ORIENTABLE RIGID CUSP TYPES COVERED BY HYPERBOLIC KNOT COMPLEMENTS

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Abstract. This paper completes a classification of the types of orientable cusps that can arise in the quotients of hyperbolic knot complements. In particular, $S^2(2, 4, 4)$ can not be the cusp cross-section of any orbifold quotient of a hyperbolic knot complement. Furthermore, if a knot complement covers an orbifold with a $S^2(2, 3, 6)$ cusp, it also covers an orbifold with a $S^2(3, 3, 3)$ cusp. We end with a discussion of non-orientable cusps, which is also informed by the main theorems.

1. Introduction

There are five orientable Euclidean 2-orbifolds:

$$T^2, S^2(2, 2, 2), S^2(2, 3, 6), S^2(3, 3, 3) \text{ and } S^2(2, 4, 4).$$

The figure 8 knot complement covers orbifolds with four of the five types of cusps (all but $S^2(2, 4, 4)$). In keeping with the standard terminology, we say that $S^2(2, 3, 6), S^2(3, 3, 3)$ and $S^2(2, 4, 4)$ are the cross-sections of rigid cusps because they have a unique geometric structure up to Euclidean similarity. We also say $T^2$ and $S^2(2, 2, 2)$ correspond to non-rigid cusps, which up to similarity have a two-dimensional (real) parameter space of possible cusp shapes. The aforementioned quotients of the figure 8 knot complement to orbifolds with $S^2(2, 3, 6)$ and $S^2(3, 3, 3)$ cusps can be easily constructed by analyzing the symmetries of the underlying space of the complement, two regular ideal tetrahedra (see [12] for example). Similarly, there are knot complements that decompose into regular ideal dodecahedra [1] and each of these knot complements also admits a quotient to an orbifold with a $S^2(2, 3, 6)$ cusp and to an orbifold with a $S^2(3, 3, 3)$ cusp (see [11, §9], and [9] for more background). Collectively, these three examples are the hyperbolic knot complements known to cover orbifolds with rigid cusps. Again, for each of the three examples, we can find a quotient with a with $T^2, S^2(2, 2, 2), S^2(2, 3, 6)$ or $S^2(3, 3, 3)$ cusp. Curiously missing from this list is a hyperbolic knot complement which covers an orbifold with a $S^2(2, 4, 4)$ cusp. The main theorem shows that such a cover cannot occur.

Theorem 1.1. Let $Q$ be an orbifold covered by a hyperbolic knot complement, then $Q$ does not have a $S^2(2, 4, 4)$ cusp.

This theorem is also relevant to the larger question of which knot complements admit hidden symmetries (see Section 2). It is conjectured by Neumann and Reid that only the figure 8 knot complement and the dodecahedral knot complements exhibit this property (see [4, Conjecture 1.1] for example). In that context, we can appeal to a similar argument to the proof of the main theorem and show that if a knot complement covers an orbifold with a $S^2(2, 3, 6)$ then it also covers an
Figure 1. The structure of quotients of a manifold covered by a knot complement that admits hidden symmetries in light of Theorem 1.2 and Theorem 1.1. Here $Q_3$ has an $S^2(3,3,3)$ cusp and $Q_T$ has a torus cusp. We point out that $n_i \geq 1$ and the degree of the cover of $Q_3$ to $Q$ could be 1 or 2.

orbifold with a $S^2(3,3,3)$ cusp. Of course, we see this phenomenon in the examples discussed above.

**Theorem 1.2.** Let $f : M \to Q$, where $M$ is a manifold covered by a hyperbolic knot complement. If $Q$ has a $(2,3,6)$ cusp, then $(\pi_{1}^{orb}(Q))^{ab} \cong \mathbb{Z}/2\mathbb{Z}$, $\text{deg}(f) = 24n$, $n \geq 1$, and $M$ covers the double cover of $Q$, which has a $S^2(3,3,3)$ cusp.

Theorem 1.2 is sharp in the sense that there is a 24-fold cover by the figure 8 knot complement to the minimum volume orientable orbifold in its commensurability class $\mathbb{H}^3/PGL(2,\mathbb{Q}_3)$. However, for each of the dodecahedral knot complements the analogous cover is degree 120.

Finally, the main results of this paper provide evidence for the Rigid Cusp Conjecture of Boileau, Boyer, Cebanu, and Walsh [4, Conjecture 1.3].

**Conjecture 1.3** (Rigid Cusp Conjecture). If a hyperbolic knot complement $S^3 \setminus K$ covers an orbifold with a rigid cusp, $S^3 \setminus K$ covers an orbifold with a $S^2(2,3,6)$ cusp.

Theorem 1.2 reduces the conjecture to answering the following question, which would imply a partial converse of Theorem 1.2.

**Question 1.4.** If $S^3 \setminus K$ covers an orbifold with a $S^2(3,3,3)$ cusp, must it also cover an orbifold with a $S^2(2,3,6)$ cusp?

We can also capture one other property about rigid cusped orbifolds that are covered by knot complements. Namely, they are unique in the following sense:

**Theorem 1.5.** If $S^3 \setminus K$ is hyperbolic, it admits at most one orbifold quotient with an $S^2(3,3,3)$ cusp and at most one quotient with a $S^2(2,3,6)$ cusp.

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2. BACKGROUND

We refer the reader to [14, Chapter 13] for more information on orbifolds and we will appeal to Thurston’s notation conventions for 2-orbifolds established in that reference. Namely for 2−orbifolds, we say \( F(a_1, \ldots a_m; b_1, \ldots b_n) \) is an orbifold with underlying space \( F \), cone points of orders \( a_i \) and corner reflectors marked by \( b_j \). A sufficiently small neighborhood of a cone point is isometric to \( D^2/(\mathbb{Z}/a_i\mathbb{Z}) \) and a sufficiently small neighborhood of a corner reflector is isometric to \( D^2/(D_{2b_j}) \) where \( D_{2b_j} \) is a dihedral group of order \( 2b_j \). The lone departure from these conventions is explained in Figure 4 and Remark 3.2, which concerns simple closed curves fixed by reflections.

A hyperbolic manifold \( M \) admits hidden symmetries if there exist a cover of \( M \), \( M \) such that \( \tilde{M} \) admits a symmetry is not the lift of a deck transformation of \( M \). For a (non-arithmetic) hyperbolic knot complement \( S^3 \setminus K \), Neumann and Reid showed that admitting hidden symmetries is equivalent to covering a rigid cusped orbifold (see [11, §9]).

2.1. The structure of the orbifold. An orientable, (finite volume) hyperbolic 3-orbifold \( O \) can be described as the quotient \( \mathbb{H}^3/\Gamma \) (with \( \text{vol}(\mathbb{H}^3/\Gamma) < \infty \)). If \( O \) is non-compact, then \( \tilde{O} \) has cusps of the form \( T^2 \times [0, \infty), S^2(2, 2, 2) \times [0, \infty), S^2(2, 4, 4) \times [0, \infty) \), \( S^2(2, 3, 6) \times [0, \infty) \) or \( S^2(3, 3, 3) \times [0, \infty) \).

The singular set of \( O \) denoted by \( \Sigma(O) \) is the set of points in the quotient \( O \cong \mathbb{H}^3/\Gamma \) with non-trivial point stabilizers in \( \Gamma \). The underlying space of \( O \) is the 3-manifold determined by ignoring labels on \( \Sigma(O) \). It is well established that \( \Sigma(O) \) is an embedded trivalent graph in \( |O| \) (see for example [5, 7]).

2.2. Elements of \( \pi_1^{orb}(S^2(2, 3, 6)) \). As this paper is relatively short and self-contained, we can be very explicit about our cusp groups and their action by isometries on the Euclidean plane. We start out by introducing relevant details of a \( S^2(2, 3, 6) \) cusp group.

Assume \( \pi_1^{orb}(S^2(2, 3, 6)) = \langle a, b | a^6, b^3, (ab)^2 \rangle \). We can find a discrete faithful representation of this group into \( \text{Isom}(E^2) \) using

\[
a \mapsto \begin{pmatrix} \cos(\pi/3) & \sin(\pi/3) \\ -\sin(\pi/3) & \cos(\pi/3) \end{pmatrix}
\]

and

\[
b \mapsto \begin{pmatrix} \cos(2\pi/3) & \sin(2\pi/3) \\ -\sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}
\]

Notice the maximal abelian subgroup (aka the subgroup of translations) in this group is generated by: \( t_1 = ba^{-2} \mapsto \left( x + \frac{1}{2}, y + \frac{\sqrt{3}}{2} \right) \), \( t_2 = b^{-1}a^2 \mapsto \left( x + 1, y \right) \).

2.3. Elements of \( \pi_1^{orb}(S^2(2, 4, 4)) \). The introduction listed the three hyperbolic knot complements known to cover orbifolds with \( S^2(2, 3, 6) \) cusps. Before we exhibit the obstruction to a knot complement covering an orbifold with \( S^2(2, 4, 4) \) cusp, we give background on this cusp type as well.

Analogously to the argument above, we will begin with a discrete faithful representation of the fundamental group of \( S^2(2, 4, 4) \) into \( \text{Isom}(E^2) \). Of course, it
is straight-forward to embed this group into $\text{Stab}(\infty) \subset \text{PSL}(2, \mathbb{C})$ in order to realize it as the peripheral subgroup subgroup of a knot complement (see [10] for example).

Assume $\pi_1^{orb}(S^2(2, 4, 4)) = \langle c, d, c^4, d^2, (cd)^4 \rangle$. We observe that if we map

$$c \mapsto \begin{pmatrix} \cos(\pi/4) & \sin(\pi/4) \\ -\sin(\pi/4) & \cos(\pi/4) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$d \mapsto \begin{pmatrix} \cos(\pi/2) & \sin(\pi/2) \\ -\sin(\pi/2) & \cos(\pi/2) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

we have our desired discrete faithful representation of this group into $\text{Isom}(E^2)$.

Notice the maximal abelian subgroup (i.e. the subgroup of translations) in this group is generated by: $t_1 = dc^{-2} \mapsto \begin{pmatrix} x \\ y + 1 \end{pmatrix}$, $t_2 = d^{-1}c^2 \mapsto \begin{pmatrix} x + 1 \\ y \end{pmatrix}$.

### 2.4. Improvements to degree bounds and a structure theorem

We establish a useful improvement to the covering degree bounds of [10, Lemma 5.5]. We point out that this result is a structure theorem in the sense that it implies that knot complements which cover $S^2(2, 3, 6)$ cusped orbifolds also cover $S^2(3, 3, 3)$ cusped orbifolds. For comparison, the Rigid Cusp Conjecture of Boileau, Boyer, Cebanu, Walsh [4, Conjecture 1.3] postulates that every hyperbolic knot complement admitting hidden symmetries covers an orbifold with a $S^3(2, 3, 6)$ cusp.

**Proof of Theorem 1.2.** Let $Q$ be a quotient of a hyperbolic knot complement $S^3 \setminus K$. Assume that $Q$ has a $S^2(2, 3, 6)$ cusp. Further assume that $S^3 \setminus K$ covers $M$ and denote by $\Gamma_K = \pi_1(S^3 \setminus K) \subset \pi_1^{orb}(Q) \subset \text{PSL}(2, \mathbb{C})$. We assume that $\Gamma_K$ and $\pi_1^{orb}(Q)$ are identified with a discrete faithful representation of $\pi_1(S^3 \setminus K)$.

Since $Q$ and $S^3 \setminus K$ have both one cusp, we can observe that $\pi_1^{orb}(Q) = P_6 \cdot \Gamma_K$ where $P_6$ is the peripheral subgroup of $Q$.

We can further assume that $\Gamma_K = \langle \mu_1, ..., \mu_n | r_1, ..., r_m \rangle$, where $\mu_i$ are meridians of $K$ and each $r_j$ is a relation of $\Gamma_K$ (e.g. coming from a Wirtinger presentation). Finally, assume $P_6 = \langle a, b | a^6, b^3, (ab)^2 \rangle$ and that $\mu_1 \in P_6$.

Thus, we appeal to the structures of $P_6$ and $\Gamma_K$ to define a left coset action on $\Gamma_K$ as a subgroup of $\pi_1^{orb}(Q)$. From this coset action, we can deduce properties of presentation for $\pi_1^{orb}(Q)$. Specifically, we have

$$\pi_1^{orb}(Q) = \langle a, b, \mu_1, ..., \mu_n | a^6, b^3, (ab)^2, r_1, ..., r_m, a\mu_j a^{-1}\gamma_{a,j}^{-1}, b\mu_j b^{-1}\gamma_{b,j}^{-1} \rangle$$

such that each $\gamma_{a,j}^{-1}$, $\gamma_{b,j}^{-1}$ is a parabolic in $\pi_1^{orb}(Q)$. Here we point out that since $\pi_1^{orb}(Q) = P_6 \cdot \Gamma_K$, these relations together with $a^6$, $b^3$, $(ab)^2$ and $r_1, ..., r_m$ form a complete set of relations for $\pi_1^{orb}(Q)$.

Notice since $\Gamma_K$ has one orbit of parabolic fixed points (which is the same set of parabolic fixed points as $\pi_1^{orb}(Q)$), $\gamma_{a,j}$ and $\gamma_{b,j}$ can be expressed as $wt_1^2t_2^2w^{-1}$ where $w \in \Gamma_K$ and $t_1 = ba_2^{-2}$, $t_2 = b^{-1}a_2^2$ (see Section 2.2 for more details). Thus, in the quotient obtained by modding out by the normal closure of all parabolic elements of $P_6\Gamma_K$ and $b\Gamma_K$ becomes trivial, and $P_6 \rightarrow \mathbb{Z}/2\mathbb{Z}$. Hence, $\pi_1^{orb}(Q)^{ab} \rightarrow \mathbb{Z}/2\mathbb{Z}$. In light of this observation and [10, Lemma 4.1.1]), we find that $\pi_1^{orb}(Q)^{ab} \cong \mathbb{Z}/2\mathbb{Z}$ and $Q$ has a 2-fold cover by an orbifold $Q_3$ with a $S^3(3, 3, 3)$ cusped. $Q_3$ is covered by $M$ since $\pi_1(Q_3)$ and $\pi_1(Q)$ have the same set of parabolic elements. By
[10, Lemma 5.5], the degree of the cover from \( M \) to \( Q_3 \) is \( 12n \) \((n \geq 1)\). Thus, the covering degree of \( M \to Q \) is \( 24n \) \((n \geq 1)\).

We now state a stronger version of [10, Theorem 1.2], which follows from a careful reading of the proof at that theorem with the fact that the relevant cover to a \( S^2(2,3,6) \) cusp orbifold is at least 24.

**Corollary 2.1.** Let \( M \) be a manifold covered by a small non-arithmetic hyperbolic knot complement \( S^3 \setminus K \) that admits two exceptional surgeries. Then \( M \) does not cover a rigid cusp orbifold.

### 2.5. Rhombic peripheral groups.

Given a rigid cusp orbifold group, we can look at subgroups of the translation group. We say a translation group is *rhombic* if it can be generated by two translations of the same length. For example, each rigid cusp orbifold group has a maximal abelian subgroup that is rhombic and that abelian subgroup in turn has a subgroup of index \((n_1^2 + 3n_2^2)\) or \((n_1^2 + n_2^2)\) which is determined by the square of the norm of algebraic integer in \( \mathbb{Z}[\frac{1}{2} + \frac{3}{\sqrt{2}}] \) or \( \mathbb{Z}[i] \). These abelian groups are rhombic and index \( 6(n_1^2 + 3n_2^2) \), \( 4(n_1^2 + n_2^2) \), or \( 3(n_1^2 + 3n_2^2) \) in the rigid cusp orbifold group. More generally, we say a cusp hyperbolic orbifold \( Q \) has a *rhombic cusp* if the peripheral subgroup associated to this cusp is generated by two translations of the same length.

Although we might expect to see an orbifold with a rhombic cusp in the commensurability class of a knot complement which covers a rigid cusped orbifold, which seems to be out of convenience, not necessity. We now state a stronger version of [10, Theorem 1.2], which follows from a careful reading of the proof at that theorem with the fact that the relevant cover to a \( S^2(2,3,6) \) cusp orbifold is at least 24.

**Lemma 2.2.** Let \( S^3 \setminus K \) be a knot compliment that covers a rigid cusped orbifold \( Q \).

1. If \( Q \) has 4-torsion on its cusp, then any orbifold \( Q_T \) with a torus cusp such that \( \Sigma(Q_T) \) is two embedded circles that is also covered by \( S^3 \setminus K \) does not have a rhombic cusp.
2. If \( Q \) has 3-torsion on its cusp, then any orbifold \( Q_T \) with a torus cusp covered by \( S^3 \setminus K \) does not have a rhombic cusp.

A key component of this proof result follows from [3, Proposition 5.8]. However, a number of the statements in that paper sometimes exclude the case that \( S^3 \setminus K \) covers a rigid cusped orbifold, which seems to be out of convenience, not necessity. We include a more or less self-contained proof below because it still remains relatively brief. We also use the notation that \( \langle \langle a_1, \ldots, a_k \rangle \rangle_G \) is the normal closure of the set \( \{a_j\} \) in \( G \).

**Proof.** The claims here can be observed directly for the figure 8 knot complement. For the remaining cases, we appeal to [13] so that we can assume that \( S^3 \setminus K \) is non-arithmetic. We then assume \( \text{Comm}^+(\pi_1(S^3 \setminus K)) \) is discrete and for simplicity we assume \( Q \cong \mathbb{H}^3/\text{Comm}^+(\pi_1(S^3 \setminus K)) \) [11, Proposition 9.1]. Assume that \( Q_T \) is covered by \( S^3 \setminus K \) and covers \( Q \) and has a rhombic cusp. Fix a subgroup \( \Gamma_K \) such that \( \Gamma_K \cong \pi_1(S^3 \setminus K) \), \( \Gamma_K \subset \pi_1^{\text{orb}}(Q_T) \subset \text{Comm}^+(\Gamma_K) \), and \( \text{Comm}^+(\Gamma_K) \) is identified with its image under a discrete faithful representation into \( PSL(2, \mathbb{C}) \). After conjugation, we may assume that \( \mathbb{H}^3/\text{Comm}^+(\Gamma_K) \) has a cusp at \( \infty \). Let \( \mu \)
be a meridian of $\Gamma_K$ fixing $\infty$ and let $r$ be a rotation of order 3 or 4 on the cusp. As a group element, $\mu, r\mu r^{-1}, r^2\mu r^{-2}$ are in $\Gamma_T = \pi_1^T(Q_T)$.

**Case 1:** If $r$ is order 4, then $r^2\mu r^{-2} = \mu^{-1}$. However, in this case, $\mu$ and $\mu' = r\mu r^{-1}$ correspond to meridians of knot groups $\Gamma_K$ and $r\Gamma_K r^{-1}$. Both of these knot groups are normal subgroups of $\Gamma_T$ by [13, Lemma 4] (see also the proof of [2, Lemma A.3]) and both quotients are cyclic. In fact, we can say $\Gamma_T/\Gamma_K \cong \Gamma_T/(r\Gamma_K r^{-1}) \cong Z/nZ$ for some fixed $n$. Furthermore, as $\Gamma_K$ and $r\Gamma_K r^{-1}$ are their own $\Gamma_T$-normal closures, $\langle \langle \mu \rangle \rangle_{\Gamma_T} = \Gamma_K$ and $\langle \langle \mu' \rangle \rangle_{\Gamma_T} = \Gamma_{K'}$. Since $\mu$ and $\mu'$ correspond to peripheral elements of $\Gamma_T$, we have that $\Gamma_T/\langle \langle \mu \rangle \rangle_{\Gamma_T} \cong \Gamma_T/\langle \langle \mu' \rangle \rangle_{\Gamma_T} \cong Z/nZ$. Thus, by the Geometrization of cyclic orbifolds [6], both of these fillings correspond to orbilens space fillings of $Q_T$. Furthermore, by assumption $\Sigma(Q_T)$ consists of two embedded circles, and so $Q_T \setminus \Sigma(Q_T)$ is a knot complement in $S^1 \times S^1 \times I$ that admits a non-trivial cosmetic surgery, which contradicts [3, Lemma 5.3].

**Case 2:** If $r$ is order 3 then $\mu, \mu' = r\mu r^{-1}, \mu'' = r^2\mu r^{-2}$ are 3 distinct elements of $\Gamma_T$. In fact, each one is a normal generator for one of the three isomorphic knot groups in $\Gamma_T$, $\Gamma_K$, $\Gamma_K' = r\Gamma_K r^{-1}$, and $\Gamma_K'' = r^2\Gamma_K r^{-2}$. Aping the argument from above, we now have $\Gamma_T/\Gamma_K \cong \Gamma_T/\Gamma_K' \cong \Gamma_T/\Gamma_K'' \cong Z/nZ$ for some fixed $n$. However when viewed as translations in the rigid cusp group (the translations for $S^2(2, 3, 6)$ and $S^2(3, 3, 3)$ cusp groups are described in Section 2.2), one of $\mu, r\mu r^{-1}, r^2\mu r^{-2}$ must be either be a homological sum of the other two. Depending on orientation, two of these having the homological quotient (here $Z/nZ$) would mean the third is either homologically trivial in $\Pi_T(Q_T) = \Gamma_T^{ab}$ or has a quotient of $Z/2nZ$. Again as above $\langle \langle \mu \rangle \rangle_{\Gamma_T} = \Gamma_K, \langle \langle \mu' \rangle \rangle_{\Gamma_T} = \Gamma_{K'},$ and $\langle \langle \mu'' \rangle \rangle_{\Gamma_T} = \Gamma_{K''},$ and so one of the quotients of $\Gamma_T$ should have infinite abelianization, contradicting that all are $Z/nZ$.

**Remark 2.3.** There are manifolds and orbifolds with rhombic cusps that both cover a rigid cusped orbifold with 3-torsion on the cusp and admit finite cyclic fillings. The Figure 8 sister manifold ‘m003’ is an example of such a manifold. Therefore, the property that $\langle \langle \mu \rangle \rangle_{\Gamma_T} = \Gamma_K$ is essential to the previous argument.

For the discussion below, we will consider an orbifold $Q$ that is covered by a knot complement and has an $S^2(2, 4, 4)$ cusp. In this case $|Q|$ is simply connected and in fact an open ball by [10, Proposition 2.3], equivalently [3, Corollary 4.11]. We will be interested in the cover $f : S^3 \setminus K \rightarrow Q$ and especially properties of the cover intrinsic to the cusp. In that case, we can denote by $Q \cong Q \setminus S^2(2, 4, 4) \times [1, \infty)$ and consider the restriction of $f : S^3 \setminus n(K) \rightarrow Q$. This is a cover of the knot exterior to the (closure of the interior) of the orbifold.

In this case, $Q$ has underlying space a closed ball and $\Sigma(Q)$ is a properly embedded trivalent graph. The vertices of this graph are fixed by non-cyclic finite subgroups of $SO(3, \mathbb{R})$. We can label a vertex of this graph by the (maximal) finite subgroup of $SO(3, \mathbb{R})$ fixing that point. Each edge of this graph is fixed by an elliptic element of $\Gamma$ and so we can label an edge using a finite cyclic group or more simply the order of the (maximal) finite cyclic group that fixes that edge. For example, $\partial(Q)$ is incident to an edge labeled 2 and two edges labeled 4.

**Proof of Theorem 1.1.** Assume that $Q$ has a $S^2(2, 4, 4)$ cusp and is covered by a knot complement $S^3 \setminus K$, then similar to the arguments used to prove Theorem 1.2
we can say $\pi_1^{orb}(Q) = P_4 \cdot \Gamma_K$, where $P_4 \cong \pi_1^{orb}(S^3(2, 4, 4)) = \langle c, d| c^4, d^2, (cd)^4 \rangle$ and $\Gamma_K \cong \pi_1(S^3 \setminus K)$.

Again just as above, we can assume that $\Gamma_K = \langle \mu_1, ..., \mu_n|r_1, ..., r_m \rangle$ and then build the following presentation

$$\pi_1^{orb}(Q) = \langle c, d, \mu_1, ..., \mu_n|r_1, ..., r_m, c^4, d^2, (cd)^4, c\mu_j c^{-1}\gamma_{c,j}^{-1}d\mu_j d^{-1}\gamma_{d,j}^{-1} \rangle$$

where (similar to the $S^3(2,3,6)$ case each $\gamma_{c,j}^{-1}$, $\gamma_{d,j}^{-1}$ is a parabolic in $\pi_1^{orb}(Q)$. The previous observation about the structure of these parabolics can be lightly adapted from that argument. Here, we still have that since $\Gamma_K$ has one orbit of parabolic fixed points (which is the same set of parabolic fixed points as $\pi_1^{orb}(Q)$), and so $\gamma_{c,j}$ and $\gamma_{d,j}$ can be expressed as $wt_1^1t_2^2w^{-1}$, where $w \in \Gamma_K$ and $t_1 = dc^{-2}$, $t_2 = d^{-1}c^2$.

There is therefore a map $h : \pi_1^{orb}(Q) \to \mathbb{Z}/2\mathbb{Z}$ given by introducing the relations that $d, c^2$ are trivial. Under $h$, $\Gamma_K$ is trivial as are all elements corresponding to peripheral 2 torsion as well as all parabolic elements. Therefore any knot complement which covers $Q$ also covers this two fold cover $Q_2$. However, $Q_2$ has no 4-torsion on the cusp. In fact, it has a $S^3(2,2,2)$ cusp. Thus, the cover $S^3 \setminus K$ is regular by [13, Lemma 4] (see also [2, Lemma A.3]).

This also shows we have the set of covering maps exhibited in Figure 2: $f_1 : S^3 \setminus K \to Q_T$, $f_2 : Q_T \to Q_2$, and $f_3 : Q_2 \to Q$. Note the degree of either $f_2$ or $f_3$ is 2.

Here, $Q_2$ is an orbifold with underlying space a ball (see [10, Proposition 2.3] and [3, Corollary 4.11]). The latter reference also shows that $Q_2$ has two-fold cover $Q_T$ which is a knot complement in an orbilens space and that $Q_T$ is cyclically covered by $S^3 \setminus K$. Thus the there are 0, 2 or 4 points fixed by non-cyclic isotropy groups of $Q_2$ which are all dihedral since $Q_2$ is a dihedral quotient of $S^3 \setminus K$. We note that just as in the proof of [10, Lemma 4.3], the 4-torsion on the cusp is either connected via a loop back to the cusp (no internal vertices), $D_4$ or $S_4$. However, $S_4$ does not have a dihedral subgroup of order 12, so if $S_4$ is an isotropy group connected to peripheral 4-torsion, then this would contradict the property that $Q_2$ is a dihedral quotient of $S^3 \setminus K$. If either set of points fixed by peripheral 4-torsion terminates at a point fixed by a $D_4$ group (a dihedral group of order 8), then both such sets terminate at points fixed by $D_4$. Moreover, both $D_4$ have $D_2$ (the dihedral group of 2)
order 2) subgroups which lift to \(\pi_1^{orb}(Q_2)\). The peripheral 2-torsion must connect to a dihedral group of order 2\((p)\) where \(p \in \mathbb{Z}\) in order to lift to a dihedral group in \(\pi_1^{orb}(Q_2)\). By cusp killing this group must be a (2, 2, \(p = 2k + 1\)) isotropy group. We observe that two conjugates of this group lifts faithfully to \(\pi_1^{orb}(Q_2)\). However, this implies that there are no other non-cyclic isotropy groups in \(\pi_1^{orb}(Q)\), i.e. the isotropy graph is a trivalent graph with 3 internal vertices. Up to homotopy of this graph, there is a unique way to connect up this graph, which corresponds to \(\pi_1^{orb}(Q)\) being non-trivial under the cusp killing map.

Therefore, the 4-torsion of \(\pi_1^{orb}(Q)\) is not part of any non-cyclic isotropy groups. In this case, we say the 4-torsion forms a loop.

After lifting to the 2-fold cover, the isotropy graph for \(Q_2\) is homotopic to what is pictured in Figure 3.

Here we can see from cusp killing that both \(m_1\) and \(m_2\) are odd and strictly bigger than 1. These edges lift to two embedded circles in the singular set \(\Sigma(Q_T)\), which by construction must be rhombic. We conclude the proof by appealing to Lemma 2.2.

There is one knot complement \(S^3 \setminus 12n706\) which is known to have cusp field \(Q(i)\) (see [8, §7.1]). This knot complement has become known as Boyd’s knot complement because Boyd showed it decomposes into regular ideal tetrahedra and regular ideal octahedra. In fact, that polyhedral decomposition is the canonical cell decomposition. A simple analysis of this decomposition shows it does not support 4-torsion on the cusp. Of course, Theorem 1.1 gives an alternate proof of this fact as well.

2.6. Uniqueness. We now show up to cusp type there is at most one rigid cusped quotient of a hyperbolic knot complement.

Proof of Theorem 1.5. We will assume that \(f_1 : S^3 \setminus K \to Q'\) and \(f_2 : Q' \to Q\). By Theorem 1.1 and Theorem 1.2, it suffices to assume \(Q'\) and \(Q\) both have \(S^2(3, 3, 3)\) cusps, and so proof reduces to showing that \(f_2\) is a trivial cover.

Let \(P'\) be the peripheral subgroup of \(\pi_1^{orb}(Q')\) and \(P\) be the peripheral subgroup of \(\pi_1^{orb}(Q)\). Just as above we assume that \(\Gamma_K = \pi_1(S^3 \setminus K), \pi_1^{orb}(Q') = P' \cdot \Gamma_K\) and \(\pi_1^{orb}(Q) = P \cdot \Gamma_K\).

If \(T'\) is the maximal abelian subgroup of \(P'\), then for any \(g \in P \cdot \Gamma_K\), \(gT'g^{-1} \subset\) \(P' \cdot \Gamma_K\) since \(g\) conjugates of parabolics all have the same length as measured against a horoball packing. Thus, \(\Gamma = \langle \langle T' \rangle \rangle_p \cdot \Gamma_K \subset P' \cdot \Gamma_K\). If \(\mathbb{H}^3/\Gamma\) has a torus cusp, it would be rhombic, contracting Lemma 2.2, so \(\mathbb{H}^3/\Gamma\) has a rigid cusp, which must be a \(S^2(3, 3, 3)\) cusp. By index considerations, \(\Gamma = P' \cdot \Gamma_K\). Also, observe that this implies \(P' \cdot \Gamma_K \leq P \cdot \Gamma_K\). Since the degree of \(f_2\) is determined by \([P : P']\), we complete the proof by observing that \(P' \leq P\) if and only if \([P : P'] = 1\). □
Thus, the orbifold quotients of the figure 8 knot complement and the dodecahedral knot complements are prototypical.

3. Non-orientable cusp types.

As noted in the introduction, it is well-established that four of the possible seventeen Euclidean 2-orbifolds show up as cusp types of orbifolds covered by knot complements. While Theorem 1.1 rules out $S^2(2, 4, 4)$ cusps, it also rules out $D^2(4; 2)$, $D^2(; 2, 4, 4)$ cusps as well since any orbifold with such a cusp type would have an orientable double cover with a $S^2(2, 4, 4)$ cusp.

The figure 8 knot complement and the dodecahedral knot complements both cover tetrahedral orbifolds. The non-orientable tetrahedral orbifolds covered by these knot complements have either $D^2(; 3, 3, 3)$ or $D^2(; 2, 3, 6)$ cusps. The figure 8 knot complement also covers the Gieseking manifold which has a Klein bottle cusp and the figure 8 knot complement modulo its full symmetry group has a $RP^2(2, 2)$ cusp.

For the six remaining cusp types (see Figure 4), each one has a reflection which would extend to a reflection in the commensurator. However, the orbifolds in the figure labelled a)-e) have a reflection, which would extend to a reflection symmetry of the knot as established by the following:

**Lemma 3.1.** Assume $S^3 \setminus K$ covers a hyperbolic orbifold $Q$ such that the cusp cross section of $Q$ admits a reflection. If all elements of the peripheral subgroup of $Q$ are order 2 or have infinite order, then $\Gamma_K = \pi_1(S^3 \setminus K) / \Gamma_Q = \pi_1^{orb}(Q)$.

**Proof.** The assumptions imply that the cusp cross-section of $Q$ is appears as one of the five orbifolds listed in Figure 4 a) - e).
If the cusp cross section of $Q$ is $D^2(2,2,2,2)$, $D^2(2;2,2)$, or $D^2(2;2;R)$, then denote by $Q$ orbifold $Q_2$ with a $S^2(2,2,2)$ cusp that corresponds to the orientation double cover. Since all parabolic elements of $\Gamma_Q$ are in $\Gamma_{Q_2} = \pi_1^{orb}(Q_2)$, $\Gamma_K \subset \Gamma_{Q_2}$. Furthermore, we can decompose $\Gamma_Q$ into cosets $\Gamma_{Q_2}, r\Gamma_{Q_2}$, where $r$ is a (fixed) reflection of the cusp cross-section $C$ of $Q$. Separately, we can denote by $Q_T$ the index 2 subgroup of $\Gamma_{Q_2}$ corresponding to a torus cusped orbifold $Q_T$ covered by $S^3 \setminus K$. Denote by $\mu$ a meridian of $\Gamma_K$ which shares a fixed point with $r$. Since $\Gamma_K$ is a characteristic subgroup of $\Gamma_T$, $r\Gamma_K r^{-1} = \Gamma_K$. Thus, the desired claim holds cusp cross-sections.

The proof for the cusp cross-sections $T_R$ and $K_R$ is analogous except $Q_T$ the index 2 subgroup corresponding to both the torus cusped cover and the orientation double cover.

\begin{remark}
We point out $T_R$ and $K_R$ in Figure 4 appear in the list of parabolic orbifolds in [14, Theorem 13.3.6] as annulus and Möbius band, respectively. Also, $D^2(2;2;R)$ seems to correspond to ‘(2;2;2)’. Assuming the last case contains a typo (the first ‘;’ should be a ‘ ,’), we have that for each case the reflection in boundary is implied.

If orbifolds with boundary were to be included the list in [14, Theorem 13.3.6], it would expand to include the seventeen quotients by wallpaper groups and all seven frieze group quotients, which would include the Möbius band and annulus, so departing from Thurston’s notation by using $T_R$ and $K_R$ is meant to avoid confusion.

We now provide a theorem which serves as a summary of the arguments of this paper.

\begin{theorem}
Let $S^3 \setminus K$ cover a orbifold $Q$. Then the cusp of $Q$ is either: orientable and one of $T^2$, $S^2(2,2,2\ell)$, $S^2(2,3,6)$, or $S^2(3,3,3\ell)$, or non-orientable and one of $K^2$, $RP^2(2,2)$, $D^2(;2,3,6)$, $D^2(;3,3,3\ell)$ or $D^2(3;3\ell)$.
\end{theorem}

\begin{proof}
First, Theorem 1.1 eliminates the orbifolds with 4-torsion on the cusp (i.e. $S^2(2,4,4)$, $D^2(4;2,4)$ and $D^2(;2,4,4)$). Then, combining Lemma 3.1 with Boileau, Boyer, Cebanu and Walsh’s observation that hyperbolic knot complements cannot admit a reflection symmetry [4, Proof of Lemma 2.1(1)], which eliminates $D^2(;2,2,2\ell)$, $D^2(2;2,2)$ or $D^2(2;2;R)$, $T_R$ or $K_R$.
\end{proof}

We point out that all but $D^2(3;3\ell)$ are observed as cusp cross-sections of orbifolds covered by hyperbolic knot complements. Although [4, Theorem 1.8] gives candidates for orbifolds with $D^2(3;3\ell)$ cusps that could be covered by knot complements, the authors of that paper conjecture no such knot complements exist.

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