Abstract

In this work, using differential Galois theory, we study the spectral problem of the one-dimensional Schrödinger equation for rational time dependent KdV potentials. In particular, we compute the fundamental matrices of the linear systems associated to the Schrödinger equation. Furthermore we prove the invariance of the Galois groups with respect to time, to generic values of the spectral parameter and to Darboux transformations.

Keywords: Differential Galois Theory, KdV hierarchy, Schrödinger operator, Darboux Transformations, Spectral curves, Rational solitons

2010 MSC: 12H05, 35Q51, 37K10
6 Spectral curves and Darboux-Crum transformations

6.1 Extended Green’s function .............................................. 20
6.1.1 Transformed Green’s functions .................................. 22
6.2 Darboux-Crum transformations for the Spectral curve ............ 25
6.3 Spectral curves and KdV hierarchy in 1 + 1 dimensions .......... 28

7 Differential Galois groups

7.1 Case $E = 0$ ............................................................... 30
7.2 Case $E \neq 0$ ............................................................... 30
7.3 Global behavior of the differential Galois groups ................. 31

Appendix A ........................................................................... 32

1. Introduction

In 1977 Airault, McKean and Moser studied in [2] some special solutions of the KdV equation,

$$u_t - 6uu_x + u_{xxx} = 0,$$

like rational and elliptic ones. Then one year later Adler and Moser studied KdV rational solutions of the KdV hierarchy by means of Darboux-Crum transformations, simplifying the proof of previous results for these solutions [1].

One of the goals of the paper is to study the invariance of the Galois group of the linear system

$$\begin{align*}
    \Phi_x &= U\Phi = \begin{pmatrix} 0 & 1 \\ u - E & 0 \end{pmatrix} \Phi, \\
    \Phi_t &= V\Phi = \begin{pmatrix} G_r(u) & F_r(u) \\ -H_r(u) & -G_r(u) \end{pmatrix} \Phi,
\end{align*}$$

associated to the KdV hierarchy, with respect to the Darboux transformations and respect to the KdV flow (ie, to time). In fact as a by-product we have obtained more than that: the Galois group is also invariant with respect to generic values of the spectral parameters (see section 7).

Thus, in some sense this paper can be considered as a continuation of our previous paper [15], where we studied the invariance of the Galois group of the AKNS systems with respect to the Darboux transformations. But one of the essential differences here is that in general we can not use the Darboux invariance result in [15], because the Darboux transformation here is not a well-defined gauge transformation, ie, it is not invertible. Thus we must use the classical Darboux transformation of the Schrödinger equation, we call it the Darboux-Crum transform; and then to verify the compatibility of this transform with the complete linear system (1.2).

In Section 3 we study the action of the Darboux transformations over the recursive relations (2.1) inside the KdV hierarchy. We point out that the results in Section 3 hold not only for rational KdV potentials but also for any \textit{arbitrary} KdV potential.

Also, in Section 6 we study the action over the spectral curve of the Darboux transformations for stationary KdV \textit{arbitrary} potentials.

Brezhnev in three papers [4, 5, 6] also consider the Galois groups associated to spectral problem for some KdV potentials. More specifically the so-called finite-gap potentials, where the spectral curve is non-singular. Here we study a completely different situation, where the spectral curves are cuspidal curves, corresponding to Adler-Moser rational type solutions.

However, the general results obtained in Sections 3 and 6 open the door to study more general families of KdV potentials, such as Rosen-Morse potentials or elliptic KdV potentials.

2. Basic facts on KdV hierarchy

Let $K$ be a differential field with compatible derivations $\partial_x, \partial_{t_1}, \partial_{t_2}, \ldots, \partial_{t_m}$ with respect to the variables $x$ and $t = (t_1, \ldots, t_m)$. Let us assume that its field of constants is the field of complex numbers $\mathbb{C}$. Let $E \in \mathbb{C}$ be a complex parameter and $u \in K$ be a fixed element of $K$. 
Let us consider the differential recursive relations:

\[ f_0 = 1, \quad f_{j,x} = \frac{1}{4} f_{j-1,x,x} + u f_{j-1,x} + \frac{1}{2} u_j f_{j-1}, \quad (2.1) \]

see [12], where the authors also provided an algorithm to compute \( \partial_x^3 (f_{j,x}) \). Functions \( f_j \) are differential polynomials in \( u \), see [12, 18]. For the first terms one finds

\[ f_0 = 1, \quad f_1 = \frac{1}{2} u + c_1, \quad f_2 = -\frac{1}{8} u_{xx} + \frac{3}{8} u^2 + \frac{1}{2} c_1 u + c_2, \quad f_3 = \frac{1}{32} u_{xxxx} - \frac{5}{16} u_{xxx} - \frac{5}{32} u_x^2 + \frac{5}{16} u_3 + c_1 \left( -\frac{1}{8} u_{xx} + \frac{3}{8} u^2 \right) + \frac{1}{2} c_2 u + c_3, \quad (2.2) \]

for some integration constants \( c_j \).

It is well known that the time dependent KdV hierarchy can be constructed as zero curvature condition of the family of integrable systems (see [13] chapter 1, section 2):

\[
\begin{align*}
\Phi_x &= U \Phi = \begin{pmatrix} 0 & 1 \\ u - E & 0 \end{pmatrix} \Phi, \\
\Phi_t &= V \Phi = \begin{pmatrix} G_r(u) & F_r(u) \\ -H_r(u) & -G_r(u) \end{pmatrix} \Phi,
\end{align*}
\]  

(2.3)

where \( F_r, G_r \) and \( H_r \) are differential polynomials of the potential \( u \) defined by

\[ F_r = \sum_{j=0}^{r} f_{r-j} E^j, \quad (2.4) \]

\[ G_r = -\frac{F_{r,x}}{2}, \quad (2.5) \]

\[ H_r = (E - u) F_r - G_{r,x} = (E - u) F_r + \frac{F_{r,xxx}}{2}. \quad (2.6) \]

Observe that the degree in \( E \) of the matrices \( V_r \) and functions \( H_r \) is \( r + 1 \). We point out that the first equation of (2.3) is equivalent to the Schrödinger equation

\[ (L - E) \phi = (-\partial_{xx} + u - E) \phi = 0 \quad (2.7) \]

with \( L = -\partial_{xx} + u \).

Now, fix a positive integer \( r \) and consider the corresponding system (2.3). Its zero curvature condition

\[ U_{t,r} - V_{r,x} + [U, V_r] = 0, \quad (2.8) \]

yields to the KdV \( r \) equation

\[ \text{KdV}_r : \quad u_{t,r} = -\frac{1}{2} F_{r,xxx} - 2(E - u) F_{r,x} + u_r F_r. \quad (2.9) \]

Using expressions (2.1) and (2.4), this equation can be rewritten as:

\[ \text{KdV}_r : \quad u_{t,r} = 2 f_{r+1,x}. \quad (2.10) \]

We recall that the equation (2.10) is called the level \( r \) equation of the KdV hierarchy. Whenever we want to specify the dependence on the potential \( u \), we will write \( f_j(u), F_j(u), G_j(u) \) and \( H_j(u) \) to emphasize this fact.

### 2.1. Adler-Moser rational potentials

In this section we review the KdV \( r \) rational potentials that Adler and Moser constructed in [1]. These are a family of rational potentials \( u_n \) for Schrödinger operator \(-\partial_{xx} + u\) of the form \( u_n = -2(\log \theta_n)_{xx} \), where \( \theta_n \) are functions in the variables \( x, t \) defined by the differential recursion:

\[ \theta_0 = 1, \quad \theta_1 = x, \quad \theta_{n+1} \theta_{n-1} - \theta_{n+1} \theta_{n-1,x} = (2n + 1) \theta_n^2, \quad (2.11) \]

The solutions of this recursion are polynomials in \( x \) with coefficients in the field \( F = C(t) \). This is an easy consequence of the next result, which is an easy extension of the proof of Lemma 2 in [1].
Lemma 2.1. Let be $F = \mathbb{C}(t)$, and $a \in \mathbb{C}$, $b \in \mathbb{C}$. Let $(F[x], \partial_x)$ be the ring of polynomials with derivative $\partial_x$, whose field of constants is $F$. Let consider the sequence defined recursively by:

$$P_0 = 1, \quad P_1 = ax + b, \quad P_{n+1,x}P_{n-1} - P_{n+1}P_{n-1,x} = (2n + 1)P_n^2.$$  \hspace{1cm} (2.12)

Then $P_n \in F[x]$ for all $n$.

Now, applying Lemma 2.1 for $a = 1$ and $b = 0$, we obtain that functions $\theta_n$ are polynomials of $x$ with coefficients in $\mathbb{C}(t)$ for all $n$. We call these polynomials Adler-Moser polynomials.

The first terms of the recursion are

$$n \quad \theta_n$$

$$0 \quad 1$$

$$1 \quad x$$

$$2 \quad x^3 + \tau_2$$

$$3 \quad x^6 + 5\tau_2x^3 + \tau_3x - 5\tau_2^2,$$

with $\tau_j \in \mathbb{C}(t)$ and $\partial_x \tau_j = 0$.

Definition 2.2. The functions

$$u_n := -2(\log \theta_n)_{xx} \hspace{1cm} (2.13)$$

defined by means of Lemma 2.1 are called KdV rational solitons.

Remark 2.3. Adler and Moser proved in [1] that, for suitable values of $\tau_j$, $j = 2, \ldots, n$, each potential $u_n$ is solution of the KdV$_r$ equation (2.9) for all $r$ and constants $c_i = 0$, $i = 1, \ldots, r$. Their theorem reads as follows:

Theorem 2.4 (Theorem 2, [1]). There is a unique choice of rational functions $\gamma_j(\tau_2, \ldots, \tau_j)$ and differential operators

$$\Xi_j = \sum_{j=1}^{\infty} \gamma_j(\tau_2, \ldots, \tau_j) \frac{\partial}{\partial \tau_j}$$

such that $2f_{r+1,i}(u_n) = \Xi_j u_n$ for $n = 0, 1, 2, \ldots$ and

$$\partial_x v_n = Y_r(v_n) = \Xi_j(v_n) \quad \text{where} \quad v_n = \frac{\theta_{n+1,i}}{\theta_{n+1}} - \frac{\theta_{n,i}}{\theta_n}.$$  \hspace{1cm} (2.14)

(Since $u_n$ and $v_n$ depend only on finitely many variables the sum breaks off.) In other words, if the $\tau_j$ satisfy

$$\frac{d\tau_j}{dt_r} = \gamma_j(\tau_2, \ldots, \tau_j), \quad j \leq n,$$

then $u_n = u_n(\tau_2, \ldots, \tau_n)$ solves the equation $u_n = 2f_{r+1,i}(u)$.

Remark 2.5. Theorem 2.4 shows that for each level $r$ the formula (2.13) for $\theta_n$ is a solution of the KdV$_r$ equation. Hence the constants $\tau_2, \ldots, \tau_j$ must be adapted to get a solution of the KdV$_r$ equation. When this is the case, we will denote adjusted polynomials as $\theta_{r,n}$ and adjusted potentials as $u_{r,n}$ to stress this fact.

Definition 2.6. The functions

$$u_{r,n} := -2(\log \theta_{r,n})_{xx} \hspace{1cm} (2.15)$$

defined by means of Lemma 2.1 and Theorem 2.4 are called KdV$_r$ rational solitons.

Example 2.7. As an example of adjusted potentials, we show the first Adler–Moser potentials for $r = 1$ with the explicit choice of functions $\tau_2, \ldots, \tau_n$. These potentials are solutions of the KdV$_1$ equation for $c_1 = 0$: $u_{i_1} =$
To simplify the notation, from now on we write \( \tau_n \) and denote them just by \( \tau \). The computations were made using SAGE. We have

\[
\begin{array}{c|c|c}
 n & u_{1,n} & (\tau_2, \cdots, \tau_n) \\
\hline
 0 & 0 & \\
 1 & 2 & \\
 2 & \frac{6x(3x^3 - 6t_1)}{(x^3 + 3t_1)^2} & (3t_1) \\
 3 & \frac{6x(2x^9 + 675x^7t_1 + 1350t_1^3)}{(x^6 + 15x^3t_1 - 45t_1^2)^2} & (3t_1, 0) \\
 4 & \frac{10p_3(x, t_1)}{(x^{10} + 45x^7t_1 + 4725x^4t_1^2)^2} & (3t_1, 0, 0) \\
 5 & \frac{30xp_5(x, t_1)}{(\theta_5)^2} & (3t_1, 0, 0, 33075t_1^2) \\
\end{array}
\]

where

\[
p_3(x, t_1) = 2x^{18} + 72x^{15}t_1 + 2835x^{12}t_1^2 - 66150x^9t_1^3 - 1190700x^6t_1^4 + 4465125t_1^6,
\]

\[
p_5(x, t_1) = x^{27} + 126x^{24}t_1 + 7560x^{21}t_1^2 + 5655825x^{18}t_1^3 + 500094000x^{15}t_1^4 + 4313310750x^9t_1^5 + 11252115000x^6t_1^6 + 29536801875x^3t_1^7,
\]

\[
\begin{align*}
\theta_5 &= x^{15} + 105x^{12}t_1 + 1575x^9t_1^2 + 33075x^6t_1^3 + 992250x^3t_1^4 + 1488375t_1^5 \\
\end{align*}
\]

We notice that the adjustment of \( \tau \), is not linear in \( t_1 \).

2.2. Spectral curves for KdV hierarchy

Now, we consider the stationary KdV hierarchy.

\[
s\text{-KdV}_r : \quad 2f_{r+1,x} = 0. \tag{2.16}
\]

We have the following result for Adler-Moser potentials \( u_{r,n} \) in the stationary case [1]:

**Lemma 2.8.** For \( \tau_j = 0, j = 2, \ldots, n \), we have

\[
\theta_n(x, 0) = \theta_n^{(0)}(x) = x^{\frac{n(n+1)}{2}} \quad \text{and} \quad u_{r,n}^{(0)}(x) = u_{r,n}(x, t_1 = 0) = n(n + 1)x^{-2}. \tag{2.17}
\]

The first level of the stationary KdV hierarchy for which potentials \( u_{r,n}^{(0)}(x) = n(n + 1)x^{-2} \) defined in the aforementioned Lemma are solutions of is level \( n \), which implies that in the stationary case we will have \( r = n \). We will denote them just by \( u_{n}^{(0)}(x) \). Therefore, the associated system will be

\[
\begin{align*}
\Phi_x &= U^{(0)}\Phi = \begin{pmatrix} 0 & 1 \\ u_n^{(0)} - E & 0 \end{pmatrix} \Phi, \\
\Phi_{u_n} &= V_n^{(0)}\Phi = \begin{pmatrix} G_n(u_n^{(0)}) & F_n(u_n^{(0)}) \\ -H_n(u_n^{(0)}) & -G_n(u_n^{(0)}) \end{pmatrix} \Phi. \tag{2.18}
\end{align*}
\]

To simplify the notation, from now on we write \( F_n^{(0)} \), \( G_n^{(0)} \) and \( H_n^{(0)} \) instead of \( F_n(u_n^{(0)}), G_n(u_n^{(0)}) \) and \( H_n(u_n^{(0)}) \). The zero curvature condition of this system is now the stationary KdV\(_n\) equation:

\[
s\text{-KdV}_n : \quad 0 = -\frac{1}{2}F_{n,xxx}^{(0)} - 2(E - u_n^{(0)})F_{n,x} + u_n^{(0)}F_{n}^{(0)}. \tag{2.19}
\]

After applying expressions (2.1) and (2.4), this equation can be rewritten as:

\[
s\text{-KdV}_n : \quad 0 = 2f_{n+1,x}^{(0)} = 2f_{n+1,x}^{(0)} = \frac{1}{5}. \tag{2.20}
\]
Of course, this coincides with equation (2.16) for these potentials for \( r = n \).

When the potential \( u^{(0)} \) is a solution of the zero curvature condition (2.19) we will say that it is a s-KdV \( n \) potential. Under this assumption, the spectral curve of system (2.18) for this potential is the characteristic polynomial of matrix \( \Gamma_n^{(0)} \):

\[
\Gamma_n : \det(\mu I_2 - \gamma_n^{(0)}) = \mu^2 + (G_n^{(0)})^2 - F_n^{(0)} H_n^{(0)} = \mu^2 - \frac{F_n^{(0)} F_n^{(0)}}{2} + (u^{(0)} - E)(F_n^{(0)})^2 + \frac{(F_n^{(0)})^2}{4} = \mu^2 - R_{2n+1}(E) = 0.
\]

(see for instance [13]). We denote by \( p_{\mu}(E, \mu) = \mu^2 - R_{2n+1}(E) \) the equation that defines the spectral curve. We will use the following notation

\[
R_{2n+1}(E) = \sum_{i=0}^{2n+1} C_i E^i,
\]

where \( C_i \) are differential polynomials in \( u^{(0)} \) with constant coefficients.

**Lemma 2.9.** We have the following equality \( \partial_{E} C_0 = -2 f_n f_{n+1} \).

**Proof.** Replacing \( E = 0 \) in (2.21) we find \( R_{2n+1}(0) = C_0 = -\frac{f_n f_{n+1}}{4} + \frac{f_{n+1}}{2} - u^{(0)} f_n \). Derivating with respecto to \( x \) and using formula (2.1) we arrive to the desire expression. \( \square \)

With this matrix presentation it is easy to prove the following result:

**Proposition 2.10** (Burchanal and Chaundy, [7]). Let \( u = u(x) \) be solution of equation (2.19), we have that \( p(E, \mu) = \mu^2 - R_{2n+1}(E) \in C[\mu, E] \). Moreover, \( R_{2n+1}(E) \) is a polynomial of degree \( 2n + 1 \) in \( C[\mu, E] \).

This proposition together with Lemma 2.9 and formula (2.1) easily implies the following result.

**Corollary 2.11.** Let \( \mu^2 - R_{2n+1}(E) = 0 \) be the spectral curve for potential \( u^{(0)} \). If the degree of \( R_{2n+1}(E) \) is \( 2n + 1 \) in \( E \) then, \( u^{(0)} \) is solution of a s-KdV \( n \) equation.

It is well known that the spectral curve associated to system (2.18) for stationary potential (2.17) is

\[
\Gamma_n : p_{\mu}(E, \mu) = \mu^2 - E^{2n+1} = 0.
\]

(2.23)

Therefore, these are the spectral curves associated to system (2.3) for Adler-Moser potentials \( u_{a,n} \).

**Remark 2.12.** Take potential \( u_{a,n} \) solution of KdV \( n \) equation, then potential \( u_{a,n}^{(0)} \) is solution of s-KdV \( n \) equation. Thus, we can link level \( r \) of the time-dependent KdV hierarchy with level \( n \) of the stationary KdV hierarchy.

3. **Darboux transformations for \( f_j \)**

In this section we will present the behavior of Darboux transformations acting on the differential polynomials \( f_j \). Therefore we will consider the Schrödinger equation

\[
(L - E_0)\phi = (\partial_{xx} + u - E_0)\phi = 0,
\]

(3.1)

where \( E_0 \) is a fixed energy level. Let \( \phi_0 \) be a solution of such equation. Recall that a Darboux transformation of a function \( \phi \) by \( \phi_0 \) is defined by the formula

\[
DT(\phi_0)\phi = \phi_x - \frac{\phi_{0,x}}{\phi_0} \phi.
\]

Then the transformed function \( \tilde{\phi} = DT(\phi_0)\phi \) is a solution of the Schrödinger equation for potential \( \tilde{u} = u - 2(\log \phi_0)_x \), whenever \( \phi \) is a solution of Schrödinger equation for potential \( u \) and energy level \( E \neq E_0 \) ([8, 9, 10, 16]). We will denote by \( DT(\phi_0)u \) the potential \( \tilde{u} \) to point out the fact that it depends on the choice of \( \phi_0 \).

Next we can observe that the Riccati equation

\[
\sigma_x = u - E_0 - \sigma^2
\]

(3.2)
has $\sigma_0 = (\log \phi_0)_x$ as solution, and then
\[ DT(\phi_0)u = u - 2\sigma_0 u_x. \] (3.3)

In this way, we retrieve a Riccati equation for $u$ as we have
\[ \ddot{u} = u - 2\sigma_x = (\sigma_x + E_0 + \sigma_x^2) - 2\sigma_x = \sigma_x^2 - \sigma_x + E_0. \] (3.4)

Moreover, whenever we have a solution $\phi$ of the Schrödinger equation (2.7), the formula $\sigma = (\log \phi)_x$ gives a solution of the Riccati equation (3.2). Hence, $\sigma$ satisfies the nonlinear differential equation
\[ \sigma_{xx} = u_x - 2\sigma_x. \] (3.5)

Next, we consider the matrix differential system (2.3). Then we perform a Darboux transformation, $DT(\phi_0)$, on it obtaining a new differential system, say $\Phi_x = \tilde{U}\Phi$, $\Phi_t = \tilde{V}\Phi$, whose zero curvature condition is still equation (2.9). Let $F_t(\tilde{u})$, $G_t(\tilde{u})$ and $H_t(\tilde{u})$ be the corresponding entries of the matrix $\tilde{V}$. These differential polynomials are given by expressions (2.4), (2.5) and (2.6) in terms of $f_j(\tilde{u})$. We will establish the relation between $f_j(\tilde{u})$ and $f_j(u)$ in the next theorem.

**Theorem 3.1.** Let $\phi$ be a solution of Schrödinger equation (3.1). Let be $\sigma = (\log \phi)_x$ and $\ddot{u} = u - 2\sigma_x$ the Darboux transformed of $u$ by $\phi$. Then, we have
\[ f_j(\tilde{u}) = f_j(u) + A_j, \quad \text{for } j = 0, 1, 2, \ldots, \]
where $A_j$ is a differential polynomial in $u$ and $\sigma$. Moreover, $A_j$ satisfies the recursive differential relations
1. $A_j = -\frac{1}{4}A_{j-1,xx} + uA_{j-1} - \frac{3}{2}\sigma_xA_{j-1} - \sigma_x f_{j-1}(u)$ and
2. $A_{j,x} + 2\sigma A_j + 2f_{j,x}(u) = 0$.

**Proof.** We will proceed by induction on $n$.

First, we prove by induction that $f_j(\tilde{u}) = f_j(u) + A_j$. For $j = 0$ we have $f_0(\tilde{u}) = f_0(u) + A_0$, where $A_0 = 0$. We suppose it true for $j$ and prove it for $j + 1$. Applying equation (2.1) and induction hypothesis we find:
\[ f_{j+1,x}(\tilde{u}) = -\frac{1}{4}f_{j,xxx}(\tilde{u}) + \ddot{u}f_{j,x}(\tilde{u}) + \frac{1}{2}\dddot{u}f_j(\tilde{u}) \]
\[ = -\frac{1}{4}f_{j,xxx}(u) + u f_{j,x}(u) + \frac{1}{2}u_4 f_j(u) - \frac{3}{4}A_{j,xxx} + uA_{j,x} - 2f_{j,x}(u)\sigma_x \]
\[ - 2A_{j,x}\sigma_x + \frac{1}{2}u_4 A_j - f_j(u)\sigma_{xx} - A_{j,xx} = f_{j+1,x}(u) + A_{j+1,x}, \]
for
\[ A_{j+1,x} = -\frac{A_{j,xxx}}{4} + uA_{j,x} - 2f_{j,x}(u)\sigma_x - 2A_{j,x}\sigma_x + \frac{u_4 A_j}{2} - f_j(u)\sigma_{xx} - A_j\sigma_{xx}. \] (3.6)

Thus, $f_{j+1}(\tilde{u}) = f_{j+1}(u) + A_{j+1}$ as we wanted to prove.

Now, we prove statements 1 and 2. We do it by induction and simultaneously. Since $A_0 = 0$ and $f_0(u) = f_0(\tilde{u}) = 1$, the case $j = 0$ is the trivial one. So, we start the induction process in $j = 1$. For this, using recursion formula (2.1) we have:
\[ f_{1,x}(\tilde{u}) = -\frac{1}{4}f_{0,xxx}(\tilde{u}) + \dddot{u}f_{0,x}(\tilde{u}) + \frac{1}{2}\dddot{u}f_0(\tilde{u}) = -\dddot{u}. \]

Hence, $f_1(\tilde{u}) = \frac{\dddot{u}}{2} + c_1 = \frac{u}{2} - \sigma_x + c_1 = f_1(u) - \sigma_x$, then $A_1 = -\sigma_x$. For $j = 1$ statements 1 and 2 read:
1. $-\frac{A_{0,xxx}}{4} + uA_0 - \frac{3}{2}\sigma_x A_0 - \sigma_x f_0(u) = -\sigma_x = A_1$ and
2. $-2f_{1,x}(u) - A_{1,xx} = -u_x + \sigma_{xx} = -2\sigma_x = 2\sigma A_1$, where $A_1 = -\sigma_x$. For $j = 1$ statements 1 and 2 read:
by equation (3.5). Now, we suppose both statements true for \( j \) and prove them for \( j + 1 \). Derivation with respect to \( x \) in the right hand side of statement 1 yields to:

\[-\frac{A_{j,xx}}{4} + u_{x}A_{j} + uA_{j,x} - \frac{3}{2}\sigma_{x}A_{j} - \frac{3}{2}\sigma_{x}A_{j,x} - \sigma_{xx}f_{j}(u) - \sigma_{x}f_{j,x}(u)\]

\[-\frac{A_{j,xx}}{4} + uA_{j,x} - \sigma_{x}f_{j}(u) - \sigma_{x}A_{j} - \frac{3}{2}\sigma_{x}A_{j,x} - \sigma_{xx}f_{j}(u)\]

\[-\frac{A_{j,xx}}{4} + uA_{j,x} - \sigma_{x}f_{j}(u) - \sigma_{x}A_{j} - \frac{3}{2}\sigma_{x}A_{j,x} - \sigma_{xx}f_{j}(u)\]

Applying equality (3.5) to the term \( \sigma_{xx}A_{j}/2 \) we get:

\[-\frac{A_{j,xx}}{4} + uA_{j,x} - \sigma_{x}f_{j}(u) - \sigma_{x}A_{j} - \frac{3}{2}\sigma_{x}A_{j,x} - \sigma_{xx}f_{j}(u)\]

which is exactly expression (3.6) for \( A_{j+1,x} \). So, we can assume that

\[
A_{j+1} = -\frac{A_{j,xx}}{4} + uA_{j} - \frac{3}{2}\sigma_{x}A_{j} - \sigma_{x}f_{j}(u).
\]

Thus, statement 1 is proved.

Finally, by equations (2.1), (3.6), (3.5) and induction hypothesis we find for statement 2:

\[
-2f_{j+1,x} - A_{j+1,x} = \frac{f_{j,xx}(u)}{2} - 2uf_{j,x}(u) - u_{x}f_{j}(u) + \frac{A_{j,xx}}{4} - uA_{j,x} + 2f_{j,x}(u)\sigma_{x} - \frac{uA_{j}}{2} + 2A_{j,xx}\sigma_{x} + f_{j}(u)\sigma_{xx} + A_{j}\sigma_{xx}
\]

\[
= \frac{f_{j,xx}(u)}{2} + \frac{A_{j,xx}}{4} + (-2f_{j,x}(u) - A_{j,x})(u - \sigma_{x}) - u_{x}f_{j}(u) - \frac{uA_{j}}{2} + A_{j,xx}\sigma_{x} + f_{j}(u)\sigma_{xx} + A_{j}\sigma_{xx}
\]

\[
= \frac{\sigma_{j,xx}}{2} + 2A_{j,xx}\sigma_{x} + A_{j}\sigma_{xx}
\]

\[
= \frac{\sigma_{j,xx}}{2} + 2A_{j,xx}\sigma_{x} + A_{j}\sigma_{xx}
\]

by statement 1. Therefore, statement 2 is also proved. This completes the proof.

\[\square\]

**Example 3.2.** To illustrate the previous theorem we will consider the following KdV potential in the system (2.3).

Take \( u = \frac{62x_{11} + 270x_{1}^{2} + 675x_{2}^{2}}{x_{1}^{2} + 45x_{2}^{2}} \) and solution \( \phi_{0} = \frac{2}{x_{1}^{2}} \). Then \( \overline{u} = \frac{2}{x_{1}^{2}} \). Observe that:

\[
f_{1}(u) = \frac{3(2x_{11} + 270x_{1}^{2} + 675x_{2}^{2})}{x_{1}^{2} + 45x_{2}^{2}}, \quad f_{2}(u) = \frac{u_{xx}}{8} + \frac{3}{8}u^{2} = \frac{45(x_{1}^{2} + 30x_{2}^{2})}{x_{1}^{2} + 45x_{2}^{2}},
\]

and also

\[
f_{1}(\overline{u}) = \frac{3}{x_{1}^{2}}, \quad f_{2}(\overline{u}) = \frac{u_{xx}}{8} + \frac{3}{8}u^{2} = \frac{9}{x_{1}^{2}}.
\]
Hence, in this case

\[ A_1 = f_1(\tilde{u}) - f_1(u) = \frac{-3(x^{10} + 360x^5t - 1350r^2)}{x^3(x^3 - 45t)^2}, \quad A_2 = f_2(\tilde{u}) - f_2(u) = \frac{-9(4x^{10} + 240x^5t - 2025r^2)}{x^3(x^3 - 45t)^2}. \]

By direct computation we can verify that the \( A_j \) satisfy the relations 1 and 2 of 3.1.

**Corollary 3.3.** For \( i \geq j \) we have the following equality

\[ \sum_{j=0}^{i} (2\sigma A_{i-j} + 2f_{i-j}(u) + A_{i-j})E^j = 0. \]  

(3.7)

Theorem 3.1 has several interesting consequences. The main ones are the relations that the transformed potential \( \tilde{u} \) produce for functions \( F_j(u) \). Next we establish some of them, which will be used in the following sections. In particular, Proposition 3.5 is specially interesting since it gives a relation between \( \sigma_x \) and \( \sigma_y \).

**Proposition 3.4.** Let \( A_i \) and \( \sigma \) be as in 3.1. For \( i = 0, 1, 2, \ldots \) we have

1. \( F_i(\tilde{u}) = F_i(u) + P_i \), where \( P_i = \sum_{j=0}^{i} E^j A_{i-j} \).
2. Moreover \( P_{x,x} + 2\sigma_x P + 2F_{x}(u) = 0 \).

**Proof.** It is an immediate consequence of Theorem 3.1.

**Proposition 3.5.** Let \( u \) be a solution of KdV equation. Let \( \phi \) be a solution of Schrödinger equation (2.7) for potential \( u \) and energy \( E_0 \). Let \( \sigma = (\log \phi)_x \). Consider \( A_{x+1} \) as defined in 3.1 and \( P \), as defined in 3.4. Then, we have:

\[ \sigma_{x} = -A_{x+1} = \frac{1}{4} P_{r,x} + EP + \sigma_x F_x(u) + \frac{1}{2} P_r(-2\sigma + 3\sigma_x). \]  

(3.8)

**Proof.** We compare the zero curvature conditions for \( u \) and \( \tilde{u} \):

\[
\begin{align*}
\tilde{u}_t &= 2f_{x+1}(u) = \frac{1}{2}F_{r,t}(u) + 2(u - E)F_{r,x}(u) + u_xF_x(u), \\
\tilde{u}_{xx} &= 2f_{x+1}(\tilde{u}) = \frac{1}{2}F_{r,xx}(\tilde{u}) + 2(\tilde{u} - E)F_{r,xx}(\tilde{u}) + \tilde{u}_xF_x(\tilde{u}).
\end{align*}
\]

We prove the first equality. For this, we have \( \tilde{u}_t = (u - 2\sigma_x)_{xx} = u_t - 2\sigma_{xx} \) and \( 2f_{x+1}(\tilde{u}) = 2f_{x+1}(u) + 2A_{x+1} \) by Theorem 3.1. Then:

\[
2\sigma_{xx} = u_t - \tilde{u}_t = 2f_{x+1}(u) - 2f_{x+1}(\tilde{u}) = -2A_{x+1}.
\]

Thus, \( \sigma_{x} = -A_{x+1} \).

Now, we prove the second equality. Using expression (3.3) for \( \tilde{u} \) and applying 3.4 (1), we obtain

\[
\tilde{u}_{xx} = -\frac{1}{2}F_{r,xxx}(u) + 2(u - E)F_{r,x}(u) + u_xF_x(u) - \frac{1}{2} P_{r,xxx} - 2(E - u)P_{r,x} \]

(3.9)

Since \( 2\sigma_{xx} = u_t - \tilde{u}_t \), we have

\[
2\sigma_{xx} = \frac{1}{2} P_{r,xxx} + 2EP_{r,x} - 2uP_{r,x} + 4\sigma_x F_{r,x}(u) + 4\sigma_x P_{r,x} - u_xP_r - 2\sigma_{xx} F_x(u) - 2\sigma_{xx} P_r.
\]

Applying 3.4 (2) to the expression \( \sigma_x P_{r,x} \), we find:

\[
2\sigma_{xx} = \frac{1}{2} P_{r,xxx} + 2EP_{r,x} - 2uP_{r,x} + 4\sigma_x F_{r,x}(u) + 3\sigma_x P_{r,x} + \sigma_x(-2\sigma_x - F_{r,x}(u))
- u_xP_r + 2\sigma_x F_x(u) + 2\sigma_{xx} P_r
= \frac{1}{2} P_{r,xxx} + 2EP_{r,x} + 2(\sigma_x F_x(u) + \sigma_x F_{r,x}(u)) + P_{r,x}(-2u + 3\sigma_x) + P_r(-2\sigma_x - u_x + 2\sigma_{xx}).
\]

Moreover, for the coefficient of \( P_r \) we have:

\[
-2\sigma_x - u_x + 2\sigma_{xx} = (-\sigma^2 - u + 2\sigma_x) = (-2u + 3\sigma_x).
\]
by (3.2). Thus, we obtain

$$2σ_{x,x} = \left( \frac{1}{2} P_{xx} + 2EP_x + 2σ_x F_x(u) + P_x(-2u + 3σ_x) \right)_{x}.$$ 

Hence we have proved the statement.

We finish this section with the following technical result. It makes a connection between differential polynomials \( f_r(u) \) and some differential polynomials \( g_r(σ) \) defined by

$$g_r(σ) := -A_{r+1} = \frac{1}{2} P_{xx} + 2EP_x + 2σ_x F_x(u) + P_x(-2u + 3σ_x).$$  \hspace{1cm} (3.10)

**Proposition 3.6.** We have the following relations:

1. \((2σ + ∂_x)g_r(σ) = 2f_{r+1,xx}(u) = -\frac{1}{2}F_{r,xx}(u) + 2(u - E)F_{r,x}(u) + u_x F_r(u) \) and
2. \((2σ - ∂_x)g_r(σ) = 2f_{r+1,x}(u) = -\frac{1}{2}F_{r,xx}(u) + 2(\bar{u} - E)F_{r,x}(\bar{u}) + \bar{u}_x F_r(\bar{u}). \)

**Proof.** Statement 1 is just statement 2 of Theorem 3.1 rewritten. For statement 2 we have:

$$2f_{r+1,x}(\bar{u}) = 2f_{r+1,x}(u) + 2A_{r+1,x} = 2σ g_r(σ) + g_{r,x}(σ) - 2g_{r,x}(σ) = 2σ g_r(σ) - g_{r,x}(σ) = (2σ - ∂_x)g_r(σ)$$

by statement 1 and equation (3.10).

4. **Fundamental matrices for KdV rational Schrödinger operators**

In this section we give a fundamental matrix for the system (2.3) depending on the energy level \( E \). The spectral curve is the tool that will allow us to understand why fundamental matrices present different behaviors according to the values of the energy.

For stationary rational potentials \( u_0(n) = n(n+1)x^2 \), it is well known that the spectral curve associated to system

\[
\begin{align*}
\Phi_x &= U^{(0)}\Phi = \begin{pmatrix} 0 & 1 \\ u_0(n) - E & 0 \end{pmatrix} \Phi, \\
\Phi_u &= V^{(0)}\Phi = \begin{pmatrix} G_0(u_0(n)) & F_0(u_0(n)) \\ -H_0(u_0(n)) & -G_0(u_0(n)) \end{pmatrix} \Phi, 
\end{align*}
\]

is the algebraic plane curve in \( \mathbb{C}^2 \) given by

$$\Gamma_n : p_n(µ, E) = µ^2 - E^{2n+1} = 0.$$  \hspace{1cm} (4.2)

Whenever an Adler-Moser potential \( u_{r,0}(x, i) \) is time dependent, we will consider \( \Gamma_n \) as the spectral curve associated to its corresponding linear differential system (2.3). Observe that \((E, µ) = (0, 0)\) is the unique affine singular point of \( \Gamma_n \). It turns out that for \( E ≠ 0 \) the behavior of the fundamental matrix associated to the system

\[
\begin{align*}
\Phi_x &= U^{0}\Phi = \begin{pmatrix} 0 & 1 \\ u_{t,n} - E & 0 \end{pmatrix} \Phi, \\
\Phi_u &= V^{0}\Phi = \begin{pmatrix} -F_{r,x}(u_{t,n}) \frac{1}{2} & F_{r}(u_{t,n}) \\ (u_{t,n} - E)F_{r}(u_{t,n}) - F_{r,xx}(u_{t,n}) \frac{1}{2} & F_{r}(u_{t,n}) \end{pmatrix} \Phi, 
\end{align*}
\]

presents a similar behavior since the point \( P = (E, µ) \) is a regular point of \( \Gamma_n \). A fundamental matrix for \( E = 0 \) can be also computed. However, it is not obtained by a specialization process from the fundamental matrix obtained for a regular point. We include some examples in this section.
4.1. Fundamental matrices for $E = 0$

In this section, we compute explicitly fundamental matrices of system (2.3) when the potential $u$ is $u_{cr} = -2(\log \theta_{cr})_{xx}$ and $E = 0$. Recall that $u_{cr}$ is a solution of KdV$\tau$ (see Remark 2.5). Hence, we study the system

$$
\begin{align*}
\Phi_e &= U\Phi = \begin{pmatrix} 0 & 1 \\ u_{cr} & 0 \end{pmatrix} \Phi, \\
\Phi_{te} &= V_{te}\Phi = \begin{pmatrix} -\frac{f_x(u_{cr})}{2} & f_x(u_{cr}) \\ u_{cr}f_x(u_{cr}) - \frac{f_{xx}(u_{cr})}{2} & \frac{f_x(u_{cr})}{2} \end{pmatrix} \Phi.
\end{align*}
$$

(4.4)

It is obvious that the zero curvature condition of this system is the KdV$\tau$ equation for $c_i = 0$, $i = 1, \ldots, r$:

$$
\partial_t u_{cr} = 2f_{r+1,x}(u_{cr}).
$$

(4.5)

From now on we will denote $u_{cr,0} = \partial_t(u_{cr})$.

We have the following result:

**Theorem 4.1.** Let $n$ be a non negative integer. For $E = 0$ and $u = u_{cr}$, a fundamental matrix for system (4.4) is:

$$
\mathcal{B}^{(r)}_{n,0} = \begin{pmatrix} \phi_{1,r,n} & \phi_{2,r,n} \\ \phi_{1,r,n} \phi_{2,r,n+1} \end{pmatrix},
$$

(4.6)

where

$$
\phi_{1,r,n}(x, t, 0) = \frac{\theta_{n-1}}{\theta_n} \quad \text{and} \quad \phi_{2,r,n}(x, t, 0) = \frac{\theta_n}{\theta_{n+1}}.
$$

(4.7)

For $n = 0$ we define $\theta_{r,-1} := 1$. We notice that $\phi_{2,r,0} = (\phi_{1,r,1})^{-1}$.

**Proof.** We prove it by induction on $n$. For $n = 0$ the definition $\theta_{r,0} = 1$ gives $u_{r,0} = 0$. So, system (4.4) reads

$$
\begin{align*}
\begin{pmatrix} \phi_{1,0,0,0} & \phi_{2,0,0,0} \\ \phi_{1,0,0,0} & \phi_{2,0,0,0} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_{1,0,0} & \phi_{2,0,0} \\ \phi_{1,0,0} & \phi_{2,0,0} \end{pmatrix} = \begin{pmatrix} \phi_{1,0,0} & \phi_{2,0,0} \\ \phi_{1,0,0} & \phi_{2,0,0} \end{pmatrix},
\end{align*}
$$

(4.8)

Thus, $\phi_{1,0,0} = 1$ and $\phi_{2,0,0} = x$ generate $\mathcal{B}^{(0)}_{0,0}$. Since $\theta_{1,1} = x$ we have that $\phi_{1,1,0} = \frac{\theta_{1,0}}{\theta_{1,1}}$ and $\phi_{2,1,0} = \frac{\theta_{1,0}}{\theta_{1,1}}$.

Now, we suppose it true for $n$ and prove it for $n + 1$. For $n$ we know that $\phi_{1,n,0} = \frac{\theta_{n,0}}{\theta_{n,n}}$ and $\phi_{2,n,0} = \frac{\theta_{n,0}}{\theta_{n,n}}$ generate $\mathcal{B}^{(r)}_{n,0}$. Therefore, $\phi_{1,r,n+1}$ and $\phi_{2,r,n+1}$ are solutions of Schrödinger equation $\phi_{xx} = u_{cr}\phi$. We apply a Darboux transformation with $\phi_{2,r,n}$ to this Schrödinger equation and we obtain:

$$
\begin{align*}
DT(\phi_{2,n})u_{cr} &= u_{cr} - 2(\log \phi_{2,n} \phi_{1,n})_{xx} = -2(\log \theta_{cr})_{xx} - 2(\log \phi_{2,n,0})_{xx} \\
&= -2(\log \phi_{2,n,0} \theta_{n,0} \phi_{1,n})_{xx} = -2(\log \theta_{n+1})_{xx} = u_{cr,n+1},
\end{align*}
$$

(4.9)

So, $\phi_{1,n+1} = \frac{\theta_n}{\theta_{n+1}}$ is a solution of $\phi_{xx} = u_{cr,n+1}\phi$ and, obviously, $(\phi_{1,n+1,1}, \phi_{1,n+1,1})^T$ is a column solution of the first equation of the system for $u_{cr,n+1}$.

Now we verify that this column matrix is also a solution of the second equation:

$$
\begin{align*}
\begin{pmatrix} \phi_{1,n+1,1} \\ \phi_{1,n+1,1} \end{pmatrix} &= \begin{pmatrix} -\frac{f_{xx}(u_{cr,n+1})}{2} & f_x(u_{cr,n+1}) \\ u_{cr,n+1}f_x(u_{cr,n+1}) - \frac{f_{xx}(u_{cr,n+1})}{2} & \frac{f_x(u_{cr,n+1})}{2} \end{pmatrix} \begin{pmatrix} \phi_{1,n+1,1} \\ \phi_{1,n+1,1} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{f_{xx}(u_{cr,n+1})}{2} \phi_{1,n+1,1} + f_x(u_{cr,n+1})\phi_{1,n+1,1} \\ u_{cr,n+1}f_x(u_{cr,n+1}) - \frac{f_{xx}(u_{cr,n+1})}{2} \phi_{1,n+1,1} + \frac{f_x(u_{cr,n+1})}{2} \phi_{1,n+1,1} \end{pmatrix}.
\end{align*}
$$
We notice that the second row is just the partial derivative with respect to $x$ of the first one. Hence, we just have to verify that expressions (4.8) and (4.9) satisfy the equation

$$\phi_{1,r,n+1,x} = -\frac{f_{r,x}(u_{r,n+1})}{2} \phi_{1,r,n+1} + f_r(u_{r,n+1}) \phi_{1,r,n+1,x}. \quad (4.10)$$

Applying expression (4.9) and induction hypothesis we obtain for the left hand side of this equation:

$$\phi_{1,r,n+1,x} = \frac{1}{2n+1} \left( \phi_{1,r,n} \frac{\phi_{2,r,n,x}}{\phi_{2,r,n}} - \phi_{1,r,n,x} \right) \left( \frac{f_{r,x}(u_{r,n})}{2} - f_r(u_{r,n}) \frac{\phi_{2,r,n,x}}{\phi_{2,r,n}} \right), \quad (4.11)$$

and for the right hand side:

$$-\frac{f_{r,x}(u_{r,n+1})}{2} \phi_{1,r,n+1} + f_r(u_{r,n+1}) \phi_{1,r,n+1,x} = \frac{1}{2n+1} \left( \phi_{1,r,n} \frac{\phi_{2,r,n,x}}{\phi_{2,r,n}} - \phi_{1,r,n,x} \right) \left( -\frac{f_{r,x}(u_{r,n+1})}{2} - f_r(u_{r,n+1}) \frac{\phi_{2,r,n,x}}{\phi_{2,r,n}} \right). \quad (4.12)$$

Now, we prove that both expressions are equal. Applying Theorem 3.1 statement 2 for $\sigma = \frac{\phi_{2,r,n,x}}{\phi_{2,r,n}}$ to expression (4.12) leads to:

$$-\frac{f_{r,x}(u_{r,n+1})}{2} - f_r(u_{r,n+1}) \frac{\phi_{2,r,n,x}}{\phi_{2,r,n}} = -\frac{f_{r,x}(u_{r,n})}{2} - (f_r(u_{r,n}) + A_r) \frac{\phi_{2,r,n,x}}{\phi_{2,r,n}} = -\frac{f_{r,x}(u_{r,n})}{2} - f_r(u_{r,n}) \phi_{2,r,n} \frac{\phi_{2,r,n,x}}{\phi_{2,r,n}} \frac{A_r}{\phi_{2,r,n}} + A_r \phi_{2,r,n} \frac{\phi_{2,r,n,x}}{\phi_{2,r,n}} \frac{A_r}{\phi_{2,r,n}} = -\frac{f_{r,x}(u_{r,n})}{2} - f_r(u_{r,n}) \phi_{2,r,n} - \frac{\phi_{2,r,n,x}}{\phi_{2,r,n}} \frac{A_r}{\phi_{2,r,n}} = \frac{f_{r,x}(u_{r,n})}{2} - f_r(u_{r,n}) \phi_{2,r,n} \frac{\phi_{2,r,n,x}}{\phi_{2,r,n}},$$

which is equal to expression (4.11). Therefore, both sides of expression (4.10) coincide.

Now we proceed as in [1]. We take another column solution $(\phi_{2,r,n+1}, \phi_{2,r,n+1,x})'$ of this system for potential $u_{r,n+1}$ which is linearly independent of the one we have just computed, i.e., $det B_{n+1,0}^{(r)}$ is a nontrivial constant. We take $\phi_{2,r,n+1}$ such that

$$det B_{n+1,0}^{(r)} = 2(n + 1) + 1.$$

We notice that with this condition we have:

$$det B_{n+1,0}^{(r)} = \phi_{2,r,n+1} \theta_{r,n+1} - \phi_{2,r,n+1} \frac{\theta_{r,n+1}}{\theta_{r,n+1}} = \frac{\theta_{r,x+1}}{\theta_{r,x+1}} + \theta_{r,n+1} - \theta_{r,n+1} = 2(n + 1) + 1,$$

multiplying both sides by $\theta_{r,n+1}^2$ and using the recursion formula (2.11) we get:

$$\phi_{2,r,n+1} \theta_{r,n+1} - \phi_{2,r,n+1} \theta_{r,n+1} - \theta_{r,n+1} = \theta_{r,n+2} \theta_{r,n} - \theta_{r,n+1} \theta_{r,n+1}.$$

Setting $\phi_{2,r,n+1} = \frac{\alpha_{2,r,n+1}}{\theta_{r,n+1}}$ yields to:

$$\alpha_{2,r,n+1} \theta_{r,n} - \alpha_{2,r,n+1} \theta_{r,n+1} = \theta_{r,n+2} \theta_{r,n} - \theta_{r,n+2} \theta_{r,n+1},$$

thus, $\alpha_{2,r,n+1} = \theta_{r,n+2}$. This concludes the proof. \qed

Adler and Moser proved in [1] that matrix $B_{n,0}^{(r)}$ is a fundamental matrix for the Schrödinger equation (2.7) for $E = 0$. But they did not prove there that this matrix is also a fundamental matrix for the second equation of the system (4.4). To do that, it is necessary to control the action of the Darboux transformations over the differential polynomials $f_r$, as we did in Section 3.
Remark 4.2. Since \( \phi_{1,i,n} = \frac{\theta_{i,n}}{\theta_{i,0}} \) and \( \phi_{2,i,n} = \frac{\theta_{i,n}}{\theta_{i,0}} \) are solutions of Schrödinger equation \((2.7)\) for \( E = 0 \), this translate into the following equation for polynomials \( \theta_{i,n} \):

\[
\theta_{i,r+1,x} \theta_{i,n} + \theta_{i,2n+1} \theta_{i,n,xx} - 2 \theta_{i,3n} \theta_{i,r+1,n} = 0.
\]

(4.13)

Theorem 4.3. We have that

\[
\det \mathcal{B}_{r,0}^{(r)} = 2n + 1.
\]

(4.14)

Example 4.4. To illustrate this case, we present explicit computations using SAGE of fundamental solutions of the system for the first values of \( n \).

1. First, we show the first examples of unadjusted fundamental solutions:

\[
\begin{array}{ccc}
 n & \phi_{1,1,n} & \phi_{2,1,n} & u_{1,n} \\
 0 & 1 & x & 0 \\
 1 & \frac{1}{x} & x^3 + \tau_2 & 2 \\
 2 & \frac{x}{x^3 + \tau_2} & \frac{x^6 + 5x^3 \tau_2 + x\tau_3 - 5\tau_2^2}{x^3 + \tau_2} & \frac{6x(x^3 - 2\tau_2)}{(x^3 + \tau_2)^2} \\
 3 & \frac{x^3 + \tau_2}{x^6 + 5x^3 \tau_2 + x\tau_3 - 5\tau_2^2} & \frac{p_1(x, \tau_2, \tau_3, \tau_4)}{x^6 + 5x^3 \tau_2 + x\tau_3 - 5\tau_2^2} & \frac{p_2(x, \tau_2, \tau_3)}{(x^6 + 5x^3 \tau_2 + x\tau_3 - 5\tau_2^2)^2}
\end{array}
\]

where \( p_1(x, \tau_2, \tau_3, \tau_4) = x^{15} + 15x^7 \tau_2 + 7x^5 \tau_3 - 35x^2 \tau_2 \tau_3 + 175x^2 \tau_2^3 - \frac{7}{3} \tau_3^2 + x^3 \tau_4 + 2\tau_2 \tau_4 \) and \( p_2(x, \tau_2, \tau_3) = 12x^{10} - 36x^8 \tau_3 + 450x^6 \tau_2^2 + 300x^4 \tau_3^2 + 2\tau_2^3 \).

2. Next, we compute fundamental solutions for potentials which are solutions of the first level of the KdV hierarchy, KdV equation: \( u_t = \frac{2}{3} uu_x - \frac{1}{3} u_{xxx} \).

We also show the explicit choice of the functions \( \tau_i \).

\[
\begin{array}{ccc}
 n & \phi_{1,1,n} & \phi_{2,1,n} & u_{1,n} \\
 0 & 1 & x & 0 \\
 1 & \frac{1}{x} & x^3 + 3\tau_1 & \frac{2}{x^2} \\
 2 & \frac{x}{x^3 + 3\tau_1} & \frac{x^6 + 15x^3 \tau_1 - 45\tau_1^2}{x^3 + 3\tau_1} & \frac{6x(x^3 - 6\tau_1)}{(x^3 + 3\tau_1)^2} \\
 3 & \frac{x^3 + 3\tau_1}{x^6 + 15x^3 \tau_1 - 45\tau_1^2} & \frac{x^{10} + 45x^7 \tau_1 + 4725x^4 \tau_1^2}{x^6 + 15x^3 \tau_1 - 45\tau_1^2} & \frac{6x(2x^9 + 675x^6 \tau_1^2 + 1350\tau_1^3)}{(x^6 + 15x^3 \tau_1 - 45\tau_1^2)^2}
\end{array}
\]

(3\tau_1, 0)

(3\tau_1, 0, 0)

4.2. Fundamental matrices for \( E \neq 0 \)

In this section, we compute explicitly fundamental matrices of system \((2.3)\) when \( u = u_{r,n} = -2(\log \theta_{r,n})_{x} \) and \( E \neq 0 \). In this case, the system is

\[
\begin{align*}
\Phi_x = U & \Phi = \begin{pmatrix} 0 & 1 \\ u_{r,n} - E & 0 \end{pmatrix} \Phi, \\
\Phi_t = V & \Phi = \begin{pmatrix} -F(r,u_{r,n}) & F(r,u_{r,n}) \\ (u_{r,n} - E)F(r,u_{r,n}) - \frac{F(r,u_{r,n})}{2} \\ F(r,u_{r,n}) - \frac{F(r,u_{r,n})}{2} \end{pmatrix} \Phi.
\end{align*}
\]

(4.15)

The zero curvature condition of this system is still the KdV equation for \( c_i = 0 \), \( i = 1, \ldots, r \):

\[
u_{,r,n+1} = 2f_{r+1,(u_{r,n})}.
\]

(4.16)

When \( E \neq 0 \), we take \( \lambda \in \mathbb{C} \) a parameter over \( K \) such that \( E + \lambda^2 = 0 \).
Next, we consider the differential systems:

\[
Q_{n,xx}^+ = Q_{n,x}^+ \left( -2 \lambda + 2 \frac{\theta_{r,n,x}}{\theta_{r,n}} \right) + Q_n^+ \left( 2 \frac{\theta_{r,n,x}}{\theta_{r,n}} - \frac{\theta_{r,n,x}}{\theta_{r,n}} \right), \tag{4.17}
\]

\[
Q_{n,b}^+ = Q_{n,b}^+ F_r(u_{r,n}) + Q_n^+ \left( -(-1)^r \lambda^{2r+1} + \lambda F_r(u_{r,n}) + \frac{\theta_{r,n+1}}{\theta_{r,n}} - \frac{F_x(u_{r,n})}{2} - F_r(u_{r,n}) \frac{\theta_{r,n+1}}{\theta_{r,n}} \right), \tag{4.18}
\]

\[
Q_{n,xx}^- = Q_{n,x}^- \left( 2 \lambda + 2 \frac{\theta_{r,n,x}}{\theta_{r,n}} \right) - Q_n^+ \left( 2 \frac{\theta_{r,n,x}}{\theta_{r,n}} + \frac{\theta_{r,n,x}}{\theta_{r,n}} \right), \tag{4.19}
\]

\[
Q_{n,b}^- = Q_{n,b}^- F_r(u_{r,n}) + Q_n^+ \left( -(-1)^r \lambda^{2r+1} - \lambda F_r(u_{r,n}) + \frac{\theta_{r,n+1}}{\theta_{r,n}} - \frac{F_x(u_{r,n})}{2} - F_r(u_{r,n}) \frac{\theta_{r,n+1}}{\theta_{r,n}} \right). \tag{4.20}
\]

We have the following relations for solutions of the differential systems (4.17)-(4.18) and (4.19)-(4.20).

**Lemma 4.5.** Functions \( Q_{n}^+ \) and \( Q_{n}^- \) recursively defined by

\[
Q_{n+1}^+ = \frac{\lambda Q_{n}^+ \theta_{r,n+1} + Q_{n}^+ \theta_{r,n+1} - Q_{n}^+ \theta_{r,n+1,x}}{\theta_{r,n}}, \tag{4.21}
\]

\[
Q_{n+1}^- = \frac{\lambda Q_{n}^- \theta_{r,n+1} - Q_{n}^- \theta_{r,n+1} + Q_{n}^- \theta_{r,n+1,x}}{\theta_{r,n}}, \tag{4.22}
\]

are solutions of the differential systems (4.17)-(4.18) and (4.19)-(4.20).

**Proof.** We prove it by induction on \( n \). For \( n = 0 \) we have \( \theta_{r,0} = 1 \), hence, \( u_{r,0} = 0 \) and \( F_r(u_{r,0}) = (-1)^r \lambda^{2r} \). So, \( Q_{0}^+ = 1 \) and \( Q_{0}^- = 1 \) are solutions of the systems.

Now, suppose it true for \( n \) and prove it for \( n + 1 \). We have to prove that expressions

\[
Q_{n+1}^+ = \frac{\lambda Q_{n}^+ \theta_{r,n+1} + Q_{n}^+ \theta_{r,n+1} - Q_{n}^+ \theta_{r,n+1,x}}{\theta_{r,n}} \quad \text{and} \quad Q_{n+1}^- = \frac{\lambda Q_{n}^- \theta_{r,n+1} - Q_{n}^- \theta_{r,n+1} + Q_{n}^- \theta_{r,n+1,x}}{\theta_{r,n}}
\]

satisfy equations (4.17), (4.18), (4.19) and (4.20) respectively, for \( n + 1 \). First, we prove that \( Q_{n+1}^+ \) satisfies (4.17) and (4.18). By induction hypothesis, we know that \( Q_{n}^+ \) satisfies (4.17), using this expression and (4.13) we have:

\[
Q_{n+1,xx}^+ = \frac{Q_{n,x}^+ p_1(x, t, \lambda) + Q_{n}^+ p_2(x, t, \lambda)}{\theta_{r,n}} + Q_{n+1}^+ \left( -2 \lambda + 2 \frac{\theta_{r,n+1,x}}{\theta_{r,n+1}} \right) + Q_{n+1}^+ \left( 2 \frac{\theta_{r,n+1,x}}{\theta_{r,n+1}} - \frac{\theta_{r,n+1,x}}{\theta_{r,n+1}} \right)
\]

where

\[
p_1(x, t, \lambda) = 2 \lambda^2 \theta_{r,n}^2 \theta_{r,n+1}^2 - 2 \lambda \theta_{r,n+1} \theta_{r,n+1} \theta_{r,n+1,x} + 2 \theta_{r,n+1,x} \theta_{r,n+1} \theta_{r,n+1,x} \theta_{r,n+1,xx},
\]

\[
p_2(x, t, \lambda) = -2 \lambda^2 \theta_{r,n} \theta_{r,n+1} \theta_{r,n+1} + 2 \lambda \theta_{r,n+1} \theta_{r,n+1} \theta_{r,n+1,x} + \theta_{r,n+1,x} \theta_{r,n+1,xx} - \theta_{r,n+1,x} \theta_{r,n+1,xx}.
\]

Thus, both expressions coincide and \( A^+ \) is solution of equation (4.17).
On the other hand, by induction hypothesis, we know that $Q^+_n$ satisfies (4.18). Using this equation, expressions

$$
\sigma_{2,n,r} = (\log \phi_{2,n}) = \frac{\theta_{2,n+1} - \theta_{2,n} - \lambda \theta_{2,n+1}}{\theta_{2,n} \theta_{2,n+1}},
$$

$$
\sigma_{2,n,r,2} = \frac{\theta_{2,n+1} \theta_{2,n}}{\theta_{2,n+1}},
$$

$$
Q^+_{n,r,2} = Q^+_{n,r} \left( -\lambda \theta_{2,n+1} - \lambda F_r(u_{r,n}) + \frac{F_r(u_{r,n})}{\theta_{2,n+1} + F_r(u_{r,n}) \frac{\theta_{2,n} - \theta_{2,n+1}}{\theta_{2,n}}} \right),
$$

the derivative with respect to $x$ of statement 2 of Corollary 3.4 and expression (3.8) for $\sigma_{2,n,r,2}$, we obtain

$$
Q^+_{n+1,2} = Q^+_{n+1,2} \frac{p_3(x, t, \lambda)}{\theta_{2,n+1}^2} + Q^+_{n} \frac{p_4(x, t, \lambda)}{\theta_{2,n}^2},
$$

where

$$
p_3(x, t, \lambda) = -\lambda \theta_{2,n+1} \theta_{2,n} + F_r(u_{r,n}) \theta_{2,n+1} - F_r(u_{r,n}) \theta_{2,n} \theta_{2,n+1} + F_r(u_{r,n}) \frac{\theta_{2,n} - \theta_{2,n+1}}{\theta_{2,n}^2} + \lambda \theta_{2,n+1} \theta_{2,n},
$$

$$
p_4(x, t, \lambda) = -\lambda \theta_{2,n+1} \theta_{2,n} + \lambda F_r(u_{r,n}) \theta_{2,n+1} + \lambda F_r(u_{r,n}) \theta_{2,n} \theta_{2,n+1} + \lambda \theta_{2,n+1} \theta_{2,n}. \quad (4.17)
$$

Finally, using relation (4.17) for $Q^+_{n}$ and statements 1 and 2 of Corollary 3.4, the right hand side of equation (4.18) for $Q^+_{n+1}$ reads

$$
Q^+_{n+1} F_r(u_{r,n+1}) + Q^+_{n} \left( \lambda^2 + \lambda F_r(u_{r,n+1}) + \frac{\theta_{2,n+1} \theta_{2,n}}{\theta_{2,n+1}} - F_r(u_{r,n+1}) \frac{\theta_{2,n} - \theta_{2,n+1}}{\theta_{2,n}^2} \right) \frac{p_3(x, t, \lambda)}{\theta_{2,n+1}^2} + Q^+_{n} \frac{p_4(x, t, \lambda)}{\theta_{2,n}^2}.
$$

Therefore, both expressions coincide and $Q^+_{n+1}$ is a solution of equation (4.18).

The proof for $Q^-_{n+1}$ is analogous.

As a consequence, we have the following result:

**Theorem 4.6.** Let $n$ be a non negative integer, then, for $E = -\lambda^2 \neq 0$ and $u = u_{r,n}$, a fundamental matrix for system (4.15) is:

$$
\mathcal{B}^{(r)}_{n} = \begin{pmatrix}
\phi^+_r & \phi^-_r \\
\phi^+_{r,n} & \phi^-_{r,n}
\end{pmatrix}
,$$

(4.23)

where

$$
\phi^+_r(x, t, \lambda) = e^{\lambda x - (1^n 2^{\lambda x})} \frac{Q^+_r(x, t, \lambda)}{\theta_{2,n}} \quad \text{and} \quad \phi^-_r(x, t, \lambda) = e^{-\lambda x - (1^n 2^{\lambda x})} \frac{Q^-_r(x, t, \lambda)}{\theta_{2,n}},
$$

where $Q^+_r$ and $Q^-_r$ are functions in $x, t, \lambda$ such that they are solutions of the differential systems (4.17)-(4.18) and (4.19)-(4.20) respectively.

**Proof.** We prove it by induction on $n$. For $n = 0$ the definition $\theta_{2,0} = 1$ leads to $u_0 = 0$. So, system (4.15) becomes

$$
\begin{pmatrix}
\phi^+_{0,x} & \phi^-_{0,x} \\
\phi^+_{0,xx} & \phi^-_{0,xx}
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
\phi^+_{0,x} & \phi^-_{0,x} \\
\phi^+_{0,xx} & \phi^-_{0,xx}
\end{pmatrix},
$$

$$
\begin{pmatrix}
\phi^+_{0,t} & \phi^-_{0,t} \\
\phi^+_{0,xt} & \phi^-_{0,xt}
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
\phi^+_{0,t} & \phi^-_{0,t} \\
\phi^+_{0,xt} & \phi^-_{0,xt}
\end{pmatrix}.
$$

(4.24)
Hence, $\phi^+_{t,0} = e^{4x(-1)^{t}t^{2^{+1}t}}$ and $\phi^-_{t,0} = e^{-4x(-1)^{t}t^{2^{+1}t}}$ generate $B_{t,0}^{(r)}$. Since $\theta_{t,0} = 1$, we find $Q_{t,0}^{+} = 1$, as in Lemma 4.5.

Next, we suppose it true for $n$ and prove it for $n+1$. Since $\phi^+_{t,n}(x, t, \lambda) = e^{4x(-1)^{t}t^{2^{+1}t}}Q_{t,n}^{+}/\theta_{t,n}$ and $\phi^-_{t,n}(x, t, \lambda) = e^{-4x(-1)^{t}t^{2^{+1}t}}Q_{t,n}^{-}/\theta_{t,n}$ are solutions of Schrödinger equation $\phi_{xx} = (u_{t,n} + \lambda^2)\phi$, we apply a Darboux transformation with $\phi_{2,n} = \frac{\theta_{t,n+1}}{\theta_{t,n}}$ to this equation and we obtain:

$$
DT(\phi_{2,n})u_{t,n} = u_{t,n} - 2\log(\phi_{2,n})_{xx} = u_{t,n} - 2\sigma_{2,n,xx} = u_{t,n+1},
$$

$$
DT(\phi_{2,n})\phi^+_{t,n} = \phi^+_{t,n+1} - \phi_{2,n,x} \frac{\phi^+_{t,n}}{\phi_{2,n,x}} = e^{4x(-1)^{t}t^{2^{+1}t}} - \theta_{t,n+1} \lambda Q_{t,n}^{+} + Q_{t,n}^{-} \theta_{t,n+1} - Q_{t,n}^{+} \theta_{t,n+1},
$$

$$
DT(\phi_{2,n})\phi^-_{t,n} = \phi^-_{t,n+1} - \phi_{2,n,x} \frac{\phi^-_{t,n}}{\phi_{2,n,x}} = e^{-4x(-1)^{t}t^{2^{+1}t}} - \theta_{t,n+1} \lambda Q_{t,n}^{-} + Q_{t,n}^{+} \theta_{t,n+1} - Q_{t,n}^{-} \theta_{t,n+1},
$$

by Lemma 4.5. Hence, $DT(\phi_{2,n})\phi^+_{t,n} = \phi^+_{t,n+1}(x, t, \lambda)$ and $DT(\phi_{2,n})\phi^-_{t,n} = \phi^-_{t,n+1}(x, t, \lambda)$ generate $B_{t,n+1}^{(r)}$. This ends the proof.

As far as we know, a general expression for fundamental matrices for system (4.15) has never been computed when $E \neq 0$. As in Theorem 4.1, the key to do that is to control the action of the Darboux transformations over the differential polynomials $f_{ij}$ as we showed in Section 3. In Section 5 we will give some examples of these fundamental solutions both in the general framework of unadjusted functions $\tau_i$ and in the particular case $r = 1$, in the same line as in Example 4.4.

**Proposition 4.7.** Functions $Q_{t,n}^{+}$, $Q_{t,n}^{-}$ and solutions $\phi_{t,n}^{+}, \phi_{t,n}^{-}$ defined in Theorem 4.6 satisfy the relations

$$
Q_{t,n}^{+}(x, t, -\lambda) = (-1)^{n}Q_{t,n}^{-}(x, t, \lambda)
$$

and

$$
\phi_{t,n}^{+}(x, t, -\lambda) = (-1)^{n}\phi_{t,n}^{-}(x, t, \lambda).
$$

**Proof.** We notice that

$$
\phi_{t,n}^{+}(x, t, -\lambda) = e^{-4x(-1)^{t}t^{2^{+1}t}}Q_{t,n}^{+}(x, t, -\lambda)
$$

since $\theta_{t,n}$ does not depend on $\lambda$. So, both relations are equivalent and it suffices to prove that $Q_{t,n}^{+}(x, t, -\lambda) = (-1)^{n}Q_{t,n}^{-}(x, t, \lambda)$. We prove it by induction on $n$. For $n = 0$, we have that $Q_{t,0}^{+} = 1 = Q_{t,0}^{-}$. Hence, $Q_{t,0}^{+}(x, t, -\lambda) = (-1)^{n}Q_{t,0}^{-}(x, t, \lambda)$.

Using expressions (4.21) and (4.22), we obtain

$$
Q_{t,n+1}^{+}(x, t, -\lambda) = (-1)^{n}(-\lambda Q_{t,n}^{+}(x, t, \lambda) \theta_{t,n+1} + Q_{t,n}^{-}(x, t, -\lambda) \theta_{t,n+1} - \theta_{t,n+1} \lambda Q_{t,n}^{+}(x, t, -\lambda))
$$

$$
= (-1)^{n+1}(\lambda Q_{t,n}^{-}(x, t, \lambda) \theta_{t,n+1} + Q_{t,n}^{+}(x, t, -\lambda) \theta_{t,n+1} + \theta_{t,n+1} \lambda Q_{t,n}^{-}(x, t, -\lambda))
$$

$$
= (-1)^{n+1}Q_{t,n+1}^{-}(x, t, \lambda),
$$

as we wanted to prove.

This corollary allows us to compute the determinant of $B_{t,n}^{(r)}$. First observe that

$$
\det B_{t,n}^{(r)} = W(\phi_{t,n}^{+}, \phi_{t,n}^{-}) = (-1)^{n}W(\phi_{t,n}^{+}, \phi_{t,n}^{-}) \phi_{t,n}(x, t, -\lambda))
$$

$$
= (-1)^{n}(-1)^{r}2Q_{t,n}^{+}(x, t, \lambda)Q_{t,n}^{-}(x, t, -\lambda) + W(Q_{t,n}^{+}(x, t, -\lambda), Q_{t,n}^{+}(x, t, -\lambda)),
$$

where $W(\phi_{1}, \phi_{2}) = \phi_{1}\phi_{2} - \phi_{1}\phi_{2}$ denotes the Wronskian of $\phi_{1}$ and $\phi_{2}$.

16
Theorem 4.8. We have

$$\det \mathcal{B}_{n,1}^{(r)} = -2 \lambda^{2n+1}.$$  

Proof. We proceed by induction on \( n \). For \( n = 0 \) we obtain \( Q_{r,0}^+ = 1 \) and \( \theta_{r,0} = 1 \), so \( \det \mathcal{B}_{0,1}^{(r)} = -2 \lambda \). Now, we suppose it true for \( n \) and prove it for \( n + 1 \). Replacing expression (4.21) for \( Q_{r,n+1}^+(x,t,\lambda) \) and \( Q_{r,n+1}^-(x,t,-\lambda) \) in formula (4.28) and using Proposition 4.7 and the induction hypothesis, we get:

$$\det \mathcal{B}_{n+1,1}^{(r)} = -2 \lambda^{2n+3} = -2 \lambda^{2(n+1)+1}.$$

As we wanted to prove. \( \square \)

Remark 4.9. Theorem 4.8 implies that matrix \( \mathcal{B}_{n,1}^{(r)} \) is not a fundamental matrix of system (2.3) for \( \lambda = E = 0 \), since it is not invertible for that value of \( E \). The reason of this is that, by Proposition 4.7, when \( \lambda = 0 \) we have \( \phi_{r,0}^+(x,t,0) = (-1)^r \phi_{r,0}^-(x,t,0) \), so, both column solutions are linearly dependent. We will detail this phenomenon in Section 6. In fact, we will show that it is not the same to set \( E = 0 \) in (2.3) and then solve the system, than to solve the system for a generic \( E \) and then replace \( E = 0 \) in the solution obtained, i.e., there is not specialization process in this sense.

Example 4.10. For \( n = 0 \) and \( n = 1 \) we obtain by direct computations the following solutions:

\[
\begin{align*}
\phi_{r,0}^+ &= e^{x(1+y)\lambda^{2n+1} t}, \\
\phi_{r,0}^- &= e^{-x(1+y)\lambda^{2n+1} t}, \\
\phi_{r,1}^+ &= e^{x(1+y)\lambda^{2n+1} t} \frac{4x - 1}{x}, \\
\phi_{r,1}^- &= e^{-x(1+y)\lambda^{2n+1} t} \frac{4x + 1}{x}.
\end{align*}
\]

In next section we will show a method to compute functions \( Q_{r,n}^+ \) and \( Q_{r,n}^- \) more efficient than solving explicitly equations (4.17), (4.18), (4.19) and (4.20) which will allow us to obtain fundamental matrices \( \mathcal{B}_{n,1}^{(r)} \). In particular \( \phi_{r,1}^+ \) and \( \phi_{r,1}^- \) are linearly independent solutions for the Schrödinger operator \( -\partial_x^2 + u_{r,1} - E = 0 \) where \( u_{r,1} = 2/x^2 \) is the constructed rational KdV, potential, as long as \( E \neq 0 \).

5. Examples of fundamental matrices for the case \( E \neq 0 \)

Along this section we prove that functions \( Q_{r,n}^\pm \) defined in Theorem 4.6 satisfy the recursion formula (2.11). This implies in particular that they are polynomials of \( x \) with coefficients in \( \mathbb{C}(\lambda,t) \). Thus, they generalized the family of Adler-Moser polynomials \( \theta_n \).

For the following computations we do not suppose that functions \( \theta_n \) and \( Q_{r,n}^\pm \) and potentials \( u_n \) are adjusted to any level of the KdV hierarchy.

5.1. Generalized Adler-Moser polynomials

In Lemma 4.5 we have obtained the recursive formulas (4.21) and (4.22) for \( Q_{r,n}^\pm \). As we have seen in the proof of Theorem 4.6, these expressions are obtained by applying Darboux–Crum transformations with \( \phi_{1,r,n}^\pm \) to \( \phi_{r,n}^\pm \), see expressions (4.25) and (4.26). For our present discussion, we consider the unadjusted relations given in Lemma 4.5:

\[
\begin{align*}
Q_{n+1}^+ &= \frac{\lambda Q_{n}^+ \theta_{n+1} + Q_{n}^- \theta_{n+1} + Q_{n}^\pm \theta_{n+1} + Q_{n}^\pm \theta_{n+1}}{\theta_n}, \\
Q_{n+1}^- &= \frac{-\lambda Q_{n}^- \theta_{n+1} - Q_{n}^+ \theta_{n+1} + Q_{n}^\pm \theta_{n+1} - Q_{n}^\pm \theta_{n+1}}{\theta_n}.
\end{align*}
\]

If we proceed in the same way performing Darboux transformations with \( \phi_{1,r,n}^\pm \) we obtain that functions

\[
\begin{align*}
DT(\phi_{1,r,n})\phi_{r,n}^+ &= \phi_{1,r,n}^+ = e^{x(1+y)\lambda^{2n+1} t} \frac{\lambda Q_{r,n}^+ \theta_{r,n-1} + Q_{r,n}^- \theta_{r,n-1} - \theta_{r,n-1} \lambda Q_{r,n}^-}{\theta_{r,n}}, \\
DT(\phi_{1,r,n})\phi_{r,n}^- &= \phi_{1,r,n}^- = e^{-x(1+y)\lambda^{2n+1} t} \frac{-\lambda Q_{r,n}^- \theta_{r,n-1} - Q_{r,n}^+ \theta_{r,n-1} + \theta_{r,n-1} \lambda Q_{r,n}^+}{\theta_{r,n}}.
\end{align*}
\]
are solutions of Schrödinger equation for $E \neq 0$ and potential

$$DT(\phi_{1,n})u_{r,n} = u_{r,n} - 2\log(\phi_{1,n})_{xx} = u_{r,n-1}. \quad (5.3)$$

In the same way that we did for functions (4.21) and (4.22), we can prove that expressions

$$Q_{r,n}^+ := \frac{\lambda Q_{r,n}^+ \theta_{r,n-1} + Q_{r,n-1}^+ \theta_{r,n-1} - \theta_{r,n-1}Q_{r,n}^+}{\lambda^2 \theta_{r,n}}$$

and

$$Q_{r,n}^- := \frac{\lambda Q_{r,n}^+ \theta_{r,n-1} - Q_{r,n-1}^+ \theta_{r,n-1} + \theta_{r,n-1}Q_{r,n}^+}{\lambda^2 \theta_{r,n}}$$

satisfy differential systems (4.17)-(4.18) and (4.19)-(4.20), respectively, for $n = 1$. So, we obtain:

$$DT(\phi_{1,n})\phi_{r,n}^+ = \phi_{r,n}^+ - \frac{\phi_{r,n+1}^+}{\phi_{r,n}} = \lambda^2 \phi_{r,n-1}^-.$$ 

For our present discussion, we just write:

$$Q_{n-1}^+ = \frac{\lambda Q_{n}^+ \theta_{n-1} + Q_{n-1}^+ \theta_{n-1} - \theta_{n-1}Q_{n}^+}{\lambda^2 \theta_{n}}, \quad (5.4)$$

$$Q_{n-1}^- = \frac{\lambda Q_{n}^+ \theta_{n-1} - Q_{n-1}^+ \theta_{n-1} + \theta_{n-1}Q_{n}^+}{\lambda^2 \theta_{n}}. \quad (5.5)$$

Now, we can prove the following result:

**Theorem 5.1.** Functions $Q_{n}^+(x, t, \lambda)$ and $Q_{n}^-(x, t, \lambda)$ satisfy the differential recursions:

$$Q_0^+ = 1, \quad Q_1^+ = \lambda x - 1, \quad Q_{n+1}^+ = Q_{n-1}^+ - Q_{n-1}^+Q_{n-1}^- = (2n + 1)Q_{n+1}^2, \quad (5.6)$$

$$Q_0^- = 1, \quad Q_1^- = \lambda x + 1, \quad Q_{n+1}^- = Q_{n+1}^-Q_{n-1}^- - Q_{n-1}^-Q_{n-1}^+ = (2n + 1)Q_{n+1}^2. \quad (5.7)$$

**Proof.** In Remark 4.10 we have computed $\phi_{n}^+$ and $\phi_{n}^-$ for $n = 0$ and $1$. We have obtained $Q_{1}^+ = 1, Q_{1}^- = \lambda x - 1$ and $Q_{1}^+ = \lambda x + 1$. So, we just have to prove the recursion formulas. First, we prove (5.6). For this, we compute $Q_{n+1}^+$ and $Q_{n+1}^-$ using expressions (5.1) and (5.4):

$$Q_{n+1}^+ = \frac{\lambda Q_{n}^+ \theta_{n+1} + Q_{n+1}^+ \theta_{n+1} - \theta_{n+1}Q_{n}^+}{\lambda^2 \theta_{n}}, \quad (5.8)$$

Replacing this expressions in the recursion formula (5.8) we get:

$$Q_{n+1}^+ = \frac{(\lambda^2 Q_{n}^+ + 2\lambda Q_{n}^+ Q_{n}^- + Q_{n}^+ Q_{n}^-)Q_{n}^+(\theta_{n+1} \theta_{n-1} - \theta_{n+1} \theta_{n-1})}{\lambda^2 \theta_{n}}, \quad (5.9)$$

We want to compute the expressions for $\theta_{n+1}$ and $\theta_{n-1}$ in brackets in terms of $\theta_{n}$. The first expression is just relation (2.11). Now, if we derive with respect to $x$ expression (2.11), we find the second one:

$$\theta_{n+1,xx} \theta_{n-1} - \theta_{n+1,xx} \theta_{n-1} = 2(2n + 1) \theta_{n} \theta_{n,xx}. \quad (5.10)$$

In order to compute

$$\theta_{n+1,xx} \theta_{n-1} - \theta_{n+1,xx} \theta_{n-1} = 2(2n + 1) \theta_{n} \theta_{n,xx}. \quad (5.11)$$

we use relation (4.13). We have:

$$\theta_{n+1,xx} = 2 \frac{\theta_{n+1,xx} \theta_{n+1} \theta_{n-1} \theta_{n}}{\theta_{n}} \quad \text{and} \quad \theta_{n-1,xx} \theta_{n-1} \theta_{n} \theta_{n} = 2 \frac{\theta_{n-1,xx} \theta_{n-1} \theta_{n}}{\theta_{n}} \theta_{n} \theta_{n,xx}. \quad (5.12)$$

Replacing both expressions in (5.12) we get the third one:

$$\theta_{n+1,xx} \theta_{n-1} - \theta_{n+1,xx} \theta_{n-1} = \frac{\theta_{n,xx}}{\theta_{n}} \theta_{n,xx} - \theta_{n+1,xx} \theta_{n-1} - \theta_{n+1,xx} \theta_{n-1} = 2(2n + 1) \theta_{n} \theta_{n,xx}. \quad (5.13)$$
Applying expressions (2.11), (5.8) and (5.10) we get:

\[ Q_{n+1,x}^r Q_{n-1}^r - Q_{n+1}^r Q_{n-1,x}^r = (2n + 1) \frac{\lambda^2 Q_n^2 \theta_n + 2\lambda Q_n^2 \theta_{n,x} + 2\lambda Q_n^2 \theta_{n,x} + 2\lambda Q_n^2 \theta_{n,x} + 2\lambda Q_n^2 \theta_{n,x} + 2\lambda Q_n^2 \theta_{n,x} + 2\lambda Q_n^2 \theta_{n,x}}{\lambda^2 \theta_n}. \]

Finally, expression (4.17) for \( Q_{n,x}^r \) yields to:

\[ Q_{n+1,x}^r Q_{n-1}^r - Q_{n+1}^r Q_{n-1,x}^r = (2n + 1)Q_n^r. \]

Analogously, the second recursion formula can be proved. So we have established our result.

Remark 5.2. By Lemma 2.1 for \( F = C(\lambda, t) \) and \( a = \lambda, b = -1 \), we can conclude from this theorem that the functions \( Q_n^\|=Q^\| \) and \( F \) are polynomials of \( x \) with coefficients in \( C(\lambda, t) \) for all \( n \). Indeed, their degree as functions of \( \lambda \) is \( n \). Thus, Theorems 4.6 and 5.1 determine the algebraic structure of \( \phi_{n,\lambda}^r \) and \( \phi_{\tau,n}^r \).

Since polynomials \( Q_n^\|=Q^\| \) are not adjusted to any level of the KdV hierarchy, when we iterate recurrences (5.6) and (5.7) we will obtain integration constants of \( x \) which may depend on \( \lambda \) and \( \tau_2, \ldots, \tau_n \). We will denote such integration constants by \( \tau_2^r, \ldots, \tau_n^r \).

Example 5.3. For the first polynomials we find

| \( n \) | \( Q_n^+ \) | \( Q_n^- \) |
|-----|---------|---------|
| 0   | \( \lambda x - 1 \) | \( \lambda x + 1 \) |
| 1   | \( \lambda x^3 - 3\lambda x^2 + 3x + \tau_2^r \) | \( \lambda x^3 + 3\lambda x^2 + 3x + \tau_2^r \) |
| 2   | \( Q_2^+ \) | \( Q_2^- \) |

where

\[ Q_3^+ = \lambda x^6 - 6\lambda x^5 + 15\lambda x^4 - 15\lambda x^3 + 5\lambda x^2 \tau_2^r - 15\lambda x^2 \tau_2^r - (\lambda \tau_3^r + 5(\tau_2^r)^2)x + \tau_3 \]

\[ Q_3^- = \lambda x^6 + 6\lambda x^5 + 15\lambda x^4 + 15\lambda x^3 + 5\lambda x^2 \tau_2^r + 15\lambda x^2 \tau_2^r + (\lambda \tau_3^r + 5(\tau_2^r)^2)x + \tau_3. \]

5.2. Examples of fundamental matrices for the case \( E \neq 0 \)

We can compute fundamental matrices for system (4.15) for any \( n \) using recursion formulas (5.6) and (5.7).

Example 5.4. We present explicit computations using SAGE for the fundamental solutions of the system (4.15) when \( E = -\lambda^2 \neq 0 \) for same potentials as in Example 4.4.

1. We first expose examples of unadjusted fundamental solutions:

| \( n \) | \( \phi_{\tau,n}^r \) | \( \phi_{\tau,n}^r \) |
|-----|---------|---------|
| 0   | \( e^{\lambda x(-1)^r \lambda^{r+1} t} \) | \( e^{-\lambda x(-1)^r \lambda^{r+1} t} \) |
| 1   | \( e^{\lambda x(-1)^r \lambda^{r+1} t} \lambda x - 1 \) | \( e^{-\lambda x(-1)^r \lambda^{r+1} t} \lambda x + 1 \) |
| 2   | \( e^{\lambda x(-1)^r \lambda^{r+1} t} \frac{\lambda^2 x^3 - 3\lambda x^2 + 3x + \tau_2^r}{\lambda^3 + \tau_2} \) | \( e^{-\lambda x(-1)^r \lambda^{r+1} t} \frac{\lambda^2 x^3 + 3\lambda x^2 + 3x + \tau_2^r}{\lambda^3 + \tau_2} \) |
| 3   | \( Q_3^+(\lambda, x, t) \) | \( Q_3^-(\lambda, x, t) \) |

where \( Q_3^+ \) and \( Q_3^- \) are the ones given in (5.11).
2. Next, we expose fundamental solutions for potentials which are solutions of the first level of the KdV hierarchy, KdV$_1$ equation: $u_t = \frac{1}{6} u u_x - \frac{1}{2} u_{xxx}$. We also show the explicit choice of the functions $\tau_i$. The choice of functions $\tau_i$ is the same as in Example 4.4.

\[
\begin{align*}
\phi^+_i & \quad e^{1x-i^3 t_1} \\
\phi^-_i & \quad e^{-1x+i^3 t_1} \\
n & = 0, 1, 2, 3 \quad \text{(r}_2, \ldots, \text{r}_n) \end{align*}
\]

where

\[
\begin{align*}
\phi^+_i(x,t_1) & = e^{1x} x^3 - 3l^2 x^2 + 3x + 3l^2 t_1 \\
\phi^-_i(x,t_1) & = e^{-1x} x^3 - 3l^2 x^2 + 3x + 3l^2 t_1 \\
\end{align*}
\]

6. Spectral curves and Darboux-Crum transformations

Let $\Gamma_n \subset \mathbb{C}^2$ be the spectral curve associated to the stationary Schrödinger operator $-\partial_{xx} + u - E$ where $u$ is a s-KdV$_n$ potential. Next we consider the Zariski closure of $\Gamma_n$, say $\overline{\Gamma_n}$, in the complex projective plane $\mathbb{P}^2$. Let be $p(E, \mu) = \mu^2 - R_{2n+1}(E) = \mu^2 - \sum_{j=0}^{2n+1} C_j E^j = 0$ an equation for $\Gamma_n$. Then an equation for $\overline{\Gamma_n}$ is

\[
p_h(E, \mu, \nu) = \mu^2 \nu^{2n+1} - \overline{R}_{2n+1}(E, \nu) = 0,
\]

where $\overline{R}_{2n+1}(E, \nu) = \nu^{2n+1} R_{2n+1}(E)$ is an homogeneous polynomial of degree $2n + 1$. Moreover, observe that the singular points of $\overline{\Gamma_n}$ are

\[
\text{Sing}\left(\overline{\Gamma_n}\right) = \{(E, 0) : E \text{ is a multiple root of } R_{2n+1}\} \cup \{P_\infty = [0 : 1 : 0]\},
\]

and also

\[
\overline{\Gamma_n} \cap \{E = 0\} = \{(0 : \mu : \nu) \in \mathbb{P}^2 : \mu^2 \nu^{2n+1} = C_0 \nu^{2n+1}\}.
\]

6.1. Extended Green’s function

Following [13], we define the Green’s function on $\Gamma_n \times \mathbb{C}$ as

\[
g(E, \mu, x) = \frac{\phi_1 \phi_2}{W(\phi_1, \phi_2)},
\]

where $\phi_1$ and $\phi_2$ are two independent solutions of Schrödinger equation

\[
(L - E)\phi = (-\partial_{xx} + u - E)\phi = 0.
\]

for the same value of $E$ and $W(\phi_1, \phi_2)$ stands for their wronskian.

Let

\[
\sigma_+ = \sigma(E, \mu) = \frac{i \mu + F_{n,1}/2}{F_n}, \quad \sigma_- = \sigma(E, -\mu) = \frac{-i \mu + F_{n,1}/2}{F_n}
\]

be functions defined over the spectral curve. We recall the following result.
Lemma 6.1 (Lemma 1.8 of [13]). Let \( u \) be solution of s-KdV\(_n\) equation (2.10). Let \( \phi_1 \) and \( \phi_2 \) be solutions of Schrödinger equation (6.5) for this potential and with corresponding functions over the spectral curve \( \sigma_+ \) and \( \sigma_- \) defined by (6.6). Then \( \sigma_+ \) and \( \sigma_- \) are solutions of the Riccati type equation:

\[
\sigma^2 + \sigma = u - E. \tag{6.7}
\]

Moreover, the following identities are satisfied:

\[
\sigma_+ + \sigma_- = \frac{F_{n,1}}{F_n} = \frac{(\phi_1 \phi_2)_x}{\phi_1 \phi_2}, \quad \sigma_+ - \sigma_- = \frac{2i\mu}{F_n} = \frac{W(\phi_1, \phi_2)}{\phi_1 \phi_2}, \quad \sigma_+ \cdot \sigma_- = \frac{H_n}{F_n} = \frac{\phi_1, \phi_2 x}{\phi_1 \phi_2}, \tag{6.8}
\]

where \( W(\phi_1, \phi_2) = \phi_1 \phi_2 x - \phi_1, \phi_2 \) denotes the wronskian of \( \phi_1 \) and \( \phi_2 \).

We remark that this lemma is essentially a reformulation of a classic result that goes back to Hermite when he was studying closed form solutions for Lamé equation ([14]). In [19] call this approach the Lindeman-Stieljes theory but, as far as we know, this approach was used for the first time by Hermite, and then by others: Halphen, Briot-Heine, Crawford, Stieljes,... The method used that the product of solutions \( X = \phi_1 \phi_2 \) is a solution of the second symmetric power of the Schrödinger equation

\[
(-\partial_{x,1} - 4(u - E)\partial_x - 2\mu_x)X = 0. \tag{6.9}
\]

Then the relations (6.8) connect the solutions of the Riccati equation with that of the second symmetric power. The fact that there is a connection between the solutions of the second symmetric product and the Riccati equation of the Schrödinger equation is relevant for the differential Galois theory, although we will not use explicitly this connection in this paper. Furthermore it is interesting to point out that the solutions of the Lamé equation obtained by Hermite in [14], are associated to other algebro-geometric solutions of KdV, finite-gap solutions with regular spectral curves, see [17] and references therein. As far as we know, the relevance of the equation (6.9) for the KdV equation was considered for the first time by Gel’fand and Dikii in their fundamental paper about the asymptotic behaviour of the resolvent of the Schrödinger equation associated to the KdV equation [12].

By Lemma 6.1, the Green’s function can be rewritten as

\[
g(E, \mu, x) = \frac{iF_n(E, x)}{2\mu} = \frac{1}{\sigma_+ - \sigma_-}. \tag{6.10}
\]

Observe that \( g \) is well defined whenever \( \mu \neq 0 \), i. e. for energy levels such that \( R_{2n+1}(E) \neq 0 \).

Next, let define a extension of \( g \) on \( \Gamma_n \times C_x \) as

\[
g_b(E, \mu, \nu, x) = \frac{i\nu^a F_n(E/\nu, x)}{2\mu^{a-1}}, \quad \text{for } [E : \mu : \nu] \in \Gamma_n \setminus \{\nu = 0\}. \tag{6.11}
\]

We call \( g_b \) the homogenized Green’s function. Next we will show that \( g_b \) is well defined and also that it extends \( g \), that is \( g_b(E, \mu, 1, x) = g(E, \mu, x) \) for \( (E, \mu, x) \in \Gamma_n \times C_x \). To do that, observe that

\[
g_b(E, \mu, 1, x) = g(E, \mu, x) \quad \text{and} \quad g_b(aE, a\mu, av, x) = g_b(E, \mu, \nu, x),
\]

for any \( a \in C, a \neq 0 \). Moreover, we have that

\[
\hat{F}_n(E, \nu, x) := \nu^a F_n(E/\nu, x) = \sum_{j=0}^{n} f_{n-j} \nu^{n-j} E^j, \tag{6.12}
\]

is an homogeneous polynomial in \( E \) of degree \( n \) and then

\[
g_b(E, \mu, \nu, x) = \frac{i\hat{F}_n(E, \nu, x)}{2\mu^{a-1}}, \quad \text{for } [E : \mu : \nu] \in \Gamma_n. \tag{6.13}
\]

Also, we get the following formula:

\[
\mu^2 \nu^{2n-2} = \nu^{2n} R_{2n+1}(E/\nu) = \frac{\nu^2 F_{n,aa}^2}{2\Gamma} - (u - E/\nu) \hat{F}_n^2 - \frac{\nu^2 \hat{F}_n^2}{4}. \tag{6.14}
\]
where
\[ \tilde{F}_{n,x} = \nu^{n-1}F_{n,x}(E/v) \quad \text{and} \quad \tilde{F}_{n,xx} = \nu^{n-1}F_{n,xx}(E/v) \] (6.15)
are homogeneous polynomials in \( E \) and \( v \) of degree \( n - 1 \).

Now, take equation (2.19):
\[ 0 = \frac{F_{n,xxx}}{2} - 2(u - E)F_{n,x} - u_x F_n, \]
after multiplication by \( F_n \) and integration, this equation reads
\[ c = \frac{F_n F_{n,xx}}{2} - (u - E)F_n^2 - \frac{F_n^4}{4}, \] (6.16)
where \( c \) is an integration constant. By (6.10) we have the following differential relation for the function \( g \):
\[ \frac{1}{2}gg_{xx} - (u - E)g^2 - \frac{1}{4}g_x^2 = -\frac{1}{4}, \]
since \( g_x = (\sigma_+ + \sigma_-)g \) and \( g_{xx} = 2(u - E + \sigma_+\sigma_-)g \).

Now let define the extensions of \( \sigma_+ \) and \( \sigma_- \) on \( \mathbb{T}_n \times \mathbb{C}_x \) as
\[ (\sigma_+)_h = \frac{i\mu^{n-1} + \nu \tilde{F}_{n,x}/2}{F_n}, \quad (\sigma_-)_h = \frac{-i\mu^{n-1} + \nu \tilde{F}_{n,x}/2}{F_n}, \] (6.17)
where we have used previous notations. Notice that the functions \((\sigma_+)_h\) and \((\sigma_-)_h\) are solutions of the Riccati type equation
\[ ((\sigma_+)_h)^2 + ((\sigma_-)_h)_x = u - E/v. \]
Moreover we have that the function
\[ g_h = \frac{i\tilde{F}_n(E, v, x)}{2\mu^{n-1}} = \frac{1}{(\sigma_-)_h - (\sigma_+)_h} \] (6.18)
is a solution of
\[ \frac{1}{2}g_h(g_{xx})h - (u - E/v)g_h^2 - \frac{1}{4}(g_x^2)_h = -\frac{1}{4}. \]

6.1.1. Transformed Green’s functions

Now, we analyze how Darboux-Crum transformations change Green’s functions \( g \) and \( g_h \). For that, we will use solutions of the Riccati type equation (6.7) as a essential tool.

Let \( u \) be solution of \( sKdV_n \) equation (2.10). Let \( \phi_1 \) and \( \phi_2 \) be solutions of Schrödinger equation (6.5) for this potential and energy level \( E \). Next we consider \( \phi_0 \) a solution of Schrödinger equation for \( u \) and \( E_0 \), with \( E_0 \neq E \) and choose as corresponding point of the spectral curve \((E_0, \mu_0)\). Recall that after applying a Darboux-Crum transformation with \( \phi_0 \) to \( u \), \( \phi_1 \) and \( \phi_2 \), we get
\[ DT(\phi_0)u = u - 2\sigma_0 x, \quad DT(\phi_0)\phi_1 = \phi_{1,x} - \sigma_0 \phi_1, \quad DT(\phi_0)\phi_2 = \phi_{2,x} - \sigma_0 \phi_2, \] (6.19)
where \( \sigma_0 = (\log \phi_0)_x \) is a solution of the Riccati equation \( \sigma^2 + \sigma_x = u - E_0 \). By Lemma 6.1, the function \( \sigma^0 \) equals
\[ \sigma^0 = \sigma(E_0, \mu_0) = \frac{i\mu_0 + F_0^{(n,x)}/2}{F_0^{(n)}}, \] (6.20)
where \( F_0^{(n)} = F_n(E_0) \), is a solution of the same Riccati equation for \( E = E_0 \). Thus, we conclude that we can perform a Darboux transformation using \( \sigma^0 \) instead of \( \sigma_0 \). The transformed functions
\[ \tilde{\phi}_1 = \phi_{1,x} - \sigma^0 \phi_1 \quad \text{and} \quad \tilde{\phi}_2 = \phi_{2,x} - \sigma^0 \phi_2 \]
are solutions of the Schrödinger equation for potential
\[ \tilde{u} = u - 2\sigma^0 x. \]
Now, we take the functions \( \sigma_1 = (\log \phi_1)_s \) and \( \sigma_2 = (\log \phi_2)_s \), which are solutions of the Riccati equation (6.7) for \( E \neq E_0 \). Then, by equations (6.8), we get the equalities

\[
\begin{align*}
\sigma_+ + \sigma_- &= \frac{2\mu}{\hat{F}_n} - \frac{W(\dot{\phi}_1, \dot{\phi}_2)}{\phi_1 \phi_2} = \frac{\phi_1 + \phi_2}{\phi_1 \phi_2} = \sigma_1 + \sigma_2, \\
\sigma_+ - \sigma_- &= \frac{F_{n,x}}{\phi_1 \phi_2} = \frac{\phi_1 \phi_2}{\phi_1 \phi_2} = \sigma_1 - \sigma_2, \\
\sigma_+ \cdot \sigma_- &= \frac{\phi_1 \phi_2}{\phi_1 \phi_2} = \sigma_1 \cdot \sigma_2.
\end{align*}
\] (6.21)

Next we define the transformed Green’s function

\[
\tilde{g}(E, \mu, x) = \frac{\tilde{\phi}_1 \tilde{\phi}_2}{W(\dot{\phi}_1, \dot{\phi}_2)}. 
\] (6.24)

The relations (6.21)-(6.23) link the Green’s functions as follows:

\[
\tilde{g}(E, \mu, x) = \frac{(\sigma_+ - \sigma_0)(\sigma_2 - \sigma_0)}{(E - E_0)} \cdot \frac{\phi_1 \phi_2}{W(\dot{\phi}_1, \dot{\phi}_2)} = \frac{(\sigma_+ - \sigma_0)(\sigma_- - \sigma_0)}{(E - E_0)} \cdot g(E, \mu, x).
\]

Hence we obtain a rational presentation of \( \tilde{g} \) as a consequence of the formulas (6.20) and (6.6). We write this formula in (6.25).

**Proposition 6.2.** The Green’s function associated to the transformed Schrödinger operator explicitly reads:

\[
\tilde{g}(E, \mu, x) = \frac{i \left( \mu^2 (F_n^0)^2 - \mu_0^2 F_n^2 - \mu_0 F_n (F_n^0 F_{n,x} - F_{n,x} F_n^0) + \frac{i \mu F_n (F_n^0 F_{n,x} - F_{n,x} F_n^0)}{4} \right)}{2\mu (E - E_0) F_n (F_n^0)^2}. 
\] (6.25)

**Remark 6.3.** Observe that for \( E_0 = 0 \) the formula (6.25) becomes:

\[
\tilde{g}(E, \mu, x) = \frac{i \left( \mu^2 f_n^2 - \mu_0^2 F_n^2 - \mu_0 F_n (f_n F_{n,x} - f_{n,x} F_n) + \frac{i \mu F_n (f_n F_{n,x} - f_{n,x} F_n)}{4} \right)}{2\mu E F_n f_n^2}. 
\] (6.26)

We will use the following result from [13].

**Proposition 6.4** (Lemma G.1 in [13]). Let \( u \) be solution of \( s \)-KdV \( n \) equation, let \( (E_0, \mu_0) \) and \( (E, \mu) \) be two different points of \( \Gamma_n \). Then the transformed Green’s function explicitly reads:

\[
\tilde{g}(E, \mu, x) = \frac{(\sigma_+ - \sigma_0)(\sigma_- - \sigma_0)}{(E - E_0)} \cdot \frac{i F_n}{2\mu} = \frac{i \tilde{F}_n(E, x)}{2\mu}, 
\] (6.27)

where \( \tilde{F}_n \) is a polynomial in \( E \) of degree \( \tilde{n} \) and \( \tilde{\mu} \) is such that \( \tilde{\mu}^2 - \tilde{R}_{2\tilde{n}+1} = 0 \) for some polynomial \( \tilde{R}_{2\tilde{n}+1}(E) \) of degree \( 2\tilde{n} + 1 \), with \( 0 \leq \tilde{n} \leq n + 1 \).

Next, for the homogenized Green’s function, choose the point of the spectral curve \( [E_0 : \mu_0 : \nu_0] \). We define the extension of \( \sigma_0 \) on \( \overline{\Gamma}_n \times \mathbb{C}_s \) as

\[
(\sigma_0)^h(E_0, \mu_0, 0) = \frac{i \mu_0 \nu_0^{n-1} + \nu_0 \tilde{F}_n^0}{\tilde{F}_n^0}, 
\] (6.28)

where \( \tilde{F}_n^0 = \tilde{F}_n(E_0, \nu_0, x) \) for \( \tilde{F}_n(E, \nu, x) \) defined by (6.12) and \( \tilde{F}_n^0 = \tilde{F}_n(E_0, \nu, x) \) for \( \tilde{F}_n \) defined in (6.15). Notice that when \( \nu_0 = 0 \) function \( (\sigma_0)^h \) vanishes. So, whenever \( \nu_0 = 0 \) we define

\[
(\sigma_0)^h(E_0, \mu_0, 0) := 0, \quad \text{for } [E_0 : \mu_0 : 0] \in \overline{\Gamma}_n.
\]

Using above notation we have the following results.
Proposition 6.5. Let assume \( C_0 = R_{2^{n+1}}(0) \neq 0 \). For \( E_0 = 0 \) and \( \mu_0 \neq 0 \), the homogenized Green’s function associated to transformed Green’s function \( \tilde{g} \) for \(-\partial_{xx} + \tilde{u} - E\) explicitly reads:

\[
(\bar{g})_h(E, \mu, \nu, x) = \frac{i}{2\mu E\nu^{n-1}} \left( \frac{\nu^2 \tilde{F}_{n,x}}{2} + (E - \nu \tilde{u}) \tilde{F}_{n} + \frac{\nu \tilde{C} \tilde{F}_{n}}{4f_n} - \frac{\nu^2 f_n \tilde{F}_{n,x}}{2f_n} - \frac{\nu \tilde{C} \tilde{F}_{n,x}}{f_n} \right) + \frac{C_0 \nu_0 (\nu f_n \tilde{F}_{n,x} - f_{n,x} \tilde{F}_{n})}{2\mu E\nu^{n-2}\mu_0 f_n^2},
\]

where \( \tilde{F}_{n}(E, \nu, x) \) is defined by (6.12) and \( \tilde{F}_{n,x}(E, \nu, x) \) are defined by (6.15).

Remark 6.6. Formula

\[
\frac{\nu^2 \tilde{F}_{n,x}}{2} + (E - \nu \tilde{u}) \tilde{F}_{n} + \frac{\nu \tilde{C} \tilde{F}_{n}}{4f_n} - \frac{\nu^2 f_n \tilde{F}_{n,x}}{2f_n} - \frac{\nu \tilde{C} \tilde{F}_{n,x}}{f_n}
\]

is an homogeneous polynomial in \( E \) and \( \nu \) of degree \( n + 1 \).

Proof. First, consider the transformed Green’s function \( \tilde{g} \) given by (6.25). Then, the homogenized Green’s function is obtained by the homogenization process as

\[
(\bar{g})_h(E, \mu, \nu, x) = \frac{(\sigma_x - \sigma^0)(\sigma_y - \sigma^0)}{(E - E_0)} \cdot \frac{i F_{n,x}}{2\mu_f} + \frac{i \tilde{u}_0^2 \tilde{F}_{n,x}}{2\mu E \nu^{n-2}\tilde{F}_{n,f_n^2}} \cdot \frac{\nu \tilde{C} \tilde{F}_{n,x}}{f_n}
\]

where \( \tilde{F}_{n}(E, \nu, x) \) is defined by (6.12), \( \tilde{F}_{n,x}(E, \nu, x) \) is defined in (6.15), \( \tilde{F}_{n} = \tilde{F}(0, \nu, 0) \) and \( \tilde{F}_{n,x} = \tilde{F}_{n,x}(0, \nu, 0) \). In particular, for \( E_0 = 0 \), we get:

\[
(\bar{g})_h(E, \mu, \nu, x) = \frac{i \tilde{u}_0^2 \tilde{F}_{n,x}}{2\mu E \nu^{n-2}\tilde{F}_{n,f_n^2}} + \frac{\nu \tilde{C} \tilde{F}_{n,x}}{f_n}
\]

Moreover, by (6.3) we have that \( \mu_0^2 = \epsilon_0 \nu_0^2 \), and then

\[
(\bar{g})_h(E, \mu, \nu, x) = \frac{i \tilde{u}_0^2 \tilde{F}_{n,x}}{2\mu E \nu^{n-2}\tilde{F}_{n,f_n^2}} + \frac{\nu \tilde{C} \tilde{F}_{n,x}}{f_n}
\]

And then the result follows.

Proposition 6.7. Let assume \( C_0 = R_{2^{n+1}}(0) \neq 0 \). For \( E_0 = 0 \) and \( \mu_0 \neq 0 \), the homogenized Green’s function associated to transformed Green’s function \( \tilde{g} \) for \(-\partial_{xx} + \tilde{u} - E\) explicitly reads:

\[
(\bar{g})_h(E, \mu, \nu, x) = \frac{i}{2\mu E\nu^{n-1}} \left( \frac{\nu^2 \tilde{F}_{n,x}}{2} + (E - \nu \tilde{u}) \tilde{F}_{n} \right),
\]

where \( \tilde{F}_{n}(E, \nu, x) \) is defined by (6.12) and \( \tilde{F}_{n,x}(E, \nu, x) \) defined in (6.15).

Remark 6.8. Formula

\[
\frac{\nu^2 \tilde{F}_{n,x}}{2} + (E - \nu \tilde{u}) \tilde{F}_{n}
\]

is an homogeneous polynomial in \( E \) and \( \nu \) of degree \( n + 1 \).
Proof. When \( C_0 = 0 \) we have that \( v_0 = 0 \) by (6.3), since \( \mu_0 \neq 0 \). Hence, the homogeneized Green’s function in this case is:

\[
(Q\mu)(E, \mu, \nu, x) = \frac{(\sigma_+)(\sigma_-)}{E/\nu}, \quad \frac{iF_n}{2\nu^{\mu-1}} = \frac{i\left(\nu^2\sigma_\nu^2 - \nu^2\sigma_\sigma^2\right)}{2\nu E\nu^{\mu-2}} = \frac{i\left(\nu^2\sigma_\nu^2 + (E - \nu u)\sigma_\sigma^2\right)}{2\nu E\nu^{\mu-1}},
\]

by (6.17) and (6.14).

6.2. Darboux-Crum transformations for the Spectral curve

In this subsection we present how Darboux-Crum transformations affect the spectral curve \( \Gamma_n \). We observe that the action of the transformation \( DT(\phi_0) \) strongly depend on the type of point \( P \) in the spectral curve we use to construct \( \phi_0 \). In fact, if \( P \) is a regular point, the curve associated with the transformed potential is the same; in the other cases the new curve is a blowing-down or a blowing-up of \( \Gamma_n \).

**Theorem 6.9** (1). Let \((E_0, \mu_0) \in \Gamma_n \) and \( b \) be a solution of s-KdV equation. Let \( \phi_0 \) be a solution of Schrödinger equation for energy \( E_0 \) and potential \( u \), i.e., \( \phi_{0,xx} = (u - E_0)\phi_0 \). Let \( \bar{u} = u - 2(\log \phi_0)_{xx} \) be the Darboux-Crum transformation of \( u \). Then, \( \bar{u} \) is a solution of s-KdV equation for

\[
\bar{\mu} = \left\{ \begin{array}{ll}
n & \text{if } (E_0, \mu_0) \text{ is a regular point of } \Gamma_n, \\
n - 1 & \text{if } (E_0, \mu_0) \text{ is an affine singular point of } \Gamma_n.
\end{array} \right.
\]

Furthermore, the spectral curve associated to \( \bar{u} \) is \( \bar{\Gamma}_n : \bar{\mu}^2 - \bar{R}_{2n+1} = 0 \), with

\[
\bar{R}_{2n+1} = \left\{ \begin{array}{ll}
R_{2n+1} & \text{if } (E_0, \mu_0) \text{ is a regular point of } \Gamma_n, \\
(E - E_0)^2R_{2n+1} & \text{if } (E_0, \mu_0) \text{ is an affine singular point of } \Gamma_n.
\end{array} \right.
\]

The idea of the proof is to compute Green’s function (6.25) associated to \( \bar{u} \) and interpret the result by means of Lemma 6.4.

**Proof.** First, we suppose that \((E_0, \mu_0) \) is a regular point and \( \mu_0 \neq 0 \). In this case, we compute

\[
(\sigma_+ - \sigma^0)(\sigma_- - \sigma^0) = \mu^2(F_n^0)^2 - \mu_0^2F_n^2 - i\mu_0F_n(F_n^0F_n - F_n^0F_n) + \frac{(\sigma_\nu^0F_n^0F_n - \sigma_\sigma^0F_n^2)}{4}.
\]

We use Corollaries Appendix A.1 and Appendix A.2 to rewrite the expressions \( F_n^0F_n - F_n^0F_n \) and \( \mu^2(F_n^0)^2 - \mu_0^2F_n^2 \). This yields to the equality

\[
(\sigma_+ - \sigma^0)(\sigma_- - \sigma^0) = (E - E_0)^2 \frac{P_{\mu \nu} + P_{\nu \mu} + P_{\mu \nu}}{F_n^0F_n}.
\]

Finally, we replace this expression in Green’s function (6.25):

\[
\bar{g}(E, \mu, x) = \frac{iF_n(\sigma_+ - \sigma^0)(\sigma_- - \sigma^0)}{2\mu(E - E_0)} = \frac{i\left(F_n + \frac{P_{\mu \nu}}{P_{\mu \nu}} - \frac{P_{\nu \mu}}{P_{\mu \nu}}\right)}{2\mu} = \frac{i\bar{F}_n}{2\mu}.
\]

Since \( \bar{F}_n = F_n + \frac{P_{\mu \nu}}{P_{\mu \nu}} - \frac{P_{\nu \mu}}{P_{\mu \nu}} \) is a polynomial in \( E \) of degree \( n \), by means of Lemma 6.4, we conclude that \( \bar{n} = n \) and \( \bar{\mu} = \mu \). Thus, \( \bar{R}_{2n+1} = R_{2n+1} \).

Now, we suppose that \((E_0, \mu_0) \) is a regular point and \( \mu_0 = 0 \). In this case, we have that \( R_{2n+1}^0 = R_{2n+1}(E_0) = 0 \) and \( R_{2n+1}^0 = \partial(E(R_{2n+1})(E_0)) \neq 0 \), thus,

\[
\bar{\mu}^2 = R_{2n+1}(E) = (E - E_0)M_{2n},
\]

25
Appendix A.1 guarantees that the equality \((E - E_0)M_{2n}(E_0) \neq 0\). Hence for \(\mu_0 = 0\), \(\mu^2 = (E - E_0)\) and Corollary Appendix A.1, the equality (6.25) becomes
\[
\tilde{g}(E, \mu, x) = \frac{i \left( (E - E_0)M_{2n}(E_0) \right) + \frac{(E-E_0)^2 \rho^2}{4}}{2\mu(E - E_0)F_n(F_n'^2)} = \frac{i \left( M_{2n} + \frac{(E-E_0)^2 \rho^2}{4F_n(F_n'^2)} \right)}{2\mu}.
\]

Now Corollary Appendix A.3 guarantees that
\[
\frac{M_{2n}}{F_n} \frac{(E - E_0)F_n^2}{4F_n(F_n'^2)}
\]
is a polynomial in \(E\) of degree \(n\). By Lemma 6.4, we obtain that \(\tilde{n} = n, \tilde{\mu} = \mu\) and \(\tilde{R}_{2n+1} = R_{2n+1}\). Therefore, for regular points \(R_{2n+1}\) is a polynomial of degree \(2n + 1\) in \(E\). By Corollary 2.11, we conclude that \(\tilde{u}\) is solution of a s-KdV equation. Thus, a Darboux-Crum transformation with a regular point preserves the spectral curve and the level of the s-KdV hierarchy.

Next, we suppose that \((E_0, \mu_0)\) is a singular point of \(\Gamma_n\), i.e., \(\mu_0 = 0\), \(R_{2n+1}^0 = R_{2n+1}(E_0) = 0\) and \(R_{2n+1,E}^0 = \partial_E(R_{2n+1}(E_0)) = 0\), thus,
\[
\mu^2 = R_{2n+1}(E) = (E - E_0)^2Z_{2n-1},
\]
where \(Z_{2n-1}(E)\) is a polynomial in \(E\) of degree \(2n - 1\). Hence for \(\mu_0 = 0\), \(\mu^2 = (E - E_0)^2Z_{2n-1}\) and Appendix A.1, the equality (6.25) becomes
\[
\tilde{g}(E, \mu, x) = \frac{i \left( (E - E_0)^2Z_{2n-1}(F_n'^2) + \frac{(E-E_0)^2 \rho^2}{4} \right)}{2\mu(E - E_0)F_n(F_n'^2)} = \frac{i \left( \frac{Z_{2n-1}}{F_n} + \frac{P_n^2}{4F_n(F_n'^2)} \right)}{2(E - E_0)^{-1}\mu}.
\]

Now Corollary Appendix A.4 guarantees that
\[
\frac{Z_{2n-1}}{F_n} + \frac{P_n^2}{4F_n(F_n'^2)}
\]
is a polynomial in \(E\) of degree \(n\). By Lemma 6.4, we obtain that \(\tilde{n} = n - 1\) and \(\tilde{\mu} = (E - E_0)^{-1}\). Therefore, \(\tilde{R}_{2n+1} = (E - E_0)^{-2}R_{2n+1}\) is a polynomial of degree \(2n - 1\) in \(E\). By Corollary 2.11, we conclude that \(\tilde{u}\) is solution of a s-KdV_{n-1} equation. So, a Darboux-Crum transformation with a singular point induces a blow-up in the spectral curve in this singular point and reduces the level of the s-KdV hierarchy in one.

Next, we will proceed to establish the situation at the point of infinity \(P_\infty = [0 : 1 : 0]\) of the spectral curve. For that, we will need to work with the Zariski closure in \(\mathbb{P}^2\) of the spectral curve to understand its behavior under Darboux transformations for the energy level \(E_0 = 0\). In addition, we will use the blowing-up map in \(\mathbb{P}^2\) to control the KdV level of the transformed potential \(\tilde{u}\).

Let \(\pi : \overline{\mathbb{P}^2} \to \mathbb{P}^2\) be the blowing-up of \(\mathbb{P}^2\) with center \([0 : 0 : 1]\). Hence, if \([E : \mu : \nu]\) are homogeneous coordinates in \(\mathbb{P}^2\), then the new ones are denoted by \([\tilde{E} : \tilde{\mu} : \tilde{\nu}]\), and \(\pi\) is given by
\[
E = \tilde{E}, \quad \mu E = \tilde{\mu}, \quad \nu = \tilde{\nu}.
\]

Theorem 6.10 (II). Let \(P_\infty = [0 : 1 : 0]\) be the infinity point of \(\Gamma_n\), and \(u\) a solution of s-KdV equation. Let \(\phi_0\) be a solution of Schrödinger equation for \(P_\infty\) (in particular \(E_0 = 0\)) and potential \(u\), i.e., \(\phi_{0,x} - u\phi_0 = 0\). Let \(u = u - 2(\log \phi_0)_x\) be the Darboux–Crum transformation of \(u\). Then, \(\tilde{u}\) is solution of s-KdV_{n-1} equation. Furthermore, the spectral curve associated to \(\tilde{u}\) is \(\Gamma_{n-1} : \mu^2 - \tilde{R}_{2n+3}(\tilde{E}) = 0\), with \(\tilde{R}_{2n+3} = E^2R_{2n+1}(E)\).

Proof. First, consider the homogeneized Green’s function associated to transformed Green’s function \(\tilde{g}\). Then, by Propositions 6.5 and 6.7, \((\tilde{g})_{\tilde{h}}\) is a well defined rational function on \(\Gamma_n\). But also we have:
\[
(\tilde{g})_{\tilde{h}} = G_{\tilde{h}} \circ \pi \quad \text{on the spectral curve.}
\]

Moreover \(G_{\tilde{h}}\) is a Green function for the curve defined by \(\mu^2 - \tilde{R}_{2n+3}(\tilde{E}) = 0\), where \(\tilde{R}_{2n+3}(\tilde{E}) = E^2R_{2n+1}(E)\); that is, for \(\Gamma_{n-1}\), the strict transform of \(\Gamma_n\). Observe that \(\tilde{R}_{2n+3} = E^2R_{2n+1}\) is a polynomial of degree \(2n + 3\) in \(E\). Then, by Corollary 2.11, we conclude that \(\tilde{u}\) is solution of a s-KdV_{n-1} equation. \(\square\)
Finally we can rewrite 6.9 and 6.10 to establish how the spectral curve \( \Gamma_n \) behaves under Darboux–Crum transformations.

**Theorem 6.11.** Let \( P = [E_0 : \mu_0 : v_0] \) be a point in \( \Gamma_n \), and \( u \) a solution of s-KdV equation. Let \( \phi_0 \) be a solution of Schrödinger equation for \( E_0 \) and potential \( u \), say \( \phi_{0,xx} = (u - E_0)\phi_0 \). Consider \( \widetilde{u} = u - 2(\log \phi_0)_{xx} \) the Darboux-Crum transformation of \( u \). Then, \( \widetilde{u} \) is solution of s-KdV equation for

\[
\widetilde{\eta} = \begin{cases} 
  n + 1 & \text{if } P = [0 : 1 : 0], \\
  n & \text{if } P \text{ is a regular point of } \Gamma_n, \\
  n - 1 & \text{if } P \text{ is an affine singular point of } \Gamma_n.
\end{cases}
\]

Furthermore, the spectral curve associated to \( \widetilde{u} \) is \( \Gamma_\widetilde{n} : \mu^2 - R_{2\widetilde{n}+1} = 0 \), with

\[
R_{2\widetilde{n}+1} = \begin{cases} 
  E^2R_{2n+1} & \text{if } P = [0 : 1 : 0], \\
  R_{2n+1} & \text{if } P \text{ is a regular point of } \Gamma_n, \\
  (E - E_0)^2R_{2n+1} & \text{if } P \text{ is an affine singular point of } \Gamma_n.
\end{cases}
\]

**Example 6.12.** Next we apply the previous theorem to a rational s-KdV potential.

Take the s-KdV potential \( u = \frac{1}{x^2} \) in the Schrödinger equation (6.5). The spectral curve associated to this potential is \( \Gamma_2 : \mu^2 - E^3 = 0 \). When \( E = 0 \), we have the fundamental solutions \( \phi_1 = x^2 \) and \( \phi_2 = x^3 \). We consider the Darboux transformations of \( u \) with these solutions:

\[
DT(\phi_1)u = u - 2(\log \phi_1)_{xx} = \frac{2}{x^2} = u_1 \quad \text{and} \quad DT(\phi_2)u = u - 2(\log \phi_2)_{xx} = \frac{12}{x^2} = u_2.
\]

We have that potential \( u_1 \) is a solution of s-KdV equation. It is well known that the spectral curve associated to this potential is \( \Gamma_2 : \mu^2 - E^3 = 0 \), the blowing-up of \( \Gamma_2 \) at \( (0, 0) \). Furthermore, potential \( u_1 \) is a solution of s-KdV equation, and its associated spectral curve \( \Gamma_1 \) is the blowing-down of \( \Gamma_2 \), that is \( \Gamma_1 : \mu^2 - E^3 = 0 \).

Now, we take a regular value of \( E \) in \( \Gamma_2 \), for instance, \( E = -1 \). Then, a solution of the Schrödinger equation (6.5) for this value of \( E \) is \( \phi^+ = \frac{e^{(x^2 - 3x + 3)}}{x} \). The Darboux transformation of \( u \) with this solution reads:

\[
DT(\phi^+)u = u - 2(\log \phi^+)_{xx} = \frac{6(x - 1)(x^3 - 3x^2 + 3x - 3)}{x(x^3 - 3x^2 + 3x - 3)} = \widetilde{u}.
\]

Then this transformed potential is a solution of s-KdV equation and the spectral curve associated to this potential is still \( \Gamma_2 : \mu^2 - E^3 = 0 \).

We sum up this example in the following diagram:

\[
(u, \Gamma_2) \xrightarrow{DT(\phi_1)} (\tilde{u}_1, \Gamma_1) \quad \text{therefore, } \phi_1 \text{ is a solution for } P_0, \\
(u, \Gamma_2) \xrightarrow{DT(\phi_2)} (\tilde{u}_2, \Gamma_1) \quad \text{therefore, } \phi_2 \text{ is a solution for } P_0, \\
(u, \Gamma_2) \xrightarrow{DT(\phi^+)} (\tilde{u}, \Gamma_2) \quad \text{therefore, } \phi^+ \text{ is a solution for a regular point,}
\]

**Remark 6.13.** The importance of Theorem 6.11 lies in the fact that we need to introduce the homogenized Green’s function to state it. This new function is the essential tool that allows us to include in our study the point of infinity \( P_\infty \) of the affine curve \( \Gamma_n \). As far as we know, this is a new approach to the understanding of the spectral curve under Darboux transformations.

Similar problems to our result 6.11 were treated by several authors, see [11, Thm 5] and [13, Thm G.2]. In [11], F. Ehlers and H. Knörrer studied the action of the Darboux transformations on the spectral curves by means of the eigenfunctions of the centralizer of the Schrödinger operator.
6.3. Spectral curves and KdV hierarchy in $1+1$ dimensions

In this section we will show how the points of the spectral curves in the stationary setting are related with the solutions of the Schrödinger operator with rational potential in the $1+1$ KdV hierarchy.

Recall that the rational soliton $u_{r,n}$ restricted to $t = 0$ is the well known $n$-soliton $u^{(0)}_n(x) = n(n+1) x^{-2}$. Let $\Gamma_n$ be its affine spectral curve. This complex plane curve has a defining equation

$$p_n(E, \mu) = \mu^2 - E^{2n+1}.$$  

Our goal was to obtain the algebraic structure of a fundamental matrix of the Schrödinger operator $-\partial_x^2 + u_{r,0} - E$ by means of the system (4.3). For this purpose we needed to use a parametric representation of the spectral curve $\Gamma_n$. Observe that $\Gamma_n$ is a rational singular plane curve, nevertheless we can have a global parametrization in the sense given in [3]. In fact, we have taken the parametrization:

$$\chi(\lambda) = (-\lambda^2, i\lambda^{2n+1})$$

and then $E = -\lambda^2$ as was taken since Section 4. Observe that the unique affine singular point of the spectral curve is reached for $\lambda = 0$. Hence, whenever $\lambda \neq 0$ we obtain regular points on $\Gamma_n$ and we can obtain the desired description of the fundamental matrix $\mathcal{B}^{(j)}_{n,j}$ as is given in Theorem 4.6. On the other hand, at the singular point $\chi(0) = (0,0)$ the fundamental matrix for the system (4.3) must be obtained in a specific way, see Theorem 4.1.

The fundamental solutions $\phi_{1,r,n}(x,t)$, $\phi_{2,r,n}(x,t)$ obtained in Theorem 4.1 were used as source to perform Darboux transformations. In particular, for $t = 0$, we get the functions:

$$\phi^{(0)}_{1,r}(x) = \phi_{1,r,n}(x, t = 0), \quad \phi^{(0)}_{2,r}(x) = \phi_{2,r,n}(x, t = 0)$$

and the corresponding potentials are transformed as is suggested in the following diagram:

$$\begin{align*}
\phi^{(0)}_{1,r,n-1} \quad DT(\phi^{(0)}_{1,r,n}) & \quad \phi^{(0)}_{1,r,n} \\
\phi^{(0)}_{2,r,n-1} \quad DT(\phi^{(0)}_{2,r,n}) & \quad \phi^{(0)}_{2,r,n+1}
\end{align*}$$

This situation is a particular case of a more general one that has been obtained in Theorem 6.11. The diagram (6.32) has its time dependent counterpart (see (4.8) and (5.3)):

$$\begin{align*}
u_{r,n-1} \quad DT(\phi_{1,r,n}) & \quad \nu_{r,n} \\
\nu_{r,n} \quad DT(\phi_{2,r,n}) & \quad \nu_{r,n+1}
\end{align*}$$

The fundamental matrix $\mathcal{B}^{(j)}_{n,j}$ associated to the functions $\phi_{1,r,n}$ and $\phi_{2,r,n}$ can not be changed by the same Darboux transformations used for the potentials since there is a loss of independent solutions; in fact we have the following diagram

$$\begin{align*}
\phi_{1,r,n-1} \quad DT(\phi_{1,r,n}) & \quad \phi_{1,r,n+1} \\
\phi_{2,r,n-1} \quad DT(\phi_{2,r,n}) & \quad \phi_{2,r,n+1}
\end{align*}$$

On the other hand, whenever the point on the spectral curve is a regular point, that is $\lambda \neq 0$, we have obtained the behavior of the fundamental matrices $\mathcal{B}^{(j)}_{n,j}$, for $j = n - 1, n, n + 1$, as it is encoded in the following diagram:

$$\begin{align*}
\phi^+_{r,n-1} \quad DT(\phi^+_{r,n}) & \quad \phi^+_{r,n+1} \\
\phi^-_{r,n-1} \quad DT(\phi^-_{r,n}) & \quad \phi^-_{r,n+1}
\end{align*}$$

All these situations are reflected in the time dependent frame coming from the stationary one, as we have seen. In particular, in the lack of specialization process from $\mathcal{B}^{(j)}_{n,j}$ to $\mathcal{B}^{(j)}_{n,0}$. According to Theorem 4.8, we have that $\det \mathcal{B}^{(j)}_{n,j} = -2\lambda^{2n+1}$, whereas we have $\det \mathcal{B}^{(j)}_{n,0} = 2n + 1$. 28
2.8. For the general case.

We have computed solutions of the Schrödinger equation for $E = 0$, they are not solutions for the same point of the spectral curve. Therefore, for each singular point of this spectral curve we can only compute one fundamental solution by means of Darboux transformations.

On the other hand, the stationary functions corresponding to $\phi_{r,n}^0$ and $\phi_{\tau,n}^0$, namely, $(\phi_{r,n}^0(x, \lambda) = \phi_{\tau,n}^0(x, \lambda, t = 0)$ and $(\phi_{r,n}^0(x, \lambda) = \phi_{\tau,n}^0(x, \lambda, t = 0)$, are fundamental solutions at regular points of the spectral curve, since they are solutions of the Schrödinger equation for $E \neq 0$. In fact, one of them, say $(\phi_{r,n}^0(x, \lambda)$, is a solution for the point $(E, \mu)$, and the other one, say $(\phi_{\tau,n}^0(x, \lambda)$, is a solution for the conjugated point $(E, -\mu)$ of the spectral curve. Then, for each value of $E = -\lambda^2$, the fundamental matrix $\mathcal{B}_{n,0}$ shows the solutions at conjugated points on the corresponding spectral curve.

Next we have computed an explicit example to illustrate the relationship between spectral curves and KdV hierarchy in $1 + 1$ dimensions for rational solitons.

Example 6.15. Consider the case $r = 1$ and $n = 2$. Let $u_{1,2}(x, t_1) = \frac{6x(\lambda^2 - 6t_1)}{(x^3 + 3t_1)^2}$ be the KdV$_1$ rational soliton obtained by taking $(\tau_2, \tau_3) = (3t_1, 0)$. Then, the corresponding stationary potential is given by $u_{1,2}^{(0)}(x) = u_{1,2}(x, t = 0) = \frac{6}{x}$ (see 2.8). Its spectral curve is $\Gamma_2 : p_2(E, \mu) = \mu^2 - E^3$.

Furthermore, the stationary Schrödinger operator presents two types of solutions a priori. In fact, when $E = 0$, the solutions are

$$\phi_{1,2}^{(0)} := \phi_{1,2}(x, t = 0) = x^{-2}, \quad \phi_{2,2}^{(0)} := \phi_{2,2}(x, t = 0) = x^3,$$

where

$$\phi_{1,2}(x, t) = \frac{x}{x^3 + 3t_1}, \quad \phi_{2,2}(x, t) = \frac{x^6 + 15x^3t_1 - 45t_1^2}{x^3 + 3t_1},$$

as they were computed in 4.4. In this case, we have the following diagram:

$$\begin{align*}
\phi_{1,2}^{(0)} &\xrightarrow{\text{DT}(\phi_{1,2}^{(0)})} u_1^{(0)} = \frac{2}{x^2} \\
\phi_{2,2}^{(0)} &\xrightarrow{\text{DT}(\phi_{2,2}^{(0)})} u_2^{(0)} = \frac{6}{x^2} \\
\phi_{3,2}^{(0)} &\xrightarrow{\text{DT}(\phi_{3,2}^{(0)})} u_3^{(0)} = \frac{12}{x^2} \\
\mu^2 - E^3 = 0 &\quad \mu^2 - E^3 = 0 \\
\mu^2 - E^3 = 0 &\quad \mu^2 - E^3 = 0
\end{align*}$$

(6.37)

When energy $E \neq 0$, in 5.4 we have computed solutions

$$\phi_{1,2}^{(0)} = e^{4\lambda x + 4t_1} \frac{x^3 - 3x\lambda^2 + 3x + 3x^2t_1}{x^3 + 3t_1}, \quad \phi_{1,2}^{(0)} = e^{-4\lambda x + 4t_1} \frac{x^3 + 3x\lambda^2 + 3x + 3x^2t_1}{x^3 + 3t_1},$$

(6.38)

where we have adjusted parameters $\tau_1 = 3\lambda^2 t_1 = \tau_2$. Next, take $t_1 = 0$ to obtain

$$\phi_1^{(0)}(x, \lambda) = \phi_{1,2}^{(0)}(x, t = 0, \lambda) = e^{4\lambda x} \frac{x^3 - 3x\lambda^2 + 3x}{x^3}, \quad \phi_2^{(0)}(x, \lambda) = \phi_{2,2}^{(0)}(x, t = 0, \lambda) = e^{-4\lambda x} \frac{x^3 + 3x\lambda^2 + 3x}{x^3}.$$

(6.39)

These functions are solutions of the Schrödinger operator for the stationary potential $u_{1,2}^{(0)} = \frac{6}{x^2}$ whenever $E \neq 0$. Observe that $\phi_1^{(0)}(x, 0) = 3/x^2 = \phi_2^{(0)}(x, 0)$, and then they are no longer independent (see 4.10 for the general case).

Next, we will show how the Darboux transformations act on time dependent potentials and solutions. First recall that for any potential $u$, we have defined the Darboux transformation as

$$\text{DT}(\phi_{i,0,0})u = u - 2(\log \phi_{i,0,0})_{xx}, \quad i = 1, 2.$$

Next, we perform the Darboux transformations by means of $\phi_{1,1,2}$ and $\phi_{2,1,2}$ to our initial potential $u_{1,2}$. In these cases we have obtained

$$\begin{align*}
u_{1,1} = \frac{2}{x^2} \xrightarrow{\text{DT}(\phi_{1,1,2})} u_{1,2} &\xrightarrow{\text{DT}(\phi_{1,2,2})} \frac{6x(x^3 - 6t_1)^2}{(x^3 + 3t_1)^2} \\
u_{1,2} = \frac{6x(x^3 - 6t_1)}{(x^3 + 3t_1)^2} &\xrightarrow{\text{DT}(\phi_{2,1,2})} \frac{6x(2x^3 + 675x^3t_1^2 + 1350x^2t_1)}{(x^3 + 15x^3t_1 - 45t_1^2)^2}.
\end{align*}$$

(6.40)

Then, we must consider the Schrödinger operators

$$-\partial_x^2 + u_{1,j}(x, t_1) - E, \quad j = 1, 2, 3.$$
Their solutions $\phi^+_1$ and $\phi^-_1$ were given in Example 5.4.

It should be noted that if the energy is not zero, these solutions inherit the same behavior as their corresponding potentials when the Darboux transformations $DT(\phi_{1,1,2})$ and $DT(\phi_{2,1,2})$ act on them. Hence we obtain the following diagram

$$
\phi^+_{1,1} = \frac{e^{4x-x^4 t}(4x - 1)}{x} \quad DT(\phi_{1,1,2}) \quad \phi^+_{1,2} = \frac{e^{4x-x^4 t}(4x^3 - 3x^2 + 3x^2 t^3 + 3x^2)}{x^3 + 3t}
$$

$$
\phi^-_{1,1} = \frac{e^{-4x-x^4 t}(4x - 1)}{x} \quad DT(\phi_{1,1,2}) \quad \phi^-_{1,2} = \frac{e^{-4x-x^4 t}(4x^3 - 3x^2 + 3x^2 t^3 + 3x^2)}{x^3 + 3t}
$$

$$
\phi^+_{1,3} = \frac{e^{4x-x^4 t}Q_1(x, x, t_1)}{x^6 + 15x^3 t_1 - 45t^2_1}
$$

where

$$
Q_1(x, x, t_1) = x^3 x^6 - 6x^6 + 15x^4 - 15x^3 + 15x^3 t^3 - 45x^2 t^3 + 45x t^3 - 45 t^3 - 45t_1,
$$

$$
Q_2(x, x, t_1) = x^3 x^6 + 6x^6 + 15x^4 + 15x^3 + 15x^3 t^3 + 45x^2 t^3 + 45x t^3 - 45t^3 + 45t_1.
$$

The zero energy case is essentially different from the point of view of the Darboux transformations. We only can partially obtain the previous diagram,

$$
\phi_{1,1,2} = \frac{x}{x^3 + 3t_1} \quad DT(\phi_{1,1,2}) \quad \phi_{1,1,3} = \frac{x^3 + 3t_1}{x^6 + 15x^3 t_1 - 45t^2_1}
$$

$$
\phi_{2,1,2} = \frac{x^3 + 3t_1}{x} \quad DT(\phi_{1,1,2}) \quad \phi_{2,1,3} = \frac{x^6 + 15x^3 t_1 - 45t^2_1}{x^3 + 3t_1}
$$

To compute fundamental matrices associated to $u_{1,1}$ and $u_{1,3}$ we have to use Theorem 4.1 (see Example 4.4).

7. Differential Galois groups

In this section we study the Picard-Vessiot extensions of the differential systems (4.4) and (4.15), obtained for energy levels $E = 0$ and $E \neq 0$ respectively. We denote the base differential field by $K_r = C(x, t_r)$ with constants field $C$.

We point out that the behavior that they present depend strongly on the affine point $P = (E, \mu)$ of the corresponding spectral curve. They present a similar behavior when the point $P = (E, \mu)$ is a regular point of $\Gamma_n$.

A fundamental matrix for $E = 0$ can be also computed. However, it is not obtained by a specialization process from the fundamental matrix obtained for a regular point.

We obtain the Picard-Vessiot extensions given by $B^{(r)}_{n,0}$ and $B^{(r)}_{n,1}$ and compute their corresponding differential Galois group, say $\mathcal{G}_{n,0}^{(r)}$ and $\mathcal{G}_{n,1}^{(r)}$ respectively.

7.1. Case $E = 0$

For this case we have the fundamental matrix

$$
\mathcal{G}_{n,0}^{(r)} = \begin{pmatrix}
\phi_{1,r,n} & \phi_{2,r,n} \\
\phi_{1,r,n,x} & \phi_{2,r,n,x}
\end{pmatrix},
$$

where $\phi_{1,r,n}, \phi_{1,r,n,x}, \phi_{2,r,n}, \phi_{2,r,n,x}$ are rational functions in $x, t$, hence they are in $K_r$. So, the Picard-Vessiot field is again $K_r$. Thus, the differential Galois group is the trivial group, $\mathcal{G}_{n,0}^{(r)} = \{\text{id}_2\}$.

7.2. Case $E \neq 0$

In this case, we compute the differential extension given for each value of $\lambda \neq 0$. For this, we fix a value of $\lambda$ different from zero, $\lambda = \lambda_0$, then the point $P = (E_0, \mu_0)$ is a regular point of $\Gamma_n$, that is $E_0 \neq 0$. The fundamental matrix is

$$
\mathcal{G}_{n,0}^{(r)} = \begin{pmatrix}
\phi_{r,n}(\lambda_0) & \phi_{r,n}(-\lambda_0) \\
\phi_{r,n,x}(\lambda_0) & \phi_{r,n,x}(-\lambda_0)
\end{pmatrix},
$$

30
for \( \phi^+_r(\lambda_0), \phi^+_r(\lambda_0), \phi^-_r(\lambda_0) \) and \( \phi^-_r(\lambda_0) \in K_r(\eta_r), \) with \( \eta_r = e^{\lambda_0 + (-1)^r \lambda^{2r+1}}. \) Then, the Picard–Vessiot field is \( L_r = K_r(\eta_r). \)

To compute the differential Galois group \( G_{n,\lambda}^{(r)} \) in this case, we just have to compute the action of \( G_{n,\lambda}^{(r)} \) on \( \eta_r. \) For this, let \( \sigma \) in \( G_{n,\lambda}^{(r)} \) be an automorphism of the differential Galois group, then

\[
\begin{align*}
\frac{\sigma(\lambda_0 \eta_r)}{\eta_r} &= \frac{\lambda_0 \sigma(\eta_r)}{\eta_r} - \lambda_0 \sigma(\eta_r) = 0, \\
\frac{\sigma(\eta_r)}{\eta_r} &= \frac{\sigma((-1)^r \lambda_0^{2r+1} \eta_r)}{\eta_r} - (-1)^r \lambda_0^{2r+1} \sigma(\eta_r) \\
&= \frac{(-1)^r \lambda_0^{2r+1} \sigma(\eta_r)}{\eta_r} - (-1)^r \lambda_0^{2r+1} \sigma(\eta_r) = 0.
\end{align*}
\]

Therefore \( \frac{\sigma(\lambda_0 \eta_r)}{\eta_r} \) is a constant in \( K_r. \) Hence \( \sigma(\eta_r) = c \cdot \eta_r \) for some \( c \in C. \) As a consequence we get that, for each \( \lambda_0 \) and every \( n, \) the differential Galois group is isomorphic to the multiplicative group, say \( G_{n,\lambda}^{(r)} = G_m = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} : c \in C \right\}. \)

**Remark 7.1.** Since the Galois groups \( G_{n,\lambda}^{(r)} \) are obtained for a particular value of \( \lambda \) by specialization process, they do not depend on \( \lambda. \) For a spectral study of the Picard–Vessiot extensions see [17].

7.3. **Global behavior of the differential Galois groups**

Let us consider the family of linear algebraic groups \( \left\{ G_{n,\lambda}^{(r)} \right\}_{\lambda \in C}. \) Then for each point in \( \Gamma_n \) we have found a linear algebraic group. As a result of our constructions we have a sheaf structure of groups on the regular points of \( \Gamma_n \)

\[
\Gamma_n \setminus \text{Sing}(\Gamma_n) \ni (-\lambda^2, i\lambda^{2n+1}) \longrightarrow G_{n,\lambda}^{(r)}
\]

For each \( \lambda \in C, \) the situation is encoded in the following diagram

\[
\begin{array}{ccc}
G_{n-1} & \longrightarrow & G_n \\
\downarrow & \searrow & \downarrow \\
\Gamma_{n-1} & \text{Blowing-up} & \Gamma_n \\
\downarrow & \searrow & \downarrow \\
L_{n-1} & \text{DT}(\phi^{(0)}_{n-1}) & L_n \\
\downarrow & \searrow & \downarrow \\
\Gamma_{n+1} & \text{Blowing-up} & \Gamma_{n+1} \\
\downarrow & \searrow & \downarrow \\
L_{n+1} & \text{DT}(\phi^{(0)}_{n+1}) & L_{n+1}
\end{array}
\]

We observe the invariance of the Galois groups with respect to:

- **Time** (ie, it is invariant by the flow of the KdV equation).
- **Generic values of the spectral parameter**, ie, moving along the regular points of the spectral curve.
- **Darboux transformations**.

Although this invariant behaviour of the Galois group is proved here for the rational solutions of Adler-Moser type, we conjectured that it is also true for arbitrary algebro geometric solutions of the KdV, ie, for solutions associated to spectral curves different of \( \mu^2 - E^{2n+1} = 0. \)
Appendix A.

We establish a series of easy corollaries of the result 6.2. They are necessary in the Subsection 6.2. We use the same notations as in Subsection 6.1.

Corollary Appendix A.1. We have

\[ F_n^0F_{n,x} - F_n^0F_n = (E - E_0)P_n, \]

where \( P_n \) is a polynomial in \( E \) of degree at most \( n - 1 \). In particular for \( E_0 = 0 \) we obtain:

\[ f_n F_{n,x} - f_n F_n = E P_n. \]

Proof. Since \( F_n = \sum_{i=0}^n f_{n-i} E^i \) and \( F_n^0 = \sum_{i=0}^n f_{n-i} E^i_0 \), we have that

\[ F_n^0 F_{n,x} - F_n^0 F_n = \sum_{i,j=0}^n f_{n-i} f_{n-j} E^i_0 E^j - \sum_{i,j=0}^n f_{n-i} f_{n-j} E^i E^j = \sum_{i,j=0}^n (E^i_0 E^j - E^i E^j) f_{n-i} f_{n-j}. \]  

(A.1)

We factor the term \( E^i_0 E^j - E^i E^j \):

\[ E^i_0 E^j - E^i E^j = (E - E_0)(E E_0)^{\min(i,j)}(-1)^{\text{sign}(i,j)} \left( \sum_{k=0}^{\min(i,j)-1} E^k E^j \sum_{k=0}^{4} \right), \]

and replace it in (A.1). We get

\[ F_n^0 F_{n,x} - F_n^0 F_n = (E - E_0) \sum_{i,j=0}^n (E E_0)^{\min(i,j)}(-1)^{\text{sign}(i,j)} \left( \sum_{k=0}^{\min(i,j)-1} E^k E^j \sum_{k=0}^{4} \right) f_{n-i} f_{n-j} = (E - E_0) P_n, \]  

(A.2)

for \( P_n \) a polynomial in \( E \) of degree at most \( n - 1 \), as it is stated.

Corollary Appendix A.2. We have

\[ \mu^2 F_n^0 - \mu_n^0 F_n^2 = (E - E_0) \left( \frac{F_n F_{n,x}^0}{2} + F_n^2(F_n^0)^2 \right) = \frac{P_n(F_n F_{n,x}^0 + F_n^0 F_n)}{4}, \]

where \( P_n \) is the polynomial obtained in Corollary Appendix A.1. In particular for \( E_0 = 0 \) we obtain

\[ \mu^2 f_n^0 - \mu_n^0 f_n^2 = E \left( \frac{F_n f_n^0 F_{n,x}^0}{2} + f_n^2 f_n^0 \right) = \frac{P_n(F_n f_n^0 + F_n^0 f_n)}{4}. \]

Proof. By (2.21) we have

\[ \mu^2 = R_{2n+1} = \frac{F_n F_{n,x}}{2} - (u - E) F_n^2 = \frac{F_n^2}{4}, \quad \mu_n^0 = R_{2n+1}(E_0) = \frac{F_n^0 F_{n,x}^0}{2} - (u - E_0)(F_n^0)^2 = \frac{(F_n^0)^2}{4}. \]

Hence,

\[ \mu^2(F_n^0)^2 - \mu_n^0 F_n^2 = \frac{F_n^0 F_{n,x}^0}{2} - (u - E) F_n^2 = \frac{F_n^0}{4}. \]

As \( F_n^0 F_{n,x} - F_n^0 F_n = (F_n^0 F_{n,x} - F_n^0 F_n) \), by Corollary Appendix A.1 we obtain

\[ \mu^2(F_n^0)^2 - \mu_n^0 F_n^2 = (E - E_0) \left( \frac{F_n F_{n,x}^0}{2} + F_n^2(F_n^0)^2 \right) = \frac{P_n(F_n F_{n,x}^0 + F_n^0 F_n)}{4}. \]

\[ \square \]
Now, let \((E_0, \mu_0)\) be a regular point of \(\Gamma_n\) and \(\mu_0 = 0\). In this case, we have that \(R_{2n+1}^0 = R_{2n+1}(E_0) = 0\) and \(\partial E(R_{2n+1})(E_0) \neq 0\), thus,
\[
\mu^2 = R_{2n+1}(E) = (E - E_0)M_{2n}, \tag{A.3}
\]
where \(M_{2n}(E)\) is a polynomial in \(E\) of degree \(2n\) such that \(M_{2n}(E_0) \neq 0\).

**Corollary Appendix A.3.** Let \((E_0, \mu_0)\) be a regular point of \(\Gamma_n\) and \(\mu_0 = 0\). We have that
\[
\frac{M_{2n}}{F_n} + \frac{(E - E_0)P_n^2}{4F_n(F_n^0)^2} = \frac{E - E_0}{2(E - E_0)} - \frac{(u - E)F_n^2}{4(E - E_0)} = \frac{F_{n,xx}^2}{2(E - E_0)} - \frac{(E - E_0)F_n^2}{4(E - E_0)} - \frac{(u - E)(F_n^0)^2F_n}{(E - E_0)^2} + \frac{(F_n^0)^2F_{n,xx} + (F_n^0)^2F_n}{(E - E_0)^2}.
\]

We replace these expressions in the formula and we get:
\[
\frac{M_{2n}}{F_n} + \frac{(E - E_0)P_n^2}{4F_n(F_n^0)^2} = \frac{2(F_n^0)^2F_{n,xx} - 4(u - E)(F_n^0)^2F_n + (F_n^0)^2F_n - 2F_n^0F_{n,xx}F_n}{4E(E - E_0)(F_n^0)^2}.
\]
The numerator of this function is a polynomial in \(E\) of degree \(n + 1\) and has a root in \(E = E_0\) as can be easily verified replacing \(E\) by \(E_0\):
\[
2(F_n^0)^2F_{n,xx} - 4(u - E)(F_n^0)^2F_n + (F_n^0)^2F_n - 2F_n^0F_{n,xx}F_n = 4F_n^0P_n^2 = 0.
\]
So, we get that
\[
2(F_n^0)^2F_{n,xx} - 4(u - E)(F_n^0)^2F_n + (F_n^0)^2F_n - 2F_n^0F_{n,xx}F_n = (E - E_0)Q_n,
\]
where \(Q_n\) denotes a polynomial in \(E\) of degree \(n\). Hence
\[
\frac{M_{2n}}{F_n} + \frac{(E - E_0)P_n^2}{4F_n(F_n^0)^2} = \frac{Q_n}{4(F_n^0)^2}
\]
and then the result follows.

Next, let \((E_0, \mu_0)\) be a singular point of \(\Gamma_n\). In this case, \(\mu_0 = 0\), \(R_{2n+1}^0 = R_{2n+1}(E_0) = 0\) and \(\partial E(R_{2n+1})(E_0) = 0\), thus,
\[
\mu^2 = R_{2n+1}(E) = (E - E_0)^2Z_{2n-1}, \tag{A.4}
\]
where \(Z_{2n-1}(E)\) is a polynomial in \(E\) of degree \(2n - 1\) such that \(Z_{2n-1}(E_0) \neq 0\).

**Corollary Appendix A.4.** Let \((E_0, \mu_0)\) be a singular point of \(\Gamma_n\). We have that
\[
\frac{Z_{2n-1}}{F_n} + \frac{P_n^2}{4F_n(F_n^0)^2}
\]
is a polynomial in \(E\) of degree \(n - 1\), with \(P_n\) the polynomial obtained in Corollary Appendix A.1 and \(Z_{2n-1}\) the polynomial defined in (A.4).

**Proof.** It follows by an analogous computation to that of Corollary Appendix A.3.

**Acknowledgments:** We kindly thank all members of the Integrability Madrid Seminar for many fruitful discussions: P. Acosta-Humánez, D. Blázquez, J.A. Capitán, R. Hernández Heredero, A. Pérez-Raposo, J. Rojo Montijano and S. Rueda.
References

[1] M. Adler and J. Moser. On a class of polynomials connected with the Korteweg-de Vries equation. *Commun. math. Phys.*, 61:1–30, 1978.
[2] H. Airault, H. P. McKean, and J. Moser. Rational and elliptic solutions of the Korteweg-de Vries equation and a related many-body problem. *Communications on Pure and Applied Mathematics*, 30(1):95–148, 1977.
[3] A. F. Beardon and T. W. Ng. Parametrizations of algebraic curves. In *Annales Academiae Scientiarum Fennicae Mathematica*, volume 31, page 541. ACADEMIA SCIENTIARUM FENNICA, 2006.
[4] Yu. V. Brezhnev. What does integrability of finite-gap or soliton potentials mean? *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 366(1867):923–945, 2008.
[5] Yu. V. Brezhnev. Spectral/quadrature duality: Picard–Vessiot theory and finite-gap potentials. Algebraic aspects of Darboux transformations, quantum integrable systems and supersymmetric quantum mechanics. *Amer. Math. Soc.*, Providence, RI, 563:1–31, 2012.
[6] Yu. V. Brezhnev. Elliptic solitons, fuchsian equations, and algorithms. *St. Petersburg Math. J.*, 24:555–574, 2013.
[7] J. L. Burchnall and T. W. Chaundy. Commutative ordinary differential operators. *Proceedings of the Royal Society of London, Series A*, 118(780):557–583, 1928.
[8] M. M. Crum. Associated Sturm–Liouville systems. *Quart. J. Math. Oxford*, 6:121–127, 1955.
[9] G. Darboux. Sur une proposition relative aux équations linéaires. *Comptes Rendus Acad. Sci.*, 94:1456–1459, 1882.
[10] G. Darboux. *Théorie des Surfaces, II*. Gauthier-Villars, Paris, 1889.
[11] F. Ehlers and H. Knörrer. An algebro-geometric interpretation of the Bäcklund-transformation for the Korteweg–de Vries equation. *Comment. Math. Helvetic*, 57:1–10, 1982.
[12] I. M. Gel’fand and L. A. Dikii. Asymptotic behaviour of the resolvent of Sturm–Liouville equations and the algebra of the Korteweg–de Vries equations. *Russian Mathematical Surveys*, 30(5):77–113, 1975.
[13] F. Gesztesy and H. Holden. *Soliton Equations and Their Algebro-Geometric Solutions, Volume 1: (1+1)-Dimensional Continuous Models*, volume 79 of *Cambridge Stud. Adv. Math*. Cambridge Univ. Press, 2003.
[14] C. Hermite. *Sur l’équation de Lamé. Oeuvres de Charles Hermite. Tome III*. Gauthier-Villars, Paris, 1912.
[15] S. Jiménez, J. J. Morales-Ruiz, R. Sánchez-Cauce, and M. A. Zurro. Differential Galois Theory and Darboux transformations for integrable systems. *Journal of Geometry and Physics*, 115:75–88, 2017.
[16] V. B. Matveev and M. A. Salle. *Darboux Transformations and Solitons*. Springer Series in Nonlinear Dynamics. Springer-Verlag, Berlin, 1991.
[17] J. J. Morales-Ruiz, S. L. Rueda, and M. A. Zurro. Algebra–geometric solitonic solutions and Differential Galois Theory. *ArXiv e-prints*: 1708.00431, 2017.
[18] P. J. Olver. *Applications of Lie groups to differential equations*, volume 107. Springer-Verlag, 1986.
[19] E. T. Whittaker and G. N. Watson. *A course of modern analysis*. Cambridge university press, 1996.