PROPERTIES OF A TWO-SPHERE SINGULARITY

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Recently Böhmer and Lobo have shown that a metric due to Florides can be extended to reveal a classical singularity that has the form of a two-sphere. Here we discuss and expand on the classical singularity properties and then show the classical singularity is not healed by a quantum analysis.

1. Introduction

The question addressed in this review is: What are the properties of the unusual two-sphere singularity discovered by Böhmer and Lobo? The answer is given for quantum as well as classical singularity structure. This conference proceeding is based on articles by Böhmer and Lobo\cite{1} and by the authors\cite{2,3}.

2. Types of Singularities

2.1. Classical Singularities

A classical singularity is indicated by incomplete geodesics or incomplete paths of bounded acceleration\cite{15} in a maximal spacetime. Since, by definition, a spacetime is smooth, all irregular points (singularities) have been excised; a singular point is a boundary point of the spacetime. There are three different types of singularity\cite{9} quasi-regular, non-scalar curvature and scalar curvature. Whereas quasi-regular singularities are topological, curvature singularities are indicated by diverging components of the Riemann tensor when it is evaluated in a parallel-propagated orthonormal frame carried along a causal curve ending at the singularity.

2.2. Quantum Singularities

A spacetime is QM (quantum-mechanically) nonsingular if the evolution of a test scalar wave packet, representing the quantum particle, is uniquely determined by
the initial wave packet, manifold and metric, without having to put boundary conditions at the singularity. Technically, a static ST (spacetime) is QM-singular if the spatial portion of the Klein-Gordon operator is not essentially self-adjoint on $C_0^\infty (\Sigma)$ in $L^2(\Sigma)$ where $\Sigma$ is a spatial slice. This is tested (see, e.g., Konkowski and Helliwell) using Weyl's limit point - limit circle criterion that involves looking at an effective potential asymptotically at the location of the singularity. Here a limit-circle potential is quantum mechanically singular, while a limit-point potential is quantum mechanically non-singular.

3. 2-Sphere Singularity – Böhmer-Lobo Space-time

The Böhmer and Lobo metric is

$$ds^2 = -\frac{dt^2}{\cos \alpha} + R^2 d\alpha^2 + R^2 \sin^2 \alpha \, d\Omega^2.$$  \hspace{1cm} (1)

where $R = \sqrt{3/8 \pi \rho_0}$ in terms of the constant energy density $\rho_0$, and $d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\phi^2$. The coordinate ranges are $-\infty < t < \infty$, $0 \leq \theta \leq \pi$, and $0 \leq \phi < 2\pi$. The radial coordinate $\alpha$ can either take the values $0 < \alpha < \pi/2$ (half a three-sphere) or $-\pi/2 \leq \alpha \leq \pi/2$ (two half three-spheres joined at $\alpha = 0$ with $\alpha = -\pi/2$ identified with $\alpha = +\pi/2$).

The Böhmer-Lobo spacetime is static, spherically symmetric, regular at $\alpha = 0$, and it has vanishing radial stresses. It is also Petrov Type D and Segre Type A1 ([11, 1, 1]), and it satisfies the strong energy condition automatically and the dominate energy condition with certain more stringent requirements. Vertical cuts through the three-sphere define latitudinal two-spheres; in particular, the equatorial cut at $\alpha = \pi/2$ is a two-sphere on which scalar polynomial invariants diverge and the tangential pressure diverges as well.

3.1. Classical singularity structure

One can show that the Böhmer-Lobo spacetime is timelike geodesically complete but null geodesically incomplete. The equatorial two-sphere is a weak, timelike, scalar curvature singularity.

3.2. Quantum singularity structure

The Klein-Gordon equation

$$|g|^{-1/2} \left( |g|^{1/2} g^{\mu\nu} \Phi_{,\nu} \right)_{,\mu} = M^2 \Phi$$  \hspace{1cm} (2)

for a scalar function $\Phi$ has mode solutions of the form

$$\Phi \sim e^{-i\omega t} F(\alpha) Y_{\ell m}(\theta, \phi)$$  \hspace{1cm} (3)
for spherically symmetric metrics, where the $Y_{\ell m}$ are spherical harmonics and $\alpha$ is the radial coordinate. The radial function $F(\alpha)$ for the Böhmer-Lobo metric obeys

$$F'' + \left( 2 \cot \alpha + \frac{1}{2} \tan \alpha \right) F' + \left[ R^2 \omega^2 \cos \alpha - \frac{\ell(\ell + 1)}{\sin^2 \alpha} - R^2 M^2 \right] F = 0,$$

and square integrability is judged by finiteness of the integral

$$I = \int d\alpha d\theta d\phi \sqrt{g} \frac{g_3}{g_0} \Phi^* \Phi,$$

where $g_3$ is the determinant of the spatial metric. A change of coordinates puts the singularity at $x = 0$ and converts the integral and differential equation to the one-dimensional Schrödinger forms $\int dx \psi^* \psi$ and

$$\frac{d^2 \psi}{dx^2} + (E - V) \psi = 0,$$

where $E = R^2 \omega^2$ with a potential that is asymptotically

$$V(x) \sim \frac{R^2 M^2 + \ell(\ell + 1)}{x^{2/3}} < \frac{3}{4x^2}.$$

It follows from Theorem X.10 in Reed and Simon that $V(x)$ is in the limit circle case, so $x = 0$ is a quantum singularity. The Klein-Gordon operator is therefore not essentially self-adjoint. Quantum mechanics fails to heal the singularity.

References

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