Brane Surgery

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Some aspects of the role of p-branes in non-perturbative superstring theory and M-theory are reviewed. It is then shown how the Chern-Simons terms in D=10 and D=11 supergravity theories determine which branes can end on which, i.e. the ‘brane-boundary rules’.

1. Introduction

Extended objects, known as ‘branes’, currently play an essential role in our understanding of the non-perturbative dynamics underlying ten-dimensional (D=10) superstring theories and the 11-dimensional (D=11) M-theory (see [1] for a recent review). In the context of the effective D=10 or D=11 supergravity theory a ‘p-brane’ is a solution of the field equations representing a p-dimensional extended source for an abelian \( (p+1) \)-form gauge potential \( A_{p+1} \) with \( (p+2) \)-form field strength \( F_{p+2} \). As such, the p-brane carries a charge

\[
Q_p = \int_{S^{D-p-2}} \ast F_{p+2} ,
\]

where \( \ast \) is the Hodge dual in the D-dimensional spacetime and the integral is over a \( (D-p-2) \)-sphere encircling the brane, as shown schematically in the figure below:

In the case of a static infinite planar p-brane this formula is readily understood as a direct generalization of the \( p=0 \) case, i.e. a point particle in electrodynamics, with the \( (D-p-1) \)-dimensional ‘transverse space’ (spanned by vectors orthogonal to the \( (p+1) \)-dimensional worldvolume) taking the place of space. In the case of a closed p-brane, static or otherwise, the charge \( Q_p \) can be understood (after suitable normalization) as the linking number of the p-brane with the \( (D-p-2) \)-sphere in the \( (D-1) \)-dimensional space.

Examples for \( p=1 \) are provided by the D=10 heterotic strings, for which

\[
Q_1 = \int_{S^7} \ast H ,
\]

where \( H \) is the 3-form field strength for the 2-form gauge potential \( B \) from the massless Neveu-Schwarz (NS) sector of the string spectrum. Further D=10 examples are provided by the type II superstrings, with the difference that \( B \) is now the 2-form of NS \( \otimes \) NS origin in type II superstring theory. A D=11 example is the charge

\[
Q_2 = \int_{S^7} \ast F
\]

carried by a supermembrane, where \( F = dA \) is the 4-form field strength for the 3-form potential \( A \) of D=11 supergravity.

The statement that a p-brane carries a charge of the above type can be rephrased as a statement about interaction terms in the effective worldvolume action governing the low-energy dynamics of the object. Consider, for example, type II and heterotic strings. Let \( \sigma^i \) be the worldsheet coordinates and let \( X^\mu(\sigma) \) describe the immersion of...
the worldsheet in the D=10 spacetime. Then the worldsheet action in a background with a non-vanishing 2-form $B$ will include the term

$$I_B = \int d^2 \sigma \, \varepsilon^{ij} \partial_i X^\mu \partial_j X^\nu \, B_{\mu\nu}(X(\sigma))$$ \hspace{1cm} (4)$$

where $\varepsilon^{ij}$ is the alternating tensor density on the worldsheet. Thus the string is a source for $B$ and, since the coupling is 'minimal', it will contribute to the charge $Q_1$ defined above. Similarly, in a D=11 background with non-vanishing 3-form potential $A$, the membrane action includes the term $^{2}$

$$I_A = \int d^3 \xi \, \varepsilon^{ijk} \partial_i X^M \partial_j X^N \partial_k X^P A_{MNP}$$ \hspace{1cm} (5)$$

where $X^M(\xi)$ describes the immersion of the supermembrane’s worldvolume in the D=11 spacetime, and $\xi^i$ are the worldvolume coordinates. This minimal interaction implies that the membrane is a source for $A$ with non-vanishing charge $Q_2$.

The actions $I_A$ and $I_B$ are actually related by double-dimensional reduction, as are the full supermembrane and IIA superstring actions $^{4}$. The dimensional reduction to D=10 involves setting $X^M = (X^\mu, y)$ where $y$ is the coordinate of the compact 11th dimension, and taking all fields to be independent of $y$. From the worldvolume perspective this amounts to a special choice of background for which $k = \partial / \partial y$ is a Killing vector field. Double-dimensional reduction is then achieved by setting $\xi = (\sigma, \rho)$ where $\rho$ is the coordinate of a compact direction of the membrane, and then setting $\partial_\rho X^\mu = 0$ and $dy = d\rho$, which is the ansatz appropriate to a membrane that wraps around the 11th dimension. The action $I_A$ then becomes $I_B$ after the identification $B = i_k A$, where $i_k$ indicates contraction with the vector field $k$.

A coupling to $B$ of the form $^{4}$ is possible only for oriented strings. Of the five D=10 superstring theories all are theories of oriented strings except the type I theory. Thus, the type I string does not couple minimally to $B$. Instead, it couples non-minimally. In the Lorentz-covariant GS formalism in which the worldsheet fermions, $\theta$, are in a spinor representation of the D=10 Lorentz group, the worldsheet interaction Lagrangian is

$$L_B = \bar{\theta} \Gamma^{\mu\nu\rho} \theta H_{\mu\nu\rho}.$$ \hspace{1cm} (6)$$

Because of the ‘derivative’ coupling of the string to $B$ through its field strength $H$, the $Q_1$ charge carried by the type I string vanishes. As the above interaction shows, the type I string theory origin of $B$ is in the $R \otimes R$ sector rather than the $NS \otimes NS$ sector. This example illustrates a general feature of string theory: $R \otimes R$ charges are not carried by the fundamental string. If there is anything that carries the charge $Q_1$ in type I string theory it must be non-perturbative. It is now known that there is such a non-perturbative object in type I string theory $^{4}$: it is just the $SO(32)$ heterotic string! This is one of the key pieces of evidence in favour of the proposed ‘duality’, i.e. non-perturbative equivalence, of the type I and $SO(32)$ heterotic string theories. Another is the fact that the two effective supergravity theories are equivalent, being related to each other by a field redefinition that takes $\phi \rightarrow -\phi$, where $\phi$ is the dilaton. Since the vacuum expectation value $\langle \phi \rangle$ is the string coupling constant $g_s$ this means that the weak coupling limit of one theory is the strong coupling limit of the other.

An important consequence of the charge $Q_p$ carried by a p-brane is that it leads to a BPS-type bound on the p-volume tension, $T_p$, of the form $T_p \geq c_p|Q_p|$, where $c_p$ is some constant characteristic of the particular supergravity theory, the choice of vacuum solution of this theory, and the value of $p$. If one considers the class of static solutions with p-fold translational symmetry then a bound of the above form follows from the requirement that there be no naked singularities. This bound is saturated by the solution that is 'extreme' in the sense of General Relativity, i.e. for which the event horizon is a degenerate Killing horizon. However, these considerations are clearly insufficient to show that the p-brane tension actually is bounded in this way because the physically relevant class of solutions is the much larger one for which only an appropriate

$^{2}$When a dilaton is present this is true only for an appropriate definition of the metric.
asymptotic behaviour is imposed. Remarkably, the attempt to establish a BPS-type bound succeeds if and only if the theory is either a supergravity theory, or a consistent truncation of one. In particular, the presence of various Chern-Simons terms in the Lagrangians of $D=10$ and $D=11$ supergravity theories is crucial to the existence of a BPS-type bound on the tensions of the $p$-brane solutions of these theories. This is so even when, as is usually the case, these Chern-Simons (CS) terms play no role in the $p$-brane solutions themselves in the sense that they are equally solutions of the (non-supersymmetric) truncated theory in which the CS terms are omitted. These facts hint at a more important role for the supergravity CS terms in determining the properties of $p$-branes than has hitherto been appreciated. This observation provided the principal motivation for this article, as will become clear.

Although the charge $Q_p$ has only a magnitude, it is associated with an object whose spatial orientation is determined by a $p$-form of fixed magnitude. Thus, a $p$-brane is naturally associated with a $p$-form charge of magnitude $Q_p$. Indeed, the supersymmetrization of terms of the form $\frac{1}{2} \epsilon^{\mu_1 \mu_2 \ldots \mu_p} F_{\mu_1 \mu_2} \cdots F_{\mu_{p-1} \mu_p}$ or $\frac{1}{2} \epsilon^{\mu_1 \mu_2 \ldots \mu_p} F^*_{\mu_1 \mu_2} \cdots F^*_{\mu_{p-1} \mu_p}$ leads to a type of super-Wess-Zumino term that implies a modification of the standard supersymmetry algebra to one of the (schematic) form

$$\{Q, Q\} = \Gamma \cdot P + \Gamma^{(p)} \cdot Z_p ,$$

where $\Gamma^{(p)}$ is an antisymmetrized product of $p$ Dirac matrices and $Z_p$ is a $p$-form charge whose magnitude is given by the coefficient of the Wess-Zumino term. For $p = 0$ this is the well-known modification that includes $Z_0 = Q_0$ as a central charge. More generally, $Q_p$ may be identified as the magnitude of $Z_p$, and an extension of the arguments used in the $p = 0$ case $[12,13]$ shows that the supersymmetry algebra (7) implies the BPS-type bound on the $p$-brane tension $T_p$. It also shows that the ‘extreme’ $p$-brane solutions of supergravity theories which saturate the bound must preserve some of the supersymmetry, and the fraction preserved is always 1/2 for $p$-brane solutions in $D=10$ and $D=11$. The heterotic and type II superstrings and the $D=11$ supermembrane are examples not only of charged $p$-branes but also of extreme charged $p$-branes. This follows from the ‘$\kappa$-symmetry’ of their Lorentz covariant and spacetime supersymmetric worldsheet/worldvolume actions (see [14] for a review). The BPS-saturated $p$-branes are important in the context of the non-perturbative dynamics of superstring theories or M-theory for essentially the same reasons that BPS-saturated solitons are important in $D=4$ field theories. In fact, most of the the latter can be understood as originating in $D=10$ or $D=11$ BPS-saturated $p$-branes. For these reasons, the BPS-saturated $p$-branes are the ones of most interest and will be the only ones considered here. It should therefore be understood in what follows that by ‘brane’ we mean ‘BPS-saturated brane’.

One of the lessons of recent years has been that much can be learned about the non-perturbative dynamics of superstring theories from the effective $D=10$ supergravity theories. One example of this is the fact that there exist $p$-brane solutions of type II supergravity theories which are charged, in the sense explained above, with respect to the $(p+1)$-form gauge fields from the $R \otimes R$ sector of the corresponding string theory. By supposing these $R \otimes R$ branes to be present in the non-perturbative string theory one can understand how otherwise distinct superstring theories might be dual versions of the same underlying theory. The basic idea is that branes can ‘improve’ string theory in the same way that strings ‘improve’ Kaluza-Klein (KK) theory. For example, the $S^1$-compactified IIA and IIB supergravity theories have an identical massless $D=9$ spectrum but are different as Kaluza-Klein theories because their massive modes differ. The corresponding string theories are the same, however, because the inclusion of the string winding modes restores the equivalence of the massive spectra. Similarly, the $K_3$-compactified type IIA superstring theory and

\[ \text{Footnote 3: In contrast, the proof of positivity of the ADM mass of asymptotically-flat spacetimes is not subject to this restriction since, for example, it is valid for arbitrary } D. \]
the $T^4$-compactified heterotic string theory have an identical massless D=6 spectrum, but since they differ in their perturbative massive spectra they are inequivalent as perturbative string theories. However, the non-perturbative massive spectrum of the IIA superstring includes ‘wrapping’ modes of 2-branes around 2-cycles of $K_3$. The inclusion of these leads to the same massive BPS spectrum in the two theories, and there is now strong evidence of a complete equivalence [15]. The inclusion of these leads to the same massive BPS spectrum in the two theories, and there is now strong evidence of a complete equivalence [15].

Non-perturbatively, all are on an equal footing since any one can be found from any other one by a combination of ‘dualities’. This feature is apparent in the IIA or IIB effective supergravity theories which treat all p-branes if we wish to specify the value of $p$: the worldvolume of a D-$p$-brane is simply a $(p+1)$-dimensional hyperplane defined by imposing $(D-p-1)$ Dirichlet boundary conditions at the boundaries of open string worldsheets.

The distinctions between the various kinds of type II p-brane, such as whether they are of NS $\otimes$ NS or R $\otimes$ R type, are not intrinsic but are rather artefacts (albeit very useful ones) of perturbation theory. Non-perturbatively, all are of equal footing since any one can be found from any other one by a combination of ‘dualities’. This feature is apparent in the IIA or IIB effective supergravity theories which treat all p-branes if we wish to specify the value of $p$: the worldvolume of a D-$p$-brane is simply a $(p+1)$-dimensional hyperplane defined by imposing $(D-p-1)$ Dirichlet boundary conditions at the boundaries of open string worldsheets.

The M-theory branes, or ‘M-branes’, consist of only the D=11 membrane and its magnetic dual, a fivebrane. We saw earlier that the classical IIA superstring action is related to that of the D=11 supermembrane by double-dimensional reduction. This was widely considered to be merely a ‘coincidence’, somewhat analogous to the fact that IIA supergravity happens to be the dimensional reduction of D=11 supergravity; after all, the quantum superstring theory has D=10 as its critical dimension. However, the critical dimension emerges from a calculation in perturbative string theory. It is still possible that the non-perturbative theory really is 11-dimensional, but if this is so the KK spectrum of the $S^1$-compactified D=11 supergravity must appear in the non-perturbative IIA supergravity spectrum. It was pointed out in [21] that the extreme black holes of IIA supergravity, now regarded as the effective field theory realization of D-0-branes, are candidates for this non-perturbative KK spectrum. This means that the IIA superstring really is an $S^1$-wrapped D=11 supermembrane, but it does not then follow that the supermembrane is also ‘fundamental’ because this adjective is meaningful only in the context of a specific perturbation theory. For example, the SO(32) heterotic string is ‘fundamental’ at weak coupling but as the coupling increases it transmutes into the D-string of the type I theory. Another example is the IIB string which is ‘fundamental’ at weak coupling but which transmutes into the D-string of a
dual IIB theory at strong coupling \cite{22}. In the IIA case the strong coupling limit is a decompactification limit in which the D=11 Lorentz invariance is restored and the effective D=10 IIA supergravity is replaced by D=11 supergravity \cite{23}. The ‘fundamental’ IIA superstring transmutes, in this limit, into the unwrapped D=11 membrane of M-theory but, because of the absence of a dilaton, there is no analogue of string perturbation theory in D=11 and so there is no analogous basis for deciding whether or not the membrane is ‘fundamental’. Nevertheless, as we shall shortly see, there is an intrinsic asymmetry between M-theory membranes and fivebranes which suggests a fundamental role for the membrane in some as yet unknown sense\cite{22}.

Given that the heterotic string appears as a D-brane in type I string theory one might wonder whether the type I string should make an appearance somewhere in the non-perturbative SO(32) heterotic string theory. As we have seen, however, the type I string carries no $Q_1$ charge, so its description in the effective supergravity theory would have to be as a non-extreme, or ‘black’, string. Infinite uncharged black strings have been shown to be unstable against perturbations that have the tendency to break the string into small segments\cite{24} (whereas extreme strings are stable because they saturate a BPS-type bound). This is exactly what one expects from string theory since a closed type I string can break, i.e. type I string theory is a theory of both closed and open strings. The reason that this is possible for type I strings, but not for heterotic or type II strings, is precisely that the type I string carries no $Q_1$ charge. To see this, suppose that a string carrying a non-zero $Q_1$ charge were to have an endpoint. One could then ‘slide off’ the 7-sphere encircling the string and contract it to a point. Provided that the integral defining $Q_1$ is homotopy invariant, which it will be if $d\ast H = 0$, the charge $Q_1$ must then vanish, in contradiction to the initial assumption. We conclude that the only breakable strings are those for which $Q_1 = 0$. Thus type II and heterotic strings cannot break. Clearly, similar arguments applied to $p$-branes carrying non-zero $Q_p$ charge lead to the conclusion that they too cannot break.

By ‘break’ we mean to imply that the $(p - 1)$-brane boundary created in this process is ‘free’ in the sense that its dynamics is determined entirely by the $p$-brane of which it is the boundary. An ‘unbreakable’ p-brane may nevertheless be open if its boundary is tethered to some other object because there may then be an obstruction to sliding the $(D - p - 2)$-sphere off the end of the p-brane. Examples of such obstructions are the D-branes on which type II superstrings can end. One way to understand how this is consistent with conservation of the charge $Q_1$ is to consider the D-brane’s effective worldvolume action, which governs its low-energy dynamics. The field content of this action is found from the massless sector of an open type II superstring with mixed Neumann/Dirichlet boundary conditions at the ends. These fields are essentially the same as those of the open type I string without Chan-Paton factors with the difference that they depend only on the D-brane’s worldvolume coordinates (see e.g. \cite{25}). In particular, these worldvolume fields include an abelian 1-form potential $V$. The bosonic sector of the effective worldvolume action, in a general NS $\otimes$ NS background, can be deduced from the requirement of conformal invariance of the type II string action for a worldsheet with a boundary\cite{26}. This effective worldvolume action is found to contain the term

$$-\frac{1}{4} \int d^{p+1} \xi |dV - B|^2 , \quad (9)$$

where the integral is over the $(p + 1)$-dimensional worldvolume $W$ and it is to be understood that the spacetime 2-form $B$ is pulled back to $W$. This shows that the D-brane is a source of $B$. If we modify the equation $d\ast H = 0$ in order to include this source we find, by integration, that

$$Q_1 = \int_{S^{p-2}} * dV , \quad (10)$$

where $Q_1$ is defined as before in \cite{27}. The integral on the right hand side of (10) is over a $(p - 2)$-sphere in the D-brane surrounding the string’s endpoint and $*$ is the worldvolume Hodge dual. This result can be interpreted as the statement...
that the charge of the string can be ‘transferred’ to an electric charge of a particle on the D-brane, so charge conservation is compatible with the existence of an open string provided that its endpoints are identified with charged particles living on a D-brane.

A similar analysis can be applied to the D=11 membrane which, we recall, is an electric-type source for the 3-form gauge potential $A$ of D=11 supergravity. In this case, the D=11 fivebrane has a worldvolume action containing the terms

$$-rac{1}{12} \int d^6 \xi \left\{ |{\mathcal F}_3|^2 - e^{ijklmn} A_{ijkl} \partial_l V_{mn} \right\}, \quad (11)$$

where $F_3 = (dV_2 - A)$ is the 3-form field strength for a worldvolume 2-form potential $V_2$, and it is again to be understood that $A$ is the pullback of the spacetime field to the worldvolume. Actually, the worldvolume 3-form $F_3$ is self-dual, but this condition must be imposed after variation of the action $\mathcal{S}$. The second term in this action is needed for consistency of the self-duality condition with the $V_2$ field equation. Apart from this subtlety, we see from its worldvolume action that the fivebrane is a source for $A$. Its inclusion in the field equation for $A$ leads, after integration, to the equation

$$Q_p = \int_{S^{q-p}} * dV_p, \quad (13)$$

where the integral in the $q$-brane is over a $(q-p)$-sphere surrounding the $(p-1)$-brane boundary of the $p$-brane. Thus, the $p$-brane charge can be transferred to the electric charge of the $(p-1)$-brane boundary living in the $q$-brane. That is, charge conservation now permits the $p$-brane to be open provided its boundary lies in a $q$-brane. The cases discussed above clearly fit this pattern, but there are drawbacks to this approach. Firstly, it is indirect because one must first determine the worldvolume field content of all relevant branes. Secondly, it is ad hoc because, in general, the worldvolume coupling is postulated rather than derived. The subtleties alluded to above in the construction of the fivebrane action show that this is not a trivial matter. In fact, even the bosonic fivebrane action is not yet fully known and until it is one cannot be completely certain that the wanted terms in this action really are present.

In this contribution I will present a new, and extremely simple, method for the determination of when $p$-branes may have boundaries on $q$-branes. Essentially, one can read off from the Chern-Simons terms in the supergravity action whether any given $p$-brane can have a boundary and, if so, in what $q$-brane the boundary must lie. As such, the method provides a further example of how much can be learned about the non-perturbative dynamics of superstring theories, or M-theory, from nothing more than the effective supergravity theory. I have called the method ‘brane surgery’ because of a notional similarity to the way in which manifolds can be ‘glued’ together by the mathematical procedure known as ‘surgery’, but it is not intended that the term should be understood here in its technical sense.

It is pleasure to dedicate this contribution to the memory of Claude Itzykson, who would surely have appreciated the remarkable confluence of ideas that has marked recent advances in the theory that is still, misleadingly, called ‘string theory’.
2. IIB brane boundaries

I shall explain the ‘brane surgery’ method initially in the context of the IIB theory. Both IIA and IIB supergravity have in common the bosonic fields \( (g_{\mu\nu}, \phi, B_{\mu\nu}) \) from the NS \( \otimes \) NS sector, all of which have already made an appearance above. The remaining bosonic fields come from the R \( \otimes \) R sector. The (massless) R \( \otimes \) R fields of the IIB theory are

\[
(\ell, B'_{\mu\nu}, C^+_{\mu\nu\rho\sigma}) ,
\]

i.e. a pseudoscalar \( \ell \), another 2-form gauge potential \( B' \) and a 4-form gauge potential \( C^+ \) with a self-dual 5-form field strength \( D^+ \). The self-duality condition makes the construction of an action problematic but, as with the self-duality condition on the D=11 fivebrane’s worldvolume field strength \( F_3 \), one can choose to impose this condition after varying the action. When the IIB action is understood in this way it contains the CS term

\[
\int_{S^7} [\star H - D^+ \wedge B'] ,
\]

where \( H = dB \), as before, and \( H' = dB' \). This CS term modifies the \( B, B' \) and \( C^+ \) field equations.

Consider first the \( B \) equation. This becomes

\[
d\star H = -D^+ \wedge H' ,
\]

where \( D^+ = dC^+ \) is the self-dual 5-form field strength for \( C^+ \). This can be rewritten as

\[
d(\star H - D^+ \wedge B') = 0 .
\]

Since \( \star H \) is no longer a closed form its integral over a 7-sphere will no longer be homotopy invariant. Clearly, the well-defined, homotopy invariant, charge associated with the fundamental IIB string is not \( Q_1 \) as defined in [1] but rather

\[
\hat{Q}_1 = -\int_{S^5} D^+ \times \int_{S^2} B' .
\]

The \( S^5 \times S^2 \) integration region is illustrated schematically by the figure below:

Observe that the \( S^5 \) integral is just the definition of the charge \( Q_3 \) carried by a 3-brane, so the IIB string has its endpoint on a 3-brane; the \( S^2 \) in-
integration surface lies within the 3-brane and surrounds the string endpoint. Let us choose \( Q_3 = 1 \). If we further suppose that \( H' \equiv dB' = 0 \) within the 3-brane, which is reasonable in the absence of any D-string source for this field, then \( B' \) is a closed 2-form which we may write, locally, as \( B' = dV' \) for some 1-form \( V' \). Then

\[
\dot{Q}_1 = - \int_{S^2} dV' .
\]  

(20)

Effectively, \( V' \) is a field living on the worldvolume of the 3-brane. Clearly, it cannot be globally defined because the right hand side of (20) is a magnetic charge on the 3-brane associated with the vector potential \( V' \).

Now consider the \( B' \) equation. Taking the CS term (15) into account we have

\[
d \ast H = D^{+} \wedge H .
\]  

(21)

By the same reasoning as before we deduce that the D-string can end on a 3-brane. Charge conservation is satisfied because the D-string charge can be expressed as

\[
\dot{Q}_1' = Q_3 \times \int_{S^2} B .
\]  

(22)

Since there is no fundamental string source in the problem we may suppose that \( H = 0 \), so that now \( B \) is a closed 2-form which we may write, locally, as \( B = dV \). For \( Q_3 = 1 \) we now have

\[
\dot{Q}_1' = \int_{S^2} dV ,
\]  

(23)

so the D-string charge has been transferred to a magnetic charge of the 1-form potential \( V \) on the 3-brane’s worldvolume.

It must be regarded as a weakness of the above analysis that it does not supply the relation between \( V \) and \( V' \), although we know that there must be one because both supersymmetry and an analysis of the small fluctuations about the 3-brane solution show that there is only one worldvolume 1-form potential. In fact, \( V \) and \( V' \) are dual in the sense that

\[
dV' = \ast dV ,
\]  

(24)

where we recall that \( \ast \) indicates the worldvolume Hodge dual. Using this relation, (23) becomes

\[
\dot{Q}_1 = \int_{S^2} \ast dV ;
\]  

(25)

i.e. the endpoint of the IIB string on the 3-brane is an electric charge associated with \( V \). We thereby recover the D-brane picture for the IIB 3-brane; the fact that the D-string can end on the magnetic charge associated with \( V \) is then a consequence of the strong/weak coupling duality in IIB superstring theory: interchanging the fundamental string with the D-string. It will be seen from the examples to follow that the need to impose a condition of the type (24) is a general feature, which is not explained by the ‘brane surgery’ method. However, the method does determine whether a given p-brane can have a boundary and, if so, the possible q-branes in which the boundary must lie.

As a further illustration we now observe that whereas (14) was previously rewritten as (17), we could instead rewrite it as

\[
d(\ast H + H' \wedge C^+) = 0 .
\]  

(26)

Thus an equivalent definition of \( \dot{Q}_1 \) is

\[
\dot{Q}_1 = \int_{S^7} (\ast H + H' \wedge C^+) .
\]  

(27)

Proceeding as before, but now deforming the \( S^7 \) into the product \( S^3 \times S^4 \), we can express \( \dot{Q}_1 \) as

\[
\dot{Q}_1 = \int_{S^3} H' \times \int_{S^4} C^+ .
\]  

(28)

We recognise the first integral as the D-5-brane charge \( \dot{Q}_5' \). Setting \( Q_5' = 1 \) and \( D^+ = 0 \), we conclude that

\[
\dot{Q}_1 = \int_{S^4} dV_3 ,
\]  

(29)

where \( V_3 \) is a locally-defined 3-form field on the 5-brane worldvolume, which can be traded for a 1-form potential \( V \) via the relation

\[
dV_3 = \ast dV .
\]  

(30)

We conclude that the CS term allows the fundamental IIB string to end on a 5-brane as well as on a 3-brane, and that the end of the string is electrically charged with respect to a 1-form potential \( V \) living on the 5-brane’s worldvolume. This is just the usual picture of the D-5-brane. Interchanging the roles of \( B \) and \( B' \) leads to the
further possibility of the D-string ending on the solitonic 5-brane.

We have not yet exhausted the implications of the CS term \([13]\) because we have still to consider how it affects the \(C^+\) equation of motion. We find that

\[
d \star D^+ = - H \wedge H'
\]  (31)

or

\[
d(\star D^+ + H' \wedge B) = 0 .
\]  (32)

This means that the 3-brane charge should be modified to

\[
\tilde{Q}_3 = \int_{S^3} [\star D^+ + H' \wedge B] .
\]  (33)

This reduces to the previously-defined 3-brane charge \(Q_3\) if the 5-sphere surrounds a 3-brane sufficiently far from the boundary. As before the 5-sphere can be slid towards, and contracted onto, the boundary, after which it emerges as the product \(S^3 \times S^2\). Setting \(B = dV\) again we arrive at the expression

\[
\tilde{Q}_3 = \int_{S^3} H' \times \int_{S^2} dV
\]  (34)

for the 3-brane charge. The singularity involved in this deformation of the 7-sphere is now the 2-brane boundary of the 3-brane within a D-5-brane, since we recognize the first integral on the right hand side of \([34]\) as \(Q_5^\prime\). Setting \(Q_5^\prime = 1\) we learn that the 3-brane charge can be transferred to a magnetic charge of a D-5-brane worldvolume 1-form potential \(V\), defined by a 2-sphere in the D-5-brane surrounding the 2-brane boundary. The main point in all this is that a 3-brane can have a boundary in a D-5-brane, as pointed out in \([29]\). In fact, this possibility follows by T-duality from the previous results: the configuration of a D-string ending on a D-3-brane is mapped to a D-3-brane ending on a D-5-brane by T-duality in two directions orthogonal to both the D-string and the D-3-brane. By interchanging the roles of \(B\) and \(B'\) in the above analysis one sees that a 3-brane can also end on a solitonic 5-brane.

\footnote{The same equation follows, given the self-duality of \(D^+\), from the ‘modified’ Bianchi identity for \(D^+\).}

We have seen that the CS term \([13]\) allows a IIB string to end on a D-3-brane or a D-5-brane, but we know from string theory that it can also end on a D-string or a D-7-brane. As we shall see shortly, these possibilities are consequences of the fact that the kinetic term for \(H'\) actually has the form

\[
- \frac{1}{6} |H' - \ell H|^2 .
\]  (35)

There is no obvious relation to CS terms yet, but if we perform a duality transformation to replace the 2-form \(B'\) by its 6-form dual \(\tilde{B}'\) with 7-form field strength \(\tilde{H}'\), so that \textit{on shell}

\[
\tilde{H}' = \star H' ,
\]  (36)

then one finds that the dualized action contains the CS term

\[
\ell \tilde{H}' \wedge H .
\]  (37)

Clearly, this modifies the \(B\) equation so that, following the steps explained previously, we end up with an expression

\[
\tilde{Q}_1 = \int_{S^7} \tilde{H}' \times \int_{S^0} \ell .
\]  (38)

The first integral can be identified, using \([29]\), as the D-string charge. The final ‘integral’ over \(S^0 \equiv Z_2\) is just the difference between the value of \(\ell\) on either side of the string boundary on the D-string; by the same logic as before we may assume that \(d\ell = 0\), \textit{locally}, but allow the constant \(\ell\) to be different on either side. Thus, the charge \(Q_1\) on the fundamental IIB string is transformed into the topological charge of a type of ‘kink’ on the D-string.

Alternatively, we can deform \(S^7\) to \(S^1 \times S^6\), so that

\[
\tilde{Q}_1 = - \int_{S^1} d\ell \times \int_{S^6} \tilde{B}' .
\]  (39)

The first integral is the charge \(Q_7\) associated with the D-7-brane. This charge can be non-zero because of the periodic identification of \(\ell\) implied by the conjectured \(SL(2; \mathbb{Z})\) invariance of IIB superstring theory \([13]\). For \(Q_7 = 1\), and setting \(B' = dV_5'\) for 5-form potential \(V_5'\) (since we may assume that \(\tilde{H}' = 0\)), we have

\[
\tilde{Q}_1 = - \int_{S^6} dV_5' .
\]  (40)
Defining the 1-form $V$ on the 7-brane’s worldvolume by
\[ dV = \ast d\tilde{V}_5 \] (41)
we can rewrite (40) as
\[ \hat{Q}_1 = \int_{S^7} \ast dV \] (42)
We conclude that the IIB string may end on an electric charge in a 7-brane. This is just the description of the D-7-brane.

3. IIA boundaries

The ‘brane surgery’ method should now be clear. We shall now apply it to IIA supergravity, for which the $R \otimes R$ gauge potentials are
\[ (C_\mu, A_{\mu\nu\rho}) \] (43)
i.e. a 1-form $C$ and a 3-form $A$. We might start by considering the CS term
\[ F \wedge F \wedge B \] (44)
where $F$ is the 4-form field strength of $A$. Consideration of this term leads to the conclusion that (i) a IIA string can end on a 4-brane, and (ii) a 2-brane can end on either a 4-brane or a 5-brane. Since the CS term (44) is so obviously related to the similar one in D=11 to be considered below we shall pass over the details. The fact that the IIA string can also end on either a 2-brane or a 6-brane follows from the fact that the field strength $F$ has a ‘modified’ Bianchi identity
\[ dF = H \wedge K \] (45)
where $K = dC$ is the field strength of $C$ (this has a Kaluza-Klein origin in D=11). We can dualize $A$ to convert this modified Bianchi into a CS term of the form\[ \tilde{F} \wedge K \wedge B \] (46)
where the 6-form $\tilde{F}$ is, on-shell, the Hodge dual of $F$. This modifies the $B$ equation to
\[ d \ast H = -\tilde{F} \wedge K \] (47)
We may therefore take the modified charge $\hat{Q}_1$ to be
\[ \hat{Q}_1 = \int_{S^7} [\ast H + \tilde{F} \wedge C] \] (48)
Now, by the identical reasoning used in the IIB case, we first deform the 7-sphere so as to arrive at the formula
\[ \hat{Q}_1 = \int_{S^6} \tilde{F} \times \int_{S^1} C \] (49)
We then identify the first integral as the charge $Q_2$ of a membrane. We then set $Q_2 = 1$ and $C = d\gamma$ for some scalar $\gamma$ defined locally on the worldvolume of the membrane to conclude that the IIA string can end on a membrane, with the string’s charge now being transferred to the magnetic-type charge
\[ \int_{S^1} d\gamma \] (50)
of a particle on the membrane\[ [[30]] \]. This charge can be non-zero if $\gamma$ is periodically identified. Clearly, from the KK origin of $C$, we should interpret $\gamma$ as the coordinate of a hidden 11th dimension. Defining the worldvolume 1-form $V$ by
\[ dV = \ast d\gamma \] (51)
we recover\[ [[31, 37, 32]] \] the usual description of the IIA D-2-brane, in which the end of the string on the membrane carries the electric charge
\[ \int_{S^1} \ast dV \] (52)
Returning to (47) we can alternatively define the modified string charge to be
\[ \hat{Q}_1 = \int_{S^7} [\ast H + K \wedge \hat{A}] \] (53)
where $\hat{A}$ is the 5-form potential associated with $\tilde{F}$, i.e. $\tilde{F} = d\hat{A}$. Since
\[ Q_6 = \int_{S^2} K \] (54)
is the 6-brane charge, similar reasoning to that above, but now setting $\hat{A} = d\tilde{V}_4$, leads to the
conclusion that a IIA string can also end on a 6-brane and that the string charge is transferred to the 6-brane magnetic charge
\[ \int_{S^5} d\tilde{V}_4 , \] (55)
which can be rewritten in the expected electric charge form
\[ \int_{S^5} s dV \] (56)
by introducing the worldvolume 1-form potential V dual to \( \tilde{V}_4 \).

The remaining IIA D-branes are the 0-brane and the 8-brane. The possibility of a IIA string ending on a 0-brane is not found by the ‘brane surgery’ method for the good reason that it is actually forbidden by charge conservation unless the 0-brane is the endpoint of two or more strings. Thus, a modification of the method will be needed to deal with this case. Neither is it clear how the method can cope with the IIA 8-brane, because of the non-generic peculiarities of this case.

Leaving aside these limitations of the method, there are further consequences to be deduced from the CS term (46). We have still to consider its effect on the \( \tilde{A} \) equation of motion. Actually it is easier to return to the modified Bianchi identity (45), which we can rewrite as
\[ d(F - K \wedge B) = 0 . \] (57)
This shows that the homotopy-invariant magnetic 4-brane charge is actually
\[ \dot{Q}_4 = \int_{S^4} [F - K \wedge B] . \] (58)
By the now familiar reasoning we deform the 4-sphere and set \( H = 0 \) to arrive at
\[ \dot{Q}_4 = - \int_{S^2} K \times \int_{S^2} dV . \] (59)
We recognise the first integral as the charge \( Q_6 \) of a 6-brane. The second integral is the magnetic charge associated with a 3-brane within the 6-brane. The 3-brane is of course the 4-brane’s boundary. Thus a 4-brane can end on a 6-brane. This is not unexpected because it follows by T-duality from the fact that a IIB 3-brane can end on a D-5-brane.

We could as well have rewritten the modified Bianchi identity (47) as
\[ d(F + H \wedge C) = 0 , \] (60)
in which case a similar line of reasoning, but setting \( K = 0 \), and so \( C = dy \), leads to the expression
\[ \dot{Q}_4 = \int_{S^3} H \times \int_{S^1} dy . \] (61)
The first integral is the magnetic 5-brane charge \( Q_5 \), so we deduce that a 4-brane can also end on a (solitonic) 5-brane. The 3-brane boundary in the 5-brane is a magnetic source for the scalar field \( y \). The KK origin of \( y \) suggests a D=11 interpretation of this possibility. It is surely closely related to the fact that two D=11 fivebranes can intersect on a 3-brane \[33\], since by wrapping one of the 5-branes (but not the other one) around the 11th dimension we arrive at a D=4-brane intersecting a solitonic 5-brane in a 3-brane. This is not quite the same as a D=4-brane \( \text{ending} \) on a 5-brane, but the intersection could be viewed as two 4-branes which happen to end on a common 3-brane boundary in the 5-brane. This illustrates a close connection between the ‘brane boundary’ rules and the ‘brane intersection rules’, which will not be discussed here.

4. M-brane boundaries

Finally, we turn to M-theory, or rather D=11 supergravity and its p-brane solutions. The bosonic fields of D=11 supergravity are the 11-metric and a 3-form gauge potential \( A \) with 4-form field strength \( F = dA \). The Bianchi identity for \( F \) is
\[ dF = 0 , \] (62)
from which we may immediately conclude that the D=11 fivebrane must be closed. The same is not true of the D=11 membrane, however, because there is a CS term in the action of the form
\[ \frac{1}{3} F \wedge F \wedge A \] (63)
which leads to the following field equation\[9\]
\[ d \star F = - F \wedge F . \] (64)
\[9\] An additional singular 5-brane source term was included in \[14\] leading to a rather different interpretation of the
We see that the well-defined membrane charge is actually
\[ \hat{Q}_2 = \int_{S^7} [\star F + F \wedge A]. \] (65)

Now consider a membrane with a boundary. Contract the 7-sphere to the boundary and deform it to the product $S^4 \times S^3$ so that the entire contribution to $\hat{Q}_2$ is given by
\[ \hat{Q}_2 = \int_{S^4} F \times \int_{S^3} A. \] (66)

The first integral is the charge $Q_5$ associated with a fivebrane. Set $Q_5 = 1$. We may also set to zero the components of $F$ ‘parallel’ to the fivebrane, so that $A = dV_2$ in the second integral. We then have
\[ \hat{Q}_2 = \int_{S^3} dV_2, \] (67)

which is the magnetic charge of the string boundary of the membrane in the fivebrane.

In fact, the 3-form field-strength $F_3 = dV_2$ (or rather $F_3 = dV_2 - A$ in a general background) is self-dual but we do not learn this fact from the ‘brane surgery’ method. As for the IIB 3-brane, where we saw that the worldvolume 1-forms $V$ and $\tilde{V}$ are related by Hodge duality of their 2-form field strengths, this information must be gleaned from a different analysis. The similarity between these constraints on the worldvolume gauge fields suggests that a deeper understanding of the phenomenon should be possible.

In this contribution I have discussed the rules governing ‘brane boundaries’ in superstring and M-theory and shown that they follow from consideration of interactions in the corresponding effective supergravity theories. It should be appreciated that brane boundaries constitute a subset of possible ‘brane interactions’, which include intersecting branes and branes of varying topologies. A reasonably complete picture is now emerging of the static aspects of brane interactions, but little is known at present about the dynamic aspects, i.e. the analogue of the splitting and joining interaction in string theory. This problem is presumably bound up with the problem of finding an intrinsic definition of M-theory, which may well require a substantially new conceptual framework. Hopefully, the current focus on branes will prove to be of some help in this daunting task.

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