SOME CURVATURE PROPERTIES ON PARACONTACT METRIC 
\((k, \mu)\)-MANIFOLDS WITH RESPECT TO THE SCHOUTEN-VAN 
KAMPEN CONNECTION

Ahmet Yıldız\(^1\) and Selcen Yüksel Perktaş\(^2\)

\(^1\) Faculty of Education, Department of Mathematics, 
Malatya, Turkey 
\(^2\) Faculty of Arts and Science, Department of Mathematics, 
Adıyaman, Turkey

Abstract. The object of the present paper is to characterize paracontact metric \((k, \mu)\)- 
manifolds satisfying certain semisymmetry curvature conditions with respect to the 
Schouten-van Kampen connection.

Key words: Paracontact metric \((k; \mu)\)-manifolds; Schouten-van Kampen connection; 
h-projective semisymmetric; \(\phi\)-projective semisymmetric.

1. Introduction

Paracontact metric structures have been introduced in \([5]\), as a natural odd-
dimensional counterpart to para-Hermitian structures, like contact metric structures 
correspond to the Hermitian ones. Paracontact metric manifolds have been studied 
by many authors in the recent years, particularly since the appearance of \([19]\). An 
important class among paracontact metric manifolds is that of the \((k, \mu)\)-manifolds, 
which satisfies the nullity condition \([2]\)

\begin{equation}
R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),
\end{equation}

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Corresponding Author: Ahmet Yıldız, Faculty of Education, Department of Mathematics, 
Malatya, Turkey | E-mail: a.yildiz@inonu.edu.tr

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for all \(X,Y\) vector fields on \(M\), where \(k\) and \(\mu\) are constants and \(h = \frac{1}{2}L_{\xi}\phi\). This class includes the para-Sasakian manifolds \([5, 19]\), the paracontact metric manifolds satisfying \(R(X,Y)\xi = 0\) for all \(X,Y\) \[20\].

Among the geometric properties of manifolds symmetry is an important one. From the local point view it was introduced by Shirokov as a Riemannian manifold with covariant constant curvature tensor \(R\), that is, with \(\nabla R = 0\), where \(\nabla\) is the Levi-Civita connection. An extensive theory of symmetric Riemannian manifolds was introduced by Cartan in 1927. A manifold is called semisymmetric if the curvature tensor \(R\) satisfies \(R(X,Y) \cdot R = 0\), where \(R(X,Y)\) is considered to be a derivation of the tensor algebra at each point of the manifold for the tangent vectors \(X,Y\). Semisymmetric manifolds were locally classified by Szabó \[16\]. Also in \[17\] and \[18\], Yildiz and De studied \(h\)-Weyl semisymmetric, \(\phi\)-Weyl semisymmetric, \(h\)-projectively semisymmetric and \(\phi\)-projectively semisymmetric non-Sasakian \((k,\mu)\)-contact metric manifolds and paracontact metric \((k,\mu)\)-manifolds respectively. Recently Mandal and De have studied certain curvature conditions on paracontact \((k,\mu)\)-spaces \[6\].

The projective curvature tensor is an important tensor from the differential geometric point of view. Let \(M\) be a \((2n+1)\)-dimensional semi-Riemannian manifold with metric \(g\). The Ricci operator \(Q\) of \((M,g)\) is defined by \(g(QX,Y) = S(X,Y)\), where \(S\) denotes the Ricci tensor of type \((0,2)\) on \(M\). If there exists a one-to-one correspondence between each coordinate neighbourhood of \(M\) and a domain in Euclidean space such that any geodesic of the semi-Riemannian manifold corresponds to a straight line in the Euclidean space, then \(M\) is said to be locally projectively flat. For \(n \geq 1\), \(M\) is locally projectively flat if and only if the well known projective curvature tensor \(P\) vanishes. Here \(P\) is defined by

\[
P(X,Y)Z = R(X,Y)Z - \frac{1}{2n} \{ S(Y,Z)X - S(X,Z)Y \},
\]

for all \(X,Y,Z \in T(M)\), where \(R\) is the curvature tensor and \(S\) is the Ricci tensor.

In fact \(M\) is projectively flat if and only if it is of constant curvature. Thus the projective curvature tensor is the measure of the failure of a semi-Riemannian manifold to be of constant curvature.

A paracontact metric \((k,\mu)\)-manifold is said to be an Einstein manifold if the Ricci tensor satisfies \(S = \lambda_1 g\), and an \(\eta\)-Einstein manifold if the Ricci tensor satisfies \(S = \lambda_1 g + \lambda_2 \eta \otimes \eta\), where \(\lambda_1\) and \(\lambda_2\) are constants.

On the other hand, the Schouten-van Kampen connection is one of the most natural connections adapted to a pair of complementary distributions on a differentiable manifold endowed with an affine connection \[1, 4, 10\]. Solov’ev has investigated hyperdistributions in Riemannian manifolds using the Schouten-van Kampen connection \[12, 13, 14, 15\]. Then Olszak has studied the Schouten-van Kampen connection to adapted to an almost (para) contact metric structure \[8\]. He has characterized some classes of almost (para) contact metric manifolds with the Schouten-van Kampen connection and he has finded certain curvature properties of this connection on these manifolds.
In the present paper we have studied certain curvature properties of a paracontact metric \((k, \mu)\)-space. The outline of the article goes as follows: After introduction, in Section 2, we recall basic facts which we will need throughout the paper. Section 3 deals with some basic results of paracontact metric manifolds with characteristic vector field \(\xi\) belonging to the \((k, \mu)\)-nullity distribution with respect to the Schouten-van Kampen connection. In section 4, we characterize paracontact metric \((k, \mu)\)-manifolds satisfying some semisymmetry curvature conditions. We prove that a \(h\)-projectively semisymmetric and \(\phi\)-projectively semisymmetric paracontact metric \((k, \mu)\)-manifold with respect to the Schouten-van Kampen connection is an \(\eta\)-Einstein manifold with respect to the Levi-Civita connection, respectively. In the all cases we assume that \(k \neq -1\).

2. Preliminaries

An \((2n + 1)\)-dimensional smooth manifold \(M\) is said to have an almost paracontact structure if it admits a \((1,1)\)-tensor field \(\phi\), a vector field \(\xi\) and a 1-form \(\eta\) satisfying the following conditions:

(i) \(\eta(\xi) = 1, \quad \phi^2 = I - \eta \otimes \xi\),

(ii) the tensor field \(\phi\) induces an almost paracomplex structure on each fibre of \(\mathcal{D} = \text{ker}(\eta)\), i.e. the \(\pm 1\)-eigendistributions, \(\mathcal{D}^{\pm} = \mathcal{D}_{\phi}(\pm 1)\) of \(\phi\) have equal dimension \(n\).

From the definition it follows that \(\phi\xi = 0, \eta \circ \phi = 0\) and the endomorphism \(\phi\) has rank \(2n\). The Nijenhuis torsion tensor field \([\phi, \phi]\) is given by

\[
[\phi, \phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].
\]

When the tensor field \(N_\phi = [\phi, \phi] - 2d\eta \otimes \xi\) vanishes identically the almost paracontact manifold is said to be normal. If an almost paracontact manifold admits a pseudo-Riemannian metric \(g\) such that

\[
g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y),
\]

for all \(X, Y \in \Gamma(TM)\), then we say that \((M, \phi, \xi, \eta, g)\) is an almost paracontact metric manifold. Notice that any such a pseudo-Riemannian metric is necessarily of signature \((n + 1, n)\). For an almost paracontact metric manifold, there always exists an orthogonal basis \(\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}\), such that \(g(X_i, X_j) = \delta_{ij}, g(Y_i, Y_j) = -\delta_{ij}, g(X_i, Y_j) = 0, g(\xi, X_i) = g(\xi, Y_j) = 0\), and \(Y_i = \phi X_i\), for any \(i, j \in \{1, \ldots, n\}\). Such basis is called a \(\phi\)-basis.

We can now define the fundamental form of the almost paracontact metric manifold by \(\theta(X, Y) = g(X, \phi Y)\). If \(d\eta(X, Y) = g(X, \phi Y)\), then \((M, \phi, \xi, \eta, g)\) is said to be paracontact metric manifold. In a paracontact metric manifold one defines a symmetric, trace-free operator \(h = \frac{1}{2} \mathcal{L}_\xi \phi\), where \(\mathcal{L}_\xi\), denotes the Lie derivative. It
is known [19] that \( h \) anti-commutes with \( \phi \) and satisfies \( h\xi = 0 \), \( trh = trh\phi = 0 \) and
\[
(\nabla_X\xi) = -\phi X + \phi h X, \tag{2.3}
\]
\[
(\nabla_X\eta) = g(X, \phi Y) - g(hX, \phi Y), \tag{2.4}
\]
where \( \nabla \) is the Levi-Civita connection of the pseudo-Riemannian manifold \((M, g)\).

Let \( R \) be Riemannian curvature operator
\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z. \tag{2.5}
\]

Moreover \( h = 0 \) if and only if \( \xi \) is Killing vector field. In this case \((M, \phi, \xi, \eta, g)\) is said to be a K-paracontact manifold. A normal paracontact metric manifold is called a para-Sasakian manifold. Also, in this context the para-Sasakian condition implies the K-paracontact condition and the converse holds only in dimension 3.

We also recall that any para-Sasakian manifold satisfies
\[
R(X, Y)\xi = \eta(X)Y - \eta(Y)X. \tag{2.6}
\]

### 3. Paracontact metric \((k, \mu)\)-manifolds with respect to the Schouten-van Kampen connection

Let \((M, \phi, \xi, \eta, g)\) be a paracontact manifold. The \((k, \mu)\)-nullity distribution of a \((M, \phi, \xi, \eta, g)\) for the pair \((k, \mu)\) is a distribution
\[
N(k, \mu) : p \rightarrow N_p(k, \mu) = \left\{ Z \in T_p M \mid R(X, Y)Z = k(g(Y, Z)X - g(X, Z)Y) + \mu(g(Y, Z)hX - g(X, Z)hY) \right\}, \tag{3.1}
\]
for some real constants \( k \) and \( \mu \). If the characteristic vector field \( \xi \) belongs to the \((k, \mu)\)-nullity distribution we have (3.1). [2] is a complete study of paracontact metric manifolds for which the Reeb vector field of the underlying contact structure satisfies a nullity condition (the condition (3.1), for some real numbers \( k \) and \( \mu \)).

**Lemma 3.1.** [2] Let \( M \) be a paracontact metric \((k, \mu)\)-manifold of dimension \( 2n + 1 \). Then the following holds:
\[
(\nabla_X h)Y - (\nabla_Y h)X = -(1 + k)(2g(X, \phi Y)\xi + \eta(X)\phi Y - \eta(Y)\phi X) + (1 - \mu)(\eta(X)\phi h Y - \eta(Y)\phi h X), \tag{3.2}
\]
\[
(\nabla_X \phi h)Y - (\nabla_Y \phi h)X = (1 + k)(\eta(X)Y - \eta(Y)X) + (\mu - 1)(\eta(X)hY - \eta(Y)hX), \tag{3.3}
\]
\[
(\nabla_X \phi)Y = -g(X, Y)\xi + g(hX, Y)\xi + \eta(Y)X - \eta(X)Y, \quad k \neq -1, \tag{3.4}
\]
for any vector fields \( X, Y \) on \( M \).
Lemma 3.2. [2] In any $(2n + 1)$-dimensional paracontact metric $(k, \mu)$-manifold $(M, \phi, \xi, \eta, g)$ such that $k \neq -1$, the Ricci operator $Q$ is given by

$$Q = (2(1 - n) + n\mu)I + (2(n - 1) + \mu)h + (2(n - 1) + n(2k - \mu))\eta \otimes \xi.$$  

(3.5)

On the other hand, we have two naturally defined distribution in the tangent bundle $TM$ of $M$ as follows:

$$H = \ker \eta, \quad V = \operatorname{span}\{\xi\}.$$  

(3.6)

Then we have $TM = H \oplus V$, $H \cap V = \{0\}$ and $H \perp V$. This decomposition allows one to define the Schouten-van Kampen connection $\tilde{\nabla}$ over an almost contact metric structure. The Schouten-van Kampen connection $\tilde{\nabla}$ on an almost (para) contact metric manifold with respect to Levi-Civita connection $\nabla$ is defined by [12]

$$\tilde{\nabla}X Y = \nabla X Y - \eta(Y)\nabla X \xi + (\nabla X \eta)(Y)\xi.$$  

(3.7)

Thus with the help of the Schouten-van Kampen connection (3.7), many properties of some geometric objects connected with the distributions $H, V$ can be characterized [12, 13, 14]. For example $g, \xi$ and $\eta$ are parallel with respect to $\tilde{\nabla}$, that is, $\tilde{\nabla}\xi = 0, \tilde{\nabla}g = 0, \tilde{\nabla}\eta = 0$. Also, the torsion $\tilde{T}$ of $\tilde{\nabla}$ is defined by

$$\tilde{T}(X, Y) = \eta(X)\nabla Y \xi - \eta(Y)\nabla X \xi + 2d\eta(X, Y)\xi.$$  

(3.8)

Now we consider a paracontact metric $(k, \mu)$-manifold with respect to the Schouten-van Kampen connection. Firstly, using (2.3) and (2.4) in (3.7), we get

$$\tilde{\nabla}X Y = \nabla X Y - \eta(Y)\phi X - \eta(Y)\phi h X + g(X, \phi Y)\xi - g(h X, \phi Y)\xi.$$  

(3.9)

Let $R$ and $\tilde{R}$ be the curvature tensors of the Levi-Civita connection $\nabla$ and the Schouten-van Kampen connection $\tilde{\nabla}$,

$$R(X, Y) = [\nabla X, \nabla Y] - \nabla_{[X,Y]}, \quad \tilde{R}(X, Y) = [\tilde{\nabla} X, \tilde{\nabla} Y] - \tilde{\nabla}_{[X,Y]}.$$  

(3.10)

If we substitute equation (3.7) in the definition of the Riemannian curvature tensor

$$\tilde{R}(X, Y)Z = \tilde{\nabla}X \tilde{\nabla}Y Z - \tilde{\nabla}Y \tilde{\nabla}X Z - \tilde{\nabla}_{[X,Y]}Z.$$  

(3.11)

Using (3.9) in (3.11), we have

$$\tilde{R}(X, Y)Z = \tilde{\nabla}X (\nabla Y Z - \eta(Z)\phi Y - \eta(Z)\phi h Y + g(Y, \phi Z)\xi - g(h Y, \phi Z)\xi)$$

$$- \nabla Y (\nabla X Z - \eta(Z)\phi X - \eta(Z)\phi h X + g(X, \phi Z)\xi - g(h X, \phi Z)\xi)$$

$$+ g(X, \phi Z)\xi - g(h X, \phi Z)\xi$$

$$- (\nabla_{[X,Y]}Z + \eta(Z)\phi [X,Y] - \eta(Z)\phi h [X,Y] + g([X,Y], \phi Z)\xi - g(h [X,Y], \phi Z)\xi).$$  

(3.12)
Using (3.2), (3.3) and (3.4) in (3.12), we obtain the following formula connecting \( R \) and \( \tilde{R} \) on \( M \)

\[
\tilde{R}(X,Y)Z = R(X,Y)Z + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + g(hY,\phi Z)\phi X \\
- g(hX,\phi Z)\phi Y + g(Y,\phi Z)\phi hX - g(X,\phi Z)\phi hY \\
+ g(hX,\phi Z)\phi hY - g(hY,\phi Z)\phi hX
\]

(3.13)

\[
+(k+1)(g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi) \\
+ k(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X) \\
+ (\mu - 1)(g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi) \\
+ \mu(\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX).
\]

Now taking the inner product in (3.13) with a vector field \( W \), we have

\[
g(\tilde{R}(X,Y)Z,W) = g(R(X,Y)Z,W) + g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W) \\
+ g(hY,\phi Z)g(\phi X,W) - g(hX,\phi Z)g(\phi Y,W) \\
+ g(Y,\phi Z)g(\phi hX,W) - g(X,\phi Z)g(\phi hY,W) \\
+ g(hX,\phi Z)g(\phi hY,W) - g(hY,\phi Z)g(\phi hX,W) \\
+ (k+1)(g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W)) \\
+ k(g(Y,W)\eta(X)\eta(Z) - g(X,W)\eta(Y)\eta(Z)) \\
+ (\mu - 1)(g(hX,Z)\eta(Y)\eta(W) - g(hY,Z)\eta(X)\eta(W)) \\
+ \mu(g(hY,W)\eta(X)\eta(Z) - g(hX,W)\eta(Y)\eta(Z)).
\]

(3.14)

If we take \( X = W = e_i, \{i = 1,...,2n+1\} \), in (3.14), where \( \{e_i\} \) is an orthonormal basis of the tangent space at each point of the manifold, we get

\[
\bar{S}(Y,Z) = S(Y,Z) - (k+2)g(Y,Z) \\
+ (k+2-2nk)\eta(Y)\eta(Z) - (\mu - 1)g(hY,Z),
\]

(3.15)

where \( \bar{S} \) and \( S \) denote the Ricci tensor of the connections \( \nabla \) and \( \tilde{\nabla} \), respectively.

As a consequence of (3.15), we get for the Ricci operator \( \bar{Q} \)

\[
\bar{Q}Y = QY - (k+2)Y + (k+2-2nk)\eta(Y)\xi - (\mu - 1)hY,
\]

(3.16)

Also if we take \( Y = Z = e_i, \{i = 1,...,2n+1\} \), in (3.16), we get

\[
\bar{r} = r - 4n(k+1),
\]

(3.17)

where \( \bar{r} \) and \( r \) denote the scalar curvatures of the connections \( \nabla \) and \( \tilde{\nabla} \), respectively.

4. Some semisymmetry curvature conditions on paracontact metric \((k,\mu)\)-manifolds

In this section we study some semisymmetry curvature conditions on paracontact metric \((k,\mu)\)-manifolds with respect to the Schouten-van Kampen connection. Firstly we give the following:
Definition 4.1. A semi-Riemannian manifold \((M^{2n+1}, g), n > 1\), is said to be \(h\)-projectively semisymmetric if
\[
P(X, Y) \cdot h = 0,
\]
holds on \(M\).

Let \(M\) be a \(h\)-projectively semisymmetric paracontact metric \((k, \mu)\)-manifold \((k \neq -1)\) with respect to the Schouten-van Kampen connection. Then above equation is equivalent to
\[
\tilde{P}(X, Y)hZ - h\tilde{P}(X, Y)Z = 0.
\]
for any \(X, Y, Z \in \chi(M)\). Thus we write
\[
\frac{1}{2n}\{\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y - \tilde{S}(Y, Z)hX + \tilde{S}(X, Z)hY\} = 0.
\]
Using (3.13) in (4.3), we have
\[
R(X, Y)hZ - hR(X, Y)Z + g(X, \phi hZ)\phi Y - g(Y, \phi hZ)\phi X
- g(hY, \phi hZ)\phi X + g(hX, \phi hZ)\phi Y + g(Y, \phi hZ)\phi hX
- g(X, \phi Z)\phi hY - g(hX, \phi Z)\phi hY + g(hY, \phi Z)\phi hX
+(k + 1)\{g(X, hZ)\eta(Y)\xi - g(Y, hZ)\eta(X)\xi\}
+(\mu - 1)\{g(hX, hZ)\eta(Y)\xi - g(hY, hZ)\eta(X)\xi\}
- g(X, \phi Z)\phi hY + g(Y, \phi Z)\phi hX - g(hY, \phi Z)\phi hX
\]
\[
+g(hX, \phi Z)\phi hY - g(Y, \phi Z)\phi hX + g(X, \phi Z)\phi hY
- g(hX, \phi Z)\phi hY + g(hY, \phi Z)\phi hX
-k\{\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX\}
-\mu\{\eta(X)\eta(Z)h^2Y - \eta(Y)\eta(Z)h^2X\}
- \frac{1}{2n}\{S(Y, hZ)X - S(X, hZ)Y - S(Y, Z)hX + S(X, Z)hY
- (k + 2)\{g(X, hZ)X - g(Y, hZ)Y + g(Y, Z)hX - g(X, Z)hY\}
+(\mu - 1)\{g(hX, hZ)X - g(hY, hZ)X + g(hY, Z)hX - g(hX, Z)hY\}
+(k + 2 - 2nk)\{\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX\}\} = 0.
\]
Yıldız and De [18] proved that
\[
R(X, Y)hZ - hR(X, Y)Z = \mu(k + 1)\{g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi
+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\}
+k\{g(hY, Z)\eta(X)\xi - g(hX, Z)\eta(Y)\xi
+ \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX
+ g(\phi Y, Z)\phi hX - g(\phi X, Z)\phi hY\}
+(\mu + k)\{g(\phi hX, Z)\phi Y - g(\phi hY, Z)\phi X\}
+ 2\mu g(\phi X, Y)\phi hZ.
\]
Again using (4.5) in (4.4), we get

\[
\begin{align*}
\mu(k + 1) & \{ g(Y, Z)\eta(Y)\xi - g(X, Z)\eta(Y)\eta(Y)\eta(Z)Y - \eta(Y)\eta(Z)X \} \\
+ k & \{ g(hY, Z)\eta(Y)\xi - g(hX, Z)\eta(Y)\eta(Y)hY \} \\
- \eta(Y)\eta(Z)hX - g(\phi Y, Z)\phi X + g(\phi X, Z)h\phi Y \\
- (\mu + k) & \{ g(h\phi X, Z)\phi Y - g(h\phi Y, Z)\phi X \} \\
- 2\mu g(\phi X, Y)h\phi Z - g(X, \phi Z)\phi Y + g(Y, \phi Z)\phi X \\
- (k + 1) & [ g(Y, \phi Z)\phi X - g(X, \phi Z)\phi Y - g(X, \phi Z)h\phi Y] \\
+ g(Y, \phi Z)\phi X - g(X, hZ)\eta(Y)\xi + g(Y, hZ)\eta(Y)\eta(X)\xi \\
+ g(Y, h\phi Z)h\phi X - g(X, h\phi Z)h\phi Y \\
+ (\mu - 1)(k + 1) & \{ g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(Y)\eta(X)\xi \} \\
- g(X, \phi Z)h\phi Y + g(Y, \phi Z)h\phi X - g(hY, \phi Z)h\phi X + g(hX, \phi Z)h\phi Y \\
+ (k + 1) & [ g(Y, \phi Z)\phi X - g(X, \phi Z)\phi Y + g(hX, \phi Z)\phi Y - g(hY, \phi Z)\phi X] \\
- k & \{ \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX \} - \mu(k + 1) \{ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \} \\
- \frac{1}{2n} & \{ S(Y, hZ)X - S(X, hZ)Y - S(Y, Z)hX + S(X, Z)hY \} \\
- (k + 2) & [ g(Y, hZ)X - g(X, hZ)Y + g(X, hZ)hY - g(Y, Z)hX] \\
+ (\mu - 1)(k + 1) & [ g(X, Z)Y - \eta(X)\eta(Z)Y - g(Y, Z)X + \eta(Y)\eta(Z)X] \\
- (k + 2 - 2nk) & [ \eta(Y)\eta(Z)hX - \eta(X)\eta(Z)hY] \\
+ (\mu - 1) & [ g(hY, Z)hX + g(hX, Z)hY] = 0,
\end{align*}
\]

which gives to

\[
\begin{align*}
\mu & \{ g(h\phi Y, Z)g(\phi Y, W) - g(h\phi X, Z)g(\phi Y, W) + 2(X, \phi Y)g(h\phi Z, W) \} \\
+ (k + 1) & \{ g(Y, Z)\eta(Y)\eta(W) - g(X, Z)\eta(Y)\eta(W) \} \\
+ g & \{ hX, Z\eta(Y)\eta(W) - g(hY, Z)\eta(X)\eta(W) \} \\
- \frac{1}{2n} & \{ S(Y, hZ)g(X, W) - S(X, hZ)g(Y, W) \} \\
+ S & \{ X, Zg(hY, W) - S(Y, Zg(hX, W) \}
\end{align*}
\]

(4.7) \[ - (k + 2)[ g(Y, hZ)g(X, W) - g(X, hZ)g(Y, W) \\
+ g(X, Z)g(hY, W) - g(Y, Z)g(hX, W) ] \\
- (\mu - 1)(k + 1) & [ g(Y, Z)g(X, W) - g(X, W)\eta(Y)\eta(Z) \\
+ g(Y, W)\eta(X)\eta(Z) - g(X, Z)g(Y, W) ] \\
- (k + 2 - 2nk) & [ g(hX, W)\eta(Y)\eta(Z) - g(hY, W)\eta(X)\eta(Z) ] \\
+ (\mu - 1) & [ g(hY, Z)g(hX, W) + g(hX, Z)g(hY, W) ] = 0.
\]

Putting \( X = W = e_i \) in (4.7), we get

\[
\begin{align*}
\mu(k + 1) & g(hZ, Y) + \mu(k + 1) \{ g(Y, Z) - \eta(Y)\eta(Z) \} - g(hY, Z) \\
- \frac{1}{2n} & \{ (2n + 1)[ S(hY, Z) - (k + 2)]g(hY, hZ) \}
\end{align*}
\]

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\[-(\mu - 1)(k + 1)(g(Y, Z) - \eta(Y)\eta(Z))] + (k + 2)g(Y, hZ) + 2(\mu - 1)(k + 1)[g(Y, Z) - \eta(Y)\eta(Z)] - (k + 2)g(hY, Z) \equiv 0.

Again putting \( Y = hY \) in (4.8) and using \( h^2 = (k + 1)\phi^2 \), we obtain

\[(k + 1)\{[2n\mu(k + 1) - 2n + (2n + 1)(k + 2)]g(Y, Z) - 2n\mu(k + 1) + 2n + (2n + 1)(k + 2) - (2n + 1)2nk\eta(Y)\eta(Z) + [2n\mu + (2n + 1)(\mu - 1) - 2(\mu - 1)]g(hY, Z) - (2n + 1)S(Y, Z)\} = 0.

As well known that

\[g(hY, Z) = \frac{1}{2(n - 1) + \mu}S(Y, Z) - \frac{2(1 - n) + n\mu}{2(n - 1) + \mu}g(Y, Z) - \frac{(2(1 - n) + n(2k - \mu))}{2(n - 1) + \mu}\eta(Y)\eta(Z).

Hence using (4.10) in (4.9), we get

\[(k + 1)\{[2n\mu(k + 1) - 2n + (2n + 1)(k + 2)]g(Y, Z) - 2n\mu(k + 1) + 2n + (2n + 1)(k + 2) - (2n + 1)2nk\eta(Y)\eta(Z) + [2n\mu + (2n + 1)(\mu - 1) - 2(\mu - 1)]g(hY, Z) - (2n + 1)S(Y, Z)\} = 0.

Hence one can write

\[S(Y, Z) = \frac{A_1}{A}g(Y, Z) + \frac{A_2}{A}\eta(Y)\eta(Z),

where

\[A_1 = 2n\mu(k + 1) - 2n + (2n + 1)(k + 2) - 2n\mu(k + 1) + 2n + (2n + 1)(k + 2) - (2n + 1)2nk\eta(Y)\eta(Z)]\frac{(2(1 - n) + n\mu)}{2(n - 1) + \mu},\]

\[A_2 = -2n\mu(k + 1) + 2n + (2n + 1)(k + 2) + 2n + (2n + 1)(\mu - 1) - 2(\mu - 1)]\frac{(2(1 - n) + n(2k - \mu))}{2(n - 1) + \mu},\]

\[A = 2n + 1 - 2n\mu + (2n + 1)(\mu - 1) - 2(\mu - 1)]\frac{1}{2(n - 1) + \mu}.

Therefore from (4.12) it follows that the manifold \( M \) is an \( \eta \)-Einstein manifold with respect to the Levi-Civita connection. Thus we have the following:
Theorem 4.1. Let $M$ be a $(2n + 1)$-dimensional $h$-projectively semisymmetric paracontact $(k, \mu)$-manifold $(k \neq -1)$ with respect to the Schouten-van Kampen connection. Then the manifold $M$ is an $\eta$-Einstein manifold with respect to the Levi-Civita connection provided $\mu \neq 2(1 - n)$.

Definition 4.2. A semi-Riemannian manifold $(M^{2n+1}, g)$, $n > 1$, is said to be $\phi$-projectively semisymmetric if

$$(4.13) \quad P(X, Y) \cdot \phi = 0 = 0,$$

holds on $M$ for all $X, Y \in \chi(M)$.

Let $M$ be a $\phi$-projectively semisymmetric paracontact metric $(k, \mu)$-manifold $(k \neq -1)$ with respect to the Schouten-van Kampen connection. Then above equation is equivalent to

$$(4.14) \quad \tilde{P}(X, Y)\phi Z - \phi \tilde{P}(X, Y)Z = 0,$$

for any $X, Y, Z, W \in \chi(M)$. Thus we have

$$(4.15) \quad \tilde{R}(X, Y)\phi Z - \phi \tilde{R}(X, Y)Z - \frac{1}{2n}\{S(Y, \phi Z)X - \tilde{S}(X, \phi Z)Y - \tilde{S}(Y, Z)\phi X + \tilde{S}(X, Z)\phi Y\} = 0,$$

Using (3.13) in (4.15), we get

$$R(X, Y)\phi Z - \phi R(X, Y)Z + g(X, Z)\phi Y - \eta(X)\eta(Z)\phi Y$$
$$-g(Y, Z)\phi X + \eta(Y)\eta(Z)\phi X + g(hY, Z)\phi X - g(hX, Z)\phi Y$$
$$+g(Y, Z)\phi hX - \eta(Y)\eta(Z)\phi hX - g(X, Z)\phi hY + \eta(X)\eta(Z)\phi hY$$
$$+g(hX, Z)\phi hY - g(hY, Z)\phi hX$$
$$+(k + 1)(g(X, \phi Z)\eta(Y)\xi - g(Y, \phi Z)\eta(X)\xi)$$
$$+(\mu - 1)(g(hX, \phi Z)\eta(Y)\xi - g(hY, \phi Z)\eta(X)\xi)$$
$$-g(X, \phi Z)Y + g(X, \phi Z)\eta(Y)\xi + g(Y, \phi Z)X - g(Y, \phi Z)\eta(X)\xi$$
$$(4.16) \quad -g(hY, \phi Z)X + g(hY, \phi Z)\eta(X)\xi + g(hX, \phi Z)Y - g(hX, \phi Z)\eta(Y)\xi$$
$$-g(hY, \phi Z)hX + g(X, \phi Z)hY - g(hX, \phi Z)hY + g(hY, \phi Z)hX$$
$$-k\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X - \mu\eta(X)\eta(Z)\phi hY - \eta(Y)\eta(Z)\phi hX$$
$$-\frac{1}{2n}\{S(Y, \phi Z)X - S(X, \phi Z)Y - S(Y, Z)\phi X + S(X, Z)\phi Y$$
$$-(k + 2)[g(Y, \phi Z)X - g(X, \phi Z)Y + g(X, Z)\phi Y - g(Y, Z)\phi X]$$
$$-\{(\mu - 1)[g(hY, \phi Z)X - g(hX, \phi Z)Y]$$
$$-(k + 2 - 2nk)[\eta(Y)\eta(Z)\phi X - \eta(Y)\eta(Z)\phi Y]$$
$$+(\mu - 1)[g(hY, Z)\phi X - g(hX, Z)\phi Y]\} = 0.$$. 
In [18], Yıldız and De proved that

\[
R(X, Y)\phi Z - \phi R(X, Y)Z = g(X, \phi Z)Y - g(Y, \phi Z)X + g(Y, Z)\phi X
- g(X, Z)\phi Y - g(X, \phi Z)hY + g(Y, \phi Z)hX
+ g(hY, \phi Z)X - g(hX, \phi Z)Y - g(Y, Z)\phi hX
\]

\[
+ g(X, Z)\phi hY - g(hY, Z)\phi X + g(hX, Z)\phi Y
\]

\[
+ \frac{-1 - \frac{\mu}{k + 1}}{k + 1} \left\{ g(hY, \phi Z)hX - g(hX, \phi Z)hY - g(hY, Z)\phi hX
\right. 
\]

\[
\left. + g(hX, Z)\phi hY \right\} - \frac{-k + \frac{\mu}{2}}{k + 1} \left\{ g(hX, Z)\phi Y - g(hY, Z)\phi hX
\right. 
\]

\[
\left. + g(hY, \phi Z)hX - g(hX, \phi Z)hY \right\}
\]

\[
+ (k + 1) \left\{ g(\phi X, Z)\eta(Y)\xi - g(\phi Y, Z)\eta(X)\xi
\right. 
\]

\[
\left. + \eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi hX \right\} + (\mu - 1) \left\{ g(\phi hX, Z)\eta(Y)\xi - g(\phi hY, Z)\eta(X)\xi
\right.
\]

\[
+ \eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi hX \right\}.
\]

Using (4.17) in (4.16), we obtain

\[
g(hY, Z)g(\phi hX, W) - g(hY, Z)g(\phi hX, W) + \eta(X)\eta(Z)g(\phi hY, W)
\]

\[
- \eta(Y)\eta(Z)g(\phi hX, W) + g(X, \phi Z)\eta(Y)\eta(W) - g(Y, \phi Z)\eta(X)\eta(W)
\]

\[
+ g(hY, \phi Z)g(hX, W) - g(hX, \phi Z)g(hY, W)
\]

\[
+ \frac{-1 - \frac{\mu}{k + 1}}{k + 1} \left\{ g(hY, \phi Z)g(hX, W) - g(hX, \phi Z)g(hY, W)
\right. 
\]

\[
\left. - g(hY, Z)g(\phi hX, W) + g(hX, Z)g(\phi hY, W) \right\}
\]

\[
- \frac{-k + \frac{\mu}{2}}{k + 1} \left\{ g(hX, Z)g(\phi hY, W) - g(hY, Z)g(\phi hX, W) \right\}
\]

\[
+ g(hY, \phi Z)g(hX, W) - g(hX, \phi Z)g(hY, W)
\]

\[
+ (\mu - 1) \left\{ g(\phi X, Z)\eta(Y)\xi - g(\phi Y, Z)\eta(X)\xi
\right.
\]

\[
- \frac{1}{2\mu} \left\{ S(Y, \phi Z)g(X, W) - S(X, \phi Z)g(Y, W) + S(X, Z)g(\phi Y, W)
\right. 
\]

\[
- S(Y, Z)g(\phi X, W) - (k + 2)\left\{ g(Y, \phi Z)g(X, W) - g(X, \phi Z)g(Y, W)
\right. 
\]

\[
+ g(X, Z)g(\phi Y, W) - g(Y, Z)g(\phi X, W) \right\} + (\mu - 1) \left\{ g(hY, Z)g(\phi X, W)
\right.
\]

\[
- g(hX, Z)g(\phi Y, W) + g(hY, \phi Z)g(X, W) - g(hX, \phi Z)g(hY, W)
\]

\[
- (k + 2 - 2nk)\left\{ \eta(Y)\eta(Z)g(\phi X, W) - \eta(X)\eta(Z)g(\phi Y, W) \right\} \right\} = 0,
\]

If we put \( Y = \phi Y \) in (4.18), we have

\[
g(h\phi Y, Z)g(hX, W) - g(hX, h\phi Z)g(h\phi^2 Y, W) - g(h\phi^2 Y, W)\eta(X)\eta(Z)
\]

\[
- g(\phi Y, \phi Z)\eta(X)\eta(W) + g(h\phi Y, \phi Z)g(hX, W) - g(X, h\phi Z)g(h\phi Y, W)
\]

\[
+ \frac{-1 - \frac{\mu}{k + 1}}{k + 1} \left\{ g(h\phi Y, \phi Z)g(hX, W) - g(X, h\phi Z)g(h\phi Y, W) \right\}
\]
\[
\begin{align*}
&+ g(h \phi Y, Z)g(hX, \phi W) - g(X, hZ)g(h\phi^2 Y, W) \\
&- \frac{k + \mu}{k + 1} \{-g(X, hZ)g(h\phi^2 Y, W) + g(h \phi Y, Z)g(hX, \phi W) \}
\end{align*}
\]

\[(4.19)\]

\[
\begin{align*}
&- g(h \phi \phi Y, Z)g(hX, W) - g(X, h \phi Z)g(h \phi Y, W) \\
&-(\mu - 1)g(h \phi Y, \phi Z)\eta(X)\eta(W) \\
&- \frac{1}{2n} \{S(\phi Y, \phi Z)g(X, W) - S(X, \phi Z)g(\phi Y, W) + S(X, Z)g(\phi^2 Y, W) \\
&- S(\phi Y, Z)g(\phi X, W) - (k + 2)[g(\phi Y, \phi Z)g(X, W) - g(X, \phi Z)g(\phi Y, W) \\
&+ g(X, Z)g(\phi^2 Y, W) - g(\phi Y, Z)g(\phi X, W)] + (\mu - 1)[g(h \phi Y, Z)g(\phi X, W) \\
&- g(h X, Z)g(\phi^2 Y, W) + g(h \phi Y, \phi Z)g(X, W) - g(h X, \phi Z)g(\phi Y, W)] \\
&+(k + 2 - 2nk)\eta(X)\eta(Z)g(h \phi^2 Y, W) \} = 0.
\end{align*}
\]

Putting \(X = W = e_i, \{i = 1, ..., 2n + 1\}\), in (4.19), we obtain

\[
\begin{align*}
S(Y, Z) &= \frac{2n}{2n - 1} \{[1 + 2k - \mu + \frac{(2n - 1)(k + 2)}{2n}]g(Y, Z) \\
&+ \{-1 - 2k + \mu + \frac{(2n - 1)(k + 2)}{2n} + (2n - 1)k\}\eta(Y)\eta(Z) \\
&- (\mu - 1)[1 + \frac{2n - 1}{2n}]g(hY, Z)\}.
\end{align*}
\]

Using (4.10) in (4.20), we obtain

\[
\begin{align*}
S(Y, Z) &= \frac{2n}{2n - 1} \{[1 + 2k - \mu + \frac{(2n - 1)(k + 2)}{2n}]g(Y, Z) \\
&- \{-1 - 2k + \mu + \frac{(2n - 1)(k + 2)}{2n} + (2n - 1)k\}\eta(Y)\eta(Z) \\
&- \{\mu - 1\}\{1 + \frac{2n - 1}{2n}\} \{\frac{1}{2(n - 1) + \mu} S(Y, Z) \\
&- \frac{(2(1 - n) + n\mu)}{2(n - 1) + \mu} g(Y, Z) - \frac{(2(n - 1) + n(2k - \mu))}{2(n - 1) + \mu} \eta(Y)\eta(Z)\} \},
\end{align*}
\]

which gives

\[
\begin{align*}
&\{1 + [(\mu - 1)(1 + \frac{2n - 1}{2n})][\frac{1}{2(n - 1) + \mu}]\}S(Y, Z) \\
&= \frac{2n}{2n - 1} \{[1 + 2k - \mu + \frac{(2n - 1)(k + 2)}{2n}] \\
&+ \{\mu - 1\}(1 + \frac{2n - 1}{2n}) \{\frac{(2(1 - n) + n\mu)}{2(n - 1) + \mu} g(Y, Z) \\
&- \frac{2n}{2n - 1} \{-1 - 2k + \mu + \frac{(2n - 1)(k + 2)}{2n} + (2n - 1)k\} \\
&+ \{\mu - 1\}(1 + \frac{2n - 1}{2n}) \frac{(2(n - 1) + n(2k - \mu))}{2(n - 1) + \mu} \eta(Y)\eta(Z)\}.
\end{align*}
\]
Hence one can write
\[(4.22) \quad S(Y, Z) = \frac{B_1}{B} g(Y, Z) + \frac{B_2}{B} \eta(Y) \eta(Z),\]
where
\[
B_1 = \frac{2n}{2n-1} \left\{1 + 2k - \mu + \frac{(2n-1)(k+2)}{2n} \right\}
+ \{\mu - 1\} \left(1 + \frac{2n-1}{2n}\right) \left\{\frac{2(1-n) + n\mu}{2(n-1) + \mu}\right\},
\]
\[
B_2 = -\frac{2n}{2n-1} \left\{-1 - 2k + \mu - \frac{(2n-1)(k+2)}{2n} + (2n-1)k\right\}
+ \{\mu - 1\} \left(1 + \frac{2n-1}{2n}\right) \left(\frac{2(n-1) + n(2k-\mu)}{2(n-1) + \mu}\right),
\]
\[
B = 1 + (\mu - 1) \left(1 + \frac{2n-1}{2n}\right) \frac{1}{2(n-1) + \mu}.
\]
Therefore from (4.22) it follows that the manifold \(M\) is an \(\eta\)-Einstein manifold with respect to the Levi-Civita connection. Thus we have the following:

**Theorem 4.2.** Let \(M\) be a \((2n+1)\)-dimensional \(\phi\)-projectively semisymmetric paracontact \((k,\mu)\)-manifold \((k \neq -1)\) with respect to the Schouten-van Kampen connection. Then the manifold \(M\) is an \(\eta\)-Einstein manifold with respect to the Levi-Civita connection provided \(\mu \neq 2(1-n)\).

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