationally intensive task. Under the used assumptions (see sect. IV A) it is possible to use either eq. (25) or eq. (26). Numerical calculation of $N_{i+1}(\bar{\alpha})/N_0 \rho_{S, i+1}(\theta_{i+1}, \bar{\alpha})$ using eq. (26) is much faster than using eq. (25), see sect. 4.6 in [28] for further comments related to computational complexity.

Calculated relative particle numbers $N_i(\bar{\alpha})/N_0$ ($i = 1, \ldots, M$) behind $(i - 1)$-th polarizer can be calculated for any value of the rotation angle of the polarizer and the rotations of all the preceding polarizers, i.e., $i$ continuous variables represented by $\bar{\alpha}$. The rotation angles are measured in discrete steps and the artificial input data, see sect. III, can be considered only in discrete steps, too. However, even if some discrete angular steps are considered it may represent many data points (depending on the width of the steps) already in the case of $M = 3$. It is, therefore, useful to simplify fitting of the model to data as much as possible.

It is not necessary to take into account all values of $\alpha_0$ due to the symmetric dependence of the relative number of transmitted particles on the value of $\alpha_0$ (see sect. III B). It is sufficient to consider only one value, e.g., $\alpha_0 = 0$ deg. This is closely related to the fact that the initial light is assumed to be unpolarized, see eq. (28).

C. Numerical results

Under the given set of assumptions, see sect. IV A, it is possible to find dependence of the parameterized functions $P_T$ and $\rho_C$ (i.e., values of the free parameters discussed in sects. IV E 2 and IV E 3) which can describe the input data.
TABLE I: The values of the free parameters of the probabilistic model fitted to the input data.

|   |   |
|---|---|
| $a_0$ | 1.693 |
| $a_1$ | 2.546 |
| $a_2$ | 0.4807 |
| $\sigma$ [rad] | 0.1887 |

The solution is, however, not unique. Therefore, only one solution corresponding to the values of the free parameters in table I is discussed in the following. We will come back to the problem of ambiguity in sect. VI. The comparison of the measured number of transmitted particles to calculated (simulated) number of transmitted particles with the help of the probabilistic model is in fig. 5. The model agrees well with the input data and can be further improved by using more flexible parameterizations of the functions $P_T$ and $\rho_C$. Our focus is, however, on conceptually important points and questions, as it has been already mentioned.

The probability $P_T$ given by eq. (29) as a function of $\theta_\text{in}$ is plotted in fig. 3 for $\alpha_0 = 0$ deg. The values of this function representing probability are in the interval from 0 to 1.

The probability density function $\rho_C$ given by eq. (31) as a function of $\theta_\text{out}$ is plotted in fig. 4 for $\alpha_0 = 0$ deg. This function is normalized to unity, see eq. (19), and the values are not in the interval from 0 to 1 (values of a probability density functions may or may not be in the interval from 0 to 1).

The probability density functions $\rho_{S,i} (\theta_i, \vec{\alpha})$ as a function of photon polarization angle $\theta_i$ ($i = 0, \ldots, 3$) are plotted in fig. 6 for one fixed combination of rotations of the polarizers: $\vec{\alpha} = (0.0, 45.0 \text{ deg, } 90.0 \text{ deg})$. The blue line in fig. 6 corresponds to eq. (28), i.e., to the assumed initial uniform distribution of photon polarization (unpolarized light). Unpolarized light transmitted through one or more linear polarizers is not uniform, see fig. 6 (and eq. (31)). The probability density functions $\rho_{S,i} (\theta_i, \vec{\alpha})$ ($i = 1, \ldots, M$) have the same shapes (differing only in position). It is the consequence of the used assumptions, see eq. (26).

The probability density function $\rho_{S,i} (\theta_i, \vec{\alpha})$ multiplied by $N_i (\vec{\alpha})/N_0$ ($i = 0, \ldots, 3$) for the same rotations of the polarizers $\vec{\alpha} = (0.0, 45.0 \text{ deg, } 90.0 \text{ deg})$ is plotted in fig. 7 similarly as the functions $\rho_{S,i} (\theta_i, \vec{\alpha})$ in fig. 6 are plotted. The function $N_i (\vec{\alpha})/N_0$ $\rho_{S,i} (\theta_i, \vec{\alpha})$ is equal to $\text{dos}_{i} (\theta_i, \vec{\alpha})/N_0$ (see eq. (22)). This function expresses not only the density of polarization angles but also the decrease of the particle numbers behind ($i - 1$)-th polarizer (if $i = 1, 2, 3$), see the normalization condition of $\rho_{S,i} (\theta_i, \vec{\alpha})$ given by eq. (18). For the value of $\vec{\alpha}$ it holds $N_1 (\vec{\alpha})/N_0 = 0.5$, $N_2 (\vec{\alpha})/N_0 = 0.25$, $N_3 (\vec{\alpha})/N_0 = 0.12$ (according to both the input data and the probabilistic model fitted to the data). This decrease of number of transmitted photons with increasing number of polarizers is visible also in fig. 7, see eqs. (2) and (21). The numerical results show that it is possible to explain the famous 3 polarizers experiment with the help of the probabilistic model and to determine the probability (density) functions characterizing the transmission of light through the polarizers. The probability function $P_T$, the probability density function $\rho_C$ and the probability density functions $\rho_{S,i} (\theta_i, \vec{\alpha})$ multiplied by $N_i (\vec{\alpha})/N_0$ ($i = 0, 1, 2, 3$) determined on the basis of the analysis of experimental data with the help of the probabilistic model characterize probabilistic transmission of individual photons through the sequence of 3 ideal identical polarizers. The functions $P_T$ and $\rho_C$ characterize interactions of polarized photons with polarizers (matter). The probability density functions $\rho_{S,i} (\theta_i, \vec{\alpha})$ multiplied by $N_i (\vec{\alpha})/N_0$ ($i = 0, 1, 2, 3$) characterize property of the photon beam before and after being transmitted through one, two or three polarizers.

The numerical results clearly show that it is possible to explain the famous 3 polarizers experiment with the help of the probabilistic model and to determine the probability (density) functions characterizing the transmission of light through the polarizers, even though the results are not unique. The open questions related to the ambiguity of the determination of the parameterized functions will be discussed in sect. VI.

The numerical results presented in this section have been obtained with the help of ROOT [38] and Matplotlib [39].
The comparison of the input data of relative particle numbers $N_i(\vec{\alpha})/N_0$ ($i = 1, 2, 3$) corresponding to transmission of unpolarized light through 3 ideal identical linear polarizers (Malus’s law, see sect. III) with the probabilistic model (describing both photon absorption in dependence on photon polarization and change of photon polarization during the transmission, see sect. IV) fitted to the data for several combinations of rotations of the polarizes $\vec{\alpha}$ (in degrees). Blue lines - the input data given by eq. (5). Orange lines - the probabilistic model, see eqs. (24) and (25). (a) - $N_1(\vec{\alpha})/N_0$ as a function of $\alpha_0$. (b) - $N_2(\vec{\alpha})/N_0$ as a function of $\alpha_1$ for fixed value of $\alpha_0 = 0$ deg. (c) - $N_3(\vec{\alpha})/N_0$ as a function of $\alpha_1$ for fixed values of $\alpha_0 = 0$ deg and $\alpha_2 = 90$ deg. (d) - $N_3(\vec{\alpha})/N_0$ as a function of $\alpha_1$ for fixed values of $\alpha_0 = 0$ deg and $\alpha_2 = 90$ deg.

FIG. 5
FIG. 6: The probability density functions $\rho_{S,i}(\theta_i, \vec{\alpha})$ as a function of photon polarization angle $\theta_i$ if the 3 polarizers are rotated by $\vec{\alpha} = (0.0, 45.0, 90.0)$ deg. In the case of $i = 0$ the function represents initial uniform polarization of the beam given by eq. (28), in the case of $i = 1, 2, 3$ it describes polarization of the beam transmitted through the one, two or three ideal identical linear polarizers. The orientations of the axes of the polarizers are specified by $\vec{\alpha}$, see the vertical dashed lines. The position of the peaks correspond to the rotation of the axis of the polarizers.

FIG. 7: Similar picture to fig. 6 but the probability density functions $\rho_{S,i}(\theta_i, \vec{\alpha})$ multiplied by $N_i(\vec{\alpha})/N_0$ are plotted, these expressions are equal to relative densities of states $\text{dos}_i(\theta_i, \vec{\alpha})/N_0$, see eq. (22).
VI. OPEN QUESTIONS (PROBLEMS) AND HOW TO ADDRESS THEM

A. List of open questions

As it has been mentioned, the dependence of the parameterized functions $P_T$ and $P_C$ shown in sect. V C is not the only one which can explain the measured (input) number of transmitted photons through 3 ideal identical linear polarizers. There are several open questions concerning mainly the determination of the probability (density) functions characterizing the polarizers:

1. **Dependence of $P_C$ on $\theta_{out}$**
   
   There is strong ambiguity on determination of value of the parameter $\sigma$, i.e., the width of the peaks characterizing the function $P_C$, see fig. 4. The input data could be fitted similarly well with value of $\sigma$ being in the interval from zero to the value specified in table I (see also the comments in sect. IV E 3).

2. **$\alpha$ and $\alpha - \pi$ (a)symmetry of $P_C$ as a function of $\theta_{out}$**
   
   The parameterization of probability density function $P_C$ given by eq. (31) (see also fig. 4) implies that $P_C$ as a function of polarization angle of outgoing photon $\theta_{out}$ has two peaks located at $\alpha$ or $\alpha - \pi$. Moreover, the peaks have the same shape. There is, therefore, the same probability that an outgoing photon has value of polarization angle $\theta_{out}$ closer to either $\alpha$ or $\alpha - \pi$. This symmetry is clearly visible in fig. 7, too.

   The input data analyzed in sect. V can be, however, fitted equally well with similar parameterizations of $P_C$ normalized to unity and having only one peak or two peaks of different heights and widths. The given input data do not allow distinguishing between these possibilities.

3. **$\alpha + \pi/2$ and $\alpha - \pi/2$ (a)symmetry of $P_C$ as a function of $\theta_{out}$**
   
   There is one more source of ambiguity. The one or two peaks in the spectrum of values of polarization angle $\theta_{out}$ of outgoing photon may be located at different positions, at $\alpha + \pi/2$ and $\alpha + \pi/2 - \pi = \alpha - \pi/2$. This situation corresponds to the axis of polarizer specified by angle $\alpha + \pi/2$ which is perpendicular to the axis specified by angle $\alpha$. The number of transmitted photons through 3 ideal identical polarizers (see sect. III) exclude the possibility that the spectrum could have two, three or four peaks corresponding to both the axes. It is, however, not possible to determine on the basis of the input data towards which of the axes the polarization of outgoing photon is predominantly aligned.

4. **Dependence of $P_C$ on $\theta_{in}$**
   
   It has been assumed that the parameterization of $P_C$ given by eq. (31) does not depend on $\theta_{in}$ and depends only on the difference $\theta_{in} - \alpha$. The possibility of more complex dependence of the function $P_C$ characterizing given polarizer on $\theta_{in}$, $\theta_{out}$ and $\alpha$ should be tested in the future.

5. **Probability function $P_T$**
   
   The parameterization of $P_T$ given by eq. (29) depends only on the difference $\theta_{in} - \alpha$. The possibility of more complex dependence of $P_T$ on $\theta_{in}$ and $\alpha$ should be tested in the future, too.

The ambiguity of determination of properties of polarizers related to the open questions 1 to 3 holds under the assumptions used in this paper, i.e., under the assumptions on which the probabilistic model is based (see sect. IV A and definition IV.8). Answering the open questions 4 and 5 necessarily requires (at least) a change in the parameterizations and additional experimental information, see sect. VI B.

B. Possibilities of answering the open questions

There are several possibilities of addressing the open questions discussed in sect. VI A:

1. **Transmission of light through various sequences of linear polarizers**
   
   Determination of the functions $P_T$ and $P_C$ characterizing a polarizer ("A") can be improved with the help of, e.g., two other different linear polarizers ("B" and "C"). One may study transmission of (unpolarized) light through 3 polarizers such as AAA, BBB, CCC, ABC, ABB, ... (any of the permutations with repetition). It is possible to analyse all the 27 different sequences (arrangements). One may try to determine the probabilistic characteristics of the 3 types of polarizers (two functions $P_T$ and $P_C$ corresponding to each of the different polarizers A, B and C; 6 functions in total) in order to describe data corresponding to all the 27 sequences. It may be useful to study real linear polarizers which transmit light similarly as the ideal polarizers (see sect. II C) as well as linear polarizers which have significantly different characteristics (see sect. II D).

   If in a polarizer some structures (e.g., needle-like crystals or molecules) are aligned predominantly in one direction which is perpendicular to the beam direction, then this direction specifies one axis of symmetry and the second axis of symmetry is perpendicular to it and the direction of the beam. The width of the peaks in the spectrum of values of polarization angle $\theta_{out}$ of outgoing photon characterized by the parameter $\sigma$ could be related to the quality of alignment of the microscopical structures of the polarizer. Polarizers manufactured with different degree of alignment of the structures could help to answer some open questions and to better understand the differences between ideal and real polarizers (see sects. II C and II D).

   Measuring and analysing transmission of light through only various linear polarizers may, however, hardly allow to answer all the open questions.

2. **Using other polarization sensitive elements than linear polarizers**
   
   Measuring transmission of light through various
sequences of polarization sensitive components (see sect. 2 in [28]) can provide essential experiment data which can be further analyzed with the help of the probabilistic approach. This should help to uniquely determine the functions $P_T$ and $\rho_C$ characterizing individual linear polarizers as well as other polarization sensitive devices.

E.g., the open questions 2 and 3 could be answered with the help of optical analog of Stern-Gerlach experiment separating photons by spins. One possible experimental method of A. Fresnel is discussed in detail in [40] (it uses quartz polyprism). Measurement and subsequent analysis of transmission of light through various sequences consisting of polarizing beam splitters and other optical elements (such as linear polarizer) with the help of a probabilistic model could significantly help to answer the open questions.

In some cases, an analysis of measured number of transmitted photons through sequences of various (polarization sensitive) optical elements can be performed with the help of STC stochastic process introduced in [28]. It may be necessary to only correctly define variables characterizing the optical elements and variables charactering photon properties needed to explain the measured number of transmitted photons.

E.g., transmission of photons through a Faraday-effect-based device (also called Faraday rotator) can be analyzed very similarly as transmission of photons through a linear polarizer. However, instead of rotation angle of a linear polarizer it is necessary to introduce magnetic field $B$ in the direction of propagation of the beam and the length $d$ of the path where the beam and magnetic field interact in the device (to explain measured number of transmitted photons in dependence on these two parameters, or one of them may be taken as fixed). The functions $P_T$ and $\rho_C$ in the case of a Faraday-effect-based device must be parameterized very differently than in the case corresponding to a linear polarizer, see eqs. (29) and (31). I.e., mainly the assumptions assumptions IV.1, IV.7 and IV.8 (and also the assumptions IV.4 and IV.5, if the beam is not transmitted through sequence of identical Faraday rotators) must be modified. In the case of a Faraday-effect-based device one can try to parameterize the function $P_T$ as a constant function having only one free parameter. The parameterization of $\rho_C$ should correspond to the fact that the polarization angle of a transmitted photon is rotated with respect to the polarization angle of the photon before the transmission by “more or less” the same amount independently on the value of the polarization of the incoming photon. The probability density function $\rho_C$ may be expressed by, e.g., Gaussian function where the value of the free parameter characterizing the width of the peak and the value of the free parameter corresponding to the position of the peak should be extracted on the basis of experimental data using the probabilistic model. The determined position of the peak divided by $Bd$ may be then compared to the Verdet constant standardly determined (using a different method) to characterize a Faraday-effect-based device.

To describe probabilistic character of transmission of light through 3 linear polarizers only one variable characterizing properties of photon, the photon polarization angle, has been taken into account in sect. V. Probabilistic description of transmission of light through other polarization elements where photon polarization states denoted as circular and elliptical are involved may require introduction of one or two additional degrees of freedom (variables). This implies more complicated parameterization of the functions $P_T$ and $\rho_C$ than in the case of linear polarizers, but the probabilistic model formulated in [28] could be useful also in this case.

STC stochastic process introduced in [28] may not be able to explain measured number of transmitted photons through some types of optical elements under the assumptions 3.3 and 3.5 to 3.7 specified in [28]. In this case one need to formulate new probabilistic model (based on different set of assumptions).

**VII. CONCLUSION**

Transmission of a photon through a linear polarizer is standardly denoted as probabilistic process but the corresponding probabilities in dependence on the polarization state of the photon and the rotation of the polarizer have never been determined in the literature (see sect. I). Explanation of well known and easily to perform 3 polarizer experiment have been, therefore, unsatisfactory.

In this paper the probability (density) functions characterizing transmission of light through 3 linear polarizers have been determined for the first time. The keyformulae of the probabilistic model (see sect. IV) used for the analysis of measured (input) relative particle numbers have been derived with the help of STC stochastic process proposed in [28].

It is worth to note that in the 3 polarizer experiment some properties of the photon beam have been measured (the relative number of transmitted photons) and some other statistical characteristics of the individual photons and their interactions with matter have been determined with the help of the probabilistic model (see sect. VC). The results of the analysis of the 3 polarizer experiment performed in sect. V clearly shows that transmission of photon (laser) beams through other optical elements may be analyzed in similar manner, too, see also sect. VI B. Experiments concerning spin-dependent and polarization dependent phenomena are essential for better understanding the connection between spin and polarization (of photon and other particles). However, the open questions formulated in sect. VIA should be answered before drawing far-reaching conclusions. Suggestions on how to address these questions have been discussed in sect. VI B.

Statistical description of particle phenomena using STC stochastic process opens up new possibilities obtaining deeper insight not only into the polarization of photons and light in general, but also into properties of other particles and their interaction with matter. With its help it is possible to analyse many experiments where other contemporary widely used theoretical approaches can provide only partial or no understanding of the underlying particle phenomena, see [28]. With the help of STC stochastic process it is possible to unify de-
The analysis of data performed in sect. V corresponds to the first stage.

The separation of an analysis of data into the two stages helps to study the involved assumptions more effectively. It has already helped enormously, e.g., in the field of genetics (biophysics) pioneered by Mendel in 1865 [42]. He performed many plant hybridization experiments. He was able to explain variability of observed properties of organisms in different generations by introducing dominant and recessive traits (characteristics). By introducing laws (additional assumptions) how these traits are inherited he was able to calculate probabilities of occurrences of organisms of given properties and in given generation (i.e., transition). The calculated probabilities agreed with the observed (measured) numbers. His analysis of data corresponded to the first stage of the separation. Genetic research is in many cases nowadays well in the second stage where determined dominant and recessive abstract traits are identified with real biophysical structures in an organisms (a gene consisting of two alleles) and related processes are further studied in greater detail. The probabilistic model formulated by Mendel [42] allowed to easily calculate the probabilities needed for comparison to data (formulation of the model itself, however, required surely far more effort).

Stochastic STC process can help in some mathematically more complicated cases with the determination of the probabilities (the first stage) which is in general delicate task [28].

In some fields of research the theory of stochastic processes and the separation of an analysis of data into the two stages has been essential to make further progress. The field of optics (studying propagation of light and interaction of light with matter) is one of the field where many very diverse theoretical approaches for description of optical phenomena have been studied for centuries. However, the potential of applicability of the theory of stochastic processes has not yet been fully explored in this field. In many analyses of experimental data not event the first stage has been achieved. This is the case of, e.g., analysis of data concerning polarized light (see the introduction). It has been shown in the presented paper that stochastic STC process open up new possibilities to help to make further progress in the field of optics.

[1] D. H. Goldstein, *Polarized light*, 3rd ed. (CRC Press, 2011).
[2] J. W. Goodman, *Statistical optics*, 2nd ed., Wiley series in pure and applied optics (Wiley, Hoboken, New Jersey, 2015).
[3] P. Rice, *An Introduction to Quantum Optics* (IOP Publishing, 2020) doi:10.1088/978-0-7503-1713-9.
[4] J. Weiner and P. Ho, *Light-Matter Interaction: Fundamentals and Applications*, Vol. 1 (Wiley-Interscience, 2003) doi:10.1002/9783527617883.
[5] R. Hartmann, *Theoretical Optics: An Introduction* (Wiley-VCH, 2005).
[6] N. Menn, *Practical Optics* (Elsevier Academic Press, 2004).
[7] K. S. Thorne and R. D. Blandford, *Modern Classical Physics: Optics, Fluids, Plasmas, Elasticity, Relativity, and Statistical Physics* (Princeton University Press, 2017).
[8] D. Meschede, *Optics, Light and Lasers*, 3rd ed. (Wiley-VCH, 2017).
[9] M. Bass, C. DeCusatis, J. Enoch, V. Lakshminarayan, G. Li, C. Macdonald, V. Mahajan, and E. Stryland, *Handbook of Optics, Third Edition Volume I: Geometrical and Physical Optics, Polarized Light, Components and Instruments*, 3rd ed. (McGraw-Hill, Inc., USA, 2009).
[10] D. L. Andrews, ed., *Photons Volume I: Fundamentals of Photonics and Physics* (Wiley, 2015).
[11] M. Born and E. Wolf, *Principles of Optics*, 7th ed. (Cambridge University Press, 2013) doi:10.1017/CBO9781139644818.
[12] W. Shurcliff, *Polarized Light: Production and Use* (Harvard University Press, 1962).
[13] D. Clarke and J. F. Grainger, *Polarized Light and Optical Measurement* (Pergamon Press, 1971).
[14] R. Azzam and N. Bashara, *Ellipsometry and polarized light* (North-Holland Publishing Company, 1977).
[15] C. Brosseau, ed., *Fundamentals of Polarized Light: A Statistical Optics Approach* (Wiley, 1998).
[16] H. G. Tompkins and E. A. Irene, *Handbook of Ellipsometry* (William Andrew, Springer, 2005).
[17] H. Fujiwara, *Spectroscopic Ellipsometry: Principles and Applications* (Wiley, 2007).
[18] G. Horváth, Polarization patterns in nature: Imaging polarimetry with atmospheric optical and biological applications (2003), Doctor of the Hung. Acad. of Sci. thesis, Loránd Eötvös University, link.
[19] K. Hentschel, *Photons: The History and Mental Models of Light Quanta* (Springer, 2018) doi:10.1007/978-3-319-95252-9.
[20] M. Planck, Zur Theorie des Gesetzes der Energieverteilung im Normalspectrum, *Verhandl. Dtsch. Phys. Ges.* 2, 237 (1900).
[21] M. Planck, Ueber das gesetz der energieverteilung im normalspectrum, *Annalen der Physik* 309, 553 (1901), doi:10.1002/andp.19013090310.
[22] A. Einstein, Über einen die erzeugung und verwandlung des lichtes betreffenden heuristischen gesichtspunkt, *Annalen der Physik* 322, 132 (1905), doi:10.1002/andp.19053220607.
[23] D. L. Andrews, ed., *Photons Volume II: Nanophotonic Structures and Materials* (Wiley, 2015).
[24] D. L. Andrews, ed., *Photons Volume III: Photonics Technology and Instrumentation* (Wiley, 2015).
[25] D. L. Andrews, ed., *Photonics Volume IV: Biomedical Photonics, Spectroscopy and Microscopy* (Wiley, 2015).
[26] O. Arteaga, Historical revision of the differential stokes–mueller formalism: discussion, J. Opt. Soc. Am. A **34**, 410 (2017), doi:10.1364/JOSAA.34.000410.
[27] R. Jones, Transmittance of a train of three polarizers, J. Opt. Soc. Am. **46**, 528 (1956), doi:10.1364/JOSA.46.000528.
[28] J. Procházka, Stochastic state-transition-change process and particle physics, Eur. Phys. J. Plus **137** (2022), doi:10.1140/epjp/s13360-022-03102-x, see also arXiv:2204.00626v1.
[29] M. V. Lokajíček, Quantum mechanics and EPR paradox (2002), arXiv:quant-ph/0211012.
[30] M. Bass, C. DeCusatis, J. Enoch, V. Lakshminarayanan, G. Li, C. Macdonald, V. Mahajan, and E. Stryland, *Handbook of Optics, Third Edition Volume II: Design, Fabrication and Testing, Sources and Detectors, Radiometry and Photometry*, 3rd ed. (McGraw-Hill, Inc., USA, 2009).
[31] É.-L. Malus, Sur une propriété des forces répulsives qui agissent sur la lumière, Mémoires de physique et de chimie de la Société d’Arcueil **2**, 254 (1809), (in French).
[32] B. Kahr and K. Claborn, The lives of Malus and his bicentennial law, ChemPhysChem **9**, 43 (2008), doi:10.1002/cphc.200700173.
[33] J. Krása, J. Jiřička, and M. Lokajíček, Depolarization of light by an imperfect polarizer, Phys. Rev. E **48**, 3184 (1993), doi:10.1103/PhysRevE.48.3184.
[34] J. Krása, M. Lokajíček, and J. Jiřička, Transmittance of a laser beam through a pair of crossed polarizers, Physics Letters A **186**, 279 (1994), doi:10.1016/0375-9601(94)91171-1.
[35] S. Kadri, D. Chong, B. Wei, and R. Juafar, Student activity: verification on Malus’s law of polarization at low cost, in *12th Education and Training in Optics and Photonics Conference*, Vol. 9289, edited by M. F. P. C. M. Costa and M. Zghal, International Society for Optics and Photonics (SPIE, 2014) pp. 549–557, doi:10.1117/12.2070727.
[36] L. Vertchenko and L. Vertchenko, Verification of Malus’s law using a LCD monitor and digital photography, Revista Brasileira de Ensino de Física **38**, e3311 (2016), doi:10.1590/1806-9126-RBEF-2016-0029.
[37] M. Monteiro, C. Stari, C. Cabeza, and A. C. Martí, The polarization of light and malus’ law using smartphones, *The Physics Teacher** **55**, 264 (2017), https://doi.org/10.1119/1.4981030.
[38] R. Brun and F. Rademakers, ROOT: An object oriented data analysis framework, Nucl. Instrum. Meth. A **389**, 81 (1997), doi:10.1016/S0168-9002(97)00048-X.
[39] J. D. Hunter, Matplotlib: A 2D graphics environment, Computing in Science & Engineering **9**, 90 (2007), doi:10.1109/MCSE.2007.55.
[40] O. Arteaga, E. García-Caurel, and R. Ossikovski, Stern-Gerlach experiment with light: separating photons by spin with the method of A. Fresnel, Opt. Express **27**, 4758 (2019), doi:10.1364/OE.27.004758.
[41] J. Procházka, Stochastic state-transition-change process and time resolved velocity spectrometry (2022), arXiv:2204.00626v2.
[42] G. Mendel, Versuche über pflanzen-hybriden, Verhandlungen des naturforschenden Vereines in Brünn **Bd.4 für das Jahr 1865**, 3 (1866), https://www.biodiversitylibrary.org/part/175272, see also English translation http://www.esp.org/foundations/genetics/classical/gm-65.pdf.