Implications of Human Irrationality for Reinforcement Learning

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Abstract

Recent work in the behavioural sciences has begun to overturn the long-held belief that human decision making is irrational, suboptimal and subject to biases. This turn to the rational suggests that human decision making may be a better source of ideas for constraining how machine learning problems are defined than would otherwise be the case. One promising idea concerns human decision making that is dependent on apparently irrelevant aspects of the choice context. Previous work has shown that by taking into account choice context and making relational observations, people can maximize expected value. Other work has shown that Partially observable Markov decision processes (POMDPs) are a useful way to formulate human-like decision problems. Here, we propose a novel POMDP model for contextual choice tasks and show that, despite the apparent irrationalities, a reinforcement learner can take advantage of the way that humans make decisions. We suggest that human irrationalities may offer a productive source of inspiration for improving the design of AI architectures and machine learning methods.

1 Introduction

Humans have long been a source of inspiration for how to build intelligent machines [1][2]. There is now a series of successful examples of where knowledge about the brain and mind has been used to develop new types of Machine Learning (ML), including artificial neural networks [3], convolutional neural networks inspired in part by the hierarchical organization of vision [4], and Reinforcement Learning (RL) which was inspired by decision making and learning under uncertainty in humans and other animals [5]. A more recent example is provided by the promise of the utility of uncertainty, which has demonstrated that incorporating human-like uncertainty about object classifications can help obtain more robust and better performing machine classification [6]. Many recent advances have come from modeling uncertainty in ML. For example, capturing uncertainty can improve model performance in regression and classification tasks [7][8], estimating uncertainty can improve deep learning algorithms [9][11], and representing the uncertainty of an agent’s policy can aid more efficient exploration in RL [12][14]. Another example of the influence of the human sciences on ML is how selective attention in human perception and neural information processing, has motivated rapid progress in object recognition [15], visual object tracking [16][18], human action recognition [19], image caption generation [20], and machine translation [21][22]. In sum, progress on multiple fronts suggests that human cognition offers a productive source of inspiration for improving ML.

However, perhaps not every aspect of human cognition should be emulated. One such aspect might be human choice behavior which has generated a long list of cognitive biases [23], leading to systematic and predictable errors. For example, humans supposedly rely too heavily on the first piece of evidence gathered (the anchoring bias), they heavily weight decisions towards more recent information (the
availability bias), and they overweight the value of outcomes that are expected to definitely occur (the certainty bias). At the last count, in a decision-making textbook, 24 lists 53 irrationalities. Together, these biases have been taken to suggest that people are not good expected value maximizers and that they are subject to irrationalities of choice that are counter to their own self-interest.

A highly critical appraisal of the “heuristics-and-biases” tradition has recently been provided by 25. Indeed, recent research has begun to show that people may be more rational than supposed 26–30. One type of bias in humans is known as a context-of-choice bias in which irrelevant options influence decisions. For example, if a person chooses an apple over a cake on the grounds of health, but then chooses the same cake when the choice is between an apple, the cake and another cake with extra sugar, then the clearly inferior (on health grounds) “cake with extra sugar” has influenced the choice between two superior alternatives. This is an example of where human choice between two options changes when a third (dominated) alternative is introduced and there is significant empirical evidence demonstrating this phenomenon 31. As the dominated choice is irrelevant to the choice between the other two options, it should have no effect on their valuation, nor on the choice. This effect has been taken by some as evidence that human cognition is irrational since it appears to violate the normative principles of independence 32–34. However, 35 has shown that these apparent irrationalities can emerge from computationally rational mechanisms.

In the current paper, we propose an agent inspired by 35 demonstration that apparent irrationalities of choice can emerge from rational processing. Our approach is an example of a broader class of analysis known as Computational Rationality 26, 36, 28. It extends 35 by modeling contextual choice tasks as sequential decision problems and formulating them as Partially Observable Markov Decision Processes (POMDPs). Previous work by 37, 38, 30, 39, 40 and others has established the value of POMDPs and related formalisms for modeling humans. In our work, a reinforcement learning agent, designed to solve a POMDP, acquires a sequential decision policy that chooses what information to gather about which options, calculates option values, and makes comparisons between options as the unfolding task demands. The agent is trained and tested on sampled choices between three gambles, each expressed as a probability and a value. The agent learns the relative value of (1) noisy calculation of option values (e.g. by multiplication of a probability by a value), (2) noisy comparisons (e.g. comparing two probabilities to see which option is riskier), and (3) acting (making a choice). The agent is not pre-programmed to gather all information but learns to gather only that information that helps it maximize utility. We contrast this agent to other simpler agents and show that the human-inspired agent performs better (achieves higher cumulative reward) than an agent that makes independent assessments of each option value, replicating the results of 35 but in the POMDP setting.

Our analysis of the new agent’s learned policy shows that it learns to use contextual information to help infer which options approximately maximize expected value while taking into account computational cost and cognitive limits. The agent’s performance shows that making use of contextual information helps it make more accurate and efficient decisions under uncertainty while also giving rise to apparently irrational and human-like decision making 35. The work demonstrates that, under choice uncertainty, there is economic value to ML agents of policies that make use of choice context and relational judgments.

This paper’s contributions are as follows:

- It replicates the previous findings of 35 with a new problem formulation based on POMDPs; it shows that preference reversals emerge from learning in a POMDP setting as well as in the exhaustive search setting reported by 35.
- It provides further evidence that cognitive biases can inform ML. Model simulations demonstrate that it is more profitable for an RL agent to take into account context and relational judgments when choosing between uncertain options than to make independent evaluations of each option, suggesting that in some circumstances RL agents should be designed with the capacity to compare options.
- It extends the analysis of 35 to account for the impact of information gathering costs on contextual choice.
- It makes novel predictions concerning optimal sequential information gathering in contextual choice tasks. In particular, it shows how the ratio of option comparisons and expected value calculations is influenced by the level of uncertainty in the observation functions.
The paper is organised as follows, we first describe experiments revealing contextual choice effects in human decision making and review theoretical accounts of human choice from the cognitive science and neuroscience perspectives. We then define a contextual choice problem as a POMDP that includes “comparison” observations and describe how to solve this POMDP with a reinforcement learning agent. We test the agent on gamble tasks for which humans are known to be influenced by choice context, and we demonstrate the correspondence between the approximately reward maximizing RL behavior and human behavior - replicating [35]. Lastly, we analyse the agent’s sequential behaviour and reveal the effect of observation noise on the frequency with which it uses option comparisons.

2 Background

2.1 The Effect of Choice Context on Humans

As we have said, the human behaviours that influence this paper are those exhibited in decision-making tasks in which people appear biased by seemingly irrelevant context. Here we look in more detail at these tasks and their effects. Contextual decision experiments have revealed many robust empirical effects and contributed to shaping cognitive theories of human decision making [32, 33, 35, 41–44]. Three of the most well known contextual decision task are the attraction, compromise and similarity tasks. These are illustrated in Figure 1a, b, c. For the attraction type task, there are two best options (the Target and the Competitor) with the very similar expected value. Each option is best on one dimension but not the other. One of these two options (the Target) dominates a third option, called the decoy, on both dimensions. It is difficult to choose between the two best options since each option dominates the other on one of the attributes. Experiments studying these three tasks have been reported by many authors. Consider the results of an experiment in which participants were asked to make decisions about criminal suspects [45]. Participants were presented with a sequence of tasks each consisting of three suspects and were asked to decide which suspect was most likely to have committed a crime. There were two types of evidence, of varying strength, about each of the three suspects, such that the suspects had likelihoods of criminality in patterns identical to the three patterns presented in Figure 1d. These three patterns were used as the materials in the three conditions of the experiment.

In the attraction condition of the experiment, there were two equally likely criminal suspects and a decoy suspect who was less likely than the other two (Figure 1a). The experimental results showed that the Target suspect who dominates the decoy was chosen more frequently than the Competitor suspect. In the compromise condition of the experiment (Figure 1b) the findings showed that the suspect who is in-between the Competitor and decoy is chosen more frequently than the Competitor. In the similarity condition (Figure 1c), the results showed that the suspect who is very similar to the decoy is chosen less frequently than the Competitor.

Figure 1: (a)(b)(c) An illustration of the options in three types of contextual choice task – called the attraction (a), compromise (b) and similarity (c) tasks. The Target T and Competitor C are two options and have equal expected value (the dotted line). Option D is a decoy designed for comparison with the Target T. In the attraction task (a), T dominates D. In the compromise task (b), T is a compromise between D and T. In the similarity task (c), D has similar expected value to T. (d) Proportion of choices of each of the three options (Target, Competitor and Decoy) in each of the three contextual choice tasks (Attraction, Compromise and Similarity). The Target is preferred in the Attraction and Compromise tasks and the Competitor is preferred in the Similarity task. Data are reproduced from [45].
Human behaviour on these tasks has been seen as biased because the sensitivity to irrelevant context (the decoy option) appears to have consequences for the choice between the other two options (34, p. 1188). The most commonly used operationalization of irrationality among decision researchers has been based on violations of value maximization. Preferring a dominated option or expressing different preferences depending on the framing of options demonstrates irrational decisions. The significance of any irrationality, if that is what they are, cannot be understated given the potential for catastrophic real world consequences. However, the conclusion that choice under uncertainty provides evidence of irrationality may be incorrect (35, 36). Substantive analysis of the value of comparing options has shown that they are in fact informative and are required, under conditions of uncertainty, for reward maximization (35). The substantive structure of these analyses has informed the design of the agent that we present below. The key cognitive strategy that is borrowed from human behaviour is the use of option comparison to inform decision making under uncertainty. Comparison was extensively explored by Stewart (47, 48) who has documented extensive of its use in a range of human decision making tasks. For example, there is eye tracking evidence (49) that people tend to make more eye movements that switch between options than eye movements that gather all of the evidence about a single option; evidence which is consistent with the use of comparisons.

2.2 Human Decision Making as a POMDP

POMDPs provide a mathematical framework for sequential decision processes (50). POMDPs have previously been used for modelling and explaining various aspects of human decision making (51, 52, 53). An early contribution was (51)’s model of the dopamine system which incorporated semi-Markov dynamics and partial observability. (53) proposed a model of neural information processing based on POMDPs and tested this model on perceptual tasks such as the random dot motion task. Further work in perceptual decision making, has used the POMDP framing to explore model confidence (52) and understand the role of priors (53). POMDPs have even been used to model social decision making (53). More recently, meta-level Markov decision processes (meta-MDP), a closely related framework, have been used for modelling higher level decision making (54). The Meta-MDP model is similar to the belief-MDP version of the POMDP, but replacing physical actions with cognitive operations. Meta-MDPs have been used to model strategy selection and heuristics in decision making (39) and attention allocation in perception (56).

Contextual preference reversals have influenced a number of models of human decision making (32, 33, 41, 30, 57, 43, 42, 41). Many of these models have focused on neurally plausible sequential processing, capturing the fact that decision making usually requires accumulation of evidence and integration of information across time (46). Other models have focused on the way that people solve this problem by sampling comparisons between option attributes and thereby impose a rank order on options (42). However, none to our knowledge, have shown that preference reversals are an emergent consequence of an RL solution to a POMDP.

3 Contextual Choice as a POMDP

We view contextual choice tasks as sequential decision making problems and formulate them as POMDPs that include, in the action space, comparison actions to assess choice option values. Given this formulation, we use a deep reinforcement learning model to discover an approximately optimal choice policy and demonstrate its capacity to simultaneously maximize reward and model humans. A consequence of an RL solution to a POMDP.

\[ s_t \sim T(s_{t+1} | s_t, a_t) \]. Then, to gather information about the state, the agent makes a partial observation \( o_{t+1} \in O \) according to the distribution \( o_{t+1} \sim Z(o_{t+1} | s_{t+1}, a_t) \). The agent received a reward \( r_{t+1} \in R \) according to the distribution \( r_{t+1} \sim R(o_{t+1} | s_{t+1}, a_t) \) after performing an action \( a_t \) in a particular state \( s_{t+1} \). The agent must rely on its observations to inform action selection since the hidden states are not directly observable. In each time step \( t \), the agent acts according to its policy \( \pi(a_t | h_t) \) which returns the probability of executing action \( a_t \), and where \( h_t = (o_0, a_0, o_1, a_1, \ldots, o_{t-1}, a_{t-1}) \) are the
histories of observations-actions pairs. The goal of the agent is to learn an optimal policy \( \pi^* \) that maximizes the expected cumulative rewards, \( \pi^* = \arg \max_{\pi} \mathbb{E} \left[ \sum_{t=0}^{T} \gamma^{t-1} r_t \right] \), where \( 0 < \gamma < 1 \) is the discount factor.

Each choice task had 3 options (X, Y, Z) which were represented with two attributes: a randomly sampled probability \( p \) and a randomly sampled value \( v \). We assumed that probabilities \( p \) were sampled from a \( \beta \)-distribution and values \( v \) were sampled from a \( t \)-distribution. These distributions represented the ecological distributions experienced by participants in the human behaviour experiments reported by [31]. We view contextual choice tasks as sequential decision making problems and formulate them as POMDPs as follows.

The state space \( S \) for each task was generated from a sampled choice task. More formally a state was \( \{(p_X, v_X), (p_Y, v_Y), (p_Z, v_Z)\} \), where probabilities \( p \) were sampled from a \( \beta \)-distribution and values \( v \) were sampled from a \( t \)-distribution. The agent selected actions from a set \( A \) which included 6 comparison actions (e.g. compute the comparison relation for \( p_X \) and \( p_Y \)), 3 calculation actions (e.g. compute the expected value for \( X \) given \( p_X \) and \( v_X \)) and the 3 choice actions (choose X, choose Y, choose Z). The reward for comparison and calculation actions was negative \( c \). The reward for a choice action was 10 if the agent chose the option with maximum expected value, otherwise, it was -10. There was therefore a trade-off between the cost of information gathering and choice accuracy. More information cost more but was more likely to lead to a better response and therefore a higher reward. As a consequence of the selected action, the subsequent observation \( o_{t+1} \) was of computing the most recent comparison or calculation with noise. Following [35] each observation of a comparison had 4 possible outcomes, which indicated that the relation was unknown, greater, equal and less. The function \( f \) represents this pairwise order relation between the two values or two probabilities of two gambles. The probability of comparison error \( P(error) \) was the probability that the relations were sampled uniformly random from the comparison set \( O = \{>, \equiv, <\} \). The observation of a calculation was computed using:

\[
E_i = p_i^\alpha \times v_i + \varepsilon \quad \varepsilon \sim N(0, \sigma^2_{\text{calc}})
\]  

(1)

where the probability \( p \) was weighted by an exponential parameter \( \alpha \). The purpose of using parameter \( \alpha \) is to model subjective probability following [58]. The exponential is used to model subjective probability is extensively in econometrics because it is mathematically well behaved.

The evidence state is the history of the partial and noisy observation of the latent state. The history of observation set \( O_h \) is the noisy encoding of the partial orderings of probabilities and values:

\[
O_h = \{f(p_X, p_Y), f(p_X, p_Z), f(p_Y, p_Z), f(v_X, v_Y), f(v_X, v_Z), f(v_Y, v_Z), E_X, E_Y, E_Z\}
\]  

(2)

It is intractable to compute a policy to solve the defined POMDP, but it is possible to approximate the optimum through learning [59][61]. We solve the POMDP by casting it as a Markov Decision Process (MDP) whose state space is the history of observation \( o_h \). We used a deep reinforcement learning method, called ACER, to find an approximately optimal policy for the POMDP [62]. For all reported experiments, we built the environments within OpenAI Gym [63] and used the OpenAI Baselines implementation of the deep RL algorithms.

4 Results

In order to test the model, we designed three different agents: The integrated agent could use both calculation and comparison selectively. States represent the results of calculation and comparison actions. The model can learn which observations are useful and not every observation needs to be made. There is no explicit integration of comparison and calculation. Instead, the results of comparison and calculation accumulate in the history and choice action values are conditional on these histories. The comparison-only agent is same as the integrated agent but could only use comparison actions, and there are states only represent the comparison information. The calculation-only agent is same as the integrated agent but could only use calculation actions and, there are states only represent the calculation information. The difference between the three models is the availability to use two kinds of observation information. All three agents learnt approximately optimal policies from experience.

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1https://github.com/openai/baselines
Figure 2: The mean expected value obtained by agents with different levels of noise: the coefficient of variation for the calculation noise (left panel) and the probability of comparison error (right panel). In the left panel, the comparison noise is fixed at $P_{\text{error}} = 0.3$. In the right panel, the calculation noise is fixed at $\delta_{\text{calc}} = 30$, that the coefficient of variation is 0.3. Results for 3 types of agent are presented in each panel: the comparison-only agent (green-doted line), calculation-only agent (blue-doted line) and integrated agent (black-doted line). This Figure replicates Figure 3 in [35].

Figure 3: The integrated agent exhibits the attraction effect. A sample of agents was tested on each of four variants of the attraction effect task (in which the decoy is in slightly different positions). People and agent exhibit more target choices than Competitor choices in task sets 1, 2, and 3. As expected, neither the integrated agent, nor people, exhibit the effect in task set 4 where the decoy was not dominated by only one of the options and was therefore in a neutral position. Task 4 thereby acts as a control. The human data is from [31]. The error bars indicate confidence interval (95%) of the predictions made by the agent. This Figure replicates Figure 8 in [35].

In what follows we first show that our new reinforcement learning model replicates previous findings [35] and then show that it makes new predictions derived from the sequential nature of the model.

Is it Beneficial to Compare Options? In order to answer this question, we first fitted the distributions of the environment to those used in a prominent human experiment [31]. The probabilities $p$ are $\beta$--distributed ($a = 1, b = 1$) and the values $v$ are $t$--distributed ($\text{location} = 19.60, \text{scale} = 5, \text{degree of freedom} = 100$). For all the experiments below, we used the same distributions. Reported results are averaged over 10 runs, each with a different seed, after training on 3 million samples. All the details on setup and learning curves can be found in the supplementary material.

All agents were tested with different levels of observation noise and the resulting performance is shown in Figure 2. The maximum expected value that could be achieved by any agent was 16.29 (horizontal upper bound in Figure 2), which was calculated by averaging the maximum expected value of 3 options across 1 million choice sets sampled from the above distributions.

In Figure 2 it can be seen that the integrated agent, using both calculation and comparison observations, can approximate the optimal policy when actions could be conducted without noise. Also, calculation-based and comparison-based agents are able to perform close to optimum when there is no noise. However, the noise has a negative effect on the performance of all types of agent. The average obtained value of choices decreases as noise increases.

Figure 2 also shows that the integrated agent combines the strengths of both noisy comparison and noisy calculation to make better decisions than the other agents in all noise conditions. The average expected value of the choices made by the integrated agent is greater than the other agents. In
other words, the human-like integrated agent performs better in accumulating reward than the agent that makes independent assessments of each option value. The results suggest that when there is observation uncertainty, both humans and artificial agents will gain higher reward if they compare options, rather than merely evaluate each option independently.

**Does the Integrated Agent Predict Human Performance?** To determine whether the integrated agent (the agent that uses both comparison and calculation) predicts human performance, we measured its behaviour on the attraction, compromise and similarity tasks. The human behaviour on these tasks is shown in Figure 4a. We used one fixed setting of the agent policy and parameter values.

Agents were trained on tasks which were randomly sampled from a β-distribution ($\alpha = 1, \beta = 1$) for the probability $p$ and $t$-distribution (location = 19.60, scale = 8.08, df = 100) for the value $v$. After 3 million training samples, the agent converged and demonstrated stable performance. The agent was repeatedly trained with adjusted values of the comparison noise, calculation noise, probability weighting parameters, cost of comparisons and calculation cost until the qualitative effects fitted the human performance [45; Figure 4b]. The fitted parameter values were: calculation noise $\sigma_{calc} = 4$, comparison error $P(error_f) = 0.1$, probability weighting parameters $\alpha = 1$, the perceived cost of comparison $C_{comparison} = -0.01$ and the calculation cost $C_{calc} = -0.1$. We do not claim to have achieved the best possible fit, nor a better fit than other models. The point of the fit was to show that the qualitative effects exhibited by humans was within the space of behaviours generated by the agent.

The results are averaged over 10 runs with different seed and shown in Figure 4a. It shows that the agent generates the three context effects using one learnt policy and one fixed set of parameter values. Comparison of Figure 4a to Figure 1d) shows that all of the qualitative effects are predicted.

To further test the agent we fitted it to variations of the attraction effect in human performance [31]. The fitted values were: calculation noise $\sigma_{calc} = 0.50$, comparison error $P(error_f) = 0.1$, probability weighting parameters $\alpha = 1.5$, the perceive cost of comparison features $C_{comparison} = -0.01$ and the calculation cost $C_{calc} = -0.1$. The results in Figure 4 show that for both agents and people, the Target is selected more often than the Competitor in three of the task sets (1, 2, and 3).
contrast, and as expected, the Target and Competitor are selected equally often in the 4th task set by both agents and people. The decoy was positioned in a neutral position in task set 4 and does not therefore have an effect on the target choice rate.

**Does the Uncertainty of Information Influence the Decision Process?** We tested the consequences of noise on choice. The results in Figure 4b, c, d show that: (1) The size of attraction effect decreases as computational cost increases, (2) the attraction effect is weaker when the agent’s accuracy of comparison is diminished with noise, (3) The effect is stronger when calculation noise is higher. While, there is no human data that directly tests the effect of noise. There are a number of studies reporting that the rate of context effect diminishes with time pressure increases [64, 57]. As shown in Figure 4b, c, d, the effects of time pressure on humans is consistent with the effect of increased noise in the model.

**What are the Effects of Noise and Computation Cost?** The effect of noise on the number of comparisons and calculation actions taken is shown in Figure 5. Increases in comparison noise leads to a selective reduction in the use of comparison and a selective increase in the use of calculation. Conversely, increases in calculation noise leads to a selective decrease in the use of calculation and an increase in the use of comparison. Increase in the cost of information gathering actions (comparison and expected value) reduces contextual effects on choice (Figure 5c) as less information is gathered.

## 5 Discussion

While ours is not the first work to demonstrate the rationality of preference reversal phenomena [35], nor the first work to use POMDPs to model humans [51, 38], it is the first to formulate the contextual choice problem as a POMDP and demonstrate that a reinforcement learning agent that uses comparison observations generates higher reward than an agent that makes independent assessments of value. These comparison actions, when deployed by people, have been thought by many to violate independence axioms. They have been shown to underpin preference reversals in humans [49]. As has previously been pointed out, this seemingly paradoxical result makes sense when it is appreciated that the comparison of options reduces the uncertainty of option values.

Further, extending [35], we have demonstrated that the same pattern of behaviours that are thought to be irrational in humans, will emerge from a learning process that attempts to maximize the cumulative reward of action. Further, our results show that comparison actions are preferred by the agent as observation noise increases (in previous models comparisons have been assumed) and we have also shown that higher information gathering costs can diminish the use of comparisons and reduce the preference reversal rate; thereby extending previous analysis to account for the economics of information gathering in contextual choice tasks.

The approach that we have taken in this paper is an example of a broader class of analysis known as Computational Rationality [26, 36, 28]. This approach starts from the assumption that people are approximately rational given the bounds imposed by the computation required for cognition [26, 39, 65]. It then seeks to discover the computational limits that give rise to boundedly optimal but apparently irrational behaviours. This aim demands that the analyst derive bounded optimal policies for well-formed decision problems. Our results suggest an answer to the paradox of why it is worth motivating machine learning algorithms with apparently biased human decision making. While the behaviour appears biased, the underlying processes and heuristics (e.g. the use of option comparison) lead to gains in efficiency and therefore reward.

## 6 Conclusion

Machine learning researchers can take inspiration from apparent human irrationalities. This claim is supported by our demonstration that reinforcement learning agents that seek to maximize cumulative reward when observations are uncertain can improve performance by selectively comparing option values and not merely making independent assessments of each option. This human-like processing appears irrational but is demonstrably rational under bounds imposed by uncertainty in the observation function.
Broader Impact

The current work is potentially influential on the future of how RL is used in systems designed to understand and interact with humans. It may have positive ethical implications by virtue of the fact that it enhances the scientific understanding of the relationship between human and machine information gathering and decision processes. We have no reason to believe that the data used to train and validate the model is biased in a way that would disadvantage protected or minority groups on the basis of race or gender.

References

[1] Demis Hassabis, Dharshan Kumaran, Christopher Summerfield, and Matthew Botvinick. Neuroscience-inspired artificial intelligence. *Neuron*, 95(2):245–258, 2017.

[2] Brenden M Lake, Tomer D Ullman, Joshua B Tenenbaum, and Samuel J Gershman. Building machines that learn and think like people. *Behavioral and brain sciences*, 40, 2017.

[3] Warren S McCulloch and Walter Pitts. A logical calculus of the ideas immanent in nervous activity. *The bulletin of mathematical biophysics*, 5(4):115–133, 1943.

[4] Yann LeCun, Yoshua Bengio, and Geoffrey Hinton. Deep learning. *nature*, 521(7553):436, 2015.

[5] Michael L Littman. Reinforcement learning improves behaviour from evaluative feedback. *Nature*, 521(7553):445–451, 2015.

[6] Joshua C Peterson, Ruairidh M Battleday, Thomas L Griffiths, and Olga Russakovsky. Human uncertainty makes classification more robust. In *Proceedings of the IEEE International Conference on Computer Vision*, pages 9617–9626, 2019.

[7] Alex Kendall, Yarin Gal, and Roberto Cipolla. Multi-task learning using uncertainty to weigh losses for scene geometry and semantics. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pages 7482–7491, 2018.

[8] Alex Kendall and Yarin Gal. What uncertainties do we need in bayesian deep learning for computer vision? In *Advances in neural information processing systems*, pages 5574–5584, 2017.

[9] Yarin Gal and Zoubin Ghahramani. Dropout as a bayesian approximation: Representing model uncertainty in deep learning. In *international conference on machine learning*, pages 1050–1059, 2016.

[10] Wesley J Maddox, Pavel Izmailov, Timur Garipov, Dmitry P Vetrov, and Andrew Gordon Wilson. A simple baseline for bayesian uncertainty in deep learning. In *Advances in Neural Information Processing Systems*, pages 13132–13143, 2019.

[11] Kazuki Osawa, Siddharth Swaroop, Mohammad Emvtyaz E Khan, Anirudh Jain, Runa Eschenhagen, Richard E Turner, and Rio Yokota. Practical deep learning with bayesian principles. In *Advances in Neural Information Processing Systems*, pages 4289–4301, 2019.

[12] Meire Fortunato, Mohammad Gheshlaghi Azar, Bilal Piot, Jacob Menick, Ian Osband, Alex Graves, Vlad Mnih, Remi Munos, Demis Hassabis, Olivier Pietquin, et al. Noisy networks for exploration. In *International Conference on Learning Representations*, 2018.

[13] Brendan O’Donoghue, Ian Osband, Remi Munos, and Volodymyr Mnih. The uncertainty bellman equation and exploration. In *International Conference on Machine Learning*, pages 3836–3845, 2018.

[14] David Janz, Jiri Hron, Przemyslaw Mazur, Katja Hofmann, José Miguel Hernández-Lobato, and Sebastian Tsiatschek. Successor uncertainties: exploration and uncertainty in temporal difference learning. In *Advances in Neural Information Processing Systems*, pages 4509–4518, 2019.

[15] Jimmy Ba, Volodymyr Mnih, and Koray Kavukcuoglu. Multiple object recognition with visual attention. *arXiv preprint arXiv:1412.7755*, 2015.

[16] J. Choi, H. J. Chang, J. Jeong, Y. Demiris, and J. Y. Choi. Visual tracking using attention-modulated disintegration and integration. In *2016 IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, pages 4321–4330, June 2016.
[17] J. Choi, H. J. Chang, S. Yun, T. Fischer, Y. Demiris, and J. Y. Choi. Attentional correlation filter network for adaptive visual tracking. In 2017 IEEE Conference on Computer Vision and Pattern Recognition (CVPR), pages 4828–4837, July 2017.

[18] J. Choi, H. J. Chang, T. Fischer, S. Yun, K. Lee, J. Jeong, Y. Demiris, and J. Y. Choi. Context-aware deep feature compression for high-speed visual tracking. In 2018 IEEE/CVF Conference on Computer Vision and Pattern Recognition, pages 479–488, June 2018.

[19] K. Lee, D. Ognibene, H. J. Chang, T. Kim, and Y. Demiris. STARE: Spatio-temporal attention relocation for multiple structured activities detection. IEEE Transactions on Image Processing, 24(12):5916–5927, Dec. 2015.

[20] Kelvin Xu, Jimmy Ba, Ryan Kiros, Kyunghyun Cho, Aaron Courville, Ruslan Salakhudinov, Rich Zemel, and Yoshua Bengio. Show, attend and tell: Neural image caption generation with visual attention. In International conference on machine learning, pages 2048–2057, 2015.

[21] Dzmitry Bahdanau, Kyunghyun Cho, and Yoshua Bengio. Neural machine translation by jointly learning to align and translate. In International Conference on Learning Representations, 2015.

[22] Ashish Vaswani, Noam Shazeer, Niki Parmar, Jakob Uszkoreit, Llion Jones, Aidan N Gomez, Łukasz Kaiser, and Illia Polosukhin. Attention is all you need. In Advances in neural information processing systems, pages 5998–6008, 2017.

[23] Amos Tversky and Daniel Kahneman. Judgment under uncertainty: Heuristics and biases. science, 185(4157):1124–1131, 1974.

[24] Jonathan Baron. Thinking and deciding. Cambridge University Press, 2008.

[25] Gerd Gigerenzer. The bias bias in behavioral economics. Review of Behavioral Economics, 5(3-4):303–336, 2018.

[26] Richard L Lewis, Andrew Howes, and Satinder Singh. Computational rationality: Linking mechanism and behavior through bounded utility maximization. Topics in cognitive science, 6(2):279–311, 2014.

[27] Xiuli Chen. An optimal control approach to testing theories of human information processing constraints. PhD thesis, University of Birmingham, 2015.

[28] Falk Lieder and Thomas L Griffiths. Resource-rational analysis: understanding human cognition as the optimal use of limited computational resources. Behavioral and Brain Sciences, pages 1–85, 2019.

[29] Peter M Todd and Gerd Ed Gigerenzer. Ecological rationality: Intelligence in the world. Oxford University Press, 2012.

[30] Peter Frazier and J Yu Angela. Sequential hypothesis testing under stochastic deadlines. In Advances in neural information processing systems, pages 465–472, 2008.

[31] Douglas H Wedell. Distinguishing among models of contextually induced preference reversals. Journal of Experimental Psychology: Learning, Memory, and Cognition, 17(4):767, 1991.

[32] Marius Usher and James L McClelland. The time course of perceptual choice: the leaky, competing accumulator model. Psychological review, 108(3):550, 2001.

[33] Robert M Roe, Jermone R Busemeyer, and James T Townsend. Multialternative decision field theory: A dynamic connectionist model of decision making. Psychological review, 108(2):370, 2001.

[34] Amos Tversky and Itamar Simonson. Context-dependent preferences. Management science, 39(10):1179–1189, 1993.

[35] Andrew Howes, Paul A Warren, George Farmer, Wael El-Deredy, and Richard L Lewis. Why contextual preference reversals maximize expected value. Psychological review, 123(4):368, 2016.

[36] Samuel J Gershman, Eric J Horvitz, and Joshua B Tenenbaum. Computational rationality: A converging paradigm for intelligence in brains, minds, and machines. Science, 349(6245):273–278, 2015.

[37] Peter Dayan and Nathaniel D Daw. Decision theory, reinforcement learning, and the brain. Cognitive, Affective, & Behavioral Neuroscience, 8(4):429–453, 2008.

[38] Rajesh PN Rao. Decision making under uncertainty: a neural model based on partially observable markov decision processes. Frontiers in computational neuroscience, 4:146, 2010.
[39] Falk Lieder, Paul M Krueger, and Tom Griffiths. An automatic method for discovering rational heuristics for risky choice. In *CogSci*, 2017.

[40] Antti Oulasvirta, Xiaojun Bi, and Andrew Howes. *Computational interaction*. Oxford University Press, 2018.

[41] Jerome R Busemeyer, Sebastian Gluth, Jörg Rieskamp, and Brandon M Turner. Cognitive and neural bases of multi-attribute, multi-alternative, value-based decisions. *Trends in cognitive sciences*, 2019.

[42] Takao Noguchi and Neil Stewart. Multialternative decision by sampling: A model of decision making constrained by process data. *Psychological review*, 125(4):512, 2018.

[43] David Ronayne and Gordon DA Brown. Multi-attribute decision by sampling: an account of the attraction, compromise and similarity effects. *Journal of Mathematical Psychology*, 81:11–27, 2017.

[44] Lena M Wollschlaeger and Adele Diederich. Similarity, attraction, and compromise effects: Original findings, recent empirical observations, and computational cognitive process models. *American Journal of Psychology*, 2019.

[45] Jennifer S Trueblood. Multialternative context effects obtained using an inference task. *Psychonomic bulletin & review*, 19(5):962–968, 2012.

[46] Konstantinos Tsetsos, Rani Moran, James Moreland, Nick Chater, Marius Usher, and Christopher Summerfield. Economic irrationality is optimal during noisy decision making. *Proceedings of the National Academy of Sciences*, 113(11):3102–3107, 2016.

[47] Neil Stewart, Nick Chater, and Gordon DA Brown. Decision by sampling. *Cognitive psychology*, 53(1):1–26, 2006.

[48] Ivo Vlaev, Nick Chater, Neil Stewart, and Gordon DA Brown. Does the brain calculate value? *Trends in cognitive sciences*, 15(11):546–554, 2011.

[49] Takao Noguchi and Neil Stewart. In the attraction, compromise, and similarity effects, alternatives are repeatedly compared in pairs on single dimensions. *Cognition*, 132(1):44–56, 2014.

[50] Leslie Pack Kaelbling, Michael L Littman, and Anthony R Cassandra. Planning and acting in partially observable stochastic domains. *Artificial intelligence*, 101(1-2):99–134, 1998.

[51] Nathaniel D Daw, Aaron C Courville, and David S Touretzky. Representation and timing in theories of the dopamine system. *Neural computation*, 18(7):1637–1677, 2006.

[52] Koosha Khalvati and Rajesh PN Rao. A bayesian framework for modeling confidence in perceptual decision making. In *Advances in neural information processing systems*, pages 2413–2421, 2015.

[53] Yanping Huang, Timothy Hanks, Mike Shadlen, Abram L Friesen, and Rajesh PN Rao. How prior probability influences decision making: A unifying probabilistic model. In *Advances in neural information processing systems*, pages 1268–1276, 2012.

[54] Koosha Khalvati, Seongmin A Park, Jean-Claude Dreher, and Rajesh PN Rao. A probabilistic model of social decision making based on reward maximization. In *Advances in Neural Information Processing Systems*, pages 2901–2909, 2016.

[55] Thomas L Griffiths, Frederick Callaway, Michael B Chang, Erin Grant, Paul M Krueger, and Falk Lieder. Doing more with less: meta-reasoning and meta-learning in humans and machines. *Current Opinion in Behavioral Sciences*, 29:24–30, 2019.

[56] Frederick Callaway, Rangel Antonio, and Griffith Tom. Fixation patterns in simple choice are consistent with optimal use of cognitive resources. *PsyArXiv preprint PsyArXiv: https://doi.org/10.31234/osf.io/S7v6k*, 2020.

[57] Jennifer S Trueblood, Scott D Brown, and Andrew Heathcote. The multiattribute linear ballistic accumulator model of context effects in multialternative choice. *Psychological review*, 121(2):179, 2014.

[58] Leonard J Savage. *The foundations of statistics*. Courier Corporation, 1972.

[59] Fiery Cushman and Adam Morris. Habitual control of goal selection in humans. *Proceedings of the National Academy of Sciences*, 112(45):13817–13822, 2015.
[60] Jane X Wang, Zeb Kurth-Nelson, Dharshan Kumaran, Dhruva Tirumala, Hubert Soyer, Joel Z Leibo, Demis Hassabis, and Matthew Botvinick. Prefrontal cortex as a meta-reinforcement learning system. *Nature neuroscience*, 21(6):860, 2018.

[61] Maximilian Igl, Luisa Zintgraf, Tuan Anh Le, Frank Wood, and Shimon Whiteson. Deep variational reinforcement learning for pomdps. In *International Conference on Machine Learning*, pages 2122–2131, 2018.

[62] Ziyu Wang, Victor Bapst, Nicolas Heess, Volodymyr Mnih, Remi Munos, Koray Kavukcuoglu, and Nando de Freitas. Sample efficient actor-critic with experience replay. *arXiv preprint arXiv:1611.01224*, 2016.

[63] Greg Brockman, Vicki Cheung, Ludwig Pettersson, Jonas Schneider, John Schulman, Jie Tang, and Wojciech Zaremba. Openai gym. *arXiv preprint arXiv:1606.01540*, 2016.

[64] Jonathan C Pettibone. Testing the effect of time pressure on asymmetric dominance and compromise decoys in choice. *Judgment and Decision Making*, 7(4):513, 2012.

[65] Andrew Howes, Richard L Lewis, and Alonso Vera. Rational adaptation under task and processing constraints: Implications for testing theories of cognition and action. *Psychological review*, 116(4):717, 2009.

[66] Stuart J Russell and Devika Subramanian. Provably bounded-optimal agents. *Journal of Artificial Intelligence Research*, 2:575–609, 1994.