AN EXPLICIT FORMULA FOR BERNOULLI POLYNOMIALS IN TERMS OF \(r\)-STIRLING NUMBERS OF THE SECOND KIND

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Abstract. In the paper, the authors establish an explicit formula for computing Bernoulli polynomials at non-negative integer points in terms of \(r\)-Stirling numbers of the second kind.

1. Introduction

It is well known that Bernoulli numbers \(B_k\) for \(k \geq 0\) may be generated by

\[
\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{x^{2k}}{(2k)!}, \quad |x| < 2\pi
\]

and that Bernoulli polynomials \(B_n(x)\) for \(n \geq 0\) and \(x \in \mathbb{R}\) may be generated by

\[
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi.
\]

In combinatorics, Stirling numbers of the second kind \(S(n, k)\) are equal to the number of partitions of the set \(\{1, 2, \ldots, n\}\) into \(k\) non-empty disjoint sets. Stirling numbers of the second kind \(S(n, k)\) for \(n \geq k \geq 0\) may be computed by

\[
S(n, k) = \frac{1}{k!} \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \ell^n.
\]

In the paper [1], among other things, Stirling numbers \(S(n, k)\) were combinatorially generalized as \(r\)-Stirling numbers of the second kind, denoted by \(S_r(n, k)\) here, for \(r \in \mathbb{N}\), which may be alternatively defined as the number of partitions of the set \(\{1, 2, \ldots, n\}\) into \(k\) non-empty disjoint subsets such that the numbers \(1, 2, \ldots, r\) are in distinct subsets.

Note that

\[
S(0, 0) = 1, \quad S_0(n, k) = S(n, k),
\]

and, when \(n \in \mathbb{N}\),

\[
S(n, 0) = 0, \quad S_1(n, k) = S(n, k),
\]

In [4, p. 536] and [5, p. 560], the simple formula

\[
B_n = \sum_{k=0}^{n} (-1)^{k} \frac{k!}{k+1} S(n, k), \quad n \in \{0\} \cup \mathbb{N}
\]

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for computing Bernoulli numbers $B_n$ in terms of Stirling numbers of the second kind $S(n, k)$ was incidentally obtained. Recently, four alternative proofs for the formula (4) were supplied in [7] and its preprint [6]. For more information on calculation of Bernoulli numbers $B_n$, please refer to the papers [8, 9, 11, 13, 14, 15], especially to the article [3], and plenty of references therein.

The aim of this paper is to generalize the formula (4). Our main result may be formulated as the following theorem.

**Theorem 1.** For all integers $n, r \geq 0$, Bernoulli polynomials $B_n(r)$ may be computed in terms of $r$-Stirling numbers of the second kind $S_r(n + r, k + r)$ by

$$B_n(r) = \sum_{k=0}^{n} \frac{(-1)^k k!}{k+1} S_r(n + r, k + r).$$

(5)

In the final section of this paper, several remarks are listed.

2. **Proof of Theorem 1**

We are now in a position to verify our main result.

For $n, r \geq 0$, let

$$F_{n,r}(x) = \sum_{k=0}^{n} k! S_r(n + r, k + r) x^k.$$  

(6)

By [1, p. 250, Theorem 16], we have

$$\sum_{n=0}^{\infty} S_r(n + r, k + r) \frac{t^n}{n!} = \sum_{n=k}^{\infty} S_r(n + r, k + r) \frac{t^n}{n!} = \frac{1}{k!} e^{rt} (e^t - 1)^k,$$

where $S_r(n, m) = 0$ for $m > n$, see [1, p. 243, (10)]. Accordingly, we obtain

$$\sum_{n=0}^{\infty} F_{n,r}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} k! x^k S_r(n + r, k + r) \frac{t^n}{n!}$$

$$= \sum_{k=0}^{\infty} k! x^k \sum_{n=k}^{\infty} S_r(n + r, k + r) \frac{t^n}{n!}$$

$$= e^{rt} \sum_{k=0}^{\infty} x^k (e^t - 1)^k$$

$$= e^{rt} \frac{1 - x(e^t - 1)}{1 - x(e^t - 1)}.$$ 

Integrating with respect to $x \in [0, s]$ for $s \in \mathbb{R}$ on both sides of the above equation yields

$$\sum_{n=0}^{\infty} \left[ \int_{0}^{s} F_{n,r}(x) \, dx \right] \frac{t^n}{n!} = -e^{rt} \frac{\ln(1 + s - se^t)}{e^t - 1}.$$ 

(7)

On the other hand,

$$\int_{0}^{s} F_{n,r}(x) \, dx = \sum_{k=0}^{n} \frac{k!}{k+1} S_r(n + r, k + r) s^{k+1}.$$
Substituting this into the equation (7) concludes that
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{k!}{k+1} S_r(n+r,k+r+1) \frac{t^n}{n!} = -e^{et} \ln(1 + s - se^t) = e^{et} - 1.
\]
Taking \( s = -1 \) in the above equation and making use of the generating function (2) result in
\[
\sum_{n=0}^{\infty} \left[ \sum_{k=0}^{n} \frac{(-1)^{k+1} k!}{k+1} S_r(n+r,k+r) \right] \frac{t^n}{n!} = -\frac{te^{rt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{[B_n(r)] t^n}{n!},
\]
which implies the formula (5). The proof of Theorem 1 is complete.

3. Remarks

Finally we would like to give several remarks on Theorem 1 and its proof.

Remark 1. Since \( B_n(0) = B_n \) and \( S_0(n,k) = S(n,k) \), when \( r = 0 \), the formula (5) becomes (4). Therefore, our Theorem 1 generalizes the formula (4).

Remark 2. It is easy to see that
\[
F_{n,0}(1) = \sum_{k=0}^{n} k! S(n,k),
\]
which are just the classical ordered Bell numbers. For more information, please refer to the papers [2, 12] and closely related references therein.

Remark 3. In the PhD thesis [10], the second author defined a variant of the polynomials \( F_{n,r}(x) \). Hence, a simple combinatorial study and interpretation of the polynomials \( F_{n,r}(x) \) is available therein.

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