GLOBAL DYNAMICS OF A CHEMOTAXIS MODEL WITH SIGNAL-DEPENDENT DIFFUSION AND SENSITIVITY

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ABSTRACT. In this paper, we shall study the initial-boundary value problem of a chemotaxis model with signal-dependent diffusion and sensitivity as follows

\begin{align*}
    &u_t = \nabla \cdot (\gamma(v)\nabla u - \chi(v)u\nabla v) + \alpha uF(w) + \theta u - \beta u^2, \quad x \in \Omega, \quad t > 0, \\
    &v_t = D\Delta v + u - v, \quad x \in \Omega, \quad t > 0, \\
    &w_t = \Delta w - uF(w), \quad x \in \Omega, \quad t > 0, \\
    &\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial \Omega, \quad t > 0, \\
    &(u(0, x), v(x, 0), w(x, 0)) = (u_0(x), v_0(x), w_0(x)), \quad x \in \Omega,
\end{align*}

in a bounded domain \( \Omega \subset \mathbb{R}^2 \) with smooth boundary, where \( \alpha, \beta, D \) are positive constants, \( \theta \in \mathbb{R} \) and \( \nu \) denotes the outward normal vector of \( \partial \Omega \). The functions \( \chi(v), \gamma(v) \) and \( F(w) \) satisfy

• \( (\gamma(v), \chi(v)) \in [C^2[0, \infty)]^2 \) with \( \gamma(v) > 0, \gamma'(v) < 0 \) and \( \frac{|\chi(v)| + |\gamma'(v)|}{\gamma(v)} \) is bounded;

• \( F(w) \in C^1([0, \infty]), F(0) = 0, F'(w) > 0 \) in \( (0, \infty) \) and \( F'(w) > 0 \) on \( (0, \infty) \).

We first prove that the existence of globally bounded solution of system (\( * \)) based on the method of weighted energy estimates. Moreover, by constructing Lyapunov functional, we show that the solution \((u, v, w)\) will converge to \((0, 0, w_\ast)\) in \( L^\infty \) with some \( w_\ast \geq 0 \) as time tends to infinity in the case of \( \theta \leq 0 \), while if \( \theta > 0 \), the solution \((u, v, w)\) will asymptotically converge to \( (\theta \beta, \theta \beta, 0) \) in \( L^\infty \)-norm provided \( D > \max_{0 \leq v \leq \infty} \frac{\|\gamma(v)\|^2}{16\beta^2 (v_1 \beta^2)} \).

1. Introduction and main results. To describe the motion of a slime mold ameba in response to the chemical signals, Keller and Segel [16] has proposed the following chemotaxis model

\begin{align}
    &u_t = \nabla \cdot (\gamma(v)\nabla u - \chi(v)u\nabla v) + \mu u(1 - u), \quad x \in \Omega, \quad t > 0, \\
    &v_t = D\Delta v + u - v, \quad x \in \Omega, \quad t > 0,
\end{align}

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where \( \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth boundary, \( u(x,t) \) denotes the cell density and \( v(x,t) \) is the concentration of chemical. The main feature of the system (1) is the motility function \( \gamma(v) > 0 \) and the chemotactic coefficient \( \chi(v) \) are functions depending on chemical signal \( v \), which are connected by the following relation

\[
\chi(v) = (\alpha - 1)\gamma'(v),
\]

(2)

where \( \alpha \) denote the ratio of effective body length to step size, and \( \gamma'(v) < 0 \) if motility decreases with concentration.

For the system (1), we can summary the results as follows.

- When \( \gamma(v) \) is a constant, the previous results on the system (1) are mostly focus on the global boundedness (cf. [39, 22, 35, 31]), stabilization ([19, 36, 5]) and pattern formation ([23, 31, 17]). One can find more details from the review papers ([6]).

- When \( \gamma(v) \) is a function satisfying (2), the existing results are mostly limited to the special case \( \chi(v) = -\gamma'(v) > 0 \), and the system (1) can be rewritten as

\[
\begin{aligned}
&u_t = \Delta(\gamma(v)u) + \mu u(1-u), \quad x \in \Omega, t > 0, \\
v_t = D\Delta v + u - v, \quad x \in \Omega, t > 0,
\end{aligned}
\]

(3)

which has also been used in [7] to study the effect of density-suppressed motility. For the system (3), it has been proved that the logistic growth source can prevent the blow-up of solution in two dimensional spaces for any \( \mu > 0 \) [11]. Moreover, if \( \mu \) is large, the constant steady state \((1, 1)\) is asymptotical stability. One can also find some further results on the global existence and large time behavior of solution for the parabolic-elliptic case with some weaker conditions [9]. The global existence results also been extended to the higher dimensions \((n \geq 3)\) for large \( \mu \) [32]. When \( \mu \) is small, the existence/nonexistene of non-constant steady states of (3) has been studied in [21] recently. When \( \mu > 0 \) and \( \gamma(v) \) is step function, the dynamics of interface was studied in [25] in one dimension.

If the logistic growth is ignored (i.e., \( \mu = 0 \)), the solution behavior is complicated. More precisely, if \( \gamma(v) \) has a positive lower and upper bound, Tao & Winkler [30] proved the existence of global classical solution in two dimensions and global weak solution in three dimensions. If \( \gamma(v) = c_0/v^k(k > 0) \), it has been proved that the global classical solution exist in two dimensions [8] and in higher dimensions \((n \geq 3)\) provided \( c_0 \) small [40] or \( 0 < k < \frac{2}{n-2} \) [1]. If \( \gamma(v) \) decays faster than algebra, the solution may blow up. In fact, if \( \gamma(v) = e^{-\chi v} \), by constructing a Lyapunov functional, it has been proved [15, 8] that there exists a critical mass \( m_* \) such that the solution exists globally if \( \int_{\Omega} u_0 dx < m_* \) and blow up if \( \int_{\Omega} u_0 dx > m_* \).

As recalled above, few results are known for the system (1) with more general \( \gamma(v) \) and \( \chi(v) \). Moreover, the nutrient consume is ignored for the above discussed system. In this paper, we shall study the global dynamics of solution of system (1) with the
behavior may become more complicated. In fact, if $\theta$ with the system (1), the nutrient is taken into and hence the dynamics of solution response function in the predator-prey systems (cf. [12, 13, 38, 34]). Compared with constants $\lambda > 0$ in (H2), the system (4) becomes the following three-component reaction-diffusion system
\[
\begin{align*}
  u_t &= \nabla \cdot (\gamma(v)\nabla u - \chi(v)u\nabla v) + \alpha uvF(w) + \theta u - \beta u^2, \quad x \in \Omega, \ t > 0, \\
  v_t &= D\Delta v + u - v, \quad x \in \Omega, \ t > 0, \\
  w_t &= \Delta w - uF(w), \quad x \in \Omega, \ t > 0, \\
  \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0, \\
  u(x, 0) &= u_0(x), \ v(x, 0) = v_0(x), \ w(x, 0) = w_0(x), \quad x \in \Omega,
\end{align*}
\] under the following assumptions:

(H1) $(\gamma(v),\chi(v)) \in [C^2(0,\infty)]^2$ with $\gamma(v) > 0$, $\gamma'(v) < 0$ and $\frac{\gamma(v)}{\gamma(v)}$ is bounded;

(H2) $F(w) \in C^1([0,\infty))$, $F(0) = 0$, $F(w) > 0$ in $(0,\infty)$ and $F'(w) > 0$ on $[0,\infty)$.

In the system (4), $u(x,t)$ denotes the cell density and $v(x,t)$ is the chemical concentration, $w(x,t)$ stands for the nutrient density. The parameters $\alpha, \theta, \beta, D$ are constants.

Let the assumptions in Theorem 1.1 hold and $u, v, w$ satisfy the conditions (H1)-(H2). Then for any $\gamma(v)$, $\chi(v) = -\gamma'(v)$, the system (4) becomes the following three-component reaction-diffusion system
\[
\begin{align*}
  u_t &= \Delta (\gamma(v)u) + \frac{\alpha w^2 u}{w^2 + \lambda}, \quad x \in \Omega, \ t > 0, \\
  v_t &= D\Delta v + u - v, \quad x \in \Omega, \ t > 0, \\
  w_t &= \Delta w - \frac{w^2 u}{w^2 + \lambda}, \quad x \in \Omega, \ t > 0,
\end{align*}
\] which has been proposed in [20] to gain a quantitative understanding of the patterning process in the experiment. For the system (5), if $\gamma(v)$ has lower and upper bound, it has been proved that the global classical solution exists, which will converge to $(u_*, u_*, 0)$ for large $D$, where $u_* = \frac{1}{|\Omega|} \int_{\Omega} u_0 dx + \frac{1}{|\Omega|} \int_{\Omega} w_0 dx$ [14]. In this paper, we shall establish the existence of global classical solution and large time behavior of system (4) with the assumptions (H1)-(H2). First, based on the weighted energy estimates, we shall prove the existence of globally bounded solutions to the system (4) in two dimensions as follows.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and the assumptions (H1)-(H2) hold. Assume $(u_0, v_0, w_0) \in [W^{1,\infty}(\Omega)]^3$ with $u_0, v_0, w_0 > 0$. Then for any $\beta > 0$, the system (4) has a unique global classical solution $(u, v, w) \in [C^0(\Omega \times [0,\infty)) \cap C^{2,1}(\Omega \times (0,\infty))]^3$ satisfying $u, v, w > 0$ for all $t > 0$. Moreover, there exists a constant $C > 0$ independent of $t$ such that
\[
\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,\infty}} + \|w(\cdot,t)\|_{W^{1,\infty}} \leq C.
\]

**Theorem 1.2.** Let the assumptions in Theorem 1.1 hold and $(u, v, w)$ be the classical solution of (4) obtained in Theorem 1.1. Then the following asymptotic stability results hold:

1. If $\theta \leq 0$, one has
\[
\lim_{t \to \infty} (\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty} + \|w(\cdot,t) - w_*\|_{L^\infty}) = 0,
\]
where \( w_0 \geq 0 \) is a constant determined by 
\[
 w_0 = \frac{1}{|\Omega|} \left( \| w_0 \|_{L^1} - \frac{1}{|\Omega|} \int_0^\infty \int_{\Omega} uF(w) \right). 
\]

Moreover, \( w_0 > 0 \) if \( \theta < 0 \).

(2) If \( \theta > 0 \), we have
\[
 \lim_{t \to \infty} (\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{L^\infty} + \|w(\cdot, t)\|_{L^\infty}) = 0, 
\]
provided that
\[
 D > \max_{0 \leq v \leq \infty} \frac{\theta |\chi(v)|^2}{16 \beta^2 \gamma(v)}. 
\]

**Remark 1.** Our results show that all the bacterial will die and the nutrition survive if \( \theta < 0 \), while if \( \theta > 0 \) all the nutrition are consumed.

2. **Local existence and Preliminaries.** The existence and uniqueness of local solutions of (4) can be readily proved by the Amann’s theorem [3, 4] (cf. also [33, Lemma 2.6]) or the fixed point theorem along with the parabolic regularity theory [11]. We omit the details of the proof for brevity.

**Lemma 2.1** (Local existence). Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary and the assumptions (H1) and (H2) hold. Assume \((u_0, v_0, w_0) \in [W^{1,\infty}(\Omega)]^3\) with \( u_0, v_0, w_0 > 0 \). Then there exists \( T_{\text{max}} \in (0, \infty] \) such that the problem (4) has a unique classical solution \((u, v, w) \in [C^0(\bar{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\Omega \times (0, T_{\text{max}}))]^3\) satisfying \( u, v, w > 0 \) for all \( t > 0 \). Moreover, if \( T_{\text{max}} < \infty \), then
\[
 \|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,\infty}} + \|w(\cdot, t)\|_{W^{1,\infty}} \to \infty \text{ as } t \nearrow T_{\text{max}}.
\]

**Lemma 2.2.** Let \( \theta \in \mathbb{R} \) and \( \beta > 0 \), then the solution \((u, v, w)\) of (4) satisfies
\[
 \|u(\cdot, t)\|_{L^1} + \|w(\cdot, t)\|_{L^1} \leq C \text{ for all } t \in (0, T_{\text{max}}),
\]
and
\[
 \|w(\cdot, t)\|_{L^\infty} \text{ is decreasing in } t.
\]

**Proof.** First, applying the maximum principle to the third equation of (4), one obtains (7) directly. Moreover, multiplying the third equation of (4) by \( \alpha \) and adding it to the first equation of (4), we have for all \( t \in (0, T_{\text{max}}) \) that
\[
 \frac{d}{dt} \left( \int_{\Omega} u + \alpha \int_{\Omega} w \right) + \beta \int_{\Omega} u^2 = \theta \int_{\Omega} u. \tag{8}
\]

Then if \( \theta \leq 0 \), we obtain (6) directly by integrating (8). On the other hand, if \( \theta > 0 \), then using the Young inequality, one has
\[
(\theta + 1) \int_{\Omega} u + \alpha \int_{\Omega} w \leq (\theta + 1) \left( \int_{\Omega} u^2 \right)^{\frac{1}{2}} |\Omega|^\frac{1}{2} + \alpha \int_{\Omega} w_0 \\
 \leq \frac{\beta}{2} \int_{\Omega} u^2 + \frac{(\theta + 1)^2 |\Omega|}{2\beta} + \alpha \int_{\Omega} w_0 \text{ for all } t \in (0, T_{\text{max}}), \tag{9}
\]
which together with (8) gives
\[
 \frac{d}{dt} \left( \int_{\Omega} u + \alpha \int_{\Omega} w \right) + \int_{\Omega} u + \alpha \int_{\Omega} w \leq C_1 \text{ for all } t \in (0, T_{\text{max}}). \tag{10}
\]

Then applying Grönwall’s inequality to (10) gives (6). \( \square \)

The following lemma will be used to show the boundedness of solution, one can see [18, Lemma 3.3] or [26, Lemma 3.4] for details.
Lemma 2.3. Let $T > 0$, $\tau \in (0, T)$, $a > 0$ and $b > 0$. Suppose that $y : [0, T) \rightarrow [0, \infty)$ is absolutely continuous and fulfills

$$y'(t) + ay(t) \leq h(t) \quad \text{for all } t \in (0, T),$$

with some nonnegative function $h \in L^1_{loc}([0, T))$ satisfying

$$\int_t^{t+\tau} h(s)ds \leq b \quad \text{for all } t \in [0, T-\tau).$$

Then

$$y(t) \leq \max \left\{ y(0) + b, \frac{b}{a\tau} + 2b \right\}, \quad \text{for all } t \in (0, T).$$

3. Boundedness of solutions: Proof of Theorem 1.1. In this section, we are devoted to studying the existence of global classical solutions for system (4). To overcome this major obstacle that the possible degeneracy of diffusion, we employ the $L^2$ energy estimate directly by treating $\gamma(v)$ as a weight function based on some ideas in [11]. This enables us to control the chemotactic advection term and derive a Grönwall type inequality:

$$\frac{d}{dt} \|u\|^2_{L^2} \leq c_1 \|u\|^2_{L^2} \|\Delta v\|^2_{L^2} + c_2 \quad \text{for all } t \in (0, T_{max}),$$

which, together with the facts $\int_t^{t+\tau} \int_\Omega u^2$ and $\int_t^{t+\tau} \int_\Omega |\Delta v|^2$ (see Lemma 3.1 and Lemma 3.2 ) derive the uniform-in-time bound of $\|u(\cdot, t)\|_{L^2}$. First, we show some basic properties of solution for the system (4).

Lemma 3.1. Suppose the assumptions in Theorem 1.1 hold. For all $\theta \in \mathbb{R}$, there exists a constant $C_1 > 0$ such that the solution to (4) satisfies

$$\int_t^{t+\tau} \int_\Omega u^2 \leq C_1, \quad \text{for all } t \in (0, \bar{T}_{max}), \quad (11)$$

where

$$\tau := \min \left\{ 1, \frac{1}{2}T_{max} \right\} \quad \text{and} \quad \bar{T}_{max} := T_{max} - \tau.$$

Proof. We multiply the third equation of (4) by $\alpha$ and add it to the first equation of (4) to have

$$\frac{d}{dt} \left( \int_\Omega u + \alpha \int_\Omega w \right) + \beta \int_\Omega u^2 = \theta \int_\Omega u \leq \frac{\beta}{2} \int_\Omega u^2 + c_1 \quad \text{for all } t \in (0, T_{max}),$$

which gives

$$\frac{d}{dt} \left( \int_\Omega u + \alpha \int_\Omega w \right) + \frac{\beta}{2} \int_\Omega u^2 \leq c_1 \quad \text{for all } t \in (0, T_{max}). \quad (12)$$

Then integrating (12) over $(t, t+\tau)$ and using (6), one has

$$\frac{\beta}{2} \int_t^{t+\tau} \int_\Omega u^2 \leq c_1 \tau + c_2 \quad \text{for all } t \in (0, \bar{T}_{max}),$$

which gives (11). \qed
Lemma 3.2. Let \((u, v, w)\) be the solution of system (4). Then there exists a constant \(C > 0\) independent of \(t\) such that
\[
\|\nabla v\|_{L^2} \leq C, \quad \text{for all } t \in (0, T_{max}),
\]
and
\[
\int_t^{t+\tau} \int_\Omega |\Delta v|^2 \leq C, \quad \text{for all } t \in (0, \overline{T}_{max}),
\]
where \(\tau\) and \(\overline{T}_{max}\) are defined in Lemma 3.1.

Proof. Testing the second equation of (4) by \(-\Delta v\), and integrating the result by parts, one has for all \(t \in (0, T_{max})\)
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla v|^2 + D \int_\Omega |\Delta v|^2 + \int_\Omega |\nabla v|^2 = -\int_\Omega u \Delta v \leq \frac{D}{2} \int_\Omega |\Delta v|^2 + \frac{1}{2D} \int_\Omega u^2,
\]
which yields
\[
\frac{d}{dt} \int_\Omega |\nabla v|^2 + D \int_\Omega |\Delta v|^2 + 2 \int_\Omega |\nabla v|^2 \leq \frac{1}{D} \int_\Omega u^2 \quad \text{for all } t \in (0, T_{max}).
\]
Then applying Lemma 2.3 to (15) and using (11), we obtain (13) directly. On the other hand, integrating (15) over \((t, t + \tau)\) and using (13), one has (14).

Next, we will show the boundedness of \(\|u(\cdot, t)\|_{L^2}\) based on the weighted energy estimate as follows.

Lemma 3.3. Let \(\Omega\) be a bounded domain in \(\mathbb{R}^2\) with smooth boundary and the hypotheses \((H1)-(H2)\) hold. Suppose \((u, v, w)\) is a solution of the system (4), then it holds that
\[
\|u(\cdot, t)\|_{L^2} \leq C, \quad \text{for all } t \in (0, T_{max}),
\]
where the constant \(C > 0\) is independent of \(t\).

Proof. Testing the first equation of (4) by \(u\) and integrating the result by parts, then we use Hölder’s inequality, Young’s inequality and the fact \(F(w) \leq F(\|w_0\|_{L^\infty}) := c_1\) to obtain
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 + \int_\Omega \gamma(v)|\nabla u|^2 + \beta \int_\Omega u^3
\]
\[
= \int_\Omega \chi(v)u \nabla u \cdot \nabla v + \alpha \int_\Omega u^2 F(w) + \theta \int_\Omega u^2
\]
\[
\leq \left( \int_\Omega \gamma(v)|\nabla u|^2 \right)^{\frac{1}{2}} \left( \int_\Omega \frac{|v|^2}{\gamma(v)} u^2 |\nabla v|^2 \right)^{\frac{1}{2}} + (\alpha c_1 + \theta) \int_\Omega u^2
\]
\[
\leq \frac{1}{2} \int_\Omega \gamma(v)|\nabla u|^2 + \frac{1}{2} \int_\Omega \frac{|v|^2}{\gamma(v)} u^2 |\nabla v|^2 + \frac{\beta}{2} \int_\Omega u^3 + c_2,
\]
for all \(t \in (0, T_{max})\), which yields
\[
\frac{d}{dt} \int_\Omega u^2 + \int_\Omega \gamma(v)|\nabla u|^2 + \beta \int_\Omega u^3 \leq \int_\Omega \frac{|v|^2}{\gamma(v)} u^2 |\nabla v|^2 + 2c_2.
\]
Based on the ideas in [11], we can derive that
\[
\gamma(v)|\nabla u|^2 = |\gamma^\frac{1}{2}(v)\nabla u|^2 = \left| \nabla (\gamma^\frac{1}{2}(v)u) - \frac{1}{2} \frac{\gamma'(v)}{\gamma^\frac{3}{2}(v)} u \nabla v \right|^2
\]
\[
\geq \frac{1}{2} |\nabla (\gamma^\frac{1}{2}(v)u)|^2 - \frac{1}{4} \frac{\gamma'(v)^2}{\gamma(v)} u^2 |\nabla v|^2,
\]
which substituted into (17) gives
\[
\frac{d}{dt} \int_\Omega u^2 + \frac{1}{2} \int_\Omega |\nabla (\gamma^{\frac{1}{2}}(v) u)|^2 + \beta \int_\Omega u^3 \\
\leq \frac{1}{4} \int_\Omega |\gamma'(v)|^2 u^2 |\nabla v|^2 + \int_\Omega \frac{|\chi(v)|^2}{\gamma(v)} u^2 |\nabla v|^2 + 2c_2 \\
= \frac{1}{4} \int_\Omega |\gamma'(v)|^2 \gamma v^2(u) |\nabla v|^2 + \int_\Omega \frac{|\chi(v)|^2}{\gamma(v)} |\gamma^{\frac{1}{2}}(v) u| |\nabla v|^2 + 2c_2 
\text{ for all } t \in (0, T_{\text{max}}).
\tag{18}
\]

Then using the hypothesis (H1), one finds a constant $K_1 > 0$ such that
\[
|\gamma'(v)| + |\chi(v)| \gamma(v) \leq K_1 \text{ for all } v \geq 0 \text{ and } t \in (0, T_{\text{max}}),
\]
which substituted into (18) gives for all $t \in (0, T_{\text{max}})$ that
\[
\frac{d}{dt} \int_\Omega u^2 + \frac{1}{2} \int_\Omega |\nabla (\gamma^{\frac{1}{2}}(v) u)|^2 + \beta \int_\Omega u^3 \leq \frac{5K_1^2}{4} \left( \int_\Omega |\gamma^{\frac{1}{2}}(v) u|^4 \right)^{\frac{1}{2}} \left( \int_\Omega |\nabla v|^4 \right)^{\frac{1}{2}} + 2c_2.
\tag{19}
\]

Then using the Gagliardo-Nirenberg inequality and the facts $\gamma(v) \leq \gamma(0) = c_3$ and $||\nabla v||_{L^2} \leq c_4$, and the Young’s inequality, we have
\[
\frac{5K_1^2}{4} \left( \int_\Omega |\gamma^{\frac{1}{2}}(v) u|^4 \right)^{\frac{1}{2}} \left( \int_\Omega |\nabla v|^4 \right)^{\frac{1}{2}} \\
= \frac{5K_1^2}{4} \|\gamma^{\frac{1}{2}}(v) u\|_{L^4}^2 \|\nabla v\|_{L^4}^2 \\
\leq c_5 \left( \|\nabla (\gamma^{\frac{1}{2}}(v) u)\|_{L^2} \|\gamma^{\frac{1}{2}}(v) u\|_{L^2} + \|\gamma^{\frac{1}{2}}(v) u\|_{L^2}^2 \right) \left( \|\Delta v\|_{L^2} \|\nabla v\|_{L^2} + \|\nabla v\|_{L^2}^2 \right) \\
\leq c_6 (\|\nabla (\gamma^{\frac{1}{2}}(v) u)\|_{L^2} \|\Delta v\|_{L^2}) (\|\nabla v\|_{L^2} + 1) \\
\leq \frac{1}{2} \|\nabla (\gamma^{\frac{1}{2}}(v) u)\|_{L^2}^2 + c_7 \|\Delta v\|_{L^2}^2.
\tag{20}
\]

Then we substitute (20) into (19), and use the estimate $c_7 \|\Delta v\|_{L^2}^2 \leq 2\delta \int_\Omega u^3 + c_8$ to obtain
\[
\frac{d}{dt} \|u\|_{L^2}^2 \leq c_7 \|u\|_{L^2}^2 \|\Delta v\|_{L^2}^2 + c_9 \text{ for all } t \in (0, T_{\text{max}}).
\tag{21}
\]

Moreover, for any $t \in (0, T_{\text{max}})$ and in both cases $t \in (0, \tau)$ and $t \geq \tau$ with $\tau = \min \left\{ 1, \frac{1}{2} T_{\text{max}} \right\}$, we can use (11) and (14) to find $t_0 = t_0(t) \in ((t - \tau), t)$ such that $t_0 \geq 0$
\[
\int_\Omega u^2(x, t_0) \leq c_{10}. 
\tag{22}
\]

and
\[
\int_{t_0}^{t_0 + \tau} \int_\Omega |\Delta v(x, s)|^2 \leq c_{11} \text{ for all } t_0 \in (0, T_{\text{max}}).
\tag{23}
\]

Then we can integrate (21) over $(t_0, t)$, and use (22) and (23) to obtain
\[
\|u(\cdot, t)\|_{L^2}^2 \leq \|u(\cdot, t_0)\|_{L^2}^2 + e^{c_7 \int_{t_0}^t \|\Delta v(\cdot, s)|_{L^2}^2 ds} + c_9 \int_{t_0}^t e^{c_7 \int_s^t \|\Delta v(\cdot, \sigma)\|_{L^2}^2 d\sigma} ds \\
\leq c_{12} \text{ for all } t \in (0, T_{\text{max}}),
\]
which gives (16).
Lemma 3.4. Let the conditions in Lemma 3.3 hold, then it holds that
\[ \|u(\cdot, t)\|_{L^\infty} \leq C \quad \text{for all } t \in (0, T_{\max}), \]
where \( C > 0 \) is a constant independent of \( t \).

Proof. Since \( \|u(\cdot, t)\|_{L^2} \) is bounded, then from the second equation of (4), we have
\[ \|v(\cdot, t)\|_{W^{1,4}} \leq c_1, \quad \text{for all } t \in (0, T_{\max}) \]
and hence
\[ \|v(\cdot, t)\|_{L^\infty} \leq c_2, \quad \text{for all } t \in (0, T_{\max}) \]
by using the Sobolev inequality. Then using the assumptions on \( \gamma(v) \) and \( \chi(v) \) and (26), we obtain
\[ \gamma(v) \geq \gamma(c_2) > 0 \quad \text{and} \quad |\chi(v)| \leq c_3. \]
Multiplying the first equation of (4) by \( u^{p-1} \) with \( p \geq 2 \) and integrating the resulting equation by parts, and using the Hörder inequality and Young’s inequality, we end up with
\[ \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + (p-1) \int_{\Omega} \gamma(v) u^{p-2} |\nabla u|^2 + \frac{1}{p} \int_{\Omega} u^p + \beta \int_{\Omega} u^{p+1} \]
\[ = (p-1) \int_{\Omega} \chi(v) u^{p-1} \nabla u \cdot \nabla v + (\theta + \frac{1}{p}) \int_{\Omega} u^p + \alpha \int_{\Omega} |\nabla u|^2 \]
\[ \leq \frac{(p-1)}{2} \int_{\Omega} \gamma(v) u^{p-2} |\nabla u|^2 + (\frac{p-1}{2}) \int_{\Omega} |\nabla v|^2 + \beta \int_{\Omega} u^{p+1} + c_4 \]
which, together with (27) and the fact \( \frac{2(p-1)\gamma(c_2)}{p} \int_{\Omega} u^{p-2} |\nabla u|^2 = \frac{2(p-1)\gamma(c_2)}{p} \int_{\Omega} |\nabla u|^2 \]
gives
\[ \frac{d}{dt} \int_{\Omega} u^p + \int_{\Omega} u^p + \frac{2(p-1)\gamma(c_2)}{p} \int_{\Omega} u^{p-2} |\nabla u|^2 \leq \frac{c_3^2(p-1)}{2\gamma(c_2)} \int_{\Omega} u^p |\nabla v|^2 + c_4 \]
for all \( t \in (0, T_{\max}) \). Then we can use Hölder’s inequality and the Gagliardo-Nirenberg inequality, (25) and the fact \( \|u(\cdot, t)\|_{L^\infty} \leq M \tilde{u} \) to obtain
\[ \frac{c_3^2(p-1)}{2\gamma(c_2)} \int_{\Omega} u^p |\nabla v|^2 \leq \frac{c_3^2(p-1)}{2\gamma(c_2)} \left( \int_{\Omega} u^2 \right)^\frac{1}{2} \left( \int_{\Omega} |\nabla v|^4 \right)^\frac{1}{4} \]
\[ \leq \frac{c_3^2(p-1)}{2\gamma(c_2)} \|u\|_{L^2}^2 + c_5 \]
\[ \leq c_5 (\|\nabla u\|_{L^2}^{2(1-\frac{1}{p})} + \|u\|_{L^\frac{p}{2}}^2 + \|u\|_{L^p}^2) \]
\[ \leq \frac{2(p-1)\gamma(c_2)}{p} \|\nabla u\|_{L^2}^2 + c_6 \quad \text{for all } t \in (0, T_{\max}). \]
Letting \( c_7 := c_4p + c_6 \), then combining (28) and (29), we obtain
\[ \frac{d}{dt} \int_{\Omega} u^p + \int_{\Omega} u^p \leq c_7, \quad \text{for all } t \in (0, T_{\max}), \]
which, together with Grönwall’s inequality, gives
\[ \|u(\cdot, t)\|_{L^p}^p \leq e^{-c_7} \|u_0\|_{L^p}^p + c_7(1 - e^{-c_7}) \leq \|u_0\|_{L^p}^p + c_7 \quad \text{for all } t \in (0, T_{\max}). \]
Then choosing some \( p \geq 3 \) in (30) and using the parabolic regularity, one can find a constant \( c_8 > 0 \) independent of \( p \) such that \( \|\nabla v(\cdot, t)\|_{L^\infty} \leq c_8. \) Then we can use
the Moser iteration procedure as in [2, 27, 28] to obtain (24). Hence we complete the proof of Lemma 3.4.

\[ \text{Proof of Theorem 1.1.} \] From Lemma 3.4, we can find a constant \( c_1 > 0 \) such that for all \( t \in (0, T_{\text{max}}) \)

\[ \| u(\cdot, t) \|_{L^\infty} \leq c_1. \]  

(31)

This along with the parabolic regularity to the second and third equation of the system (4) gives

\[ \| v(\cdot, t) \|_{W^{1,\infty}} + \| w(\cdot, t) \|_{W^{1,\infty}} \leq c_2 \]  

(32)

for all \( t \in (0, T_{\text{max}}) \).

Then the combination of (31), (32) and Lemma 2.1 gives Theorem 1.1 directly.

4. \text{Large time behavior.} In this section, we will derive the asymptotic behavior of solutions as shown in Theorem 1.2. First, we shall improve the regularity of \( u, v \) and \( w \) by using the standard parabolic property.

**Lemma 4.1.** Let \( (u, v, w) \) be the nonnegative global classical solution of (4) obtained in Theorem 1.1. Then there exist \( \sigma \in (0, 1) \) and \( C > 0 \) such that

\[ \| u(\cdot, t) \|_{C^\sigma, \sigma} \leq C \]  

for all \( t > 1 \).

(33)

**Proof.** In fact, we can rewrite the first equation of the system (4) as follows

\[ u_t = \nabla \cdot A(x, t, u, \nabla u) + B(x, t, u), \]

where \( A(x, t, u, \nabla u) = \gamma(v) \nabla u - \chi(v) u \nabla v \) and \( B(x, t, u) = \alpha u F(w) + \theta u - \beta u^2 \). From Theorem 1.1, we know there exists a constant \( c_1 > 0 \) such that

\[ \| u(\cdot, t) \|_{L^\infty} + \| v(\cdot, t) \|_{W^{1,\infty}} \leq c_1. \]  

Then one has

\[ A(x, t, u, \nabla u) \cdot \nabla u = (\gamma(v) \nabla u - \chi(v) u \nabla v) \cdot \nabla u \]

\[ \leq \gamma(v) |\nabla u|^2 - \chi(v) u \nabla u \cdot \nabla v \]

\[ \leq \frac{3\gamma(v)}{2} |\nabla u|^2 + \frac{|\chi(v)|^2}{2\gamma(v)} u^2 |\nabla v|^2 \]

\[ \leq \frac{3\gamma(0)}{2} |\nabla u|^2 + c_2 \]  

(34)

and

\[ |A(x, t, u, \nabla u)| = |\gamma(v) \nabla u - \chi(v) u \nabla v| \]

\[ \leq |\gamma(v)| |\nabla u| + |\chi(v)||u| L^\infty \| \nabla v \|_{L^\infty} \]

\[ \leq \gamma(0) |\nabla u| + c_3. \]  

(35)

At last, using the properties of \( F(w) \), we can derive that

\[ |B(x, t, u)| = |\alpha F(w) u + \theta u - \beta u^2| \]

\[ \leq |\alpha F(w)||u|_{L^\infty} + \theta |u|_{L^\infty} + \beta |u|^2_{L^\infty} \]

\[ \leq c_4. \]  

(36)

Then the combination of (34), (35) and (36), we obtain (33) by using [24, Theorem 1.3].
4.1. Case of $\theta \leq 0$. In this subsection, we will study the large time behavior of solutions in the case of $\theta \leq 0$. In fact, if $\theta \leq 0$, we can directly obtain $\int_0^\infty \int_\Omega u^2 < \infty$, which combined with the relative compactness of $(u(\cdot, t))_{t \geq 1}$ in $C(\Omega)$ (see Lemma 4.1) implies some decay information for $u$ and hence the decay properties of $v$ from the second equation of (4). More precisely, we have the following results.

**Lemma 4.2.** Let the conditions in Theorem 1.2 hold, and suppose $\theta \leq 0$ and $(u, v, w)$ is the solution of the system (4). Then it follows that

$$\|u(\cdot, t)\|_{L^\infty} \to 0 \quad \text{as} \quad t \to \infty,$$

and

$$\|v(\cdot, t)\|_{L^\infty} \to 0 \quad \text{as} \quad t \to \infty.$$

**Proof.** Multiplying the third equation of (4) by $\alpha$ and adding it to the first equation of (4), we have

$$\frac{d}{dt} \left( \int_\Omega u + \alpha \int_\Omega w \right) + \beta \int_\Omega u^2 = \theta \int_\Omega u \leq 0,$$

due to $\theta \leq 0$ and $u > 0$. Then from (39), we obtain

$$\int_0^\infty \int_\Omega u^2 \leq c_1.$$

Next, we shall use some ideas in [29, Lemma 3.10] to prove (37) by the argument of contradiction. In fact, suppose that (37) is wrong, then for some constant $c_4 > 0$, one can find some sequences $(x_j)_{j \in \mathbb{N}} \subset \Omega$ such that $(t_j)_{j \in \mathbb{N}} \subset (0, \infty)$ satisfying $t_j \to \infty$ as $j \to \infty$ such that

$$u(x_j, t_j) \geq c_2, \quad \text{for all} \quad j \in \mathbb{N}.$$

From Lemma 4.1, we know $u$ is uniformly continuous in $\Omega \times (1, \infty)$. Then for any $j \in \mathbb{N}$ it holds that for some $r > 0$ and $T_1 > 0$,

$$u(x, t) \geq \frac{c_2}{2} \quad \text{for all} \quad x \in B_r(x_j) \cap \Omega \quad \text{and} \quad t \in (t_j, t_j + T_1). \quad (41)$$

Noting the smoothness of $\partial \Omega$, one can find a constant $c_3 > 0$ such that

$$|B_r(x_j) \cap \Omega| \geq c_3, \quad \text{for all} \quad x_j \in \Omega. \quad (42)$$

Then combining (41) and (42), for all $j \in \mathbb{N}$, we have

$$\int_{t_j}^{t_j + T_1} \int_\Omega u^2(x, t) dx dt \geq \int_{t_j}^{t_j + T_1} \int_{B_r(x_j) \cap \Omega} u^2(x, t) dx dt \geq \int_{t_j}^{t_j + T_1} |B_r(x_j) \cap \Omega| \cdot \left( \frac{c_2}{2} \right)^2 dt \geq \frac{c_2^2 c_3 T_1}{4}.$$ \hspace{1cm} (43)

However, noting the fact $t_j \to \infty$ as $j \to \infty$, from (40) one has

$$\int_{t_j}^{t_j + T_1} \int_\Omega (u(x, t) - \bar{u}_0)^2 dx dt \leq \int_{t_j}^\infty \int_\Omega (u(x, t) - \bar{u}_0)^2 dx dt \to 0 \quad \text{as} \quad j \to \infty,$$

which contradicts (43). Hence (37) holds by the argument of contradiction.
Next, we show (38) holds to complete the proof of this lemma. In fact, from the second equation of (4), we have
\[
\frac{d}{dt} \int_\Omega v + \int_\Omega v = \int_\Omega u,
\]
which combined with the fact \( \int_\Omega u \to 0 \) as \( t \to \infty \) gives
\[
\int_\Omega v \to \infty, \quad \text{as} \quad t \to \infty. \tag{44}
\]
Then using the fact \( \|\nabla v\|_{L^4} \leq c_6 \) and applying the Gagliardo-Nirenberg inequality, one has
\[
\|v\|_{L^\infty} \leq c_7 \|\nabla v\|_{L^4}^{\frac{1}{4}} \|v\|_{L^1}^{\frac{1}{4}} + \|v\|_{L^1} \leq c_8 \|v\|_{L^1},
\]
which combined with (44) gives (38).

\[\square\]

**Lemma 4.3.** Suppose the conditions in Lemma 4.2 hold. Let \((u, v, w)\) be the solution of the system (4). Then we have the following result
\[
\|w(\cdot, t) - w_*\|_{L^\infty} \to 0 \quad \text{as} \quad t \to \infty, \tag{46}
\]
where \(w_* \geq 0\) is a constant determined by
\[
w_* = \frac{1}{|\Omega|} \|w_0\|_{L^1} - \frac{1}{|\Omega|} \int_0^\infty \int_\Omega uF(w).
\]
Moreover, \(w_* > 0\) if \(\theta < 0\).

**Proof.** We can rewrite the third equation of (4) as
\[
(w - \bar{w})_t = \Delta (w - \bar{w}) - uF(w) + \bar{u}F(w), \tag{47}
\]
where \(\bar{f}(t) = \frac{1}{|\Omega|} \int_\Omega f.\) Then applying the variation-of-constants formula to (47) and using the fact \(\|w(\cdot, t)\|_{L^\infty} \leq c_1\), one has
\[
\|w(\cdot, t) - \bar{w}(t)\|_{L^\infty}
\leq \|e^{\frac{t}{\lambda_1}} (w(\cdot, t/2) - \bar{w}(t/2))\|_{L^\infty} + \int_0^t \|e^{(t-s)\lambda_1} (uF(w) - \bar{u}F(w))\|_{L^\infty} ds
\leq c_2 e^{\frac{t}{\lambda_1}} \|w(\cdot, t/2) - \bar{w}(t/2)\|_{L^\infty} + c_3 \int_0^t e^{-(t-s)\lambda_1} \|u(\cdot, s)\|_{L^\infty} ds \tag{48}
\leq c_4 e^{\frac{t}{\lambda_1}} + c_5 \sup_{t/2 \leq s \leq t} \|u(\cdot, s)\|_{L^\infty},
\]
which combined with the decay property of \(u\) in (37) gives
\[
\lim_{t \to \infty} \|w(\cdot, t) - \bar{w}(t)\|_{L^\infty} = 0. \tag{49}
\]
We integrate the third equation of (4) over \(\Omega \times (0, t)\) to have
\[
\bar{w}(t) = \frac{1}{|\Omega|} \|w_0\|_{L^1} - \frac{1}{|\Omega|} \int_0^\infty \int_\Omega uF(w) + \frac{1}{|\Omega|} \int_t^\infty \int_\Omega uF(w)
= w_* + \frac{1}{|\Omega|} \int_t^\infty \int_\Omega uF(w)
\]
which implies
\[
\|\bar{w}(t) - w_*\|_{L^\infty} \leq c_6 \int_t^\infty \|u(\cdot, s)\|_{L^1} ds \to 0 \quad \text{as} \quad t \to \infty. \tag{50}
\]
Then the combination of (49) and (50) gives
\[
\|w(\cdot, t) - w_*\|_{L^\infty} \leq \|w(\cdot, t) - \bar{w}(t)\|_{L^\infty} + \|\bar{w}(t) - w_*\|_{L^\infty} \to 0, \quad \text{as} \quad t \to \infty,
\]
which completes the proof. \(\square\)
which yields (46).

Next, we shall show that \( w_\ast > 0 \) in the case of \( \theta < 0 \). Using the boundedness of \( u, w \) and the properties of \( F(w) \), we can derive that

\[
u F(w) = u F'(\xi)w \leq c_6 w.
\] (51)

Then suppose that \( \tilde{w}(x,t) \) is the solution of the problem

\[
\begin{cases}
\tilde{w}_t - \Delta \tilde{w} = -c_6 \tilde{w}, & x \in \Omega, t > 0, \\
\frac{\partial \tilde{w}}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
\tilde{w}(x,0) = w_0(x), & x \in \Omega,
\end{cases}
\]

Then by using the comparison principle, \( \tilde{w}(x,t) \) is a sub-solution of \( w(x,t) \) and hence

\[
w(x,t) \geq \tilde{w}(x,t).
\] (52)

Then from [10, Lemma 3.1], we know that there a constant \( \Gamma_0 > 0 \) such that for all \( t \geq 1 \)

\[
\tilde{w}(x,t) = e^{-c_6 t} e^{\Delta t} w_0 \geq e^{-c_6 t} \Gamma_0 \int_\Omega w_0,
\]

which together with (52) gives

\[
w(x,t) \geq e^{-Kt} \Gamma_0 \int_\Omega w_0, \text{ for all } t \geq 1.
\] (53)

Then we multiply the third equation of the system (4) by \( \frac{1}{w} \), and integrate by parts with respect to \( x \in \Omega \) to have

\[
\frac{d}{dt} \int_\Omega \ln w(x,t) = \int_\Omega \frac{\nabla w}{w^2} - \int_\Omega \frac{F(w)}{w} u \\
\geq -c_7 \int_\Omega u.
\] (54)

Then integrating (54) by parts over \((1,t)\), we obtain

\[
\int_\Omega \ln w(x,t) \geq \int_\Omega \ln w(x,1) - c_7 \int_1^t \int_\Omega u.
\] (55)

On other hand, from the system (4), we have

\[
\frac{d}{dt} \left( \int_\Omega u + \alpha \int_\Omega w \right) + \beta \int_\Omega u^2 = \theta \int_\Omega u,
\]

which implies \( \int_0^t \int_\Omega u \leq c_8 \) in the case of \( \theta < 0 \). Then the combination of (53), (55) and the fact \( \int_0^t \int_\Omega u \leq c_8 \) gives

\[
\int_\Omega \ln w(x,t) \geq -c_9, \text{ for all } t \geq 1,
\]

which together with the fact (46) implies \( w_\ast > 0 \).
4.2. Case of $\theta > 0$. In this subsection, we will study the large time behavior of solution for the system (4) with $\theta > 0$. To this end, we shall show the following energy functional

$$ E(t) := \int_\Omega \left( u - \frac{\theta}{\beta} - \frac{\theta}{\beta} \ln \frac{\beta u}{\theta} \right) + \frac{\delta}{2} \int_\Omega (v - \frac{\theta}{\beta})^2 + \alpha \int_\Omega w $$

(56)
can act as Lyapunov functional under some conditions on the parameters based on some ideas in [11].

Lemma 4.4. Let $(u, v, w)$ be the solution of (4). Assume $E(t)$ is defined by (56). Then we can derive that $E(t) \geq 0$. Furthermore, if $D > \max_{0 \leq v \leq \infty} \frac{\theta |\chi(v)|^2}{16\beta^2 \gamma(v)}$, then there exists a positive constant $\beta$ such that for all $t > 0$

$$ E'(t) \leq -F(t), $$

(57)

where

$$ F(t) := \zeta \{ \int_\Omega (u - \frac{\theta}{\beta})^2 + \int_\Omega (v - \frac{\theta}{\beta})^2 \} + \frac{\alpha \theta}{\beta} \int_\Omega F(w). $$

Proof. The non-negativity of $E(t)$ can be verified by noting $\phi(u) := u - \frac{\theta}{\beta} - \ln \frac{2u}{\theta}$, $u > 0$ is non-negativity. In fact applying the Taylor’s formula to $\phi(u)$ to gives

$$ \phi(u) = \frac{1}{2} \phi''(\bar{u})(u - \frac{\theta}{\beta})^2 = \frac{1}{2} \frac{\theta}{u^2} (u - \frac{\theta}{\beta})^2 \geq 0, $$

(58)

which implies $E(t) \geq 0$.

Next, we prove (57) to complete the proof of this lemma. In fact, from the system (4), we have

$$ \frac{d}{dt} E(t) = \int_\Omega \frac{u - \frac{\theta}{\beta}}{u} u_t + \delta \int_\Omega \left( v - \frac{\theta}{\beta} \right) v_t + \alpha \int_\Omega w_t $$

$$ = -\frac{\theta}{\beta} \int_\Omega \gamma(v) \frac{\nabla u^2}{u^2} - \delta D \int_\Omega |\nabla v|^2 + \frac{\theta}{\beta} \int_\Omega \chi(v) \frac{\nabla u \cdot \nabla v}{u} $$

$$ - \beta \int_\Omega (u - \frac{\theta}{\beta})^2 - \delta \int_\Omega (v - \frac{\theta}{\beta})^2 + \delta \int_\Omega (u - \frac{\theta}{\beta})(v - \frac{\theta}{\beta}) - \frac{\alpha \theta}{\beta} \int_\Omega F(w) $$

(59)

Then letting

$$ \Theta_1 = \begin{pmatrix} \nabla u \\ \nabla v \end{pmatrix}, \quad A_1 = \begin{pmatrix} \frac{\theta \gamma(v)}{\beta u^2} & -\frac{\theta \chi(v)}{2\beta u} \\ -\frac{\theta \chi(v)}{2\beta u} & \delta D \end{pmatrix}, $$

we can rewrite $I_1 = -\Theta_1^T A_1 \Theta_1$ where $\Theta_1^T$ denotes the transpose of $\Theta_1$. Then it is easy to check that $A_1$ is non-negative definite and hence $I_1 \leq 0$ if and only if

$$ \delta \geq \max_{0 \leq v \leq \infty} \frac{\theta |\chi(v)|^2}{4\beta^2 \gamma(v)}. $$

(60)

Similarly, we can also rewrite $I_2$ as

$$ I_2 = -\Theta_2^T A_2 \Theta_2, \quad \Theta_2 = \begin{pmatrix} \frac{u - \frac{\theta}{\beta}}{v} \\ \frac{\theta}{\beta} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \beta & 0 \\ 0 & \delta \end{pmatrix}. $$
One can check that $A_2$ is positive definite if and only if
\[ \beta > \frac{\delta}{4}. \]  
(61)

Then if $D > \max_{0 \leq v \leq \infty} \frac{\theta |x(v)|^2}{16\beta^2 \gamma(v)}$ there exists a positive constant $\delta$ such that (60) and (61). Hence from (59), we can find a constant $\zeta > 0$ such that (57) holds. Then we complete the proof of this lemma.

**Lemma 4.5.** Suppose that $D > \max_{0 \leq v \leq \infty} \frac{\theta |x(v)|^2}{16\beta^2 \gamma(v)}$ and let $(u, v, w)$ be the global classical solution of the system (4). Then it follows that
\[ \|u(\cdot, t) - \frac{\theta}{\beta}\|_{L^\infty} + \|v(\cdot, t) - \frac{\theta}{\beta}\|_{L^\infty} \to 0, \quad \text{as} \quad t \to \infty \]  
(62)

and
\[ \|w(\cdot, t)\|_{L^\infty} \to 0, \quad \text{as} \quad t \to \infty. \]  
(63)

**Proof.** Since $\mathcal{E}(t)$ is nonnegative, we can integrate (57) over $[1, t]$ to obtain
\[ \int_1^t \mathcal{F}(s)ds \leq \mathcal{E}(1) - \mathcal{E}(t) \leq \mathcal{E}(1), \]
which combined with the definition of $\mathcal{F}(t)$, gives
\[ \int_1^\infty \int_\Omega \left[ \left( u - \frac{\theta}{\beta} \right)^2 + \left( v - \frac{\theta}{\beta} \right)^2 \right] < \infty \]  
(64)

and
\[ \int_1^\infty \int_\Omega F(w) < \infty. \]  
(65)

Then by the similar arguments as in Lemma 4.2, we can use (64) and Lemma 4.1 to obtain (62).

Next, we shall show (63) from (65) based on some ideas in [37]. In fact, (65) implies
\[ \int_j^{j+1} \int_\Omega F(w) \to 0, \quad \text{as} \quad j \to \infty. \]  
(66)

Next, we will show that (66) implies (63). In fact, if we define $w_j(x, s) := w(x, j + s), (x, s) \in \Omega \times (0, 1), j \in \mathbb{N}$, (66) implies
\[ \int_0^1 \int_\Omega F(w_j(x, s)) \to 0, \quad \text{as} \quad j \to \infty. \]  
(67)

Then from (67), we can extract a subsequence $(j_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ such that $j_k \to \infty$ and $F(w_{j_k}) \to 0$ almost everywhere in $\Omega \times (0, 1)$ as $k \to \infty$. Noting that the function $F$ is positive on $(0, \infty)$, which entails that $w_{j_k} \to 0$ almost everywhere in $\Omega \times (0, 1)$ as $k \to \infty$. Furthermore, the fact $\|w\|_{L^\infty} \leq C$ for all $t > 0$ implies that the sequence $(w_{j_k})_{k \in \mathbb{N}} \to 0$ in $L^1(\Omega \times (0, 1))$ as $k \to \infty$. Choosing $t_k := j_k$, one has
\[ \int_{t_k}^{t_k+1} \int_\Omega w \to 0 \quad \text{as} \quad k \to \infty. \]  
(68)
On the other hand, we can use the Gagliardo-Nirenberg inequality to find a constant $c_1 > 0$ such that

$$\|w(\cdot, t)\|_{L^\infty} \leq c_1 \|\nabla w\|_{L^4}^{\frac{3}{4}} \|w\|_{L^1}^{\frac{1}{4}} + c_1 \|w\|_{L^1},$$  

(69)

where $\delta_1 > 0$ is arbitrary, $c_2 > 0$ is a constant depends on $\delta$. Noting that $\|\nabla w\|_{L^4} \leq c_3$, then from (69) we can derive

$$\int_{t_k}^{t_{k+1}} \|w(\cdot, t)\|_{L^\infty} dt \leq \delta_1 \int_{t_k}^{t_{k+1}} \|\nabla w(\cdot, t)\|_{L^4} dt + c_2 \int_{t_k}^{t_{k+1}} \|w(\cdot, t)\|_{L^1} dt$$

$$\leq \delta_1 c_3 + c_2 \int_{t_k}^{t_{k+1}} \|w(\cdot, t)\|_{L^1} dt,$$

which together with (68) implies

$$\lim_{k \to \infty} \sup_{t \in (t_k, t_{k+1})} \|w(\cdot, t)\|_{L^\infty} dt < \delta_1 c_3 + c_2 \lim_{k \to \infty} \sup_{t \in (t_k, t_{k+1})} \|w(\cdot, t)\|_{L^1} dt = \delta_1 c_3. \quad (70)$$

Since $\delta_1 > 0$ can be arbitrary, (70) implies

$$\int_{t_k}^{t_{k+1}} \|w(\cdot, t)\|_{L^\infty} dt \to 0, \quad \text{as } k \to \infty,$$

which combined with the monotonicity of $t \to \|w(\cdot, t)\|_{L^\infty}$ (see Lemma 2.2) gives (63).

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