The global stability investigation of the mathematical design of a fractional-order HBV infection

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Abstract
This work presents approximate solutions of a fractional-order design for hepatitis B virus infection. The numerical solution of the system is given by using an implicit fractional linear multi-step method of the second order. Here, Caputo fractional derivative is considered for fractional derivative. Basic theoretical properties are discussed. We prove the global stability analysis of the fractional-order model. Numerical simulations are demonstrated to display our theoretical results. This current study is to reveal that the order of the fractional derivative $\beta$ does not affect the regular state’s stability concerning both theoretical and numerical results. Besides, if the fractional-order $\beta$ increases, the solutions converge more rapidly to the regular states. Finally, we note that this study can provide beneficial outcomes for understanding and estimating the dissipation of distinct epidemics.

Keywords Systems of fractional differential equations · Hepatitis B virus · Multi-step methods · Fractional trapezoidal formula · Global stability

1 Introduction

One of the serious viral infections and global health problems is called Hepatitis B which is created by HBV that tacks the liver cells. This virus can cause many diseases such as carcinoma, cirrhosis, chronic liver disease, primary hepatocellular [1–4]. The prevalence of HBV changes importantly around the world. Especially, HBV infection is one of the major global health problems in Asia, Africa, Southern Europe, and Latin America. While the disease is endemic in China, it is epidemic in Asia and Africa [5]. The widespread transmission routes of HBV include perinatal, early inapparent, childhood infection, tribal tattooing, scarification contaminated water and
food, unsafe injection. Although most of the HBV-positive people have no symptoms in the acute stage, in some cases people have acute illness displaying symptoms including vomiting, abdominal pain, fatigue, jaundice [6]. There is no specific treatment for acute HBV but chronic HBV can be controlled using medicines, including oral antiviral agents [7].

Mathematical designs are important tools to comprehend better the dynamics of infectious diseases. Thus, describing virus dynamics like HIV, HBV, and HCV infections are established and developed over recent years. Firstly, Nowak and Bangham have established the main model of the virus dynamics for HIV infection [8]. Afterward, the mathematical design was transcribed to HBV and HCV infection [9–12]. The researchers [4] have redesignated the HBV infection model which introduces a reversal rate constant to the uninfected case into the mentioned system [9]. Indeed, this model indicates the returning of the infected hepatocytes to be the uninfected case with the loss of all cccDNA from their nucleus [13]. The dynamics and control of HBV transmission in China are presented [5]. The transmission designs of HBV with diffusion and incorporating two-time delays can be studied in [14]. The integer-order derivative in all of these models is used but in order to capture the complexity nature of the virus, the integer-order derivative is not sufficient.

On the other hand, since fractional-order models offer an exhaustive description of the long memory behavior than the integer-order models, they have gained significance and attention in recent years. Recently, fractional derivatives have been used to generalize models that depict the epidemic disease like COVID-19 [15, 16], Zika virus dynamics [17], Chagas disease [18], biological pest control in tea plants [19]. Analysis of fractional-order model to HBV is proposed in [6, 20–23]. Due to the lack of analytic solutions of most nonlinear fractional differential equations (FDE), numerical approaches like Adams-type predictor-correctors scheme, generalized Euler method, the multi-step method can be used to acquire an approximation solution. Askar et. al. [15] study a fractional-order SITR model to forecast the epidemics of COVID-19 in India. Firstly, they analyze the existence and uniqueness of the solution, boundedness, and nonnegativity for the fractional-order COVID-19 system. Then, the local and global stability analyses of fractional-order systems are investigated. In [16] the authors describe the fractional-order novel COVID-19 model to explain the dynamics of the epidemic. They use the generalized Adams-Bashforth-Moulton method to obtain the numerical simulations. The dynamics of the fractional partial differential equation model of the Zika virus are analyzed in [17]. The HBV system is solved by using the PECE (Predict, Evaluate, Correct, Evaluate) scheme in the study [20]. They also have illustrated the properties of the solution and examined the local stability of the equilibrium points. A fractional HBV model is also studied by Saif et. al. [6]. They demonstrate the fractional model HBV with the Atangana-Balenau derivative and then obtain the numerical solutions by using the Adams-Bashforth method. A fractional differential system of HBV infection with antibody immune response is described by Danane et. al. [21]. They present the positivity and boundedness of all solutions, the numerical simulation, global stability. Besides, the stability analysis of the FDE is examined in many studies [23–27]. The local asymptotical stability of the equilibria system is given by Zhou and Sun [24]. Barbalat’s lemma and Lyapunov functions are used for discussing the global stability analysis in the studies [23, 25–27].

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In the current paper, we will concern numerical solutions of fractional-order mathematical design HBV infection which is formulated as [24]:

\[ D^\beta P = \lambda_1 - \mu P - \frac{\lambda_2 PS}{P + R} + \delta R, \]  
\[ D^\beta R = \frac{\lambda_2 PS}{P + R} - (\eta + \delta)R, \]  
\[ D^\beta S = \sigma R - \gamma S - \frac{\lambda_2 PS}{P + R}, \]  

(I. C.) \[ P(0) = p_0 \geq 0, \quad R(0) = r_0 \geq 0, \quad S(0) = s_0 \geq 0 \]  

in here \( D^\beta \) denotes the CFD and \( \beta \in (0, 1) \). Also, \( P, R, S \) denotes the densities of hepatocytes that are uninfected, infected, and disease-free virions, respectively. It is assumed that the birth rate of sensitive hepatocytes is at the constant rate \( \lambda_1 \), the death rate constant is \( \mu \), and parameter \( \lambda_2 \) is the infection rate constant. \( \eta \) refers to the rate of death of infected hepatocytes, the output rate of free virions from infected hepatocytes is \( \sigma \), the recovery rate of viral particles is \( \gamma \). Infected hepatocytes are cured by noncytolytic stages at a constant rate \( \delta \) per cell [28]. The readers can read more details about the proposed model (1) in [24]. This model (1) is a generalization of an integer-order which is depicted by Su et. al. [28].

The main aim of this research is to propose, analyze, and simulate the HBV model using fractional calculus. The model yet has not been solved numerically by using the fractional trapezoidal method FTM. Numerical solutions are obtained under the MATLAB environment using the code \textit{flmm2} by R. Garrappa, see details [29, 30]. This code contains optionally three methods, but we will choose FTM to obtain the approximate solutions of the mentioned design. To the best of our knowledge, no one yet has also examined the analysis of the global stability for this present epidemiological model. Thus, we will investigate the global stability analysis of the fractional structure by using the Lyapunov stability theorem. Another point that is worthy of being emphasized is that the order of the fractional derivative \( \beta \) does not affect the regular state’s stability concerning both theoretical and numerical results. Besides, if the fractional-order \( \beta \) increases, the solutions converge more rapidly to the regular states. That’s why when we compare to the other existing results in the available literature, we would also like to say that this study will contribute to the literature.

The designation of the current study is as follows: We will first present some principal concepts of fractional calculus in Sect. 2. The existence and uniqueness of the proposed system will be discussed in Sect. 3. In the next section, the description of the FLMS techniques for the system of fractional differential equations (SFDEs) will be constructed. Then, we will carry out the global stability analysis. Finally, we will illustrate the numerical experiments in Sect. 6 and end the study with a short outcome that will be dedicated in Sect. 7.
2 Fundamental concepts of fractional calculus

The part gives some fundamental definitions. Firstly, we want to give the definitions of CFD and Riemann-Liouville integral (RLI). Then we want to recall the notions of Mittag-Leffler function (MLF).

**Definition 1** Let \( g \in C^n \) and \( t, \beta \in \mathbb{R} \). Then the CFD \( D^\beta g(t) \) of order \( \beta \) is introduced by

\[
D^\beta g(t) = \begin{cases} 
\frac{1}{\Gamma(s-\beta)} \int_0^t g^{(s)}(\xi)(t-\xi)^{s-\beta-1}d\xi, & s - 1 < \beta < s \\
g^{(s)}(t), & \beta = s, \ s \in \mathbb{N},
\end{cases}
\]

where the function \( \Gamma(.) \) is named a Gamma function.

CFD was introduced by the Italian mathematician Caputo in 1967 [31]. It is so useful and flexibility for resolving the initial value problems with classical initial conditions.

**Definition 2** The RLI for a function \( g \in L^1([0, T]) \), \( L^1 \) represents the set of Lebesque integrable functions), of order \( \beta \in (0, 1) \) is introduced as

\[
J^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\xi)^{\beta-1}g(\xi)d\xi
\]

provide that the integral exists on the right hand side. Notice that also \( D^\beta \) is a left inverse of the RLI, namely \( D^\beta J^\beta g = g \), but not its right inverse [32] since

\[
J^\beta D^\beta f(t) = f(t) - \sum_{i=0}^{p-1} \left( \frac{(t-t_0)^i}{i!} f^{(i)}(t_0) \right). \tag{4}
\]

The book by Diethelm [32] can be examined by the readers for more informations of above results.

**Definition 3** [33] The function with one-parameter is introduced as

\[
E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)} \tag{5}
\]

in here \( \alpha > 0 \). The MLF with two parameter is constructed as follows

\[
E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)} \tag{6}
\]

The MLF was described by Mittag-Leffler in 1903 [33]. Since the MLF frequently is used to compute the solutions of FDE, it is of great importance in fractional calculus.
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The textbooks [32, 34, 35] are recommended to readers for more comprehensive introduction.

**Definition 4** [36] A dynamical system with Caputo fractional operator is introduced as

\[ D^\beta u(t) = g(t, u), \quad u(0) = u_0 \]  (7)

in here \(0 < \beta \leq 1\).

The equilibrium points of the given system (7) are computed when \(g(t, u) = 0\).

**Theorem 1** [37] If all eigenvalues \(\lambda_j\) of the Jacobian matrix \(J = \frac{\partial g}{\partial u}\) assessed at the equilibrium points fulfill \(|\arg(\lambda_j)| > \frac{\alpha \pi}{2}\), then the points are called local asymptotical stability.

We state the Arithmetic-Geometric means inequality for demonstrating the fractional derivative of Lyapunov function is non-positive.

**Lemma 1** [38] Let \(v_1, v_2, \ldots, v_n\) be positive real numbers. Then

\[ \sqrt[n]{v_1 v_2 \cdots v_n} \leq \frac{v_1 + v_2 + \cdots + v_n}{n}. \]

Besides, exact equality only occurs when \(v_1 = v_2 = \cdots = v_n\).

### 3 Examination of the fractional model of HBV

We will display the properties of the solution for the proposed design (1). The following outcomes will be used for displaying the solutions are non-negative and bounded, respectively.

**Lemma 2** [39] Assume that \(h(x) \in C[a, b]\) and \(D^\beta h(x) \in C[a, b]\) for \(\beta \in (0, 1]\), then we get \(h(x) = h(a) + \frac{1}{\Gamma(\beta)}(D^\beta h(\xi))(x - a)^\beta\), in here \(a \leq \xi \leq x,\ \forall x \in [a, b]\).

**Lemma 3** Let \(h(x) \in C[a, b]\) and \(D^\beta h(x) \in C[a, b]\) for \(\beta \in (0, 1]\). If \(D^\beta h(x) \geq 0,\ \forall x \in (a, b)\), then \(h(x)\) is nondecreasing \(\forall x \in [a, b]\). If \(D^\beta h(x) \leq 0,\ \forall x \in (a, b)\), then \(h(x)\) is nonincreasing \(\forall x \in [a, b]\).

The next lemma is significant to indicate the uniform boundedness of the solution.

**Lemma 4** [40] Let \(u(t)\) be a continuous function on \([t_0, \infty)\). If \(u(t)\) holds

\[ D^\beta u(t) \leq -\alpha u(t) + \xi, \quad u(t_0) = u_0 \in \mathbb{R}, \]

where \(0 < \beta \leq 1,\ \alpha, \xi \in \mathbb{R}\) and \(\alpha \neq 0,\) and \(t_0 \geq 0\) is the initial time. Then

\[ u(t) \leq \left( u_0 - \frac{\xi}{\alpha} \right) E^\beta \left[ -\alpha (t - t_0)^\beta \right] + \frac{\xi}{\alpha}. \]
The following theorems show the existence and uniqueness of non-negative and the uniform bounded solutions, respectively. The existence and uniqueness of non-negative solutions are demonstrated by Zhou and Sun [24].

**Theorem 2** [24] There is a unique solution to fractional order model (1) with initial conditions (2) and the solution will remain in $\mathbb{R}^3_+ = \{(P, R, S) \in \mathbb{R}^3, P \geq 0, R \geq 0, S \geq 0\}$.

The following result indicates the solutions of the system (1) are uniformly bounded.

**Proposition 1** The solutions of the proposed model (1) which start in $\mathbb{R}^3_+$ are uniformly bounded.

**Proof** We will utilize a strategy given in [40]. Let’s define the function $H(t) = 2P(t) + R(t) - S(t)$. Then

$$D^\beta H(t) = 2D^\beta (P(t)) + D^\beta (R(t)) - D^\beta (S(t))$$

$$= \left[2\lambda_1 - 2\mu P - 2\lambda_2 \frac{PS}{P + R} + 2\delta R\right] + \left[\lambda_2 \frac{PS}{P + R} - (\eta + \delta)R\right]$$

$$- \left[\sigma R - \gamma S - \lambda_2 \frac{PS}{P + R}\right]$$

$$= 2\lambda_1 - 2\mu P + (\delta - \eta - \sigma)R + \gamma S$$

$$\leq -\alpha H(t) + 2\lambda_1$$

where $\alpha = \min\{\mu, \eta + \sigma - \delta, -\gamma\}$. Now, we will apply Lemma 4. We have

$$0 \leq H(t) \leq H(0)E_\beta(-\alpha(t)^\beta) + 2\lambda_1(t)^\beta E_{\beta, \beta+1}(-\alpha(t)^\beta).$$

According to Lemma 5 and Corollary 6 in [41], by taking $t \to \infty$, we get $0 \leq H(t) \leq \frac{2\lambda_1}{\alpha}$. Consequently, the solutions of the system (1) starting in $\mathbb{R}^3_+$ are uniformly bounded within the region $\Omega_1 = \{(P, R, S) \in \mathbb{R}^3_+ : H(t) \leq \frac{2\lambda_1}{\alpha} + \epsilon, \epsilon > 0\}$. \qed

### 4 Description of FLMS methods for SFDEs

The elementary techniques such as one-step and multi-step methods are utilized for obtaining the approximate solutions of ordinary differential equations. In the one-step method, using only one approximation of the solution at the preceding stage can be calculated the solution. But, we use more previously obtained approximations to calculate the solution in the multi-step methods.

In recent years, with the widespread use of fractional differential equations, many researchers have worked to obtain analytical and numerical solutions to such differential equations. The First pioneering study is proposed by Lubich [42]. He proposed a reliable and effective methodology for FLMS. This strategy is, which a generalization of the classical linear multi-step methods, a powerful strategy for a SFDEs.
Now, we will focus on the initial value problem for a SFDEs in the following as

\[
\begin{align*}
D^\beta u(t) &= g(t, u(t)) \\
u^{(j)}(t_0) &= u_{0,j}, \quad j = 0, 1, 2, \ldots, p - 1
\end{align*}
\] (8)

in here \( p = \lceil \beta \rceil \) is the smallest integer such that \( p > \beta \), \( g : [t_0, T] \times \mathbb{R}^q \rightarrow \mathbb{R}^q \) is supposed to be sufficiently smooth, \( u : [t_0, T] \rightarrow \mathbb{R}^q \) is the unknown solution and \( u^{(j)} \) indicates the classical derivative of integer order \( j \).

The RLI (3) is implied to the both sides of the Eq. (8) with Eq. (4), the fractional differential equation can be redescribed in terms of the weakly-singular Volterra integral equation as

\[
u(t) = \sum_{i=0}^{p-1} \frac{(t-t_0)^i}{i!} u^{(i)}(t_0) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-\xi)^{\beta-1} g(\xi, u(\xi)) d\xi
\] (9)

The integral formulation (9) is certainly convenient, as it yields using the theoretical and numerical results already available for the Volterra integral equations class to study and solve FDE.

The main idea in FLMS technique is the approximation of the RLI (3) in terms of the convolution quadrature

\[
I^\beta g(t_n) = h^\beta \sum_{j=0}^{k} w_{n,j} g(t_j) + h^\beta \sum_{j=0}^{n} \omega_{n-j} g(t_j)
\] (10)

on uniform grids \( t_n = t_0 + nh, \ h > 0 \), and in which convolution and starting quadrature weights \( w_{n,j}, \ \omega_{n-j} \) do not depend on \( h \). We note that starting weights \( w_{n,j} \) are very significant in the first part integration interval to overcome the possibly singular character of integrand function at \( t_0 \). We would also like to say that the convolution quadrature weights \( \omega_{n-j} \) are the major piece of the quadrature rule and qualify the certain FLMS methods [30].

Now, let us remember the first and second characteristic polynomials \( \rho(\zeta) \) and \( \sigma(\zeta) \), which are defined as follows

\[
\rho(\zeta) = \rho_0 \zeta^k + \rho_1 \zeta^{k-1} + \cdots + \rho_k
\]

and

\[
\sigma(\zeta) = \sigma_0 \zeta^k + \sigma_1 \zeta^{k-1} + \cdots + \sigma_k
\]

for a linear multistep method. A generating function is obtained as the coefficients of the formal power series

\[
\omega(\zeta) = \sum_{n=0}^{\infty} \omega_n \zeta^n, \quad \omega(\zeta) = \frac{\sigma(1/\zeta)}{\rho(1/\zeta)}
\]
for the linear multistep method. In order to generate a quadrature rule for fractional problems (8) by evaluating the convolution weights as the coefficients in the formal power series of the fractional-order power of the generating function

$$\omega_\beta(\zeta) = \sum_{n=0}^{\infty} \omega_n \zeta^n, \quad \omega_\beta = \left( \frac{\sigma(1/\zeta)}{\rho(1/\zeta)} \right)^\beta. \quad (11)$$

This method named as FLMS methods, when applied to (8) read as

$$u_n = \sum_{i=0}^{p-1} \frac{(t-t_0)^i}{i!} u^{(i)}(t_n) + h^\beta \sum_{j=0}^{k} w_{n,j} g_j + h^\beta \sum_{j=0}^{n} \omega_{n-j} g_j \quad (12)$$

The convergence properties are given in the following studies [42, 43]. The readers can be read more detailed information for the computation of the weights by investigating the studies [30, 42–44].

### 4.1 Fractional trapezoidal rule

In this study, we will use the fractional trapezoidal rule, which is one of the FLMS methods to solve the numerically proposed model. This rule is the generalization of the standard trapezoidal rule to fractional differential equations. Firstly, let us consider its classical formulation for ordinary differential equations

$$u_{n+1} - u_n = \frac{h}{2} (g_n + g_{n+1}) \quad (13)$$

with characteristic polynomials $\rho(\zeta) = \zeta - 1$ and $\sigma(\zeta) = (\zeta + 1)/2$ and generating function

$$\omega(\zeta) = \frac{\sigma(1/\zeta)}{\rho(1/\zeta)} = \frac{(1 + \zeta)}{2(1 - \zeta)} = \frac{1}{2} \left( 1 + 2 \sum_{n=1}^{\infty} \zeta^n \right).$$

Evaluating the weights $\omega_n$ in FLMS (12) as the coefficients in the formal power series (11) is one of the major difficulties. Despite of establishing some sophisticated algorithms for manipulating formal power series [45], for most methods an efficient tool is the J. C. P. Miller formula stated by the following theorem [46].

**Theorem 3** Let $\varphi(\zeta) = 1 + \sum_{n=1}^{\infty} a_n \zeta^n$ be a formal power series. Then for any $\beta \in \mathbb{C}$,

$$(\varphi(\zeta))^\beta = \sum_{n=0}^{\infty} \nu_n^{(\beta)} \zeta^n,$$
where coefficients \( v_n^{(β)} \) can be recursively evaluated as

\[
v_0^{(β)} = 1, \quad v_n^{(β)} = \sum_{j=1}^{n} \left( \frac{(β + 1) j}{n} - 1 \right) a_j v_{n-j}^{(β)}.
\]

In the most general case, this formula allows to evaluate the first \( N \) coefficients of \( (ω(ζ))^β \) with a number of operations proportional to \( N^2 \). In the study [30], the author advises the twice application of the Miller formula to \((1 + ζ)^β \) and \((1 - ζ)^{-β} \) and thus evaluate the coefficient of their product by a Fast Fourier transform algorithm, with a number of operations proportional to \( 3N \log_2 4N \) when \( N^2 \) [45].

5 Global stability analysis

We will focus on the global stability analysis of the mentioned design (1) equilibria. But first, we will give the results of the local asymptotical stability of the equilibrium points. The authors in [24] demonstrated the basic reproductive number. It is obtained as

\[
R_0 = \frac{λ_2 (σ - η - δ)}{γ(η + δ)}.
\]

This number tells us that the predicted number of secondary cases produced, in a completely susceptible population, by a typical infective individual [47]. It is also so important tool mathematically because of indicates the spread of the disease. The disease-free equilibrium point \( E^* = (λ_1 / μ, 0, 0) \) and endemic equilibrium point \( E^{**} = \left( \frac{λ_1}{η(R_0 - 1) + μ}, \frac{λ_1(σ - η - δ)(R_0 - 1)}{η(η - 1) + μγ}, \frac{λ_1(σ - η - δ)(R_0 - 1)}{ηγ(R_0 - 1) + μγ} \right) \) are found in [24]. Then, they obtained Theorem 3.1, and Proposition 3.1-3.2 in [24] in order to prove local asymptotical stability of the given fractional system (1). If \( R_0 < 1 \), the disease-free equilibrium is local stability which means that the disease does not spread. When \( R_0 > 1 \), the disease perennially exists in the society, and then transforms an epidemic. When the reproductive number passes the unity, the endemic equilibrium is local stability.

A most significant concern for the FDE is about that the global stability of the solution. To our knowledge, the global stability of the disease-free equilibrium point \( E^* \) and endemic equilibrium point \( E^{**} \) of the present design (1) has not been proved yet. Now, we will use the extended Barbalat’s lemma and Lyapunov functions to demonstrate the stability of fractional systems. To prove the global stability of the disease-free equilibrium and endemic equilibrium points, motivated by the studies in [23, 25, 27].

**Theorem 4** [25] If \( v : \mathbb{R} \to \mathbb{R} \) is a uniformly continuous function on \([t_0, \infty)\) and

\[
J^β \|v\| ≤ N, \quad ∀t > t_0 \quad \text{with} \quad β \in (0, 1), \quad s \quad \text{and} \quad N \quad \text{are positive constants, then} \quad v(t) \to 0 \quad \text{as} \quad t \to \infty.
\]

We will utilize the given Lemma to find Lyapunov candidate functions to demonstrate the stability of SFDEs.

**Lemma 5** [26] Assume that \( z(t) \in \mathbb{R}^+ \) is a continuous function. Then for any time \( t \geq t_0 \),

\[
\frac{1}{2} D^β z^2 (t) \leq D^β z(t), \quad ∀β \in (0, 1).
\]
Now, let’s give the following Lemma which denotes the extended Volterra-type Lyapunov function to SFDEs through an inequality for approximating the CFD of the function. This Lemma is described by Leon [27].

**Lemma 6** [27] Assume that \( z(t) \in \mathbb{R}^+ \) is a continuous function. Then for any time \( t \geq t_0 \),

\[
D^\beta \left[ z(t) - z_* - z_* \ln \frac{z(t)}{z_*} \right] \leq \left( 1 - \frac{z_*}{z(t)} \right) D^\beta z(t), \quad z_* \in \mathbb{R}^+ \forall \beta \in (0, 1). \tag{15}
\]

The next outcome indicates the solutions of model (1) are uniformly continuous. The proof is done in a similar argument as in [23].

**Lemma 7** The solutions \( P, R \) and \( S \) of model (1) are uniformly continuous functions on \([0, \infty)\).

**Theorem 5** Assume \( R_0 < 1 \) and \( \inf_{t \geq 0} P(t) > 0 \). The disease-free equilibrium point \( E_* \) is globally asymptotically stable on the interior of \( \Omega \).

**Proof** We describe a Lyapunov function \( V : \{(P, R, T) \in \Omega : P > 0 \} \to \mathbb{R} \) such as

\[
v(P, R, T) = \left( P - P_0 - P_0 \ln \frac{P}{P_0} \right) + \frac{\delta}{2(\eta + \delta)} \left[(P - P_0) + R \right]^2 + \frac{(\eta + \delta)}{\lambda_2(\sigma - (\eta + \delta))} S
\]

By using Lemmas 5 and 6, we have

\[
d^\beta V \leq \left( \frac{P - P_0}{P} \right) \left[ \mu P_0 - \mu P - \lambda_2 \frac{PS}{P + R} + \delta R \right] + \frac{\delta}{(\eta + \delta)} \left[(P - P_0) + R \right] \left[ [\lambda_1 - \mu P - \eta R] \right]
\]

Since \( P_0 = \frac{\lambda_1}{\mu} \), we get

\[
d^\beta V \leq \left( \frac{P - P_0}{P} \right) \left[ \mu P_0 - \mu P - \lambda_2 \frac{PS}{P + R} + \delta R \right] + \frac{\delta}{(\eta + \delta)} \left[(P - P_0) + R \right] \left[ [\lambda_1 - \mu P - \eta R] \right]
\]

By equation \( \delta R \left( \frac{P - P_0}{P} \right) = -\delta R \left( \frac{P - P_0}{P} \right)^2 + \delta R \left( \frac{P - P_0}{P} \right), \)

\[
d^\beta V \leq \left( \frac{P - P_0}{P} \right) \left[ \mu P_0 - \mu P - \lambda_2 \frac{PS}{P + R} - \delta R \left( \frac{P - P_0}{P} \right)^2 \right] + \delta R \left( \frac{P - P_0}{P} \right)
\]
Then,
\[ D^βV ≤ \left[ μP_0 + δR + \frac{μδP}{(η + δ)} \right] \left( \frac{P - P_0}{P_0} \right)^2 - \frac{ηβ}{(η + δ)P_0} \left( \frac{P - P_0}{P_0} \right) R^2 - \left[ \frac{(η + δ)γ}{λ_2(σ - (η + δ))} - 1 \right] \frac{λ_2 PS}{P + R} \]

Let’s describe
\[ F(t) = \left[ μP_0 + δR + \frac{μδP}{(η + δ)} \right] \left( \frac{P - P_0}{P_0} \right)^2 + \frac{ηβ}{(η + δ)P_0} R^2 + \left[ \frac{(η + δ)γ}{λ_2(σ - (η + δ))} - 1 \right] \frac{λ_2 PS}{P + R} \]

Since \( R_0 < 1 \), we have \[ \left[ \frac{(η + δ)γ}{λ_2(σ - (η + δ))} - 1 \right] > 0 \]. Hence, \( F(t) \) is a positive defined function. We can observe that \( D^βV ≤ -F(t) \), which implies that \( V(t) - V(0) ≤ -J^βF(t) \). Then, we have \( V(t) + J^βH(t) ≤ V(0) = C \). It implies \( J^β \left( \frac{(P-P_0)^2}{P} \right) ≤ C \), \( J^β R^2 ≤ C \) and \( J^β \frac{PS}{P+R} ≤ C \). By using the Lemma (7) and the assumption of \( \inf P(t) > 0 \), we have the uniform continuity of \( \frac{(P-P_0)^2}{P} \), \( R^2 \) and \( \frac{PS}{P+R} \). According to Theorem 4, we get \( \frac{(P-P_0)^2}{P} → 0 \), \( R^2 → 0 \) and \( \frac{PS}{P+R} → 0 \) as \( t → ∞ \). We have \( P → P_0 \), \( R → 0 \) and \( S → 0 \) as \( t → ∞ \). Therefore, \( \lim_{t→∞} (P, R, S) = (P_0, 0, 0) \) independent of the initial data in the interior of \( Ω \). This result indicates that \( E_σ \) is globally asymptotically stable in the interior of \( Ω \).

**Theorem 6** The endemic equilibrium point \( E_{σσ} \) of the system (1) is globally asymptotically stable on the interior of \( Ω \) if \( R_0 > 1 \).

**Proof** We assume that \( R_0 > 1 \), so that the concerned endemic equilibrium exists. Now, we will define a Lyapunov function \( W : \{(P, R, T) ∈ Ω : P > 0\} → ℝ \) such as
\[ W(P, R, T) = \left( P - P_1 - P_1 \ln \frac{P}{P_1} \right) + \left( R - R_1 - R_1 \ln \frac{R}{R_1} \right) + \left( S - S_1 - S_1 \ln \frac{S}{S_1} \right) \]

By using Lemma 6, we have
\[ D^βW ≤ \left( 1 - \frac{P_1}{P} \right) D^βP + \left( 1 - \frac{R_1}{R} \right) D^βR + \left( 1 - \frac{S_1}{S} \right) D^βS \]
\[ = \left( 1 - \frac{P_1}{P} \right) \left[ λ_1 - μP - λ_2 \frac{PS}{P+R} + δR \right] + \left( 1 - \frac{R_1}{R} \right) \left[ λ_2 \frac{PS}{P+R} - (η + δ)R \right] \]
\[ + \left( 1 - \frac{S_1}{S} \right) \left[ σR - γS - λ_2 \frac{PS}{P+R} \right] \]

It can be displayed from (1) the endemic steady state,
\[ λ_1 = μP_1 + λ_2 \frac{P_1 S_1}{P_1 + R_1} - δR_1, \quad η + δ = λ_2 \frac{P_1 S_1}{P_1 + R_1}, \quad σ = \frac{γ S_1}{R_1} + λ_2 \frac{P_1 S_1}{R_1 (P_1 + R_1)} \]
We will use the above relation in (16), then we get
\[
D^\beta W \leq \mu P_1 \left( 2 - \frac{P}{P_1} - \frac{P_1}{P} \right) + \lambda_2 \frac{P_1 S_1}{P_1 + R_1} \left( \frac{P_1}{P} - \frac{S_1 R}{S_1} \right) \\
+ \lambda_2 \frac{P_1 S_1}{P_1 + R} \left( \frac{S}{S_1} - \frac{P R_1 S}{P_1 R S_1} + \frac{P}{P_1} - \frac{P S}{P_1 S_1} \right) \\
+ R_1 \left( \frac{R}{R_1} \left( 1 - \frac{P}{P_1} \right) + \frac{P_1}{P} - 1 \right) \\
+ \gamma \frac{S_1}{R_1} \left( R_1 + R \left( 1 - \frac{S_1}{S} \right) - \frac{S R_1}{S_1} \right)
\]

By using Lemma 1, since the arithmetic mean is greater than geometric mean it follows that
\[
2 - \frac{P}{P_1} - \frac{P_1}{P} \leq 0, \quad \frac{R}{R_1} \left( 1 - \frac{P}{P_1} \right) + \frac{P_1}{P} - 1 \leq 0, \\
R_1 + R \left( 1 - \frac{S_1}{S} \right) - \frac{S R_1}{S_1} \leq 0, \\
\frac{S}{S_1} - \frac{P R_1 S}{P_1 R S_1} + \frac{P}{P_1} - \frac{P S}{P_1 S_1} \leq 0, \quad \frac{P_1}{P} - \frac{S_1 R}{S R_1} \leq 0.
\]

Therefore, \( D^\beta W \leq 0 \). Assume that \( N \) is the largest invariant set in \( \{(P, R, S); D^\beta W = 0\} \). We notice that \( D^\beta W = 0 \) iff \( P = P_1, R = R_1, S = S_1 \) for any time \( t \). Hence, we can say that \( N = \{E_{**}\} = \{(P_1, R_1, S_1)\} \). Afterwards, by the Lyapunov-LaSalle invariance principle, the model is globally asymptotically stable at \( E_{**} \) when \( R_0 > 1 \). \( \square \)

### 6 Numerical results

This current section gives the numerical simulations for the present system (1). Also, our theoretical results are validated by the some numerical simulations. The numerical simulations of different values of \( \beta \) are illustrated in Figs. 1 and 2. We would like to say that the spreading of the infection during the first 1500 days converges toward the disease-free equilibrium point \( E_* = (2.5 \times 10^9, 0, 0) \). In this step, \( R_0 = 0.4952 < 1 \) implies our theoretical result about the stability of \( E_* \). In the Fig. 2, the numerical solutions converge to the endemic equilibrium point \( E_{**} = (2.1077 \times 10^8, 1.0923 \times 10^9, 4.1651 \times 10^{10}) \). In this case, we have \( R_0 = 6.1824 > 1 \). Thus this situation supports the stability result of \( E_{**} \). It is observed that from the numerical outcomes, the order of the fractional derivative \( \beta \) has no affect on the stability of the two equilibria. It should be noted that for higher values of \( \beta \), which describes the long memory behavior, the solutions converge more quickly to the regular states. Besides, to \( R(t) \), we can see the smallest values to \( \beta \) imply a wider period infectiously, so the disease takes a long time to be eradicated. This property is significant from the health point of view because it reflects a long period in which the infected individuals can affect the health system.
Fig. 1 Simulation of the infection as function of time (days) with the parameters $\lambda_1 = 2.5 \times 10^7$, $\lambda_2 = 1.67 \times 10^{-3}$, $\mu = 0.01$, $\delta = 0.08$, $\gamma = 3.8$, $\eta = 0.053$, $\sigma = 150$, which correspond to the stability of the disease-free equilibrium point $E_*$.
Fig. 2 Simulation of the infection as function of time (days) with the parameters
\( \lambda_1 = 6 \times 10^7 \), \( \lambda_2 = 1.67 \times 10^{-2} \), \( \mu = 0.01 \), \( \delta = 0.05 \), \( \gamma = 3.8 \), \( \eta = 0.053 \), \( \sigma = 145 \), which correspond to the stability of the endemic equilibrium point \( E_{ss} \).
7 Conclusion

This work proposes the numerical solutions of a fractional-order design of HBV infection by using the fractional trapezoidal formula. The mathematical representation of the fractional model is demonstrated by using the Caputo fractional derivative. Since the extension of Barbalat’s lemma to the fractional situation is an effective tool to examine the asymptotic stability analysis of the fractional dynamical systems, we have used this lemma to prove the global stability of equilibrium points are discussed by composing a suitable Lyapunov function. Then, we have shown that the global asymptotical stability of the equilibrium points $E_*$ and $E_{**}$ for $R_0 < 1$, and $R_0 > 1$, respectively. All obtained numerical experiments are illustrated by graphs. Another point that is worthy of being emphasized is that the order of the fractional derivative $\beta$ does not affect the regular state’s stability concerning both theoretical and numerical results. Besides, if the fractional-order $\beta$ increases, the solutions converge more rapidly to the regular states. As a concluding remark, we note that this study can provide beneficial outcomes for understanding and estimating the dissipation of distinct epidemics.

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