Solving Problems in Scalar Algebras of Reduced Powers

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Abstract

Following our previous work, we suggest here a large class of algebras of scalars in which simultaneous and correlated computations can be performed owing to the existence of surjective algebra homomorphisms. This may replace the currently used traditional computations in which only real or complex scalars are used, or occasionally, non-standard ones. The usual real, complex, or nonstandard scalars are included in the mentioned large class of algebras. Such simultaneous and correlated computations offer a depth of insight which has so far been missed when only using the few traditional kind of scalars.

1. A Large Class of Scalar Algebras of Reduced Powers

The following large class of algebras of scalars can be obtained easily as reduced powers, Rosinger. This reduced power construction, in its more general forms, is one of the fundamental tools in Model Theory, see Hodges. Historically, it has been used in the 19th century in a particular case, by the classical Cauchy-Bolzano construction of the set \( \mathbb{R} \) of real numbers from the set \( \mathbb{Q} \) of rational ones.

Let \( \Lambda \) be any infinite set, then the power \( \mathbb{R}^\Lambda \) is in a natural way an associative and commutative algebra. Namely, the elements \( x \in \mathbb{R}^\Lambda \) can be seen as mappings \( x: \Lambda \rightarrow \mathbb{R} \), and as such, they can be added to, and multiplied with one another point-wise. In the same way, the
elements $x \in \mathbb{R}^\Lambda$ can be multiplied with scalars from $\mathbb{R}$.

The well known remarkable fact connected with such a power algebra $\mathbb{R}^\Lambda$ is that there is a one-to-one correspondence between the *proper ideals* in it, and on the other hand, the *filters* on the infinite set $\Lambda$, Rosinger. Indeed, this one-to-one correspondence operates as follows

\begin{align*}
\mathcal{I} & \mapsto \mathcal{F}_\mathcal{I} = \{ Z(x) \mid x \in \mathcal{I} \} \\
\mathcal{F} & \mapsto \mathcal{I}_\mathcal{F} = \{ x \in \Lambda \to \mathbb{R} \mid Z(x) \in \mathcal{F} \}
\end{align*}

where $\mathcal{I}$ is an ideal in $\mathbb{R}^\Lambda$, $\mathcal{F}$ is a filter on $\Lambda$, while for $x : \Lambda \to \mathbb{R}$, we denoted $Z(x) = \{ \lambda \in \Lambda \mid x(\lambda) = 0 \}$, that is, the zero set of $x$.

Important properties of the one-to-one correspondence in (1.1) are as follows. Given two ideals $\mathcal{I}, \mathcal{J}$ in $\mathbb{R}^\Lambda$, and two filters $\mathcal{F}, \mathcal{G}$ on $\Lambda$, we have

\begin{align*}
\mathcal{I} \subseteq \mathcal{J} & \implies \mathcal{F}_\mathcal{I} \subseteq \mathcal{F}_\mathcal{J} \\
\mathcal{F} \subseteq \mathcal{G} & \implies \mathcal{I}_\mathcal{F} \subseteq \mathcal{I}_\mathcal{G}
\end{align*}

Furthermore, the correspondences in (1.1) are idempotent when iterated, namely

\begin{align*}
\mathcal{I} & \mapsto \mathcal{F}_\mathcal{I} \mapsto \mathcal{I}_{\mathcal{F}_\mathcal{I}} = \mathcal{I} \\
\mathcal{F} & \mapsto \mathcal{I}_\mathcal{F} \mapsto \mathcal{F}_{\mathcal{I}_\mathcal{F}} = \mathcal{F}
\end{align*}

It follows that every *reduced power algebra*

\begin{equation}
A = \mathbb{R}^\Lambda / \mathcal{I}
\end{equation}

where $\mathcal{I}$ is an ideal in $\mathbb{R}^\Lambda$, is of the form

\begin{equation}
A = A_\mathcal{F} \overset{\text{def}}{=} \mathbb{R}^\Lambda / \mathcal{I}_\mathcal{F}
\end{equation}

for a suitable unique filter $\mathcal{F}$ on $\Lambda$.

Needless to say, the *advantage* of the representation of reduced power algebras given in (1.5) is in the fact that filters $\mathcal{F}$ on $\Lambda$ are *simpler*
mathematical structures, than ideals $\mathcal{I}$ in $\mathbb{R}^\Lambda$.

We shall call $\Lambda$ the index set of the reduced power algebra $A_F = \mathbb{R}^\Lambda / \mathcal{I}_F$, while $\mathcal{F}$ will be called the generating filter which, we recall, is a filter on that index set.

Obviously, we can try to relate various reduced power algebra $A_F = \mathbb{R}^\Lambda / \mathcal{I}_F$ according to the two corresponding parameters which define them, namely, their index sets and their generating filters. We start here by relating them with respect to the latter.

Namely, a direct consequence of the second implication in (1.2) is the following one. Given two filters $\mathcal{F} \subseteq \mathcal{G}$ on $\Lambda$, we have the surjective algebra homomorphism

\[(1.6) \quad A_F \ni x + \mathcal{I}_F \rightarrow x + \mathcal{I}_G \in A_G\]

This obviously means that the algebra $A_G$ is smaller than the algebra $A_F$, the precise meaning of it being that

\[(1.6^*) \quad A_G \text{ and } A_F / (\mathcal{I}_G / \mathcal{I}_F) \text{ are isomorphic algebras}\]

which follows from the so called third isomorphism theorem for rings, a classical result of undergraduate Algebra.

Here we note that in the particular case when the filter $\mathcal{F}$ on $\Lambda$ is generated by a nonvoid subset $I \subseteq \Lambda$, that is, when we have

\[(1.7) \quad \mathcal{F} = \{ J \subseteq \Lambda \mid J \supseteq I \} \]

then it follows easily that

\[(1.8) \quad A_F = \mathbb{R}^I\]

which means that we do not in fact have a reduced power algebra, but only a power algebra.

For instance, in case $I$ is finite and has $n \geq 1$ elements, then $A_F = \mathbb{R}^n$ is in fact the usual $n$-dimensional Euclidean space.
Consequently, in order to avoid such a degenerate case of reduced power algebras, we have to avoid the filters of the form (1.7). This can be done easily, since such filters are obviously characterized by the property

\[(1.9) \quad \bigcap_{J \in \mathcal{F}} J = I \neq \emptyset\]

It follows that we shall only be interested in filters \( \mathcal{F} \) on \( \Lambda \) which satisfy the condition

\[(1.10) \quad \bigcap_{J \in \mathcal{F}} J = \emptyset\]

or equivalently

\[(1.11) \quad \forall \lambda \in \Lambda : \exists J_\lambda \in \mathcal{F} : \lambda \notin J_\lambda\]

which is further equivalent with

\[(1.12) \quad \forall I \subset \Lambda, \ I \text{ finite} : \Lambda \setminus I \in \mathcal{F}\]

We recall now that the Frechét filter on \( \Lambda \) is given by

\[(1.13) \quad \mathcal{F}_{\text{re}}(\Lambda) = \{ \Lambda \setminus I \mid I \subset \Lambda, \ I \text{ finite} \}\]

In this way, condition (1.10) - which we shall ask from now on about all filters \( \mathcal{F} \) on \( \Lambda \) - can be written equivalently as

\[(1.14) \quad \mathcal{F}_{\text{re}}(\Lambda) \subseteq \mathcal{F}\]

This in particular means that
\[ \forall I \in \mathcal{F} : \]
\[
I \text{ is infinite} \tag{1.14*}
\]
Indeed, if we have a finite \( I \in \mathcal{F} \), then \( \Lambda \setminus I \in \mathcal{F}_{re}(\Lambda) \), hence (1.14) gives \( \Lambda \setminus I \in \mathcal{F} \). But \( I \cap (\Lambda \setminus I) = \emptyset \), and one of the axioms of filters is contradicted.

In view of (1.6), it follows that all reduced power algebras considered from now on will be homomorphic images of the reduced power algebra \( A_{\mathcal{F}_{re}(\Lambda)} \), through the surjective algebra homomorphisms
\[
A_{\mathcal{F}_{re}(\Lambda)} \ni x + I_{\mathcal{F}_{re}(\Lambda)} \mapsto x + I \in A_{\mathcal{F}} \tag{1.15}
\]
or in view of (1.6*), we have the isomorphic algebras
\[
A_{\mathcal{F}} \cong A_{\mathcal{F}_{re}(\Lambda)}/(I_{\mathcal{F}}/I_{\mathcal{F}_{re}(\Lambda)}) \tag{1.15*}
\]

Let us note that the nonstandard reals \( {}^*\mathbb{R} \) are a particular case of the above reduced power algebras (1.4). Indeed, \( {}^*\mathbb{R} \) can be defined by using free ultrafilters \( \mathcal{F} \) on \( \Lambda \), that is, ultrafilters which satisfy (1.10), or equivalently (1.14).

We note that the field of real numbers \( \mathbb{R} \) can be embedded naturally in each of the reduced power algebras (1.4), by the injective algebra homomorphism
\[
\mathbb{R} \ni \xi \mapsto x_\xi + I \in A \tag{1.16}
\]
where \( x_\xi(\lambda) = \xi \), for \( \lambda \in \Lambda \). Indeed, if \( x_\xi \in I \) and \( \xi \neq 0 \), then the ideal \( I \) must contain \( x_1 \), which means that it is not a proper ideal, thus contradicting the assumption on it.

For simplicity of notation, from now on we shall write \( x_\xi = \xi \), for \( \xi \in \mathbb{R} \), thus (1.16) will take the form
\[
\mathbb{R} \ni \xi \mapsto \xi + I \in A \tag{1.17}
\]
which in view of the injectivity of this mapping, we may further sim-
There is also the issue to relate reduced power algebras corresponding to different index sets. Namely, let $\Lambda \subseteq \Gamma$ be two index sets which, as always in this paper, are assumed to be both infinite. Then we have the obvious surjective algebra homomorphism

\[
(1.19) \quad \mathbb{R}^\Gamma \ni x \mapsto x|_\Lambda \in \mathbb{R}^\Lambda
\]

since the elements $x \in \mathbb{R}^\Gamma$ can be seen as mappings $x : \Gamma \rightarrow \mathbb{R}$. Consequently, given any ideal $\mathcal{I}$ in $\mathbb{R}^\Gamma$, we can associate with it the ideal in $\mathbb{R}^\Lambda$, given by

\[
(1.20) \quad \mathcal{I}|_\Lambda = \{ x|_\Lambda \mid x \in \mathcal{I} \}
\]

As it happens, however, such an ideal $\mathcal{I}|_\Lambda$ need not always be a proper ideal in $\mathbb{R}^\Lambda$, even if $\mathcal{I}$ is a proper ideal in $\mathbb{R}^\Gamma$. For instance, if we take $\gamma \in \Gamma \setminus \Lambda$, and consider the proper ideal in $\mathbb{R}^\Gamma$ given by $\mathcal{I} = \{ x \in \mathbb{R}^\Gamma \mid x(\gamma) = 0 \}$, then we obtain $\mathcal{I}|_\Lambda = \mathbb{R}^\Lambda$, which is not a proper ideal in $\mathbb{R}^\Lambda$.

We can avoid that difficulty by noting the following. Given a filter $\mathcal{F}$ on $\Gamma$ which satisfies (1.14), that is, $\mathcal{F}re(\Gamma) \subseteq \mathcal{F}$, then

\[
(1.21) \quad \mathcal{F}|_\Lambda = \{ I \cap \Lambda \mid I \in \mathcal{F} \}
\]

satisfies the corresponding version of (1.14), namely $\mathcal{F}re(\Lambda) \subseteq \mathcal{F}|_\Lambda$. Indeed, let us take $J \subseteq \Lambda$ such that $\Lambda \setminus J$ is finite. Then clearly $\Gamma \setminus (J \cup (\Gamma \setminus \Lambda))$ is finite, hence $J \cup (\Gamma \setminus \Lambda) \in \mathcal{F}$. However, $J = (J \cup (\Gamma \setminus \Lambda)) \cap \Lambda$, thus $J \in \mathcal{F}|_\Lambda$.

Now in order for $\mathcal{F}|_\Lambda$ to be a filter on $\Lambda$, it suffices to show that $\phi \notin \mathcal{F}|_\Lambda$. Assume on the contrary that for some $I \in \mathcal{F}$ we have $I \cap \Lambda = \phi$, then $I \subseteq \Gamma \setminus \Lambda$, thus $\Lambda \notin \mathcal{F}$.

It follows that
(1.22) \( \mathcal{F}|\Lambda \) is a filter on \( \Lambda \) which satisfies (1.14) \( \iff \Lambda \in \mathcal{F} \)

In view of (1.19) - (1.22), for every filter \( \mathcal{F} \) on \( \Gamma \), such that

(1.23) \( \Lambda \in \mathcal{F} \)

we obtain the surjective algebra homomorphism

(1.24) \( A_{\mathcal{F}} = \mathbb{R}^\Gamma / \mathcal{I}_\mathcal{F} \ni x + \mathcal{I}_\mathcal{F} \mapsto x|_\Lambda + \mathcal{I}_{\mathcal{F}|\Lambda} \in A_{\mathcal{F}|\Lambda} = \mathbb{R}^\Lambda / \mathcal{I}_{\mathcal{F}|\Lambda} \)

and in particular, we have the following relation between the respective proper ideals

(1.25) \( (\mathcal{I}_\mathcal{F})|_\Lambda = \mathcal{I}_{\mathcal{F}|\Lambda} \)

2. Zero Divisors and the Archimedean Property

It is an elementary fact of Algebra that a quotient algebra (1.4) has zero divisors, unless the ideal \( \mathcal{I} \) is prime. A particular case of that is when a quotient algebra (1.4) is a field, which is characterized by the ideal \( \mathcal{I} \) being maximal. And in view of (1.5), (1.2), this means that the filter \( \mathcal{F} \) generating such an ideal must be an ultrafilter.

On the other hand, none of the reduced power algebras (1.5) which correspond to filters satisfying (1.14) are Archimedean. And that includes the nonstandard reals \(*\mathbb{R} \) as well.

Let us elaborate in some detail in this regard. First we note that on reduced power algebras (1.5), one can naturally define a partial order as follows. Given two elements \( x + \mathcal{I}_\mathcal{F}, y + \mathcal{I}_\mathcal{F} \in A_{\mathcal{F}} = \mathbb{R}^\Lambda / \mathcal{I}_\mathcal{F} \), we define

(2.1) \( x + \mathcal{I}_\mathcal{F} \leq y + \mathcal{I}_\mathcal{F} \iff \{ \lambda \in \Lambda \mid x(\lambda) \leq y(\lambda) \} \in \mathcal{F} \)

Now, with this partial order, the algebra \( A_{\mathcal{F}} \) is called Archimedean, if and only if
\[ \exists \ u + \mathcal{I}_F \in A_F, \ u + \mathcal{I}_F \geq 0 : \]
\[ \forall \ x + \mathcal{I}_F \in A_F, \ x + \mathcal{I}_F \geq 0 : \]
\[ \exists \ n \in \mathbb{N} : \]
\[ x + \mathcal{I}_F \leq n(u + \mathcal{I}_F) \]

However, in view of (1.14), we can take an infinite \( I \in \mathcal{F} \). Thus we can define a mapping \( v : \Lambda \rightarrow \mathbb{R} \) which is unbounded for above on \( I \). And in this case taking \( x + \mathcal{I}_F = (u + v) + \mathcal{I}_F \), it follows easily that condition (2.2) is not satisfied.

We note that the reduced power algebras (1.5) are Archimedean only in the degenerate case (1.7), (1.8), when in addition the respective sets \( I \) are finite, thus as noted, the respective algebras reduce to finite dimensional Euclidean spaces.

3. Simultaneous Correlated Computations in Scalar Algebras of Reduced Powers

Instead of the traditional way which confines all computations with scalars to the real numbers in \( \mathbb{R} \) or to the complex numbers in \( \mathbb{C} \), one can do a simultaneous and correlated scalar computation in all of the reduced power algebras, by using the surjective algebra homomorphisms (1.15) and (1.24).

Such a computation offers a depth of insight which has so far been missed when computing only with scalars in \( \mathbb{R}, \mathbb{C} \), or even in the non-standard \( ^*\mathbb{R} \), all of which are included as particular cases in the large, two parameter class of reduced power algebras \( A_F = \mathbb{R}^\Lambda / \mathcal{I}_F \), where \( \Lambda \) can be any infinite set, while \( \mathcal{F} \) can be any filter on \( \Lambda \) which satisfies (1.14).

Let us consider in some more detail this two parameter family of reduced power algebras.

Let us start by fixing any given infinite index set \( \Lambda \). Then, when \( \mathcal{F} \) ranges over all the filters on \( \Lambda \) which satisfy (1.14), the largest corresponding reduced power algebra of type (1.4) is given by, see (1.15)
and all other reduced power algebras $A_{\mathcal{F}e}(\Lambda) = \mathbb{R}^\Lambda / \mathcal{I}_{\mathcal{F}e}(\Lambda)$, with $\mathcal{F}$ satisfying (1.14), are images of it through the surjective algebra homomorphism (1.15).

Now, let us allow the index set $\Lambda$ to range over all infinite sets. Then the algebras (3.1) become ordered by the surjective algebra homomorphisms (1.24), namely

$$A_{\mathcal{F}e}(\Gamma) \rightarrow A_{\mathcal{F}e}(\Lambda)$$

whenever two infinite index sets $\Lambda$ and $\Gamma$ are in the relation

$$\Lambda \subseteq \Gamma, \quad \Gamma \setminus \Lambda \text{ is finite}$$

in which case we clearly have

$$\mathcal{F}e(\Lambda) = \mathcal{F}e(\Gamma)|_{\Lambda}$$

Indeed, for $\Lambda \subseteq \Gamma$, we obviously have, see (1.23)

$$\Lambda \in \mathcal{F}e(\Gamma) \iff \Gamma \setminus \Lambda \text{ is finite}$$

thus according to (1.23), (1.24), the surjective algebra homomorphism (3.2) holds.

In this way we obtain the following two directional table of surjective algebra homomorphisms
whenever the following conditions are satisfied

\[ \Gamma \supseteq \Lambda, \quad \Gamma \setminus \Lambda \text{ is finite} \]

\[ \mathcal{F}, \mathcal{G} \text{ are filters on } \Gamma, \quad \mathcal{F}(\Gamma) \subseteq \mathcal{F} \subseteq \mathcal{G} \]

\[ (3.6^*) \]

\[ \mathcal{H}, \mathcal{K} \text{ are filters on } \Lambda, \quad \mathcal{F}(\Lambda) \subseteq \mathcal{H} \subseteq \mathcal{K} \]

\[ \mathcal{F}|_{\Lambda} \subseteq \mathcal{H}, \quad \mathcal{G}|_{\Lambda} \subseteq \mathcal{K} \]

Furthermore, as follows easily from (1.15), (1.24), under the above conditions \((3.6^*)\), the following diagrams of surjective algebra homomorphisms are *commutative*

\[ (3.7) \]

\[ A_{\mathcal{F}(\Gamma)} \rightarrow A_{\mathcal{F}} \rightarrow A_{\mathcal{G}} \]

\[ A_{\mathcal{F}(\Lambda)} \rightarrow A_{\mathcal{H}} \rightarrow A_{\mathcal{K}} \]

Finally, if we disregard the leftmost column in (3.6), then under weaker conditions than in \((3.6^*)\), we still obtain the following commutative diagrams of surjective algebra homomorphisms

\[ (3.8) \]

\[ \ldots \rightarrow A_{\mathcal{F}} \rightarrow \ldots \rightarrow A_{\mathcal{G}} \rightarrow \ldots \]

\[ \ldots \rightarrow A_{\mathcal{H}} \rightarrow \ldots \rightarrow A_{\mathcal{K}} \rightarrow \ldots \]
which hold whenever
\[ \Gamma \supseteq \Lambda \]
\[ \mathcal{F}, \mathcal{G} \text{ are filters on } \Gamma, \quad \mathcal{F}(\Gamma) \subseteq \mathcal{F} \subseteq \mathcal{G}, \quad \Lambda \in \mathcal{F} \]
\[ (3.8^*) \]
\[ \mathcal{H}, \mathcal{K} \text{ are filters on } \Lambda, \quad \mathcal{F}(\Lambda) \subseteq \mathcal{H} \subseteq \mathcal{K} \]
\[ \mathcal{F}|_\Lambda \subseteq \mathcal{H}, \quad \mathcal{G}|_\Lambda \subseteq \mathcal{K} \]

In this case, instead of (3.7), we obtain the following commutative diagrams of surjective algebra homomorphisms

\[ (3.9) \quad \begin{array}{ccc}
A_\mathcal{F} & \rightarrow & A_\mathcal{G} \\
\downarrow & & \downarrow \\
A_\mathcal{H} & \rightarrow & A_\mathcal{K}
\end{array} \]

In subsequent papers, we shall apply the above method of simultaneous and correlated scalar computation to several important problems in Theoretical Physics.

References

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