Axisymmetric metrics in arbitrary dimensions

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Abstract: We consider axially symmetric static metrics in arbitrary dimension, both with and without a cosmological constant. The most obvious such solutions have an $SO(n)$ group of Killing vectors representing the axial symmetry, although one can also consider abelian groups which represent a flat ‘internal space’. We relate such metrics to lower dimensional dilatonic cosmological metrics with a Liouville potential. We also develop a duality relation between vacuum solutions with internal curvature and those with zero internal curvature but a cosmological constant. This duality relation gives a solution generating technique permitting the mapping of different spacetimes. We give a large class of solutions to the vacuum or cosmological constant spacetimes. We comment on the extension of the C-metric to higher dimensions and provide a novel solution for a braneworld black hole.

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1. Overview

Most systems of physical interest in nature exhibit a certain amount of symmetry: a star is roughly spherically symmetric, a galaxy roughly axisymmetric, our universe has roughly constant spatial curvature. These three examples share the feature that the dominant interaction governing their large scale structure is gravity. Einstein’s theory of general relativity in principle describes a highly nonlinear interaction, yet, if one applies a few physically motivated coordinate choices reflecting the symmetry
of the system, the gravitational equations become easier, and sometimes straightforward, to solve. Indeed, in four dimensions we have a huge range of known solutions covering a wide selection of physically relevant situations [1].

Four dimensions are of course very special for gravity. It is the smallest spacetime dimension in which general relativity becomes nontrivial, in the sense that gravity can propagate through spacetime and objects interact with one another – it is when the nonlinearities of gravity truly start to show up. Yet four dimensions are also a suitably low enough number that we have a pretty good handle on gravitational interactions, and can in fact construct an encyclopaedia [1] of exact solutions rich enough that we can find metrics for all but the most complicated physical systems.

It seems however that string theory is compelling us to have more than four – in fact eleven – dimensions. While the true nature of this eleven dimensional M-theory has yet to be elucidated, there is likely to be some energy range in which physics is described well by a classical (super)gravity field theory in more than four dimensions. As a result, an increasingly important role has emerged for the study of supergravity solutions in 10 and 11 dimensions. Indeed, the celebrated adS/CFT correspondence [2] arises from consideration of the consistency of stringy versus supergravity pictures of a D-brane.

Furthermore in the last few years, and largely motivated by string theory, there has been a huge interest in toy models where our universe is a submanifold, or braneworld, embedded in a higher dimensional spacetime [3, 4, 5, 6]. Many interesting scenarios and ideas have been put forward such as the one by Randall and Sundrum [7], where the “extra” fifth dimension, can be infinite provided that the bulk spacetime is negatively curved. Of course, given that these models allegedly describe our universe, it is crucial to study their cosmology, as well as other strongly gravitating phenomena such as black holes. Brane cosmology of codimension one (i.e., where there is only one extra dimension) is well understood [7, 8, 9], and cosmological perturbations of such models have been studied [10], although there still remain many important unanswered questions.

On the other hand, progress in finding the five dimensional solution to a black hole which is localised on the four dimensional brane-Universe has proved much more elusive. While one can describe a black hole on the brane by simply extending the Schwarzschild solution into the bulk [11], this solution is singular at the adS horizon, and in addition is unstable [12]. Clearly the physically correct solution will be localized in the bulk, and should correspond to a correction or extension of a Schwarzschild black hole in a four dimensional Universe much like the standard FLRW equation of cosmology is extended in the cosmological version of the RS model [1]. Progress in this direction has been realised numerically [13], but it would clearly be preferable to have an analytic solution to the problem if at all possible.

An interesting indirect investigation by Emparan, Horowitz and Myers [14] gave a lower dimensional solution to the problem, namely a three-dimensional black hole...
in a three-dimensional braneworld living in four-dimensional adS spacetime. They realised that the bulk metric should describe an accelerating black hole, the C-metric [16] (for a full reference list see [1]-for the adS case more recently see [17]). To realise why this is so, one must remember that a domain wall, [18], generically has a Rindler horizon i.e., it has an accelerating trajectory in an otherwise constant curvature spacetime. Any black hole residing on the wall (or braneworld universe) must also accelerate by the same amount to “keep up” with the wall’s motion. Unfortunately, to date it has been impossible to find the generalisation of the C-metric in higher than four dimensions. Recently Emparan, Fabbri and Kaloper [19] (see also [20]) put forward a conjecture relating the above problem to the adS/CFT correspondence. A classical bulk solution describing a black hole localised on the brane corresponds to a quantum-corrected black hole for the four-dimensional observer living on the boundary-brane. If this conjecture is true it is not at all surprising that we have difficulty in finding the extension of the C-metric.

The C-metric in four dimensions belongs to the so called Weyl class of metrics: static and axisymmetric solutions of Einstein’s equations. General Relativity in more than four dimensions is rather more rich and complicated. Weyl metrics, which we shall be discussing here, correspond to an integrable system in four dimensions but not in higher dimensions. Furthermore, in four dimensions event horizons are topologically spherical, but in higher dimensions we can have solutions with not only spherical (for early work in the context of string theory see [21]), but also hyper-cylindrical (for the canonical string theoretic p-brane solutions see [22]), and even genuinely toroidal event horizons [23], although it would seem that many of the more exotic topologies have instabilities [24]. However, the story does not stop there, in fact, a consideration of the endpoint of these instabilities leads us to suspect that there might even be ripply horizons [25] (first pointed out in [26]), as well as the more obvious possibility of periodic black hole solutions [13].

While we have an excellent catalogue of solutions in four dimensions, the situation in higher dimensions is more patchy. The known solutions are by and large mostly one-dimensional, in the sense that the metric depends on only one coordinate. In the case of the Horowitz-Strominger black branes [22], or the cosmic p-branes [27], (a set of Poincaré boost-symmetric solutions corresponding to pure gravitating brane-like sources), the solutions depend on a single coordinate transverse to the brane. While it is true that most physical systems can indeed be reduced to effectively depend on only one variable (such as time in the case of cosmology, or ‘radius’ in the case of a static star) in practise, we also want to have a system which depends on two variables - such as an axially symmetric mass distribution, like a galaxy, an anisotropic cosmology or (more recently) a braneworld metric. The known “two-dimensional” solutions tend to have some special symmetry which allows their integrability, isotropy in the case of cosmological braneworld solutions [4], or supersymmetry in the case of ‘intersecting’ brane solutions [28] – although as
these solutions tend to be delocalized they are not as intuitive as might first appear. Genuinely two-dimensional solutions, such as would represent a higher dimensional galaxy, are much harder to find. First steps in this direction are the work of Emparan and Reall [29], who analyse the case where the Killing vectors representing the symmetries of the spacetime commute, and also the work of reference [30], where special classes of dependence were considered.

In this paper, we are interested in the general axisymmetric metric, which we will call a Weyl metric, after the work of Weyl [31] on four dimensional static axisymmetric metrics. (Note that by analytic continuation, we could also consider metrics depending on time and one of the spatial coordinates, which we will call cosmological or Thorne metrics after the work of Thorne [32] on cylindrically symmetric metrics.) The fact that the metric depends on two coordinates is not itself a guarantee that the spacetime is genuinely two dimensional, as the situation with braneworlds so eloquently illustrates [9, 33]. Rather, it is the fact that the ‘transverse’ space, or that part of the metric which does not depend on the two main coordinates, is split into (at least) two separate subspaces spanned by mutually commuting sets of Killing vectors. Each such subspace introduces an additional degree of freedom into the metric – a breathing mode or modulus – and if one of these subspaces does not have zero curvature (Emparan and Reall [29] dealt with the zero curvature case) then, roughly speaking, this curvature introduces a source term into the equation of motion for that particular breathing mode.

To see why this is problematic, it is useful to think of this multidimensional spacetime as an effective two-dimensional field theory, by simply dimensionally reducing over the nonessential coordinates. In this case, we get a scalar field which represents the two dimensional part of the metric, and each breathing mode introduces an additional scalar into this two-dimensional theory. If all we have are kinetic terms, then this is an integrable system, and we can prescriptively solve the general metric. If however there is a curvature term present, then this has the effect of introducing a Liouville potential into the theory, which renders the problem a great deal more subtle and complex.

What we try to do in this paper is to determine to what extent we can give a prescription for solving the general axisymmetric problem. We find a transformation which reduces the Einstein equations to a relatively simple form and give a set of methods for their solution, remarking on the level of completeness of our prescription. We describe a duality relation which relates vacuum spacetimes with ‘internal’ curvature and cosmological constant spacetimes without this ‘internal’ curvature. We present a large range of solutions, some already discovered, and some new ones. The layout of the paper is as follows: after first reviewing the four-dimensional case for reference, we examine the general axisymmetric metric with curvature, and present our reduction of the Einstein equations into two canonical ‘dual’ forms. In the following sections we treat the cases of zero curvature and cosmological constant
(the Emparan-Reall case) presenting some extensions of four-dimensional solutions, and remarking on the intuition these simpler metrics give. We then discuss vacuum spacetimes with internal curvature, and cosmological constant spacetimes without internal curvature, finding a new range of solutions. Finally, we apply our results to braneworld black holes and the higher dimensional C-metric. We present a linearized gravity prescription for general sources, and a new regular exact solution for black holes on adS branes.

2. Four dimensional Weyl metrics: A review

In four dimensions, time and axisymmetry mean that the metric depends only on two remaining coordinates, \( r \) and \( z \) say, and we can write the metric into the block diagonal form

\[
ds^2 = e^{2\lambda}dt^2 - e^{2(\nu-\lambda)}(dr^2 + dz^2) - \alpha^2 e^{-2\lambda}d\phi^2
\]

called the Weyl canonical form. Note that although we have three metric functions, the transverse \( t \) and \( \phi \) spaces are both intrinsically Ricci flat (somewhat trivially), which turns out to be crucial in the integrability of the Einstein equations:

\[
\Delta \alpha = -\alpha e^{2(\nu-\lambda)} [T_{rr} + T_{zz}]
\]

\[
\Delta \lambda + \nabla \lambda \cdot \nabla \alpha - \frac{1}{\alpha} e^{2(\nu-\lambda)} [T_t^t - T_t^r - T_z^z - T_\phi^\phi]
\]

\[
\Delta \nu + (\nabla \lambda)^2 = -e^{2(\nu-\lambda)}T_\phi^\phi
\]

\[
\frac{\partial^2 \alpha}{\alpha} + 2(\partial_{\pm} \lambda)^2 - 2\partial_{\pm} \nu \frac{\partial_{\pm} \alpha}{\alpha} = T_{rr} - T_{zz} \pm 2iT_{rz}
\]

where \( T^b_a \) is the energy momentum tensor (with the factor \( 8\pi G \) absorbed), \( \Delta \) is the two dimensional Laplacian \( (\partial_r^2 + \partial_z^2 = \partial_+ \partial_-) \), with \( \partial_{\pm} = \partial_r \mp i \partial_z \) the derivatives with respect to the complex coordinates \( \zeta = (r + iz)/2 \) and \( \overline{\zeta} \).

In the absence of matter or a cosmological constant, these have a very elegant solution: one simply fixes the conformal gauge freedom remaining in the metric (2.1) by setting \( \alpha \equiv r \), which is consistent with (2.2). This then means (2.3) becomes a cylindrical Laplace equation for \( \lambda \), with solution

\[
\lambda = -\frac{1}{4\pi} \int \frac{S(r')d^3r'}{|r - r'|}
\]

for a source with energy density \( S(r) \). Note then that the metric component \( \lambda \), is nothing but the Newtonian source of axial symmetry. In turn \( \nu \) is determined from \( \lambda \) via (2.5). Since the \( \lambda \) equation is linear, its solutions can obviously be superposed – the nonlinear nature of Einstein gravity showing up in the solution of \( \nu \). Note that as regularity of the \( r \)-axis requires \( \nu(0, z) = 0 \), in general there will be conical singularities when regular solutions are superposed. These can be interpreted as
strings or struts supporting the static sources in equilibrium. Let us now describe briefly some solutions of physical interest which we would like to be able to reproduce in more than four dimensions.

2.1 Black hole spacetimes

Physical solutions of particular interest are of course black hole and multi-black hole solutions, which correspond to line mass sources in (2.6) for the Newtonian picture (for a clear and concise description see [34]) – a semi-infinite line mass source actually corresponding to an acceleration horizon.

To see this, input into (2.6) a line source with unit mass per unit length, \( S(r) = \delta(r)/r \) for \( z \in [c_3, c_4] \):

\[
\lambda_S = -\frac{1}{2} \int_{c_3}^{c_4} \frac{dz'}{[r^2 + (z - z')^2]^{1/2}} = \frac{1}{2} \ln \frac{R_3 - z_3}{R_4 - z_4} = \frac{1}{2} \ln \frac{X_3}{X_4}
\]

(2.7)

where

\[
z_i = z - c_i \quad , \quad R_i^2 = r^2 + z_i^2
\]

(2.8)

Integration of (2.5) then gives

\[
\nu_S = \frac{1}{2} \ln \frac{(R_3 R_4 + z_3 z_4 + r^2)}{2R_3 R_4} = \frac{1}{2} \ln \frac{Y_{34}}{2R_3 R_4}
\]

(2.9)

Although this does not appear to be a Schwarzschild black hole, defining \( M = c_4 - c_3 \), the simple transformation

\[
z = (\rho - M) \cos \theta \quad , \quad r^2 = \rho(\rho - 2M) \sin^2 \theta
\]

(2.10)

in fact returns the metric to its standard spherical form.

Now we can consider superposing solutions for \( \lambda \), to build up multi-black hole solutions in the way first described by Israel and Khan [35]. For the simplest case of adding a second black hole to (2.7) of mass \( M' = c_2 - c_1 \) (with \( c_1 < c_2 < c_3 < c_4 \)) we have:

\[
\lambda = \frac{1}{2} \ln \frac{X_1 X_3}{X_2 X_4}
\]

(2.11)

\[
\nu = \frac{1}{2} \ln \frac{Y_{21} Y_{34} Y_{23} Y_{41}}{4R_1 R_2 R_3 R_4 Y_{13} Y_{24}} + \nu_0
\]

(2.12)

where \( Y_{ij} \) was defined implicitly in (2.9).

The reason for the constant \( \nu_0 \) is that for regularity of the \( z \)-axis, we require \( \nu(0, z) = 0 \), as already mentioned. For a single rod potential, the Schwarzschild black hole, we can make the axis everywhere nonsingular except for the event horizon, \( z \in [c_3, c_4] \). However, once we add another mass source, due to their mutual attraction,
we can no longer have a static spacetime and a nonsingular z-axis. Calculating \( \nu \) on the axis gives

\[
\nu(0, z) = \begin{cases} 
\nu_0 & z > c_4 \text{ or } z < c_1 \\
\ln \left( \frac{(c_3 - c_2)(c_4 - c_1)}{(c_3 - c_1)(c_4 - c_2)} \right) + \nu_0 & c_2 < z < c_3.
\end{cases}
\] (2.13)

Thus if \( \nu_0 = 0 \), meaning that the z-axis is regular away from the black holes, then inbetween the two black holes \( \nu(0, z) < 0 \), hence we have a conical excess or strut, which can be interpreted as supporting the black holes in their static equilibrium. Similarly, if we choose \( \nu_0 \) to have a regular axis inbetween the black holes, we have a conical deficit extending from each black hole out to infinity, which can be interpreted as a string suspending the black holes in their (unstable) equilibrium.

### 2.2 The Rindler and C-metrics

Now consider a semi-infinite line mass (SILM) which can formally be obtained from (2.7) by letting \( c_3, c_4 \to \infty \) with \( c_3/c_4 = \hat{c}_3 \) a constant. Then simultaneously rescaling the dimensionful coordinates \( \hat{t} = t/(2c_4A) \), \( \hat{r} = 2c_4Ar \), \( \hat{z} = 2c_4Az \), gives the Rindler metric:

\[
ds^2 = A\hat{X}_3d\hat{t}^2 - \frac{1}{2AR_3} \left( d\hat{r}^2 + d\hat{z}^2 \right) - \frac{\hat{r}^2}{A\hat{X}_3} d\phi^2
\] (2.14)

One can put this in a more transparent ‘C-metric’ form with the coordinate transformation

\[
\hat{r} = \frac{\sqrt{1 - x^2} \sqrt{y^2 - 1}}{A(x + y)^2}, \quad \hat{z} = \frac{1 + xy}{A(x + y)^2}
\] (2.15)

where \( A = 1/(2\hat{c}_3) \) is the acceleration parameter for the Rindler metric, and which gives

\[
ds^2 = \frac{1}{(x + y)^2} \left[ (y^2 - 1)A^{-2}dt^2 - \frac{dy^2}{(y^2 - 1)} - \frac{dx^2}{(1 - x^2)} - (1 - x^2)d\phi^2 \right]
\] (2.16)

To get flat space use

\[
T = \frac{\sqrt{y^2 - 1}}{A(x + y)} \sinh At \quad Y = \frac{\sqrt{1 - x^2}}{A(x + y)} \cos \phi
\]

\[
X = \frac{\sqrt{y^2 - 1}}{A(x + y)} \cosh At \quad Z = \frac{\sqrt{1 - x^2}}{A(x + y)} \sin \phi
\] (2.17)

Thus we see how flat space can be written in a variety of forms, the ‘C-metric’ form (2.16) being the most obvious way of tailoring a coordinate system to an accelerating observer. On the other hand the canonical Weyl form (2.14) allows us to make contact with Newtonian potential theory and building up spacetimes with acceleration horizons and black holes - the simplest of course being the C-metric [16] which we turn to next.
As with the Rindler metric, if we take $c_4 \to \infty$, and rescale $t, r, z$ as before, then we get the C-metric in Weyl coordinates:

$$ds^2 = \frac{X_1 \dot{X}_3}{AX_2} dt^2 - \frac{\hat{Y}_{12} \hat{Y}_{23}}{4AR_1 R_2 R_3 Y_{13}} (d\dot{r}^2 + d\dot{z}^2) - \frac{\dot{X}_2}{AX_1 X_3} d\phi^2$$

(2.18)

The C-metric has the canonical form (see also more recently [36]),

$$ds^2 = A^{-2}(x + y)^{-2}[F(y)dt^2 - F^{-1}(y)dy^2 - G(x)d\phi^2 - G^{-1}(x)dx^2],$$

(2.19)

where

$$G(x) = 1 - x^2 - 2mA x^3, \quad F(y) = -1 + y^2 - 2mA y^3$$

(2.20)

Here, $m$ represents the mass of the black holes, and $A$ their acceleration. In the flat space limit, $A^{-1}$ represents half the distance of closest approach. Let us write $x_1 < x_2 < x_3$ for the roots of $G$. Then, in order to obtain the correct signature, we must have $x_2 < x < x_3$ and $-x_2 < y < -x_1$. The coordinates cover only one patch of the full spacetime corresponding to the exterior spacetime of one accelerating hole up to its acceleration horizon, which is located at $y = -x_2$. The coordinate singularity at $y = -x_1$ corresponds to the event horizon of the black hole, whereas the black hole singularity itself is screened and resides at $y \to \infty$. The conical deficit sits along $x = x_2$, while $x = x_3$ points towards the other black hole, which means that $\phi$ has periodicity $4\pi/|G'(x_3)|$. We will return to the C-metric in the conclusions.

3. General set-up in arbitrary dimensions and the equivalence

In more than four dimensions, an axisymmetric solution will in general have a non-abelian group of Killing symmetries (the abelian case was formally analyzed in [29]), usually corresponding to $SO(n)$ symmetry for some $n$. It turns out that this has a significant effect on the solubility of the Einstein equations, as the metric now contains closed Killing surfaces of constant curvature, which give rise to source terms in the Einstein equations somewhat analogous to cosmological terms in the four-dimensional equations (2.2-2.5). This is what we mean by internal curvature. To see this explicitly, consider the Weyl metric corresponding to an axisymmetric $p$-brane living in $D = p + n + 3$ dimensions:

$$ds^2 = A^2 [dt^2 - dy_p^2] - B^2 (dr^2 + dz^2) - C^2 dx_{n,\kappa}^2$$

(3.1)

This represents a Poincare $p$-brane with $(n + 2)$-orthogonal directions, and the $x$-space has curvature $\kappa = 0, -1, 1$. In essence, the $p$-spatial directions of the brane are superfluous, and we are primarily interested in $p = 0$, but for generality, we will maintain the parameter $p$ while deriving the equations of motion and many of the solutions.
In seeking the equivalent of the Weyl canonical form we first calculate the Ricci curvature of this metric:

\[
R^i_i = \frac{1}{B^2} \left[ \frac{\Delta A}{A} + p \left( \frac{\nabla A}{A} \right)^2 + n \frac{\nabla A \cdot \nabla C}{AC} \right] \tag{3.2}
\]

\[
R^x_x = \frac{1}{B^2} \left[ \frac{\Delta C}{C} + (n-1) \left( \frac{\nabla C}{C} \right)^2 + (p+1) \frac{\nabla A \cdot \nabla C}{AC} \right] - \frac{(n-1) \kappa}{C^2} \tag{3.3}
\]

\[
R^z_z = \frac{p+1}{B^2} \left[ \frac{\dddot{A}}{A} + \frac{\dot{A}'}{AB} - \frac{\dddot{B}}{AB} \right] + \frac{n}{B^2} \left[ \frac{\dddot{C}}{C} + \frac{\dot{C}'}{CB} - \frac{\dddot{B}}{CB} \right] + \frac{\Delta \ln B}{B^2} \tag{3.4}
\]

\[
R^r_r = \frac{p+1}{B^2} \left[ \frac{\dddot{A}}{A} - \frac{\dot{A}'}{AB} + \frac{\dddot{B}}{AB} \right] + \frac{n}{B^2} \left[ \frac{\dddot{C}}{C} - \frac{\dot{C}'}{CB} + \frac{\dddot{B}}{CB} \right] + \frac{\Delta \ln B}{B^2} \tag{3.5}
\]

\[
R_{rz} = -n \frac{\dot{C}'}{C} - (p+1) \frac{\dot{A}'}{A} + n \frac{\dot{C}'}{CB} + n \frac{\dot{C}'}{CB} + (p+1) \frac{\dot{A}'}{AB} + (p+1) \frac{\dot{B}'}{AB} \tag{3.6}
\]

where a prime denotes \( \partial/\partial r \), and a dot \( \partial/\partial z \). It is immediately apparent that the appropriate generalisation of the \( \alpha \) variable from the four-dimensional case is

\[
\alpha = A^{p+1}C^n, \tag{3.7}
\]

however, we have some freedom in how we spread \( \alpha \) across \( A \) and \( C \). To see why this might be relevant, consider first writing

\[
\ln A = a \phi \tag{3.8}
\]

\[
\ln B = \chi - \frac{\gamma}{2} \phi - \frac{n-1}{2n} \ln \alpha \tag{3.9}
\]

where we have set

\[
a = \pm \sqrt{\frac{n}{2(p+1)(n+p+1)}}, \quad \gamma = \pm \sqrt{\frac{2(p+1)}{n(n+p+1)}} \tag{3.10}
\]

In other words, we have rewritten the metric as:

\[
ds^2 = e^{2a\phi} \left[ dt^2 - dy_p^2 \right] - e^{-\gamma \phi} \left\{ \alpha^{-(n-1)/n} e^{2\chi} (d\tau^2 + dz^2) + \alpha^{2/n} dx_{n,\kappa} \right\} \tag{3.11}
\]

By rewriting the metric specifically in this way, the connection to cosmological dilatonic metrics can be made apparent, since upon “dimensional reduction” (ignoring the space or time-like nature of the dimensions being reduced over) over \( t \) and \( y \) we obtain a cosmological-type metric in an \( (n+2) \)-dimensional spacetime with a Liouville potential if the cosmological constant in \( D \)-dimensions does not vanish:

\[
S_D = \int d^D X \left[ -R_D + 2 \Lambda \right] \propto \int d^{n+2} x \left[ -R_{n+2} + \frac{1}{2} (\nabla \phi)^2 + 2 \Lambda e^{-\gamma \phi} \right] \tag{3.12}
\]

Note that \( \gamma \) in (3.10) relates to the exponential Liouville coupling in (3.12). Of course, strictly speaking to get a metric with a cosmological interpretation we have
to double-analytically continue $t \rightarrow i\chi$ and $r \rightarrow i\tau$, however the overall symmetries of the equations of motion remain essentially the same, simply swapping operators from Laplacian to D’Alembertian. Such metrics arise in many string-inspired cosmological models (see for example [37], [38], [39].

Using (3.8-3.9), the equations of motion take the form

\[
\Delta \alpha = -2\Lambda \alpha B^2 + n(n-1)\kappa \alpha G \tag{3.13}
\]
\[
a \left( \Delta \phi + \nabla \phi \cdot \frac{\nabla \alpha}{\alpha} \right) = -\frac{2\Lambda B^2}{(D-2)} \tag{3.14}
\]
\[
\Delta \chi + \frac{1}{4}(\nabla \phi)^2 = -\frac{\Lambda B^2 n - (n-1)\kappa G}{n} \tag{3.15}
\]
\[
\frac{\partial^2 \alpha}{\alpha} + \frac{1}{2}(\partial_{\pm} \phi)^2 - 2\partial_{\pm} \chi \frac{\partial_{\pm} \alpha}{\alpha} = 0 \tag{3.16}
\]

where as before $2\partial_{\pm} = \partial/\partial(r \pm iz)$, and

\[
B^2 = e^{2\chi} \alpha^{-(n-1)/n} e^{-\gamma \phi} \tag{3.17}
\]
\[
G = B^2/C^2 = e^{2\chi} \alpha^{-(n+1)/n} \tag{3.18}
\]

The operator expressions appearing on the left hand side of (3.13-16) are independent of dimension unlike the expressions, (3.2-3.4). Therefore upon switching off the curvature scales $\kappa$ and $\Lambda$ we immediately obtain the equivalent of the Weyl canonical form in arbitrary dimensions. Note also that if $\Lambda = 0$, then (3.14) states that $\alpha \nabla \phi$ is divergence free in the $(r, z)$ plane.

The writing of the field equations in a canonical form is not unique, we could instead choose to write

\[
\ln A = \bar{a} \bar{\phi} + \frac{1}{D-2} \ln \alpha \tag{3.19}
\]
\[
\ln B = \bar{\chi} - \frac{D-3}{2(D-2)} \ln \alpha \tag{3.20}
\]

for a $D$-dimensional spacetime. The metric then takes quite a different form from (3.11) and reads,

\[
ds^2 = \alpha^{\frac{2}{D-2}} \left\{ e^{2\bar{a} \bar{\phi}} \left[ dt^2 - dy_\rho^2 \right] - e^{-\gamma \bar{\phi}} dx_{\bar{a}, \bar{\eta}}^2 \right\} - \alpha \left( \frac{\partial}{\partial \bar{\phi}} \right)^2 e^{2\bar{\chi}} (dr^2 + dz^2) \tag{3.21}
\]

Then the equations of motion are:

\[
\Delta \alpha = -2\Lambda \alpha B^2 + \bar{n}(\bar{n} - 1)\kappa \alpha G \tag{3.22}
\]
\[
\bar{a} \left( \Delta \bar{\phi} + \nabla \bar{\phi} \cdot \frac{\nabla \alpha}{\alpha} \right) = -\frac{\bar{n}(\bar{n} - 1)}{(D-2)} \kappa G \tag{3.23}
\]
\[
\Delta \bar{\chi} + \frac{1}{4}(\nabla \bar{\phi})^2 = -\frac{\Lambda B^2}{D-2} - \frac{\bar{n}(\bar{n} - 1)\kappa G}{2(D-2)} \tag{3.24}
\]
\[
\frac{\partial^2 \alpha}{\alpha} + \frac{1}{2}(\partial_{\pm} \bar{\phi})^2 - 2\partial_{\pm} \bar{\chi} \frac{\partial_{\pm} \alpha}{\alpha} = 0 \tag{3.25}
\]
It is important now to emphasize that although (3.11) and (3.21) are different, the constraint equations (3.16) and (3.25) are unchanged, as is the form of the $\alpha$ equation. Furthermore the expressions for $B^2$ and $G$ are now

$$B^2 = e^{2\tilde{\chi}_\alpha^{-(\bar{D}-3)/(\bar{D}-2)}}$$  \hspace{1cm} (3.26)$$

$$G = B^2/C^2 = e^{2\tilde{\chi}e^{2\gamma\bar{\phi}^{-(\bar{D}-1)/(\bar{D}-2)}}}$$  \hspace{1cm} (3.27)$$

With the new definition of the variables (3.19-3.20) there is therefore a natural parallel, or duality, between the $\kappa = 0$, $\Lambda \neq 0$ system and the $\Lambda = 0$, $\kappa \neq 0$ barred system. Indeed by performing the mapping, $\chi \leftrightarrow \tilde{\chi}, \phi \leftrightarrow \tilde{\phi}$ and formally identifying,

$$n \leftrightarrow -\bar{D} + 2, \quad \kappa \leftrightarrow \frac{2\Lambda}{(\bar{D}-1)(\bar{D}-2)}$$  \hspace{1cm} (3.28)$$

the expressions for $B^2$ and $G$ are exchanged from (3.17)-(3.18) to (3.26)-(3.27) and vice-versa. Therefore the field equations (3.13-3.16) and (3.22-3.25) are exactly equivalent. In practical terms this means that once we have a set of solutions (in arbitrary dimension $D$) for the $\kappa = 0$, $\Lambda \neq 0$ $\phi$-Weyl system we can, using (3.28), obtain the set of solutions for the $\Lambda = 0$, $\kappa \neq 0$ $\phi$- Weyl system and vice-versa. The duality relates vacuum $D$-dimensional solutions with an internal curvature source $\kappa$, to constant curvature $\Lambda$ spacetimes of dimension $\bar{D}$. Of course this is not a relation between physical spacetimes, since for $n \geq 2, \bar{D} \leq 0$. However, many of the equations that follow, and their corresponding solutions, are written formally in terms of $n$ and $D$ and so can be dualized. The same of course holds for the $n + 2$-dimensional scalar field counterparts of these Weyl spacetimes. The duality in that case relates free scalar field solutions with internal curvature $\kappa$ to scalar field solutions with a Liouville potential. We shall be making use of this in section 4 (for an explicit example see section 6.2).

4. Vacuum spacetimes with a flat transverse space

If $\kappa$ and $\Lambda$ both vanish, this is clearly of the Emparan-Reall [29] form, and writing $\phi = 2\lambda$, we see that the equations are identical to the four-dimensional Weyl-vacuum equations. Thus any four-dimensional vacuum solution automatically generates a higher dimensional solution. While these solutions are implicitly contained in [29], it is useful to consider a few simple examples to understand the appropriate generalization of the four-dimensional Weyl solutions.

First of all, note that the metric (3.11) as written is Poincare invariant in the $(t, y)$ directions. This means that any solution has the interpretation of a boost symmetric
or cosmic $p$-brane (in the sense of [27]). So let us consider the case of $n = 1$, where the space orthogonal to the $p$-brane is three dimensional. Since $n = 1$, naturally $\kappa = 0$, and so we can write

$$\alpha = r, \quad \phi = 2\lambda_4, \quad \chi = \nu_4$$

or

$$ds^2 = (e^{2\lambda_4})\sqrt{\frac{2}{(p+1)(p+2)}}[dt^2 - dy^2_p] - (e^{-2\lambda_4})\sqrt{\frac{2(p+1)}{(p+2)}}[e^{2\nu_4}(dr^2 + dz^2) + r^2d\theta^2]$$

(4.1)

A natural four-dimensional potential to consider is that of the Schwarzschild black hole, ([2.7]), which, after unravelling back to a spherical coordinate system via (2.10), gives

$$ds^2 = \left(1 - \frac{2M}{\rho}\right)\sqrt{\frac{\rho}{(p+1)(p+2)}}[dt^2 - dy^2_p] - \left(1 - \frac{2M}{\rho}\right)^{-\sqrt{\frac{2(p+1)}{(p+2)}}} [d\rho^2 + \rho(\rho - 2M)d\Omega^2_4]$$

(4.2)

in agreement with the results (see eqn (3.8)) in [27]. This method therefore provides a separate confirmation that the appropriate Poincaré symmetric $p$-brane solutions are of this singular form.

The existence of a singular solution seems disturbing after our experience of four-dimensional gravity, where singularities are mostly cloaked by event horizons. However, in higher dimensions with extended solutions, the natural nonsingular black branes of Horowitz and Strominger are unfortunately generally unstable [24], and it seems likely that the true stable extended solution simply is this singular one. Indeed, the conical deficit of the extended string solution in four dimensions is strictly speaking singular, in that the Ricci scalar has a delta-function singularity at the string core. This is nonetheless tolerated as field theory is well defined on the remaining spacetime – wave operators are self-adjoint and the propagator is well defined. In this case, these higher dimensional extended solutions, while containing null singularities, also have the property that wave operators are self-adjoint, with Green’s functions being well defined [40]. It seems likely therefore that despite our qualms, these singular solutions should be accepted as legitimate additions to the family of extended gravitating solutions and indeed perhaps the only appropriate ones. It is in fact quite likely that these are the endpoints of any black string instability.

We can also construct more complicated black brane metrics, such as two $p$-branes held in equilibrium at a fixed distance apart by $(p+1)$-brane conical deficits by simply inputting the Israel-Khan potentials $\lambda_4$, $\nu_4$, from (2.11,2.12). However, perhaps a more interesting physical set-up to consider is that of an accelerating $p$-brane obtained by putting in the C-metric potential:

$$ds^2 = \left(\frac{F(y)}{A^2(x+y)^2}\right)^\frac{\sqrt{2}}{\sqrt{(p+1)(p+2)}} [dt^2 - dy^2_p]$$
\[- \left( \frac{F(y)}{A^2(x+y)^2} \right)^{1-\sqrt{\frac{2(\mu+1)}{(p+2)}}} \left( \frac{1}{A^2(x+y)^2} \left[ \frac{dy^2}{F(y)} + \frac{dx^2}{G(x)} + G(x) d\phi^2 \right] \right) \] (4.4)

The interesting point to note about this metric is that it is now not only singular at the \(p\)-brane horizon, but also at the acceleration horizon. In part, this is because of geometry – Rindler spacetime is a transformation of flat space, and it is not possible to slice flat space in such a way as to have a flat accelerating brane. Another way of putting this is to see that using the Rindler potentials (2.14) for (4.2) gives a non-flat metric. In order to have zero Weyl curvature, we must have curvature on the \(p\)-brane, \(i.e.,\) an Einstein-de Sitter metric.

Does this mean that (4.4) has no physical significance? This is an interesting question. In the case of domain walls or global vortices, imposing a Poincaré symmetry on the defect leads to a singular metric, whereas allowing a dS worldbrane removes this singularity. This is also the case for a pure gravitating “spherically symmetric” \(p\)-brane [41]. Although we are interested here in general axisymmetric pure gravity solutions, the fact that formerly regular horizons become singular as dimensionality is altered, and also the possibility of removing those singularities by altering the intrinsic geometry of the brane is a key point, and one we will return to later. It may well be that if we wish a cosmic censor to be operative, we are severely restricted as to the types of geometry we can consider. Unfortunately, only the existence of the regular inflating branes of [41] are known, not an explicit solution.

We can also extend the C-metric in arbitrary dimensions for \(\kappa = \Lambda = 0\). To do this, rather than bringing the C-metric to the Weyl form, we coordinate transform (3.21) for \(D = 4\) and \(\Lambda = 0\),

\[ ds^2 = \alpha \left\{ e^\delta dt^2 - e^{-\bar{\phi}} dx^2 \right\} - \alpha^{-1/2} e^{2\chi} \left( \frac{df^2}{f^2} + \frac{dg^2}{g^2} \right) \] (4.5)

in a “canonical form” where we set

\[
\left( \frac{df}{dr} \right)^2 = f'^2 = F(f), \quad \left( \frac{dg}{dr} \right)^2 = g'^2 = G(g) \] (4.6)

with \(F\) and \(G\) given by (2.20). Then identifying (2.13) and (4.7) we get,

\[
\alpha = \frac{f'g'}{A^2(f + g)^2}, \quad \bar{\phi} = \ln \frac{f'}{g'} \] (4.7)

and the field equations (3.22-3.25) are satisfied with,

\[
e^{2\chi} = \frac{\sqrt{f'g'}}{A^3(f + g)^3} \] (4.8)

Of course for the moment we have achieved nothing new, this is just the C-metric for \(D = 4\) in the variables we defined in the previous section. However in these variables
the solution can now be extended to arbitrary dimension $D$ as long as $\Lambda = 0$ and $\kappa = 0$. The solution takes the rather complicated form,

$$ds^2 = \left(\frac{1}{A(f+g)}\right)^{\frac{1}{D-2}} \left\{ -(FG)^{\frac{D-2}{2}} \left(\frac{df^2}{F} + \frac{dy^2}{G}\right) + (FG)^{\frac{1}{2-D}} \left[ \frac{F}{G} a [dt^2 - dy^2] - \frac{F}{G} \frac{2}{n} dx^2 \right] \right\}$$

(4.9)

This solution is regular only for $D = 4$. Indeed note then from (3.13-3.16) that we can have a source term $\kappa = 1$ (and thus a well defined horizon) since $n = 1$. For $D > 4$ the requirements of planar topology, and the absence of a cosmological constant, do not permit the screening of a singularity, analogous to the case of the ordinary planar symmetric four-dimensional spacetime [42] which is also singular.

5. Weyl metrics with subspaces of constant curvature

In the general vacuum case, where $\Lambda = 0$, but $\kappa = 1$, we would ideally like to classify all solutions. Note that this case is closely related to braneworlds with bulk scalars via the dimensional reduction discussed earlier. Clearly, there is a large class of solutions which are effectively one-dimensional following from earlier work in braneworld scalar solutions [38], as well as a more general (though still essentially one-dimensional) analysis in [30]. These cases can be considered as special within the context of a more general analysis which we now present.

The key is to consider the $\phi$ equation (3.14) which is now homogeneous. This equation states that $\ast\alpha d\phi$ is a closed form. This in turn suggests that writing $\phi = \phi(z)$ should pick up a large class of solutions. Inputting this form for $\phi$ into (3.14) implies that $\alpha$ is separable:

$$\alpha = f(r)g(z),$$

(5.1)

where

$$g(z) = c/\dot{\phi}$$

(5.2)

and $c$ is an arbitrary nonzero constant for $\dot{\phi} \neq 0$, if $\dot{\phi} = 0$ we set $c = 0$, $g \equiv 1$. Therefore under the hypothesis $\phi = \phi(z)$ we can find the general solution to the field equations once we solve for $f$, $g$ and $\chi$ using (3.13-3.16). We have three classes of possible solutions: Class I with $f' = 0$, Class II with $g' = 0$, or, Class III where neither $f'$ or $g'$ vanish. Let us deal with these in turn.

5.1 Class I solutions

Given that $f' = 0$, Class I solutions are characterised by the fact that they depend only on one variable. Equivalently we can note that Class I solutions manifestly
have an extra Killing vector, therefore, we can expect solutions of greater than Weyl
symmetry appearing in this class. For \( \dot{\phi} \neq 0 \) \([5.2]\), after some algebra we obtain a
static \( p \)-brane solution:

\[
ds^2 = V(\xi) \frac{n^2}{M(n-1)(p+1)} (dt^2 - dy_p^2) - V(\xi) \frac{n}{M(n-1)} \# d\tau^2 - V(\xi)^{\frac{p+n+M}{(n-1)M}} \left( \frac{d\xi^2}{V(\xi)} + \xi^2 d_n^2 \right)
\]

(5.3)
of axial symmetry with \( \xi \) the radial coordinate. Here of course we are restricted to
\( n \neq 1 \). We have defined a spacetime dimension dependent parameter

\[
M^2 = \mu^2 + \frac{n^2(n+p+1)}{(n-1)(p+1)},
\]

(5.4)
and \( \mu \) is an integration constant. The potential \( V(\xi) \) reads,

\[
V(\xi) = 1 + \frac{M}{\xi^{n-1}}
\]

(5.5)
where \( M \) is not necessarily positive. Note that while class I solutions are asymptotically flat, the solution has curvature singularities at \( \xi = 0 \) and \( V(\xi) = 0 \). This is the most general axisymmetric solution depending on one coordinate only, and generalizes the cosmic \( p \)-brane solutions of \([27]\).2

Analytic continuation between the component independent coordinates can yield
different solutions. With little effort for example we can obtain a de-Sitter or inflating
\((n - 1)\)-brane solution,

\[
ds^2 = V^{\frac{n+n+M}{M(n-1)(p+1)}} \left[ \xi^2 (d\tau^2 - e^{2\sqrt{\kappa} \tau} dx_{n-1}^2) - \frac{d\xi^2}{V} \right] - V^{\frac{n}{M(n-1)}} \# d\tau^2 - V^{-\frac{n^2}{M(n-1)(p+1)}} dy_{p+1}^2
\]

(5.6)
where the curvature radius of the spherical sections \((5.3)\) translates into the de-Sitter
expansion of the \((n - 1)\)-brane.

Alternatively, to obtain the class I Thorne vacuum solution simply take \( z \leftrightarrow i\tau \),
\( \tau \leftrightarrow iy \),

\[
ds^2 = V^{\frac{n}{M(n-1)}} \# Kd\tau^2 - V^{-\frac{n^2}{M(n-1)(p+1)}} dy_{p+1}^2 - V^{\frac{n+n+M}{(n-1)M}} \left( \frac{\kappa d\xi^2}{V} + \xi^2 dx_{n,m}^2 \right)
\]

(5.7)
It is now apparent that these solutions are the extensions of the cosmic \((p+1)\)-brane
solutions \([27]\) (see also \([40]\) for an application of these spacetimes to braneworld models). Indeed here the cosmic \((p+1)\)-brane exhibits ‘cosmological’ rather than Poincaré symmetry. To recover the flat case \([27]\) one simply equates the metric components obtaining

\[
\mu_{flat} = \frac{n(n+p+1)}{(n-1)(p+1)}
\]

(5.8)

\[2\]It is worth noting here that \( \xi = 0 \) is an event horizon for \( c \neq 0 \) only if \( \mu = \frac{n}{n-1} \) and \( p = 0 \). Then the solution reduces to the black string solution given by \((5.11)\).
The solutions (5.7) are asymptotically flat and singular at the origin just like their flat counterparts [27].

Thorne class I solutions are all related via (3.12) to a free scalar field spacetime of dimension \( d = n + 2 \). The class I solution (for arbitrary \( c \)) reads,

\[
ds^2 = V^{-\mu \kappa} \kappa dt^2 - V^{\mu - c(n-1)/2} \kappa d\xi^2 - V^{\mu + c(n-1)/2} \xi^2 dx_{n,\kappa}^2 \tag{5.9}
\]

with,

\[
\phi = \frac{cn}{M(n-1)^\gamma} \ln|V| \tag{5.10}
\]

For \( \phi = \text{const.} \) the class I solution (5.7) is simply the \( d \)-dimensional Schwarzschild black hole extended in \((p+1)\) dimensions

\[
ds^2 = \left( \kappa - \frac{\mu}{\xi^{n-1}} \right) dt^2 - \left( \kappa - \frac{\mu}{\xi^{n-1}} \right) - \xi^2 dx_{n,\kappa}^2 - dy_{p+1}^2 \tag{5.11}
\]

with ADM mass proportional to \( \mu \) i.e., the Horowitz-Strominger black \((p+1)\)-brane. These solutions are the only regular solutions for this class of metrics, however, it has been shown that they are unstable [24] and hence unphysical.

### 5.2 Class II solutions

For Class II solutions, i.e., \( g' = 0 \), the \( \phi \) field depends linearly on \( z \), but every other variable depends only on \( r \). These solutions are therefore minimally two-dimensional. The field equations then reduce to a single third order non-linear differential equation for \( f(r) = \alpha/g \):

\[
\frac{f''}{f} - \frac{f'}{f} \left( \frac{f'}{nf} + \frac{f'''}{f''} \right) = \frac{1}{2g^2} \tag{5.12}
\]

Without loss of generality we can set \( g = 1 \), and by writing \( X = f'/f \), \( Y = f''/f'' \), this equation can be recast as a two dimensional dynamical system which allows us to analyse the general form of the solution:

\[
X' = XY - \frac{(n-1)}{n} X^2 + \frac{1}{2} \tag{5.13}
\]

\[
Y' = \frac{(n-2)}{n^2} X^2 - \frac{2}{n} XY - \frac{(n+2)}{2n} \tag{5.14}
\]

This has no finite critical points, however, there are four very clear asymptotes, two for increasing \( r \) and two for decreasing \( r \). The phase plane is plotted in figure [ ] for \( n = 2 \), although the picture is qualitatively the same for all \( n \).

Note that since \( \alpha = f \), we must have \( f'' = f(2XY + \frac{X^2}{n} + \frac{1}{2}) \neq 0 \) for a solution of the Einstein equations with \( \kappa \neq 0 \). The phase plane therefore splits into two regions corresponding to positive and negative \( f'' \). These are separated by the invariant hyperboloid \( Y = -X/n - 1/2X \), shown as the thick line in figure [ ] for \( n = 2 \). The connected region between the two branches of this hyperboloid corresponds to positive \( \kappa \).
**Figure 1**: The Weyl class II phase plane. The thick lines represent the invariant hyperboloid $\mathcal{C} = XY + \frac{X^2}{2} + \frac{1}{2} = 0$ which corresponds to $\kappa = 0$. The positive $\kappa$ region lies inbetween the two branches of $\mathcal{C}$.

To find these asymptotes, note first that by plotting the isoclines we see that $Y \sim -\frac{1}{2X}$ is an asymptotic solution which, solving for $X$, hence $f$ and $\alpha$, corresponds to the metric

$$ds^2 = e^{2az} \left[ dt^2 - dY_p^2 \right] - e^{-2(p+1)az/n} \left[ r^{\frac{n}{2}} d\Omega_n^2 + \frac{e^{-r^2/4}}{n(n-1)|r|^{n-2}} (dr^2 + dz^2) \right] \quad (5.15)$$

for large $|r|$. In other words, this asymptotic solution corresponds to a singular
infinity for \(|r| \to \infty\).

For the other asymptotes, note that for large \(X\) and \(Y\), a solution to \((5.13, 5.14)\) must have \(Y = \lambda_\pm X\), where \(\lambda_\pm = [(n - 3) \pm (n - 1)]/(2n)\). The first of these, \(\lambda_+\) corresponds to a separatrix between solutions which asymptote \((5.13)\) and those which asymptote \(Y = \lambda_-X = -X/n\), which correspond to the metric

\[
    ds^2 = e^{2a_{c_0}z} \left[ dt^2 - dy^2 \right] - e^{-2(p+1)ac_0z/n} \left[ R^2 d\Omega^2_n + \frac{dR^2}{1 + \mu R^{-n-1}} + \left( 1 + \frac{\mu}{R^n} \right) dz \right]
\]

for \(R \to 0\), where \(\mu > 0\) and \(c_0\) are integration constants. Although this solution is reminiscent of the Euclidean black hole cigar, since \(\mu > 0\) and \(z\) is not periodically identified this solution is singular as \(R \to 0\).

Now that we have these asymptotic forms, we can see that there are two distinct types of class II spacetime solutions, one for \(\kappa = 1\), which asymptotes \((5.15)\) for both large negative and large positive \(r\), and one for \(\kappa = -1\), which for small finite \(r\) looks like the Euclidean Schwarzschild style solution \((5.16)\) and asymptotes \((5.13)\) for large \(r\). (The trajectory starting from \((5.15)\) and terminating on \((5.16)\) is simply this spacetime reversing \(r\).)

If however, we are dealing with a Thorne rather than a Weyl metric, then the sign of the RHS of \((5.12)\) changes, and hence the signs of the constant terms in \((5.13, 5.14)\) change. The main effect of this is that it introduces a pair of critical points \(P_\pm = \pm(\sqrt{n/2}, \sqrt{n/2})\), \(P_+\) an attractor, \(P_-\) a repellor, which are focal in nature for \(n < 8\). The phase plane is shown in figure 2 for \(n = 2\). The critical point solution is obtained by setting \(f = e^r\) (restoring the constant \(g\) in \((5.12)\)),

\[
    ds^2 = e^{2r} \left( dt^2 - dr^2 \left/ \left( n(n-1)\kappa \right) \right. \right) - e^{-2t} (p+1)(p+1) d\Omega^2_n - e^{-2t} (p+1)(p+1) dx^2_{p+1} \tag{5.17}
\]

The coordinates \(t\) and \(r\) vary on the whole real line and the metric is singular as \(r \to -\infty\) and as \(r \to -\infty\). For the dilaton spacetime we have simply,

\[
    ds^2 = e^{2r} \left( (dt^2 - dr^2) / (n(n-1)\kappa) \right) - dx^2_{n,\kappa} \tag{5.18}
\]

\[
    \phi = -\sqrt{2/n} t + \phi_0 \tag{5.19}
\]

the linear dilaton metric solution. The general form of the Class II Thorne metric is given by,

\[
    ds^2 = e^{2r} / (n(n-1)\kappa) f''(r)(dt^2 - dr^2) - f(r) e^{2r} / (p+1)(p+1) d\Omega^2_n - e^{-2t} / (p+1)(p+1) dx^2_{p+1} \tag{5.20}
\]

where \(f\) is a solution of \((5.12)\).
Figure 2: The Thorne class II phase plane for \( n = 2 \). Once again the thick lines are the invariant hyperboloid \( \mathcal{C} = XY + \frac{X^2}{2} - \frac{1}{2} = 0 \), with the region inbetween the two branches now corresponding to negative \( \kappa \). The critical points have \( \kappa = 1 \), and the attractor solution \( P_+ \) therefore corresponds to the generic asymptotic spacetime.

5.3 Class III solutions

In the case of class III, where both \( f \) and \( g \) are nontrivial, the field equations (3.13-3.16) give after some algebra two possible families of equations for \( f \) and \( g \):

\[
\begin{align*}
 f^2 &= f_0 f^{2-2/n} - c_0 f^2 \\
 g^2 &= \frac{n}{2(n+1)} + c_0 g^2
\end{align*}
\]  (5.21)
or \(f'^2 = d_0 f^2\)
\(g'^2 = \frac{n}{2(n - 1)} + g_0 g^{2-2/n} - d_0 g^2\) \hspace{1cm} (5.22)

The constants \(f_0\) and \(g_0\) are both different from zero since otherwise \(\Delta \alpha = 0\).

The first family of equations (5.21) are readily solved (with \(c_0 \neq 0\)) to give

\[
f = \left(\frac{f_0}{c_0}\right)^\frac{2}{n} \sin^n \frac{c_0 r}{n} \quad g = \frac{\sqrt{n}}{\sqrt{2c_0(n + 1)}} \sinh \sqrt{c_0} z
\]

or, writing

\[
\xi^n = \frac{f_0^{\frac{2}{n}} \sqrt{n}}{c_0^{\frac{2}{n}} \sqrt{2(n + 1)}} [1 + \cosh \sqrt{c_0} z] = \frac{M}{2} [1 + \cosh \sqrt{c_0} z] \quad (5.24)
\]
gives the metric

\[
ds^2 = \left(1 - \frac{M}{\xi^n}\right)^{\frac{(n+1)}{(p+1)(p+n+1)}} \left[dt^2 - d\eta^2\right]
- \left(1 - \frac{M}{\xi^n}\right)^{\frac{(n+1)(p+1)}{(p+n+1)}} \left[\xi^2 d\Omega^2_{n+1} + \frac{d\xi^2}{\frac{M}{\xi^n}}\right] \quad (5.25)
\]

which is in fact a massive boost symmetric \(p\)-brane solution [27]. The case \(p = 0\) corresponds to the Schwarzschild solution in \(n + 3\) dimensions. Notice here, that unlike the previous cases, \(n = 1\) is permitted. In the case that \(c_0 < 0\), one gets a hyperbolic black brane solution in the region inside the event horizon. Furthermore using (3.11) and by analytic continuation we obtain the “dimensionally reduced” free scalar field metric,

\[
ds^2 = \left(1 - \frac{M}{\xi^n}\right)^{\frac{1}{n}} \left[\xi^2 (d\tau^2 - e^{2\sqrt{\kappa} r} dx^2_n) - \frac{d\xi^2}{1 - \frac{M}{\xi^n}}\right] \quad (5.26)
\]

\[
\phi = \sqrt{\frac{n + 1}{2n}} \ln \left(1 - \frac{M}{\xi^n}\right) + \phi_0 \quad (5.27)
\]

which is singular at \(\xi = 0\) and \(\xi^n = M\). Note that this metric is conformally an angular analytic continuation of a Schwarzschild wormhole. The presence of the scalar field here however, renders the usual horizon \(\xi^n = M\) singular.

For \(c_0 = 0\), (5.24) is easily integrated and we get a different type of solution:

\[
ds^2 = (z)^{\frac{2}{2\sqrt{\eta^2} + 1}} \left[dt^2 - d\eta^2\right]
- (z)^{\frac{2}{n}} \left(1 - \frac{(n+1)(p+1)}{(p+n+1)}\right) \left[dz^2 + dr^2 + r^2 d\Omega^2_n\right] \quad (5.28)
\]
This is reminiscent of a Rindler metric, although if \( p \neq 0 \) it is not a flat solution and in fact has singularities at \( z = 0 \) and \( z = \infty \). For \( p = 0 \) writing \( T = z \sinh t \), \( Z = z \cosh t \) we get the Minkowski metric. This shows that this is a flat space solution, giving the Rindler metric in \( n + 3 \) dimensions tailored to an accelerating particle.

The second family of solutions do not have an interpretation in terms of known spacetimes. For simplicity let us examine (5.22) for \( n = 2 \), for which \( g \) is readily integrated. Writing \( R = \sqrt{g_0/d_0 e^{\pm \sqrt{d_0} \, r/2}} \), \( 2 \theta = \sqrt{d_0} z - \pi/2 \), and \( \lambda_0 = 2 \sqrt{d_0/g_0} \), we have:

\[
e^\phi = G(\theta) = \frac{(1 + \lambda_0 + \sqrt{1 + \lambda_0^2}) \sin \theta + (1 - \lambda_0 - \sqrt{1 + \lambda_0^2}) \cos \theta}{(1 - \lambda_0 + \sqrt{1 + \lambda_0^2}) \sin \theta + (1 + \lambda_0 - \sqrt{1 + \lambda_0^2}) \cos \theta} \tag{5.29}
\]

with the metric

\[
ds^2 = G^{2\alpha} \left[ dt^2 - dy^2 \right] - G^{-\alpha(p+1)} \left\{ dR^2 + R^2 \left[ d\theta^2 + \left( \sqrt{1 + \lambda_0^2 \sin^2 \theta + \frac{1 - \sqrt{1 + \lambda_0^2}}{2}} \right) d\Omega^2_{II} \right] \right\} \tag{5.30}
\]

\( \lambda_0 \to 0 \) is clearly the flat space limit. For nonzero \( \lambda_0 \) however, we have an angular distortion of the spacetime, with singularities at \( \sin^2 \theta = \frac{1 + \lambda_0^2 - 1}{2 \sqrt{1 + \lambda_0^2}} \), one of which, with \( G = 0 \) is a null singularity, and the other an asymptotic singularity with \( G \to \infty \).

6. Weyl metrics in a constant curvature spacetime

Let us now consider the case where both subspaces are of planar topology (\( \kappa = 0 \)) and, in contrast, switch on a cosmological constant \( \Lambda \). Using the dual field equations (3.22, 3.25) we can obtain as before three classes of solutions. As before, if we write \( \phi = \phi(z) \), then for \( \kappa = 0 \) the general solution can be found in the form \( \alpha = f(r)g(z) \), with \( g \) defined by (5.2). As before for class I we have \( f' = 0 \), for class II \( g' = 0 \), and for class III \( f' \neq 0 \) and \( g' \neq 0 \). In the subsections that follow we use extensively the duality relation (3.28) to map the \( \kappa \neq 0 \) solutions to the cosmological constant solutions, \( \Lambda \neq 0 \). Here, unlike the case (\( \kappa \neq 0 \)) studied in the previous section, we can set \( D = 4 \) finding the four-dimensional versions of the solutions with a cosmological constant.

6.1 Class I solutions

The class I solution for \( c \neq 0 \) reads,

\[
ds^2 = \xi^2 V_{(n+p+3)/M} \left[ V^{n(n+1)(n+p+1)} (dt^2 - dy^2) - V^{-M} dz^2 - V^{M(p+1)} dz d\xi - V^{n(n+p+1)/M} d\alpha^2 \right] - \frac{\xi^2}{k^2 \xi^2 V} \tag{6.1}
\]
with
\[ M^2 = \mu^2 + \frac{n(n+p+1)^2}{(n+p+2)(p+1)}, \]
(6.2)
The potential is now given by,
\[ V(\xi) = 1 + \frac{M}{\xi^{D-1}} \]
and we have set \(-2\Lambda = (D-1)(D-2)k^2\). Solution (6.1) asymptotically approaches adS spacetime and is singular for \(V = 0\), and at \(\xi = 0\). The Thorne spacetimes are obtained with \(z \leftrightarrow i\tau, \ t \leftrightarrow iy\) where for \(\Lambda < 0\) (adS) the coordinate \(\xi\) is timelike (and hence \(t\) spacelike) inbetween the critical points \(\xi = 0\) and \(\xi = -(M)^{\frac{1}{D-1}}\). The coordinate is spacelike on the exterior of the interval 3.

Starting from the Class I Thorne metric one can make the connection with the dilaton system in \(d = n + 2\) dimensions with potential \(V = 2\Lambda e^{\gamma\phi}\) using (3.12). The solution corresponds to the Class I solutions found in \([30]\).

Starting from a Thorne metric and for \(c = 0\) we obtain,
\[ ds^2 = -(k^2\xi^2 - \frac{\mu}{\xi^{D-3}})dt^2 + \frac{d\xi^2}{k^2\xi^2 - \frac{\mu}{\xi^{D-3}}} + \xi^2 dx_{D-2}^2 \]
(6.3)
the planar topological black hole with cosmological constant. Note then that this solution is dual to the black brane solution (5.11). The corresponding Weyl metric gives naturally the Euclidean version of (6.3). The dilatonic black hole solution is then rather nicely seen to map to the black hole solution (5.3) via (3.12), (see ref. [43]).

6.2 Class II solutions

To obtain the Class II solutions we again make use of the duality relation (3.28). In order to spell out the relation let us consider to start with the simple case where \(f = e^r\) is solution to (5.12) treated in section 5. We saw that in this case we can obtain the Class II vacuum solution (5.17) with components,
\[ \alpha = e^r, \quad \phi = -\sqrt{\frac{2}{n+1}}t, \quad e^{2\chi} = \frac{e^{\frac{4\pi}{n+1}r}}{n(n-1)\kappa} \]
(6.4)
All we have to do now is set \(n = -D + 2, \ \kappa = \frac{-2\Lambda}{(D-1)(D-2)}\) and insert the above components in (3.21). The form of the scalar field in (6.4) dictates that we will have to analytically continue on applying the duality map (3.28). We then obtain with ease,
\[ ds^2 = e^{\frac{2r}{n+p+1}} \left( e^{\frac{-2\pi}{n+p+1}}(dt^2 - dy_k^2) - e^{\frac{2\pi}{n(n+p+1)}}dx_n^2 \right) - \frac{dr^2 + dz^2}{-2\Lambda} \]
(6.5)
Figure 3: The Weyl phase plane for class II solutions with a cosmological constant. The thick lines are the invariant hyperboloid $\mathcal{C} = XY - \frac{X^2}{Z} + \frac{1}{2} = 0$ which corresponds formally to $\Lambda = 0$. The connected region between the two branches of $\mathcal{C}$ corresponds to negative cosmological constant and hence the saddle critical point has $\Lambda < 0$.

where note that now we have a Weyl solution in a negatively curved spacetime.

Using again (3.28) we have that $f$ must satisfy the ODE,

$$
\frac{f''}{f} + \frac{f'}{f} \left( \frac{f'}{(D-2)f} - \frac{f''}{f''} \right) = \frac{1}{2g^2}
$$

(6.6)

\(^3\)The situation is reversed for a De Sitter cosmological constant
As in the previous section we can set $g = 1$, and by writing $X = f' / f$, $Y = f'' / f''$, this equation can be recast as a two dimensional dynamical system:

\[
\begin{align*}
X' &= XY - \frac{(D - 1)}{D - 2}X^2 + \frac{1}{2} \\
Y' &= -\frac{D}{(D - 2)^2}X^2 + \frac{2}{D - 2}XY - \frac{D - 4}{2(D - 2)}
\end{align*}
\] (6.7)

A representative phase plot is shown in figure 3 for $D = 4$. Now it is the Weyl system which has two critical points $P_{\pm} = \pm \left(\sqrt{\frac{D - 2}{2}}, \sqrt{\frac{D - 2}{2}}\right)$; however, unlike the Thorne vacuum $\kappa = 1$ spacetime, these critical points are now saddles, and lie in the $\Lambda < 0$ connected region of the phase plane. The asymptotic solutions can be obtained from (5.15,5.16) by setting $n = -(D - 2)$.

Generically the Class II solutions take the form,

\[
ds^2 = -\frac{f''}{f} \frac{dr^2 + dz^2}{2\Lambda} = f \frac{2}{\sqrt{n + p + 1}} \left(e^{\frac{2}{\sqrt{n + p + 1}}} dx^2_n - e^{-\frac{2}{\sqrt{n + p + 1}}} \left(-dt^2 + dy_p^2\right)\right)
\] (6.9)

where $f$ is a solution of (6.6).

### 6.3 Class III solutions

In the case of Class III where $f' \neq 0$ and $g' \neq 0$, the field equations (3.22-3.23) give after some algebra two possible sets of solutions for $f$ and $g$:

\[
f'^2 = f_0 f^2 \frac{D - 1}{2} - c_0 f^2 \\
g'^2 = \frac{D - 2}{2(D - 3)} + c_0 g^2
\] (6.10)

or

\[
f'^2 = d_0 f^2 \\
g'^2 = \frac{D - 2}{2(D - 1)} + g_0 g^2 \frac{D - 3}{2} - d_0 g^2
\] (6.11)

with $\dot{\phi}(z) = \frac{1}{g(z)}$.

Taking the dual of solution (5.25), the first family of solutions (5.21) are easily obtained after a bit of algebra for $c_0 \neq 0$,

\[
ds^2 = \frac{1}{(\sin kr)^2} \left[\xi^2 V \frac{n + p + 1}{(n + p + 1)}(1 + \sqrt{\frac{n + p + 1}{n + p + 1}})dt^2 - dy_p^2\right]
\] (6.12)

\[
- \xi^2 V \frac{n + p + 1}{(n + p + 1)}\left(1 - \sqrt{\frac{p + 1(n + p + 1)}{n}}\right)dx^2_n - dr^2 - \frac{d\xi^2}{V k^2 \xi^2}
\]

with the potential $V$ given by,

\[
V(\xi) = 1 - \frac{M}{\xi n + p + 1}
\] (6.13)

This solution is singular for $\xi = 0$ and $V = 0$. For $p = 0$ however this solution is regular for $V = 0$, and reads

\[
ds^2 = (\cosh(kz))^2 \left[\xi^2 V dt^2 - \frac{d\xi^2}{V k^2 \xi^2} - \xi^2 dx^2_n\right] - dz^2
\] (6.14)
where $z$ is now the proper distance coordinate. $V = 0$ is now an horizon, and this metric describes an $(n + 1)$-dimensional planar adS black hole embedded in an $(n + 2)$-dimensional adS spacetime. As $\xi \to 0$ (or $M = 0$) we get an adS spacetime where the $n$-dimensional slicings are also of adS geometry with the same curvature $k$. This solution interestingly is dual to the usual Schwarzschild black hole (5.25). The black hole singularity at $\xi = 0$ is screened by what is now a horizon, $V = 0$, of planar topology. Furthermore unlike the case studied in [11] the solution is well defined in the adS horizon since,

$$\left(\text{Riemann}\right)^2 \sim k^4 \left(1 + \nu \frac{M^2 \left( \text{sech}(kz) \right)^4}{\xi^6} \right)$$ (6.15)

where $\nu$ is some numerical coefficient depending on the spacetime dimension.

For $c_0 = 0$ we obtain,

$$ds^2 = \frac{1}{r^2} \left[ z^{\frac{2}{n+p+1}} (1 + \sqrt{\frac{(p+1)(p+n)}{n}})(dt^2 - dy_p^2) - z^{\frac{2}{n+p+1}} (1 - \sqrt{\frac{(p+1)(p+n)}{n}}) dx_n^2 - \frac{(dr^2 + dz^2)}{k^2} \right]$$ (6.16)

which for $p = 0$ (6.16) reduces to adS spacetime. This solution is quite naturally seen to be dual to the Rindler solution (5.28).

The second family of solutions is given implicitly by,

$$ds^2 = g_0^{\frac{D-2}{2}} \left[ f^{\frac{2}{D-2}} \left( e^{2\phi} (dt^2 - dy_p^2) - e^{-\gamma \phi} dx_n^2 \right) - \frac{g_0}{(D-2)^2 k^2} \left( df^2 + \frac{dg_0^2}{2(D-1)} + \frac{g_0^2}{2(D-1)} - d_0 g_0^2 \right) \right]$$ (6.17)

where the component $\phi$ is given by,

$$\phi = c \int \frac{dg}{g \sqrt{\frac{D-2}{2(D-1)} + g_0^2 g_0^2 - d_0 g_0^2}}$$ (6.18)

7. Linearization, the C-metric and braneworld black holes

In this section, we would like to remark upon two open, related, questions: where is the C-metric, and what can we say about black holes on branes? Before embarking upon this however, we would like to note that if a far-field description of a physical system is all that is required, then there is an extremely straightforward linearization prescription for the case where one of $\Lambda$ or $\kappa$ vanishes. We will discuss $\Lambda = 0$ for definiteness.

Start by observing that the flat space solution to (3.13-3.16) is

$$\alpha = \alpha_0 = r^n; \quad \phi = 0 \quad e^{2\chi} = e^{2\chi_0} = r^{n-1}$$ (7.1)
We now expand the metric functions in the usual way: $\alpha = \alpha_0 + \varepsilon \alpha_1 + \ldots$. Since $\phi_0 = 0$, we see that the equations of motion actually decouple at first order:

$$\Delta \alpha = n(n-1)\alpha^{-1/n} e^{2\chi}$$
$$\Delta \chi = \frac{(n-1)}{2} \alpha^{-(n+1)/2} e^{2\chi}$$

(7.2)

and

$$\frac{\partial^2 \alpha}{\alpha} = 2\partial_\pm \chi \frac{\partial_\pm \alpha}{\alpha}$$

and

$$\Delta \phi + \nabla \phi \cdot \nabla \frac{\alpha}{\alpha} = 0$$

(7.3)

The first set of equations can actually be solved to all orders by a (Euclidean) black hole solution:

$$\alpha = R(r)^n, \quad e^{2\chi} = R^{n-1} - C$$

(7.4)

where $R(r)$ is defined implicitly by $r = \int ne^{-2\chi} d\alpha$. Usually, the Euclidean black hole solution is characterised by the periodic identification of the angular variable, in this case $z$. However, this periodicity is imposed to make the solution regular at the analytically continued event horizon, $R^n = C$. In our case, we only require a far-field solution, therefore we do not need to impose any periodicity on the $z$-coordinate, and simply note that it is possible for $\alpha$ and $\chi$ to receive first order corrections of this form, independent of the value of $\phi$.

The $\phi$-field is independently given to first order by the solution of the Laplace equation (7.3)

$$\phi = -2a \int \frac{S(r') \ d^{n+2}r'}{|r - r'|^n}$$

(7.5)

where $S(r')$ is a source term which can be thought of as the matter source, and is exactly the higher dimensional equivalent of (2.6).

As an example, consider the five-dimensional axisymmetric metric with $p = 0$ and $n = 2$. Then for a line source lying between $z_1$ and $z_2$, we have the corresponding linear $\phi$ potential:

$$\phi = \frac{a}{r} \left[ \tan^{-1} \frac{z - z_2}{r} - \tan^{-1} \frac{z - z_1}{r} \right]$$

(7.6)

Not surprisingly this agrees to leading order with the black hole solution, but any mass source ought to have its leading monopole order agreeing with the black hole.

In this case, to get the next order correction it is necessary to return to a full system of PDE’s, however, for some systems of physical interest, the linear order term will be sufficient.

7.1 The C-metric

One of the original motivations of undertaking this study of static axisymmetric metrics in higher dimensions was to try to generalise the C-metric to greater than
four dimensions. It would appear that this attempt has not been successful. Let us first recap what the C-metric physically represents.

In four dimensions, the C-metric [16] represents two black holes (in general charged) uniformly accelerating away from one another. In the usual “relativists” coordinates, (2.19), the metric depends on two variables, $x$ and $y$, roughly speaking splitting into two pieces – an angular part $(x, \phi)$, and a ‘black hole’ part $(t, y)$. As described in section 2.2, the $y$ coordinate, which pairs up with the time coordinate, is effectively a radial variable in the spacetime, and runs from the black hole to the acceleration horizon; the $x$ coordinate is an angular variable, and runs from the conical singularity driving the acceleration to the direction pointing directly towards the other black hole.

A more transparent form of the C-metric is obtained if we set

$$\bar{t} = A^{-1}t , \quad r = 1/ Ay , \quad \text{and} \quad \theta = \int_{x}^{x3} dx/\sqrt{G} \quad (7.7)$$

when

$$ds^2 = [1 + Ar\theta(\theta)]^{-2} \left[(1 - \frac{2m}{r} - A^2 r^2)dt^2 - \frac{dr^2}{(1 - \frac{2m}{r} - A^2 r^2)} - r^2 d\theta^2 - r^2 G d\phi^2 \right]. \quad (7.8)$$

This is almost conformally equivalent to the Kottler, or Schwarzschild de-Sitter metric as might be expected from the acceleration horizon and clearly shows the black hole nature of the spacetime, as well as the existence of the acceleration horizon (which is the de-Sitter horizon of the Kottler metric). From this form of the solution it is clear that we reduce to the Schwarzschild metric as $A \to 0$, whereas the canonical form (2.19) shows how we reduce to the Rindler metric as $m \to 0$.

The key characteristics of the C-metric we glean from this four-dimensional known solution are therefore the following:

- We expect two horizons, one corresponding to the black hole, and one corresponding to the acceleration horizon.
- We expect to have two parameters, representing mass and acceleration, under which our solution reduces to the known Rindler, or Schwarzschild, solutions as each parameter (or two linearly independent combinations of said parameters) is set to zero.
- We expect to have the higher dimensional equivalent of a conical singularity meeting the black hole and event horizons for one limit of the coordinates, which gives the physical impetus for the acceleration of the black hole.

Unfortunately, we did not find any solutions with two horizons, and many of the horizons we did discover were in fact singular. As we will discuss presently, we believe that singularities may well be a necessary part of the higher dimensional C-metric. In addition, a frustrating aspect of the work on the class III solutions was
that we found a family of solutions which contained the Rindler and Schwarzschild metrics, but only those two metrics. Although we have made various Ansätze which attempt to preserve features common to both Rindler and Schwarzschild, all of these attempts give only Rindler and Schwarzschild as possible solutions.

Dealing with the third bullet point we can make more progress however. There is of course a ready generalization of the conical singularity to higher dimensions – the cosmic $p$-brane [27] for $p = 1$. These spacetimes are Poincaré boost symmetric like the conical singularity, and like the conical singularity propagators are well defined on the spacetime [10]. On the other hand, a striking feature of these spacetimes is that they are singular – the conical singularity turns into a null curvature singularity. It is difficult to know whether this is a feature that is to be accepted or avoided. If one wishes instead a nonsingular boost symmetric source, one is forced to add intrinsic curvature to the worldbrane – to have a de Sitter type of induced metric parallel to the source [11]. However, if one modifies the geometry parallel to this source, then the Rindler limit would also contain such a modification, and the fact that Rindler spacetime must be flat restricts us severely in the slicings that we are allowed to take (see the discussion after (4.4)).

An obvious first step in finding the C-metric would be to attempt to find the generalization of the Aryal-Ford-Vilenkin (AFV) solution [46]. This solution in four dimensions represents a cosmic string threading a black hole, and therefore a natural extension would be to take a black hole in higher dimensions and allow a cosmic 1-brane to thread it. We have not found this solution within our metrics, (most likely it will require a numerical integration) but it would however give a first step of intuition as to the likely geometry of the black hole horizon.

It has been suggested that allowing a singular 1-brane to interact with the black hole horizon would render the horizon singular: that the null singularity of the $p$-brane would somehow ‘infect’ the black hole horizon. One way of exploring this issue is to take the limit in which the black hole has an infinitely large mass compared to the 1-brane. In other words, the 1-brane intersects a planar Rindler horizon. This is readily obtained from the 1-brane metric (5.25) by a Rindler transformation:

\[ t = Z \sinh T \quad ; \quad y = Z \cosh T \quad \text{(7.9)} \]

under which the metric becomes:

\[
\begin{align*}
    ds^2 &= \left(1 - \frac{M}{\xi^n}\right)^{\frac{1}{n}}\left[Z^{\frac{(n+1)}{2(n+2)}}\left(Z^2 T^2 - dZ^2\right)
    \right. \\
    &\left. - \left(1 - \frac{M}{\xi^n}\right)^{\frac{(n+1)}{n}}\left[Z^{\frac{(n+1)}{n}(n+2)}\right]^{\frac{1}{n}}\left[\xi^2 d\Omega_{n+1}^2 + \frac{d\xi^2}{1 - \frac{M}{\xi^n}}\right]\right] \quad \text{(7.10)}
\end{align*}
\]

\(^4\text{We would like to thank Roberto Emparan for suggesting this limit.}\)
Although this is not in Weyl form, it is easy to see that the only singularity remains the null 1-brane singularity. The Rindler horizon $Z = 0$ remains nonsingular.

Of course this is not a definitive result as to the nonsingularity of the black hole event horizon. The Rindler horizon has planar topology just as a very large mass black hole looks roughly planar to a thin strong locally piercing it. A finite mass black hole horizon would of course have finite intrinsic curvature, and just as the Poincaré $p$-brane null singularity becomes an horizon upon adding intrinsic curvature, it is possible that here adding intrinsic curvature to the Rindler horizon could yet render it a null singularity.

7.2 Black holes on branes

Although without the C-metric we cannot give a completely satisfactory resolution to the problem of braneworld black holes, we would like to draw attention to a pragmatic alternative provided by (6.14). This solution describes a $(D-1)$-dimensional planar adS black hole solution embedded in $D$-dimensional adS space. The adS curvature of the $(D-1)$-dimensional slicing is the same as the global adS curvature. It is easy to generalise this solution to

$$ds^2 = \left(cosh(kz)\right)^2 \left[V dt^2 - \frac{d\xi^2}{V} - \xi^2 dx_{(D-3),\kappa}^2\right] - dz^2$$  \hspace{1cm} (7.11)

where $V(\xi) = \kappa + k^2 \xi^2 - \frac{M}{\xi^{D-3}}, \kappa = 0, \pm 1$.

To better understand the geometry, let $\kappa = M = 0$, and consider the coordinate transformation

$$ku = \xi^{-1} \text{sech} kz, \quad kv = \xi^{-1} \tanh kz$$  \hspace{1cm} (7.12)

This transforms (6.14) into conformal planar adS coordinates:

$$ds^2 = \frac{1}{k^2 u^2} \left[dt^2 - dv^2 - du^2 - dx_n^2\right]$$  \hspace{1cm} (7.13)

Thus the adS horizon $u \to \infty$ corresponds to $\xi \to 0$, and the adS boundary $u \to 0$ to $|z| \to \infty$ (see figure 4). Lines of constant $z$ are radial lines in the $(u, v)$ plane, and constant $\xi$ are semi-circles centered on $u = v = 0$. An adS braneworld corresponds to a line at constant $z$. As pointed out in [47, 48], it is possible for two positive tension adS braneworlds to localize gravity, which would correspond to two constant $z$-trajectories, one with $z > 0$ and the other $z < 0$.

Now consider $M \neq 0$. Then since $\xi =$constant corresponds to a semi-circle centered on $u = v = 0$, the black brane solution (6.14) corresponds to an horizon at fixed $(u^2 + v^2)$. In other words, the adS conformal plane has now been truncated by the horizon at fixed radius, and in particular, the horizon of the black hole extends out all the way to the adS boundary. The former adS horizon now corresponds to the location of the singularity and is cloaked by the black hole event horizon. Since
Figure 4: The braneworld black hole spacetime in conformal coordinates. The horizon is at \((u^2 + v^2) = k^{-2}M^{-2/(n+1)}\), and the physical spacetime exterior to the horizon corresponds to the *interior* of the semi-circle.

Clearly the properties of such black holes are interesting, particularly the issue of the holographic interpretation of such a truncated spacetime, with an horizon on the adS boundary, and will be the topic of a future study.
8. Conclusions and further questions

In this paper we have found and studied extensively solutions to the Einstein equations in arbitrary dimensions under the assumption of axial symmetry and staticity. We analysed the field equations, finding a duality relation which permitted us to map between solutions of different dimensionality and geometrical characteristics. The duality relation maps vacuum solutions with ‘internal’ curvature, i.e., where there is a nonabelian $\text{SO}(n)$ symmetry group of the spacetime, to $\text{adS}$ spacetimes with no internal curvature by quite simply associating the unique curvature (or length) scale of one solution to the other. For example we saw that the usual black brane solution with cylindrical horizon was dual to the planar $\text{adS}$ black hole.

Three classes of exact solutions were found in both vacuum and in the presence of a bulk cosmological constant. These solutions are related to lower dimensional spacetimes with a dilaton. The first class depend on only one variable and represent the most general solutions of this type. They represent an axisymmetric generalization of the cosmic $p$-brane solution of [27]. The second class of solutions depend on both variables, although the dependence on one of these is linear. The solutions were not all given explicitly, rather, their existence demonstrated and asymptotic properties derived from a dynamical systems analysis of the equations of motion. There was however, a single stable critical point exact solution for the Thorne vacuum $\kappa = 1$ system, and the Weyl cosmological constant system first discovered in [39] in the context of braneworld cosmology and later discussed in [30]. Finally, the most general two-dimensional solution with the simplest form of $\phi$ was found to be either the cosmic $p$-brane/Rindler spacetime or one of a second family of spacetimes which have a singular angular dependence in the metric.

What has also become quite clear within our analysis is that higher dimensional spacetimes are richer in solutions, which is not surprising, but also that they are generically more singular. Furthermore, solutions which seem more physically acceptable such as the black brane solutions of Horowitz and Strominger [22] are unstable to small perturbations [24] and therefore should be considered as unphysical. Should one then question cosmic censorship in higher than four dimensions? Or simply non-supersymmetric gravity in higher dimensions? Or perhaps Einstein gravity, rather than Einstein-Gauss-Bonnet gravity [14] (see for example [15] and references within in the context of braneworld cosmology) in more than four dimensions? Since we have not found an exhaustive classification of solutions we cannot answer this question, although it is possible that we are simply looking for solutions with the wrong sets of Killing symmetries. By this, we mean that it might be possible for null singularities to be replaced by null horizons by a change in the geometry of the internal spaces, just as the cosmic $p$-branes become nonsingular as their worldbranes are allowed to inflate [11].
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References

[1] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, Exact Solutions of Einstein’s Field Equations (Cambridge Monographs on Mathematical Physics).

[2] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

[3] A. Lukas, B. A. Ovrut, K. S. Stelle and D. Waldram, Phys. Rev. D 59, 086001 (1999) [arXiv:hep-th/9803235].

[4] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B 429, 263 (1998) [arXiv:hep-ph/9803315].
   I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B 436, 257 (1998) [arXiv:hep-ph/9804398].

[5] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999) [arXiv:hep-ph/9905221].

[6] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999) [arXiv:hep-th/9906064].

[7] P. Binetruy, C. Deffayet and D. Langlois, Nucl. Phys. B 565, 269 (2000) [arXiv:hep-th/9905012].
   C. Csaki, M. Graesser, C. F. Kolda and J. Terning, Phys. Lett. B 462, 34 (1999) [arXiv:hep-ph/9906513].
   J. M. Cline, C. Grojean and G. Servant, Phys. Rev. Lett. 83, 4245 (1999) [arXiv:hep-ph/9906523].

[8] P. Kraus, JHEP 9912, 011 (1999) [arXiv:hep-th/9910149].
   P. Binetruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. B 477, 285 (2000) [arXiv:hep-th/9910219].
   S. S. Gubser, Phys. Rev. D 63, 084017 (2001) [arXiv:hep-th/9912001].
   P. Brax and C. van de Bruck, “Cosmology and brane worlds: A review,” arXiv:hep-th/0303095.

[9] P. Bowcock, C. Charmousis and R. Gregory, Class. Quant. Grav. 17, 4745 (2000) [arXiv:hep-th/0007177].
[10] S. Mukohyama, Phys. Rev. D 62, 084015 (2000) [arXiv:hep-th/0004067].  
H. Kodama, A. Ishibashi and O. Seto, Phys. Rev. D 62, 064022 (2000) [arXiv:hep-th/0004160].  
D. Langlois, Phys. Rev. D 62, 126012 (2000) [arXiv:hep-th/0005025].  
C. van de Bruck, M. Dorca, R. H. Brandenberger and A. Lukas, Phys. Rev. D 62, 123515 (2000) [arXiv:hep-th/0005032].  
K. Koyama and J. Soda, Phys. Rev. D 62, 123502 (2000) [arXiv:hep-th/0005239].  
A. Neronov and I. Sachs, Phys. Lett. B 513, 173 (2001) [arXiv:hep-th/0011254].  
A. Riazuelo, F. Vernizzi, D. Steer and R. Durrer, “Gauge invariant cosmological perturbation theory for braneworlds,” arXiv:hep-th/0205220.  

[11] A. Chamblin, S. W. Hawking and H. S. Reall, Phys. Rev. D 61, 065007 (2000) [arXiv:hep-th/9909205].  

[12] R. Gregory, Class. Quant. Grav. 17, L125 (2000) [arXiv:hep-th/0004101].  

[13] T. Wiseman, Phys. Rev. D 65, 124007 (2002) [arXiv:hep-th/0111057].  
T. Harmark and N. A. Obers, JHEP 0205, 032 (2002) [arXiv:hep-th/0204047].  
T. Wiseman, Class. Quant. Grav. 20, 1137 (2003) [arXiv:hep-th/0209051].  

[14] H. Kudoh, T. Tanaka and T. Nakamura, “Small localized black holes in braneworld: Formulation and numerical method,” arXiv:gr-qc/0301089.  

[15] R. Emparan, G. T. Horowitz and R. C. Myers, JHEP 0001, 007 (2000) [arXiv:hep-th/9911043].  

[16] W. Kinnersley and M. Walker, Phys. Rev. D 2, 1359 (1970).  

[17] O. J. Dias and J. P. Lemos, “Pair of accelerated black holes in anti-de Sitter background: AdS C-metric and AdS Ernst solution,” arXiv:hep-th/0210065.  

[18] A. Vilenkin, Phys. Lett. B 133, 177 (1983).  
J. Ipser and P. Sikivie, Phys. Rev. D 30, 712 (1984).  

[19] R. Emparan, A. Fabbri and N. Kaloper, JHEP 0208, 043 (2002) [arXiv:hep-th/0206155].  

[20] T. Tanaka, Prog. Theor. Phys. Suppl. 148, 307 (2003) [arXiv:gr-qc/0203082].  

[21] R. C. Myers and M. J. Perry, Annals Phys. 172, 304 (1986).  

[22] G. T. Horowitz and A. Strominger, Nucl. Phys. B 360, 197 (1991).  

[23] R. Emparan and H. S. Reall, Phys. Rev. Lett. 88, 101101 (2002) [arXiv:hep-th/0110260].  

[24] R. Gregory and R. Laflamme, Phys. Rev. Lett. 70, 2837 (1993) [arXiv:hep-th/9301052].  
R. Gregory and R. Laflamme, Nucl. Phys. B 428, 399 (1994) [arXiv:hep-th/9404071].
[25] G. T. Horowitz and K. Maeda, Phys. Rev. Lett. 87, 131301 (2001) [arXiv:hep-th/0105111].
S. S. Gubser, Class. Quant. Grav. 19, 4825 (2002) [arXiv:hep-th/0110193].

[26] R. Gregory and R. Laflamme, Phys. Rev. D 51, 305 (1995) [arXiv:hep-th/9410050].

[27] R. Gregory, Nucl. Phys. B 467, 159 (1996) [arXiv:hep-th/9510202].

[28] J. P. Gauntlett, “Intersecting branes,” arXiv:hep-th/9705011.
D. J. Smith, Class. Quant. Grav. 20, R233 (2003) [arXiv:hep-th/0210157].

[29] R. Emparan and H. S. Reall, Phys. Rev. D 65, 084025 (2002) [arXiv:hep-th/0110258].

[30] C. Charmousis, Class. Quant. Grav. 19, 83 (2002) [arXiv:hep-th/0107126].

[31] H. Weyl, Annalen Physik 54, 117 (1917).

[32] K. S. Thorne, Phys. Rev. 135 B251 (1965)

[33] R. Gregory and A. Padilla, Phys. Rev. D 65, 084013 (2002) [arXiv:hep-th/0104262].

[34] H. F. Dowker and S. N. Thambyahpillai, Class. Quant. Grav. 20, 127 (2003) [arXiv:gr-qc/0105044].

[35] W. Israel and K. A. Khan, Nuovo Cim. 33, 331 (1964).

[36] K. Hong and E. Teo, arXiv:gr-qc/0305089.

[37] H. A. Chamblin and H. S. Reall, Nucl. Phys. B 562, 133 (1999) [arXiv:hep-th/9903225].
O. DeWolfe, D. Z. Freedman, S. S. Gubser and A. Karch, Phys. Rev. D 62, 046008 (2000) [arXiv:hep-th/9909134].
S. Kachru, M. Schulz and E. Silverstein, Phys. Rev. D 62, 045021 (2000) [arXiv:hep-th/0001206].
P. Brax and A. C. Davis, JHEP 0105, 007 (2001) [arXiv:hep-th/0104023].

[38] K. i. Maeda and D. Wands, Phys. Rev. D 62, 124009 (2000) [arXiv:hep-th/0008188].
A. Mennim and R. A. Battye, Class. Quant. Grav. 18, 2171 (2001) [arXiv:hep-th/0008192].
S. C. Davis, JHEP 0203, 054 (2002) [arXiv:hep-th/0106271].
G. N. Felder, A. V. Frolov and L. Kofman, Class. Quant. Grav. 19, 2983 (2002) [arXiv:hep-th/0112165].

[39] D. Langlois and M. Rodriguez-Martinez, Phys. Rev. D 64, 123507 (2001) [arXiv:hep-th/0106245].

[40] C. Charmousis, R. Emparan and R. Gregory, JHEP 0105, 026 (2001) [arXiv:hep-th/0101198].

[41] R. Gregory, “Inflating p-branes,” arXiv:hep-th/0304262.
[42] A. H. Taub, Annals Math. 53, 472 (1951).

[43] K. C. Chan, J. H. Horne and R. B. Mann, Nucl. Phys. B 447, 441 (1995) [arXiv:gr-qc/9502042].

[44] D. Lovelock, J. Math. Phys. 12 (1971) 498.

[45] J. E. Lidsey, “Inflation and Braneworlds,” arXiv:astro-ph/0305528.
C. Charmousis and J. F. Dufaux, Class. Quant. Grav. 19 (2002) 4671 [arXiv:hep-th/0202107].
J. P. Gregory and A. Padilla, “Braneworld holography in Gauss-Bonnet gravity,”
arXiv:hep-th/0304250.

[46] M. Aryal, L. H. Ford and A. Vilenkin, Phys. Rev. D 34, 2263 (1986).
A. Achucarro, R. Gregory and K. Kuijken, Phys. Rev. D 52, 5729 (1995) [arXiv:gr-qc/9505039].

[47] R. Emparan, G. T. Horowitz and R. C. Myers, JHEP 0001, 021 (2000) [arXiv:hep-th/9912135].

[48] I. I. Kogan, S. Mouslopoulos and A. Papazoglou, Phys. Lett. B 501, 140 (2001)
[arXiv:hep-th/0011141].

[49] A. Karch and L. Randall, JHEP 0105, 008 (2001) [arXiv:hep-th/0011156].