Cartier’s first theorem for Witt vectors on $\mathbb{Z}_{\geq 0}^n - 0$

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Abstract. We show that the dual of the Witt vectors on $\mathbb{Z}_{\geq 0}^n - 0$ as defined by Angeltveit, Gerhardt, Hill, and Lindenstrauss represent the functor taking a commutative formal group $G$ to the maps of formal schemes $\hat{\mathbb{A}}^n \to G$, and that the Witt vectors are self-dual for $\mathbb{Q}$-algebras or when $n = 1$.

1. Introduction

Hesselholt and Madsen computed the relative $K$-theory of $k[\![x]/\!(x^a)\!]$ for $k$ a perfect field of positive characteristic in [HM], and give the answer in terms of the Witt vectors of $k$. In the analogous computation for the ring

$$A = k[\![x_1, \ldots, x_n]/\!(a_1^{x_1}, \ldots, x_n^{a_n})\!]$$

Angeltveit, Gerhardt, Hill, and Lindenstrauss define an $n$-dimensional version of the Witt vectors, which they use to express the relative $K$-theory and topological cyclic homology of $A$ [AGHL].

We show that the Cartier dual of the additive group underlying their Witt vectors on the truncation set $\mathbb{Z}_{\geq 0}^n - 0$, denoted $\mathbb{W}_{\mathbb{Z}_{\geq 0}^n - 0}$, represents the functor taking a commutative formal group $G$ to the pointed maps of formal schemes $\hat{\mathbb{A}}^n \to G$ (Theorem 2.2). We also show that the additive group of $\mathbb{W}_{\mathbb{Z}_{\geq 0}^n - 0}$ is self dual (Lemma 2.3) when $n = 1$ or $R$ is a $\mathbb{Q}$-algebra. Combining these results implies that the additive formal group of $\mathbb{W}_{\mathbb{Z}_{\geq 0}^n - 0}$ represents the functor sending $G$ to the group of maps $\hat{\mathbb{A}}^n \to G$ when $n = 1$ or $R$ is a $\mathbb{Q}$-algebra. The case of $n = 1$ is Cartier’s first theorem [C] [H Th. 27.1.14] on the classical Witt vectors.

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2. Cartier’s first theorem for Witt vectors on $\mathbb{Z}_{\geq 0}^n - 0$

Here is Angeltveit, Gerhardt, Hill, and Lindenstrauss’s $n$-dimensional version of the Witt vectors, defined in Section 2 of [AGHL]: a set $S \subseteq \mathbb{Z}_{\geq 0}^n - 0$ is a truncation

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set if \( \{k_1, k_2, \ldots, k_n\} \) in \( S \) for \( k \in \mathbb{N} = \mathbb{Z}_{>0} \) implies that \( \{j_1, j_2, \ldots, j_n\} \) is in \( S \).

For \( \vec{f} = (j_1, \ldots, j_n) \) in \( \mathbb{Z}_{\geq 0}^n - 0 \), let \( \gcd(\vec{f}) \) denote the greatest common divisor of the non-zero \( j_i \). Given a ring \( R \) and a truncation set \( S \), let the Witt vectors \( \mathbb{W}_S(R) \) be the ring with underlying set \( R^S \) and addition and multiplication defined so that the ghost map

\[
\mathbb{W}_S(R) \to R^S
\]

that takes \( \{r_I : I \in S\} \) to \( \{w_I : I \in S\} \) where

\[
w_I = \sum_{k_I = I} \gcd(\vec{f}) r_I^k
\]

is a ring homomorphism, where in the above sum, \( k \) ranges over \( \mathbb{N} \) and \( \vec{f} \) is in \( S \). In \textbf{AGHL}, one requires \( S \) to be a subset of \( \mathbb{N}^n \), but the same proof that there is a unique functorial way to define such a ring structure \textbf{AGHL} Lem 2.3 holds for \( S \subseteq \mathbb{Z}_{\geq 0}^n - 0 \). Note that

\[
\mathbb{W}_S(R) = \prod_{Z \subseteq \{1, \ldots, n\}} \mathbb{W}_{S_Z}(R)
\]

where \( S_Z \) is defined \( S_Z = \{(j_1, \ldots, j_n) \in S : j_i = 0 \text{ if and only if } i \in Z\} \), and that for \( S = \mathbb{Z}_{\geq 0}^n - 0 \), we have \( \mathbb{W}_{S_Z}(R) \cong \mathbb{W}_{S_t}(R) \) with \( m = n - |Z| \).

Let \( R \) be a ring. For any truncation set \( S \), the additive group underlying the ring \( \mathbb{W}_S(R) \) determines a commutative group scheme and formal group over \( R \).

Let \( \hat{\mathbb{A}}^n = \text{Spf} R[[t_1, t_2, \ldots, t_n]] \) be formal affine \( n \)-space and consider \( \hat{\mathbb{A}}^n \) as a pointed formal scheme, equipped with the point \( \text{Spf} R \to \hat{\mathbb{A}}^n \) corresponding to the ideal \( (t_1, \ldots, t_n) \). Let \( \text{Mor}_{\mathbb{A}}(\hat{\mathbb{A}}^n, G) \) denote the morphisms of pointed formal schemes over \( R \) from \( \hat{\mathbb{A}}^n \) to a pointed formal \( R \)-scheme \( G \). The identity of a formal group \( G \) gives \( G \) the structure of a pointed formal scheme.

For commutative formal groups \( G_1 \) and \( G_2 \) over \( R \), let \( \text{Mor}_{\mathbb{A}}(G_1, G_2) \) denote the corresponding morphisms.

**Theorem 2.1.** Suppose \( R \) is a \( \mathbb{Q} \)-algebra or \( n = 1 \). The additive formal group of \( \mathbb{W}_{\mathbb{Z}_{\geq 0}^n - 0}(R) \) represents the functor

\[
G \mapsto \text{Mor}_{\mathbb{A}}(\hat{\mathbb{A}}^n, G)
\]

from commutative formal groups over \( R \) to groups, i.e. there is a natural identification

\[
\text{Mor}_{\mathbb{A}}(\mathbb{W}_{\mathbb{Z}_{\geq 0}^n - 0}(R), G) \cong \text{Mor}_{\mathbb{A}}(\hat{\mathbb{A}}^n, G)
\]

for commutative formal groups \( G \) over \( R \).

Theorem 2.1 is proven by combining Theorem 2.2 and Lemma 2.6 below.

Cartier duality gives a contravariant equivalence between certain topological \( R \)-algebras and \( R \)-coalgebras \textbf{H} Prop 37.2.7. For such a topological \( R \)-algebra (respectively coalgebra) \( B \), let \( B^* \) denote its Cartier dual

\[
B^* = \text{Mor}_R(B, R)
\]

where \( \text{Mor}_R(B, R) \) denotes the continuous \( R \)-module homomorphisms from \( B \) to \( R \) (respectively the \( R \)-module homomorphisms from \( B \) to \( R \)). Say that an algebra or coalgebra is augmented if it is equipped with a splitting of the unit or counit map. It is straightforward to see that Cartier duality induces an equivalence between

\[\quad\]
augmented topological $R$-algebras satisfying the conditions of [H 37.2.4] and augmented $R$-coalgebras satisfying the conditions of [H 37.2.5]. Denote the morphisms in the former category by $\text{Mor}_{\text{top alg}} (−, −)$ and the morphisms in the latter category by $\text{Mor}_{\text{coalg}} (−, −)$.

The commutative formal group scheme determined by the additive group underlying $W_S(R)$ has a Cartier dual $W_S(R)^*$ which is a topological Hopf algebra or formal group.

**Theorem 2.2.** The Cartier dual of the additive group scheme of $W_{\geq 0}^0(R)$ represents the functor

$$G \mapsto \text{Mor}_{\text{fs}}(\hat{A}^n, G)$$

from commutative formal groups over $R$ to groups, i.e. there is a natural identification

$$\text{Mor}_{\text{fs}}(W_{\geq 0}^0(R)^*, G) \cong \text{Mor}_{\text{fs}}(\hat{A}^n, G)$$

for commutative formal groups $G$ over $R$.

**Proof.** First assume that the formal group $G$ is affine. Let $A$ denote the functions of $G$, so $A$ is a Hopf algebra and $G = \text{Spf } A$.

$$\text{Mor}_{\text{fs}}(\hat{A}^n, G) = \text{Mor}_{\text{top alg}}(A, \mathbb{R}[[t_1, t_2, \ldots, t_n]])$$

By Cartier duality,

$$\text{Mor}_{\text{top alg}}(A, \mathbb{R}[[t_1, t_2, \ldots, t_n]]) = \text{Mor}_{\text{coalg}}(\mathbb{R}[[t_1, t_2, \ldots, t_n]]^*, A^*)$$

Let $F$ denote the left adjoint to the functor taking a Hopf algebra (as defined [H 37.1.7]) to its underlying augmented coalgebra. Since $A$ is a Hopf algebra, so is $A^*$. Therefore,

$$\text{Mor}_{\text{coalg}}(\mathbb{R}[[t_1, t_2, \ldots, t_n]]^*, A^*) = \text{Mor}_{\text{Hopf alg}}(F(\mathbb{R}[[t_1, t_2, \ldots, t_n]]^*), A^*) = \text{Mor}_{\text{Hopf alg}}(A, F(\mathbb{R}[[t_1, t_2, \ldots, t_n]]^*)) = \text{Mor}_{\text{Hopf alg}}(\text{Spf } F(\mathbb{R}[[t_1, t_2, \ldots, t_n]]^*), G)$$

where $\text{Mor}_{\text{Hopf alg}} (−, −)$ denotes morphisms of topological Hopf algebras whose underlying topological $R$-algebra is as before.

By Lemma [2.3] proven below, the formal group $\text{Spf } F(\mathbb{R}[[t_1, t_2, \ldots, t_n]]^*)$ is isomorphic to the Cartier dual of the additive group scheme of $W_{\geq 0}^0(R)$.

Thus $W_{\geq 0}^0(R)^*$ represents the functor $G \mapsto \text{Mor}_{\text{fs}}(\hat{A}^n, G)$ restricted to affine commutative formal groups $G$. Since $W_{\geq 0}^0(R)^*$ is an affine formal group, the identity morphism determines an element of $\text{Mor}_{\text{fs}}(\hat{A}^n, W_{\geq 0}^0(R)^*)$, which in turn defines a natural transformation

$$\eta : \text{Mor}_{\text{fs}}(W_{\geq 0}^0(R)^*, −) \to \text{Mor}_{\text{fs}}(\hat{A}^n, −).$$

For any formal group $G$, the sets $\text{Mor}_{\text{fs}}(W_{\geq 0}^0(R)^*, G)$ and $\text{Mor}_{\text{fs}}(\hat{A}^n, G)$ extend to sheaves on $\text{Spf } R$. Since locally on $\text{Spf } R$, every formal group $G$ is affine, $\eta$ is a natural isomorphism.

**Lemma 2.3.** The group scheme determined by the Hopf algebra

$$F(\mathbb{R}[[t_1, t_2, \ldots, t_n]]^*)$$

is isomorphic to the additive group scheme of $W_{\geq 0}^0(R)$. 

\qed
Proof. For notational convenience, given \( \vec{I} = (i_1, i_2, \ldots, i_n) \) and \( \vec{J} = (j_1, \ldots, j_n) \) in \( \mathbb{Z}_{\geq 0}^n \), let \( t^\vec{I} = t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n} \), and write \( \vec{I} \leq \vec{J} \) when \( i_k \leq j_k \) for all \( k \).

\( \mathbb{R}[[t_1, t_2, \ldots, t_n]]^* \) is a free \( \mathbb{R} \)-module on the basis \( \{ b_\vec{I} : \vec{I} = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}_{\geq 0}^n \} \) where \( b_\vec{I} \) is dual to \( t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n} \). The \( \mathbb{R} \)-co-algebra structure is given by the comultiplication

\[
(2.1) \quad b_\vec{I} \mapsto \sum_{0 \leq f \leq \vec{I}} b_f \otimes b_{\vec{I}-f},
\]

and the augmentation \( \mathbb{R} \to \mathbb{R}[[t_1, t_2, \ldots, t_n]]^* \) sends \( r \) to \( rb_\vec{0} \).

It follows that \( F(\mathbb{R}[[t_1, t_2, \ldots, t_n]]^*) \) is the polynomial algebra

\[
\mathbb{R}[b_\vec{I} : \vec{I} \in \mathbb{Z}_{\geq 0}^n]/(b_\vec{0} - 1)
\]

with comultiplication equal to the \( \mathbb{R} \)-algebra morphism determined by \( (2.1) \). Thus, for any \( \mathbb{R} \)-algebra \( B \)

\[
\text{Mor}_{\text{alg}}(F(\mathbb{R}[[t_1, t_2, \ldots, t_n]]^*), B)
\]

is the group under multiplication of power series in \( \mathbb{n} \) variables \( t_1, t_2, \ldots, t_n \) with leading coefficient 1 and coefficients in \( B \)

\[
(2.2) \quad \{ 1 + \sum_{\vec{I} \in \mathbb{Z}_{\geq 0}^n-0} b_\vec{I} t^\vec{I} : b_\vec{I} \in B \}.
\]

Any such power series can be written uniquely in the form

\[
(2.3) \quad \prod_{\vec{I} \in \mathbb{Z}_{\geq 0}^n-0} (1 - a_\vec{I} t^\vec{I})
\]

with \( a_\vec{I} \in B \). It follows that \( F(\mathbb{R}[[t_1, t_2, \ldots, t_n]]^*) \) is isomorphic as a Hopf algebra to the polynomial algebra \( \mathbb{R}[a_\vec{I} : \vec{I} \in \mathbb{Z}_{\geq 0}^n - \vec{0}] \) with comultiplication determined by multiplication of power series of the form \( (2.3) \). By the definition of the Witt vectors, it suffices to show that the Witt polynomials \( \sum_{k\vec{I}=\vec{I}} \gcd(\vec{J})a_\vec{I}^k \) are primitives for this comultiplication for all \( \vec{I} \) in \( \mathbb{Z}_{\geq 0}^n - \vec{0} \). To show this, we may assume that \( \mathbb{R} \) is a free ring, since every ring is a quotient of a free ring. Then \( \mathbb{R} \) embeds into its field of fractions, so we may further assume that \( k \) is invertible for all \( k \in \mathbb{Z}_{>0} \).

Note that

\[
\log \prod_{\vec{I} \in \mathbb{Z}_{\geq 0}^n-0} (1 - a_\vec{I} t^\vec{I}) = - \sum_{\vec{I} \in \mathbb{Z}_{\geq 0}^n-0} \sum_{k \in \mathbb{N}} \frac{a_\vec{I}^k}{k} t^{k\vec{I}}
\]

\[
= - \sum_{\vec{I} \in \mathbb{Z}_{\geq 0}^n-0} \sum_{k\vec{I}=\vec{I}} \frac{a_\vec{I}^k}{k} t^{\vec{I}}
\]

\[
= \sum_{\vec{I} \in \mathbb{Z}_{\geq 0}^n-0} \left( \sum_{k\vec{I}=\vec{I}} \gcd(\vec{J})a_\vec{I}^k \right) \frac{-t^{\vec{I}}}{\gcd(\vec{I})}.
\]

Thus the group under multiplication with elements \( (2.3) \) is isomorphic to the group with elements \( (a_\vec{I} \in B) \) and whose group operation is such that

\[
(\sum_{k\vec{I}=\vec{I}} \gcd(\vec{J})a_\vec{I}^k)
\]
is an additive homomorphism, i.e. the Witt polynomials $\sum_{k \geq 1} \gcd(j) a_j^k$ are indeed primitives as desired.

The additive group scheme of $\mathcal{W}_{\mathbb{Z}_{\geq 0} - 0}(R)$ corresponds to a graded Hopf algebra, meaning that there is a grading on the underlying $R$-module such that the structure maps are maps of graded $R$-modules. This grading can be defined by giving $a_j$ as in Lemma 2.3 degree $j_1 + j_2 + \ldots + j_n$. A graded Hopf algebra $B$ whose underlying graded $R$-module is free and finite rank in each degree has a graded Hopf algebra dual $B^*$ which we define to have $m$th graded piece $\text{Gr}_m B^* = \text{Hom}_R(\text{Gr}_m B, R)$ and $B^* = \oplus_m \text{Gr}_m B^*$.

Note the difference with the Cartier dual

$B^* = \prod_m \text{Gr}_m B^*$.

Say that a graded Hopf algebra $B$ is self-dual if there is an isomorphism $B \cong B^*$. An affine group scheme corresponding to a graded Hopf algebra will be called self-dual if its corresponding graded Hopf algebra is self-dual.

**Lemma 2.4.** The graded additive group scheme of $\mathcal{W}_{\mathbb{Z}_{\geq 0} - 0}(R)$ is self-dual if $R$ is a $\mathbb{Q}$-algebra or if $n = 1$.

**Proof.** We give an isomorphism of graded Hopf algebras

$$F(\mathbb{R}[[t_1, t_2, \ldots, t_n]]) \cong F(\mathbb{R}[[t_1, t_2, \ldots, t_n]]^*)$$

which is equivalent to the claim by Lemma 2.3.

We saw above that $F(\mathbb{R}[[t_1, t_2, \ldots, t_n]]^*)$ is the polynomial algebra

$$R[\bar{b}_{\bar{i}} : \bar{i} \in \mathbb{Z}_{\geq 0}^n] / (b_0 - 1)$$

with comultiplication determined by (2.1). Thus, an $R$-basis for $F(\mathbb{R}[[t_1, t_2, \ldots, t_n]]^*)$ is given by the collection of monomials $b_{\bar{i}}^{m_1} b_{\bar{i}_2}^{m_2} b_{\bar{i}_3}^{m_3} \cdots b_{\bar{i}_k}^{m_k}$ in the variables $\{b_{\bar{i}} : \bar{i} \in \mathbb{Z}_{\geq 0}^n - 0\}$. Let $C = \{c_{\bar{i}_1, \bar{i}_2, \bar{i}_3, \ldots, \bar{i}_k} : m_j > 0, \bar{i}_j \in \mathbb{Z}_{\geq 0}^n - 0\}$ denote the dual basis of $F(\mathbb{R}[[t_1, t_2, \ldots, t_n]]^*)^*$. For notational convenience, we will also write $c_{\bar{i}_1, \bar{i}_2, \bar{i}_3, \ldots, \bar{i}_k}$ even when some of the $m_j$ are 0; it is to be understood that such an expression is identified with the corresponding expression with the $\bar{i}_j$th terms with $m_j = 0$ removed.

Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{Z}_n$, so $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$ etc. For notational convenience, for $M = (m_1, m_2, \ldots, m_n)$ in $\mathbb{Z}_{\geq 0}^n - 0$, let $C_M$ abbreviate $c_{e_1^{m_1}, e_2^{m_2}, \ldots, e_n^{m_n}}$.

Note that

$$\mu(C_M) = \sum_{0 \leq \bar{j} \leq \bar{M} \bar{i}} C_{\bar{j}} \otimes C_{\bar{M} - \bar{j}}$$

where $\mu$ denotes the comultiplication of $F(\mathbb{R}[[t_1, t_2, \ldots, t_n]]^*)^*$.

Sending $b_{\bar{i}}$ to $C_{\bar{i}}$ thus defines a morphism of Hopf algebras

$$F(\mathbb{R}[[t_1, t_2, \ldots, t_n]]^*) \to F(\mathbb{R}[[t_1, t_2, \ldots, t_n]]^*)^*,$$

and to prove the lemma it suffices to see that the $C_{\bar{i}}$ are free $R$-algebra generators of $F(\mathbb{R}[[t_1, t_2, \ldots, t_n]]^*)^*$ when either $n = 1$ or $\mathbb{Q} \subseteq R$.

We first show that the $C_{\bar{i}}$ generate $F(\mathbb{R}[[t_1, t_2, \ldots, t_n]]^*)^*$ as an $R$-algebra in both cases:
There is an induced map which is determined by the following calculation of where the sum runs over non-negative \(d\) that the subalgebra generated by the \(F\) and we identify and length greater than \(k\) is in the subalgebra. So we may assume by induction that any element of \(C\) of degree \(d\) and length greater than \(k\) is in the subalgebra. The multiplication on \(F(\{t_1\})^*\) is dual to

\[
b_{i_1}b_{i_2}b_{i_3} \cdots b_{i_k} \mapsto \prod_{j=1}^k (\sum_{0 \leq j \leq i_j} b_j \otimes b_{i_j-j}).
\]

Thus the difference

\[
c - c_{i_1-1,i_2-1,\ldots,i_k-1}C_k
\]

is a sum of terms of degree \(d\) and length greater than \(k\). It follows by induction that the \(C_m = c_{e_{m}}^\ast\) generate \(F(\{t_1\})^*\) as claimed.

Now let \(n\) be arbitrary. Consider the map \(f : R[\{t\}] \to R[[t_1, \ldots, t_n]]\) defined by

\[
f(t) = t_1 + t_2 + \ldots + t_n.
\]

There is an induced map

\[
f : F(R[[t_1, \ldots, t_n]]^*) \cong R[b_\Gamma : \bar{\Gamma} \in \mathbb{Z}_{\geq 0}^n - \emptyset] \to R[b_m : m \in \mathbb{Z}_{>0}] \cong F(\{t\})^*
\]

which is determined by the following calculation of \(f(b_\Gamma)\) for \(\bar{\Gamma} = (i_1, i_2, \ldots, i_n).

\[
f(b_\Gamma)(t^m) = b_\Gamma(f(t^m)) = b_\Gamma(t_1 + \ldots + t_n)^m =
\]

\[
b_\Gamma\left(\sum_{a_1, \ldots, a_n \geq 0} \left(\begin{array}{c} m \\ a_1 a_2 \cdots a_n \end{array}\right) t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}\right),
\]

where the sum runs over non-negative \(a_i\) whose sum is \(m\) and where

\[
\left(\begin{array}{c} m \\ a_1 a_2 \cdots a_n \end{array}\right) = \frac{m!}{a_1! a_2! \cdots a_n!}.
\]

Thus

\[
f(b_\Gamma) = \left(\begin{array}{c} d \\ i_1 i_2 \cdots i_n \end{array}\right) b_d,
\]

where \(d = \sum_{j=1}^n i_j\). There is likewise an induced map

\[
f : F(R[[t]]^*) \to F(R[[t_1, t_2, \ldots, t_n]]^*)^*,
\]

and we identify

\[
F(\{t\})^* \cong R[c_{i_1,i_2,\ldots,i_k} : i_j \in \mathbb{Z}_{>0}]
\]

and

\[
F(\{t_1, t_2, \ldots, t_n\})^* \cong R[c_{i_1,i_2,\ldots,i_k} : \bar{\Gamma} \in \mathbb{Z}_{\geq 0}^n - \emptyset].
\]

By calculation as above, this map satisfies

\[
f(C_m) = \sum_{\text{degree } \bar{\Gamma} = m} C_\bar{\Gamma}
\]
where the sum runs over \( \bar{I} \in \mathbb{Z}_{\geq 0}^n \) of degree \( m \), and

\[
f(c_m) = \sum_{\text{degree } \bar{I} = m} \binom{m}{\bar{I}} c_{\bar{I}},
\]

where

\[
\binom{m}{\bar{I}} = \begin{pmatrix} m \\ i_1, i_2, \ldots, i_n \end{pmatrix}
\]

when \( \bar{I} = (i_1, i_2, \ldots, i_n) \). By the \( n = 1 \) case, \( f(c_m) \) is in the \( R \)-subalgebra generated by the \( f(C_m) \). Since \( F[R[[t_1, t_2, \ldots, t_n]]]^* \) is a \( \mathbb{Z}_n \)-graded Hopf algebra, it follows that the homogenous pieces of \( f(c_m) \) are in the \( R \)-subalgebra generated by the homogeneous pieces of \( f(C_m) \). Thus \( \binom{m}{\bar{I}} c_{\bar{I}} \) is in the \( R \)-subalgebra generated by the \( C_{\bar{I}} \). Since \( \binom{m}{\bar{I}} \) is invertible in \( R \), it follows that \( c_{\bar{I}} \) is in this subalgebra.

An arbitrary element \( c \) of \( C \) is of the form \( c_{\bar{I}_1 \bar{I}_2 \cdots \bar{I}_k} \). The multiplication on \( F[R[[t_1, t_2, \ldots, t_n]]]^* \) is dual to

\[
b_{\bar{I}_1} b_{\bar{I}_2} b_{\bar{I}_3} \cdots b_{\bar{I}_k} \mapsto \prod_{j=1}^k \left( \sum_{0 \leq f \leq \bar{I}_j} b_f \otimes b_{\bar{I}_j - f} \right).
\]

It follows that the difference \( c - c_{\bar{I}_1 \bar{I}_2 \cdots \bar{I}_{k-1}} c_{\bar{I}_k} \) is a linear combination of elements of \( C \) of length less than \( k \). Thus \( c \) is in the \( R \)-subalgebra generated by the \( C_{\bar{I}} \) by induction on the length \( k \).

We now show that there are no relations among the \( C_{\bar{I}} \), i.e. that the distinct monomials \( C_{\bar{I}_1} C_{\bar{I}_2} \cdots C_{\bar{I}_k} \) form an \( R \)-linearly independent subset of

\[F[R[[t_1, t_2, \ldots, t_n]]]^*:\]

Fix \( \bar{M} \) in \( \mathbb{Z}_{\geq 0}^n \). Let \( \mathcal{I} \) denote the set of finite sets \( \{\bar{I}_1, \bar{I}_2, \ldots, \bar{I}_k\} \) with \( \bar{I}_1 \) in \( \mathbb{Z}_{\geq 0}^n \) and \( \sum_{i=1}^k \bar{I}_i = \bar{M} \). For \( S \) in \( \mathcal{I} \) with \( S = \{\bar{I}_1, \bar{I}_2, \ldots, \bar{I}_k\} \), let \( C_S = \prod_{i=1}^k C_{\bar{I}_i} \) in \( F[R[[t_1, t_2, \ldots, t_n]]]^* \) and let \( c_S = c_{\bar{I}_1 \bar{I}_2 \cdots \bar{I}_k} \) in \( C \). Note that for all \( S \) in \( \mathcal{I} \), \( C_S \) is in the sub-\( R \)-module \( F_{\bar{M}} \) spanned by \( \{c_S : S \in \mathcal{I}\} \). By the above, \( \{C_S : S \in \mathcal{I}\} \) spans \( F_{\bar{M}} \). Since \( F_{\bar{M}} \) is isomorphic to \( \mathbb{R}^N \) where \( N \) is the (finite) cardinality of \( \mathcal{I} \), any spanning set of size \( N \) is also a basis \([\mathbb{A}, \mathbb{M}] \) Ch 3 Exercise 15). In particular \( \{C_S : S \in \mathcal{I}\} \) is an \( R \)-linearly independent set. Since any monomial in the \( C_{\bar{I}} \) is of the form \( C_S \) for some \( \bar{M} \), it follows that the distinct monomials in the \( C_{\bar{I}} \) form a linearly independent set.

\[\square\]

Remark 2.5. The \( C_{\bar{I}} \) do not generate \( F[R[[t_1, t_2, t_3]]]^* \) when \( 2 \) is not invertible in \( R \) as can be checked by computing that the homogenous degree-(1,1,1) component of the \( R \)-subalgebra generated by the \( C_{\bar{I}} \) is the span of the following five vectors

\[
\begin{align*}
C_{e_1} C_{e_2} C_{e_3} &= c_{(1,1,1)} + c_{(1,1,0)(0,0,1)} + c_{(1,0,1)(1,0,0)} + c_{(0,1,1)(1,0,0)} + c_{e_1 e_2 e_3}, \\
C_{(0,1,1)} C_{e_1} &= c_{(1,1,0)(0,0,1)} + c_{(1,0,1)(1,0,0)} + c_{e_1 e_2 e_3}, \\
C_{(1,0,1)} C_{e_2} &= c_{(0,1,1)(1,0,0)} + c_{(1,1,0)(0,0,1)} + c_{e_1 e_2 e_3}, \\
C_{(1,1,0)} C_{e_3} &= c_{(1,0,1)(1,0,0)} + c_{(0,1,1)(1,0,0)} + c_{e_1 e_2 e_3}, \\
C_{(1,1,1)} &= c_{e_1 e_2 e_3}.
\end{align*}
\]
Lemma 2.6. If $R$ is a $\mathbb{Q}$-algebra or if $n = 1$, the Cartier dual of the additive group scheme of $\mathbb{W}_{\mathbb{Z}_{\geq 0}}(R)$ is the formal group associated to the additive group of $\mathbb{W}_{\mathbb{Z}_{\geq 0}}(R)$.

Proof. By Lemma 2.3, the claim is equivalent to showing that the topological Hopf algebra $F(R[[t_1, t_2, \ldots, t_n]])^*$ is the ring of functions of the formal group associated to the additive group of $\mathbb{W}_{\mathbb{Z}_{\geq 0}}(R)$.

The Cartier dual $F(R[[t_1, t_2, \ldots, t_n]])^*$ of the Hopf algebra $F(R[[t_1, t_2, \ldots, t_n]]^*)$ is the product

$$F(R[[t_1, t_2, \ldots, t_n]]^*) = \prod_{m=0}^{\infty} \text{Gr}_m F(R[[t_1, t_2, \ldots, t_n]]^*)$$

over $m$ of the $m$th graded pieces of the graded Hopf algebra dual. By Lemma 2.4

$$F(R[[t_1, t_2, \ldots, t_n]]^*) = F(R[[t_1, t_2, \ldots, t_n]]^*) \cong R[\bar{b}^\vec{I} : \vec{I} \in \mathbb{Z}_{\geq 0}^n]/(\bar{b}^0 - 1),$$

with comultiplication determined by (2.1). So

$$\prod_{m=0}^{\infty} \text{Gr}_m F(R[[t_1, t_2, \ldots, t_n]]^*) = R[\bar{b}^\vec{I} : \vec{I} \in \mathbb{Z}_{\geq 0}^n]/(\bar{b}^0 - 1),$$

and applying Lemma 2.3 completes the proof.

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