Direct evaluation of measurement uncertainties by feedback compensation of decoherence

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It is shown that measurement uncertainties can be observed directly by evaluating the feedback compensation of the decoherence induced by the measured system on a probe qubit in a weak interaction occurring between state preparation and measurement. The uncompensated decoherence is described by the measurement uncertainties introduced by Ozawa in Phys. Rev. A 67, 042105 (2003), confirming the empirical validity of measurement theories that combine the initial information of the input state with the additional information provided by each measurement outcome.

As new quantum technologies are being developed, fundamental questions may obtain new and unexpected practical significance. An interesting question concerns the measurement error associated with the observation of a physical property in an uncertainty limited measurement [1–3]. Since it is not possible to go back in time to perform a precise measurement of the target observable, the uncertainty principle itself seems to prevent any observable effects of the measurement error. A mathematical definition of the error based on the representation of values by their corresponding operators was proposed by Ozawa [4], but this mathematical definition has been criticized precisely because it refers to hypothetical properties that do not appear in the observable statistics of quantum states [5–9]. In fact, the insistence on concepts considered to be “useful” in quantum information protocols might have done more harm than good in the objective and scientific discussion of the issue. Specifically, the consistency of Ozawa’s theory with the results of weak measurements and the fact that the results of error free measurements can be anomalous weak values have not been sufficiently recognized as convincing evidence in favor of these theories, possibly because it is assumed that there are no practical implications of these theories [10, 11]. It is therefore important to consider the possibility that some practical effects associated with a physical property between state preparation might have been overlooked.

Taking inspiration from recent implementations of quantum feedback protocols [12–14, 16, 17], I would like to propose the idea of using the outcome of a measurement as a feedback signal on a quantum probe that weakly interacted with the system between state preparation and measurement. In this case, the uncertainty of the target observable causes a small but detectable amount of decoherence in the probe system. This decoherence corresponds to a random unitary operation, the parameter of which is determined by the value of the uncertain observable. If more information on that observable is obtained in a quantum measurement, this information can be used to implement a negative feedback to compensate the decoherence suffered by the probe. It is then possible to directly observe the uncertainty of the quantum measurement in the amount of uncompensated decoherence of the probe. A standardized setup for feedback compensation of decoherence using a qubit as a probe can thus be used to implement an operational definition of measurement uncertainty. The measurement uncertainty determined in this standardized manner is a technically relevant property of the quantum measurement that provides practical information on the performance of the measurement within a larger quantum circuit.

In the following analysis, I show that the uncompensated decoherence in the feedback compensation scenario described above is given by the Ozawa uncertainty of the measurement. The optimal definition of the measurement outcomes is given by the weak values of the target observable for each measurement outcome [11]. Complete feedback compensation is possible when the weak values of pure state inputs are all positive and real. Weak values therefore provide an accurate representation of quantum fluctuations in the absence of direct measurements of the fluctuating observable. The present scenario thus highlights a fundamental difference between the nature of quantum fluctuations and the statistical fluctuations of classical signals. Importantly, this difference emerges from empirical standards that do not distinguish between the physical effects of quantum fluctuations and classical fluctuations. The method proposed here thus provides an objective benchmark test of measurement uncertainties, independent of any theoretical assumptions.

The scenario to be considered is shown in Fig. 1. A probe qubit is prepared in an eigenstate of \( \hat{X} \), which means that it is now maximally sensitive to phase shifts generated by the operator \( \hat{Z} \). The probe qubit then interacts weakly with the target observable \( \hat{A} \) of the quantum system under investigation. The interaction is described by the unitary
FIG. 1. Feedback compensation of the decoherence in a probe qubit caused by a weak interaction with the noisy property \( \hat{A} \) of a quantum system. The value of the property \( \hat{A} \) is estimated based on the measurement outcome \( m \) and a corresponding negative feedback of \( -\hat{A}(m) \) is applied to the probe qubit. The amount of decoherence observed in the output of the probe qubit is a directly observable quantitative measure of the error in the estimate \( \hat{A}(m) \) of \( \hat{A} \).

The interaction with the system depends on the value of the property \( \hat{A} \) in the input state \( \hat{\rho}_S \) of the system. In general, the statistics of \( \hat{A} \) can be represented by the probabilities of the eigenstates of \( \hat{A} \) in the state \( \hat{\rho}_S \). The uncertainty associated with this probability distribution results in decoherence, since the phase changes in the probe qubit superposition of \( Z = -1 \) and \( Z = +1 \) depend on the random eigenvalue \( A_a \) of \( \hat{A} \). Without any feedback, the uncompensated decoherence can be expressed by

\[
\langle \hat{X} \rangle_{\text{out}} = \sum \langle a | \hat{\rho}_S | a \rangle \cos \left( \frac{2\epsilon}{\hbar} A_a \right).
\] (2)

For sufficiently small values of \( \epsilon \), the decoherence is approximately determined by the uncertainty \( \Delta A^2 \) of the property \( \hat{A} \) in the input state \( \hat{\rho}_S \). If the expectation value of \( \hat{A} \) is zero,

\[
1 - \langle \hat{X} \rangle_{\text{out}} \approx \frac{2\epsilon^2}{\hbar^2} \Delta A^2.
\] (3)

The decoherence in the probe qubit thus provides us with direct evidence of the statistical fluctuations of the property \( \hat{A} \) in the system state \( \hat{\rho}_S \).

We can now investigate whether the outcome \( m \) of a quantum measurement described by a positive operator valued measure \( \{ \hat{E}(m) \} \) contains any information about the quantity \( \hat{A} \) before the measurement. The interaction between the system and the probe qubit created a correlation between the phase rotation of the qubit and the value of \( \hat{A} \) in the system. A correct estimate of the value of \( \hat{A} \) can undo the effects of the interaction and restore maximal phase coherence to the probe qubit. The experimental test of the quality of the measurement based estimate of \( \hat{A} \) is shown in Fig. 1. It consists of a feedback signal that compensates the effects of the estimated value \( \hat{A}(m) \) on the probe qubit,

\[
\hat{U}_Z(m) = \exp \left( -i\frac{\epsilon}{\hbar} (-\hat{A}(m)) \hat{Z} \right).
\] (4)
We can combine this operation together with the original interaction to arrive at a more direct expression of the feedback compensation,

\[ \hat{U}_Z(m)\hat{U}_{SP} = \exp\left(-i\frac{\epsilon}{\hbar}(\hat{A} - A(m)) \otimes \hat{Z}\right). \]  

(5)

Since the unitary operation now depends on the measurement outcome, it is necessary to sum over all the possible outcomes \(m\) to find out the net effect of the different feedback operations associated with the estimates \(A(m)\). The modified expectation value of the probe qubit output is then given by

\[ \langle \hat{X} \rangle_{(\text{out})} = \sum_m \text{Tr} \left( (\hat{E}(m) \otimes \hat{X})\hat{U}_Z(m)\hat{U}_{SP}(\hat{\rho}_S \otimes \hat{\rho}_{X=+1})\hat{U}_{SP}^\dagger\hat{U}_Z^\dagger(m) \right). \]  

(6)

This equation is greatly simplified by the fact that the unitary operators commute with the \(\hat{Z}\) operator of the probe qubit. Since the operator \(\hat{X}\) exchanges the eigenstates of \(\hat{Z}\), the equation is a sum of two complex conjugate terms in which opposite eigenvalues of \(\hat{Z}\) appear in the unitary operations between which the input state is sandwiched. The result is an expression that refers only to the Hilbert space of the system,

\[ \langle \hat{X} \rangle_{(\text{out})} = \text{Re} \left( \sum_m \text{Tr} \left( \hat{E}_S(m) \exp(i\frac{\epsilon}{\hbar}(\hat{A} - A(m)))\hat{\rho}_S \exp(i\frac{\epsilon}{\hbar}(\hat{A} - A(m))) \right) \right). \]  

(7)

Note the similarity of this equation with Eq.(2), where the sum ran over eigenstates of \(\hat{A}\). In the feedback compensated result, the sum must run over the actual measurement outcomes, and the value of \(\hat{A}\) is still represented by an operator. Eq.(7) evaluates the fluctuations of \(\hat{A}\) without assigning error free values of \(\hat{A}\) to each outcome \(m\). For sufficiently weak interactions, the decoherence that remains after feedback compensation is given by

\[ 1 - \langle \hat{X} \rangle_{(\text{out})} \approx \frac{2\epsilon^2}{\hbar^2} \sum_m \text{Tr} \left( \hat{E}_S(m)(\hat{A} - A(m))\hat{\rho}_S(\hat{A} - A(m)) \right). \]  

(8)

The uncompensated decoherence represents the amount by which the estimates \(A(m)\) differ from the actual phase shifts induced by the operator \(\hat{A}\) in the initial interaction. The amount of uncompensated decoherence can therefore be used to evaluate the error of the estimates \(A(m)\). Comparison with Eq.(3) shows that the quantitative error corresponds to a quantum uncertainty \(\eta_A\) of the measurement outcomes \(A(m)\) given by

\[ \eta_A^2 = \sum_m \text{Tr} \left( \hat{E}_S(m)(\hat{A} - A(m))\hat{\rho}_S(\hat{A} - A(m)) \right). \]  

(9)

Remarkably, this formula is identical to the general definition of measurement uncertainties given by Ozawa in [4] even though it describes the directly observable amount of decoherence in a feedback compensation scenario. Previous discussions of measurement uncertainties have been based on theoretical speculations regarding the unobservable error-free values of the physical property \(\hat{A}\) [9, 18]. With regard to Ozawa’s proposal, experimental confirmations have been based on theoretical arguments regarding the measurement outcomes obtained with input states different from the one for which the uncertainty was determined [19, 22]. The present result shows that the Ozawa uncertainty characterizes an actual physical phenomenon associated with a specific combination of input state \(\hat{\rho}_S\) and measurement \(\{\hat{E}(m)\}\). The Ozawa uncertainty is experimentally observable as uncompensated decoherence following a feedback compensation based on the estimate \(A(m)\) of \(\hat{A}\) for an outcome \(m\) in a measurement of the system after its interaction with the probe qubit.

There are a number of very important consequences to this result associated with the previously known properties of Ozawa uncertainties [11, 23]. Most importantly, the measurement results \(A(m)\) assigned to the outcomes \(m\) can be optimized to reduce the decoherence to its minimal value. The result of this optimization corresponds to an assignment of weak values to the measurement outcomes,

\[ A_{\text{opt.}}(m) = \text{Re} \left( \frac{\text{Tr}(\hat{E}_S(m)\hat{\rho}_S)}{\text{Tr}(\hat{E}_S(m)\hat{\rho}_S)} \right). \]  

(10)

Weak values therefore provide the best estimate for a compensation of decoherence induced by the quantum fluctuations of \(\hat{A}\). Note that this result is obtained from the analysis of a feedback compensation procedure that is independent of the original definitions of weak values. The emergence of weak values in the present context thus shows that weak values correspond to empirically valid measures of a physical quantity between state preparation
and measurement and can be defined operationally without any assumptions about their quantum mechanical origin, revealing a serious flaw in arguments that seek to explain weak values in terms of their quantum mechanical origin. In addition, the Ozawa uncertainty $\eta_A$ associated with the weak values describes the remaining fluctuations. It is therefore possible to evaluate how well the weak values describe the actual values of a quantity $\hat{A}$ between state preparation and measurement. As shown in previous works, the Ozawa uncertainty drops to zero for projective measurements of pure states if all of the weak values are real,

$$\eta_A = 0 \quad \text{for} \quad A(m) = \frac{\langle m | \hat{A} | \psi \rangle}{\langle m | \psi \rangle}.$$  \hspace{1cm} (11)

This condition is satisfied by a wide range of projective measurements $\{| m \rangle\}$, indicating that error free measurements can be very different from the conventional projections onto eigenstates of the operator $\hat{A}$. The fluctuations of $\hat{A}$ in $| \psi \rangle$ are not defined by the eigenvalues of $\hat{A}$ and their probabilities, but take contextual values depending on the type of measurement made to complement the information about $\hat{A}$ already available in $| \psi \rangle$. This observation may be of particular importance when the decoherence effects of different non-commuting observables need to be compensated at the same time. In this case the uncertainty relations given by Ozawa in [4] and later improved upon by Branciard in [21] provide the correct limit of a joint compensation protocol. The analysis of feedback compensation thus reveals how quantum contextuality works in a wide range of practically relevant situations.

In conclusion, the analysis presented above shows that feedback compensation of decoherence caused by sufficiently weak interactions of the system with a probe qubit can be used as a direct experimental evaluation of the measurement uncertainty for the measurement outcomes $A(m)$ assigned to each individual result $m$. The analysis of the feedback compensation scenario shows that the theory derived by Ozawa which was entirely based on the mathematical description of physical properties by the operator algebra correctly describes the amount of uncompensated decoherence. This result demonstrates that the Ozawa uncertainties can be observed directly in an experimental test of the actual measurement procedure. Different from previous methods [19–22], no tomographic reconstruction is needed. In addition, the optimal feedback is obtained when the weak values associated with each measurement result are used as outcomes of the measurement. Weak values therefore represent the optimal estimates of physical properties between preparation and measurements and provide an accurate description of the quantum fluctuations of an observable $\hat{A}$ in the pure state limit where the Ozawa uncertainties drop to zero [11, 23]. This result demonstrates that weak values are not just a mathematical artifact, but describe technically relevant effects that are meaningful outside of their quantum theoretical description. Feedback compensation of decoherence thus confirms the empirical validity of both Ozawa’s generalization of uncertainty and of weak values without any appeal to untestable assumptions.

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