Dynamical gauge symmetry breaking in strongly coupled lattice theories

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We show how a strongly coupled lattice theory consisting of just fermions and gauge fields can exhibit a dynamical Higgs mechanism through the formation of a gauge invariant four fermion condensate. Furthermore, we argue that this lattice Higgs phase may survive into the continuum limit.

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I. INTRODUCTION

The idea that the Higgs mechanism can occur through the formation of fermionic condensates is an attractive one when constructing many theories of Beyond Standard Model (BSM) physics and finds application in technicolor, composite Higgs models, tumbling and grand unification schemes [1–4]. Lattice realizations of these scenarios thus potentially give a rigorous setting for understanding how non-perturbative dynamics in models without elementary scalars can spontaneously break gauge symmetries and potentially can give us new tools to analyze such theories.

Of course the terminology dynamical or spontaneous gauge symmetry breaking is strictly misleading; the gauge symmetry never truly breaks but instead becomes hidden and the theory enters a Higgs phase characterized by a Yukawa-like behavior for the potential between static sources. Indeed, in a lattice theory, Elitzur’s theorem [5] guarantees that any condensate which is not invariant under the gauge symmetry must necessarily have vanishing expectation value. Instead, the gauge invariant way to understand the operation of the Higgs mechanism in such theories is that it proceeds via the condensation of a gauge invariant four fermion operator. The formation of this four fermion condensate can be thought of as equivalent to the development of a non-trivial effective potential for $\phi^2$ where $\phi$ is a composite Higgs field corresponding to a bilinear fermionic condensate carrying gauge charges. With this caveat in mind we will often use the terminology of spontaneous or dynamical gauge symmetry breaking when discussing the appearance of a Higgs phase in these theories since it is commonly employed in continuum, gauged-fixed discussions of the Higgs mechanism.

The construction of lattice theories which exhibit such a Higgs phase has traditionally proven difficult and indeed a folklore has developed that it is impossible. This belief derives from two well known results; the Vafa Witten theorem [6] which prohibits spontaneous breaking of global vector symmetries and the well known difficulties of constructing non-perturbative lattice chiral gauge theories (see the reviews [7–10] and references therein). However, there is a loophole in these arguments; one can imagine a lattice theory describing Dirac fermions which develops a condensate which Higgses a vector gauge symmetry at non-zero lattice spacing or equivalently strong coupling. Furthermore there will be no contradiction with the Vafa Witten theorem if this symmetry reappears as an axial symmetry in the continuum limit.

In this paper we construct a lattice gauge theory based on reduced staggered fermions which appears to possess precisely this structure and we will show via explicit simulations that indeed the resultant lattice theory develops a Higgs phase as a result of strong dynamics. Furthermore, since we observe no phase transitions separating this phase from the weak coupling region we infer that this lattice phase likely survives in the continuum limit where we argue the symmetry takes on an axial character.

We start our construction by considering a free, massless, continuum theory consisting of four Dirac fields. We then show how to twist the fermions so that the action can be rewritten in terms of fermionic matrices. This twisted action can then be decomposed into two independent pieces corresponding to the projection of the matrix fermions into two independent components. The key observation is that the kinetic terms for each of these components may then be gauged separately under any internal symmetries. This is completely analogous to the procedure one follows to build chiral gauge theories. Once this is done the theory no longer admits gauge invariant mass terms which are also invariant under the twisted Lorentz symmetry. It is important to recognize, however, that the theory still contains an equal number of left and right handed fermions in any given representation of the gauge group and hence it is not a chiral gauge theory.

The importance of these twisted continuum theories becomes apparent when one uses them to derive lattice theories which is done in section III. In this case each continuum projected matrix fermion yields a reduced staggered fermion in the lattice theory and the kinetic terms for these lattice fields are uncoupled as in the continuum. Subsequently gauging these terms under independent internal symmetries leads to a result, analogous to the continuum, that single site fermion bilinear terms will, in general, break gauge invariance. We use this feature to build interesting models by assuming that the gauge interactions factorize into a strong and weakly coupled sector; the strong interaction favoring the generation of condensates which can break the weakly coupled gauge...
II. CONTINUUM TWISTED THEORY

Consider first a continuum (Euclidean) theory comprising four degenerate massless Dirac fields $\psi^i$ where the index runs from $i = 1 \ldots 4$. The action for this theory is invariant under the global symmetry $SO_{\text{Lorentz}}(4) \times SU_{\text{flavor}}(4)$. Let us focus on the $SO(4)$ subgroup of the flavor symmetry. Under flavor and Lorentz symmetries the fermions transform like

$$\psi_{\alpha i} \rightarrow L_{\alpha \beta} \psi_{\beta j} F_{ji}$$

(1)

Under the action of the diagonal subgroup $SO'(4) = \text{diag} (SO_{\text{Lorentz}} \times SO_{\text{flavor}}(4))$ corresponding to choosing $F = L$ above the fermions transform like matrices $\Psi$ and the action may be written as

$$S = \int d^4x \text{Tr} (\overline{\Psi} \gamma_{\mu} \partial_{\mu} \Psi)$$

(2)

This is the called the twisted representation of the original theory and is only possible when a subgroup of the flavor symmetry matches the Euclidean Lorentz symmetry.

Now let us define the projectors $P_\pm X = \frac{1}{2} (X \pm \gamma_5 X \gamma_5)$ for some $4 \times 4$ matrix $X$. It is easy to show that $P_+^2 = P_+$, $P_-^2 = P_-$ and $P_+ P_- = P_- P_+ = 0$ as usual. Starting from the field $\Psi$ we can define the projected field $\Psi_{\pm} = P_{\pm} \Psi$ with the property that $P_\pm \Psi_{\mp} = 0$. We will assume that $\Psi$ transforms in some representation of an internal global symmetry group. In the same way we can introduce a new twisted matrix field $\Lambda$ transforming in some other representation and project it down to $\Lambda_{\pm} = P_{\pm} \Lambda$. Using this we can construct a more general action than that given in eqn. (2) which is the sum of two independent projected fields carrying different internal quantum numbers.

$$S = \int \text{Tr} (\overline{\Psi}_+ \gamma_{\mu} \partial_{\mu} \Psi_-) + \text{Tr} (\overline{\Lambda}_- \gamma_{\mu} \partial_{\mu} \Lambda_+)$$

(3)

In a chiral basis these projected matrices take the explicit $2 \times 2$ block form

$$\Psi_+ = \begin{pmatrix} \overline{\psi}_L & 0 \\ 0 & \psi_R \end{pmatrix}, \quad \Lambda_- = \begin{pmatrix} 0 & \overline{\lambda}_R \\ \lambda_L & 0 \end{pmatrix}, \quad \Lambda_+ = \begin{pmatrix} \lambda_R & 0 \\ 0 & \lambda_L \end{pmatrix}, \quad \Psi_- = \begin{pmatrix} 0 & \psi_R \\ \overline{\psi}_L & 0 \end{pmatrix}$$

(4)

The action written in terms of these $2 \times 2$ blocks is then

$$S = \int d^4x \text{tr} \left( \overline{\Psi}_L \sigma_\mu \partial_\mu \psi_L + \psi_R \overline{\sigma}_\mu \partial_\mu \psi_R \right)$$

$$+ \int d^4x \text{tr} \left( \overline{\Lambda}_L \sigma_\mu \partial_\mu \lambda_L + \lambda_R \overline{\sigma}_\mu \partial_\mu \lambda_R \right)$$

(5)

with $\sigma_\mu = (I, \sigma_i)$ and the $\text{tr}$ symbol denotes the trace over the remaining $2 \times 2$ blocks. The twisted fields transform under their internal symmetries according to

$$\Psi_+ \rightarrow \Psi_+ G^I$$

$$\Psi_- \rightarrow G \Psi_-$$

$$\Lambda_- \rightarrow \Lambda_- H^I$$

$$\Lambda_+ \rightarrow H \Lambda_+$$

(6)

In Weyl components the transformed matrix fields read

$$\Psi'_+ = \begin{pmatrix} \overline{\psi}_L G^I & 0 \\ 0 & \psi_R G^I \end{pmatrix}, \quad \Lambda'_- = \begin{pmatrix} 0 & \overline{\lambda}_R H^I \\ \lambda_L H^I & 0 \end{pmatrix}$$

$$\Lambda'_+ = \begin{pmatrix} H \lambda_R & 0 \\ 0 & H \lambda_L \end{pmatrix}, \quad \Psi'_- = \begin{pmatrix} 0 & G \psi_R \\ G \overline{\psi}_L & 0 \end{pmatrix}$$

(7)

These internal symmetries can be made local by introducing appropriate covariant derivatives. However, notice that unless $G = H$ it is not possible to write down gauge invariant mass terms which are simultaneously twisted Lorentz invariant. The natural fermion bilinear invariant under the twisted Lorentz symmetry is

$$\text{Tr} (\overline{\Psi}_+ \Lambda_+ + \overline{\Lambda}_- \Psi_-)$$

(8)

which clearly corresponds to a set of degenerate Dirac mass terms when written out in components

$$\text{tr} \left( \overline{\Psi}_L \lambda_R + \overline{\psi}_R \lambda_L + \overline{\lambda}_R \psi_L + \overline{\lambda}_L \psi_R \right)$$

(9)

Such a term breaks the original $G$ and $H$ symmetries since it transforms as

$$\overline{\Psi}_+ G^I H \Lambda_+ + \overline{\Lambda}_- H^I G \Psi_-$$

(10)

Notice that if a condensate of this type forms in the theory it implies that the low energy theory can be written in terms of Dirac spinors with the broken symmetries being realized as axial transformations of these spinors.

From this analysis it should be clear that the fermionic interaction term that is invariant under both gauge and twisted Lorentz symmetries is a four fermion operator of the form

$$\text{Tr} (\overline{\Psi}_+ \Lambda_+) \text{Tr} (\overline{\Lambda}_- \Psi_-)$$

(11)

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1 It is the same representation used in recent constructions of supersymmetric lattice theories.

2 Gauge invariant bilinear mass terms which break twisted Lorentz invariance but maintain the usual Lorentz symmetry are also possible. However, in section VII and section VIII we shall present evidence that condensates with this structure do not form in this lattice model.
Notice that the explicit traces appearing in this operator act only on (twisted) spin and flavor indices. Each fermion bilinear appearing in this term will also carry internal gauge indices which are suppressed in this expression. Later we will assume that the gauge interactions factor into a strong and weakly coupled sector and that the fermion bilinears appearing in the four fermi term are singlets under the strong force but in general transform under the weak gauge group. As can be seen from eqn. \[10\] the four fermi term is nevertheless fully gauge invariant. If a condensate of this form appears in the vacuum it implies that the bilinear terms themselves must be non-zero. Viewing the latter as composite scalar fields it is clear that the formation of such a four fermion condensate is equivalent to the statement that the effective potential for the composite scalars must have developed a minimum away from the origin and hence that the theory is in a Higgs phase. In addition, this four fermion operator simultaneously acts as an order parameter for the breaking of the global symmetry of the twisted theory $SO(4) \times SO(4) \to SO(4)$ (each $SO(4)$ factor in the unbroken theory corresponds to the usual $SU(2) \times SU(2)$ chiral symmetry expected for two Dirac fermions). Notice that this global symmetry of the twisted theory is much smaller than the usual $SU(4) \times SU(4)$ chiral symmetry expected for four massless Dirac fermions but follows once one gauges the kinetic terms for the projected fields differently.

Finally, to complete the action of this continuum theory, we add the usual field strengths corresponding to the gauge fields associated with the symmetries $G$ and $H$. In the next section we will discretize this theory using the staggered fermion prescription and derive a lattice model with similar properties.

### III. LATTICE MODEL

We start with the matrix fields $\Psi$ and $\Lambda$ introduced in the last section and expand these matrices on a basis corresponding to products of gamma matrices and associate these products with staggered fields.

\[
\Psi(x) = \frac{1}{8} \sum_b \gamma^{x+b} \chi(x+b)
\]
\[
\Lambda(x) = \frac{1}{8} \sum_b \gamma^{x+b} \xi(x+b),
\]
where $\gamma^{x+b} = \prod_{i=1}^{4} \gamma_{i}^{x_{i}+b_{i}}$ and the sums correspond to the vertices in an elementary hypercube associated with lattice site $x$ as the components vary $b_{i} = 0, 1$. It is easy to see that the projected matrix fields $\Psi_{-}$ and $\Lambda_{+}$ introduced in the continuum construction corresponds to restricting the staggered field $\chi$ and $\xi$ to even and odd parity lattice sites. With a small abuse of notation we can write

\[
\Psi_{+}(x) \to \frac{1}{2} (1 + \epsilon(x)) \chi(x) = \overline{\psi}_{+}(x)
\]
\[
\overline{\Lambda}_{-}(x) \to \frac{1}{2} (1 - \epsilon(x)) \xi(x) = \overline{\lambda}_{-}(x)
\]
\[
\Lambda_{+}(x) \to \frac{1}{2} (1 + \epsilon(x)) \xi(x) = \lambda_{+}(x)
\]
\[
\Psi_{-}(x) \to \frac{1}{2} (1 - \epsilon(x)) \chi(x) = \psi_{-}(x)
\]

where the parity of a lattice site is given by $\epsilon(x) = (-1)^{\sum_{i=1}^{4} x_{i}}$. The fields $\psi$ and $\lambda$ are termed reduced staggered fermions since each contains half the number of degrees of freedom of the usual staggered fermion and corresponds to two rather than four Dirac fermions in the continuum limit \[12\]-\[15\]. Since the lattice theory results from discretization of a continuum theory with equal numbers of left and right handed fields in any given representation of the gauge group it clearly does not violate the Nielsen-Niomiya theorem \[16\].

The free continuum action given in eqn. \[8\] can then be recast as a lattice action by substituting the matrix expressions given in Eq. \[12\] into the twisted action having replaced the derivative with a symmetric difference operator and evaluating the trace using the orthogonality properties of the gamma matrices. The result is the usual one expected for staggered fermions here written as the explicit sum of two reduced staggered fermion actions

\[
\sum_{x,\mu} \eta_{\mu}(x) \overline{\psi}_{+}(x) (\psi_{-}(x + \mu) - \psi_{-}(x - \mu))
\]
\[
\sum_{x,\mu} \eta_{\mu}(x) \overline{\Lambda}_{-}(x) (\lambda_{+}(x + \mu) - \lambda_{+}(x - \mu))
\]
where the phase $\eta_{\mu}(x)$ is the usual staggered quark phase given by

\[
\eta_{\mu}(x) = (-1)^{\sum_{i=1}^{4} x_{i}}.
\]

Just as before we can now take the staggered fields to transform in different representations of one or more symmetry groups. Following the continuum we take

\[
\overline{\psi}_{+} \to \overline{\psi}_{+} G \dagger
\]
\[
\psi_{-} \to G \psi_{-}
\]
\[
\overline{\lambda}_{-} \to \overline{\lambda}_{-} H \dagger
\]
\[
\lambda_{+} \to H \lambda_{+}
\]

Again, these symmetries can be made local by inserting appropriate gauge links between the $\psi$ and $\lambda$ fields on neighboring sites. However, it is then impossible to write down a single site mass term that preserves these symmetries. If I take the matrix expression given in eqn. \[12\] and substitute it into the expression for the mass term one easily derives the usual single site staggered mass term

\[
(\overline{\psi}_{+}(x) \lambda_{+}(x) + \overline{\lambda}_{-}(x) \psi_{-}(x))
\]

\[3\] It is important to note that this $SO(4)$ flavor symmetry automatically enhances to $SU(4)$ if $\Lambda_{+}$ and $\Psi_{-}$ carry identical representations of the internal symmetry group. In this case they can be thought of as arising by projection from a single twisted field. This is the case for staggered quark simulations of QCD.
which, in general, is no longer gauge invariant.

However, it is possible to write down a gauge invariant lattice four fermion term by analogy with the continuum. As in our discussion of the twisted continuum theory we will assume that the interactions factorize into a strongly coupled sector and a weakly coupled sector and that the $\psi$ and $\lambda$ fields transform differently under the weak symmetries. We can then introduce composite scalar fields which are singlets under the strong gauge symmetry but transform under the weak group:

$$u_+(x) = \overline{\psi}_+(x)\lambda_+(x)$$
$$u_-(x) = \overline{\lambda}_-(x)\psi_-(x)$$  \hspace{1cm} (18)

A gauge invariant four fermi term can then be constructed by connecting these composite scalar fields defined in neighboring lattice sites through an appropriate set of weak gauge links.

As in the continuum the formation of a condensate of this form signals a Higgsing of lattice gauge symmetry. But as we saw earlier this symmetry has an axial character in the usual (untwisted) continuum theory since it acts differently on left and right handed components of a given Dirac spinor. This is at first sight puzzling; how can a manifestly vector lattice symmetry be reconciled with an axial symmetry in the continuum limit? The resolution to this puzzle is easily found; recall that the even parity lattice fields $\lambda_+$ are contained in the continuum twisted matrix field $\Lambda_+$ and transform according to $H$ while odd parity lattice fields $\psi_-$ live in $\Psi_-$ and transform according to $G$. The condensate we examine couples $\Psi_+$ to $\Lambda_+$ and contains terms like $\overline{\psi}_L\lambda_R + \overline{\lambda}_R\psi_L$. This in turn implies that the physical fields surviving at long distance comprise Dirac spinors whose chiral components transform differently under these broken internal symmetries. This chiral or axial behavior cannot however be transferred to the lattice since the natural candidate for a lattice Dirac spinor would be assembled from $\psi$ and $\lambda$ fields at different sites. Such an object does not transform covariantly under lattice gauge transformations and hence no exact axial symmetry is possible in the lattice model. However, formally these constraints disappear at vanishing lattice spacing.

Notice that the single site condensate discussed here is similar to that encountered in conventional staggered lattice simulations of QCD. The usual $U(1)_V \times U(1)_A$ of staggered fermions is then equivalent to a global $U(1) \times U(1)$ vector symmetry of the system of two reduced staggered fermions that we study. The condensate breaks this symmetry down to its diagonal $U(1)$ subgroup which then becomes fermion number. The orthogonal combination is then to be interpreted as the usual broken axial symmetry of staggered fermions. The reader should notice the structure of this argument; a vector symmetry of the system of 2 reduced staggered fermions is reinterpreted as a (broken) axial symmetry in the usual staggered fermion model.

Finally, as for the continuum, we will need to introduce kinetic terms for the lattice gauge fields by adding the usual Wilson plaquette terms built from the corresponding gauge links.

We will illustrate these ideas by constructing technicolor-like models in which the gauge interactions factorize into strong and weakly coupled sectors with corresponding gauge groups $S = SU(N)$ and $W = SU(M)$ respectively. As an explicit example consider the case where the twisted matrix fermion $\Psi_-$ transforms in the fundamental representation of both $S$ and $W$ while $\Lambda_+$ transforms as a fundamental under $S$ but is sterile under $W$. Thus any bilinear of the form $\text{Tr} \left( \Psi_+ \Lambda_+ \right)$ transforms as an anti-fundamental under $W$ and hence a condensate with this structure will break $SU(M) \rightarrow SU(M-1)$.

### IV. GLOBAL SYMMETRY BREAKING

Before turning to the problem of breaking local or gauge symmetries it is instructive to first consider the simpler case where the would be broken symmetry is global. It is well know how to search for spontaneous symmetry breaking in this case. One measures the expectation value of some suitable order parameter in a modified version of the theory which incorporates a symmetry breaking external field coupled to that order parameter. The symmetry breaking is then signaled by a nonzero value for the expectation value of the order parameter as the external field is sent to zero after the thermodynamic limit is taken. In this case the natural order parameter is the condensate $< \overline{\psi}_\lambda >$.

We can look for such a condensate using numerical simulations of the associated model based on reduced staggered fermions as described in section III. The gauging of the lattice kinetic term is then given explicitly as

\begin{equation}
\sum_{x,\mu} \overline{\psi}_+(x) \left( U_\mu(x)V_\mu(x)\psi_-(x+\mu) - U_\mu^\dagger(x-\mu)V_\mu^\dagger(x-\mu)\psi_-(x-\mu) \right)
+ \sum_{x,\mu} \overline{\lambda}_-(x) \left( V_\mu(x)\lambda_+(x+\mu) - V_\mu^\dagger(x-\mu)\lambda_+(x-\mu) \right) \hspace{1cm} (19)
\end{equation}

where we set the weak gauge links $U_\mu(x) = 1$ for the case of a global symmetry. As an external symmetry breaking perturbation we add to the action the following term

\begin{equation}
\sum_x g \left[ \phi \overline{\psi}_+(x)\lambda_+(x) + \phi^* \psi_-(x)\overline{\lambda}_-(x) \right] \hspace{1cm} (20)
\end{equation}
with $\phi$ a constant external field transforming in the fundamental of the $SU(M)$ group. We break the $SU(2)$ symmetry explicitly by setting $\phi = (1,0)$ and in this case the Yukawa coupling $g$ gives the magnitude of the symmetry breaking external field.

We have studied this model for the case $N = 3$ and $M = 2$ using the standard RHMC algorithm and working in the phase quenched approximation (the reduced staggered fermion models we consider here in general have a sign problem) Fig. 1 shows the results for the bilinear condensate $<\overline{\psi}\lambda>$ as the strong gauge coupling $\beta_S$ is varied for a fixed value of the external field $g = 0.1$ on a lattice of size $L = 4$. On the same plot we show the Polyakov line corresponding to the strongly coupled $SU(3)$ gauge field. The latter is small at strong coupling (small $\beta_S$) and rises for $\beta_S > 5.5$ corresponding to the onset of a deconfinement in the $SU(3)$ sector for weak enough coupling. The condensate shows a complementary behavior saturating at a non-zero value for small $\beta_S$ and dropping quickly for large values of $\beta_S$. Indeed, one can see that the change in behavior for both these quantities is highly correlated; as soon as the strong sector of the model deconfines the condensate starts to fall. The rate at which the condensate falls to zero is governed by the magnitude of the symmetry breaking field - the plot shows data for $g = 0.1$ but we have observed that the fall off with $\beta_S$ appears more abrupt for smaller $g$. Notice that the effective number of fermion flavors in these simulations is given by $N_f = 6 = 2 \times (2_{SU(2)} + 1_{sterile})$ since each reduced field yields two continuum Dirac fermions. We expect confinement for this number of flavors in $SU(3)$.

Of course the question of whether we are seeing spontaneous symmetry breaking hinges on the behavior of the condensate as we send the external field $g$ to zero. Fig. 2 shows the condensate for lattices of size $L = 4$ and $L = 6$ as we scan in the symmetry breaking field $g$ at a fixed value of $\beta_S = 5.5$ representative of the scaling window for these small lattices. A broad plateau is visible which declines slowly for large $g$ and turns over to approach zero for very small $g$. This turnover point moves to smaller $g$ as the lattice size is increased. This result is consistent with the expected behavior for a system in which symmetry is spontaneously broken; while the order parameter must vanish on any finite system one can obtain a nonzero value at ever decreasing values of the external field as the volume of the system increases. In principle one should perform an extrapolation of the condensate to infinite volume and then examine how this extrapolated value varies as the external field is sent to zero. This is beyond the scope of this initial work but would be required in any followup study. However, the behavior we see is certainly consistent with existence of a non-zero condensate in the thermodynamic limit.

We have also checked the volume dependence of the condensate; fig. 3 shows data for both $L = 4$ and $L = 6$. Notice that we have scaled the external field as the inverse of the lattice volume to produce these curves. This is the scaling expected for systems with chiral symmetry breaking where the lattice condensate is expected to be a function of $gV$. The larger lattice data then lies close to the results for $L = 4$. Deviations from this naive scaling likely reflect both finite volume and lattice spacing (running) effects. Notice that the data collapse closely on a single curve in the confined phase while the deviation is larger in the deconfined phase. We think that the former likely indicates the effects of the running coupling while the latter effect is likely due to finite volume. To test these ideas would require simulations on larger volumes and for smaller lattice spacings than in our current study and we hope to pursue this in future work.

The circles visible in this figure correspond to the measurement of an exactly gauge invariant condensate based on the one link operator

$$O_1 = \overline{\psi}_+(x)V_\mu(x)\psi_-(x + \mu)$$ (21)
In terms of the twisted theory this corresponds to an operator of the form $\text{Tr} (\overline{\psi} \gamma_4 \psi)$. While this term is invariant under the usual Lorentz symmetry it is clearly not invariant under the twisted Lorentz symmetry. The numerical results that are shown correspond to the Monte Carlo average of the absolute value of this operator measured on a given configuration (the naive average is statistically consistent with zero). The numerical results show that the one-link condensate is much smaller than the single site condensate and, perhaps more importantly, shows little dependence on the magnitude of the strong coupling $\beta_s$.

Actually, this should not be surprising. Once the weak interactions are set to zero one can interpret the field content as corresponding to a single conventional staggered fermion built from $(\psi, \overline{\psi})$ with conventional single site mass term together with an additional massless reduced staggered fermion $\lambda^2$ where the explicit indices correspond to the exact weak $SU(2)$ symmetry. The condensate we observe is then just the usual single site condensate expected for staggered fermions. Following our experience with QCD we expect that this survives to the continuum limit and breaks all the axial symmetries of the theory which must hence include this exact $SU(2)$ symmetry.

To distinguish such a Higgsed phase from the weakly coupled one one would like to have an order parameter. Elitzur’s theorem tells us that the vacuum expectation value of any gauge variant local observable will vanish even for non-zero $g$ if the action is gauge invariant \cite{17}. Instead one should distinguish a Higgs phase by analysing the asymptotic behavior of non local quantities such as Wilson loops or Polyakov lines. In our numerical work we will base our conclusions partly on the behavior of the one-link operator $O_1$.

V. GAUGE SYMMETRY BREAKING

The breaking of symmetries in a gauge theory is somewhat more subtle; in this case the global external field must be replaced by a local field $\phi(x)$ which also transforms under the gauge symmetry in such a way that this term is exactly gauge invariant. This requires that $\phi$ transforms as a fundamental of $SU(2)$

$$\phi(x) \rightarrow W(x)\phi(x)$$

where $W$ corresponds to weak gauge transformations. To render the path integral well defined after integration over $\phi(x)$ one must then also add a suitable action for $\phi(x)$. We choose an additional simple term $\sum_\mu \phi^\dagger(x)\phi(x)$. The effect of these Yukawa terms is to add a small gauge invariant four fermion interaction to the action that favors the conjectured symmetry breaking pattern. In addition we now allow the lattice gauge field $U_\mu(x)$ appearing in eqn. \cite{19} to fluctuate and add an associated Wilson plaquette term to the action.

To summarize: the numerical results in this section lend strong support to the idea that this staggered fermion system spontaneously breaks an exact global $SU(2)$ symmetry as a result of strong gauge dynamics. Under rather general assumptions such a system should therefore exhibit a Higgs mechanism once this global symmetry is gauged. We will show direct evidence for this in the next section.
condensate studied in the case of a global symmetry.

\[ \sum_\mu < \bar{\psi}(x) \lambda_+(x) > U_\mu(x) < \bar{\lambda}_-(x+\mu) \psi_-(x+\mu) > \]  

(23)

While this quantity is not strictly an order parameter we might expect it to be strongly enhanced in a Higgs phase. It is important to notice that the fermion bilinears appearing in the above Wick contractions are singlets under the strong gauge interactions and carry only weak gauge indices. A final gauge invariant expression is then constructed by tying these objects together using the weak gauge link \( U_\mu \). Again we have tested these ideas for the model with \( N = 3 \) and \( M = 2 \) where one expects complete breaking of the weak \( SU(2) \) for sufficiently strong \( \beta_S \). Figure 4 shows a plot of the four fermion condensate versus the auxiliary Yukawa coupling for \( \beta_S = 5.5 \) and \( \beta_W = 10.0 \). Similar to the case where the \( SU(2) \) symmetry is global we see that the magnitude of the condensate remains approximately constant as we decrease \( g \) until very small \( g \) and that this effect is enhanced as the volume increases.

To verify that the appearance of this condensate is a direct result of the strong \( SU(3) \) dynamics we have also examined the four fermion condensate as a function of \( \beta_S \) for fixed auxiliary coupling \( g = 0.1 \) and \( L = 4 \). The results are shown in figure 5. Clearly the condensate is enhanced at strong coupling falling to small values as \( \beta_S \to \infty \). Most importantly we see no sign of a phase transition as we vary the strong coupling. This is evidence that the Higgs phase of the lattice theory may survive the continuum limit \( \beta_S \to \infty \).

We can see a signal of the appearance of a Higgs phase even more clearly by looking at the plot in fig. 6 of the Polyakov line corresponding to the weak gauge field as we vary the strong gauge coupling \( \beta_S \). For large \( \beta_S \) the weak Polyakov line \( P_W \) is large but as \( \beta_S \) is lowered it rapidly crosses over to fluctuate around a much smaller but non zero value. Since the weak Polyakov line \( P_W = e^{-FT} \) measures the free energy \( F \) of an isolated static quark in the fundamental representation of the weak gauge group a value approaching unity is associated with a deconfined phase. This is to be expected for a bare weak gauge coupling \( \beta_W = 10.0 \) on such a small lattice. Conversely, a confining phase typically would be associated with a small value of \( P_W \) (see the \( SU(3) \) Polyakov line data in fig. 1). What is observed for small \( \beta_S \) is intermediate between these two regimes and may be a signal of a Higgs phase. Furthermore, the crossover between these two regimes corresponds precisely to the switching on of the four fermion condensate and the observation of deconfinement in the strong sector. Thus our numerical results are at least consistent with the dynamical generation of a non-zero four fermion condensate and the appearance of a Higgs phase as a result of strongly coupled dynamics. Clearly to be sure of this interpretation requires much larger simulation volumes and an analysis.
of the full static quark potential which we hope to turn to later.

At this point it is legitimate to ask whether the effect we are seeing is any way connected to the use of the phase quenched approximation. We can answer this by considering results from the same model when we employ an SU(2) strong sector. Figure 4 shows a plot of the condensate versus auxiliary coupling at βs = 1.8 (see [15] for details on this choice of coupling) and βV = 10.0 for this model. Since SU(2) is pseudoreal the fermion measure for this theory is positive definite for vanishing auxiliary coupling. Clearly the condensate still rises to a non-zero plateau even at small g where we have observed that the phase is negligible.

VI. BREAKING TO A DIAGONAL SUBGROUP

It is possible to explore other models in an effort to understand how generic is this symmetry breaking scenario. One simple generalization of the sterile model discussed earlier again assumes that ψ and λ transform in the fundamental representation of a strong SU(N) interaction. But now the weak sector has an SU(M) × SU(M′) structure with the fermions ψ and λ transforming in the (1, □) and (□, 1) representations of this product group. The kinetic terms are now given explicitly as

\[
\sum_{x,\mu} \overline{\psi}_+ (x) (U_\mu (x) V_\mu (x) \psi_-(x + \mu) - U_\mu^\dagger (x - \mu) V_\mu^\dagger (x - \mu) \psi_-(x - \mu)) \\
\sum_{x,\mu} \overline{\lambda}_- (x) (W_\mu (x) V_\mu (x) \lambda_+ (x + \mu) - W_\mu^\dagger (x - \mu) V_\mu^\dagger (x - \mu) \lambda_+ (x - \mu))
\]

where the gauge field \(V_\mu (x)\) corresponds to the strongly coupled SU(N) sector while \(U_\mu (x)\) and \(W_\mu (x)\) are the gauge fields for the two independent SU(M) symmetries. It is clear that a non-zero condensate will now indicate a breaking of the two SU(M)’s down to their diagonal SU_D(M) = diag (SU(M) × SU(M′)). We will focus on the case \(N = 3\) and \(M = 2\) so that the weak symmetries are just SU(2) × SU(2).

Let us first examine the case where the weak gauge coupling is set to zero and the weak symmetry is purely global. Since each reduced staggered field contributes 2 Dirac fermions in the continuum limit the theory will contain eight Dirac fields with global symmetry SUV(8) × SUA(8). As a result of strong interactions we expect that this symmetry will break according to the pattern

\[SU_V(8) × SU_A(8) → SU_V(8)\]  

The SU(2) × SU(2) weak symmetries of the staggered lattice theory constitute a subgroup of this continuum global symmetry. The precise embedding of these weak symmetries in the continuum limit can be obtained by the following argument. \(^4\)

We start by assuming that each staggered field yields 2 Dirac fermions in the continuum eg. \(ψ_1 \rightarrow \{Ψ_1^1, Ψ_1^2\}\) with the upper indices reflecting this extra factor of two. We can then arrange these continuum fields into an eight component vector

\[
(Ψ_1^1 \Lambda_1^1, Ψ_1^2 \Lambda_1^2, Ψ_2^1 \Lambda_2^1, Ψ_2^2 \Lambda_2^2)
\]

In this representation the broken generators \(τ_α, a = 1 \ldots 3\) take the explicit form

\[
τ_1 = \begin{pmatrix} 0 & σ_3 \\ σ_3 & 0 \end{pmatrix} \times I_2 \\
τ_2 = \begin{pmatrix} 0 & iσ_3 \\ -iσ_3 & 0 \end{pmatrix} \times I_2 \\
τ_3 = \begin{pmatrix} σ_3 & 0 \\ 0 & -σ_3 \end{pmatrix} \times I_2
\]

where the two dimensional unit matrix \(I_2\) represents this extra factor of two degeneracy associated with the upper indices of \(Ψ, Λ\). We will neglect the \(I_2\) factor in what follows since it enters trivially in our analysis. On the lattice the usual single site fermion condensate takes the form

\[
\sum_{a=1}^{\times 2} \overline{\psi}_a^+ \lambda_+^a + \overline{\psi}_-^a \psi_-^a
\]

This clearly breaks the lattice SU(2) × SU(2)

---

\(^4\) We thank Maarten Golterman and Yigal Shamir for pointing out this argument and sharing their notes with us [19].
symmetry down to its diagonal subgroup. We then expect that this breaking pattern corresponds to a continuum condensate of the form

$$\Sigma = \left( \begin{array}{cc} \sigma_1 & 0 \\ 0 & \sigma_1 \end{array} \right) \times I_2$$  \hspace{1cm} (29)

Standard universality arguments tell us that it should be possible to change basis for our fermion fields to force this condensate to take the canonical flavor symmetric form $\Sigma = I_8$. To accomplish this first diagonalize $M \rightarrow P^\dagger M P$ using the unitary transformation

$$P = \left( \begin{array}{cc} \frac{1}{\sqrt{2}} (\sigma_3 + \sigma_1) & 0 \\ 0 & \frac{1}{\sqrt{2}} (\sigma_3 + \sigma_1) \end{array} \right)$$  \hspace{1cm} (30)

This results in a condensate of the form

$$\Sigma' = \left( \begin{array}{cc} \sigma_3 & 0 \\ 0 & \sigma_3 \end{array} \right)$$  \hspace{1cm} (31)

To transform this to the unit matrix we employ the non-anomalous chiral transformation $M' \rightarrow Q M' Q$ with

$$Q = \left( \begin{array}{cc} D & 0 \\ 0 & D^\dagger \end{array} \right)$$  \hspace{1cm} (32)

where the $2 \times 2$ matrix

$$D = \left( \begin{array}{cc} 1 & 0 \\ 0 & i\gamma_5 \end{array} \right)$$  \hspace{1cm} (33)

To find the explicit form of the broken generators in this new basis we transform them according to the rule $Q^\dagger P^\dagger \gamma_5 PQ$. It is straightforward to show that the broken generators acquire an axial character in the new basis (we have reinserted the $I_2$ at this point)

$$\tau'_1 = \gamma_5 \left( \begin{array}{cc} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{array} \right) \times I_2 \quad \tau'_2 = \gamma_5 \left( \begin{array}{cc} 0 & \sigma_1 \\ \sigma_1 & 0 \end{array} \right) \times I_2 \quad \tau'_3 = \gamma_5 \left( \begin{array}{cc} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{array} \right) \times I_2$$  \hspace{1cm} (34)

This confirms our earlier arguments that the broken generators do indeed correspond to axial symmetries in the continuum limit. We see that the staggered fermion action we use picks out a particular breaking direction corresponding to a specific embedding of the weak symmetries into the global symmetry group. The fact that the broken symmetries are axial in the continuum limit is a necessary condition, according to the Vafa-Witten theorem, for this broken phase of the lattice theory to survive the continuum limit.

As in the sterile case we have checked these ideas using explicit simulations. In this case we utilize an auxiliary scalar field which is a bi-fundamental in the two weak groups. For a scan in the auxiliary coupling $g$ we again set the strong gauge coupling $\beta_S = 5.5$ corresponding to a confining regime of the $SU(3)$ gauge interactions and use weak couplings $\beta_U = \beta_W = 10.0$. As before these results were obtained in the phase quenched approximation. The plots (figs. 8 and 9) show, once again, that a nonzero four fermion condensate develops, which is insensitive to the auxiliary Yukawa coupling $g$, and whose magnitude is determined by the strong gauge coupling $\beta_S$. We observe that it falls towards zero once the $SU(3)$ sector deconfines\[5\]

---

\[5\] One might wonder whether one could break a $U(1) \times U(1) \rightarrow U(1)$ gauge symmetry along these lines. However, the measure for a single reduced staggered field is not invariant under a local $U(1)$ transformation and so this lattice symmetry suffers from an anomaly.

FIG. 8. Absolute value of the four fermion condensate vs $g$ with $\beta_S = 5.5$ for the case of breaking to the diagonal subgroup

VII. DISCUSSION

We have shown that lattice theories comprising two reduced staggered fermion fields can be constructed in such a way that they can naturally generate non-perturbative condensates that break exact weakly coupled gauge symmetries as a result of strongly coupled gauge dynamics. The key element that allows for a Higgs mechanism to operate in these models is the absence of single site fermion
bilinears which are invariant under the gauge symmetries. This implies that the first gauge invariant object that can condense is a four fermion operator. Remarkably, these broken symmetries, which start out as vector symmetries in the staggered lattice theory, can be identified with axial symmetries in the continuum limit and the result is hence compatible with the Vafa-Witten theorem. The vector-like nature of the continuum theory ensures that it is trivially free of gauge anomalies [20].

These features arise as a consequence of the fact that the lattice theory is obtained by discretization of a continuum theory in which the original Lorentz symmetry has been twisted with an internal chiral-flavor symmetry. The free action of these twisted theories then naturally decomposes into two independent components which may be gauged independently. The resultant models, while having a vector like field content, can act nevertheless like chiral gauge theories by not admitting mass terms that are simultaneously both gauge invariant and twisted Lorentz invariant.

While preparing this paper we discovered a similar construction had been proposed many years ago by Banks et al. [21] for the case of breaking to a diagonal subgroup. This earlier work employed arguments based on a strong coupling expansion to argue for a Higgs phase at non-zero lattice spacing. Our numerical work provides evidence that this Higgs phase may survive into the continuum limit which is also consistent with our interpretation that the broken symmetries behave as axial symmetries in that limit.

In the continuum limit the formation of such a condensate will break the usual global axial symmetries in addition to any internal symmetries. Thus the continuum theory naturally contains additional light Goldstone bosons in addition to any massive gauge bosons associated with the breaking of gauge symmetry. In addition, one expects that any fermions participating in the condensate gain mass on the order of the symmetry breaking scale and decouple from the low energy spectrum.

In conclusion, it seems that the lattice models discussed here may serve as useful toy models for understanding the possibilities for dynamical symmetry breaking in strongly coupled gauge theories and can be used to test ideas such as the maximal attractive channel hypothesis and tumbling scenarios. In many respects the behavior of these theories mimicks that of chiral fermions although in the continuum limit they more closely resemble left-right symmetric models such as Patti-Salam [22]. It would be interesting to investigate such models via strong coupling expansions which would avoid possible sign problems.

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