ENTIRE SOLUTIONS ORIGINATING FROM MONOTONE FRONTS FOR NONLOCAL DISPERAL EQUATIONS WITH BISTABLE NONLINEARITY

FANG-DI DONG AND WAN-TONG LI

School of Mathematics and Statistics, Lanzhou University
Lanzhou, Gansu 730000, P.R. China

SHI-LIANG WU

School of Mathematics and Statistics, Xi’dian University
Xi’an, Shaanxi 710071, P.R. China

LI ZHANG

School of Science, Chang’an University
Xi’an, Shaanxi 710064, P.R. China

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Abstract. This paper mainly focuses on the entire solutions of nonlocal dispersal equations with bistable nonlinearity. Under certain assumptions of wave speed, firstly constructing appropriate super- and sub-solutions and applying corresponding comparison principle, we established the existence and related properties of entire solutions formed by the collision of three and four traveling wave solutions. Then by introducing the definition of terminated sequence, it is proved that there has no entire solutions formed by \( k \) traveling wave solutions that collide with each other as long as \( k \geq 5 \). Finally, based on the classical weighted energy approach, we obtain the global exponentially stability of the entire solutions in some weighted space.

1. Introduction and main results. In this paper, we are considered with the following nonlocal evolution equation

\[ u_t(x, t) = (J \ast u)(x, t) - u(x, t) + f(u(x, t)), \quad x \in \mathbb{R}, t \in \mathbb{R}, \]

where \( u(x, t) \) denotes the population density of species at space \( x \) and time \( t \). \( \int_\mathbb{R} J(x - y)u(y, t)dy - u(x, t) \) is called the nonlocal dispersal and can represent transportation due to long range dispersion mechanisms, one can refer to [2, 3, 9–11, 19, 21–23, 39, 40, 52]. Further, \( J : \mathbb{R} \rightarrow \mathbb{R} \) is a function satisfying:

\[ J \in C^1(\mathbb{R}), J(-x) = J(x) \geq 0, \int_\mathbb{R} J(x)dx = 1 \] and \( J \) is compactly supported.

The nonlinearity is induced by the function \( f \), which we assume satisfies the following condition:

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* Corresponding author: Wan-Tong Li.
(F) \( f \in C^2(\mathbb{R}), f(0) = f(1) = f'(0) = 0, f'(1) < 0, f'(a) > 0 \), and
\( f(s) < 0 \) for \( s \in (0, a) \), \( f(s) > 0 \) for \( s \in (a, 1) \), \( f'(a) < 0 \) for \( s \in (0, 1) \),
\( \int_0^a f(s) ds > 0 \). \( f \) has only one zero \( a \) in \((0, 1)\), and no zeros outside \([0, 1]\).

It is easy to see that the constant states \( u = 0 \) and \( u = 1 \) are stable and
\( u = a \) is unstable for the kinetic equation (i.e., (1) without diffusion term), since
\( f'(0) < 0, f'(1) < 0, f'(a) > 0 \).

It is well known that the movements and interactions of all individuals described
in equation (1) are free and large-range, and we call this phenomenon nonlocal dispersal. Correspondingly, when the individuals move randomly between the adjacent spatial locations in the continuous space, we can obtain the following classical reaction-diffusion equation:

\[
 u_t(x, t) = \Delta u(x, t) + f(u(x, t)), \quad x \in \mathbb{R}, t \in \mathbb{R}. \quad (2)
\]

Moreover, if the species live in a patchy environment, we get the corresponding spatially discrete form of (2)

\[
 u_k(x, t) = d(u(x + 1, t) + u(x - 1, t) - 2u(x, t)) + f(u(x, t)), \quad x \in \mathbb{R}, t \in \mathbb{R}. \quad (3)
\]

Traveling wave solutions are some important dynamical issues about (1),(2) and
(3). Especially, there has been significant progress in the study of traveling wave solutions for equation (1), one can refer to [1,4,5,11–13,40] and references. Although traveling wave solutions play an important role in describing the large-time behavior of solutions, to understand the whole dynamics is far from enough. In fact, traveling wave solutions are only special examples of the entire solutions. Here the entire solutions are meant by solutions defined for all \((x, t) \in \mathbb{R} \times \mathbb{R}\). Our focus in current paper is to consider the existence and stability of entire solutions for (1). On the one hand, the study of entire solutions can help us mathematically understand the structures of the global attractors which consist of entire solutions. On the other hand, from a biological point of view, different entire solutions can represent different invasion patterns of the population or different ways of transmission of the disease, such as, [6, 7,14–18, 26–28, 56]. Therefore, the study of entire solutions is meaningful and valuable, and the investigation of entire solutions has attracted widespread attention since the pioneering works on the classic KPP equation by Hamel-Nadirashvili [17, 18] and bistable equation by Yagisita [53]. We refer the readers to [6, 16, 27, 37, 45] for scalar reaction-diffusion equations with or without delay, [46–48, 50] for lattice dynamical systems, [14, 24, 31] for reaction-advection-diffusion equations, [28, 38, 44, 51] for reaction diffusion systems, and [15, 25, 26, 28, 42, 43, 49, 55, 57] for nonlocal dispersal problems, where, with some combinations of traveling wave solutions and special solutions, various types of entire solutions have been established.

Very recently, under certain assumptions on wave speed, some entire solutions originating from three fronts have been constructed by Chen [7] for discrete diffusive equation (3). Further, for the classical reaction-diffusion equation (2), Chen et al. [8] established the existence of entire solutions formed by collisions of three or four fronts and the absence of entire solutions resulting from the collisions of five or more fronts. However, the issue of the existence of entire solutions originating from \( k(k \geq 3) \) traveling wave solutions for nonlocal dispersal equations (1) is still open. Resolving this issue represents a main contribution of our current study. In order to extend the above work [7, 8] to nonlocal dispersal equations (1), we shall study entire solutions \( u \) originating from \( k \) traveling wave solutions \( \{(c_j, \phi_j), j = \)}
for some constants \( \theta \) the Sobolev space of the \( L \) proving the stability of entire solutions. Letting global exponentially stable. solutions is in a suitable space, we established that the entire solutions of (1) is also an extremely important subject. The commonly used methods for obtaining the stability are the (technical) weighted energy method, the super- and sub-solutions method and squeezing technique, and the combination of the comparison principle and the weighted energy method, for the details, one can refer to [5, 20, 29, 30, 32–36, 41, 54]. In our paper, we will apply the third method to investigate the stability are the (technical) weighted energy method, the super- and sub-solutions method and squeezing technique, and the combination of the comparison principle and the weighted energy method, for the details, one can refer to [5, 20, 29, 30, 32–36, 41, 54]. In our paper, we will apply the third method to investigate the stability of entire solutions, that is, when the initial perturbation around the entire solutions is in a suitable space, we established that the entire solutions of (1) is global exponentially stable.

We end this part with a series of notations which will be used frequently in proving the stability of entire solutions. Letting \( I \) be an interval, especially \( I = \mathbb{R} \), \( L^2(\mathbb{R}) \) is the space of the square integrable function on \( I \), and \( H^k(I)(k \geq 0) \) is the Sobolev space of the \( L^2 \)-function \( f(x) \) defined on \( I \) whose derivatives \( \frac{d^i}{dx^i} f \) for \( i = 1, ..., k \) also belong to \( L^2(I) \). \( L^2_w(I) \) represents the weighted \( L^2 \)-space with the weighted function \( w(x) > 0 \) and its norm is defined by

\[
\|f(x)\|_{L^2_w} = \left( \int_I w(x)f^2(x)dx \right)^{\frac{1}{2}}.
\]

\( H^k_w(I) \) is the weighted Sobolev space with the norm

\[
\|f(x)\|_{H^k_w} = \left( \sum_{i=0}^{k} \int_I w(x)\left| \frac{d^i}{dx^i} f(x) \right|^2 \, dx \right)^{\frac{1}{2}}.
\]

Letting \( T > 0 \) and \( \mathcal{B} \) is a space, we denote by \( C^0([0, T]; \mathcal{B}) \) as the space of the \( \mathcal{B} \)-valued continuous function on \([0, T] \), and \( L^2([0, T]; \mathcal{B}) \) as the space of the \( \mathcal{B} \)-valued \( L^2 \)-function on \([0, T] \). The corresponding spaces of the \( \mathcal{B} \)-valued function on \([0, \infty) \) are defined similarly.

Set \( -L := \min_{0 \leq s \leq 1} f'(s) \) and define

\[
\mathcal{W}(\eta) := \frac{1}{2} - \int_{\mathbb{R}} J(y)e^{-\eta y}dy + 2L, \quad \eta \in \mathbb{R}.
\]

(5)

It is obvious that \( \mathcal{W}(0) > 0 \) if \( L > 1/4 \). A bistable nonlinearity function that satisfies \( L > 1/4 \) is \( f(s) = s(s-a)(1-s) \) with \( 0 < a < 1/2 \). By the continuity and
Therefore, for any \( \psi \) where \( \psi \in \mathbb{R} \), after, a traveling wave solution of equation (1) refers to a pair \((c, \phi)\), where \( \phi = \phi(\xi) \) is a function on \( \mathbb{R} \) and \( c \) is a constant, such that \( u(x, t) := \phi(x + ct) = \phi(\xi) \) is a solution of the equation

\[
\begin{aligned}
&c \phi' = J \ast \phi - \phi + f(\phi), \\
&\phi(\infty) = 0, \quad \phi(-\infty) = 1.
\end{aligned}
\] (7)

We call \( c \) the traveling wave speed and \( \phi \) the profile of such a traveling wave solution. By virtue of assumption (F), it is easy to get \( c > 0 \).

Besides, Carr et al. [4] considered the monostable case for (1) and proved that if \( J \) satisfies (J) and \( f(0) = f(1) = 0, f(s) > 0 \) in \((0, 1)\) and \( f'(s) \leq f'(0) \) for \( s \in (0, 1) \), then (1) admits a unique (up to translation) and an increasing traveling wave solution \( \varphi(x + ct) \) satisfying

\[
\begin{aligned}
&\tilde{c} \varphi' = J \ast \varphi - \varphi + f(\varphi), \\
&\varphi(-\infty) = 0, \quad \varphi(\infty) = 1,
\end{aligned}
\]
for any \( \tilde{c} \geq \overline{c}_* \), where

\[
\overline{c}_* := \min \left\{ \lambda > 0 : \frac{\int_{\mathbb{R}} J(y)e^{\lambda y} dy - 1 + f'(0)}{\lambda} > 0 \right\}. \tag{8}
\]

Therefore, for any \( v_2 \geq \overline{c}_* \), (1) admits a unique (up to translation) and an increasing traveling wave solution \( \psi_2 := \psi_2(x + v_2 t) \) satisfying

\[
\begin{aligned}
v_2 \psi_2' &= J \ast \psi_2 - \psi_2 + f(\psi_2), \\
\psi_2(-\infty) &= a, \quad \psi_2(\infty) = 1.
\end{aligned}
\]

where

\[
\overline{c}_* := \min \left\{ \lambda > 0 : \frac{\int_{\mathbb{R}} J(y)e^{\lambda y} dy - 1 + f'(a)}{\lambda} > 0 \right\}. \tag{9}
\]

Next let us consider (1) on [0, a]. By taking \( v(x, t) = a - u(x, t) \), (1) reduces to

\[
v_1(x, t) = (J \ast v)(x, t) - v(x, t) + g(v(x, t)), \quad x \in \mathbb{R}, t \in \mathbb{R}, \tag{10}
\]

where \( g(v) = -f(a - v) \). Then from the assumption (F), we get \( g(0) = g(a) = 0, g(v) > 0 \) in \( v \in (0, a) \) and \( g'(0) = f'(a) \geq g'(v) \) for all \( v \in (0, a) \). Thus for any \( c_0 \geq \overline{c}_* \), equation (10) admits an unique traveling wave solution \( \varphi_1(x + c_0 t) \), which satisfies \( \varphi_1' > 0 \) and

\[
\begin{aligned}
c_0 \varphi_1' &= J \ast \varphi_1 - \varphi_1 + g(\varphi_1), \\
\varphi_1(-\infty) &= 0, \quad \varphi_1(\infty) = a.
\end{aligned}
\]
where \( c_* \) is given by (9). Letting \( v_1 := -c_0 \) and \( \psi_1(x + v_1 t) := a - \varphi_1(-x + c_0 t) \). Then by simple calculations, we get that \( \psi_1(x + v_1 t) \) is an increasing traveling wave solution to (1) with \( v_1 \leq -c_* \) and satisfying
\[
\begin{align*}
v_1 \psi_1' &= J * \psi_1 - \psi_1 + f(\psi_1), \\
\psi_1(-\infty) &= 0, \quad \psi_1(\infty) = a.
\end{align*}
\]
(11)

The remaining part of this paper is organized as follows. In Section 2, by super- and sub-solutions and comparison principle, we establish the existence of entire solutions originating from three traveling wave solutions for equation (1). In Section 3, the existence result of entire solutions originating from four traveling wave solutions is firstly studied by the same method as in Section 2. Further, by using careful analysis on the sign of the wave speed of the traveling fronts, the nonexistence result of entire solutions originating from \( k \) traveling wave solutions if \( k \geq 5 \) is obtained in Section 3. In the last Section 4, we consider the global exponentially stability of entire solutions established in Sections 2 and 3 in weighted space.

2. Existence of entire solutions originating from 3 traveling wave solutions

In this section, we establish the existence results of entire solutions which originate from 3 traveling wave solutions. First, we state some preparing works, which will be used frequently. Then by constructing appropriate super- and sub-solutions and applying comparison principle, we get the desired results.

2.1. I-type entire solutions originating from 3 waves

2.1.1. Preparing works.

Consider the following auxiliary function (c.f. [7, 8]):
\[
Q(y, z, w) := z + (1 - z) \frac{(1 - y)z(w - a) + y(a - z)(1 - w)}{(1 - y)z(1 - a) + (a - z)(1 - w)},
\]
where \( (y, z, w) \in [0, 1] \times [0, a] \times [a, 1] \setminus \{(1, a, w)\mid a \leq w \leq 1\} \cup \{(1, z, 1)\mid 0 \leq z \leq a\} \cup \{(y, 0, 1)\mid 0 \leq y \leq 1\} \). By simple calculations, the following assertions hold:
\[
\begin{align*}
Q_y(y, z, w) &= \frac{y + (1 - y)z(1 - y)z(1 - a)(1 - w) + y(a - z)(1 - w)}{[(1 - y)z(1 - a) + (a - z)(1 - w)]^2}, \\
Q_z(y, z, w) &= \frac{a(1 - a)(1 - y)(1 - w)(w - y)}{[(1 - y)z(1 - a) + (a - z)(1 - w)]^2}, \\
Q_w(y, z, w) &= \frac{a(1 - a)(1 - y)^2 z(1 - z)}{[(1 - y)z(1 - a) + (a - z)(1 - w)]^2}, \\
Q_{yy}(y, z, w) &= \frac{2a(1 - a)(1 - y)z(a - z)(1 - z)(1 - w)^2}{[(1 - y)z(1 - a) + (a - z)(1 - w)]^3}, \\
Q_{zz}(y, z, w) &= -\frac{a(1 - a)(1 - w)^2 [(y - w)z + a(1 - 2y - z + w + yz)]}{[(1 - y)z(1 - a) + (a - z)(1 - w)]^3}, \\
Q_{yw}(y, z, w) &= -\frac{2a(1 - a)(1 - y)z(a - z)(1 - z)(1 - w)^2}{[(1 - y)z(1 - a) + (a - z)(1 - w)]^3}, \\
Q_{zz}(y, z, w) &= -\frac{2a(1 - a)(1 - y)(1 - w)(w - y)[w - a - y(1 - a)]}{[(1 - y)z(1 - a) + (a - z)(1 - w)]^3},
\end{align*}
\]
\[ Q_{zw}(y, z, w) = -\frac{a(1 - a)(1 - y)^2((w - y)z + a(-1 + w + z - 2wz + yz))}{[(1 - y)z(1 - a) + (a - z)(1 - w)]^3}, \]
\[ Q_{ww}(y, z, w) = \frac{2a(1 - a)(1 - y)^2z(a - z)(1 - z)}{[(1 - y)z(1 - a) + (a - z)(1 - w)]^3}. \]

Then we have following results, since the proof is standard, the details are omitted.

**Lemma 2.1.** For the above defined function \( Q(y, z, w) \), there hold:

(i) There exist functions \( Q_1, Q_2, Q_3 \) such that

\[ Q_y(y, z, w) = (a - z)(1 - w)Q_1(y, z, w), \]
\[ Q_z(y, z, w) = (1 - y)(1 - w)Q_2(y, z, w), \]
\[ Q_w(y, z, w) = (1 - y)zQ_3(y, z, w). \]

(ii) There exist functions \( R_j, j = 1, 2, ..., 16 \), such that

\[ Q_{yy}(y, z, w) = zR_1(y, z, w) = (a - z)R_2(y, z, w) = (1 - w)R_3(y, z, w), \]
\[ Q_{zz}(y, z, w) = (1 - y)R_4(y, z, w) = (1 - w)R_5(y, z, w), \]
\[ Q_{ww}(y, z, w) = (a - z)R_6(y, z, w), \]
\[ Q_{yz}(y, z, w) = (1 - y)R_7(y, z, w), \]
\[ Q_{zw}(y, z, w) = (1 - y)R_8(y, z, w), \]
\[ Q_{yw}(y, z, w) = (1 - y)R_9(y, z, w) = zR_{10}(y, z, w) = (a - z)R_{11}(y, z, w), \]
\[ Q_{yw}(y, z, w) = (1 - y)R_{12}(y, z, w), \]
\[ Q_{yw}(y, z, w) = (1 - y)R_{13}(y, z, w) = zR_{14}(y, z, w) = (a - z)R_{15}(y, z, w) = (1 - w)R_{16}(y, z, w). \]

Letting \( c_1 := -c, c_2 := v_1, c_3 := v_2, \phi_i := \phi_i(x + c_i t), \) where \( \phi_2(z) = \psi_1(z), \phi_3(z) = \psi_2(z) \) and \( \phi_1(z) = \phi(-z) \), then

\[
\begin{align*}
&c_i \phi_i' = J * \phi_i - \phi_i + f(\phi_i), \\
&\phi_i(-\infty) = \alpha_i, \quad \phi_i(\infty) = \omega_i,
\end{align*}
\]

where \( (\alpha_1, \omega_1, \alpha_2, \omega_2, \alpha_3, \omega_3) = (1, 0, 0, a, a, 1) \). Here the prime denotes the derivative with respect to \( z \). In this subsection, we always assume

\[ \phi_1(0) = \phi_2(0) = \frac{a}{2}, \quad \phi_3(0) = \frac{1 + a}{2}. \]

By a similar argument as [7, Lemma 2.3], there holds

**Lemma 2.2.** For \( p_2 \leq p_1 \leq -\sigma < 0 \), there exist positive constants \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) such that

\[ Q_y(\phi_1(x - p_1), \phi_2(x + p_1), \phi_3(x + p_2)) \geq \varepsilon_1, \quad \forall x \leq -p_1, \]
\[ Q_z(\phi_1(x - p_1), \phi_2(x + p_1), \phi_3(x + p_2)) \geq \varepsilon_2, \quad \forall p_1 \leq x \leq -p_2, \]
\[ Q_w(\phi_1(x - p_1), \phi_2(x + p_1), \phi_3(x + p_2)) \geq \varepsilon_3, \quad \forall x \geq -p_1. \]

From [42] and [4], the following statements hold.

**Lemma 2.3.** Assume (J) and (F) hold, then there exist positive constants \( R_0, C_0, \tau_1, \tau_2, \delta, \delta_1 \) such that

(i) For \( x \leq R_0, \)

\[ |\phi'_1(x)|, |\phi'_2(x)|, |\phi'_3(x)| \leq C_0e^{\tau_1 x}, \]

(ii) For \( x \geq -R_0, \)

\[ |\phi'_1(x)|, |\phi'_2(x)|, |\phi'_3(x)| \leq C_0e^{\tau_2 x}, \]

(iii) For \( x \geq -R_0, \)

\[ |\phi''_1(x)|, |\phi''_2(x)|, |\phi''_3(x)| \leq C_0e^{\delta_1 x}. \]
It is easily check that (12) has traveling wave solutions as follows:

**Proof.**

\[ \delta \leq \frac{|\phi'_{1}(x)|}{1 - \phi_{1}(x)} \frac{|\phi'_{2}(x)|}{\phi_{2}(x)} \frac{|\phi'_{3}(x)|}{a - \phi_{3}(x)} \leq \delta_{1}. \]

(ii) For \( x \geq -R_{0} \),

\[ |\phi'_{1}(x)|, |\phi'_{2}(x)|, |\phi'_{3}(x)| \leq C_{0}e^{-\gamma_{x}}, \]

\[ \delta \leq \frac{|\phi'_{1}(x)|}{\phi_{1}(x)}, \frac{|\phi'_{2}(x)|}{\phi_{2}(x)}, \frac{|\phi'_{3}(x)|}{a - \phi_{3}(x)} \leq \delta_{1}. \]

Moreover, for some constant \( M > 0 \), there hold

\[ \frac{1 - \phi_{1}(x + y)}{1 - \phi_{1}(x)}, \frac{\phi_{2}(x + y)}{\phi_{2}(x)}, \frac{a - \phi_{3}(x + y)}{a - \phi_{3}(x)} \leq M, \quad \forall x \in \mathbb{R}, \quad y \in \text{supp}(J). \]

\[ \frac{\phi_{1}(x + y)}{\phi_{1}(x)}, \frac{\phi_{2}(x + y)}{\phi_{2}(x)}, \frac{1 - \phi_{3}(x + y)}{1 - \phi_{3}(x)} \leq M, \quad \forall x \in \mathbb{R}, \quad y \in \text{supp}(J). \]

### 2.1.2. Construction of super- and sub-solutions.

Let \( \xi := x + \tau_{t}, \bar{c} := (c_{1} + c_{2})/2 \), then (1) becomes

\[ U_{t} + \tau U_{\xi} = J * U - U + f(U). \]  

(12)

It is easily check that (12) has traveling wave solutions as follows:

\[ \phi_{1}(\xi - s_{1}t), \phi_{2}(\xi + s_{1}t), \phi_{3}(\xi + s_{2}t) \]

where \( s_{1} := (c_{2} - c_{1})/2 > 0 \), \( s_{2} := [2c_{3} - (c_{1} + c_{2})]/2 > 0 \), and \( s_{2} - s_{1} > 0 \).

Denote

\[ A(\phi_{1}, \phi_{2}, \phi_{3}) := -Q_{y}(\phi_{1}(\xi - p_{1}), \phi_{2}(\xi + p_{1}), \phi_{3}(\xi + p_{2}))\phi'_{1}(\xi - p_{1}) + Q_{z}(\phi_{1}(\xi - p_{1}), \phi_{2}(\xi + p_{1}), \phi_{3}(\xi + p_{2}))\phi'_{3}(\xi + p_{2}), \phi_{3}(\xi + p_{2})\phi'_{3}(\xi + p_{2}) \]

Similar to [8, Lemma 3.3], we can obtain the following result.

**Lemma 2.4.** Assume (J) and (F) hold, then for any \( p_{2} \leq p_{1} \leq -\sigma < 0 \), it holds

\[ 2A(\phi_{1}, \phi_{2}, \phi_{3}) \geq [\begin{align*}
|Q_{y}(\phi_{1}(\xi - p_{1}), \phi_{2}(\xi + p_{1}), \phi_{3}(\xi + p_{2}))\phi'_{1}(\xi - p_{1})|, & \xi \leq p_{1}, \\
|Q_{y}(\phi_{1}(\xi - p_{1}), \phi_{2}(\xi + p_{1}), \phi_{3}(\xi + p_{2}))\phi'_{1}(\xi - p_{1})| & + [Q_{z}(\phi_{1}(\xi - p_{1}), \phi_{2}(\xi + p_{1}), \phi_{3}(\xi + p_{2}))\phi'_{2}(\xi + p_{1})], & p_{1} \leq \xi \leq -p_{1}, \\
|Q_{z}(\phi_{1}(\xi - p_{1}), \phi_{2}(\xi + p_{1}), \phi_{3}(\xi + p_{2}))\phi'_{2}(\xi + p_{1})| & + [Q_{w}(\phi_{1}(\xi - p_{1}), \phi_{2}(\xi + p_{1}), \phi_{3}(\xi + p_{2}))\phi'_{3}(\xi + p_{2})], & -p_{1} \leq \xi \leq -p_{2}, \\
|Q_{w}(\phi_{1}(\xi - p_{1}), \phi_{2}(\xi + p_{1}), \phi_{3}(\xi + p_{2}))\phi'_{3}(\xi + p_{2})|, & \xi \geq -p_{2},
\end{align*}] \]

where \( \sigma \) is a sufficiently large constant.

**Proof.** From above calculations, there hold \( Q_{y} > 0, Q_{w} > 0 \). Since \( \phi_{1}(0) = c_{0}(0) = a/2 \) and \( \phi_{3}(0) = (1 + a)/2 \), it follows from \( \phi'_{3} < 0, \phi'_{3} > 0 \) and \( \phi_{1}(0) < \phi_{3}(0) \) that there exists \( p_{0} < p_{1} \) such that \( \phi_{1}(p_{0} - p_{1}) = \phi_{3}(p_{0} + p_{2}) \). Then

\[ \phi_{1}(\xi - p_{1}) > \phi_{1}(p_{0} - p_{1}) = \phi_{3}(p_{0} + p_{2}) > \phi_{3}(\xi + p_{2}), \quad \forall \xi < p_{0}, \]

and

\[ \phi_{1}(\xi - p_{1}) < \phi_{3}(\xi + p_{2}), \quad \forall p_{0} \leq \xi < p_{1}. \]
When $\xi < p_0$, we have $Q_z < 0$, by Lemma 2.3 and (13), then for $p_1$ sufficiently small,

$$A(\phi_1, \phi_2, \phi_3) - \frac{1}{2}Q_y|\phi'_1|$$

$$\geq \frac{1}{2}Q_y|\phi'_1| + Q_z|\phi'_2|$$

$$= \frac{1}{2} a(1 - \phi_2)(a - \phi_2)(1 - \phi_3)^2 \left(1 - \phi_1\right)$$

$$+ \frac{a(1 - a)(1 - \phi_1)|\phi'_2|(1 - \phi_1)}{((1 - \phi_1)\phi_2(1 - a) + a - \phi_2)(1 - \phi_3)^2} (\phi_3 - \phi_1)$$

$$\geq \frac{a(1 - \phi_1)(1 - \phi_3)}{((1 - \phi_1)\phi_2(1 - a) + a - \phi_2)(1 - \phi_3)^2} \left[\frac{\delta}{2} (1 - \phi_3)(a - \phi_2)(1 - \phi_2) + (1 - a)(\phi_3 - \phi_1)C_0e^{\tau_1p_1}\right]$$

$$\geq \frac{a(1 - \phi_1)(1 - \phi_3)}{((1 - \phi_1)\phi_2(1 - a) + a - \phi_2)(1 - \phi_3)^2} \left[\frac{\delta}{2} (1 - \phi_3)(a - \phi_2)(1 - \phi_2) - (1 - a)^2C_0e^{\tau_1p_1}\right]$$

$$= \frac{a(1 - \phi_1)(1 - \phi_3)}{((1 - \phi_1)\phi_2(1 - a) + a - \phi_2)(1 - \phi_3)^2} \left[\frac{\delta}{2} (1 - \phi_3)(0)(a - \phi_2(0))(1 - \phi_2(0)) - (1 - a)^2C_0e^{\tau_1p_1}\right]$$

$$\geq 0.$$
where $\theta \in [0, 1]$, $\zeta \in \text{supp}(J)$, and $p_2 \leq p_1 \leq -\sigma < 0$. Then there is $M_2 > 0$ such that

$$
\left| G_1 \left( \tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3; \theta \right) \right| \leq \begin{cases} 
M_2\left( |\phi'_2(\xi + p_1 + \zeta)| + |\phi'_3(\xi + p_2 + \zeta)| \right), & \forall \xi \leq 0, \\
M_2\left( |\phi'_1(\xi - p_1 + \zeta)| + |\phi'_3(\xi + p_2 + \zeta)| \right), & \forall \xi \leq -\frac{p_1 + p_2}{2}, \\
M_2\left( |\phi'_1(\xi - p_1 + \zeta)| + |\phi'_2(\xi + p_1 + \zeta)| \right), & \forall \xi \geq -\frac{p_1 + p_2}{2},
\end{cases}
$$

(15)
i = 1, 2, ..., 6 and $\zeta \in \text{supp}(J)$, where $\sigma$ is given by Lemma 2.4.

Proof. We only prove (15) for $i = 1$, since the other cases can be treated similarly.

When $\xi \leq p_1$, by Lemmas 2.1, 2.4, 2.2 and 2.3, then

$$
\left| G_1 \left( \tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3; \theta \right) \right| \leq \frac{|\tilde{\phi}_2(1)R_1(\theta \tilde{\phi}_1(1) + (1 - \theta)\tilde{\phi}_1(0), \tilde{\phi}_2(1), \tilde{\phi}_3(1))(\tilde{\phi}_1(1) - \tilde{\phi}_1(0))^2|}{A(\phi_1, \phi_2, \phi_3)}
$$

(16)

$$
\leq 2C\phi_2(\xi + p_1 - \zeta)|\phi'_1(\xi - p_1 - \theta_1 \xi)|
$$

$$
\leq \frac{C_\phi_2(\xi + p_1 - \zeta)}{Q_y(\phi_1, \phi_2, \phi_3)|\phi'_1(\xi - p_1)| + Q_x(\phi_1, \phi_2, \phi_3)|\phi'_2(\xi + p_1)|}
$$

$$
\leq 2C\phi_2(\xi + p_1 - \zeta)
$$

(17)

When $p_1 \leq \xi \leq 0$, by Lemmas 2.1, 2.4, 2.2 and 2.3, then

$$
\left| G_1 \left( \tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3; \theta \right) \right| \leq \frac{|\tilde{\phi}_2(1)R_1(\theta \tilde{\phi}_1(1) + (1 - \theta)\tilde{\phi}_1(0), \tilde{\phi}_2(1), \tilde{\phi}_3(1))(\tilde{\phi}_1(1) - \tilde{\phi}_1(0))^2|}{A(\phi_1, \phi_2, \phi_3)}
$$

(19)

$$
\left| G_1 \left( \tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3; \theta \right) \right| \leq \frac{C\phi_2(\xi + p_1 - \zeta)|\phi'_1(\xi - p_1 - \theta_2 \zeta)|}{|\phi_1(\xi - p_1 - \theta_2 \zeta)| + |\phi_1(\xi - p_1)|}
$$

(18)
\[
\leq 2C |\phi_1'(\xi - p_1 - \theta_{35})| \cdot \frac{1}{Q_z(\phi_1, \phi_2, \phi_3)} \frac{|\phi_2(\xi + p_1 - \varsigma)|}{|\phi_2'(\xi + p_1)|}
\leq \frac{2C}{\varepsilon_2 \delta} |\phi_1'(\xi - p_1 - \theta_{35})| \phi_1 \cdot M \cdot \frac{1}{\delta} \leq M_6 |\phi_1'(\xi - p_1 - \theta_{35})|. \tag{20}
\]

When \(-p_1 \leq \xi \leq -\frac{p_1 + p_2}{2}\), by Lemmas 2.1, 2.4, 2.2 and 2.3, then
\[
\left| G_1 \left( \tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3; \theta \right) \right| \frac{A(\phi_1, \phi_2, \phi_3)}{A(\phi_1, \phi_2, \phi_3)}
= \left| (a - \tilde{\phi}_2(1)) R_2(\theta \tilde{\phi}_1(1) + (1 - \theta) \tilde{\phi}_1(0), \tilde{\phi}_2(1), \tilde{\phi}_3(1))(\tilde{\phi}_1(1) - \tilde{\phi}_1(0))^2 \right|
\leq 2 \frac{C |a - \phi_2(\xi + p_1 - \varsigma)| |\phi_1'(\xi - p_1 - \theta_{45})|}{Q_z(\phi_1, \phi_2, \phi_3)|\phi_2'(\xi + p_1)| + Q_w(\phi_1, \phi_2, \phi_3)|\phi_3'(\xi + p_2)|}
\leq 2 \frac{C |\phi_1'(\xi - p_1 - \theta_{45})|}{\varepsilon_2} \frac{1}{Q_z(\phi_1, \phi_2, \phi_3)} \frac{|a - \phi_2(\xi + p_1 - \varsigma)|}{|\phi_2'(\xi + p_1)|}
\leq 2 \frac{C |\phi_1'(\xi - p_1 - \theta_{45})|}{\varepsilon_2} \frac{|a - \phi_2(\xi + p_1)|}{|\phi_2'(\xi + p_1)|} \leq \frac{2C}{\varepsilon_2} |\phi_1'(\xi - p_1 - \theta_{45})| \cdot \frac{1}{\delta} \cdot M \leq M_7 |\phi_1'(\xi - p_1 - \theta_{45})|. \tag{21}
\]

When \(-\frac{p_1 + p_2}{2} \leq \xi \leq -p_2\), by Lemmas 2.1, 2.4, 2.2 and 2.3, then
\[
\left| G_1 \left( \tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3; \theta \right) \right| \frac{A(\phi_1, \phi_2, \phi_3)}{A(\phi_1, \phi_2, \phi_3)}
= \left| (a - \tilde{\phi}_2(1)) R_2(\theta \tilde{\phi}_1(1) + (1 - \theta) \tilde{\phi}_1(0), \tilde{\phi}_2(1), \tilde{\phi}_3(1))(\tilde{\phi}_1(1) - \tilde{\phi}_1(0))^2 \right|
\leq 2 \frac{C |a - \phi_2(\xi + p_1 - \varsigma)| |\phi_1'(\xi - p_1 - \theta_{55})|}{Q_z(\phi_1, \phi_2, \phi_3)|\phi_2'(\xi + p_1)| + Q_w(\phi_1, \phi_2, \phi_3)|\phi_3'(\xi + p_2)|}
\leq 2 \frac{C |\phi_1'(\xi - p_1 - \theta_{55})|}{\varepsilon_2} \frac{1}{Q_z(\phi_1, \phi_2, \phi_3)} \frac{|a - \phi_2(\xi + p_1 - \varsigma)|}{|\phi_2'(\xi + p_1)|}
\leq 2 \frac{C |\phi_1'(\xi - p_1 - \theta_{55})|}{\varepsilon_2} \frac{|a - \phi_2(\xi + p_1)|}{|\phi_2'(\xi + p_1)|} \leq \frac{2C}{\varepsilon_2} |\phi_1'(\xi - p_1 - \theta_{55})| \cdot \frac{1}{\delta} \cdot M \leq M_8 |\phi_1'(\xi - p_1 - \theta_{55})|. \tag{22}
\]

When \(\xi \geq -p_2\), by Lemmas 2.1, 2.4, 2.2 and 2.3, then
\[
\left| G_1 \left( \tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3; \theta \right) \right| \frac{A(\phi_1, \phi_2, \phi_3)}{A(\phi_1, \phi_2, \phi_3)}
= \left| (1 - \tilde{\phi}_3(1)) R_3(\theta \tilde{\phi}_1(1) + (1 - \theta) \tilde{\phi}_1(0), \tilde{\phi}_2(1), \tilde{\phi}_3(1))(\tilde{\phi}_1(1) - \tilde{\phi}_1(0))^2 \right|
\leq 2 \frac{C |\phi_1'(\xi - p_1 - \theta_{65})|}{Q_w(\phi_1, \phi_2, \phi_3)|\phi_3'(\xi + p_2)|} \frac{1 - \phi_3(\xi + p_2 - \varsigma)}{|\phi_3(\xi + p_2)|}
\]
Some calculations yield that
Thus for some functions $F$ for $p$ \( \theta \)
where \( \begin{align}
\text{Lemma 2.6.} \end{align} \)
written explicitly as

Let
\[
F(\phi_1, \phi_2, \phi_3) := f(Q(\phi_1, \phi_2, \phi_3)) - Q_y f(\phi_1) - Q_z f(\phi_2) - Q_w f(\phi_3).
\]

Some calculations yield that
\[
F(1, \phi_2, \phi_3) = F(\phi_1, 0, \phi_3) = F(\phi_1, a, \phi_3) = F(\phi_1, \phi_2, 1) = F(0, \phi_2, a) = 0.
\]

Thus for some functions $F_{ij}, i, j = 1, 2$, we have following expression
\[
F(\phi_1, \phi_2, \phi_3) = (1 - \phi_1)\phi_2[\phi_1F_{11} + (\phi_3 - a)F_{12}],
\]
\[
F(\phi_1, \phi_2, \phi_3) = (1 - \phi_3)(\phi_2)[\phi_1F_{21} + (\phi_3 - a)F_{22}].
\]

Similar to Lemma 2.5, we have following result.

**Lemma 2.6.** Assume (J) and (F) hold, then there exists $M_3 > 0$ such that
\[
\left| \frac{F(\phi_1, \phi_2, \phi_3)}{A(\phi_1, \phi_2, \phi_3)} \right| \leq \begin{cases} 
M_3(|\phi'_1(\xi)| + |\phi'_2(\xi)|), & \forall \xi \leq 0, \\
M_3(|\phi'_1(\xi)| + |\phi'_3(\xi)|), & \forall 0 \leq \xi \leq -\frac{p_1 + p_2}{2}, \\
M_3(|\phi'_1(\xi)| + |\phi'_3(\xi)|), & \forall \xi \geq -\frac{p_1 + p_2}{2}
\end{cases}
\]
for $p_2 \leq p_1 < -\sigma < 0$ with $\sigma$ is given by Lemma 2.4.

Consider the following initial value problems (c.f. [7,8]):
\[
p_1' = s_1 + Le^{\kappa p_1}, \quad -\infty < t < 0, \quad p_1(0) = p_0; \quad (24)
p_2' = s_2 + Le^{\kappa p_2}, \quad -\infty < t < 0, \quad p_2(0) = p_0; \quad (25)
r_1' = s_1 - Le^{\kappa p_1}, \quad -\infty < t < 0, \quad r_1(0) = r_0; \quad (26)
r_2' = s_2 - Le^{\kappa p_2}, \quad -\infty < t < 0, \quad r_2(0) = r_0; \quad (27)
\]
where $L$ will be determined in the later and
\[
\kappa := \min\{\tau_1, \tau_2, (s_2 - s_1)\tau_1/4s_1\}, \quad (28)
\]
where $\tau_1$ and $\tau_2$ are given in Lemma 2.3. In fact, the solutions of (24)-(27) can be written explicitly as
\[
p_1(t) = s_1 t - \frac{1}{\kappa} \log \left( e^{-\kappa p_0} + \frac{L(1 - e^{\kappa s_1 t})}{s_1} \right),
p_2(t) = s_2 t - \frac{1}{\kappa} \log \left( e^{-\kappa p_0} + \frac{L(1 - e^{\kappa s_1 t})}{s_1} \right),
r_1(t) = s_1 t - \frac{1}{\kappa} \log \left( e^{-\kappa r_0} - \frac{L(1 - e^{\kappa s_1 t})}{s_1} \right),
r_2(t) = s_2 t - \frac{1}{\kappa} \log \left( e^{-\kappa r_0} - \frac{L(1 - e^{\kappa s_1 t})}{s_1} \right).
\]

Now, we take the initial value $p_0$ and $r_0$ satisfying
\[
r_0 < -\frac{1}{\kappa} \log \left( \frac{2L}{s_1} + e^{\kappa \sigma} \right), \quad p_0 = -\frac{1}{\kappa} \log \left( e^{-\kappa r_0} - \frac{2L}{s_1} \right) < -\sigma,
\]
where \(\sigma\) is given in Lemma 2.4. Then we have
\[
\lim_{t \to -\infty} (p_1(t) - r_1(t)) = \lim_{t \to -\infty} (p_2(t) - r_2(t)) = 0. \tag{29}
\]
\[
\lim_{t \to -\infty} (p_1(t) - s_1 t) = \lim_{t \to -\infty} (p_2(t) - s_2 t) = -\frac{1}{\kappa} \log \left( e^{-\kappa p_0} + \frac{L}{s_1} \right). \tag{30}
\]
\[
\lim_{t \to -\infty} (r_1(t) - s_1 t) = \lim_{t \to -\infty} (r_2(t) - s_2 t) = -\frac{1}{\kappa} \log \left( e^{-\kappa r_0} - \frac{L}{s_1} \right). \tag{31}
\]

\[0 < p_1(t) - r_1(t) = p_2(t) - r_2(t) \leq N e^{\kappa s_1 t}, \quad \forall t \leq 0 \tag{32}\]
and \(p_1(t), p_2(t), r_1(t), r_2(t) \leq -\sigma\) for all \(t \leq 0\).

**Lemma 2.7.** Assume that (J) and (F) hold. Then there exist \(C_4 > 0, t_0 < 0\) such that
\[
\frac{|F(\phi_1, \phi_2, \phi_3)|}{A(\phi_1, \phi_2, \phi_3)} \leq C_4 e^{\kappa p_1(t)}, \quad \forall \xi \in \mathbb{R}, \quad t \leq t_0, i = 1, 2, ..., 6. \tag{33}
\]

**Proof.** By the expression of \(F(\phi_1, \phi_2, \phi_3)\) and \(G_i(\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3)\), we only need to prove that there exist \(C_4 > 0, t_0 < 0\) such that \(|G_i(\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3)| \leq C_4 A(\phi_1, \phi_2, \phi_3) e^{\kappa p_1(t)}\), for any \(\xi \in \mathbb{R}\) and \(t \leq t_0\). Since, by the asymptotic behavior Lemma 2.3, (28) and the fact \(p_2(t) \leq p_1(t)\) for \(t \leq 0\). When \(\xi \leq 0\),
\[
|\phi_2'(\xi p_1(t) + \varsigma) + |\phi_3'(\xi + p_2(t) + \varsigma)| \\
\leq C_0 e^{\tau_1(\xi + p_1(t) + \varsigma)} + C_0 e^{\tau_1(\xi + p_2(t) + \varsigma)} \\
\leq C_1 e^{\tau_1 p_1(t)} + e^{\tau_1 p_2(t)} \leq C_1 e^{\kappa p_1(t)}. \tag{34}
\]

When \(0 \leq \xi \leq -\frac{p_1(t) + p_2(t)}{2}\), by Lemma 2.3 and (28), then
\[
|\phi_1'(\xi - p_1(t) + \varsigma)| + |\phi_3'(\xi + p_2(t) + \varsigma)| \\
\leq C_0 e^{-\tau_2(\xi - p_1(t) + \varsigma)} + C_0 e^{\tau_1(\xi + p_2(t) + \varsigma)} \\
\leq C_2 e^{\kappa p_1(t)} + e^{\frac{(p_2(t) - p_1(t))^2}{2} \tau_1}. \tag{35}
\]

When \(\xi \geq -\frac{p_1(t) + p_2(t)}{2}\), by Lemma 2.3 and (28), then
\[
|\phi_1'(\xi - p_1(t) + \varsigma)| + |\phi_2'(\xi + p_1(t) + \varsigma)| \\
\leq C_0 e^{-\tau_2(\xi - p_1(t) + \varsigma)} + C_0 e^{-\tau_2(\xi + p_1(t) + \varsigma)} \\
\leq C_3 e^{\tau_2 p_1(t)} + e^{\tau_2 p_2(t)} \leq C_3 e^{\kappa p_1(t)}. \tag{36}
\]

By (28), we claim for some \(t_0 < 0\), there holds
\[
\frac{(p_2(t) - p_1(t)) r_1}{2} \leq \kappa p_1(t), \quad \forall t \leq t_0.
\]

We give a simple proof. Since \(\kappa p_1(t) \geq \frac{(t_2 - t_1 t_1 r_1) p_1(t)}{2}, \frac{(p_2 - p_1) r_1}{2} = \frac{(t_2 - t_1) t_1}{2},\) then if we can prove \(p_1(t) \geq 2s_1 t, t \leq t_0\) for some \(t_0 < 0\), then the conclusion hold. Let \(r(t) := r_1(t) - 2s_1 t\). It follows from \(r'(t) = 0, r(-\infty) = \infty, r(0) > 0\) that there exists \(t_0 < 0\) such that \(r(t) > 0\) for \(t \leq t_0\). That is, \(p_1(t) > r_1(t) \geq 2s_1 t, t \leq t_0\).
Hence the conclusion hold. Therefore, from (34)-(36), for the above $t_0$, there exists a constant $C_4 := \max\{C_1, 2C_2, C_3\}$ such that
\[
\left| G_i(\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3) \right| \leq C_4 e^{\kappa \int t_0(t)}, \quad \forall \xi \in \mathbb{R}, \ t \leq t_0.
\]

\[\square\]

Lemma 2.8. Assume (J) and (F) hold, then for any $\xi \in \mathbb{R}, t \leq t_0$, define
\[
U(\xi, t) := Q(\phi_1(\xi - p_1(t)), \phi_2(\xi + p_2(t)), \phi_3(\xi + p_2(t))),
\]
\[
\tilde{U}(\xi, t) := Q(\phi_1(\xi - r_1(t)), \phi_2(\xi + r_1(t)), \phi_3(\xi + r_2(t))).
\]
Then $\tilde{U}(\xi, t)$ and $U(\xi, t)$ is a pair of super- and sub-solution of (12) if $L$ (given in (24)-(27)) is appropriately large. Further, for some $M_4 > 0$, there holds
\[
0 < \tilde{U}(\xi, t) - U(\xi, t) \leq M_4 e^{\kappa \int t}, \quad \forall \xi \in \mathbb{R}, \ t \leq t_0.
\]

Proof. Let
\[
F(U) := U_t + \tau U - J \ast U + U - f(U).
\]
We first prove $F(\tilde{U}) \geq 0$. From the expression of $A(\phi_1, \phi_2, \phi_3), F(\phi_1, \phi_2, \phi_3)$ and $s_1, s_2$, some calculations yield that
\[
F(\tilde{U}) = U_t + \tau U - J \ast \tilde{U} + U - f(\tilde{U})
\]
\[
= Q_y \phi'_1(p'_1 + Q_z \phi'_2 + Q_w \phi'_3 + \frac{c_1 + c_2}{2} Q_y \phi'_1 + Q_z \phi'_2 + Q_w \phi'_3) - J \ast \tilde{U} + U - f(\tilde{U})
\]
\[
= -Q_y \phi'_1(p'_1 - \frac{c_1 + c_2}{2}) + Q_z \phi'_2(p'_2 - \frac{c_1 + c_2}{2}) + Q_w \phi'_3(p'_2 - c_3)
\]
\[
+ Q_y p'_1 + Q_z c_2 p'_2 + Q_w c_3 p'_3 - J \ast \tilde{U} + U - f(\tilde{U})
\]
\[
= -Q_y \phi'_1(p'_1 - s_1) + Q_z \phi'_2(p'_2 - s_1) + Q_w \phi'_3(p'_2 - s_2)
\]
\[
+ f(\phi_1) + f(\phi_2) + f(\phi_3) - J \ast \tilde{U} + U - f(\tilde{U})
\]
\[
= -Q_y \phi'_1(p'_1 - s_1) + Q_z \phi'_2(p'_2 - s_1) + Q_w \phi'_3(p'_2 - s_2)
\]
\[
+ [f(\phi_1) - f(\phi_2) + f(\phi_3) - f(\phi_3)] - J \ast \tilde{U} + U - f(\tilde{U})
\]
\[
= Le^{\kappa \int t} A(\phi_1, \phi_2, \phi_3) - G(\phi_1, \phi_2, \phi_3) - F(\phi_1, \phi_2, \phi_3)
\]
where
\[
G(\phi_1, \phi_2, \phi_3) := J \ast (Q_y(J \ast \phi_1 - \phi_1) - Q_z(J \ast \phi_2 - \phi_2) - Q_w(J \ast \phi_3 - \phi_3).
\]
For convenience, we denote $\tilde{\phi}_1(\theta) := \phi_1(\xi - p_1 - \theta)$, $\tilde{\phi}_2(\theta) := \phi_2(\xi + p_1 - \theta)$, $\tilde{\phi}_3(\theta) := \phi_3(\xi + p_2 - \theta)$. Then by Lemma 2.5, we get
\[
\left| G(\phi_1, \phi_2, \phi_3) \right|
\]
\[
= \left| \int_\mathbb{R} J(\xi) [Q(\tilde{\phi}_1(1), \tilde{\phi}_2(1), \tilde{\phi}_3(1)) - Q(\tilde{\phi}_1(0), \tilde{\phi}_2(0), \tilde{\phi}_3(0))] d\xi - \int_\mathbb{R} J(\xi) [\tilde{\phi}_1(1) - \tilde{\phi}_1(0)] dq - Q_z \int_\mathbb{R} J(\xi) [\tilde{\phi}_2(1) - \tilde{\phi}_2(0)] dq - Q_w \int_\mathbb{R} J(\xi) [\tilde{\phi}_3(1) - \tilde{\phi}_3(0)] dq \right|
\]
\[
\begin{align*}
&= \int_{\mathbb{R}} J(\zeta) Q_y(\theta_1 \tilde{\phi}_1(1) + (1 - \theta_1) \tilde{\phi}_1(0), \tilde{\phi}_2(1), \tilde{\phi}_3(1)) (\tilde{\phi}_1(1) - \tilde{\phi}_1(0)) d\zeta \\
&\quad + \int_{\mathbb{R}} J(\zeta) Q_z(\tilde{\phi}_1(0), \theta_2 \tilde{\phi}_2(1) + (1 - \theta_2) \tilde{\phi}_2(0), \tilde{\phi}_3(1)) (\tilde{\phi}_2(1) - \tilde{\phi}_2(0)) d\zeta \\
&\quad + \int_{\mathbb{R}} J(\zeta) Q_w(\tilde{\phi}_1(0), \tilde{\phi}_2(0), \theta_3 \tilde{\phi}_3(1) + (1 - \theta_3) \tilde{\phi}_3(0)) (\tilde{\phi}_3(1) - \tilde{\phi}_3(0)) d\zeta \\
&\quad - \int_{\mathbb{R}} J(\zeta) Q_y(\tilde{\phi}_1(0), \tilde{\phi}_2(0), \tilde{\phi}_3(0)) (\tilde{\phi}_1(1) - \tilde{\phi}_1(0)) d\zeta \\
&\quad - \int_{\mathbb{R}} J(\zeta) Q_z(\tilde{\phi}_1(0), \tilde{\phi}_2(0), \tilde{\phi}_3(0)) (\tilde{\phi}_2(1) - \tilde{\phi}_2(0)) d\zeta \\
&\quad - \int_{\mathbb{R}} J(\zeta) Q_w(\tilde{\phi}_1(0), \tilde{\phi}_2(0), \tilde{\phi}_3(0)) (\tilde{\phi}_3(1) - \tilde{\phi}_3(0)) d\zeta \\
&= \int_{\mathbb{R}} J(\zeta) Q_{yy}(\theta_1 \theta_4 \tilde{\phi}_1(1) + (1 - \theta_1 \theta_4) \tilde{\phi}_1(0), \tilde{\phi}_2(1), \tilde{\phi}_3(1)) \theta_1 (\tilde{\phi}_1(1) - \tilde{\phi}_1(0))^2 d\zeta \\
&\quad + \int_{\mathbb{R}} J(\zeta) Q_{yz}(\tilde{\phi}_1(0), \theta_5 \tilde{\phi}_2(1) + (1 - \theta_5) \tilde{\phi}_2(0), \tilde{\phi}_3(1)) (\tilde{\phi}_1(1) - \tilde{\phi}_1(0)) (\tilde{\phi}_2(1) - \tilde{\phi}_2(0)) d\zeta \\
&\quad + \int_{\mathbb{R}} J(\zeta) Q_{yw}(\tilde{\phi}_1(0), \tilde{\phi}_2(0), \theta_6 \tilde{\phi}_3(1) + (1 - \theta_6) \tilde{\phi}_3(0)) (\tilde{\phi}_1(1) - \tilde{\phi}_1(0)) d\zeta \\
&\quad + \int_{\mathbb{R}} J(\zeta) Q_{zz}(\tilde{\phi}_1(0), \tilde{\phi}_2(0), \tilde{\phi}_3(1)) (\tilde{\phi}_2(1) - \tilde{\phi}_2(0))^2 d\zeta \\
&\quad + \int_{\mathbb{R}} J(\zeta) Q_{zw}(\tilde{\phi}_1(0), \tilde{\phi}_2(0), \theta_8 \tilde{\phi}_3(1) + (1 - \theta_8) \tilde{\phi}_3(0)) d\zeta \\
&\quad + \int_{\mathbb{R}} J(\zeta) Q_{ww}(\tilde{\phi}_1(0), \tilde{\phi}_2(0), \theta_9 \tilde{\phi}_3(1) + (1 - \theta_9) \tilde{\phi}_3(0)) d\zeta \\
&\quad - \int_{\mathbb{R}} \theta_1 J(\zeta) |G_1(\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3; \theta_1 \theta_4)| d\zeta \\
&\quad + \int_{\mathbb{R}} \theta_2 J(\zeta) |G_2(\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3; \theta_5)| d\zeta \\
&\quad + \int_{\mathbb{R}} \theta_3 \theta_4 J(\zeta) |G_3(\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3; \theta_6)| d\zeta \\
&\quad + \int_{\mathbb{R}} \theta_3 \theta_7 J(\zeta) |G_4(\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3; \theta_8)| d\zeta \\
&\quad + \int_{\mathbb{R}} \theta_3 \theta_9 J(\zeta) |G_5(\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3; \theta_9)| d\zeta \\
&\quad \leq \sum_{i=1}^{6} |G_i(\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3; \theta_i)|,
\end{align*}
\]

where \( \theta_1 = \theta_1 \theta_4, \theta_2 = \theta_5, \theta_3 = \theta_6, \theta_4 = \theta_2 \theta_7, \theta_5 = \theta_8, \theta_6 = \theta_3 \theta_9 \) with \( \theta_i \in [0, 1], i = 1, \ldots, 9. \)

Thus by Lemma 2.7, if we choose \( L \geq 7C_4 \), we have

\[
\mathcal{F}(\mathcal{U}) = Le^{p_1(t)} A(\phi_1, \phi_2, \phi_3) - F(\phi_1, \phi_2, \phi_3) - G(\phi_1, \phi_2, \phi_3)
\geq A(\phi_1, \phi_2, \phi_3) \left[ Le^{p_1(t)} - \sqrt{\frac{|F(\phi_1, \phi_2, \phi_3)|}{A(\phi_1, \phi_2, \phi_3)}} \right]
\geq A(\phi_1, \phi_2, \phi_3) \left[ Le^{p_1(t)} - C_4 e^{p_1(t)} - \sum_{i=1}^{6} C_4 e^{p_1(t)} \right]
\]
Existence of entire solutions.

2.1.3. Hence we complete the proof.

where

Also, there holds

\[ T \rightarrow -\infty \]

\[ F(U) = -Qy\phi'_1(r'_1 - s_1) + Qz\phi'_2(r'_2 - s_2) - F(\phi_1, \phi_2, \phi_3) \]

\[ = -Qy\phi'_1(-Le^{\kappa r_1(t)}) + Qz\phi'_2(-Le^{\kappa r_1(t)}) \]

\[ = -A(\phi_1, \phi_2, \phi_3) \left[ Le^{\kappa r_1(t)} - \frac{F(\phi_1, \phi_2, \phi_3) + G(\phi_1, \phi_2, \phi_3)}{-A(\phi_1, \phi_2, \phi_3)} \right] \]

\[ \leq -A(\phi_1, \phi_2, \phi_3) \left[ Le^{\kappa r_1(t)} - 7C' e^{\kappa r_1(t)} \right] \leq 0 \]

if \( L \geq 7C' \) with some positive constant \( C' \).

Next we prove (37), by the fact \( p_2(t) - r_2(t) = p_1(t) - r_1(t) \)

\[ U(\xi, t) - U_0(\xi, t) \]

\[ = Q(\phi_1(\xi - p_1(t)), \phi_2(\xi + p_1(t)), \phi_3(\xi + p_2(t))) - Q(\phi_1(\xi - r_1(t)), \phi_2(\xi + r_1(t)), \phi_3(\xi + r_2(t))) \]

\[ = \int_0^1 A(\phi_1(\xi - \theta p_1(t) - (1 - \theta)r_1(t)), \phi_3(\xi + \theta p_1(t) + (1 - \theta)r_1(t)), \phi_3(\xi + \theta p_2(t) + (1 - \theta)r_2(t)))d\theta \cdot (p_1(t) - r_1(t)) > 0. \]

Further, by (32), it is obvious there exists some \( M_4 > 0 \) such that

\[ \sup_{\xi \in \mathbb{R}}(U(\xi, t) - U_0(\xi, t)) \leq M_4 e^{\kappa_4 t}, \quad \forall t \leq t_0. \]

\[ \square \]

2.1.3. Existence of entire solutions.

In this subsection, we state the main existence results and then prove them.

**Theorem 2.9.** Assume (J) and (F) and \( c_1 < c_2 \). Then (12) admits a entire solution \( u(x,t) \) such that

\[ \bar{U}(x + \bar{c}t, t) \leq u(x,t) \leq \bar{U}(x + \bar{c}t, t), \quad \forall x \in \mathbb{R}, t \leq t_0. \]

Also, there holds

\[ \lim_{t \rightarrow -\infty} \left\{ \sup_{x \leq \omega_1(t)} |u(x, t) - \phi_1(x + c_1 t - \theta)| + \sup_{\omega_1(t) \leq x \leq \omega_2(t)} |u(x, t) - \phi_2(x + c_2 t + \theta)| + \sup_{x \geq \omega_2(t)} |u(x, t) - \phi_3(x + c_3 t + \theta)| \right\} = 0, \quad (38) \]

where \( \omega_1(t) = -(c_1 + c_2) t / 2, \omega_2(t) = -(c_2 + c_3) t / 2 \) and \( \theta = -\frac{1}{\kappa} \log(e^{-\kappa r_0} - \frac{L}{s_1}). \)

**Proof.** By the standard method (see, e.g., [56]), we can easily get the existence result. Using the same method in [8, Theorem 1.1], one can easily prove (38). Hence we complete the proof.

When \( x \leq -\bar{c}t = -\frac{c_1 + c_2}{2} t \), then \( \xi \leq 0 \), by Lemmas 2.8 and 2.3, we have

\[ |u(x, t) - \phi_1(x + c_1 t - \theta)| = |U(\xi, t) - \phi_1(\xi - s_1 t - \theta)| \]
\[ \leq |U(\xi,t) - \overline{U}(\xi,t)| + |U(\xi,t) - \phi_1(\xi - s_1 t - \theta)| \\
\leq |U(\xi,t) - \overline{U}(\xi,t)| + |\phi_3(\xi + r_2(t)) - \phi_3(\xi + r_1(t))| \\
\leq |\overline{U}(\xi,t) - \overline{U}(\xi,t)| + \sup_{\xi \in \mathbb{R}}|\phi_1'(|r_1(t) - s_1 t - \theta| + C_1|\phi_2(\xi + r_1(t))| \\
\leq M_4 e^{\kappa s_1 t} + \sup_{\xi \in \mathbb{R}}|\phi_1'(|r_1(t) - s_1 t - \theta| + C_1 C_0 e^{\tau r_1(t)} \rightarrow 0 \text{ as } t \to -\infty. \\
\]

When \(-\frac{s_1 + s_2}{2} \leq x \leq -\frac{s_1 + s_2}{2} t\), then \(0 \leq \xi \leq -\frac{s_1 + s_2}{2} t\), by Lemmas 2.8 and 2.3, we have

\[ |u(x,t) - \phi_2(x + c_2 t + \theta)| \\
\leq |U(\xi,t) - \phi_2(\xi + s_1 t + \theta)| \\
\leq |U(\xi,t) - \overline{U}(\xi,t)| + |U(\xi,t) - \phi_2(\xi + s_1 t + \theta)| \\
\leq |\overline{U}(\xi,t) - \overline{U}(\xi,t)| + |\phi_2(\xi + r_1(t)) - \phi_2(\xi + s_1 t + \theta)| + (\phi_3(\xi + r_2(t)) - a) \\
\]

as \(t \to -\infty\), since \(r_2(t) - (s_1 + s_2)t/2 \to -\infty\) as \(t \to -\infty\).

When \(x \geq -\frac{s_1 + s_2}{2} t\), then \(\xi \geq -\frac{s_1 + s_2}{2} t\), by Lemmas 2.8 and 2.3, we have

\[ |u(x,t) - \phi_3(x + c_3 t + \theta)| \\
\leq |U(\xi,t) - \phi_3(\xi + s_2 t + \theta)| \\
\leq |U(\xi,t) - \overline{U}(\xi,t)| + |\overline{U}(\xi,t) - \phi_3(\xi + s_3 t + \theta)| \\
\leq |\overline{U}(\xi,t) - \overline{U}(\xi,t)| + |\phi_3(\xi + r_2(t)) - \phi_3(\xi + s_2 t + \theta)| \\
\]

as \(t \to -\infty\), since \(r_2(t) - (s_1 + s_2)t/2 \to -\infty\) as \(t \to -\infty\).
Therefore, we have

**Theorem 2.10.** Assume (J) and (F) and \(-c < v_1\). Then (1) admits an entire solution \(u(x, t)\) satisfying

\[
\lim_{t \to -\infty} \left\{ \sup_{x \leq \omega_1(t)} |u(x, t) - \phi(-x + ct + \theta)| + \sup_{\omega_1(t) \leq x \leq \omega_2(t)} |u(x, t) - \psi_1(x + v_1 t + \theta)| + \sup_{x \geq \omega_2(t)} |u(x, t) - \psi_2(x + v_2 t + \theta)| \right\} = 0,
\]

where \(\omega_1(t) = -(c + v_1)t/2\), \(\omega_2(t) = -(v_1 + v_2)t/2\) and \(\theta = -\frac{1}{2} \log(e^{-kr_0} - \frac{1}{2})\).

Further, if \(f'(a) < 1\), then for some constant \(C > 0\), there hold

\[
|u(x + \eta, t) - u(x, t)| \leq C\eta,
\]

\[
\left| \frac{\partial u(x + \eta, t)}{\partial t} - \frac{\partial u(x, t)}{\partial t} \right| \leq C\eta,
\]

for any \(x, t \in \mathbb{R}, \eta > 0\) and

\[
\lim_{t \to -\infty} \sup_{x \in \mathbb{R}} |u(x, t) - 1| = 0.
\]

**Proof.** Based on the Theorem 2.9 and the relationship between \(\phi_1, \phi_2, \phi_3\) and \(\psi_1, \psi_2\), we have (39). Next, we only prove (40)-(42). Fix a positive constant \(n_0\), for any \(n \geq n_0\), by the boundedness of \(Q_y, Q_z, Q_w\), there exists \(M > 0\) such that \(|u_{n_0}(x) - u_{n_0}(y)| \leq M|x - y|\). Under the assumption \(f'(a) < 1\), for any \(x \in \mathbb{R}\) and \(\eta > 0\), it follows from [25, Proposition 2.5] that

\[
|u_n(x + \eta, t) - u_n(x, t)| \leq M_1 \eta,
\]

\[
\left| \frac{\partial u_n(x + \eta, t)}{\partial t} - \frac{\partial u_n(x, t)}{\partial t} \right| \leq M_1 \eta,
\]

where \(u_n(x, t)\) is the unique solution of

\[
\begin{cases}
(u_n)_t = J * u_n - u_n + f(u_n), & x \in \mathbb{R}, t > -n, \\
u_n(x, -n) = U(x - cn, -n), & x \in \mathbb{R},
\end{cases}
\]

In view of \(0 < u_n(x, t) < 1, 0 < u(x, t) < 1\) for any \((x, t) \in \mathbb{R} \times \mathbb{R}\) by comparison principle. Set \(D_1 = 2 + M_1\), where \(M_1\) is defined by (43). Fix \((x, t) \in \mathbb{R} \times \mathbb{R}\) and \(\eta > 0\). Let \(K \subset \mathbb{R} \times \mathbb{R}\) with \((x, t) \in K\) and \((x + \eta, t) \in K\) be a compact subset. Then there exists \(I_0 \in \mathbb{N}\) such that for any \(i > I_0\),

\[
|u(x, t) - u_{n_i}(x, t)| \leq \eta \quad \forall (x, t) \in K.
\]

Therefore, we have

\[
|u(x + \eta, t) - u(x, t)| \leq |u(x + \eta, t) - u_n(x + \eta, t)| + |u_n(x + \eta, t) - u_{n_i}(x, t)| + |u_{n_i}(x, t) - u(x, t)| \leq \eta + M_1 \eta + \eta := C\eta.
\]

Similarly, we have

\[
\left| \frac{\partial u(x + \eta, t)}{\partial t} - \frac{\partial u(x, t)}{\partial t} \right| \leq C\eta, \text{ for any } (x, t) \in \mathbb{R} \times \mathbb{R} \text{ and } 0 < \eta < \infty.
\]

Therefore, we have (40)-(41).

Now, under the assumption \(f'(a) < 1\), we prove (42). Since \(U(\xi, t) \leq U(\xi, t) \leq \bar{U}(\xi, t)\), \(t \leq t_0 < 0, \xi \in \mathbb{R}\). For any fixed \(t_1 \leq t_0\), it follows from

\[
\phi_1(\xi - r_1(t_1)) \to 0, \phi_2(\xi + r_1(t_1)) \to 0, \phi_3(\xi - r_2(t_1)) \to 0, \text{ as } \xi \to \infty
\]


that $U(\xi,t_1) \to 1$ as $\xi \to \infty$. Therefore, we get \( \liminf_{\xi \to \infty} U(\xi,t_1) > a \). Similarly,
$$
\phi_1(\xi - r_1(t_1)) \to 1, \quad \phi_2(\xi + r_1(t_1)) \to a, \quad \phi_3(\xi - r_2(t_1)) \to a, \quad \text{as } \xi \to -\infty,
$$
then $U(\xi,t_1) \to 1$ as $\xi \to -\infty$, thus \( \liminf_{\xi \to -\infty} U(\xi,t_1) > a \). By virtue of [5, Remark 3.4], there holds \( ||U(t) - 1||_{L^\infty(\mathbb{R})} = ||u(t) - 1||_{L^\infty(\mathbb{R})} \to 0 \) as $t \to \infty$. Thus (42) holds. This completes the proof. \( \square \)

2.2. II-type entire solutions originating from 3 waves.
Consider the following auxiliary function (c.f. [7, 8]):
$$
\tilde{Q}(y,z,w) := z + \frac{(1 - y)z(a - w)(-z) + y(a - z)w(1 - z)}{(1 - y)za + (a - z)w},
$$
where \((y, z, w) \in [0, 1] \times [0, a] \times [0, a] \backslash \{(1, a, w)|0 \leq w \leq a\} \cup \{(1, z, a)|0 \leq z \leq a\} \cup \{(y, 0, 0)|0 \leq y \leq 1\}). Simple calculations yield that
$$
\tilde{Q}(y,z,w) = \begin{cases} 
\frac{y + (1 - y)z}{1 - y} \frac{a(w-y)}{(1-y)za + (a-z)w}, \\
\frac{w}{1 - y} \frac{w-y}{(1-y)za + (a-z)w}.
\end{cases}
$$

**Lemma 2.11.** For the above defined function $\tilde{Q}(y,z,w)$, then there exist functions $\tilde{Q}_i, i = 1, 2, 3$ such that
$$
\tilde{Q}_y(y,z,w) = (a - z)w\tilde{Q}_1(y,z,w),
$$
$$
\tilde{Q}_z(y,z,w) = (1 - y)w\tilde{Q}_2(y,z,w),
$$
$$
\tilde{Q}_w(y,z,w) = (1 - y)z\tilde{Q}_1(y,z,w),
$$
and functions $\tilde{R}_j, j = 1, 2, \ldots, 14$ such that
$$
\tilde{Q}_{yy}(y,z,w) = z\tilde{R}_1(y,z,w) = (a - z)\tilde{R}_2(y,z,w) = w\tilde{R}_3(y,z,w),
$$
$$
\tilde{Q}_{zz}(y,z,w) = (1 - y)\tilde{R}_7(y,z,w) = w\tilde{R}_8(y,z,w),
$$
$$
\tilde{Q}_{ww}(y,z,w) = (1 - y)\tilde{R}_{12}(y,z,w) = z\tilde{R}_{13}(y,z,w) = (a - z)\tilde{R}_{14}(y,z,w).
$$
Consider the following initial problem:
$$
\tilde{p}_1' = s_1 + Le^{\kappa \tilde{p}_1}, \quad -\infty < t < 0, \quad \tilde{p}_1(0) = \tilde{p}_0, \quad (45)
$$
$$
\tilde{p}_2' = s_2 + Le^{\kappa \tilde{p}_2}, \quad -\infty < t < 0, \quad \tilde{p}_2(0) = \tilde{r}_0, \quad (46)
$$
$$
\tilde{r}_1' = s_1 - Le^{\kappa \tilde{r}_1}, \quad -\infty < t < 0, \quad \tilde{r}_1(0) = \tilde{r}_0, \quad (47)
$$
$$
\tilde{r}_2' = s_2 + Le^{\kappa \tilde{r}_2}, \quad -\infty < t < 0, \quad \tilde{r}_2(0) = \tilde{p}_0. \quad (48)
$$
with $\tilde{p}_0 = p_0, \tilde{r}_0 = r_0$. Then solutions to (45)-(48) can be expressed as follows
$$
\tilde{p}_1(t) = s_1 t - \frac{1}{\kappa} \log \left[ e^{-\kappa \tilde{p}_0} + \frac{L(1 - e^{s_1 t})}{s_1} \right],
$$
$$
\tilde{p}_2(t) = s_2 t + \frac{1}{\kappa} \log \left[ e^{-\kappa \tilde{p}_0} + \frac{L(1 - e^{s_1 t})}{s_1} \right] + \tilde{p}_0 + \tilde{r}_0,$$
\[ \tilde{r}_1(t) = s_1 t - \frac{1}{\kappa} \log \left[ e^{-\kappa \tilde{r}_0} - \frac{L(1 - e^{\kappa s_1 t})}{s_1} \right], \]
\[ \tilde{r}_2(t) = s_2 t + \frac{1}{\kappa} \log \left[ e^{-\kappa \tilde{r}_0} - \frac{L(1 - e^{\kappa s_1 t})}{s_1} \right] + \tilde{p}_0 + \tilde{r}_0. \]

Moreover, \( \tilde{p}_2(t) - \tilde{p}_1(t) \to -\infty, \tilde{r}_2(t) - \tilde{r}_1(t) \to -\infty \) as \( t \to -\infty \). Then there exists \( t_1 < 0 \) such that
\[ \tilde{p}_2(t) - \tilde{p}_1(t) < -\sigma, \quad \tilde{r}_2(t) - \tilde{r}_1(t) < -\sigma, \quad \forall t \leq t_1. \]

Further, there exists constant \( N > 0 \) such that
\[ 0 < \tilde{p}_1(t) - \tilde{r}_1(t) = \tilde{r}_2(t) - \tilde{p}_2(t) \leq Ne^{\kappa s_1 t}, \quad \forall t \leq 0. \]

Let \( c_1 := -c, c_2 := c_1, c_3 := \tilde{c}_1 \) and \( \tau := (c_1 + c_2)/2, s_1 := (c_2 - c_1)/2, s_2 := c_3 - \tau > s_1, \) and \( \phi_1 := \phi_1(x + c_1 t), \phi_2 := \phi_2(x + c_2 t), \phi_3 := \phi_3(x + c_3 t) \). Then \( \phi_1(z) = \phi(-z), \phi_2(z) = \psi_1(z), \phi_3(z) = \tilde{\psi}_1(z) \), where \( \psi_1(x + \tilde{\tau} t) \) denotes another traveling wave solution of (11), which satisfies \( \tilde{\psi}_1(-\infty) = 0, \tilde{\psi}_1(\infty) = a \) with \( \tilde{\psi}_1 \leq -c_\ast \). By translation if necessary, in this subsection, we assume
\[ \phi_1(0) = \frac{\alpha}{4}, \quad \phi_2(0) = \phi_3(0) = \frac{\alpha}{2}. \]

Similar to Lemma 2.3, we have following asymptotic behavior.

**Lemma 2.12.** Assume \((J),(F)\) hold. Then there exist positive constants \( R_1, C_0, \tau_1, \tau_2, \delta, \delta_1 \) such that
\[ \text{(i)} \] For \( x \leq R_1, \]
\[ |\phi'_1(x)|, |\phi'_2(x)|, |\phi'_3(x)| \leq C_0 e^{\mu_1 x}, \]
\[ \delta_2 \leq \frac{|\phi'_1(x)|}{1 - \phi_1(x)} \frac{|\phi'_2(x)|}{\phi_2(x)} \frac{|\phi'_3(x)|}{a - \phi_3(x)} \leq \delta_3. \]

\[ \text{(ii)} \] For \( x \geq -R_1, \]
\[ |\phi'_1(x)|, |\phi'_2(x)|, |\phi'_3(x)| \leq C_0 e^{-\mu_2 x}, \]
\[ \delta_2 \leq \frac{|\phi'_1(x)|}{\phi_1(x)} \frac{|\phi'_2(x)|}{a - \phi_2(x)} \frac{|\phi'_3(x)|}{\phi_3(x)} \leq \delta_3. \]

Moreover, for some \( M' > 0 \), there hold
\[ \frac{1 - \phi_1(x + y)}{1 - \phi_1(x)} \frac{\phi_2(x + y)}{\phi_2(x)} \frac{a - \phi_3(x + y)}{a - \phi_3(x)} \leq M', \quad \forall x \in \mathbb{R}, y \in \text{supp}(J), \]
\[ \frac{\phi_1(x + y)}{\phi_1(x)} \frac{a - \phi_2(x + y)}{a - \phi_2(x)} \frac{\phi_3(x + y)}{\phi_3(x)} \leq M', \quad \forall x \in \mathbb{R}, y \in \text{supp}(J). \]

Then applying the similar argument as the above section, we have

**Lemma 2.13.** Assume \((J)\) and \((F)\). For any \( x \in \mathbb{R}, t \leq \bar{t} < 0 \). Define
\[ \bar{U}(x, t) := \bar{Q}(\phi_1(x + \tau t - \bar{p}_1(t)), \phi_2(x + \tau t + \bar{p}_1(t)), \phi_3(x + \tau t + \bar{p}_2(t))). \]
\[ \bar{U}(x, t) := \bar{Q}(\phi_1(x + \tau t - \bar{r}_1(t)), \phi_2(x + \tau t + \bar{r}_1(t)), \phi_3(x + \tau t + \bar{r}_2(t))). \]

Then \((\bar{U}(x, t), \bar{U}(x, t))\) is a pair of super- and sub-solution of (12) for \( x \in \mathbb{R}, t \leq \bar{t} < 0 \).

Further,
\[ 0 < \bar{U}(x, t) - \bar{U}(x, t) \leq M_5 e^{\kappa_1}, \quad \forall x \in \mathbb{R}, t \leq \bar{t}, \]
for some constant \( M_5 > 0 \).
By the standard method as Theorem 2.10, we have

**Theorem 2.14.** Assume (J) and (F) and \(-c < v_1\). Then (1) admits a solution \(u(x, t)\) such that

\[
\lim_{t \to -\infty} \left\{ \sup_{x \leq \omega_1(t)} |u(x, t) - \phi(-x + ct + \theta_1)| + \sup_{\omega_1(t) \leq x \leq \omega_2(t)} |u(x, t) - \psi_1(x + v_1 t + \theta_1)| + \sup_{x \leq \omega_2(t)} |u(x, t) - \tilde{\psi}_1(-x + \tilde{v}_1 t - \theta_2)| \right\} = 0,
\]

where \(\theta_1 = -\frac{1}{\kappa} \log \left( e^{-\kappa \tilde{r}_0} - \frac{e^{-\kappa r_0}}{s_1} \right), \theta_2 = \frac{1}{\kappa} \log \left( e^{-\kappa \tilde{r}_0} - \frac{e^{-\kappa r_0}}{s_1} \right) + \tilde{p}_0 + \tilde{r}_0\), and

\[
\omega_1(t) = -\frac{(c + v_1)t}{2}, \quad \omega_2(t) = -\frac{(v_1 - \tilde{v}_1)t}{2}.
\]

Also, if \(f'(a) < 1\), then there exists constant \(C > 0\) such that

\[
|u(x + \eta, t) - u(x, t)| \leq C \eta,
\]

\[
\left| \frac{\partial u(x + \eta, t)}{\partial t} - \frac{\partial u(x, t)}{\partial t} \right| \leq C \eta,
\]

for any \(x, t \in \mathbb{R}, \eta > 0\), and there exists \(\theta \in \mathbb{R}\) such that

\[
\lim_{t \to -\infty} \sup_{x \in \mathbb{R}} |u(x, t) - \phi(-x + ct + \theta)| = 0.
\]

3. Entire solutions originating from \(k(k \geq 4)\) waves. This section contains two parts, in the first part, we deal with the existence of entire solutions which originate from \(4\) traveling wave solutions. In second part, the nonexistence theorem for entire solutions originating from \(k\) traveling wave solutions if \(k \geq 5\) is established.

### 3.1. Existence of entire solutions originating from 4 waves.

By super- and sub-solution method and comparison principle, this subsection deals with the existence of entire solutions originating from four traveling wave solutions.

Let \(\tau := \frac{c - v_1}{2}, c_0 := \frac{c + v_1}{2} > 0\) and consider the ordinary differential equations:

\[
\begin{align*}
p_3'(t) &= c_0 - N_1 e^{\sigma_1 p_3(t)}, \quad t \leq 0, \quad p_3(0) < \min \{0, \frac{1}{\sigma} \log \frac{c_0}{2N} \}, \\
p_4'(t) &= c_0 + N_1 e^{\sigma_1 p_4(t)}, \quad t \leq 0, \quad p_4(0) < 0,
\end{align*}
\]

(49)

where \(N_1, \sigma_1\) are two positive constants and the initial value \(p_3(0) \leq p_4(0)\). We can solve (49) explicitly as

\[
\begin{align*}
p_3(t) &= p_3(0) + c_0 t - \frac{1}{\sigma_1} \log \left( 1 - \frac{N_1}{c_0} e^{\sigma_1 p_3(0)} (1 - e^{-\sigma_1 t}) \right), \\
p_4(t) &= p_4(0) + c_0 t - \frac{1}{\sigma_1} \log \left( 1 + \frac{N_1}{c_0} e^{\sigma_1 p_4(0)} (1 - e^{-\sigma_1 t}) \right).
\end{align*}
\]

Define

\[
\begin{align*}
U_1(x, t) &:= Q(\phi(x + \tau t + p_4(t)), \psi_1(-x + \tau t - p_4(t)), \alpha), \\
U_2(x, t) &:= Q(\phi(x + \tau t + p_3(t)), \psi_1(-x + \tau t - p_3(t)), \alpha).
\end{align*}
\]

Then by [56, Lemma 4.3], \((U_1(x, t), U_2(x, t))\) is a pair of super- and sub-solution of (1) for \(x \geq 0, t \leq 0\). Moreover, for some constant \(\theta_3 \in \mathbb{R}\), it follows from [56,
Theorem 1.3] that (1) admits an entire solution \( u(x, t) \) such that

\[
\lim_{t \to -\infty} \left\{ \sup_{0 \leq x \leq \omega_1(t)} |u(x, t) - \psi_1(-x + v_1 t - \theta_3)| + \sup_{x \geq \omega_1(t)} |u(x, t) - \phi(x + ct + \theta_3)| \right\} = 0, 
\]

where

\[
\omega_1(t) = \frac{(c - v_1)t}{2}. 
\]

Some calculations yield that

\[
U(x, t) = U_1(|x|, t), \quad \bar{U}(x, t) = U_2(|x|, t), \quad x \in \mathbb{R}, t \leq 0, 
\]

is also a pair of super- and sub-solution of (1) for \( x \in \mathbb{R}, t \leq 0 \). Thus by applying the standard method as in section 2, as well as (50)-(52), there holds

**Theorem 3.1.** Assume \((J),(F)\) hold and \(-c < v_1\), then (1) admits an entire solution \( u(x, t) \) such that

\[
\lim_{t \to -\infty} \left\{ \sup_{x \leq -\omega_1(t)} |u(x, t) - \phi(-x + ct + \theta_3)| + \sup_{-\omega_1(t) \leq x \leq 0} |u(x, t) - \psi_1(x + v_1 t - \theta_3)| 
\]

\[
+ \sup_{0 \leq x \leq \omega_1(t)} |u(x, t) - \psi_1(-x + v_1 t - \theta_3)| + \sup_{x \geq \omega_1(t)} |u(x, t) - \phi(x + ct + \theta_3)| \right\} = 0, 
\]

where \(\omega_1(t)\) is given by (51).

Also, if \(f'(a) < 1\), then there exists constant \(C > 0\) such that

\[
|u(x + \eta, t) - u(x, t)| \leq C \eta, \\
\left| \frac{\partial u(x + \eta, t)}{\partial t} - \frac{\partial u(x, t)}{\partial t} \right| \leq C \eta, 
\]

for any \(x, t \in \mathbb{R}, \eta > 0\) and \(\lim_{t \to \infty} \sup_{x \in \mathbb{R}} |u(x, t) - 1| = 0\).

**Remark 1.** We now give a schematic diagram of the entire solutions in theorems 2.10, 2.14 and 3.1 as time \(t \to -\infty\), respectively. Although we know that these diagrams are only schematic, not the evolution of real solutions, they can still help us to better understand the new entire solutions. Besides, let \(U := \psi\) just for drawing convenience. The following three figures correspond to Theorem 2.10, Theorem 2.14 and Theorem 3.1 in order.
Theorem 4.1. Assume that (H) holds and we make the following additional hypothesis: the entire solutions established in Theorem 2.10 or Theorem 2.14 or Theorem 3.1.

3.2. Nonexistence of entire solutions originating from $k$ waves if $k \geq 5$. This subsection consider the nonexistence of entire solutions originating from $k$ traveling wave solutions if $k \geq 5$.

First we recall a sequence $(\alpha_1, \omega_1, ..., \alpha_k, \omega_k)$ with $\alpha_i \in \{0, a, 1\}, \alpha_{i+1} = \omega_i$ and $\omega_{i+1} \in \{0, a, 1\} \setminus \{\omega_i\}$ for $i = 1, ..., k-1$ is determined by an entire solution $u(x, t)$ in Section 1. The sequence $(\alpha_1, \omega_1, ..., \alpha_k, \omega_k)$ is called a terminated sequence, if the corresponding wave speed $c_1, ..., c_k$ satisfies $c_1 < c_2 < ... < c_k$, and for any sequence $(\alpha_{k+1}, \omega_{k+1})$ with $\alpha_{k+1} = \omega_k, \omega_{k+1} \in \{0, a, 1\} \setminus \{\omega_k\}$, the wave speed satisfies $c_{k+1} \geq c_k$. That is, for a terminated sequence $(\alpha_1, \omega_1, ..., \alpha_k, \omega_k)$, there has no entire solutions originating from $k+1$ traveling wave solutions with the sequence $(\alpha_1, \omega_1, ..., \alpha_{k+1}, \omega_{k+1})$ for any $\alpha_{k+1}, \omega_{k+1} \in \{0, a, 1\}$.

Theorem 3.2. Under the condition $(4)$, $(1)$ admits no entire solutions originating from $k$ traveling wave solutions if $k \geq 5$.

Proof. In this paper, since we only investigate the existence results of entire solutions originating from $k$ traveling wave solutions for $k \geq 3$, hence we only concerned about the terminated sequence $(\alpha_1, \omega_1, ..., \alpha_k, \omega_k)$ with $k \geq 3$. By simple calculations, under the condition $(4)$, the terminated sequences start from $0, a$ and 1 are $\{0, a, a, 0, 0, 1\}, \{1, 0, 0, a, a, 1\}$ and $\{1, 0, 0, a, a, 0, 0, 1\}$. Therefore, the longest terminated sequence is $\{1, 0, 0, a, a, 0, 0, 1\}$. Hence, by the definition of terminated sequence, there is no entire solutions originating from $k$ traveling wave solutions if $k \geq 5$. \qed

4. Stability of entire solutions. In this section, we prove the stability result of the entire solutions established in Theorem 2.10 or Theorem 2.14 or Theorem 3.1. We make the following additional hypothesis:

$$(H): -L := \min_{s \in [0, 1]} f'(s) < -1/4.$$ 

Theorem 4.1. Assume that $(H)$ holds and $\max_{s \in [0, 1]} |f''(s)|$ is appropriate small. Let $u(x, t)$ be an entire solution established in Theorem 2.10 or Theorem 2.14 or Theorem 3.1. If the initial value $\varphi(x)$ satisfies

$$0 \leq \varphi(x) \leq 1, \quad x \in \mathbb{R},$$

and assume the initial perturbations satisfy

$$u(x, 0) - \varphi(x) \in H_w^1(\mathbb{R}).$$

Then the nonnegative solution $u(x, t; \varphi)$ of the Cauchy problem:

$$
\begin{cases}
  u_t = J * u - u + f(u), \\
  u(x, 0) = \varphi(x),
\end{cases}
$$

uniquely exists and satisfies $0 \leq u(x, t; \varphi) \leq 1, x \in \mathbb{R}, t > 0$ and

$$u(x, t; \varphi) - u(x, t) \in C([0, \infty); H_w^1(\mathbb{R})) \cap L^2([0, \infty); H_w^2(\mathbb{R})),
$$

and

$$\sup_{x \in \mathbb{R}} |u(x, t; \varphi) - u(x, t)| \leq Ce^{-\mu t}, \quad t > 0$$

for some positive constant $C$ and $\mu$.

In the sequel, we always assume all the assumptions in Theorem 4.1 hold. Before proving the main result, we first state some lemmas. Set

$$A_w(x) := 1 - \int_{\mathbb{R}} J(y) \frac{w(x+y)}{w(x)} \, dy + 2L, \quad x \in \mathbb{R}. \quad (54)$$
Lemma 4.2. There exists some constant $C > 0$ such that
$$A_w(x) \geq C, \quad \forall x \in \mathbb{R},$$
where $w(x)$ is given by (6).

Proof. By the definition of $w(x)$ and $A_w(x)$. When $x \leq 0$, $w(x) = e^{-\eta_0 x}$, then we have
$$A_w(x) = 1 - \int_{-\infty}^{-x} J(y) e^{-\eta_0 (x+y)} dy - \int_{-x}^{\infty} J(y) e^{\eta_0 x} dy + 2L$$
$$= 1 - \int_{-\infty}^{-x} J(y) e^{-\eta_0 y} dy - \int_{-x}^{\infty} J(y) e^{\eta_0 y} dy + 2L$$
$$\geq 1 - \int_{\mathbb{R}} J(y) e^{-\eta_0 y} dy - \frac{1}{2} + 2L = \mathcal{W}(\eta_0) > 0.$$ The last inequality comes from the choice of $\eta_0$.

When $x \geq 0$, $w(x) = 1$, then we have
$$A_w(x) = 1 - \int_{-\infty}^{-x} J(y) e^{-\eta_0 (x+y)} dy - \int_{-x}^{\infty} J(y) dy + 2L$$
$$\geq 1 - \int_{-\infty}^{0} J(y) e^{-\eta_0 y} dy - \int_{-x}^{\infty} J(y) dy + 2L$$
$$= \left[ 1 - \int_{\mathbb{R}} J(y) e^{-\eta_0 y} dy + 2L \right] + \frac{1}{2} + \int_{0}^{\infty} J(y) e^{-\eta_0 y} dy - \int_{-x}^{\infty} J(y) dy$$
$$= \mathcal{W}(\eta_0) + \frac{1}{2} + \int_{0}^{\infty} J(y) e^{-\eta_0 y} dy - \int_{-x}^{\infty} J(y) dy$$
$$\geq \mathcal{W}(\eta_0) + \int_{0}^{\infty} J(y) e^{-\eta_0 y} dy - \frac{1}{2} > 0.$$ The last inequality comes from the fact that we can choose $\eta_0 > 0$ sufficiently small.

Based on the Lemma 4.2, there have

Lemma 4.3. Then there exists some constant $C_1 > 0$ such that
$$B_{\mu,w}(x) := A_w(x) - 2\mu \geq C_1$$
for all $x \in \mathbb{R}$, and $0 < \mu < C/2$, where the constant $C$ is given by Lemma 4.2.

Denote
$$u^-_0(x) := \min\{ \phi(x), u(x,0) \}, \quad x \in \mathbb{R},$$
$$u^+_0(x) := \max\{ \phi(x), u(x,0) \}, \quad x \in \mathbb{R}.$$ Then we have
$$\left\{ \begin{array}{ll}
0 \leq u^-_0(x) \leq \phi(x) \leq u^+_0(x) \leq 1, & x \in \mathbb{R},
0 \leq u^-_0(x) \leq u(x,0) \leq u^+_0(x) \leq 1, & x \in \mathbb{R}.
\end{array} \right. \quad (55)$$
Let $u^\pm(x,t)$ be the solutions of (1) with initial value $u^\pm(\cdot,0) = u^\pm_0(\cdot)$. By comparison principle, we get
$$\left\{ \begin{array}{ll}
0 \leq u^-(x,t) \leq u(x,t) \leq \phi \leq u^+(x,t) \leq 1, & x \in \mathbb{R}, t \geq 0,
0 \leq u^-(x,t) \leq u(x,t) \leq u^+(x,t) \leq 1, & x \in \mathbb{R}, t \geq 0.
\end{array} \right. \quad (56)$$
Denote \( v(x, t) := u^+(x, t) - u(x, t), \) \( v_0(x) := v(x, 0) = u^+_0(x) - u(x, 0). \) By virtue of Lemmas 4.2-4.3, it holds

**Lemma 4.4.** There exists some constant \( C_2 > 0 \) such that

\[
\|v(t)\|_{L^2_x}^2 + \int_0^t e^{-2\mu(t-s)}\|v(s)\|_{L^2_x}^2 \, ds \leq C_2 e^{-2\mu t}\|v(0)\|_{L^2_x}^2, \tag{57}
\]

for any \( t \geq 0. \)

**Proof.** By (55) and (56), we have

\[
0 \leq v_0(x) \leq 1, \quad 0 \leq v(x, t) \leq 1, \quad x \in \mathbb{R}, t \geq 0.
\]

Then \( v(x, t) \) satisfies

\[
\frac{\partial v}{\partial t} = J \ast v - v + f(u^+) - f(u),
\]

with the initial data \( v_0. \)

It is obvious that if \( v_0(x) \in H^2_w(\mathbb{R}), \) then we have \( v(x, t) \in C([0, \infty); H^2_w(\mathbb{R})), \) for the details, one can see [54]. In order to establish the energy estimates, sufficient regularity of the solutions to (58) and (1) is required. We thus mollify the initial data as follows

\[
v_{0\varepsilon}(x) = \Gamma_\varepsilon \ast v_0(x), \quad u_{0\varepsilon}(x, 0) = \Gamma_\varepsilon \ast u_0(x),
\]

where

\[
\Gamma_\varepsilon \ast \varrho_0(x) = \int_\mathbb{R} \Gamma_\varepsilon(x-y)\varrho_0(y) \, dy \in H^2_w(\mathbb{R}), \quad \varrho_0(x) = v_0(x), \text{ or } u_0(x),
\]

and \( \Gamma_\varepsilon(x) \) is the mollifier. Let \( v_\varepsilon(x, t), u_\varepsilon(x, t) \) be the solutions to (58) and (1) with these mollified initial data. Then we have

\[
v_{\varepsilon}(x, t), u_{\varepsilon}(x, t) \in C([0, \infty); H^2_w(\mathbb{R})).
\]

By taking the limit \( \varepsilon \to 0, \) we can obtain the corresponding energy estimate for the original solution \( v(x, t). \) For the sake of simplicity, below we formally use \( v(x, t) \) to establish the desired energy estimates.

Multiplying the equation (58) by \( e^{2\mu t}w(x)v(x, t), \) we obtain

\[
\left(\frac{1}{2} e^{2\mu t}w(x)v^2(x, t)\right)_t - e^{2\mu t}w(x)v(x, t) \int_\mathbb{R} J(y)v(x-y, t) \, dy - \mu e^{2\mu t}w(x)v^2(x, t)
\]

\[
= - e^{2\mu t}w(x)v^2(x, t) + (f(u^+(x, t)) - f(u(x, t)))e^{2\mu t}w(x)v(x, t).
\]

Hence

\[
\left(\frac{1}{2} e^{2\mu t}w(x)v^2(x, t)\right)_t - e^{2\mu t}w(x)v(x, t) \int_\mathbb{R} J(y)v(x-y, t) \, dy 
\]

\[
+ (1-\mu)e^{2\mu t}w(x)v^2(x, t)
\]

\[
= (f(u^+(x, t)) - f(u(x, t)))e^{2\mu t}w(x)v(x, t). \tag{59}
\]

Integrating (59) over \( \mathbb{R} \times [0, t] \) with respect to \( x \) and \( t, \) we get

\[
e^{2\mu t}\|v(t)\|_{L^2_x}^2 - 2 \int_0^t \int_\mathbb{R} e^{2\mu s}w(x)v(x, s) \int_\mathbb{R} J(y)v(x-y, s) \, dy \, dx \, ds
\]

\[
+ 2 \int_0^t \int_\mathbb{R} (1-\mu)e^{2\mu s}w(x)v^2(x, s) \, dx \, ds
\]

\[
= \|v(0)\|_{L^2_x}^2 + 2 \int_0^t (f(u^+(x, s)) - f(u(x, s)))e^{2\mu s}w(x)v(x, s) \, dx \, ds. \tag{60}
\]
By the Cauchy-schwarz inequality \( 2xy \leq x^2 + y^2 \), we have
\[
2 \int_0^t \int_R e^{2\mu s} \omega(x)v(x,s) J(y)v(x-y,s) dy dx ds \\
\leq \int_0^t \int_R e^{2\mu s} \omega(x)v^2(x,s) dx ds + \int_0^t \int_R e^{2\mu s} \omega(x) J(y)v^2(x-y,s) dy dx ds \\
= \int_0^t \int_R e^{2\mu s} \omega(x)v^2(x,s) dx ds + \int_0^t \int_R e^{2\mu s} \omega(x)v^2(x,s) J(y) \frac{w(x+y)}{w(x)} dy dx ds
\]
Hence the inequality (60) becomes
\[
e^{2\mu t} \|v(t)\|_{L^2}^2 + \int_0^t \int_R \left[ 1 - \int_R J(y) \frac{w(x+y)}{w(x)} dy - 2\mu \right] e^{2\mu s} \omega(x)v^2(x,s) dx ds \\
\leq \|v(0)\|_{L^2}^2 + 2 \min_{0 \leq u \leq 1} f'(u) \int_0^t \int_R e^{2\mu s} \omega(x)v^2(x,s) dx ds \\
+ \int_0^t \int_R f''(\xi) e^{2\mu s} \omega(x)v^3(x,s) dx ds, \\
\leq \|v(0)\|_{L^2}^2 + 2L \int_0^t \int_R e^{2\mu s} \omega(x)v^2(x,s) dx ds \\
+ \max_{\xi \in [0,1]} |f''(\xi)| \int_0^t \int_R e^{2\mu s} \omega(x)v^2(x,s) dx ds
\]
(61)
where \( \xi \in [u, u^+] \). Since \( u, v \in [0, 1] \), therefore, we have
\[
e^{2\mu t} \|v(t)\|_{L^2}^2 + \int_0^t \int_R \left[ 1 - \int_R J(y) \frac{w(x+y)}{w(x)} dy - 2\mu \right] e^{2\mu s} \omega(x)v^2(x,s) dx ds \\
\leq \|v(0)\|_{L^2}^2 + 2 \min_{0 \leq u \leq 1} f'(u) \int_0^t \int_R e^{2\mu s} \omega(x)v^2(x,s) dx ds \\
+ \int_0^t \int_R f''(\xi) e^{2\mu s} \omega(x)v^3(x,s) dx ds, \\
\leq \|v(0)\|_{L^2}^2 - 2L \int_0^t \int_R e^{2\mu s} \omega(x)v^2(x,s) dx ds \\
+ \max_{\xi \in [0,1]} |f''(\xi)| \int_0^t \int_R e^{2\mu s} \omega(x)v^2(x,s) dx ds
\]
(62)
And obviously, inequality (62) is equivalent to
\[
e^{2\mu t} \|v(t)\|_{L^2}^2 + \int_0^t \int_R e^{2\mu s} B_{u,w}(x)w(x)v^2(x,s) dx ds \\
\leq \|v(0)\|_{L^2}^2 + \max_{\xi \in [0,1]} |f''(\xi)| \int_0^t \int_R e^{2\mu s} \omega(x)v^2(x,s) dx ds.
\]
By virtue of Lemma 4.3, as long as the absolute value of the second derivative of the \( f \) on \([0, 1]\) is appropriately small, there exists positive constant \( C_2 \) such that (57) holds. This completes the proof. \( \square \)

**Lemma 4.5.** There exists constant \( C_0 > 0 \) such that
\[
\|v_x(t)\|_{L^2}^2 + \int_0^t \int_R e^{-2\mu(t-s)} \|v_x(s)\|_{L^2}^2 ds \leq C_0 e^{-2\mu t} \|v(0)\|_{H^1_0}^2.
\]

**Proof.** Similarly, differentiating the equation (58), we get
\[
v_{tx} = J * v_x - v_x + (f(u^+) - f(u))x.
\]
(63)
Multiplying the equation (63) by \( e^{2\mu t} w(x)v_x(x,t) \), it holds
\[
\left( \frac{1}{2} e^{2\mu t} w(x)v_x^2(x,t) \right)_t - \mu e^{2\mu t} w(x)v_x^2(x,t) - e^{2\mu t} w(x)v_x(x,t)
\]
\[ \times \int_{\mathbb{R}} J(y)v_x(x-y,t)dy + e^{2\mu t}w(x)v_x(x,t) \]
\[ = (f(u^+))_x e^{2\mu t}w(x)v_x(x,t). \]  

Integrating (64) over \( \mathbb{R} \times [0,t] \) with respect to \( x \) and \( t \), we have

\[ e^{2\mu t}||v_x(t)||_{L_w^2}^2 - 2 \int_0^t \int_{\mathbb{R}} e^{2\mu s}w(x)v_x(x,s) \int_{\mathbb{R}} J(y)v_x(x-y,s)dydxds \]
\[ + 2 \int_0^t \int_{\mathbb{R}} (1-\mu)e^{2\mu s}w(x)v_x^2(x,s)dxds \]
\[ = ||v_x(0)||_{L_w^2}^2 + 2 \int_0^t \int_{\mathbb{R}} (f(u^+(x,s)) - f(u(x,s)))_x e^{2\mu s}w(x)v_x(x,s)dxds. \]  

Since

\[ (f(u^+(x,s)) - f(u(x,s)))_x \]
\[ = f'(u^+(x,s))u_x^+(x,s) - f'(u(x,s))u_x(x,s) \]
\[ = f'(u^+(x,s))v_x(x,s) + u_x(x,t)f''(\theta u^+(x,s) + (1-\theta)u(x,s))v(x,t), \]  

where \( \theta \in [0,1] \). By the Cauchy-schwarz inequality \( 2xy \leq x^2 + y^2 \), we get

\[ 2 \int_0^t \int_{\mathbb{R}} e^{2\mu s}w(x)v_x(x,s) \int_{\mathbb{R}} J(y)v_x(x-y,s)dydxds \]
\[ \leq \int_0^t \int_{\mathbb{R}} e^{2\mu s}w(x)v_x^2(x,s)dxds + \int_0^t \int_{\mathbb{R}} e^{2\mu s}w(x)J(y)v_x^2(x-y,s)dydxds \]
\[ = \int_0^t \int_{\mathbb{R}} e^{2\mu s}w(x)v_x^2(x,s)dxds + \int_0^t \int_{\mathbb{R}} e^{2\mu s}w(x)v_x^2(x,s) \int_{\mathbb{R}} J(y)\frac{w(x+y)}{w(x)}dydxds. \]  

Bring formula (66) and (67) into formula (65), we get

\[ e^{2\mu t}||v_x(t)||_{L_w^2}^2 + \int_0^t \int_{\mathbb{R}} \left[ 1 - 2\mu - \int_{\mathbb{R}} J(y)\frac{w(x+y)}{w(x)}dy \right] e^{2\mu s}w(x)v_x^2(x,s)dxds \]
\[ \leq ||v_x(0)||_{L_w^2}^2 + 2 \int_0^t \int_{\mathbb{R}} f'(u^+(x,s))e^{2\mu s}w(x)v_x^2(x,s)dxds \]
\[ + 2 \int_0^t \int_{\mathbb{R}} u_x(x,s)f''(\theta u^+(x,s) + (1-\theta)u(x,s))e^{2\mu s}w(x)v_x(x,s)v(x,s)dxds, \]  

where \( \theta \in [0,1] \). Since \( u^+ \in [0,1] \), therefore, we have

\[ e^{2\mu t}||v_x(t)||_{L_w^2}^2 + \int_0^t \int_{\mathbb{R}} \left[ 1 - 2\mu - \int_{\mathbb{R}} J(y)\frac{w(x+y)}{w(x)}dy \right] e^{2\mu s}w(x)v_x^2(x,s)dxds \]
\[ \leq ||v_x(0)||_{L_w^2}^2 + 2 \min_{0 \leq u^+ \leq 1} f'(u^+) \int_0^t \int_{\mathbb{R}} e^{2\mu s}w(x)v_x^2(x,s)dxds \]
\[ + 2 \int_0^t \int_{\mathbb{R}} u_x(x,s)f''(\theta u^+(x,s) + (1-\theta)u(x,s))e^{2\mu s}w(x)v_x(x,s)v(x,s)dxds \]
\[ = ||v_x(0)||_{L_w^2}^2 - 2L \int_0^t \int_{\mathbb{R}} e^{2\mu s}w(x)v_x^2(x,s)dxds \]
+ \int_{\mathbb{R}} \int_{0}^{t} u_{x}(x, s) f''(\theta u^{+}(x, s) + (1 - \theta) u(x, s)) e^{2\mu s} w(x)v_{x}(x, s)v(x, s)dxds.

(69)

Then the inequality (69) is equivalent to

\[ e^{2\mu t} ||v_{x}(t)||_{L_{w}^{2}}^{2} + \int_{0}^{t} \int_{\mathbb{R}} B_{\mu, w}(x)e^{2\mu s} w(x)v_{x}^{2}(x, s)dxds \]

\[ \leq ||v_{x}(0)||_{L_{w}^{2}}^{2} + 2 \int_{0}^{t} \int_{\mathbb{R}} Q(x, s)e^{2\mu s} w(x)dxds, \]

where

\[ Q(x, t) = u_{x}(x, t)f''(\theta u^{+}(x, t) + (1 - \theta) u(x, t))v_{x}(x, t)v(x, t). \]

Now we estimate the last term in the right-hand side of (70). On one hand, from Based on Lemmas 4.4-4.5, we have

According to the standard Sobolev embedding inequality

\[ H_{w}^{1}(\mathbb{R}) \hookrightarrow H^{1}(\mathbb{R}) \hookrightarrow C(\mathbb{R}) \]

for any \( \eta > 0 \) such that \( \frac{C_{\eta}}{\eta} < C_{1} \) (C1 is given by Lemma 4.3) and combining with (70), we have

\[ ||v_{x}(t)||_{L_{w}^{2}}^{2} + \int_{0}^{t} e^{-2\mu(t-s)} ||v_{x}(s)||_{L_{w}^{2}}^{2} ds \leq C_{6}e^{-2\mu t} ||v(0)||_{H_{w}^{1}}^{2}, \]

for some constant \( C_{6} > 0 \). This completes the proof.

Based on Lemmas 4.4-4.5, we have

**Lemma 4.6. (Priori estimate)** There exists constant \( C_{7} > 0 \) such that

\[ ||v(t)||_{H_{w}^{1}} \leq C_{7}e^{-\mu t} ||v(0)||_{H_{w}^{1}}, \]

for \( t > 0 \) and \( \mu \) is given by Lemma 4.3.

According to the standard Sobolev embedding inequality

\[ H_{w}^{1}(\mathbb{R}) \hookrightarrow H^{1}(\mathbb{R}) \hookrightarrow C(\mathbb{R}) \]

for \( w(x) \geq 1 \), one has

\[ \sup_{x \in \mathbb{R}} |v(x, t)| \leq C_{8} ||v(t)||_{H_{w}^{1}}^{2} \leq C_{8} ||v(t)||_{H_{w}^{1}}^{2}, \]
for any \( t \geq 0 \) and for some constant \( C_8 > 0 \). Based on the above discussion and Lemma 4.6, it holds

**Lemma 4.7.** For some constant \( C_9 > 0 \), there holds
\[
\sup_{x \in \mathbb{R}} |u^+(x, t) - u(x, t)| = \sup_{x \in \mathbb{R}} |v(x, t)| \leq C_9 e^{-\mu t},
\]
for any \( t \geq 0 \).

As Lemma 4.7, repeating the above discussion, we get

**Lemma 4.8.** There exists constant \( C_{10} > 0 \) such that
\[
\sup_{x \in \mathbb{R}} |u^-(x, t) - u(x, t)| \leq C_{10} e^{-\mu t}, \quad \forall t \geq 0.
\]

Now we prove Theorem 4.1.

**Proof of Theorem 4.1.** Since \( u^-(x, t) \leq u(x, t; \varphi(x) \leq u^+(x, t) \text{ for any } x, t \in \mathbb{R} \).

Then we get
\[
|u(x, t; \varphi) - u(x, t)| \leq \max\{|u^+(x, t) - u(x, t)|, |u^-(x, t) - u(x, t)|\}.
\]

In view of Lemmas 4.7-4.8, there exists constant \( C_{10} \) such that
\[
\sup_{x \in \mathbb{R}} |u(x, t; \varphi) - u(x, t)| \leq C_{10} e^{-\mu t}, \quad t \geq 0.
\]

The proof is completed.

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**REFERENCES**

[1] P. W. Bates, P. C. Fife, X. Ren and X. Wang, Traveling waves in a convolution model for phase transitions, *Arch. Rational Mech. Anal.*, 138 (1997), 105–136.

[2] P. W. Bates and G. Zhao, Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal, *J. Math. Anal. Appl.*, 332 (2007), 428–440.

[3] J. F. Cao, Y. Du, F. Li and W. T. Li, The dynamics of a Fisher-KPP nonlocal diffusion model with free boundaries, *J. Funct. Anal.*, 277 (2019), 2772–2814.

[4] J. Carr and A. Chmaj, Uniqueness of travelling waves for nonlocal monostable equations, *Proc. Amer. Math. Soc.*, 132 (2004), 2433–2439.

[5] X. Chen, Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations, *Adv. Differential Equations*, 2 (1997), 125–160.

[6] X. Chen and J. S. Guo, Existence and uniqueness of entire solutions for a reaction-diffusion equation, *J. Differential Equations*, 212 (2005), 62–84.

[7] Y. Y. Chen, Entire solutions originating from three fronts for a discrete diffusive equation, *Tamkang Journal of Mathematics*, 48 (2017), 215–226.

[8] Y. Y. Chen, J. S. Guo, N. Ninomiya and C. H. Yao, Entire solutions originating from monotone fronts to the Allen-Cahn equation, *Phys. D*, 378/379 (2018), 1–19.

[9] C. Cortazar, M. Elgueta, J. D. Rossi and N. Wolanski, Boundary fluxes for nonlocal diffusion, *J. Differential Equations*, 234 (2007), 360–390.

[10] C. Cortazar, M. Elgueta, J. D. Rossi and N. Wolanski, How to approximate the heat equation with Neumann boundary conditions by nonlocal diffusion problems, *Arch. Ration. Mech. Anal.*, 187 (2008), 137–156.
[11] J. Coville, Traveling Fronts in Asymmetric Nonlocal Reaction Diffusion Equation: The Bistable and Ignition Case, Prépublication du CMM, Hal-00606208.

[12] J. Coville, J. Dávila and S. Martínez, Nonlocal anisotropic dispersal with monostable nonlinearity, *J. Differential Equations*, **244** (2008), 3080–3118.

[13] J. Coville and L. Dupaigne, On a nonlocal equation arising in population dynamics, *Proc. Roy. Soc. Edinburgh Sect. A*, **137** (2007), 727–755.

[14] E. C. M. Crooks and J. C. Tsai, Front-like entire solutions for equations with convection, *J. Differential Equations*, **253** (2012), 1206–1249.

[15] F. D. Dong, W. T. Li and J. B. Wang, Asymptotic behavior of traveling waves for a three-component system with nonlocal dispersal and its application, *Discrete Contin. Dyn. Syst.*, **37** (2017), 6291–6318.

[16] J. S. Guo and Y. Morita, Entire solutions of reaction-diffusion equations and an application to discrete diffusive equations, *Discrete Contin. Dyn. Syst.*, **12** (2005), 193–212.

[17] F. Hamel and N. Nadirashvili, Entire solutions of the KPP equation, *Comm. Pure Appl. Math.*, **52** (1999), 1255–1276.

[18] F. Hamel and N. Nadirashvili, Travelling fronts and entire solutions of the Fisher-KPP equation in $\mathbb{R}^N$, *Arch. Rational Mech. Anal.*, **157** (2001), 91–163.

[19] R. Huang, M. Mei and Y. Wang, Planar traveling waves for nonlocal dispersion equation with monostable nonlinearity, *Discrete Contin. Dyn. Syst.*, **32** (2012), 3621–3649.

[20] R. Huang, M. Mei, K. J. Zhang and Q. F. Zhang, Asymptotic stability of non-monotone traveling waves for time-delayed nonlocal dispersion equations, *Discrete Contin. Dyn. Syst.*, **36** (2016), 1331–1353.

[21] V. Hutson and M. Grinfeld, Non-local dispersal and bistability, *European J. Appl. Math.*, **17** (2006), 221–232.

[22] V. Hutson, S. Martinez, K. Mischaikow and G. T. Vickers, The evolution of dispersal, *J. Math. Biol.*, **47** (2003), 483–517.

[23] L. I. Ignat and J. D. Rossi, A nonlocal convection-diffusion equation, *J. Funct. Anal.*, **251** (2007), 399–437.

[24] W. T. Li, Y. J. Sun and Z. C. Wang, Entire solutions in the Fisher-KPP equation with nonlocal dispersal SIR epidemic model, *Dyn. Partial Differ. Equ.*, **15** (2018), 1531–1560.

[25] W. T. Li, J. B. Wang and L. Zhang, Entire solutions of nonlocal dispersal equations with monostable nonlinearity in space periodic habitats, *J. Differential Equations*, **261** (2016), 2472–2501.

[26] W. T. Li, Z. C. Wang and J. Wu, Entire solutions in monostable reaction-diffusion equations with delayed nonlinearity, *J. Differential Equations*, **245** (2008), 102–129.

[27] W. T. Li, Z. Zhang and G. B. Zhang, Invasion entire solutions in a competition system with nonlocal dispersal, *Discrete Contin. Dyn. Syst.*, **35** (2015), 1531–1560.

[28] Y. Li, W. T. Li and G. B. Zhang, Stability and uniqueness of traveling waves of a nonlocal dispersal SIR epidemic model, *Dyn. Partial Differ. Equ.*, **14** (2017), 87–123.

[29] C. K. Lin, C. T. Lin, Y. P. Lin and M. Mei, Exponential stability of nonmonotone traveling waves for Nicholson’s blowflies equation, *SIAM J. Math. Anal.*, **46** (2014), 1053–1084.

[30] N. W. Liu, W. T. Li and Z. C. Wang, Entire solutions of reaction-advection-diffusion equations with bistable nonlinearity in cylinders, *J. Differential Equations*, **246** (2009), 4249–4267.

[31] M. Mei, C. K. Lin, C. T. Lin and J. W. H. So, Traveling wavefronts for time-delayed reaction-diffusion equation. I. Local nonlinearity, *J. Differential Equations*, **247** (2009), 511–529.

[32] M. Mei, C. K. Lin, C. T. Lin and J. W. H. So, Traveling wavefronts for time-delayed reaction-diffusion equation. I. Local nonlinearity, *J. Differential Equations*, **247** (2009), 495–510.

[33] M. Mei, C. Ou and X. Q. Zhao, Global stability of monostable traveling waves for nonlocal time-delayed reaction-diffusion equations, *SIAM J. Math. Anal.*, **42** (2010), 2762–2790.

[34] M. Mei and J. W. H. So, Stability of strong travelling waves for a non-local time-delayed reaction-diffusion equation, *Proc. Roy. Soc. Edinburgh Sect. A*, **138** (2008), 551–568.

[35] M. Mei, J. W. H. So, M. Y. Li and S. S. Shen, Asymptotic stability of travelling waves for Nicholson’s blowflies equation with diffusion, *Proc. Roy. Soc. Edinburgh Sect. A*, **134** (2004), 579–594.

[36] Y. Morita and H. Ninomiya, Entire solutions with merging fronts to reaction-diffusion equations, *J. Dynam. Differential Equations*, **18** (2006), 841–861.
30 FANG-DI DONG, WAN-TONG LI, SHI-LIANG WU AND LI ZHANG

[38] Y. Morita and K. Tachibana, An entire solution to the Lotka-Volterra competition-diffusion equations, *SIAM J. Math. Anal.*, 40 (2009), 2217–2240.

[39] S. Pan, W. T. Li and G. Lin, Travelling wave fronts in nonlocal reaction–diffusion systems and applications, *Z. Angew. Math. Phys.*, 60 (2009), 377–392.

[40] K. Schumacher, Travelling-front solutions for integro-differential equations, I, *J. Reine Angew. Math.*, 1980 (2009), 54–70.

[41] H. L. Smith and X. Q. Zhao, Global asymptotic stability of traveling waves in delayed reaction-diffusion equations, *SIAM J. Math. Anal.*, 31 (2000), 514–534.

[42] Y. J. Sun, W. T. Li and Z. C. Wang, Entire solutions in nonlocal dispersal equations with bistable nonlinearity, *J. Differential Equations, 251* (2011), 551–581.

[43] Y. J. Sun, L. Zhang, W. T. Li and Z. C. Wang, Entire solutions in nonlocal monostable equations: Asymmetric case, *Comm. Pure Appl. Anal.*, 18 (2019), 1049–1072.

[44] M. Wang and G. Lv, Entire solutions of a diffusive and competitive Lotka-Volterra type system with nonlocal delays, *Nonlinearity*, 23 (2010), 1609–1630.

[45] Z. C. Wang, W. T. Li and S. Ruan, Entire solutions in bistable reaction-diffusion equations with nonlocal delayed nonlinearity, *Trans. Amer. Math. Soc.*, 361 (2009), 2047–2084.

[46] Z. C. Wang, W. T. Li and S. Ruan, Entire solutions in lattice delayed differential equations with nonlocal interaction: bistable cases, *Math. Model. Nat. Phenom.*, 8 (2013), 78–103.

[47] Z. C. Wang, W. T. Li and J. Wu, Entire solutions in delayed lattice differential equations with monostable nonlinearity, *SIAM J. Math. Anal.*, 40 (2009), 2392–2420.

[48] S. L. Wu and C. H. Hsu, Entire solutions with merging fronts to a bistable periodic lattice dynamical system, *Discrete Contin. Dyn. Syst.*, 36 (2016), 2329–2346.

[49] S. L. Wu and S. Ruan, Entire solutions for nonlocal dispersal equations with spatio-temporal delay: Monostable case, *J. Differential Equations, 258* (2015), 2435–2470.

[50] S. L. Wu, Z. X. Shi and F. Y. Yang, Entire solutions in periodic lattice dynamical systems, *J. Differential Equations*, 255 (2013), 3505–3535.

[51] S. L. Wu, Y. J. Sun and S. Y. Liu, Traveling fronts and entire solutions in partially degenerate reaction-diffusion systems with monostable nonlinearity, *Discrete Contin. Dyn. Syst.*, 33 (2013), 921–946.

[52] T. Xu, S. Ji, R. Huang, M. Mei and J. Yin, Theoretical and numerical studies on global stability of traveling waves with oscillations for time-delayed nonlocal dispersion equations, *Int. J. Numer. Anal. Model.*, 17 (2020), 68–86.

[53] H. Yagisita, Backward global solutions characterizing annihilation dynamics of travelling fronts, *Publ. Res. Inst. Math. Sci.*, 39 (2003), 117–164.

[54] G. B. Zhang and R. Ma, Spreading speeds and traveling waves for a nonlocal dispersal equation with convolution type crossing-monostable nonlinearity, *Z. Angew. Math. Phys.*, 65 (2014), 819–844.

[55] L. Zhang, W. T. Li and Z. C. Wang, Entire solution in an ignition nonlocal dispersal equation: Asymmetric kernel, *Sci. China Math.*, 60 (2017), 1791–1804.

[56] L. Zhang, W. T. Li, Z. C. Wang and Y. J. Sun, Entire solutions for nonlocal dispersal equations with bistable nonlinearity: asymmetric case, *Acta Math. Sin. English Ser.*, 35 (2019), 1771–1794.

[57] L. Zhang, W. T. Li and S. L. Wu, Multi-type entire solutions in a nonlocal dispersal epidemic model, *J. Dynam. Differential Equations*, 28 (2016), 189–224.

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E-mail address: dongfd15@lzu.edu.cn (F.-D. Dong)
E-mail address: wtli@lzu.edu.cn (W.-T. Li)
E-mail address: slwu@xidian.edu.cn (S.-L. Wu)
E-mail address: zhangli@chd.edu.cn (L. Zhang)