Hierarchical deposition and deterministic scale-free networks: a visibility algorithm approach

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(Dated: April 29, 2022)

The deterministic growth of an interface formed by the synchronous hierarchical deposition of particles of unequal size, where the number of particles in a generation \( n \) obeys a hyperbolic scaling law, is studied. In the framework of a dynamical network generated by a horizontal visibility algorithm (HVA), the discrete deposition sites of the interface constitute network nodes, while the edges are determined by means of a mutual visibility criterion. We show analytically that the resulting sparse network is scale-free with a dominant degree exponent \( \gamma_\nu = \ln 3/\ln 2 \) and a transient exponent \( \gamma_\nu = 1 \). Through an exact calculation of the average clustering coefficient we show that the network inherits the modular hierarchical nature of the underlying deposition process. It is shown numerically that the eigenvalue spectrum of the adjacency matrix possesses a fractal structure, the exact shape of which remains undetermined.

The deposition of particles onto a substrate is a subject of considerable practical importance, and it has found applications in various fields. A plethora of different deposition processes exists within the area of nonequilibrium statistical physics, which often reduce to variations of either the random deposition process, or the ballistic deposition process. These discrete surface growth models are understood to be the microscopic discrete versions of the continuous models of respectively Edwards and Wilkinson [1], and Kardar, Parisi and Zhang [2, 3]. Through experimental and theoretical studies performed over the past decades, it is known that such surfaces are rough and generally display fractal behavior. One common assumption that is often made in the discrete variations is that the particles are geometrically identical in every aspect, and that the deposition steps occur sequentially, i.e. one-by-one. This assumption is obviously not generally true for real-world applications. Here, we will consider the deterministic deposition model (HDM) that was introduced in Ref. [4], which assumes that the particles are deposited according to their size in a synchronous fashion, where particles of the same size are all deposited at once in “generations”. The HDM has been extended to include different scaling of particle size and deposition probability [5–7], and has been used to model the deposition of particles with spins [8]. Furthermore, it has found applications in the description of bacterial biofilms [9, 10] and has been used to describe coastline formation and fractal percolation [11].

In this Letter, we map the topology of the surface formed by a deterministic version of the HDM into a complex network by means of a visibility algorithm (VA). The essence of the VA is to create a complex network from a set of data by assigning a node to each datum and assigning edges based on the mutual visibility between two data, i.e., if a line of visibility is not “intersected” by any intermediate data. This algorithm was originally developed [12] to uncover hidden structures in time series data and has found applications in astrophysics [13], medicine [14], fluid mechanics [15] and several other fields [16]. Generally, the VA comes in two types: the Natural Visibility Algorithm (NVA) and the Horizontal Visibility Algorithm (HVA). The former considers the visibility line as a direct connection between two data and hence it is inclined with respect to the chosen common baseline. The latter is based on whether or not two data can have mutual horizontal visibility (illustrated in the middle panel of Fig. 1) and has no inclination with respect to the baseline. It is not difficult to see that the graph generated by the HVA is always a subgraph of the NVA, as horizontal visibility by definition implies natural visibility between two data. While one might lose some information by choosing the HVA, it often allows for analytical tractability. An extensive comparison between the two choices is performed in Ref. [12].

Let us start by briefly introducing the deterministic HDM. We consider the deposition of rigid particles on a one-dimensional substrate, i.e., on a line, where we assume that the deposited particles are squares with different sizes. If the deposition occurs in a viscous medium such as air or water, the larger particles are deposited first, followed by the smaller particles. Additionally, we assume that the number of particles \( N(s) \) of size \( s \) follows a hyperbolic distribution

\[
N(s) = \lambda^{-1} N(s/\lambda),
\]

where \( \lambda \in \mathbb{R}^+ \) and \( \lambda > 1 \). For simplicity, let us now only consider \( \lambda = 3 \) and divide the unit interval \([0, 1]\) into segments of length \(1/3\). Deposit a square with side length \(1/3\) on the middle segment and nothing on the left and right segments. Repeating this procedure of the segmentation of each horizontal line and the subsequent deposition of particles results in a “castle” landscape as shown in Fig. 1(a), where one step of segmentation and deposition corresponds to a single generation \( n \).

In Ref. [4], it was shown that this construction results in a logarithmic fractal, whereby the surface length increment quickly saturates to the constant \(2/3\), resulting...
Figure 1: (a) Iterative construction of the fractal landscape for the first three generations of the HDM. (b) Height profile for the fractal landscape. The shortest path length between the leftmost and central nodes is indicated by dashed blue lines for n = 3. (c) The associated complex network based on the HVA as defined by equation (4). The different stages of each figure are indicated by colors: generation n = 1 is black, n = 2 is green, and n = 3 is orange.

in a linear increase in the surface length. This logarithmic fractal character remains present even for a random model with deposition probability P.

From the above construction, it is clear that not every value for the interface height, measured with respect to the common baseline, is accessible. In particular, the set of accessible height values (in ascending order) in a generation n is the subset of the first $2^n$ values of the set $3^{-n}S(0,1)$, i.e., $H(n) = 3^{-n}S(n)(0,1) \subset 3^{-n}S(0,1)$ where $S(0,1) = \{0, 1, 3, 4, 9, 10, 12, 13, \ldots \}$ is the Stanley sequence (OEIS A005836). The height distribution function, supported on $H(n)$, is then

$$P^{(n)}(y = h, i \in H(n)) = \left(\frac{2}{3}\right)^n 2^{-s(i)}, \quad (2)$$

where $i$ is the index of $h$ in the set $H(n)$, starting from $i = 0$, and the fractal sequence $s(i)$ indicates the number of 1’s in a binary expansion of $i$, defined through the following recurrence

$$s(0) = 0, \quad s(2i) = s(i), \quad s(2i + 1) = s(i) + 1. \quad (3)$$

This distribution is shown in Fig. 2(a), together with the set $H(n)$ on which it is supported. Note that by construction, the height at the central point of the interface grows to the highest value $h_{\text{max}} = \sum_{i=1}^{n} 3^{-i}$, which is bounded by $1/2$ for large $n$.

We now show that the deterministic HDM can be mapped to a scale-free network, the so-called horizontal visibility graph (HVG), by means of the HVA [12].

The HVA converts a sequence of real data into a graph according to the following visibility criterion: two data points $x_i$ and $x_j$ possess mutual horizontal visibility if the following criterion holds:

$$x_i, x_j > x_k \quad \text{for all } k \quad \text{such that } i < k < j. \quad (4)$$

If two data points $x_i, x_j$ have mutual visibility, an edge exists between the two nodes $i, j$ in the network created by the HVA. This algorithm has been used to characterize time series [17] by converting them to complex networks, enabling one to study the series combinatorially. In particular, the HVG is connected, undirected, invariant under affine transformations of the data, and irreversible, i.e., one cannot deduce the exact series from the graph, as there occurs some coarse graining during the application of the HVA.

To the author’s knowledge, VAs have not received much attention in the context of deposition processes, with only one relevant work [18] that deduces Hurst exponents for the Edwards-Wilkinson (EW), Kardar-Parisi-Zhang (KPZ) and Molecular Beam Epitaxy (MBE) equations, as well as for the discrete Random Deposition (RD), Random Deposition with Surface Relaxation (RDSR) and Eden models. Another related work [19] studies the random deposition of patchy particles partly in the context of VAs.

Degree distribution – When the HVA is applied to the deterministic HDM, the node degree $k_j(n)$ can be found for every individual node $j = 1,...,3^n$ in a generation $n$ by summing all entries in the $j$-th column of the adjacency matrix $A^{(n)}$, i.e., $k_j(n) = \sum_{i=1}^{3^n} A_{ij}^{(n)}$. After some
algebra, which is outlined in more detail in the supplemental material [20], and by making use of the hierarchical structure of $A^{(n)}$, the $j$th node degree can be found as follows

$$
k_j(n) = (1 + \delta_{1,\alpha_1}) + \sum_{k=1}^{n-1} \left( \delta_{1,\alpha_{k+1}} \sum_{i=0}^{2^{k-1}-1} \left( \delta_{j-\sigma_{nk},\Xi_{k+1}+1/2} + \delta_{j-\sigma_{nk},\Xi_{k+1}+3/2} \right) + \delta_{2,\alpha_{k+1}} \sum_{i=0}^{2^{k-1}-1} \left( \delta_{3^{\infty}+1-j+\sigma_{nk},\Xi_{k+1}+1/2} + \delta_{3^{\infty}+1-j+\sigma_{nk},\Xi_{k+1}+3/2} \right) + \delta_{1,\alpha_{k+1}} \cdot 2^k \left( \delta_{j-\sigma_{nk},1} + \delta_{j-\sigma_{nk},3^k} \right) \right).\tag{5}$$

where $\sigma_{nk}$ and $\Xi_{ki}$ are defined in the following manner:

$$\sigma_{nk} = \sum_{i=k}^{n} \alpha_{i+1} 3^{i}, \quad \Xi_{ki} = \frac{3^k}{2} + \frac{3i}{2} + \sum_{m=1}^{i} \frac{3^m}{2}. \tag{6}$$

The $\alpha_i$’s, where $i \in \{1,\ldots,n\}$ are the digits in the base 3 representation of the node index $j$. For example, for $j = 17$ in generation $n = 4$, the number $j$ can be written as $17 = 2 \cdot 3^3 + 2 \cdot 3^2 + 1 \cdot 3^1 + 0 \cdot 3^0$, hence, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (2, 2, 1, 0)$. Furthermore, the function $a(m)$ is the so-called ruler function (OEIS A001511) [21], where the $n$th term in this integer sequence is the highest power of 2 that divides $2n$.

In Fig. 2(b), the node index $j$ is mapped to $\theta_j = 2\pi(j - 1)/(3^n - 1)$ in $[0, 2\pi)$ and the points $(k_j, \theta_j)$ are visualized on the circle. This representation enables us to deduce the following: i) the degree sequence is symmetric around the axis $\theta = 0$, which follows from the construction of the hierarchical interface that is invariant under the transformation $x \rightarrow 1 - x$, where $x \in [0, 1]$, ii) the nodes at $\theta \approx 2\pi/3$ and $\theta \approx 4\pi/3$ possess the largest degree, indicating that they constitute the two main hubs of the network, and iii) a significant portion of nodes possesses degree $k = 2$ as a consequence of shielding by their direct neighbors, preventing them from seeing other nodes.

It is clear from equation (5) that not every possible integer value of $k$ occurs in every generation. In particular, the number of distinct values for the degree grows as $2n + 1$ for $n > 1$. In a generation $n > 1$, $k$ can only take values in the set $K^{(n)}$, where $K^{(n)} = \{1, 2, \Lambda_{\text{odd}}^{(n)}, \Lambda_{\text{even}}^{(n)}\}$, and where the sets of odd and even degrees are defined, respectively, as $\Lambda_{\text{odd}}^{(n)} = \left\{1 + 2\frac{j-1}{2^i}\right\}_{j, \text{odd}}$ and $\Lambda_{\text{even}}^{(n)} = \left\{2 + 2\frac{j-1}{2^i}\right\}_{j, \text{even}}$, with $j = 3, 4, \ldots, 2n - 1$. For the first generation, the set of possible degrees is $K^{(1)} = \{1, 2\}$.

After some algebra one can deduce the $n$th generation degree distribution $P^{(n)}(k)$ from the node degree as described by equation (5), i.e.,

$$\begin{align*}
P^{(n)}(1) &= \frac{2}{3^n}, \quad P^{(n)}(2) = \frac{1}{3}, \quad P^{(n)}(3) = \left(\frac{2}{3}\right)^{n-1} - \frac{2}{3^n}, \\
P^{(n)}(4) &= \frac{5}{9} - \left(\frac{2}{3}\right)^{n-1} + \frac{1}{3^n}, \quad P^{(n)}(\Lambda_{\text{odd}}^{(n)}) = 2^{1+2i}, \\
P^{(n)}(\Lambda_{\text{even}}^{(n)}) &= 2 \left(3^{-\frac{1}{2}} - 2^{-\frac{1}{2}} \left(\frac{2}{3}\right)^{n}\right). \tag{7}
\end{align*}$$

Note that the probability that a node possesses degree $k = 2$ is exactly equal to $1/3$ in every generation, justifying the large number of $k = 2$ nodes in Fig. 2(b). In Fig. 3, the numerically determined degree distribution is shown for $n = 10$ (red crosses), together with the theoretically predicted even (odd) degree distributions $P_e (P_o)$, for values of $n = 10$ (purple dots) and $n = 20$ (orange squares). For visualization purposes, lines are shown for the even (full) and odd (dashed) distributions.

Note that the degree distribution behaves as a power law, i.e., $P_e \sim k^{-\gamma_e}$, and $P_o \sim k^{-\gamma_o}$, with exponents $\gamma_e$ and $\gamma_o$. The exact values of $\gamma_e$ and $\gamma_o$ can be calculated in the large $n$ limit to be $\gamma_e = \log 3/\log 2 \approx 1.585$ and $\gamma_o = 1$. Both values were confirmed numerically, as indicated in the inset in Fig. 3. The value of $\gamma_e = \log (3)/\log (2)$ has been found previously for the deterministic network introduced by Barabási, Ravasz, and Vicsek [22, 23].

Eliminating $j$ from $k = 1 + 2\frac{j-1}{2^i}$ and $P^{(n)}(k \in \Lambda_{\text{odd}}^{(n)})$, one finds that for large $k$, the odd degree distribution is $P^{(n)}(k \in \Lambda_{\text{odd}}^{(n)}) \sim (2/3)^n k^{-1}$. It follows that this vanishes for large $n$. A similar calculation reveals that no such behavior is present for the even degrees, which will consequently dominate $P(k)$ for large $n$. 


Figure 3: The numerical distribution (red crosses) for the \( n = 10 \) HVG. \( P_r(k) \) and \( P_o(k) \) (7) are shown (respectively black full and dashed lines) for \( n = 10 \) (black dots) and \( n = 20 \) (orange squares). Inset: exponents \( \gamma_e \) (dots) and \( \gamma_o \) (stars) as a function of \( n \).

Moments – Equipped with the exact expression for the degree distribution (7), one can calculate a number of properties. The moments of the distribution can be found as

\[
\langle k^m(n) \rangle = \sum_k k^m P^{(n)}(k).
\]  

(8)

The average degree \( \langle k \rangle \) can be determined exactly from equation (8) for \( m = 1 \) through

\[
\langle k \rangle = \sum_k k P^{(n)}(k) = 4 \left( 1 - \left( \frac{2}{3} \right)^n \right),
\]  

(9)

where the \( n \) is suppressed. This expression converges to \( \langle k^\infty \rangle = 4 \), indicating that the network becomes sparse for large \( n \), a property that could also be determined from the increasing sparsity of the adjacency matrix of the HVG.

Through a similar calculation we find that the second moment \( \langle k^2 \rangle \), i.e.,

\[
\langle k^2 \rangle = \frac{32}{3} + \frac{2}{3^n} - 2(n + 8) \left( \frac{2}{3} \right)^n + 2 \left( \frac{4}{3} \right)^n,
\]  

(10)

diverges for large generations \( n \) as \( \langle k^2(n) \rangle \sim 2(4/3)^n \), as expected for scale-free networks.

Graph diameter – The diameter \( D \) of a graph is defined as \( D = \max_{ij} l_{ij} \), where \( l_{ij} \) is the shortest path distance between nodes \( i \) and \( j \). For the HVG, the diameter \( D(n) \) in generation \( n \) is

\[
D(n) = 2(2^n - 1).
\]  

(11)

This can easily be shown by considering that in the fractal landscape the outermost nodes \( i = 1 \) and \( j = 3^n \) are the two most distant nodes in the corresponding HVG. The total path distance between these nodes can hence be split into two identical distances by virtue of the reflection symmetry about the central node. Therefore, we only consider the left side of the landscape for the calculation of the diameter. Assume now that we know the shortest path distance between the outermost left node and the central node in generation \( n-1 \). Then, again by virtue of the symmetry, we only need to consider the left side of the different copies of the \( n-1 \) landscape, because the leftmost node of such a copy is already visible from the central node of the copy to the left.

This is shown graphically with blue lines in Fig. 1. The resulting diameter of the left side of the landscape is twice the left diameter from the previous generation plus one from the connection between the two copies. From this construction, it is clear that \( D(1) = 2, D(2) = 6, D(3) = 14, \ldots \) Hence, the recurrence for the total diameter is \( 2D(n+1) = 4D(n) + 2 \). Solving for \( D(n) \) results in equation (11). It is clear that for large \( n \), the diameter increases as a power law of the number of nodes \( N(n) = 3^n \), i.e., \( D(n) \sim N(n)^{\beta} \). The exponent \( \beta \) can easily be found to be the inverse of the degree exponent for the even degree nodes \( \beta = \gamma_e^{-1} \).

Distribution of clustering – By inspecting the clustering coefficient \( C(k) \) for the nodes with degree \( k \), it becomes clear that there is a one-to-one correspondence between the clustering coefficient of a vertex and its degree: \( C(k) \) is proportional to \( k^{-1} \) and stays stationary, i.e., nodes with degree \( k \) always have the same clustering coefficient \( C(k) \), independent of the generation. A simple geometrical argument reveals that the clustering coefficient is the ratio of the number of triangles that a node \( i \) with degree \( k \) is part of, normalized by the maximal number of triangles, i.e., \( \binom{k}{3} \). So for e.g. \( k = 2 \), the number of triangles is one (except for the central node of the network, which is not a part of any triangle). Hence, \( C(k = 2) = 1 \). For \( k = 3 \), one of the connections will be one of the neighbor data, while the other two form one triangle. Hence, \( C(k = 3) = 1/3 \).

We list the local clustering coefficients in the supplemental material [20]. We find that the nodes with even and odd degree have the following form, respectively

\[
C \left( k \in \Lambda_{\text{even}}^{(n)} \right) = \frac{2}{k}, \quad C \left( k \in \Lambda_{\text{odd}}^{(n)} \right) = \frac{2(k-2)}{k(k-1)},
\]  

(12)

which are independent of the generation \( n \). By construction, we also have \( C(k = 1) = 0 \), as nodes with only a single connection (edge nodes) can never form triangles. Hence, for large \( k \), \( C(k) \sim 2k^{-1} \), as was previously found to indicate hierarchical clustering [24, 25]. The scaling law indicates that a hierarchy of nodes with different degrees of modularity coexists in the network.

The mean clustering coefficient \( \langle C(n) \rangle \) can now be
found by
\[ \langle C^{(n)} \rangle = \sum_k C(k)P^{(n)}(k), \quad (13) \]
For large \( n \), the mean clustering coefficient saturates to a constant value \( \langle C^{\infty} \rangle \). It can easily be shown that the contributions from the odd degrees vanish for large \( n \), as well as the majority of factors from the even degrees. The terms that remain are
\[ \langle C^{\infty} \rangle = \lim_{n \to \infty} \langle C^{(n)} \rangle = \frac{11}{18} + 8 \sum_{l=3}^{\infty} \frac{3^{-l}}{4 + 2^l}, \quad (14) \]
where in the sum over the even degrees we have made the change of variables \( l = j/2 \). It can be checked numerically that \( \langle C^{\infty} \rangle \) converges to the value \( \langle C^{\infty} \rangle \approx 0.642 \). This is shown in Fig. 4.

Figure 4: Numerical (red) and exact (black) mean clustering coefficient \( \langle C^{(n)} \rangle \) as a function of \( n \). The black dashed line indicates \( \langle C^{\infty} \rangle \approx 0.64185 \). Inset: \( C(k) \) for even (full line) and odd degrees (dashed line).

Adjacent matrix and the eigenvalue spectrum – The adjacency matrix \( A^{(n)} \) is the matrix with values \( a_{ij} = 1 \) if the nodes \( i \) and \( j \) have mutual visibility, and value \( a_{ij} = 0 \) otherwise. The \( A^{(n+1)} \) matrix is a block tridiagonal matrix with on the diagonal the adjacency matrix \( A^{(n)} \) from the previous generation, repeated three times. Furthermore, the matrix \( B \) is related to \( C \) by \( B = JC^TJ \), where \( J \) is the exchange matrix. As the matrix is symmetric, it has a complete set of real eigenvalues and an orthogonal eigenvector basis.

For the adjacency matrix spectrum, we follow the representation used in [26] and rescale the eigenvalue index for every generation \( n \) such that the smallest eigenvalue corresponds to 0 and the largest corresponds to 1. This way, we can show that the spectrum converges to a fixed shape with a hierarchical structure. In Fig. 5, we show the spectrum in the rescaled coordinates for the first seven generations, where the point size is inversely related to the generation number. Note that the shape of the spectrum appears to be fractal.

Figure 5: Eigenvalue spectrum \( \{ \lambda_i \} \) as a function of the rescaled coordinate \( i \) for generations \( n = 1, ..., 7 \). The spectrum converges to a fixed shape for large \( n \). Inset: Enlarged image around the middle part of the spectrum. The fine structure is clearly visible.

Discussion – A mapping between the deterministic HDM for surface growth and complex networks was introduced, based on the HVA. We have shown that the resulting network is sparse and scale-free through the calculation of the degree distribution and exponents, and by calculating the first two moments of this distribution. Subsequently, we calculated the graph diameter and derived an exact expression for the clustering coefficient, which indicates that the network exhibits hierarchical clustering. Finally, the spectrum of the adjacency matrix was determined numerically and it was shown that it converges to a yet undetermined fractal shape.

The connection between surface growth phenomena and complex networks is still largely unexplored. The results from this work indicate that the structure of the growing surface is encoded in the topology of the associated visibility network, and that it can be discovered through careful analysis of different network measures. Subsequent research could extend the above results to more complex situations, such as the fully random HDM with a general scaling parameter \( \lambda \geq 2 \), to determine whether the modular hierarchical structure persists.

More general, one can construct a classification of “network universality classes” for surface growth phenomena based on topological properties and exponents determined from visibility graphs. This can complement or refine the existing theory of universality classes for which the exponents are well-known.
ACKNOWLEDGEMENTS

The author is grateful for the spirited discussions with J. O. Indekeu and R. Tielemans, whose insights were very enlightening, and acknowledges the support of the G-Research grant.

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Supplemental Material: Hierarchical deposition and deterministic scale-free networks: a visibility algorithm approach

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(Dated: April 29, 2022)

I. DERIVATION OF THE INDIVIDUAL NODE DEGREE

Consider the adjacency matrix of the visibility graph in generation \( n \), i.e., \( A^{(n)} \) with dimensions \( 3^n \times 3^n \), see e.g., Fig. 1. There are \( 3^n \) nodes with degree \( k_j \), where \( j = 1, 2, \ldots, 3^n \) denotes the node index. Let us for convenience shift this index \( j \) by one, \( j' = j - 1 \) and expand the result into base 3, writing the digits to a vector \( \vec{\alpha} \). For example, for the eighteenth node, \( j = 18 \), hence \( j' = 17 \) in generation \( n = 4 \), and the number \( j' \) can be written as \( 17 = 2 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + 0 \cdot 3^3 \), hence, \( \vec{\alpha}_{18} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (2, 2, 1, 0) \), where every \( \alpha_i \in \{0, 1, 2\} \) for \( i = 1, 2, \ldots, n \).

Thus, every degree \( k_j \) acquires a unique vector \( \vec{\alpha}_j \), which is associated with its position in the adjacency matrix. We now consider the lowest-order adjacency matrix \( A^{(1)} \), which is given by

\[
A^{(1)} = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
\end{pmatrix}.
\]

For every subsequent generation \( n \), the adjacency matrix \( A^{(1)} \) represents a “unit cell”, which is repeated along the diagonal of every \( A^{(n)} \). From a physical standpoint, this can be attributed to the mutual visibility of neighbors, where every node can always see its direct neighbors.

![Figure 1: (a)-(c) Adjacency matrices for generations \( n = 2, 3, 4 \), respectively. Black squares indicate matrix elements \( a_{ij} = 1 \), while white regions indicate that \( a_{ij} = 0 \). (d) A general depiction of the hierarchical structure of the adjacency matrix \( A^{(n+1)} \).](image)

The first element in the vector \( \vec{\alpha} \) indicates the column of the node index within this unit cell, i.e., \( \alpha_1 = 0, 1, 2 \) indicates the left, middle and right columns, respectively. The contribution of the position in this unit cell to the total node degree is the sum of elements in the column \( \alpha_1 \), and can be written consisely as \( 1 + \delta_{\alpha_1,1} \), where \( \delta_{ij} \) is the Kronecker delta, i.e., left and right columns contribute 1 to the degree, while the middle column contributes 2, which can easily be deduced from equation (1).

We now look at higher “levels” \( k \) in the adjacency matrix through the subsequent values in the \( \vec{\alpha} \) vector; when \( \alpha_k = 0, 1, 2 \), the index is located in the left, middle or right column of the \( k \)th level of the adjacency matrix. Let us...
look at the following example for $n = 2$:

$$A^{(2)} = \begin{pmatrix} A^{(1)} & (C^{(1)})^t & 0 \\ C^{(1)} & A^{(1)} & J^{(1)}C^{(1)}J^{(1)} \\ 0 & J^{(1)}(C^{(1)})^tJ^{(1)} & A^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

(2)

where the first ($k = 1$) and second ($k = 2$) levels are indicated by black and green boxes, respectively. This simple example of the second generation of the network already illustrates the nested structure of the adjacency matrix for a general generation $n$.

Consider now for example the column with index $j = 8$, which corresponds to $j' = j - 1 = 7$, and which has the following representation in our base 3 notation: $\overline{\alpha}_8 = (\alpha_1, \alpha_2) = (1, 2)$. From this, one can read off that the index $j = 8$ is located in the right column of the $k = 2$ level, which is the uppermost level, and in the middle column of the $k = 1$ level, or unit cell. From the latter, the node degree $k_8$ gains a contribution of $1 + \delta_{\alpha_1, 1} = 2$. The remaining factor of one needed to obtain the exact result $k_8 = 3$ will be discussed now.

For levels $k \geq 2$, the block matrices located on the sub- and superdiagonals are nonzero and are given by transposing the matrix $C^{(n)}$ either along the main diagonal, or along the antidiagonal. For $n = 2$, the block matrices are given by

$$C^{(1)} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (C^{(1)})^t = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J^{(1)}C^{(1)}J^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad J^{(1)}(C^{(1)})^tJ^{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

(3)

where $J^{(n)}$ is the $3^n \times 3^n$ exchange matrix, i.e., for $n = 1$, this is

$$J^{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(4)

The first row in the $C^{(n)}$ matrix will be denoted as the vector $\vec{v}^{(n)} = (\vec{0}, \vec{v}^{(n-1)}, \vec{v}^{(n-1)})$, where $\vec{0} = (0, ..., 0)$ is the zero vector with $3^{n-1}$ elements. We list here the first four $\vec{v}^{(n)}$:

$$\vec{v}^{(0)} = (1)$$

$$\vec{v}^{(1)} = (0, 1, 1)$$

$$\vec{v}^{(2)} = (0, 0, 0, 0, 1, 0, 1, 1)$$

$$\vec{v}^{(3)} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1)$$

(5)

From the structure of (5), it can be seen that for $n \geq 2$ the first $(3^n - 1)/2$ elements in the vector $\vec{v}^{(n)}$ are identically zero. The first two nonzero elements are then located at positions $j = (3^n + 1)/2$ and $j = (3^n + 3)/2$. Subsequent nonzero elements are separated by gaps of unequal size.

With moderate effort, one can deduce that subsequent nonzero elements are located at indices that are shifted by a factor of $(3i + \sum_{m=1}^{i} 3^{a(m)})/2$ with respect to the positions $j = (3^n + 1)/2$ and $j = (3^n + 3)/2$. Herein, $a(m)$ is the so-called ruler function (OEIS A001511), the elements of which are the exponent of the largest power of 2 which divides a given number $2m$ [? ]. This sequence can be characterized by the following recurrence relations:

$$a(2m + 1) = 1 \quad a(2m) = 1 + a(m).$$

(6)

A plot of the magnitude of the elements of the ruler function $a(m)$ is shown for $m = 1, 2, \ldots, 255$ in Fig. 2. One can now easily see why it is called as such, as the rising and falling of the sequence as a function of the index resembles the markings on a classical ruler.

Hence, after some simple algebra, the elements $v_j^{(n)}$, with $j = 1, ..., 3^n$ can be fully determined by

$$v_j^{(n)} = \sum_{i=0}^{2^n-1} \left( \delta_{j, \Xi_{m_i+\frac{1}{2}}} + \delta_{j, \Xi_{m_i+\frac{3}{2}}} \right),$$

(7)
Figure 2: Graphical representation of the ruler sequence $a(m)$ generated by the recurrence relations (6) for $m = 1, 2, \ldots, 255$. The pattern resembles the markings on a classical ruler.

with

$$\Xi_{ni} = \frac{1}{2}[3^n + 3i + \sum_{m=1}^{i} 3^{a(m)}]. \quad (8)$$

The number of nonzero elements in $v^{(n)}$ can be deduced by noticing that in a generation $n + 1$, this number has doubled with respect to generation $n$, and the first generation possesses two nonzero elements. Hence, the total number of nonzero elements is equal to $2^n$.

When the index $j$ is located in either the left or right column of the $k$th level (i.e., when $\alpha_k = 0$ or $\alpha_k = 2$, respectively), it picks up an extra factor of $+1$ for the node degree $k_j$ based on its position within the vectors $\vec{v}^{(k-1)}$ for $\alpha_k = 0$ or $\vec{v}^{(k-1)} \cdot J^{(k-1)}$ for $\alpha_k = 2$. Moreover, when $j$ is located in the middle column ($\alpha_k = 1$) at positions $j = 3^{k-1} + 1$ or $j = 2 \cdot 3^{k-1}$, the node degree $k_j$ picks up an extra $2^{k-1}$.

Summing over levels $k$ and shifting the indices for simplicity, we arrive after some cumbersome algebra at the final result for the node degree $k_j(n)$:

$$k_j(n) = (1 + \delta_{1,\alpha_1}) + \sum_{k=1}^{n-1} \left\{ \delta_{0,\alpha_{k+1}} \cdot \sum_{i=0}^{2^{k-1}-1} \left( \delta_{j-\sigma_{nk}, \Xi_{k+1}+1/2} + \delta_{j-\sigma_{nk}, \Xi_{k+1}+3/2} \right) + \delta_{2,\alpha_{k+1}} \cdot \sum_{i=0}^{2^{k-1}-1} \left( \delta_{2^{k-1}+1-j+\sigma_{nk}, \Xi_{k+1}+1/2} + \delta_{2^{k-1}+1-j+\sigma_{nk}, \Xi_{k+1}+3/2} \right) + \delta_{1,\alpha_{k+1}} \cdot 2^k \left( \delta_{j-\sigma_{nk}, 1} + \delta_{j-\sigma_{nk}, 3^k} \right) \right\} \quad (9)$$

where $\sigma_{nk}$ is defined in the following manner:

$$\sigma_{nk} = \sum_{l=k}^{n} \alpha_l + 1 \cdot 3^l. \quad (10)$$

The first three degree vectors $\vec{k}^{(n)} = (k_1(n), k_2(n), \ldots, k_{3^n}(n))$ are given by

$$\vec{k}^{(1)} = (1, 2, 1)$$
$$\vec{k}^{(2)} = (1, 3, 2, 3, 2, 3, 2)$$
$$\vec{k}^{(3)} = (1, 3, 2, 3, 4, 2, 4, 2, 5, 3, 2, 3, 2, 3, 5, 2, 4, 2, 4, 3, 3, 2, 3, 1), \quad (11)$$

which can easily be checked numerically from simulations to be the correct expressions.

II. CLUSTERING COEFFICIENT

In Table I, the clustering coefficients are listed as a function of the node degree $k$ and generation $n$. The full
Table I: Clustering coefficient $C^{(n)}(k)$ for the first five generations of the visibility graph.

| $k$ | $C^{(1)}(k)$ | $C^{(2)}(k)$ | $C^{(3)}(k)$ | $C^{(4)}(k)$ | $C^{(5)}(k)$ |
|-----|---------------|---------------|---------------|---------------|---------------|
| 2   | 0             | 1             | 1             | 1             | 1             |
| 3   | -             | 1/3           | 1/3           | 1/3           | 1/3           |
| 4   | -             | -             | 1/2           | 1/2           | 1/2           |
| 5   | -             | -             | 3/10          | 3/10          | 3/10          |
| 6   | -             | -             | 1/3           | 1/3           | 1/3           |
| 9   | -             | -             | 7/36          | 7/36          |               |
| 10  | -             | -             | -             | 1/5           |               |
| 17  | -             | -             | -             | -             | 15/136        |

Expression for the mean clustering coefficient $\langle C \rangle$ is

$$
\langle C^{(n)} \rangle = \sum_k C(k) P^{(n)}(k)
$$

$$
= C(2) \left( P^{(n)}(2) - \frac{1}{3^n} \right) + C(3) P^{(n)}(3) + C(4) P^{(n)}(4)
$$

$$
+ \sum_{j=5 \atop j \text{ odd}}^{2n-1} \left( \frac{4 - 2^{\frac{n-j}{2}}}{2 + 2^{\frac{n-j}{2}}} \right) \cdot \frac{j}{2} \left( \frac{2}{3} \right)^n
+ 2 \sum_{j=6 \atop j \text{ even}}^{2n-2} \left( \frac{4}{4 + 2^{n-j}} \right) \left( 3^{-\frac{j}{2}} - 2^{-\frac{j}{2}} \left( \frac{2}{3} \right)^n \right),
$$

(12)

where in the second line a factor $3^{-n}$ is removed from $P^{(n)}(2)$ in order to compensate for the fact that the central node has degree $k = 2$ but does not contribute to the clustering coefficient, as it is not a part of any closed triangle of nodes.