The vacuum induced Berry phase beyond rotating-wave approximation

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With reference to the vacuum induced Berry phase (VIBP) obtained in the interaction of a spin-1/2 particle with quantized irradiation field under rotating-wave approximation (RWA), we present completely different treatment for the VIBP by a fully quantum mechanical treatment beyond the RWA, which gives a new definition of the VIBP and indicates the validity of RWA to be relevant not only to the energy conservation and comparison of characteristic parameters, but also to more subtle physics, such as the geometric property of the state evolution. Our result is of conceptual importance and also of significant relevance to quantum information processing.

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The Berry phase presents the geometric and topological property of quantum system under an adiabatic cyclic evolution\(^1\), which has been widely investigated over past decades from the viewpoints of fundamental physics and the application for quantum information processing (QIP). A recent work\(^2\) on Berry phase by a fully quantum mechanical treatment demonstrated the existence of the Berry phase relevant to vacuum field, which could be tested by a single spin-1/2 interacting with a quantized irradiation field in terms of Jaynes-Cummings (JC) model\(^3\). This vacuum induced Berry phase (VIBP), due to no counterpart in classical treatment, has brought about a series of studies for more interesting physical interpretation and application.

We argue in this Letter that the VIBP defined in\(^2\) strongly depends on the employed rotating-wave approximation (RWA). Despite success for tens of years for light-matter interaction, the JC model has been debated due to employment of the RWA\(^4\), although the RWA has been justified from the viewpoint of energy conservation and works very well if the light-matter coupling is weak enough compared to other characteristic parameters in the system. Our remarkable results include: (1) The VIBP defined in\(^2\) is actually relevant to both the vacuum state and the one-boson state of the quantized field. We thereby define a new VIBP which is only associated with vacuum quantized field. We show this newly defined VIBP to be constantly zero under the RWA but different from zero or any integer multiple of 2\(\pi\) beyond the RWA; (2) With reference to the VIBP defined in\(^2\), the counterparts beyond the RWA are actually the Berry phases for the lowest excited states, whose values are much different from those in\(^2\). This reminds us to pay more attention to identification of the generated Berry phase, particularly in some recent schemes for QIP using geometric phases based on JC model\(^3\).

The key point of our work is to solve the JC model in the absence of the RWA by a method called coherent-state diagonalization\(^5\), which enables us to obtain eigensolutions in a very good approximation and to compare with the results under the RWA one by one. In contrast to the conventional viewpoint that validity of the RWA depends on the energy conservation and the strength comparison of the parameters in the system, we demonstrate that breakdown of the RWA is related to more subtle physics, such as the geometric property of the state evolution.

We first review briefly the idea in Ref.\(^2\) by considering a two-level system interacting with a single-mode quantum field under the RWA, written in units of \(\hbar\) as,

\[ H_R = \frac{\omega_0}{2}\sigma_z + \omega a^\dagger a + g(a^\dagger \sigma_- + a\sigma_+), \]

where \(\omega_0\) is the resonant frequency of the two levels (i.e., the upper level \(|e\rangle\) and the lower level \(|g\rangle\)), and \(\omega\) and \(a\) \((a^\dagger)\) are, respectively, the frequency and the annihilation (creation) operator of the single mode. \(g\) means the coupling and \(\sigma_{z,+,-}\) are usual Pauli operators. Under the rotation with respect to \(\omega(a^\dagger a + \sigma_z/2)\) and then under a unitary transformation \(U(\varphi) = e^{-i\varphi a^\dagger a}\), Eq. (1) is rewritten as,

\[ H'_R = \frac{\Delta}{2}\sigma_z + g(a^\dagger e^{-i\varphi}\sigma_- + ae^{i\varphi}\sigma_+), \]

with \(\Delta = \omega_0 - \omega\). Considering the eigenstates \(|\Psi_n^\pm(\varphi)\rangle\) of Eq. (2) to traverse a loop \(C\) on the associated Bloch sphere with \(\varphi\) varied adiabatically from 0 to \(2\pi\), we obtain the Berry phase \(\gamma_{n,\pm} = i \int_C d\varphi \langle \Psi_{n,\pm}(\varphi) | d/d\varphi | \Psi_{n,\pm}(\varphi) \rangle\) to be

\[ \gamma_{n,\pm} = \pi(1 \mp \cos \theta_n) + 2\pi n, \]

with \(\cos \theta_n = \Delta/\sqrt{\Delta^2 + 4g^2(n+1)}\). This implies the VIBP defined in\(^2\) to be different from zero or any integer multiple of \(2\pi\) in the case of \(n = 0\). In above
treatment, we have used the eigensolutions \( |\Psi_{n,\pm}(\varphi)\rangle = e^{i\varphi a^d |\Psi_{n,\pm}\rangle} \) with
\[
|\Psi_{n,\pm}\rangle = c_{n,\pm}|n\rangle|e\rangle + d_{n,\pm}|n+1\rangle|g\rangle,
\]
(3)
where \( c_{n,\pm} = \cos \theta_n/2, d_{n,\pm} = \sin \theta_n/2, c_{n,-} = \sin \theta_n/2 \)
and \( d_{n,-} = -\cos \theta_n/2 \). As shown in Appendix, the Hamiltonian in Eq. (1) commutes with the parity operator
\( P = -\sigma_z e^{i\pi a^d a} \), and thereby \( \gamma_{0,\pm} \), the VIBPs defined
in [2], both correspond to the wavefunctions of odd parity. This will be more clarified in our later comparison with the treatment beyond the RWA. Another point we have to emphasize is the eigenfunction \( \langle \Psi_{GS}\rangle = |0\rangle|g\rangle \)
omitted in Eq. (3), which was conventionally considered as a trivial solution due to decoupling from the interaction. In fact, \( |\Psi_{GS}\rangle \) is energetically lower than \( |\Psi_{0,\pm}\rangle \) and should thereby be the real ground state of the system. More importantly, it is only associated with the vacuum quantized field. Instead, \( |\Psi_{0,\pm}\rangle \) employed in [2] to define the VIBP is actually related to both \( |0\rangle \) and \( |1\rangle \) of the field. In this sense, we prefer to define the VIBP in the present Letter based on \( |\Psi_{GS}\rangle \), and call the VIBP defined in [2] as original VIBP. It is easy to obtain our defined VIBP \( \gamma_{GS} = 0 \) which implies no VIBP under RWA. This consideration would be more interesting when we go beyond the RWA below.

Elimination of the RWA from Eq. (1) yields
\[
H_{NR} = \frac{\omega_0}{2} \sigma_x + \omega a^d a + g(a^d + a)\sigma_z,
\]
(4)
where the appearance of the counter-rotating terms \( \sigma_+ a^d \)
and \( \sigma_- a \) makes it harder to solve Eq. (4) exactly. Conventional approaches to Eq. (4) have been developed under different approximations yielding solutions within some particular ranges of parameters. What we adopt in this Letter is the solution by diagonalizing Eq. (4) using displaced coherent states [6], which could obtain eigensolutions of Eq. (4) in a very good approximation. For convenience to employ the results from [6], we first perform a unitary transformation on Eq. (4) by \( V = e^{-i\pi \sigma_x}/4 \), which yields
\[
H_{NRX} = -\frac{\omega_0}{2} \sigma_x + \omega a^d a + g(a^d + a)\sigma_z.
\]
(5)
Eq. (5) has the same eigenenergies as Eq. (4) and owns the eigenfunctions written as [6],
\[
|\Phi^\pm\rangle = \frac{1}{\sqrt{2}} \sum_n f_n |n\rangle_{\alpha}|e\rangle \pm (-1)^n |n\rangle_{-\alpha}|g\rangle,
\]
(6)
with \( \alpha = g/\omega \) and the coefficient \( f_n \) to be determined. \( |n\rangle_{\alpha} \) is a displaced coherent state [6] defined as \( |n\rangle_{\alpha} = (1/\sqrt{n})|a^d + \alpha|^n \exp\{-\alpha^d - \alpha^2/2\}|0\rangle \). It is easily checked that \( |\Phi^+\rangle \) (\( |\Phi^-\rangle \)) is the eigenfunction with even (odd) parity under the parity operator \( \Pi = \sigma_z e^{i\pi a^d a} \) with \( \Pi|\Phi^\pm\rangle = \pm|\Phi^\pm\rangle \). Considering the unitary transformation \( V \) for Eq. (5), we have \( \Pi = V^d PV \), implying that the parity operator \( P \) commutes with both the Hamiltonian \( H_R \) under the RWA and the Hamiltonian \( H_{NR} \) beyond the RWA.

Putting Eq. (6) into the Schrödinger equation of Eq. (5) yields,
\[
\omega(m - \alpha^2)f_m + \frac{\omega_0}{2} \sum_n f_n D_{mn} = Ef_m,
\]
(7)
where \( D_{mn} \) is given by [6]
\[
D_{mn} = e^{-2\alpha^2} \sum_{k=0}^{\min(m,n)} (-1)^k \sqrt{m!n!} [2(2\alpha)^{m+n-2k}(m-k)!(n-k)!] k!.
\]
By solving Eq. (7), we may immediately obtain the Berry phases as
\[
\gamma'_{\pm} = \pm \int_C d\varphi \langle \Phi^\pm |V^d U(\varphi) \frac{d}{d\varphi} U(\varphi)V|\Phi^\pm\rangle
\]
\[
= 2\pi \{\alpha^2 + \sum_{n=0}^M [n(f^+_n)^2 - 2\alpha\sqrt{n}f^+_nf^-_{n-1}]\},
\]
(8)
where we have used the prime here to distinguish from the result under the RWA, and the superscript ' + ' (‘-’) is for the even (odd) parity. \( M \) is the integer relevant to the size of the truncated space spanned by our employed eigenfunctions [6] and \( f^\pm_k \) \( (k = n, n - 1) \) is to be determined later.

Despite the possibility to solve Eq. (7) numerically, we would like to solve Eq. (7) analytically in order to show the physics more clearly. We consider the expansion with \( m, n = k, k + 1 \), which is actually the first-order approximation in [6]. This leads to
\[
(E^\pm)^2 - (\eta^\pm_k + \eta^\pm_{k+1}) E^\pm + \eta^\pm_k \eta^\pm_{k+1} + \omega_0^2 D^2_{k,k+1}/4 = 0,
\]
(9)
with \( \eta^\pm_k = \omega(k - \alpha) \mp \omega_0 D_{k,k}/2 \). For \( k = 0 \), we found that the trivial solution under the RWA is not trivial any more due to involvement of the counter-rotating terms and it evolves as the ground state of the system meeting even parity with the eigenenergy as
\[
E_{GS} = (1/2)[\eta^+_0 + \eta^-_{1} - \sqrt{(\eta^+_0 - \eta^-_{1})^2 + \omega_0^2 D^2_{0,1}}].
\]
Taking the corresponding eigenfunction, we obtain the VIBP
\[
\gamma_{GS}' = 2\pi \{\alpha^2 + \sum_{f^+_0}^M [2\alpha f^+_0 f^-_1] \}
\]
\[
= 2\pi \{\alpha^2 + \sum_{f^+_0}^M [2\alpha f^+_0 f^-_1] \}
\]
(10)
the counterparts of the original VIBPs beyond the RWA are of odd parity, i.e., \( \gamma_{0,-} = \frac{2}{\gamma} (\alpha^2 + \frac{1}{1+\gamma} \mu_{0,-}) \) and \( \gamma_{0,+} = \frac{2}{\gamma} (\alpha^2 + \frac{1}{1+\gamma} \nu_{0,-}) \), with \( \mu_{0,-} \) and \( \nu_{0,-} \) defined in Appendix.

![Comparison of the eigenenergies under the RWA (a) with beyond the RWA (b) depending on the dimensionless coupling \( g \), where we set \( \Delta = 0 \).](image)

The comparison in Fig. 1 for the eigenenergies solved under the RWA with those beyond the RWA demonstrates the increasing difference with the coupling \( g \). Particularly, it shows that \( |\Psi_{GS}(t)\rangle \) under the RWA is really the ground state of the system with counterpart in the case beyond the RWA. As a result, it is reasonable for us to define the VIBP based on \( |\Psi_{GS}(t)\rangle \).

More importantly, by removing the RWA, we have very different VIBPs from those under the RWA. As plotted in Fig. 2, our defined VIBP varies from zero under the RWA to non-zero values beyond the RWA, and the original VIBPs also change significantly. Although our new definition of the VIBP might be only of conceptual interest, our treatment of the original VIBPs is of practical application. From Fig. 2 we know that the values of Berry phases strongly depend on the RWA. As long as \( g \) is not strictly zero, the Berry phases obtained under the RWA are different from those beyond the RWA, particularly for the case of large detuning \( \Delta \). Consequently, when considering the quantum logic gating based on the Berry phase in JC-type model \( \mathcal{H}_J \), employment of the RWA, yielding inaccurate Berry phase, would introduce errors in the logic gating. Since the idea in \( \mathcal{H}_J \) there have been many proposals and experiments using Berry phases for universal QIP \( \mathcal{H}_J \) because of the robustness of the geometric phases under some kinds of noise. Among those works JC model has been extensively employed, in which the qubits are encoded in the spin-1/2 states and the quantized field is used as data bus. For an efficient operation \( \mathcal{H}_J \), the light-matter coupling (i.e., \( g \) used above) is required to be reasonably large, and in some cases detuning (e.g., \( \Delta \) used above) is necessary to be large enough for removing effectively the quantized field from the qubit coupling. In terms of our present results, employment of the RWA should be considered with great caution because the generated geometric phases would be deviated from the expectation due to contributions of counter-rotating terms. This is the intrinsic error in the models involving the RWA, which might be incorrectly resorted to imperfections due to diabaticity or operations in experiments.

We have noticed the experimental reports for quantum gating operations using geometric phases in trapped ions \( \mathcal{H}_J \), in which the RWA was employed in both the JC-type models. Taking the latter experiment \( \mathcal{H}_J \) as an example. We find the Rabi frequency \( \Omega = 2\pi \times 0.21 \) MHz, the Lamb-Dicke parameter \( \eta = 0.056 \) and the trap frequency being \( 6\Omega \). This implies that the measured geometric phase, although under the RWA, is nearly accurate because the corresponding dimensionless coupling is smaller than 0.01. The key point in this case is the tiny Lamb-Dicke parameter which significantly reduces the effective Rabi frequency, although it is disadvantageous for large-scale QIP. This also means that the future large-scale trapped-ion QIP using geometric phases, if considering faster operations, should seriously justify the RWA.

![Comparison of the Berry phases under the RWA (blue curves) with beyond the RWA (red bold curves) vs the dimensionless coupling \( g/\omega \) and detuning \( \Delta' = \Delta/\omega \), where the solid, dashed and dashed-dotted curves are for \( \gamma_{GS}(\gamma_{0,s}), \gamma_{0,-}(\gamma_{0,-}), \) and \( \gamma_{0,+}(\gamma_{0,+}) \), respectively.](image)

Our result about the VIBPs seems opposite to that in a recent paper \( \mathcal{H}_J \) which pointed out that the original VIBPs beyond the RWA are strictly zero. The main difference of our work from \( \mathcal{H}_J \) is the fully quantum mechanical treatment presenting reasonable comparison between the solutions with and without the RWA, which enables us to understand the contribution from the counter-rotating terms. We have demonstrated in Figs. 1 and 2 the gradually increasing differences with the coupling \( g \) due to involvement of the counter-rotating terms. The
The contribution from the counter-rotating terms is also reflected in our defined VIBP, which, beyond the RWA, is associated with \(|n\rangle_{\alpha}\) and \(|n\rangle_{-\alpha}\) instead of the vacuum state of the quantized field any more. This is due to additional excitations of the system by the counter-rotating terms, which enhances the Berry phases and also makes the situation more complicated. In contrast, in the semi-classical treatment in [13], the addition of the counter-rotating terms cancels the original Berry phases, for which it is not clear whether the changes of the VIBPs are sudden or continuous with deletion of the RWA. In addition, it is also not clear in the semi-classical treatment to which quantized field states the considered Berry phases correspond.

In summary, we have defined a new VIBP based on the eigensolutions of JC model and investigated analytically the VIBP and the original VIBPs by removing the RWA from the JC model. From the analytical expressions, we might be able to understand how and what physical parameters bring about the changes, and it also helps us to realize that justification of employing the RWA should be seriously taken by considering both energies of the system and the geometric property of the state evolution. Particularly, more attention should be paid to identification of the geometric phase if we employ a JC model for designing quantum gating schemes. Since the JC model has been widely employed in different fields and is essential to QIP designs, our results should be of general interests in both concept and application. Moreover, our study on deletion of the RWA is not restricted to the condition of adiabaticity. So we believe that our results could be applied to both adiabatic geometric phase - Berry phase, and non-adiabatic geometric phase, such as the Ahaonov- Anandon phase [14]. In one word, our results open up a new arena to the study of breakdown of the RWA in the geometric evolution of quantum systems.

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Appendix

We compare the eigensolutions of the JC model in the presence and absence of the RWA.

Under the RWA For Eq. (1), the eigenfunctions conventionally considered are \(|\Psi_{n,\pm}\rangle\) in Eq. (3) with corresponding eigenenergies \(E_{n,\pm} = (n + 1/2)\omega \pm \sqrt{\Delta^2 + 4\Omega^2(n + 1/2)/2}\). In fact, there is another eigenstate decoupled from the interaction, i.e., \(|\Psi_{GS}\rangle = |0\rangle\) with corresponding eigenenergy \(E_G = -\omega_0/2\).

Introducing the parity operator \(P = -\sigma_z e^{i\pi a^\dagger a}\), we have \([P, \hat{H}_R] = 0\) with \(P|\Psi_{n,\pm}\rangle = (-1)^{n+1}|\Psi_{n,\pm}\rangle\) and \(P|\Psi_{GS}\rangle = |\Psi_{GS}\rangle\). So the eigenfunctions are not only related to the upper or lower level of the spin-1/2, but also associated with the even or odd parity.

Beyond the RWA The first-order expansion of Eq. (7) leads to Eq. (9), from which we may obtain analytical eigensolutions of Eq. (5). Except the ground state written in the text, the excited states, like in the RWA situation, are expressed to be related both to the upper or lower level, and to the even or odd parity. For the excited states with even parity, the eigenenergies are

\[
E_{k,\pm}^+ = (1/2)[\eta_{k,\pm}^+ + \eta_{k+1,k+1,\pm}^+] \\
\sqrt{(\eta_{k,\pm}^+-\eta_{k+1,k+1,\pm}^+)^2+\omega_0^2D_{k,k+1}^2},
\]

(11)

corresponding to eigenfunctions with coefficients \(f_{k,\pm}^+ = \mu_{k,\pm}/\sqrt{1+\mu_{k,\pm}^2}, \quad f_{k+1,\pm}^+ = 1/\sqrt{1+\mu_{k,\pm}^2}\), and \(\mu_{k,\pm} = \omega_0D_{k+1,k}/[2(\eta_{k,\pm}^+-E_{k,\pm}^+)]\). The corresponding Berry phases are \(\gamma_{k,\pm}^+ = 2\pi(\alpha^2+k+1-2\sqrt{\Delta^2+4\Omega^2(k+1/2)})\).

In contrast, for the excited eigenstates with odd parity, the eigenenergies are

\[
E_{k,\pm}^- = (1/2)[\eta_{k,\pm}^- + \eta_{k+1,k+1,\pm}^-] \\
\sqrt{(\eta_{k,\pm}^-\eta_{k+1,k+1,\pm}^-)^2+\omega_0^2D_{k,k+1}^2},
\]

(12)

corresponding to eigenfunctions with coefficients \(f_{k,\pm}^- = \nu_{k,\pm}/\sqrt{1+\nu_{k,\pm}^2}, \quad f_{k+1,\pm}^- = 1/\sqrt{1+\nu_{k,\pm}^2}\), and \(\nu_{k,\pm} = -\omega_0D_{k+1,k}/[2(\eta_{k,\pm}^-\eta_{k+1,k+1,\pm}^-)]\). The corresponding Berry phases are \(\gamma_{k,\pm}^- = 2\pi(\alpha^2+k+1-2\sqrt{\Delta^2+4\Omega^2(k+1/2)})\).

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