Central Extensions of the families of Quasi-unitary Lie algebras

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Abstract

The most general possible central extensions of two whole families of Lie
algebras, which can be obtained by contracting the special pseudo-unitary al-
gebras $su(p,q)$ of the Cartan series $A_l$ and the pseudo-unitary algebras $u(p,q)$,
are completely determined and classified for arbitrary $p$, $q$. In addition to the
$su(p,q)$ and $u(p,q)$ algebras, whose second cohomology group is well known
to be trivial, each family includes many non-semisimple algebras; their central
extensions, which are explicitly given, can be classified into three types as far
as their properties under contraction are involved. A closed expression for the
dimension of the second cohomology group of any member of these families
of algebras is given.

1 Introduction

This paper investigates the Lie algebra cohomology of the unitary Cayley–Klein
(CK) families of Lie algebras in any dimension. These families, also called ‘quasi-
unitary’ algebras, include both the special (pseudo-)unitary $su(p,q)$ and (pseudo-)uni-
tary $u(p,q)$ algebras —which have only trivial central extensions—, as well as
many other obtained from these by a sequence of contractions, which are no longer
semisimple and may have non-trivial central extensions.
The paper can be considered as a further step in a series of studies on the CK families of Lie algebras. These have both mathematical interest and physical relevance. The families of CK algebras provide a frame to describe the behaviour of mathematical properties of algebras under contraction; in physical terms this is related to some kind of approximation. The central extensions for the family of quasi-orthogonal algebras, also in the general situation and for any dimension, have been determined in a previous paper [1]. We refer to this work for references and for physical motivations; we simply remark here that there are three main reasons behind the interest in the second cohomology groups for Lie algebras. First, in any quantum theory the relevant representations of any symmetry group are projective instead of linear ones. Second, homogeneous symplectic manifolds under a group appear as orbits of the coadjoint representation of either the group itself or of a central extension. And third, quasi-invariant Lagrangians are also directly linked to the central extensions of the group; these can be related also to Wess–Zumino terms. In addition to the references in [1], we may add that Wess–Zumino–Witten models leading to central extensions have also been studied (see e.g. [2, 3] and references therein).

The knowledge of the second cohomology group for a Lie algebra relies on the general solution of a set of linear equations, yet some general results allow to bypass the calculations in special cases. For instance, the second cohomology group is trivial for semisimple Lie algebras. But once a contraction is made, the semisimple character disappears, and the contracted algebra might have non-trivial central extensions. Instead of finding the general solution for the extension equations on a case-by-case basis, our approach is to do these calculations for a whole family including a large number of algebras simultaneously. This program has been developed for the quasi-orthogonal algebras, and here we discuss the ‘next’ quasi-unitary case. There are two main advantages in this approach. First, it allows to record, in a form easily retrievable, a large number of results which can be needed in applications, both in mathematics and in physics. This avoids at once and for all the case-by-case type computation of the central extensions of algebras included in the unitary families. And second, it sheds some further light on the interrelations between cohomology and contractions, by discussing in particular examples how and when a contraction increases the cohomology of the algebra: central extensions can be classed into three types, with different behaviour under contraction.

The section 2 is devoted to the description of the two families of unitary CK algebras. We show how to obtain these as graded contractions of the compact algebras $su(N+1)$ and $u(N+1)$, and we provide some details on their structure. It should be remarked that the CK unitary algebras are associated to the complex hermitian spaces with metrics of different signatures and to their contractions. In section 3 the general solution to the central extension problem for these algebras is given; this includes the completely explicit description of all possible central extensions and the discussion of their triviality. A closed formula for the dimension of the second cohomology group is also obtained. Computational details on the procedure to solve the central extension problem are given in an Appendix. The results are
illustrated in section 4 for the lowest dimensional examples. Finally, some remarks close the paper.

2 The CK families of quasi-unitary algebras

The family of special quasi-unitary algebras, which involves the simple Lie algebras $su(p, q)$, as well as many non-simple algebras obtained by İnönü–Wigner [4] contraction from $su(p, q)$ can be easily described in terms of graded contraction theory [5, 6], taking the compact real form $su(N + 1)$ of the simple algebras in the series $A_N$ as starting point. As it is well known, the special unitary algebra can be realised by complex antihermitian and traceless matrices, and is the quotient of the algebra of all complex antihermitian matrices by its center (generated by the pure imaginary multiples of the identity). It will be convenient to consider the family of quasi-unitary algebras altogether; these can be similarly described in terms of graded contractions of $u(N + 1)$, and will include algebras obtained from $u(p, q)$ by İnönü–Wigner contractions. Let us consider the (fundamental) matrix representation of the algebras $su(N + 1)$ and $u(N + 1)$, as given by the complex matrices $J_{ab}$, $M_{ab}$, $B_l$ and $J_{ab}$, $M_{ab}$, $B_l$, $I$:

$$J_{ab} = -e_{ab} + e_{ba} \quad M_{ab} = i(e_{ab} + e_{ba}) \quad B_l = i(e_{l-1,l-1} - e_{ll}) \quad I = i \sum_{a=0}^{N} e_{aa},$$

where $a < b$, $a, b = 0, \ldots, N$, $l = 1, \ldots, N$, and where $e_{ab}$ means the $(N+1) \times (N+1)$ matrix with a single 1 entry in row $a$ and column $b$. The commutation relations involved in either of these algebras are given by

$$[J_{ab}, J_{ac}] = J_{bc}, \quad [J_{ab}, J_{bc}] = -J_{ac}, \quad [J_{ac}, J_{bc}] = J_{ab},$$
$$[M_{ab}, M_{ac}] = J_{bc}, \quad [M_{ab}, M_{bc}] = J_{ac}, \quad [M_{ac}, M_{bc}] = J_{ab},$$
$$[J_{ab}, M_{ac}] = M_{bc}, \quad [J_{ab}, M_{bc}] = -M_{ac}, \quad [M_{ac}, M_{bc}] = -M_{ab},$$
$$[J_{ab}, J_{de}] = 0, \quad [M_{ab}, M_{de}] = 0, \quad [J_{ab}, M_{de}] = 0,$$

$$[J_{ab}, B_l] = (\delta_{a,l-1} - \delta_{b,l-1} + \delta_{bl} - \delta_{al})M_{ab},$$
$$[M_{ab}, B_l] = -(\delta_{a,l-1} - \delta_{b,l-1} + \delta_{bl} - \delta_{al})J_{ab}$$

$$[J_{ab}, M_{ab}] = -2 \sum_{s=a+1}^{b} B_s, \quad [B_k, B_l] = 0$$
$$[J_{ab}, I] = 0, \quad [M_{ab}, I] = 0, \quad [B_l, I] = 0.$$

The algebra $su(N + 1)$ has a grading by a group $\mathbb{Z}_2^\otimes N$ related to a set of $N$ commuting involutions in the subalgebra $so(N + 1)$ generated by $J_{ab}$ [7, 8]. If $S$ denotes any subset of the set of indices $\{0, 1, \ldots, N\}$, and $\chi_S(a)$ denotes the characteristic function over $S$, then each of the linear mappings given by

$$S_SJ_{ab} = (-1)^{\chi_S(a) + \chi_S(b)}J_{ab} \quad S_SM_{ab} = (-1)^{\chi_S(a) + \chi_S(b)}M_{ab} \quad S_SB_l = B_l$$

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is an involutive automorphism of the algebra $su(N+1)$; by considering all possible subsets of indices we get $2^N$ different automorphisms defining a $\mathbb{Z}_2^N$ grading for this algebra. The corresponding graded contractions of $su(N+1)$ constitute a large set of Lie algebras, but there exists a particular subset or family of these graded contractions, nearer to the simple ones, which essentially preserves the properties associated to simplicity, and which belong to the so termed [9, 10] ‘quasi-simple’ algebras. This family, to be defined below, encompasses the special pseudo-unitary algebras (in the $A_N$ Cartan series) as well as their nearest non-simple contractions. By taking the generator $I$ as invariant under all involutions, this grading can be extended to the algebra $u(N+1)$, whose graded contractions include the pseudo-unitary algebras as well as many non-semisimple algebras; again a particular family of these graded contractions, to be introduced below, preserves properties associated to semi-simplicity. Collectively, all these algebras (special or not) are called quasi-unitary; these are also called Cayley–Klein algebras of unitary type, or unitary CK algebras, since they are exactly those algebras behind the geometries of a complex hermitian space with a projective metric in the CK sense [10]. Another view to these algebras is given in [11].

The overall details on the structure of this family are similar to the orthogonal case. The set of unitary CK algebras is parametrised by $N$ real coefficients $\omega_a$ ($a = 1, \ldots, N$), whose values codify in a convenient way the pertinent information on the Lie algebra structure [12, 13]. In terms of the $N(N+1)/2$ two-index coefficients $\omega_{ab}$ defined by

$$\omega_{ab} := \omega_{a+1} \omega_{a+2} \cdots \omega_b \quad a, b = 0, 1, \ldots, N \quad a < b \quad \omega_{aa} := 1$$

which verify

$$\omega_{ac} = \omega_{ab} \omega_{bc} \quad a \leq b \leq c \quad \omega_a = \omega_{a-1} a = 1, \ldots, N,$$

the algebras to be denoted $su(\varnothing)(N+1)$ and $u(\varnothing)(N+1)$, $\varnothing \equiv (\omega_1, \ldots, \omega_N)$, of dimensions $(N+1)^2 - 1$ and $(N+1)^2$, are generated by $J_{ab}$, $M_{ab}$, $B_l$ and $J_{ab}$, $M_{ab}$, $B_l$, $I$ ($a < b$), with commutators:

$$[J_{ab}, J_{ac}] = \omega_{ab} J_{bc} \quad [J_{ab}, J_{bc}] = -J_{ac} \quad [J_{ac}, J_{bc}] = \omega_{bc} J_{ab}$$
$$[M_{ab}, M_{ac}] = \omega_{ab} J_{bc} \quad [M_{ab}, M_{bc}] = J_{ac} \quad [M_{ac}, M_{bc}] = \omega_{bc} M_{ab}$$
$$[J_{ab}, J_{de}] = 0 \quad [M_{ab}, M_{de}] = 0 \quad [J_{ab}, M_{de}] = 0$$

$$[J_{ab}, B_l] = (\delta_{a,l-1} - \delta_{b,l-1} + \delta_{bl} - \delta_{al}) M_{ab}$$
$$[M_{ab}, B_l] = -((\delta_{a,l-1} - \delta_{b,l-1} + \delta_{bl} - \delta_{al}) J_{ab}$$

$$[J_{ab}, M_{ab}] = -2\omega_{ab} \sum_{s=a+1}^{b} B_s \quad [B_k, B_l] = 0$$

$$[J_{ab}, I] = 0 \quad [M_{ab}, I] = 0 \quad [B_l, I] = 0$$

where $a, b, c, d, e = 0, \ldots, N$ and $k, l = 1, \ldots, N$; we assume $a < b < c$ for each set of three indices $\{a, b, c\}$, and $a < b, d < e$ for each set of four indices $\{a, b, d, e\}$ which are also assumed to be different.
2.1 The unitary CK groups

The connection with groups of isometries of a hermitian metric is as follows: for a generic choice, with all \( \omega_a \neq 0 \), let us consider the space \( \mathbb{C}^{N+1} \) endowed with a hermitian (sesqui)linear form \( \langle \cdot | \cdot \rangle_\omega : \mathbb{C}^{N+1} \times \mathbb{C}^{N+1} \to \mathbb{C} \) associated to the matrix

\[
\mathcal{I}_\omega = \text{diag} (1, \omega_{01}, \omega_{02}, \ldots, \omega_{0N}) = \text{diag} (1, \omega_1, \omega_2, \ldots, \omega_N); \tag{2.11}
\]

this is, for any pair of vectors \( a, b \in \mathbb{C}^{N+1}, \)

\[
\langle a | b \rangle_\omega := \bar{a}^0 b^0 + \bar{a}^1 \omega_1 b^1 + \bar{a}^2 \omega_1 \omega_2 b^2 + \ldots = \sum_{i=0}^{N} \bar{a}^i \omega_0^i b^i. \tag{2.12}
\]

Let us define the group \( U_{\omega_1,\ldots,\omega_N}(N+1) \equiv U_\omega(N+1) \) as the group of linear isometries of the hermitian metric (2.11). The isometry condition

\[
\langle U_a | U_b \rangle_\omega = \langle a | b \rangle_\omega \quad \forall a, b \in \mathbb{C}^{N+1}, \tag{2.13}
\]

implies for the matrix \( U \in U_\omega(N+1) \) the condition

\[
U^\dagger \mathcal{I}_\omega U = \mathcal{I}_\omega \quad \forall U \in U_\omega(N + 1). \tag{2.14}
\]

For the corresponding Lie algebra the above relation leads to

\[
X^\dagger \mathcal{I}_\omega + \mathcal{I}_\omega X = 0 \quad \forall X \in u_\omega(N + 1). \tag{2.15}
\]

This Lie algebra is generated by the complex matrices (cf. (2.1))

\[
J_{ab} = -\omega_{ab} e_{ab} + e_{ba} \quad M_{ab} = i(\omega_{ab} e_{ab} + e_{ba}) \quad B_l = i(e_{1-l-1} - e_{ll}) \quad I = i \sum_{a=0}^{N} e_{aa} \tag{2.16}
\]

with \( a < b, \ a, b = 0, \ldots, N, \ l = 1, \ldots, N. \)

The group \( SU_{\omega_1, \ldots, \omega_N}(N + 1) \equiv SU_\omega(N + 1) \) is defined similarly by adding the unimodularity condition \( \det(U) = 1 \); this leads for the Lie algebra to the condition \( \text{trace}(X) = 0 \), so the algebra \( su_\omega(N + 1) \) is generated by \( J_{ab}, M_{ab}, B_l \) alone.

The action of the groups \( U_\omega(N + 1) \) and \( SU_\omega(N + 1) \) in \( \mathbb{C}^{N+1} \) is not transitive, and the ‘sphere’ with equation

\[
\langle x | x \rangle_\omega := \sum_{i=0}^{N} \bar{x}^i \omega_0^i x^i = 1 \tag{2.17}
\]

is stable. For the action of \( SU_\omega(N + 1) \), the isotropy subgroup of a reference point in this sphere, say \( (1, 0, \ldots, 0) \), is easily shown to be isomorphic to \( SU_{\omega_2, \omega_3, \ldots, \omega_N}(N) \), and the isotropy subgroup of the ray of a reference point is \( U_{\omega_2, \omega_3, \ldots, \omega_N}(N) \), locally isomorphic to \( U(1) \otimes SU_{\omega_2, \omega_3, \ldots, \omega_N}(N) \). The quotient spaces \( SU_{\omega_1, \omega_2, \omega_3, \ldots, \omega_N}(N + 1)/(U(1) \otimes SU_{\omega_2, \omega_3, \ldots, \omega_N}(N)) \) are a family of hermitian spaces which includes examples.
with non-definite and/or degenerate hermitian metrics; the CK scheme provides a common frame to discuss all them jointly. The most familiar corresponds to \( \omega_2 = \omega_3 = \ldots = \omega_N = 1 \), and depends on a single parameter \( \omega_1 = K \); when \( K > 0 \) or \( K < 0 \) these are the usual elliptic or hyperbolic complex hermitian spaces of (holomorphic constant) curvature \( K \); when \( \omega_1 = 0 \) we get the ‘Euclidean’ flat hermitian space (finite-dimensional Hilbert space).

When the constants \( \omega_a \) are allowed to vanish, the set of isometries of the hermitian metric (2.11) is larger than the group generated by the matrices \( J_{ab}, M_{ab}, B_l, I \). In this case, there are additional geometric structures in \( \mathbb{C}^{N+1} \) (related to the existence of additional invariant foliations similar to the one implied by (2.17)), and the proper definition of the automorphism group of these structures leads again to the group generated by the matrix Lie algebra (2.16) with the commutation relations (2.8)–(2.10). These matrix realisations can be considered as the fundamental representation of the unitary CK Lie algebras \( su_\omega(N+1) \) and \( u_\omega(N+1) \).

Since each coefficient \( \omega_a \) can be positive, negative or zero, each unitary CK family comprises \( 3^N \) Lie algebras although some of them may be isomorphic. For instance, the map

\[
J_{ab} \rightarrow J'_{ab} = -J_{N-b,N-a} \quad M_{ab} \rightarrow M'_{ab} = -M_{N-b,N-a} \quad B_l \rightarrow B'_l = B_{N+1-l} \quad (2.18)
\]

provides an isomorphism

\[
su_{\omega_1,\omega_2,\ldots,\omega_{N-1},\omega_N}(N+1) \simeq su_{\omega_N,\omega_N-1,\ldots,\omega_2,\omega_1}(N+1). \quad (2.19)
\]

### 2.2 Structure of the unitary CK algebras

The unitary CK algebras \( su_\omega(N+1) \) contain many subalgebras isomorphic to algebras in both families \( su_\omega(M+1) \) and \( u_\omega(M+1), M < N \). To best describe this, we introduce a new set of Cartan subalgebra generators for \( su_\omega(N+1), G_a \) \((a = 1, \ldots, N)\), defined by

\[
G_a := \frac{1}{a}(B_1 + 2B_2 + \ldots + (a-1)B_{a-1}) + B_a + \frac{1}{N+1-a}(N-a)B_{a+1} + (N-a-1)B_{a+2} + \ldots + B_N. \quad (2.20)
\]

In the matrix realisation (2.16) \( G_a \) is given by

\[
G_a = i\left(\frac{1}{a} \sum_{s=0}^{a-1} e_{ss} - \frac{1}{N+1-a} \sum_{s=a}^{N} e_{ss}\right). \quad (2.21)
\]

so each \( G_a \) appears as a direct sum of two blocks, each proportional with a pure imaginary coefficient to the identity matrix.

Denoting by \( X_{ij} \) the pair of generators \( \{J_{ij}, M_{ij}\} \), we can check that the set \( \langle X_{ij}, i, j = 0, 1, \ldots, a-1; B_l, l = 1, \ldots, a-1 \rangle \) closes a Lie subalgebra \( su_{\omega_1,\ldots,\omega_{a-1}}(a) \).
Furthermore, $G_a$ commutes with all the generators in this subalgebra, so that the former generators plus $aG_a$ close an algebra isomorphic to $u_{\omega_1,...,\omega_{a-1}}(a)$.

Similarly, the set $\langle X_{ij}, i, j = a, a + 1, \ldots, N; B_l, l = a + 1, \ldots, N \rangle$ closes the special unitary CK Lie algebra $su_{\omega_{a+1},\ldots,\omega_N}(N+1-a)$, and by adding $-(N+1-a)G_a$ we get an algebra isomorphic to $u_{\omega_{a+1},\ldots,\omega_N}(N+1-a)$.

This structure can be visualised by arranging the basis generators as in Fig. 2.1.

\begin{center}
\begin{tabular}{cccccccc}
$X_{01}$ & $X_{02}$ & $\ldots$ & $X_{0a-1}$ & $X_{0a}$ & $X_{0a+1}$ & $\ldots$ & $X_{0N}$ \\
B_1 & X_{12} & $\ldots$ & $X_{1a-1}$ & $X_{1a}$ & $X_{1a+1}$ & $\ldots$ & $X_{1N}$ \\
$\vdots$ & $\vdots$ & $\ddots$ & $\vdots$ & $\vdots$ & $\vdots$ & $\ddots$ & $\vdots$ \\
$B_{a-1}$ & $X_{a-2a-1}$ & $\ldots$ & $X_{a-1}$ & $X_{a-2a}$ & $X_{a-2a+1}$ & $\ldots$ & $X_{a-2N}$ \\
$B_a$ & $\vdots$ & $\ddots$ & $\vdots$ & $\vdots$ & $\vdots$ & $\ddots$ & $\vdots$ \\
$B_{a+1}$ & $\vdots$ & $\ddots$ & $\vdots$ & $\vdots$ & $\vdots$ & $\ddots$ & $\vdots$ \\
$B_{N-1}$ & $X_{N-2N-1}$ & $X_{N-2N}$ & $\ldots$ & $X_{aN}$ & $\ldots$ & $X_{aN}$ & $\ldots$ \\
$B_N$ & & & & & & & \\
\end{tabular}
\end{center}

Figure 2.1: Generators of the (special) unitary CK algebras

The special unitary subalgebras $su_{\omega_1,...,\omega_{a-1}}(a)$ and $su_{\omega_{a+1},\ldots,\omega_N}(N+1-a)$ correspond, in this order, to the two triangles to the left and below the rectangle, both excluding the generator $G_a$. The unitary subalgebras $u_{\omega_1,...,\omega_{a-1}}(a)$ and $u_{\omega_{a+1},\ldots,\omega_N}(N+1-a)$ correspond, in this order, to the two triangles to the left and below the rectangle, both including the generator $G_a$. This generator $G_a$ closes a $u(1)$ subalgebra.

We sum up the details relative to the structure of the special unitary CK algebras in two statements

- When all $\omega_a$ are different from zero, $su_{\omega}(N+1)$ is a pseudo-unitary simple Lie algebra $su(p,q)$ in the Cartan series $A_N$ ($p$ and $q$ are the number of positive and negative signs in diagonal of the metric matrix (2.11), $p + q = N + 1$).

- If a coefficient $\omega_a$ vanishes, the CK algebra is a non-simple Lie algebra which has a semidirect structure

$$su_{\omega_1,...,\omega_{a-1},\omega_a=0,\omega_{a+1},\ldots,\omega_N}(N+1) \equiv t \circ (su_{\omega_1,...,\omega_{a-1}}(a) \oplus u(1) \oplus su_{\omega_{a+1},\ldots,\omega_N}(N+1-a)),$$

where the subalgebras appearing in (2.22) are generated by

$$t = \langle X_{ij}, i = 0, 1, \ldots, a - 1, j = a, a + 1, \ldots, N \rangle$$
$$su_{\omega_1,...,\omega_{a-1}}(a) = \langle X_{ij}, i, j = 0, 1, \ldots, a - 1 ; B_l, l = 1, \ldots, a - 1 \rangle$$
$$u(1) = \langle G_a \rangle$$
$$su_{\omega_{a+1},\ldots,\omega_N}(N+1-a) = \langle X_{ij}, i, j = a, a + 1, \ldots, N; B_l, l = a + 1, \ldots, N \rangle.$$

(2.23)
We note that \( t \) is an abelian subalgebra of dimension \( 2a(N + 1 - a) \). In terms of the triangular arrangement of generators (Fig. 2.1), \( t \) is spanned by the generators inside the rectangle; we remark that these generators do not close a subalgebra when \( \omega_a \neq 0 \). The three remaining sets are always subalgebras, no matter of whether or not \( \omega_a = 0 \).

For the particular case \( \omega_1 = 0 \) (or, \textit{mutatis mutandis}, \( \omega_N = 0 \)) the contracted algebra is a quasi-unitary inhomogeneous algebra,

\[
su_{0,\omega_2,\ldots,\omega_N}(N + 1) \equiv t_{2N} \oplus u_{\omega_2,\ldots,\omega_N}(N).
\]

The subindex \( 2N \) in \( t \) denotes the real dimension of \( t \equiv \mathbb{C}^N \) which can be identified with the space \( SU_{0,\omega_2,\ldots,\omega_N}(N + 1)/U_{\omega_2,\ldots,\omega_N}(N) \), with the natural action of \( U_{\omega_2,\ldots,\omega_N}(N) \) (locally isomorphic to \( U(1) \otimes SU_{\omega_2,\ldots,\omega_N}(N) \)) over \( \mathbb{C}^N \). This direct product appeared as the isotropy subalgebra of a ray for the natural action of \( SU_{0,\omega_2,\ldots,\omega_N}(N+1) \) on \( \mathbb{C}^{N+1} \) discussed after (2.17). In the case where \( \omega_2, \omega_3, \ldots, \omega_N \) are all different from zero, the algebra is an ordinary inhomogeneous pseudo-unitary (not special) algebra:

\[
t_{2N} \oplus u_{\omega_2,\ldots,\omega_N}(N) \equiv iu(p, q) \quad p + q = N
\]

and in this case \( t_{2N} \) can be identified to the \( N \)-dimensional flat complex hermitian space with signature \( p, q \) determined as the number of positive and negative terms in the sequence \( (1, \omega_2, \omega_2 \omega_3, \ldots, \omega_2 \ldots \omega_N) \).

When several coefficients \( \omega_a \) are equal to zero the algebra \( su_{\omega_1,\omega_2,\ldots,\omega_N}(N+1) \) has simultaneously several such decompositions. The more contracted case corresponds to taking all \( \omega_a \) equal to zero; this gives rise to the special unitary flag algebra.

### 3 Central extensions

Now we proceed to compute in a unified way all the central extensions for the two unitary families of CK algebras, for arbitrary choices of the constants \( \omega_a \) and in any dimension. Let \( \mathcal{G} \) be an arbitrary \( r \)-dimensional Lie algebra with generators \( \{X_1, \ldots, X_r\} \) and structure constants \( C_{ij}^k \). A central extension \( \mathcal{T} \) of the algebra \( \mathcal{G} \) by the one-dimensional algebra generated by \( \Xi \) will have \( (r + 1) \) generators \( (X_i, \Xi) \) with commutation relations given by

\[
[X_i, X_j] = \sum_{k=1}^r C_{ij}^k X_k + \xi_{ij} \Xi \quad [\Xi, X_i] = 0.
\]

The extension coefficients or central charges \( \xi_{ij} \) must be antisymmetric in the indices \( i, j \), \( \xi_{ji} = -\xi_{ij} \) and must fulfill the following conditions coming from the Jacobi identities in the extended Lie algebra:

\[
\sum_{k=1}^r (C_{ij}^k \xi_{kl} + C_{ji}^k \xi_{ki} + C_{li}^k \xi_{kj}) = 0.
\]
These extension coefficients are the coordinates \( \xi(X_i, X_j) = \xi_{ij} \) of the anti-symmetric two-tensor \( \xi \) which is the two-cocycle of the specific extension being considered, and (3.2) is the two-cocycle condition for the Lie algebra cohomology.

Let us consider the ‘abstract’ extended Lie algebra \( \overline{G} \) with the Lie brackets (3.1) and let us perform a change of generators:

\[
X_i \rightarrow X'_i = X_i + \mu_i \Xi,
\]

where \( \mu_i \) are arbitrary real numbers. The commutation rules for the generators \( \{X'_i\} \) become

\[
[X'_i, X'_j] = \sum_{k=1}^{r} C_{ij}^k X'_k + (\xi_{ij} - \sum_{k=1}^{r} C_{ij}^k \mu_k) \Xi.
\]

Thus, the general expression for the two-coboundary \( \delta \mu \) generated by \( \mu \) is

\[
\delta \mu(X_i, X_j) = \sum_{k=1}^{r} C_{ij}^k \mu_k.
\]

Two two-cocycles differing by a two-coboundary lead to equivalent extensions; the classes of equivalence of non-trivial two-cocycles associated with the tensors \( \xi \) determine the second cohomology group \( H^2(\mathcal{G}, \mathbb{R}) \).

### 3.1 The general solution to the extension problem for the unitary CK algebras

In a previous paper [1] we have given the general solution to the extension equations for the case of the orthogonal CK algebras. The same approach can be used for the family of quasi-unitary algebras. However, and in order not to burden the exposition, the main details on the procedure have been placed in the Appendix. The results obtained there give the general solution to the problem of finding the central extensions for the unitary CK algebras. They are summed up as

**Theorem 3.1**

The most general central extension \( \mathfrak{su}_\omega(N+1) \) of any algebra in the family of special unitary CK algebras \( \mathfrak{su}_\omega(N+1) \) is determined by the following basic coefficients:

**Type I.** \( N(N+1)/2 \) basic extension coefficients \( \eta_{ab} \) and \( N(N+1)/2 \) basic extension coefficients \( \tau_{ab} \) \( (a < b, a, b = 0, 1, \ldots, N) \). These coefficients are not subjected to any further relationship.

**Type II.** \( N \) basic extension coefficients \( \alpha_k \) \( (k = 1, \ldots, N) \), not subjected to any further relationship.

**Type III.** \( N(N-1)/2 \) basic extension coefficients \( \beta_{kl} \) \( (k < l, k, l = 1, \ldots, N) \) which must satisfy the conditions

\[
\omega_k \beta_{kl} = 0 \quad \omega_l \beta_{kl} = 0.
\]
Theorem 3.2
The most general central extension $\mathfrak{u}_\omega(N+1)$ of any algebra in the unitary CK family $u_\omega(N+1)$, is determined by the basic extension coefficients given in Theorem 3.1, and by an additional set of

Type III. $N$ basic extension coefficients $\gamma_k$ ($k = 1, \ldots, N$), subjected to the relation

$$\omega_k \gamma_k = 0.$$  \hspace{1cm} (3.7)

For any given choice of the constants $\omega_k$, these basic extension coefficients determine two-cocycles for the algebras $su_\omega(N+1)$ and $u_\omega(N+1)$. The Lie brackets of the extended algebras $\mathfrak{su}_\omega(N+1)$ and $\mathfrak{u}_\omega(N+1)$ are given by

$$\begin{align*}
[J_{ab}, J_{ac}] &= \omega_{ab}(J_{bc} + \eta_{ac}\Xi) \\
[J_{ab}, J_{bc}] &= -(J_{ac} + \eta_{ac}\Xi) \\
[J_{ac}, J_{bc}] &= \omega_{bc}(J_{ab} + \eta_{ab}\Xi) \\
[J_{ab}, J_{mn}] &= 0 \\
[J_{ab}, M_{ac}] &= \omega_{ab}(M_{bc} + \tau_{bc}\Xi) \\
[M_{ab}, M_{ac}] &= \omega_{ab}(J_{bc} + \eta_{bc}\Xi) \\
[M_{ab}, M_{bc}] &= J_{ac} + \eta_{ac}\Xi \\
[M_{ab}, M_{bc}] &= \omega_{bc}(J_{ab} + \eta_{ab}\Xi) \\
[M_{ab}, M_{mn}] &= 0 \\
[J_{ac}, M_{bc}] &= -\omega_{bc}(M_{ab} + \tau_{ab}\Xi) \\
[M_{ac}, M_{bc}] &= J_{ab} + \eta_{ab}\Xi \\
[J_{ac}, M_{bc}] &= -\omega_{bc}(M_{ab} + \tau_{ab}\Xi) \\
[M_{ac}, M_{bc}] &= \omega_{bc}(J_{ab} + \eta_{ab}\Xi) \\
[J_{ab}, M_{ab}] &= -2\omega_{ab} \sum_{s=a+1}^b B_s + \sum_{s=a+1}^b \omega_{as} \omega_{sb} \alpha_s \Xi \\
[J_{ab}, I] &= 0 \\
[M_{ab}, I] &= 0 \\
[B_k, I] &= \gamma_k \Xi,
\end{align*}$$  \hspace{1cm} (3.8)

where $a < b < c$, $k < l$, $m < n$ and $a, b, m, n$ are all different.

The complete expression for the two-cocycles for $su_\omega(N+1)$ and $u_\omega(N+1)$ can be read directly from these commutators; for future convenience, we collect some expressions relating the basic extension coefficients with particular values of the two-cocycles determining the extensions (however, and as it can be seen in (3.8), most of these basic coefficients appears related to the values of the cocycle in several ways)

$$\begin{align*}
\eta_{ac} &= -\xi(J_{ab}, J_{bc}) & \tau_{ac} &= -\xi(J_{ab}, M_{bc}) \\
\alpha_k &= \xi(J_{k-1, k}, M_{k-1, k}) & \beta_{kl} &= \xi(B_k, B_l) \\
\gamma_k &= \xi(B_k, I). 
\end{align*}$$  \hspace{1cm} (3.9)

3.2 Equivalence of extensions

According to the general discussion in the beginning of this section, we now look for the more general coboundary for $su_\omega(N+1)$ or $u_\omega(N+1)$. We write a change of basis (see (3.3)) for the generators as

$$\begin{align*}
J_{ab} &\to J_{ab}' = J_{ab} + \sigma_{ab}\Xi & M_{ab} &\to M_{ab}' = M_{ab} + \rho_{ab}\Xi & B_k &\to B_k' = B_k + \nu_k\Xi
\end{align*}$$  \hspace{1cm} (3.10)
\[ I \rightarrow I + \zeta \Xi \quad (3.13) \]

where \( \sigma_{ab}, \rho_{ab}, \upsilon_k, \zeta \) are the values of \( \mu \) on the generators \( J_{ab}, M_{ab}, B_k, I \). By using (3.5) and the structure constants of the algebras \( su_\omega(N+1) \) or \( u_\omega(N+1) \) read from (2.8)–(2.10), we find for the associated coboundaries \( \delta \mu \),

\[
\begin{align*}
\delta \mu(J_{ab}, J_{bc}) &= -\sigma_{ac} \\
\delta \mu(J_{k-1l}, M_{k-1l}) &= -2\omega_k \upsilon_k \\
\delta \mu(J_{ab}, M_{bc}) &= -\rho_{ac} \\
\delta \mu(B_k, B_l) &= 0 \\
\delta \mu(B_k, I) &= 0 .
\end{align*}
\]

We shall not need the remaining values of the coboundaries \( \delta \mu \) for \( su_\omega(N+1) \) or \( u_\omega(N+1) \); each \( \delta \mu \) being a two-cocycle, it must necessarily appear as a particular case of the most general two-cocycles which are completely determined by the basic extension coefficients (3.10).

The question of whether a general two-cocycle for a CK algebra in Theorem 3.1 defines a trivial extension amounts to checking whether it is a coboundary, which will allow to eliminate the central \( \Xi \) term from (3.8). This may depend on the values of the constants \( \omega_a \). In fact, the three types of extensions behave in three different ways, which mimics the pattern found in the orthogonal case [1]:

- **Type I extensions** can be done for all unitary CK algebras, since there is not any \( \omega_a \)-dependent restriction to the basic Type I coefficients \( \tau_{ab}, \eta_{ab} \). However, as seen in (3.14), these extensions are always trivial. A considerable simplification of all expressions can be gained if these trivial extensions are simply discarded, as we shall do from now on. Hence for the extended algebra, the whole block of commutation relations in (2.8) will hold and only those commutators in (2.9) or (2.10) may change.

- **Type II extensions** appear also in all unitary CK algebras, as there is not any \( \omega_a \)-dependent restriction to the basic Type II coefficients \( \alpha_k \). The triviality of these extensions is \( \omega_a \)-dependent, and (3.14) shows that the extension determined by the coefficient \( \alpha_k \) is non-trivial if \( \omega_k = 0 \), and trivial otherwise. It is within this type of extensions that a pseudoextension (trivial extension by a two-coboundary) may become a non-trivial extension by contraction [14, 15].

- **Type III extensions** behave in a completely different way. Due to the additional conditions (3.6) and (3.7) that Type III extension coefficients must fulfil, some of them might be necessarily equal to zero. Hence, these extensions do not exist for all unitary CK algebras. But those allowed (one \( \beta_{kl} \) for each pair of vanishing constants \( \omega_k = \omega_l = 0 \) and for the (non-special) unitary case one additional \( \gamma_k \) for each vanishing constant \( \omega_k = 0 \) are always non-trivial, as the last equation in (3.14) and (3.15) show. Therefore, Type III extensions do not appear through the pseudoextension mechanism.
3.3 The second cohomology groups of the unitary CK algebras

If we disregard Type I extensions, which are trivial for all members in the two CK families of unitary algebras, the above results can be summarised in the following

**Theorem 3.3**

The commutation relations of any central extension \( \mathfrak{su}_\omega(N+1) \) of the special unitary CK algebra \( \mathfrak{su}_\omega(N+1) \) can be written as the commutation relations in (2.8), together with:

\[
[J_{ab}, M_{ab}] = -2\omega_{ab} \sum_{s=a+1}^{b} B_s + \sum_{s=a+1}^{b} \omega_{a,s-1} \omega_{ab} \alpha_s \Xi \quad [B_k, B_l] = \beta_{kl} \Xi \quad k < l
\]

which will replace those in (2.9). The extension is completely characterised by

- \( N \) Type II coefficients \( \alpha_k \) (\( k = 1, \ldots, N \)). Each of them gives rise to a non-trivial extension if \( \omega_k = 0 \) and to a trivial one otherwise.
- \( N(N-1)/2 \) Type III extension coefficients \( \beta_{kl} \) (\( k < l \) and \( k, l = 1, \ldots, N \)), satisfying

\[
\omega_k \beta_{kl} = 0 \quad \omega_l \beta_{kl} = 0.
\]

Thus, \( \beta_{kl} \) must be equal to zero when at least one of the constants \( \omega_k, \omega_l \) is different from zero. When \( \beta_{kl} \) is non-zero, the extension that it determines is always non-trivial.

**Theorem 3.4**

The commutation relations of any central extension \( \mathfrak{su}_\omega(N+1) \) of the unitary CK algebra \( \mathfrak{u}_\omega(N+1) \) can be written as the commutation relations in the preceding statement, together with

\[
[J_{ab}, I] = 0 \quad [M_{ab}, I] = 0 \quad [B_k, I] = \gamma_k \Xi
\]

which will replace those in (2.10). In addition to the extension coefficients \( \alpha_k \) and \( \beta_{kl} \), the extension is completely characterised by

- \( N \) Type III coefficients \( \gamma_k \) (\( k = 1, \ldots, N \)) satisfying

\[
\omega_k \gamma_k = 0.
\]

When \( \gamma_k \) is non-zero, the extension that it determines is non-trivial.

All Type II extensions come from the pseudocohomology mechanism [14, 15]. We can write (3.16) as

\[
[J_{ab}, M_{ab}] = -2\omega_{ab} \sum_{s=a+1}^{b} (B_s - \frac{\alpha_s}{2\omega_s}) \Xi
\]

(3.20)
which is well defined even if any of the $\omega_s (s = a+1, a+2, \ldots, b)$ is equal to zero. This clearly shows that when a given $\omega_s$ is different from zero, the extension coefficient $\alpha_s$ gives rise to a trivial extension, which can be removed by the one-cochain $\mu(B_s) = -\frac{\omega_s}{\omega_s} (all other coordinates of the one-cochain being zero). However, when $\omega_s$ goes to zero, the corresponding extension is non-trivial, as the cochain defined above diverges, but the term $\omega_{ab}/\omega_s$ in (3.20) does not.

In terms of the triangular arrangement for the generators of $su_\omega(N + 1)$ (see Fig. 2.1), it is also worth remarking that Type III extensions only affect the commutators of the Cartan generators in the outermost ‘$B$’ diagonal, while the Type II extension $\alpha_a$ only modifies the commutators of each those pairs $\{J_{ij}, M_{ij}\} \equiv X_{ij}$ with $i < a \leq j$, i.e. those pairs contained inside a rectangle with left-down corner $X_{a-1a}$.

As a by-product of these results we can give closed expressions for the dimension of the second cohomology group of any Lie algebra in the unitary CK families.

**Proposition 3.1**

Let $su_\omega(N + 1)$ or $u_\omega(N + 1)$ be a Lie algebra belonging to a family of unitary CK algebras, and let $n$ be the number of coefficients $\omega_k$ equal to zero. The dimension of its second cohomology group is given by

\[
\dim (H^2(su_\omega(N + 1), \mathbb{R}) = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2} \quad (3.21)
\]

\[
\dim (H^2(u_\omega(N + 1), \mathbb{R}) = n + \frac{n(n-1)}{2} + n = \frac{n(n+3)}{2} \quad (3.22)
\]

The first term $n$ in the sum of (3.21), (3.22) corresponds to the central extensions $\alpha_k$, the second term $\frac{n(n-1)}{2}$ to the $\beta_{kl}$ and the third term $n$ in (3.22) to the central extensions $\gamma_k$. We recall that the analogous expression for the quasi-orthogonal case is far more complicated, and depends not only on the number of constants equal to zero, but also on the detailed arrangement of zeros in the sequence $\omega_1, \ldots, \omega_N$ [1].

As expected for the simple $su(p, q)$ or the semisimple $u(p, q)$ algebras, which appear within the two unitary CK families when all $\omega_a \neq 0$, the second cohomology group is trivial. The inhomogeneous $iu(p, q)$ algebras, appearing in the special unitary family when either $\omega_1 = 0$ or $\omega_N = 0$, with all other constants $\omega_a \neq 0$, have, in any dimension, a single non-trivial extension: $\alpha_1$ when $\omega_1 = 0$ or $\alpha_N$ if $\omega_N = 0$. The special unitary flag algebra (when all $\omega_a = 0$) has the maximum number of non-trivial extensions within the special unitary family, that is, $N(N+1)/2$.

### 4 Examples

Let us illustrate the general results of the above section for the $su_\omega(N + 1)$ algebras in the three lowest dimensional cases, $N = 1, 2, 3$. A completely similar discussion can be performed for the $u_\omega(N + 1)$ algebras.
4.1 \( \mathbf{su}_{\omega_1}(2) \)

We simply mention this example for the sake of completeness. The results for the extensions of \( su_{\omega_1}(2) \) could be also obtained from those in [1] by using the isomorphism 
\[ su_{\omega_1}(2) \cong so_{\omega_1, +}(3, \mathbb{R}) \]
provided by \( J_0/2 \leftrightarrow \Omega_0, M_0/2 \leftrightarrow \Omega_2, -B_1/2 \leftrightarrow \Omega_2 \). The most general extension is defined by the extension coefficient \( \alpha_1 \) and the non-zero Lie brackets

\[
\begin{align*}
[J_{01}, M_{01}] &= -2\omega_1 B_1 + \alpha_1 \Xi \\
[J_{01}, B_1] &= 2M_{01} \\
[M_{01}, B_1] &= -2J_{01}. 
\end{align*}
\]

The extension is non-trivial for \( \omega_1 = 0 \) and trivial otherwise, the triviality being exhibited by the redefinition

\[
B_1 \rightarrow B_1 - \frac{\alpha_1}{2\omega_1} \Xi. \tag{4.2}
\]

4.2 \( \mathbf{su}_{\omega_1, \omega_2}(3) \)

The most general extended special unitary CK algebra \( \mathbf{su}_{\omega_1, \omega_2}(3) \) has nine generators \( \{J_{01}, J_{02}, J_{12}, M_{01}, M_{02}, M_{12}, B_{1}, B_{2}, \Xi\} \), and it is determined by three possible extension coefficients \( \{\alpha_1, \alpha_2, \beta_{12}\} \), with \( \omega_1 \beta_{12} = \omega_2 \beta_{12} = 0 \). Their commutators are:

\[
\begin{align*}
[J_{01}, J_{02}] &= \omega_1 J_{12} \\
[J_{01}, M_{02}] &= \omega_1 M_{12} \\
[M_{01}, M_{02}] &= \omega_1 J_{12} \\
[J_{01}, J_{12}] &= -J_{02} \\
[J_{01}, M_{12}] &= J_{02} \\
[M_{01}, J_{12}] &= -M_{02} \\
[01, M_{12}] &= \omega_2 M_{01} \\
[J_{01}, B_{1}] &= 2M_{01} \\
[J_{02}, B_{1}] &= M_{02} \\
[M_{02}, B_{1}] &= J_{12} \\
[M_{12}, B_{1}] &= J_{12} \\
[J_{01}, B_{2}] &= -M_{01} \\
[J_{02}, B_{2}] &= M_{02} \\
[M_{02}, B_{2}] &= J_{12} \\
[M_{12}, B_{2}] &= J_{12} \\
[M_{12}, B_{2}] &= -J_{12} \\
[J_{01}, M_{12}] &= -2\omega_1 B_1 + \alpha_1 \Xi \\
[J_{12}, M_{12}] &= -2\omega_2 B_2 + \alpha_2 \Xi \\
[J_{02}, M_{02}] &= \omega_2 (-2\omega_1 B_1 + \alpha_1 \Xi) + \omega_1 (-2\omega_2 B_2 + \alpha_2 \Xi) \\
[B_{1}, B_{2}] &= \beta_{12} \Xi. \tag{4.4}
\end{align*}
\]

The triviality of Type II extensions is governed by the values of the constants \( \omega_1, \omega_2 \). We analyse this problem for each specific CK algebra within \( \mathbf{su}_{\omega_1, \omega_2}(3) \). The extension determined by \( \alpha_1 \) is trivial when \( \omega_1 \neq 0 \), and the extension determined by \( \alpha_2 \) is trivial when \( \omega_2 \neq 0 \), the triviality being exhibited by the redefinitions

\[
\begin{align*}
B_1 \rightarrow B_1 - \frac{\alpha_1}{2\omega_1} \Xi \\
B_2 \rightarrow B_2 - \frac{\alpha_2}{2\omega_2} \Xi. \tag{4.5}
\end{align*}
\]

Thus, \( \dim (H^2(su_{\omega_1, \omega_2}(3), \mathbb{R})) \) is equal to

- 0 when both \( \omega_1, \omega_2 \neq 0 \). Here both \( \alpha_1, \alpha_2 \) produce trivial extensions, and \( \beta_{12} \) must vanish. This case corresponds to the extensions of \( su(3) \) for \( (\omega_1, \omega_2) = (+, +) \), and \( su(2, 1) \) for \( (\omega_1, \omega_2) = \{ (+, -), (-, +), (-, -) \} \) and the result is in agreement with Whitehead’s lemma, according to which simple algebras have no non-trivial extensions.
• 1 for the inhomogeneous unitary algebras $iu(2)$ and $iu(1, 1)$. These algebras appear twice in the CK family, namely for $\omega_1 = 0$, $\omega_2 \neq 0$ and for $\omega_1 \neq 0$, $\omega_2 = 0$. In the first case the only non-trivial extension coefficient is $\alpha_1$ and the extended Lie brackets (4.4) reduce to

$$[J_{01}, M_{01}] = \alpha_1 \Xi \quad [J_{02}, M_{02}] = \omega_2 \alpha_1 \Xi \quad [J_{12}, M_{12}] = -2 \omega_2 B_2 \quad [B_1, B_2] = 0.$$  

(4.6)

The second case is related to the former one due to the isomorphism (2.19). Here there is a single non-trivial extension coefficient $\alpha_2$ and the extended Lie brackets are

$$[J_{01}, M_{01}] = -2 \omega_1 B_1 \quad [J_{02}, M_{02}] = \omega_1 \alpha_2 \Xi \quad [J_{12}, M_{12}] = \alpha_2 \Xi \quad [B_1, B_2] = 0.$$  

(4.7)

• 3 for the special unitary flag algebra $su_{0, 0}(3)$ when $\omega_1 = \omega_2 = 0$. The three extensions are non-trivial

$$[J_{01}, M_{01}] = \alpha_1 \Xi \quad [J_{02}, M_{02}] = 0 \quad [J_{12}, M_{12}] = \alpha_2 \Xi \quad [B_1, B_2] = \beta_{12} \Xi.$$  

(4.8)

4.3 $\overline{su}_{\omega_1, \omega_2, \omega_3}(4)$

We consider now the extensions $\overline{su}_{\omega_1, \omega_2, \omega_3}(4)$ of the CK algebra $su_{\omega_1, \omega_2, \omega_3}(4)$. There are six possible basic extension coefficients, $\{\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}, \beta_{23}\}$, which must satisfy the conditions

$$\omega_1 \beta_{12} = \omega_2 \beta_{12} = 0 \quad \omega_1 \beta_{13} = \omega_3 \beta_{13} = 0 \quad \omega_2 \beta_{23} = \omega_3 \beta_{23} = 0,$$

(4.9)

and the Lie brackets of the extension are given by the non-extended ones in (2.8) and by the extended ones

$$[J_{01}, M_{01}] = -2 \omega_1 B_1 + \alpha_1 \Xi$$

$$[J_{02}, M_{02}] = \omega_2 (-2 \omega_1 B_1 + \alpha_1 \Xi) + \omega_1 (-2 \omega_2 B_2 + \alpha_2 \Xi)$$

$$[J_{03}, M_{03}] = \omega_2 \omega_3 (-2 \omega_1 B_1 + \alpha_1 \Xi) + \omega_1 \omega_3 (-2 \omega_2 B_2 + \alpha_2 \Xi) + \omega_1 \omega_2 (-2 \omega_3 B_3 + \alpha_3 \Xi)$$

$$[J_{12}, M_{12}] = -2 \omega_2 B_2 + \alpha_2 \Xi$$

$$[J_{13}, M_{13}] = \omega_3 (-2 \omega_2 B_2 + \alpha_2 \Xi) + \omega_2 (-2 \omega_3 B_3 + \alpha_3 \Xi)$$

$$[J_{23}, M_{23}] = -2 \omega_3 B_3 + \alpha_3 \Xi$$

$$[B_1, B_2] = \beta_{12} \Xi \quad [B_1, B_3] = \beta_{13} \Xi \quad [B_2, B_3] = \beta_{23} \Xi.$$  

(4.10)

The results for each one of the 27 CK algebras $\overline{su}_{\omega_1, \omega_2, \omega_3}(4)$ are displayed in Table 4.1. The columns in this Table show, in this order, the number of coefficients $\omega_a$ set equal to zero (number of contractions), the centrally extended Lie algebras, the signs $+, -, 0$ of each coefficient $(\omega_1, \omega_2, \omega_3)$ together with the non-trivial central extensions allowed for the algebra with these signs for the coefficients, and finally, the dimension of the second cohomology group as a sum of the number of non-trivial extensions of Types II and III, coming respectively from the coefficients $\alpha_k$ and $\beta_{kl}$. In the table $+$ $(-)$ denotes a positive (negative) $\omega_a$ coefficient which could be rescaled to 1 $(-1)$.
Extended algebra

\[ \text{dim} H^2 \]

su(4) \( (+, +, +) \)

su(3, 1) \( (-, +, +), (-, -, +), (+, +, -), (+, -, -) \)

su(2, 2) \( (+, +, -), (-, +, -), (-, -, -) \)

\[ \text{dim} H^2 \]

i\(u(3) \)

i\(u(2, 1) \)

\( \tilde{\mathfrak{s}}(u(2) \oplus u(1) \oplus u(2)) \)

\( \tilde{\mathfrak{s}}(u(2) \oplus u(1) \oplus u(1, 1)) \)

\( \tilde{\mathfrak{s}}(u(1, 1) \oplus u(1) \oplus u(1, 1)) \)

\[ \text{dim} H^2 \]

\( \text{Flag algebra} \)

\[ \text{dim} H^2 \]

Table 4.1: Non-trivial central extensions su_{\omega_1, \omega_2, \omega_3}(4).

5 Conclusions and outlook

We restrict here to a couple of remarks. First, the pattern of three types of extensions behaving under contractions in three different ways, first found for the quasi-orthogonal family [1], appears also in the quasi-unitary case. This seems likely to be a general phenomenon, not restricted to a single family of contractions of some Lie algebras. The analysis of the extensions for the third CK main series of algebras, which embraces the symplectic \( sp(p, q) \) in the \( C_l \) series and their contractions, would be required to complete the study of the relationships between cohomology and contractions undertaken in [1] and continued in this paper. These algebras can be adequately realised by quaternionic antihermitian matrices, or, alternatively, by quaternionic antihermitian traceless matrices plus the Lie algebra of derivations of the quaternion division algebra. Work in this area is in progress. Second, as compared to the quasi-orthogonal case, the quasi-unitary algebras have a comparatively smaller set of extensions, whose description in terms of the values taken by the CK constants \( \omega_a \) is straightforward. The suitability of a CK approach to the study of the central extensions of a complete family is therefore put forward more clearly than in the orthogonal case. While the ordinary inhomogeneous orthogonal algebras \( iso(p, q) \) associated to the real orthogonal \( N = p + q \) dimensional flat spaces have non-trivial extensions only in the case \( N = 2 \), the algebras \( iu(p, q) \) associated to the complex pseudo-Euclidean hermitian flat spaces have a single non-trivial extension, in any dimension. The relevance of this fact in relation with the classical limit of quantum mechanics will be discussed elsewhere.
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Appendix: The general solution to the Jacobi identities

In order to get the general solution of the set of linear equations determining the possible extensions of the unitary CK algebras, we first introduce a suitable notation for the central extension coefficients, which is ‘adapted’ to the structure of the algebras $su_{\omega}(N + 1)$ (2.8)–(2.9) and $u_{\omega}(N + 1)$ (2.8)–(2.10) whose basic generators come naturally divided in either three or four ‘kinds’ $J_{ab}$, $M_{ab}$, $B_k$, $I$. The symbol corresponding to $\xi(X,Y)$ will have one or two letters taken from $j, m, b, i$, determined by the kind of the basis generators $X$, $Y$. To this symbol we append two groups of indices, each coming from those of the corresponding generators. The complete list of all extension coefficients as written in this notation is

\[
\begin{align*}
& j_{ab,de} \quad m_{ab,de} \quad jm_{ab,de} \quad mj_{ab,de} \\
& jb_{ab,k} \quad mb_{ab,k} \quad b_{k,l} \quad jm_{ab} \\
& j_{i,ab} \quad mi_{ab} \quad bi_{l}
\end{align*}
\]  

where we implicitly assume $a < b$, $d < e$, $a, b, d, e = 0, \ldots N$, $k < l$, $k, l = 1, \ldots N$. We remark that $jm$, $mj$, $jb$, $mb$, $ji$, $mi$, $bi$ are single, unbreakable symbols, and are not products. In the course of the derivation we will find useful to sort these coefficients into several subsets, as follows

- Coefficients $j_{ab,de}$, $m_{ab,de}$, $jm_{ab,de}$, $mj_{ab,de}$ involving four different indices. If we write these four indices as $a < b < c < d$ the coefficients are

\[
\begin{align*}
& j_{ab,cd} \quad m_{ab,cd} \quad jm_{ab,cd} \quad mj_{ab,cd} \\
& j_{ac,bd} \quad m_{ac,bd} \quad jm_{ac,bd} \quad mj_{ac,bd} \\
& j_{ad,bc} \quad m_{ad,bc} \quad jm_{ad,bc} \quad mj_{ad,bc}
\end{align*}
\]  

- Coefficients $j_{ab,de}$, $m_{ab,de}$, $jm_{ab,de}$, $mj_{ab,de}$ involving three different indices. If we write the three indices as $a < b < c$ these coefficients are

\[
\begin{align*}
& j_{ab,ac} \quad m_{ab,ac} \quad jm_{ab,ac} \quad mj_{ab,ac} \\
& j_{ab,bc} \quad m_{ab,bc} \quad jm_{ab,bc} \quad mj_{ab,bc} \\
& j_{ac,bc} \quad m_{ac,bc} \quad jm_{ac,bc} \quad mj_{ac,bc}
\end{align*}
\]
• Coefficients $jm_{ab}$ involving two different indices

\[ jm_{ab} \] (A.4)

• Coefficients $jb_{ab,i}, mb_{ab,i}$ with two different indices $a < b$ and a third index $i \in \{a, a+1, b, b+1\}$

\[ jb_{ab,i} \quad mb_{ab,i} \] (A.5)

• Coefficients $jb_{ab,j}, mb_{ab,j}$ with two different indices $a < b$ and a third index $j \notin \{a, a+1, b, b+1\}$

\[ jb_{ab,j} \quad mb_{ab,j} \] (A.6)

• Coefficients $b_{k,l}$ with two different indices $k < l$

\[ b_{k,l} \] (A.7)

• Coefficients $ji_{ab}$ and $mi_{ab}$ with two different indices $a < b$

\[ ji_{ab} \quad mi_{ab} \] (A.8)

• Coefficients $bi_l$ with a single index

\[ bi_l \] (A.9)

The Lie brackets of the extended CK algebra $\overline{\mathfrak{g}}(N+1)$ and $\overline{\mathfrak{g}}(N+1)$ read

\[
\begin{align*}
[J_{ab}, J_{ac}] &= \omega_{ab} J_{bc} + j_{ab,ac}\Xi \\
[J_{ab}, J_{bc}] &= -J_{ac} - j_{ab,bc}\Xi \\
[J_{ac}, J_{bc}] &= -J_{ab} + j_{ac,bc}\Xi \\
[J_{ab}, J_{de}] &= j_{ab,de}\Xi \\
[J_{ab}, M_{ac}] &= \omega_{ab} M_{bc} + j m_{ab,ac}\Xi \\
[J_{ab}, M_{bc}] &= -M_{ac} - j m_{ab,bc}\Xi \\
[J_{ac}, M_{bc}] &= -\omega_{bc} M_{ab} - j m_{ac,bc}\Xi \\
[J_{ab}, M_{de}] &= j m_{ab,de}\Xi \\
[J_{ab}, B_a] &= -M_{ab} - j b_{ab,a}\Xi \\
[J_{ab}, B_{a+1}] &= M_{ab} + j b_{a,a+1}\Xi \\
[J_{a+1}, B_{a+1}] &= 2M_{a,1} + 2j b_{a+1,1}\Xi \\
[J_{ab}, B_b] &= M_{ab} + j b_{ab,b}\Xi \\
[J_{ab}, B_{b+1}] &= -M_{ab} - j b_{ab,b+1}\Xi \\
[J_{ab}, B_j] &= j b_{ab,j}\Xi
\end{align*}
\] (A.10)

\[
\begin{align*}
[M_{ab}, B_a] &= J_{ab} + m b_{ab,a}\Xi \\
[M_{ab}, B_{a+1}] &= -J_{ab} - m b_{ab,a+1}\Xi \\
[M_{a+1}, B_{a+1}] &= -2J_{a+1} - 2m b_{a+1,a+1}\Xi \\
[M_{ab}, B_b] &= -J_{ab} - m b_{ab,b}\Xi \\
[M_{ab}, B_{b+1}] &= J_{ab} + m b_{ab,b+1}\Xi \\
[M_{ab}, B_j] &= m b_{ab,j}\Xi
\end{align*}
\] (A.11)

\[
\begin{align*}
[J_{ab}, M_{ab}] &= -2\omega_{ab} \sum_{s=a+1}^b B_s + j m_{ab}\Xi \\
[B_k, B_l] &= b_{k,l}\Xi
\end{align*}
\] (A.12)

\[
\begin{align*}
[J_{ab}, I] &= j i_{ab}\Xi \\
[M_{ab}, I] &= m i_{ab}\Xi \\
[B_l, I] &= b i_l\Xi
\end{align*}
\] (A.13)
where as indicated before, the relations $a < b < c$, $a < d$, $d < e$, $j \notin \{a, a+1, b, b+1\}$, $k < l$ for the indices $a, b, c, d, e = 0, \ldots, N$, $j, k, l = 1, \ldots, N$ and $a, b, d, e$ are all different, will be assumed without saying.

Our strategy here will be to enforce the complete set of Jacobi identities first for $su_\omega(N+1)$ and then for $u_\omega(N+1)$, in a carefully selected order which actually allows to explicitly solve the rather large set of linear equations. The first stage will be to identify many extension coefficients which are forced to vanish; the remaining Jacobi equations will drastically simplify and will either produce relations allowing to express certain derived extension coefficients in terms of the so-called basic ones, or further relations to be satisfied by the basic extension coefficients.

To begin with, we show that all coefficients in (A.2) vanish. Denoting by $\{X, Y, Z\}$ the Jacobi identity for the generators $X, Y$ and $Z$, we display several choices for them and the equations ensuing from these choices:

\[
\begin{align*}
\{J_{ab}, M_{cd}, B_d\} : & \quad j_{ab,cd} = 0 \\
\{J_{ab}, M_{cd}, B_b\} : & \quad m_{ab,cd} = 0 \\
\{J_{ab}, J_{cd}, B_d\} : & \quad j_{mab,cd} = 0 \\
\{J_{ab}, J_{cd}, B_b\} : & \quad m_{jab,cd} = 0 \\
\{J_{ad}, M_{bc}, B_e\} : & \quad j_{ad,bc} = 0 \\
\{M_{ad}, J_{bc}, B_c\} : & \quad m_{ad,bc} = 0 \\
\{J_{ad}, J_{bc}, B_c\} : & \quad j_{mad,bc} = 0 \\
\{M_{ad}, M_{bc}, B_c\} : & \quad m_{jad,bc} = 0 \quad \text{(A.14)}
\end{align*}
\]

\[
\begin{align*}
\{J_{ab}, M_{bc}, B_d\} : \quad & \quad \omega_{bc}j_{ab,cd} + j_{ac,bd} - j_{ad,bc} = 0 \\
\{J_{ab}, M_{bc}, M_{bd}\} : \quad & \quad \omega_{bc}m_{ab,cd} + m_{ac,bd} - m_{ad,bc} = 0 \\
\{J_{ab}, J_{bc}, M_{bd}\} : \quad & \quad \omega_{bc}j_{mab,cd} + j_{mab,cd} - m_{jad,bc} = 0 \\
\{J_{ab}, M_{bc}, J_{bd}\} : \quad & \quad \omega_{bc}j_{mab,cd} - m_{jac,bd} + j_{mad,bc} = 0. \quad \text{(A.15)}
\end{align*}
\]

By substituting (A.14) and (A.15) in (A.16), we find that all coefficients in (A.2) are necessarily equal to zero. From now on, substitution of the already known information in further equations will be automatically assumed.

The coefficients in (A.6) turn out to be also equal to zero:

\[
\begin{align*}
\{M_{ab}, B_b, B_j\} : & \quad j_{b_{ab,j}} = 0 \quad \{J_{ab}, B_b, B_j\} : \quad m_{b_{ab,j}} = 0 \quad j \notin \{a, a+1, b, b+1\}. \quad \text{(A.17)}
\end{align*}
\]

Now we look for equations involving the coefficients $b_{k,l}$ in (A.7). We find:

\[
\begin{align*}
\{J_{a_{a+1}}, M_{a_{a+1}}, B_k\} : \quad & \quad \omega_{a_{a+1}}b_{k,a+1} = 0 \quad 1 \leq k \leq a \quad a = 1, \ldots, N-1 \\
\{J_{b_{b-1}}, M_{b_{b-1}}, B_l\} : \quad & \quad \omega_{b_{b-1}}b_{b,l} = 0 \quad b + 1 \leq l \leq N \quad b = 1, \ldots, N-1 \quad \text{(A.18)}
\end{align*}
\]

so the $N(N-1)/2$ coefficients of the type $b_{k,l}$ might be different from zero. We denote them as

\[
\beta_{kl} := b_{k,l} \quad \text{(A.19)}
\]

and from (A.18) they must fulfil two additional conditions

\[
\omega_k \beta_{kl} = 0 \quad \omega_l \beta_{kl} = 0. \quad \text{(A.20)}
\]
We now look for Jacobi identities leading to equations which involve the extension coefficients in (A.5):

\[
\begin{align*}
\{M_{ab}, B_a, B_{a+1}\} : & \quad j b_{ab,a+1} = j b_{ab,a} & \{J_{ab}, B_a, B_{a+1}\} : & \quad mb_{ab,a+1} = mb_{ab,a} \\
\{M_{ab}, B_a, B_b\} : & \quad j b_{ab,b} = j b_{ab,a} & \{J_{ab}, B_a, B_b\} : & \quad mb_{ab,b} = mb_{ab,a} \\
\{M_{ab}, B_a, B_{a+1}\} : & \quad j b_{ab,b+1} = j b_{ab,a} & \{J_{ab}, B_a, B_{a+1}\} : & \quad mb_{ab,b+1} = mb_{ab,a} \\
\{M_{ab}, B_{a+1}, B_b\} : & \quad j b_{ab,a+1} = j b_{ab,b} & \{J_{ab}, B_{a+1}, B_b\} : & \quad mb_{ab,a+1} = mb_{ab,b} \\
\{M_{ab}, B_{a+1}, B_{a+1}\} : & \quad j b_{ab,a+1} = j b_{ab,b+1} & \{J_{ab}, B_{a+1}, B_{a+1}\} : & \quad mb_{ab,a+1} = mb_{ab,b+1} \\
\{M_{ab}, B_b, B_{b+1}\} : & \quad j b_{ab,b} = j b_{ab,b+1} & \{J_{ab}, B_b, B_{b+1}\} : & \quad mb_{ab,b} = mb_{ab,b+1}
\end{align*}
\]

which hold no matter of either \( b = a + 1 \) or \( b \neq a + 1 \). These equations show that

\[
\begin{align*}
jb_{ab,a} = j b_{ab,a+1} = j b_{ab,b} = j b_{ab,b+1} \\
mb_{ab,a} = mb_{ab,a+1} = mb_{ab,b} = mb_{ab,b+1}
\end{align*}
\]

and therefore these coefficients only depend on the first pair of indices. These common values must be considered as another set of basic coefficients

\[
\tau_{ab} := j b_{ab,i}, \quad \eta_{ab} := mb_{ab,i} \quad i \in \{a, a + 1, b, b + 1\}.
\]

Now we consider Jacobi identities leading to equations which involve the coefficients in (A.3), those \( j_{ab,de}, m_{ab,de}, j m_{ab,de}, m j_{ab,de} \) with three different indices. This is the most tedious part of the process, due to the need of paying minute attention to the index ranges. Let us first look for equations involving the coefficients with indices \( \{ab, bc\} \), which appear in the middle line of (A.3)

\[
\begin{align*}
\{J_{ab}, M_{bc}, B_{c+1}\} : & \quad j_{ab, bc} = \eta_{ac} & c < N \\
\{J_{ab}, M_{bN}, B_N\} : & \quad j_{ab, bN} = \eta_{aN} & b < N - 1 \\
\{J_{aN-1}, M_{N-1N}, B_a\} : & \quad m_{a,N-1,N-1} = \eta_{aN} & a > 0 \\
\{J_{0N-1}, M_{N-1N}, B_1\} : & \quad m_{0,N-1,N-1} = \eta_{0N} & \\
\{M_{ab}, J_{bc}, B_{c+1}\} : & \quad m_{ab, bc} = \eta_{ac} & c < N \\
\{M_{ab}, J_{bN}, B_N\} : & \quad m_{ab, bN} = \eta_{aN} & b < N - 1 \\
\{M_{aN-1}, J_{N-1N}, B_a\} : & \quad j_{a,N-1,N-1} = \eta_{aN} & a > 0 \\
\{M_{0N-1}, J_{N-1N}, B_1\} : & \quad j_{0,N-1,N-1} = \eta_{0N}
\end{align*}
\]

\[
\begin{align*}
\{J_{ab}, J_{bc}, B_{c+1}\} : & \quad j m_{ab, bc} = \tau_{ac} & c < N \\
\{J_{ab}, J_{bN}, B_N\} : & \quad j m_{ab, bN} = \tau_{aN} & b < N - 1 \\
\{J_{aN-1}, J_{N-1N}, B_a\} : & \quad m j_{a,N-1,N-1} = \tau_{aN} & a > 0 \\
\{J_{0N-1}, J_{N-1N}, B_1\} : & \quad m j_{0,N-1,N-1} = \tau_{0N} & \\
\{M_{ab}, M_{bc}, B_{c+1}\} : & \quad m j_{ab, bc} = \tau_{ac} & c < N \\
\{M_{ab}, M_{bN}, B_N\} : & \quad m j_{ab, bN} = \tau_{aN} & b < N - 1 \\
\{M_{aN-1}, M_{N-1N}, B_a\} : & \quad j m_{a,N-1,N-1} = \tau_{aN} & a > 0 \\
\{M_{0N-1}, M_{N-1N}, B_1\} : & \quad j m_{0,N-1,N-1} = \tau_{0N},
\end{align*}
\]

so in all cases, and no matter on the value of the middle index \( b \), we have

\[
\begin{align*}
jb_{ab, bc} = m_{ab, bc} = \eta_{ac} & \quad j m_{ab, bc} = m j_{ab, bc} = \tau_{ac}.
\end{align*}
\]
For the coefficients in the first line of (A.3) we obtain that

\[
\begin{align*}
\{J_{ab}, M_{ac}, B_{c+1}\} : \quad j_{ab,ac} &= \omega_{ab}\eta_{bc} & c < N \\
\{J_{ab}, M_{aN}, B_N\} : \quad j_{ab,aN} &= \omega_{ab}\eta_{bN} & b < N - 1 \\
\{J_{aN-1}, M_{aN}, B_{N-1}\} : \quad m_{aN-1,aN} &= \omega_{aN-1}\eta_{N-1N} & a < N - 2 \\
\{M_{ab}, J_{ac}, B_{c+1}\} : \quad m_{ab,ac} &= \omega_{ab}\eta_{bc} & c < N \\
\{M_{ab}, J_{aN}, B_N\} : \quad m_{ab,aN} &= \omega_{ab}\eta_{bN} & b < N - 1 \\
\{M_{aN-1}, J_{aN}, B_{N-1}\} : \quad j_{aN-1,aN} &= \omega_{aN-1}\eta_{N-1N} & a < N - 2 \\
\{J_{N-2N-1}, M_{N-2N}, B_{N-1}\} : \\
\quad j_{N-2N-1,aN} &= \omega_{N-2N-1}\eta_{N-1N} - 2m_{N-2N-1,aN} = 0 \\
\{M_{N-2N-1}, J_{N-2N}, B_{N-1}\} : \\
\quad -2j_{N-2N-1,aN} + \omega_{N-2N-1}\eta_{N-1N} + m_{N-2N-1,aN} = 0 & \quad (A.27)
\end{align*}
\]

These equations are summarised in

\[
\begin{align*}
\{J_{ab}, J_{ac}, B_{c+1}\} : \quad jm_{ab,ac} &= \omega_{ab}\tau_{bc} & c < N \\
\{J_{ab}, J_{aN}, B_N\} : \quad jm_{ab,aN} &= \omega_{ab}\tau_{bN} & b < N - 1 \\
\{J_{aN-1}, J_{aN}, B_{N-1}\} : \quad mj_{aN-1,aN} &= \omega_{aN-1}\tau_{N-1N} & a < N - 2 \\
\{M_{ab}, M_{ac}, B_{c+1}\} : \quad mj_{ab,ac} &= \omega_{ab}\tau_{bc} & c < N \\
\{M_{ab}, M_{aN}, B_N\} : \quad mj_{ab,aN} &= \omega_{ab}\tau_{bN} & b < N - 1 \\
\{M_{aN-1}, M_{aN}, B_{N-1}\} : \quad jm_{aN-1,aN} &= \omega_{aN-1}\tau_{N-1N} & a < N - 2 \\
\{J_{N-2N-1}, J_{N-2N}, B_{N-1}\} : \\
\quad jm_{N-2N-1,aN} &= \omega_{N-2N-1}\tau_{N-1N} + 2mj_{N-2N-1,aN} = 0 \\
\{M_{N-2N-1}, M_{N-2N}, B_{N-1}\} : \\
\quad -2jm_{N-2N-1,aN} + \omega_{N-2N-1}\tau_{N-1N} - mj_{N-2N-1,aN} = 0 & \quad (A.28)
\end{align*}
\]

so again these are derived extension coefficients, expressible in terms of \(\eta_{bc}\) and \(\tau_{bc}\).

For the coefficients in the third line of (A.3) with indices \(\{ac, bc\}\) we get

\[
\begin{align*}
\{J_{ac}, M_{bc}, B_a\} : \quad m_{ac,bc} &= \omega_{bc}\eta_{ab} & a > 0 \\
\{J_{bc}, M_{bc}, B_1\} : \quad m_{0c,bc} &= \omega_{bc}\eta_{0b} & b > 1 \\
\{J_{bc}, M_{1c}, B_2\} : \quad j_{bc,1c} &= \omega_{1c}\eta_{b0} & c > 2 \\
\{M_{ac}, J_{bc}, B_a\} : \quad j_{ac,bc} &= \omega_{bc}\eta_{ab} & a > 0 \\
\{M_{0c}, J_{bc}, B_1\} : \quad j_{0c,bc} &= \omega_{bc}\eta_{0b} & b > 1 \\
\{M_{0c}, J_{1c}, B_2\} : \quad m_{0c,1c} &= \omega_{1c}\eta_{b0} & c > 2 \\
\{J_{02}, M_{12}, B_2\} : \quad -2j_{02,12} + \omega_{12}\eta_{01} + m_{02,12} = 0 \\
\{M_{02}, J_{12}, B_2\} : \quad 2m_{02,12} - \omega_{12}\eta_{01} - j_{02,12} = 0 & \quad (A.30)
\end{align*}
\]

\[
\begin{align*}
\{J_{ac}, J_{bc}, B_a\} : \quad jm_{ac,bc} &= \omega_{bc}\tau_{ab} & a > 0 \\
\{J_{bc}, J_{bc}, B_1\} : \quad jm_{0c,bc} &= \omega_{bc}\tau_{0b} & b > 1 \\
\{J_{bc}, J_{1c}, B_2\} : \quad jm_{0c,1c} &= \omega_{1c}\tau_{0b} & c > 2 \\
\{M_{ac}, M_{bc}, B_a\} : \quad jm_{ac,bc} &= \omega_{bc}\tau_{ab} & a > 0 \\
\{M_{0c}, M_{bc}, B_1\} : \quad jm_{0c,bc} &= \omega_{bc}\tau_{0b} & b > 1 \\
\{M_{0c}, M_{1c}, B_2\} : \quad jm_{0c,1c} &= \omega_{1c}\tau_{0b} & c > 2 \\
\{J_{02}, J_{12}, B_2\} : \quad 2jm_{02,12} - \omega_{12}\tau_{01} - mj_{02,12} = 0 \\
\{M_{02}, M_{12}, B_2\} : \quad -jm_{02,12} - \omega_{12}\tau_{01} + 2mj_{02,12} = 0 & \quad (A.31)
\end{align*}
\]
These equations lead to
\[ j_{ac,be} = m_{ac,be} = \omega_{be} \eta_{ab} \quad m_{ac,be} = m_{jc,be} = \omega_{bc} \tau_{ab} \; , \]
so these coefficients are also derived.

Finally we look for equations involving the coefficients in (A.4), this is \( j_{mc} \). Whenever there exists an index \( b \) between \( a \) and \( c \), the choice
\[
\{ J_{ab}, J_{ac}, M_{bc} \} : \quad j_{ac} = \omega_{be} j_{mb} + \omega_{ab} j_{mc} \quad (A.33)
\]
leads to an expression for \( j_{mc} \) in terms of \( j_{ma} \) and \( j_{mb} \). By iterating while possible, we find that the coefficients \( j_{mc} \) with \( a \) and \( c \) not contiguous can be written in terms of \( j_{ma} \) with \( a \) and \( b \) contiguous. These must be considered as basic ones
\[
\alpha_k := j_{m_{k-1}k} \quad k = 1, \ldots, N \; , \quad (A.34)
\]
and the remaining coefficients in (A.4) are given, recalling that \( \omega_{ii} \equiv 1 \), by
\[
j_{ma} = \sum_{s=a+1}^{b} \omega_{as} \omega_{sb} \alpha_s \quad b \geq a + 2. \quad (A.35)
\]

As far as \( su_{\omega}(N+1) \) is concerned, the final step in this process is to ascertain that there is no any relation for the extensions coefficients further to the ones yet considered. It can be checked that all remaining Jacobi equations involving the generators \( J_{ab}, M_{ab}, B_l \) are identically satisfied, so the process has indeed terminated.

Now we deal with the \( u_{\omega}(N+1) \) case; as Jacobi equations involving \( J_{ab}, M_{ab}, B_l \) have been already considered, we must take into account only the extra generator \( I \) and the associated extension coefficients. For these, successively we obtain
\[
\{ J_{ab}, B_b, I \} : \quad m_{ib} = 0 \quad m_{ib} = 0 \quad (A.36)
\]
so the extension coefficients in (A.8) are equal to zero, and those in (A.9) are basic, to be denoted as
\[
\gamma_k := b_i k \quad (A.37)
\]
and must satisfy
\[
\omega_k \gamma_k = 0. \quad (A.38)
\]

Again in this case, it is easy to check that all remaining Jacobi equations involving the generator \( I \) are satisfied and do not lead to any further relation.

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