Comparison of different models for stress singularities in higher order finite element methods for elastic waves

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In this contribution, we present three models to capture singularities in combination with the Spectral Element Method. The first model, the continued-fraction-based Scaled Boundary Finite Element Method, the second model, a new approach based on enrichment with static modes, and the third model, which uses an hp-refinement near the singularity, are compared among each other and evaluated in terms of their respective efficiency and accuracy.

The Spectral Element Method (SEM) with p-refinement has been proven to be an efficient numerical method for solving the wave equation in linear elastic bodies. This efficiency is reduced if a stress singularity is present in the body [1]. Additionally, the most common refinement strategies, a uniform h- or p-refinement in all elements, do not lead to a significant reduction of the error because the order of the stress singularity limits the convergence rate under uniform h- or p-refinement [2]. Special solution strategies are required to preserve the efficiency of the SEM. There are many approaches which consider stress singularities, but comparisons are rare for dynamic problems. Finding an efficient model is an essential step for many applications.

Below, we present several models to treat the singularity in combination with the SEM for an L-shaped domain (Fig. 2). The re-entrant corner of an L-shaped domain is a classic example of a problem with a point of singularity in the stress field. We assume the linear elastic wave equation inside the domain, i.e. \(\partial_t^2 u = \nabla \cdot \sigma(u)\), where \(u\) is the displacement and \(\sigma\) is the linearized stress operator. Fig. 2 shows an overview of the domain, the boundary conditions, and material properties, where \(\nu\) is the Poisson’s ratio and \(\rho\) the density of the isotropic material. A time-dependent pressure \(\tau\) in form of a sine modulated Gaussian pulse

\[
\tau(t) = \sin(2\pi tf_0) \cdot \exp\left(-0.5(t - 0.25f_0^{-1})^2/(f_0^{-1})^2\right)
\]

with a center frequency \(f_0\) is applied to the left side. The simulation advances in time for \(25f_0^{-1}\) using the Newmark scheme. A very small time step \(\Delta t = 7.81 \cdot 10^{-4}\) is utilized to minimize the time integration error [1]. The four different methods are compared at point \(P\) (Fig. 2). The error with respect to a reference solution is measured as follows

\[
\text{error} = \frac{\sqrt{\sum_i (u^h(P, t_i) - u^{\text{ref}}(P, t_i))^2}}{\sqrt{\sum_i (u^{\text{ref}}(P, t_i))^2}} \Delta t.
\]

**Continued-fraction-based SBFEM** The first approach consists in replacing the elements around the point of singularity \(x_0 = (0, 0)\) by a scaled boundary polynomial element. Fig. 1a shows this treatment. The essence of this semi-analytical method is the scaled boundary coordinate system

\[
x = \xi (\gamma(\eta) - x_0) + x_0,
\]

where \(\gamma(\eta)\) is the curve between the polynomial element and the classic elements (Fig. 1a). One-dimensional spectral shape functions \(N(\eta)\) approximate the curve \(\gamma(\eta) = N(\eta)x_b\). These shape functions are used in a continued fraction approach to compute the stiffness matrix of this element. A detailed derivation can be found in [3,4]. The number of fraction steps is optimized for the comparison. This method is abbreviated with SBFEM in Fig. 3.

**Enrichment with static SBFEM** For the second treatment, we construct an enrichment function instead of replacing the elements. An enriched finite element approach approximates the displacement, i.e. \(u(x) = \sum_j M_j(x) + \sum_i M_i^e(x)\), where \(M_j\) are the classic spectral shape functions [5] and \(M_i^e\) are the shape functions to approximate the singular stress. For an arbitrarily small region around the point of singularity, the dynamic problem should act like a loaded, static problem in every time step. Following this argumentation, the time-dependent term can be neglected. The static modes associated with the singular stress field in the SBFEM are of the form \(\phi_i(x(x, \eta)) = x^{-\lambda_i}N(\eta)m_i\) with \(-1 < \Re(\lambda_i) < 0\), where the vector \(m_i\) and the number \(\lambda_i\) are derived from a quadratic eigenvalue problem [6]. For these modes, the shifted enrichment functions are

\[
M_i^e(x) = \begin{cases} \phi_i(x) - \sum_k \phi_i(x_k)M_k(x_k) & \text{x inside the polygon} \\ 0 & \text{x outside the polygon} \end{cases}
\]
**Fig. 1:** Example meshes for the four methods: a) SBFEM  b) P-eSEM  c) HP-SEM  d) P-SEM

where $x_k$ are the nodes of the spectral elements. The elements are integrated by the approach of Schwab [7]. The support of these enrichment functions is the polygon and they vanish on the curve $\gamma$, which is an advantage over classic enrichment functions. This is indicated by the gray area in Fig. 1b. The method is abbreviated as P-eSEM in Fig. 3.

Both methods, continued-fraction-based SBFEM and enrichment with static SBFEM, can treat anisotropic materials which vary arbitrarily in $\eta$ (eq. (3)). Moreover, both methods can handle zero Dirichlet and traction free boundary conditions.

The third model uses an hp-refinement near the crack tip (Fig. 1c). The hp-refined grid is strongly graded toward the singularity. As the elements become smaller, it is more efficient to reduce the degree to achieve the same error [8]. A fourth model has no treatment of the stress singularity (Fig. 1d). Fig. 3 shows a comparison of the respective errors (eq. (2)) for two center frequencies $f_0$ (eq. (1)).

**Conclusion**  The influence of points of singularity is frequency-dependent - see Fig. 3. For this setup, the singularity is more dominant in the lower frequency range. SBFEM is the most efficient out of the investigated methods. SBFEM, P-eSEM, and HP-SEM all show exponential convergence. For the presented frequencies, a combination of a graded h-refinement and p-refinement will be nearly as efficient as the hp-approach, because only a few elements have a lower degree than the maximum degree. The hp-approach becomes more relevant if the number of smaller elements near the singularity is higher. The static enrichment is sufficient to reach an error under 1% efficiently in all tested examples – not only the presented L-shaped domain. The enrichment by static SBFEM-modes is an attractive alternative to continued-fraction-based SBFEM because it is easy to extend to non-linear problems.

**References**

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