Abstract. Eigenfunctions of the Laplace-Beltrami operator on a hyperboloid are studied in the spirit of the treatment of the spherical harmonics by Stein and Weiss. As a special case, a simple self-contained proof of Laplace’s integral for a Legendre function is obtained.

In [2, Chapter IV, Section 2], Stein and Weiss described the spectral decomposition of the Laplace-Beltrami operator on the unit sphere. Their approach was to identify the eigenfunctions with homogeneous harmonic functions on Euclidean space.

In this article the eigenfunctions of the Laplace-Beltrami operator on a hyperboloid are identified with homogeneous harmonic functions (with respect to a Laplacian of type \((p, q)\)) on an open cone. In the case treated by Stein and Weiss, Liouville’s theorem implies that the degree of homogeneity must be a non-negative integer, whereas here the degree of homogeneity can be any complex number. This identification is used to compute spherical functions for \(O(1, q)\), and consequently Laplace’s integral formula for Legendre functions is obtained. Laplace’s integral formula can also be obtained by using the residue theorem [3, §15.23]. Spherical functions for semisimple Lie groups in general are obtained using different methods (see, e.g., [1, Chapter IV]).

Let \(n = p + q\). Let \(\mathbb{R}^{p,q}\) denote the space of real \(n\)-dimensional vectors equipped with the indefinite scalar product of signature \((p, q)\):

\[
x \cdot y = x^T Q y
\]

where \(Q\) is the diagonal matrix with \(p\) 1’s followed by \(q\) \((-1)\)’s along the diagonal. Write \(|x|^2\) for \(x \cdot x\). There should be no confusion with the usual positive definite dot product and norm as they are never used in this paper.

Let \(\mathbb{R}^{p,q}_+\) denote the subset of \(\mathbb{R}^{p,q}\) consisting of those vectors for which \(|x|^2 > 0\). For \(x \in \mathbb{R}^{p,q}_+\), let \(|x|\) denote the positive square root of \(|x|^2\). Let \(O(p, q)\) denote the group consisting of matrices such that \(A^T Q A = Q\). Denote by \(O(p, q)_0\) the connected component of the identity element of \(O(p, q)\). Let \(S^{p,q}\) denote the connected component of \((1, 0, \ldots, 0)\) in the hyperboloid

\[
\{x : |x| = 1, x_1 > 0\}.
\]

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Let $\rho$ be any complex number. Let $\mathcal{P}_\rho$ denote the space of all functions $f \in C^2(\mathbb{R}_{p,q}^n)$ which are homogeneous of degree $\rho$, i.e., functions such that
\[ f(\lambda x) = \lambda^\rho f(x) \] for all $x \in \mathbb{R}_{p,q}^n$, $\lambda > 0$.

Denote by $\Delta$ the differential operator $|\nabla|^2 = \nabla \cdot \nabla$ (using the indefinite dot product), where $\nabla$ is the gradient operator $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$.

Thus, $\Delta = \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} \right) - \left( \frac{\partial^2}{\partial x_{p+1}^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)$.

Define $H_\rho = \{ f \in \mathcal{P}_\rho : \Delta f = 0 \}$.

A function $u \in C^2(S^{p,q})$ is called a spherical harmonic\footnote{Perhaps a more apt name would be hyperboloidal harmonic.} of degree $\rho$ if $u$ is the restriction to $S^{p,q}$ of a function in $H_\rho$. Let $H_\rho$ denote the space of spherical harmonics of degree $\rho$:
\[ H_\rho = \{ f |_{S^{p,q}} : f \in H_\rho \} . \]

The Laplace-Beltrami operator $\Delta_{S^{p,q}}$ on $S^{p,q}$ is defined by
\[ \Delta_{S^{p,q}} u = \Delta_{|x|} \tilde{u} |_{S^{p,q}} , \]
where $\tilde{u} : \mathbb{R}_{p,q}^n \to \mathbb{C}$ is defined by $\tilde{u}(x) = u(x/|x|)$ (the degree zero homogeneous extension of $u$).

Let $x' = Qx$. The following is easily verified:

**Lemma 1.** Let $x \in \mathbb{R}_{p,q}^n$. Then
\[ \nabla |x| = x'/|x| . \] \hspace{1cm} (1)
\[ \nabla |x|^{\rho} = \rho |x|^{\rho-2} x' . \] \hspace{1cm} (2)
\[ |x'| = |x| . \] \hspace{1cm} (3)
\[ x' \cdot \nabla \tilde{u}(x) = 0 \text{ for any } u \in C^1(S^{p,q}) . \] \hspace{1cm} (4)
\[ \nabla \cdot x' = n . \] \hspace{1cm} (5)

**Lemma 2.** If $u \in H_\rho$, then $\Delta_{S^{p,q}} u = -\rho(\rho + n - 2) u$.

**Proof.** Since $u \in H_\rho$, $|x|^{\rho} \tilde{u}(x) \in H_\rho$. Therefore (using the formulas in Lemma 1),
\[ 0 = \Delta (|x|^{\rho} \tilde{u}(x)) = \nabla \cdot (\nabla (|x|^{\rho} \tilde{u}(x))) = \nabla \cdot (\rho |x|^{\rho-2} x' \tilde{u}(x) + |x|^{\rho} \nabla \tilde{u}(x)) \]
Setting $|x| = 1$ in the result of the above calculation yields

$$0 = \rho(\rho - 2 + n) \hat{u}(x) + \Delta \hat{u}(x),$$

from which the lemma follows.

The following proposition gives a construction of spherical harmonics when $p = 1$:

**Proposition 3.** Suppose $c \in R^1_+^q$ is an isotropic vector, (meaning that $|c|^2 = 0$) such that $c_1 > 0$. Then $c \cdot x > 0$ for all $x \in S^1_q$. Let $f(x) = (c \cdot x)^p$. Then $f \in \mathcal{H}_p$.

**Proof.** The set of points where $c \cdot x = 0$ form a hyperplane tangential to the cone $|c|^2 = 0$. For fixed $x$, the sign of $c \cdot x$ can change only when $c$ crosses this hyperplane. However, the entire half-cone

$$\{c: |c|^2 = 0, \ c_1 > 0\}$$

lies on one side of the hyperplane, because the cone is quadratic. Therefore, for each $x \in S^1_q$, it suffices to verify that $c \cdot x > 0$ for $c = (1, 1, 0, \ldots, 0)$. In this case, $c \cdot x = x_1 - x_2$, which is positive since $x_1 > 0$ and $x_1^2 - x_2^2 - \cdots - x_n^2 = 1$, so that $x_1 > |x_1|$ for each $i > 1$.

If $g \in C^2(R^{p,q})$ and $\varphi \in C^2(R)$, then

$$\Delta(\varphi \circ g)(x) = \varphi''(g(x))|\nabla g(x)|^2 + \varphi'(g(x))\Delta g(x).$$

Let $g(x) = c \cdot x$, then $\nabla g(x) = c$, so that $|\nabla g(x)|^2 = 0$. Since $g$ is linear, $\Delta g(x) = 0$. Therefore $\Delta f(x) = 0$.

Let $e = (1, 0, \ldots, 0)$. Then $K = \text{Stab}_{O(1,q)}(e)$ is isomorphic to $SO(q)$ and is a maximal compact subgroup of $O(1,q)$. The action of $O(1,q)$ on $S^1_q$ is transitive, and the $K$-invariant spherical harmonics on $S^1_q$ are precisely the $K$-invariant spherical functions for $O(1,q)$. It follows from Proposition 3 that

**Proposition 4.** Let $c$ be any isotropic vector in $R^1_+^q$. Then

$$\int_K (kc \cdot x)^p\, dk$$

is a $K$-invariant spherical harmonic of degree $\rho$ on $S^1_q$.

Since $K$ acts transitively on the slices of $S^1_q$ by the hyperplanes on which the first coordinate $x_1$ is constant, the value of a $K$-invariant spherical harmonic is simply a function of $x_1$, which will be denoted by $P(x_1)$. A $K$-invariant spherical harmonic may be viewed as a solution to an ordinary differential equation in $x_1$:
Theorem 5. Suppose that $P_\rho(x_1)$ is the value of a $K$-invariant spherical harmonic which is homogeneous of degree $\rho$. Then $P_\rho$ is a solution to the differential equation

$$(1 - x_1^2)P''_\rho(x_1) + (1 - n)x_1 P'_\rho(x_1) + \rho(\rho - 2 + n)P(x_1) = 0. \quad (6)$$

Proof. For any $f \in C^2(S^{1,q})$ we have

$$\nabla f(x_1/|x|) = \nabla(x_1/|x|) f'(x_1/|x|)$$

$$= \frac{\nabla(x_1) - x_1 \nabla x}{|x|^2} f'(x_1/|x|)$$

$$= \frac{e|x| - (x_1/|x|)x^\theta}{|x|^2} f'(x_1/|x|)$$

$$= \nu v,$$

where $u = |x|^{-3} f'(x_1/|x|)$ and $v = e|x|^2 - x_1 x^\theta$. Since $\Delta = |\nabla|^2$,

$$\Delta f(x_1/|x|) = (\nabla u) \cdot v + u \nabla \cdot v. \quad (7)$$

Now,

$$\nabla u = -3 |x|^{-5} x^\theta f'(x_1/|x|) + |x|^{-3} \nabla(x_1/|x|) f''(x_1/|x|)$$

$$= -3 |x|^{-5} x^\theta f'(x_1/|x|) + |x|^{-6} (e|x|^2 - x_1 x^\theta) f''(x_1/|x|).$$

and

$$\nabla \cdot v = e \cdot \nabla |x|^2 - (e \cdot x^\theta + nx_1) = (1 - n)x_1.$$

Suppose there exists a function $P$ such that $P(x_1) = f(x)$ for each $x$ such that $|x| = 1$. Substituting the above values of $\nabla u$ and $\nabla \cdot v$ in (7) and then setting $|x| = 1$ we have,

$$\Delta_{S^{1,q}} f|_{S^{1,q}}(x) = (-3x^\theta P'(x_1) + (e - x_1 x^\theta) P''(x_1)) \cdot (e - x_1 x^\theta)$$

$$+ (1 - n)x_1 P'(x_1).$$

When $|x| = 1$, $|e - x_1 x^\theta|^2 = (1 - x_1^2)$ and $(e - x_1 x^\theta) \cdot x^\theta = 0$ so that the above equality simplifies to

$$\Delta_{S^{1,q}} f|_{S^{1,q}}(x) = (1 - x_1^2) P''(x_1) + (1 - n)x_1 P'(x_1).$$

Combining this with Lemma 2 gives (6).

Corollary 8. For $n \geq 3$, there is (up to scaling) a unique $K$-invariant spherical function of degree $\rho$ given by

$$\int_K (k \cdot c)^\rho \, dk,$$

where $c$ is any non-zero isotropic vector in $R^{1,q}$.

Proof. The ordinary differential equation (6) is linear of degree 2 with a regular singular point at $x_1 = 1$. The indicial equation at this point is

$$m(m + (n - 1)/2 - 1) = 0.$$
Therefore, it has (up to scaling) at most one solution defined on $[1, \infty)$. This solution is known by Proposition 4.

The classical integral formula due to Laplace for Legendre functions is readily derived from the preceding analysis:

**Corollary 9.** Every solution of the ordinary differential equation

$$(1 - x^2)P''(x) - 2xP'(x) + \rho(\rho + 1)P(x) = 0$$

that is defined on $[1, \infty)$ is a scalar multiple of

$$P_\rho(x) = \frac{1}{2\pi} \int_0^{2\pi} (x + \sqrt{x^2 - 1} \cos \theta)^\rho \, d\theta.$$

**Proof.** Evaluate the formula from Corollary 8 taking $q = 2$, $c = (1, 0, -1)$ and $x = (x, 0, \sqrt{x^2 - 1})$.

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