Research Article
Explicit Formulas Involving $q$-Euler Numbers and Polynomials

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We deal with $q$-Euler numbers and $q$-Bernoulli numbers. We derive some interesting relations for $q$-Euler numbers and polynomials by using their generating function and derivative operator. Also, we derive relations between the $q$-Euler numbers and $q$-Bernoulli numbers via the $p$-adic $q$-integral in the $p$-adic integer ring.

1. Preliminaries

Imagine that $p$ is a fixed odd prime number. Throughout this paper we use the following notations, where $\mathbb{Z}_p$ denotes the ring of $p$-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_q$ denotes the field of $p$-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

The $p$-adic absolute value is defined by

$$|p|_p = \frac{1}{p}. \quad (1.1)$$

In this paper, we will assume that $|q-1|_p < 1$ as an indeterminate. $[x]_q$ is a $q$-extension of $x$, which is defined by

$$[x]_q = \frac{1-q^x}{1-q}. \quad (1.2)$$

We note that $\lim_{q \to 1}[x]_q = x$ (see [1–12]).
We say that $f$ is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, if the difference quotient

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y}, \quad (1.3)$$

has a limit $f'(a)$ as $(x, y) \to (a, a)$ and denote this by $f \in \text{UD}(\mathbb{Z}_p)$.

Let $\text{UD}(\mathbb{Z}_p)$ be the set of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in \text{UD}(\mathbb{Z}_p)$, let us start with the expression

$$\frac{1}{[p^N]} \sum_{0 \leq \xi < p^N} f(\xi)q^\xi = \sum_{0 \leq \xi < p^N} f(\xi)\mu_q(\xi + p^N\mathbb{Z}_p), \quad (1.4)$$

which represents $p$-adic $q$-analogue of Riemann sums for $f$. The integral of $f$ on $\mathbb{Z}_p$ will be defined as the limit $(N \to \infty)$ of these sums, when it exists. The $p$-adic $q$-integral of function $f \in \text{UD}(\mathbb{Z}_p)$ is defined by Kim

$$I_q(f) = \int_{\mathbb{Z}_p} f(\xi)d\mu_q(\xi) = \lim_{N \to \infty} \frac{1}{[p^N]} \sum_{\xi = 0}^{p^N-1} f(\xi)q^\xi. \quad (1.5)$$

The bosonic integral is considered as a bosonic limit $q \to 1$, $I_1(f) = \lim_{q \to 1} I_q(f)$. Similarly, the fermionic $p$-adic integral on $\mathbb{Z}_p$ is introduced by Kim as follows:

$$I_{-q}(f) = \lim_{q \to -q} I_q(f) = \int_{\mathbb{Z}_p} f(\xi)d\mu_{-q}(\xi) \quad (1.6)$$

(for more details, see [9–12]).

In [6], the $q$-Euler polynomials with weight 0 are introduced as

$$\bar{E}_{n,q}(x) = \int_{\mathbb{Z}_p} (x + y)^n d\mu_{-q}(y). \quad (1.7)$$

From (1.7), we have

$$\bar{E}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} q^l \bar{E}_{n-l,q}, \quad (1.8)$$

where $\bar{E}_{n,q}(0) = \bar{E}_{n,q}$ are called $q$-Euler numbers with weight 0. Then, $q$-Euler numbers are defined as

$$q(\bar{E}_q + 1)^n + \bar{E}_{n,q} = \begin{cases} [2]_q^n & \text{if } n = 0, \\ 0 & \text{if } n \neq 0, \end{cases} \quad (1.9)$$

where the usual convention about replacing $(\bar{E}_q)^n$ by $\bar{E}_{n,q}$ is used.
Similarly, the $q$-Bernoulli polynomials and numbers with weight 0 are defined, respectively, as

$$\tilde{B}_{n,q}(x) = \lim_{n \to \infty} \frac{1}{[p^n]_q} \sum_{y=0}^{p^n-1} (x+y)^n q^y$$

$$= \int_{\mathbb{Z}_p} (x+y)^n d\mu_q(y), \quad (1.10)$$

(1.10)

where $x$ is a fixed parameter. Thus, by expression of (2.1), we can readily see the following:

$$q e^t D^m (D + I)^k F^q_x + D^k F^q_x = [2]_q x^k e^{xt}.$$  \hspace{1cm} (2.3)

where $k \in \mathbb{N}^*$ and $I$ is identity operator. By multiplying $e^{-t}$ on both sides of (2.3), we get

$$q(D + I)^k F^q_x + e^{-t} D^k F^q_x = [2]_q x^k e^{(x-1)t}.$$  \hspace{1cm} (2.4)

Let us take derivative operator $D^m (m \in \mathbb{N})$ on both sides of (2.4). Then we get

$$q e^t D^m (D + I)^k F^q_x + D^k (D - I)^m F^q_x = [2]_q x^k (x-1)^m e^{xt}.$$  \hspace{1cm} (2.5)
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Let $G[0]$ (not $G(0)$) be the constant term in a Laurent series of $G(t)$. Then, from (2.5), we get

$$\sum_{j=0}^{k} \binom{k}{j} (q e^{D^{k+m-j} F_2^q(t)})[0] + \sum_{j=0}^{m} \binom{m}{j} (-1)^j (D^{k+m-j} F_2^q(t))[0] = [2]_q x^k (x - 1)^m. \quad (2.6)$$

By (2.1), we see

$$\left( D^N F_2^q(t) \right)[0] = \bar{E}_{N,q}(x), \quad \left( e^t D^N F_2^q(t) \right)[0] = \bar{E}_{N,q}(x). \quad (2.7)$$

By expressions of (2.6) and (2.7), we see that

$$\sum_{j=0}^{\max\{k,m\}} q \binom{k}{j} + (-1)^j \binom{m}{j} \bar{E}_{k+m-j,q}(x) = [2]_q x^k (x - 1)^m. \quad (2.8)$$

From (2.1), we note that

$$\frac{d}{dx} \left( \bar{E}_{n,q}(x) \right) = n \sum_{l=0}^{n-1} \binom{n-1}{l} \bar{E}_{l,q} x^{n-1-l} = n \bar{E}_{n-1,q}(x). \quad (2.9)$$

By (2.9), we easily see

$$\int_0^1 \bar{E}_{n,q}(x) dx = \frac{\bar{E}_{n+1,q}(1) - \bar{E}_{n+1,q}}{n+1} = - \frac{[2]_q (-1)^m B(k+1, m+1)}{n+1} \bar{E}_{n+1,q}. \quad (2.10)$$

Now, let us consider definition of integral from 0 to 1 in (2.8), then we have

$$- [2]_q \sum_{j=0}^{\max\{k,m\}} q \binom{k}{j} + (-1)^j \binom{m}{j} \frac{\bar{E}_{k+m-j+1,q}}{\bar{E}_{k+m-j,q}} = [2]_q (-1)^m \frac{\Gamma(k+1)\Gamma(m+1)}{\Gamma(k+m+2)},$$

$$= \frac{\Gamma(k+1)\Gamma(m+1)}{\Gamma(k+m+2)}, \quad (2.11)$$

where $B(m, n)$ is beta function which is defined by

$$B(m, n) = \int_0^1 x^{m-1}(1 - x)^{n-1} dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad m > 0, \ n > 0. \quad (2.12)$$

As a result, we obtain the following theorem.
**Theorem 2.1.** For \( n \in \mathbb{N} \), one has

\[
\sum_{j=1}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \frac{\tilde{E}_{k+m-j+1,q}}{k+m-j+1} = q \frac{(-1)^m}{(k+m+1)(k+m)} - [2]_q \tilde{E}_{k+m+1,q}.
\]

(2.13)

Substituting \( m = k + 1 \) into Theorem 2.1, we readily get

\[
\sum_{j=1}^{k+1} \left[ q \binom{k}{j} + (-1)^j \binom{k+1}{j} \right] \frac{\tilde{E}_{2k+2-j,q}}{2k+2-j} = q \frac{(-1)^k}{(2k+2)(2k+1)} - [2]_q \tilde{E}_{2k+2,q}.
\]

(2.14)

By (2.1), it follows that

\[
\sum_{j=0}^{\max\{k,m\}} (k+m-j) \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \tilde{E}_{k+m-j,q}(x) = [2]_q x^k (x-1)^{m-1}((k+m)x-k).
\]

(2.15)

Let \( m = k \) in (2.1), we see that

\[
\sum_{j=0}^{k} \left[ q \binom{k}{j} + (-1)^j \binom{k}{j} \right] \tilde{E}_{2k-j,q}(x) = [2]_q x^k (x-1)^k.
\]

(2.16)

Last from equality, we discover the following:

\[
[2]_q \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j} \tilde{E}_{2k-2j,q}(x) + (q-1) \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j+1} \tilde{E}_{2k-2j-1,q}(x) = [2]_q x^k (x-1)^k.
\]

(2.17)

Here \([\cdot]\) is Gauss’ symbol. Then, taking integral from 0 to 1 in both sides of last equality, we get

\[
- [2]_q \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j} \frac{\tilde{E}_{2k-2j+1,q}}{2k-2j+1} + [2]_q \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j+1} \frac{\tilde{E}_{2k-2j-1,q}}{2k-2j} = [2]_q (-1)^k B(k+1,k+1)
\]

(2.18)

Consequently, we derive the following theorem.
Theorem 2.2. The following identity

\[
[2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \tilde{E}_{2k-2j+1,q} + (q-1) \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \tilde{E}_{2k-2j,q} = \frac{q(-1)^{k+1}}{(2k+1)(\frac{2k}{k})}
\]

(2.19)

is true.

In view of (2.1) and (2.17), we discover the following applications:

\[
= \sum_{j=0}^{k+1} \binom{k}{j} (-1)^j \binom{k+1}{j} \tilde{E}_{2k+1-j,q}(x)
\]

\[
= [2]_q \tilde{E}_{2k+1,q}(x) + \sum_{j=1}^{[(k+1)/2]} \left[ q \binom{k}{2j} \binom{k}{2j} + \binom{k}{2j-1} \right] \tilde{E}_{2k+2j,q}(x)
\]

\[
+ \sum_{j=0}^{[(k+1)/2]} \left[ q \binom{k}{2j} - \binom{k}{2j+1} - \binom{k}{2j} \right] \tilde{E}_{2k-2j,q}(x)
\]

(2.20)

By expressions (2.17) and (2.20), we have the following theorem.

Theorem 2.3. For \( k \in \mathbb{N} \), one has

\[
[2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \tilde{E}_{2k+1-2j,q}(x) + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \tilde{E}_{2k+1-2j,q}(x)
\]

\[
+ (q-1) \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \tilde{E}_{2k-2j,q}(x) + \frac{q-1}{1+q} \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \tilde{E}_{2k-2j+1,q}(x)
\]

(2.21)

\[
= x^k(x-1)^k ([2]_qx - q).
\]
3. *p*-adic Integral on $\mathbb{Z}_p$ Associated with Kim’s $q$-Euler Polynomials

In this section, we consider Kim’s $q$-Euler polynomials by means of $p$-adic $q$-integral on $\mathbb{Z}_p$. Now we start with the following assertion.

Let $m, k \in \mathbb{N}$. Then by (2.8),

$$I_1 = [2]_q \int_{\mathbb{Z}_p} x^k(x-1)^m d\mu_q(x)$$

$$= [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \int_{\mathbb{Z}_p} x^{2+l} d\mu_q(x)$$

$$= [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \tilde{E}_{l+k,q}. \tag{3.1}$$

On the other hand, in right hand side of (2.8),

$$I_2 = \sum_{j=0}^{\max\{k,m\}} \left[ q\binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k-m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \int_{\mathbb{Z}_p} x^{k+l} d\mu_q(x)$$

$$= \sum_{j=0}^{\max\{k,m\}} \left[ q\binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k-m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{E}_{l,q}. \tag{3.2}$$

Equating $I_1$ and $I_2$, we get the following theorem.

**Theorem 3.1.** For $m, k \in \mathbb{N}$, one has

$$\sum_{j=0}^{\max\{k,m\}} \left[ q\binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k-m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{E}_{l,q}$$

$$= [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \tilde{E}_{l+k,q}. \tag{3.3}$$

Let us take fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ in left hand side of (2.21), we get

$$I_3 = \int_{\mathbb{Z}_p} x^k(x-1)^k \left[ [2]_q x - q \right] d\mu_q(x)$$

$$= [2]_q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l+1} d\mu_q(x) - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_q(x) \tag{3.4}$$

$$= [2]_q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \tilde{E}_{k+l+1,q} - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \tilde{E}_{k+l,q}.$$
In other words, we consider right hand side of (2.21) as follows:

\[
I_4 = [2q] \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} E_{2k+1-2j-l,q} \int_{Z_p} x^l d\mu_q(x)
\]

\[
+ \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} E_{2k+1-2j-l,q} \int_{Z_p} x^l d\mu_q(x)
\]

\[
+ \sum_{j=1}^{[k/2]} \binom{k}{2j+1} \left[ (q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \bar{E}_{2k-2j-q} \int_{Z_p} x^l d\mu_q(x) \right]
\]

\[
+ \sum_{j=0}^{[k/2]} \frac{q-1}{1+q} \sum_{j=0}^{2k-2j+1} \frac{2k-2j+1}{l} \bar{E}_{2k-2j-q} E_{l,q}
\]

Equating \( I_3 \) and \( I_4 \), we get the following theorem.

**Theorem 3.2.** For \( k \in \mathbb{N} \), one has

\[
\sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left[ [2q] E_{k+l+1,q} - q\bar{E}_{k+l,q} \right]
\]

\[
= [2q] \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} E_{2k+1-2j-l,q} E_{l,q}
\]

\[
+ \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} E_{2k+1-2j-l,q} E_{l,q}
\]

\[
+ \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left[ (q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \bar{E}_{2k-2j-q} \bar{E}_{l,q} \right]
\]

\[
+ \sum_{j=0}^{[k/2]} \frac{q-1}{1+q} \sum_{j=0}^{2k-2j+1} \frac{2k-2j+1}{l} \bar{E}_{2k-2j-q} E_{l,q}
\]
Now, we consider (2.8) and (2.1) by means of $q$-Volkenborn integral. Then, by (2.8), we see

\[ [2]_q \int_{\mathbb{Z}_q} x^k (x - 1)^m d\mu_q(x) \]

\[ = [2]_q \sum_{l=0}^{m \leq l} \binom{m}{l} (-1)^{m-l} \int_{\mathbb{Z}_q} x^{l+k} d\mu_q(x) \]

\[ = [2]_q \sum_{l=0}^{m \leq l} \binom{m}{l} (-1)^{m-l} \tilde{B}_{k+m-q, l}. \tag{3.7} \]

On the other hand,

\[ \sum_{j=0}^{\max\{k, m\}} q^j \binom{k}{j} + (-1)^j \binom{m}{j} \sum_{l=0}^{k + m - j} \binom{k + m - j}{l} \tilde{E}_{k+m-j-l, q}\int_{\mathbb{Z}_q} x^l d\mu_q(x) \]

\[ = \sum_{j=0}^{\max\{k, m\}} q^j \binom{k}{j} + (-1)^j \binom{m}{j} \sum_{l=0}^{k + m - j} \binom{k + m - j}{l} \tilde{E}_{k+m-j-l, q} \tilde{B}_{l, q}. \tag{3.8} \]

Therefore, we get the following theorem.

**Theorem 3.3.** For $m, k \in \mathbb{N}$, one has

\[ [2]_q \sum_{l=0}^{m \leq l} \binom{m}{l} (-1)^{m-l} \tilde{B}_{k+m-q, l} \]

\[ = \sum_{j=0}^{\max\{k, m\}} q^j \binom{k}{j} + (-1)^j \binom{m}{j} \sum_{l=0}^{k + m - j} \binom{k + m - j}{l} \tilde{E}_{k+m-j-l, q} \tilde{B}_{l, q}. \tag{3.9} \]

By using fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ in left hand side of (2.21), we get

\[ I_5 = [2]_q \int_{\mathbb{Z}_p} x^k (x - 1)^k ([2] x - q) d\mu_q(x) \]

\[ = [2]_q \sum_{l=0}^{k \leq l} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l+1} d\mu_q(x) - q \sum_{l=0}^{k \leq l} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_q(x) \]

\[ = [2]_q \sum_{l=0}^{k \leq l} \binom{k}{l} (-1)^{k-l} \tilde{B}_{k+l+1, q} - q \sum_{l=0}^{k \leq l} \binom{k}{l} (-1)^{k-l} \tilde{B}_{k+l, q}. \tag{3.10} \]
Also, we consider right hand side of (2.21) as follows:

\[
I_6 = [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \frac{k!}{2k - 2j + 1} E_{2k+1-2j-l,q} \int_{z_p} x^l d\mu_q(x)
+ \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left( \frac{q}{2} \right)_j \sum_{l=0}^{2k-2j+1} \frac{k!}{2k - 2j + 1} E_{2k+1-2j-l,q} \int_{z_p} x^l d\mu_q(x)
\]

\[
= [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \frac{k!}{2k - 2j + 1} E_{2k+1-2j-l,q} B_{l,q}
+ \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left( \frac{q}{2} \right)_j \sum_{l=0}^{2k-2j+1} \frac{k!}{2k - 2j + 1} E_{2k+1-2j-l,q} B_{l,q}
\]

Equating \(I_5\) and \(I_6\), we get the following corollary.

**Corollary 3.4.** For \(k \in \mathbb{N}\), one gets

\[
\sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left\{ [2]_q B_{k+l+1,q} - qB_{k+l,q} \right\}
= [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \frac{k!}{2k - 2j + 1} E_{2k+1-2j-l,q} B_{l,q}
+ \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left( \frac{q}{2} \right)_j \sum_{l=0}^{2k-2j+1} \frac{k!}{2k - 2j + 1} E_{2k+1-2j-l,q} B_{l,q}
\]

\[
= [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \frac{k!}{2k - 2j + 1} E_{2k+1-2j-l,q} B_{l,q}
+ \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left( \frac{q}{2} \right)_j \sum_{l=0}^{2k-2j+1} \frac{k!}{2k - 2j + 1} E_{2k+1-2j-l,q} B_{l,q}
\]

\[
+ \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left( \frac{q}{2} \right)_j \sum_{l=0}^{2k-2j+1} \frac{k!}{2k - 2j + 1} E_{2k+1-2j-l,q} B_{l,q}
\]

\[
+ \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left( \frac{q}{2} \right)_j \sum_{l=0}^{2k-2j+1} \frac{k!}{2k - 2j + 1} E_{2k+1-2j-l,q} B_{l,q}
\]
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