Lattices of lattice paths*

Luca Ferrari† Renzo Pinzani‡

Abstract

We consider posets of lattice paths (endowed with a natural order) and begin the study of such structures. We give an algebraic condition to recognize which ones of these posets are lattices. Next we study the class of Dyck lattices (i.e., lattices of Dyck paths) and give a recursive construction for them. The last section is devoted to the presentation of a couple of open problems.

Keywords - ECO method, lattice paths, posets.

1 Introduction

When a class of objects is introduced in mathematics, one of the first problems that naturally arises is to count how many objects there are. Typically, one recognizes an interesting numerical parameter and tries to enumerate the objects according to it, so obtaining a sequence of nonnegative integers. This is the main problem of enumerative combinatorics.

The second step (after mere enumeration) is to look for some “mathematical” structure (like, e.g., operations) the class of objects naturally possesses. As a matter of fact, one of the simplest structure that can be found is an order relation, as Gian-Carlo Rota suggested in his masterful paper [17]. By the way, his theory of Möbius functions of posets leads to many deep enumerative results, so proving that a better knowledge of the mathematical structure of a set gives a better insight even on his combinatorial and enumerative properties.

In this work we deal with classes of lattice paths (like, e.g., Dyck paths, Motzkin paths, Schröder paths, Łukasiewicz paths, etc.) which can be ordered in a completely natural manner. We postpone the formal definitions of these classes of paths at the beginning of section [2]. However, in fig. 1 we give

---

*This work was partially supported by MIUR project: Linguaggi formali e automi: teoria e applicazioni.
†Dipartimento di Matematica “U. Dini”, Viale Morgagni 67/A, 50135 Firenze, Italy ferrari@math.unifi.it
‡Dipartimento di Sistemi e Informatica, via Lombroso 6/17, 50135 Firenze, Italy pinzani@dsi.unifi.it
some instances of the paths which will more frequently occur in the present work. More precisely, we have drawn: (a) a Dyck path, (b) a Motzkin path, (c) a Schröder path and (d) a Lukasiewicz path.

Using a standard vocabulary, we say that Dyck paths only use rise (or \((1, 1)\)) steps and fall (or \((1, -1)\)) steps, whereas Motzkin and Schröder paths also use horizontal steps of length 1 (or \((1, 0)\)) steps and of length 2 (or \((2, 0)\)) steps), respectively. The case of Lukasiewicz paths is quite different, since they use an infinite set of possible steps, namely rise step of any type \((1, k)\) and (simple) fall steps of type \((1, -1)\).

Our goal is to begin a systematic study of the posets of paths arising in this way. A similar point of view has been undertaken in [1], where the authors study the posets arising from Delannoy paths; however, the present work deals with essentially different classes of paths. The only general problem we tackle here is to determine in which cases a poset of paths is a lattice, and we propose a possible solution to this problem. We also point out that the first works in this direction have been done by Narayana (see, for example, [14]); in [15] it is proved a very interesting result, described at the end of section 2. Next, we focus on the study of a single type of paths, namely Dyck paths, and we provide an explicit (recursive) construction for the lattices of Dyck paths. It is then easy to see that a suitable, slight modification of such a construction can be successfully applied also to lattices of Schröder paths. We remark that in [13, 12] some similar problems are studied: the authors provide the construction of the lattice of partitions of a given integer \(n\) (with the dominance order) starting from the knowledge of the lattice of partitions of \(n - 1\). However, the methods used in the present work are completely different from those employed in [13, 12].

Quite surprisingly, our study sheds new light on a method of enumeration, usually called ECO method (ECO stands for “Enumeration of Com-
binatorial Objects”), which has proved fruitful in many problems of enum-
meration. A rough description of this method is given in section (mainly
with the help of the example of Dyck paths). What is important to point
out here is that such a method provides (among other things) a partition of
the objects of size \( n \) of a given set of combinatorial objects (when a suitable
definition of size is provided); this fact is obviously crucial in trying to enu-
merate the objects of the class according to the size. What is typical of the
ECO method when applied to the construction of a class of paths (Dyck,
Motzkin, Schröder, and so on) is that the equivalence classes of the above
mentioned partition have a nice order structure: they are chains (i.e., totally
ordered sets of paths) with respect to the natural order of paths introduced
in this paper. So the ECO method, if applicable, provides a chain partition
of the lattice of paths under consideration.

Throughout the whole paper \( \mathbb{N} \) will denote the set of nonnegative in-
tegers and \( \mathbb{Z} \) the set of all integers. For \( x, y \in \mathbb{Z} \), the expression \([x, y]\) will
always indicate the interval of integers between \( x \) and \( y \).

Given a lattice \( L \), a filter of \( L \) is a subset \( F \) such that, if \( x, y \in F \), then
\( x \land y \in F \) (that is, \( F \) is closed for meet) and such that, if \( x \in F \) and \( x \leq y \),
then \( y \in F \). Dually, an ideal of \( L \) is a subset \( I \) such that, if \( x, y \in I \), then
\( x \lor y \in I \) (that is, \( F \) is closed for join) and such that, if \( x \in I \) and \( y \leq x \),
then \( y \in I \). If \( L \) has minimum \( 0 \), \( L \) is said to be ranked when there exists
a map \( r : L \rightarrow \mathbb{N} \) such that \( r(0) = 0 \) and, if \( y \) covers \( x \) (i.e., \( x < y \) and
there is no element \( z \) such that \( x < y < z \)), then \( r(y) = r(x) + 1 \). The
function \( r \) is called the rank function of \( L \). For any other poset and lattice
concept, we refer to the texts [7, 8]. In [18] a whole chapter is devoted to
the study of posets, especially in connection with enumerative and algebraic
combinatorics.

Our last remark concerns the use of the term “isomorphism” (and simi-
lar ones): in this paper it has to be considered a synonym of lattice isomor-
phism, i.e. a bijective map between two lattices which is both join-preserving
and meet-preserving.

2 Definitions and a first characterization

Let \( \Gamma \) be a finite subset of \( \mathbb{Z} \). We call \( \Gamma \)-path of length \( n \) every element
of the set:

\[
C_n^\Gamma = \{ f : [0, n] \rightarrow \mathbb{N} \mid f(0) = f(n) = 0; f(k + 1) - f(k) \in \Gamma, \forall k < n \}.
\]

The elements of the set \( C_n^\Gamma = \bigcup_n C_n^\Gamma \) are called \( \Gamma \)-paths. The elements
of the set \( \Gamma \) are called steps. This definition is completely equivalent to the
usual definition of a lattice path starting from the origin, ending on the

3
x-axis and using steps of a prescribed type. Observe that this definition allows us to consider only lattice paths whose steps have length 1 (that is, of the form \((1, k)\)); so, for example, Schröder paths do not fall within our definition (since horizontal steps are of type \((2, 0)\)). The following examples can be immediately checked by the reader:

- \{-1, 1\}-paths are the ordinary Dyck paths;
- \{-1, 0, 1\}-paths are the ordinary Motzkin paths.

It is possible to introduce a natural order between the elements of each set \(C_n^\Gamma\), by defining \(f \leq g\) in \(C_n^\Gamma\) whenever \(f(i) \leq g(i)\), for any \(i \leq n\). The fact that \([C_n^\Gamma; \leq]\) is a poset is immediate. Instead, it could be interesting to wonder whether such an order is a lattice order, that is: for which \(\Gamma \subseteq \mathbb{Z}\) the poset \([C_n^\Gamma; \leq]\) is a lattice? In this section we will be concerned precisely with this problem.

A first remark to be done is that, among the various lattices of \(\Gamma\)-paths, there are some cases which are particularly nice. Indeed, for some choices of \(\Gamma\), the meet and join operations induced on \(C_n^\Gamma\) by the order introduced above are defined pointwise, i.e.:

\[
(f \lor g)(i) = f(i) \lor g(i),
\]
\[
(f \land g)(i) = f(i) \land g(i),
\]

(2)
where the join and meet in the r.h.s. are computed in \(\mathbb{N}\). Therefore, the lattices arising in such cases are distributive. Unfortunately, this is not true in general, since for some choices of \(\Gamma\) it happens that \(C_\Gamma^n\) is actually a lattice but the operations are not defined as in (2). The following example will clarify this statement.

**Example.** Consider \(\Gamma = \{-1, 0, 2\}\). The poset \(C_4^\Gamma\) contains 5 paths and its Hasse diagram is depicted in fig. 2. As it is well-known, it is a lattice which is not distributive (it is not even modular), so lattice operations on such paths cannot be defined pointwise as in (2). Indeed, consider the two paths \(b\) and \(c\): their meet in \(C_4^\Gamma\) is the minimum 0 of the lattice, which is not the coordinatewise meet of the two paths.

The next example shows that there are also cases in which \(C_\Gamma^n\) is not a lattice.

**Example.** Take \(\Gamma = \{-1, 1, 2\}\), and consider the poset \(C_5^\Gamma\). Its Hasse diagram is the following:

![Fig.3](image)

which is not the Hasse diagram of a lattice.

Our next theorem will give an answer to the above problem in the case of lattices with coordinatewise meet and join. To state and prove our result we need to introduce some notation. We set \(\gamma_+ = \max \Gamma, \gamma_- = \min \Gamma\); it is clear that \(\gamma_+ \geq 0\) and \(\gamma_- \leq 0\) (that is, \(\Gamma\) must contain both up and down steps, otherwise the set of \(\Gamma\)-paths would be empty). We call diameter of \(\Gamma\) the difference between the maximum and the minimum of \(\Gamma\), that is \(\overline{\gamma} = \operatorname{diam} \Gamma = \gamma_+ - \gamma_-\). Finally, we define the set \(\Delta_\Gamma^n \subseteq \mathbb{Z}\) as follows (for any \(n \in \mathbb{N}\)):

\[
\Delta_\Gamma^n = \left\{ \sum_{i=1}^{n} (x_i - y_i) \mid x_i, y_i \in \Gamma, x_i \neq y_j, \forall i, j \right\},
\]

and we set \(\Delta_\Gamma = \bigcup_{n \in \mathbb{N}} \Delta_\Gamma^n\).

**Example.** Take \(\Gamma = \{-1, 2\}\). Then \(\Delta_2^\Gamma\) contains all the sums having \(n\) summands, each of the form \(2 - (-1) = 3\) or \(-1 - 2 = -3\). If we let \(n\) running over \(\mathbb{N}\), then we obtain \(\Delta_\Gamma = 3\mathbb{Z}\) (set of all integers divisible by 3).
Now we are ready to state our main result concerning the algebraic characterization of a particular class of lattices of lattice paths.

**Theorem 2.1** \( \mathcal{C}_\Gamma^\Gamma \) is a (finite) distributive lattice with respect to coordinate-wise meet and join if and only if

\[
(\Delta^\Gamma + \Gamma) \cap [\gamma_-, \gamma_+] \subseteq \Gamma.
\]  

*(4)*

**Proof.** It is clear that all the axioms of distributive lattices are satisfied by \( \mathcal{C}_\Gamma^\Gamma \). The only difficult thing to check in order to prove that \( \mathcal{C}_\Gamma^\Gamma \) is a lattice is that it is closed with respect to the coordinatewise join and meet operations described above, i.e., if \( f, g \in \mathcal{C}_\Gamma^\Gamma \), then \( f \wedge g, f \vee g \in \mathcal{C}_\Gamma^\Gamma \). So take \( f, g \in \mathcal{C}_\Gamma^\Gamma \), and suppose that, for a given \( k \in \mathbb{N} \), \( f(k) < g(k) \) and \( f(k+1) > g(k+1) \) (this is the only difficult case to study).

For suitable \( \gamma_1, \gamma_2 \in \Gamma \), we have

\[
f(k+1) = f(k) + \gamma_1, \\
g(k+1) = g(k) + \gamma_2.
\]

In order that the path \( f \vee g \) belongs to \( \Gamma \), we must impose that the step \( f(k+1) - g(k) \) is contained in \( \Gamma \). Now \( f(k+1) - g(k) = (f(k) - g(k)) + \gamma_1 \). Both \( f(k) \) and \( g(k) \) can be expressed as a sum of \( k \) elements of \( \Gamma \); more precisely, we can set:

\[
f(k) = \sum_{i=1}^{k} \tilde{x}_i, \quad g(k) = \sum_{i=1}^{k} \tilde{y}_i,
\]

where \( \tilde{x}_i, \tilde{y}_i \in \Gamma \). Thus we can express the difference \( f(k) - g(k) \) as a sum of differences between elements of \( \Gamma \); therefore, possibly deleting pairs of equal steps appearing both in the expressions of \( f(k) \) and \( g(k) \), for a suitable \( h \in \mathbb{N} \), we have:

\[
f(k) - g(k) = \sum_{i=1}^{h} (x_i - y_i),
\]

where \( x_i \neq y_j \), for every \( i, j \leq h \) (\( x_i, y_j \in \Gamma \)).

Now we are very close to completing our proof. Imposing that \( f(k+1) - g(k+1) = (f(k) - g(k)) + \gamma_1 \in \Gamma \) leads in general to the condition

\[
((\Delta^\Gamma \cap [\gamma_, \gamma_+]) + \Gamma) \cap [\gamma_-, \gamma_+] \subseteq \Gamma,
\]  

*(5)*

(recall the definition of the set \( \Delta^\Gamma \) given above). The reasons for which we have considered the intersection \( \Delta^\Gamma \cap [\gamma_, \gamma_+] \subseteq \Gamma \), are the following:

- \( f(k) - g(k) \) must be negative, since we have supposed that \( f(k) < g(k) \);
the paths $f$ and $g$ must cross, so the difference between $f(k)$ and $g(k)$ has to allow the crossing: this is why such difference must be greater than $-\gamma$.

Also the fact that we consider the intersection with $[\gamma-, \gamma+]$ needs an explanation. Since we are supposing there is a crossing, necessarily we have $f(k + 1) \geq g(k + 1)$ and $f(k) \leq g(k)$, whence:

$$\gamma_- \leq g(k + 1) - g(k) \leq f(k + 1) - g(k) \leq f(k + 1) - f(k) \leq \gamma_+.$$  

Thus, condition (5) is necessary for the existence of the join $f \vee g$ in $C^\Gamma_n$. Analogously, it can be shown that a similar condition must be verified for the existence of the meet $f \wedge g$, and precisely:

$$(\Delta^\Gamma \cap [0, \gamma]) + \Gamma \cap [\gamma-, \gamma+] \subseteq \Gamma. \quad (6)$$

Putting things together, we have proved that, if $C^\Gamma_n$ is a lattice, then:

$$(\Delta^\Gamma + \Gamma) \cap [\gamma-, \gamma+] \subseteq \Gamma. \quad (7)$$

Conversely, suppose that condition (4) is satisfied. Then, if the $\Gamma$-paths $f$ and $g$ cross between $k$ and $k + 1$, the fact that $f(k + 1) - g(k)$ and $g(k + 1) - f(k)$ both belong to $\Gamma$ is ensured by the hypothesis, and we are done. ■

Corollary 2.1 If $\Gamma$ is an interval (i.e. $\Gamma = [\gamma_1, \gamma_2]$), then $C^\Gamma_n$ is a lattice.

Proof. Indeed, in this case $\Gamma = [\gamma-, \gamma+]$, and so we get $(\Delta^\Gamma + \Gamma) \cap \Gamma \subseteq \Gamma$. ■

Corollary 2.2 If $\Gamma = \{-b, a\}$ ($a, b \in \mathbb{N}$), then $C^\Gamma_n$ is a lattice.

Proof. First observe that, in the assumed hypothesis, we have:

$$\Delta^\Gamma = \{(a + b)z \mid z \in \mathbb{Z}\},$$

and so

$$\Delta^\Gamma + \Gamma = \{(a + b)z + a, (a + b)z - b \mid z \in \mathbb{Z}\}.$$  

Now consider the set $(\Delta^\Gamma + \Gamma) \cap [-b, a]$, i.e. the elements of $\Delta^\Gamma + \Gamma$ contained in the integer interval $[-b, a]$. If $c$ is such an element then two cases are possible.

i) $c = (a + b)z + a$: then $z$ must necessarily be of the form

$$z = \frac{c - a}{a + b} \in \mathbb{Z};$$

the r. h. s. is an increasing function of $c$, moreover $c = -b$ implies that $z = -1$ and $c = a$ implies that $z = 0$. Since $z \in \mathbb{Z}$, this means that no value of $c$ in $[-b, a]$ is allowed other than $c = -b$ and $c = a$, and so $c \in \{-b, a\} = \Gamma$.  

7
ii) \( c = (a + b)z - b \): a similar argument leads to the same conclusion, i.e.
\[ c \in \{ -b, a \} = \Gamma. \]

**Remark.** Each of the previous results holds also if \( \Gamma \) is infinite (on the right, on the left, on both sides). In this case, we have \( \gamma_+ = +\infty \) and/or \( \gamma_- = -\infty \); to obtain the corresponding statements, one has to replace the interval \([\gamma_-, \gamma_+]\) in the text of theorem 2.1 with \([\gamma_-, +\infty[\), \([ -\infty, \gamma_+ ]\) or \(] - \infty, +\infty[\), depending on the cases.

**Examples.**

1. Motzkin paths of a given length \( n \) have a lattice structure. Indeed, as we have seen before, Motzkin paths correspond to \( \Gamma \)-paths for \( \Gamma = \{-1, 0, 1\} \), so in this case \( \Gamma \) is an integer interval and we can apply corollary 2.1.

2. Dyck paths are precisely \( \{-1, 1\}\)-paths, so, thanks to corollary 2.2, they constitute a lattice for every even length \( 2n \). Analogously, also \( \{2, -1\}\)-paths of any given length (necessarily of the form \( 3n \)) constitute a lattice.

3. Recalling the preceding remark, we can consider also paths with an infinite set of steps as, for example, the so-called Lukasiewicz paths (see, for instance, [2]). According to our vocabulary, they are \( \Gamma \)-paths for \( \Gamma = \{-1\} \cup \mathbb{N} \). By translating corollary 2.1 into its corresponding version for the infinite set \( \Gamma \), we can conclude that Lukasiewicz paths of any given length form a lattice.

**Remark.** As we have said at the beginning, in this paper we will be concerned with paths using exclusively single steps (i.e., of length 1). Anyway, any class of paths of the same length can be endowed with the natural order we have defined, therefore it makes sense to ask whether such an order is a lattice order. For example, it is natural to wonder whether Schröder paths of a given length possess a lattice structure. In this case, it is possible to show that the answer is positive, even if we prefer not to go into details. The reason for which everything works is that, if \( f, g \) are Schröder paths (of the same length) and \( g(k) - f(k) = 1 \), it means that either \( f \) or \( g \) possesses a double horizontal step having its middle point in \( k \), so \( f \) and \( g \) can not cross at \( k + 1 \).

In closing this section, we recall a work of Narayana and Fulton [15], in which it is shown that the set of the (minimal) lattice paths from the origin to \( (m, n) \) endowed with the **dominance order** is a distributive lattice. Setting \( m = n \) and translating into our language, this means that the set of Grand-Dyck paths (see [16] for the definition) with the order we have defined is a distributive lattice. To the best of our knowledge, this is the first result towards an order-theoretic investigation of classes of lattice paths.
3 Dyck lattices

In the previous section we have shown that a lot of classes of paths often used in combinatorics can be endowed with a lattice structure in a natural way. In this section we try to examine in details the lattice structure of a very well-known class of paths, namely Dyck paths. For a very detailed survey on Dyck paths and their enumeration, see [9].

We have just proved that, for any given $n \in \mathbb{N}$, Dyck paths of length $2n$ constitute a finite distributive lattice: we will denote it by $\mathcal{D}_n$. Throughout this whole section we will always denote the minimum and the maximum of $\mathcal{D}_n$ by $0$ and $1$, respectively (independently from $n$). In fig. 4 we have drawn the Hasse diagrams of $\mathcal{D}_n$ for small values of $n$.

![Hasse diagrams of $\mathcal{D}_n$](image)

Our goal is to give a description of the shape of the lattices $\mathcal{D}_n$ and then to provide an efficient way for constructing $\mathcal{D}_n$ starting from the knowledge of $\mathcal{D}_{n-1}$, in the same spirit of [13], where the same problem is solved (in a completely different way) for lattices of integer partitions.

The first thing we observe is that $\mathcal{D}_n$ has a rank function: this follows immediately from the fact that it is a distributive lattice (see [8]). The determination of the rank of an element of $\mathcal{D}_n$ is the object of the next proposition.

**Proposition 3.1** If we denote by $r_n$ the rank function of $\mathcal{D}_n$, then, given $f \in \mathcal{D}_n$, we have:

$$r_n(f) = \frac{\alpha(f) - n}{2},$$

where $\alpha(f)$ denotes the area of $f$, that is, by definition, the sum of the values of $f$:

$$\alpha(f) = \sum_{k=1}^{n} f(k).$$

9
Proof. The minimum \(0 \in \mathcal{D}_n\) is the function defined as follows (for \(k \leq 2n\)):
\[
0(k) = \begin{cases} 
0, & \text{if } k \text{ is even;} \\
1, & \text{if } k \text{ is odd.}
\end{cases}
\]

Thus we have \(r_n(0) = 0 = \frac{n-n}{2} = \frac{\alpha(0)-n}{2}\). Now, by induction, suppose that \(r_n(f) = \rho = \frac{\alpha(f)-n}{2}\) for a given \(f \in \mathcal{D}_n\), and take \(g \in \mathcal{D}_n\) such that \(f \prec g\) (this is a standard notation to mean that \(f\) is covered by \(g\)). Then, there exists precisely one \(k\) such that \(2 \leq k \leq 2n-2\), \(f(h) = g(h)\) for every \(h \neq k\) and \(g(k) = f(k) + 2\). Therefore, by the definition of area given above, \(\alpha(g) = \alpha(f) + 2\) and, by the definition of rank function,
\[
\alpha(g) - \frac{n}{2} = \frac{\alpha(g) - n}{2} = \frac{\alpha(f) - n}{2} + 1 = \frac{\alpha(f) + 2 - n}{2} = \frac{\alpha(g) - n}{2}.
\]

Remark. Thanks to the above proposition, we can assert that two Dyck paths of the same length have the same rank if and only if they have the same area.

There is a very classical and natural way of investigating the structure of finite distributive lattices. A famous result by Birkhoff asserts that each finite distributive lattice is isomorphic to the lattice of the down-sets (or order ideals) of the poset of its join-irreducible elements. In the case of Dyck lattices, join-irreducible elements are quite easy to describe: they are precisely those paths having exactly one hill of height \(>1\) (where by “hill” of height \(h\) we mean a sequence of \(h\) rise steps followed by a sequence of \(h\) fall steps). A careful analysis of the situation shows that the poset of the join-irreducible elements of \(\mathcal{D}_n\) is precisely the poset of the intervals of the \(n\)-element chain. For \(n = 4\), for instance, the Hasse diagram of this poset is the following:

![Fig.5](image)

This point of view will be considered in a future work [11], in which it will be applied to many classes of lattice paths.

Now we come to describe our construction of \(\mathcal{D}_{n+1}\) starting from \(\mathcal{D}_n\). The first thing to observe is that it is possible to determine a very natural chain partition of any Dyck lattice, suggested by a particular method.
of enumeration, called the \textit{ECO method}, which in this context reveals some unexpected peculiarities in exploring order-theoretic properties of a combinatorial structure.

The ECO method was introduced in a series of articles, due to Pinzani et al., in which they fruitfully apply a particular tool, namely \textit{succession rules} (deeply studied by West and others, see e.g. \cite{19}), to give a purely combinatorial construction performing a local recursive expansion of a given class of objects. Such a construction often allows to determine a functional equation satisfied by the generating function of the combinatorial structure under consideration. This, in turn, can be solved (in many cases) to give an explicit expression for the desired generating function. For a detailed survey concerning the ECO method, we suggest \cite{3}.

In this work, we are mainly interested in the ECO construction of Dyck paths and in its properties with respect to the order introduced on Dyck paths. So let us briefly recall how such construction works.

Consider a Dyck path $P$ of length $2n$, and suppose that the length of its last descent (i.e., of its last sequence of fall steps) is $k$. Then we construct $k + 1$ Dyck paths of length $2n + 2$ starting from $P$ (they will be called the \textit{sons} of $P$) simply by inserting a peak (i.e., a rise step followed by a fall step) in every point of its last descent. In fig. 6 it is shown in a concrete example how this construction works.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6.png}
\caption{Fig.6}
\end{figure}

By performing this construction on all Dyck paths of length $2n$, one obtains every Dyck path of length $2n + 2$ exactly once. Moreover, such a construction possesses strong recursive properties, which can be encoded by the following succession rule:

$$\Omega : \begin{cases} (2) \\ (k) \leadsto (2)(3) \ldots (k)(k + 1). \end{cases}$$  \hfill (9)
In the above expression it has to be intended that every Dyck path whose last descent has length \( k - 1 \) (encoded by the label \( (k) \)) produces \( k \) sons (through the above ECO construction) having last descents of length \( 1, 2, \ldots, k - 1, k \) respectively. A possible way of visualizing this succession rule is to draw its generating tree, that is the infinite rooted labelled tree in which the root is labelled \((2)\) and every node labelled \((k)\) has \((k)\) sons labelled \((2), (3), \ldots, (k), (k+1)\) respectively. Thus the number of Dyck paths of length \( 2n \) is precisely the number of nodes at level \( n \) of the generating tree. In this situation, we say that the sequence of Catalan numbers is the sequence defined by (or related to, or associated with) the rule (9).

The above ECO construction of Dyck paths provides a partition of the sets of Dyck paths of any fixed length. So one can consider the equivalence classes of those Dyck paths having the same father. The following result is stated without proof, since it is completely trivial; nevertheless, it contains the main order-theoretic feature of the ECO construction of Dyck paths.

**Theorem 3.1** The set of the sons of a Dyck path of length \( 2n - 2 \) forms a saturated chain in the lattice \( D_n \) (i.e. a chain such that for any two elements \( x \) and \( y \) in it, \( x < y \) implies that \( x \preceq y \)). Therefore, the ECO method provides a partition into saturated chains of every Dyck lattice, which we will refer to as the ECO-partition of \( D_n \).

In fig. 7 the ECO-partition of \( D_4 \) is drawn.

![Fig.7](image)

Observe that the rule in (9) is not the only possible rule defining Catalan numbers; in 4 many different rules for Catalan numbers are derived, and the same thing can be done for any other numerical sequence. However, it turns out that this is the "best" one from an order-theoretical point of view, meaning that the nice order structure of the equivalence classes gives a considerable help in the description of Dyck lattices.

To understand the structure of Dyck lattices it remains now to investigate how the saturated chains of the ECO-partition of \( D_n \) are linked together. This is precisely what we are going to do in the next pages.
First of all we describe another partition of $\mathcal{D}_n$ into sublattices (not necessarily chains) which follows in a very natural way from some elementary geometric properties of Dyck paths. More precisely, the sublattices of such a partition are certain filters of $\mathcal{D}_{n-1}$.

Denote by $\mathcal{D}_{nk}$ the set of all Dyck paths starting with precisely $k$ rise steps (so that the $(k+1)$-st step is a fall step). In symbols:

$$\mathcal{D}_{nk} = \{ f \in \mathcal{D}_n \mid \forall i \leq k, f(i) = i, f(k+1) = k-1 \}.$$  \hspace{1cm} (10)

This definition is meaningful only when $k \leq n$. In particular, $\mathcal{D}_{n0} = \emptyset$ and $\mathcal{D}_{nn} = \{ 1 \}$. In fig. 8 it is shown how a path in $\mathcal{D}_{n5}$ starts.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig8.png}
\caption{Fig.8}
\end{figure}

**Proposition 3.2** For every $k \leq n$, $\mathcal{D}_{nk}$ is a sublattice of $\mathcal{D}_n$. Moreover, the family of sublattices $\{ \mathcal{D}_{nk} \mid 0 < k \leq n \}$ constitutes a partition of $\mathcal{D}_n$.

*Proof.* Observe that, if two paths starts with the same sequence of steps, then this happens for their join and meet as well: hence the first part of the proposition follows. The second part is obvious, since the starting sequence of rise steps is uniquely determined for every Dyck path. \hspace{1cm} ■

In particular, for $\mathcal{D}_{n1}$ we have the following, nice result.

**Proposition 3.3** $\mathcal{D}_{n1} \simeq \mathcal{D}_{n-1}$.

*Proof.* Consider the function mapping each element $f \in \mathcal{D}_{n1}$ into the element of $\mathcal{D}_{n-1}$ obtained by removing the first two steps of $f$: this is a bijection which preserves joins and meets. \hspace{1cm} ■

The above, very simple proposition states that, for every $n$, $\mathcal{D}_n$ contains a copy of $\mathcal{D}_{n-1}$, which is precisely the ideal of the elements of $\mathcal{D}_n$ starting with a peak.

Now consider the sublattice $\mathcal{D}_{nk}$ of $\mathcal{D}_n$, for $k > 1$. The next proposition asserts that this lattice “lives” also inside $\mathcal{D}_{n-1}$ as a particular filter.
Proposition 3.4 Denote by $F_{(n-1)(k-1)}$ the subset of $D_{n-1}$ of all Dyck paths starting with $k - 1$ rise steps (without any further hypothesis on the $k$-th step). Then $F_{(n-1)(k-1)}$ is a filter, and is isomorphic to $D_{nk}$.

Proof. The fact that $F_{(n-1)(k-1)}$ is a filter of $D_{n-1}$ is immediate (the argument is the same as that of proposition 3.2). Now consider any path in $D_{nk}$ and delete its first peak (which, by definition, occurs precisely after the first $k - 1$ rise steps). In this way we obtain a path of $D_{n-1}$ starting with $k - 1$ rise steps, that is an element of $F_{(n-1)(k-1)}$. This correspondence is a meet- and join-preserving bijection between $D_{nk}$ and $F_{(n-1)(k-1)}$, so the proof is complete. ■

Remark. Notice that, as a filter of $D_{n-1}$, $F_{(n-1)(k-1)}$ is the principal filter generated by the least Dyck path starting with a hill of height $k - 1$, in the sense that it contains precisely all the paths greater or equal than the above mentioned one.

Next we show how the sublattices $D_{nk}$ are linked together to form the whole $D_n$.

Proposition 3.5 Consider the function $\varphi_{nk} : D_{n(k+1)} \longrightarrow D_{nk}$ defined as follows. Let $f \in D_{n(k+1)}$, so that $f(k+1) = k+1$, $f(k+2) = k$; we set by definition $\varphi_{nk}(f) = g \in D_{nk}$, where $f(i) = g(i)$, $\forall i \neq k + 1$, and $g(k+1) = k - 1$. Then $\varphi_{nk}$ is a lattice monomorphism (i.e., injective homomorphism). Moreover, if $g = \varphi_{nk}(f)$, then $f \prec g$ in $D_n$.

Proof. First observe that $\varphi_{nk}$ is well-defined, since $\varphi_{nk}(f)$ is an element of $D_{nk}$. If $\varphi_{nk}(f_1) = \varphi_{nk}(f_2)$, then $f_1(i) = f_2(i)$ at least for every $i \neq k + 1$; moreover $f_1(k + 1) = k + 1 = f_2(k + 1)$ (since $f_1, f_2 \in D_{n(k+1)}$), so that $f_1 \equiv f_2$. Now consider $f, f' \in D_{n(k+1)}$, so that $f$ and $f'$ coincide for $i \leq k + 2$. Then also $f \lor f'$ coincides with both $f$ and $f'$ for $i \leq k + 2$. Therefore, for $i \neq k + 1$, we have $\varphi_{nk}(f \lor f')(i) = (f \lor f')(i) = (\varphi_{nk}(f) \lor \varphi_{nk}(f'))(i)$, whereas $\varphi_{nk}(f \lor f')(k + 1) = k - 1 = (\varphi_{nk}(f) \lor \varphi_{nk}(f'))(k + 1)$, and so $\varphi_{nk}(f \lor f') = \varphi_{nk}(f) \lor \varphi_{nk}(f')$. The same argument can be used also for meet. Finally, we have that $f \prec \varphi_{nk}(f)$, since $\alpha(f) + 2 = \alpha(\varphi_{nk}(f))$ and $f \prec g$ (recall that $\alpha(f)$ denotes the area of $f$, as defined in proposition 3.1). ■

Fig. 9 illustrates how the map $\varphi_{n3}$ works.

Fig. 9
Remark. Observe that $\varphi_{nk}(D_{n(k+1)})$ is a filter of $D_{nk}$. More precisely, it is the filter of all the paths of $D_{nk}$ whose $(k + 2)$-nd step is a rise step (the reader can check that this is actually a filter of $D_{nk}$).

Let us summarize what we have done until now. We have discovered a special partition of $D_n$ into sublattices (the $D_{nk}$'s). We have shown that a certain filter of $D_{n-1}$ is isomorphic to $D_{nk}$, which, in turn, is isomorphic to a filter of $D_{n(k-1)}$. As a consequence of this fact, we have that $D_n$ can be constructed starting from $D_{n-1}$ or, which is the same, from $D_{n1}$ by gluing together some of its filters. Notice that this is all we need to draw the Hasse diagram of $D_n$, since $x \prec y$ in $D_n$ if and only if either $x \prec y$ in $D_{nk}$, for some $k$, or $x \in D_{nk}$, $y \in D_{n(k+1)}$ and $y = \varphi_{nk}(x)$.

Now, to give a complete formalization of our construction we need to define a particular operation on lattices.

Given a lattice $L$, let $F$ be a filter of $L$. We define the $F$-filtered doubling of $L$ to be the lattice $L \times F \equiv \{(x, n) \in L \times \{0, 1\} \mid n = 1 \Rightarrow x \in F\}$ (11) endowed with coordinatewise meet and join.

Remark. We wish to point out that this construction is not new at all. In fact, this is merely a particular instance of a subdirect product of lattices. Recall that a subdirect product of two lattices $L$ and $M$ is, by definition, any sublattice $N$ of the direct product $L \times M$ such that both the projections of $N$ onto $L$ and $M$ are surjective homomorphisms (see, for example, [7]). So, an $F$-filtered doubling of $L$ is nothing more than a particular subdirect product of the lattices $L$ and $2 = \{0, 1\}; \lor, \land$. By the way, this also proves that $L \times F \equiv L \times 2$.

Now consider, for every positive integer $k < n$, all the filters $F_{(n-1)k}$ of $D_{n-1}$ as defined above. So, for example, $F_{(n-1)1}$ is the whole lattice $D_{n-1}$ and $F_{(n-1)(n-1)}$ is the singleton of the maximum of $D_{n-1}$. We introduce the following recursive notation:

$$\beta_1(D_{n-1}) = D_{n-1} \times F_{(n-1)1} \equiv D_{n-1} \times 2,$$

$$\beta_k(D_{n-1}) = \beta_{k-1}(D_{n-1}) \times F_{(n-1)k} \equiv \beta_{k-1}(D_{n-1}) \times 2, \quad \text{for } k < n.$$

Observe that the operations performed make sense since, in general, $F_{(n-1)k}$ is a filter of $\beta_{k-1}(D_{n-1})$ (this is a consequence of propositions 3.4 and 3.5).

The following theorem condenses in a single formula the whole construction of $D_n$ from $D_{n-1}$ described in this section, making use of the notations introduced above.

1In particular, $D_{n1}$ is isomorphic to the whole $D_{n-1}$.
Theorem 3.2 $D_n = \beta_{n-1}(D_{n-1})$.

3.1 Schröder lattices

As we have said in section 2, Schröder paths do not fall within the class of paths captured by our definition. Recall that a Schröder path is a lattice path starting from the origin, ending on the $x$-axis, never falling below the $x$-axis and using steps of the type $(1,1)$ (rise), $(1,-1)$ (fall) and $(2,0)$ (double horizontal). Nevertheless, they can be naturally ordered in the same way, and it can be shown that they constitute a lattice. It turns out that the study of Schröder lattices follows essentially the same lines as those for Dyck lattices. Also in this case, there is a privileged rule among all those defining Schröder numbers, which is precisely

$$\Omega : \left\{ \begin{array}{c} \binom{2}{2} \\
(2k) \rightarrow (2)(4)^2(6)^2 \ldots (2k - 2)^2(2k)^2(2k + 2). \end{array} \right\}$$

(12)

The ECO-construction of Schröder paths encoded by the above rule provides a partition of any Schröder lattice $S_n$ into saturated chain. Besides, the construction of $S_n$ starting from $S_{n-1}$ can be carried out in an analogous way to Dyck lattices: the interested reader will find no difficulties in reproducing it. In fig. 10 we give the Hasse diagram of $S_3$ (lattice of Schröder paths of length 6).

Fig.10

16
4 Conclusions and open problems

In this paper we would like to begin a deep investigation on the order-theoretic aspects of the ECO method, so it should be intended only as a first step in this direction. We hope that the present work will be followed soon by analogous ones concerning several other classes of combinatorial objects naturally definable by means of some ECO-construction. As far as paths are concerned, the following problems seems to be particularly challenging.

1. Recall that a finite sequence of numbers \( a_0, \ldots, a_n \) is said to be unimodal whenever there exists an index \( k \) such that \( a_0 \leq a_1 \leq \ldots \leq a_k \) and \( a_k \geq a_{k+1} \geq \ldots \geq a_n \). It seems natural to think that Dyck lattices are rank-unimodal (i.e. the sequence given by the elements of the same rank is unimodal), even if we have not succeeded in proving it yet. This problem was also mentioned in [5], both for Dyck and Schröder lattices, where it is presented as an “intriguing open question”. Obviously the same question can be asked for any other lattice of paths. In [6] the author studies problems related with the area and the moments of Dyck paths, and in [10], exercise 5.2.12 gives formulas for the area under various paths: these references could be of some help in tackling this problem.

2. The approach used in the study of Dyck and Schröder lattices does not work for Motzkin lattices. In particular, the ECO decomposition for Motzkin lattices does not provide a good description, as it happens for Dyck and Schröder lattices. In fact, it is still true that the sons of a Motzkin path constitute a chain of the Motzkin lattice they belong to, but such a chain is not saturated (see [3] for a precise description of the usual ECO construction of Motzkin paths). Therefore we are not helped by the ECO method in drawing the Hasse diagrams of Motzkin lattices. So the study of such lattices (as well as of other lattices of paths of some interest in combinatorics) should be done, maybe using different techniques from those developed in the present work.

Acknowledgments. The authors wish to thank Sri Gopal Mohanty for drawing their attention to the references [14, 15]. They also thank the referees for useful suggestions and comments.

References

[1] J.-M. Autebert, M. Latapy, S. R. Schwer  Le treillis des chemins de Delannoy, Discr. Math. 258, 225-234, 2002.

[2] C. Banderier, P. Flajolet Basic analytic combinatorics of directed lattice paths, Theoret. Comput. Sci. 281, 37-80, 2002.

17
[3] E. Barcucci, A. Del Lungo E. Pergola, R. Pinzani  
*ECO: a methodology for the Enumeration of Combinatorial Objects*,  
J. Difference Eq. Appl. 5, 435-490, 1999.

[4] S. Brlek, E. Duchi, E. Pergola, S. Rinaldi  
*On the equivalence problem for succession rules*,  
preprint.

[5] J. Bonin, L. Shapiro, R. Simion  
*Some q-analogues of the Schröder numbers arising from combinatorial statistics on lattice paths*,  
J. Statis. Plann. Inference 34, 35-55, 1993.

[6] R. Chapman  
*Moments of Dyck paths*,  
Discr. Math. 204, 113-117, 1999.

[7] P. Crawley, R. P. Dilworth  
*Algebraic Theory of Lattices*,  
Prentice-Hall, N.J., 1973.

[8] B. A. Davey, H. A. Priestley  
*Introduction to Lattices and Order*,  
Cambridge University Press, Cambridge, 1990.

[9] E. Deutsch  
*Dyck path enumeration*,  
Discr. Math. 204, 167-202, 1999.

[10] I. P. Goulden, D. M. Jackson  
*Combinatorial Enumeration*,  
Wiley, New York, 1983.

[11] L. Ferrari, E. Munarini  
*On the representation theory for path lattices*,  
in preparation.

[12] M. Latapy  
*Partitions of an integer into powers*,  
Discr. Math. Theor. Comp. Sci. AA, 215-228, 2001.

[13] M. Latapy, H. D. Phan  
*The lattice of integer partitions and its infinite extension*,  
Discr. Math. Theor. Comp. Sci.  (to appear).

[14] T. V. Narayana  
*Lattice Path Combinatorics with Statistical Applications*,  
University of Toronto Press, Toronto, 1979.

[15] T. V. Narayana, G. E. Fulton  
*A note on the compositions of an integer*,  
Canad. Math. Bull. 1, 169-173, 1958.

[16] E. Pergola  
*Two bijections for the area of Dyck paths*,  
Discr. Math. 241, 435-447, 2001.

[17] G.-C. Rota  
*On the foundations of Combinatorial Theory I: Theory of Möbius functions*,  
Z. Wahrscheinlichkeitsrechnung u. verw. Geb. 2, 340-368, 1964.

[18] R. P. Stanley  
*Enumerative Combinatorics, vol. 1*,  
Wadsworth, Belmont, CA, 1986.
[19] J. West *Generating trees and the Catalan and Schröder numbers*,
Discr. Math. 146, 247-262, 1995.