SUMMATION IDENTITIES AND TRANSFORMATIONS FOR HYPERGEOMETRIC SERIES

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Abstract. We find summation identities and transformations for the McCarthy’s $p$-adic hypergeometric series by evaluating certain Gauss sums which appear while counting points on the family $Z_\lambda : x_1^d + x_2^d = d\lambda x_1 x_2^{d-1}$ over a finite field $\mathbb{F}_p$. A. Salerno expresses the number of points over a finite field $\mathbb{F}_p$ on the family $Z_\lambda$ in terms of quotients of $p$-adic gamma function under the condition that $d|p-1$. In this paper, we first express the number of points over a finite field $\mathbb{F}_p$ on the family $Z_\lambda$ in terms of McCarthy’s $p$-adic hypergeometric series for any odd prime $p$ not dividing $d(d-1)$, and then deduce two summation identities for the $p$-adic hypergeometric series. We also find certain transformations and special values of the $p$-adic hypergeometric series. We finally find a summation identity for the Greene’s finite field hypergeometric series.

1. Introduction and statement of results

It is a well known result that the number of points over a finite field on the Legendre family of elliptic curves can be written in terms of a hypergeometric function modulo $p$. In [17], A. Salerno extends this result to a family of monomial deformations of a diagonal hypersurface. She finds explicit relationships between the number of points and generalized hypergeometric functions as well as their finite field analogues. Let $X_\lambda$ denote the family of monomial deformations of diagonal hypersurfaces

$$X_\lambda : x_1^d + x_2^d + \cdots + x_n^d = d\lambda x_1^{h_1} x_2^{h_2} \cdots x_n^{h_n},$$

where $\sum h_i = d$ and $\gcd(d, h_1, \ldots, h_n) = 1$. For $\lambda \in \mathbb{Z}$, let $N_{\mathbb{F}_q}(X_\lambda)$ denote the number of points on $X_\lambda$ in $\mathbb{F}_q^{n-1}$, where $\mathbb{F}_q$ is the finite field of $q = p^e$-elements. Under the condition that $dh_1 \cdots h_n (q-1)$, A. Salerno [17] Thm. 4.1] expresses $N_{\mathbb{F}_q}(X_\lambda) - N_{\mathbb{F}_q}(X_0)$ as a sum of finite field analogues of hypergeometric functions defined by N. Katz [12]. She studies the special case Dwork family $X_4^d : x_4^d + x_2^d + \cdots + x_d^d = d\lambda x_1 x_2 \cdots x_d$ when $d = 3, 4$. In [6], H. Goodson gives an expression for the number of points on the family of Dwork K3 surfaces $X_4^4 : x_1^4 + x_2^4 + x_3^4 + x_4^4 = 4\lambda x_1 x_2 x_3 x_4$ over a finite field $\mathbb{F}_q$ in terms of Greene’s finite field hypergeometric functions under the condition that $q \equiv 1 (\text{mod } 4)$. She further gives an expression for the number of points on the family $X_4^4$ in terms of
McCarthy’s $p$-adic hypergeometric series $\phi_n, G_n[\cdots]$ (defined in Section 2) under the condition that $p \equiv 1 \pmod{4}$. Recently, the authors with H. Rahman [1] express the number of $F_p$-points on $X^d_f$ in terms of McCarthy’s $p$-adic hypergeometric series when $d$ is an odd prime such that $p \nmid d$ and $q \equiv 1 \pmod{d}$, which gives a solution to a conjecture of H. Goodson [6].

The aim of this paper is to find summation identities and transformations for the McCarthy’s $p$-adic hypergeometric series and Greene’s finite field hypergeometric series. In [2], the authors with D. McCarthy find eight summation identities for the number of $F_p$-points on a conjecture of H. Goodson [6].

Let $\lambda$ be an odd prime such that $p \nmid d$. A. Salerno expresses the number of points on a finite field $F_p$ on the family $Z_\lambda$ in terms of quotients of $p$-adic gamma function (for example, see [17, Lemma 5.4]). In the following theorem, we express the number of points over a finite field $F_p$ on the family $Z_\lambda$ in terms of McCarthy’s $p$-adic hypergeometric series for any odd prime $p$ not dividing $d(d-1)$.

**Theorem 1.1.** Let $p$ be an odd prime such that $p \nmid d(d-1)$. If $\lambda \neq 0$, then the number of $F_p$-points $N_{\phi_p}(Z_\lambda)$ on the 0-dimensional variety $Z^d_\lambda : x^d_1 + x^d_2 = d\lambda x_1 x_2^{d-1}$ is given by

$$N_{\phi_p}(Z_\lambda) = 1 + d-1G_{d-1}\left[\frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d} \right] \left[\lambda^d(d-1)^{d-1}\right].$$

We evaluate certain Gauss sums which appear while counting points on $Z_\lambda$ over $F_p$ and deduce the following two summation identities. Let $\phi$ denote the quadratic character on $F_p$.

**Theorem 1.2.** Let $d \geq 3$ be odd and $p$ an odd prime such that $p \nmid d(d-1)$. For $x \in F_p^\times$ we have

$$\sum_{t \in F_p^\times} \phi(t(t-1)) \times \left[d-1G_{d-1}\left[\frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d} \right] \left[\frac{\lambda^d(d-1)^{d-1}}{d} \right] \right] = -1 - p \cdot d-1G_{d-1}\left[\frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d} \right] \left[\frac{\lambda^d(d-1)^{d-1}}{d} \right]$$

**Theorem 1.3.** Let $d > 2$ be even and $p$ an odd prime such that $p \nmid d(d-1)$. For $x \in F_p^\times$ we have

$$\sum_{t \in F_p^\times} \phi(1-t)_{d-2}G_{d-2}\left[\frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d} \right] \left[\frac{\lambda^d(d-1)^{d-1}}{d} \right] = -d-1G_{d-1}\left[\frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d} \right] \left[\frac{\lambda^d(d-1)^{d-1}}{d} \right].$$

Using the summation identities, we obtain the following two point count formulas for $Z_\lambda$. 

Corollary 1.4. Let $d > 2$ be even and $p$ an odd prime such that $p \mid d(d-1)$. Then

$$N_{F_p}(Z_k) = 1 - \sum_{t \in \mathbb{F}_p} \phi(1-t) \times d-2G_{d-2} \left[ \frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d}, \frac{d+1}{d}, \frac{d+2}{d}, \ldots, \frac{d-2}{d}, \frac{d-1}{d} \bigg| \alpha \right],$$

where $\alpha = \lambda^d(d-1)^{d-1}$.

Corollary 1.5. Let $d \geq 3$ be odd and $p$ an odd prime such that $p \mid d(d-1)$. Then

$$pN_{F_p}(Z_k) = p - 1 - \sum_{t \in \mathbb{F}_p} \phi(t(t-1)) \times d-1G_{d-1} \left[ \frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d}, \frac{d+1}{d}, \frac{d+2}{d}, \ldots, \frac{d-3}{d}, \frac{d-2}{d}, \frac{d-1}{d} \bigg| \alpha \right],$$

where $\alpha = \lambda^d(d-1)^{d-1}$. Hence,

$$\sum_{t \in \mathbb{F}_p} \phi(t(t-1)) \times d-1G_{d-1} \left[ \frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d}, \frac{d+1}{d}, \frac{d+2}{d}, \ldots, \frac{d-3}{d}, \frac{d-2}{d}, \frac{d-1}{d} \bigg| \alpha \right] \equiv p - 1 \pmod{p}.$$ 

In the following example, we take some values of $d$ to show how our results are applied to particular cases.

Example 1.6. We put $d = 5$ and $d = 4$ in Theorem 1.2 and Theorem 1.3 respectively. Then, for $x \in \mathbb{F}_p^\times$, we have the following summation identities.

$$\sum_{t \in \mathbb{F}_p} \phi(t(t-1))_4G_4 \left[ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{x} \right] = -1 - p \cdot G_4 \left[ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{x} \right],$$

$$\sum_{t \in \mathbb{F}_p} \phi(1-t)_2G_2 \left[ \frac{1}{3}, \frac{2}{3}, \frac{3}{3} \right] \equiv -3G_3 \left[ \frac{1}{3}, \frac{2}{3}, \frac{3}{3} \right].$$

The first identity is valid for $p = 3$ and all $p > 5$; whereas the second identity is valid for all prime $p > 3$.

In [8], J. Fussell and D. McCarthy establish certain transformations and identities for the $G$-function, and use them to prove a supercongruence conjecture of Rodriguez-Villegas between a truncated $_4F_3$ classical hypergeometric series and the $p$-th Fourier coefficients of a weight four modular form, modulo $p^3$. Here, we prove that the $G$-function satisfies the following transformations.
Theorem 1.7. Let $d \geq 2$ and $p$ an odd prime such that $p \not| d(d-1)$. For $\lambda \in \mathbb{F}_p^\times$ we have

$$d-1G_{d-1} \left[ \frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d}, 0, \frac{1}{d-1}, \ldots, \frac{d-2}{d-1} \mid \lambda \right] = \phi(-\lambda(d-1))$$

$$\chi_{d-1}G_{d-1} \left[ \frac{1}{2(d-1)}, \frac{2}{2(d-1)}, \ldots, \frac{d-1}{2(d-1)}, \frac{2}{d-1}, \frac{4}{d+1}, \ldots, \frac{2d-3}{2d-1} \mid \lambda \right]$$

For example, if we put $d = 6$, then for all prime $p > 5$, we have

$$5G_5 \left[ \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, 0 \mid \lambda \right] = \phi(-5\lambda)5G_5 \left[ \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{5}{10}, \frac{7}{10}, \frac{9}{10} \mid \frac{1}{\lambda} \right].$$

Theorem 1.8. For $p > 7$ and $p \neq 23$ we have

$$4G_4 \left[ \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10} \mid -\frac{5}{4} \right] = \phi(-1) + \phi(3) + \phi(-1) 2G_2 \left[ \frac{1}{5}, \frac{2}{5}, \frac{4}{27} \right];$$

$$4G_4 \left[ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \mid -\frac{4}{5} \right] = 1 + \phi(-3) + 2G_2 \left[ \frac{1}{5}, \frac{1}{5}, \frac{27}{4} \right];$$

From Theorem 1.2 and Theorem 1.8 we have the following summation identities.

Corollary 1.9. For $p > 7$ and $p \neq 23$ we have

$$\sum_{t \in \mathbb{F}_p} \phi(t(t-1)) 2G_2 \left[ \frac{1}{3}, \frac{2}{3}, 0, \frac{4t}{27} \right] = p - 1 + p\phi(-3) - p\phi(-1)4G_4 \left[ \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{3}{10}, \frac{9}{10} \mid -\frac{5}{4} \right];$$

$$\sum_{t \in \mathbb{F}_p} \phi(t(t-1))4G_4 \left[ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \mid -\frac{4t}{5} \right] = -1 - p + p\phi(-3) - p \cdot 2G_2 \left[ \frac{1}{6}, \frac{1}{6}, \frac{27}{4} \right];$$

Finally, we find a summation identity for the Greene’s finite field hypergeometric series. We first recall some definitions to state our results. Let $q = p^e$ be a power of an odd prime $p$ and $\mathbb{F}_q^\times$ the finite field of $q$ elements. Let $\mathbb{F}_q^\times$ be the group of all multiplicative characters $\chi : \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times$. We extend the domain of each $\chi \in \mathbb{F}_q^\times$
to $\mathbb{F}_q$ by setting $\chi(0) := 0$ including the trivial character $\varepsilon$. If $A$ and $B$ are two characters on $\mathbb{F}_q$, then $\binom{A}{B}$ is defined by

\[
\binom{A}{B} := \frac{B(-1)}{q} \sum_{x \in \mathbb{F}_q} A(x) \overline{B}(1 - x),
\]

where $\overline{B}$ is the character inverse of $B$. In [9], J. Greene introduced the notion of hypergeometric series over finite fields which are also known as Gaussian hypergeometric series. For any positive integer $n$ and characters $A_0, A_1, \ldots, A_n$ and $B_1, B_2, \ldots, B_n \in \mathbb{F}_q^\times$, the Gaussian hypergeometric series $n+1F_n$ is defined to be

\[
n+1F_n \left( \begin{array}{c} A_0, \ A_1, \ldots, \ A_n \\ B_1, \ldots, \ B_n \end{array} \mid x \right) := \frac{q}{q - 1} \sum_{\chi \subseteq \mathbb{F}_q^\times} \binom{A_0\chi}{B_1\chi} \cdots \binom{A_n\chi}{B_n\chi} \chi(x),
\]

where the sum is over all multiplicative characters $\chi$ on $\mathbb{F}_q$.

The motivation for deriving summation identities for Greene’s hypergeometric series is the following summation identity due to Greene [9, Theorem 3.13]. Let $A_0, A_1, \ldots, A_n, B_1, \ldots, B_n$ be multiplicative characters on $\mathbb{F}_q$ and let $x \in \mathbb{F}_q$. Greene proved that

\[
(1.1) \quad n+1F_n \left( \begin{array}{c} A_0, \ A_1, \ldots, \ A_n \\ B_1, \ldots, \ B_n \end{array} \mid x \right) = \frac{A_nB_n(-1)}{q} \sum_{y \in \mathbb{F}_q} nF_{n-1} \left( \begin{array}{c} A_0, \ A_1, \ldots, A_{n-1} \\ B_1, \ldots, B_{n-1} \end{array} \mid xy \right) A_n(y) \overline{A_nB_n}(1 - y). 
\]

We first express the number of $\mathbb{F}_q$-points on $Z\lambda$ in terms of Greene’s hypergeometric series in the following result.

**Theorem 1.10.** Let $p$ be an odd prime and $q = p^e$ for some $e > 0$. Let $d \geq 3$ be odd such that $q \equiv 1 \pmod{d(d - 1)}$. For $\lambda \neq 0$, the number of $\mathbb{F}_q$-points $N_{\mathbb{F}_q}(Z\lambda)$ on the 0-dimensional variety $Z_d^d : x_1^d + x_2^d = d\lambda x_1 x_2^{d-1}$ is given by

\[
q \cdot N_{\mathbb{F}_q}(Z\lambda) = q - 1 + q^{\frac{d-1}{2}} \sum_{t \in \mathbb{F}_q^\times} \phi(1 - t) \times d-1F_{d-2} \left( \begin{array}{c} \chi \frac{d-1}{2}, \ldots, \chi \frac{d-1}{2} - 1, \chi \frac{d-1}{2} - 1 + 1, \chi \frac{d-1}{2} - 1 + 2, \ldots, \chi d-1 \\ \psi, \ldots, \psi \frac{d-1}{2} - 1, \varepsilon, \psi \frac{d-1}{2} - 1, \ldots, \psi d-2 \end{array} \mid \frac{t}{\alpha} \right),
\]

where $\chi$ and $\psi$ are characters of order $d$ and $d - 1$ respectively, and $\alpha = \lambda^d(d-1)^{d-1}$.

Using the above point-count formula, we prove the following summation identity. Unlike to (1.1), our summation identity contains characters of specific orders. It would be interesting to know if the identity could be derived from (1.1).

**Theorem 1.11.** Let $p$ be an odd prime and $q = p^e$ for some $e > 0$. Let $d \geq 3$ be odd such that $q \equiv 1 \pmod{d(d - 1)}$. For $\lambda \in \mathbb{F}_q^\times$ we have

\[
\sum_{t \in \mathbb{F}_q^\times} \phi(1 - t)_{d-1}F_{d-2} \left( \begin{array}{c} \chi \frac{d-1}{2}, \chi \frac{d-1}{2} - 1, \chi \frac{d-1}{2} - 1 + 1, \chi \frac{d-1}{2} - 1 + 2, \ldots, \chi d-1 \\ \psi, \ldots, \psi \frac{d-1}{2} - 1, \varepsilon, \psi \frac{d-1}{2} - 1, \ldots, \psi d-2 \end{array} \mid \lambda t \right) = 1 - \phi(-\lambda) \frac{q^{\frac{d-1}{2}}}{q^{\frac{d-1}{2}}} + q \phi(-1)_{d-1}F_{d-1} \left( \begin{array}{c} \phi, \chi, \ldots, \chi \frac{d-1}{2}, \chi \frac{d-1}{2} - 1, \chi \frac{d-1}{2} - 1 + 1, \ldots, \chi d-1 \\ \psi, \ldots, \psi \frac{d-1}{2}, \psi \frac{d-1}{2} - 1, \ldots, \psi d-2 \end{array} \mid \lambda \right),
\]

where $\chi$ and $\psi$ are characters of order $d$ and $d - 1$ respectively.
If we put $d = 3$ in Theorem 1.11, then, for $\lambda \neq 0$, we have

$$\sum_{t \in \mathbb{F}_q} \phi(1 - t) \phi_1 \left( \chi_3^3, \chi_3^2 | t \right) = \frac{1 - \phi(-\lambda)}{q} + q \phi(-1) \phi_2 \left( \phi, \chi_3^3, \chi_3^2 | \lambda \right),$$

where $\chi_3$ is a character of order 3. In particular, if we take $\lambda = -1$, then we have

$$\sum_{t \in \mathbb{F}_q} \phi(1 + t) \phi_1 \left( \chi_3^3, \chi_3^2 | t \right) = q \phi(-1) \phi_2 \left( \phi, \chi_3^3, \chi_3^2 | -1 \right).$$

If we apply (1.1), then we have

$$\sum_{t \in \mathbb{F}_q} \chi_3(t) \chi_3 \phi(1 + t) \phi_1 \left( \chi_3^3, \chi_3^2 | t \right) = q \chi_3 \phi(-1) \phi_2 \left( \phi, \chi_3^3, \chi_3^2 | -1 \right).$$

Remark 1.12. When $d$ is even, we are unable to simplify certain Gauss sums which appear while counting points on the family $Z_\lambda$. It would be interesting to know if similar results like Theorem 1.11 and Theorem 1.10 exist when $d$ is even.

2. Preliminaries

2.1. Gauss sums and Davenport-Hasse relation. Recall that $\widehat{\mathbb{F}_q}$ denotes the group of all multiplicative characters on $\mathbb{F}_q$. The orthogonality relations for multiplicative characters are listed in the following lemma.

Lemma 2.1. ([11, Chapter 8]). We have

1. $\sum_{x \in \mathbb{F}_q} \chi(x) = \begin{cases} q - 1 & \text{if } \chi = \varepsilon; \\ 0 & \text{if } \chi \neq \varepsilon. \end{cases}$

2. $\sum_{\chi \in \widehat{\mathbb{F}_q}} \chi(x) = \begin{cases} q - 1 & \text{if } x = 1; \\ 0 & \text{if } x \neq 1. \end{cases}$

We now introduce some properties of Gauss sums. For further details, see [5] noting that we have adjusted results to take into account $\varepsilon(0) = 0$. Define the additive character $\theta : \mathbb{F}_q \to \mathbb{C}^\times$ by

$$\theta(\alpha) = \zeta_p^{\text{tr}(\alpha)}$$

where $\zeta_p = e^{2\pi i/p}$ and $\text{tr} : \mathbb{F}_q \to \mathbb{F}_p$ is the trace map given by

$$\text{tr}(\alpha) = \alpha + \alpha^p + \alpha^{p^2} + \cdots + \alpha^{p^{e-1}}.$$

For $\chi \in \widehat{\mathbb{F}_q}$, the Gauss sum is defined by

$$g(\chi) := \sum_{x \in \mathbb{F}_q} \chi(x) \zeta_p^{\text{tr}(x)} = \sum_{x \in \mathbb{F}_q} \chi(x) \theta(x).$$

It is easy to see that $\theta(a + b) = \theta(a)\theta(b)$ and

$$\sum_{x \in \mathbb{F}_q} \theta(x) = 0.$$

Using (2.3) one easily finds that $g(\varepsilon) = -1.$
The following lemma provides a formula for the multiplicative inverse of a Gauss sum. Let $T$ be a generator of the cyclic group $\mathbb{F}_q^\times$.

**Lemma 2.2.** ([9] Eqn. 1.12). If $k \in \mathbb{Z}$ and $T^k \not\equiv 1$, then
\[ g(T^k)g(T^{-k}) = q \cdot T^k(-1). \]

Using orthogonality, we can write $\theta$ in terms of Gauss sums as given in the following lemma.

**Lemma 2.3.** ([7] Lemma 2.2). For all $\alpha \in \mathbb{F}_q^\times$,
\[ \theta(\alpha) = \frac{1}{q-1} \sum_{m=0}^{q-2} g(T^{-m})T^m(\alpha). \]

For $\chi, \psi \in \mathbb{F}_q^\times$ we define the Jacobi sum by $J(\chi, \psi) := \sum_{t \in \mathbb{F}_q} \chi(t)\psi(1-t)$. We will use the following relationship between Gauss and Jacobi sums (for example, see [9] Eqn 1.14). For $\chi, \psi \in \mathbb{F}_q^\times$ not both trivial, we have

\[ J(\chi, \psi) = \begin{cases} \frac{\gamma(\chi)\gamma(\psi)}{\gamma(1)}, & \text{if } \chi \psi \not\equiv 1; \\ \frac{\gamma(\chi)\gamma(\psi)}{q}, & \text{if } \chi \psi \equiv 1. \end{cases} \]

**Lemma 2.4.** ([9] Eqn. 1.14]). If $T^{m-n} \not\equiv 1$, then
\[ g(T^m)g(T^{-n}) = q \left( \frac{T^m}{T^n} \right) g(T^{m-n})T^n(-1) = J(T^m, T^n)g(T^{m-n}). \]

**Theorem 2.5.** ([14] Davenport-Hasse relation]). Let $p$ be an odd prime and $q = p^e$ for some $e > 0$, and let $m$ be a positive integer such that $q \equiv 1 \pmod{m}$. For multiplicative characters $\chi, \psi \in \mathbb{F}_q^\times$, we have
\[ \prod_{\chi^m = 1} \gamma(\chi) = -g(\psi^m)\psi(m^{-m}) \prod_{\chi^m = 1} \gamma(\chi). \]

2.2. $p$-adic Gamma function, Gross-Koblitz formula and McCarthy’s $p$-adic hypergeometric series. Let $\mathbb{Z}_p$ denote the ring of $p$-adic integers, $\mathbb{Q}_p$ the field of $p$-adic numbers, $\mathbb{Q}_p^\wedge$ the algebraic closure of $\mathbb{Q}_p$, and $\mathbb{C}_p$ the completion of $\mathbb{Q}_p$. It is known that $\mathbb{Z}_p^\wedge$ contains all the $(p-1)$-th roots of unity. Therefore, we can consider multiplicative characters on $\mathbb{F}_p^\times$ to be maps $\chi : \mathbb{F}_p^\times \rightarrow \mathbb{Z}_p^\wedge$. Let $\omega : \mathbb{F}_p^\times \rightarrow \mathbb{Z}_p^\wedge$ be the Teichmüller character. For $a \in \mathbb{F}_p^\times$, the value $\omega(a)$ is just the $(p-1)$-th root of unity in $\mathbb{Z}_p$ such that $\omega(a) \equiv a \pmod{p}$. Also, $\mathbb{F}_p^\wedge = \{ \omega^j : 0 \leq j \leq p-2 \}$. Thus, in the $p$-adic setting the Gauss sum $g(\chi)$ takes value in $\mathbb{Q}_p(\zeta_p)$ for any $\chi \in \mathbb{F}_p^\times$.

We now recall the definition of $p$-adic gamma function. For further details, see [13]. The $p$-adic gamma function $\Gamma_p$ is defined by setting $\Gamma_p(0) = 1$, and for positive integer $n$ by
\[ \Gamma_p(n) := (-1)^n \prod_{0 < j < n, p \nmid j} j. \]

If $x$ and $y$ are two positive integers satisfying $x \equiv y \pmod{p^k\mathbb{Z}}$, then $\Gamma_p(x) \equiv \Gamma_p(y) \pmod{p^k\mathbb{Z}}$. Therefore, the function has a unique extension to a continuous function
\[ \Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p^\times. \] If \( x \in \mathbb{Z}_p \) and \( x \neq 0 \), then \( \Gamma_p(x) \) is defined as
\[ \Gamma_p(x) := \lim_{x_n \to x} \Gamma_p(x_n), \]
where \( x_n \) runs through any sequence of positive integers \( p \)-adically approaching \( x \).

We now introduce Gross-Koblitz formula, which allows us to relate Gauss sum and the \( p \)-adic Gamma function. Let \( \pi \in \mathbb{C}_p \) be the fixed root of \( x^{p-1} + p = 0 \) which satisfies \( \pi \equiv \zeta_p - 1 \pmod{\langle \zeta_p - 1 \rangle^2} \). For \( x \in \mathbb{Q} \) we let \( \lfloor x \rfloor \) denote the greatest integer less than or equal to \( x \) and \( \langle x \rangle \) denote the fractional part of \( x \). We have \( \langle x \rangle = x - \lfloor x \rfloor \) and \( 0 \leq \langle x \rangle < 1 \). Recall that \( \overline{\pi} \) denotes the character inverse of the Teichmüller character \( \omega \).

**Theorem 2.6.** ([10] Gross-Koblitz). For \( a \in \mathbb{Z}_p \), we have
\[ g(a^p) = -\pi^{(p-1)(\frac{a}{p-1})} \Gamma_p \left( \left\langle \frac{a}{p-1} \right\rangle \right). \]

We also need the following lemma to prove the main results.

**Lemma 2.7.** ([2] Eqn 3.4, Lemma 3.4). For odd prime \( p \) and \( 0 < l \leq p - 2 \), we have
\[ \Gamma_p \left( \frac{l}{p-1} \right) \Gamma_p \left( \left\langle 1 - \frac{l}{p-1} \right\rangle \right) = -\overline{\pi}(-1). \]

We now state a product formula for the \( p \)-adic Gamma function.

**Lemma 2.8.** ([16] Lemma 4.1). Let \( p \) be an odd prime. For \( 0 \leq l \leq p - 2 \) and \( t \in \mathbb{Z}_p^+ \) with \( p \nmid t \), we have
\[
\omega(t^l) \Gamma_p \left( \left\langle \frac{l}{p-1} \right\rangle \right) \prod_{h=1}^{t-1} \Gamma_p \left( \left\langle \frac{h}{t} \right\rangle \right) = \prod_{h=0}^{t-1} \Gamma_p \left( \left\langle \frac{h + l}{p-1} \right\rangle \right),
\]
\[
\omega(-t^l) \Gamma_p \left( \left\langle \frac{-l}{p-1} \right\rangle \right) \prod_{h=1}^{t-1} \Gamma_p \left( \left\langle \frac{h}{t} \right\rangle \right) = \prod_{h=0}^{t-1} \Gamma_p \left( \left\langle \frac{1+h}{t} - \frac{l}{p-1} \right\rangle \right).
\]

In [15] [10], D. McCarthy introduces the notion of hypergeometric series in the \( p \)-adic setting which are now famously known as \( p \)-adic hypergeometric series. The McCarthy’s \( p \)-adic hypergeometric series \( \_G_n[\cdots] \) is defined as follows.

**Definition 2.9.** ([10] Definition 1.1) Let \( p \) be an odd prime and let \( t \in \mathbb{F}_p \). For positive integer \( n \) and \( 1 \leq i \leq n \), let \( a_i, b_i \in \mathbb{Q} \cap \mathbb{Z}_p \). Then the function \( \_G_n[\cdots] \) is defined by
\[
_{n}G_n \left[ a_1, a_2, \ldots, a_n \mid t \right] := \frac{-1}{p-1} \sum_{j=0}^{p-2} (-1)^j \overline{\pi}^j(t)
\]
\[
\times \prod_{i=1}^{n} (-p)^{-\lfloor (a_i - \frac{1}{p-1}) \rfloor - \langle -b_i + \frac{1}{p-1} \rangle} \frac{\Gamma_p((a_i - \frac{1}{p-1})) \Gamma_p((-b_i + \frac{1}{p-1}))}{\Gamma_p(a_i) \Gamma_p(-b_i)}.
\]

**3. Counting points on \( Z_\lambda : x_1^d + x_2^d = d\lambda x_1 x_2^{d-1} \)**

In this section, we prove Theorem [17] which expresses the number of points over a finite field \( \mathbb{F}_p \) on the 0-dimensional variety \( Z_\lambda : x_1^d + x_2^d = d\lambda x_1 x_2^{d-1} \) in terms of \( p \)-adic hypergeometric series. We first prove a lemma which will be used to derive the point count formula.
Lemma 3.1. Let $p$ be an odd prime. Then for $0 < l \leq p - 2$ we have

$$\frac{l}{p-1} + \binom{(d-1)l}{p-1} + \binom{-dl}{p-1} = 1 - \sum_{h=1}^{d-2} \left\lfloor \frac{h}{d} - \frac{l}{p-1} \right\rfloor - \sum_{h=1}^{d-1} \left\lfloor \frac{h}{d-1} + \frac{l}{p-1} \right\rfloor.$$

Proof. We have

$$\frac{l}{p-1} + \binom{(d-1)l}{p-1} + \binom{-dl}{p-1} = \frac{l}{p-1} + \binom{(d-1)l}{p-1} - \binom{dl}{p-1} - \binom{(d-1)l}{p-1}$$

(3.1)

$$= - \left\lfloor \frac{(d-1)l}{p-1} \right\rfloor - \left\lfloor \frac{-dl}{p-1} \right\rfloor.$$

Now, it is enough to prove that

$$\left\lfloor \frac{(d-1)l}{p-1} \right\rfloor = d-2 \sum_{h=1}^{d-2} \left\lfloor \frac{h}{d} - \frac{l}{p-1} \right\rfloor,$$

(3.2)

$$\left\lfloor \frac{-dl}{p-1} \right\rfloor = d-1 \sum_{h=1}^{d-1} \left\lfloor \frac{h}{d-1} - \frac{l}{p-1} \right\rfloor - 1.$$  

(3.3)

Since $0 < \frac{l}{p-1} < 1$, we have $0 < \frac{(d-1)l}{p-1} < d-1$. Therefore, $\left\lfloor \frac{(d-1)l}{p-1} \right\rfloor \in \{0, 1, 2, \ldots, d-2\}$. We now prove the lemma by considering some cases.

Case 1: If $\left\lfloor \frac{(d-1)l}{p-1} \right\rfloor = 0$, then $\frac{(d-1)l}{p-1} \neq 0$ by the choice of $l$, which yields $0 < \frac{(d-1)l}{p-1} < 1$. So, $0 < \frac{l}{p-1} < \frac{1}{d-1}$. Therefore, $\left\lfloor \frac{h}{d-1} + \frac{l}{p-1} \right\rfloor = 0$ for $h = 1, 2, \ldots, d-2$, which gives

$$\sum_{h=1}^{d-2} \left\lfloor \frac{h}{d-1} + \frac{l}{p-1} \right\rfloor = 0.$$  

(3.4)

Thus, (3.2) is true in this case.

Case 2: Let $\left\lfloor \frac{(d-1)l}{p-1} \right\rfloor = s$, where $0 < s < d - 2$. Then we have

$$s \leq \frac{(d-1)l}{p-1} < s + 1,$$

and this implies

$$\frac{s}{d-1} \leq \frac{l}{p-1} < \frac{s+1}{d-1}.$$  

(3.5)

Therefore, (3.4) implies that whenever $1 \leq h \leq d - s - 2$ we have $\left\lfloor \frac{h}{d-1} + \frac{l}{p-1} \right\rfloor = 0$, which yields

$$\sum_{h=1}^{d-s-2} \left\lfloor \frac{h}{d-1} + \frac{l}{p-1} \right\rfloor = 0.$$  

Also, (3.4) implies that for $d - s - 1 \leq h \leq d - 2$, we have

$$\left\lfloor \frac{h}{d-1} + \frac{l}{p-1} \right\rfloor = 1.$$
which yields

\[
\sum_{h=d-s-1}^{d-2} \left\lfloor \frac{h}{d-1} + \frac{l}{p-1} \right\rfloor = s.
\] (3.6)

Combining (3.5) and (3.6) we find that (3.2) is also true in this case. This completes the proof of (3.2).

Now, we are going to prove (3.3) using similar arguments. Since \(0 < \frac{l}{p-1} < 1\) in the given range of \(l\), so we have \(-d < \frac{-dl}{p-1} < 0\). Hence, \(\left\lfloor \frac{-dl}{p-1} \right\rfloor \in \{-d, -d + 1, \ldots, -1\}\). Case 1: Let \(\left\lfloor \frac{-dl}{p-1} \right\rfloor = -d\), then by the choice of \(l\), \(\frac{-dl}{p-1} \neq -d\), which yields \(-d < \frac{-dl}{p-1} < -d + 1\). Thus, we have

\[
-1 < -\frac{l}{p-1} < -1 + \frac{1}{d}.
\] (3.7)

Using (3.7) we find that \(\left\lfloor \frac{h}{d} - \frac{l}{p-1} \right\rfloor = -1\) for \(1 \leq h \leq d - 1\), and this gives

\[
\sum_{h=1}^{d-1} \left| \frac{h}{d} - \frac{l}{p-1} \right| = -(d - 1).
\]

Therefore, (3.3) is true in this case.

Case 2: Let \(\left\lfloor \frac{-dl}{p-1} \right\rfloor = -s\), where \(s = 1, 2, \ldots, d - 1\). Then we have \(-s \leq \frac{-dl}{p-1} < -s + 1\), which implies that

\[
-\frac{s}{d} \leq -\frac{l}{p-1} < -\frac{s}{d} + \frac{1}{d}.
\] (3.8)

Using (3.8) we deduce that \(\left\lfloor \frac{h}{d} - \frac{l}{p-1} \right\rfloor = -1\) for \(1 \leq h \leq s - 1\) and \(\left\lfloor \frac{h}{d} - \frac{l}{p-1} \right\rfloor = 0\) for \(s \leq h \leq d - 1\). Thus, we have

\[
\sum_{h=1}^{d-1} \left| \frac{h}{d} - \frac{l}{p-1} \right| = -(s - 1).
\]

Hence, (3.3) is also true in this case. Finally, combining (3.2) and (3.3) we complete the proof of the lemma. \(\square\)

We now prove the point count formula for the family \(Z_\lambda: x_1^d + x_2^d = d\lambda x_1 x_2^{d-1}\).

\textbf{Proof of Theorem 1.1.} Let \(P(x_1, x_2) = x_1^d + x_2^d - d\lambda x_1 x_2^{d-1}\). Let \(\#Z_\lambda(\mathbb{F}_p) = \#\{(x_1, x_2) \in \mathbb{F}_p^2 : x_1^d + x_2^d = d\lambda x_1 x_2^{d-1}\}\) be the number of \(\mathbb{F}_p\)-points on \(Z_\lambda\). If \(N_{\mathbb{F}_p}(Z_\lambda)\) denotes the number of points on \(Z_\lambda\) in \(\mathbb{F}_p^1\) then

\[
N_{\mathbb{F}_p}(Z_\lambda) = \frac{\#Z_\lambda(\mathbb{F}_p) - 1}{p-1}.
\] (3.9)

Using the identity

\[
\sum_{z \in \mathbb{F}_p} \theta(z P(x_1, x_2)) = \begin{cases} p, & \text{if } P(x_1, x_2) = 0; \\ 0, & \text{otherwise}, \end{cases}
\] (3.10)
we have

\[
p \cdot \# Z_\lambda(F_p) = \sum_{z, x_1, x_2 \in F_p} \theta(zP(x_1, x_2))
\]

\[
= p^2 + \sum_{z \in F_p^*} \theta(0) + \sum_{z, x_1 \in F_p^*} \theta(zx_1^d) + \sum_{z, x_2 \in F_p^*} \theta(zx_2^d)
\]

\[
+ \sum_{z, x_1, x_2 \in F_p^*} \theta(zx_1^d)\theta(zx_2^d)\theta(-d\lambda z x_1 x_2^{d-1})
\]

\[
\sum_{p} \theta(zx_1^d)\theta(zx_2^d)\theta(-d\lambda z x_1 x_2^{d-1})
\]

\[
(3.11) = p^2 + p - 1 + B + A,
\]

where \(B = 2 \sum_{z, x_1 \in F_p^*} \theta(zx_1^d)\) and \(A = \sum_{z, x_1, x_2 \in F_p^*} \theta(zx_1^d)\theta(zx_2^d)\theta(-d\lambda z x_1 x_2^{d-1})\).

Using Lemma 2.3 and Lemma 2.1 we obtain \(B = -2(p - 1)\).

Again, using Lemma 2.3 we obtain

\[
A = \sum_{z, x_1, x_2 \in F_p^*} \theta(zx_1^d)\theta(zx_2^d)\theta(-d\lambda z x_1 x_2^{d-1})
\]

\[
= \frac{1}{(p-1)^3} \sum_{z, x_1, x_2 \neq 0} p-2 \sum_{l,m,n=0} g(T^{-l})g(T^{-m})g(T^{-n})T^l(zx_1^d)^m(zx_2^d)^n(-d\lambda z x_1 x_2^{d-1})
\]

\[
= \frac{1}{(p-1)^3} \sum_{l,m,n=0} g(T^{-l})g(T^{-m})g(T^{-n})T^l\sum_{z \neq 0} T^{dm+(d-1)n}(x_2) \sum_{z \neq 0} T^{l+m+n}(z).
\]

From Lemma 2.1 we observe that the inner sums are non zero only if \(n = -dl\) and \(m = (d-1)l\). Substituting these values in the above sum we have

\[
(3.12) A = \sum_{l=0}^{p-2} g(T^{-l})g(T^{-(d-1)l})g(T^{dl})(-d\lambda).
\]

Now, taking \(T = \omega\) and applying Gross-Koblitz formula we obtain

\[
A = -\sum_{l=0}^{p-2} \epsilon^{-l} \sum_{l=0}^{p-1} \Gamma_p\left(\frac{1}{p-1}\right) \Gamma_p\left(\frac{(d-1)l}{p-1}\right) \Gamma_p\left(\frac{dl}{p-1}\right) \Gamma_p\left(\frac{dl}{p-1}\right).
\]
Applying Lemma 2.8 we deduce that
\[
A = -p^{-2} \sum_{l=0}^{p-2} \pi^{p-1}(\frac{h}{p} + \frac{(d-1)l}{d}) \times (-d^l) \omega^d(d-1) \omega^d(d)
\]
\[
\times \Gamma_p \left( \frac{l}{p} - 1 \right) \prod_{h=1}^{d-2} \frac{\Gamma_p((\frac{h}{d} + \frac{l}{p-1}))}{\Gamma_p((\frac{h}{d}) - \frac{l}{p-1})} \prod_{h=1}^{d-1} \frac{\Gamma_p((\frac{h}{d} - \frac{l}{p-1}))}{\Gamma_p(\frac{h}{d})}
\]
\[
= -p^{-2} \sum_{l=0}^{p-2} \pi^{p-1}(\frac{h}{p} + \frac{(d-1)l}{d}) \times (-d^l) \lambda^d(d-1)^{d-1} \Gamma_p \left( \frac{l}{p} - 1 \right)
\]
\[
\times \Gamma_p \left( \frac{l}{p} - 1 \right) \prod_{h=1}^{d-2} \frac{\Gamma_p((\frac{h}{d} + \frac{l}{p-1}))}{\Gamma_p((\frac{h}{d}) - \frac{l}{p-1})} \prod_{h=1}^{d-1} \frac{\Gamma_p((\frac{h}{d} - \frac{l}{p-1}))}{\Gamma_p(\frac{h}{d})}
\]
\[
(3.13)
\]
Using Lemma 3.11 and Lemma 2.7 we have
\[
A = -1 + \sum_{l=1}^{p-2} (-p)^{-1} \sum_{h=1}^{d-1} \frac{h}{d} \times \sum_{l=1}^{d-1} \frac{l}{p} \times \omega^d((d-1) \lambda^d(d-1)^{d-1})
\]
\[
\times \Gamma_p \left( \frac{l}{p} - 1 \right) \prod_{h=1}^{d-2} \frac{\Gamma_p((\frac{h}{d} + \frac{l}{p-1}))}{\Gamma_p((\frac{h}{d}) - \frac{l}{p-1})} \prod_{h=1}^{d-1} \frac{\Gamma_p((\frac{h}{d} - \frac{l}{p-1}))}{\Gamma_p(\frac{h}{d})}
\]
\[
= -1 - p \sum_{l=1}^{p-2} (-p)^{-1} \sum_{h=1}^{d-1} \frac{h}{d} \times \sum_{l=1}^{d-1} \frac{l}{p} \times \omega^d((-1)^{d-1} \lambda^d(d-1)^{d-1})
\]
\[
\times \Gamma_p \left( \frac{l}{p} - 1 \right) \prod_{h=1}^{d-2} \frac{\Gamma_p((\frac{h}{d} + \frac{l}{p-1}))}{\Gamma_p((\frac{h}{d}) - \frac{l}{p-1})} \prod_{h=1}^{d-1} \frac{\Gamma_p((\frac{h}{d} - \frac{l}{p-1}))}{\Gamma_p(\frac{h}{d})}
\]
Adding and subtracting the term under summation for \( l = 0 \), we obtain
\[
A = -1 + p + p \sum_{l=1}^{p-2} (-p)^{-1} \sum_{h=1}^{d-1} \frac{h}{d} \times \sum_{l=1}^{d-1} \frac{l}{p} \times \omega^d((-1)^{d-1} \lambda^d(d-1)^{d-1})
\]
\[
\times \Gamma_p \left( \frac{l}{p} - 1 \right) \prod_{h=1}^{d-2} \frac{\Gamma_p((\frac{h}{d} + \frac{l}{p-1}))}{\Gamma_p((\frac{h}{d}) - \frac{l}{p-1})} \prod_{h=1}^{d-1} \frac{\Gamma_p((\frac{h}{d} - \frac{l}{p-1}))}{\Gamma_p(\frac{h}{d})}
\]
Since \( \omega^d((-1)^l) = (-1)^l \), we have the following expression for \( A \) in terms of the \( G \)-function.
\[
(3.14)
\]
\[
A = p - 1 + p(p - 1)G_{d-1} \left[ \frac{\frac{1}{d}}{\frac{1}{d}}, \frac{\frac{2}{d}}{\frac{1}{d}}, \ldots, \frac{\frac{d-1}{d}}{\frac{1}{d}} \right] \alpha,
\]
where \( \alpha = \lambda^d(d-1)^{d-1} \). Finally, substituting the expressions for \( A \) and \( B \) in (3.11), and then using (3.9) we complete the proof. \( \square \)
Lemma 4.1. For $1 \leq l \leq p - 2$ we have

$$(-p)^{-(\frac{l}{2} + \frac{1}{p-1})} \frac{\Gamma_p \left( (1 - \frac{l}{p-1}) \right) \Gamma_p \left( \frac{1}{2} + \frac{1}{p-1} \right)}{\Gamma_p \left( \frac{1}{2} \right)} = \frac{1}{p} \sum_{t \in \mathbb{F}_p} \frac{\omega'(t)}{t} (t(t-1)).$$

Proof. We have

$$(-p)^{-(\frac{l}{2} + \frac{1}{p-1})} \frac{\Gamma_p \left( (1 - \frac{l}{p-1}) \right) \Gamma_p \left( \frac{1}{2} + \frac{1}{p-1} \right)}{\Gamma_p \left( \frac{1}{2} \right)}$$

$$= \frac{\pi^{-(p-1)(\frac{l}{2} + \frac{1}{p-1})} \Gamma_p \left( (1 - \frac{l}{p-1}) \right) \Gamma_p \left( \frac{1}{2} + \frac{1}{p-1} \right)}{\Gamma_p \left( \frac{1}{2} \right)}$$

$$= \pi^{-(p-1)(\frac{l}{2} + \frac{1}{p-1})} \Gamma_p \left( \frac{1}{2} \right)$$

$$= \pi^{-(p-1)(\frac{l}{2} + \frac{1}{p-1})} \Gamma_p \left( \frac{1}{2} \right) \frac{\left( \frac{1}{2} + \frac{1}{p-1} \right)^{\frac{1}{p-1}} \left( \frac{1}{2} - \frac{1}{p-1} \right)^{\frac{1}{p-1}}}{\pi^{(p-1)(\frac{l}{2} + \frac{1}{p-1})}}.$$  

Using Gross-Koblitz formula we find that

$$(4.1) \quad (-p)^{-(\frac{l}{2} + \frac{1}{p-1})} \Gamma_p \left( (1 - \frac{l}{p-1}) \right) \Gamma_p \left( \frac{1}{2} + \frac{1}{p-1} \right) = \frac{-g(\phi^{\omega'}) g(\omega')}{\pi^{(p-1)\phi(\omega')}}.$$  

Since $1 \leq l \leq p - 2$, Lemma 2.2 gives $g(\omega') g(\omega') = p \omega'(1)$. Then (4.1) reduces to

$$(4.2) \quad (-p)^{-(\frac{l}{2} + \frac{1}{p-1})} \Gamma_p \left( (1 - \frac{l}{p-1}) \right) \Gamma_p \left( \frac{1}{2} + \frac{1}{p-1} \right) = \frac{-\omega'(1) g(\phi^{\omega'}) g(\phi)}{\pi^{(p-1)\phi(\omega')}}.$$  

Now, using (2.4) we deduce that

$$(4.3) \quad \frac{\omega'(1) g(\phi^{\omega'}) g(\phi)}{\pi^{(p-1)\phi(\omega')}} = \frac{\omega'(1)}{p} J(\phi^{\omega'}, \phi)$$

$$= \frac{\omega'(1)}{p} \sum_{t \in \mathbb{F}_p} \phi^{\omega'}(t) \phi(1 - t)$$

Finally, combining (4.2) and (4.3) we obtain the desired result. \qed
Lemma 4.2. Let \(0 \leq l \leq p - 2\). Then we have

\[
(-p)^{-\frac{d}{2} - \frac{l}{p-1}} \Gamma_p \left(\frac{l}{p-1}\right) \Gamma_p \left(\frac{d}{2} - \frac{l}{p-1}\right) = - \sum_{t \in \mathbb{F}_p} \omega^t (-t) \phi(t(t - 1)).
\]

Proof. If we put \(l = 0\) in both the sides of (4.4) then we obtain that the left hand side is 1 and the right hand side is equal to \(2 \sum_{t \in \mathbb{F}_p} \phi(t(t - 1))\). Using (2.2) and Lemma 2.2 we easily find that \(2 \sum_{t \in \mathbb{F}_p} \phi(t(t - 1)) = -1\), and hence the right hand side of (4.4) is also 1. Thus, (4.4) is true for \(l = 0\). For \(1 \leq l \leq p - 2\), the proof proceeds along similar lines to the proof of Lemma 4.1 so we omit the details for reasons of brevity. \(\square\)

Proof of Theorem 4.2. For \(x \in \mathbb{F}_p^\times\), we consider the sum

\[
A_x = -1 - \sum_{l=1}^{p-2} n^{(p-1)} \left(\frac{l}{p-1} + \frac{(d-1)l}{p-1} + \frac{d l}{p-1}\right) \omega^l (-x) \Gamma_p \left(\frac{l}{p-1}\right) \Gamma_p \left(\frac{d}{p-1}\right)
\]

(4.5) \(\times \Gamma_p \left(\frac{l}{p-1}\right) \prod_{h=1}^{d-2} \frac{\Gamma_p \left(\frac{h}{p-1} + \frac{l}{p-1}\right) \prod_{h=1}^{d-1} \Gamma_p \left(\frac{h}{p-1}\right)}{\Gamma_p \left(\frac{h}{p-1}\right)}\).

Since \(d\) is odd, the term \(A\) given in (3.13) is equal to \(A_\alpha\) with \(\alpha = \lambda d(d - 1)^{d-1}\). Thus, proceeding similarly as shown in the proof of Theorem 4.1 we deduce that

\[
A_x = p - 1 + p(p-1)G_{d-1} \left[\frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d} - 1, \frac{1}{d-1}, \ldots, \frac{d-2}{d-1}, |x|\right].
\]

Also,

\[
A_x = -1 - \sum_{l=1}^{p-2} n^{(p-1)} \left(\frac{l}{p-1} + \frac{(d-1)l}{p-1} + \frac{d l}{p-1}\right) \omega^l (-x) \Gamma_p \left(\frac{l}{p-1}\right) \Gamma_p \left(\frac{d}{p-1}\right)
\]

(4.6) \(\times \Gamma_p \left(\frac{l}{p-1}\right) \prod_{h=1}^{d-2} \frac{\Gamma_p \left(\frac{h}{p-1} + \frac{l}{p-1}\right) \prod_{h=1}^{d-1} \Gamma_p \left(\frac{h}{p-1}\right)}{\Gamma_p \left(\frac{h}{p-1}\right)}\).

\(\times \frac{\Gamma_p \left(\frac{d}{p-1}\right) \Gamma_p \left(\frac{1}{d-1} + \frac{l}{p-1}\right) \prod_{h=1}^{d-2} \frac{\Gamma_p \left(\frac{1}{d-1} + \frac{l}{p-1}\right) \prod_{h=1}^{d-1} \Gamma_p \left(\frac{1}{d-1} + \frac{l}{p-1}\right)}{\Gamma_p \left(\frac{1}{d-1} + \frac{l}{p-1}\right)}\).

\[
= -1 - \sum_{l=1}^{p-2} n^{(p-1)} \left(\frac{l}{p-1} + \frac{(d-1)l}{p-1} + \frac{d l}{p-1}\right) \omega^l (-x) \Gamma_p \left(\frac{l}{p-1}\right) \Gamma_p \left(\frac{d}{p-1}\right)
\]

\(\times \Gamma_p \left(\frac{d}{p-1}\right) \prod_{h=1}^{d-2} \frac{\Gamma_p \left(\frac{h}{p-1} + \frac{l}{p-1}\right) \prod_{h=1}^{d-1} \Gamma_p \left(\frac{h}{p-1} + \frac{l}{p-1}\right)}{\Gamma_p \left(\frac{h}{p-1} + \frac{l}{p-1}\right)}\).
Using Lemma 3.1 we have

\[ A_x = -1 - \sum_{t=1}^{p-2} (-p)^{1 - \sum_{h=1}^{d-1} \left\lfloor \frac{h}{p-1} \right\rfloor - \sum_{h=1}^{d-2} \frac{1}{p-1} + \frac{1}{p-1}} \omega'(-x) \times \frac{\Gamma_p \left( (1 - \frac{1}{p-1}) \right) \Gamma_p \left( \frac{l}{p-1} - \frac{1}{p-1} \right)}{\Gamma_p \left( \frac{l}{p-1} \right)} \times \prod_{h=1}^{d-2} \frac{\Gamma_p (\frac{h}{p-1} + \frac{l}{p-1})}{\Gamma_p (\frac{h}{p-1})} \times \prod_{h=1}^{d-1} \frac{\Gamma_p (\frac{h}{p-1} - \frac{l}{p-1})}{\Gamma_p (\frac{h}{p-1})} \times \prod_{h=1}^{d-2} \frac{\Gamma_p (\frac{h}{d-1} + \frac{l}{p-1})}{\Gamma_p (\frac{h}{d-1})} \times \prod_{h=1}^{d-1} \frac{\Gamma_p (\frac{h}{d-1} - \frac{l}{p-1})}{\Gamma_p (\frac{h}{d-1})}. \]

Lemma 4.1 yields

\[ A_x = -1 + \sum_{t=1}^{p-2} (-p) ^ \left( 1 - \sum_{h=1}^{d-1} \left\lfloor \frac{h}{p-1} \right\rfloor - \sum_{h=1}^{d-2} \frac{1}{p-1} + \frac{1}{p-1} \right) \omega'(-x) \times \sum_{t \in \mathbb{F}_p} (t) \phi(t(t-1)) \frac{\Gamma_p \left( \frac{l}{p-1} \right)}{\Gamma_p \left( \frac{l}{p-1} - \frac{1}{p-1} \right)} \times \prod_{h=1}^{d-2} \frac{\Gamma_p (\frac{h}{d-1} + \frac{l}{p-1})}{\Gamma_p (\frac{h}{d-1})} \times \prod_{h=1}^{d-1} \frac{\Gamma_p (\frac{h}{d-1} - \frac{l}{p-1})}{\Gamma_p (\frac{h}{d-1})}. \]

The term under summation for \( l = 0 \) is \( \sum_{t \in \mathbb{F}_p} \phi(t(t-1)) \). Using 2.4 and Lemma 2.2 we easily find that \( \sum_{t \in \mathbb{F}_p} \phi(t(t-1)) = -1 \). Thus,

\[ A_x = \sum_{t \in \mathbb{F}_p} \phi(t(t-1)) \sum_{t=1}^{p-2} (-p) ^ \left( 1 - \sum_{h=1}^{d-1} \left\lfloor \frac{h}{p-1} \right\rfloor - \sum_{h=1}^{d-2} \frac{1}{p-1} + \frac{1}{p-1} \right) \omega'(xt) \times \Gamma_p \left( \frac{l}{p-1} \right) \Gamma_p \left( \frac{l}{p-1} - \frac{1}{p-1} \right) \times \prod_{h=1}^{d-2} \frac{\Gamma_p (\frac{h}{d-1} + \frac{l}{p-1})}{\Gamma_p (\frac{h}{d-1})} \times \prod_{h=1}^{d-1} \frac{\Gamma_p (\frac{h}{d-1} - \frac{l}{p-1})}{\Gamma_p (\frac{h}{d-1})} \]

\[ = -(p-1) \sum_{t \in \mathbb{F}_p} \phi(t(t-1)) \times d^{-1} G_{d-1} \begin{bmatrix} \frac{1}{d}, & \frac{2}{d}, & \ldots, & \frac{d-1}{d}, & \frac{d}{d}, & \frac{d+1}{d}, & \ldots, & \frac{d-3}{d}, & \frac{d-2}{d}, & \frac{d-1}{d} \end{bmatrix} \]
Finally, combining (4.6) and the above expression for $A_x$ we derive the required summation identity. \qed

**Proof of Theorem 4.3** For $x \in \mathbb{F}_p^*$, we consider the sum

\begin{equation}
A_x = \frac{1}{p-1} \sum_{l=1}^{p-2} \pi(l-1) \Gamma_p \left( \frac{l}{p-1} \right) \Gamma_p \left( 1 - \frac{l}{p-1} \right)
\end{equation}

(4.7) $\times \Gamma_p \left( \frac{l}{p-1} \right) \prod_{h=1}^{d-2} \Gamma_p \left( \frac{h}{d-1} + \frac{l}{p-1} \right) \prod_{h=1}^{d-1} \Gamma_p \left( \frac{h}{d} - \frac{l}{p-1} \right).

Since $d$ is even, the term $A$ given in (4.13) is equal to $A_\alpha$ with $\alpha = \lambda^d(d-1)^{d-1}$. Thus, proceeding similarly as shown in the proof of Theorem 4.1 we deduce that

\begin{equation}
A_x = p - 1 + p(p-1)G_{d-1} \left[ \frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d-2}, x \right]
\end{equation}

Also,

\begin{align*}
A_x &= -1 - \sum_{l=1}^{p-2} \pi(l-1) \Gamma_p \left( \frac{l}{p-1} \right) \prod_{h=1}^{d-2} \Gamma_p \left( \frac{h}{d-1} + \frac{l}{p-1} \right) \prod_{h=1}^{d-1} \Gamma_p \left( \frac{h}{d} - \frac{l}{p-1} \right) \\
&\times \frac{\Gamma_p \left( \frac{l}{p-1} \right) \Gamma_p \left( \frac{h}{d} - \frac{l}{p-1} \right) \prod_{h=1}^{d-2} \Gamma_p \left( \frac{h}{d-1} + \frac{l}{p-1} \right) \prod_{h=1}^{d-1} \Gamma_p \left( \frac{h}{d} - \frac{l}{p-1} \right)}{\Gamma_p \left( \frac{1}{d} \right) \prod_{h=1}^{d-1} \Gamma_p \left( \frac{h}{d} - \frac{l}{p-1} \right)}
\end{align*}

We now apply Lemma 3.1 and Lemma 2.2 to obtain

\begin{align*}
A_x &= -1 + \sum_{l=1}^{p-2} (-p)^{1 - \sum_{h=1}^{d-1} \left( \frac{h}{d-1} - \frac{l}{p-1} \right) - \sum_{h=1}^{d-2} \left( \frac{h}{d-2} + \frac{l}{p-1} \right)} \pi(l-1) \\
&\times \frac{\Gamma_p \left( \frac{l}{p-1} \right) \Gamma_p \left( \frac{h}{d} - \frac{l}{p-1} \right) \prod_{h=1}^{d-2} \Gamma_p \left( \frac{h}{d-1} + \frac{l}{p-1} \right) \prod_{h=1}^{d-1} \Gamma_p \left( \frac{h}{d} - \frac{l}{p-1} \right)}{\Gamma_p \left( \frac{1}{d} \right) \prod_{h=1}^{d-1} \Gamma_p \left( \frac{h}{d} - \frac{l}{p-1} \right)}
\end{align*}

\begin{align*}
&= -1 + \sum_{l=1}^{p-2} (-p)^{-\sum_{h=1}^{d-1} \left( \frac{h}{d-1} - \frac{l}{p-1} \right) - \sum_{h=1}^{d-2} \left( \frac{h}{d-2} + \frac{l}{p-1} \right)} \pi(l-1) \\
&\times \left( \frac{(-p)^{\frac{1}{d} - \frac{l}{p-1}} \Gamma_p \left( \frac{1}{d} - \frac{l}{p-1} \right) \prod_{h=1}^{d-1} \Gamma_p \left( \frac{h}{d} - \frac{l}{p-1} \right)}{\Gamma_p \left( \frac{1}{d} \right) \prod_{h=1}^{d-1} \Gamma_p \left( \frac{h}{d} - \frac{l}{p-1} \right)} \right)
\end{align*}

Finally, combining (4.6) and the above expression for $A_x$ we derive the required summation identity. \qed
Adding and subtracting the term under summation for \( l = 0 \), and then applying Lemma 4.2 we deduce that

\[
A_x = -1 + p + p \sum_{t \in \mathbb{F}_p^*} \phi(t(t - 1)) - \sum_{h=1}^{d-1} \sum_{k \neq \frac{d}{h} \in \mathbb{Z}/d\mathbb{Z}} \frac{1}{2} \left( \frac{\lambda}{h} - \frac{p+1}{p-h} \right) \sum_{l=0}^{p-2} \frac{\Gamma_p\left(\frac{h}{d-1} + 1\right)}{\Gamma_p\left(\frac{h}{d-1}\right)} \prod_{h \neq \frac{d}{h} \in \mathbb{Z}/d\mathbb{Z}} \Gamma_p\left(\frac{h}{d-1}\right)
\]

\[
\times \sum_{l=0}^{d-2} \left[ \frac{1}{d-1}, \frac{2}{d}, \ldots, \frac{d-2}{d} \right] \left[ \frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d} \right].
\]

Finally, combining (1.8) and the above expression, and then replacing \( 1/t \) by \( t \) we complete the proof.

5. Transformations and special values of the \( p \)-adic hypergeometric series

In this section, we derive transformations for the \( p \)-adic hypergeometric series. We use these transformations to find certain special values of the \( G \)-function. In [3], we express the number of distinct zeros of the polynomials \( x^d + ax + b \) and \( x^d + ax^{d-1} + b \) over a finite field in terms of McCarthy’s \( p \)-adic hypergeometric series. We use certain Gauss sums evaluations from [3] in the proof of Theorem 1.7 below.

**Proof of Theorem 1.7.** For \( \lambda \in \mathbb{F}_p^* \), we consider the sum

\[
A_\lambda = \sum_{l=0}^{p-2} \frac{g(T)^l g(T^{-(d-1)l}) g(T^{d}T^l)}{d^d \lambda^{l+1}} \left( \frac{(-1)^d(d-1)^{d-1}}{d^d \lambda} \right).
\]

Since (3.12) and (5.1) contain the same Gauss sums, so proceeding similarly as shown in the proof of Theorem 1.1 we deduce that

\[
A_\lambda = p - 1 + p(p - 1) \left[ \frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d} \right] \left[ 0, \frac{1}{d-1}, \ldots, \frac{d-1}{d-1} \right].
\]

Now, if \( d \) is even, then replacing \( l \) by \( l - \frac{d-1}{2} \) in (5.1) we have

\[
A_\lambda = p - 1 + p(p - 1) \left[ \frac{1}{d}, \frac{3}{2(d-1)}, \ldots, \frac{d-1}{2(d-1)}, \frac{d+1}{2(d-1)}, \ldots, \frac{2(d-1)-1}{2(d-1)} \right].
\]

We observe that the Gauss sums present in (5.3) are the same Gauss sums appeared in [3, Eqn 11]. Therefore, proceeding similarly as shown in the proof of [3, Theorem 1.2], we deduce that

\[
A_\lambda = p - 1 + p(p - 1) \phi(-\lambda(d-1))
\]

\[
\times d^{-1} G_{d-1} \left[ \frac{1}{2(d-1)}, \frac{3}{2(d-1)}, \ldots, \frac{d-1}{2(d-1)}, \frac{d+1}{2(d-1)}, \ldots, \frac{2(d-1)-1}{2(d-1)} , \frac{1}{d} \right].
\]
Combining (5.2) and (5.4) we obtain the desired transformation when $d$ is even.

If $d$ is odd, then replacing $l$ by $l - \frac{d-1}{2}$ in (5.1) we have

$$(5.5) \quad A_\lambda = \phi(-d\lambda) \sum_{l=0}^{p-2} g(T^{-l+\frac{d-1}{2}}) g(T^{-(d-1)l}) g(T^{d\lambda + \frac{d-1}{2}}) T^{l} \left( -\frac{(d-1)^{d-1}}{d^d \lambda} \right).$$

Again, we observe that the Gauss sums present in (5.5) are the same Gauss sums appeared in [3, Eqn 22]. Therefore, proceeding similarly as shown in the proof of [3, Theorem 1.3], we deduce that

$$A_\lambda = p - 1 + p(p-1)\phi(d\lambda)$$

$$(5.6) \quad \times \quad d^{-1} G_{d-1} \left[ \frac{1}{d}, \frac{1}{2d}, \ldots, \frac{d-3}{(d-1)2d}, \frac{d-1}{2d}, \frac{d}{2d}, \ldots, \frac{d-2}{2d}, \frac{d-1}{2d} \right].$$

Finally, combining (5.2) and (5.6) we obtain the desired transformation when $d$ is odd. This completes the proof of the theorem. $\square$

**Remark 5.1.** For $\lambda \neq 0$, the number of points in $\mathbb{F}_q$ over a finite field $\mathbb{F}_q$ on the family $Z_\lambda: x_1^d + x_2^d = d\lambda x_1 x_2^{d-1}$ is equal to the number of distinct zeros of the polynomial $x^d - d\lambda x + 1$ over $\mathbb{F}_q$. Therefore, using [3, Theorem 1.2 and Theorem 1.3] and Theorem 1.7 we obtain the transformations stated in Theorem 1.7 for certain values of $\lambda$, namely $\lambda^d(d-1)^{d-1}$.

**Proof of Theorem 1.8.** Putting $d = 3$ in Theorem 1.7, we find that

$$(5.7) \quad 2G_2 \left[ \frac{1}{4}, \frac{3}{2} \mid \frac{4}{27} \right] = 2G_2 \left[ \frac{1}{4}, \frac{3}{2} \mid \frac{27}{4} \right]$$

for $p > 3$. Now, from [2, Theorem 4.6] we have

$$4G_4 \left[ \frac{1}{10}, \frac{1}{10}, \frac{7}{10}, \frac{3}{10} \mid \frac{5^5}{44} \right] = \phi(-1) + \phi(3) + \phi(-1)2G_2 \left[ \frac{1}{5}, \frac{3}{2} \mid \frac{27}{4} \right]$$

$$(5.8) \quad = \phi(-1) + \phi(3) + \phi(-1)2G_2 \left[ \frac{1}{5}, \frac{3}{2} \mid \frac{27}{4} \right]$$

for $p > 7$ and $p \neq 23$. Combining (5.7) and (5.8) we readily obtain the first identity.

Again, if we apply Theorem 1.7 for $d = 5$, then for $p = 3$ and $p > 5$ we have

$$4G_4 \left[ \frac{1}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5} \mid - \frac{4^4}{5^5} \right]$$

$$(5.9) \quad = \phi(-1)4G_4 \left[ \frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{9}{15} \mid - \frac{5^5}{44} \right] .$$

Combining (5.8), (5.9) and (5.7) we obtain the second set of transformations. $\square$

In [2], the authors with D. McCarthy find certain special values of the $G$-function. We use the transformations given in Theorem 1.7 to find some new values of the $G$-function.
Theorem 5.2. Let $a, b, c \in \mathbb{F}_p^*$ be such that $a + b + c = 0$ and $ab + bc + ca \neq 0$. Then, for $p \geq 5$, we have

$$\binom{2G_2}{\frac{1}{p}, \frac{2}{3}, \frac{3}{2}} = A,$$

where $A = 2$ if all of $a, b, c$ are distinct and $A = 1$ if exactly two of $a, b, c$ are equal.

If $a, b, c \in \mathbb{F}_p^*$ are such that $ab + bc + ca = 0$ and $a + b + c \neq 0$, then, for $p \geq 5$, we have

$$\binom{2G_2}{\frac{1}{p}, \frac{2}{3}, \frac{3}{2}} = A,$$

Proof. Let $a + b + c = 0$ and $ab + bc + ca \neq 0$. Then, from [2] Theorem 4.1, for $p \geq 5$, we have

$$\binom{2G_2}{0, \frac{1}{5}, \frac{2}{5}} = A \cdot \phi(- (ab + bc + ca)).$$

Now, applying Theorem 1.7 for $d = 3$ and $\lambda = -\frac{4(ab + bc + ca)^3}{27abc}$, and then comparing the result with (5.12) we derive (5.10). Again, if $ab + bc + ca = 0$ and $a + b + c \neq 0$, then [2] Theorem 4.1 gives

$$\binom{2G_2}{0, \frac{1}{5}, \frac{2}{5}} = A \cdot \phi(- abc(a + b + c))$$

for $p \geq 5$. We now apply Theorem 1.7 for $d = 3$ and $\lambda = -\frac{4(ab + bc + ca)^3}{27abc}$, and then compare the result with (5.13) to derive (5.11). This completes the proof of the theorem.

Example 5.3. If we put $a = b = 1$ and $c = -2$ in (5.10), then for $p \geq 5$ we have

$$\binom{2G_2}{\frac{1}{3}, \frac{2}{3}, \frac{3}{2}} = 1.$$

If we put $a = 1, b = 2$ and $c = -3$ in (5.11), then for $p \geq 5$ we have

$$\binom{2G_2}{\frac{1}{5}, \frac{2}{5}} = 2.$$

Theorem 5.4. If $p \geq 5$, then we have

$$\binom{3G_3}{\frac{1}{3}, \frac{2}{3}, \frac{3}{2}} = 1 + \phi(-2).$$

Proof. If $p \geq 5$, then from [2] Theorem 4.5] we have

$$\binom{3G_3}{\frac{1}{5}, \frac{2}{5}} = \phi(-3) + \phi(6).$$

If we use Theorem 1.7 for $d = 4$ and $\lambda = 1$, then we have

$$\binom{3G_3}{\frac{1}{4}, \frac{2}{4}} = \phi(-3)3G_3\left[\frac{1}{6}, \frac{2}{6}, \frac{3}{6} \mid 1\right].$$

Now, (5.14) and (5.15) readily gives us the desired special value. 

□
6. Summation identities for Greene’s hypergeometric series

In this section we prove the point count formula for the family $Z_\lambda$ and the summation identity for Greene's hypergeometric series. We first prove two lemmas which will be used to prove our main results. The following lemma is a special case of Davenport-Hasse relation.

**Lemma 6.1.** Let $d$ be a positive integer and let $p$ be an odd prime and $q = p^r$ such that $q \equiv 1 \pmod{d}$. Then for $t \in \{1, -1\}$ and $l \in \mathbb{Z}$ we have

$$
\prod_{i=0}^{d-1} g(T^{i+t\frac{i(i+1)}{2}}) = \begin{cases}
q^{\frac{d-1}{2}} T^{\frac{(d-1)(d+1)(q-1)}{8}} (-1)^{T^{-ld}(d)g(T^{ld})}, & \text{if } d \geq 1 \text{ is odd} ; \\
q^{\frac{d-2}{2}} g(\phi) T^{\frac{(d-2)(q-1)}{8}} (-1)^{T^{-ld}(d)g(T^{ld})}, & \text{if } d \geq 2 \text{ is even}.
\end{cases}
$$

**Proof.** The lemma readily follows by putting $m = d$ in Theorem 2.5 and then applying Lemma 2.2. □

**Lemma 6.2.** Let $0 \leq l \leq q - 2$. Then we have

$$
\frac{g(T^l)g(T^{-l}\phi)}{g(\phi)} = \sum_{t \in \mathbb{F}_q} \phi(t(t - 1))T^{-l}(-t).
$$

**Proof.** If we put $l = 0$ on the left hand side of (6.1), then we have $\frac{g(\epsilon)g(\phi)}{g(\phi)} = -1$.
Also, if we simplify the expression on the right hand side of (6.1) for $t = 0$, then we have

$$
\sum_{t \in \mathbb{F}_q} \phi(t(t - 1)) = \phi(-1) \sum_{t \in \mathbb{F}_q} \phi(t)\phi(1 - t) = \phi(-1)J(\phi, \phi).
$$

Using (2.4) and then Lemma 2.2 we obtain that the above sum is equal to $-1$. Thus the right hand side of (6.1) is also $-1$. So, the result is true for $l = 0$. Now, for $l \neq 0$ using Lemma 2.2 and then (2.4) we have

$$
\frac{g(T^l)g(T^{-l}\phi)}{g(\phi)} = \frac{\phi T^l(-1)g(T^{-l}\phi)g(\phi)}{g(T^{-l})} = \phi T^l(-1)J(T^{-l}\phi, \phi) = \phi T^l(-1) \sum_{t \in \mathbb{F}_q} T^{-l}\phi(t)\phi(1 - t) = \sum_{t \in \mathbb{F}_q} \phi(t(t - 1))T^{-l}(-t).
$$

This completes the proof of the lemma. □

We now prove Theorem 1.10 which will be used to deduce the summation identity.

**Proof of Theorem 1.10** Let $\#Z_\lambda(\mathbb{F}_q) = \# \{(x_1, x_2) \in \mathbb{F}_q^2 : x_1^d + x_2^d = d\lambda x_1 x_2^{d-1}\}$ denote the number of $\mathbb{F}_q$-points on the 0-dimensional variety $Z_\lambda^d : x_1^d + x_2^d = d\lambda x_1 x_2^{d-1}$. If $N_{\mathbb{F}_q}(Z_\lambda)$ denotes the number of points on $Z_\lambda$ in $\mathbb{F}_q^1$ then

$$
N_{\mathbb{F}_q}(Z_\lambda) = \frac{\#Z_\lambda(\mathbb{F}_q) - 1}{q - 1}.
$$

From the proof of Theorem 1.1 we have

$$
q \cdot \#Z_\lambda(\mathbb{F}_q) = q^2 + q - 1 + B + A,
$$

where $B = 2\sum_{z, x_1, x_2 \in \mathbb{F}_q^d} \theta(zx_1^d) \theta(zx_2^d)\theta(-d\lambda x_1 x_2^{d-1})$. Using Lemma 2.3 and Lemma 2.1 we obtain $B = -2(q - 1)$. Also, proceeding
simply as shown in the proof of Theorem 6.3 we have

\[ A = \sum_{l=0}^{q-2} g(T^{-l}) g(T^{-(d-1)l}) g(T^{dl}) T^{-dl} (-d\lambda). \]

Here \( d \geq 3 \) is odd. From Lemma 6.1 we have

\[
\begin{align*}
g(T^{dl}) &= \frac{\prod_{i=0}^{d-1} g(T^{l+\frac{i(q-1)}{d}})}{q^{\frac{d-1}{2} T^{\frac{(d-1)(d+1)(q-1)}{4d}}} (-1)} T^{dl}(d),
g(T^{-(d-1)l}) &= \frac{\prod_{i=0}^{d-2} g(T^{-l-\frac{i(q-1)}{d}})}{q^{\frac{d}{2} g(\lambda) T^{\frac{(d-2)(q-1)}{2d}}} (-1)} T^{-(d-1)l}(d-1).
\end{align*}
\]

Plugging these two expressions in (6.4) we deduce that

\[
A = \frac{T^{\frac{(d-1)(q-1)}{sd}}}{q^{2d^2 g(\lambda)}} (-1) \sum_{l=0}^{q-2} g(T^{-l}) \prod_{i=0}^{d-1} g(T^{l+\frac{i(q-1)}{d}}) \prod_{i=0}^{d-2} g(T^{-l-\frac{i(q-1)}{d}}) T^l \left( -\frac{1}{\alpha} \right)
\]

\[
= \frac{T^{\frac{(d-1)(q-1)}{sd}}}{q^{2d^2 g(\lambda)}} (-1) \sum_{l=0}^{q-2} g(T^l) g(T^{-l-\frac{2(q-1)}{d}}) g(T^{-l})^2 \prod_{i=1}^{d-1} g(T^{l+\frac{i(q-1)}{d}})
\]

\[
\times \prod_{i \neq \frac{1}{d}}^{d-2} g(T^{-l-\frac{i(q-1)}{d}}) T^l \left( -\frac{1}{\alpha} \right),
\]

where \( \alpha = \lambda^d (d-1)^{d-1} \).

Now, pairing the terms under summation we obtain

\[
A = \frac{T^{\frac{(d-1)(q-1)}{sd}}}{q^{2d^2 g(\lambda)}} (-1) \sum_{l=0}^{q-2} \{ g(T^l) g(T^{-l-\frac{2(q-1)}{d}}) \} g(T^{-l})^2 \prod_{i=1}^{d-1} g(T^{l+\frac{i(q-1)}{d}})
\]

\[
\times \prod_{i \neq \frac{1}{d}}^{d-2} g(T^{-l-\frac{i(q-1)}{d}}) T^l \left( -\frac{1}{\alpha} \right)
\]

\[
= \frac{T^{\frac{(d-1)(q-1)}{sd}}}{q^{2d^2 g(\lambda)}} (-1) \sum_{l=0}^{q-2} T^l \left( -\frac{1}{\alpha} \right) \{ g(T^l) g(T^{-l-\frac{2(q-1)}{d}}) \} \{ g(T^{l+\frac{2(q-1)}{d}}) g(T^{-l-\frac{2(q-1)}{d}}) \}
\]

\[
\times \{ g(T^{l+\frac{2(q-1)}{d}}) g(T^{-l-\frac{2(q-1)}{d}}) \} \ldots \{ g(T^{l+(d-1)(q-1)}}{d}) g(T^{-l-(d-1)(q-1)}) \}
\]

\[
\times \{ g(T^{l+(d-1)(q-1)}}{d}) g(T^{-l-(d-1)(q-1)}) \} \ldots \{ g(T^{l+(d-2)(q-2)}}{d}) g(T^{-l-(d-2)(q-2)}) \}.
\]
Applying Lemma 2.4 and Lemma 2.2 we deduce that

\[
A = \frac{q^{d+1}}{g(\phi)} \sum_{t=0}^{q-2} T^{l} \left( -\frac{1}{\alpha} \right) \left\{ g(T^{l})g(T^{-l-\frac{2}{d+1}}) \right\} \left( T^{l+\frac{2}{d+1}} \right) \\
\times \left( T^{l+\frac{2(q-1)}{d+1}} \right) \cdots \left( T^{l+\frac{2(q-1)}{d+1}} \right) \left( T^{l+(\frac{d-1}{d+1})(q-1)} \right)
\]

\[
= q^{d+1} \sum_{t \in \mathbb{F}_{q}} \phi(t(t-1)) \sum_{t=0}^{q-2} T^{l} \left( -\frac{1}{\alpha} \right) \left( T^{l+\frac{2}{d+1}} \right) \left( T^{l+\frac{2}{d+1}} \right) \\
\times \left( T^{l+\frac{2(q-1)}{d+1}} \right) \cdots \left( T^{l+\frac{2(q-1)}{d+1}} \right) \left( T^{l+(\frac{d-1}{d+1})(q-1)} \right)
\]

\[
(6.5) \quad \times \left( T^{l+\frac{2(q-1)}{d+1}} \right) \cdots \left( T^{l+\frac{2(q-1)}{d+1}} \right) \left( T^{l+(\frac{d-1}{d+1})(q-1)} \right)
\]

Lemma 6.2 yields

\[
A = q^{d+1} \sum_{t \in \mathbb{F}_{q}} \phi(t(t-1)) \sum_{t=0}^{q-2} T^{l} \left( -\frac{1}{\alpha} \right) \left( T^{l+\frac{2}{d+1}} \right) \\
\times \left( T^{l+\frac{2(q-1)}{d+1}} \right) \cdots \left( T^{l+\frac{2(q-1)}{d+1}} \right) \left( T^{l+(\frac{d-1}{d+1})(q-1)} \right)
\]

\[
\times \left( T^{l+\frac{2(q-1)}{d+1}} \right) \cdots \left( T^{l+\frac{2(q-1)}{d+1}} \right) \left( T^{l+(\frac{d-1}{d+1})(q-1)} \right)
\]

\[
= q^{d+1} (q-1) \sum_{t \in \mathbb{F}_{q}} \phi(t(t-1)) \\
\times d^{-1} F_{d-2} \left( \frac{\chi^{d+1}}{\psi}, \frac{\chi^{d+1}}{\psi}, \frac{\chi^{d+1}}{\psi}, \frac{\chi^{d+1}}{\psi}, \cdots, \frac{\chi^{d+1}}{\psi}, \frac{\chi^{d+1}}{\psi}, \cdots, \frac{\chi^{d-1}}{\psi} \right).
\]
Now, substituting the values of $A$ and $B$ in (6.3), and then using (6.2) we deduce that
\[ q \cdot N_{F_q}(Z_\lambda) = q - 1 + q^{d-1} \sum_{t \in \mathbb{F}_q^*} \phi(t(t-1)) \]
(6.6)\]
\[ \times_{d-1} F_{d-2} \left( \begin{array}{c} \lambda \frac{d-1}{2}, \chi, \ldots, \chi \frac{d-3}{2}, \chi \frac{d-1}{2}, \chi \frac{d+1}{2}, \ldots, \chi \frac{d-1}{2}, \\ \psi, \ldots, \psi \frac{d-1}{2}, \varepsilon, \psi \frac{d+1}{2}, \ldots, \psi \frac{d-1}{2} \end{array} \right) \]
Finally, replacing $t$ by $\frac{1}{t}$ in (6.6) we derive the required result. \hfill \Box

Proof of Theorem 1.11 For $\lambda \in \mathbb{F}_q^*$, we consider
\[ A_\lambda = \sum_{l=0}^{q-2} g(T^{-l})g(T^{-(d-1)l})g(T^{d})T^l \left( \frac{-(d-1)^{d-1}\lambda}{d^2} \right). \]
(6.7)

We observe that (6.4) and (6.7) contain the same Gauss sums. Therefore, proceeding similarly as shown in the proof of Theorem 1.10 we deduce that
\[ A_\lambda = q^{d-1}(q-1) \sum_{t \in \mathbb{F}_q^*} \phi(t(t-1)) \]
(6.8)\]
\[ \times_{d-1} F_{d-2} \left( \begin{array}{c} \lambda \frac{d-1}{2}, \chi, \ldots, \chi \frac{d-3}{2}, \chi \frac{d-1}{2}, \chi \frac{d+1}{2}, \ldots, \chi \frac{d-1}{2}, \\ \psi, \ldots, \psi \frac{d-1}{2}, \varepsilon, \psi \frac{d+1}{2}, \ldots, \psi \frac{d-1}{2} \end{array} \right) , \]
where $\chi$ and $\psi$ are characters of order $d$ and $d-1$, respectively.

In [4] we express the number of distinct zeros of the polynomial $x^d + ax^i + b$ over a finite field $\mathbb{F}_q$ in terms of Greene’s hypergeometric function under the condition that $i|d$ and $q \equiv 1 \pmod{\frac{d(d-1)}{2}}$. In [4] Eqn 17, we consider the following term.

\[ B = \frac{1}{q-1} \sum_{l=0}^{q-2} g(T^{-l})g(T^{d})T^l \left( \frac{b^{d-1}}{d^2} \right) . \]
(6.9)

When $i = 1$, the Gauss sums present in (6.7) and (6.9) are the same. Therefore, proceeding similarly as shown in the proof of [4] Thm. 1.3 for $i = 1$ we deduce that
\[ A_\lambda = q - 1 - \phi(-\lambda)(q-1) + (q-1)q^{d-1} \phi(-1) \]
(6.10)\]
\[ \times_{d} F_{d-1} \left( \begin{array}{c} \phi, \chi, \ldots, \chi \frac{d-1}{2}, \chi \frac{d+1}{2}, \ldots, \chi \frac{d-1}{2}, \\ \psi, \ldots, \psi \frac{d-1}{2}, \varepsilon, \psi \frac{d+1}{2}, \ldots, \psi \frac{d-1}{2} \end{array} \right) , \]
where $\chi$ and $\psi$ are characters of order $d$ and $d-1$, respectively. Finally, combining (6.8) and (6.10), and then replacing $1/t$ by $t$ we deduce the desired summation identity. This completes the proof of the theorem. \hfill \Box

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