Zeroth-Order Algorithms for Stochastic Distributed Nonconvex Optimization

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Abstract

In this paper, we consider a stochastic distributed nonconvex optimization problem with the cost function being distributed over $n$ agents having access only to zeroth-order (ZO) information of the cost. This problem has various machine learning applications. As a solution, we propose two distributed ZO algorithms, in which at each iteration each agent samples the local stochastic ZO oracle at two points with a time-varying smoothing parameter. We show that the proposed algorithms achieve the linear speedup convergence rate $O(\sqrt{p/(nT)})$ for smooth cost functions under the state-dependent variance assumptions which are more general than the commonly used bounded variance and Lipschitz assumptions, and $O(p/(nT))$ convergence rate when the global cost function additionally satisfies the Polyak–Łojasiewicz (P–Ł) condition in addition, where $p$ and $T$ are the dimension of the decision variable and the total number of iterations, respectively. To the best of our knowledge, this is the first linear speedup result for distributed ZO algorithms, which enables systematic processing performance improvements by adding more agents. We also show that the proposed algorithms converge linearly under the relative bounded second moment assumptions and the P–Ł condition. We demonstrate through numerical experiments the efficiency of our algorithms on generating adversarial examples from deep neural networks in comparison with baseline and recently proposed centralized and distributed ZO algorithms.

Index Terms—Distributed Nonconvex Optimization, Gradient-Free, Linear Speedup, Polyak-Łojasiewicz Condition, Stochastic Optimization.
I. Introduction

We consider stochastic distributed nonconvex optimization with zeroth-order (ZO) information feedback. Specifically, consider a network of $n$ agents/machines collaborating to solve the following optimization problem

$$\min_{x \in \mathbb{R}^p} f(x) := \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{\xi_i}[F_i(x, \xi_i)],$$

(1)

where $x \in \mathbb{R}^p$ is the decision variable, $\xi_i$ is a random variable, and $F_i(\cdot, \xi_i) : \mathbb{R}^p \mapsto \mathbb{R}$ is a stochastic component function (not necessarily convex). Each agent $i$ only has information about its own stochastic ZO oracle $F_i(x, \xi_i)$. In other words, for any given $x$ and $\xi_i$, each agent $i$ can sample $F_i(x, \xi_i)$ as a stochastic approximation of the true local cost function value $f_i(x) = \mathbb{E}_{\xi_i}[F_i(x, \xi_i)]$, but other information such as the first-order oracle cannot be observed. Agents can communicate with their neighbors through an underlying communication network. The network is modeled by an undirected graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, n\}$ is the agent set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the edge set, and $(i, j) \in \mathcal{E}$ if agents $i$ and $j$ can communicate with each other. The neighboring set of agent $i$ is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. The ZO information feedback setting has wide usage in applications, particularly when explicit expressions of the gradients are unavailable or difficult to obtain [1]–[3]. For example, the cost functions of many big data problems that deal with complex data generating processes cannot be explicitly defined [4]. Moreover, the distributed setting is a core aspect of many important applications in view of flexibility and scalability to large-scale datasets and systems, data privacy and locality, communication reduction to the central entity, and robustness to potential failures of the central entity [5]–[7].

A. Literature Review

The study of gradient-free (derivative-free) optimization has a long history, which can be traced back at least to the 1960’s [8]–[10]. It has recently gained renewed attention by the machine learning community. Classical gradient-free optimization methods can be classified into direct-search and model-based methods. For example, stochastic direct-search and model-based trust-region algorithms have been proposed in [11]–[14] and [15]–[17], respectively. In recent years, the more popular gradient-free optimization methods are ZO methods, which are gradient-free counterparts of first-order optimization methods and thus easy to implement. In ZO optimization methods, the full or stochastic gradients are approximated by directional derivatives, which are
calculated through sampled function values. A commonly used method to calculate directional
derivatives is simply using the function difference at two points \([18]–[20]\).

Various ZO optimization methods have been proposed, e.g., ZO (stochastic) gradient descent
algorithms \([20]–[31]\), ZO stochastic coordinate descent algorithms \([32]\), ZO (stochastic) variance
reduction algorithms \([24]\), \([25]\), \([29]\), \([30]\), \([33]–[45]\), ZO (stochastic) proximal algorithms \([33]\),
\([41]\), \([46]\), \([47]\), ZO Frank-Wolfe algorithms \([24]\), \([43]\), \([45]\), \([48]\), ZO mirror descent algorithms
\([18]\), \([39]\), \([49]\), ZO adaptive momentum methods \([47]\), \([50]\), ZO methods of multipliers \([34]\),
\([35]\), \([51]\), \([52]\), ZO stochastic path-integrated differential estimator \([37]\), \([42]\), \([52]\). Convergence
properties of these algorithms have been analyzed in detail. For instance, the typical convergence
result for deterministic centralized optimization problems is that first-order stationary points can
be found at an \(O(p/T)\) convergence rate by the two-point sampling based ZO algorithms \([20]\),
\([28]\), while under stochastic settings the convergence rate is reduced to \(O(\sqrt{p/T})\) \([22]\), \([32]\),
where \(T\) is the total number of iterations.

Aforementioned ZO optimization algorithms are all centralized and thus not suitable to solve
distributed optimization problems. Recently distributed ZO algorithms have been proposed,
e.g., distributed ZO gradient descent algorithms \([53]–[58]\), distributed ZO push-sum algorithm
\([59]\), distributed ZO mirror descent algorithm \([60]\), distributed ZO gradient tracking algorithm
\([58]\), distributed ZO primal–dual algorithms \([61]–[63]\), distributed ZO sliding algorithm \([64]\),
privacy-preserving distributed ZO algorithm \([65]\), distributed ZO Frank-Wolfe algorithm \([66]\).
Among these algorithms, the algorithms in \([53]–[55]\), \([58]–[60]\), \([63]\) are suitable to solve the
deterministic form of (1), while the algorithm in \([62]\) can be directly applied to solve the
stochastic optimization problem (1). However, the algorithm in \([62]\) requires each agent to have
an \(O(T)\) sampling size per iteration, which is not favorable in practice, although it was shown
that first-order stationary points can be found at an \(O(p^2n/T)\) convergence rate.

From the discussion above, three core theoretical questions arise when considering stochastic
distributed optimization problems:

(Q1) Can distributed ZO algorithms achieve similar convergence properties as centralized ZO
algorithms? For instance, can distributed ZO algorithms based on two-point sampling have an
\(O(\sqrt{p/T})\) convergence rate as their centralized counterparts in \([22]\), \([32]\)?

(Q2) As shown in \([67]\), distributed stochastic gradient descent (SGD) algorithms can achieve
linear speedup with respect to the number of agents compared with centralized SGD algorithms.
Can distributed ZO algorithms also achieve linear speedup? In particular, can distributed ZO al-
gorithms based on two-point sampling achieve the linear speedup convergence rate $O(\sqrt{p/nT})$?

(Q3) For deterministic optimization problems, centralized and distributed ZO algorithms can achieve faster convergence rates under more stringent conditions such as strong convexity or Polyak–Łojasiewicz (P–Ł) conditions, as shown in [20], [26], [28], [42], [44], [46] and [58], [63], respectively. For stochastic optimization problems, can ZO algorithms also achieve faster convergence rates under strong convexity or P–Ł conditions?

B. Main Contributions

This paper provides positive answers to the above three questions. We propose two distributed ZO algorithms, one primal–dual and one primal algorithm, to solve the stochastic optimization problem (1). In both algorithms, at each iteration each agent communicates its local primal variables to its neighbors through an arbitrarily connected communication network. Moreover, each agent samples its local stochastic ZO oracle at two points with a time-varying smoothing parameter. The contributions of this paper are summarized as follows.

(C1) We show in Theorems 2 and 8 that our algorithms find a stationary point with the linear speedup convergence rate $O(\sqrt{p/nT})$ for nonconvex but smooth cost functions under the state-dependent variance assumptions which are more general than the commonly used bounded variance and Lipschitz assumptions. This rate is faster than that achieved by the centralized ZO algorithms in [22], [24], [29]–[32], [50] and the distributed ZO algorithm in [58]. To the best of our knowledge, this is the first linear speedup result for distributed ZO algorithms; thus (Q1) and (Q2) are answered.

(C2) We show in Theorems 4, 5, 10, and 11 that our proposed algorithms find a global optimum with an $O(p/(nT))$ convergence rate when the global cost function satisfies the P–Ł condition in addition. This rate is faster than that achieved by the centralized ZO algorithms in [21], [23] and the distributed ZO algorithms in [54], [58], although [21], [23], [54] assumed strongly convex cost functions and only considered additive sampling noise, and [58] only considered the deterministic problem. This paper presents the first performance analysis for ZO algorithms to solve stochastic optimization problems under P–Ł or strong convexity assumptions; thus (Q3) is answered.

(C3) We show in Theorems 6 and 12 that our algorithms with constant algorithm parameters linearly converges to a neighborhood of a global optimum under the P–Ł condition. Moreover, a precise global optimum can be linearly found if the relative bounded second moment assumptions...
also hold, see Corollaries 1 and 2. It should be mentioned that the P–Ł constant is not used to
design the algorithm parameters when showing these results. Compared with [20], [26], [28],
[42], [44], [46] which also achieved linear convergence, we use weaker assumptions on the cost
function and/or less samplings per iteration.

The detailed comparison between this paper and the literature is summarized in TABLE I.

C. Outline

The rest of this paper is organized as follows. Section II introduces some preliminaries.
Sections III and IV provide the distributed primal–dual and primal ZO algorithms, respectively,
and analyze their convergence properties. Numerical evaluations for an image classification
problem from the literature are given in Section V. Finally, concluding remarks are offered
in Section VI. All the proofs are given in Appendix.

Notations: \( \mathbb{N}_+ \) denotes the set of positive integers. \([n]\) denotes the set \{1, \ldots, n\} for any \( n \in \mathbb{N}_+ \).
\( \| \cdot \| \) represents the Euclidean norm for vectors or the induced 2-norm for matrices. \( \mathbb{B}_p \) and \( \mathbb{S}_p \) are the unit ball and sphere centered around the origin in \( \mathbb{R}^p \) under Euclidean norm, respectively.

Given a differentiable function \( f \), \( \nabla f \) denotes its gradient.

II. Preliminaries

In this section, we introduce the P–Ł condition, the random gradient estimator, and the
assumptions used in this paper.

A. Polyak–Łojasiewicz Condition

**Definition 1.** [68] A differentiable function \( f(x) : \mathbb{R}^p \mapsto \mathbb{R} \) satisfies the Polyak–Łojasiewicz
(P–Ł) condition with constant \( \nu > 0 \) if \( f^* > -\infty \), where \( f^* = \min_{x \in \mathbb{R}^p} f(x) \), and

\[
\frac{1}{2} \| \nabla f(x) \|^2 \geq \nu (f(x) - f^*), \quad \forall x \in \mathbb{R}^p.
\]

It is straightforward to see that every (essentially, weakly, or restricted) strongly convex
function satisfies the P–Ł condition. The P–Ł condition implies that every stationary point is
a global minimizer. But unlike (essentially, weakly, or restricted) strong convexity, the P–Ł
condition alone does not imply convexity of \( f \). Moreover, it does not imply that the set of
global minimizers is a singleton [68], [69]. Examples of nonconvex functions which satisfy the
P–Ł condition can be found in [68], [69].
TABLE I: Summary of existing works on ZO optimization, where NoSPPI denotes the number of sampled points per iteration, and the sampling complexity is the total number of function samplings to achieve $\mathbb{E}[||\nabla f(x_T)||^2] \leq \epsilon$ for nonconvex problems or $\mathbb{E}[f(x_T) - f^*] \leq \epsilon$ for (strongly) convex problems or problems satisfying the P–Ł condition.

| Reference | Problem settings | NoSPPI | Convergence rate | Sampling complexity |
|-----------|------------------|--------|------------------|---------------------|
| 10        | Deterministic, centralized, unconstrained, nonconvex, smooth | Two   | $O(p^2/T)$ | $O(p^2/s)$ |
|           | Strongly convex in addition | | Linear | $O(p \log(1/\epsilon))$ |
| 11        | Deterministic, centralized, strongly convex, unconstrained, smooth, Lipschitz Hessian | $p$  | Linear | $O(p \log(1/\epsilon))$ |
| 12        | Deterministic, centralized, unconstrained, nonconvex, smooth | Two   | $O(p^2/T)$ | $O(p^2/s)$ |
|           | P–Ł condition in addition | | Linear | $O(p \log(1/\epsilon))$ |
| 13        | Deterministic, centralized, restricted strongly convex, unconstrained, smooth, $s$-sparse gradient | $4s \log(p/s)$ | Linear | $O(s \log(p/s) \log(1/\epsilon))$ |
| 14        | Deterministic, centralized, quadratic, unconstrained, additive sampling noise | One  | $O(p^2/T)$ | $O(p^2/s)$ |
|           | Two              | $O(p^3/\sqrt{T})$ | $O(p^2/s^2)$ |
| 15        | Deterministic, centralized, unconstrained, nonconvex, Lipschitz, smooth | One  | $O(p^2/T^{1/3})$ | $O(p^3/s^{1/3})$ |
|           | Stochastic, centralized, unconstrained, nonconvex, Lipschitz, smooth | Two  | $O(p^3/3^{T/3})$ | $O(p^3/s^2)$ |
| 16        | Stochastic, centralized, unconstrained, nonconvex, smooth, additive sampling noise | Two  | $O(s \log(p/s) \sqrt{T})$ | $O(s \log(p/s)^2/\epsilon^2)$ |
|           | Stochastic, centralized, constrained, nonconvex, Lipschitz, smooth | $O(pT)$ | $O(1/T)$ | $O(p^2/s)$ |
|           | Stochastic, centralized, constrained, Lipschitz, smooth | Two  | $O(p^2/\sqrt{T})$ | $O(p^2/s^2)$ |
| 17        | Deterministic, finite-sum, nonconvex, constrained, Lipschitz, smooth | $O(\sqrt{T})$ | $O(p^2/\sqrt{T})$ | $O(p^3/s^2)$ |
|           | Deterministic, finite-sum, nonconvex, constrained, Lipschitz, smooth | One  | $O(pT)$ | $O(p^2/s)$ |
|           | Deterministic, finite-sum, nonconvex, constrained, Lipschitz, smooth, the original and mixture gradients are proportional | $2b$ | $O(pn^2/(bT))$ | $O(pn^2/s^2)$ |
|           | Deterministic, finite-sum, nonconvex, unconstrained, smooth, bounded variance | $O(pn^{1/2})$ | $O(1/T)$ | $O(pn^{1/2}/s)$ |
| 19        | Deterministic, finite-sum, nonconvex, unconstrained, bounded variance | $2n$ | $O(pT)$ | $O(pn/s)$ |
| 20        | Deterministic, finite-sum, nonconvex, constrained, bounded variance | $O(pn^{2/3})$ | $O(pn^{2/3}/s)$ | $O(pn^{2/3}/s^{1/3})$ |
| 21        | Deterministic, finite-sum, nonconvex, unconstrained, bounded variance | $O(pn^{1/2})$ | $O(1/T)$ | $O(pn^{1/2}/s)$ |
| 22        | P–Ł condition in addition | Linear | $O(pn^2 \log(1/\epsilon))$ |
| 23        | Deterministic, finite-sum, strongly convex, constrained, Lipschitz, smooth | $O(nT)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
| 24        | Stochastic, finite-sum, nonconvex, constrained, Lipschitz, smooth | $O(T)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
| 25        | Deterministic, distributed, convex, constrained, Lipschitz, smooth | $O(p \sqrt{n}/T^2)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
|           | Deterministic, distributed, convex, unconstrained, Lipschitz | $O(p \sqrt{n}/T^2)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
| 26        | Deterministic, distributed, convex, compact constrained, Lipschitz | $O(p \sqrt{n}/T^2)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
|           | Deterministic, distributed, strongly convex, constrained, Lipschitz | $O(p \sqrt{n}/T)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
|           | Deterministic, distributed, strongly convex, unconstrained, smooth, additive sampling noise | $O(p \sqrt{n}/T)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
|           | Deterministic, distributed, convex, compact constrained, Lipschitz, additive sampling noise | $O(p \sqrt{n}/T)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
| 27        | Deterministic, distributed, convex, constrained, Lipschitz | $O(p \sqrt{n}/T)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
| 28        | Stochastic, distributed, convex, constrained, Lipschitz | $O(p \sqrt{n}/T)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
|           | Deterministic, distributed, convex, unconstrained, Lipschitz | $O(p \sqrt{n}/T)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
|           | Deterministic, distributed, nonconvex, constrained, Lipschitz, smooth | $O(p \sqrt{n}/T)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
|           | Deterministic, distributed, nonconvex, unconstrained, smooth, P–Ł condition | $O(p \sqrt{n}/T)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
|           | Deterministic, distributed, nonconvex, unconstrained, smooth, P–Ł condition | $2pn$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
|           | Deterministic, distributed, nonconvex, unconstrained, smooth, P–Ł condition in addition (without using the P–Ł constant) | $(p + 1)n$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
|           | Stochastic, distributed, nonconvex, unconstrained, Lipschitz, smooth | $O(p \sqrt{n}/T)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
|           | Stochastic, distributed, nonconvex, unconstrained, Lipschitz, smooth, state-dependent variance | $O(p \sqrt{n}/T)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
|           | Stochastic, distributed, nonconvex, unconstrained, smooth, P–Ł condition | $O(p \sqrt{n}/T)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
|           | Stochastic, distributed, nonconvex, unconstrained, smooth, relative bounded second moment | $O(p \sqrt{n}/T)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
|           | P–Ł condition in addition (without using the P–Ł constant) | $2n$ | $O(p \sqrt{n}/(nT))$ | $O(p^2/nT^2)$ |
|           | Stochastic, distributed, nonconvex, unconstrained, smooth, P–Ł condition | $O(p \sqrt{n}/T)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
|           | Stochastic, distributed, nonconvex, constrained, Lipschitz, smooth | $O(p \sqrt{n}/T)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
|           | Stochastic, distributed, nonconvex, unconstrained, smooth, P–Ł condition | $O(p \sqrt{n}/T)$ | $O(p^2/nT^2)$ | $O(p^2/nT^2)$ |
|           | Stochastic, distributed, nonconvex, unconstrained, smooth, P–Ł condition in addition (without using the P–Ł constant) | $2n$ | $O(p \sqrt{n}/(nT))$ | $O(p^2/nT^2)$ |

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B. Gradient Estimator

Let \( f(x) : \mathbb{R}^p \mapsto \mathbb{R} \) be a function. The authors of \cite{18} proposed the following random gradient estimator:

\[
\hat{\nabla}_2 f(x, \delta, u) = \frac{p}{\delta} (f(x + \delta u) - f(x))u,
\]

where \( \delta > 0 \) is the smoothing/exploration parameter and \( u \in \mathbb{S}^p \) is a uniformly distributed random vector. This gradient estimator can be calculated by sampling the function \( f \) at two points (e.g., \( x \) and \( x + \delta u \)). The intuition of this estimator follows from directional derivatives \cite{18}. From a practical point of view, the larger the smoothing parameter \( \delta \) the better, since in this case it is easier to distinguish the two sampled function values.

C. Assumptions

The following assumptions are made.

**Assumption 1.** The undirected graph \( \mathcal{G} \) is connected.

**Assumption 2.** The optimal set \( \mathcal{X}^* \) is nonempty and \( f^* > -\infty \), where \( \mathcal{X}^* \) and \( f^* \) are the optimal set and the minimum function value of the optimization problem \cite{1}, respectively.

**Assumption 3.** For almost all \( \xi_i \), the stochastic ZO oracle \( F_i(\cdot, \xi_i) \) is smooth with constant \( L_f > 0 \).

**Assumption 4.** Each stochastic gradient \( \nabla_x F_i(x, \xi_i) \) has state-dependent variance, i.e., there exist two constants \( \sigma_0 \) and \( \sigma_1 \) such that \( \mathbb{E}_{\xi_i}[\|\nabla_x F_i(x, \xi_i) - \nabla f_i(x)\|^2] \leq \sigma_0^2 \|\nabla f_i(x)\|^2 + \sigma_1^2, \forall i \in [n], \forall x \in \mathbb{R}^p \).

**Assumption 5.** Each local gradient \( \nabla f_i(x) \) has state-dependent variance, i.e., there exists two constants \( \tilde{\sigma}_0 \) and \( \sigma_2 \) such that \( \|\nabla f_i(x) - \nabla f(x)\|^2 \leq \tilde{\sigma}_0^2 \|\nabla f(x)\|^2 + \sigma_2^2, \forall i \in [n], \forall x \in \mathbb{R}^p \). Here \( \nabla f_i(x) \) can be viewed as a stochastic gradient of \( \nabla f(x) \) by randomly picking an index \( i \in [n] \).

**Assumption 6.** The global cost function \( f(x) \) satisfies the P–Ł condition with constant \( \nu > 0 \).

**Remark 1.** It should be highlighted that no convexity assumptions are made. Assumption 1 is common in distributed optimization, e.g., \cite{58, 62, 64, 70–73}. Assumption 2 is basic.
Assumption 3 is standard in stochastic optimization with ZO information feedback, e.g., [22], [24], [32]–[35], [39], [40], [62]. When $\sigma_0 = 0$, Assumption 4 recovers the bounded variance assumption, which is commonly used in the literature studying stochastic ZO optimization, e.g., [22], [24], [32]–[35], [39], [48], [62]. Therefore, Assumption 4 is more general. When $\tilde{\sigma}_0 = 0$, Assumption 5 becomes the bounded variance assumption, i.e., $\|\nabla f_i(x)\|$ is globally bounded, and is weaker than the Lipschitz assumption, i.e., $\|\nabla f_i(x)\|$ is globally bounded. Both the bounded variance and Lipschitz assumptions are normally used in the literature studying ZO optimization, e.g., [37], [38], [42] and [18], [29], [30], [34]–[36], [41], [51]–[53], [58], [59], [62], respectively. However, the Lipschitz assumption is too restricted since the simple quadratic functions are not Lipschitz. Moreover, the bounded variance assumption is also restricted, for instance it is impractical to assume this assumption for distributed learning problems with local cost functions being constructed by heterogenous data collected locally by agents. In contrast, Assumption 5 is more general due to the state-dependent term $\tilde{\sigma}_0^2 \|\nabla f(x)\|^2$, and it is not needed when Assumption 6 holds and the constant $\nu$ is known in advance as shown in Theorems 5 and 11 in the later sections. Assumption 6 is weaker than the assumption that the global or each local cost function is (restricted) strongly convex. It plays a key role to guarantee that a global optimum can be found and to show that faster convergence rate can be achieved.

To end this section, we introduce the stronger alternatives of Assumptions 4 and 5, which are the key to show faster convergence for the proposed algorithms as shown in the later sections.

**Assumption 4.** The second moment of each stochastic gradient $\nabla_x F_i(x, \xi_i)$ is relative bounded, i.e., there exists a constant $\tilde{\sigma}_0$ such that $E_{\xi_i}[\|\nabla_x F_i(x, \xi_i)\|^2] \leq \tilde{\sigma}_0^2 \|\nabla f_i(x)\|^2$, $\forall i \in [n]$, $\forall x \in \mathbb{R}^p$.

**Assumption 5.** The second moment of each local gradient $\nabla f_i(x)$ is relative bounded, i.e., there exists a constant $\tilde{\sigma}_0$ such that $\|\nabla f_i(x)\|^2 \leq \tilde{\sigma}_0^2 \|\nabla f(x)\|^2$, $\forall i \in [n]$, $\forall x \in \mathbb{R}^p$.

It is straightforward to check that Assumption 4 (Assumption 5) is equivalent to Assumption 4 (Assumption 5) when $\sigma_1 = 0$ ($\sigma_2 = 0$). Assumption 4 is satisfied trivially when the deterministic ZO information is available. Assumption 5 holds when $\nabla f_i(x)$ is proportional to $\nabla f(x)$, for example when all the random variables $\xi_i$ have a common probability distribution and the local stochastic component functions are the same, which is a common setup in distributed empirical risk minimization problems. Moreover, for deterministic centralized optimization problems, Assumption 4 and 5 hold trivially.
III. DISTRIBUTED ZO PRIMAL–DUAL ALGORITHM

In this section, we propose a distributed ZO primal–dual algorithm and analyze its convergence properties.

When gradient information is available, in [63] the following distributed first-order primal–dual algorithm was proposed to solve (1):

\[ x_{i,k+1} = x_{i,k} - \eta \left( \alpha \sum_{j=1}^{n} L_{ij} x_{j,k} + \beta v_{i,k} + \nabla f_i(x_{i,k}) \right), \]  
\[ v_{i,k+1} = v_{i,k} + \eta \beta \sum_{j=1}^{n} L_{ij} x_{j,k}, \quad \forall x_{i,0} \in \mathbb{R}^p, \quad \sum_{j=1}^{n} v_{j,0} = 0_p, \quad \forall i \in [n], \]  

(4a)  
(4b)

where \( \alpha, \beta, \) and \( \eta \) are positive algorithm parameters, and \( L = [L_{ij}] \) is the weighted Laplacian matrix associated with the undirected communication graph \( G \). As pointed out in [63] the distributed first-order algorithm (4) is a special form of several existing first-order algorithms in the literature, e.g., [70], [74], and it has been shown that this algorithm can find a stationary point with an \( O(1/k) \) convergence rate.

Noting that we consider the scenario where only stochastic ZO oracles rather than the explicit expressions of the gradients are available, we need to estimate the gradients used in the distributed first-order algorithm (4). Inspired by (3), we introduce

\[ g_{i,k}^e = \frac{p(F_i(x_{i,k} + \delta_{i,k} u_{i,k}, \xi_{i,k}) - F_i(x_{i,k}, \xi_{i,k}))}{\delta_{i,k}} u_{i,k}, \]  

(5)

where \( \delta_{i,k} > 0 \) is a time-varying smoothing parameter and \( u_{i,k} \in \mathbb{S}^p \) is a uniformly distributed random vector chosen by agent \( i \) at iteration \( k \); \( \xi_{i,k} \) is a random variable sampled by agent \( i \) at iteration \( k \) according to the distribution of \( \xi_i \); and \( F_i(x_{i,k} + \delta_{i,k} u_{i,k}, \xi_{i,k})\) and \( F_i(x_{i,k}, \xi_{i,k}) \) are the values sampled by agent \( i \) at iteration \( k \). We replace the gradient and fixed algorithm parameters in (4) with the stochastic gradient estimator (5) and time-varying parameters, respectively. Then we get the following ZO algorithm:

\[ x_{i,k+1} = x_{i,k} - \eta_k \left( \alpha_k \sum_{j=1}^{n} L_{ij} x_{j,k} + \beta_k v_{i,k} + g_{i,k}^e \right), \]  
\[ v_{i,k+1} = v_{i,k} + \eta_k \beta_k \sum_{j=1}^{n} L_{ij} x_{j,k}, \quad \forall x_{i,0} \in \mathbb{R}^p, \quad \sum_{j=1}^{n} v_{j,0} = 0_p, \quad \forall i \in [n]. \]  

(6a)  
(6b)

We write the distributed ZO algorithm (6) in pseudo-code as Algorithm 1. In this algorithm,
Algorithm 1 Distributed ZO Primal–Dual Algorithm

1: Input: positive sequences \(\{\alpha_k\}, \{\beta_k\}, \{\eta_k\}\), and \(\{\delta_{i,k}\}\).
2: Initialize: \(x_{i,0} \in \mathbb{R}^p\) and \(v_{i,0} = 0_p, \forall i \in [n]\).
3: for \(k = 0, 1, \ldots\) do
4:   for \(i = 1, \ldots, n\) in parallel do
5:     Broadcast \(x_{i,k}\) to \(N_i\) and receive \(x_{j,k}\) from \(j \in N_i\);
6:     Generate \(u_{i,k} \in S^p\) independently and uniformly at random;
7:     Generate \(\xi_{i,k}\) independently and randomly according to the distribution of \(\xi_i\);
8:     Sample \(F_i(x_{i,k}, \xi_{i,k})\) and \(F_i(x_{i,k} + \delta_{i,k} u_{i,k}, \xi_{i,k})\);
9:     Update \(x_{i,k+1}\) by (6a);
10:    Update \(v_{i,k+1}\) by (6b).
11: end for
12: end for
13: Output: \(\{x_k\}\).

from the way to generate \(u_{i,k}\) and \(\xi_{i,k}\), we know that \(u_{i,k}, \xi_{j,l}, \forall i, j \in [n], k, l \in \mathbb{N}_+\) are mutually independent. Let \(\mathcal{L}_k\) denote the \(\sigma\)-algebra generated by the independent random variables \(u_{1,k}, \ldots, u_{n,k}, \xi_{1,k}, \ldots, \xi_{n,k}\) and let \(\mathcal{L}_k = \bigcup_{t=0}^{k} \mathcal{L}_t\). From the independence property of \(u_{i,k}\) and \(\xi_{i,l}\), we can see that \(x_{i,k}\) and \(v_{i,k+1}, i \in [n]\) depend on \(\mathcal{L}_{k-1}\) and are independent of \(\mathcal{L}_t\) for all \(t \geq k\).

**Remark 2.** In Algorithm 1, each agent \(i\) maintains two local sequences, i.e., the local primal and dual variable sequences \(\{x_{i,k}\}\) and \(\{v_{i,k}\}\), and communicates its local primal variables to its neighbors through the network. Moreover, at each iteration each agent samples its local stochastic ZO oracle at two points to estimate the gradient of its local cost function. It should be highlighted that the agent-wise smoothing parameter \(\delta_{i,k}\) is time-varying. It can in many situations be chosen larger than the fixed smoothing parameter used in existing ZO algorithms. For example, in the following we use an \(O(1/k^{1/4})\) smoothing parameter, which is larger than the \(O(1/T^{1/2})\) smoothing parameter used in [22].

A. Find stationary points

Let us consider the case when Algorithm 1 is able to find stationary points. We first have the following convergence result.

**Theorem 1.** Suppose Assumptions 4–5 hold. Let \(\{x_k\}\) be the sequence generated by Algorithm 1.
with

\[ \alpha_k = \kappa_1 \beta_k, \quad \beta_k = \kappa_0 (k + t_1)^{\theta}, \quad \eta_k = \frac{\kappa_2}{\beta_k}, \quad \delta_{i,k} \leq \frac{\kappa_\delta \sqrt{p} k}{\sqrt{n + p}}, \quad \forall k \in \mathbb{N}_0, \] (7)

where \( \kappa_1 > c_1, \ \kappa_2 \in (0, c_2(\kappa_1)), \ \theta \in (0.5, 1), \ t_1 \geq (\sqrt{p c_3(\kappa_1, \kappa_2)})^\frac{1}{\theta}, \ \kappa_0 \geq \frac{c_0(\kappa_1, \kappa_2)}{t_1^2}, \ \text{and} \ \kappa_\delta > 0 \)

with \( c_0(\kappa_1, \kappa_2), \ c_1, \ c_2(\kappa_1), \ \text{and} \ c_3(\kappa_1, \kappa_2) \) being given in Appendix B. Then, for any \( T \in \mathbb{N}_+, \)

\[ \frac{1}{T} \sum_{k=0}^{T-1} E[\|\nabla f(\bar{x}_k)\|^2] = O\left(\frac{\sqrt{p}}{T^{1-\theta}} + \frac{p}{T}\right), \] (8a)

\[ E[f(\bar{x}_T)] - f^* = O(1), \] (8b)

\[ E\left[\frac{1}{n} \sum_{i=1}^{n} \|x_{i,T} - \bar{x}_T\|^2\right] = O\left(\frac{1}{T^{2\theta}}\right), \] (8c)

\[ \lim_{T \to +\infty} E[\|\nabla f(\bar{x}_T)\|^2] = 0, \] (8d)

where \( \bar{x}_k = \frac{1}{n} \sum_{i=1}^{n} x_{i,k}. \)

**Proof:** The proof is given in Appendix B.

If the total number of iterations \( T \) and the number of agents \( n \) are known in advance, then, as shown in the following, Algorithm 1 can find a stationary point of (1) with an \( O\left(\sqrt{p/n T}\right) \) convergence rate, and thus achieves linear speedup with respect to the number of agents compared to the \( O\left(\sqrt{p/T}\right) \) convergence rate achieved by the centralized stochastic ZO algorithms in [22], [32]. The linear speedup property enables us to scale up the computing capability by adding more agents into the algorithm [75].

**Theorem 2 (Linear speedup).** Suppose Assumptions 1–5 hold. For any given \( T \geq \max\{\frac{n \tilde{c}_0(\kappa_1, \kappa_2)^2}{pn^2}, \ \frac{n^3}{p}\} \), let \( \{x_k, k = 0, \ldots, T\} \) be the output generated by Algorithm 1 with

\[ \alpha_k = \kappa_1 \beta_k, \ \beta_k = \kappa_0 (k + t_1)^{\theta}, \ \eta_k = \frac{\kappa_2}{\beta_k}, \ \delta_{i,k} \leq \frac{p^2 n^4 lasic_\delta}{\sqrt{n + p(k + 1)^2}}, \ \forall k \leq T, \] (9)

where \( \tilde{c}_0(\kappa_1, \kappa_2) \) is given in Appendix C. \( \kappa_1 > c_1, \ \kappa_2 \in (0, c_2(\kappa_1)), \ \text{and} \ \kappa_\delta > 0 \) with \( c_1 \) and \( c_2(\kappa_1) \) being given in Appendix B. Then,

\[ \frac{1}{T} \sum_{k=0}^{T-1} E[\|\nabla f(\bar{x}_k)\|^2] = O\left(\frac{\sqrt{p}}{\sqrt{nT}}\right) + O\left(\frac{n}{T}\right), \] (10a)

\[ E[f(\bar{x}_T)] - f^* = O(1), \] (10b)
\[ E \left[ \frac{1}{n} \sum_{i=1}^{n} \| x_{i,T} - \bar{x}_T \|^2 \right] = O \left( \frac{n}{T} \right), \quad (10c) \]

\[ \lim_{T \to +\infty} E[\| \nabla f(\bar{x}_T) \|^2] = 0. \quad (10d) \]

**Proof:** The proof is given in Appendix C. It should be highlighted that the omitted constants in the first term on the right-hand side of (10a) do not depend on any parameters related to the communication network.

**Remark 3.** To the best of our knowledge, Theorem 2 is the first result to establish linear speedup for a distributed ZO algorithm to solve stochastic optimization problems. The achieved rate is faster than that achieved by the centralized ZO algorithms in [22], [24], [29]–[32], [50], and the distributed ZO gradient descent algorithm in [58]. The rate is slower than that achieved by the centralized ZO algorithms in [33], [35]–[38], [40]–[42], which is reasonable since these algorithms not only are centralized but also use variance reduction techniques. The distributed ZO gradient tracking algorithm in [58] and the distributed ZO primal–dual algorithms in [62], [63] also achieved faster convergence rates than ours. However, in [36]–[38], [41], [42], [58], [63], the considered problems are deterministic; in [58], [63], the sampling size of each agent at each iteration is \( O(p) \), which results in a heavy sampling burden when \( p \) is large; in [33], [35], [62], the sampling size of each agent at each iteration is \( O(T) \), which is difficult to execute in practice. One of our future research directions is to establish faster convergence with reduced sampling complexity by using variance reduction techniques.

**B. Find global optimum**

Let us next consider cases when Algorithm 1 finds global optimum.

**Theorem 3.** Suppose Assumptions [1]–[3] hold. Let \( \{ x_k \} \) be the sequence generated by Algorithm 1 with

\[ \alpha_k = \kappa_1 \beta_k, \quad \beta_k = \kappa_0 (k + t_1)^\theta, \quad \eta_k = \frac{\kappa_2}{\beta_k}, \quad \delta_{i,k} \leq \frac{\kappa_\delta \sqrt{p \eta_k}}{\sqrt{n+p}}, \quad \forall k \in \mathbb{N}_0, \quad (11) \]

where \( \kappa_1 > c_1, \quad \kappa_2 \in (0, c_2(\kappa_1)), \quad \theta \in (0, 1), \quad t_1 \in [(pc_3(\kappa_1, \kappa_2))^{\frac{1}{p}}, (pc_4c_3(\kappa_1, \kappa_2))^{\frac{1}{p}}], \quad \kappa_0 \geq \frac{c_0(\kappa_1, \kappa_2)}{\sqrt{c_1}}, \quad \kappa_\delta > 0, \quad \text{and} \quad c_4 \geq 1 \] with \( c_0(\kappa_1, \kappa_2), \quad c_1, \quad c_2(\kappa_1), \quad \text{and} \quad c_3(\kappa_1, \kappa_2) \) being given in Appendix B. Then,
for any $T \in \mathbb{N}_+$,
\[
E\left[\frac{1}{n} \sum_{i=1}^{n} \|x_{i,T} - \bar{x}_T\|^2\right] = \mathcal{O}\left(\frac{p}{T^2}\right),
\]
\[
E[f(\bar{x}_T) - f^*] = \mathcal{O}\left(\frac{p}{nT^2}\right) + \mathcal{O}\left(\frac{p}{T^2}\right).
\] (12a) (12b)

**Proof:** The proof is given in Appendix D. It should be highlighted that the omitted constants in the first term on the right-hand side of (12b) do not depend on any parameters related to the communication network.

From Theorem 3, we see that the convergence rate is strictly slower than $\mathcal{O}(p/(nT))$. In the following we show that the $\mathcal{O}(p/(nT))$ convergence rate can be achieved if the P–Ł constant $\nu$ is known in advance. Information about the total number of iterations $T$ is not needed.

**Theorem 4** (Linear speedup). Suppose Assumptions 1–6 hold and the P–Ł constant $\nu$ is known in advance. Let $\{x_k\}$ be the sequence generated by Algorithm 1 with
\[
\alpha_k = \kappa_1 \beta_k, \quad \beta_k = \kappa_0 (k + t_1), \quad \eta_k = \frac{\kappa_2}{\beta_k}, \quad \delta_{i,k} \leq \frac{\kappa_3 \sqrt{p\eta_k}}{\sqrt{n + p}}, \quad \forall k \in \mathbb{N}_0,
\]
where $\kappa_1 > c_1$, $\kappa_2 \in (0, c_2(\kappa_1))$, $\kappa_0 \in \left[\frac{3\nu_2\kappa_2}{16}, \frac{3\nu_2}{16}\right]$, $t_1 \geq \hat{c}_3(\kappa_0, \kappa_1, \kappa_2)$, $\kappa_3 > 0$, and $\hat{c}_0 \in (0, 1)$ with $c_1$ and $c_2(\kappa_1)$ being given in Appendix B and $\hat{c}_3(\kappa_0, \kappa_1, \kappa_2)$ being given in Appendix E.

Then, for any $T \in \mathbb{N}_+$,
\[
E\left[\frac{1}{n} \sum_{i=1}^{n} \|x_{i,T} - \bar{x}_T\|^2\right] = \mathcal{O}\left(\frac{p}{T^2}\right),
\]
\[
E[f(\bar{x}_T) - f^*] = \mathcal{O}\left(\frac{p}{nT}\right) + \mathcal{O}\left(\frac{p}{T^2}\right).
\] (14a) (14b)

**Proof:** The proof is given in Appendix E. It should be highlighted that the omitted constants in the first term on the right-hand side of (14b) do not depend on any parameters related to the communication network.

Although Assumption 5 is weaker than the bounded gradient assumption, it can be further relaxed by a mild assumption. Specifically, if each $f_i^* > -\infty$, where $f_i^* = \min_{x \in \mathbb{R}^p} f_i(x)$, then without Assumption 5 the convergence results stated in (14a) and (14b) still hold, as shown in the following.

**Theorem 5** (Linear speedup). Suppose Assumptions 1–4 and 6 hold, and the P–Ł constant $\nu$ is
known in advance, and each $f_i^* > -\infty$. Let $\{x_k\}$ be the sequence generated by Algorithm 1 with

$$
\alpha_k = \kappa_1 \beta_k, \quad \beta_k = \kappa_0 (k + t_1), \quad \eta_k = \frac{\kappa_2}{\beta_k}, \quad \delta_{i,k} \leq \frac{\kappa_\delta \sqrt{p/n}}{\sqrt{n + p}}, \quad \forall k \in \mathbb{N}_0,
$$

(15)

where $\kappa_1 > c_1$, $\kappa_2 \in (0, c_2(\kappa_1))$, $\kappa_0 \in \left[\frac{3c_0 \nu \kappa_2}{16}, \frac{3 \nu \kappa_2}{16}\right)$, $t_1 \geq \tilde{c}_3(\kappa_0, \kappa_1, \kappa_2)$, $\kappa_\delta > 0$, and $\tilde{c}_0 \in (0, 1)$ with $c_1$ and $c_2(\kappa_1)$ being given in Appendix B and $\tilde{c}_3(\kappa_0, \kappa_1, \kappa_2)$ being given in Appendix F.

Then, for any $T \in \mathbb{N}_+$,

$$
\mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} \|x_{i,T} - \bar{x}_T\|^2\right] = \mathcal{O}\left(\frac{p}{T^2}\right),
$$

(16a)

$$
\mathbb{E}[f(\bar{x}_T) - f^*] = \mathcal{O}\left(\frac{p}{nT}\right) + \mathcal{O}\left(\frac{p}{T^2}\right).
$$

(16b)

Proof: The proof is given in Appendix F. It should be highlighted that the omitted constants in the first term on the right-hand side of (16b) do not depend on any parameters related to the communication network.

Remark 4. To the best of our knowledge, Theorems 3–5 are the first performance analysis results for ZO algorithms to solve stochastic optimization problems under the P–Ł condition or strong convexity assumption. In [21], a centralized ZO algorithm based on one-point sampling with additive sampling noise was proposed and an $\mathcal{O}(p^2/T)$ convergence rate was achieved for deterministic optimization problems with strongly convex quadratic cost functions. In [23], a centralized ZO algorithm based on two-point sampling with additive noise was proposed and an $\mathcal{O}(p/\sqrt{T})$ convergence rate was achieved for deterministic strongly convex and smooth optimization problems. In [54], a distributed ZO gradient descent algorithm based on 2p-point sampling with additive noise was proposed and an $\mathcal{O}(pn^2/\sqrt{T})$ convergence rate was achieved for deterministic strongly convex and smooth optimization problems. In [58], a distributed ZO gradient descent algorithm based on two-point sampling was proposed and an $\mathcal{O}(p/T)$ convergence rate was achieved for deterministic smooth optimization problems under the P–Ł condition. It is straightforward to see that aforementioned convergence rates achieved in [21], [23], [54], [58] are slower than that achieved by our distributed stochastic ZO primal–dual algorithm as stated in Theorem 5. Moreover, we consider stochastic optimization problems and use the P–Ł condition, which is slightly weaker than the strong convexity condition. The distributed ZO gradient tracking algorithm in [58] and the distributed ZO primal–dual algorithms
in [6,3] achieved linear convergence under the P–Ł condition. However, both algorithms require each agent at each iteration to sample $O(p)$ points, which results in a heavy sampling burden when $p$ is large.

As shown in Theorems 3–5, in expectation, the convergence rate of Algorithm 1 with diminishing stepsizes is sublinear. The following theorem establishes that, in expectation, the output of Algorithm 1 with constant algorithm parameters linearly converges to a neighborhood of a global optimum.

**Theorem 6.** Suppose Assumptions [1–5] hold. Let $\{x_k\}$ be the sequence generated by Algorithm 1 with

$$\alpha_k = \alpha = \kappa_1 \beta, \quad \beta_k = \beta, \quad \eta_k = \eta = \frac{\kappa_2}{\beta}, \quad \delta_{i,k} \leq \kappa_3 \tilde{\varepsilon}^k, \quad \forall k \in \mathbb{N}_0,$$

where $\kappa_1 > c_1$, $\kappa_2 \in (0, c_2(\kappa_1))$, $\beta \geq \tilde{c}_0(\kappa_1, \kappa_2)$, $\tilde{\varepsilon} \in (0, 1)$, and $\kappa_3 > 0$ with $\tilde{c}_0(\kappa_1, \kappa_2)$ being given in Appendix C and $c_1$ and $c_2(\kappa_1)$ being given in Appendix B. Then, for any $T \in \mathbb{N}_+$,

$$\begin{aligned}
\frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \|x_{i,k} - \bar{x}_k\|^2 \right] &= \mathcal{O} \left( \frac{1}{T} + (\sigma_1^2 + 2(1 + \sigma_0^2)\sigma_2^2)pn^2 \right), \quad (18a) \\
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \|x_{i,T} - \bar{x}_T\|^2 \right] &= \mathcal{O} \left( pn^2 + (\sigma_1^2 + 2(1 + \sigma_0^2)\sigma_2^2)p^2\eta^4 \left( \frac{1}{n} + \eta \right) T \right), \quad (18b) \\
\frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E} \left[ \|\nabla f(\bar{x}_k)\|^2 \right] &= \mathcal{O} \left( \frac{1}{\eta^2} + (\sigma_1^2 + 2(1 + \sigma_0^2)\sigma_2^2) \left( \frac{pn}{n} + pn^2 \right) \right). \quad (18c)
\end{aligned}$$

Moreover, if Assumption 6 also holds, then

$$\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \|x_{i,k} - \bar{x}_k\|^2 + f(\bar{x}_k) - f^* \right] = \mathcal{O} (\varepsilon^k + (\sigma_1^2 + 2(1 + \sigma_0^2)\sigma_2^2)pn), \quad \forall k \in \mathbb{N}_+, \quad (19)$$

where $\varepsilon \in (0, 1)$ is a positive constant given in Appendix C.

**Proof:** The proof is given in Appendix C. \hfill ■

If Assumption 4–5 hold, then $\sigma_1 = \sigma_2 = 0$. In this case, from Theorem 6 we have the following results.

**Corollary 1** (Linear convergence). Under the same setup as Theorem 6 and suppose Assump-
tion 4–5 hold, then, for any $T \in \mathbb{N}_+$,

\[
\frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \|x_{i,k} - \bar{x}_k\|^2 \right] = O \left( \frac{1}{T} \right), \quad (20a)
\]

\[
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \|x_{i,k} - \bar{x}_k\|^2 \right] = O(p\eta^2), \quad (20b)
\]

\[
\frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E} \|\nabla f(\bar{x}_k)\|^2 = O \left( \frac{1}{\eta T} \right). \quad (20c)
\]

Moreover, if Assumption 6 also holds, then

\[
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \|x_{i,k} - \bar{x}_k\|^2 + f(\bar{x}_k) - f^* \right] = O(\varepsilon^k), \quad \forall k \in \mathbb{N}_+. \quad (21)
\]

**Remark 5.** The result stated in (20c) shows that a stationary point can be found with a rate $O(p/T)$. This rate is the same as that achieved by the ZO algorithms in [20], [28], [36], [38], [41]. Although the ZO variance reduced algorithms in [37], [42] and the stochastic direct-search algorithms in [11]–[13] achieved a faster rate $O(1/T)$, these algorithms require three or more samplings at each iteration, while our proposed algorithm requires only two samplings.

Moreover, the result stated in (21) shows that a global optimum can be found linearly. The ZO algorithms in [20], [26], [28], [42], [44], [46] and the stochastic direct-search algorithms in [11]–[14] also achieved linear convergence. However, the algorithms in [11]–[14], [26], [42], [44] require three or more samplings at each iteration; the P–Ł constant needs to be known in advance in [28], [42], which is not needed in Theorem 6; and the cost functions in [11]–[14], [20], [26], [44], [46] are (restricted) strongly convex, which is stronger than the P–Ł condition used in Theorem 6.

To end this section, we would like to briefly explain the challenges when analyzing the performance of Algorithm 1. Algorithm 1 is simple in the sense that it is a combination of the first-order algorithm proposed in [63] with zeroth-order gradient estimators. For such a kind of combination, the standard technique to handle the bias in the ZO gradients is using smoothing function, which is also used in our proofs. However, there still is a gap between the smoothing function and the original function. This gap complicates the proof details, especially under the distributed and stochastic setting. As a result, one needs to make an assumption on the relation between the local and global gradients, such as the Lipschitz assumption, i.e., $\|\nabla f_i(x)\|$ is
**Algorithm 2 Distributed ZO Primal Algorithm**

1: **Input**: positive constant $\gamma$ and positive sequences $\{\eta_k\}$ and $\{\delta_{i,k}\}$.
2: **Initialize**: $x_{i,0} \in \mathbb{R}^p$, $\forall i \in [n]$.
3: for $k = 0, 1, \ldots$ do
4:   for $i = 1, \ldots, n$ in parallel do
5:     Broadcast $x_{i,k}$ to $\mathcal{N}_i$ and receive $x_{j,k}$ from $j \in \mathcal{N}_i$;
6:     Generate vector $u_{i,k} \in \mathbb{S}^p$ independently and uniformly at random;
7:     Generate $\xi_{i,k}$ independently and randomly according to the distribution of $\xi_i$;
8:     Sample $F_i(x_{i,k}, \xi_{i,k})$ and $F_i(x_{i,k} + \delta_{i,k} u_{i,k}, \xi_{i,k})$;
9:     Update $x_{i,k+1}$ by (22).
10: end for
11: end for
12: **Output**: $\{x_k\}$.

Globally bounded, the bounded variance assumption, i.e., $\|\nabla f_i(x) - \nabla f(x)\|_2$ is globally bounded, or the weaker Assumption 5 used in this paper. Moreover, to the best of our knowledge, how to show linear speedup for distributed ZO algorithms is an open problem in the literature. A key point to show linear speedup is to guarantee that the omitted constants in the dominate term in the convergence rate do not depend on any parameters related to the communication network. In addition, the proofs are much more complicated due to weaker assumptions.

**IV. DISTRIBUTED ZO PRIMAL ALGORITHM**

In this section, we propose a distributed ZO primal algorithm and analyze its convergence rate. Inspired by distributed first-order (sub)gradient descent algorithm proposed in [76], we propose the following distributed ZO primal algorithm:

$$x_{i,k+1} = x_{i,k} - \gamma \sum_{j=1}^{n} L_{ij} x_{j,k} - \eta_k g_{i,k}^e,$$  \hspace{1cm} (22)

where $\gamma$ is a positive constant, $\{\eta_k\}$ is a positive sequence to be specified later, and $g_{i,k}^e$ is the stochastic gradient estimator defined in (5).

We write the distributed random ZO algorithm (22) in pseudo-code as Algorithm 2. Compared with Algorithm 1 in Algorithm 2 each agent only computes the primal variable. Similar results as stated in Theorems 1-6 and Corollary 1 also hold for Algorithm 2.
A. Find stationary points

**Theorem 7.** Suppose Assumptions 1–5 hold. Let \( \{x_k\} \) be the sequence generated by Algorithm 2 with

\[
\gamma \in (0, d_1), \quad \eta_k = \frac{\kappa_\eta}{(k + t_1)^\theta}, \quad \delta_{i,k} \leq \frac{\kappa_\delta \sqrt{\eta_k}}{\sqrt{n + p}}, \quad \forall k \in \mathbb{N}_0,
\]

(23)

where \( \kappa_\delta > 0, \kappa_\eta \in (0, d_2(\gamma)^{t_1^\theta}), \theta \in (0.5, 1), \) and \( t_1 \geq \frac{p^{1/2}}{1} \) with \( d_1 \) and \( d_2(\gamma) \) being given in Appendix H. Then, for any \( T \in \mathbb{N}_+ \),

\[
\frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E}[\|\nabla f(\bar{x}_k)\|^2] = \mathcal{O}\left(\frac{\sqrt{p}}{T^{1-\theta}} + \frac{p}{T}\right),
\]

(24a)

\[
\mathbb{E}[f(\bar{x}_T)] - f^* = \mathcal{O}(1),
\]

(24b)

\[
\mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} \|x_{i,T} - \bar{x}_T\|^2\right] = \mathcal{O}\left(\frac{1}{T^{2\theta}}\right),
\]

(24c)

\[
\lim_{T \to +\infty} \mathbb{E}[\|\nabla f(\bar{x}_T)\|^2] = 0.
\]

(24d)

**Proof:** The proof is given in Appendix H. \[\blacksquare\]

**Theorem 8** (Linear speedup). Suppose Assumptions 1–5 hold. For any given \( T \geq \max\{\frac{n\eta}{p\delta(\gamma)}, \frac{n^3}{p}\} \), let \( \{x_k, k = 0, \ldots, T\} \) be the output generated by Algorithm 2 with

\[
\gamma \in (0, d_1), \quad \eta_k = \frac{\sqrt{n}}{\sqrt{p}T}, \quad \delta_{i,k} \leq \frac{p^{1/4} n^{1/4} \kappa_\delta}{\sqrt{n + p(k + 1)^2}}, \quad \forall k \leq T,
\]

(25)

where \( \kappa_\delta > 0, \) and \( d_1 \) as well as \( d_2(\gamma) \) are given in Appendix H then

\[
\frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E}[\|\nabla f(\bar{x}_k)\|^2] = \mathcal{O}\left(\frac{\sqrt{p}}{\sqrt{nT}}\right) + \mathcal{O}\left(\frac{n}{T}\right),
\]

(26a)

\[
\mathbb{E}[f(\bar{x}_T)] - f^* = \mathcal{O}(1),
\]

(26b)

\[
\mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} \|x_{i,T} - \bar{x}_T\|^2\right] = \mathcal{O}\left(\frac{n}{T}\right),
\]

(26c)

\[
\lim_{T \to +\infty} \mathbb{E}[\|\nabla f(\bar{x}_T)\|^2] = 0.
\]

(26d)

**Proof:** The proof is given in Appendix I. It should be highlighted that the omitted constants in the first term on the right-hand side of (26a) do not depend on any parameters related to the communication network. \[\blacksquare\]
B. Find global optimum

Theorem 9. Suppose Assumptions 1–6 hold. Let \( \{x_k\} \) be the sequence generated by Algorithm 2 with

\[
\gamma \in (0, d_1), \quad \eta_k = \frac{\kappa_\eta}{(k + t_1)^	heta}, \quad \delta_{i,k} \leq \frac{\kappa_\delta \sqrt{n\eta_k}}{\sqrt{n + p}}, \quad \forall k \in \mathbb{N}_0, \tag{27}
\]

where \( \kappa_\delta > 0, \kappa_\eta \in (0, d_2(\gamma)t_1^\theta], \theta \in (0, 1), t_1 \in [p^{1/2}, d_3p^{3/2}], \) and \( d_3 \geq 1 \) with \( d_1 \) and \( d_2(\gamma) \) being given in Appendix H. Then, for any \( T \in \mathbb{N}_+ \),

\[
\mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \|x_{i,T} - \bar{x}_T\|^2 \right] = \mathcal{O}\left( \frac{p}{T^{2\theta}} \right), \tag{28a}
\]

\[
\mathbb{E}[f(\bar{x}_T) - f^*] = \mathcal{O}\left( \frac{p}{nT^{\theta}} \right) + \mathcal{O}\left( \frac{p}{T^{2\theta}} \right). \tag{28b}
\]

Proof: The proof is given in Appendix J. It should be highlighted that the omitted constants in the first term on the right-hand side of (28b) do not depend on any parameters related to the communication network.

Theorem 10 (Linear speedup). Suppose Assumptions 1–6 hold and the P–Ł constant \( \nu \) is known in advance. Let \( \{x_k\} \) be the sequence generated by Algorithm 2 with

\[
\gamma \in (0, d_1), \quad \eta_k = \frac{\kappa_\eta}{(k + t_1)^	heta}, \quad \delta_{i,k} \leq \frac{\kappa_\delta \sqrt{n\eta_k}}{\sqrt{n + p}}, \quad \forall k \in \mathbb{N}_0, \tag{29}
\]

where \( \kappa_\delta > 0, \kappa_\eta \in (\frac{\delta}{p\nu}, \frac{8d_3}{\nu}], t_1 > \hat{d}_2(\gamma) \) and \( d_3 > 1 \) with \( d_1 \) and \( \hat{d}_2(\gamma) \) being given in Appendices H and K respectively. Then, for any \( T \in \mathbb{N}_+ \),

\[
\mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \|x_{i,T} - \bar{x}_T\|^2 \right] = \mathcal{O}\left( \frac{p}{T^2} \right), \tag{30a}
\]

\[
\mathbb{E}[f(\bar{x}_T) - f^*] = \mathcal{O}\left( \frac{p}{nT^{\theta}} \right) + \mathcal{O}\left( \frac{p}{T^2} \right). \tag{30b}
\]

Proof: The proof is given in Appendix K. It should be highlighted that the omitted constants in the first term on the right-hand side of (30b) do not depend on any parameters related to the communication network.

Theorem 11 (Linear speedup). Suppose Assumptions 1–4 and 6 hold, and the P–Ł constant \( \nu \) is known in advance, and each \( f_i^* > -\infty \). Let \( \{x_k\} \) be the sequence generated by Algorithm 2
where $\gamma \in (0, d_1)$, $\eta_k = \frac{\kappa_\eta}{k + t_1}$, $\delta_{i,k} \leq \frac{\kappa_\delta \sqrt{\eta k}}{\sqrt{n + p}}$, $\forall k \in \mathbb{N}_0$, (31)

with

$$
\gamma \in (0, d_1), \quad \eta_k = \frac{\kappa_\eta}{k + t_1}, \quad \delta_{i,k} \leq \frac{\kappa_\delta \sqrt{\eta k}}{\sqrt{n + p}}, \quad \forall k \in \mathbb{N}_0,
$$

where $\kappa_\delta > 0$, $\kappa_\eta \in \left( \frac{8}{\nu}, \frac{8d_3}{\nu} \right]$, $t_1 > \tilde{d}_2(\gamma)$, and $d_3 > 1$ with $d_1$ and $\tilde{d}_2(\gamma)$ being given in Appendices $H$ and $L$, respectively. Then, for any $T \in \mathbb{N}_+$,

$$
\mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \|x_{i,T} - \bar{x}_T\|^2 \right] = O\left( \frac{p}{T^2} \right), \quad (32a)
$$

$$
\mathbb{E}[f(\bar{x}_T) - f^*] = O\left( \frac{p}{nT} \right) + O\left( \frac{p}{T^2} \right). \quad (32b)
$$

**Proof:** The proof is given in Appendix $L$. It should be highlighted that the omitted constants in the first term on the right-hand side of (32b) do not depend on any parameters related to the communication network.

**Theorem 12.** Suppose Assumptions $[H,L]$ hold. Let $\{x_k\}$ be the sequence generated by Algorithm $2$ with

$$
\gamma \in (0, d_1), \quad \eta_k = \eta, \quad \delta_{i,k} \leq \kappa_\delta \tilde{c}_k, \quad \forall k \in \mathbb{N}_0,
$$

where $\eta \in (0, d_2(\gamma))$ and $\tilde{c} \in (0, 1)$ with $d_1$ and $d_2(\gamma)$ being given in Appendix $H$. Then, for any $T \in \mathbb{N}_+$,

$$
\mathbb{E}\left[ \frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \|x_{i,k} - \bar{x}_k\|^2 \right] \right] = O\left( \frac{1}{T^2} + (\sigma_1^2 + 2(1 + \sigma_0^2)\sigma_2^2)p\eta^2 \right), \quad (34a)
$$

$$
\mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \|x_{i,T} - \bar{x}_T\|^2 \right] = O\left( p\eta^2 + (\sigma_1^2 + 2(1 + \sigma_0^2)\sigma_2^2)p^2\eta^4 \left( \frac{1}{n} + \eta \right) T \right), \quad (34b)
$$

$$
\mathbb{E}\left[ \frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E}[\|\nabla f(\bar{x}_k)\|^2] \right] = O\left( \frac{1}{\eta T} + (\sigma_1^2 + 2(1 + \sigma_0^2)\sigma_2^2) \left( \frac{pm}{n} + pn^2 \right) \right). \quad (34c)
$$

Moreover, if Assumption $[L]$ also holds, then

$$
\mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \|x_{i,k} - \bar{x}_k\|^2 + f(\bar{x}_k) - f^* \right] = O\left( \epsilon^k + (\sigma_1^2 + 2(1 + \sigma_0^2)\sigma_2^2)p\eta \right), \quad \forall k \in \mathbb{N}_+, \quad (35)
$$

where $\epsilon \in (0, 1)$ is a positive constant given in Appendix $M$.

**Proof:** The proof is given in Appendix $M$.

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Corollary 2 (Linear convergence). Under the same setup as Theorem 12 and suppose Assumption 4′–5′ hold, then, for any $T \in \mathbb{N}_+$,

\[
\frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \|x_{i,k} - \bar{x}_k\|^2 \right] = O\left( \frac{1}{T} \right), \tag{36a}
\]

\[
\mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \|x_{i,k} - \bar{x}_k\|^2 \right] = O(\eta k^2), \tag{36b}
\]

\[
\frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E}[\|\nabla f(\bar{x}_k)\|^2] = O\left( \frac{1}{\eta T} \right). \tag{36c}
\]

Moreover, if Assumption 6 also holds, then

\[
\mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \|x_{i,k} - \bar{x}_k\|^2 + f(\bar{x}_k) - f^* \right] = O(\epsilon_k), \quad \forall k \in \mathbb{N}_+. \tag{37}
\]

V. Simulations

In this section, we verify the theoretical results through numerical simulations. Specifically, we evaluate the performance of Algorithms 1 and 2 in generating adversarial examples from black-box deep neural networks (DNNs).

In image classification tasks, DNNs are vulnerable to adversarial examples [77] even under small perturbations, which leads to misclassifications. Considering the setting of ZO attacks in [38], [78], the model is hidden and no gradient information is available. We treat this task of generating adversarial examples as a ZO optimization problem. The black-box attack loss function [38], [78] is given as

\[
f_i(x) = \max \left\{ F_{y_i} \left( \frac{1}{2} \tanh(\tanh^{-1} 2a_i + x) \right) - \max_{j \neq y_i} \left\{ F_j \left( \frac{1}{2} \tanh(\tanh^{-1} 2a_i + x) \right) \right\}, 0 \right\}
\]

\[
+ c \left\| \frac{1}{2} \tanh(\tanh^{-1} 2a_i + x) - a_i \right\|_2^2,
\]

where $c$ is a constant, $(a_i, y_i)$ denotes the pair of the $i$th natural image $a_i$ and its original class label $y_i$. The output of function $F(z) = \text{col}(F_1(z), \ldots, F_m(z))$ is the well-trained model prediction of the input $z$ in all $m$ image classes.

The well-trained DNN model[7] on the MNIST handwritten dataset has 99.4% test accuracy on natural examples [38]. We compare the proposed distributed primal–dual ZO algorithm

[https://github.com/carlini/nn_robust_attacks](https://github.com/carlini/nn_robust_attacks)
(Algorithm 1) and distributed primal ZO algorithm (Algorithm 2) with state-of-the-art centralized and distributed ZO algorithms: RSGF [22], SZO-SPIDER [37], ZO-SVRG [38], SZVR-G [40], and ZO-SPIDER-Coord [42], ZO-GDA [58], and ZONE-M [62].

We consider $n = 10$ agents and assume the communication network is generated randomly following the Erdős–Rényi model with probability of 0.4. All the hyper-parameters used in the experiment are given in TABLE II.

| Algorithm                  | Distributed | Parameters                              |
|----------------------------|-------------|-----------------------------------------|
| Algorithm 1                | ✓           | $\eta = 0.5/k^{10^{-5}}$, $\alpha = 0.5k^{10^{-5}}$, $\beta = 0.1k^{10^{-5}}$ |
| Algorithm 2                | ✓           | $\gamma = 0.01$, $\eta = 0.08/k^{10^{-5}}$ |
| ZO-GDA                     | ✓           | $\eta = 0.08/k^{10^{-5}}$                |
| ZONE-M                     | ✓           | $\mu = 1\sqrt{k}$, $\rho = 0.4\sqrt{k}$ |
| RSGF                       | ×           | $\mu = 0.01$                            |
| SZO-SPIDER                 | ×           | $\mu = 0.01$                            |
| ZO-SVRG                    | ×           | $\mu = 0.01$                            |
| SZVR-G                     | ×           | $\mu = 0.01$                            |
| ZO-SPIDER-Coord            | ×           | $\mu = 0.01$                            |

Fig. 1 and Fig. 2 show the evolutions of the black-box attack loss achieved by each ZO algorithm with respect to the number of iterations and function value queries, respectively. From these two figures, we can see that our proposed distributed ZO algorithms are as efficient as ZO-GDA [58] in terms of both convergence rate and sampling complexity, and more efficient than the other algorithms. The least $\ell_2$ distortions of the successful adversarial perturbations are listed in TABLE III. We can see that the adversarial examples generated by the distributed algorithms in general have slightly larger $\ell_2$ distortions than those generated by the centralized algorithms. TABLE IV provides a comparison of generated adversarial examples from the DNN on the MNIST dataset: digit class “4”.

In order to verify the result that linear speedup convergence is achieved with respect to the number of agents, we also consider $n = 100$ agents. To illustrate the linear speedup results in a more clear manner, we plot the loss in log scale and draw the extensive lines along the convergence lines in Fig. 3. The slopes of 10-node lines (blue and red lines) are approximately $-0.025$ and the slopes of 100-node lines (blue and red dash lines) are approximately $-0.079$. 
Fig. 1: Evolutions of the black-box attack loss with respect to the number of iterations.

| Algorithm          | $\ell_2$ distortion |
|--------------------|----------------------|
| Algorithm 1        | 6.44                 |
| Algorithm 2        | 5.77                 |
| ZO-GDA             | 7.23                 |
| ZONE-M             | 9.96                 |
| RSGF               | 5.69                 |
| SZO-SPIDER         | 6.19                 |
| ZO-SVRG            | 4.76                 |
| SZVR-G             | 5.16                 |
| ZO-SPIDER-Coord    | 5.76                 |
which implies the linear speedup results since $-0.079/\sqrt{10} \approx -0.025$. This simulation shows that linear speedup is achieved by our proposed two algorithms even though the optimization problem is nonsmooth.

VI. Conclusions

In this paper, we studied stochastic distributed nonconvex optimization with ZO information feedback. We proposed two distributed ZO algorithms and analyzed their convergence properties. More specifically, linear speedup convergence rate $O(\sqrt{p/(nT)})$ was established for smooth nonconvex cost functions under arbitrarily connected communication networks. The convergence rate was improved to $O(p/(nT))$ when the global cost function satisfies the P–Ł condition. It was also shown that the output of the proposed algorithms linearly converges to a neighborhood of
TABLE IV: Comparison of generated adversarial examples from a black-box DNN on MNIST: digit class “4”.

| Image ID | Original | Algorithm 1 | Algorithm 2 | ZO-GDA | ZONE-M | RSGF | SZO-SPIDER | ZO-SVRG | SZVR-G | ZO-SPIDER-Coord |
|----------|----------|-------------|-------------|--------|--------|------|------------|---------|--------|----------------|
|          | 4 4 4 4 | 9 8 2 7 2 9 9 9 9 | 9 9 7 9 9 9 9 9 9 | 9 9 2 9 9 9 9 9 9 | 9 9 7 9 9 9 9 9 9 | 9 9 2 9 9 9 9 9 9 | 9 9 7 9 9 9 9 9 9 | 9 8 2 9 9 9 9 9 9 | 9 8 2 9 9 9 9 9 9 | 9 9 2 9 9 9 9 9 9 |

a global optimum. Interesting directions for future work include establishing faster convergence with reduced sampling complexity by using variance reduction techniques, and considering communication reduction with asynchronous, periodic, or compressed communication.

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Fig. 3: Evolutions of the black-box attack loss with respect to the number of iterations when using different numbers of agents.

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APPENDIX

A. Notations, Definitions, and Useful Lemmas

1) Notations: \(1_n (0_n)\) denotes the column one (zero) vector of dimension \(n\). \(\text{col}(z_1, \ldots, z_k)\) is the concatenated column vector of vectors \(z_i \in \mathbb{R}^{p_i}, \ i \in [k]\). \(I_n\) is the \(n\)-dimensional identity matrix. Given a vector \([x_1, \ldots, x_n]^\top \in \mathbb{R}^n\), \(\text{diag}(x_1, \ldots, x_n)\) is a diagonal matrix with the \(i\)-th diagonal element being \(x_i\). The notation \(A \otimes B\) denotes the Kronecker product of matrices \(A\) and \(B\). \(\text{null}(A)\) is the null space of matrix \(A\). Given two symmetric matrices \(M, N\), \(M \succeq N\) means that \(M - N\) is positive semi-definite. \(\rho(\cdot)\) stands for the spectral radius for matrices and \(\rho_2(\cdot)\) indicates the minimum positive eigenvalue for matrices having positive eigenvalues. For
any square matrix $A$, $\|x\|_A^2$ denotes $x^T A x$. $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ denote the ceiling and floor functions, respectively. For any $x \in \mathbb{R}$, $[x]_+$ is the positive part of $x$. $1_\{\cdot\}$ is the indicator function.

2) Graph Theory: For an undirected graph $G = (\mathcal{V}, \mathcal{E})$, let $\mathcal{A} = (a_{ij})$ be the associated weighted adjacency matrix with $a_{ij} > 0$ if $(i, j) \in \mathcal{E}$ if $a_{ij} > 0$ and zero otherwise. It is assumed that $a_{ii} = 0$ for all $i \in [n]$. Let $\deg_i = \sum_{j=1}^{n} a_{ij}$ denotes the weighted degree of vertex $i$. The degree matrix of graph $G$ is $\text{Deg} = \text{diag}(\deg_1, \ldots, \deg_n)$. The Laplacian matrix is $L = (L_{ij}) = \text{Deg} - \mathcal{A}$. A path of length $k$ between vertices $i$ and $j$ is a subgraph with distinct vertices $i_0 = i, \ldots, i_k = j \in [n]$ and edges $(i_j, i_{j+1}) \in \mathcal{E}$, $j = 0, \ldots, k-1$. An undirected graph is connected if there exists at least one path between any two distinct vertices.

For a connected undirected graph, we have the following results.

**Lemma 1.** (Lemmas 1 and 2 in [79]) Let $L$ be the Laplacian matrix of the connected graph $G$ and $K_n = I_n - \frac{1}{\sqrt{n}} 1_n 1_n^T$. Then $L$ and $K_n$ are positive semi-definite, $\text{null}(L) = \text{null}(K_n) = \{1_n\}$, $L \leq \rho(L) I_n$, $\rho(K_n) = 1$,

\begin{align*}
K_n L &= L K_n = L, \\
0 \leq \rho_2(L) K_n &\leq L \leq \rho(L) K_n.
\end{align*}

Moreover, there exists an orthogonal matrix $[r \ R] \in \mathbb{R}^{n \times n}$ with $r = \frac{1}{\sqrt{n}} 1_n$ and $R \in \mathbb{R}^{n \times (n-1)}$ such that

\begin{align*}
R \Lambda_1^{-1} R^T L &= L R \Lambda_1^{-1} R^T = K_n, \\
\frac{1}{\rho(L)} K_n &\leq R \Lambda_1^{-1} R^T \leq \frac{1}{\rho_2(L)} K_n,
\end{align*}

where $\Lambda_1 = \text{diag}(\lambda_2, \ldots, \lambda_n)$ with $0 < \lambda_2 \leq \cdots \leq \lambda_n$ being the eigenvalues of the Laplacian matrix $L$.

3) Smooth Functions:

**Definition 2.** [80] A function $f(x) : \mathbb{R}^p \mapsto \mathbb{R}$ is smooth with constant $L_f > 0$ if it is differentiable and

\[ \| \nabla f(x) - \nabla f(y) \| \leq L_f \| x - y \|, \quad \forall x, y \in \mathbb{R}^p. \]
From Lemma 1.2.3 in [80], we know that (40) implies

\[ |f(y) - f(x) - (y - x)\top \nabla f(x)| \leq \frac{L_f}{2} \|y - x\|_2^2, \quad \forall x, y \in \mathbb{R}^p, \]  

which further implies

\[ \|\nabla f(x)\|_2^2 \leq 2L_f(f(x) - f^*), \quad \forall x, y \in \mathbb{R}^p, \]  

where \( f^* = \min_{x \in \mathbb{R}^p} f(x) \).

4) Properties of Gradient Approximation: The random gradient estimator \( \hat{\nabla}^2 f \) defined in (3) is an unbiased gradient estimator of \( f^s \), where \( f^s \) is the uniformly smoothed version of \( f \) defined as

\[ f^s(x, \delta) = \mathbb{E}_{u \in \mathcal{B}^p}[f(x + \delta u)], \]  

with the expectation is taken with respect to uniform distribution.

From Lemma 2 in [81], Lemma 5 in [58], and Proposition 7.6 in [34], we have the following properties of \( f^s \) and \( \hat{\nabla}^2 f \).

**Lemma 2.** (i) The uniform smoothing \( f^s(x, \delta) \) is differentiable with respect to \( x \), and

\[ \nabla f^s(x, \delta) = \mathbb{E}_{u \in \mathcal{S}^p}[\hat{\nabla}^2 f(x, \delta, u)]. \]  

(ii) If \( f \) is smooth with constant \( L_f > 0 \), then

\[ \|\nabla f^s(x, \delta) - \nabla f(x)\| \leq \delta L_f, \]  

\[ \mathbb{E}_{u \in \mathcal{S}^p}[\|\hat{\nabla}^2 f(x, \delta, u)\|_2^2] \leq 2p\|\nabla f(x)\|_2^2 + \frac{1}{2}p^2\delta^2 L_f^2. \]

5) Useful Lemmas on Series:

**Lemma 3.** Let \( a, b \in (0, 1) \) be two constants, then

\[ \sum_{\tau=0}^{k} a^\tau b^{k-\tau} \leq \begin{cases} 
\frac{a^{k+1}}{a-b}, & \text{if } a > b \\
\frac{b^{k+1}}{b-a}, & \text{if } a < b \\
\frac{c^{k+1}}{c-b}, & \text{if } a = b,
\end{cases} \]  

where \( c \) is any constant in \((a, 1)\).
Proof: If $a > b$, then
\[
\sum_{\tau=0}^{k} a^{\tau} b^{k-\tau} = a^{k} \sum_{\tau=0}^{k} \left( \frac{b}{a} \right)^{k-\tau} \leq \frac{a^{k+1}}{a-b}.
\]
Similarly, when $a < b$, we have
\[
\sum_{\tau=0}^{k} a^{\tau} b^{k-\tau} = b^{k} \sum_{\tau=0}^{k} \left( \frac{a}{b} \right)^{\tau} \leq \frac{b^{k+1}}{b-a}.
\]
If $a = b$, then for any $c \in (a, 1)$, we have
\[
\sum_{\tau=0}^{k} a^{\tau} b^{k-\tau} \leq \sum_{\tau=0}^{k} c^{\tau} b^{k-\tau} = c^{k} \sum_{\tau=0}^{k} \left( \frac{b}{c} \right)^{k-\tau} \leq \frac{c^{k+1}}{c-b}.
\]
Hence, this lemma holds.

Lemma 4. Let $k$ and $\tau$ be two integers and $\delta$ be a constant. Suppose $k \geq \tau \geq 1$, then
\[
\sum_{l=\tau}^{k} l^{\delta} \leq \begin{cases} 
\frac{(k+1)^{\delta+1}}{\delta+1}, & \text{if } \delta > -1 \\
\ln(k), & \text{if } \delta = -1 \\
\frac{-(\tau-1)^{\delta+1}}{\delta+1}, & \text{if } \delta < -1 \text{ and } \tau \geq 2.
\end{cases}
\] (47)

Proof: If $\delta \geq 0$, then $h(t) = t^{\delta}$ is an increasing function in the interval $[1, +\infty)$. Hence,
\[
\sum_{l=\tau}^{k} l^{\delta} \leq \int_{\tau}^{k+1} t^{\delta} dt = \frac{(k+1)^{\delta+1} - \tau^{\delta+1}}{\delta+1} \leq \frac{(k+1)^{\delta+1}}{\delta+1}.
\] (48)

If $\delta < 0$, then $h(t) = t^{\delta}$ is a decreasing function in the interval $[1, +\infty)$. Hence,
\[
\sum_{l=\tau}^{k} l^{\delta} \leq \int_{\tau-1}^{k} t^{\delta} dt = \begin{cases} 
\ln\left( \frac{k}{\tau-1} \right), & \text{if } \delta = -1, \\
\frac{k^{\delta+1} - (\tau-1)^{\delta+1}}{\delta+1}, & \text{if } -1 < \delta < 0, \\
\frac{k^{\delta+1} -(\tau-1)^{\delta+1}}{\delta+1}, & \text{if } \delta < -1 \text{ and } \tau \geq 2,
\end{cases}
\] (49)

Finally, (48) and (49) yield (47).
Lemma 5. Let \( \{z_k\}, \{r_{1,k}\}, \text{ and } \{r_{2,k}\} \) be sequences. Suppose there exists \( t_1 \in \mathbb{N}_+ \) such that

\[
z_k \geq 0, \tag{50a}
\]

\[
z_{k+1} \leq (1 - r_{1,k})z_k + r_{2,k}, \tag{50b}
\]

\[
1 > r_{1,k} \geq \frac{a_1}{(k + t_1)^{\delta_1}}, \tag{50c}
\]

\[
r_{2,k} \leq \frac{a_2}{(k + t_1)^{\delta_2}}, \quad \forall k \in \mathbb{N}_0, \tag{50d}
\]

where \( a_1 > 0, a_2 > 0, \delta_1 \in [0, 1], \) and \( \delta_2 > \delta_1 \) are constants.

(i) If \( \delta_1 \in (0, 1) \), then

\[
z_k \leq \phi_1(k, t_1, a_1, a_2, \delta_1, \delta_2, z_0), \quad \forall k \in \mathbb{N}_+, \tag{51}
\]

where

\[
\phi_1(k, t_1, a_1, a_2, \delta_1, \delta_2, z_0) = \frac{1}{s_1(k + t_1)} \left( s_1(t_1)z_0 + \frac{[t_2 - 1 - t_1] + s_1(t_1 + 1)a_2}{t_1^{\delta_2}} + \frac{a_2}{(k + t_1 - 1)^{\delta_2}} + \frac{1_{(k+t_1-1\geq t_2)}(t_1+1)^{\delta_2}a_2\delta_2}{a_1\delta_1(k + t_1)^{\delta_2-\delta_1}} \right), \tag{52}
\]

\[
s_1(k) = e^{\frac{a_1}{a_1}k^{1-\delta_1}} \quad \text{and} \quad t_2 = \left\lceil \left(\frac{\delta_2}{a_1}\right)^{1-\delta_1} \right\rceil.
\]

(ii) If \( \delta_1 = 1 \), then

\[
z_k \leq \phi_2(k, t_1, a_1, a_2, \delta_2, z_0), \quad \forall k \in \mathbb{N}_+, \tag{53}
\]

where

\[
\phi_2(k, t_1, a_1, a_2, \delta_2, z_0) = \frac{t_1^{a_1}z_0}{(k + t_1)^{a_1}} + \frac{a_2}{(k + t_1 - 1)^{\delta_2}} + \left(\frac{t_1 + 1}{t_1}\right)^{\delta_2}a_2s_2(k + t_1), \tag{54}
\]

and

\[
s_2(k) = \begin{cases} 
\frac{1}{(a_1 - \delta_2 + 1)^{k^{\delta_2}}} & \text{if } a_1 - \delta_2 > -1, \\
\ln(k-1) & \text{if } a_1 - \delta_2 = -1, \\
\frac{-t_1^{a_1-\delta_2+1}}{(a_1 - \delta_2 + 1)^{a_1}} & \text{if } a_1 - \delta_2 < -1.
\end{cases}
\]

(iii) If \( \delta_1 = 0 \), then

\[
z_k \leq \phi_3(k, t_1, a_1, a_2, \delta_2, z_0), \quad \forall k \in \mathbb{N}_+, \tag{55}
\]
where

$$
\phi_3(k, t_1, a_1, a_2, \delta_2, z_0) = (1 - a_1)^k z_0 + a_2(1 - a_1)^{k+t_1-1}([t_3 - t_1]_+ s_3(t_1)
+ ([t_4 - t_1]_+ - [t_3 - t_1]_+) s_3(t_4))
+ \frac{1_{(k+t_1-1\geq t_0)}}{2a_2} \ln(1 - a_1)(k + t_1)^{\delta_2}(1 - a_1),
$$

(56)

$$
s_3(k) = \frac{1}{k^{\delta_2(1-a_1)}} t_3 = \lceil \frac{-\delta_2}{\ln(1-a_1)} \rceil, \text{ and } t_4 = \lceil \frac{-2\delta_2}{\ln(1-a_1)} \rceil.
$$

**Proof:** This proof is inspired by the proof of Lemma 25 in [82].

From (50a)–(50c), for any $k \in \mathbb{N}_+$, it holds that

$$
z_k \leq \prod_{\tau=0}^{k-1} (1 - r_{1,\tau}) z_0 + r_{2, k-1} + \sum_{l=0}^{k-2} \prod_{\tau=l+1}^{k-1} (1 - r_{1,\tau}) r_{2, l}.
$$

(57)

For any $t \in [0, 1]$, it holds that $1 - t \leq e^{-t}$ since $s_4(t) = 1 - t - e^{-t}$ is a non-increasing function in the interval $[0, 1]$ and $s_4(0) = 0$. Thus, for any $k > l \geq 0$, it holds that

$$
\prod_{\tau=l}^{k-1} (1 - r_{1,\tau}) \leq e^{-\sum_{\tau=l}^{k-1} r_{1,\tau}}.
$$

(58)

We also have

$$
\sum_{\tau=l}^{k-1} r_{1,\tau} \geq \sum_{\tau=l}^{k-1} \frac{a_1}{(\tau + t_1)^{\delta_1}} = \sum_{\tau=l+t_1}^{k-1} \frac{a_1}{\tau^{\delta_1}} \geq \int_{t=t_1}^{k+t_1} \frac{a_1}{\tau^{\delta_1}} d\tau
$$

$$
= \begin{cases} 
\frac{a_1}{\ln(1+t_1)} ((k+t_1)^{1-\delta_1} - (l+t_1)^{1-\delta_1}), & \text{if } \delta_1 \in (0, 1), \\
\frac{a_1}{\ln(1+t_1)} - a_1 \ln(\frac{k+t_1}{l+t_1}), & \text{if } \delta_1 = 1,
\end{cases}
$$

(59)

where the first inequality holds due to (50c) and the second inequality holds since $s_5(t) = \frac{a_1}{t^{\delta_1}}$ is a decreasing function in the interval $[1, +\infty)$.

Hence, (58) and (59) yield

$$
\prod_{\tau=l}^{k-1} (1 - r_{1,\tau}) \leq e^{-\sum_{\tau=l}^{k-1} r_{1,\tau}} \leq \begin{cases} 
\frac{s_1(l+t_1)}{s_1(k+t_1)}, & \text{if } \delta_1 \in (0, 1), \\
\frac{(l+t_1)^{\delta_1}}{(k+t_1)^{\delta_1}}, & \text{if } \delta_1 = 1.
\end{cases}
$$

(60)

(i) When $\delta_1 \in (0, 1)$, from (60) and (50d), we have

$$
\sum_{l=0}^{k-2} \prod_{\tau=l+1}^{k-1} (1 - r_{1,\tau}) r_{2, l} \leq \sum_{l=0}^{k-2} \frac{s_1(l+t_1 + 1)}{s_1(k+t_1)} \frac{a_2}{(l+t_1)^{\delta_2}}
$$
Thus, for any $k \in [1, t_2 - 1]$, we have

$$
\sum_{l=1}^{t_2-1} \frac{s_1(l)}{t^{\delta_2}} \leq (t_2 - k) \frac{s_1(k)}{k^{\delta_2}}.
$$

Noting that $s_0(t) = \frac{s_1(t)}{t^{\delta_2}}$ is an increasing function in the interval $[t_2, +\infty)$, for any $k \geq t_2$, we have

$$
\sum_{l=k}^{k} \frac{s_1(l)}{t^{\delta_2}} \leq \int_{t_2}^{k+1} \frac{s_1(t)}{t^{\delta_2}} dt.
$$

We have

$$
\int_{t_2}^{k+1} \frac{s_1(t)}{t^{\delta_2}} dt = \int_{t_2}^{k+1} \frac{1}{a_1 t^{\delta_2 - \delta_1}} ds_1(t)
$$

$$
= \frac{s_1(k+1)}{a_1 (k+1)^{\delta_2 - \delta_1}} - \frac{s_1(t_2)}{a_1 t_2^{\delta_2 - \delta_1}} + \int_{t_2}^{k+1} \frac{(\delta_2 - \delta_1) s_1(t)}{a_1 t^{\delta_2 - \delta_1 + 1}} dt
$$

$$
\leq \frac{s_1(k+1)}{a_1 (k+1)^{\delta_2 - \delta_1}} + \int_{t_2}^{k+1} \frac{(\delta_2 - \delta_1) s_1(t)}{a_1 t^{\delta_2 - \delta_1 + 1}} dt
$$

$$
\leq \frac{s_1(k+1)}{a_1 (k+1)^{\delta_2 - \delta_1}} + \frac{\delta_2 - \delta_1}{a_1 t_2^{\delta_2 - \delta_1}} \int_{t_2}^{k+1} \frac{s_1(t)}{t^{\delta_2}} dt
$$

$$
\leq \frac{s_1(k+1)}{a_1 (k+1)^{\delta_2 - \delta_1}} + \frac{\delta_2 - \delta_1}{\delta_2} \int_{t_2}^{k+1} \frac{s_1(t)}{t^{\delta_2}} dt.
$$
where the second inequality holds since $s_2(t) = \frac{1}{t^{1-a_1}}$, $s_1(t) = 1 - t^{1-\delta_1}$ is a decreasing function in the interval $[1, +\infty)$; and the last inequality holds due to $t_2^{1-\delta_1} \geq \frac{\delta_2}{a_1}$.

From (63) and (64), for any $k \geq t_2$, we have

$$\sum_{l=t_2}^{k} \frac{s_1(l)}{t^{\delta_2}} \leq \int_{t_2}^{k+1} \frac{s_1(t)}{t^{\delta_2}} \, dt \leq -\frac{\delta_2 s_1(k + 1)}{a_1 \delta_1 (k + 1)^{\delta_2 - \delta_1}}. \quad (65)$$

From (61), (62), and (65), we have

$$\sum_{l=0}^{k-2} \sum_{\tau=l+1}^{k-1} (1 - r_{1,\tau}) r_{2,l} \leq \frac{(t_1 + 1)^{\delta_2} a_2}{s_1(k + t_1)} \left( t_2 - 1 + t_1 \right) s_1(t_1 + 1) \left( t_1 + 1 \right)^{\delta_2} + 1_{(k + t_1 - 1 \geq t_2)} \left( \frac{\delta_2 s_1(k + t_1)}{a_1 \delta_1 (k + t_1)^{\delta_2 - \delta_1}} \right). \quad (66)$$

Then, (67), (60), and (66) yield (51).

(ii) When $\delta_1 = 1$, from (60) and (50d), we have

$$\sum_{l=0}^{k-2} \sum_{\tau=l+1}^{k-1} (1 - r_{1,\tau}) r_{2,l} \leq \sum_{l=0}^{k-2} \frac{(l + t_1 + 1)^{a_1}}{(k + t_1)^{a_1}} \frac{a_2}{(l + t_1)^{\delta_2}} \leq \sum_{l=0}^{k-2} \frac{(l + t_1 + 1)^{a_1}}{(k + t_1)^{a_1}} \frac{a_2}{(t_1 + 1)^{\delta_2}} = \frac{(t_1 + 1)^{\delta_2} a_2}{s_1(k + t_1)} \sum_{l=0}^{k-2} (l + t_1 + 1)^{a_1} \left( t_1 + 1 \right)^{\delta_2} \leq a_2 \left( t_1 + 1 \right)^{a_1 - \delta_2}, \quad (67)$$

where the first inequality holds due to (60) and (50d).

From (57), (60), (67), and (47), we have (53).

(iii) Denote $a = 1 - a_1$. From (50c) and $\delta_1 = 0$, we know that $a_1 \in (0, 1)$. Thus, $a \in (0, 1)$.

From (50a)–(50d) and $\delta_1 = 0$, for any $k \in \mathbb{N}_+$, it holds that

$$z_k \leq (1 - a_1)^k z_0 + \sum_{\tau=0}^{k-1} (1 - a_1)^{k-1-\tau} r_{2,\tau},$$

$$\leq a^k z_0 + a_2 a^{k+t_1-1} \sum_{\tau=0}^{k-1} \frac{1}{(\tau + t_1)^{\delta_2 a^{\tau+t_1}}}. \quad (68)$$
We have
\[ \sum_{\tau=0}^{k-1} \frac{1}{(\tau + t_1)^{\delta_2} a^{\tau + t_1}} = \sum_{\tau=t_1}^{k-t_1} \frac{1}{\tau^{\delta_2} a^{\tau}} = \sum_{\tau=t_1}^{t_3-1} s_3(\tau) + \sum_{\tau=t_3}^{t_4-1} s_3(\tau) + \sum_{\tau=t_4}^{k+t_1-1} s_3(\tau). \] (69)

We know that \( s_3(t) = \frac{1}{t^{\delta_2} a^t} \) is decreasing and increasing in the intervals \([1, t_3 - 1]\) and \([t_3, +\infty)\), respectively, since
\[
\frac{ds_3(t)}{dt} = -s_3(t) \left( \frac{\delta_2}{t} + \ln(a) \right) \leq 0, \quad \forall t \in \left(0, \frac{-\delta_2}{\ln(a)}\right];
\]
\[
\frac{ds_3(t)}{dt} = -s_3(t) \left( \frac{\delta_2}{t} + \ln(a) \right) \geq 0, \quad \forall t \in \left[\frac{-\delta_2}{\ln(a)}, +\infty\right).
\]

Thus, we have
\[
\sum_{\tau=k_1}^{t_3-1} s_3(\tau) \leq (t_3 - k_1)s_3(k_1), \quad \forall k_1 \in [1, t_3 - 1], \quad (70a)
\]
\[
\sum_{\tau=k_2}^{t_4-1} s_3(\tau) \leq (t_4 - k_2)s_3(t_4), \quad \forall k_2 \in [t_3, t_4 - 1], \quad (70b)
\]
\[
\sum_{\tau=t_4}^{k_3} s_3(\tau) \leq \int_{t_4}^{k_3+1} s_3(t) dt, \quad \forall k_3 \geq t_4. \quad (70c)
\]

Denote \( b = \frac{1}{a} \). We have
\[
\int_{t_4}^{k_3+1} s_3(t) dt = \int_{t_4}^{k_3+1} \frac{b^t}{t^{\delta_2} \ln(b)^{t^{\delta_2}}} dt = \int_{t_4}^{k_3+1} \frac{1}{\ln(b)^{t^{\delta_2}}} db^t = \frac{b^{k_3+1}}{\ln(b)(k_3 + 1)^{\delta_2}} - \frac{b^{t_4}}{\ln(b)^{t_4^{\delta_2}}} + \int_{t_4}^{k_3+1} \frac{\delta_2 b^t}{\ln(b)^{t^{\delta_2}+1}} dt \]
\[
\leq \frac{b^{k_3+1}}{\ln(b)(k_3 + 1)^{\delta_2}} + \frac{\delta_2 b^{t_4}}{\ln(b)^{t_4^{\delta_2}}} + \int_{t_4}^{k_3+1} \frac{\delta_2 s_3(t)}{\ln(b)^{t^{\delta_2}} s_3(t)} dt \]
\[
\leq \frac{b^{k_3+1}}{\ln(b)(k_3 + 1)^{\delta_2}} + \frac{\delta_2 b^{t_4}}{\ln(b)^{t_4^{\delta_2}}} + \frac{\delta_2}{\ln(b)} \int_{t_4}^{k_3+1} s_3(t) dt \]
\[
\leq \frac{b^{k_3+1}}{\ln(b)(k_3 + 1)^{\delta_2}} + \frac{1}{2} \int_{t_4}^{k_3+1} s_3(t) dt, \quad (71)
\]

where the last inequality holds due to \( t_4 = \left[ \frac{-2\delta_2}{\ln(1-a_1)} \right] \geq \frac{-2\delta_2}{\ln(1-a_1)} = \frac{2\delta_2}{\ln(b)} \).

From (70c) and (71), we have
\[
\sum_{\tau=t_4}^{k_3} s_3(\tau) \leq \frac{2}{-\ln(a)(k_3 + 1)^{\delta_2} a^{k_3+1}}, \quad \forall k_3 \geq t_4. \quad (72)
\]
B. Proof of Theorem 1

Denote \( L = L \otimes I_p, K = K_n \otimes I_p, H = \frac{1}{n}(1_n 1_n^T \otimes I_p), Q = RA_1^{-1}R^T \otimes I_p, \delta_k = \max_{i \in [n]} \{ \delta_{i,k} \}, x = \text{col}(x_1, \ldots, x_n), \bar{f}(x) = \sum_{i=1}^{n} f_i(x_i), \bar{x}_k = \frac{1}{n}(1_n^T \otimes I_p)x_k, \bar{x}_k = 1_n \otimes \bar{x}_k, g_k = \nabla \bar{f}(x_k), \bar{g}_k = H g_k, g^0_k = \nabla \bar{f}(\bar{x}_k), \bar{g}^0_k = H g^0_k = 1_n \otimes \nabla \bar{f}(\bar{x}_k), g^c_k = \text{col}(g^c_{1,k}, \ldots, g^c_{n,k}), \bar{g}^c_k = \frac{1}{n}(1_n^T \otimes I_p)g^c_k, \bar{g}^c_k = 1_n \otimes \bar{g}^c_k = H g^c_k, f_i^s(x, \delta_{i,k}) = E_{u \in \mathbb{R}^p}[f_i(x + \delta_{i,k}u)], g^s_{i,k} = \nabla f^s_i(x_{i,k}, \delta_{i,k}), g^s_k = \text{col}(g^s_{1,k}, \ldots, g^s_{n,k}), \text{and } \bar{g}^s_k = H g^s_k.

We also denote the following notations.

\[
c_0(\kappa_1, \kappa_2) = \max\left\{ \varepsilon_1, \frac{2\varepsilon_5}{\varepsilon_4}, \left( \frac{2p(1 + \sigma_0^2)(1 + \bar{\sigma}_0^2)\varepsilon_7}{\varepsilon_4} \right)^{\frac{1}{2}}, \frac{\varepsilon_8}{2\varepsilon_6}, \frac{24(1 + \bar{\sigma}_0^2)\kappa_4}{\kappa_2}, 128p(1 + \sigma_0^2)(1 + \bar{\sigma}_0^2)\kappa_2\varepsilon_10 \right\},
\]

\[
c_1 = \frac{1}{\rho_2(L)} + 1,
\]

\[
c_2(\kappa_1) = \min\left\{ \frac{\varepsilon_2}{\varepsilon_3}, \frac{1}{5} \right\},
\]

\[
c_3(\kappa_1, \kappa_2) = \frac{24(1 + \bar{\sigma}_0^2)\kappa_3}{\kappa_2},
\]

\[
\kappa_3 = \frac{1}{\rho_2(L)} + \kappa_1 + 1,
\]

\[
\kappa_4 = \frac{1}{\rho_2(L)} + \kappa_1,
\]

\[
\kappa_5 = \frac{1}{\rho_2(L)} + \kappa_1 + \frac{3}{2},
\]

\[
\kappa_6 = \frac{\kappa_1 + 1}{2} + \frac{1}{2\rho_2(L)},
\]

\[
\kappa_7 = \min\left\{ \frac{1}{2\rho_2(L)}, \frac{\kappa_1 - 1}{2\kappa_1} \right\},
\]

\[
\varepsilon_1 = \max\{1 + 3L_1^2, (8 + 8p(1 + \sigma_0^2)(1 + \bar{\sigma}_0^2)(6 + L_f))^\frac{1}{2}L_f, p\kappa_3 \},
\]

\[
\varepsilon_2 = (\kappa_1 - 1)\rho_2(L) - 1,
\]

\[
\varepsilon_3 = \rho(L) + (2\kappa_1^2 + 1)\rho(L^2) + 1,
\]

\[
\varepsilon_4 = \frac{1}{2}(\varepsilon_2\kappa_2 - \varepsilon_3\kappa_2^2),
\]

\[
\varepsilon_5 = \frac{1}{2} - \kappa_1\kappa_2\rho_2(L) + \kappa_2^2\rho(L) + \frac{1}{2}(1 + 3\kappa_1\kappa_2 + 2\kappa_2)\kappa_1\kappa_2\rho(L^2),
\]

\[
\varepsilon_6 = \frac{1}{4}(\kappa_2 - 5\kappa_2^2),
\]

From (68), (69), (70a), (70b), and (72), we get (55).\[\blacksquare\]
\[ \varepsilon_7 = 8(7 + 6\kappa_2 + 2\kappa_4 + 10\kappa_2\kappa_4)\kappa_2 L_f^4 + \frac{(1 + 2L_f^2)\kappa_2}{2p(1 + \sigma_0^2)(1 + \bar{\sigma}_0^2)} + \left(\frac{5}{p(1 + \sigma_0^2)(1 + \bar{\sigma}_0^2)} + 24\right) L_f^2 \kappa_2^2, \]

\[ \varepsilon_8 = \kappa_4 + \kappa_1\kappa_2 + 3\kappa_2^2 + \kappa_2\kappa_4, \]

\[ \varepsilon_9 = \frac{3\kappa_0}{2\kappa_2^2}(2\kappa_4 + 1), \]

\[ \varepsilon_{10} = 10 + L_f + \frac{1}{\kappa_2}(2\kappa_4 + 1)L_f^2 + (10\kappa_4 + 6)L_f^2, \]

\[ \varepsilon_{11} = L_f^2 \left(\frac{1}{512(1 + \sigma_0^2)(1 + \bar{\sigma}_0^2)} + \frac{13\kappa_2 + 4}{p}\right), \]

\[ \varepsilon_{12} = 2\varepsilon_{10}\sigma_1^2 + \frac{\varepsilon_9\sigma_2^2}{\bar{\sigma}_0^2} + 4(1 + \sigma_0^2)\varepsilon_{10}\sigma_2^2, \]

\[ \varepsilon_{13} = \frac{(1 + \bar{\sigma}_0^2)\varepsilon_9}{\sigma_0^2} + 8(1 + \sigma_0^2)(1 + \bar{\sigma}_0^2)\varepsilon_{10}, \]

\[ \varepsilon_{14} = \frac{W_0}{n} + \frac{p(\varepsilon_{11}\kappa_3^2 + \varepsilon_{12})\kappa_2^2}{2(\theta - 1)\kappa_0^2}, \]

\[ \varepsilon_{15} = 2(\sigma_1^2 + 2(1 + \sigma_0^2)\sigma_2^2)L_f, \]

\[ a_1 = \frac{1}{\kappa_6} \min\{\varepsilon_4, \varepsilon_6\}, \]

\[ a_2 = np(\varepsilon_{11}\kappa_3^2 + \varepsilon_{12} + 2L_f\varepsilon_{13}\varepsilon_{14})\kappa_2^2. \]

To prove Theorem 1, the following three lemmas are used.

**Lemma 6.** Suppose Assumption 3 holds. Let \( \{x_k\} \) be the sequence generated by Algorithm 1, then

\[ g_k^* = E_{\mathcal{C}_k}[g_k^*], \]  

\[ \|g_0^0 - g_k^*\|^2 \leq 2L_f^2\|x_k\|^2 + 2nL_f\delta_k^2, \]  

\[ \|g_0^0 - g_k^*\|^2 \leq 2L_f^2\|x_k\|^2 + 2nL_f\delta_k^2, \]  

\[ E_{\mathcal{C}_k}[\|g_k^*\|^2] \leq \frac{1}{n}E_{\mathcal{C}_k}[\|g_k^0\|^2] + \|g_k^*\|^2, \]  

\[ E_{\mathcal{C}_k}[\|g_k^0 - g_k^*\|^2] \leq 4L_f^2\|x_k\|^2 + 4nL_f\delta_k^2 + 2E_{\mathcal{C}_k}[\|g_k^0\|^2], \]  

\[ \|g_{k+1}^0 - g_k^0\|^2 \leq \eta_k^f L_f^2\|g_k^0\|^2 \leq \eta_k^f L_f^2\|g_k^*\|^2, \]  

\[ \|g_k^0\|^2 \leq 2nL_f(f(\bar{x}_k) - f^*). \]
If Assumptions 4 and 5 also hold, then

\[ E_{\xi_k}[\|\mathbf{g}_k^c\|^2] \leq 16p(1 + \sigma_0^2)(1 + \bar{\sigma}_0^2)(\|\mathbf{g}_0^c\|^2 + L_f^2\|\mathbf{x}_k\|^2_K) + 4np\sigma_1^2 + 8np(1 + \sigma_0^2)\bar{\sigma}_0^2 + \frac{1}{2}np^2L_f^2\delta_k^2, \]

\[ (74a) \]

\[ \|\mathbf{g}_{k+1}^0\|^2 \leq 3(n_k^2L_f^2\|\mathbf{g}_k^c\|^2 + n\sigma_2^2 + (1 + \bar{\sigma}_0^2)\|\mathbf{g}_0^c\|^2). \]

\[ (74b) \]

**Proof:** (i) From \( u_{i,k} \) and \( \xi_{i,k} \) are independent, \( x_{i,k} \) is independent of \( u_{i,k} \) and \( \xi_{i,k} \), and (44), we have

\[ E_{\xi_k}[\mathbf{g}_{i,k}^c] = E_{u_{i,k}}\left[E_{\xi_{i,k}}\left[\frac{p}{\delta_{i,k}}(F_i(x_{i,k} + \delta_{i,k}u_{i,k},\xi_{i,k}) - F_i(x_{i,k},\xi_{i,k}))u_{i,k}\right]\right] \]

\[ = E_{u_{i,k}}\left[\frac{p}{\delta_{i,k}}(f_i(x_{i,k} + \delta_{i,k}u_{i,k}) - f_i(x_{i,k}))u_{i,k}\right] \]

\[ = E_{u_{i,k}}[\hat{\nabla}^2 f_i(x_{i,k},\delta_{i,k},u_{i,k})] = \nabla f_i^s(x_{i,k},\delta_{i,k}) = g_{i,k}^s, \]

which gives (73a).

(ii) From Assumption 3, we know that each \( f_i(x) = E_{\xi_i}[F_i(x,\xi_i)] \) is smooth with constant \( L_f \) since

\[ \|\nabla f_i(x) - \nabla f_i(y)\| = \|E_{\xi_i}[\nabla_x F_i(x,\xi_i) - \nabla_x F_i(y,\xi_i)]\| \]

\[ \leq E_{\xi_i}[\|\nabla_x F_i(x,\xi_i) - \nabla_x F_i(y,\xi_i)\|] \]

\[ \leq E_{\xi_i}[L_f\|x - y\|] = L_f\|x - y\|, \quad \forall x, y \in \mathbb{R}^p. \]

(75)

From (75), we have

\[ \|\mathbf{g}_0^c - \mathbf{g}_k\|^2 = \|\nabla \bar{f}(\mathbf{x}_k) - \nabla \bar{f}(\mathbf{x}_k)\|^2 \]

\[ = \sum_{i=1}^n \|\nabla f_i(\bar{x}_k) - \nabla f_i(x_{i,k})\|^2 \leq \sum_{i=1}^n L_f^2\|\bar{x}_k - x_{i,k}\|^2 \]

\[ = L_f^2\|\bar{x}_k - \mathbf{x}_k\|^2 = L_f^2\|\mathbf{x}_k\|^2_K. \]

(76)

From (75) and (45a), we have

\[ \|g_{i,k}^s - g_{i,k}\| \leq L_f\delta_{i,k}. \]
Thus,

\[
\|g_k^s - g_k\|^2 = \sum_{i=1}^{n} \|g_{i,k}^s - g_{i,k}\|^2 \leq nL^2\delta_k^2. \tag{77}
\]

Noting \(\|g_k^0 - g_k^e\|^2 \leq 2\|g_k^0 - g_k\|^2 + 2\|g_k^0 - g_k^e\|^2\), from (76) and (77), we know (73b) holds. (iii) Noting \(\|g_k^0 - \bar{g}_k\|^2 = \|H(g_k^0 - g_k)\|^2\), from \(\rho(H) = 1\) and (73b), we know (73c) holds. (iv) We have

\[
E_{\ell_k} [\|\bar{g}_k^e\|^2] = E_{\ell_k} \left[ \left( \sum_{i=1}^{n} \frac{1}{n} g_{i,k}^e \right)^2 \right] = \frac{1}{n^2} E_{\ell_k} \left[ \sum_{i=1}^{n} \|g_{i,k}^e\|^2 + \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \langle g_{i,k}^e, g_{j,k}^e \rangle \right] = \frac{1}{n^2} E_{\ell_k} [\|g_k^e\|^2] + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \langle g_{i,k}^e, g_{j,k}^e \rangle = \frac{1}{n^2} E_{\ell_k} [\|g_k^e\|^2] + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \langle g_{i,k}^e, g_{j,k}^e \rangle = \frac{1}{n^2} E_{\ell_k} [\|g_k^e\|^2] + \|\bar{g}_k\|^2 - \frac{1}{n^2} \|g_k^e\|^2, \tag{78}
\]

where the third equality holds since \(u_{i,k}\) and \(\xi_{i,k}\), \(\forall i \in [n], k \in \mathbb{N}_+\) are mutually independent; and the fourth equality holds due to (73a).

From (78), \(E_{\ell_k} [\|g_k^e\|^2] = nE_{\ell_k} [\|\bar{g}_k^e\|^2]\) and \(\|\bar{g}_k\|^2 = n\|\bar{g}_k^e\|^2\), we know that (73d) holds. (v) We have

\[
E_{\ell_k} [\|g_k^0 - g_k^e\|^2] \leq 2\|g_k^0 - g_k^e\|^2 + 2E_{\ell_k} [\|g_k^s - g_k^e\|^2] = 2\|g_k^0 - g_k^e\|^2 + 2E_{\ell_k} [\|g_k^e\|^2] - 2\|g_k^e\|^2, \tag{79}
\]

where the inequality holds due to the Cauchy–Schwarz inequality; and the equality holds since (73a) and \(x_k\) is independent of \(\ell_k\).

From (79) and (73b), we know (73e) holds. (vi) The distributed ZO algorithm (6) can be rewritten as

\[
\begin{align*}
x_{k+1} &= x_k - \eta_k (\alpha_k L x_k + \beta_k v_k + g_k^e), \tag{80a} \\
v_{k+1} &= v_k + \eta_k \beta_k L x_k, \quad \forall x_0 \in \mathbb{R}^{np}, \sum_{i=1}^{n} v_{i,0} = 0_p. \tag{80b}
\end{align*}
\]
From (80b), we know that
\[ \bar{v}_{k+1} = \bar{v}_k. \] (81)

Then, from (81), \( \sum_{i=1}^{n} v_{i,0} = 0_p \), and (80a), we know that \( \bar{v}_k = 0_p \) and
\[ \bar{x}_{k+1} = \bar{x}_k - \eta_k \bar{g}_k^e. \] (82)

Then, we have
\[
\| \bar{g}_{k+1}^0 - \bar{g}_k^0 \|^2 = \| \nabla \tilde{f}(\bar{x}_{k+1}) - \nabla \tilde{f}(\bar{x}_k) \|^2 \\
\leq L_f^2 \| \bar{x}_{k+1} - \bar{x}_k \|^2 = \eta_k^2 L_f^2 \| \bar{g}_k^e \|^2 \leq \eta_k^2 L_f^2 \| \bar{g}_k^e \|^2;
\]
where the first inequality holds due to (75); the last equality holds due to (82); and the last equality holds due to \( \bar{g}_k^e = H g_k^e \) and \( \rho(H) = 1 \). Thus, (73f) holds.

(vii) From (42), we have
\[
\| \bar{g}_k^0 \|^2 = n \| \nabla f(\bar{x}_k) \|^2 \leq 2n L_f (f(\bar{x}_k) - f^*),
\] (83)
which yields (73g).

(viii) From Assumption 3, \( x_{i,k} \) and \( \xi_{i,k} \) are independent of \( u_{i,k} \), and (45b), we know that for almost every \( \xi_{i,k} \) it holds that
\[
E_{u_{i,k}}[\| g_{i,k}^e \|^2] \leq 2p \| \nabla f_i(x_{i,k}, \xi_{i,k}) \|^2 + \frac{1}{2} p^2 L_f^2 \delta_{i,k}^2.
\] (84)

Then,
\[
E_{\xi_{i,k}}[\| g_{i,k}^e \|^2] \leq 2p E_{\xi_{i,k}}[\| \nabla f_i(x_{i,k}, \xi_{i,k}) \|^2] + \frac{1}{2} p^2 L_f^2 \delta_{i,k}^2 \\
= 2p E_{\xi_{i,k}}[\| \nabla f_i(x_{i,k}, \xi_{i,k}) - \nabla f_i(x_{i,k}) + \nabla f_i(x_{i,k}) \|^2] + \frac{1}{2} p^2 L_f^2 \delta_{i,k}^2 \\
\leq 4p E_{\xi_{i,k}}[\| \nabla f_i(x_{i,k}, \xi_{i,k}) \|^2 + \| \nabla f_i(x_{i,k}) \|^2 + \| \nabla f_i(x_{i,k}) \|^2] + \frac{1}{2} p^2 L_f^2 \delta_{i,k}^2 \\
\leq 4p (1 + \sigma_0^2) \| \nabla f_i(x_{i,k}) \|^2 + 4p \sigma_1^2 + \frac{1}{2} p^2 L_f^2 \delta_{i,k}^2,
\] (85)
where the first inequality holds due to (84); the second inequality holds due to the Cauchy–Schwarz inequality; and the last inequality holds since Assumption 4 and \( x_{i,k} \) is independent of \( \xi_{i,k} \).
Then, we have
\[
\|\nabla f(x) - \nabla f(y)\|^2 = \left\| \frac{1}{n} \sum_{i=1}^{n} (\nabla f_i(x) - \nabla f_i(y)) \right\|^2 \\
\leq \frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(x) - \nabla f_i(y)\|^2 \\
\leq L_f^2 \|x - y\|^2, \ \forall x, y \in \mathbb{R}^p. \quad (86)
\]

From (75), we have
\[
\|\nabla f_i(x_{i,k})\|^2 = \|\nabla f_i(x_{i,k}) - \nabla f(x_{i,k}) + \nabla f(x_{i,k})\|^2 \\
\leq 2(\|\nabla f_i(x_{i,k}) - \nabla f(x_{i,k})\|^2 + \|\nabla f(x_{i,k})\|^2) \\
\leq 2(\sigma_2^2 + (1 + \tilde{\sigma}_0^2)\|\nabla f(x_{i,k})\|^2) \\
= 2(\sigma_2^2 + (1 + \tilde{\sigma}_0^2)(\|\nabla f(x_{i,k}) - \nabla f(\bar{x}_k)\|^2 + \|\nabla f(\bar{x}_k)\|^2) \\
\leq 2\sigma_2^2 + 4(1 + \tilde{\sigma}_0^2)(\|\nabla f(x_{i,k}) - \nabla f(\bar{x}_k)\|^2 + \|\nabla f(\bar{x}_k)\|^2) \\
\leq 2\sigma_2^2 + 4(1 + \tilde{\sigma}_0^2)(L_f^2\|x_{i,k} - \bar{x}_k\|^2 + \|\nabla f(\bar{x}_k)\|^2), \quad (87)
\]

where the first and third inequalities hold due to the Cauchy–Schwarz inequality; the second inequality holds due to Assumption 5, and the last inequality holds due to (86).

From (85) and (87), we know (74a) holds.

(ix) From the Cauchy–Schwarz inequality, we have
\[
\|g_{k+1}\|^2 = \|g_{k+1} - g_k + g_k - g_0 + g_0\|^2 \leq 3(\|g_{k+1} - g_k\|^2 + \|g_k - g_0\|^2 + \|g_0\|^2). \quad (88)
\]

From Assumption 5, we have
\[
\|g_0 - g_k\|^2 = \sum_{i=1}^{n} \|f_i(\bar{x}_k) - f(\bar{x}_k)\|^2 \leq n\sigma_2^2 + n\tilde{\sigma}_0^2\|f(\bar{x}_k)\|^2. \quad (89)
\]

From (88), (89), and (73f), we know (74b) holds.

**Lemma 7.** Suppose Assumptions 1–5 hold. Suppose \(\{\beta_k\}\) is non-decreasing, \(\alpha_k = \kappa_1\beta_k\), and \(\eta_k = \frac{\eta_2}{\beta_k}\), where \(\kappa_1 > 1\) and \(\kappa_2 > 0\) are constants. Moreover, suppose \(\beta_k \geq \varepsilon_1\). Let \(\{x_k\}\) be the sequence generated by Algorithm 7 then
\[
\mathbb{E}_{x_k}[W_{k+1}] \leq W_k - \|x_k\|^2(2\varepsilon_4 - \varepsilon_5\omega_k - b_{1,k})K - \left\| v_k + \frac{1}{\beta_k^2} g_k \right\|^2_{b_{2,k}K} \\
- \eta_k \left( 1 - (1 + \tilde{\sigma}_0^2)(b_{3,k} + 8\rho(1 + \sigma_0^2)b_{4,k})\eta_k \right) \|g_k\|^2 + 2\rho n\sigma_1^2 b_{4,k}\eta_k^2
\]
where $W_k = \sum_{i=1}^{4} W_{i,k}$, $\tilde{W}_k = \sum_{i=1}^{3} W_{i,k}$, $\omega_k = \frac{1}{\beta_k} - \frac{1}{\beta_{k+1}}$ and

$$\begin{align*}
W_{1,k} &= \frac{1}{2} \|x_k\|_K^2, \\
W_{2,k} &= \frac{1}{2} \|v_k + \frac{1}{\beta_k} g_k^0\|_{Q+\kappa_k \mathbf{I}}^2, \\
W_{3,k} &= x_k^T K \left(v_k + \frac{1}{\beta_k} g_k^0\right), \\
W_{4,k} &= n(f(\bar{x}_k) - f^*) = \bar{f}(x_k) - f^*, \\
b_{1,k} &= 8p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2) \kappa_3 L_f^4 \eta_k \kappa_2 + 16p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2) \kappa_5 L_f^4 \eta_k \kappa_2^2 \\
&\quad + \left(\frac{1}{2} + L_f^2\right) \eta_k \omega_k + 8p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2) \kappa_4 L_f^4 \eta_k \omega_k \kappa_2 \\
&\quad + (5 + 32p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2) + 24p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2) \kappa_3 L_f^2) L_f^2 \eta_k^2 \omega_k \\
&\quad + 24p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2) \kappa_4 L_f^4 \eta_k^2 \omega_k^2 + 16p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2) \kappa_4 L_f^4 \eta_k^2 \omega_k^2,
\end{align*}$$

$$\begin{align*}
b_{2,k} &= 2\varepsilon_0 - \frac{1}{2} \omega_k (\kappa_1 \kappa_2 + 3\kappa_2^2 + \kappa_4) - \frac{1}{2} \omega_k \kappa_2 \kappa_4, \\
b_{3,k} &= \frac{3}{2} \kappa_2 \omega_k + \frac{3}{2} \kappa_4 \omega_k^2 - \frac{3}{2} \kappa_2 \eta_k^2, \\
b_{4,k} &= 6 + L_f + \frac{\kappa_3 L_f^2}{\kappa_2} \frac{1}{\beta_k} + 2\kappa_5 L_f^2 \frac{1}{\beta_k} + \frac{\kappa_4 L_f^2}{\kappa_2} \omega_k \\
&\quad + (4 + 3\kappa_3 L_f^2) \omega_k + 3\kappa_4 L_f^2 \omega_k + 2\kappa_4 L_f^2 \omega_k \kappa_2, \\
b_{5,k} &= n L_f^2 \left(\frac{1}{4} p^2 b_{4,k} \eta_k + 3 + \omega_k + 8\eta_k + 5\eta_k \omega_k\right).
\end{align*}$$

**Proof**: Note that $W_{4,k}$ is well defined due to $f^* > -\infty$ as assumed in Assumption 2. Thus, $W_k$ is well defined.

(i) We have

$$\mathbf{E}_{\mathcal{L}_k} [W_{1,k+1}] = \mathbf{E}_{\mathcal{L}_k} \left[ \frac{1}{2} \|x_{k+1}\|_K^2 \right]$$
inequality holds due to the Cauchy–Schwarz inequality and
\( \rho \) the fourth equality holds since (73a) and that
\( \omega \) the second equality holds due to (80a); the third equality holds due to (38a) in Lemma 1;
\( \gamma \) the last inequality holds due to (73b) and (73c).

where the second equality holds due to (80a); the third equality holds due to (38a) in Lemma 1;
the fourth equality holds since (73a) and that \( x_{i,k} \) and \( v_{i,k} \) are independent of \( \Omega_k \); the first
inequality holds due to the Cauchy–Schwarz inequality and \( \rho(K) = 1 \); and the last inequality
holds due to (73b) and (73c).
(ii) We know that $\omega_k \geq 0$ since $\{\beta_k\}$ is non-decreasing. We have

$$W_{2,k+1} = \frac{1}{2}\left\| v_{k+1} + \frac{1}{\beta_k} g_{k+1} \right\|_{Q+\kappa_1K}^2$$
$$= \frac{1}{2}\left\| v_{k+1} + \frac{1}{\beta_k} g_{k+1} + \frac{1}{\beta_k} - \frac{1}{\beta_k} g_{k+1} \right\|_{Q+\kappa_1K}^2$$
$$\leq \frac{1}{2}(1 + \omega_k)^2\left\| v_{k+1} + \frac{1}{\beta_k} g_{k+1} \right\|_{Q+\kappa_1K}^2 + \frac{1}{2}(\omega_k + \omega_k^2)\|g_{k+1}\|_{Q+\kappa_1K}^2,$$

(92)

where the inequality holds due to the Cauchy–Schwarz inequality.

For the first term on the right-hand side of (92), we have

$$\frac{1}{2}\left\| v_{k+1} + \frac{1}{\beta_k} g_{k+1} \right\|_{Q+\kappa_1K}^2 = \frac{1}{2}\left\| v_{k+1} + \frac{1}{\beta_k} g_{k+1} + \eta_k \beta_k L x_k + \frac{1}{\beta_k} (g_{k+1} - g_k) \right\|_{Q+\kappa_1K}^2$$
$$= W_{2,k} + \eta_k \beta_k x_k^T (K + \kappa_1 L) (v_k + \frac{1}{\beta_k} g_k)$$
$$+ \|x_k\|^2 \frac{\eta_k\beta_k^2}{\lambda_k} (L + \kappa_1 L^2) + \frac{1}{2\beta_k^2} \|g_{k+1} - g_k\|_{Q+\kappa_1K}^2$$
$$+ \frac{1}{\beta_k} (v_k + \frac{1}{\beta_k} g_k + \eta_k \beta_k L x_k)^\top \left( Q + \kappa_1 K \right) (g_{k+1} - g_k)$$
$$\leq W_{2,k} + \eta_k \beta_k x_k^T (K + \kappa_1 L) (v_k + \frac{1}{\beta_k} g_k)$$
$$+ \|x_k\|^2 \frac{\eta_k\beta_k^2}{\lambda_k} (L + \kappa_1 L^2) + \|v_k + \frac{1}{\beta_k} g_k\|_{Q+\kappa_1K}^2 + \frac{1}{2\eta_k \beta_k^2} \|g_{k+1} - g_k\|_{Q+\kappa_1K}^2$$
$$+ \frac{1}{\beta_k^2} \left(1 + \frac{1}{2\eta_k}\right) \|g_{k+1} - g_k\|_{Q+\kappa_1K}^2$$
$$\leq W_{2,k} + \eta_k \beta_k x_k^T (K + \kappa_1 L) (v_k + \frac{1}{\beta_k} g_k)$$
$$+ \|x_k\|^2 \frac{\eta_k\beta_k^2}{\lambda_k} (L + \kappa_1 L^2) + \|v_k + \frac{1}{\beta_k} g_k\|_{Q+\kappa_1K}^2 + \frac{1}{\beta_k^2} \left(1 + \frac{1}{2\eta_k}\right) \kappa_1 \|g_{k+1} - g_k\|_{Q+\kappa_1K}^2$$
$$\leq W_{2,k} + \eta_k \beta_k x_k^T (K + \kappa_1 L) (v_k + \frac{1}{\beta_k} g_k)$$
\[ + \|x_k\|_{\beta_k \alpha_k (L+\kappa_1 L^2)}^2 + \left\| v_k + \frac{1}{\beta_k} g_k^0 \right\|^2_{\frac{\eta_k}{2}(Q+\kappa_1 K)} \]
\[ + \frac{\eta_k}{\beta_k} \left( \eta_k + \frac{1}{2} \right) \kappa_4 L_f^2 \|g_k^c\|^2, \]  
(93)

where the first equality holds due to (80b); the second equality holds due to (38a) and (39a) in Lemma 1; the first inequality holds due to the Cauchy–Schwarz inequality; the last equality holds due to (38a) and (39a); the second inequality holds due to \( \rho(Q+\kappa_1 K) \leq \rho(Q) + \kappa_1 \rho(K), \) (39b), \( \rho(K) = 1; \) and the last inequality holds due to (73f).

For the second term on the right-hand side of (92), we have
\[ \|g_{k+1}^0\|^2_{Q+\kappa_1 K} \leq \kappa_4 \|g_{k+1}^0\|^2. \]  
(94)

Also note that
\[ \left\| v_k + \frac{1}{\beta_k} g_k^0 \right\|^2_{Q+\kappa_1 K} \leq \kappa_4 \left\| v_k + \frac{1}{\beta_k} g_k^0 \right\|^2_K. \]  
(95)

Then, from (92)–(95), we have
\[ W_{2,k+1} \leq W_{2,k} + (1 + \omega_k) \eta_k \beta_k x_k^\top (K + \kappa_1 L) \left( v_k + \frac{1}{\beta_k} g_k^0 \right) \]
\[ + \frac{1}{2} (\eta_k + \omega_k + \eta_k \omega_k) \kappa_4 \left\| v_k + \frac{1}{\beta_k} g_k^0 \right\|^2_K \]
\[ + \|x_k\|_{(1+\omega_k)\eta_k^2 \beta_k^2 (L+\kappa_1 L^2)}^2 + \frac{\eta_k}{\beta_k} \left( \eta_k + \frac{1}{2} \right) (1 + \omega_k) \kappa_4 L_f^2 \|g_k^c\|^2 \]
\[ + \frac{1}{2} \kappa_4 (\omega_k + \omega_k^2) \|g_{k+1}^0\|^2. \]  
(96)

(iii) We have
\[ W_{3,k+1} = x_{k+1}^\top K \left( v_{k+1} + \frac{1}{\beta_{k+1}} g_{k+1}^0 \right) \]
\[ = x_{k+1}^\top K \left( v_{k+1} + \frac{1}{\beta_k} g_{k+1}^0 + \left( \frac{1}{\beta_{k+1}} - \frac{1}{\beta_k} \right) g_{k+1}^0 \right) \]
\[ = x_{k+1}^\top K \left( v_{k+1} + \frac{1}{\beta_k} g_{k+1}^0 \right) - \omega_k x_{k+1}^\top K g_{k+1}^0 \]
\[ \leq x_{k+1}^\top K \left( v_{k+1} + \frac{1}{\beta_k} g_{k+1}^0 \right) + \frac{1}{2} \omega_k \left( \|x_{k+1}\|^2_K + \|g_{k+1}^0\|^2 \right). \]  
(97)

For the first term on the right-hand side of (97), we have
\[ E_k \left[ x_{k+1}^\top K \left( v_{k+1} + \frac{1}{\beta_k} g_{k+1}^0 \right) \right] = E_k \left[ (x_k - \eta_k (\alpha_k L x_k + \beta_k v_k + g_k^0 + g_k^c - g_k^0))^\top \right] \]
\[
\times K \left( v_k + \frac{1}{\beta_k} g_k^0 + \eta_k \beta_k L x_k + \frac{1}{\beta_k} (g_k^0 - g_k) \right)
\]
\[
= x_k^T (K - \eta_k (\alpha_k + \eta_k \beta_k^2) L) \left( v_k + \frac{1}{\beta_k} g_k^0 \right)
\]
\[
+ \|x_k\|_{\eta_k \beta_k (L - \eta_k \alpha_k L^2)}^2
\]
\[
+ \frac{1}{\beta_k} x_k^T (K - \eta_k \alpha_k L) E_{\alpha_k} [g_{k+1}^0 - g_k^0]
\]
\[
- \eta_k \beta_k \left\| v_k + \frac{1}{\beta_k} g_k^0 \right\|^2_K
\]
\[
- \eta_k \left( v_k + \frac{1}{\beta_k} g_k^0 \right)^T K E_{\alpha_k} [g_{k+1}^0 - g_k^0]
\]
\[
- \eta_k (g_k^0 - g_k^0)^T K \left( v_k + \frac{1}{\beta_k} g_k^0 + \eta_k \beta_k L x_k \right)
\]
\[
- \frac{1}{\beta_k} E_{\alpha_k} [\eta_k (g_k^0 - g_k^0)^T K (g_{k+1}^0 - g_k^0)]
\]
\[
\leq x_k^T (K - \eta_k \alpha_k L) \left( v_k + \frac{1}{\beta_k} g_k^0 \right) + \frac{1}{2} \eta_k \beta_k^2 \left\| L x_k \right\|^2
\]
\[
+ \frac{1}{2} \eta_k \beta_k^2 \left\| v_k + \frac{1}{\beta_k} g_k^0 \right\|^2_K + \frac{1}{2} \eta_k \beta_k \left\| x_k \right\|^2_K
\]
\[
+ \frac{1}{2} \eta_k \beta_k \left\| g_k^0 \right\|^2 + \frac{1}{2} \eta_k \beta_k \left\| v_k + \frac{1}{\beta_k} g_k^0 \right\|^2_K
\]
\[
+ \frac{1}{2} \eta_k \beta_k \left\| g_k^0 - g_k \right\|^2 + \frac{1}{2} \eta_k \beta_k \left\| L x_k \right\|^2
\]
\[
+ \frac{1}{2} \eta_k \beta_k \left\| g_k^0 - g_k^0 \right\|^2 + \frac{1}{2} \eta_k \beta_k \left\| E_{\alpha_k} [ || g_{k+1}^0 - g_k^0 ||^2 ] \right\|
\]
\[
= x_k^T (K - \eta_k \alpha_k L) \left( v_k + \frac{1}{\beta_k} g_k^0 \right)
\]
\[
+ \frac{1}{2} (\eta_k + \eta_k^2) || g_k^0 - g_k ||^2 + \frac{1}{2} \eta_k \beta_k \left\| E_{\alpha_k} [ || g_k^0 - g_k ||^2 ] \right\|
\]
\[
+ \left( \frac{1}{2} \eta_k \beta_k^2 + \frac{3}{2} \beta_k^2 \right) E_{\alpha_k} [ || g_{k+1}^0 - g_k^0 ||^2 ]
\]
where the first equality holds due to (80); the second equality holds since (38a), (73a), and (73b), (73e), and (73f).

Then, from (97)–(99), we have

(iv) We have

\[ \mathbb{E}_{\mathcal{L}_k}[W_{3,k+1}] \leq W_{3,k} - (1 + \omega_k)\eta_k\alpha_k\|x_k\|^2 \]
\[= W_{4,k} - \eta_k (\bar{g}_k^s)^\top \bar{g}_k^0 + \frac{1}{2} \eta_k^2 L_f \mathbf{E}_{\mathcal{C}_k} [||g_k^e||^2] \]

\[= W_{4,k} - \frac{1}{2} \eta_k (\bar{g}_k^s)^\top (\bar{g}_k^0 + \bar{g}_k^s) - \frac{1}{2} \eta_k (\bar{g}_k^s - \bar{g}_k^0)^\top \bar{g}_k^0 + \frac{1}{2} \eta_k^2 L_f \mathbf{E}_{\mathcal{C}_k} [||g_k^e||^2] \]

\[\leq W_{4,k} - \frac{1}{4} \eta_k (||\bar{g}_k^s||^2 - ||\bar{g}_k^0 - \bar{g}_k^s||^2 + ||\bar{g}_k^0||^2 \]

\[= W_{4,k} - \frac{1}{4} \eta_k ||\bar{g}_k^s||^2 + \frac{1}{2} \eta_k ||\bar{g}_k^0 - \bar{g}_k^s||^2 \]

\[\leq W_{4,k} - \frac{1}{4} \eta_k ||\bar{g}_k^s||^2 + \frac{1}{2} \eta_k ||\bar{g}_k^0||^2 \]

\[+ n L_f^2 \eta_k \delta^2_k - \frac{1}{4} \eta_k ||\bar{g}_k^0||^2 + \frac{1}{2} \eta_k^2 L_f \mathbf{E}_{\mathcal{C}_k} [||g_k^e||^2], \quad (101) \]

where the first inequality holds since that \(\bar{f}\) is smooth as shown in (75), (41) and (82); the third equality holds since (73a) and that \(x_{i,k}\) and \(v_{i,k}\) are independent of \(\mathcal{L}_k\); the fourth equality holds due to \((\bar{g}_k^0)^\top \bar{g}_k^0 = (\bar{g}_k^s)^\top H \bar{g}_k^0 = (\bar{g}_k^s)^\top H \bar{g}_k^0 = (\bar{g}_k^s)^\top \bar{g}_k^0\); the second inequality holds due to the Cauchy–Schwarz inequality; and the last inequality holds due to (73c).

(v) Denote

\[M_{1,k} = (\alpha_k - \beta_k) \mathbf{L} - (1 + 3 L_f^2) \mathbf{K}, \]

\[M_{2,k}^0 = \beta_k^2 \mathbf{L} + (2 \alpha_k^2 + \beta_k^2) \mathbf{L}^2 + 8 L_f^2 \mathbf{K}, \]

\[M_{2,k} = M_{2,k}^0 + 8 p(1 + \sigma_0^2)(1 + \sigma_0^2)(6 + L_f) L_f^2 \mathbf{K}, \]

\[M_3 = \frac{1}{2} \mathbf{K} - \kappa_1 \kappa_2 \mathbf{L} + \frac{1}{2} \kappa_1 \kappa_2 \mathbf{L}^2 + \frac{3}{2} \kappa_1^2 \kappa_2^2 \mathbf{L}^2 + \kappa_2^2 (\mathbf{L} + \kappa_1 \mathbf{L}^2), \]

\[b_{2,k} = \frac{1}{2} \eta_k (2 \beta_k - \beta_3) - \frac{5}{2} \kappa_2^2 - \frac{1}{2} \omega_k (\kappa_1 \kappa_2 + 3 \kappa_2^2 + \kappa_4) - \frac{1}{2} \omega_k \eta_k \kappa_4, \]

\[b_{4,k} = \frac{1}{2} (6 + L_f) + \frac{\kappa_3 \kappa_2 L_f^2}{2 \kappa_2} \frac{1}{\beta_k} + 2 \omega_k + \kappa_5 \frac{L_f}{\beta_k} + \frac{\kappa_4 L_f^2 \omega_k}{2 \beta_k} + \frac{\kappa_4 L_f^2 \omega_k}{\beta_k^2}. \]

We have

\[E_{\mathcal{C}_k}[W_{k+1}] \leq W_k + \frac{1}{2} \omega_k ||x_k||^2_{\mathbf{K}} - (1 + \omega_k) ||x_k||^2_{\eta_k \alpha_k \mathbf{L} - \frac{1}{2} \eta_k \mathbf{K} - \frac{1}{2} \eta_k^2 \alpha_k^2 \mathbf{L}^2 - \eta_k (1 + 5 \eta_k) L_f^2 \mathbf{K}} \]

\[+ (1 + \omega_k) \left\| v_k + \frac{1}{\beta_k} g_k^0 \right\|^2_{\frac{1}{2} \eta_k^2 \beta_k^2 \mathbf{K}} + (1 + \omega_k) n L_f^2 \eta_k (1 + 5 \eta_k) \delta_k^2 \]
where the first inequality holds due to (91), (96), (100), and (101); the second inequality holds due to $\alpha = \kappa_1 \beta_k$, $\eta_k = \frac{\alpha_k}{\beta_k}$ and $\|\mathbf{g}_k^0\|^2 \leq \|\mathbf{g}_k\|^2$; the last inequality holds due to (74a) and (74b).

From (38b), $\alpha_k = \kappa_1 \beta_k$, $\kappa_1 > 1$, $\beta_k \geq \varepsilon_1 \geq 1 + 3L_f^2$, and $\eta_k = \frac{\alpha_k}{\beta_k}$, we have

$$\eta_k M_{1,k} \geq \varepsilon_2 \kappa_2 K. \tag{104}$$

From (38b), $\alpha_k = \kappa_1 \beta_k$, $\beta_k \geq \varepsilon_1 \geq (8 + 8p(1 + \sigma_0^2)(1 + \bar{s}_0^2)(6 + L_f)) \frac{1}{2} L_f$, and $\eta_k = \frac{\alpha_k}{\beta_k}$, we have

$$\eta_k^2 M_{2,k} \leq \varepsilon_3 \kappa_2^2 K. \tag{105}$$

From (38b), $\alpha_k = \kappa_1 \beta_k$, and $\eta_k = \frac{\alpha_k}{\beta_k}$, we have

$$M_3 \leq \varepsilon_5 K. \tag{106}$$
From \( \beta_k \geq \varepsilon_1 \geq p\kappa_3 \geq \kappa_3 \) and \( \eta_k = \frac{\kappa_2}{\beta_k} \), we have
\[
b_0 \geq b_{2,k}. \tag{107}
\]

From (103)–(107), we know that (90a) holds.

Similar to the way to get (90a), we have (90b).

**Lemma 8.** Suppose Assumptions [7,5] hold. Suppose \( \alpha_k = \kappa_1\beta_k \), \( \beta_k = \kappa_0(k + t_1)^\theta \), and \( \eta_k = \frac{\kappa_2}{\beta_k} \), where \( \theta \in [0, 1] \), \( \kappa_0 \geq \frac{c_0(\kappa_1,\kappa_2)}{t_1^\theta} \), \( \kappa_1 > c_1 \), \( \kappa_2 \in (0, c_2(\kappa_1)) \), and \( t_1 \geq (c_3(\kappa_1,\kappa_2)) \frac{1}{\beta} \). Let \( \{x_k\} \) be the sequence generated by Algorithm [7] then
\[
E_{x_k}[W_{k+1}] \leq W_k - \varepsilon_4\|x_k\|^2_k - \varepsilon_6\|v_k + \frac{1}{\beta_k}g_0\|^2_k - \frac{1}{16}\eta_k\|g_0\|^2 + pn\varepsilon_{12}\eta^2_k + pn\varepsilon_{13}\eta_k\delta_k^2,
\]
\[
E_{x_k}[\tilde{W}_{k+1}] \leq \tilde{W}_k - \varepsilon_4\|x_k\|^2_k - \varepsilon_6\|v_k + \frac{1}{\beta_k}g_0\|^2_k + p\varepsilon_{13}\eta^2_k\|g_0\|^2 + pn\varepsilon_{12}\eta^2_k + pn\varepsilon_{11}\eta_k\delta_k^2,
\]
\[
E_{x_k}[W_{4k+1}] \leq W_{4k} + \|x_k\|^2_{2\eta_kL_2^2} - \frac{3}{16}\eta_k\|g_0\|^2 + pn\varepsilon_{15} + (n + p)L_2^2\eta_k\delta_k^2.
\]

**Proof:** (i) Noting that \( \kappa_1 > c_1 > 1 \) and \( \beta_k = \kappa_0(k + t_1)^\theta \geq \kappa_0 t_1^\theta \geq c_0(\kappa_1,\kappa_2) \geq \varepsilon_1 \geq 1 \), we know that all conditions needed in Lemma [7] are satisfied, so (90a) and (90b) hold.

From \( \kappa_1 > c_1 = \frac{1}{\rho^2(\beta)} + 1 \), we have
\[
\varepsilon_2 > 0. \tag{109}
\]

From (109) and \( \kappa_2 \in (0, \min\{\frac{\varepsilon_2}{\varepsilon_3}, \frac{1}{5}\}) \), we have
\[
\varepsilon_4 > 0 \text{ and } \varepsilon_6 > 0. \tag{110}
\]

From \( t_1 \geq (c_3(\kappa_1,\kappa_2))^\frac{1}{\beta} \) and \( c_3(\kappa_1,\kappa_2) = \frac{24(1 + \delta^2)\kappa_3}{\kappa_2} \), we have
\[
\frac{3\kappa_3}{2\kappa_2 t_1^\theta} \leq \frac{1}{16(1 + \delta^2)}.
\]

From \( \kappa_0 \geq \frac{c_0(\kappa_1,\kappa_2)}{t_1^\theta} \geq \frac{24(1 + \delta^2)\kappa_4}{\kappa_2 t_1^\theta} \), we have
\[
\frac{3\kappa_4}{2\kappa_2 \kappa_0 t_1^3} \leq \frac{1}{16(1 + \delta^2)}. \tag{112}
\]
From $\beta_k = \kappa_0(k + t_1)^{\theta}$, we have

$$\omega_k = \frac{1}{\beta_k} - \frac{1}{\beta_{k+1}} = \frac{1}{\kappa_0} \left( \frac{1}{(k + t_1)^{\theta}} - \frac{1}{(k + t_1 + 1)^{\theta}} \right) \leq \frac{1}{\kappa_0(k + t_1)^{\theta}(k + t_1 + 1)^{\theta}} \leq \frac{\kappa_0}{\beta_k^2} \leq 1. \quad (113)$$

From (113), $\eta_k = \frac{\kappa_2}{\beta_k}$, $\beta_k \geq 1$, $\omega_k \leq 1$, and $\kappa_0 \geq \frac{c_0(\kappa_1, \kappa_2)}{\epsilon_1^{m_1}} \geq (\frac{2p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2)\epsilon_2}{\epsilon_4 \epsilon_1^{m_1}})^{1/2}$, we have

$$b_{1,k} \leq \frac{p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2)\epsilon_2}{\kappa_0^{2/3} \eta_k} \leq \frac{\epsilon_4}{2}. \quad (114)$$

From (113), (114), $\kappa_0 \geq \frac{c_0(\kappa_1, \kappa_2)}{\epsilon_1^{m_1}} \geq \frac{2\epsilon_4}{\epsilon_4 \epsilon_1^{m_1}}$, and (110), we have

$$2\epsilon_4 - \epsilon_5 \omega_k - b_{1,k} \geq 2\epsilon_4 - \frac{\epsilon_5}{\kappa_0^{2/3}} \geq \frac{\epsilon_4}{2} \geq \epsilon_4 > 0. \quad (115)$$

From (113), $\eta_k = \frac{\kappa_2}{\beta_k}$, $\kappa_0 \geq \frac{c_0(\kappa_1, \kappa_2)}{\epsilon_1^{m_1}} \geq \frac{\epsilon_8}{2\kappa_0 \epsilon_1^{m_1}}$, and (110), we have

$$b_{2,k} \geq 2\epsilon_6 - \frac{\epsilon_8}{2\kappa_0 \epsilon_1^{2m}} \geq \epsilon_6 > 0. \quad (116)$$

From (111)–(113) and $\eta_k = \frac{\kappa_2}{\beta_k}$, we have

$$b_{3,k} \eta_k \leq \frac{3\kappa_3}{2\kappa_2 \epsilon_1^{m_1}} + \frac{3\kappa_4}{2\kappa_2 \epsilon_1^{2m_1}} \leq \frac{1}{8}. \quad (117)$$

From $\beta_k \geq 1$ and $\omega_k \leq 1$, we have

$$b_{3,k} \leq \epsilon_9, \quad (118a)$$

$$b_{4,k} \leq \epsilon_{10}. \quad (118b)$$

From (117), (118b), and $\kappa_0 \geq \frac{c_0(\kappa_1, \kappa_2)}{\epsilon_1^{m_1}} \geq \frac{128p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2)\kappa_2 \epsilon_{10}}{\epsilon_1^{m_1}}$, we have

$$\frac{1}{4} - (1 + \tilde{\sigma}_0^2)(b_{3,k} + 8p(1 + \sigma_0^2)b_{4,k}) \eta_k \geq \frac{1}{8} - 8p(1 + \tilde{\sigma}_0^2)(1 + \sigma_0^2)b_{4,k} \eta_k \geq \frac{1}{8} - \frac{8p(1 + \tilde{\sigma}_0^2)(1 + \sigma_0^2)\kappa_2 \epsilon_{10}}{\kappa_0 \epsilon_1^{m_1}} \geq \frac{1}{16}. \quad (119)$$

From (119), $\eta_k = \frac{\kappa_2}{\beta_k}$, $\beta_k \geq 1$, and $\omega_k \leq 1$, we have

$$b_{5,k} \leq pm \epsilon_{11}. \quad (120)$$

From (90a), (115), (116), and (118a)–(120), we know that (108a) holds.
where the first inequality holds due to (39b) and the Cauchy–Schwarz inequality; and the last

\[ (ii) \text{ From (90b), (115), (116), (118a), (118b), and (120), we have (108b).} \]

\[ (iii) \text{ From (101), (73d), and (74a), we have (108c).} \]

\[ \text{Denote } \eta_k = \sigma_k^2 \leq \sigma_k^2 \text{ and } \kappa_0 t_1^0 \geq \kappa_0 t_1^0 \geq 0, \]

\[ \text{Now it is ready to prove Theorem 1.} \]

\[ \text{Denote } \tilde{V}_k = \|x_k\|_K^2 + \|v_k + \frac{1}{\beta_k} g_k^0\|_K^2 + n(f(\bar{x}_k) - f^*). \]

\[ \text{We have } \]

\[ W_k = \frac{1}{2} \|x_k\|_K^2 + \frac{1}{2} \|v_k + \frac{1}{\beta_k} g_k^0\|_K^2 + x_k^\top K (v_k + \frac{1}{\beta_k} g_k^0) + n(f(\bar{x}_k) - f^*) \]

\[ \geq \frac{1}{2} \|x_k\|_K^2 + \frac{1}{2} \left( \frac{1}{\rho(L)} + \kappa_1 \right) \|v_k + \frac{1}{\beta_k} g_k^0\|_K^2 - \frac{1}{2} \|x_k\|_K^2 - \frac{1}{2} \kappa_1 \|v_k + \frac{1}{\beta_k} g_k^0\|_K^2 + n(f(\bar{x}_k) - f^*) \]

\[ \geq \kappa_7 \left( \|x_k\|_K + \|v_k + \frac{1}{\beta_k} g_k^0\|_K^2 \right) + n(f(\bar{x}_k) - f^*) \]

\[ \geq \kappa_7 \tilde{V}_k \geq 0, \]

where the first inequality holds due to (39b) and the Cauchy–Schwarz inequality; and the last
inequality holds due to $0 < \kappa_7 < \frac{1}{2}$. Similarly, we have

$$W_k \leq \kappa_6 \hat{V}_k. \quad (125)$$

From (108a) and (110), we have

$$\mathbb{E}_{e_k}[W_{k+1}] \leq W_k - \varepsilon_4 \|x_k\|^2_K + \frac{1}{16} \eta_k \|g_k^0\|^2 + pn\varepsilon_{12} \eta_k^2 + pn\varepsilon_{11} \eta_k \delta_k. \quad (126)$$

Then, taking expectation in $L_T$, summing (126) over $k \in [0, T-1]$, noting $\eta_k = \frac{\kappa_2}{\kappa_0(k+t_1)}$, $\theta \in (0.5, 1)$, and $\delta_k \leq \frac{\kappa_5 \eta_k}{\sqrt{n+p}}$ as stated in (7), and using (47) yield

$$\mathbb{E}[W_T] + \sum_{k=0}^{T-1} \mathbb{E} \left[ \varepsilon_4 \|x_k\|^2_K + \frac{1}{16} \eta_k \|g_k^0\|^2 \right] \leq W_0 + \frac{pn(\varepsilon_{11}\kappa_0^2 + \varepsilon_{12})\kappa_2^2}{\kappa_0^2} \sum_{k=0}^{T-1} \frac{1}{(k+t_1)^{2\theta}} \leq n\varepsilon_{14}. \quad (127)$$

Noting that $t_1^\theta = O(\sqrt{p})$, we have

$$\kappa_0 = O \left( \frac{p}{t_1^\theta} \right) = O(\sqrt{p}). \quad (128)$$

From $W_0 = O(n)$ and (128), we have

$$\varepsilon_{14} = \frac{W_0}{n} + \frac{p(\varepsilon_{11}\kappa_0^2 + \varepsilon_{12})\kappa_2^2}{(2\theta - 1)\kappa_0^2} = O(1). \quad (129)$$

From (127), (123), and $\kappa_7 > 0$, we have

$$\mathbb{E}[f(\bar{x}_T)] - f^* \leq \frac{1}{n} \mathbb{E}[W_T] \leq \varepsilon_{14}, \quad \forall T \in \mathbb{N}_0, \quad (130)$$

which gives (8b).

From (127), (124), and (110), we have

$$\sum_{k=0}^{T-1} \mathbb{E}[\|x_k\|^2_K] \leq \frac{n\varepsilon_{14}}{\varepsilon_4}, \quad \forall T \in \mathbb{N}_+. \quad (131)$$

From (73a) and (130), we have

$$\mathbb{E}[\|g_k^0\|^2] \leq 2nL_f(\mathbb{E}[f(\bar{x}_k)] - f^*) \leq 2nL_f\varepsilon_{14}. \quad (132)$$

From (74a), (131), and (132), we know that $\mathbb{E}[\|g_k^\star\|^2]$ is bounded. Then, same as the proof of the first part of Theorem 1 in [58], we have (8d).
From (123) and (125), we have

\[ 0 \leq 2\kappa_7(W_{1,k} + W_{2,k}) \leq \tilde{W}_k \leq 2\kappa_6(W_{1,k} + W_{2,k}). \]  

(133)

Denote \( \tilde{z}_k = \mathbb{E}[\tilde{W}_k] \). From (108b), (132), (133), and (7), we have

\[ \tilde{z}_{k+1} \leq (1 - a_1)\tilde{z}_k + \frac{a_2}{(k + t_1)^{2\theta}}. \]  

(134)

From \( \kappa_1 > 1 \), we have \( \kappa_6 > 1 \). From \( 0 < \kappa_2 < \frac{1}{5} \), we have \( \varepsilon_6 = \frac{1}{3}(\kappa_2 - 5\kappa_2^2) \leq \frac{1}{80} \). Thus,

\[ 0 < a_1 \leq \frac{\varepsilon_6}{\kappa_6} \leq \frac{1}{80}. \]  

(135)

From (110), we know that

\[ a_1 > 0 \text{ and } a_2 > 0. \]  

(136)

From (133)–(136) and (55), we have

\[ 2\kappa_7\mathbb{E}[\|x_k\|_\mathcal{K}^2] \leq \tilde{z}_k \leq \phi_3(k, t_1, a_1, a_2, 2\theta, \tilde{z}_0), \forall k \in \mathbb{N}_+, \]  

(137)

where the function \( \phi_3 \) is defined in (56).

Noting that \( \phi_3(k, t_1, a_1, a_2, 2\theta, \tilde{z}_0) = \mathcal{O}(\frac{n}{k^{2\theta}}) \) due to (128), from (137), we have (8c).

From (108c), we have

\[ \left( \frac{1}{\eta_k} - \frac{1}{\eta_{k+1}} + \frac{1}{\eta_{k+1}} \right)\mathbb{E}_{\mathcal{E}_k}[W_{4,k+1}] \leq \frac{W_{4,k}}{\eta_k} + \|x_k\|_\mathcal{K}^2 - \frac{3}{16}\|\bar{g}_k^0\|^2 + p\eta_k\varepsilon_{15} + (n + p)L_f^2\delta_k^2. \]  

(138)

Then, taking expectation in \( \mathcal{L}_T \), summing (138) over \( k \in [0, T - 1] \), noting (130), \( \eta_k = \frac{\kappa_2}{\kappa_0(k + t_1)^{2\theta}} \), \( \theta \in (0.5, 1) \), and \( \delta_k \leq \frac{\kappa_2\eta_k}{\sqrt{n + p}} \) as stated in (7), and using (47) yield

\[ \frac{3}{16} \sum_{k=0}^{T-1} \mathbb{E}[\|\bar{g}_k^0\|^2] \leq \frac{W_{4,0}}{\eta_0} + \sum_{k=0}^{T-1} \left( \frac{1}{\eta_{k+1}} - \frac{1}{\eta_k} \right)\mathbb{E}[W_{4,k+1}] + \sum_{k=0}^{T-1} \mathbb{E}[\|x_k\|_\mathcal{K}^2] + \frac{p(\varepsilon_{15} + L_f^2\delta_0^2)(T + t_1)^{1-\theta}}{\kappa_0(1 - \theta)} \]  

\[ \leq \frac{n\varepsilon_{14}}{\eta_0} + \sum_{k=0}^{T-1} \left( \frac{1}{\eta_{k+1}} - \frac{1}{\eta_k} \right)n\varepsilon_{14} + \sum_{k=0}^{T-1} \mathbb{E}[\|x_k\|_\mathcal{K}^2] + \frac{p(\varepsilon_{15} + L_f^2\delta_0^2)(T + t_1)^{1-\theta}}{\kappa_0(1 - \theta)}. \]
and (101), similar to the way to get (103), we have
\[ C. \text{Proof of Theorem 2} \]

(i) Substituting
\[ \theta \]
From (139), (137), (128), and \( \theta \in (0.5, 1) \), we have (8a).

\[ \text{C. Proof of Theorem 2} \]

In addition to the notations defined in Appendix B, we also denote the following notations.

\[ \bar{\alpha}, \bar{\beta}, \bar{\gamma} \]
\[ \tilde{c}_0(\kappa_1, \kappa_2) = \max \left\{ \varepsilon, \left( \frac{p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2)\varepsilon}{\varepsilon_4} \right)^{\frac{1}{4}}, 64p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2)\kappa_2\varepsilon_{10} \right\}, \]
\[ \varepsilon_7 = 8(1 + 3\kappa_2 + \kappa_4 + 2\kappa_2\kappa_4)\kappa_2L_f^4, \]
\[ \varepsilon_{10} = 6 + L_f + \frac{1}{\kappa_2}(\kappa_4 + 1)L_f^2 + (3\kappa_4 + 3)L_f^2, \]
\[ \varepsilon_{11} = L_f^2 \left( \frac{1}{256(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2)} + \frac{8\kappa_2 + 3}{p} \right), \]
\[ \varepsilon_{12} = 2(\sigma_2^2 + 2(1 + \sigma_0^2)\sigma_2^2)\varepsilon_{10}, \]
\[ \varepsilon_{13} = 8(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2)\varepsilon_{10}, \]
\[ \bar{a}_1 = \frac{1}{\kappa_6} \min\{\varepsilon_4, 2\varepsilon_6\}. \]

To prove Theorem 2, the following lemma is used.

**Lemma 9.** Suppose Assumptions \( 7 \)-\( 13 \) hold. Suppose \( \alpha_k = \alpha = \kappa_1\beta, \beta_k = \beta, \text{and} \eta_k = \eta = \frac{\kappa_2}{\beta} \),
where \( \beta \geq \tilde{c}_0(\kappa_1, \kappa_2), \kappa_1 > c_1, \text{and} \kappa_2 \in (0, c_2(\kappa_2)) \) are constants. Let \( \{x_k\} \) be the sequence generated by Algorithm 7 then

\[ \mathbb{E}_{\xi_1}[W_{k+1}] \leq W_k - \varepsilon_4\|x_k\|^2_K - 2\varepsilon_6\|v_k + \frac{1}{\beta}g_k^0\|^2_K - \frac{1}{8}\eta\|g_k^0\|^2 + pn\tilde{\varepsilon}_{12}\eta^2 + pn\tilde{\varepsilon}_{11}\eta\delta_k^2, \quad (140a) \]
\[ \mathbb{E}_{\xi_1}[\tilde{W}_{k+1}] \leq \tilde{W}_k - \varepsilon_4\|x_k\|^2_K - 2\varepsilon_6\|v_k + \frac{1}{\beta}g_k^0\|^2_K + p\tilde{\varepsilon}_{13}\eta^2\|g_k^0\|^2 + pn\tilde{\varepsilon}_{12}\eta^2 + pn\tilde{\varepsilon}_{11}\eta\delta_k^2, \quad (140b) \]
\[ \mathbb{E}_{\xi_1}[W_{4,k+1}] \leq W_{4,k} + \|x_k\|^2_{2\eta L_f^2K} - \frac{1}{8}\eta\|g_k^0\|^2 + p\varepsilon_{15}\eta^2 + (n + p)L_f^2\eta\delta_k^2. \quad (140c) \]

**Proof:** (i) Substituting \( \alpha_k = \alpha = \kappa_1\beta, \beta_k = \beta, \eta_k = \eta = \frac{\kappa_2}{\beta} \), and \( \omega_k = 0 \) into (91), (96), (100), and (101), similar to the way to get (103), we have

\[ \mathbb{E}_{\xi_1}[W_{4,k+1}] \leq W_{4,k} + \|x_k\|^2_{\eta M_1 - \eta^2 M_2 - b_1 K} - \|v_k + \frac{1}{\beta}g_k^0\|^2_{K}. \]
where

\[ \tilde{M}_1 = (\alpha - \beta) L - (1 + 3L^2) K, \]
\[ \tilde{M}_2 = \beta^2 L + (2\alpha^2 + \beta^2) L^2 + 8L^2 K + 8p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2)(6 + L_f) L^2 K, \]
\[ b_1 = 8p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2) \kappa_3 L^4 \frac{\eta}{\beta^2} + 16p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2) \kappa_5 L^4 \frac{\eta^2}{\beta^2}, \]
\[ b_2^0 = \frac{1}{2} \eta(2\beta - \kappa_3) - \frac{5}{2} \kappa_2, \]
\[ b_4 = 6 + L_f + \frac{\kappa_3}{\kappa_2} L^2 \frac{1}{\beta} + 2\kappa_5 L^2 \frac{1}{\beta^2}, \]
\[ b_5 = nL^2 \left( \frac{1}{4} p^2 b_4 \eta + 3 + 8\eta \right). \]

From (141), similar to the way to get (90a), we have

\[ \mathbb{E}_{k}[W_{k+1}] \leq W_k - \|x_k\|_{(2\varepsilon_1 - b_1)K} - \|v_k + \frac{1}{\beta_k} g_k^0\|_{2\varepsilon_6 K} \]
\[ - \eta \left( \frac{1}{4} - 8p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2) b_1 \eta \right) \|g_k^0\|^2 + 2pn(\sigma_1^2 + 2(1 + \sigma_0^2) \tilde{\sigma}_0^2) b_4 \eta^2 + b_5 \eta \delta_k^2. \]

From (142), similar to the way to get (108a), we have (140a).

(ii) Similarly, we know that (140b) holds.

(iii) Noting \( \eta_k = \eta, \beta \geq \bar{c}_0(\kappa_1, \kappa_2) \geq 64p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2) \kappa_2 \varepsilon_{10} \geq 64p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2) \kappa_2 L_f, \)
and \( \eta = \frac{\sigma_1^2}{\beta}, \)
similar to the way to get (108c), we have (140c).

We are now ready to prove Theorem 2.

From \( \beta_k = \beta = \frac{\kappa_2 \sqrt{p} T}{\sqrt{n}} \) and \( T \geq \frac{n(\bar{c}_0(\kappa_1, \kappa_2))^2}{\sqrt{p} n^2} \), we have \( \beta \geq \bar{c}_0(\kappa_1, \kappa_2) \). Thus, all conditions needed in Lemma 9 are satisfied. So (140a–140c) hold.

Taking expectation in \( L_T \), summing (140a) over \( k \in [0, T - 1] \), noting \( \eta_k = \eta = \frac{\sqrt{n}}{\sqrt{p} T} \) and \( \delta_{i,k} \leq \frac{p\frac{3}{4} n \frac{1}{4} \kappa_5 \varepsilon_{11}}{\sqrt{n + \frac{1}{p}(k+1)^2}} \) as stated in (9), and using (47) yield

\[ \frac{1}{nT} \sum_{k=0}^{T-1} \mathbb{E}[\|x_k\|^2] \leq \frac{1}{\varepsilon_3} \left( \frac{W_0}{nT} + \frac{n\varepsilon_{12}}{T} + \frac{2n\varepsilon_{11} \kappa_4^2}{T} \right). \]
Similarly, from (140c) and (9), we have
\[
\frac{1}{T} \sum_{k=0}^{T} \mathbb{E}[\|\nabla f(\bar{x}_k)\|^2] = \frac{1}{nT} \sum_{k=0}^{T} \mathbb{E}[\|g_k^0\|^2] \leq 8 \left( \frac{W_{4,0}}{nT\eta} + \frac{2L_f \sqrt{n}}{nT} \sum_{k=0}^{T} \mathbb{E}[\|x_k\|^2] \right) + \frac{2\sqrt{p}L_f^2\kappa_8^2}{\sqrt{nT}}.
\]
(144)

Noting that \( \eta = \frac{n\beta_k}{\sqrt{pT}} \), from (144) and (143), we have
\[
\frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E}[\|\nabla f(\bar{x}_k)\|^2] = 8(\bar{f}(\bar{x}_0) - f^* + \varepsilon_{15} + 2L_f^2\kappa_8^2) \sqrt{p} T + O\left( \frac{n}{T} \right),
\]
which gives (10a).

Taking expectation in \( L_T \), summing (140c) over \( k \in [0, T] \), and using (9) yield
\[
n(\mathbb{E}[f(\bar{x}_T)] - f^*) = \mathbb{E}[W_{4,T}] \leq W_{4,0} + \frac{2L_f \sqrt{n}}{\sqrt{pT}} \sum_{k=0}^{T-1} \mathbb{E}[\|x_k\|^2] + n\varepsilon_{15} + 2nL_f^2\kappa_8^2.
\]
(145)

Noting that \( W_{4,0} = O(n) \) and \( \frac{\sqrt{n}}{\sqrt{pT}} \leq 1 \) due to \( T \geq \frac{n^3}{p} \), from (143) and (145), we have (10b).

Then, from (73g), we know that there exists a constant \( \tilde{c}_g > 0 \), such that
\[
\mathbb{E}[\|g_k^0\|^2] \leq n\tilde{c}_g, \quad \forall k \in \mathbb{N}_0.
\]
(146)

From (140b), (146), (133), and (9), we have
\[
\tilde{z}_{k+1} \leq (1 - \tilde{a}_1)\tilde{z}_k + \frac{n^2\varepsilon_{11}\kappa_2^2}{\sqrt{T}(k+1)} + \frac{n^2(\varepsilon_{13}\tilde{c}_g + \varepsilon_{12})}{T}, \quad \forall 0 \leq k \leq T.
\]
(147)

From (135), we have
\[
0 < \tilde{a}_1 \leq \frac{2\varepsilon_6}{\kappa_6} \leq \frac{1}{40}.
\]
(148)

From (147) and (148), similar to the way to get (55), we have \( \tilde{z}_T = O\left( \frac{n^2}{T} \right) \). Then, from (133), we have (10c).

Similar to the proof of (8d), we have (10d).

D. Proof of Theorem 3

In addition to the notations defined in Appendix B, we also denote the following notations.
\[
\varepsilon_{16} = \frac{1}{\kappa_6} \min \left\{ \frac{\varepsilon_4\kappa_0\ell_1^\theta}{\kappa_2}, \frac{\varepsilon_6\kappa_0\ell_1^\theta}{\kappa_2}, \frac{\nu}{8} \right\},
\]
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\[ \varepsilon_{17} = \frac{16\theta^4\kappa_2 (\varepsilon_{15} + L_j^2 \kappa_5^2)}{3\nu \kappa_0}, \]
\[ a_2 = pn (\varepsilon_{11} \kappa_3^2 + \varepsilon_{12} + \varepsilon_{13} \cdot \varepsilon_{14}) \frac{\kappa_2^2}{\kappa_0^2}, \]
\[ a_3 = \frac{\kappa_2 \varepsilon_{16}}{\kappa_0}, \]
\[ a_4 = pn (\varepsilon_{11} \kappa_3^2 + \varepsilon_{12}) \frac{\kappa_2^2}{\kappa_0^2}. \]

All conditions needed in Lemma 8 are satisfied, so (108a)–(108c) hold. From (2), we have that
\[ \|\bar{g}_0\|^2 = n \|
abla f(\bar{x}_k)\|^2 \geq 2\nu n (f(\bar{x}_k) - f^*) = 2\nu W_{4,k}. \quad (149) \]
From (108a), (149), (124), and (125), we have
\[ \mathbb{E}_{\mathcal{E}_k}[W_{k+1}] \leq W_k - \varepsilon_4 \|x_k\|^2 - \varepsilon_6 \|v_k\| + \frac{1}{\beta_k} \|g_0\|^2 - \frac{\eta_k \nu n}{8} W_{4,k} + pn \varepsilon_{12} \eta_k^2 + pn \varepsilon_{11} \eta_k \delta_k^2 \]
\[ \leq W_k - \frac{\eta_k}{\kappa_6} \min \left\{ \frac{\varepsilon_4}{\eta_k}, \frac{\varepsilon_6}{\eta_k}, \frac{\nu}{8} \right\} W_k + pn \varepsilon_{12} \eta_k^2 + pn \varepsilon_{11} \eta_k \delta_k^2 \]
\[ \leq W_k - \eta_k \varepsilon_{16} W_k + pn \varepsilon_{12} \eta_k^2 + pn \varepsilon_{11} \eta_k \delta_k^2, \quad \forall k \in \mathbb{N}_0. \quad (150) \]
Denote \( z_k = \mathbb{E}[W_k], \) \( r_{1,k} = \eta_k \varepsilon_{16} \), and \( r_{2,k} = pn \varepsilon_{12} \eta_k^2 + pn \varepsilon_{11} \eta_k \delta_k^2 \). From (150), we have
\[ z_{k+1} \leq (1 - r_{1,k}) z_k + r_{2,k}, \quad \forall k \in \mathbb{N}_0. \quad (151) \]
From (11), we have
\[ r_{1,k} = \eta_k \varepsilon_{16} = \frac{a_3}{(k + t_1)^\theta}, \quad (152) \]
\[ r_{2,k} = pn \varepsilon_{12} \eta_k^2 + pn \varepsilon_{11} \eta_k \delta_k^2 \leq \frac{a_4}{(k + t_1)^{2\theta}}. \quad (153) \]
From (135), we have
\[ r_{1,k} \leq \frac{\varepsilon_6}{\kappa_6} \leq \frac{1}{80}. \quad (154) \]
From (110), we know that
\[ a_3 > 0 \text{ and } a_4 > 0. \quad (155) \]
Then, from $\theta \in (0, 1)$, (151)–(155), and (51), we have
\[ z_k \leq \phi_1(k, t_1, a_3, a_4, \theta, 2\theta, z_0), \forall k \in \mathbb{N}_+, \] (156)
where the function $\phi_1$ is defined in (52).

From $t_1 \geq (pc_3(\kappa_1, \kappa_2))^{1/\theta}$, we have
\[ t_1^\theta = \mathcal{O}(p). \] (157)

From $\kappa_0 \geq \frac{c_0(\kappa_1, \kappa_2)}{t_1}, t_1 \leq (pc_4c_3(\kappa_1, \kappa_2))^{\frac{1}{\theta}}, c_0(\kappa_1, \kappa_2) \geq \epsilon_1 \geq p\kappa_3$, and $c_3(\kappa_1, \kappa_2) = \frac{24(1 + \sigma_0^2)\kappa_3}{\kappa_2}$, we have
\[ \frac{\kappa_2}{\kappa_0} \leq \frac{\kappa_2 t_1^\theta}{c_0(\kappa_1, \kappa_2)} \leq \frac{\kappa_2 pc_4c_3(\kappa_1, \kappa_2)}{p\kappa_3} \leq 24(1 + \sigma_0^2)c_4. \] (158)

Thus,
\[ \phi_1(k, t_1, a_3, a_4, \theta, 2\theta, z_0) = \mathcal{O}\left(\frac{pm}{(k + t_1)^{\theta}}\right). \] (159)

From (124), we have
\[ \|x_k\|_K^2 + W_{4,k} \leq \hat{V}_k \leq \frac{W_k}{\kappa_7}. \] (160)

From (73g), (156), (159), and (160), we get
\[ \mathbb{E}[\|\tilde{g}_k\|^2] = \mathcal{O}\left(\frac{pm}{(k + t_1)^{\theta}}\right), \forall k \in \mathbb{N}_+. \] (161)

From (157) and (161), we know that there exists a constant $\bar{c}_g > 0$, such that
\[ \mathbb{E}[\|\tilde{g}_k\|^2] \leq n\bar{c}_g, \forall k \in \mathbb{N}_0. \] (162)

From (108b), (162), (133), and (11), we have
\[ \hat{z}_{k+1} \leq (1 - a_1)\hat{z}_k + \frac{\tilde{a}_2}{(t + t_1)^{2\theta}}. \] (163)

Using (55), from (135) and (163), we have
\[ \hat{z}_k \leq \phi_3(k, t_1, \hat{a}_2, 2\theta, \hat{z}_0), \forall k \in \mathbb{N}_+, \] (164)
where the function $\phi_3$ is defined in (56). From (164), (133), (56), and (158), we have
\[
E[\|x_k\|_K^2] \leq \frac{1}{\kappa_7} \tilde{z}_k \leq \frac{1}{\kappa_7} \phi_3(k, t_1, \tilde{a}_1, 2\theta, \tilde{z}_0) = O\left(\frac{pn}{(k + t_1)^{2\theta}}\right),
\] (165)
which yields (12a).

From (108c), (149), and $\delta_k \leq \frac{\kappa_1 \sqrt{pn}}{\sqrt{n + p}}$ as stated in (11), we have
\[
E[W_{4,k+1}] \leq E[W_{4,k}] - \frac{3\nu}{8} \eta_k E[W_{4,k}] + \|x_k\|_{L_f^2}^2 + p\varepsilon_{15}^2 \eta_k^2 + pL^2_f \kappa_3^2 \eta_k^2.
\] (166)

Similar to the way to prove (51), from (165) and (166), we have
\[
E[f(\bar{x}_T) - f^\star] \leq \frac{\varepsilon_{17}p}{n(T + t_1)^{2\theta}} + O\left(\frac{p}{(T + t_1)^{2\theta}}\right).
\] (167)

From (158), we have
\[
\varepsilon_{17} = \frac{16\theta^4 \kappa_2 (\varepsilon_{15} + L_f^2 \kappa_5^2)}{3\nu \kappa_0} \leq \frac{128\theta^4 \kappa_2 (\varepsilon_{15} + L_f^2 \kappa_5^2)(1 + \tilde{a}_0^2)c_4}{\nu}.
\] (168)

Thus, from (167) and (168), we have (12b).

**E. Proof of Theorem 4**

In addition to the notations defined in Appendices B and D, we also denote the following notations.

\[
\hat{c}_3(\kappa_0, \kappa_1, \kappa_2) = \max \left\{ \frac{c_0(\kappa_1, \kappa_2)}{\kappa_0}, \frac{c_3(\kappa_1, \kappa_2)}{\kappa_0}, \frac{2}{3\varepsilon_4}, \frac{2}{3\varepsilon_6} \right\},
\]
\[
\hat{\varepsilon}_{17} = \frac{4\kappa^2_2 (\varepsilon_{15} + L_f^2 \kappa_5^2)}{\kappa^2_0 \left(\frac{3\varepsilon_6}{8\kappa_0} - 1\right)},
\]
\[
\hat{a}_2 = \frac{pn(\varepsilon_{11} \kappa_5^2 + \varepsilon_{12} + \varepsilon_{13} \hat{c}_g \kappa_5^2)}{\kappa^2_0},
\]
\[
\hat{a}_3 = \frac{2}{3\kappa_6}.
\]

From $t_1 \geq \hat{c}_3(\kappa_0, \kappa_1, \kappa_2) \geq \frac{c_0(\kappa_1, \kappa_2)}{\kappa_0}$, we have
\[
\kappa_0 \geq \frac{c_0(\kappa_1, \kappa_2)}{t_1}.
\]

Thus, all conditions needed in Lemma 8 are satisfied, so (151)–(155) still hold when $\theta = 1.$
From \( t_1 \geq \hat{c}_3(\kappa_0, \kappa_1, \kappa_2) \geq \frac{2}{3\varepsilon_4} \), we have
\[
\epsilon t_1 \leq \frac{2}{3\kappa_6}.
\] (169)

From \( t_1 \geq \hat{c}_3(\kappa_0, \kappa_1, \kappa_2) \geq \frac{2}{3\varepsilon_6} \), we have
\[
\epsilon t_1 \leq \frac{2}{3\kappa_6}.
\] (170)

From \( \kappa_0 \in \left( \frac{3\nu\kappa_2}{16}, \frac{3\nu\kappa_2}{16} \right) \), we have
\[
\frac{16}{3\nu} < \frac{\kappa_2}{\kappa_0} \leq \frac{16}{3\hat{c}_0\nu}.
\] (171)

Thus,
\[
\frac{\nu\kappa_2}{8\kappa_0\kappa_0} > \frac{2}{3\kappa_6}.
\] (172)

Hence, from (169), (170), (172), and \( \kappa_6 > 1 \) due to \( \kappa_1 > 1 \), we have
\[
a_3 > \hat{a}_3 > 0 \quad \text{and} \quad \hat{a}_3 < \frac{2}{3}.
\] (173)

Then from \( \theta = 1 \), (151)—(155), (173), and (53), we have
\[
z_k \leq \phi_2(k, t_1, \hat{a}_3, a_4, 2, z_0, \forall k \in \mathbb{N}_+),
\] (174)

where the function \( \phi_2 \) is defined in (54).

From (173) and (171), we have \( \phi_2(k, t_1, \hat{a}_3, a_4, 2, z_0) = \mathcal{O}\left(\frac{n_1^{a_3}}{(k + t_1)^{a_3}} + \frac{m}{(k + t_1)^{a_3}t_1^{1-a_3}}\right) \). Hence, from (73g), (174), and (160), we get
\[
\mathbb{E}[\|g_k^0\|^2] = \mathcal{O}\left(\frac{n_1^{a_3}}{(k + t_1)^{a_3}} + \frac{m}{(k + t_1)^{a_3}t_1^{1-a_3}}\right), \quad \forall k \in \mathbb{N}_+.
\] (175)

Noting that \( t_1 > \hat{c}_3(\kappa_0, \kappa_1, \kappa_2) \geq \frac{c_0(\kappa_1, \kappa_2)}{\kappa_0} \geq \frac{\nu\kappa_2}{\kappa_0} \), from (175) and (171), we know that there exists a constant \( \hat{c}_g > 0 \), such that
\[
\mathbb{E}[\|g_k^0\|^2] \leq n\hat{c}_g, \quad \forall k \in \mathbb{N}_0.
\] (176)

From (108b), (176), (133), and (13), we have
\[
\tilde{z}_{k+1} \leq (1 - a_1)\tilde{z}_k + \frac{\hat{a}_2}{(t + t_1)^2}.
\] (177)
Using (55), from (135) and (177), we have
\[ \dot{z}_k \leq \phi_3(k, t_1, a_1, \hat{a}_2, 2, \dot{z}_0), \quad \forall k \in \mathbb{N}_+, \] (178)
where the function \( \phi_3 \) is defined in (56). From (178), (133), (56), and (171), we have
\[ \mathbb{E}[\|x_k\|^2] \leq \frac{1}{\kappa_7} \dot{z}_k \leq \frac{1}{\kappa_7} \phi_3(k, t_1, a_1, \hat{a}_2, 2, \dot{z}_0) = o\left(\frac{pn}{(k + t_1)^2}\right), \] (179)
which yields (14a).

From \( \kappa_0 < \frac{3\nu_2}{16} \), we have
\[ \frac{3\nu_2}{8\kappa_0} > 2. \] (180)

Same to the way to prove (53), from (179), (180), and (166), we have
\[ \mathbb{E}[f(x_T) - f^*] \leq \frac{\dot{\varepsilon}_17p}{n(T + t_1)} + o\left(\frac{p}{(T + t_1)^2}\right). \] (181)

From (171), we have
\[ \dot{\varepsilon}_17 = \frac{4\kappa_2^2(\varepsilon_{15} + L_f^2\kappa_2^2)}{2\kappa_0(\varepsilon_{15} + L_f^2\kappa_2^2)} \leq \frac{1024(\varepsilon_{15} + L_f^2\kappa_2^2)}{9\hat{c}_0(2 - \hat{c}_0)p^2}. \] (182)

Thus, from (181) and (182), we have (14b).

\[ F. \ Proof \ of \ Theorem 5 \]

In addition to the notations defined in Appendices B, D, and E, we also denote the following notations.
\[ \hat{\varepsilon}_3(\kappa_0, \kappa_1, \kappa_2) = \max \left\{ \frac{2}{3\kappa_4}, \frac{2}{3\kappa_6}, \frac{\dot{\varepsilon}_1}{\kappa_0}, \frac{2\varepsilon_5}{\kappa_0\varepsilon_4}, \left(\frac{2p(1 + \sigma_0^2)\dot{\varepsilon}_7}{\kappa_0^2\varepsilon_4}\right)^\frac{1}{2}, \left(\frac{\varepsilon_8}{2\kappa_0\varepsilon_6}\right)^\frac{1}{2}, \frac{16L_f\kappa_3}{\nu\kappa_2}, \left(\frac{16L_f\kappa_4}{\nu\kappa_2}\right)^\frac{1}{4}, \frac{64p(1 + \sigma_0^2)L_f\kappa_2\dot{\varepsilon}_{10}}{\nu\kappa_0}, \frac{p(1 + \sigma_0^2)\kappa_2\dot{\varepsilon}_{10}}{4\kappa_0} \right\}, \]
\[ \dot{\varepsilon}_1 = \max\{1 + 3L_f^2, (8 + 4p(1 + \sigma_0^2))(6 + L_f))^\frac{1}{2}L_f, p\kappa_3\}, \]
\[ \dot{\varepsilon}_7 = 4(6 + 5\kappa_2 + 2\kappa_4 + 8\kappa_2\kappa_4)\kappa_2L_f^4 + \frac{(1 + 2L_f^2)\kappa_2}{2p(1 + \sigma_0^2)} + \left(\frac{5}{p(1 + \sigma_0^2)} + 16\right)L_f^2\kappa_2^2, \]
\[ \dot{\varepsilon}_{10} = 10 + L_f + \frac{1}{\kappa_2}(2\kappa_4 + 1)L_f^2 + (8\kappa_4 + 5)L_f^2, \]
\[ \dot{\varepsilon}_{11} = L_f^2\left(\frac{1}{1 + \sigma_0^2} + \frac{13\kappa_2 + 4}{p}\right). \]
\[ \delta_{12} = 2\varepsilon_{10}\sigma_1^2 + \frac{2\varepsilon_9\sigma_2^2}{3p} + 8(1 + \sigma_0^2)\varepsilon_{10}\sigma_2^2, \]
\[ \delta_{13} = \frac{4L_f\varepsilon_9}{3p} + 8L_f(1 + \sigma_0^2)\varepsilon_{10}, \]
\[ \sigma_2^2 = 2L_f f^* - \frac{2L_f}{n} \sum_{i=1}^{n} f_i^* \geq 0. \]

To prove Theorem 5, the following lemma is used.

**Lemma 10.** Suppose Assumptions 7–10 hold and each \( f_i^* > -\infty \). Suppose \( \alpha_k = \kappa_1\beta_k, \beta_k = \kappa_0(k + t_1)\theta, \) and \( \eta_k = \frac{n^2}{\beta_k}, \) where \( \theta \in [0, 1], \kappa_0 > 0, \kappa_1 > c_1, \kappa_2 \in (0, c_2(\kappa_1)), \) and \( t_1^0 \geq \hat{c}_3(\kappa_0, \kappa_1, \kappa_2) \) with any constant \( \nu > 0. \) Let \( \{x_k\} \) be the sequence generated by Algorithm 7 then

\[ \mathbb{E}_{\xi_k}[W_{k+1}] \leq W_k - \varepsilon_4\|x_k\|^2_K - \varepsilon_6\|v_k + \frac{1}{\beta_k}g_k^0\|^2_K - \frac{1}{4}\eta_k\|g_k^0\|^2 + \frac{3\nu}{8}\eta_k W_{4,k} + \frac{p\varepsilon_{13}\eta_k^2}{2} + \frac{p\varepsilon_{11}\eta_k\delta_k^2}{2}, \]
\[ \mathbb{E}_{\xi_k}[\tilde{W}_{k+1}] \leq \tilde{W}_k - \varepsilon_4\|x_k\|^2_K - \varepsilon_6\|v_k + \frac{1}{\beta_k}g_k^0\|^2_K + \frac{p\varepsilon_{13}\eta_k^2}{2} + \frac{p\varepsilon_{11}\eta_k\delta_k^2}{2}, \]
\[ \mathbb{E}_{\xi_k}[W_{4,k+1}] \leq W_{4,k} + \|x_k\|^2_{2\eta_k L_f^2 K} - \frac{1}{4}\eta_k\|g_k^0\|^2 + \frac{\nu}{8}\eta_k W_{4,k} + 2p\eta_k^2 L_f(\sigma_1^2 + 2(1 + \sigma_0^2)\sigma_2^2) + (n + p)L_f^2\eta_k\delta_k^2. \]

**Proof:** The proof of this lemma is similar to the proof of Lemma 8. For the sake of completeness, the proof is included here.

We know that (73a)–(73g) and (85) still hold since Assumptions 3 and 4 hold. Moreover, (102) still holds.

We have

\[ \|g_k^0\|^2 = \sum_{i=1}^{n} \|\nabla f_i(\bar{x}_k)\|^2 \leq \sum_{i=1}^{n} 2L_f(f_i(\bar{x}_k) - f^*) = 2L_f n(f(\bar{x}_k) - f^*) + n\sigma_2^2, \]

where the inequality holds due to (42).

We have

\[ \|g_k\|^2 = \|g_k - g_k^0 + g_k^0\|^2 \leq 2(\|g_k - g_k^0\|^2 + \|g_k^0\|^2) \leq 2(L_f^2\|x_k\|^2_K + 2L_f W_{4,k} + n\sigma_2^2), \]
where the first inequality holds due to the Cauchy–Schwarz inequality; and the last inequality holds due to (76) and (184).

From (85) and (185), we have

$$E_{\xi_k}[\|g_k^0\|^2] \leq 16p(1 + \sigma_0^2)L_fW_{4,k} + 8p(1 + \sigma_0^2)L_f^2\|x_k\|^2 + 4n\rho\sigma_1^2 + 8n(1 + \sigma_0^2)\sigma_2^2 + \frac{1}{2}np^2L_f^2\delta_k^2. \tag{186}$$

From the Cauchy–Schwarz inequality, (73f), and (184), we have

$$\|g_{k+1}^0\|^2 \leq \|g_{k+1}^0 - g_k^0 + g_k^0\|^2 \leq 2(\|g_{k+1}^0 - g_k^0\|^2 + \|g_k^0\|^2) \leq 2(\eta_k^2L_f^2\|g_k^0\|^2 + 2L_fW_{4,k} + n\sigma_2^2). \tag{187}$$

Then, from (102), (186), and (187), we get

$$E_{\xi_k}[W_{k+1}] \leq W_k - \|x_k\|^2_{\eta_k\tilde{M}_{1,k} - \eta_k^2\tilde{M}_{2,k}} - \|v_k + \frac{1}{\beta_k}g_k^0\|^2_{\beta_k^2K} - \frac{1}{4}\eta_k\|g_k^0\|^2 + L(L_f(\frac{4}{3}b_{3,k} + 8p(1 + \sigma_0^2)b_{4,k})\eta_k^2W_{4,k} + 2p\mu\sigma_1^2b_{4,k}\eta_k^2 + n\sigma_2^2(\frac{2}{3}b_{3,k} + 4p(1 + \sigma_0^2)b_{4,k})\eta_k^2 + b_{5,k}\eta_k\delta_k^2, \tag{188}$$

where

$$\tilde{M}_{2,k} = M_{2,k}^0 + 4p(1 + \sigma_0^2)(6 + L_f)L_f^2K,$$

$$b_{1,k} = 4p(1 + \sigma_0^2)\kappa_3L_f^4\frac{\eta_k^2}{\beta_k^2} + 8p(1 + \sigma_0^2)\kappa_5L_f^4\frac{\eta_k^2}{\beta_k^2} + \left(\frac{1}{2} + L_f^2\right)\eta_k\omega_k + 4p(1 + \sigma_0^2)\kappa_4L_f^4\frac{\eta_k\omega_k}{\beta_k^2} + (5 + 16p(1 + \sigma_0^2) + 8p(1 + \sigma_0^2)\kappa_3L_f^2)\eta_k^2L_f^2\eta_k\omega_k + 8p(1 + \sigma_0^2)\kappa_4L_f^4\frac{\eta_k\omega_k^2}{\beta_k^2},$$

$$b_{4,k} = 6 + L_f + \frac{3\kappa_3L_f^2}{\beta_k^2} + 2\kappa_5L_f^2\frac{1}{\beta_k^2} + \frac{\kappa_4L_f^2}{\beta_k^2}\omega_k + (4 + 2\kappa_3L_f^2)\omega_k + 2\kappa_4L_f^2\omega_k + 2\kappa_4L_f^2\omega_k + 2\kappa_4L_f^2\omega_k + 2\kappa_4L_f^2\omega_k, \quad b_{4,k} = \mu L_f^2\left(\frac{1}{4}p^2\eta_k\right)$$

Then, from (188) and $t_1^0 \geq \check{c}_3(\kappa_0, \kappa_1, \kappa_2) \geq \frac{1}{2}\kappa_0^2$, similar to the way to get (90a)–(90b), we have

$$E_{\xi_k}[W_{k+1}] \leq W_k - \|x_k\|^2_{(2\epsilon_4 - \epsilon_5\omega_k - b_{1,k})K} - \|v_k + \frac{1}{\beta_k}g_k^0\|^2_{\beta_k^2K} - \frac{1}{4}\eta_k\|g_k^0\|^2 \tag{188}$$
\[ + \frac{2}{3} b_{3,k} + 8p(1 + \sigma_0^2) b_{4,k} \eta_k^2 W_{4,k} + 2pm\sigma_1^2 \bar{b}_{4,k} \eta_k^2 \]
\[ + n\sigma_2^2 \left( \frac{2}{3} b_{3,k} + 4p(1 + \sigma_0^2) b_{4,k} \right) \eta_k^2 + \bar{b}_{5,k} \eta_k \delta_k^2, \quad (189a) \]
\[ \mathbb{E}_{\mathbb{C}_k}[\tilde{W}_{k+1}] \leq \tilde{W}_k - \left\| x_k \right\|^2_{2\varepsilon_4 - \varepsilon_5\omega_k - b_{1,k}^2} - \left\| v_k + \frac{1}{\beta_k} g_k^0 \right\|^2_{b_{2,k}} \]
\[ + \frac{2}{3} b_{3,k} + 8p(1 + \sigma_0^2) b_{4,k} \eta_k^2 W_{4,k} + 2pm\sigma_1^2 \bar{b}_{4,k} \eta_k^2 \]
\[ + n\sigma_2^2 \left( \frac{2}{3} b_{3,k} + 4p(1 + \sigma_0^2) b_{4,k} \right) \eta_k^2 + \bar{b}_{5,k} \eta_k \delta_k^2. \quad (189b) \]

From \( t_1^0 \geq \tilde{c}_3(\kappa_0, \kappa_1, \kappa_2) \geq \max \left\{ \frac{2\varepsilon_4}{\kappa_0 t_1^0}, \frac{(2p + \sigma_0^2) \varepsilon_7}{\kappa_0^2 t_1^0} \right\}, \) similar to the way to prove \( [115], \) we have
\[ 2\varepsilon_4 - \varepsilon_5\omega_k - \bar{b}_{1,k} \geq 2\varepsilon_4 - \frac{\varepsilon_5}{\kappa_0 t_1^0} \leq \frac{p(1 + \sigma_0^2) \varepsilon_7}{\kappa_0^2 t_1^0} \geq \varepsilon_4 > 0. \quad (190) \]

From \( t_1^0 \geq \tilde{c}_3(\kappa_0, \kappa_1, \kappa_2) \geq \left( \frac{\varepsilon_8}{2\kappa_0 t_1^0} \right)^2, \) similar to the way to prove \( [116], \) we have
\[ b_{2,k} \geq 2\varepsilon_6 - \frac{\varepsilon_8}{2\kappa_0 t_1^0} \geq \varepsilon_6 > 0. \quad (191) \]

From \( \beta_k \geq 1 \) and \( \omega_k \leq 1, \) we have
\[ \bar{b}_{4,k} \leq \varepsilon_{10}. \quad (192) \]

From \( t_1^0 \geq \tilde{c}_3(\kappa_0, \kappa_1, \kappa_2) \geq \max \left\{ \frac{16L_f \kappa_3}{\nu \kappa_2}, \left( \frac{16L_f \kappa_4}{\nu \kappa_3} \right)^{\frac{1}{2}}, \frac{64p(1 + \sigma_0^2) L_f \kappa_2 \varepsilon_{10}}{\nu \kappa_0} \right\}, \) similar to the way to get \( [119], \) we have
\[ \frac{4}{3} L_f (b_{3,k} + 6p(1 + \sigma_0^2) b_{4,k}) \eta_k \leq \frac{2\kappa_3}{\kappa_2 t_1^0} + \frac{2\kappa_4}{\kappa_0 \kappa_2 \kappa_1^\theta} + \frac{8p(1 + \sigma_0^2) L_f \kappa_2 \varepsilon_{10}}{\kappa_0 \kappa_1^\theta} \leq \frac{3\nu}{8}. \quad (193) \]

From \( t_1^0 \geq \tilde{c}_3(\kappa_0, \kappa_1, \kappa_2) \geq \frac{p(1 + \sigma_0^2) \kappa_2 \varepsilon_{10}}{2\kappa_0}, \) we have
\[ \bar{b}_{5,k} \leq \varepsilon_{11}. \quad (194) \]

From \( [189a] - [189b] \) and \( [190] - [194], \) we have \( [183a] - [183b]. \)

From \( [101], \) \( [73d], \) and \( [186], \) we have
\[ \mathbb{E}_{\mathbb{C}_k}[W_{4,k+1}] \leq W_{4,k} - \frac{1}{4} \eta_k \| g_k^0 \|^2 + \| x_k \|^2_{\eta_k L_f^2 K} + nL_f^2 \eta_k \delta_k^2 \]
\[ - \frac{1}{4} \eta_k \| g_k^0 \|^2 + \frac{1}{4} \eta_k L_f \left( \frac{16p(1 + \sigma_0^2) L_f W_{4,k}}{n} + \frac{8p(1 + \sigma_0^2) L_f^2 \| x_k \|^2_{K}}{n} \right) + \frac{4}{2} \sigma_1^2 + 8p(1 + \sigma_0^2) \sigma_2^2 + \frac{1}{2} \eta_k^2 \| g_k^0 \|^2. \quad (195) \]
From $\eta_k = \frac{\kappa_2}{\beta_k} \leq \frac{\kappa_2}{\kappa_0} \beta_k$, $t_1^\theta \geq \tilde{c}_3(\kappa_0, \kappa_1, \kappa_2) \geq \max\{\frac{64p(1+\sigma_0^2)L_f\kappa_2\tilde{\epsilon}_{10}}{\nu\kappa_0}, \frac{p(1+\sigma_0^2)\kappa_2\tilde{\epsilon}_{10}}{4\kappa_0}\}$, and $\tilde{\epsilon}_{10} > (1 + 2\sqrt{10(8\kappa_4 + 5)})L_f > 23L_f$ due to $\kappa_4 > 1$, we have

\[
\frac{8p(1+\sigma_0^2)L_f^2}{n} \eta_k^2 \leq \frac{8p(1+\sigma_0^2)\kappa_2 L_f^2}{\kappa_0 \beta_k \eta_k} \frac{\nu}{8\eta_k}, \tag{196a}
\]

\[
\frac{4p(1+\sigma_0^2)L_f^3}{n} \eta_k^2 \leq \frac{4p(1+\sigma_0^2)\kappa_2 L_f^2}{\kappa_0 \beta_k \eta_k} \eta_k L_f^2 \leq \frac{16}{23} \eta_k L_f^2, \tag{196b}
\]

\[
\frac{1}{2} \frac{\eta_k^2}{L_f} < \frac{2}{23} \eta_k, \tag{196c}
\]

\[
\frac{1}{4} \frac{p^2 \eta_k^2}{L_f^3} < p L_f \eta_k. \tag{196d}
\]

From (195)–(196d), we have (183c).

Now we are ready to prove Theorem 5.

All conditions needed in Lemma [10] are satisfied, so (183a)–(183c) still hold when $\theta = 1$.

From (183a)–(183c), (149), (133), (124), (125), (15), and $t_1 \geq \tilde{c}_3(\kappa_0, \kappa_1, \kappa_2) \geq \max\{\frac{2}{3\tilde{\epsilon}_4}, \frac{2}{3\tilde{\epsilon}_6}\}$, similar to the way to get (14a) and (14b), we have (16a) and (16b).

G. Proof of Theorem 6

In addition to the notations defined in Appendix C, we also denote the following notations.

\[
\varepsilon = \frac{1}{2} + \frac{1}{2} \max\{1 - \tilde{\varepsilon}_{16}, 2\tilde{\varepsilon}^2\},
\]

\[
\tilde{\varepsilon}_{16} = \frac{1}{4\kappa_6} \min\{4\varepsilon_4, 8\varepsilon_6, \eta \nu\}.
\]

All conditions needed in Lemma [9] are satisfied, so (140a) still holds.

(i) Taking expectation in $\mathcal{L}_T$, summing (140a) over $k \in [0, T - 1]$, and using $\delta_{t,k} \in (0, \kappa_6 \tilde{\varepsilon}^k]$ yield

\[
E[W_T] + \varepsilon_4 \sum_{k=0}^{T-1} E[\|x_k\|^2_K] + \frac{1}{8} \eta \sum_{k=0}^{T-1} E[\|g_k^0\|^2] \leq W_0 + pn\tilde{\varepsilon}_{12}\eta^2 T + \frac{pn\tilde{\varepsilon}_{11}\kappa_2^2 \eta}{1 - \tilde{\varepsilon}^2},
\]

which further implies

\[
\sum_{k=0}^{T-1} E[\|x_k\|^2_K] \leq \frac{1}{\varepsilon_4} \left(W_0 + pn\tilde{\varepsilon}_{12}\eta^2 T + \frac{pn\tilde{\varepsilon}_{11}\kappa_2^2 \eta}{1 - \tilde{\varepsilon}^2}\right). \tag{197}
\]

Therefore, (18a) holds due to $\eta = \mathcal{O}(\frac{1}{p})$. 

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From (140c), we have
\[
E[W_{4,T}] \leq W_{4,0} + \sum_{k=0}^{T-1} E[\|x_k\|^2_{2nL^2_\delta K}] - \frac{1}{8} \sum_{k=0}^{T-1} E[\|g_k^n\|^2] + p \varepsilon_{15} \eta^2 T + \frac{(n+p)L^2_j \kappa \eta}{1 - \varepsilon^2}. \tag{198}
\]

From (197) and (198), we have
\[
\sum_{k=0}^{T-1} E[\|g_k^n\|^2] \leq \frac{8W_{4,0}}{\eta} + \sum_{k=0}^{T-1} E[\|x_k\|^2_{16L^2_\delta K}] + 8p \varepsilon_{15} \eta T + \frac{8(n+p)L^2 \kappa^2}{1 - \varepsilon^2}
\]
\[
\leq \frac{8W_{4,0}}{\eta} + \frac{16L^2_\delta}{\varepsilon_4} \left( W_0 + p \varepsilon_{12} \eta^2 T + \frac{p \varepsilon_{11} \kappa \eta^2}{1 - \varepsilon^2} \right) + 8p \varepsilon_{15} \eta T + \frac{8(n+p)L^2 \kappa^2}{1 - \varepsilon^2},
\]
which gives (18c).

From (73g), (197), and (198), we have
\[
E[\|g_k^n\|^2] \leq 2L_f E[W_{4,k}]
\]
\[
\leq 2L_f \left( W_{4,0} + p \varepsilon_{15} \eta^2 k + \frac{(n+p)L^2 \kappa^2}{1 - \varepsilon^2} \right) + \frac{4L^2_\delta \eta}{\varepsilon_4} \left( W_0 + p \varepsilon_{12} \eta^2 k + \frac{p \varepsilon_{11} \kappa \eta^2}{1 - \varepsilon^2} \right), \quad \forall k \in \mathbb{N}_0. \tag{199}
\]

From (140b), (133), (148), \( \delta_{i,k} \in (0, \kappa \varepsilon^k) \), (199), and , we have
\[
E[\tilde{W}_{k+1}] \leq E[(1 - \tilde{a}_1) \tilde{W}_k + p \varepsilon_{13} \eta^2 \|g_k^n\|^2 + p \varepsilon_{12} \eta^2 + p \varepsilon_{11} \eta \kappa^2 \tilde{\varepsilon}^2 k]
\]
\[
\leq (1 - \tilde{a}_1)^k \tilde{W}_0 + p \varepsilon_{13} \eta^2 \sum_{\tau=0}^{k} (1 - \tilde{a}_1)^\tau E[\|g_k^n\|^2]
\]
\[
+ p \varepsilon_{12} \eta^2 \sum_{\tau=0}^{k} (1 - \tilde{a}_1)^\tau + p \varepsilon_{11} \eta \kappa^2 \sum_{\tau=0}^{k} (1 - \tilde{a}_1)^\tau \tilde{\varepsilon}^{2(k-\tau)}
\]
\[
= O(p \eta \varepsilon_1^2 + \varepsilon_{15} \eta^2 (k+1) + \varepsilon_{12} \eta \varepsilon_1^2 (k+1)),
\]
which gives (18b).

(ii) If Assumption 6 also holds, then (149) holds. From (140a), (149), and (125), for any \( k \in \mathbb{N}_0 \), we have
\[
E[W_{k+1}] \leq E \left[ W_k - \varepsilon_4 \|x_k\|^2_{K} - 2 \varepsilon_6 \|v_k + \frac{1}{\beta} g_k^n\|^2_{K} - \frac{\eta m}{4} (f(x_k) - f^*) \right]
\]
\[
+ 2pn(\sigma^2 + 2(1 + \sigma_0^2) \sigma_2^2) \varepsilon_{10} \eta^2 + p n \varepsilon_{11} \eta \delta_k^2
\]
\[
\leq E[W_k - \varepsilon_{10} W_k] + 2pn(\sigma_1^2 + 2(1 + \sigma_0^2) \sigma_2^2) \varepsilon_{10} \eta^2 + p n \varepsilon_{11} \eta \delta_k^2. \tag{200}
\]
From (135), we have

\[ 0 < \bar{\epsilon}_{16} \leq \frac{2\bar{\epsilon}_6}{\kappa_6} \leq \frac{1}{40}. \]  

(201)

From (200), (124), (201), and \( \delta_{i,k} \in (0, \kappa_\delta \bar{\epsilon}^k] \), we have

\[
E[W_{k+1}] \leq (1 - \bar{\epsilon}_{16})^{k+1} W_0 + 2pn(\sigma_1^2 + 2(1 + \sigma_0^2)\bar{\epsilon}_{10}\eta^2 \sum_{\tau=0}^{k} (1 - \bar{\epsilon}_{16})^\tau
\]

\[ + pn\bar{\epsilon}_{11}\kappa_7^2 \eta \sum_{\tau=0}^{k} (1 - \bar{\epsilon}_{16})^\tau \bar{\epsilon}_{2(k-\tau)}, \forall k \in \mathbb{N}_0. \]  

(202)

From (124), (202), (46), and \( \varepsilon > \max\{1 - \bar{\epsilon}_{16}, \bar{\epsilon}^2\} \), we have

\[
E[\|\mathbf{x}_k\|_K^2 + n(f(\bar{x}_k) - f^*)] \leq \frac{1}{\kappa_7} E[W_k]
\]

\[ \leq \frac{n}{\kappa_7} \left( W_0 + \frac{p\bar{\epsilon}_{11}\kappa_7^2 \eta}{\varepsilon - \bar{\epsilon}^2} \right) \varepsilon^k + \frac{2n\bar{\epsilon}_{10}\eta}{\kappa_7 \bar{\epsilon}_{16}} (\sigma_1^2 + 2(1 + \sigma_0^2)\sigma_2^2) p\eta, \forall k \in \mathbb{N}_0, \]  

(203)

which gives (19).

**H. Proof of Theorem 7**

We denote the following notations.

\[ d_1 = \frac{\rho_2(L)}{2\rho(L^2)}, \]

\[ d_2(\gamma) = \min\left\{ \frac{4\epsilon_1}{9L_f^2}, \frac{1}{64p(1 + \sigma_0^2)(1 + \sigma_0^2)(2\epsilon_2 + L_f)} \right\}, \]

\[ \epsilon_1 = \frac{1}{2} \gamma \rho_2(L) - \gamma^2 \rho(L^2), \]

\[ \epsilon_2 = \frac{1 + 2\gamma \rho_2(L)}{2\gamma \rho_2(L)}, \]

\[ \epsilon_3 = \left( 2\epsilon_2 + \frac{L_f}{n} \right) \epsilon_5, \]

\[ \epsilon_4 = \frac{L_f^2}{4} \left( \frac{1}{64(1 + \sigma_0^2)(1 + \sigma_0^2)} + \frac{4}{p} \right), \]

\[ \epsilon_5 = 2(\sigma_1^2 + 2(1 + \sigma_0^2)\sigma_2^2), \]

\[ \epsilon_6 = \frac{W_{1.0} + W_{4.0}}{n} + p(\epsilon_3 + \kappa_8^2 \epsilon_4) \kappa_7^2 \]

\[ \epsilon_7 = pn\kappa_8^2 (32(1 + \sigma_0^2)(1 + \sigma_0^2)L_f \epsilon_2 \epsilon_6 + 2\epsilon_2 \epsilon_5 + \epsilon_4 \kappa_8^2). \]
To prove Theorem 7, the following lemma is used.

**Lemma 11.** Suppose Assumptions 1–5 hold. Suppose $\gamma \in (0, d_1)$ and $\eta_k \in (0, d_2(\gamma))$. Let $\{x_k\}$ be the sequence generated by Algorithm 2, then

\[
E_{\Sigma_k}[W_{1,k+1} + W_{4,k+1}] \leq W_{1,k} + W_{4,k} - \|x_k\|^2_{\frac{1}{2}K} - \frac{1}{8}\eta_k\|g_k^0\|^2 + pn\epsilon_3\eta_k^2 + pn\epsilon_4\eta_k\delta_k^2, \tag{204a}
\]

\[
E_{\Sigma_k}[W_{1,k+1}] \leq W_{1,k} - \|x_k\|^2_{\frac{1}{2}K} + 16p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2)\epsilon_2\eta_k^2\|g_k^0\|^2 + 2pn\epsilon_5\eta_k^2 + pn\epsilon_4\eta_k\delta_k^2, \tag{204b}
\]

\[
E_{\Sigma_k}[W_{4,k+1}] \leq W_{4,k} + \|x_k\|^2_{2L_f^2\eta_kK} - \frac{1}{8}\eta_k\|g_k^0\|^2 + pL_f\epsilon_5\eta_k^2 + (p + n)L_f^2\eta_k\delta_k^2. \tag{204c}
\]

**Proof:** It is straightforward to see that for $\{x_k\}$ generated by Algorithm 2, Lemma 6 and (101) still hold. Thus, (121) still holds.

We have

\[
E_{\Sigma_k}[W_{1,k+1}] = E_{\Sigma_k} \left[ \frac{1}{2} \|x_{k+1}\|^2_K \right]
\]

\[
= E_{\Sigma_k} \left[ \frac{1}{2} \|x_k - (\gamma L x_k + \eta_k g_k^0)\|^2_K \right]
\]

\[
= E_{\Sigma_k} \left[ \frac{1}{2} \|x_k\|^2_K - \gamma\|x_k\|^2_L + \frac{1}{2}\gamma^2\|x_k\|^2_{L^2}
\]

\[
- \eta_k x_k^T(I_{np} - \gamma L)Kg_k^0 + \frac{1}{2}\eta_k^2\|g_k^0\|^2_K \right]
\]

\[
\leq E_{\Sigma_k} \left[ \frac{1}{2} \|x_k\|^2_K - \gamma\|x_k\|^2_L + \frac{1}{2}\gamma^2\|x_k\|^2_{L^2}
\]

\[
+ \frac{1}{2}\gamma\rho_2(L)\|x_k\|^2_K + \frac{1}{2}\gamma^2\|x_k\|^2_{L^2} + \frac{1}{2}\eta_k^2\|g_k^0\|^2 + \frac{1}{2}\eta_k\|g_k^0\|^2_K \right]
\]

\[
\leq E_{\Sigma_k} \left[ \frac{1}{2} \|x_k\|^2_K - \gamma\|x_k\|^2_{\rho_2(L)K} + \gamma^2\|x_k\|^2_{\rho(L^2)K}
\]

\[
+ \frac{1}{2}\gamma\rho_2(L)\|x_k\|^2_K + \frac{1}{2}\gamma^2\rho_2(L)\eta_k^2\|g_k^0\|^2 \right]
\]

\[
= \frac{1}{2}\|x_k\|^2_K - \|x_k\|^2_{\epsilon_1K} + \epsilon_2\eta_k^2 E_{\Sigma_k}[\|g_k^0\|^2] \tag{205}
\]

\[
\leq \frac{1}{2}\|x_k\|^2_K - \|x_k\|^2_{\epsilon_1K} + \epsilon_2\eta_k^2 \left( 16p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2)(\|g_k^0\|^2 + 2L_f^2\|x_k\|^2_K) \right)
\]

\[
+ 4np\sigma_1^2 + 8np(1 + \sigma_0^2)\sigma_2^2 + \frac{np^2L_f^2}{2}\delta_k^2 \right)
\]

\[
= \frac{1}{2}\|x_k\|^2_K - \|x_k\|^2_{\epsilon_1K} + \epsilon_2\eta_k^2 \left( 16p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2)\|g_k^0\|^2 \right)
\]

\[
+ 4np\sigma_1^2 + 8np(1 + \sigma_0^2)\sigma_2^2 + \frac{np^2L_f^2}{2}\delta_k^2 \right) + \epsilon_2\eta_k^2 \left( 16p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2)\|g_k^0\|^2 \right)
\]
where the second equality holds due to (22); the third equality holds due to (38a); the first inequality holds due to the Cauchy–Schwarz inequality and \( \rho(K) = 1 \); the second inequality holds due to (38b); the second last equality holds since \( x_{i,k} \) is independent of \( \mathcal{L}_k \); and the last inequality holds due to (74a).

From (121) and (206), we have

\[
E_{\mathcal{L}_k}[W_{1,k+1} + W_{4,k+1}] \leq W_{1,k} + W_{4,k} - \|x_k\|^2 - \frac{1}{4} (1 - 64p(1 + \sigma_0^2)(1 + \sigma_0^2)\epsilon_2 \eta_k - \frac{32p(1 + \sigma_0^2)(1 + \sigma_0^2)L_f \eta_k}{n}) \|g_k^0\|^2 
- \frac{1}{4} (1 - 2L_f \eta_k) \|g_k^0\|^2 + 2pn\left(2\epsilon_2 + \frac{L_f}{n}\right)(\sigma_1^2 + 2(1 + \sigma_0^2)\sigma_0^2)\eta_k 
+ \frac{pmL_f^2}{4} \left(2p\epsilon_2 \eta_k + 2pL_f \eta_k + \frac{4}{p}\right) \eta_k \sigma_k^2.
\]  

From (207)–(210), we have

\[
(204a).
\]

Now it is ready to prove Theorem 7. The proof is similar to the proof of Theorem 1. For the
sake of completeness, the proof is included here.

From $\kappa_\eta \in (0, d_2(\gamma) t_1^0)$ and $\eta_k = \frac{\kappa_\eta}{(k+t_1)\theta}$, we have $\eta_k \leq d_2(\gamma)$. Thus, all conditions needed in Lemma 11 are satisfied. So (204a)–(204b) hold.

Taking expectation in $\mathcal{L}_T$, summing (204a) over $k \in [0, T-1]$, noting $\eta_k = \kappa_\eta (k+t_1)\theta$, $\theta \in (0.5, 1)$, and $\delta_k \leq \frac{\kappa_\eta}{\sqrt{n+p}}$ as stated in (23), and using (47) yield

$$E[W_{1,T} + W_{4,T}] + \sum_{k=0}^{T-1} E\left[\frac{\epsilon_1}{2} \|x_k\|^2 + \frac{1}{8} \eta_k \|\hat{g}_k^0\|^2\right]$$

$$\leq W_{1,0} + W_{4,0} + pn(\epsilon_3 + \kappa_\gamma^2 \epsilon_4)\kappa_\eta^2 \sum_{k=0}^{T-1} \frac{1}{(k+t_1)^2\theta} \leq n\epsilon_6. \quad (211)$$

Noting that $t_1^0 = O(\sqrt{p})$, we have

$$\kappa_\eta = O\left(\frac{t_1^0}{p}\right) = O\left(\frac{1}{\sqrt{p}}\right). \quad (212)$$

From $W_{1,0} + W_{4,0} = O(n)$ and (212), we have

$$\epsilon_6 = \frac{W_{1,0} + W_{4,0}}{n} + \frac{p(\epsilon_3 + \kappa_\gamma^2 \epsilon_4)\kappa_\eta^2}{2\theta - 1} = O(1). \quad (213)$$

From (211) and (208), we have

$$E[f(\bar{x}_T)] - f^* \leq \frac{1}{n} W_{4,T} \leq \epsilon_6. \quad (214)$$

From (214) and (213), we have (24b).

From (211) and (208), we have

$$\sum_{k=0}^{T-1} E[\|x_k\|^2] \leq \frac{2n\epsilon_6}{\epsilon_1}. \quad (215)$$

From (73g) and (214), we have

$$\|\hat{g}_k^0\|^2 \leq 2nL_f \epsilon_6. \quad (216)$$

From (74a), (215), and (216), we know that $E[\|g_k\|^2]$ is bounded. Then, same as the proof of the first part of Theorem 1 in [58], we have (24d).

From (204b), (216), and (23), we have

$$E[W_{1,k+1}] \leq (1 - \epsilon_1) E[W_{1,k}] + \frac{\epsilon_7}{(t+t_1)^{2\theta}}. \quad (217)$$
From (217), (208), and (55), we have

\[ E[W_{1,k}] \leq \phi_3(k, t_1, \epsilon_1, \epsilon_7, 2\theta, W_{1,0}), \quad \forall k \in \mathbb{N}_+ , \quad (218) \]

where the function \( \phi_3 \) is defined in (56).

Noting that \( \phi_3(k, t_1, \epsilon_1, \epsilon_7, 2\theta, W_{1,0}) = O\left( \frac{n}{k^{\theta}} \right) \) due to (212), from (218), we have (24c).

From (204c), we have

\[
\left( \frac{1}{\eta_k} - \frac{1}{\eta_{k+1}} + \frac{1}{\eta_{k+1}} \right) E_{\xi_k}[W_{4,k+1}] \leq \frac{W_{4,k}}{\eta_k} + \|x_k\|_{2L_f^2K}^2 - \frac{1}{8} \|g_k^0\|^2 + pL_f\epsilon_5\eta_k + (p + n)L_f^2\delta_k^2.
\]  

(219)

Then, taking expectation in \( \mathcal{L}_T \), summing (219) over \( k \in [0, T-1] \), noting (214), \( \eta_k = \frac{\kappa_\theta}{(k+t_1)^\theta} \), \( \theta \in (0.5, 1) \), and \( \delta_k \leq \frac{n_{\epsilon_3}\sqrt{p}}{\sqrt{n+\theta}} \) as stated in (23), and using (47) yield

\[
\frac{1}{8} \sum_{k=0}^{T-1} E[\|g_k^0\|^2] \leq \frac{W_{4,0}}{\eta_0} + \sum_{k=0}^{T-1} \left( \frac{1}{\eta_{k+1}} - \frac{1}{\eta_k} \right) E[W_{4,k+1}] + \sum_{k=0}^{T-1} E[\|x_k\|_{2L_f^2K}^2] + \frac{pL_f(\epsilon_5 + L_f\kappa_\theta^2)\kappa_\theta(T + t_1)^{1-\theta}}{1 - \theta} \left( \frac{n_{\epsilon_6}T^{1-\theta}}{\kappa_\eta} + \sum_{k=0}^{T-1} E[\|x_k\|_{2L_f^2K}^2] + \frac{pL_f(\epsilon_5 + L_f\kappa_\theta^2)\kappa_\theta(T + t_1)^{1-\theta}}{1 - \theta} \right).
\]  

(220)

From (220), (218), (212), and \( \theta \in (0.5, 1) \), we have (24a).

I. Proof of Theorem 8

We use the notations defined in Appendix H.

From \( \eta_k = \eta = \frac{\sqrt{n}}{\sqrt{T}} \) and \( T \geq \frac{n}{\rho_{d_2}(\gamma)} \), we have \( \eta_k \leq d_2(\gamma) \). Thus, all conditions needed in Lemma 11 are satisfied. So (204a)–(204c) hold.

Taking expectation in \( \mathcal{L}_T \), summing (204a) over \( k \in [0, T-1] \), noting \( \eta_k = \eta = \frac{\sqrt{n}}{\sqrt{T}} \) and \( \delta_{i,k} \leq \frac{\gamma_{nT}^4\kappa_\gamma}{\sqrt{n+\rho}(k+1)^2} \) as stated in (25), and using (47) yield

\[
\frac{1}{nT} \sum_{k=0}^{T-1} E[\|x_k\|_{\mathcal{K}}^2] \leq \frac{2}{\epsilon_1} \left( \frac{W_{1,0} + W_{4,0}}{nT} + \frac{n_{\epsilon_3}}{T} + \frac{2n\kappa_\theta^2\epsilon_4}{T} \right).
\]  

(221)
Similarly, from (204c) and (25), we have
\[
\frac{1}{T} \sum_{k=0}^{T-1} E[\|\nabla f(\bar{x}_k)\|^2] = \frac{1}{nT} \sum_{k=0}^{T} E[\|g_k^0\|^2] \leq 8 \left( \frac{W_{4,0}}{nT} + \frac{2L_f^2}{nT} \sum_{k=0}^{T} E[\|x_k\|^2] \right) + \frac{pL_f\epsilon_5\eta}{n} + \frac{2\sqrt{p}L_f^2\kappa_5^2}{\sqrt{nT}}. \tag{222}
\]

Noting that \(\eta = \frac{\sqrt{n}}{\sqrt{pT}}\), from (221) and (222), we have
\[
\frac{1}{T} \sum_{k=0}^{T-1} E[\|\nabla f(\bar{x}_k)\|^2] = 8(f(\bar{x}_0) - f^* + \epsilon_5L_f + 2L_f^2\kappa_5^2) \frac{\sqrt{p}}{\sqrt{nT}} + O(\frac{n}{T}),
\]
which gives (26a).

Taking expectation in \(L_T\), summing (204c) over \(k \in [0, T-1]\), and using (25) yield
\[
n(E[f(\bar{x}_T)] - f^*) = E[W_{4,T}] \leq W_{4,0} + \frac{2L_f^2\sqrt{n}}{\sqrt{pT}} \sum_{k=0}^{T-1} \|x_k\|^2 + nL_f\epsilon_5 + 2nL_f^2\kappa_5^2. \tag{223}
\]

Noting that \(W_{4,0} = O(n)\) and \(\frac{\sqrt{n}}{\sqrt{pT}} \leq 1\) due to \(T \geq \frac{n^3}{p}\), from (221) and (223), we have (26b).

Then, from (73g), we know that there exists a constant \(\tilde{d}_g > 0\), such that
\[
E[\|g_k^0\|^2] \leq n\tilde{d}_g, \quad \forall k \in \mathbb{N}_0. \tag{224}
\]

From (204b), (224), and (25), we have
\[
E[W_{1,k+1}] \leq (1 - \epsilon_1)E[W_{1,k}] + \frac{n^2\epsilon_4\kappa_5^2}{\sqrt{T(k+1)}} + \frac{n^2(16(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2)\epsilon_2\tilde{d}_g + 2\epsilon_2\epsilon_5)}{T}, \quad \forall 0 \leq k \leq T. \tag{225}
\]

From (225) and (208), similar to the way to get (55), we have \(E[W_{1,T}] = O(\frac{n^2}{T})\), which gives (26c).

Similar to the proof of (24d), we have (26d).

**J. Proof of Theorem 9**

In addition to the notations defined in Appendix [H] we also denote the following notations.
\[
\tilde{\epsilon}_7 = p\kappa_7^2(16(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2)\epsilon_2\tilde{d}_g + 2\epsilon_2\epsilon_5 + \epsilon_4\kappa_5^2),
\]
\[
\epsilon_8 = \min \left\{ \frac{\epsilon_1T^\theta}{\kappa_\eta}, \frac{\nu}{4} \right\},
\]
\[
b_1 = \epsilon_8\kappa_\eta,
\]

\[ b_2 = pn(\epsilon_3 + \epsilon_4k^2)\kappa_\eta. \]

All conditions needed in Lemma 11 are satisfied, so (204a)--(204c) hold. Denote \( W_k = W_{1,k} + W_{4,k} \). From (204a) and (149), we have
\[
E_{g_k}[\tilde{W}_{k+1}] \leq \tilde{W}_k - \|x_k\|^2_{\frac{\nu}{4}k} - \frac{\nu}{4}\eta_kW_{4,k} + pn\epsilon_3\eta^2_k + pn\epsilon_4\eta_k\delta^2_k \\
\leq \left( 1 - \eta_k \min \left\{ \frac{\epsilon_1}{\eta_k}, \frac{\nu}{4} \right\} \right) \tilde{W}_k + pn\epsilon_3\eta^2_k + pn\epsilon_4\eta_k\delta^2_k \\
\leq \left( 1 - \eta_k\epsilon_8 \right) \tilde{W}_k + pn\epsilon_3\eta^2_k + pn\epsilon_4\eta_k\delta^2_k, \ \forall k \in \mathbb{N}_0. \tag{226}
\]

Denote \( \tilde{z}_k = E[\tilde{W}_k] \), \( s_{1,k} = \eta_k\epsilon_8 \), and \( s_{2,k} = pn\epsilon_3\eta^2_k + pn\epsilon_4\eta_k\delta^2_k \). From (226), we have
\[
\tilde{z}_{k+1} \leq (1 - s_{1,k})\tilde{z}_k + s_{2,k}, \ \forall k \in \mathbb{N}_0. \tag{227}
\]

From (27), we have
\[
s_{1,k} = \eta_k\epsilon_8 = \frac{b_1}{(k + t_1)^\theta}. \tag{228}
\]
\[
s_{2,k} = pn\epsilon_3\eta^2_k + pn\epsilon_4\eta_k\delta^2_k \leq \frac{b_2}{(k + t_1)^{2\theta}}. \tag{229}
\]

From (208), we have
\[
0 < s_{1,k} \leq \epsilon_1 \leq \frac{1}{16}. \tag{230}
\]

Then, from \( \theta \in (0, 1) \), (227)--(230), and (51), we have
\[
\tilde{z}_k \leq \phi_1(k, t_1, b_1, b_2, \theta, 2\theta, \tilde{z}_0), \ \forall k \in \mathbb{N}_+, \tag{231}
\]

where the function \( \phi_1 \) is defined in (52).

From \( \kappa_\eta \in (0, d_2(\gamma)t_1^\theta) \), \( \theta \in (0, 1) \), and \( t_1 \in [p^{\frac{1}{2}}, d_3p^{\frac{1}{2}}] \), we have
\[
0 < \kappa_\eta \leq d_3pd_2(\gamma) < \frac{d_3}{64(1 + \sigma_0^2)(1 + \sigma_0^2)L_f}. \tag{232}
\]

From (73g), (231), and (232), we get
\[
E[\|g^0_k\|^2] = O\left( \frac{pn}{(k + t_1)^\theta} \right), \ \forall k \in \mathbb{N}_+. \tag{233}
\]

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From (233) and $t_1^g = \mathcal{O}(p)$, we know that there exists a constant $\tilde{d}_g > 0$, such that

$$E[\|\vec{g}_k^0\|^2] \leq n\tilde{d}_g, \forall k \in \mathbb{N}_0. \quad (234)$$

From (204b), (234), and (27), we have

$$E[W_{1,k+1}] \leq (1 - \epsilon_1)E[W_{1,k}] + \frac{\hat{\epsilon}_7}{(t + t_1)^{2\theta}}. \quad (235)$$

Using (55), from (208) and (235), we have

$$E[W_{1,k}] \leq \phi_3(k,t_1,\epsilon_1,\hat{\epsilon}_7,2\theta,W_{0,k}), \forall k \in \mathbb{N}_+, \quad (236)$$

where the function $\phi_3$ is defined in (56). From (236), (56), and (232), we have

$$E[\|x_k\|^2_K] \leq 2E[W_{1,k}] \leq 2\phi_3(k,t_1,\epsilon_1,\hat{\epsilon}_7,2\theta,W_{0,k}) = \mathcal{O}\left(\frac{pn}{(k+t_1)^{2\theta}}\right), \quad (237)$$

which yields (28a).

From (204c), (149), and $\delta_k \leq \frac{\kappa\sqrt{p\eta}}{\sqrt{n+P}}$ we have

$$E[W_{4,k+1}] \leq E[W_{4,k}] - \frac{\nu}{4}\eta_k E[W_{4,k}] + \|x_k\|^2_{2L_f^2\eta_kK} + pL_f\epsilon_5\eta_k^2 + pL_f^2\kappa_2^2\eta_k^2. \quad (238)$$

Similar to the way to prove (51), from (237) and (238), we have

$$E[f(\bar{x}_T) - f^*) \leq \frac{\epsilon_9p}{n(T+t_1)^\theta} + \mathcal{O}\left(\frac{p}{(T+t_1)^{2\theta}}\right). \quad (239)$$

From (232), we have

$$\epsilon_9 = \frac{8\theta4^\theta\kappa_\eta(\epsilon_5 + L_f^2\kappa_\delta^2)}{\nu} < \frac{8\theta4^\theta(\epsilon_5 + L_f^2\kappa_\delta^2)d_3}{64\nu(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2)L_f}. \quad (240)$$

Thus, from (239) and (240), we have (28b).

**K. Proof of Theorem 10**

In addition to the notations defined in Appendices H and J, we also denote

$$\hat{d}_2(\gamma) = \max \left\{ \frac{1}{\epsilon_1}, \frac{\kappa_\eta}{d_2(\gamma)} \right\}. \quad (241)$$

From $t_1 > \hat{d}_2(\gamma) \geq \frac{\kappa_\eta}{d_2(\gamma)}$, we have

$$\eta_k = \frac{\kappa_\eta}{k + t_1} \leq \frac{\kappa_\eta}{t_1} < d_2(\gamma). \quad (242)$$
Thus, all conditions needed in Lemma 11 are satisfied, so (227)–(230) still hold when \( \theta = 1 \).

From \( t_1 > \hat{d}_2(\gamma) \geq \frac{1}{\epsilon_1} \), we have

\[
\epsilon_1 t_1 > 1. \tag{241}
\]

From \( \kappa_\eta \in \left( \frac{8}{v}, \frac{8d_3}{v} \right) \), we have

\[
2d_3 \geq \frac{\nu \kappa_\eta}{4} > 2. \tag{242}
\]

Hence, from (241) and (242), we have

\[
b_1 = \epsilon_8 \kappa_\eta > 1. \tag{243}
\]

Then from \( \theta = 1 \), (227)–(230), (243), and (53), we have

\[
\hat{z}_k \leq \phi_2(k, t_1, b_1, b_2, 2, \hat{z}_0), \quad \forall k \in \mathbb{N}_+, \tag{244}
\]

where the function \( \phi_2 \) is defined in (54).

From (242) and (243), we know \( \phi_2(k, t_1, b_1, b_2, 2, \hat{z}_0) = \mathcal{O}(n + \frac{pm}{k + t_1}) \). Hence, from (73g) and (244), we get

\[
\mathbb{E}[\|\hat{g}_k^0\|^2] = \mathcal{O}\left(n + \frac{pm}{k + t_1}\right), \quad \forall k \in \mathbb{N}_+. \tag{245}
\]

Noting that \( t_1 > \hat{d}_2(\gamma) \geq \frac{\kappa_\eta}{\hat{d}_2(\gamma)} \geq 64p \kappa_\eta (1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2)(2\epsilon_2 + L_f) \), from (245) and (242), we know that there exists a constant \( \hat{d}_g > 0 \), such that

\[
\mathbb{E}[\|\hat{g}_k^0\|^2] \leq n \hat{d}_g, \quad \forall k \in \mathbb{N}_0. \tag{246}
\]

Then, similar to the way to get (28a) and (28b), we get (30a) and (30b).

L. Proof of Theorem 11

In addition to the notations defined in Appendices H, J, and K, we also denote

\[
\tilde{d}_2(\gamma) = \min \left\{ \frac{\epsilon_1}{4L_f}, \frac{1}{4p(1 + \sigma_0^2)(2\epsilon_2 + L_f)} \right\},
\]

\[
\hat{d}_2(\gamma) = \max \left\{ \frac{1}{\epsilon_1}, \frac{\kappa_\eta}{\hat{d}_2(\gamma)}, \frac{4\kappa_\eta \epsilon_{10}}{\nu} \right\},
\]

\[
\hat{\epsilon}_3 = 2\left(2\epsilon_2 + \frac{L_f}{n}\right)\hat{\epsilon}_5,
\]
Lemma 12. Suppose Assumptions 7–4 hold and each \( f_i^* > -\infty \). Suppose \( \gamma \in (0, d_1) \) and \( \eta_k \in (0, \tilde{d}_2(\gamma)] \). Let \( \{x_k\} \) be the sequence generated by Algorithm 2 then

\[
E_{\Omega_k}[W_{1,k+1} + W_{4,k+1}] \leq W_{1,k} + W_{4,k} - \|x_k\|^2 + \frac{1}{2} \eta_k \|g_k^0\|^2
+ \epsilon_1 \eta_k^2 W_{4,k} + pm \epsilon_3 \eta_k^2 + pm \epsilon_4 \eta_k^2
\]

\[
E_{\Omega_k}[W_{1,k+1}] \leq W_{1,k} - \|x_k\|^2 + 16 p(1 + \sigma_0^2) \epsilon_2 L_f \eta_k^2 W_{4,k}
+ 2 p \epsilon_2 \eta_k^2 + pm \epsilon_4 \eta_k^2
\]

\[
E_{\Omega_k}[W_{4,k+1}] \leq W_{4,k} + \|x_k\|^2 - \frac{1}{2} \eta_k \|g_k^0\|^2
+ \frac{8 \eta_k \|g_k^0\|^2}{n} + \frac{p L_f \epsilon_3 \eta_k^2 + (p + n) L_f^2 \eta_k^2}{2}
\]

(247a)

(247b)

(247c)

Proof: We know that (73a)–(73g) and (186) still hold since Assumptions 3 and 4 hold, and each \( f_i^* > -\infty \). Therefore, (195) also holds.

From (195), (205), and (186), we have

\[
E_{\Omega_k}[W_{1,k+1} + W_{4,k+1}] \leq W_{1,k} + W_{4,k} - \|x_k\|^2
- \frac{1}{4} \eta_k \|g_k^0\|^2
+ 8 p(1 + \sigma_0^2) \left( \epsilon_2 + \frac{L_f}{n} \right) L_f \eta_k^2 W_{4,k}
- \frac{1}{4} \left( 1 - 2 L_f \eta_k \|g_k^0\|^2 + 2 p \left( \epsilon_2 + \frac{L_f}{n} \right) (\sigma_1^2 + 2(1 + \sigma_0^2) \delta_2^2) \eta_k^2
+ \frac{p n L_f^2}{4} \left( 2 \epsilon_2 \eta_k + \frac{L_f p}{n} \eta_k + \frac{4}{p} \right) \eta_k^2\right)
\]

(248)

Then, similar to the rest of the proof of Lemma 11 we get Lemma 12.

Now we are ready to prove Theorem 11.

From \( t_1 > \tilde{d}_2(\gamma) \geq \max \{ \frac{\kappa_\eta}{\tilde{d}_2(\gamma)}, \frac{4 \epsilon_1 \eta_{10}}{\nu} \} \), we have

\[
\eta_k = \frac{\kappa_\eta}{k + t_1} \leq \frac{\kappa_\eta}{t_1} < \min \left\{ \tilde{d}_2(\gamma), \frac{\nu}{4 \epsilon_{10}} \right\}.
\]

(249)
Thus, all conditions needed in Lemma 12 are satisfied, so (247a)–(247c) hold.

From (247a), (149), and (249), we know that (226) still holds when $\epsilon_3$ and $\epsilon_4$ are replaced by $\tilde{\epsilon}_3$ and $\tilde{\epsilon}_4$, respectively.

Then, similar to the way to get (30a) and (30b), we have (32a) and (32b).

**M. Proof of Theorem 12**

In addition to the notations defined in Appendix H, we also denote the following notations.

\[\epsilon = \frac{1}{2} + \frac{1}{2} \max \{1 - \tilde{\epsilon}_8, \tilde{\epsilon}_8^2\},\]

\[\tilde{\epsilon}_8 = \min \{\epsilon_1, \frac{\nu \eta}{4}\}.\]

All conditions needed in Lemma 11 are satisfied, so (204a) still holds.

(i) Taking expectation in $L_T$, summing (204a) over $k \in [0, T - 1]$, and using $\eta_k = \eta$ and $\delta_{i,k} \in (0, \frac{\tilde{\epsilon}_k}{\sqrt{T}})$ yield

\[\mathbb{E}[W_{1,T} + W_{4,T}] + \frac{\epsilon_1}{2} \sum_{k=0}^{T-1} \|x_k\|_K^2 + \frac{1}{8} \eta \sum_{k=0}^{T-1} \|g_k^0\|^2 \leq W_{1,0} + W_{4,0} + p\epsilon_3\eta^2 T + \frac{p\epsilon_4\kappa_3^2 \eta}{1 - \tilde{\epsilon}^2},\]

which further implies

\[\sum_{k=0}^{T-1} \mathbb{E}[\|x_k\|_K^2] \leq \frac{2}{\epsilon_1} \left( W_{1,0} + p\epsilon_3\eta^2 T + \frac{p\epsilon_4\kappa_3^2 \eta}{1 - \tilde{\epsilon}^2} \right). \tag{250}\]

Therefore, (34a) holds due to $\eta = O(\frac{1}{p})$.

From (204c), we have

\[\mathbb{E}[W_{4,T}] \leq W_{4,0} + \sum_{k=0}^{T-1} \mathbb{E}[\|x_k\|_{2\eta L_f^2 \kappa_1}^2] - \frac{1}{8} \eta \sum_{k=0}^{T-1} \mathbb{E}[\|g_k^0\|^2] + pL_f\epsilon_5\eta^2 T + \frac{(n + p)L_f^2\kappa_5^2 \eta}{1 - \tilde{\epsilon}^2}. \tag{251}\]

From (250) and (251), we have

\[\sum_{k=0}^{T-1} \mathbb{E}[\|g_k^0\|^2] \leq \frac{8W_{4,0}}{\eta} + \sum_{k=0}^{T-1} \mathbb{E}[\|x_k\|_{16L_f^2 \kappa_1}^2] + 8pL_f\epsilon_5\eta T + \frac{8(n + p)L_f^2\kappa_5^2 \eta}{1 - \tilde{\epsilon}^2} \leq \frac{8W_{4,0}}{\eta} + \frac{16L_f^2}{\epsilon_1} \left( W_{1,0} + p\epsilon_3\eta^2 T + \frac{p\epsilon_4\kappa_3^2 \eta}{1 - \tilde{\epsilon}^2} \right) + 8pL_f\epsilon_5\eta T + \frac{8(n + p)L_f^2\kappa_5^2 \eta}{1 - \tilde{\epsilon}^2},\]

which gives (34c).
From (73g), (250), and (251), we have
\[
\mathbb{E}[\|g_k^0\|^2] \leq 2L_f \mathbb{E}[W_{4,k}]
\]
\[
\leq 2L_f \left( W_{4,0} + pL_f \varepsilon_5 \eta^2 k + \frac{(n + p) L_f^2 \kappa_5^2 \eta}{1 - \varepsilon^2} \right) + \frac{4L_f^3 \eta}{\epsilon_1} (W_{1,0} + p n \varepsilon_3 \eta^2 k + \frac{p n \varepsilon_4 \kappa_5^2 \eta}{1 - \varepsilon^2}), \quad \forall k \in \mathbb{N}_0.
\]
(252)

From (204b), (208), \( \delta_{i,k} \in (0, \kappa_\delta \varepsilon^k) \), (252), and , we have
\[
\mathbb{E}[W_{1,k+1}] \leq \mathbb{E}[(1 - \epsilon_1) W_{1,k} + 16p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2)\varepsilon_2 \eta^2 \|g_k^0\|^2 + 2p n \varepsilon_5 \eta^2 + p n \varepsilon_4 \kappa_5^2 \tilde{\varepsilon}^k]
\]
\[
\leq (1 - \epsilon_1)^k \bar{W}_0 + 16p(1 + \sigma_0^2)(1 + \tilde{\sigma}_0^2)\varepsilon_2 \eta^2 \sum_{\tau=0}^{k} (1 - \epsilon_1)^\tau \mathbb{E}[\|g_{k-\tau}^0\|^2]
\]
\[
+ 2p n \varepsilon_5 \eta^2 \sum_{\tau=0}^{k} (1 - \epsilon_1)^\tau + p n \varepsilon_4 \kappa_5^2 \sum_{\tau=0}^{k} (1 - \epsilon_1)^\tau \tilde{\varepsilon}^{2(k-\tau)}
\]
\[
= \mathcal{O}(p n \eta^2 + \varepsilon_5 p^2 \eta^4 (k + 1) + \epsilon_3 p^2 n \eta^5 (k + 1)),
\]
which gives (34b).

(ii) If Assumption 6 also holds, then (149) holds. Thus, (226) also holds when \( \eta_k = \eta \). From (226) and \( \eta_k = \eta \), for all \( k \in \mathbb{N}_0 \), we have
\[
\mathbb{E}[\bar{W}_{k+1}] \leq (1 - \tilde{\epsilon}_8) \bar{W}_k + p n \varepsilon_3 \eta^2 + p n \varepsilon_4 \delta_k^2.
\]
(253)

From (208)
\[
0 < \tilde{\epsilon}_8 \leq \epsilon_1 < \frac{1}{16}.
\]
(254)

From (253), (254), and \( \delta_{i,k} \in (0, \kappa_\delta \varepsilon^k) \), we have
\[
\mathbb{E}[\bar{W}_{k+1}] \leq (1 - \tilde{\epsilon}_8)^{k+1} \bar{W}_0 + p n \varepsilon_3 \eta^2 \sum_{\tau=0}^{k} (1 - \tilde{\epsilon}_8)^\tau + p n \varepsilon_4 \kappa_5^2 \eta \sum_{\tau=0}^{k} (1 - \tilde{\epsilon}_8)^\tau \tilde{\varepsilon}^{2(k-\tau)}, \quad \forall k \in \mathbb{N}_0.
\]
(255)

From (255), (46), and \( \epsilon > \max\{1 - \tilde{\epsilon}_8, \varepsilon^2\} \), we have
\[
\mathbb{E}[W_{k+1}] \leq \left( \frac{W_{1,0} + W_{4,0}}{n} + \frac{p n \varepsilon_4 \kappa_5^2 \eta}{\epsilon - \tilde{\epsilon}^2} \right) n \epsilon^{k+1} + \frac{2n}{\epsilon_1} \left( 2\epsilon_2 + \frac{1}{n} L_f \right) n (\sigma_1^2 + 2(1 + \sigma_0^2) p \eta), \quad \forall k \in \mathbb{N}_0,
\]
(256)

which gives (35).