Random perturbations of dynamical systems with reflecting boundary and corresponding PDE with a small parameter

Wenqing Hu*, Lucas Tcheuko†

Abstract

We study the asymptotic behavior of a diffusion process with small diffusion in a domain $D$. This process is reflected at $\partial D$ with respect to a co-normal direction pointing inside $D$. Our asymptotic result is used to study the long time behavior of the solution of the corresponding parabolic PDE with Neumann boundary condition.

Keywords: PDE with a small parameter, large deviations, Freidlin-Wentzell theory, diffusion process with reflection.

2010 Mathematics Subject Classification Numbers: 60J60, 60F10, 60H30.

1 Introduction

Consider the following parabolic initial-boundary value problem

\begin{equation}
\begin{aligned}
\frac{\partial u^\varepsilon}{\partial t} &= L^\varepsilon u^\varepsilon \equiv \frac{\varepsilon^2}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i(x) \frac{\partial u^\varepsilon}{\partial x_i} , \\
\varepsilon &> 0 ; \\
u^\varepsilon(x,0) &= g(x) , \\
\frac{\partial u^\varepsilon}{\partial n}(x,t) &= 0 ,
\end{aligned}
\tag{1.1}
\end{equation}

Here $D$ is a $d$-dimensional bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial D$. The initial condition $g(\bullet)$ is smooth in $D \cup \partial D$. The matrix $a(x) = (a_{ij}(x))_{1 \leq i,j \leq d}$ is positive definite. The functions $a_{ij}(x)$ are smooth and uniformly bounded, with uniformly bounded derivatives. There is a constant $\theta > 0$ such that for any $\xi = (\xi^1, ..., \xi^d)$ we have $\theta^2 |\xi|_{\mathbb{R}^d}^2 \leq \sum_{i,j=1}^{d} a_{ij}(x) \xi^i \xi^j \leq \theta^{-2} |\xi|_{\mathbb{R}^d}^2$. The vector field $b(x) = (b^1(x), ..., b^d(x))$ have terms which are uniformly bounded, smooth in $|D|$ (here and below $|D|$ is the closure of $D$ in Euclidean metric), and have uniformly bounded derivatives. The vector field

*Department of Mathematics, University of Maryland at College Park, huwenqing@math.umd.edu
†Department of Mathematics, University of Maryland at College Park, lucast@math.umd.edu
\(\gamma(x) = (\gamma^1(x), ..., \gamma^d(x))\) is the inward co-normal unit vector field on \(\partial D\) with respect to the matrix \(a^{-1}(x) = (a^{ij}(x))_{1 \leq i,j \leq d}\). That is to say, for any vector \(v(x) = (v^1(x), ..., v^d(x))\) tangent to \(\partial D\) we have 
\[
(\gamma, v)_{a^{-1}(x)} = \sum_{i,j=1}^d a^{ij}(x)\gamma^i(x)v^j(x) = 0.
\]
(Here and below \((\gamma, v)_{a^{-1}(x)}\) is the inner product with respect to the matrix \(a^{-1}(x)\). For a detailed discussion of the co-normal condition we refer to \cite{3} Section 2.5.) We also have \(|\gamma|_{\partial D} = 1\).

Let us assume that the vector field \(b(x)\) is pointing outward to \(D\) on a connected subset \(\partial_1 D\) of \(\partial D\), and it is pointing inward on \(\partial_2 D \equiv \partial D \setminus \partial_1 D\). (It is never tangent to \(\partial D\).) Let \(\bar{b}(x)\) be the field coinciding with \(b(x)\) everywhere except at those points of \(\partial_1 D\). At these points \(\bar{b}(x)\) is defined as the projection of \(b(x)\) onto the direction of the boundary. Suppose the dynamical system \(\bar{x}_t = \bar{b}(x_t)\) has all its \(\omega\)-limit sets on \(\partial_1 D\). These \(\omega\)-limit sets are points \(O_1, ..., O_l\) (\(l \geq 1\)).

Our goal in this paper is to describe the long-time behavior of the solution \(u^\varepsilon(x,t)\) of (1.1) as \(\varepsilon \to 0\) and \(t \to \infty\). One can relate problem (1.1) with a certain diffusion process \(X^\varepsilon_t\) with small diffusion and reflection with respect to \(\gamma\) on \(\partial D\). This process can be described as a solution of the following stochastic differential equation:

\[
dX^\varepsilon_t = b(X^\varepsilon_t)dt + \varepsilon \sigma(X^\varepsilon_t)dW_t + \mathbf{1}_{\partial D}(X^\varepsilon_t)\gamma(X^\varepsilon_t)d\xi^\varepsilon_t, \quad X^\varepsilon_0 = x, \quad \xi^\varepsilon_0 = 0. \tag{1.2}
\]

Here \(\sigma(x)\) is a \(d \times d\) matrix with smooth terms (and bounded derivatives) that satisfies \(\sigma(x)\sigma^T(x) = \sigma^T(x)\sigma(x) = a(x)\). The function \(\mathbf{1}_{\partial D}(\bullet)\) is the indicator function of \(\partial D\). The processes \(X^\varepsilon_t\) and \(\xi^\varepsilon_t\) are continuous time stochastic processes, adapted to the filtration \(\mathcal{F}_t\) \(t \geq 0\). They satisfy the following assumptions with probability 1:

1. The process \(X^\varepsilon_t \in [D]\);
2. The process \(\xi^\varepsilon_t\) is non-decreasing in \(t\) and increases only at \(\Delta = \{t : X^\varepsilon_t \in \partial D\}\);
3. The set \(\Delta\) has Lebesgue measure zero.

Under these assumptions, it was proved in \cite{1} (also see \cite{11}) that such a pair of processes \((X^\varepsilon_t, \xi^\varepsilon_t)\) exist and is unique (in the sense of probability 1). The process \(\xi^\varepsilon_t\) is called the local time of the process \(X^\varepsilon_t\) on \(\partial D\). (We remark here that this notion of the local time for the multidimensional diffusion process extends the classical 1-dimensional local time in \cite{8}. See \cite{11} for a discussion based on SDE approach. For other discussions of the local time for multidimensional diffusion process we also refer to \cite{9} and \cite{10}.) The process \(X^\varepsilon_t\) is a strong Markov process in \([D]\) and it satisfies the Doeblin condition, which leads to the existence and uniqueness of an invariant measure in \([D]\).

It turns out that the solution \(u^\varepsilon(x,t)\) of (1.1) can be represented as 
\[
u^\varepsilon(x,t) = E_x g(X^\varepsilon_t)\quad (\text{see Section 4 for details})
\]
as $\varepsilon \to 0, t \to \infty$ is determined by the asymptotic behavior of the process $X_t^\varepsilon$. However, the latter can be calculated using the Freidlin-Wentzell large deviation theory (see [5], [4]).

In Section 2 of the present paper we will give an expression of the action functional $S_{0T}^+$ of the process $X_t^\varepsilon$. By using the large deviation principle for the family of processes $\{X_t^\varepsilon\}_{\varepsilon > 0}$ we will give a description of the asymptotic behavior of $X_t^\varepsilon$ in Section 3. Since the proof is based on the method of [5, Ch.6] and [4], we will only prove some key technical lemmas and sketch the result. In particular, we give the algorithm on the calculation of metastable states. Section 4 provides the corresponding result for problem (1.1). We point out that a related question for elliptic boundary value problems was already considered in [6] (also see [5, Section 10.3]). An example is given in Section 5.

2 Calculation of the action functional

In this section we give an expression of the action functional corresponding to the large deviation principle of the process $X_t^\varepsilon$. The main proofs and justifications of our results are contained in [1] (also see [5, Section 10.3]), so we just summarize the results we need.

In [1], the authors have constructed the process $(X_t^\varepsilon, \xi_t^\varepsilon)$ corresponding to (1.2) by first realize it in the space $\mathbb{R}^d_+$ using the following stochastic differential equation:

$$dY_t^\varepsilon = b(\Gamma(Y_t^\varepsilon))dt + \varepsilon \sigma(\Gamma(Y_t^\varepsilon))dW_t, \quad Y_0^\varepsilon = x \in \mathbb{R}^d_+. \quad (2.1)$$

Here $\Gamma : C_{(0,\infty)}(\mathbb{R}^d) \to C_{(0,\infty)}(\mathbb{R}_+^d)$ is a functional defined by

$$\Gamma(\psi)_t \equiv (\Gamma(\psi))_t \equiv \Gamma_t(\psi) = (\psi_t^1 - 0 \wedge \inf_{0 \leq s \leq t} \psi_s^1, \psi_t^2, ..., \psi_t^d) \quad (2.2)$$

for $\psi_t = (\psi_t^1, ..., \psi_t^d) \in C_{(0,\infty)}(\mathbb{R}^d)$. It was proved in [1] that in the case of a half space $\mathbb{R}^d_+$ one can take $(X_t^\varepsilon, \xi_t^\varepsilon) = (\Gamma(Y_t^\varepsilon), (\Gamma(Y_t^\varepsilon) - Y_t^\varepsilon)^1)$.

In the general case when $D$ is a bounded region in $\mathbb{R}^d$ with smooth boundary one can take a finite covering of $D$ by a set of open neighborhoods $\{U_1, ..., U_N\}$. Within each $U_i$ ($i = 1, ..., N$) the process can be constructed via a homeomorphism between $U_i$ and $\mathbb{R}^d$, or between $U_i \cap D$ and $\mathbb{R}^d_+$ (when $U_i \cap \partial D \neq \emptyset$). In the latter case we use the construction of the process in half space as above. By appropriately ”glue” these pieces of the trajectories together one can construct the processes $(X_t^\varepsilon, \xi_t^\varepsilon)$. The process $X_t^\varepsilon$ is the diffusion process with reflection in $D$ and the process $\xi_t^\varepsilon$ is the local time on $\partial D$. For details of this construction we refer to [1], [3] Section 1.6.
It was shown in [1] Section 1.2 that the corresponding action functional for the family of processes \( \{X^\varepsilon_t\}_{\varepsilon > 0} \) as \( \varepsilon \downarrow 0 \) is given by the formula

\[
S^{+\varepsilon}_{0T}(\varphi) = \begin{cases} 
\frac{1}{2} \int_0^T \| \dot{\varphi}_s - b(\varphi_s) - 1_{\partial D}(\varphi_s) \omega(s) \gamma(\varphi_s) \|_{a^{-1}(\varphi_s)}^2 ds, \\
+\infty,
\end{cases}
\]

for \( \varphi \in C_{[0,T]}([D]) \) absolutely continuous, \( \varphi_0 = x \); \( \varphi_0 = x \) \( \text{for the rest of } \varphi \in C_{[0,T]}([D]) \).

Here

\[
\omega(s) = \frac{(\dot{\varphi}_s - b(\varphi_s), \gamma(\varphi_s))_{a^{-1}(\varphi_s)}}{\| \gamma(\varphi_s) \|_{a^{-1}(\varphi_s)}^2} \vee 0,
\]

and \( \|v\|_{a^{-1}(x)} = \langle v, v \rangle_{a^{-1}(x)}^{1/2} \) for vector \( v \in \mathbb{R}^d \).

We have

\[
1_{\partial D}(\varphi_s) \omega(s) \gamma(\varphi_s)
= 1_{\partial D}(\varphi_s) \gamma(\varphi_s) \left( \frac{(\dot{\varphi}_s, \gamma(\varphi_s))_{a^{-1}(\varphi_s)}}{\| \gamma(\varphi_s) \|_{a^{-1}(\varphi_s)}^2} - \frac{(b(\varphi_s), \gamma(\varphi_s))_{a^{-1}(\varphi_s)}}{\| \gamma(\varphi_s) \|_{a^{-1}(\varphi_s)}^2} \right) \vee 0
= -1_{\partial D}(\varphi_s) \frac{\gamma(\varphi_s)}{\| \gamma(\varphi_s) \|_{a^{-1}(\varphi_s)}^2} [0 \wedge (b(\varphi_s), \gamma(\varphi_s))_{a^{-1}(\varphi_s)}] \text{ for a.s. } s \in [0,T].
\]

Define

\[
\tilde{b}(x) = b(x) - 1_{\partial D}(x) \frac{\gamma(x)}{\| \gamma(x) \|_{a^{-1}(x)}^2} [0 \wedge (b(x), \gamma(x))_{a^{-1}(x)}].
\]

We see that \( \tilde{b}(x) \) is the field coinciding with \( b(x) \) everywhere except at those points of \( \partial_1 D \). (Recall that \( \partial_1 D \) is the part of the boundary \( \partial D \) on which \( b(x) \) is pointing outward.) At these points \( \tilde{b}(x) \) is defined as the projection of \( b(x) \) onto the direction of the boundary. The action functional for the family of processes \( \{X^\varepsilon_t\}_{\varepsilon > 0} \) can now be formulated as

\[
S^{+\varepsilon}_{0T}(\varphi) = \begin{cases} 
\frac{1}{2} \int_0^T \| \dot{\varphi}_s - \tilde{b}(\varphi_s) \|_{a^{-1}(\varphi_s)}^2 ds, \\
+\infty,
\end{cases}
\]

for \( \varphi \in C_{[0,T]}([D]) \) absolutely continuous, \( \varphi_0 = x \); \( \text{for the rest of } \varphi \in C_{[0,T]}([D]) \).

The deterministic trajectory \( X^0_t \) at which the above action functional is 0 is also calculated in [1]. It is given by the system \( \dot{x}_t = \tilde{b}(x_t), \ x_0 = x \), i.e., it coincides with the deterministic trajectory given by the vector field \( b(x) \) everywhere except at those points of \( \partial_1 D \), and at points of \( \partial_1 D \) it follows the projection of \( b(x) \) onto the direction of the boundary.
We formulate below the large deviation principle for the family of processes \( \{X^\varepsilon_t\}_{\varepsilon > 0} \).

**Theorem 2.1.** (Large deviation principle) For the process \( X^\varepsilon_t \), we have

1. The set \( \Phi(s) = \{ \varphi \in C_{[0,T]}([D]) : S^+_0(\varphi) \leq s \} \) is compact for every \( s \geq 0 \);
2. Given \( \varphi \in C_{[0,T]}([D]) \). For any \( \delta > 0 \) and any \( \gamma > 0 \) there exist an \( \varepsilon_0 > 0 \) such that for any \( 0 < \varepsilon < \varepsilon_0 \) we have
   \[
   P\{\rho_{0T}(X^\varepsilon, \varphi) < \delta \} \geq \exp[-\varepsilon^{-2}(S^+_0(\varphi) + \gamma)] ,
   \]
   where \( T > 0 \) and \( \rho_{0T}(\cdot, \cdot) \) denotes the uniform distance between functions in \( C_{[0,T]}([D]) \);
3. For any \( \delta, \gamma > 0 \) and any \( s > 0 \) there exists an \( \varepsilon_0 > 0 \) such that for any \( 0 < \varepsilon < \varepsilon_0 \) we have
   \[
   P\{\rho_{0T}(X^\varepsilon, \Phi(s)) \geq \delta \} \leq \exp[-\varepsilon^{-2}(s - \gamma)] ,
   \]
   where \( \rho_{0T}(\varphi, \Phi(s)) = \inf_{\psi \in \Phi(s)} \rho_{0T}(\varphi, \psi) \).

3 Asymptotic behavior of \( X^\varepsilon_t \)

3.1 Estimates on the time to converge to \( \omega \)-limit sets on the boundary

We now begin our study of the asymptotic behavior of the process \( X^\varepsilon_t \). First, since the dynamical system \( \dot{x}_t = \tilde{b}(x_t) \) does not have any \( \omega \)-limit set within \( D \), we shall expect that as \( \varepsilon \) is small, the trajectories of \( X^\varepsilon_t \) come to the boundary \( \partial_1 D \) within finite time. (Notice that at points of \( \partial_1 D \) the vector field \( b(x) \) is pointing outward and at points of \( \partial_2 D \) it is pointing inward. Therefore the deterministic trajectory \( X^0_t \) will not come to \( \partial D \).)

For any \( x, y \in [D] \), we define

\[
V^+(x, y) = \inf_{\varphi \in C_{[0,T]}([D])} \{ S^+_0(\varphi), \varphi_0 = x, \varphi_T = y, \varphi_t \in D \cup \partial D, 0 \leq t \leq T \} .
\]

Recall that the dynamical system \( \dot{x}_t = \tilde{b}(x_t) \) has all its \( \omega \)-limit sets on \( \partial_1 D \). These \( \omega \)-limit sets are points \( O_1, ..., O_l \) \( (l \geq 1) \). Let us suppose, that for any \( x \) and \( y \) in \( [D] \), \( x \neq y \) we have at least one of \( V^+(x, y) \) and \( V^+(y, x) \) being > 0.

For each \( O_i, i = 1, 2, ..., l \), by an \( \alpha \)-neighborhood \( E_\alpha(O_i) \) of \( O_i \), we refer to the intersection of \( D \) with an open ball having center \( O_i \) and radius \( \alpha > 0 \). We use the symbol \( \partial E_\alpha(O_i) \) to mean the intersection of \( [D] \) with the boundary of the open \( \alpha \)-ball centered at \( O_i \). We call \( \partial E_\alpha(O_i) \) the boundary of the \( \alpha \)-neighborhood of \( O_i \). Let us choose \( \alpha > 0 \) such that the \( \alpha \)-neighborhoods \( E_\alpha(O_i) \) for all \( O_i, i = 1, 2, ..., l, \) does not intersect each other. We now prove the following:
**Theorem 3.1.** There exist positive constants $c$ and $T_0$ such that for all sufficiently small $\varepsilon > 0$ and any $x \in [D], X_0^\varepsilon = x$ we have
\[
P_x\{\zeta_\alpha > T\} \leq \exp[-\varepsilon^{-2}c(T - T_0)],
\]
where $\zeta_\alpha = \inf\{t : X_t^\varepsilon \in \bigcup_{i=1}^l [E_\alpha(O_i)]\}.$

**Proof.** We consider the dynamical system $\dot{x}_t = b(x_t)$ on the whole domain $D \cup \partial D$, where vector field $b(x)$ is defined as before. Since system $\dot{x}_t = b(x_t)$ does not have any $\omega$-limit set in $D$ and $(b(x), \gamma(x))a^{-1}(x)|_{\partial x} > 0$, we can say that the time $T_1(x)$ that the trajectory $x_t(x)$ spends until reaching $\partial_1 D$ is finite (if $x \in \partial D$, let $T_1(x) = 0$).

Let $y(x)$ be the point where trajectory first hits $\partial_1 D$. Starting from $y(x)$, the time $T_2(y(x), \alpha) = T_2(x, \alpha)$ that the trajectory of system $\dot{x}_t = b(x_t)$ on $\partial_1 D$ spend to come into \(\bigcup_{i=1}^l [E_{\Phi}(O_i)]\) is also finite (as is the same, if $y(x) \in \bigcup_{i=1}^l [E_{\Phi}(O_i)]$, then $T_2(x, \alpha) = 0$).

The function $T(x, \alpha) = T_1(x) + T_2(x, \alpha)$ is upper semi-continuous in $x$ (i.e., for $x, x_0 \in [D]$ we have $\lim_{x \to x_0} T(x, \alpha) \leq T(x_0, \alpha)$ because $x_t(x)$ depends continuously on $x$). Thus there exists $T_0 = \max T(x, \alpha) < \infty$. The set of functions in $C_{[0,T_0]}([D])$ assuming their values in $[D]\setminus \left(\bigcup_{i=1}^l E_{\Phi}(O_i)\right)$ is closed and thus $S_{0,T_0}^+$ attains a minimum $A$ on this set.

Taking into account the construction of $T_0$ and the form of $S_{0,T_0}^+$ in (2.5), we see that $A > 0$. Let $0 < \delta < \frac{A}{2}$. Let $\Phi_2(A/2) = \{\varphi \in C_{[0,T_0]}([D]), \varphi_0 = x, S_{0,T_0}^+(\varphi) \leq A/2\}$. We see that trajectories for which $\zeta_\alpha > T_0$ are at a distance $\geq \delta$ from $\Phi_2(A/2)$. Thus by the part (iii) of the large deviation principle we have
\[
P_x\{\zeta_\alpha > T_0\} \leq \exp[-\varepsilon^{-2}(A/2 - \gamma)]
\]
for some $0 < \gamma < A/2$.

Thus by strong Markov property,
\[
P_x\{\zeta_\alpha > (n + 1)T_0\} = \mathbb{E}_x[\zeta_\alpha > nT_0; P_{X_{nT_0}^\varepsilon}\{\zeta_\alpha > T_0\}] \
\leq P_x\{\zeta_\alpha > nT_0\} \exp[-\varepsilon^{-2}(A/2 - \gamma)].
\]
So by induction we see that
\[
P_x\{\zeta_\alpha > T\} \leq P_x\{\zeta_\alpha > \left\lfloor \frac{T}{T_0} \right\rfloor T_0\} \
\leq \exp\{-\varepsilon^{-2}(A/2 - \gamma) \left\lfloor \frac{T}{T_0} \right\rfloor \} \
\leq \exp\{-\varepsilon^{-2} \left( \frac{T}{T_0} - 1 \right) (A/2 - \gamma) \}.
\]

Putting $c = \frac{A/2 - \gamma}{T_0}$, we get as desired. $\square$
3.2 Transition probabilities between neighborhoods of the $O_i$’s

In this section we study the asymptotic transition probabilities between neighborhoods of the $\omega$-limit sets $\{O_1,\ldots, O_l\}$. We first provide several auxiliary lemmas.

**Lemma 3.1.** There exists a constant $L > 0$ such that for any $x, y \in [D]$ sufficiently close to each other, there exists a function $\varphi \in C_{[0,T]}([D])$, $\varphi_0 = x$, $\varphi_T = y$, such that we have $S^{+}_{0T}(\varphi) < L \cdot |x - y|_{\mathbb{R}^d}$.

**Proof.** Let $x$ and $y$ be so close to each other that they can be covered by one coordinate chart $U$. Let this coordinate chart correspond to a coordinate function $u : U \to \mathbb{R}^d$ (or $\mathbb{R}^d_+$). The function $u$ is smooth with bounded derivatives. Let us take $T = |x - y|_{\mathbb{R}^d}$,

$$\varphi_t = u^{-1}\left(u(x) + \frac{t}{T}(u(y) - u(x))\right).$$

We have, for some constant $M > 0$,

$$S^{+}_{0T}(\varphi_s) = \frac{1}{2} \int_0^T \|\dot{\varphi}_s - \tilde{b}(\varphi_s)\|^2_{a^{-1}(\varphi_s)} ds \leq \frac{\theta^2}{2} \int_0^T |\dot{\varphi}_s - \tilde{b}(\varphi_s)|^2_{\mathbb{R}^d} ds \leq \frac{\theta^2}{2} \left( \frac{1}{T} \left|\frac{1}{T}(u^{-1})^T(u(x) + \frac{t}{T}(u(y) - u(x))) \circ (u(y) - u(x)) - \tilde{b}(\varphi_s)\right|^2_{\mathbb{R}^d} ds \right).$$

Taking into account that $T = |y - x|_{\mathbb{R}^d}$, we are done. □

**Lemma 3.2.** For any $\gamma > 0$ and any compact subset $K \subseteq [D]$ there exists $T_0$ such that for any $x, y \in K$ there exists a function $\varphi_t, 0 \leq t \leq T, \varphi_0 = x, \varphi_T = y, T \leq T_0$ such that $S^{+}_{0T}(\varphi) \leq V^{+}(x, y) + \gamma$.

**Proof.** We choose a finite $\delta$-net $\{x_i\}$ of points in $K$; we connect them with curves at which the action functional assumes values differing from the infimum by less than $\frac{\delta \gamma}{2}$ and complete them with end sections using Lemma 3.1: from $x$ to a point $x_i$ near $x$ and then from $x_i$ to a point $x_j$ near $y$, and from $x_j$ to $y$. By choosing $\delta$ small enough we get as desired. □

We define
\[ \hat{V}^+(O_i, O_j) = \inf_{\varphi \in C_{[0,T]}([D])} \{ S^+_{0,t}(\varphi) : \varphi_0 = O_i, \varphi_T = O_j, \varphi_t \in [D] \setminus \bigcup_{s \neq i,j} \{O_s\} , 0 < t < T \} . \]

A “\( \hat{V}^+(O_i, O_j) \)" version of the above Lemma can be proved similarly: one can take the curve \( \varphi \) in such a way that it avoids \( \bigcup_{s \neq i,j} \{O_s\} \) and such that \( S^+_{0,T}(\varphi) \leq \hat{V}^+(O_i, O_j) + \gamma \). We omit the proof.

Let constant \( \rho_0 > 0 \) be small. Let constant \( 0 < \rho_1 < \rho_0 \). We denote by \( C \) the set \( D \cup \partial D \) from which we delete the \( \rho_0 \)-neighborhoods of the \( O_i, i = 1, 2, ..., l \); by \( \Gamma_i \) the boundaries of the \( \rho_0 \)-neighborhoods of \( O_i \); \( \Gamma_i = \partial E_{\rho_0}(O_i) \); by \( g_i \) the \( \rho_1 \)-neighborhoods of the \( O_i \), and by \( g \) the union of all the \( g_i \).

We introduce the following random times \( \tau_0 = 0, \sigma_n = \inf\{t \geq \tau_n, X^e_t \in C\}, \tau_n = \inf\{t \geq \sigma_{n-1}, X^e_t \in \partial g\} \). We consider the Markov chain \( Z_n = X^e_{\tau_n} \) for \( n \geq 0 \). We see that from \( n = 1 \) on \( Z_n \in \partial g \). Also, \( X^e_{\sigma_n} \) can be any point of \( C \); all the following \( X^e_{\sigma_n} \) belong to one of the \( \Gamma_i \)'s. The chain never stops.

We are now ready to prove:

**Theorem 3.2.** For any \( \gamma > 0 \) there exists \( \rho_0 > 0 \) (which can be chosen arbitrary small) such that for any \( \rho_2, 0 < \rho_2 < \rho_0 \), there exists \( \rho_1, 0 < \rho_1 < \rho_2 \) such that for all \( x \) in the \( \rho_2 \)-neighborhood of \( O_i (i = 1, ..., l) \) the one-step transition probabilities of \( Z_n, \ Z_0 = x \) satisfy the inequality

\[ \exp[-\varepsilon^{-2}(\hat{V}^+(O_i, O_j) + \gamma)] \leq P(x, \partial g_j) \leq \exp[-\varepsilon^{-2}(\hat{V}^+(O_i, O_j) - \gamma)] \]

for some \( 0 < \varepsilon < \varepsilon_0 \).

**Proof.** We can assume \( \hat{V}^+(O_i, O_j) < \infty \). Set \( \hat{V}^+_0 = \max_{i,j=1,2,\ldots,l} \hat{V}^+(O_i, O_j) \). Choose positive \( \rho_0 \) small enough. For every pair \( O_i, O_j \) for which \( \hat{V}^+(O_i, O_j) < \infty \) we choose a function \( \varphi^{i,j}_t \in C_{[0,T]}([D]), 0 \leq t \leq T = T(O_i, O_j) \), such that \( \varphi_0^{i,j} = O_i, \varphi_T^{i,j} = O_j, \varphi_t^{i,j} \) does not touch \( \bigcup_{s \neq i,j} \{O_s\} \), and such that (by Lemma 3.2)

\[ S^+_{0,t}(\varphi^{i,j}) \leq \hat{V}^+(O_i, O_j) + 0.5\gamma . \]

We choose positive \( \rho_1 \) smaller than \( \frac{\rho_0}{2}, \rho_2 \) and

\[ \frac{1}{2} \min\{\rho(\varphi^{i,j}_t, \bigcup_{s \neq i,j} \{O_s\}) : 0 \leq t \leq T, i, j = 1, 2, ... , l \} . \]

For every \( x \) in a \( \rho_2 \)-neighborhood of \( O_i \) we take a curve connecting \( x \) with \( O_i \) and for which the value of \( S^+ \) does not exceed \( 0.3\gamma \) (by Lemma 3.1). We combine this curve
with the curve $\varphi_{t}^{i,j}$ and obtain a function $\varphi_{t}, 0 \leq t \leq T, \varphi_{0} = x, \varphi_{T} = O_{j}$ (with a possible small change of $T$ from $T = T(O_{i}, O_{j})$) such that

$$S_{0T}^{+}(\varphi) \leq \tilde{V}^{+}(O_{i}, O_{j}) + 0.8\gamma.$$  

From Lemma 3.2 we choose a $T_{0} \geq T$, and extend the curve $\varphi_{t}$ to $T \leq t \leq T_{0}$ by using a trajectory of the dynamical system $\dot{x}_{t} = \tilde{b}(x_{t})$ on $\partial_{1}D$, without changing the value of $S_{0T_{0}}^{+}(\varphi)$ from that of $S_{0T}^{+}(\varphi)$. We choose positive $\delta$ less than $\rho_{1}, \rho_{0} - \rho_{2}$. For a trajectory of $X_{t}^{i}$ starting from $x$, passing at a distance from $\varphi_{t}$ smaller than $\delta$ for $0 \leq t \leq T_{0}$, it must intersect with $\Gamma_{i}$ and reaches the $\delta$-neighborhood of $O_{j}$ without getting closer than $\delta$ from any of the other $O_{s}, s \neq i, j$. Moreover, $X_{t}^{i} \in \partial g_{j}$, thus

$$P(x, \partial g_{j}) \geq P_{x}\{\rho_{0T_{0}}(X^{\varepsilon}, \varphi) < \delta\} \geq \exp[-\varepsilon^{-2}(S_{0T_{0}}^{+}(\varphi)+0.1\gamma)] > \exp[-\varepsilon^{-2}(\tilde{V}^{+}(O_{i}, O_{j})+\gamma)].$$

Now we turn to the proof of the upper estimates. For any curve $\varphi_{t}, 0 \leq t \leq T$ beginning at $x$, touching the $\delta$-neighborhood of $\partial g_{j}$, not touching any of the $O_{s}, s \neq i, j$, we have

$$S_{0T}^{+}(\varphi) \geq \tilde{V}^{+}(O_{i}, O_{j}) - 0.7\gamma.$$  

We use Theorem 3.1 to choose $T_{1}$ such that for any $x \in [D] \setminus g$ we have $P_{x}\{\tau_{1} > T_{1}\} \leq \exp(-\varepsilon^{-2}V_{0}^{+})$ for some $V_{0}^{+} > 0$.

Any trajectory $X_{t}^{i}$ beginning at $x$ and being in $\partial g_{j}$ at time $\tau_{1}$ either spends time $T_{1}$ without touching $\partial g$ or reaches $\partial g_{j}$ over time $T_{1}$, in this case

$$\rho_{0T_{1}}(X^{\varepsilon}, \Phi_{x}(\tilde{V}^{+}(O_{i}, O_{j}) - 0.7\gamma)) \geq \delta.$$

Therefore we have

$$P_{x}\{X_{t}^{i} \in \partial g_{j}\} \leq P_{x}\{\tau_{1} > T_{1}\} + P_{x}\{\rho_{0T_{1}}(X^{\varepsilon}, \Phi_{x}(\tilde{V}^{+}(O_{i}, O_{j}) - 0.7\gamma)) \geq \delta\} \leq \exp(-\varepsilon^{-2}V_{0}^{+}) + \exp[-\varepsilon^{-2}(\tilde{V}^{+}(O_{i}, O_{j}) - 0.9\gamma)] \leq \exp[-\varepsilon^{-2}(\tilde{V}^{+}(O_{i}, O_{j}) - \gamma)]$$

for sufficiently small $\varepsilon$. □

In an exactly similar way one can also formulate the estimate on transition probability based on the quantities

$$\tilde{V}^{+}(x, O_{j}) = \inf_{\varphi \in C_{[0,T]}([D])} \{S_{0T}^{+}(\varphi) : \varphi_{0} = x, \varphi_{T} = O_{j}, \varphi_{t} \in [D] \setminus \bigcup_{s \neq j} O_{s}, 0 < t < T\}.$$  

We have
**Theorem 3.3.** For any $\gamma > 0$ there exists $\rho_0 > 0$ (which can be chosen arbitrary small) such that for any $\rho_2, 0 < \rho_2 < \rho_0$, there exists $\rho_1, 0 < \rho_1 < \rho_2$ such that for all $x$ outside the $\rho_2$-neighborhood of $O_i (i = 1, \ldots, l)$ the one-step transition probabilities of $Z_n$, $Z_0 = x$ satisfy the inequality

$$\exp[-\varepsilon^2(\tilde{V}^+(x, O_j) + \gamma)] \leq P(x, \partial g_j) \leq \exp[-\varepsilon^2(\tilde{V}^+(x, O_j) - \gamma)]$$

for some $0 < \varepsilon < \varepsilon_0$.

### 3.3 The invariant measure of $X_t^\varepsilon$; sublimiting distribution

In this section we study the invariant measure of the process $X_t^\varepsilon$. Based on the estimates on transition probabilities given above, the proof of the asymptotic result is the same as that of [5, Ch.6] and [4]. Let us formulate and prove two more technical lemmas, after which the rest of the proof is just a study of Markov chains on graphs. The latter part will be omitted since it is the same as [5, Ch.6] and [4].

**Lemma 3.3.** For $i \in \{1, 2, \ldots, l\}$, define

$$\tau_{E_\delta(O_i)} = \inf \{t, X_t^\varepsilon = x, X_t^\varepsilon \in \partial E_\delta(O_i)\}.$$

For any $\gamma > 0$, there exist $\delta > 0$ such that for all sufficiently small $\varepsilon$ and $x \in E_\delta(O_i)$ we have

$$E_x^\varepsilon \tau_{E_\delta(O_i)} < \exp(\gamma \varepsilon^{-2}).$$

**Proof.** Choose point $z \in D$ close to $O_i$. Put $\delta = \frac{|z - O_i|}{2}$. Connect $x$ with $O_i$ and $O_i$ with $z$ with the values of $S^+$ not exceeding $\tilde{\gamma}$ and $\tilde{\eta}$, the resulting function is called $\tilde{\varphi}_t$. The length of the time interval of $\tilde{\varphi}_t$ is uniformly bounded by $T_0$ for all $x \in G$. We extend $\tilde{\varphi}_t$ up to $T_0$ by using a trajectory of $\dot{x}_t = \tilde{b}(x_t)$ in $D \cup \partial D$ without making $S^+$ larger.

Now we have for $x \in E_\delta(O_i)$,

$$P_x^\varepsilon \{\tau_{E_\delta(O_i)} < T_0\} \geq P_x^\varepsilon \{\rho_0 T_0(X^\varepsilon, \tilde{\varphi}) < \delta\} \geq \exp(-0.9\gamma \varepsilon^{-2}).$$

Using the Markov property we see that

$$P_x^\varepsilon \{\tau_{E_\delta(O_i)} \geq nT_0\} \leq [1 - \exp(-0.9\gamma \varepsilon^{-2})]^n.$$

This yields

$$E_x^\varepsilon \tau_{E_\delta(O_i)} \leq T_0 \sum_{n=0}^\infty [1 - \exp(-0.9\gamma \varepsilon^{-2})]^n = T_0 \exp(0.9\gamma \varepsilon^{-2}).$$
Sacrificing $0.1\gamma$ in order to get rid of $T_0$ we get the desired result. □

**Lemma 3.4.** For any $\gamma > 0$ there exist $\rho_1 > 0$ such that for all sufficiently small $\varepsilon$ and $y \in \partial g_i$ we have

$$E_y^\varepsilon \int_0^{\sigma_0} \chi_{\mathcal{E}_{\rho_0}}(X_t^\varepsilon)dt > \exp(-\gamma \varepsilon^{-2}) .$$

**Proof.** Choose $\rho_1$ small. We connect $y \in \partial g_i$ with $O_i$ using a curve $\varphi_t$, extend it using the trajectory of $\dot{x}_t = \bar{b}(x_t)$ on $\partial D$ till first exit time $\sigma_0$ from $\mathcal{E}_{\rho_0}(O_i)$, with corresponding $S^+$ less than $0.5\gamma$. All the trajectories at a distance less than $\rho_1$ spends a time at least $t_0 > 0$ within $g_i$, uniformly for all $y \in \partial g_i$. The probability of all such trajectories is no less than $\exp(-0.9\gamma \varepsilon^{-2})$. Thus the expected value is no less than $t_0 \exp(-0.9\gamma \varepsilon^{-2})$. By sacrificing $0.1\gamma$ we can get rid of $t_0$. □

The rest of this section is devoted to the description of the algorithm for the calculation of the invariant measure and the metastable states. The proof we shall omit here follows [5, Ch.6] and [4].

Let $L$ be a finite set (in our case $L = \{1, 2, ..., l\}$), whose elements are denoted by letters $i, j, k, m, n$, etc. Let a subset $W$ be selected in $L$. A graph consisting of arrows $m \to n$ ($m \in L \setminus W, n \in L, n \neq m$) is called a $W$-graph if it satisfies the following conditions:

1. every point $m \in L \setminus W$ is the initial point of exactly one arrow;
2. there are no cycles in the graph.

Intuitively, a $W$-graph is a graph consisting of arrows starting from each point $m \in L \setminus W$, and going along a sequence of arrows leading to some point $n \in W$.

The set of $W$-graphs is denoted by $G(W)$. We shall use the letter $g$ to denote graphs.

Let $W(O_i) = \min_{g \in G(i)} \sum_{(m \to n) \in g} \bar{V}^+(O_m, O_n)$. It can be proved that

$$W(O_i) = \min_{g \in G(i)} \sum_{(m \to n) \in g} V^+(O_m, O_n) .$$

We have

**Theorem 3.4.** Let $\mu^\varepsilon$ be the normalized invariant measure of the process $X_t^\varepsilon$. Then for any $\gamma > 0$ there exists $\rho_1 > 0$ such that we have

$$\exp[-\varepsilon^{-2}(W(O_i) - \min_i W(O_i) + \gamma)] \leq \mu^\varepsilon(g_i) \leq \exp[-\varepsilon^{-2}(W(O_i) - \min_i W(O_i) - \gamma)]$$

for sufficiently small $\varepsilon > 0$. 

11
We shall say that a set $N \subset [D]$ is stable if for any $x \in N$, $y \not\in N$ we have $V^+(x,y) > 0$. One can show that for an unstable $O_j$ ($j = 1, ..., l$) there exist a stable $O_i$ ($i \neq j, i = 1, ..., l$) such that $V^+(O_i, O_j) = 0$.

**Theorem 3.5.** For $x \in [D]$ set

$$W(x) = \min \{W(O_i) + V(O_i, x)\},$$

where the minimum can be taken over either all of $O_1, ..., O_l$ or only stable ones. Let $\mu^\varepsilon$ be the normalized invariant measure of the process $X_t^\varepsilon$. Then for any $\gamma > 0$ there exists $\bar{\rho} > 0$ such that for any $0 < \rho < \bar{\rho}$ we have

$$\exp[-\varepsilon^{-2}(W(x) - \min_i W(O_i) + \gamma)] \leq \mu^\varepsilon(\mathcal{E}_\rho(x)) \leq \exp[-\varepsilon^{-2}(W(x) - \min_i W(O_i) - \gamma)]$$

for sufficiently small $\varepsilon > 0$.

Here $\mathcal{E}_\rho(x)$ is a $\rho$-neighborhood of $x$.

The above two theorems roughly say that as first $t \to \infty$ and then $\varepsilon \to 0$, the process $X_t^\varepsilon$ will be situated in one of the $O_i$'s which minimizes the values of $W(O_i)$ (it can be calculated either via all $O_1, ..., O_l$ or only via the stable ones). In generic case, when $\min_i W(O_i)$ is attained at some unique point $i$, we have for any $\delta > 0$,

$$\lim_{\varepsilon \to 0} \lim_{t \to \infty} P_{x}^\varepsilon \{|X_t^\varepsilon - O_i| > \delta\} = 0. \quad (3.1)$$

A natural question is that how the limiting distribution behaves when we take the limit in a coordinated way, i.e. take $\varepsilon \to 0$ and $t = t(\varepsilon^{-2}) \to \infty$. This is the problem of metastability and sublimiting distributions (see [2]). Let us assume that $T = \frac{T(\varepsilon)}{\varepsilon^{2}}$ and we consider $\lim_{\varepsilon \to 0} P_{x}^\varepsilon \{X_T^\varepsilon \in \Gamma\}$. In the generic case one can define a function $K^\varepsilon(x, \lambda) \in \{1, 2, ..., l\}$ such that

$$\lim_{\varepsilon \to 0} P_{x}^\varepsilon \{|X_T^\varepsilon - O_{K^\varepsilon(x, \lambda)}| > \delta\} = 0 \quad (3.2)$$

for any $\delta > 0$.

The algorithm to determine $K^\varepsilon(x, \lambda)$ is as follows. First we consider for each $O_i$ (the rank 0 cycle) the ”next” most probable $\omega$-limit set $N(O_i)$ that we are going to jump to. Continuing this determination of ”next” states we form the rank 1 cycle $O_i \to N(O_i) \to N^2(O_i) \to ... \to N^{m_i}(O_i)$. We stop once we get a repetition $N(N^{m_i}(O_i)) = O_i$. Cycles generated by distinct initial points $i \in \{1, ..., l\}$ either do not intersect each other or coincide: in the latter case the cycle order on them is one and the same.
We continue by recurrence. Let the cycles of rank \((k-1)\) be \(\pi_{k-1}^1, ..., \pi_{n_k-1}^{k-1}\). Starting from each \((k-1)\)-cycle \(\pi_{i}^{k-1}\) one can determine the "next" most probable \((k-1)\)-cycle \(\mathcal{N}(\pi_{i}^{k-1})\) that we will first jump to. Continuing this determination we form a rank \(k\) cycle \(\pi_{i}^{k-1} \rightarrow \mathcal{N}(\pi_{i}^{k-1}) \rightarrow ... \rightarrow \mathcal{N}^{m_{i}^{k-1}}(\pi_{i}^{k-1})\). We stop once we get a repetition \(\mathcal{N}(\mathcal{N}^{m_{i}^{k-1}}(\pi_{i}^{k-1})) = \pi_{i}^{k-1}\). Cycles of rank \(k\) generated by distinct cycles of rank \(k-1\) either do not intersect each other or coincide.

In this way we can continue until the last cycle which is the whole of \(\{O_1, ..., O_l\}\). The metastable states are determined by the timescale of the cycles that we traverse.

Let us be more precise. Starting from a cycle \(\pi\), to determine the "next" cycle \(\mathcal{N}(\pi)\) that we first jump to, we calculate

\[
A(\pi) = \min_{g \in G(L \setminus \pi)} \sum_{(m \rightarrow n) \in g} V^+(O_m, O_n). \tag{3.3}
\]

Here \(L = \{1, 2, ..., l\}\). The minimum of the above expression determines a \(L \setminus \pi\) graph consisting of chains of arrows leading to the first state in \(L \setminus \pi\) we jump to.

We put

\[
C(\pi) = A(\pi) - \min_{i \in \pi} \min_{g \in G_{\pi \{i\}}} \sum_{(m \rightarrow n) \in g} V^+(O_m, O_n). \tag{3.4}
\]

Here \(G_{\pi \{i\}}\) is the set of \(\{i\}\)-graphs restricted to \(\pi\). Then the asymptotic exit time from \(\pi\) is of order \(\propto \exp\left(\frac{C(\pi)}{\varepsilon^2}\right)\).

Starting from \(i = i(x)\) (which is the label for the first equilibrium among \(O_1, ..., O_l\) that we approach in finite time, starting from \(x\)), let \(\pi, \pi', ..., \pi^{(s)}\) be cycles of next to the last rank, unified into the last cycle, which exhausts \(\{1, 2, ..., l\}\). If the constant \(\lambda\) is greater than \(C(\pi), C(\pi'), ..., C(\pi^{(s)})\), then over time of order \(\exp(\lambda \varepsilon^{-2})\) the process can traverse all these cycles many times (and all cycles of smaller rank inside them) and the limiting distribution is concentrated on that one of the cycles for which \(C(\pi), C(\pi'), ..., C(\pi^{(s)})\) is the greatest. Within this cycle, it is concentrated on that one of the subcycles for which the corresponding constant \(C(\bullet)\) in (3.4) is the greatest possible, and so on up to points (one point in the generic case) \(O^{K^*}_{K^*(x, \lambda)}\). This point \(O^{K^*}_{K^*(x, \lambda)}\) is the metastable state in (3.2).

### 4 Application to PDE

The solution of (1.1) can be represented through process (1.2) by the formula

\[
u^\varepsilon(x, t) = E_{x}g(X^\varepsilon_{t}).
\]

This is an immediate consequence of the following generalized Itô's formula:
Lemma 4.1. Assume process $(X^\varepsilon_t, \xi^\varepsilon_t)$ is given by (1.2), $X^\varepsilon_0 = x$. Let $u(x,t)$ be of class $C^{2,1}([\mathbb{R}^d \times \mathbb{R}_+])$ with uniform bounded derivatives up to the second order in $x$ and up to the first order in $t$. Then we have

$$u(X^\varepsilon_t, t) - u(x, 0) = \int_0^t \left( \frac{\partial}{\partial s} + \mathcal{L}^\varepsilon \right) u(X^\varepsilon_s, s) ds + \int_0^t \nabla u(X^\varepsilon_s, s) \cdot \gamma(X^\varepsilon_s) d\xi^\varepsilon_s + \int_0^t \nabla u(X^\varepsilon_s, s) \cdot \sigma(X^\varepsilon_s) dW_s.$$

For a proof of this theorem see [7, Section 3].

Our answer to the problem (1.1) is

Theorem 4.1. Under all our assumptions, in generic case, for $T(\varepsilon) \asymp \exp(\frac{\lambda}{\varepsilon^2})$, we have

$$\lim_{\varepsilon \to 0} u^\varepsilon(x, T(\varepsilon)) = g(O_{K^*(x, \lambda)}),$$

where $K^*(x, \lambda)$ is defined as in Section 3.3.

5 Example

Consider an example. Let the domain $D$ be a unit disk $B(1) = \{(y_1, y_2); y_1^2 + y_2^2 < 1\}$ in $\mathbb{R}^2$. Let the smooth vector field $b_y(y_1, y_2)$ be given such that $\bar{b}_y(y_1, y_2) = (\bar{b}_{y_1}(y_1, y_2), \bar{b}_{y_2}(y_1, y_2))$ is as in Fig.1. We consider the problem

$$\begin{cases}
\frac{\partial u^\varepsilon(y_1, y_2, t)}{\partial t} = \frac{\varepsilon^2}{2} \Delta_{y_1, y_2} u^\varepsilon(y_1, y_2, t) + b_y(y_1, y_2) \cdot \nabla u^\varepsilon(y_1, y_2, t), & \varepsilon > 0; \\
u^\varepsilon(y_1, y_2, 0) = g(y_1, y_2), & y_1^2 + y_2^2 \leq 1; \\
\frac{\partial u^\varepsilon}{\partial r}(y_1, y_2, t) = 0, & y_1^2 + y_2^2 = 1, \ t \geq 0.
\end{cases}$$

(5.1)

Here $\frac{\partial}{\partial r}$ is the derivative with respect to the inward unit normal. The action functional takes the form

$$S_{0T}^\varepsilon(\varphi) = \begin{cases} 
\frac{1}{2} \int_0^T |\dot{\varphi}_s - \bar{b}_y(\varphi_s)|_{\mathbb{R}^2}^2 ds, & \varphi \in C_{[0,T]}([D]) \text{ absolutely continuous}, \varphi_0 = x; \\
+\infty, & \text{for the rest of } \varphi \in C_{[0,T]}([D]).
\end{cases}$$

(5.2)

We calculate the "quasi-potential" using (5.2)
\[
V^+(x, y) = \inf_{\varphi \in C([0,T]([D]))} \{ S^+_{0T}(\varphi), \varphi_0 = x, \varphi_T = y, \varphi_t \in D \cup \partial D, 0 \leq t \leq T < \infty \}.
\]

The \( \omega \)-limit sets of the dynamical system \( \dot{x}_t = \vec{b}_y(x_t) \) are the zeros of the vector field \( \vec{b}_y(x) \) on \( \partial D = S^1 \). (And also the origin but it is unstable so that we neglect it.) In Fig.1 the points \( O_1, O_3 \) and \( O_5 \) are stable equilibriums are the points \( O_2, O_4 \) and \( O_6 \) are unstable ones. We can consider only the quasi-potentials between the stable ones.

Suppose we have \( V^+(O_1, O_3) = 1, V^+(O_3, O_1) = 2, V^+(O_1, O_5) = 6, V^+(O_5, O_1) = 7, V^+(O_3, O_3) = 3, V^+(O_3, O_5) = 4. \)

We are concerned with the limit \( \lim_{\varepsilon \downarrow 0} u_\varepsilon(y_1, y_2, T(\varepsilon)) \) for \( T(\varepsilon) \approx \exp(\frac{1}{\varepsilon^2}) \). Starting from the initial point \((y_1, y_2)\), we suppose that we are attracted to \( O_1 \) first. By calculating \( \min_{g \in G(L\{1\})} \sum_{(m \to n) \in g} V^+(O_m, O_n) = 1 \) we see that over time \( \exp(\frac{1}{\varepsilon^2}) \) we are going to jump to \( O_3 \) first. We then calculate \( \min_{g \in G(L\{3\})} \sum_{(m \to n) \in g} V^+(O_m, O_n) = 2 \) and we see that over time \( \exp(\frac{2}{\varepsilon^2}) \) we will jump from \( O_3 \) back to \( O_1 \) and we form a cycle \( \pi^{(1)} = \{1, 3\} \) of rank 1. We then calculate \( A(\pi^{(1)}) = \min_{g \in G(L\{\pi^{(1)}\})} \sum_{(m \to n) \in g} V^+(O_m, O_n) = V^+(O_1, O_3) + V^+(O_3, O_5) = 5 \) and the first state out of cycle \( \pi^{(1)} \) that we are going to jump to is \( O_5 \). Within cycle \( \pi^{(1)} \) we are mostly staying in \( O_3 \). We calculate \( C(\pi^{(1)}) = 5 - \min_{i \in \{1,3\}} \min_{g \in G(L\{\{i\}\})} \sum_{(m \to n) \in g} V^+(O_m, O_n) = 4. \) This means, that over time \( \exp(\frac{4}{\varepsilon^2}) \) we are jumping from \( O_3 \) to \( O_5 \). We then calculate \( \min_{g \in G(L\{5\})} \sum_{(m \to n) \in g} V^+(O_m, O_n) = 3 \) and we see that we are jumping from \( O_5 \) out to \( O_3 \) in time \( \exp(\frac{3}{\varepsilon^2}) \). This implies that within the cycle \( \pi^{(2)} = \{1, 3, 5\} \) which exhausts all \( \omega \)-limit sets, we are mostly staying in \( \pi^{(1)}, \)
and within $\pi^{(1)}$ it is $O_3$.

Our result can be summarized as

$$\lim_{\varepsilon \downarrow 0} u^{\varepsilon}(y_1, y_2, T(\varepsilon)) = g(O_1) \text{ for } T(\varepsilon) \asymp \exp\left(\frac{\lambda}{\varepsilon^2}\right) \text{ and } 0 < \lambda < 1;$$

$$\lim_{\varepsilon \downarrow 0} u^{\varepsilon}(y_1, y_2, T(\varepsilon)) = g(O_3) \text{ for } T(\varepsilon) \asymp \exp\left(\frac{\lambda}{\varepsilon^2}\right) \text{ and } 1 \leq \lambda.$$

Acknowledgement: We would like to thank our advisor Professor Mark Freidlin for posing this problem to us and for many useful discussions.

References

[1] Anderson, R.F., Orey, S., Small random perturbations of dynamical systems with reflecting boundary, Nagoya Math J., 60 (1976), 189–216.

[2] Freidlin, M., Sublimiting Distributions and Stabilization of Solutions of Parabolic Equations with a Small Parameter, Soviet Math. Dokl., 235, 5, 1042–1045, 1977.

[3] Freidlin, M., Functional integration and partial differential equations, Annals of Mathematical Studies, Princeton University Press, 1985.

[4] Freidlin, M., Wentzell, A., On small random perturbations of dynamical systems, Russ. Math. Surv., 25 (1970), No.1, 1–56.

[5] Freidlin, M., Wentzell, A., Random perturbations of dynamical systems, Second Edition, Springer, 1998.

[6] Freidlin, M., Zhivoglyadova, L., Boundary value problems with a small parameter for a diffusion process with reflection, Russ. Math. Surv., 31 (1976), No.5, 241–242 (in Russian).

[7] Gikhman, I., Skorokhod, A., The Theory of Stochastic Processes, III, Springer, 1979.

[8] Itô, K., McKean, H.P. Jr., Diffusion processes and their sample paths, Springer, 1974.

[9] Sato, K., Tanaka, H., Local times on the boundary for multidimensional reflecting diffusion, Proc. Japan Acad., 38, 10 (1962), 699–702.

[10] Sato, K., Ueno, T., Multidimensional diffusion and Markov processes on the boundary, J. Math. Kyoto U., 4 (1965), 529–605.
[11] Watanabe, S., On stochastic differential equations for multidimensional diffusion processes with boundary, I, II, *J. Math. Kyoto U.*, 11 (1971), 169–180, 545–551.