The Longest Queue Drop Policy for Shared-Memory Switches is 1.5-competitive

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Abstract

We consider the Longest Queue Drop memory management policy in shared-memory switches consisting of \( N \) output ports. The shared memory of size \( M \geq N \) may have an arbitrary number of input ports. Each packet may be admitted by any incoming port, but must be destined to a specific output port and each output port may be used by only one queue.

The Longest Queue Drop policy is a natural online strategy used in directing the packet flow in buffering problems. According to this policy and assuming unit packet values and cost of transmission, every incoming packet is accepted, whereas if the shared memory becomes full, one or more packets belonging to the longest queue are preempted, in order to make space for the newly arrived packets.

It was proved in 2001 [Hahne et al., SPAA '01] that the Longest Queue Drop policy is 2-competitive and at least \( \sqrt{2} \)-competitive. It remained an open question whether a \( (2 - \epsilon) \) upper bound for the competitive ratio of this policy could be shown, for any positive constant \( \epsilon \). We show that the Longest Queue Drop online policy is 1.5-competitive.

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1 Introduction

Memory management policies constitute a large area of research for online algorithms. In the case of shared memory switches, memory may completely fill up, therefore an online policy should accept packets trying to maximize the total number of served packets, towards the total number of served packets of an optimal offline policy. The online policy has no knowledge of the future packet arrivals, contrary to the offline strategy which has the advantage of knowing the whole sequence of incoming packet flow in advance. Ensuring Quality of Service (QoS) in scheduling the packet traffic of buffering problems, is a task an online algorithm should aim at [7, 3, 11]. Two cases of online policies have been analyzed extensively: the preemptive policies where the rejection of packets already accepted is possible and the nonpreemptive policies where accepted packets cannot be rejected later. We consider packets of unit values and cost of transmission, organized in FIFO queues in case they are admitted.

The idea of preempting packets from the longest queue when the shared memory becomes full, was proposed by Wei et al. [15]. According to this policy, every packet is accepted as long as the buffer is not full. If it is full and the packet waiting for admission is destined to the longest queue currently in the buffer, this packet is rejected, otherwise a packet from the longest queue is preempted and the packet waiting for admission is accepted. Since all packets are considered uniform in values and cost of transmission, we may assume that a packet from the longest queue is preempted in case an incoming packet is destined to the same queue and the latter packet is accepted. When we have two or more queues of lengths equal to the maximum one in the buffer, one of them is chosen arbitrarily for a packet rejection, for each accepted packet. This preemptive online policy is called Longest Queue Drop.

Time is assumed to be discrete. For every timestep, the packet positioned in the head of every active queue is sent to an output link and afterwards, but before the next timestep begins, the policy decides which packets to accept between those placed in the input ports. An input port may accept any number of packets at the same timestep and the packets not accepted by the buffer are rejected, clearing up the input ports for the next timestep. When we refer to the buffer content at some timestep, we will be taking under account the packet admissions from the incoming packet flow of the same timestep.

The throughput is defined as the number of served packets per timestep and the total transmission as the total number of served packets, from the first until the last timestep that at least one packet exists in the buffer. When we mention ratios of throughputs or total transmissions, we refer to the respective ratios of the optimal offline policy (OPT) to that of the Longest Queue Drop policy (LQD).

Former research on shared-memory switches [1], restricts them having an equal number of input and output ports. But, this buffer model is equivalent to the one having an arbitrary number of input ports, since we assume that there is no limit on the size of the input port queues. Also, since \( M \geq N \), we may work on the case \( M = N \), leaving vacant the output ports that we do not intent to use.

We will be using competitive analysis [3, 5]. It has been shown in [10, 1] that LQD is 2-competitive, where a lower bound of \( \sqrt{2} \) is, also, obtained. In [10] it is, also, shown that no online policy can achieve a competitive ratio smaller than 4/3 for this problem. No \( (2 - \epsilon) \) upper bound (for some positive constant \( \epsilon \)) has been shown for the LQD competitive ratio, except for the special case of memory switches with two output ports, where an upper bound of \( (4M - 4)/(3M - 2) < 4/3 \) is given in [12]. In [12], an upper bound of \( 2 - o(1) \) is, also, shown for the general case of \( N \) output ports. We show that LQD is 1.5-competitive, closing significantly the gap with the lower bound of \( \sqrt{2} \).

2 Analysis

2.1 Definitions

We denote as \( p_i^{t,LQD} \) the length of queue \( i \) in the LQD buffer at timestep \( t \) and as \( p_i^{t,OPT} \) the length of the same queue in the OPT buffer at the same timestep, where \( i \in [N] \). We refer to an overflow as
the situation where the buffer filled up and at least one packet was rejected from it. We shall, also, call as overflowed the queues that a policy rejected packets from. Finally, we define as bursty flow the simultaneous arrival of at least \( M \) packets destined to the same queue, in the input ports.

We denote as \( d^t_i \) the difference \( (p^t_{i,OPT} - p^t_{i,LQD}) \). We may skip the subscript or superscript of \( d^t_i \) when these are understood by the context. The total number of timesteps from the first timestep that at least one packet exists in any buffer until the last one that at least one packet exists in any buffer is denoted \( T^{tot} > 0 \). We call free the queues having a greater length in the LQD buffer than that in OPT at the same timestep. As dominating we shall call the queues with a greater or equal length in the OPT buffer at timestep \( t \) than that in LQD, where \( p^t_{i,LQD} > 0 \) and \( d^t_{i-1} \leq d^t_i \) (\( i \) refers to a dominating queue, here). If \( d^t_{i-1} > d^t_i \), we will refer to the \( i^{th} \) queue as semi-dominating and everything else stays the same as in the case of dominating queues. We gave these names to the last two sets of queues, since OPT dominates LQD in packet quantity, emphasizing on whether the difference of quantities between them has a decreasing tendency or not. Note that for a queue retaining packets in any of the two buffers at timestep \( t \) but not at timestep \( t - 1 \), we assume that \( d^{t-1} = 0 \). Finally, if \( p^t_{i,LQD} = 0 \) and \( p^t_{i,OPT} > 0 \), we call this \( i^{th} \) queue established at timestep \( t \). No queue may belong to two or more sets between the four sets we have defined, at the same timestep. We will denote as \( F^t \), \( D^t \), \( S^t \) and as \( E^t \) the numbers of free, dominating, semi-dominating and established queues at timestep \( t \), respectively. Under these assumptions, LQD’s competitive ratio is upper-bounded by the maximum over all possible incoming flows of \( R^{t^{tot}} = \sum_{t=1}^{t^{tot}} (F^t + E^t + D^t + S^t) / \sum_{t=1}^{t^{tot}} (F^t + D^t + S^t) \).

Before proceeding, we should mention that we may use the same variable names for different queues or timesteps in the analysis to follow. However, we will always refer, explicitly, to their current meaning. Also, in some cases we will be adding the name of the sequence of incoming flow as a subscript to the length of a queue or to the number of queues of a specific type, as defined before, when this may not be fully understood by the context.

2.2 An optimal offline policy

It would be convenient to express the queues in mathematical formulae, using the first letter of their names, as in the following inequality. For any timestep \( t \) of a LQD overflow we have:

\[
\sum_{i \in D \cup S} (d^t_i) + \sum_{i \in OPT} (p^t_{i,OPT}) \leq \sum_{i \in F} (|d^t_i|) \tag{1}
\]

since OPT dedicates more packets to specific queues than LQD does, letting LQD admit more packets in at least one other queue. Recall that both buffers have the same size \( M \). In case the OPT buffer is full after the LQD buffer overflow, too, the equality holds in (1).

A LQD packet holding a position in its queue that is not held by an OPT packet at the same timestep will be called a free packet and it is always located in free queues. The right part of (1) refers to the free packets at timestep \( t \). A LQD packet that is not free will be called common and a free packet may become common before its transmission, or the opposite. Following \([1]\), we call extra packet, a packet that is transmitted by OPT when its corresponding queue is inactive in the LQD buffer. A potential extra packet is a packet occupying a higher queue position in the OPT buffer than the same queue’s length in the LQD buffer, at the same timestep. The potential extra packets may be held by all queues in the OPT buffer except for the free queues and upper-bounding their number by a percentage of the number of transmitted LQD packets, may give us an upper bound for the LQD competitive ratio, since the total number of extra packets is upper-bounded by the total number of potential extra packets, when considering all timesteps in \([1, T^{tot}]\).

\footnote{We assume that each strategy uses its own buffer and, for the analysis to follow, we compare the two buffer contents at the same timestep.}
At a timestep \( t \) of an overflow of one or more dominating or semi-dominating queues, it would be natural to define a threshold line in the two-dimensional combined buffer snapshot of \( LQD \) and \( OPT \), indicating that no \( LQD \) packet may be located at a queue position above it, at the same timestep. This line defines the length of each of the overflowed queues at any overflow timestep in the \( LQD \) buffer. The only exception to this, is when we have more than one queues of equal lengths and \( LQD \) chooses arbitrarily some of them for the preemptions. This will cause one or more overflowed queues to surpass the threshold line by one packet, at the overflow timestep. In terms of defining the competitive ratio, we will assume that dominating or semi-dominating queues are always picked first to drop packets from at the same timestep, before the first free queue is chosen, in order to maximize the number of potential extra packets available in the \( OPT \) buffer and consequently, the additional number of packets that \( OPT \) may potentially send, compared to \( LQD \). Wrapping up, we will allow free queues to surpass the current threshold line by one packet, in case a dominating queue overflows. An illustration of a two-dimensional combined buffer snapshot along with a threshold line is given in Figure 3 (see Appendix), though the specific figure illustrates further attributes, showed later on.

**Lemma 1.** There exists an optimal offline strategy \( q_1 \) that will never choose to have an empty queue at the same timestep this queue is not empty in the \( LQD \) buffer.

**Proof.** (see Appendix) \( \square \)

The following corollary is derived by Lemma 1 and (1):

**Corollary 1.** Inequality (1) holds for any timestep \( t \).

Also, the next corollary comes directly from Lemma 1:

**Corollary 2.** There will always exist at least one common packet in every active free queue in the \( LQD \) buffer.

Next lemma shows that there exists an optimal offline strategy that will never choose to have semi-dominating queues in its buffer:

**Lemma 2.** Assume that at some timestep \( t \) we have \( p_{i,OPT}^t > p_{i,LQD}^t > 0 \), for some queue \( i \). Then, there exists an optimal offline strategy \( q_2 \) which will never choose to have \( d_i^t > d_i^t' \), for any subsequent timestep \( t' > t \), until queue \( i \) becomes established for the first time after \( t \).

**Proof.** Assume that the lemma does not hold for an optimal offline policy and \( d_i^t > d_i^t' \) for a timestep \( t' \) after \( t \), whereas the \( i \)th queue does not become established in \([t,t']\). But then, there exists another offline policy \((q_2)\) which can dedicate a smaller packet quantity to queue \( i \) at timestep \( t \) and gain more buffer space for, possibly, admitting at least one packet destined to some other of its queues, until the point it holds that \( d_i^t \neq d_i^t' \). Policy \( q_2 \) does not lack in total throughput compared to the optimal offline policy, since their buffer contents become identical at \( t' \) and after. Therefore, \( q_2 \) may be regarded as \( OPT \). \( \square \)

Due to Lemma 2, the competitive ratio of the \( LQD \) policy is upper-bounded by the maximum of

\[
R^{T_{\text{tot}}} = \sum_{t=1}^{T_{\text{tot}}}(F^t + E^t + D^t) / \sum_{t=1}^{T_{\text{tot}}}(F^t + D^t), \quad \text{over all sequences of incoming packet flow.}
\]

In case we refer to a specific incoming packet sequence \( \sigma \), then we may define \( r_{\sigma} = \sum_{t=1}^{T_{\text{tot}}}(F^t_{\sigma} + E^t_{\sigma} + D^t_{\sigma}) / \sum_{t=1}^{T_{\text{tot}}}(F^t_{\sigma} + D^t_{\sigma}) \) restricting all queue numbers per timestep, to the ones occurring for \( \sigma \). According to that, we get:

\[
R^{T_{\text{tot}}} = \max_{\sigma}\{r_{\sigma}\} \quad (2)
\]
Note that we may have $T^{tot} \to \infty$ for a sequence of incoming packet flow. In such a case, an upper bound of the competitive ratio is equal to $\lim_{T^{tot} \to \infty}(\sup R^{tot})$.

Next lemma shows that the properties of the optimal offline policies $q_1$ and $q_2$ do not conflict.

**Lemma 3.** There exists an optimal offline strategy $q_o$ encapsulating the properties of both $q_1$ and $q_2$.

**Proof.** (see Appendix)

We will use $q_o$ as our optimal offline strategy from now on. Apart from that, we define as compact, a time period $[t_x, t_y]$ ($t_x \leq t_y$) for a queue, if there exists a presence of at least one of its packets in both buffers at any timestep of it, but not at $t_x - 1$ and $t_y + 1$.

**Definition 1.** We call a dominating queue $i$ immediate at a specific compact period when either 1) the queue attains a positive value for $d_i$ at the first timestep of that compact period and there exists no more incoming packet flow (not necessarily accepted) for this queue until the end of this compact period, or 2) $d_i = 0$ during the whole compact period.

**Definition 2.** We call a dominating queue $j$ urgent for a specific compact period, when either 1) there exists no further incoming flow for this queue (not necessarily accepted) after the first timestep it attains its maximum $d_j > 0$ at that compact period and until the last timestep of this compact period or 2) $d_j = 0$ during the whole compact period.

Note that an immediate queue is always urgent and that when we refer to the maximum $d_i$ or $d_j$ as in the above definitions, we refer to their maximum values attained at specific compact periods.

**Lemma 4.** Let a sequence of incoming packet flow giving rise to urgent dominating queues only, while two compact periods of two different urgent queues $a$ and $b$ overlap for at least one timestep. Assuming that $a$ attains its maximum $d_a > 0$, for the first time in its compact period, before $b$ attains its maximum $d_b > 0$ for the first time in its compact period, then the compact period of $a$ cannot end after the compact period of $b$ ends.

**Proof.** (see Appendix)

Before proceeding we should briefly give a sketch of our analysis to follow. According to Corollary 1 and since we want to maximize the number of potential extra packets per timestep, we must maximize the number of free packets located in the $LQD$ buffer at the same timestep. In order to achieve that, the $OPT$ buffer must keep as few as possible packets of free queues in its buffer, while the $LQD$ buffer should accept as many packets for them as possible, when dominating queue overflows take place. A reasonable thought would be to constantly provide bursty incoming flows to the free queues, making $OPT$ accept only one packet for each free queue per timestep and derive an upper bound for the ratio of total transmissions for this sequence of incoming flow.

**Lemma 5.** Consider a sequence of incoming flow $\kappa$, giving rise to at least one non-urgent dominating queue. Then, there exists another sequence of incoming flow $\kappa_1$, where $r_{\kappa_1} \geq r_\kappa$, which gives rise to urgent dominating queues only.

**Proof.** Assume that for the non-urgent dominating queue $l$, the maximum $d_l > 0$ is attained at timestep $t_m$, the current compact period ends at timestep $t_f > t_m$ and there exists incoming packet flow for this queue -since it is non-urgent- for at least one timestep in $[t_m + 1, t_f]$. Since, we referred to the maximum $d_l$ for this compact period, both $LQD$ and $OPT$ should accept an equal number of packets at any timestep of $[t_m + 1, t_f]$, regarding the non-urgent $l$. We are going to design a new sequence of incoming flow $\kappa'$, where the packet flow ceases for $l$ in $[t_m + 1, t_f]$, introducing currently inactive urgent dominating queues in both buffers.
Figure 1: On the upper part we have two consecutive timesteps for the non-urgent queue $l$. According to Lemma 5, we may cease any further incoming flow to $l$, introducing a new urgent queue $n$ (below). The buffer snapshots below refer to the incoming sequence $\kappa_1$ and correspond to the same timesteps, as for the snapshots above each of them (for $\kappa$).

Regarding $\kappa'$, the incoming packet flow for $l$ ceases in $[t_m + 1, t_f]$ making $l$ urgent for the current compact period, while for the first timestep in it (if such a timestep exists, we call it $t_u \leq t_f$) that another $LQD$ queue overflows (for $\kappa$), an incoming packet flow is introduced for an urgent dominating queue $n$, currently inactive in both buffers. The number of packets arrived at $t_u$ for the new queue $n$ is set to be $(p_{t, LQD, \kappa}^u - p_{t, LQD, \kappa'}^u)$.

The $LQD$ buffer will accept all packets for $n$ at $t_u$, since the sum of the lengths of $n$ and $l$ (for $\kappa'$) is equal to the length of $l$ (for $\kappa$) at the same timestep; therefore there exists available space for all the $n$th packets in its buffer. Also, $LQD$’s queue lengths in $[t_u, t_f]$, except for $n$ and $l$, do not change (for $\kappa'$) compared to the respective $LQD$ queue lengths for $\kappa$, since for any $t \in [t_u, t_f]$ it holds that:

$$p_{l, LQD, \kappa'}^t + p_{n, LQD, \kappa'}^t = p_{l, LQD, \kappa}^t$$

(3)

In case we have more $LQD$ overflows of other queues until $t_f$, $\kappa'$ keeps feeding $n$ with packet flow at the same timesteps that the overflows take place for $\kappa$, keeping (3) valid at any timestep $t$ until $t_f$. Note that there is no possibility for $n$ to overflow, because of (3) and since we assumed that $l$ already attained its maximum $d_l$ for this compact period, when considering $\kappa$; therefore $l$ overflowed for the last time in the current compact period.

Queue $l$ is active in the $LQD$ buffer (regarding $\kappa'$) for $(p_{l, LQD, \kappa'}^m - 1)$ timesteps after $t_m$, while it was active for $(t_f - t_m)$ timesteps, for $\kappa$, after $t_m$ again. The urgent queue $n$ cannot be active (for $\kappa'$) for more than $(t_f - t_m) - (p_{l, LQD, \kappa'}^m - 1)$ timesteps, since we subtract from the total number of timesteps of this compact period after $t_m$, the maximum number of $LQD$ packets that $l$ and $n$ together may have (due to (3)). Hence, $D_{\kappa'}^t$ is decreased by 1 compared to $D_{\kappa}^t$, for a number of timesteps not less than the number of timesteps it increases by 1. Apart from that, we have $E_{\kappa'}^t = E_{\kappa}^t$ for any $t \in [t_u, t_f]$. Therefore, we obtain $r_{\kappa'} \geq r_{\kappa}$.

Starting from $\kappa'$ now, we continue applying this procedure until no non-urgent dominating queues are left. We call the last attained incoming sequence $\kappa_1$ and we have $r_{\kappa_1} \geq r_{\kappa'} \geq r_{\kappa} \Rightarrow r_{\kappa_1} \geq r_{\kappa}$.

Note that we will always have available inactive queues to use as above, since in order for $OPT$ to retain more packets in dominating queues than $LQD$ by at least one packet per queue, we must have
Figure 2: According to Lemma 6, a non-immediate urgent queue (which we can see above in 4 different timesteps, assuming a sorting in chronological order) can be substituted by an immediate queue at the timestep \( t_e \) the former queue attains its maximum \( d \) for the first time in this compact period. Note that in the leftmost timestep (above), the queue is free and becomes dominating later on.

at most \( M/2 \) dominating queues per timestep (recall that we assumed \( M = N \)). This leaves us with at least \( M/2 \) inactive queues per timestep.

**Lemma 6.** Consider a sequence of incoming flow \( \sigma \), giving rise to urgent dominating queues only, where at least one of them is non-immediate. Then, there exists another sequence of incoming flow \( \sigma_1 \) which gives rise to immediate dominating queues only and it holds that \( r_{\sigma_1} \geq r_{\sigma} \).

**Proof.** We arrange the dominating queues according to the timesteps they become established queues. Regarding the urgent dominating queues that do not become established, keeping \( d = 0 \) for their whole compact period, we provide an alternative incoming flow enough to keep their queue lengths as they already are for \( \sigma \) in both buffers, but no more than that. Therefore, in case an overflow takes place for these queues in both buffers, we subtract their additional incoming flow that is being rejected. Every packet flow to any other queue stays as it is for \( \sigma \). Obviously, all queue lengths for the new incoming flow stay the same at any timestep. We call the new incoming flow \( \sigma' \) and it holds \( r_{\sigma'} = r_{\sigma} \).

Regarding the rest of the urgent dominating queues, we note that any of them may appear more than once in the arrangement we fixed, since it may be defined in more than one compact periods. We pick the first dominating queue that becomes established and we call it \( j \). We denote as \( t_b \) the timestep the incoming flow is triggered for \( j \) at the current compact period, and \( d_j \) is assumed to take its maximum value for the first time in this compact period at a timestep denoted as \( t_e \).

Assuming \( j \) is non-immediate and \( d_j^{t_e} \neq 0 \), we provide a new incoming sequence for which there is no packet flow for queue \( j \) in \([t_b, t_e - 1]\), while at \( t_e \) we have the arrival of a number of packets for \( j \), equal to \( p_{j,OPT}^{t_e} \). No more incoming flow for this queue shall exist at this compact period. The \( OPT \) buffer will accept all of the packets for \( j \) at \( t_e \), since there exists available space for them in its buffer, as it existed for \( \sigma' \). The \( LQD \) buffer will accept an amount of packets at least equal to \( p_{j,LQD}^{t_e,\sigma'} \), because the lengths of the rest \( LQD \) queues at \( t_e \) cannot be smaller than the respective queue lengths for \( \sigma' \), at the same timestep \( t_e \). We, also, know that this is the maximum possible amount of packets...
the \( LQD \) buffer can accept for \( j \) at \( t_e \), since at timestep \( t_e \) we have a \( LQD \) overflow, for \( \sigma' \) (because \( d_j \) gets its maximum value at this timestep). So, the length of \( j \) in the \( LQD \) buffer for the new sequence will be equal to \( p_{j,LQD,\sigma'}^{t_e} \) and this queue will overflow at \( t_e \). Because of that, all other \( LQD \) queue lengths will adjust to their respective values for the new sequence of incoming flow at \( t_e \), as they were for \( \sigma \) at the same timestep. The new sequence has one less occurrence of a non-immediate dominating queue; it is the \( j^{th} \) queue. The new \( LQD \) buffer contents may change in \([t_b, t_e - 1]\), due to the absence of packets for queue \( j \), making other queues accept more packets, compared to \( \sigma' \). However, since all queue lengths adjust to their values at \( t_e \) and for any subsequent timestep take their respective values, as for \( \sigma' \), the new ratio of total transmissions is not smaller than \( r_{\sigma'} \).

Assuming \( j \) is immediate and \( d_j^b \neq 0 \), we provide a new incoming flow for it, existing of \( p_{j,OPT,\sigma'}^{t_e} \) \( j^{th} \) packets at timestep \( t_e \). No more incoming flow for this queue shall exist at this compact period.

All of these packets will be accepted by \( OPT \), while \( LQD \) will accept the number of packets it would have accepted for \( \sigma' \) at \( t_e \), for the same reason we discussed in the case of non-immediate queues.

We apply the same procedure, iteratively, on the new sequence of incoming flow we obtain, until no non-immediate queues have been left in the buffer. We call the final alternative incoming flow \( \sigma_1 \). It holds that \( r_{\sigma_1} \geq r_{\sigma'} = r_\sigma \), since \( D_{\sigma_1}^t \) is decreased equally, both in the numerator and the denominator of \( r_{\sigma_1} \) compared to \( D_{\sigma'}^t \), for one or more timesteps \( t \).

Without Corollary 3 we would not be able to define the exact amount of new incoming flow or the timestep that the new incoming flow would have taken place, for every unsettled queue in the arrangement.

In terms of defining an upper bound of the \( LQD \) competitive ratio and due to Lemmas 5 and 6, we shall focus only on those sequences of incoming flow for which every dominating queue is immediate. We shall call these sequences of incoming flow as \textit{fine} sequences. Due to the definition of immediate dominating queues, we obtain the following corollary.

\textbf{Corollary 3.} No free queue may transform into a dominating queue during the same compact period, for any fine sequence.

\textbf{Lemma 7.} The free queues that are active at a timestep \( t_e \), for any fine sequence of incoming flow, will be active for any subsequent timestep of \( t_e \), too.

\textit{Proof.} (see Appendix)

From Lemma 7 and Corollary 2, we have that for every fine sequence it holds \( T^{tot} \to \infty \).

\textbf{Lemma 8.} Assume we have a fine sequence \( \phi_1 \) of incoming flow that, either 1) there exists one free queue of non-unit length in the \( OPT \) buffer at any timestep, or 2) at a timestep a dominating queue overflows, at least one free queue length in the \( LQD \) buffer is strictly smaller than the length of the overflowed dominating queue in the same buffer. Then, there exists a fine sequence \( \phi_2 \) for which neither of these two conditions hold and \( r_{\phi_1} \leq r_{\phi_2} \).

\textit{Proof.} We will describe \( \phi_2 \): it provides a bursty flow to all active free queues at any timestep, instead of the flow that \( \phi_1 \) provides for the same queues. Certainly, all free queue lengths in the \( OPT \) buffer stay unit at any timestep, while all free queues overflow in the \( LQD \) buffer, at any timestep. The incoming flow to the dominating queues stays the same as for \( \phi_1 \). Therefore, we get \( \sum_{i \in F_{\phi_1}} |d_i^t| \leq \sum_{i \in F_{\phi_2}} |d_i^t| \) for any timestep \( t \) and due to Corollary 2 we are able to obtain \( \sum_{i \in D_{\phi_1}} (d_i^t) + \sum_{i \in F_{\phi_1}} (p_{i,OPT}^t) \leq \sum_{i \in D_{\phi_2}} (d_i^t) + \sum_{i \in F_{\phi_2}} (p_{i,OPT}^t) \).
\[ \sum_{i \in D_{\phi_1}}(d_i^t) + \sum_{i \in E_{\phi_1}}(p_i^{t,OPT}). \] Note that we deduce the last inequality, since each bursty flow overflows the OPT buffer, too, making (1) hold as an equality.

Hence, for each dominating queue overflow timestep, this queue’s length in the LQD buffer cannot be greater for \( \phi_2 \) than its respective value for \( \phi_1 \) and the same queue’s length in the OPT buffer cannot be smaller for \( \phi_2 \) than its respective value for \( \phi_1 \). According to that, we have \( E_{\phi_1}^t \leq E_{\phi_2}^t \), \( D_{\phi_1}^t \geq D_{\phi_2}^t \) and \( F_{\phi_1}^t \leq F_{\phi_2}^t \), for any \( t \). It follows that \( r_{\phi_1} \leq r_{\phi_2} \).

Note that we may have a different sequence of incoming flow \( \phi_2' \), than the described \( \phi_2 \), for which the two conditions of the lemma do not hold (as an example, consider a sequence providing bursty flows to the free queues only when the dominating queues overflow). However, we cannot have \( r_{\phi_2'} > r_{\phi_2} \) at any case, since obviously \( F_{\phi_2'}^t = F_{\phi_2}^t \) for any timestep \( t \) and since by keeping every free queue overflowing at any timestep a dominating queue overflows (in order for the second condition to not hold), we will have \( D_{\phi_2'}^t = D_{\phi_2}^t \) and \( E_{\phi_2'}^t = E_{\phi_2}^t \), at any \( t \) again. \( \square \)

**Definition 3.** An ideal fine sequence of incoming packet flow is a fine sequence providing bursty flow to each of its free queues, at any timestep.

Providing an upper bound for the ratio of total transmissions for any ideal fine sequence, gives automatically an upper bound for the LQD competitive ratio, due to Lemmas 5, 6, 8 and since an upper bound of the competitive ratio is equal to \( \lim_{T \to \infty} (\sup R^{T, \text{tot}}) \), as described before. For the analysis to follow, we shall restrict further our attention on ideal fine sequences, only.

Finally, since \( M = N \), we easily derive the next corollary.

**Corollary 4.** Every established queue was dominating at the last timestep of its previous compact period.

### 2.3 The packet connections – in brief

The way we are going to bound the competitive ratio is by matching the additional packets that OPT may send, with packets that LQD sends, resembling the way the upper bound of 2 is proved in [10].

We define as a **valid** connection, a matching of a potential extra packet with a LQD packet located at a strictly smaller queue position than it, not necessarily in the same queue. All matched LQD packets, will be, therefore, leaving the buffer earlier than their matched potential extra packets do, when they have been matched with valid connections. Note that in case LQD decides to drop a connected packet from its buffer, the packet that will take its position keeps this connection which stays valid, according to the LQD preemption policy.

We will show that, for every ideal fine sequence of incoming flow, we can validly connect every potential extra packet with two different LQD packets, each of which will be connected with at most one potential extra packet. Since the number of extra packets is upper-bounded by the number of potential extra packets, the additional number of packets that OPT may send cannot be more than half the packets that LQD sends, for any ideal fine sequence. Therefore, we obtain an upper bound on the total transmission ratio of any ideal fine sequence, equal to 1.5.

Each connection is assigned at the last overflow timestep of any dominating queue \( i \), for its current compact period, since by Lemma 2, \( d_i \) is non-decreasing during each compact period.

We separate the dominating queues into two categories, regarding the ideal fine sequences. A primary dominating queue \( i \) is a dominating queue which at the timestep \( t_1 \) of its last overflow for each of its compact periods, it holds that \( d_{i}^{t_1} \leq p_i^{t_1,LQD} \). Any dominating queue that is not primary at the timestep of its last LQD overflow, will be called **secondary**. The potential extra packets of any secondary queue \( i' \) that are located more than \( 2p_i^{t_1,LQD} \) positions in their queues away from the output port at the queue’s last overflow timestep \( t_1 \), will be called **pending** potential extra packets.
Our connection assignment algorithm is trivial and it is described in Table 1, where $t_2$ refers to the timestep each dominating queue becomes established, due to Corollary 4. The detailed justification on the assignment procedure for the primary and secondary queues will follow. Figure 3 (see Appendix) gives an illustration of the connection assignment procedure for a secondary dominating queue.

Table 1: The connection assignment algorithm between the potential extra packets and $LQD$ packets

| Queue  | First Connection                                                                 | Second Connection                                                                 |
|--------|---------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|
| Primary| Assigned at timestep $t_1$ with common packets located at the same dominating queue. The order we pick packets for connections is arbitrary. | Assigned at timestep $t_1$ with free packets. We start with the highest potential extra packets connecting them with the highest free packets that are located below the threshold line and move gradually to lower queue positions. |
| Secondary| (i) Assigned at timestep $t_1$ for the non-pending potential extra packets with common packets located at the same dominating queue (order does not matter) and (ii) assigned at timestep $t_1$ for the pending packets with free packets, connecting the highest pending packets with the highest free packets and moving gradually to lower queue positions. | (i) Assigned at timestep $t_1$ for the non-pending potential extra packets with free packets and (ii) assigned at timestep $t_2$ for the pending packets with free packets. In both cases we pick the highest potential extra packets connecting them with the highest unmatched free packets that give valid connections and gradually move to lower queue positions. |

In case an established queue becomes dominating before it becomes inactive in the $OPT$ buffer, we will have applied connections to potential extra packets that will not become extra packets, but this overestimates, only, the total transmission ratio upper bound of any ideal fine sequence.

2.4 The double connection of the potential extra packets for the primary dominating queues

The first connection of any potential extra packet belonging to a primary queue is assigned at the timestep $t_1$ that the queue overflows for the last time and for each of its compact periods. The potential extra packet is connected with a free packet. We assign the connections, starting from the highest free packets and potential extra packets (breaking ties arbitrarily between the potential extra and free packets respectively that occupy equal queue positions), connecting them together and gradually reach packets located at lower queue positions. The number of free packets suffices for valid connections, due to Corollary 1, unless we have arbitrary queue drops, which we must examine separately.

So, in case we have arbitrary drops and, therefore, at most one free packet per free queue is located above the threshold line at $t_1$, we may assign connections with (unmatched) common packets, if needed. Recall that the maximum number of free packets located above the threshold line is one packet per free queue and therefore it is at most the number of common packets located in free queues at that timestep, due to Corollary 2. We assign the connections as before, starting from the highest free packets located below the threshold line and continue as described before. Even if all common packets located in free queues are already matched before $t_1$, the free packets located above the threshold line are not needed for connections, since we will need $F_{t_1}$ less free packets for connections. According to this connection procedure, for every connected free packet there will exist at least one $LQD$ packet located at some dominating queue, at an equal queue position. The same applies to the case of secondary queues.

The second connection is assigned, again at $t_1$, with a common $LQD$ packet belonging to the same primary dominating queue. Obviously, such a connection is valid and the number of common $LQD$ packets suffices. The order we assign connections is arbitrary.
There is no possibility a LQD packet located in a dominating queue to be an accepted packet in a position of a formerly connected LQD packet that was preempted from a free queue in the previous timestep, since for every connected free packet there will exist at least one LQD packet located at some dominating queue, at an equal queue position, as stated before. But in such a case, the LQD packet belonging to the dominating queue should have been preempted first, since we assumed that LQD packets of dominating queues are always chosen first for preemptions, in case of arbitrary drops. That would increase \( d \) for this dominating queue and this is not allowed for the dominating queues which we have, already, applied connections, since we assign connections at each dominating queue’s last overflow timestep. The same applies to the case of secondary dominating queues.

2.5 The double connection of the potential extra packets for the secondary dominating queues

In the case of secondary dominating queues, we do not have the required number of common packets belonging to the same queue, to apply both connections at \( t_1 \). The pending packets are matched only once at this timestep. We will have to wait until timestep \( t_2 \), in order to apply the second connections to the pending potential extra packets, when all previously matched LQD packets with potential extra packets of this queue will have left the LQD buffer, in order to avoid duplicated connections.

Since we refer to ideal fine sequences, all LQD free queue lengths cannot differ for more than one packet at any timestep. Therefore, whether the smallest free queue length at \( t_2 \) in the LQD buffer has decreased or not compared to the smallest free queue length at \( t_1 \) in the same buffer, choosing the highest potential extra packets and connecting them with the highest free packets that give valid connections at \( t_2 \), while gradually moving to lower queue positions and applying connections to all pending packets, completes the assignment of the second connection to every pending packet. Note that we know we can assign valid connections at \( t_2 \) to all pending packets, due to Lemma 7.

**Corollary 5.** Every pending packet can be validly connected twice with formerly unmatched LQD packets, when we refer to ideal fine sequences. The first connection takes place at its secondary queue’s last overflow timestep for its current compact period and the second when this queue becomes established.

This completes the proof for the double connection of any potential extra packet belonging to a secondary dominating queue, regarding the ideal fine sequences of incoming packet flow.

2.6 An upper bound of the LQD competitive ratio

We showed that every potential extra packet can be connected with two unique LQD packets, for any ideal fine sequence. Hence, the number of packets that OPT may additionally send is at most half the number of packets that LQD sends for any such sequence. We, also, showed that for every sequence of incoming flow that is not fine, there exists a fine sequence attaining at least the same ratio of total transmissions and that for every fine sequence that is not ideal, there exists an ideal fine sequence attaining at least the same ratio of total transmissions. Therefore, the ratio of total transmissions for any sequence of incoming flow is upper-bounded by 1.5 and LQD is 1.5-competitive, due to (2).

3 Concluding Remarks

We proved that the Longest Queue Drop policy for shared-memory switches is 1.5-competitive, closing significantly the gap with the lower bound of \( \sqrt{2} \). Regarding future research, the application of randomization in the queues that are chosen for packet rejections would be very interesting, as well as the use of non-uniform packet transmission cost and values.
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A Omitted Proofs

A.1 Proof of Lemma 1

Proof. We need to show that for every optimal offline policy for which the lemma does not hold, if such one policy exists, there exists another offline policy, which we call \( q_1 \), for which the lemma holds and this latter policy has at least the total transmission of the former one, for any possible sequence of packet arrivals \( \sigma \).

Suppose that, without loss of generality, \( LQD \) accepts a set of packets \( S_2 \) belonging to the same queue \( i \) at timestep \( t \), while an optimal offline policy \( q_3 \) chooses to accept a set of packets \( S_1 \) of the same queue at the same timestep (where \( |S_2| > |S_1| \geq 0 \)) which will cause the optimal buffer to become empty of \( i^{th} \) packets earlier than \( LQD \)’s buffer does. The \( i^{th} \) queue may be already inactive in both buffers at timestep \( t \). Note that we may have more than \( |S_2| \) incoming flow for this queue at timestep \( t \), but we need only to concentrate on the amount of packets that \( LQD \) accepts. Also, there may exist later incoming flow for this queue, before it becomes inactive in the \( OPT \) buffer, assuming that \( q_3 \) accepts at least the amount of packets that \( LQD \) accepts for it, for every timestep of a later incoming flow.

Strategy \( q_3 \) may admit \( |S_2| - |S_1| \) different packets in position of the packets that \( LQD \) decided to accept for this specific queue and we know that the queues the \( |S_2| - |S_1| \) packets may belong to, already appear in the \( LQD \) buffer, otherwise \( LQD \) would have accepted them, too. However, since we assumed that \( LQD \) accepted \( S_2 \), the length of the \( i^{th} \) queue in the \( LQD \) buffer at timestep \( t \) is not greater than each of the lengths of its queues that the \( |S_2| - |S_1| \) packets belong to, due to the \( LQD \) policy. Hence, \( q_3 \) cannot transmit, compared to \( LQD \), more than \( (|S_2| - |S_1|) \) packets after the timestep the \( i^{th} \) queue becomes inactive in the \( LQD \) buffer. Until this timestep, \( q_3 \) will be lacking in throughput of one packet per timestep, after this queue becomes inactive in its buffer. So, \( q_3 \) cannot gain something more in total transmission than the amount of total transmission it will have lost until this timestep.

Finally, in case we have two or more queues staying inactive in the \( q_3 \)’s buffer but not in the \( LQD \) one at one or more timesteps, the result stays the same by applying repetitively the aforementioned procedure.

Therefore, offline strategy \( q_1 \) does not lack in total transmission, compared to the optimal offline strategy \( q_3 \).

\( \Box \)

A.2 Proof of Lemma 3

Proof. Assume strategy \( q_1 \) cannot have a non-empty queue (which we shall call \( q \)) at some timestep \( t \) when the specific queue is active in the \( LQD \) buffer, since otherwise it should be \( d_{z}^{t'} > d_{z}^{t} \) for at least one of its dominating queues, which we call \( z \), at some timestep \( t' < t \).

If \( q \) becomes empty at timestep \( t \) in the optimal buffer while it is active in the \( LQD \) buffer, then we know from Corollary 1 that at some timestep \( t'' \) for which \( OPT \) accepted less packets than \( LQD \) did for \( q \) \((t'' \leq t' < t)\), \( OPT \) admitted more packets than \( LQD \) did for \( z \). But, \( OPT \) could have chosen to accept a smaller amount of packets for \( z \), still having \( d_{z}^{t''} \geq d_{z}^{t''-1} \). Hence, \( OPT \) could have dedicated this gained amount of packets for increasing its packet quantity in \( q \) and decreasing \( |d_{q}^{t''}| \).

By applying this procedure for -if necessary- more than one dominating queues at \( t'' \) or more timesteps before \( t' \), we can obtain an equal quantity of packets for \( q \) in the two buffers at timestep \( t \) or after but, certainly, before \( q \) becomes inactive in the \( OPT \) buffer. Therefore, there exists an optimal offline strategy retaining the properties of both \( q_1 \) and \( q_2 \) and we shall call it \( q_o \).

\( \Box \)
Figure 3: A secondary dominating queue at the timestep $t_1$ of its last overflow, on the left. We have an ideal fine sequence, keeping each free queue length unit in the $OPT$ buffer (there are two free queues on the left and three on the right). The pending potential extra packets (in dark) can be assigned only one connection and the rest potential extra packets (in light colour) two connections, with free packets. At timestep $t_2$, on the right, when all previously matched $LQD$ packets with potential extra packets of this queue have left the buffer (dashed-dotted arrows), we apply the second connections to the pending packets. If we had a primary queue, both connections would be assigned at $t_1$. The dotted horizontal lines represent the current threshold lines and we assume that no dominating queue overflows on the right, causing only the free queues to overflow, since we have an ideal fine sequence.

A.3 Proof of Lemma 4

Proof. We assume that the urgent dominating queue $a$ attains its maximum $d_a$ for the first time at $t_o$ and the urgent dominating queue $b$ attains its maximum $d_b$ for the first time at $t_p > t_o$. Since there exists no more incoming packet flow for $a$ after $t_o$ and $d_a$ is maximized at this timestep (therefore no more preemptions take place for this queue in the $LQD$ buffer and for this compact period), it holds that $p_{a,LQD}^t \leq p_{b,LQD}^t$ for any timestep $t$ until $a$ becomes established. This completes the proof.

A.4 Proof of Lemma 7

Proof. In order for a free queue to become inactive in the $LQD$ buffer at some timestep $t_c > t_v$, its length in the $OPT$ buffer, due to Corollary 2, should become at least equal to that in the $LQD$ buffer, at a timestep before $t_c$. But then, a free queue becomes dominating during the same compact period and this cannot happen, due to Corollary 3.