Topological susceptibility and information theory

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In this note we address the question of the $\theta$ dependence in non abelian gauge theories from a pure quantum information point of view. The main result is that the topological susceptibility is the quantum Fisher information of the ground state and that the maximally efficient quantum estimator of $\theta$ can be identified with the physical axion. In this setup the low energy dynamics of the axion is fully determined by quantum estimation theory.

I. INTRODUCTION

A typical situation in quantum physics is to have observable probability distributions depending on some external free parameters that cannot be directly measured. In this case although the quantum state of the system, for instance the ground state, depends on this set of external parameters we don’t have in the algebra of self adjoint observables anyone representing them. For a given state, pure or mixed, the amount of information we have about the actual value of these external parameters is encoded in the corresponding quantum Fisher information. The simplest and more basic among these external parameters is time itself. In this case the quantum Fisher information defines a metric on the Hilbert space of states on the basis of which we can define the distinguishability between quantum states at different times. Another example of external parameter, on which we shall focus in this note, is the $\theta$ parameter in QCD.

In those cases where the external parameter is a gauge artifact i.e. a parameter whose value can be changed using the symmetries of the theory, the corresponding quantum Fisher information is zero. A radical example is time itself in quantum gravity where we can use general covariance to make time a gauge artifact. In this case the corresponding quantum Fisher information vanishes.

This situation changes in Cosmology where we include a physical clock defined, for instance, in terms of the inflaton dynamics (see for instance). Another related issue is the case of superselection charges. In case of a superselection charge we should expect a divergent quantum Fisher information that can be only regularized including non vanishing amplitudes between different superselection sectors i.e. violating the super selection rule itself. We will not touch this issue in the note.

What we will suggest here is the simplest possible way to associate with some external parameter a full fledged quantum field. This will be done using the most efficient quantum estimator of the corresponding parameter. This can be done only if we have a finite and non vanishing associated quantum Fisher function. In this way we shall suggest the following conjecture:

In quantum gravity any external parameter, that is not a gauge artifact, is associated with a finite and non vanishing quantum Fisher function.

In this note we will focus on the $\theta$ parameter of pure Yang Mills theory. The main result is that in this case the quantum Fisher function for the ground state associated with the $\theta$ parameter is the well known topological susceptibility. The associated axion is defined using the quantum estimator of $\theta$ that encodes in its definition the standard theorems defining the low energy dynamics of the axion. These results are well grounded. However this line of thought is unable to fix the axion decay constant.

In order to estimate the axion decay constant we will use a very qualitative approach where we map the quantum Fisher information defined in terms of the topological susceptibility of a GUT dynamics with the cosmological quantum Fisher function we expect in an inflationary setup with the axion playing the role of the inflaton. The logic underlying this view is that in quantum gravity not only any external physical parameter should be promoted to a dynamical field but also that such a field defines a cosmological clock with a finite associated cosmological quantum Fisher information in general completely determined by slow roll parameters.

II. TOPOLOGICAL SUSCEPTIBILITY

One of the main non perturbative characterization of the QCD vacuum is the gluon condensate defin-
ing the topological susceptibility \(16\). This condensate parametrizes the \(\theta\) dependence of the vacuum energy. For pure \(SU(N)\) Yang Mills theory the topological susceptibility is given by

\[
\frac{d^2E(\theta)}{d^2\theta}|_{\theta=0} = C
\]

with \(E(\theta)\) the vacuum energy density. In terms of the partition function

\[
Z(\theta) = \int dA e^{\frac{1}{4}d^4x L(A, \theta)}
\]

for \(L(A, \theta) = L(A) + \theta \frac{g^2}{16\pi^2N} F \wedge F\), the topological susceptibility can be formally written as

\[
C = \frac{1}{VT} \frac{d^2Z(\theta)}{d^2\theta}|_{\theta=0}
\]

that up to contact terms reduces to the correlator

\[
C = (\frac{g^2}{16\pi^2N})^2 \int d^4x (T(F \wedge F(0)F \wedge F(x))
\]

evaluated at \(\theta = 0\).

### III. A QUANTUM INFORMATION DESCRIPTION

Let us define for the standard \(\theta\) vacuum \(|\theta\rangle = \sum_n e^{in\theta}|n\rangle\) the density matrix

\[
\rho_\theta = |\theta\rangle \langle \theta|
\]

Let us now define for the density matrix \(\rho_0\) corresponding to the \(\theta = 0\) vacuum the modular Hamiltonian by

\[
\rho_0 = e^{-\mathcal{H}}
\]

The quantum distinguishability between different \(\theta\) vacua can be measured in terms of the relative entanglement entropy \(17\ [18]\)

\[
S(\rho_0|\rho_0) = - \text{tr} \rho_0 \ln \frac{\rho(\theta)}{\rho(0)}
\]

Thus for \(|\theta\rangle\) a pure state we get

\[
S(\rho_0|\rho_0) = \langle \theta| \mathcal{H} |\theta\rangle
\]

Defining the vacuum energy density as \(\langle \theta| \mathcal{H} |\theta\rangle = E(\theta)VT\) we get

\[
\frac{1}{VT} \frac{d^2S(\rho_0|\rho_0)}{d^2\theta}|_{\theta=0} = C
\]

This is the desired relation between the topological susceptibility \(C\) and the quantum Fisher information density \(F\) defined as

\[
F = \frac{1}{TV} \frac{d^2S(\rho_0|\rho_0)}{d^2\theta}|_{\theta=0}
\]

Hence we get the basic correspondence:

**Topological susceptibility ⇔ Quantum Fisher information for \(\theta\) parameter**

In particular this means that vanishing topological susceptibility implies quantum indistinguishability between different \(\theta\) vacua.

In order to make contact with the definition of the topological susceptibility as the correlator \(16\) we simply need to remember the representation of the quantum Fisher information in quantum estimation theory as

\[
F = \frac{1}{VT} \text{tr}(\rho_0 \hat{L}^2)
\]

for \(\hat{L}\) defined by

\[
\frac{d\rho(\theta)}{d\theta} = \frac{\hat{L}\rho(\theta) + \rho(\theta)\hat{L}}{2}
\]

From the definition of the \(\theta\) vacuum the operator \(\hat{L}\) is easy to derive using the temporal gauge \(A_0 = 0\)

\[
\hat{L} = \int d^3x \frac{g^2}{16\pi^2N} T \pi_i B_i
\]

for \(\pi_i = \partial_0 A_i\) and \(B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}\). This leads to

\[
\frac{1}{VT} \text{tr}(\rho_0 \hat{L}^2) = C
\]

**A. Vafa-Witten theorem**

The previous relation leads to a simple interpretation of the Vafa-Witten theorem \(19\) as the *The first law of entanglement*. Namely this law in our case means that

\[
\frac{\partial S(\rho_0|\rho_0)}{\partial \theta}|_{\theta=0} = 0
\]

implying that the \(\theta = 0\) vacuum has minimal energy and that parity violating condensates are zero.

### IV. AXION AS A QUANTUM ESTIMATOR

Let us think on \(\theta\) as a parameter on which the different physical correlators depend. The quantum Fisher information or equivalently as shown above the topological susceptibility of the vacuum encodes how much information we have to estimate the physical value of \(\theta\). As it is standard in estimation theory we can define a quantum estimator for this parameter. Around \(\theta = 0\) the quantum estimator is simply

\[
\hat{A}(x) = \frac{\hat{L}(x)}{F}
\]

As it is defined this operator has zero dimension. Let us define the associated axion as \(\phi \equiv f_\alpha \hat{A}\) for \(f_\alpha\) with units of mass.
Since this is a maximally efficient quantum estimator the corresponding two point function for the so defined axion field, at zero momentum, is given by

\[ \Delta^2(\phi) = \frac{f_a^2}{F^2} \langle \hat{L}_k \hat{L}_0 \rangle \big|_{k=0} = \frac{f_a^2}{F} \]  

(17)

giving an effective mass for \( \phi \)

\[ m^2 = \frac{F}{f_a^2} = \frac{C}{f_a^2} \]  

(18)

The very definition of the axion field \( \phi \) as the quantum estimator of \( \theta \) encodes the well known low energy theorems. In fact from [16] it follows

\[ \langle \phi | \hat{L} | 0 \rangle = \frac{F}{f_a} \]  

(19)

Using the form of \( \hat{L} \) and \( F \) this is the standard low energy theorem defining the effective coupling between the axion field \( \phi \) and \( F \wedge F \). In this form the topological susceptibility can be written, using (17), as

\[ \langle F \wedge F | \phi \rangle \Delta^2(\phi) \langle \phi | F \wedge F \rangle = \frac{F^2}{f_a^2} \frac{f_a^2}{F} = F \]  

(20)

In summary we get the following two basic correspondences:

Axion ⇔ Quantum estimator of \( \theta \) parameter

and

Axion low energy theorems ⇔ Axion as a maximally efficient quantum estimator

It is important to stress the differences between the former approach and the one initially developed by Witten in the solution to the \( U(1) \) problem. In that case the fermionic contribution, in the case of massless fermions, to the Fourier transform of the topological susceptibility

\[ U(k) = \int d^4x e^{ikx} T( \phi F F \wedge F(0) ) \]

should compensate the pure Yang Mills contribution. Denoting \( F \) the pure Yang Mills topological susceptibility this implies the existence of the \( \eta' \) meson contributing to \( U(k = 0) \) as

\[ c_{\eta'}^2 \sim \frac{m_{\eta'}^2}{C} \]  

(21)

leading to the basic formula [16] [20]

\[ m_{\eta'}^2 = \frac{c_{\eta'}^2}{C} \]  

(22)

The low energy theorem is now represented as

\[ c_{\eta'}^2 = m_{\eta'}^2 f_a^2 \]  

(23)

leading to

\[ m_{\eta'}^2 = \frac{C}{f_a^2} \]  

(24)

In our approach and for pure Yang Mills we define the quantum estimator operator with units of mass that we introduce by hand using \( f_a \). Once this quantum estimator \( \phi \) is defined everything is determined, namely \( \langle \phi | \hat{L} \rangle \) as well as \( \langle \phi \phi \rangle \) in the \( k = 0 \) limit. In other words the constraint used in [16] to relate \( U(k = 0) \) and \( c_{\eta'}^2 \) using the low energy theorems is automatically implemented for the quantum estimator that plays the role of the effective axion field. What we normally discover in QCD is that the \( \eta' \) meson plays the role of the quantum estimator of \( \theta \). This is very natural since the operator defining the quantum estimator is the one defining transformations of \( \theta \) that in the case of massless fermions is just the \( U(1) \) axial charge.

Of course in this case the scale \( f_a \) is just the analog of \( f_\pi \) for the corresponding Goldstone boson. In the case we have not available massless fermions to define \( \hat{L} \) the corresponding quantum estimator is an extra ingredient which is nothing else but the Peccei Quinn axion for the solution of the strong CP problem. In this case \( f_a \) is completely undetermined by the low energy dynamics.

V. COSMOLOGICAL ESTIMATION OF THE AXION DECAY CONSTANT

Even if \( F \) is different from zero we can have decoupling of axions in the case \( f_a \to 0 \). So the question is how we can set the value of \( f_a \). It is natural to assume that once the theory is coupled to gravity the quantum estimator becomes a full-fledged local quantum field. This is the main quantum gravity dogma briefly discussed in the introduction. Moreover coupling to gravity a full fledged quantum estimator ( axion ) field naturally leads us to a situation similar to the one we face in the inflationary approach to the early Universe with the axion playing a role similar to the inflaton.

Using the meaning of \( F \) as a quantum Fisher information we could very naively guess that once we UV complete with gravity we must have a relation of the type

\[ \frac{f_a^2}{F} \sim \frac{1}{M_P F_{\cos}} \]  

(25)

For \( F_{\cos} \) a cosmological Fisher information accounting for the slow variation in time of the axionic contribution to the primordial vacuum energy. Using the recent information approach to cosmology [21] we could make the guess ( conjecture )

\[ F_{\cos} \sim c_a^2 \]  

(26)

for \( c_a \) an axionic analog of the slow roll parameter. Using this bold conjecture we get as the estimation of \( f_a \)

\[ f_a = \left( \frac{\Lambda}{M_P} \right)^2 \frac{1}{c_a} M_P \]  

(27)

where we have used \( F = \Lambda^4 \). This expression for \( \Lambda \) defined at the GUT grand unified scale \( 10^{16} \text{Gev} \) and for
a value of $\epsilon$ based on imposing a reasonable amount of e-foldings $O(10^3)$ leads to

$$f_a \sim \Lambda_{\text{GUT}} = 10^{16}\text{GeV} \quad (27)$$

Needless to say that these comments should be interpreted as simply a possible way to use gravity to estimate the axion decay constant $f_a$.

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