COMBINATORIAL MINIMAL SURFACES IN PSEUDOMANIFOLDS

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ABSTRACT. We define combinatorial analogues of stable and unstable minimal surfaces in the setting of weighted pseudomanifolds. We prove that, under mild conditions, such combinatorial minimal surfaces always exist. We use a technique, adapted from work of Johnson and Thompson, called thin position. Thin position is defined using orderings of the cells of a pseudomanifold. In addition to defining and finding combinatorial minimal surfaces, from thin orderings, we derive invariants of even-dimensional closed simplicial pseudomanifolds called width and trunk. We study the additivity properties of these invariants under connected sum and prove theorems analogous to theorems in knot theory and 3-manifold theory.

1. INTRODUCTION

Given a Riemannian n-manifold M, a co-dimension 1 submanifold $U \subset M$ is a minimal surface if it is a critical point of the area (i.e. $(n - 1)$-dimensional volume) functional on all compactly supported variations of $U$ [MP Definition 2.1.4]. The quest of discrete differential geometry is to convert definitions and theorems from differential geometry into definitions and results concerning discrete objects, such as simplices and triangulations. Typically, these discrete objects inherit a geometry from an embedding in some (typically low-dimensional) Euclidean space. In this paper, we embark on the related (but different) quest to convert differential geometric ideas into combinatorial analogues. In this paper we adapt ideas of Thompson [T2] and Johnson [J] to produce examples of combinatorial minimal surfaces in weighted pseudomanifolds.

Deferring our definitions until Sections 2 and 3 in this paper we do the following:

- Show how an ordering $O$ of the top-dimensional cells of an n-dimensional pseudomanifold having non-negative weights on the $(n - 1)$ faces can be put into a “local thin position” $O'$. (Definition 3.7)
- Prove that in a locally thin ordering, local minima (of a certain function) are stable combinatorial minimal (hyper)-surfaces and local maxima are either stable or unstable combinatorial minimal surfaces (Theorem 4.1).
- Use thin position to define invariants (called width and trunk) of simplicial pseudomanifolds and study the additivity properties of these invariants under connected sum.
- Pose a list of conjectures and questions.

We begin with background on our motivation and techniques.

1.1. Background. Given a concept in differential geometry, there may be several ways of defining a combinatorial analogue. For instance, given a triangulated surface, a “curve” might be considered to be either a cycle in the 1-skeleton of the triangulation (as in [T2]) or as a normal curve transverse to the triangulation (as in [M]). Similarly, given a triangulation of a 3-manifold, a surface might be considered either as a subcomplex of the triangulation or as a certain kind of normal surface (as in [JR]). Additionally, when deciding what additional criteria should be imposed on a curve or surface in order to be considered a geodesic or minimal surface, certain aspects of the differential theory are selected to be of primary importance. For instance, least weight normal surfaces function much as least area surfaces in differential geometry (see [JR]). Similarly, in [B][BS], the isothermic properties of minimal surfaces are chosen as the key feature to adapt to the discrete
geometric setting. In this paper, we elect to adapt the notion of minimal surface to the combinatorial setting by adapting Pitts’ construction of differential geometric minimal surfaces using sweepouts.

Consider orientable surfaces in a closed orientable 3-manifolds endowed with a Riemannian metric. One view of minimal surfaces is that they are critical points of certain functionals on parameterized families of surfaces; the number of parameters used to define the family should correspond to the index of the minimal surface. In 3-manifold topology, this view of minimal surfaces has been reinterpreted in terms of compressing discs for surfaces (see [P] and [B] to begin). A surface without compressing discs (and which is not a 2-sphere bounding a 3-ball) is the topological analogue of a stable minimal surface. A surface which has compressing discs on both sides, but which does not have a pair of disjoint compressing discs on opposite sides, (a weakly compressible surface) is the analogue of an index 1 minimal surface. When the surface is in an irreducible 3-manifold and can be compressed down to trivial 2-spheres in both directions and is weakly compressible, it is called a strongly irreducible Heegaard surface. Recently Ketover and Liokumovich [KL] confirmed a conjecture of Pitts and Rubinstein by showing that these “topological minimal surfaces” are equivalent (e.g. isotopic to) “geometric minimal surfaces” of index 0 or 1.

Inspired by Rubinstein’s view of topological notions of minimal surfaces, Bachman [B] defined a topological index for surfaces in 3-manifolds in terms of the compressing discs for those 3-manifolds. Similarly, Thompson considered cycles in the 1-skeleton of a triangulated 2–sphere. She identified certain “shortening moves” as the combinatorial analogue of both the geometric notion of creating a parameterized family of curves and the topological notion of a compressing disc for a surface in the 3-manifold. More precisely, in Thompson’s work, if $\mathcal{T}$ is a (simplicial) triangulation of a surface and if $\gamma$ is a path in the 1-skeleton of $\mathcal{T}$, then a triangle $T \in \mathcal{T}$ is a shortening move for $\gamma$ if two adjacent edges of $\gamma$ are two sides $a, b$ of $T$ and $\gamma$ does not contain the third side $c$ of $T$. See Figure 1 for an example. Replacing $a, b$ in $\gamma$ with $c$ produces a new path $\gamma'$ containing one less edge than $\gamma$. For Thompson, a stable closed geodesic is a cycle in the 1-skeleton of $\mathcal{T}$ which does not have a shortening move and which is not the boundary of a single triangle. A closed unstable geodesic, on the other hand, is a cycle $\gamma$ in the 1-skeleton which has shortening moves, at least one on each side of $\gamma$, but for which any two shortening moves on opposite sides of $\gamma$ share an edge. The path in Figure 1 is an unstable geodesic in the triangulated sphere obtained by doubling the disc along its boundary. Thompson uses a technique called “thin position” to produce stable and unstable closed geodesics in $\mathcal{T}$.

The main point of this paper is to generalize Thompson’s ideas to weighted pseudomanifolds $(M, \omega)$ of all dimensions. An $n$-dimensional simplicial manifold where each $(n-1)$-face is given a weight $\omega = 1$, is an example of a weighted pseudomanifold. If $M$ is a triangulated Riemannian manifold, it is also natural to consider the weight function to be the $(n-1)$-volume of each $(n-1)$-dimensional face. In our context, however, the weight function need not have any actual geometric meaning attached to it. Additionally, all that matters for our construction is that the space $M$ is made up of $n$-dimensional spaces glued together along $(n-1)$-dimensional subspaces in their boundaries.

Our work can also be see as a version of Johnson’s work [J] using thin position techniques to create a clustering algorithm. Johnson considers graphs with weighted edges and uses thin position to produce subsets of the vertices (called “pinch clusters”) which are more connected to each other than to vertices not in the subset. (See [BHJ] for an improved version.) We can convert our setting to that of Johnson’s by considering the dual graph to the $n$-dimensional cells in $M$. That is, consider the graph where each $n$-dimensional cell corresponds to a vertex of the graph and two vertices are joined by an edge with weight $w$ if they share an $(n-1)$-dimensional face of weight $w$. Johnson considers only the significance of the vertex subsets corresponding to our stable minimal surfaces. Since we also are interested in unstable minimal surfaces, we prefer to develop the theory from first principles in a way analogous to [T2].

Our main technique, derived from [J] and [T2], is called “thin position.” Thin position was first defined in the setting of knots in the 3-sphere by Gabai [G]. Gabai used thin position to show that the local maxima of
a certain function (called width) had certain desirable properties, analogous to those of an unstable minimal surface. Thompson [T1] later showed that the global minimum of width also has certain desirable properties, analogous to those of a stable minimal surface. Thin position has since been adapted to numerous other settings by other authors. The most significant adaptation is likely that of Scharlemann and Thompson [ST] who defined thin position for a 3-manifold. They show that the local minima and local maxima of width have analogous nice properties.

Scharlemann and Thompson’s idea is the main antecedent for the ideas in this paper. They consider handle structures on 3-manifolds and associate an ordering (of sorts) to each handle structure. An ordering is thinned by swapping the order of a 3-dimensional 1-handle and 2-handle, so that the 2-handle is attached before the 1-handle. In our setting, the cells (i.e. top-dimensional faces) of the pseudomanifold play the role of the handles in a handle decomposition. As the cells are attached, the boundary of all the cells attached up to a certain point forms a level surface. The attachment of some cells cause the weight of the level surface to increase and others to decrease. In contrast to the analogous situation with knots and 3-manifolds, some of the cells may also cause the weight of the level surface to remain the same. Our basic aim (again following [J] and [T2]) will be to swap adjacent cells in the ordering so that, if possible, the weight of the sublevel surface decreases before it increases.

2. Combinatorial minimal surfaces

A single point (i.e. a 0-dimensional ball) is a faceted 0-cell. For \( n \geq 1 \), a faceted \( n \)-cell is a compact \( n \)-dimensional manifold \( T \) with \( \partial T \) tiled by finitely many faceted \((n-1)\)-dimensional cells, called facets. The recursive nature of this definition allows us to refer to the faces of a faceted \( n \)-cell. For instance, an \((n-2)\)-dimensional face is a facet of one of the facets in the tiling of the boundary of the faceted \( n \)-cell.

It’s most natural to consider the case when a faceted \( n \)-cell is a compact \( n \)-ball, but other situations may also be useful. A gluing map from a faceted \( n \)-cell \( T_1 \) to a distinct faceted \( n \)-cell \( T_2 \) is a homeomorphism from
the union $X$ of some facets of $T_1$ to the union $Y$ of some facets of $T_2$ such that the restriction of the map to each face in $X$ takes that face to a face in $Y$. A pseudomanifold $(M, \mathcal{T})$ consists of a topological space $M$, a collection of faceted $n$-cells $\mathcal{T}$ together with gluing maps such that:

1. For each cell $T \in \mathcal{T}$ and each facet $F$ of $T$, there is at most one gluing map $\phi$ such that $T$ is either in the domain or range of $\phi$.
2. $M$ is homeomorphic to the quotient space obtained by gluing together the cells in $\mathcal{T}$ using the gluing maps.

**Remark 2.1.** We will often refer to a pseudo-manifold $(M, \mathcal{P})$ using only $M$. This obscures not only the gluing maps, but also the faceted cells. However, in this paper, no confusion should result. We also refer to the images in $M$ of the faceted cells as cells and the images in $M$ of the facets as facets.

Natural examples of pseudomanifolds include pure simplicial complexes and, more generally, pure polypetal complexes $[Z]$ of dimension $n$ where each $(n - 1)$-face is incident to at most two $n$-dimensional faces. Unlike in some definitions of pseudomanifold we do not allow a cell to be incident to itself along a facet. Most of the important features of our work are easily visualized in the case when $n \in \{2, 3\}$.

**Example 2.2.** A compact 2-dimensional manifold with a triangulation is a pseudomanifold. Each solid triangle is a cell and the edges of the triangulation are facets.

**Example 2.3.** A compact 3-manifold with a triangulation is a pseudomanifold. Each solid tetrahedron is a cell and the faces of the triangulation are the facets.

**Example 2.4.** A non-compact finite-volume hyperbolic 3-manifold having an ideal triangulation such that no ideal tetrahedron has two of its faces identified produces a pseudomanifold when each cusp is collapsed to a point.

**Example 2.5.** A compact 2-dimensional manifold with a pants decomposition is a pseudomanifold, where each pair of pants is a cell and the cuffs of a pair of pants are the facets.

Let $\mathcal{T}$ denote the set of cells in $M$ and $\mathcal{F}$ the set of facets in $M$. We denote by $\partial M$ the set of facets adjacent to a single cell. If $A$ is a cell, we let $\partial_1 A$ denote the facets (called the interior facets) of $A$ which do not lie in $\partial M$.

A surface $S \subset M$ is the union of facets. We refer to the facets of $M$ belonging to $S$ as facets in $S$. Observe that since facets are faceted $(n - 1)$-cells, it is natural to consider $S$ as a cell-complex, though it may not be a pseudomanifold. A surface is proper if no facet in $S$ is also a facet in $\partial M$ and if for every $(n - 2)$-dimensional face $\alpha$ in $S$ which does not lie in $\partial M$, the face $\alpha$ is a face of an even number of facets in $S$. If $S_1$ and $S_2$ are surfaces, we let $S_1 \cap S_2$ denote the facets in common to both surfaces and $S_1 \setminus S_2$ to be the facets in $S_1$ which are not facets of $S_2$. For a cell $A$ and a proper surface $S$, we let $\partial_2 A$ denote the union of the facets of $\partial_1 A$ in $S$ and $\partial_3 A$ the union of the facets of $\partial_1 A$ not in $S$.

As a combinatorial proxy for area, we will consider weight systems on pseudomanifolds. A weight system for a pseudomanifold $M$ is a function $\omega : \mathcal{F} \to [0, \infty) \subset \mathbb{R}$. If $S$ is the union of facets in $M$, we define the weight $\omega(S)$ of $S$ to be the sum of the weights of the facets in $S$.

**Example 2.6.** When $M$ is a compact 2-dimensional Riemannian manifold with a triangulation, it is natural to take the weight function to be the length of the facets (i.e. edges) in the boundary of each cell. Similarly, if $M$ is a simplicial or polypetal complex, it is natural to take the weight function to be the $(n - 1)$-dimensional volume of each facet. Alternatively, we could take the weight of each facet simply to be equal to 1.

**Example 2.7.** When $M$ is a 2-dimensional Riemannian manifold with a pants decomposition, it is natural to take the weight function to be the length of each cuff of each pair of pants.
Example 2.8. Suppose that $\tau$ is a regular train track (see [PH]) on a compact 2-dimensional manifold with a transverse measure. Let $N(\tau)$ be the fibered neighborhood of $\tau$. Decompose $N(\tau)$ into cells by using fibers based at the midpoints of each edge. Each cell is a 2-dimensional disc whose boundary has been partitioned into arcs, which are the facets. Three of the arcs correspond to the fibers used in the decomposition; give those facets a weight corresponding to the measure of that branch of the train track. The other edges (i.e. facets) belong to the horizontal and vertical boundary of $N(\tau)$.

In what follows, we only require that the weights be non-negative; they need not arise from any geometric considerations. We call $(M,\omega)$ a weighted pseudomanifold. We will not be varying $\omega$; henceforth we denote a weighted pseudomanifold simply by $M$.

**Definition 2.9.** The strength of a cell $A$ relative to a surface $S$ is

$$\sigma(A;S) = \omega(\partial\mathcal{A}) - \omega(\partial\mathcal{S}A).$$

We can form a new surface $S_A$ by replacing the faces of $A$ on $S$ with the faces of $\partial A$ not on $S$. We say that $A$ is a variation for $S$ and that $S_A$ is obtained by varying $A$. If $A \cap IS \neq \emptyset$ and if $\sigma(S_A;\Delta) \leq 0$, then $A$ is a shortening move for $S$. If the inequality is strict, $A$ is a strict shortening move for $S$.

Observe that if $A$ is a variation of $S$, then $\omega(S_A) = \omega(S) + \sigma(S;A)$. Thus, $A$ is a strict shortening move if and only if $\omega(S_A) < \omega(S)$. Notice also that if $S = \partial \Delta$ for some simplex $\Delta \subset M$, then $\Delta$ is a shortening move on $S$ and the empty surface results from shortening $S$ using $\Delta$.

**Remark 2.10.** Johnson [J] calls his version of $\sigma(A;S)$ (for graphs with weighted edges), the slope of $A$.

**Lemma 2.11.** If $A$ is a variation of a proper surface $S$, then $S_A$ is a proper surface.

**Proof.** Assume that $S$ is a proper surface and that $A \subset M$ is a cell. By definition, no facet in $S$ or $\partial A$ lies in $\partial M$. Thus, no facet in $S_A$ lies in $\partial M$.

Consider an $(n-2)$-dimensional face $\alpha$ in $S_A$ such that $\alpha$ does not lie in $\partial M$. We will show that the number $\deg(\alpha;S_A)$ of facets in $S_A$ containing $\alpha$ is even. Since $S$ is proper, $\deg(\alpha;S)$ is even (possibly 0). If $\alpha$ does not belong to $\partial A$, then we are done, so suppose that $\alpha$ lies in $\partial A$. Observe that $\partial A$ is a proper surface. Thus, there exist precisely two facets $F$ and $F'$ in $\partial A$ containing $\alpha$. If both $F$ and $F'$ lie in $S$, then $\deg(\alpha;S_A) = \deg(\alpha;S) - 2$ is still even. If precisely one of $F$ and $F'$ lies in $S$, then that facet does not lie in $S_A$, but the one other does. Hence, $\deg(\alpha;S_A) = \deg(\alpha;S)$ is still even. If neither $F$ nor $F'$ lie in $S$, then $\deg(\alpha;S_A) = \deg(\alpha;S) + 2$, and so $\deg(\alpha;S_A)$ is still even. Thus, $S_A$ is a proper surface.

We now come to the central definition of this paper.

**Definition 2.12.** A proper surface $S \subset M$ is:

- a stable minimal surface if $S$ has no strict shortening moves,
- an unstable minimal surface if there is a partition of the cells of $M$ into two non-empty sets $\mathcal{A}$ and $\mathcal{B}$ such that:
  1. If $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $A \cap B \subset S$;
  2. Each of $\mathcal{A}$ and $\mathcal{B}$ contain a shortening move for $S$ and one of them contains a strict shortening move;
  3. For each strict shortening move $A \in \mathcal{A}$, there exists a shortening move $B \in \mathcal{B}$ such that
      $$2\omega(A \cap B) \geq |\sigma(A;S)| + |\sigma(B;S)|.$$
      and vice versa; and
  4. For all strict shortening moves $A \in \mathcal{A}$ and strict shortening moves $B \in \mathcal{B}$,
      $$2\omega(A \cap B) \geq |\sigma(A;S)| + |\sigma(B;S)|.$$
To motivate this definition for minimal surfaces, consider a proper surface $S$ with two variations $A$ and $B$ such that $A \cap B \subset S$. Let $S_1 = S_A$ and $S_2 = S_B$. Then we say that $S_1, S, S_2$ is a variation sequence for $S$. If $S$ is a stable minimal surface, then $S$ is a local minimum of $\omega$ for every variational sequence of $S$.

Similarly, consider the surfaces

$$S_{AB} = (S_A)_B$$
$$S_{BA} = (S_B)_A$$

be the surfaces obtained by varying $S$ using $A$ and then $B$ and by varying $S$ using $B$ and then $A$. See Figure 2 for an example where the surface is a curve in a 2-dimensional manifold. Notice that $S_{AB} = S_{BA}$. We have

$$\omega(S_{AB}) = \omega(S) + \sigma(A;S) + \sigma(B;S) + 2\omega(F).$$

Suppose that $A$ and $B$ are both shortening moves for $S$. Then both $\sigma(A;S)$ and $\sigma(B;S)$ are non-positive. In which case $\omega(S_{AB}) < \omega(S)$ if and only if

$$|\sigma(A;S)| + |\sigma(B;S)| > 2\omega(F).$$

Hence, if $S$ is an unstable minimal surface then $\omega(S_{AB}) \geq \omega(S)$. That is, $S$ is a local minimum of $\omega$ when varying using two variations.

![Figure 2](image-url)
3. Orderings and Local Thin Position

For $N \in \mathbb{N} \cup \{0\}$, we let $[N] = \{1, \ldots, N\}$ and $[N]^* = [N] \cup \{0\}$. Suppose that $M$ is a weighted pseudomanifold with $N = |\mathcal{T}|$. An ordering of $M$ is a function $\mathcal{O} : \mathcal{T} \to [N]$. If the ordering $\mathcal{O}$ is clear from context, we let $T_i = \mathcal{O}^{-1}(i)$. Adapting terminology from Morse theory, for $j \in [N]^*$ we let

$$M_j = \bigcup_{i \leq j} T_i$$

be the sublevel set of $\mathcal{O}$ at height $j$. Notice, $M_0 = \emptyset$. We let $M_j^C$ be the union of all the polytopes $T_i$ with $i > j$. The level set $S_j$ at height $j$ consists of all facets $F$ of $M_j \cap M_j^C$. Let $\Lambda(j) = \omega(S_j)$.

The next lemma is where we use the fact that $M$ is a pseudomanifold and not a general cell complex.

**Lemma 3.1.** For all $j \in [N]$, $S_j$ is obtained by varying $S_{j-1}$ across $T_j$. In particular, we have

$$\Lambda(j) - \Lambda(j-1) = \sigma(T_j; S_{i-1}).$$

Also, each surface $S_j$ is a proper surface.

**Proof.** Notice that $S_0 = \emptyset$ is a proper surface with $\omega(S_0) = 0$. For $j \in [N]$, $M_j$ is obtained by including $T_j$ into $M_{j-1}$. Each facet of $\partial T_j$ which is not also a facet of $M_{j-1}$ thus lies in $S_j$. Likewise, any facet of $\partial T_j$ which does lie in $M_{j-1}$ lies in $S_{j-1}$ but not in $S_j$. Hence, $S_j$ is obtained by varying $S_{j-1}$ across $T_j$. The result follows from Lemma 2.11.

A number $t \in [N-1]$ is a (local) maximum if there exist $t_0 < t_1 < \ldots < t_n$ with each $t_i \in [N]^*$ and with $t = t_k$ for some $k \in [n-1]$ such that:

1. $\Lambda(t_0) < \Lambda(t_1)$,
2. $\Lambda(t_n) < \Lambda(t_{n-1})$, and
3. $\Lambda(t_1) = \Lambda(t_2) = \cdots = \Lambda(t_{n-1})$

We define (local) minima similarly, reversing the inequalities in (1) and (2). The maximum or minimum is extremal if $k = 1$ or $k = n - 1$.

Observe that $\Lambda(\emptyset) = \Lambda(N) = 0$ since both $S_0$ and $S_N$ are empty. Consequently, we refer to “maxima” and “minima” rather than “local maxima” and “local minima.” Also observe that if at least one weight on an interior facet is positive, then every ordering will have a maximum.

**Definition 3.2** (Thompson). The width of an ordering is the sequence whose terms are the values of $\Lambda$ at the local maxima of $\Lambda$, arranged in non-increasing order (with repetitions). Widths of orderings are compared lexicographically.

**Remark 3.3.** Our definition of width is due to Thompson [12]. In [1], Johnson defines width using all the values of $\Lambda$, not only the maxima. In that paper, he is primarily concerned with the properties of local minima of width. Using Thompson’s width allows us, like Thompson, to consider both local maxima and local minima.

3.1. Legal swaps and thinning. In the remainder of the section, we consider the effect of interchanging two simplices in an ordering. We establish some notation. In the group of permutations of $[N]$, let $\tau_i$ be the transposition interchanging $i$ and $i + 1$, for $i < N$. Suppose $\mathcal{O}$ is an ordering and that $i \in [N-1]$. Let $\mathcal{O}' = \tau_i \circ \mathcal{O}$. Set $A = \mathcal{O}^{-1}(i)$ and $B = \mathcal{O}^{-1}(i+1)$. Let $F = \partial A \cap \partial B$. For any $j$, let $S_j$ and $S'_j$ be the level sets at height $j$ for $\mathcal{O}$ and $\mathcal{O}'$ respectively. Observe that $S'_j = S_j$ whenever $j \leq i - 1$ or $j \geq i + 1$. The surface $S'_j$ is the variation of $S_{i-1}$ along $B$ and $S_{i+1}$ is the variation of $S'_i$ along $A$ and of $S_i$ along $B$. Let $\Lambda'$ be the function used to define width for $\mathcal{O}'$. 

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Observe that:
\[
\Lambda'(i) = \Lambda(i) - \sigma(A;S_{i-1}) + \sigma(B;S_{i-1}).
\]
Since \( \partial_{S_{i+1}}B = \partial_{S_{i+1}}(B \cup F) \), and since \( F \cap S_{i+1} = \emptyset \), we have
\[
\sigma(B;S_i) = \sigma(B;S_{i-1}) - 2\omega(F).
\]
Observe also that
\[
\sigma(A;S_i) = -\sigma(A;S_{i-1})
\]
since each facet of \( \partial_i A \) belongs either to \( S_{i-1} \) or to \( S_i \). Thus,
\[
\Lambda'(i) = \Lambda(i) + \sigma(A;S_i) + \sigma(B;S_i) + 2\omega(F).
\]

**Lemma 3.4** (The swap lemma). If
\[
\sigma(A;S_i) + \sigma(B;S_i) + 2\omega(F) \leq 0 \quad (*)
\]
then \( \Omega(\Theta') \leq \Omega(\Theta) \). Furthermore, if \( i \) is a maximum for \( \Theta \) and if \( (*) \) is strict, then \( \Omega(\Theta') < \Omega(\Theta) \).

The proof is similar to other proofs we will encounter. The basic idea is as follows. We begin by showing that any index \( i \) which is a maximum for both \( \Theta \) and \( \Theta' \) has \( \Lambda'(i) \leq \Lambda(i) \). We then consider \( i - 1 \) and \( i + 1 \). Each of these may be a maximum for \( \Theta \) but not for \( \Theta' \); if so, it does not contribute to increased width. Each may also be a maximum for \( \Theta' \) but not for \( \Theta \). If such is the case, we show that \( i \) is a maximum for \( \Theta \) but not for \( \Theta' \) and \( \Lambda(i+1) < \Lambda(i) \). We can then conclude that \( \Omega(\Theta') \leq \Omega(\Theta) \).

**Proof.** Assume that \( (*) \) holds. If \( (*) \) is an equality, then \( \Lambda'(i) = \Lambda(i) \) and so \( \Lambda' = \Lambda \). The fact that \( \Omega(\Theta') = \Omega(\Theta) \) follows immediately.

Assume, therefore, that \( \Lambda'(i) < \Lambda(i) \). If \( i \) a maximum for both \( \Theta \) and \( \Theta' \), we obtain \( \Omega(\Theta') \) from \( \Omega(\Theta) \) by removing (one copy of) \( \Lambda(i) \) and replacing it with at least one copy of \( \Lambda'(i) \). It may, in fact be replaced with more than one copy, as it may now be the case that \( \Lambda(i+1) = \Lambda'(i) \), in which case \( \Lambda(i+1) \) are now maxima. In any case, we have replaced \( \Lambda(i) \) with one or more integers strictly smaller than it and so \( \Omega(\Theta') < \Omega(\Theta) \).

Suppose now that \( i \) is a maximum for \( \Theta \), but not for \( \Theta' \). In this case we remove \( \Lambda(i) \) from \( \Omega(\Theta) \) in the process of forming \( \Omega(\Theta') \). We need to consider the possibility though that \( i - 1 \) or \( i + 1 \) is now a maximum for \( \Theta' \). If \( i + 1 \) was not a maximum for \( \Theta \), then \( \Lambda(i+1) < \Lambda(i) \), as \( i \) is a maximum for \( \Theta \). Consequently, even if \( i + 1 \) was not a maximum for \( \Theta \) but is for \( \Theta' \), we again have \( \Omega(\Theta') < \Omega(\Theta) \), as desired.

Finally, assume that \( i \) is not a maximum for \( \Theta \). Since \( \Lambda'(i) < \Lambda(i) \), the index \( i \) is not a maximum for \( \Theta' \), either. Suppose, therefore, that \( i + 1 \) is also not a maximum for \( \Theta \), but is for \( \Theta' \). Since it is a maximum for \( \Theta' \), there exists \( j < i - 1 \) such that \( \Lambda(j) < \Lambda(i-1) \) and \( \Lambda \) is constant on the interval \( (j, i-1] \). Since \( i - 1 \) is not a maximum for \( \Theta \), we must have \( \Lambda(i) \geq \Lambda(i-1) \). Since \( i \) is not a maximum for \( \Theta \), we must have \( \Lambda(i) \geq \Lambda(i) \). Thus, \( \sigma(B;S_i) \geq 0 \).

Since \( i - 1 \) is a maximum for \( \Theta' \) and \( i \) is not a maximum for \( \Theta' \), we have \( \Lambda'(i) < \Lambda(i-1) \). Hence, \( \sigma(B;S_{i-1}) < 0 \). But then
\[
0 > \sigma(B;S_{i-1}) = \sigma(B;S_i) + 2\omega(F) \geq 0,
\]
a contradiction. Hence, if \( i - 1 \) is a maximum for \( \Theta' \) then it for \( \Theta \), as well.

Now suppose that \( i + 1 \) is a maximum for \( \Theta' \) and not for \( \Theta \). As in the previous argument, we can conclude that \( \Lambda(i) \geq \Lambda(i+1) \) and that \( \Lambda(i+1) \geq \Lambda(i) \). This implies that \( \sigma(A;S_{i-1}) \leq 0 \). Also, we have that \( \Lambda'(i) < \Lambda(i+1) \), and so \( \sigma(A;S_i') > 0 \). Therefore, we have
\[
0 < \sigma(A;S_i') = \sigma(A;S_{i-1}) - 2\omega(F) \leq 0,
\]
a contradiction. Thus, if \( i + 1 \) is a maximum for \( \Theta' \) then it is also a maximum for \( \Theta \). Hence, \( \Omega(\Theta') \leq \Omega(\Theta) \).
Proof. shares no facet with Lemma 3.8 of knot theory that a maximum for a knot can be pushed higher and a minimum can be pushed lower. If 3.2. Delaying and Advancing. As Thompson observes in her context, there are times when it is advantageous to delay or advance the position of a cell in an ordering. We present two useful lemmas. They correspond to Thompson’s reordering principle [12]. They are similar to the principle in the thin position of knot theory that a maximum for a knot can be pushed higher and a minimum can be pushed lower.

Definition 3.6. If \( \Omega(\mathcal{O}''') \leq \Omega(\mathcal{O}) \) where \( \mathcal{O}''' = \tau_1 \circ \mathcal{O} \) for some \( i \in [N - 1] \), then we say that \( \mathcal{O} \) thins to \( \mathcal{O}''' \) and that \( \tau_1 \) is a legal swap. We extend the terminology so that “thins to” is reflexive and transitive. Hence, if \( \mathcal{O} \) thins to \( \mathcal{O}'' \), then there exists a sequence of legal swaps converting \( \mathcal{O} \) to \( \mathcal{O}'' \) such that width is non-increasing under each of the transpositions.

Definition 3.7. An ordering \( \mathcal{O} \) is locally thin if it does not thin to any ordering with strictly smaller width. It is thin if there is no ordering of strictly smaller width.

Remark 3.5. Suppose that \( \mathcal{O} \) is not a maximum for \( \Lambda \), then \( \mathcal{O} \) thins to \( \mathcal{O} = (i j i - 1 \cdots i + 1) \circ \mathcal{O} \).

Proof. Let \( B = T_{i+1} \) and \( F = A \cap B \). Since \( M \) is a pseudomanifold, \( F \) does not share any facet with \( S_j \). It also shares no facet with \( S_j \). Thus, \( \partial S_j \cup S_j \subset \partial A \). Similarly, each facet of \( \partial A \) is not a facet of \( S_j \). Thus, \( A \subset \partial S_j \cup \partial S_j \). Consequently,

\[
0 < \omega(\partial A) - \omega(\partial A) \\
\leq \omega(\partial S_j) - \omega(F) - \omega(\partial S_j - \omega(F)) \\
\leq \sigma(A; S_j) - 2\omega(F)
\]

Hence,

\[
\sigma(A; S_j) + \sigma(B; S_j) + 2\omega(F) = -\omega(A; S_j) + \sigma(B; S_j) + 2\omega(F) < \sigma(B; S_j)
\]

If \( i \) is a maximum for \( \mathcal{O} \), then \( \sigma(B; S_j) \leq 0 \), in which case by Lemma 3.4, \( \Omega(\mathcal{O}'') < \Omega(\mathcal{O}) \), as desired.

Suppose that \( i \) is not a maximum for \( \mathcal{O} \). Since \( \sigma(A; S_j) > 2\omega(F) \geq 0 \), we have \( \Lambda(i) > \Lambda(i - 1) \). Furthermore,

\[
\sigma(A; S_j) = \sigma(A; S_j) - 2\omega(F) > 0.
\]

Thus, \( \Lambda(i + 1) > \Lambda(i) \). Hence, \( i \) is not a maximum for \( \mathcal{O}'' \) either.

Suppose, for the moment, that \( \Lambda(i + 1) \) is a maximum for \( \mathcal{O}'' \). Since \( \sigma(A; S_j) > 0 \), we have \( \Lambda(i) > \Lambda(i - 1) \). Since \( i \) is not a maximum for \( \mathcal{O} \), then \( \Lambda(i + 1) > \Lambda(i) \). Hence, \( i + 1 \) is a maximum for \( \mathcal{O} \) as well.

Suppose that \( i - 1 \) is a maximum for \( \mathcal{O}'' \). We will derive a contradiction. Thus, \( \Lambda'(i) \leq \Lambda(i - 1) \). If equality holds, then \( i \) would also be a maximum for \( \mathcal{O}'' \), which it is not. Thus, \( \Lambda'(i) < \Lambda(i - 1) \). Thus, \( \sigma(B; S_j) < 0 \). Hence, \( \sigma(B; S_j) < 0 \). This implies \( \Lambda(i + 1) > \Lambda(i) \). Since \( \sigma(A; S_j) > 0 \), we also have \( \Lambda(i) > \Lambda(i - 1) \). But this implies that \( i \) is a maximum for \( \mathcal{O} \); a contradiction. Thus, \( i - 1 \) is not a maximum for \( \mathcal{O}'' \).

We conclude that \( \Omega(\mathcal{O}'') = \Omega(\mathcal{O}) \) when \( i \) is not a maximum for \( \mathcal{O} \). Hence, \( \mathcal{O} \) thins to \( \mathcal{O}'' \) as desired.

If \( i < j - 1 \), then performing the swap does not change the fact that \( \mathcal{O} \) holds, and so we may continue performing swaps, thinning the ordering each time, until \( A \) is at height \( j \).

We also need a version of Lemma 3.8 which allows us to advance cells in the ordering. We could adapt the previous proof, but instead we introduce the reverse ordering. It is similar to turning a Morse function upside down.
**Definition 3.9.** Suppose that $\mathcal{O} : \mathcal{R} \to [N]$ is an ordering. The reverse ordering $r\mathcal{O}$ is defined by
\[
    r\mathcal{O}(k) = \mathcal{O}(N - k + 1)
\]
for each $k \in [N]^*$. 

Observe that the level surface for $\mathcal{O}$ at height $k$ is equal to the level surface for $r\mathcal{O}$ at height $n - k$. Thus, the widths of $\mathcal{O}$ and $r\mathcal{O}$ are the same.

**Lemma 3.10** (The Advancing Lemma). Let $i \in [N - 2]$ and let $j + 1 < i$. For the polytope $A = T_i$, let $\partial_j A = \partial A \cap S_j$ and $\partial'_j A = \partial A \setminus \partial_j A$. Suppose that
\[
    \omega(\partial_j A) > \omega(\partial'_j A)
\]
Then $\mathcal{O}$ thins to the ordering $\mathcal{O}' = (j j + 1 j + 2 \cdots i) \circ \mathcal{O}$.

**Proof.** Let $B = T_{i-1}$. Consider the reverse ordering $r\mathcal{O}$. Let $T'_i = r\mathcal{O}^{-1}(k)$. Set $i' = N - i + 1$ and $j' = N - j$. Thus, $(i' + 1) = N - (i - 1) + 1$. Thus $A = T'_i$ and $B = T'_{i+1}$). Also,
\[
    r(\tau_{i-1} \circ \mathcal{O}) = \tau_{i'} \circ r\mathcal{O}.
\]
Let $S'_k = S_{N-k}$ be the level surface for $r\mathcal{O}$ at height $k$. Thus, $S'_{j - 1} = S_j$. By hypothesis,
\[
    \omega(\partial_{j'} A) > \omega(\partial'_{j'} A).
\]
Since $j + 1 \leq i - 1$, we have $N - j - 1 \geq N - i + 1$. Hence, $j' = N - j \geq N - i + 2 = i' + 1$. By Lemma 3.8 we have
\[
    \Omega(\tau_{i'} \circ r\mathcal{O}) \leq \Omega(\mathcal{O}).
\]
Hence,
\[
    \Omega(\mathcal{O}') \leq \Omega(\mathcal{O}),
\]
as desired. \qed

**Remark 3.11.** Observe how the thinning sequences produced by the proofs of Lemmas 3.8 and 3.10 do not move any cells not between heights $i$ and $j$ in the ordering.

4. Local Thin Orderings and Minimal Surfaces

Our main theorem states that we can use locally thin orderings to produce stable and unstable minimal surfaces. Its simplicity should be contrasted with existence results for minimal surfaces in differential geometry. It extends [12, Theorem 6] to pseudomanifolds of arbitrary dimension.

**Theorem 4.1.** Suppose that $(M, \omega)$ is a weighted pseudomanifold. Let $\mathcal{O}$ be a locally thin ordering. The following hold:

1. If $j$ is a maximum of $\mathcal{O}$, then $S_j$ is either a stable or an unstable minimal surface. If $j$ is an extremal maximum then $S_j$ is an unstable minimal surface.
2. If $j$ is a minimum of $\mathcal{O}$, then $S_j$ is a stable minimal surface.

**Proof.** We begin by proving the statement for local maxima. Suppose that $j$ is a maximum of $\mathcal{O}$. If $j$ is an extremal maximum, then either $\Lambda(j - 1) < \Lambda(j)$ or $\Lambda(j + 1) < \Lambda(j)$. Such would imply either $\sigma(T_j; S_j) < 0$ or $\sigma(T_{j+1}; S_j) < 0$. In which case, either $T_j$ or $T_{j+1}$ is a shortening move. If $S_j$ has no shortening moves, then it is a stable minimal surface, and we are done. Suppose, therefore, that $S_j$ has a shortening...
move \(T_i\). Let \(\mathcal{A}\) be the set of cells of \(M\) with height at most \(j\) and \(\mathcal{B}\) be the set of cells of height at least \(j + 1\). By definition,
\[
\omega(\partial_S T_i) > \omega(\partial'_S T_i).
\]

By Lemma 3.8 if \(i < j\), then we may thin \(\mathcal{O}\) to an ordering \(\mathcal{O}'\) in such a way that each thinning move preserves the set \(\mathcal{A}\) and is constant on \(\mathcal{B}\) and so that \(T_i\) is moved to be at height \(j\). Since \(\mathcal{O}\) is locally thin, width does not change. Similarly, if \(i > j + 1\), then by Lemma 3.10 we may similarly thin \(\mathcal{O}\) to an ordering \(\mathcal{O}'\) preserving the set \(\mathcal{B}\), remaining constant on \(\mathcal{A}\) and moving \(T_i\) to height \(j + 1\). Since \(\mathcal{O}\) is locally thin, width does not change. Indeed, given strict shortening moves \(A\) and \(B\) for \(S_j\) with \(\mathcal{O}(A) \leq j\) and \(\mathcal{O}(B) \geq j + 1\), we may thin \(\mathcal{O}\) to an ordering \(\mathcal{O}'\) with \(\mathcal{O}'(A) = j\) and \(\mathcal{O}'(B) = j + 1\). This may be done so that the surface \(S_j\) is unchanged. Furthermore, since \(A\) and \(B\) are strict shortening moves, \(j\) remains a maximum.

Given a strict shortening move \(A \in \mathcal{A}\) for \(S_j\), we perform the first of our reorderings to ensure that \(A\) is at height \(j\). Since \(j\) is a maximum, \(B = T_{j+1} \in \mathcal{B}\) is a shortening move. If
\[
\sigma(A; S_j) + \sigma(B; S_j) + 2\omega(F) < 0,
\]
then the ordering obtained by interchanging \(A\) and \(B\) has strictly smaller width. The ordering \(\mathcal{O}\) thins to this ordering, a contradiction. Thus,
\[
\sigma(A; S_j) + \sigma(B; S_j) + 2\omega(F) \geq 0.
\]

Since both \(\sigma(A; S_j)\) and \(\sigma(B; S_j)\) are non-positive,
\[
2\omega(F) \geq |\sigma(A; S_j)| + |\sigma(B; S_j)|.
\]

Similarly, if \(B \in \mathcal{B}\) is a strict shortening move for \(S_j\), then letting \(A = T_j\) we have that \(A \in \mathcal{A}\) is a shortening move for \(S_j\) and
\[
2\omega(F) \geq |\sigma(A; S_j)| + |\sigma(B; S_j)|.
\]

Finally, if \(A \in \mathcal{A}\) and \(B \in \mathcal{B}\) are strict shortening moves, we may thin \(\mathcal{O}\) as above to an ordering in which \(A\) is at height \(j\), \(B\) is at height \(j + 1\), and we deduce again that
\[
2\omega(F) \geq |\sigma(A; S_j)| + |\sigma(B; S_j)|.
\]

Thus, \(S_j\) is an unstable minimal surface.

Now suppose that \(k\) is a minimum for \(\mathcal{O}\) and that \(A = T_i\) is a strict shortening move for \(S_k\). We will derive a contradiction. Suppose, first, that \(i \leq k\). We deduce that
\[
\sigma(A; S_{i-1}) > 0.
\]
Thus, \(\Lambda(i - 1) < \Lambda(i)\). Since \(k\) is a minimum for \(\Lambda\), there is a maximum \(j \in [i, k)\). If \(j > i\), observe that
\[
\omega(\partial_S A) \geq \omega(\partial'_S A) > \omega(\partial'_S A) \geq \omega(\partial'_S A)
\]
Therefore, we may delay \(A\) to height \(j\), without increasing width. Furthermore, by the swap lemma (Lemma 3.4) we may then interchange \(A = T_j\) and \(B = T_{j+1}\) resulting in an ordering of strictly decreased width. This contradicts the choice of \(\mathcal{O}\) to be locally thin. If \(k < i\), we may adapt the previous argument or directly apply the previous argument to the reverse ordering to derive a contradiction. \(\square\)

**Corollary 4.2.** Suppose that \((M, \omega)\) is a weighted pseudomanifold such that there is an internal facet of \(M\) on which \(\omega\) is non-zero. Then \(M\) contains a stable or unstable minimal surface.
In this section we consider \(n\)-dimensional closed triangulated pseudomanifolds \(M\); that is, simplicial complexes whose geometric realizations have the property that each \((n-1)\)-dimensional face is incident to exactly two \(n\)-dimensional faces (cells). For such a triangulated manifold, define \(\Omega(M)\) to be the minimal width among all orderings of the cells (i.e. top-dimensional simplices) of \(M\). An ordering is thin if its width is equal to the width of the manifold.

Given two (distinct) such pseudomanifolds \(M_1\) and \(M_2\), we form a connected sum as follows. For \(i = 1, 2\), choose a cell \(C_i \subset M_i\). Let \(M_1 \#_2 M_2\) be the result of removing \(C_i\) from \(M_i\) and gluing together the resulting complexes via some simplicial homeomorphism \(\partial C_1 \to \partial C_2\). The resulting triangulated manifold is not uniquely defined (as a simplicial complex); the result depends on the choice of cells \(C_1\) and \(C_2\) and gluing map. The main result of this section is:

**Theorem 5.1.** Consider a finite closed simplicial \(n\)-dimensional pseudomanifold \(M = M_1 \# M_2\) with \(n \geq 2\) even. Let \(\omega(F) = 1\) for every facet \(F\) of \(M\). We have:

\[
\max(\Omega(M_1), \Omega(M_2)) \leq \Omega(M)
\]

This result should be compared with [SS, Corollary 6.4], where Scharlemann and Schultens prove the analogous statement for the width of the connected sum of knots in \(S^3\). In [BT], Blair and Tomova show that this is the best result possible for knots. However, in [TT], Taylor and Tomova show how a modification of the definition of width for knots leads to it being additive under connected sum. In the next section, we explore possible parallels between that result and the setting of this paper.

The reason for restricting to even dimensional manifolds is that, when \(n\) is even, each \(n\)-simplex has an odd number of facets in its boundary. Consequently, when each facet has weight 1, then for any ordering and every index \(\Lambda(i) \neq \Lambda(i \pm 1)\). This makes the study of maxima (and, therefore, width) much easier.

**proof of Theorem 5.1.** Let \(C_1\) and \(C_2\) be the cells whose interiors are removed from \(M_1\) and \(M_2\) to form \(M = M_1 \# M_2\). Let \(P\) be the homeomorphic image of their boundaries in \(M\). Observe that by the definition of simplicial complex, for each facet of \(P\), there is exactly one cell in \(M_1\) and one cell in \(M_2\) containing \(P\).

Let \(X_1\) and \(X_2\) be the two sides of \(P\) in \(M\) (so \(X_1\) is obtained by removing the interior of \(C_j\) from \(M_i\) for \(i = 1, 2\)). Let \(\emptyset\) be the set of thin orderings \(\mathcal{O}\) on \(M\). For \(\mathcal{O} \in \emptyset\), for an index \(j \in [N]^*\) and \(i = 1, 2\), let \(x_i(j)\) be the weight of the facets of \(S_j\) which lie in \(X_i\) but not in \(P\). Let \(T_{n_1}, T_{n_2}, \ldots, T_{n_n}\) be the subsequence of \((T_n)_{n=1}^{N}\) consisting of cells in \(X_1\) and let \(T_{u_1}, \ldots, T_{u_r}\) be the subsequence consisting of cells in \(X_2\). Without loss of generality, we may assume that \(n_1 = 1\). (If not, exchange labels in what follows.)

If it exists, let \(r + 1 \in [m]\) be the smallest index such that every cell of the component of \(X_2\) containing \(P\) is contained in \(M_{n_{r+1}}\). If such an \(r + 1\) does not exist, let \(r = m\).

Consider the following orderings \(\mathcal{O}_1\) and \(\mathcal{O}_2\) of \(M_1\) and \(M_2\) respectively:

\[
\mathcal{O}_1 = T_{n_1}, T_{n_2}, \ldots, T_{n_r}, C_1, T_{n_{r+1}}, \ldots, T_{n_m}
\]

\[
\mathcal{O}_2 = C_2, T_{u_1}, T_{u_2}, \ldots, T_{u_r}
\]

We consider how the maxima of \(\mathcal{O}\) relate to those of \(\mathcal{O}_1\) and \(\mathcal{O}_2\). We follow the strategy outlined before the proof of Lemma 3.4. Let \(\Lambda_1\) and \(\Lambda_2\) be the functions used in defining the widths of those orderings. Observe that for \(i \leq r\):

\[
\Lambda_1(i) = \Lambda(n_i) - x_2(n_i) + \epsilon_1 - \epsilon_2
\]

where \(\epsilon_1\) is the number of facets of \(P\) contained in \(M_{n_i}\) but not \(S_{n_i}\) and \(\epsilon_2\) is the number of facets of \(P \cap M_{n_i}\) incident to a cell in \(M_{n_i} \cap X_2\) but not to a cell of \(M_{n_i} \cap X_1\). We say that \(i\) and \(n_i\) are corresponding indices of \(\mathcal{O}_1\) and \(\mathcal{O}\) respectively.
Also, if \( i = r + j + 1 \) for some \( j \geq 1 \), then
\[
\Lambda_1(i) = \Lambda(n_{r+j}) - x_2(n_{r+j}) - \varepsilon_3 + \varepsilon_4
\]
where \( \varepsilon_3 \) is the number of facets of \( P \cap M_{n_{r+j}} \) which are either incident to a cell of \( X_2 \cap M_{n_{r+j}} \) and not to a cell of \( X_1 \cap M_{n_{r+j}} \) or vice versa. Also, \( \varepsilon_4 \) is the number of facets of \( P \) incident to no cell of \( X_1 \cap M_{n_{r+j}} \) or \( X_2 \cap M_{n_{r+j}} \). We say that \( i = r + j + 1 \) and \( n_{r+j} \) are **corresponding** indices of \( \mathcal{O}_1 \) and \( \mathcal{O} \) respectively.

Finally,
\[
\Lambda_1(r+1) = \Lambda(n_r) - x_2(n_r) + \varepsilon_5 - \varepsilon_6
\]
where \( \varepsilon_5 \) is the number of facets of \( P \) which do not belong to \( M_{n_r} \) and \( \varepsilon_6 \) is the number of facets of \( P \) which belong to \( M_{n_r} \cap X_1 \) but not \( M_{n_r} \cap X_2 \).

Similarly, for \( i \in [\nu] \) we say that \( i + 1 \) and \( u_i \) are **corresponding** indices of \( \mathcal{O}_2 \) and \( \mathcal{O} \) respectively. Observe
\[
\Lambda_2(i+1) = \Lambda(u_i) - x_1(u_i) + \varepsilon_7 - \varepsilon_8
\]
where \( \varepsilon_7 \) is the number of facets of \( P \) which do not belong to \( M_{u_i} \) and \( \varepsilon_8 \) is the number of facets which belong to \( M_{u_i} \cap X_2 \) but not \( M_{u_i} \cap X_1 \).

**Claim 1:** Suppose that \( \mu \) and \( n_r \) are corresponding indices of \( \mathcal{O}_1 \) and \( \mathcal{O} \) respectively. If both are maxima for their respective orderings, then \( \Lambda_1(\mu) \leq \Lambda(n_r) \).

Suppose first, that \( \ell \leq r \). Let \( \ell_1 \) be the number of facets of \( P \) contained in \( M_{n_r} \), but not \( S_{n_r} \), and \( \ell_2 \) be the number of facets of \( P \) incident to a cell in \( M_{n_r} \cap X_2 \) but not to a cell of \( M_{n_r} \cap X_1 \). For a contradiction, assume that \( \Lambda_1(\ell_1) > \Lambda(n_r) \).

Thus, \( \ell_1 \geq 1 \). In particular, some cell of \( X_2 \) incident to \( P \) belongs to \( M_{n_r} \). By the definition of \( r \), not every cell of the component of \( X_2 \) containing \( P \) is contained in \( M_{n_r} \). Hence, \( x_2(n_r) > 0 \). Therefore, there is a facet \( F \subset S_{n_r} \) contained in \( X_2 \) and not in \( P \). Since \( M_{n_r} \) is a simplicial complex, \( F \) contains at most one \( (n-2) \)-dimensional face which is also contained in \( P \). Each other \( (n-2) \)-dimensional face \( \alpha \) is incident to at least one other facet \( F_{\alpha} \neq F \) of \( S_{n_r} \). By the definition of simplicial complex, if \( \alpha \neq \beta \) then \( F_{\alpha} \neq F_{\beta} \). Hence, consider these \( F_{\alpha} \) along with \( F \) we see that \( S_{n_r} \) contains at least \( n \) facets in \( X_2 \) which are not in \( P \). Thus, \( x_2(n_r) \geq n \). Furthermore, if \( F \) does not share an \( (n-2) \)-dimensional face with \( P \), then the inequality is strict. If the inequality is strict, \( x_2(n_r) = n + 1 = \omega(P) \geq \ell_1 \). In which case, \( \Lambda_1(\ell_1) \leq \Lambda(n_r) \). Thus, \( x_2(n_r) = n \). In particular, every facet of \( S_{n_r} \) in \( X_2 \) but not in \( P \), has exactly one \( (n-2) \)-dimensional face in \( P \). Also, \( \ell_1 + n + 1 = \omega(P) \) and \( \ell_2 = 0 \), as otherwise \( \Lambda_1(\ell_1) \leq \Lambda(n_r) \).

Let \( \gamma \) be the \( (n-2) \)-dimensional face of \( F \) contained in \( P \). Since \( F \subset S_{n_r} \), there exists a facet \( \rho \) in \( P \) containing \( \gamma \) and incident to a cell \( A_0 \subset X_2 \) which does not contain \( F \). Since \( \ell_1 = \omega(P) \), the cell \( A_0 \) belongs to \( M_{n_r} \). Observe the facet \( F_0 \) of boundary \( \partial A_0 \) containing \( \gamma \) is not equal to \( F \) and cannot share an \( (n-2) \)-dimensional face with \( F \) other than \( \gamma \) (since \( M \) is a simplicial complex). Since \( x_2(n_r) = n \) and \( \omega(F \cup \bigcup_{\alpha} F_{\alpha}) = n \), the facet \( F_0 \) does not belong to \( S_{n_r} \). Thus, there is another cell \( A_1 \neq A_0 \) having \( \gamma \) as a face and sharing a facet with \( A_0 \) such that \( A_1 \subset M_{n_r} \). There is a facet \( F_1 \neq F_0 \) of \( A_1 \) containing \( \gamma \). It must also must differ from \( F \) and lie in \( M_{n_r} \). There are then another cell \( A_2 \neq A_0, A_1 \) in \( M_{n_r} \cap X_2 \) having \( \gamma \) as a face. Continuing on in this way we define an infinite sequence of facets \( F_0, F_1, \ldots \), contradicting the local finiteness of the simplicial complex \( M \).

Now suppose that \( \ell \geq r + 1 \). By definition of \( \mathcal{O}_1 \), there is a \( j \in [\nu] \) such that \( \mu = r + j + 1 = r + j \). Let \( \varepsilon_3 \) be the number of facets of \( P \cap M_{n_r} \) which are either incident to a cell of \( X_2 \cap M_{n_{r+j}} \) and not to a cell of \( X_1 \cap M_{n_{r+j}} \), or vice versa. Let \( \varepsilon_4 \) be the number of facets of \( P \) incident to no cell of \( X_1 \cap M_{n_r} \) or \( X_2 \cap M_{n_r} \). By the definition of \( r \), every facet of \( P \) belongs to \( M_{n_r} \). Thus, \( \varepsilon_4 = 0 \). Hence, \( \Lambda_1(\mu) \leq \Lambda(n_r) \), as desired.

**Claim 2:** \( r + 1 \) is not a maximum for \( \mathcal{O}_1 \).
We have $\Lambda_1(r + 1) = \Lambda(r) - x_2(n_r) + \varepsilon_5 - \varepsilon_6$, where $\varepsilon_5$ is the number of facets of $P$ not belonging to $M_{n_r}$ and $\varepsilon_6$ is the number of facets of $P$ belonging to a cell in $M_{n_r} \cap X_1$ but not to a cell in $M_{n_r} \cap X_2$. Assume, for a contradiction, that $x_2(n_r) + \varepsilon_6 \leq \varepsilon_5$. If $x_2(n_r) = 0$, then, as in Claim 1, no cell of $M_{n_r}$ lies in $X_2$. This contradicts the definition of $r + 1$. Thus, $x_2(n_r) \geq 1$. As in Claim 1, if there is a facet $F$ of $S_{n_r} \cap X_2$ not sharing an $(n - 2)$-face with $P$, then

$$x_2(n_r) \geq n + 1 \geq \omega(P) \geq \varepsilon_5.$$

Thus, every facet $F$ of $S_{n_r} \cap X_2$ shares exactly one $(n - 2)$-face with $P$. (It cannot share more than one since $M_2$ is a simplicial complex.) As before, we may also conclude that $x_2(n_r) = n$ that $\varepsilon_6 = 0$, and that $\varepsilon_5 = n + 1$. Thus, no facet of $P$ belongs to $M_{n_r}$. Let $\gamma$ be an $(n - 2)$-dimensional face of $P$ which is also a face of some cell $A \subset M_{n_r} \cap X_2$. The face $\gamma$ is a face of two distinct facets $F$ and $F'$ of $\partial A$. Neither lies in $P$. Without loss of generality, we may assume that $F$ lies in $S_{n_r}$. The other $n - 1$ faces of dimension $(n - 2)$ of $F$ (besides $\gamma$) are incident to distinct facets of $S_{n_r}$, none of which is $F'$. If $F'$ was in $S_{n_r}$, then we would have $x_2(n_r) \geq n + 1$, a contradiction. Thus, $F'$ does not lie in $S_{n_r}$. The cell $A_1$ adjacent to $A$ across $F'$ must, therefore, lie in $M_{n_r}$. Let $F_1 \neq F'$ be the other face of $A_1$, containing $\gamma$. As before, $F_1$ cannot lie in $S_{n_r}$. Continuing on in this way, we construct an infinite sequence of facets having $\gamma$ as a face. This contradicts the fact that $M$ is a simplicial complex.

**Claim 3**: Suppose $\mu$ and $n_1$ are corresponding indices of $\mathcal{O}_1$ and $\mathcal{O}$ such that $\mu$ is a maximum for $\mathcal{O}_1$ but $n_1$ is not a maximum for $\mathcal{O}_2$, then there exists a maximum $s$ for $\mathcal{O}$ in the interval $[n_1 + 1, n_{i+1} - 1]$ such that $\Lambda(s) > \Lambda_1(\mu)$.

Suppose, first, that $i \leq r$, so that $\mu = i$. Since $\mu$ is a maximum for $\mathcal{O}_1$, $\Lambda_1(i) > \Lambda_1(i - 1)$. This implies that $\Lambda(n_i) > \Lambda(n_{i-1})$. If $n_{i-1} = n_i - 1$, then we also have $\Lambda(n_i) > \Lambda(n_i - 1)$. If $n_{i-1} < n_i - 1$, then $T_{n_i}$ is a cell in $X_1$ and $T_{n_i} - n_{i-1}$ is a cell in $X_2$. They share at most one facet (which, if it exists, is in $P$). Let $\zeta$ be the number of facets shared between $T_{n_i}$ and $M_{n_{i-1}} \cap X_1$. Notice that $\zeta \leq n/2$. Since $T_{n_i}$ has $n + 1$ facets, we have

$$\Lambda(n_i) - \Lambda(n_i - 1) = (n + 1) - \zeta - 1 \geq n/2 \geq 1.$$  

Thus, $\Lambda(n_i) > \Lambda(n_i - 1)$, in this case also.

Since $n_i$ is not a maximum for $\Lambda$, we must have $\Lambda(n_i + 1) > \Lambda(n_i)$. Suppose that $i \leq r - 1$. Then this implies $n_{i+1} > n_i + 1$. Since $\Lambda(n_{i+1}) > \Lambda(n_i)$, there must be a maximum $s \in [n_i + 1, n_{i+1} - 1]$ for $\Lambda$ with $\Lambda(s) > \Lambda(n_i)$, as desired. If $i = r$, then by the definition of $r$, we again have $n_{i+1} > n_i + 1$. The same argument then produces our desired maximum.

Suppose now that $i \geq r + 1$. The argument is similar to what we had before. We have $\mu = i + 1$. Since $i$ is a maximum for $\Lambda_1$, $\Lambda(n_i) > \Lambda(n_i + 1)$. If $i \geq r + 2$, then for the same reason we also have $\Lambda(n_i) > \Lambda(n_i - 1)$. If $n_{i-1} = n_i - 1$, then tautologically $\Lambda(n_i) > \Lambda(n_i - 1)$. If $n_{i-1} \leq n_i - 1$, then $T_{n_i}$ is a cell in $X_1$, but $T_{n_i} - n_{i-1}$ is a cell in $X_2$. These cells share at most one facet in common, so again $\Lambda(n_i) > \Lambda(n_i - 1)$. Since $i$ is not a maximum for $\Lambda$, $\Lambda(n_i + 1) > \Lambda(n_i)$. Since $\Lambda(n_{i+1}) < \Lambda(n_i)$, there is a maximum $s \in [n_i + 1, n_{i+1} - 1]$ for $\Lambda$ with $\Lambda(s) > \Lambda(n_i)$.

**Claim 4**: $\Omega(\mathcal{O}_1) \leq \Omega(\mathcal{O})$

This follows immediately from Claims 1 - 3.

**Claim 5**: Suppose that $i + 1$ and $u_i$ are corresponding indices of $\mathcal{O}_2$ and $\mathcal{O}$ respectively. If both are maxima for their respective orderings, then $\Lambda_2(i + 1) \leq \Lambda(u_i)$.

This is nearly identical to Claim 1. Let $e_7$ be the number of facets of $P$ which do not belong to $M_{u_i}$ and $e_8$ be the number of facets which belong to $M_{u_i} \cap X_2$ but not $M_{u_i} \cap X_1$. Let $W = M_{u_i} \cap X_1$. An argument similar to the previous ones shows that $x_1(u_i) \geq e_7$. Thus, $\Lambda_2(i + 1) \leq \Lambda(u_i)$.

**Claim 6**: 1 is not a maximum for $\mathcal{O}_2$. 


We have $\Lambda_2(1) = \omega(P) = n + 1$. Each cell of $X_2$ is incident to at most one facet of $C_2$. Thus, either $\Lambda_2(2) = \Lambda_2(1) + (n + 1)$ or $\Lambda_2(2) = \Lambda(1) + n - 1$. Thus, in either case $\Lambda_2(2) > \Lambda_2(1)$.

**Claim 7:** Suppose $\mu = i + 1$ and $u_i$ are corresponding indices of $\mathcal{O}_2$ and $\mathcal{O}$ such that $\mu$ is a maximum for $\mathcal{O}_2$ but $u_i$ is not a maximum for $\mathcal{O}_2$, then there exists a maximum $s$ for $\mathcal{O}$ in the interval $[u_i + 1, u_{i+1} - 1]$ such that $\Lambda(s) > \Lambda_2(\mu)$.

Assume that $\Lambda_2(i + 1) \geq \Lambda_2((i + 1) \pm 1)$. Consequently, for $i \geq 1$, we have $\Lambda(u_i) > \Lambda(u_{i+1})$ and for all $i \geq 2$, we have $\Lambda(u_i) \geq \Lambda(u_{i-1})$. If, for $i \geq 2$, we have $u_{i-1} = u_i - 1$, then $\Lambda(u_i - 1) < \Lambda(u_i)$, tautologically. If, for $i \geq 2$, we have $u_{i-1} < u_i - 1$, then $T_{u_i}$ lies in $X_1$ and $T_{u_i - 1}$ lies in $X_2$. Since every facet that $T_{u_i}$ shares with $X_1$ is also shared with $B$, we have $\Lambda(u_i) > \Lambda(u_{i-1})$. Since $u_i$ is not a maximum for $\mathcal{O}$, we must have $\Lambda(u_i + 1) > \Lambda(u_i)$. Since $\Lambda(u_{i+1}) < \Lambda(u_i)$, there exists $s \in [u_i + 1, u_{i+1} - 1]$ such that $s$ is a maximum for $\Lambda$ and $\Lambda(s) > \Lambda(u_i)$.

**Claim 8:** $\Omega(\mathcal{O}_2) \leq \Omega(\mathcal{O})$ This follows immediately from Claims 5 - 7.

For widths $\omega(V_1)$ and $\omega(V_2)$, let $\omega(V_1) \cup \omega(V_2)$ be the sequence with the same terms as the sequences $\omega(V_1)$ and $\omega(V_2)$, but arranged in non-increasing order. As a consequence, we have:

**Corollary 5.2.** Suppose that $V_1$ and $V_2$ are two closed $n$-dimensional simplicial pseudomanifolds with $n \geq 2$ even. Then there is a connected sum $V_1 \# V_2$ such that

$$\max(\Omega(V_1), \Omega(V_2)) \leq \Omega(V_1 \# V_2) \leq \Omega(V_1) \cup \Omega(V_2)$$

**Proof.** Suppose that

$$\mathcal{O}_1 : T_1, T_2, \ldots, T_a$$

and

$$\mathcal{O}_2 : U_1, U_2, \ldots, U_b$$

are thin orderings of $V_1$ and $V_2$ respectively. Let $M = V_1 \# V_2$ be the $n$-manifold obtained by removing the interiors of $T_a$ and $U_1$ and choosing a simplicial homeomorphism $\partial T_a \to \partial U_1$ to glue the resulting spaces together. Consider the ordering

$$\mathcal{O} : T_1, T_2, \ldots, T_{a-1}, U_2, \ldots, U_{b-1}, U_{b+1}, \ldots, U_b.$$

of $M$. Observe that for $i \leq a - 2$, $i$ is a maximum for $\mathcal{O}$ if and only if it is a maximum for $\mathcal{O}_1$. Furthermore, for such an $i$, $\Lambda(i) = \Lambda_1(i)$. For $i \geq a + 1$, since $\partial U_1$ and $\partial M_{a-1}$ contain exactly the same facets, $i$ is a maximum for $\mathcal{O}$ if and only if $i - a + 2$ is a maximum for $\mathcal{O}_2$. Furthermore, the maxima take exactly the same values.

Consider $i = a - 1$. We have $\Lambda_1(i) > \Lambda_1(i + 1)$, so if $i$ is a maximum for $\mathcal{O}$, it is also a maximum for $\mathcal{O}_1$ of exactly the same value. Now consider $i = a$. Since $U_1$ and $U_2$ share at most one facet by the definition of simplicial complex, $\Lambda_2(2) > \Lambda_2(1)$. Consequently, if $i = a$ is a maximum for $\mathcal{O}$, then $i - a + 2$ is also a maximum for $\mathcal{O}_2$ of exactly the same value.

Hence,

$$\Omega(\mathcal{O}) \leq \Omega(\mathcal{O}_1) \cup \Omega(\mathcal{O}_2) = \Omega(M_1) \cup \Omega(M_2).$$

From Theorem [5.1] we see that

$$\max(\Omega(M_1), \Omega(M_2)) \leq \Omega(M).$$

In [O], Ozawa defines an invariant of knots in $S^3$, called trunk. He shows that the trunk of the connected sum of two knots is at most the maximum of the trunk of the factors. Problem 1.8 of [O] asks if equality
necessarily holds. In [DZ], Davies and Zupan show that it does. We adapt Ozawa’s definition to our setting and prove the corresponding theorem.

**Definition 5.3** (cf. [O]). Suppose that $M$ is a simplicial $n$-dimensional pseudomanifold and let $\mathcal{O}$ be an ordering of $M$. The **trunk** of $\mathcal{O}$ is equal to the maximum value $\text{tr}(\mathcal{O})$ achieved by $\Lambda$. The **trunk** $\text{tr}(M)$ of $M$ is the minimum of $\text{tr}(\mathcal{O})$ over all orderings $\mathcal{O}$ of $M$.

Recall that for an ordering $\mathcal{O}$, $\text{tr}(\mathcal{O})$ is the initial entry in the sequence $\Omega(\mathcal{O})$. Since we compare width lexicographically and since thin orderings minimize width, $\text{tr}(M) = \text{tr}(\mathcal{O})$ for every thin ordering $\mathcal{O}$. An easy adaptation of the proof of Theorem 5.2 yields the following version of the Davies-Zupan theorem.

**Corollary 5.4.** Let $M_1$ and $M_2$ be an $n$-dimensional simplicial pseudomanifolds with $n \geq 2$ even. Then there exists a connected sum $M_1 \# M_2$ such that

$$\text{tr}(M_1 \# M_2) = \max(\text{tr}(M_1), \text{tr}(M_2))$$

6. **Questions and Conjectures**

6.1. **Comparisons with Thompson’s work.** Theorem 13 of [T2] shows that (using her definitions) if a sphere has a triangulation with a thin ordering having a single maximum, then the triangulation is the tetrahedral triangulation.

**Question 6.1.** Suppose that $M$ is an $n$-dimensional strongly-connected pseudo-manifold without boundary such that there is a thin ordering having a single maximum. What special topological properties does $M$ have? Must it be an $n$-ball?

Thompson [T2] uses the fact that all triangulations of a disc are shellable to construct *embedded* closed geodesics in triangulated 2-spheres. In our context, this raises the question:

**Question 6.2.** Do the stable and unstable combinatorial minimal surfaces constructed in this paper correspond in some natural way to embedded stable and unstable minimal surfaces?

Section 6 of [T2] proves a combinatorial analogue of the “3 geodesics theorem” from differential geometry. As the author pointed out to us [T3], the result is somewhat weakened since generally the link of a vertex in a triangulated 2-sphere will be a stable geodesics. This means that it is generally easy to find at least 3 closed stable geodesics. Nevertheless, we can ask:

**Question 6.3.** If $P$ is a triangulated closed 2-dimensional manifold, does the genus of $P$ give a lower bound on the number of closed, embedded combinatorial geodesics?

6.2. **Comparison with Bachman’s work.** We can try to interpret stable and unstable minimal surfaces using Bachman’s theory of topological index. For simplicity, we phrase this only for embedded separating surfaces (i.e. curves) in a compact 2-dimensional manifold with a triangulation where every edge has weight 1. Notice that in such a setting, every shortening move is a strict shortening move. Given an embedded surface (i.e. curve) $S \subset M$ which separates $M$, let $\mathcal{D}$ denote the simplicial complex defined as follows. Each shortening move for $S$ is a vertex of $\mathcal{D}$. If we have $m \geq 2$ distinct shortening moves for $S$, they span an $m-1$ simplex if they pairwise do not share any facets of $S$. The **topological index** of $S$ is 0 if $\mathcal{D}$ is empty and is $k \geq 1$, if $k$ is the least number such that the $k$th homotopy group $\pi_{k-1}(\mathcal{D})$ is trivial.

**Lemma 6.4.** Assume that $M$ is a triangulated compact 2-dimensional manifold which is a simplicial complex, where every facet (i.e. edge of the triangulation has weight 1). Let $S$ be a proper embedded separating surface (i.e. curve) such that $S$ does not contain the boundary of the union of two triangles sharing a single edge not in $S$. Then $S$ is a stable minimal surface if and only if it is index 0. Furthermore, $S$ is an unstable minimal surface if and only if $\mathcal{D}$ is non-empty and disconnected, i.e. $\pi_0(\mathcal{D})$ is non-trivial.
Proof. The condition “$S$ has index 0” is equivalent to $S$ having no shortening moves. Since in our context, all shortening moves are strict shortening moves, this is equivalent to $S$ being a stable minimal surface.

Suppose, therefore, that $S$ is an unstable minimal surface. Let $\mathcal{A}$ and $\mathcal{B}$ be the partition of the cells of $M$ from the definition. Since all shortening moves are strict shortening moves, by part (2) of Definition 2.12 $\mathcal{A}$ and $\mathcal{B}$ each contain a strict shortening move for $S$. By part (4), every strict shortening move in $\mathcal{A}$ shares a facet with every strict shortening move in $\mathcal{B}$. If $A$ and $A'$ are shortening moves in $\mathcal{A}$, then they each contain two or three edges in $S$ and so either share zero or one edges. If they share one edge, then $S$ contains the boundary of the union of two triangles sharing a single edge. Thus, by hypothesis, $A$ and $A'$ are disjoint. Thus, in $D$ the corresponding vertices are joined by an edge. A similar argument applies to any two shortening moves contained in $\mathcal{B}$. Thus, the shortening moves in $\mathcal{A}$ and in $\mathcal{B}$ correspond to distinct components of $D$. In particular, $\pi_0(D)$ is non-trivial.

Suppose, on the other hand, that $\pi_0(S)$ is non-trivial. Let $A$ and $B$ be cells of $M$ lying in distinct components of $D$. Since both are shortening moves, they each have two or three of their edges lying in $S$. Since they lie in distinct components of $D$, they must also share at least one edge. By the definition of simplicial complex, they share exactly one edge. If that edge does not lie in $S$, then $S$ contains the boundary of $A \cup B$, a contradiction. Thus, $A \cap B \subset S$. Since $S$ is separating, $A$ and $B$ lie on opposite sides of $S$. Let $\mathcal{A}$ be all the triangles of $M$ on the same side of $S$ as $A$ and $\mathcal{B}$ be all the triangles of $M$ on the same side of $S$ as $B$. As in the previous paragraph, any two shortening moves for $S$ in $\mathcal{A}$ (respectively, $\mathcal{B}$) must be disjoint and, therefore, in the same component of $D$ as $A$ (respectively, $B$).

Question 6.5. What is the “right” definition of index when $S$ is non-separating or non-embedded? Are there minimal surfaces of arbitrarily high index?

6.3. Connected Sums. In the previous section, we gave bounds on the width of a connect sum in terms of the widths of the factors. The upper bound relied on constructing the sum using particular choices of cells. Perhaps the choice can be made in such a way that the upper bound no longer holds?

Conjecture 6.6. There exists $n$-dimensional simplicial manifolds $M_1$ and $M_2$ and a choice of connected sum $M_1 \# M_2$ such that

$$\Omega(M_1 \# M_2) > \Omega(M_1) \cup \Omega(M_2).$$

For many years, it was an open question as to whether Gabai’s width for knots in $S^3$ satisfied an additivity property under connected sum. Blair and Tomova [BT] answered the question negatively. But in [TT], Taylor and Tomova slightly modified the definition of width so as to make it additive under connected sum of (3-manifold, graph) pairs.

Conjecture 6.7. There is a modification of Thompson’s width, so that for any connected sum of simplicial $n$-manifolds $M_1$ and $M_2$, we have

$$\Omega(M_1 \# M_2) = \Omega(M_1) \cup \Omega(M_2).$$

As further evidence for this (admittedly vague) conjecture, we remark that in Taylor and Tomova’s construction, height functions on 3-manifolds (in particular, $S^3$) are allowed to have multiple minima and that this is crucial (see [TT, Section 6]) to making width additive. In a different direction, Bowman, Heistercamp, and Johnson [BHJ] were able to improve Johnson’s clustering algorithm by using a version of width based on partial orders, rather than linear orders. Perhaps there is a version of the complexity in that paper which would be useful?

Finally, even in our setting, we have not addressed the case of odd dimensional manifolds.

Conjecture 6.8. If $M_1$ and $M_2$ are closed odd-dimensional triangulated manifolds, then

$$\max(\Omega(M_1), \Omega(M_2)) \leq \Omega(M_1 \# M_2)$$
Each new simplex has \( n \) triangulated manifold. (The manifolds \( M \) are ordered into account the number of cells incident to each facet. One natural choice of complexity is as follows. Let \( \mathcal{O} \) be an ordering of the cells of \( M \). Let \( M_i \) be the union of all cells \( T \) such that \( \mathcal{O}(T) \leq i \). For a facet \( F \), let \( \deg(F; M_i) \) be the number of cells in \( M_i \) having \( F \) as a facet and let \( \deg(F; M_i^G) \) denote the number of cells not in \( M_i \) having \( F \) as a facet. Let \( \Lambda(i) = \sum_{F} \deg(F; M_i) \deg(F; M_i^G)\omega(F) \) where the sum is over all facets in \( M \). Let \( S_i \) be the union of all facets \( F \) with \( \deg(F; M_i) \deg(F; M_i^G) \neq 0 \). Define \( \Omega(\mathcal{O}) \) as before, but using this \( \Lambda \). Observe that when \( \deg(F; M) \leq 2 \) for every facet \( F \), then \( M \) is a pseudo-manifold and the new definitions for \( \Lambda \) and \( S_i \) coincide with the previous definitions. Some of the results of this paper can be extended to this new setting, but the fundamental difficulty in obtaining completely satisfactory answers is finding the “right” notion of what it means to vary a surface across a cell.

A Pachner move on a triangulation is one of a set of local moves converting the triangulation into another triangulation. The exact definition of the set of moves depends on the dimension of the simplicial complex. It is known that any two PL-homeomorphic triangulated manifolds are related by a sequence of Pachner moves. See [L]. One of the Pachner moves is often called a stabilization. Given an \( n \)-simplex \( A \) in an \( n \)-dimensional triangulated manifold \( M \), insert a new vertex \( x \) in the barycenter of \( A \), remove \( A \), and insert \((n+1)\)-simplices. Each new simplex has \( n \) vertices lying in a facet of \( \partial A \) as well as having \( x \) as a vertex. Let \( M' \) be the new triangulated manifold. (The manifolds \( M \) and \( M' \) are homeomorphic, though the triangulations differ.) If \( \mathcal{O} \) is an ordering on \( M \), there is a natural ordering \( \mathcal{O}' \) given by replacing \( A \) with the new cells \( A_1, \ldots, A_{n+1} \) and then shifting the subsequent cells of \( M \) by \( n \). Different choices of orderings on the new cells of course give different ordenings \( \mathcal{O}' \). We call any of these new orderings a stabilization of \( \mathcal{O} \).

The following is an analogue of the Reidemeister-Singer theorem for Heegaard splittings of 3-manifolds.

**Question 6.9.** Is there a version of Theorem 4.4 for weighted pure simplicial complexes? or polytopal complexes?

It seems likely that to adapt this paper to the setting of pure-simplicial complexes, we would need to take into account the number of cells incident to each facet. One natural choice of complexity is as follows. Let \( \mathcal{O} \) be an ordering of the cells of \( M \). Let \( M_i \) be the union of all cells \( T \) such that \( \mathcal{O}(T) \leq i \). For a facet \( F \), let \( \deg(F; M_i) \) be the number of cells in \( M_i \) having \( F \) as a facet and let \( \deg(F; M_i^G) \) denote the number of cells not in \( M_i \) having \( F \) as a facet. Let \( \Lambda(i) = \sum_{F} \deg(F; M_i) \deg(F; M_i^G)\omega(F) \) where the sum is over all facets in \( M \). Let \( S_i \) be the union of all facets \( F \) with \( \deg(F; M_i) \deg(F; M_i^G) \neq 0 \). Define \( \Omega(\mathcal{O}) \) as before, but using this \( \Lambda \). Observe that when \( \deg(F; M) \leq 2 \) for every facet \( F \), then \( M \) is a pseudo-manifold and the new definitions for \( \Lambda \) and \( S_i \) coincide with the previous definitions. Some of the results of this paper can be extended to this new setting, but the fundamental difficulty in obtaining completely satisfactory answers is finding the “right” notion of what it means to vary a surface across a cell.

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The following is an analogue of the Reidemeister-Singer theorem for Heegaard splittings of 3-manifolds.

**Question 6.10.** Given orderings \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) of a triangulated manifold \( M \), does there exist a sequence of stabilizations of \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) to arrive at triangulated manifold \( M' \) with orderings \( \mathcal{O}_1' \) and \( \mathcal{O}_2' \) respectively, such that there is an ordering \( \mathcal{O} \) of \( M' \) which thins to both \( \mathcal{O}_1' \) and \( \mathcal{O}_2' \)?

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