Generalized Models and Local Invariants of Kohn–Nirenberg Domains

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Vienna, Preprint ESI 1866 (2006) November 15, 2006

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at
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ABSTRACT. This paper studies the Kohn-Nirenberg phenomenon - existence of weakly pseudoconvex but locally nonconvexifiable hypersurfaces. We give a characterization of such hypersurfaces in terms of a generalized model, which captures behaviour of the hypersurface also in the complex nontangential direction. As an application we obtain a new class of nonconvexifiable pseudoconvex hypersurfaces with convex models.

1. Introduction

Although local convexity of the boundary of a weakly pseudoconvex domain in \( \mathbb{C}^n \) is not a biholomorphic invariant, it is often used as an assumption which provides useful tools to study biholomorphically invariant objects (see e.g. [4], [5], [16]). The natural invariant condition for results of this kind is then just that the domain be locally biholomorphic to a convex domain. This property is usually called local convexifiability, and one would like to find verifiable conditions for it to hold.

The history of this problem starts in 1973 with the work of J. J. Kohn and L. Nirenberg ([10]). It was then a well known fact that both strongly pseudoconvex and Levi flat hypersurfaces are locally convexifiable. In the intermediate case, when the Levi form vanishes at the point, but not identically, the situation turned out to be much more interesting. The example of a nonconvexifiable pseudoconvex domain in \( \mathbb{C}^2 \), discovered by Kohn and Nirenberg, opened the problem how to characterize locally convexifiable domains.

This question was considered in [11] and [13], with results giving a satisfactory answer for the class of model domains. They are based on exact conditions computed for domains of Kohn-Nirenberg type (see also Proposition C below). These results do not take into account behavior of the defining function in the complex non-tangential direction. Moreover, convexifiability of the model domain is necessary, but not sufficient for convexifiability of the domain itself. Hence the question of other possible obstruction to convexifiability remained open.

Our aim in this paper is to extend the results of [11], [13], by means of introducing a generalized model domain which almost always carries the information about local

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The author was supported by a grant of the GA ČR no. 201/05/2117.
convexifiability of the domain itself. In this way, the problem which apriori requires to consider the whole infinite dimensional group of germs of local biholomorphisms is reduced to a simple finite dimensional problem, which is often solved by an application of the explicit results of [11].

As an application, we find a new class of examples of nonconvexifiable domains. While all previous examples are on the level of model domains, we describe nonconvexifiable pseudoconvex domains whose model domains are convexifiable.

Major part of the work on this paper was done while the author was visiting the ESI program Complex Analysis, Operator Theory, and Applications to Mathematical Physics. He would like to thank the organizers, Friedrich Haslinger, Emil Straube and Hans Upmeier for the invitation, and for the support received from ESI.

2. Generalized model domains

We will consider a pseudoconvex domain \( D \subseteq \mathbb{C}^2 \) with real analytic boundary \( M \), and a point \( p \in M \) of finite type \( k \), where \( k > 2 \). Recall that the type of a point measures the maximal order of contact of \( M \) with complex curves passing through \( p \). Strongly pseudoconvex points correspond to points of type two. It follows from pseudoconvexity that \( k \) is an even integer. We will use local holomorphic coordinates \((z, w), z = x + iy, w = u + iv\), centered at \( p \), such that the direction of the positive \( v \)-axis is the inner normal direction to \( D \) at \( p \) and write the defining equation for \( M \) in the form

\[
v = F(z, \bar{z}, u).
\]

It follows directly from the definition of finite type that we can choose the coordinates so that the above equation takes form

\[
v = P_1(z, \bar{z}) + o(|z|^k, u),
\]

where \( P_1 \) is a real valued homogeneous polynomial of degree \( k \)

\[
P_1(z, \bar{z}) = a_0|z|^k + \sum_{j=0,2,\ldots,k} |z|^{k-j} \text{Re}(a_j z^j)
\]

for some \( a_j \in \mathbb{C} \) and \( a_0 \in \mathbb{R}^+ \). The domain

\[
M_D = \{(z, w) \in \mathbb{C}^2 \mid v > P_1(z, \bar{z})\}
\]

is called a model domain to \( D \) at \( p \). Its boundary is a model hypersurface to \( M \) at \( p \). It is determined uniquely up to a linear change of variables and addition of a harmonic term \( \text{Re} \alpha z^k \) (see e.g. [7] or [13]).

**Definition 1.** \( M \) is called locally convexifiable at \( p \in M \) if there exist local holomorphic coordinates in a full neighbourhood of \( p \) such that \( M \) is convex with respect to the underlying linear space.

The polynomial \( P_1 \) in (2.1) captures local behavior of \( M \) in the complex tangential direction. In order to study convexifiability of \( M \) near \( p \) we wish to take into account also behavior in complex nontangential directions, involving the variable \( u \). A natural tool for this is provided by weighted coordinates. We will assign weights to the coordinates \( z, w \) and \( u \). The following construction is similar to Catlin’s definition of multitype in \( \mathbb{C}^n \) (see [1]), where only complex tangential variables are considered.
Definition 2. A weight vector is a pair of rational numbers \( \lambda = (\lambda_1, \lambda_2) \) such that \( \lambda_1 = \frac{1}{n} \) for some \( n \in \mathbb{N} \) and there exist integers \( k_1, k_2 \), with \( k_2 > 0 \) such that \( k_1 \lambda_1 + k_2 \lambda_2 = 1 \).

The weight of monomials \( c_{ijk} z^j \bar{z}^k \) and \( d_{ij} z^j w^k \) is defined to be \((i + j) \lambda_1 + k \lambda_2\) respectively. A real valued polynomial \( P(z, \bar{z}, u) \) is \( \lambda \)-homogeneous of weight \( \gamma \) if it is a sum of monomials of weight \( \gamma \), and similarly for a holomorphic polynomial \( P(z, w) \). The weight of a function \( h(z, w) \) is the lowest of the weights of the terms in its Taylor expansion at the origin.

Given a domain and a point on its boundary, we first assign weight to the variable \( z \), equal to the reciprocal of the type of the point, i.e., \( \lambda_1 = \frac{1}{k} \). The weight of \( w \) and \( u \) will be denoted by \( \mu \), and is determined as follows. We will call a weight vector \( \lambda = (\frac{1}{k}, \lambda_2) \) allowable, if there exist local holomorphic coordinates such that the defining equation can be written in the form

\[
v = P(z, \bar{z}, u) + o_{wt}(1),
\]

where

\[
P(z, \bar{z}, u) = \sum_{\frac{m}{k} + \lambda_2 l = 1}^{\infty} \sum_{j=0}^{m} a_{mjl} z^j \bar{z}^{m-j} u^l
\]

is a \( \lambda \)-homogeneous polynomial of weight one, and \( o_{wt}(1) \) denotes terms of weight bigger than one with respect to the weight \((\frac{1}{k}, \lambda_2)\).

Clearly, for any real \( \delta > 0 \) there are only finitely many rational numbers \( \lambda_2 \) bigger than \( \delta \) such that \((\frac{1}{k}, \lambda_2)\) is a weight. When the set of allowable values for \( \lambda_2 \) is finite, we set \( \mu \) to be the smallest of them. If the set is not bounded away from zero, we set \( \mu = 0 \).

When \( \mu > 0 \), we fix local holomorphic coordinates \((z, w)\) which correspond to the weight \( \lambda = (\frac{1}{k}, \mu) \). Hence the defining equation has form

\[
(2.4) \quad v = \sum_{\frac{m}{k} + \mu l = 1}^{\infty} \sum_{j=0}^{m} a_{mjl} z^j \bar{z}^{m-j} u^l + o_{wt}(1),
\]

where, as before, the leading term will be denoted by \( P(z, \bar{z}, u) \).

The hypersurface given by

\[
v = P(z, \bar{z}, u)
\]

will be called a generalized model to \( M \) at \( p \).

If \( \mu = 0 \), a generalized model is defined to be

\[
v = P_1(z, \bar{z}).
\]

Hence in this case it coincides with the standard model hypersurface.

3. Characterizations of convexifiability

Let \( \mathbb{H}_{k, \mu} \) denote the set of all real valued \( \lambda \)-homogeneous polynomials in \( z, u \) of weight one which are harmonic in the \( z \) variable. Clearly, \( h(z, u) \in \mathbb{H}_{k, \mu} \) if and only if

\[
h(z, u) = \text{Re}(\sum_{\frac{m}{k} + \mu l = 1} a_m z^m u^l)
\]
for some \( \alpha_m \in \mathbb{C} \).

For \((z, u) \in \mathbb{C} \times \mathbb{R}\) and \(\zeta \in \mathbb{C}\) we will use the following notation for the value of the real Hessian restricted to the \(z\)-direction:

\[
D_z^2 P(z, u; \zeta) = \sum_{i,j=1}^{2} \frac{\partial^2 P}{\partial x_i \partial x_j}(z, u)\zeta_i \zeta_j,
\]

where \(z = x_1 + ix_2\) and \(\zeta = \xi_1 + i\xi_2\). Since \(P\) is weighted homogeneous, positivity of \(D_z^2 P\) is determined by its restriction to the set \(S_2 \times S_1\), where \(S_2 = \{(z, u) \in \mathbb{C} \times \mathbb{R} : |z|^2 + u^2 = 1\}\) and \(S_1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}\).

Our aim is to prove the following pair of conditions for convexifiability of \(M\).

**Proposition A.** If there exists \(h \in \mathbb{H}_{k,\mu}\) such that \(D_z^2 (P + h) > 0\) on \(S_2 \times S_1\), then \(M\) is locally convexifiable.

**Proof.** Let \(h \in \mathbb{H}_{k,\mu}\) be such a function and let \(\tilde{P} = P + h\). Considering first the condition on the \(u\)-axis, it follows from the assumption that \(\frac{\mu}{1 - \frac{2}{k}}\) is an even integer and that \(P\) contains a term \(|z|^2 u_x^{(1 - \frac{2}{k})}\) with a positive coefficient. By homogeneity, we have

\[
D_z^2 \tilde{P}(z, u; \zeta) \geq \epsilon \left(|z|^{k-2} + |u|^{\frac{1}{\mu}(1 - \frac{2}{k})}\right)|\zeta|^2
\]

for a sufficiently small \(\epsilon > 0\). By the change of coordinates

\[
w^* = w + iw^2
\]

we get

\[F^*(z, \bar{z}, u - 2uF(z, \bar{z}, u)) = F(z, \bar{z}, u) + u^2 - (F(z, \bar{z}, u))^2.\]

Comparing terms of weight \(\leq 1\) on both sides we obtain

\[F^*(z^*, \bar{z}^*, u^*) = \tilde{P}(z, \bar{z}, u) + u^2 + o_{wt}(1).\]

We will prove that \(F^*\) is a convex function in a neighbourhood of the origin.

Denote \(P^*(z, \bar{z}, u) = \tilde{P}(z, \bar{z}, u) + u^2\) and consider its \(3 \times 3\) Hessian matrix with respect to the variables \(x, y, u\). The \(2 \times 2\) submatrix of the Hessian formed by the derivatives with respect to \(x\) and \(y\) satisfies (3.1), hence is positive semidefinite. It remains to prove that the determinant of the Hessian is also nonnegative in a neighbourhood of the origin. We calculate the weights of the terms entering into the determinant. On the one hand, \(wt(P^*_{uu} P^*_{xy} P^*_{yy}) = wt(P^*_{uu} P^*_{xy} P^*_{yy}) = 2 - \frac{4}{k}\). On the other hand, \(wt((P^*_{uu})^2 P^*_{yy}) = 3 - 2\mu - \frac{4}{k}\) and the same for all the remaining terms. Since \(k > 2\) and \(\mu < \frac{1}{2}\), we have \(3 - 2\mu - \frac{4}{k} > 2 - \frac{4}{k}\), and the results follows from (3.1).

Proposition A is complemented by the following

**Proposition B.** If \(M\) is locally convexifiable, then there exists \(h \in \mathbb{H}_{k,\mu}\) such that \(D_z^2 (P + h) \geq 0\) on \(S_2 \times S_1\).

**Proof.** We will assume that \(D_z^2 (P + h)\) is not nonnegative on \(S_2 \times S_1\) for any \(h \in \mathbb{H}_{k,\mu}\), and show that \(M\) cannot be convex in any other coordinates. If \(\mu = 0\), the claim follows from Lemma 3 in [13]. Let \(\mu > 0\). Recall that we are considering local holomorphic
coordinates in which the defining equation has form (2.4). Consider a holomorphic transformation

\begin{align*}
    z^* &= z + g(z, w) \\
    w^* &= w + f(z, w).
\end{align*}

(3.2)

We may restrict ourselves to transformations which preserves our description of \( M \), i.e., \( v^* = 0 \) is tangent to \( M \) at \( p = 0 \) and the positive \( v^* \)-axis points inside. Also, without any loss of generality we may assume that the linear part of the transformation is normalized. More precisely, we require that

\[
    f = 0, \quad g = 0, \quad f_z = 0, \quad g_z = 0.
\]

and

\[
    g_z = 0, \quad f_w = 0 \quad \text{at} \quad z = w = 0.
\]

We will call \( f \) subhomogeneous if \( wt(f) \leq wt(w) \) and superhomogeneous if \( wt(f) \geq wt(w) \). Similarly, \( g \) is subhomogeneous if \( wt(g) \leq wt(z) \) and superhomogeneous if \( wt(g) \geq wt(z) \). If the preceding inequalities are strict, we speak of strict subhomo-
geneity and superhogeneity.

Let \( F^* \) be the function describing \( M \) in new coordinates. Substituting (3.2) into \( v^* = F^*(z^*, \bar{z}^*, u^*) \), we get a formula relating coefficients of \( F^* \) and \( F, f, g \),

\[
    F^*(z + g, \bar{z} + \bar{g}, u + \text{Re } f) = F(z, \bar{z}, u) + \text{Im } f(z, u + iF(z, \bar{z}, u)),
\]

(3.3)

where \( g \) and \( \text{Re } f \) are also evaluated at \((z, u + iF(z, \bar{z}, u))\), i.e. on \( M \). We write the transformation in the form

\[
    z^* = z + \sum_{j=1}^{\infty} \alpha_j w^j + \sum_{i=1, j=0}^{\infty} \beta_{ij} \bar{z}^i w^j
\]

and

\[
    w^* = w + \sum_{j=1}^{\infty} \epsilon_j w^j + \sum_{i=1, j=0}^{\infty} \gamma_{ij} \bar{z}^i w^j.
\]

Consider another transformation of the form

\[
    z^{**} = z^* + \sum_{j=1}^{\frac{1}{m!}} \delta_j (w^*)^j
\]

(3.4)

\[
    w^{**} = w^*.
\]

We will show that for suitable values of \( \delta_j \) the defining equation in coordinates \((z^{**}, w^{**})\) has either form (2.4), or the leading weighted homogeneous term is harmonic.

We denote by

\[
    z^{**} = z + g^{**}(z, w) \\
    w^{**} = w + f^{**}(z, w)
\]
the composition of the two transformations, which is given by

\[
\begin{align*}
    z^{**} &= z + \sum_{j=1}^{\infty} \alpha_j w^j + \sum_{i=1,j=0}^{\infty} \beta_{i,j} z^i w^j + \sum_{m=1}^{\infty} \delta_m (w + \sum_{j=1}^{\infty} \epsilon_j w^j + \sum_{i,j=0}^{\infty} \gamma_{i,j} z^i w^j)^m \\
    w^{**} &= w + \sum_{j=1}^{\infty} \epsilon_j w^j + \sum_{i,j=0}^{\infty} \gamma_{i,j} z^i w^j .
\end{align*}
\]

Now we determine inductively the coefficients \( \delta_j \) in such a way that \( g^{**} \) is strictly superhomogeneous. We obtain \( \delta_1 = -\alpha_1, \delta_2 = -\alpha_2 - \delta_1 \epsilon_2 \), and so on up to index \( \left[ \frac{1}{k\mu} \right] \).

Now we use (3.3) and consider terms of weight less or equal to one. There are two possibilities. Either \( f^{**} \) contains terms of weight strictly less than one, in which case the leading weighted homogeneous term of \( F^{**} \) is harmonic and the domain cannot be locally convex. Or all terms in \( f^{**} \) are of weight at least one, then \( F^{**} \) has form (2.4), with the leading term equal to \( P + h \) for some \( h \in \mathbb{H}_{k,\mu} \). Hence, by assumption, \( F^{**} \) is not convex. Since the transformation (3.4) only shifts complex lines parallel to the \( z \)-axis, it does not influence convexity in the \( z \) direction. It follows that the defining function \( F^* \) in coordinates \((z^*, w^*)\) is not convex either, which proves the statement. \( \Box \)

4. Convexifiability and Kohn-Nirenberg invariants

For applications of Propositions A and B we will need the explicit results obtained in [11] for model domains. In order to state those results, we recall the definition of a Kohn-Nirenberg domain of type \( k, l \). For two even integers \( k, l \) and a positive real number \( a \) we denote

\[
M_{a}^{k,l} = \{(z, w) \in \mathbb{C}^2 \mid \text{Im } w > P_{a}^{k,l}(z)\},
\]

where

\[
P_{a}^{k,l}(z, \bar{z}) = |z|^k + a|z|^{k-l} \text{Re } z^l,
\]

and call \( M_{a}^{k,l} \) a Kohn-Nirenberg domain of type \( k, l \). The Kohn-Nirenberg example is a domain of this type, where \( k = 8, l = 6 \) and \( a = \frac{15}{7} \).

**Proposition C.** \( M_{a}^{k,l} \) is convex if and only if

\[
a \leq \gamma_{lk},
\]

where

\[
\gamma_{lk} = \frac{k}{l^2 - k}
\]

if \( l^2 \geq 3k - 2 \) and

\[
\gamma_{lk} = \sqrt{\frac{(4k - l^2 - 4)k^2}{(4k - 4)(k^2 - l^2)}}
\]
if $l^2 \leq 3k - 2$. Moreover, if $l$ is not a divisor of $k$, then this condition is equivalent to convexifiability of $M^{k,l}_a$.

Proof of Proposition C is contained in [11]. Now we consider the general case, when the hypersurface is given by
\begin{equation}
(4.1) \quad v = P_1(z, \bar{z}) + O(|z|^{k+1}, u),
\end{equation}
where $P_1(z, \bar{z})$ is a polynomial of the form (2.2). For such domains we now define the Kohn-Nirenberg invariants.

**Definition 3.** Let the model hypersurface to $M$ at $p$ be given by (2.3), and let $l$ be an even integer, $0 < l < k$. We will call the real number $\kappa_l = \frac{|a_l|}{a_0}$ the Kohn-Nirenberg invariant of order $l$.

The first appearance of such an invariant is the seminal paper [10]. The originally mysterious constant $\frac{15}{7}$, which appears in the Kohn-Nirenberg example is an invariant of order six.

We can view these invariants as 3-rd level local biholomorphic invariants, the first level being the signature of the Levi form, with values in $\{-1, 0, 1\}$, the second level being the type of $p$, taking integer values.

We now show that $\kappa_l$ is indeed an absolute biholomorphic invariant. The argument is based on the following fact. The polynomial $P_1$ in (2.1) is determined uniquely up to biholomorphic transformations
\begin{equation}
(4.2) \quad z^* = \alpha z, \quad w^* = w + \beta z^k,
\end{equation}
where $\alpha, \beta \in \mathbb{C}$. This was proved in Lemma 2 in [11]. For the reader’s convenience, we give the argument here. Consider holomorphic transformations of the form (3.2) and let $F^*$ be the function describing $M$ in new coordinates. Consider again the change of variables formula:
\begin{align*}
F^*(z + g(z, u + iF(z, \bar{z}, v)), z + g(z, u + iF(z, \bar{z}, v)), v + Im \ f(z, u + iF(z, \bar{z}, v))) = \\
= F(z, \bar{z}, u) + Re \ f(z, u + Im \ f(z, u + iF(z, \bar{z}, v))).
\end{align*}
We have to show that if a transformation (3.2) preserves the form given by (2.3), (2.1), then the new leading polynomial $P^*$ of $F^*$ can be obtained from $P$ by a transformation (4.2). We assign again weights to the variables $z, w, u$. Weight 1 is given to $z, \bar{z}$ and weight $k$ to $u$ and $w$. $F$ starts with terms of weight $k$, so the terms of weight $\nu$ for $\nu \leq k$ in $Im \ f(z, u + iF(z, \bar{z}, u))$ come from corresponding terms of weight $\nu$ in $f(z, w)$. It follows that $f$ does not contain $z^2, \ldots, z^{k-1}$, for all other entries in the change of variables formula are of weight $\geq k$. So $Re \ f$ has also weight $\geq k$, and the change of variables formula becomes
\begin{align*}
P^*(z + g, \bar{z} + \bar{g}) + \cdots = P(z, \bar{z}) + Im \ f(z, u + iF(z, \bar{z}, u)) + \cdots
\end{align*}
where dots stand for terms of weight $> k$. Now we compare terms of weight $k$ on both sides, which shows that $P^*$ depends only on terms of weight 1 in $g$ and weight $k$ in $f$. In other words, it is obtained from $P$ by a transformation (4.2). Now we finish the argument by observing that the numbers $\kappa_l$ are preserved by this transformation.

In the general case we have a sufficient and a necessary condition for local convexifiability. The sufficient condition was proved in [12].
Theorem 1. Let the model at \( p \in M \) be given by (2.3). If
\[
\sum_{j=2,4,\ldots,k-2} \gamma_{jk}^{-1} \kappa_j < 1,
\]
then \( M \) is locally convexifiable at \( p \).

The necessary condition was proved in [11]

Theorem 2. Let the model at \( p \in M \) be given by (2.3). If \( M \) is locally convexifiable at \( p \), then
(i) \( \kappa_j \leq \gamma_{jk} \) for all \( j > \frac{k}{2} \)
and
(ii) \( \kappa_j \leq 2\gamma_{jk} \) for all \( j \leq \frac{k}{2} \).

We note that the Kohn-Nirenberg phenomenon does not occur for model domains of type four. Here \( P \) is of the form
\[
P(z, \bar{z}) = |z|^4 + a|z|^2 \text{Re} z^2,
\]
where \( a \geq 0 \). \( M_D \) is pseudoconvex if and only if \( a \leq \frac{4}{3} \). It is easily verified that the case \( a = \frac{4}{3} \) corresponds to the tube domain \( v = x^4 \), which is convex. It follows immediately that for \( a < \frac{4}{3} \), \( M_D \) are convexifiable by adding a suitable harmonic fourth order term.

On the other hand, it is not known if all type four domains are locally convexifiable. Partial results were obtained in [13] and [14]. The first one shows that if \( a < \frac{4}{3} \) then the domain is convexifiable.

If \( a = \frac{4}{3} \), then the model is a tube. Convexifiability was proved under the additional assumption that the domain is rigid, i.e. in some coordinates it can be written as
\[
v = x^4 + \sum_{i+j \geq 5} a_{ij} z^i \bar{z}^j,
\]
where \( \bar{a}_{ij} = a_{ji} \).

5. Examples of nonconvexifiable domains with convex models

Using Proposition B and C we can easily give examples of pseudoconvex domains which are not convexifiable, although their model domains are.

For \( a > 0 \) we will denote
\[
P_{6,4}^a(z, \bar{z}) = |z|^6 + a|z|^2 \text{Re} z^4
\]
and consider the hypersurface \( M_a \) defined by
\[
v = |z|^8 + P_{6,4}^a(z, \bar{z}) u^2 + |z|^2 u^8.
\]

Using (3.3) we verify that in this case \( \mu = \frac{1}{k} = \frac{1}{8} \). The generalized model is given by \( v = P(z, \bar{z}, u) \), where
\[
P(z, \bar{z}, u) = |z|^8 + (|z|^6 + a|z|^2 \text{Re} z^4) u^2
\]
Now we calculate the Levi form. To obtain the pseudoconvexity condition in terms of \( F \), we take \( r(z, w) = F - v \) as the defining function and use the coordinate expression
of the Levi form

\[ L(r) = |r_w|^2 r_{z\overline{z}} + |r_z|^2 r_{w\overline{w}} - 2Re(r_{z\overline{w}}r_{z\overline{w}}). \]

We get

\[ L(F - v) = F_{z\overline{z}}(1 + |F_u|^2) + F_{uu}|F_z|^2 + 2Re F_z u(F_u + i)F_z. \]

Hence we have

\[ L(F - v) = \frac{1}{4} \Delta F + o(1 - \frac{2}{k}). \]

It follows that \( M_a \) is pseudoconvex if \( \Delta P_6^{6,4} > 0 \), i.e., if \( a < \frac{9}{5} \). Since \( D^2 \) of \( P_6^{6,4}u^2 \) dominates \( D^2 \) of \( P \) along the \( u \) axis, we obtain from Proposition B and Proposition C that \( M_a \) is not convexifiable if \( a > \frac{3}{5} \). Hence for \( \frac{3}{5} < a < \frac{9}{5} \) the hypersurface \( M_a \) is nonconvexifiable. On the other hand, the model at zero is the convex domain \( v = |z|^8 \), and all other points in a neighbourhood of zero are strongly pseudoconvex, hence their models are also convex.

**References**

[1] Catlin, D., *Boundary invariants of pseudoconvex domains*, Ann. Math. 120 (1984), 529–586.
[2] D’Angelo, J., *Orders of contact, real hypersurfaces and applications*, Ann. Math. 115 (1982), 615–637.
[3] S.S.Chern and J.Moser: *Real hypersurfaces in complex manifolds*, Acta Math. 133 (1974), p. 219-271
[4] Diederich, K. and Fornaess, J. E., *Support functions for convex domains of finite type*, Math. Z. 230 (1999), 145–164.
[5] Diederich, K. and McNeal, J. D., *Pointwise nonisotropic support functions on convex domains*, Progress Math. 188 (2000), 184–192.
[6] P.Ebenfelt : *New invariant tensors in CR structures and a normal form for real hypersurfaces at a generic Levi degeneracy*, J.Diff.Geometry 50 (1998), p. 207-247
[7] Fornaess, J. E. and Stensones, B., *Lectures on Counterexamples in Several Complex Variables*, Princeton Univ. Press 1987.
[8] H.Jacobowitz : *An introduction to CR structures*, Mathematical Surveys and Monographs 32, AMS, 1990
[9] Kohn, J. J., *Boundary behaviour of \( \partial \) on weakly pseudoconvex manifolds of dimension two*, J. Differential Geom. 6 (1972), 523–542.
[10] Kohn, J. J. and Nirenberg, L., *A pseudoconvex domain not admitting a holomorphic support function*, Math. Ann. (1973), 265–268.
[11] Kolář, M., *Convexifiability and supporting functions in \( \mathbb{C}^2 \)*, Math. Res. Lett. 2 (1995), 505–513.
[12] Kolář, M., *On local convexifiability of type four domains in \( \mathbb{C}^2 \)*, Differential Geometry and Applications, Proceeding of Satellite conference of ICM in Berlin (1999), 361–371.
[13] Kolář, M., *A necessary conditions for local convexifiability of pseudoconvex domains in \( \mathbb{C}^2 \)*, Rend. Circ. Mat. Palermo (2001).
[14] Kolář, M., *Local convexifiability of some rigid domains*, Rend. Circ. Mat. Palermo 75 (2005), 251–257.
[15] M.Kolář : *Normal forms for hypersurfaces of finite type in \( \mathbb{C}^2 \)*, Math. Res. Lett. 12 (2005) p. 897-910
[16] McNeal, J. D., *Estimates on the Bergman Kernels on Convex Domains*, Adv. Math. **109** (1994), 108–139.

[17] McNeal, J. D., *Uniform subelliptic estimates on scaled convex domains of finite type*, Proc. Amer. Math. Soc. **130** (2002), 39–47 (electronic).

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