Degree three cohomology of function fields of surfaces

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Let \( k \) be a global field or a local field. Class field theory says that every central division algebra over \( k \) is cyclic. If \( k \) contains \( l^{th} \) roots of unity, for a prime \( l \) not equal to the characteristic of \( k \), then every element in \( H^2(k, \mu_l) \) is a symbol. A natural question is a higher dimensional analogue of this result: let \( F \) be a function field in one variable over \( k \) (\( k \) being a global field or a local field) which contains \( l^{th} \) roots of unity. Is every element in \( H^3(F, \mu_l) \) a symbol? This question, for \( F = \mathbb{Q}_p(t) \) and \( l = 2 \) was raised by Serre ([Sc], §8.3) in the context of the study of \( G_2 \)-torsors over this field. He remarks that if every quadratic form in at least nine variables over \( \mathbb{Q}_p(t) \) has a non-trivial zero, then indeed every element in \( H^3(\mathbb{Q}_p(t), \mu_2) \) is a symbol. In view of ([PS2], Heath-Brown/Leep [L]), every quadratic form over \( \mathbb{Q}_p(t) \) in at least nine variables has a non-trivial zero so that every element in \( H^3(\mathbb{Q}_p(t), \mu_2) \) is a symbol. In ([PS1], [PS2]), we proved that every element in \( H^3(F, \mu_1) \) is a symbol for a function field in one variable over a \( p \)-adic field, \( l \neq p \). This was a key ingredient in the final proof ([PS2]) that over a function field in one variable over a non-dyadic \( p \)-adic field, every quadratic form in at least nine variables has a non-trivial zero.

We observe that if \( k \) is a global field of positive characteristic \( p \neq 2 \) and \( F \) a function field in one variable over \( k \), \( F \) is a \( C_3 \) field and every quadratic form in 9 variables over \( F \) has a non-trivial zero. In particular every element in \( H^3(F, \mu_2) \) is a symbol. In this paper, we prove that if \( l \) is a prime not equal to \( p \) and \( F \) contains a primitive \( l^{th} \) root of unity, then every element in \( H^3(F, \mu_1) \) is a symbol. If \( F \) is the function field of a curve over a \( p \)-adic field \( k \) with \( p \neq l \), a local-global principle for elements of \( H^3(F, \mu_1) \) in terms of symbols in \( H^2(F, \mu_1) \) with respect to discrete valuations of \( F \) is proved in ([PS2]). The main idea in this paper is to extend this local-global principle...
to the setting where \( k \) is a global field of positive characteristic. Surprisingly this local global principle is equivalent to the vanishing of certain unramified cohomology groups of 3-folds over finite fields.

Let \( X \) be a smooth projective surface over a finite field \( F \) of characteristic \( p \) not equal to 2. Let \( f : Y \rightarrow X \) be a surjective morphism, \( Y \) being a smooth, projective 3-fold over \( F \), with the generic fibre of \( f \) a smooth conic over \( F(X) \). Using the local-global principle described earlier, we prove that the unramified cohomology group \( H^{3}_{nr}(F(Y)/Y, Q_{l}/Z_{l}(2)) \) is zero if \( l \neq p \).

The vanishing of the unramified cohomology groups has consequences in the study of integral Tate conjecture and Brauer-Manin obstruction for existence of zero-cycles (cf. §6).

We conclude (§6, 6.2) that if \( X \) is a smooth projective geometrically integral ruled surface over \( F \) and \( Y \rightarrow X \) a surjective morphism with the generic fibre a smooth conic, then the cycle map \( CH^{2}(Y) \otimes Z_{l} \rightarrow H^{4}(Y, Z_{l}(2)) \) is surjective. If \( X = C \times P^{1} \), \( C \) being a smooth projective curve and \( Y \rightarrow C \) the composite morphism \( Y \rightarrow C \times P^{1} \rightarrow C \), then Brauer-Manin obstruction is the only obstruction to local-global principle for the existence of zero-cycles of degree one on the generic fibre \( Y_{\eta} \), \( \eta \) being the generic point of \( C \) (6.2).

We would like to thank Colliot-Thélène for bringing to our attention questions related to the unramified cohomology of varieties over finite fields and their connections to the integral Tate conjecture. We also thank him for helpful discussions when this work was carried out. We also thank Saltman for discussions on splitting ramification on surfaces.

1. Some Preliminaries

In this section we recall a few basic facts from the theory of Galois cohomology. We refer the reader to (§CT1)).

Let \( F \) be a field and \( l \) a prime not equal to the characteristic of \( F \). Let \( \mu_{l} \) be the group of \( l^{th} \) roots of unity. For \( i \geq 1 \), let \( \mu_{l}^{(i)} \) be the Galois module given by the tensor product of \( i \) copies of \( \mu_{l} \). For \( n \geq 0 \), let \( H^{n}(F, \mu_{l}^{(i)}) \) be the \( n^{th} \) Galois cohomology group with coefficients in \( \mu_{l}^{(i)} \).

We have the Kummer isomorphism \( F^{*}/F^{*l} \cong H^{1}(F, \mu_{l}) \). For \( a \in F^{*} \), its class in \( H^{1}(F, \mu_{l}) \) is denoted by \( (a) \). If \( a_{1}, \cdots, a_{n} \in F^{*} \), the cup product \( (a_{1}) \cdots (a_{n}) \in H^{n}(F, \mu_{l}^{(n)}) \) is called a symbol. We have an isomorphism \( H^{2}(F, \mu_{l}) \) with the \( l \)-torsion subgroup \( lBr(F) \) of the Brauer group of \( F \).
We define the \textit{index} of an element $\alpha \in H^2(F, \mu_l)$ to be the index of the corresponding central simple algebra in $\text{Br}(F)$.

Suppose $F$ contains all the $l$th roots of unity. We fix a generator $\rho$ for the cyclic group $\mu_l$ and identify the Galois modules $\mu_l^\otimes i$ with $\mu_l$. This leads to an identification of $H^n(F, \mu_l^\otimes m)$ with $H^n(F, \mu_l)$. The element in $H^n(F, \mu_l)$ corresponding to the symbol $(a_1) \cdots (a_n) \in H^n(F, \mu_l^\otimes m)$ through this identification is again denoted by $(a_1) \cdots (a_n)$.

For a discrete valuation $\nu$ of $F$, let $\kappa(\nu)$ denote the residue field at $\nu$, $\mathcal{O}_\nu$ the ring of integers in $F$ at $\nu$ and $F_\nu$ the completion of $F$ at $\nu$. Let $\alpha \in H^n(F, \mu_l^\otimes m)$. We say that $\alpha$ is \textit{unramified} at $\nu$ if $\alpha$ is in the image of the restriction map $H^n_{\text{rig}}(\mathcal{O}_\nu, \mu_l^\otimes m) \to H^n(F, \mu_l^\otimes m)$. We say that $\alpha$ is \textit{ramified} at $\nu$ if it is not in the image of the restriction map.

Suppose $\text{char}(\kappa(\nu)) \neq l$. Then there is a \textit{residue} homomorphism $\partial_\nu : H^n(F, \mu_l^\otimes m) \to H^{n-1}(\kappa(\nu), \mu_l^\otimes (m-1))$. For an $\alpha \in H^n(F, \mu_l^\otimes m)$, $\alpha$ is unramified at $\nu$ if and only if $\partial_\nu(\alpha) = 0$. Suppose $\alpha$ is unramified at $\nu$. Let $\pi \in F^*$ be a parameter at $\nu$ and $\zeta = \alpha \cdot (\pi) \in H^{n+1}(F, \mu_l^\otimes (m+1))$. Let $\overline{\nu} = \partial_\nu(\zeta) \in H^n(\kappa(\nu), \mu_l^\otimes m)$. The element $\overline{\nu}$ is independent of the choice of the parameter $\pi$ and is called the \textit{specialization} of $\alpha$ at $\nu$.

Let $\mathcal{X}$ be a regular integral scheme of dimension $d$, with field of fractions $F$. Assume that $l$ is a unit on $\mathcal{X}$. Let $\mathcal{X}^1$ be the set of points of $\mathcal{X}$ of codimension 1. A point $x \in \mathcal{X}^1$ gives rise to a discrete valuation $\nu_x$ on $F$. The residue field of this discrete valuation ring is denoted by $\kappa(x)$. The corresponding residue homomorphism is denoted by $\partial_x$. We say that an element $\zeta \in H^n(F, \mu_l^\otimes m)$ is \textit{unramified} at $x$ if $\partial_x(\zeta) = 0$; otherwise it is said to be \textit{ramified} at $x$. We define the ramification divisor $\text{ram}\mathcal{X}(\zeta) = \sum x$ as $x$ runs over the points of $\mathcal{X}^1$ where $\zeta$ is ramified. The unramified cohomology on $\mathcal{X}$, denoted by $H^n_{\text{nr}}(F/\mathcal{X}, \mu_l^\otimes m)$, is defined as the intersection of kernels of the residue homomorphisms $\partial_x : H^n(F, \mu_l^\otimes m) \to H^{n-1}(\kappa(x), \mu_l^\otimes (m-1))$, $x$ running over $\mathcal{X}^1$. We say that $\zeta \in H^n(F, \mu_l^\otimes m)$ is \textit{unramified on} $\mathcal{X}$ if $\zeta \in H^n_{\text{nr}}(F/\mathcal{X}, \mu_l^\otimes m)$; if $\mathcal{X} = \text{Spec}(R)$, then we say that $\zeta$ is unramified on $R$ if it is unramified on $\mathcal{X}$. If $\mathcal{R}$ is a local ring at some point $P$ of $\mathcal{X}$, then we say that $\alpha$ is unramified at $P$ if it is unramified on $R$. Suppose $C$ is an irreducible subscheme of $\mathcal{X}$ of codimension 1. Then the generic point $x$ of $C$ belongs to $\mathcal{X}^1$ and we set $\partial_C = \partial_x$. If $\alpha \in H^n(F, \mu_l^\otimes m)$ is unramified at $x$, then we say that $\alpha$ is \textit{unramified} at $C$. The group of elements of $H^n(F, \mu_l^\otimes m)$ which are unramified at all discrete valuation of $F$ is denoted by $H^n_{\text{nr}}(F, \mu_l^\otimes m)$. If $C$
is an integral curve (not necessarily regular) with function field $\kappa(C)$, then $H^n_{nr}(\kappa(C)/C, \mu_l^\otimes m)$ denotes the subgroup of $H^n(\kappa(C), \mu_l^\otimes m)$ consisting of those elements which are unramified at all discrete valuations of $\kappa(C)$ which lie over a closed point of $C$. Note that if $C$ is regular this notation coincides with earlier one.

Let $\mathcal{X}$ be a regular, integral surface and $K$ its function field. Let $D$ be a divisor on $\mathcal{X}$. Then by the resolution of singularities for surfaces (cf. [Li1] and [Li2]), there exists a regular, integral surface $\mathcal{X}'$ with a proper birational morphism $\mathcal{X}' \to \mathcal{X}$, such that the total transform of $D$ is a union of regular curves with normal crossings (cf. [Sh], Theorem, p.38 and Remark 2, p. 43). We use this result throughout this paper without further reference.

We now recall some facts concerning ramifications of division algebra on surfaces from ([S1], [S2], [S3]). Let $K$ be the function field of regular integral surface $\mathcal{X}$. Let $l$ be a prime which is a unit on $\mathcal{X}$. Assume that $k$ contains a primitive $l^{th}$ root of unity. Let $\alpha \in H^2(K, \mu_l)$. Assume that $\text{ram}_\mathcal{X}(\alpha)$ is a union of regular curves with normal crossings. Let $P \in \mathcal{X}$ be a closed point. If $P$ is not on any curve in the support of $\text{ram}_\mathcal{X}(\alpha)$, then $\alpha$ is unramified on $\mathcal{O}_{\mathcal{X}, P}$. Suppose that $P$ is on a curve in the support of $\text{ram}_\mathcal{X}(\alpha)$.

If $P$ is only on one irreducible curve $C$ in the support of $\text{ram}_\mathcal{X}(\alpha)$, then we say that $P$ is a curve point for $\alpha$. Let $P$ be a curve point of $\alpha$ and $\pi \in \mathcal{O}_{\mathcal{X}, P}$ define the curve $C$ at $P$. Since $P$ is a curve point of $\alpha$, there exists a unit $u \in \mathcal{O}_{\mathcal{X}, P}$ such that $\partial_C(\alpha) = (\pi) \in H^1(\kappa(C), \mu_l)$. Further for any such unit $u \in \mathcal{O}_{\mathcal{X}, P}$, we have ([S1], 1.1) $\alpha = \alpha' + (\pi) \cdot (u)$ where $\alpha' \in H^2(K, \mu_l)$ is unramified at $P$. Let $D$ be a curve on $\mathcal{X}$ which is not in the support of $\text{ram}_\mathcal{X}(\alpha)$ and which passes through the point $P$ and $\delta$ a prime in $R = \mathcal{O}_{\mathcal{X}, P}$ defining the $D$ at $P$. Let $\overline{R}_\delta$ be the integral closure of $R/(\delta)$ in its field of fractions. We say that $D$ is an unramified prime at $P$ with respect to $\alpha$ if the specialisation of $\alpha$ at $D$ is unramified at all the discrete valuation of $\overline{R}_\delta$.

Suppose that $P$ is on more than one curve in the support of $\text{ram}_\mathcal{X}(\alpha)$. Since the support of $\text{ram}_\mathcal{X}(\alpha)$ is a union of regular curves with normal crossings, $P$ is only on two curves, say $C$ and $E$, in the support of $\text{ram}_\mathcal{X}(\alpha)$. Such a point $P$ is called a nodal point. Let $\pi$ and $\delta$ in $\mathcal{O}_{\mathcal{X}, P}$ define $C$ and $E$ at $P$ respectively so that the maximal ideal of $\mathcal{O}_{\mathcal{X}, P}$ is generated by $\pi$ and $\delta$. Suppose that $\partial_C(\alpha) \in H^1(\kappa(C), \mu_l)$ and $\partial_E(\alpha) \in H^1(\kappa(E), \mu_l)$ are unramified at $P$. Let $u(P), v(P) \in H^1(\kappa(P), \mu_l)$ be the specialisations at $P$ of $\partial_C(\alpha)$ and $\partial_E(\alpha)$ respectively. Following Saltman ([S2], §2), we say that
$P$ is a cool point if $u(P), v(P)$ are trivial and a chilli point if $u(P)$ and $v(P)$ generate the same subgroup of $H^1(\kappa(P), \mathbb{Z}/l\mathbb{Z})$ and neither of them is trivial. If $u(P)$ and $v(P)$ do not generate the same subgroup of $H^1(\kappa(P), \mathbb{Z}/l\mathbb{Z})$, then $P$ is said to be a hot point. The condition that $\partial_C(\alpha) \in H^1(\kappa(C), \mathbb{Z}/l\mathbb{Z})$ and $\partial_E(\alpha) \in H^1(\kappa(E), \mathbb{Z}/l\mathbb{Z})$ are unramified at $P$ is equivalent to the condition $\alpha = \alpha' + (u) \cdot (\pi) + (v) \cdot (\delta)$ for some units $u, v \in \mathcal{O}_{\mathcal{X}, P}$ and $\alpha'$ unramified on $\mathcal{O}_{\mathcal{X}, P}$ ([S2], §2). The specialisations of $\partial_C(\alpha)$ and $\partial_E(\alpha)$ in $H^1(\kappa(P), \mathbb{Z}/l\mathbb{Z}) \simeq \kappa(P)^*/\kappa(P)^{\times}$ are given by the images of $u$ and $v$ in $\kappa(P)$. We say that $P$ is a cold point if $\partial_C(\alpha)$ and $\partial_E(\alpha)$ are ramified at $P$.

Suppose $P$ is a chilli point. Then $v(P) = u(P)^s$ for some $s$ with $1 \leq s \leq l - 1$ and $s$ is called the coefficient of $P$ with respect to $C$. Saltman ([S2]) defined a graph using chilli points and showed that we can blow up $\mathcal{X}$ and assume that there are no loops in this graph and further one may assume that $\alpha$ has no cool points on $\mathcal{X}$. Assume that there are no chilli loops on $\mathcal{X}$ for $\alpha$. Let $\{C_1, \ldots, C_n\}$ be the support of the ram $\mathcal{X}(\alpha)$. Then Saltman ([S2]), showed that there exist integers $s_i$, $1 \leq i \leq n$, such that for every chilli point which is on $C_i$ and $C_j$ for some $i \neq j$, then the coefficient of $P$ with respect to $C_i$ is given by $s_i/s_j$. We call these $s_i$’s as coefficients of $\alpha$ at $C_i$.

2. The Brauer group of a surface

Let $\mathcal{X}$ be a regular integral surface which is quasi projective over an affine scheme. Let $K$ be the field of fractions of $\mathcal{X}$. Let $l$ be a prime which is a unit on $\mathcal{X}$. Assume that $K$ contains a primitive $l^{th}$ root of unity.

Let $\alpha \in H^2(K, \mu_l)$ with index $l$. Suppose that the ramification locus $\text{ram}_\mathcal{X}(\alpha)$ is a union of regular curves $C_1, \ldots, C_n$ with only normal crossings and $\alpha$ has no cool points and no chilly loops on $\mathcal{X}$. Let $s_i$ be the corresponding coefficients attached to $C_i$ (cf. §1). Let $F_1, \ldots, F_r$ be irreducible regular curves on $\mathcal{X}$ which are not in the support of $\text{ram}_\mathcal{X}(\alpha)$ and such that $\{F_1, \ldots, F_r, C_1, \ldots, C_n\}$ have only normal crossings. Let $m_1, \ldots, m_r$ be integers.

Suppose that $E = \sum t_jE_j$ is a divisor on $\mathcal{X}$ such that all the $t_j$’s are coprime with $l$, $E_j$’s are distinct from $C_i$, $F_s$ and do not pass through the intersection points of $C_1, \ldots, C_n, F_1, \ldots, F_r$. Further assume that the intersection multiplicity $(E \cdot C_i)_P$ at any point $P$ is a multiple of $l$ for all $1 \leq i \leq n$. 

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We begin with the following Lemma due to Saltman ([S3], 7.12). We record here a proof due to Saltman, which bridges missing details in the original proof.

**Lemma 2.1** There is an element $x \in K^*$ such that $(x) = \sum t_jE_j + E''$ where $E''$ does not contain in its support any $C_i$, $E_s$, $F_j$ and does not pass through the points of intersection of $C_1, \ldots, C_n, F_1, \ldots, F_r$. Further every curve $D$ in the support of $E''$ with prime to $l$ coefficient is an unramified prime at all points of $D \cap C_i$ for all $i$ with respect to $\alpha$.

**Proof.** Let $\mathcal{P}$ be a finite set of closed points consisting of all the points of intersection of $C_1, \ldots, C_n, E_1, \ldots, E_m, F_1, \ldots, F_r$ and at least one point from each components $C_i, F_j$ and $E_s$. By ([S3], 7.11), there exists $\delta \in K^*$ such that $(\delta) = \sum t_jE_j + E'$ satisfying the following properties:

1) $E'$ avoids $C_i$ and $E_j$ in its support, does not pass through any point of $\mathcal{P}$ except possibly some points of intersection of $C_i$ and $C_j$ for $i \neq j$.

2) $\partial_{C_i}(\alpha) = (\overline{\delta})$, where $\overline{\delta}$ denotes the image in $\kappa(C_i)$.

Let $R$ be the semi-local ring at $\mathcal{P}$. Let $T$ be the integral closure of $R$ in $L = K(\sqrt{\overline{\delta}})$. Note that there is a unique prime $\overline{E}_j$ in $\text{Spec}(T)$ lying over $E_j$ because $L/K$ is ramified at $E_j$ for all $j$.

Since $T$ is normal, it is regular at codimension one points. We claim that this divisor $\sum t_j\overline{E}_j$ is principle on $T$. Since $T$ is semi-local, it is enough to verify that $\sum t_j\overline{E}_j$ is principle at each maximal ideal $m$ of $T$. Let $P \in \mathcal{P}$ be the point corresponding to the maximal ideal of $R$ lying below $m$. If $P$ is not on any of the $E_j$’s, then clearly $\sum t_j\overline{E}_j$ is trivial at $m$. Suppose that $P$ is on $E_j$ for some $j$. Since $E'$ passes through at most $C_i \cap C_j$, $i \neq j$, among the points of $\mathcal{P}$ and $E_j$ avoids all such points, $E'$ does not pass through $P$. Hence the divisor of $\delta$ at $P$ is $t_jE_j$. In particular, the divisor of $\sqrt{\overline{\delta}}$ at $m$ is $\sum t_j\overline{E}_j$. We conclude that $\sum t_j\overline{E}_j$ is principle at $m$.

Let $w \in T$ be such that the divisor of $w$ on $T$ is $\sum t_j\overline{E}_j$. Let $x = N_{L/K}(w)$. We claim that $x$ has the required properties. The divisor of $x$ on $R$ is $\sum t_jE_j$. Hence the divisor $(x)$ of $x$ on $\mathcal{X}$ is of the form $\sum t_jE_j + E''$ for some $E''$ which avoids all the points of $\mathcal{P}$.

Let $D$ be a curve in $E''$ with prime to $l$ coefficient in $(x)$. Let $P$ be a point of intersection of $D$ and $C_i$ for some $i$. Let $\pi, \theta \in \mathcal{O}_{\mathcal{X}, P}$ be primes defining $C_i$ and $D$ at $P$ respectively. Since $E''$ avoids all the point of $\mathcal{P}$, $P$ is not on
any $E_j$. Since the divisor of $\delta$ on $R$ is $\sum t_j E_j$, $\delta$ is a unit at $P$ and $L/K$ is unramified at $P$. Since the coefficient of $D$ in $(x)$ is coprime to $l$ and $x$ is a norm from $L$, $L/K$ being Galois, $D$ splits completely in $L$. Thus the image of $\delta$ in $\kappa(D)$ is an $l^{th}$ power. Since $E''$ avoids all points of intersection of $C_i$ and $C_j$, $P$ is a curve point for $\alpha$. Since $\delta$ is a unit in $\mathcal{O}_{\mathcal{X}, P}$ and $\delta$ is the lift of the $\partial_{C_i}(\alpha)$, we have $\alpha = \alpha' + (\pi) \cdot (\delta)$ for some $\alpha'$ unramified on $\mathcal{O}_{\mathcal{X}, P}$. Thus the specialisation of $\alpha$ at $D$ is $\overline{\alpha'} + (\pi') \cdot (\overline{\delta})$. Since $\alpha$ is unramified at $P$ and $\overline{\delta}$ is an $l^{th}$ power in $\kappa(D)$, the specialisation of $\alpha$ at $D$ is unramified on the integral closure of $\mathcal{O}_{\mathcal{X}, P}/(\theta)$ in $\kappa(D)$. Thus $D$ is an unramified prime at $P$ with respect to $\alpha$. 

**Proposition 2.2** Let $\mathcal{X}$, $K$ and $l$ be as above. Let $\alpha \in H^2(K, \mu_l)$ with index $l$. Suppose that the ramification locus $\text{ram}_{\mathcal{X}}(\alpha)$ is a union of regular curves with only normal crossings and $\alpha$ has no cool points and no chilly loops on $\mathcal{X}$. Let $s_i$ be the corresponding coefficients with respect to $\alpha$. Let $F_1, \ldots, F_r$ be irreducible regular curves on $\mathcal{X}$ which are not in $\text{ram}_{\mathcal{X}}(\alpha) = \{C_1, \ldots, C_n\}$ and such that $\{F_1, \ldots, F_r\} \cup \text{ram}_{\mathcal{X}}(\alpha)$ have only normal crossings. Let $m_1, \ldots, m_r$ be integers. Then there exists $g \in K^*$ such that

$$\text{div}_{\mathcal{X}}(g) = \sum s_i C_i + \sum m_s F_s + \sum n_j D_j + lE',$$

where $D_1, \ldots, D_t$ are irreducible curves which are not equal to $C_i$ or $F_s$ for all $i, s$, $(n_j, l) = 1$ and the specialisation of $\alpha$ at $D_j$ is in $H^2_{nr}(\kappa(D_j)/D_j, \mu_l)$ for all $j$.

**Proof.** Let $T$ be a finite set of closed points of $\mathcal{X}$ containing all the points of intersection of distinct $C_i$ and $F_s$ and at least one point from each $C_i$ and $F_s$. By a semilocal argument, we choose $f \in K^*$ such that $\text{div}_{\mathcal{X}}(f) = \sum s_i C_i + \sum m_s F_s + G$ where $G$ is a divisor on $\mathcal{X}$ whose support does not contain any of $C_i$ or $F_s$ and does not intersect $T$.

Since $\alpha$ has no cool points and no chilly loops on $\mathcal{X}$, by ([S2], Prop. 4.6, see also [PS2], proof of 3.2), there exists $u \in \hat{K}^*$ such that $\text{div}_{\mathcal{X}}(uf) = \sum s_i C_i + \sum m_s F_s + \sum t_j E_j$, where $E = \sum t_j E_j$ is a divisor on $\mathcal{X}$ whose support does not contain any $C_i$ or $F_s$, does not pass through the points in $T$ and either $E$ intersect $C_i$ at a point $P$ where the specialization of $\partial_{C_i}(\alpha)$ is 0 or the intersection multiplicity $(E \cdot C_i)_P$ is a multiple of $l$. By ([S3], 7.8), we assume that $E$ intersects all points of the $C_i$ with multiplicity a multiple of
$l$, by blowing up, if necessary, the points of intersection of $E$ and $C_i$ with specialisation of $\partial C_i(\alpha)$ equal to zero.

By (2.1), there exists $x \in K^*$ such that $(x) = \sum t_j E_j + E''$ with $E''$ does not contain any $C_i$, $F_j$ or any of the nodal points of $\text{ram}_X(\alpha)$ and $F_j$'s. Further, all the curves in $E''$ with prime to $l$ coefficients in $(x)$ are unramified with respect to $\alpha$ at any intersection point with a $C_i$.

Let $D$ be a curve which is in the support of $E''$ with prime to $l$ coefficient. Since $D$ is not equal to $C_i$ for an $i$, $\alpha$ is unramified at $D$. Let $\overline{\alpha}$ be the specialisation of $\alpha$ at $D$. Let $P$ be a point of $D$. If $P$ is not on any of $C_i$, $\alpha$ is unramified at $P$ and hence $\overline{\alpha}$ is unramified on $D$ at $P$. In particular $\overline{\alpha}$ is unramified at all those discrete valuations of $\kappa(D)$ lying over the local ring $\mathcal{O}_{D,P}$. Suppose that $P$ is on $C_i$ for some $i$. Then by the choice of $x$, $D$ is an unramified prime with respect to $\alpha$ at $P$. Thus $\overline{\alpha}$ is unramified at all those discrete valuations of $\kappa(D)$ lying over $\mathcal{O}_{D,P}$. Thus $g = x^{-1}uf \in K^*$ has the required properties and the proof is complete. $\square$.

3. A local-global principle

Let $\mathcal{X}$ be a regular integral surface and $K$ its function field. Let $l$ be a prime which is a unit on $\mathcal{X}$. We assume that $K$ contains a primitive $l^{th}$ root of unit. We say that $(K, \mathcal{X})$ satisfies a local-global principle for $H^3(K, \mu_l)$ in terms of symbols in $H^2(K, \mu_l)$ if the following holds: Given $\zeta \in H^3(K, \mu_l)$ and a symbol $\alpha \in H^2(K, \mu_l)$, if for every $x \in \mathcal{X}^1$ there exists $f_x \in K_x^*$ such that $\zeta - \alpha \cdot (f_x) \in H^3_{nr}(K_x, \mu_l)$, then there exists $f \in K^*$ such that $\zeta - \alpha \cdot (f) \in H^3_{nr}(K/\mathcal{X}, \mu_l)$.

The following theorem follows on parallel lines along the proof of ([PS2], 3.4) with results from §2 substituting the ingredients in the case of function fields of $p$-adic curves.

**Theorem 3.1** Let $\mathcal{X}$ be a regular integral surface and $K$ its field of fractions. Let $l$ be a prime which is a unit on $\mathcal{X}$. Suppose that for every $D \in \mathcal{X}^1$, $H^3_{nr}(\kappa(D), \mu_l) = 0$. Then $(K, \mathcal{X})$ satisfies the local-global principle for $H^3(K, \mu_l)$ in terms of symbols in $H^2(K, \mu_l)$.

**Proof.** Let $\zeta \in H^3(K, \mu_l)$ and $\alpha \in H^2(K, \mu_l)$ which is a symbol. Suppose that for every $x \in \mathcal{X}^1$ there exists $f_x \in K_x^*$ such that $\zeta - \alpha \cdot (f_x) \in H^3_{nr}(K_x, \mu_l)$. 

Let $\mathcal{C}$ be a finite set of codimension one points of $\mathcal{X}$ containing $\text{ram}_\mathcal{X}(\alpha) \cup \text{ram}_\mathcal{X}(\zeta)$. By the weak approximation, we find $f \in K^*$ such that $(f) = (f_x) \in H^1(K_x, \mu_l)$ at all $x \in \mathcal{C}$. Let 

$$(f)_X = C' - \sum m_i F_i + lE,$$

where $C'$ is a divisor with support contained in $\mathcal{C}$, $F_i$’s are distinct irreducible curves which are not in $\mathcal{C}$, $m_i$ are positive integers coprime to $l$ and $E$ some divisor on $\mathcal{X}$.

Let $\mathcal{P}$ be a finite set of closed points of $\mathcal{X}$ containing all the points of intersections of curves in $\mathcal{C}$ and at least one point from each curve in $\mathcal{C}$. Let $A$ be the semi-local ring at $\mathcal{P}$.

Let $C \in \mathcal{C}$. If $C \in \text{ram}_\mathcal{X}(\zeta) \setminus \text{ram}_\mathcal{X}(\alpha)$, let $u_C \in \kappa(C)^*$ which is not an $l^\text{th}$ power. If $C \in \text{ram}_\mathcal{X}(\alpha)$, let $u_C \in \kappa(C)^*$ be such that $\partial_C(\alpha) = (u_C)$. If $C \not\in \text{ram}_\mathcal{X}(\zeta) \cup \text{ram}_\mathcal{X}(\alpha)$, then let $u_C \in \kappa(C)^*$ be any element.

By Chinese remainder theorem, there exists $u \in K^*$ with $\pi = u_C$ for all $C \in \mathcal{C}$ and $\nu_{F_i}(u) = m_i$. In particular $u$ is a unit at $C$ for any $C \in \mathcal{C}$.

Let $L = K(\sqrt[l]{u})$. Let $\eta : \mathcal{Y} \to \mathcal{X}$ be the normalization of $\mathcal{X}$ in $L$. Since $\nu_{F_i}(u) = m_i$ and $m_i$ is coprime to $l$, $\eta : \mathcal{Y} \to \mathcal{X}$ is ramified at $F_i$. In particular there is a unique irreducible curve $\tilde{F}_i$ in $\mathcal{Y}$ such that $\eta(\tilde{F}_i) = F_i$ and $\kappa(F_i) = \kappa(\tilde{F}_i)$.

Let $\pi : \tilde{\mathcal{Y}} \to \mathcal{Y}$ be a proper birational morphism such that $\tilde{\mathcal{Y}}$ is regular and $\text{ram}_\tilde{\mathcal{Y}}(\alpha)$, $\text{ram}_\tilde{\mathcal{Y}}(\zeta)$ and the strict transform of $\tilde{F}_i$ on $\tilde{\mathcal{Y}}$ are a union of regular curves with normal crossings. We denote the strict transforms of $\tilde{F}_i$ by $\tilde{F}_i$ again.

Let $\tilde{C}$ be an irreducible curve in $\tilde{\mathcal{Y}}$. Suppose that $\eta_*\pi_*(\tilde{C}) = d\tilde{C}$ for some irreducible curve in $\mathcal{X}$. Since $L/K$ is a Galois extension and $[\mathcal{L} : K] = l$ is a prime, $d = 1$ or $l$. Suppose $\tilde{C}$ is an irreducible curve in $\text{ram}_\tilde{\mathcal{Y}}(\alpha)$ and $\eta(\pi(\tilde{C})) = C$ is an irreducible curve in $\mathcal{X}$. By the choice of $u$, $C$ is inert in $\tilde{\mathcal{Y}}$ and hence $\eta_*\pi_*(\tilde{C}) = lC$.

Since $\text{ram}_\tilde{\mathcal{Y}}(\alpha)$ and $\tilde{F}_i$, $1 \leq i \leq r$ have only normal crossings, $\tilde{F}_i$’s do not pass through the nodal points of $\text{ram}_\tilde{\mathcal{Y}}(\alpha)$. Thus, by (2.2), there exists $g \in L^*$ such that

$$\text{div}_{\tilde{\mathcal{Y}}}(g) = C'' + \sum m_i \tilde{F}_i + \sum n_j D_j + lG,$$

where $C''$ is a divisor with support contained in $\text{ram}_\tilde{\mathcal{Y}}(\alpha)$, $G$ is a divisor on $\tilde{\mathcal{Y}}$, integers $n_j$ coprime to $l$ and irreducible curves $D_j$’s not equal to $\tilde{F}_i$, $\alpha$ is
unramified at \( D_j \) and the specialisation of \( \alpha \) in \( \kappa(D_j) \) is in \( H^2_{nr}(\kappa(D_j)/D_j, \mu_l) \). Note that \( \eta_\pi(F'') \) is a multiple of \( l \).

We now claim that \( \zeta - \alpha \cdot (fN_{L/K}(g)) \) is unramified at all codimension 1 points of \( X \). Let \( x \) be a codimension one point of \( X \).

Suppose \( x \) is not in \( \text{ram}_X(\alpha) \cup \text{ram}_X(\zeta) \cup \text{Supp}(fN_{L/K}(g)) \). Then \( \zeta \) and \( \alpha \cdot (fN_{L/K}(g)) \) are unramified at \( x \).

Suppose that \( x \) is in \( \text{ram}_X(\alpha) \cup \text{ram}_X(\zeta) \). Then by the choice of \( f \) we have \( \langle f \rangle = \langle f_x \rangle \in H^1(K_x, \mu_l) \). Hence \( \zeta - \alpha \cdot (f) \) is unramified over \( K_x \).

By the choice of \( u, \alpha \cdot (N_{L/K}(g)) \) is unramified at \( x \) (cf. [PS2], 3.3). Hence \( \zeta - \alpha \cdot (fN_{L/K}(g)) \) is unramified at \( x \).

Suppose that \( x \) is in the support of \( \text{div}_X(fN_{L/K}(g)) \) and not in \( \text{ram}_X(\alpha) \cup \text{ram}_X(\zeta) \). Then \( \alpha \) and \( \zeta \) are unramified at \( x \). We have

\[
\text{div}_X(fN_{L/K}(g)) = \text{div}_X(f) + \text{div}_X(N_{L/K}(g)) = C' - \sum m_i F_i + lE + \eta_\pi(C'') + \sum m_i \tilde{F}_i + \sum n_j D_j + lG = C' + \sum n_j \eta_\pi(D_j) + lE'.
\]

for some \( E' \). If \( D_j \) maps to a point, then \( \eta_\pi(D_j) = 0 \). The point \( x \) is in the support of \( \eta_\pi(D_j) \) for some \( j \) or \( x \) is in the support of \( E' \). In the later case, \( \alpha \cdot (fN_{L/K}(g)) \) is unramified at \( x \).

Suppose \( x \) is in the support of \( \eta_\pi(D_j) \) for some \( j \). If \( x \) occurs with multiplicity a multiple of \( l \) in \( \text{div}_X(fN_{L/K}(g)) \), then \( \alpha \cdot (fN_{L/K}(g)) \) is unramified at \( x \). Suppose \( x \) occurs with multiplicity coprime to \( l \) in \( \text{div}_X(fN_{L/K}(g)) \). In this case, \( x = \eta_\pi(D_j) \) for some \( j \) and \( x \) is a split prime in \( L/K \). In particular \( \kappa(x) = \kappa(D_j) \). By the choice of \( g, \alpha_L \) is unramified at \( D_j \) with specialisation of \( \alpha \) in \( \kappa(D_j) \) is in \( H^2_{nr}(\kappa(D_j)/D_j, \mu_l) \). Since by the assumption \( H^2_{nr}(\kappa(D_j), \mu_l) = 0 \), the specialisation of \( \alpha \) in \( \kappa(D_j) \) is zero. Since the residue of \( \alpha \cdot (fN_{L/K}(g)) \) at \( x \) is the specialisation of \( \alpha \) at \( D_j \), the residue is trivial. This completes the proof. \( \square \)

**Corollary 3.2** ([PS2], 3.4) Let \( k \) be a \( p \)-adic field and \( l \) a prime not equal to \( p \). Let \( X \) be a curve over \( k \) and \( K \) its field of fractions. Assume that \( K \) contains a primitive \( l \)-th root of unity. Let \( X \) be a regular proper curve over the ring of integers in \( k \) with the function field \( K \). Let \( \zeta \in H^3(K, \mu_l) \) and \( \alpha \in H^2(K, \mu_l) \). If for every codimension one point \( x \) of \( X \) there exists \( f_x \in K_x^* \) such that \( \zeta = \alpha \cdot (f_x) \in H^3(K_x, \mu_l) \), then there exists \( f \in K^* \) such that \( \zeta = \alpha \cdot (f) \in H^3(K, \mu_l) \).
Proof. Let $D$ be a curve in $\mathcal{X}$. Then the residue field $\kappa(D)$ is either a local field or a function field of curve over a finite field. In either case we have $\ell\text{Br}(D) = 0$. By (3.1), there exists $f \in K^*$ such that $\zeta - \alpha \cdot (f) \in H^3_{nr}(K/\mathcal{X}, \mu_l)$. By a theorem of Kato ([Ka], 5.2), we have $H^3_{nr}(K/\mathcal{X}, \mu_l) = 0$. In particular $\zeta = \alpha \cdot (f)$.

Corollary 3.3  Let $k$ be a finite field and $l$ a prime not equal to the characteristic of $k$. Let $X$ be a smooth projective surface over $k$ and $K$ its field of fractions. Assume that $K$ contains a primitive $l$th root of unity. Let $\zeta \in H^3(K, \mu_l)$ and $\alpha \in H^2(K, \mu_l)$. If for every codimension one point $x$ of $\mathcal{X}$ there exists $f_x \in K_x^*$ such that $\zeta = \alpha \cdot (f_x) \in H^3(K_x, \mu_l)$, then there exists $f \in K^*$ such that $\zeta = \alpha \cdot (f) \in H^3(K, \mu_l)$.

Proof. Let $D$ be a curve in $X$. Then $D$ is a projective curve over a finite field. In particular $\ell\text{Br}(D) = 0$. By (3.1), there exists $f \in K^*$ such that $\zeta - \alpha \cdot (f) \in H^3_{nr}(K/\mathcal{X}, \mu_l)$. By ([CTSS], [Ka]), we have $H^3_{nr}(K/\mathcal{X}, \mu_l) = 0$. In particular $\zeta = \alpha \cdot (f)$.

4. The $H^3$ of the function field of a surface

Let $K$ be a field of characteristic not equal to 2. Suppose that for every discrete valuation $\nu$ of $K$, the characteristic of $\kappa(\nu)$ is not equal to 2 and every element in $H^2(\kappa(\nu), \mu_2)$ is a symbol. Since $H^3_{nr}(K_{\nu}, \mu_l)$ is the kernel of the residue homomorphism $\partial_\nu : H^3(K_{\nu}, \mu_2) \to H^2(\kappa(\nu), \mu_2)$, every element $\zeta \in H^3(K_{\nu}, \mu_2)$ is of the form $\zeta' + (u) \cdot (v) \cdot (\pi)$ for some $\zeta' \in H^3_{nr}(K_{\nu}, \mu_2)$, $u, v$ units in $O_{\nu}$ and $\pi \in O_{\nu}$ a parameter (cf. [PS2], proof of 1.1).

Theorem 4.1  Let $\mathcal{X}$ be regular integral surface and $K$ its field of fractions. Let $l$ be a prime not equal to the characteristic of $K$. Suppose that $K$ contains a primitive $l$th root of unit. Assume that for $x \in \mathcal{X}^1$, every element in $H^2(\kappa(x), \mu_l)$ is a symbol. If the local-global principle holds for $H^3(K, \mu_l)$ in terms of symbols in $H^2(K, \mu_l)$, then every element in $H^3(K, \mu_l)$ is of the form $\beta + (a) \cdot (b) \cdot (c)$ for some $a, b, c \in K^*$ and $\beta \in H^3_{nr}(K/\mathcal{X}, \mu_l)$.

Proof. Let $\zeta \in H^3(K, \mu_l)$. Let $x \in \mathcal{X}$ be a point of codimension 1. Then $\zeta$ is unramified at almost all codimension one points of $\mathcal{X}$. Let $x_1, \ldots, x_n$
be all the codimension one points of $\mathcal{X}$ where $\zeta$ is ramified. Since, by the assumption, every element in $H^2(\kappa(i), \mu_i)$ is a symbol, we have $\zeta = \beta_i + (a_i) \cdot (b_i) \cdot (c_i) \in H^3(K_{x_i}, \mu_i)$ for some $a_i, b_i, c_i \in K_{x_i}$ and $\beta_i \in \text{H}^3_{nr}(K_{x_i}, \mu_i)$.

By the weak approximation, there exist $a, b \in K^*$ such that $(a) = (a_i)$ and $(b) = (b_i)$ in $H^1(K_{x_i}, \mu_i)$. Let $\alpha = (a) \cdot (b) \in H^2(K, \mu_i)$. Then we have $\alpha = (a_i) \cdot (b_i) \in H^2(K_{x_i}, \mu_i)$ for all $i$.

Let $x$ be a codimension one point of $\mathcal{X}$. We set $f_x = c_i$ if $x = x_i$ for some $i$ and $f_x = 1$ otherwise. By the choice of $\alpha$ and $f_x$, we have $\zeta - \alpha \cdot (f_x) = \beta_i \in \text{H}^3_{nr}(K_{x_i}, \mu_i)$ for $1 \leq i \leq n$. If $x \neq x_i$ for any $i$, then $\zeta \in \text{H}^3_{nr}(K_x, \mu_i)$ and $\alpha \cdot (f_x) = 1$, by the choice of $f_x$. Thus $\zeta - \alpha \cdot (f_x) \in \text{H}^3_{nr}(K_x, \mu_i)$ for all $x \in \mathcal{X}$. Since, by the assumption, the local-global principle for $H^3(K, \mu_i)$ in terms of symbols in $H^2(K, \mu_i)$ holds for $(K, \mathcal{X})$, there exists $f \in K^*$ such that $\zeta - \alpha \cdot (f) = \zeta - (a) \cdot (b) \cdot (f)$ is unramified at every codimension one point of $\mathcal{X}$.

$\square$.

**Corollary 4.2** ([PS2], 3.5) Let $k$ be a $p$-adic field and $l$ a prime not equal to $p$. Suppose that $k$ contains a primitive $l$th of unity. Let $X$ be a curve over $k$ and $K$ its function field. Then every element in $H^3(K, \mu_i)$ is a symbol.

**Proof.** Let $\mathcal{O}_k$ be the ring of integers in $k$ and $\mathcal{X}$ a regular integral surface, proper over $\text{Spec}(\mathcal{O}_k)$ with function field $K$. Let $x$ be a codimension one point of $\mathcal{X}$. Then $\kappa(x)$ is either a $p$-adic field or a function field of a curve over a finite field. In either case, every element in $H^2(\kappa(x), \mu_i)$ is a symbol. Since $H^3_{nr}(K/\mathcal{X}, \mu_i) = 0$ ([Ka], 5.2), the corollary follows from (4.1).

$\square$.

**Corollary 4.3** Let $k$ be a finite field and $X$ a smooth, projective, geometrically integral surface over $k$. Let $K$ be the function field of $X$. Let $l$ be a prime not equal to $\text{char}(k)$. Assume that $k$ contains all the $l$th roots of units. Then every element in $H^3(K, \mu_i)$ is a symbol.

**Proof.** Let $x$ be a codimension one point of $X$. Then $\kappa(x)$ a function field of a curve over a finite field and hence every element in $H^2(\kappa(x), \mu_i)$ is a symbol. Since $H^3_{nr}(K/\mathcal{X}, \mu_i) = 0$ ([CTSS], [Ka]), corollary follows from (4.1).

$\square$
5. The unramified $H^3$ of function fields of conic fibrations over surfaces

Let $K$ be a field of characteristic not equal to 2. For a discrete valuation $\nu$ of $K$, let $K_\nu$ be the completion of $K$ at $\nu$ and $\kappa(\nu)$ the residue field at $\nu$.

We say that a local-global principle for $H^3(K, \mu_2)$ holds in terms of symbols in $H^2(K, \mu_2)$ if for every $\zeta \in H^3(K, \mu_2)$, $\alpha \in H^2(K, \mu_2)$ representing a quaternion algebra over $K$ and $\nu$ a discrete valuation of $K$, if there exists $f_\nu \in K_\nu^*$ such that $\zeta - \alpha \cdot (f_\nu) \in H^3_{nr}(K_\nu, \mu_2)$ then there exists $f \in K^*$ such that $\zeta - \alpha \cdot (f) \in H^3_{nr}(K, \mu_2)$.

**Theorem 5.1** Let $K$ be a field of characteristic not equal to 2. Suppose that for every discrete valuation $\nu$ of $K$, the characteristic of $\kappa(\nu)$ is not equal to 2. The local-global principle for $H^3(K, \mu_2)$ in terms of symbols in $H^2(K, \mu_2)$ holds if and only if for every smooth conic $C$ over $K$, the restriction map

$$H^3_{nr}(K, \mu_2) \to H^3_{nr}(K(C), \mu_2) \cap \text{image}(H^3(K, \mu_2))$$

is onto.

**Proof.** Suppose that the local-global principle for $H^3(K, \mu_2)$ in terms of symbols in $H^2(K, \mu_2)$ holds. Let $\beta \in H^3_{nr}(K(C), \mu_2)$ which is in the image of $H^3(K, \mu_2)$. Let $\zeta \in H^3(K, \mu_2)$ which maps to $\beta$. Let $\alpha \in H^2(K, \mu_2)$ be the element representing the quaternion algebra associated to the conic $C$.

Since $\alpha \cdot (f) \in H^3(K, \mu_2)$ maps to zero in $H^3(K(C), \mu_2)$ for any $f \in K^*$, it is enough to show that there exists $f \in K^*$ such that $\zeta - \alpha \cdot (f) \in H^3_{nr}(K, \mu_2)$. Since the local-global principle for $H^3(K, \mu_2)$ in terms of symbols in $H^2(K, \mu_2)$ holds, it suffices to show that for every discrete valuation $\nu$ of $K$, there exists $f_\nu \in K_\nu^*$ such that $\zeta - \alpha \cdot (f_\nu) \in H^3_{nr}(K_\nu, \mu_2)$.

Let $\nu$ be a discrete valuation of $K$. If $\zeta$ is trivial over $K_\nu$, then $\zeta = \alpha \cdot (1)$ over $K_\nu$. Assume that $\zeta$ is non-trivial over $K_\nu$.

Suppose that $\alpha$ is unramified at $\nu$. Then there exists a discrete valuation $w$ on $K(C)$ extending $\nu$ on $K$ with residue field $\kappa(\nu)(\overline{C})$, where $\overline{C}$ is the reduction of $C$ at $\nu$. Since $\beta \in H^3_{nr}(K(C), \mu_2)$ is the image of $\zeta \in H^3(K, \mu_2)$, we have $\partial_\nu(\zeta) = \partial_w(\beta) = 0$ over $\kappa(\nu)(\overline{C})$. Since $\partial_\nu(\zeta) \neq 0$ in $H^2(\kappa(\nu), \mu_2)$
and \( \overline{C} \) is the conic associated to the specialisation \( \overline{\alpha} \) of \( \alpha \) at \( \nu \), by a theorem of Amitsur, we have \( \partial_{\nu}(\zeta) = \overline{\alpha} \). Since \( K_{\nu} \) is a complete discrete valued field with residue field \( \kappa(\nu) \), we have \( \zeta - \alpha \cdot (\pi) \in H^3_{nr}(K_{\nu}, \mu_2) \).

Suppose that \( \alpha \) is ramified at \( \nu \). Then we can write \( \alpha = (a) \cdot (b\pi) \) over \( K_{\nu} \) for some \( a, b \in K_{\nu}^* \) which are units in \( \mathcal{O}_{\nu} \). There is an extension \( w \) of \( \nu \) to \( K(C) \) with residue field \( \kappa(\nu)(\sqrt{a}) \). As above since the residue \( \partial_w(\zeta) \) is zero over \( \kappa(w) = \kappa(\nu)(\sqrt{a}) \), it follows that \( \partial_w(\zeta) \) is split over \( K(\nu)(\sqrt{a}) \) and hence \( \partial_w(\zeta) = (\overline{\pi})(\overline{\alpha}) \) for some unit \( c \in \mathcal{O}_{\nu} \). Hence \( \zeta' = \zeta - (a) \cdot (\pi) \) is unramified at \( \nu \). We have over \( K_{\nu} \)

\[
\zeta - \alpha \cdot (c) = \zeta - (a) \cdot (c) \cdot (\pi) + (a) \cdot (c) \cdot (\pi) - (a) \cdot (c) \cdot (b\pi) = \zeta' - (a) \cdot (b) \cdot (c).
\]

Since \( a, b, c \) are units at \( \nu \), \( (a) \cdot (c) \cdot (b) \) is unramified at \( \nu \). In particular \( \zeta - \alpha \cdot (c) \) is unramified at \( \nu \).

Conversely, suppose that the restriction map \( H^3_{nr}(K, \mu_2) \to H^3_{nr}(K(C), \mu_2) \cap \text{image}(H^3(K, \mu_2)) \) is onto.

Let \( \zeta \in H^3(K, \mu_2) \) and \( \alpha \in H^2(K, \mu_2) \) representing a quaternion algebra. Suppose that for every discrete valuation \( \nu \) there exists \( f_{\nu} \in K_{\nu}^* \) such that \( \zeta - \alpha \cdot (f_{\nu}) \in H^3_{nr}(K_{\nu}, \mu_2) \).

Let \( C \) be the conic associated to \( \alpha \). Let \( w \) be a discrete valuation on \( K(C) \). Suppose that \( w \) is trivial on \( K \). Since \( \zeta \) is defined over \( K \), \( \zeta \) is unramified at \( w \). Assume that \( w \) is non-trivial over \( K \). Let \( \nu \) be the restriction of \( w \) to \( K \). Then we have \( K_{\nu} \subset K_{\nu}(C) \subset K(C)_w \). Since \( \zeta - \alpha \cdot (f_{\nu}) \) is unramified at \( \nu \) and \( \alpha \cdot (f_{\nu}) \) is zero over \( K(C)_w \), \( \zeta \) is unramified at \( w \).

In particular \( \zeta \in H^3_{nr}(K(C), \mu_2) \cap \text{image}(H^3(K, \mu_2)) \). By the hypothesis, there exists \( \beta \in H^3_{nr}(K, \mu_2) \) which maps to \( \zeta \) in \( H^3(K(C), \mu_2) \). Since \( \zeta - \beta \in H^3(K, \mu_2) \) maps to 0 in \( H^3(K(C), \mu_2) \), we have \( \zeta = \beta + \alpha \cdot (f) \) for some \( f \in K^* \). Since \( \beta \in H^3_{nr}(K, \mu_2) \) we are done.

We thank Piruka for for the observation that no condition on the Brauer group of the residue fields is required in the hypothesis of 5.1.

**Corollary 5.2** Let \( k \) be a finite a field of characteristic not equal to 2. Let \( X \) be a smooth, projective, integral surface over \( k \) and \( K \) its function field. Let \( n \) be a natural number coprime to the characteristic of \( K \) and \( C \) a conic over \( K \). Then \( H^3_{nr}(K(C), \mu_n) = 0 \).
Proof. Suppose that \( C \) is the projective line. Then we have \( H^3_{nr}(K(C), \mu_n) \cong H^3_{nr}(K, \mu_n) = 0 \) ([CTSS], [Ka]).

Assume that \( C \) is an anisotropic conic. Since \( C \) is isomorphic to the projective line over a degree 2 extension of \( K \), it follows that the cokernel of the map \( H^3_{nr}(K, \mu_n) \to H^3_{nr}(K(C), \mu_n) \) is 2-torsion. Since \( H^3_{nr}(K, \mu_n) = 0 \) ([CTSS], [Ka]), \( H^3_{nr}(K(C), \mu_n) \) is 2-torsion. Hence \( H^3_{nr}(K(C), \mu_n) \subset H^3_{nr}(K(C), \mu_2) \). Thus it is enough to show that \( H^3_{nr}(K(C), \mu_2) = 0 \).

Let \( \beta \in H^3_{nr}(K(C), \mu_2) \). We identify \( H^3(K, \mu_2) \) with its image in \( H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \) ([MS]). Since \( C \) is a smooth conic, there exists \( \zeta \in H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \) which maps to \( \beta \) (KRS, Thm. 5). Let \( v \) be a discrete valuation of \( K \). As in the proof of (5.1), there exists a discrete valuation \( w \) of \( K(C) \) with \( \kappa(w) \) either the function field of a conic over \( \kappa(v) \) or the function field of the projective line over a quadratic extension of \( \kappa(v) \). We have \( \partial_v(\zeta) \otimes \kappa(w) = \partial_w(\beta) = 0 \). Since \( \kappa(w) \) is either the function field of a conic over \( \kappa(v) \) or the function of the projective line over a quadratic extension of \( \kappa(v) \), it follows that \( 2\partial_v(\zeta) = 0 \). Hence \( 2\zeta \) is unramified at \( v \). Since \( H^3_{nr}(K, \mathbb{Q}/\mathbb{Z}(2)) = 0 \) ([CTSS], [Ka]), \( 2\zeta = 0 \), i.e. \( \zeta \in H^3(K, \mu_2) \). Hence \( \beta \in H^3_{nr}(K(C), \mu_2) \cap \text{image}(H^3(K, \mu_2)) \). By (3.3) and (5.1), the map \( H^3_{nr}(K, \mu_2) \to H^3_{nr}(K(C), \mu_2) \cap \text{image}(H^3(K, \mu_2)) \) is onto. Since \( H^3_{nr}(K, \mu_2) = 0 \) ([CTSS], [Ka]), the theorem follows. \( \square \).

Corollary 5.3 Let \( k \) be a \( p \)-adic field with \( p \neq 2 \). Let \( X \) be a smooth, projective, integral curve over \( k \) and \( K \) its function field. Let \( l \) be a prime not equal to 2 and \( C \) a conic over \( K \). Then \( H^3_{nr}(K(C), \mu_l) = 0 \).

Proof. Since \( H^3_{nr}(K, \mu_2) = 0 \) ([Ka], 5.2), the theorem follows (3.2) and (5.1) as above. \( \square \).

6. Local-global principle for zero-cycles and integral Tate conjecture

Let \( F \) be a finite field and \( X \) a smooth, projective, geometrically integral variety over \( F \) of dimension \( d \). Let \( l \) be a prime not equal to the characteristic of \( F \). We have the cycle map \( CH^i(X) \otimes_{\mathbb{Z}} \mathbb{Z}_l \to H^{2i}(X, \mathbb{Z}_l(i)) \). The integral Tate Conjecture states that this map is surjective.
Let $C$ be a smooth, projective, geometrically integral curve over $F$. Let $X$ be a smooth projective, geometrically integral variety over $F$ of dimension $d+1$ with a flat morphism $X \to C$ with generic fibre $X_\eta /F(C)$ smooth and geometrically integral. Let $l$ be a prime not equal to the characteristic of $F$. A theorem of Saito/Colliot-Thélène ([Sa], [CT2]) asserts that if the cycle map $CH^d(X) \otimes Z_l \to H^{2d}(X, Z_l(d))$ is onto, then, the Brauer-Manin obstruction is the only obstruction for the local-global principle for the existence of zero-cycles of degree 1 on $X_\eta$.

For a certain class $B_{Tate}(F)$ of smooth, projective varieties $Y$, Bruno Kahn ([K]) showed that the cycle map $CH^2(Y) \otimes Z_l \to H^4(Y, Z_l(2))$ is onto if and only if $H^3_{nr}(F(Y)/Y, Q_l/Z_l(2)) = 0$ ([K]). Suppose $Y$ is a smooth projective variety with a dominant morphism $Y \to X$, where $X$ is a smooth geometrically ruled surface, with the generic fibre a smooth conic over $F(X)$. By a theorem of Soulé ([So]) $Y$ is in $B_{Tate}(F)$. We have the following

**Theorem 6.1** Let $F$ be a field of characteristic not equal to 2. Let $l$ be a prime not equal to the characteristic of $F$. Let $X$ be a smooth, projective, geometrically ruled surface over $F$ and $Y \to X$ a surjective morphism with the generic fibre a smooth conic over $F(X)$. Then the cycle map $CH^2(Y) \otimes Z_l \to H^4(Y, Z_l(2))$ is surjective.

**Proof.** By the theorem of Soulé ([So]), $Y$ is in $B_{Tate}(F)$ and by the theorem of Kahn ([K]), surjectivity of the cycle map follows from the vanishing of the group $H^3_{nr}(F(Y)/Y, Q_l/Z_l(2))$ which is proved in (5.2).

**Corollary 6.2** Let $F$ be a finite field of characteristic not equal to 2. Let $X$ a smooth, projective, geometrically ruled surface over $F$ and $Y$ a smooth, geometrically integral 3-fold over $F$ with a surjective morphism $Y \to X$ with generic fibre a smooth conic over $F(X)$. Let $C$ be a smooth, projective geometrically integral curve over $F$ together with a surjective morphism $X \to C$. Let $Y_\eta$ be the generic fibre of the composite morphism $Y \to X \to C$. Then the Brauer-Manin obstruction is the only obstruction to local global principle for the existence of zero-cycles of degree one on $Y_\eta$.

**Proof.** Suppose that there is no Brauer-Manin obstruction for the existence of zero-cycles of degree one on $Y_\eta$. Suppose that $Y_\eta$ has a zero-cycle of degree
one over each completion of $\mathbf{F}(C)$. Since the characteristic of $\mathbf{F}$ is not equal to 2, by (6.1) and (Saito/CT), there exists a zero-cycle on $Y_\eta$ of odd degree.

Let $X_\eta$ be the generic fibre of the projection $X \to C$. Then $Y_\eta \to X_\eta$ is a conic fibration. In particular $Y_\eta$ has a point of degree 2 over $\mathbf{F}(C)$. Hence $Y_\eta$ has a zero-cycle of degree 1. \hfill \Box

**Example 6.3.** Let $E$ be a smooth, projective geometrically integral non-rational curve over a finite field $\mathbf{F}$ of characteristic not equal to 2. Let $X = E \times \mathbf{P}^1$. Let $Y \to X$ be as in (6.2) and $X \to \mathbf{P}^1$ be the projection. Let $Y_\eta$ be the generic fibre of the composite $Y \to X \to \mathbf{P}^1$. Then $Y_\eta$ is a non-rational surface over $\mathbf{F}(t)$ and Brauer-Manin obstruction is the only obstruction to local global principle for the existence of zero-cycles of degree one on $Y_\eta$.

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