The number of eigenvalues of a tensor

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\textsuperscript{1}joint with Bernd Sturmfels
Eigenvalues of a tensor

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From this we have the operator on \( x \in \mathbb{C}^n \) written \( Ax^{m-1} \) and defined by

\[
(Ax^{m-1})_j = \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} a_{j i_2 \ldots i_m} x_{i_2} \cdots x_{i_m}
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**Definition (Qi, Lim)**

The \((E-)\)eigenvectors of \( A \) are the fixed points (up to scaling) of this operator:

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Ax^{m-1} = \lambda x \quad \text{ for } x \neq 0
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We will call \((\lambda, x)\) an eigenpair and consider them up to the equivalence:

\[ (\lambda, x) \sim (t^{m-2}\lambda, tx) \quad \text{for any } t \neq 0 \]
The number of eigenpairs

Theorem (Ni-Qi-Wang-Wang, C-Sturmfels)

The number of equivalence classes of eigenpairs of $A$ is either infinity or

$$\frac{(m-1)^n - 1}{m-2} = \sum_{i=0}^{n-1} (m-1)^i$$

when counted with multiplicity. For a generic tensor, there are exactly this many equivalence classes of eigenpairs.
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Given the first sentence, the second can be seen by considering the diagonal tensor

$$A_{i_1, \ldots, i_m} = \begin{cases} 1 & \text{if } i_1 = \cdots = i_m \\ 0 & \text{otherwise.} \end{cases}$$

It has exactly as many eigenvalues up to equivalence as the quantity in (1).
A proof for generic tensors

\[ Ax^{m-1} = \lambda x \]

Substitute \( \lambda = \mu^{m-2} \):

\[ Ax^{m-1} = \mu^{m-2}x \] (2)

\( n \) equations, homogeneous of degree \( m - 1 \) in \( \mu, x_1, \ldots, x_n \).
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One solution is trivial:

\[(\mu : x_1 : \ldots : x_n) = (1 : 0 : \ldots : 0)\]

An equivalence class of eigenpairs with \( \lambda \neq 0 \) corresponds to \( m - 2 \) solutions of (2) by taking \( \mu \) to be the \( m - 2 \) roots of \( \lambda \). Thus, we get

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Unfortunately, this approach cannot directly treat the case when \( \lambda = 0 \).
A geometric reinterpretation

\[ \mathbb{P}^n = \left\{ (\mu : x_1 : \ldots : x_n) \neq 0 \right\} / (\mu : x_1 : \ldots : x_n) \sim (t\mu : tx_1 : \ldots : tx_n) \]

\[ \downarrow \]

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Quotient of projective space by group which multiplies $\mu$ by $e^{2\pi i/(m-2)}$.

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Summary of previous slide: lift the eigenvalue problem from $\tilde{\mathbb{P}}^n$ to $\mathbb{P}^n$, used Bézout’s theorem, and the fact that the quotient is $(m-2)$-to-1 except:

- when $x_1 = \cdots = x_n = 0$ (subtract this out anyways)
- when $\lambda = \mu = 0$ (as noted, exceptional from this perspective)
A geometric reinterpretation

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Any quotient of a smooth variety is smooth where the group action is free. In this case, it so happens that the quotient is also smooth where $\mu = 0$. 
Intersections in $\mathbb{P}^n$

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The intersection of the eigenvalue equations in the rational Chow ring is

$$\frac{(m-1)^n}{m-2}.$$  \hfill (3)

(not an integer if $m > 3$.)
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Intersections in $\tilde{\mathbb{P}}^n$

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The unique singular point $x = 0$ counts for

$$\frac{1}{m-2}$$

so the number of equivalence classes of eigenvalues is

$$\frac{(m-1)^n}{m-2} - \frac{1}{m-2}$$
Corollaries

Theorem (Ni-Qi-Wang-Wang, C-Sturmfels)

The number of equivalence classes of eigenpairs of $A$ is either infinity or

$$\frac{(m - 1)^n - 1}{m - 2} = \sum_{i=0}^{n-1} (m - 1)^i$$  \hspace{1cm} (1)

when counted with multiplicity. For a generic tensor, there are exactly this many equivalence classes of eigenpairs.

Corollary

For $m \geq 3$, the number of equivalence classes of eigenvalues grows exponentially in $n$. 
Theorem (Ni-Qi-Wang-Wang, C-Sturmfels)

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Corollary

If the entries of $A$ are real and either $m$ or $n$ is odd, then $A$ has at least one real eigenpair.

If $m$ or $n$ is odd, then the sum in (1) is odd.
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Corollary

The characteristic polynomial $\phi_A(\lambda)$ has degree

$$\frac{(m - 1)^n - 1}{m - 2} = \sum_{i=0}^{n-1} (m - 1)^i$$
The characteristic polynomial

The coefficients of the characteristic polynomial $\phi_A(\lambda)$ are polynomials in the entries of the tensor. It vanishes on eigenvalues whose eigenvector has been normalized to have $x \cdot x = 1$. 
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More precisely: let $I \subset \mathbb{C}[a_{i_1,\ldots,i_m}, x_j, \lambda]$ be the ideal generated by:

$$\lambda x_1 - \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} a_{1i_2\ldots i_m} x_{i_2} \cdots x_{i_n}$$

$$\vdots$$

$$\lambda x_n - \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} a_{ni_2\ldots i_m} x_{i_2} \cdots x_{i_n}$$

$$x_1^2 + \cdots + x_n^2 - 1$$
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$$

The elimination ideal $I \cap \mathbb{C}[a_{i_1,...,i_m}, \lambda]$ is a principal ideal. If $m$ is even, characteristic polynomial $\phi_A(\lambda)$ is the the generator of this ideal. If $m$ is odd, the generator is $\phi_A(\lambda^2)$. 

The roots of the characteristic polynomial

Normalized eigenvalues are roots of the characteristic polynomial but not necessarily conversely.

Example

Let $A$ be the $2 \times 2 \times 2$ tensor with

$$a_{111} = a_{221} = 1 \quad \text{and} \quad a_{122} = a_{222} = \sqrt{-1}$$

and 0 entries elsewhere. The eigenvalue problem for $A$ is:

$$x_1^2 + ix_1x_2 = \lambda x_1 \quad \text{and} \quad x_1x_2 + ix_2^2 = \lambda x_2.$$

This has a normalized eigenvalue of $\lambda$ if and only $\lambda \neq 0$. Therefore, the characteristic polynomial is identically zero. Thus $\phi_A(0) = 0$, even though $\lambda = 0$ is not a (normalized) eigenvalue.
The roots of the characteristic polynomial

Example

Let $A$ be the symmetric $2 \times 2 \times 2$ tensor with

\[
\begin{align*}
    a_{111} &= -2i \\
    a_{222} &= 1
\end{align*}
\]

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\begin{align*}
    a_{112} &= a_{121} = a_{211} = 1 \\
    a_{122} &= a_{212} = a_{221} = 0
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Then the characteristic polynomial of $A$ vanishes identically, but $A$ has only a single normalized eigenvalue, namely 1.
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Then the characteristic polynomial of $A$ vanishes identically, but $A$ has only a single normalized eigenvalue, namely 1.

Why? Because $A$ also has a non-normalized eigenpair with eigenvalue 1 and eigenvector $(1, i)$. For small perturbations of $A$, this can take on any value as a normalized eigenvalue.
Symmetric tensors

A tensor is symmetric if it is invariant under all permutations of the $m$ factors.

**Theorem**

*If $A$ is symmetric, then $A$ has at most*

$$\frac{(m - 1)^n - 1}{m - 2}$$

*distinct normalized eigenvalues.*
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Two caveats:

- As in the example on the previous slide, the characteristic polynomial of $A$ may still vanish identically.
- There may also be infinitely many eigenvectors with the same normalized eigenvalue, and with a different method of normalization, these may yield infinitely many eigenvalues.
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