Some properties of a class of coupled Langevin equations

Parthapratim Biswas*, Arnab Majumdar†, Anita Mehta‡
S.N. Bose National Centre For Basic Sciences
Block JD, Sector III, Salt Lake City, Calcutta-700 091, INDIA
J.K. Bhattacharjee§
Department of Theoretical Physics
Indian Association For Cultivation of Sciences
Jadavpur, Calcutta-700 032, INDIA

We present a set of coupled continuum equations with a specific coupling between mobile grains \(\rho\) and clusters \(h\) on the surface of a sandpile. The equations are analysed self-consistently; we demonstrate that Edwards’ infrared divergence is responsible for the unexpected critical exponents we find, which are verified by simulations.

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A certain class of coupled Langevin equations [1–3] has been useful of late in describing the dynamics of sandpile surfaces [4]. The fluctuations on the sandpile surface are described by the local height \(h(x, t)\) and the density \(\rho(x, t)\) of flowing grains – the flow being initiated by the formation of bumps. The coupling between the two variables is described in terms of a transfer term which converts static grains to mobile grains and vice-versa. In this communication we point out that a singularity discovered by Edwards [5] three decades ago in the context of fluid turbulence is present in such models due to the specific choice of the transfer term. This singularity largely controls the dynamics and produces unexpected exponents. Our contention is supported by numerical evidence.

We begin with a model introduced by Mehta, Luck and Needs (MLN) [2] where the coupled Langevin equations are

\[
\frac{\partial h}{\partial t} = D_h \nabla^2 h - T(h, \rho) + \eta_h(x, t) \\
\frac{\partial \rho}{\partial t} = D_\rho \nabla^2 \rho + T(h, \rho) + \eta_\rho(x, t) \\
T(h, \rho) = -\mu \rho (\nabla h)
\]

(1)

where the terms \(\eta_h(x, t)\) and \(\eta_\rho(x, t)\) represent Gaussian white noise. A variant of the above is the model due to Bouchaud et al. (BCRE) [3] where

\[ T = -\nu \nabla h - \mu \rho (\nabla h) \]

and the noise is present only in the equation of motion for \(h\). All the results which we will find here for the MLN system also holds for BCRE which we have checked both analytically and numerically.

A simple physical picture of the coupling or ‘transfer’ term \(T(h, \rho)\) between \(h\) and \(\rho\) is the following: flowing grains are added to regions of the interface which are at less than the critical slope, and vice versa, provided that the local density of flowing grains is always non-zero and in proportion to the local density. This is the simplest possible form of exchange between the species that appears intuitively reasonable during the process of avalanching and we analyse in what follows the profiles of both species consequent on this form.

Before this, we review some well-known facts about interfacial roughening [6]. Three critical exponents, \(\alpha\), \(\beta\), and \(z\), characterise the spatial and temporal scaling behaviour of a rough interface. They are conveniently defined by considering the (connected) two-point correlation function of the heights

\[ S(x - x', t - t') = \langle h(x, t)h(x', t') \rangle - \langle h(x, t) \rangle \langle h(x', t') \rangle. \]
We have
\[ S(x, 0) \sim |x|^{2\alpha} \quad (|x| \to \infty) \quad \text{and} \quad S(x = 0, t) \sim |t|^{2\beta} \quad (|t| \to \infty), \]
and \( z = \alpha/\beta. \)

In recent years a self-consistent mode coupling analysis used hitherto in dynamic critical phenomena has been used to look at in particular the Kardar-Parisi-Zhang (KPZ) equation and we extend its use to the case of the coupled equations presented here.

In this method we set up equations (to one-loop order) for the correlation functions and self-energies in terms of the full Green’s functions, correlation functions and vertices using assumed scaling forms for each. The critical exponents \( \alpha \) and \( \beta \) defined above are obtained from the self-consistent solutions of these equations after setting \( D_h \) and \( D_\rho \) to unity.

The analysis of these functions will be in terms of a weak scaling hypothesis which states
\[ G_h(k, \omega) = k^{-z_h} f_h \left( \frac{\omega}{k^{z_h}}, \frac{\omega}{\pi} \right) \]
along with a similar scaling relation for \( G_\rho(k, \omega) \). A strong scaling would imply the existence of a single time scale \( \text{i.e.} \ z_h = z_\rho \). As we show below, this cannot be the case here. The absence of strong scaling implies that the roughness exponents \( \alpha_h \) and \( \alpha_\rho \) may become functions of \( k \).

We consider the full Green’s function \( G_h(k, \omega) \), which is given via the well-known Dyson equation,
\[ G_h^{-1}(k, \omega) = G_h^0(k, \omega) + \Sigma_h(k, \omega) \]
Here, the zeroth order Green’s function is
\[ G_h^0(k, \omega) = (-i\omega + k^2)^{-1} \]

In the scaling limit \( k^2 \) can be dropped in comparison with \( \Sigma_h(k, \omega) \). To one-loop order, the self-energy \( \Sigma_h(k) \) is given by
\[ \Sigma_h(k, \omega) = \mu^2(2\pi)^{-2} \int dq \int d\Omega G_h(k - q, \omega - \Omega) S_\rho(q, \Omega) k(k - q) \quad \text{(2)} \]
We note that due to the presence of the term \( S_\rho(q, \Omega) \), the integral is dominated by the singularity in the integrand at \( q \to 0 \). This ‘infrared divergence’ which relates to the divergence of the internal momenta \( q \), is very different from the usual divergences encountered in critical phenomena where the latter occur for small wave numbers and are associated with long wavelength instabilities in the external momenta. In this case due to the infrared divergence in the above equation in the internal momenta \( q \), the integral diverges for any value of the external momenta \( k \), so long as \( \alpha_\rho > 0 \). This is the divergence found by Edwards for the Navier-Stokes equation. We thus need either to evaluate the integral with a lower cut-off \( k_0 \) or to introduce a suitable regulator. We follow the first of these procedures for the above equation.

We then proceed to evaluate the self-energy at zero external frequency, \( \text{i.e.} \Sigma_h(k, \omega = 0) \) from Eq.(2). The integral in Eq.(3) becomes in the limit of zero external frequencies
\[ \Sigma_h(k) = \mu^2 k^2 \frac{dq}{\Sigma_h(k)} \int \frac{d\Omega}{2\pi} S_\rho(q, \Omega) \]
We have to evaluate the integral by cutting off the momentum integration at \( k_0 \ll 1 \), \( \text{i.e.} \) we follow the first of the procedures given above to handle the infrared divergence. This gives, after some simplification,
\[ \Sigma_h^2(k) = \mu^2 k^2 \frac{k_0^{-2\alpha_\rho} C_\rho}{4\pi \alpha_\rho} \]
From the above equation with the scaling relation \( \Sigma_h(k) \sim k^{z_h} \) we find, on equating powers of \( k \),
\[ z_h = 1 \]
We note here that the presence of the term \( \rho \nabla h \) could in principle cause the vertex \( \mu \) to renormalize, leading to a correction to \( z_h \). We have checked that this correction vanishes as \( k \to 0 \) in the lowest order.

The structure factor at one-loop level is given by
\[ S_h(k, \Omega) = \frac{1}{\omega^2 + |\Sigma_h(k, \Omega)|^2} \left[ 1 + \mu^2 \int \frac{dq}{2\pi} \int \frac{d\Omega}{2\pi} |k - q|^2 S_h(k - q, \Omega) S_h(q, \Omega) \right] \tag{3} \]

which on integrating with respect to \( \omega \) gives \( S_h(k, t = 0) \) as

\[ S_h(k, t = 0) = \int S_h(k, \Omega) d\omega = \frac{A_0}{k} + \frac{B_0}{k^{1+2\alpha_h}} \tag{4} \]

Recognising that the scaling form of \( S_h(k, t = 0) \sim k^{-1-2\alpha_h} \), we notice that \( \alpha_h \) cannot in general be determined from \( \text{Eq.} (4) \). This is because the second term on the right-hand-side of \( \text{Eq.} (4) \) dominates at small momenta \( k \) provided \( \alpha_h > 0 \).

We turn now to the critical exponents in \( \rho \). The single loop self-energy \( \Sigma_\rho(k, \omega) \) is given by

\[ \Sigma_\rho(k, \omega = 0) = -\mu^2 (2\pi)^{-2} \int dq \int d\Omega G_\rho(k - q, -\Omega) S_h(q, \Omega) q^2 \tag{5} \]

This gives, on performing the integral over internal frequency \( \Omega \),

\[ \Sigma_\rho(k, \omega = 0) = -\mu^2 \int dq \int \frac{q^2}{2\pi q^{1+2\alpha_h}} \frac{1}{|k - q|^s + q^{2\alpha_h}} \tag{6} \]

We see from the above that \( \Sigma_\rho(k, 0) \), the relaxation rate for \( \rho \) fluctuations, is negative and finite as \( k \to 0 \), and we need to add a positive constant, \( \Sigma_0 \), to the self-energy (\( \Sigma_\rho > |\Sigma_\rho(k \to 0)| \)) for regulatory purposes. This divergence in the relaxation rate, needing regulation, is reflected in the divergence we have encountered in our numerical investigations below; we have there followed an analogous procedure by introducing a numerical regulator which replaces divergent values of the transfer term by suitably defined cutoffs \( [2] \). The resulting constancy of \( \Sigma_\rho \) implies \( z_\rho \approx 0 \) for the regulated equations and will be used in the following.

The correlation function \( S_\rho(k, \omega) \) is given by

\[ S_\rho(k, \omega) = (\omega^2 + k^{2z_\rho})^{-1} (2\pi)^{-2} \int dq \int d\Omega (k - q)^2 S_h(k - q, \omega - \Omega) S_\rho(q, \Omega) \tag{7} \]

which on iteration over \( \omega \) gives

\[ S_\rho(k, t = 0) \sim k^{-(1+2\alpha_\rho)} \sim k^{1-2\alpha_h + z_h} \frac{1}{k^{2z_\rho + 2z_h}(k^{2\alpha_h} + k^{2\alpha_h})} \]

Finally using \( z_\rho \approx 0 \) we have

\[ \alpha_\rho = \frac{\alpha_h + \frac{z_h}{2} - 1}{2} \quad \text{for large } k \tag{8} \]

\[ \alpha_\rho = \frac{\alpha_h - 1}{2} \quad \text{for small } k \tag{9} \]

Given our numerical result of \( \alpha_h = 0.5 \), the above predicts a negative \( \alpha_\rho \), at small \( k \). This is consistent with, and validates our assumption of, a cutoff \( k_0 \) which arises naturally as the wavevector separating the region of \( \alpha_\rho < 0 \) and \( \alpha_\rho > 0 \).

The coupled equations have been numerically integrated by using the method of finite differences. Our grids in time and space were kept as fine-grained as computational constraints allowed. This is in order to avoid the instabilities associated with the discretisation of nonlinear continuum equations. Convergence has been checked by keeping \( \Delta t \) small enough such that the quantities under investigation are independent of further discretisation. In all our calculations, we chose \( D_h = D_\rho = \mu = 1 \).

On discretising the equations \( \text{Eqs.}(4) \) we found once again the divergences that were previously observed in \( [2] \). These divergences are in our view a direct representation of the negativity of \( \Sigma_\rho \). We follow here a parallel course in regulating these via an explicit regulator. In earlier work \( [2] \), a regulator was introduced which replaced the function \( \mu \rho \nabla h \) by the following:

\[ T = +1 \quad \text{for } \mu \rho (\nabla h) > 1 \]

\[ = \mu \rho (\nabla h) \quad \text{for } -1 \leq \mu \rho (\nabla h) \leq 1 \]

\[ = -1 \quad \text{for } \mu \rho (\nabla h) < -1 \]
In addition in this paper, we have introduced noise reduction to the regulated equations which has led to a more accurate evaluation of all our critical exponents.

Our results for the single Fourier transforms for the \((hh)\) correlation function are

(i) The Fourier transform \(S_h(k,t = 0)\) (Fig. 1) is consistent with a spatial roughening exponent \(\alpha_h \sim 0.5 \pm 0.02\) via our observation of \(S_h(k, t = 0) \sim k^{-2.00 \pm 0.04}\).

(ii) The Fourier transform \(S_h(x = 0, \omega)\) (Fig. 2) is consistent with a temporal roughening exponent \(\beta \sim 0.48 \pm 0.01\) via our observation of \(S_h(x = 0, \omega) \sim \omega^{-1.96 \pm 0.02}\). Hence \(z_h \sim 1.00 \pm 0.02\) consistent with our prediction.

The full structure factor \(S_h(k, \omega)\) has been calculated at three different \(k\) points and Fig. 3 displays a fit of our results to an appropriately scaled form of Eq. (3). The spatial structure factor \(S_h(k, \omega = 0)\) shows a power-law behaviour (Fig. 4) given by \(S_h(k, \omega = 0) \sim k^{-3.0 \pm 0.00}\) in qualitative accord with our expression, which predicts an exponent of \(-3\). The temporal structure factor \(S_h(k = 0, \omega)\) shows a power-law behaviour (Fig. 5) given by \(S_h(k = 0, \omega) \sim \omega^{-1.97 \pm 0.03}\) which is in agreement with our prediction of \(\omega^{-2}\) from the expression of the structure factor (Eq. (3)).

Given our values of \(\alpha_h \sim 0.5\) and \(z_h \sim 1\), Eqs. (5) and (6) predict a crossover in \(\alpha_h\) from 0.0 at large \(k\) to -0.5 as \(k \to 0\). We observe that the single Fourier transform \(S_{\rho}(k, t = 0)\) (Fig. 6) shows a crossover behaviour from \(S_{\rho}(k, t = 0) \sim k^{-2.00 \pm 0.08}\) for large wavevectors to \(S_{\rho}(k, t = 0) \sim \) constant as \(k \to 0\). This indicates a crossover from \(\alpha_\rho = 0.5\) for large \(k\) to -0.5 as \(k \to 0\), which shows the same trend as the prediction above. Note however that the simulations also manifest in addition to the above the normal diffusive behaviour represented by \(\alpha_\rho = 0.5\) at large wavevectors. This anomalous smoothing behaviour in \(\rho\) is the direct consequence of the infrared divergence in Eq. (2) discussed in the preceding.

We have analysed in the above via perturbative methods a model of sandpile dynamics which was presented without analysis in earlier work [2]. The good agreement between our theoretical predictions and numerical simulations confirms our contention that Edwards' infrared divergence [3] plays a crucial role in producing unexpected critical exponents in our model.

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FIG. 1. The single Fourier transform \(S_h(k, t = 0)\). with a power-law fit to an exponent of \(2.0 \pm 0.04\).

FIG. 2. The single Fourier transform \(S_h(x = 0, \omega)\). with a power-law fit to an exponent of \(1.96 \pm 0.02\).

FIG. 3. The double Fourier transforms \(S_h(k_i, \omega)\) evaluated at three different \(k_i = 0.1, 0.2, 0.3\) fitted to Eq. (4).

FIG. 4. The double Fourier transform \(S_h(k, \omega = 0)\) with a power-law fit to an exponent of \(3.3 \pm 0.05\).

FIG. 5. The double Fourier transform \(S_h(k = 0, \omega)\) with a power-law fit to an exponent of \(1.97 \pm 0.03\).

FIG. 6. The single Fourier transform \(S_{\rho}(k, t = 0)\) with a power-law fit to an exponent of \(2.0 \pm 0.08\) at large \(k\) crossing over to an exponent of zero at small \(k\).
Fig. 1

$S_h(k, t=0)$
Fig. 2

$S_h(x=0, w)$
Fig. 3

$S_{h(k_i, w)}$
Fig. 4

$S_{h(k, w=0)}$
Fig. 6

\[ S_r(k,t=0) \]