LEMNISCATE CONVEXITY AND OTHER PROPERTIES
OF GENERALIZED BESSEL FUNCTIONS

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Abstract. Sufficient conditions on associated parameters \( p, b, \) and \( c \) are obtained so that the general-ized and “normalized” Bessel function \( u_{p}(z) = u_{p,b,c}(z) \) satisfies \(|1 + (zu_{p}^{\prime}(z)/u_{p}(z))| < 1 \) or \(|(zu_{p}(z)/u_{p}(z))^{2} - 1| < 1 \). We also determine the condition on these parameters so that \(-4(p + b + 1)/2/c)u_{p}(z) < \sqrt{1 + z} \). Relations between the parameters \( \mu \) and \( p \) are obtained such that the normalized Lommel function of first kind \( h_{\mu,p}(z) \) satisfies the subordination \( 1 + (zh_{\mu,p}^{\prime}(z)/h_{\mu,p}(z)) < \sqrt{1 + z} \). Moreover, the properties of Alexander transform of the function \( h_{\mu,p}(z) \) are discussed.

1. Introduction

We consider analytic functions \( f \) defined on the unit disk \( \mathbb{D} = \{ z : |z| < 1 \} \), and normalized by \( f(0) = 0 = f'(0) - 1 \). The class of all these functions is denoted by \( \mathcal{A} \) and its subclass consisting of univalent \((\equiv \text{one-to-one})\) functions is denoted by \( \mathcal{S} \). For two analytic functions \( f \) and \( g \) on \( \mathbb{D} \), \( f \) is said to be subordinate to \( g \), written as \( f(z) \prec g(z) \) (or \( f \prec g \)), if there is an analytic function \( w : \mathbb{D} \to \mathbb{D} \) with \( w(0) = 0 \) satisfying \( f = g \circ w \). If \( g \) is a univalent function, then \( f(z) \prec g(z) \) if and only if \( f(0) = g(0) \) and \( f(\mathbb{D}) \subset g(\mathbb{D}) \). It is worth to mention that this concept of subordination is a natural generalization of inequalities to complex plane. The class of convex functions (respectively starlike functions) consists of all those functions \( f \in \mathcal{A} \) for which \( f(\mathbb{D}) \) is convex (respectively starlike with respect to origin) and is denoted by \( \mathcal{K} \) (respectively \( \mathcal{S}^{*} \)). An analytic description of class \( \mathcal{K} \) and \( \mathcal{S}^{*} \) is as follows:

\[
\mathcal{K} := \left\{ f \in \mathcal{A} : \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in \mathbb{D} \right\} \quad \text{and} \quad \mathcal{S}^{*} := \left\{ f \in \mathcal{A} : \text{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbb{D} \right\}.
\]

A function \( f \in \mathcal{A} \) is lemniscate convex if \( 1 + (zf''(z)/f'(z)) \) lies in the region bounded by right half of lemniscate of Bernoulli given by \( \{ w : |w^{2} - 1| = 1 \} \). In terms of subordination, the function \( f \) is called lemniscate convex if \( 1 + (zf''(z)/f'(z)) \prec \sqrt{1 + z} \) and similarly, the function \( f \) is lemniscate starlike if \( zf'(z)/f(z) \prec \sqrt{1 + z} \). On the other hand, the function \( f \in \mathcal{A} \) is lemniscate Carathéodory if \( f'(z) \prec \sqrt{1 + z} \). Since \( \text{Re} \sqrt{1 + z} > 0 \), a lemniscate Carathéodory function is a Carathéodory function and hence is univalent. Note that a lemniscate convex function \( f \) satisfies

\[
\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\pi}{4}
\]

and hence it is strongly convex of order \( 1/2 \).

The function \( w(z) := w_{p,b,c}(z) \) is a particular solution of the second order linear differential equation

\[
z^{2}w''(z) + bw'(z) + (cz^{2} - p^{2} + p(1 - b))w(z) = 0, \quad p, b, c \in \mathbb{C}
\]

and is given by

\[
w_{p,b,c}(z) = \sum_{n \geq 0} \frac{(-c)^{n}}{n! \Gamma(n + (b + 1)/2)} \left( \frac{z}{2} \right)^{2n + p}, \quad z \in \mathbb{C}.
\]
The function $w_{p,b,c}(z)$ is called the generalized Bessel function of first kind of order $p$. For some particular values of $b$ and $c$, the equation (1.1) reduces to Bessel ($b = 1, c = 1$), modified Bessel ($b = 1, c = -1$), spherical Bessel ($b = 2, c = 1$), modified spherical Bessel ($b = 2, c = -1$) differential equations.

To study the geometric properties such as univalence, starlikeness and convexity of Bessel function, modified Bessel function, spherical Bessel function and modified spherical Bessel function of first kind of order $p$, we consider the normalization of $u_{p,b,c}(z)$ which is defined by the transformation $u_{p,b,c}(z) = 2^p \Gamma(p + (b + 1)/2)z^{-p/2}w_{p,b,c}(\sqrt{z})$. Let $(\lambda)_n$ denote the Pochhammer (or Appell) symbol defined in terms of the Euler gamma function by

$$
(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \lambda(\lambda + 1)\ldots(\lambda + n - 1)
$$

and $(\lambda)_0 = 1$. Using the Pochhammer symbol, the expression $u_{p,b,c}(z)$ becomes

$$
u_{p}(z) = u_{p,b,c}(z) = \sum_{n \geq 0} \frac{(-c/4)^n z^n}{(\kappa)_n}, \quad z \in \mathbb{C}
$$

where $p, b, c \in \mathbb{C}$ and $\kappa = p + (b + 1)/2 \neq 0, -1, -2, \ldots$. The function $u_{p,b,c}(z)$ is called generalized and “normalized” Bessel function of first kind of order $p$. Let $b_n = (-c/4)^n/((\kappa)_n n!)$. Note that the function $u_{p}(z)$ is not normalized according to the usual definition of normalization but $(u_{p} - b_n)/b_1$ is. Therefore, the word normalized has been put in quotes. Also, the series given in (1.3) is convergent in the whole complex plane and hence the function $u_{p}(z)$ is an entire function. Note that the function $u_{p}(z)$ satisfies the differential equation

$$4z^2 u_{p}''(z) + 4\kappa z u_{p}'(z) + cz u_{p}(z) = 0, \quad z \in \mathbb{C}.
$$

For a detailed study about the Bessel functions, one may refer [1,3,4,6,10,17]. Let $S(\alpha, \beta, \lambda)$ be a subclass of $A$ satisfying $z/f(z) \neq 0$ and

$$
\left| f'(z) \left( \frac{z}{f(z)} \right)^2 - \beta z^3 \left( \frac{z}{f(z)} \right)^{''} - (\alpha + \beta)z^2 \left( \frac{z}{f(z)} \right)^{''} - 1 \right| \leq \lambda, \quad z \in \mathbb{D}.
$$

Baricz et al. [2] obtained sufficient conditions on the constants $\alpha > -1$ and $\beta$ such that the function $z/u_{p,b,c}(z) \in S(\alpha, \beta, \lambda)$. Prajapat [14] determined conditions for generalized bessel function (with a different normalization that one considered in this paper) to be univalent in the open unit disk. Kanas et al. [11] used the method of differential subordination to obtain sufficient conditions which imply that the function $u_{p,b,c}(z)$ is Janowski convex and $zu_{p,b,c}(z)$ is Janowski starlike. The method of differential subordination was formulated by Miller and Mocanu [13]. Radhika et al. [15] established sufficient conditions for Bessel function to be in class of Janowski starlike and Janowski convex functions. In [5], Baricz determined the conditions which imply that the function $u_{p,b,c}(z)$ is convex and $zu_{p,b,c}(z)$ is starlike of order 1/2 in $\mathbb{D}$. Bohra et al. [8] obtained the conditions so that the functions $u_{p,b,c}(z)$ and $z u_{p,b,c}(z)$ are strongly convex of order 1/2 and strongly starlike of order 1/2 respectively in $\mathbb{D}$.

The Lommel function of first kind $s_{\mu,p}$ is a particular solution of inhomogeneous Bessel differential equation

$$z^2 w''(z) + zw'(z) + (z^2 - p^2)w(z) = z^{\mu+1}.
$$

The function $s_{\mu,p}(z)$ can be expressed in terms of hypergeometric function

$$
s_{\mu,p}(z) = \frac{z^{\mu+1}}{\Gamma(\mu + p + 1)(\mu + p + 1)} F_{2}\left(1, \frac{\mu - p + 3}{2}, \frac{\mu + p + 1}{2}, \frac{-z^2}{4}\right),
$$

where $\mu \pm p$ is not a negative odd integer. Note that the Lommel function $s_{\mu,p}$ does not belong to the class $A$. Hence, we consider the following normalization of the Lommel function of the first kind:

$$h_{\mu,p}(z) = (\mu - p + 1)(\mu + p + 1)z^{1+\mu} s_{\mu,p}(\sqrt{z}) = z + \sum_{n=1}^{\infty} \frac{(-1/4)^n}{(K)_n (F)_n} z^{n+1},$$
where $K = (\mu - p + 3)/2$ and $F = (\mu + p + 3)/2$. Clearly the function $h_{\mu,p} \in \mathcal{A}$ and satisfies the differential equation

\begin{equation}
(1.5) \quad z^2 h''_{\mu,p}(z) + \mu z h'_{\mu,p}(z) + \left(\frac{(\mu + 1)^2 - p^2}{4} + \frac{z}{4}\right) h_{\mu,p}(z) = (\mu + 1 - p)(\mu + 1 + p)\frac{z}{4},
\end{equation}

Yağmur [18] obtained conditions on the parameters $\mu$ and $p$ such that the function satisfies $\text{Re}(h_{\mu,p}(z)/z) > \alpha$ for $0 \leq \alpha < 1$. Baricz et al. [7] studied the zeroes of some normalization of Lommel and Struve function and hence determined the radius of convexity of these functions.

In this paper, sufficient conditions on parameters $p, b$ and $c$ (and $\mu, p$) are derived so that the generalized and “normalized” Bessel function $u_{p,b,c}(z)$ (and the normalization $h_{\mu,p}(z)$ of Lommel function $s_{\mu,p}(z)$) is lemniscate convex in $\mathbb{D}$. As an application of lemniscate convexity of generalized and “normalized” Bessel function, a relation between the parameters $p, b$ and $c$ is obtained such that the function $zu_{p,b,c}(z)$ becomes lemniscate starlike in $\mathbb{D}$. Moreover, sufficient conditions are obtained on $p, b$ and $c$ for which the function $(-4\kappa/c)u_{p,b,c}(z)$ is lemniscate Carathéodory in $\mathbb{D}$, hence becoming close-to-convex and therefore univalent. Also, relations between the constants $\mu$ and $p$ are obtained that implies the Alexander transform of the function $h_{\mu,p}(z)$ is lemniscate convex and lemniscate Carathéodory in $\mathbb{D}$. The method of admissibility conditions for differential subordination formulated by Miller and Mocanu [13] has been used to prove the stated results.

2. Main Results

The following theorem describes the conditions on $\kappa$ and $c$ such that $(-4\kappa/c)u'_{\mu}(z) \prec \sqrt{1 + z}$.

**Theorem 2.1.** Let $\kappa, c \in \mathbb{C}$ be such that $c \neq 0$ and satisfy

\begin{equation}
(2.1) \quad \text{Re} \kappa > \max\{0, |c| - 3/4\},
\end{equation}

then $(-4\kappa/c)u'_{\mu}(z) \prec \sqrt{1 + z}$.

The next result gives sufficient conditions on the parameters $\kappa$ and $c$ so that the generalized and “normalized” Bessel function $u_{\mu}(z)$ is lemniscate convex in $\mathbb{D}$.

**Theorem 2.2.** If $b, p, c \in \mathbb{C}$ are such that $c \neq 0$ and

\begin{equation}
(2.2) \quad \sqrt{3}\kappa - 2|c| + \frac{|c|}{4} < \sqrt{\frac{9}{8} + \frac{1}{\sqrt{2}}},
\end{equation}

then the function $u_{\mu}(z)$ is lemniscate convex in $\mathbb{D}$.

Baricz proved the recursive relation satisfied by $u_{\mu}(z)$ as given in

**Lemma 2.3.** [7] Lemma 1.2, p. 14] If $b, p, c \in \mathbb{C}$ and $\kappa \neq 0, -1, -2, \ldots$, then the function $u_{\mu}(z)$ satisfies the relation $4\kappa u'_{\mu}(z) = -cu_{\mu+1}(z)$ for all $z \in \mathbb{C}$.

If $\sqrt{3}\kappa - 3|+|c|/4 < \sqrt{\frac{9}{8} + \frac{1}{\sqrt{2}}}$, then from Theorem 2.2 it follows that $1 + (zu''_{p-1}(z)/u'_{p-1}(z)) \prec \sqrt{1 + z}$ and hence $(zu'_{p-1}(z))'/u'_{p-1}(z) \prec \sqrt{1 + z}$ which means that $zu'_{p-1}(z)$ is lemniscate starlike in $\mathbb{D}$. Also Lemma 2.3 gives that $czu_{\mu}(z) = -4(\kappa - 1)zu'_{p-1}(z)$. Therefore $zu_{\mu}(z)$ is lemniscate starlike in $\mathbb{D}$. Thus we have the following:

**Corollary 2.4.** If $b, p, c \in \mathbb{C}$ are such that

\begin{equation}
(2.3) \quad \sqrt{3}\kappa - 3|+|c|/4 < \sqrt{\frac{9}{8} + \frac{1}{\sqrt{2}}},
\end{equation}

then the function $zu_{\mu}(z)$ is lemniscate starlike in $\mathbb{D}$.
For $b = 1 = c$, the generalized Bessel function $w_p(z)$, as given in (12), reduces to the Bessel function of first kind of order $p$, denoted by $J_p(z)$ is given by

$$J_p(z) = \sum_{n \geq 0} \frac{(-1)^n}{n! \Gamma(p + n + 1)} \left(\frac{z}{2}\right)^{2n+p}.$$  \hspace{1cm} (2.4)

With $b = 1, c = -1$ the function $w_p(z)$ reduces to the modified Bessel function of first kind of order $p$, denoted by $I_p(z)$, is given by

$$I_p(z) = \sum_{n \geq 0} \frac{1}{n! \Gamma(p + n + 1)} \left(\frac{z}{2}\right)^{2n+p}.$$  \hspace{1cm} (2.5)

With the values $b = 1, c = 1$ using Theorem 2.2 and Corollary 2.4, we get the following

**Corollary 2.5.** Let $p \in \mathbb{C}$. For the function

$$J_p(z^{1/2}) = 2^p \Gamma(p+1) z^{-p/2} J_p(z^{1/2}),$$

where $J_p$ is the Bessel function as defined in (2.4), the following holds:

(i) If $|p - 1| \sqrt{3} < \frac{\sqrt{9 + 4 \sqrt{2}}}{4} - \frac{1}{4}$, then $J_p(z^{1/2})$ is lemniscate convex in $D$.

(ii) If $|p - 2| \sqrt{3} < \frac{\sqrt{9 + 4 \sqrt{2}}}{4} - \frac{1}{4}$, then $J_p(z^{1/2})$ is lemniscate starlike in $D$.

If $b = 1, c = -1$ in Theorem 2.2 and Corollary 2.4, then the function

$$I_p(z^{1/2}) = 2^p \Gamma(p+1) z^{-p/2} I_p(z^{1/2}),$$

where $I_p$ is the modified Bessel function of the first kind of order $p$, has the properties same as that for the function $J_p(z^{1/2})$, because $|c| = 1$ in this case.

Since some of the Bessel functions of first kind of order $p$ can also be expressed in terms of trigonometric functions like $\cos, \sin, \cosh, \text{and sinh}$, we have some relations for some trigonometric ratios to be lemniscate convex or lemniscate starlike in $D$ which are as follows:

Since

$$J_{1/2}(z^{1/2}) = \sqrt{\frac{z}{2}} z^{-1/4} J_{1/2}(\sqrt{z}) = \frac{\sin \sqrt{z}}{\sqrt{z}} \quad \text{and} \quad I_{1/2}(z^{1/2}) = \sqrt{\frac{z}{2}} z^{-1/4} I_{1/2}(\sqrt{z}) = \frac{\sinh \sqrt{z}}{\sqrt{z}},$$

the functions $(\sin \sqrt{z})/\sqrt{z}$ and $(\sinh \sqrt{z})/\sqrt{z}$ are lemniscate convex in $D$. Also, since

$$z^3 J_{3/2}(z^{1/2}) = 3 \left( \frac{\sin \sqrt{z}}{\sqrt{z}} - \cos \sqrt{z} \right),$$

the function $(\sin \sqrt{z} - \sqrt{z} \cos \sqrt{z})/\sqrt{z}$ is lemniscate starlike in $D$.

Now we obtain conditions on $\mu$ and $p$ such that the function $h_{\mu,p}(z)$ is lemniscate convex in $D$.

**Theorem 2.6.** Let $\mu, p \in \mathbb{R}$ be such that $\mu \pm p$ is not an odd negative integer. If

$$\frac{3\mu}{2\sqrt{2}} - \sqrt{3} \left| \frac{(\mu + 1)^2 - p^2}{4} - 2\mu - 2 \right| > \frac{13}{4} \sqrt{3} - \frac{15}{8\sqrt{2}} + \frac{1}{2},$$  \hspace{1cm} (2.6)

then the function $h_{\mu,p}(z)$ is lemniscate convex in $D$.

As the function $h_{\mu,p}(z)$ is lemniscate convex in $D$ for $\mu$ and $p$ satisfying (2.6), the function $z h'_{\mu,p}(z)$ is lemniscate starlike in $D$ for $\mu, p$ as in (2.6).

The convolution or Hadamard product of two functions $f$ and $g$ having power series expansion as $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$
Shanmugam [16] proved that the for a function \( g \in \mathcal{A} \) and a convex univalent function \( h \) satisfying \( h(0) = 1 \) and \( \text{Re} \ h(z) > 0 \), the class \( K_g(h) := \left\{ f \in \mathcal{A} | (g * f)'(z) \neq 0 \right\} \) is closed with respect to convolution with convex functions. In particular, for \( g(z) = z/(1 - z) \) and \( h(z) = \sqrt{1 + z} \), the class of lemniscate convex functions is closed with respect to convolution with convex functions.

The Libera operator \( L : \mathcal{A} \to \mathcal{A} \) is defined by

\[
L[f](z) := \frac{2}{z} \int_0^z f(t) \, dt = -2(z + \log[1 - z]) * f(z).
\]

The Libera operator \( L : \mathcal{A} \to \mathcal{A} \) is defined by

Theorem 2.7. If \( \mu, p \) satisfy (2.6), then \( (h_{\mu, p} * f)(z) \) is lemniscate convex in \( \mathbb{D} \) and thus the functions \( A[h_{\mu, p}](z) \) and \( L[h_{\mu, p}](z) \) are lemniscate convex in \( \mathbb{D} \).

Now, consider the Alexander transform of the function \( h_{\mu, p}(z) \) named as \( f_{\mu, p} : \mathbb{D} \to \mathbb{C} \) by

\[
f_{\mu, p}(z) := \int_0^z \frac{h_{\mu, p}(t)}{t} \, dt.
\]

The function \( f_{\mu, p}(z) \) is analytic in \( \mathbb{D} \). Moreover, \( f_{\mu, p} \in \mathcal{A} \). As \( h_{\mu, p}(z) \) satisfies the differential equation (1.5), \( f_{\mu, p}(z) \) satisfies the differential equation

\[
z^2 f''_{\mu, p}(z) + (\mu + 2)z f'_{\mu, p}(z) + \left( \frac{(\mu + 1)^2 - p^2}{4} + \frac{z}{4} \right) f_{\mu, p}(z) = \frac{(\mu + 1)^2 - p^2}{4}.
\]

Differentiating and dividing by \( f'_{\mu, p}(z) \) and multiplying by \( z \), we get

\[
z^3 f^{(4)}_{\mu, p}(z) f_{\mu, p}(z) + (\mu + 4)z^2 f'''_{\mu, p}(z) f_{\mu, p}(z) + \left( \frac{(\mu + 1)^2 - p^2}{4} + \frac{z}{4} + \mu \right) zf''_{\mu, p}(z) f_{\mu, p}(z) + \frac{z}{4} = 0.
\]

Using Theorem 2.7 the function \( f_{\mu, p}(z) \) is lemniscate convex in \( \mathbb{D} \) for \( \mu, p \) satisfying (2.6). Furthermore, the next theorem admits the conditions so that the function \( f_{\mu, p}(z) \) defined above is lemniscate convex in \( \mathbb{D} \).

Theorem 2.8. Let \( \mu, p \in \mathbb{R} \) such that \( \mu \pm p \) is not a negative odd integer. If \( \mu, p \) satisfy

\[
\frac{3\mu}{2\sqrt{2}} - \sqrt{3} \left| \frac{(\mu + 1)^2 - p^2}{4} - 2\mu - 2 \right| > \frac{13\sqrt{3}}{4} - \frac{15}{8\sqrt{2}} + \frac{1}{4},
\]

then the function \( f_{\mu, p}(z) \) is lemniscate convex in \( \mathbb{D} \).

The following lemma gives conditions on the constants \( \mu, p \) such that the function \( f_{\mu, p}(z) \) is lemniscate Carathéodory.

Theorem 2.9. Let \( \mu, p \in \mathbb{C} \) be such that \( \mu \pm p \) is not a negative odd integer and \( \text{Re} \mu > -1 \) and satisfy

\[
\left| \frac{(\mu + 1)^2 - p^2}{2} \right| > \sqrt{3} \left( \frac{\text{Re} \mu}{2\sqrt{2}} + \frac{3}{8\sqrt{2}} \right),
\]

then the function \( f'_{\mu, p}(z) = \frac{h_{\mu, p}(z)}{z} \sim \sqrt{1 + z} \).
3. PROOF OF THE MAIN RESULTS

The proofs of our theorems are based on the theory of first and second order differential subordination and the following results are needed to prove our results.

Lemma 3.1. [12] Let $p \in \mathcal{H}[1, n]$ with $p(z) \neq 1$ and $n \geq 1$. Let $\Omega \subset \mathbb{C}$ and $\psi : D \subset \mathbb{C}^3 \times D \rightarrow \mathbb{C}$ satisfy

$$\psi(r, s, t; z) \notin \Omega \quad \text{whenever } z \in \mathbb{D},$$

$$r = \sqrt{2\cos 2\theta} e^{i\theta}, \quad s = me^{3i\theta}/(2\sqrt{2\cos 2\theta}) \text{ and } \Re((t + s)e^{-3i\theta}) \geq 3m^2/(8\sqrt{2\cos 2\theta}) \text{ where } m \geq n \geq 1 \text{ and } -\pi/4 < \theta < \pi/4. \text{ If } (p(z), zp'(z), z^2p''(z); z) \in D \text{ for } z \in \mathbb{D} \text{ and } \psi(p(z), zp'(z), z^2p''(z); z) \in \Omega, \quad z \in \mathbb{D}, \text{ then } p(z) \prec \sqrt{1+z}.$$

In the case $\psi : \mathbb{C}^2 \times \mathbb{D} \rightarrow \mathbb{C}$, the condition in Lemma 3.1 reduces to $\psi(r, s; z) \notin \Omega$ whenever $r = \sqrt{2\cos 2\theta} e^{i\theta}$, $s = me^{3i\theta}/(2\sqrt{2\cos 2\theta})$ for $m \geq n \geq 1$, $-\pi/4 < \theta < \pi/4$ and $z \in \mathbb{D}$.

Baricz in [1] determined the following conditions on $\kappa$ and $c$ for which the function $u_p(z)$ is univalent in $\mathbb{D}$.

Lemma 3.2. [1, Theorem 2.9, p. 29] If $b, p, c \in \mathbb{C}$ are such that $\Re \kappa > |c|/4 + 1$, then $\Re u_p(z) > 0$ for all $z \in \mathbb{D}$. Further, if $\Re \kappa > |c|/4$ and $c \neq 0$, then $u_p$ is univalent in $\mathbb{D}$.

Proof of Theorem 2.1 Define the function $p : \mathbb{D} \rightarrow \mathbb{C}$ by

$$p(z) = -\frac{4\kappa}{c} u_p'(z).$$

Then the function $p(z)$ is analytic in $\mathbb{D}$ and $p(0) = 1$. As the function $u_p(z)$ satisfies the differential equation $4z^2u_p''(z) + 4\kappa z u_p'(z) + cz u_p(z) = 0$, the function $p(z)$ satisfies

$$4z^2p''(z) + 4(\kappa + 1)zp'(z) + czp(z) = 0.$$ 

Define the function $\psi : \mathbb{C}^3 \times \mathbb{D} \rightarrow \mathbb{C}$ by $\psi(r, s, t; z) = 4t + 4(\kappa + 1)s + czr$ and let $\Omega := \{0\}$. Then, for all $z \in \mathbb{D}$ we have that $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$. For $r, s, t$ as in Lemma 3.1 we have

$$\left| \frac{\psi(r, s, t; z)}{4} \right| = \left| t + s + \kappa \frac{me^{3i\theta}}{2\sqrt{2\cos 2\theta}} + \frac{c}{4} \sqrt{2\cos 2\theta} e^{i\theta} z \right| \geq \left| (t + s)e^{-3i\theta} + \kappa \frac{m}{2\sqrt{2\cos 2\theta}} \right| - \frac{|c|}{4} |z| \sqrt{2\cos 2\theta} \geq \Re((t + s)e^{-3i\theta}) + \frac{m \Re \kappa}{2\sqrt{2\cos 2\theta}} - \frac{|c|}{4} \sqrt{2} \geq \frac{3m^2}{8\sqrt{2\cos 2\theta}} + \frac{m \Re \kappa}{2\sqrt{2\cos 2\theta}} - \frac{|c|}{4} \sqrt{2}.$$

By hypothesis, $\Re \kappa > 0$ and since $m \geq 1$ so we have

$$\left| \frac{\psi(r, s, t; z)}{4} \right| \geq \frac{3}{8\sqrt{2}} + \frac{\Re \kappa}{2\sqrt{2}} - \frac{|c|}{4} \sqrt{2}.$$ 

Therefore, we get $\psi(r, s, t; z) \neq 0$ for $r = \sqrt{2\cos 2\theta} e^{i\theta}, s = me^{3i\theta}/(2\sqrt{2\cos 2\theta}), \Re((t + s)e^{-3i\theta}) \geq 3m^2/(8\sqrt{2\cos 2\theta})$ for $m \geq n \geq 1, -\pi/4 < \theta < \pi/4$ and $z \in \mathbb{D}$ if $\Re \kappa > \max\{0, |c| - 3/4\}$. Hence, by Lemma 3.1 the theorem follows.

Proof of Theorem 2.2 Define the function $p : \mathbb{D} \rightarrow \mathbb{C}$ by

$$p(z) = 1 + \frac{zu_p''(z)}{u_p'(z)}.$$
Since, for \( \kappa, c \in \mathbb{C} \) satisfying (2.2), \( \text{Re} \kappa > |c|/4 \) also holds, Lemma 3.2 implies that \( u_p \) is univalent in \( \mathbb{D} \) and thus \( u'_p(z) \neq 0 \) for all \( z \in \mathbb{D} \). The function \( p(z) \) therefore is analytic in \( \mathbb{D} \) and \( p(0) = 1 \). Since \( u_p \) satisfies the equation \( 4z^2u''_p(z) + 4\kappa z u'_p(z) + cz u_p(z) = 0 \), the function \( p(z) \) satisfies the differential equation

\[
4zp'(z) - (p(z) - 1) + (p(z) - 1)^2 + 4(\kappa + 1)(p(z) - 1) + cz = 0.
\]

Let \( \psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C} \) be defined by

\[
\psi(r, s; z) = 4(s - (r - 1) + (r - 1)^2) + 4(\kappa + 1)(r - 1) + cz
\]

and let \( \Omega := \{0\} \). Then \( \psi(p(z), zp'(z); z) \in \mathbb{D} \) for all \( z \in \mathbb{D} \). For \( r, s \) as in Lemma 3.1 we have

\[
\left| \frac{\psi(r, s; z)}{4} \right| = \left| s + r^2 - 1 + (\kappa - 2)(r - 1) + \frac{c^2}{4} \right|.
\]

(3.1)

We first note that, using Lemma 3.1,

\[
|s + r^2 - 1|^2 = \left| \frac{me^{3i\theta}}{2\sqrt{2} \cos 2\theta} + e^{4i\theta}\right|^2 = \left| \frac{m}{2\sqrt{2} \cos 2\theta} + e^{i\theta}\right|^2
\]

\[
= \frac{m^2}{8 \cos 2\theta} + 1 + \frac{m \cos \theta}{\sqrt{2} \cos 2\theta} \geq \frac{9}{8} + \frac{1}{\sqrt{2}}
\]

and \( |r - 1|^2 = 2 \cos 2\theta + 1 - 2\sqrt{2} \cos 2\theta \cos \theta \leq 3 \). Using these in equation (3.1), we get

\[
\left| \frac{\psi(r, s; z)}{4} \right| \geq \sqrt{\frac{9}{8} + \frac{1}{\sqrt{2}} - \sqrt{3}|k - 2| - \frac{|c|}{4}}.
\]

Hence, we get \( \psi(r, s; z) \neq 0 \) for \( r = \sqrt{2 \cos 2\theta} e^{i\theta} \), \( s = m e^{3i\theta}/(2\sqrt{2} \cos 2\theta) \) for \( m \geq n \geq 1 \), \( -\pi/4 < \theta < \pi/4 \) and \( z \in \mathbb{D} \) if \( \sqrt{3}|k - 2| + |c|/4 < \sqrt{\frac{9}{8} + \frac{1}{\sqrt{2}}} \) which holds by given hypothesis. By Lemma 3.1 the theorem follows.

**Proof of Theorem 2.6.** Define the function \( p : \mathbb{D} \to \mathbb{C} \) by

\[
p(z) = 1 + \frac{zh''_{\mu,p}(z)}{h'_{\mu,p}(z)}.
\]

As in [13] Theorem 2.1,

\[
|h'_{\mu,p}(z)| > \frac{2MN - 4M - 3N}{N(2M - 3)} \quad (M > 3/2)
\]

where \( M = (\mu + 5)^2 - p^2 \) and \( N = (\mu + 3)^2 - p^2 \). Since for \( \mu, p \) satisfying (2.2), \( M > 3/2 \) and

\[
|h'_{\mu,p}(z)| > \frac{2MN - 4M - 3N}{N(2M - 3)} > 0,
\]

the function \( p(z) \) is analytic in \( \mathbb{D} \) and \( p(0) = 1 \).

On differentiating the equation (1.5), dividing by \( h'_{\mu,p}(z) \) and multiplying by \( z \), we get

\[
z^3 \frac{h''_{\mu,p}(z)}{h'_{\mu,p}(z)} + \left( \mu + 4 \right) \frac{zh''_{\mu,p}(z)}{h'_{\mu,p}(z)} + \left( \mu + 2 + \left( \mu + 1 \right)^2 - p^2 \right) \frac{zh''_{\mu,p}(z)}{h'_{\mu,p}(z)} + \frac{z^2}{4} = 0.
\]

(3.2)
Using (3.2), we see that the function \( p(z) \) satisfies
\[
\begin{align*}
&z^2p''(z) - 2(zp'(z) - (p(z) - 1)) - 2(p(z) - 1)^3 + 3(p(z) - 1) \\
&(zp'(z) - (p(z) - 1) + (p(z) - 1)^2) + (\mu + 4)(zp'(z) - (p(z) - 1) + (p(z) - 1)^2) \\
&\quad + \left( \mu + 2 + \frac{(\mu + 1)^2 - p^2}{4} + \frac{z}{4} \right) (p(z) - 1) + \frac{z}{2} = 0.
\end{align*}
\]

Define \( \psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C} \) by
\[
\psi(r, s, t; z) = t - 2(s - (r - 1)) - 2(r - 1)^3 + 3(r - 1)
\]
\[
(s - (r - 1) + (r - 1)^2) + (\mu + 4)(s - (r - 1) + (r - 1)^2)
\]
\[
+ \left( \mu + 2 + \frac{(\mu + 1)^2 - p^2}{4} + \frac{z}{4} \right) (r - 1) + \frac{z}{2}
\]
\[
= t + s + 3rs + (\mu - 2)s + (\mu + 1)(r^2 - 1) + (r - 1)^3
\]
\[
+ (r - 1) \left( \frac{(\mu + 1)^2 - p^2}{4} - 2\mu - 2 + \frac{z}{4} \right) + \frac{z}{2}.
\]

Let \( \Omega := \{0\} \). Then we have that \( \psi(p(z), zp'(z), z^2p''(z); z) \in \Omega \) for all \( z \in \mathbb{D} \). For \( r, s, t \) as in Lemma 6.1, we have
\[
|\psi(r, s, t; z)| \geq \left| (t + s)e^{-3i\theta} + 3m^2 \frac{2\sqrt{2\cos 2\theta}}{4} + (\mu + 1)e^{3i\theta} \right|
\]
\[
- \left| \sqrt{2\cos 2\theta}e^{i\theta} - 1 \right|^3 - \left| (\mu + 1)^2 - p^2 - 2\mu - 2 \right| \left| \sqrt{2\cos 2\theta}e^{i\theta} - 1 \right|
\]
\[
\geq \left| \frac{1}{2} Re(t + s)e^{-3i\theta} + 3m^2 \frac{2\cos \theta}{4} + (\mu + 1) \cos \theta \right|
\]
\[
- \frac{13\sqrt{3}}{4} \left| \frac{(\mu + 1)^2 - p^2}{4} - 2\mu - 2 \right| \frac{1}{2}
\]
\[
\geq \frac{3m^2}{8\sqrt{2\cos 2\theta}} + \frac{3m^2}{2\sqrt{2}} + (\mu - 2) \frac{m}{2\sqrt{2\cos 2\theta}} + (\mu + 1) \frac{1}{4} - \frac{13\sqrt{3}}{4}
\]
\[
- \frac{(\mu + 1)^2 - p^2}{4} - 2\mu - 2 \sqrt{3} - \frac{1}{2}
\]

For \( \mu, p \) as in (2.4), \( \mu > 2 \), we have
\[
|\psi(r, s, t; z)| \geq \frac{15}{8\sqrt{2}} + \frac{3\mu}{2\sqrt{2}} - \frac{(\mu + 1)^2 - p^2}{4} - 2\mu - 2 \sqrt{3} - \frac{13\sqrt{3}}{4} - \frac{1}{2}
\]

Hence, we get \( \psi(r, s, t; z) \neq 0 \) for \( r = \sqrt{2\cos 2\theta}e^{i\theta}, \ s = me^{3i\theta}/(2\sqrt{2\cos 2\theta}), \) and \( Re((t + s)e^{-3i\theta}) \geq 3m^2/(8\sqrt{2}\cos 2\theta) \) for \( m \geq n \geq 1, \ -\pi/4 < \theta < \pi/4 \) and \( z \in \mathbb{D} \) if
\[
\frac{3\mu}{2\sqrt{2}} - \sqrt{3} \left| \frac{(\mu + 1)^2 - p^2}{4} - 2\mu - 2 \right| > \frac{13\sqrt{3}}{4} - \frac{15}{8\sqrt{2}} + \frac{1}{2},
\]
which holds by given hypothesis. By Lemma 6.1, the theorem follows.
**Proof of Theorem** Define the function \( p : \mathbb{D} \to \mathbb{C} \) by
\[
p(z) = 1 + \frac{z f'_{\mu,p}(z)}{f'_{\mu,p}(z)}.
\]
As \( f'_{\mu,p}(z) = h_{\mu,p}(z)/z \), using Corollary 2.4, \( p(z) \) is analytic in \( \mathbb{D} \) for \( \mu, p \) satisfying (2.7) and \( p(0) = 1 \). Then the function \( p(z) \) satisfies the differential equation
\[
\begin{align*}
z^2 p''(z) &- 2(zp'(z) - (p(z) - 1) + (p(z) - 1)^2) + 3(zp'(z) - (p(z) - 1) + (p(z) - 1)^2) \\
(p(z) - 1) + 2(p(z) - 1)^2 &- 2(p(z) - 1)^3 + (\mu + 4)(zp'(z) - (p(z) - 1) + (p(z) - 1)^2) \\
+ \left( \frac{(\mu + 1)^2 - p^2}{4} + \frac{z}{4} + \mu \right) (p(z) - 1) + \frac{z}{4} = 0.
\end{align*}
\]
Define \( \psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C} \) by
\[
\psi(r, s, t; z) = t - 2(s - (r - 1) + (r - 1)^2) + 3(s - (r - 1) + (r - 1)^2)(r - 1) \\
+ 2(r - 1)^2 - 2(r - 1)^3 + (\mu + 4)(s - (r - 1) + (r - 1)^2) \\
+ \left( \frac{(\mu + 1)^2 - p^2}{4} + \frac{z}{4} + \mu \right) (r - 1) + \frac{z}{4} \\
= t + s + 3sr + (\mu - 2)s + (\mu + 1)(r^2 - 1) + (r - 1)^3 \\
+ (r - 1) \left( \frac{(\mu + 1)^2 - p^2}{4} + \frac{z}{4} - 2\mu - 2 \right) + \frac{z}{4}.
\]
Let \( \Omega := \{0\} \). Clearly \( \psi(p(z), zp'(z), z^2p''(z); z) \in \Omega \) for all \( z \in \mathbb{D} \). For \( r, s, t \) as in Lemma 3.1 we have
\[
|\psi(r, s, t; z)| = \left| t + s + \frac{3m}{2} e^{i\theta} + (\mu - 2) \frac{me^{3i\theta}}{2 \sqrt{2} \cos 2\theta} + (\mu + 1) e^{4i\theta} + (\sqrt{2} \cos 2\theta) e^{i\theta} - 1 \right|^3
\]
\[
+ \left( \frac{(\mu + 1)^2 - p^2}{4} + \frac{z}{4} - 2\mu - 2 \right) \left( \sqrt{2} \cos 2\theta e^{i\theta} - 1 \right) - \frac{z}{4}
\]
\[
\geq \left| (t + s)e^{-3i\theta} + \frac{3m}{2} e^{i\theta} + (\mu - 2) \frac{m}{2 \sqrt{2} \cos 2\theta} + (\mu + 1) e^{i\theta} \right| - \left| \sqrt{2} \cos 2\theta e^{i\theta} - 1 \right| - \frac{|z|}{4}
\]
\[
\geq \operatorname{Re}(t + s)e^{-3i\theta} + \frac{3m}{2} \cos \theta + (\mu - 2) \frac{m}{2 \sqrt{2} \cos 2\theta} + (\mu + 1) \cos \theta - \frac{13\sqrt{3}}{4}
\]
\[
- \left| (\mu + 1)^2 - p^2 - 2\mu - 2 \right| \sqrt{3} - \frac{1}{4}.
\]
Since for \( \mu, p \) satisfying (2.7), \( \mu > 2 \) and hence we have
\[
|\psi(r, s, t; z)| \geq \frac{15}{8\sqrt{2}} + \frac{3\mu}{2\sqrt{2}} - \left| (\mu + 1)^2 - p^2 - 2\mu - 2 \right| \sqrt{3} - \frac{13\sqrt{3}}{4} - \frac{1}{4}.
\]
Hence, we get \( \psi(r, s, t; z) \neq 0 \) for \( r = \sqrt{2} \cos 2\theta e^{i\theta}, s = me^{3i\theta}/(2\sqrt{2} \cos 2\theta) \), and \( \operatorname{Re}(t + s)e^{-3i\theta} \geq 3m^2/(8\sqrt{2} \cos 2\theta) \) for \( m \geq n \geq 1, -\pi/4 < \theta < \pi/4 \) and \( z \in \mathbb{D} \) if
\[
\frac{3\mu}{2\sqrt{2}} - \sqrt{3} \left| (\mu + 1)^2 - p^2 - 2\mu - 2 \right| > \frac{13\sqrt{3}}{4} - \frac{15}{8\sqrt{2}} + \frac{1}{4},
\]
which holds by given hypothesis. By Lemma 3.1, the theorem follows. \( \square \)
Proof of Theorem 2.9. Define the function \( p : \mathbb{D} \to \mathbb{C} \) by
\[
p(z) = f'_{\mu,p}(z) = \frac{h_{\mu,p}(z)}{z}.
\]
Then the function \( p(z) \) is analytic in \( \mathbb{D} \) and \( p(0) = 1 \). As the function \( h_{\mu,p}(z) \) satisfies equation (1.3), so \( p(z) \) satisfies
\[
z^2 p''(z) + (\mu + 2)zp'(z) + \left( \frac{z}{4} + \frac{(\mu + 1)^2 - p^2}{4} \right) p(z) - \left( \frac{(\mu + 1)^2 - p^2}{4} \right) = 0.
\]
Define \( \psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C} \) by
\[
\psi(r, s, t; z) = t + (\mu + 2)s + \left( \frac{z}{4} + \frac{(\mu + 1)^2 - p^2}{4} \right) r - \left( \frac{(\mu + 1)^2 - p^2}{4} \right) = 0
\]
and let \( \Omega := \{0\} \). Then for all \( z \in \mathbb{D} \) we have that \( \psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega \). For \( r, s, t \) given in Lemma 3.1 we have
\[
\psi(r, s, t; z) = t + s + (\mu + 1) \frac{me^{3i\theta}}{2\sqrt{2\cos 2\theta}} - \left( \frac{(\mu + 1)^2 - p^2}{4} \right) \sqrt{2\cos 2\theta} e^{i\theta}
\]
\[
\geq \left| (t + s)e^{-3i\theta} + \frac{(\mu + 1)m}{2\sqrt{2\cos 2\theta}} - \frac{|z|}{4} \sqrt{2\cos 2\theta} \right| - \left| \frac{(\mu + 1)^2 - p^2}{4} \right| \sqrt{3}
\]
\[
\geq \frac{3m^2}{8\sqrt{2\cos 2\theta}} + \frac{(\Re \mu + 1)m}{2\sqrt{2\cos 2\theta}} - \frac{1}{2\sqrt{2}} - \left| \frac{(\mu + 1)^2 - p^2}{4} \right| \sqrt{3}.
\]
According to given hypothesis, \( \Re \mu > -1 \) and \( m \geq 1 \), we have
\[
|\psi(r, s, t; z)| \geq \frac{(\Re \mu + 1)}{2\sqrt{2}} - \frac{1}{8\sqrt{2} \sqrt{3}} - \left| \frac{(\mu + 1)^2 - p^2}{4} \right| \sqrt{3}.
\]
It is clear from the hypothesis that for \( r = \sqrt{2\cos 2\theta} e^{i\theta}, s = me^{3i\theta}/(2\sqrt{2\cos 2\theta}) \) and \( t \) such that \( \Re((t + s)e^{-3i\theta}) \geq 3m^2/(8\sqrt{2\cos 2\theta}) \) for \( m \geq n \geq 1, -\pi/4 < \theta < \pi/4 \) and \( z \in \mathbb{D} \), we have \( \psi(r, s, t; z) \neq 0 \). Hence, by Lemma 3.1 the theorem follows.

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