The Inhomogeneous Phase of Dense Skyrmion Matter

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Abstract

It was predicted qualitatively in ref. \cite{1} that skyrmion matter at low density is stable in an inhomogeneous phase where skyrmions condense into lumps while the remaining space is mostly empty. The aim of this paper is to proof quantitatively this prediction. In order to construct an inhomogeneous medium we distort the original FCC crystal to produce a phase of planar structures made of skyrmions. We implement mathematically these planar structures by means of the ’t Hooft instanton solution using the Atiyah-Manton ansatz. The results of our calculation of the average density and energy confirm the prediction suggesting that the phase diagram of the dense skyrmion matter is a lot more complex than a simple phase transition from the skyrmion FCC crystal lattice to the half-skyrmion CC one. Our results show that skyrmion matter shares common properties with standard nuclear matter developing a skin and leading to a binding energy equation which resembles the Weiszäcker mass formula.

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1 Introduction

The Skyrme model [2] describes baryons as topological solitons of an effective meson Lagrangian, which can be interpreted as a large $N_c$ limit of QCD [3]. Once the topological winding number is identified with the baryon number the model provides an excellent framework for hadron physics leading to a reasonable description of the properties of single baryons [4], the baryon-baryon interaction [5, 6], the baryon-meson interaction [7], the properties of light nuclei [9, 10], and even of their excited states [12].

The baryon number one particles associated with this lagrangians are called skyrmions. Due to its classical nature, which may be subject to quantization if required, the Skyrme model is also able to describe dense matter, the so called skyrmion matter, free from any obstacles such as the “sign problem” in lattice QCD [13, 14]. Moreover, the strong interaction generated by the skyrmions can be treated nonperturbatively. The great advantage of the model is that the same Skyrme Lagrangian can describe both infinite baryonic matter and its mesonic fluctuations in a unified way [15].

We have carried out several studies on dense matter with this scheme [17]. Our interest has been in understanding the phase transitions of skyrmion matter and the associated changes in the hadron properties. As the density of the matter increases, skyrmion matter undergoes in some models a phase transition from a FCC (face centered cubic) structure made of localized single skyrmions to a CC (cubic crystal) made of localized “half-skyrmions”, i.e. that is skyrmion like particles with baryon number $1/2^1$. In all cases, when matter is made of half-skyrmions, the value of the half-skyrmions at the lattice sites is such that the average value of the non linear field $\langle U \rangle = \exp(-i\vec{\tau} \cdot \vec{\pi}/f_\pi)$, vanishes. The vanishing of $\langle U \rangle$ was interpreted as a symptom of chiral symmetry restoration in a dense medium. However, in ref. [15], it was pointed out that $\langle U \rangle = 0$ does not mean the vanishing of the pion decay constant in medium and therefore does not imply the restoration of chiral symmetry. The mechanism that takes place is well known under the name of “pseudo-gap” scenario [18]. At higher density $f_\pi$ does vanish and chiral symmetry is restored. Thus the chiral restoration phase transition in the most naive Skyrme model happens in a two step process. This unconventional realization is resolved by incorporating a dilaton field into the model Lagrangian, which is associated with the scale anomaly of QCD [15]. Moreover, by incorporating more mesons such as the $\rho$ and the $\omega$ into the model Lagrangian via a hidden local gauge symmetry [19], the description improves resembling closely the standard scenario [20] leading to quantitative predictions on physical quantities such as the critical density of the phase transition [21].

However, a fundamental question on the phase transition remains. From the very beginning the conventional approach adopted to study skyrmion matter assumes as ansatz for the minimum energy configuration a homogeneous crystal structure with certain symmetry [13]. Although it is far from the Fermi liquid phase, which is generally accepted as a state for the normal nuclear matter, this solid crystaline structure is a reasonable starting point. However, the problem in this approach is that the phase transition occurs at a density where the pressure of the system is negative. Thus the system is unstable against collapse and some mechanism should be provided to keep the system in that homogeneous phase. As suggested in ref. [1] at this density, where the system has negative pressure, the homogeneous symmetric configuration has no advantage in comparison to some inhomogeneous phase where the skyrmions that are close to each other form lumps and leave large portions of space empty. This reasoning is supported by astrophysics where it is known that nuclear matter inside a neutron star appears in inhomogeneous phases known due to their structure by pasta names: gnocchi, spaghetti, lasagna, antispaghetti and

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1 In other models the transition goes from single skyrmion CC to half-skyrmion BCC (body centered cubic).
antignocchi [22]. In here, as a first step in the description of inhomogeneous skyrmion matter, we construct mathematically and study a few planar structures made of skyrmions.

Contrary to the case of homogeneous skyrmion matter with adequate symmetries, the construction of inhomogeneous skyrmion matter structures is not easy. In the case of a few skyrmions, one can find the minimum energy configuration numerically [9], or by means of useful mathematical tools such as the rational map ansatz [10]. In ref. [11], a rational map ansatz is modified and applied to study a two-dimensional skyrmion lattice with hexagonal symmetry. In the present work, as in ref. [1], we adopt the Atiyah-Manton ansatz [23] and construct a few layers of “skyrmion sheets” as a model for inhomogeneous skyrmion matter. We recall that in the Atiyah-Manton ansatz, a multi-skyrmion configuration can be generated from an instanton-like function in four dimensions. The instanton-like functions provide an intuitive way of constructing complex skyrmion configurations by matching the singular points of the instanton-like function to the single skyrmion positions. By modifying the instanton-like function we can incorporate also the relative orientations to each skyrmion. By following this procedure we can construct a single sheet of skyrmions where one skyrmion is lodged in a particular lattice site and its nearest skyrmion neighbors have relative orientation leading to the most attractive configuration. We study in here single, double and triple arrays of sheets, which provide information on how the skyrmions behave in the bulk of a matter lump and on its surface.

In the next section we describe our model lagrangian and the use of the Atiyah-Manton ansatz to construct skyrmion matter structures. In Section 3 we describe how two construct two dimensional matter structures, which we call sheets, its symmetries and calculate average densities for the minimum average energy per baryon for each of the structures. These sheets describe our model for an inhomogeneous phase. Finally in section 4 we draw some conclusions.

2 Model Lagrangian and Atiyah-Manton Ansatz

Our starting point is the simplest Skyrme Lagrangian [2], which reads

\[ \mathcal{L} = \frac{f_\pi^2}{4} \text{Tr} \left( \partial_\mu U^\dagger \partial^\mu U \right) + \frac{1}{32e^2} \text{Tr} \left[ U^\dagger \partial_\mu U, U^\dagger \partial_\nu U \right]^2, \]  

(1)

where \( U = \exp(i \vec{\pi} \cdot \vec{\tau})/f_\pi \) with the pion fields \( \vec{\pi} \) and \( SU(2) \) Pauli matrices \( \vec{\tau} \). The model Lagrangian contains two parameters: the pion decay constant \( f_\pi \) and the Skyrme parameter \( e \). By expressing the energy in units of \((6\pi^2)f_\pi/e\) and the length in units of \(1/ef_\pi\), the Lagrangian \( \mathcal{L} \) can be rewritten as

\[ \mathcal{L} = -\frac{1}{24\pi^2} \text{Tr} \left( L_\mu L^\mu \right) + \frac{1}{192\pi^2} \text{Tr} \left[ L_\mu, L_\nu \right]^2 \]  

(2)

where we have introduced the “left-current” \( L_\mu \) defined as

\[ L_\mu \equiv \partial_\mu U U^\dagger, \]  

(3)

\( L_\mu \) are traceless and their spatial components are of the form \( L_i = \partial_i U U^\dagger = i \vec{\tau} \cdot \vec{\ell}_i \), which serves as definition of \( \vec{\ell}_i \). The Faddeev-Bogomol’ny bound of the soliton solution carrying baryon number (winding number) \( B \) results simply as

\[ E \geq |B|, \]  

(4)

where \( E \) is the energy of the static soliton

\[ E = \int d^3x \left( \mathcal{E}_2 + \mathcal{E}_4 \right) = \int d^3x \left\{ \frac{1}{12\pi^2} \vec{\ell}_i \cdot \vec{\ell}_i + \frac{1}{24\pi^2} (\vec{\ell}_i \times \vec{\ell}_j)^2 \right\}, \]  

(5)
and the baryon number is given by

$$B = \int d^3x \rho_B = \varepsilon_{ijk} \int d^3x \left\{ \frac{1}{24\pi^2} \vec{\ell}_i \cdot (\vec{\ell}_j \times \vec{\ell}_k) \right\}. \quad (6)$$

For later convenience, we have introduced the densities $E_2, E_4$ and $\rho_B$. The spherical hedgehog solution, i.e. the lowest energy solution for the $B = 1$ skyrmion, solved exactly has an energy in our units of $E = 1.23$, which is much greater than the bound. The lowest value found so far is $E/B = 1.04$ in the half-skyrmion CC phase [17][1].

The Atiyah-Manton ansatz [23], provides a skyrmion configuration carrying baryon number $B$ from the time component of the gauge potential of an instanton carrying the same charge in four dimensional Euclidean space. Explicitly, it reads

$$U(x,t) = CS \left\{ \mathcal{P} \exp \left[ \int_{-\infty}^{\infty} -A_4(\vec{x},t)dt \right] \right\} C^\dagger, \quad (7)$$

where $A_4(\vec{x},t)$ is the time component of the gauge potential of an instanton of charge $N(= B)$. The integration is along the time direction keeping the time-ordering as denoted by the symbol $\mathcal{P}$. In eq. (7), $S$ is a constant $SU(2)$ matrix whose mission is to let $U$ approach the vacuum value 1 at infinity and $C$ is also an $SU(2)$ matrix that provides the global rotation of $U$ in the isospin space $^2$. By treating $C$ as collective variables and quantizing its degrees of freedom, one may introduce isospin to the system and evaluate the symmetry energy of matter [24].

An exact solution to $A_4(\vec{x},t)$ satisfying the equation of motion $\partial_\mu F^{\mu\nu} = 0$ together with the self-duality condition can be found in the form of

$$A_4(\vec{x},t) = \frac{i}{2} \vec{\tau} \cdot \vec{\nabla} (\ln \Phi), \quad (8)$$

with a scalar function $\Phi$. This ansatz reduces the equation of motion for $A_\mu$ to $\partial^2 \Phi = 0$ [25]. Various solutions to $\Phi$ carrying instanton charge $N$ are available and each yields different soliton configurations. The one that better suits our purposes is ’t Hooft’s instanton solution [26][27], where $\Phi$ is given by

$$\Phi = 1 + \sum_{n=1}^{B} \frac{\lambda_n^2}{(x - X_n)^2}. \quad (9)$$

When this $\Phi$ is substituted into eq. (7) with enough separations between the instantons, the spatial components of $X_n$, namely the singular points of the instanton solution, become the positions of the centers of the skyrmions. Thus, this solution provides an intuitive way of arranging the skyrmions in three dimensional space.

Two neighboring skyrmions are in the most attractive configuration when they have a specific relative orientation: one skyrmion is rotated in isospin space with respect to the other an angle $\pi$ with respect to an axis that is perpendicular to the line joining their centers. In order to incorporate such adequate orientations for each the skyrmions we use a modified $A_4(\vec{x},t)$ developed in the liquid instanton model [28]. There Eqs. (8) and (9) are modified to

$$A_4(\vec{x},t) = \frac{i}{2} \sum_{n=1}^{B} C_n \vec{\tau} \cdot \vec{\nabla} (\ln \phi(x, X_n, \lambda_n)) C_n^\dagger, \quad (10)$$

$^2$ In the description of skyrmion matter the value at infinity is taken at the points of the lattice chosen by the symmetry of the crystal so that the system has the proper baryon number. For example, in the half-skyrmion phase, the points with $U = +1$ become the center of the half-skyrmions, while the original skyrmion centers with $U = -1$ remain as the center of the other half-skyrmion.
Figure 1: (a) The most attractive arrangement of skyrmions in an FCC cell. The unrotated skyrmions are occupying each of the vertices of the cubic cell. The skyrmions denoted by \( x, y, z \) are located at the center of the sides parallel to the \( y-z, z-x, x-y \) planes respectively, and they are rotated by an angle \( \pi \) with respect to the axis of the label. (b) Euler rotation of the coordinates to let the \((1,1,1)\) plane be parallel to the new \((x'',y'')\) plane. (c) Arrangement of the skyrmions in a unit cell of the triangular lattice. 1, 2, 3 denote the skyrmions rotated by an angle \( \pi \) with respect to the axis \( x, y, z \) in the \textit{old} coordinate system respectively.

\[
\phi(x, X_n, \lambda_n) = 1 + \frac{\lambda_n^2}{(x - X_n)^2}, \tag{11}
\]

The change has been motivated because the rotation does not act on the scalar function but does act on \( \vec{\nabla}\phi \). Although eq. (10) is no more an exact solution of the equation of motion for the gauge fields, the homotopy still provides us a \( U \) with the correct baryon number. What plays the most important role in this solution are the \( N \) singularities in eq. (7) since they characterize the lattice structure mathematically. The \( SU(2) \) rotation for each instanton done by \( C_n \) correspond exactly to the rotations for each skyrmion in the skyrmion configuration.

In ref. [1], the Atiyah-Manton ansatz with the modified instanton configuration was applied to construct an homogeneous FCC crystal. As shown in Fig. 1, to have the most attractive relative orientations we have to locate skyrmions at each site of the FCC crystal in a specific way: the unrotated skyrmions (denoted by 0) are put at the vertices of a cubic cell and the ones rotated by an angle \( \pi \) with respect to an axis (denoted by \( x, y, z \)) perpendicular to the corresponding sides of the cube are put at the centers of the corresponding sides. The advantage of the ansatz \( A_4(\vec{x}, t) \) defined as in eqs. (10) and (11) is that this structure can be mathematically implemented by taking the space components of \( X_n \) at skyrmion sites and their orientations by \( C_n \). When the size of the cell is large enough, that is, at low densities, the resulting skyrmion configuration has just the FCC structure made of individual single skyrmions arranged in cells as in Fig. 1. As we increase the density beyond the critical value, matter has a lower energy per baryon in a different configuration. The crystal structure becomes CC as each skyrmion of the previous configuration is separated into two well localized lumps which are almost half-skyrmions (B=1/2). The fact that they are not exactly half-skyrmions has to do with the limitations of
using an approximate ansatz.

As already mentioned the problem with this phase transition is that it occurs at a density where matter has negative pressure. The system, instead of remaining in an unstable homogeneous state, prefers to form stable lumps of higher density with large portions of space empty, i.e. a distribution of matter resembling a thin soup of pasta. It is known that in the core of neutron stars, depending on the density of nuclear matter, several inhomogeneous phases of this type have been suggested [22].

Let us model some inhomogeneous phases in skyrmion matter to guide ulterior massive numerical computations. We may get some insight in the inhomogeneous phases by distorting the cell dimensions in Fig. 1. For example, by enlarging the cell dimension in the $x$ and $y$ directions while keeping that in the $z$ direction, we construct strings of skyrmions, while if we enlarge the cell dimension only in $z$ direction, we will obtain skyrmion sheets, where skyrmions are lodged in square lattice sites and one skyrmion has four nearest neighbors. The more energetically favorable planar structures are obtained by taking the $(1,1,1)$ plane shown explicitly in Fig. 1. There the skyrmions are in triangular lattice sites and each of them has six nearest neighbors with the most attractive relative orientations. The transition from the BCC to FCC associated with the $(1,1,1)$ plane has been studied in ref. [8] to describe the three dimensional structural phase transition. However, our main interest here is the the study of the two dimensional structure itself.

With our Atiyah-Manton based approach using eqs. (7) and (10), we can obtain such linear or planar structures rather simply by taking the space components of $X_n$ as the lattice sites for lines or planes and by choosing proper $C_n$ to characterize the rotations. The resulting dimensional hexagonal skyrmion lattice has been studied in ref. [11] using rational Donaldson maps by solving the full equations leading to the lowest energy configuration. We will take their results as a reference value for our study and will go beyond to describe double and triple layers of planes which we call skyrmion sheets.

### 3 Models for an inhomogeneous phases in Skyrme matter

We are going to construct different inhomogeneous phases based on parallel skyrmion sheets leading to, given our previous culinary metaphor, a Lasagna type phase. For convenience, we rotate the coordinate system so that the $(1,1,1)$ plane in Fig. 1(a) is taken as $(x,y)$ plane in the new coordinate system. As shown explicitly in Fig. 1(b) this can be done by a two step Euler rotation. At first, we rotate by an angle $\pi/4$ about the $z$ axis which yields new coordinates system $(x', y', z' = z)$. Next, we rotate by an angle $\cos^{-1}(1/\sqrt{3})$ about the $y'$ axis to get the coordinates system $(x''$, $y'' = y'$, $z')$. Explicitly, the corresponding rotation matrix is

$$R = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & +\frac{\sqrt{2}}{\sqrt{3}} \\ 0 & +1 & 0 \\ +\frac{1}{\sqrt{3}} & 0 & +\frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} +\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} & 0 \\ +\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} +\frac{1}{\sqrt{6}} & +\frac{1}{\sqrt{6}} & -\frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} & 0 \\ +\frac{1}{\sqrt{3}} & +\frac{1}{\sqrt{3}} & +\frac{1}{\sqrt{3}} \end{pmatrix}$$

(12)

Fig. 1(c) shows how the skyrmions are arranged in the new $z'' = L_F/\sqrt{3}$ plane. The $x$, $y$, $z$ axes in the old coordinate system are now transformed to new axis parallel to the unit vectors, whose components are

$$\hat{n}_1 = \begin{pmatrix} +\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{3}} \end{pmatrix}, \quad \hat{n}_2 = \begin{pmatrix} +\frac{1}{\sqrt{6}} \\ +\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{3}} \end{pmatrix}, \quad \hat{n}_3 = \begin{pmatrix} -\frac{\sqrt{2}}{\sqrt{3}} \\ 0 \\ +\frac{1}{\sqrt{3}} \end{pmatrix},$$

(13)
Figure 2: (a) Lattice points for the spatial components of $X_n$ and the relative orientations in the most attractive planar structure of skyrmions. A unit cell is denoted by a shaded diamond and the hexagonal prism (unbound in $z$ direction) is shown as a hexagon. The 2-fold symmetry about the $x$-axis is shown explicitly. We denote the distance between the centers of two nearest neighboring skyrmions as $L$. In terms of the FCC cell dimension $L_F$, $L = L_F/\sqrt{2}$. (b) We show $\rho_{max}$ appearing in eqs. (17) and (18).

3.1 Inhomogeneous phase of a single skyrmion sheet

As discussed above, the minimum energy configuration of the skyrmions in a sheet-like structure is the one of the (1,1,1) plane. Fig. 2(a) explicitly shows the location of skyrmions, from which we read the translational symmetries as

$$U(x, y + L, z) = (\vec{\tau} \cdot \hat{n}_3) U(x, y, z)(\vec{\tau} \cdot \hat{n}_3), U(x + \frac{\sqrt{3}}{2}L, y + \frac{1}{2}L, z) = (\vec{\tau} \cdot \hat{n}_1) U(x, y, z)(\vec{\tau} \cdot \hat{n}_1).$$  \hspace{1cm} (14)

It is straightforward to formulate this skyrmion arrangement in our approach: we take the spatial components of $X_n$ in eq. (10) as the lattice points and the rotations in the isospin space $C_n$ as shown in Fig.2(a).

Note that the summation in eq. (10) runs over all the triangular lattice sites $\tilde{X}_n$ that expand to infinity. However, the summation converges since for $X_n \gg L$

$$\partial_i \ln \phi \sim \frac{(x - X_n)_i}{(x - X_n)^2}; \hspace{1cm} (15)$$

and

$$\sum_n \partial_i \ln \phi \sim \int d^2X \frac{1}{|x - X|^2}. \hspace{1cm} (16)$$

Furthermore, the symmetry arising from the lattice structure provides one order higher suppression in the large $X_n$ contributions to the summation since, as can be seen in Fig. 2(a), the
skyrmions 0, 1, 2, 3 are located radially opposite with respect to the skrymion 0 at the origin. These symmetry make the leading order term in eq. (15) \(X_{n,i}/(x - X_n)^4\) vanish. There are two more parameter sets in our ansatz: \(T_n\), the time component of \(X_n\) and the instanton size parameters \(\lambda_n\). To satisfy the symmetries (14), \(T_n\) and \(\lambda_n\) should not depend on \(n\). Let’s denote these two numbers \(T\) and \(\lambda\). \(T\) can be absorbed in the integration process over \(t\) in Eq.(7) by changing to \(t' = t - T\) as the new variable.

In this single sheet configuration, the matter containing a single baryon is located inside a hexagonal prism (unbound in \(z\) direction) denoted by the dashed line. This arrangement yields the densities in the integrals of eqs. (5) and (6) with 3-fold symmetry with respect to \(2\pi/3\) rotations about the \(z\) axis and a 2-fold symmetry with respect to a rotation by \(\pi\) about the \(x\) axis. There is no reflection symmetry about the \(z = 0\) plane. Thus, we need to integrate only over the triangular prism of \(0 \leq z < \infty\). In cylindrical coordinates \((\rho, \varphi, z)\), the baryon number and the energy per baryon can be expressed as

\[
1 = 12 \int_0^\infty \int_\Delta dxdy \rho B = 12 \int_0^\infty \int_0^{\pi/2} d\varphi \int_0^{\rho_{\text{max}}} \rho d\rho \rho B, \tag{17}
\]

and

\[
E/B = 12 \int_0^\infty \int_\Delta dxdy (\mathcal{E}_2 + \mathcal{E}_4) = 12 \int_0^\infty \int_0^{\pi/2} d\varphi \int_0^{\rho_{\text{max}}} \rho d\rho (\mathcal{E}_2 + \mathcal{E}_4), \tag{18}
\]

with \(\rho_{\text{max}}\) given by

\[
\rho_{\text{max}} = \frac{L}{2} \cos\left(\frac{\varphi - \frac{\pi}{6}}{6}\right). \tag{19}
\]

Our problem becomes to minimize this eq. (18) by adjusting the distance \(L\) and the size parameter \(\lambda\). However, we can reduce our problem to a single parameter minimization. Let’s rescale all the lengths by \(L\)

\[
x_i = L\tilde{x}_i, \quad t = Lt \quad X_{n,\mu} = L\tilde{X}_{n,\mu}, \quad \lambda = L\tilde{\lambda}. \tag{20}
\]

The function \(\phi\) in (10) remains exactly of the same form under this scaling,

\[
\phi = 1 + \frac{\tilde{\lambda}_n^2}{(\tilde{x} - X_n)^2}. \tag{21}
\]

Thus the \(L\) dependence of the integrals in eq.(18) becomes explicit as

\[
E/B = L \left(\tilde{E}/B\right)_2 + \frac{1}{L} \left(\tilde{E}/B\right)_4, \tag{22}
\]

where \((\tilde{E}/B)_{2,4}\) are the integrals of the densities \(\mathcal{E}_{2,4}\) evaluated with the tilded variables. We can now do the minimization with respect to the two parameters as follows. We evaluate \((\tilde{E}/B)_{2,4}\) for a given \(\tilde{\lambda}\), thereafter we minimize easily eq.(22) by varying \(L\), which leads us to

\[
(E/B)_{\text{min}} = 2\sqrt{(\tilde{E}/B)_2}, \quad L = \sqrt{(\tilde{E}/B)_4/(\tilde{E}/B)_2} \tag{23}
\]

with the value of \(\lambda\) as \(L\tilde{\lambda}\). Thus, all the expressions are now written in terms of a unique parameter \(\tilde{\lambda}\) with respect to which we have to vary.

We show in Fig. 3 the results of such a minimization process. The \(\tilde{\lambda}\) values used in the calculation are given at each point. The minimum energy per baryon is \(E/B = 1.14\) at \(L = 2.95\).\footnote{This symmetry makes the summation converge even in three dimensional lattice.}
Although this \((E/B)_{\text{min}}\) is larger than 1.076 of the exact solution to hexagonal lattice [11], it is not too bad if we take into account that it is obtained by varying just a single parameter. We may obtain better values by modifying the instanton-like function \(\phi\) in eq. (11).

It is reasonable to compare our result with those of ref. [1] obtained by using the same Atiyah-Manton ansatz. There, the phase transition from the single skyrmion FCC to half-skyrmion CC occurs when the energy per baryon is \((E/B) \sim 1.16\). That is, our two dimensional planar structure has lower \((E/B)\) than any homogeneous FCC and even the half-skyrmion CC near the critical density. We infer that skyrmion matter prefers to be in an inhomogeneous phase, where skyrmions condense to form planar structures surrounded by empty space.

In Fig. 4, the minimum energy configuration is depicted by contour plots of baryon number densities on the plane (a) \(z = 0\), (b) \(z = 0.3\) and (c) \(z = 1.0\). In each plot, the color of contours denote only relative densities. The triangular lattice is explicitly shown by the white lines. At the central \(z = 0\) plane, matter is concentrated in narrow regions around the lattice points in the form of circular contours. As \(z\) increases, the density distribution expands to a wider region and the contour shape is distorted to curved triangles. Finally, in the tail region far from the central plane, the contours of higher density become hexagons, similar to the results of ref. [11].

Three vertices of the hexagon are the lattice points of the triangular lattice, while the other three are the centers of the triangles. Matter gets accumulated in these full regions by the overlapping of the tails of the skyrmions. Although this overlapping happens in the low density tail region, it shows an interesting feature of the skyrmion approach, namely that it can describe naturally shape distortion of objects due to the interaction of neighbors.

In order to see the density distributions along \(z\) direction, in Fig. 5(a), we present the densities integrated over the triangle region for a given \(z\). They can be interpreted as the averaged densities over that area. Contrary from what we had expected, they do not have maximum values at \(z = 0\). These results tell us that skyrmions in our minimum energy configuration are not single objects sitting at each triangular lattice points. As can be seen in the rough three dimensional plot in Fig. 5(b), a single skyrmion is distributed into what looks as two recognizable objects. These
Figure 4: Baryon number distributions of the minimum energy configuration on the plane (a) $z = 0$, (b) $z = 0.5$ and (c) $z = 1.0$. In each graph, the color of the contours denote relative densities only. The white lines show the triangular lattice.

Figure 5: (a) Averaged density distributions along the $z$ direction. (b) A rough three dimensional plot of baryon number distribution in a hexagonal prism. We draw the positions where the density is larger than 10% of the maximum values.
two distinguishable density distributions may be taken as primogenial half-skyrmions.

3.2 Inhomogeneous phase of a double layer of skyrmion sheets

Now, let us consider two parallel skyrmion sheets. It is reasonable to assume that the minimum energy configuration is made by two nearest (1,1,1) planes of an FCC crystal. It can be constructed by arranging skyrmions as shown in Fig. 6. In order to make nearest neighboring skyrmions at two layers equally separated, for example, 0 in the lower layer and 1, 2, 3 in the upper layer (shown in the figure by shaded circles), the distance between the two central planes is $d = L/\sqrt{6}$. If we place the $z = 0$ plane at the center between the two sheets, the matter carrying baryon number one is located inside the semi-infinite hexagonal prism defined by $z \geq 0$ or $z \leq 0$.

A similar numerical procedure as described in detail before for the single sheet phase leads to $(E/B)_{\text{min}} = 1.11$ at $L = 3$. The interaction with three more nearest neighbors in the other sheet reduces $E/B$ to 1.11 from 1.14 of the single skyrmion sheet. Compared to the reduction from 1.24 from the spherical hedgehog solution using the Atiyah-Manton ansatz, to 1.14 going to the single sheet with six nearest neighbors, this reduction is rather small. Fig. 7 provides information on the minimum energy configuration, where we show (a) the averaged densities over the triangular region for a given $z$ (not scaled) and (b) a three dimensional plot where the region with density greater than 10% the maximum value. The dashed line in Fig. 7(a) denotes one of the lattice planes, $z/d = \pm 0.5$. The averaged densities outside the lattice plane show great similarity to those for the single sheet in Fig. 5(a). It may look natural, because we just stack the instanton-like objects on two layers. However, if we take into account the fact that the instanton-like function $\phi$ in eq. (10) is long-range, it seems quite unnatural that the other layer does not affect the density distributions in this region. In Fig. 7(a), the baryon number in the hexagonal prism defined by $0 \leq z \leq d/2$ is exactly 0.5 and that in the semi-infinite hexagonal prism defined by $z \geq d/2$ is the remaining half baryon number. On the other hand, the energies in each hexagonal prisms are 0.53 and 0.58, respectively. This implies that the
“half”-skyrmions in between two lattice planes carry less energy than those outside the lattice planes. It is also interesting to see that two times 0.53, that is, 1.06 is the \( (E/B) \) of the half skyrmion CC obtained using the Atiyah-Manton ansatz in ref. \[1\]. Furthermore, twice of 0.58 is close to what we have obtained as \( (E/B)_{\text{min}} \) of single skyrmion sheet.

In Fig. 7(b), besides the lumps around the lattices points, one can see newly generated lumps at the shaded \( z = 0 \) plane. Although they are not fully symmetric, these generated lumps can be interpreted as half-skyrmions. That is, we have arranged the instanton-like objects on two sheets, but we end up with a triple layer of “half”-skyrmions. Of course, in the configuration obtained with larger \( L \) (and larger \( (E/B) \) consequently), the lumps appear only around the lattice points of the two sheets.

Our double layer results provide enough informations on how the skyrmions behave inside the dense matter and on its surface. That is, the lumps inside matter have all the possible nearest neighbors and have lower energy, while those on the surface have less neighbors and larger energy. This discussion resembles the liquid drop model arguments, i.e. common statements to all matter made of particles interacting with strong but short-range forces. In our case these interactions are just a consequence of the Lagrangian eq.(1) even with the massless pions. Furthermore, it is extremely interesting that we have obtained such a result by using the long-range instanton-like function \( \phi \).

\[ \text{In ref. [1], the Atiyah-Manton ansatz could not produce perfectly symmetric half-skyrmions even in the FCC configuration.} \]

\[ \text{We should say half-skyrmions instead of skyrmions, because there is no inside skyrmion in the double layer configuration. Only once they separate into half-skyrmions we can distinguish between inside the layer objects and those on the surface.} \]
3.3 Inhomogeneous phase with a triple layer of skyrmion sheets

The calculation follows a similar procedure as above. Again, we fix the distance between the sheets to \( d = L/\sqrt{6} \) and the lattice planes are \( z = \pm d \) and \( z = 0 \). The minimum energy configuration is obtained for \( \tilde{\lambda}^2 = 0.053 \) with \( E/B = 1.1 \), which is slightly less than that of the double layer. The reason for this little reduction in \( E/B \) is that the surface effect is still strong in triple layer configuration. That is, although a skyrmion in the middle layer has 12 nearest neighbors, the skyrmion in the outer layer has only 9.

Shown in Fig. 8 are the average densities and a three dimensional plot for the configuration. Again, average densities outside of the lattice plane are almost the same as those of double layer and single sheet. This reminds us of the shape of heavy nuclei which have a similar density distribution near the surface and have the same surface thickness. The area below the curve tells us how much baryon number and energy are carried by matter in the hexagonal prism in a given range. Exactly half baryon number is lodged in each of the three hexagonal prisms \( 0 \leq z \leq d/2 \) and \( d/2 \leq z \leq d \) and \( z \geq d \), while the energies carried by matter are 0.53, 0.53 and 0.58, in each location respectively. This is exactly what we had expected. As shown naively in Fig. 8(b) half-skyrmions have emerged. The half-skyrmions inside matter carry \( E/B = 1.06 \) and those in the surface carry \( E/B = 1.16 \). By generalizing this to any finite system, we can extract a mass formula which reads

\[
E = 0.53 \times N_{h_i}^i + 0.56 \times N_{h_s}^s
\]

where \( N_{h_i}^i \) is the number of half-skyrmions in the bulk of matter and \( N_{h_s}^s \) denote those on the surface.

4 Conclusions

It has been shown in the past that the Skyrme model has a phase transition from an FCC crystal at low densities to a CC half-skyrmion crystal at high densities. At that time we pointed out that the FCC phase was unstable due to negative pressure and predicted that inhomogeneous
phases in line of the Fermi drop model would be preferable. We have constructed such a phase in here which resembles very much some of the phases suggested for nuclear matter in the core of nuclear stars. In our case it is a phase made of matter sheets, i.e. a Lasagna type phase. We have been able to construct this phase mathematically by means of the Atiyah-Manton ansatz using the 't Hooft instanton function showing that its energy is lower than that of the FCC crystal and with a characteristic mass distribution whose support is a planar hexagonal lattice.

This model for the low energy phase satisfies properties similar to those of conventional nuclear matter: it develops a skin, a skin depth and a bulk, with similar profiles as conventional nuclear matter. It leads to a mass formula which resembles a simplified version of the Weisszäcker semi-empirical mass formula. Most important dynamically this result implies that our skyrmion system shares a common behavior to systems of particles interacting through strong short-range forces.

A careful analysis of our calculation shows that the long-range instanton profile function through its long range tails transforms into a short-range interaction among the skyrmions. This phenomenon is even more surprising since in our lagrangian the pion is massless. This observation seems to indicate that the short-range character of the strong nuclear force is not crucially dependent on the mass of the pion but arises from confinement which manifests itself though the chiral SU(2) structure of the effective lagrangian. The Skyrme model is describing in an effective way how the massless gluons of QCD through confinement can give rise to manifestations of a strong short-range interaction with the limitations associated to the large $N_c$ limit.

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