THREE MANIFOLDS AND GRAPH INVARIANTS

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ABSTRACT

We show how the Turaev–Viro invariant can be understood within the framework of Chern–Simons theory with gauge group SU(2). We also describe a new invariant for certain class of graphs by interpreting the triangulation of a manifold as a graph consisting of crossings and vertices with three lines. We further show, for $S^3$ and $RP^3$, that the Turaev-Viro invariant is the square of the absolute value of their respective partition functions in SU(2) Chern–Simons theory and give a method of evaluating the later in a closed form for lens spaces $L_{p,1}$.
Following the discovery of the Jones invariants in knot theory there has been renewed interest in trying to find new invariants associated with knots, graphs and 3–manifolds. A significant step was taken by Witten when he was able to interpret the Jones invariants by using ideas from quantum field theory [1]. Recently Turaev and Viro introduced [2] a new invariant of $M_3$ ($M_d$ denotes a d–manifold). This invariant is combinatorial in nature and is defined as a state sum computed on a triangulation of the manifold and is based on the quantum 6–j symbols associated with the quantised universal enveloping algebra, $U_q(SL(2, C))$. Here $q = \exp(i \frac{2\pi}{k+2})$, $k > 0$ is a complex root of unity. The Turaev–Viro (TV) invariant is also of interest for the following reason. If one regards $j_i$’s of the classical 6–j symbol as the lengths of the sides (colors) of a tetrahedron, then in the large $j$ limit the positive frequency part of the 6–j symbol becomes $e^{iS_R}$ where the Regge action $S_R$ is the discretised version of the Euclidean Einstein–Hilbert action $\int d^3x \sqrt{g} R$ for 3–d gravity [3]. However, the sum over the coloring of the tetrahedra is divergent in the classical case but the quantum 6–j symbols provide a natural cutoff $j \leq \frac{k}{2}$ and regulate the divergent behaviour [4] [5]. Thus the TV invariant is seen to be closely related to the partition function of the 3-d gravity action.

A few remarks on the relation between the TV invariant $I_{TV}$ and other known invariants of $M_3$ are also in order. For orientable $M_3$ of the type $M_2 \times [0, 1]$ where $M_2$ is closed, the authors of [4] mention that they have examined the partition functions of the TV model for lower genus $M_2$ and found them to be equal to the absolute–value–square of the partition function of the SU(2) Chern–Simons theory. They also mention that the equality of these two partition functions has been proved by Turaev in a rather different approach.

In this paper we will show how the TV invariant can be understood within the framework of Chern–Simons (CS) theory in which the gauge group is SU(2). We evaluate the TV invariant for the manifolds $S^3$ and $RP^3$ and show that for
these manifolds $I_{TV}(M) = I_W^2(M)$, where $I_W(M)$ is the invariant for the manifold $M$ obtained by calculating the absolute value of the partition function for a CS gauge theory with gauge group SU(2). We will also attempt to understand the TV invariant in terms of graphs and describe how one obtains a new invariant for certain class of graphs using these ideas. In our discussions we will always take $M_3$ to be a compact orientable 3–manifold with no boundary.

We start by assuming that a given $M_3$ has been triangulated and we consider the graph $G$ associated with the triangulation. For example if two tetrahedra $T_1, T_2$ are glued together along the face $F$ as shown in figure 1 then we regard $T_1, T_2$ as graphs rather than as 3–dimensional objects. The important observation we make regarding $G$ is that $G$ is not an arbitrary graph; it is obtained by gluing together a collection of tetrahedra $\{T_i\}$, represented symbolically as

$$G = \bigcup_{\{g_{ij}\}} T_i$$

(1)

where $\{g_{ij}\}$ encodes gluing information. Because of this if we want to evaluate the invariant associated with $G$ by evaluating $G$ with respect to a CS measure we have to specify what exactly the gluing process means within this framework. First observe that a given tetrahedron can be regarded as a collection of Wilson lines, joined together at each vertex by an appropriate invariant coupling factors as discussed by Witten [6] . Each line may carry a different representation of SU(2). Gluing two tetrahedra along face $F$ we will take to mean that the faces are connected by “walls” all carrying the trivial representation of SU(2) and the representations on the common glued face are summed over. By repeatedly using the factorisation technique of Witten [1] [6] , and observing that if a given tetrahedron is enclosed in a ball with surface $S^2$ which intersects these trivial representation–walls then the Hilbert space of $S^2$ is one dimensional, it is easy to see that

$$I(G) = \sum_{rep} \prod T_i$$

(2)

where $I(G)$ is the graph invariant and $T_i$ is an object associated with the tetrahe-
dron $T_i$ defined by its six sides each of which carries a representation of SU(2). At this point we would like to point out the factorised nature of the invariant $I(G)$ over the constituent tetrahedra $T_i$. We will see shortly that there is a solution for $T_i$, an object associated with the tetrahedron $T_i$ which respects the factorisable property. However this solution will not have the required tetrahedral symmetry and will lead to a modified prescription for the graph invariant. Since $G$ does not have any boundary each “face” of a $T_i$ in $G$ is glued to some other face of a $T_j$ in $G$. Hence all the representations appearing in $\prod_i T_i$ in equation (2) are to be summed over. Note that the graph $G$ is essentially the spine of a triangulated manifold [2] and hence is inherently 3–dimensional. In particular the lines in $G$ never cross and an arbitrary number ($\geq 3$) of them can meet at a vertex.

We will now attempt to understand the TV invariant in terms of planar graphs with crossings and vertices with fixed number of lines, and their invariants. As before we consider the triangulation of $M_3$ as a set of tetrahedra and “gluing instructions”. The set of tetrahedra with a given coloring can be viewed as a set of graphs and the gluing instructions a way of joining the graphs. First, a tetrahedron with the colors $(a, b, c, d, e, f)$ labelling its sides can be represented as a graph in any of the four ways shown in figure 2 where the crossings have a relative phase factor [7]. In this graph each line and region is assigned a color. The colors of a line and its neighbouring two regions as also the colors of the three lines joining at a vertex, which we call a 3–joint, form an unordered triplet. Thus in figure 2 the triplets are $(abc)$, $(ae f)$, $(bdf)$ and $(cef)$. A triplet $(abc)$ is said to be admissible if $|a - b| \leq c \leq (a + b)$. In what follows we will consider only those colorings with admissible triplets.

We implement the gluing instructions in this case as follows. When two tetrahedra with colors $(a, b, c, d, e, f)$ and $(a, b, c, l, m, n)$ are glued along their common face with colors $(a, b, c)$, the graphs representing them are joined together such that the common area and the lines will have the same colors as shown in figure 3. In this way the gluing instructions will lead to planar graphs with crossings and 3–joints only. Since the TV invariant does not have any phase factors to be
associated with crossings we will implement the gluing instructions in such a way that in the resulting graph these phase factors will cancel out. This will restrict us to a certain class of graphs only as described below. It can be easily seen that representing the tetrahedra by crossings and joints and gluing them may result in more than one graph. It is also clear that the graph corresponding to a manifold without boundary will be closed consisting only of closed lines. The characterisation of the graphs obtained by the triangulations of a given 3–manifold as described above is a difficult subject and is under further study.

For future use we now define several quantities in a given graph. First we define a quantity we call character for each given crossing. It is a sum total of the colors of the four regions around the crossing each with an appropriate sign which is determined to be + or − according to whether the region with that color comes to the right or left, respectively, as one moves towards the crossing along the over–line. Thus the characters of the crossings in figures 2B and 2C are \((b + e - c - f)\) and \((c + f - b - e)\) respectively. Note that switching one crossing into another (allowed by tetrahedral symmetry) reverses the sign of the character. We call the sum of the characters of all the crossings in a graph as the character of the graph, assigning 3–joints a zero character. The character of the graph is closely related to the sum total of the phase factors associated with each crossings. In particular these phase factors will cancel out whenever the character of the graph vanishes. Since the TV invariant does not have any phase factors associated with the crossings it is necessary for the graph to have a vanishing character. Henceforth we will consider only such graphs.

We also define \(C_3\) and \(C_4\) to be the total number of 3–joints and crossings respectively; \(L\) the total number of lines; \(l_i\) the total number of pieces one obtains by cutting a given line \(i\) at all crossings and 3–joints; \(\lambda\) the sum of \(l_i\)’s and \(R\) the total number of regions. When a given set of three lines all begin and end in 3–joints we call them the “Baryon Orbits” (following [6] ) and we define \(B\) to be the total number of baryon orbits. The significance of these quantities is the following. Let \(N_0\), \(N_1\), \(N_2\) and \(N_3\) denote the number of vertices, lines, faces and tetrahedra
respectively in the triangulation of the manifold that gave rise to the given graph. Then \(N_3 = C_3 + C_4\), \(N_2 = C_3 + \lambda - B\), \(N_1 = L + R\) and \(N_0 = \chi + R + C_4 + L + B - \lambda\), where \(\chi\) is the Euler characteristic of the 3–manifold. Note that \(\chi\) vanishes for \(M_3\), a compact orientable 3–manifold with no boundary that we are considering here.

Now given such a graph we would like to associate an object \(T\) for each 3–joint and crossing (along with a phase factor \([7]\)) and define a quantity as in (2) which will be invariant under Reidemeister (R–)moves. An invariant for which these properties are true is obtained by identifying for each tetrahedron

\[
T \equiv T_W = \{ \} \quad \text{q}, \tag{3}
\]

where \(\{ \} \text{q}\), denotes the quantum 6–j symbol. However this object has a geometrical shortcoming. It does not have the symmetries of a tetrahedron. There is an object related to \(\{ \} \text{q}\) with the symmetries of a tetrahedron which is none other than the quantum Racah–Wigner coefficient upto some symmetry–preserving phase factors and is given by

\[
T \equiv T_TV = (-1)^{-\frac{1}{2}(a+b+c+d+e+f)} \left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\}^{RW} \tag{4}
\]

for each tetrahedron colored \((a, b, c, d, e, f)\) as in figure 2A. But \(I(G)\) in (2) calculated using this object is not invariant under R–moves. The problem which we consider is this: can we modify the RHS of (2) so that the tetrahedral symmetries are present and it represents an invariant of the graph \(G\) i.e. the R–moves leave \(I(G)\) invariant? This problem can be fixed, while preserving the tetrahedral symmetry, by associating a factor \((-1)^{2j} \frac{S_{0j}}{S_{00}}\) with each line or region colored \(j\) and by requiring the graph to have vanishing character. We then take the sum of this quantity over all possible coloring with admissible triplets only. Thus the quantity

\[
I'_G = \sum_{\text{coloring}} \prod_{\text{lines,regions}} (-1)^{2j} \frac{S_{0j}}{S_{00}} \prod_{\text{crossings,3–joints}} T \tag{5}
\]

satisfies our requirements of tetrahedral symmetry and is invariant under Reidemeister moves. Notice, however, that in this process the original factorisation
property is lost.

In the above analysis we required tetrahedral symmetry for each crossing and 3–joint because they represent actual tetrahedra in the triangulation of the manifold which was the origin of our graph. Invariance under Reidemeister moves are necessary because any graph is defined only upto these moves. However since the graphs we consider are obtained from (or can be interpreted as) the triangulations of a 3–manifolds it follows that $I_G'$ must be the same for any set of graphs corresponding to different triangulations of the same 3–manifold. For the case of 3–manifolds it is known that any two triangulations describe the same manifold if these two are connected by a finite number of $(k,l)$ moves and their inverses where $k + l = 5$ [8]. This set of moves is equivalent to Alexander moves and to Matveev moves and bubble moves [2] [8]. The $(2, 3)$ move corresponds to splitting two tetrahedra $OABC$ and $XABC$ into three tetrahedra $OXAB$, $OXBC$ and $OXCA$ by joining the two vertices $O$ and $X$. In terms of graphs this move corresponds to the Reidemeister–3 move. The $(1, 4)$ move corresponds to obtaining four tetrahedra from one by adding a new vertex inside a tetrahedron and connecting it to its four vertices. In terms of graphs this corresponds to the “bubble move” shown in figure 4. (The inverses of the above moves are obvious).

Thus we require the quantity $I_G'$ to be invariant not only under the Reidemeister moves but also under the bubble moves. However invariance under the bubble move requires that we add a vertex dependent factor and hence, the right candidate is found to be

$$I_G = S_{00}^{2N_0} I_G'$$

(6)

where $N_0 = R + C_4 + L + B - \lambda$ is the total number of vertices as defined before. Thus we obtain $I_G$, an invariant of a given graph with vanishing character. We consider in the following the examples of $S^3$ and $RP^3$ in which we check explicitly that two different triangulations lead to the same result. That $I_G$ is indeed independent of triangulation for any manifold $M_3$ has been proved by Turaev and Viro [2], which can also be seen easily in our approach using the $(k, l)$ moves.
Thus we see that interpreting the triangulation of a manifold as a planar graph with crossings and vertices with fixed number of lines one obtains a new invariant which is defined for all such graphs with vanishing character. This quantity remains invariant under Reidemeister moves and bubble moves for the graphs and gives TV invariant when the graph is viewed as a triangulation of a manifold. This suggests that any graph with vanishing character can be interpreted as a triangulation of 3–manifold and has $I_G$ as its invariant. (However, as mentioned earlier, the full classification of graphs which correspond to the triangulation of a 3–manifold is a difficult subject and is under further study). As can be seen from such an interpretation $I_G$ is invariant not only under Reidemeister moves and bubble moves but also under ‘cutting off the crossings and 3–joints and interconverting them and gluing them back preserving the “gluing instructions” and the character of the graph’ – such a process will in general lead to a different graph. This process of reducing by symmetry transformations the original graph to a set of standard simple objects whose invariants can be evaluated easily is quite akin to the skein relations in calculating the usual graph invariants. To evaluate $I_G$ completely we need one further identity

$$\sum_{a,b} b \quad a \quad c \quad \frac{S_{0a}}{S_{00}} \frac{S_{0b}}{S_{00}} = \frac{1}{S_{00}^2} \frac{S_{0c}}{S_{00}}$$

(7)

where $a$, $b$ and $c$ label the line and its neighbouring regions respectively.

We illustrate the above procedures for a simple case of two tetrahedra glued together as in figure 5a. (Actually, this figure represents one way of triangulating $S^3$). This graph has a vanishing character. Replacing the two tetrahedra (3–joints) by two different crossings we obtain the graph of figure 5b which has a non vanishing character and hence is not admissible. Switching one of the crossings lead to an admissible graph with vanishing character (figure 5c) which, under a Reidemeister move (same as summing over $b_1$), leads to the graph in figure 5d. Note that one could have obtained the graph in figure 5d from that in figure 5a in a single step by using the Reidemeister move for the 3–joint. Thus this example,
as well as all the others that we have tried, indicates that irrespective of how one initially represents the given set of tetrahedra as a graph preserving the gluing instructions one always gets the same result for $I_G$ after a sufficient number of symmetry operations – as one must if $I_G$ were a genuine invariant of a graph with vanishing character (or equivalently of a triangulated manifold).

We now evaluate the TV invariant for $S^3$ and $RP^3$. The 3–sphere $S^3$ can be viewed as the boundary of a 4–simplex with the vertices denoted by 0, 1, 2, 3 and 4. Thus $S^3$ consists of five tetrahedra whose vertices are (0123), (0124), (0134), (0234) and (1234). The graph corresponding to this triangulation is given in figure 6. The graph invariant $I_{S^3}$ (which is the same as TV invariant) for $S^3$ can be evaluated easily using the Reidemeister moves and the result is

$$I_{S^3} = S_{00}^2. \quad (8)$$

The 3–projective space $RP^3$ can be triangulated as shown in figure 7 where the lines labelled the same, and the corresponding faces, are to be identified. The TV invariant for $RP^3$ can be calculated easily and is given by

$$I_{RP^3} = \frac{2}{k+2}(1 + (-1)^k)\sin^2\frac{\pi}{2(k+2)} \quad (9)$$

where $k$ is related to $q$, the root of unity, as defined earlier.

One can obtain another triangulation of $S^3$ and $RP^3$ by first triangulating a lens space $L_{p,q}$ since the manifolds $S^3$ and $RP^3$ are special cases of $L_{p,q}$ with $(p, q) = (1, 0)$ and $(2, 1)$ respectively. The lens space $L_{p,q}$ is obtained as follows [9]. Consider a region of 3–space bounded by two spherical caps meeting in an equatorial circle. Rotate the lower cap onto itself through an angle of $\frac{2\pi q}{p}$ radians and then reflect it about the equatorial plane onto the upper cap. The resulting manifold thus obtained is the lens space $L_{p,q}$. One possible triangulation $L_{p,q}$, which suffices for our purposes, is shown in figure 8 with the lines labelled the same and the corresponding faces identified [10]. In figure 8, $i \in \mathbb{Z}/p$ where $i$ is
the subscript of the label $\beta_i$. Using this triangulation the expression for the TV invariant $\mathcal{I}_{L_{p,q}}$ can be written in terms of $\mathcal{T}$ for each tetrahedron. For $(p, q) = (1, 0)$ and $(2, 1)$, this expression can be evaluated and the results agree with equations (8) and (9), as they should. However, this agreement would not be there if the vertex factor in equation (6) were absent. Though we find it hard to evaluate the expression for $\mathcal{I}_{L_{p,q}}$ in general, we are able to evaluate it for another case, $(p, q) = (3, 1)$. But in this particular instance, with the triangulation for $L_{3,1}$ given as in figure 8, the proof for the invariance of $\mathcal{I}_{L_{p,q}}$ is not valid for reasons given in [2] and hence $\mathcal{I}_{L_{3,1}}$ evaluated as above is not the right answer.

In a different context Danielsson [11] has evaluated $I_W(M)$ for $M = S^3$ and $L_{p,1}$, where $I_W(M)$ is the invariant for the manifold $M$ obtained by calculating the absolute value of the partition function for a CS gauge theory with gauge group SU(2). They can be written as

$$I_W(S^3) = \sqrt{\frac{2}{k+2}} \sin \frac{\pi}{k+2}$$

$$I_W(L_{p,1}) = \frac{2}{k+2} \left| \sum_{n=0}^{k+1} \sin^2 \frac{\pi n}{k+2} \exp(i \frac{\pi p n^2}{2(k+2)}) \right|. \quad (10)$$

Comparing our results for $\mathcal{I}_{S^3}$ and $\mathcal{I}_{R^3}$ with equation (10) after setting $p = 2$ (see below for evaluation of $I_W(L_{p,1})$) we find that

$$\mathcal{I}_M = I_W^2(M) \quad (11)$$

for $M = S^3$ and $R^3$.

The sum in equation (10), denoted by $\frac{1}{4} \sigma_{p,r}$, $r = k + 2$ can be evaluated as follows. (This procedure is due to R. Balasubramanian [12].) First, it can be seen that

$$\sigma_{p,r} = \frac{1}{2} (G(p, 0, 4r) - G(p, 4, 4r)) \quad (12)$$

where $G(a, b, l) = \sum_{n=0}^{l-1} \exp(i \frac{\pi}{4} (an^2 + bn))$. Now we state the following properties
of $G(a, b, l)$ [12]. In the following all the variables are integer valued and $(x, y)$ denotes the greatest common divisor of two integers $x$ and $y$.

1. If $(a, l)$ does not divide $(b, l)$ then $G(a, b, l) = 0$.

2. $G(ca, cb, cl) = cG(a, b, l)$.

Using the properties (1) and (2) we will restrict ourselves to the case where $(a, l) = 1$.

3. Let $\omega$ be an integer such that $2a\omega + b$ is a multiple of $l$. Such an $\omega$ is guaranteed to exist for any $a, b$ and $l$ provided $(2a, l)$ divides $(b, l)$. Since $(a, l) = 1$, this implies that $\omega$ exists if either $l$ is odd or if both $b$ and $l$ are even. Then $G(a, b, l) = \exp(-i\frac{2\pi}{l}a\omega^2)G(a, 0, l)$.

4. If $l$ is even and $b$ is odd then $G(a, b, l) = \exp(-i\frac{2\pi}{l}a\mu^2)G(a, 0, l)$ where $\mu$ is an integer such that $2a\mu + (b - 1)$ is a multiple of $l$.

5. For $l$ odd $|G(a, 0, l)| = \sqrt{l}$.

6. Let $l = 2\lambda$. Then $|G(a, 0, l)|^2 + |G(a, 1, l)|^2 = 2l$ and $|G(a, 0, l)| = 0$ (or $\sqrt{2l}$) if $a\lambda$ is odd (or even).

Using the above properties (1)–(6) of $G(a, b, l)$, $I_W(L_{p,1})$ in (10) can be evaluated for any $p$ and $r$. For example, one can obtain all the values of $I_W(L_{p,1})$ listed in [11] for some specific values of $p$ and $k$.

Moreover, assuming that equation (11) is true for any manifold (see [4]), the TV invariant for the lens space $L_{p,1}$ can also be obtained by the above method.

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FIGURE CAPTIONS

Figure 1: Gluing procedure.

Figure 2: Representation of a tetrahedron by graphs.

Figure 3: Gluing procedure for graphs.

Figure 4: Bubble move.

Figure 5: One example of graph manipulations.

Figure 6: Graph of $S^3$.

Figure 7: Triangulation of $RP^3$.

Figure 8: Triangulation of $L_{p.q}$ (subscripts for $\beta \in \mathbb{Z}/p$).