$L^p$-estimates for the Schrödinger equation associated to the harmonic oscillator

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Abstract: In this paper we obtain some Strichartz estimates for the Schrödinger equation associated to the harmonic oscillator and the Laplacian. Our main tool will be some embeddings between Lebesgue spaces and suitable Triebel-Lizorkin spaces. MSC 2010. Primary: 42B35, Secondary: 42C10, 35K15. To appear in Electron. J. Differential Equations. Received: Feb 6-2018; Accepted: Aug 8-2018.

1. Introduction

Let us consider the quantum harmonic oscillator $H := -\Delta + |x|^2$ on $\mathbb{R}^n$ where $\Delta$ is the standard Laplacian. In this paper we obtain regularity for the Schrödinger equation (associated to $H$) given by

$$iu_t(t, x) - Hu(x, t) = 0,$$

with initial data $u(0, \cdot) = f$. As it is well known, this is an important model in quantum mechanics (R. P. Feynman, and A.R. Hibbs, [6]). As consequence of such estimates we also provide estimates for the classical Schrödinger equation

$$iu_t(t, x) + \Delta u(x, t) = 0.$$  (2)

The regularity for the problem (1) has been extensively developed, some works on the subject are S. Thangavelu [17, Section 5], B. Bongioanni and J. L. Torrea [2], B. Bongioanni and K. M. Rogers [3] and K. Yajima [19] and references therein. On the other hand, regularity properties for (2) can be found in the seminal work of J. Ginibre and G. Velo [8], and in A. Moyua and L. Vega [9], M. Keel and T. Tao [11] and references therein. The works L. Carleson [4] and B. Dahlberg and C. Kenig [5] include pointwise convergence theorems for the solution $u(x, t) = e^{it\Delta}f$.

It was proved in [9] the following sharp theorem: for $\frac{2(n+2)}{n} \leq p \leq \infty$, and $2 \leq q < \infty$ with $\frac{1}{q} \leq \frac{2}{p} \left( \frac{1}{2} - \frac{1}{p} \right)$,

$$\|u(t, x)\|_{L^p_t(\mathbb{R}^n, L^q_t([0,2\pi]))} \leq C_s\|f\|_{\mathcal{H}^s(\mathbb{R}^n)}$$

holds true for all $s \geq s_{n,p,q} := n\left( \frac{1}{2} - \frac{1}{p} \right) - \frac{2}{q}$. If $s < s_{n,p,q}$ then (3) is false. In the result above $\mathcal{H}^s$ is the Sobolev space associated to $H$ and with norm $\|f\|_{\mathcal{H}^s} := \|H^{s/2}f\|_{L^2}$. The proof of (3) involves Strichartz estimates of M. Keel and T. Tao [11] and Wainger’s Sobolev embedding theorem. It is important to mention that the machinery of the work M. Keel and T. Tao [11] implies the following estimate

$$\|u(t, x)\|_{L^p_t([0,2\pi], L^p\mathcal{H}^{s}(\mathbb{R}^n))} \leq C_p\|f\|_{L^2(\mathbb{R}^n)},$$  (4)
for $2 \leq q < \infty$ and $\frac{1}{q} = \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$, excluding the case $(p, q, n) = (\infty, 2, 2)$. On the other hand, H. Koch and D. Tataru have proved the estimate (4) for Schrödinger type operators in more general contexts, but including the operator $H$, and they have proved that estimates of this type can be not obtained for $2 \leq p < \frac{2n}{n-2}$.

A remarkable formula that links the solution of (1) to that of the classical Schrödinger equation (see P. Sjögren, and J.L. Torrea [16]) is the following

$$\|e^{-it((-(\Delta+|x|^2))f)}\|_{L^q([0,\frac{4}{3}),L^p(R^d))] = \|e^{it\Delta}f\|_{L^q([0,\infty),L^p(R^d)]}$$

for $1 \leq p, q \leq \infty$ and $\frac{2}{q} = n(\frac{1}{2} - \frac{1}{p})$. As it was pointed out in [16], the interval of integration in the $t$ variable is now bounded, (4) remains true if the equality in (5) is replaced by the inequality $n(\frac{1}{2} - \frac{1}{p}) \leq \frac{2}{q}$, and the interval $(0, \frac{4}{3})$ can be replaced by $(0, \frac{2}{3})$ and in a such case the two norms are equivalents for real functions $f$. In particular, (5) shows that (4) is equivalent to the following Strichartz estimate (see [12])

$$\|e^{it\Delta}f\|_{L^q([0,\infty),L^p(R^d)]} \leq C\|f\|_{L^2(R^n)}$$

which holds if and only if $n = 1$ and $2 \leq p \leq \infty$, $n = 2$ and $2 \leq p < \infty$ and $2 \leq p < \frac{2n}{n-2}$ for $n \geq 3$.

The novelty of this paper is that we provide regularity results for the Schrödinger equation associated to $H$, involving $L^p$-Sobolev norms for the initial data instead of the $L^2$ and $L^2$-Sobolev bounds mentioned above. Our main result in this paper is the following theorem.

**Theorem 1.** Let us assume $n > 2$, $2 \leq q < \infty$ and $1 < p < 2$ satisfying $|\frac{1}{2} - \frac{1}{p}| < \frac{1}{2n}$. Then, the following estimate

$$\|u(t, x)\|_{L^p_{\nu}[\mathbb{R}^n, L^q_{\nu}[0,2\pi])] \leq C\|f\|_{W^{2s-p,\nu}(\mathbb{R}^n)}$$

holds true for every $s \geq s_q := \frac{1}{2} - \frac{1}{q}$. In particular, if $q = 2$ we have

$$\|u(t, x)\|_{L^p_{\nu}[\mathbb{R}^n, L^2_{\nu}[0,2\pi])] \leq C\|f\|_{L^p(\mathbb{R}^n)}.$$ (7)

Moreover, for $n > 2$, $1 < p < 2$, and $1 \leq q \leq p'$, we have

$$\|u(t, x)\|_{L^p_{\nu}[\mathbb{R}^n, L^q_{\nu}[0,2\pi])] \leq C\|f\|_{L^p(\mathbb{R}^n)},$$

provided that $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{nq}$.

Now, in the following remarks, we briefly discuss some consequences of our main result.

**Remark 1.** The main contribution in Theorem 1 is the estimate (7) which in particular implies an analogue of the Littlewood-Paley theorem (see [26]). Littlewood-Paley type results can be understood as substitutes of the Plancherel identity on $L^p$-spaces.

**Remark 2.** An important consequence of Theorem 1 is the following estimate

$$\|e^{it\Delta}f\|_{L^q([0,\infty),L^p(R^d))] \leq C\|f\|_{\tilde{W}^{s,p,\nu}(\mathbb{R}^n)}, \quad s \geq s_q,$$

for $2 \leq p \leq q < \infty$, $\frac{2}{q} = n(\frac{1}{2} - \frac{1}{p})$, (see Theorem 4), the inequality

$$\|e^{it\Delta}f\|_{L^q([0,\infty),L^p(R^d))] \leq C\|f\|_{\tilde{W}^{2s-p,\nu}(\mathbb{R}^n)}, \quad s \geq s_q,$$

for $2 \leq p < q < \infty$. (11)
for $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n}, 1 < p < 2, n > 2$ and $\frac{2}{q} = n(\frac{1}{p} - \frac{1}{2})$, (please, let us compare (11) and (4)) as well as the estimate

$$\|f\|_{\mathcal{F}^s_{p,2}(\mathbb{R}^n)} \leq C\|e^{it\Delta}f\|_{L^q([0,\infty),L^p_\nu(\mathbb{R}^n))} \asymp C\|u(t,x)\|_{L^q([0,2\pi],L^p_\nu(\mathbb{R}^n))}$$

(12)

when $2 \leq q \leq p < \infty$ provided that $n(\frac{1}{2} - \frac{1}{p}) = \frac{2}{q}$. In the results above the spaces $\mathcal{F}^s_{p,2}$ are Triebel-Lizorkin spaces associated to $H$ and they will be introduced in the next section.

**Remark 3.** Finally, (10) links our results with those in [11] and [16]. For $\frac{1}{q} = \frac{n}{2}(\frac{1}{2} - \frac{1}{p})$ we show in Corollary 1 that the following estimate

$$\|u(t,x)\|_{L^p_\nu(\mathbb{R}^n),L^q_\nu([0,2\pi])} \leq C_s\|f\|_{L^2(\mathbb{R}^n)}$$

(13)

holds true provided that $n = 1$ and $2 \leq p \leq \infty$, $n = 2$ and $2 \leq p < \infty$ and $2 \leq p < \frac{2n}{n-2}$ for $n \geq 3$. As a consequence of the embedding $\mathcal{H}^s \hookrightarrow L^2$ for $s \geq 0$, the estimate (13) improves (3) in any case above.

This paper is organized as follows. In section 2 we present some basics on the spectral decomposition of the harmonic oscillator and we discuss our analogue of the Littlewood-Paley theorem. Finally, in the last section we provide our regularity results.

## 2. Spectral decomposition of the harmonic oscillator and a result of type Littlewood-Paley

Let $H = -\Delta + |x|^2$ be the Hermite operator or (quantum) harmonic oscillator. This operator extends to an unbounded self-adjoint operator on $L^2(\mathbb{R}^n)$, and its spectrum consists of the discrete set $\lambda_\nu := 2|\nu| + n, \nu \in \mathbb{N}_0^n$, with a set of real eigenfunctions $\phi_\nu, \nu \in \mathbb{N}_0^n$, (called Hermite functions) which provide an orthonormal basis of $L^2(\mathbb{R}^n)$. Every Hermite function $\phi_\nu$ on $\mathbb{R}^n$ has the form

$$\phi_\nu = \Pi_{j=1}^n \phi_{\nu_j}, \quad \phi_{\nu_j}(x_j) = (2^{\nu_j}\nu_j!\sqrt{\pi})^{-\frac{1}{2}} H_{\nu_j}(x_j)e^{-\frac{1}{2}x_j^2}$$

(14)

where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{N}_0^n$, and

$$H_{\nu_j}(x_j) := (-1)^{\nu_j}e^{x_j^2} \frac{d^{\nu_j}}{dx_j^{\nu_j}}(e^{-x_j^2})$$

denotes the Hermite polynomial of order $\nu_j$. By the spectral theorem, for every $f \in \mathcal{D}(\mathbb{R}^n)$ we have

$$Hf(x) = \sum_{\nu \in \mathbb{N}_0^n} \lambda_\nu \hat{f}(\phi_\nu)\phi_\nu(x),$$

(15)

where $\hat{f}(\phi_\nu)$ is the Hermite-Fourier transform of $f$ at $\nu$ defined by

$$\hat{f}(\phi_\nu) := \langle f, \phi_\nu \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(x)\phi_\nu(x) \, dx.$$ 

(16)

The main tool in the harmonic analysis of the harmonic oscillator is the Hermite semigroup, which we introduce as follows. If $P_\ell, \ell \in 2\mathbb{N}_0 + n$, is the projection on $L^2(\mathbb{R}^n)$ given by

$$P_\ell f(x) := \sum_{2|\nu|+n=\ell} \hat{f}(\phi_\nu)\phi_\nu(x),$$

(17)
then, the Hermite semigroup (semigroup associated to the harmonic oscillator) $T_t := e^{-tH}$, $t > 0$ is given by
\[ e^{-tH} f(x) = \sum_{\ell} e^{-\ell t} P_\ell f(x). \] (18)

For every $t > 0$, the operator $e^{-tH}$ has Schwartz kernel given by
\[ K_t(x, y) = \sum_{\nu \in \mathbb{N}_0^n} e^{-t(2|\nu| + n)} \phi_\nu(x) \phi_\nu(y). \] (19)

In view of Mehler’s formula (see Thangavelu [18]) the above series can be summed up and we obtain
\[ K_t(x, y) = (2\pi)^{-\frac{n}{2}} \sinh(2t) e^{-(\frac{1}{4}|x|^2 + |y|^2) \cot(2t) + x \cdot y - \text{csch}(2t)}. \] (20)

In this paper we want to estimate the mixed norms $L^p(L^q)$ of solutions to Schrödinger equations by using the following version of Triebel-Lizorkin associated to $H$.

**Definition 1.** Let us consider $0 < p \leq \infty$, $r \in \mathbb{R}$ and $0 < q \leq \infty$. The Triebel-Lizorkin space associated to $H$, to the family of projections $P_\ell$, $\ell \in 2\mathbb{N} + n$, and to the parameters $p, q$ and $r$ is defined by those complex functions $f$ satisfying
\[
\|f\|_{\mathcal{F}_{p,q}^r} := \left( \sum_{\ell} \ell^r |P_\ell f|^q \right)^{\frac{1}{q}} \lesssim \infty.
\] (21)

The definition considered above differs from those arising with dyadic decompositions (see e.g. [1] and [13]). The following are natural embedding properties of such spaces. $H^s$ denotes the Sobolev space associated to $H$ and defined by the norm $\|f\|_{H^s} := \|H^{s/2} f\|_{L^2}$. Sobolev spaces $W^{2s,p,H}$ in $L^p$-spaces and associated to $H$, can be defined by the norm $\|f\|_{W^{2s,p,H}} := \|H^s f\|_{L^p}$.

1. $\mathcal{F}_{p,1}^r \subset \mathcal{F}_{p,1}^s \subset \mathcal{F}_{p,1}^\infty$, $\varepsilon > 0$, $0 < p \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$.
2. $\mathcal{F}_{p,1}^r \subset \mathcal{F}_{p,2}^s$, $\varepsilon > 0$, $0 < p \leq \infty$, $1 \leq q_2 < q_1 < \infty$.
3. $\mathcal{F}_{2,2}^0 = L^2$ and consequently, for every $s \in \mathbb{R}$, $H^{2s} = \mathcal{F}_{2,2}^s$. Other properties associated to Sobolev spaces of the harmonic oscillator can be found in [1, 2] and [13].

Now we discuss a close relation between $\mathcal{F}_{p,2}^0$ and Lebesgue spaces. If $\psi$ is a smooth function supported in $[\frac{1}{4}, 2]$, such that $\psi = 1$ on $[\frac{1}{4}, 1]$,
\[
\sum_{k=0}^{\infty} \psi_k(t) = 1, \quad \psi_k(t) := \psi(2^{-k} t),
\] (22)

and $A$ is an elliptic pseudo-differential operator on $\mathbb{R}^n$ of order $\nu > 0$, the (dyadic) Triebel-Lizorkin space $F_{p,q,\delta}^\nu(\mathbb{R}^n)$ associated to $A$ is defined by the norm
\[
\|f\|_{F_{p,q,\delta}^\nu} := \| \{ 2^{kr/\nu} \| \psi_k(A) f \|_{L^p} \} \|_{\ell^q}
\] (23)

where $r \in \mathbb{R}$ and $0 < p, q \leq \infty$. For $A = H$ or $A = \Delta_x$ is known the Littlewood-Paley theorem (see [7]) which stands that $F_{p,2}^0(A) = L^p$ for all $1 < p < \infty$. If $A = \Delta_x$, one also have
\[
\left\| \left( \sum_k |1(k,k+1)(\Delta_x) f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p}, \quad 2 < p < \infty,
\] (24)
with \( C \) depending only on \( p \). However, such inequality is false for \( 1 < p < 2 \). For \( \ell \in 2\mathbb{N} + n \), \( P_\ell = 1_{[\ell,\ell+1)}(H) \) and

\[
\|f\|_{\mathcal{F}_{p,2}^0} = \left\| \left( \sum_\ell |1_{[\ell,\ell+1)}(H)f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}.
\] (25)

Although in Remark 4, we explain in detail that we have not a Littlewood-Paley theorem for \( \mathcal{F}_{p,q}^0 \), in the proof of our main theorem we obtain the following estimate for \( 1 < p < 2 \) (see equation (45))

\[
\|f\|_{\mathcal{F}_{p,2}^0} = \left\| \left( \sum_\ell |1_{[\ell,\ell+1)}(H)f|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}(\mathbb{R}^n)} \leq C\|f\|_{L^p}
\] (26)

provided that \( \frac{1}{p} - \frac{1}{2} < \frac{1}{2n} \). Such inequality is indeed, an analogue of (24). An immediate consequence is the estimate:

\[
\|f\|_{\mathcal{F}_{p,2}^0} = \|H^*f\|_{\mathcal{F}_{p,2}^0} \leq C\|H^*f\|_{L^p} =: C\|f\|_{W^{2,p};H}
\] (27)

provided that \( \frac{1}{p} - \frac{1}{2} < \frac{1}{2n} \).

3. Regularity properties

In order to analyze the mixed norms of solutions of the Schrödinger equation we need the following multiplier theorem. The space \( L^2_j(\mathbb{R}^n) \) consists of those finite linear combinations of Hermite functions on \( \mathbb{R}^n \).

**Theorem 2.** Let us assume that \( m \in L^\infty(\mathbb{N}_0) \) is a bounded function. Then the multiplier \( m(H) \) extends to a bounded operator on \( \mathcal{F}_{p,q}^0(\mathbb{R}^n) \) for all \( 0 < p \leq \infty \) and \( 0 < q \leq \infty \). Moreover

\[
\|m(H)\|_{\mathcal{B}(\mathcal{F}_{p,q}^0)} = \|m\|_{L^\infty}.
\] (28)

In particular if \( m := 1_{[0,\epsilon]} \), then \( S_\epsilon = 1_{[0,\epsilon]}(H) \), \( \|S_\epsilon\|_{\mathcal{B}(\mathcal{F}_{p,q}^0)} = 1 \) and

\[
\lim_{\epsilon \to \infty} \|S_\epsilon f - f\|_{\mathcal{F}_{p,q}^0} = 0
\] (29)

uniformly on the \( \mathcal{F}_{p,q}^0 \)-norm.

**Proof.** Let us consider \( f \in \mathcal{F}_{p,q}^0 \). Then, \( P_\ell(m(H)f) = m(\ell)P_\ell f \) and

\[
\|m(H)f\|_{\mathcal{F}_{p,q}^0} = \left\| \left( \sum_\ell |m(\ell)|^q |P_\ell f|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \leq \sup_\ell |m(\ell)|\|f\|_{\mathcal{F}_{p,q}^0}.
\] (30)

As consequence

\[
\|m(H)\|_{\mathcal{B}(\mathcal{F}_{p,q}^0)} \leq \|m\|_{L^\infty}.
\] (31)

Now, for the reverse inequality, let us choose \( f = \phi_\nu, \ell' = 2|\nu| + n \). Then, \( \|m(H)f\|_{\mathcal{F}_{p,q}^0} = |m(\ell')|\|f\|_{\mathcal{F}_{p,q}^0} \) and as consequence \( \|m(H)\|_{\mathcal{B}(\mathcal{F}_{p,q}^0)} \geq \sup_\ell |m(\ell)|. \) The second part is consequence of the uniform boundedness principle. \( \square \)
Lemma 1. Let us consider a function $f \in \mathcal{F}^{0}_{p,2}(\mathbb{R}^{n})$, then for all $0 < p \leq \infty$ we have

$$
\|u(t, x)\|_{L_{t}^{\infty}(\mathbb{R}^{n},L_{x}^{2}[0,2\pi])} = \sqrt{2\pi} \|f\|_{\mathcal{F}^{0}_{p,2}(\mathbb{R}^{n})}.
$$

(32)

Proof. In view of (29) we consider by density, $f \in L_{f}^{2}(\mathbb{R}^{n})$. The solution $u(t, x)$ for (1) is given by

$$
u
u
u
u

\begin{align*}
\|u(t, x)\|_{L_{t}^{2}[0,2\pi]}^{2} &= \sum_{\ell} 2\pi \cdot |P_{\ell}f(x)|^{2},
\end{align*}

(33)

Then, we have (see [9])

in fact, it can be proved by using the orthogonality of trigonometric polynomials. So, we conclude the following fact

$$
\|u(t, x)\|_{L_{t}^{2}[0,2\pi]} = \left(\sum_{\ell} 2\pi \cdot |P_{\ell}f(x)|^{2}\right)^{\frac{1}{2}}, f \in L_{f}^{2}(\mathbb{R}^{n}).
$$

(34)

Consequently

$$
\|u(t, x)\|_{L_{t}^{p}(\mathbb{R}^{n},L_{x}^{2}[0,2\pi])} = \sqrt{2\pi} \|f\|_{\mathcal{F}^{0}_{p,2}(\mathbb{R}^{n})}.
$$

(35)

Lemma 2. Let $0 < p \leq \infty$, $2 \leq q < \infty$ and $s_{q} := \frac{1}{2} - \frac{1}{q}$. Then

$$
C_{p}^{q} \|f\|_{\mathcal{F}^{0}_{p,2}} \leq \|u(t, x)\|_{L_{t}^{p}(\mathbb{R}^{n},L_{x}^{q}[0,2\pi])} \leq C_{p,s} \|f\|_{\mathcal{F}^{0}_{p,2}},
$$

(36)

holds true for every $s \geq s_{q}$.

Proof. We consider, by a density argument, $f \in L_{f}^{2}(\mathbb{R}^{n})$. By following the approach in [3], in order to estimate the norm $\|u(t, x)\|_{L_{t}^{p}(\mathbb{R}^{n},L_{x}^{q}[0,2\pi])}$ we use the Wainger Sobolev embedding Theorem:

$$
\left\| \sum_{\ell \in \mathbb{Z}, \ell \neq 0} |\ell|^{-\alpha} \hat{F}(\ell)e^{-i\ell t} \right\|_{L_{t}^{q}[0,2\pi]} \leq C \|F\|_{L_{t}^{r}[0,2\pi]}, \ \alpha := \frac{1}{r} - \frac{1}{q}.
$$

(37)
Theorem 3. For \( s > s_q := \frac{1}{2} - \frac{1}{q} \) we have

\[
\|u(t, x)\|_{L^q[0, 2\pi]} = \left\| \sum_{\nu \in \mathbb{N}_0^n} e^{-it(2|\nu|+n)} \hat{f}(\phi_\nu) \phi_\nu(x) \right\|_{L^q[0, 2\pi]} = \left\| \sum_{\ell} e^{-it\ell \hat{f}(\phi_\ell)} \right\|_{L^q[0, 2\pi]}
\]

\[
\leq C \left\| \sum_{\ell} \ell^{s_q} e^{-it\ell} P_\ell f(x) \right\|_{L^2[0, 2\pi]} = C \left\| \sum_{\ell} e^{-it\ell} P_\ell [H^{s_q} f(x)] \right\|_{L^2[0, 2\pi]}
\]

\[
= C \left( \sum_{\ell} |P_\ell [H^{s_q} f(x)]|^2 \right) := T'(H^{s_q} f)(x).
\]

So, we have

\[
\|u(t, x)\|_{L^p_{x}[\mathbb{R}^n, L^q_{t}[0, 2\pi]]} \leq C \|T'(H^{s_q} f)\|_{L^p(\mathbb{R}^n)} \leq C_p \|H^{s_q} f\|_{\mathcal{F}_{p,2}(\mathbb{R}^n)} = C_p \|f\|_{\mathcal{F}_{p,2}(\mathbb{R}^n)}.
\] (38)

We end the proof by taking into account the embedding \( \mathcal{F}_{p,2} \hookrightarrow \mathcal{F}_{p,2} \) for every \( s > s_q \) and the following inequality for \( 2 \leq q < \infty \)

\[
\|f\|_{\mathcal{F}_{p,2}} = \frac{1}{\sqrt{2\pi}} \|T'f\|_{L^p} = \frac{1}{\sqrt{2\pi}} \|u(t, x)\|_{L^p_{x}[\mathbb{R}^n, L^q_{t}[0, 2\pi]]} \lesssim \|u(t, x)\|_{L^p_{x}[\mathbb{R}^n, L^q_{t}[0, 2\pi]]}.
\] (39)

Proof. First, we want to prove the case \( q = 2 \) and later we extend the proof for \( 2 < q < \infty \) by using a suitable embedding. Our main tool will be the following dispersive inequality (see [15] pg. 114.)

\[
\|u(t, x)\|_{L^p_x(\mathbb{R}^n)} \leq C|t|^{-\frac{n}{2}-\frac{1}{q'}} \|f\|_{L^q(\mathbb{R}^n)}, \quad 1 < p < 2.
\] (43)

As consequence we have

\[
\|u(t, x)\|_{L^p_x([0, 2\pi], L^q_x(\mathbb{R}^n))} \leq C \|\| t|^{-\frac{n}{2}-\frac{1}{q'}} \|_{L^q_x([0, 2\pi])} \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < 2.
\] (44)
We need \( \frac{1}{p} - \frac{1}{2} < \frac{1}{2n} \) in order that \( \| \| \cdot \| \cdot^{n(1 - \frac{1}{2p})} \|_{L^2[0,2\pi]} < \infty \). Because \( p' \geq 2 \) we can use Minkowski integral inequality in order to obtain

\[
\| f \|_{p',2} = \| u(t,x) \|_{L^p(\mathbb{R}^n,L^2_{\pi}(\mathbb{R}^n))} \leq \| u(t,x) \|_{L^2([0,2\pi],L^p(\mathbb{R}^n))} \lesssim \| f \|_{L^p(\mathbb{R}^n)}. \tag{45}
\]

In fact we have,

\[
\| u(t,x) \|_{L^p(\mathbb{R}^n,L^2_{\pi}(\mathbb{R}^n))}^2 = \left( \int_{\mathbb{R}^n} \left( \int_0^{2\pi} |u(t,x)|^2 dt \right)^{\frac{p'}{2}} dx \right)^{\frac{2}{p'}} \leq \left( \int_{\mathbb{R}^n} \left( \int_0^{2\pi} |u(t,x)|^{p'} dx \right)^{\frac{p'}{2}} dt \right)^{\frac{2}{p'}} =: \| u(t,x) \|_{L^2([0,2\pi],L^p(\mathbb{R}^n))}.
\]

Now (45) can be obtained from (44) for \( 1 < p < 2 \) and \( \frac{1}{p} - \frac{1}{2} < \frac{1}{2n} \). The estimate (45) proves the theorem for \( q = 2 \). The result for \( 2 < q < \infty \) now follows, as in the proof of Theorem 2 by using the Wainger Sobolev embedding Theorem as in (38) together with (27):

\[
\| u(t,x) \|_{L^p(\mathbb{R}^n,L^q_{\pi}(\mathbb{R}^n))} \leq C \| T'(H^{s_q}f) \|_{L^p(\mathbb{R}^n)} \leq C_{p'} \| H^{s_q}f \|_{p',2} \]

\[
= C_{p'} \| f \|_{s',2} \leq C \| f \|_{W^{2q,p}H(\mathbb{R}^n)}.
\]

So, we end the proof of the first announcement. Now, in order to proof (42) we observe that

\[
\| u(t,x) \|_{L^p(\mathbb{R}^n)} \leq C |t|^{-\frac{nq}{2} - \frac{1}{2}} \| f \|_{L^p(\mathbb{R}^n)}, \quad 1 < p < 2,
\]

implies

\[
\| u(t,x) \|_{L^q([0,2\pi],L^p(\mathbb{R}^n))} \leq C \cdot I_{p,n,q} \| f \|_{L^p(\mathbb{R}^n)}, \quad 1 < p < 2,
\]

where

\[
I_{p,n,q} = \left( \int_0^{2\pi} |t|^{-nq\left(\frac{1}{2} - \frac{1}{p}\right)} \right)^{\frac{1}{q}} < \infty
\]

for \( |1/2 - 1/p| < 1/nq \). Since, \( q \leq p' \), by using the Minkowski inequality we have

\[
\| u(t,x) \|_{L^p(\mathbb{R}^n,L^q_{\pi}(\mathbb{R}^n))} \leq \| u(t,x) \|_{L^q([0,2\pi],L^p(\mathbb{R}^n))}
\]

and consequently

\[
\| u(t,x) \|_{L^p(\mathbb{R}^n,L^q_{\pi}(\mathbb{R}^n))} \leq C \| f \|_{L^p}.
\]

Theorem 4. Let us assume, for some \( s \), that \( f \in \mathcal{F}_{s,2}^\alpha(\mathbb{R}^n) \) is a real function and \( u(\cdot,t) = e^{-itH} f(\cdot) \). Let us assume \( 2 \leq p \leq q < \infty \) and \( \frac{2}{q} = n\left(\frac{1}{2} - \frac{1}{p}\right) \). Then, the following estimate

\[
\| e^{it\Delta} f \|_{L^q([0,\infty),L^p(\mathbb{R}^n))} \lesssim \| u(t,x) \|_{L^q([0,2\pi],L^p(\mathbb{R}^n))} \leq C \| f \|_{s',2(\mathbb{R}^n)}, \quad s \geq s_q,
\]

\( \Box \)
holds true. Consequently we have
\[
\|e^{it\Delta}f\|_{L^q([0,\infty),L^p_x(\mathbb{R}^d))} \lesssim \|u(t,x)\|_{L^p([0,2\pi],L^q_x(\mathbb{R}^n))} \leq C\|f\|_{W^{2n,p-1,1}(\mathbb{R}^n)}, \ s \geq s_q, 
\]
(50)
for \(|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n}, 1 < p < 2, n \geq 2\) and \(\frac{2}{q} = n\left(\frac{1}{p} - \frac{1}{2}\right)\). Moreover, for \(2 \leq q \leq p < \infty\) and \(\frac{2}{q} = n\left(\frac{1}{2} - \frac{1}{p}\right)\) we have
\[
\|f\|_{\mathcal{F}^0_{p,2}(\mathbb{R}^n)} \leq C\|e^{it\Delta}f\|_{L^q([0,\infty),L^p_x(\mathbb{R}^n))}, C\|u(t,x)\|_{L^p([0,2\pi],L^q_x(\mathbb{R}^n))}. 
\]
(51)

Proof. From the Minkowski integral inequality applied to \(L^\frac{p}{q}\), we deduce the inequality
\[
\|u(t,x)\|_{L^p([0,2\pi],L^q_x(\mathbb{R}^n))} \leq \|u(t,x)\|_{L^p([\mathbb{R}^n],L^q_x([0,2\pi]))}. 
\]
(52)
In fact,
\[
\|u(t,x)\|_{L^p([0,2\pi],L^q_x(\mathbb{R}^n))} := \left(\int_0^{2\pi} \left(\int_{\mathbb{R}^n} |u(t,x)|^p dx\right)^{\frac{2}{q}} dt\right)^{\frac{1}{\frac{2}{q}}}, 
\]
\[
\leq \left(\int_{\mathbb{R}^n} \left(\int_0^{2\pi} |u(t,x)|^q dt\right)^{\frac{2}{q}} dx\right)^{\frac{1}{\frac{2}{q}}} =: \|u(t,x)\|_{L^p([\mathbb{R}^n],L^q_x([0,2\pi]))}. 
\]

Now, we only need to apply Lemma 2 and the equivalence given by (5). The estimate (50) is consequence of (27) and (49) applied to \(p'\) instead of \(p\). On the other hand, for \(2 \leq q \leq p < \infty\), by using the Minkowski integral inequality on \(L^\frac{p}{q}\) we have
\[
\|f\|_{\mathcal{F}^0_{p,2}(\mathbb{R}^n)} = \|u(t,x)\|_{L^p([\mathbb{R}^n],L^q_x([0,2\pi]))} \lesssim \|u(t,x)\|_{L^p([\mathbb{R}^n],L^q_x([0,2\pi]))} \leq \|u(t,x)\|_{L^p([0,2\pi],L^q_x(\mathbb{R}^n))}. 
\]
(53)
So, by using the equivalence expressed in (5) we obtain
\[
\|f\|_{\mathcal{F}^0_{p,2}(\mathbb{R}^n)} \leq C\|e^{it\Delta}f\|_{L^q([0,\infty),L^p_x(\mathbb{R}^n))} \approx C\|u(t,x)\|_{L^p([0,2\pi],L^q_x(\mathbb{R}^n))} 
\]
which end the proof of the theorem. \(\square\)

Corollary 1. Let \(1 < q \leq p \leq \infty\) and \(\frac{1}{q} = \frac{n}{2}\left(\frac{1}{2} - \frac{1}{p}\right)\). Then,
\[
\|u(t,x)\|_{L^p([\mathbb{R}^n],L^q_x([0,2\pi]))} \leq C_s\|f\|_{L^2(\mathbb{R}^n)}, 
\]
(54)
holds true provided that, \(n = 1\) and \(2 \leq p \leq \infty\), \(n = 2\) and \(2 \leq p < \infty\) and \(2 \leq p < \frac{2n}{n+2}\) for \(n \geq 3\).

Proof. Same as in Theorem 4 by using the Minkowski integral inequality on \(L^\frac{p}{q}\), for \(1 < q \leq p < \infty\), we have the inequality
\[
\|u(t,x)\|_{L^p([\mathbb{R}^n],L^q_x([0,2\pi]))} \leq \|u(t,x)\|_{L^p([0,2\pi],L^q_x(\mathbb{R}^n))}. 
\]
(55)
Finally (54) now follows by using (6) and the equivalence (5). \(\square\)
4. References

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