CRITICAL EXPONENT FOR NONLINEAR WAVE EQUATIONS WITH DAMPING AND POTENTIAL TERMS

MASAKAZU KATO AND HIDEO KUBO

Abstract. The aim of this paper is to determine the critical exponent for the nonlinear wave equations with damping and potential terms of the scale invariant order, by assuming that these terms satisfy a special relation. We underline that our critical exponent is different from the one for related equations such as the nonlinear wave equation without lower order terms, only with a damping term, and only with a potential term. Moreover, we study the effect of the decaying order of initial data at spatial infinity. In fact, we prove that not only the lower order terms but also the order of the initial data affects the critical exponent, as well as the sharp upper and lower bounds of the maximal existence time of the solution.

1. Introduction

This paper is concerned with the Cauchy problem for the nonlinear wave equation with damping and potential terms:

\[
\begin{cases}
(\partial_t^2 + 2w(r)\partial_t - \Delta + V(r))U = |U|^p & \text{in } (0, T) \times \mathbb{R}^3, \\
U(0, x) = \varepsilon f_0(x), \quad (\partial_t U)(0, x) = \varepsilon f_1(x) & \text{for } x \in \mathbb{R}^3,
\end{cases}
\]

where \( r = |x|, \ p > 1, \ \varepsilon > 0, \) and \( f_0, f_1 \) are given functions vanishing at spatial infinity, like

\[
f_0(x) = O(|x|^{-\kappa}), \quad f_1(x) = O(|x|^{-\kappa-1}) \quad \text{as } |x| \to \infty.
\]

Here \( \kappa \) is a positive constant.

The Cauchy problem for the wave equation with power-type nonlinearity has a long history. The starting point is the study for the case where both damping and potential terms are absent. In this case, the critical exponent has been determined for general space dimensions \( n \) with \( n \geq 2 \). The exponent is obtained as the positive root of

\[
\gamma_S(p, n) := 2 + (n + 1)p - (n - 1)p^2 = 0
\]

and denoted by \( p_S(n) \). We call it Strauss exponent after the Strauss conjecture. The conjecture says that if \( p > p_S(n) \), then there exists uniquely a global
solution to
\[(\partial_t^2 - \Delta)U = |U|^p \quad \text{in} \ (0, T) \times \mathbb{R}^n\]
for sufficiently small initial data, and that if \(1 < p \leq p_S(n)\), then the solution blows up in finite time even for the small initial data (see also [25], [14], [9], [8], [4], [29], [20], or references in [5]).

We underline that if we add a damping term \(\mu(1 + |x|^2)^{-\alpha/2}\partial_t U\) with \(\mu > 0\) and \(0 \leq \alpha \leq 1\) in the above equation, i.e.
\[(1.4) \quad (\partial_t^2 + \mu(1 + |x|^2)^{-\alpha/2}\partial_t - \Delta)U = |U|^p, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n,\]
then the critical exponent drastically changes. In fact, when \(\alpha = 0\), Todorova and Yordanov [27] showed that the critical exponent becomes the Fujita exponent \(p_F(n) = 1 + 2/n\). This is because the solution behaves rather similar to that of the heat equation \(\partial_t v - \Delta v = v^p\). When \(0 < \alpha < 1\), Ikeda, Todorova and Yordanov [12] obtained that the critical exponent shifts to \(p_F(n - \alpha) = 1 + 2/(n - \alpha)\). For the scale invariant case \(\alpha = 1\), Li [24] showed the non-existence result of the global solution for \(1 < p \leq p_F(n - 1)\) with \(p_F(0) = \infty\). On the one hand, an interesting blow-up result was derived by Ikeda and Sobajima [10] for the case where \(0 \leq \mu < (n - 1)^2/(n + 1)\), \(n/(n - 1) < p \leq p_S(n + \mu)\) and \(n \geq 3\). Actually, since \(p_S(n) > p_F(n - 1)\) and \(p_S(n)\) is monotonically decreasing to 1, we see that the critical exponent is of the Strauss type if \(\mu\) is small. This type of shift is an analogue to the results due to D’Abbicco, Lucente and Reissig [3], Ikeda, Sobajima [11], Kato, Sakuraba [16], and Lai [23] in which the Cauchy problem for the wave equation with damping term of time dependent coefficient:
\[(\partial_t^2 + 2(1 + t)^{-1}\partial_t - \Delta)U = |U|^p \quad \text{in} \ (0, T) \times \mathbb{R}^3\]
was studied. Indeed, the problem admits a global solution for sufficiently small initial data if \(p > p_S(5)\), and that the solution blows up in finite time if \(1 < p \leq p_S(5)\). This means that the critical exponent \(p_S(3)\) is shifted to \(p_S(3 + 2)\), by virtue of the presence of the damping term.

On the other hand, the critical exponent does not change when we add only potential term \(V(x)U\) with a non-negative function \(V\) as
\[(\partial_t^2 - \Delta + V(x))U = |U|^p \quad \text{in} \ (0, T) \times \mathbb{R}^n.\]
Indeed, it was shown in [6] that there exists small global solutions if \(n = 3, \ p > p_S(3)\) and \(V \in C_0^\infty(\mathbb{R}^3)\). When \(V = O(|x|^{-2-\delta})\) as \(|x| \to \infty\) for some \(\delta > 0\), the blow-up result was obtained by [29], provided \(n \geq 3\) and \(1 < p < p_S(n)\) (see also [26], [18] for potentials with small magnitude).

Therefore, it is natural to ask whether the critical exponent might change or not if both damping and potential terms are in presence as in (1.1). In Georgiev, Kubo and Wakasa [7], by assuming that \(w(r)\) takes the form of \(1/r\) for large values of \(r\) and that \(w(r), V(r)\) have the following relation:
\[(1.5) \quad V(r) = -w'(r) + w(r)^2 \quad \text{for} \ r > 0,\]
the critical exponent was shown to be shifted from $p_S(3)$ to $p_S(5)$. Namely, the effect of the damping term appears as the shift of the critical exponent. But it is not still clear if the size of the coefficient of $w(r)$ comes into play or not. For this reason, we assume that $w(r)$ is a function in $C([0, \infty)) \cap C^1(0, \infty)$ satisfying
\begin{equation}
2w(r) = \frac{\mu}{r} + \tilde{w}(r), \quad |\tilde{w}(r)| \lesssim r^{-1-\delta} \quad \text{for } r \geq r_0
\end{equation}
with some positive numbers $r_0$, $\mu \geq 0$ and $\delta > 0$. Then, our question reformulate as follows: Does the critical exponent $p_S(3)$ shift to $p_S(3+\mu)$ for any $\mu \geq 0$?

In this paper, we shall give an affirmative answer to this question. To be more precise, one more issue concerning the decaying order of the initial data.
In fact, in [7], the initial data is assumed to vanish sufficiently fast order at spatial infinity. However, in view of the work of Asakura [1], the self-similarity becomes another important factor which determines the blow-up and global existence for small initial data (see also [22], [2], [28], [19]). Namely, the global behavior would be different between the cases $\kappa \geq 2/(p-1)$ and $\kappa < 2/(p-1)$, where $\kappa$ is the number from (1.2). As a matter of fact, we prove a blow-up result for $1 < p \leq p_S(3+\mu)$ in Theorem 2.1 and a global existence result for $p > p_S(3+\mu)$ and $\kappa \geq 2/(p-1)$ in Theorem 2.2. Moreover, we obtain lower bounds of the lifespan of the solution when $1 < p \leq p_S(3+\mu)$ or $\kappa < 2/(p-1)$ in Theorem 2.3. Finally, we show the optimality of the lifespan estimates by deriving upper bounds of lifespan for slowly decreasing initial data in Theorem 2.4. In conclusion, we find from these results that our equation (1.1) has different nature from the wave equation only with damping term, like (1.4).

In the next section, we formulate our problem under the assumption of the radial symmetry and describe the statements mentioned in the above, precisely.

2. Formulation of the Problem and Results

Since we are interested in spherically symmetric solutions to the problem (1.1), we set
\[ u(t, r) = rU(t, r, \omega) \quad \text{with } r = |x|, \quad \omega = x/|x|. \]
Then, by the relation (1.5) we obtain
\begin{equation}
\begin{cases}
(\partial_t - \partial_r + w(r))(\partial_t + \partial_r + w(r))u = |u|^p/r^{p-1} & \text{in } (0, T) \times (0, \infty), \\
u(0, r) = \varepsilon \varphi(r), \quad (\partial_t u)(0, r) = \varepsilon \psi(r) & \text{for } r > 0, \\
u(t, 0) = 0 & \text{for } t \in (0, T),
\end{cases}
\end{equation}
where we put $\varphi(r) = r f_0(r)$ and $\psi(r) = r f_1(r)$.

In order to express the solution of (2.1), we set $W(r) = \int_0^r w(\tau)d\tau$ for $r \geq 0$ and define
\begin{equation}
E_-(t, r, y) = e^{-W(r)}e^{2W(2^{-1}(y-t+r))}e^{-W(y)} \quad \text{for } t, r \geq 0, \quad y \geq t-r.
\end{equation}
From the assumption (1.6) we can deduce
\[ e^{W(r)} \sim \langle r \rangle^{\mu/2}, \quad r > 0. \]
Then the definition (2.2) of \( E_- \) implies
\[ (2.3) \quad E_-(t, r, y) \sim \langle r - t + y \rangle^{\mu} / \langle r \rangle^{\mu/2} \langle y \rangle^{\mu/2}. \]

Following the argument in [7], we see that the problem (2.1) can be written in the integral form
\[ (2.4) \quad u(t, r) = \varepsilon u_L(t, r) + \frac{1}{2} \int \int_{\Delta_-(t, r)} E_-(t - \sigma, r, y) \frac{|u(\sigma, y)|^p}{y^{p-1}} \, dy \, d\sigma \]
for \( t > 0, r > 0 \), where we have set
\[ \Delta_-(t, r) = \{ (\sigma, y) \in (0, \infty) \times (0, \infty); \, |t - r| < \sigma + y < t + r, \, \sigma - y < t - r \}. \]
Besides, \( u_L \) is the free solution defined by
\[ (2.5) \quad u_L(t, r) = \frac{1}{2} \int_{|t-r|}^{t+r} E_-(t, r, y) (\psi(y) + \varphi'(y) + w(y) \varphi(y)) \, dy \]
\[ + \chi(r - t) E_-(t, r, r - t) \varphi(r - t), \]
where \( \chi(s) = 1 \) for \( s \geq 0 \), and \( \chi(s) = 0 \) for \( s < 0 \).

First of all, we extend the blow-up result in [7] where \( \mu = 2 \) is assumed as follows. Its proof can be found in [15].

**Theorem 2.1.** Suppose that (1.6) holds. Let \( \varphi, \psi \in C([0, \infty)) \) satisfy
\[ \varphi(r) \equiv 0, \quad \psi(r) \geq 0 \, (\neq 0) \quad \text{for} \, r \geq 0. \]
If \( 1 < p \leq p_S(3 + \mu) \), then
\[ T(\varepsilon) \leq \begin{cases} \exp(C \varepsilon^{-p(p-1)}) & \text{if} \, \quad p = p_S(3 + \mu), \\ C \varepsilon^{-2p(p-1)/\gamma_S(p, 3+\mu)} & \text{if} \, \quad 1 < p < p_S(3 + \mu). \end{cases} \]
Here \( T(\varepsilon) \) denotes the lifespan of the problem (2.4).

On the other hand, when \( p > p_S(3 + \mu) \), we prove that the solution exists globally, analogously to [7]. But the pointwise estimate for the solution is improved as in (2.9) below, by refining the basic estimates. In fact, the decaying order is stronger than the one in [7] in the region away from the light cone, due to the factor \( (t + r)^{-1} \).

To state our results, we introduce the following parameters:
\[ (2.6) \quad \nu = \kappa - \mu/2 - 1, \]
\[ (2.7) \quad \eta = (\mu/2 + 1)p - (\mu/2 + 2). \]
Besides, \( p_S(n) \) denotes the positive root of (1.3) and we put \( p_F(\kappa) := 1 + 2/\kappa. \)
Theorem 2.2. Let $\kappa > \mu/2$. Suppose that (1.6) holds and $f_0 \in C^1([0,\infty))$, $f_1 \in C^0([0,\infty))$ satisfy
\begin{equation}
|f_0(r)| \leq \langle r \rangle^{-\kappa}, \quad |f'_0(r)| + |f_1(r)| \leq \langle r \rangle^{-\kappa - 1} \quad \text{for } r \geq 0.
\end{equation}

If $p > p_S(3 + \mu)$ and $p \geq p_F(\kappa)$, then there exists $\varepsilon_0 > 0$ so that for any $\varepsilon \in (0, \varepsilon_0]$ the corresponding integral equation (2.4) to the problem (2.1) has a unique global solution satisfying
\begin{equation}
|u(t, r)| \lesssim \varepsilon r \langle r \rangle^{-\mu/2} \times \begin{cases} 
\langle t + r \rangle^{-1-\nu}, & (-1 < \nu < 0), \\
\langle t + r \rangle^{-1}\Psi(t-r, t+r), & (\nu = 0), \\
\langle t + r \rangle^{-1}\langle t-r \rangle^{-\min\{\nu,\eta\}} & (\nu > 1)
\end{cases}
\end{equation}
for $t > 0$, $r > 0$. Here, for $|\beta| \leq \alpha$, we put
\begin{equation}
\Psi(\beta, \alpha) := 1 + \log \frac{1 + \alpha}{1 + |\beta|}.
\end{equation}

When either $1 < p \leq p_S(3 + \mu)$ or $1 < p < p_F(\kappa)$, we obtain the following lower bounds of the lifespan.

Theorem 2.3. Let $\kappa > \mu/2$. Suppose that (1.6) holds. Let $f_0 \in C^1([0,\infty))$, $f_1 \in C^0([0,\infty))$ satisfy (2.8). If $1 < p \leq p_S(3 + \mu)$ or $1 < p < p_F(\kappa)$, then there exist $C > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$
\begin{equation}
T(\varepsilon) \geq \begin{cases} 
\exp(C\varepsilon^{-(p-1)}) & (p = p_S(3 + \mu) \text{ and } p\nu > 1), \\
C\varepsilon^{-2p(p-1)/\gamma_S(p,3+\mu)} & (1 < p < p_S(3 + \mu) \text{ and } p\nu > 1), \\
\exp(C\varepsilon^{-(p-1)}) & (p = p_S(3 + \mu) \text{ and } p\nu = 1), \\
Cb(\varepsilon) & (1 < p < p_S(3 + \mu) \text{ and } p\nu = 1), \\
C\varepsilon^{-(p-1)/\gamma_F(p,\kappa)} & (1 < p < p_F(\kappa) \text{ and } p\nu < 1).
\end{cases}
\end{equation}
Here we put $\gamma_F(p, \kappa) = 2 - (p - 1)\kappa$, and $b(\varepsilon)$ is defined by
\begin{equation}
b(\varepsilon) \{\log(1 + b(\varepsilon))\}^{2(p-1)/\gamma_S(p,3+\mu)} = \varepsilon^{-2p(p-1)/\gamma_S(p,3+\mu)}.
\end{equation}

We remark that $b(\varepsilon)$ is well-defined, because $\gamma_S(p, 3 + \mu) > 0$ for $1 < p < p_S(3 + \mu)$. Moreover, it is easy to see that $b(\varepsilon)$ is a decreasing function and tends to $0$ as $\varepsilon \to 0+0$. Also, we notice that the last case where $1 < p < p_F(\kappa)$ and $p\nu < 1$ includes the case where $1 < p \leq p_S(3 + \mu)$ and $p\nu < 1$. To conclude the optimality of the lower bounds with respect to $\varepsilon$, the upper bounds given in Theorem 2.1 are not enough for the last three cases. Indeed, both $b(\varepsilon)$ and $\varepsilon^{-(p-1)/\gamma_F(p,\kappa)}$ is smaller than $\varepsilon^{-2p(p-1)/\gamma_S(p,3+\mu)}$, if we choose $\varepsilon$ is suitably small and $p\nu \leq 1$, because
\[
\frac{\gamma_S(p,3+\mu)}{2p(p-1)} = \frac{\gamma_F(p,\kappa)}{p-1} + \frac{p\nu - 1}{p}.
\]
However, the following result tells us the optimality in those cases.

Theorem 2.4. Suppose that (1.6) holds. Let $\varphi, \psi \in C([0,\infty))$ satisfy
\begin{equation}
\varphi(r) \equiv 0, \quad \psi(r) \geq (1 + r)^{-\kappa} \quad \text{for } r \geq 0.
\end{equation}
for some $\kappa > 0$. If either $1 < p \leq p_S(3 + \mu)$ and $p\nu = 1$, or $1 < p < p_F(\kappa)$ and $p\nu < 1$, then there exist $C > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$,

$$T(\varepsilon) \leq \begin{cases} 
\exp(C\varepsilon^{-(p-1)}) & (p = p_S(3 + \mu) \text{ and } p\nu = 1), \\
Cb(\varepsilon) & (1 < p < p_S(3 + \mu) \text{ and } p\nu = 1), \\
C\varepsilon^{-(p-1)/\gamma_F(p,\kappa)} & (1 < p < p_F(\kappa) \text{ and } p\nu < 1).
\end{cases}$$

This paper is organized as follows. In the section 3, we give preliminary facts. In the section 4, we derive a priori upper bounds and complete the proofs of Theorem 2.2 and Theorem 2.3. In particular, it is essential to linearize the problem around the free solution from the nonlinearity as (4.30). The section 5 is devoted to the proof of a blow-up result given in Theorem 2.4 based on the argument due to John [14].

3. Preliminaries

In this section we prepare lemmas which will be used in the proofs of the theorems. For the proofs of Lemma 3.1 and Lemma 3.2, see [15], Lemma 3 and Lemma 4, respectively.

**Lemma 3.1.** Let $0 \leq a \leq b$ and $k \in \mathbb{R}$. Then we have

$$\int_a^b (x)^{-k} dx \lesssim (b - a) \times \begin{cases} 
(b)^{-k} & (k < 1), \\
(b)^{-1}\Phi(a,b) & (k = 1), \\
(b)^{-1}(a)^{-k+1} & (k > 1).
\end{cases}$$

**Lemma 3.2.** Let $k_1, k_2, k_3 \geq 0$ and $\alpha \geq 0$. Then we have

$$\int_{-\alpha}^{\alpha} (\alpha + \beta)^{-k_1-k_2} (\beta)^{-k_1-k_3} d\beta \lesssim \langle \alpha \rangle^{-k_1} \times \begin{cases} 
\Phi_{1-(k_1+k_2+k_3)}(\alpha) & (k_1 + k_2 + k_3 \neq 1), \\
\log(2 + \alpha) & (k_1 + k_2 + k_3 = 1).
\end{cases}$$

Here, we put

$$\Phi_\rho(s) := \max\{1, \langle s \rangle^\rho\} \quad (s, \rho \in \mathbb{R}).$$

(3.1)

**Lemma 3.3.** Let $0 \leq a \leq b$ and $k > 1$. Then we have

$$\int_a^b (1 + x)^{-k} \log(1 + x) dx \lesssim (b - a)(b)^{-1}\langle a \rangle^{-k+1} \log(2 + a).$$

**Proof.** Integrating by parts, we get

$$\int_a^b (1 + x)^{-k} \log(1 + x) dx = \frac{1}{1-k} \left[ (1 + x)^{1-k} \log(1 + x) \right]_a^b$$

$$+ \frac{1}{k-1} \int_a^b (1 + x)^{-k} dx$$

$$\lesssim \log(1 + a) (1 + a)^{k-1} \left[ 1 - \left( \frac{1 + a}{1 + b} \right)^{k-1} \right]$$

$$+ \int_a^b (1 + x)^{-k} dx.$$
Using the inequality:
\[ 1 - s^l \leq \max\{1, l\}(1 - s) \quad \text{for} \quad l \geq 0, \ 0 \leq s \leq 1 \]
in the first term, and Lemma 3.1 in the second term, we get (3.2). This completes the proof.

\[ \square \]

**Lemma 3.4.** Let \( 0 \leq a \leq \alpha \) and \( k \geq 0 \). Then we have
\[ f_k(\alpha, a) := \int_a^\alpha \{\Psi(\beta, \alpha)\}^k d\beta \leq C_k(\alpha - a), \]
where \( \Psi(\beta, \alpha) \) is defined in (2.10).

**Proof.** It suffices to prove the estimate when \( k \) is a non-negative integer, because \( \Psi(\beta, \alpha) \geq 1 \) for \( a \leq \beta \leq \alpha \). It is clear that \( f_0(\alpha, a) = \alpha - a \). Suppose that \( f_{k-1}(\alpha, a) \leq C_{k-1}(\alpha - a) \) holds. Then, by the integration by parts, we have
\[
f_k(\alpha, a) = \left[ \beta \left\{ 1 + \log \left( \frac{1 + \alpha}{1 + \beta} \right) \right\}^k \right]_a^\alpha
+ k \int_a^\alpha \frac{\beta}{1 + \beta} \left\{ 1 + \log \left( \frac{1 + \alpha}{1 + \beta} \right) \right\}^{k-1} d\beta
\leq \alpha - a + kf_{k-1}(\alpha, a).
\]
Thus we get the desired estimate by the induction. This completes the proof. \( \square \)

**Lemma 3.5.** Let \( r_1 \neq 1 \), \( r_2 \geq 0 \) and \( \alpha \geq 0 \). Then we have
\[
\int_{-\alpha}^\alpha \langle \alpha + \beta \rangle^{-r_1} \{\Psi(\beta, \alpha)\}^{r_2} d\beta \lesssim \Phi_{1-r_1}(\alpha).
\]

**Proof.** We set
\[
I_1 := \int_{-\alpha}^{-\alpha/2} \langle \alpha + \beta \rangle^{-r_1} \{\Psi(\beta, \alpha)\}^{r_2} d\beta,
I_2 := \int_{-\alpha/2}^\alpha \langle \alpha + \beta \rangle^{-r_1} \{\Psi(\beta, \alpha)\}^{r_2} d\beta.
\]
We have from (2.10)
\[
I_1 \lesssim (1 + \log 2)^{r_2} \int_{-\alpha}^\alpha \langle \alpha + \beta \rangle^{-r_1} d\beta \lesssim \Phi_{1-r_1}(\alpha).
\]
From Lemma 3.4 we obtain
\[
I_2 \lesssim \langle \alpha \rangle^{-r_1} \int_{-\alpha}^\alpha \{\Psi(\beta, \alpha)\}^{r_2} d\beta \lesssim \langle \alpha \rangle^{1-r_1}.
\]
Combining these estimates, we finish the proof. \( \square \)
Lemma 3.6. Let $\rho_1 \geq 0$ and $\rho_2 \geq 0$. Then there exists a constant $C = C(\rho_1, \rho_2) > 0$ such that
\[
\int_a^b \frac{(y-a)^{\rho_2}}{y^{\rho_1}} dy \geq \frac{C}{a^{\rho_1-\rho_2-1}} \left(1 - \frac{a}{b}\right)^{\rho_2+1}
\]
for $0 < a < b$.

Lemma 3.7. Let $C_1, C_2 > 0$, $\alpha, \beta \geq 0$, $\theta \leq 1$, $\varepsilon \in (0, 1]$, and $p > 1$. Suppose that $f(y)$ satisfies
\[
f(y) \geq C_1 \varepsilon^\alpha, \quad f(y) \geq C_2 \varepsilon^\beta \int_y^\infty \left(1 - \frac{\xi}{y}\right) \frac{f(\xi)^p}{\xi^\theta} d\xi, \quad y \geq 1.
\]
Then, $f(y)$ blows up in a finite time $T_*(\varepsilon)$. Moreover, there exists a constant $C^* = C^*(C_1, C_2, p, \theta) > 0$ such that
\[
T_*(\varepsilon) \leq \begin{cases} \exp(C^* \varepsilon^{-(p-1)(1/\theta)}) & \text{if } \theta = 1, \\ C^* \varepsilon^{-(p-1)(1/\theta)} & \text{if } \theta < 1. \end{cases}
\]

4. Global existence and lower bounds of the lifespan

4.1. Proof of Theorem 2.2 and Theorem 2.3 for $1 < p < p_F(\kappa)$ and $p \nu < 1$. Our first step is to obtain the estimates for the homogeneous part of the solution to the problem (2.4).

We put
\[
(4.1) \quad w_1(t, r) = r \langle r \rangle^{-\mu/2} \times \begin{cases} \langle t + r \rangle^{-1-\nu} & (\nu < 0), \\ \langle t + r \rangle^{-1} \Psi(t - r, t + r) & (\nu = 0), \\ \langle t + r \rangle^{-1} \langle t - r \rangle^{-\nu} & (\nu > 0), \end{cases}
\]
where $\nu$ and $\Psi(\beta, \alpha)$ are defined in (2.6) and (2.10), respectively.

We define the following weighted $L^\infty$-norm:
\[
(4.2) \quad \|u\|_1 = \sup_{(t, r) \in I \times [0, \infty)} \{w_1(t, r)^{-1} |u(t, r)|\},
\]
where we put
\[
I := \begin{cases} [0, \infty) & (p > p_S(3 + \mu) \text{ and } p \geq p_F(\kappa)), \\ [0, T) & (1 < p \leq p_S(3 + \mu) \text{ or } 1 < p < p_F(\kappa)). \end{cases}
\]

For the proof of Lemma 4.1 see [15], Lemma 6.

Lemma 4.1. Assume that (1.6) holds and $\varphi \in C^1([0, \infty))$, $\psi \in C^0([0, \infty))$ satisfy
\[
|\varphi(r)| \lesssim r \langle r \rangle^{-\kappa}, \quad |\psi(r) + \varphi'(r) + w(r) \varphi(r)| \lesssim \langle r \rangle^{-\kappa} \text{ for } r \geq 0
\]
with some positive constant $\kappa$. Then we have
\[
(4.3) \quad |u_L(t, r)| \leq \tilde{C}_0 w_1(t, r), \quad (t, r) \in [0, \infty) \times [0, \infty)
\]
with some positive constant \( \widetilde{C}_0 \). Here \( u_L \) is defined in (2.5).

Our next step is to consider the integral operator appeared in (2.4):
\[
I_-(F)(t, r) = \frac{1}{2} \int \int_{\Delta_-(t, r)} E_-(t - \sigma, r, y) F(\sigma, y) dy d\sigma.
\]

For \((\sigma, y) \in \Delta_-(t, r)\) we have \( y \geq |t - r - \sigma| \), so that (2.3) yields
\[
|E_-(t - \sigma, r, y)| \lesssim \langle r \rangle^{-\mu/2} \langle y \rangle^{\mu/2} \quad \text{for} \quad (\sigma, y) \in \Delta_-(t, r).
\]

Hence we get
\[
|I_-(F)(t, r)| \lesssim \langle r \rangle^{-\mu/2} \int \int_{\Delta_-(t, r)} \langle y \rangle^{\mu/2} |F(\sigma, y)| dy d\sigma.
\]

We set
\[
D_1(T) = \left\{ \begin{array}{ll}
1 & (p \geq p_F(\kappa)), \\
(1 + T)^{2 - (p - 1)} & (1 < p < p_F(\kappa)).
\end{array} \right.
\]

In order to prove Theorem 2.2 and Theorem 2.3 with \( p \nu < 1 \), we prepare the following Lemma 4.2 and Lemma 4.3.

**Lemma 4.2.** Let \(-1 < \nu < 0\) and \( p > 1 \). Then, there exists a positive constant \( \widetilde{C}_1 \) such that
\[
\|I_-(F)\|_1 \leq \widetilde{C}_1 \|u\|_1^p D_1(T)
\]
with \( F(t, r) = |u(t, r)|^p / r^{p-1} \).

**Proof.** From (4.1) and (4.2), we obtain
\[
\langle r \rangle^{\mu/2} (t + r)^{p(1 + \nu)} |F(t, r)| \leq r \|u\|^p_1.
\]

It follows from (4.3) that
\[
|I_-(F)(t, r)| \lesssim \langle r \rangle^{-\mu/2} \|u\|^p_1 I(t, r),
\]
where we put
\[
I(t, r) = \int \int_{\Delta_-(t, r)} \frac{y}{\langle y \rangle^{(p-1)\mu/2} \langle \sigma + y \rangle^{p(1+\nu)}} dy d\sigma.
\]

From (4.2), (4.1) and (4.5), in order to (4.6), it is enough to prove
\[
I(t, r) \lesssim \frac{r}{\langle t + r \rangle^{1 + \nu}} \times \left\{ \begin{array}{ll}
1 & (p \geq p_F(\kappa)), \\
(1 + T)^{2 - (p - 1)} & (1 < p < p_F(\kappa)).
\end{array} \right.
\]

To evaluate the integral (4.7), we pass to the coordinates
\[
\alpha = \sigma + y, \quad z = y
\]
and deduce
\[
I(t, r) \lesssim \int_{|r-t|}^{t+r} \frac{1}{\langle \alpha \rangle^{p(1+\nu)}} \int_{(\alpha-t+r)/2}^{\alpha} \frac{z}{\langle z \rangle^{(p-1)\mu/2}} dz d\alpha.
\]
First, we suppose \( t \geq r/2 \) and divide the proof into two cases.

(i) \( p \neq p_F(\kappa) \).

First of all, we note that (2.6) implies

\[
-(p - 1)\mu/2 = (p - 1)(1 + \nu) - (p - 1)\kappa.
\]

Since \((p - 1)(1 + \nu) > 0\) and \((p - 1)\kappa \neq 2\), we get from (4.9), (4.10) and (3.1)

\[
I(t, r) \lesssim \int_{|r-t|}^{t+r} \frac{1}{\langle \alpha \rangle^{1+\nu}} \int_0^\alpha \frac{1}{\langle z \rangle^{(p-1)\kappa-1}} dz d\alpha
\]

\[
\lesssim \int_{|r-t|}^{t+r} \frac{1}{\langle \alpha \rangle^{1+\nu}} \Phi_{2-(p-1)\kappa}(\alpha) d\alpha
\]

\[
\lesssim \Phi_{2-(p-1)\kappa}(t) \int_{|r-t|}^{t+r} \frac{1}{\langle \alpha \rangle^{1+\nu}} d\alpha.
\]

Here, \( \Phi_\rho(s) \) is defined in (3.1). Since \( \nu < 0 \), it follows from Lemma 3.1 that

\[
I(t, r) \lesssim \frac{r}{(t + r)^{1+\nu}} \Phi_{2-(p-1)\kappa}(t).
\]

(ii) \( p = p_F(\kappa) \).

Taking \( \delta > 0 \) so small that \((p - 1)(1 + \nu) - \delta > 0\). Since \((p - 1)\kappa = 2\), from (4.9), (4.10) and Lemma 3.1, we have

\[
I(t, r) \lesssim \int_{|r-t|}^{t+r} \frac{1}{\langle \alpha \rangle^{1+\nu+\delta}} \int_0^\alpha \frac{1}{\langle z \rangle^{1-\delta}} dz d\alpha
\]

\[
\lesssim \int_{|r-t|}^{t+r} \frac{1}{\langle \alpha \rangle^{1+\nu}} d\alpha
\]

\[
\lesssim \frac{r}{(t + r)^{1+\nu}}.
\]

Therefore, from (4.11) and (4.12), we obtain (4.8) for \( t \geq r/2 \).

Next, suppose \( r/2 \geq t \). Since \((r + t)/3 \leq r - t\), we obtain from (4.9) and (4.10)

\[
I(t, r) \lesssim \int_{r-t}^{t+r} \frac{\alpha(\alpha + t - r)}{\langle \alpha \rangle^{\rho(1+\nu)}(\alpha - t + r)^{(p-1)\mu/2}} d\alpha
\]

\[
\lesssim \frac{t + r}{(t + r)^{1+\nu+(p-1)\kappa}} \int_{r-t}^{t+r} (\alpha + t - r) d\alpha
\]

\[
\lesssim \frac{r}{(t + r)^{1+\nu}(1 + t)^{2-(p-1)\kappa}}.
\]

Hence we get (4.8) for \( r/2 \geq t \). This completes the proof.

**Lemma 4.3.** Let \( 0 \leq \nu < 1/p \) if \( 1 < p \leq p_S(3 + \mu) \) and \( 0 \leq \nu \leq \eta \) if \( p > p_S(3 + \mu) \). Then, there exists a positive constant \( \widetilde{C}_1 \) such that

\[
\|I_-(F)\|_1 \leq \widetilde{C}_1\|u\|_p^p \times D_1(T)
\]
with \( F(t, r) = |u(t, r)|^p/r^{p-1} \). Here \( \eta = (\mu/2 + 1)p - (\mu/2 + 2) \).

**Remark.** When \( p > p_S(3 + \mu) \) (resp. \( 1 < p < p_S(3 + \mu) \)), we have \( \eta > 1/p \) (resp. \( \eta < 1/p \)).

**Proof.** It follows from (4.1) and (4.2) that

\[
\langle r \rangle^{\mu/2} \langle t + r \rangle^p F(t, r) \leq r \{\Psi(t - r, t + r)\} \|u\|_1^p,
\]

where the number \( q_1 \) is defined as follows:

\[ q_1 = 1 \text{ for } \nu = 0, \quad \text{and } q_1 = 0 \text{ for } \nu \neq 0. \]

We get from (4.4)

\[
|I - (F)(t, r)| \lesssim \langle r \rangle^{-\mu/2} \|u\|_1^p I(t, r),
\]

where we put

\[
I(t, r) = \int_{\Delta_{-(t, r)}} \frac{y \{\Psi(\sigma - y, \sigma + y)\}}{\langle y \rangle^{(p-1)\mu/2} \langle \sigma + y \rangle^p \langle \sigma - y \rangle^{p\nu}} dy d\sigma.
\]

In order to show (4.13), from (4.1) and (4.2), it is enough to show

\[
I(t, r) \lesssim \langle r \rangle^p \{\Psi(t - r, t + r)\}^{q_1} \times \left\{ \begin{array}{ll}
1 & (p \geq p_F(\kappa)), \\
(1 + t)^{2(p-1)\kappa} & (1 < p < p_F(\kappa)).
\end{array} \right.
\]

To evaluate the integral (4.14), we pass to the coordinates

\[ \alpha = \sigma + y, \quad \beta = y - \sigma, \]

and deduce

\[
I(t, r) \lesssim \int_{|\alpha - \beta|}^{t+r} \frac{\{\Psi(\beta, \alpha)\}}{\langle \alpha \rangle^{1+n} \langle \beta \rangle^{p\nu}} d\beta d\alpha.
\]

First, suppose \( r \geq t \). Then, we get from (4.16) and (2.7)

\[
I(t, r) \lesssim \int_{r-t}^{t+r} \frac{1}{\langle \alpha \rangle^{1+n}} \int_{\alpha}^{\alpha + \beta} \{\Psi(\beta, \alpha)\} \langle \beta \rangle^{p\nu} d\beta d\alpha.
\]

For \( \alpha \geq r - t \geq 0 \), we have from Lemma 3.1 and Lemma 3.4

\[
\int_{r-t}^{\alpha} \frac{\{\Psi(\beta, \alpha)\}}{\langle \beta \rangle^{p\nu}} d\beta \lesssim \left\{ \begin{array}{ll}
\frac{\alpha - r + t}{\langle \alpha \rangle^{p\nu}} & (p\nu < 1), \\
1 + \log(1 + \alpha) & (p\nu = 1), \\
1 & (p\nu > 1).
\end{array} \right.
\]

We divide the proof into two cases.

(i) \( 1/p \leq \nu \leq \eta \) and \( p > p_S(3 + \mu) \).

Since \( \nu < \eta \) if \( p\nu = 1 \), we obtain from (4.17), (4.18) and Lemma 3.1

\[
I(t, r) \lesssim \int_{r-t}^{t+r} \frac{1}{\langle \alpha \rangle^{1+n}} d\alpha \lesssim \frac{r}{\langle t + r \rangle^{(t - r)\nu}}.
\]
\[(ii) \quad 0 \leq \nu < \frac{1}{p}.\]

Since \(\eta + p\nu = \nu + (p - 1)\kappa - 1\), we have from (4.17) and (4.18)

\[I(t,r) \lesssim \int_{t-r}^{t+r} \frac{\alpha - r + t}{\langle \alpha \rangle^{\nu+(p-1)\kappa}} d\alpha. \tag{4.19}\]

If \((p - 1)\kappa \geq 2\), it follows from (4.19) and Lemma 3.1 that

\[I(t,r) \lesssim \int_{t-r}^{t+r} \frac{1}{\langle t + r \rangle^{\nu+(p-1)\kappa}} \times \left\{ \begin{array}{ll} 1 & \text{if } (p - 1)\kappa > 2 \text{ or } \nu > 0, \\ \Psi(t-r,t+r) & \text{if } (p - 1)\kappa = 2 \text{ and } \nu = 0. \end{array} \right. \tag{4.20}\]

On the other hand, if \(0 < (p - 1)\kappa < 2\), we obtain from (4.19) and Lemma 3.1

\[I(t,r) \lesssim \int_{t-r}^{t+r} \frac{1}{\langle t - r \rangle^{\nu+(p-1)\kappa}} \times \left\{ \begin{array}{ll} \frac{1}{t(t+r)^{1-(p-1)\kappa}} & \text{if } (p - 1)\kappa < 2, \\ t^2(t+r)^{-(p-1)\kappa} & \text{if } (p - 1)\kappa < 1. \end{array} \right. \tag{4.21}\]

Let \(r \geq 1\). Since \(\langle r \rangle \sim \langle t + r \rangle\) for \(r \geq t\), we get from (4.21)

\[I(t,r) \lesssim \frac{r}{\langle t + r \rangle^{(p-1)\kappa}}. \tag{4.22}\]

Let \(0 \leq r \leq 1\). Since \(\langle r \rangle \sim 1\) for \(r \geq t\), we have from (4.21)

\[I(t,r) \lesssim \frac{r}{\langle t + r \rangle^{(p-1)\kappa}}. \tag{4.23}\]

Hence, from (4.20), (4.22) and (4.23), we obtain (4.15) for \(r \geq t\).

Next, we suppose \(t \geq r\) and divide the proof into three cases. We remark that those cases assure the assumptions on \(\nu\), because \(1 < p \leq p_S(3 + \mu)\) and \(p\nu < 1\) implies \(p < p_F(\kappa)\).

(i) \(0 \leq \nu \leq \eta\) and \(p \neq p_F(\kappa)\).

Since \(\langle \alpha \rangle \geq \langle (\alpha + \beta)/2 \rangle\) for \(-\alpha \leq \beta \leq \alpha\), we have from (4.16) and (2.7)

\[I(t,r) \lesssim \int_{t-r}^{t+r} \frac{1}{\langle \alpha \rangle^{\nu+\mu}} \int_{-\alpha}^{\alpha} \Psi(\beta,\alpha) \frac{1}{\langle \alpha + \beta \rangle^{\eta\nu}} d\beta d\alpha. \]

We shall use Lemma 3.5 when \(\nu = 0\), and Lemma 3.2 with \(k_1 = \nu\), \(k_2 = \eta - \nu\), \(k_3 = (p - 1)\nu\) when \(\nu > 0\). Then we get

\[I(t,r) \lesssim \int_{t-r}^{t+r} \frac{1}{\langle \alpha \rangle^{1+\mu}} \Phi_{2-(p-1)\kappa}(\alpha) d\alpha, \]

because \((p - 1)\kappa \neq 2\) when \(p \neq p_F(\kappa)\), and

\[\eta + (p - 1)\nu = -1 + (p - 1)\kappa. \tag{4.24}\]
Now, Lemma 3.1 yields

\[(4.25) \quad I(t, r) \leq \frac{r \{\Psi(t - r, t + r)\}^q_1}{(t + r)(t - r)\nu_2} \Phi_{2 - (p - 1)\kappa}(t),\]

(ii) \(0 \leq \nu \leq \eta, p = p_F(\kappa)\) and \(p > p_S(3 + \mu)\).

Note that \(p = p_F(\kappa)\) and \(p > p_S(3 + \mu)\) leads to \(\nu < 1/p < \eta\). Therefore, we can take \(\delta > 0\) so small that \(p - 1 - \delta > 0\) and \(\eta - \delta > \nu\). Then, similarly to the previous case, we get from (4.16), (4.24), Lemma 3.5 and Lemma 3.2

\[(4.26) \quad I(t, r) \leq \frac{1}{(t + r)(t - r)\nu} \int_{t_r}^{t+r} \frac{1}{\langle \alpha \rangle^{\frac{1}{\nu} + p_0}} \Phi_{\delta}(\alpha) d\alpha \]

(iii) \(\max\{\eta, 0\} \leq \nu < 1/p\) and \(1 < p \leq p_S(3 + \mu)\).

Since \(p - 1 - \eta + \nu > 0\) and \((p - 1)(\mu/2 + 1) = 1 + \eta\), we get from (4.16)

\[(4.27) \quad I(t, r) \leq \frac{1}{(t + r)(t - r)^\nu} \int_{t_r}^{t+r} \frac{1}{\langle \alpha \rangle^{\frac{1}{\nu} + p_0}} \Phi_{\delta}(\alpha) d\alpha \]

Using Lemma 3.4 when \(\nu = 0\), and Lemma 3.2 with \(k_1 = \nu, k_2 = 0, k_3 = (p - 1)\nu\) when \(\nu > 0\), we obtain

\[(4.28) \quad I(t, r) \leq \int_{t_r}^{t+r} \frac{1}{\langle \alpha \rangle^{\eta + p_0}} d\alpha = \int_{t_r}^{t+r} \frac{1}{\langle \alpha \rangle^{\nu + 1 - (p - 1)\kappa}} d\alpha,\]

by \(p\nu < 1\) and (4.24). Recalling the fact that \((p - 1)\kappa < 2\) in this case, we find

\[(4.29) \quad I(t, r) \leq (t + r)^2 \int_{t_r}^{t+r} \frac{1}{\langle \alpha \rangle^{\nu + 1}} d\alpha \]

Hence, from (4.26), (4.28) and (4.29), we obtain (4.15) for \(t \geq r\). This completes the proof. \(\square\)

**End of the proof of Theorem 2.2 and Theorem 2.3** for \(1 < p < p_F(\kappa)\) and \(p\nu < 1\). Let \(X\) be the linear space defined by

\[X = \{u(t, x) \in C(I \times (0, \infty)) ; \|u\|_1 < \infty\}.\]

We can verify easily that \(X\) is complete with respect the norm \(\|\cdot\|_1\). We define the sequence of functions \(\{u_n\}\) by

\[u_0 = \varepsilon u_L, \quad u_{n+1} = \varepsilon u_L + I_+(|u_n|^p/r^{p-1}) \quad (n = 0, 1, 2, \cdots).\]

It follows from Lemma 4.1 that \(\|u_0\|_1 \leq \varepsilon C_0\). Hence \(u_0 \in X\).
In the following, we assume that
\[ 2^p p \tilde{C}_1 D_1(T)(\varepsilon \tilde{C}_0)^p \leq 1 \quad \text{and} \quad \varepsilon \tilde{C}_0 \leq \frac{1}{2}, \]
where \( \tilde{C}_1 \) is the constant in Lemma 4.2 and Lemma 4.3. Then, we have
\[ 2^p p \tilde{C}_1 D_1(T)\|u_0\|_{L^1}^{p-1} \leq 1 \quad \text{and} \quad \|u_0\|_1 \leq 1. \]
(4.29)

Now, we conclude the proof of the theorems. First, we prove Theorem 2.2.
Suppose that \( \kappa > \mu/2, p > p_S(3 + \mu) \) and \( p \geq p_F(\kappa) \) hold. Then \( \nu > -1 \) and \( D_1(T) = 1 \). In addition, we replace \( \kappa \) by \( \tilde{\kappa} = \min\{\kappa, (\mu/2 + 1)p - 1\} \) in (2.8), so that \( \tilde{\nu} := \tilde{\kappa} - \mu/2 - 1 = \min\{\nu, \eta\} \). As in [14], we see that if \( u_0 \) satisfies (4.28), then it follows from Lemma 4.2 and Lemma 4.3 that \( \{u_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, there exists \( u \in X \) such that \( u_n \) converges uniformly. Clearly \( u \) satisfies (2.4). Thus we get Theorem 2.2.

Next, we move on to the proof of Theorem 2.3 for the case where \( \kappa > \mu/2, 1 < p < p_F(\kappa) \) and \( p\nu < 1 \) hold. In this case, we have \( \nu > -1 \) and \( D_1(T) = (1 + T)^{2-(p-1)}\kappa \). Similarly to the above, we obtain a unique local solution of (2.4), provided (4.28) holds. This means (2.11) with \( 1 < p < p_F(\kappa) \) and \( p\nu < 1 \) is valid. This competes the proof.

4.2. Proof of Theorem 2.3 with \( p\nu \geq 1 \). It is convenient to transform the equation (2.4) to the following integral equation:
\[ v = I_{-}(|\varepsilon u_L + v|/r^{p-1}), \]
(4.30)
where \( w_2(t, r) := \frac{r}{\langle r \rangle^{\mu/2}} \times \begin{cases} \frac{\Psi(t - r, t + r)}{\langle t + r \rangle^{1+\eta}} & (\eta \leq 0), \\ \frac{\log(2 + |t - r|)}{\langle t + r \rangle \langle t - r \rangle^{\eta}} & (\eta > 0), \end{cases} \]
(4.31)
where we put
\[ q_2 = \begin{cases} 1 & (\eta = 0), \\ 0 & (\eta \neq 0), \end{cases} \quad q_3 = \begin{cases} 1 & (p\nu = 1), \\ 0 & (p\nu > 1). \end{cases} \]
By using the following weighted \( L^\infty \) norm:
\[ \|v\|_2 := \sup_{(t, r) \in [0, T] \times [0, \infty)} w_2(t, r)^{-1}|v(t, r)|, \]
(4.32)
we shall prove Lemma 4.4 and Lemma 4.5 below.
Lemma 4.4. If $pv \geq 1$ and $1 < p \leq p_S(3 + \mu)$, then there exists a positive constant $\tilde{C}_1$ such that

$$\| I_-(|uL|^{p/r^{p-1}}) \|_2 \leq \tilde{C}_1 \| uL \|_1^p. \quad (4.33)$$

Proof. Since $\nu > 0$, similarly to the proof of Lemma 4.3 we get

$$\| I_-(|uL|^{p/r^{p-1}})(t, r) \| \lesssim \langle r \rangle^{\mu/2} \| uL \|_1^p I(t, r),$$

where $I(t, r)$ is the one defined in (4.14). In view of (4.31) and (4.32), for getting (4.33), it is enough to prove

$$I(t, r) \lesssim \langle r \rangle^{\mu/2} w_2(t, r). \quad (4.34)$$

It follows from (4.16) and (2.7) that

$$I(t, r) \lesssim \int_{|r-t|}^{t+r} \frac{1}{(\alpha)} \int_{-\alpha}^\alpha \frac{1}{(\alpha + \beta)\langle \beta \rangle^{pv}} d\beta d\alpha.$$

From $1 < p \leq p_S(3 + \mu)$, we have $\eta \leq 1/p$, so that $p\nu > \nu \geq 1/p \geq \eta$. Therefore, using Lemma 3.2 with $k_1 = \eta$, $k_2 = 0$, $k_3 = p\nu - \eta$, we get

$$\int_{-\alpha}^\alpha \frac{1}{(\alpha + \beta)\langle \beta \rangle^{pv}} d\beta \lesssim \langle \alpha \rangle^{-\eta} \{ \log(2 + \alpha) \}^{q_3}.$$

From Lemma 3.1 and Lemma 3.3 we obtain

$$I(t, r) \lesssim \int_{|t-r|}^{t+r} \frac{\{ \log(2 + \alpha) \}^{q_3}}{(\alpha)^{1+\eta}} d\alpha \lesssim r \times \left\{ \begin{array}{ll} \langle t + r \rangle^{-1+\eta} \{ \Psi(t - r, t + r) \}^{q_2} \{ \log(2 + t + r) \}^{q_3} & (\eta \leq 0), \\
\langle t + r \rangle^{-1} \{ t - r \}^{-\eta} \{ \log(2 + |t - r|) \}^{q_3} & (\eta > 0). \end{array} \right. $$

Hence (4.34) is deduced. This completes the proof. \hfill \Box

We set

$$D_2(T) = \left\{ \begin{array}{ll} (1 + T)^{\gamma_S(p,3+\mu)/2} \{ \log(2 + T) \}^{(p-1)q_3} & (1 < p < p_S(3 + \mu)), \\
\{ \log(2 + T) \}^{1+(p-1)q_3} & (p = p_S(3 + \mu)). \end{array} \right. \quad (4.35)$$

Lemma 4.5. Let $p_1 = 1$ or $p$. If $pv \geq 1$ and $1 < p \leq p_S(3 + \mu)$, then there exists a positive constant $\tilde{C}_2$ such that

$$\| I_-(G) \|_2 \leq \tilde{C}_2 \| uL \|_1^{p-p_1} \| v \|_2^{p_1} D_2(T)^{p_1/p}. \quad (4.36)$$

with $G(t, r) = |uL(t, r)|^{p-p_1} |v(t, r)|^{p_1} / r^{p-1}$.

Proof. At first, we put

$$\eta_1 = p_1 \eta + (p - p_1)\nu, \quad \eta_2 = p_1 \eta + p, \quad \eta_3 = (p - p_1)/p.$$ 

Noting that $1 - p\eta = \gamma_S(p,3 + \mu)/2$, we get

$$1 - \eta_1 = \gamma_S(p,3 + \mu)p_1/(2p) - (p - p_1)(pv - 1)/p, \quad (4.37)$$

$$1 - p_1 \eta - \eta_3 = \gamma_S(p,3 + \mu)p_1/(2p). \quad (4.38)$$
Besides, (2.7) implies

\[(4.39) \quad (p-1)\mu/2 + \eta_2 = 2 + (1+p_1)\eta.\]

In the following, let \((t, r) \in [0,T] \times (0, \infty), \ p\nu \geq 1, \) and either \(p_1 = 1\) or \(p_1 = p.\) We divide the proof into two cases.

(i) \(\eta > 0.\)

Since \(\nu > 0,\) we have from (4.1), (4.2) and (4.3)

\[(4.40) \quad \langle r \rangle^{\mu/2}(t + r)^\eta u_L(t, r) \leq r \|u_L\|_1.\]

Therefore, we get from (4.31) and (4.32)

\[(4.41) \quad \langle r \rangle^{p\mu/2}(t + r)^p(t - r)^\eta |G(t, r)| \leq r \{ \log(2 + |t - r|) \}^{p_1q_3} \|u_L\|_{1}^{p-p_1} \|v\|_{2}^{p_1}.\]

Then, it follows from (4.4) that

\[|I_{-}(G)(t, r)| \lesssim \langle r \rangle^{-\mu/2}\|u_L\|_{1}^{p-p_1} \|v\|_{2}^{p_1} I(t, r),\]

where we put

\[I(t, r) := \int \int_{\Delta_{(t, r)}} \frac{y \{ \log(2 + |\sigma - y|) \}^{p_1q_3}}{\langle y \rangle^{(p-1)\mu/2} \langle \sigma + y \rangle^{p} \langle \sigma - y \rangle^{\eta}} dyd\sigma.\]

In order to get (4.36), by (4.31) and (4.32), it is enough to prove

\[(4.41) \quad I(t, r) \lesssim \frac{r \{ \log(2 + |t - r|) \}^{q_3}}{\langle t + r \rangle \langle t - r \rangle^\eta} D_2(T)^{p_1/p}.\]

To evaluate the integral, we pass to the coordinates

\[\alpha = \sigma + y, \quad \beta = y - \sigma,\]

and deduce

\[(4.42) \quad I(t, r) \lesssim \int_{r-t}^{t+r} \int_{r-t}^{\alpha} \frac{\{ \log(2 + |\beta|) \}^{p_1q_3}}{\langle \alpha \rangle^{1+\eta} \langle \beta \rangle^{p} \langle \beta \rangle^{\eta}} d\beta d\alpha.\]

First, suppose \(r \geq t.\) Then, we get from (4.42) and (2.7)

\[(4.43) \quad I(t, r) \lesssim \int_{r-t}^{t+r} \frac{\{ \log(2 + \alpha) \}^{p_1q_3}}{\langle \alpha \rangle^{1+\eta} \langle \alpha \rangle^{p}} d\alpha \times \int_{r-t}^{t+r} \frac{1}{\langle \beta \rangle^{\eta}} d\beta.\]

Since \(\eta > 0,\) we have from Lemma 3.1 and Lemma 3.3

\[(4.44) \quad \int_{r-t}^{t+r} \frac{\{ \log(2 + \alpha) \}^{p_1q_3}}{\langle \alpha \rangle^{1+\eta} \langle \alpha \rangle^{p}} d\alpha \lesssim t \{ \log(2 + |t - r|) \}^{q_3} \langle t + r \rangle \langle t - r \rangle^\eta \{ \log(2 + t + r) \}^{(p_1-1)q_3}.\]

On the other hand, to evaluate the \(\beta\)-integral, we shall use the following facts about \(\eta_1\) which can be deduced by (4.37): When \(p < p_S(3 + \mu), \) \(\eta_1 = 1 - \gamma_S(p, 3 + \mu)p_1/(2p) + (p - p_1)(p\nu - 1)/p \geq 1 - \gamma_S(p, 3 + \mu)p_1/(2p). \) When
\[ p = p_S(3 + \mu), \quad p_1 = 1 \text{ and } p \nu > 1, \quad \eta_1 = 1 + (p - 1)(p \nu - 1)/p > 1. \] And, when \( p = p_S(3 + \mu) \) and either \( p_1 = p \) or \( p \nu = 1, \quad \eta_1 = 1. \) These facts lead to

\[
\int_{r-t}^{t+r} \frac{1}{\langle \beta \rangle^n} d\beta \lesssim \begin{cases} 
\langle t + r \rangle^{\gamma_S(p,3+\mu)p_1/(2p)} & (p < p_S(3 + \mu)), \\
1 & (p = p_S(3 + \mu), \quad p_1 = 1 \text{ and } p \nu > 1), \\
\log(2 + t + r) & (p = p_S(3 + \mu), \quad \text{either } p_1 = p \text{ or } p \nu = 1)
\end{cases}
\]

where we defined

\[
q_4 = \begin{cases} 
0 & \text{(either } p < p_S(3 + \mu) \text{ or } p = p_S(3 + \mu), \quad p_1 = 1 \text{ and } p \nu > 1), \\
1 & \text{(either } p_1 = p \text{ or } p \nu = 1)\nonumber.
\end{cases}
\]

Therefore, from (4.43), (4.44) and (4.45), we find

\[
I(t, r) \lesssim \frac{t \{\log(2 + |r - t|)\}^{q_3 + q_4}}{(t - r)^\eta} \times \frac{\{\log(2 + t + r)\}^{(p_1 - 1)q_3 + q_4}}{(t + r)^{1 - \gamma_S(p,3+\mu)p_1/(2p)}}. \tag{4.46}
\]

We note that if \( r_1 > 0, \quad r_2 \geq 0, \) then \( g(s) = s^{-r_1} \{\log(1 + s)\}^{r_2} \) \((s \geq 1) \) is decreasing for large \( s. \) Since \( 1 - \gamma_S(p,3+\mu)p_1/(2p) = p_1 \eta + (p-p_1)/p > \eta > 0, \) by recalling (4.35), we then get

\[
\frac{\langle t \rangle \{\log(2 + t + r)\}^{(p_1 - 1)q_3 + q_4}}{(t + r)^{1 - \gamma_S(p,3+\mu)p_1/(2p)}} \lesssim D_2(t)^{p_1/p}.
\]

Moreover, for \( r \geq t, \) we have

\[
\frac{t}{\langle t \rangle} \lesssim \frac{r}{\langle t + r \rangle}.
\]

Indeed, if \( r \geq 1, \) then we have \( r \sim \langle t + r \rangle, \) so that the estimate holds. On the other hand, if \( 0 \leq r \leq 1, \) then \( \langle t + r \rangle \sim 1, \) thus we obtain the needed estimates. Now, from (4.46) we get (4.41) for \( r \geq t. \)

Next, suppose \( t \geq r. \) We get from (4.42) and (2.7)

\[
I(t, r) \lesssim \int_{t-r}^{t+r} \frac{\{\log(2 + \alpha)\}^{p_1q_3}}{\langle \alpha \rangle} \frac{1}{\langle \alpha + \beta \rangle^{\eta} \langle \beta \rangle^n} d\beta d\alpha.
\]

Since \( 0 < \eta < 1 - \gamma_S(p,3+\mu)p_1/(2p) \leq \eta_1, \) it follows from Lemma 3.2 with \( k_1 = \eta, \quad k_2 = 0, \quad k_3 = 1 - \gamma_S(p,3+\mu)p_1/(2p) - \eta \) (resp. \( k_3 = \eta_1 - \eta \)) when \( p < p_S(3 + \mu) \) (resp. \( p = p_S(3 + \mu) \)) that

\[
\int_{t-r}^{t+r} \frac{1}{\langle \alpha + \beta \rangle^{\eta} \langle \beta \rangle^n} d\beta \lesssim \langle \alpha \rangle^{-\eta} \langle \alpha \rangle^{\gamma_S(p,3+\mu)p_1/(2p)} \{\log(2 + \alpha)\}^{q_4}.
\]
Therefore, we get from (4.35), Lemma 3.1 and Lemma 3.3

\[ I(t, r) \lesssim \int_{t-r}^{t+r} \frac{\langle \alpha \rangle^{\eta} \{ \log(2 + \alpha) \}^{p_1 q_3 q_4}}{\langle \alpha \rangle^{1+\eta}} \ d\alpha \]

\[ \lesssim D_2(t + r)^{p_1/p} \int_{t-r}^{t+r} \frac{\{ \log(2 + \alpha) \}^{\eta}}{\langle \alpha \rangle^{1+\eta}} \ d\alpha \]

\[ \lesssim r \{ \log(2 + |t - r|) \}^{q_1} \langle t + r \rangle^{1+\eta} D_2(T)^{p_1/p}. \]

Hence we obtain (4.41).

(ii) \( \eta \leq 0. \)

Since \( p_\nu \geq 1, \) we obtain from (4.40)

\[ \langle r \rangle^{\eta/2} \langle t + r \rangle^{1/p} |u_L(t, r)| \leq r \|u_L\|_1. \]

Similarly to the argument in the case of \( \eta > 0, \) we obtain

\[ |I_-(G)(t, r)| \lesssim \langle r \rangle^{-\eta/2} \|u_L\|^{p_1/p} \|v\|^{p_1} I(t, r), \]

where we put

\[ I(t, r) := \int \int_{\Delta_{r,t}(r)} \frac{y \{ \Psi(\sigma - y, \sigma + y) \}^{p_1 q_2} \{ \log(2 + \sigma + y) \}^{p_1 q_3}}{\langle \sigma + y \rangle^{(p-1)\mu/2} \langle \sigma - y \rangle^{\eta}} \ dy \ d\sigma \]

\[ \lesssim \int_{|r-t|}^{t+r} \int_{r-t}^{\alpha} \frac{(\alpha + \beta) \{ \Psi(\beta, \alpha) \}^{p_1 q_2} \{ \log(2 + \alpha) \}^{p_1 q_3}}{\langle \alpha + \beta \rangle^{(p-1)\mu/2} \langle \alpha \rangle^{\eta_2} \langle \beta \rangle^{\eta_3}} \ d\beta \ d\alpha. \] (4.47)

We shall show

\[ I(t, r) \lesssim r \{ \log(2 + t + r) \}^{q_3} \langle t + r \rangle^{1+\eta} D_2(T)^{p_1/p}. \] (4.48)

First, suppose \( r/2 \geq t. \) Since \( \eta > -1 \) for \( p > 1, \) we have \( \eta_2 = p_1 \eta + p \geq 0. \)

Besides, we have \( \eta_3 \geq 0. \) Therefore, we get from (4.37), (4.39), and Lemma 3.4

\[ I(t, r) \lesssim \frac{r \{ \log(2 + t + r) \}^{p_1 q_3}}{\langle r - t \rangle^{2+(1+p_1)\eta_2+\eta_3}} \int_{r-t}^{t+r} \int_{r-t}^{\alpha} \{ \Psi(\beta, \alpha) \}^{p_1 q_2} d\beta d\alpha \]

\[ \lesssim \frac{r t^2 \{ \log(2 + t + r) \}^{p_1 q_3}}{\langle r - t \rangle^{2+(1+p_1)\eta_2+\eta_3}} \]

\[ \lesssim \frac{r \langle t \rangle^2 \{ \log(2 + t + r) \}^{q_3}}{\langle t + r \rangle^{2+(1+p_1)\eta_2+\eta_3}} \{ \log(2 + t + r) \}^{(p_1-1)q_3} \]

\[ \lesssim \frac{r \{ \log(2 + t + r) \}^{q_3}}{\langle t + r \rangle^{1+\eta}} \langle t \rangle^{1-p_1 \eta - \eta_3} \{ \log(2 + t) \}^{(p_1-1)q_3}, \]

because \( (r + t)/3 \leq r - t \) for \( r/2 \geq t, \) and \( 1 + p_1 \eta + \eta_3 > 0 \) which follows from the equality \( 1 + p_\nu = (p - 1)((\mu/2 + 1)p - 1). \) Then, from (4.38), we obtain (4.48).
Next, suppose \( t \geq r/2 \). We note that \( \eta \leq 0 \) leads to \( (p - 1)\mu/2 \leq 1 \). Then, we get from (4.47)

\[
I(t, r) \lesssim \int_{|r-t|}^{t+r} \int_{-\alpha}^{\alpha} \frac{\langle \alpha \rangle^{1-(p-1)\mu/2} \{\Psi(\beta, \alpha)\}_{\eta_2} \{\log(2 + \alpha)\}_{\eta_3}}{\langle \alpha \rangle^{\eta_2} \langle \beta \rangle^{\eta_3}} d\beta d\alpha.
\]

Since \( \{\Psi(\beta, \alpha)\}_{\eta_2} / \langle \beta \rangle^{\eta_3} \) is an even function of \( \beta \), we obtain

\[
I(t, r) \lesssim \{\log(2 + t + r)\}_{\eta_3} \int_{|r-t|}^{t+r} \frac{1}{\langle \alpha \rangle^{1+(1+p_1)\eta_1}} \int_0^\alpha \frac{\{\Psi(\beta, \alpha)\}_{\eta_2}}{\langle \beta \rangle^{\eta_3}} d\beta d\alpha,
\]

by (4.39). When \( p_1 = p \), we have \( \eta_3 = 0 \), so that Lemma 3.3 can be applied. On the other hand, when \( p_1 = 1 \), we have \( 0 < \eta_3 < 1 \) and use the following inequality

\[
(4.49) \quad \int_0^\alpha \log \left( \frac{1 + \alpha}{1 + \beta} \right)(1 + \beta)^{-\eta_3} d\beta \lesssim (1 + \alpha)^{1-\eta_3}, \quad \alpha \geq 0, \quad 0 < \eta_3 < 1,
\]

which is verified by the integration by parts. Then we get from (2.10) and (4.49)

\[
(4.50) \quad \int_0^\alpha \frac{\{\Psi(\beta, \alpha)\}_{\eta_2}}{\langle \beta \rangle^{\eta_3}} d\beta \lesssim \langle \alpha \rangle^{1-\eta_3}.
\]

Hence, recalling (4.38), we get from (4.50) and Lemma 3.1

\[
I(t, r) \lesssim \{\log(2 + t + r)\}_{\eta_3} \int_{|r-t|}^{t+r} \frac{1}{\langle \alpha \rangle^{1+(1+p_1)\eta_1-\eta_3}} d\alpha
\]

\[
\lesssim \{\log(2 + t + r)\}_{\eta_3} \langle t + r \rangle^{1-p_1\eta-\eta_3} \int_{|r-t|}^{t+r} \frac{1}{\langle \alpha \rangle^{1+\eta}} d\alpha
\]

\[
\lesssim \frac{r \{\Psi(t-r, t+r)\}_{\eta_2} \{\log(2 + t + r)\}_{\eta_3}}{\langle t + r \rangle^{1+\eta}} (1 + t)^{\eta_3} \langle \beta \rangle^{\gamma_3(p+3+p_1)/(2p)}
\]

\[
\lesssim \frac{r \{\Psi(t-r, t+r)\}_{\eta_2} \{\log(2 + t + r)\}_{\eta_3}}{\langle t + r \rangle^{1+\eta}} D_2(T)^{p_{1/p}}.
\]

Therefore we obtain (4.38). This completes the proof. □

End of the proof of Theorem 2.3 with \( \nu \geq 1 \). Let \( Y \) be the linear space defined by

\[
Y = \{v(t, r) \in C([0, T] \times (0, \infty)) ; \|v\|_2 < \infty \}.
\]

We shall construct a local solution of integral equation (1.30) in the Banach space \( (Y, \| \cdot \|_2) \). We define the sequence of functions \( \{v_n\} \) by

\[
v_1 = 0, \quad v_{n+1} = I_- (|\varepsilon u_L + v_n|^{p_{1/p-1}}) \quad (n = 1, 2, 3, \ldots).
\]

We set

\[
M_0 = 2^{p-1} \tilde{C}_1 \tilde{C}_0^p,
\]

\[
\tilde{C}_3 = (2^p)^p p \max \{\tilde{C}_2 M_0^{-1}, (\tilde{C}_1 \tilde{C}_3^{-1})^p\},
\]

\[
\tilde{C}_2 = \tilde{C}_1 \tilde{C}_0^p.
\]
where $\tilde{C}_i$ ($0 \leq i \leq 2$) are positive constants given in Lemma 4.1, Lemma 4.4 and Lemma 4.5. Then, analogously to the proof of Theorem 2.1 in [17] and Theorem 2.2 in [13], we see that $\{v_n\}$ is a Cauchy sequence in $Y$ provided that the inequality

$$
(4.51) \quad \tilde{C}_3 \epsilon^{p(p-1)} D_2(T) \leq 1
$$

holds, where $D_2(T)$ is defined in (4.35). Since $Y$ is complete, there exists a function $v \in Y$ such that $v_n$ converges to $v$ in $Y$. Therefore $v$ satisfies (4.30).

Note that (2.11) follows from (4.51). We shall show this fact only in the case of $1 < p < p_S(3 + \mu)$ and $p \nu = 1$, since the other cases can be proved similarly. By definition of $b(\epsilon)$ in (2.12), we know that $b(\epsilon)$ is decreasing in $\epsilon$ and $\lim_{\epsilon \to 0+0} b(\epsilon) = \infty$. Let us fix $\epsilon_0 > 0$ as

$$
(4.52) \quad 1 < \tilde{C}_4 b(\epsilon_0),
$$

where $\tilde{C}_4 := \min\{2^{-1}, (2\tilde{C}_3^{2/\gamma_S(p,3+\mu)})^{-1}\}$. For $0 < \epsilon \leq \epsilon_0$, we take $T$ to satisfy

$$
(4.53) \quad 1 \leq T < \tilde{C}_4 b(\epsilon),
$$

so that $1+T \leq 2T \leq 2\tilde{C}_4 b(\epsilon)$. Then, since $q_3 = 1$ by $p \nu = 1$, $\tilde{C}_3(2\tilde{C}_4)^{\gamma_S(p,3+\mu)/2} \leq 1$, and $2\tilde{C}_4 \leq 1$, it follows that

$$
\tilde{C}_3 \epsilon^{p(p-1)} D_2(T) \leq \tilde{C}_3 \epsilon^{p(p-1)} (2T)^{\gamma_S(p,3+\mu)/2} \{\log(1+2T)\}^{p-1}
$$

$$
\leq \tilde{C}_3 (2\tilde{C}_4)^{\gamma_S(p,3+\mu)/2} \epsilon^{p(p-1)} b(\epsilon)^{\gamma_S(p,3+\mu)/2} \{\log(1+2\tilde{C}_4 b(\epsilon))\}^{p-1}
$$

$$
\leq \epsilon^{p(p-1)} b(\epsilon)^{\gamma_S(p,3+\mu)/2} \{\log(1+b(\epsilon))\}^{p-1} = 1,
$$

by (2.12). Hence, if we assume (4.52) and (4.53), then (4.51) holds. Therefore (2.11) in the case is obtained for $0 < \epsilon \leq \epsilon_0$. This completes the proof.

5. Upper bounds of the lifespan

Let $u$ denote the solution of the problem (2.4) in what follows. When $\varphi \equiv 0$ and $\psi \geq 0$, it follows from (2.4), (2.5) and (2.3) that

$$
(5.1) \quad u(t, r) \gtrsim \epsilon u_L(t, r) + \widetilde{I}_-(|u|^p/y^{p-1})(t, r),
$$

$$
(5.2) \quad u_L(t, r) \gtrsim \widetilde{J}_-(\psi)(t, r)
$$

hold for $t, r > 0$, where we put

$$
(5.3) \quad \widetilde{I}_-(F)(t, r) = \int_{\Delta_-(t, r)} \frac{(-t + \sigma + r + y)^\mu}{(r)^{\mu/2}(y)^{\mu/2}} F(\sigma, y) dy d\sigma,
$$

$$
(5.4) \quad \widetilde{J}_-(\psi)(t, r) = \int_{|t-r|}^{t+r} \frac{r - t + y)^\mu}{(r)^{\mu/2}(y)^{\mu/2}} \psi(y) dy.
$$

Our first step is to obtain basic lower bounds of $u_L$ defined by (2.5).
Lemma 5.1. Assume that (2.13) holds. Then there exists \( M = M(\mu, \kappa) > 0 \) such that
\[
u(t, r) \geq \frac{M}{r^{\mu/2}(t - r)^\nu} \quad \text{for} \quad t \geq r + 1 \quad \text{and} \quad t \leq 2r.
\]
Here \( \nu \) is defined by (2.6), i.e., \( \nu = \kappa - (\mu/2) - 1 \).

Proof. Let \( t \geq r + 1 \) and \( t \leq 2r \). By (2.13), (5.2) and (5.4) we have
\[
u(t, r) \geq \int_{t-r}^{t+r} \frac{\langle t - r + y \rangle^\mu}{\langle r \rangle^{\mu/2} \langle y \rangle^{\mu/2} (1 + y)^\kappa}) \frac{1}{1 + y^\kappa} dy 
\geq \frac{1}{\langle r \rangle^{\mu/2}} \int_{t-r}^{t+r} \frac{(y - t + r)^\mu}{y^{\kappa + (\mu/2)}} dy,
\]
because \( y \geq t - r \geq 1 \). Applying Lemma 3.6, we get
\[
u(t, r) \geq \frac{1}{\langle r \rangle^{\mu/2} (t - r)^\nu} \left( 1 - \frac{t - r}{t + r} \right)^{\mu + 1},
\]
which implies the desired estimate, because \( r \geq 1 \) and \( (t - r)/(t + r) \leq 1/3 \) for \( t \geq r + 1, \ t \leq 2r \). This completes the proof. \( \square \)

Our next step is to derive iterative lower bounds of the solution to (2.4).

Lemma 5.2. For \( T_0 > 1 \) we set
\[
\Sigma := \left\{ (t, r) \in [0, \infty) \times [0, \infty); \ t \geq r + T_0, \ t \leq 2r \right\}.
\]
Let \( u(t, r) \) be the continuous solution of (2.4) and \( a, b \geq 0, M_1 > 0 \). If \( u_L(t, r) \geq 0 \) and
\[
u(t, r) = \frac{M_1(t - r - T_0)^a}{r^{\mu/2}(t - r)^b} \quad \text{for} \quad (t, r) \in \Sigma,
\]
then there exists \( C > 0 \), which is independent of \( a, b \) and \( M_1 \), such that
\[
u(t, r) \geq \frac{CM_1^p(t - r - T_0)^{pa+2}}{(pa + 2)^{2p^{\mu/2}r^{\mu/2}(t - r)^{pb+(p-1)(\mu/2)+1}}} \quad \text{for} \quad (t, r) \in \Sigma.
\]

Proof. For \( (t, r) \in \Sigma \), we put
\[
Q := \left\{ (\sigma, y) \in [0, \infty) \times [0, \infty); \ t - r \leq y, \ \sigma + y \leq 3(t - r), \ T_0 \leq \sigma - y \leq t - r \right\}.
\]
In the following, let \( (t, r) \in \Sigma \). Since \( Q \subset \Delta_-(t, r) \) and \( Q \subset \Sigma \), we have from (5.1), (5.3) and (5.5)
\[
u(t, r) \geq \iint_Q \frac{(t - \sigma + r + y)^\mu}{r^{\mu/2} y^{\mu/2}} \frac{|u(\sigma, y)|^p}{y^{p-1}} dyd\sigma 
\geq \frac{CM_1^p}{\langle r \rangle^{\mu/2}} \iint_Q \frac{(t - \sigma + r + y)^\mu}{(y)^{p+1/2+p-1} \sigma - y} dyd\sigma.
\]
Changing the variables by $\alpha = \sigma + y$, $\beta = \sigma - y$, since $y \geq t - r \geq T_0 > 1$ for $(\sigma, y) \in Q$, we get

$$u(t, r) \gtrsim \frac{M^p_1}{(r)^{\frac{\mu}{2}}} \int_{T_0}^{t-r} d\beta \int_{2(t-r)+\beta}^{3(t-r)} \frac{(\alpha - t + r)^{p\alpha}(\beta - T_0)^{p\alpha}}{(\alpha - \beta)^{(p+1)^{\frac{\mu}{2}} + p - 1}\beta^b} d\alpha$$

$$\geq \frac{M^p_1}{(r)^{\frac{\mu}{2}}(t-r)^{b_0 + (p-1)(\frac{\mu}{2} + 1)}} \int_{T_0}^{t-r} (\beta - T_0)^{p\alpha}(t - r - \beta)d\beta$$

$$= \frac{M^p_1(t - r - T_0)^{p\alpha + 2}}{(pa + 1)(pa + 2)(r)^{\frac{\mu}{2}}(t - r)^{b_0 + (p-1)(\frac{\mu}{2} + 1)}}$$

by the integration by parts. Thus we get the desired estimate, because $r \geq T_0 > 1$. This completes the proof. \(\square\)

**End of the proof of Theorem 2.4** We divide the argument into three cases.

(i) $1 < p < p_F(\kappa)$.

From (5.1) and Lemma 5.1

$$u(t, r) \geq \frac{M\varepsilon(t - r - T_0)^{\frac{\mu}{2} + 1}}{r^{\frac{\mu}{2}}(t - r)^{\kappa}}$$

for $(t, r) \in \Sigma$.

Therefore, by Lemma 5.2, we obtain

$$u(t, r) \geq \frac{C_n(t - r - T_0)^{a_n}}{r^{\frac{\mu}{2}}(t - r)^{b_n}}$$

for $(t, r) \in \Sigma$, $n = 0, 1, \ldots$, \(5.6\)

where we put

$$a_{n+1} = pa_n + 2, \quad a_0 = \frac{\mu}{2} + 1,$$

$$b_{n+1} = pb_n + \left(\frac{\mu}{2} + 1\right)(p - 1), \quad b_0 = \kappa,$$

$$C_{n+1} \geq \frac{CC_n^p}{(pa_n + 2)^2}, \quad C_0 = M\varepsilon,$$

where $C$ is the number from Lemma 5.2. It is easy to see that

$$a_n = \left(\frac{2}{p-1} + \frac{\mu}{2} + 1\right)p^n - \frac{2}{p-1}, \quad b_n = (\kappa + \mu/2 + 1)p^n - (\mu/2) - 1,$$

and hence there exists a constant $0 < D < 1$ such that

$$C_{n+1} \geq \frac{CC_n^p}{(a_{n+1})^2} \geq \frac{DC_n^p}{p^{2n}}.$$
Then, we get

\[
\log C_n \geq \log D \sum_{j=0}^{n-1} p^j - (2 \log p) \sum_{j=1}^{n-1} j p^{n-j-1} + p^n \log C_0
\]

\[
\geq p^n \left\{ \frac{\log D}{p - 1} - (2 \log p) \sum_{j=1}^{\infty} j p^{-j-1} + \log(M \varepsilon) \right\}
\]

\[
\geq p^n \log(\varepsilon E)
\]

with a suitable positive constant \(E\). Therefore, from (5.6) we get

\[
(5.7) \quad u(t, r) \geq \frac{(t - r - T_0)^{-2/(p-1)}(t - r)^{\mu/2+1}}{r^{\mu/2}} \exp(p^n \log J(t, r))
\]

for \((t, r) \in \Sigma\), where we set

\[
J(t, r) = \varepsilon E(t - r - T_0)^{2/(p-1)+\mu/2+1}(t - r)^{-\kappa-(\mu/2)-1}.
\]

Let \((t, r) = (\tau, \tau/2)\) and \(\tau \geq 4T_0\). Then we get \(t - r - T_0 \geq \tau/4\) and \((t, r) \in \Sigma\), so that \(J(\tau, \tau/2) \geq \varepsilon E_1 \gamma_F(p, \kappa)/(p-1)\), where \(\gamma_F(p, \kappa) = 2 - (p-1)\kappa\) and we put \(E_1 = 2^{\kappa-4/(p-1)-(\mu/2)-1} E\). Now we specify

\[
\tau = (2^{-1} E_1 \varepsilon)^{-(p-1)/\gamma_F(p, \kappa)}.
\]

Since \(\gamma_F(p, \kappa) > 0\) for \(p < p_F(\kappa)\), we may assume \(\tau \geq 4T_0\), by choosing \(\varepsilon\) suitably small. Thus we get \(J(\tau, \tau/2) \geq 2\). Then, we see from (5.7) that \(u(\tau, \tau/2) \to \infty\) as \(n \to \infty\). This means that the lifespan \(T(\varepsilon) \leq \tau\), which yields the desired estimate.

(ii) \(1 < p < p_3(3 + \mu)\) and \(p \nu = 1\).

Since \(p \nu = 1\), we get from (5.1) and Lemma 5.1

\[
(5.8) \quad u(t, r) \geq \frac{M \varepsilon}{r^{\mu/2}(t - r)^{1/p}} \quad \text{for } r + 1 \leq t \leq 2r.
\]

Then, similarly to the argument in the proof of Lemma 5.2 we obtain

\[
u(t, r) \geq \frac{\varepsilon \rho}{(r \rho)^{\mu/2}} \int_1^{t-r} \int_{2(t-r) + \beta}^{3(t-r)} \frac{(\alpha - t + r)^\mu}{(\alpha - \beta)^{(p+1)\mu/2+p-1} \beta} d\alpha d\beta
\]

\[
\geq \frac{\varepsilon \rho}{r^{\mu/2}(t - r)^{(p+1)/(p-1)}} \int_1^{t-r} \frac{t - r - \beta}{\beta} d\beta \quad \text{for } r + 1 \leq t \leq 2r.
\]

Let \((t, r) \in \Sigma\) and \(T_0 > 2\) in the following. Then, we have

\[
\int_1^{t-r} \frac{t - r - \beta}{\beta} d\beta = \int_1^{t-r} \log \beta d\beta \geq \int_{T_0/2}^{t-r} \log \beta d\beta
\]

\[
\geq (t - r - T_0/2) \log(T_0/2) \geq \frac{t - r}{2} \log(T_0/2).
\]
Hence, we get from (5.9) and (5.10)
\[ u(t, r) \geq \frac{\bar{M} \log(T_0/2)\varepsilon^p(t - r - T_0)}{r^{\mu/2}(t - r)^{(\mu/2+1)(p-1)}} \text{ for } (t, r) \in \Sigma, \]
where $\bar{M}$ is a suitable positive constant. Therefore, by Lemma 5.2, we obtain (5.6) with
\[ a_0 = 1, \ b_0 = (\mu/2 + 1)(p - 1) \text{ and } C_0 = \bar{M} \log(T_0/2)\varepsilon^p, \]
Then, similar computation as in the previous case leads to
\[ u(t, r) \geq \frac{(t - r - T_0)^{-2/(p-1)}}{r^{\mu/2}} \exp(p \log \tilde{J}(t, r)), \quad (t, r) \in \Sigma, \]
where we set
\[ \tilde{J}(t, r) = E_1 \varepsilon^p \log(T_0/2)(t - r - T_0)^{(p+1)/(p-1)}(t - r)^{-(\mu/2+1)p} \]
with a suitable positive constant $E_1$.

Let $(t, r) = (\tau, \tau/2)$ and $\tau = 4T_0$. Then we get $t - r - T_0 = 0$ and $(t, r) \in \Sigma$. Therefore, we find a constant $E_1 \in (0, 1)$, which is independent of $\varepsilon$ and $T_0$, such that
\[ (5.11) \quad \tilde{J}(\tau, \tau/2) \geq 2E_1 \varepsilon^p (T_0/4)^{\gamma(p, 3+\mu)/(2(p-1))} \log(T_0/2), \]
where $\gamma(p, n)$ is defined by (1.3). Now we define $T_0 = T_0(\varepsilon)$ by
\[ T_0 = 4E_1^{2(p-1)/\gamma(p, 3+\mu)} b(\varepsilon). \]
Recalling the fact that $b(\varepsilon)$ is monotonically decreasing in $\varepsilon$ and $\lim_{\varepsilon \to 0^+} b(\varepsilon) = \infty$, we may assume that $E_1^{2(p-1)/\gamma(p, 3+\mu)} b(\varepsilon) \geq 1$, by choosing $\varepsilon$ sufficiently small. Then, we see that $T_0 > 2$ and
\[ T_0/2 = 2E_1^{2(p-1)/\gamma(p, 3+\mu)} b(\varepsilon) \geq 1 + b(\varepsilon), \]
because $\gamma(p, 3 + \mu) > 0$ when $1 < p < p_S(3 + \mu)$. Therefore, we get from (5.11) and (2.12)
\[ \tilde{J}(\tau, \tau/2) \geq 2E_1 \varepsilon^p b(\varepsilon)^{\gamma_S(p, 3+\mu)/(2(p-1))] \log(1 + b(\varepsilon)) = 2. \]
Hence, as in the previous case, we obtain the desired estimate.

(iii) $p = p_S(3 + \mu)$ and $p\nu = 1$.

For $\rho > 0$, we define the following quantity:
\[ \langle u \rangle(\rho) = \inf \{ \langle y \rangle^{\mu/2} (\sigma - y)^{\rho} | u(\sigma, y); 0 \leq \sigma \leq 2y, \ \sigma - y \geq \rho \}. \]
Since $p\nu = 1$ and $\nu = \eta$ by $p = p_S(3 + \mu)$, it follows from (5.8) that
\[ \langle u \rangle(\eta) \geq C_1 \varepsilon, \quad (\eta \geq 1). \]
As is shown in Section 4 in [15], it holds that
\[ \langle u \rangle(\eta) \geq C_2 \int_{\eta}^{y} \left( 1 - \frac{\xi}{y} \right) \frac{[\langle u \rangle(\xi)]^{p}}{\xi^{pn}} d\xi, \quad (\eta \geq 1). \]
Therefore, using Lemma 3.7 with $\alpha = 1, \beta = 0, \theta = p\eta = 1$, we get

$$T_*(\varepsilon) \leq \exp(C\varepsilon^{-(p-1)}).$$

Since $T(\varepsilon) \leq T_*(\varepsilon)$, we obtain the desired estimate. Summing up the conclusions in (i), (ii) and (iii), we finish the proof of Theorem 2.4.

References

[1] F. Asakura, Existence of a global solution to a semilinear wave equation with slowly decreasing initial data in three space dimensions, Comm. Partial Differential Equations 11 (1986), no. 13, 1459–1487.
[2] R. Agemi, H. Takamura, The lifespan of classical solutions to nonlinear wave equations in two space dimensions, Hokkaido Math. J. 21 (1992), no. 3, 517–542.
[3] M. D’Abbicco, S. Lucente, M. Reissig, A shift in the Strauss exponent for semilinear wave equations with a not effective damping, J. Differential Equations 259 (2015), no. 10, 5040–5073.
[4] P. D’Ancona, V. Georgiev, H. Kubo, Weighted decay estimates for the wave equation, J. Differential Equations 177 (2001), no. 1, 146–208.
[5] V. Georgiev, Semilinear hyperbolic equations, With a preface by Y. Shibata, Second edition. MSJ Memoirs, 7. Mathematical Society of Japan, Tokyo, 2005.
[6] V. Georgiev, Ch. Heiming, H. Kubo, Supercritical semilinear wave equation with non-negative potential, Comm. Partial Differential Equations 26 (2001), no. 11-12, 2267–2303.
[7] V. Georgiev, H. Kubo, K. Wakasa, Critical exponent for nonlinear damped wave equations with non-negative potential in 3D, J. Differential Equations 267 (2019), no. 5, 3271–3288.
[8] V. Georgiev, H. Lindblad, Ch. Sogge, Weighted Strichartz estimates and global existence for semilinear wave equations, Amer. J. Math. 119 (1997), no. 6, 1291–1319.
[9] R.T. Glassey, Finite-time blow-up for solutions of nonlinear wave equations, Math. Z. 177 (1981), no. 3, 323–340.
[10] M. Ikeda, M. Sobajima, Life-span of blowup solutions to semilinear wave equation with space-dependent critical damping, Funkcial. Ekvac. 64 (2021), no. 2, 137–162.
[11] M. Ikeda, M. Sobajima, Life-span of solutions to semilinear wave equation with time-dependent critical damping for specially localized initial data, Math. Ann. 372 (2018), no. 3-4, 1017–1040.
[12] R. Ikehata, G. Todorova, B. Yordanov, Critical exponent for semilinear wave equations with space-dependent potential, Funkcial. Ekvac. 52 (2009), no. 3, 411–435.
[13] T. Imai, M. Kato, H. Takamura, K. Wakasa, The lifespan of solutions of semilinear wave equations with the scale-invariant damping in two space dimensions, J. Differential Equations 269 (2020), no. 10, 8387–8424.
[14] F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, Manuscripta Math. 28 (1979), no. 1-3, 235–268.
[15] M. Kato, H. Kubo, On the Cauchy problem for the nonlinear wave equation with damping and potential, submitted to the proceedings of the 13th ISAAC Congress.
[16] M. Kato, M. Sakuraba, Global existence and blow-up for semilinear damped wave equations in three space dimensions, Nonlinear Anal. 182 (2019), 209–225.
[17] M. Kato, H. Takamura, K. Wakasa, The lifespan of solutions of semilinear wave equations with the scale-invariant damping in one space dimension, Differential Integral Equations 32 (2019), no. 11-12, 659–678.
[18] P. Karageorgis, Existence and blow up of small-amplitude nonlinear waves with a sign-changing potential, J. Differential Equations, 219 (2005), 259–305.
[19] H. Kubo, Slowly decaying solutions for semilinear wave equations in odd space dimensions, Nonlinear Anal. 28 (1997), no. 2, 327–357.
[20] H. Kubo, M. Ohta, On the global behavior of classical solutions to coupled systems of semilinear wave equations, New trends in the theory of hyperbolic equations, 113–211, Oper. Theory Adv. Appl., 159, Adv. Partial Differ. Equ. (Basel), Birkhäuser, Basel, 2005.
[21] H. Kubo, M. Ohta, Blowup for systems of semilinear wave equations in two space dimensions, Hokkaido Math. J. 35 (2006), no. 3, 697–717.
[22] K. Kubota, Existence of a global solution to a semi-linear wave equation with initial data of noncompact support in low space dimensions. Hokkaido Math. J. 22 (1993), no. 2, 123–180.
[23] N.A. Lai, Weighted $L^2$-$L^2$ estimate for wave equation in $\mathbb{R}^3$ and its applications, The role of metrics in the theory of partial differential equations, 269–279, Adv. Stud. Pure Math., 85, Math. Soc. Japan, Tokyo, 2020.
[24] X. Li, Critical exponent for semilinear wave equation with critical potential, NoDEA Nonlinear Differential Equations Appl. 20 (2013), no. 3, 1379–1391.
[25] W.A. Strauss, Nonlinear wave equations, CBMS Regional Conference Series in Mathematics, vol. 73, American Math. Soc., Providence, RI, 1989.
[26] W.A. Strauss, K. Tsutaya, Existence and blow up of small amplitude nonlinear waves with a negative potential, Discrete Contin. Dynam. Systems 3 (1997), no. 2, 175–188.
[27] G. Todorova, B Yordanov, Critical exponent for a nonlinear wave equation with damping, J. Differential Equations 174 (2001), no. 2, 464–489.
[28] K. Tsutaya, Global existence and the life span of solutions of semilinear wave equations with data of noncompact support in three space dimensions, Funkcial. Ekvac. 37 (1994), no. 1, 1–18.
[29] B. Yordanov, Q. Zhang, Finite-time blowup for wave equations with a potential, SIAM J. Math. Anal. 36 (2005), no. 5, 1426–1433.

M. Kato
Faculty of Science and Engineering, Muroran Institute of Technology, Muroran 050-8585, Japan
Email address: mkato@mmm.muroran-it.ac.jp

H. Kubo
Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060-0810, Japan
Email address: kubo@math.sci.hokudai.ac.jp