Upper capacity entropy and packing entropy of saturated sets for amenable group actions

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ABSTRACT
Let $(X, G)$ be a $G$-action topological system, where $G$ is a countable infinite discrete amenable group and $X$ a compact metric space. In this paper we study the upper capacity entropy and packing entropy for systems with weaker version of specification. We prove that the upper capacity always carries full entropy while there is a variational principle for packing entropy of saturated sets.

KEYWORDS
Amenable group actions, packing entropy, upper capacity entropy, specification

1. Introduction

The study of the size of saturated sets for dynamical systems was initialized by Bowen and later was developed by Sigmund, Pfister and Sullivan et. al. This paper focuses on establishing upper capacity and packing entropy formulae of saturated sets for countable amenable group actions with specification or weaker version.

Throughout this paper, $X$ will be a compact metric space with metric $\rho$ and $G$ a discrete countable amenable group. Let $M(X)$, $M(X, G)$ and $E(X, G)$ denote the sets of all Borel probability measures, $G$-invariant Borel probability measures and ergodic $G$-invariant Borel probability measures induced with the weak$^*$ topology, respectively. When $G = \mathbb{Z}$, we write the system $(X, G)$ as $(X, T)$, when $T : X \to X$ is a homeomorphism. For finite subsets $\Gamma \subset X$, we denote $\#\Gamma$ the cardinality of $\Gamma$.

Let $\{F_n\}$ be a Følner sequence. The \textit{empirical measure} of $x \in X$ with respect to...
Upper capacity entropy and packing entropy of saturated sets

one finite subset \( F \subset G \) is the Borel probability measure

\[
\mathcal{E}_F(x) := \frac{1}{|F|} \sum_{s \in F} \delta_{sx},
\]

where \( \delta_y \) is the Dirac measure at point \( y \). Let \( V(x, \{F_n\}) \) be the limit-point set of \( \{\mathcal{E}_{F_n}(x)\} \), which is always a compact connected subset of \( M(X, G) \). For each compact connected non-empty subset \( K \subset M(X, G) \), denote by \( Y_K(\{F_n\}) = \{x \in X \mid V(x, \{F_n\}) = K\} \) which is called the saturated set of \( K \). We write \( Y_K(\{F_n\}) \) as \( Y_K \) for simplicity. In particular, when specializing \( K \) to be a singleton measure \( \mu \in M(X, G) \), the set \( Y_K \) coincides with the set of generic points of \( \mu \) with respect to \( \{F_n\} \) and we write \( Y_{\mu} \) as \( Y_\mu \).

Bowen introduced a definition of topological entropy of subsets in [2] which is known as Bowen topological entropy and obtained a remarkable single-saturated property

\[
h_{\text{top}}^B(G_\mu, T) = h_\mu(X, T)
\]

when \( \mu \) is ergodic, where \( h_\mu(T, X) \) is the metric entropy of \( \mu \). The interesting part of Bowen’s formula is that the metric entropy is affine while Bowen topological entropy can be viewed as dynamically analogous to Hausdorff dimension.

After Bowen’s work, a series of papers pursued Bowen’s formula further into more general saturated set \( Y_K \), with \( K \) to be a compact connected subset in \( M(X, T) \). Pister and Sullivan [18] pointed out that it is not difficult to provide examples with \( h_\mu(T, X) > 0 \) but the generic set of \( \mu \) is empty. However, Denker, Grillenberg and Sigmund [5] proved that \( Y_K \) is not empty for systems with hyperbolicity or specification and this result is generalized to non-uniformly hyperbolic systems in [14]. Pítre and Sullivan proved a saturated property for \( T \) satisfying the \( g \)-almost product property and uniform separation property (see the definitions in Section 2) the following holds:

\[
h_{\text{top}}^B(Y_K, T) = \inf \{h_\mu(X, T) \mid \mu \in K\}. \tag{1.1}
\]

The \( g \)-almost product property is weaker than specification and uniform separation is weaker than expansiveness. Also the equation [1.1] is generalized into non-uniformly hyperbolic systems in [13] and non-uniformly expanding maps in [20].

Besides Bowen topological entropy, packing topological entropy and upper capacity topological entropy are another two important concepts for characterizing the size of non-compact subset \( Z \subset X \). Actually, the upper capacity topological entropy is a direct generalization of the classical topological entropy [1]. The packing topological entropy was introduced by Feng and Huang [8] in a way by resembling packing dimension in dimension theory.

It is natural to ask if there are different saturated properties for Bowen topological entropy, packing topological entropy and upper capacity topological entropy. The authors in [9] gave answers to this question for \( \mathbb{Z} \)-actions. In this paper we are dealing with this question for more general group actions.

In contrast to amenable group \( \mathbb{Z} \), a more general countable amenable may have more complicated structure and new phenomena and difficulties may arise. In 1987, Ornstein and Weiss [17] developed the so-called quasi-tiling method, which has been a basic tool in the study of amenable group actions. Many researchers have made lots of progress in many directions of ergodic theory for amenable group actions. For example Kieffer [12] proved the Shannon-McMillan theorem for amenable group
actions; Lindenstrauss [15] later established this pointwise result for tempered Følner sequence with superlogarithmic growth, which can be found in any amenable group. Downarowicz, Huczek and Zhang [7] recently have a new result on the quasi-tiling. Also Zhang [21] used the new quasi-tiling techniques to calculate topological pressure of generic points. For more information of amenable group actions, readers may refer [10, 16, 11].

Our main results are the following theorems.

**Theorem 1.1.** Suppose the topological dynamical system \((X, G)\) satisfies the \(g\)-almost product property. Let \(K \subset M(X, G)\) be a compact connected nonempty set and \(\{F_n\}\) be a Følner sequence. Then we have

\[ h_{top}^{UC}(Y_K, \{F_n\}) = h_{top}(X, \{F_n\}) = h_{top}(X, G). \]

**Theorem 1.2.** Suppose the topological dynamical system \((X, G)\) satisfies the \(g\)-almost product property and has uniform separation property. Let \(K \subset M(X, G)\) be a compact convex nonempty set and \(\{F_n\}\) be a Følner sequence with \(\frac{|F_n|}{\log n} \to \infty\). Then we have

\[ h_{top}^P(Y_K, \{F_n\}) = \sup\{h_\mu(X, G) \mid \mu \in K\}. \]

From Theorem 1.1 and Theorem 1.2, we notice that the upper capacity topological entropy and the packing topological entropy show quite different behavior from saturated property viewpoint.

The paper is organized as follows. In Section 2, we introduce some notions and lemmas. In Section 3, we prove the Theorem 1.1 and Theorem 1.2. In section 4, we give an application of the main theorems. In Appendix, we give the proof of Theorem 2.15.

2. Preliminary

In this section, we will introduce some notions and recall some lemmas.

Let \(F(G)\) be the collection of all finite subsets of \(G\). Let \(K \in F(G)\). The \(K\) boundary of \(F \in G\) is defined by

\[ \partial_K(F) := \{ c \in G \mid Kc \cap F \neq \emptyset, Kc \cap (G \setminus F) \neq \emptyset \}. \]

A set \(F \in F(G)\) is called \((K, \delta)\) - invariant if \(\frac{|\partial_K(F)|}{|F|} < \delta\). A sequence \(\{F_n\} \subset F(G)\) is called a Følner sequence if for any \(s \in G\),

\[ \lim_{n \to \infty} \frac{|g_{F_n} \triangle F_n|}{|F_n|} = 0. \]

And \(\{F_n\}\) is a Følner sequence if and only if for any \(K \in F(G)\) and \(\delta > 0\), there exists \(N > 0\) such that when \(n > N\), \(F_n\) is \((K, \delta)\)-invariant.

A Følner sequence \(\{F_n\}_{n=1}^{\infty}\) is said to be tempered if there exists \(C > 0\) which is independent of \(n\) such that

\[ \left| \bigcup_{k<n} F_{-1}^{-1} F_n \right| \leq C |F_n|, \]
A Følner sequence is said to have the superlogarithmic growth if
\[ \lim_{n \to \infty} \frac{|F_n|}{\log n} = \infty. \]  
(2.1)

For \( F \in F(G) \), let \( \rho_F \) be the metric defined by
\[ \rho_F(x, y) := \max_{s \in F} \rho(sx, sy). \]

For \( \varepsilon > 0, x \in X \) and \( F \in F(G) \), we denote by
\[ B_F(x, \varepsilon) := \{ y \in X \mid \rho_F(x, y) < \varepsilon \} \]
and
\[ \mathcal{B}_F(x, \varepsilon) := \{ y \in X \mid \rho_F(x, y) \leq \varepsilon \} \]
which are the open and closed \( F \)-Bowen balls with center \( x \) and radius \( \varepsilon \), respectively.

For \( \varphi \in C(X, \mathbb{R}) \) and \( \mu \in M(X) \), we denote \( \langle \varphi, \mu \rangle = \int_X \varphi d\mu \). There is a countable and separating set of continuous functions \( \{ \varphi_1, \varphi_2, \ldots \} \) with \( 0 \leq \varphi_i \leq 1, i = 1, 2, \ldots \) such that
\[ D(\mu, \nu) := \sum_{i=1}^{\infty} \frac{|\langle \varphi_i, \mu - \nu \rangle|}{2^i} \]  
(2.2)

which defines a compatible metric for the weak*-topology on \( M(X) \). For \( \xi > 0 \), we denote a ball in \( M(X) \) by
\[ B(\mu, \xi) := \{ \nu \in M(X) \mid D(\mu, \nu) < \xi \}. \]

We mention that in the rest of this paper, we will always use the equivalent metric
\[ \rho(x, y) := D(\delta_x, \delta_y) \]  
(2.3)
as the metric on \( X \), where \( \delta_x \) is the Dirac mass at \( x \in X \).

### 2.1. Entropies for subsets

In this part, we introduce three important types of topological entropies for subsets. They are the Bowen topological entropy, packing topological entropy and the upper capacity entropy. The former two are dimension-like concepts and the third one is a topological concept.

**Definition 2.1** (Bowen topological entropy). Let \( Z \subset X \) and \( \{ F_n \} \) be a Følner sequence. For \( \varepsilon > 0 \) and \( N \in \mathbb{N} \), denote \( C_N(Z, \varepsilon, \{ F_n \}) \) the collection of all finite or countable cover \( \mathcal{C} = \{ B_{F_n}(x, \varepsilon) \} \) of \( Z \) with \( n_i \geq N \). For \( s > 0 \), denote
\[
M(Z, \varepsilon, N, s, \{ F_n \}) := \inf_{\mathcal{C} \in C_N(Z, \varepsilon, \{ F_n \})} \sum_{B_{F_n}(x, \varepsilon) \in \mathcal{C}} e^{-s|F_n|}.
\]
The value $M(Z, \varepsilon, N, s, \{F_n\})$ does not decrease as $N$ increases, hence the following limit exists
\[ M(Z, \varepsilon, s, \{F_n\}) = \lim_{N \to \infty} M(Z, \varepsilon, N, s, \{F_n\}). \]

There exists a critical value of $s$ such that $M(Z, \varepsilon, s, \{F_n\})$ jumps from $+\infty$ to 0. Let
\[ h^{B}_{\text{top}}(Z, \varepsilon, \{F_n\}) = \inf \{ s \mid M(Z, \varepsilon, s, \{F_n\}) = 0 \} \]
\[ = \sup \{ s \mid M(Z, \varepsilon, s, \{F_n\}) = \infty \}. \]

Clearly, $h^{B}_{\text{top}}(Z, \varepsilon, \{F_n\})$ does not decrease as $\varepsilon$ decreases, hence the following limit exists
\[ h^{B}_{\text{top}}(Z, \{F_n\}) = \lim_{\varepsilon \to 0} h^{B}_{\text{top}}(Z, \varepsilon, \{F_n\}), \]
and we call it Bowen topological entropy of $Z$ with respect to the sequence $\{F_n\}$.

The packing topological entropy for amenable group actions was introduced by Dou, Zheng and Zhou [6].

**Definition 2.2 (Packing topological entropy).** For $Z \subset X, s \geq 0, N \in \mathbb{N}$ and $\varepsilon > 0$, define
\[ P(Z, \varepsilon, N, s, \{F_n\}) = \sup \sum \exp(-s|F_n|), \]
where the supremum is taken over all finite or countable pairwise disjoint families $\{B_{F_n}(x_i, \varepsilon)\}$ with $x_i \in Z, n_i \geq N$ for all $i$. The quantity $P(Z, \varepsilon, N, s, \{F_n\})$ does not increase as $N$ increases, hence the following limit exists:
\[ P(Z, \varepsilon, s, \{F_n\}) = \lim_{N \to \infty} P(Z, \varepsilon, N, s, \{F_n\}). \]

Define
\[ P(Z, \varepsilon, s, \{F_n\}) = \inf \{ \sum_{i=1}^{\infty} P(Z_i, \varepsilon, s, \{F_n\}) : \bigcup_{i=1}^{\infty} Z_i \supset Z \}. \]

There exists a critical value of the parameter $s$, which we will denote by $h^P_{\text{top}}(Z, \varepsilon, \{F_n\})$, where $P(Z, \varepsilon, s, \{F_n\})$ jumps from $+\infty$ to 0, i.e.
\[ P(Z, \varepsilon, s, \{F_n\}) = \begin{cases} 0 & s > h^P_{\text{top}}(Z, \varepsilon, \{F_n\}) \\ +\infty & s < h^P_{\text{top}}(Z, \varepsilon, \{F_n\}). \end{cases} \]
It is not hard to see that $h_{\text{top}}^P(Z, \varepsilon, \{F_n\})$ increases when $\varepsilon$ decreases. We call

$$h_{\text{top}}^P(Z, \{F_n\}) := \lim_{\varepsilon \to 0} h_{\text{top}}^P(Z, \varepsilon, \{F_n\})$$

the amenable packing topological entropy (packing entropy for short) of $Z$ (with respect to the Følner sequence $\{F_n\}$).

Packing entropy has the following properties.

**Proposition 2.3.** [6, Proposition 2.1]

1. If $Z' \subset Z \subset X$, then

$$h_{\text{top}}^P(Z', \{F_n\}) \leq h_{\text{top}}^P(Z, \{F_n\});$$

2. If $Z \subset \bigcup_{i=1}^{\infty} Z_i$, then

$$h_{\text{top}}^P(Z, \{F_n\}) \leq \sup_{i \geq 1} h_{\text{top}}^P(Z_i, \{F_n\});$$

3. If $\{F_{n_k}\}$ is a subsequence of $\{F_n\}$, then

$$h_{\text{top}}^P(Z, \{F_{n_k}\}) \leq h_{\text{top}}^P(Z, \{F_n\}).$$

Let $Z \subset X$ be a non-empty set and $F \subset G$ a non-empty finite set. For $\varepsilon > 0$, a set $E \subset Z$ is called $(F, \varepsilon)$-separated if $x \neq y \in E$ implies $\rho_F(x, y) > \varepsilon$; a set $E$ is said to $(F, \varepsilon)$ span $Z$ if for any $y \in Z$, there exists $x \in E$ such that $\rho_F(x, y) \leq \varepsilon$. Let $s_F(Z, \varepsilon)$ denote the largest cardinality of $(F, \varepsilon)$-separated sets for $Z$ and $r_F(Z, \varepsilon)$ the smallest cardinality of any $(F, \varepsilon)$-spanning set for $Z$.

**Definition 2.4** (Upper capacity topological entropy). Let $\{F_n\}$ be a Følner sequence. Then upper capacity topological entropy of $Z$ is defined by

$$h_{\text{top}}^{UC}(Z, \{F_n\}) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{|F_n|} \log s_{F_n}(Z, \varepsilon) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{|F_n|} \log r_{F_n}(Z, \varepsilon).$$

The following propositions show order relations of $h_{\text{top}}^B, h_{\text{top}}^P$ and $h_{\text{top}}^{UC}$.

**Proposition 2.5.** [6, Proposition 2.2] For any $Z \subset X$ and any Følner sequence $\{F_n\}$,

$$h_{\text{top}}^B(Z, \{F_n\}) \leq h_{\text{top}}^P(Z, \{F_n\}).$$

**Proposition 2.6.** [6, Proposition 2.4] If the Følner sequence $\{F_n\}$ satisfies $\frac{|F_n|}{\log n} \to \infty$, then for any subset $Z \subset X$,

$$h_{\text{top}}^P(Z, \{F_n\}) \leq h_{\text{top}}^{UC}(Z, \{F_n\}).$$

From measure theoretical viewpoint, Brin and Katok [3] introduced the local measure-theoretical lower and upper entropies of $\mu \in M(X)$. We give the definitions with related to one Følner sequence $\{F_n\}$. 

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6
Definition 2.7. For $\mu \in M(X), x \in X, \varepsilon > 0$ and a Følner sequence $\{F_n\}$, let
\[
\overline{h}_\mu^{loc}(x, \varepsilon; \{F_n\}) = \limsup_{n \to \infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x, \varepsilon)),
\]
and
\[
\underline{h}_\mu^{loc}(x, \varepsilon; \{F_n\}) = \liminf_{n \to \infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x, \varepsilon)).
\]

For $Z \in \mathcal{B}(X)$ (the Borel $\sigma$-algebra of $X$), denote by
\[
\overline{h}_\mu^{loc}(Z; \{F_n\}) = \int_{Z} \lim_{\varepsilon \to 0} \overline{h}_\mu^{loc}(x, \varepsilon; \{F_n\}) \, d\mu
\]
and
\[
\underline{h}_\mu^{loc}(Z; \{F_n\}) = \int_{Z} \lim_{\varepsilon \to 0} \underline{h}_\mu^{loc}(x, \varepsilon; \{F_n\}) \, d\mu,
\]
which are called the upper local entropy and the lower local entropy of $\mu$ over $Z$, respectively.

The following theorem is an amenable version of [8, Theorem 1.3]. It shows a variational principle between packing entropy and local measure-theoretical upper entropy.

Theorem 2.8. [6, Theorem 1.3] Let $(X, G)$ be a $G$-action topological dynamical system and $G$ a discrete countable amenable group. Let $\{F_n\}$ be a Følner sequence with $\frac{|F_n|}{\log n} \to \infty$. Then for any non-empty Borel subset $Z \subset X$,
\[
h_{top}^P(Z; \{F_n\}) = \sup \{\overline{h}_\mu^{loc}(Z; \{F_n\}) \mid \mu \in M(X), \mu(Z) = 1\}.
\]

2.2. The $g$-almost product property

The specification property was first introduced by Bowen. The $g$-almost product property was introduced by Pfister and Sullivan in [18] and it turns out to be a weaker concept than specification property. For instance, it is well known that the $g$-almost product property holds for all $\beta$-shifts. However, the specification property is not satisfied for every parameter of $\beta$-shifts. The $g$-almost product property for amenable group actions was introduced by Zhang in [21].

Definition 2.9. A map $g : (0, 1) \to (0, 1)$ is called a mistake-density function if
\[
\lim_{r \to 0} g(r) = 0 \text{ and } g(r) \leq g(s), \quad \forall 0 < r < s < 1.
\]

For $F \in F(G), \varepsilon > 0$ and $x \in X$, define the dynamical ball with respect to the mistake-density function $g$ by
\[
B(g; F, x, \varepsilon) := \{y \in X : |\{s \in F \mid \rho(sx, sy) > \varepsilon\}| \leq g(\varepsilon)|F| \}.
\]

Next we will recall the $g$–almost product property for group actions.
**Definition 2.10.** Let \( g \) be a mistake-density function. The system \((X, G)\) satisfies the \( g \)-almost product property if there exists a map \( m : (0,1) \to F(G) \times (0,1) \) such that for any \( k \in \mathbb{N} \), any \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \) and any \( x_1, x_2, \ldots, x_k \in X \), if \( \{F_i\}_{i=1}^k \) are pairwise disjoint and \( F_i \) is \( m(\varepsilon_i) \)-invariant, \( i = 1, 2, \ldots, k \), then

\[
\bigcap_{i=1}^k B(g; F_i, x_i, \varepsilon_i) \neq \emptyset.
\]

**Lemma 2.11.** Assume the system \((X, G)\) has the \( g \)-almost product property. Let \( x_1, x_2, \ldots, x_k \in X \) and \( \varepsilon_1 > 0, \varepsilon_2 > 0, \ldots, \varepsilon_k > 0 \) and \( F_1, F_2, \ldots, F_k \in F(G) \) be given satisfying each \( F_j \) is \( m(\varepsilon_j) \)-variant for \( j = 1, 2, \ldots, k \) and \( F_i \cap F_j = \emptyset \) for \( 1 \leq i \neq j \leq k \). Let \( F = \bigcup_{i=1}^k F_i \). Assume that \( \mathcal{E}_F(x_j) \in B(\mu_j, \xi_j), j = 1, 2, \ldots, k \). Then for any \( y \in \bigcap_{j=1}^k B(g; F_j, x_j, \varepsilon_j) \) and any probability measure \( \alpha \),

\[
D(\mathcal{E}_F(y), \alpha) \leq \sum_{j=1}^k \frac{|F_j|}{|F|} (D(\mu_j, \alpha) + \xi_j + \varepsilon_j + g(\varepsilon_j)).
\]

**Proof.** We have

\[
\mathcal{E}_F(y) = \sum_{j=1}^k \frac{|F_j|}{|F|} \mathcal{E}_{F_j}(y).
\]

Because of the distance \([2.3]\) on \( X \),

\[
D(\mathcal{E}_{F_j}(x_j), \mathcal{E}_{F_j}(y)) < g(\varepsilon_j) + \varepsilon_j.
\]

The result follows from the triangle inequality and the definition of the distance \([2.3]\): 

\[
D(\mathcal{E}_F(y), \alpha) \leq \sum_{j=1}^k \frac{|F_j|}{|F|} D(\mathcal{E}_{F_j}(y), \alpha) \leq \sum_{j=1}^k \frac{|F_j|}{|F|} (D(\mu_j, \alpha) + g(\varepsilon_j) + \varepsilon_j + \xi_j).
\]

\( \square \)

### 2.3. Uniform separation property and entropy dense

Let \( F \in F(G) \) and \( \delta > 0, \varepsilon > 0 \). A subset \( \Gamma \subset X \) is \((\delta, F, \varepsilon) \)-separated if for \( x \neq y \in \Gamma \),

\[
\left| \left\{ s \in F \mid \rho(sx, sy) > \varepsilon \right\} \right| \geq \delta.
\]

Let \( C \subset M(X) \) be a neighborhood of \( \mu \in M(X, G) \). Define

\[
X_{F,C} := \{ x \in X \mid \mathcal{E}_F(x) \in C \},
\]

8
and
\[ N(C; F, \varepsilon) := \text{largest cardinality of an } (F, \varepsilon) - \text{separated subset of } X_{F,C} \]

and
\[ N(C; \delta, F, \varepsilon) := \text{largest cardinality of a } (\delta, F, \varepsilon) - \text{separated subset of } X_{F,C}. \]

With this convention, we have the following proposition.

**Proposition 2.12.** [19, Proposition 3.7]

Let \( \{F_n\} \) be a Følner sequence and \( \mu \) be an ergodic measure. Then for \( h^* < h_\mu(X,G) \), there exist \( \delta^* > 0 \) and \( \varepsilon^* > 0 \) such that for any neighborhood \( C \) of \( \mu \), there exists \( n^*_C \), such that for any \( n \geq n^*_C \), there exists a \( (\delta^*, F_n, \varepsilon^*) \)-separated set \( \Gamma_n \) of \( X_{F_n,C} \) satisfying
\[ \#\Gamma_n \geq e^{h^*[F_n]} . \]

The uniform separation property for \( \mathbb{Z} \)-actions was introduced by Pfister and Sullivan in [18]. The concept for amenable group actions was defined in [19, Section 3].

**Definition 2.13.** The system \((X, G)\) has uniform separation property if the following holds. Let \( \{K_n\} \) be a tempered Følner sequence. For any \( \eta > 0 \), there exists \( \varepsilon^* > 0 \) and \( \delta^* > 0 \) such that for \( \mu \in E(X,G) \) and any neighborhood \( C \subset M(X) \) of \( \mu \), there exists \( n^*_C; \mu, \eta \in \mathbb{N} \), such that for \( n \geq n^*_C; \mu, \eta \),
\[ N(C; \delta^*, K_n, \varepsilon^*) \geq e^{[K_n](h_\mu(X,G)-\eta)} . \]

**Remark:** The uniform separation property implies \( h_{\text{top}}(X,G) < \infty \).

At last, we will recall the concept of entropy dense.

**Definition 2.14.** The measure \( \nu \in M(X,G) \) is entropy-approachable by ergodic measures if for any neighborhood \( C \) of \( \nu \) and each \( h^* < h_\nu(X,G) \), there exists a measure \( \mu \in E(X,G) \cap C \) such that \( h_\mu(X,G) > h^* \). The ergodic measures are entropy-dense if each \( \nu \in M(X,G) \) is entropy-approachable by ergodic measures.

For amenable group actions, we prove the following result.

**Theorem 2.15.** Suppose the system \((X, G)\) has the \( g \)-almost product property. Then the ergodic measures are entropy dense.

**Proof.** We will prove this theorem in the Appendix part. \( \square \)

The following proposition is [19, Proposition 3.4]

**Proposition 2.16.** Let \((X, G)\) be a dynamical system. Assume the system has the uniform separation property and that the ergodic measures are entropy dense. Let \( \{K_n\} \) be a tempered Følner sequence. For any \( \eta > 0 \), there exist \( \delta^* > 0 \) and \( \varepsilon^* > 0 \) so that for \( \mu \in M(X,G) \) and any neighborhood \( C \subset M(X) \) of \( \mu \), there exists \( n^*_C; \mu, \eta \) such that for \( n \geq n^*_C; \mu, \eta \)
\[ N(C; \delta^*, K_n, \varepsilon^*) \geq e^{[K_n](h_\mu(X,G)-\eta)} . \]
The following lemmas will be needed.

**Lemma 2.17.** [19, Lemma 3.6] Let $\mu \in M(X, G)$ and $\{F_n\}$ be a Følner sequence. For $\varepsilon > 0$, let $E_n$ be a sequence of $(F_n, \varepsilon)$-separated subsets. Define

$$\nu_n := \frac{1}{|E_n|} \sum_{x \in E_n} \varepsilon_{F_n}(x).$$

Assume $\nu_n \to \mu$. Then

$$\limsup_{n \to \infty} \frac{1}{|F_n|} \log |E_n| \leq h_\mu(X, G).$$

**Lemma 2.18.** [21, Lemma 2.6] Let $(X, G)$ be a dynamical system. Let $\mu \in M(X, G), \delta^* > 0, \varepsilon^* > 0, \xi > 0$. Let $0 < \delta < \min\{\frac{1}{2}, \frac{\xi}{3}, \frac{\delta^*}{2}\}, F \in F(G)$ and $\Gamma \subset X_{F, B(\mu, \xi)}$ be a $(\delta^*, F, \varepsilon^*)$-separated set. Then for any $F' \subset F$ with $\frac{|F'|}{|F|} > 1 - \delta$, $\Gamma$ is a $(\frac{\delta^*}{2}, F', \varepsilon^*)$-separated set and $\Gamma \subset X_{F', B(\mu, 2\xi)}$.

We remark that the statement of Lemma 2.18 is a little bit different from [21, Lemma 2.6]. But from the proof of [21, Lemma 2.6], it is the same thing.

### 3. Proofs of main theorems

The ideas of the proof is to obtain a decomposition for $\bigcup_{n=1}^{\infty} F_n$ which will be used to find suitable orbit segments. Then we construct subsets based on the orbit segments by using the $g$-almost product property and we show those subsets have properties as we want.

First we will recall the quasi-tiling method and a recent result on tilings of amenable groups [17]. The ideas to find the decomposition of $\bigcup_{n=1}^{\infty} F_n$ is from [21].

Let $\{A_1, \cdots, A_k\} \subset F(G), \alpha > 0, \varepsilon > 0$ and $A \in F(G)$. The collection $\{A_1, \cdots, A_k\}$ is called $\varepsilon$-disjoint if there exists $B_i \subset A_i, i = 1, \cdots, k$ with $B_i \cap B_j = \emptyset, 1 \leq i \neq j \leq k$ and $\frac{|B_i|}{|A_i|} > 1 - \varepsilon$ for $i = 1, \cdots, k$. We say $\{A_1, \cdots, A_k\}$ is an $\alpha$-covering of $A$ if $|A \cap \bigcup_{i=1}^{k} A_i| \geq \alpha|A|$.

We say $\{A_1, \cdots, A_k\} \subset F(G)$ is an $\varepsilon$-quasi-tile of $A$ if there are $C_1, \cdots, C_k \in F(G)$ satisfying:

1. $A_iC_i \subset A, i = 1, \cdots, k$;
2. $A_iC_i \cap A_jC_j = \emptyset, 1 \leq i \neq j \leq k$;
3. $\{A_iC_i \mid c \in C_i \}$ are $\varepsilon$-disjoint for $i = 1, \cdots, k$;
4. $\{A_iC_i \mid i = 1, \cdots, k\}$ is a $(1 - \varepsilon)$-covering of $A$.

Such $\{C_1, \cdots, C_k\}$ are called tiling centers.

The following lemma is a fundamental tool in the quasi-tiling theory for amenable groups.

**Lemma 3.1** ([4]). Let $G$ be a countable amenable discrete group and $\{e_G\} \subset F_1 \subset F_2 \subset \cdots$ be a Følner sequence. Then for any $\varepsilon \in (0, \frac{1}{4})$ and $N \in \mathbb{N}$, there exist $n_1, n_2, \cdots, n_k \in \mathbb{N}$ and $\delta > 0$ with $N \leq n_1 < \cdots < n_k$ such that for $F \in F(G)$ which is $(\delta, F, n_kF^{-1})$-invariant and $\frac{|F_n|}{|F|} < \delta$, then $F$ can be $\varepsilon$-quasi tiled by $F_{n_1}, \cdots, F_{n_k}$.
Remark 3.1. Let $K_1, K_2, \ldots, K_s$ be an $\varepsilon$-quasi-tile of $D \subset G$ and $\{C_1, \ldots, C_s\}$ be the tiling centers. We can modify the tile to get a disjoint $(1 - \varepsilon)^2$-covering of $D$ by shrinking every translation of $K_i, i = 1, 2, \ldots, s$. In fact, for each $j$, since \{\$K_j c_j \mid c_j \in C_j\}$ are $\varepsilon$-disjoint, we can choose $K_j(c_j) \subset K_j$ with $\frac{|K_j(c_j)|}{|K_j|} \geq 1 - \varepsilon$ and the elements in $\{K_j(c_j) c_j\}$ are pairwise disjoint. Thus the elements in the collection \{\$K_j(c_j) c_j \mid c_j \in C_j, j = 1, 2, \ldots, s\} are pairwise disjoint and

$$\frac{|\bigcup_{j=1}^{s} \bigcup_{c_j \in C_j} K_j(c_j) c_j|}{|D|} \geq (1 - \varepsilon) \frac{|\bigcup_{j=1}^{s} \bigcup_{c_j \in C_j} K_j c_j|}{|D|} \geq (1 - \varepsilon)^2.$$ 

Also we need a recent tiling result in [7].

Definition 3.2. $\mathcal{T}$ is called a tiling of $G$ if there exist a shape set $\mathcal{S} = \{S_j \in F(G) \mid 1 \leq j \leq k\}$ and tiling centers $\{C_1, C_2, \ldots, C_k\}$ such that

$$\mathcal{T} := \{S_j g \mid g \in C_j, j = 1, 2, \ldots, k\}$$

with $G = \bigcup \mathcal{T}$ and $A \cap B = \emptyset$ for $A \neq B \in \mathcal{T}$. Let $\{\mathcal{T}_k\}_{k \geq 1}$ be a sequence of tilings of $G$, we say $\{\mathcal{T}_k\}_{k \geq 1}$ is congruent if for each $k \geq 1$, each element in $\mathcal{T}_{k+1}$ is a union of elements in $\mathcal{T}_k$.

The following lemma is part of [7, Lemma 5.1].

Lemma 3.3. Fix a positive converging to zero sequence $\{\varepsilon_k\}$ and a sequence $\{K_k\}$ of finite subsets of $G$. There exists a congruent sequence of tilings $\{\tilde{\mathcal{T}}_k\}$ of $G$ such that shapes of $\tilde{\mathcal{T}}_k$ are $(K_k, \varepsilon_k)$-invariant.

Now we will construct the decomposition of $\bigcup_{n=1}^{\infty} F_n$. Let $\{\beta_k\}_{k=1}^{\infty}$ be a strictly decreasing to 0 positive sequence. We will choose an increasing sequence $M(k) \in \mathbb{N}$ and a sequence of subsets $H_0 \subset H_1 \subset \cdots \subset G$ in the following way.

1. Let $H(0) = \emptyset$. Choose $M(0) \in \mathbb{N}$ such that $F_n$ is $(\cup S_1, \frac{\beta_1}{|S_1|})$-invariant for any $n \geq M(0)$.

2. Choose $M(1) > M(0)$ such that for any $n \geq M(1)$, $F_n$ is $(\cup S_2, \frac{\beta_2}{|S_2|})$-invariant.

Let $\tilde{F}_1 = \bigcup_{i=M(0)+1}^{M(1)} F_i$, $\tilde{T}_2 = \{T \in \mathcal{T}_2 \mid T \cap \tilde{F}_1 \neq \emptyset\}$ and $H(1) = \cup \tilde{T}_2$.

3. Choose $M(2) > M(1)$ such that for any $n \geq M(2)$, $F_n$ is $(\cup S_3, \frac{\beta_3}{|S_3|})$-invariant and $|H(1)| < \beta_3 |F_n|$. Let $\tilde{F}_2 = \bigcup_{i=M(2)+1}^{M(2)} F_i$, $\tilde{T}_3 = \{T \in \mathcal{T}_3 \mid T \cap \tilde{F}_2 \neq \emptyset\}$ and $H(2) = \cup \tilde{T}_3$.

4. Assume that $M(0) < M(1) < \cdots < M(k-1)$ and $H(0) \subset H(1) \subset \cdots \subset H(k-1)$ have been chosen, then choose $M(k) > M(k-1)$ such that for any $n \geq M(k)$, $F_n$ is $(\cup S_{k+1}, \frac{\beta_{k+1}}{|S_{k+1}|})$-invariant and $|H(k-1)| < \beta_{k+1} |F_n|$. Let $\tilde{F}_k = \bigcup_{i=M(k)+1}^{M(k)} F_i$, $\tilde{T}_{k+1} = \{T \in \tilde{T}_{k+1} \mid T \cap \tilde{F}_k \neq \emptyset\}$ and $H(k) = \cup \tilde{T}_{k+1}$.

For $k \geq 1$, denote

$$H_k := \{T \in \mathcal{T}_k \mid T \subset H(k) \setminus H(k-1)\} \text{ and } H_k := \cup H_k.$$ 

Then $H(k) \setminus H(k-1) = H_k$ for $k \geq 1$. 


We call elements in each $T_k$ the standard bricks. Next we will use standard bricks to cover each $F_n$. The following lemma shows that most part of $F_n$ can be covered by standard bricks.

**Lemma 3.4.** [273] For any $k$ and $M(k-1) < n \leq M(k)$, let

$$\Lambda_n^1 = \{T \in \mathcal{H}_k \mid T \subset F_n\}$$

and

$$\Lambda_n^2 = \{T \in \mathcal{H}_{k-1} \mid T \subset F_n\}.$$ 

Let $\Lambda_n = \Lambda_n^1 \cup \Lambda_n^2$ and $F'_n = \cup \Lambda_n$. Then

$$F'_n \subset F_n \text{ and } |F'_n| > (1-2\beta_k)|F_n|.$$ 

### 3.1. Upper capacity entropy formula

In this part, we will give the proof of Theorem [141]

**Proof.** Since $Y_K \subset X$, we have $h^{UC}(Y_K, \{F_n\}) \leq h_{top}(X, \{F_n\}) = h_{top}(X, G)$. Next we will construct a closed subset of $Y_K$ whose upper capacity topological entropy is close to $h_{top}(X, G)$.

For each $\varepsilon > 0$, there exist a finite sequence $\alpha_1, \cdots, \alpha_n$ in $K$ such that each point in $K$ is within $\varepsilon$ of some $\alpha_i$. As $K$ is connected, by repeating some $\alpha_i$, we can choose this sequence $\alpha_1, \cdots, \alpha_n$ so that each point in $K$ is within $\varepsilon$ of some $\alpha_i$ and $D(\alpha_j, \alpha_{j+1}) < \varepsilon$ for each $j$. Extending this argument, we deduce that there exists a sequence $\{\alpha_j : j = 1, 2, \cdots\}$ in $K$ so that the closure of $\{\alpha_j : j > n\}$ for each $n$ equals $K$ and

$$\lim_{j \to \infty} D(\alpha_j, \alpha_{j+1}) = 0.$$ 

We define the stretched sequence $\{\alpha'_n\}$ by

$$\alpha'_n = \alpha_k \text{ if } M(k-1) < n \leq M(k).$$

The sequences $\{\alpha_n\}$ and $\{\alpha'_n\}$ have the same limit-point set.

By the variational principle, for any $\eta > 0$, there exists $\mu \in E(X, G)$ such that $h_\mu(X, G) > h_{top}(X, G) - \eta$.

Let $\{e_G\} \subset K_1 \subset K_2 \subset \cdots$ be a tempered Følner sequence. Let $\{\xi_k\}_{k=0}^\infty$ be a strictly decreasing positive sequence with $\xi_k \to 0$. By Proposition [121] there exists $\delta^* > 0$ and $\varepsilon^* > 0$ such that for the neighborhood $B(\mu, \xi_0)$, there exists $n^*_{B(\mu, \xi_0), \mu, \eta} \in \mathbb{N}$ such that for any $n \geq n^*_{B(\mu, \xi_0), \mu, \eta}$, there is a $(\delta^*, K_\eta, 3\varepsilon^*)$-separated set $\Gamma_n \subset X_{K_\eta, B(\mu, \xi_0)}$ with

$$\#\Gamma_n \geq e^{h_\mu(X, G) - \eta}|K_n|.$$ 

(3.1)

Let $\{\gamma_k\}_{k=1}^\infty$ be a strictly decreasing positive sequence with $\gamma_1 < \min\{\frac{1}{2}, \frac{\delta^*}{3}, \frac{\varepsilon^*}{3}\}$ and $\gamma_k \to 0$. By Lemma [3.2] and Lemma [3.3] there exists a sequence of congruent tilings $\mathcal{T}_k$ with shape set $\mathcal{S}_k$ and $N_1 \leq n_{1,1} < \cdots < n_{1,t_1} < N_2 \leq n_{2,1} < \cdots < n_{2,t_2} < N_3 < \cdots$
such that the following hold: Any $S \in \mathcal{S}_k$ can be $\gamma_k$-quasi tiled by $\{K_{n_k,1}, \cdots, K_{n_k,t_k}\}$ with tiling centers $\{C_{k,S,1}, \cdots, C_{k,S,t_k}\}$ and we denote the $\gamma_k$-quasi-tiling by

$$\mathcal{T}_{k,S} = \{K_{c_k,S,j} | c_k,S,j \in C_{k,S,j}, 1 \leq j \leq t_k\}. \quad (3.2)$$

Here we assume $N_1 \geq n^*_B(\mu, \xi_0, \eta)$. By the pointwise ergodic theorem, we can assume that for all $k \geq 1$ and $n \geq N_k$, the set $X_{K_{n_k,B}(\alpha_k, \xi_k)}$ is not empty.

From Remark 3.1, we can modify $\mathcal{T}_{k,S}$ to get a new $\gamma_k$-quasi tile $\tilde{\mathcal{T}}_{k,S} = \{\tilde{K}_{c_k,S,j} | \tilde{K}_{c_k,S,j} \subset K_{c_k,S,j} \in \mathcal{T}_{k,S}\}$ (3.3)

such that the following hold:

1. Elements in $\tilde{\mathcal{T}}_{k,S}$ are pairwise disjoint and $|\tilde{K}_{c_k,S,j}| > 1 - \gamma_k$;
2. $\bigcup \tilde{\mathcal{T}}_{k,S} \subset S$ and $|\bigcup \tilde{\mathcal{T}}_{k,S}| > (1 - \gamma_k)^2 |S|$.

Note that the sequence obtained from $\tilde{\mathcal{T}}_{k,S}$

$$\bigcup_{k \geq 1} \{\tilde{K}_{c_k,S,j}\}$$

is still a Følner sequence. Let $g : (0,1) \rightarrow (0,1)$ be the mistake-density function as described in the $g$-almost product property and

$$m : (0,1) \rightarrow F(G) \times (0,1)$$

be the map (with respect to the mistake-density function $g$) in Definition 2.9. Choose a decreasing to 0 sequence $\{\varepsilon_n\}_{n=1}^\infty$ with

$$\varepsilon_1 < \frac{\varepsilon^*}{4} \text{ and } g(\varepsilon_1) < \frac{\delta^*}{4}.$$  

By taking a subsequence of $\bigcup_{k \geq 1} \{\tilde{K}_{c_k,S,j}\}$, we can assume for each $n \in \mathbb{N}$, any

$$A \in \bigcup_{k \geq n} \{\tilde{K}_{c_k,S,j} | c_k,S,j \in C_{k,S,j} 1 \leq j \leq t_k, S \in \mathcal{S}_k\}$$

is $m(\varepsilon_n)$-variant.

Let $\mathcal{N}$ be any integer larger than $M(1)$. Then there exists $\kappa \geq 2$ such that $M(\kappa-1) < \mathcal{N} \leq M(\kappa)$.

For $k = \kappa - 1, \kappa$ and $S \in \mathcal{S}_k$, let $\mathcal{T}_{k,S}$ and $\tilde{\mathcal{T}}_{k,S}$ be as in (3.2) and (3.3), respectively. For $K_{c_k,S,j} \subset \mathcal{T}_{k,S}$, there is a $(\delta^*, K_{c_k,S,j}, 3\varepsilon^*)$-separated set $\Gamma_{c_k,S,j} \subset X_{K_{c_k,S,j},B(\mu, \xi_0)}$ with $\# \Gamma_{c_k,S,j} \geq e^{\delta^*|K_{c_k,S,j}|(h_n(X,G)-\eta)}$. By Lemma 2.18, the set $\Gamma_{c_k,S,j}$ is also $(\delta^*/2, K_{c_k,S,j}, 3\varepsilon^*)$-separated and contained in $X_{K_{c_k,S,j},B(\mu,2\xi_0)}$.  

13
Define
\[ \Gamma(S) := \prod_{\tilde{K}_{k,S,j} \in \tilde{T}_{k,S}} \Gamma_{c_k,S,j} \]
\[ := \{ \bar{\bar{x}} = (x_{c_k,S,j}) \mid x_{c_k,S,j} \in \Gamma_{c_k,S,j} \}. \]

For each \( \bar{\bar{x}} \in \Gamma(S) \), denote
\[ B(g; S, \bar{\bar{x}}, \varepsilon_k) := \bigcap_{\tilde{K}_{k,S,j} \in \tilde{T}_{k,S}} B(g; \tilde{K}_{c_k,S,j} c_k S,j, c_k S,j x_{c_k,S,j}, \varepsilon_k), \] (3.4)

where the dynamical ball \( B(g; \tilde{K}_{c_k,S,j} c_k S,j, c_k S,j x_{c_k,S,j}, \varepsilon_k) \) is defined in Definition 2.9.

By the \( g \)-almost product property, \( B(g; S, \bar{\bar{x}}, \varepsilon_k) \neq \emptyset \).

For \( k \geq \kappa + 1 \) and \( S \in S_k \), since the set \( X_{K_n,B(\alpha_k, \gamma_k)} \) is not empty for \( n \geq N_k \), we pick \( x_{c_k,S,j} \in X_{K_n,B(\alpha_k, \xi_k)} \) for \( K_{c_k,S,j} c_k S,j \in \tilde{T}_{k,S} \). By a simple calculation, we have \( x_{c_k,S,j} \in X_{\tilde{K}_{c_k,S,j} B(\alpha_k, \xi_k + 2\gamma_k)} \). Let \( \Gamma_{c_k,S,j} = \{ x_{c_k,S,j} \} \).

Let
\[ \Gamma(S) := \prod_{\tilde{K}_{k,S,j} \in \tilde{T}_{k,S}} x_{c_k,S,j} = \{ \bar{\bar{x}} = (x_{c_k,S,j}) \mid x_{c_k,S,j} \in \Gamma_{c_k,S,j}, \tilde{K}_{c_k,S,j} c_k S,j \in \tilde{T}_{k,S} \}. \]

For each \( \bar{\bar{x}} \in \Gamma(S) \), we denote
\[ B(g; S, \bar{\bar{x}}, \varepsilon_k) := \bigcap_{\tilde{K}_{c_k,S,j} c_k S,j \in \tilde{T}_{k,S}} B(g; \tilde{K}_{c_k,S,j} c_k S,j, c_k S,j x_{c_k,S,j}, \varepsilon_k). \]

By the \( g \)-almost product property, \( B(g; S, \bar{\bar{x}}, \varepsilon_k) \neq \emptyset \).

For any integer \( q > 0 \), let
\[ Y_q^{(N)} := \bigcap_{i=\kappa-1}^{\kappa+q} \bigcup_{S \in H_i} \bigcup_{\bar{\bar{x}} \in \Gamma(S)} B(g; S d^{-1} \bar{\bar{x}}, \varepsilon_i), \]

where \( B(g; S d^{-1} \bar{\bar{x}}, \varepsilon_i) := \bigcap_{\tilde{K}_{c_k,S,j} c_k S,j \in \tilde{T}_{k,S}} B(g; \tilde{K}_{c_k,S,j} c_k S,j, d^{-1} c_k S,j x_{c_k,S,j}, \varepsilon_i) \). By the \( g \)-almost product property, the set \( Y_q^{(N)} \) is not empty. Let \( Y^{(N)} = \bigcap_{q \geq 1} Y_q^{(N)} \), then \( Y^{(N)} \) is not empty.

Next we will show some properties of \( Y^{(N)} \):

(1) \( Y^{(N)} \subset Y_K \);

(2) There exists \( Y \subset Y^{(N)} \) which is \( (F_N, \varepsilon^+) \)-separated and \#\( Y \geq e^{(1-3q) |F_N| (\text{htop}(X,G) - 2\eta)} \).

**Proof of Item (1):** Choose \( y \in Y^{(N)} \) and \( F_n \). Suppose \( n \) large such that \( M(k-1) < n \leq M(k) \) and \( k \geq \kappa + 2 \). Let \( \Lambda_1^{(1)} \) and \( \Lambda_2^{(2)} \) be as described in Lemma 3.24. Using the triangle inequality, Lemma 2.11 and Lemma 2.18.
$$D(\mathcal{E}_{F_n}(y), \alpha'_n) \leq D(\mathcal{E}_{F_n}(y), \alpha_k) + D(\mathcal{E}_{F_n}(y), \mathcal{E}_{F_n}(y))$$

$$\leq 4\beta_k + \sum_{Sd \in \Lambda_n^1} \frac{|Sd|}{|F'_n|} D(\mathcal{E}_{Sd}(y), \alpha_k) + \sum_{Sd \in \Lambda_n^2} \frac{|Sd|}{|F'_n|} D(\mathcal{E}_{Sd}(y), \alpha_k)$$

$$\leq 4\beta_k + \sum_{Sd \in \Lambda_n^1} \frac{|Sd|}{|F'_n|} (D(\mathcal{E}_{Sd}(y), \mathcal{E}_{Sd}(y)) + D(\mathcal{E}_{Sd}(y), \alpha_k))$$

$$+ \sum_{Sd \in \Lambda_n^2} \frac{|Sd|}{|F'_n|} (D(\mathcal{E}_{Sd}(y), \mathcal{E}_{Sd}(y)) + D(\mathcal{E}_{Sd}(y), \alpha_k))$$

$$\leq 4\beta_k + \sum_{Sd \in \Lambda_n^1} \frac{|Sd|}{|F'_n|} (4\gamma_k + D(\mathcal{E}_{Sd}(y), \alpha_k))$$

$$+ \sum_{Sd \in \Lambda_n^2} \frac{|Sd|}{|F'_n|} (4\gamma_k - 1 + D(\mathcal{E}_{Sd}(y), \alpha_k))$$

$$\leq 4\gamma_k - 1 + 4\beta_k + \sum_{Sd \in \Lambda_n^1} \frac{|Sd|}{|F'_n|} (\xi_k + \varepsilon_k + g(\varepsilon_k))$$

$$+ \sum_{Sd \in \Lambda_n^2} \frac{|Sd|}{|F'_n|} (\xi_k - 1 + \varepsilon_k - 1 + g(\varepsilon_k) + D(\alpha_k - 1, \alpha_k))$$

$$\leq 4\gamma_k - 1 + 4\beta_k + \xi_k - 1 + \varepsilon_k - 1 + g(\varepsilon_k) + D(\alpha_k - 1, \alpha_k).$$

This implies $\lim_{n \to \infty} D(\mathcal{E}_{F_n}(y), \alpha'_n) = 0$. Hence we have $Y^{(\mathcal{N})} \subset Y_K$.

**Proof of Item (2):** Define

$$\Gamma(F_N) := \prod_{Sd \in \Lambda_N} \Gamma(S)$$

$$= \{ \bar{x} = (x_S) | x_S \in \Gamma(S), Sd \in \Lambda_N \}$$

$$= \{ \bar{x} = (x_{c_k,S,j}) | x_{c_k,S,j} \in \Gamma_{c_k,S,j}, K_{c_k,S,j} \in \mathcal{T}_{K,S,j}, Sd \in \Lambda_N \}.$$  (3.10)

From the definition, we have

$$\# \Gamma(F_N) = \prod_{Sd \in \Lambda_N} \# \Gamma(S).$$

Write $\Gamma(F_N) = \{ \bar{x}^1, \ldots, \bar{x}^r \}$, where $r = \# \Gamma(F_N)$. For $\bar{x}^u \neq \bar{x}^v \in \Gamma(F_N)$, there exists $Sd \in \Lambda_N$ such that $\bar{x}^u_S \neq \bar{x}^v_S$ which implies there exists $K_{c_k,S,j} \in \mathcal{T}_{K,S,j}$ such that $x_{c_k,S,j}^u$ and $x_{c_k,S,j}^v$ are $(\delta^*, K_{c_k,S,j}, 3e^*)$-separated.

For each $\bar{x}^u \in \Gamma(F_N)$, there is $x^u \in Y^{(\mathcal{N})}$ such that $x^u \in B(g; Sd, d^{-1}x^u_S, \varepsilon)$ for every $Sd \in \Lambda_N$, where $\varepsilon = \varepsilon_K$ if $Sd \in \Lambda_N^1$ and $\varepsilon = \varepsilon_{K-1}$ if $Sd \in \Lambda_N^2$. Denote $Y = \{ x^u | \bar{x}^u \in \Gamma(F_N) \}$. Thus there is a one-to-one map $\Phi$ from $\Gamma(F_N)$ to $Y$.

To this end, fix $x^u \neq x^v \in Y$. Let $\bar{x}^u = (x_{c_k,S,j}^u) = \Phi^{-1}(x^u) \in \lambda_N$ and $\bar{x}^v = (x_{c_k,S,j}^v) = \Phi^{-1}(x^v) \in \Gamma(F_N)$. Then there is $Sd \in \Lambda_N$ and $K_{c_k,S,j} \in \mathcal{T}_{K,S,j}$ such that $x_{c_k,S,j}^u$ and $x_{c_k,S,j}^v$ are $(\delta^*, K_{c_k,S,j}, 3e^*)$-separated. And also $x^u \in B(g; Sd, d^{-1}x_S^u, \varepsilon), x^v \in$
Upper capacity entropy and packing entropy of saturated sets

Expression: $B(g; Sd, d^{-1}x^v_G, \tilde{e})$ implies

$$\frac{|\{s \in \tilde{K}_{k,s,j} \mid \rho(sx^u, sx^v_{c_k,s,j}) > \tilde{e}\}|}{|\tilde{K}_{c_k,s,j}|} > \delta^* \text{ and } \frac{|\{s \in \tilde{K}_{c_k,s,j} \mid \rho(sx^v, sx^v_{c_k,s,j}) > \tilde{e}\}|}{|\tilde{K}_{c_k,s,j}|} > \delta^*.$$ 

Then there is $s \in \tilde{K}_{c_k,s,j}$ such that

$$\rho(sx^u, sx^v) \geq 3\tilde{e} - 2\tilde{e} > \varepsilon^*.$$ 

Based on items (1) and (2), we get that $Y^{(\mathcal{N})} \subset Y_K$ and for each $\eta > 0$, $h_{top}^{UC}(Y_K, \{F_n\}) \geq h_{top}^{UC}(Y^{(\mathcal{N})}, \{F_n\}) > h_\mu(X, G) > h_{top}(X, G) - 2\eta$. By the arbitrariness of $\eta$, we have $h_{top}^{UC}(X, G) \geq h_{top}(X, G)$, which finish the proof of Theorem 1.1.

3.2. Packing entropy formula

In this part, we will proceed the proof of Theorem 1.2.

First, we will construct a closed subset of $Y_K$ whose packing topological entropy is close to $\sup\{h_\mu(X, G) \mid \mu \in K\}$. Let $\eta > 0$ and $H^* = \sup\{h_\mu(X, G) \mid \mu \in K\} - \eta$ and $h^* = \inf\{h_\mu(X, G) \mid \mu \in K\} - \eta$. Pick $\mu_{\max} \in K$ such that $h_{\mu_{\max}}(X, G) > H^*$. Since $K$ is convex and compact, for each $n \in \mathbb{N}$, we can choose $\{\mu_{n,1}, \cdots, \mu_{n,M_n}\} \subset K$ such that

$$K \subset \bigcup_i B(\mu_{n,i}, \frac{1}{n}), D(\mu_{n,i}, \mu_{n,i+1}) \leq \frac{1}{n} \quad \text{and} \quad \mu_{n,M_n} = \mu_{n,M_{n-1}} = \cdots = \mu_{n,M_1} = \mu_{\max}.$$ 

Here we repeat $\mu_{\max}$ twice since two levels of standard bricks are needed to cover each $F_n$. Denote $\{\alpha_j\} = \{\mu_{1,1}, \mu_{1,2}, \cdots, \mu_{1,M_1}, \mu_{2,1}, \cdots, \mu_{2,M_2}, \cdots\}$. Then $K = \{\alpha_j\}_{j \geq n}$ for each $n \geq 1$ and $D(\alpha_j, \alpha_{j+1}) \to 0$ as $j \to \infty$. Define the stretched sequence $\{\alpha'_n\}$ by

$$\alpha'_n = \alpha_k \text{ if } M(k - 1) < n \leq M(k).$$

The sequences $\{\alpha_n\}$ and $\{\alpha'_n\}$ have the same limit-point set.

Let $\{\xi_k\}, \{\eta_k\}$ and $\{\gamma_k\}$ be sequences of strictly decreasing positive numbers with $\gamma_1 < \min\{\frac{1}{3}, \frac{3}{4}, \frac{2}{3}\}, \eta_1 < \eta$ and $\xi_k \to 0, \eta_k \to 0, \gamma_k \to 0$. Let $\{K_n\}$ be a tempered Følner sequence with $\epsilon_G \in K_1 \subset K_2 \subset \cdots$. For the given $\eta > 0$, by Lemma 2.16, we can find $\delta^* > 0$ and $\varepsilon^* > 0$ such that for $\mu \in M(X, G)$ and any neighborhood $C \subset M(X)$ of $\mu$, there exists $n_{C,\mu,\eta}^*$ such that for $n \geq n_{C,\mu,\eta}^*$

$$N(C; \delta^*, K_n, \varepsilon^*) \geq e^{K_n|h_\mu(X, G) - \eta|}.$$  \hspace{1cm} \text{(3.11)}

By Lemma 3.1 and Lemma 3.3, there exists a sequence of congruent tilings $T_k$ with shape set $S_k$ and $N_1 \leq n_{1,1} < \cdots < n_{1,t_1} \leq N_2 \leq n_{2,1} < \cdots < n_{2,t_2} < N_3 \leq \cdots$ such that the following hold: Any $S \in S_k$ can be $\gamma_k$- quasi tiled by $\{K_{k,1}, \cdots, K_{k,t_k}\}$ with tiling centers $\{C_{k,s,1}, \cdots, C_{k,s,t_k}\}$ and we denote the $\gamma_k$-quasi-tiling by

$$T_{k,S} = \{K_{k,j}c_{k,s,j} \mid c_{k,s,j} \in C_{k,s,j}, 1 \leq j \leq t_k\}.$$ \hspace{1cm} \text{(3.12)}

We also assume that $N_k \geq n_{\mu_{\max}}^* B(\alpha_k, \xi_k, \alpha_k, \eta_k)$ for each $k$. 


From Remark 3.1, we can modify $T_{k,S}$ to get a new $\gamma_k$-quasi tile
\[ \tilde{T}_{k,S} = \{ \tilde{K}_{c_k,s_j}c_k,s_j \mid K_{c_k,s_j}c_k,s_j \in T_{k,S} \} \] (3.13)
such that the following hold:

1. Elements in $\tilde{T}_{k,S}$ are pairwise disjoint and $|\tilde{K}_{c_k,s_j}| > 1 - \gamma_k$;
2. $\bigcup \tilde{T}_{k,S} \subset S$ and $|\bigcup \tilde{T}_{k,S}| > (1 - \gamma_k)^2 |S|$.

From (3.11), we get the existence of a $(\delta^*, K_{c_k,s_j}, \epsilon^*)$-separated set $\Gamma_{c_k,s_j} \subset X_{K_{c_k,s_j}B(\alpha_k, \xi_k)}$ with $\# \Gamma_{c_k,s_j} \geq e^{k(S,G - \eta_k)}$. By Lemma 2.18, the set $\Gamma_{c_k,s_j}$ is also $(\delta^*/2, K_{c_k,s_j}, \epsilon^*)$-separated and contained in $X_{\tilde{K}_{c_k,s_j}B(\alpha_k, 2\xi_k)}$.

Let
\[ \Gamma(S) := \prod_{K_{c_k,s_j}c_k,s_j \in \tilde{T}_{k,S}} \Gamma_{c_k,s_j} \]
\[ = \{ \vec{x} = (x_{c_k,s_j}) \mid x_{c_k,s_j} \in \Gamma_{c_k,s_j}, K_{c_k,s_j}c_k,s_j \in \tilde{T}_{k,S} \} \]
for $S \in T_k$, let $\Gamma(S) = \Gamma(S)$. For each $\vec{x} \in \Gamma(S)$, denote
\[ B(g; S, \vec{x}, \epsilon_k) := \bigcap_{\tilde{K}_{c_k,s_j}c_k,s_j \in \tilde{T}_{k,S}} B(g; \tilde{K}_{c_k,s_j}c_k,s_j, c_{k,s_j}^{-1}x_{c_k,s_j}, \epsilon_k). \]

For $q \geq 1$, let
\[ Y^q_\eta := \bigcap_{j=0}^q \left( \bigcup_{S \in H_j} \bigcup_{\vec{x} \in \Gamma(S)} B(g; S, d^{-1}\vec{x}, \epsilon_k) \right), \]
where $B(g; S, d^{-1}\vec{x}, \epsilon_k) = B(g; S, \vec{x}, \epsilon_k)$. By the $g$-almost product property, the set $Y^q_\eta$ is a non-empty closed set.

For each $q \geq 1$, define
\[ \Gamma_q := \prod_{j=1}^q \prod_{S \in H_j} \Gamma(S) \]
\[ = \{ \vec{x} = (\vec{x}_{j,S}) \mid \vec{x}_{j,S} \in \Gamma(S), S \in H_j \text{ and } j = 1, 2, \cdots, q \}. \] (3.14)

For each $\vec{x}_q \in \Gamma_q$, pick $x_q \in X$ such that $x_q \in B(g; S, d^{-1}x_j, S_d)$ for all $S_d \in H_j, j = 1, 2, \cdots, q$. Let
\[ Z_q := \{ x_q \mid \vec{x}_q \in \prod_{j=1}^q \prod_{S \in H_j} \Gamma(S) \}. \]
To this end, define
\[ \mu_q := \frac{1}{\# Z_q} \sum_{\vec{x} \in Z_q} \delta_{\vec{x}}, \]
where \( \delta_x \) is the Dirac measure at \( x \). For any fixed \( l \geq 0 \) and \( p \geq 0 \), we have 
\[
\mu_{l+p}(Y^l_\eta) = 1 \text{ since } Y^l_\eta \supset Y^{l+p}_\eta \text{ and } \mu_{l+p}(Y^{l+p}_\eta) = 1.
\]
Let \( \mu \) be a limit measure of \( \{\mu_q\} \) i.e. there exists a sequence \( \{q_n\} \) such that \( \mu_{q_n} \to \mu \). Since \( Y^l_\eta \) is closed, we have 
\[
\mu(Y^l_\eta) \geq \lim_{n \to \infty} \mu_{q_n}(Y^l_\eta) = 1.\]
It follows that \( \mu(Y^l_\eta) = 1 \).

Take \( x \in Y^l_\eta \). For \( i \geq 1 \), let \( L_i = \sum_{j=1}^i M_j \). Then for \( q_n \geq L_i \), one has 
\[
\mu_{q_n}(B_{F_{M(L_i)}}(x, \varepsilon)) \leq \frac{1}{\prod_{S \in \Lambda_{M(L_i)}} \#\Gamma(S)} \leq \frac{1}{\prod_{S \in \Lambda_{M(L_i)}} \prod_{\delta_{k,S,j} \in \mathcal{F}_k,S,j} \#\Gamma_{\delta_{k,S,j}}} \leq \frac{1}{\prod_{S \in \Lambda_{M(L_i)}} \prod_{\delta_{k,S,j} \in \mathcal{F}_k,S,j} e^{|K_{\delta_{k,S,j}}|(h_{\mu_{max}(X,G)-\eta})} \leq e^{-(\gamma_{L_{i-1}})^2(1-2\beta_{L_{i-1}})(h_{\mu_{max}(X,G)-\eta})}.
\]

Therefore \( \mu(B_{F_{M(L_i)}}(x, \varepsilon), \{F_n\}) \geq \limsup_{i \to \infty} - \frac{1}{|F_{M(L_i)}|} \log \mu(B_{F_{M(L_i)-1}}(x, \varepsilon)) \geq \limsup_{i \to \infty} (1-\gamma_{L_{i-1}})^2(1-2\beta_{L_{i-1}})(h_{\mu_{max}(X,G)-\eta}) \geq H^* - \eta, \)

where \( \mu_{loc}(x, \varepsilon, \{F_n\}) \) is defined in Definition 2.8.

By Theorem 2.8, we have 
\[
h_{top}(Y_K, \{F_n\}) \geq h_{top}(Y_\eta, \{F_n\}) \geq H^* - \eta = \sup\{h_{\mu}(X,G) \mid \mu \in K\} - 2\eta.
\]
By the arbitrariness of \( \eta \), we have 
\[
h_{top}(Y_K, \{F_n\}) \geq \sup\{h_{\mu}(X,G) \mid \mu \in K\}.
\]
Next we will show 
\[
h_{top}(Y_K, \{F_n\}) \leq \sup\{h_{\mu}(X,G) \mid \mu \in K\}. \]
Recall that \( V(x, \{F_n\}) \) is the limit-point set of \( \{E_{F_n}(x)\} \). Define 
\[
Y^K := \{y \in X \mid V(y, \{F_n\}) \cap K \neq \emptyset\}.
\]
From the definition, \( Y_K \subset Y^K \). For \( \delta > 0 \) and \( n \in \mathbb{N} \), set 
\[
\mathcal{R}(K, \delta, n) := \{x \in X \mid E_{F_n}(x) \in B(K, \delta)\},
\]
where \( B(K, \delta) = \{\nu \in M(X) \mid \exists \mu \in K \text{ such that } D(\mu, \nu) \leq \delta\} \).

Fix \( \varepsilon > 0 \) and let \( N(K, \delta, F_n, \varepsilon) \) denote the smallest number of balls \( B_{F_n}(x, \varepsilon) \) required to cover \( \mathcal{R}(K, \delta, n) \). Notice that \( N(K, \delta, F_n, \varepsilon) \) does not increase as \( \delta \) decreases and does not decrease as \( \varepsilon \) decreases. As a result, the following limit denoted by \( \Theta(Y^K, \{F_n\}) \) exists

\[
\Theta(Y^K, \{F_n\}) := \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{|F_n|} \log N(K, \delta, F_n, \varepsilon). \tag{3.15}
\]

Let \( R(K, \delta, k) = \bigcap_{n=k}^{\infty} \mathcal{R}(K, \delta, n) = \bigcap_{n=k}^{\infty} \{y \in X \mid E_{F_n}(x) \in B(K, \delta)\} \). Let \( \tilde{N}(K, \delta, F_n, \varepsilon) \) be the smallest number of balls \( B_{F_n}(x, \varepsilon) \) used to cover \( R(K, \delta, n) \). From
the definition,
\[ \tilde{N}(K, \delta, F_n, \varepsilon) \leq N(K, \delta, F_n, \varepsilon). \]  
(3.16)

For each \( \delta > 0 \), we have
\[ Y^K = \{ x \in X \mid \lim_{n \to \infty} D(E_{F_n}(x), K) = 0 \} \subset \bigcup_{k=1}^{\infty} R(K, \delta, k). \]  
(3.17)

For each \( k \), let \( R(K, k) = \bigcap_{\delta > 0} R(K, \delta, k) \). Then \( Y^K \subset \bigcup_{k=1}^{\infty} R(K, k) \). By Proposition 3.16, we have
\[ h_{top}^P(Y^K, \{ F_n \}) \leq \sup_{k \geq 1} h_{top}^P(R(K, k)). \]  
(3.18)

By \( 3.16 \), we have
\[ h_{top}^{UC}(R(K, k), \{ F_n \}) \leq \Theta(Y^K, \{ F_n \}). \]  
(3.19)

By Proposition 2.6 and \( 3.18 - 3.19 \), \( h_{top}^P(Y^K, \{ F_n \}) \leq \Theta(Y^K, \{ F_n \}). \) By Proposition 2.3, we have
\[ h_{top}^P(Y^K, \{ F_n \}) \leq \Theta(Y^K, \{ F_n \}). \]  
(3.20)

We just need to show that
\[ \Theta(Y^K, \{ F_n \}) \leq \sup_{\mu \in K} h_{\mu}(X, G). \]

By \( 3.15 \), for any \( \gamma > 0 \), there exists \( \varepsilon_0 \) small such that for all \( 0 < \varepsilon < \varepsilon_0 \) the following holds
\[ \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{|F_n|} \log N(K, \delta, F_n, \varepsilon) \geq \Theta(G^K, \{ F_n \}) - \frac{\gamma}{3}. \]  
(3.21)

For each \( k \geq 1 \), set \( \varepsilon_k = \frac{\varepsilon_0}{2} \). From \( 3.21 \), there is \( \delta_k \) small with \( \delta_k \to 0 \) and
\[ \limsup_{n \to \infty} \frac{1}{|F_n|} \log N(K, \delta_k, F_n, \varepsilon_k) \geq \Theta(Y^K, \{ F_n \}) - \frac{2\gamma}{3}. \]  
(3.22)

Choose \( n_k \in \mathbb{N} \) such that
\[ N(K, \delta_k, F_{n_k}, \varepsilon_k) \geq e^{\lceil F_{n_k}|(\Theta(Y^K, \{ F_n \}) - \gamma) \rceil}. \]  
(3.23)

Let \( C_k \) be the centers of one covering of \( R(K, \delta_k, n_k) \) with balls \( B_{F_{n_k}}(x, \varepsilon_k) \) and \( \#C_k = N(K, \delta_k, F_{n_k}, \varepsilon_k) \). For each \( x \in M(X) \) be defined by \( \mu_n = \frac{1}{\#C_k} \sum_{x \in C_k} E_{F_{n_k}}(x) \). Let \( \mu \in M(X) \) be a limit point of the sequence \( \{ \mu_n \} \). Since \( \{ F_n \} \) is a Følner sequence, the measure \( \mu \) is in \( M(X, G) \). For each \( x \in C_k \), we can pick one point \( y(x) \in R(K, \delta_k, n_k) \cap \)
Then $D(\mathcal{E}_{F_{n_k}}(x), K) \leq D(\mathcal{E}_{F_{n_k}}(y(x)), K) + \delta_k \leq \varepsilon_k + \delta_k$. As a result we have $D(\mu, K) = 0$. Since $K$ is closed, $\mu \in K$.

By Lemma 2.17

$$h_\mu(X, G) \geq \limsup_{k \to \infty} \frac{1}{|F_{n_k}|} \log N(K, \delta_k, F_{n_k}, \varepsilon_k).$$ (3.24)

Combining (3.23) and (3.24), we have

$$h_\mu(X, G) \geq \Theta(Y^K, \{F_n\}) - \gamma.$$ By the arbitrariness of $\gamma$, we have

$$h_\mu(X, G) \geq \Theta(Y^K, \{F_n\}).$$ (3.25)

We get $h^{UC}_{top}(G_K, \{F_n\}) \leq \sup_{\mu \in K} h_\mu(X, G)$ by combining (3.20) and (3.25).

4. An application in multi-fractal analysis

We present one application of the above results.

Let $(X, G)$ be a dynamical system and $\{F_n\}$ be a Følner sequence. Let $\varphi : X \to \mathbb{R}$ be a continuous function. Denote by

$$L_\varphi = \left[ \inf_{\mu \in M(X, G)} \int \varphi d\mu, \sup_{\mu \in M(X, G)} \int \varphi d\mu \right].$$

For any $a \in L_\varphi$, define the level set with respect to $\{F_n\}$

$$R_\varphi(a) = \{x \in X \mid \lim_{n \to \infty} \frac{1}{|F_n|} \sum_{s \in F_n} \varphi(sx) = a\}.$$

For $\mu \in M(X, G)$, let $Y_\mu = \{x \in X \mid V(x, \{F_n\}) = \{\mu\}\}$ be the saturated set with respect to $\{\mu\}$.

**Proposition 4.1.** Let $(X, G)$ be a system satisfying the $g$-almost product property and $\{F_n\}$ be a Følner sequence Let $\varphi : X \to \mathbb{R}$ be a continuous function. Then for any $a \in L_\varphi$, we have

$$h^{UC}_{top}(R_\varphi(a), \{F_n\}) = h_{top}(X, G).$$

Moreover, if the system also has the uniform separation property and the sequence $\{F_n\}$ satisfies $\frac{|F_n|}{|\log n|} \to \infty$, we have

$$h^P_{top}(R_\varphi(a), \{F_n\}) = \sup_{\mu \in M(X, G)} \{h_\mu(X, G) \mid \mu \in M(X, G), \int \varphi d\mu = a\}.$$

**Proof.** Let $F(a) = \{\nu \in M(X, G) \mid \int \varphi d\nu = a\}$. It is clear that for any $\nu \in F(a)$, $Y_\nu \subset R_\varphi(a)$. So

$$h^{UC}_{top}(Y_\nu, \{F_n\}) \leq h^{UC}_{top}(R_\varphi(a), \{F_n\}), \forall \nu \in F(a).$$

20
Combining Theorem 1.1 we have $h_{top}^{UC}(R\varphi(a), \{F_n\}) = h_{top}(X, G)$.

Now we prove the moreover part. It is obvious that

$$R\varphi(a) = \{x \in X | V(x, \{F_n\}) \subset F(a)\}.$$ 

With the similar discussion in the proof of the upper bound of $h_{top}^{P}(Y_K, \{F_n\})$, we get that $h_{top}^{P}(R\varphi(a), \{F_n\}) \leq \sup\{h_\mu(X, G) | \mu \in F(a)\}$. For any $\mu \in F(a)$, one has $Y_\mu \subset R\varphi(a)$ which implies $h_{top}^{P}(Y_\mu, \{F_n\}) \leq h_{top}^{P}(R\varphi(a), \{F_n\})$. By Theorem 1.2 we have $h_{top}^{P}(Y_\mu, \{F_n\}) = h_\mu(X, G)$. Then we have $h_{top}^{P}(R\varphi(a), \{F_n\}) = \sup\{h_\mu(X, G) | \mu \in M(X, G), \int \varphi d\mu = a\}$.

5. Appendix

Before proceeding the proof, we give the definition of $f$-neighborhood.

**Definition 5.1.** An $f$-neighborhood of $\mu \in M(X)$ is the set of the form

$$F(\alpha) := \{\nu \in M(X) | |\langle f_i, \mu \rangle - \langle f_i, \nu \rangle| \leq \alpha \varepsilon_i\},$$

where $\alpha > 0, \varepsilon_i > 0, f_i \in C(X, \mathbb{R}), i = 1, \cdots, k$ and $\|f_i\| \leq 1$ for each $i$, where $\|f_i\| = \sup_{x \in X} |f_i(x)|$.

The $f$-neighborhoods form a neighborhood base for the weak* topology on $M(X)$, which is the topology we use.

Following ideas in the Appendix part of [19], we just need to show the proposition below.

**Proposition 5.2.** Let $(X, G)$ be a dynamical system and $\{K_n\}$ a tempered Følner sequence and $\mu \in M(X, G)$. Suppose the system has the $g$-almost product property and $\mu$ verifies the conclusions of Proposition 2.12. Let $0 < h' < h_\mu(X, G)$. Then there exists $\varepsilon' > 0$, such that for any neighborhood $C$ of $\mu$, there exists a $G$-invariant closed subset $Y \subset X$ satisfying the following properties.

1. There exists $n_C' \in \mathbb{N}$, such that $E_{K_n}(y) \in C$ for all $y \in Y$ and $n \geq n_C'$;
2. There exists $n_C'' \in \mathbb{N}$, such that there exists a subset $\Gamma_n$ of $Y$ which is $(K_n, \varepsilon')$-separated and $|\Gamma_n| \geq e^{|K_n|h'}$ for all $n \geq n_C''$.

In particular, $h_{top}(Y, G) \geq h'$.

**Proof.** Take $h' < h^* < h_\mu(X, G)$. Given the neighborhood $C$ of $\mu$, take an $f$-neighborhood $F^{(1)} \subset C$ of $\mu$ with fixed $\{f_j, \varepsilon_j : j = 1, 2, \cdots, p\}$. Denote $\varepsilon_{\text{min}} = \min\{\varepsilon_j | j = 1, 2, \cdots, p\}$. Let $\delta^*, \varepsilon^*$ and $n_{F^{(1/5)}}$ correspond to $h^*$ in the conclusion of Proposition 2.12. Set $n^* = n_{F^{(1/5)}}$.

Because $\{f_j \in C(X, \mathbb{R})\}$ are uniformly continuous on $X$, there exists $\triangle > 0$ such that

$$\triangle < \varepsilon^*/3 \text{ and } \rho(x, y) < \triangle \implies |f_j(x) - f_j(y)| < \varepsilon_j/5$$

for each $f_j$ associated with $F^{(1)}$. 

21
Let \( \{\gamma_n\} \) and \( \{\tau_n\} \) be two sequences of decreasing positive numbers with \( \gamma_n, \tau_n \to 0 \) and \( \tau_1 < \Delta \). Let \( g : (0, 1) \to (0, 1) \) be the mistake-density function as in the \( g \)-almost product property and \( m : (0, 1) \to F(G) \times (0, 1) \) be the map in Definition 2.19. By Lemma 3.1 and Lemma 3.3, there exist \( N_1 < n_{1,1} < n_{1,2} < N_2 < n_{2,1} < \cdots \) and a congruent tilings \( \{T_k\} \) with shape sets \( \{S_k\} \) such that the following hold: Any \( S \in S_k \) can be \( \gamma_k \)-quasi tiled by \( \{K_{n_1,1}, \ldots, K_{n_k,1}\} \) with tiling centers \( \{C_k,s,1, \ldots, C_k,s,t_k\} \). Denote the \( \gamma_k \)-quasi tile by

\[
T_{k,S} = \{ K_{k,j}c_{k,S,j} \mid c_{k,S,j} \in C_{k,S,j}, 1 \leq j \leq t_k \}.
\] (5.1)

Here we also assume \( N_1 \geq n^* \).

From Remark 3.1, we can modify \( T_{k,S} \) to get a new \( \gamma_k \)-quasi tile

\[
\tilde{T}_{k,S} = \{ \tilde{K}_{k,S,j}c_{k,S,j} \mid K_{n_k,j}c_{k,S,j} \in T_{k,S} \}
\] (5.2)
such that the following hold:

1. Elements in \( \tilde{T}_{k,S} \) are pairwise disjoint and \( \frac{|\tilde{K}_{k,S,j}|}{|K_{k,S,j}|} > 1 - \gamma_k \);
2. \( \mathcal{U} \tilde{T}_{k,S} \subset S \) and \( |\mathcal{T}_{k,S}| > (1 - \gamma_k)^2|S| \).

Note that the sequence obtained from \( \tilde{T}_{k,S} \) is still a Følner sequence. By taking subsequence, we can assume for each \( n \in \mathbb{N} \), any

\[
A \in \bigcup_{k \geq n} \{ \tilde{K}_{k,S,j} \mid c_{k,S,j} \in C_{k,S,j}, 1 \leq j \leq t_k, S \in S_k \}
\]

is \( m(\tau_n) \)-variant.

From the conclusion of Proposition 2.12, we get the existence of \( (\delta^*, K_{c_k,S,j}, \varepsilon^*) \)-separated set \( \Gamma_{c_k,S,j} \subset X_{K_{c_k,S,j}F^{(1/5)}} \) with \( |\Gamma_{c_k,S,j}| \geq e^{|K_{c_k,S,j}|h^*} \) for \( K_{c_k,S,j} \in T_{k,S} \). By Lemma 2.18, the set \( \Gamma_{c_k,S,j} \) is also \( (\delta^*/2, K_{c_k,S,j}, \varepsilon^*) \)-separated and contained in \( X_{K_{c_k,S,j}F^{(2/5)}} \).

We mention that for \( x_{c_k,S,j} \in \Gamma_{c_k,S,j} \subset X_{K_{c_k,S,j}F^{(2/5)}} \),

\[
| < f_j, \mathcal{E}_{K_{c_k,S,j}}(x_{c_k,S,j}) > - < f_j, \mu > | < \frac{2}{5} \varepsilon_j.
\] (5.3)

For \( S \in \mathbb{T}_k \), let

\[
\Gamma(Sd) = \Gamma(S) := \prod_{K_{c_k,S,j}c_{k,S,j} \in \tilde{T}_{k,S}} \Gamma_{c_k,S,j} := \{ \tilde{x} = (x_{c_k,S,j}) \mid x_{c_k,S,j} \in \Gamma_{c_k,S,j} \}.
\]

Consider \( Z_{F^{(1)}_k}^\# \) defined by the requirement that \( x \in Z_{F^{(1)}_k}^\# \) if and only if for all \( Sd \in \mathbb{T}_k \) there exists \( \tilde{x} \in \Gamma(Sd) \) such that

\[
\rho_{K_{c_k,S,j}}(c_{k,S,j}dx, x_{c_k,S,j}) \leq \tau_k.
\] (5.4)

Since the system has the \( g \)-almost product property, the set \( Z_{F^{(1)}_k}^\# \) is not empty.
Let $\beta > 0$ with $\beta < \frac{\varepsilon_{min}}{20}$ and $\beta < \frac{h^* - h'}{3h^*}$. Choose $m_k$ large such that for $m \geq m_k$, $K_m$ is $(\cup S_k, \frac{3}{\varepsilon})$-invariant.

Let $m \geq m_k$. Define

$$Y_{m,k} := \{x \in X \mid s x \in X_{K_m,F(4/5)}, \forall s \in G\}.$$

By the definition, $Y_{m,k}$ is a closed $G$-invariant subset. Next we will show that $Z_{F(1),k}^\# \subset Y_{m,k}$.

Take $s \in G$ and let $\Lambda_{K_{m,s}} = \{T \in \mathcal{T}_k \mid T \subset K_{m,s}\}$ and $\overline{K_{m,s}} = \bigcup \Lambda_{K_{m,s}}$. Let

$$I_n = \{s \in F_n : \exists T \in \mathcal{T}_k \text{ such that } s \in T, T \cap (G \setminus F_n) \neq \emptyset\} \subset \bigcup \{Sc : \exists S \in \mathcal{S}_k, c \in G \text{ such that } Sc \cap F_n \neq \emptyset, Sc \cap (G \setminus F_n) \neq \emptyset\} \subset \bigcup \{(\cup S_k) c : c \in \partial \cup \mathcal{S}_k F_n\}.$$ 

Hence $|I_n| \leq |\cup \mathcal{S}_k| |\partial \cup \mathcal{S}_k F_n| \leq \beta |F_n|$. Then we have

$$|\Lambda_{K_{m,s}}| \geq (1 - \beta)|F_n|. \quad (5.5)$$

For $x \in Z_{F(1),k}^\#$ and $\overline{K_{c_k,s,j}} \subset \overline{F_{(1)}}$, by (5.4) and the choice of metric (2.3), we have

$$D(E_{K_{c_k,s,j}}(x_{c_k,s,j}), E_{\overline{K_{c_k,s,j}}}(x_{c_k,s,j})) \leq \tau_k. \quad (5.6)$$

For $sd \in \Lambda_{K_{m,s}}$, let $\tilde{S} = \bigcup \mathcal{T}_k$. From (5.3) and (5.6)

$$\left| \frac{1}{|sd|} \sum_{s \in sd} f_j(sx) - \int f_j d\mu \right| \leq \frac{1}{|sd|} \left| \sum_{s \in sd} f_j(sx) - \frac{1}{|sd|} \sum_{s \in sd} f_j(sx) \right|$$

$$+ \frac{1}{|sd|} \left| \sum_{s \in sd} f_j(sx) - \int f_j d\mu \right|$$

$$\leq \|f_j\| \cdot 2\gamma_k + \frac{2}{5}\varepsilon_j + \|f_j\| \tau_k \leq \frac{3}{5}\varepsilon_j. \quad (5.7)$$

By (5.5) and (5.7),

$$\left| \frac{1}{|K_{m,s}|} \sum_{t \in K_{m,s}} f_j(tx) - \int f_j d\mu \right| \leq \frac{1}{|K_{m,s}|} \left| \sum_{t \in K_{m,s}} f_j(tx) - \frac{1}{|K_{m,s}|} \sum_{t \in K_{m,s}} f_j(tx) \right|$$

$$+ \frac{1}{|K_{m,s}|} \left| \sum_{t \in K_{m,s}} f_j(tx) - \int f_j d\mu \right|$$

$$\leq 2\beta \|f_j\| + \frac{3}{5}\varepsilon_j \leq \frac{4}{5}\varepsilon_j. \quad (5.8)$$

From (5.8), we get $Z_{F(1),k}^\# \subset Y_{m,k}$. 

23
Define

\[ Y := \bigcap_{m \geq m_k} Y_{m,k}. \]

Then \( Y \) is a non-empty closed \( G \)-invariant subset of \( X \). Set \( n'_C = m_k \). For \( m \geq n'_C \), we have \( Y \subset Y_{m,k} \), which implies that for \( y \in Y \), the measure \( \mathcal{E}_{K_m}(y) \in F(4/5) \subset C \).

Then statement (1) is true.

Now we prove the statement (2) of this proposition. We set \( n''_C = m_k \) and \( \varepsilon' = \frac{\varepsilon^*}{3} \).

Let \( \Lambda_{K_n} = \{ T \in T_k \mid T \subset K_n \} \) and \( \tilde{K}_n = \cup \Lambda_{K_n} \). From Lemma 5.5,

\[ |\tilde{K}_n| > (1 - 2\beta)|K_n|. \] (5.9)

For \( S \in S_k \), denote \( \Gamma(S) = \prod_{t_k} \prod_{c_{k,S,i} \in C_{k,S,i}} \Gamma_{c_{k,S,i}} \). Denote \( \Gamma(K_n) = \prod_{S \in \Lambda_n} \Gamma(S) \), where \( \Lambda_n \) is as described in Lemma 3.3.

For each \( n \geq n''_C \), we will consider a subset \( Z_n^\# \subset Z_{F(1)}^\# \) with the following property: for each \( \bar{x} = \{ x_{S,d,c_{k,S,i}} \} \in \Gamma(K_n) \), there exists exact one point \( x \in Z_n^\# \) such that

\[ \rho_{K_{c_{k,S,j}}}(c_{k,S,j} dx, x_{S,d,c_{k,S,j}}) \leq \tau_k. \] (5.10)

Define a map \( \Phi \) from \( \Gamma(K_n) \) to \( Z_n^\# \) such that \( \Phi(\bar{x}) \) satisfies (5.10). For \( \bar{x} \neq \bar{y} \in \Gamma(K_n) \), we have

\[ \rho_{K_n}(\Phi(\bar{x}), \Phi(\bar{y})) \geq \varepsilon^* - 2\tau_k > \frac{\varepsilon^*}{3} = \varepsilon'. \]

Then \( Z_n^\# \) is \((K_n, \varepsilon')\)-separated. By the definition of \( \Gamma(K_n) \) and \( |\Gamma_{c_{k,S,i}}| \geq e^{h^*|K_{c_{k,S,i}}|} \)

\[ |\Gamma(K_n)| = \prod_{S \in \Lambda_n} \prod_{t_k} \prod_{c_{k,S,i} \in C_{k,S,i}} |\Gamma_{c_{k,S,i}}| \]

\[ \geq e^{h^* \sum_{S \in \Lambda_n} \sum_{t_k} \sum_{c_{k,S,i} \in C_{k,S,i}} |K_{c_{k,S,i}}|} \]

\[ \geq e^{h^*(1-2\beta)(1-4\gamma_k)|K_n|} \geq e^{h'|K_n|}. \] (5.11)

Thus statement (2) is true which implies \( h_{top}(Y, G) \geq h' \).

\[ \square \]

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Upper capacity entropy and packing entropy of saturated sets

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