Necessary and sufficient conditions for the existence of a classical solution of the mixed problem for the wave equation on a graph

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Abstract. We study a mixed problem for the wave equation with integrable potential on the simplest geometric graph consisting of two ring edges that touch at a point. We use a new approach in the Fourier method to obtain necessary and sufficient conditions for the existence of a classical solution. We do not use refined asymptotic formulas for the eigenvalues and any information on the eigenfunctions. The solution is represented by a rapidly convergent series.

1. Introduction
We consider the simplest geometric graph consisting of two ring edges that touch at a point (at the node of the graph). Parametrizing each edge by the interval \([0, 1]\), we study the following mixed problem for the wave equation on this graph:

\[
\frac{\partial^2 u_j(x, t)}{\partial t^2} = \frac{\partial^2 u_j(x, t)}{\partial x^2} - q_j(x) u_j(x, t),
\]

\(x \in [0, 1], t \in (-\infty, +\infty), \quad (j = 1, 2),\)

\(u_1(0, t) = u_1(1, t) = u_2(0, t) = u_2(1, t),\)

\(u_1'(0, t) - u_1'(1, t) + u_2'(0, t) - u_2'(1, t) = 0,\)

\(u_1(x, 0) = \varphi_1(x), \quad u_2(x, 0) = \varphi_2(x), \quad u_1'(x, 0) = u_2'(x, 0) = 0.\)

We assume that \(q_j(x) \in L[0, 1]\) are complex-valued. Conditions (2), (3) are generated by the structure of the graph \([1, 2]\). Research some problems on geometric graphs can also be found in \([3]–[8]\).

In this problem the application of the Fourier method causes difficulties associated with the fact that the eigenvalues of the corresponding spectral problem might be multiple. These difficulties can be coped with by applying the resolvent approach \([9]\). By the resolvent approach a sufficient conditions for the existence of classical solution of a mixed problem on such a graph is obtained in \([10]–[11]\). Note that we did not use refined asymptotic formulas for the eigenvalues and any information on the eigenfunctions. We used Krylov’s ideas \([12, \text{Chapter VI}]\) concerning the convergence acceleration of Fourier-like series (see also \([13]–[15]\).
Sufficient conditions on the function \( \varphi_j(x) \) for the existence of a classical Fourier method solution of problem (1)–(4) (Eq. (1) is satisfied almost everywhere) were obtained in [11]. These conditions are as follows:

\[
\varphi_1(0) = \varphi_1(1) = \varphi_2(0) = \varphi_2(1), \quad \varphi'_1(0) - \varphi'_1(1) + \varphi'_2(0) - \varphi'_2(1) = 0,
\]

and in addition \( L\varphi \in L^2_2[0,1] \), where \( L \) is the operator corresponding to the problem, \( L^2_2[0,1] \) denotes the space of vector functions \( f(x) = (f_1(x), f_2(x))^T \) such that \( f_k(x) \in L^2[0,1] \) (\( k = 1, 2 \)), \( T \) denotes the transpose. These conditions, except for condition \( L\varphi \in L^2_2[0,1] \), are necessary for the existence of a classical solution of the problem. Here we remove this additional condition and thus obtain the necessary and sufficient conditions for the existence of the classical solution.

Here we will use a different approach proposed by A. P. Khromov in [16, 17], again based on the ideas of accelerating the convergence of series, but implying a different transformation of the formal series. This approach allows us to obtain the necessary and sufficient conditions for the existence of classical solutions. In this case, the solution is represented as a series that converges at an exponential rate.

2. Uniqueness of a classical solution

Here we consider the following problem, which is more general than (1)–(4):

\[
\frac{\partial^2 u_j(x,t)}{\partial t^2} = \frac{\partial^2 u_j(x,t)}{\partial x^2} - q_j(x)u_j(x,t) + f_j(x,t),
\]

\[
x \in [0,1], \ t \in (-\infty, +\infty), \ (j = 1, 2),
\]

\[
u_1(0,t) = u_1(1,t) = u_2(0,t) = u_2(1,t),
\]

\[
u_{1x}(0,t) - u_{1x}(1,t) + u_{2x}(0,t) - u_{2x}(1,t) = 0,
\]

\[
u_1(0,0) = \varphi_1(x), \quad u_2(0,0) = \varphi_2(x), \quad u_{1x}(0,0) = \psi_1(x), \quad u_{2x}(0,0) = \psi_2(x).
\]

We assume all functions in (1)–(4) are complex-valued, \( q_j(x) \in L[0,1] \), and \( f_j(x,t) \in L(Q_T) \) for each \( T > 0 \), where \( Q_T = [0,1] \times [0,T] \). Further we will also use the notation: \( \varphi(x) = (\varphi_1(x), \varphi_2(x))^T \), \( \psi(x) = (\psi_1(x), \psi_2(x))^T \), \( f(x,t) = (f_1(x,t), f_2(x,t))^T \), \( Q(x) = \text{diag}(q_1(x), q_2(x)) \).

A classical solution is defined as a function \( u(x,t) = (u_1(x,t), u_2(x,t))^T \) such that \( u_j(x,t) \) and their first derivatives with respect to \( x \) and \( t \) are absolutely continuous, and satisfies the boundary and initial conditions (6)–(8) and the differential equation (5) almost everywhere. Therefore, we assume that the vector functions \( \varphi(x), \psi(x) \) and \( \varphi'(x) \) are absolutely continuous and such that satisfy the following conditions: \( \varphi_1(0) = \varphi_1(1) = \varphi_2(0) = \varphi_2(1), \varphi_1'(0) - \varphi_1'(1) + \varphi_2'(0) - \varphi_2'(1) = 0 \), \( \psi_1(0) = \psi_1(1) = \psi_2(0) = \psi_2(1) \).

**Theorem 1** If \( u(x,t) \) is a classical solution of problem (5)–(8) such that \( \frac{\partial^2 u(x,t)}{\partial t^2} \in L[Q_T] \) for each \( T > 0 \) (the uniqueness condition), then this solution is unique and can be represented by the formula

\[
u(x,t) = -\frac{1}{2\pi i} \left( \int_{|\lambda|=r} + \sum_{n \geq n_0} \int_{(a,b)} \right) \left( (R_\lambda \varphi) \cos \rho t + (R_\lambda \psi) \sin \rho t + \int_0^t R_\lambda(f(\tau), \tau) \sin \rho (t - \tau) \frac{d\tau}{\rho} \right) d\lambda,
\]

where the series on the right-hand side converges absolutely and uniformly with respect to \( x \in [0,1] \) for each \( t \geq 0 \).
Here \( R_\lambda = (L - \lambda E)^{-1} \) is the resolvent of the operator 
\[
Ly = (-y''(x) - q(x)y(x), -y''(x) - q(x)y(x))^T, \quad y = y(x) = (y_1(x), y_2(x))^T
\]
with boundary conditions 
\[
y(0) = y(1) = y_2(0) = y_2(1), \quad y_1(0) - y_1(1) + y_2(0) - y_2(1) = 0
\]
(\( \lambda \) is the spectral parameter, and \( E \) is the identity operator); the notation \( R_\lambda(f(\cdot, \tau)) \) indicates that the operator \( R_\lambda \) is applied to \( f(x, \tau) \) with respect to the variable \( x \; ; \quad \lambda = \rho^2 \), \( \Re \rho \geq 0 \), \( \gamma_n \) is the image of the circle \( \gamma_n = \{\rho||\rho - n\pi| = \delta\} \) in the \( \lambda \)-plane, \( \delta > 0 \) is sufficiently small, \( r \) is sufficiently large and fixed; and \( n_0 \) is a number such that for each \( n \geq n_0 \) the contour \( \gamma_n \) lies outside the circle \( |\lambda| = r \) and all eigenvalues of \( L \) are inside \( \gamma_n \).

The proof of Theorem 1 is analogous to the proof of Theorem 1 in [16] (see also in [18, 19]).

3. Transformation of the formal solution

Let’s go back to problem (1)–(4). The Fourier method is related to the spectral problem \( Ly = \lambda y \) for operator \( L \). By \( R_\lambda = (L - \lambda E)^{-1} \), \( R_0^\lambda = (L^0 - \lambda E)^{-1} \) denote the resolvents of the operators \( L \) and \( L^0 \), where \( L^0 \) is \( L \) with \( q_j(x) = 0 \).

The formal solution \( u(x, t) = (u_1(x, t), u_2(x, t))^T \) of problem (1)–(4) produced by the Fourier method can be represented as
\[
u(x, t) = -\frac{1}{2\pi i} \left( \int_{|\lambda| = r} + \sum_{n \geq n_0} \int_{\gamma_n} \right) (R_\lambda \varphi)(x) \cos \rho t d\lambda, \tag{10}\]
where \( r > 0 \) is fixed and such that all the eigenvalues \( \lambda_n \), with \( n < n_0 \), belong to the disk \( |\lambda| < r \), and there are no eigenvalues of \( L \) on the contour \( |\lambda| = r \); \( \gamma_n \) are the contours of sufficiently small radius in \( \lambda \)-plane such that all the eigenvalues of operator \( L \) and \( L^0 \) with \( n \geq n_0 \) are only inside \( \gamma_n \) (see [9, 18]).

The formal solution can be represented as 
\[
u(x, t) = U_0(x, t) + U_1(x, t),
\]
where \( U_0(x, t) \) is (10) with \( R_0^\lambda \) instead of \( R_\lambda \), and \( U_1(x, t) \) is the series (10) in which \( R_\lambda \) is replaced with \( R_0^\lambda - R_\lambda R_0^\lambda \).

The series \( U_0(x, t) \) is solution of the problem (1)–(4) with \( q_j(x) = 0 \) and it converges absolutely and uniformly over all \( x \) and \( t \) [11]. Let’s denote its sum \( A_0(x, t) \). According to [11] follows the statement

**Theorem 2** It is true that 
\[
u_0(x, t) = A_0(x, t) = \frac{1}{2} \left( \tilde{F}(x + t) + \tilde{F}(x - t) \right),
\]
where the following relations hold for \( \tilde{F}(x) = (F_1(x), F_2(x))^T \):
\[
F_1(-x) = [F_1(1 - x) + F_2(1 - x) - F_1(x) + F_2(x)]/2, \\
F_2(-x) = [F_1(1 - x) + F_2(1 - x) + F_1(x) - F_2(x)]/2, \\
F_1(1 + x) = [F_1(x) - F_1(1 - x) + F_2(x) + F_2(1 - x)]/2, \\
F_2(1 + x) = [F_1(x) + F_1(1 - x) + F_2(x) - F_2(1 - x)]/2,
\]
and \( \tilde{F}(x) = \varphi(x) \) for \( x \in [0, 1] \).
4. Study of $U_1(x,t)$.
In [9]–[11] the series $U_1(x,t)$ was investigated using resolvent estimates. Now, as in [16], we consider it as a solution to the problem

$$
\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} - Q(x)u(x,t) + F^0(x,t),
$$

(12)

$$
u_1(0, t) = u_1(1, t) = u_2(0, t) = u_2(1, t),
$$

(13)

$$
u_1'(0, t) - u_1'(1, t) + u_2'(0, t) - u_2'(1, t) = 0,
$$

(14)

$$
u_1(x, 0) = 0, \quad u_2(x, 0) = 0, \quad u_1'(x, 0) = u_2'(x, 0) = 0,
$$

(15)

where $F^0(x,t) = -Q(x)A^0(x,t)$.

First, consider the problem (12)–(15) with an arbitrary right-hand side $f(x,t)$ and $Q(x) = 0$.

**Theorem 3** If the function $f(x,t)$ is continuously differentiable with respect to $x$ and $t$ and $f_1(0,t) = f_2(0,t) = f_1(1,t) = f_2(1,t)$, then the series for the formal solution of problem

$$
\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),
$$

$$
u_1(0, t) = u_1(1, t) = u_2(0, t) = u_2(1, t),
$$

$$
u_1'(0, t) - u_1'(1, t) + u_2'(0, t) - u_2'(1, t) = 0,
$$

$$
u_1(x, 0) = 0, \quad u_2(x, 0) = 0, \quad u_1'(x, 0) = u_2'(x, 0) = 0,
$$

converges for all $x$ and $t$, and its sum can be represented in the form

$$
u(x,t) = \frac{1}{2} \int_0^t d\tau \int_{x-t+\tau}^{x+t-\tau} \tilde{F}(\eta, \tau) d\eta,
$$

(16)

where the function $\tilde{F}(\eta, \tau) = f(\eta, \tau)$ for $\eta \in [0,1]$ and $\tilde{F}(\eta, \tau)$ continues on the entire axis using the relations (11).

Next, we represent $U_1(x,t)$ as:

$$
U_1(x,t) = A_1(x,t) + U_2(x,t),
$$

where now the function $A_1(x,t)$ has the form (16) in which $f(x,t)$ is replaced with $F^0(x,t)$.

Just as in [16, Theorem 3], one can prove

**Theorem 4** If $u(x,t)$ is a classical solution of problem (1)–(4) with the uniqueness condition, then the function $A_1(x,t)$ is continuous with respect to $(x,t) \in (-\infty, +\infty) \times [0, \infty)$, and continuously differentiable with respect to $x$ and $t$, and its derivative $A_1'(x,t)$ ($A_1''(x,t)$) is absolutely continuous with respect to $x$ (respectively, $t$); conditions (13)–(15) are satisfied for $A_1(x,t)$; and almost everywhere with respect to $(x,t) \in (-\infty, +\infty) \times [0, \infty)$ one has the relation

$$
\frac{\partial^2 A_1(x,t)}{\partial t^2} = \frac{\partial^2 A_1(x,t)}{\partial x^2} - \tilde{Q}(x)A_0(x,t),
$$

where $\tilde{Q}(x)$ continues on the entire axis using the relations (11). The equation being true for all $x$ and $t$ such that the functions

$$
\frac{d}{d\xi} \int_0^\xi q_j(\tau) d\tau, \quad \frac{d}{d\xi} \int_0^\xi |q_j(\tau)| d\tau
$$

remain finite for $\xi = x, x-t, x+t$. Here $\frac{\partial^2 A_1(x,t)}{\partial x^2} \in L[Q_T]$. 


5. The classical solution

We will study the function \( U_2(x, t) \) by analogy with the function \( U_1(x, t) \) above, i.e., based on the fact that \( U_2(x, t) \) is a classical solution of problem (12)–(15) with the condition \( \frac{\partial^2 U_2(x,t)}{\partial t^2} \in L[Q_T] \). As a result, we obtain

\[
U_2(x, t) = A_2(x, t) + U_3(x, t),
\]

where the same formula holds for the function \( A_2(x, t) \) as for the function \( A_1(x, t) \) above with \( F^0(\eta, \tau) \) replaced with \( F^1(\eta, \tau) = -Q(\eta)A_1(\eta, \tau) \) and \( U_3(x, t) \) is the classical solution of problem (12)–(15) with \( U_2(x, t) \) and \( F^2(x, t) \) replaced with \( U_3(x, t) \) and \( F^2(x, t) \), respectively. Continuing this process ad infinitum, we obtain

\[
U_n(x, t) = A_n(x, t) + U_{n+1}(x, t),
\]

where the same formula takes place for the function \( A_n(x, t) \) as for the function \( A_1(x, t) \) above with \( F^0(\eta, \tau) \) replaced with \( F^{n-1}(\eta, \tau) \) \( F^k(\eta, \tau) = -Q(\eta)A_k(\eta, \tau) \) and \( U_{n+1}(x, t) \) is the solution of the problem obtained from (12)–(15) by the replacement of \( U_1(x, t) \) and \( F^0(\eta, \tau) \) with \( U_{n+1}(x, t) \) and \( F^n(x, t) = -Q(x)A_n(x, t) \), respectively.

By induction, for \( A_n(x, t) \) we obtain the theorem similar to Theorem 4.

Thus, the representation is true:

\[
u(x, t) = \sum_{k=0}^{m} A_k(x, t) + \Omega_m(x, t) \quad (m = 2, \ldots),
\]

where

\[
\Omega_m(x, t) = -\frac{1}{2\pi i} \left( \int_{|\lambda|=\epsilon} + \sum_{n \geq n_0} \int_{|\lambda|=\epsilon} \right) \left[ \int_0^t (R_\lambda - R_{\lambda,n}) (F^{m-1}(\cdot, \tau)) \frac{\sin(\rho(t - \tau)}{\rho} d\tau \right] d\lambda.
\]

Using Theorem 3 and \([16, \text{Lemma 10}]\) we can prove

**Lemma 1** Let \( T \) be an arbitrary positive number, and let \( N \) be the least positive integer such that \( T \leq N \). Then

\[
\| A_n(x, t) \|_{C[Q_T]} \leq M_1 \left( \frac{M_2}{2} \right)^{n-1} \frac{T^{n-1}}{(n-1)!} \quad (n = 1, 2, \ldots)
\]

where \( M_1 = \| A_n(x, t) \|_{C[Q_T]} \), \( M_2 = (2N + 1) \| q \|_1 \). Here \( \| f \|_1 = \int_0^T (|f_1(t)| + |f_2(t)|) dt \) is the norm in \( L[0, T] \). Besides \( M_1 \leq C_T \| \varphi \|_1 \) and the constant \( C_T \) is independent of \( \varphi(x) \).

**Theorem 5** If \( u(x, t) \) is a classical solution of problem (1)–(4) such that \( \frac{\partial^2 u(x,t)}{\partial t^2} \in L[Q_T] \), then

\[
u(x, t) = A(x, t) = \sum_{n=0}^{\infty} A_n(x, t),
\]

where

\[
A_0(x, t) = \frac{1}{2}[\tilde{F}(x + t) + \tilde{F}(x - t)],
\]

\[
A_n(x, t) = \frac{1}{2} \int_0^t d\tau \int_{x+t-\tau}^{x+t+\tau} \tilde{F}^{n-1}(\eta, \tau) d\eta, \quad n \geq 1
\]

and \( \tilde{F}^n(x, t) = -Q(x)A_n(x, t) \) for \( x \in [0, 1] \), and continues on the entire axis using the relations (11). Here the series converges absolutely and uniformly with respect to \( x, t \in Q_T \) for each \( T > 0 \).
Proof. As follows from Lemma 1, the series $A(x, t)$ converges absolutely and uniformly with respect to $x, t \in Q_T$ for each $T > 0$. Further, for $\rho \in \tilde{\gamma}_n$ we have the estimates
\[
\left| \left( R_\lambda - R_\lambda^\ast \right) (f_n(\cdot, \tau)) \right|_k \sin \frac{\rho(t - \tau)}{\rho} \leq C \frac{1}{n^3} \int_0^1 \left( \left| F^n(x, \tau) \right|_1 + \left| F^n(x, \tau) \right|_2 \right) d\tau
\]
where $[\cdot]_k$ denotes the k-th component of the vector function, the constant $C$ is independent of $\eta, \tau$ and $n$, and, by Lemma 1, we have the estimates
\[
\int_0^1 \left| F^n(x, \tau) \right|_k d\tau \leq M_1 \left( \frac{M_2 T}{2} \right)^{n-1} \frac{1}{(n-1)!} \| q \|_1.
\]
The proof of the theorem is complete.

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