Abstract. We show that if $B$ is a block of a finite group algebra $kG$ over an algebraically closed field $k$ of prime characteristic $p$ such that $HH^1(B)$ is a simple Lie algebra and such that $B$ has a unique isomorphism class of simple modules, then $B$ is nilpotent with an elementary abelian defect group $P$ of order at least 3, and $HH^1(B)$ is in that case isomorphic to the Jacobson-Witt algebra $HH^1(kP)$. In particular, no other simple modular Lie algebras arise as $HH^1(B)$ of a block $B$ with a single isomorphism class of simple modules.

1. Introduction

Let $p$ be a prime and $k$ an algebraically closed field of characteristic $p$. The purpose of this note is to illustrate close connections between the Lie algebra structure of $HH^1(B)$ and the structure of $B$, where $B$ is a block of a finite group algebra $kG$. The main motivation for this is the fact that the Lie algebra structure of $HH^1(B)$ is invariant under stable equivalences of Morita type (cf. [3, Theorem 10.7]). We consider two extreme cases for a block $B$ with a single isomorphism class of simple modules. The first result describes the circumstances under which $HH^1(B)$ a simple Lie algebra.

**Theorem 1.1.** Let $G$ be a finite group and let $B$ be a block algebra of $kG$ having a unique isomorphism class of simple modules. Then $HH^1(B)$ is a simple Lie algebra if and only if $B$ is nilpotent with an elementary abelian defect group $P$ of order at least 3. In that case, we have a Lie algebra isomorphism $HH^1(B) \cong HH^1(kP)$.

Theorem 1.1 implies in particular that no simple modular Lie algebras other than the Jacobson-Witt algebras occur as $HH^1(B)$ of some block algebra of a finite group with the property that $B$ has a single isomorphism class of simple modules. See [6], [7] for details and further references on the classification of simple Lie algebras in positive characteristic. We do not know whether the hypothesis on $B$ to have a single isomorphism class of simple modules is necessary in Theorem 1.1. For the sake of completeness, the second result rules out the case of the trivial one-dimensional Lie algebra for blocks with one isomorphism class of simple modules.

**Theorem 1.2.** Let $G$ be a finite group and let $B$ be a block algebra of $kG$ having a nontrivial defect group and a unique isomorphism class of simple modules. Then $\dim_k(HH^1(B)) \geq 2$.

The hypothesis that $B$ has a single isomorphism class of simple modules is necessary in Theorem 1.2 for instance, if $P$ is cyclic of order $p \geq 3$ and if $E$ is the cyclic automorphism group of order $p - 1$ of $P$, then $HH^1(k(P \rtimes E))$ has dimension one.
2. Quoted results

We collect in this section results needed for the proof of Theorem 1.1.

**Theorem 2.1** (Okuyama and Tsushima [4]). Let $G$ be a finite group and $B$ a block algebra of $kG$. Then $B$ is a nilpotent block with an abelian defect group if and only if $J(B) = J(Z(B))B$.

Let $A$ be a finite-dimensional (associative and unital) $k$-algebra. A derivation on $A$ is a $k$-linear map $f : A \to A$ satisfying $f(ab) = f(a)b + af(b)$ for all $a, b \in A$. The set $\text{Der}(A)$ of derivations on $A$ is a Lie subalgebra of $\text{End}_k(A)$, with respect to the Lie bracket $[f, g] = f \circ g - g \circ f$, for any $f, g \in \text{End}_k(A)$. For $c \in A$, the map sending $a \in A$ to the additive commutator $[c, a] = ca - ac$ is a derivation on $A$; any derivation arising this way is called an inner derivation on $A$. The set $\text{IDer}(A)$ of inner derivations is a Lie ideal in $\text{Der}(A)$, and we have a canonical identification $HH^1(A) \cong \text{Der}(A)/\text{IDer}(A)$. See [8] Chapter 9 for more details on Hochschild cohomology. If $A$ is commutative, then $HH^1(A) \cong \text{Der}(A)$. A $k$-algebra $A$ is symmetric if $A$ is isomorphic to its $k$-dual $A^*$ as an $A$-$A$-bimodule; this definition implies that $A$ is finite-dimensional.

**Theorem 2.2** ([1] Theorem 3.1). Let $A$ be a symmetric $k$-algebra and let $E$ be a maximal semisimple subalgebra. Let $f : A \to A$ be an $E$-$E$-bimodule homomorphism satisfying $E + J(A)^2 \subseteq \ker(f)$ and $\text{Im}(f) \subseteq \text{soc}(A)$. Then $f$ is a derivation on $A$ in $\text{soc}_{Z(A)}(\text{Der}(A))$, and if $f \neq 0$, then $f$ is an outer derivation of $A$. In particular, we have

$$\sum_S \dim_k(\text{Ext}_A^1(S, S)) \leq \dim_k(\text{soc}_{Z(A)}(HH^1(A)))$$

where in the sum $S$ runs over a set of representatives of the isomorphism classes of simple $A$-modules.

**Corollary 2.3** ([1] Corollary 3.2). Let $A$ be a local symmetric $k$-algebra. Let $f : A \to A$ be a $k$-linear map satisfying $1 + J(A)^2 \subseteq \ker(f)$ and $\text{Im}(f) \subseteq \text{soc}(A)$. Then $f$ is a derivation on $A$ in $\text{soc}_{Z(A)}(\text{Der}(A))$, and if $f \neq 0$, then $f$ is an outer derivation of $A$. In particular, we have

$$\dim_k(J(A)/J(A)^2) \leq \dim_k(\text{soc}_{Z(A)}(HH^1(A)))$$

**Theorem 2.4** (Jacobson [2] Theorem 1). Let $P$ be a finite elementary abelian $p$-group of order at least 3. Then $HH^1(kP)$ is a simple Lie algebra.

The converse to this theorem holds as well.

**Proposition 2.5.** Let $P$ be a finite abelian $p$-group. If $HH^1(kP)$ is a simple Lie algebra, then $P$ is elementary abelian of order at least 3.

**Proof.** Suppose that $P$ is not elementary abelian; that is, its Frattini subgroup $Q = \Phi(P)$ is nontrivial. We will show that the set of derivations with image contained in $I(kQ)kP = \ker(kP \to kP/Q)$ is a nonzero Lie ideal in $\text{Der}(kP)$, where $I(kQ)$ is the augmentation ideal of $kQ$. Indeed, every element in $Q$ is equal to $x^p$ for some $x \in P$, and hence every element in $I(kQ)$ is a linear combination of elements of the form $(x - 1)^p$, where $x \in P$. Every derivation on $kP$ annihilates all elements of this form (using the fact that $k$ has characteristic $p$), and hence every derivation on $kP$ preserves $I(kQ)kP$. Thus there is a canonical Lie algebra homomorphism $\text{Der}(kP) \to \text{Der}(kP/Q)$, which is easily seen to be nonzero, with nonzero kernel, and hence $HH^1(kP)$ is not simple. The result follows. \(\square\)
Remark 2.6. Theorem [1,4] implies that in fact for any finite $p$-group $P$ the Lie algebra $HH^1(kP)$ is simple if and only if $P$ is elementary abelian of order at least 3. The special case with $P$ abelian, as stated in [25] will be needed in the proof of [1,4].

3. Auxiliary results

In order to exploit the hypothesis on $HH^1$ being simple in the statement of Theorem [1,1] we consider Lie algebra homomorphisms into the $HH^1$ of subalgebras and quotients.

Lemma 3.1. Let $A$ be a finite-dimensional $k$-algebra and $f$ a derivation on $A$. Then $f$ sends $Z(A)$ to $Z(A)$, and the map sending $f$ to the induced derivation on $Z(A)$ induces a Lie algebra homomorphism $HH^1(A) \to HH^1(Z(A))$.

Proof. Let $z \in Z(A)$. For any $a \in A$ we have $az = za$, hence $f(az) = f(a)z + af(z) = f(z)a + zf(a) = f(z)a$. Comparing the two expressions, using $zf(a) = f(a)z$, yields $af(z) = f(z)a$, and hence $f(z) \in Z(A)$. The result follows. □

Lemma 3.2. Let $A$ be a local symmetric $k$-algebra such that $J(Z(A))A \neq J(A)$. Then the canonical Lie algebra homomorphism $HH^1(A) \to HH^1(Z(A))$ is not injective.

Proof. Since $J(Z(A))A < J(A)$, it follows from Nakayama’s lemma that $J(Z(A))A + J(A)^2 < J(A)$. Thus there is a nonzero linear endomorphism $f$ of $A$ which vanishes on $J(Z(A))A + J(A)^2$ and on $k \cdot 1_A$, with image contained in $soc(A)$. In particular, $f$ vanishes on $Z(A) = k \cdot 1_A + J(Z(A))$. By [2,3] the map $f$ is an outer derivation on $A$. Thus the class of $f$ in $HH^1(A)$ is nonzero, and its image in $HH^1(Z(A))$ is zero, whence the result. □

Lemma 3.3. Let $A$ be a local symmetric $k$-algebra and let $f$ be a derivation on $A$ such that $Z(A) \subseteq \ker(f)$. Then $f(J(A)) \subseteq J(A)$.

Proof. Since $A$ is local and symmetric, we have $soc(A) \subseteq Z(A)$, and $J(A)$ is the annihilator of $soc(A)$. Let $x \in J(A)$ and $y \in soc(A)$. Then $xy = 0$, hence $0 = f(xy) = f(x)y + xf(y)$. Since $y \in soc(A) \subseteq Z(A)$, it follows that $f(y) = 0$, hence $f(x)y = 0$. This shows that $f(x)$ annihilates $soc(A)$, and hence that $f(x) \in J(A)$. □

Lemma 3.4. Let $A$ be a finite-dimensional $k$-algebra and $J$ an ideal in $A$.

(i) Let $f$ be a derivation on $A$ such that $f(J) \subseteq J$. Then $f(J^n) \subseteq J^n$ for any positive integer $n$.

(ii) Let $f, g$ be derivations on $A$ and let $m$, $n$ be positive integers such that $f(J) \subseteq J^m$ and $g(J) \subseteq J^n$. Then $[f, g](J) \subseteq J^{m+n-1}$.

Proof. In order to prove (i), we argue by induction over $n$. For $n = 1$ there is nothing to prove. If $n > 1$, then $f(J^n) \subseteq f(J)J^{n-1} + Jf(J^{n-1})$. Both terms are in $J^n$, the first by the assumptions, and the second by the induction hypothesis $f(J^{n-1}) \subseteq J^{n-1}$. Let $y \in J$. Then $[f, g](y) = f(g(y)) - g(f(y))$. We have $g(y) \in J^n$; that is, $g(y)$ is a sum of products of $n$ elements in $J$. Applying $f$ to any such product shows that the image is in $J^{m+n-1}$. A similar argument applied to $g(f(y))$ implies (ii). □

Proposition 3.5. Let $A$ be a finite-dimensional $k$-algebra. For any positive integer $m$ denote by $\text{Der}_m(A)$ the $k$-subspace of derivations $f$ on $A$ satisfying $f(J(A)) \subseteq J(A)^m$.

(i) For any two positive integers $m$ and $n$ we have $[\text{Der}_m(A), \text{Der}_n(A)] \subseteq \text{Der}_{m+n-1}(A)$.

(ii) The space $\text{Der}_1(A)$ is a Lie subalgebra of $\text{Der}(A)$. 

□
(iii) For any positive integer m, the space $\text{Der}_{(m)}(A)$ is an ideal in $\text{Der}_{(1)}(A)$.

(iv) The space $\text{Der}_{(2)}(A)$ is a nilpotent Lie subalgebra of $\text{Der}(A)$.

Proof. Statement (i) follows from (ii). The statements (ii) and (iii) are immediate consequences of (i). Statement (iii) follows from (i) and the fact that $J(A)$ is nilpotent. □

4. PROOF OF THEOREM 1.1

Let $G$ be a finite group and $B$ a block of $kG$. Suppose that $B$ has a single isomorphism class of simple modules. If $B$ is nilpotent and $P$ a defect group of $B$, then by [5], $B$ is Morita equivalent to $kP$, and hence there is a Lie algebra isomorphism $HH^1(B) \cong HH^1(kP)$. Thus if $B$ is nilpotent with an elementary abelian defect group $P$ of order at least 3, then $HH^1(B)$ is a simple Lie algebra by [2,4].

Suppose conversely that $HH^1(B)$ is a simple Lie algebra. If $J(B) = J(Z(B))B$, then $B$ is nilpotent with an abelian defect group $P$ by [2,1]. As before, we have $HH^1(B) \cong HH^1(kP)$, and hence [2,4] implies that $P$ is elementary abelian of order at least 3.

Suppose that $J(Z(B))B \neq J(B)$. Let $A$ be a basic algebra of $B$. Then $J(Z(A))A \neq J(A)$. Moreover, $A$ is local symmetric, since $B$ has a single isomorphism class of simple modules. Thus $\text{soc}(A)$ is the unique minimal ideal of $A$. We have $J(A)^2 \neq \{0\}$. Indeed, if $J(A)^2 = \{0\}$, then $\text{soc}(A)$ contains $J(A)$, and hence $J(A)$ has dimension 1, implying that $A$ has dimension 2. In that case $B$ is a block with defect group of order 2. But then $HH^1(A) \cong HH^1(kC_2)$ is not simple, a contradiction. Thus $J(A)^2 \neq \{0\}$, and hence $\text{soc}(A) \subseteq J(A)^2$. By [3,2], the canonical Lie algebra homomorphism $HH^1(A) \rightarrow HH^1(Z(A))$ is not injective. Since $HH^1(A)$ is a simple Lie algebra, it follows that this homomorphism is zero. In other words, every derivation on $A$ has $Z(A)$ in its kernel. It follows from [6,3] that every derivation on $A$ sends $J(A)$ to $J(A)$. Thus, by [3,4], every derivation on $A$ sends $J(A)^2$ to $J(A)^2$. This implies that the canonical surjection $A \rightarrow A/J(A)^2$ induces a Lie algebra homomorphism $HH^1(A) \rightarrow HH^1(A/J(A)^2)$. Note that the algebra $A/J(A)^2$ is commutative since $A$ is local. Since $J(A)^2$ contains $\text{soc}(A)$, it follows that the kernel of the canonical map $HH^1(A) \rightarrow HH^1(A/J(A)^2)$ contains the classes of all derivations with image in $\text{soc}(A)$.

Since there are outer derivations with this property, it follows from the simplicity of $HH^1(A)$ that the canonical map $HH^1(A) \rightarrow HH^1(A/J(A)^2)$ is zero. Using that $A/J(A)^2$ is commutative, this implies that every derivation on $A$ has image in $J(A)^2$. But then [6,4] implies that $\text{Der}(A) = \text{Der}_{(2)}(A)$ is a nilpotent Lie algebra. Thus $HH^1(A)$ is nilpotent, contradicting the simplicity of $HH^1(A)$. The proof of Theorem 1.1 is complete.

Proof of Theorem 1.2. Denote by $A$ a basic algebra of $B$. Since $B$ has a unique isomorphism class of simple modules and a nontrivial defect group, it follows that $A$ is a local symmetric algebra of dimension at least 2. By [2,3], we have $\dim_k(\text{HH}^1(A)) \geq \dim_k(J(A)/J(A)^2)$. Thus $\dim_k(\text{HH}^1(A)) \geq 1$. Moreover, if $\dim_k(\text{HH}^1(A)) = 1$, then $\dim_k(J(A)/J(A)^2) = 1$, and hence $A$ is a uniserial algebra. In that case $B$ is a block with a cyclic defect group $P$ and a unique isomorphism class of simple modules, and hence $B$ is a nilpotent block. Thus $A \cong kP$. We have $\dim_k(\text{HH}^1(kP)) = |P|$, a contradiction. The result follows. □

Remark 4.1. All finite-dimensional algebras in this paper are split thanks to the assumption that $k$ is algebraically closed. It is not hard to see that one could replace this by an assumption requiring $k$ to be a splitting field for the relevant algebras. The statements [5,1] and [3,4] do not require any hypothesis on $k$. 

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