On Deformations of Lie Algebroids

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Abstract. For any Lie algebroid $A$, its 1-jet bundle $JA$ is a Lie algebroid naturally and there is a representation $\pi : JA \rightarrow DA$. Denote by $d_3$ the corresponding coboundary operator. In this paper, we realize the deformation cohomology of a Lie algebroid $A$ introduced by M. Crainic and I. Moerdijk as the cohomology of a subcomplex $(\Gamma(\text{Hom}(\text{\wedge}^\bullet JA, A)_DA), d_3)$ of the cochain complex $(\Gamma(\text{Hom}(\text{\wedge}^\bullet JA, A)), d_3)$.

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1. Introduction

The notion of Lie algebroid was introduced by Pradines in 1967, it is a generalization of Lie algebras and tangent bundles. A Lie algebroid over a manifold $M$ is a vector bundle $A \rightarrow M$ together with a Lie bracket $[,]$ on the section space $\Gamma(A)$ and a bundle map $a : A \rightarrow TM$, called the anchor, satisfying the compatibility condition:

$$[X, fY] = f[X, Y] + a(X)(f)Y, \quad \forall X, Y \in \Gamma(A), \ f \in C^\infty(M).$$

We usually denote $[X, fY]$ by $(A, [, ], a)$, or $A$ if there is no confusion. See [10] [18] for more details about Lie algebroids.

In [9], Crainic and Moerdijk studied the cohomology theory underlying deformations of Lie algebroids, where they defined the deformation cohomology of a Lie algebroid $(A, [, ], a)$ and denote by $H^\bullet_{\text{def}}(A)$. Any deformation of the Lie bracket $[,]$ gives rise to a cohomology class in $H^2_{\text{def}}(A)$. But in general, this cohomology does not come from a representation of the Lie algebroid $(A, [, ], a)$. The deformation complex was also given by Grabowska,

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Grabowski and Urbański in [15], where the authors studied Lie brackets on affine bundles.

In general, there is no natural adjoint representation for Lie algebroids. For a Lie algebroid $A$, the action of $\Gamma(A)$ on itself via the bracket is generally not $C^\infty(M)$-linear in the first entry. There is a natural Lie algebroid structure on the 1-jet bundle $\mathcal{J}A$. We call $\mathcal{J}A$ the jet Lie algebroid of $A$. In fact, there is a natural Lie algebroid structure on the $k$-jet bundle $\mathcal{J}^kA$. There is also a representation of the jet Lie algebroid $\mathcal{J}A$ on $A$, which we denote by $\pi : \mathcal{J}A \rightarrow \mathcal{D}A$, where $\mathcal{D}A$ is the gauge Lie algebroid of $A$. This representation was first given in [8], where the authors call it the jet adjoint representation of Lie algebroids. It was further studied by Blaom in [2, 3], where the author call this representation the adjoint representation of Lie algebroids. Recently, Camilo Arias Abad and Marius Crainic suggest to define the adjoint representation of Lie algebroids using representations up to homotopy [1]. A similar notion under the name “super representations” was introduced independently by Alfonso Gracia-Saz and Rajan Amit Mehta in [16].

The main purpose of this paper is to realize the deformation cohomology of $A$ as some cohomology related to a representation. We will see that the cohomology of the cochain complex $(\Gamma(\text{Hom}(\bigwedge \cdot \mathcal{J}A, A)_{\mathcal{D}A}), d_3)$ is isomorphic to the deformation cohomology, where $d_3$ is decided by the representation $\pi : \mathcal{J}A \rightarrow \mathcal{D}A$ (Theorem 4.1).

The paper is organized as follows. In Section 2 we recall the definition of the deformation cohomology and we proved that for a transitive Lie algebroid $(A, [\cdot, \cdot], \alpha)$, the deformation cohomology is isomorphic to the cohomology of the Lie algebroid $A$ with coefficients in the adjoint representation. In Section 3 we give the notion of the $k$-th differential operator bundle Hom$(\bigwedge^k \mathcal{J}E, E)_{\mathcal{D}E}$ and establish the $k$-th differential operator bundle sequence. In particular, for a Lie algebroid $A$, we obtain a subcomplex $(\Gamma(\text{Hom}(\bigwedge^k \mathcal{J}A, A)_{\mathcal{D}A}), d_3)$ of the cochain complex $(\Gamma(\text{Hom}(\bigwedge \mathcal{J}A, A)), d_3)$ associated with the representation $\pi : \mathcal{J}A \rightarrow \mathcal{D}A$. In Section 4 we prove that the cohomology of the subcomplex $(\Gamma(\text{Hom}(\bigwedge^k \mathcal{J}A, A)_{\mathcal{D}A}), d_3)$ is isomorphic to the deformation cohomology and we also give some interesting examples.

2. The deformation cohomology

In this paper, $E \rightarrow M$ is a vector bundle with the base manifold $M$. $d$ is the usual differential on forms. $d$ is the coboundary operator associated with the complex $(\Gamma(\text{Hom}(\bigwedge \cdot \mathcal{D}E, E)_{\mathcal{J}E}), d_1)$. $d_3$ is the coboundary operator associated with the complex $(\Gamma(\text{Hom}(\bigwedge \cdot \mathcal{J}A, A)_{\mathcal{D}A}), d_3)$.

Recall that a multiderivation of degree $n$ of a vector bundle $E$ is a skew-symmetric multi-linear map $D : \Gamma(\bigwedge^n E) \rightarrow \Gamma(E)$, such that for any $f \in C^\infty(M)$ and $u_1, \ldots, u_n \in \Gamma(E)$, we have

$$D(u_1, \ldots, f u_n) = f D(u_1, \ldots, u_n) + \sigma_D(u_1, \ldots, u_{n-1})(f) u_n,$$
where $\sigma_D : \wedge^{n-1} E \rightarrow TM$ is the symbol of $D$. Denote by $D^n(E)$ the set of multiderivations of degree $n$. It is known that $[9]$ $D^n(E)$ is the space of sections of a vector bundle $\mathfrak{D}^n E \rightarrow M$ which fits into a short exact sequence of vector bundles:

$$0 \rightarrow \text{Hom}(\wedge^n E, E) \rightarrow \mathfrak{D}^n E \rightarrow \text{Hom}(\wedge^{n-1} E, TM) \rightarrow 0.$$  \hfill (1)

In particular, $\mathfrak{D}^1 E = \mathfrak{D} E$ is the gauge Lie algebroid of the frame bundle $\mathcal{F}(E)$, which is also called the covariant differential operator bundle of $E$ (see [18, Example 3.3.4]). The corresponding Atiyah sequence is as follows:

$$0 \rightarrow \text{gl}(E) \overset{i}{\rightarrow} \mathfrak{D} E \overset{j}{\rightarrow} TM \rightarrow 0.$$  \hfill (2)

In [9], the deformation complex of a Lie algebroid $(A, [\cdot, \cdot], a)$ is defined as the complex $(C^\bullet_{\text{def}}(A), \delta)$ in which the $n$-cochains $D \in C^\bullet_{\text{def}}(A)$ are multi-linear skew-symmetric maps

$$D : \Gamma(\wedge^n A) \rightarrow \Gamma(A),$$

which are multiderivations and the coboundary operator is given by

$$\delta(D)(u_0, \cdots, u_n) = \sum_i (-1)^i [u_i, D(u_0, \cdots, \hat{u}_i, \cdots, u_n)] + \sum_{i<j} (-1)^{i+j} D([u_i, u_j], u_0, \cdots, \hat{u}_i, \cdots, \hat{u}_j, \cdots, u_n).$$

The deformation cohomology of a Lie algebroid $A$, denote by $H^\bullet_{\text{def}}(A)$, is the cohomology of the cochain complex $(C^\bullet_{\text{def}}(A), \delta)$.

Recall that a Lie algebroid $(A, [\cdot, \cdot], a)$ is called a transitive Lie algebroid if the anchor $a : A \rightarrow TM$ is surjective, and we have the following exact sequence of Lie algebroids:

$$0 \rightarrow L \rightarrow A \overset{a}{\rightarrow} TM \rightarrow 0,$$  \hfill (3)

where, $L = \ker(a)$ is a bundle of Lie algebras. For transitive Lie algebroids, there is a well defined adjoint representation $\text{ad} : A \rightarrow \mathfrak{D} L$ of the Lie algebroid $A$ on the vector bundle $L$, which is given by

$$\text{ad}_u X = [u, X], \quad \forall \ u \in \Gamma(A), \ X \in \Gamma(L).$$

Let $d_A$ be the coboundary operator associated with the adjoint representation. Denote the corresponding cochain complex by $(C^\bullet(A; \text{ad}), d_A) = (\Gamma(\text{Hom}(\wedge^n A, L)), d_A)$, and the cohomology by $H^\bullet(A; \text{ad})$. In the following, we will show that $H^\bullet(A; \text{ad})$ is isomorphic to $H^\bullet_{\text{def}}(A)$.

For any $n$-cochain $D \in C^n_{\text{def}}(A)$, denote by $D_a : \Gamma(\wedge^n A) \rightarrow \mathfrak{X}(M)$ the composition of the anchor $a$ and the multiderivation $D$, i.e.

$$D_a(u_1, \cdots, u_n) = a(D(u_1, \cdots, u_n)), \quad \forall \ u_1, \cdots, u_n \in \Gamma(A).$$

Denote by $C^n_a(A)$ the set of $D_a$, i.e.

$$C^n_a(A) = \{ D_a \mid \forall \ D \in C^n_{\text{def}}(A) \}.$$  \hfill (4)
Obviously, for any \( f \in C^\infty(M) \), we have
\[
D_a(u_1, \cdots, f u_n) = f D_a(u_1, \cdots, u_n) + \sigma_D(u_1, \cdots, u_{n-1})(f)a(u_n),
\]
which implies that the symbol \( \sigma_{D_a} \) of \( D_a \) and the symbol \( \sigma_D \) are equal.

Furthermore, we can define the differential \( \delta : C^n_a(A) \rightarrow C^{n+1}_a(A) \) by setting (we use the same notation of the coboundary operator of the deformation complex)
\[
\delta(D_a)(u_0, \cdots, u_n) = \sum_{i=0}^n (-1)^i [a(u_i), D_a(u_0, \cdots, \tilde{u}_i, \cdots, u_n)] + \sum_{i<j} (-1)^{i+j} D_a([u_i, u_j], u_0, \cdots, \tilde{u}_i, \cdots, \tilde{u}_j, \cdots, u_n).
\]
Then we have
\[
\delta(D_a) = (\delta(D))_a.
\]

**Proposition 2.1.** With the above notations, for a transitive Lie algebroid \( A \), \( C^n_a(A) \) which is defined by \( \mathcal{D}^n_aA \) is the space of sections of a vector bundle which we denote by \( \mathcal{D}^n_aA \) and fits into the following exact sequence:
\[
0 \rightarrow \text{Hom}(\wedge^n A, TM) \rightarrow \mathcal{D}^n_aA \rightarrow \text{Hom}(\wedge^{n-1} A, TM) \rightarrow 0. \quad (5)
\]
Furthermore, the complex \( (C^*_a(A), \delta) \) is acyclic.

**Proof.** The exact sequence (5) follows from applying the anchor \( a \) to the exact sequence (1). For any multiderivation \( D \in D^n_a \), we have
\[
\sigma_{\delta(D)} = \delta(\sigma_D) + (-1)^{n+1} a \circ D = \delta(\sigma_D) + (-1)^{n+1} D_a.
\]
If \( \delta(D_a) = 0 \), we have \( (\delta(D))_a = 0 \). Since \( \sigma_D = \sigma_{D_a} \), we have
\[
\sigma_{\delta(D)} = \sigma_{(\delta(D))_a} = 0.
\]
Therefore, we obtain
\[
D_a = (-1)^n \sigma(\sigma_D) = (-1)^n \delta(\sigma_{D_a}),
\]
which implies that \( D_a \) is exact and this completes the proof. □

Therefore, we can obtain the following result which is also given in [9].

**Corollary 2.2.** With the above notations, for a transitive Lie algebroid \( A \), the cohomology of \( A \) with coefficients in the adjoint representation is isomorphic to the deformation cohomology, i.e. we have
\[
H^\bullet(A, \text{ad}) \cong H^\bullet_{\text{def}}(A).
\]

**Proof.** Obviously, the cochain complex \( (C^\bullet(A; \text{ad}), d_A) \) is a subcomplex of \( (C^\bullet_{\text{def}}(A), \delta) \). For any \( D \in C^n_{\text{def}}(A) \), write \( D = D_a + D_L \), for some \( D_L \in \Gamma(\text{Hom}(\wedge^n A, L)) \). It is straightforward to see that
\[
\delta(D) = \delta(D_a) + d_A(D_L).
\]
Therefore, If \( \delta(D) = 0 \), we have \( \delta(D_a) = 0 \) and \( d_A(D_L) = 0 \). By Proposition 2.1 the complex \( (C^*_a(A), \delta) \) is acyclic, thus we have \( H^\bullet(A, \text{ad}) \cong H^\bullet_{\text{def}}(A) \). □
3. The complex \((\text{Hom}(\wedge^k J A, A)_{\mathcal{D} A}, d_\gamma)\)

The 1-jet vector bundle \(\mathcal{J} E\) of the vector bundle \(E\) (see [19] for more details about jet bundles) is defined as follows. For any \(m \in M\), \((\mathcal{J} E)_m\) is defined as a quotient of local sections of \(E\). Two local sections \(u_1\) and \(u_2\) are equivalent and we denote this by \(u_1 \sim u_2\) if

\[
0 = u_1(m) = u_2(m) \quad \text{and} \quad d\langle u_1, \xi \rangle_m = d\langle u_2, \xi \rangle_m, \quad \forall \xi \in \Gamma(E^*).
\]

So any \(\mu \in (\mathcal{J} E)_m\) has a representative \(u \in \Gamma(E)\) such that \(\mu = [u]_m\). Let \(p\) be the projection which sends \([u]_m\) to \(u(m)\). Then \(\text{Ker} p \cong \text{Hom}(TM, E)\) and there is a short exact sequence, called the jet sequence of \(E\),

\[
0 \longrightarrow \text{Hom}(TM, E) \xrightarrow{\mu} \mathcal{J} E \xrightarrow{p} E \longrightarrow 0.
\]

from which it is straightforward to see that \(\mathcal{J} E\) is a finite dimensional vector bundle. Moreover, \(\Gamma(\mathcal{J} E)\) is isomorphic to \(\Gamma(E) \oplus \Gamma(T^*M \otimes E)\) as an \(\mathbb{R}\)-vector space.

In [5], the authors proved that the jet bundle \(\mathcal{J} E\) may be considered as an \(E\)-dual bundle of \(\mathcal{D} E\), i.e.

\[
\mathcal{J} E \cong \{ \nu \in \text{Hom}(\mathcal{D} E, E) \mid \nu(\Phi) = \Phi \circ \nu(1_E), \quad \forall \Phi \in \mathfrak{gl}(E) \}.
\]

A natural nondegenerate symmetric \(E\)-valued pairing \(\langle \cdot, \cdot \rangle_E\) between \(\mathcal{J} E\) and \(\mathcal{D} E\) is given by

\[
\langle \mu, \xi \rangle_E = \langle \xi, \mu \rangle_E \triangleq \xi u, \quad \forall \mu = [u]_m \in \mathcal{J} E, \ u \in \Gamma(E), \ \xi \in \mathcal{D} E.
\]

Moreover, this pairing is \(C^\infty(M)\)-linear and satisfies the following properties:

\[
\langle \mu, \Phi \rangle_E = \Phi \circ p(\mu), \quad \forall \Phi \in \mathfrak{gl}(E), \ \mu \in \mathcal{J} E;
\]

\[
\langle \eta, \xi \rangle_E = \eta \circ j(\xi), \quad \forall \eta \in \text{Hom}(TM, E), \ \xi \in \mathcal{D} E.
\]

For \(k \geq 2\), the \(k\)-th skew-symmetric jet bundle \(\text{Hom}(\wedge^k \mathcal{D} E, E)_{\mathcal{J} E}\) is defined in [7]:

\[
\text{Hom}(\wedge^k \mathcal{D} E, E)_{\mathcal{J} E} \triangleq \{ \mu \in \text{Hom}(\wedge^k \mathcal{D} E, E) \mid \text{Im}(\mu_2) \subset \mathcal{J} E \},
\]

in which \(\mu_2 : \wedge^{k-1} \mathcal{D} E \longrightarrow \text{Hom}(\mathcal{D} E, E)\) is given by

\[
\mu_2(\xi_1, \cdots, \xi_{k-1})(\xi_k) = \mu(\xi_1, \cdots, \xi_{k-1}, \xi_k), \quad \forall \xi_1, \cdots, \xi_k \in \mathcal{D} E.
\]

Furthermore, the authors proved that \((\Gamma(\text{Hom}(\wedge^k \mathcal{D} E, E)_{\mathcal{J} E}), d)\) is a subcomplex of the cochain complex \((\Gamma(\text{Hom}(\wedge^k \mathcal{D} E, E), d)\), where \(d\) is the coboundary operator associated with the gauge Lie algebroid \(\mathcal{D} E\) with the obvious action on the vector bundle \(E\). In particular, \(d : \Gamma(E) \longrightarrow \Gamma(\mathcal{J} E)\) satisfies the following formula which is very useful:

\[
d(f X) = df \otimes X + f dX, \quad \forall X \in \Gamma(E), \ f \in C^\infty(M).
\]

Furthermore, \(\Gamma(\mathcal{J} E)\) is an invariant subspace of the Lie derivative \(L_\xi\) for any \(\xi \in \Gamma(\mathcal{D} E)\), which is defined as follows:

\[
\langle L_\xi \mu, \xi' \rangle_E \triangleq \langle \xi' \mu, \xi \rangle_E - \langle \mu, [\xi, \xi'] \rangle_E, \quad \forall \mu \in \Gamma(\mathcal{J} E), \ \xi', \xi' \in \Gamma(\mathcal{D} E).
\]

Considering the corresponding cohomology groups of the cochain complex \((\Gamma(\text{Hom}(\wedge^k \mathcal{D} E, E)_{\mathcal{J} E}), d)\), we have
Theorem 3.1. For the cochain complex $C(E) = (\Gamma(\text{Hom}(\bigwedge^* \mathfrak{D}E, E)_{\mathcal{D}E}), \partial)$, we have $H^k(C(E)) = 0$, for all $k = 0, 1, 2, \ldots$. In other words, there is a long exact sequence:

$$0 \to \Gamma(E) \overset{\partial}{\longrightarrow} \Gamma(\mathfrak{J}E) \overset{\partial}{\longrightarrow} \Gamma(\text{Hom}(\bigwedge^2 \mathfrak{D}E, E)_{\mathcal{D}E}) \overset{\partial}{\longrightarrow} \cdots \overset{\partial}{\longrightarrow} \Gamma(\text{Hom}(\bigwedge^n \mathfrak{D}E, E)_{\mathcal{D}E}) \to 0,$$

where $n = \dim M + 1$.

By this theorem, the authors studied the deformation of omni-Lie algebroids as well as its automorphism groups in [7].

Assume that, for the moment, the rank of $E$ is $r \geq 2$. Similar to (7), we can define the $k$-th differential operator bundle $\text{Hom}(\bigwedge^k \mathfrak{J}E, E)_{\mathcal{D}E}$ by

$$\text{Hom}(\bigwedge^k \mathfrak{J}E, E)_{\mathcal{D}E} \triangleq \{ \mathfrak{d} \in \text{Hom}(\bigwedge^k \mathfrak{J}E, E) \mid \text{Im}(\mathfrak{d}_i) \subset \mathfrak{D}E \}, \quad (k \geq 2).$$

Next we study the property of the bundle $\text{Hom}(\bigwedge^k \mathfrak{J}E, E)_{\mathcal{D}E}$ and give its corresponding exact sequence of vector bundles.

Proposition 3.2. For any $\mathfrak{d} \in \text{Hom}(\bigwedge^k \mathfrak{J}E, E)_{\mathcal{D}E}$, there is a unique bundle map $\lambda_0 \in \text{Hom}(\bigwedge^{k-1} E, TM)$ such that for any $\eta \in \text{Hom}(TM, E)$, $\mu_i \in \mathfrak{J}E$, we have

$$\mathfrak{d}(\mu_1 \wedge \cdots \wedge \mu_{k-1} \wedge \eta) = \eta \circ \lambda_0(\mathfrak{D}\mu_1 \wedge \cdots \wedge \mathfrak{D}\mu_{k-1}). \quad (10)$$

Proof. For any $\alpha \otimes u$, $\beta \otimes v \in \text{Hom}(TM, E)$, where $\alpha$, $\beta \in \Omega^1(M)$ and $u$, $v \in \Gamma(E)$, since $\mathfrak{d}$ is skew-symmetric, we have

$$\mathfrak{d}(\mu_1, \cdots, \mu_{k-2}, \alpha \otimes u, \beta \otimes v) = \langle j \circ \mathfrak{d}_z(\mu_1, \cdots, \mu_{k-2}, \alpha \otimes u), \beta \rangle v = -\langle j \circ \mathfrak{d}_z(\mu_1, \cdots, \mu_{k-2}, \beta \otimes v), \alpha \rangle u,$$

where the notation $\cdot_z$ is given by (8). Since the rank of $E$ was assumed to be bigger than one, it follows that

$$j \circ \mathfrak{d}_z(\mu_1, \cdots, \mu_{k-2}, \eta) = 0, \quad \forall \eta \in \text{Hom}(TM, E).$$

Therefore, $j \circ \mathfrak{d}_z$ factors though $\mathfrak{D}$, i.e. there is a unique $\lambda_0 \in \text{Hom}(\bigwedge^{k-1} E, TM)$ such that

$$j \circ \mathfrak{d}_z(\mu_1 \wedge \cdots \wedge \mu_{k-1}) = \lambda_0(\mathfrak{D}\mu_1 \wedge \cdots \wedge \mathfrak{D}\mu_{k-1}),$$

which yields the conclusion.  

We will write $j(\mathfrak{d}) = \lambda_0$ by (10). For any $\Phi \in \text{Hom}(\bigwedge^k E, E)$, $i(\Phi) \in \text{Hom}(\bigwedge^k \mathfrak{J}E, E)_{\mathcal{D}E}$ is given by

$$i(\Phi)(\mu_1, \cdots, \mu_k) = \Phi(\mathfrak{D}\mu_1, \cdots, \mathfrak{D}\mu_k). \quad (11)$$

Theorem 3.3. For any $k \geq 1$, we have the following exact sequence:

$$0 \to \text{Hom}(\bigwedge^k E, E) \overset{i}{\longrightarrow} \text{Hom}(\bigwedge^k \mathfrak{J}E, E)_{\mathcal{D}E} \overset{j}{\longrightarrow} \text{Hom}(\bigwedge^{k-1} E, TM) \to 0. \quad (12)$$
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Proof. For any $\lambda \in \text{Hom}(\wedge^{k-1} E, TM)$, define $\hat{\lambda} \in \text{Hom}(\wedge^k \mathcal{J} E, E)_{\mathcal{D} E}$ by

$$\hat{\lambda}(\mu_1, \ldots, \mu_k) = \sum_i (-1)^{i+1} (\mu_i - \gamma \mu_i)(\lambda(p_{\mu_1}, \ldots, p_{\mu_i}, \ldots, p_{\mu_k})),$$

for any split $\gamma : E \to \mathcal{J} E$ of $[1]$. Evidently, $\mu_i - \gamma \mu_i \in \text{Hom}(TM, E)$ and $\hat{\lambda} \in \text{Hom}(\wedge^k \mathcal{J} E, E)$. Furthermore, for any $\eta \in \text{Hom}(TM, E)$, we have

$$\hat{\lambda}(\mu_1, \ldots, \mu_{k-1}, \eta) = (-1)^{k+1} \eta \circ \lambda(p_{\mu_1}, \ldots, p_{\mu_{k-1}}),$$

which means that $\hat{\lambda} \in \text{Hom}(\wedge^k \mathcal{J} E, E)_{\mathcal{D} E}$ and $j((-1)^{k+1} \hat{\lambda}) = \lambda$, i.e. the bundle map $j$ is surjective. By (11), the definition of $i$, it is obvious that $j \circ i = 0$.

If $\delta \in \text{Hom}(\wedge^k \mathcal{J} E, E)_{\mathcal{D} E}$ satisfies $j(\delta) = 0$, we have

$$\partial(\mu_1, \ldots, \mu_{k-1}, \eta) = \eta \circ j(\delta)(p_{\mu_1}, \ldots, p_{\mu_{k-1}}) = 0,$$

which implies $\delta$ factors through $p$, i.e. there is a unique $\Phi \in \text{Hom}(\wedge^k E, E)$ such that

$$\partial(\mu_1, \ldots, \mu_k) = \Phi(p_{\mu_1}, \ldots, p_{\mu_k}).$$

This completes the proof of the exactness of (12). \[ \Box \]

We call exact sequence (12) the $k$-th differential operator bundle sequence.

Remark 3.4. If the rank of the vector bundle is $r = 1$, when $n \geq 2$, we should extend the definition of $\text{Hom}(\wedge^n \mathcal{J} E, E)_{\mathcal{D} E}$ to satisfy the exact sequence (12).

Associated with any Lie algebroid $(A, [\cdot, \cdot], a)$, there is a bundle map $\pi : \mathcal{J} A \to \mathcal{D} A$ which is given by [5]

$$\pi(\text{du})(v) = [u, v], \quad \forall u, v \in \Gamma(A), \quad (13)$$

and a bracket $[\cdot, \cdot]_\pi$ on $\Gamma(\mathcal{J} A)$ by setting

$$[\mu, \nu]_\pi \triangleq \frac{1}{i} \left[ \pi(\mu), \pi(\nu) \right] - \frac{1}{i} \left[ \pi(\nu), \pi(\mu) \right] - \text{d} \left( \frac{1}{i} \left[ \pi(\mu), \pi(\nu) \right] \right)_A = \frac{1}{i} \left[ \pi(\mu), \pi(\nu) \right] - \pi(\nu) \text{d} \pi(\mu).$$

(14)

It turns out that $(\mathcal{J} A, [\cdot, \cdot]_\pi, j \circ \pi)$ is a Lie algebroid together with the representation $\pi$. We give a list of several useful formulas here which will be used later. The proof is straightforward by (9), (13), (14) and we leave it to readers.

Lemma 3.5. For any $u, v \in \Gamma(A)$, $\omega, \theta \in \Omega^1(M)$, $f \in C^\infty(M)$, we have

$$\left[ \text{d} u, \text{d} v \right]_\pi = \text{d} [u, v], \quad (15)$$

$$\left[ \text{d} u, \omega \otimes v \right]_\pi = \left. L_a(u) \omega \otimes v + \omega \otimes [u, v] \right|_{\theta = 0}, \quad (16)$$

$$\left[ \omega \otimes u, \theta \otimes v \right]_\pi = \left. \langle a(u), \theta \rangle \omega \otimes v - \langle a(v), \omega \rangle \theta \otimes u \right|_{\theta = 0}, \quad (17)$$

$$L_\theta (\text{d} f \otimes v) = \text{d} f \otimes \theta v + \text{d} (j(\delta) f) \otimes v, \quad (18)$$

$$\pi(\text{d} f \otimes v)(u) = -a(u)(f) v, \quad (19)$$

and

$$j(\pi(\text{d} u)) = a(u). \quad (20)$$
For more information about [14], see [5]. Denote by $d_3$ the associated coboundary operator in the cochain complex $(\Gamma(\text{Hom}(\wedge^iJ\mathcal{A}, A)), d_3)$. Furthermore, in [7], by using the theory of Manin pairs, the authors proved that $(\mathcal{D}A, J\mathcal{A})$ is an A-Lie bialgebroid. Therefore, $(\Gamma(\text{Hom}(\wedge^iJ\mathcal{A}, A)\mathcal{D}A), d_3)$ is a subcomplex of the cochain complex $(\Gamma(\text{Hom}(\wedge^iJ\mathcal{A}, A)), d_3)$. In fact, we have

**Proposition 3.6.** For any $\mathfrak{d} \in \text{Hom}(\wedge^kJ\mathcal{A}, A)\mathcal{D}A$, we have

$$j(d_3\mathfrak{d}) = \delta(j(\mathfrak{d})) + (-1)^{k+1}a \circ \mathfrak{d} \circ d.$$  

More precisely, for any $u_1, \ldots, u_k \in \Gamma(A)$, we have

$$j(d_3\mathfrak{d})(u_1, \ldots, u_k) = \delta(j(\mathfrak{d}))(u_1, \ldots, u_k) + (-1)^{k+1}a \circ \mathfrak{d}(du_1, \ldots, du_k), \quad (21)$$

where $\delta$ is given by [5].

**Proof.** For any $\mu_1, \ldots, \mu_k, df \otimes v \in \Gamma(J\mathcal{A})$, we have

$$d_3\mathfrak{d}(\mu_1, \ldots, \mu_k, df \otimes v) = \sum_{i=1}^{k} (-1)^{i+1} \pi(\mu_i)\mathfrak{d}(\mu_1, \ldots, \mu_i, \ldots, \mu_k, df \otimes v)$$

$$+ (-1)^k \pi(df \otimes v)\mathfrak{d}(\mu_1, \ldots, \mu_k)$$

$$+ \sum_{i<j \leq k} (-1)^{i+j} \mathfrak{d}([\mu_i, \mu_j]_{\pi}, \mu_1, \ldots, \mu_i, \ldots, \mu_j, \ldots, \mu_k, df \otimes v)$$

$$+ \sum_{i} (-1)^{i+k+1} \mathfrak{d}([\mu_i, df \otimes v]_{\pi}, \mu_1, \ldots, \mu_i, \ldots, \mu_k).$$

By straightforward computations, we have

$$\sum_{i=1}^{k} (-1)^{i+1} \pi(\mu_i)\mathfrak{d}(\mu_1, \ldots, \mu_i, \ldots, \mu_k, df \otimes v)$$

$$= \sum_{i=1}^{k} (-1)^{i+1} \pi(\mu_i)(j(\mathfrak{d})(\hat{p}\mu_1, \ldots, \hat{p}\mu_i, \ldots, \hat{p}\mu_k)(f)v)$$

$$= \sum_{i=1}^{k} (-1)^{i+1} j(\mathfrak{d})(\hat{p}\mu_1, \ldots, \hat{p}\mu_i, \ldots, \hat{p}\mu_k)(f)\pi(\mu_i)(v)$$

$$+ \sum_{i=1}^{k} (-1)^{i+1} j(\pi(\mu_i))(j(\mathfrak{d})(\hat{p}\mu_1, \ldots, \hat{p}\mu_i, \ldots, \hat{p}\mu_k)(f)v).$$

By (19), we obtain

$$(-1)^k \pi(df \otimes v)\mathfrak{d}(\mu_1, \ldots, \mu_k) = (-1)^{k+1}a \circ \mathfrak{d}(\mu_1, \ldots, \mu_k)(f)v.$$

It is obvious that

$$\sum_{i<j \leq k} (-1)^{i+j} \mathfrak{d}([\mu_i, \mu_j]_{\pi}, \mu_1, \ldots, \mu_i, \ldots, \mu_j, \ldots, \mu_k, df \otimes v)$$

$$= \sum_{i<j \leq k} (-1)^{i+j} j(\mathfrak{d})(\hat{p}[\mu_i, \mu_j]_{\pi}, \hat{p}\mu_1, \ldots, \hat{p}\mu_i, \ldots, \hat{p}\mu_j, \ldots, \hat{p}\mu_k)(f)v.$$
By (18), we have
\[ [\mu_i, df \otimes v]_\pi = L_{\pi(\mu_i)}(df \otimes v) - i_{\pi(df \otimes v)}d\mu_i = df \otimes \pi(\mu_i)(v) + d \circ j(\pi(\mu_i))(f) \otimes v - i_{\pi(df \otimes v)}d\mu_i. \]

Consequently, we have
\[
\sum_i (-1)^{i+k+1}d([\mu_i, df \otimes v]_\pi, \mu_1, \cdots, \widehat{\mu_i}, \cdots, \mu_k) \\
= \sum_i (-1)^{i+k+1}\left((-1)^{k-1}d(\mu_1, \cdots, \widehat{\mu_i}, \cdots, \mu_k, df \otimes \pi(\mu_i)(v))
+ (-1)^{k-1}d(\mu_1, \cdots, \widehat{\mu_i}, \cdots, \mu_k, d \circ j(\pi(\mu_i))(f) \otimes v)
+ (-1)^{k}d(\mu_1, \cdots, \widehat{\mu_i}, \cdots, \mu_k, i_{\pi(df \otimes v)}d\mu_i)\right) \\
= \sum_i (-1)^{i}j(d)(\partial\mu_1, \cdots, \widehat{\mu_i}, \cdots, \partial\mu_k)(f)\pi(\mu_i)(v)
+ \sum_i (-1)^{i}j(d)(\partial\mu_1, \cdots, \widehat{\mu_i}, \cdots, \partial\mu_k)(j(\pi(\mu_i))(f)v)
+ \sum_i (-1)^{i+1}j(d)(\partial\mu_1, \cdots, \widehat{\mu_i}, \cdots, \partial\mu_k, i_{\pi(df \otimes v)}d\mu_i)_A.
\]

Therefore, we have
\[
d_3 d(\partial(\mu_1, \cdots, \mu_k, df \otimes v)) \\
= (-1)^{k+1}a \circ d(\mu_1, \cdots, \mu_k)(f)v
+ \sum_{i<j} (-1)^{i+j+1}j(d)(\partial[\mu_i, \mu_j], \partial\mu_1, \cdots, \widehat{\mu_i}, \cdots, \partial\mu_k)(f)v
+ \sum_i (-1)^{i+1}j(d)(\partial(\mu_1, \cdots, \widehat{\mu_i}, \cdots, \partial\mu_k))(f)v
+ \sum_i (-1)^{i+1}j(d)(\partial(\mu_1, \cdots, \widehat{\mu_i}, \cdots, \partial\mu_k), i_{\pi(df \otimes v)}d\mu_i)_A.
\]

Now assume that \( \mu_1 = du_1, \cdots, \mu_k = du_k \), we get
\[
d_3 d(du_1, \cdots, du_k, df \otimes v) \\
= (-1)^{k+1}a \circ d(du_1, \cdots, du_k)(f)v
+ \sum_{i<j} (-1)^{i+j+1}j(d)([u_i, u_j], u_1, \cdots, \widehat{u_i}, \cdots, u_k)(f)v
+ \sum_i (-1)^{i+1}a(u_i, j(d)(u_1, \cdots, \widehat{u_i}, \cdots, u_k))(f)v
= \left(\delta(j(d))(u_1, \cdots, u_k) + (-1)^{k+1}a \circ d(du_1, \cdots, du_k)\right)(f)v.
\]

This implies that
\[
j(d_3 d) = \delta(j(d)) + (-1)^{k+1}a \circ d \circ d. \]
4. Infinitesimal Deformations of Lie algebroids

Denote by \( H^\bullet(\mathfrak{J}A; A) \) the resulting cohomology of \( (\Gamma(\text{Hom}(\wedge^\bullet \mathfrak{J}A, A)_D A), d_3) \). The main result in this section is

**Theorem 4.1.** Given a Lie algebroid \( A \), the cohomology \( H^\bullet(\mathfrak{J}A; A) \) is isomorphic to the deformation cohomology of \( A \), i.e.

\[
H^\bullet(\mathfrak{J}A; A) \cong H^\bullet_{\text{def}}(A).
\]

**Proof.** First we prove that there is a one-to-one correspondence between \( \Gamma(\text{Hom}(\wedge^k \mathfrak{J}A, A)_D A) \) and \( C^k_{\text{def}}(A) \) at the level of cochains. For any \( \mathfrak{A} \in \Gamma(\text{Hom}(\wedge^k \mathfrak{J}A, A)_D A) \), define \( D_\mathfrak{A} \in C^k_{\text{def}}(A) \) by

\[
D_\mathfrak{A}(u_1, \cdots, u_k) = \mathfrak{A}(du_1, \cdots, du_k).
\]

Follow from

\[
D_\mathfrak{A}(u_1, \cdots, fu_k) = \mathfrak{A}(du_1, \cdots, dfu_k)
= f\mathfrak{A}(du_1, \cdots, du_k) + \mathfrak{A}(du_1, \cdots, df \otimes u_k)
= fD_\mathfrak{A}(u_1, \cdots, u_k) + \mathfrak{J}(\mathfrak{A})(u_1, \cdots, u_{k-1})(fu_k),
\]

we know that \( D_\mathfrak{A} \) is well defined and the following equality holds:

\[
\sigma_{D_\mathfrak{A}} = \mathfrak{J}(\mathfrak{A}).
\]

Conversely, for any \( D \in C^k_{\text{def}}(A) \), define \( \mathfrak{A}_D \in \Gamma(\text{Hom}(\wedge^k \mathfrak{J}A, A)_D A) \) by

\[
\mathfrak{A}_D(du_1, \cdots, du_k) = D(u_1, \cdots, u_k),
\]

and

\[
\mathfrak{A}_D(du_1, \cdots, df \otimes u_k) = \sigma_{D}(u_1, \cdots, u_{k-1})(fu_k).
\]

By (9), it is straightforward to see that \( \mathfrak{A}_D \) is well defined and satisfies

\[
\mathfrak{J}(\mathfrak{A}_D) = \sigma_{D}.
\]

Furthermore, obviously we have

\[
\mathfrak{A}_{D_\mathfrak{A}} = \mathfrak{A}, \quad D_{\mathfrak{A}_D} = D,
\]

which implies that, at the level of cochains, there is a one-to-one correspondence between \( \Gamma(\text{Hom}(\wedge^k \mathfrak{J}A, A)_D A) \) and \( C^k_{\text{def}}(A) \).
If $\mathfrak{d}$ is closed, i.e. $d_3 \mathfrak{d} = 0$, then follows from $\pi(du)(v) = [u, v]$ and $d[u, v] = [du, dv]_\pi$, we have
$$
\delta(D_0)(u_0, \cdots, u_k)
= \sum_i (-1)^i [u_i, D_\mathfrak{d}(u_0, \cdots, \widehat{u_i}, \cdots, u_k)]
+ \sum_{i<j} (-1)^{i+j} D_\mathfrak{d}([u_i, u_j], u_0, \cdots, \widehat{u_i}, \cdots, \widehat{u_j}, \cdots, u_k))
= \sum_i (-1)^i [u_i, \mathfrak{d}(du_0, \cdots, \widehat{du_i}, \cdots, du_k)]
+ \sum_{i<j} (-1)^{i+j} \mathfrak{d}(du_i, u_j, du_0, \cdots, \widehat{du_i}, \cdots, \widehat{du_j}, \cdots, du_k))
= d_3 \mathfrak{d}(du_0, \cdots, du_k)
= 0.
$$

If $\mathfrak{d}$ is exact, i.e. there is some $t \in \text{Hom}(\wedge^{k-1} \mathfrak{J} \mathfrak{A}, A)_{\mathfrak{D} A}$ such that $\mathfrak{d} = d_3 t$, then we have
$$
D_\mathfrak{d}(u_1, \cdots, u_k)
= \mathfrak{d}(du_1, \cdots, du_k)
= (d_3 t)(du_1, \cdots, du_k)
= \sum_i (-1)^{i+1} [u_i, t(du_1, \cdots, \widehat{du_i}, \cdots, du_k)]
+ \sum_{i<j} (-1)^{i+j} t(du_i, u_j, du_1, \cdots, \widehat{du_i}, \cdots, \widehat{du_j}, \cdots, du_k)
= \delta(D_1)(u_1, \cdots, u_k).
$$

Conversely, if $D \in C^k_{\text{def}}(A)$ is closed, first we have
$$(d_3 \mathfrak{d}_D)(du_0, \cdots, du_k) = \delta(D)(u_0, \cdots, u_k) = 0.$$ Then for any $m \leq k$, any $f_l$, $l = m, \cdots, k$, by [9], we have
$$(d_3 \mathfrak{d}_D)(du_0, \cdots, du_{m-1}, df_m \otimes u_m, \cdots, df_k \otimes u_k)
= (d_3 \mathfrak{d}_D)(du_0, \cdots, du_{m-1}, df_m u_m, \cdots, df_k u_k)
- f_m \cdots f_k (d_3 \mathfrak{d}_D)(du_0, \cdots, du_{m-1}, du_m, \cdots, du_k)
= 0,$$
which implies that $\mathfrak{d}_D$ is closed. Similarly, if $D$ is exact, $\mathfrak{d}_D$ is also exact. The proof of the theorem is completed. \(\blacksquare\)

**Corollary 4.2.** $\mathfrak{d} \in \Gamma(\text{Hom}(\wedge^{k-1} \mathfrak{J} \mathfrak{A}, A)_{\mathfrak{D} A})$ is closed if and only of $\mathfrak{d} |_{d_1 \Gamma(A)}$ is closed, i.e. for any $u_0, \cdots, u_k$,
$$(d_3 \mathfrak{d})(du_0, \cdots, du_k) = 0.$$
Next we examine the cohomology $H^\bullet(\mathfrak{A}; A)$ in low degrees.

- In degree 0, $u \in \Gamma(A)$ is closed means that $u$ belongs to the center $Z(\Gamma(A))$ of the infinite-dimensional Lie algebra $\Gamma(A)$, i.e.
  \[ H^0(\mathfrak{A}; A) = Z(\Gamma(A)). \]

  In fact, for any $\mu \in \mathfrak{A}$, $d_3 u(\mu) = 0$ is equivalent to the condition that for any $v \in \Gamma(A)$, $\omega \in \Omega^1(M)$,
  \[ d_3 u(\omega(v)) = 0, \quad d_3 u(\omega \otimes v) = 0. \]

  On the other hand, we have
  \[ d_3 u(\omega \otimes v) = \pi(\omega \otimes v)(u) = -\langle \omega, a(u) \rangle v. \]

  Thus we have
  \[ d_3 u = 0 \iff \begin{cases} [u, v] = 0, \forall v \in \Gamma(A) \\ a(u) = 0. \end{cases} \]

  However, if $[u, v] = 0$, for any $v \in \Gamma(A)$, then for any $f \in C^\infty(M)$, we have $[u, f v] = 0$, which implies that $a(u)(f) = 0$, for any $f \in C^\infty(M)$. This happens exactly when $a(u) = 0$. Thus we have
  \[ d_3 u = 0 \iff u \in Z(\Gamma(A)). \]

- In degree 1, $\mathfrak{d} \in \Gamma(\mathfrak{D}; A)$ is closed if and only if $\mathfrak{d} \in \text{Der}(A)$, where $\text{Der}(A)$ denotes the set of derivatives of the Lie algebroid $A$. In fact, $\mathfrak{d} \in \Gamma(\mathfrak{D}; A)$ is closed if and only if for any $u, v \in \Gamma(A)$, $\omega, \theta \in \Omega^1(M)$, the following equalities hold:
  \[ (d_3 \mathfrak{d})(d u, d v) = 0, \quad (d_3 \mathfrak{d})(d u, \theta \otimes v) = 0, \quad (d_3 \mathfrak{d})(\omega \otimes u, \theta \otimes v) = 0. \]

  On the other hand, by Lemma 3.5, we have
  \[
  (d_3 \mathfrak{d})(d u, d v) &= \pi(d u) \langle \mathfrak{d}, d v \rangle_A - \pi(d v) \langle \mathfrak{d}, d u \rangle_A - \langle \mathfrak{d}, [d u, d v]_\pi \rangle_A \\
  &= [u, \mathfrak{d}(v)] - [v, \mathfrak{d}(u)] - \mathfrak{d}([u, v]), \\
  (d_3 \mathfrak{d})(d u, \theta \otimes v) &= \pi(d u) \langle \mathfrak{d}, \theta \otimes v \rangle_A - \pi(\theta \otimes v) \langle \mathfrak{d}, d u \rangle_A - \langle \mathfrak{d}, [d u, \theta \otimes v]_\pi \rangle_A \\
  &= [u, \langle \theta, j \mathfrak{d} \rangle v] + \langle \theta, a(\mathfrak{d}(u)) \rangle v - \langle j \mathfrak{d}, L a(u) \theta \rangle v - \langle \theta, j \mathfrak{d} \rangle [u, v] \\
  &= (\langle \theta, a(\mathfrak{d}(u)) \rangle + \langle \theta, [a(u), j \mathfrak{d}] \rangle) v,
  \]

  and
  \[
  (d_3 \mathfrak{d})(\omega \otimes u, \theta \otimes v) &= \pi(\omega \otimes u) \langle \mathfrak{d}, \theta \otimes v \rangle_A - \pi(\theta \otimes v) \langle \mathfrak{d}, \omega \otimes u \rangle_A - \langle \mathfrak{d}, [\omega \otimes u, \theta \otimes v]_\pi \rangle_A \\
  &= -\langle j \mathfrak{d}, \theta \rangle \langle a(v), \omega \rangle u + \langle j \mathfrak{d}, \omega \rangle \langle a(u), \theta \rangle v \\
  &\quad - \langle \mathfrak{d}, \langle a(u), \theta \rangle \omega \otimes v - \langle a(v), \omega \rangle \theta \otimes u \rangle_A \\
  &= 0.
  \]
Thus we have this 2-cocycle should also define a Lie bracket. In fact only a 2-cocycle can not contain all the information of the deformation, i.e., \( \mathfrak{d} \in \text{Der}(A) \). Furthermore, \( \mathfrak{d} \in \text{Der}(A) \) also implies that \( (d_{\mathfrak{d}})(du, \theta \otimes v) = 0 \). This follows from the next lemma.

**Lemma 4.3.** If \( \mathfrak{d} \in \text{Der}(A) \), i.e. \( \mathfrak{d} \) is a derivation with respect to the Lie bracket of \( A \), then we have

\[
a(\mathfrak{d}(u)) + [a(u), \mathfrak{j}\mathfrak{d}] = 0, \quad \forall u \in \Gamma(A).
\]

**Proof.** Since \( \mathfrak{d} \) is a derivation, for any \( f \in C^\infty(M) \), we have

\[
\mathfrak{d}[u,fv] = [\mathfrak{d}(u),fv] + [u,\mathfrak{d}(fv)].
\]

Furthermore, we have

\[
\mathfrak{d}[u, fv] = \mathfrak{d}(f[u,v] + a(u)(f)v) = j\mathfrak{d}(f)[u,v] + f\mathfrak{d}[u,v] + a(u)(f)\mathfrak{d}(v) + j\mathfrak{d}(a(u)(f))v,
\]

and

\[
[\mathfrak{d}(u),fv] + [u,\mathfrak{d}(fv)] = f[\mathfrak{d}(u),v] + a(\mathfrak{d}(u))(f)v + a(u)(f)\mathfrak{d}(v) + f[u,\mathfrak{d}(v)] + j\mathfrak{d}(f)[u,v] + a(u)j\mathfrak{d}(f)v.
\]

Thus we have

\[
a(\mathfrak{d}(u)) + [a(u), \mathfrak{j}\mathfrak{d}] = 0, \quad \forall u \in \Gamma(A). \tag{13}
\]

Therefore, \( d_{\mathfrak{d}} = 0 \) if and only if \( \mathfrak{d} \in \text{Der}(A) \). If \( \mathfrak{d} \) is exact, i.e. \( \mathfrak{d} = d_{\mathfrak{d}}u \) for some \( u \in \Gamma(A) \), we have

\[
\mathfrak{d}(dv) = \langle d_{\mathfrak{d}}u, dv \rangle_A = \pi(dv)(u) = -[u,v],
\]

and

\[
\mathfrak{d}(\omega \otimes v) = \langle d_{\mathfrak{d}}u, \omega \otimes v \rangle_A = \pi(\omega \otimes v)(u) = -\langle a(u), \omega \rangle v,
\]

which implies that \( \mathfrak{d} = -\text{ad}_u \). Thus we have

\[
H^1(\mathfrak{j}A; A) = \text{Out}(A) = \text{Der}(A)/\text{Inn}(A).
\]

Let \( (A, [\cdot, \cdot], a) \) be a fix Lie algebroid over the base manifold \( M \) and \( I \subset \mathbb{R} \) be an integral. A 1-parameter infinitesimal deformation of the Lie algebroid \( A \) over \( I \) is a collection \( A_t \) of Lie algebroids \( A_t = (A, [\cdot, \cdot], a_t) \) varying smoothly with respect to \( t \). In \([9]\), for a deformation \( A_t = (A, [\cdot, \cdot], a_t) \) of the Lie algebroid \( A \), the authors proved that there is an associated 2-cocycle \( c_0 \in C^2_{\text{def}}(A) \) which is defined by

\[
c_0(X,Y) = \frac{d}{dt}[X,Y]_t \bigg|_{t=0}, \quad \forall \; X, Y \in \Gamma(A).
\]

In fact only a 2-cocycle can not contain all the information of the deformation, this 2-cocycle should also define a Lie bracket.

Next we consider the 1-parameter infinitesimal deformation of the Lie algebroid \( A \) of the following form:

\[
[X,Y]_t = [X,Y] + t\mathfrak{d}(dx,dY), \quad \forall \; X, Y \in \Gamma(A), \tag{23}
\]
where $\mathfrak{d} \in \Gamma(\text{Hom}(\wedge^2 \mathfrak{J}A, A)_{\mathcal{D}A})$. It is easy to see that the anchors vary as follows:

$$a_t = a + t\mathfrak{J}(\mathfrak{d}).$$

Since $[\cdot, \cdot]_t$ should satisfy the Jacobi identity, we can obtain

$$\mathfrak{d}(d[X, Y], dZ) + [\mathfrak{d}(dX, dY), Z] + c.p. = 0,$$

and

$$D_{\mathfrak{d}}(D_{\mathfrak{d}}(X, Y), Z) + c.p. = 0,$$

which implies that $d_{\mathfrak{d}}\mathfrak{d} = 0$ and $D_{\mathfrak{d}}$ (see (22)) itself defines a Lie bracket.

We summarize the discussion in the following proposition.

**Proposition 4.4.** For any 1-parameter infinitesimal deformation of the Lie algebroid $A$ of the form (23), $\mathfrak{d} \in \Gamma(\text{Hom}(\wedge^2 \mathfrak{J}A, A)_{\mathcal{D}A})$ is a 2-cocycle such that $D_{\mathfrak{d}} : \Gamma(\wedge^2 A) \to \Gamma(A)$ defines a Lie algebroid structure on $A$.

**Remark 4.5.** In general, we can study higher order deformations and versa deformations, see [11, 12] for more details. However, for our objects we leave this study for later consideration.

By (13), we can write (23) as

$$[X, Y]_t = \pi(dX)(Y) + t\mathfrak{d}(dX)(Y) = (\pi + t\mathfrak{d})(dX)(Y).$$

Thus, for any $t \in I$, $[\cdot, \cdot]_t$ is a Lie bracket iff

$$[\pi + t\mathfrak{d}, \pi + t\mathfrak{d}] = 0,$$

which holds if and only if

$$[\pi, \mathfrak{d}] = 0, \quad [\mathfrak{d}, \mathfrak{d}] = 0.$$  

It is straightforward to see that it is equivalent to (24) and (25). In particular, if we only condition the deformation in the following form:

$$[X, Y]_{\mathfrak{d}} = [X, Y] + \mathfrak{d}(dX, dY),$$

obviously we have

**Theorem 4.6.** With the above notations, (26) defines a deformation of the Lie algebroid $(A, [\cdot, \cdot], a)$ for some $\mathfrak{d} \in \Gamma(\text{Hom}(\wedge^2 \mathfrak{J}A, A)_{\mathcal{D}A})$ if and only if $\mathfrak{d}$ satisfies the Maurer-Cartan equation:

$$d_{\mathfrak{d}}\mathfrak{d} + \frac{1}{2}\mathfrak{d} \wedge \mathfrak{d} = 0.$$  

**Remark 4.7.** In fact, for any $\mathfrak{d} \in \Gamma(\text{Hom}(\wedge^2 \mathfrak{J}A, A)_{\mathcal{D}A})$, if we consider the graph $\mathcal{G}_{\mathfrak{d}}$ which is given by

$$\mathcal{G}_{\mathfrak{d}} = \{\mathfrak{d}(\mu) + \mu | \forall \mu \in \mathfrak{J}A\} \subset \mathcal{D}A \oplus \mathfrak{J}A,$$

(27) also means that $\mathcal{G}_{\mathfrak{d}}$ is a Dirac structure. See [6] and Theorem 7.8 in [7] for more details about Dirac structures.
Example 4.8. If the Lie algebroid $A$ is a Lie algebra $\mathfrak{g}$, we have $\mathfrak{J}\mathfrak{g} = \mathfrak{g}$, $\mathcal{D}\mathfrak{g} = \mathfrak{gl}(\mathfrak{g})$ and $\text{Hom}(\wedge^2 \mathfrak{J}\mathfrak{g}, \mathfrak{g}) \mathcal{D}\mathfrak{g} = \text{Hom}(\mathfrak{g} \wedge \mathfrak{g}, \mathfrak{g})$. The resulting cohomology $H^\bullet(\mathfrak{J}\mathfrak{g}, \mathfrak{g})$ turns out to be the cohomology of the Lie algebra $\mathfrak{g}$ with coefficients in the adjoint representation. See [11, 12] for more details.

Example 4.9. If the Lie algebroid $A$ is the tangent Lie algebroid $\mathcal{T}M$, we have already known that all the deformations are trivial [9]. In fact, in this case, it is evident that the gauge Lie algebroid $\mathcal{D}(\mathcal{T}M)$ is isomorphic to the jet Lie algebroid $\mathfrak{J}(\mathcal{T}M)$. Therefore, the cohomology of cochain complex $(\Gamma(\text{Hom}(\wedge^\bullet \mathfrak{J}(\mathcal{T}M), \mathcal{T}M) \mathcal{D}(\mathcal{T}M), d))$ is isomorphic to the cohomology of cochain complex $(\Gamma(\text{Hom}(\wedge^\bullet \mathcal{D}(\mathcal{T}M), \mathcal{T}M) \mathfrak{J}(\mathcal{T}M), d))$. By Theorem 3.1, we know that all the deformations are trivial. It also implies that the tangent Lie algebroid is rigid.

Example 4.10. We consider the deformation of Lie algebroid $(A, [\cdot, \cdot], a)$ by a two cocycle $d = d_3 N$, where $N \in \Gamma(\mathfrak{gl}(A)) \subset \Gamma(\mathcal{D}A)$. For all $u, v \in \Gamma(A)$, we have

$$\mathcal{D}(du, dv) = d_3 N(du, dv) = [u, Nv] + [Nu, v] - N[u, v] = [u, v]_N.$$ 

If $N$ is a Nijenhuis operator, we can obtain the deformation of $A$ as follows:

$$[u, v]_t = [u, v] + t[u, v]_N, \quad a_t = a + ta \circ N, \quad \forall u, v \in \Gamma(A). \quad (28)$$

The deformation of the Lie algebroid $A$ by a Nijenhuis operator is trivial, i.e. $1 + tN$ is an isomorphism from the Lie algebroid $(A, [\cdot, \cdot], a_t)$ to the Lie algebroid $(A, [\cdot, \cdot], a)$.

In particular, if $A$ is the cotangent bundle Lie algebroid of some Poisson manifold, we can consider the compatibility condition of Poisson structures and Nijenhuis structures, i.e. Poisson-Nijenhuis structures. For more information about Poisson-Nijenhuis structures on oriented 3D-manifolds, see [4].

Example 4.11. For a Lie algebra $\mathfrak{g}$, there is a Lie-Poisson structure $\pi_1$ on $\mathfrak{g}^*$, we can consider the deformation of the corresponding Lie algebroid by a quadratic Poisson structure $\pi_2$. The corresponding 2-cocycle $\Omega_{\pi_2}$ is given by

$$\Omega_{\pi_2}(\xi, \eta) = L_{\pi_2}(\xi)\eta - L_{\pi_2}(\eta)\xi - d\pi_2(\xi, \eta), \quad \forall \, \xi, \eta \in \Omega^1(\mathfrak{g}^*).$$

Obviously, $\Omega_{\pi_2}$ defines a Lie bracket if and only if $\pi_2$ is a Poisson structure, $\Omega_{\pi_2}$ is closed is equivalent to the condition that $\pi_2$ and $\pi_1$ are compatible:

$$[\pi_1, \pi_2] = 0.$$ 

For more information about quadratic deformation of Lie-Poisson structures on $\mathbb{R}^3$, see [17].

Example 4.12. We consider a special deformation of a 4-dimensional Lie algebra $\mathfrak{h}$, which is the direct sum of a 3-dimensional Lie subalgebra $\mathfrak{g}$ and a 1-dimensional center $\mathbb{R}e$. As shown in Example 4.8, $H^\bullet(\mathfrak{J}\mathfrak{h}, \mathfrak{h})$ is just the Lie
algebra cohomology of $\mathfrak{h}$ with coefficient in the adjoint representation. For any $D \in \text{Der}(\mathfrak{g})$, $\Omega_D : \wedge^2 \mathfrak{h} \to \mathfrak{h}$ is given by

$$\Omega_D(X, Y) = 0; \quad \Omega_D(X, e) = De, \quad \forall X, Y \in \mathfrak{g}. \quad (29)$$

Obviously, $\Omega_D$ defines a Lie bracket and since $e$ is a center of $\mathfrak{h}$, we have $\Omega_D$ is closed if and only if $D$ is a derivation of $\mathfrak{g}$. Therefore, this problem can also be considered as the extension of a 3-dimensional Lie algebra by a derivation. In [20], the author gives a classification of such extensions using Poisson geometry method and therefore obtains the classification of 4-dimensional Lie algebras at the end (see also [13, 14]).

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References

[1] C. Arias Abad and M. Crainic, Representations up to homotopy of Lie algebroids, arXiv:0901.0319.
[2] A. Blaom, Geometric structures as deformed infinitesimal symmetries, Trans. Amer. Math. Soc., 358 (2006), 3651-3671.
[3] A. Blaom, Lie algebroids and Cartan’s method of equivalence, arXiv:math/0509071v3.
[4] B. Chen and Y. Sheng, Poisson-Nijenhuis structures on oriented 3D-manifolds, Rep. Math. Phys. 61 (3) (2008), 361-380.
[5] Z. Chen and Z. Liu, Omni-Lie algebroids, J. Geom. Phys. 60 (2010), 799-808.
[6] Z. Chen, Z. Liu and Y. Sheng, Dirac structures of omni-Lie algebroids, arXiv:0802.3819, to appear in International J. Math.
[7] Z. Chen, Z. Liu and Y. Sheng, $E$-Courant algebroids, Int. Math. Res. Not. Vol. 2010, No. 22, pp. 4334-4376.
[8] M. Crainic and R. L. Fernandes, Secondary characteristic classes of Lie algebroids. Quantum field theory and noncommutative geometry, 157–176, Lecture Notes in Phys., 662, Springer, Berlin, 2005.
[9] M. Crainic and I. Moerdijk, Deformation of Lie brackets: cohomological aspects, J. Eur. Math. Soc. (JEMS) 10 (2008), no. 4, 1037-1059.
[10] R. Fernandes, Lie algebroids, holonomy and characteristic classes. Adv. Math. 170 (2002), no. 1, 119–179.
[11] A. Fialowski, Deformations of Lie algebras, Math. USSR Sbornik, Vol. 55 (1986), No. 2, 467-473.
[12] A. Fialowski, An example of formal deformations of Lie algebras, NATO Conf. Proceedings, Kluwer (1988), 375-401.
[13] A. Fialowski and M. Penkava, Deformations of Four Dimensional Lie Algebras, Comm. Contemp. Math. 9 (2007), 41-79.
[14] A. Fialowski and M. Penkava, Moduli spaces of low dimensional real Lie algebras, *J. Math. Phys.* 49 (2008), 073507.

[15] K. Grabowska, J. Grabowski and P. Urbański, Lie brackets on affine bundles, *Ann. Global Anal. Geom.* 24 (2003), 101-130.

[16] A. Gracia-Saz and R.A. Mehta, Lie algebroid structures on double vector bundles and representation theory of Lie algebroids, *Adv. Math.* 223 (2010), 1236-1275.

[17] Q. Lin, Z. Liu and Y. Sheng, Quadratic deformations of Lie-Poisson structures on $\mathbb{R}^3$, *Lett. Math. Phys.* 83 (2008), 217-229.

[18] K. Mackenzie, *General theories of Lie groupoids and Lie algebroids*, Cambridge University Press, 2005.

[19] D. J. Saunders, *The Geometry of Jet Bundles*, Cambridge University Press, Cambridge, 1989.

[20] Y. Sheng, Linear Poisson structures on $\mathbb{R}^4$, *J. Geom. Phys.* 57 (2007), 2398-2410.

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