Minimal energy for the traveling waves of the Landau–Lifshitz equation

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Abstract

We consider nontrivial finite energy traveling waves for the Landau–Lifshitz equation with easy-plane anisotropy. Our main result is the existence of a minimal energy for these traveling waves, in dimensions two, three and four. The proof relies on a priori estimates related with the theory of harmonic maps and the connection of the Landau–Lifshitz equation with the kernels appearing in the Gross–Pitaevskii equation.

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1 Introduction

In this work we consider the Landau–Lifshitz equation

\[ \partial_t m + m \times (\Delta m + \lambda m_3 e_3) = 0, \quad m(t, x) \in S^2, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^N, \] (1.1)

where \( e_3 = (0, 0, 1), \lambda \in \mathbb{R} \) and \( m = (m_1, m_2, m_3) \). This equation was originally introduced by L. Landau and E. Lifshitz in [34] to describe the dynamics of magnetization in ferromagnetic materials. Here the parameter \( \lambda \) takes into account the anisotropy of such material. More precisely, the value \( \lambda = 0 \) corresponds to the isotropic case, meanwhile \( \lambda > 0 \) and \( \lambda < 0 \) correspond to materials with an easy-axis and an easy-plane anisotropy, respectively (see [31, 27]).

The isotropic case \( \lambda = 0 \) recovers the Schrödinger map equation, which has been intensively studied due to its applications in several areas of physics and mathematics (see [19, 40]). For \( \lambda > 0 \), the existence of solitary waves periodic in time have been established in [22, 43]. Moreover, Pu and Guo [44] showed that \( \lambda \neq 0 \) is a necessary condition to the existence of these types of solutions.

In this paper we are interested in the case of easy-plane anisotropy \( \lambda < 0 \). By a scaling argument we can suppose from now on that \( \lambda = -1 \). Then the energy of (1.1) is given by

\[ E(m) = \int_{\mathbb{R}^N} e(m) \, dx \equiv \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla m|^2 + m_3^2) \, dx, \]

that it is formally conserved due to the Hamiltonian structure of (1.1). If \( m \) is smooth, by differentiating twice the condition \( |m(t, x)|^2 = 1 \) we obtain \( m \cdot \Delta m = -|\nabla m|^2 \), so that taking cross product of \( m \) and (1.1), we can recast (1.1) as

\[ m \times \partial_t m = \Delta m + |\nabla m|^2 m - (m_3 e_3 - m_2^2 m). \] (1.2)

Using formal developments and numerical simulations, Papanicolaou and Spathis [12] found in dimensions \( N \in \{2, 3\} \) nonconstant finite energy traveling waves of (1.2), propagating with speed \( c \in (0, 1) \) along the \( x_1 \)-axis, i.e. of the form

\[ m_c(x, t) = u(x_1 - ct, x_2, \ldots, x_N). \]
By substituting $m_c$ in (1.2), the profile $u$ satisfies

$$-\Delta u = |\nabla u|^2 u + u_3^2 u - u_3 e_3 + cu \times \partial_1 u.$$  \hspace{1cm} (TW_c)

Notice that if $u$ satisfies $\text{(TW}_c\text{)}$ with speed $c$, so does $-u$ with speed $-c$, therefore we can assume that $c \geq 0$. Also, we see that any constant in $S^1 \times \{0\}$ satisfies $\text{(TW}_c\text{)}$, so that we refer to them as the trivial solutions. Since we are interested in finite energy solutions, the natural energy space to work in is

$$\mathcal{E}(\mathbb{R}^N) = \{ v \in L^1_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^3) : \nabla v \in L^2(\mathbb{R}^N), \; v_3 \in L^2(\mathbb{R}^N), \; |v| = 1 \text{ a.e. on } \mathbb{R}^N \}.$$

### 1.1 The minimal energy

Our main theorem is in the same spirit as the result proved by the author for the Gross–Pitaevskii equation in [11] (see also [2]). Precisely, we show the existence of a minimal value for the energy for the nontrivial traveling waves.

**Theorem 1.1.** Let $N \in \{2, 3, 4\}$. There exists a universal constant $\mu > 0$ such that if $u \in \mathcal{E}(\mathbb{R}^N)$ is a nontrivial solution of $\text{(TW}_c\text{)}$ with $c \in (0, 1]$, satisfying in addition that $u$ is uniformly continuous if $N \in \{3, 4\}$, then

$$E(u) \geq \mu. \hspace{1cm} (1.3)$$

As noticed in [23] in dimension two, there is no smooth static solution of $\text{(TW}_c\text{)}$, i.e. with speed $c = 0$. More generally, we obtain the following result for static waves.

**Proposition 1.2.** Let $N \geq 2$. Assume that $u \in \mathcal{E}(\mathbb{R}^N)$ is a solution of $\text{(TW}_c\text{)}$ with $c = 0$. Suppose also that $u$ is uniformly continuous if $N \geq 3$. Then $u$ is a trivial solution.

Theorem 1.1 shows that there are no small energy traveling wave solutions in dimensions two, three and four (assuming that they are uniformly continuous in dimensions three and four). This opens the door to have a scattering theory for equation (1.1) with $\lambda = -1$, similarly to the theory developed for the Gross–Pitaevskii equation by Gustafson, Nakanishi and Tsai [20, 21].

The one-dimensional case is different. If $N = 1$, $\text{(TW}_c\text{)}$ is completely integrable and we can compute the solutions in $\mathcal{E}(\mathbb{R})$ explicitly. More precisely,

**Proposition 1.3.** Let $N = 1$, $c \geq 0$ and $u \in \mathcal{E}(\mathbb{R})$ be solution of $\text{(TW}_c\text{)}$.

(i) If $c \geq 1$, then $u$ is a trivial solution.

(ii) If $0 \leq c < 1$ and $u$ is nontrivial, then, up to invariances, $u$ is given by

$$u_1 = c \text{sech}(\sqrt{1-c^2} x), \; u_2 = \tanh(\sqrt{1-c^2} x), \; u_3 = \sqrt{1-c^2} \text{sech}(\sqrt{1-c^2} x).$$

Moreover, if $0 < c < 1$,

$$E(u) = 2\sqrt{1-c^2} \quad \text{and} \quad E(p(u)) = 2\sin(p(u)/2),$$

where $p(u)$ denotes the momentum of $u$.

We notice that equation $\text{(TW}_c\text{)}$ is invariant under translations and under the action of $S^1$ by a rotation around the $e_3$-axis, that is if $u = (u_1, u_2, u_3)$ is a solution of $\text{(TW}_c\text{)}$, so is

$$(u_1 \cos(\varphi) - u_2 \sin(\varphi), u_1 \sin(\varphi) + u_2 \cos(\varphi), u_3),$$

for any $\varphi \in \mathbb{R}$. Also, if $u = (u_1, u_2, u_3)(x)$ is a solution, so is $u = (u_1, u_2, -u_3)(-x)$. These are the invariances that we refer to in Proposition 1.3.

We provide the proof of Proposition 1.3 in Section 6 as well as the precise definition of momentum. The relation $E = 2\sin(p/2)$ is showed in Figure 1. In particular we note that there are solutions of small energy, but there is a maximum value for the energy and the momentum. We also remark that Proposition 1.3 provides a solution with $c = 0$, meanwhile Proposition 1.2 states that this is not possible in the case $N \geq 2$. 


1.2 From Gross–Pitaevskii to Landau–Lifshitz

The results of this paper have been motivated by the numerical simulations in [42], where the authors determine a branch of nontrivial solutions of (TW\(c\)), axisymmetric around the \(x_1\)-axis, for any speed \(c \in (0, 1)\) in dimensions two and three. They also conjecture that there is no nontrivial finite energy solution of (TW\(c\)) for \(c \geq 1\). For \(c\) small, the existence of these traveling waves has been proved rigorously by Lin and Wei [35]. The branch of solutions is depicted in Figure 2. We see that the curve has a nonzero minimum, which represents the minimal energy in Theorem 1.1.

![Figure 1: Curve of energy \(E\) as a function of the momentum \(p\) in the one-dimensional case.](image1)

![Figure 2: Curve of energy \(E\) as a function of the momentum \(p\) in the two-dimensional case.](image2)

Some properties of the solutions found in [42] are very similar to those of the traveling waves for the Gross–Pitaevskii equation obtained numerically by Jones, Putterman and Roberts [29, 28] and studied rigorously in [1, 2, 36]. In fact, if \(u\) is a solution of (TW\(c\)), the stereographic variable

\[
\psi = \frac{u_1 + iu_2}{1 + u_3},
\]

satisfies

\[
\Delta \psi + \frac{1 - |\psi|^2}{1 + |\psi|^2} \psi - ic\partial_1 \psi = \frac{2\bar{\psi}}{1 + |\psi|^2} (\nabla \psi)^2,
\]

that seems like a perturbed equation for the traveling waves for the Gross–Pitaevskii equation, namely

\[
\Delta \Psi + (1 - |\Psi|^2) \Psi - ic\partial_1 \Psi = 0.
\]
However, other properties of the solutions are very different. For instance, the energy-momentum curve for the Gross–Pitaevskii equation tends to zero as the momentum goes to zero in the two-dimensional case, but there exists a minimal energy if \( N \geq 3 \) (see [2, 11]).

From a mathematical point of view, (1.4) is a quasilinear Schrödinger equation meanwhile (1.5) is a semilinear Schrödinger equation. Therefore, it is not clear how to relate both equations. One of the purposes of this paper is to clarify this connection, to show how to exploit the similarities between (1.4) and (1.5), and how to deal with the extra difficulties of equation (1.5). In particular, we will discuss the regularity of the solutions of (1TW), some a priori bounds and their asymptotic behavior as \( |x| \to \infty \).

### 1.3 Sketch of the proof of Theorem 1.1

The starting point of our analysis is that for any solution \( u \in \mathcal{E}(\mathbb{R}^N) \) of (1TW), there exists \( R \equiv R(u) \) such that we have the lifting

\[
\hat{u} \equiv u_1 + iu_2 = \rho e^{i\theta}, \quad \text{on } B(0,R)^c, \tag{1.6}
\]

where \( \rho \equiv \sqrt{u_1^2 + u_2^2} = \sqrt{1-u_3^2} \) and \( \rho, \theta \in \dot{H}^1(B(0,R)^c) \) (see Lemma 2.4). Let \( \chi \in C_\infty(\mathbb{R}^N) \) be such that \( |\chi| \leq 1, \chi = 0 \) on \( B(0,2R) \) and \( \chi = 1 \) on \( B(0,3R)^c \), if \( R > 0 \). In the case that \( R = 0 \), we let \( \chi = 1 \) on \( \mathbb{R}^N \). In this way, we can assume that the function \( \chi \theta \) and

\[
G = (G_1, \ldots, G_N) \equiv u_1 \nabla u_2 - u_2 \nabla u_1 - \nabla(\chi \theta), \tag{1.7}
\]

is well-defined on \( \mathbb{R}^N \). For \( u = (u_1, u_2, u_3) \), equation (1TW) reads

\[
\begin{align*}
\Delta u_1 &= 2c(u_1)u_1 + c(u_2\partial_1 u_1 - u_3 \partial_1 u_2), \tag{1.8} \\
\Delta u_2 &= 2c(u_2)u_2 + c(u_3\partial_1 u_1 - u_1 \partial_1 u_3), \tag{1.9} \\
\Delta u_3 &= 2c(u_3)u_3 - u_3 + c(u_1 \partial_1 u_2 - u_2 \partial_1 u_1). \tag{1.10}
\end{align*}
\]

Then, using (1.8) and (1.9),

\[
\begin{align*}
\text{div}(G) &= u_1 \Delta u_2 - u_2 \Delta u_1 - \Delta(\chi \theta) \\
&= c(\partial_1 u_3 - u_3 u \cdot \partial_1 u) - \Delta(\chi \theta) \\
&= c\partial_1 u_3 - \Delta(\chi \theta), \tag{1.11}
\end{align*}
\]

where we used the fact that \( u \cdot \partial_1 u = 0 \). By combining with (1.10), we obtain

\[
\Delta^2 u_3 - \Delta u_3 + c^2 \partial_{11}^2 u_3 = -\Delta F + c \partial_1 (\text{div } G), \quad \text{on } \mathbb{R}^N. \tag{1.12}
\]

At this point we remark that the differential operator

\[
\Delta^2 - \Delta + c^2 \partial_{11}^2
\]

is elliptic if and only if \( c \leq 1 \), which shows that \( c = 1 \) is a critical value for the equation (1TW).

Taking Fourier transform in (1.12), we get

\[
(|\xi|^4 + |\xi|^2 - c^2 \xi_1^2) \hat{u}_3(\xi) = |\xi|^2 \hat{F}(\xi) - c \sum_{j=1}^N \xi_j \xi_j \hat{G}_j(\xi), \tag{1.13}
\]

and hence

\[
\hat{u}_3(\xi) = L_c(\xi) \left( \hat{F}(\xi) - c \sum_{j=1}^N \frac{\xi_j \xi_j}{|\xi|^2} \hat{G}_j(\xi) \right), \tag{1.14}
\]

where

\[
L_c(\xi) = \frac{|\xi|^2}{|\xi|^4 + |\xi|^2 - c^2 \xi_1^2}.
\]

Equivalently, we can write (1.14) as the convolution equation

\[
u_3 = L_c * F - c \sum_{j=1}^N L_{c,j} * G_j, \tag{1.15}
\]
where $\hat{L}_c = L_c$ and
\[
\hat{L}_{c,j} = \frac{\xi_j}{|\xi|^4 + |\xi|^2 - c^2 \xi_j^2}.
\] (1.16)

Similarly, from (1.11) and (1.14), for $j \in \{1, \ldots, N\}$,
\[
\partial_j (\chi \theta) = c L_{c,j} * F - c^2 \sum_{k=1}^{N} T_{c,j,k} * G_k - \sum_{k=1}^{N} R_{j,k} * G_k,
\] (1.17)

where
\[
\hat{T}_{c,j,k} = \frac{\xi_j \xi_k}{|\xi|^4 + |\xi|^2 - c^2 \xi_j^2} \quad \text{and} \quad \hat{R}_{j,k} = \frac{\xi_j \xi_k}{|\xi|^2},
\]
for all $j, k \in \{1, \ldots, N\}$.

The kernels $L_c, L_{c,j}, R_{j,k}$ and $T_{c,j,k}$ are the same as those appearing in the Gross–Pitaevskii equation (1.5). As a consequence, all the properties valid for the (1.5) depending only in the structure of these kernels, can be transfer to the Landau–Lifshitz equation. For instance, the asymptotic behavior theory developed in [16, 18, 4, 9] can be applied, after proving some algebraic decay of the solutions. We provide a precise statement in Theorem 1.6 at the end of this introduction.

Roughly speaking, the principle used to find a minimal energy for the traveling waves of (1.5) in [2, 11], written in the context of the equation (TW), is that on one hand a convolution equation such as (1.15) should imply that
\[
\|u_3\|_{L^p} \leq C(\|u\|_{W^{k,q}}) E(u)^{\gamma},
\] (E1)
for some $q, k \in \mathbb{N}$, and $\gamma > 0$. On the other hand, using (1.8), (1.9), (1.10) and integrating by parts one should get an a priori bound for the energy of the form:
\[
E(u) \leq C(\|u\|_{W^{l,r}}) \|u_3\|_{L^p}^\delta,
\] (E2)
for some $l, r \in \mathbb{N}$, and $\delta > 0$. By putting together (1.11) and (1.2),
\[
E(u) \leq C(\|u\|_{W^{l,r}}) C(\|u\|_{W^{k,p}})^\delta E(u)^{\gamma}. 
\]
Notice that we can assume that $E(u) > 0$ because $u$ is not constant. If $\gamma \delta > 1$ and if
\[
C(\|u\|_{W^{l,r}}) C(\|u\|_{W^{k,p}})^\delta \leq M,
\] (E3)
for some constant $M$ independent of $u$ and $c$, we can conclude that
\[
\frac{1}{M^{1/(\gamma\delta - 1)}} \leq E(u),
\]
so that we have the existence of a minimal energy.

In conclusion, we have reduced the proof of Theorem 1.1 to the proof of the estimates (E1), (E2) and (E3), for some $\gamma, \delta > 0$ such that $\gamma \delta > 1$.

1.3.1 Estimate (E3) and regularity of traveling waves

Let us consider the quasilinear elliptic system,
\[
\Delta u = f(x, u, \nabla u), \quad \text{in } \Omega,
\]
where $\Omega$ is a smooth domain and $f$ is a smooth function with quadratic growth
\[
|f(x, z, p)| \leq A + B|p|^2.
\]

We notice that the square-gradient term prevents us from invoking the usual elliptic regularity estimates. However, well-known regularity results imply that every continuous solution in $H^1(\Omega)$ belongs to $H^{2,2}_{\text{loc}}(\Omega) \cap C^{0,\alpha}_{\text{loc}}(\Omega)$ (see [14, 33, 5, 39]), but in general we do not have nice a priori estimates such as in
the $L^p$-regularity theory because the $H^{2,2}_0(\Omega)$-norm depends on the modulus of continuity of the $u$. To exemplify this point, let us consider the harmonic map equation
\begin{equation}
-\Delta v = |\nabla v|^2 v, \quad \text{in } \Omega, \, v \in \mathbb{S}^2.
\end{equation}
Let $\Omega = \mathbb{R}^N$, $N \geq 2$, and assume that there exists $C_1, C_2, \alpha > 0$ such that
\begin{equation}
\|\nabla v\|_{L^\infty(\mathbb{R}^N)} \leq C_1 + C_2\|\nabla v\|^\alpha_{L^2(\mathbb{R}^N)}.
\end{equation}

Note that $\|\nabla v\|^2_{L^2(\mathbb{R}^N)}$ is the energy associated to (1.18). Since the function $v_\lambda(x) = v(\lambda x)$, $\lambda > 0$, also solves (1.18), we conclude that $v_\lambda$ satisfies (1.19), but this implies
\begin{equation}
\|\nabla v\|_{L^\infty(\mathbb{R}^N)} \leq \frac{1}{\lambda} \left(C_1 + \frac{C_2}{\lambda^\alpha(N-2)/2}\|\nabla v\|^\alpha_{L^2(\mathbb{R}^N)}\right).
\end{equation}

Then, letting $\lambda \to \infty$, we deduce that $v$ is constant. Therefore an estimate such as (1.19) does not hold for (1.18) and probably neither for (1.W). This a big difference with the semilinear equation (1.5). Indeed, if $\Psi$ is a solution of (1.5), then (see [13, 2])
\begin{equation}
\|\Psi\|_{C^\infty(\mathbb{R}^N)} \leq C(c, k, N).
\end{equation}

In dimension $N = 2$, Hélein [24, 25] proved that any finite energy solution of (1.18) is continuous and therefore smooth. In dimension $N \geq 3$, this result is false. In fact, if $N \geq 3$, $v(x) = x/|x|$ is a discontinuous finite energy solution and Rivièrè [145] proved that (1.18) has almost everywhere discontinuous solutions with finite energy.

For these reasons, we need to treat differently the cases $N = 2$ and $N \geq 3$. In any case, we establish in Section 2 a bound of $\|\nabla u\|_{L^\infty(\mathbb{R}^N)}$ in terms of the energy, provided that the energy is small enough.

**Proposition 1.4.** Let $c \geq 0$ and $u \in E(\mathbb{R}^2)$ be a solution of (1.W). Then $u \in C^\infty(\mathbb{R}^2)$, $u_3 \in L^p(\mathbb{R}^2)$ for all $p \in [2, \infty]$ and $\nabla u \in W^{k,p}(\mathbb{R}^2)$ for all $k \in \mathbb{N}$ and $p \in [2, \infty]$. Moreover, there exist constants $\varepsilon_0 > 0$ and $K > 0$, independent of $u$ and $c$, such that
\begin{equation}
\|u_3\|_{L^\infty(\mathbb{R}^2)} \leq K(1 + c)E(u)^{1/2},
\end{equation}
\begin{equation}
\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \leq K(1 + c)E(u)^{1/4},
\end{equation}
provided that $E(u) \leq \varepsilon_0$.

Denoting by $UC(\mathbb{R}^N)$ the set of uniformly continuous functions, in the higher dimensional case, we have

**Proposition 1.5.** Let $N \geq 3$, $c \geq 0$ and $u \in E(\mathbb{R}^N) \cap UC(\mathbb{R}^N)$ be a solution of (1.W). Then $u \in C^\infty(\mathbb{R}^N)$ and $\nabla u \in W^{k,p}(\mathbb{R}^N)$ for all $k \in \mathbb{N}$ and $p \in [2, \infty]$. Moreover, if $N \in \{3, 4\}$ and $c \in [0, 1]$, there exist $\varepsilon_0, K, \alpha > 0$, independent of $u$ and $c$, such that
\begin{equation}
\|u_3\|_{L^\infty(\mathbb{R}^N)} \leq KE(u)^{\alpha},
\end{equation}
\begin{equation}
\|\nabla u\|_{L^\infty(\mathbb{R}^N)} \leq KE(u)^{\alpha},
\end{equation}
provided that $E(u) \leq \varepsilon_0$.

As we will show, Propositions 1.4 and 1.5 will be enough to get the universal constant $M$ in estimate (1.23). The proof of Proposition 1.5 is the only point of the paper where the condition $N \leq 4$ is used. It is straightforward to verify that if the estimates (1.22) and (1.23) are satisfy for some dimension $N$, then Theorem 1.1 holds for this $N$.

**1.3.2 Estimates (E1) and (E2)**

In order to prove Theorem 1.1, we can assume that $E(u)$ is small. Then, by (1.20) and (1.22), we only need to prove (E1) and (E2) for traveling waves such that $\|u_3\| \leq 1/2$.

In Section 3, we will prove that estimate (E1) holds with $p = 4$, $\gamma = 1$ if $N = 2$, $p = 2$, $\gamma = \frac{2N+3}{2(N-1)}$ if $N \geq 3$, and $k = 1$, $q = \infty$ in both cases. The main element in the proof is the study of the Fourier multiplier $L_0$ done by the author in [11] if $N \geq 3$ and by Chiron and Maris [7] if $N = 2$.

Sections 4 and 5 are devoted to establish some Pohozaev identities and a priori bounds that allow us to obtain estimate (E2). More precisely, under the condition $\|u_3\| \leq 1/2$, we show that
\begin{equation}
E(u) \leq K\|u_3\|^\delta_{L^p},
\end{equation}
with $p = 4$, $\delta = 4$ if $N = 2$, and $p = 2$, $\delta = 2$ if $N \geq 3$.
1.4 Asymptotic behavior at infinity

As remarked before, the arguments given by Gravejet in [15] [13] apply to (1.15) and [14] [17], since they rely mainly on the structure of the kernels. This allows to establish the precise limit at infinity of the finite energy solutions of (TW).

**Theorem 1.6.** Let \( N \geq 2 \) and \( c \in (0, 1) \). Assume that \( u \in \mathcal{E}(\mathbb{R}^N) \) is a solution of (TW). Suppose further that \( u \in UC(\mathbb{R}^N) \) if \( N \geq 3 \). Then there exist a constant \( \lambda_\infty \in \mathbb{C} \) of modulus one and two functions \( \bar{u}_\infty, u_{3, \infty} \in C(\mathbb{S}^{N-1}; \mathbb{R}) \) such that

\[
|x|^{N-1}(\bar{u}(x) - \lambda_\infty) - i\lambda_\infty \bar{u}_\infty \left( \frac{x}{|x|} \right) \to 0, \tag{1.24}
\]

\[
|x|^N u_{3, \infty} \left( \frac{x}{|x|} \right) \to 0, \tag{1.25}
\]

uniformly as \( |x| \to \infty \). Moreover, assuming without loss of generality that \( \lambda_\infty = 1 \), we have

\[
\bar{u}_\infty(\sigma) = \frac{\alpha \sigma_1}{(1 - c^2 + c^2 \sigma_1^2)^\frac{N}{2}} + \sum_{j=2}^N \frac{\beta_j \sigma_j}{(1 - c^2 + c^2 \sigma_j^2)^\frac{N}{2}}, \tag{1.26}
\]

\[
u_{3, \infty}(\sigma) = \alpha c \left( \frac{1}{(1 - c^2 + c^2 \sigma_1^2)^\frac{N}{2}} - \frac{N \sigma_1^2}{(1 - c^2 + c^2 \sigma_1^2)^\frac{N}{2}} \right) - \sum_{j=2}^N \beta_j \frac{N \sigma_1 \sigma_j}{(1 - c^2 + c^2 \sigma_j^2)^\frac{N}{2}}, \tag{1.27}
\]

where \( \sigma = (\sigma_1, \ldots, \sigma_N) \in \mathbb{S}^{N-1} \),

\[
\alpha = \frac{\Gamma \left( \frac{N}{2} \right)}{2\pi \frac{N}{2}} (1 - c^2)^\frac{N-2}{2} \left( 2c \int_{\mathbb{R}^N} e(u) u_3 \, dx - (1 - c^2) \int_{\mathbb{R}^N} G_1(x) \, dx \right)
\]

and

\[
\beta_j = -\frac{\Gamma \left( \frac{N}{2} \right)}{2\pi \frac{N}{2}} (1 - c^2)^\frac{N-2}{2} \int_{\mathbb{R}^N} G_j(x) \, dx.
\]

In particular, since the solutions found in [35] are uniformly continuous, Theorem 1.6 applies to those solutions. For the sake of completeness we sketch the proof of Theorem 1.6 in Section 7.

**Notations.** We use the standard notations "·" and "×" for the inner and cross product, respectively.

For \( y \in \mathbb{R}^N \) and \( r \geq 0 \), \( B(y, r) \) or \( B_r(y) \) denote the open ball of center \( y \) and radius \( r \) (which is empty for \( r = 0 \)). In the case that there is no confusion, we simply put \( B_r \).

Given \( x = (x_1, x_2), f : \mathbb{R}^2 \to \mathbb{R}^2, f = (f_1, f_2) \), we set \( x^\perp = (-x_2, x_1) \), and \( \text{curl}(f) = \partial_1 f_2 - \partial_2 f_1 \). We also use the skew gradient \( \nabla^\perp = (-\partial_2, \partial_1) \).

For a function \( g : \mathbb{R}^3 \to \mathbb{R}^3, g = (g_1, g_2, g_3) \), we define \( \hat{g} \) as the complex-valued function \( \hat{g} = g_1 + ig_2 \).

We identify \( \nabla g \) with the matrix in \( \mathbb{R}^{3 \times 3} \) whose columns are \( \nabla g_1, \nabla g_2 \) and \( \nabla g_3 \).

Let \( A = [A_1 | A_2 | A_3] \) and \( \tilde{A} = [\tilde{A}_1 | \tilde{A}_2 | \tilde{A}_3] \) be two matrices in \( \mathbb{R}^{3 \times 3} \), then

\[
A : \tilde{A} = \sum_{j=1}^3 A_j \cdot \tilde{A}_j,
\]

and for any vector \( b \in \mathbb{R}^3, A b \in \mathbb{R}^3 \), denotes the standard matrix-vector product.

\( F(f) \) or \( \hat{f} \) stand for the Fourier transform of \( f \), namely

\[
F(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^N} f(x)e^{-ix \cdot \xi} \, dx.
\]

We also adopt the standard notation \( K(\cdot, \ldots, \cdot) \) to represent a generic constant that depends only on each of its arguments.
2 Estimates for $\|u_3\|_{L^\infty(\mathbb{R}^N)}$ and $\|\nabla u\|_{L^\infty(\mathbb{R}^N)}$

In this section we use some of the elements developed to the study of the harmonic map equation. In particular, the next lemma is a consequence of the Wente lemma [19] [18] and Hélein’s trick [24] [25].

**Lemma 2.1.** Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and $g \in L^2(\Omega)$. Assume that $u \in H^1(\Omega, \mathbb{S}^2)$ satisfies

$$-\Delta u = |\nabla u|^2 u + g, \quad \text{in } \Omega. \quad (2.1)$$

Let $r > 0$ and $x \in \Omega$ such that $B(x, r) \subseteq \Omega$. Then for any $i \in \{1, 2, 3\}$ we have

$$\text{osc}_{B(x, r/2)} u_i \leq K \left( \min \left\{ \|\nabla u\|_{L^2(B(x, r))}, \|u_i\|_{L^\infty(\partial B_r)} \right\} + \|\nabla u\|^2_{L^2(B(x, r))} \right) \leq K \left( \|u_i\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \right) \leq K \left( \|u_i\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \right), \quad (2.2)$$

for some universal constant $K > 0$. In particular $u \in C(\Omega)$. Moreover, if the trace of $u$ on $\partial \Omega$ belongs to $C(\partial \Omega)$, then $u \in C(\Omega)$ and

$$\|u_i\|_{L^\infty(\Omega)} \leq \|u_i\|_{L^\infty(\partial \Omega)} + K(\Omega)(\|\nabla u\|^2_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}(1 + \|\nabla u\|_{L^2(\Omega)})), \quad (2.3)$$

for some constant $K(\Omega)$ depending only on $\Omega$.

**Proof.** As for the standard harmonic maps, we recast (2.1) as

$$-\Delta u_i = \sum_{j=1}^3 v_{i,j} \cdot \nabla u_j + g_i, \quad \text{in } \Omega, \quad i = 1, 2, 3, \quad (2.4)$$

where $v_{i,j} = u_i \nabla u_j - u_j \nabla u_i$. Then

$$\text{div}(v_{i,j}) = u_j g_i - u_i g_j \quad \text{and} \quad \|\text{div}(v_{i,j})\|_{L^2(\Omega)} \leq 2\|g\|_{L^2(\Omega)}. \quad (2.5)$$

Let us consider $h_{i,j} \in H^2(\Omega)$ the solution of

$$\begin{cases} 
\Delta h_{i,j} = \text{div}(v_{i,j}), & \text{in } \Omega, \\
h_{i,j} = 0, & \text{on } \partial \Omega. 
\end{cases} \quad (2.6)$$

Thus

$$\|\nabla h_{i,j}\|_{L^2(\Omega)} \leq \|v_{i,j}\|_{L^2(\Omega)} \leq 2\|\nabla u\|_{L^2(\Omega)}. \quad (2.7)$$

Since $\text{div}(v_{i,j} - \nabla h_{i,j}) = 0$, with $v_{i,j} - \nabla h_{i,j} \in L^2(\Omega)$, there exists $w_{i,j} \in H^1(\Omega)$ (see e.g. [15] Thm 2.9)) such that

$$v_{i,j} = \nabla h_{i,j} + \nabla w_{i,j}, \quad \text{in } \Omega. \quad (2.8)$$

Now we decompose $u$ as $u_i = \phi_i + \varphi_i + \psi_i$, where $\phi_i, \varphi_i, \psi_i$ are the solutions of the equations

$$\begin{cases} 
-\Delta \phi_i = 0, & \text{in } U, \\
\phi_i = u_i, & \text{on } \partial U, 
\end{cases} \quad (2.9)$$

$$\begin{cases} 
-\Delta \varphi_i = \nabla h_i : \nabla u + g_i, & \text{in } U, \\
\varphi_i = 0, & \text{on } \partial U, 
\end{cases} \quad (2.10)$$

$$\begin{cases} 
-\Delta \psi_i = \nabla^2 u_i : \nabla u, & \text{in } U, \\
\psi_i = 0, & \text{on } \partial U, 
\end{cases} \quad (2.11)$$

where $\nabla h_i = [\nabla h_{i,1} \mid \nabla h_{i,2} \mid \nabla h_{i,3}], \nabla^2 u_i = [\nabla^2 u_{i,1} \mid \nabla^2 u_{i,2} \mid \nabla^2 u_{i,3}]$, and $U$ is an open smooth domain such that $B_r \subseteq U \subseteq \Omega$. We now prove (2.2) for $r = 1$, supposing that $B_1 \subseteq U$, since then (2.2) follows from a scaling argument. First, invoking Theorem A.1 we have that

$$\text{osc}_{B_{1/2}} \phi_i \leq K \min \left\{ \|\nabla \phi_i\|_{L^2(B_{1/2})}, \|\phi_i\|_{L^2(B_{1/2})} \right\}. \quad (2.12)$$
Also, some standard computations and the maximum principle yield
\[ \|\nabla \phi_i\|_{L^2(U)} \leq \|\nabla u_i\|_{L^2(U)} \quad \text{and} \quad \|\phi_i\|_{L^\infty(U)} \leq \|u_i\|_{L^\infty(\partial U)}. \tag{2.13} \]
Thus from (2.12) and (2.13), we conclude that
\[ \text{osc}_{B_{1/2}} \phi_i \leq K \min \left\{ \|\nabla u_i\|_{L^2(U)}, \|u_i\|_{L^\infty(\partial U)} \right\}. \tag{2.14} \]
For \( \varphi_1 \), Theorem A.2 gives
\[ \text{osc}_{B_i} \varphi_i \leq K (\|\nabla h \cdot \nabla u\|_{L^{3/2}(B_i)} + \|g\|_{L^2(B_i)}). \tag{2.15} \]
To estimate the first term in the r.h.s. of (2.15), we use the Hölder inequality
\[ \|\nabla h \cdot \nabla u\|_{L^{3/2}(B_i)} \leq \|\nabla h\|_{L^6(B_i)} \|\nabla u\|_{L^2(B_i)}, \tag{2.16} \]
and the Sobolev embedding theorem
\[ \|\nabla h\|_{L^6(B_i)} \leq K (\|\nabla h\|_{L^2(B_i)} + \|D^2 h\|_{L^2(B_i)}). \tag{2.17} \]
By using (2.5), (2.15), (2.16), (2.17) and \( L^2 \)-regularity estimates for (2.6), we are led to
\[ \text{osc}_{B_i} \varphi_i \leq K \|g\|_{L^2(B_i)} (1 + \|\nabla u\|_{L^2(B_i)}). \tag{2.18} \]
Similarly, since \( W^{2,p}(U) \hookrightarrow C(\bar{U}) \), for all \( p > 1 \), we also have
\[ \|\varphi_i\|_{C(\bar{U})} \leq K (U) \|g\|_{L^2(U)} (1 + \|\nabla u\|_{L^2(U)}). \tag{2.19} \]
To estimate \( \psi_i \) we invoke the Wente estimate (see [18], [25]), so that
\[ \|\psi_i\|_{C(\bar{U})} + \text{osc}_{B_i} \psi_i \leq K \|\nabla u\|_{L^2(B_i)} \|\nabla u\|_{L^2(U)} \leq K \|\nabla u\|_{L^2(U)}^2, \tag{2.20} \]
where we have used (2.7) and (2.8) for the last inequality.
Therefore, taking \( U = B_1 \) and putting together (2.14), (2.18) and (2.20), we conclude (2.22) with \( r = 1 \).

If the trace of \( u \) on \( \partial \Omega \) belongs to \( C(\partial \Omega) \), we take \( \Omega = U \) and then from (2.9) we deduce that \( \phi_i \in C^2(\Omega) \cap C(\bar{\Omega}) \). Since \( \varphi_i, \psi_i \in C(\bar{\Omega}) \), we conclude that \( u_i \in C(\bar{\Omega}) \) and (2.5) follows from (2.13), (2.19) and (2.20).

**Lemma 2.2.** Let \( y \in \mathbb{R}^2 \), \( r > 0 \) and \( B_r \equiv B(y, r) \). Assume that \( u \in H^1(B_r, \mathbb{S}^2) \) satisfies
\[ -\Delta u = |\nabla u|^2 u + f(x, u(x), \nabla u(x)), \quad \text{in } B_r, \tag{2.21} \]
where \( f \) is a continuous function such that \( |f(x, z, p)| \leq C_1 + C_2|p| \), for some constants \( C_1, C_2 \geq 0 \), for a.e. \( x \in B_r, \ z \in \mathbb{R}^3, \ p \in \mathbb{R}^{3 \times 3} \). Suppose that
\[ A \equiv A(u, r) \equiv \frac{\text{osc}_{B_r} u (1 + r^2(C_1 + C_2^2))}{1 - 3 \text{osc}_{B_r} u} \leq \frac{1}{32}. \tag{2.22} \]
Then
\[ \|D^2 u\|_{L^1(B_{r/2})} + \|u\|_{L^2(B_{r/2})}^2 \leq K r^{-1} \left( \|\nabla u\|_{L^2(B_r)} + \|g\|_{L^2(B_r)} \right), \tag{2.23} \]
where \( g(x) = f(x, u(x), \nabla u(x)) \). Assume further that \( f(x, z, p) = \tilde{f}(x) + R_f(x, z, p) \), for some continuous functions \( \tilde{f}, R_f \), such that \( |R_f(x, z, p)| \leq C_3|p| \), for some constant \( C_3 \geq 0 \), for a.e. \( x \in B_r, \ z \in \mathbb{R}^3, \ p \in \mathbb{R}^{3 \times 3} \). Then,
\[ \|\nabla u\|_{L^\infty(B_{r/4})} \leq K r^{-1} \|\nabla u\|_{L^2(B_r)} + K r^{-2/3} \left( \|\nabla u\|_{L^2(B_r)}^2 (r^{-2} + r^{-4/3}) + \|g\|_{L^2(B_r)}^2 \right) \]
\[ + \|\tilde{f}\|_{L^2(B_r)} + C_3 r^{-1/3} \|\nabla u\|_{L^2(B_r)}^{1/3} (\|\nabla u\|_{L^2(B_r)}^{1/3} + \|g\|_{L^2(B_r)}^{1/3}), \tag{2.24} \]
where \( K \) is some universal constant.
Proof. As mentioned before, Lemma 2.1 and the quadratic growth of the r.h.s. of (2.21) imply that \( u \in H_{0 \text{loc}}^2(\Omega) \). In fact, this could be seen by repeating the following arguments with finite differences instead of weak derivatives. As standard in the analysis of this type of equations, we let \( \rho \in (0,r) \) and \( \chi \in C_0^\infty(B_r) \), with \( \chi(x) = 1 \) if \( |x| \leq \rho \).

\(|\chi| \leq 1 \quad \text{and} \quad |\nabla \chi| \leq K/(r-\rho), \quad \text{on} \ B_r. \tag{2.25}\)

Then setting \( \eta = \chi|\nabla u| \), taking inner product in (2.21) with \((u - u(x_0))\eta^2\) and integrating by parts we obtain

\[
\int_{B_r} |\nabla u|^2 \eta^2 + 2 \int_{B_r} (\nabla u \cdot (u - u(x_0))) \cdot (\eta \nabla \eta) = \int_{B_r} |\nabla u|^2 u \cdot (u - u(x_0)) \eta^2 + \int_{B_r} \eta^2 g \cdot (u - u(x_0)). \tag{2.26}\]

Then, using the elementary inequality \( 2ab \leq a^2 + b^2 \),

\[
|\eta^2 g \cdot (u - u(x_0))| \leq \eta^2(C_1 + C_2|\nabla u|) \text{osc}(u) \leq C_1 \eta^2 \text{osc}(u) + C_2 \eta^2|\nabla u| \text{osc}(u) + \frac{1}{4} C_2^2 \eta^2 \text{osc}(u).
\]

In a similar fashion, we estimate the remaining terms in (2.26). Then, using the Poincaré inequality

\[
||\eta||_{L^2(B_r)} \leq \frac{r}{j_0} ||\nabla \eta||_{L^2(B_r)},
\]

where \( j_0 \approx 2.4048 \) is the first zero of the Bessel function, and that \(|u| = 1\), we conclude that

\[
\int_{B_r} |\nabla u|^2 \eta^2 \leq \frac{\text{osc}(u)}{1 - 3 \text{osc}(u)} \int_{B_r} |\nabla \eta|^2,
\]

where we bounded \(1/j_0\) and \(1/(4j_0)\) by 1 to simplify the estimate. Thus,

\[
\int_{B_r} |\nabla u|^4 \chi^2 \leq A \int_{B_r} (|D^2 u|^2 \chi^2 + |\nabla u|^2 |\nabla \chi|^2). \tag{2.27}\]

On the other hand, taking inner product in (2.21) with \( \partial_i (\chi^2 \partial_k u) \), integrating by parts and summing over \( k = 1, 2 \), we have

\[- \int_{B_r} \chi^2 |D^2 u|^2 - 2 \int_{\Omega} \sum_{i \in \{1,2\}} \sum_{j,k \in \{1,2,3\}} \partial_j u \chi \partial_k \chi \partial_k u_i = \sum_{i \in \{1,2\}} \int_{B_r} (|\nabla u|^2 u + g_i)(2 \chi \nabla \chi \nabla u_i + \chi^2 \Delta u_i). \tag{2.28}\]

Using again the inequalities \(2ab \leq \varepsilon a^2 + b^2/\varepsilon\) and \(ab \leq \varepsilon a^2 + b^2/4\varepsilon\), we are led to

\[
\int_{B_r} \chi^2 |D^2 u|^2 \leq \frac{1}{1 - 3\varepsilon} \int_{B_r} ((2 + \varepsilon^{-1})|\nabla u|^2 |\nabla \chi|^2 + (1 + 4\varepsilon^{-1})|\nabla u|^4 \chi^2 + (1 + 4\varepsilon^{-1})|\nabla u|^2 |g|^2). \tag{2.29}\]

Then, minimizing with respect to \( \varepsilon \), it follows that

\[
\int_{B_r} \chi^2 |D^2 u|^2 \leq 16 \int_{B_r} (|\nabla u|^2 |\nabla \chi|^2 + |\nabla u|^4 \chi^2 + \chi^2 |g|^2). \tag{2.30}\]

By combining (2.26), (2.27) and (2.30), we infer that

\[
\int_{B_r} |\nabla u|^4 \leq \frac{KA}{1-16A} \left( \frac{1}{(r-\rho)^2} \int_{B_r} |\nabla u|^2 + \int_{B_r} |g|^2 \right), \tag{2.31}\]

\[
\int_{B_r} |D^2 u|^2 \leq \frac{K}{1-16A} \left( \frac{1+A}{(r-\rho)^2} \int_{B_r} |\nabla u|^2 + \int_{B_r} |g|^2 \right). \tag{2.32}\]

Taking \( \rho = r/2 \) and using that \( A \leq 1/32 \), (2.25) follows.
Now we decompose \( u \) as \( u_i = \phi_i + \psi_i \), where
\[
\begin{cases}
-\Delta \phi_i = 0, & \text{in } B_{r/2}, \\
\phi_i = u_i, & \text{on } \partial B_{r/2}, \\
-\Delta \psi_i = |\nabla u_i|^2 u_i + \hat{f}_i + (R_f(x, u, \nabla u))_i, & \text{in } B_{r/2}, \\
\psi_i = 0, & \text{on } \partial B_{r/2},
\end{cases}
\tag{2.33}
\]
Since \( \phi_i \) is a harmonic function,
\[
\|\nabla \phi_i\|_{L^\infty(B_{r/4})} \leq K r^{-1/2} \|\nabla \phi_i\|_{L^2(B_{r/2})},
\]
so that using also (2.13), we obtain the estimate
\[
\|\nabla \phi_i\|_{L^\infty(B_{r/2})} \leq K r^{-1/2} \|\nabla u_i\|_{L^2(B_{r/2})}. \tag{2.35}
\]
For \( \psi_i \), we recall that using the \( L^p \)-regularity theory for the Laplacian and a scaling argument, the solution \( v \in H^1_0(B_R) \) of the equation \(- \Delta v = h\) satisfies
\[
\|\nabla v\|_{L^\infty(B_R)} \leq K(p) R^{1-2/p} \|h\|_{L^p(B_R)}, \quad \text{for all } p > 2.
\]
Applying this estimate with \( p = 3 \) to (2.34), we get
\[
\|\nabla \psi_i\|_{L^\infty(B_{r/2})} \leq C r^{-2/3} \left( \|\nabla u_i\|_{L^6(B_{r/2})}^2 + \|\hat{f}\|_{L^6(B_{r/2})} + C_3 \|D^2 u\|_{L^6(B_{r/2})} \right). \tag{2.36}
\]
Also, by the Sobolev embedding theorem, we have
\[
\|\nabla u\|_{L^6(B_{r/2})} \leq K \left( \|D^2 u\|_{L^6(B_{r/2})} + r^{-2/3} \|\nabla u\|_{L^6(B_{r/2})} \right). \tag{2.37}
\]
Therefore, by putting together (2.33), (2.35), (2.36) and (2.37) and the interpolation inequality
\[
\|\nabla u\|_{L^4(B_{r/2})} \leq \|\nabla u\|_{L^6(B_{r/2})}^{1/3} \|\nabla u\|_{L^4(B_{r/2})}^{2/3},
\]
we deduce (2.24).

Now we turn back to equation (TW). By setting
\[
E_{x,r}(u) = \int_{B(x,r)} e(u),
\]
we obtain the following result.

**Corollary 2.3.** There exist \( \varepsilon_0 > 0 \) and a positive constant \( K(\varepsilon_0) \), such that for any \( c \geq 0 \) and any \( u \in \mathcal{E}(\mathbb{R}^2) \) solution of (TW) satisfying
\[
E_{x,r}(u) \leq \varepsilon_0,
\]
for some \( x \in \mathbb{R}^2 \) and \( r \in (0, 1] \), we have
\[
\text{osc}_{B(x,r/2)} u \leq K(\varepsilon_0)(1 + c)E_{x,r}(u)^{1/2}, \tag{2.38}
\]
\[
\|\nabla u\|_{L^\infty(B(x,r/4))} \leq K(\varepsilon_0)(1 + c)E_{x,r}(u)^{1/4} r^{-2/3}. \tag{2.39}
\]
In particular, if \( E(u) \leq \varepsilon_0 \), then (1.20) and (1.21) hold.

**Proof.** Estimates (2.38) and (2.39) follow from Lemmas 2.1 and 2.2. Then, taking \( r = 1 \), we conclude that (1.21) holds. Now we turn to (1.20). For any \( y \in \mathbb{R}^2 \) we have
\[
2E(u) \geq \int_{B(y,1/2)} u_3^2 \geq \frac{7}{4} \min_{B(y,1/2)} |u_3|^2. \tag{2.40}
\]
On the other hand, by Lemma 2.1
\[
\max_{B(y,1/2)} |u_3| \leq \text{osc}_{B(y,1/2)} u_3 + \min_{B(y,1/2)} u_3 \leq K(1 + c)E(u)^{1/2} + \min_{B(y,1/2)} |u_3|. \tag{2.41}
\]
By combining (2.40) and (2.41), we are led to (1.20).
Proof of Proposition 1.4. Since \( u \) has finite energy, for every \( \varepsilon > 0 \), there exists \( \rho > 0 \) such that for all \( y \in \mathbb{R}^2 \)
\[
E_{y,\rho}(u) \leq \varepsilon. \tag{2.42}
\]
In fact, since \( e(u) \in L^1(\mathbb{R}^2) \), by Lemma A.3 for every \( \varepsilon > 0 \) we can decompose \( e(u) = e_{1,\varepsilon}(u) + e_{2,\varepsilon}(u) \) such that
\[
\|e_{1,\varepsilon}(u)\|_{L^1(\mathbb{R}^2)} \leq \varepsilon/2 \quad \text{and} \quad \|e_{2,\varepsilon}(u)\|_{L^\infty(\mathbb{R}^2)} \leq K_\varepsilon,
\]
for some constant \( K_\varepsilon \). Then for any \( y \in \mathbb{R}^2 \),
\[
\|e_{2,\varepsilon}(u)\|_{L^1(B(y,\rho))} \leq K_\varepsilon \rho^2.
\]
Taking
\[
\rho = \left(\frac{\varepsilon}{2K_\varepsilon \pi}\right)^{1/2},
\]
we obtain (2.42). Thus, invoking Corollary 2.5 with \( \varepsilon = \varepsilon_0 \) and \( \tau = \min\{1, \rho\} \), we conclude that
\[
\|\nabla u\|_{L^\infty(B(y,\tau/4))} \leq K_0(1+c)E(u)^{1/4}, \quad \text{for all } y \in \mathbb{R}^2.
\]
Therefore \( u \in W^{1,\infty}(\mathbb{R}^2) \), with \( \|\nabla u\|_{L^\infty(\mathbb{R}^2)} \leq K_0(1+c)E(u)^{1/4} \). Differentiating (TW), we find that \( v = \partial_j u \), \( j = 1, 2, \) satisfies
\[
L_\lambda(v) \equiv -\Delta v - 2(\nabla u : \nabla v)u - c(u \cdot \partial_1 v) + \lambda v = |\nabla u|^2 u + 2u_3v_3u + u_3^2v - v_3c_3 + c(v \cdot \partial_1 u) + \lambda v,
\]
in \( \mathbb{R}^2 \). Since \( \nabla u \in L^\infty(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \), we deduce that the r.h.s. of the formula above belongs to \( L^2(\mathbb{R}^2) \). Therefore taking \( \lambda > 0 \) large enough, we can invoke the elliptic regularity theory for linear systems and deduce that \( v \in W^{2,2}(\mathbb{R}^2) \). Then, by the Sobolev embedding theorem, \( D^2u \in L^p(\mathbb{R}^2) \), for all \( p \in [2, \infty) \) and a bootstrap argument allows us to conclude that \( \nabla u \in W^{k,p}(\mathbb{R}^2) \) for all \( k \in \mathbb{N} \) and \( p \in [2, \infty] \).

The estimates (1.20) and (1.21) are given by Corollary 2.3.

Proposition 1.4 shows that \( u_3 \) is uniformly continuous, so that \( u_3(x) \to 0 \), as \( |x| \to \infty \). In particular \( u \) belongs to the space \( \tilde{E}(\mathbb{R}^2) \), where
\[
\tilde{E}(\mathbb{R}^N) = \{ v \in E(\mathbb{R}^N) : \exists R \geq 0 \text{ s.t. } \|v_3\|_{L^\infty(B(0,R))} < 1 \}.
\]
In the case \( N \geq 3 \), we always suppose that \( u \in \tilde{E}(\mathbb{R}^N) \cap UC(\mathbb{R}^N) \) and then it is immediate that \( u \in \tilde{E}(\mathbb{R}^N) \).

Now we recall a well-known result (see e.g. [37] Proposition 2.5) that provides the existence of the lifting for any function in \( \tilde{E}(\mathbb{R}^N) \).

Lemma 2.4. Let \( N \geq 2 \) and \( v \in \tilde{E}(\mathbb{R}^N) \). Then there exists \( R \geq 0 \) such that \( v \) admits the lifting
\[
\tilde{v}(x) = \rho(x)e^{i\theta(x)}, \quad \text{on } B(0,R), \tag{2.43}
\]
where \( \rho = \sqrt{1 - u_3^2} \) and \( \theta \) is a real-valued function. Moreover, \( \rho, \theta \in H_{\text{loc}}^1(B(0,R)) \) and \( \nabla \rho, \nabla \theta \in L^2(B(0,R)) \).

Corollary 2.5. Let \( c \geq 0 \) and \( u \in E(\mathbb{R}^2) \) be a solution of (TW). Then there is \( R \geq 0 \) such that the lifting \( \tilde{u}(x) = \rho(x)e^{i\theta(x)} \) holds on \( B(0,R) \) and satisfies \( \nabla \rho, \nabla \theta \in W^{k,p}(B(0,R)) \) for any \( k \geq 2 \) and \( p \in [2, \infty) \). Moreover, there exists a constant \( \varepsilon(c) > 0 \), depending only on \( c \), such that if \( E(u) \leq \varepsilon(c) \), then we can take \( R = 0 \).

Proof. By Proposition 1.4 \( u \in \tilde{E}(\mathbb{R}^2) \) and then Lemma 2.4 gives us the existence of the lifting, whose properties follow from Proposition 1.4 and the identity
\[
|\nabla \tilde{u}|^2 = \rho^2|\nabla \theta|^2 + |\nabla \rho|^2, \quad \text{on } B(0,R), \tag{2.44}
\]
noticing that \( 1 - \|v_3\|^2_{L^\infty(B(0,R))} = \inf\{\rho(x)^2 : x \in B(0,R)\} > 0 \). The last assertion is an immediate consequence of (1.20).

In the case \( N \geq 3 \), some regularity for the solutions of the equation (2.1) can be obtained considering that \( u \) is a stationary solution in the sense introduced by R. Moser in [39].
Definition 2.6. Let \( \Omega \subset \mathbb{R}^N \) be a smooth bounded domain and \( g \in L^p(\Omega; \mathbb{R}^3) \). A solution \( u \in H^1(\Omega; \mathbb{S}^2) \) of (2.1) is called stationary if
\[
\text{div}(|\nabla u|^2 e_j - 2 \nabla u \cdot \partial_j u) = 2\partial_j u \cdot g, \quad \text{in } \Omega,
\]
for all \( j \in \{1, \ldots, N\} \) in the distributional sense.

If we suppose that \( u \) is a smooth solution of (2.1), then
\[
\text{div}(|\nabla u|^2 e_j - 2 \nabla u \cdot \partial_j u) = -2\Delta u \cdot \partial_j u = 2g \cdot \partial_j u,
\]
so it is a stationary solution. However not every solution \( u \in H^1(\Omega; \mathbb{S}^2) \) of (2.1) satisfies (2.4). The advantage of stationary solutions is that they satisfy a monotonicity formula that allows to generalize some standard results for harmonic maps. However, when \( g \) belongs only to \( L^2(\Omega) \), the regularity estimates hold only for \( N \leq 4 \).

Lemma 2.7 ([30]). Let \( N \leq 4 \) and \( y \in \mathbb{R}^N \). Assume that \( u \in H^1(B(y,1); \mathbb{S}^2) \cap W^{1,4}(B(y,1)) \) is a stationary solution of (2.1), with \( \Omega = B(y,1) \) and \( g \in L^2(B(y,1)) \). Then there exist \( K > 0 \) and \( \varepsilon_0 > 0 \), depending only on \( N \), such that if
\[
\|\nabla u\|_{L^2(B(y,1))} + \|g\|_{L^2(B(y,1))} = \varepsilon \leq \varepsilon_0,
\]
we have
\[
\|\nabla u\|_{L^4(B(y,1/4))} \leq K \varepsilon^4.
\]

Applying this result to equation (1W.4), we are led to the following estimate.

Lemma 2.8. Let \( N \leq 4 \). There exist \( K > 0 \) and \( \varepsilon_0 > 0 \), depending only on \( N \), such that for any solution \( u \in \mathcal{E}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N) \) of (1W.4), with \( c \in [0,1] \), satisfying \( E(u) \leq \varepsilon_0 \), we have
\[
\|\nabla u\|_{L^4(B(x,1))} \leq K E(u)^{1/2}.
\]

Now we are in position to complete the regularity result in higher dimensions stated in the introduction.

Proof of Proposition 1.6. Recalling again a classical results for elliptic systems with quadratic growth (see [5, 33, 30]), \( u \in UC(\mathbb{R}^N) \) yields that \( u \in C^\infty(\mathbb{R}^N) \). This is due to the fact that now we are assuming that \( u \) is uniformly continuous and then we can choose \( r > 0 \) small such that the oscillation of \( u \) on the ball \( B(y,r) \) is small, uniformly in \( y \). Then we can make the quantity \( A(u,r) \) defined in (2.22) as small as needed and repeat the first part of the proof of Lemma 2.2 to conclude that for all \( y \in \mathbb{R}^N \)
\[
\|D^2 u\|_{L^2(B(y,r/2))} + \|\nabla u\|^2_{L^2(B(y,r/2))} \leq K(N)^{-1} \left( \|\nabla u\|_{L^2(B(y,r))} + \|u_3\|_{L^2(B(y,r))} \right),
\]
for some constant \( K(N) \) and \( r > 0 \) small enough, independent of \( y \). At this stage we note that we cannot follow the rest of the argument of Lemma 2.2, since it relies on the two-dimensional Sobolev embeddings. However, it is well-known that using (2.40) it is possible to deduce that \( \nabla u \in L^p(\mathbb{R}^N) \), for all \( p \geq 2 \). More precisely, as discussed before, there exists \( r \in (0,1] \) such that
\[
\text{osc}_{B(y,2Nr)} u \leq \frac{1}{8(1+c)(2N-1)}, \quad \text{for all } y \in \mathbb{R}^N.
\]

Then, by iterating Lemma 2.3, we have
\[
\int_{B(y,r)} |\nabla u|^{2N+2} \leq K(N)(1+c)^{2N} \frac{E(u)}{r^{2N}}, \quad \text{for all } y \in \mathbb{R}^N.
\]

By proceeding as in the proof of Lemma 2.2 we decompose \( u_i \) as \( u_i = \phi_i + \psi_i \), where
\[
\begin{cases}
-\Delta \phi_i = 0, & \text{in } B(y,r), \\
\phi_i = u_i, & \text{on } \partial B(y,r),
\end{cases}
\]
we conclude that 
\[
\nabla \phi \quad \text{as in the proof of Proposition 1.4, we conclude that} 
\]
(A.2) is satisfied for any \( y \) depending only on \( i \) 
\[
\text{i.e.} \quad \text{which implies (1.22).} 
\]
\[
\nabla \phi \quad \text{depending only on} \quad y, r \quad \text{so it does (2.48) (with} \quad r = 1/16 \quad \text{and then (1.22) and (1.23) follow as before.}
\]

In view of (2.48), elliptic regularity estimates imply that 
\[
\psi \quad \text{in (2.48),} 
\]
\[
\text{same argument shows that} 
\]
\[
\text{Thus, using (2.51),} 
\]
\[
\text{osc}_{B(y,1/8)} u \leq K E(u)^{1/3}, \quad \text{for all} \ y \in \mathbb{R}^4, 
\]
and then (1.22) and (1.23) follow as before. 

\section{Properties related to the kernels and the convolution equations}

Through this section, we fix \( u \in \tilde{E}(\mathbb{R}^N) \cap UC(\mathbb{R}^N) \) a solution of \( \text{TW}_c \) for a speed \( c \in [0,1] \). We also use the notation introduced in Subsection 1.3.

We start recalling the following result for \( L_c \).
Lemma 3.1. \[\text{[11\cite{11}]}\] For any \(c \in (0, 1)\), we have
\[
\|L_c\|_{L^{1} \cap (R^2)} \leq 11\|f\|_{L^{1}(R^2)},
\] (3.1)
and
\[
\|L_c\|_{L^{2}(R^N)} \leq K(N)\|f\|_{L^{2\frac{2N}{N+2}}(R^N)}, \quad \text{if } N \geq 3.
\] (3.2)

Proof. By the Plancherel identity, the estimate (3.2) is exactly \[\text{[11\cite{11} Lemma 4.3]}\]. To prove (3.1), we note that
\[
\|L_c\|_{L^{1} \cap (R^2)} \leq \|L_c\|_{L^{2}(R^N)}\|f\|_{L^{2\frac{2N}{N+2}}(R^N)},
\]

Then it only remains to compute \(\|L_c\|_{L^{2}(R^2)}\). Using polar coordinates, we have
\[
\|L_c\|_{L^{2}(R^2)}^{1/3} = 4 \int_{0}^{\pi/2} \int_{0}^{\pi/2} \frac{r \, dr \, d\theta}{(r^2 + 1 - c^2 \cos^2(\theta))^{1/3}} = 6 \int_{0}^{\pi/2} \frac{d\theta}{(1 - c^2 \cos^2(\theta))^{1/3}} = 3B\left(\frac{1}{6}, \frac{1}{2}\right),
\]
where \(B\) denotes the Beta function. Using that \(B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)\), we conclude that
\[
\|L_c\|_{L^{2}(R^2)} \leq \left(3\frac{\Gamma(1/6)\Gamma(1/2)}{\Gamma(2/3)}\right)^{3/4} \leq 11.
\] (3.4)

From (3.3) and (3.4), (3.1) follows.

Now we are able to prove the exact form of estimate \[\text{[11]}\] stated in the introduction and also further integrability for \(u_3\).

Proposition 3.2. Let \(N \geq 2\) and \(c \in (0, 1)\). Then \(u_3 \in L^p(R^N)\), for all \(p \in (1, 2)\). Moreover, if \(c \in (0, 1]\) and \(\|u_3\|_{L^\infty(R^N)} \leq 1/2\), we have
\[
\|u_3\|_{L^4(R^2)} \leq 54\|u_3\|_{L^\infty(R^N)} E(u),
\] (3.5)
and
\[
\|u_3\|_{L^2(R^2)} \leq K(N)\|u_3\|_{L^\infty(R^N)} \left(1 + \|\nabla u\|_{L^{N-1}(R^N)}^{\frac{N-1}{2}}\right) E(u) \frac{2N+1}{N-1}, \quad \text{if } N \geq 3.
\] (3.6)

Proof. Let us recall that by Propositions \[\text{[11\cite{11} and 1.4]}\] and noticing that
\[
G = -u_3^3 \nabla \theta, \quad \text{on } B(0, 3R)c,
\] (3.7)
we infer that \(F, G_1, G_2 \in L^p(R^N)\), for all \(p \in [1, \infty)\). On the other hand, from the Riesz-operator theory, the functions \(\xi \mapsto \xi_i \xi_j/|\xi|^2\) are \(L^2\)-multipliers for any \(q \in (1, \infty)\) and \(1 \leq i, j \leq N\). Since \(L_c\) is also an \(L^2\)-multiplier for any \(q \in (1, \infty)\) (see \[\text{[15]}\]), from \[\text{[1.14]}\] we conclude that \(u_3 \in L^q(R^N)\), for all \(q \in (1, \infty)\).

We turn now to the proof \[\text{[3.3]}\]. Using \[\text{[1.14]}\] and the Hausdorff–Young inequality
\[
\|\eta\|_{L^p(R^N)} \leq p^{1/p}q^{-1/2q}\|\eta\|_{L^{p'}(R^N)}, \quad p \in [1, 2], \ quad q = p/(p-1),
\]
with \(p = 4/3\), and \[\text{[3.1]}\] we obtain
\[
\|u_3\|_{L^4(R^2)} \leq \|u_3\|_{L^{4/3}(R^2)} \leq 11 \left(\|F\|_{L^1(R^2)} + \|G_1\|_{L^1(R^2)} + \frac{1}{2}\|G_2\|_{L^1(R^2)}\right),
\] (3.8)
where we have used that \(\xi_i^2/|\xi|^2 \leq 1\) and \(\xi_i \xi_j/|\xi|^2 \leq (\xi_i^2 + \xi_j^2)/2|\xi|^2 \leq 1/2\) for the last inequality.

On the other hand, since \(\|u_3\|_{L^\infty(R^N)} \leq 1/2\), the inequality \[\text{[4.6]}\] implies that
\[
\|G_j\|_{L^{1}(R^N)} \leq \frac{2}{\sqrt{3}}\|u_3\|_{L^\infty(R^N)} E(u).
\] (3.9)

From \[\text{[3.3]}\] and \[\text{[3.9]}\], since \(F = 2c(u)u_3 + cG_1\) and \(11(2 + 5/\sqrt{3}) < 54\), \[\text{[3.5]}\] follows.
Let us prove now (3.6). By applying the Plancherel identity to (1.14) and using (3.2) and (3.9), we are led to
\[
\|u_3\|_{L^2(\mathbb{R}^N)} \leq K(N) \left( \|F\|_{L^{\frac{2(N-1)}{N+1}}(\mathbb{R}^N)} + \sum_{j=1}^{N} \|G_j\|_{L^{\frac{2(N-1)}{N+1}}(\mathbb{R}^N)} \right) 
\leq K(N)\|u_3\|_{c} \|u(c)\|_{L^{\frac{2(N-1)}{N+1}}(\mathbb{R}^N)} 
\leq K(N)\|u_3\|_{L^\infty(\mathbb{R}^N)} \|c(u)\|_{L^2(\mathbb{R}^N)},
\]
which gives (3.6). \hfill \Box

**Lemma 3.3.** For all \( k \in \mathbb{N} \) and \( p \in (1, \infty) \), we have \( u_3, \nabla(\chi \theta) \in W^{k,p}(\mathbb{R}^N) \).

**Proof.** By Propositions 1.4 and 1.5, it remains only to treat the case \( p \in (1, 2) \). Differentiating (1.14) and (1.17), we have
\[
\partial^\alpha u_3 = L_{c,j} \partial^\alpha F - c \sum_{j=1}^{N} L_{c,j} \partial^\alpha G_j,
\]
\[
\partial^\alpha \partial_j (\chi \theta) = c L_{c,j} \partial^\alpha F - c^2 \sum_{j=1}^{N} \mathcal{T}_{c,j,k} \partial^\alpha G_k - \sum_{k=1}^{N} \mathcal{R}_{j,k} \partial^\alpha G_k,
\]
for all \( \alpha \in \mathbb{N}^N \). The conclusion follows by observing that \( L_{c,j}, \mathcal{T}_{c,j,k} \) and \( \mathcal{R}_{j,k} \) are \( L^p \)-multipliers for all \( p \in (1, \infty) \), that \( u_3, \nabla(\chi \theta), \nabla u \in W^{k,p}(\mathbb{R}^N) \) for all \( k \in \mathbb{N} \) and \( p \in [2, \infty) \) and using the Leibniz rule. \hfill \Box

**Corollary 3.4.** Let \( N \geq 2 \) and \( c \in [0,1) \). Then the function \( \theta \) is bounded on \( B(0,R)^c \) and there exists \( \tilde{\theta} \in \mathbb{R}^c \) such that
\[
\theta(x) \to \tilde{\theta}, \quad \text{as } |x| \to \infty.
\]
\hfill (3.10)

**Proof.** By Lemma 3.3, \( \nabla \theta \in L^p(\mathbb{R}^N) \), for all \( 1 < p \leq \infty \). Then there exists \( \tilde{\theta} \in \mathbb{R} \) such that \( \theta - \tilde{\theta} \in L^{\frac{N}{N-2}}(\mathbb{R}^N) \) (see e.g. [26] Theorem 4.5.9). Since \( \nabla \theta \in L^\infty(\mathbb{R}^N) \), we have \( \partial v \in UC(\mathbb{R}^N) \) and therefore (3.10) follows.

**Proof of Proposition 1.2.** For \( c = 0 \), we deduce from (1.11) and (1.12) that \( \|u_3\|_{L^2(\mathbb{R}^N)} = 0 \), so that \( u_3 \equiv 0 \). Thus \( \bar{u} = e^{i\theta} \) on \( \mathbb{R}^N \) and using (1.2) we deduce that \( \Delta \theta \equiv 0 \) on \( \mathbb{R}^N \). Therefore, by Corollary 3.4 we obtain that \( \theta \) is a bounded harmonic function, which implies that it is constant and so that \( \bar{u} \) is constant.

## 4 Pohozaev identities

We start establishing the following Pohozaev identities for (1.14). For this purpose, we introduce the notation
\[
w_k(v) \equiv v \cdot (\partial_1 v \times \partial_k v), \quad k \in \{2, \ldots, N\}.
\]

**Proposition 4.1.** Let \( u \in E(\mathbb{R}^N) \cap C^2(\mathbb{R}^N) \) be a solution of (1.14). Then there exists a sequence \( r_n \to \infty \) such that
\[
E(u) = \int_{\mathbb{R}^N} |\partial_1 u|^2 \, dx,
\]
\hfill (4.1)
\[
E(u) = \int_{\mathbb{R}^N} |\partial_1 u|^2 \, dx - c \lim_{r_n \to \infty} \int_{B(0,r_n)} x_k w_k(u) \, dx, \quad \text{for all } k \in \{2, \ldots, N\}.
\]
\hfill (4.2)

**Proof.** Taking inner product between (1.14) and \( x_k \partial_k u \), \( 1 \leq k \leq N \), integrating by parts in the ball \( B(0,R) \) and using that \( u \cdot \partial_k u = 0 \), we obtain
\[
\int_{B(0,R)} |\partial_k u|^2 - \frac{1}{2} \int_{B(0,R)} |\nabla u|^2 - \int_{\partial B(0,R)} \frac{\partial u}{\partial v} \partial_k u + \int_{\partial B(0,R)} |\nabla u|^2 x_k v_k =
\]
\[
\frac{1}{2} \int_{B(0,R)} u_k^2 - \frac{1}{2} \int_{\partial B(0,R)} u_k^2 x_k v_k + c \int_{B(0,R)} x_k w_k(u),
\]
\[
\}
\]

\[
16
\]
where \(\nu\) denotes the exterior normal of the ball \(B(0, R)\) and \(\frac{\partial u}{\partial \nu} = (\nabla u_1, \nu, \nabla u_2, \nu, \nabla u_3, \nu)\). By Lemma 4.4, there is a sequence \(r_n \to \infty\) such that

\[
- \int_{\partial B(0, r_n)} \frac{\partial u}{\partial \nu} \cdot \partial_k x_k + \int_{\partial B(0, r_n)} |\nabla u|^2 x_k \nu_k + \frac{1}{2} \int_{\partial B(0, r_n)} u_3^2 x_k \nu_k \to 0, \quad n \to \infty.
\]

Therefore

\[
E(u) = \int_{\mathbb{R}^N} |\partial_k u|^2 - c \lim_{r_n \to \infty} \int_{B(0, r_n)} x_k \nu_k(u),
\]

which completes the proof.

Let us now discuss the definition of momentum in the two-dimensional case. Formally, the first component of the vectorial momentum is given by (see [42])

\[
p(v) = -\int_{\mathbb{R}^2} x_2 w_2(u) \, dx,
\]

but it is not clear that this quantity is well-defined in \(\mathcal{E}(\mathbb{R}^2)\). In general, it is a delicate task to define the momentum as a functional in the energy space. This difficulty also appears in the context of the Gross–Pitaevskii equation (see e.g. [12]). For the purpose of this paper, we will only define \(p\) for smooth solutions of \((\mathbf{TW}_\nu)\). In fact, from Proposition 4.1, there exists a sequence \(r_n \to \infty\) such that the limit

\[
\lim_{r_n \to \infty} \int_{B(0, r_n)} x_2 x_2 w_2(u) \, dx,
\]

exists. Moreover, (4.2) shows that this limit does not depend on the sequence \(r_n\) and therefore we will define this quantity as the momentum

\[
p(u) = -\lim_{r_n \to \infty} \int_{B(0, r_n)} x_2 w_2(u) \, dx. \tag{4.3}
\]

With this notation we have the following consequence of Proposition 4.1.

**Corollary 4.2.** Let \(u \in \mathcal{E}(\mathbb{R}^2)\) be a solution of \((\mathbf{TW}_\nu)\). Then

\[
\int_{\mathbb{R}^2} u_3^2 \, dx = cp(u). \tag{4.4}
\]

**Proof.** Writing

\[
\int_{\mathbb{R}^2} u_3^2 \, dx = 2E(u) - \int_{\mathbb{R}^2} |\partial_1 u|^2 \, dx - \int_{\mathbb{R}^2} |\partial_2 u|^2 \, dx,
\]

since \(u \in C^2(\mathbb{R}^2)\) by Proposition 1.4, the result is a direct consequence of Proposition 4.1.

In the case that \(u\) admits a global lifting, we obtain

**Lemma 4.3.** Let \(u \in \mathcal{E}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)\) such that \(|u_3|_{L^\infty(\mathbb{R}^2)} < 1\). Then

\[
p(u) = \int_{\mathbb{R}^2} u_3 \partial_1 \theta, \tag{4.5}
\]

where \(u_1 + iu_2 = \sqrt{1 - u_3^2}e^{i\theta}\).

**Proof.** First we notice that

\[
|u_3 \partial_1 \theta| \leq \frac{|u_3|_{L^1(\mathbb{R}^2)}|\partial_1 \theta|}{(1 - |u_3|_{L^\infty(\mathbb{R}^2)})^{1/2}} \leq \frac{e(u)}{(1 - |u_3|_{L^\infty(\mathbb{R}^2)})^{1/2}}, \tag{4.6}
\]

so that the integral in (4.5) is well-defined in \(\mathcal{E}(\mathbb{R}^2)\). We notice that

\[
u \cdot (\partial_1 u \times \partial_2 u) = u_3 \theta (\partial_1 \varrho \partial_2 \theta - \partial_2 \varrho \partial_1 \theta) + \varrho^2 (\partial_1 \theta \partial_2 u_3 - \partial_2 \theta \partial_1 u_3) = \partial_2 (u_3 \partial_1 \theta) - \partial_1 (u_3 \partial_2 \theta),
\]

where we have used that \(u_3^2 = 1 - \varrho^2\) for the last equality. Then, multiplying by \(x_2\), integrating by parts and using the definition of \(p(u)\), (4.5) follows.

From (4.6), we see that integral in (4.5) is well-defined in \(\mathcal{E}(\mathbb{R}^2)\). Actually, integrating on \(\mathbb{R}^N\) instead of \(\mathbb{R}^2\), this expression provides a general definition of momentum, for functions that admit a global lifting, in any dimension. We will see this for \(N = 1\) in Section 8.
5 Properties of solutions satisfying \( \|u_3\|_{L^\infty(\mathbb{R}^N)} \leq 1/2 \)

In this section we assume that \( u \in \tilde{E}(\mathbb{R}^N) \cap UC(\mathbb{R}^N) \) is a nontrivial solution of (TW) with \( c \in (0,1] \) and

\[
\|u_3\|_{L^\infty(\mathbb{R}^N)} \leq \frac{1}{2}. \tag{5.1}
\]

We have chosen \( 1/2 \) to simplify the estimates. The main assumption here is that \( \|u_3\|_{L^\infty(\mathbb{R}^N)} < 1 \), which implies that \( \bar{u} = \theta e^{it} \) on \( \mathbb{R}^N \). Hence we can recast (TW) as

\[
\begin{align*}
\delta & \text{ follows.} \\
\text{(5.5)}
\end{align*}
\]

\[
\begin{align*}
\text{Thus from (5.5),} \\
\text{we multiply (5.2) by} \ 	heta \ \text{and integrate by parts on the ball} \ B(0,R_n). \ \text{Using the fact that} \ u_3, \nabla \theta \in L^2(\mathbb{R}^N) \text{ and} \ u_3, \theta \in L^\infty(\mathbb{R}^N), \text{we can choose} \ R_n \text{ as in (5.9) such that the integrals on} \ \partial B(0,R_n) \text{ go to zero and (5.3)} \ \text{follows.}
\end{align*}
\]

\[
\begin{align*}
\text{To obtain (5.6), (5.7) and (5.8), we multiply (5.3) by} \ \theta, \ (5.3) \ \text{by} \ u_3^2 \ \text{and (5.4) by} \ u_3, \ \text{and proceed in a similar way.}
\end{align*}
\]

The following result corresponds to the estimate (5.2) in the case \( N \geq 3 \).

**Proposition 5.2.**

\[
E(u) \leq 3\|u_3\|^2_{L^2(\mathbb{R}^N)}. \tag{5.10}
\]

**Proof.** Let \( \delta = \|u_3\|_{L^\infty(\mathbb{R}^N)} \in [0,1/2] \). By the Cauchy–Schwarz inequality we have

\[
\int_{\mathbb{R}^N} u_3 \partial_t \theta \leq \left( \int_{\mathbb{R}^N} u_3^2 \right)^{1/2} \left( \int_{\mathbb{R}^N} (\partial_t \theta)^2 \right)^{1/2} \leq \frac{1}{\sqrt{1 - \delta^2}} \left( \int_{\mathbb{R}^N} u_3^2 \right)^{1/2} \left( \int_{\mathbb{R}^N} \theta^2 |\nabla \theta|^2 \right)^{1/2}.
\]

Thus from (5.5),

\[
\int_{\mathbb{R}^N} \rho^2 |\nabla \theta|^2 \leq \frac{c^2}{1 - \delta^2} \int_{\mathbb{R}^N} u_3^2 \leq \frac{4}{3} \int_{\mathbb{R}^N} u_3^2.
\tag{5.11}
\]

On the other hand, from (5.7) and the Cauchy–Schwarz inequality, we obtain

\[
2\sqrt{1 - \delta^2} \int_{\mathbb{R}^N} |\nabla \theta|^2 \leq \frac{\delta^2}{\sqrt{1 - \delta^2}} \int_{\mathbb{R}^N} \theta^2 |\nabla \theta|^2 + c\delta^2 \left( \int_{\mathbb{R}^N} u_3^2 \right)^{1/2} \left( \int_{\mathbb{R}^N} \theta^2 |\nabla \theta|^2 \right)^{1/2}.
\tag{5.12}
\]

By combining (5.11) and (5.12), with \( \delta \leq 1/2 \), we are led to

\[
\int_{\mathbb{R}^N} |\nabla \rho|^2 \leq \frac{7}{18} \int_{\mathbb{R}^N} u_3^2.
\tag{5.13}
\]
From (5.8), using the Cauchy–Schwarz inequality, we have

\[
(1 - \delta^2) \int_{\mathbb{R}^N} (|\nabla u_3|^2 + u_3^2) \leq \delta^2 \left( \int_{\mathbb{R}^N} |\nabla \rho|^2 + \int_{\mathbb{R}^N} \rho^2 |\nabla \theta|^2 \right) + \left( \int_{\mathbb{R}^N} u_3^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \rho^2 |\nabla \theta|^2 \right)^{\frac{1}{2}},
\]

so that, by (5.11) and (5.13),

\[
\int_{\mathbb{R}^N} |\nabla u_3|^2 \leq \left( \frac{4}{3} \left( \frac{1}{18} + \frac{4}{3} \right) + \left( \frac{4}{3} \right)^{1/2} \right) \int_{\mathbb{R}^N} u_3^2 \leq 2 \int_{\mathbb{R}^N} u_3^2.
\]

Finally, by putting together (5.11), (5.13) and (5.14),

\[
E(u) \leq \frac{1}{2} \left( \frac{4}{3} \left( \frac{1}{18} + \frac{7}{18} + 3 \right) \int_{\mathbb{R}^N} u_3^2 \right) \leq 3 \int_{\mathbb{R}^N} u_3^2.
\]

In the two-dimensional case, Corollary 4.2 allows us also to estimate the energy in terms of $\|u_3\|_{L^1(\mathbb{R}^2)}$. To this purpose, as remarked in [5], it is useful to study the norm of $\partial_1 \theta - cu_3$.

**Lemma 5.3.**

\[
\|\partial_1 \theta - cu_3\|_{L^2(\mathbb{R}^2)}^2 + \|\partial_2 \theta\|_{L^2(\mathbb{R}^2)}^2 \leq \frac{9}{4} \|u_3\|^4_{L^4(\mathbb{R}^2)}.
\]

**Proof.** By adding (5.1) and (5.3), we obtain

\[
\int_{\mathbb{R}^2} u_3^2 + \int_{\mathbb{R}^2} \partial_1 \theta \leq c \int_{\mathbb{R}^2} u_3 \partial_1 \theta. \tag{5.15}
\]

Since $|\nabla \theta|^2 = (\partial_1 \theta)^2 + (\partial_2 \theta)^2$, by defining the function $\varphi = \partial_1 \theta - cu_3$, we have $\partial_1 \theta = \varphi + cu_3$ and then we recast (5.15) as

\[
\int_{\mathbb{R}^2} (1 - u_3^2)(\varphi^2 + (\partial_2 \theta)^2) + (1 - c^2) \int_{\mathbb{R}^2} u_3^2 = 2c \int_{\mathbb{R}^2} u_3^2 \partial_1 \theta.
\]

Letting $\delta = \|u_3\|_{L^\infty(\mathbb{R}^2)}$, and using that $c \in [0, 1]$, we conclude that

\[
(1 - \delta^2)(\|\varphi\|_{L^2(\mathbb{R}^2)}^2 + \|\partial_2 \theta\|_{L^2(\mathbb{R}^2)}^2) \leq 2\|u_3\|^4_{L^4(\mathbb{R}^2)} + \|u_3\|^4_{L^4(\mathbb{R}^2)}. \tag{5.16}
\]

For the first term in the r.h.s., we use the H"{o}lder inequality

\[
\|u_3\|^4_{L^4(\mathbb{R}^2)} \leq \|u_3\|_{L^\infty(\mathbb{R}^2)} \|u_3\|^4_{L^4(\mathbb{R}^2)} \leq \delta \|u_3\|^4_{L^2(\mathbb{R}^2)} \|\varphi\|_{L^2(\mathbb{R}^2)}. \tag{5.17}
\]

Then (5.10), (5.17) and the inequality $ab \leq a^2/4 + b^2$ imply that

\[
\left( 1 - \delta^2 - \frac{\delta}{2} \right) \|\varphi\|_{L^2(\mathbb{R}^2)}^2 + (1 - \delta^2) \|\partial_2 \theta\|_{L^2(\mathbb{R}^2)}^2 \leq \left( 1 + \frac{\delta}{4} \right) \|u_3\|^4_{L^4(\mathbb{R}^2)}.
\]

Since $\delta \leq 1/2$, the conclusion follows.

**Lemma 5.4.**

\[
\|\nabla \theta\|_{L^2(\mathbb{R}^2)}^2 \leq 6 \|u_3\|^4_{L^4(\mathbb{R}^2)}. \tag{5.18}
\]

**Proof.** Since $\rho = \sqrt{1 - u_3^2} \in [\sqrt{3}/2, 1]$, from (5.7) we have

\[
\sqrt{3} \int_{\mathbb{R}^2} |\nabla \rho|^2 + \sqrt{3} \int_{\mathbb{R}^2} c(u_3)^2 \leq \int_{\mathbb{R}^2} \rho u_3^2 (|\nabla \theta|^2 + cu_3 \partial_1 \theta). \tag{5.19}
\]

As in the proof of Lemma 5.3, we define $\varphi = \partial_1 \theta - cu_3$ so that

\[
|\nabla \theta|^2 + cu_3 \partial_1 \theta = \varphi^2 + 3cu_3 \varphi + 2c^2 u_3^2 + (\partial_2 \theta)^2. \tag{5.20}
\]
By using Lemma 5.4, (5.21) and the fact that \( \kappa \) Proposition 5.5.

\[
\| u_3 \|_{L^4(\mathbb{R}^2)} \leq \frac{35}{4} \| u_3 \|_{L^4(\mathbb{R}^2)}^4,
\]

where we have used Lemma 5.3 for the last inequality. By combining (5.19), (5.21) and the fact that \( 35/(4\sqrt{3}) \leq 6 \), we obtain (5.18).

Finally, we get estimate (5.22) for \( N = 2 \).

**Proposition 5.5.**

\[
E(u) \leq 10 \| u_3 \|_{L^4(\mathbb{R}^2)}^4.
\]

**Proof.** From (5.6) we obtain

\[
2 \int_{\mathbb{R}^2} e(u) \varphi^2 = \int_{\mathbb{R}^2} |\nabla \varphi|^2 + \int_{\mathbb{R}^2} \varphi^2 (|\nabla \varphi|^2 + cu_3 \partial_1 \varphi).
\]

By using Lemma 6.4, (5.21) and the fact that \( \varphi^2 [3/4, 1] \), we conclude that

\[
E(u) \leq \frac{59}{6} \| u_3 \|_{L^4(\mathbb{R}^2)}^4 \leq 10 \| u_3 \|_{L^4(\mathbb{R}^2)}^4.
\]

At this point we dispose of all the elements to prove our result, as was sketched in the introduction.

**Proof of Theorem 1.1.** By virtue of Propositions 1.4 and 1.5, we can fix \( \varepsilon_0 > 0 \) such that if \( E(u) \leq \varepsilon_0 \), then \( \| u_3 \|_{L^\infty(\mathbb{R}^N)} \leq 1/2 \) and \( \| u \|_{L^\infty(\mathbb{R}^N)} \) is uniformly bounded. Then Propositions 5.2, 5.5 imply that

\[
E(u) \leq 10 \| u_3 \|_{L^4(\mathbb{R}^N)}^4 \leq 10(54)^4 E(u)^4, \text{ if } N = 2,
\]

and

\[
E(u) \leq 3 \| u_3 \|_{L^2(\mathbb{R}^N)}^2 \leq KE(u)^{\frac{2N+3}{N-1}}, \text{ if } N \in \{3, 4\}.
\]

Thus, since \( u \) is nonconstant, \( E(u) > 0 \) and we can divide by \( E(u) \). Therefore, from (5.22) and (5.23) we conclude that \( K \leq E(u) \), for some constant \( K > 0 \). Taking \( \mu = \min \{\varepsilon_0, K\} \), the proof is complete.

\section{6 The one-dimensional case}

In this section we consider the case \( N = 1 \). Then equation (TW) is integrable and the solutions can be computed explicitly as was noticed in [38, 41, 46]. More precisely, we have

**Proposition 6.1.** Let \( N = 1, c \geq 0 \) and \( u \in \mathcal{E}(\mathbb{R}) \) be solution of (TW).

(i) If \( c \geq 1 \), then \( u \) is a trivial solution.

(ii) If \( 0 \leq c < 1 \) and \( u \) is nontrivial, then, up to invariances, \( u \) is given by

\[
\begin{align*}
u_1 &= c \sech(\sqrt{1-c^2} x), 
\nu_2 &= \tanh(\sqrt{1-c^2} x), 
\nu_3 &= \sqrt{1-c^2} \sech(\sqrt{1-c^2} x).
\end{align*}
\]

(iii) If \( 0 < c < 1 \), we can write

\[
\begin{align*}
\ddot{u} &= \sqrt{1-u_3^2} \exp(i\theta), 
\theta &= \arctan\left(\frac{\sinh(\sqrt{1-c^2} x)}{c}\right).
\end{align*}
\]
Proof. We first remark that since \( N = 1 \), it is simply to verify that \( u \) is smooth and then the condition \( u \in \mathcal{E}(\mathbb{R}) \) implies that \( u' \) and \( u_3 \) vanish at infinity. Let us write (6.10) in coordinates

\[
-u'_3 = 2e(u)u_1 + c(u_2 u'_3 - u_3 u'_2),
\]

(6.6)

\[
-u'_2 = 2e(u)u_2 + c(u_3 u'_1 - u_1 u'_3),
\]

(6.7)

\[
-u'_1 = 2e(u)u_3 + c(u_1 u'_2 - u_2 u'_1).
\]

(6.8)

Also, as in (1.11), we have

\[
(u_1 u'_2 - u'_1 u_2)' = cu'_3.
\]

(6.9)

Integrating (6.5), we obtain

\[
u_1 u'_2 - u'_1 u_2 = cu_3.
\]

(6.10)

Then, replacing (6.9) in (6.8), we get

\[
u''_3 + 2e(u)u_3 - (1 - c^2)u_3 = 0.
\]

(6.11)

Now, multiplying (6.6), (6.7), (6.11) by \( u'_1, u'_2, u'_3 \), respectively, adding these relations and using again (6.10),

\[-((u')^2)' = 2e(u)(u_1^2 + u_2^2 + u_3^2)' - (u_3^2)'.
\]

(6.12)

Since \((u_1^2 + u_2^2 + u_3^2)' = (|u|^2)' = 0\), integrating (6.12) we conclude that

\[|u|^2 = u_3^2,
\]

(6.13)

so that \(e(u) = u_3^2\) and equation (6.11) reduces to

\[
u''_3 - 2u_3^3 - (1 - c^2)u_3 = 0.
\]

(6.14)

As before, multiplying (6.14) by \( u'_3 \) and integrating, we conclude that

\[(u'_3)^2 = u_3^3((1 - c^2) - u_3^2),\]

(6.15)

If \( u_3 \) is identically zero, (6.13) implies that \( u \) is a trivial solution. Therefore, we suppose from now on that \( u_3 \) not identically zero. Since equation (6.14) is invariant under translation, we can assume that

\[|u_3(0)| = \max\{|u_3(x)| : x \in \mathbb{R}\} > 0.
\]

Therefore

\[u'_3(0) = 0,
\]

(6.16)

and from (6.15) and (6.16), \( u_3(0) = 1 - c^2 \). In particular we deduce that if \(c \geq 1\), \( u_3 \equiv 0 \), which implies that \(u_1\) and \(u_2\) are constant, which completes the proof of (i). If \(0 \leq c < 1\), by the Cauchy–Lipschitz theorem, equation (6.14) with initial conditions (6.16) and \( u_3(0) = \sqrt{1 - c^2} \) or \( u_3(0) = -\sqrt{1 - c^2} \) has a unique maximal solution. It is straightforward to check that

\[u_3(x) = \pm \sqrt{1 - c^2} \text{sech}(\sqrt{1 - c^2} x)
\]

(6.17)

is the desired solution. Moreover, (6.17) shows that \( \|u_3\|_{L^\infty(\mathbb{R})} < 1 \) if \(c \in (0, 1)\). Hence, for \(c \in (0, 1)\), we can write \( \tilde{u} = (1 - u_3^2)^{1/2} e^{i\theta} \), and then (6.9) yields

\[\theta' = \frac{cu_3}{1 - u_3^2}.
\]

(6.18)

From (6.17) and (6.18), we are led to

\[\theta = \theta_0 + \text{arctan} \left( \frac{\sinh(\sqrt{1 - c^2} x)}{c} \right),
\]

for some constant \( \theta_0 \in \mathbb{R} \), which proves (6.4)–(6.5). Using some standard identities for trigonometric and hyperbolic functions, we also obtain (6.1)–(6.3), for \(c \in (0, 1)\). It only remains to show that for \(c = 0\,
and (6.2) are the unique solutions of (6.6)–(6.8). Indeed, since \( e(u)(x) = u_2^2(x) = \text{sech}^2(x) \), we recast (6.1) and (6.2) as
\[
-\ddot{u}'' = 2\, \text{sech}^2(x)\dot{u},
\]
(6.19)
and from (6.13) we can assume that, up to a multiplication by a complex number of modulus one,
\[
\dot{u}'(0) = 1.
\]
(6.20)
Then the Cauchy–Lipschitz theorem provides the existence of a unique solution of (6.19)–(6.20) in a neighborhood of \( x = 0 \), and it is immediate to check that \( \dot{u}(x) = \tanh(x) \) is the solution, which concludes the proof.

In the one-dimensional case, the momentum is formally given by
\[
p(u) = \int_{\mathbb{R}} \frac{u_3(u_1u_2' - u_2u_1')}{1 - u_3^2}.
\]
If \( \|u_3\|_{L^\infty(\mathbb{R})} < 1 \), we see that
\[
p(u) = \int_{\mathbb{R}} u_3\theta',
\]
and therefore it agrees with the corresponding expression in dimension two.

**Corollary 6.2.** Assume that \( c \in [0,1) \) and let \( u \in \mathcal{E}(\mathbb{R}) \) be a nontrivial solution of \((TW_2)\). Then
\[
E(u) = 2\sqrt{1 - c^2}.
\]
(6.21)
Moreover,
\[
p(u) = \int_{\mathbb{R}} u_3\theta' = 2\arctan \left( \frac{\sqrt{1 - c^2}}{c} \right), \quad \text{for } c \in (0,1).
\]
(6.22)
In particular, we can write explicitly \( E \) as a function of \( p \) as
\[
E(p) = 2\sin \left( \frac{p}{2} \right)
\]
(6.23)
and
\[
\frac{dE}{dp} = \cos \left( \frac{p}{2} \right) = c,
\]
(6.24)
for \( c \in (0,1) \).

**Proof.** Using (6.3) and (6.13), we have
\[
E(u) = \int_{\mathbb{R}} u_3^2 = \sqrt{1 - c^2} \int_{\mathbb{R}} \text{sech}^2(x) \, dx = 2\sqrt{1 - c^2}.
\]
For the momentum, (6.18) yields
\[
p(u) = \int_{\mathbb{R}} u_3\theta' = c \int_{\mathbb{R}} \frac{u_3^2}{1 - u_3^2} = c(1 - c^2) \int_{\mathbb{R}} \frac{\text{sech}^2(\sqrt{1 - c^2}x)}{1 - (1 - c^2)\text{sech}^2(\sqrt{1 - c^2}x)} \, dx.
\]
Then, using the change of variables \( y = \frac{\sqrt{1 - c^2}}{c} \tanh(\sqrt{1 - c^2}x) \), we obtain (6.22), from where we deduce that
\[
c^2 = \frac{1}{\tan^2(p/2) + 1} = \cos^2(p/2).
\]
(6.25)
Finally, from (6.21) and (6.25), we establish (6.23), from where (6.24) is an immediate consequence.  

Proposition 1.3 follows from Proposition 6.1 and Corollary 6.2.
7 Decay at infinity

In this section we provide a sketch of the proof of Theorem 4.10. The first step is to obtain some algebraic decay at infinity of the solutions of \(TW_\alpha\). This can be achieved following an argument of 3.

Proposition 7.1. Assume that \(c \in (0, 1)\). Let \(u \in \mathcal{E}(\mathbb{R}^N)\) be a solution of \(TW_\alpha\). Suppose further that \(u \in UC(\mathbb{R}^N)\) if \(N \geq 3\). Then there exist constants \(R_1, \alpha > 0\) such that for all \(R \geq R_1\),

\[
\int_{B_1(0,R)^c} e(u) \leq \left(\frac{R_1}{R}\right)^\alpha \int_{B_1(0,R)} e(u). \tag{7.1}
\]

Proof. By Corollary 2.5 there exists \(R_0 > 0\) such that equations (5.2)–(5.4) hold on \(B(0, R_0)^c\). Let \(\rho > r \geq R_0\) and

\[
\Omega_{\rho,r} = \{r \leq |x| \leq \rho\}.
\]

Multiplying (5.2) by \(\rho - \rho_r\), with \(\rho_r = \frac{1}{|\partial B_r|} \int_{\partial B_r} \theta\), and integrating by parts, we get

\[
\int_{\Omega_{\rho,r}} \rho^2 \nabla \theta^2 = c \int_{\Omega_{\rho,r}} u_3 \partial_1 \theta + \int_{\partial \Omega_{\rho,r}} (\rho - \rho_r) \rho^2 \partial_\nu \theta - c \int_{\partial \Omega_{\rho,r}} (\rho - \rho_r) u_3 \nu_1, \tag{7.2}
\]

where \(\nu\) denotes the outward normal to \(\Omega_{\rho,r}\).

We recall that the Poincaré inequality for \(\partial B_r\) reads

\[
\int_{\partial B_r} (\rho - \rho_r)^2 \leq r^2 \int_{\partial B_r} |\nabla \theta|^2.
\]

Then we obtain

\[
\left| \int_{\partial B_r} (\rho - \rho_r) \rho^2 \partial_\nu \theta \right| \leq r \left( \int_{\partial B_r} |\nabla \theta|^2 \right)^{1/2} \left( \int_{\partial B_r} |\rho \nabla \theta|^2 \right)^{1/2} \leq \frac{r}{\sqrt{1 - \delta^2}} \int_{\partial B_r} |\rho \nabla \theta|^2,
\]

where \(\delta = \|u_3\|_{L^\infty(B_r^c)}\). Similarly, using also the inequality \(ab \leq a^2/2 + b^2/2\),

\[
\left| \int_{\partial \Omega_{\rho,r}} (\rho - \rho_r) u_3 \nu_1 \right| \leq \frac{r}{\sqrt{1 - \delta^2}} \int_{\partial B_r} e(u) \quad \text{and} \quad \left| \int_{B_r^c} u_3 \partial_1 \theta \right| \leq \frac{1}{\sqrt{1 - \delta^2}} \int_{B_r^c} e(u).
\]

On the other hand, by Lemma 3.34 and Corollary 3.31

\[
(\rho - \rho_r) \rho^2 \partial_\nu \theta, (\rho - \rho_r) u_3 \nu_1 \in L^2(B(0, R_0)^c).
\]

Then by Lemma A.4 we conclude that there exists a sequence \(\rho_n \to \infty\) such that

\[
\int_{\partial B_{\rho_n}} (\rho - \rho_r) \rho^2 \partial_\nu \theta \to 0 \quad \text{and} \quad \int_{\partial B_{\rho_n}} (\rho - \rho_r) u_3 \nu_1 \to 0. \tag{7.3}
\]

Therefore, taking \(\rho = \rho_n\), using (7.2)–(7.3) and the dominated convergence theorem we conclude that

\[
\int_{B_r^c} \rho^2 \nabla \theta^2 \leq \frac{c}{\sqrt{1 - \delta^2}} \int_{B_r^c} e(u) + \frac{3r}{\sqrt{1 - \delta^2}} \int_{\partial B_r} e(u).
\]

In the same way, multiplying (5.3) by \(u_3\), integrating by parts on the set \(\Omega_{r,\rho_n}\), for a suitable sequence \(\rho_n \to \infty\), we are led to

\[
\int_{B_r^c} (|\nabla u_3|^2 + u_3^2) \leq (2\delta^2 + c) \int_{B_r^c} e(u) + \int_{\partial B_r} e(u).
\]

Since \(c < 1\), we can choose \(r\) large enough such that

\[
\frac{1}{2(1 - \delta^2)} \left(2\delta^2 + c \left(1 + \frac{1}{\sqrt{1 - \delta^2}}\right)\right) < 1.
\]
Therefore, noticing that
\[ e(u) \leq \frac{1}{2(1-\delta^2)}(|\nabla u_3|^2 + g^2|\nabla \theta|^2 + u_3^2), \]
we conclude that there exists a constant \( K(\delta,c) > 0 \) such that
\[ \int_{B^c_r} e(u) \leq K(\delta,c) r \int_{\partial B_r} e(u). \] (7.4)

Since
\[ \frac{d}{dr} \int_{B^c_r} e(u) = - \int_{\partial B_r} e(u), \]
we can integrate inequality (7.4) to conclude that
\[ \int_{B^c_r} e(u) \leq \left( \frac{r}{R} \right)^{1/K(c,\delta)} \int_{B^c} e(u), \quad \text{for all } R \geq r, \]
which completes the proof. \( \square \)

**Corollary 7.2.** Under the hypotheses and notations of Proposition 7.1, we have
\[ |\cdot|^\beta e(u) \in L^1(\mathbb{R}^N) \quad \text{and} \quad |\cdot|^\beta (|F| + |G_1| + \cdots + |G_N|) \in L^1(\mathbb{R}^N), \]
for all \( \beta \in [0,\alpha). \)

**Proof.** Since \( u \in C^\infty(\mathbb{R}^N) \), the fact that \( |\cdot|^\beta e(u) \in L^1(\mathbb{R}^N) \) is a direct consequence of Proposition 7.1 (see e.g. [16, Proposition 28]). On the other hand, we take \( R \) large enough such that \( \|u_3\|_{L^\infty(B^c_R)} \leq 1/2 \). Then using that \( |u_3| \leq 1 \), (7.6) and (1.6), we deduce that for all \( j \in \{1,\ldots,N\}, \)
\[ |F| + |G_j| \leq 2e(u) + |u_3^2 \partial_1 \theta + |u_3^2 \partial_j \theta| \leq \frac{|\nabla \theta|^2}{2} \leq \left( 2 + \frac{4}{\sqrt{3}} \right) e(u), \]
and then the conclusion follows. \( \square \)

The properties of the kernels appearing in equations (1.15) and (1.17) has been extensively studied in [16]. Indeed, using the sets
\[ \mathcal{M}_k(\mathbb{R}^N) = \left\{ f : \mathbb{R}^N \to \mathbb{C} : \sup_{x \in \mathbb{R}^N} |x|^k |f(x)| < \infty \right\}, \quad k \in \mathbb{N}, \]
\[ \mathcal{M}(\mathbb{R}^N) = \left\{ f \in C^\infty(\mathbb{R}^N \setminus \{0\}; \mathbb{C}) : D^k f \in \mathcal{M}_k(\mathbb{R}^N) \cap \mathcal{M}_{k+2}(\mathbb{R}^N), \text{ for all } k \in \mathbb{N} \right\}, \]
it is proved that
\[ D^\alpha \mathcal{L}_c, D^\alpha \mathcal{L}_{c,j}, D^\alpha \mathcal{T}_{c,j,k} \in \mathcal{M}_{\alpha+n}(\mathbb{R}^N), \quad \text{for all } 1 \leq j,k \leq N, \quad n \in \mathbb{N}, \quad \alpha \in (N-2,N], \] (7.5)
and also that
\[ \hat{\mathcal{L}}_c, \hat{\mathcal{L}}_{c,j}, \hat{\mathcal{L}}_{c,j,k} \in \mathcal{M}(\mathbb{R}^N). \] (7.6)

Similar results hold for the composed Riesz kernels \( R_{j,k} \). By combining these results with Corollary 7.2, equations (1.15) and (1.17) allow us to obtain the following algebraic decay.

**Lemma 7.3.** For any \( n \in \mathbb{N}, \)
\[ u_3, D^n(\nabla (\chi \theta)), D^n(\nabla \bar{u}) \in \mathcal{M}_N(\mathbb{R}^N) \quad \text{and} \quad D^n u_3 \in \mathcal{M}_{N+1}(\mathbb{R}^N). \]

**Proof.** In view of Corollary 7.2 the proof follows using the same arguments in [16, Theorem 11]. \( \square \)
Proposition 7.4. Let $N \geq 2$ and $c \in (0,1)$. Assume that $u \in \mathcal{E}(\mathbb{R}^N)$ is a solution of \textbf{(TW)}. Suppose further that $u \in UC(\mathbb{R}^N)$ if $N \geq 3$. Then there exists constants $R(u), K(c,u) \geq 0$ such that
\begin{align}
|u_3(x)| + |\nabla \theta(x)| + |\nabla \hat{u}(x)| &\leq \frac{K(c,u)}{1 + |x|^N}, \\
|\nabla u_3(x)| + |D^2 \theta(x)| + |D^2 \hat{u}(x)| &\leq \frac{K(c,u)}{1 + |x|^{N+1}}, \\
|D^2 u_3(x)| &\leq \frac{K(c,u)}{1 + |x|^{N+2}},
\end{align}
for all $x \in B(0, R(u))^c$.

Proof. Inequality (7.7) and the estimate for $\nabla u_3$ in (7.8) are particular cases of Lemma 7.3. A slightly improvement of Lemma 7.3 is necessary for the decay of the second derivatives in (7.8) and (7.9). This can be done by following the lines in [17, Theorem 6], which completes the proof.

The pointwise convergence at infinity follows from general arguments in [17], valid for all functions satisfying (7.6).

Lemma 7.5 ([17]). Assume that $T$ is a tempered distribution whose Fourier transform $\hat{T} = P/Q$ is a rational fraction which belongs to $\mathcal{M}(\mathbb{R}^N)$ and such that $Q \neq 0$ on $\mathbb{R}^N \setminus \{0\}$. Then there exists a function $T_\infty \in L^\infty(\mathbb{S}^{N-1}; \mathbb{C})$ such that
\[ R^N T(R\sigma) \to T_\infty(\sigma), \quad \text{as } R \to \infty, \quad \text{for all } \sigma \in \mathbb{S}^{N-1}. \]
Moreover, assume that $f \in C^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap \mathcal{M}_2(\mathbb{R}^N)$. Then $g \equiv T \ast f$ satisfies
\[ R^N g(R\sigma) \to T_\infty(\sigma) \int_{\mathbb{R}^N} f(x) \, dx, \quad \text{as } R \to \infty, \quad \text{for all } \sigma \in \mathbb{S}^{N-1}. \]

Roughly speaking, it only remains to pass to the limit in the terms associated to the Riesz kernels $\mathcal{R}_{i,j}$. For this purpose, we also recall the following.

Lemma 7.6 ([17]). Assume that $f \in C^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap \mathcal{M}_2(\mathbb{R}^N)$ with $\nabla f \in L^\infty(\mathbb{R}^N) \cap \mathcal{M}_2(\mathbb{R}^N)$. Then $g \equiv R_{j,k} \ast f$ satisfies for all $j, k \in \{1, \ldots, N\}$,
\[ R^N g(R\sigma) \to (2\pi)^{-\frac{N}{2}} \Gamma \left( \frac{N}{2} \right) (\delta_{j,k} - N\sigma_j \sigma_k) \int_{\mathbb{R}^N} f(x) \, dx, \quad \text{as } R \to \infty, \quad \text{for all } \sigma \in \mathbb{S}^{N-1}. \]

Finally, we have all the elements to provide the sketch of the proof of Theorem 1.6.

Proof of Theorem 1.6. In view of (1.15), (7.6) and Lemma 7.6, we can apply Lemma 7.5 to the function $u_3$ to conclude that there exists $u_{3,\infty} \in L^\infty(\mathbb{S}^{N-1}; \mathbb{R})$ such that
\[ R^N u_3(R\sigma) \to u_{3,\infty}(\sigma), \quad \text{as } R \to \infty, \quad \text{for all } \sigma \in \mathbb{S}^{N-1}, \quad \text{(7.10)} \]
where
\[ u_{3,\infty}(\sigma) = \mathcal{L}_{c,\infty}(\sigma) \int_{\mathbb{R}^N} F - c \sum_{j=1}^N \mathcal{L}_{c,j,\infty}(\sigma) \int_{\mathbb{R}^N} G_j, \quad \text{(7.11)} \]
for some functions $\mathcal{L}_{c,\infty}, \mathcal{L}_{c,j,\infty}$. Moreover, adapting [18, Proposition 2], we obtain
\begin{align}
\mathcal{L}_{c,\infty}(\sigma) &= \frac{\Gamma \left( \frac{N}{2} \right)}{2\pi^\frac{N}{2}} \frac{(1 - c^2) \delta_{j,j} e^2}{(1 - c^2 + c^2 \sigma_j^2)^\frac{N}{2}} \left( 1 - \frac{N \sigma_j^2}{1 - c^2 + c^2 \sigma_j^2} \right), \\
\mathcal{L}_{c,j,\infty}(\sigma) &= \frac{\Gamma \left( \frac{N}{2} \right)}{2\pi^\frac{N}{2}} \frac{(1 - c^2) \delta_{j,j} e^2}{(1 - c^2 + c^2 \sigma_j^2)^\frac{N}{2}} \left( \delta_{j,j} (1 - c^2) \frac{N \sigma_j^2}{1 - c^2 + c^2 \sigma_j^2} - \frac{N (1 - c^2)^2 \delta_{j,j} \sigma_j^2}{1 - c^2 + c^2 \sigma_j^2} \right),
\end{align}
which gives (1.27).
Now we turn to equation (1.17). Proceeding as before and using also Lemma 7.6 we infer that there exist functions $\theta_\infty^j \in L^\infty(S^{N-1}; \mathbb{R})$, $j \in \{1, \ldots, N\}$ such that

$$R^N \partial_j \theta(R\sigma) \to \theta_\infty^j(\sigma), \quad \text{as } R \to \infty,$$

(7.13)

for all $j \in \{1, \ldots, N\}$, and also that $\theta_\infty^j$ is given by

$$\theta_\infty^j(\sigma) = c \mathcal{L}_{c,j,\infty}(\sigma) \int_{R^N} F - \sum_{k=1}^N \left( c^2 \mathcal{T}_{c,j,k,\infty}(\sigma) + \frac{\Gamma(N)}{2\pi^N} (\delta_{j,k} - N \sigma_j \sigma_k) \right) \int_{R^N} G_k. \quad (7.14)$$

As before, adapting [18] Proposition 2 we have

$$\mathcal{T}_{c,j,k,\infty} = \frac{\Gamma(N)}{2\pi^N c^2} \left((1 - c^2)^{\frac{\delta_{j,k} + \delta_{k,j}}{2}} - \frac{N(1 - c^2)^{-\delta_{j,k} - \delta_{k,j} + \frac{1}{2} \sigma_j \sigma_k}}{(1 - c^2 + c^2 \sigma_1^2)^{\frac{N-1}{2}}} \right) - \delta_{j,k} + N \sigma_j \sigma_k. \quad (7.15)$$

At this stage, we invoke Corollary 3.4 and suppose that $\bar{\theta} = 0$. Then by [17] Lemma 10,

$$R\bar{\theta}(R\sigma) \to \theta_\infty(\sigma) \equiv -\frac{1}{N-1} \sum_{j=1}^N \sigma_j \theta_\infty^j, \quad \text{as } R \to \infty. \quad (7.16)$$

A further analysis shows that the convergence in (7.10) and (7.10) are uniform, which implies that

$$R^{N-1}(\bar{u}(R\sigma) - 1) = R^{N-1} \left( \sqrt{1 - u_0^2(R\sigma) \exp(i\theta(R\sigma))} - 1 \right) \to i\theta_\infty(\sigma), \quad \text{in } L^\infty(S^{N-1}).$$

By combining with the expression for $\theta_\infty^j$ above, (1.24) follows with $\lambda_\infty = 1$ and $\bar{u}_\infty = \theta_\infty$, provided that $\bar{\theta} = 0$. Moreover, using (7.11), (7.16) and that

$$\sum_{j=1}^N \sigma_j \mathcal{L}_{c,j,\infty}(\sigma) = -\frac{\Gamma(N)}{2\pi^N (1 - c^2 + c^2 \sigma_1^2)^{\frac{N-1}{2}}},$$

$$\sum_{j=1}^N \sigma_j \mathcal{T}_{c,j,k,\infty}(\sigma) = -\frac{\Gamma(N)}{2\pi^N c^2} \frac{(1 - c^2)^{\frac{-\delta_{j,k} + \delta_{k,j}}{2}} - \frac{N(1 - c^2)^{-\delta_{j,k} - \delta_{k,j} + \frac{1}{2} \sigma_j \sigma_k}}{(1 - c^2 + c^2 \sigma_1^2)^{\frac{N-1}{2}}}}{1},$$

we obtain (1.26).

In the case that $\bar{\theta} \neq 0$, it is enough to redefine the function $G$ in (1.7) as

$$G = u_1 \nabla u_2 - u_2 \nabla u_1 - \nabla(\chi(\theta - \bar{\theta})), $$

since then we can establish an equation such as (1.17) for $\partial_j(\chi(\theta - \bar{\theta}))$. Since $\theta(x) - \bar{\theta} \to 0$, as $x \to \infty$, we conclude as before that there exists $\theta_\infty \in L^\infty(S^{N-1}; \mathbb{R})$ such that

$$R^{N-1} \left( \sqrt{1 - u_0^2(R\sigma) \exp(i\theta(R\sigma) - \bar{\theta})} - 1 \right) \to i\theta_\infty(\sigma), \quad \text{in } L^\infty(S^{N-1}).$$

Since $\sqrt{1 - u_0^2(R\sigma) \exp(i\theta(R\sigma) - \bar{\theta})} = \bar{u}(R\sigma) \exp(-i\bar{\theta})$, taking $\lambda_\infty = \exp(i\bar{\theta})$, we conclude that

$$R^{N-1}(\bar{u}(R\sigma) - \lambda_\infty) \to i\lambda_\infty \theta_\infty, \quad \text{in } L^\infty(S^{N-1}),$$

which completes the proof of Theorem 1.6.

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Appendix

For the convenience of the reader we recall some well-known results used in this paper. We assume $\Omega$ to be a smooth open bounded domain of $\mathbb{R}^N$.

**Theorem A.1** ([17, 32]). Let $u \in H^1(\Omega)$, such that $\Delta u = 0$ on $\mathcal{D}'(\Omega)$. Then there are constants $0 < \alpha \leq 1$, $\alpha = \alpha(N)$, and $K > 0$ such that if $x \in \Omega$ and $0 < \rho < r < \text{dist}(x, \Omega)$,

$$\text{osc}_{B_r} u \leq K \left( \frac{\rho}{r} \right)^{\alpha} \frac{\|u\|_{L^2(B_r)}}{r^{N/2}}.$$ 

Moreover, if $N = 2$, then

$$\text{osc}_{B_r} u \leq K (\ln(\rho/r))^{-1/2} \|\nabla u\|_{L^2(B_r)},$$

for some $K > 0$.

**Theorem A.2** ([17]). Let $p > N/2$ and $f \in L^p(\Omega)$. Assume that $u \in H^1_0(\Omega)$ is solution of $-\Delta u = f$, in $\Omega$. Then $u$ is Hölder continuous in $\Omega$. Moreover, for $\rho > 0$, there exists a constant $K(\rho)$ such that

$$\text{osc}_{B_{\rho/2}} u \leq K(\rho)\|f\|_{L^p(\Omega)}.$$ 

**Lemma A.3.** Let $f \in L^1(\mathbb{R}^N)$. Then for every $\varepsilon > 0$ there exists a constant $K(\varepsilon)$ such that $f = f_1 + f_2$ a.e. on $\mathbb{R}^N$ and

$$\|f_2\|_{L^1(\mathbb{R}^N)} \leq \varepsilon, \quad \|f_1\|_{L^\infty(\mathbb{R}^N)} \leq K(\varepsilon).$$

**Proof.** Let

$$f_{1,k} = \begin{cases} 
  k, & \text{if } f \geq k, \\
  f, & \text{if } |f| \leq k, \\
  -k, & \text{if } f \leq -k.
\end{cases}$$

and $f_{2,k} = f - f_{1,k}$. Then

$$\|f_{2,k}\|_{L^1(\mathbb{R}^N)} \leq 2 \int_{\{|f| \geq k\}} |f|. \quad (A.1)$$

Since

$$|\{|f| \geq k\}| = \int_{\{|f| \geq k\}} 1 \leq \frac{1}{k} \|f\|_{L^1(\mathbb{R}^N)} \to 0, \quad \text{as } k \to \infty,$$

invoking the dominated convergence theorem and (A.1), we conclude that $\|f_{2,k}\|_{L^1(\mathbb{R}^N)} \to 0$, as $k \to \infty$ and the conclusion follows.

**Lemma A.4.** Let $N \geq 1$. Assume that $f \in L^p(\mathcal{B}(0, R_0)^c)$, for some $R_0 \geq 0$ and $p \in [1, \infty)$. Then there exists a sequence $R_n \to \infty$ such that for all $s \in [0, N/p - N + 1]$ we have

$$R_n^s \int_{\partial \mathcal{B}(0, R_n)} |f| d\sigma \leq \frac{K(p, N)}{\ln R_n}^p, \quad \text{as } n \to \infty,$$

for some constant $K(p, N) > 0$.

**Proof.** Since $f \in L^p(\mathcal{B}(0, R_0)^c)$,

$$\int_{R_0}^{\infty} \left( \int_{\mathcal{B}(0, r)} |f|^p \right) dr < \infty,$$

and thus there is a sequence $R_n \to \infty$, as $n \to \infty$, such that

$$\int_{\partial \mathcal{B}(0, R_n)} |f|^p \leq \frac{1}{R_n \ln R_n}.$$ 

Then, using the Hölder inequality we obtain

$$\int_{\partial \mathcal{B}(0, R_n)} |f| \leq (K(N)R_n^{N-1})^{1-1/p} \frac{1}{(R_n \ln R_n)^{1/p}},$$

from where the result follows.
Lemma A.5. Let \( c \geq 0 \) and \( u \in C^\infty(\mathbb{R}^N) \cap UC(\mathbb{R}^N) \) be a solution of (TW). Assume that

\[
\text{osc}_{B(y,r)} u \leq \frac{1}{8(1 + c)(2s + 1)},
\]

for some \( y \in \mathbb{R}^N, r > 0 \) and \( s \geq 1 \). Then

\[
\int_{B(y,r/2)} |\nabla u|^{2s+1} \leq 4(1 + c)^2 \left( 1 + \frac{16}{r^2} \right) \int_{B(y,r)} |\nabla u|^{2s}.
\]

Proof. The ideas of the proof are based on classical computations for elliptic equations with quadratic growth (see e.g. [33, 5, 30]). Therefore we only provide the main ideas, in order to show the dependence on \( u, c, s \) and \( N \) as stated. We set \( B_r \equiv B(y, r) \) and \( \eta \in C^\infty_0(B_r) \) a function such that \( 0 \leq \eta \leq 1 \),

\[
|\nabla \eta| \leq \frac{4}{r} \quad \text{on } B_r \quad \text{and} \quad \eta \equiv 1 \text{ on } B_{r/2}.
\]

Finally, we fix \( w = |\nabla u|^2 \), which is smooth by hypothesis, so that

\[
|\nabla w| \leq 2w^{1/2}|D^2 u|.
\]

We now divide the computations in several steps.

**Step 1.** If \( \text{osc}_{B_r} u \leq 1/4 \), we have

\[
\int_{B_r} \eta^2 w^{s+1} \leq 2 \text{osc}_{B_r} u \left( \int_{B_r} |\nabla \eta|^2 w^s + \frac{2s + 1}{2} \int_{B_r} \eta^2 |D^2 u|w^{s-1} \right).
\]

Indeed, since

\[
\int_{B_r} \eta^2 w^{s+1} = \int_{B_r} \eta^2 w^s \nabla(u - u(y)) \cdot \nabla u,
\]

integrating by parts and using (A.5), we deduce that

\[
\int_{B_r} \eta^2 w^{s+1} \leq \text{osc}_{B_r} u \left( 2 \int_{B_r} \eta |\nabla \eta|^2 w^{s+1/2} + (2s + 1) \int_{B_r} \eta^2 |D^2 u|w^s \right).
\]

Using the elementary inequalities \( 2ab \leq a^2 + b^2 \) and \( ab \leq a^2 + b^2/4 \) in the first and second integrals in the r.h.s. of (A.6), we obtain

\[
(1 - 2 \text{osc}_{B_r} u) \int_{B_r} \eta^2 w^{s+1} \leq \text{osc}_{B_r} u \left( \int_{B_r} |\nabla \eta|^2 w^s + \frac{(2s + 1)^2}{4} \int_{B_r} \eta^2 |D^2 u|w^{s-1} \right).
\]

Since \( \text{osc}_{B_r} u \leq 1/4 \), we conclude Step 1.

**Step 2.** We have

\[
\frac{1}{2} \int_{B_r} \eta^2 |D^2 u|^2 w^{s-1} \leq 2 \int_{B_r} |\nabla \eta|^2 w^s - \sum_{k=1}^N \int_{B_r} \partial_k (\Delta u) \cdot \partial_k \eta w^{s-1}.
\]

Let \( k \in \{1, \ldots, N\} \) and \( \phi_k = \eta^2 w^{s-1}\partial_k u \in C^\infty_0(B_r) \). On one hand, integrating by parts,

\[
\sum_{j=1}^N \int_{B_r} \partial_{jk}^2 u \cdot \partial_j \phi_k = - \int_{B_r} \partial_k (\Delta u) \cdot \phi_k.
\]

On the other hand, using

\[
\partial_j w = 2 \sum_{k=1}^N \partial_k u \cdot \partial_{jk}^2 u,
\]

\[
\frac{1}{2} \int_{B_r} \eta^2 |D^2 u|^2 w^{s-1} \leq 2 \int_{B_r} |\nabla \eta|^2 w^s - \sum_{k=1}^N \int_{B_r} \partial_k (\Delta u) \cdot \partial_k \eta w^{s-1}.
\]
and developing the term $\partial_j \phi_k$,

$$
\sum_{j,k=1}^{N} \int_{B_r} \partial^2_{j,k} u \cdot \partial_j \phi_k = \int_{B_r} \eta^2 |D^2 u|^2 w^{s-1} + 2 \sum_{j,k=1}^{N} \eta \partial_j \eta w^{s-1} \partial_k u \cdot \partial^2_{j,k} u + \frac{s-1}{2} \int_{B_r} \eta^2 w^{s-2} |\nabla w|^2. \tag{A.8}
$$

Then the conclusion of this step follows combining (A.7) and (A.8), noticing that the last integral in the r.h.s. of (A.8) is nonnegative, and that

$$
2 \sum_{j,k=1}^{N} |\eta \partial_j \eta w^{s-1} \partial^2_{j,k} u \cdot \partial_k u| \leq 2 \sum_{j,k=1}^{N} \eta |\partial^2_{j,k} u| w^{s-2} \cdot |\partial_j \eta| |\partial_k u| w^{s-1} \leq \frac{1}{2} \eta^2 |D^2 u|^2 w^{s-1} + 2 |\nabla \eta|^2 w^{s}.
$$

**Step 3.** For all $\delta > 0$, we have

$$
\sum_{j=1}^{N} |\partial_j \Delta u \cdot \partial_j u| \leq \delta (c + 1) |D^2 u|^2 + \left( c + \frac{c}{\delta} + 4 \right) w + (1 + c) w^2.
$$

Using (TW$_r$) and the fact that $|u| = 1$, it is simple to check that

$$
\sum_{j=1}^{N} |\partial_j \Delta u \cdot \partial_j u| \leq 2 |D^2 u|^2 + w^2 + 4w + 2cw^{3/2} + 2c|D^2 u|w^{1/2}.
$$

By combining with the fact that $2ab \leq \delta a^2 + \delta^{-1} b^2$, for all $\delta > 0$, we finish Step 3.

**Step 4.**

$$
\frac{1}{T} \int_{B_r} \eta^2 |D^2 u|^2 w^{s-1} \leq 2 \int_{B_r} |\nabla \eta|^2 w^{s} + (4c^2 + 5c + 4) \int_{B_r} \eta^2 w^s + (c + 1) \int_{B_r} \eta^2 w^{s+1}.
$$

Step 4 follows immediately from Steps 2 and 3, taking $\delta = (4(c + 1))^{-1}$.

Now we are in position to finish the proof of Lemma A.5. In fact, by combining Steps 1 and 4, we are led to

$$(1 - 4(c + 1)(2s + 1) \text{ osc } u) \int_{B_r} \eta^2 w^{s+1} \leq 2 \text{ osc } u \left( 8s + 5 \right) \int_{B_r} |\nabla \eta|^2 w^{s} + 2(4c^2 + 5c + 4)(2s + 1) \int_{B_r} \eta^2 w^{s+1}. 
$$

Also, we see that (A.2) implies that

$$
1/2 \leq (1 - 4(c + 1)(2s + 1) \text{ osc } u),
$$

and that

$$
2 \text{ osc } u \cdot \max\{8s + 5, 2(4c^2 + 5c + 4)(2s + 1)\} \leq \frac{2(4c^2 + 5c + 4)}{4(1 + c)} \leq 2(1 + c^2).
$$

By combining with (A.4), we conclude (A.3). \(\square\)

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