Spectral correlations of the massive QCD Dirac operator at finite temperature

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Abstract

We use the graded eigenvalue method, a variant of the supersymmetry technique, to compute the universal spectral correlations of the QCD Dirac operator in the presence of massive dynamical quarks. The calculation is done for the chiral Gaussian unitary ensemble of random matrix theory with an arbitrary Hermitian matrix added to the Dirac matrix. This case is of interest for schematic models of QCD at finite temperature.

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1 Introduction

By now, it has been firmly established that the spectrum of the QCD Dirac operator possesses a number of universal features which can be described by chiral random matrix theory (RMT) \cite{1,2}. In particular, the RMT predictions agree very well with data from lattice gauge simulations, both for the eigenvalue correlations in the bulk of the spectrum on the scale of the mean level spacing \cite{3–5} and for the distribution and correlations of the low-lying eigenvalues \cite{6–9}. It is the key assumption of chiral RMT that the matrix elements of the Dirac operator in an appropriate energy basis behave as random numbers. This concept has already been very successful in many other areas of physics, see the detailed review in Ref. \cite{10}. Thus, it is fair to say that RMT approaches can be viewed as thermodynamics for spectral fluctuations and related properties. In the context of QCD, deviations from pure RMT statistics have also been found in the microscopic \cite{11,12} and in the bulk region \cite{5}.
These findings provide evidence for the conjecture that lattice QCD may have much in common with disordered systems [13,14,11]. In the present work, however, we focus on some formal and theoretical aspects of chiral RMT. Hence, we shall not discuss in any detail the physical applications but refer to the existing literature [15,16].

Our aim here is to compute the universal spectral correlations of the QCD Dirac operator in the presence of massive dynamical quarks for the chiral Gaussian unitary ensemble of RMT. Such a calculation has been done previously at zero temperature using orthogonal polynomials [17,18] and the finite volume partition function [19,20]. In Ref. [21], the connection between these two results was established in the framework of partially quenched chiral perturbation theory. In this paper, we employ the graded eigenvalue method [22–24], which is a special variant of the supersymmetry method [25,26], for the following two reasons. First, it allows us to extent the calculation to the case where an arbitrary deterministic Hermitian matrix is added to the Dirac matrix. This is relevant for schematic random-matrix models of QCD at finite temperature [27,28]. The standard orthogonal-polynomial method cannot easily be applied in this case, because a certain rotation invariance in the space of the random matrices is lost. Second, the present problem leads to an interesting extension of the graded eigenvalue method in the context of chiral RMT. In ordinary RMT, this method was developed to calculate spectral correlations in crossover transitions from regularity to chaos [23,24]. This method was then extended to chiral RMT and used to compute the universal spectral correlations of the Dirac operator in the quenched approximation, i.e., without dynamical quarks [29,30]. For an application of the supersymmetry method to the case of one flavor at zero temperature, see Ref. [31].

In Refs. [17,32] it was shown that the RMT results are invariant under deformations of the distribution of the random matrix. While we suspect this statement to hold also for the results computed in the present paper, a rigorous proof would require an extension of the work of Ref. [33]. We shall not address this issue here.

The outline of this paper is as follows. In Sec. 2, we define the problem and outline the main idea for the solution. The graded eigenvalue method is applied in Sec. 3. The special case of energies and masses on the scale of the mean level spacing near zero, which is also the most interesting case for physical applications, is considered in Sec. 4. We conclude with a summary in Sec. 5. Technical details of the calculation are discussed in two appendixes.
2 Setup of the calculation

Since this paper is a natural extension of Ref. [29], we shall attempt to use a similar notation. The QCD Dirac operator in Euclidean space is defined by
\[ D = \gamma_\mu \partial_\mu + ig \gamma_\mu A_\mu, \]
where \( g \) is the coupling constant and the \( A_\mu \) are the gauge fields. Note that \( D \) is anti-Hermitian. In a random matrix model in a chiral basis, the matrix \( A \) representing the Dirac operator has the form [2]
\[
D \longrightarrow iA = i \begin{bmatrix}
0 & W + Y \\
W^\dagger + Y & 0
\end{bmatrix}, \tag{1}
\]
where \( W \) is a square random matrix of dimension \( N \) and \( Y \) is an arbitrary Hermitian matrix. Expression (1) is a schematic model for the QCD Dirac operator at finite temperature. The matrix \( Y \) represents the effects of the temperature on the Dirac spectrum. Its specific form depends on the choice of basis states [27,28]. Since we consider an arbitrary Hermitian matrix \( Y \), we cover all possible choices of basis states.

One could also consider the more general problem of a rectangular matrix \( W \), giving rise to exact zero modes of the Dirac operator whose number can be identified with the topological charge. At \( Y = 0 \), this is not necessary since this problem is equivalent to introducing additional massless flavors [34]. Although it is not obvious, we expect this equivalence to hold also for nonzero \( Y \). We hope to address this problem in future work.

We will be interested in the correlations of the eigenvalues of the matrix \( A \). In this paper, we study the chiral Gaussian unitary ensemble (chGUE) appropriate for QCD with three or more colors for which \( W \) is a complex matrix without any symmetries [34]. The probability distribution of \( W \) is given by
\[
P(W) = \frac{1}{\mathcal{N}} P_0(W) \prod_{f=1}^{N_f} \det(i m_f - A), \tag{2}
\]
with
\[
P_0(W) = \exp \left( -N \Sigma^2 \text{tr} WW^\dagger \right). \tag{3}
\]
Here, \( N_f \) is the number of quark flavors with masses \( m_f \), \( \mathcal{N} \) is a normalization factor (see below), and \( \Sigma \) is a real parameter which will turn out to be equal to the chiral condensate at zero temperature. Note that \( \mathcal{N} \) depends on the quark masses. To be precise, the argument of the determinant in (2) should have been \( m_f + D = m_f + iA \). For convenience, we have pulled out a factor of \(-i\) and absorbed it in \( \mathcal{N} \).
We are interested in the \( k \)-point spectral correlation functions, defined as the probability of finding energies in infinitesimal intervals around the points \( x_1, \ldots, x_k \), regardless of labeling. Apart from some trivial contributions involving \( \delta \)-functions [10] they are given by

\[
R_k(x_1, \ldots, x_k) = \left(-\frac{1}{\pi}\right)^k \int d[W] P(W) \prod_{p=1}^{k} \text{Im tr} \frac{1}{x_p^+ - A},
\]

where \( x_p^+ = x_p + i\varepsilon \) with \( \varepsilon \) positive infinitesimal. The measure \( d[W] \) is simply the product of the differentials of the real and imaginary parts of the elements of \( W \), i.e., of all independent variables. The integrations extend from \(-\infty\) to \(+\infty\). The normalization factor \( \mathcal{N} \) is determined by the requirement \( \int d[W] P(W) = 1 \). Since both the distribution (2) and the measure \( d[W] \) are invariant under unitary transformations of \( W \), only the relative unitary rotation between \( W \) and \( Y \) matters and we can, without loss of generality, write the Hermitian matrix \( Y \) in diagonal form, \( Y = \text{diag}(y_1, \ldots, y_N) \). Advantageously, the \( k \)-point functions can be obtained as

\[
R_k(x_1, \ldots, x_k) = \left(-\frac{1}{\pi}\right)^k \frac{\partial^k}{\prod_{p=1}^{k} \partial J_p} Z_k(J) \bigg|_{J_p=0},
\]

with a generating function given by

\[
Z_k(J) = \int d[W] P(W) \prod_{p=1}^{k} \det(x_p - A) \text{Im} \frac{1}{\det(x_p^+ - J_p - A)} ,
\]

where \( J \) stands for \( J_1, \ldots, J_k \). The starting point of the graded eigenvalue method is to rewrite the determinants in (6) as Gaussian integrals over commuting and anti-commuting variables. Note that the distribution \( P(W) \) contains \( N_f \) determinants in the numerator. To retain the determinant structure of the problem, it is highly desirable to have an equal number of determinants in numerator and denominator so that the bosonic and fermionic blocks in the supersymmetric representation of the generating function have the same size. Therefore, we introduce \( N_f \) additional determinants in the denominator and write, in the large-\( N \) limit,

\[
Z_k(J) = \lim_{\{a_f\} \to \infty} \tilde{Z}_k(J)
\]

with

\[
\tilde{Z}_k(J) = \frac{1}{\mathcal{N}} \int d[W] P_0(W) \prod_{p=1}^{k} \det(x_p - A) \text{Im} \frac{1}{\det(x_p^+ - J_p - A)}
\times \prod_{f=1}^{N_f} \det(im_f - A) \text{Im} \frac{1}{\det(a_f - A)} ,
\]
where the $a_f$ ($f = 1, \ldots, N_f$) are dummy real variables. The modified normalization $\tilde{N}$ depends on the $a_f$ and is given by

$$
\tilde{N} = \int d[W] \, P_0(W) \prod_{f=1}^{N_f} \det(im_f - A) \, \text{Im} \frac{1}{\det(a_f^+ - A)}.
$$

(9)

The fact that Eq. (7) holds in the limit $N \to \infty$ is proved in App. A. The introduction of the dummy determinants in (8) is the main idea of the present calculation. As we shall see below, it allows us to use the results of Ref. [29] so that the generalization from the quenched approximation to the case with $N_f > 0$ can be obtained with moderate effort.

We observe that (9) is essentially a special case of (8) with $k = 0$. To simplify the notation, we will compute the generic quantity

$$
G_\gamma(t) = \int d[W] \, P_0(W) \prod_{j=1}^{\gamma} \det(t_{j2} - A) \, \text{Im} \frac{1}{\det(t_{j1}^+ - A)},
$$

(10)

where $\gamma$ is a nonnegative integer and $t = \text{diag}(t_{11}, \ldots, t_{\gamma 1}, t_{11}, \ldots, t_{\gamma 2})$ is a diagonal graded matrix of dimension $2\gamma$. Both (8) and (9) can be obtained from (10) by choosing $\gamma$ and $t$ appropriately.

3 Supersymmetric representation and graded eigenvalue method

Since we can employ the results of Ref. [29] to compute the function (10) very efficiently, we only review the major steps in the derivation. First, the determinants in (10) are rewritten as Gaussian integrals over commuting and anti-commuting variables which are arranged in a graded (or super) vector $\psi$. Then, the integration over $W$ can be performed, resulting in fourth-order terms in the $\psi$-variables. These terms can be removed by a Hubbard-Stratonovitch transformation at the expense of introducing additional integration variables which can be arranged in a complex graded (or super) matrix $\sigma$. The order of the integrations over $\sigma$ and $\psi$ can then be interchanged, provided that one is only interested in the imaginary parts in (10). This point has been discussed in Refs. [29,30,35]. The $\psi$-integration can then be performed trivially. The graded matrix $\sigma$ can be written in spherical coordinates as $us\bar{v}$ with graded (or super) unitary matrices $u$ and $\bar{v}$ and radial coordinates $s$. The integration over $u$ and $\bar{v}$ can be performed using the supersymmetric generalization of the Berezin-Karpelevich integral [36]. It can be viewed as the extension of the supersymmetric Itzykson-Zuber integral [22] to complex graded matrices. The final result for the function (10) then becomes (see Eqs. (33) through (36) of
Ref. [29])

\[
G_\gamma(t) = \left( \frac{\pi}{N\Sigma^2} \right)^{N^2} \frac{\exp(-N\Sigma^2 \text{trg} t^2)}{B_\gamma(t^2)} \det[C_N(t_{i1}, t_{j2})]_{i,j=1,\ldots,\gamma}
\]

with

\[
C_N(x_1, x_2) = (2N\Sigma^2)^2 \int_0^\infty \int_0^\infty ds_1 ds_2 \frac{s_1 s_2}{s_1^2 + s_2^2} \exp\left(-N\Sigma^2(s_1^2 + s_2^2)\right)
\]

\[
\times I_0(2N\Sigma^2 s_1 x_1) J_0(2N\Sigma^2 s_2 x_2) \text{Im} \prod_{n=1}^N \frac{y_n^2 + s_2^2}{y_n^2 - (s_1^+)^2},
\]

where \( J \) and \( I \) denote the Bessel and modified Bessel function, respectively. In Eq. (11), the symbol \( \text{trg} \) denotes the graded trace, and

\[
B_\gamma(t^2) = \frac{\Delta_\gamma(t_1^2) \Delta_\gamma(t_2^2)}{\prod_{ij}(t_{i1}^2 - t_{j2}^2)} \quad \text{with} \quad \Delta_\gamma(x) = \prod_{i>j}(x_i - x_j).
\]

Equations (8) and (9) can now be obtained by choosing \( \gamma = k + N_f \) and \( \gamma = N_f \), respectively, and substituting appropriate values for the entries of \( t \). Performing the differentiations according to Eq. (5) (with \( Z_k(J) \) replaced by \( \tilde{Z}_k(J) \), see Eq. (7)) then yields the \( k \)-point functions. We obtain after some algebra

\[
R_k(x_1, \ldots, x_k) = (\frac{2}{\pi})^k \left( \prod_{p=1}^k x_p \right) \lim_{\{a_f\} \to \infty} \det[C_N(z_p, \zeta_q)]_{p,q=1,\ldots,k+N_f} \det[C_N(a_f, im_g)]_{f,g=1,\ldots,N_f}
\]

with

\[
z_p = \begin{cases} 
  x_p & \text{for } p = 1, \ldots, k, \\
  a_{p-k} & \text{for } p = k + 1, \ldots, k + N_f,
\end{cases}
\]

\[
\zeta_q = \begin{cases} 
  x_q & \text{for } q = 1, \ldots, k, \\
  im_{q-k} & \text{for } q = k + 1, \ldots, k + N_f.
\end{cases}
\]

Note that Eq. (14), just like Eq. (7), holds in the limit \( N \to \infty \) (which is the interesting limit for physical applications) but not for finite \( N \). Instead of introducing dummy determinants in Eq. (8), there is an alternative way to proceed which is exact for finite \( N \). We briefly explain the idea for readers familiar with the graded eigenvalue method. Without the introduction of the dummy determinants in Eq. (8), the transformation from ordinary space to superspace leads to a graded (or super) matrix \( \sigma \) whose boson-boson and fermion-fermion blocks have dimension \( k \) and \( k + N_f \), respectively. The transformation of \( \sigma \) to spherical coordinates involves a Berezinian which cannot be written as a determinant. The idea now is to enlarge the boson-boson block
of $\sigma$ to dimension $k + N_f$ by introducing dummy integration variables in superspace. Then, the Berezinian resulting from the enlarged $\sigma$-matrix can be written as a determinant which is the prerequisite for expressing the $k$-point function in form of a determinant. However, we will not discuss this alternative way since we are only interested in the limit $N \to \infty$ for which the method of Eqs. (7) through (9) appears to be more economic.

Our conventions are such that the support of the spectral density is of order $\mathcal{O}(1)$ and the typical level spacing is of order $\mathcal{O}(1/N)$. While Eq. (14) holds for all values of the $x_p$ and $m_f$, we are particularly interested in the microscopic region where the $x_p$ and $m_f$ are of order $\mathcal{O}(1/N)$. We now turn to this limit.

4 Microscopic limit

If $x_1$ and $x_2$ in Eq. (12) are of order $\mathcal{O}(1/N)$, the integrals can be performed in saddle-point approximation in the large-$N$ limit. This was done in Ref. [29], and we obtain the Bessel kernel (see Eq. (63) of [29])

$$C_N(x_1, x_2) = \pi N \Xi \frac{x_1 J_1(2N\Xi x_1) J_0(2N\Xi x_2) - x_2 J_0(2N\Xi x_1) J_1(2N\Xi x_2)}{x_1^2 - x_2^2},$$

(17)

where $\Xi$ is the only real and positive solution of

$$1 = \frac{1}{N} \sum_{n=1}^{N} \frac{1}{(\Sigma y_n)^2 + (\Xi/\Sigma)^2}$$

(18)

or zero if no such solution exists [29]. As we shall show below, $\Xi = \Xi(Y)$ can be identified with the chiral condensate in the presence of the arbitrary offset $Y$. We now rescale the energies, the masses, and the dummy variables by $2N\Xi$ and define $u_p = 2N\Xi x_p$, $\mu_f = 2N\Xi m_f$, and $\alpha_f = 2N\Xi a_f$. We thus obtain from (14)

$$R_k(x_1, \ldots, x_k) = (2N\Xi)^k \left( \prod_{p=1}^{k} u_p \right) \lim_{\{\alpha_f\} \to \infty} \frac{\det[C(\tilde{z}_p, \tilde{z}_q)]_{p,q=1,\ldots,k+N_f}}{\det[C(\alpha_f, i\mu_g)]_{f,g=1,\ldots,N_f}}$$

(19)

with a kernel given by

$$C(u_1, u_2) = \frac{u_1 J_1(u_1) J_0(u_2) - u_2 J_0(u_1) J_1(u_2)}{u_1^2 - u_2^2},$$

(20)
where we have used the notation \( \tilde{z}_p = 2N\Xi z_p \) and \( \tilde{\zeta}_p = 2N\Xi \zeta_p \). We shall also need this kernel for the second argument purely imaginary,

\[
C(u, i\mu) = \frac{u J_1(u) I_0(\mu) + \mu J_0(u) I_1(\mu)}{u^2 + \mu^2},
\]

and in the special case \( u_1 = u_2 = u \),

\[
C(u, u) = \frac{1}{2} \left[ J_0^2(u) + J_1^2(u) \right].
\]

The major difficulty now is to perform the limit \( \{\alpha_f\} \to \infty \) in (19). For this purpose, it is convenient to divide both numerator and denominator of the last term in (19) by \( \Delta_{N_f}(\alpha) \) and to perform the limits separately in numerator and denominator. Since the derivation is somewhat technical it is presented in App. B. The final result is

\[
\lim_{\{\alpha_f\} \to \infty} \frac{\det[C(\tilde{z}_p, \tilde{\zeta}_q)]_{p,q=1,...,k+N_f}}{\det[C(\alpha_f, i\mu_g)]_{f,g=1,...,N_f}} = \frac{\det[A_{pq}]_{p,q=1,...,k+N_f}}{\det[B_{fg}]_{f,g=1,...,N_f}},
\]

where

\[
\begin{bmatrix}
I_0(\mu_1) & \cdots & I_0(\mu_{N_f}) \\
-\mu_1 I_1(\mu_1) & \cdots & -\mu_{N_f} I_1(\mu_{N_f}) \\
\vdots & \ddots & \vdots \\
(-\mu_1)^{N_f-1} I_{N_f-1}(\mu_1) & \cdots & (-\mu_{N_f})^{N_f-1} I_{N_f-1}(\mu_{N_f})
\end{bmatrix}^{-1}.
\]

and

\[
A = \begin{bmatrix}
C(u_1, u_1) & \cdots & C(u_1, u_k) & C(u_1, i\mu_1) \cdots C(u_1, i\mu_{N_f}) \\
\vdots & \ddots & \vdots & \vdots \\
C(u_k, u_1) & \cdots & C(u_k, u_k) & C(u_k, i\mu_1) \cdots C(u_k, i\mu_{N_f}) \\
J_0(u_1) & \cdots & J_0(u_k) \\
u_1 J_1(u_1) & \cdots & u_k J_1(u_k) \\
\vdots & \ddots & \vdots & \vdots \\
u_1^{N_f-1} J_{N_f-1}(u_1) & \cdots & u_k^{N_f-1} J_{N_f-1}(u_k)
\end{bmatrix}
\]

\[
B
\]

\[
8
\]
In compact notation, we have

\[
A_{pq} = \begin{cases} 
    C(u_p, u_q) & \text{for } 1 \leq p, q \leq k; \\
    C(u_p, i\mu_{q-k}) & \text{for } 1 \leq p \leq k; k+1 \leq q \leq k+N_f; \\
    u_q^{p-k-1} J_{p-k-1}(u_q) & \text{for } k+1 \leq p \leq k+N_f; 1 \leq q \leq k; \\
    (-\mu_{q-k})^{p-k-1} J_{p-k-1}(\mu_{q-k}) & \text{for } k+1 \leq p, q \leq k+N_f 
\end{cases}
\]  

(26)

and

\[
B_{fg} = (-\mu_g)^{f-1} I_{f-1}(\mu_g) \quad \text{for } 1 \leq f, g \leq N_f. 
\]  

(27)

Rescaling the \(R_k\) by \((2N\Xi)^{-k}\), we arrive at the final result for the microscopic spectral correlations,

\[
\rho_k(u_1, \ldots, u_k) \equiv \frac{1}{(2N\Xi)^k} R_k(x_1, \ldots, x_k) = \left( \prod_{p=1}^{k} u_p \right) \frac{\det[A_{pq}]}{\det[B_{fg}]}_{p,q=1,\ldots,k+N_f}. 
\]  

(28)

It remains to be shown that \(\Xi\) can be identified with the absolute value of the chiral condensate \(\langle \bar{\psi}\psi \rangle\). According to the Banks-Casher relation [37], we have \(V|\langle \bar{\psi}\psi \rangle| = \pi R_1(0)\), where \(R_1(0)\) is the spectral density at zero. The space-time volume \(V\) can be identified with \(2N\). Furthermore, we have

\[
R_1(0) = 2N\Xi \lim_{u \to \infty} \rho_1(u). 
\]  

(29)

To compute this limit, we use the matrix \(\mathcal{A}\) in (25) with \(k = 1\) and \(u_1 = u\). We first observe that \(C(u, u) \to 1/(\pi u)\) as \(u \to \infty\), see (22). In addition, the denominator of the entries \(C(u, i\mu_f)\), see (21), can be written as a geometric series in \(1/u^2\) so that \(C(u, i\mu_f)\) is given as an expansion in \(1/u^2\) with coefficients containing \(I_0(\mu_f), \mu_f I_1(\mu_f), \) etc. By subtracting appropriate multiples of rows 2 to \(N_f + 1\), the first \(N_f\) of these terms can be eliminated, see also the discussion of Eq. (B.8) in App. B. The leading large-\(u\) behavior of \(C(u, u)\) is not modified by these subtractions. Higher-order terms in the expansion of \(C(u, i\mu_f)\) are suppressed by powers of \(1/u\), the leading term being of order \(J_{(N_f+1)\text{mod}2}(u)/u^{N_f+1}\). Even when multiplied by the largest entry in the lower-left corner of \(\mathcal{A}\), \(u^{N_f-1} J_{N_f-1}(u)\), the result is suppressed compared to \(1/(\pi u)\). In the large-\(u\) limit, the determinant of (25) with \(k = 1\) thus becomes \(1/(\pi u) \cdot \det \mathcal{B}\), and we obtain

\[
2N\Xi \lim_{u \to \infty} \rho_1(u) = 2N\Xi u \frac{1}{\pi u} = \frac{1}{\pi} 2N\Xi
\]  

(30)

and, hence, \(2N\Xi = \pi R_1(0)\) as desired. Therefore, we have shown that the functional form of the microscopic spectral correlations in the presence of
massive dynamical quarks does not change if a deterministic matrix $Y$ is added to the matrix of the Dirac operator, provided that $\Xi$, i.e., the chiral condensate, is nonzero. The only dependence of the final result on the matrix $Y$ appears in form of a rescaling of the energy scale, from $\Sigma$ at $Y = 0$ to $\Xi$ at $Y \neq 0$.

Our final result (28) is given in terms of the determinant of a $(k+N_f) \times (k+N_f)$ matrix whose entries are simple functions. This structure arises naturally in the graded eigenvalue method. Two other forms for the microscopic spectral correlations have been obtained previously. In the orthogonal-polynomial method, the result is given as the determinant of a $k \times k$ matrix whose entries are $(N_f + 2) \times (N_f + 2)$ matrices [17,18]. From the finite volume partition function, the result is given as the determinant of a $(2k + N_f) \times (2k + N_f)$ matrix whose entries are simple functions [20]. At the present time, we do not have a closed mathematical proof that these three results are identical. However, we have performed extensive checks for a large number of different values of $k$ and $N_f$, both numerically and using computer algebra. In all cases, the three results agree perfectly so that we do not have any doubt that they are identical. One of the virtues of the present method is that it allows us to include the deterministic matrix $Y$ as well. Moreover, it appears to lead to the most economical representation of the final result.

In Ref. [18], the distribution of the smallest eigenvalue, $P(\lambda_{\text{min}})$, was also computed. Its universality with respect to deformations of $P_0(W)$ was shown in Ref. [38]. Since $P(\lambda_{\text{min}})$ follows directly from the microscopic spectral density $\rho_1(u)$ [18], and since we have shown in this paper that the functional form of the latter quantity is not affected by the addition of $Y$, it follows that the functional form of $P(\lambda_{\text{min}})$ also remains unchanged.

5 Summary

In this paper, we have extended the graded eigenvalue method to the case where massive dynamical quarks enter the distribution of the random matrix. The virtue of this approach is twofold. First, we have obtained a novel representation of results computed previously with other methods. Our representation appears to be the most economical one. It is also very stable numerically, compared to the representations obtained from the orthogonal-polynomial method and the finite-volume partition function. Second, our method allows us to perform the calculation with a deterministic matrix added to the Dirac matrix. This is not easily possible in the standard orthogonal-polynomial method because this approach rests on the rotation invariance of the matrix ensemble which is fully broken due to the presence of the deterministic offset.
We point out again that the microscopic correlation functions (28) computed in RMT are universal in the sense that they are expected to agree with the microscopic spectral correlations of the Dirac operator in full QCD. We hope to be able to compare the random matrix results with data from lattice gauge simulations with dynamical fermions in the near future.

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A Derivation of Eq. (7)

To show that Eq. (7) holds, we write the matrix \( W \) in spherical coordinates, \( W = U \Lambda \bar{V} \) with \( U \in U(N) \), \( \bar{V} \in U(N)/U(N) \), and \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N) \), where the \( \lambda_n \) are real and nonnegative [36]. The integration measure transforms according to

\[
d[W] = J(\Lambda)d[\Lambda]d\mu(U)d\mu(\bar{V})
\]

with

\[
d[\Lambda] = \prod_{n=1}^{N} d\lambda_n \quad \text{and} \quad J(\Lambda) = \Delta_N^2(\Lambda^2) \prod_{n=1}^{N} \lambda_n.
\]

The integrations over the \( \lambda_n \) extend from 0 to \( \infty \), \( d\mu(U) \) and \( d\mu(\bar{V}) \) are the invariant Haar measures, and the Vandermonde determinant is defined in Eq. (13). In the integration over \( d[W] \) in Eq. (6), we shift \( W \) and \( W^\dagger \) by \( -Y \) and obtain

\[
\begin{align*}
Z_k(J) &= \frac{1}{N} \int d[\Lambda]d\mu(U)d\mu(\bar{V})J(\Lambda)e^{-N\Sigma^2 \text{tr}(W-Y)(W^\dagger-Y)} \prod_{j=1}^{Nf} \det \begin{bmatrix} \text{Im} & -W \\ -W^\dagger & \text{Im} \end{bmatrix} \\
&\times \prod_{p=1}^{k} \det \begin{bmatrix} x_p & -W \\ -W^\dagger & x_p \end{bmatrix} \frac{1}{\det \begin{bmatrix} x_p^+ - J_p & -W \\ -W^\dagger & x_p^+ - J_p \end{bmatrix}} \\
&= \frac{e^{-N\Sigma^2 \text{tr} Y^2}}{N} \int d[\Lambda]F_N(\Lambda)G_N(\Lambda)H_N(\Lambda)
\end{align*}
\]
with
\[ F_N(\Lambda) = \Delta_N^2(\Lambda^2) \prod_{n=1}^{N} \lambda_n e^{-N\Sigma^2 \lambda_n^2} \prod_{f=1}^{N_f} (-m_f^2 - \lambda_n^2) , \quad (A.4) \]
\[ G_N(\Lambda) = \prod_{p=1}^{k} \text{Im} \prod_{n=1}^{N} \frac{x_p^2 - \lambda_n^2}{(x_p^+ - J_p)^2 - \lambda_n^2} , \quad (A.5) \]
\[ H_N(\Lambda) = \int d\mu(U) d\mu(\bar{V}) e^{N\Sigma^2 \text{tr}(W+W^\dagger)Y} . \quad (A.6) \]

Analogously, we obtain for Eq. (8)
\[ \tilde{Z}_k(J) = \frac{e^{-N\Sigma^2 \text{tr}Y^2}}{\mathcal{N}} \int d[\Lambda] F_N(\Lambda) G_N(\Lambda) H_N(\Lambda) \prod_{f=1}^{N_f} \text{Im} \prod_{n=1}^{N} \frac{1}{(a_f^+)^2 - \lambda_n^2} . \quad (A.7) \]

The normalization factors \( \mathcal{N} \) and \( \tilde{\mathcal{N}} \) follow by setting \( G_N(\Lambda) \) to unity in Eqs. (A.3) and (A.7), respectively. Throughout the remainder of this section, we assume that the dummy variables \( a_f \) are pairwise different.

We proceed by converting the product over \( n \) in (A.7) to a sum,
\[ \prod_{n=1}^{N} \frac{1}{(a_f^+)^2 - \lambda_n^2} = \frac{1}{\Delta_N(\Lambda^2)} \sum_{n=1}^{N} (-1)^{N-n} \frac{\Delta_{N-1}(\Lambda_{(n)}^2)}{(a_f^+)^2 - \lambda_n^2} , \quad (A.8) \]
where the subscript \( (n) \) means that \( \lambda_n \) is omitted in \( \Lambda \). We now have
\[ \text{Im} \frac{1}{(a_f^+)^2 - \lambda_n^2} = -\frac{\pi}{2a_f} [\delta(\lambda_n - a_f) + \delta(\lambda_n + a_f)] \]
\[ \rightarrow -\frac{\pi}{2a_f} \delta(\lambda_n - a_f) , \quad (A.9) \]
where in the last step we have used the fact that the integrations over the \( \lambda_n \) extend from 0 to \( \infty \) and that \( a_f > 0 \) (since we consider the limit \( a_f \rightarrow \infty \)). Thus, out of the \( N \) integrations over the \( \lambda_n \) in Eq. (A.7), \( N_f \) can be done using the \( \delta \)-functions. Using the symmetry of the integrand with respect to the labeling of the \( \lambda_n \), we can choose to integrate over the last \( N_f \) variables, \( \lambda_{N-N_f+1}, \ldots, \lambda_N \), by relabeling the \( \lambda_n \) appropriately and multiplying by a combinatorial factor. (The functions \( F_N(\Lambda) \) and \( G_N(\Lambda) \) are obviously symmetric under interchanges \( \lambda_n \leftrightarrow \lambda_m \), for \( H_N(\Lambda) \) this follows from Eq. (A.14) below.)

We briefly pause to explain the essence of the proof of Eq. (7). After performing the \( N_f \) integrations in Eq. (A.7) and taking the limit \( \{a_f\} \rightarrow \infty \), one obtains a result which is essentially equal to the expression for \( Z_k(J) \) in Eq. (A.3), the only difference being that the integration is over \( N-N_f \) variables \( \lambda_n \) instead of over \( N \) variables. In the limit \( N \rightarrow \infty \), this difference can be neglected. (There
are some additional prefactors which will be canceled by identical factors in the normalization $\tilde{N}$.)

By performing the $N_f$ integrations in Eq. (A.7) using the $\delta$-functions of Eq. (A.9) the variables $\lambda_{N-N_f+1}, \ldots, \lambda_N$ are replaced by $a_1, \ldots, a_{N_f}$, respectively. We define $\Lambda' = \text{diag}(\lambda_1, \ldots, \lambda_{N-N_f})$ and $\bar{\Lambda} = \text{diag}(\lambda_1, \ldots, \lambda_{N-N_f}, a_1, \ldots, a_{N_f})$. The various contributions in the integrand of (A.7) become

$$d[\Lambda] \longrightarrow d[\Lambda'], \quad (A.10)$$

$$F_N(\Lambda) \longrightarrow F_{N-N_f}(\Lambda') \Delta_{N_f}^2 (a^2) \prod_{f=1}^{N_f} a_f e^{-N_Y^2 a_f^2} \prod_{n=1}^{N-N_f} (a_f^2 - \lambda_n^2)^2 \prod_{f'=1}^{N_f} (-m_f^2 - a_f^2)$$
$$\longrightarrow C_1(a, m) F_{N-N_f}(\Lambda') \quad \text{as } \{a_f\} \rightarrow \infty, \quad (A.11)$$

$$G_N(\Lambda) \longrightarrow \prod_{p=1}^{k} \text{Im} \prod_{n=1}^{N-N_f} \frac{x_p^2 - \lambda_n^2}{(x_p^2 - J_p)^2 - \lambda_n^2} \prod_{f=1}^{N_f} \frac{x_p^2 - a_f^2}{(x_p^2 - J_p)^2 - a_f^2}$$
$$\longrightarrow G_{N-N_f}(\Lambda') \quad \text{as } \{a_f\} \rightarrow \infty. \quad (A.12)$$

Here, $C_1(a, m)$ is a function which no longer depends on the $\lambda_n$. From the product over the imaginary parts in Eq. (A.7) we obtain with Eqs. (A.8) and (A.9) and after appropriate relabeling of the $\lambda_n$

$$\int_0^\infty d\lambda_{N-N_f+1} \cdots d\lambda_N \prod_{f=1}^{N_f} \text{Im} \prod_{n=1}^{N} \frac{1}{(a_f^+)^2 - \lambda_n^2}$$
$$\longrightarrow \frac{1}{\Delta_N^2(\bar{\Lambda})} \prod_{f=1}^{N_f} (-1)^{N_f-f} \Delta_{N-1}(\Lambda_{N-N_f+f})$$
$$\longrightarrow C_2(a) \quad \text{as } \{a_f\} \rightarrow \infty, \quad (A.13)$$

where $C_2(a)$ is a function which depends only on the $a_f$.

Consider now the angular integrals in Eq. (A.6). Using Eq. (2.3) of Ref. [36], we have

$$H_N(\Lambda) = c \frac{\det[ I_0(\lambda_n \tilde{y}_m)]_{n,m=1,...,N} }{\Delta_N(\Lambda^2) \Delta_N(Y^2) }, \quad (A.14)$$

where we have defined $\tilde{y}_m = 2 N_Y^2 y_m$. The constant $c$ depends neither on $\Lambda$ nor on $Y$. After the integration over $\lambda_{N-N_f+1}, \ldots, \lambda_N$, $\Lambda$ is replaced by $\bar{\Lambda}$, i.e., the last $N_f$ entries are $a_1, \ldots, a_{N_f}$. We now expand the determinant in the numerator of Eq. (A.14) with respect to the last row. Without loss of generality we can assume that $y_1 < y_2 < \ldots < y_N$. Using the asymptotic behavior of $I_0$, only the term proportional to $I_0(a_{N_f} \tilde{y}_{N_f})$ remains in the limit $a_{N_f} \rightarrow \infty$. The other terms are suppressed by factors of $\exp(-a_{N_f}(\tilde{y}_N - \tilde{y}_m))$, where $n = 1, \ldots, N - 1$. Using the same argument for the remaining $N_f - 1$
bottom rows, we obtain for \( \{a_f\} \rightarrow \infty \)

\[
\det[I_0(\tilde{\lambda}_n \tilde{y}_m)]_{n,m=1,\ldots,N} \rightarrow \det[I_0(\lambda_n \tilde{y}_m)]_{n,m=1,\ldots,N-N_f} \\
\times \det[I_0(a_f \tilde{y}_{N-N_f+g})]_{f,g=1,\ldots,N_f}.
\]  

(A.15)

Denoting \( Y' = \text{diag}(y_1, \ldots, y_{N-N_f}) \) and \( Y'' = \text{diag}(y_{N-N_f+1}, \ldots, y_N) \), we have

\[
\Delta_N(A^2) = \Delta_{N-N_f}(A^2) \Delta_{N_f}(a^2) \prod_{f=1}^{N_f} \prod_{n=1}^{N-N_f} (a_f^2 - \lambda_n^2),
\]  

(A.16)

\[
\Delta_N(Y^2) = \Delta_{N-N_f}(Y'^2) \Delta_{N_f}(Y''^2) \prod_{f=1}^{N_f} \prod_{n=1}^{N-N_f} (y_{N-N_f+f}^2 - y_n^2),
\]  

(A.17)

and thus obtain

\[
H_N(\Lambda) \rightarrow C_3(a,Y) H_{N-N_f}(\Lambda') \quad \text{as} \quad \{a_f\} \rightarrow \infty.
\]  

(A.18)

The function \( C_3(a,Y) \) no longer depends on the \( \lambda_n \). Note that the last \( N_f \) entries of the diagonal matrix \( \tilde{Y} \) have effectively disappeared from the problem. However, this effect is negligible in the limit \( N \rightarrow \infty \).

Collecting the various terms, we finally obtain for \( \{a_f\} \rightarrow \infty \)

\[
\tilde{Z}_k(J) = \frac{1}{\tilde{N}} e^{-\Sigma^2 \text{tr} Y^2} C_1(a, m) C_2(a) C_3(a, Y) \\
\times \int d[\Lambda'] F_{N-N_f}(\Lambda') G_{N-N_f}(\Lambda') H_{N-N_f}(\Lambda').
\]  

(A.19)

The normalization factor \( \tilde{N} \) is obtained by setting \( G_{N-N_f}(\Lambda') \) to unity so that

\[
\lim_{\{a_f\} \rightarrow \infty} \tilde{Z}_k(J) = \frac{\int d[\Lambda'] F_{N-N_f}(\Lambda') G_{N-N_f}(\Lambda') H_{N-N_f}(\Lambda')}{\int d[\Lambda'] F_{N-N_f}(\Lambda') H_{N-N_f}(\Lambda')}.
\]  

(A.20)

which is equal to \( Z_k(J) \) in Eq. (A.3) with \( N \) replaced by \( N-N_f \). In the limit \( N \rightarrow \infty \), this difference is negligible. This completes the proof.

**B Derivation of Eq. (23)**

We wish to compute the \( \{\alpha_f\} \rightarrow \infty \) limit in Eq. (19). It is convenient to divide both numerator and denominator by the Vandermonde determinant \( \Delta_{N_f}(\alpha) \), see Eq. (13), and to compute the \( \{\alpha_f\} \rightarrow \infty \) limit of the quantity

\[
\mathcal{R} = \frac{\det[C(\tilde{z}_p, \tilde{z}_q)]]_{p,q=1,\ldots,k+N_f}}{\Delta_{N_f}(\alpha)}
\]  

(B.1)
with a kernel $C$ given in Eq. (20), $\tilde{z}_p$ and $\tilde{\zeta}_q$ defined after Eq. (20), and $z_p$ and $\zeta_q$ given in Eqs. (15) and (16), respectively. The denominator in Eq. (19) then follows immediately by setting $k = 0$. The $\alpha$-dependence in the numerator determinant of $R$ is found in the last $N_f$ rows,

$$R = \frac{1}{\Delta_{N_f}(\alpha)} \begin{vmatrix} C(\alpha_1, u_1) & \cdots & C(\alpha_1, u_k) & C(\alpha_1, i\mu_1) & \cdots & C(\alpha_1, i\mu_{N_f}) \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ C(\alpha_{N_f}, u_1) & \cdots & C(\alpha_{N_f}, u_k) & C(\alpha_{N_f}, i\mu_1) & \cdots & C(\alpha_{N_f}, i\mu_{N_f}) \end{vmatrix},$$

or, indicating rows and their $\alpha_i$-dependence schematically by $r(\alpha)$,

$$R = \frac{1}{\Delta_{N_f}(\alpha)} \begin{vmatrix} \cdots \cdots \\ \cdots r(\alpha) \cdots \\ \vdots \\ \cdots r(\alpha_{N_f}) \cdots \end{vmatrix}. \quad (B.2)$$

Before taking the $\{\alpha_f\} \to \infty$ limit, we need to take the limits $\alpha_f \to \alpha_g$ for all $f < g$. Let us start with $\alpha_1$. We subtract the row $r(\alpha_1)$ from all following rows, $r(\alpha_2)$ to $r(\alpha_{N_f})$, without changing the value of the determinant. The Vandermonde determinant supplies factors of the kind $1/(\alpha_i - \alpha_1)$ for $2 \leq i \leq N_f$. Thus, for the $i$-th row $r(\alpha_i)$, we arrive at

$$r(\alpha_i) \to r(\alpha_i) - r(\alpha_1) \to \frac{r(\alpha_i) - r(\alpha_1)}{\alpha_i - \alpha_1} \xrightarrow{\alpha_1 \to \alpha_i} \partial_{\alpha_i} r(\alpha_i). \quad (B.3)$$

Next, the resulting second row $\partial_{\alpha_2} r(\alpha_2)$ is subtracted from all following rows, $\partial_{\alpha_3} r(\alpha_3)$ to $\partial_{\alpha_{N_f}} r(\alpha_{N_f})$. Including factors supplied by the Vandermonde determinant, the $j$-th row $(3 \leq j \leq N_f)$ thus becomes

$$\partial_{\alpha_j} r(\alpha_j) \to \partial_{\alpha_j} r(\alpha_j) - \partial_{\alpha_2} r(\alpha_2) \to \frac{\partial_{\alpha_j} r(\alpha_j) - \partial_{\alpha_2} r(\alpha_2)}{\alpha_j - \alpha_2} \xrightarrow{\alpha_2 \to \alpha_j} \partial_{\alpha_j}^2 r(\alpha_j). \quad (B.4)$$

Analogous steps for the subsequent rows lead to higher derivatives, at the same time eating up all factors contained in the Vandermonde determinant.
Finally, we obtain

\[
\lim_{\{f\} \to \infty} R = \lim_{\alpha \to \infty} \left| \begin{array}{ccc}
\ldots & \ldots & \ldots \\
\ldots & r(\alpha) & \ldots \\
\vdots & & \\
\ldots & \partial_\alpha^{N_f-1}r(\alpha) & \ldots \\
\end{array} \right|
\]

where \( \alpha \) is now a simple number.

We now consider a generic derivative appearing in (B.5). Recall that \( r(\alpha) \) stands for \( C(\alpha, x) \), where \( x \) can be one of the \( u_p \) or one of the \( i\mu_f \). The following manipulations will become more transparent by considering the first few special cases. We have

\[
\partial_\alpha^0 C(\alpha, x) = C(\alpha, x) \xrightarrow{\alpha \to \infty} \frac{1}{\alpha} J_1(\alpha)J_0(x) .
\]

The point is that this result is proportional to \( J_0(x) \) in the large-\( \alpha \) limit. The first derivative becomes

\[
\partial^1_\alpha C(\alpha, x) \xrightarrow{\alpha \to \infty} \left[ \frac{1}{\alpha} J_0(\alpha) - \frac{2}{\alpha^2} J_1(\alpha) \right] J_0(x) + \frac{1}{\alpha^2} J_0(\alpha)xJ_1(x)
\]

\[
\xrightarrow{\alpha \to \infty} \frac{1}{\alpha^2} J_0(\alpha)xJ_1(x) ,
\]

where the term proportional to \( J_0(x) \) has been eliminated by subtracting an appropriate multiple of the row (B.6). Thus, the row containing the first derivative is proportional to \( xJ_1(x) \) in the large-\( \alpha \) limit. For the second derivative, we proceed analogously and obtain, in the large-\( \alpha \) limit, terms proportional to \( J_0(x), xJ_1(x), \) and \( x^2J_0(x) \). The first two of these terms can be eliminated by subtracting appropriate multiples of the rows (B.6) and (B.7), respectively. Thus, the row containing the second derivative is proportional to \( x^2J_0(x) \) in the large-\( \alpha \) limit. We now proceed to a general \( m \)-fold derivative for which it is useful to expand the denominator of \( C \) in a geometric series,
leading to

\[
\partial^m_\alpha C(\alpha, x) = \sum_{l=0}^{m} \binom{m}{l} \partial^l_\alpha \left( \frac{1}{\alpha^2 - x^2} \right) \partial^{m-l}_\alpha \left[ \alpha J_1(\alpha) J_0(x) - x J_0(\alpha) J_1(x) \right]
\]

It will not be necessary to perform the derivatives of \( J_0(\alpha) \) and \( J_1(\alpha) \). The expression (B.8) contains terms proportional to \( J_0(x) \), \( x J_1(x) \), \( \ldots \) up to \( x^m J_0(x) \) (if \( m \) is even) or \( x^m J_1(x) \) (if \( m \) is odd). All higher order terms in \( x \) are suppressed by powers of \( \alpha \) in the large-\( \alpha \) limit, see the sum over \( n \). Subtracting appropriate multiples of previous rows, only the term proportional to \( x^m J_{m \mod 2}(x) \) remains. By using Bessel function recursion relations and adding appropriate multiples of previous rows, \( J_{m \mod 2}(x) \) can be replaced by \( J_m(x) \). We thus obtain

\[
\partial^m_\alpha C(\alpha, x) \xrightarrow{\alpha \to \infty} x^m J_m(x) \times f_m(\alpha)
\]

with unspecified functions \( f_m(\alpha) \). Defining

\[
\mathcal{F} = \lim_{\alpha \to \infty} \prod_{m=0}^{N_f-1} f_m(\alpha)
\]

and noting that \( (i\mu)^m J_m(i\mu) = (-\mu)^m I_m(\mu) \), we arrive at

\[
\lim_{\{\alpha_f\} \to \infty} \frac{\det[C(\bar{\zeta}_p, \bar{\zeta}_q)|_{p,q=1,\ldots,k+N_f}]}{\Delta_{N_f}(\alpha)} = \mathcal{F} \cdot \det \mathcal{A}
\]

with \( \mathcal{A} \) given in Eq. (25). Setting \( k = 0 \), we obtain

\[
\lim_{\{\alpha_f\} \to \infty} \frac{\det[C(\alpha_f, i\mu_g)|_{f,g=1,\ldots,N_f}]}{\Delta_{N_f}(\alpha)} = \mathcal{F} \cdot \det \mathcal{B}
\]

with \( \mathcal{B} \) given in Eq. (24). Taking the ratio of (B.11) and (B.12), we finally arrive at (23). Note that it is not necessary to evaluate (B.10) since \( \mathcal{F} \) drops out of the final result.

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