Research article

Existence of nonradial positive and nodal solutions to a critical Neumann problem in a cone

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Abstract: We study the critical Neumann problem

\[
\begin{align*}
-\Delta u &= |u|^{2^*-2}u \quad \text{in } \Sigma_\omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Sigma_\omega,
\end{align*}
\]

in the unbounded cone \( \Sigma_\omega := \{tx : x \in \omega \text{ and } t > 0\} \), where \( \omega \) is an open connected subset of the unit sphere \( S^{N-1} \) in \( \mathbb{R}^N \) with smooth boundary, \( N \geq 3 \) and \( 2^* := \frac{2N}{N-2} \). We assume that some local convexity condition at the boundary of the cone is satisfied. If \( \omega \) is symmetric with respect to the north pole of \( S^{N-1} \), we establish the existence of a nonradial sign-changing solution. On the other hand, if the volume of the unitary bounded cone \( \Sigma_\omega \cap B_1(0) \) is large enough (but possibly smaller than half the volume of the unit ball \( B_1(0) \) in \( \mathbb{R}^N \)), we establish the existence of a positive nonradial solution.

Keywords: semilinear elliptic equation; critical nonlinearity; conical domain; Neumann boundary condition; nonradial solution
1. Introduction

We consider the Neumann problem
\[
\begin{align*}
-\Delta u &= |u|^{2^*-2}u & \text{in } \Sigma_{\omega}, \\
\frac{\partial u}{\partial \nu} &= 0 & \text{on } \partial \Sigma_{\omega},
\end{align*}
\] (1.1)
in the unbounded cone \( \Sigma_{\omega} := \{tx : x \in \omega \text{ and } t > 0\} \), where \( \omega \) is an open connected subset of the unit sphere \( S^{N-1} \) in \( \mathbb{R}^N \) with smooth boundary, \( N \geq 3 \), and \( 2^* := \frac{2N}{N-2} \) is the critical Sobolev exponent.

It is well known that, if \( \omega = S^{N-1} \), i.e., if \( \Sigma_{\omega} \) is the whole space \( \mathbb{R}^N \), then the only positive solutions to the critical problem
\[
-\Delta w = |w|^{2^*-2}w, \quad w \in D^{1,2}(\mathbb{R}^N),
\] (1.2)
are the rescalings and translations of the standard bubble \( U \) defined in (2.3). Moreover, they are the only nontrivial radial solutions to (1.2), up to sign. It is immediately deduced that, up to sign, the restriction of the bubbles (3.1) to \( \Sigma_{\omega} \) are the only nontrivial radial solutions of (1.1) in any cone; see Proposition 3.4. In addition, if the cone \( \Sigma_{\omega} \) is convex, it was shown in [8, Theorem 2.4] that these are the only positive solutions to (1.1). The convexity property of the cone is crucial in the proof of this result, and it is strongly related to a relative isoperimetric inequality obtained in [7].

The aim of this paper is to establish the existence of nonradial solutions to (1.1), both positive and sign-changing. As mentioned above, the positive ones can only exist in nonconvex cones. On the other hand, nodal radial solutions to (1.1) do not exist, as this would imply the existence of a nontrivial solution to problem (2.5) in the bounded cone \( \Lambda_{\omega} := \{tx : x \in \omega \text{ and } t \in (0,1)\} \), which is impossible because of the Pohozaev identity (2.6) and the unique continuation principle.

For the problem (1.2) in \( \mathbb{R}^N \) various types of sign-changing solutions are known to exist; see [2–5]. In particular, a family of entire nodal solutions, which are invariant under certain groups of linear isometries of \( \mathbb{R}^N \), were exhibited in [2]. These solutions arise as blow-up profiles of symmetric minimizing sequences for the critical equation in a ball, and are obtained through a fine analysis of the concentration behavior of such sequences.

Here we use some ideas from [2] to produce sign-changing solutions to (1.1), but we exploit a different kind of symmetry. Our main result shows that, if \( \omega \) is symmetric with respect to the north pole of \( S^{N-1} \) and if the cone \( \Sigma_{\omega} \) has a point of convexity in the sense of Definition 2.6, then the problem (1.1) has an axially antisymmetric least energy solution, which is nonradial and changes sign; see Theorem 2.8. As far as we know, this is the first existence result of a nodal solution to (1.1).

Next, we investigate the existence of positive nonradial solutions. In this case we do not require the cone to have any particular symmetry. We establish the existence of a positive nonradial solution to (1.1) under some conditions involving the local convexity of \( \Sigma_{\omega} \) at a boundary point and the measure of the bounded cone \( \Lambda_{\omega} \); see Corollary 3.5 and Theorem 3.6. We refer to Section 3 for the precise statements and further remarks.

2. A nonradial sign-changing solution

If \( \Omega \) is a domain in \( \mathbb{R}^N \) we consider the Sobolev space
\[
D^{1,2}(\Omega) := \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega, \mathbb{R}^N)\}
\]
with the norm
\[ \|u\|_{\Omega}^2 := \int_{\Omega} |\nabla u|^2. \]

We denote by \( J_\Omega : D^{1,2}(\Omega) \to \mathbb{R} \) the functional given by
\[ J_\Omega(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{2^*} \int_{\Omega} |u|^{2^*}, \]
and its Nehari manifold by
\[ N(\Omega) := \left\{ u \in D^{1,2}(\Omega) : u \neq 0 \text{ and } \int_{\Omega} |\nabla u|^2 = \int_{\Omega} |u|^{2^*} \right\}. \]

For \( u \in D^{1,2}(\Omega) \setminus \{0\} \) let \( t_u \in (0, \infty) \) be such that \( t_u u \in N(\Omega) \). Then,
\[ J_\Omega(t_u u) = \frac{1}{N} [Q_\Omega(u)]^\frac{2}{N}, \quad \text{where} \quad Q_\Omega(u) := \left( \frac{\int_{\Omega} |\nabla u|^2}{(\int_{\Omega} |u|^{2^*})^{2/N}} \right)^N. \] (2.1)

Hence,
\[ c_\Omega := \inf_{u \in N(\Omega)} J_\Omega(u) = \inf_{u \in D^{1,2}(\Omega) \setminus \{0\}} \frac{1}{N} [Q_\Omega(u)]^\frac{2}{N}. \] (2.2)

We set \( c_\infty := c_{\mathbb{R}^N} \). It is well known that this infimum is attained at the function
\[ U(x) = a_N \left( \frac{1}{1 + |x|^2} \right)^{\frac{N^2}{2}}, \quad a_N := (N(N - 2))^{\frac{N}{N^2}}, \] (2.3)
which is called the standard bubble, and at every rescaling and translation of it, and that
\[ c_\infty = J_{\mathbb{R}^N}(U) = \frac{1}{N} S^\frac{N}{2}, \]
where \( S \) is the best constant for the Sobolev embedding \( D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \).

Let \( \mathbb{S}^{N-1} \) be the unit sphere in \( \mathbb{R}^N \) and let \( \omega \) be a smooth domain in \( \mathbb{S}^{N-1} \) with nonempty boundary, i.e., \( \omega \) is connected and open in \( \mathbb{S}^{N-1} \) and its boundary \( \partial \omega \) is a smooth \( (N - 2) \)-dimensional submanifold of \( \mathbb{S}^{N-1} \). The nontrivial solutions to the Neumann problem (1.1) in the unbounded cone
\[ \Sigma_\omega := \{ tx : x \in \omega \text{ and } t > 0 \} \]
are the critical points of \( J_{\Sigma_\omega} \) on \( N(\Sigma_\omega) \).

To produce a nonradial sign-changing solution for (1.1) we introduce some symmetries. We write a point in \( \mathbb{R}^N \) as \( x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \), and consider the reflection \( \varrho(x', x_N) := (-x', x_N) \). Then, a subset \( X \) of \( \mathbb{R}^N \) will be called \( \varrho \)-invariant if \( \varrho x \in X \) for every \( x \in X \), and a function \( u : X \to \mathbb{R} \) will be called \( \varrho \)-equivariant if
\[ u(\varrho x) = -u(x) \quad \forall x \in X. \]

Note that every nontrivial \( \varrho \)-equivariant function is nonradial and changes sign.
Throughout this section we will assume that \( \omega \) is \( \varphi \)-invariant. Note that \((0, \pm 1) \notin \partial \omega \) because \( \partial \omega \) is smooth. Hence, \( \varphi x \neq x \) for every \( x \in \partial \Sigma_\omega \setminus \{0\} \). Our aim is to show that \((1.1)\) has a \( \varphi \)-equivariant solution. We set
\[
D^{1,2}_\varphi(\Sigma_\omega) := \{u \in D^{1,2}(\Sigma_\omega) : u \text{ is } \varphi \text{-equivariant}\},
\]
\[
N^0(\Sigma_\omega) := \{u \in N(\Sigma_\omega) : u \text{ is } \varphi \text{-equivariant}\}
\]
and
\[
c^\varphi_{\Sigma_\omega} := \inf_{u \in N^0(\Sigma_\omega)} J_{\Sigma_\omega}(u) = \inf_{u \in D^{1,2}_\varphi(\Sigma_\omega) \setminus \{0\}} \frac{1}{N}\{Q_{\Sigma_\omega}(u)\}^{\frac{2}{N}}.
\tag{2.4}
\]

Define
\[
\Lambda_\omega := \{tx : x \in \omega \text{ and } 0 < t < 1\}
\]
and set \( \Gamma_1 := \partial \Lambda_\omega \setminus \overline{\omega} \). In \( \Lambda_\omega \) we consider the mixed boundary value problem
\[
\begin{cases}
-\Delta u = |u|^{2^* - 2} u & \text{in } \Lambda_\omega, \\
u = 0 & \text{on } \omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1.
\end{cases}
\tag{2.5}
\]

We point out that \((2.5)\) does not have a nontrivial solution. Indeed, by the well known Pohozaev identity, a solution to \((2.5)\) must satisfy
\[
0 = \int_{\Gamma_1 \cup \omega} \left( \nabla u \cdot s \frac{\partial u}{\partial \nu} - \frac{|\nabla u|^2}{2} s \cdot v + F(u)s \cdot v \right) ds
= \int_\omega \left( \nabla u \cdot s \frac{\partial u}{\partial \nu} - \frac{|\nabla u|^2}{2} s \cdot v \right) ds = \frac{1}{2} \int_\omega |\nabla u|^2 ds
\tag{2.6}
\]
because \( s \cdot v = 0 \) for every \( s \in \Gamma_1 \) and \( s \cdot v = 1 \) for every \( s \in \omega \). Therefore \( \frac{\partial u}{\partial \nu} \) vanishes on \( \omega \). But then the trivial extension of \( u \) to the infinite cone \( \Sigma_\omega \) solves \((1.1)\), contradicting the unique continuation principle.

Let \( V(\Lambda_\omega) \) be the space of functions in \( D^{1,2}(\Lambda_\omega) \) whose trace vanishes on \( \omega \). Note that \( V(\Lambda_\omega) \subset D^{1,2}(\Sigma_\omega) \) via trivial extension. Let \( J_{\Lambda_\omega} : V(\Lambda_\omega) \to \mathbb{R} \) be the restriction of \( J_{\Sigma_\omega} \) to \( V(\Lambda_\omega) \) and set
\[
N^0(\Lambda_\omega) := N^0(\Sigma_\omega) \cap V(\Lambda_\omega) \quad \text{and} \quad c^\varphi_{\Lambda_\omega} := \inf_{u \in N^0(\Lambda_\omega)} J_{\Lambda_\omega}(u).
\]

To produce a sign-changing solution for the problem \((1.1)\) we will study the concentration behavior of \( \varphi \)-equivariant minimizing sequences for \((2.5)\). We start with the following lemmas.

**Lemma 2.1.** \( 0 < c^\varphi_{\Lambda_\omega} = c^\varphi_{\Sigma_\omega} \leq c_\infty \).

**Proof.** It is shown in [8, Theorem 2.1] that \( c^\varphi_{\Lambda_\omega} > 0 \).

Since \( N^0(\Lambda_\omega) \subset N^0(\Sigma_\omega) \), we have that \( c^\varphi_{\Lambda_\omega} \geq c^\varphi_{\Sigma_\omega} \). To prove the opposite inequality, let \( \varphi_k \in N^0(\Sigma_\omega) \cap C^\omega(\Sigma_\omega) \) be such that \( \varphi_k \) has compact support and \( J(\varphi_k) \to c^\varphi_{\Sigma_\omega} \) as \( k \to \infty \). Then, we may choose \( \varepsilon_k > 0 \) such that the support of \( \varphi_k(x) := \varepsilon_k^{-(N-2)/2} \varphi_k(\varepsilon_k^{-1} x) \) is contained in \( \Lambda_\omega \setminus \overline{\omega} \). Thus, \( \varphi_k \in N^0(\Lambda_\omega) \) and, hence,
\[
c^\varphi_{\Lambda_\omega} \leq J(\varphi_k) = J(\varphi_k) \quad \text{for all } k.
\]
Then, after passing to a subsequence, one of the following statements holds true:

Letting \( k \to \infty \) we conclude that \( c_{\Lambda_{\omega}}^k \leq c_{\Sigma_{\omega}}^k \).

To prove that \( c_{\Sigma_{\omega}}^k \leq c_{\omega}^k \) we fix a point \( \xi \in \partial \Sigma_{\omega} \setminus \{0\} \) and a sequence of positive numbers \( \varepsilon_k \to 0 \), and we set \( \Sigma_k := \varepsilon_k^{-1}(\Sigma_{\omega} - \xi) \). Since \( \partial \Sigma_{\omega} \setminus \{0\} \) is smooth, the limit of the sequence of sets \( (\Sigma_k) \) is the half-space

\[
\mathbb{H}_\nu := \{ z \in \mathbb{R}^N : z \cdot \nu < 0 \},
\]

where \( \nu \) is the exterior unit normal to \( \Sigma_{\omega} \) at \( \xi \). Let \( u_k(x) := \varepsilon_k^{(2-N)/2} U(\frac{x - \xi}{\varepsilon_k}) \), where \( U \) is the standard bubble. Then,

\[
\lim_{k \to \infty} \int_{\Sigma_{\omega}} |\nabla u_k|^2 = \lim_{k \to \infty} \int_{\Sigma_k} |\nabla U|^2 = \int_{\mathbb{H}_\nu} |\nabla U|^2 = \frac{1}{2N} \mathcal{S}^\mathbb{H},
\]

\[
\lim_{k \to \infty} \int_{\Sigma_{\omega}} |u_k|^2 = \lim_{k \to \infty} \int_{\Sigma_k} |U|^2 = \int_{\mathbb{H}_\nu} |U|^2 = \frac{1}{2N} \mathcal{S}^\mathbb{H}.
\]

The function

\[
\hat{u}_k(x) = u_k(x) - u_k(qx) = \varepsilon_k^{\frac{3-N}{2}} U \left( \frac{x - \xi}{\varepsilon_k} \right) - \varepsilon_k^{\frac{3-N}{2}} U \left( \frac{x - \xi}{\varepsilon_k} \right)
\]

is \( \varphi \)-equivariant, and from (2.4), (2.8) and (2.9) we obtain

\[
c_{\Sigma_{\omega}}^k \leq \lim_{k \to \infty} \frac{1}{N} [Q_{\Sigma_{\omega}}(\hat{u}_k)]^\mathbb{H} = \frac{1}{N} \mathcal{S}^\mathbb{H} = c_{\omega}.
\]

This concludes the proof. \( \square \)

**Lemma 2.2.** Given a domain \( \Omega \) in \( \mathbb{R}^N \) and \( \varepsilon > 0 \), we set \( \Omega_\varepsilon := \{ \varepsilon^{-1} x : x \in \Omega \} \). If \( \partial \Omega \) is Lipschitz continuous, then there exist linear extension operators \( P_\varepsilon : W^{1,2}(\Omega_\varepsilon) \to D^{1,2}(\mathbb{R}^N) \) and a positive constant \( C \), independent of \( \varepsilon \), such that

(i) \( (P_\varepsilon u)(x) = u(x) \) for every \( x \in \Omega_\varepsilon \).

(ii) \( \int_{\mathbb{R}^N} |\nabla (P_\varepsilon u)|^2 \leq C \int_{\Omega} |\nabla u|^2 \).

(iii) \( \int_{\mathbb{R}^N} |P_\varepsilon u|^2 \leq C \int_{\Omega_\varepsilon} |u|^2 \).

(iv) If \( \Omega \) is \( \varphi \)-invariant, then \( P_\varepsilon u \) is \( \varphi \)-equivariant if \( u \) is \( \varphi \)-equivariant.

**Proof.** The existence of an extension operator \( P_\varepsilon : W^{1,2}(\Omega_\varepsilon) \to D^{1,2}(\mathbb{R}^N) \) satisfying (i) – (iii) is well known, and the fact that the constant \( C \) does not depend on \( \varepsilon \) was proved in [6, Lemma 2.1]. To obtain (iv) we replace \( P_\varepsilon u \) by the function \( x \mapsto \frac{1}{2}[(P_\varepsilon u)(x) - (P_\varepsilon u)(qx)] \). \( \square \)

The following proposition describes the behavior of minimizing sequences for \( J_{\Lambda_{\omega}} \) on \( N^0(\Lambda_{\omega}) \).

**Proposition 2.3.** Let \( u_k \in N^0(\Lambda_{\omega}) \) be such that

\[
J_{\Lambda_{\omega}}(u_k) \to c_{\Lambda_{\omega}}^k \quad \text{and} \quad J'_{\Lambda_{\omega}}(u_k) \to 0 \quad \text{in} \quad (V(\Lambda_{\omega}))'.
\]

Then, after passing to a subsequence, one of the following statements holds true:

(i) There exist a sequence of positive numbers \( (\varepsilon_k) \), a sequence of points \( (\xi_k) \) in \( \Gamma_1 \) and a function \( w \in D^{1,2}(\mathbb{R}^N) \) such that \( \varepsilon_k^{-1} \text{dist}(\xi_k, \partial \Lambda_{\omega} \cup \{0\}) \to \infty \), \( w\big|_{\mathbb{H}} \) solves the Neumann problem

\[
-\Delta w = |w|^{2^*-2} w, \quad w \in D^{1,2}(\mathbb{H}), \quad \text{and} \quad \varepsilon_k^{\frac{3-N}{2}} U \left( \frac{x - \xi_k}{\varepsilon_k} \right) \to w \quad \text{in} \quad V(\mathbb{R}^N).
\]

\[
\hat{u}_k(x) = u_k(x) - u_k(qx) = \varepsilon_k^{\frac{3-N}{2}} U \left( \frac{x - \xi_k}{\varepsilon_k} \right) - \varepsilon_k^{\frac{3-N}{2}} U \left( \frac{x - \xi_k}{\varepsilon_k} \right).
\]
Proof. Since the sequence \( \{u_k\} \) is bounded and, after passing to a subsequence, \( u_k \rightharpoonup u \) weakly in \( V(\Lambda_\omega) \). Then, \( J'_{\Lambda_\omega}(u) = 0 \). Since the problem (2.5) does not have a nontrivial solution, we conclude that \( u = 0 \).

Fix \( \delta \in (0, \frac{N}{2}\epsilon^2) \). As

\[
\int_{\Lambda_\omega} |u_k|^2 \, d\omega = N\left(J_{\Lambda_\omega}(u_k) - \frac{1}{2}J'_{\Lambda_\omega}(u_k)u_k\right) \leq C + o(1)||u_k||_{\Lambda_\omega},
\]

the sequence \( \{u_k\} \) is bounded and, after passing to a subsequence, \( u_k \rightarrow u \) weakly in \( V(\Lambda_\omega) \). Then, \( J'_{\Lambda_\omega}(u) = 0 \). Since the problem (2.5) does not have a nontrivial solution, we conclude that \( u = 0 \).

there are bounded sequences \( \{\epsilon_k\} \) in \((0, \infty)\) and \( \{x_k\} \) in \( \mathbb{R}^N \) such that, after passing to a subsequence,

\[
\delta = \sup_{x \in \mathbb{R}^N} \int_{\Lambda_\omega \cap B_{\epsilon_k}(x)} |u_k|^2 \, d\omega = \int_{\Lambda_\omega \cap B_{\epsilon_k}(x_k)} |u_k|^2 \, d\omega,
\]

where \( B_r(x) := \{y \in \mathbb{R}^N : |y - x| < r\} \). Note that, as \( \delta > 0 \), we have that \( \text{dist}(x_k, \Lambda_\omega) < \epsilon_k \). We claim that, after passing to a subsequence, there exist \( \xi_k \in \Lambda_\omega \) and \( C_0 > 0 \) such that

\[
\epsilon_k^{-1}|x_k - \xi_k| < C_0 \quad \forall k \in \mathbb{N},
\]

and one of the following statements holds true:

(a) \( \xi_k = 0 \) for all \( k \in \mathbb{N} \).
(b) \( \xi_k \in \partial \omega = \overline{\omega} \cap \Gamma_1 \) for all \( k \in \mathbb{N} \).
(c) \( \xi_k \in \Gamma_1 \) for all \( k \in \mathbb{N} \) and \( \epsilon_k^{-1}\text{dist}(\xi_k, \partial \omega) \rightarrow 0 \).
(d) \( \xi_k \in \omega \) for all \( k \in \mathbb{N} \) and \( \epsilon_k^{-1}\text{dist}(\xi_k, \Gamma_1) \rightarrow \infty \).
(e) \( \xi_k \in \Lambda_\omega \) for all \( k \in \mathbb{N} \), \( \epsilon_k^{-1}\text{dist}(\xi_k, \partial \Lambda_\omega) \rightarrow \infty \) and, either \( \epsilon_k^{-1}|\xi_k - \varrho \xi_k| \rightarrow \infty \), or \( \xi_k = \varrho \xi_k \) for all \( k \in \mathbb{N} \).

This can be seen as follows: If the sequence \( \{\epsilon_k^{-1}|x_k|\} \) is bounded, we set \( \xi_k := 0 \). Then, (2.12) and (a) hold true. If \( (\epsilon_k^{-1}\text{dist}(x_k, \partial \omega)) \) is bounded, we take \( \xi_k \in \partial \omega \) such that \( |x_k - \xi_k| = \text{dist}(x_k, \partial \omega) \). Then, (2.12) and (b) hold true. If both \( (\epsilon_k^{-1}|x_k|) \) and \( (\epsilon_k^{-1}\text{dist}(x_k, \partial \omega)) \) are unbounded and \( (\epsilon_k^{-1}\text{dist}(x_k, \Gamma_1)) \) is bounded, we take \( \xi_k \in \Gamma_1 \) with \( |x_k - \xi_k| = \text{dist}(x_k, \Gamma_1) \). Then, (2.12) and (c) hold true. If \( (\epsilon_k^{-1}\text{dist}(x_k, \Gamma_1)) \)
is unbounded and \((e_k^{-1}\text{dist}(x_k, \omega))\) is bounded, we take \(\xi_k \in \omega\) with \(|x_k - \xi_k| = \text{dist}(x_k, \omega)\). Then, (2.12) and (d) hold true. Finally, if \((e_k^{-1}\text{dist}(x_k, \partial \Lambda_\omega))\) is unbounded, we set \(\xi_k := \frac{x_k + \omega}{2}\) if \((e_k^{-1}|x_k - \omega x_k|)\) is bounded and \(\xi_k := x_k\) if \((e_k^{-1}|x_k - \omega x_k|)\) is unbounded. Then, (2.12) and (e) hold true.

Set \(C_1 := C_0 + 1\). Inequality (2.12) yields

\[
\delta = \int_{\Lambda_\omega \cap B_{\psi}(x_k)} |u_k|^2 \leq \int_{\Lambda_\omega \cap B_{C_1}(\xi_k)} |u_k|^2.
\]  

(2.13)

We consider \(u_k\) as a function in \(D^{1,2}(\Sigma_\omega)\) via trivial extension, and we define \(\tilde{u}_k \in D^{1,2}(\Sigma_\omega)\) as \(\tilde{u}_k(z) := e_k^{(N-2)/2} u_k(e_k z)\). Since \(\tilde{u}_k\) is \(\varrho\)-equivariant, so is its extension \(P_x \tilde{u}_k \in D^{1,2}(\mathbb{R}^N)\) given by Lemma 2.2. Let

\[
w_k(z) := (P_x \tilde{u}_k)(z + e_k^{-1} \xi_k) \in D^{1,2}(\mathbb{R}^N).
\]

Then,

\[
w_k(z) = e_k^{1/2} u_k(e_k z + \xi_k) \quad \text{if } z \in \Lambda_k := e_k^{-1}(\Lambda_\omega - \xi_k),
\]

(2.14)

\[
w_k(z - e_k^{-1} \xi_k) = -w_k(e_k z - e_k^{-1} \xi_k) \quad \text{for every } z \in \mathbb{R}^N,
\]

(2.15)

\[
\delta = \sup_{z \in \mathbb{R}^N} \int_{\Lambda_k \cap B_{\psi}(z)} |w_k|^2 \leq \int_{\Lambda_k \cap B_{C_1}(0)} |w_k|^2,
\]

(2.16)

and \((w_k)\) is bounded in \(D^{1,2}(\mathbb{R}^N)\). Hence, a subsequence satisfies that \(w_k \rightarrow w\) weakly in \(D^{1,2}(\mathbb{R}^N)\), \(w_k \rightarrow w\) a.e. in \(\mathbb{R}^N\) and \(w_k \rightarrow w\) strongly in \(L^2(\mathbb{R}^N)\). Choosing \(\delta\) sufficiently small and using (2.16), a standard argument shows that \(w \neq 0\); see, e.g., [10, Section 8.3]. Moreover, we have that \(\xi_k \rightarrow \xi\) and \(e_k \rightarrow 0\), because \(u_k \rightarrow 0\) weakly in \(V(\Lambda_\omega)\) and \(w \neq 0\).

Let \(\mathcal{E}\) be the limit of the domains \(\Lambda_k\). Since \((w_k)\) is bounded in \(D^{1,2}(\mathbb{R}^N)\), using Hölder’s inequality we obtain

\[
\left| \int_{\mathcal{E} \setminus \Lambda_k} \nabla w_k \cdot \nabla \varphi \right| \leq C \left( \int_{\mathcal{E} \setminus \Lambda_k} |\nabla \varphi|^2 \right)^{1/2} = o(1),
\]

\[
\left| \int_{\mathcal{E} \setminus \Lambda_k} |w_k|^{2-2} w_k \varphi \right| \leq C \left( \int_{\mathcal{E} \setminus \Lambda_k} |\varphi|^2 \right)^{1/2} = o(1),
\]

for every \(\varphi \in C_c^\infty(\mathbb{R}^N)\), and similarly for the integrals over \(\Lambda_k \setminus \mathcal{E}\). Therefore, as \(w_k \rightarrow w\) weakly in \(D^{1,2}(\mathcal{E})\), rescaling and using (2.14) we conclude that

\[
\int_{\mathcal{E}} \nabla w \cdot \nabla \varphi - \int_{\mathcal{E}} |w|^{2-2} w \varphi = \int_{\mathcal{E}} \nabla w_k \cdot \nabla \varphi - \int_{\mathcal{E}} |w_k|^{2-2} w_k \varphi + o(1)
\]

\[
= \int_{\Lambda_k} \nabla w_k \cdot \nabla \varphi - \int_{\Lambda_k} |w_k|^{2-2} w_k \varphi + o(1)
\]

\[
= \int_{\Lambda_\omega} \nabla u_k \cdot \nabla \varphi - \int_{\Lambda_\omega} |u_k|^{2-2} u_k \varphi + o(1),
\]

(2.17)

where \(\varphi_k(x) := e_k^{(2-N)/2} \varphi(\frac{x - \xi_k}{e_k})\). Next, we analyze all possibilities, according to the location of \(\xi_k\).
(a) If \( \xi_k = 0 \) for all \( k \in \mathbb{N} \), then \( \overline{\mathcal{B}} = \Sigma_\omega \) and \( w_k \) is \( \varrho \)-equivariant. Hence, \( w \) is \( \varrho \)-equivariant. Let \( \varphi \in C^\infty_c(\mathbb{R}^N) \). Then, \( \varphi|_{\Lambda_\omega} \in V(\Lambda_\omega) \) for large enough \( k \), and from (2.17) we obtain

\[
J'_{\Sigma_\omega}(w)[\varphi|_{\Sigma_\omega}] = \int_{\Sigma_\omega} \nabla w \cdot \nabla \varphi - \int_{\Sigma_\omega} |w|^{2^* - 2}w \varphi = J'_{\Lambda_\omega}(u_k)[\varphi|_{\Lambda_\omega}] = o(1).
\]

This shows that \( w|_{\Sigma_\omega} \) solves (1.1). Therefore,

\[
c^0_{\Sigma_\omega} \leq \frac{4}{N} ||w||_{\Sigma_\omega}^2 \leq \liminf_{k \to \infty} \frac{1}{N} ||w_k||_{\Sigma_\omega}^2 = \lim_{k \to \infty} \frac{1}{N} ||u_k||_{\Lambda_\omega}^2 = c^0_{\Lambda_\omega}.
\]

Together with Lemma 2.1, this implies that \( J_{\Sigma_\omega}(w) = c^0_{\Sigma_\omega} = c^0_{\Lambda_\omega} \leq c_\infty \) and

\[
o(1) = ||w_k - w||_{\Sigma_\omega} = \left\| u_k - \varepsilon_k^{2_\mathcal{B}} w \left( \cdot \varepsilon_k^{-1} \right) \right\|_{\Sigma_\omega}.
\]

So, in this case, we obtain statement (ii).

(b) If \( \xi_k \in \partial \omega \) for all \( k \in \mathbb{N} \), then \( \overline{\mathcal{B}} = \mathbb{H}_\xi \cap \mathbb{H}_\nu \), where \( \xi = \lim_{k \to \infty} \xi_k \), \( \nu \) is the exterior unit normal to \( \Sigma_\omega \) at \( \xi \), and \( \mathbb{H}_\xi \) and \( \mathbb{H}_\nu \) are half-spaces defined as in (2.7). If \( \varphi \in C^\infty_c(\mathbb{H}_\xi) \), then \( \varphi|_{\Lambda_\omega} \in V(\Lambda_\omega) \) for large enough \( k \), and using (2.17) we conclude that \( w|_{\mathcal{B}} \) solves the mixed boundary value problem

\[-\Delta w = |w|^{2^* - 2}w, \quad w = 0 \text{ on } \partial \mathcal{B} \cap \partial \mathcal{B}_\xi, \quad \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial \mathcal{B} \cap \partial \mathcal{B}_\nu.\]

Since \( \xi \) and \( \nu \) are orthogonal, extending \( w|_{\mathcal{B}} \) by reflection on \( \partial \mathcal{B} \cap \partial \mathcal{B}_\nu \), yields a nontrivial solution to the Dirichlet problem

\[-\Delta w = |w|^{2^* - 2}w, \quad w \in D_0^{1,2}(\mathbb{H}_\xi).\]

(2.18)

It is well known that this problem does not have a nontrivial solution, so (b) cannot occur.

(c) If \( \xi_k \in \Gamma_1 \) for all \( k \in \mathbb{N} \) and \( \varepsilon_k^{-1} \text{dist}(\xi_k, \mathbb{O} \cup \{0\}) \to \infty \), then \( \overline{\mathcal{B}} = \mathbb{H}_\nu \), where \( \nu \) is the exterior unit normal to \( \Sigma_\omega \) at \( \xi = \lim_{k \to \infty} \xi_k \). Using (2.17) we conclude that \( w|_{\mathcal{B}_\xi} \) solves the Neumann problem (2.10) in \( \mathbb{H}_\nu \). Since \( \varepsilon_k^{-1} |\xi_k| \to \infty \), we have that \( \varepsilon_k^{-1} |\xi_k| = \varrho |\xi_k| \to \infty \). Therefore,

\[w_k - (w \circ \varrho)(\cdot - \varepsilon_k^{-1} \xi_k) \to w \quad \text{weakly in } D^{1,2}(\mathbb{R}^N).\]

Note also that \( w_k \circ \varrho \to w \circ \varrho \) weakly in \( D^{1,2}(\mathbb{R}^N) \). Using these facts and performing suitable rescalings and translations we obtain:

\[
\begin{align*}
&\left\| u_k - \varepsilon_k^{2_\mathcal{B}} w \left( \cdot - \xi_k \right) + \varepsilon_k^{2_\mathcal{B}} w \left( \cdot - \xi_k \right) \right\|_{\Sigma_\omega}^2
= \left\| u_k - w \left( \cdot - \varepsilon_k^{-1} \xi_k \right) + (w \circ \varrho)(\cdot - \varepsilon_k^{-1} \xi_k) \right\|_{\Sigma_\omega}^2
= \left\| w_k - w + (w \circ \varrho)(\cdot + \varepsilon_k^{-1} (\xi_k - \xi_k)) \right\|_{\Sigma_\omega - \varepsilon_k^{-1} \xi_k}^2
= \left\| w_k + (w \circ \varrho)(\cdot + \varepsilon_k^{-1} (\xi_k - \xi_k)) \right\|_{\Sigma_\omega - \varepsilon_k^{-1} \xi_k}^2 - \left\| w \right\|_{\Sigma_\omega}^2 + o(1)
= \left\| -w_k \circ \varrho + w \circ \varrho \right\|_{\Sigma_\omega - \varepsilon_k^{-1} \xi_k}^2 - \left\| w \right\|_{\Sigma_\omega}^2 + o(1).
\end{align*}
\]
The mean curvature of \( \partial \omega \) to (1.2) is a solution to (1.1). 

Definition 2.6. A point \( \xi \in \partial \omega \) is a point of convexity of \( \Sigma_{\omega} \) (of radius \( r > 0 \)) if \( B_{r}(\xi) \cap \Sigma_{\omega} \subset \mathbb{H}_{r} \) and the mean curvature of \( \partial \Sigma_{\omega} \) at \( \xi \) with respect to the exterior unit normal \( \nu \) at \( \xi \) is positive.

Equality is not enough, as the following example shows. Set \( \xi_{k} = \xi_{k} \) for every \( k \). Since \( w \) is a sign-changing solution to (1.2) we have that

\[
2c_{\infty} < \frac{1}{N} \| w \|_{L^{2}(\mathbb{R}^{N})}^{2} \leq \lim_{k \to \infty} \frac{1}{N} \| w_{k} \|_{L^{2}(\Sigma_{\omega})}^{2} = \lim_{k \to \infty} \frac{1}{N} \| u_{k} \|_{L^{2}(\Sigma_{\omega})}^{2} = \epsilon_{\omega},
\]

contradicting Lemma 2.1. On the other hand, if \( \epsilon_{k} \rightarrow \infty \), then, arguing as in case (c), we conclude that

\[
2c_{\infty} \leq \frac{2}{N} \| w \|_{L^{2}(\mathbb{R}^{N})}^{2} \leq \lim_{k \to \infty} \frac{1}{N} \| w_{k} \|_{L^{2}(\Sigma_{\omega})}^{2} = \lim_{k \to \infty} \frac{1}{N} \| u_{k} \|_{L^{2}(\Sigma_{\omega})}^{2} = \epsilon_{\omega},
\]

contradicting Lemma 2.1 again. So (e) cannot occur.

We are left with (a) and (c). This concludes the proof. \( \square \)

Proposition 2.3 immediately yields the following result.

Corollary 2.4. If \( \epsilon_{\omega} < c_{\infty} \), then the problem (1.1) has a \( \rho \)-equivariant least energy solution in \( D^{1,2}(\Sigma_{\omega}) \).

Equality is not enough, as the following example shows. Set

\[
S^{N-1}_{+} := \{(x_{1}, \ldots, x_{N}) \in S^{N-1} : x_{N} > 0\}.
\]

Example 2.5. If \( \omega = S^{N-1}_{+} \), then problem (1.1) does not have a \( \rho \)-equivariant least energy solution in \( D^{1,2}(\Sigma_{\omega}) \).

Proof. \( \Sigma_{\omega} \) is the upper half-space \( \mathbb{R}^{N}_{+} := \{(x_{1}, \ldots, x_{N}) \in \mathbb{R}^{N} : x_{N} > 0\} \). If \( u \) were a \( \rho \)-equivariant least energy solution to (1.1) in \( \mathbb{R}^{N}_{+} \) then, extending \( u \) by reflection on \( \partial(\mathbb{R}^{N}_{+}) \), would yield a sign-changing solution \( \bar{u} \) to the problem (1.2) in \( \mathbb{R}^{N} \) with \( J_{\mathbb{R}^{N}}(\bar{u}) \leq 2c_{\infty} \). But the energy of any sign-changing solution to (1.2) is \( > 2c_{\infty} \); see [9]. \( \square \)

The following local geometric condition guarantees the existence of a minimizer. It was introduced by Adimurthi and Mancini in [1].

Definition 2.6. A point \( \xi \in \partial \omega \) is a point of convexity of \( \Sigma_{\omega} \) (of radius \( r > 0 \)) if \( B_{r}(\xi) \cap \Sigma_{\omega} \subset \mathbb{H}_{r} \) and the mean curvature of \( \partial \Sigma_{\omega} \) at \( \xi \) with respect to the exterior unit normal \( \nu \) at \( \xi \) is positive.
As in [1] we make the convention that the curvature of a geodesic in \(\partial \Sigma_\omega\) is positive at \(\xi\) if it curves away from the exterior unit normal \(\nu\). The half-space \(\mathbb{H}_1\) is defined as in (2.7). Examples of cones having a point of convexity are given as follows.

**Proposition 2.7.** If \(\overline{\omega} \subset S^{N-1}_+\), then \(\Sigma_\omega\) has a point of convexity.

*Proof.* Let \(\beta\) be the smallest geodesic ball in \(S^{N-1}_+\), centered at the north pole \((0, \ldots, 0, 1)\), which contains \(\omega\). Then, \(\partial \omega \cap \partial \beta \neq \emptyset\) and \(\overline{\beta} \subset S^{N-1}_+\). Hence, every point on \(\partial \beta\) is a point of convexity of \(\Sigma_\beta\). As \(\omega \subset \beta\), we have that any point \(\xi \in \partial \omega \cap \partial \beta\) is a point of convexity of \(\Sigma_\omega\). \(\Box\)

**Theorem 2.8.** If \(\Sigma_\omega\) has a point of convexity, then \(c^\rho_{\Sigma_\omega} < c_\omega\). Consequently, the problem (1.1) has a \(\rho\)-equivariant least energy solution in \(D^{1,2}(\Sigma_\omega)\). This solution is nonradial and changes sign.

*Proof.* Let \(\xi \in \partial \omega\) be a point of convexity of \(\Sigma_\omega\) of radius \(r > 0\). It is shown in [1, Lemma 2.2] that, after fixing \(r\) small enough and a radial cut-off function \(\psi \in C^\infty_c(\mathbb{R}^N)\) with \(\psi(x) = 1\) if \(|x| \leq \frac{r}{2}\) and \(\psi(x) = 0\) if \(|x| \geq \frac{r}{2}\), the function \(u_{\varepsilon, \xi}(x) := \psi(x - \xi)e^{2(2-N)/2}(\varepsilon^{-1}(x - \xi))\), with \(U\) as in (2.3), satisfies

\[
Q_{\Sigma_\omega}(u_{\varepsilon, \xi}) = \begin{cases} 
\frac{N}{2N} - d_N H_\omega(\xi) S \varepsilon \ln(\varepsilon^{-2}) + O(\varepsilon) & \text{if } N = 3, \\
\frac{N}{2N} - d_N H_\omega(\xi) S \varepsilon + O(\varepsilon^2 \ln(\varepsilon^{-2})) & \text{if } N \geq 4,
\end{cases}
\]

(2.19)

where \(d_N\) is a positive constant depending only on \(N\) and \(H_\omega(\xi)\) is the mean curvature of \(\partial \Sigma_\omega\) at \(\xi\). Hence, for \(\varepsilon\) small enough,

\[
J_{\Sigma_\omega}(t_{\varepsilon, \xi} u_{\varepsilon, \xi}) = \frac{1}{N} [Q_{\Sigma_\omega}(u_{\varepsilon, \xi})]_2^\frac{N}{2} < \frac{1}{2N} S_2^\frac{N}{2} = \frac{1}{2} c_\omega,
\]

where \(t_{\varepsilon, \xi} > 0\) is such that \(t_{\varepsilon, \xi} u_{\varepsilon, \xi} \in N(\Sigma_\omega)\); see (2.1). Choosing \(r\) so that \(B_r(\xi) \cap B_r(\varrho \xi) = \emptyset\) we conclude that \(t_{\varepsilon, \xi}(u_{\varepsilon, \xi} - u_{\varepsilon, \xi} \circ \varrho) \in N(\Sigma_\omega)\) and

\[
c^\rho_{\Sigma_\omega} \leq J_{\Sigma_\omega}(t_{\varepsilon, \xi}(u_{\varepsilon, \xi} - u_{\varepsilon, \xi} \circ \varrho)) < c_\omega.
\]

The existence of a \(\rho\)-equivariant least energy solution to (1.1) follows from Corollary 2.4. \(\Box\)

### 3. A positive nonradial solution

In this section \(\omega\) is not assumed to have any symmetries.

We are interested in positive solutions to the problem (1.1). Note that this problem has always a positive *radial* solution given by the restriction to \(\Sigma_\omega\) of the standard bubble \(U\) defined in (2.3). The question we wish to address in this section is whether problem (1.1) has a positive *nonradial* solution.

Recall the notation introduced in Section 2 and set

\[
c_{\Sigma_\omega} := \inf_{u \in N(\Sigma_\omega)} J_{\Sigma_\omega}(u) = \inf_{u \in D^{1,2}(\Sigma_\omega) \setminus \{0\}} \frac{1}{N} [Q_{\Sigma_\omega}(u)]_2^\frac{N}{2},
\]

\[
N(\Lambda_\omega) := N(\Sigma_\omega) \cap V(\Lambda_\omega) \quad \text{and} \quad c_{\Lambda_\omega} := \inf_{u \in N(\Lambda_\omega)} J_{\Lambda_\omega}(u).
\]

It is shown in [8, Theorem 2.1] that \(c_{\Lambda_\omega} > 0\). As in Lemma 2.1 one shows that \(c_{\Sigma_\omega} = c_{\Lambda_\omega} \leq \frac{1}{2} c_\omega\). We start by describing the behavior of minimizing sequences for \(J_{\Lambda_\omega}\) on \(N(\Lambda_\omega)\).
Proposition 3.1. Let $u_k \in \mathcal{N}(\Lambda_\omega)$ be such that
\[ J_{\Lambda_\omega}(u_k) \to c_{\Lambda_\omega} \quad \text{and} \quad J'_{\Lambda_\omega}(u_k) \to 0 \quad \text{in} \quad (V(\Lambda_\omega))'. \]

Then, after passing to a subsequence, one of the following statements holds true:

(i) There exist a sequence of positive numbers $(\epsilon_k)$, a sequence of points $(\xi_k)$ in $\Gamma_1$ and a function $w \in D^{1,2}(\mathbb{R}^N)$ such that $\epsilon_k^{-1} \text{dist}(\xi_k, \bar{\omega} \cup \{0\}) \to \infty$, $w|_{\bar{\Omega}}$ solves the Neumann problem
\[ -\Delta w = |w|^{p-2}w, \quad w \in D^{1,2}(\bar{\Omega}), \]

in some half-space $\bar{\Omega}$, $J_{\bar{\Omega}}(w) = \frac{1}{2} c_{\bar{\Omega}}$, and $c_{\Sigma_\omega} = c_{\Lambda_\omega} = \frac{1}{2} c_{\bar{\Omega}}$.

(ii) There exist a sequence of positive numbers $(\epsilon_k)$ with $\epsilon_k \to 0$ and a solution $w \in D^{1,2}(\Sigma_\omega)$ to the problem (1.1) such that
\[ \lim_{k \to \infty} \left\| u_k - \epsilon_k^{-\frac{N}{2}} w \left( \frac{\cdot - \xi_k}{\epsilon_k} \right) \right\|_{\Sigma_\omega} = 0, \]

and $J_{\Sigma_\omega}(w) = c_{\Sigma_\omega} = c_{\Lambda_\omega} \leq \frac{1}{2} c_{\bar{\Omega}}$.

Proof. The proof is similar, but simpler than that of Proposition 2.3. \qed

The following statement is an immediate consequence of this proposition.

Corollary 3.2. If $c_{\Sigma_\omega} < \frac{1}{2} c_{\bar{\Omega}}$, then the problem (1.1) has a positive least energy solution in $D^{1,2}(\Sigma_\omega)$.

Theorem 3.3. If $\Sigma_\omega$ has a point of convexity, then $c_{\Sigma_\omega} < \frac{1}{2} c_{\bar{\Omega}}$. Consequently, the problem (1.1) has a positive least energy solution in $D^{1,2}(\Sigma_\omega)$.

Proof. The proof is similar to that of Theorem 2.8. \qed

Let $D_{\text{rad}}^{1,2}(\Sigma_\omega)$ be the subspace of radial functions in $D^{1,2}(\Sigma_\omega)$, and define $\mathcal{N}_{\text{rad}}(\Sigma_\omega) := \mathcal{N}(\Sigma_\omega) \cap D_{\text{rad}}^{1,2}(\Sigma_\omega)$ and
\[ c_{\Sigma_\omega}^{\text{rad}} := \inf_{u \in \mathcal{N}_{\text{rad}}(\Sigma_\omega)} J_{\Sigma_\omega}(u) = \inf_{u \in D_{\text{rad}}^{1,2}(\Sigma_\omega) \setminus \{0\}} \frac{1}{N} |Q_{\Sigma_\omega}(u)|^\frac{N}{2}. \]

It was shown in [8, Theorem 2.4] that, if $\Sigma_\omega$ is convex, then $c_{\Sigma_\omega}^{\text{rad}} = c_{\Sigma_\omega}$ and the only positive minimizers are the restrictions of the rescalings
\[ U_\omega(x) = d_N \left( \frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{N+2}{2}}, \quad \epsilon > 0, \]

of the standard bubble to $\Sigma_\omega$. In fact, the proof of [8, Theorem 2.4] shows that these are the only positive solutions of (1.1) in a convex cone. Moreover, the following statement holds true.
nontrivial radial solutions to the problem (1.2) in $\mathbb{R}^N$. If $\omega$ only nontrivial radial solutions to (1.1).

This last problem does not depend on $\omega$. For any cone $c$ minimizers of $J_{\Sigma}(\omega)$. Proposition 3.4. A radial function $u$ solves (1.1) in $\Sigma$ if and only if the function $\tilde{u}$ given by $\tilde{u}(r) := u(x)$ with $r = \|x\|$ solves

$$\frac{d}{dr}(r^{N-1}\tilde{u}'(r)) = r^{N-1}|\tilde{u}(r)|^{N-2}\tilde{u}(r) \text{ in } (0, \infty), \quad \tilde{u}(0) = u(0), \quad \tilde{u}'(0) = 0.$$ 

The same formula holds true when we replace $\omega$ by $\Lambda$ in Lemma 2.1 one shows that $|\tilde{u}(r)|^{N-2}\tilde{u}(r)$ in (0, $\infty$).

Therefore, $\Lambda$ $c$ has a positive least energy solution in $D$. Therefore, $\Lambda$ $c$ $\omega$ $\Lambda$ $\omega$ $\omega$).

The same formula holds true when we replace $\omega$ by $\mathcal{S}^{N-1}$. In this case, the left-hand side is $c_{\omega}$. Hence, $\Lambda$ $\omega$ $\omega$.

**Corollary 3.5.** If $\Sigma$ has a point of convexity and $|\Lambda| \geq \frac{1}{2}|B_1(0)|$, then

(i) the problem (1.1) has a positive least energy solution in $D^{1,2}(\Sigma)$,

(ii) every least energy solution of (1.1) is nonradial.

**Proof.** From Theorem 3.3 and Proposition 3.4 we get that $c_{\Sigma}$ is attained and

$$c_{\Sigma} \leq \frac{1}{2}c_\epsilon = c_{\Sigma,\omega}^\text{rad} = \frac{b_N}{2}|B_1(0)| \leq b_N|\Lambda| = c_{\Sigma,\omega}^\text{rad},$$

where $\mathbb{R}_+^N := \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_N > 0\}$. So every least energy solution is nonradial.

Note that the hypothesis that $|\Lambda| \geq \frac{1}{2}|B_1(0)|$ implies that $\Sigma$ is not convex.

A closer look at the estimate (2.19) allows to refine Corollary 3.5 and to produce examples of cones $\Sigma$ with $|\Lambda| < \frac{1}{2}|B_1(0)|$ for which the problem (1.1) has a positive nonradial solution.
To this end, we fix a smooth domain $\omega_0$ in $\mathbb{S}^{N-1}$ for which $\Sigma_{\omega_0}$ has a point of convexity $\xi \in \partial \omega_0$ of radius $r > 0$, and we define

$$\ell(\omega_0, \xi, r) := \{ \omega : \omega \text{ is a smooth domain in } \mathbb{S}^{N-1}, B_\epsilon(\xi) \cap \Sigma_{\omega_0} \subset B_r(\xi) \cap \Sigma_\omega$$

and

$$\text{dist}(B_\epsilon(\xi) \cap \Sigma_{\omega_0}, B_r(\xi) \cap \Sigma_{\omega_0}) > 0 \}.$$

Then, we have the following result.

**Theorem 3.6.** There exists $\alpha_\xi \in (0, \frac{1}{2}|B_1(0)|)$, depending only on $B_r(\xi) \cap \Sigma_{\omega_0}$, such that, for every $\omega \in \ell(\omega_0, \xi, r)$ with $|\Lambda_\omega| > \alpha_\xi$, the following statements hold true:

(i) the problem (1.1) has a positive least energy solution in $D^{1,2}(\Sigma_\omega)$,

(ii) every least energy solution of (1.1) is nonradial,

(iii) $\Sigma_{\omega_\xi}$ is not convex.

**Proof.** Recall that the functions $u_{\epsilon, \xi}$, introduced in the proof of Theorem 2.8, vanish outside the ball $B_{r/2}(0)$. Moreover, the value $Q_{\Sigma_{\omega_0}}(u_{\epsilon, \xi})$ and the estimate (2.19) depend only on the value of $u_{\epsilon, \xi}$ in $B_r(\xi) \cap \Sigma_{\omega_0}$. We fix $\epsilon_0 > 0$ small enough so that

$$Q_\xi := Q_{\Sigma_{\omega_0}}(u_{\epsilon_0, \xi}) < \frac{S}{2^{2/N}},$$

and we set $\alpha_\xi := \frac{1}{N b_N} Q_{\xi}^{N/2}$ with $b_N$ as in Proposition 3.4. Then,

$$\alpha_\xi < \frac{1}{2N b_N} S^{\frac{2}{N}} = \frac{1}{2} |B_1(0)|.$$

Given $\omega \in \ell(\omega_0, \xi, r)$, we fix a function $\tilde{u}_{\epsilon_0, \xi} \in C^\infty(B_1(0))$ such that $\tilde{u}_{\epsilon_0, \xi}(x) = u_{\epsilon_0, \xi}(x)$ if $x \in B_r(\xi) \cap \Sigma_{\omega_0}$ and $\tilde{u}_{\epsilon_0, \xi}(x) = 0$ if $x \in B_r(\xi) \cap (\Sigma_\omega \setminus \Sigma_{\omega_0})$. So, if $|\Lambda_\omega| > \alpha_\xi$, we have that

$$c_{\Sigma_{\omega_\xi}} \leq \frac{1}{N} [Q_{\Sigma_{\omega_0}}(\tilde{u}_{\epsilon_0, \xi})]^{\frac{N}{2}} = \frac{1}{N} Q_{\xi}^\frac{N}{2} = b_N \alpha_\xi < b_N |\Lambda_\omega| = c_{\Sigma_\omega}^{\text{rad}}.$$

Note that $\xi$ is a point of convexity of $\omega$. Hence, by Theorem 3.3 and the previous inequality, $c_{\Sigma_{\omega_\xi}}$ is attained at a nonradial solution of (1.1). Finally, recall that, if $\Sigma_\omega$ were convex, then $c_{\Sigma_{\omega_\xi}} = c_{\Sigma_\omega}^{\text{rad}}$; see [8, Theorem 2.4]. This completes the proof. \qed

**Corollary 3.7.** There exists a smooth domain $\omega \subset \mathbb{S}^{N-1}$ such that the problem (1.1) has a positive nonradial solution in $\Sigma_{\omega_\xi}$.

**Proof.** Let $\omega_0$ be the geodesic ball in $\mathbb{S}^{N-1}$ of radius $\pi/4$ centered at the north pole and let $\xi$ be any point on $\partial \omega_0$. Fix $r > 0$ such that $B_r(\xi) \cap \mathbb{S}^{N-1} \subset \mathbb{S}^{N-1}$. Clearly, $\xi$ is a point of convexity of $\Sigma_{\omega_0}$ of radius $r$, so we may fix $\alpha_\xi > 0$ as in Theorem 3.6. As $\alpha_\xi < \frac{1}{2} |B_1(0)|$, there exists $\omega \in \ell(\omega_0, \xi, r)$ with $\omega \subset \mathbb{S}^{N-1}$ and $|\Lambda_\omega| > \alpha_\xi$. Now, Theorem 3.6 yields a positive nonradial solution to problem (1.1) in $\Sigma_{\omega_\xi}$. \qed

**Remark 3.8.** Let $\omega \neq \mathbb{S}^{N-1}$ be such that $\Sigma_\omega$ is convex. Then, every point $\xi \in \partial \omega$ is a point of convexity of radius $r$ for any $r > 0$. Fix $r = 1$, and fix $\epsilon > 0$ such that

$$Q_\xi := Q_{\Sigma_{\omega_\xi}}(u_{\epsilon, \xi}) < \frac{S}{2^{2/N}} \quad \forall \xi \in \partial \omega.$$
Now, define \( \alpha_\xi := \frac{1}{Nb_\xi} Q_\xi^{N/2} \), as in Theorem 3.6. Since \( \Sigma_\omega \) is convex, we must have that

\[ |\Lambda_\omega| \leq \alpha_\xi = \frac{|B_1(0)|}{S^{N/2}} Q_\xi^{N/2}, \quad \forall \xi \in \partial \omega, \]

where the equality follows from the definition of \( b_\xi \); see Proposition 3.4. Hence, for any convex cone \( \Sigma_\omega \), we obtain the upper bound

\[ |\Lambda_\omega| \leq \left( \frac{|B_1(0)|}{S^{N/2}} \right) \min_{\xi \in \partial \omega} Q_\xi \]

for the measure of \( \Lambda_\omega \), which is given in terms of the Sobolev constant and the local energy of the standard bubbles.

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Conflict of interest

The authors declare no conflict of interest.

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