Various characterizations of Besov-Dunkl spaces

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Abstract: In this paper, different characterizations of the Besov-Dunkl spaces, previously considered in [1, 2, 3, 11], are given. We provide equivalence between these characterizations, using the Dunkl translation, the Dunkl transform and the Peetre $K$-functional.

AMS Subject Classification: 46E30, 44A15, 44A35.
Key Words: Dunkl operators, Dunkl transform, Dunkl translation operators, Dunkl convolution, Besov-Dunkl spaces.

1. Introduction

On the real line, we consider the first-order differential-difference operator defined by

$$\Lambda_\alpha(f)(x) = \frac{df}{dx}(x) + \frac{2\alpha + 1}{x} \left[ \frac{f(x) - f(-x)}{2} \right], \quad f \in \mathcal{E}(\mathbb{R}), \quad \alpha > -\frac{1}{2},$$

which is called Dunkl operator. Such operators have been introduced in 1989, by C. Dunkl in [8]. The Dunkl kernel $E_\alpha$ is used to define the Dunkl transform $\mathcal{F}_\alpha$ which was introduced by C. Dunkl in [9]. Rösler in [17] shows that the Dunkl kernel verify a product formula. This allows us to define the Dunkl translation $\tau_x$, $x \in \mathbb{R}$. As a result, we have the Dunkl convolution.

There are many ways to define Besov spaces (see [4, 5, 15, 21]). This paper deals with Besov-Dunkl spaces (see [1, 2, 3, 11]). Let $1 \leq p < +\infty$,
1 ≤ q ≤ +∞ and β > 0, the Besov-Dunkl space denoted by $BD_{p,q}^{\beta,\alpha}$ is the subspace of functions $f \in L^p(\mu_\alpha)$ satisfying
\[
\int_0^{+\infty} \left( \frac{w_{p,\alpha}(f, x)}{x^\beta} \right)^q \frac{dx}{x} < +\infty \quad \text{if} \quad q < +\infty
\]
and
\[
\sup_{x \in (0, +\infty)} \frac{w_{p,\alpha}(f, x)}{x^\beta} < +\infty \quad \text{if} \quad q = +\infty,
\]
where $w_{p,\alpha}(f, x) = \sup_{|t| \leq x} \| \tau_t(f) - f \|_{p,\alpha}$ and $\mu_\alpha$ is a weighted Lebesgue measure on $\mathbb{R}$ (see next section).

We consider the subspace $K\mathcal{D}_{p,q}^{\beta,\alpha}$ of functions $f \in L^p(\mu_\alpha)$ satisfying
\[
\int_0^{+\infty} \left( \frac{K_{p,\alpha}(f, x)}{x^\beta} \right)^q \frac{dx}{x} < +\infty \quad \text{if} \quad q < +\infty
\]
and
\[
\sup_{x \in (0, +\infty)} \frac{K_{p,\alpha}(f, x)}{x^\beta} < +\infty \quad \text{if} \quad q = +\infty,
\]
where $K_{p,\alpha}$ is the Peetre $K$-functional (see[12]) given by
\[
K_{p,\alpha}(f, x) = \inf \left\{ \| f_0 \|_{p,\alpha} + x \| \Lambda_\alpha f_1 \|_{p,\alpha} ; f_0, f_1 \in L^p(\mu_\alpha), f_1 \in \mathcal{D}_{p,\alpha}, f = f_0 + f_1 \right\}.
\]

We denote by $\mathcal{E}\mathcal{D}_{p,q}^{\beta,\alpha}$ the subspace of functions $f \in L^p(\mu_\alpha)$ satisfying
\[
\int_1^{+\infty} \left( x^\beta E_{p,\alpha}(f, x) \right)^q \frac{dx}{x} < +\infty \quad \text{if} \quad q < +\infty
\]
and
\[
\sup_{x \in (1, +\infty)} x^\beta E_{p,\alpha}(f, x) < +\infty \quad \text{if} \quad q = +\infty,
\]
where $E_{p,\alpha}(f, x) = \inf \left\{ \| f - g \|_{p,\alpha} ; \text{supp}(F_\alpha(g)) \subset [-x, x] \right\}, x > 0.$
Our objective will be to prove that $\mathcal{BD}^{\beta,\alpha}_{p,q} = \mathcal{KD}^{\beta,\alpha}_{p,q}$ and when $1 \leq p \leq 2$, $1 \leq q < +\infty$, $0 < \beta < 1$ then $\mathcal{BD}^{\beta,\alpha}_{p,q} = \mathcal{ED}^{\beta,\alpha}_{p,q}$.

Analogous results have been obtained by Betancor, Méndez and Rodríguez-Mesa in [6] for the Bessel operator on $(0, +\infty)$.

The contents of this paper are as follows.

In section 2, we collect some basic definitions and results about harmonic analysis associated with Dunkl operators.

In section 3, we prove the results about inclusion and coincidence between the spaces $\mathcal{BD}^{\beta,\alpha}_{p,q}$, $\mathcal{KD}^{\beta,\alpha}_{p,q}$ and $\mathcal{ED}^{\beta,\alpha}_{p,q}$.

In the sequel $c$ represents a suitable positive constant which is not necessarily the same in each occurrence. Furthermore, we denote by

- $\mathcal{D}_e(\mathbb{R})$ the space of even $C^\infty$-functions on $\mathbb{R}$ with compact support.
- $\mathcal{S}_e(\mathbb{R})$ the space of even Schwartz functions on $\mathbb{R}$.

## 2. Preliminaries

Let $\mu_\alpha$ the weighted Lebesgue measure on $\mathbb{R}$ given by

$$
d\mu_\alpha(x) = \frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)}dx.
$$

For every $1 \leq p \leq +\infty$, we denote by $L^p(\mu_\alpha)$ the space $L^p(\mathbb{R}, d\mu_\alpha)$ and we use $\| \|_{p,\alpha}$ as a shorthand for $\| \|_{L^p(\mu_\alpha)}$.

The Dunkl transform $\mathcal{F}_\alpha$ which was introduced by C. Dunkl in [9], is defined for $f \in L^1(\mu_\alpha)$ by

$$
\mathcal{F}_\alpha(f)(x) = \int_{\mathbb{R}} E_\alpha(-ixy)f(y)d\mu_\alpha(y), \quad x \in \mathbb{R},
$$

where for $\lambda \in \mathbb{C}$, the Dunkl kernel $E_\alpha(\lambda \cdot)$ is given by

$$
E_\alpha(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha + 1)}j_{\alpha+1}(i\lambda x), \quad x \in \mathbb{R},
$$

with $j_\alpha$ the normalized Bessel function of the first kind and order $\alpha$ (see [22]).

The Dunkl kernel $E_\alpha(\lambda \cdot)$ is the unique solution on $\mathbb{R}$ of initial problem for the Dunkl operator (see [8]). We have for all $x, y \in \mathbb{R}$,

$$
|E_\alpha(-ixy)| \leq 1. \quad (1)
$$

According to [7], we have the following results:
i) For all $f \in L^1(\mu_\alpha)$, we have $\|F_\alpha(f)\|_\infty,\alpha \leq \|f\|_{1,\alpha}$.

ii) For all $f \in L^1(\mu_\alpha)$ such that $F_\alpha(f) \in L^1(\mu_\alpha)$, we have the inversion formula

$$f(x) = \int_{\mathbb{R}} E_\alpha(i\lambda x)F_\alpha(f)(\lambda)d\mu_\alpha(\lambda), \text{ a.e } x \in \mathbb{R}. \quad (2)$$

iii) For every $f \in L^2(\mu_\alpha)$, we have the Plancherel formula

$$\|F_\alpha(f)\|_{2,\alpha} = \|f\|_{2,\alpha}.$$  

For all $x, y, z \in \mathbb{R}$, consider

$$W_\alpha(x, y, z) = \frac{(\Gamma(\alpha + 1)^2)}{2^{\alpha-1}\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})}(1 - b_{x,y,z} + b_{z,x,y} + b_{z,y,x})\Delta_\alpha(x, y, z) \quad (3)$$

where

$$b_{x,y,z} = \begin{cases} \frac{x^2+y^2-z^2}{2xy} & \text{if } x, y \in \mathbb{R}\setminus\{0\}, \; z \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Delta_\alpha(x, y, z) = \begin{cases} \frac{((|x|+|y|)^2-z^2)(z^2-(|x|-|y|)^2)^{\alpha-\frac{1}{2}}}{|xy|^2\alpha} & \text{if } |z| \in S_{x,y} \\ 0 & \text{otherwise} \end{cases}$$

where

$$S_{x,y} = \left[||x| - |y||, |x| + |y|\right].$$

The kernel $W_\alpha$ (see [17]), is even and we have

$$W_\alpha(x, y, z) = W_\alpha(y, x, z) = W_\alpha(-x, z, y) = W_\alpha(-z, y, -x)$$

and

$$\int_{\mathbb{R}} |W_\alpha(x, y, z)|d\mu_\alpha(z) \leq 4.$$  

In the sequel we consider the signed measure $\gamma_{x,y}$, on $\mathbb{R}$, given by

$$d\gamma_{x,y}(z) = \begin{cases} W_\alpha(x, y, z)d\mu_\alpha(z) & \text{if } x, y \in \mathbb{R}\setminus\{0\} \\ d\delta_x(z) & \text{if } y = 0 \\ d\delta_y(z) & \text{if } x = 0. \quad (4)$$
For $x, y \in \mathbb{R}$ and $f$ a continuous function on $\mathbb{R}$, the Dunkl translation operator $\tau_x$ is given by

$$\tau_x(f)(y) = \int_{\mathbb{R}} f(z) d\gamma_{x,y}(z).$$

It was shown in [13] that for $x \in \mathbb{R}$, $\tau_x$ is a continuous linear operator from $\mathcal{E}(\mathbb{R})$ into itself and for all $f \in \mathcal{E}(\mathbb{R})$, we have

$$\tau_0(f)(x) = f(x), \quad \tau_x \circ \tau_y = \tau_y \circ \tau_x$$

and for $x, y \in \mathbb{R}$,

$$\tau_x(f)(y) = \tau_y(f)(x), \quad \Lambda_\alpha \circ \tau_x = \tau_x \circ \Lambda_\alpha, \quad x, y \in \mathbb{R},$$

(5)

where $\mathcal{E}(\mathbb{R})$ denotes the space of $C^\infty$-functions on $\mathbb{R}$.

According to [19], the operator $\tau_x$ can be extended to $L^p(\mu_\alpha)$, $1 \leq p \leq +\infty$ and for $f \in L^p(\mu_\alpha)$ we have

$$\|\tau_x(f)\|_{p,\alpha} \leq 4\|f\|_{p,\alpha},$$

(6)

and for all $x, \lambda \in \mathbb{R}$, $f \in L^1(\mu_\alpha)$, we have

$$\mathcal{F}_\alpha(\tau_x(f))(\lambda) = E_\alpha(i\lambda x)\mathcal{F}_\alpha(f)(\lambda).$$

(7)

Using the change of variable $z = (x, y)_\theta = \sqrt{x^2 + y^2 - 2xy \cos \theta}$, we have also

$$\tau_x(f)(y) = \int_0^\pi \left[ f_e((x, y)_\theta) + \frac{x + y}{(x, y)_\theta} f_o((x, y)_\theta) \right] d\nu_\alpha(\theta)$$

(8)

where

$$f_e((x, y)_\theta) = f((x, y)_\theta) + f(-(x, y)_\theta), \quad f_o((x, y)_\theta) = f((x, y)_\theta) - f(-(x, y)_\theta)$$

and

$$d\nu_\alpha(\theta) = \frac{\Gamma(\alpha + 1)}{2\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})} (1 - \cos \theta) \sin^{2\alpha} \theta d\theta.$$

The Dunkl convolution $f *_{\alpha} g$, of two continuous functions $f$ and $g$ on $\mathbb{R}$ with compact support, is defined by

$$(f *_{\alpha} g)(x) = \int_{\mathbb{R}} \tau_x(f)(-y)g(y) d\mu_\alpha(y), \quad x \in \mathbb{R}.$$

The convolution $*_{\alpha}$ is associative and commutative (see [17]).

We have the following results (see [18]).
i) Assume that $p, q, r \in [1, +\infty]$ satisfying $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ (the Young condition). Then the map $(f, g) \rightarrow f \ast_{\alpha} g$ defined on $C_c(\mathbb{R}) \times C_c(\mathbb{R})$, extends to a continuous map from $L^p(\mu_{\alpha}) \times L^q(\mu_{\alpha})$ to $L^r(\mu_{\alpha})$ and we have

$$\|f \ast_{\alpha} g\|_{r,\alpha} \leq 4\|f\|_{p,\alpha}\|g\|_{q,\alpha}. \quad (9)$$

ii) For all $f \in L^1(\mu_{\alpha})$ and $g \in L^2(\mu_{\alpha})$, we have

$$\mathcal{F}_{\alpha}(f \ast_{\alpha} g) = \mathcal{F}_{\alpha}(f)\mathcal{F}_{\alpha}(g) \quad (10)$$

and for $f \in L^1(\mu_{\alpha})$, $g \in L^p(\mu_{\alpha})$ and $1 \leq p < \infty$, we get

$$\tau_t(f \ast_{\alpha} g) = \tau_t(f) \ast_{\alpha} g = f \ast_{\alpha} \tau_t(g), \quad t \in \mathbb{R}. \quad (11)$$

3. Characterizations of the Besov-Dunkl spaces

In this section, we provide equivalence between different characterizations of the Besov-Dunkl spaces.

**Theorem 1.** Let $1 \leq p < +\infty$, $1 \leq q \leq +\infty$ and $\beta > 0$, then

$$\mathcal{BD}^{\beta,\alpha}_{p,q} = \mathcal{KD}^{\beta,\alpha}_{p,q}.$$  

**Proof.** For $x > 0$ and $0 < |z| \leq x$, put

$$\Theta(x, z) = \frac{1}{2x^{2\alpha+1}} + \frac{\text{sgn}(z)}{2|x|^{2\alpha+1}}.$$  

We start with the proof of the inclusion $\mathcal{BD}^{\beta,\alpha}_{p,q} \subset \mathcal{KD}^{\beta,\alpha}_{p,q}$. For $f \in \mathcal{BD}^{\beta,\alpha}_{p,q}$ and $x > 0$, we take

$$f_1 = \frac{1}{x} \int_{-x}^{x} \Theta(x, z) \tau_z(f) \, d\mu_{\alpha}(z).$$

Using the Minkowski’s inequality for integrals and (6), we have

$$\|f_1\|_{p,\alpha} \leq \frac{1}{x} \int_{-x}^{x} |\Theta(x, z)| \|\tau_z(f)\|_{p,\alpha} \, d\mu_{\alpha}(z)$$

$$\leq c \frac{\|f\|_{p,\alpha}}{x} \int_{-x}^{x} |\Theta(x, z)| \, d\mu_{\alpha}(z) \leq c\|f\|_{p,\alpha}.$$  

By (5) and the generalized Taylor formula with integral remainder (see[14], Theorem 2, p. 349), we get

$$\Lambda_{\alpha} f_1 = \frac{1}{x} \int_{-x}^{x} \Theta(x, z) \tau_z(\Lambda_{\alpha} f) \, d\mu_{\alpha}(z)$$

$$= \frac{1}{x}(\tau_x(f) - f).$$
then we obtain,
\[ x\|\Lambda_{\alpha}f_1\|_{p,\alpha} \leq c \ w_{p,\alpha}(f, x). \] (12)

On the other hand, put \( f_0 = f - 2^{\alpha+2}\Gamma(\alpha + 2)f_1 \), we can write
\[ f_0 = -\frac{2^{\alpha+2}\Gamma(\alpha + 2)}{x} \int_{-x}^{x} \Theta(x, z)(\tau_z(f) - f) d\mu_\alpha(z), \]
by the Minkowski’s inequality for integrals, we get
\[
\|f_0\|_{p,\alpha} \leq c \int_{-x}^{x} |\Theta(x, z)| \|\tau_z(f) - f\|_{p,\alpha} d\mu_\alpha(z) \\
\leq c \frac{w_{p,\alpha}(f, x)}{x} \int_{-x}^{x} |\Theta(x, z)| d\mu_\alpha(z) \\
\leq c \ w_{p,\alpha}(f, x). \] (13)

Hence by (12) and (13), we deduce that
\[ K_{p,\alpha}(f, x) \leq c \ w_{p,\alpha}(f, x). \] (14)

Let prove now the inclusion \( \mathcal{K}\mathcal{D}^{\beta,\alpha}_{p,q} \subset \mathcal{B}\mathcal{D}^{\beta,\alpha}_{p,q} \). For \( f \in \mathcal{K}\mathcal{D}^{\beta,\alpha}_{p,q}, x > 0 \) and \( f_0 \in L^p(\mu_\alpha), f_1 \in D_{p,\alpha} \) such that \( f = f_0 + f_1 \), we have by (6)
\[ w_{p,\alpha}(f_0, x) \leq c \|f_0\|_{p,\alpha}, \] (15)
on the other hand, using ([14], Theorem 2) we can write for \( t \) such that \( |t| \leq x \)
\[ \tau_t(f_1) - f_1 = \int_{-|t|}^{\ |t|} \Theta(t, z) \tau_z(\Lambda_{\alpha}f_1) d\mu_\alpha(z), \]
by the Minkowski’s inequality for integrals and (6) again, we get
\[
\|\tau_t(f_1) - f_1\|_{p,\alpha} \leq \int_{-|t|}^{\ |t|} |\Theta(t, z)| \|\tau_z(\Lambda_{\alpha}f_1)\|_{p,\alpha} d\mu_\alpha(z) \\
\leq c \|\Lambda_{\alpha}f_1\|_{p,\alpha} \int_{-|t|}^{\ |t|} |\Theta(t, z)| d\mu_\alpha(z) \\
\leq c \ |t| \|\Lambda_{\alpha}f_1\|_{p,\alpha} \leq c x \|\Lambda_{\alpha}f_1\|_{p,\alpha},
\]
then we obtain,
\[ w_{p,\alpha}(f_1, x) \leq c x \|\Lambda_{\alpha}f_1\|_{p,\alpha}, \] (16)
since
\[ w_{p,\alpha}(f, x) \leq w_{p,\alpha}(f_0, x) + w_{p,\alpha}(f_1, x), \]
by (15) and (16), we deduce that
\[ w_{p,\alpha}(f, x) \leq c K_{p,\alpha}(f, x). \tag{17} \]
Our theorem is proved. \[\square\]

**Theorem 2.** Let \(1 \leq p \leq 2, 1 \leq q \leq +\infty\) and \(\beta > 0\), then
\[ BD^{\beta,\alpha}_{p,q} \subset ED^{\beta,\alpha}_{p,q}. \]

**Proof.** Let \(f \in BD^{\beta,\alpha}_{p,q}\) and \(\lambda, x > 0\), by (14) and (17) we have
\[ w_{p,\alpha}(f, \lambda x) \leq c K_{p,\alpha}(f, \lambda x) \leq c \max\{1, \lambda\} K_{p,\alpha}(f, x) \leq c \max\{1, \lambda\} w_{p,\alpha}(f, x). \tag{18} \]
Choose \(\varphi \in S_*(\mathbb{R})\) with \(\text{supp}(\mathcal{F}_\alpha(\varphi)) \subset [-1, 1]\) and \(\int_{\mathbb{R}} \varphi(x)d\mu_\alpha(x) = 1\).

From (10), we get for \(t > 0\)
\[ \mathcal{F}_\alpha(f \ast_\alpha \varphi_\frac{1}{t}) = \mathcal{F}_\alpha(f)\mathcal{F}_\alpha(\varphi_\frac{1}{t}) \]
where \(\varphi_\frac{1}{t}(x) = t^{2(\alpha+1)}\varphi(tx)\), which implies \(\text{supp}(\mathcal{F}_\alpha(f \ast_\alpha \varphi_\frac{1}{t})) \subset [-t, t]\) and
\[ E_{p,\alpha}(f, t) \leq \|f - f \ast_\alpha \varphi_\frac{1}{t}\|_{p,\alpha}. \tag{19} \]

On the other hand, by the Minkowski’s inequality for integrals
\[
\|f - f \ast_\alpha \varphi_\frac{1}{t}\|_{p,\alpha} = \left( \int_{\mathbb{R}} \left| f(y) - \int_{\mathbb{R}} \varphi_\frac{1}{t}(z) \tau_\gamma(f)(y) d\mu_\alpha(z) \right|^p d\mu_\alpha(y) \right)^{1/p}
\leq \left( \int_{\mathbb{R}} |\varphi_\frac{1}{t}(z)| d\mu_\alpha(y) \int_{\mathbb{R}} \tau_\gamma(f)(y) d\mu_\alpha(z) \right)^{1/p}
\leq \int_{\mathbb{R}} |\varphi_\frac{1}{t}(z)| \|\tau_\gamma(f) - f\|_{p,\alpha} d\mu_\alpha(z)
\leq \int_{\mathbb{R}} |\varphi_\frac{1}{t}(z)| w_{p,\alpha}(f, |z|) d\mu_\alpha(z),
\]
using (18), we obtain
\[
\| f - f *_1 \|_{p,\alpha} \leq c \, w_{p,\alpha}(f, \frac{1}{t}) \int | \varphi_1(z) | (1 + |z|) \, d\mu_\alpha(z)
\]
\[
\leq c \, w_{p,\alpha}(f, \frac{1}{t}) \int | \varphi(z) | (1 + |z|) \, d\mu_\alpha(z)
\]
\[
\leq c \, w_{p,\alpha}(f, \frac{1}{t}).
\]  
(20)

Thus, (19) and (20) imply
\[
\int_1^{+\infty} (t^\beta E_{p,\alpha}(f, t)) \frac{dt}{t} \leq c \int_0^{+\infty} (t^\beta w_{p,\alpha}(f, \frac{1}{t}))^q \frac{dt}{t}
\]
\[
\leq c \int_0^{+\infty} \left( \frac{w_{p,\alpha}(f, t)}{t^\beta} \right)^q \frac{dt}{t}, \quad \text{if } q < +\infty
\]
and the same is true for \( q = +\infty \).

This completes the proof of the inclusion. \( \square \)

Now, in order to establish that \( \mathcal{BD}^{\beta,\alpha}_{p,q} = \mathcal{ED}^{\beta,\alpha}_{p,q} \) for \( 1 \leq p \leq 2, 1 \leq q < +\infty \) and \( 0 < \beta < 1 \), we need to show some useful results.

In the following lemma, we prove a Bernstein-type inequality for the Dunkl translation operators. An analogous result has been proved by [6, 10] for the generalized translation operators associated with the Bessel operator.

**Lemma 1.** For \( 1 \leq p < +\infty \), there exists a constant \( c > 0 \) such that for \( h \in L^p(\mu_\alpha) \) an even differentiable function on \( \mathbb{R} \) with \( h' \in L^p(\mu_\alpha) \) and \( y_1, y_2 > 0 \), we have
\[
\| \tau_{y_1}(h) - \tau_{y_2}(h) \|_{p,\alpha} \leq c |y_1 - y_2| \| h' \|_{p,\alpha}.
\]

**Proof.** Using (8) and the fact that \( h \) is even, we can assert that
\[
\| \tau_{y_1}(h) - \tau_{y_2}(h) \|_{p,\alpha}^p = \int_{\mathbb{R}} |[\tau_{y_1}(h) - \tau_{y_2}(h)](x)|^p \, d\mu_\alpha(x)
\]
\[
= \int_{\mathbb{R}} \left[ \int_0^\pi [2h((x, y_1)_\theta) - 2h((x, y_2)_\theta)] \, d\nu_\alpha(\theta) \right]^p \, d\mu_\alpha(x)
\]
\[
\leq c \int_{\mathbb{R}} \left( \int_0^\pi |h((x, y_1)_\theta) - h((x, y_2)_\theta)| \, d\nu_\alpha(\theta) \right) d\mu_\alpha(x)
\]
\[
\leq c \int_{\mathbb{R}} \left( \int_0^\pi \int_0^1 \frac{d}{ds} [h((x, y_2 + s(y_1 - y_2))_\theta)] \, ds \right)^p \, d\nu_\alpha(\theta) d\mu_\alpha(x),
\]
By \( \frac{d}{ds} |(x, y_2 + s(y_1 - y_2))_\theta| \leq |y_1 - y_2| \), then we can write

\[
\| \tau_{y_1}(h) - \tau_{y_2}(h) \|_{p, \alpha}^p 
\]

\[
\leq c |y_1 - y_2|^p \int_0^\pi \int_0^1 \left| \int_0^1 h((x, y_2 + s(y_1 - y_2))_\theta) ds \right|^p d\nu(\theta) d\mu(\theta),
\]

\[
\leq c |y_1 - y_2|^p \int_0^\pi \left( \int_0^1 \left| h((x, y_2 + s(y_1 - y_2))_\theta) \right|^p \sin^{2\alpha} \theta d\theta \right) d\mu(\theta) ds.
\]

Then from (21), (22) and (23), we obtain

\[
\int_R \left( \int_0^\pi \left| h((x, y_2 + s(y_1 - y_2))_\theta) \right|^p \sin^{2\alpha} \theta d\theta \right) d\mu(\theta) = 2c_\alpha \int_0^\infty T_{y_2+s(y_1-y_2)}(|h'|^p)(x) dx
\]

where \( T_y, y \geq 0 \) is the generalized translation operator associated with the Bessel operator and \( c_\alpha = \sqrt{\pi \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)}} \).

On the other hand, by the change of variable \( \theta' = \pi - \theta \), we get for \( x \leq 0 \),

\[
\int_0^\pi \left| h((x, y_2 + s(y_1 - y_2))_\theta) \right|^p \sin^{2\alpha} \theta d\theta = c_\alpha T_{y_2+s(y_1-y_2)}(|h'|^p)(-x).
\]

Then from (21), (22) and (23), we obtain

\[
\int_R \left( \int_0^\pi \left| h((x, y_2 + s(y_1 - y_2))_\theta) \right|^p \sin^{2\alpha} \theta d\theta \right) d\mu(\theta)
\]

\[
= 2c_\alpha \int_0^\infty T_{y_2+s(y_1-y_2)}(|h'|^p)(x) d\mu(\theta)
\]

\[
\leq c \int_0^\infty |h'|^p(x) d\mu(\theta) \leq c \|h'\|_{p, \alpha}^p.
\]
Hence, we deduce
$$\|\tau_{y_1}(h) - \tau_{y_2}(h)\|_{p,\alpha} \leq c|y_1 - y_2| \|h'\|_{p,\alpha},$$
which proves the result.

**Lemma 2.** For $1 \leq p \leq 2$, there exists a constant $c > 0$ such that for any $x > 0$, any function $g \in L^p(\mu_\alpha)$ with $\text{supp}(\mathcal{F}_\alpha(g)) \subset [-x, x]$ and $y_1, y_2 > 0$, we have
$$\|\tau_{y_1}(g) - \tau_{y_2}(g)\|_{p,\alpha} \leq c x |y_1 - y_2| \|g\|_{p,\alpha}.$$  

**Proof.** Let $g \in S(\mathbb{R})$ with $\text{supp}(\mathcal{F}_\alpha(g)) \subset [-x, x]$. Choose $\varphi \in D_*(\mathbb{R})$ such that $\varphi(t) = 1$ if $|t| \leq 1$ and $\varphi(t) = 0$ if $|t| \geq 2$. Then by the inversion formula (2), we have $\varphi = \mathcal{F}_\alpha(h)$ for some $h \in S_*(\mathbb{R})$. Put $h_x(y) = x^{2(\alpha+1)}h(xy)$ for $y \in \mathbb{R}$, then $\mathcal{F}_\alpha(h_x)(y) = \varphi(\frac{y}{x}) = 1$ for $|y| \leq x$. Note that $\text{supp}(\mathcal{F}_\alpha(g)) \subset [-x, x]$, then using (1), (7) and (10), we can write
$$\mathcal{F}_\alpha(\tau_{y_1}(g) - \tau_{y_2}(g)) = \mathcal{F}_\alpha(h_x \ast_{\alpha} (\tau_{y_1}(g) - \tau_{y_2}(g))),$$
by (2) and (9), we obtain
$$\tau_{y_1}(g) - \tau_{y_2}(g) = h_x \ast_{\alpha} (\tau_{y_1}(g) - \tau_{y_2}(g)) = (\tau_{y_1}(h_x) - \tau_{y_2}(h_x)) \ast_{\alpha} g.$$
The change of variable $t' = xt$ in (3) gives
$$W_\alpha(xy, xz, t') x^{2(\alpha+1)} = W_\alpha(y, z, t),$$
then from (4), we get
$$d\gamma_{xy,xz}(t') = d\gamma_{y,z}(t) \text{ and } \tau_{y}(h_x)(z) = x^{2(\alpha+1)}\tau_{xy}(h)(xz).$$
Therefore, using the lemma 1 , we have
$$\|\tau_{y_1}(g) - \tau_{y_2}(g)\|_{p,\alpha} \leq 4 \|\tau_{y_1}(h_x) - \tau_{y_2}(h_x)\|_{1,\alpha} \|g\|_{p,\alpha}$$
$$= 4 \|\tau_{xy_1}(h) - \tau_{xy_2}(h)\|_{1,\alpha} \|g\|_{p,\alpha}$$
$$\leq c x |y_1 - y_2| \|h'\|_{p,\alpha} \|g\|_{p,\alpha}$$
$$\leq c x |y_1 - y_2| \|g\|_{p,\alpha}.$$  

Since $S(\mathbb{R})$ is a dense subset of $L^p(\mu_\alpha)$ for $1 \leq p < +\infty$ and by (6), we obtain the result. \qed
Theorem 3. Let \(1 \leq p \leq 2, 1 \leq q < +\infty\) and \(0 < \beta < 1\), then
\[
\mathcal{ED}^{\beta,\alpha}_{p,q} = \mathcal{BD}^{\beta,\alpha}_{p,q}.
\]

Proof. We have only to show that \(\mathcal{ED}^{\beta,\alpha}_{p,q} \subseteq \mathcal{BD}^{\beta,\alpha}_{p,q}\). Assume \(f \in \mathcal{ED}^{\beta,\alpha}_{p,q}\), we can consider \(f \neq 0\) a.e., then we get
\[
\left( \int_0^1 (t^{-\beta} w_{p,\alpha}(f,t))^q \frac{dt}{t} \right)^{\frac{1}{q}} = \left( \sum_{n=0}^{+\infty} \int_{2^{-n-1}}^{2^{-n}} (t^{-\beta} w_{p,\alpha}(f,t))^q \frac{dt}{t} \right)^{\frac{1}{q}}
\leq 2^\beta \left( \sum_{n=0}^{+\infty} (2^n \beta w_{p,\alpha}(f,2^{-n}))^q \right)^{\frac{1}{q}}
= 2^\beta \sum_{n=0}^{+\infty} \lambda_n 2^n \beta w_{p,\alpha}(f,2^{-n}),
\]
where \(\lambda_n = \frac{(2^n \beta w_{p,\alpha}(f,2^{-n}))^{\frac{q}{q'}}}{\left( \sum_{n=0}^{+\infty} (2^n \beta w_{p,\alpha}(f,2^{-n}))^q \right)^{\frac{1}{q'}}}\) with \(q'\) the conjugate of \(q\).

By reasoning as in the proof on ([16], Proposition 3.1, p. 88) and using the lemma 2, we have for \(0 < \beta < 1\),
\[
\left( \int_0^1 (t^{-\beta} w_{p,\alpha}(f,t))^q \frac{dt}{t} \right)^{\frac{1}{q}} \leq 2^\beta c \left( \|f\|_p + \left( \sum_{m=1}^{+\infty} (2^n \beta E_{p,\alpha}(f,2^{-m}))^q \right)^{\frac{1}{q'}} \right).
\]

Since \(E_{p,\alpha}(f,t)\) is decreasing in \(t\) and by (19),
\[
\left( \sum_{m=1}^{+\infty} (2^n \beta E_{p,\alpha}(f,2^{-m}))^q \right)^{\frac{1}{q'}} = 2^\beta E_{p,\alpha}(f,1) + \left( \sum_{m=2}^{+\infty} (2^n \beta E_{p,\alpha}(f,2^{-m-1}))^q \right)^{\frac{1}{q'}}
\leq c \left( \|f\|_p + \left( \int_1^{+\infty} \left( t^\beta E_{p,\alpha}(f,t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q'}} \right).
\]

The result of the two inequalities above is
\[
\left( \int_0^1 (t^{-\beta} w_{p,\alpha}(f,t))^q \frac{dt}{t} \right)^{\frac{1}{q}} \leq c \left( \|f\|_p + \left( \int_1^{+\infty} \left( t^\beta E_{p,\alpha}(f,t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q'}} \right).
\]

On the other hand, we easily obtain,
\[
\left( \int_1^{+\infty} (t^{-\beta} w_{p,\alpha}(f,t))^q \frac{dt}{t} \right)^{\frac{1}{q}} \leq c \|f\|_p \left( \int_1^{+\infty} t^{-1 - \beta q} dt \right)^{\frac{1}{q}} \leq c \|f\|_p.
\]
Hence, we conclude that
\[
\left( \int_0^{+\infty} (t^{-\beta} w_{p,\alpha}(f, t))^q \frac{dt}{t} \right)^\frac{1}{q} \leq c \left( \|f\|_p + \left( \int_1^{+\infty} (t^{\beta} E_{p,\alpha}(f, t))^q \frac{dt}{t} \right)^\frac{1}{q} \right).
\]
This completes the proof. \qed

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