WITT’S THEOREM FOR NONCOMMUTATIVE CONIC CURVES

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Abstract. Let $k$ be a field. We extend the main result in [11] to show that all homogeneous noncommutative curves of genus zero over $k$ are noncommutative $\mathbb{P}^1$-bundles over a (possibly) noncommutative base. Using this result, we compute complete isomorphism invariants of homogeneous noncommutative curves of genus zero, allowing us to generalize a theorem of Witt.

1. Introduction

Throughout this paper, $k$ will denote a field over which everything is defined. In particular, all categories and equivalences of categories will be $k$-linear, and all bimodules will be $k$-central.

Recall that if $X$ is a smooth projective curve of genus zero with canonical bundle $\omega_X$ and a $k$-rational point $P$, then there is a commutative diagram of isomorphisms (1-1)

$$
\begin{array}{ccc}
X & \rightarrow & \text{Proj}(\bigoplus_j H^0(X, \omega_X^{\otimes j})) \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & \cong \text{Proj}(\bigoplus_i H^0(X, \mathcal{O}_X(P)^{\otimes i})) & \rightarrow & \text{Proj}(\bigoplus_j H^0(X, \mathcal{O}_X(P)^{\otimes 2j}))
\end{array}
$$

whose top horizontal is the anti-canonical embedding, whose left vertical is the embedding induced by the invertible sheaf $\mathcal{O}_X(P)$, whose bottom horizontal is the Veronese embedding, and whose right vertical is induced by an isomorphism $\mathcal{O}_X(P)^{\otimes 2} \cong \omega_X^*$. The purpose of this paper is to show that a version of (1-1) exists when $X$ is any noncommutative curve of genus zero (see (1-3) below), and to use this fact to generalize a theorem of Witt to the noncommutative context (Corollary 4.2). It will follow that even if $X$ is a smooth commutative curve of genus zero without a $k$-rational point, it is isomorphic to a noncommutative projective line, in contrast to what happens in the purely commutative situation. This illustrates the increased flexibility present in the noncommutative setting.

In order to make sense of the noncommutative version of (1-1), we proceed to describe noncommutative versions of the commutative objects in (1-1). A noncommutative curve of genus zero $\mathbb{S}$ is a small abelian, noetherian category $\mathbb{S}$ which has a tilting bundle, an object of infinite length, an Auslander-Reiten translation and has the property that for all simple objects $S$ in $\mathbb{S}$, we have $\text{Ext}^1_{\mathbb{S}}(S, S) \neq 0$. In $\mathbb{S}$, this last condition makes the curve homogeneous, but we shall omit this term for simplicity. The motivating example is the category of coherent sheaves over a smooth projective curve of genus zero.

2010 Mathematics Subject Classification. Primary 14A22, 14H45; Secondary 16S38.

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Next, we recall the definition of a noncommutative version of \( \mathbb{P}^1 \) due to M. Van den Bergh [18]. If \( K \) and \( L \) are finite extensions of \( k \) and \( N \) is a \( K - L \)-bimodule of finite dimension as both a \( K \)-module and an \( L \)-module, then one can form the \( \mathbb{Z} \)-algebra \( \mathbb{S}^n_{nc}(N) \), the noncommutative symmetric algebra of \( N \). The noncommutative \( \mathbb{P}^1 \)-bundle generated by \( N \), \( \mathbb{P}^n_{nc}(N) \), is defined to be the quotient of the category of graded right \( \mathbb{S}^n_{nc}(N) \)-modules modulo the full subcategory of direct limits of right bounded modules (see [16] for more details on this quotient category construction).

Below, we will write \( \text{coh} \mathbb{P}^n_{nc}(N) \) for the full subcategory of noetherian objects.

In order to describe the noncommutative analogue of the isomorphism in the lower left corner of (1-1), we first recall the main result of [11], which relates the concept of a noncommutative curve of genus zero to the notion of noncommutative \( \mathbb{P}^1 \)-bundle. To this end, we state some preliminaries. Let \( H \) be a noncommutative curve of genus zero. The full subcategory \( H_0 \) of objects of \( H \) of finite length is a Serre subcategory, and the quotient category \( H/H_0 \) is equivalent to \( \text{mod} k(H) \) [9, Section 1]. The rank of an object in \( H \) is defined to be the dimension of the image of the object under the quotient functor. A vector bundle in \( H \) is an object without a simple subobject, and a line bundle is a rank one vector bundle. Suppose \( L \) is a line bundle on \( H \). Then there exists an indecomposable bundle \( \mathcal{L} \) and an irreducible morphism \( \mathcal{L} \to \mathcal{E} \) coming from an AR sequence starting at \( \mathcal{L} \). Following [8], we let \( M \) denote the underlying bimodule of \( H \), defined as \( \text{End}(\mathcal{L}) \cdot \text{Hom}_H(\mathcal{L}, \mathcal{L})/\text{End}(\mathcal{L}) \). The main result in [11] is that if \( \text{End}(L) \) and \( \text{End}(\mathcal{L}) \) are commutative, then

\[
\tilde{H} \equiv \mathbb{P}^n_{nc}(M)
\]

where \( \tilde{H} \) is the unique locally noetherian category whose subcategory of noetherian objects is \( H \) and \( \equiv \) denotes equivalence of categories.

There are many interesting noncommutative curves of genus zero not described by this result. For example, if \( X \) is a (commutative) nondegenerate conic without a \( k \)-rational point, then the indecomposable bundle \( \mathcal{E} \) has rank two, and \( \text{End}(\mathcal{E}) \) is a non-split quaternion algebra. Furthermore, there exists a degree two extension \( k' \) of \( k \) such that \( \text{End}(\mathcal{E}) \otimes_k k' \cong M_2(k') \) and \( C_{k'} \cong \mathbb{P}^1_{k'} \) [2, Remark 3.7].

In this paper, we first show, in Section 2, that the notion of \( \mathbb{P}^1 \)-bundle still makes sense when the finite extensions \( K \) and \( L \) of \( k \) are replaced by arbitrary noetherian \( k \)-algebras. We then observe, in Section 3, that the proof of [11, Theorem 3.10] shows that the equivalence (1-2) still holds in case \( \text{End}(L) \) and \( \text{End}(\mathcal{L}) \) are division rings finite dimensional over \( k \). It will follow that (1-2) applies to all noncommutative curves of genus zero, yielding the noncommutative analogue of the lower left isomorphism in (1-1). In fact, we have the following result (see Sections 2 and 3 for more details): If \( H \) is a noncommutative curve of genus zero with underlying bimodule \( M \), then there exists a \( \mathbb{Z} \)-algebra \( C \) such that the diagram

\[
\begin{array}{ccc}
\text{H} & \rightarrow & \text{proj}(\bigoplus_{j \geq 0} \text{Hom}_H(i^j\mathcal{L}, \mathcal{L})) \\
\downarrow & & \downarrow \\
\text{coh} \mathbb{P}^n_{nc}(M) \equiv \text{proj}(\bigoplus_{ij} C_{ij}) & \rightarrow & \text{proj}(\bigoplus_{ij} C_{2ij})
\end{array}
\]

whose top horizontal is the noncommutative anticanonical embedding, whose left vertical is the equivalence in [16, Theorem 11.1.1], whose bottom horizontal is the
As a consequence of our generalization of (1-2), we import from [18] an intrinsic notion of base change for noncommutative curves of genus zero, which allows us to generalize [2] Remark 3.7 (see Example 6.5). We then use (1-2) to classify noncommutative curves of genus zero up to isomorphism (Theorem 4.1), generalizing [13, Theorem 5.2]. Specializing this result allows us to classify noncommutative conics (that is, noncommutative curves of genus zero over \( k \) with underlying (1, 4)-bimodule, i.e. whose underlying bimodule is one-dimensional as a left module and four-dimensional as a right module) in terms of their underlying bimodules (Corollary 4.2):

**Corollary 1.1.** For \( i = 1, 2 \), let \( D_i \) and \( E_i \) be division rings finite dimensional over \( k \), let \( M \) be a \( D_1 - D_2 \)-bimodule of left right dimension \((1, 4)\) and let \( N \) be an \( E_1 - E_2 \)-bimodule of left right dimension \((1, 4)\). There is an equivalence

\[
P^{nc}(M) \rightarrow P^{nc}(N)
\]

if and only if there exist isomorphisms \( \phi_1 : D_1 \rightarrow E_1 \) and \( \psi : M \rightarrow N \) such that

\[
\psi(a \cdot m \cdot b) = \phi_1(a) \cdot \psi(m) \cdot \phi_2(b).
\]

As we shall prove in Corollary 4.3, this is a generalization of a theorem of Witt which states that if \( \text{char } k \neq 2 \), then two quaternion \( k \)-algebras are isomorphic over \( k \) if and only if their associated conics are isomorphic.

**Acknowledgement:** I thank D. Kussin for his helpful comments.

2. **Noncommutative symmetric algebras over noetherian rings and noncommutative \( \mathbb{P}^1 \)-bundles**

2.1. **Duality and admissible bimodules.** Let \( R \) and \( S \) be noetherian \( k \)-algebras, so that every surjective endomorphism of \( R^n \) or \( S^n \) (as either right or left modules) is an isomorphism. It follows that \( R \) and \( S \) are IBN rings [4, p. 216], so that the notion of rank of a left or right free module is well defined. In this section, following [18], we define the noncommutative symmetric algebra of certain \( R - S \)-bimodules. The exposition is adapted from [11, Section 2].

We assume throughout this section that \( N \) is an \( R - S \)-bimodule which is free of rank \( m \) as a left \( R \)-module and free of rank \( n \) as a right \( S \)-module. We will sometimes say that \( N \) is a \((m,n)\)-bimodule and we will sometimes write \( R \otimes_S N \) to recall the left and right scalars. It is straightforward to check that if \( R, S, \) and \( T \) are noetherian \( k \)-algebras and \( L \) is an \( S - T \) bimodule which is free of rank \( o \) as a left \( S \)-module and is free of rank \( p \) as a right \( T \)-module, then \( N \otimes_S L \) is free of rank \( mo \) as a left \( R \)-module and is free of rank \( np \) as a right \( T \)-module.

We next recall the notion of left and right dual of \( N \). The **right dual** of \( N \), denoted \( N^* \), is the \( S - R \)-bimodule whose underlying set is \( \text{Hom}_S(N_S, S) \), with action

\[
(a \cdot \psi \cdot b)(n) = a \psi(bn)
\]

for all \( \psi \in \text{Hom}_S(N_S, S) \), \( a \in S \) and \( b \in R \).

The **left dual** of \( N \), denoted \( {}^*N \), is the \( S - R \)-bimodule whose underlying set is \( \text{Hom}_R(RN, R) \), with action

\[
(a \cdot \phi \cdot b)(n) = b \phi(na)
\]
for all \( \phi \in \text{Hom}_R(\mu N, R) \), \( a \in S \) and \( b \in R \). This assignment extends to morphisms between \( R - S \)-bimodules in the obvious way.

We set

\[
N^i := \begin{cases} 
N & \text{if } i = 0, \\
(N^{i-1})^* & \text{if } i > 0, \\
^*(N^{i+1}) & \text{if } i < 0.
\end{cases}
\]

We note that although \( N^* \) (resp. \(*N\)) is free of rank \( n \) on the left (resp. free of rank \( m \) on the right), it is not clear that \( N^* \) is free of rank \( m \) on the right. Therefore, we make the following

\[\text{Definition 2.1.} \text{ Let } l_i := \begin{cases} m & \text{if } i \text{ is even,} \\
n & \text{if } i \text{ is odd} \end{cases} \text{ and let } r_i := \begin{cases} n & \text{if } i \text{ is even,} \\
m & \text{if } i \text{ is odd}.\end{cases} \]

We say \( N \) is \textit{admissible} if \( N^* \) is free of rank \( l_i \) on the left and free of rank \( r_i \) on the right.

We remark that if \( R \) and \( S \) are finite dimensional simple rings over \( k \), then \( N \) is automatically admissible.

Next, let

\[F_i = \begin{cases} R & \text{if } i \text{ is even,} \\
S & \text{if } i \text{ is odd.}\end{cases}\]

We check that, if \( N \) is admissible, then for each \( i \), both pairs of functors

\[- \otimes_{F_i} N^i, - \otimes_{F_{i+1}} N^{i+1}\]

and

\[- \otimes_{F_i} *(N^{i+1}), - \otimes_{F_{i+1}} N^{i+1}\]

between the category of right \( F_i \)-modules and the category of right \( F_{i+1} \)-modules, have adjoint structures. While it is easy to prove this, in the sequel we will need an adjoint structure with the specific unit maps \((2-1)\) and \((2-2)\). We remark that even if \( N \) is not admissible, the above statement still holds true for \( i = 0 \). In what follows, we let \( \text{Mod} F_i \) and \( \text{Mod} F_{i+1} \) denote categories of right \( F_i \)-modules and right \( F_{i+1} \)-modules.

We suppose \( i \) is even (the case \( i \) is odd is similar). \( \{\phi_1, \ldots, \phi_n\} \) is a right basis for \( N^i \) and \( \{f_1, \ldots, f_n\} \) is the corresponding dual left basis for \( N^{i+1} \). We suppose \( \{\phi'_1, \ldots, \phi'_n\} \) is the right basis for \( *(N^{i+1}) \) dual to \( \{f_1, \ldots, f_n\} \).

We define \( \eta_i : F_i \rightarrow N^i \otimes_{F_{i+1}} N^{i+1} \) by

\[
(2-1) \quad \eta_i(a) = a \sum_j \phi_j \otimes f_j
\]

and we define \( \eta'_i : F_i \rightarrow *(N^{i+1}) \otimes_{F_{i+1}} N^{i+1} \) by

\[
(2-2) \quad \eta'_i(a) = a \sum_j \phi'_j \otimes f_j.
\]

We denote the image of \( \eta_i \) by \( Q_i \) and the image of \( \eta'_i \) by \( Q'_i \).

By the Eilenberg-Watts theorem, these maps correspond to natural transformations \( \eta_i : \text{id}_{\text{Mod} F_i} \rightarrow - \otimes_{F_i} N^i \otimes_{F_{i+1}} N^{i+1} \) and \( \eta'_i : \text{id}_{\text{Mod} F_i} \rightarrow - \otimes_{F_i} *(N^{i+1}) \otimes_{F_{i+1}} N^{i+1} \).
Proposition 2.2. Let \( i \in \mathbb{Z} \). There exist natural transformations
\[
\epsilon_i : N^{i+1} \otimes_{F_i} N^i \otimes_{F_{i+1}} \to \text{id}_{\text{Mod} F_{i+1}}
\]
and
\[
\epsilon'_i : N^{i+1} \otimes_{F_i} (N^{i+1}) \otimes_{F_{i+1}} \to \text{id}_{\text{Mod} F_{i+1}}
\]
such that
\[
(- \otimes_{F_i} N^i, - \otimes_{F_{i+1}} N^{i+1}, \eta, \epsilon)
\]
and
\[
(- \otimes_{F_i} (N^{i+1}), - \otimes_{F_{i+1}} N^{i+1}, \eta', \epsilon')
\]
are adjunctions.

\textbf{Proof.} We prove the first assertion. The proof of the second follows easily from the
first by [6, Corollary 3.5]. Let
\[
(- \otimes_{F_i} N^i, \text{Hom}_{F_{i+1}}(N^i, -), \eta, \epsilon)
\]
denote the canonical adjunction between \( \text{Mod} F_i \) and \( \text{Mod} F_{i+1} \). If \( P \) is a right
\( F_i \)-module, then the assignment
\[
\psi_P : \text{Hom}_{F_{i+1}}(N^i, P) \to P \otimes_{F_{i+1}} N^{i+1}
\]
defined by \( \psi_P(\delta) := \sum_j \delta(\phi_j) \otimes f_j \) is a surjective right \( F_i \)-module map [6, Lemma
3.4] and induces a natural transformation of functors from
\( \text{Mod} F_{i+1} \) to \( \text{Mod} F_i \). Furthermore, if \( P \cong F_{i+1}^l \) for \( l < \infty \), then \( \psi_P \) induces a surjection \( (N^{i+1})^\oplus_l \to
(N^{i+1})^\oplus_l \) of right \( F_i \)-modules. Since \( N^{i+1} \) has finite \( F_i \)-rank and \( F_i \) is noetherian,
\( \psi_P \) is an isomorphism in this case. Therefore, by a diagram chase, it follows that
if \( P \) is finitely generated, then \( \psi_P \) is an isomorphism. Finally, since the functors in
question commute with direct limits, \( \psi \) defines an isomorphism of functors.

We conclude that there is an adjunction
\[
(- \otimes_{F_i} N^i, - \otimes_{F_{i+1}} N^{i+1}, \eta, \epsilon)
\]
with \( \tilde{\eta} = (\psi \circ (- \otimes N^i)) \circ \eta \) and \( \tilde{\epsilon} = \epsilon \circ ((- \otimes N^i) \otimes \psi^{-1}) \) where \( \circ \) denotes
the horizontal composition of natural transformations. It remains to explicitly
compute \( \tilde{\eta}_{F_i}(1) \) and show it may be identified with \( \eta_i(1) \). We leave this elementary
computation to the reader. \( \square \)

The proof of the following fact, which is implicit in [18], is almost identical to
[13] p. 201-202.

\textbf{Lemma 2.3.} Suppose \( L \) is an \( S \)-\( R \)-bimodule and \( P_1 \) and \( P_2 \) are \( R \)-\( S \)-bimodules
such that
\[
(- \otimes_S L, - \otimes_R P_1)
\]
and
\[
(- \otimes_R P_2, - \otimes_S L)
\]
are adjoint pairs between \( \text{Mod} R \) and \( \text{Mod} S \). Then
\[
(N \otimes_S L)^* \cong P_1 \otimes_S N^*
\]
and
\[
^*(N \otimes_S L) \cong P_2 \otimes_S ^*N.
\]
A similar formula holds for the right and left duals of \( L \otimes_R N \).
Finally, suppose $k'$ is an extension field of $k$. We let $N_{k'}$ denote the $k'$-central $R \otimes_k k' - S \otimes_k k'$-bimodule whose underlying set is $N \otimes_k k'$ and whose action is inherited from the $R - S$-bimodule action on $N$. We note that base change on bimodules is compatible with taking duals (as in [18 Lemma 3.1.9]). Therefore, if $N$ is admissible, so is $N_{k'}$.

2.2. Noncommutative symmetric algebras and $\mathbb{P}^1$-bundles. The purpose of this section is to define the notions of noncommutative symmetric algebra and noncommutative $\mathbb{P}^1$-bundle of an admissible bimodule, following M. Van den Bergh. We first introduce a convention that will be in effect throughout the remainder of this paper: all unadorned tensor products will be bimodule tensor products over the appropriate base ring.

We next recall the definition of $\mathbb{Z}$-algebra from [19 Section 2]: a $\mathbb{Z}$-algebra is a ring $A$ with decomposition $A = \bigoplus_{i,j \in \mathbb{Z}} A_{ij}$ into $k$-vector spaces, such that multiplication has the property $A_{ij} A_{jk} \subseteq A_{ik}$ while $A_{ij} A_{kl} = 0$ if $j \neq k$. Furthermore, for $i \in \mathbb{Z}$, there is a local unit $e_i \in A_{ii}$, such that if $a \in A_{ij}$, then $e_i a = a = ae_j$. Just as with $\mathbb{Z}$-graded algebras, we define $\text{proj} A$ to be the quotient category $\text{gr} A/\text{tors} A$, where $\text{gr} A$ denotes the category of finitely generated graded right $A$-modules and $\text{tors} A$ denotes the full subcategory of $\text{gr} A$ consisting of right bounded modules.

We will need the following terminology regarding $\mathbb{Z}$-algebras, from [19]: Let $i \in \mathbb{Z}$. If $A$ is a $\mathbb{Z}$-algebra, we let $A(i)$ (the shift of $A$ by $i$) denote the $\mathbb{Z}$-algebra with $A(i)_{jk} = A_{i+j,k+i}$, and with multiplication induced from that of $A$. A $\mathbb{Z}$-algebra is called $i$-periodic if $A \cong A(i)$. If $B$ is a $\mathbb{Z}$-graded ring, we let $\tilde{B}$ denote the $\mathbb{Z}$-algebra with $\tilde{B}_{ij} = B_{j-i}$.

We have the following

**Lemma 2.4.** [19 Lemma 2.4] If $A$ is a 1-periodic $\mathbb{Z}$-algebra, then $A \cong \tilde{B}$ for a $\mathbb{Z}$-graded ring $B$.

Let $A^{(2)}$ denote the 2-Veronese of $A$, i.e., the $\mathbb{Z}$-subalgebra of $A$ consisting of all even components. We will need the fact that if $A$ is noetherian (i.e. the category of graded right $A$-modules is locally noetherian), then inclusion $A^{(2)} \to A$ induces an equivalence $\text{proj} A \to \text{proj} A^{(2)}$ [19 Lemma 2.5].

Now we are finally ready to recall (from [19]) the definition of the noncommutative symmetric algebra of $N$. The **noncommutative symmetric algebra of $N$**, denoted $S^{nc}(N)$, is the $\mathbb{Z}$-algebra $\bigoplus_{i,j \in \mathbb{Z}} A_{ij}$ with components defined as follows:

- $A_{ij} = 0$ if $i > j$.
- $A_{ii} = R$ for $i$ even,
- $A_{ii} = S$ for $i$ odd, and
- $A_{ii+1} = N^{i*}$.

In order to define $A_{ij}$ for $j > i + 1$, we introduce some notation: we define $T_{ii+1} := A_{ii+1}$, and, for $j > i + 1$, we define

$$T_{ij} := A_{ii+1} \otimes A_{i+1i+2} \otimes \cdots \otimes A_{j-1j}.$$  

We let $R_{ii+1} := 0$, $R_{ii+2} := Q_i$,

$$R_{ii+3} := Q_i \otimes N^{i+2*} + N^{i*} \otimes Q_{i+1},$$

and, for $j > i + 3$, we let

$$R_{ij} := Q_i \otimes T_{i+2j} + T_{ii+1} \otimes Q_{i+1} \otimes T_{i+3j} + \cdots + T_{ij-2} \otimes Q_{j-2}.$$
For \( j > i + 1 \), we define \( A_{ij} \) as the quotient \( T_{ij}/R_{ij} \).

Multiplication in \( S^\text{nc}(N) \) is defined as follows:

- if \( x \in A_{ij} \), \( y \in A_{lk} \) and \( j \neq l \), then \( xy = 0 \),
- if \( x \in A_{ij} \) and \( y \in A_{jk} \), with either \( i = j \) or \( j = k \), then \( xy \) is induced by the usual scalar action,
- otherwise, if \( i < j < k \), we have
  \[
  A_{ij} \otimes A_{jk} = \frac{T_{ij}}{R_{ij}} \otimes \frac{T_{jk}}{R_{jk}} \cong \frac{T_{ik}}{R_{ij} \otimes T_{jk} + T_{ij} \otimes R_{jk}}.
  \]

Since \( R_{ij} \otimes T_{jk} + T_{ij} \otimes R_{jk} \) is a submodule of \( R_{ik} \), there is thus an epi\
\( \mu_{ijk} : A_{ij} \otimes A_{jk} \rightarrow A_{ik} \).

The following is an easy consequence of [12, Lemma 6.6]:

**Lemma 2.5.** If \( \psi : M \rightarrow N \) is an isomorphism of admissible \( R - S \)-bimodules, then \( \psi \) induces an isomorphism \( S^\text{nc}(M) \rightarrow S^\text{nc}(N) \).

We define \( P^\text{nc}(N) \), to be the quotient of the category of graded right \( S^\text{nc}(N) \)-modules modulo the full subcategory of direct limits of right bounded modules [15 Definition 1.1]. For the motivation behind this definition, we refer the reader to [16 Section 2].

Furthermore, if \( k' \) is a field extension of \( k \), we define \( P^\text{nc}(N)_{k'} := P^\text{nc}(N_{k'}) \).

### 2.3. Morita equivalence.

We end this section by studying the behavior of \( P^\text{nc}(N) \) under Morita equivalence of bases. We introduce notation that will be used throughout the rest of the section. Let \( S' \) be a \( k \)-algebra and let \( F : \text{Mod} S \rightarrow \text{Mod} S' \) be an equivalence. By the Eilenberg-Watts theorem, we may write

\[
F = - \otimes_S L
\]

where \( L \) is an \( S - S' \)-bimodule such that there exists an \( S' - S \)-bimodule \( L' \) such that \( L \otimes_S L' \) and \( L' \otimes_S L \) are trivial. Similarly, let \( R' \) be a \( k \)-algebra and let \( G : R - \text{Mod} \rightarrow R' - \text{Mod} \) be an equivalence given by

\[
G = P \otimes_R -
\]

where \( P \) is an \( R' - R \)-bimodule such that there exists an \( R - R' \)-bimodule \( P' \) such that \( P' \otimes_{R'} P \) and \( P \otimes_R P' \) are trivial.

**Proposition 2.6.** Suppose \( R, R', S \) and \( S' \) are noetherian \( k \)-algebras. If the bimodules \( N, N \otimes_S L \) and \( P \otimes_R N \) are admissible then there are equivalences

\[
P^\text{nc}(N) \rightarrow P^\text{nc}(N \otimes_S L)
\]

and

\[
P^\text{nc}(N) \rightarrow P^\text{nc}(P \otimes_R N).
\]

In particular, if \( R \) and \( S \) are division rings finite dimensional over \( k \), there are \( k \)-algebra isomorphisms \( \phi_1 : R \rightarrow R' \) and \( \phi_2 : S \rightarrow S' \), and if \( N' \) is the \( R' - S' \)-bimodule whose underlying set is \( N \) whose action is induced by \( \phi_1 \) and \( \phi_2 \), then there is an equivalence \( P^\text{nc}(N) \rightarrow P^\text{nc}(N') \).
Proof. The hypothesis ensures that the noncommutative spaces in the conclusion exist.

The proof that the stated equivalences exist is almost identical to the proof of \cite{10} Lemma 4.1 and Theorem 4.1 in light of Lemma 2.3, so we omit the details.

To prove the last statement, under the hypotheses given, both $P$ and $L$ are free of rank one on both sides. Therefore, by Lemma 2.3, $N \otimes_S L$ and $P \otimes_R N$ are admissible so the first conclusion holds. □

Remark 2.7. Even if $R$ and $R'$ are simple finite dimensional $k$-algebras which are Morita equivalent via $G: R \rightarrow \text{Mod} \rightarrow R' \rightarrow \text{Mod}$, it may not be the case that the $R'$-$R$ module $P$ is admissible. For example, if $G: \text{Mod} \rightarrow \text{Mod}$ is the equivalence $P \otimes M_2(k) \rightarrow k \otimes k$, where $P$ is the bimodule whose underlying set is $e_{11} M_2(k)$ and whose right $M_2(k)$-module structure is the usual one, then $P$ is not obviously not free on the right. However, it may still be the case that if $N$ is admissible, then $P \otimes N$ is admissible. In the example above, if we let $N$ denote the $M_2(k)$-$k$-bimodule whose underlying set is $M_2(k)$ with standard left and right actions, then $N$ is admissible and $P \otimes M_2(k) \otimes k \oplus k$ is an admissible $k-k$-bimodule.

3. Tsen’s theorem over noncommutative bases and commutative conics

Throughout the rest of this paper, if $H$ is a noncommutative curve of genus zero and $\tau$ is an AR-translation functor on $H$, then we let $\tau^{-1}$ denote a right (and left) adjoint to $\tau$.

3.1. Tsen’s theorem and an immediate consequence. Our goal in this section is to prove the following theorem, and to deduce some straightforward consequences.

Theorem 3.1. Let $H$ be a noncommutative curve of genus zero with underlying bimodule $M$ and let $\tilde{H}$ denote the unique locally noetherian category whose subcategory of noetherian objects is $H$. Then there is an equivalence

$$\tilde{H} \rightarrow \mathbb{P}^{nc}(M).$$

Conversely, if $M$ is a $(2,2)$- or $(1,4)$-bimodule over division rings finite dimensional over $k$, then $\mathbb{P}^{nc}(M)$ is a noncommutative curve of genus zero with $M$ as an underlying bimodule.

Proof. As a consequence of Section 2 the space $\mathbb{P}^{nc}(M)$ is well defined. The only part of the proof of \cite{11} Theorem 3.10 that uses commutativity of $\text{End}(L)$ and $\text{End}(\mathbb{Z})$ is the proof of \cite{11} Proposition 3.8 in case $M$ is a $(2,2)$-bimodule. However, it is proven in \cite{13} that the commutativity assumption is not necessary. □

In order to prove Theorem 4.1 below, we will need Corollary 3.3. To prove it, we will invoke the $\mathbb{Z}$-algebra which we now define. We let

$$\mathcal{O}(n) := \begin{cases} \tau^{-n} L & \text{if } n \text{ is even} \\ \tau^{-n-1} \mathcal{Z} & \text{if } n \text{ is odd}. \end{cases}$$

We define a $\mathbb{Z}$-algebra $C$ by setting

$$C_{ij} = \begin{cases} \text{Hom}(\mathcal{O}(-j), \mathcal{O}(-i)) & \text{if } j \geq i \\ 0 & \text{if } i > j \end{cases}$$
and defining multiplication as composition. The $\mathbb{Z}$-algebra $C$ is equal to the algebra $\mathcal{H}(1)$ defined in [11, Section 3].

**Lemma 3.2.** If $M$ is an underlying bimodule of $\mathcal{H}$, then there is an isomorphism of $\mathbb{Z}$-algebras $S^{nc}(M^*) \rightarrow C$. Therefore,

$$\tilde{\mathcal{H}} \equiv \text{Proj} \equiv \text{Proj}^{nc}(M^*).$$

**Proof.** There is an isomorphism $C(-1) \rightarrow S^{nc}(M)$ by [11, Theorem 3.10], which induces an isomorphism $C \rightarrow S^{nc}(M)(1)$. Furthermore, by repeated application of [11, Lemma 2.1], one can show that there is an isomorphism $S^{nc}(M)(1) \rightarrow S^{nc}(M^*)$.

To prove the second result, we note that the sequence $\{\mathcal{O}(n)\}_{n \in \mathbb{Z}}$ is ample in the sense of [11, Section 3] by the proof of [11, Proposition 3.9]. Therefore, by [16, Theorem 11.1.1], there is an equivalence $\tilde{\mathcal{H}} \equiv \text{Proj} \equiv \text{Proj}^{nc}(M^*)$. The last equivalence follows from the first paragraph. □

The next result follows immediately from Theorem 3.1 and Lemma 3.2:

**Corollary 3.3.** If $M$ is either a $(2,2)$- or $(1,4)$-bimodule over division rings finite dimensional over $k$, then there is an equivalence $\text{Proj}(M) \rightarrow \text{Proj}(M^*)$.

### 3.2. Commutativity of diagram (1-3)

In this section, we explicitly define diagram (1-3) and prove that it is commutative. First recall that if $\sigma$ is an auto-equivalence of $\mathcal{H}$ then the abelian group $\bigoplus_{j \geq 0} \text{Hom}_{\mathcal{H}}(\sigma^{-j} L, L)$ can be made into a $\mathbb{Z}$-graded $k$-algebra by defining multiplication as follows: for $a \in \text{Hom}_{\mathcal{H}}(\sigma^{-i} L, L)$ and $b \in \text{Hom}_{\mathcal{H}}(\sigma^{-j} L, L)$, we let $a \cdot b := a \circ \sigma^{-j}(b)$.

We let the top functor in (1-3),

$$\mathcal{H} \rightarrow \text{proj}(\bigoplus_{j \geq 0} \text{Hom}_{\mathcal{H}}(\tau^j L, L)),$$

be defined by sending an object $\mathcal{M}$ to the object in the quotient category corresponding to the graded right module $\bigoplus_{j \geq 0} \text{Hom}_{\mathcal{H}}(\tau^j L, \mathcal{M})$ with obvious action, and by sending a morphism to the usual induced morphism (compare with the equivalence from [1, Theorem 4.5]). The fact that this functor is an equivalence will follow from the commutativity of (1-3).

We define the left vertical to be the equivalence from Lemma 3.2. In particular, it takes $\mathcal{M}$ to the image (under the quotient functor) of the graded right $C$-module $\bigoplus_{j} \text{Hom}_{\mathcal{H}}(\mathcal{O}(-j), \mathcal{M})$ with obvious action by $C$ and with the obvious assignment on morphisms. The fact that the lower left category in (1-3) is equivalent to the full subcategory of noetherian objects of $\mathbb{H}^{nc}(M)$ follows from Theorem 3.1.

Since the lower left category is noetherian, its 2-Veronese is an equivalence by the remark following 2.4 and is induced by sending graded modules and morphisms to their even components. This is the bottom horizontal functor of (1-3).

We now define the right vertical functor. One can check explicitly that $C^{(2)}$ is one-periodic and that there is thus an induced equivalence

$$\text{proj}(C^{(2)}) \rightarrow \text{proj}(\bigoplus_{j \geq 0} \text{Hom}_{\mathcal{H}}(\tau^j L, L))$$

by Lemma 2.4. We define the right vertical to be this equivalence.
The fact that \((1-3)\) commutes is now a straightforward computation which we leave to the reader.

3.3. **Commutative conics from the noncommutative perspective.** Now suppose \(\text{char } k \neq 2\). Let \(a, b \in k\) be nonzero such that \((a, b)\) is the non-split quaternion algebra with \(i^2 = a\) and \(j^2 = b\). Let \(C(a, b)\) denote its associated conic in \(\mathbb{P}^2_k\) defined as the zero set of \(aX^2 + bY^2 - Z^2\).

In light of Theorem 3.1, the following result is implicit in [7]. For the reader's convenience, we include a proof.

**Lemma 3.4.** The category \(\text{coh}\mathbb{C}(a, b)\) is equivalent to \(\text{coh}\mathbb{P}^{nc}(a, b)\).

**Proof.** We begin by noting that since \(\mathbb{C}(a, b)\) has an indecomposable vector bundle, \(\mathcal{L}\), of rank two [7, Theorem 2.2], Theorem 3.1 implies that \(\text{coh}\mathbb{C}(a, b)\) is \(k\)-equivalent to \(\text{coh}\mathbb{P}^{nc}(\text{End}(\mathcal{L}))\). Now, the endomorphism rings of rank two indecomposable vector bundles over \(C(a, b)\) are all isomorphic by [7, Corollary 3.5]. Therefore, by [7, Proposition 4.3], or by [17, Lemma 3] and the fact that \(C(a, b)\) is the Brauer-Severi variety of \((a, b)\), we deduce \(\text{End}(\mathcal{L}) \cong (a, b)\). The result now follows from Lemma 2.5 and Proposition 2.6. \(\square\)

**Example 3.5.** Suppose \(\text{char } k \neq 2\), let \(R := k\), and let \(S := D\) be a non-split quaternion algebra over \(k\). Let \(N := DD_k\). Then there exists a degree two extension \(k'\) over \(k\) such that \(D_{k'}\) is split [5, Proposition 1.2.3]. It follows that if \(\text{Qcoh}\mathbb{P}^1_{k'}\) denotes the category of quasi-coherent sheaves over \(\mathbb{P}^1_{k'}\), then

\[
\mathbb{P}^{nc}(N)_{k'} \equiv \mathbb{P}^{nc}(N)_{k'} \\
\equiv \mathbb{P}^{nc}(M_2(k')) \\
\equiv \mathbb{P}^{nc}(k' \oplus k') \\
\equiv \text{Qcoh}\mathbb{P}^1_{k'},
\]

where the second equivalence is the equivalence from Lemma 2.5 and Proposition 2.6 induced by the isomorphism of bimodules

\[D_{k'} \to M_2(k')M_2(k'),\]

the third equivalence is the equivalence from Proposition 2.6 induced by the canonical equivalence

\[G : M_2(k') - \text{Mod} \to k' - \text{Mod}\]

sending an object \(M\) to \(e_{11}M\) and sending a morphism \(f : M \to N\) to its restriction \(e_{11}M \to e_{11}N\), and the last equivalence is from Lemma 2.4.

4. **Isomorphism invariants and noncommutative Witt’s theorem**

In [13, Section 5], isomorphism invariants of noncommutative curves of genus zero whose underlying bimodule is a \((2, 2)\)-bimodule over a pair of isomorphic commutative fields are found. In this section, using Theorem 3.1 we obtain a similar result for arbitrary noncommutative curves of genus zero (Theorem 4.1), generalizing [8, Proposition 5.1.4]. We will then use this result to obtain isomorphism invariants of noncommutative conics (those noncommutative curves of genus zero whose underlying bimodule is a \((1, 4)\)-bimodule). This provides a generalization of Witt’s theorem, as we shall see in Corollary 4.3.
Theorem 4.1. For \( i = 1, 2 \), let \( D_i \) and \( E_i \) be division rings finite dimensional over \( k \), let \( M \) be a \( D_1 - D_2 \)-bimodule of left right dimension \( (2, 2) \) or \( (1, 4) \) and let \( N \) be an \( E_1 - E_2 \)-bimodule of left right dimension \( (2, 2) \) or \( (1, 4) \). There is an equivalence

\[
H_1 = \mathbb{P}^{nc}(M) \longrightarrow \mathbb{P}^{nc}(N) = H_2
\]

if and only if either

1. there exist isomorphisms \( \phi_i : D_i \rightarrow E_i \) and \( \psi : M \rightarrow N \) such that
   \[
   \psi(a \cdot m \cdot b) = \phi_1(a) \cdot \psi(m) \cdot \phi_2(b),
   \]
   or

2. there exist isomorphisms \( \phi_1 : D_1 \rightarrow E_2, \phi_2 : D_2 \rightarrow E_1 \) and \( \psi : M \rightarrow N^* \) such that
   \[
   \psi(a \cdot m \cdot b) = \phi_1(a) \cdot \psi(m) \cdot \phi_2(b).
   \]

Proof. First, suppose there is an equivalence \( F : \mathbb{P}^{nc}(M) \rightarrow \mathbb{P}^{nc}(N) \), where \( M = \text{Hom}_{H_1}(\mathcal{L}, \overline{\mathcal{L}}) \) and \( N = \text{Hom}_{H_2}(\mathcal{L}', \overline{\mathcal{L}}') \). Then \( F \) induces an isomorphism of \( k \)-modules

\[
(4-1) \quad \text{Hom}_{H_1}(\mathcal{L}, \overline{\mathcal{L}}) \rightarrow \text{Hom}_{H_2}(F(\mathcal{L}), F(\overline{\mathcal{L}})).
\]

We break the proof of the forward direction into two cases:

Case 1: \( M \) is a \((1, 4)\)-bimodule. By comparing ranks and using the fact that there are exactly two \( \tau \)-orbits of indecomposable bundles in \( H_2 \) \cite[Section 1.1]{Witt'sThm}, we know there exist \( i, j \in \mathbb{Z} \) such that

\[
(4-2) \quad F(\mathcal{L}) \cong \tau^i \mathcal{L}'
\]

and

\[
(4-3) \quad F(\overline{\mathcal{L}}) \cong \tau^j \overline{\mathcal{L}}'.
\]

Furthermore, since the map \((4-1)\) is a \( k \)-module isomorphism, it follows from \cite[Corollary 3.6]{K-theory} that \( i = j \). Therefore \( F \) together with the isomorphisms \((4-2)\) and \((4-3)\) induces \( k \)-algebra isomorphisms

\[
\phi_1 : \text{End}_{H_1}(\overline{\mathcal{L}}) \rightarrow \text{End}_{H_2}(\tau^i \overline{\mathcal{L}}') \quad \tau^{-i} \rightarrow \text{End}_{H_2}(\overline{\mathcal{L}}')
\]

and

\[
\phi_2 : \text{End}_{H_1}(\mathcal{L}) \rightarrow \text{End}_{H_2}(\tau^i \mathcal{L}') \quad \tau^{-i} \rightarrow \text{End}_{H_2}(\mathcal{L}')
\]

whose last maps send \( f \) to \( \eta(\tau^{-i}f)\eta^{-1} \), where \( \eta \in \text{Hom}_{H_2}(\tau^{-i} \mathcal{L}', \overline{\mathcal{L}}') \) is induced by the adjointness of the pair \( (\tau^{-1}, \tau) \).

Finally, we define \( \psi \) as the composition

\[
\text{Hom}_{H_1}(\mathcal{L}, \overline{\mathcal{L}}) \xrightarrow{\phi_1} \text{Hom}_{H_2}(F(\mathcal{L}), F(\overline{\mathcal{L}})) \xrightarrow{\psi} \text{Hom}_{H_2}(\mathcal{L}', \overline{\mathcal{L}}')
\]

whose second arrow is induced by \((4-2)\) and \((4-3)\). It is straightforward to check that \( \psi, \phi_1 \) and \( \phi_2 \) satisfy the conclusion of the result.

The second possibility in this case is impossible since \( N^* \) is a \((4, 1)\)-bimodule.

Case 2: In the case that \( M \) is a \((2, 2)\)-bimodule, the proof is similar, with one exception. The module \( F(\mathcal{L}) \) can be any indecomposable bundle, since all are line
Corollary 4.3. With the notation above \( D_1 \) is isomorphic to \( D_2 \) if and only if \( C(a_1, b_1) \) is isomorphic to \( C(a_2, b_2) \).
Proof. First, suppose there is a $k$-algebra isomorphism $\phi_1: D_1 \to D_2$. We define $\phi_2: k \to k$ to be the identity map and we let $\psi: D_1 \to D_2$ equal $\phi_1$. It follows that $\psi$ is compatible with $\phi_1$ and $\phi_2$ so that Corollary 4.2 implies that there is an equivalence

$$\mathbb{P}^{nc}(D_1D_1k) \to \mathbb{P}^{nc}(D_2D_2k).$$

Therefore, by Lemma 3.4, $\text{coh}C(a_1, b_1)$ is equivalent to $\text{coh}C(a_2, b_2)$ so that $C(a_1, b_1)$ is $k$-isomorphic to $C(a_2, b_2)$ by Rosenberg’s reconstruction theorem [14].

Conversely, if $C(a_1, b_1)$ is $k$-isomorphic to $C(a_2, b_2)$, then by Lemma 3.4 again, there is an equivalence

$$\mathbb{P}^{nc}(D_1D_1k) \to \mathbb{P}^{nc}(D_2D_2k),$$

so that the result follows from Corollary 4.2. □

References

[1] M. Artin and J. J. Zhang, Noncommutative projective schemes, *Adv. Math.* 109 (1994), no. 2, 228-287.
[2] I. Biswas and D.S. Nagaraj, Vector bundles over a nondegenerate conic, *J. Aust. Math. Soc.* 86 (2009), 145-154.
[3] D. Chan and A. Nyman, Representations of species via noncommutative $\mathbb{P}^1$-bundles, in progress.
[4] P. M. Cohn, Some remarks on the invariant basis property, *Topology* 5 (1966), 215-228.
[5] P. Gille and T. Szamuely, *Central Simple Algebras and Galois Cohomology* Cambridge Studies in Advanced Mathematics, vol. 101, Cambridge University Press, Cambridge, 2006.
[6] J. Hart and A. Nyman, Duals of simple two-sided vector spaces, *Comm. Algebra* 40 (2012), 2405-2419.
[7] D. Kussin, Factorial algebras, quaternions and preprojective algebras, *Algebras and modules, II (Geiranger, 1996)*, 393-402, CMS Conf. Proc., 24, *Amer. Math. Soc.*, Providence, RI, 1998.
[8] D. Kussin, Noncommutative curves of genus zero: related to finite dimensional algebras, *Mem. Amer. Math. Soc.* 942 (2009), x+128pp.
[9] H. Lenzing and I. Reiten, Hereditary Noetherian categories of positive Euler characteristic, *Math. Z.* 254 (2006), 133-171.
[10] I. Mori, On the Classification of Decomposable Quantum Ruled Surfaces, *Ring Theory 2007*, World Sci. Publ. (2009), 126-140.
[11] A. Nyman, Noncommutative Tsen’s theorem in dimension one, *J. Algebra*, to appear.
[12] A. Nyman, Serre duality for non-commutative $\mathbb{P}^1$-bundles, *Trans. Amer. Math. Soc.* 357 (2005), 1349-1416.
[13] A. Nyman, The geometry of arithmetic noncommutative projective lines, *J. Algebra* 414 (2014), 190-240.
[14] A. L. Rosenberg, Reconstruction of Schemes, MPI Preprints Series, 1996 (108).
[15] S. P. Smith, Maps between non-commutative spaces, *Trans. Amer. Math. Soc.* 356 (2004), 2927-2944.
[16] J.T. Stafford and M. Van den Bergh, Noncommutative curves and noncommutative surfaces, *Bull. Amer. Math. Soc. (N.S.)* 38 (2001), no. 2, 171-216.
[17] M. Van den Bergh, The Brauer-Severi scheme of the trace ring of generic matrices, *Perspectives in ring theory* (Antwerp, 1987), 333-338, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 233, Kluwer Acad. Publ., Dordrecht, 1988.
[18] M. Van den Bergh, Non-commutative $\mathbb{P}^1$-bundles over commutative schemes, *Trans. Amer. Math. Soc.* 364 (2012), 6279-6313.
[19] M. Van den Bergh, Noncommutative quadrics, *Int. Math. Res. Not.* (2011), no. 17, 3983-4026.

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