HOLE PROBABILITY FOR ENTIRE FUNCTIONS
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ABSTRACT. Consider the Gaussian entire function

\[ f(z) = \sum_{n=0}^{\infty} \xi_n a_n z^n, \]

where \( \{\xi_n\} \) is a sequence of independent and identically distributed standard complex Gaussians and \( \{a_n\} \) is some sequence of non-negative coefficients, with \( a_0 > 0 \). We study the asymptotics (for large values of \( r \)) of the hole probability for \( f(z) \), that is the probability \( P_H(r) \) that \( f(z) \) has no zeros in the disk \(|z| < r\). We prove that

\[ \log P_H(r) = -S(r) + o(S(r)), \]

where

\[ S(r) = 2 \cdot \sum_{n \geq 0} \log^+ (a_n r^n), \]

as \( r \) tends to \( \infty \) outside a deterministic exceptional set of finite logarithmic measure.

1. INTRODUCTION

We study entire functions of the form

\[ f(z) = \sum_{n=0}^{\infty} \xi_n a_n z^n, \]

where \( \{\xi_n\} \) are independent standard complex Gaussians and \( \{a_n\} \) are (non-negative) deterministic coefficients, with \( a_0 > 0 \). It is well known (see [BKPV][Ka]) that, almost surely, the series (1.1) has an infinite radius of convergence if and only if the (non-random) Taylor series \( \sum a_n z^n \) has infinite radius of convergence, that is

\[ \lim_{n \to \infty} \frac{\log a_n}{n} = -\infty. \]

We use the following notation

\[ S(r) = 2 \cdot \sum_{n \geq 0} \log^+ (a_n r^n) \]

for the weight of the 'important' coefficients.

We study the probability of the event where the function \( f(z) \) has no zeros inside \( \{ |z| < r \} \):

\[ p_H(r) = -\log P(f(z) \neq 0 \text{ inside } |z| < r), \]

We derive the asymptotics of \( p_H(r) \) for large values of \( r \).

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Theorem 1. There exists an exceptional set $E \subset [1, \infty)$ which depends only on the coefficients $a_n$ such that $\int_E \frac{dt}{t} < \infty$. For $r \to \infty$ not belonging to the set $E$

$$p_H(r) = S(r) + o(S(r)),$$  \hspace{1cm} (1.2)

In the case where the coefficients $a_n$ satisfy some regularity conditions, this result was proved in [Ni2]. The general case obtained here uses ideas that were introduced in [Ni2, ST], as well as some new ingredients, including accurate lower bounds for determinants of some large covariance matrices, which might be of independent interest. Some exceptional set $E$ in Theorem 1 seems to be unavoidable if we do not require additional regularity conditions from the coefficients $a_n$. In many cases, it can be dropped and the error term in (1.2) can be shown to be of order $O\left(\sqrt{S(r) \log S(r)}\right)$, see Section 2.6 below.

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2. Preliminaries

2.1. Notation and assumptions. We denote by $r\mathbb{D}$ the disk $\{z \mid |z| \leq r\}$ and by $r\mathbb{T}$ its boundary $\{z \mid |z| = r\}$, with $r \geq 1$. The letters $c$ and $C$ denote positive absolute constants (which can change across lines). We also use the standard notation:

$$M(r) = \max_{z \in r\mathbb{D}} |f(z)|.$$

In order to simplify some of the expressions in the paper, we will assume from now on that

$$a_0 = 1.$$

2.2. A growth lemma by Hayman. A set $E \subset \mathbb{R}^+$ is of finite logarithmic measure (FILM) if

$$\int_E \frac{dt}{t} < \infty.$$

The following lemma is a general lemma about the growth of functions (taken from [Hay]).

Lemma 2. Suppose that $N(r)$ is a positive increasing function of $r$ for $r \geq r_0$. Then if $\alpha > 0$, and $|h| < N(r)^{-\alpha}$, we have

$$|N(re^h) - N(r)| < \alpha N(r),$$

for all $r$ outside a set of FILM.

2.3. Further notations, definition of the exceptional set. We use the following notations:

$$b_n = \begin{cases} \frac{\log a_n}{n} + \log r, & a_n > 0, \\ -\infty, & a_n = 0, \end{cases}$$

for the regularized coefficients, and

$$N(r) = \{n \mid b_n \geq 0\},$$

$$S(r) = 2 \cdot \sum_{n \in N(r)} \log (a_n r^n) = 2 \cdot \sum_{n \in N(r)} nb_n.$$
Since the coefficients $a_n$ satisfy $\frac{\log a_n}{n} \to -\infty$, we have that $b_n \to -\infty$ as $n \to \infty$ and the set $N(r)$ is finite for every $r \in \mathbb{R}$.

We now write

\[
\begin{align*}
n (r) &= \# N(r), \\
m (r) &= \sum_{n \in N(r)} n, \\
N_\delta (r) &= \{ n \mid b_n (r) \geq -\delta \}, \\
n_\delta (r) &= \# N_\delta (r).
\end{align*}
\]

Note that $b_n (r)$ is increasing with $r$ and therefore $n (r)$ and $m (r)$ are increasing functions of $r$. Also notice the following relations:

\[
(2.1) \quad N_\delta (r) \subset N_0 (r) = N (r), \\
N_\delta (r) = N (re^{-\delta}).
\]

Applying Lemma 2 to $m (r)$, with $\delta = m^{-1/4} (r)$,

we have that outside an exceptional set of FILM:

\[
(2.2) \quad m (re^{-\delta}) > \frac{3}{4} \cdot m (r),
\]

\[
\frac{5}{4} \cdot m (r).
\]

We will use the term normal for non-exceptional values of $r$ (the choice of $\delta$ remains the same throughout the paper). In general if the inequalities (2.2) hold for all large values of $r$, then there is no exceptional set in Theorem 1.

2.4. Estimates for $S(r)$. First we find relations between $S (r)$ and the functions $m (r), n (r)$ which appear in the error terms for the hole probability.

**Lemma 3.** For normal values of $r$ we have

\[
S(r) \geq \frac{3}{2} \cdot (m (r))^{3/4} \geq c \cdot n (r)^{3/2}.
\]

**Proof.** First notice that

\[
(2.3) \quad m (r) = \sum_{n \in N(r)} n \geq c \cdot (n (r))^2,
\]

since $m (r)$ is minimal when $N (r) = \{0, 1, \ldots, n (r) - 1\}$. Now, using the relation (2.1) between $N (r)$ and $N_\delta (r)$, we get:

\[
\begin{align*}
\frac{S (r)}{2} &= \sum_{n \in N(r)} nb_n (r) \geq \sum_{n \in N_\delta (r)} nb_n (r) \\
&\geq \sum_{n \in N(re^{-\delta})} n\delta \geq \delta \cdot m (re^{-\delta}) \geq \\
&\geq \frac{3}{4} \cdot \frac{m (r)}{(m (r))^{1/4}} \geq c \cdot n (r)^{3/2}.
\end{align*}
\]
We now estimate the rate of growth of the function \( S(r) \).

**Lemma 4.** For \( \gamma \in (0, \frac{1}{2}) \) we have,
\[
S((1 - \gamma) r) \geq S(r) - 4\gamma \cdot m(r).
\]

**Proof.** Write \( r' = (1 - \gamma) r \) and notice that for \( \gamma < \frac{1}{2} \) we have
\[
\log(1 - \gamma) \geq -\gamma - \frac{\gamma^2}{2} \geq -2\gamma.
\]
It follows that (since \( N(r') \subset N(r) \))
\[
\frac{S(r) - S(r')}{2} = \sum_{n \in N(r) \setminus N(r')} \log(a_n r^n) + \sum_{n \in N(r')} \{\log(a_n r^n) - \log(a_n (r')^n)\}
\]
\[
= \sum_{n \in N(r) \setminus N(r')} \log(a_n r^n) + \sum_{n \in N(r')} n \cdot \log\left(\frac{1}{1 - \gamma}\right)
\]
\[
= \sum_1 + \sum_2.
\]
For the first sum we notice that if \( n \in N(r) \setminus N(r') \) then
\[
0 \geq \log(a_n (r')^n) \geq \log(a_n r^n) - 2n\gamma
\]
\[
\downarrow
\]
\[
\log(a_n r^n) \leq 2n\gamma
\]
and so
\[
\sum_1 \leq 2\gamma \cdot \sum_{n \in N(r) \setminus N(r')} n \leq 2\gamma \cdot [m(r) - m(r')]
\]
For the second sum we have
\[
\sum_2 \leq 2\gamma \cdot m(r').
\]
Overall we have
\[
S(r) - S(r') \leq 4\gamma \cdot m(r).
\]
\(\square\)

2.5. **Gaussian Distributions.** Many times we use the fact that if \( a \) has a \( N \mathcal{C}(0, 1) \) distribution, then
\[
P(|a| \geq \lambda) = \exp(-\lambda^2),
\]
and for \( \lambda \leq 1, \)
\[
P(|a| \leq \lambda) \in \left[\frac{\lambda^2}{2}, \lambda^2\right].
\]

2.6. **Strategy of the proof.** We had the following

**Definition.** The value \( r \geq 1 \) is normal if
\[
cm(r) \leq m(re^h) \leq Cm(r), \quad |h| \leq (m(r))^{-1/4}.
\]
We show that for normal values of \( r \)
\[
p_H(r) \leq S(r) + C\sqrt{m(r)} \log m(r)
\]
(Proposition 5) and also that
\[
p_H(r) \geq S(r) - Cn(r) \log S(r)
\]
Combining these bounds with Lemma 3, we get Theorem 1 (with an error term $O\left(S(r)^{2/3+\epsilon}\right)$). If the coefficients $a_n$ are such that each value $r \geq 1$ is normal, then the exceptional set $E$ in Theorem 1 is not needed. In some cases, where the coefficients $a_n$ have a regular asymptotic behaviour (for instance $a_n = \frac{1}{\sqrt{n}}$, or more generally, $a_n \sim \frac{1}{\Gamma(\alpha n + 1)}$, with some $\alpha > 0$), then $m(r) \leq CS(r)$ and we obtain

$$p_H(r) = S(r) + O\left(\sqrt{S(r) \log S(r)}\right), \quad r \to \infty.$$ 

### 3. Upper Bound for $p_H(r)$

In this section we prove the following Proposition 5.

**Proposition 5.** For normal values of $r$, we have

$$p_H(r) \leq S(r) + C \cdot \sqrt{m(r) \log m(r)},$$

with $C$ some positive absolute constant.

**Remark.** We note that $r$ is assumed to be large.

The simplest case where $f(z)$ has no zeros inside $r\mathbb{D}$ is when the constant term dominates all the others. We therefore study the event $\Omega_r$, which is the intersection of the events (i),(ii) and (iii), ($C$ will be selected in an appropriate way)

- (i) : $|\xi_0| \geq C (m(r))^{1/4}$,
- (ii) : $\bigcap_{n \in N(r) \setminus \{0\}} (\text{ii})_n$,
- (iii) : $\bigcap_{n \in \tilde{N}_\delta(r) \cap N(r)} (\text{iii})_n$,
- (iv) : $\bigcap_{n \in (\tilde{N}_\delta(r))^c} (\text{iv})_n$,

where $\tilde{N}_\delta(r) = N_\delta(r) \cup \left\{ n \mid n < \sqrt{m(r)} \right\}$ and

- (ii)$_n$ : $|\xi_n| \leq \frac{(a_n r^n)^{-1}}{\sqrt{m(r)}}$,
- (iii)$_n$ : $|\xi_n| \leq \frac{1}{\sqrt{m(r)}}$,
- (iv)$_n$ : $|\xi_n| \leq e^{\frac{2}{\delta}}$.

We notice that by (2.2) and (2.3) we have that $\# \tilde{N}_\delta(r) \leq C \sqrt{m(r)}$, for $r$ large enough.

**Lemma 6.** If $\Omega_r$ holds for a normal value of $r$, then $f$ has no zeros inside $r\mathbb{D}$.

**Proof.** We remind that $\delta = m(r)^{-1/4}$. To see that $f(z)$ has no zeros inside $r\mathbb{D}$ we note that

$$|f(z)| \geq |\xi_0| - \sum_{n=1}^{\infty} |\xi_n| a_n r^n.$$ 

\footnote{In that case we get $p_H(r) = \frac{\epsilon^2 r^4}{4} + O(r^2 \log r)$. Due to a computational error, in [N1] we gave the main term in this asymptotics with the extra factor 3.}
First, we estimate the sum over the terms in \( N(r) \setminus \{0\} \),
\[
\sum_{n \in N(r) \setminus \{0\}} |\xi_n| a_n r^n \leq \sum_{n \in N(r)} \frac{1}{\sqrt{m(r)}} \leq c_1,
\]
by (2.3). Second, we estimate the sum over the terms in \( \tilde{N}_q (r) \setminus N(r) \), (notice that here \( b_n \leq 0 \) and \( a_n r^n = e^{\log(a_n r^n)} = e^{nb_n} \))
\[
\sum_{n \in \tilde{N}_q (r) \setminus N(r)} |\xi_n| a_n r^n = \sum_{n \in \tilde{N}_q (r) \setminus N(r)} |\xi_n| e^{nb_n} \leq \sum_{n \in \tilde{N}_q (r) \setminus N(r)} \frac{1}{\sqrt{m(r)}} \leq c_2,
\]
using (2.2). Now the rest of the tail is bounded by: (here \( b_n \leq -\delta \))
\[
\sum_{n \in (\tilde{N}_q (r))^c} |\xi_n| a_n r^n = \sum_{n \in (\tilde{N}_q (r))^c} |\xi_n| e^{nb_n} \leq \sum_{n \in (\tilde{N}_q (r))^c} e^{\frac{4n}{\delta}}e^{-\delta n} \leq \sum_{n \geq 0} e^{-\frac{4n}{\delta}} \leq \frac{3}{\delta} \leq 3 \cdot (m(r))^{1/4} .
\]
So overall from (3.1)
\[
|f(z)| > C (m(r))^{1/4} - c_1 - c_2 - 3 (m(r))^{1/4},
\]
if we select \( C = 4 \) then for \( r \) large enough we have that \( f(z) \neq 0 \) inside \( r \mathbb{D} \). \( \square \)

**Lemma 7.** The probability of the event \( \Omega_r \) is bounded from below by
\[
\log \mathbb{P} (\Omega_r) \geq - S(r) - C \cdot \sqrt{m(r)} \log m(r),
\]
for normal values of \( r \) which are large enough.

**Proof.** By the properties of \( \xi_n \) (see (2.5)) we get
\[
\mathbb{P} ((i)) = \exp \left( -C^2 \cdot \sqrt{m(r)} \right)
\]
and
\[
\mathbb{P} ((ii)_{n}) \geq \frac{(a_n r^n)^{-2}}{2m(r)},
\]
therefore
\[
\mathbb{P} ((ii)) \geq \prod_{n \in N(r)} \frac{(a_n r^n)^{-2}}{2m(r)} = \prod_{n \in N(r)} e^{-2nb_n} \cdot \exp (-n (r \log (2m(r)))) \geq \exp \left( -2 \cdot \sum_{n \in N(r)} nb_n \right) \cdot \exp (-Cn (r \log m(r)) \geq \exp \left( -S(r) - C \sqrt{m(r)} \log m(r) \right).
\]
Similarly we have
\[
\mathbb{P} ((iii)_{n}) \geq \frac{1}{2m(r)}
\]
and so (by (2.2) and (2.3))
\[
\mathbb{P} ((iii)) \geq \exp \left( -C \sqrt{m(r)} \log m(r) \right).
\]
Finally we have

\[ P((iv)_n) = \exp(-e^{\delta n}) , \]

we use the following inequality (for some positive sequence \( \{A_n\} \))

\[ P(\forall n : |\xi_n| \leq A_n) = 1 - P(\exists n : |\xi_n| > A_n) \geq 1 - \sum P(|\xi_n| > A_n). \]

Now

\[
P((iv)) \geq 1 - \sum_{n \in (\tilde{N}(r))^c} \exp(-e^{\delta n}) \geq 1 - \sum_{n \geq \sqrt{m(r)}} \exp(-e^{\delta n}) \geq 1 - \sum_{n \geq \sqrt{m(r)}} \exp(-\delta n) \geq 1 - \frac{C}{\delta} \exp\left(-\frac{1}{\delta}\right) \geq \frac{1}{2} .
\]

for \( r \) large enough (since \( \sqrt{m(r)} = \frac{1}{\sqrt{2}} \)). Since all the events are independent, in total the probability is bounded by:

\[
P(\Omega_r) = P((i)) \cdot P((ii)) \cdot P((iii)) \cdot P((iv)) \geq \exp\left(-S(r) - C\sqrt{m(r)}\log m(r)\right). \]

Proposition 5 now follows from the previous lemmas.

4. Bounds for Gaussian Entire Functions

In this section we get some bounds for the modulus and logarithmic derivative of Gaussian entire functions, which hold with high probability (we use this term for events where the exceptional set is of small probability in relation to the hole probability). These results will be used in the next section in the proof of the lower bound.

4.1. Bounds on the modulus of Gaussian entire functions. We first bound the probability of the events when \( M(r) \) is relatively large or small.

**Lemma 8.** For normal values of \( r \) which are large enough, we have

\[
P\left(M(r) \geq e^{3S(r)}\right) \leq e^{-S^2(r)}. \]  

**Proof.** We use the notation \( \tilde{N}(r) = N_3(r) \cup \{n < S^2(r)\} \). The proof is similar to the one of Proposition 5. We define the following event \( E \), which is the intersection of the events (i) and (ii)

(i) : \( \bigcap_{n \in N(r)} (i)_n \),

(ii) : \( \bigcap_{n \in (\tilde{N}(r))^c} (ii)_n \).
and

\[(i)_n : \quad |ξ_n| \leq (a_n r^n)^{-1} e^{2S(r)}, \]

\[(ii)_n : \quad |ξ_n| \leq \exp \left( \frac{1}{2} δn \right). \]

We have the following estimate for \( M(r) \):

\[|f(z)| \leq \sum_{n \in \mathbb{N}(r)} |ξ_n| a_n r^n + \sum_{n \in (\mathbb{N}(r))^c} |ξ_n| a_n r^n \leq \]

\[\leq \# \mathbb{N}(r) \cdot e^{2S(r)} + \sum_{n \geq S^2(r)} e^{\frac{\delta n}{2}} \cdot e^{-δn} \leq \]

\[\leq C \cdot S^2(r) \cdot e^{2S(r)} + \frac{C}{δ} \cdot e^{-\frac{S^2(r)}{2}} \leq \]

\[\leq e^{3S(r)}, \]

for \( r \) large enough.

Now we estimate the probability of the complement of \( E \). We have:

\[P \left( |ξ_n| \geq \frac{e^{2S(r)}}{a_n r^n} \right) = \exp \left( \frac{e^{4S(r)}}{(a_n r^n)^2} \right) \leq \exp \left( -e^{2S(r)} \right), \]

\[P \left( |ξ_n| \geq e^{\frac{\delta n}{2}} \right) = \exp \left( -\exp (\delta n) \right). \]

By the union bound:

\[P (\bar{(i)}) \leq C S^2(r) \cdot \exp \left( -\exp (2S(r)) \right), \]

\[P (\bar{(ii)}) \leq \sum_{n \geq S^2(r)} \exp \left( -\exp (\delta n) \right) \leq \]

\[\leq C \cdot \exp \left( -\exp (S(r)) \right). \]

So overall

\[P \left( M(r) \geq e^{3S(r)} \right) \leq \exp \left( -S^2(r) \right). \]

□

In the other direction we have the following

**Lemma 9.** The probability of deviation from the lower bound can be bounded by

\[P (M(r) \leq \exp (-S(r))) \leq \exp (-S(r) \cdot n(r)). \]

**Proof.** By Cauchy’s estimate:

\[|ξ_n| a_n r^n \leq M(r) \leq e^{-S(r)}. \]

For \( n \in \mathbb{N}(r) \) we have

\[P \left( |ξ_n| \leq (a_n r^n)^{-1} e^{-S(r)} \right) \leq e^{-2S(r)}, \]

and we get

\[P \left( M(r) \leq \exp (-S(r)) \right) \leq \prod_{n \in \mathbb{N}(r)} e^{-2S(r)} \leq \exp (-S(r) \cdot n(r)). \]

□

Notice that there are no assumptions on (the normality of) \( r \).
4.2. **Bounds for the logarithmic derivative.** In this section we assume that $f(z) \neq 0$ inside $R \mathbb{D}$, and therefore $\log |f|$ is harmonic there. First we find a bound for the average value of $|\log |f||$. $m$ denotes the normalized angular measure on $r \mathbb{T}$.

Under these conditions we have

**Lemma 10.** For normal values of $R$ and outside an exceptional set of probability at most

$$2 \cdot \exp (-S(r) \cdot n(r)),$$

we have

$$\int_{R \mathbb{T}} |\log |f|| \, dm \leq C \left(1 - \frac{r}{R}\right)^{-2} \cdot S(R).$$

**Proof.** Denote by $P_j(z) = P(z, z_j)$ the Poisson kernel for the disk $R \mathbb{D}$, $|z| = R$, $|z_j| < R$. Using Lemma 9, we may suppose that there is a point $a \in r \mathbb{T}$ such that $\log |f(a)| \geq -S(r)$ (discarding an exceptional event of probability at most $\exp (-S(r) \cdot n(r))$). Then we have

$$-S(r) \leq \int_{R \mathbb{T}} P(z, a) \log |f(z)| \, dm(z),$$

and hence

$$\int_{R \mathbb{T}} P(z, a) \log^{-} |f(z)| \, dm(z) \leq \int_{R \mathbb{T}} P(z, a) \log^{+} |f(z)| \, dm(z) + S(r).$$

For $|z| = R$ and $|a| = r$ we have,

$$\frac{R - r}{2R} \leq \frac{R - r}{R + r} \leq P(z, a) \leq \frac{R + r}{R - r} \leq \frac{2R}{R - r}.$$

By Lemma 8 outside a very small exception set (of the order $e^{-S^2(R)}$), we have $\log M(R) \leq 3 \cdot S(R)$. Therefore

$$\int_{R \mathbb{T}} \log^{+} |f| \, d\mu \leq 3 \cdot S(R).$$

Now we have

$$\int_{R \mathbb{T}} \log^{-} |f| \, d\mu \leq \frac{2R}{R - r} \cdot S(r) + \frac{12R^2}{(R - r)^2} \cdot S(R).$$

Finally we get

$$(4.1) \quad \int_{R \mathbb{T}} |\log |f|| \, d\mu \leq \frac{CR^2}{(R - r)^2} \cdot S(R) \leq C \left(1 - \frac{r}{R}\right)^{-2} \cdot S(R).$$

$\square$
Now we find an upper bound for the (angular) logarithmic derivative of $\log |f|$ inside $R \mathbb{D}$.

**Lemma 11.** Let $r < R$, then
\[
\left| \frac{d \log |f(re^{i\phi})|}{d\phi} \right| \leq C \left( 1 - \frac{r}{R} \right)^{-5} \cdot S(R),
\]
for normal values of $R$ and outside an exceptional set of probability at most
\[
2 \cdot \exp \left(-S(r) \cdot n(r)\right).
\]

**Proof.** We start with the equation
\[
\log |f(re^{i\phi})| = \int_0^{2\pi} \frac{R^2 - r^2}{|Re^{i\theta} - re^{i\phi}|^2} \cdot \log |f(Re^{i\theta})| \frac{d\theta}{2\pi},
\]
so taking the derivative under the integral we get:
\[
\frac{d \log |f(re^{i\phi})|}{d\phi} = \int_0^{2\pi} \frac{rR (R^2 - r^2) \sin (\theta - \phi)}{|Re^{i\theta} - re^{i\phi}|^4} \cdot \log |f(Re^{i\theta})| \frac{d\theta}{2\pi},
\]
taking absolute value:
\[
\left| \frac{d \log |f(re^{i\phi})|}{d\phi} \right| \leq \frac{C (R + r)}{(R - r)^3} \int_0^{2\pi} \left| \log |f(Re^{i\theta})| \right| \frac{d\theta}{2\pi},
\]
using the previous lemma we get the required result. $\square$

5. **Lower Bound for $p_H(r)$**

In order to find the lower bound for $p_H(r)$ we now assume that $f(z) \neq 0$ inside $r \mathbb{D}$, for normal value of $r$. We choose $\rho < r$, and write $\gamma = 1 - \frac{\rho}{r}$, where $\gamma$ will be small, depending on $r$.

The function $\log |f(z)|$ is harmonic in $r \mathbb{D}$. Therefore for $\rho < r$
\[
\log |f(0)| = \int_0^{2\pi} \log |f(\rho e^{i\alpha})| \frac{d\alpha}{2\pi}.
\]

Now, if we select $n$ points $z_j = \rho e^{i\theta_j}$, we have:
\[
\int_0^{2\pi} \frac{1}{n} \sum_{j=1}^{n} \log |f(\rho e^{i\theta_j} \cdot e^{i\alpha})| \frac{d\alpha}{2\pi} = \frac{1}{n} \sum_{j=1}^{n} \int_0^{2\pi} \log |f(\rho e^{i\theta_j} \cdot e^{i\alpha})| \frac{d\alpha}{2\pi} =
\]
\[
= \frac{1}{n} \sum_{j=1}^{n} \log |f(0)| = \log |f(0)|.
\]

Since $\log |f(z)|$ is continuous inside $r \mathbb{D}$, we conclude that there exists some $\alpha^*$ such that
\[
\frac{1}{n} \sum_{j=1}^{n} \log \left| f(\rho e^{i\theta_j} \cdot e^{i\alpha^*}) \right| = \log |f(0)|.
\]
By Lemma 11 the logarithmic derivative of $f$ is not too large with high probability. Therefore, if $\alpha$ satifies
\[ |\alpha - \alpha^*| < \Delta \alpha = \frac{c \gamma^5}{S(r)} \]
then:
\[ \frac{1}{n} \sum_{j=1}^{n} \log |f(\rho e^{i\theta_j} \cdot e^{i\alpha})| \leq \log |f(0)| + 1. \]

In this section we will show that the probability of the previous event is very small. In particular we prove

**Proposition 12.** For normal values of $r$, we have
\[ p_H(r) \geq S(r) - C n (r) \log S(r), \]
with $C$ some positive absolute constant.

5.1. **Reduction to an estimate of a multivariate Gaussian event.** In this section we reduce the problem to an estimate of a probability of an event with multivariate (complex) Gaussian distribution. We first note that we work in the product space $\{(\alpha, \omega) \in [0, 2\pi] \times \Omega\}$, where $\alpha$ is chosen uniformly in $[0, 2\pi]$ and $\Omega$ is the probability space for our Gaussian entire function $f$ (we denote the probability measures by $m$ and $\mu$, respectively). We define the following events (all depend on $r$):

- $H = \{(\alpha, \omega) \mid f(z) \neq 0 \text{ in } r\mathbb{D}\}$
  - The hole event.
- $L = \{(\alpha, \omega) \mid \left| \frac{d\log|f(\rho e^{i\phi})|}{d\phi} \right| \leq C \cdot (1 - \rho/\rho_0)^{-5} \cdot S(r), \forall \phi \in [0, 2\pi]\}$
  - The non-exceptional event of Lemma 11 w.r.t $r$ and $\rho$
- $C = \{(\alpha, \omega) \mid \frac{1}{n} \sum \log |f(z_j e^{i\alpha})| \leq \log |f(0)| + 1\}$
- $D = \{(\alpha, \omega) \mid |\alpha - \alpha^* (\omega)| < \Delta \alpha\}$

We note that $\alpha^*$ is measurable with respect to $\Omega$, also the events $H$ and $L$ do not depend on the choice of $\alpha$.

By the previous discussion, we have:
\[ 1_{C}(\alpha, \omega) \geq 1_{H \cap L \cap D}(\alpha, \omega) = 1_{H \cap L}(\alpha, \omega) \cdot 1_{D}(\alpha, \omega) = 1_{H \cap L}(\omega) \cdot 1_{D}(\alpha, \omega), \]
where $1_A(\alpha, \omega)$ is the indicator function of the event $A$. We now have by Fubini
\[ P(C) = \int \int 1_{C}(\alpha, \omega) \, dm(\alpha) \, d\mu(\omega) \geq \int \int 1_{H \cap L}(\omega) \, dm(\alpha) \, d\mu(\omega) = 2\Delta \alpha \cdot \int 1_{H \cap L}(\omega) \, d\mu(\omega) \geq \Delta \alpha \cdot P(H \cap L) \geq \Delta \alpha \cdot (P(H) - P(L^c)), \]
and so
\[ P(H) \leq \frac{1}{\Delta \alpha} \cdot P(C) + P(L^c). \]
We now use the following events:

- $A = \{(\alpha, \omega) \mid \log |f(0)| \leq \log S(r)\}$
- $B = \{(\alpha, \omega) \mid M(r) = \max_{z \in rD} |f(z)| \leq e^{3S(r)}\}$

To estimate $P(C)$ from above we use:

$$P(C) \leq P(A \cap B \cap C) + P(A^c) + P(B^c).$$

By (2.4) and Lemma 8, we have

$$P(A^c) \leq e^{-S^2(r)},$$
$$P(B^c) \leq e^{-cS^2(r)},$$

note that these probabilities are very small with respect to the main term.

In the next section, we will show that for a good selection of the set of points $\{z_j\}$ we have

$$P(A \cap B \cap C) \leq \exp \left(-S(\rho) + Cn(r) \log S(r)\right).$$

Therefore, we get (using Lemma 11),

$$P(H) \leq P(L^c) + \frac{1}{\Delta \alpha} \cdot (P(A \cap B \cap C) + P(A^c) + P(B^c)) \leq$$
$$\leq 2e^{-S(\rho)n(\rho)} + \exp \left(-S(\rho) + \log \frac{1}{\Delta \alpha} + Cn(r) \log S(r) + O(1)\right) \leq$$
$$\leq \exp \left(-S(\rho) + c \log \frac{1}{\gamma} + Cn(r) \log S(r) + O(1)\right).$$

Finally, by Lemma 3 if we select $\gamma = \frac{1}{m(\rho)}$, we have

$$P(H) \leq \exp (-S(r) + Cn(r) \log S(r)),$$
thus proving Proposition 12.

5.2. Estimates for the probabilities. We now turn to find an upper bound for the probability of the event $A \cap B \cap C$, defined in the previous section, that is the event when:

$$\log |f(0)| \leq \log S(r),$$
$$M(r) = \max_{z \in rD} |f(z)| \leq e^{3S(r)},$$
$$\frac{1}{n} \sum \log |f(z_j e^{i\alpha})| \leq \log |f(0)| + 1 \leq S(r) + 1.$$
Note that the covariance matrix is invariant with respect to rotations \( f (e^{i\alpha} z) \). By Fubini we get

\[
P (A \cap B \cap C) = \int \int 1_{A \cap B \cap C} (\alpha, \omega) \, dm (\alpha) \, d\mu (\omega) = \int \left[ \int 1_{A \cap B \cap C} (\alpha, \omega) \, d\mu (\omega) \right] \, dm (\alpha)
\]

\[
\leq \int_E \frac{1}{\pi^n \det \Sigma} \exp (-\zeta^\ast \Sigma^{-1} \zeta) \, d\zeta,
\]

where \( E \) is the following set/event:

\[
E = \left\{ (f (z_1), \ldots, f (z_n)) \mid \frac{1}{n} \sum_{j=1}^n \log |f (z_j)| \leq \log S (r) + 1, |f (z_j)| \leq e^{3S (r)}, 1 \leq j \leq n \right\}.
\]

We have the following upper bound for the probability of the event \( E \):

\[
P (E) = \int_E \frac{1}{\pi^n \det \Sigma} \exp (-\zeta^\ast \Sigma^{-1} \zeta) \, d\zeta \leq \frac{1}{\pi^n \det \Sigma} \int_E \frac{1}{\pi^n \det \Sigma} \, d\zeta = \frac{\text{vol}_{E} (E)}{\pi^n \det \Sigma}.
\]

We start by finding a good lower bound for \( \det \Sigma \), this depends on a special selection of the points \( \{z_j\} \).

**Lemma 13.** Let \( f_1, \ldots, f_{n-1} \in \mathbb{N} \). There exist \( n \) points \( \{z_j\} \) on \( r \mathbb{T} \) such that the determinant of the following generalized Vandermonde matrix

\[
A = \begin{pmatrix}
1 & z_{j_1}^1 & \cdots & z_{j_{n-1}}^{j_{n-1}} \\
& & & \\
& & & \\
& & & \\
& & & \\
1 & z_{j_1}^n & \cdots & z_{j_{n-1}}^{j_{n-1}}
\end{pmatrix}
\]

satisfies \( |\det A| \geq r^{\sum_{k=1}^{n-1} j_k} \).

**Proof.** Start with the formal expression for the determinant

\[
\det A = \sum_{\sigma} \text{sgn} (\sigma) \prod_{m=0}^{n-1} z_{j_m}^{j_m (m)}.
\]

Write \( z_j = r e^{i\theta_j} \), now we have:

\[
\int_{\mathbb{T}^n} |\det A|^2 \, d\theta_1 \ldots d\theta_n = \int_{\mathbb{T}^n} \sum_{\sigma} \text{sgn} (\sigma) \prod_{m=1}^n z_{j_m}^{j_m (m)} \cdot \sum_{\sigma} \text{sgn} (\sigma) \prod_{m=1}^n z_{j_m}^{j_m (m)} \, dz_1 \ldots dz_n = r^2 \sum_{k=1}^{n-1} j_k \cdot \left( (2\pi)^n \cdot \sum_{\sigma} 1 + \int_{\mathbb{T}^n} \sum_{\sigma \neq \tau} \text{sgn} (\sigma) \cdot \text{sgn} (\tau) e^{i \sum a_{\sigma,\tau,j} \theta_j} \right) \geq r^2 \sum_{k=1}^{n-1} j_k \cdot n!,
\]

since at least one of the \( a_{\sigma,\tau,j} \) is different than 0 in each sum.

We now have the following

**Corollary 14.** Using the configuration of the points \( \{z_j\} \) given in the previous lemma, we have

\[
\log (\det \Sigma) \geq S (r).
\]
Proof. Notice that we can represent $\Sigma$ in the following form
$$\Sigma = V \cdot V^*$$
where
$$V = \begin{pmatrix} a_0 & a_1 \cdot z_1 & \ldots & a_n \cdot z_1^n & \ldots \\ \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1 \cdot z_N & \ldots & a_n \cdot z_N^n & \ldots \end{pmatrix}$$

We can estimate the determinant of $\Sigma$ by projecting $V$ on $N(r) = \{ a_0, a_{j_1}, \ldots, a_{j_n-1} \}$ coordinates (let’s denote this projection by $P$). Since $\det \Sigma$ is the square of the product of the singular values of $V$, and these values are only reduced by the projection, we have
$$\det \Sigma \geq (\det PV)^2 = \prod_{j \in N_1(r)} a_j^2 \cdot r^{\sum_{j \in N_1(r)} j}$$
and so by the previous lemma
$$\det \Sigma \geq \prod_{j \in N_1(r)} a_j^2 \cdot r^{\sum_{j \in N_1(r)} j} = \exp(S(r))$$

We now want to estimate the integral
$$I = \int_E \frac{1}{\pi^n} d\zeta,$$
that is to estimate the volume of the following set:
$$E = \left\{ \zeta \in \mathbb{C}^n \mid \frac{1}{n} \sum_{j=1}^{n} \log |\zeta_j| \leq \log S(r) + 1 \text{ and } |\zeta_j| \leq e^{3S(r)}, 1 \leq j \leq n \right\},$$
with respect to the Lebesgue measure on $\mathbb{C}^n$. We will use the following lemma (see [Ni1, Lemma 11])

Lemma 15. Set $s > 0$, $t > 0$ and $N \in \mathbb{N}^+$, such that $\log \left( t^n / s \right) \geq N$. Denote by $C_N$ the following set
$$C_N = C_N(t, s) = \left\{ r = (r_1, \ldots, r_N) : 0 \leq r_j \leq t, \prod_{j=1}^{N} r_j \leq s \right\}.$$
Then
$$\text{vol}_{\mathbb{R}^N}(C_N) \leq \frac{s}{(N-1)!} \log^N \left( t^n / s \right).$$
Now we have the almost immediate

**Corollary 16.** Suppose that $r$ is normal and large enough, then we have

$$I \leq \exp \left( Cn \left( r \right) \log S \left( r \right) \right).$$

**Proof.** To shorten the expressions, we write

$$n = n(r),$$
$$s = \exp \left( n \left( r \right) \left( \log S \left( r \right) + 1 \right) \right),$$
$$t = \exp (3S(r)).$$

We want to translate the integral $I$ into an integral in $\mathbb{R}^n$, using the change of variables $\zeta_j = r_j \cos(\theta_j) + ir_j \sin(\theta_j)$. Integrating out the variables $\theta_j$, we get

$$I' = 2^n \int_C \prod r_j \, dr,$$

where the new domain is

$$C = \left\{ r = (r_1, \ldots, r_n) : 0 \leq r_j \leq t, \prod_{j=1}^n r_j \leq s \right\}.$$

We can find an explicit expression for this integral, but, instead we will simplify it even more to

$$I' \leq 2^n s \cdot \text{vol}_{\mathbb{R}^n}(C) \tag{5.2}$$

Now, in order to use the previous lemma, we have to check the condition $\log \left( t^n / s \right) \geq n$, or (where $C > 0$)

$$3n(r)S(r) - n(r) \left( \log S(r) + 1 \right) \geq n(r),$$

which is satisfied under our assumptions, for $r$ large enough. After applying the lemma, we get (for $r$ large enough)

$$I' \leq \frac{n \cdot 2^n s^2}{n!} \log^n \left( t^n / s \right) \leq \frac{s^2 e^{2n}}{n^n} \log^n \left( t^n / s \right) = \exp \left( 2 \log s + n \log_2 t + 2n - n \log_2 s \right) \leq \exp \left( 2 \log s + n \log_2 t \right).$$

Recalling the definitions of $n, s$ and $t$, we finally get

$$\log I' \leq 2n \left( r \right) \left( \log S \left( r \right) + 1 \right) + n \left( r \right) \log S \left( r \right) + Cn \left( r \right) \leq Cn \left( r \right) \log S \left( r \right).$$

□

Now, the estimate (5.1) follows, since the original points $\{z_j\}$ satisfy $|z_j| = \rho$ and by Corollaries 14 and 10

$$\mathbb{P}(E) \leq \exp \left( -S \left( \rho \right) + Cn \left( r \right) \log S \left( r \right) \right).$$

This completes the proof of Proposition 12.
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