Mu-Tau Reflection Symmetry in the Standard Parametrization and Contributions from Charged Lepton Sector

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Abstract

The $\mu - \tau$ reflection symmetry of the lepton mixing matrix accommodates maximal atmospheric mixing ($\theta_{23} = \pi/4$) as well as maximal Dirac CP phase ($\delta = \pm \pi/2$) for the Dirac case. In the standard parametrization of the PMNS matrix the reflection symmetric nature is not directly visible while substituting the maximal values of the mixing parameters. This issue has been addressed in this paper. It is found that the reflection symmetry in the 'standard' PMNS matrix can be restored by allowing maximal values of the Majorana CP phases ($\alpha, \beta$) as well, along with maximal $\delta$. With the proposed scheme, the reflection symmetry is realized in the neutrino mixing matrix and the perturbation from charged lepton sector is studied. CKM like contributions are found to be consistent with current oscillation data.

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1 Introduction

The measurement of the non-zero reactor angle $\theta_{13}$ [1–3] elevates neutrino physics research one step ahead. It also initiates the exploration of leptonic CP violation in oscillation experiments. The Dirac CP violating phase $\delta$ is likely to be determined soon with good accuracy whereas the problems of octant degeneracy and mass ordering still require their solutions. Recent data from T2K [4], NO$\nu$A [5] and IceCube [6] experiments indicates a preference for the atmospheric angle $\theta_{23}$ to lie in the second octant which is also reflected in the global analysis of neutrino oscillation data made in Refs. [7, 8]. The global analysis also indicates that the value of $\delta$ is close to $-\pi/2$.

The approximate mixing pattern revealed by the oscillation data stems the main motivation towards $\mu - \tau$ flavour symmetry to understand the theory of lepton mixing. The near maximal value of the atmospheric mixing angle $\theta_{23}$ predicted by oscillation data is the key point behind $\mu - \tau$ flavour symmetry. The flavour symmetry, mostly exercised in lepton flavour models, is the so called $\mu - \tau$ permutation symmetry. It accommodates the well known predictions: $\theta_{23} = \pi/4$ and $\theta_{13} = 0$, that dominates the field of neutrino physics research over a long period of time. The permutation symmetry embedded with a CP conjugation of the lepton sector is referred to as the $\mu - \tau$ reflection symmetry. This concept of $\mu - \tau$ reflection symmetry was first put forwarded by Harrison and Scott [9]. Subsequently the mass matrix bearing the reflection symmetry property is realized in a $A_4$ based model by Babu, Ma and Valle [10] and a general treatment of the reflection symmetry is rendered in Ref. [11]. A review of $\mu - \tau$ flavour symmetry is also available in Ref. [12].

The prediction $\theta_{23} = \pi/4$ is a common feature of $\mu - \tau$ symmetry and of course $\theta_{12}$ remains arbitrary in either cases. However the two types of symmetry differ by their predictions on $\theta_{13}$ and the CP phase $\delta$. In case of $\mu - \tau$ permutation symmetry $\delta$ is washed out in the standard parametrization of PMNS matrix, as a consequence of $\theta_{13} = 0$. Thereby $\mu - \tau$ permutation symmetry naturally corresponds to CP conservation. In contrast to $\mu - \tau$ permutation symmetry, the reflection symmetry is featured with a non-zero $\theta_{13}$ and in addition, it corresponds to a maximal value of the CP phase ($\delta = \pm \pi/2$). We can also note down that in the standard parametrization, if we restrict $\theta_{13}$ to be zero, $\mu - \tau$ reflection symmetry readily reproduces the properties of permutation symmetry. In that sense $\mu - \tau$ reflection symmetry is a more general symmetry that can accommodate non-zero $\theta_{13}$ as well as CP violation.

Bi-maximal (BM) mixing and tri-bimaximal (TBM) mixing are two special cases of $\mu - \tau$ permutation symmetry. It is obvious that these special mixing schemes or in general the permutation symmetric models are mostly explored in the phenomenological studies of lepton mixing. In the present scenario, after the discovery of non zero $\theta_{13}$, permutation symmetry is seemingly an inadequate theory as it can not accommodate non-zero $\theta_{13}$ and the CP violation. In this regard $\mu - \tau$ reflection symmetry might serve a precious role in neutrino physics research. In comparison to the permutation symmetry, $\mu - \tau$ reflection symmetry and its possible phenomenological implications are less studied. The predictions $\theta_{23} = \pi/4$ and $\delta = \pm \pi/2$ are the central point of any $\mu - \tau$ reflection symmetric model. Such models incorporated with non abelian discrete symmetries and their significance have been discussed in Refs. [13–18]. Phenomenological consequences of $\mu - \tau$ reflection symmetry in the formalisms like texture zero and see-saw mechanism are studied in Refs. [19, 20].

Though $\mu - \tau$ reflection symmetry corresponds to the predictions- $\theta_{23} = \pi/4$ and $\delta = \pm \pi/2$, we can notice that the reflection symmetric nature of the mixing matrix can not be viewed by direct substitution of these values in the lepton mixing matrix in standard parametrization. This is in
contrast to $\mu - \tau$ permutation symmetry where its predictions ($\theta_{23} = \pi/4$ and $\theta_{13} = 0$) directly results into a $\mu - \tau$ (permutation) symmetric mixing matrix upon substitution. We have addressed this issue in this work and seek possible solutions to restore the symmetry in the standard parametrization. A full parametrization of the lepton mixing matrix with three mixing angles and six phases is considered and specific choice of the phases is found to serve the purpose of this work. Besides the maximal Dirac phase $\delta$, as accommodated by reflection symmetry itself, we are led to additionally set maximal values of Majorana phases ($\alpha$ and $\beta$) too in order to restore reflection symmetry property of the lepton mixing matrix.

In view of the oscillation data, a small deviation of $\theta_{23}$ from its maximal value is also notable. A perturbation which can break the symmetry is necessary to account for the desired deviations. We consider the contributions from charged lepton sector as a possible scheme to deviate $\theta_{23}$ and $\delta$ from their maximal values. In a basis where charged lepton mass matrix is non-diagonal, the charged lepton mixing matrix is allowed to break the reflection symmetry of the neutrino mixing matrix. To analyse the results of mixing parameters we assume the charged lepton contributions to be CKM like and parametrize the charged lepton mixing matrix in terms of the Wolfenstein parameter $\lambda$. As the atmospheric angle and the Dirac phase are fixed at maximal values in the neutrino mixing matrix, the remaining two mixing angles act as free parameters in our set up. These two free parameters mainly plays the governing role in determining the viability of the charged lepton correction scheme considered.

The paper is organised as follow: in section 2 we outline the basic ingredients of lepton mixing which are necessary for analysis. In section 3 we briefly review $\mu - \tau$ reflection symmetry and discuss the ambiguity addressed. Section 4 contains the scenario of broken reflection symmetry under the charged lepton correction scheme. For convenience the analysis is divided into two parts: at first we consider the case of $\theta_{23} = \pi/4$ and $\delta = +\pi/2$ and then we go for the other ($\theta_{23} = \pi/4$, $\delta = -\pi/2$). The summary and conclusions of the work are presented in section 5.

## 2 Ingredients of lepton mixing

Standard model charged current interaction Lagrangian for the leptons in flavour basis is given by

$$\mathcal{L}_{\text{int}} = - \frac{g}{\sqrt{2}} l'_L T^\mu \nu'_L W^{-}_\mu + \text{h.c.},$$

where $l'_L = (e' \mu' \tau')^T_L$ and $\nu'_L = (\nu'_e \nu'_\mu \nu'_\tau)^T_L$ represent the left handed charged lepton flavour states and neutrino flavour states respectively. In transforming to mass basis we get the lepton mixing $U$, also known as the PMNS matrix, in the Lagrangian :

$$\mathcal{L}_{\text{int}} = - \frac{g}{\sqrt{2}} l_L T^\mu U \nu_L W^{-}_\mu + \text{h.c.}$$

The un-primed fields, viz. $l_L = (e \mu \tau)^T_L$ and $\nu_L = (\nu_1 \nu_2 \nu_3)^T_L$, denote the respective mass eigenstates. We define the diagonalizing matrices $U_l$ and $U_\nu$, for the charged lepton and Majorana neutrino mass matrices respectively as : $U_l^\dagger M_l^\dagger M_l U_l = M_{ld}^2 \equiv \text{Diag}(m_e^2, m_\mu^2, m_\tau^2)$ and $U_\nu^\dagger M_\nu U_\nu^* = M_{\nu d} \equiv \text{Diag}(m_1, m_2, m_3)$, such that the PMNS matrix is given by

$$U = U_l^\dagger U_\nu.$$
\begin{center}
\begin{tabular}{|c|c|c|}
\hline
Mixing Parameter & Best fit & 3 $\sigma$ \\
\hline
$\sin^2 \theta_{12}$ & 0.310 & 0.275 - 0.350 \\
$\sin^2 \theta_{23}$ & 0.580 & 0.418 - 0.627 \\
$\sin^2 \theta_{13}$ & 0.0224 & 0.0204 - 0.0244 \\
$\delta$ & 215$^\circ$ & 125$^\circ$ - 392$^\circ$ \\
\hline
\end{tabular}
\end{center}

Table 1: Best fit and 3$\sigma$ values of mixing parameters for normal hierarchy (NH) from global analysis \cite{8}.

If we choose the basis where flavour eigenstates and mass eigenstates of the charged leptons are identical, the charged lepton mixing matrix $U_l$ in Eq.(3) becomes an identity matrix. The PMNS matrix $U$, which is a unitary matrix, can be parametrized in terms three mixing angles and six phases. In a familiar parametrization $U$ can be expressed as

$$U = P_1 V P_2,$$

where the mixing matrix $V$ is parametrized in terms of the three mixing angles ($\theta_{12}, \theta_{12}, \theta_{12}$) and the Dirac CP phase $\delta$. The diagonal phase matrix $P_2 = \text{diag}(e^{i\alpha}, e^{i\beta}, 1)$ contains two Majorana phases $\alpha$ and $\beta$ while $P_1 = \text{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})$ contains the remaining three phases. The three phases in $P_1$ are un-physical which can be eliminated from the mixing matrix $U$ by phase redefinition of the charged lepton fields. In the standard parametrization we have

$$V = \begin{pmatrix}
c_{12}c_{13}
c_{12}s_{23}c_{13}e^{i\delta}
s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta}
s_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta}
c_{12}c_{23} - s_{12}c_{23}s_{13}e^{i\delta}
s_{12}c_{13}
s_{13}e^{-i\delta}
c_{23}c_{13}
\end{pmatrix},$$

where $s_{ij} = \sin \theta_{ij}$ and $c_{ij} = \cos \theta_{ij}$ with $ij = 12, 23, 13$. So far physical observables are concerned, one may simply drop $P_1$ from Eq.(4). If neutrinos are considered as Dirac particles $P_2$ can further be dropped in a particular study.

Turning to the mixing angles, the sine of the angles can be expressed in terms of the absolute values of the elements of $U$ as follows :

$$\sin^2 \theta_{13} = |U_{e3}|^2, \quad \sin^2 \theta_{12} = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2}, \quad \sin^2 \theta_{23} = \frac{|U_{\mu3}|^2}{1 - |U_{e3}|^2}. $$

The measure of CP violation is expressed in terms of parametrization independent quantities called rephasing invariants. We consider the Jarlskog invariant \cite{21} given by

$$J = \text{Im}[U_{e2} U_{\mu3} U_{e3}^* U_{\mu3}^*],$$

for our analysis. For the mixing matrix $V$ in Eq.(5), Eq.(7) yields

$$J = s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^2 \sin \delta.$$ 

The best fit and 3$\sigma$ values of the three mixing angles and the Dirac CP phase for normal hierarchy (NH) are presented in Table 1 from the global analysis \cite{8}.
3 $\mu - \tau$ reflection symmetry

The original formulation of $\mu - \tau$ reflection symmetry, introduced by Harrison and Scott \[9\], concerns the Dirac phase $\delta$ only where Majorana phases are dropped from the lepton mixing matrix. In the present consideration it is represented by the mixing matrix $V$ in Eq.(5). They were motivated from the observation that the modulus of each $\mu$-flavour element of the mixing matrix is approximately equal to that of the corresponding $\tau$-flavour element (i.e. $|V_{\mu i}| \simeq |V_{\tau i}|$), as revealed by the neutrino oscillation data. They follow a specific parametrization of the mixing matrix based on the assumption $|U_{\mu i}| = |U_{\tau i}|$, and arrive at the mixing matrix

$$V_{HS} = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ v_1^* & v_2^* & v_3^* \end{pmatrix}, \quad (9)$$

where $u_i$’s are taken as real and $v_i$’s as complex. This mixing matrix is symmetric under a combined operation of interchanging $\nu_\mu$ and $\nu_\tau$ flavour states and complex conjugation of the mixing matrix. This combined operation of symmetry is referred to as $\mu - \tau$ reflection symmetry. The corresponding mass matrix is required to invariant under the $\mu - \tau$ reflection operation which can be expressed as

$$(A_{\mu \tau} M A_{\mu \tau})^* = M, \quad (10)$$

where

$$A_{\mu \tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (11)$$

is the $\mu - \tau$ exchange operator. The mass matrix satisfying Eq.(10) is given by

$$M = \begin{pmatrix} M_{ee} & M_{e\mu} & M_{e\tau}^* \\ M_{e\mu} & M_{\mu\mu} & M_{\mu\tau} \\ M_{e\tau}^* & M_{\mu\tau} & M_{\mu\mu} \end{pmatrix}, \quad (12)$$

where the elements $M_{ee}$ and $M_{\mu\tau}$ are real. This mass matrix were reproduced in an $A_4$ based model by Babu, Ma and Valle, one year after the concept of reflection symmetry introduced. The crucial thing about the mixing matrix $V_{HS}$ is that it is linked with the aforementioned predictions $\theta_{23} = \pi/4$ and $\delta = \pm \pi/2$. To see this connection let us consider the Jarlskog’s invariant for $V_{HS}$ which is given by $J = \frac{1}{2} u_1 u_2 u_3$, as obtained from Ref. \[9\]. In terms of mixing matrix elements modulus of $J$ can be written as

$$|J| = \frac{1}{2} |V_{e1} V_{e2} V_{e3}|. \quad (13)$$

Again in the standard parametrization, from Eq.(8) we have

$$|J| = \frac{1}{2} |V_{e1} V_{e2} V_{e3}| |\sin \delta| \sin 2\theta_{23}. \quad (14)$$

For non zero $\theta_{13}$, comparison of Eqs.(13) and (14) gives $|\sin \delta| \sin 2\theta_{23} = 1$. For $\theta_{23} = \pi/4$, this implies $\delta = \pm \pi/2$. 

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Conversely we may now wish to see whether the mixing matrix \( V \) in Eq.(5), with \( \theta_{23} = \pi/4 \) and \( \delta = \pm \pi/2 \), reflect the reflection symmetric nature of \( V_{HS} \) or not. To be explicit, we have

\[
V = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & \mp is_{13} \\
\frac{1}{\sqrt{2}}(-s_{12} \mp ic_{12}s_{13}) & \frac{1}{\sqrt{2}}(c_{12} \mp is_{12}s_{13}) & \frac{\mp is_{13}}{\sqrt{2}} \\
\frac{1}{\sqrt{2}}(s_{12} \mp ic_{12}s_{13}) & \frac{1}{\sqrt{2}}(-c_{12} \mp is_{12}s_{13}) & \frac{\mp is_{13}}{\sqrt{2}}
\end{pmatrix},
\tag{15}
\]

from Eq.(5), where \( \mp \) sign corresponds to \( \delta = \pm \pi/2 \). We can see that the matrix reproduces the presumed conditions : \( |U_{\mu}| = |U_{\tau}| \), as expected. However \( \mu \) and \( \tau \)-flavour mixing elements of \( V \) do not satisfy \( V_{\tau j} = V_{\mu j}^* \) (followed from Eq.(9)), instead we have \( V_{\tau j} = -V_{\mu j}^* \) for \( j = 1, 2 \) while \( V_{\tau 3} = V_{\mu 3}^* \). Further, most significantly, first row elements of the mixing matrix are not real which necessarily violates the inherent reflection symmetric nature carried by \( V_{HS} \). That means the PMNS matrix in the standard parametrization does not exhibit reflection symmetry under the constraints \( \theta_{23} = \pi/4 \) and \( \delta = \pm \pi/2 \). It is however necessary to point out that the mass matrix diagonalized by \( V \) in Eq.(15) respects reflection symmetry (Eq.(12)).

To realize the properties of reflection symmetry in the 'standard' PMNS matrix consistent with \( \theta_{23} = \pi/4 \) and \( \delta = \pm \pi/2 \), we find it useful to consider the full parametrization defined in Eq.(4). All the six phases taken into account, we get the PMNS matrix from Eq.(4) as

\[
U = \begin{pmatrix} V_{e1}e^{i(\alpha + s_1\phi_1)} & V_{e2}e^{i(\beta + \phi_1)} & V_{e3}e^{i\phi_1} \\
V_{\mu 1}e^{i(\alpha + \phi_2)} & V_{\mu 2}e^{i(\beta + \phi_2)} & V_{\mu 3}e^{i\phi_2} \\
V_{\tau 1}e^{i(\alpha + \phi_3)} & V_{\tau 2}e^{i(\beta + \phi_3)} & V_{\tau 3}e^{i\phi_3}
\end{pmatrix},
\tag{16}
\]

with the elements \( V_{ij} \) \( (l = e, \mu, \tau; \ j = 1, 2, 3) \) defined through Eq.(5). Compared to the 'Dirac like' mixing matrix \( V \) (Eq.(5)), concerned with the original formulation of reflection symmetry, we now have five additional phases under consideration. With \( \theta_{23} = \pi/4 \) and \( \delta = \pm \pi/2 \) the elements \( V_{ij} \) are given in Eq.(15). We find that the reflection symmetric nature of \( V_{HS} \) can be restored in \( U \) with proper choice of these additional phases. Let us first consider the case \( \delta = \pi/2 \). We then conveniently choose \( \phi_1 = \pi/2 \) and \( \alpha = \beta = -\pi/2 \) to make the first row elements all real. In addition the remaining two phases are constrained to zero \( (\phi_2 = \phi_3 = 0) \). With these specific values of the phases the PMNS matrix in Eq.(16) becomes

\[
U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\
\frac{1}{\sqrt{2}}s(-c_{12}s_{13}) + is_{12} & \frac{1}{\sqrt{2}}(s_{12}s_{13} - ic_{12}) & \frac{s_{13}}{\sqrt{2}} \\
\frac{1}{\sqrt{2}}(-c_{12}s_{13}) - is_{12} & \frac{1}{\sqrt{2}}(s_{12}s_{13} + ic_{12}) & \frac{s_{13}}{\sqrt{2}}
\end{pmatrix}.
\tag{17}
\]

This matrix is now exactly similar to \( V_{HS} \) with the first row elements all real and second and third row elements satisfying the condition \( U_{\tau j} = U_{\mu j}^* \) for all \( j = 1, 2, 3 \). The specific values of the un-physical phases so chosen may be attributed to the arbitrariness in their values. Besides, the values of the Majorana phases are remarkable. It is meant that, in addition to maximal \( \delta \), Majorana phases are also enforced to pick up maximal values in order to restore the symmetry. For the other case with \( \delta = -\pi/2 \), we may have the choices: \( \phi_1 = -\pi/2 \) and \( \alpha = \beta = (\pi/2) \), which differ by a negative sign in comparison to the previous set of values. The other two phases \( \phi_2 \) and \( \phi_3 \) should be kept fixed at zero as before. The resulting PMNS matrix, containing the reflection symmetry, is similar to that in Eq.(17) but with the elements \( U_{\mu 1} \) and \( U_{\mu 2} \) replaced by complex.
conjugation of the respective elements of $U$ in Eq.(17). In other sense the two mixing matrices are related by

$$U_{\delta=\pi/2} = U^*_{\delta=\pi/2},$$

where $U_{\delta=\pi/2}$ represents the matrix in Eq.(17).

In the basis where charged lepton mass matrix is already diagonal, the Majorana neutrino mass matrix can be obtained from $M = UM_{\nu d}U^T$. The mixing matrix in Eq.(17) leads to a mass matrix satisfying the reflection symmetry as shown by $M$ in Eq.(12). The elements are given by

$$M_{ee} = \left(m_1 c_{12}^2 + m_2 s_{12}^2\right) s_{13}^2 + m_3 s_{13},$$

$$M_{\mu\tau} = \frac{1}{2} m_1 \left(c_{12}^2 s_{13}^2 + s_{12}^2\right) + \frac{1}{2} m_2 \left(s_{12}^2 s_{13}^2 + c_{12}^2\right) + \frac{1}{2} m_3 c_{13}^2,$$

$$M_{e\mu} = \frac{1}{\sqrt{2}} \left(-m_1 c_{12}^2 - m_2 s_{12}^2 + m_3\right) s_{13} c_{13} + \frac{i}{\sqrt{2}} (m_1 - m_2) s_{12} c_{12} c_{13},$$

$$M_{\mu\mu} = \frac{1}{2} \left[m_1 \left(c_{12}^2 s_{13}^2 - s_{12}^2\right) + m_2 \left(s_{12}^2 s_{13}^2 - c_{12}^2\right) + m_3 c_{13}^2\right] - i(m_1 - m_2) s_{12} c_{12} s_{13}.$$

For the case $\theta_{23} = \pi/4$ and $\delta = -\pi/2$, the mass matrix obtained from $U_{\delta=\pi/2}$ follows a similar connection as presented in Eq.(18). The mass matrices of the two cases are complex conjugate of each other ($M_{\delta=-\pi/2} = M^*_{\delta=\pi/2}$).

### 4 Charged lepton contributions

If the values of $\theta_{23}$ and $\delta$ are not exactly maximal, one has to deviate from the reflection symmetry in some way. Possible corrections from the charged lepton sector are often considered in this regard \[\text{22}\text{,}\text{32}\]. To employ charged lepton corrections we recall Eq.(3) and consider the basis where charged lepton mass matrix is non-diagonal. Under this basis the lepton mixing matrix will contain contributions from both $U_l$ and $U_\nu$. The common idea of this approach is to assume a perfect symmetry in either of the two sectors ($U_l$ or $U_\nu$) and then perturb this symmetry by the other leading to a desired lepton mixing matrix. A treatment involving both the alternate cases is available in Ref. [22]. The symmetry considered in most works is the $\mu - \tau$ permutation symmetry which incorporates maximal atmospheric angle and zero reactor angle while solar angle is left arbitrary. Since Bimaximal mixing and Tri-bimaximal mixing are two special cases of $\mu - \tau$ permutation symmetry, deviations from such special mixing through charged lepton correction is most common. However corrections to special mixing based on $\mu - \tau$ reflection symmetry from charged lepton sector is very rare in the literature.

Each of $U_l$ and $U_\nu$ is a unitary matrix and can be parametrized in terms of three mixing angles and six phases as well. We invoke the parametrization set up in Eq.(4) and define

$$U_l = P_1^l V_l^T P_2^l, \quad U_\nu = P_1^\nu V_\nu^T P_2^\nu,$$

so that the resulting PMNS matrix becomes

$$U = (P_2^\dagger)^T (V_l^T)^T (P_1^l)^T P_1^\nu V_\nu^T P_2^\nu.$$ (21)

The diagonal phase matrices are defined as : $P_1^l = \text{diag}(e^{i\alpha_1^l}, e^{i\alpha_2^l}, e^{i\alpha_3^l})$, $P_2^l = \text{diag}(e^{i\alpha_1^l}, e^{i\beta_1}, 1)$, $P_1^\nu = \text{diag}(e^{i\alpha_1^\nu}, e^{i\alpha_2^\nu}, e^{i\alpha_3^\nu})$, $P_2^\nu = \text{diag}(e^{i\alpha_\nu}, e^{i\beta_\nu}, 1)$; while the matrices $V_l^T$ and $V_\nu^T$ resemble $V$ in
Eq.(5), given by

\[
V' = \begin{pmatrix}
-c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta'} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta'} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta'} & s_{13}' \\
-c_{12}c_{23}' - s_{12}s_{23}'s_{13}'e^{i\delta'} & c_{12}c_{23}' - s_{12}s_{23}'s_{13}'e^{i\delta'} & s_{13}'
\end{pmatrix}, \quad (22)
\]

\[
V'' = \begin{pmatrix}
-c_{12}'c_{13}' & s_{12}'c_{13}' & s_{13}'e^{-i\delta''} \\
-s_{12}'c_{23}' - c_{12}'s_{23}'s_{13}'e^{i\delta''} & c_{12}'c_{23}' - s_{12}'s_{23}'s_{13}'e^{i\delta''} & s_{13}' \\
-c_{12}'c_{23}' - s_{12}'s_{23}'s_{13}'e^{i\delta''} & c_{12}'c_{23}' - s_{12}'s_{23}'s_{13}'e^{i\delta''} & s_{13}'
\end{pmatrix}. \quad (23)
\]

The arbitrariness in the values of un-physical phases can be exploited here, and one may, in general, set \(\phi_j^l = \phi_j^\nu\) for all \(j = 1, 2, 3\), such that the total number of phases in the resulting PMNS matrix in Eq.(21) reduces to six.

We now assume an exact \(\mu - \tau\) reflection symmetry in the neutrino sector and let this symmetry perturb by the mixing matrix \(U_l\) of the charged lepton sector. Let us divide the analysis in two separate cases.

**Case I :** \(\theta_{23}^\nu = \pi/4\) and \(\delta'' = \pi/2\)

To realize the exact symmetry (Eq.(17)) in this case we have to set \(\phi_1^\nu = -(\pi/2) + \phi_1^\nu\) and \(\phi_j^\nu = \phi_j^\nu\) for \(j = 1, 2\), instead of the general relation \((\phi_j^\nu = \phi_j^\nu\) for all \(j = 1, 2, 3\)). The additional factor \(-\pi/2\) in \(\phi_1^\nu\), along with \(\alpha'' = \beta'' = -\pi/2\) will maintain \(\mu - \tau\) reflection symmetry in the neutrino sector. Under these considerations, the PMNS matrix in Eq.(21) can be expressed as

\[
U = (\bar{U}_l)^\dagger \bar{U}_\nu, \quad (24)
\]

with \((\bar{U}_l)^\dagger = (P_2^l)^\dagger (V')^\dagger\) and the neutrino mixing matrix given by

\[
\bar{U}_\nu = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\
\frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{-i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\]

Above mixing matrix of the neutrino sector thus contains the reflection symmetry as presented in Eq.(17).

A natural possibility to account charged lepton corrections to neutrino mixing is that the charged lepton mixing matrix is CKM like [33][35]. We consider this natural choice and use Wolfenstein parameter \(\lambda\) [36] to parametrize \(V'\). Taking \(s_{12} = \lambda, s_{23} = A\lambda^2\) and \(s_{13} = B\lambda^3\), we consider the following mixing matrix for the lepton sector given by

\[
V' = \begin{pmatrix}
1 - \frac{\lambda^2}{2} & \frac{\lambda}{2} & B\lambda^3e^{-i\delta'} \\
y^2 & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\
\lambda^3(A - Be^{i\delta'}) & -A\lambda^2 & 1
\end{pmatrix}, \quad (26)
\]

for the analysis, where the higher order contributions from \(\lambda \sim O(n \geq 4)\) are dropped. From Ref. [37] the values of \(\lambda, A\) and \(B\) are taken to be \(\lambda = 0.2245, A = 0.836, B = 0.3482\). From
Figure 1: (a) Variation of $\sin^2 \theta_{13}$ wrt $\sin \theta_{13}'$. Horizontal coloured band represents the $3\sigma$ range of $\sin^2 \theta_{13}$ obtained from global data. (b) Variation of $\sin^2 \theta_{23}$ wrt $\sin \theta_{23}'$. Vertical coloured bands represent the allowed ranges of $\sin \theta_{23}'$ corresponding to (a) (left stands for $\sin \theta_{13}' \sim 0.01$, right for $\sin \theta_{13}' \sim 0.30$).

Eqs.(24), (25) and (26) we can now compute the elements of the desired PMNS matrix. Let us write down four relevant elements which are useful for our analysis:

$$U_{e2} = \left(1 - \frac{\lambda^2}{2}\right)s_{12}'c_{13}' - \frac{\lambda}{\sqrt{2}} Z + \frac{\lambda^3}{\sqrt{2}} Z^* (A - B e^{-i\delta^l}) e^{-i\alpha^l}, \quad (27)$$

$$U_{e3} = \left(1 - \frac{\lambda^2}{2}\right)s_{12}'c_{13}' - \frac{\lambda}{\sqrt{2}} Z + \frac{\lambda^3}{\sqrt{2}} c_{13}' (A - B e^{-i\delta^l}) e^{-i\alpha^l}, \quad (28)$$

$$U_{\mu 2} = \left[\lambda s_{13}' c_{13}' + \frac{1}{\sqrt{2}} (1 - \frac{\lambda^2}{2})Z - A \frac{\lambda^2}{\sqrt{2}} Z^* \right] e^{-i\beta^l}, \quad (29)$$

$$U_{\mu 3} = \left[\lambda s_{13}' c_{13}' + \frac{1}{\sqrt{2}} (1 - \frac{\lambda^2}{2})c_{13}' - A \frac{\lambda^2}{\sqrt{2}} c_{13}' \right] e^{-i\beta^l}, \quad (30)$$

with $Z = -s_{12}' s_{13}' + i c_{12}'$. From Eq.(28) we have

$$|U_{e3}|^2 \simeq \sin^2 \theta_{13}' - \frac{\lambda}{\sqrt{2}} \sin 2\theta_{13}' + \frac{\lambda^2}{2} \left(1 - 3(s_{13}')^2\right)$$

$$+ \frac{\lambda^3}{2\sqrt{2}} \sin 2\theta_{13}' \left[1 + 2(A - B \cos \delta^l)\right], \quad (31)$$

where higher order terms with $\lambda \sim O(n \geq 4)$ are neglected. For the best fit value of $|U_{e3}|^2 = 0.0224$ (Table 1) and $\lambda = 0.2245$ we can solve Eq.(31) for $\sin \theta_{13}'$. Note that influence of the phase $\delta^l$ of charged lepton sector in this expression is suppressed by $O(\lambda^3)$. For $\delta^l = 0$ say, we find that there exists two possible values of $\sin \theta_{13}'$ contributing to $|U_{e3}|^2$: one vanishingly small ($\sin \theta_{13}' \approx 0.01$) and other relatively large ($\sin \theta_{13}' \approx 0.3$). The variation of $\sin^2 \theta_{13}$ with $\sin \theta_{13}'$ is shown in Fig.1(a).
The allowed values of $\sin^{\nu}\theta_{13}$ also shows that the terms dominated by higher orders of $s_{13}^{\nu}$ can be neglected to simplify the analytic expressions. Following this, expression (31) can be further reduced and we get

$$\sin^{2}\theta_{13} = |U_{e3}|^{2} \approx \sin^{2}\theta_{13}^\nu - \frac{\lambda}{\sqrt{2}} \sin 2\theta_{13}^\nu + \frac{\lambda^{2}}{2}. \quad (32)$$

We emphasize that the above expression is almost consistent with that of Eq.(31) with the solutions $\sin^{\nu}\theta_{13} \approx 0.009, 0.294$. These two solutions in fact signify the following two relations-

$$\sin \theta_{13} \approx -\sin^{\nu}\theta_{13} + \frac{\lambda}{\sqrt{2}},$$

$$\sin \theta_{13} \approx \sin^{\nu}\theta_{13} - \frac{\lambda}{\sqrt{2}}, \quad (33)$$

respectively, that can be verified approximately by taking square root of Eq.(32). The presence of the factor $\frac{\lambda^{2}}{2}$ on the right hand side of Eq.(32) is important to note as its value is very much close to $\sin^{2}\theta_{13}$. The significance of this factor has been pointed out in Ref. [38].

From Eqs.(30) and (27) and using Eq.(6) we have

$$\sin^{2}\theta_{23} \approx \frac{1}{2} + \left[ \frac{\lambda}{2\sqrt{2}} \sin 2\theta_{13}^\nu \left( 1 + 3(s_{13}^\nu)^{2} \right) - \left( \frac{1}{4} + A \right) \lambda^{2} - \left( \frac{1}{2}(s_{13}^\nu)^{2} + \lambda^{2} \right) (s_{13}^\nu)^{2} \right], \quad (34)$$

$$\sin^{2}\theta_{12} \approx \sin^{2}\theta_{12}^\nu \left[ 1 + \frac{\lambda}{\sqrt{2}} \sin 2\theta_{13}^\nu - \frac{\lambda^{2}}{2} \left( 3 + 4(s_{13}^\nu)^{2} \right) - (s_{13}^\nu)^{4} \right]$$

$$+ \frac{\lambda^{2}}{2} - B \frac{\lambda^{3}}{\sqrt{2}} \sin 2\theta_{12}^\nu c_{13}^\nu \sin \delta_{l}. \quad (35)$$

The bracketed quantity in the expression (34) accounts for the deviation from the maximal value of $\theta_{23}$. The expression also shows that the prediction on $\theta_{23}$ depends only on the neutrino sector angle $\theta_{13}^\nu$. For the allowed values of $\sin^{\nu}\theta_{13}$ obtained from Eq.(32) we can calculate the values of $\sin^{2}\theta_{23}$. For $\sin^{\nu}\theta_{13} = 0.009$, we get $\sin^{2}\theta_{23} = 0.4467$ while the other possibility $\sin^{\nu}\theta_{13} = 0.294$ yields $\sin^{2}\theta_{23} = 0.4933$. Although these predictions lie in the first octant still they are consistent with the $3\sigma$ range of global analysis data (Table 1). A summary of the calculated values of $\sin^{2}\theta_{23}$ corresponding to the two possibilities ($\sin^{\nu}\theta_{13} \sim 0.01$ and $\sin^{\nu}\theta_{13} \sim 0.3$) are presented in Table 2. Variation of $\sin^{2}\theta_{23}$ with $\sin^{\nu}\theta_{13}$ is depicted in Fig.1(b).

From the expression (35) we see that $\theta_{12}$ depends on both the neutrino sector angles $\theta_{12}^\nu$ and

| $\sin^{\nu}\theta_{13}$ | $\sin^{2}\theta_{23}$ | $\sin_{23}$ | $\sin^{\nu}\theta_{13}$ | $\sin^{2}\theta_{23}$ | $\sin_{23}$ |
|------------------------|------------------------|-------------|------------------------|------------------------|-------------|
| 0.009                  | 0.4467                 | 0.4452      | 0.294                  | 0.4933                 | 0.4944      |
| 0.5309                 | 0.5320                 | 0.5299      | 0.5775                 | 0.5764                 | 0.5786      |

Table 2: Allowed values of $\sin^{\nu}\theta_{13}$ corresponding to the best fit and $3\sigma$ values of $\sin^{2}\theta_{13}$ (Table 1) obtained from Eq.(32) and those of $\sin^{2}\theta_{23}$ that predicted from Eq.(34).
\[ \theta_{13}^\nu. \] However its dependence on \( \delta^l \) is suppressed by \( O(\lambda^3) \). The correlation between \( \sin^2 \theta_{12} \) and \( \sin \theta_{13}^\nu \) is shown in Fig.2(a). From the plot we can determine an approximate allowed range of \( \sin \theta_{12} \) corresponding to the \( 3\sigma \) range of \( \sin^2 \theta_{12} \). For \( \sin \theta_{13}^\nu = 0.01 \) and \( \delta^l = 0 \) we get an allowed range \( 0.52 \lesssim \sin \theta_{12} \lesssim 0.59 \), while for the other possibility, \( \sin \theta_{13}^\nu = 0.30 \) along with \( \delta^l = 0 \), we get \( 0.50 \lesssim \sin \theta_{12}^\nu \lesssim 0.57 \) corresponding to the \( 3\sigma \) values of \( \sin^2 \theta_{12} \) \((0.275 - 0.350)\).

Using Eq.(7) the Jarlskog invariant from Eqs.(27)-(30) is obtained as

\[
J_+ \simeq \frac{1}{2} s_{12}^\nu c_{12}^\nu s_{13}^\nu (c_{13}^\nu)^2 + \frac{\lambda}{2\sqrt{2}} s_{12}^\nu c_{12}^\nu c_{13}^\nu [3(s_{13}^\nu)^2 - 1] - \lambda^2 s_{12}^\nu c_{12}^\nu s_{13}^\nu (c_{13}^\nu)^2 + \frac{\lambda^3}{2\sqrt{2}} \left[ s_{12}^\nu c_{12}^\nu c_{13}^\nu \left( \frac{1}{2} + A - B \cos \delta^l \right) - B(c_{12}^\nu)^2 s_{13}^\nu c_{13}^\nu \sin \delta^l \right].
\] (36)

The first term of above equation represents the contribution that corresponds to the case of maximal CP violation \( (\delta^\nu = \pi/2) \) and maximal atmospheric mixing \( (\theta_{23}^\nu = \pi/4) \) in the neutrino sector. The correlation between \( J_+ \) and \( \sin \theta_{12}^\nu \) for the allowed ranges of \( \sin \theta_{13}^\nu \) (Table 2) and \( \delta^l \) \((0 - 2\pi)\) is shown in Fig.2(b). For \( \sin \theta_{13}^\nu = 0.01 \) and \( \delta^l = 0 \), the allowed range of \( \sin \theta_{12}^\nu = 0.52 - 0.59 \) corresponds to \(-0.03111 \lesssim J_+ \lesssim (-0.03346)\). For the other possibility, \( \sin \theta_{13}^\nu = 0.30 \) along with \( \delta^l = 0 \), we get \(-0.03159 \lesssim J_+ \lesssim (-0.03398)\) corresponding to \( \sin \theta_{12}^\nu = 0.50 - 0.57 \). The predictions on \( J_+ \) are thus well fitted with the maximal value of the Jarlskog invariant \((J_{\text{max}} = 0.0333)\) as provided by the global analysis [8]. The global analysis also presented a best fit value \( J_{\text{best}} = -0.019 \) for non-maximal \( \delta \). In the present analysis we have constrained the values of \( \sin \theta_{13}^\nu \) and \( \sin \theta_{12}^\nu \) using the experimental bounds on \( \sin \theta_{13} \) and \( \sin \theta_{12} \) in Eqs.(32) and (35) respectively. We have seen that the values of \( \sin \theta_{13}^\nu \) and \( \sin \theta_{12}^\nu \) so obtained are able to predict \( J_+ \) which is closed to \( J_{\text{max}} = 0.0333 \), for \( \delta^l = 0 \). To estimate non-maximal value of \( J_+ \), we rely on non-zero values of \( \delta^l \). However, as the effect of \( \delta^l \) in Eq.(36) is suppressed by \( O(\lambda^3) \), it does not cause significant change in the prediction of \( J_+ \). We have calculated the values of \( J_+ \) for \( \delta^l = \pi/2 \) and are given in Table 3. To lower the value of \( J_+ \) to the order of \( J_{\text{best}} = -0.019 \) some effective theoretical treatment will be required.
The mixing matrix $V$ on $\theta$ to this problem in the present charged lepton correction scheme. It is found that the prediction of $\theta$ affects those elements of $V$. Eq.(38) now predicts $\sin \theta_{12}$.

Table 3: Allowed values of $\sin \theta_{12}$ obtained from Eq.(35) and values of $J_+$ predicted from Eq.(36).

| $\sin \theta'_{13}$ | $\delta' = 0$ | $\sin \theta'_{13}$ | $\delta' = 0$ |
|---------------------|---------------|---------------------|---------------|
| Best fit            | $3\sigma$ range | Best fit            | $3\sigma$ range |
| $\sin \theta_{12}$ | 0.5541        | 0.5189 - 0.5917     | 0.5341        | 0.5002 - 0.5704 |
| $-J_+$              | 0.03235       | 0.03111 - 0.03346   | 0.03286       | 0.03159 - 0.03398 |
| $-J_+$              | 0.03578       | 0.03441 - 0.03701   | 0.02992       | 0.02877 - 0.03094 |
| $\sin \theta_{12}$ | 0.5566        | 0.5215 - 0.5941     | 0.5365        | 0.5027 - 0.5727 |
| $-J_+$              | 0.03259       | 0.03136 - 0.03369   | 0.03296       | 0.03168 - 0.03410 |
| $-J_+$              | 0.03568       | 0.03432 - 0.03688   | 0.03002       | 0.02885 - 0.03106 |

As per the recent indication that $\theta_{23}$ prefers the second octant we search for possible solution to this problem in the present charged lepton correction scheme. It is found that the prediction on $\theta_{23}$ can be lifted up to the desired octant by making a transition: $\theta_{23}' \rightarrow -\theta_{23}'$. This transition affects those elements of $V'$ in Eq.(26) that involve $\theta_{23}$ by causing a change of $A$ by a negative sign. The mixing matrix $V'$ in Eq.(26) thus becomes

$$V' = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & B\lambda^3 e^{-i\delta'} \\ -\lambda & 1 - \frac{\lambda^2}{2} & -A\lambda^2 \\ -\lambda^3(A + Be^{i\delta'}) & A\lambda^2 & 1 \end{pmatrix}.$$  

This mixing matrix together with $\hat{U}_\nu$ in Eq.(25) will now generate the desired PMNS matrix. Algebraic calculations show that the transition stated above, causes an overall effect that can be accounted to the change of the parameter $A$ by a negative sign ($A \rightarrow -A$). Thus the predictions on $\theta_{13}$ and $\theta_{12}$ are not affected under this transition and are given by Eqs.(32) and (35) respectively. But that on $\theta_{23}$ is now given by

$$\sin^2 \theta_{23} \simeq \frac{1}{2} + \left[ \frac{\lambda}{2\sqrt{2}} \sin 2\theta_{13} (1 + 3(s_{13}'^2)^2) - \left( \frac{1}{4} - A \right) \lambda^2 - \left( \frac{1}{2} (s_{13}'^2 + \lambda^2) (s_{13}'^2)^2 \right) \right].$$  

and the Jarlskog invariant becomes

$$J_+ \simeq \frac{1}{2} s_{12}' c_{12}' s_{13}' (c_{13}')^2 + \frac{\lambda}{2\sqrt{2}} s_{12}' c_{12}' c_{13}' \left[ 3(s_{13}'^2)^2 - 1 \right] - \lambda^2 s_{12}' c_{12}' s_{13}' (c_{13}')^2$$

$$+ \frac{\lambda^3}{2\sqrt{2}} \left[ s_{12}' c_{12}' c_{13}' \left( \frac{1}{2} - A - B \cos \delta' \right) - B(c_{12}')^2 s_{13}' c_{13}' \sin \delta' \right].$$  

Eq.(38) now predicts $\sin^2 \theta_{23} = 0.531$ for $\sin \theta'_{13} = 0.01$ and $\sin^2 \theta_{23} = 0.577$ for $\sin \theta'_{13} = 0.30$. These predictions are in good agreement with the global analysis data (Table 1). The prediction on $J_+$ suffers a slight change compared to the previous prediction which can be read from Table 3.

**Case II**: $\theta'_{23} = \pi/4$ and $\delta'' = -\pi/2$

To realize the reflection symmetry (Eq.(17)) in the neutrino sector in this case, we set $\phi_{1}' = (\pi/2) + \phi_1''$ and $\phi_{j}' = \phi_j''$ for $j = 1, 2$. The factor $\pi/2$ in $\phi_1'$, along with $\alpha'' = \beta'' = \pi/2$ will now
maintain $\mu - \tau$ reflection symmetry in the neutrino sector as before. The relevant neutrino mixing matrix containing the symmetry can be obtained by taking complex conjugation of the neutrino mixing matrix in Eq.(25) (the corresponding matrix of case I). The corresponding charged lepton mixing matrix $V^l$ will remain same as that of Case I (Eq.(26)). We compute the new PMNS matrix for this case and find that the expressions for $\theta_{13}$ and $\theta_{23}$ remain unaltered and are given by Eqs.(32) and (34) respectively. However the expression for solar angle becomes

$$\sin^2 \theta_{12} \simeq \sin^2 \theta'_{12} \left[ 1 + \frac{\lambda}{\sqrt{2}} \sin 2\theta'_{13} - \frac{\lambda^2}{2} \left( 3 + 4(s_{13}^\nu)^2 \right) - (s_{13}^\nu)^4 \right]$$

$$+ \frac{\lambda^2}{2} + B \frac{\lambda^3}{\sqrt{2}} \sin 2\theta_{12} c_{13}^\nu \sin \delta^l.$$  \hspace{1cm} (40)

The expression for Jarlskog invariant remains same except for a negative sign and is given by $J^- = -J^+$ with $J^+$ given in Eq.(36). If we compare the expressions for $\sin^2 \theta_{12}$ of case I and case II (Eqs(35) and (40)), it can be seen that they only differ by a negative sign in the last term involving the factor $\sin \delta^l$. As these last terms are suppressed by $O(\lambda^3)$, the numerical predictions on mixing parameters in this case almost remain same as those of case I.

To lift $\theta_{23}$ to the second octant we can employ the same transition ($\theta_{23}^l \rightarrow -\theta_{23}^l$) in this case also. Under this transition $\theta_{13}$ and $\theta_{12}$ will respectively be given by Eqs.(32) and (40), while $\theta_{23}$ is given by Eq.(38). The Jarlskog invariant follows the same relation: $J^- = -J^+$, with $J^+$ given by Eq.(39). Relevant predictions are found almost similar to those of case I.

5 Summary and conclusion

The role of $\mu - \tau$ reflection symmetry, as it features a non zero $\theta_{13}$ besides maximal $\theta_{23}$, is significant in the study of lepton flavour models. In this work we point out that the reflection symmetric nature of the lepton mixing matrix is not reflected back while substituting the maximal values of $\theta_{23}$ and $\delta$ in the standard parametrization. Motivated by this observation we look for possible solution to this ambiguity and find that the symmetry can be restored by assigning maximal values of the Majorana phases as well in addition to maximal Dirac phase $\delta$. A noteworthy contribution from the un-physical phases is remarked in the symmetry realization.

We have exercised the scenario under a broken symmetry such that deviated values of maximal $\theta_{23}$ and maximal $\delta$ can be accommodated. The contributions from charged lepton sector is considered as a possible scheme to generate the broken symmetry. We implant the reflection symmetry in the neutrino mixing matrix $\tilde{U}_\nu$ while the charged lepton contributions ($\tilde{U}_l$) are parametrized in terms of Wolfenstein parameter $\lambda$. This leaves the neutrino mixing angles $\theta_{13}^\nu$ and $\theta_{12}^\nu$ as free parameters in the set up. Besides these we have another free parameter $\delta^l$ from the charged lepton sector. We find that the reactor angle $\theta_{13}$ depends on $\theta_{13}^\nu$ only in the leading order and $\theta_{13}^\nu$ in turn governs the prediction on $\sin^2 \theta_{23}$. On the other hand the solar angle and the Jarlskog invariant are found to be related with all the three free parameters. The effect of $\delta^l$ is however negligible in predicting the mixing parameters. We use the experimental bounds of $\theta_{13}$ and $\theta_{12}$ to constrain the values of $\sin \theta_{13}^\nu$ and $\sin \theta_{12}^\nu$. These constrained values are in turn used to examine the numerical predictions on $\sin^2 \theta_{23}$ and $J$. Consistency of the predictions on $\sin^2 \theta_{23}$ and $J$ with observed data would imply viability of the charged lepton correction scheme adopted. We observe two possible
scenarios corresponding to $\sin^2 \theta_{13} \sim 0.01$ and $\sin^2 \theta_{13} \sim 0.30$. For the generic parametrization of $\tilde{U}_l$ with positive $\theta^l_{23}$, the atmospheric angle $\theta_{23}$ finds its location in the first octant with a value closed to the maximal. We find that a negative argument of the charged lepton atmospheric mixing angle ($-\theta^l_{23}$) can lift the value of $\theta_{23}$ to the second octant. In view of the best fit value of $\theta_{23} = 0.58$, the choice $\sin^2 \theta_{13} \sim 0.30$ is preferable as can be seen from table 2. Regarding the prediction on CP violation, the maximal values of $J$ computed through the constrained values of $\theta^\nu_{13}$ and $\theta^\nu_{12}$, are found to be consistent with the observed value $J_{max} = 0.0333$ of global data $[8]$. The charged lepton phase $\delta^l$ can impart non maximal values of $J$ in our present set up. But the effect of $\delta^l$ is suppressed by the higher orders in $\lambda$ and thereby values of $J$ does not change significantly from its maximal values for non zero values of $\delta^l$. As per the non maximal best fit value $J_{best} = -0.019$, provided by the global analysis $[8]$, some theoretical refinements to the present charged lepton contribution scheme will be required in this regard. We planned it for a future study. In conclusion, besides the prediction on non maximal $J$, the model studied is in pretty good agreement with the current data.

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