Transitive decomposition of symmetry groups for the \( n \)-body problem

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March 29, 2022

Abstract

Periodic and quasi-periodic orbits of the \( n \)-body problem are critical points of the action functional constrained to the Sobolev space of symmetric loops. Variational methods yield collisionless orbits provided the group of symmetries fulfills certain conditions (such as the rotating circle property). Here we generalize such conditions to more general group types and show how to constructively classify all groups satisfying such hypothesis, by a decomposition into irreducible transitive components. As examples we show approximate trajectories of some of the resulting symmetric minimizers.

1 Introduction and main results

Periodic and quasi-periodic for the \( n \)-body problem have received much of attention over the last years, also because of the success of variational and topological methods. The starting point can be traced back to the nonlinear analysis works of A. Ambrosetti, A. Bahri, V. Coti-Zelati, P. Majer, J. Mawin, P.H. Rabinowitz, E. Serra and S. Terracini (among others) issued around 1990 [4, 26, 1, 2, 3, 24, 23, 31]; methods were developed that could deal with singular potential and particular symmetry groups of the functional. For other approaches one can see also I. Stewart [32] and C. Moore [28]. The next new wave of results has followed the remarkable A. Chenciner and R. Montgomery’s proof of the “figure-eight” periodic solution of the three-body problem in the case of equal masses [13], where collisions and singularities were excluded by the computation of the action level on test curves and a non-commutative finite group of symmetries was taken as the constraint for a global equivariant variational approach. In order to generalize the equivariant variational method (that is, to restrict the action functional to the space of equivariant loops) to a new range of applicability S. Terracini and the author in [19] could make use of C. Marchal’s averaging idea [25] and prove that local minimizers of the action functional are collisionless, provided an algebraic condition on the symmetry group (termed the rotating circle property) holds. Meanwhile, symmetry groups and various approaches to level estimates or local variations have been found, together with the corresponding symmetric minimizers, and published by many authors (see for example [6, 7, 18, 8, 9, 14, 29, 30, 33, 17, 12], and [21]. The aim of this article is to provide a unified framework for the construction and classification problem.
and at the same time to extend the application range of the averaging and blow-up techniques. More precisely, when dealing with the problem of classifying in a constructive way finite symmetry groups for the $n$-body problem one has to face three issues. First, it is of course preferable to have an equivalence relation defined between groups, which rules out differences thought as non–substantial. Second, one has to find a suitable decomposition of a symmetry group into a sum of (something like) irreducible components. The way of decomposing things depends upon the context. In our settings we could choose an orthogonal representations decomposition (as direct sum of $G$-modules) or as permutations decomposition, or a mixture of the two. Third, it would be interesting to deduce from the irreducible components, from their (algebraic and combinatorial) properties, some consequent properties of action-minimizing periodic orbits (like being collisionless, existence, being homographic or non-homographic, and the like). The purpose of the paper is to give a procedure for constructing all symmetry groups of the $n$-body problem (in three-dimensional space, but of course the planar case can be done as a particular case) according to these three options, with the main focus on the existence of periodic or quasi-periodic non-colliding solutions. The main result can be used to list groups that might be considered as the elementary building blocks for generic symmetry groups yielding collisionless minimizers.

The first reduction will be obtained by defining the cover of a symmetry group (that is, the group acting on the time line instead of the time circle) and considering equivalent groups with the same equivariant periodic trajectories (up to repeating loops). Also, it is possible to consider equivalent those symmetry groups that differ by a change in the action functional (the angular speed). Using this simple escamotage it is possible to dramatically reduce the cardinality of the symmetry groups and to deal with a finite number of (numerable) families of groups for every $n$. The next step is to exploit the fact that any finite permutation representation can be decomposed à la Burnside into the disjoint sum of transitive (or, equivalently, homogeneous) permutation representations. This decomposition requires the definition of a suitably crafted sum of Lagrangean symmetry groups, which will be written in term of $Krh$ and $\hat{K}rh$ data (to be defined later) yielded by the group. The transitive decomposition allows to state the main result, which can be written as follows. Definition and notation of course refer to the body of the paper.

**Theorem A.** Let $G$ be a symmetry group with a colliding $G$-symmetric Lagrangean local minimizer. If $G_\ast$ is the $\mathbb{T}$-isotropy group of the colliding time restricted to the index subset of colliding bodies, then $G_\ast$ cannot act trivially on the index set; if the permutation isotropy of a transitive component of $G_\ast$ is trivial, then the image of $G_\ast$ in $O(3)$ cannot be one of the following: $I$, $C_p$ (for $p \geq 1$), $D_p$ (for $p \geq 2$), $T$, $O$, $Y$, $P'_{2p}$, $C_{ph}$.

This result allows to clarify and to extend the above-mentioned rotating circle property; in the proof we show how with a simple application of the averaging Marchal technique on space equivariant spheres one can deduce that for the group actions listed in the statement minimizers are collisionless. It is also worth mentioning that the transitive decomposition approach has two interesting consequences: from one hand it is possible to determine whether the hypothesis of Theorem A is fulfilled

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simply by computing the space-representations of the transitive decomposition of the maximal T-isotropy subgroups of the symmetry group (thus making the task of finding rotating circles unnecessary); on the other hand a machinery for finding examples of symmetry groups can be significantly improved by allowing the construction of actions using smaller and combinatorial components. Even if feasible, a complete classification of all symmetry groups satisfying the hypotheses which imply collision-less minimizers and coercivity would just result into an unreasonably long unreadable list. We decided to formulate only the method that can be used for such generation, leaving a few examples in the last section to illustrate it in simple cases. Therefore the paper is basically organized as a multi-step proof of Theorem A, together with the introduction and explanation of the necessary preliminaries, results and definitions. In sections 2 and 3 we will review the notation about Euclidean symmetry groups and the main properties of Lagrangean symmetry groups. In section 4 the definition of transitive decomposition and disjoint sum of symmetry groups is carried out: this is the one of the main step in the construction process. Furthermore, in section 5 a simple proof allows to extend the averaging technique to all orientation-preserving finite isotropy groups. Together with the rotating circle property and the classification of finite subgroups of SO(3) this will yield the method on avoiding collisions. The analysis of possible transitive component is then carried out in section 6, according to the previous definition and results. At the end, in section 7 the few examples mentioned above are shown, together with pictures of the corresponding approximate minimizers.

2 Preliminaries and notation

We denote by $O(d)$ the orthogonal group in dimension $d$, that is, the group of $d \times d$ orthogonal matrices over the real field. The symbol $\Sigma_n$ denotes the permutation group on $n$ elements $\{1, \ldots, n\}$. Space isometries are named rotation, reflection, central inversion and rotatory reflection (actually a central inversion is a particular rotatory reflection). We recall, following the terminology and notation of [16] (page 99, pages 270–277) and [15] (appendix A, pages 351–367; see also [5] and [27]), that the non-trivial finite subgroups of $SO(3)$ are the following: $C_p$ (the cyclic group generated by a rotation of order $p$, for $p \geq 2$, with a single $p$-gonal axis), $D_p$ (the dihedral group of order $2p$, with $p$ horizontal digonal axes and a vertical $p$-gonal axis, with $p \geq 2$), $T \cong A_4$ (the tetrahedral group of order 12, with 4 trigonal axes and 3 mutually orthogonal digonal axes), $O \cong S_4$ (the octahedral group of order 24, with 4 trigonal axes, the same as $T$, and 3 mutually orthogonal tetragonal axes; it is isomorphic to the orientation-preserving symmetry group of the cube and contains the tetrahedral group as a normal subgroup of index 2) and $Y \cong A_5$ (the icosahedral group of order 60, with 6 pentagonal axes, 10 trigonal axes and 15 digonal axes). The dihedral group $D_2$ is a normal subgroup of $T$ of index 3.

The finite subgroups of $O(3)$ are index 2 extensions of the groups listed above. Let $I$ denote the group generated by the central inversion $-1 \in O(3)$. Since $O(3) = I \times SO(3)$ and $I$ is the center of $O(3)$, finite groups containing the central inversion are $I$, $I \times C_p$, $I \times D_p$, $I \times T$, $I \times O$ and $I \times Y$.

The remaining mixed groups are those not containing the central inversion: $C_{2p}C_p$.
Table 1: Finite subgroups of $SO(3)$, their normalizers in $SO(3)$ and generators (the generators of the normalizer are obtaining adding the generator of the fourth column to the generators of the second column)

| Name                      | Symbol | Order | Gen. | $N_{SO(3)}G$ | Gen. |
|---------------------------|--------|-------|------|---------------|------|
| Rotation Cyclic           | $C_p$  | $p \geq 2$ | $\zeta_p$ | $O(2)$ | $\zeta_s, \kappa$ |
| Rotation Three Axes       | $D_2$  | 4     | $\zeta_2, \kappa$ | $O$ | $\zeta_4, \pi_3$ |
| Rotation Dihedral         | $D_p$  | $2p \geq 6$ | $\zeta_p, \kappa$ | $D_{2p}$ | $\zeta_{2p}$ |
| Rotation Tetrahedral      | $T \cong A_4$ | 12     | $\zeta_2, \pi_3$ | $O$ | $\zeta_4$ |
| Rotation Octahedral       | $O \cong S_4$ | 24     | $\zeta_4, \pi_3$ | $O$ | $\zeta_4$ |
| Rotation Icosahedral      | $Y \cong A_5$ | 60     | $\pi_3, \pi_3'$ | $Y$ | $\zeta_4$ |

Table 2: Finite subgroups of $O(3)$ containing the central inversion

| Name                        | Symbol | Order | Gen. | $N_{O(3)}G$ | Gen. |
|-----------------------------|--------|-------|------|---------------|------|
| Central Inversion           | $I \times C_p$ | 2     | $-1$ | $O(3)$ | |
| Prism/Antiprism             | $I \times C_p$ | $2p \geq 4$ | $-1, \zeta_p$ | $I \times O(2)$ | $\zeta_s, \kappa$ |
| Three planes                | $I \times D_2$ | 8     | $-1, \zeta_2, \kappa$ | $I \times O$ | $\zeta_4, \pi_3$ |
|                            | $I \times D_p$ | $2p \geq 6$ | $-1, \zeta_p, \kappa$ | $I \times D_{2p}$ | $\zeta_{2p}$ |
|                            | $I \times T$ | 24     | $-1, \zeta_2, \pi_3$ | $I \times O$ | $\zeta_4$ |
| Full octahedron             | $I \times O$ | 48     | $-1, \zeta_4, \pi_3$ | $I \times O$ | |
| Full icosahedron            | $I \times Y$ | 120    | $-1, \pi_3, \pi_3'$ | $I \times Y$ | |

(of order $2p$), $D_p C_p$ (of order $2p$, it is a Coxeter group, i.e. generated by plane reflections; it is the full symmetry group of a $p$-gonal pyramid), $D_{2p} D_p$ (of order $4p$; it is a Coxeter group if $p$ is odd, full symmetry group of a $p$-gonal prism or a $p$-gonal dipyramid) and $S_4 A_4 = OT$ (of order 24, it is a Coxeter group: the full symmetry group of a tetrahedron). One word about notation: mixed groups are denoted by a pair $GH$, where $G$ is a finite rotation group of table 1, which turns out to be isomorphic to the group under observation but not conjugated to it, and $H$ a subgroup of index 2 in $G$. Given such a pair, a group not containing $I$ is obtained as the union (of sets) $H \cup (-1(G \setminus H))$. Let $\zeta_p$ and $\kappa$ be the rotations

$$\zeta_p = \begin{bmatrix} \cos 2\pi/p & -\sin 2\pi/p & 0 \\ \sin 2\pi/p & \cos 2\pi/p & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \kappa = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

and, if $\varphi = (\sqrt{5}+1)/2$ denotes the golden ratio, let $\pi_3$ and $\pi_3'$ be the rotations defined by the following matrices.

$$\pi_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \pi_3' = \begin{bmatrix} \varphi/2 & (1-\varphi)/2 & 1/2 \\ (\varphi-1)/2 & -1/2 & -\varphi/2 \\ 1/2 & \varphi/2 & (1-\varphi)/2 \end{bmatrix}.$$

Then the generators and normalizers of finite subgroups of $SO(3)$ are listed in table 1. For more data on the icosahedral group, see also [22].

Note that other symbols might be used: $O^- = \bar{O} = OT$, $D_{2p}^d = D_{2p} D_p$, $D_p^z = D_p C_p$, $Z_{2p}^- = \bar{Z}_{2p} = C_{2p} C_p$, $Z_2^c = I$, $Z_p = C_p$, $I = Y$ (here there is a notation
clash with $I = (-1)$; the Schönflies notation for crystallographic point groups (or the equivalent Hermann-Mauguin notation) is also another option: for example, $T_d = OT$, $T_h = I \times T$, $O_h = I \times O$, $Y_h = I \times Y$ or $D_pC_p = C_{pv}$. Groups generated by reflections (that is, Coxeter groups) are $D_pC_p$ (with $p \geq 1$), $D_{2p}D_p$ (with $p$ odd), $I \times D_p$ (with $p$ even), $OT$, $I \times O$, $I \times Y$.

Finally, note that the $G$-orbit of a point in general position in $\mathbb{R}^3$ is a regular $p$-agon for $G = C_p$ but it is not a regular polygon for $G = D_p$ or if $G$ is a polyhedral group (full or rotation). For the groups $I \times C_p$ and $C_{2p}C_p$, the $G$-orbit of a point (in general position in $\mathbb{R}^3$) is the set of vertices of a prism in $p$ is even and $G = I \times C_p$ or if $p$ is odd and $G = C_{2p}C_p$. It is an antiprism (also known as twisted prism) if $p$ is odd and $G = I \times C_p$ or $p$ is even and $G = C_{2p}C_p$. Therefore such groups might be called prism/antiprism groups correspondingly. In the Schoenflies notation the antiprism group of order $2p$ is denoted by $S_{2p}$ and the prism group of order $2p$ by $C_{ph}$. To avoid possible confusion, we define for $p \geq 1$ the antiprism group $S_{2p}$ also as

$$P_{2p} = \begin{cases} I \times C_p & \text{if } p \text{ is odd} \\ C_{2p}C_p & \text{if } p \text{ is even.} \end{cases}$$

It is a cyclic group generated by a rotatory reflection of order $2p$. The prism group on the other hand is defined for $p \geq 1$ as

$$C_{ph} = \begin{cases} I \times C_p & \text{if } p \text{ is even} \\ C_{2p}C_p & \text{if } p \text{ is odd} \end{cases}$$

and is generated by a rotation of order $p$ together with a reflection (with fixed plane orthogonal to the rotation axis).

### 3 Symmetry groups and Lagrangean action

Let $X$ be configuration space of $n$ point particles in $\mathbb{R}^3$: $X = (\mathbb{R}^3)^n$. Let $\mathbb{T}$ be the circle of length $T = |\mathbb{T}|$. A function $\mathbb{T} \to X$ is a $T$-periodic path in $X$. By loops in $X$ we mean the elements of the Sobolev space $\Lambda = H^1(\mathbb{T}, X)$, i.e. all $L^2$ functions $\mathbb{T} \to X$ with $L^2$-derivative. The aim is to find periodic (in an inertial frame or in a uniformly rotating frame) orbits for the $n$-body problem: they can be obtained as critical points of the Lagrangian action functional

$$A_\omega = \int_\mathbb{T} \left( \sum_{i \in \mathbb{N}} \frac{m_i}{2} |\dot{x}_i(t) + \Omega x_i(t)|^2 + \sum_{i < j} m_i m_j |x_i(t) - x_j(t)|^{-\alpha} \right) dt,$$

(3.1)
where $\Omega$ is the anti-symmetric $3 \times 3$ matrix defined by the relation $\Omega v = \omega \times v$ for every $v \in \mathbb{R}^3$, with the vector $\omega \in \mathbb{R}^3$ representing the rotation axis of the rotating frame and its norm $|\omega|$ the angular velocity. The domain of the functional $\mathcal{A}_\omega$ is $\Lambda = H^1(\mathbb{T}, X)$ (of course, allowing a range with infinite value). Any collisionless critical point is in fact a $C^2$ solution of the corresponding Euler-Lagrange, or Newton, equations under a homogeneous gravitational potential of degree $-\alpha$, which is periodic in the rotating frame.

Now consider a group $G$ acting orthogonally on $\mathbb{T}, \mathbb{R}^3$ and permuting the indices $n$. In other words, consider three homomorphisms $\tau, \rho$ and $\sigma$ from $G$ to $O(\mathbb{T})$, $O(3)$ and $\Sigma_n$ respectively. The group $G$ can be seen as subgroup (possibly mod a normal subgroup) of the direct product $O(\mathbb{T}) \times O(3) \times \Sigma_n$ under the monomorphism $\tau \times \rho \times \sigma$, and the three homomorphisms can be recovered as projections onto the first, second and third factor of the direct product. Given $\rho$ and $\sigma$, it is customary to define an action on the configuration space $X$ by the rule $(\forall g \in G), x_{(g \sigma \rho)} = \rho(g)x_\sigma$. We will denote simply by $gx$ the value of $g \cdot x$ under this action of $G$. In the same way, the action of $G$ on $\mathbb{T}$ and $X$ induces an action on the functions $\mathbb{T} \to X$ by the rule $(\forall g \in G), x(\tau(g)t) = gx$, and therefore $\Lambda$ is a $G$-equivariant vector space (the action of $G$ is orthogonal under the standard Hilbert metric on $\Lambda$).

(3.2) Definition. A subgroup of $O(\mathbb{T}) \times O(3) \times \Sigma_n$ is termed symmetry group. It will be termed a symmetry group of the Lagrangian action functional $\mathcal{A}_\omega$ if it leaves the value of the action $\mathcal{A}_\omega$ (3.1) invariant.

Note that if $i, j \in n$ are indices and $gi = j$ for some element $g \in G$, then it has to be $m_i = m_j$. More generally, consider the decomposition of $n$ into (transitive) $G$-orbits, also known as transitive decomposition. Indices in the same $G$-orbit must share the value of the mass and, furthermore, the transitive decomposition yields an orthogonal splitting of the configuration space:

$$(3.3) \quad X = (X_1 + X_{g_1} + \ldots) \oplus (X_2 + X_{g_2} + \ldots) \oplus \ldots,$$

where each $X_j$ is a copy of $\mathbb{R}^3$ and each summand grouped by brackets is given by a transitive $G$-orbit in $n$. This transitive decomposition is nothing but the standard decomposition of a permutation representation in the Burnside ring $A(G)$.

(3.4) Definition. Consider a symmetry group $G$. A vector $v \in \mathbb{R}^3$ is a rotation axis for $G$ if $(\forall g \in G)gv \in \{\pm v\}$ (that is, the line $\langle v \rangle \subset \mathbb{R}^3$ is $G$-invariant) and the orientation $G$-representation on the time circle (that is, $\det(\tau)$) coincides with the orientation representation on the orthogonal plane of $v$ (that is, $\det(\rho)\det(v)$).

We recall from [18] (proposition 2.15) that if $\omega$ is a rotation axis for a symmetry group $G$ (and the values of the masses are compatible with the transitive decomposition (3.3)) then $G$ is a symmetry group of the action functional $\mathcal{A}_\omega$. The converse holds, after a straightforward proof, for linear or orthogonal actions.

(3.5) In case the group has a rotation axis it is termed group of type $R$. If the symmetry group $G$ is not of type $R$, then all $G$-equivariant loops have zero angular momentum.
Proof. The proof an analogous proposition for 3 bodies can be found in [18], proposition 4.2; the details are given for 3 bodies, but it can be trivially generalized to the case of \( n \) bodies: if \( J \) denotes the angular momentum of the \( G \)-equivariant path \( x(t) \), for every \( g \in G \) the formula

\[
J(gt) = \det(\rho(g)) \det(\tau(g))\rho(g)J(t)
\]

holds, and hence the angular momentum \( J \) (which is constant) belongs to the subspace \( V \) in \( \mathbb{R}^3 \) fixed by the \( G \)-representation \( \det(\tau) \det(\rho) \). But if \( V \neq 0 \), then there is a non-trivial vector \( v \in \mathbb{R} \) with the property that for every \( g \in G \), \( \det(\tau(g))\det(\rho(g))v = v \). The orientation representation on the plane orthogonal if \( v \) denotes the representation on \( \langle v \rangle \) and \( \rho_2 \) the representation on its orthogonal complement, it follows that \( \det(\tau(g))\det(\rho_2(g))\det(v) = 1 \), and hence that \( \det(\tau) = \det(\rho_2) \): the direction spanned by \( v \) is a rotation axis, which contradicts the hypothesis.

\[
q.e.d.
\]

Let \( \text{Iso}(\mathbb{R}) \) denote the group of (affine) isometries of the time line \( \mathbb{R} \), generated by translations and reflections. For every \( T > 0 \) there is a surjective projection \( \text{Iso}(\mathbb{R}) \to O(\mathbb{T}) \), where \( \mathbb{T} = \mathbb{R}/\mathbb{T} \). Let \( G \) be a symmetry group and \( \tilde{G} \) its cover in \( \text{Iso}(\mathbb{R}) \times O(3) \times \Sigma_n \), that is the pre-image of \( G \) via the projection

\[
\text{Iso}(\mathbb{R}) \times O(3) \times \Sigma_n \to O(\mathbb{T}) \times O(3) \times \Sigma_n.
\]

It is easy to see that there is a canonical isomorphism

\[
H^1(\mathbb{R}, X)^{\tilde{G}} \cong H^1(\mathbb{T}, X)^G
\]

and hence that we can consider solutions of the \( n \)-body problem which are \( \tilde{G} \)-equivariant loops instead of the periodic solutions of the \( n \)-body problem which are \( G \)-equivariant. Assume now that the symmetry group \( G \) has a rotating axis, and therefore that \( \mathcal{A}_\omega \) is \( G \)-invariant. For a fixed angular speed \( \theta \), the equation \( x(t) = e^{i\theta t}q(t) \) induces an isomorphism \( \theta_* q \mapsto x \)

\[
\theta_* : H^1(\mathbb{R}, X) \to H^1(\mathbb{R}, X).
\]

The image \( \theta_* \left( H^1(\mathbb{R}, X)^{\tilde{G}} \right) \) can be seen as

\[
\theta_* \left( H^1(\mathbb{R}, X)^{\tilde{G}} \right) = H^1(\mathbb{R}, X)^{\tilde{G}'}
\]

for a new symmetry group \( \tilde{G}' \) (still of type R) with the property that if \( g \) is a time translation, then \( \rho(g) \) is trivial. Since the following diagram commutes

\[
\begin{array}{ccc}
H^1(\mathbb{R}, X)^{\tilde{G}} & \xrightarrow{\sim} & H^1(\mathbb{R}, X)^{\tilde{G}'} \\
\mathcal{A}_\omega \downarrow & & \mathcal{A}_\omega' \downarrow \\
\mathbb{R} & \xrightarrow{\omega'} & \mathbb{R}
\end{array}
\]

(where \( \omega' \) is chosen as suggested above) a suitable change of angular speed allows one to reduce the size of the symmetry group \( G \) and assume that (if it is of type R, of course) \( \ker(\det \tau) \subset \ker \tau \cup \ker \rho \).
(3.6) Definition. Consider the following terms. A symmetry group $G$ is:

**bound to collision:** if every $G$-equivariant loop has collisions;

**homographic:** if every $G$-equivariant loop is homographic, i.e. constant up to Euclidean similarities;

**transitive:** if the permutation action of $G$ on the index set is transitive;

**fully uncoercive:** if for every possible rotation vector $\omega$ the corresponding action functional $A_\omega$ is coercive if restricted to the space of $G$-equivariant loops $\Lambda^G$.

(3.7) Definition. The kernel $\ker \tau$ is termed the core of the symmetry group.

4. Transitive groups and transitive decomposition

Consider a symmetry group $G \subset O(\mathbb{T}) \times O(3) \times \Sigma_n$ and its cover $\tilde{G} \subset \text{Iso}(\mathbb{R}) \times O(3) \times \Sigma_n$ (which anyhow is a discrete group acting on the time line $\mathbb{R}$ as time-shifts and time-reflections); the kernel of the projection $p: \tilde{G} \to G$ is a free abelian group of rank 1. By composition with the projection $p: \tilde{G} \to G$ it is possible to define the homomorphisms $\tilde{\tau} = \tau p$, $\tilde{\rho} = \rho p$ and $\tilde{\sigma} = \sigma p$. Let us note that the diagram

\[
\begin{array}{ccc}
\tilde{G} & \xrightarrow{\tilde{\tau} \times \tilde{\rho}} & \text{Iso}(\mathbb{R}) \times O(3) \\
\downarrow & & \downarrow \\
G & \xrightarrow{\tau \times \rho} & O(\mathbb{T}) \times O(3)
\end{array}
\]

commutes: the horizontal arrows are monomorphisms and the vertical arrows are epimorphisms. The projection onto the second factor of $\tilde{G} \subset \text{Iso}(\mathbb{R}) \times O(3)$ has as image a finite space point group $F \subset O(3)$. Now consider the core of $G$, $\ker \tau = K \subset G$. Its pre-image $\tilde{K} = p^{-1}(K) = \ker \tilde{\tau} \subset \tilde{G}$ is isomorphic to $K$ via $p$ and the restriction of $\rho$ to $K$ is a monomorphism. With an abuse of notation we can identify $K \cong \tilde{K} \subset \tilde{G} \subset G \subset \text{Iso}(\mathbb{R}) \times O(3)$ with its image in $O(3)$ under $\tilde{\rho}$ Consider the normalizer $N_{O(3)} K$ of $K$ in $O(3)$. If follows that:

(4.1) There is a homomorphism $\tilde{G}/\tilde{K} \to G/K \to W_{O(3)} K$ of $G/K$ with image in the Weyl group of $K$ in $O(3)$.

Now note that $p^{-1}(\ker \det \tau) = \ker \det \tilde{\tau}$ and consider the fact that the quotient $\det \ker \tilde{\tau}/K$ is isomorphic to $\ker p \cong \mathbb{Z}$. In fact, $G/K$ is projected onto a subgroup of $O(\mathbb{T}) \times W_{O(3)} K$, while $\tilde{G}/\tilde{K}$ is projected onto a subgroup of $\text{Iso}(\mathbb{R}) \times W_{O(3)} K$.

(4.2) Definition. Let $G$ be a symmetry group. Then define

(i) $K = \ker \tau$,

(ii) $[r] \in W_{O(3)} K$ as the image in the Weyl group of the generator mod $K$ of $\ker \det \tau \subset G/K$ (corresponding to the time-shift with minimal angle). If $\ker \det \tau = K$, then $[r] = 1$. 

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(iii) \([h] \in W_{O(3)} K\) as the image in the Weyl group of one of the time-reflections mod \(K\) in \(G/K\), in the cases such an element exists. Otherwise it is not defined.

In short, the triple
\[
(K, [r], [h])
\]
is said the \(Krh\) data of \(G\). \(^1\)

Now assume that the action of \(G\) on the index set is transitive. Then the following easy results hold.

**4.3** For all \(i \in n\), the isotropy subgroups \(H_i = \{g \in G \mid gi = i\}\) are mutually conjugated in \(G\). Left multiplication by elements in \(G\) yields a bijection \(G/H_1 \cong n\) of the index set \(n\) and the set of left cosets \(G/H_1\), which is \(G\)-equivariant (that is, a \(G\)-bijection).

We can here define the last piece of information needed for a classification of the symmetry groups: let \(H_1\) denote one of the isotropy subgroups defined above in **4.3**.

**4.4** Assume that \(\ker \tau \neq 1\). Then one (and only one) of the following cases can occur:
\[
\begin{cases}
\ker \tau \cap H_1 = 1 \\
\ker \tau \cap H_1 = \langle \text{reflection along a plane} \rangle \\
\ker \tau \cap H_1 = \ker \tau.
\end{cases}
\]

*Proof.* We tacitly assumed that the core \(\ker \tau\) is not a reflection along a plane, since otherwise the problem would be a planar \(n\)-body problem or bound to collisions. Furthermore, since we assume \(\ker \tau \neq 1\), the only one part is trivial. Suppose on the other hand that \(\ker \tau \cap H_1 \neq 1\). Let \(E \subseteq \mathbb{R}^3\) be the linear fixed subspace
\[
E = (\mathbb{R}^3)^{\ker \tau \cap H_1}.
\]
Let \(K\) be as above \(K = \ker \tau\). The configuration space \(X\) can be seen as the space of maps \(G/H_1 \to \mathbb{R}^3\), where \(G/H_1\) is seen as a \(G\)-set with \([G : H_1]\) elements and \(\mathbb{R}^3\) is of course a \(G\)-space via \(\rho\). The action on \(X\) (as space of maps) is the diagonal action, and configurations in \(X^K\) correspond to \(K\)-equivariant maps \(G/H_1 \to \mathbb{R}^3\). Now, the number of \(K\)-orbits in \(G/H_1\) is also the number of the double cosets \(K \setminus G/H_1\); since \(K\) is normal in \(G\), it coincides with the number of \(H_1\)-orbits in \(G/K\), which is \([G : H_1K]\). Any \(K\)-map \(x: G/H_1 \to \mathbb{R}^3\) (i.e. an element in \(X^K\)) can therefore be decomposed into a sum of \([G : H_1K]\) disjoint parts (more precisely, its domain can be) which are the \(K\)-orbits in \(G/H_1\). Each map defined on a \(K\)-orbit is conjugated via an element of \(G\) to a \(K\)-map of type
\[
K/(K \cap H_1) \to (\mathbb{R}^3)^{|K \cap H_1|} = E,
\]
(thus yielding \([K : K \cap H_1]\) particles in \(E\)). The space \(X^K\) is isomorphic to a direct sum of \([G : H_1K]\) copies of \(E\), over which the action of \(G\) acts via conjugation (actually, it is the induced/inflated module).

\(^1\)For the sake of simplicity we omit such square brackets when unnecessary.
Now, consider the hypothesis that \( K \neq K \cap H_1 \). The dimension \( \dim E \) can be 0, 1 or 2 (it cannot be 3 since by assumption \( K \cap H_1 \neq 1 \) and the action of \( K \) on \( \mathbb{R}^3 \) is faithful). If it is 0, then \( K = K \cap H_1 \) (since otherwise at each time a collision would occur. If \( \dim E = 1 \) (and \( K \neq K \cap H_1 \), assumption above) then it is easy to show that either the group \( G \) is fully uncoercive or it is bound to collisions: in fact for one-dimensional \( E \) there cannot exist rotation axes, and any symmetry element yielding coercivity would make the group bound to collisions (the complementary of the collision set in \( E \) is not connected). It is left the case \( \dim E = 2 \), i.e. where \( K \cap H_1 \) is the group generated by a single plane reflection. If the plane \( r \) fixed by \( K \cap H_1 \) is \( K \)-invariant (that is, \( Kr = r \)), then \( K \cap H_1 \) is normal in \( K \) and the \( K \)-representation given by \( \rho \) is one of the following:

1) \( C_{ph} \) with \( p \geq 1 \) (the group generated by the reflection around the plane \( r \) and \( p \) rotations orthogonal to \( r \)),

2) \( I \times D_p \) with \( p \geq 2 \) even (the Coxeter group generated by the reflection around \( r \) and \( p \) “vertical” plane reflections),

3) \( D_pC_p \) with \( p \geq 2 \) (the Coxeter group generated by \( p \) plane reflections) and

4) \( D_{2p}D_p \) with \( p \geq 1 \) (generated by \( D_p \) and \(-\zeta_{2p}\): it is a Coxeter group for \( p \) odd).

Cases 2), 3) and 4) do not possibly have rotation axes, and a symmetry group extending \( K \) and not coercive would be fully uncoercive. Since the bodies are constrained to belong to \( r \) (\( H_1 \) is the isotropy of the permutation action) and the singular set of \( r \) cuts \( r \) into different components, a symmetry group extending such \( K \) cannot be coercive without being bound to collisions. Case 1) is of a different type: the \( p = \left[ K : K \cap H_1 \right] \) bodies are constrained to be vertices of a regular \( k \)-agon centered at the origin and contained in \( r \). The direction orthogonal to \( r \) is a rotation axis. This is the case in which the reflection along a plane yields possible periodic orbits.

q.e.d.

Note that if \( \ker \tau \cap H_1 = 1 \), then the isotropy \( H_1 \) is isomorphic to its image under \( \rho \) (after the composition with the projection onto the \( K \)-quotient) in the Weyl group \( W_{O(3)}K \).

(4.5) Definition. Let \( G \) be a symmetry group acting transitively on the index set and \( H_1 \subset G \) the isotropy subgroup with respect to the index permutation action, defined in (4.3). If \( \tilde{H}_1 \) is the cover of \( H \) (that is, \( p^{-1}H_1 \subset G \subset \text{Iso}(\mathbb{R}) \times O(3) \)) Consider the following pieces of data:

(i) \( \tilde{K} = K \cap \tilde{H}_1 \subset \tilde{G} \) (note that \( \tilde{K} \cong \ker \tau \cap H_1 \)); that is, elements of \( \tilde{K} \) are the elements in \( \tilde{H}_1 \) fixing the time.

(ii) Consider the image of \( H \) in \( G/K \). Its intersection with the cyclic group \( \ker \tau / K \) is a cyclic group with a distinguished non-trivial generator (if not trivial), say \( r \mod K \). One of its pre-images \( p^{-1}r \) in \( \text{Iso}(\mathbb{R}) \times N_{O(3)}K \subset \text{Iso}(\mathbb{R}) \times O(3) \) is an element \( (k, \tilde{r}) \) in \( \tilde{H}_1 \cap \ker \det \tilde{\tau} \), defined up to multiplication with elements in \( \tilde{K} \). Without loss of generality one can assume \( k \) to be an integer.
If the set \( \tilde{H} \cap (\tilde{G} \setminus \ker \det \tilde{\tau}) \) is non-empty, then let \( \tilde{h} \) be the projection in \( N_{O(3)}K \) of one of its elements.

Then the triple \((\tilde{K}, (k, \tilde{r}), \tilde{h})\) is said to be the \( \tilde{K}rh \) data of \( G \).

Let \( G \) be a transitive symmetry group and \((\tilde{K}, (k, \tilde{r}), \tilde{h})\) the matrix with as first row that \( \tilde{K}rh \) data defined above in (4.5) and as second row the \( Krh \) data defined in (4.2). Then the cover \( \tilde{G} \) of \( G \) is, up to conjugacy, defined in \( \text{Iso}(\mathbb{R}) \times O(3) \) by the \( Krh \) data. The cover of the isotropy \( \tilde{H}_1 \) is defined by the \( \tilde{K}rh \) data in the first row. Its permutation representation on indices can be deduced by considering that \( G/H_1 \cong \tilde{G}/\tilde{H}_1 \).

This allows to properly define a decomposition of the permutation action of \( G \) into transitive components, using the following sum.

\begin{align*}
\text{(4.6) Definition.} & \text{ Let } G_1 \text{ and } G_2 \text{ two groups with the same } Krh \text{ data. Then the disjoint sum } G_1 + G_2 \text{ is defined as follows: the covers } \tilde{G}_1 \text{ and } \tilde{G}_2 \text{ are isomorphic and generated in } \text{Iso}(\mathbb{R}) \times O(3) \text{ by the (common) } Krh \text{ data. The action of such a resulting } \tilde{G} \text{ on the index set can be defined by taking the disjoint union of the } \tilde{G}-\text{sets } \tilde{G}/\tilde{H}_1 + \tilde{G}/\tilde{H}_2. \text{ Now to find the projection } p: \tilde{G} \to G, \text{ it suffices to consider the least common multiplier of the integers } k_1, k_2, |r_1 \mod \tilde{K}_1| \text{ and } |r_2 \mod \tilde{K}_2|. \end{align*}

\section{5 Local variations and averaging techniques over equivariant spheres}

The sum defined in the previous section allows one to build and generate all symmetry groups using their transitive components. Now we come to the problem of collisions. Let \( \tilde{G} \) be (the cover of) a symmetry group and \( x = x(t) \in \Lambda = H^1(\mathbb{R}, X)^G \) a local minimizer. Assume that at time \( t = 0 \in \mathbb{R} \) the trajectory \( x(t) \) collides, and all bodies in a cluster \( k \subset n \) collide (which means that other bodies might collide, but not with bodies in \( k \)). In sections 7–9 of \cite{19} the blow-up and the averaging technique are developed for equivariant trajectories; we refer to it for details. We now extend the range of applicability of the averaging technique to symmetry groups that do not need to fulfill the rotating circle property. The blow-up of \( x(t) \) centered at \( 0 \) \( \bar{q} \) is a local minimizer of the Lagrangian action \( A \) restricted to the space \( H^1(\mathbb{R}, X_k)^{G_*} \), where \( G_* \) is the restriction of \( \tilde{G} \) to the subgroup

\[ \text{Iso}_*(\mathbb{R}) \times O(3) \times \Sigma_k \subset \text{Iso}(\mathbb{R}) \times O(3) \times \Sigma_n, \]

\( \text{Iso}_*(\mathbb{R}) \) is the group of order 2 consisting in the isometries of \( \mathbb{R} \) fixing 0 and \( \Sigma_k \) the permutation group on the indices in \( k \subset n \). In other words, \( G_* \) is the symmetry group with \( Krh \) data \((K, 1, h)\), the transitive decomposition obtained restricting the permutation action to the colliding particles in \( k \) and \( X_k \) denotes the configuration.
space of the particles in $k$. Another way to define $G_*$ is to consider it as the subgroup of all elements in $\tilde{G}$ (or, equivalently, $G$) fixing the colliding time (that is, its isotropy subgroup, a maximal isotropy subgroup).

Now it comes to the standard variation. Let us define

$$S(s, \delta) = \int_0^\infty \left[ \frac{1}{|t^{2/(2+\alpha)}s + \delta|^\alpha} - \frac{1}{|t^{2/(2+\alpha)}s|^\alpha} \right] dt$$

The following lemma follows from section 9 of [19].

(5.1) Let $\vec{q}(t)$ be a colliding blow-up trajectory and $\vec{s}$ the limiting central configuration in $X_k$. If there exists a symmetric configuration $\delta \in X_{G_*}^k$ (that is, $\delta$ is fixed by the isotropy $G_*$) such that for every $i, j \in k$

$$S(\vec{s}_i - \vec{s}_j, \delta_i - \delta_j) \leq 0$$

and for at least a pair of indices the inequality is strict, then the colliding blow-up trajectory $\vec{q}(t)$ is not a minimizer.

Now we consider three different procedures that can be used to find such a $\delta$. A symmetric variation $\delta$ that let the action functional $A$ decrease on the standard variation is called $V$-variation.

The following proposition is contained in the proof of theorem (10.10) of [19].

(5.2) If $G_*$ acts trivially on $k$, then a $V$-variation always exists.

We recall that we say that a circle $\mathcal{S} \subset \mathbb{R}^3$ (with center in the origin 0) is called rotating under a group $G_*$ for an index when it is $G_*$-invariant and $\mathcal{S} \subset (\mathbb{R}^3)^{H_i}$, where $H_i \subset G_*$ is the isotropy of $i$ with respect to the the permutation action of $G_*$ on the index set, via $\sigma$. Now, proposition (9.8) of [19] can be re-phrased as follows.

(5.3) If there is an index $i \in k$ and a circle $\mathcal{S} \subset \mathbb{R}^3$ which is rotating under $G_*$ for the index $i$, then the average

$$\int_{\delta \in \iota_i\mathcal{S}} \int_{\delta \in \iota_i\mathcal{S}} \sum_{j \neq i} S(\vec{s}_i - \vec{s}_j, \delta_i - \delta_j) < 0,$$

is strictly negative, where $\iota_i\mathcal{S} \subset X_k$ is the image of the rotating circle $\mathcal{S}$ under the inclusion $\iota_i$ defined as the inclusions in the proof of (4.4). In other words, if there is a rotating circle under $H_i$ then by averaging it is possible to find a $V$-variation.

Note that [5.3] holds true if and only if the hypothesis of the claim is true for even just one of the transitive components of the index set. In other words, a $V$-variation obtained by averaging over a circle exists if and only if it is possible to obtain a $V$-variation by averaging over a circle only in one of the transitive components in which $G_*$ can be subdivided. The same will be true for the next proposition, which is a new generalization of the rotating circle property.

(5.4) Let $G_*$ be the symmetry group of a blow up solution $\vec{q}$ as above. If $\det \rho(G_*) = 1$ (i.e. $G_*$ acts orientation-preserving on the space $\mathbb{R}^3$) and for one of the indices $i \in k$ the permutation isotropy $H_i$ (restricted to $G_*$) is trivial, then there exists a $V$-variation, obtained by averaging over a 2-sphere.
Proof. Let $S^2 \subset \mathbb{R}^3$ be a 2-sphere centered in 0. If $H_i = H_i \cap G_\ast = 1$, then the space $E = (\mathbb{R}^3)^{G_\ast \cap H}$ defined as in the proof of [4.4] is equal to $\mathbb{R}^3$ and it contains the sphere $S^2$. As explained in the same proof, the fixed configuration space $X^{G_\ast}$ can be decomposed into a sum of some copies of $E$ (exactly $\left|G_\ast\right|$ since the isotropy is trivial) and a remainder (which depends on the indices which are not in the same homogeneous part of $i$); there is hence an embedding $\iota_i : S^2 \to X_k$ defined by the group action. Now, all elements of $G_\ast$ by hypotheses act by rotations on $\mathbb{R}^3$. Now consider the average

$$A = \int_{\delta_i \in S^2} \sum_{i<j} S(\bar{s}_i - \bar{s}_j, \delta_i - \delta_j)$$

Each term in the sum is equal to the sum of terms like

$$A_g = \int_{\delta_i \in S^2} S(\bar{s}_i - \bar{s}_j, (1-g)\delta_i)$$

where $g$ ranges in $G_\ast$. But since $g$ acts as rotation in $\mathbb{R}^3$, $(1-g)$ is a projection onto a plane composed with a dilation: for each $g \in G_\ast$ there is a positive constant $c_g > 0$ such that

$$A_g = c_g \int_{\delta_i \in S} S(\bar{s}_i - \bar{s}_j, \delta_i)$$

obtained exactly as in the case of the integration of a disc. Since such terms are strictly negative, the conclusion follows.

q.e.d.

6 Transitive components of groups with non-colliding minimizers

(6.1) Definition. We say that a group $G$ has property (6.1) if it is: (i) not bound to collision, (ii) not fully uncoercive, (iii) not homographic and at last that (iv) for all maximal time-isotropy subgroups $G_\ast \subset G$ at least one of the propositions (5.2), (5.3) or (5.4) can be applied (that is, either $G_\ast$ acts trivially on indexes, or there is a transitive component with a rotating circle or $G_\ast$ acts by rotations).

If the group is not fully uncoercive, then possibly considering a non-zero angular velocity $\omega$ it is possible to show that local minima always exist. We exclude the groups bound to collisions and homographic simply because we are looking for collisionless solutions which are not homographic. Now, if furthermore property (iv) (which can be easily tested only on the transitive components, as noted above) holds, the existence of a V-variation implies that all local minimizers are collisionless, which is our goal. We start by considering the possible cores for $G$ (not considering at the moment the permutations on the indices).

All finite subgroups of $SO(3)$ listed in table I (and the trivial group, not listed) can be cores by (5.4) as far as the isotropy ($\hat{K}$ in the $\hat{K}r\hat{h}$ data of the corresponding component) of one of the indices is trivial. Then, of the groups of table II the central inversion group $I$ and the central prism/antiprism $I \times C_p$ group have a rotating circle and can be considered. The groups $I \times D_p$, with $p \geq 2$ are generated by plane
reflections for \( p \) even and do not contain rotating circles: the action restricted to invariant planes is never consisting of rotations. The only possible hypothesis for the existence of a V-variation is the triviality of the permutation action: but the subspace of \( \mathbb{R}^3 \) fixed is 0, and hence with more than one particle the group would be bound to collisions. The remaining groups \( I \times T, I \times O \) and \( I \times Y \) of the table act on \( \mathbb{R}^3 \) without invariant planes (the representation is irreducible) and hence they must be excluded. The same is true for the full tetrahedron group \( OT \) of table 3. Of the three remaining groups in this table, the prism/antisprisms group \( C_{2p}C_p \) clearly has a rotating circle and must be added to the list. The groups \( D_pC_p \) (the \( p \)-gonal planes reflection group) and \( D_{2p}D_p \) (for \( p \geq 2 \)) do not have rotating circles and have reflections: not only none of the (5.2), (5.3) and (5.4) can be applied: all the symmetry groups having this core would result to be bound to collisions or fully uncoercive.

\[(6.2)\] The groups satisfying (6.1) are the following: 1) \( C_p \) (for \( p \geq 1 \)), 2) \( I \times C_p \) (for \( p \geq 1 \)), 3) \( C_{2p}C_p \) (for \( p \geq 2 \)), 4) \( D_p \) (for \( p \geq 2 \)), 5) \( T \), 6) \( O \), 7) \( Y \).

Of course for the same reason this is also the list of projections on \( O(3) \) of the (possible) maximal time-isotropy groups and of the cores. The same argument yields the proof of Theorem A. Now we consider the extensions (of index 2) of such cores as possible time-istropy for times fixed by reflections. The method used for obtaining the existence of V-variations sets constraints on the type of admissible extensions: a group with V-variations only by averaging on spheres and without rotating circles cannot be extended other than in \( SO(3) \), as in the case of the last four items in the list. On the other hand, the two prism/antiprism family of groups \( I \times C_p \) and \( C_{2p}C_p \) have a rotating plane but they are not orientation-preserving: hence can be extended without restrictions on the orientation but keeping the rotating plane. Also, we need to rule out from the list of possible cores the groups that do not occur as cores of symmetry groups not bound to collisions and not fully uncoercive. By the same argument used in the proof of (4.4), if \( \mathbb{R}^3 \) is disconnected by the collision subspaces, then it is not possible to assume coercivity and being collisionless. Hence those groups with fixed planes must be eliminated: of the two families \( I \times C_p \) and \( C_{2p}C_p \) only the anti-prism family of groups \( P_{2p}' \) survives, with normalizer \( I \times C_{2p} (= C_{2p h}) \).

Simple geometric and algebraic considerations lead us to the following conclusions:

\[(6.3)\] The index 2 extensions satisfying (6.1) of cores satisfying (6.1) are the following: 1) \( C_1 \): \( I \), \( C_2C_1 \), \( C_2 \). 2) \( C_p \) (for \( p \geq 2 \)): \( C_{2p} \), \( D_p \), \( I \times C_p \), \( C_{2p}C_p \). 3) \( P_{2p}' \) (for \( p \geq 1 \)): \( I \times C_{2p} \), 4) \( D_p \) (for \( p \geq 2 \)): \( D_{2p} \). 5) \( T \). 6) \( O \): nothing. 7) \( Y \): nothing.

Recall that the matrices of \( Krh \) and \( \tilde{K}rh \) data are \( \left( \tilde{K} \begin{array}{c} (k, \tilde{r}) \\ [r] \end{array} \right) \) (for the cyclic type) or \( \left( \tilde{K} \begin{array}{c} (k, \tilde{r}) \\ [r] \end{array} \right) \) (for brake or dihedral type), as defined in (4.6).

**Trivial core**

Let us now consider the simpler case of trivial core. By definition \( K = 1 \) and \( \tilde{K} = 1 \), which implies \( Z \cong \tilde{G} = \langle (1, r) \rangle \subset \text{Iso}(\mathbb{R}) \times O(3) \). About the pair \((k, \tilde{r})\) generating
the cover $\tilde{\mathcal{H}}_1$ of the permutation isotropy, it must be a power of the generator $(1, r)$ and hence of the form $(k, r^k)$.

If the action is of cyclic type, then the $Krh$ can be written as

\[
\begin{pmatrix}
1 & (k, r^k) \\
1 & r
\end{pmatrix},
\]

where up to rotating frames $r$ can be chosen with order at most 2 (it is not difficult to see that every cyclic symmetry group is of type R). Since if $r = 1$, then it must be $\equiv 1$, we have for every $k \geq 1$ the choreographic symmetry

\[
\begin{pmatrix}
1 & (k, 1) \\
1 & 1
\end{pmatrix},
\]

which acts transitively on the set of $k$ bodies. Of course, the constraints can be written also as the better known form $x_i(t + i) = x_i(t)$ for $i = 1, \ldots, k$ for $k$-periodic loops.

If $r$ is the reflection $-\zeta_2$, then the $Krh$ is

\[
\begin{pmatrix}
1 & (k, (-\zeta_2)^k) \\
1 & -\zeta_2
\end{pmatrix},
\]

which acts again on set of $k$ indices, but with a resulting cyclic group $G$ with $2k$ elements. Any other choice of $r$ would give rise to one of these groups, up to a change of rotating frame.

Following the same argument as in section 6 of [18], one can see that the $Krh$ for a dihedral group of type R can be chosen of the following forms (for $h_1$ and $h_2$ integers):

\[
\begin{pmatrix}
1 & (k, 1) \\
1 & 1
\end{pmatrix}^* (-1)^{h_1} (\zeta_2)^{h_2} 	ext{ or } \begin{pmatrix}
1 & (k, (-\zeta_2)^k) \\
1 & -\zeta_2
\end{pmatrix}^* (-1)^{h} (\zeta_2)^{h_2}.
\]

Groups not of type R can be found in a similar fashion.

7 A few examples

(7.1) Example. Consider the icosahedral group $Y$ of order 60. The group $G$ with $Krh$ data

\[
\begin{pmatrix}
1 & (1, -1) \\
Y & -1
\end{pmatrix}
\]

is isomorphic to the direct product $I \times Y$ of order 120, and acts on the euclidean space $\mathbb{R}^3$ as the full icosahedron group. The action on $T$ is cyclic and given by the fact that ker $\tau = 1 \times Y$. The isotropy is generated by the central inversion $-1$, and hence the set of bodies is $G/I \cong Y$. Thus at any time $t$ the 60 point particles are constrained to be a $Y$-orbit in $\mathbb{R}^3$ (which does not mean they are vertices of a icosahedron, simply that the configuration is $Y$-equivariant). After half period every body is in the antipodal position: $x_i(t + T/2) = -x_i$ (in other words, the group contains the anti-symmetry, also known as Italian symmetry – see [3, 2, 10, 11]. Of course, the group $Y$ is just an example: one can choose also the tetrahedral group $T$ or the octahedral $O$ and obtain anti-symmetric orbits for 12 (tetrahedral) or 24
Figure 1: 60-icosahedral $Y$, 12-tetrahedral $T$ and 24-octahedral $O$ periodic minimizers (chiral)

Figure 2: 4-dihedral $D_2$ and 6-dihedral $D_6$ symmetric periodic minimizers

(7.2) Example. Let $G$ be the group with $Krh$ data

$$
\begin{pmatrix}
1 & (1,-1) \\
D_k & -1
\end{pmatrix},
$$

where $D_k$ is the rotation dihedral group of order $2k$. As in the previous example, the action is such that the action functional is coercive and its local minima collisionless. At every time instant the bodies are $D_k$-equivariant in $\mathbb{R}^k$ and the anti-symmetry holds. Approximations of minima can be seen in figure 2.

(7.3) Example. To illustrate the case of non-transitive symmetry group, consider the following (cyclic) $Krh$ data:

$$
\begin{pmatrix}
1 & (3,-1) \\
1 & -1
\end{pmatrix},
$$

which yields a group of order 6 acting cyclically on 3 bodies, and with the antipodal map on $\mathbb{R}^3$. Since $\ker \tau$ is trivial and the group is of cyclic type, local minima are
collisionless. Now, by adding $k$ copies of such group one obtains a symmetry group having $k$ copies of it as its transitive components, where still local minimizers are collisionless and the restricted functional is coercive. Some possible minima can be found in figure 3 for $k = 3, 4$.

Remark. The planar case can be dealt exactly as we did for the spatial case, with a significative simplification: only when the permutation action is trivial or there exists a rotating circle (that is, under these hypotheses the maximal $\mathbb{T}$-isotropy group of all possible colliding times has transitive components which act on the position space as rotations). A transitive decomposition of such planar symmetry group, also, is much simpler since the core has to be a (regular polygon) cyclic group. Nevertheless, also in the planar case many examples can be built using these simple building block. It is still an open problem whether there are symmetry groups not bound to collisions with (local or global?) minimizers which are colliding trajectories. It has been proved in [7] that it cannot happen for $n = 3$, but to the author’s knowledge there is not yet a general result.

References

[1] Antonio Ambrosetti. Critical points and nonlinear variational problems. *Mém. Soc. Math. France (N.S.),* (49):139, 1992.

[2] Antonio Ambrosetti and Vittorio Coti Zelati. *Periodic solutions of singular Lagrangian systems.* Birkhäuser Boston Inc., Boston, MA, 1993.

[3] Antonio Ambrosetti and Vittorio Coti Zelati. Non-collision periodic solutions for a class of symmetric 3-body type problems. *Topol. Methods Nonlinear Anal.*, 3(2):197–207, 1994.

[4] A. Bahri and P. H. Rabinowitz. Periodic solutions of Hamiltonian systems of 3-body type. *Ann. Inst. H. Poincaré Anal. Non Linéaire,* 8(6):561–649, 1991.

[5] Thomas Bartsch. *Topological methods for variational problems with symmetries.* Springer-Verlag, Berlin, 1993.
[6] V. Barutello and S. Terracini. Action minimizing orbits in the $n$-body problem with simple choreography constraint. *Nonlinearity*, 17(6):2015–2039, 2004.

[7] Vivina Barutello, Davide L. Ferrario, and Susanna Terracini. Symmetry groups of the planar 3-body problem and action–minimizing trajectories, 2004. Preprint: www.arxiv.org, PaperId: math.DS/0404514.

[8] Kuo-Chang Chen. Binary decompositions for planar $N$-body problems and symmetric periodic solutions. *Arch. Ration. Mech. Anal.*, 170(3):247–276, 2003.

[9] Kuo-Chang Chen. Variational methods on periodic and quasi-periodic solutions for the $N$-body problem. *Ergodic Theory Dynam. Systems*, 23(6):1691–1715, 2003.

[10] Alain Chenciner. Action minimizing solutions of the Newtonian $n$-body problem: from homology to symmetry. In *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)*, pages 279–294, Beijing, 2002. Higher Ed. Press.

[11] Alain Chenciner. Simple non-planar periodic solutions of the $n$-body problem. In *Proceedings of the NDDS Conference, Kyoto*. 2002.

[12] Alain Chenciner, Jacques Féjoz, and Richard Montgomery. Rotating eights. I. The three $\Gamma_i$ families. *Nonlinearity*, 18(3):1407–1424, 2005.

[13] Alain Chenciner and Richard Montgomery. A remarkable periodic solution of the three-body problem in the case of equal masses. *Ann. of Math. (2)*, 152(3):881–901, 2000.

[14] Alain Chenciner and Andrea Venturelli. Minima de l’intégrale d’action du problème newtonien de 4 corps de masses égales dans $\mathbb{R}^3$: orbites “hip-hop”. *Celestial Mech. Dynam. Astronom.*, 77(2):139–152 (2001), 2000.

[15] Pascal Chossat, Reiner Lauterbach, and Ian Melbourne. Steady-state bifurcation with $O(3)$-symmetry. *Arch. Rational Mech. Anal.*, 113(4):313–376, 1990.

[16] H. S. M. Coxeter. *Introduction to geometry*. John Wiley & Sons Inc., New York, second edition, 1969.

[17] Jacques Féjoz. Quasiperiodic motions in the planar three-body problem. *J. Differential Equations*, 183(2):303–341, 2002.

[18] Davide L. Ferrario. Symmetry groups and non-planar collisionless action-minimizing solutions of the three-body problem in three-dimensional space, 2005. to appear on *Archive for Rational Mechanics and Analysis*.

[19] Davide L. Ferrario and Susanna Terracini. On the existence of collisionless equivariant minimizers for the classical $n$-body problem. *Invent. Math.*, 155(2):305–362, 2004.
[20] Wu-Yi Hsiang. On the kinematic geometry of many body systems. *Chinese Ann. Math. Ser. B*, 20(1):11–28, 1999. A Chinese summary appears in Chinese Ann. Math. Ser. A **20** (1999), no. 1, 141.

[21] Tomasz Kapela and Piotr Zgliczyński. The existence of simple choreographies for the $N$-body problem—a computer-assisted proof. *Nonlinearity*, 16(6):1899–1918, 2003.

[22] Felix Klein. *Lectures on the icosahedron and the solution of equations of the fifth degree*. Dover Publications Inc., New York, N.Y., revised edition, 1956. Translated into English by George Gavin Morrice.

[23] Pietro Majer and Susanna Terracini. Periodic solutions to some problems of $n$-body type. *Arch. Rational Mech. Anal.*, 124(4):381–404, 1993.

[24] Pietro Majer and Susanna Terracini. On the existence of infinitely many periodic solutions to some problems of $n$-body type. *Comm. Pure Appl. Math.*, 48(4):449–470, 1995.

[25] C. Marchal. How the method of minimization of action avoids singularities. *Celestial Mechanics and Dynamical Astronomy*, 83:325–353, 2002.

[26] Jean Mawhin and Michel Willem. *Critical point theory and Hamiltonian systems*, volume 74 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1989.

[27] Willard Miller, Jr. *Symmetry groups and their applications*. Academic Press, New York, 1972. Pure and Applied Mathematics, Vol. 50.

[28] Cristopher Moore. Braids in classical dynamics. *Phys. Rev. Lett.*, 70(24):3675–3679, 1993.

[29] Michael Nauenberg. Periodic orbits for three particles with finite angular momentum. *Phys. Lett. A*, 292(1-2):93–99, 2001.

[30] Matthew Salomone and Zhihong Xia. Non-planar minimizers and rotational symmetry in the $N$-body problem. *J. Differential Equations*, 215(1):1–18, 2005.

[31] Enrico Serra and Susanna Terracini. Noncollision solutions to some singular minimization problems with Keplerian-like potentials. *Nonlinear Anal.*, 22(1):45–62, 1994.

[32] I. Stewart. Symmetry methods in collisionless many-body problems. *J. Nonlinear Sci.*, 6(6):543–563, 1996.

[33] Shiqing Zhang and Qing Zhou. Periodic solutions for planar $2N$-body problems. *Proc. Amer. Math. Soc.*, 131(7):2161–2170 (electronic), 2003.