FINITE DOMINATION AND NOVIKOV HOMOLOGY
OVER STRONGLY $\mathbb{Z}^2$-GRADED RINGS

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Dedicated to the memory of Andrew Ranicki

Abstract. Let $R$ be a strongly $\mathbb{Z}^2$-graded ring, and let $C$ be a bounded chain complex of finitely generated free $R$-modules. The complex $C$ is $R(0,0)$-finitely dominated, or of type $FP$ over $R(0,0)$, if it is chain homotopy equivalent to a bounded complex of finitely generated projective $R(0,0)$-modules. We show that this happens if and only if $C$ becomes acyclic after taking tensor product with a certain eight rings of formal power series, the graded analogues of classical Novikov rings. This extends results of Ranicki, Quinn and the first author on Laurent polynomial rings in one and two indeterminates.

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Date: August 1, 2022.

2020 Mathematics Subject Classification. Primary 18G35; Secondary 16W60, 16W50, 16P99, 55U15.
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Part I. Finite Domination over strongly $\mathbb{Z}^2$-graded rings

I.1. Introduction

Let $L$ be a unital ring, and let $K$ be a subring of $L$. A bounded chain complex $C$ of (right) $L$-modules is $K$-finitely dominated if $C$, considered as a complex of $K$-modules, is a retract up to homotopy of a bounded complex of finitely generated free $K$-modules; this happens if and only if $C$ is homotopy equivalent, as a $K$-module complex, to a bounded complex of finitely generated projective $K$-modules [Ran85, Proposition 3.2. (ii)]. The following result of Ranicki gives a complete homological characterisation of finite domination in an important special case:

Theorem I.1.1 (Ranicki [Ran95, Theorem 2]). Let $K$ be a unital ring, and let $K[t, t^{-1}]$ denote the Laurent polynomial ring in the indeterminate $t$. Let $C$ be a bounded chain complex of finitely generated free $K[t, t^{-1}]$-modules. The complex $C$ is $K$-finitely dominated if and only if both

$$C \otimes_{K[t, t^{-1}]} K((t^{-1})) \quad \text{and} \quad C \otimes_{K[t, t^{-1}]} K((t))$$

have vanishing homology in all degrees. Here we use the notation $K((t)) = K[[t]][t^{-1}]$ for the ring of formal Laurent series in $t$, and similarly $K((t^{-1})) = K[[t^{-1}]][t]$ stands for the ring of formal Laurent series in $t^{-1}$.

The cited paper [Ran95] also contains a discussion of the relevance of finite domination in topology. — The rings $K((t))$ and $K((t^{-1}))$ are known as Novikov rings. The theorem can be formulated more succinctly: The chain complex $C$ is $K$-finitely dominated if and only if it has trivial Novikov homology.

This result was extended by Quinn and the first author to Laurent polynomial rings in two variables; contrary to appearance, this is not a straight-forward modification of the original result, introducing additional levels of complication in homological algebra.

Theorem I.1.2 (Hüttemann and Quinn [HQ15, Theorem I.1.2]). Let $C'$ be a bounded chain complex of finitely generated free $L$-modules,
where $L = K[x, x^{-1}, y, y^{-1}]$ is a Laurent polynomial ring in two variables over the unital ring $K$. The following two statements are equivalent:

1. The complex $C$ is $K$-finitely dominated, i.e., $C$ is homotopy equivalent, as an $K$-module chain complex, to a bounded chain complex of finitely generated projective $K$-modules.

2. The eight chain complexes listed below are acyclic (all tensor products are taken over $L$):

$$
\begin{align*}
C \otimes K[x, x^{-1}][y] & \quad C \otimes K[x, x^{-1}][y^{-1}] \\
C \otimes K[y, y^{-1}][x] & \quad C \otimes K[y, y^{-1}][x^{-1}] \\
C \otimes K((x, y)) & \quad C \otimes K((x^{-1}, y)) \\
C \otimes K((x, y^{-1})) & \quad C \otimes K((x^{-1}, y))
\end{align*}
$$

(I.1.2.1a)

$$
\begin{align*}
C \otimes K((x, y)) & \quad C \otimes K((x, y^{-1})) \\
C \otimes K((x^{-1}, y)) & \quad C \otimes K((x^{-1}, y))
\end{align*}
$$

(I.1.2.1b)

Here $K((x, y)) = K[[x, y]][1/xy]$ is a localisation of the ring of formal power series in $x$ and $y$, and the other rings are defined analogously.

The authors of the present paper generalised Theorem I.1.1 in an entirely different direction, exhibiting the graded structure of Laurent polynomial rings as the crucial property for setting up the theory.

**Theorem I.1.3 (Hüttemann and Steers [HS17, Theorem 1.3]).** Let $R^*_[t, t^{-1}] = \bigoplus_{k \in \mathbb{Z}} R_k$ be a strongly $\mathbb{Z}$-graded ring, and let $C$ be a bounded chain complex of finitely generated free $R^*_[t, t^{-1}]$-modules. The complex $C$ is $R_0$-finitely dominated if and only if both

$$
\begin{align*}
C \otimes_{R^*_[t, t^{-1}]} R^*_((t^{-1})) & \quad C \otimes_{R^*_[t, t^{-1}]} R^*_((t))
\end{align*}
$$

have vanishing homology in all degrees. Here the rings

$$
R^*_((t^{-1})) = \bigcup_{n \geq 0} \prod_{k \leq n} R_k \quad \text{and} \quad R^*_((t)) = \bigcup_{n \geq 0} \prod_{k \geq -n} R_k
$$

are used as formal analogues of the usual Novikov rings.

The notion of a strongly graded ring will be discussed in detail below. In the first instance, $R = \bigoplus_{k \in \mathbb{Z}} R_k$ is a $\mathbb{Z}$-graded ring. The (usual) Laurent polynomial ring $R[t, t^{-1}]$ has a $\mathbb{Z}$-graded subring $R^*_[t, t^{-1}]$ with $k$th component the set of monomials $r_k t^k$ with $r_k \in R_k$. In fact, we may identify $R^*_[t, t^{-1}]$ with $R$ itself. In a similar spirit, the Novikov rings $R((t^{-1}))$ and $R((t))$ have subrings $R^*_((t^{-1}))$ and $R^*_((t))$ determined by the condition that the coefficient of $t^k$ be an element of $R_k$, for any $k \in \mathbb{Z}$. It may be worth pointing out that these subrings do not contain the indeterminate $t$, which should be considered a purely notational device.
I.2. The main theorem

In the present paper we take the step to strongly $\mathbb{Z}^2$-graded rings, combining ideas from both of the aforementioned publications [HQ15] and [HS17]. Roughly speaking, a bounded chain complex $C$ of finitely generated free modules over a strongly $\mathbb{Z}^2$-graded ring is finitely dominated over the degree-0 subring if and only if certain eight complexes induced from $C$ are acyclic. Indeed, from a $\mathbb{Z}^2$-graded ring $R = \bigoplus_{\sigma \in \mathbb{Z}^2} R_\sigma$ we construct the following eight Novikov-type rings:

\[
\begin{align*}
R_*(x, x^{-1})((y)) &= \bigcup_{n \geq 0} \prod_{y \geq -n} \bigoplus_{x \in \mathbb{Z}} R(x, y) \\
R_*(x, x^{-1})((y^{-1})) &= \bigcup_{n \geq 0} \prod_{y \geq -n} \bigoplus_{x \in \mathbb{Z}} R(x, -y) \\
R_*[y, y^{-1}](x) &= \bigcup_{n \geq 0} \prod_{x \geq -n} \bigoplus_{y \in \mathbb{Z}} R(x, y) \\
R_*[y, y^{-1}](x^{-1}) &= \bigcup_{n \geq 0} \prod_{x \geq -n} \bigoplus_{y \in \mathbb{Z}} R(-x, y) \\
R_*(x, y) &= \bigcup_{n \geq 0} \prod_{x, y \geq -n} R(x, y) \\
R_*(x, y^{-1}) &= \bigcup_{n \geq 0} \prod_{x, y \geq -n} R(x, -y) \\
R_*[(x^{-1}, y^{-1})] &= \bigcup_{n \geq 0} \prod_{x, y \geq -n} R(-x, -y) \\
R_*[(x^{-1}, y)] &= \bigcup_{n \geq 0} \prod_{x, y \geq -n} R(-x, y)
\end{align*}
\]

Similar to notation used earlier, the symbols $x$ and $y$ do not stand for actual indeterminates; they are purely notational devices, emphasising a formal similarity with Laurent polynomial rings and their associated Novikov rings. The ring $R_*[x, x^{-1}]((y))$ is the subring of $R[x, x^{-1}]((y))$ with elements $\sum_{(a, b) \in \mathbb{Z}^2} r_{a,b} x^a y^b$ such that $r_{a,b} \in R(a,b)$, and similar for the other seven cases. — With this notational camouflage we obtain a perfect analogue of Theorem I.2.

**Theorem I.2.1.** Let $R = \bigoplus_{k \in \mathbb{Z}} R_k$ be a strongly $\mathbb{Z}^2$-graded ring, and let $C$ be a bounded chain complex of finitely generated free $R$-modules. The complex $C$ is $R_{(0,0)}$-finitely dominated if and only if all of the eight complexes

\[
\begin{align*}
C \otimes_R R_*[x, x^{-1}]((y)) &\quad C \otimes_R R_*[x, x^{-1}]((y^{-1})) \\
C \otimes_R R_*[y, y^{-1}](x) &\quad C \otimes_R R_*[y, y^{-1}](x^{-1})
\end{align*}
\]

(I.2.1.1a)
and
\[
\begin{align*}
C \otimes R_s((x, y)) & \quad C \otimes R_s((x^{-1}, y^{-1})) \\
C \otimes R_s((x, y^{-1})) & \quad C \otimes R_s((x^{-1}, y)),
\end{align*}
\]
(I.2.1.1b)

have vanishing homology in all degrees.

To illustrate the extent of the generalisation we describe a strongly \( \mathbb{Z}^2 \)-graded ring \( \hat{K} \) that is not a crossed product (and in particular not a Laurent polynomial ring), and a complex of finitely generated free \( K \)-modules that is \( \hat{K} \)-finitely dominated by the above criterion.

Let \( \overline{K} = K[a, b, c, d]/ab + cd - 1 \); this is a strongly \( \mathbb{Z} \)-graded ring if we let \( a, c \) have degree 1 and \( b, d \) have degree \(-1\). It is not a Laurent polynomial ring, in fact not a crossed product, since the only units in \( \overline{K} \) are the elements of \( K^\times \) in degree 0, as can be shown using ideas from Gröbner basis theory. In particular, there is no unit in \( \overline{K}_1 \).

The \( \mathbb{Z}^2 \)-graded ring \( \overline{K} = \overline{K} \otimes_{\mathbb{Z}_0} \overline{K} \), where \( \overline{K}_{(k,\ell)} = \overline{K}_k \otimes_{\mathbb{Z}_0} \overline{K}_\ell \), can be seen to be strongly graded as the necessary partitions of unity are present as in point (3) of Proposition I.4.5. It is not a crossed product since, for example, there is no unit in \( \overline{K}_{(1,0)} = \overline{K}_1 \).

We will define a four-fold chain complex \( V \) concentrated in degrees \((\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \in \mathbb{Z}^4 \) with \( \epsilon_j = 0, 1 \). The entries are \( \hat{K} \), a free \( \hat{K} \)-module of rank 1; the non-trivial differentials are multiplication by

\[
\begin{align*}
1 - a \otimes 1 & \quad \text{in 1-direction}, \\
1 - 1 \otimes c & \quad \text{in 2-direction}, \\
1 - d \otimes 1 & \quad \text{in 3-direction}, \\
1 - 1 \otimes bcd & \quad \text{in 4-direction}.
\end{align*}
\]

That is, the map from position \((1, \epsilon_2, \epsilon_3, \epsilon_4) \) to position \((0, \epsilon_2, \epsilon_3, \epsilon_4) \) is multiplication by \( 1 - a \otimes 1 \), the map from position \((\epsilon_1, 1, \epsilon_3, \epsilon_4) \) to position \((\epsilon_1, 0, \epsilon_3, \epsilon_4) \) is multiplication by \( 1 - 1 \otimes c \), and so on.

The totalisation \( C \) of \( V \) can be obtained by taking “iterated mapping cones”, and up to isomorphism it does not matter in which order we choose the different directions. Now the map \( 1 - a \otimes 1 \) is an isomorphism with inverse

\[
(1 - a \otimes 1)^{-1} = 1 + a \otimes 1 + a^2 \otimes 1 + a^3 \otimes 1 + \ldots
\]

over the rings \( \hat{K}_s[y, y^{-1}][(x)] \) and \( \hat{K}_s((x, y^{\pm 1})) \), by the usual telescoping sum argument familiar from the geometric series. Note that the series defines an element in each of these rings since \( a^k \otimes 1 = (a \otimes 1)^k \) has degree \((k, 0)\). Thus after tensoring \( V \) with one of these rings the mapping cones in 1-direction will be acyclic, hence the tensor product of \( C \) with one of these rings will be acyclic.
Similarly, the map $1 - 1 \otimes bcd$ is an isomorphism over $\hat{K}_*[x, x^{-1}]((y^{-1}))$ and $\hat{K}_*[x^{\pm 1}, y^{-1})$, with inverse

$$(1 - 1 \otimes bcd)^{-1} = 1 + 1 \otimes bcd + 1 \otimes (bcd)^2 + 1 \otimes (bcd)^3 + \ldots ;$$

this is an element of the rings in question since the degree of $1 \otimes (bcd)^k$ is $(0, -k)$. Consequently, after tensoring $V$ with one of these rings the mapping cones in 4-direction will be acyclic, hence the tensor product of $C$ with one of these rings will be acyclic. The cases of $1 - 1 \otimes c$ and $1 - d \otimes 1$ are similar, involving the rings $\hat{K}_*[x, x^{-1}]((y))$ and $\hat{K}_*[x^{\pm 1}, y]$ in the former case, the rings $\hat{K}_*[y, y^{-1}]((x^{-1}))$ and $\hat{K}_*[x^{-1}, y^{\pm 1})$ in the latter.

**Structure of the paper.** The paper can be seen as an amalgamation of the two publications [HQ15] and [HS17], combining the graded viewpoint of the latter with the elaborate homological algebra of the former. Replacing central indeterminates by inherently non-commutative structures is a non-trivial task, and it is rather surprising that with the right set-up the overall pattern of proof remains virtually unchanged.

We start with recalling basic concepts from the theory of strongly graded rings. The “if” implication of the main theorem is verified in Part II, building on [HS17] and [HQ15]; the main point is to relate the given chain complex with a complex of diagrams which are the analogues of well-known line bundles on the scheme $\mathbb{P}^1 \times \mathbb{P}^1$. Part III focuses on the “only if” implication. The main technical device is the algebraic torus (a “two-dimensional” version of the algebraic mapping torus), and the MATHER trick which allows to replace the complex $C$ with an algebraic torus of a complex $D$ consisting of finitely generated projective $R_{(0,0)}$-modules.

**I.3. Graded rings and Novikov rings**

**Definition I.3.1.** A $\mathbb{Z}^2$-graded ring is a (unital) ring $L$ equipped with a direct sum decomposition into additive subgroups $L = \bigoplus_{k \in \mathbb{Z}^2} L_k$ such that $L_k L_\ell \subseteq L_{k+\ell}$ for all $k, \ell \in \mathbb{Z}^2$, where $L_k L_\ell$ consists of the finite sums of ring products $xy$ with $x \in L_k$ and $y \in L_\ell$. The summands $L_k$ are called the (homogeneous) components of $L$; elements of $L_k$ are called homogeneous of degree $k$.

Relevant examples are the polynomial ring $K[x, y]$ and the LAURENT polynomial ring $R = R_0[x, x^{-1}, y, y^{-1}]$ in two variables, equipped with the usual $\mathbb{Z}^2$-grading by the exponents of $x$ and $y$, respectively; here $K$ may be any unital ring.
From a unital ring $R$, for the moment not equipped with a grading, we can construct various “Novikov rings” of formal \textsc{Laurent} series as follows:

$$R[x, x^{-1}]((y)) = R[x, x^{-1}][[y]][y^{-1}]$$
$$R[y, y^{-1}]((x)) = R[y, y^{-1}][[x]][x^{-1}]$$
$$R[x, x^{-1}]((y^{-1})) = R[x, x^{-1}][[y^{-1}]][y]$$
$$R[y, y^{-1}]((x^{-1})) = R[y, y^{-1}][[x^{-1}]][x]$$
$$R((x, y)) = R[[x, y]][(xy)^{-1}]$$
$$R((x^{-1}, y^{-1})) = R[[x^{-1}, y^{-1}]][xy]$$
$$R((x, y^{-1})) = R[[x, y^{-1}]]((xy)^{-1})^{-1}$$
$$R((x^{-1}, y)) = R[[x^{-1}, y]][(x^{-1}y)^{-1}]$$

Here $x$ and $y$ are central indeterminates which commute with each other, and with the elements of $R$. The slightly informal notation $S[[x]][x^{-1}]$ is meant to denote the ring of formal power series in the indeterminate $x$ with coefficients in $S$, localised at the multiplicative system $\{x^k \mid k \geq 0\}$.

Returning to the case of a $\mathbb{Z}^2$-graded ring $R$, we can introduce subrings of rings of formal power or \textsc{Laurent} series, denoted by adding the decoration “$*$” as a subscript to the ring $R$, consisting of those elements such that the coefficient of $x^s y^t$ is an element of $R_{(s,t)}$. The resulting unital subrings will not contain the elements $x$ and $y$, as, for example, $x = 1 \cdot x$ has coefficient $1 \in R_0$, whereas the monomial $x$ has degree $(1,0)$. At this point we may treat the “variables” as a purely notational device keeping track of degrees of various elements. As an illustration we put forward the ring $R_*[x, x^{-1}, y, y^{-1}]$ which has elements the finite sums of terms $r_{s,t} x^s y^t$, with $s, t \in \mathbb{Z}$ and $r_{s,t} \in R_{(s,t)}$. We may in fact identify this ring with the (graded) ring $R$ itself. —

When applied to the Novikov rings listed above, we arrive at the identifications listed in (I.2.0.1).

1.4. Partitions of unity and strongly graded rings

\textbf{Definition 1.4.1.} Let $S = \bigoplus_{\sigma \in \mathbb{Z}^2} S_{\sigma}$ be a $\mathbb{Z}^2$-graded unital ring. For $\rho \in \mathbb{Z}^2$, an finite sum expression of the form $1 = \sum_j u_j v_j$ with $u_j \in S_{\rho}$ and $v_j \in S_{-\rho}$ is called a partition of unity of type $(\rho, -\rho)$.

By a straightforward computation one can show:

\textbf{Lemma 1.4.2.} For $\rho, \rho' \in \mathbb{Z}^2$ let $1 = \sum_j u_j v_j$ and $1 = \sum_k u'_k v'_k$ be partitions of unity of type $\rho$ and $\rho'$, respectively. Then

$$1 = \sum_k \sum_j (u_j u'_j)(v'_j v_j)$$

is a partition of unity of type $\rho + \rho'$. \hfill $\Box$
If \( S_\rho S_{-\rho} = S_{(0,0)} \) then, since \( 1 \in S_{(0,0)} \), there exists a partition of unity of type \((\rho, -\rho)\). Conversely, if a partition of unity of type \((\rho, -\rho)\) exists then \( 1 \in S_\rho S_{-\rho} \) and thus \( S_{(0,0)} \subseteq S_\rho S_{-\rho} \) so that \( S_{(0,0)} = S_\rho S_{-\rho} \); this implies moreover
\[
S_\sigma = S_\sigma S_{(0,0)} = S_\sigma S_\rho S_{-\rho} \subseteq S_{\sigma + \rho} S_{-\rho} \subseteq S_\sigma
\]
so that \( S_\sigma = S_{\sigma + \rho} S_{-\rho} \) for any \( \sigma \in \mathbb{Z}^2 \), and similarly \( S_\sigma = S_\rho S_{\sigma - \rho} \).

**Proposition I.4.3.** Let \( R \) be a \( \mathbb{Z}^2 \)-graded ring. Let \( \lambda, \rho \in \mathbb{Z}^2 \) be such that there exists a partition of unity \( 1 = \sum_j \alpha_j \beta_j \) of type \((\lambda, -\lambda)\). The multiplication map
\[
\pi_{\lambda,\rho}: R_\lambda \otimes_{R_0} R_\rho \longrightarrow R_{\lambda + \rho}, \quad x \otimes y \mapsto xy
\]
is an isomorphism of \( R_{(0,0)} \)-bimodules; its inverse can be written as
\[
\mu_{\lambda,\rho} = \pi_{\lambda,\rho}^{-1}: R_{\lambda + \rho} \longrightarrow R_\lambda \otimes_{R_{(0,0)}} R_\rho, \quad z \mapsto \sum_j \alpha_j \otimes \beta_j z.
\]
The map \( \mu_{\lambda,\rho} \) does not depend on the choice of partition of unity.

*Proof.* The map \( \pi_{\lambda,\rho} \) is an \( R_{(0,0)} \)-balanced, thus well defined, \( R_{(0,0)} \)-bi-module homomorphism. Hence it is enough to show that \( \mu_{\lambda,\rho} \), considered as a homomorphism of right \( R_{(0,0)} \)-modules, is its inverse. For \( z \in R_{\lambda + \rho} \) we calculate
\[
\pi_{\lambda,\rho} \circ \mu_{\lambda,\rho}(z) = \sum_j \alpha_j \beta_j z = z
\]
so that \( \pi_{\lambda,\rho} \circ \mu_{\lambda,\rho} \) is the identity map. Similarly, for \( x \in R_\lambda \) and \( y \in R_\rho \) we have
\[
\mu_{\lambda,\rho} \circ \pi_{\lambda,\rho}(x \otimes y) = \sum_j \alpha_j \otimes (\beta_j x) y = \sum_j \alpha_j \beta_j x \otimes y = x \otimes y
\]
(since \( \beta_j x \in R_{(0,0)} \)) so that \( \mu_{\lambda,\rho} \circ \pi_{\lambda,\rho} \) is the identity map. \( \square \)

**Definition I.4.4 (Dade [Dad80, §1]).** A \( \mathbb{Z}^2 \)-graded ring \( L = \bigoplus_{\rho \in \mathbb{Z}^2} L_\rho \) is called a strongly \( \mathbb{Z}^2 \)-graded ring if \( L_\kappa L_\lambda = L_{\kappa + \lambda} \) for all \( \kappa, \lambda \in \mathbb{Z}^2 \).

The establishing example is the Laurent polynomial ring in two variables, \( R = R_0[x, x^{-1}, y, y^{-1}] \). This is a strongly \( \mathbb{Z}^2 \)-graded ring when \( x \) and \( y \) are given degrees \((1, 0)\) and \((0, 1)\), respectively.

Using [Dad80] Proposition 1.6] and Lemma [I.4.2, one obtains the following characterisation of strongly \( \mathbb{Z}^2 \)-graded rings:

**Proposition I.4.5.** Let \( R \) be a \( \mathbb{Z}^2 \)-graded ring. The following statements are equivalent:

1. The ring \( R \) is strongly graded.
2. For every \( \rho \in \mathbb{Z}^2 \) there is at least one partition of unity of type \((\rho, -\rho)\).
3. There is at least one partition of unity of each of the types \((\rho, -\rho)\) for some \( \rho \in \mathbb{Z}^2 \).
• \((1, 0), (−1, 0))\) and \((−1, 0), (1, 0))\),
• \((0, 1), (0, −1))\) and \((0, −1), (0, 1))\).

(4) For every \(\rho \in \mathbb{Z}^2\) we have \(L_{\rho}L_{−\rho} = L_{(0,0)}\).

Corollary I.4.6. If \(L = \bigoplus_{\rho \in \mathbb{Z}^2} L_\rho\) is strongly graded, then all \(L_\rho\) are invertible \(L_{(0,0)}\)-bimodules. In particular, the \(L_\rho\) are finitely generated projective as right and as left \(L_{(0,0)}\)-modules.

Part II. Contractibility of Novikov homology implies finite domination

II.1. Rings and modules associated to faces of a square

We denote by the symbol \(\mathcal{S}\) the set of non-empty faces of the polytope \(S = [−1, 1] \times [−1, 1] \subset \mathbb{R}^2\), partially ordered by inclusion. In Fig. 1 we show our chosen labelling and orientation of the faces (the orientations will be used later to define incidence numbers).

![Figure 1. Labelling and orientation of the faces of the square S](image)

With each face \(F \in \mathcal{S}\) we associate its barrier cone

\[ T_F = \operatorname{cone}\{x − y \mid x \in S, y \in F\} \subseteq \mathbb{R}^2, \]

where \(\operatorname{cone}(X)\) denotes the set of finite linear combinations of elements of \(X\) with non-negative real coefficients. For example, \(T_{\text{vul}}\) is the first quadrant and \(T_{el}\) is the left half-plane. — Given a \(\mathbb{Z}^2\)-graded ring \(R = \bigoplus_{\rho \in \mathbb{Z}^2} R_\rho\) we let \(A_F = R_*[T_F]\) denote the subring of \(R = R_*[x, x^{-1}, y, y^{-1}]\) consisting of elements with support in \(T_F\); explicitly,

\[ A_F = R_*[T_F] = \bigoplus_{\rho \in T_F \cap \mathbb{Z}^2} R_\rho. \]

Thus, for example,

\[ A_{vbr} = R_*[T_{vbr}] = R_*[x^{-1}, y], \]
\[ A_{el} = R_*[T_{el}] = R_*[x, y, y^{-1}], \]
\[ A_S = R_*[T_S] = R_*[x, x^{-1}, y, y^{-1}] = R. \]

For \(k \in \mathbb{Z}\) we introduce the \(R_*[T_F]\)-bimodules

\[ R_*[kF + T_F] = \bigoplus_{\sigma \in (kF + T_F) \cap \mathbb{Z}^2} R_\sigma, \]

where \(kF + T_F = \{kx + y \mid x \in F\text{ and } y \in T_F\}\). In different notation,
As a final bit of notation for now, we let \( p_F \in F \cap \mathbb{Z}^2 \) denote the barycentre of \( F \). For example, \( p_{e_i} = (0, 1) \) and \( p_S = (0, 0) \). Then \( kF + T_F = kp_F + T_F \), and every element \( \sigma \) of \( kF + T_F \) can in fact be written in the form \( kp_F + \tau \) for some \( \tau \in T_F \). Also, given faces \( F \subseteq G \) the barrier cones satisfy \( T_G = T_F + \mathbb{R} \cdot (p_G - p_F) \) so that, in particular, \( k(p_G - p_F) \in T_G \).

**Lemma II.1.1.** Let \( R \) be a strongly \( \mathbb{Z}^2 \)-graded ring. For all \( k \in \mathbb{Z} \) and for all \( F, G \in \mathcal{S} \) with \( F \subseteq G \), the \( R_\ast[T_F] \)-linear inclusion map \( \alpha_{F,G}: R_\ast[kF + T_F] \to R_\ast[kG + T_G] \) is such that its adjoint map

\[
\alpha_{F,G}^\ast: R_\ast[kF + T_F] \otimes_{R_\ast[T_F]} R_\ast[T_G] \to R_\ast[kG + T_G], \quad r \otimes s \mapsto rs
\]

is an isomorphism.

**Proof.** Choose a partition of unity of type \((kp_F, -kp_F)\), say \( 1 = \sum_j u_j v_j \) with \( u_j \in R_{kp_F} \) and \( v_j \in R_{-kp_F} \). The map

\[
\beta_{F,G}: R_\ast[kG + T_G] \to R_\ast[kF + T_F] \otimes_{R_\ast[T_F]} R_\ast[T_G], \quad x \mapsto \sum_j u_j \otimes v_j x
\]

is well-defined. First, \( u_j \) has degree \( kp_F \in kF \subseteq kF + T_F \) whence \( u_j \in R_\ast[kF + T_F] \). Second, every element \( \sigma \) of \((kG + T_G) \cap \mathbb{Z}^2\) can be written in the form \( kp_G + \tau \), with \( \tau \in T_G \cap \mathbb{Z}^2 \). Thus

\[
\sigma - kp_F = kp_G + \tau - kp_F = \tau + k(p_G - p_F) \in T_G.
\]

It follows that for \( x \in R_\ast[kG + T_G] \) the product \( v_j x \) is an element of \( R_\ast[T_G] \). — The map \( \beta_{F,G} \) is \( R_\ast[T_G] \)-linear and satisfies \( \alpha_{F,G}^\ast \circ \beta_{F,G} = \text{id} \) by direct calculation, using \( 1 = \sum_j u_j v_j \). We also have

\[
\beta_{F,G} \circ \alpha_{F,G}^\ast(r \otimes s) = \beta_{F,G}(rs) = \sum_j u_j \otimes v_j rs = \sum_j u_j v_j r \otimes s = r \otimes s
\]

where the equality labelled \((*)\) is true since \( v_j r \in R_\ast[T_F] \) for any \( r \in R_\ast[kF + T_F] \), whence \( \beta_{F,G} \circ \alpha_{F,G}^\ast = \text{id} \).

\[\square\]

II.2. Čech complexes

**Incidence numbers and Čech complexes.** Suppose \( P \) is a poset equipped with a strictly increasing “rank” function

\[
\text{rk}: P \to \mathbb{N}
\]

and with a notion of “incidence numbers”

\[
\langle \cdot, \cdot \rangle: P \times P \to \mathbb{Z}
\]

satisfying the following conditions:

(D11) \( \langle x: y \rangle = 0 \) unless \( x > y \) and \( \text{rk}(y) = \text{rk}(x) - 1 \).
(DI2) For all \( z < x \) with \( \text{rk}(z) = \text{rk}(x) - 2 \), the set
\[
P(z < x) = \{ y \in P \mid z < y < x \}
\]
is finite, and
\[
\sum_{y \in P(z < x)} [x: y] \cdot [y: z] = 0.
\]

(DI3) For \( x \in P \) with \( \text{rk}(x) = 1 \) the set
\[
P(< x) = \{ y \in P \mid y < x \}
\]
is finite, and
\[
\sum_{y \in P(< x)} [y: z] = 0.
\]

We can then associate a “Čech complex” with a diagram of \( K \)-modules (\( K \) a unital ring) indexed by \( P \):

**Definition II.2.1.** Given diagram \( \Phi : P \longrightarrow \text{K-Mod} \) with structure maps \( \varphi_{x,y} : \Phi_y \longrightarrow \Phi_x \), define its Čech complex \( \Gamma(\Phi) = \Gamma_P(\Phi) \) to be the chain complex concentrated in non-positive degrees with chain modules
\[
\Gamma(\Phi)_{-n} = \bigoplus_{x \in \text{rk}^{-1}(n)} \Phi_x \quad (n \geq 0)
\]
with differential induced by \( [x : y]\varphi_{x,y} \) for \( x > y \).

It is easy to check that by virtue of (DI1) and (DI2) this is indeed a chain complex. Condition (DI3) ensures that a cone \( M \longrightarrow \Phi \) on \( \Phi \), that is, a collection of \( K \)-module maps \( \kappa_x : M \longrightarrow \Phi_x \) satisfying the compatibility condition \( \kappa_x = \varphi_{x,y} \circ \kappa_y \), yields a chain complex
\[
\ldots \longleftarrow \Gamma_P(\Phi)_{-1} \longrightarrow \Gamma_P(\Phi)_0 \longleftarrow M
\]
with co-augmentation \( \kappa \) induced by the maps \( \kappa_x \) for \( x \in \text{rk}^{-1}(0) \).

Čech complexes of diagrams of chain complexes. The assignment \( \Phi \mapsto \Gamma_P(\Phi) \) is in fact an exact functor from the category of \( P \)-indexed diagrams of \( K \)-modules to the category of chain complexes of \( K \)-modules. Hence it can be prolongated, by levelwise application, to a functor \( \Gamma_P \) from the category of \( P \)-indexed diagrams of \( K \)-module chain complexes to the category of double chain complexes of \( K \)-modules. The resulting double complexes has rows \( \Gamma_P(C_t) \), that is, \( \Gamma_P(C)_{s,t} = \Gamma_P(C_t)_s \), and commuting differentials. By construction, \( \Gamma_P(C)_{s,t} = 0 \) for \( s > 0 \). — Passage to Čech complexes is homotopy invariant in the following sense:

**Lemma II.2.2.** Let \( \chi : \Phi \longrightarrow \Psi \) be a natural transformation of functors \( \Phi, \Psi : P \longrightarrow \text{Ch}(K\text{-Mod}) \). Suppose that for each \( p \in P \) the component \( \chi_p : \Phi_p \longrightarrow \Psi_p \) is a quasi-isomorphism. Suppose also that
the rank function is bounded above. Then the induced map of chain complexes
\[
\text{Tot}\Gamma(\chi) : \text{Tot}\Gamma(\Phi) \longrightarrow \text{Tot}\Gamma(\Psi)
\]
is also a quasi-isomorphism.

**Proof.** This is a standard result, we include an elementary proof for convenience. For \( p \geq 0 \) let \( \Gamma^p \) denote the horizontal truncation at \(-p\) of \( \Gamma \). That is, \( \Gamma^p(\Phi) \) is defined to be the double chain complex that agrees with \( \Gamma(\Phi) \) in columns \(-p, -p+1, \ldots, 0\), and is trivial otherwise. As the rank function is bounded above, \( \Gamma^p = \Gamma \) for sufficiently large \( p \).

The obvious surjection \( \Gamma^p(\Phi) \longrightarrow \Gamma^p(\Phi) \), given by identity maps in horizontal degrees \(-p\) and higher, has kernel the vertical chain complex
\[
K^p(\Phi) = \bigoplus_{x \in \text{rk}^{-1}(p+1)} \Phi_x,
\]
considered as a double chain complex concentrated in column \(-(p+1)\). Its totalisation is then the shift suspension \( K^p(\Phi)[-p-1] \) of the chain complex \( K \). By induction on \( p \geq 0 \), using the five lemma for ladder diagrams of long exact homology sequences induced by the diagram of short exact sequences below,
\[
\begin{array}{cccc}
0 & \longrightarrow & K_{p+1}(\Phi)[-p-1] & \longrightarrow & \text{Tot}\Gamma^{p+1}(\Phi) & \longrightarrow & \text{Tot}\Gamma^p(\Phi) & \longrightarrow & 0 \\
0 & \longrightarrow & K_{p+1}(\Psi)[-p-1] & \longrightarrow & \text{Tot}\Gamma^{p+1}(\Psi) & \longrightarrow & \text{Tot}\Gamma^p(\Psi) & \longrightarrow & 0
\end{array}
\]
the induced maps \( \text{Tot}\Gamma^p(\chi) : \text{Tot}\Gamma^p(\Phi) \longrightarrow \text{Tot}\Gamma^p(\Psi) \) are seen to be quasi-isomorphisms for all \( p \in \mathbb{N} \). \( \square \)

**II.3. Quasi-coherent diagrams**

As before we denote by the symbol \( \mathcal{S} \) the poset of non-empty faces of the square \( S = [-1, 1] \times [-1, 1] \). For the purpose of taking Čech complexes we equip \( \mathcal{S} \) with the rank function \( \text{rk}(F) = \text{dim}(F) \), and the usual incidence numbers coming from the orientations indicated in Fig. \( \mathbb{1} \). For example, \([e_r : v_{tr}] = 1\) and \([e_r : v_{br}] = -1\), and \([S : e_l] = 1\) for any decoration \(? \in \{t,l,b,r\}\). Let \( R \) be a \( \mathbb{Z}^2 \)-graded ring.

**Definition II.3.1.** A quasi-coherent diagram of modules is a functor
\[
M : \mathcal{S} \longrightarrow R_{(0,0)}\text{-Mod} , \quad F \mapsto M^F
\]
as depicted in Fig. \( \mathbb{2} \). In addition, for each \( F \) the entry \( M^F \) is to be equipped with a specified structure of an \( R_s[T_F] \)-module, extending the given \( R_{(0,0)} \)-module structure. For an inclusion of faces \( F \subseteq G \) we require the structure map \( \alpha^{F,G} : M^F \longrightarrow M^G \) to be \( R_s[T_F] \)-linear, such that the adjoint map
\[
\alpha^{F,G}_s : M^F \otimes_{R_s[T_F]} R_s[T_G] \longrightarrow M^G , \quad m \otimes x \mapsto \alpha^{F,G}(m) \cdot x
\]
is an isomorphism of $R_e[T_G]$-modules.—A quasi-coherent diagram of chain complexes is a chain complex of quasi-coherent diagrams of modules.

\[
\begin{array}{ccc}
M_{\nu_l} & \xrightarrow{\alpha_{\nu_l,\nu_l}} & M_{\nu_l} \\
| & | & | \\
\alpha_{\nu_l,\nu_l} & \alpha_{\nu_l,\nu_l} & \alpha_{\nu_l,\nu_l} \\
| & | & | \\
M_{\nu_l} & \xrightarrow{\alpha_{\nu_l,\nu_l}} & M_{\nu_l} \\
| & | & | \\
\alpha_{\nu_l,\nu_l} & \alpha_{\nu_l,\nu_l} & \alpha_{\nu_l,\nu_l} \\
| & | & | \\
M_{\nu_l} & \xrightarrow{\alpha_{\nu_l,\nu_l}} & M_{\nu_l} \\
| & | & | \\
\alpha_{\nu_l,\nu_l} & \alpha_{\nu_l,\nu_l} & \alpha_{\nu_l,\nu_l} \\
| & | & | \\
M_{\nu_l} & \xrightarrow{\alpha_{\nu_l,\nu_l}} & M_{\nu_l} \\
| & | & | \\
\alpha_{\nu_l,\nu_l} & \alpha_{\nu_l,\nu_l} & \alpha_{\nu_l,\nu_l} \\
| & | & | \\
M_{\nu_l} & \xrightarrow{\alpha_{\nu_l,\nu_l}} & M_{\nu_l} \\
| & | & | \\
\alpha_{\nu_l,\nu_l} & \alpha_{\nu_l,\nu_l} & \alpha_{\nu_l,\nu_l} \\
| & | & | \\
M_{\nu_l} & \xrightarrow{\alpha_{\nu_l,\nu_l}} & M_{\nu_l} \\
\end{array}
\]

**Figure 2.** Quasi-coherent diagram

We remark that a quasi-coherent diagram of chain complexes can be considered as a functor defined on $S$ with values in the category of chain complexes of $R_{(0,0)}$-modules, subject to conditions as above specified levelwise. Moreover, any quasi-coherent diagram of modules can be considered as a quasi-coherent diagram of complexes concentrated in chain degree 0.

In case $R$ is a Laurent polynomial ring in two variables with coefficients in a commutative ring $K$, a quasi-coherent diagram of modules is nothing but a quasi-coherent sheaf of modules on the product $\mathbb{P}_K^1 \times \mathbb{P}_K^1$ of the projective line over $R_K$ with itself.

Given a $\mathbb{Z}^2$-graded ring $R$ and $k \in \mathbb{Z}$, we denote by $D(k)$ the diagram depicted in Fig. 3. The arrows are inclusion maps. For a strongly $\mathbb{Z}^2$-graded ring this diagram is quasi-coherent as the adjoint maps are isomorphisms in light of Lemma 1.1. These diagrams will play a central role later on. They are the analogues of certain line bundles on $\mathbb{P}_K^1 \times \mathbb{P}_K^1$, viz., the external tensor square of $\mathcal{O}(k)$. We thus are led to expect the following calculation of its “global sections” and (trivial) “higher cohomology” (note $H_{-n}\Gamma_S \cong \lim^n$ as shown, for example, in [Hut04, Corollary 2.19]):

**Proposition II.3.2.** For $k \geq 0$ the complex

\[
\Gamma_S(D(k)) \xleftarrow{i} \bigoplus_{x,y=-k}^k R(x,y) \xrightarrow{k} 0
\]

is exact, where $i$ is the diagonal embedding, induced from the inclusions of its source into $D(k)^e$ with $v$ a vertex of $S$. 
More explicitly, the sequence

\[ 0 \xrightarrow{\Gamma_0(D(k))} \xrightarrow{d} \xrightarrow{d^{-1}} \Gamma_0(D(k)) \xrightarrow{k} \xrightarrow{\bigoplus_{x,y=-k} R(x,y)} \xrightarrow{\iota} \xrightarrow{\kappa} 0 \]  (II.3.2.1)

is exact.

**Proof.** The complex consists of \(\mathbb{Z}^2\)-graded \(R_{(0,0)}\)-modules and degree-preserving maps, so exactness can be checked in each degree separately. In degree \((x,y) \in [-k,k]^2\), the complex is the dual of the augmented cellular chain complex of the square (equipped with its obvious cellular structure), which is contractible, tensored with \(R(x,y)\). For all other degrees \((x,y)\), we see the dual of the cellular complex associated to the complement of a visibility subcomplex of the boundary of \(S\). In either case, the resulting complex is acyclic. — More details on the computation can be found in the appendix of §2.5 in [Hüt09], for example. □

II.4. Extending chain complexes of modules to quasi-coherent diagrams of chain complexes

Suppose that \(M\) is a finitely generated free module over the \(\mathbb{Z}^2\)-graded ring \(R\). Then there exists a quasi-coherent diagram which has \(M\) as its middle entry. More precisely, we fix an isomorphism between \(M\) and \(R^n\), for some \(n \geq 0\), and have \((\bigoplus_n D(k))^S \cong M\) for any \(k \in \mathbb{Z}\). This shows that finitely generated free \(R\)-modules can be extended to quasi-coherent diagrams. We need the following chain complex version of this fact:

**Proposition II.4.1.** Let \(R\) be a \(\mathbb{Z}^2\)-graded ring, and let \(C\) be a bounded above chain complex of finitely generated free \(R\)-modules. Then there
exists a complex $\mathcal{Y}$ of diagrams of the form $\bigoplus_r D(k)$, for various $r \geq 0$, such that $C \cong \mathcal{Y}^S$ as $R$-module complexes. More precisely, suppose that $C$ is concentrated in degrees $\leq t$. There are numbers

$$0 \leq k_t \leq k_{t-1} \leq k_{t-2} \leq \ldots$$

and numbers $r_n \geq 0$, with $r_n = 0$ whenever $C_n = 0$, such that we can choose $\mathcal{Y}_n = \bigoplus_{r_n} D(k_n)$. If $C$ is bounded, then so is $\mathcal{Y}$. — If the ring $R$ is strongly $\mathbb{Z}^2$-graded then $\mathcal{Y}$ consists of quasi-coherent diagrams.

**Proof.** We identify the chain module $C_n$ with $R^{r_n}$, for suitable numbers $r_n \geq 0$, where $r_n = 0$ if and only if $C_n = 0$; this amounts to choosing basis elements for the free modules $C_n$. The action of the differential $C_n \xrightarrow{\partial} C_{n-1}$ is then given by left multiplication by a matrix $D_n$ of size $r_n \times r_n$, with entries in $R$.

Given a homogeneous non-zero element $x \in R(s,t)$ we write $a(x) = \max(|s|,|t|)$; we also agree $a(0) = 0$. For a general element $x \in R$ let $a(x)$ denote the maximum of the values $a(x_i)$, where the $x_i$ are the homogeneous components of $x$. We denote by $a_n$ the maximum of the values $a(x)$ where $x$ varies over the entries of the matrix $D_n$, with the convention that $a_n = 0$ if $D_n$ is the empty matrix (i.e., if $r_n = 0$ or $r_{n-1} = 0$).

Suppose that $C$ is concentrated in chain degrees $\leq t$. We set $k_n = 0$ for $n \geq t$, and

$$k_n = \sum_{j=n+1}^{t} a_j$$

for $k < t$. We then define the chain complex $\mathcal{Y}$ of quasi-coherent diagrams by setting $\mathcal{Y}_n = \bigoplus_{r_n} D(k_n)$; the differentials $d_n : \mathcal{Y}_n \xrightarrow{\partial} \mathcal{Y}_{n-1}$ are given by left multiplication by the matrix $D_n$ in each component, so that

$$d_n^F : \mathcal{Y}_n^F = \left( \bigoplus_{r_n} D(k_n) \right)^F \xrightarrow{\partial} \left( \bigoplus_{r_{n-1}} D(k_{n-1}) \right)^F = \mathcal{Y}_{n-1}^F.$$

(This is well defined: The numbers $k_n$ are chosen precisely to ensure that $d_n^F$ maps $\mathcal{Y}_n^F$ to $\mathcal{Y}_{n-1}^F$.) As $D_{n-1} \cdot D_n = 0$ we have $d_{n-1} \circ d_n = 0$, and $C = \mathcal{Y}^S$ by construction.

It was observed before that the diagrams $D(k)$ are quasi-coherent if $R$ is strongly $\mathbb{Z}^2$-graded (this follows from Lemma [II.1.1]); hence $\mathcal{Y}$ consists of quasi-coherent diagrams in this case. \hfill $\square$

**Corollary II.4.2.** Let $R$, $C$, $\mathcal{Y}$, $r_n$ and $k_n$ be as in Proposition [II.4.1] with $C$ bounded. Then $D = \lim \mathcal{Y}$ is a bounded complex of $R(0,0)$-modules with chain modules

$$D_n = \left( \bigoplus_{x,y=-k_n}^{k_n} R(x,y) \right)^{r_n}.$$
If $R$ is strongly $\mathbb{Z}^2$-graded these $R_{(0,0)}$-modules are finitely generated projective. The natural map $D \rightarrow \text{Tot} \Gamma_S(\mathcal{V})$ is a homotopy equivalence of $R_{(0,0)}$-module complexes so that $\text{Tot} \Gamma_S(\mathcal{V})$ is $R_{(0,0)}$-finitely dominated.

Proof. Consider the double complex concentrated in columns $-2, -1, 0$ and $1$ with $n$th row given by

$$\Gamma_S(\mathcal{V}_n) \xrightarrow{i} \bigoplus_{x, y = -k_n} R_{(x,y)}^{r_n}.$$ 

Column 1 is the chain complex $\lim \mathcal{Y}$, see Proposition II.3.2, which consists of finitely generated projective $R_{(0,0)}$-modules if $R$ is strongly graded by Corollary I.4.6. As the rows are exact (by Proposition II.3.2 again), standard homological algebra asserts that the natural map $\lim \mathcal{Y} \rightarrow \text{Tot} \Gamma(\mathcal{V})$ is a quasi-isomorphism. In the strongly graded case, this is a homotopy equivalence (all complexes are bounded below and consist of projective $R_{(0,0)}$-modules) so that $\text{Tot} \Gamma(\mathcal{V})$ is $R_{(0,0)}$-finitely dominated. 

II.5. Flags and stars, and their associated rings

Definition II.5.1. For $F \in S$ define the star of $F$, denoted $\text{st}(F)$, as the set of faces of $S$ that contain $F$; this is a sub-poset of $S$. Let $\mathcal{N}_F$ be the set of flags in $\text{st}(F)$; the elements of $\mathcal{N}_F$ are strictly increasing sequences $\tau = (P_0 \subset P_1 \subset \cdots \subset P_k)$ of faces of $S$ with $F \subseteq P_0$. We consider $\mathcal{N}_F$ as a poset with order given by inclusion (or refinement) of flags, the smaller flag having fewer elements.

For each $F \in S$ we equip $\mathcal{N}_F$ with the rank function

$$\text{rk}(P_0 \subset P_1 \subset \cdots \subset P_k) = k.$$ 

and with standard simplicial incidence numbers: $[A : B] = 0$ unless the flag $B$ is obtained from $A = (P_0 \subset P_1 \subset \cdots \subset P_k)$ by omitting the entry $P_j$, in which case $[A : B] = (-1)^j$. Whenever we form a Čech complex of a diagram indexed by $\mathcal{N}_F$, we will use these data.

We now attach a ring $A_\tau$ to each flag $\tau = (F_0 \subset \cdots \subset F_k)$ of faces in $S$, as follows:

$$A(v_b) = R_n[[x, y]]$$

$$A(e_b) = R_n[x, x^{-1}][[y]]$$

$$A(S) = R_n[x, x^{-1}, y, y^{-1}] = R$$

$$A(v_b, e_b) = R_n((x))[[y]]$$

$$A(v_b, S) = R_n((x, y))$$

$$A(e_b, S) = R_n[x, x^{-1}][((y))]$$

$$A(v_b, e_b, S) = R_n((x))((y))$$

The effect of replacing “left” by “right” (i.e., replacing the subscript “$l$” by “$r$” throughout) is to replace $x$ by $x^{-1}$, while replacing “bottom”
by "top" (i.e., replacing the subscript "b" by "t" throughout) means replacing $y$ by $y^{-1}$. Swapping "bottom" and "left" amounts to swapping $x$ with $y$, and swapping "top" and "right" results in $x^{-1}$ and $y^{-1}$ swapping places. Thus, for example,

$$A(v_{bl}, e_t) = R_*(y^{-1})[[x]] \quad \text{and} \quad A(v_{tr}, e_r, S) = R_*(y^{-1})(x^{-1}) .$$

By direct inspection, this collection of rings is seen to have the following properties:

1. If $\sigma \subseteq \tau$ then $A(\sigma) \subseteq A(\tau)$. In particular, if $S \in \tau$ then $R = R_*[x, x^{-1}, y, y^{-1}] = R(S)$ is a subring of $A(\tau)$.

2. If $\tau \in N_F$ (that is, if all faces in $\tau$ contain $F$) then $A_F \subseteq A(\tau)$.

Fix $F \in \mathcal{G}$. The rings $A(\tau)$ defined above fit into a commutative diagram

$$E_F : N_F \longrightarrow A_F \text{-Mod}, \quad \tau \mapsto A(\tau)$$

of $A_F$-modules with structure maps given by inclusions. For $F = S$, the diagram $E_S$ has a single entry indexed by the flag \{S\}, and takes value $A(S) = R$ there. For $F = e_t$, the left-hand edge of $S$, the diagram $E_{e_t}$ looks like this:

$$R_*[y, y^{-1}][[x]] \quad R_*[y, y^{-1}][(x)] \quad R_*[x, x^{-1}, y, y^{-1}]$$

$$A(e_t) \quad A(e_t, S) \quad A(S)$$

(II.5.1.1)

And finally, for $F = v_{bl}$ the bottom left vertex of $S$, the diagram $E_{v_{bl}}$ is depicted in Fig. 4.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig4}
\caption{The diagram $E_{v_{bl}}$}
\end{figure}
We let $\Gamma_{N_F}(E_F)$ denote the Čech complex of $E_F$. Considering $A_F$ as a chain complex concentrated in chain level 0 we have a chain map

$$\xi^F: A_F \longrightarrow \Gamma_{N_F}(E_F)$$

given by the diagonal embedding $A_F \longrightarrow \bigoplus_{G \supseteq F} A(G)$. This is indeed a chain map thanks to property (DI3) of incidence numbers.

**Lemma II.5.2.** The map $\xi^F$ is a quasi-isomorphism of chain complexes of $A_F$-modules.

**Proof.** Note that for $F = S$ there is actually nothing to show as the category $N_S$ has a single object; for $F = e_l$ the claim is equivalent to saying that the sequence

$$0 \longrightarrow R_*[x, y, y^{-1}] \xrightarrow{(1)} R_*[y, y^{-1}][[x]] \oplus R_*[x, x^{-1}, y, y^{-1}] \xrightarrow{(1-1)} R_*[y, y^{-1}][(x)] \longrightarrow 0$$

is exact, and it is not too hard to see that it is actually split exact as a sequence of $R_{(0,0)}$-modules. The case of $F = v$, a vertex is somewhat more demanding in terms of book-keeping. Full details are worked out in the proof of Lemma 4.5.1 in [HQ15], writing $R_*[x, x^{-1}][(y)]$ in place of $R[x, x^{-1}][(y)]$ (and similar for the other rings); the proof carries over mutatis mutandis since the claim can be checked in each degree separately.

For faces $G \supseteq F$ let $E^G_F$ denote the $N_F$-indexed diagram which agrees with $E_G$ on $N_G$, and is zero everywhere else. The obvious map of diagrams $E_F \longrightarrow E^G_F$, given by identities where possible, results in a map of chain complexes

$$\Gamma_{N_F}(E_F) \longrightarrow \Gamma_{N_F}(E^G_F) = \Gamma_{N_G}(E_G),$$

given by projecting away from all those summands in the source which involve flags in $N_F \setminus N_G$. These maps are functorial on the poset $\mathcal{G}$ so that there results a commutative diagram

$$E: \mathcal{G} \longrightarrow R_{(0,0)}\text{-Mod}, \quad F \mapsto \Gamma_{N_F}(E_F).$$

**Proposition II.5.3.** The maps $\xi^F$ constructed above assemble to a natural transformation

$$\xi: D(0) \longrightarrow E.$$ 

Its components $\xi^F: A_F \longrightarrow \Gamma_{N_F}(E_F)$ are quasi-isomorphisms of complexes of $A_F$-modules.

**Proof.** This is the content of Lemma II.5.2 together with the observation that the maps $\xi^F$ are natural in $F$ with respect to the structure maps in $E$.

□
Given a flat $A_F$-module $M_F$ we thus obtain a quasi-isomorphism
\[ \chi^F : M_F \cong M_F \otimes_{A_F} \Gamma_{N_F}(E_F) \cong \Gamma_{N_F}(M_F \otimes_{A_F} E_F) ; \]
here $M_F \otimes_{A_F} E_F$ stands for the pointwise tensor product of the module $M_F$ with the entries of the diagram $E_F$. In other words, the sequence
\[ 0 \longrightarrow M_F \longrightarrow \Gamma_{N_F}(M_F \otimes_{A_F} E_F)_0 \longrightarrow \Gamma_{N_F}(M_F \otimes_{A_F} E_F)_{-1} \longrightarrow \Gamma_{N_F}(M_F \otimes_{A_F} E_F)_{-2} \longrightarrow 0 \quad (\text{II.5.3.1}) \]
is exact.

For $M$ an arbitrary quasi-coherent diagram we obtain a natural transformation
\[ \chi : M \cong M \otimes_{D(0)} D(0) \longrightarrow M \otimes_{D(0)} E \cong M' \quad (\text{II.5.3.2}) \]
where the target $M'$ is the diagram of chain complexes
\[ M' : F \mapsto \Gamma_{N_F}(M_F \otimes_{A_F} E_F) , \]
$M \otimes_{D(0)} D(0)$ denotes the pointwise tensor product
\[ M \otimes_{D(0)} D(0) : F \mapsto M_F \otimes_{A_F} D(0)^F = M_F \otimes_{A_F} A_F , \]
and $M \otimes_{D(0)} E$ stands similarly for the pointwise tensor product of the diagrams $M$ and $E$. If moreover $M$ is such that the entry $M_F$ is a flat $A_F$-module, the components of $\chi$ are quasi-isomorphisms.

II.6. From trivial Novikov homology to finite domination

We will now prove the “if” implication of Theorem 1.2.1. So let $R$ be a strongly $\mathbb{Z}^2$-graded ring. Suppose that $C$ is a bounded complex of finitely generated free $R$-modules, and further that the complexes listed in (1.2.1.1a) and (1.2.1.1b) are acyclic. In view of the assumed freeness of $C$, these eight complexes are then actually contractible. They are of the form
\[ C \otimes_{R} A\langle e, S \rangle \quad \text{and} \quad C \otimes_{R} A\langle v, S \rangle \]
where $e$ and $v$ denote an edge and a vertex of $S$, respectively. As tensor products preserve contractions, it follows that for all eight maximal flags $v \subset e \subset S$, for $v$ a vertex of $S$ and $e$ an edge incident to $v$, the complex
\[ C \otimes_{R} A\langle v, e, S \rangle \cong C \otimes_{R} A\langle v, S \rangle \otimes_{A\langle v, S \rangle} A\langle v, e, S \rangle \cong 0 \quad (\text{II.6.0.1}) \]
is contractible (and hence acyclic) as well.

Extend the complex $C$ to a complex of sheaves $\mathcal{Y} : F \mapsto \mathcal{Y}_F$, which can be done by Proposition II.4.1. In more detail, this means that we find a complex of sheaves such that $C$ is identified with $\mathcal{Y}^S$, and
such that there is a quasi-isomorphism $\lim_{\rightarrow} Y \rightarrow \text{Tot } \Gamma_{\mathcal{S}}(Y)$ with $R_{(0,0)}$-finitely dominated source (Corollary II.4.2).

Let, for the moment, $F \neq \emptyset$ denote a fixed face of $S$. We observe that if $F \neq S$,

\[
Y_F \otimes A(F, S) \cong Y_F \otimes A_S \otimes A(F, S) \cong C \otimes A(F, S) \cong 0 . \tag{II.6.0.2}
\]

The isomorphism labelled (†) combines two facts: first, the entries of the complex $Y$ are quasi-coherent diagrams (Proposition II.4.1) so that $Y_F \otimes \Lambda_F \Lambda_S \Lambda_{\langle F, S \rangle} \cong Y_S \otimes \Lambda_S \Lambda_{\langle F, S \rangle} \cong C \otimes \Lambda_{\langle F, S \rangle} \cong 0$. (II.6.0.3)

For arbitrary $F \neq \emptyset$, exactness of the sequence (II.5.3.1) implies that the double complex $Y_F \rightarrow \Gamma_{\mathcal{N}_F}(Y_F \otimes E_F)$ has exact rows (using the fact that $Y_F$ consists of projective $A_F$-modules), so there results a quasi-isomorphism

\[
\text{Tot}(\chi^F) : Y_F \rightarrow \text{Tot } \Gamma_{\mathcal{N}_F}(Y_F \otimes E_F) .
\]

We will now make use of the fact that some entries of the diagram $Y_F \otimes A_F E_F$ are known to be acyclic. Let $Z_F$ denote the $\mathcal{N}_F$-indexed diagram which agrees with $Y_F \otimes A_F E_F$ on those flags not containing $S$, and is zero elsewhere. Let $K_F$ denote the diagram which takes the value $C$ at $\{S\}$, and is zero otherwise. The obvious surjective map $Y_F \otimes A_F E_F \rightarrow Z_F \oplus K_F$ is a pointwise quasi-isomorphism, by (II.6.0.2) and (II.6.0.3), and hence induces a quasi-isomorphism

\[
\text{Tot } \Gamma_{\mathcal{N}_F}(Y_F \otimes E_F) \rightarrow \text{Tot } \Gamma_{\mathcal{N}_F}(Z_F \oplus K_F) .
\]

We compute further that the target of this map is

\[
\text{Tot } \Gamma_{\mathcal{N}_F}(Z_F \oplus K_F) = \text{Tot } \Gamma_{\mathcal{N}_F}(Z_F) \oplus C .
\]

In total this yields a quasi-isomorphism

\[
\Xi^F : Y_F \rightarrow \text{Tot } \Gamma_{\mathcal{N}_F}(Z_F) \oplus C .
\]

Allowing $F$ to vary again, and making use of the naturality of the constructions above, we see that we obtain a natural quasi-isomorphism of diagrams

\[
\Xi : \mathcal{Y} \rightarrow \text{Tot } \Gamma_{\mathcal{N}_F}(Z_{\langle - \rangle}) \oplus \text{con } (C) ,
\]
with the $\mathcal{S}$-indexed diagram
\[
\Gamma_{\mathcal{N}(-)}(Z(\cdot)) : F \mapsto \Gamma_{\mathcal{N}F}(Z_F)
\]
and the constant diagram
\[
\text{con} (C) : F \mapsto C.
\]
It follows that $\text{Tot} \Gamma_{\mathcal{S}}(Y)$, which is an $R(0,0)$-finitely dominated by Corollary II.4.2, is quasi-isomorphic to
\[
\text{Tot} \Gamma_{\mathcal{S}}(\text{Tot} \Gamma_{\mathcal{N}(-)}(Z(\cdot))) \oplus \text{Tot} \Gamma_{\mathcal{S}}(\text{con} (C)).
\]
The second summand is, in turn, quasi-isomorphic to $C$. (In effect, this is true since the nerve of $\mathcal{S}$ is contractible; for a more explicit argument, observe that the double complex
\[
\Gamma_{\mathcal{S}}(\text{con} (C)) \xrightarrow{C} C
\]
concentrated in columns $-2, -1, 0$ and $1$ has acyclic rows so that its totalisation is acyclic. But the totalisation is, up to isomorphism, the mapping cone of the map $C \xrightarrow{C} \text{Tot} \Gamma_{\mathcal{S}}(\text{con} (C))$ which is thus a weak equivalence. See also [HQ15, Lemma 4.6.4].)

Thus in the derived category of the ring $R(0,0)$ the complex $C$ is a retract of $\text{Tot} \Gamma_{\mathcal{S}}(Y)$, and as both complexes are bounded and consist of projective $R(0,0)$-modules, this makes $C$ a retract up to homotopy of $\text{Tot} \Gamma_{\mathcal{S}}(Y)$. As the latter is $R(0,0)$-finitely dominated so is $C$, as was to be shown.

Part III. Finite domination implies triviality of Novikov homology.

III.1. Algebraic tori

Definition III.1.1. Let $R = \bigoplus_{\sigma \in \mathbb{Z}^2} R_{\sigma}$ be a strongly $\mathbb{Z}^2$-graded unital ring. Given an $R$-module $M$ and an element $\rho \in \mathbb{Z}^2$, we define the map of right $R$-modules
\[
\chi_{\rho} : M \otimes_{R(0,0)} R \xrightarrow{R(0,0)} M \otimes_{R(0,0)} R, \quad m \otimes r \mapsto \sum_j m u_j \otimes v_j r,
\]
where $1 = \sum_j u_j v_j$ is any partition of unity of type $(-\rho, \rho)$.

The map $\chi_{\rho}$ is $R(0,0)$-balanced and independent of the choice of partition of unity as its restriction to $M \otimes_{R(0,0)} R_{\sigma}$ can be re-written as
\[
M \otimes_{R(0,0)} R_{\sigma} \xrightarrow{\cong} M \otimes_{R(0,0)} R_{-\rho} \otimes_{R(0,0)} R_{\sigma+\rho} \xrightarrow{\omega} M \otimes_{R(0,0)} R
\]
with $\mu_{-\rho,\sigma+\rho}$ the $R(0,0)$-bimodule isomorphism from Proposition I.4.3 and with $\omega(m \otimes x \otimes y) = (mx) \otimes y$. — In case $M = R$, the map $\chi_{\rho}$ is an $R$-bimodule homomorphism.

Of course $\chi_{(0,0)} = \text{id}$. As $\chi_{\rho}$ does not depend on the choice of partition of unity, Lemma I.4.2 implies:
Corollary III.1.2. For all \( \rho, \sigma \in \mathbb{Z}^2 \) there are equalities of maps \( \chi_\rho \chi_\sigma = \chi_{\rho + \sigma} = \chi_\sigma \chi_\rho \). The maps \( \chi_\rho \) are isomorphisms of \( R \)-modules with \( \chi_\rho^{-1} = \chi_{-\rho} \). \( \square \)

We can consider

\[
M \otimes R = \bigoplus_{\rho \in \mathbb{Z}^2} M \otimes R_{\rho, (0,0)}
\]
as a \( \mathbb{Z}^2 \)-graded \( R \)-module. With respect to this grading, we observe:

Lemma III.1.3. The map \( \chi_\rho \) maps homogeneous elements of degree \( \sigma \) to homogeneous elements of degree \( \rho + \sigma \). For the homogeneous element \( m \otimes r \) of degree \( -\rho \), the formula \( \chi_\rho(m \otimes r) = mr \otimes 1 \) holds. \( \square \)

Let \( C \) be a chain complex of \( R \)-modules. Then \( \chi_\rho \), applied in each chain level, defines a chain map

\[
\chi_\rho : C \otimes R \rightarrow C \otimes R.
\]

Let \( D \) be an additional \( R_{(0,0)} \)-chain complex, let \( \alpha : C \rightarrow D \) and \( \beta : D \rightarrow C \) be \( R_{(0,0)} \)-linear chain maps, and let \( H : id_C \simeq \beta \alpha \) be a homotopy from \( \beta \alpha \) to \( id \) such that \( dH + Hd = \beta \alpha - id_C \). Then the outer square in the diagram \( T(\alpha, \beta; H) \)

\[
\begin{array}{ccc}
D \otimes R & \text{id}^* - \alpha^* \chi_{e_2} \beta^* & D \otimes R \\
\text{id}^* - \alpha^* \chi_{e_2} \beta^* & & \text{id}^* - \alpha^* \chi_{e_2} \beta^* \\
D \otimes R & & D \otimes R
\end{array}
\]

(writing \( f^* = f \otimes id \), for any map \( f \)) is homotopy commutative, with the diagonal arrow recording a preferred homotopy of the two possible compositions. — Note that if \( \beta \alpha = id_C \) we can choose \( H = 0 \) and in this case the outer squares of \( T(\alpha, \beta; 0) \) commutes by Corollary III.1.2.

Definition III.1.4. The algebraic torus \( \mathfrak{T}(\alpha, \beta; H) \) is defined as the totalisation of the homotopy commutative diagram \( T(\alpha, \beta; H) \). That is, \( \mathfrak{T}(\alpha, \beta; H) \) is the complex with chain modules

\[
\mathfrak{T}(\alpha, \beta; H)_n = (D_{n-2} \otimes R_{(0,0)}) \oplus (D_{n-1} \otimes R_{(0,0)}) \oplus (D_{n-1} \otimes R_{(0,0)}) \oplus (D_n \otimes R_{(0,0)})
\]

and boundary \( d_{\mathfrak{T}(\alpha, \beta; H)} \) given by the following matrix:

\[
\begin{pmatrix}
d^* \\
\text{id}^* - \alpha^* \chi_{e_1} \beta^* & -d^* & 0 & 0 \\
\text{id}^* - \alpha^* \chi_{e_2} \beta^* & 0 & -d^* & 0 \\
(\alpha^* (\chi_{e_1} H \chi_{e_2} - \chi_{e_2} H \chi_{e_1}) \beta^*) & \text{id}^* - \alpha^* \chi_{e_2} \beta^* & -\text{id}^* + \alpha^* \chi_{e_1} \beta^* & d^*
\end{pmatrix}
\]
III.2. Canonical resolutions

Let $C$ be a chain complex of $R$-modules. The commutative diagram

$$
\begin{array}{ccc}
C \otimes_{R_{(0,0)}} R & \xrightarrow{\text{id} - \chi_{e_1}} & C \otimes_{R_{(0,0)}} R \\
\downarrow \text{id} - \chi_{e_2} & & \downarrow \text{id} - \chi_{e_2} \\
C \otimes_{R_{(0,0)}} R & \xrightarrow{\text{id} - \chi_{e_2}} & C \otimes_{R_{(0,0)}} R \\
\end{array}
$$

$T(id_C, id_C; 0)$

gives rise, via totalisation, to the algebraic torus $T(id_C, id_C; 0)$. The $R$-module structure map $\gamma: (x, r) \mapsto xr$ induces a map of $R$-module chain complexes

$$
\kappa: T(id_C, id_C; 0) \longrightarrow C.
$$

**Proposition III.2.1** (Canonical resolution). The map $\kappa$ is a quasi-isomorphism. If $C$ is a bounded below complex of projective $R$-modules then $\kappa$ is a chain homotopy equivalence.

In preparation of the proof, we note that $T(id_C, id_C; 0)$ can be described as the totalisation (in the usual sense) of a double complex of the form

$$
\begin{array}{ccc}
C \otimes_{R_{(0,0)}} R & \xrightarrow{(\text{id} - \chi_{e_1})^2} & C \otimes_{R_{(0,0)}} R \\
\end{array}
$$

(III.2.1.1)

The $R$-module structure map $\gamma: (x, r) \mapsto xr$ of $C$ can be used to augment this to the $R$-module double complex

$$
\begin{array}{ccc}
0 & \longrightarrow & C \otimes_{R_{(0,0)}} R \longrightarrow \left( C \otimes_{R_{(0,0)}} R \right)^2 \longrightarrow C \otimes_{R_{(0,0)}} R \\
& & \downarrow \gamma \\
& & C \otimes_{R_{(0,0)}} R \longrightarrow 0.
\end{array}
$$

(III.2.1.2)

We will consider a single row of this double complex, that is, a sequence of the form

$$
\begin{array}{ccc}
0 & \longrightarrow & M \otimes_{R_{(0,0)}} R \longrightarrow \left( M \otimes_{R_{(0,0)}} R \right)^2 \longrightarrow M \otimes_{R_{(0,0)}} R \longrightarrow M \longrightarrow 0 \\
& & \alpha \downarrow \beta \downarrow \gamma \\
& & M \otimes_{R_{(0,0)}} R \longrightarrow M
\end{array}
$$

(III.2.1.3)

where $M$ is a right $R$-module. Here the maps $\alpha$ and $\beta$ are given by the matrices

$$
\alpha = \begin{pmatrix} \text{id} - \chi_{e_1} \\ \text{id} - \chi_{e_2} \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} \text{id} - \chi_{e_2} & -(\text{id} - \chi_{e_1}) \end{pmatrix}.
$$

**Lemma III.2.2.** The complex (III.2.1.3) is exact.

**Proof.** We allow $M$ to be an arbitrary $R_{(0,0)}$-module initially. We have the direct sum decomposition $M \otimes_{R_{(0,0)}} R = \bigoplus_{(i,j)} M \otimes_{R_{(0,0)}} R_{(i,j)}$; in
fact, \( M \otimes_{R_{(0,0)}} R \) is a \( \mathbb{Z}^2 \)-graded \( R \)-module in this way. Any element \( z \in M \otimes_{R_{(0,0)}} R \) can be expressed uniquely in the form

\[
z = \sum_{i,j \in \mathbb{Z}} m_{i,j} \quad \text{where} \quad m_{i,j} \in M \otimes_{R_{(0,0)}} R_{(i,j)},
\]

such that \( m_{i,j} = 0 \) for almost all pairs \((i, j)\). We say that \( z \) has \textit{x-amplitude in the interval} \([a, b]\) if \( m_{i,j} = 0 \) for \( i \not\in [a, b] \). The support of \( z \) is the (finite) set of all pairs \((i, j)\) with \( m_{i,j} \neq 0 \).

After these initial comments, we proceed to verify exactness of the sequence. We remark first that \( \gamma \) is surjective since \( \gamma(p \otimes 1) = p \). As a matter of fact, \( \sigma(p) = p \otimes 1 \) defines an \( R_{(0,0)} \)-linear section \( \sigma \) of \( \gamma \).

Next, we show that \( \alpha \) is injective. Let \( z = \sum_{i,j \in \mathbb{Z}} m_{i,j} \) be an element of \( \ker(\alpha) \). Since \( \chi_{e_2}(m_{i,j}) \in M \otimes_{R_{(0,0)}} R_{(i,j+1)} \), the equality

\[
\alpha(z) = \left( \sum_{i,j \in \mathbb{Z}} (m_{i,j} - \chi_{e_1}(m_{i,j})) \right) \sum_{i,j \in \mathbb{Z}} (m_{i,j} - \chi_{e_2}(m_{i,j})) = 0
\]

implies, by considering the homogeneous component of degree \((i, j)\) of the second entry, that \( m_{i,j} - \chi_{e_2}(m_{i,j-1}) = 0 \). If \( z \neq 0 \) there exists \((i, j)\) in the support of \( z \) such that \( m_{i,j-1} = 0 \). For these indices we have \( m_{i,j} = \chi_{e_2}(m_{i,j-1}) = \chi_{e_2}(0) = 0 \), a contradiction. This enforces \( z = 0 \) whence \( \alpha \) is injective.

To show that \( \im \alpha = \ker \beta \) we will take an element \((z_1, z_2)\) of \( \ker \beta \) and show that it can be reduced to 0 by subtracting a sequence of elements of \( \im \alpha \subseteq \ker \beta \). This clearly implies that \((z_1, z_2) \in \im \alpha \) as required.

So let \((z_1, z_2) \in \ker \beta \), where

\[
z_1 = \sum_{i,j \in \mathbb{Z}} m_{i,j} \quad \text{and} \quad z_2 = \sum_{i,j \in \mathbb{Z}} n_{i,j}, \quad \text{with} \quad m_{i,j}, n_{i,j} \in M \otimes_{R_{(0,0)}} R_{(i,j)}.
\]

Choose integers \(a, b\) and \(k\) such that \( z_1 \) has \textit{x-amplitude in} \([a, k]\) and \( z_2 \) has \textit{x-amplitude in} \([a, b]\). If \( k > a \) we define

\[
u = \sum_{j \in \mathbb{Z}} \chi_{-e_1}(m_{k,j}) \in M \otimes_{R_{(0,0)}} R, \quad \text{with} \quad \chi_{-e_1}(m_{k,j}) \in M \otimes_{R_{(0,0)}} R_{(k-1,j)},
\]

and set \((z_1', z_2') = (z_1, z_2) - \alpha(u)\). The \((k, j)\)-homogeneous component of \( z_1' \) vanishes by construction of \( u \) and Corollary \([11.1.2]\) so that \( z_1' \) has \textit{x-amplitude in} \([a, k - 1]\) while the \textit{x-amplitude} of \( z_2 \) remains in \([a, b]\). The element \((z_1', z_2')\) will be in \( \im \alpha \) if and only if \((z_1, z_2) \in \im \alpha \).

By iteration, we may thus assume that our initial pair \((z_1, z_2)\) is such that the \textit{x-amplitude} of \( z_1 \) is in \([a] = [a, a]\). This actually necessitates \( z_2 = 0 \). To see this, assume \( z_2 \neq 0 \). We can then choose \((i, j)\) in the support of \( z_2 \) (so that in particular \( i \geq a \)) such that \((i + 1, j)\) is not
in the support of $z_2$, i.e., such that $n_{i+1,j} = 0$. Since $\beta(z_1, z_2) = 0$ we have, by considering the homogeneous component of degree $(i + 1, j)$, the equality

$$m_{i+1,j} - \chi_{e_2} (n_{i+1,j-1}) - n_{i+1,j} + \chi_{e_1} (n_{i,j}) = 0.$$  

But the first three terms vanish, by choice of $i$ and our hypothesis on $z_1$, so that $\chi_{e_1} (n_{i,j}) = 0$ whence $n_{i,j} = 0$, contradicting the choice of $i$.

Thus $z_2 = 0$. This in turn implies that $\beta(z_1, 0) = 0$ so that $m_{a,\ell} - \chi_{e_2} (m_{a,\ell-1}) = 0$ for any $\ell$. If $z_1$ is non-zero, we let $\ell$ be minimal with $m_{a,\ell} \neq 0$. Then $m_{a,\ell} = \chi_{e_2} (m_{a,\ell-1}) = \chi_{e_2} (0) = 0$, a contradiction. We conclude that $(z_1, z_2) = 0 \in \text{im} \alpha$ as required.

To show $\text{im} \beta = \ker \gamma$ it is enough to produce an $R_{(0,0)}$-linear map

$$\varphi: M \otimes R_{(0,0)} \rightarrow \left( M \otimes R_{(0,0)} \right)^2$$

such that $\beta \varphi + \sigma \gamma = \text{id}$, where $\sigma$ is the section of $\gamma$ given by $\sigma(p) = p \otimes 1$. It is enough to define $\varphi$ on homogeneous primitive tensors $x = m \otimes r$ with $r \in R_p$, for the various $p \in {\mathbb Z}^2$.

- If $a, b \geq 0$:

$$\varphi(x) = \left( - \sum_{k=1}^{b} \chi_{(0,-k)} (x) \sum_{\ell=1}^{a} \chi_{(-\ell,-b)} (x) \right)$$

- If $a, b < 0$:

$$\varphi(x) = \left( \sum_{k=0}^{|b|-1} \chi_{(0,k)} (x) - \sum_{\ell=0}^{|a|-1} \chi_{(\ell,b)} (x) \right)$$

- If $a < 0$ and $b \geq 0$:

$$\varphi(x) = \left( - \sum_{k=1}^{b} \chi_{(0,-k)} (x) - \sum_{\ell=0}^{a} \chi_{(\ell,b)} (x) \right)$$

- If $a \geq 0$ and $b < 0$:

$$\varphi(x) = \left( \sum_{k=0}^{|b|-1} \chi_{(0,k)} (x) \sum_{\ell=1}^{a} \chi_{(-\ell,-b)} (x) \right)$$

In view of Corollary \textbf{III.1.2} computing $\beta \varphi (m \otimes r)$, for $r \in R$ a homogeneous element of degree $p$, results in a telescoping sum simplifying to $m \otimes r - \chi_{-p} (m \otimes r)$. But $\chi_{-p} (m \otimes r) = m r \otimes 1$ by Lemma \textbf{III.1.3} whence $\chi_{-p} (m \otimes r) = \sigma \gamma (m \otimes r)$ so $\beta \varphi + \sigma \gamma = \text{id}$ as required. 

\textit{Proof of Proposition \textbf{III.2.1}}. The map $\kappa$ is a quasi-isomorphism if and only if its mapping cone is acyclic. But this mapping cone is precisely the totalisation of the double complex \textbf{(III.2.1.2)}. As it is concentrated in a finite vertical strip, and as its rows are acyclic by Lemma \textbf{III.2.2} the totalisation is acyclic by standard results for double complexes. 

\hfill $\square$
III.3. The Mather trick

Let $C$ be a chain complex of $R$-modules, and let $D$ be a chain complex of $R_{(0,0)}$-modules. Suppose we have $R_{(0,0)}$-linear chain maps $\alpha: C \to D$ and $\beta: D \to C$ and a chain homotopy $H: \text{id}_C \simeq \beta \alpha$ such that $dH + HD = \beta \alpha - \text{id}_C$. A calculation shows:

**Lemma III.3.1.** The matrix

\[
\begin{pmatrix}
\alpha^* & 0 & 0 & 0 \\
-\alpha^* \mu_1 H^* & \alpha^* & 0 & 0 \\
-\alpha^* \mu_2 H^* & 0 & \alpha^* & 0 \\
K & \alpha^* \mu_2 H^* & -\alpha^* \mu_1 H^* & \alpha^*
\end{pmatrix},
\]

where we abbreviate

\[K = \alpha^* (\mu_1 H^* \mu_2 - \mu_2 H^* \mu_1) H^* : C_n \otimes R \to D_{n+2} \otimes R ,\]

defines a chain map of algebraic tori

\[\lambda: \Sigma(\text{id}_C, \text{id}_C; 0) \to \Sigma(\alpha, \beta; H) .\]

**Theorem III.3.2 (Mather trick).** If the map $\alpha$ is a quasi-isomorphism then $C$ is quasi-isomorphic to $\Sigma(\alpha, \beta; H)$. More precisely, the maps

\[C \xleftarrow{\kappa} \Sigma(\text{id}_C, \text{id}_C; 0) \xrightarrow{\lambda} \Sigma(\alpha, \beta; H)\]

are quasi-isomorphism.

**Proof.** The map $\kappa$ is a quasi-isomorphism by Lemma [III.2.1]. The map $\lambda$, defined in Lemma [III.3.1] is a quasi-isomorphism since the representing matrix of $\lambda$ is lower triangular with diagonal terms $\alpha^*$. Note that $\alpha^* = \alpha \otimes \text{id}_R$ is a quasi-isomorphism since $R$ is strongly graded and hence is a projective $R_{(0,0)}$-module.

III.4. Novikov homology

Let $K$ be a $\mathbb{Z}^2$-graded $R$-module, with $R$ a $\mathbb{Z}^2$-graded ring as usual. In analogy to (I.2.0.1), we can define

\[K_*([x, x^{-1}](y)) = \bigcup_{n \geq 0} \prod_{y \geq -n} \bigoplus_{x \in \mathbb{Z}} K(x, y),\]

which is an $R_*(x, x^{-1})(y)$-module in a natural way; similarly, we can define

\[K_*((x, y)) = \bigcup_{n \geq 0} \prod_{x, y \geq -n} K(x, y),\]

which is an $R_*((x, y))$-module in a natural way; and so on.

For $R$-modules of the form $K = M \otimes_{R_{(0,0)}} R$, with $M$ an $R_{(0,0)}$-module, these constructions are close to the usual induction functors; for example:
**Proposition III.4.1.** Let $M$ be a right $R_{(0,0)}$-module. There is a natural map

$$
\Psi_M : M \otimes_{R_{(0,0)}} R_*(x, y) \rightarrow (M \otimes_{R_{(0,0)}} R_*)^*(x, y),
$$

$$
m \otimes \sum_{x,y} r_{x,y} \mapsto \sum_{x,y} m \otimes r_{x,y}
$$

which is an isomorphism if $M$ is finitely presented.

**Proof.** This can be verified by standard techniques: establish the result for $M = R_{(0,0)}$ first (in which case it is trivial), extend to the case $M = R_{(0,0)}^n$, and then treat the general case using a finite presentation of $M$. \hfill \Box

From now on, assume that $C$ is a bounded below complex of finitely generated projective $R$-modules which is $R_{(0,0)}$-finitely dominated. Thus we choose, once and for all, a bounded complex of finitely generated projective $R_{(0,0)}$-modules $D$ and mutually inverse chain homotopy equivalences $\alpha : C \rightarrow D$ and $\beta : D \rightarrow C$, and a chain homotopy $H : \text{id}_C \simeq \beta \alpha$ so that $dH + Hd = \beta \alpha - \text{id}_C$.

The MATHER trick [III.3.2] guarantees that $C$ and $\mathfrak{T}(\alpha, \beta; H)$ are quasi-isomorphic; as both $R$-module complexes are bounded below and consist of projective $R$-modules, they are in fact chain homotopy equivalent.

**Lemma III.4.2.** The chain complex $C \otimes_R R_*(x, y)$ is contractible.

**Proof.** By the previous remarks, it is enough to show that the chain complex $\mathfrak{T}(\alpha, \beta; H) \otimes_R R_*(x, y)$ is acyclic. Recall that $\mathfrak{T}(\alpha, \beta; H)$ is obtained from the diagram $[\mathfrak{T}(\alpha, \beta; H)]$ by a totalisation process, which commutes with induction. Thus we can realise $\mathfrak{T}(\alpha, \beta; H) \otimes_R R_*(x, y)$ by totalising the following diagram:

$$
\begin{array}{ccc}
D \otimes_{R_{(0,0)}} R_*(x, y) & \xrightarrow{id^* - \alpha^* \chi_{x_1} \beta^*} & D \otimes_{R_{(0,0)}} R_*(x, y) \\
\downarrow{id^* - \alpha^* \chi_{x_2} \beta^*} & & \downarrow{id^* - \alpha^* \chi_{x_2} \beta^*} \\
D \otimes_{R_{(0,0)}} R_*(x, y) & \xrightarrow{id^* - \alpha^* \chi_{x_1} \beta^*} & D \otimes_{R_{(0,0)}} R_*(x, y)
\end{array}
$$

(We have used implicitly that the two functors $\otimes_{R_{(0,0)}} R_*(x, y)$ and $\otimes_{R_{(0,0)}} R \otimes_R R_*(x, y)$ are naturally isomorphic.) By Lemma [III.4.1]
we can re-write this further as the totalisation of the diagram

\[
\begin{array}{ccc}
D_\ast((x, y)) & \xrightarrow{id^\ast - \alpha^\ast \chi_{e_1}} & D_\ast((x, y)) \\
\downarrow{id^\ast - \alpha^\ast \chi_{e_2}} & & \downarrow{id^\ast - \alpha^\ast \chi_{e_2}} \\
D_\ast((x, y)) & \xrightarrow{id^\ast - \alpha^\ast \chi_{e_1}} & D_\ast((x, y)) \\
\end{array}
\] (III.4.2.1)

with maps suitably interpreted. To wit, an element \( z \) of \( D_\ast((x, y)) \) (in some fixed chain level) has the form

\[ z = \sum_{x, y \geq a} z_{x, y} \]

for some \( a \in \mathbb{Z} \) and certain \( z_{x, y} \in \mathcal{D} \otimes \mathbb{R}(0, 0) \mathcal{R}_{x,y} \), and

\[
(id^\ast - \alpha^\ast \chi_{e_1}, \beta^\ast) (z) = \sum_{x, y \geq a} z_{x, y} - \alpha^\ast \chi_{e_1}(z_{x-1, y}) ,
\]

see Lemma III.4.1; similar formulæ can be written out for the other maps.

The point is that the self-map \( id^\ast - \alpha^\ast \chi_{e_1}, \beta^\ast \) of \( D_\ast((x, y)) \) is an isomorphism with inverse given by the “geometric series”

\[
P(z) = \sum_{k \geq 0} (\alpha^\ast \chi_{e_1}, \beta^\ast)^k (z)
\]

\[= z + \alpha^\ast \chi_{e_1}(z) + \alpha^\ast \chi_{e_1}(z) + \cdots .\]

This follows immediately from the usual telescoping sum argument, once it is understood that the series actually defines a well-defined self-map of \( D_\ast((x, y)) \). But this is the case because \( \alpha^\ast \chi_{e_1}, \beta^\ast \) maps an element of degree \((x, y)\) to an element of degree \((x+1, y)\).

It is now a matter of computation to verify that the matrix

\[
p = \begin{pmatrix}
0 & P & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -P \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\] (III.4.2.2)

almost defines a contraction of the totalisation of (III.4.2.1), in the sense that \( dp + pd \) is an automorphism \( q \) (in each chain level) — in fact, \( q \) is given by a triagonal matrix with identity diagonal entries — where \( d \) denotes the boundary map of the totalisation. It follows that \( q^{-1}p \) is a chain contraction as required. \(\square\)

**Lemma III.4.3.** The complex \( \mathcal{C} \otimes_\mathbb{R} \mathbb{R}_*[y, y^{-1}](x) \) is contractible.

**Proof.** This is shown just like the previous Lemma. It is enough to demonstrate that the chain complex \( \mathfrak{T}(\alpha, \beta; H) \otimes_\mathbb{R} \mathbb{R}_*[y, y^{-1}](x) \) is acyclic; this complex can be written, using a variant of Lemma III.4.1
as the totalisation of the following diagram:

\[
\begin{array}{ccc}
D_*[y, y^{-1}](\langle x \rangle) & \xrightarrow{id^*-\alpha^*\chi_{e_1}\beta^*} & D_*[y, y^{-1}](\langle x \rangle) \\
& & \\
\text{id}^*-\alpha^*\chi_{e_2}\beta^* & \text{id}^*-\alpha^*\chi_{e_2}\beta^* & \text{id}^*-\alpha^*\chi_{e_2}\beta^* \\
D_*[y, y^{-1}](\langle x \rangle) & \xrightarrow{id^*-\alpha^*\chi_{e_2}\beta^*} & D_*[y, y^{-1}](\langle x \rangle)
\end{array}
\]

(III.4.3.1)

The self-map \(\text{id}^* - \alpha^*\chi_{e_1}\beta^*\) of \(D_*[y, y^{-1}](\langle x \rangle)\) is an isomorphism with inverse \(P(z) = \sum_{k \geq 0} (\alpha^*\chi_{e_1}\beta^*)^k(z)\); the matrix \(p\) from (III.4.2.2) is such that \(dp + pd\) is an automorphism \(q\) (in each chain level), where \(d\) denotes the boundary map of the totalisation. Consequently, \(q^{-1}p\) is a chain contraction.

\[\square\]

Proof of “only if” in Theorem I.2.1. By Lemma III.4.2, the complex \(C \otimes_R R_*[y, y^{-1}]\) is acyclic. Replacing \(R\) by the strongly \(\mathbb{Z}^2\)-graded ring \(\bar{R} \equiv R[-x,y]\), effectively substituting \(x^{-1}\) for \(x\), gives that \(C \otimes_{\bar{R}} R_*[x^{-1}, y]\) is acyclic. The remaining cases of (I.1.2.1b) are dealt with by similar re-indexing.

By Lemma III.4.3, the complex \(C \otimes_R R_*[y, y^{-1}]\) is acyclic. Replacing \(R\) by the strongly \(\mathbb{Z}^2\)-graded ring \(\bar{R} \equiv R[(-x,y)]\) gives that \(C \otimes_{\bar{R}} R_*[y, y^{-1}]\) is acyclic. The other cases of (I.1.2.1a) are dealt with by using \(\bar{R}(x,y) = R(y,x)\) and \(\bar{R}(x,y) = R(-y,x)\), respectively.

\[\square\]

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