A combined approximation method for nonlinear foam drainage equation

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May 18, 2021

Abstract
The aim of this study is to develop a combined approximative technique to find a numerical solution to the foam drainage equation arising in various absorption and distillation processes. In this approach, first, the discretization of time is performed with the aid of the Taylor expansion series. Hence, a collocation method based on novel Bessel polynomials is utilized for the space variable. Thus the solution is found by solving a linear system of algebraic equations at each time step in contrast to solving a nonlinear system. Numerical simulations are provided to check the accuracy and efficiency of the presented algorithm. The numerical results are compared with exact solutions as well as with the outcomes of other existing available numerical methods.

Key Words: Foam drainage equation, Bessel functions, Collocation points, Nonlinear PDE, Taylor series expansion.

2010 Mathematics Subject Classification: 26A33, 65L60, 42C05, 65L05.

1 Introduction

This research aimed to develop an efficient approximation algorithm to solve the nonlinear foam drainage equation [2]

\[
\frac{\partial W}{\partial t} + \frac{\partial}{\partial x} \left( W^2 - \frac{\sqrt{W}}{2} \frac{\partial W}{\partial x} \right) = 0,
\]

with initial condition \( W|_{t=0} = W_0(x) \).

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Here, $W(x,t)$ (with $x$ as the scaled coordinates and $t$ as the time position) denotes the the cross section of a channel formed where three films meet, usually indicated as “Plateau border”. Foams are naturally appearing in numerous applications and technological processes and due to their importance, have attracted many researchers, see cf. [2]-[3].

By substituting $W(x,t) := w^2(x,t)$ in (1)-(2), we arrive at the following nonlinear initial-value problem

$$\begin{cases}
w_t + 2w^2 w_x - w_x^2 - \frac{1}{2} w w_{xx} = 0, & 0 \leq x \leq L, \: 0 \leq t \leq T, \\
w|_{t=0} = w_0(x) := \sqrt{W_0(x)}, & 0 < x < L,
\end{cases}$$

(3)

where $T > 0$ is a given final time and $L > 0$ is a real constant. In addition, the following boundary conditions are supplemented with the initial-value problem (3)

$$w(0,t) = h_0(t), \quad w(L,t) = h_1(t), \quad 0 \leq t \leq T,$$

(4)

where $h_0(t), h_1(t)$ are two prescribed functions. In the last few decades, researchers have proposed several analytical techniques as well as approximative algorithms to solve the foam drainage equation. Among these methods, we mention the Tahn and Adomian decomposition methods [4], the homotopy perturbation method [5], the symmetry Lie group approaches [6]-[7], the series solution based on the homotopy analysis method [8], the Exp-function approach [9], the homotopy perturbation transform method [10], a semi-analytical approach based upon the quasilinearization and the Haar wavelet bases, and a hybrid computational approach based on the generalized Chebyshev polynomials and quasilinearization technique [11].

The main goal of this research study is to derive a new combined approximation technique based on a combination of Taylor-series approach and spectral collocation scheme for the numerical treatment of the nonlinear foam-drainage equation. In one hand, the considered equation is a time-dependent model problem and thus it is of interest to develop an accurate time-marching algorithm for our model. On the other hand, collocation-based method have been applied successfully to many model problems in science and engineering due to their efficiency as well as simple applicability while giving high accuracy. Among many existing approaches based on the collocation strategy, let us mention the meshfree methods [12]-[13], the spectral collocation approach based on diverse polynomial bases such as Legendre, Chebyshev, Chelyshkov, etc. were utilized in [14]-[17], over the past decades.

Roughly speaking, we first employ the Taylor approach with second-order accuracy to discretize the time variable. Then in each time step, we suppose that the underlying model problem has a solution in terms of the novel Bessel series expansion of unknown function. Afterwards, representing all involved unknowns in the Bessel matrix form together with the proper usage of a suitable set of collocation points leads to determine the unknown series coefficients through solving a linear system of matrix
equation. Indeed, the Bessel polynomial of order \( \ell \) is defined explicitly as
\[
\mathcal{B}_\ell(x) = \sum_{\kappa=0}^\ell \frac{(\ell + \kappa)!}{(\ell - \kappa)! 2^{\kappa} \kappa!} x^\kappa, \quad \ell = 0, 1, \ldots,
\] (5)
see also [19]-[22] for a recent applications. Besides the fact that all coefficients of \( \mathcal{B}_\ell(x) \) are positive, they also satisfy the second-order differential equation
\[
x^2 \mathcal{B}_\ell''(x) + (1 - \ell(x+1)) \mathcal{B}_\ell'(x) - \ell(\ell+1) \mathcal{B}_\ell(x) = 0.
\]

It should be noted that the considered Bessel functions \( \mathcal{B}_\ell(x) \) are different from the traditional Bessel functions of the first kind, which have previously been utilized in various research papers, see cf. [17, 23].

2 Taylor scheme for time discretization

We are firstly aimed to discretize the foam drainage equation with respect to time variable. In this respect, we subdivide the interval \([0, T]\) into \((M + 1)\) grid points
\[
0 =: t_0 < t_1 = \Delta t < \ldots < t_M := M\Delta t = T,
\]
being \( \Delta t = t_n - t_{n-1} \) the uniform time step. To get a time-accurate discretization scheme, according to the Taylor series representation for \( w^n_t = w(x, t_n) \) we obtain
\[
w^n_t = \frac{w^{n+1}_t - w^n_t}{\Delta t} - \frac{1}{2} \Delta t \frac{w^n_{tt}}{\Delta t} + O(\Delta t^2).
\] (6)

To proceed, we differentiate equation (6) with respect to \( t \) to get
\[
w^n_{tt} = \frac{1}{2} \frac{w^n_{tt}}{\Delta t} + \frac{1}{2} \frac{w^n_{t} w^n_\ell}{\Delta t} + \frac{1}{2} \frac{w^n_{x} w^n_\ell}{\Delta t} + \frac{1}{2} \frac{w^n_{x} w^n_\ell}{\Delta t} - 4 w^n_{x} w^n_\ell - 2 (w^n)_{xx}.
\]

By replacing the first order derivatives \( w^n_t \approx (w^{n+1} - w^n)/\Delta t \) in all occurrences, we may write \( w^n_t \) as
\[
\Delta t w^n_{tt} = \frac{1}{2} w^n_{xx} - 4 w^n_{x} w^n_\ell (w^{n+1} - w^n) + 2 [w^n_{x} - (w^n)^2] (w^{n+1} - w^n) + \frac{1}{2} w^n (w^{n+1} - w^n)_\ell. \] (7)

We next insert equation (7) into the right-hand side of equation (6) and using the time discretized form of (6), i.e.,
\[
w^n_t = -2 (w^n)^2 w^n_\ell + \frac{1}{2} w^n w^n_\ell,
\]
for the left-hand side of equation (3). After some manipulations, the following time discretized equation for equation (3) with second-order accuracy in time is obtained

\[
\Delta t \left( 2w^n w^n_x - \frac{1}{4} w^n_{xx} \right) + 1 \right] w^{n+1} - \Delta t \left[ w^n_x - (w^n)^2 \right] w_x^{n+1} = w^n \left[ 1 + \Delta t w^n w^n_x \right],
\]

for \( n = 0, 1, \ldots \). To start computations in equation (3), we need \( w_0 = w_0(x) \), which is obtained from the initial condition (2). Moreover, the boundary conditions obtained from (4) at \( x = 0 \), \( L \) are

\[
w^{n+1}(0) := h_0^{n+1} = h_0(t_{n+1}), \quad w^{n+1}(L) := h_1^{n+1} = h_1(t_{n+1}), \quad n = 0, 1, \ldots \quad (9)
\]

3 Bessel functions: Basic matrix relations

Now, the first stage in discretizing the foam drainage equation in time is carried out by equation (3). In the second stage, we require to approximate the solution of the original model (1) with respect to the space variable through solving equation (8). To do so, we assume that the solution \( w^{n+1} \) of equation (3) can be written as a combination of \( B_\ell(x) \). In the first time step, i.e., for \( n = 0 \), we use the initial condition \( w_0(x) \) to determine \( w^0 \) exactly. Let us denoted \( W_{n,N}(x) \) to be the approximate solution of \( w^n \) at time level \( t_n \). Then, we looking for the \( W_{n+1,N}(x) \) at the next time level \( t_{n+1} \) as

\[
W_{n+1,N}(x) = \sum_{\ell=0}^{N} a_{\ell,n} B_\ell(x), \quad x \in [0, L],
\]

for \( n = 0, 1, \ldots, M \). Here, \( a_{\ell,n}, \ell = 0, 1, \ldots, N \) as the unknown Bessel coefficients must be found. Let us introduce the Bessel vector \( B_N(x) \) as well as the unknown vector \( A_{n,N} \) in the forms

\[
B_N(x) = [B_0(x) \ B_1(x) \ \ldots \ B_N(x)], \quad A_{n,N} = [a_{0,n} \ a_{1,n} \ \ldots \ a_{n,n}]^T.
\]

With the help of these vectors, we are able to rewrite relation (10) in a compact representation as follows

\[
W_{n+1,N}(x) = B_N(x) A_{n,N},
\]

(11)
Additionally, by introducing the the matrix $D$

$$
D = \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 & 0 \\
1 & 1 & 0 & \ldots & 0 & 0 \\
1 & 3 & 3 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
1 & \frac{N!}{(N-2)!1!2!} & \frac{(N+1)!}{(N-3)!2!2^2} & \ldots & \frac{(2N-2)!}{0!(N-1)!2^{N-1}} & 0 \\
1 & \frac{(N+1)!}{(N-1)!1!2!} & \frac{(N+2)!}{(N-2)!2!2^2} & \ldots & \frac{(2N-1)!}{1!(N-1)!2^{N-1}} & \frac{(2N)!}{0!N!2^N}
\end{bmatrix}_{(N+1) \times (N+1)}
$$

and the monomial vector $X_N(x) = [1, x, x^2, \ldots, x^N]$, we shall express $B_N(x)$ as

$$
B_N(x) = X_N(x) D^T,
$$

(12)

We are left with finding a relationship between $X_N(x)$ and $\frac{d}{dx}X_N(x)$. A straightforward calculation shows that

$$
\frac{d}{dx}X_N(x) = X_N(x) M^T,
$$

(13)

$$
M^T = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ddots & N \\
0 & 0 & 0 & \ldots & 0
\end{bmatrix}_{(N+1) \times (N+1)}
$$

We are able to mention a result on the convergence of the Bessel functions as $N \to \infty$. This property indicates that the Bessel function are convergent exponentially in the weighted $L_2$ norm with respect to weigh function $r(x) = \exp(-2L/x)$.

**Theorem 3.1 ([22])** Let us denote by $Z_N(x) = B_N(x) A_{n,N}$ the best square approximation to $Z(x)$. Under the assumptions $Z(x) \in C^{N+1}(0,L)$ and $M_\infty := \max_{x \in (0,L)} |Z^{(N+1)}(x)|$, we have the following error bound

$$
\|Z(x) - Z_N(x)\|_r \leq \frac{M_\infty}{\sqrt{2N+3}} \frac{L^{M+\frac{3}{2}}}{(N+1)!} \exp(1).
$$

## 4 Taylor-Bessel collocation method

Now, we are going to approximate the solution the discretized model problem (8) in the form (10). To do so, we first define the set of collocation points $\{x_q\}_{q=0}^N$ on $[0,L]$ with

$$
x_q = \frac{L}{N}q, \quad q = 0, 1, \ldots, N,
$$

(14)
Next, we express the unknown functions $u^{p+1}, u^{p+1}_x,$ and $u^{p+1}_{xx}$ in equation (8) in a matrix form. By placing the collocation points (14) into the resultant equation, we get a linear matrix equation.

Our next task is to combine two previously asserted relation (11) and (12). In this way, we rewrite relation (11) in the matrix expression as

$$W_{n+1,N}(x) = X_N(x) D^T A_{n,N}. \quad (15)$$

After evaluating the preceding equation at the collocation points (14) we arrive at

$$W_{n+1} = Y D^T A_{n,N}, \quad W_{n+1} = \begin{bmatrix} W_{n+1,N}(x_0) \\ W_{n+1,N}(x_1) \\ \vdots \\ W_{n+1,N}(x_N) \end{bmatrix}, \quad Y = \begin{bmatrix} X_N(x_0) \\ X_N(x_1) \\ \vdots \\ X_N(x_N) \end{bmatrix} \quad (16)$$

By means of relations (15) and (13) we can represent the first and second-orders derivatives in equation (8) in the matrix forms

$$\begin{cases} u^{p+1}_x \approx W^{(1)}_{n+1,N}(x) = X_N(x) M^T D^T A_{n,N}, \\ u^{p+1}_{xx} \approx W^{(2)}_{n+1,N}(x) = X_N(x) (M^T)^2 D^T A_{n,N}. \end{cases} \quad (17)$$

Similarly, by evaluating them at the collocation points, the first and second derivatives in relations (17) can be written in the matrix forms

$$\begin{aligned} \dot{W}_{n+1} &= Y M^T D^T A_{n,N}, \\ \dot{W}_{n+1} &= \begin{bmatrix} W^{(1)}_{n+1,N}(x_0) \\ W^{(1)}_{n+1,N}(x_1) \\ \vdots \\ W^{(1)}_{n+1,N}(x_N) \end{bmatrix} \quad , \quad (18) \\ \ddot{W}_{n+1} &= Y (M^T)^2 D^T A_{n,N}, \\ \ddot{W}_{n+1} &= \begin{bmatrix} W^{(2)}_{n+1,N}(x_0) \\ W^{(2)}_{n+1,N}(x_1) \\ \vdots \\ W^{(2)}_{n+1,N}(x_N) \end{bmatrix}. \quad (19) \end{aligned}$$

By introducing the following functions

$$p_{n,0}(x) = 2\Delta t u^p w_x - \frac{1}{4} \Delta t w_{xx}^p + 1, \quad p_{n,1}(x) = -\Delta t w_x^p + \Delta t (w^p)^2, \quad p_{n,2}(x) = -\frac{1}{4} \Delta t w^p, \quad g_n(x) = w^p + \Delta t (w^p)^2 w_x^p,$$

and using the approximations $W_{n+1,N}(x), W^{(1)}_{n+1,N}(x), W^{(2)}_{n+1,N}(x),$ we may rewrite equation (8) as

$$p_{n,2}(x) W^{(2)}_{n+1,N}(x) + p_{n,1}(x) W^{(1)}_{n+1,N}(x) + p_{n,0}(x) W_{n+1,N}(x) = g_n(x), \quad 0 \leq x \leq L. \quad (20)$$
By inserting the collocation points into equation (20) to get the system

\[ P_{n,2} \mathbf{\ddot{W}}_{n+1} + P_{n,1} \mathbf{\ddot{W}}_{n+1} + P_{n,0} \mathbf{\ddot{W}}_{n+1} = \mathbf{G}_n. \] (21)

In equation (21), the matrices \( P_{n,l} \), and the vector \( \mathbf{G}_n \) take the forms

\[
P_{n,l} = \begin{bmatrix}
p_{n,l}(x_0) & 0 & \cdots & 0 \\
p_0(x_1) & p_{n,l}(x_1) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & p_{n,l}(x_N)
\end{bmatrix}_{(N+1) \times (N+1)}, \quad \mathbf{G}_n = \begin{bmatrix}
g_n(x_0) \\
g_n(x_1) \\
\vdots \\
g_n(x_N)
\end{bmatrix}_{(N+1) \times 1}
\]

for \( \ell = 0, 1, 2 \). Let us place the relations (16), (18)-(19) into (21). This yields the fundamental matrix equation

\[
\mathbf{V}_n \mathbf{A}_{n,N} = \mathbf{G}_n, \tag{22}
\]

where

\[
\mathbf{V}_n := \{ P_{n,2} \mathbf{Y} (\mathbf{M}^T)^2 + P_{n,1} \mathbf{Y} \mathbf{M}^T + P_{n,0} \mathbf{Y} \} \mathbf{D}^T.
\]

Clearly, the fundamental matrix equation (22) is a set of \((N + 1)\) linear equations in terms of \((N + 1)\) unknown coefficients \(a_{0,t}, a_{1,t}, \ldots, a_{N,t}\) to be found.

To consider the boundary conditions (2), we must also convert them into matrix form. Based on the representation (15), these conditions i.e., \( \mathbf{W}_{n+1,N}(0) = h_0^{n+1} \) and \( \mathbf{W}_{n+1,N}(1) = h_1^{n+1} \) can be expressed in the matrix notation

\[
\mathbf{\hat{V}}_{n,0} \mathbf{A}_{n,N} = h_0^{n+1}, \quad \mathbf{\hat{V}}_{n,0} := \mathbf{X}_N(0) \mathbf{D}^T = [\hat{v}_{0,0}, \hat{v}_{0,1}, \ldots, \hat{v}_{0,N}],
\]

\[
\mathbf{\hat{V}}_{n,1} \mathbf{A}_{n,N} = h_1^{n+1}, \quad \mathbf{\hat{V}}_{n,1} := \mathbf{X}_N(1) \mathbf{D}^T = [\hat{v}_{1,0}, \hat{v}_{1,1}, \ldots, \hat{v}_{1,N}].
\]

Next, we substitute the first two rows of the augmented matrix \([\mathbf{V}_n; \mathbf{F}_n]\) by the vectors \([\mathbf{\hat{V}}_{n,0}; h_0^{n+1}]\) and \([\mathbf{\hat{V}}_{n,1}; h_1^{n+1}]\), for convenience. Thus, the following modified linear system of equations is obtained

\[
\begin{bmatrix}
\hat{v}_{0,0} & \hat{v}_{0,1} & \hat{v}_{0,2} & \hat{v}_{0,3} & \cdots & \hat{v}_{0,N} & h_0^{n+1} \\
\hat{v}_{1,0} & \hat{v}_{1,1} & \hat{v}_{1,2} & \hat{v}_{1,3} & \cdots & \hat{v}_{1,N} & h_1^{n+1} \\
u_{2,0} & u_{2,1} & u_{2,2} & u_{2,3} & \cdots & u_{2,N} & g_n(x_2) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
u_{N-1,0} & u_{N-1,1} & u_{N-1,2} & u_{N-1,3} & \cdots & u_{N-1,N} & g_n(x_{N-1}) \\
u_{N,0} & u_{N,1} & u_{N,2} & u_{N,3} & \cdots & u_{N,N} & g_n(x_N)
\end{bmatrix} = 0. \quad (23)
\]

Now, by solving the above linear system we are able to obtain the unknown Bessel coefficients in relation (15).

### 5 Numerical simulations

To testify the performance of the combine Taylor and Bessel-collocation approach, numerical simulations based on two test cases are given for the nonlinear initial and
boundary value problem (3)-(4). Furthermore, comparisons of numerical results with the outcomes of diverse existing schemes are also made for validation. For implementations, we utilize MATLAB software version 2017a.

**Test problem 5.1** We first consider the foam drainage equation (3) with the following initial condition [4, 7, 11]

\[ w_0(x) = -\tanh(x). \]

The exact solution is given by \( w(x, t) = -\tanh(x - t) \).

We first employ \( \Delta t \) equals to \( T = 0.01 \). Considering relation (10) with \( N = 5 \), the following approximation for \( 0 \leq x \leq L = 1 \) is obtained

\[ W_{1,5}(x) = 0.01807890048x^5 - 0.1934987886x^4 + 0.4424756341x^3 - 0.03689672509x^2 - 0.9975210119x + 0.00999966668. \]

We plot the above obtained solution as an approximation to \( w(x, T) \) in Fig. 1. We also show the corresponding absolute errors \( |w(x, T) - W_{1,5}(x)| \) at \( x \in [0, 1] \) in Fig. 2. Besides \( \Delta t = 0.01 \), we use \( \Delta t = 0.1, 0.01 \) to show the impact of different values of time step size on the computations.

To validate our results, some comparisons are carried out in Tables 1-2, which show the numerical solutions obtained by the presented scheme evaluated at \( t = 0.01, 0.001 \) and various \( x \in [0, 1] \). The corresponding absolute errors (A.E.) are also reported in the second column of these tables. Furthermore, analogue results of the previously well-established methods are displayed in Table 1. These include the collocation method based on bivariate Chebyshev functions (BCF) [11], the Adomian decomposition method (ADM) [3], the homotopy perturbation method (HPM) [5], the Haar wavelet quasilinearization approach (HWQA) [24], and the homotopy perturbation transform method (HPTM) [10]. It can be observed that our numerical results are in good agreement with the corresponding exact solutions. However, our approach is more straightforward compared to other existing methods.

**Test problem 5.2** As the second example, we consider the following initial condition [4, 7, 11]

\[ w_0(x) = (1 + e^x)^{-1} - \frac{1}{2}. \]

It is shown that the exact solution is given by \( w(x, t) = (1 + e^{x-\frac{1}{2}})^{-1} - \frac{1}{2} \).

For this test problem, we consider \( T = 0.1, \Delta t = 0.001 \), and \( N = 5 \). The snapshots of numerical solutions at different time instants \( t = s\Delta t, s = 1, 2, \ldots, 100 \) are shown in Fig. 5. In addition, the corresponding absolute errors are also plotted in Fig. 6. In this case, the approximated solutions at \( t = \Delta t \) and \( t = T \) are obtained as follow

\[ W_{1,5}(x) = -0.001109924646x^5 - 0.001404371504x^4 + 0.02164565497x^3 - 0.0002217269879x^2 - 0.2499815546x + 0.00006249999967, \]
$W_{100,5}(x) = -0.000731548639 x^5 - 0.00211132455 x^4 + 0.0224345299 x^3$

$- 0.002035416473 x^2 - 0.249920896 x + 0.0062496745.$

We next compute the maximum absolute errors will be denoted by $L_{\infty}$ as well as the $L_2$ error norms evaluated at the final time $t = T$ via

$L_{\infty} := \max_{0 \leq x \leq 1} |w(x, T) - W_{M+1,N}(x)|, \quad L_2 := \left( \frac{1}{N + 1} \int_0^1 |w(x, T) - W_{M+1,N}(x)|^2 dx \right)^{\frac{1}{2}}.$

We utilize various $N = 4, 5, \ldots, 8$ and report the results of errors in Table 3. Also, different final times $T = 0.1, 0.5$, and $T = 1$ are used with the step sizes $\Delta t = 0.001, 0.001$, and $T = 0.1$.

Finally, our numerical results and computations are verified through a comparison with well-established numerical models and simulations. Tables 4-5 show these comparisons with the methods used in the previous Tables 1-2 in Example 5.1. However, in the following tables we employ the Laplace decomposition method (LDM) [25] instead of HPM.

### 6 Conclusions

In this work, a space and time-accurate approximation technique was presented to solve the foam drainage equation. For the temporal discretization, the Taylor series expansion approach with order $O(\Delta t^2)$ was employed. Afterwards at each time step, the novel Bessel based collocation approach with exponential accuracy was utilized to approximate the space variable. By using the matrix representations of these polynomials in conjunction with the collocation points, the scheme converts the underlying model problem into an algebraic linear system of equations. The utility and accuracy of the presented technique were examined by using numerical experiments. Comparisons with earlier computational and experimental studies have also been made. The presented results demonstrated the reliability and the applicability of the presented combined algorithm for the nonlinear time-dependent foam drainage equation. The combined technique with inherited simplicity and ease in implementation can be easily extended to other nonlinear model problems in diverse discipline of engineering and sciences.

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**Nomenclature:**

A.E. : Absolute errors

BCF : Bivariate Chebyshev functions

ADM : Adomian decomposition method

HPM : Homotopy perturbation method

HWQA: Haar wavelet quasilinearization approach

HPTM: Homotopy perturbation transform method

LDM : Laplace decomposition method
Biography:

Mohammad Izadi received his Ph.D. degree from Leipzig University (2008-2012) in the group of “scientific computing” at Max-Planck Institute for Mathematics in the Sciences, Leipzig, Germany. Since 2013, he has served as an assistant professor in the department of applied mathematics at Shahid Bahonar University of Kerman. His interest area includes numerical analysis, numerical methods for (fractional) ordinary and partial differential equations, and spectral methods.

List of Figure captions:

Figure 1: Graphs of exact and solutions at different time instants \( t = \Delta t \) for \( \Delta t = 0.1, 0.01, 0.001, N = 5 \) in test problem (5.1).

Figure 2: Graphs of absolute errors at different time instants \( t = \Delta t \) for \( \Delta t = 0.1, 0.01, 0.001, N = 5 \) in test problem (5.1).

Figure 3: Graphs of numerical solutions in test problem (5.2) at different time instants \( t = s\Delta t, s = 1, 2, \ldots, 100 \) for \( \Delta t = 0.001, T = 0.1, \) and \( N = 5 \).

Figure 4: Graphs of absolute errors in test problem (5.2) at different time instants \( t = s\Delta t, s = 1, 2, \ldots, 100 \) for \( \Delta t = 0.001, T = 0.1, \) and \( N = 5 \).

List of Table captions:

Table 1: The comparison of numerical results in test problem (5.1) for \( N = 8 \) and various \( x \in [0, 1] \) at time \( t = 0.01 \).

Table 2: The comparison of numerical results in test problem (5.1) for \( N = 8 \) and various \( x \in [0, 1] \) at time \( t = 0.001 \).

Table 3: The comparison of \( L_2 \) and \( L_\infty \) error norms in test problem (5.2) for diverse \( N = 4, 5, \ldots, 8, \Delta t = 0.001, 0.01, 0.1 \) evaluated at the final times \( t = T, \) with \( T = 0.1, 0.5, 1 \).

Table 4: The comparison of numerical results in test problem (5.2) for \( N = 8 \) and various \( x \in [0, 1] \) at time \( t = 0.1 \).

Table 5: The comparison of numerical results in test problem (5.2) for \( N = 8 \) and various \( x \in [0, 1] \) at time \( t = 0.01 \).
Figure 1: Graphs of exact and solutions at different time instants $t = \Delta t$ for $\Delta t = 0.1, 0.01, 0.001, N = 5$ in test problem (5.1).

Figure 2: Graphs of absolute errors at different time instants $t = \Delta t$ for $\Delta t = 0.1, 0.01, 0.001, N = 5$ in test problem (5.1).

Figure 3: Graphs of numerical solutions in test problem at different time instants $t = s\Delta t$, $s = 1, 2 \ldots, 100$ for $\Delta t = 0.001$, $T = 0.1$, and $N = 5$. 

13
Figure 4: Graphs of absolute errors in test problem \( S_2 \) at different time instants \( t = s\Delta t, s = 1, 2, \ldots, 100 \) for \( \Delta t = 0.001, T = 0.1, \) and \( N = 5. \)

Table 1: The comparison of numerical results in test problem \( S_1 \) for \( N = 8 \) and various \( x \in [0, 1] \) at time \( t = 0.01. \)

| \( \frac{x}{\Delta t} \) | Present | A.E.  | BCF  \[11\] | HWQA \[24\] | ADM  \[3\] | HPM  \[5\] | HPTM \[10\] |
|------------------------|---------|-------|--------|--------|-------|--------|--------|
| 1                      | -0.0056255222 | 5.82\_7 | -0.005626 | -0.005626 | -0.004253 | -0.004358 | -0.048341 |
| 3                      | -0.0368592824 | 9.87\_7 | -0.036858 | -0.036874 | -0.009002 | -0.027011 | -0.017108 |
| 5                      | -0.0680207213 | 9.16\_7 | -0.068019 | -0.068085 | -0.015399 | -0.058439 | -0.014053 |
| 7                      | -0.0990498931 | 7.28\_7 | -0.099049 | -0.099199 | -0.023360 | -0.089857 | -0.045083 |
| 9                      | -0.1298876683 | 5.79\_7 | -0.129887 | -0.130158 | -0.032796 | -0.121194 | -0.075921 |
| 27                     | -0.3900638141 | 2.53\_7 | -0.390063 | -0.392716 | -0.168432 | -0.387936 | -0.336218 |
| 29                     | -0.4162316016 | 2.05\_7 | -0.416231 | -0.419230 | -0.187449 | -0.414778 | -0.362463 |
| 31                     | -0.4417276182 | 1.71\_7 | -0.441727 | -0.445063 | -0.206944 | -0.440899 | -0.388074 |
| 33                     | -0.4665295588 | 1.49\_7 | -0.466529 | -0.470208 | -0.226840 | -0.466274 | -0.413043 |
| 35                     | -0.4906189456 | 1.30\_7 | -0.490618 | -0.494646 | -0.247065 | -0.490881 | -0.437369 |
| 55                     | -0.6907427887 | 2.57\_8 | -0.690742 | -0.697478 | -0.453650 | -0.693486 | -0.651564 |
| 57                     | -0.7067322567 | 3.28\_8 | -0.706732 | -0.713630 | -0.473606 | -0.709521 | -0.671453 |
| 59                     | -0.7220308393 | 8.46\_8 | -0.722030 | -0.729056 | -0.493271 | -0.724843 | -0.691471 |
| 61                     | -0.7366545782 | 1.55\_7 | -0.736654 | -0.743775 | -0.512612 | -0.739469 | -0.711761 |
| 63                     | -0.7506204287 | 1.27\_7 | -0.750620 | -0.757808 | -0.531603 | -0.753420 | -0.732485 |
Table 2: The comparison of numerical results in test problem [5,1] for $N = 8$ and various $x \in [0, 1]$ at time $t = 0.001$.

| $\frac{\Delta t}{N}$ | Present | A.E.  | BCF [11] | HWQA [24] | ADM [3] | HPM [5] | HPTM [10] |
|-----------------------|---------|-------|----------|-----------|---------|---------|-----------|
| 1                     | -0.0146242559 | 2.99\_7 | -0.014624 | -0.014624 | -0.000433 | -0.013626 | -0.039344 |
| 3                     | -0.0458432514 | 4.06\_7 | -0.045843 | -0.045847 | -0.002700 | -0.044858 | -0.008125 |
| 5                     | -0.0769726894 | 2.46\_7 | -0.076972 | -0.076984 | -0.006885 | -0.076014 | -0.023004 |
| 7                     | -0.1079527574 | 6.67\_8 | -0.107953 | -0.107973 | -0.012948 | -0.107033 | -0.053984 |
| 9                     | -0.1387246409 | 4.28\_8 | -0.138725 | -0.138769 | -0.020837 | -0.137855 | -0.084756 |
| 11                    | -0.3976673000 | 1.64\_8 | -0.397667 | -0.398034 | -0.160042 | -0.397455 | -0.343819 |
| 13                    | -0.4236441749 | 1.58\_9 | -0.423644 | -0.423939 | -0.181144 | -0.423499 | -0.369872 |
| 15                    | -0.4489424537 | 6.57\_9 | -0.448942 | -0.449301 | -0.202947 | -0.448860 | -0.395283 |
| 17                    | -0.4735409398 | 6.06\_9 | -0.473541 | -0.473964 | -0.225333 | -0.473515 | -0.420046 |
| 19                    | -0.4974222452 | 1.56\_8 | -0.497422 | -0.497826 | -0.248187 | -0.497448 | -0.444159 |
| 21                    | -0.6954195363 | 5.22\_8 | -0.695419 | -0.696107 | -0.480881 | -0.695694 | -0.655872 |
| 23                    | -0.7112084249 | 3.79\_9 | -0.711208 | -0.711883 | -0.502829 | -0.711487 | -0.675449 |
| 25                    | -0.7263108975 | 1.93\_7 | -0.726311 | -0.726999 | -0.524303 | -0.726592 | -0.695130 |
| 27                    | -0.7407432348 | 3.37\_7 | -0.740744 | -0.741464 | -0.545268 | -0.741025 | -0.715055 |
| 29                    | -0.7545229508 | 2.54\_7 | -0.754523 | -0.755247 | -0.565692 | -0.754803 | -0.735378 |

Table 3: The comparison of $L_2$ and $L_\infty$ error norms in test problem [5,2] for diverse $N = 4, 5, \ldots, 8$, $\Delta t = 0.001, 0.01, 0.1$ evaluated at the final times $t = T$, with $T = 0.1, 0.5, 1$.

| $N$ | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ |
|-----|------------|-------|------------|-------|------------|-------|
| 4   | 6.137\_6   | 1.533\_6 | 1.534\_4   | 2.034\_5 | 6.512\_4   | 3.349\_4 |
| 5   | 6.264\_6   | 2.606\_7 | 1.607\_4   | 4.955\_5 | 1.133\_3   | 5.212\_4 |
| 6   | 5.918\_6   | 2.316\_7 | 1.391\_4   | 5.211\_5 | 1.577\_3   | 7.531\_4 |
| 7   | 6.033\_6   | 6.318\_6 | 2.164\_4   | 7.232\_5 | 3.833\_3   | 1.668\_3 |
| 8   | 6.076\_6   | 2.276\_7 | 3.382\_4   | 9.884\_5 | 1.249\_3   | 4.766\_3 |

15
Table 4: The comparison of numerical results in test problem \( N = 8 \) and various \( x \in [0, 1] \) at time \( t = 0.1 \).

| \( x \) | Present | A.E. | BCF \( [11] \) | HWQA \( [24] \) | ADM \( [4] \) | LDM \( [5] \) | HPTM \( [10] \) |
|------|--------|------|----------------|----------------|---------|--------|---------|
| 1    | +0.0023435127 2.20 \(-7\) | +0.002344 | +0.002344 | +0.002098 | +0.002083 | +0.002084 |
| 3    | −0.0054690211 4.89 \(-7\) | −0.005469 | −0.005468 | −0.005793 | −0.005785 | −0.005787 |
| 5    | −0.0132787383 6.11 \(-7\) | −0.013278 | −0.013278 | −0.013664 | −0.013650 | −0.013651 |
| 7    | −0.0210818974 6.53 \(-7\) | −0.021081 | −0.021081 | −0.021512 | −0.021508 | −0.021509 |
| 9    | −0.0288747443 6.56 \(-7\) | −0.028874 | −0.028874 | −0.029357 | −0.029354 | −0.029353 |
| 27   | −0.0979371596 5.45 \(-7\) | −0.097937 | −0.097935 | −0.098832 | −0.098830 | −0.098831 |
| 29   | −0.1054263741 5.27 \(-7\) | −0.105426 | −0.105424 | −0.106358 | −0.106357 | −0.106358 |
| 31   | −0.1128664061 5.08 \(-7\) | −0.112866 | −0.112863 | −0.113834 | −0.113833 | −0.113833 |
| 33   | −0.1202541435 4.88 \(-7\) | −0.120254 | −0.120251 | −0.121255 | −0.121254 | −0.121254 |
| 35   | −0.1275865681 4.69 \(-7\) | −0.127586 | −0.127583 | −0.128620 | −0.128619 | −0.128619 |
| 55   | −0.1972794716 2.66 \(-7\) | −0.197279 | −0.197274 | −0.198532 | −0.198532 | −0.198532 |
| 57   | −0.2038347870 2.50 \(-7\) | −0.203834 | −0.203830 | −0.205099 | −0.205099 | −0.205099 |
| 59   | −0.2103071180 2.27 \(-7\) | −0.210307 | −0.210304 | −0.211582 | −0.211582 | −0.211582 |
| 61   | −0.2166949149 1.81 \(-7\) | −0.216695 | −0.216691 | −0.217979 | −0.217979 | −0.217979 |
| 63   | −0.2229967352 8.29 \(-8\) | −0.222997 | −0.222994 | −0.224288 | −0.224288 | −0.224288 |
Table 5: The comparison of numerical results in test problem for $N = 8$ and various $x \in [0, 1]$ at time $t = 0.01$.

| $\frac{x}{0.64}$ | Present | A.E. | BCF [11] | HWQA [24] | ADM [4] | LDM [5] | HPTM [10] |
|------------------|---------|------|----------|-----------|---------|---------|-----------|
| 1                | -0.0032812030 | 1.18$_{-10}^+$ | -0.003281 | -0.003281 | -0.003309 | -0.003307 | -0.003307 |
| 3                | -0.0110919303 | 3.50$_{-10}^+$ | -0.011092 | -0.011091 | -0.011126 | -0.011123 | -0.011123 |
| 5                | -0.0188972450 | 5.21$_{-10}^+$ | -0.018897 | -0.018897 | -0.018939 | -0.018935 | -0.018934 |
| 7                | -0.0266933472 | 6.16$_{-10}^+$ | -0.026693 | -0.026693 | -0.026741 | -0.026737 | -0.026736 |
| 9                | -0.0344764548 | 6.52$_{-10}^+$ | -0.034476 | -0.034476 | -0.034529 | -0.034525 | -0.034524 |
| 27               | -0.1033336837 | 5.28$_{-10}^+$ | -0.103334 | -0.103333 | -0.103430 | -0.103424 | -0.103423 |
| 29               | -0.1107878235 | 5.10$_{-10}^+$ | -0.110788 | -0.110787 | -0.110888 | -0.110883 | -0.110880 |
| 31               | -0.1181905313 | 4.91$_{-10}^+$ | -0.118191 | -0.118190 | -0.118297 | -0.118289 | -0.118287 |
| 33               | -0.1255387617 | 4.71$_{-10}^+$ | -0.125539 | -0.125538 | -0.125649 | -0.125641 | -0.125638 |
| 35               | -0.1328295665 | 4.52$_{-10}^+$ | -0.132830 | -0.132829 | -0.132943 | -0.132932 | -0.132932 |
| 55               | -0.2020073393 | 2.59$_{-10}^+$ | -0.202007 | -0.202007 | -0.202149 | -0.202136 | -0.202132 |
| 57               | -0.2085030871 | 2.27$_{-10}^+$ | -0.208503 | -0.208502 | -0.208650 | -0.208636 | -0.208629 |
| 59               | -0.2149147407 | 1.71$_{-10}^+$ | -0.214915 | -0.214914 | -0.215065 | -0.215047 | -0.215041 |
| 61               | -0.2212408533 | 9.18$_{-11}^+$ | -0.221241 | -0.221240 | -0.221389 | -0.221376 | -0.221368 |
| 63               | -0.2274800989 | 1.51$_{-11}^+$ | -0.227480 | -0.227480 | -0.227639 | -0.227618 | -0.227608 |