ALL ABELIAN SYMMETRIES OF LANDAU–GINZBURG POTENTIALS

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ABSTRACT

We present an algorithm for determining all inequivalent abelian symmetries of non-degenerate quasi-homogeneous polynomials and apply it to the recently constructed complete set of Landau–Ginzburg potentials for $N = 2$ superconformal field theories with $c = 9$. A complete calculation of the resulting orbifolds without torsion increases the number of known spectra by about one third. The mirror symmetry of these spectra, however, remains at the same low level as for untwisted Landau–Ginzburg models. This happens in spite of the fact that the subclass of potentials for which the Berglund–Hübsch construction works features perfect mirror symmetry. We also make first steps into the space of orbifolds with $\mathbb{Z}_2$ torsions by including extra trivial fields.
1 Introduction

Landau–Ginzburg (LG) models \([1,2]\) represent a fairly general framework for constructing \(N = 2\) superconformal field theories, which are needed for supersymmetric string vacua \([3]\). They provide, for example, a link between exactly solvable models and Calabi–Yau compactifications \([4]\), and also contain large classes of such models as special cases. In general, LG theories cannot be solved exactly. Still, some basic information on the massless spectrum of the resulting string models can be extracted in a simple way from the superpotential owing to non-renormalization properties. As a bonus, on the other hand, a very efficient algorithm for the calculation of the number of non-singlet representations of \(E_6\) in (2,2) vacua can be derived from the formulae for charge degeneracies of LG orbifolds given in refs. \([5,6]\). We therefore believe that it is worthwhile to invest some effort into the classification of the string vacua that can be obtained in this way.

Recently the basis for this work has been laid by the enumeration of all (deformation classes of) non-degenerate potentials with central charge \(c = 9\) \([7,8]\). In the present paper, as a second step, we calculate all abelian orbifolds that can result from manifest symmetries at non-singular points in the moduli spaces of these potentials, disregarding however the possibility of discrete torsions \([9,10]\) (except for a modest probe into \(\mathbb{Z}_2\) torsions). To do so, we extend the results of Vafa and Intriligator \([5,6]\) for the calculation of the chiral ring in orbifolds and prove an analogue of a well-known theorem for Calabi–Yau manifolds \([11]\) in the LG context.

Theoretically, our results are interesting for the question of mirror symmetry \([12,13,14]\) and for the classification of \(N = 2\) models. Unfortunately, for both of these (related) issues, our results are, in a sense, negative, as even many spectra with a large Euler number remain without mirror. From a phenomenological point of view, however, extensions of our calculations towards including torsion and non-abelian orbifolds look very promising, because new spectra mainly arise in the realm of small particle content in the effective field theory.

In section 2 we show how the (non-singlet) massless spectrum of an orbifold can, in general, be obtained from the index and the dimension of the chiral ring in a very efficient way. The fact that the computation of neither of these quantities requires explicit knowledge of a basis for the ring is vital for a complete investigation of all non-degenerate cases. Only if there are states with left-right charges \((q_L, q_R) = (1, 0)\) or \((0, 1)\) do we need extra information. We show how to extract the number of such states and that they can exist only if Witten’s index vanishes. Section 3 is quite technical and details how we construct, based upon the classification of potentials \([15]\), all possible (linear) abelian symmetries. The reader who is only interested in the results may wish to proceed to section 4. Implications of our findings are then discussed in the final section.

2 Hodge numbers

In this section we review the results of refs. \([5,6]\) and use them to derive an efficient algorithm for the calculation of the spectrum in case of vanishing torsions. We do, however, keep the discussion general as long as possible.
The basic information about the massless spectrum of $(2, 2)$ heterotic string models is contained in the chiral ring, i.e. the non-singular operator product algebra of chiral primary fields \[16\]. These fields have a linear relation between their conformal dimensions and their $U(1)$ charges. Their charge degeneracies are conveniently summarized by the Poincaré polynomial which, for left- and right-chiral fields, is defined as

$$P(t, \bar{t}) = \text{tr}_{(c, c)} t^{J_0} \bar{t}^{\bar{J}_0}. \quad (1)$$

Analogous generating functions can be defined for the charge degeneracies of Ramond ground states and anti-chiral fields; they are related to one another by spectral flow \[2\].

For $N = 2$ supersymmetric LG models, defined by a non-degenerate quasi-homogeneous superpotential

$$W(\lambda^a X_i) = \lambda^d W(X_i), \quad (2)$$

the Poincaré polynomial is given by \[17\]

$$P(t, \bar{t}) = P(t, \bar{t}) = \prod \frac{1 -(t\bar{t})^{1-q_i}}{1-(t\bar{t})^{q_i}}, \quad (3)$$

with $q_i = n_i/d$. Strictly speaking, this is a polynomial in $t^{1/d}$ and $\bar{t}^{1/d}$ rather than in $t$ and $\bar{t}$.

In order to obtain a model with integer charges we thus need to project onto states invariant under the transformation

$$j = e^{2\pi i J_0} = \mathbb{Z}_d [n_1, \ldots, n_N], \quad (4)$$

which rotates the field $X_i$ by a phase $2\pi i q_i$ in the complex plane (we will use the same symbol for the generator of a cyclic group and for the group itself). We can, of course, use any symmetry that contains this projection to twist the original LG conformal field theory. Only under certain conditions, however, will all the charges in the twisted sectors of the resulting orbifold be integer and thus space-time supersymmetry be unbroken \[6\] (see below). We will restrict ourselves to models where both left and right $U(1)$ charges are integer.

Even though some of these models definitely do not have an interpretation in terms of a string propagating on a Calabi–Yau (CY) manifold,\[4\] we will use CY terminology and call the coefficients of the Poincaré polynomial $P(t, \bar{t}) = p_{ij} t^i \bar{t}^j$, i.e. the numbers $p_{ij}$ of chiral primary fields with charge $(q_L, q_R) = (i, j)$, Hodge numbers. Let $n_{27}$ and $n_{\overline{27}}$ denote the numbers of $27$ and $\overline{27}$ representations of $E_6$ in the corresponding heterotic string model. If we have a CY interpretation, then $h_{11} = p_{12} = n_{27}$, $h_{12} = p_{11} = n_{27}$, and the Euler number of the manifold is $\chi = 2(h_{11} - h_{12}) = 2(p_{12} - p_{11}) = 2(n_{27} - n_{27})$.

In the twisted sectors of a LG orbifold only the fields that are invariant under the twist should contribute to Ramond ground states and to chiral primary fields. It can then be shown \[3, 4\] that the left/right charges of the Neveu–Schwarz ground state $|h\rangle$ in the sector twisted by an element $h$ of the symmetry group are given by

$$\sum_{\theta_i^h > 0} \frac{1}{2} - q_i \pm (\theta_i^h - \frac{1}{2}), \quad (5)$$

\[1\]Namely those which would have $h_{11} = 0$, in contradiction with the requirement of the existence of a Kähler form.
and that the action of a group element $g$ on that state is

$$g|h| = (-1)^{K_g K_h} \varepsilon(g, h) \frac{\det g_{ij}}{\det g}|h|,$$

(6)

where $h$ is assumed to act diagonally with phases $0 \leq \theta_i^h < 1$ on the fields $X_i$ and $g$ commutes under $h$. Here $\det g_{ij}$ denotes the determinant of the action of $g$ on the fields that are invariant under $h$. There is a certain freedom in the phase of the action of $g$ on the $h$-twisted sector, which is parametrized by the discrete torsions $\varepsilon(g, h)$ \cite{ref} and by the integers $K_g$ mod 2 satisfying $K_{gh} = K_g + K_h$ ($K_g$ determines the sign of the action of $g$ in the Ramond sector \cite{ref}).

For space-time supersymmetry, and hence integer left charges in the internal conformal field theory, the symmetry group used for the modding must contain the canonical $\mathbb{Z}_d$ symmetry \cite{ref} of the potential. Furthermore, the state $|j^{-1}\rangle$ is the analogue of the holomorphic 3-form, should be invariant under the complete group, which fixes the torsions $\varepsilon(j, g) = (-1)^{K_g K_j}$, det $g$ and $K_j = N - D$, where $N$ is the number of fields and $D = c/3 = \sum_{i \leq N} (1 - 2 \theta_i)$. We will also demand $(-1)^{K_g} = \det g$ to ensure that both left and right charges are integer and that the left-right symmetric spectral flow between the Neveu–Schwarz and the Ramond sector is local \cite{ref}.

### 2.1 Index and dimension of the chiral ring

With the above information it is, in principle, straightforward to compute the Poincaré polynomial for any given LG orbifold by summing over all invariant states for all possible twisted sectors. To do so, however, we would explicitly need a basis for the chiral ring, i.e. for the polynomial for any given LG orbifold by summing over all invariant states for all possible twisted states have been inferred from modular invariance of Witten’s index $\text{tr}_R (-1)^{J_0 - J_0}$, which is shown to be given by

$$P(-1, -1) = -\chi = \frac{1}{|G|} \sum_{gh=hg} (-1)^{N + K_g K_h + K_{gh} \varepsilon(g, h)} \prod_{\theta_i^g = \theta_i^h = 0} n_i - \frac{d}{n_i}.$$

(7)

As usual, this formula can be interpreted in two different ways: We can think of it as the sum over the contributions with boundary conditions $g$ and $h$ in the space and time direction of the torus, respectively. The last factor $\prod (d - n_i)/n_i$ in \cite{ref} is just the dimension of the ring restricted to fields invariant under $g$ and $h$, as is obvious from formula \cite{ref} for the Poincaré polynomial. Alternatively, we may consider the sum over $h$ to be the sum over twists, with the sum over $g$, normalized by the dimension $|G|$ of the group, implementing the projection.

Using the latter interpretation, it is easy to obtain from \cite{ref} a formula for the sum of all entries in the Hodge diamond, which we denote by $\bar{\chi} = P(1, 1) = 4 + 2n_{\overline{27}} + 2n_{\overline{27}} + 8p_{01}$ (for the second equality we have assumed $p_{01} = p_{02} = p_{10} = p_{20}$ and the Poincaré duality $P(t, \overline{t}) = (t \overline{t})^D P(1/t, 1/\overline{t})$). As the chiral fields $X_i$ have left-right symmetric charges, all states in a given twisted sector contribute with the same sign to the index \cite{ref}. This sign can be computed from the ground state, for which, according to \cite{ref}, $J_0 - \overline{J_0} = \sum_{\theta_i > 0} (2\theta_i^h - 1)$. As $\det h = (-1)^{K_h}$, this is nothing but $K_h + N - N_h \mod 2$, where $N_h$ is the number of untwisted
fields in that sector. Correspondingly, \((-1)^{K_h + N - N_h}\) is the sign of the contribution with \(g = 1\) to the index (7). We thus obtain

\[
P(1, 1) = \bar{\chi} = \frac{1}{|G|} \sum_{gh = hg} (-1)^{N_h + K_g K_h + K_{\varepsilon}} \varepsilon(g, h) \prod_{\theta_i^g = \theta_i^h = 0} \frac{n_i - d}{n_i} \tag{8}
\]

for the dimension of the \((c, c)\)-ring of the orbifold.

Given the analogue of the first Betti number, \(p_1 = p_{01} + p_{10}\), we can use this information to compute \(n_{27}\) and \(n_{\overline{27}}\). At first sight, eqs. (7) and (8) seem to have the disadvantage that we need to go twice over the group, implying that the computation time for an orbifold grows with the square of the group order. In fact, however, this need not be the case if we restrict ourselves to vanishing torsions. We will now show how this comes about, and then determine under what conditions \(p_1\) need not vanish and how to compute this number.

Our first simplification to this end is to assume \(K_g = 0\) for all \(g\). This is, in fact, no restriction, since we can always make a determinant positive by including an additional trivial superfield \(X_{N+1}\) with \(q_{N+1} = 1/2\), and letting \(g\) act non-trivially on that field. This modification does not change the central charge of a theory, nor does it change the ring, because the additional field can be eliminated by its equation of motion. In particular, with \(j\) contained in the group, \(\det j = 1\) implies that for \(D = 3\) we require the number of fields to be odd. With several trivial fields, different actions of group elements \(g\) and \(h\) of even order on these fields correspond to generically different choices of \(\mathbb{Z}_2\) torsions \(\varepsilon(g, h) = \pm 1\).

Restricting ourselves to \(\varepsilon(g, h) = 1\) for all \(g\) and \(h\), we obtain

\[
\chi = |G|^{-1} \sum_{g \in G} \sum_{h \in G} \prod_{\theta_i^g = \theta_i^h = 0} \frac{q_i - 1}{q_i}, \tag{9}
\]

\[
\bar{\chi} = |G|^{-1} \sum_{g \in G} (-1)^{N_g} \sum_{h \in G} \prod_{\theta_i^g = \theta_i^h = 0} \frac{q_i - 1}{q_i}, \tag{10}
\]

where \(N_g\) denotes the number of \(X_i\) that are invariant under \(g\). Special care has to be taken in their evaluation. A naïve application would imply a number of operations proportional to \(|G|^2\). If we assign to each subset \(\mathcal{M} \subset \{X_i\}\) of the set of fields the number \(n_{\mathcal{M}}\) of group elements that leave the elements of \(\mathcal{M}\) invariant while acting non-trivially on all other \(X_i\), we can rewrite these formulae as

\[
\chi = |G|^{-1} \sum_{\mathcal{M} \subset \{X_i\}} \sum_{\mathcal{\overline{M}} \subset \{X_i\}} n_{\mathcal{M}} n_{\mathcal{\overline{M}}} \prod_{i : X_i \in \mathcal{M} \cap \mathcal{\overline{M}}} \frac{q_i - 1}{q_i}, \tag{11}
\]

\[
\bar{\chi} = |G|^{-1} \sum_{\mathcal{M} \subset \{X_i\}} \sum_{\mathcal{\overline{M}} \subset \{X_i\}} (-1)^{|\mathcal{M}|} n_{\mathcal{M}} n_{\mathcal{\overline{M}}} \prod_{i : X_i \in \mathcal{M} \cap \mathcal{\overline{M}}} \frac{q_i - 1}{q_i}. \tag{12}
\]

The advantage of this formulation is that we can avoid the double summation over the group by first creating a list of the \(n_{\mathcal{M}}\)'s (evaluating each group element only once). Then the double summation takes place over the subsets \(\mathcal{M}\) of \(\{X_i\}\), whose number is between 32 for \(N = 5\) and 512 for \(N = 9\), to be compared with frequently occurring group orders of several thousands. Besides, we only have to go over the \(\mathcal{M}\)'s for those \(\mathcal{M}\) where \(n_{\mathcal{M}} \neq 0\). There is
a natural bitwise representation for the sets \( \mathcal{M} \), namely setting the \( i^{th} \) bit to 1 if \( \mathcal{M} \) contains \( X_i \) and to 0 otherwise, and of course this bit pattern can be identified with an integer. The operator \& provided by the programming language \( C \) (bitwise logical and) then represents the intersection of \( \mathcal{M} \) and \( \bar{\mathcal{M}} \).

A fast algorithm using these tricks works the following way:

1. Create an array of length \( 2^N \) containing the numbers \( n_{\mathcal{M}} \).
2. Create arrays with entries \( k_{\mathcal{M}} = \sum_{M \cap \mathcal{M} = \mathcal{M}} n_{\mathcal{M}} n_{\bar{\mathcal{M}}} \) and \( \bar{k}_{\mathcal{M}} = \sum_{M \cap \mathcal{M} = \bar{\mathcal{M}}} (-1)^{|M|} n_{\mathcal{M}} n_{\bar{\mathcal{M}}} \).
3. Calculate
   \[
   \chi = |G|^{-1} \sum_{\mathcal{M} \subset \{X_i\}} k_{\mathcal{M}} \prod_{i : X_i \in \mathcal{M}} \frac{q_i - 1}{q_i},
   \]
   \[
   \bar{\chi} = |G|^{-1} \sum_{\mathcal{M} \subset \{X_i\}} \bar{k}_{\mathcal{M}} \prod_{i : X_i \in \mathcal{M}} \frac{q_i - 1}{q_i},
   \]
evaluating the time-consuming product over rational numbers only once for each \( \mathcal{M} \) with \( (k_{\mathcal{M}}, \bar{k}_{\mathcal{M}}) \neq (0, 0) \).

### 2.2 The first Betti number

The final ingredient we need for the calculation of the Hodge diamond is the number \( p_{01} \). In [7] we have shown that this number can be non-zero only if there is a subset \( \mathcal{M}_1 \) of the set of fields \( \{X_i\} \) and an element \( j_1 \) of the symmetry group such that \( j_1 \) acts like \( j \) on the fields \( X_i \in \mathcal{M}_1 \) and does not act at all on the remaining fields. Furthermore, the contribution of the fields in \( \mathcal{M}_1 \) to the central charge has to be 3, i.e. \( \sum_{X_i \in \mathcal{M}_1} (1 - 2q_i) = 1 \). The proof of this statement in [7] applies without modification to the general case with arbitrary torsions and \( K_g \). In addition, we have also shown there that, as for CY manifolds, \( p_1 > 0 \) implies \( \chi = 0 \), because any LG model with \( p_1 > 0 \) factorizes into the product of a torus times a conformal field theory with \( c = 6 \). Although the factorization property does not generalize to the case of an arbitrary orbifold, we will now show that \( \chi = 0 \) can still be concluded, so that we need to calculate \( p_{01} \) only if the index vanishes. (Note that only twisted vacua can contribute to \( p_{01} \); thus the calculation of this number does not require explicit knowledge of the ring either.)

In order to show this, we consider a state \( |j_1 \rangle \) contributing to \( p_{01} \). Of course, it has to be invariant under the whole group, implying
\[
(-1)^{K_{j_1} K_g \varepsilon}(g, j_1) = \frac{\det g}{\det g_{|j_1}} = \det g_{|\mathcal{M}_1}. \tag{15}
\]

In case of trivial torsions and \( K_g = 0 \) this implies that any group element must have determinant 1 on \( \mathcal{M}_1 \). We are going to show that, as a consequence, all twists \( h \) that can contribute to the Poincaré polynomial coincide with \( (j_1)^p \), where \( p \in \{-1, 0, 1\} \), on \( \mathcal{M}_1 \), which in turn will imply that the Poincaré polynomial factorizes. In particular, \( p_{01} \) is the number of different subsets \( \mathcal{M}_1 \) of \( \{X_i\} \) that contribute 3 to the central charge, for which there is a group element that acts like \( j \) on \( \mathcal{M}_1 \) and trivially on the remaining fields, and with all determinants of group elements equal to one on \( \mathcal{M}_1 \). For \( D = 3 \) this number can only be 0, 1 or 3, with 3 corresponding to the 3-torus.
To prove the assertion, consider a symmetry \( g \) of \( W(X_i) \) and let \( \{X_i\}_g \) denote the set of those \( X_i \) that are themselves invariant under \( g \). The restriction of \( W \) to \( \{X_i\}_g \), i.e. setting all other fields to 0, is also non-degenerate (in the language of [13], no fields in \( \{X_i\}_g \), and thus no links between these fields can point out of that set). Then, with \( g = j(j_1)^{-1} \), \( W \) restricted to \( M_1 \) is a non-degenerate potential with \( D = 1 \). Any twist \( h \) can be decomposed as \( h_1 h_2 \), where \( h_1 \) acts only on \( M_1 \) and \( h_2 \) acts only on the other fields. Consider the \( D = 1 \) torus twisted by \( j_1 \) and \( h_1 \). The Poincaré polynomial of the \( D = 1 \) torus is unique and all its entries come from states twisted by powers \( j_1^n \) of \( j_1 \) (unless there are at least two trivial fields leading to a doubling of the ground state; this would not affect our arguments, however). Therefore the \( h_1 \)-twisted states cannot survive the \( j_1 \)-projection unless \( h_1 \) is itself a power of \( j_1 \) (\( h_1 \) does not project itself out).

According to eq. (\[1\]) the action of \( j_1 \) on the \( h_1 \) twisted sector for \( D = 1 \) is the same as the action of \( j_1 \) on the \( h \) twisted sector for \( D = 3 \). Thus the twists \( h \) that contribute to \( P(t, \bar{t}) \) belong to one of the following classes: Either \( p = \pm 1 \), then the fields in \( M_1 \) contribute a factor \( t \) or \( \bar{t} \), respectively, or \( p = 0 \). In the latter case the fields in \( M_1 \) contribute two states: \( q_L = q_R = 0 \) and \( q_L = q_R = 1 \). If any of these four contributions is present, then the other three, with identical contributions from the remaining fields, will also occur. Therefore the complete Poincaré polynomial has the form \( P(t, \bar{t}) = (1 + t)(1 + \bar{t})Q(t, \bar{t}) \), and \( \chi = -P(-1, -1) = 0 \).

This factorization of the Poincaré polynomial does not mean that the LG orbifold is a product of a \( c = 3 \) and a \( c = 6 \) theory. For the latter there seem to be only two possible spectra, namely those corresponding to the torus \( T^2 \) or the K3 surface, whereas we find several different \( c = 9 \) spectra with \( p_{01} \neq 0 \). The reason is that not all states of a “would-be product” survive the group projections. Consider, for example, the \( 1^9 \) with the symmetries \( j_1 = \mathbb{Z}_3[1, 1, 1, 0, 0, 0, 0, 0, 0] \), \( j_2 = \mathbb{Z}_3[0, 0, 0, 1, 1, 1, 1, 1, 1] \) and \( g = \mathbb{Z}_3[0, 1, 2, 0, 1, 2, 0, 0, 0] \). Without \( g \), the Poincaré polynomial would be

\[
P(t, \bar{t}) = (1 + t)(1 + \bar{t}) \left( (1 + t^2)(1 + \bar{t}^2) + 20t\bar{t} \right),
\]

where the \( 20t\bar{t} \) stand for the \( \binom{6}{3} = 20 \) states \( X_iX_jX_k|0\rangle \) with \( 4 \leq i, j, k \leq 9 \) in the untwisted sector. The \( g \) projection reduces this number to \( 4 + \binom{4}{3} = 8 \), coming from states \( X_4X_5X_6|0\rangle \) and \( X_iX_jX_k|0\rangle \), \( 6 \leq i, j, k \leq 9 \).

### 3 Finding the symmetries

This section is based on the criterion for non-degeneracy of a configuration \( C_{(n_1, \ldots, n_N)}[d] \), i.e. the existence of a non-degenerate polynomial that is quasi-homogeneous with respect to the weights \( q_i = n_i/d \), given in ref. [13]. It is easy to see that for any such polynomial there has to be a monomial of the form \( X_i^{a_i} \) or \( X_i^{a_i}X_j \) for each field \( X_i \). We call the second type of monomial a pointer from \( X_i \) to \( X_j \) and refer to the sum of \( N \) such monomials as a skeleton for the non-degenerate polynomial. Such a skeleton is in general not unique, and if there is more than one pointer at the same field, additional monomials are required for non-degeneracy [13]. We call

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[1] These features cannot be generalized to orbifolds with torsion. It is straightforward to construct examples where twists with complex determinants on \( M_1 \) contribute to the chiral ring. Still, in all the examples we know, the conclusion \( \chi = 0 \) is true.
the skeletons without such double pointers invertible. They correspond to the polynomials that are non-degenerate with only $N$ monomials and play an important role in mirror symmetry. Note that a skeleton already determines a configuration and also fixes a maximal abelian phase symmetry, which of course contains any phase symmetry of the complete potential.

The crucial simplification in considering abelian LG orbifolds is that we can assume any abelian symmetry to be diagonalized, i.e. to act as a phase symmetry. Thus we construct all possible inequivalent abelian symmetries by first constructing the inequivalent skeletons of a configuration. Then we determine the maximal phase symmetry of such a skeleton. Finally we construct all subgroups of this symmetry that satisfy the relevant set of conditions.

3.1 All skeletons of a configuration

Constructing all skeletons of a configuration is straightforward by choosing all possible combinations of pointers: $X_i$ can point at $X_j$ iff $n_i$ divides $d - n_j$ (here it is convenient to say a field points at itself iff it corresponds to a Fermat-type monomial $X_i^{n_i}$; these “pointers”, however, do not count for the non-degeneracy criterion). In the case of permutation symmetries of the configuration, i.e. if some $n_i$ are equal, this procedure, however, can generate many equivalent skeletons. It is convenient to represent each consistent skeleton for a given configuration by an integer whose $i^{th}$ digit is the index of the target of the $i^{th}$ field. This implies a natural ordering. A simple concept for eliminating the redundancy is to compute for each skeleton the set of all permutations consistent with the quasi-homogeneity of the configuration. Then we keep only those skeletons whose integer representation is minimal in the respective set of equivalent skeletons. This (admittedly crude) method requires only a few minutes to produce 106144 inequivalent skeletons for the 10838 non-degenerate configurations with up to 8 fields, which were constructed in ref. [7, 8]. For the unique configuration with 9 fields, however, which has the maximal permutation symmetry, this method wastes about a day of computing time. In that case we can alternatively use the algorithm of ref. [7] to compute all topologically inequivalent skeletons, which takes about a second. As a check for our programs we have used both algorithms to construct the 2615 different skeletons with 9 fields.

3.2 The maximal abelian symmetry of a skeleton

We will see that for a given skeleton the maximal abelian phase symmetry is of order $\mathcal{O} = \prod_i \alpha_i \prod_j (\prod_{k \leq l_j} \alpha_{jk}) - (-1)^{l_j}$, where the first product extends over all exponents of fields that are not members of a loop, while the second product extends over all loops $j$ with $l_j$ respective fields. This symmetry group can be represented by at most $N$ generators $g_i$, which we construct recursively.

We start with the fields that are not members of a loop and always consider the origin of a pointer before its target. When arriving at the field $Y$ with exponent $\beta$, the typical situation is that the $Y$-dependence of the skeleton polynomial is given by

$$Z Y^\beta + \sum_{i=1}^{l} Y X_i^{\alpha_i}. \tag{17}$$
As an induction hypothesis we assume that for each such field $X_i$ there is, so far, only one generator $g_i$ under which $X_i$ transforms, and that $X_i$ transforms with a phase $1/\alpha_i$, i.e. $g_iX_i = \exp(2\pi i/\alpha)X_i$ (for convenience we omit the obvious factor $2\pi$ and thus have rational “phases”).

Let $a_i$ be the order of $g_i$ and $U$ be the least common multiple of the $a_i$. Choose a set of divisors $t_i$ of the $a_i$ such that $U = \prod t_i$ and $\text{gcd}(t_i, a_i) = \text{gcd}(t_i, a_i/t_i) = 1$. Given the prime decomposition $U = \prod p_j^{n_j}$, this means that each number $p_j^{n_j}$ divides a particular $t_i$. The group generated by $g_i$ is therefore the direct product of the groups generated by $g_i^{t_i}$ and by $(g_i)^{a_i/t_i}$.

Allowing now $Y$ to transform, but keeping $Z$ fixed, we obviously enlarge the group order by the factor $\beta$, because the $X_i^a$ may have $\beta$ different phases under a group transformation. We now construct a generator $g_Y$ of order $U\beta$, which generates, together with the $g_i^{t_i}$ and all generators $g_k$ already constructed, the maximal phase symmetry that keeps $Z$ fixed. Let the phases of $X_i$ and $Y$ under $g_Y$ be $-1/(\alpha_i\beta)$ and $1/\beta$, respectively. All other fields, pointing at some $X_i$, transform with $-1/\beta$ times their phase under $g_i$. Note that we never use a relation involving the group order to change a phase to an equivalent one, so that all monomials are manifestly invariant, even if we enlarge the group order by a factor. Thus, if a path of length $n$ points from some field $X_k$ to $Y$, then the inverse phase of $X_k$ under $g_Y$ is $(-1)^n$ times the product of all exponents along the path. The order of $g_Y$ is $\beta$ times $U$, which is equal to the least common multiple of the products of the exponents along maximal paths pointing at $Y$. Furthermore, there are no non-trivial relations between $g_Y$ and the $g_i^{t_i}$, so that the group we have constructed has the correct order.

There is a slight complication if we eventually hit a field belonging to a loop. Within a loop, the transformation of all fields is fixed by the phase of any particular one. Thus, for our purpose, a loop acts like a single field. Let $Y_j$ point at $Y_{j+1}$ for $j < J$ and $Y_J$ point at $Y_1$ with respective exponents $\beta_j$. The maximal phase symmetry of the loop is generated by $g_Y$ acting on $Y_1, Y_2, \ldots, Y_J$ with phases $1/O, -\beta_1/O, \beta_1\beta_2/O, \ldots$, where $O = \prod \beta_i - (-1)^J$. In the final step of the calculation of the maximal abelian symmetry of a connected component of the skeleton we now have to consider all fields $X_i$ pointing at the loop and the respective generators $g_i$ of orders $a_i$. The generators $g_i^{t_i}$ are constructed as above. If we used the same recipe as above to construct the action of $g_Y$ on $X_i$ pointing at $Y_j$, $X_k$ pointing at $X_i$, etc., we would define phases

$$\frac{-b_j}{\mathcal{O}}\left(\frac{1}{\alpha_i}, \frac{-1}{\alpha_i\alpha_k}, \ldots\right), \quad b_j = \prod_{i \leq j}(-\beta_i),$$

with $\alpha_i$ being the least common denominator of the fractions in parenthesis. Unfortunately, if $b_j$ and $a_i$ have a common divisor, it is no longer guaranteed that the order of $g_Y$ is a multiple of $\mathcal{O}a_i$. We may, however, multiply $g_Y$ by any power $c_i$ of $g_i$ without changing the action of $g_Y$ on the fields in the loop. In this way, we replace $(-b_j)$ by $(c_i\mathcal{O} - b_j)$ in the above formula for the phases under $g_Y$. To ensure that the order of $g_Y$ becomes $\mathcal{O}U$, we require $\text{gcd}(c_i\mathcal{O} - b_j, a_i) = 1$. This is the case, for example, if $c_i$ is the product of all primes dividing $a_i$, but not $b_j$ (in fact, $\text{gcd}(c_i\mathcal{O} - b_j, t_i) = 1$ would be sufficient for our purpose). This completes the construction of the maximal phase symmetry of a skeleton polynomial.
3.3 All abelian symmetries of a skeleton

Obviously the result of the procedure described above is a direct product of cyclic groups \( G = \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_k} \). Now we want to find all subgroups \( G \) of this group that fulfill the following criteria:

(1) \( \det g = 1 \) for all \( g \in G \),

(2) \( \mathbb{Z}_d \subset G \),

(3) There is a non-degenerate polynomial that is invariant under \( G \).

Since \( \mathbb{Z}_a \times \mathbb{Z}_b = \mathbb{Z}_{a \times b} \) if \( \gcd(a, b) = 1 \) our problem reduces to the construction of all subgroups of \( \mathbb{Z}_{p^l_1} \times \ldots \times \mathbb{Z}_{p^l_k} \), for some prime number \( p \), that fulfill these criteria. Then, for the second condition, \( \mathbb{Z}_d \) has to be replaced by its maximal subgroup whose order is a power of \( p \). In addition, the existence of a non-degenerate polynomial with the symmetry group \( G \) has to be checked again after its subgroups corresponding to different prime numbers have been combined. If we denote the generators of \( \mathbb{Z}_{p^l_1} \times \ldots \times \mathbb{Z}_{p^l_k} \) by \( g_1, \ldots, g_k \), they will have determinants \( \det g_i = \exp(2\pi a_i/p^m_i) \) with \( \gcd(a_i, p) = 1 \) and \( 0 \leq m_i \leq l_i \).

We use the following algorithm for constructing the maximal subgroup with \( \det = 1 \):

(1) Find the maximal \( m_i \).

(2) If there are \( i, i' \) with \( m_i = m_{i'} \) and (say) \( l_i \geq l_{i'} \), replace \( g_i \) by \( g_i g_{i'}^{-x} \), where \( x \) is chosen in such a way that \( xa_{i'} = a_i \mod p^{m_i} \).

(3) Repeat this until there is only one maximal \( m_i \), then replace \( g_i \) by \( (g_i)^p \).

(4) Repeat the whole procedure until \( \det g_i = 1 \) for all \( g_i \).

Denoting the result of this construction again by \( \mathbb{Z}_{p^l_1} \times \ldots \times \mathbb{Z}_{p^l_k} \), any subgroup will be of the form \( \mathbb{Z}_{p^l_1} \times \ldots \times \mathbb{Z}_{p^l_k} \) with generators \( \hat{g}_i = \prod_{j=1}^{k} g_j^{\lambda_{ij}}, 1 \leq i \leq k, 0 \leq \lambda_{ij} < p^{l_j} \). Of course we severely overcount the subgroups in this way, for two reasons: A generator \( \hat{g} \) will be equivalent to \( \hat{g}^\lambda \) if \( \lambda \) is not divisible by \( p \), and the set of generators \( \{\hat{g}_a, \hat{g}_b\} \) is equivalent to \( \{\hat{g}_a g_1^\lambda, \hat{g}_b\} \) for any \( \lambda \). The first source of overcounting can be overcome in the following way: We note that the order of \( \hat{g}_i \) is given by \( p^{\hat{l}_i} = \max_{j}(p^{l_j}/\gcd(\lambda_{ij}, p^{l_j})) \). If we assume the \( j \)'s to be ordered, we can denote by \( \tilde{j}(i) \) the first \( j \) for which \( p^{l_j}/\gcd(\lambda_{ij}, p^{l_j}) = p^{\hat{l}_i} \). This allows us to choose the “normalization” \( \lambda_{ij}(i) = p^{\hat{l}_i} - l_i \). Overcountings of the second type can be avoided by demanding \( \lambda_{ij} < \max(1, p^{\hat{l}_i} - l_i) \) if \( j = \tilde{j}(i) \).

This implies the following algorithm for constructing each subgroup exactly once:

(1) Fix the type \( (l_1, \ldots, l_k) \) of the subgroup, with some ordering (e.g. \( \hat{l}_i \geq \hat{l}_{i+1} \)). Of course \( \hat{l}_i \leq l_i \), if the same ordering for the \( l_i \)'s is assumed.

(2) Choose \( \tilde{j}(i) \) for each \( i \), with some ordering if \( l_i = l_{i+1} \) (e.g. \( \tilde{j}(i) < \tilde{j}(i+1) \) if \( l_i = l_{i+1} \)). Thereby the \( \lambda_{ij}(i) \)'s are determined.

(3) Choose all other \( \lambda_{ij} \)'s subject to \( \lambda_{ij} < \max(1, p^{\hat{l}_i - l_i}) \) if \( j = \tilde{j}(i) \).

At this point we check for the \( \mathbb{Z}_d \) by explicitly calculating all elements of \( \mathbb{Z}_{p^l_1} \times \ldots \times \mathbb{Z}_{p^l_k} \) and comparing them with the \( p \)-projection of the generator of the \( \mathbb{Z}_d \). Putting together the subgroups is straightforward. The check for non-degeneracy is based on the results of \[15\] and follows the route indicated in \[7\].
4 Results

We have implemented these ideas in a C program, which we had running for about 6 days (real time) on a workstation to produce the results presented below. Because of this rather reasonable computing time we did not worry about the redundancy of calculating the same orbifolds several times for different skeletons or even calculating equivalent moddings for a particular skeleton in case of permutation symmetries of the potential. In general this redundancy is not too bad, since there is an average of only 10 skeletons per configuration and the overlap occurs mainly for groups of low orders. It is particularly bad, however, for the $1^9$, i.e. the potential $W = \sum_{i=1}^{9} X_i^3$ – one out of 108759 skeletons – which alone consumed more than 80% of our computation time and required the calculation of about 2 million orbifolds, producing eventually only 23 spectra, none of which was new. Obviously, for including all torsions one will have to work harder on this part of the calculation.

In table I we give a detailed statistics of our results. According to our organization of the computation we list, in the first 6 columns, the results for a fixed number of non-trivial fields, and finally combine the individual figures. Our starting point was the list of non-degenerate configurations with $c = 9$, obtained in refs. [7, 8]. Using the algorithms described in section 3 we have then calculated for each configuration all inequivalent skeletons, and for each skeleton all subgroups of the maximal phase symmetry that contain the canonical $\mathbb{Z}_d$ and allow a non-degenerate invariant polynomial. As the $\mathbb{Z}_d$ has negative determinant for even $N$, we have added a trivial field in that case before restricting to determinant 1. The numbers of different skeletons and symmetries that arise in this way are given in lines 2 and 3. Then we list the maximal numbers of symmetries and different orbifold spectra that can come from a single skeleton.

In line denoted by “spectra” we list the total number of different spectra obtained from all configurations with the respective number of non-trivial fields. The majority of these spectra appear in pairs with $n_{27}$ and $n_{27}^\prime$ exchanged: the numbers of singles are given in the next line. As the invertible skeletons play an important role in this “mirror symmetry”, the numbers of such skeletons and their spectra are listed in the last two lines of table I.

In fig. 1 we show the 800 new spectra that we found in addition to the 2998 spectra of canonical orbifolds [4, 8]. All their Euler numbers are between $-276$ and 480, and the new spectrum with the largest number of non-singlet $E_6$ representations has $n_{27} = 116$ and $n_{27}^\prime = 230$. Note that the spectra with the largest numbers of particles have large positive Euler number, in agreement with the expectation that orbifolding generically increases the Euler number.

Among the new models there is a number of spectra with $p_1 = p_{01} + p_{10} > 0$, namely $n_{27} = n_{27}^\prime = \{3, 5, 9, 13, 21\}$ for $p_1 = 2$ and $n_{27} = n_{27}^\prime = 9$ for $p_1 = 6$ (we count, in this paper, spectra with different $p_1$ as different; our plots may thus in fact show up to 5 points fewer than is indicated in the figure captions). In accordance with the results of section 2,
Table I: Statistics of the calculation and results. (I)SK and SP denote (invertible) skeletons and spectra.

|                        | 4  | 5  | 6  | 7  | 8  | 9  | total |
|------------------------|----|----|----|----|----|----|-------|
| configurations         | 2390 | 5165 | 2567 | 669 | 47  | 1   | 10839 |
| skeletons (SK)         | 7674 | 30575 | 31216 | 29257 | 7422 | 2615 | 108759 |
| symmetries             | 17833 | 53282 | 139696 | 111692 | 187641 | 2324394 | 2834538 |
| orbifolds per SK       | ≤140 | ≤140 | ≤13506 | ≤2664 | ≤56632 | ≤2052656 | ≤2052656 |
| spectra per SK         | ≤43  | ≤28  | ≤63  | ≤39  | ≤47  | ≤47  | ≤63  |
| spectra (SP)           | 2278 | 3182 | 2002 | 1015 | 289  | 85   | 3798 |
| SP without mirror      | 258  | 675  | 199  | 77   | 17   | 1    | 816  |
| invertible skeletons   | 4556 | 11053 | 7605 | 3406 | 564  | 115  | 27299 |
| spectra from ISK       | 1910 | 2259 | 1651 | 793  | 242  | 73   | 2730 |

all these spectra have $\chi = 0$. In contrast with the canonical orbifolds of [7], however, they do not all correspond to tensor products of conformal field theories, as has been discussed in section 2. This conclusion can also be drawn from the fact that the factors would have to be either the torus, with all coefficients of the Poincaré polynomial equal to 1, or a model with $D = 2$. By applying our program to the 922 inequivalent skeletons of the 124 non-degenerate configurations with $D = 2$ we have checked that, within our class of orbifolds, the Hodge diamond of the K3 surface and the 2-torus are the only possible spectra. This implies that only the last two of the above spectra can be products and is another check for the remarkably successful geometric interpretation of $D = 2$ models [18]. Some of our spectra with $p_1 > 0$ have recently also been obtained from $CP_m$ coset models with non-diagonal modular invariants [20]. They can be compared with the 80 non-chiral spectra we have found with $p_1 = 0$, which exist for $n_{27} = n_{27} \in \{3, 7, 9, 10, 11, \ldots, 48, 49, 52, 53, 55, \ldots, 179, 223, 251\}$.

Let us now return to the discussion of trivial fields. In addition to compensating negative determinants, they can be used to simulate $\mathbb{Z}_2$ torsion, i.e. $\varepsilon(g, h) = -1$, between group elements of even order. To get an idea of what one might expect from such torsions we have added 2 (3) trivial fields for potentials with an odd (even) number of fields, respectively. The corresponding additional $\mathbb{Z}_2$ symmetries were added to the set of generators of the maximal group before constructing all subgroups with determinant 1. In this way we could generate models with several different $\mathbb{Z}_2$ torsions without much extra effort. For 4, 5, 6 and 7 non-trivial fields, we thus obtained 114, 393, 69 and 309 new spectra as compared with the respective cases with no additional fields (for 8 and 9 fields the calculation was stopped because it could not be expected to finish within a reasonable time). The total number of spectra, however, rises only from 3798 to 3837 since there is a large overlap of spectra for different numbers of fields.

The 39 new spectra resulting from $\mathbb{Z}_2$ torsions are shown in fig. 2 as little circles. They all have $n_{27} + n_{27} \leq 66$. To give a more detailed picture we have also included in that plot all other spectra with $n_{27} + n_{27} \leq 80$ by dots, the ones for orbifolds bigger than the ones for untwisted models. Among these 39 models we find the one with the smallest dimension of the chiral ring. It comes from the Fermat skeleton in $\Omega_{(1,1,1,1,1,1,2,2,2)}[4]$ with 6 non-trivial and 3 trivial fields.
and has the spectrum $\chi = b_1 = 0$, $n_{27} = n_{37} = 3$. The symmetry that generates this spectrum is a product of 3 cyclic groups: $g_1 = \mathbb{Z}_4[0,1,0,3,2,0,0,2,0]$, $g_2 = \mathbb{Z}_4[0,1,0,1,2,2,0,0,2]$ and $g_3 = \mathbb{Z}_2[0,0,0,0,1,0,1,0,1]$. Alternatively, we can omit the trivial fields, i.e. start with the configuration $\mathcal{C}^{(1,1,1,1,1)}[4]$, and twist by the group $(\mathbb{Z}_4)^2 \times \mathbb{Z}_2$ with the same action on the non-trivial fields, but now with torsions $\varepsilon(g_1, g_2) = \varepsilon(g_2, g_3) = -1$ and $\varepsilon(g_1, g_3) = 1$. Of course, we also have to use the appropriate signs $K_g$ and torsions with the canonical $\mathbb{Z}_4$ as discussed in section 2. These results give us a clear hint that realistic spectra with very low numbers of fields can be expected in the set of models with non-trivial torsions.

To check the construction of the mirror model by Berglund and Hübsch (BH) \[14\], which applies exactly to the invertible skeletons, we have verified for a number of such skeletons that the inverted (or “transposed”) skeleton yields exactly the mirror spectra. In addition, we have examined the mirror symmetry of the complete set of spectra that come from invertible skeletons (let us recall that we use the term mirror symmetry in its naïve sense of just rotating the Hodge diamond; we do not check the fusion rules). The BH construction does not imply that this space is exactly symmetric, since the mirror of a potential containing $X^\alpha + XY^2$ would contain a trivial field. Indeed, we found 12 spectra violating the mirror symmetry of the list of spectra of invertible skeletons we produced. An explicit check of these models shows that they are of the type described above, and that the inverted skeleton, which contains trivial fields, produces the correct spectrum. In fact, 6 of the missing mirror spectra are already contained in the set of spectra from non-invertible skeletons, whereas the other 6 are part of the above 39 spectra with $\mathbb{Z}_2$ torsions.

The situation is drastically different for non-invertible models. Of the 1068 spectra that cannot be obtained from invertible skeletons, 810 have no mirror spectrum. Figure 3 shows the 258 remaining spectra in this class, which do have mirrors. The ones without mirror are indicated by small dots in fig. 3, and all of them are plotted separately in fig. 4. It should be noted that the 258 spectra with mirrors all occur in the “dense” region of all spectra in fig. 5. It is, therefore, not unlikely that their mirror pairings are purely accidental. A look at fig. 2 shows that indeed in some of the low-lying regions of the plot of all spectra a high percentage of all possible combinations of Euler numbers divisible by 4 and even $n_{27} + n_{37}$ occur.

Finally, we come to the presentation of what seem to be the most interesting models. In table II we list the 25 new spectra with a net number of 3 generations, together with a representation as a LG orbifold. The 40 untwisted 3-generation models were already given in refs. \[13\]: they have 29 different spectra with $n_{27} = n_{37} - 3 \in \{13, 15, 16, 17, 20, 26, 27, 29, 31, 34, 37, 42, 47, 67, 74\}$ and $n_{27} = n_{37} - 3 \in \{13, 17, 20, 21, 23, 26, 29, 32, 35, 37, 40, 47, 48, 57\}$ for positive and negative Euler number, respectively. In view of the much smaller increase of the total number of spectra, this shows again that the percentage of “realistic” models is larger for more general constructions.

The first model in table II is the one with the lowest $n_{27} + n_{37}$. It belongs to the configuration $\mathcal{C}^{(2,3,3,4,4,5,5)}[13]$ and is represented by the non-invertible potential

$$W = X^5 Y + Y^3 U + Z^3 V + U^2 W + V^2 W + W^2 Z + T^2 Z + \varepsilon_1 U V T + \varepsilon_2 W T Y + \varepsilon_3 W X^4. \quad (19)$$

The twist by $\mathbb{Z}_2[1 1 0 1 0 0 1]$ with $X_i = \{X, Y, Z, U, V, W, T\}$ transforms its spectrum from $(29, 5, -48)$ into $(14, 11, -6)$. This indicates that one should be careful in applying empirical “quantization rules” in the search for 3-generation models. Actually, this model corresponds to two different skeletons, both of which belong to the same non-degenerate potential (in the
| $n_{27}$ | $n_{17}$ | $\chi$ | configuration | twist |
|------|------|-----|--------------|------|
| 14   | 11   | 6   | $C_{(2,3,3,4,4,5,5)}[13]$ | $\mathbb{Z}_2[1 0 1 0 0 1]$ |
| 17   | 14   | 6   | $C_{(2,2,2,3,3,3,3)}[9]$ | $(\mathbb{Z}_3)^2 \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}$ |
| 18   | 15   | 6   | $C_{(1,2,3,3,3,3,3)}[9]$ | $\mathbb{Z}_4[1 0 2 0 3 0 0]$ |
| 21   | 18   | 6   | $C_{(2,3,5,8,9)}[27]$ | $\mathbb{Z}_2[0 1 0 1 0]$ |
| 25   | 22   | 6   | $C_{(2,3,4,9,9)}[27]$ | $\mathbb{Z}_2[0 0 1 0 1]$ |
| 30   | 27   | 6   | $C_{(1,4,5,5,10)}[25]$ | $\mathbb{Z}_2[1 0 0 0 1]$ |
| 31   | 28   | 6   | $C_{(1,3,4,4,9)}[21]$ | $\mathbb{Z}_3[0 0 1 2 0]$ |
| 34   | 31   | 6   | $C_{(1,2,3,3,8)}[17]$ | $(\mathbb{Z}_2)^2 \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$ |
| 37   | 34   | 6   | $C_{(2,3,5,15,20)}[45]$ | $\mathbb{Z}_2[0 1 0 0 1]$ |
| 45   | 42   | 6   | $C_{(1,3,9,14,18)}[45]$ | $\mathbb{Z}_2[1 0 0 0 1]$ |
| 47   | 44   | 6   | $C_{(2,3,7,21,30)}[63]$ | $\mathbb{Z}_2[1 0 1 0 0]$ |
| 49   | 46   | 6   | $C_{(1,2,5,10,17)}[35]$ | $\mathbb{Z}_3[0 2 0 1 0]$ |
| 54   | 51   | 6   | $C_{(1,2,4,13,19)}[39]$ | $\mathbb{Z}_4[2 2 3 0 1]$ |
| 66   | 63   | 6   | $C_{(1,3,11,18,32)}[65]$ | $\mathbb{Z}_2[0 1 1 1 1]$ |
| 70   | 67   | 6   | $C_{(1,3,15,20,36)}[75]$ | $\mathbb{Z}_2[1 0 0 0 1]$ |
| 14   | 17   | 6   | $C_{(1,1,1,1,2,2,2)}[5]$ | $\mathbb{Z}_{16}[8 12 6 13 14 10 1]$ |
| 18   | 21   | 6   | $C_{(2,3,5,8,9)}[27]$ | $\mathbb{Z}_2[1 0 1 0 0]$ |
| 19   | 22   | 6   | $C_{(2,3,3,5,5,6,6)}[15]$ | $\mathbb{Z}_2[0 0 1 0 0 1 0]$ |
| 21   | 24   | 6   | $C_{(1,1,2,2,5)}[11]$ | $\mathbb{Z}_5[0 1 1 1 2]$ |
| 23   | 26   | 6   | $C_{(1,2,4,5,7)}[19]$ | $\mathbb{Z}_3[1 2 0 0 0]$ |
| 32   | 35   | 6   | $C_{(3,3,10,14,15)}[45]$ | $\mathbb{Z}_2[0 1 0 1 0]$ |
| 36   | 39   | 6   | $C_{(1,2,3,9,12)}[27]$ | $(\mathbb{Z}_2)^2 \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| 40   | 43   | 6   | $C_{(1,1,5,8,10)}[25]$ | $(\mathbb{Z}_2)^2 \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ |
| 46   | 49   | 6   | $C_{(1,6,7,21,28)}[63]$ | $\mathbb{Z}_2[1 0 0 0 1]$ |
| 48   | 51   | 6   | $C_{(1,5,6,18,25)}[55]$ | $\mathbb{Z}_2[1 0 1 1 1]$ |

Table II: New 3-generation models. A superscript $I$ indicates that the twist can be applied at an invertible point in the moduli space of the configuration.
Table III: 1-generation models

| \(n_{27}\) | \(n_{27}\) | \(\chi\) | configuration | twist |
|----|----|----|----------------|-------|
| 12  | 13  | 2  | \(\mathcal{C}_{(5,6,7,8,9,11,12)}[29]\) | untwisted |
| 16  | 17  | 2  | \(\mathcal{C}_{(3,3,4,5,7,8,8)}[19]\) | \(\mathbb{Z}_2[0 1 0 0 1 0]\) |
| 20  | 21  | 2  | \(\mathcal{C}_{(3,3,4,5,10)}[25]\) | \(\mathbb{Z}_2[0 1 0 0 1]\) |
| 22  | 23  | 2  | \(\mathcal{C}_{(3,4,5,7,16)}[35]\) | \(\mathbb{Z}_2[0 1 0 0 1]\) |
| 34  | 35  | 2  | \(\mathcal{C}_{(1,1,4,6)}[13]\) | \(\mathbb{Z}_6[0 2 1 4 5]\) |
| 22  | 21  | \(-2\) | \(\mathcal{C}_{(2,3,4,5,9)}[23]\) | \(\mathbb{Z}_2[1 0 0 0 1]\) |
| 33  | 32  | \(-2\) | \(\mathcal{C}_{(1,3,4,4,11)}[23]\) | \(\mathbb{Z}_5[0 0 2 1 0]\) |
| 46  | 45  | \(-2\) | \(\mathcal{C}_{(1,5,6,12,23)}[47]\) | \(\mathbb{Z}_2[0 1 0 1 0]\) |

language of [13], each of the monomials \(X^5Y\) and \(X^4W\) in \([19]\) can be interpreted as a pointer; then the other one belongs to the set of required links).

In table III we list the 1-generation models. None of them has a mirror spectrum, and hence none of them comes from an invertible skeleton. With two exceptions, all values of the original inverse charge quantum \(d\) are prime. In fact, as for the untwisted case [7], all 1- and 3-generation models have odd \(d\), and hence an odd number of fields. The number of 2-generation models in our list is 33; 26 of them do not require a twist and 18 have negative Euler numbers.

It is significant that the spectra without mirror that have the largest values of \(|\chi|\) all come from untwisted LG models (compare figs. 1 and 4). The first 12 of these, with spectra \((13, 433), (17, 341), (20, 326), \ldots\), have large positive Euler numbers. As all our new spectra have much smaller particle content, it appears to be most unlikely that these spectra will eventually find their mirrors in the realm of (more general) LG orbifolds.

5 Discussion and outlook

Considering the complete set of abelian symmetries of Landau–Ginzburg potentials, we have studied approximately 250 times as many models as previously, with canonically twisted theories (there is, however, some redundancy in our constructions owing to the possible occurrence of the same symmetries for different skeletons in a specific configuration and to permutation symmetries of some skeletons). Doing so without pushing computer time to astronomical heights was only possible with an algorithm that was extremely efficient, at least at its central part, i.e. at the calculation of the numbers of chiral generations and anti-generations from a given potential and symmetry. The numbers of spectra obtained, and the number of new features, however, did not rise in a comparable manner. Less than 25% of the spectra we found were new compared with \([7, 8]\), and the overall impression of the plot of spectra in fig. 5 is the same as in the pioneering work ref. \([12]\). Although the new spectra arise primarily in
the range of low generation and anti-generation numbers, we have not found any models that look particularly promising from a phenomenological point of view. Yet, we believe that our results are quite interesting from the following points of view:

We have established now what we already found in [7], namely that mirror symmetry in the context of Landau–Ginzburg orbifolds occurs regularly for those and only for those models for which the Berglund–Hübsch construction works. In particular, among the spectra that remain without mirrors, there are a number of canonical LG models whose Euler numbers are much larger than what any orbifold contributed. This makes it appear very unlikely that a generalization of the BH construction exists for LG orbifolds or for the related Calabi–Yau manifolds. In accordance with the results of [21], the lack of mirror symmetry also indicates that, for a complete classification of rational \( N = 2 \) theories, we need to go beyond LG models. Still, in addition to their phenomenological use, the lessons we learn from them may be helpful also in that direction.

Obviously non-abelian symmetries and twists with non-trivial torsion [10] are good candidates for providing phenomenologically more realistic spectra. This is clear from the fact that our “smallest” 3-generation model requires nearly twice as many fermions of opposite chirality as the well-known model of [22], which can be interpreted as a non-abelian Landau–Ginzburg orbifold. Furthermore, the few new spectra that we obtained with our excursion into the space of models with non-trivial \( \mathbb{Z}_2 \) torsion are all in the area of very small particle numbers, and this set contains the model with the least value of \( n_{27} + n_{27} \) that we have found.

There are severe obstacles to a complete classification of models with torsion along the lines of this work: Our fast algorithm for the calculation of spectra cannot be generalized to the case of torsion. It relies on the fact that, without torsion, the only characteristic of a group element required for the calculation of the spectrum is the information about which subset of the fields it leaves invariant. Besides, the number of possible different torsions for a set of \( n \) generators of order \( p \) is given by \( p^\binom{n}{2} \). For the \( 1^9 \) with the maximal symmetry we have, with unit determinants and the restrictions on torsions with \( \mathbb{Z}_d \), \( p = 3 \) and \( n = 7 \), thus yielding \( 3^{21} \) cases, which makes a naïve calculation completely impossible. Still, our analysis and computation of abelian symmetries provides the necessary first step for such an investigation, be it complete or not.

With our present knowledge, it seems that the number of consistent spectra of \( N = 2 \) superconformal field theories with \( c = 9 \) and integer charges is restricted to a few thousands. This raises the question of whether it might not be possible to classify all (2,2) vacua by enumeration, with a scheme based only on the axioms of \( N = 2 \) superconformal field theory. The mathematical tools for such a task, however, are still waiting to be discovered.

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Fig. 1: $n_{27} + n_{27}$ vs. Euler number $\chi$ for the 800 new LGO spectra
Fig. 2: \( n_{27} + n_{27}^{-1} \) vs. Euler number for the 39 spectra with \( \mathbb{Z}_2 \) torsion (circles) and for the LG/LGO spectra with \( n_{27} + n_{27}^{-1} \leq 80 \) (small/big dots)

Fig. 3: \( n_{27} + n_{27}^{-1} \) vs. Euler number \( \chi \) for the 258 spectra that do have a mirror spectrum but cannot be obtained from an invertible skeleton (the ones without mirror and with \( n_{27} + n_{27}^{-1} \leq 180 \) are indicated by small dots)
Fig. 4: \( n_{27} + n_{27}^{-} \) vs. Euler number \( \chi \) for the 810 spectra without mirror
Fig. 5: $n_{27} + n_{27}^{-}$ vs. Euler number $\chi$ for all 3837 different spectra