THE CENTRAL NILRADICAL OF
NONNOETHERIAN DIMER ALGEBRAS

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Abstract. Let $Z$ be the center of a nonnoetherian dimer algebra $A$ on a torus. We show that the nilradical $\text{nil } Z$ of $Z$ is prime, may be nonzero, and consists precisely of the central elements that vanish under a cyclic contraction of $A$. This implies that the nonnoetherian scheme $\text{Spec } Z$ is irreducible. We also show that the reduced center $\hat{Z} = Z/\text{nil } Z$ embeds into the center $R$ of the corresponding ghor algebra, and that their normalizations are equal. Finally, we give three characterizations of the normality of $R$, and show that if $\hat{Z}$ is normal, then it has the special form $k + J$ where $J$ is an ideal of the cycle algebra of $A$.

1. Introduction

The main objective of this article is to establish certain key algebraic and geometric properties of the centers of nonnoetherian dimer algebras and ghor algebras on a torus. A dimer algebra is a type of quiver algebra whose quiver embeds into a compact surface with homotopy-like relations defined on its paths. Dimer algebras originated in string theory in 2005 [HK, F-K, F-W, HV], and since have found wide application to many areas of mathematics. Among these areas are noncommutative algebraic geometry [B5, B7, Br, CQ, D, IN, IU, MR], cluster algebras and categories [BKM, GK, K, MS, P, RW], mirror symmetry [F-V, FU, HN], and number theory [BGH, H]. The torus is special among other surfaces without boundary in that only on a torus do noetherian dimer algebras have exceptionally nice algebraic and homological properties as noncommutative crepant resolutions. Ghor algebras, in contrast, remain suitably nice on higher genus surfaces [BB1, BB2]. Throughout, we will restrict our attention to dimer and ghor algebras on a torus.

A dimer algebra is noetherian if and only if its center is noetherian [B4, Theorem 1.1]. Regardless of noetherianity, the Krull dimension of the center is three; if nonnoetherian, then the center is the coordinate ring for a three-dimensional affine toric variety with a single ‘positive dimensional’ point [B6, Theorem 1.1]. Recall that an integral domain is normal if it is integrally closed in its field of fractions. A well known property of noetherian dimer algebras—being noncommutative crepant resolutions—is that their centers are normal integral domains. In this article we
consider the questions: Is the center of a nonnoetherian dimer algebra necessarily normal, or necessarily a domain? If not, what can be said about its normalization, zero divisors, and nilradical?

We will use two fundamental tools to answer these questions, both introduced in [B2]: cyclic contractions and ghor algebras. A cyclic contraction $\psi : A \to A'$ of a dimer algebra $A = kQ/I$ is the contraction of a particular set of arrows of $Q$ to vertices such that the cycles of $Q$ are preserved and the resulting dimer algebra $A'$ is noetherian (see (3)). Remarkably, every (nondegenerate) dimer algebra admits a cyclic contraction [B1, Theorem 1.1]. The ghor algebra of $Q$ is the quotient
\[
\Lambda := A/(p - q \mid p, q \text{ a non-cancellative pair}),
\]
where a pair of distinct paths $p, q$ is said to be non-cancellative if there is a path $r$ such that $rp = rq \neq 0$ or $pr = qr \neq 0$. A dimer algebra equals its ghor algebra if and only if it is noetherian [B4, Theorem 1.1]. The center $R = Z(\Lambda)$ of $\Lambda$ will play an essential role in deciphering the structure of the center $Z$ of $A$. Our main theorem is the following.

**Theorem 1.1.** (3.5, 4.1, 4.2, 5.5, 5.8, 6.6, 6.7) Let $A = kQ/I$ be a nonnoetherian dimer algebra on a torus with center $Z$, let $R$ be the center of its ghor algebra $\Lambda$, and let $\psi : A \to A'$ be any cyclic contraction of $A$.

1. The nilpotent central elements of $A$ are precisely the central elements in the kernel of $\psi$,
\[
\text{nil } Z = Z \cap \ker \psi.
\]
2. The reduced center $\hat{Z} := Z/\text{nil } Z$ is an integral domain. The scheme $\text{Spec } Z$ is therefore irreducible.
3. $\hat{Z}$ is a subalgebra of $R$, and their normalizations are equal and nonnoetherian.
4. $R$ is normal if and only if $R = k + J$ for some ideal $J$ of the center $Z'$ of $A'$.

Consequently, if $\hat{Z}$ is normal, then $\hat{Z} = k + J$.

We give examples of dimer algebras exhibiting various properties of the central nilradical. Notably, we show that $\text{nil } Z$ may be nonzero (Example 3.1); the containment $\hat{Z} \subseteq R$ may be proper (Example 4.3); and $\hat{Z}$ may not be normal (Proposition 6.4).

2. Preliminaries

Throughout, $k$ is an algebraically closed field of characteristic zero. Given a quiver $Q$, we denote by $kQ$ the path algebra of $Q$, and by $Q_\ell$ the paths of length $\ell$. The vertex idempotent at vertex $i \in Q_0$ is denoted $e_i$, and the head and tail maps are denoted $h, t : Q_1 \to Q_0$. By monomial, we mean a non-constant monomial.
Definition 2.1.

- A dimer quiver $Q$ is a quiver whose underlying graph $\overline{Q}$ embeds into a compact surface $\Sigma$ such that each connected component of $\Sigma \setminus \overline{Q}$ is simply connected and bounded by an oriented cycle of length at least 2, called a unit cycle. The dimer algebra $A$ of $Q$ is the quotient $kQ/I$, where $I$ is the ideal
  \[ I := \langle p - q \mid \exists a \in Q_1 \text{ s.t. } pa \text{ and } qa \text{ are unit cycles} \rangle \subset kQ, \]
  and $p, q$ are paths. Throughout, we will take $\Sigma$ to be a real two-torus.
- A perfect matching $x$ of $Q$ is a set of arrows such that each unit cycle contains precisely one arrow in $x$. A perfect matching $x$ is simple if there is an oriented path between any two vertices in $Q \setminus x$. In particular, $x$ is a simple matching if $Q \setminus x$ supports a simple $A$-module of dimension 1.
- A dimer quiver is nondegenerate if each arrow is contained in a perfect matching. Throughout, we will take all dimer quivers to be nondegenerate.
- If $p$ is a path in $Q$, then we refer to $p + I$ as a path in $A$ since each representative of $p + I$ is a path. If $p, q$ are paths in $Q$ (resp. $A$) that are equal in $A$, then we will write $p \equiv q$ (resp. $p = q$).
- $A$ and $Q$ are non-cancellative if there are paths $p, q, r \in A$ for which $p \neq q$, and $pr = qr \neq 0$ or $rp = rq \neq 0$; in this case, $p, q$ is called a non-cancellative pair. Otherwise, $A$ and $Q$ are cancellative; cancellativity was introduced in [D]. A (nondegenerate) dimer algebra is cancellative if and only if it is noetherian [B4, Theorem 1.1].

Notation 2.2. Let $\pi : \mathbb{R}^2 \to T^2$ be a covering map such that for some $i \in Q_0$,
  \[ \pi(\mathbb{Z}^2) = i. \]
Denote by $Q^+ := \pi^{-1}(Q) \subset \mathbb{R}^2$ the covering quiver of $Q$. For each path $p$ in $Q$, denote by $p^+$ the unique path in $Q^+$ with tail in the unit square $[0, 1) \times [0, 1) \subset \mathbb{R}^2$ satisfying $\pi(p^+) = p$.
For paths $p, q$ satisfying
\begin{equation}
(2) \quad t(p^+) = t(q^+) \quad \text{and} \quad h(p^+) = h(q^+),
\end{equation}
denote by $R_{p,q}$ the compact region in $\mathbb{R}^2 \supset Q^+$ bounded by (representatives of) $p^+$ and $q^+$, and denote by $R_{p,q}^\circ$ the interior of $R_{p,q}$.

Notation 2.3. By a cyclic subpath of a path $p$, we mean a subpath of $p$ that is a nontrivial cycle. Consider the following sets of cycles in $A$:

- Let $C$ be the set of cycles in $A$ (i.e., cycles in $Q$ modulo $I$).

\[ \text{The dual graph of a dimer quiver is called a dimer model or brane tiling, or, if on a disc, a plabic (}=\text{planar bicolored}) \text{ graph } [P]. \]
• For \( u \in \mathbb{Z}^2 \), let \( C^u \) be the set of cycles \( p \in C \) such that 
\[ h(p^+) = t(p^+) + u \in Q_0^+ \.
• For \( i \in Q_0 \), let \( C_i \) be the set of cycles in the vertex corner ring \( e_i Ae_i \).
• Let \( \hat{C} \) be the set of cycles \( p \in C \) such that \((p^2)^+\) does not have a cyclic subpath; or equivalently, the lift of each cyclic permutation of \( p \) does not have a cyclic subpath.

We denote the intersection \( \hat{C} \cap C^u \cap C_i \), for example, by \( \hat{C}^u_i \). Note that \( C_0 \) is the set of cycles whose lifts are cycles in \( Q^+ \). In particular, \( \hat{C}_0 = Q_0 \). Furthermore, the lift of any cycle \( p \) in \( \hat{C} \) has no cyclic subpaths, although \( p \) itself may have cyclic subpaths.

**Lemma 2.4.** \([B2, \text{Lemma 4.13.2}]\) Suppose paths \( p^+, q^+ \) have no cyclic subpaths modulo \( I \), satisfy (2), and bound a region \( R_{p,q} \) that contains no vertices in its interior. Then \( p \equiv q \).

Let \( A = kQ/I \) be a dimer algebra. For each perfect matching \( x \) of \( A \), consider the map
\[ n_x : Q_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0} \]
defined by sending a path \( p \) to the number of arrow subpaths of \( p \) that are contained in \( x \). Observe that \( n_x \) is additive on concatenated paths. Furthermore, if \( p, p' \in Q_{\geq 0} \) are paths satisfying \( p + I = p' + I \), then \( n_x(p) = n_x(p') \), by \([B2, \text{Lemma 2.1}]\). In particular, \( n_x \) induces a well-defined map on the paths of \( A \).

Now consider dimer algebras \( A = kQ/I \) and \( A' = kQ'/I' \), and suppose \( Q' \) is obtained from \( Q \) by contracting a set of arrows \( Q_1^* \subset Q_1 \) to vertices. This contraction defines a \( k \)-linear map of path algebras
\[ \psi : kQ \rightarrow kQ' \]
If \( \psi(I) \subseteq I' \), then \( \psi \) induces a \( k \)-linear map of dimer algebras \( \psi : A \rightarrow A' \), called a contraction.\(^2\)

To specify the structure we wish \( \psi \) to preserve, consider the polynomial ring \( k[S'] \) generated by the simple matchings \( S' \) of \( A' \). To each path \( p \in A' \), associate the monomial
\[ \bar{\tau}(p) := \prod_{x \in S'} x^{n_x(p)} \in k[S'] \]
For each \( i, j \in Q_0' \), this association may be extended to a \( k \)-linear map \( \bar{\tau} : e_j A' e_i \rightarrow k[S'] \), and is an algebra homomorphism if \( i = j \). Given \( p \in e_j Ae_i \) and \( q \in e_i A' e_k \), we shall write
\[ \bar{\tau}(p) := \tau(p) := \bar{\tau}(\psi(p)) \text{ and } \bar{q} := \bar{\tau}(q). \]
\( \psi \) is called a cyclic contraction if \( A' \) is cancellative and
\[ S := k \left[ \bigcup_{e_i \in Q_0} \bar{\tau}(e_i A e_i) \right] = k \left[ \bigcup_{e_i \in Q_0'} \bar{\tau}(e_i A' e_i) \right] =: S'. \]
\(^2\)If, for example, \( \psi \) contracts a unit cycle to a vertex, then \( \psi(I) \not\subseteq I' \) by \([B2, \text{Lemma 3.5}]\).
The algebra $S$, called the cycle algebra, is independent of the choice of cyclic contraction $\psi$ [B3, Theorem 3.14]. $S$ is also isomorphic to the center of $A'$, and is a depiction of both the reduced center of $A$ and the center of $\Lambda$ [B6, Theorem 1.1]. Surprisingly, every nondegenerate dimer algebra admits a cyclic contraction [B1, Theorem 1.1]. Cyclic contractions and the cycle algebra were introduced in [B2].

In addition to the cycle algebra $S$, we will also consider the ghor center of $A'$, $R := k \left[ \cap_{i \in Q_0} \tau_\psi(e_i A e_i) \right] = \bigcap_{i \in Q_0} \tau_\psi(e_i A e_i)$.

$R$ is isomorphic to the center of the ghor algebra $\Lambda$, given in [1] [B2, Theorem 1.1].

For $g, h$ in $R$ or $S$, we shall write $g \mid h$ if $g$ divides $h$ in the polynomial ring $k[S']$.

The following lemmas will be useful.

**Lemma 2.5.** Let $\psi : A \to A'$ be a cyclic contraction.

1. If $p$ and $q$ are paths in $A$ (or $A'$) satisfying $qp \neq 0$, then $\overline{qp} = \overline{q} \overline{p}$.

2. For each $i, j \in Q'_0$, the $k$-linear map $\overline{\tau} : e_j A e_i \to k[S']$ is injective.

**Proof.** (1) holds since for each simple matching $x \in S$, the map $n_x$ is additive on concatenated paths. (2) holds by [B2, Proposition 4.30].

**Lemma 2.6.** If $\sigma_i, \sigma'_i$ are unit cycles at $i \in Q_0$, then $\sigma_i = \sigma'_i$ in $A$. Furthermore, the element $\sum_{i \in Q_0} \sigma_i \in A$ is central.

**Proof.** Clear.

We denote by $\sigma_i$ the (unique) unit cycle at $i \in Q_0$ modulo $I$, and by $\sigma$ the monomial

$$\sigma := \sigma_i = \prod_{x \in S'} x.$$  

**Lemma 2.7.** [B2, Lemma 4.3.1] If $p, q \in e_j A e_i$ are paths satisfying (2), then there is an $m, n \geq 0$ such that $p \sigma^m = q \sigma^n$.

**Lemma 2.8.** Suppose $Q$ admits a cyclic contraction. Let $u, v \in \mathbb{Z}^2$ and let $p \in C^u$, $q \in C^v$ be cycles.

1. $u = 0$ if and only if $p = \sigma^m$ for some $m \geq 1$.

2. $u = v$ if and only if $p = q \sigma^m$ for some $m \in \mathbb{Z}$.

3. $p \notin \hat{C}$ if and only if $\sigma \mid \overline{p}$.

**Proof.** (1, $\Rightarrow$): [B2, Lemma 5.2].

(2, $\Rightarrow$): [B2, Lemma 4.19].

(2, $\Leftarrow$): Suppose $\overline{p} = \overline{q} \sigma^m$ for some $m \in \mathbb{Z}$. Let $r$ (resp. $s$) be a path whose lift $r^+$ ($s^+$) has tail (head) $h(p^+)$ and head (tail) $h(q^+)$. Since $\overline{p} = \overline{q} \sigma^m$, there is some $\ell \in \mathbb{Z}$ such that $\tau = \sigma^\ell$. In particular, $\tau(\psi(r)) = \sigma^\ell$. Thus, $\psi(r)$ is in $C^0$ since $A'$ is cancellative [B2, Lemma 4.29]. Whence, $\psi$ contracts $s$ to a vertex. Furthermore, $\psi$ does not contract any cycle to a vertex since $\psi$ is a cyclic contraction [B2, Lemma...
Figure 1. Examples for Remarks 2.9 and Proposition 6.4. The quivers are drawn on a torus, the contracted arrows are drawn in green, and the 2-cycles have been removed from $Q'$. In each example, the cycle in $Q$ formed from the red arrows is not equal to a product of unit cycles (modulo $I$). However, in example (i) this cycle is mapped to a unit cycle in $Q'$ under $\psi$.

3.6.1. Therefore $s$ is a vertex. But then $r$ is in $C^0$ and $h(p^+) = h(q^+)$. Consequently, $u = v$.

$(1, \Leftarrow)$: Follows from $(2, \Leftarrow)$ with $q = e_{t(p)}$.

$(3, \Rightarrow)$: [B2, Lemma 5.2].

$(3, \Leftarrow)$: First recall that if $\psi(p)$ is not in $\hat{C}'$, then the lift $(\psi(p)^2)^+$ has a cyclic subpath (by definition). Whence, the lift $(p^2)^+$ has a cyclic subpath (though the converse need not hold). Thus, $p$ is not in $\hat{C}$. Equivalently, if $p$ is in $\hat{C}$, then $\psi(p)$ is in $\hat{C}'$. But then $\sigma \not| \bar{\tau}(\psi(p)) = \bar{p}$ since $A'$ is cancellative, by [B2, Proposition 4.21.1].

Remark 2.9. Let $p^+$ be a cycle in $Q^+$; then $\bar{p} = \sigma^m$ for some $m \geq 0$ by Lemma 2.8.1. However, $p$ may not necessarily equal a power of the unit cycle $\sigma_{t(p)}$ (modulo $I$). Two examples are given by the red cycles in Figures (i) and (ii).

Furthermore, it is possible for two cycles in $Q^+$, distinct modulo $I$, and one of which is properly contained in the region bounded by the other, to have equal $\bar{\tau}_\psi$-images. Indeed, consider Figure (i): the red cycle and the unit cycle in its interior both have $\bar{\tau}_\psi$-image $\sigma$.

3. The central nilradical from cyclic contractions

Let $A = kQ/I$ be a dimer algebra with center $Z$. In this section we will show that the nilpotent elements in $Z$ are precisely the central elements that vanish under
Figure 2. The nonnoetherian dimer algebra $A = kQ/I$ cyclically contracts to the noetherian dimer algebra $A' = kQ'/I'$. Both quivers are drawn on a torus and the contracted arrow is drawn in green. Here, the cycle algebra of $A$ is $S = k[x^2, y^2, xy, z] \subset k[S'] = k[x, y, z]$, and the ghor center of $A$ is $R = k + (x^2, y^2, xy)S$.

Noetherian dimer algebras are prime [B2, Theorem 4.31, Corollary 5.12], and therefore their centers are reduced. In the following two examples we show that nonnoetherian dimer algebras may have non-reduced centers, and consequently that dimer algebras need not be prime.

**Example 3.1.** Consider the nonnoetherian dimer algebra $A$ with quiver $Q$ given in Figure 3. (A cyclic contraction of $A$ is given in Figure 2.) The paths $p$, $q$, $a$ satisfy

$$z := (p - q)a + a(p - q) \in \text{nil } Z.$$ 

In particular, $\text{nil } Z \neq 0$. $A$ is therefore not prime since

$$zAz = z^2A = 0.$$ 

We note that $A$ also contains non-central elements $a, b$ with the property that $aAb = 0$; for example, $(p - q)Ae_1 = 0$.

**Example 3.2.** Let $Q$ be a dimer quiver containing the subquiver given in Figure 4. Given any cyclic contraction $\psi : A \to A'$, the $\psi$-image of the cycle $st$ is a unit cycle in $Q'$. Set $p := cbtba$. Then

$$p + \sum_{j \in Q_0 \setminus \{i\}} \sigma_j^2 \quad \text{and} \quad z := p - \sigma_i^2$$

are nonzero central elements of $A$, by Lemma 2.6. Furthermore, $z^2 = 0$, and so $z$ is in the central nilradical of $A$.

**Lemma 3.3.** Let $i \in Q_0$, and suppose $z \in A$ is a central element for which $ze_i = p - q$. Then

$$pq = qp.$$
Figure 3. The dimer algebra $A$ given in Figure 2 has a non-vanishing central nilradical, $\text{nil } Z \neq 0$. A fundamental domain of $Q$ is shown on the left and a larger region of $Q^+$ is shown on the right. The paths $p, q, a$ are drawn in red, blue, and teal respectively. The element $(p - q)a + a(p - q)$ is central and squares to zero.

Figure 4. The subquiver of $Q$ in Example 3.2. The paths $a$ and $c$ are arrows in $Q$, and all other paths are paths of positive length. Setting $p := cbtba$, the elements $p + \sum_{j \in Q_0 \setminus \{i\}} \sigma_j^2$ and $z := p - \sigma_i^2$ are in the center of $A$. Furthermore, $z^2 = 0$, and so $A$ has a nonvanishing central nilradical.

**Proof.** Since $z$ is central, we have

$$p^2 - pq = p(p - q) = pz = zp = (p - q)p = p^2 - qp.$$  

Whence $pq = qp$.  

**Proposition 3.4.** Let $z \in Z$ and $i \in Q_0$, and suppose there is a non-cancellative pair of cycles $p, q \in e_iAe_i$ such that $ze_i = p - q$. Then

$$p^2 = pq = qp = q^2.$$  

**Proof.** In the following, by ‘path’ or ‘cycle’ we mean a path or cycle in $Q$ (not modulo $I$). If $a$ is an arrow and $s, t$ are paths such that $as, at$ are unit cycles, then $s$ is called an ‘arc’ and $t$ its ‘complementary arc’.
Let $p, q \in kQ$ be representative paths of $p + I, q + I \in A$. To prove the lemma, it suffices to show that $p^2 \equiv p q$, since $q p \equiv p q$ by Lemma 3.3. If $p = \alpha^n$ for some $n \geq 1$, then $p^2 \equiv p q$, by Lemma 2.6. (Such cases exist; see Example 3.2.) So suppose $p$ is not a power of a unit cycle.

Since $q p \equiv p q$ by Lemma 3.3 we may assume that the representatives $p, q$ factor into paths

$$p = \alpha' p' \alpha \quad \text{and} \quad q = \beta q' \beta',$$

where $\alpha, \alpha', \beta, \beta' \in Q_{\geq 1}$ are subpaths of unit cycles and $\alpha \beta$ and $\beta' \alpha'$ are arcs. Let $\gamma, \gamma'$ be their complementary arcs:

$$\alpha \beta \equiv \gamma \quad \text{and} \quad \beta' \alpha' \equiv \gamma'.$$
There are two main cases to consider.

(a) First assume that $p^{2^+}$ does not intersect itself.

(a.i) Consider the setup given in Figure 5, where $p, q$ factor into paths

$$p = ap''r \quad \text{and} \quad q = q''br,$$

with $a, b \in Q_1$ and $r \in Q_{\geq 0}$.

If $rp \equiv rq$, then

$$p^2 = ap''rp \equiv ap''rq = pq,$$

which is what we wanted to show.

So suppose $p^2 \not\equiv pq$; then $rp \not\equiv rq$. Thus, by our choice of representatives $p, q$ satisfying (4), $bra$ must be a subpath of a unit cycle. However, it is clear from the figure that this is not possible.

(a.ii) Since case (i) is not possible and $p, q$ factor into the paths (4), we have the setup shown in Figure 5. From (5) we have

$$qp \equiv \beta q'\gamma \alpha \quad \text{and} \quad pq \equiv \alpha' p' \gamma q' \beta'. $$

But $(\beta q'\gamma \alpha) ^+ \quad \text{and} \quad (\alpha' p' \gamma q' \beta') ^+$ have moved away from $(pq) ^+$ and $(qp) ^+$ respectively. Similarly, further applications of the dimer relations $I$ only homotope $(pq) ^+$ and $(qp) ^+$ further away from each other. Consequently, it is not possible that $qp \equiv pq$ in this case, a contradiction.

(b) Now assume that $p^{2^+}$ intersects itself.

(b.i) First suppose $p, q$ share a common leftmost (or rightmost) nontrivial subpath $\beta \in Q_{\geq 1}$. Then there are paths $p', q' \in Q_{\geq 1}$ such that

$$p_1 := p = \beta p' \quad \text{and} \quad q_1 := q = \beta q'. $$

Set

$$p_2 := p' \beta \quad \text{and} \quad q_2 := q' \beta.$$ 

Let $z \in Z$ be such that $z e_{h(\beta)} = p_1 - q_1 + I$. Then, since $z \beta = \beta z$, we have $z e_{i(\beta)} = p_2 - q_2 + I$. Therefore, by Lemma 3.3,

$$p_1 q_1 \equiv q_1 p_1 \quad \text{and} \quad p_2 q_2 \equiv q_2 p_2.$$ 

It thus suffices to assume that $p'$ factors into paths $p' = \alpha' p'' \alpha$, that is,

$$p_1 = \beta \alpha' p'' \alpha \quad \text{and} \quad p_2 = \alpha' p'' \alpha \beta,$$

where $\alpha \beta$ is an arc subpath of $p_1 q_1$ and $\beta \alpha'$ is an arc subpath of $q_2 p_2$.

Since $\alpha \beta$ and $\beta \alpha'$ are both arcs, $\alpha \beta \alpha'$ must be a unit cycle with $\alpha, \alpha'$ arrows. We therefore have the setup shown in Figure 5.b.i. Here, $\gamma, \alpha \beta$ are complementary arcs, and $\gamma', \beta \alpha'$ are complementary arcs.

In order to homotope the path $p_1 q_1$ to $q_1 p_1$, we first use the relation $\alpha \beta \equiv \gamma$. Continuing, we obtain

$$p_1 q_1 = (\beta \alpha' p'' \alpha)(\beta q') \equiv \beta \alpha' p'' \gamma q' \equiv \beta \alpha' p'' p \alpha \sigma_{h(\beta)}$$
for some \( \ell \geq 0 \), by Lemma \( \ell \). But \( p_1 = q_1 \) since \( p_1, q_1 \) is a non-cancellative pair. Whence, \( \ell = 1 \). Therefore,

\[
pq = p_1q_1 \equiv \beta\alpha'p''p'^{\sigma_{h(\beta)}} \equiv \beta\alpha'p''p_{h(\beta)}p'^{\sigma_{h(\beta)}} = \beta\alpha'p''(\alpha\beta\alpha')p'' = p_1^2 = p^2,
\]

which is what we wanted to show. Similarly, \( p_2q_2 \equiv p_2^2 \).

(b.ii) Finally, suppose \( p, q \) do not share a common leftmost or rightmost nontrivial subpath. Then, since \( p, q \) factor into the paths \( (4) \), we have the setup shown in Figure 5b.ii. (Although not drawn, \( p^{2+} \) and \( q^{2+} \) may intersect themselves multiple times.)

Factor \( p \) into arrow subpaths, \( p = a_n \cdots a_2a_1, a_j \in Q_1 \). Denote by

\[
p_j := a_{j-1} \cdots a_{j+1}a_j
\]

the cyclic permutation of \( p \) starting with arrow \( a_j \). Since \( z \in Z \) is central and the relations \( I \) are generated by binomials in paths of \( Q \), for each \( j \in [1, m] \) there are cycles \( p_j', q_j \) at \( h(a_{j-1}) \) for which

\[
(p'_j - q_j)(a_{j-1} \cdots a_2a_1) = z(a_{j-1} \cdots a_1) \\
= (a_{j-1} \cdots a_1)z = (a_{j-1} \cdots a_1)(p - q) = p_j(a_{j-1} \cdots a_1) - (a_{j-1} \cdots a_1)q.
\]

Upon setting \( p'_j(a_{j-1} \cdots a_1) = p_j(a_{j-1} \cdots a_1) \), we have

\[
(6) \quad q_j(a_{j-1} \cdots a_1) \equiv (a_{j-1} \cdots a_1)q.
\]

In particular, without loss of generality we may assume

\[
(7) \quad z_{e_t(p_j)} = p_j - q_j + I.
\]

Whence,

\[
(8) \quad q_{j+1}a_j \equiv a_jq_j
\]

since \( p_{j+1}a_j = a_j \cdots a_{j+1}a_j = a_jp_j \) and

\[
(p_{j+1} - q_{j+1})a_j = za_j \equiv a_jz = a_j(p_j - q_j).
\]

Suppose \( p_j \equiv q_j \) for some \( j \in [1, m] \). Then (6) implies

\[
pq = (a_m \cdots a_j)(a_{j-1} \cdots a_1)q \equiv (a_m \cdots a_j)q_j(a_{j-1} \cdots a_1) \\
\equiv (a_m \cdots a_j)p_j(a_{j-1} \cdots a_1) = p^2,
\]

which is what we wanted to show.

So suppose \( p_j \not\equiv q_j \) for each \( j \in [1, m] \). Then \( p_j, q_j \) is a non-cancellative pair: \( p, q \)

is a non-cancellative pair, so there is a path \( r \) such that \( (p - q)r \equiv 0 \), and therefore

\[
(p_j - q_j)(a_{j-1} \cdots a_1)r \equiv (a_{j-1} \cdots a_1)(p - q)r \equiv 0.
\]

We may assume that \( q_j \) is a representative of \( q_j + I \) for which the region \( \mathcal{R}_{p_j,q_j} \) contains a minimal number of unit cycles.

By assumption, there are minimal indices \( 1 \leq k < \ell \leq n \) such that \( t(a_k) = h(a_\ell) \).

Factor \( q_j \) into paths \( q_j = \beta_jq_j' \), where \( \beta_j \) is the maximal leftmost subpath of \( q_j \) that is a subpath of a unit cycle. Then, by the minimality of \( \mathcal{R}_{p_j,q_j} \) and the minimality
of the index $k$, (b.i) implies that $a_j \beta_j$ is an arc for $j \in [1, k-1]$. This is shown in Figure 6 where each $c_j \in Q_1$ is an arrow and $c_ja_j \beta_j$ is a unit cycle. Observe that the complementary arc to $(a_j \beta_j)^+$ lies in the region $R_{p_j,q_j}$.

Since $a_j$ is a rightmost arrow subpath of $p_j$ and $ze_{c(p_j)} = p_j - q_j + I$ by (7), we may assume that $a_j$ is not a rightmost arrow subpath of any representative of $q_j + I$ since otherwise we have case (b.i) with $p_j, q_j$ in place of $p, q$. In particular, $q_j$ does not have a unit cycle subpath modulo $I$, by Lemma 2.6.

Now consider (8) with $j = k$:

$$a_kq_k \equiv q_{k+1}a_k.$$  

As we have just shown, the arrow $a_k$ is not a rightmost subpath of $q_k$ modulo $I$. But it is clear from Figure 6 that $(a_k \beta_k)^+$ cannot be an arc whose complementary arc lies in $R_{p_k,q_k}$. Therefore, by the minimality of $R_{p_k,q_k}$ we have $a_kq_k \not= sa_k$ for all paths $s$. Consequently $ze_{c(p_k)} \not= p_k - q_k + I$, contrary to (7). \hfill \square

**Theorem 3.5.** Let $A$ be a nonnoetherian dimer algebra with center $Z$ and $\psi : A \to A'$ a cyclic contraction. Then

$$Z \cap \ker \psi = \text{nil } Z.$$  

*Proof.* (i) We first claim that if $z \in Z \cap \ker \psi$, then $z^2 = 0$, and in particular $z \in \text{nil } Z$. Consider a central element $z$ in $\ker \psi$. Since $z$ is central it commutes with the vertex idempotents, and so $z$ is a $k$-linear combination of cycles. Therefore, since $\psi$ sends paths to paths and $I'$ is generated by certain differences of paths, it suffices to suppose that $z$ is of the form

$$z = \sum_{i \in Q_0} (p_i - q_i),$$  

Figure 6. Case (b.ii) in the proof of Proposition 3.4.
where $p_i$, $q_i$ are cycles in $e_i Ae_i$ with equal $\psi$-images modulo $I'$. Note that there may be vertices $i \in Q_0$ for which $p_i = q_i = 0$.

By Proposition 3.4, we have

$$p_ip_i = p_iq_i = q_ip_i.$$  

Therefore

$$z^2 = \left( \sum_{i \in Q_0} (p_i - q_i) \right)^2 = \sum_{i \in Q_0} (p_i - q_i)^2 = 0.$$  

(ii) We now claim that if $z \in \text{nil } Z$, then $z \in \ker \psi$.

Suppose $z^n = 0$. Then for each $i \in Q_0$, we have

$${\bar{\tau}}(z^n e_i) = {\bar{\tau}}(z^n e_i) = 0,$$

where (i) holds since $\bar{\tau}$ is an algebra homomorphism on $e_i Ae_i$, and (ii) holds since $z$ is central. But $\bar{\tau}(e_i Ae_i)$ is contained in the integral domain $k[S']$. Whence

$${\bar{\tau}}(\psi(z e_i)) = {\bar{\tau}}(z e_i) = 0.$$  

Thus $\psi(z e_i) = 0$ since $\bar{\tau}$ is injective, by Lemma 2.5.2. Therefore

$$\psi(z) = \psi \left( z \sum_{i \in Q_0} e_i \right) = \sum_{i \in Q_0} \psi(z e_i) = 0,$$

where (i) holds since the vertex idempotents form a complete set and (ii) holds since $\psi$ is a $k$-linear map.  

\begin{proof}

\end{proof}

4. The central nilradical is prime

Let $Z$ and $S$ be the center and cycle algebra of a nonnoetherian dimer algebra $A = kQ/I$, let $\psi : A \to A'$ be a cyclic contraction, and let $R \subset S$ be the center of the ghor algebra $\Lambda$ of $Q$. In this section we will show that the reduced center $\hat{Z} := Z/ \text{nil } Z$ of $A$ is an integral domain.

**Theorem 4.1.** There is an exact sequence of $Z$-modules

\[
0 \longrightarrow \text{nil } Z \longrightarrow Z \xrightarrow{\tilde{\psi}} R,
\]

where $\tilde{\psi}$ is an algebra homomorphism. Therefore $\hat{Z} := Z/ \text{nil } Z$ is isomorphic to a subalgebra of $R$.

**Proof.** (i) We first claim that for each $i \in Q_0$, the map

\[
\tilde{\psi} : Z \to R, \quad z \mapsto \overline{z e_i},
\]

is a well-defined algebra homomorphism and independent of the choice of $i$.  

\begin{proof}

\end{proof}
Consider a central element $z \in Z$ and vertices $j, k \in Q_0$. Since $Q$ is a dimer quiver, there is a path $p$ from $j$ to $k$. For $i \in Q_0$, set $z_i := ze_i \in e_iAe_i$. Recall that $\bar{\tau}_\psi$ is an algebra homomorphism on each vertex corner ring $e_iAe_i$. Thus
\[ pz_j = p\bar{\tau}_\psi(z) = \bar{\tau}_\psi(pz) = \bar{\tau}_\psi cz_kp \in k[S']. \]
But $\bar{p} = \bar{\tau}(\psi(p))$ is nonzero since $\bar{\tau}$ is injective by Lemma 2.5.2, and the $\psi$-image of any path is nonzero. Thus, since $k[S']$ is an integral domain,
\[ z_j = z_k. \]
Therefore, since $j, k \in Q_0$ were arbitrary, $z_j \in k[\cap_{i \in Q_0} \bar{\tau}_\psi(e_iAe_i)] = R$.

(ii) Let $z \in Z$, $i \in Q_0$, and suppose $\psi(ze_i) = 0$. We claim that $\psi(z) = 0$.
For each $j \in Q'_0$, denote by
\[ c_j := |\psi^{-1}(j) \cap Q_0| \]
the number of vertices in $\psi^{-1}(j)$. Since $\psi$ maps $Q_0$ surjectively onto $Q'_0$, we have $c_j \geq 1$. Furthermore, if $k \in \psi^{-1}(j)$, then
\[ \psi(z)e_j = c_j \psi(ze_k). \]
Set
\[ z'_j := c_j^{-1} \psi(z)e_j. \]
Then
\[ z' := \sum_{j \in Q'_0} z'_j \]
is in the center $Z'$ of $A'$ by [11] and [B2] (6) and Theorem 5.9.1[3]. Whence, for each $j \in Q_0$,
\[ \bar{\tau}(z'_j) = \bar{\tau}(z'_{\psi(i)}) = \bar{\tau}(c_j^{-1} \psi(z)e_{\psi(i)}) = \bar{\tau}(\psi(ze_i)) = 0, \]
where (i) holds by [11] and (ii) holds by [12]. Thus, each $z'_j$ vanishes since $\bar{\tau}$ is injective, by Lemma 2.5.2. Therefore
\[ \psi(z) = \sum_{j \in Q'_0} \psi(z)e_j = \sum_{j \in Q'_0} c_j z'_j = 0. \]

(iii) We now claim that the homomorphism [10] can be extended to the exact sequence [9]. Let $z \in \ker \psi$. Then for each $i \in Q_0$,
\[ \bar{\tau}(\psi(ze_i)) = ze_i = \bar{\psi}(z) = 0. \]
Whence $\psi(ze_i) = 0$ since $\bar{\tau}$ is injective. Thus $\psi(z) = 0$ by Claim (ii). Therefore, by Theorem 3.5
\[ z \in \ker \psi \cap Z = \text{nil } Z. \]

[3] Note that $\psi(z)$ is not in $Z'$ if there are vertices $i, j \in Q'_0$ for which $c_i \neq c_j$. Therefore, in general $\psi(Z)$ is not contained in $Z'$. 
Corollary 4.2. The algebras \( \hat{Z} \) and \( R \) are integral domains. Therefore, the central nilradical \( \text{nil } Z \) of \( A \) is a prime ideal of \( Z \). In particular, the schemes \( \text{Spec } Z, \text{Spec } \hat{Z}, \) and \( \text{Spec } R \) are irreducible.

Proof. \( R \) and \( S \) are domains since they are subalgebras of the domain \( k[\mathcal{S}'] \). Therefore \( \hat{Z} \) is a domain since it isomorphic to a subalgebra of \( R \) by Theorem 4.1.

For brevity, we will identify \( \hat{Z} \) with its isomorphic \( \bar{\psi} \)-image in \( R \) (Theorem 4.1), and thus write \( \hat{Z} \subseteq R \).

The following example shows that it is possible for the reduced center \( \hat{Z} \) to be properly contained in the ghor center \( R \). However, they determine the same non-noetherian variety \([B6]\), and we will show below that their normalizations are equal (Theorem 5.5).

Example 4.3. Dimer algebras exist for which the containment \( \hat{Z} \hookrightarrow R \) is proper. Indeed, consider the contraction given in Figure 7. This contraction is cyclic since the cycle algebra is preserved:

\[
S = k[x^2, xy, y^2, z] = S'.
\]

We claim that the reduced center \( \hat{Z} \) of \( A = kQ/I \) is not isomorphic to \( R \). By the exact sequence (9), it suffices to show that the homomorphism \( \bar{\psi} : Z \rightarrow R \) is not surjective.

We claim that the monomial \( z\sigma \) is in \( R \), but is not in the image \( \bar{\psi}(Z) \). It is clear that \( z\sigma \) is in \( R \) from the \( \bar{\tau}_\psi \) labeling of arrows given in Figure 7.

Assume to the contrary that \( z\sigma \in \bar{\psi}(Z) \). Then, by (11), for each \( j \in Q_0 \) there is an element in \( Z e_j \) whose \( \bar{\tau}_\psi \)-image is \( z\sigma \). Consider the vertex \( i \in Q_0 \) shown in Figure 8. The set of cycles in \( e_i A e_i \) with \( \bar{\tau}_\psi \)-image \( z\sigma \) are drawn in red. As is shown in the figure, none of these cycles ‘commute’ with both of the arrows with tail at \( i \). Therefore \( \hat{Z} \not\cong R \).

Example 4.3 raises the following question.

Question 4.4. Are there necessary and sufficient conditions for the isomorphism \( \hat{Z} \cong R \) to hold?

5. Normalization of the reduced center

Let \( Z \) and \( S \) be the center and cycle algebra of a nonnoetherian dimer algebra \( A = kQ/I \), let \( \psi : A \rightarrow A' \) be a cyclic contraction, and let \( R \subseteq S \) be the center of the ghor algebra \( \Lambda \) of \( Q \). In this section we will show that the normalizations of \( \hat{Z} = Z/\text{nil } Z \) and \( R \) are equal, nonnoetherian, and properly contained in the cycle algebra \( S \). We denote by \( \bar{Z} \) and \( \bar{R} \) their respective normalizations. Recall that for \( g, h \in S \), we write \( g \mid h \) if \( g \) divides \( h \) in the polynomial ring \( k[\mathcal{S}'] \).
Figure 7. A cyclic contraction $\psi : A \to A'$ for which the containment $\hat{Z} \hookrightarrow R$ is proper. $Q$ and $Q'$ are drawn on a torus, and the contracted arrows are drawn in green. The arrows drawn in blue form removable 2-cycles under $\psi$. The arrows in $Q$ are labeled by their $\bar{\tau}_\psi$-images, and the arrows in $Q'$ are labeled by their $\bar{\tau}$-images.

Figure 8. There are six cycles $p_1, \ldots, p_6$ (or five distinct cycles modulo $I$) at $i$ with $\bar{\tau}_\psi$-image $z\sigma = xyz^2$, drawn in red. There are two arrows with tail at $i$, labeled $a$ and $b$. Observe that if the rightmost arrow subpath of $p_j$ is $a$ (resp. $b$), then $bp_j$ (resp. $ap_j$) cannot homotope to a path whose rightmost arrow subpath is $b$ (resp. $a$). Therefore there is no cycle $q_j$ for which $bp_j \equiv q_j b$ (resp. $ap_j \equiv q_j a$). Consequently, $z\sigma$ is in $R \setminus \hat{Z}$. 
Lemma 5.1. Let $u \in \mathbb{Z}^2 \setminus 0$. Suppose $a \in Q_1$, $p \in \hat{C}_t(a)$, and $q \in \hat{C}_h(a)$. If $R_{o,p,qa}$ contains no vertices, then $ap = qa$. Consequently, $p = q$.

Proof. Suppose that there are representatives of $(ap) +$ and $(qa) +$ that bound a compact region $R_{o,p,qa}$ with no vertices in its interior. If $(ap) +$ and $(qa) +$ have no cyclic subpaths (modulo $I$), then $ap = qa$ by Lemma 2.4.

So suppose $(qa) +$ contains a cyclic subpath. The path $q +$ has no cyclic subpaths since $q$ is in $\hat{C}$. Thus $q$ factors into paths $q = q_2q_1$, where $(q_1a) +$ is a cycle. In particular,

$$t(p^+) = t((q_1q_2)^+) \quad \text{and} \quad h(p^+) = h((q_1q_2)^+).$$

Whence $p$ and $q_1q_2$ bound a compact region $R_{p,q_1q_2}$. Furthermore, its interior $R_{o,p,q_1q_2}$ contains no vertices.

The path $(q_2^2)^+$ has no cyclic subpaths, again since $q$ is in $\hat{C}$. Thus $(q_1q_2)^+$ also has no cyclic subpaths. Furthermore, $p^+$ has no cyclic subpaths since $p$ is in $\hat{C}$. Therefore $p = q_1q_2$, again by Lemma 2.4.

Since there are no vertices in $R_{o,p,qa}$, there are also no vertices in the interior of the region bounded by the cycle $(aq_1)^+$. Thus, there is some $\ell \geq 1$ such that

$$aq_1 = \sigma_{h(a)}^\ell \quad \text{and} \quad q_1a = \sigma_{t(a)}^\ell.$$

Therefore,

$$ap = aq_1q_2 = \sigma_{h(a)}^\ell q_2 = q_2\sigma_{t(a)}^\ell = q_2q_1a = qa,$$

where (i) holds by Lemma 2.6.

Finally, $ap = qa$ and $\bar{a} \neq 0$ together imply $\bar{p} = \bar{q}$ by Lemma 2.5.1. \hfill \Box

Lemma 5.2. If $g$ is a monomial in $k[S']$ and $g\sigma$ is in $S$, then $g$ is also in $S$.

Proof. Suppose $g$ is a monomial in $k[S']$ for which $g\sigma$ is in $S$. Then there is a cycle $p \in A'$ such that $\bar{p} = g\sigma$. Let $u \in \mathbb{Z}^2$ be such that $p \in C'_{nu}$. Since $A'$ is cancellative, $C'_{nu} \neq \emptyset$ by [B2, Proposition 4.11]; fix $q \in C'_{nu}$. Then $\sigma \nmid \bar{q}$ by [B2, Proposition 4.21.1]. Thus, there is some $m \geq 1$ such that

$$\bar{q}\sigma^m \in R = g\sigma,$$

by [B2, Lemma 4.19]. Therefore,

$$g = (g\sigma)\sigma^{-1} = \bar{q}\sigma^{m-1} \in S' \overset{(1)}{=} S,$$

where (i) holds since $\psi$ is cyclic. \hfill \Box

Lemma 5.3. There is some $n \geq 1$ for which

$$(13) \quad \sigma^{n-1}S \notin R \quad \text{and} \quad \sigma^nS \subset R.$$

Proof. Let $s \in S$. Since $S$ is generated over $k$ by a set of monomials in $k[S']$, we may assume that $s$ is a monomial. In particular, there is a cycle $p$ for which $\bar{p} = s$. 

By Lemma 2.6, there is some \( n \geq 1 \) such that for each \( i \in Q_0 \), the unit cycle \( \sigma_i^n \) is equal (modulo \( I \)) to a cycle \( q_i \) that passes through each vertex of \( Q \). Thus, the concatenated cycle \( q_{(p)j} \) passes through each vertex of \( Q \). Whence \( \sigma^n p = \overline{t(p)j} \) is in \( R \). Therefore \( \sigma^n S \subset R \). Since \( S \not\subset R \), there is a minimal such \( n \geq 1 \). \( \square \)

**Proposition 5.4.**

(1) If \( r \in R \) and \( \sigma \nmid r \), then \( r \in \hat{Z} \).

(2) If \( s \in S \), then there is some \( n \geq 0 \) such that for each \( m \geq 1 \), \( s^m \sigma^n \in \hat{Z} \).

(3) If \( r \in R \), then there is some \( n \geq 1 \) such that \( r^n \in \hat{Z} \).

**Proof.** (1) Since \( R \) is generated over \( k \) by a set of monomials in \( k[S'] \), it suffices to consider a monomial \( r \in R \) for which \( \sigma \nmid r \). For each vertex \( i \in Q_0 \), then, there is a cycle \( c_i \in e_i \mathcal{A} c_i \) satisfying \( \overline{c_i} \). Fix \( a \in Q_1 \) and set

\[
p := c_{i(a)} \quad \text{and} \quad q := c_{b(a)}.
\]

See Figure 9. We claim that \( ap = qa \).

Let \( u, v \in \mathbb{Z}^2 \) be such that

\[
p \in C^u \quad \text{and} \quad q \in C^v.
\]

Then \( u = v \) since \( p = r = q \), by Lemma 2.8.2. Furthermore, \( u \neq 0 \) since \( \sigma \nmid r \), by Lemma 2.8.1. Therefore, \((ap)^+ \) and \((qa)^+ \) bound a compact region \( \mathcal{R}_{ap, qa}^0 \) in \( \mathbb{R}^2 \).

We proceed by induction on the number of vertices in the interior \( \mathcal{R}_{ap, qa}^0 \).

First suppose there are no vertices in \( \mathcal{R}_{ap, qa}^0 \). Since \( \sigma \nmid r = \overline{p} = \overline{q} \), the cycles \( p \) and \( q \) are in \( \hat{C} \), by Lemma 2.8.3. Therefore \( ap = qa \), by Lemma 5.1.

So suppose \( \mathcal{R}_{ap, qa}^0 \) contains at least one vertex \( i^+ \). Let \( w \in \mathbb{Z}^2 \) be such that \( c_i \in C^w \).

Then \( w = u = v \), again by Lemma 2.8.2. Therefore \( c_i \) intersects \( p \) at least twice or \( q \) at least twice. Suppose \( c_i \) intersects \( p \) at vertices \( j \) and \( k \).

Then \( p \) factors into paths

\[
p = p_2 e_k b c_{j} p_1 = p_2 b p_1.
\]

Let \( d^+ \) be the subpath of \((c_i^2)^+ \) from \( j^+ \) to \( k^+ \). Then

\[
t(d^+) = t(b^+) = j^+ \quad \text{and} \quad h(d^+) = h(b^+) = k^+.
\]

In particular, \( d^+ \) and \( b^+ \) bound a compact region \( \mathcal{R}_{d^+, b^+} \).

Since we are free to choose the vertex \( i^+ \) in \( \mathcal{R}_{ap, qa}^0 \), we may suppose \( \mathcal{R}_{d^+, b^+} \) contains no vertices. Furthermore, \( c_i^+ \) and \( p^+ \) have no cyclic subpaths since \( \sigma \nmid r \), by Lemma 2.8.1. Thus, their respective subpaths \( d^+ \) and \( b^+ \) have no cyclic subpaths. Whence \( d = b \), by Lemma 2.4.

Furthermore, since \( \mathcal{R}_{ap, qa}^0 \) contains less vertices than \( \mathcal{R}_{ap, qa}^0 \), it follows by induction that

\[
ap_2 d p_1 = qa.
\]

Therefore

\[
ap = a(p_2 b p_1) = a(p_2 d p_1) = qa,
\]

proving our claim.
Finally, since \( a \in Q_1 \) was arbitrary, the sum \( \sum_{i \in Q_0} c_i \) is central in \( A \).

(2) Let \( s \in S \) be a monomial. By Lemma 5.3, there is an \( N \geq 0 \) such that \( s^m \sigma^N \) is in \( R \) for each \( m \geq 1 \). Fix \( m \geq 1 \) and set \( r := s^m \sigma^N \). Then, for each \( i \in Q_0 \), there is a cycle \( c_i \in e_i A e_i \) for which \( c_i = r \).

Fix an arrow \( a \in Q_1 \). Set \( i := t(a) \) and \( j := h(a) \). Let \( t^+ \) be a path in \( Q^+ \) from \( h((ac_i)^+) \) to \( t((ac_i)^+) \). Then by Lemma 2.7, there is some \( \ell, m_i, m_j \geq 0 \) such that

\[
 tc_j a \sigma_i^\ell = \sigma_j^{m_i} \quad \text{and} \quad ac_i t \sigma_j^\ell = \sigma_j^{m_j}.
\]

Thus,

\[
 \sigma_j^{m_i} = \overline{\tau} \psi(tc_j a \sigma_i^\ell) = t c_j a \sigma_i^\ell = ac_i t \sigma_j^\ell = \overline{\tau} \psi(ac_i t \sigma_j^\ell) = \sigma_j^{m_j}.
\]

Furthermore, \( \sigma \neq 1 \) since \( \overline{\tau} \) is injective by Lemma 2.5.2. Whence \( m := m_i = m_j \) since \( k[S'] \) is an integral domain. Therefore,

\[
 \sigma_j^{m_i} = \overline{\tau} \psi(tc_j a \sigma_i^\ell) = t c_j a \sigma_i^\ell = ac_i t \sigma_j^\ell = \overline{\tau} \psi(ac_i t \sigma_j^\ell) = \sigma_j^{m_j}.
\]

For each \( a \in Q_1 \) there is an \( m = m(a) \) such that (14) holds. Set

\[
 n := \max \{ m(a) \mid a \in Q_1 \}.
\]

Then (14) implies that the element \( \sum_{i \in Q_0} c_i \sigma_i^n \) is central. Furthermore, for each \( k \in Q_0 \),

\[
 \overline{\tau} \psi(c_k \sigma_k^n) = r \sigma^n = s^m \sigma^{N+n}.
\]

The claim then follows since \( m \geq 1 \) was arbitrary.

(3) By Claim (1), it suffices to suppose \( \sigma \mid r \). Then there is a monomial or scalar \( g \in k[S'] \) such that \( r = g \sigma \). By Lemma 5.2, \( g \) is in \( S \) since \( g \sigma = r \in R \subset S \). Therefore, by Claim (2), there is some \( n \geq 1 \) such that

\[
 r^n = g^n \sigma^n \in \hat{Z}.
\]

\[\square\]

**Theorem 5.5.** The normalizations of the reduced and ghor centers are isomorphic,

\[
 \hat{Z} \cong \hat{R}.
\]

**Proof.** By [5, Theorem 1.1], the fraction fields of \( \hat{Z} \), \( R \), and \( S \) coincide,

\[
 \text{Frac} \hat{Z} = \text{Frac} R = \text{Frac} S.
\]

Thus the inclusion \( \hat{Z} \subseteq R \) implies

\[
 \hat{Z} \subseteq \hat{R}.
\]

To show \( \hat{Z} \supseteq \hat{R} \), consider \( r \in R \). Then there is some \( n \geq 1 \) such that \( r^n \in \hat{Z} \), by Proposition 5.4.3. Whence, \( r \) is a root of the monic polynomial

\[
 x^n - r^n \in \hat{Z}[x].
\]
Thus $r$ is in $\tilde{Z}$. Therefore
\begin{equation}
R \subseteq \tilde{Z}.
\end{equation}
But then
\[
\bar{R} \subseteq \tilde{Z} \subseteq \bar{R},
\]
where (i) holds by (17) and (ii) holds by (16). Therefore $\bar{R} \cong \tilde{Z}$.

\begin{proposition}
If $s \in S \setminus R$ is a monomial and $\sigma \nmid s$, then $s^n$ is not in $R$ for each $n \geq 1$. Moreover, there exists such a monomial $s$.
\end{proposition}
\begin{proof}
Holds by [134, Proposition 3.14] since $A$ is nonnoetherian.
\end{proof}

\begin{proposition}
Let $s \in S \setminus R$ be a monomial. Then $\sigma | s$ if and only if $s \in \bar{R}$.
\end{proposition}
\begin{proof}
First suppose $\sigma | s$. Then $s = gs$ for some monomial $g \in k[S']$. Whence $g$ is in $S$, by Lemma 5.2. Thus for sufficiently large $n \geq 1$, $(gs)^n$ is in $\bar{R}$, by Lemma 5.3. But then $x^n - (gs)^n$ is in $\bar{R}[x]$. Furthermore,
\[
gs \in S \subseteq \text{Frac} S = \text{Frac} R,
\]
where the last equality holds by (15). Consequently, $s = gs$ is in the normalization $\bar{R}$.

Now suppose $\sigma \nmid s$. Then $s^n$ is in not in $R$ for each $n \geq 1$, by Proposition 5.6. Furthermore, since $R$ is generated by monomials in a polynomial ring (in particular, $R$ is toric), each monomial in $\bar{R}$ is a root of a monic binomial $x^n - r \in R[x]$ for some $n \geq 1$ and $r \in R$. But then $s$ is not in $\bar{R}$ since $s^n$ is not in $R$ for each $n \geq 1$.
\end{proof}

\begin{theorem}
The normalizations $\bar{R} \cong \tilde{Z}$ are nonnoetherian and properly contained in the cycle algebra $S$.
\end{theorem}
\begin{proof}
Since $A$ is nonnoetherian, there is a monomial $s \in S \setminus R$ for which $\sigma \nmid s$, by Proposition 5.6. Thus $s$ is not in $\bar{R}$, by Proposition 5.7. Whence $\bar{R} \neq S$. But
$S = S' = R'$ is the center of the noetherian (equivalently, cancellative) dimer algebra $A'$ and so is normal. Consequently, the inclusion $R \subset S$ implies $\bar{R} \subseteq \bar{S} = S$. Therefore $\bar{R}$ is properly contained in $S$.

Moreover, since $\sigma = \prod_{x \in S'} x$, we also have $\sigma \nmid s^n$ for each $n \geq 1$. Thus, $s^n$ is not in $\bar{R}$ for each $n \geq 1$, again by Proposition 5.7. It follows that $\bar{R}$ is nonnoetherian by [B4, Claims (i) and (iii) in the proof of Theorem 3.16].

Alternatively, recall that an element $s$ of the fraction field $\text{Frac} \, T$ of an integral domain $T$ is said to be ‘almost integral over $T$’ if there is some nonzero $t \in T$ such that $s^m t$ is in $T$ for all $m \geq 0$. It is well known that if $T$ is noetherian, then every almost integral element $s \in \text{Frac} \, T$ over $T$ is integral over $T$ [S, Lemma 10.37.4].

Now let $s \in S \setminus \bar{R}$ be as above with $\sigma \nmid s$. Then $s \in \text{Frac} S = \text{Frac} R = \text{Frac} \bar{R}$, by (15). Thus, for $n \geq 1$ sufficiently large, $s^m s^n \in R \subseteq \bar{R}$ for all $m \geq 0$, by Lemma 5.3. Whence, $s$ is almost integral over both $R$ and $\bar{R}$. But $s \notin \bar{R}$, so $s$ is not integral over $R$ or $\bar{R}$. Therefore both $R$ and $\bar{R}$ are nonnoetherian. \[\square\]

6. Three characterizations of normality

Let $Z$ and $S$ be the center and cycle algebra of a nonnoetherian dimer algebra $A = kQ/I$, let $\psi : A \rightarrow A'$ be a cyclic contraction, and let $R \subset S$ be the center of the ghor algebra $\Lambda$ of $Q$. In this section we will introduce three equivalent conditions for $R$ to be normal. These conditions provide an explicit description of the reduced center $\hat{Z} = Z/\text{nil} Z$ if it is normal.

Lemma 6.1. Suppose $r \in R$ and $s \in S$ are monomials. If $r \neq \sigma^n$ for any $n \geq 1$, then $rs \in R$.

Proof. Suppose the hypotheses hold. Fix $i \in Q_0$. Since $r$ is a monomial in $R$ and $s$ is a monomial in $S$, there are $u, v \in \mathbb{Z}^2$ and cycles

$$p \in \mathcal{C}_i^u \quad \text{and} \quad q \in \mathcal{C}_i^v$$

such that

$$\bar{p} = r \quad \text{and} \quad \bar{q} = s.$$ 

The assumption $r \neq \sigma^n$ for $n \geq 1$ implies $u \neq 0$, by Lemma 2.8.1. Furthermore, if $v = 0$, then $\bar{q} = \sigma^m$ for some $m \geq 1$, again by Lemma 2.8.1. But then $s = \bar{q}$ is in $R$ since $\sigma$ is in $R$. Whence, $rs$ is in $R$.

We may thus suppose that $u, v$ are both nonzero.

(i) If $v = u$, then $\bar{p} = \bar{q} \sigma^m$ for some $m \in \mathbb{Z}$, by Lemma 2.8.2. 

(i.a) If $m \leq 0$, then $s = r \sigma^{|m|}$ is in $R$ since $r$ and $\sigma$ are both in $\bar{R}$. Whence, $rs = r^2 \sigma^{|m|}$ is in $R$.

(i.b) So suppose $m > 0$; then $r = s \sigma^{|m|}$. In particular, $\sigma \mid r = \bar{p}$. Thus, $p$ is not in $\hat{C}$, by Lemma 2.8.3. Consequently, there is a cyclic subpath $c^+$ of the lift $p^{2+}$. It suffices to suppose that $p$ factors into paths

$$p = d_2 p' d_1$$
with \( c = d_1d_2 \in C^0 \) and \( p' \in \hat{C} \); otherwise, if \( p' \) is not in \( \hat{C} \), then repeat the argument with \( p' \) in place of \( p \). Since \( p' \) is in \( \hat{C} \), we have \( \sigma \nmid \tilde{p}' \), by Lemma 2.8.3. Furthermore, since \( c \) is in \( C^0 \), we have \( \tilde{p}' \in C^u \). Thus \( \tilde{p} = \tilde{p}'\sigma^\ell \) for some \( \ell \in \mathbb{Z} \), by Lemma 2.8.2.

Whence, \( \tilde{p}'\sigma^\ell = \tilde{p} = r = s\sigma^m \).

Without loss of generality we may assume \( \sigma \nmid s \). Then, since \( \sigma \nmid \tilde{p}' \), we have \( \ell = m \) and \( \tilde{p}' = s \). Therefore the cycle \( d_2p'\sigma^2d_1 \) has tail at \( i \) and \( \tau_\psi \)-image \( s^2\sigma^m \). Since \( i \in Q_0 \) was arbitrary, it follows that \( rs = s^2\sigma^m \) is in \( R \).

(ii) Finally, suppose \( v \neq u \). Then, since \( u, v \neq 0 \) (and the surface is a torus), the lifts \( p^+ \) and \( q^+ \) intersect at some vertex \( j^+ \) in \( Q^+ \). Consequently, \( p \) and \( q \) factor into paths

\[
p = p_2e_jp_1 \quad \text{and} \quad q = q_2e_jq_1.
\]

We may thus form the cycle

\[
c = p_2q_1q_2p_1 \in e_iAe_i
\]

with \( \tau_\psi \)-image

\[
\tilde{c} = \tilde{p}\tilde{q} = rs.
\]

But \( i \in Q_0 \) was arbitrary, and so it again follows that \( \tilde{c} = rs \) is in \( R \).

\[\square\]

**Proposition 6.2.** A ghor center \( R \) is normal if and only if \( \sigma S \subset R \).

**Proof.** (1) First suppose \( \sigma S \subset R \).

It is well known that cancellative dimer algebras (on a torus) are noncommutative crepant resolutions, and in particular that their centers are normal domains (e.g., [Br, D]). Moreover, \( A' \) is cancellative and its center is isomorphic to \( S \) [B2, Theorem 1.1.3]. Thus, \( S \) is a normal domain. Therefore, since \( R \) is a subalgebra of \( S \), we have \( R \subseteq S \).

Now let \( s \in S \setminus R \). We claim that \( s \) is not in \( \tilde{R} \). Indeed, assume otherwise. Since \( S \) is generated by monomials in the polynomial ring \( k[S'] \), there are monomials \( s_1, \ldots, s_\ell \in S \) and scalars \( s_0, c_1, \ldots, c_\ell \in k \) such that

\[
s = s_0 + c_1s_1 + \cdots + c_\ell s_\ell.
\]

Since \( s \notin R \), there is some \( 1 \leq k \leq \ell \) such that \( s_k \notin R \). Choose \( s_k \) to have maximal degree among the subset of monomials in \( \{s_1, \ldots, s_\ell\} \) which are not in \( \tilde{R} \).

If \( \sigma \mid s_k \) in \( k[S'] \), then there would be a monomial \( g \in k[S'] \) such that \( s_k = \sigma g \). Furthermore, \( g \) would be in \( S \) by Lemma 5.2. Whence, \( s_k = \sigma g \) would be in \( R \) by our assumption that \( \sigma S \subset R \). But this is not possible since \( s_k \) is not in \( \tilde{R} \). Therefore \( \sigma \nmid s_k \).

By assumption \( s \) is in \( \tilde{R} \), and so there is some \( n \geq 1 \) and \( r_0, \ldots, r_{n-1} \in R \) for which

\[
s^n + r_{n-1}s^{n-1} + \cdots + r_1s = -r_0 \in R.
\]
The summand $s_k^n$ of $s^n$ is not in $R$ since $\sigma \nmid s_k$ [B4, Proposition 3.14]. Thus $-s_k^n$ is a summand of the left-hand side of (19). In particular, for some $1 \leq m \leq n$, there are monomial or scalar summands $r'$ of $r_m$ and $s' = s_{j_1} \cdots s_{j_m}$ of $s^m$, and a nonzero scalar $c \in k$, such that

$$r's' = cs_k^n.$$  

Since $\sigma \nmid s_k$, we have $\sigma \nmid s_k^n$, and thus $\sigma \nmid r'$. Whence $r' \neq \sigma^m$ for any $m \geq 1$. Thus, $r'$ is a nonzero scalar since $r' \in R$, $s' \in S$, and $s_k^n \notin R$, by Lemma 6.1. Therefore

$$s_{j_1} \cdots s_{j_m} = s' = (c/r')s_k^n.$$ 

Consequently, $s_{j_1}, \ldots, s_{j_m}$ is not in $R$. But $m \leq n$ and the monomial $s_k$ was chosen to have maximal degree, and so (20) is not possible. Hence,

$$\bar{R} \cap S \subseteq R.$$ 

Therefore $R = \bar{R}$ is normal.

(2) Now suppose $\sigma S \not\subseteq R$. Then there is a monomial $s \in S \setminus R$ for which $\sigma \mid s$. Furthermore, $s$ is in $\bar{R}$ by Proposition 5.7. Consequently, $s$ is in $\bar{R} \setminus R$ and so $R \neq \bar{R}$. 

Corollary 6.3.

(1) If the head or tail of each contracted arrow has indegree 1, then $R$ is normal.

(2) If $\psi$ contracts precisely one arrow, then $R$ is normal.

Proof. In both cases (1) and (2), clearly $\sigma S \subseteq R$. 

Proposition 6.4. For each $n \geq 1$, there exist ghor algebras for which (13) holds. Consequently, there are ghor algebras whose centers are not normal.

Proof. Recall the conifold quiver $Q$ with one nested square given in Figure 11.i. Clearly $\sigma S \subseteq R$. More generally, the conifold quiver with $n \geq 1$ nested squares satisfies (13); see Figure 10. The corresponding ghor center $R$ is therefore not normal for $n \geq 2$ by Proposition 6.2.

Let $m_0 \in \text{Max } R$ be the maximal ideal generated by all monomials in $R$. Let $\tilde{m}_0 \subseteq m_0$ be the ideal of $R$ generated by all monomials in $R$ which are not powers of $\sigma$. Then

$$m_0 = (\tilde{m}_0, \sigma)R.$$ 

Proposition 6.5.

(1) $m_0 = \tilde{m}_0S$, and thus $\tilde{m}_0$ is an ideal of both $R$ and $S$.

(2) Let $n \geq 1$, and suppose $\sigma^nS \subseteq R$. Then

$$R = k[\sigma] + (\tilde{m}_0, \sigma^n)S.$$
Figure 10. Examples for Proposition 6.4. Each quiver is drawn on a torus. In each case, set \( p := ba \) and let \( n \geq 0 \) be the minimum for which \( \tilde{p}\sigma^n \) is in the ghor center \( R \). In (i) we have \( n = 1 \); (ii) \( n = 2 \); and (iii) \( n = 3 \). More generally, these values yield \( \sigma^n S \subset R \) and \( \sigma^{n-1} S \not\subset R \). Consequently, only the ghor center of (i) is normal; the ghor centers of (ii) and (iii) are not normal.

Proof. The equality \( \tilde{m}_0 = \tilde{m}_0 S \) follows from Lemma 6.1. Thus,

\[
R \subseteq k[\sigma] + \tilde{m}_0 \subseteq k[\sigma] + (\tilde{m}_0, \sigma^n) S \subseteq R,
\]

where (i) holds since \( R \) is generated by monomials and (ii) holds since \( \tilde{m}_0 S = \tilde{m}_0 \subset R \).

Theorem 6.6. Let \( R \) and \( S \) be the center and cycle algebra of a ghor algebra. The following are equivalent:

1. \( R \) is normal.
2. \( \sigma S \subset R \).
3. \( R = k + m_0 S \).
4. \( R = k + J \) for some ideal \( J \) in \( S \).
Proof. If the ghor algebra is noetherian, then the conditions trivially hold since in this case $R = S$ and $R$ is normal. So suppose the ghor algebra is nonnoetherian.

(1) $\iff$ (2) holds by Proposition 6.2.
(2) $\Rightarrow$ (3): Suppose $\sigma S \subset R$. Then

$$R \overset{(i)}{=} k + m_0 \subseteq k + m_0 S \overset{(ii)}{\subseteq} R,$$

where (i) holds since $R$ is generated over $k$ by a set of monomials in $k[S']$, and $m_0 \subset R$ is generated over $R$ by all monomials in $R$. To show (ii), let $r \in m_0$ and $s \in S$; we claim that $rs \in R$. Since $m_0$ is generated by the monomials in $R$, we may assume that $r$ is a monomial. Thus, if $r \neq \sigma^m$ for all $m \geq 1$, then $rs \in R$ by Lemma 6.1. Otherwise $rs \in \sigma S$. But $\sigma S \subset R$ by assumption, and so $rs \in R$, proving our claim. Therefore $R = k + m_0 S$.

(3) $\Rightarrow$ (2): Suppose $R = k + m_0 S$. Then, since $\sigma \in m_0$, we have $\sigma S \subset m_0 S \subset R$.

(3) $\Rightarrow$ (4): Clear.

(4) $\Rightarrow$ (2): Suppose $R = k + J$ for some ideal $J$ of $S$. By Lemma 5.3, there is some $n \geq 1$ such that $\sigma^{n-1} S \not\subseteq R$ and $\sigma^n S \subset R$. Fix $g \in S$ for which $g \sigma^{n-1} \not\in R$.

Since $\sigma \in R = k + J$, there is some $c \in k$ such that $c + \sigma \in J$. Then

$$cg\sigma^{n-1} = (c + \sigma)g\sigma^{n-1} - g\sigma^n \in JS + R = J + R = R.$$ 

Whence $c = 0$ since $g\sigma^{n-1} \not\in R$. Thus $\sigma \in J$, and therefore $\sigma S \subset JS = J \subset R$. $\square$

Corollary 6.7. If the reduced center $\hat{Z} = Z/\text{nil} Z$ of a (noetherian or nonnoetherian) dimer algebra is normal, then $\hat{Z} = k + m_0 S$.

Proof. First observe that $\hat{Z} = k + m_0 S$ holds trivially if the dimer algebra is noetherian: in this case, $\hat{Z} = Z = S$ and $m_0 \subset Z$ is the maximal ideal generated by all monomials in $Z$.

So suppose the dimer algebra is nonnoetherian. If $\hat{Z}$ is normal, then

$$R \subseteq R \overset{(i)}{=} Z = \hat{Z} \overset{(ii)}{\subseteq} R,$$

where (i) holds by Theorem 5.5 and (ii) holds by Theorem 4.1. Thus $R = \hat{Z}$, and so $R$ is normal. But then

$$\hat{Z} = R \overset{(i)}{=} k + m_0 S,$$

where (i) holds by Theorem 6.6. $\square$

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