Vortex in a $d$-wave Superconductor at Low Temperatures

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A systematic perturbation theory is developed to describe the magnetic field-induced subdominant $s$- and $d_{xy}$-wave order parameters in the mixed state of a $d_{x^2-y^2}$-wave superconductor, enabling us to obtain, within weak-coupling BCS theory, analytic results for the free energy of a $d$-wave superconductor in an applied magnetic field $H_{d1} \lessgtr H \ll H_{d2}$ from $T_c$ down to very low temperatures. Known results for a single isolated vortex in the Ginzburg-Landau regime are recovered, and the behavior at low temperatures for the subdominant component is shown to be qualitatively different. In the case of subdominant $d_{xy}$ pair component, superfluid velocity gradients and an orbital Zeeman effect are shown to compete in determining the vortex state, but for realistic field strengths the latter appears to be irrelevant. On this basis, we argue that recent predictions of a low-temperature phase transition in connection with recent thermal conductivity measurements are unlikely to be correct.

PACS Numbers: 74.25.Nf., 74.20.Fg

I. INTRODUCTION

Vortices in classic superconductors involve a winding of $2\pi$ of the order parameter phase $\phi(r)$ around the vortex line. Since the order parameter $\Delta_k = |\Delta_k| e^{i\phi}$ in a $d_{x^2-y^2}$-wave superconductor like the high-$T_c$ cuprates is also simply a complex scalar with single global phase $\phi$, it was initially expected that the vortex state in the cuprates might be structurally quite similar to the textbook case. One remarkable difference was pointed out by Joynt [1]: in the $d$-wave case, if subdominant pair potential components of different symmetry exist, corresponding order parameter components can be induced at $T_c$ by any probe which couples to gradients of the order parameter, even if the subdominant zero field “bare” critical temperatures are very small or zero. Thus the structure of a $d$-wave vortex generically involves admixtures of different symmetry order parameters. Secondly, as noted by Volovik [2], the traditional roles played by extended and localized quasiparticle states in classic superconductors are reversed in the $d$-wave case. In classic superconductors at low temperatures, Caroli-de Gennes-Matricon bound states in the vortex core [3] dominate the electronic density of states because extended states are fully gapped and therefore unoccupied. In the $d$-wave case, the existence of order parameter nodes inhibits the formation of the bound states (their very existence is questionable [4,5]), and populates the extended ones, which are found to dominate thermodynamics at low temperatures and magnetic fields.

Several authors [6–13] have attacked the $d$-wave mixed state structure problem in recent years, armed with these ideas. Early studies focussed on an isolated $d$-wave vortex, allowing for an induced $s$-wave order parameter component and solving the Bogoliubov-de Gennes equations on a lattice. [4] Within the Ginzburg-Landau (GL) theory, similar results were obtained. [11] The $s$ component was shown to have opposite winding number to that of the parent $d_{x^2-y^2}$ in the core regions. Far away from the core center, it decays as $1/r^2$, and its winding number becomes 3, [6–10], implying that there are four extra vortices in the $s$-field at large distances from the main core. [6,7] These results were confirmed by numerical solutions of the Elienberger equations by Ichioka et al. [11] The possibility of an induced $d_{xy}$ component was also allowed for in Ref. [11], which concluded that the structure in this case was similar to that of the (induced) $s$-wave case, except that the induced order parameter at large distances was found to decay more rapidly, roughly as $1/r^4$, and have opposite winding number 5. Koyama and Tachiki [12] pointed out, however, that if the calculation is done in a gauge invariant manner there is an additional term in the free energy not found by Ichioka et al., involving a Zeeman coupling of the field to an intrinsic orbital magnetic moment in a state with structure close to a uniform $d_{x^2-y^2} + id_{xy}$. They furthermore showed, within a GL framework with coefficients determined by BCS weak-coupling theory, that this term is proportional to the particle-hole asymmetry of the normal metal from which the superconductor condenses, and dominates sufficiently far from the vortex core.

Interest in the possibility of order-parameter mixing in the vortex state was heightened by the experimental observation [14,15] of a plateau in the thermal conductivity as a function of the magnetic field $H$ when $H$ is above some critical value $H^*$. Krishana et al., [14] in particular, speculated that their observation of a sharp kink at $H^*$ might be explained by the sudden onset of an out-of-phase $d_{xy}$ component at this critical field. The new high-field
this type should be valid only down to a scale $\Delta \sim \nabla^2 \phi$, several authors \cite{22-25}. Initially Volovik \cite{2} and Kopnin and Volovik \cite{26} proposed that a semiclassical analysis of velocity, local order parameter magnitude fluctuations, and their gradients. We thus neglect states possibly localized this effect, and we show here in fact that the orbital Zeeman coupling in the current problem is quite small.

Physically, $\Delta$ might account for the phase transition, and put forward a $T$- and $H$-dependent free energy functional driving the phase transition, and found a critical field $H^* \sim T^2$ similar to experiment. This special free energy functional does not have a microscopic basis and it is not clear yet whether there is such a field-induced secondary phase transition. To date, no magnetic-field induced transition has been found in relevant numerical studies of the vortex lattice. \cite{11,12}

Laughlin’s argument ignored the physics of the core region, thought to be negligible, but Ramakrishnan pointed out that quite similar effects are to be expected due to a combination of superfluid velocity Doppler shifts and Andreev reflection near the vortex cores. \cite{18} In higher fields, he proposed that these local $d_{x^2-y^2} + id_{xy}$ patches might overlap, causing a transition to a uniform gapped state. This scenario is similar to one proposed by Movshovich et al. \cite{19} in the related case of magnetic impurities in a $d$-wave superconductor in zero field. Finally, Balatsky \cite{20} has recently investigated the effect of the orbital Zeeman term near the upper critical field, and argued that the $d$-wave state is always unstable to a $d_{x^2-y^2} + id_{xy}$ mixture. An unusual collective clapping mode in association with the relative phase of $d_{xy}$ to $d_{x^2-y^2}$ was predicted in this superconductor. \cite{21}

Several important questions have not been addressed in the analyses of these issues thus far and have motivated this work. First and foremost, we would like to understand whether a phase transition of the type proposed by Krishana et al. is possible. Analyses in the GL regime are not applicable, and numerical calculations \cite{5,6,9,11} are not always useful to understand competing physical effects. We have therefore developed a systematic calculational approach capable of treating carefully the relevant quasiparticle states in the presence of spatially varying superflow together with the relevant subdominant order parameter components on an equal footing in the low-temperature phase. Our theory works for $H \ll H_c$, in which case the vortex core region can be safely neglected. Secondly, we would like to understand the structure of the vortex state and the role of the quasiparticles as a preliminary to the yet-unsolved problem of quasiparticle transport in applied magnetic field. In addition to being inapplicable at low temperatures, the GL calculations on which most of one’s intuition for this problem is based are unable to predict magnitudes of physical effects since they are based entirely on symmetry considerations. This is particularly important in the case of the orbital Zeeman coupling in the $d_{x^2-y^2}, d_{xy}$ mixing problem. The magnitude of the induced orbital moment is a very difficult quantity to estimate properly, as one might \textit{a priori} deduce by analogy to the intrinsic orbital angular momentum problem in the $^3$He $\lambda$-phase. In this case, it was found that naive calculations dramatically overestimated this effect, and we show here in fact that the orbital Zeeman coupling in the current problem is quite small.

In this paper, we adopt a semiclassical approach, expanding the BCS free energy in powers of the local superfluid velocity, local order parameter magnitude fluctuations, and their gradients. We thus neglect states possibly localized in the vortex core, and other quasiparticle bandstructure effects in a periodic vortex lattice discussed recently by several authors \cite{22,23}. Initially Volovik \cite{2} and Kopnin and Volovik \cite{26} proposed that a semiclassical analysis of this type should be valid only down to a scale $\left(\frac{\Delta_0}{E_F}\right)\sqrt{H/H_c}$. Recent numerical work \cite{21,22} indicated, however, that the true crossover scale is much smaller for realistic systems with $\Delta_0/E_F \ll 1$. Such fine details of the true quantum quasiparticle band structure will also be smeared out by impurity effects. We believe, therefore, that our neglect of the vortex core and quasiparticle bandstructure will be justified for cuprate superconductors at low fields ($H \ll H_c$) and temperatures, and that the current analysis will thus be adequate.

We begin by presenting in Sec. II the method, which involves a functional integral representation of the BCS free energy $F$, which we then expand in powers of slow superfluid velocity gradients and small subdominant order parameter components. Analytical results for $F$ in the GL regime, the low temperature regime, and, for $s$-wave case, an ultralow temperature regime where nonlinear superflow effects dominate, are given. This allows us in Sec. III to calculate the order parameter fluctuations directly. We then apply these results to the comparison of structure of a single isolated vortex with $s$ or $d_{xy}$ subdominant pairing at various temperatures in Sec. IV, and go on in Sec. V to discuss the prospects for observing a low temperature field-induced transition of this structure. In Sec. VI we discuss existing experiments and make some comments on the various available scenarios. In Appendix A, a detailed derivation of the free energy is presented, while Appendix B is devoted to a general calculation of the spontaneous magnetization in a $d_{x^2-y^2} + id_{xy}$-wave superconductor.
II. FREE ENERGY

We start from a two-dimensional (2D) phenomenological BCS mean-field Hamiltonian in the mixed state: \[ H_{MF} = \sum_{\sigma} \int d^2r c^\dagger_{\sigma}(r) \left\{ \frac{1}{2m} \left[ -i \nabla - \frac{\mathbf{e}}{c} \mathbf{A}(r) \right]^2 - \mu \right\} c_{\sigma}(r) + \int d^2rd^2r' \left[ \Delta(r, r') c^\dagger_{\uparrow}(r') c^\dagger_{\downarrow}(r') + h.c. \right] - \int d^2rd^2r' V(r - r') |b(r, r')|^2, \] where \[ V(r) = V_d \Phi_d(r) + V_s + V_{xy} \Phi_{xy}(r) \] with \( \Phi_i \) characterizing the irreducible representations \( d_{x^2-y^2}, s, \) and \( d_{xy}, \) for which the lowest order basis functions over a circular Fermi surface are

\[
\Phi_{ik} = \begin{cases} 
\cos 2\varphi, & i = d_{x^2-y^2} \\
1, & i = s \\
\sin 2\varphi, & i = d_{xy}
\end{cases},
\]

\[
\Delta(r, r') = V(r - r') b(r, r')
\]
is the pairing order parameter with \( b(r, r') = \langle c_{\uparrow}(r') c_{\downarrow}(r) \rangle, \) and \( \mathbf{A}(r) \) the magnetic vector potential. Throughout the paper \( h = k_B = 1 \) units are chosen. The magnetic field in the problem is perpendicular to the 2D plane, i.e., along the \( \hat{z} \) direction. We assume that \( V_d < 0, \) and \( V_d, V_s \) and \( V_{xy} \) take such values that in the absence of the magnetic field, the superconducting state is of \( d_{x^2-y^2}-\text{wave symmetry with} \Delta(r, r') = \Delta_0 \Phi_d(r - r'). \)

In the mixed state, \( \Delta(r, r') \) takes the form of \( \Delta(r, r') = e^{i\phi(r, r')} \Delta(\frac{r + r'}{2}) \Theta(\frac{r - r'}{2}), \) where \( \Delta(\mathbf{R}, \mathbf{\rho}) = \Delta_d(\mathbf{R}) \Phi_d(\mathbf{\rho}) + D_s(\mathbf{R}) + D_{xy}(\mathbf{R}) \Phi_{xy}(\mathbf{\rho}), \) with \( \Delta_d(\mathbf{R}) = \Delta_0 + D_d(\mathbf{R}). \) The \( D_i(\mathbf{r}) \) are the magnetic field-induced pairing order parameter deviations from their values in zero field.

The partition function of Hamiltonian \([17] \), after making a canonical transformation \([28] \) to eliminate the phase field of the pairing order parameter \( \phi(r), \) \([27] \) is:

\[
Z = Z_0 \exp \left\{ \sum_n \text{Tr} \ln \left[ \tilde{M}_0 + \left( \frac{\tilde{V}_1}{\tilde{D}} \right) \right] \right\},
\]

where

\[
Z_0 = \exp(-F_0/T),
\]

\[
F_0 = -\int d^2r \left( \frac{|	ilde{\Delta}_d(r)|^2}{V_d} + \frac{|D_s(r)|^2}{V_s} + \frac{|D_{xy}(r)|^2}{V_{xy}} \right),
\]

and, in the momentum representation,

\[
(\tilde{M}_0)_{k,k'} = \begin{pmatrix} -i\omega_n + \epsilon_k & \Delta_k \\ \Delta_k & -i\omega_n - \epsilon_k \end{pmatrix} \delta_{k,k'}
\]

\[
(\tilde{V}_1)_{k,k'} = \int d^2r e^{i(k' - k) \cdot r} \left[ -\mathbf{v}_s(r) \cdot \mathbf{k} + \frac{1}{2} m \nu_s^2(r) \right],
\]

\[
(\tilde{V}_2)_{k,k'} = \int d^2r e^{i(k' - k) \cdot r} \left[ -\mathbf{v}_s(r) \cdot \mathbf{k} - \frac{1}{2} m \nu_s^2(r) \right],
\]

\[
\tilde{D}_{k,k'} = \sum_i (\tilde{D}_i)_{k,k'} = \sum_i \int d^2r e^{i(k' - k) \cdot r} D_i(r) \Phi_{d,i} \frac{\mathbf{k} \cdot \mathbf{k}'}{2m}
\]

with \( \omega_n = (2n + 1)\pi T \) the fermion Matsubara frequency, \( \Delta_k = \Delta_0 \Phi_{d,k}, \) and

\[
\mathbf{v}_s(r) = \frac{e \mathbf{A}(r)}{mc} - \frac{\nabla \phi(r)}{2m}
\]
the supercurrent velocity. In writing down Eq. (11) we have used the relation \( \vec{\nabla} \cdot \mathbf{v}_s(\mathbf{r}) = 0 \) which corresponds to the conservation of the supercurrent. Otherwise, \( mv_s^2(\mathbf{r})/2 \) in Eqs. (7) and (8) has no significant effect and will be neglected hereafter.

The free energy resulting from Eq. (4) is
\[
F = -T \text{Tr} \ln Z = F_0 - T \text{Tr} \ln \hat{M}_0 + T \sum_{m=1}^{\infty} \frac{1}{m} \text{Tr} \hat{p}^m,
\]
where
\[
\hat{p} = \hat{g} \cdot \left( \frac{\hat{V}_1}{\hat{D}^1} \frac{\hat{D}}{\hat{V}_2} \right),
\]
with \( \hat{g} \) the Green-function matrix which, in the momentum representation, is
\[
\hat{g}_k = \left( \begin{array}{cc} g_{1k} & g_{2k} \\ g_{2k} & g_{4k} \end{array} \right) = - \left( \begin{array}{cc} \frac{i\omega_n + q_n}{W_{nk}} & \Delta_k \\ \frac{i\omega_n - q_n}{W_{nk}} & \frac{\Delta_k}{W_{nk}} \end{array} \right).
\]
Here,
\[
W_{nk} = \omega_n^2 + E_k^2, \quad E_k = \sqrt{c_k^2 + \Delta_k^2}.
\]
The calculation of the trace of \( \hat{p}^m \) in Eq. (11) can be done by noting that
\[
\text{Tr} \hat{p}^m = \prod_{j=1}^{m} \int \frac{d^2k_j}{(2\pi)^2} \int d^2r_j \epsilon^{i(k_j - k_{j+1}) \cdot r_j} \text{Tr} \left( \prod_{j=1}^{m} \hat{p}_{k_j, k_{j+1}}(r_j) \right),
\]
where \( k_{m+1} = k_1 \). In most of the bulk region not close to a vortex core, \( v_s \) as well as the \( \mathbf{v}_s \)-induced \( D_i \) are spatially slowly varying functions. Thus we are allowed to expand \( \mathbf{v}_s(r_j) \) and \( D_i(r_j) \) in \( \hat{p}_{k, k_1}(r) \) in Eq. (15) as power series in their derivatives.

\[
\begin{align*}
\mathbf{v}_s(r_j) & \approx \epsilon^{i(r_j - r)} \bar{\mathbf{v}}_s \mathbf{v}_s(r) = \mathbf{v}_s(r) + [i(r_j - r) \cdot \bar{\mathbf{v}}_s] \mathbf{v}_s(r) + \cdots, \\
D_i(r_j) & \approx \epsilon^{i(r_j - r)} \bar{\mathbf{D}}_i D_i(r) = D_i(r) + [i(r_j - r) \cdot \bar{\mathbf{D}}_i] D_i(r) + \cdots,
\end{align*}
\]
the first few terms of which make main contribution to \( \text{Tr} \hat{p}^m \). This property enables us to develop a perturbation theory to obtain the free energy. The resulting calculation is straightforward but tedious, and is summarized in Appendix A. The final result for the free energy with respect to the pairing order parameters \( \bar{F} = \int d^2r \bar{f}(r) \), where the free energy density \( \bar{f}(r) \), keeping terms up to quadratic in \( D_i \), is
\[
\bar{f}(r) = f_{\Delta_0}(r) + f_s(r) + f_{xy}(r) + \delta f(r),
\]
\[
f_{\Delta_0}(r) = -\frac{|\Delta_0(r)|^2}{V_d} - T \sum_n \int \frac{d^2k}{(2\pi)^2} \ln \left( \frac{W_{nk}(r)}{W_{nk}} \right),
\]
\[
f_s(r) = \Delta_0^2(T) N_0 \left[ L'_s(T, \mathbf{v}_s) \bar{D}_s(r) + L''_s(T, \mathbf{v}_s) \bar{D}_s''(r) \right] + \left[ c_s + \epsilon_s(T, \mathbf{v}_s) \right] \left[ D_s'(r) \right]^2 + \left[ c_s + \epsilon_s''(T, \mathbf{v}_s) \right] \left[ D_s''(r) \right]^2,
\]
\[
f_{xy}(r) = F_0(r) + F_{OZ}(r) + \Delta_0^2(T) N_0 \left[ c_{xy} + \epsilon_{xy}(T, \mathbf{v}_s) \right] \bar{D}_{xy}'(r) \left[ D_{xy}'(r) \right]^2 + \left[ c_{xy} + \epsilon_{xy}''(T, \mathbf{v}_s) \right] \bar{D}_{xy}''(r) \left[ D_{xy}''(r) \right]^2,
\]
\[
F_{OZ}(r) = \Delta_0^2(T) N_0 Q^{OZ}(T) \frac{e}{mc} B(r) \bar{D}_{xy}'(r),
\]
where \( ' \) and \( '' \) indicate real and imaginary parts, respectively, \( \bar{D}_i = D_i/\Delta_0(T), \ i = s, d_{xy}, c_s = (-V_{xy} N_0)^{-1} - 2c_d, \ c_{xy} = (-V_{xy} N_0)^{-1} - c_d \), with \( c_d = (-V_{xy} N_0)^{-1} \), \( B(r) \) is the magnetic induction, and \( W_{nk}(r) \), \( Q^{OZ}(T) \), \( L_i(T, \mathbf{v}_s) \) and \( \eta_i(T, \mathbf{v}_s) \) are defined in Appendix A. \( f_{\Delta_0} \) in Eq. (18) is the free energy density in association with the dominant \( d_{x^2-y^2} \) component in the absence of \( D_s \) and \( D_{xy} \). \( f_s \) and \( f_{xy} \) are the free energy densities for the \( s \) and \( d_{xy} \) components, respectively. \( \delta f \) involves terms of high orders in \( v_s, D_s \), derivatives of \( v_s \), and mixed \( s \) and \( d_{xy} \) terms. Note in Eqs. (19) and (20) the quadratic terms in \( D_i, i = s, d_{xy} \) can be easily reformulated to coincide with the familiar GL form:
\((c_i + \frac{d_i}{2})|\mathcal{D}_i(r)|^2 = \frac{d_i}{2}|\mathcal{D}_i(r)|^2 + \frac{d_i}{2}|\mathcal{D}_i(r)|^2\). In addition, derivatives of \(\mathcal{D}_i\) terms are absorbed into the powers of \(\mathcal{D}_i\) terms by partial integration, as indicated in Appendix A.

With the free energy in Eqs. (17)–(22) we are in a position to investigate the vortex state. We will first show general results for an arbitrary \(v_s(r)\) distribution \(\mathcal{D}_s\) in Sec. III, and apply the theory to the single vortex case in Sec. IV.

III. ORDER PARAMETERS

A. \(d\)-wave pairing order parameter \(\bar{\Delta}_d(r) = \Delta_0 + \mathcal{D}_d(r)\)

In studying the dominant \(d_{x^2-y^2}\) component, \(\mathcal{D}_s\) and \(\mathcal{D}_{xy}\) can be set to zero since the \(\bar{\Delta}_d\cdot\mathcal{D}_s\) and \(\bar{\Delta}_d\cdot\mathcal{D}_{xy}\) mixing terms appear in higher orders of \(v_s\) or its derivatives. The gap equation \(\partial f_{\bar{\Delta}_d}/\partial \bar{\Delta}_d(r) = 0\) produces

\[
- \frac{\bar{\Delta}_d(r)}{V_d} = T \sum_n \int \frac{d^2k}{(2\pi)^2} \frac{\bar{\Delta}_d(r) \cos 2\varphi}{W_{nk}(r)},
\]

where \(W_{nk}(r)\) is defined in Eq. (A6) in Appendix A. At \(H = 0\), Eq. (23) reduces to

\[
1 = -V_d T \sum_n \int \frac{d^2k}{(2\pi)^2} \frac{\cos 2\varphi}{W_{nk}(r)} = -V_d \int \frac{d^2k}{(2\pi)^2} \frac{\cos 2\varphi \tanh(E_k/2T)}{2E_k},
\]

which yields the well-known \(T_c\) formula \(c_d = (1/2)\ln(2\gamma \omega_D/\pi T_c)\), with \(\omega_D\) the BCS cutoff and \(\gamma \approx 0.5772\) the Euler constant, as well as the asymptotic behaviors of the gap maximum

\[
\Delta_0(T) \simeq \begin{cases} 
3.54[T_c(T - T_c)]^{1/2}, & T \gg \Delta_0(T), \\
2.14T_c - 0.39(T/T_c)^3 T_c, & T \ll \Delta_0(T).
\end{cases}
\]

The presence of vortices depletes the pairing order parameter \(\Delta_0\) by \(\mathcal{D}_d(r)\). Sufficiently far from the vortex cores, either \(T \neq \Delta_0(T)\) is larger than \(v_s(r)k_F\) in the whole \(T\) regime, and thus we can treat \(v_s(r)k_F\) and \(\mathcal{D}_d(r)\) perturbatively. It is straightforward to show that the normalized real part of \(\mathcal{D}_d(r)\) is

\[
\mathcal{D}_d(r) \simeq \left\{ \begin{array}{ll}
-\frac{1}{3} \frac{\epsilon_F m v_s^2(r)}{(2\ln 2)\Delta_0^2(T)} \frac{\Delta_0^2(T)}{\Delta_0(T)} & \text{GL regime}, \\
-\frac{1}{3} [\zeta(3)^2 - 4 \ln(2)] \frac{\epsilon_F m v_s^2(r)}{\Delta_0^2(T)} \frac{\Delta_0^2(T)}{\Delta_0(T)} & v_s k_F \ll T \ll \Delta_0(T), \\
-\frac{3}{5} \frac{\epsilon_F m v_s^2(r)}{\Delta_0^2(T)} \frac{\Delta_0^2(T)}{\Delta_0(T)} & T \ll v_s k_F \ll \Delta_0(T)
\end{array} \right.,
\]

where \(\epsilon_F\) is the Fermi energy and \(\zeta(3)\) is the Riemann function. The imaginary part of \(\mathcal{D}_d(r)\) driven by the derivatives of the supercurrent is unimportant compared with \(\mathcal{D}_d(r)\) in the whole temperature region, for its driving term \(F_d^{(1)}\) in Eq. (A13) is very small as discussed in Appendix A.

B. Field-Induced \(\mathcal{D}_s(r)\)

From Eqs. (7) and (11) we obtain the gap equation for the \(s\) component, \(\partial f_s(r)/\partial \mathcal{D}_s(r) = 0\), which gives

\[
\mathcal{D}_s'(r) \simeq -\frac{L_s'(T, v_s)}{2[c_s + \eta_s'(T, v_s)]}, \quad \mathcal{D}_s''(r) \simeq -\frac{L_s''(T, v_s)}{2[c_s + \eta_s''(T, v_s)]},
\]

where \(L_s'(T, v_s), L_s''(T, v_s), \eta_s'(T, v_s)\) and \(\eta_s''(T, v_s)\) are defined in Appendix A.

For \(T > v_s, k_F\), we can plug \(L_s\) and \(\eta_s\) obtained in Eqs. (A13)–(A18) into Eq. (27) to get the scaling functions of \(\mathcal{D}_s'(r)\) and \(\mathcal{D}_s''(r)\),

\[
\mathcal{D}_s'(r) \simeq G_{s1} \left( \frac{\Delta_0(T)}{T}, \frac{\epsilon_F m v_s^2(r)}{T^2} \right) \frac{\epsilon_F m v_s^2(r) - v_s^2(r)}{T^2},
\]

\[
\mathcal{D}_s''(r) \simeq G_{s2} \left( \frac{\Delta_0(T)}{T}, \frac{\epsilon_F m v_s^2(r)}{T^2} \right) \frac{\epsilon_F \partial_z v_s^2(r)}{T^2},
\]

5
\[
G_{s_1}(d, z) = -\frac{h_{1s}(d)}{c_s - h_{3s}(d) + 2d^2h_{2s}(d) + zh_{4s}(d)},
\]
\[
G_{s_2}(d, z) = -\frac{2h_{2s}(d)}{c_s - h_{3s}(d) + zh_{4s}(d)}.
\]

with \(h_{1s}, \ldots, h_{4s}\) defined in Eqs. (A11) and (A20).

It is easy to see that in the GL regime,
\[
\mathcal{D}_{s_1}^{GL}(r) \simeq -\frac{1}{c_s + \frac{3}{4} \frac{\Delta_2(T)}{\pi^2 T^2} + \frac{\epsilon_F m v_{s,xy}(r)}{\pi^2 T^2}} \epsilon_F m [v_{s,xy}^2(r) - v_{s,y}^2(r)],
\]
\[
\mathcal{D}_{s_2}^{GL}(r) \simeq -\frac{1}{c_s - \frac{\Delta_2(T)}{4\pi^2 T^2} + \frac{\epsilon_F m v_{s,xy}(r)}{\pi^2 T^2}} \epsilon_F \partial_x v_{s,xy}(r).
\]

It follows that the ratio between the real and imaginary parts of \(\mathcal{D}_s\)
\[
\frac{\mathcal{D}_{s_1}^{GL}(r)}{\mathcal{D}_{s_2}^{GL}(r)} \simeq \frac{m [v_{s,xy}^2(r) - v_{s,y}^2(r)]}{\partial_x v_{s,xy}(r)}.
\]

is of order unity in most of the region outside the core where \(v_s(r) \sim 1/r_0\) and \(\partial_x v_{s,xy}(r) \sim 1/r_0^2\) with \(r_0\) the distance to the closest vortex core center.

In the case of \(v_s \cdot k_F < T < \Delta_0(T),\)
\[
\mathcal{D}_{s_1}^{low T} (r) \simeq -\frac{(\ln 2)}{c_s + \frac{1}{2} + \frac{\epsilon_F m v_{s,xy}(r)}{4T \Delta_0}} \frac{T}{\Delta_0} \frac{\epsilon_F m [v_{s,xy}^2(r) - v_{s,y}^2(r)]}{\Delta_0^2},
\]
\[
\mathcal{D}_{s_2}^{low T} (r) \simeq -\frac{1}{c_s - \frac{1}{2} + 2(\ln 2) \frac{T}{\Delta_0} + \frac{\epsilon_F m v_{s,xy}(r)}{4T \Delta_0}} \frac{\epsilon_F \partial_x v_{s,xy}(r)}{\Delta_0^2}.
\]

So the ratio between the real and imaginary parts becomes
\[
\frac{\mathcal{D}_{s_1}^{low T} (r)}{\mathcal{D}_{s_2}^{low T} (r)} \simeq (\ln 2) \frac{T}{\Delta_0} \frac{m [v_{s,xy}^2(r) - v_{s,y}^2(r)]}{\partial_x v_{s,xy}(r)}.
\]

Comparing Eqs. (34) and (37) we find an extra prefactor \(T/\Delta_0\) is acquired at low \(T\), indicative of a suppressed real part of the \(s\) component with decreasing \(T\). This is an interesting observation in the present work, the consequence of which on the structure of a single vortex will be discussed in Sec. IV.

For \(T < v_s \cdot k_F\), which can be achieved either by lowering \(T\) in a certain spatial position or approaching the core region at a certain \(T\), the nonlinear effects dominate over the thermal effects and the prefactor \(T/\Delta_0\) in Eq. (37) is expected to be replaced by \(v_s k_F/\Delta_0\). So the real part is negligibly small, and the imaginary part at \(T = 0\) is
\[
\mathcal{D}_{s_1}^{T=0} (r) \simeq -\frac{1 - S_0 v_s(r) k_F / \Delta_0}{c_s - 1/2 + S_0 v_s(r) k_F / 2 \Delta_0} \frac{\epsilon_F \partial_x v_{s,xy}(r)}{\Delta_0^2},
\]
where \(S_0 = \sum_{l=\pm 1} |\cos(\theta + l \frac{\pi}{2})|\). It is interesting to find from Eqs. (38) and (39) that the factor \(T/\Delta_0\) at \(v_s \cdot k_F < T\) is replaced by \(v_s k_F / \Delta_0\) (up to some prefactor) at \(T < v_s \cdot k_F\), reflecting the nonlinear effect due to the Doppler energy shift. Its effect on the free energy and the penetration depth in the Meissner state of a \(d\)-wave superconductor was extensively discussed in the previous papers by the present authors. [27]

C. Field-Induced \(D_{xy}(r)\)

As shown in Eq. (24), there are two terms, \(F^{OZ}\) and \(F_4\), competing in driving the induced \(d_{xy}\) component. The first term \(F^{OZ}\), the so-called orbital Zeeman term, can be rewritten as
\[
F^{OZ}(r) = -M(r) \cdot B(r),
\]
with $M(\mathbf{r})$ the effective magnetic moment associated with the internal orbital current of Cooper pairs in a $d_{x^2-y^2} + id_{xy}$ wave superconductor. A general derivation of this spontaneous magnetization is presented in Appendix B. As also discussed there, $M(\mathbf{r})$ is proportional to the particle-hole asymmetry $\alpha$ (see Eq. (A1)), making the Zeeman orbital effects very small for general density of states. It is therefore worth pointing out that analyses of the $d_{x^2-y^2} - d_{xy}$ mixing problem based solely on a symmetry-based analysis of the orbital moment may lead to unrealistic results.

Besides, it is important to note the very weak temperature dependence of the coefficient $Q^{OZ}$ of $F^{OZ}$, as shown in Fig. 1.

The second term, $F_4$, is in the order $\mathcal{O}(v'^s)$ and contains driving terms for $D'_{xy}$ as well as $D'_{xy'}$. This term can be significant particularly if the system has small or zero particle-hole asymmetry. Note that in the $s$-wave case, it is irrelevant except for very short length scales of order the core size, due to the nonvanishing leading ($\mathcal{O}(v'^s)$) term.

![Normalized $Q^{OZ}$ as a functions of $T/T_c$. $c_d = 3$ is chosen.](image)

The orbital Zeeman term has been invoked by Laughlin and Balatsky as the effect driving a putative transition to a time reversal symmetry-breaking $d_{x^2-y^2} + id_{xy}$ state induced by field. It is therefore particularly interesting to investigate the relative importance of $F^{OZ}$ and $F_4$ within the current approach. For fields $H \gg H_{c1}$, the overlap of vortices leads to nearly homogeneous $B(\mathbf{r}) \approx H$ in space and hence $F^{OZ}/|D_{xy}|$ can be taken as a constant over the bulk. On the other hand, $F_4/|D_{xy}|$, which scales as $(\xi/r)^4$, is strongly space dependent. It increases rapidly when approaching the vortex core, but decays into the bulk. Thus we expect a critical radius $r^*$ beyond which $F^{OZ}$ dominates over $F_4$, but for $r < r^*$ $F_4$ becomes more important and determines the structure. $r^*$ can be estimated from $F_4(r^*)/F^{OZ}(r^*) \sim 1$. It turns out to grow with increasing $H$ and decreasing $T$. In the GL and low $T$ regime, we have

$$r^*(T, H) = \begin{cases} \xi \left( \frac{\epsilon_F}{eH/mc} \right)^{1/4} \sim \xi \left( \frac{R_H}{\lambda_F} \right)^{1/2}, & \text{GL} \\ \xi \left( \frac{\Delta_0(T)}{T} \right)^{1/4} \sim \xi \left( \frac{\Delta_0(T)}{T} \right)^{1/2}, & \text{low } T \end{cases}$$

with $\lambda_F$ the Fermi wavelength, $\xi = v_F/\pi \Delta_0$ the superconducting coherence length, and $R_H = \sqrt{c/eH}$ the average intervortex distance. We summarize the above estimates in Table I, and display schematically the competition between the two driving terms in Fig. 2. Note the “low $T$” results for $F_4$ obtain only down to temperatures above the local superfluid velocity Doppler shift $v_s k_F$. For lower $T$, the perturbation calculation for $F_4$ that we present in Appendix A.3 breaks down because of the zero modes in $\tilde{E}_k$. Instead one has to get a full expression for the local Doppler shift and the derivatives of the supercurrents which is suitable for perturbation expansion after integrating over momenta. This is apparently too complicated to be achieved analytically and thus requires a self-consistent numerical work, which is beyond the scope of the present paper. However, based on the nonlinear results for the $s$ component and $Q^{OZ}$, we do not expect anything qualitatively new in the extremely low $T$ case compared with the $v_s \cdot \mathbf{k}_F \ll T \ll \Delta_0(T)$ case.

| TABLE I. Order of magnitude of terms driving $d_{xy}$ in free energy outside of core region |
|---------------------------------------------------------------|
| Term | GL regime ($\Delta_0 \ll T$) | Low $T$ ($v_s k_F \ll T \ll \Delta_0$) |
| $F^{OZ}$ | $\frac{\xi v_F}{eH/mc} \Delta_0 D_{xy}$ | $\frac{\xi v_F}{eH/mc} \Delta_0 D_{xy}$ |
| $F_4$ | $N_0 \left( \frac{\xi}{\lambda} \right)^2 \Delta_0 D_{xy}$ | $N_0 \left( \frac{\xi}{\lambda} \right)^2 \left( \frac{\epsilon}{\lambda} \right) \Delta_0 D_{xy}$ |
The resulting $d_{xy}$-wave order parameter, for $\xi < r < r^*$ where $F_4$ dominates, is
\[ \tilde{D}_{xy}^r(r) \simeq -\frac{L_{xy}'(T, v_s)}{2[c_{xy} + \eta_{xy}'(T, v_s)]}, \quad \tilde{D}_{xy}''(r) \simeq -\frac{L_{xy}''(T, v_s)}{2[c_{xy} + \eta_{xy}''(T, v_s)]}, \]
(41)
where $L_{xy}'(T, v_s)$, $L_{xy}''(T, v_s)$, $\eta_{xy}'(T, v_s)$, and $\eta_{xy}''(T, v_s)$ are defined in Eqs. (A38), (A39), (A42) and (A43), and their asymptotic behaviors can be found in Table II. In the region $r^* < r < R_H$ where $F^{OZ}$ dominates, the $d_{xy}$ component reads
\[ \tilde{D}_{xy}^r(r) \simeq 0, \quad \tilde{D}_{xy}''(r) \simeq -\frac{Q^{OZ}(T)}{2[c_{xy} + \eta_{xy}''(T, v_s)]} e_{H/mc}, \]
(42)
where $Q^{OZ}(T)$ is defined in Eq. (A40). Eq. (42) and the asymptotic behaviors of $Q^{OZ}$, $\eta_{xy}'(T, v_s)$, and $\eta_{xy}''(T, v_s)$ shown in Table II lead to
\[ \tilde{D}_{xy}''(r) \simeq \left\{ \begin{array}{ll}
\frac{1}{2} \alpha (2c_d - 1) |c_{xy} - \frac{3}{8} \Delta_0^2(T) + \frac{\epsilon_{F}m_{z}^2(r)}{\lambda^2 \eta_{xy}} | - \frac{1}{8} \frac{e_{H}}{mc} - \frac{\Delta_0(T)}{2} & \text{if } T < \Delta_0(T) \\
\frac{1}{8} \alpha \ln \left( \frac{4e_{H} \Delta_0}{\Delta_0} \right) - \frac{1}{2} - 6 \ln(2) \frac{T}{\Delta_0} |c_{xy} - \frac{1}{2} + 2 \ln(2) \frac{T}{\Delta_0} + \frac{\epsilon_{F}m_{z}^2(r)}{4 \Delta_0^2 T} | - \frac{1}{8} \frac{e_{H}}{mc} & \text{if } T > \Delta_0(T)
\end{array} \right. \]
(43)
It is interesting to estimate the value of $r^*$ in a real material. It follows from Eq. (41) that for $T \sim T_c$, $r^*_{GL}/R_H \simeq \sqrt{\xi / \lambda_F (H/H_c)^2} \simeq 10^{1/4}$. For high-$T_c$ cuprates, $\xi / \lambda_F \sim 10$, and thus $r^*_{GL} > R_H$ for fields $H > 0.01 H_c$. In materials with larger $\xi / \lambda_F$, $r^*_{GL}$ becomes order of $R_H$ for smaller fields. Since from Eq. (41) $r^*$ increases with decreasing temperatures, it seems unlikely that the orbital Zeeman free energy plays an important role in determining the local order parameter in the vortex state for fields in the Tesla range.

We would like to make some further remarks on the magnetic field dependence of $\tilde{D}_{xy}^r$ for $r > r^*$ at low temperatures. $\tilde{D}_{xy}''$ shown in Eqs. (42) and (43) are obtained from minimizing the free energy density $f_{xy}(r)$ in Eq. (24). This free energy density is up to quadratic in $\tilde{D}_{xy}''$, which is sufficient for low field and for generic $c_{xy} \gg 1$. In the case of larger field and/or special case of $c_{xy}$ close to or smaller than one, one has to include the free energy density term cubed in $\tilde{D}_{xy}''$. This term can be easily found to be $(\tilde{D}_{xy}''/3)^{3}$ with the spatial dependence of the coefficient neglected. Thus the free energy density up to cubed in $\tilde{D}_{xy}''$ for $r > r^*$ at low temperatures can be written as
\[ \tilde{f}_{xy} = -\gamma B \tilde{D}_{xy}'' + c_0 (\tilde{D}_{xy}''^3)^{1/3}, \]
(44)
where $c_0 = c_{xy} + \eta_{xy}''(T, v_s)$ and $\gamma = \Delta_0^3 N_0 Q^{OZ} e/mc$. Minimizing $\tilde{f}_{xy}$ with respect to $\tilde{D}_{xy}''$ we immediately get
\[ \tilde{D}_{xy}'' \simeq \sqrt{c_0^2 + \gamma B/2} - c_0. \]
(45)
It is obvious that there is a crossover of the linear-$B$ dependence of $\tilde{D}_{xy}''$ for $\gamma B/2 \ll c_0^2$ to square root of $B$ dependence of $\tilde{D}_{xy}''$ for $\gamma B/2 \gg c_0^2$. This interesting behavior is shown in Fig. 8.

In a periodic vortex lattice, the supercurrent field $v_s(r)$ in the London approximation reads
\[ v_s(r) = \frac{\pi}{n} \sum_{K \in G} i \tilde{\Delta}_e^{*} e^{-iK \cdot r} \frac{e^{-i K \cdot r}}{K^2 + \lambda^2}, \]
(46)
and thus there will be special symmetry points where $v_s = 0$. At these points, a careful examination shows that the coefficients $L_{xy}^r$ and $L_{xy}''$ in $F_4$ (Eq. (22)) vanish, and that $\tilde{D}_{xy}$ is driven entirely by $F^{OZ}$, as shown in Eq. (43). This coincides with the numerical result of Yasui and Kita. [3]

| Term | GL regime ($\Delta_0 \ll T$) | Low T ($v, k_F \ll T \ll \Delta_0$) |
|------|-----------------------------|----------------------------------|
| $L_{xy}^r$ | $(\frac{1}{4} U_1 + \frac{1}{4} U_2 - \frac{1}{4} U_3)/(\pi T)^3$ | $(\frac{1}{4} U_1 + \frac{1}{4} U_2 + \frac{1}{4} U_3)/\Delta_0 T$ |
| $L_{xy}''$ | $(-\frac{1}{4} U_1 + \frac{1}{4} U_2 + \frac{1}{4} U_3)/(\pi T)^3$ | $(-\frac{1}{4} U_1 + \frac{1}{4} U_2 + \frac{1}{4} U_3)/\Delta_0 T$ |
| $Q^{OZ}$ | $-(\alpha / \epsilon / F)(2c_d - 1)$ | $-\alpha / \epsilon / F \ln(4e_{H} / \Delta_0) - 1/2 - 6 \ln(2)(T / \Delta_0)$ |
| $\eta_{xy}$ | $-(1/8) \Delta_0^2(T)/(\pi T)^2 + \epsilon_{F}m_{z}^2(r)/(\pi T)^2$ | $\epsilon_{F}m_{z}^2(r)/4 \Delta_0 T$ |
| $\eta_{xy}''$ | $-(3/8) \Delta_0^2(T)/(\pi T)^2 + \epsilon_{F}m_{z}^2(r)/(\pi T)^2$ | $-1/2 + 2 \ln(2)(T / \Delta_0 + \epsilon_{F}m_{z}^2(r)/4 \Delta_0 T$ |
Up to now, we have not included the effect of the $d_{x^2-y^2}$ order-parameter suppression $D_d$ on the $d_{xy}$ component, since we have shown in Sec. III A that $D_d$ is negligibly small in the bulk region, in which we are primarily interested. However, the mechanism for a transition to a $d_{x^2-y^2} + id_{xy}$ state proposed by Ramakrishnan \[18\] involves precisely the interplay between $D_d$ and the supercurrent near the core, leading potentially to local $d_{x^2-y^2} + id_{xy}$ patches which then overlap at some critical field. Motivated by this suggestion, we examine the free energy terms including this effect within our approximation. These terms, which we refer to as the order parameter suppression terms $F_{OPS}$, are obtained from Eq. (A9-A12) (and have been already neglected in arriving at Eq. (20)). It can easily be seen that the $D_{xy}$ component couples to derivatives of either $D_d(r)$ or of $v_s$ in these terms, indicative of a pure nonlocal effect as found by Ramakrishnan. However, for $v_s \cdot k_F < T$, our surprising finding is that up to leading order, terms including derivatives of $D_d(r)$ vanish, leaving

$$F_{OPS} = -\int d^2r \left( T \sum_{n} \int \frac{d^2k}{(2\pi)^2} \frac{\epsilon_k \cos^2\frac{\epsilon_k}{W_{nk}}}{W_{nk}} D_d(r) |\nabla r \times v_s(r)|_z D''_{xy}(r). \right)$$

(47)

Since the $|\nabla r \times v_s(r)|_z$ factor is dominated by the vector potential $A(r)$ rather than $\nabla \phi$ part in $v_s(r)$, it may be replaced by $eB(r)/(mc)$. Comparing Eqs. (47) and (21), we see that $F_{OPS}$ in the core region is of the same order as $F^{OZ}$ in the bulk (in particular, it is also proportional to the particle-hole asymmetry of the normal state), and may be viewed as the leading correction to $F^{OZ}$, if $\Delta_0(T)$ in $F^{OZ}$ is replaced by $\Delta_d(r) = \Delta_0(T) + D_d(r)$. It is therefore clear from our previous discussion of $F^{OZ}$ that $F_{OPS}$ also gives in fact a very small effect even near the vortex core.

FIG. 2. Schematic comparison of $F^{OZ}$ and $F_4$.

IV. SINGLE VORTEX

The results obtained in Sec. III enable us to compare the subdominant order parameters, at various temperatures, in the presence of vortices characterized by a certain superfluid velocity field $v_s(r)$. [31] Our purpose in this section
is to test these results in the concrete case of a single isolated vortex.

We use cylindrical coordinates \( r = (r, \theta) \) with the origin located at the vortex core center. The phase field of the order parameter \( \phi(r) = \theta \) leading to \( \nabla \phi(r) = -(1/2mr)\hat{\theta} \). In the spatial regime of interest, the magnetic field is roughly homogeneous and thus \( A(r) = Br\hat{\theta} \). For \( r \ll R_H \), \( |\nabla \phi(r)/2m| \gg eA(r)/(mc) \), so we can neglect \( A \), and simply write the supercurrent components and their derivatives as

\[
\begin{align*}
v_{sx} &\approx \sin \theta, \\
v_{sy} &\approx -\cos \theta, \\
\partial_x v_{sx}(r) &\approx -\sin 2\theta,
\end{align*}
\]

except for studying the orbital Zeeman term, in which case \( A(r) \) is no longer negligible because \( \nabla \times A \) enters.

\[
D_{s}^{\text{GL}}(r) \simeq 0.67 c_{s}^{-1} \left( \frac{\xi}{r} \right)^{2} \left( \cos 2\theta + 2i \sin 2\theta \right) = 0.37 c_{s}^{-1} \left( \frac{\xi}{r} \right)^{2} (3e^{i2\theta} - e^{-2i\theta})
\]

\[
(49)
\]

**FIG. 4.** Single vortex structure with \( d_{x^2-y^2} \) and \( s \)-wave symmetries only in the GL regime (“High T”) and for \( v_{s}k_{F} \ll T \ll \Delta_{0} \) (“Low T”). \( c_{d} = 3 \) and \( c_{s} = 4 \) are chosen throughout, and distances are given in units of \( \xi \). a) and b) Relative phase of \( s \) and \( d_{x^2-y^2} \) order parameters with long arrows corresponding to \( d_{x^2-y^2} \), short to \( s \), and angle between them to relative phase. c)-f) Arguments and magnitudes of normalized subdominant \( s \) order parameter \( \overline{D}_{s} \).

**A. \( d_{x^2-y^2} \)-s mixing**

In the GL regime, the \( s \)-wave subdominant order parameter has been investigated by many authors, with substantial agreement. The existing results in this limit are easily shown to be recovered in the present theory. Inserting Eq. (48) into Eqs. (32) and (33) leads, in the generic case of \( c_{s} \gg 1 \), to
which is consistent with the results in Ref. [13] in the bulk asymptotic region \( r \gg \xi \). As a result, the \( s \) component is of four-fold symmetry, and the relative winding of \( s \) and \( d \) components is uniform across the whole vortex as shown in Fig. 3.

In the low \( T \) case, as discussed in Sec. III B, \( \mathcal{D}'(r) \) is smaller than \( \mathcal{D}''(r) \) by a factor of \((\ln 2)/\Delta_0\), and the vortex structure at low \( T \) is thus expected to be qualitatively different from that in the GL regime. From Eqs. (48), (35), and (36), one finds that

\[
\mathcal{D}''_{s}(r) \simeq 1.23 c_s^{-1} \left( \frac{\xi}{r} \right)^2 \left[ (\ln 2) \frac{T}{\Delta_0} \right] \cos 2\theta + 2i \sin 2\theta.
\]

The magnitude and phase of \( \mathcal{D}''_{s}(r) \) are shown in Fig. 4. The relative winding of \( s \) and \( d \) components at low \( T \) takes place in a very narrow region of real space near antinode directions set by \( \max(T/\Delta_0, v_s k_F/\Delta_0) \). In Fig. 4, we show the \( T \) dependence of \( \mathcal{D}_{s}'' \) and \( \mathcal{D}_{s}' \) in a spatial position in the bulk.

**FIG. 5.** Real and imaginary parts of \( \mathcal{D}_s \) as functions of \( T/T_c \) for \( \mathbf{v}_s \cdot \mathbf{k}_F < T, \ r/\xi = 10, \ \theta = \pi/8, \ c_d = 3, \) and \( c_s = 4 \) are chosen.

### B. \( d_{x^2-y^2}-d_{xy} \) mixing

As discussed in Sec. III C, the competition between \( F^OZ \) and \( F_4 \) divides the outside-vortex-core region into two rings: the outer ring region \( r^* < r < R_H \) is dominated by \( F^OZ \) resulting in a rigid \( d_{x^2-y^2} + id_{xy} \) superconducting state with spatially nearly constant \( \mathcal{D}_{xy}'' \) obtained in Eqs. (12) and (13); In the inner ring \( \xi < r < r^* \), \( F_4 \) is more important and a spatially varying \( \mathcal{D}_{xy}'' \) is expected. Now we show the single vortex structure in the inner ring region. We first insert Eq. (48) into \( U(\mathbf{v}_s) \) defined in Eqs. (A33), (A34), (A26), (A29), and (A37) to find that

\[
U_1'(\mathbf{v}_s) = 8 U_2'(\mathbf{v}_s) = -32 U_3'(\mathbf{v}_s) = -\frac{2 \epsilon_F^2 \sin 4\theta}{m^2 r^4},
\]

\[
U_1''(\mathbf{v}_s) = -\frac{1}{96} U_2''(\mathbf{v}_s) = -\frac{\epsilon_F^2 \cos 4\theta}{4m^2 r^4}.
\]

Equations (41), (51) and (52) together with the asymptotic behaviors of \( L_{xy} \) in Tabel II imply that in the single vortex case,

\[
L_{xy}' \simeq Y'(T) \left( \frac{\xi}{r} \right)^4 \sin 4\theta, \quad L_{xy}'' \simeq Y''(T) \left( \frac{\xi}{r} \right)^4 \cos 4\theta,
\]

where \( Y' \simeq -7.4 \) in the GL regime, and \( -7.1 \Delta_0(T)/T \) for \( T \ll \Delta_0 \), while \( Y'' \simeq 9.8 \) in the GL regime, and \( 25.9 \Delta_0(T)/T \) for \( T \ll \Delta_0 \). Equations (41), (53) and the asymptotic behaviors of \( n_{xy} \) in Tabel II yield

\[
\mathcal{D}_{xy}^{GL}(r) \simeq 3.7 \left[ c_{xy} - \frac{\Delta_0^2(T)}{8(\pi T)^2} + \frac{\epsilon_F m v_F^2(\mathbf{r})}{(\pi T)^2} \right]^{-1} \left( \frac{\xi}{r} \right)^4 \sin 4\theta,
\]

\[
\mathcal{D}_{xy}''^{GL}(r) \simeq -4.9 \left[ c_{xy} - \frac{3 \Delta_0^2(T)}{8(\pi T)^2} + \frac{\epsilon_F m v_F^2(\mathbf{r})}{(\pi T)^2} \right]^{-1} \left( \frac{\xi}{r} \right)^4 \cos 4\theta,
\]
and hence for \(c_{xy} \gg 1\)

\[
\bar{D}_{xy}^{\text{GL}}(r) \simeq -0.61 i c_{xy}^{-1} \left( \frac{\xi}{r} \right)^4 (e^{-4i\theta} + 7 e^{4i\theta}).
\] (56)

This result coincides with that of Ref. [11].

At low \(T\), we obtain

\[
\bar{D}_{xy}^{\text{low}T}(r) \simeq 3.5 \left[ c_{xy} + \frac{\epsilon F_{\text{MV}}^2(r)}{4\Delta_0 T} \right]^{-1} \left( \frac{\Delta_0}{T} \right) \left( \frac{\xi}{r} \right)^4 \sin 4\theta,
\] (57)

\[
\bar{D}_{xy}^{\text{low}T}(r) \simeq -12.9 \left[ c_{xy} - \frac{1}{2} + 2(\ln 2) \frac{T}{\Delta_0} + \frac{\epsilon F_{\text{MV}}^2(r)}{4\Delta_0 T} \right]^{-1} \left( \frac{\Delta_0}{T} \right) \left( \frac{\xi}{r} \right)^4 \cos 4\theta,
\] (58)

leading, for \(c_{xy} \gg 1\), to

\[
\bar{D}_{xy}^{\text{low}T}(r) \simeq -4.69 i c_{xy}^{-1} \left( \frac{\Delta_0}{T} \right) \left( \frac{\xi}{r} \right)^4 (e^{-4i\theta} + 1.76 e^{4i\theta}).
\] (59)

In Fig. 6, we show the winding of \(d_{xy}\) component and the magnitude of the normalized \(\bar{D}_{xy}\) at high \(T\) (GL) and low \(T\) respectively. The eight-fold symmetry is more obvious at low \(T\).

Since many current theories postulate a homogeneous \(d_{x^2-y^2}\) \(+ id_{xy}\) state without justification, it is interesting to consider the size of the spatially varying part of \(D_{xy}\) relative to its homogeneous component. The smallest relevant value of the ratio \(D_{xy}^{F_4}/D_{xy}^{OZ}\), where the superscripts \(F_4\) and \(OZ\) indicate the relevant driving terms in Eqs. (41) and (43), respectively, is attained at \(r = R_H\) in the physically relevant regime \(R_H < r^*\). The value is \((E_F/\Delta_0)^2(H/H_{c2})(\Delta_0/T)\) at low temperatures; a simple estimate then shows that the spatially fluctuating component is always at least an order of magnitude larger than the homogeneous component at experimentally relevant temperatures and fields.

![Fig. 6](image-url)

**FIG. 6.** Single vortex structure with \(d_{x^2-y^2}\) and \(d_{xy}\) symmetries only. (a) Relative winding of \(d_{xy}\) (short arrows) and \(d_{x^2-y^2}\) (long arrows) components in both the outer and inner rings. (b) Relative phase of \(d_{xy}\) to \(d_{x^2-y^2}\) component in GL and low \(T\) regimes, and magnitude of the normalized \(\bar{D}_{xy}\) in (c) high \(T\) and (d) low \(T\), respectively, in the inner ring \(\xi < r < r^*\). \(c_d = 3\) and \(c_{xy} = 7\) are chosen. Distances are given in units of \(\xi\).
V. IS THERE MAGNETIC FIELD-INDUCED PSEUDO-PHASE TRANSITION?

The results presented in Sections III and IV suggest that there always exist field-induced \( s \) and \( d_{xy} \) components in a parent \( d_{x^2-y^2} \)-wave superconducting state where the electronic interactions in the \( s \) and \( d_{xy} \) channels are nonzero. These subdominant components are spatially inhomogeneous and, in a general case of small \( V_s \) and \( V_{xy} \) compared with \( V_d \), are negligibly small as far as any bulk physical quantity is concerned. However, there is also a special situation in which either \( V_s \) or \( V_{xy} \) is nearly degenerate with \( V_d \), leading to a small \( c_s \) or \( c_{xy} \). In this case, the denominator(s) of \( \mathcal{D}_i \), \( i = s, d_{xy} \) may vanish at some critical temperature \( T^*_c \), a singularity marking a second pseudo-phase transition into a \( d_{x^2-y^2} + \mathcal{D}_i \) state with \( \mathcal{D}_i \) a homogeneous bulk quantity. Investigation of such a possible pseudo-phase transition is particularly interesting in association with the experimentally observed thermal-conductivity plateau as mentioned in the Introduction. Since in high-\( T_c \) cuprates no such phase transition has been reported to be found in the absence of magnetic field, we focus on the question whether there can be magnetic field-driven phase transition. This is equivalent to searching for a nonzero \( T^*_c \) at a finite field which vanishes at zero field.

We first study the \( s \) component. From Eqs. (52), (53), (55), and (56), we see that the denominator of \( \mathcal{D}_s \) can never be zero, and the singularity may occur in \( \mathcal{D}_s' \). In the GL regime, the critical temperatures are

\[
T^*_c(s)(\mathbf{v}_s) = \frac{1}{\pi} \sqrt{c_s^{-1}[1/4 - 2eFm\nu^2/\Delta_0^2(T^*_c)\Delta_0(T^*_c)]}.
\]

(60)

It follows that \( T^*_c(s)(\mathbf{v}_s) < T^*_c(s)(0) \), implying there is no instability at nonzero field leading to a homogeneous subdominant \( \mathcal{D}_s \). At low \( T \), \( T^*_c \) becomes

\[
T^*_c(s)(\mathbf{v}_s) = \frac{1}{4(\ln 2)} \left[ 1 - 2c_s - \frac{eFm\nu^2}{2T^*_c\Delta_0} \right] \Delta_0.
\]

(61)

Again, \( T^*_c(s)(\mathbf{v}_s) < T^*_c(s)(0) \) is found, meaning that the magnetic field does not favor such a second-order phase transition from a \( d_{x^2-y^2} \) to \( d_{x^2-y^2} + is \) state. Accounting for nonlinear effects does not affect this conclusion.

A similar analysis can be made in the \( d_{xy} \) situation. From the asymptotic behaviors of \( \eta'_{xy} \) and \( \eta''_{xy} \) listed in Table II, we find that there is also a singularity in \( \mathcal{D}_{xy}' \) in the GL regime, but its corresponding critical temperatures are smaller than that in \( \mathcal{D}_{xy}'' \). Except for this point, we reach the same conclusion as the \( s \) case: No magnetic field-induced phase transition is found.

At this stage, we may examine the prospects of a possible low-\( T \) field-induced first-order phase transition into a time-reversal symmetry breaking state \( d_{x^2-y^2} + id_{xy} \) as suggested by Laughlin [17]. The free energy functional of the \( d_{xy} \) pairing order parameter \( \phi_{xy} \) assumed by Laughlin takes the form

\[
F = \frac{1}{6\pi} \left( \frac{\mathcal{D}_{xy}''}{\nu} \right)^3 - \frac{1}{\pi} \frac{eB}{c} \mathcal{D}_{xy}' \tanh^2 \left( \frac{\mathcal{D}_{xy}''}{2T} \right) \left[ 1 - \frac{4}{\pi} \int \frac{(\mathcal{D}_{xy}'')^2}{2T^2} \ln[1 + e^{-\mathcal{D}_{xy}''/T}] + \int_{\mathcal{D}_{xy}''/T}^{\infty} \ln[1 + e^{-x}] dx \right] + \frac{1}{12c^2} \mathcal{D}_{xy}' \partial E_{loc},
\]

(62)

where \( \nu = \sqrt{2\Delta_0/m} \). The special \( T \) dependences of the second and third terms lead to a weak first-order phase transition at \( T \approx 0.52\nu/\sqrt{2eB/c} \). We may check the justification of this free energy based on the present theory. The first and third terms in the right-hand side of Eq. (52) are understood to come from the local free energy \( F_{loc} \) in Eq. (45). The second one, as the coupling of the unusual magnetization to the field, is naturally related to the orbital Zeeman term \( F^{OZ} \) in Eq. (37) with \( M(r) \) shown in Eq. (38). But the assumed temperature dependence crucial for the first-order phase transition, is inconsistent with that of \( F^{OZ} \), which we have shown to be a weak function of \( T \) for \( T \ll \Delta_0 \) (see Fig. 1) and a linear function of \( \mathcal{D}_{xy}'' \) in the limit \( \mathcal{D}_{xy}'' \rightarrow 0 \). Therefore, the microscopic calculation in the present work does not confirm Laughlin’s free energy functional which formed the basis of the first-order phase transition found in his work. In the present theory, a small homogeneous \( \mathcal{D}_{xy}'' \) is found for any magnetic field and temperature in the superconducting state. It is swamped by a larger (but still much smaller than \( \Delta_0 \)) \( \mathcal{D}_{xy}' \) spatially fluctuating component over almost all of the vortex lattice for physically relevant fields.

VI. CONCLUSIONS

In this paper, we have formulated a perturbation theory to investigate the magnetic-field induced subdominant order parameters of a clean \( d \)-wave superconductor in the presence of the gauge-invariant spatially varying supercurrent field in the mixed state. With the assumption of slowly spatially varying supercurrents and their induced \( s \) and \( d_{xy} \) components in the bulk sample, we are able to derive the free energy as power series in the Doppler energy shift, the
derivatives of the supercurrents, and the subdominant components. The free energy is valid from $T_c$ down to very low temperatures, enabling us to compare the resulting $s$ and $d_{xy}$ components at low $T$ with the existing results in the GL regime. To leading order, the real and imaginary parts of the $s$ component, driven by local $m_0(v_x^2 - v_y^2)$ and the derivative of the supercurrent $\partial_x v_{sx}$, respectively, were shown to have very different temperature dependences. In the GL regime, both the real and imaginary parts are in the same order. But at low $T$, the real part acquires an extra small prefactor $T/\Delta_0$ and the resulting $s$ winding happens within a very small region near antinodal directions, leaving a rigid $d_{x^2 - y^2} + is$ state over most of the vortex. It is important to note, however, that this structure does not imply a gap in the quasiparticle spectrum, due to the small size of the $s$ component compared with the large Doppler shifts near the core.

To leading order, the $d_{xy}$ component is driven by two terms of different symmetries competing over different parts of the vortex lattice. The first is the orbital Zeeman term $F^{OZ}$ arising from the coupling of the spontaneous magnetization to the magnetic field. Its significance is limited by its small magnitude due to the particle-hole asymmetric effects. The second driving term $F_4$ scales as $(\xi/r)^4$ for $\xi \ll r \ll R_H$. The crossover scale $r^*$ divides the bulk region into inner and outer regions dominated by the two distinct physical effects. In the inner region, $F_4$ determines the vortex structure, and the $d_{xy}$ component has eight-fold symmetry and its relative phase winds 4 times that of the $d_{x^2 - y^2}$ component. In the outer region, $F^{OZ}$ is more important, leading to a rigid $d_{x^2 - y^2} + id_{xy}$ superconducting state, where the $d_{xy}$ component is spatially nearly homogeneous and a weak function of temperature. We have shown that the crossover scale of order $r^* = \xi(R_H/\lambda_F)^{1/2}$ in the GL regime, and becomes $(\Delta_0/T)^{1/4}$ times larger at low $T$. Our best estimate for the high-$T_c$ cuprates suggests that $r^*$ is greater than the intervortex separation $R_H$ for fields above 0.01 $R_{c2}$, so that it appears that vortex lattice structure in fields of order Tesla is governed by $F_4$ and the orbital Zeeman effect is irrelevant. The relative phase between the $d_{x^2 - y^2}$ and $d_{xy}$ components, as well as the magnitude of the induced $d_{xy}$, are therefore strongly space dependent everywhere in the sample.

Our results have implications for several scenarios which have been proposed to create a state without quasiparticle excitations at low temperatures via the creation of a finite out-of-phase subdominant pair component. No such bulk state is found for generic values of the pair potentials $V_s$ and $V_d$. It remains possible that induced core $d_{xy}$ patches overlap with increasing field, as proposed by Ramakrishnan [18], leading to a gapped state at high fields beyond the scope of our analysis. However, analyzing the interplay between order parameter suppression $D_d(r)$ around the vortex cores and $v_{s}(r)$, we found it to be a very small effect on the induced $d_{xy}$ component; it therefore seems unlikely that the field scale where such a transition may occur can be significantly less than $H_{c2}$.

Finally, we searched for a possible second order magnetic field-induced phase transition for special values of the coupling constants, and reached a negative conclusion. No such phase transition into a bulk $d_{x^2 - y^2} + s$ or a $d_{x^2 - y^2} + d_{xy}$ state is found unless the transition has already taken place in the absence of the field, which is apparently not the case in the high-$T_c$ cuprates. Examining the Laughlin free energy driving a first-order phase transition, [17] we found that the crucial field-dependent term has an assumed temperature dependence inconsistent with the BCS theory; thus this phase transition picture is not supported in the present work, consistent with numerical results of Yasui and Kita. [8]

The apparent phase transition observed by Krishana et al. [14] has not been clearly reproduced by other groups, and the possibility exists that the effect is due to inhomogeneously trapped flux. It is still interesting, however, to ask what kinds of intrinsic phase transitions might be possible in a $d$-wave superconductor. We have shown that it is unlikely that any bulk phase transition can be induced by a magnetic field, at least in the low-field regime where our approach is valid. Since our model neglects the vortex core regions, it is conceivable that vortex core transitions such as those observed in the $^3He$ system might still be relevant. However, since the number of bound states in the core region is small and possibly zero for the present case, it seems unlikely that such a transition would have an important effect on the quasiparticles responsible for heat transport at low $T$. A final possibility, currently under investigation within the present framework, is that transitions occur in the vortex lattice structure as a function of field.

**ACKNOWLEDGMENTS**

The authors are grateful to N. Andrei, M. Fogelström, M. Franz, C. D. Gong, T. Kita, T. Kopf, S. Sachdev, S. H. Simon, and Y.-J. Wang for helpful communications. Partial support was provided by the A. v. Humboldt Foundation, NSF Grants No. DMR-9974396 and INT-9815833, and “Graduiertenkolleg Anwendungen der Supraleitung” of the Deutsche Forschungsgemeinschaft.
APPENDIX A: DERIVATION OF FREE ENERGY

In this Appendix we derive the free energy $\tilde{F}$ in Eqs. (17-22) which is valid for space region where $v_s$ varies slowly. We begin with rewriting $\text{Tr} \hat{p}^m$ in Eq. (11) according to Eqs. (A5) and (11),

$$\text{Tr} \hat{p}^m \simeq (\text{Tr} \hat{p}^m)^{\text{(loc)}} + (\text{Tr} \hat{p}^m)^{\text{(der)}}, \quad (A1)$$

where $(\text{Tr} \hat{p}^m)^{\text{(loc)}}$ corresponds to $v_s(r_j)$ and $D(r_j)$ for all $j = 1, \cdots, m-1$ taking the local values $v_s(r)$ and $D(r)$, respectively, and becomes

$$(\text{Tr} \hat{p}^m)^{\text{(loc)}} = \int \frac{d^2k}{(2\pi)^2} \int d^2r \text{Tr} [\hat{p}_{k,k}(r)]^m, \quad (A2)$$

and $(\text{Tr} \hat{p}^m)^{\text{(der)}}$ contains derivatives of $v_s(r_j)$ and/or $D(r_j)$. Inserting Eq. (A1) into Eq. (11) leads to

$$F \simeq F^{\text{(loc)}} + F^{\text{(der)}}, \quad (A3)$$

where $F^{\text{(loc)}}$ is just the local free energy obtained in the semiclassical approximation,

$$F^{\text{(loc)}} = F_0 - T \text{Tr} \ln \tilde{M}_0 + T \sum_{m=1}^{\infty} \frac{1}{m} (\text{Tr} \hat{p}^m)^{\text{(loc)}} = F_0 - \int d^2r T \sum_{m=1}^{\infty} \int \frac{d^2k}{(2\pi)^2} \ln [-\tilde{W}_{nk}(r) - \eta_k(r)], \quad (A4)$$

$$= F_0 - \int d^2r \int \frac{d^2k}{(2\pi)^2} \sqrt{E_k^2(r) + \eta_k(r)} - \int d^2r \int \frac{d^2k}{(2\pi)^2} \sum_{l=\pm 1} \ln \left\{ 1 + e^{-|v_s(r)\cdot k_F + \sqrt{E_k^2(r) + \eta_k(r)}/T} \right\}, \quad (A5)$$

with

$$\tilde{W}_{nk}(r) = -[i\omega_n + v_s(r) \cdot k_F]^2 + E_k^2(r), \quad \tilde{E}_k(r) = \sqrt{\epsilon_k^2 + |\Delta(r)|^2 \Phi_{dk}} \Phi_{ik} + |D_i(r)|^2 \Phi_{ik}^2 \right\}, \quad (A6)$$

Expanding $F^{\text{(loc)}}$ in power series in $D_i$ will give linear-$D_i$ term as well as quadratic of $D_i$ terms. The prefactors are not universal in the whole temperature regimes, as will be discussed and shown below. $F^{\text{(der)}}$ in Eq. (A3) reads

$$F^{\text{(der)}} = T \sum_{m=1}^{\infty} \frac{1}{m} (\text{Tr} \hat{p}^m)^{\text{(der)}} = \sum_j F^{(j)}, \quad (A8)$$

where we expanded $(\text{Tr} \hat{p}^m)^{\text{(der)}}$ as power series in the $j$th derivatives of $v_s$ or $D_i$ with respect to $r$: $(\text{Tr} \hat{p}^m)^{\text{(der)}} = \sum_j (\text{Tr} \hat{p}^m)^{(j)}$. $F^{\text{(der)}}$ reflects nonlocal couplings of the subdominant order parameters to the supercurrent fields. We examine the formal leading term, $F^{(1)}$ which includes $\nabla_r v_s(r)$ or $\nabla_r D_i(r)$, in Eq. (A8),

$$F^{(1)} = \sum_{m=1}^{\infty} \frac{1}{m} (\text{Tr} \hat{p}^m)^{(1)} = \sum_{m=1}^{\infty} \frac{1}{m} (\rho_{m1} + \rho_{m2} + \rho_{m3}), \quad (A9)$$

$\rho_{m1} = \sum_{\nu=0}^{m-2} \int \frac{d^2k}{(2\pi)^2} \int d^2r \text{Tr} \left\{ [\nabla_r \hat{p}_{k,k}(r)] \cdot \hat{p}_{k,k}^{m-2-\nu}(r) \cdot [\nabla_q \hat{p}_{q,k}(r)]_{q=k} \cdot \hat{p}_{k,k}(r) \right\}, \quad (A10)$

$\rho_{m2} = \sum_{\nu=0}^{m-2} \nu \int \frac{d^2k}{(2\pi)^2} \int d^2r \text{Tr} \left\{ [\nabla_r \hat{p}_{k,k}(r)] \cdot \hat{p}_{k,k}^{m-2-\nu}(r) \cdot [\nabla_k \hat{p}_{q,k}(r)] \cdot \hat{p}_{k,k}(r) \right\}, \quad (A11)$

$\rho_{m3} = i(m-1) \int \frac{d^2k}{(2\pi)^2} \int d^2r \text{Tr} \left\{ ([\nabla_r \cdot \nabla_q] \hat{p}_{k,q}(r)]_{q=k} \cdot \hat{p}_{k,k}(r) \right\}. \quad (A12)$

$F^{(1)}$ can be expanded as power series in $D_i$. Note in terms including $\nabla_r D_i(r)$ the derivative can be transferred to that of $v_s$ by partial integral. The resulting linear-$D_i$ term in $F^{(1)}$ is $F_i^{(1)} = \sum_{i=d_s,d_{xy}} F_i^{(1)}$, where
\[ F^{(1)}_s \approx i \int d^2r T \sum_n \int \frac{d^2k}{(2\pi)^2} \Phi_{ik} \{ \nabla_r [v_s(r) \cdot k_F] \} \cdot \sum_{m=2}^{\infty} \sum_{\nu=1}^{m-1} \nu \text{Tr} \left\{ (\nabla_k \hat{g}_k) \cdot \hat{g}_k^\nu \cdot \hat{D}_i(r) \cdot \hat{g}_k^{m-\nu-1} \right\} \]

\[ = \int d^2r D''_i(r) T \sum_n \int \frac{d^2k}{(2\pi)^2} \Phi_{ik} 2\Delta_k [\nabla_k \epsilon_k \cdot \nabla_r] [v_s(r) \cdot k_F] Z_{nk}(v_s) \]

\[ - \int d^2r D''_i(r) T \sum_n \int \frac{d^2k}{(2\pi)^2} \Phi_{ik} 2\epsilon_k [\nabla_k \Delta_k \cdot \nabla_r] [v_s(r) \cdot k_F] Z_{nk}(v_s), \]  

(A13)

with \( \hat{D}_i(r) = \begin{pmatrix} 0 & D_i(r) \\ D_i^*(r) & 0 \end{pmatrix} \) and

\[ Z_{nk}(v_s) = \frac{\{ [v_s(r) \cdot k_F]^2 - W_{nk} \}^2 - 4\omega_k^2 [v_s(r) \cdot k_F]^2}{\{ [v_s(r) \cdot k_F]^2 - W_{nk} \}^2 + 4\omega_k^2 [v_s(r) \cdot k_F]^2} \]  

(A14)

It is easy to see that \( F^{(1)}_d \) is negligibly small, so \( F_{\Delta} \) in Eq. (18) is simply obtained from \( F^{(loc)} \) in Eq. (A4) by setting \( D_s, D_{xy} = 0 \). We argue that \( F^{(loc)} \) and \( F^{(1)}_s \) in Eqs. (A4) and (A13) are sufficient for studying the \( s \) component up to leading orders. Expanding \( F^{(loc)} \) as power series in \( D_s \) and \( v_s k_F \), we will see that the driving term for \( D_s' \) is in leading order of \( m(v_{sx}^2 - \nu_{sx}^2) \), which, in the spatial regime of interest \( \xi < r < R_H \), scales as \( 1/r^2 \). \( F^{(1)}_s \) gives a driving term for \( D_s'' \) scaling as \( \partial_x v_{sx} \sim 1/r^2 \). Clearly, terms contained in \( \sum_{i > 1} F^{(i)} \) are of higher orders of \( v_s \) and derivatives of \( v_s \). We first show results for \( s \) component. The \( d_{xy} \) situation is more complicated and will be discussed later.

1. Scaling expressions for \( L_s(T, v_s) \) and \( \eta_s(T, v_s) \) at \( T > v_s \cdot k_F \)

In this temperature regime, we can expand \( \hat{W}_{nk}(r) \) in Eqs. (A4) and (A13) as power series in \( v_s k_F, D_i'(r), \) and \( D''_s(r) \) to find explicit expressions for \( L_s(T, v_s) \) and \( \eta_s(T, v_s) \) in Eq. (19).

\[ L'_s(T, v_s) = N_0^{-1} T \sum_n \int \frac{d^2k}{(2\pi)^2} 2 \cos^2 2\varphi \frac{\epsilon_{Fm} [v_{sx}^2 - v_{sy}^2]}{W_{nk}^2} h_1_s(d), \]  

(A15)

\[ L''_s(T, v_s) = T \sum_n \int \frac{d^2k}{(2\pi)^2} 4 \cos^2 2\varphi \frac{\epsilon_F \partial_x v_{sx}}{W_{nk}^2} h_2_s(d), \]  

(A16)

\[ \eta'_s(T, v_s) = \eta''_s(T, r) + 2d h_2_s(d), \]  

(A17)

\[ \eta''_s(T, v_s) = -h_3_s(d) + \frac{\epsilon_{Fmv_r^2(r)}}{T^2} h_4_s(d), \]  

(A18)

where Eq. (24) was used, \( d = \Delta_0(T)/T \), and

\[ h_1_s(d) = \frac{8}{\pi} \int_0^\infty dx \int_0^{\pi/4} d\varphi \cos 2\varphi \Phi_{ik}^2 \frac{1}{2\eta} \frac{d^2 f(\eta)}{d\eta^2}, \]  

\[ h_2_s(d) = \frac{8}{\pi} \int_0^\infty dx \int_0^{\pi/4} d\varphi \cos 2\varphi \Phi_{ik}^2 \frac{t(\eta)}{4\eta^3}, \]  

(A19)

\[ h_3_s(d) = \frac{8}{\pi} \int_0^\infty dx \int_0^{\pi/4} d\varphi (1 - 2 \cos^2 2\varphi) \Phi_{ik}^2 \frac{\tanh(\eta/2)}{2\eta}, \]  

\[ h_4_s(d) = \frac{8}{\pi} \int_0^\infty dx \int_0^{\pi/4} d\varphi \Phi_{ik}^2 \frac{1}{2\eta} \frac{d^2 f(\eta)}{d\eta^2}. \]  

(A20)

with \( \eta = \sqrt{x^2 + d^2 \cos^2 2\varphi}, f(\eta) = 1/(e^{\eta} + 1), \) and \( t(\eta) = 1 - 2f(\eta) + 2\eta f(\eta)/d\eta. \)

2. \( L_s(T, v_s) \) and \( \eta_s(T, v_s) \) at \( T = 0 \)

At extremely low \( T \), the perturbation expansions of the free energy as power series in \( v_s(r) k_F \) and \( D'_s(r) \) by expanding \( \hat{W}_{nk}(r) \) before integrating over \( k \) breaks down due to the existence of the zero modes in \( E_k \). The correct approach is to expand the free energy after doing the integrals over \( k \). For simplicity, we only show the \( T = 0 \) results. It is easy to find that the driving term for \( D'_s \) is negligibly small compared with that for \( D''_s \). We may simply set \( D'_s = 0 \). The local free energy in Eq. (A4) becomes
where

\[ F_{T=0}^{(loc)} = F_0 - \int d^2 r \int \frac{d^2 k}{(2\pi)^2} \sqrt{E_k^2(r) + \eta_k(r)} + 2 \int d^2 r \int \frac{d^2 k}{(2\pi)^2} Y_k(r) \Theta(-Y_k(r)) \]  

(A21)

where \( Y_k(r) = \sqrt{E_k^2(r) + [D'_s(r)]^2} - v_s(r) \cdot k_F \) and \( \Theta(x) \) is the Heaviside step function. We obtain \( F_{T=0}^{(loc)} = \Delta F + \int d^2 r [c_s + \eta'_s(T = 0, v_s)] [D'_s(r)]^2 \), where \( \Delta F \) is the local free energy at \( D_s = 0 \), and

\[ \eta'_s(T = 0, v_s) \simeq -\frac{1}{2} + S_\theta v_s(r) k_F \]  

(A22)

with \( S_\theta = \sum_{l=\pm 1} |\cos(\theta + l\pi)| \).

From \( F_s^{(1)} \) in Eq. (A13), we get the driving term for \( D'_s \), from which we find

\[ L'_s(T = 0, v_s) \simeq \frac{2e_F m[v_{xx}^2 - v_{yy}^2]}{\Delta_0^2} \left[ 1 - \frac{S_\theta v_s(r) k_F}{\Delta_0} \right] . \]  

(A23)

3. Free energy with respect to \( d_{xy} \) component

We first analyze the driving terms at \( T > v_s \cdot k_F \). The leading-order linear-\( D_{xy} \) term from \( F^{(loc)} \) in Eq. (A4) turns out to be

\[ F_4^{(loc)} \simeq \int d^2 r \Delta_0^2(T) N_0 Q'_0(T) U'_0(v_s) \mathcal{D}'_{xy}, \]  

(A24)

where

\[ Q'_0(T) = N_0^{-1} T \sum_n \int \frac{d^2 k}{(2\pi)^2} \sin^2 4\varphi \left\{ -\frac{5}{2W_{nk}^3} + \frac{10E_k^2}{W_{nk}^4} - \frac{8E_k^4}{W_{nk}^5} \right\}, \]  

(A25)

\[ U'_0(v_s) = 4e^2 F m^2 [v_{xx}^2(r) - v_{yy}^2(r)] v_{xx}(r) v_{yy}(r). \]  

(A26)

For \( \xi < r < R_H \), \( F_4^{(loc)} \) is of order \( 1/r^4 \). As for \( F_{xy}^{(1)} \) in Eq. (A13), the two terms in the right-hand side are qualitatively different. We leave the discussion on the second term for a bit later. The first term is nonzero in the leading order of \( (mv_{xy}^2) \partial_x v_{xx} \) which scales as \( 1/r^4 \) too. For \( T > v_s k_F \), this term is

\[ F_{xy}^{(1)} \simeq \int d^2 r \Delta_0^2(T) N_0 Q''_0(T) U''_0(v_s) \mathcal{D}''_{xy}, \]  

(A27)

where

\[ Q''_0(T) = N_0^{-1} T \sum_n \int \frac{d^2 k}{(2\pi)^2} \sin^2 4\varphi \left\{ -\frac{5}{W_{nk}^3} + \frac{6E_k^2}{W_{nk}^4} \right\}, \]  

(A28)

\[ U''_0(v_s) = c_F^2 \left\{ m [v_{xx}^2(r) - v_{yy}^2(r)] \partial_x v_{xy}(r) + \partial_y v_{xx}(r) \right\} + 4mv_{xy}(r) v_{xy}(r) \partial_x v_{xx}(r). \]  

(A29)

A simple analysis shows that there are terms coming from \( F^{(2)} \) and \( F^{(3)} \) in Eq. (A8) which are of the same order. After some algebra, we find that the terms of order \( 1/r^4 \) in \( F^{(2)} \) are

\[ F_{xy}^{(2)} \simeq \int d^2 r \Delta_0^2(T) N_0 \left\{ Q'_1(T) U'_1(v_s) + Q'_2(T) U'_2(v_s) \right\} \mathcal{D}'_{xy}, \]  

(A30)

where

\[ Q'_1(T) = N_0^{-1} T \sum_n \int \frac{d^2 k}{(2\pi)^2} \sin^2 4\varphi \left\{ \frac{3}{W_{nk}^3} - \frac{4(E_k^2 + 3c_F^2)}{W_{nk}^4} + \frac{16c_F^2 E_k^2}{W_{nk}^5} \right\}, \]  

(A31)

\[ Q'_2(T) = N_0^{-1} T \sum_n \int \frac{d^2 k}{(2\pi)^2} \sin^2 4\varphi \left\{ \frac{2}{W_{nk}^3} + \frac{24c_F^2 E_k^2}{W_{nk}^4} - \frac{32c_F^2 E_k^2}{W_{nk}^5} \right\}, \]  

(A32)

\[ U'_1(v_s) = c_F^2 \left\{ v_{xx}(r) [\partial_x v_{xy}(r) + 3\partial_y v_{xx}(r)] + v_{xy}(r) [-\partial_y v_{xx}(r) + 3\partial_x v_{xx}(r)] \right\}, \]  

(A33)

\[ U'_2(v_s) = c_F^2 [\partial_x v_{xx}(r)] \partial_x v_{xy}(r) + \partial_y v_{xx}(r)], \]  

(A34)
and in $F^{(3)}$ are

\[
F^{(3)}_{xy} \simeq \int d^2r \Delta_0^2(T) N_0 Q''_2(T) U''_x(v_y) \overline{D}_{xy}^0,
\]

where

\[
Q''_2(T) = N_0^{-1} T \sum_n \int \frac{d^2k}{(2\pi)^2} \sin^2 \frac{2\varphi}{m} \left\{ \frac{1}{2W_{nk}^2} - \frac{2k^2}{W_{nk}^2} \right\},
\]

\[
U''_x(v_y) = \epsilon_F \left[ \frac{1}{m} \partial_x^3 v_{xy}(r) - \frac{1}{m} \partial_y^3 v_{xz}(r) + \frac{6}{m} \partial_x^2 \partial_y v_{xz}(r) \right].
\]

The sum of $F^{(loc)}_4$, $F^{(1)}_{xy}$, $F^{(2)}_{xy}$, and $F^{(3)}_{xy}$ leads to $F_4(r)$ term in Eq. (22), with

\[
L'_{xy}(T, v_x) = Q'_1(T) U'_1(v_x) + Q'_2(T) U'_2(v_x) + Q'_3(T) U'_3(v_x),
\]

\[
L''_{xy}(T, v_x) = Q''_1(T) U''_1(v_x) + Q''_2(T) U''_2(v_x) + Q''_3(T) U''_3(v_x).
\]

The second term on the right-hand side of Eq. (A13) is nonvanishing by noting that $[\nabla_k \Delta_k \cdot \nabla_r] [v_x(r) \cdot k_F] = \Delta_0 \sin 2\varphi [\nabla_r \times v_x(r)]_z$ which picks up the magnetic vector potential $A(r)$ in $v_x(r)$. This term is nothing but $F^{OZ}$ in Eq. (21), with

\[
Q^{OZ}(T) = -N_0^{-1} T \sum_n \int \frac{d^2k}{(2\pi)^2} 2\epsilon_k \sin^2 \frac{2\varphi}{m} Z_{nk}(v_x) \simeq -N_0^{-1} T \sum_n \int \frac{d^2k}{(2\pi)^2} \frac{2\epsilon_k \sin^2 \frac{2\varphi}{m}}{W_{nk}^2}.
\]

$F^{OZ}$ is very different from $F_4$. Discussions about comparative importance of these two terms can be found in Sec. III.C. The integrand of $Q^{OZ}$ includes odd power in $\epsilon_k$, implying nonzero contribution only in a particle-hole asymmetric system. In order to proceed further, we adopt the following density of states near the Fermi surface in the normal state,

\[
N(\epsilon) \simeq N_0 + \epsilon \frac{dN(\epsilon)}{d\epsilon} \bigg|_{\epsilon=0} \simeq N_0(1 + \alpha \epsilon),
\]

to take into account both the particle-hole symmetric and asymmetric contributions. Here $\alpha$ is of order unity, and $\epsilon$ is typically order of $T$ or $\Delta_0$, implying that the second term on the right-hand side of Eq. (A41) may be negligible when the contribution from the particle-hole symmetric mode does not vanish.

The quadratic free energy terms in Eq. (21) are extracted from $F^{(loc)}$, where

\[
\eta'_x(T, v_x) = \eta''_x(T, v_x) + 2d^2 h_{2dxy}(d),
\]

\[
\eta''_x(T, v_x) = -h_{2dxy}(d) + \frac{\mu_{mk} v_{sx}(r)}{T^2} h_{4dxy}(d),
\]

with $h_{idxy}$ defined in Eqs. (A19) and (A20).

**APPENDIX B: SPONTANEOUS MAGNETIZATION IN THE $D_{X^2-Y^2} + ID_{XY}$ STATE**

To proceed with a general derivation of the spontaneous magnetization, we rewrite the free energy in Eq. (11) as

\[
F = F_0 - T \sum_n \text{Tr} \ln \left( -\mathcal{G}^{-1} \right) - T \sum_n \text{Tr} \ln \left[ 1 - \mathcal{G} \left( \begin{array}{cc} \tilde{V}_1 & 0 \\ 0 & \tilde{V}_2 \end{array} \right) \right],
\]

where $\mathcal{G} = -\left( \mathcal{M}_0 + \tilde{D} \right)^{-1}$ with $\tilde{D}$ the field-induced off-diagonal pairing order parameter matrix. The orbital Zeeman term coming from the linear-$V$ term is

\[
F^{OZ} \simeq -T \sum_n \int \frac{d^2k d^2k_1}{(2\pi)^4} \int d^2r d^2r_1 e^{i(k_1 - k) \cdot (r_1 - r)} [G_{11k,k_1}(r) + G_{22k,k_1}(r)] \left\{ [r_1 - r] \cdot \nabla_r \right\} [v_s(r) \cdot k_F]
\]

\[
\simeq T \sum_n \int \frac{d^2k}{(2\pi)^2} \int d^2r \lim_{k_1 \to k} \{i
abla_k [G_{11k,k_1}(r) + G_{22k,k_1}(r)] \cdot \nabla_r \} [v_s(r) \cdot k_F]
\]

\[
\simeq -\int d^2r \frac{e}{2mc} T \sum_n \int \frac{d^2k}{(2\pi)^2} \lim_{k_1 \to k} \{i(k \times \nabla_k) [G_{11k,k_1}(r) + G_{22k,k_1}(r)] \} \cdot B(r).
\]

(B2)
Comparing Eq. (B2) with Eq. (B3) we see that the magnetization is just the expectation value of the magnetic moment operator \((e/2mc)(ik \times \nabla_k)\),

\[
M(r) = \frac{e}{2mc} \int \frac{d^2k}{(2\pi)^2} \text{Tr}(d^0_k (ik \times \nabla_k) \hat{d}_k) = \frac{e}{2mc} T \sum_n \int \frac{d^2k}{(2\pi)^2} \lim_{k_i \to k} (ik \times \nabla_k) \text{Tr} \hat{G}_{k,k_1}(r),
\]

with \(\hat{d}_k = (c_{k\uparrow}, c_{k\downarrow}^\dagger)\). It is easy to check that \(M = 0\) for a pure \(d_{x^2-y^2}\) wave superconductor, and also for a \(d_{x^2-y^2} + is\)-wave one in the order of linear \(s\). For a \(d_{x^2-y^2} + id_{xy}\)-wave superconductor with \(d_{xy}\) component perturbatively small compared to the \(d_{x^2-y^2}\) component, we use the Dyson equation,

\[
\hat{G}_{k,k_1} \approx \hat{g}_k \delta(k - k_1) + \hat{g}_k \left( \begin{array}{cc} 0 & D_{xy}(r) \Phi_{xy}^{k \rightarrow k_1} + \frac{g_{xy}(k \times \nabla_k) g_{y\downarrow}}{mc} \right) \hat{g}_{k_1},
\]

where \(\hat{g}_k\) is defined in Eq. (13). Inserting Eq. (B4) into Eq. (B3) yields

\[
M(r) = \frac{e}{2mc} T \sum_n \int \frac{d^2k}{(2\pi)^2} \Phi_{xy} \left[ D_{xy}(r) g_{y\downarrow} + D_{xy}^\dagger(r) g_{y\uparrow} \right] (k \times \nabla_k) g_{y\downarrow}
\]

\[
= -\frac{e}{2mc} T \sum_n \int \frac{d^2k}{(2\pi)^2} D_{xy}''(r) \frac{4\Delta_0 \epsilon_k \sin^2 2\phi}{W_{nk}},
\]

which, up to the leading order of \(D_{xy}\) and \(\nu_s\), is consistent with the second term on the right-hand side of Eq. (A13). A particle-hole asymmetric system is required to obtain a nonvanishing \(M(r)\). To understand the physics, we note \(M(r)\) in Eq. (B5) is in fact

\[
M(r) = \frac{e}{2mc} \sum_{n,k} 2D_{xy}'' \left\{ \left( u_k^2 - v_k^2 \right) \frac{1}{i\omega_n + E_k} - \left( u_k^2 - v_k^2 \right) \frac{1}{i\omega_n - E_k} \right\} (k \times \nabla_k) g_{y\downarrow},
\]

with \(u_k^2 = (1 + \epsilon_k/E_k)/2\) and \(v_k^2 = (1 - \epsilon_k/E_k)/2\) measuring the particle and hole populations, respectively. Eq. (B7) can be interpreted as a magnetic moment contributed from number currents of both particles and holes flowing in the opposite directions, which cancel in a particle-hole symmetric system.

The spontaneous magnetization \(M(r)\) as a full expression of \(D_{xy}''\) can also be obtained from Eq. (B3). After some straightforward algebra we get

\[
M(r) \approx \frac{eT}{2mc} \sum_n \int \frac{d^2k}{(2\pi)^2} \frac{4\Delta_0(T) D_{xy}''(r) \epsilon_k \Phi_{xy}^2}{W_{nk} \{ W_{nk} + [D_{xy}''(r)]^2 \Phi_{xy}^2 \}}.
\]

\(M(r)\) in Eq. (B8) is important for the direct comparison with Laughlin’s free energy as discussed in Sec. V.

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We use a symmetric gauge transformation criticized by Franz and Tesanovic [24] which we believe to be justified in the current application. This follows because the free energy involves only the energy eigenvalues, which do not depend on the choice of gauge.

We note that the second term $\sim (\overline{D}_i(r)^2 - \overline{D}_i^*(r)^2)$ is allowed by symmetry, but was neglected in the GL analysis of the collective mode in the $d_{x^2-y^2} + id_{xy}$ state of Ref. [21], and may in principle substantially change the collective mode frequency calculated therein. Within the weak coupling analysis presented here, the prefactor is of order $c_{xy}$, which is small for generic values of the coupling constants $V_d, V_{xy}$. Therefore, the neglect of this term in Ref. [21] is generally justified within weak-coupling theory, except in cases of near degeneracy of the $d_{x^2-y^2}, d_{xy}$ coupling constants.

S. Sachdev, private communication.

In a vortex crystal, however, the quasiparticles take the form of Bloch waves in the presence of the periodic supercurrents; thus the perturbation theory presented in the present paper breaks down. Even in this case, however, the semiclassical approach we have adopted may yield sensible results. [32]

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