Exact solutions for charged spheres and their stability. I. Perfect Fluids.

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Abstract. We study exact solutions of the Einstein-Maxwell equations for the interior gravitational field of static spherically symmetric charged compact spheres. The spheres are composed of a perfect fluid with a charge distribution that creates a static radial electric field. The inertial mass density of the fluid has the form \( \rho(r) = \rho_0 + \alpha r^2 \) (\( \rho_0 \) and \( \alpha \) are constants) and the total charge \( q(r) \) within a sphere of radius \( r \) has the form \( q = \beta r^3 \) (\( \beta \) is a constant). We evaluate the critical values of \( M/R \) for these spheres as a function of \( Q/R \) and compare these values with those given by the Andréasson formula.

Keywords: Einstein-Maxwell equations - exact solutions - black-hole physics

1. Introduction

The Reissner-Nordström metric (Reissner [1], Weyl [2] and Nordström [3]) is a solution of the Einstein-Maxwell equations that represents the exterior gravitational field of a spherically symmetric charged body. In fact, it is the unique asymptotically flat vacuum solution around any charged spherically symmetric object, irrespective of how that body may be composed or how it may evolve in time (Carter [4], Ruback [5] and Chruściel [6]). It contains no information about its source other than its total mass and total charge, and probably most interesting it imposes no constraints on its internal structure.

The internal structure of a charged sphere is established by solving the coupled Einstein-Maxwell equations. A solution of these equations describes the gravitational field inside the sphere as a function of the distribution of matter and energy within the sphere. The Einstein-Maxwell equations for a charged static spherically distribution of matter reduce to a set of three independent non-linear second order differential equations that connect the metric coefficients \( g_{tt} \) and \( g_{rr} \) with the physical quantities that represent the matter content of the sphere. A description of the matter content of a charged perfect fluid sphere includes functions that defines its inertial mass density \( \rho_{fl} \), its pressure \( p \) and the electric charge distribution \( q \) or the electromagnetic energy density \( \rho_{em} \). Thus, the complete system of equations to be solved consists of three linearly independent equations with five unknowns. This situation gives total freedom in choosing two of the five functions and then solving for the remaining three. It is therefore hardly surprising...
that there exists very many exact solutions of the Einstein-Maxwell equations for charged spheres. A comprehensive review of the various methods of solving the Einstein-Maxwell equations for charged spheres was done by Ivanov [7].

An important question that arises in the study of the structure of spherically symmetric compact objects is the following: what is the maximum of value $M/R$ (the total mass of the body divided its radius) that is allowed before gravitational collapse occurs? It is well known that for neutral perfect fluid spheres that the stability limit is given by the Buchdahl limit [8]:

$$\frac{M}{R} \leq \frac{4}{9} \quad \text{for uncharged perfect fluid spheres.} \quad (1.1)$$

In case of charged compact objects this quantity becomes dependent on the total charge $Q$, since the addition of charge to the system increases its total energy and hence its total mass. Andréasson [9], has published a remarkable result that claims that for any compact spherically symmetric charged distribution the following relationship between $M/R$ and $Q/R$ holds:

$$\frac{M}{R} \leq \left( \frac{1}{3} + \sqrt{\frac{1}{9} + \frac{1}{3} \frac{Q^2}{R^2}} \right)^2 \quad \text{for all charged spheres.} \quad (1.2)$$

This formula generalizes the Buchdahl result for uncharged perfect fluid spheres. It is worth noting that in his derivation, Buchdahl placed the following constraints on the physical properties of the fluid: (i) $d\rho/dr < 0$ i.e., the density decreases outward from the center of the sphere and (ii) the pressure is isotropic. In the derivation of his formula, Andréasson made the following assumptions about the matter content of the sphere: (i) $p_r + 2p_t < \rho$ (in a spherically symmetric system it is possible to have the radial pressure, $p_r$, different from the tangential pressure, $p_t$) and (ii) $\rho > 0$. We note that unlike the neutral case Andréasson did not require the condition $d\rho/dr < 0$. Numerical investigations ([10] and [11]), of the upper bound of $M/R$ as a function of $Q/R$ have verified that the Andréasson formula provides a valid upper bound of $M/R$ for charged spheres. One of our aims in this study is to continue the investigation of the validity of the Andréasson limit using a mixture of analytical and numerical techniques.

A class of charge spheres that have received considerable attention in theoretical physics are extremal black holes. Extremal non-rotating charged black holes are interior solutions of the Reissner-Nordstrom metric in which the mass is equal to the charge in geometric units. These objects have zero surface gravity and their horizon structure is different from that of regular black holes. In general relativity extremal black holes are important in the study and proof of the third law of black hole thermodynamics. In string theory they are ubiquitous, in particular they were used to derived the Bekenstein-Hawking entropy formula and in general they are easier to describe than regular black holes quantum mechanically because of their vanishing surface gravity and consequently vanishing temperature for Hawking radiation [12]. In many solutions for charged spheres extremality is only achieved when $M/R = Q/R = 1$. In our study we will look for cases where extremality can occur before the upper bound on $Q/R$ and $M/R$ is reached.
This paper is organized as follows in the next section we develop the full Einstein-Maxwell equations for the interior gravitational field of a charged sphere. In section 3 we solve the equations and consider their stability properties and the formation of extremal black holes in these solutions. In section 4 we discuss our results and formulate our conclusions. We note that a prime (‘) or a comma (,) denotes derivatives with respect \( r \), and a semi-colon (;) is used to represent covariant derivatives. We will work in units where \( c = G = 1 \).

The material presented in the next section is well known. However, for consistency and clarity we have chosen to show the full derivation of the electromagnetic energy-momentum tensor for a static radial electric field. A complete pedagogical description of the derivation of the general form of the perfect fluid and electromagnetic energy-momentum tensors is given in Alder, Bazin and Schiffer [13].

2. The field equations

We are interested in studying the interior gravitational field of charged spheres, therefore we will assume a spherically symmetric metric of the form

\[
ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + r^2d\theta^2 + r^2\sin^2 \theta d\phi^2.
\]  
(2.1)

In this paper we are concerned only with spherically symmetric static solutions of the coupled Einstein-Maxwell equations, therefore \( \nu = \nu(r) \) and \( \lambda = \lambda(r) \) are functions of \( r \) only. The Einstein field equations are

\[
G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi T_{\alpha\beta}.
\]  
(2.2)

The spheres that we will study are composed of a perfect fluid with a charge distribution that creates a static radial electric field. The energy-momentum tensor \( T_{\alpha\beta} \), will thus be written as

\[
T_{\alpha\beta} = (T_{\alpha\beta})_{pf} + (T_{\alpha\beta})_{em},
\]  
(2.3)

with \( (T_{\alpha\beta})_{pf} \) the energy-momentum tensor for a perfect fluid and \( (T_{\alpha\beta})_{em} \) the energy-momentum tensor associated with the electric field.

The energy momentum tensor for a static spherically symmetric perfect fluid with a rest-frame energy-mass density \( \rho \) and isotropic pressure \( p \) [13] is

\[
(T_{\alpha\beta})_{f} = (\rho + p)u_{\alpha}u_{\beta} + pg_{\alpha\beta}.
\]  
(2.4)

Here, \( \rho \) and \( p \) are functions of \( r \) only and \( u_{\alpha} \) is the 4-velocity of the fluid.

The four velocity \( u_{\alpha} \) is defined such that

\[
g_{\alpha\beta}u^{\alpha}u^{\beta} = -1.
\]  
(2.5)

Since we are considering a fluid at rest, we will take

\[
u_t = u_\theta = u_\phi = 0, \quad \text{and} \quad u_t = -(-g^{tt})^{-\frac{1}{2}} = -e^{\nu(r)}
\]  
(2.6)

thus,

\[
u_t = -\delta^t_\alpha e^{\nu}
\]  
(2.7)
and the energy-momentum tensor for a perfect fluid is
\[(T_{αβ})_{pf} = \text{diag}(-ρe^{-2ν}, p e^{-2λ}, pr^2, pr^2 \sin^2 θ).\] (2.8)

The energy-momentum tensor of the electric field, \((T_{αβ})_{em}\) is constructed from the electromagnetic field tensor \(F_{αβ}\):
\[(T_{αβ})_{em} = \frac{1}{4π}(F_{α}^{β}F_{β}^{τ} - \frac{1}{4}g_{αβ}F_{τκ}F_{τκ}).\] (2.9)

We are interested in studying a fluid whose charge distribution gives rise to static radial electric field therefore the only non-zero components of \(F_{αβ}\) are \(F_{rt} = -F_{tr} = E(r)\), thus
\[F_{αβ} = δ_{r}^{α}δ_{t}^{β}E(r).\] (2.10)

Maxwell’s equations in curved space-time are:
\[F_{αβ};β = 4πj_{α}, \quad \text{and} \quad F_{[αβ;τ] + F_{[rα;β]} + F_{[βτ;α]} = 0}.\] (2.11)

Given the form of \(F_{αβ}\) (2.10), the second set of equations is identically zero, also since \(F_{αβ}\) is anti-symmetric the first set of equations can be written as
\[\frac{1}{\sqrt{-g}}(\sqrt{-g}F_{αβ})_{,β} = 4πj_{α},\] (2.12)
where \(j_{α}\) is the current 4-vector. If the charge density \(σ\) is given then
\[j_{α} = σu^{α}.\] (2.13)

Using the explicit forms of \(F_{αβ}\) and \(u_{α}\) in equation (2.12) we find that
\[(r^2e^{-(ν+λ)}E(r))_{,r} = 4πr^2σe^{λ}.\] (2.14)

Integrating this equation gives an expression for \(E(r)\):
\[E(r) = \frac{e^{ν+λ}q(r)}{r^2},\] (2.15)
with
\[q(r) = \int_{0}^{r} 4πr^2σe^{λ}dr.\] (2.16)

We note that since \(σ = j^{t}e^{ν}\), \(q(r)\) can be written as
\[q(r) = \int_{0}^{r} \int_{0}^{π} \int_{0}^{2π} j^{t}e^{ν+λ}r^2 \sin θdθdφdr = \int_{V} j^{t} \sqrt{-g} dV.\] (2.17)

Thus \(q(r)\) represents the total charge contained in a sphere of radius \(r\). Using (2.15) we can write
\[F_{αβ} = δ_{r}^{α}δ_{t}^{β} \frac{e^{ν+λ}q(r)}{r^2},\] (2.18)
and the electromagnetic energy-momentum tensor takes the from
\[(T_{αβ})_{em} = \frac{q(r)^2}{8πr^4} \text{diag}(e^{-2ν}, -e^{2λ}, r^2, r^2 \sin^2 θ).\] (2.19)
We can now write out complete set of equations that describe the interior gravitational field of static charged spheres. They are:

\[ G^t_t = e^{-2\lambda}\left(\frac{2\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi \rho + \frac{q^2}{r^4}, \]  

(2.20)

\[ G^r_r = e^{-2\lambda}\left(\frac{2\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi p - \frac{q^2}{r^4}, \]  

(2.21)

and

\[ G^\theta_\theta = G^\phi_\phi = e^{-2\lambda}\left(\nu'' + \nu'^2 - \nu'\lambda' + \frac{\nu'}{r} - \frac{\lambda'}{r}\right) = 8\pi p + \frac{q^2}{r^4}. \]  

(2.22)

3. Solutions for the field equations

We will now develop solutions for the field equations. We start by noting that (2.20), can be written as

\[ \frac{d}{dr}(re^{-2\lambda}) = 1 - 8\pi \rho r^2 - \frac{q^2}{r^2}. \]  

(3.1)

This equation can be immediately integrated to give

\[ e^{-2\lambda(r)} = 1 - \frac{2m_i(r)}{r} - f(r), \]  

(3.2)

with

\[ m_i(r) = 4\pi \int_0^r \rho(r')r'^2 dr' \quad \text{and} \quad f(r) = \int_0^r \frac{q(r')^2}{r'^2}dr'. \]  

(3.3)

The quantity \( m_i(r) \) is the inertial mass of the fluid in a sphere of radius \( r \). A related quantity is the total gravitational mass \( m_g(r) \) of the fluid in a sphere of radius \( r \). Bekenstein [16] in his study of charged spheres introduced \( m_g(r) \) in an ad hoc manner. Since the exterior metric of a charged sphere, the Reissner-Nordström solution has

\[ e^{-2\lambda(r)} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \]  

(3.4)

where \( M \) is the total gravitational mass of the sphere and \( Q \) is the total charge of the sphere, Bekenstein proposed that in the interior of a charged sphere, \( e^{-2\lambda(r)} \) should have following form:

\[ e^{-2\lambda(r)} = 1 - \frac{2m_g(r)}{r} + \frac{q^2(r)}{r^2}. \]  

(3.5)

The function \( q(r) \) is the total charge in a sphere of radius \( r \). It is the charge function defined in (2.16) that we have been studying. In order for (3.2) to be equal to (3.5), \( m_g(r) \) must be defined in the following manner:

\[ m_g(r) \equiv \frac{1}{2} \int_0^r \left(8\pi \rho r'^2 + \frac{q^2(r')}{r'^2}\right) dr' + \frac{q(r)^2}{2r}. \]  

(3.6)

We note that in the absence of the electric field \( m_g = m_i \).
The requirement that $e^{-2\lambda(r)}$ matches the Reissner-Nordström metric at surface of the sphere, $r = R$ gives
\[
1 - \frac{M}{R} + \frac{Q^2}{R^2} = 1 - \frac{1}{R} \int_0^R (8\pi \rho r^2 + \frac{q^2}{r^2})dr. \tag{3.7}
\]
An expression for the total mass can be found from this equation:
\[
M = \frac{1}{2} \int_0^R (8\pi \rho r^2 + \frac{q^2}{r^2})dr + \frac{Q^2}{2R}. \tag{3.8}
\]
In this paper we will study charged spheres with the following inertial mass density $\rho(r)$ and charge distribution $q(r)$ profiles:
\[
\rho(r) = \rho_o - \frac{b}{8\pi} \frac{Q^2}{R^6} r^2 \quad \text{and} \quad q(r) = \frac{Q}{R^3} r^3, \tag{3.9}
\]
in the expression for $\rho(r)$, $b$ is a number. In our models
\[
e^{-2\lambda(r)} = 1 - \frac{8\pi \rho_o}{3} r^2 + \frac{(b - 1)}{5} \frac{Q^2}{R^6} r^4 \tag{3.10}
\]
and
\[
m_g(r) = \frac{4\pi \rho_o}{3} r^3 + \frac{(6 - b)}{10} \frac{Q^2}{2R} \frac{r^5}{R^5}. \tag{3.11}
\]
and
\[
M = \frac{4\pi \rho_o}{3} R^3 + \frac{(6 - b)}{10} \frac{Q^2}{2R}. \tag{3.12}
\]
With the assumed mass and charge distributions here, when $b = 6$
\[
m_g(r) \frac{r^3}{R^3} = \frac{M}{R^3} = \text{const}, \tag{3.13}
\]
thus the $b = 6$ model is a sphere with a constant gravitational mass density. Also all models have
\[
q(r) \frac{r^3}{R^3} = \left(\frac{Q}{R^3}\right) = \text{const}. \tag{3.14}
\]
thus the charge density is constant for all our models. The models that we will study in detail here include following three charged configurations:

(i) $b = 0$ - a sphere with a constant inertial mass density and constant charge density.

(ii) $b = 1$ - a sphere with constant total energy density and constant charge density.

(iii) $b = 6$ - a sphere with constant gravitational mass density and constant charge density.

We can solve for $\rho_o$ from (3.12) and rewrite (3.10) as
\[
e^{-2\lambda(r)} = 1 - \left(\frac{2M}{R} + \frac{(b - 6)}{5} \frac{Q^2}{R^2} \right) r^2 + \frac{(b - 1)}{5} \frac{Q^2}{R^2} \frac{r^4}{R^4}. \tag{3.15}
\]
Thus, if we are given $\rho(r)$ and $q(r)$ we have found $\lambda(r)$.

We now need to solve for $\nu(r)$. We start by transforming (2.22). First we subtract (2.21) from (2.22) to get
\[
e^{-2\lambda} \left(\nu'' + \nu' - \frac{\nu'}{r} - \frac{\lambda e^{-2\lambda}}{r} - \frac{e^{-2\lambda}}{r^2} + \frac{1}{r^2} \right) = 2\frac{q^2}{r^4}. \tag{3.16}
\]
Then we substitute for $1/r^2$ from (3.20) to get the following equation

$$\nu'' + \nu'^2 - \nu' = 3\lambda' - \left(8\pi\rho - \frac{q^2}{r^4}\right)e^{2\lambda}. \tag{3.17}$$

We now multiply both sides of this equation by $e^{-\lambda+\nu}/r$, and we find that the left hand-side becomes an exact differential: $(\nu e^{-\lambda+\nu}/r)'.$ Introducing $\zeta(r) \equiv e^{\nu(r)}$, we can write (3.17) in the following form

$$\left(\frac{1}{r} e^{-\lambda} \zeta'\right)' = \left[\frac{3\lambda' e^{-2\lambda}}{r^2} - \frac{8\pi\rho}{r} + \frac{2q}{r^5}\right] e^{\lambda} \zeta. \tag{3.18}$$

This equation was derived by Giuliani and Rothman 15 in their study of charged spheres. The transformation of the left hand-side (3.17) into that of (3.18) was introduced by Weinberg 14 in deriving the equation he used to prove the Buchdahl limit of $M/R$ for neutral perfect fluid spheres.

Substituting for $\rho, q$ (from (3.9)) and $\lambda' e^{-2\lambda}$ (from (3.71)) we find that (3.18) becomes

$$\left(\frac{1}{r} e^{-\lambda} \zeta'\right)' = \left(\frac{11}{5} - \frac{b}{5}\right) \frac{Q^2}{R^6} re^{\lambda} \zeta. \tag{3.19}$$

We now define a new variable

$$\tilde{\zeta}(u(r)) \equiv \zeta(r) \quad \text{with} \quad u(r) = \frac{1}{R^2} \int_0^r se^{\lambda(s)} ds. \tag{3.20}$$

Then (3.19) becomes

$$\frac{d^2 \tilde{\zeta}}{du^2} = \left(\frac{11}{5} - \frac{b}{5}\right) \frac{Q^2}{R^2} \tilde{\zeta}(u). \tag{3.21}$$

This is our master equation. Our task now is to find solutions for it given various values of $b$. There are three types of solutions of the master equation depending on the value of $b$:

(i) $b < 11 : \quad \tilde{\zeta}(u(r)) = Ae^{\frac{Q}{R}u(r)} + Be^{-\frac{Q}{R}u(r)} \quad \text{with} \quad a = \left(\frac{11}{5} - \frac{b}{5}\right)^{\frac{1}{2}}, \tag{3.22}$

(ii) $b = 11 : \quad \tilde{\zeta}(u(r)) = A + Bu(r), \tag{3.23}$

(iii) $b > 11 : \quad \tilde{\zeta}(u(r)) = A \sin \left(\frac{Q}{R} u(r)\right) + B \cos \left(-\frac{Q}{R} u(r)\right) \quad \text{with} \quad d = \left|\frac{11}{5} - \frac{b}{5}\right|^{\frac{1}{2}}. \tag{3.24}$

The values of the constants $A$ and $B$ in (3.22), (3.23) and (3.24) are fixed by the boundary conditions imposed on $\tilde{\zeta}(u)$:

$$\tilde{\zeta}(u(R)) = \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{1}{2}} \tag{3.25}$$

and

$$\frac{d\tilde{\zeta}(u(R))}{du(R)} = \left(\frac{M}{R} - \frac{Q^2}{R^2}\right). \tag{3.26}$$

The condition for stability is $\zeta(r = 0) > 0$. The critical values of $M/R$ as a function of $Q/R$ are found from solving the equation $\zeta(r = 0) = 0$. 

3.1. The $b = 0$ solution.

We start developing solutions for (3.21) by studying the case with $b = 0$. This choice of $b$ corresponds to a fluid with constant inertial density and a constant charge density. This model will serve as an important prototype for comparing the effects of changing $b$ on the physical properties of the system. When $b = 0$, we find that

$$e^{-2\lambda(r)} = 1 - \left(\frac{2M}{R} - \frac{6Q^2}{5R^2}\right)\frac{r^2}{R^2} - \frac{1}{5}\frac{Q^2}{R^2}r^4,$$

(3.27)

$$u(r) = \frac{1}{R^2} \int_0^r e^{\lambda(s)} \frac{sd}{R^2} = \frac{1}{R^2} \int_0^r \frac{sd}{R^2} \left[1 - \left(\frac{2M}{R} - \frac{6Q^2}{5R^2}\right)\frac{s^2}{R^2} - \frac{1}{5}\frac{Q^2}{R^2}s^4\right]^{\frac{1}{2}},$$

(3.28)

$$= \frac{\sqrt{5}}{2} \frac{R}{Q} \left[\tan^{-1}\left(\left(\sqrt{\frac{M}{Q}} - \frac{3Q}{\sqrt{5}R} + \frac{1}{\sqrt{5}R^3}\right)e^{\lambda(r)}\right)\right]$$

$$- \tan^{-1}\left(\sqrt{\frac{M}{Q}} - \frac{3Q}{\sqrt{5}R}\right),$$

(3.29)

and the master equation becomes

$$\frac{d^2\tilde{\zeta}}{du^2} = \frac{11}{5} \frac{Q^2}{R^2} \tilde{\zeta}(u).$$

(3.30)

The solution for this equation is

$$\tilde{\zeta}(u(r)) = A\exp\left(\sqrt{\frac{11}{5}} \frac{Q}{R} u(r)\right) + B\exp\left(-\sqrt{\frac{11}{5}} \frac{Q}{R} u(r)\right).$$

(3.31)

The constants $A$ and $B$ are found using the boundary conditions. Solving for them we find we find,

$$A = \frac{1}{2} \left[\left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{1}{2}} + \sqrt{\frac{5}{11}} \left(\frac{M}{Q} - \frac{Q}{R}\right)\right] \exp\left(-\sqrt{\frac{11}{5}} \frac{Q}{R} u(R)\right)$$

(3.32)

and

$$B = \frac{1}{2} \left[\left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{1}{2}} - \sqrt{\frac{5}{11}} \left(\frac{M}{Q} - \frac{Q}{R}\right)\right] \exp\left(\sqrt{\frac{11}{5}} \frac{Q}{R} u(R)\right).$$

(3.33)

The critical values of $M/R$ vs $Q/R$ are found from condition $\tilde{\zeta}(u(r = 0)) = 0$. Here this condition requires $A + B = 0$ and leads to the following transcendental equation for $M/R$ as function of $Q/R$:

$$\frac{\sqrt{\frac{5}{11}} \left(\frac{M}{Q} - \frac{Q}{R}\right)}{\sqrt{\frac{5}{11} \left(\frac{M}{Q} - \frac{Q}{R}\right) - \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{1}{2}}}}$$

$$= \exp\left(\sqrt{11}\left[\tan^{-1}\left(\left(\sqrt{\frac{M}{Q}} - \frac{2Q}{\sqrt{5}R}\right)e^{\lambda(R)}\right) - \tan^{-1}\left(\sqrt{\frac{5}{11}} \frac{Q}{\sqrt{5}R}\right)\right]\right)$$

(3.34)

We solved this equation numerically and the results are plotted in Figure 1. For comparison we also plotted the corresponding values of $M/R$ vs $Q/R$ from Andréasson’s
Figure 1: The critical values of $M/R$ vs $Q/R$ for $b = 0$ (---) and the Andréasson formula ( ).

We find that critical values of $M/R$ from this model almost saturates the Andréasson limit but does not exceed it. We note that we were able to integrate only up to $Q/R \approx 0.9$ with this model.

3.2. The $b = 1$ solution.

When $b = 1$,

$$e^{-2\lambda(r)} = 1 - \left(\frac{2M}{R} - \frac{Q^2}{R^2}\right) \frac{r^2}{R^2},$$

(3.35)

$$u(r) = \frac{1}{R^2} \int_0^r e^{\lambda(s)} s ds = \left(\frac{2M}{R} - \frac{Q^2}{R^2}\right)^{-1} \left[1 - \left[1 - \left(\frac{2M}{R} - \frac{Q^2}{R^2}\right) \frac{r^2}{R^2}\right]^2\right].$$

(3.36)

and the master equation becomes

$$\frac{d^2\tilde{\zeta}}{du^2} = 2 \frac{Q^2}{R^2} \tilde{\zeta}(u).$$

(3.37)

The solutions here are

$$\tilde{\zeta}(u(r)) = A \exp\left(\sqrt{2} \frac{Q}{R} u(r)\right) + B \left(-\sqrt{2} \frac{Q}{R} u(r)\right).$$

(3.38)
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Figure 2: The critical values of $M/R$ vs $Q/R$ for $b = 1$ (---) and the Andréasson formula (—).

with

$$A = \frac{1}{2} \left[ \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} + \frac{1}{\sqrt{2}} \left( \frac{M}{Q} - \frac{Q}{R} \right) \right] \exp \left( -\sqrt{2} \frac{Q}{R} u(R) \right), \quad (3.39)$$

$$B = \frac{1}{2} \left[ \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} - \frac{1}{\sqrt{2}} \left( \frac{M}{Q} - \frac{Q}{R} \right) \right] \exp \left( \sqrt{2} \frac{Q}{R} u(R) \right) \quad (3.40)$$

and

$$u(R) = \left( \frac{2M}{R} - \frac{Q^2}{R^2} \right)^{-1} \left[ 1 - \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} \right]. \quad (3.41)$$

Here the critical values of $M/R$ are found from the following equation:

$$\frac{\frac{1}{\sqrt{2}} \left( \frac{M}{Q} - \frac{Q}{R} \right) + \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}}}{\frac{1}{\sqrt{2}} \left( \frac{M}{Q} - \frac{Q}{R} \right) - \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}}} = \quad (3.42)$$

$$\exp \left[ 2\sqrt{2} \frac{Q}{R} \left( 1 - \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} \right) \right]$$

The critical values from (3.42) for the $b = 1$ model and the Andréasson formula are compared in figure 2. We see that $b = 1$ the critical values are less than the corresponding values from the Andréasson formula.
3.3. The b = 6 solution.

When \( b = 6 \),

\[
e^{-2\lambda(r)} = 1 - \frac{2M}{R} \frac{r^2}{R^2} + \frac{Q^2}{R^2} \frac{r^4}{R^4},
\]

\[
u(r) = \frac{R}{2Q} \left[ \log \left( \frac{M}{Q} + 1 \right) - \log \left( \frac{M}{Q} - \frac{Q}{R} \frac{r^2 + \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}}} \right) \right],
\]

and the master equation becomes

\[
\frac{d^2 \tilde{\zeta}}{du^2} = \frac{Q}{R^2} \tilde{\zeta}(u),
\]

The solution for (3.45) is

\[
\tilde{\zeta}(u(r)) = Ae^{\frac{Q}{R}u(r)} + Be^{-\frac{Q}{R}u(r)}.
\]

The boundary conditions require that

\[
\left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{1}{2}} = Ae^{\frac{Q}{R}u(R)} + Be^{-\frac{Q}{R}u(R)}. \tag{3.47}
\]

and

\[
\left(\frac{M}{R} - \frac{Q^2}{R^2}\right) = \frac{Q}{R} \left(Ae^{\frac{Q}{R}u(R)} - Be^{-\frac{Q}{R}u(R)}\right). \tag{3.48}
\]

Solving for \( A \) and \( B \) we find that

\[
A = \frac{1}{2} \left(\frac{M}{Q} + 1\right)^{\frac{1}{2}} \left[\frac{M}{Q} - \frac{Q}{R} \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{1}{2}}\right]^{\frac{3}{2}}, \tag{3.49}
\]

and

\[
B = \frac{1}{2} \left(\frac{M}{Q} + 1\right)^{\frac{1}{2}} \left[\frac{Q}{R} - \frac{M}{Q} \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{1}{2}}\right]^{\frac{3}{2}} \left[\frac{M}{Q} - \frac{Q}{R} \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{1}{2}}\right]^{\frac{3}{2}}. \tag{3.50}
\]

The stability condition \( \zeta(r = 0) = 0 \) leads to the following critical values equation:

\[
\left[\frac{M}{Q} - \frac{Q}{R} \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{1}{2}}\right]^{2} = \left(\frac{M}{Q} + 1\right) \left[\frac{M}{Q} - \frac{Q}{R} \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{1}{2}}\right]^{2}, \tag{3.51}
\]

or

\[
2 \left(\frac{4Q^4}{R^4} - 12 \frac{M}{R} \frac{Q^2}{R^2} + 9 \frac{M^2}{R^2} + 3 \frac{Q^2}{R^2} - 4 \frac{M}{R}\right) \frac{Q^2}{R^2} \left(\frac{M}{R} + \frac{Q}{R}\right) = 0. \tag{3.52}
\]

The term that is quadratic in \( M/R \) has exact solutions for \( M/R \) as a function of \( Q/R \). These solutions are

\[
\frac{M}{R} = \frac{2}{9} + \frac{2Q^2}{3R^2} - \frac{1}{9} \left(4 - 3\frac{Q^2}{R^2}\right)^{\frac{1}{2}} \quad \text{and} \quad \frac{M}{R} = \frac{2}{9} + \frac{2Q^2}{3R^2} + \frac{1}{9} \left(4 - 3\frac{Q^2}{R^2}\right)^{\frac{1}{2}}. \tag{3.53}
\]

The first of these two solutions gives \( M/R = 0 \) when \( Q/R = 0 \). This result does not correspond to properties the model that we are studying here. Here, when \( Q/R = 0 \),
Figure 3: The critical values of $M/R$ vs $Q/R$ for $b = 6$ (−−−) and the Andréasson formula (——).

$M/R$ should be equal to $4/9$, the Buchdahl limit for neutral perfect fluids. The second solution does give the correct $M/R = 4/9$ value when $Q/R = 0$. Thus we will take it to represent the critical values for the $b = 6$ model.

This model was studied extensively by Giuliani and Rothman [15]. However they arrived at this model using a different approach from us. It is useful here to review their development of this model. Giuliani and Rothman [15] showed that $3.18$ can be written in the following form:

$$\left(1 - \lambda \zeta' \right)' = \left[ \left( \frac{m_g(r)}{r^3} \right)' - q(r) \left( \frac{q(r)}{r^4} \right) \right] e^{\lambda \zeta}. \quad (3.54)$$

They then proposed the following ansatz: $m_g = M(r/R)^3$ and $q = Q(r/R)^3$. The substitution of their ansatz into (3.54) brings into the following form:

$$\left(\frac{1}{r} e^{-\lambda \zeta'} \right)' = r \frac{Q^2}{R^6} e^{\lambda \zeta}. \quad (3.55)$$

This is our $3.19$ for $b = 6$.

We plotted the critical values of $M/R$ vs $Q/R$ for this model in Figure 3 along with the values from the Andréasson formula. We find that this model ($b = 6$) does not saturate the Andréasson bound. At first glance, the critical values curves for the $b = 0$, the $b = 1$ and the $b = 6$ models do not appear differ from each other. In order to clearly show that the critical values of $M/R$ from these models are different from each other
we plotted the results for all three models in figure 4 with the axes magnified. We find from figure 4 that for spheres with a constant charge density, a sphere with constant inertial mass density (the $b = 0$ model) is more stable than a sphere with constant total energy density (the $b = 1$ model) which is turn more stable than a sphere with constant gravitational mass density (the $b = 6$ model).

3.4. The $b = 11$ solution

For $b = 11$

$$e^{-2\lambda(r)} = 1 - \left(\frac{2M}{R} + \frac{Q^2}{R^2}\right) \frac{r^2}{R^2} + 2 \frac{Q^2}{R^2} \frac{r^4}{R^4}$$ (3.56)

$$u(r) = \frac{\sqrt{2} R}{4 Q} \left[ \ln \left( \sqrt{2} \left( \frac{M}{2Q} + \frac{Q}{4R} \right) + 1 \right) - \ln \left( \sqrt{2} \left( \frac{M}{2Q} + \frac{Q}{4R} - \frac{Q^2}{R^3} \right) + e^{-\lambda(r)} \right) \right]$$ (3.57)

and the master equation is

$$\frac{d^2 \tilde{\zeta}}{du^2} = 0.$$ (3.58)

The solution here is

$$\tilde{\zeta}(u) = A + Bu(r)$$ (3.59)

Applying the boundary conditions we find that
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Figure 5: The critical values of $M/R$ vs $Q/R$ for $b = 11$ ($\ldots$) and the Andréasson formula ($\ldots$). The extremal line $M/R = Q/R$ is also drawn ($\ldots$).

$$
\zeta(r) = \frac{\sqrt{2}}{4} \left( \frac{M}{Q} - \frac{Q}{R} \right) \ln \left[ \frac{\sqrt{2}}{4} \left( \frac{M}{2Q} - \frac{3Q}{4R} \right) + e^{-\lambda(R)} \right] + \left( 1 - 2\frac{M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}}. 
$$  \hspace{1cm} (3.60)

The critical values equation here is

$$
\sqrt{1 - \frac{2M}{R} + \frac{Q^2}{R^2}} = \frac{\sqrt{2}}{4} \left( \frac{Q}{R} - \frac{M}{Q} \right) \ln \left[ \frac{\sqrt{2}}{4} \left( \frac{M}{2Q} - \frac{3Q}{4R} \right) + e^{-\lambda(R)} \right]. 
$$  \hspace{1cm} (3.61)

A plot the critical values of $M/R$ vs $Q/R$ for this model is given in Figure 5. We find from Figure 5 that the critical values of $M/R$ in this model are less than the values given by the Andréasson limit. We were only able to integrate up to $Q/R \approx 0.88$. We also found here that the extremal value $M/R = Q/R$ is achieved before the $Q/R = 1$ limit. Here $M/R = Q/R$ when $Q/R = 0.77$.

3.5. The $b = 16$ solution

When $b \geq 11$, the master equation becomes

$$
\frac{d^2\tilde{\zeta}}{du^2} = -d^2 \frac{Q^2}{R^2} \tilde{\zeta}(u).
$$  \hspace{1cm} (3.62)
with \( d = |(11 - b)/5|^{1/2} \) and the solution is

\[
\tilde{\zeta}(u(r)) = A \sin \left( d \frac{Q}{R} u(r) \right) + B \cos \left( d \frac{Q}{R} u(r) \right),
\]

with

\[
A = e^{-\lambda(R)} \sin \left( d \frac{Q}{R} u(R) \right) + \frac{1}{d} \left( \frac{M}{Q} - \frac{Q}{R} \right) \cos \left( d \frac{Q}{R} u(R) \right),
\]

\[
B = e^{-\lambda(R)} \cos \left( d \frac{Q}{R} u(R) \right) - \frac{1}{d} \left( \frac{M}{Q} - \frac{Q}{R} \right) \sin \left( d \frac{Q}{R} u(R) \right).
\]

The stability condition \( \zeta(r = 0) = 0 \) requires

\[
A \sin \left( d \frac{Q}{R} u(0) \right) + B \cos \left( d \frac{Q}{R} u(0) \right) = 0,
\]

however, since \( u(0) \equiv 0 \), then \( B \) must be equal to zero here. This results in the following critical values equation:

\[
e^{-\lambda(R)} \cos \left( d \frac{Q}{R} u(R) \right) = \frac{1}{d} \left( \frac{M}{Q} - \frac{Q}{R} \right) \sin \left( d \frac{Q}{R} u(R) \right)
\]

Here for \( b = 16 \)

\[
e^{-2\lambda(r)} = 1 - \left( \frac{2M}{R} + \frac{2Q^2}{R^2} \right) r^2 + 3 \frac{Q^2}{R^2} r^4.
\]

and

\[
u(r) = \frac{1}{R^2} \int_{0}^{r} \frac{s ds}{\sqrt{1 - \left( \frac{2M}{R} + \frac{2Q^2}{R^2} \right) \frac{s^2}{R^2} + \frac{3Q^2}{R^2} \frac{s^4}{R^4}}}
\]

\[
= \frac{\sqrt{3}}{6} \frac{R}{Q} \left( \log \left[ 1 + \sqrt{3} \left( \frac{M}{3Q} + \frac{Q}{3R} \right) \right] - \log \left[ \sqrt{3} \left( \frac{M}{3Q} + \frac{Q}{3R} - \frac{Qr^2}{R^3} \right) + e^{-\lambda(r)} \right] \right).
\]

Thus the critical values of \( M/R \) vs \( Q/R \) for \( b = 16 \) are solutions of the following equation:

\[
\left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{1/2} = \left( \frac{M}{Q} - \frac{Q}{R} \right) \tan \left[ \frac{\sqrt{3}}{6} \log \left( \frac{1 + \sqrt{3} \left( \frac{M}{3Q} + \frac{Q}{3R} \right)}{e^{-\lambda(R)} + \sqrt{3} \left( \frac{M}{3Q} - \frac{2Q}{3R} \right)} \right) \right]
\]

We solved this equation numerically and the values are plotted in Figure 6. The critical values of \( M/R \) vs \( Q/R \) for \( b = 21 \) and 31 are also plotted in Figure 6. We note that the critical values curves for these cases (\( b > 11 \)) are distinctly different from the previous cases (\( b \leq 11 \)) that we have already studied. The previous critical values curves essentially follow the shape of the Andréasson curve. In the current case we see an initial similarity with previous curves, however now the curves with \( b > 11 \) attains a peak before \( Q/R = 1 \) and then decreases to a final value of \( M/R < 1 \) when \( Q/R = 1 \). These curves have extremal values of \( M/R \) less than one.

Our investigation of cases with \( b \geq 11 \) found that both the peak values, the extremal values and the range of values of \( Q/R \) for which a non-zero value of \( M/R \)}
Figure 6: The critical values of $M/R$ vs $Q/R$ from the $b = 31$ (---), $b = 21$ (-----), $b = 16$ (·····) models and the Andréasson formula (----) are plotted. The extremal line $M/R = Q/R$ (---) is also drawn.

Table 1: This table shows the change in various $M/R$ and $Q/R$ values with increasing $b$. In column (ii) $(M/R)_{\text{ext}}$ are the extremal values of $M/R$, in column (iii) $(Q/R)_{\text{max}}$, gives the maximum value of $Q/R$ for which a non zero value of $M/R$ exists, in column (iv) the quantity $(Q/R)_{\text{peak}}$ is the value of $Q/R$ when the critical values curve attains its peak and in column (v) $(M/R)_{\text{peak}}$ are the corresponding values $M/R$ at the peak.

| $b$ | $(M/R)_{\text{ext}}$ | $(Q/R)_{\text{max}}$ | $(Q/R)_{\text{peak}}$ | $(M/R)_{\text{peak}}$ |
|-----|----------------------|-----------------------|-----------------------|-----------------------|
| 16  | 0.66                 | 1.0                   | 0.92                  | 0.744                 |
| 21  | 0.60                 | 1.0                   | 0.75                  | 0.63                  |
| 26  | 0.558                | 1.0                   | 0.68                  | 0.564                 |
| 31  | 0.523                | 0.952                 | 0.56                  | 0.524                 |
| 36  | 0.495                | 0.851                 | 0.50                  | 0.499                 |
| 41  | 0.471                | 0.766                 | 0.45                  | 0.482                 |

exists continuously decreases with increasing $b$ values. In particular we found that when $b \approx 29.25$ the limiting value of $M/R$ is zero for $Q/R = 1$. This result dictates that a stable configuration with $b \approx 29.25$ and $Q/R = 1$ does not exist. We also found that for $b > 29.25$ the maximum value of $Q/R$ for which a stable final configuration can exist decreases. These behaviors described here is shown in Figure 6 and some values are
Figure 7: The critical values of $M/R$ vs $Q/R$ from the $b = -69$ (........) and $b = -9$ (−−−−) models and the Andréasson formula (____).

given in Table 1.

3.6. Models with $b < 0$

Andréasson, in deriving his formula for the critical values of $M/R$ as a function of $Q/R$ for charged sphere, placed the following constraints on the matter content of the sphere: (i) $p_r + 2p_t < \rho$ and (ii) $\rho > 0$.

Buchdahl in his derivation of the critical value of $M/R$ for neutral perfect fluid spheres explicitly required that $d\rho/dr > 0$. Andréasson did not have such a requirement in the derivation of his formula, in order to check the necessity of such a condition we studied charged spheres with $d\rho/dr > 0$.

The density profile for the spheres that we are studying is given by (3.9), thus when $b < 0$, $d\rho/dr > 0$. We consider the case $b = -9$ in detail here and then comment on other solutions with $b < 0$.

When $b = -9$

$$e^{-2\lambda(r)} = 1 - \left(\frac{2M}{R} - 3\frac{Q^2}{R^2}\right)\frac{r^2}{R^2} - 2\frac{Q^2}{R^2} \frac{r^4}{R^4},$$

(3.71)

and

$$u(r) = \frac{1}{R^2} \int_0^r \frac{sd\sigma}{\sqrt{1 - \left(\frac{2M}{R} - 3\frac{Q^2}{R^2}\right)\frac{s^2}{R^2} - 2\frac{Q^2}{R^2} \frac{s^4}{R^4}}}$$

(3.72)
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\[
\frac{2^4 R}{4 Q} \left[ \arctan \left( 2^\frac{3}{2} \left( \frac{M}{2Q} - 3 \frac{Q}{4 R} + \frac{r^2 Q}{R^3} \right) e^{\lambda(r)} \right) - \arctan \left( 2^\frac{3}{2} \left( \frac{M}{2Q} - 3 \frac{Q}{4 R} \right) \right) \right].
\]

The master equation here is

\[ \frac{d^2 \tilde{\zeta}}{du^2} = a^2 \frac{Q^2}{R^2} \tilde{\zeta}(u) \quad \text{with} \quad a^2 = \left( \frac{11}{5} - b \right) \]  

The solution for this equation with \( b = -9 \) is

\[ \tilde{\zeta}(u(r)) = Ae^{\frac{2Q}{R} u(r)} + Be^{-\frac{2Q}{R} u(r)}. \]  

Applying the boundary conditions we find

\[ A = \frac{1}{2} \left[ \sqrt{1 - \frac{2M}{R} + \frac{Q^2}{R^2}} + \frac{1}{2} \left( \frac{M}{Q} - \frac{Q}{R} \right) \right] e^{-\frac{2Q}{R} u(R)}, \]  

\[ B = \frac{1}{2} \left[ \sqrt{1 - \frac{2M}{R} + \frac{Q^2}{R^2}} - \frac{1}{2} \left( \frac{M}{Q} - \frac{Q}{R} \right) \right] e^{\frac{2Q}{R} u(R)}. \]  

and the equation from which the critical values of \( M/R \) vs \( Q/R \) are obtained is:

\[ \left[ \frac{1}{2} \left( \frac{M}{Q} - \frac{Q}{R} \right) + \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right) \right] = \left[ \frac{1}{2} \left( \frac{M}{Q} - \frac{Q}{R} \right) - \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right) \right] \]  

\[ \times \exp \left[ \sqrt{2} \left( \arctan \left( \left( \frac{1}{2} \frac{M}{\sqrt{2} Q} + \frac{1}{2\sqrt{2} R} \right) e^{\lambda(R)} \right) - \arctan \left( \frac{1}{\sqrt{2} Q} - \frac{3}{2\sqrt{2} R} \right) \right) \right]. \]  

The critical values curve for \( b = -9 \) or \( a = 2 \) is shown in Figure 7. We see that this model saturates the Andréasson limit. In our investigation of solutions for \( b \geq 0 \) we found that the increase in the critical values of \( M/R \) with increasing negative \( b \) values is relatively slow. Therefore to show a noticeable increase in the \( M/R \) values we needed to plot a relatively large negative \( b \) value. We plotted the \( b = -69 \) or \( a = 4 \) curve in Figure 7 and we observe that this model does violate the Andréasson limit. Thus it is possible to exceed the upper limit for \( M/R \) from the Andréasson formula if \( d\rho/dr > 0 \).

4. Conclusion

In this paper we studied exact solutions of the Einstein-Maxwell equations for the interior gravitational field of static spherically symmetric compact objects. We considered spheres that consists of a perfect fluid with a charge distribution that gives rise to a radial static electric field. The inertial mass density of the fluid and the charged density associated with the electric field are given in (3.9). We explored in detail models with \( b = 0, 1, 6, 11, 16 \) and \( -9 \). We computed the stability curves for these models and compared them with the stability curve from the Andréasson formula. A summary of our results are:

(i) All models with \( b \geq 0 \) have critical values of \( M/R \) for a given \( Q/R \) less than those predicted by the Andréasson formula.
(ii) The critical values of $M/R$ for a given $Q/R$ decreases with increasing $b$ ($b > 0$).
   In particular we found that a sphere with a constant inertial mass density and constant charge density (the $b = 0$ model) is more stable than a sphere with constant gravitational mass density and constant charge density (the $b = 6$ model).

(iii) The critical values curves for models with $0 \leq b \leq 11$ closely follows the contour of the Andréasson curve.

(iv) The critical values curves for models with $b > 11$ initially follows the contour of the Andréasson curve but reaches a peak before $Q/R = 1$ and then turns sharply away from the Andréasson curve. The value of $M/R$ when $Q/R = 1$ continuously decreases with increasing $b$ until $b \approx 29.25$ then $M/R = 0$ for $Q/R = 1$. For $b > 29.25$ the maximum value of $Q/R$ for which a stable configuration can exist continuously decreases.

(v) For $b > 11$ extremal models can be formed with $Q/R = M/R \neq 1$.

(vi) For $b < -9$ the critical values of $M/R$ as function of $Q/R$ are larger than those predicted by the Andréasson formula, thus if the density of the fluid $\rho(r)$ has $d\rho(r)/dr > 0$ it is possible to violate the Andréasson limit.

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