Covariant Linear Response Theory of Relativistic QED Plasmas

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Abstract

We start from the QED Lagrangian to describe a charged many-particle system coupled to the radiation field. A covariant density matrix approach to kinetic theory of QED plasmas, subjected to a strong external electro-magnetic field has recently been developed [1,2]. We use the hyperplane formalism in order to perform a manifest covariant quantization and to implement initial correlations to the solution of the Liouville-von Neumann equation. A perturbative expansion in orders of the fine structure constant for the correlation functions as well as the statistical operator is applied. The non-equilibrium state of the system is given within generalized linear response theory. Expressions for the susceptibility tensor, describing the plasma response, are calculated within different approximations, like the RPA approximation or considering collisions within the Born-approximation. In particular, the process of relativistic inverse bremsstrahlung in a plasma is discussed.

Key words: relativistic kinetic theory; QED plasma; hyperplane formalism; inverse bremsstrahlung, relativistic linear response theory

1 Introduction

In recent years the theoretical study of dense relativistic plasmas is of increasing interest. Such plasmas are not only limited to astrophysics, but can nowadays be produced by high-intense short-pulse lasers [3–5]. In view of the

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inertial confinement fusion, one has to consider a plasma under extreme conditions which is created by a strong external field. This new experimental progress needs a systematic approach based on quantum electrodynamics and methods of non-equilibrium statistical mechanics.

Considerable attention has been focused on a kinetic approach, formulated for the fermionic Wigner function. Using the Wigner operator defined in four-dimensional momentum space [6–8], a manifestly covariant mean-field kinetic equation can be derived from the Heisenberg equations of motion for the field operators. In this approach, however, it is difficult to formulate an initial value problem for the kinetic equation since the four-dimensional Fourier transformation of the covariant Wigner function includes integration of two-point correlation functions over time. Describing modern pump-and-probe laser experiments by an initial value problem can lead to a significant simplification, since the pump and probe process can be described separately. However, at short time scales the plasma response depends on its initial correlation, which have to be included appropriately.

An initial value problem can be formulated within a one-time formulation, where the field operators are taken at the same time and only the spatial Fourier transformation is performed. In the context of QED, the one-time formulation was proposed by Bialynicki-Birula et al. [9] and used successfully in their study of the QED vacuum. Within this approach the one-time Wigner function has a direct physical interpretation and allows to calculate local observables, such as the charge density and the current density. The description in terms of one-time quantities is quite natural in kinetic theory based on the von Neumann equation for the statistical operator and provides a consistent account of causality in collision integrals.

It should be noted, however, that the one-time description does not contain the complete information about one-particle dynamics. Spectral properties of correlation functions are naturally described in terms of two-point Green’s functions which are closely related to the covariant Wigner function. In non-relativistic kinetic theory, where two-time correlation functions can, in principle, be reconstructed from the one-time Wigner function by solving integral equations which follow from the Dyson equation for non-equilibrium Green’s functions [10], this problem can be overcome. In a relativistic theory it was suggested [11] that an expansion in terms of energy-moments can recover the complete spectral information within a one-time formulation. The discussion of spectral properties will not be issued in this work. Recently the aspect of relativistic kinetic theory was studied within the mean-field approximation [11,12].

In this work we follow the studies presented in [1,2] where a density matrix approach to kinetic theory of QED plasmas subjected to a strong electromag-
netic field was considered. The BGR scheme [9] was generalized in two aspects. First, the one-time formalism was presented in a covariant form. This removes a drawback of the BGR theory which is not manifestly covariant. The covariant formulation will be performed using the hyperplane formalism. Second, we demonstrate how the non-equilibrium statistical operator can be expressed within the generalized linear response theory in a hyperplane formalism. The linear response theory was successfully applied in many fields of modern physics and can serve as tool to investigate the rather complicated structure of general kinetic equations. Here we are interested in the response of the plasma to an external perturbation, caused by the laser pulse. Considering the external field and the response of the system as a small perturbation, the non-equilibrium statistical operator is expanded up to linear order with respect to the external field. It is clear, that this approximation is only reliable for moderate laser intensities, but will break down for strong fields. The method applied here was successfully used in non-relativistic theory [13,14].

Having determined the non-equilibrium statistical operator, response functions like the susceptibility tensor is given. As an illustration, the relativistic susceptibility tensor is derived in the mean-field approximation. In the present work we also give first results beyond the RPA where interaction processes between the charged particles and the EM field are considered. The susceptibility tensor, given by the current-current correlation function, is most appropriately expressed in terms of the force-force correlation function, which allows for a well defined perturbation expansion with respect to the interaction. It is shown how the absorption coefficient is related to the imaginary part of the force-force correlation function in lowest order perturbation theory. In this approximation the absorption of an external field is treated. As well known, this first order process is possible only in combination with the Coulomb interaction in order to obey conservation laws. In Born approximation, considered here, the inverse bremsstrahlung is obtained. We give explicit expressions for plasmas near the equilibrium. A final expression for the inverse bremsstrahlung can be compared with other results, like relativistic S-matrix calculations in vacuum [15] or with non-relativistic results [16].

The paper is organized as follows. In Section 2 we demonstrate how the quantum Hamiltonian defined on a hyperplane can be obtained from the classical QED Lagrangian. Major points like the gauge-fixing or the canonical quantization on the hyperplane is reviewed. Further the relativistic Liouville-von Neumann equation is formulated on the plane. In Section 3 it is shown how the Liouville-von Neumann equation can be solved in linear response, fixing some initial distribution. The self-consistency relation leads to a response equation, which defines the four dimensional susceptibility tensor. The result for the susceptibility tensor in RPA is presented. In section 4 the inclusion of collisions is discussed. The electron-ion collisions are considered in lowest order perturbation theory. In this approximation the absorption coefficient for inverse
bremsstrahlung is related to the imaginary part of the force-force correlation function. Finally Section 5 concludes with a discussion of the results and gives a short outlook.

2 Canonical Description in the Hyperplane Formalism

In this section we demonstrate the derivation of a quantum Hamiltonian starting from the classical QED Lagrangian. The main issues like gauge fixing, or canonical quantization on the hyperplane are discussed to some extend.

2.1 The Lagrangian Formulation of the Plasma

We consider a charge neutral system consisting of two species of fermions, like for instance electrons (mass $m$, charge $Ze$, $Z = -1$) and ions (mass $m_i$, charge $Z_ie$). In particular, we consider protons ($Z_i = 1$), but the generalization to arbitrary charged particles is straightforward. The standard covariant formulation of a QED plasma is given in terms of the Lagrangian $L$

$$L'(x) = L_D(x) + L_{EM}(x) + L_{int}(x) \quad (2.1)$$

where $L_D(x)$ is the Dirac part, describing the fermionic components of the plasma (e.g. electrons and protons), $L_{EM}(x)$ is the electro-magnetic part and $L_{int}(x)$ describes the interaction in the plasma. In standard notation [17,18] we can write the different terms as

$$L_D(x) = \sum_{c=e,i} \bar{\psi}^c(x) \left( \frac{i}{2} \gamma^\mu \overset{\rightarrow}{\partial}_\mu - m \right) \psi^c(x), \quad (2.2)$$

$$L_{EM}(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x), \quad (2.3)$$

$$L_{int}(x) = - \sum_{c=e,i} Z_c \epsilon^c j^c_\mu(x) A^\mu(x), \quad (2.4)$$

where $\overset{\rightarrow}{\partial}_\mu = \partial_\mu - \partial^\mu$. In the following the electro-magnetic field tensor is taken in the form $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The current density four-vector will be expressed as $j^c_\mu = \psi^c \gamma_\mu \psi^c$ with $e < 0$.

Additionally to the terms given in Eq. (2.5) we consider the influence of an external field $A^\mu_{ext}(x)$ on the system. This external field is not a dynamical field variable, but is some given function of space and time. This implies that
there is no back reaction mechanism of the system onto $A^\mu_{\text{ext}}(x)$. The external field couples to the fermion current and therefore the complete Lagrangian takes the form

$$\mathcal{L}(x) = \mathcal{L}'(x) + \mathcal{L}_{\text{ext}}(x),$$

(2.5)

$$\mathcal{L}_{\text{ext}}(x) = -\sum_{c=e,i} Z_c e j^c_\mu(x) A^\mu_{\text{ext}}(x).$$

(2.6)

2.2 Gauge fixing

The Lagrangian given in Eq. (2.5) is covariant and gauge invariant. However, as well known, the gauge invariance leads to non-physical degrees of freedom, which have to be eliminated from the description. One possible way to perform this elimination is to apply an additional gauge condition. We will follow this way and use the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, which is the most natural choice for Coulomb systems, since the Coulomb interaction appears naturally in this description.

The disadvantage of this method, however, is that the explicit covariance is lost. In particular, after application of some further approximations it is hard to control, if the final result will be expressible in covariant form. For this reason it is most advantageous to use the hyperplane formalism [19], which singles out time-like and space-like parts in a covariant way.

Beside the natural appearance of the instantaneous Coulomb interaction due to the breaking of the explicit covariance by the Coulomb gauge constraint, there is a second advantage for this method. In view of laser plasma interactions in general we have to deal with a highly correlated plasma, which is subject to the external laser field. In order to describe the correlated plasma initially, the time coordinate has to be singled out, which leads to the breaking of covariance. As already mentioned above, in the hyperplane formalism the inclusion of initial correlation as well as the gauge fixing leading to the Coulomb interaction can be performed in a manifest covariant manner. The main issues of this formalism will be discussed in the next section.

2.3 Introduction of space-like hyperplanes

A space-like hyperplane $\sigma \equiv \sigma_{n,\tau}$ in Minkowski space can be characterized by a unit time-like normal vector $n^\mu$ and a scalar parameter $\tau$ which may be
interpreted as an “invariant time”. The equation of the hyperplane $\sigma_{n,\tau}$ reads

$$x \cdot n = \tau, \quad n^2 = n^\mu n_\mu = 1. \quad (2.7)$$

In the special Lorentz frame where $n^\mu = (1, 0, 0, 0)$ and consequently Eq. (2.7) reads $x^0 = \tau$ the parameter $\tau$ coincides with the time variable $t = x^0$. We will refer to this special frame as the “instant frame”, since only here observables are measured at the same instant of time $t$. Expressing the field variables as functionals of the hyperplane $\psi[\sigma_{n,\tau}]$ or as functions of $n$ and $\tau$, The Lagrangian can be expressed on the hyperplane $\sigma_{n,\tau}$. The gauge condition is expressed on the plane according to

$$\nabla_\mu A^\mu_\perp = 0, \quad (2.8)$$

where the following decomposition of four vectors is used

$$V^\mu = n^\mu V_\parallel + V_\perp^\mu, \quad V_\parallel = n_\nu V^\nu, \quad V_\perp^\mu = \Delta^\mu_\nu V^\nu, \quad (2.9)$$

$$\partial_\mu = n_\mu \frac{\partial}{\partial \tau} + \nabla_\mu, \quad \nabla_\mu = \Delta_\mu^\nu \partial_\nu = \Delta_\mu^\nu \frac{\partial}{\partial x_\nu^\perp}, \quad (2.10)$$

with the transverse projector $\Delta^\mu_\nu$

$$\Delta^\mu_\nu = \delta^\mu_\nu - n^\mu n_\nu. \quad (2.11)$$

Applying the decomposition (2.9) – (2.10) to the Lagrangian (2.5), we obtain

$$\mathcal{L} = -\frac{1}{4} F_\perp^\mu_\nu F^\mu_\nu_\perp - \frac{1}{2} \left( \nabla^\mu A_\parallel - \dot{A}_\perp^\mu \right) \left( \nabla_\mu A_\parallel - \dot{A}_\perp^\mu \right)$$

$$- \sum_c Z_c e \left( j^c_\parallel A_\parallel - j^c_\perp \right) A^\mu_\perp + \mathcal{L}_D + \mathcal{L}_{ext} , \quad (2.12)$$

where we have introduced the notation

$$F_\perp^\mu_\nu = \nabla^\mu A^\nu_\perp - \nabla^\nu A^\mu_\perp. \quad (2.13)$$

Due to the gauge condition (2.8) we find a constraint equation, similar to the Poisson equation

$$\nabla_\mu \nabla^\mu A_\parallel = \sum_c Z_c e j^c_\parallel. \quad (2.14)$$
The solution of Eq. (2.14) is

$$A_\parallel(\tau,x_\perp) = \sum_c Z_c e \int_{\sigma_n} d\sigma' \ G(x_\perp - x'_\perp) \ j_c^c(\tau,x'_\perp) ,$$

(2.15)

where the Green function $G(x_\perp)$ satisfies the equation

$$\nabla_\mu \nabla^\mu G(x_\perp) = \delta^3(x_\perp)$$

(2.16)

and the three-dimensional delta function on a hyperplane $\sigma_n$ defined as

$$\delta^3(x_\perp) = \int \frac{d^4 p}{(2\pi)^3} e^{-ip \cdot x} \delta(p \cdot n) .$$

(2.17)

The solution of Eq. (2.16) for $G(x_\perp)$ is given by

$$G(x_\perp) = -\int \frac{d^4 p}{(2\pi)^3} e^{-ip \cdot x} \delta(p \cdot n) \frac{1}{p^2_\perp} .$$

(2.18)

The variable $A_\parallel$ can now be eliminated from the Lagrangian density (2.12) using Eq. (2.15) imposing appropriate boundary conditions. Then a straightforward algebra leads to

$$\mathcal{L} = -\frac{1}{4} F_{\perp\mu\nu} F_{\perp}^{\mu\nu} - \frac{1}{2} \dot{A}_\perp - \sum_c Z_c e j_c^c A_\perp$$

$$+ \mathcal{L}_D + \mathcal{L}_{\text{ext}} - \frac{\epsilon^2}{2} \sum_{c,c'} Z_c Z_{c'} \int_{\sigma_n} d\sigma' \ j_c^c(\tau,x_\perp) G(x_\perp - x'_\perp) j_{c'}(\tau,x'_\perp) .$$

(2.19)

2.4 The commutation and anti-commutation relations

In the canonical quantization scheme, which will be applied here, the commutation and anti-commutation relations of the canonical field operators have to be derived. We follow the method, originated by Dirac [20,21]. The momentum $\Pi_{\perp\mu}$ canonical to the electro-magnetic field variable $A_{\perp\mu}$ is defined as

$$\Pi_{\perp\mu} = \frac{\partial \mathcal{L}}{\partial \dot{A}_\perp^\mu} = -\dot{A}_{\perp\mu} ,$$

(2.20)
with the \( \tau \)-derivative \( \dot{A}_{\perp} \). Similarly, for the fermionic field variables we define the canonical momenta \( \bar{\pi} \) and \( \pi \) according to

\[
\bar{\pi} \equiv \frac{\partial L_D}{\partial \dot{\bar{\psi}}} = \frac{i}{2} \bar{\psi} \gamma_\parallel, \quad \pi \equiv \frac{\partial L_D}{\partial \dot{\psi}} = -\frac{i}{2} \gamma_\parallel \psi,
\]

(2.21)

with the decomposition of Dirac’s \( \gamma \)-matrices

\[
\gamma^\mu = n^\mu \gamma_\parallel(n) + \gamma_\perp(n), \quad \gamma_\parallel(n) = n_\nu \gamma^\nu, \quad \gamma_\perp(n) = (\delta^\mu_\nu - n^\mu n_\nu) \gamma^\nu.
\]

(2.22)

The dynamical fields and its canonical momenta will now be interpreted as operators, satisfying commutator and anti-commutator relations. These relations can be obtained by calculating the Dirac brackets, which account for constraints, like the gauge-fixing constraint Eq. (2.8). In Appendix A we shortly review the calculation, leading to the following non-vanishing relations

\[
[\hat{A}_\perp^\mu(\tau,x_\perp), \hat{\Pi}_\perp^\nu(\tau,x'_\perp)] = i e^{\mu\nu}(x_\perp - x'_\perp),
\]

(2.23)

\[
[\hat{A}_\perp^\mu(\tau,x_\perp), \hat{A}_\perp^\nu(\tau,x'_\perp)] = [\hat{\Pi}_\perp^\mu(\tau,x_\perp), \hat{\Pi}_\perp^\nu(\tau,x'_\perp)] = 0,
\]

(2.24)

where

\[
e^{\mu\nu}(x_\perp - x'_\perp) = \int \frac{d^4 p}{(2\pi)^3} e^{-ip \cdot (x - x') } \delta(p \cdot n) \left[ \Delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right].
\]

(2.25)

For the Dirac field operators the anti-commutation relations on the hyperplane take the form

\[
\left\{ \hat{\psi}_{ac}(\tau,x_\perp), \hat{\bar{\psi}}_{a'c'}(\tau,x'_\perp) \right\} = \left[ \gamma_\parallel(n) \right]_{aa'} \delta_{c,c'} \delta^3(x_\perp - x'_\perp),
\]

(2.26)

\[
\left\{ \hat{\psi}_{ac}(\tau,x_\perp), \hat{\psi}_{a'c'}(\tau,x'_\perp) \right\} = \left\{ \hat{\psi}_{ac}(\tau,x_\perp), \hat{\bar{\psi}}_{a'c'}(\tau,x'_\perp) \right\} = 0,
\]

(2.27)

where \( a, a' \) are the spinor indices and \( c, c' \) denote the different species. In the special Lorentz frame where \( x^\mu = (t, \mathbf{r}) \) and \( n^\mu = (1, 0, 0, 0) \), we have \( \gamma_\parallel = \gamma^0 \) and \( \delta^3(x_\perp - x'_\perp) = \delta(r - r') \), so that Eq. (2.26) reduces to the well-known anticommutation relation for the quantized Dirac field.
2.5 The Hamiltonian

The quantum Hamiltonian can be constructed by a Legendre transformation, defined on the hyperplane as

$$H(n) = \int d\sigma \left\{ \Pi_{\perp \mu} \dot{A}_\perp^\mu + \pi \dot{\psi} + \dot{\bar{\psi}} \bar{\psi} - \mathcal{L} \right\}, \quad (2.28)$$

where $\mathcal{L}$ is given by Eq. (2.19). The Hamiltonian can be written in the form

$$\hat{H}_\tau(n) = \hat{H}_D(n) + \hat{H}_EM(n) + \hat{H}_{\text{int}}(n) + \hat{H}_{\text{ext}}^\tau(n), \quad (2.29)$$

where $\hat{H}_D(n)$ and $\hat{H}_EM(n)$ are the Hamiltonians for free fermions and the polarization EM field respectively, $\hat{H}_{\text{int}}(n)$ is the interaction term, and $\hat{H}_{\text{ext}}^\tau(n)$ describes the external EM field effects. In the Schrödinger picture the explicit expressions for these terms are

$$\hat{H}_D(n) = \sum_c \int d\sigma \hat{\psi}_c \left( -\frac{i}{2} \gamma_\perp^\mu(n) \dot{\nabla}_\mu + m_c \right) \hat{\psi}_c, \quad (2.30)$$

$$\hat{H}_EM(n) = \int d\sigma \left( \frac{1}{4} \hat{F}_{\perp \mu \nu} \hat{F}^{\mu \nu}_\perp - \frac{1}{2} \hat{\Pi}_{\perp \mu} \hat{\Pi}^\mu_\perp \right), \quad (2.31)$$

$$\hat{H}_{\text{int}}(n) = \frac{e^2}{2} \sum_{cc'} Z_c Z_{c'} \int d\sigma \int d\sigma' \int d\sigma' \int d\sigma' \int d\sigma' \int d\sigma' \int d\sigma' \int d\sigma' \hat{j}_\perp^c(x_\perp) G(x_\perp - x'_\perp) \hat{j}_\perp^{c'}(x'_\perp)$$

$$+ \sum_c \int d\sigma Z_c e j_{\mu}^c \hat{A}_\perp^\mu, \quad (2.32)$$

$$\hat{H}_{\text{ext}}^\tau(n) = \sum_c \int d\sigma Z_c e j_{\mu}^c(x_\perp) A_{\text{ext}}^\mu(\tau, x_\perp). \quad (2.33)$$

The field strength tensor $\hat{F}_{\perp \mu \nu}$ and the transverse field operators $\hat{A}_\perp^\mu$ and $\hat{\Pi}_\perp^\mu$ in Eq. (2.31) are defined according to their classical relations (2.9) and (2.13). The longitudinal part $\hat{A}_\parallel$ has been eliminated in the interaction Hamiltonian (2.32) by the operator version of Eq. (2.14). In Eqs. (2.30) – (2.33) normal ordering in operators is implied. The self-energy contribution to the last term in Eq. (2.32) is omitted, so that the product $:j_\perp^c(x_\perp) :j_\perp^{c'}(x'_\perp)$: is understood. The generalization of the Hamiltonian to a many-component case is obvious.

It should be mentioned, that the same expressions for the Hamiltonian Eq.(2.30) – (2.33) can be obtained from the symmetrized energy-momentum tensor, the
so called Belinfante tensor $T_{\mu \nu}$ [22] via

$$H(n) = P_{\mu} n^\mu \equiv \int d\sigma n'' T_{\mu \nu} n'' \,.$$ 

(2.34)

2.6 The relativistic von Neumann equation

As already noted, the state of the system $|\Psi[\sigma_{n,\tau}]\rangle$ is taken as a functional of $\sigma_{n,\tau}$. A specific frame of reference can now be related to a family of space-like hyperplanes with a fixed normal vector $n$. The relation between different frames of reference is given by a homogeneous Lorentz transformation $\Lambda$

$$\sigma_{n,\tau} \rightarrow \sigma'_{n',\tau} = \Lambda \sigma_{n,\tau} : \, x \rightarrow x' = \Lambda x \,.$$ 

(2.35)

Eq. (2.35) means that $x_{\mu} n^\mu = x'_{\mu} n'^\mu = \tau$, i.e. $x$ is located at $\sigma$ and $x'$ at $\sigma'$.

With a unitary representation of the homogeneous Lorentz group $U(\Lambda)$ state vectors on different planes are related by [23]

$$U(\Lambda) |\Psi[\Lambda \sigma]\rangle = |\Psi[\sigma]\rangle \,.$$ 

(2.36)

Having $n_{\mu}$ fixed, the evolution of the state in this frame of reference is governed by a representation $U(a)$ of time-like translations $a_{\mu}$ in the direction of the normal $n_{\mu}$. The generator of this transformation is $\hat{P}_{\mu}$, the energy-momentum vector. $U(a)$ is given by

$$U(a) = \exp \left\{ i \hat{P}_{\mu} a^\mu \right\} \,.$$ 

(2.37)

Writing the states as functions of $n$ and $\tau$, we find for an infinitesimal time-like translation $a^\mu = n^\mu \delta \tau$

$$|\Psi(n, \tau + \delta \tau)\rangle + i \delta \tau \left( \hat{P}_{\mu} n^\mu \right) |\Psi(n, \tau)\rangle = |\Psi(n, \tau)\rangle \,,$$ 

(2.38)

from which we obtain the relativistic Schrödinger equation

$$i \frac{\partial}{\partial \tau} |\Psi(n, \tau)\rangle = \hat{H}(n) |\Psi(n, \tau)\rangle \,$$ 

(2.39)

with the Hamiltonian on the hyperplane given by

$$\hat{H}(n) = \hat{P}_{\mu} n^\mu \,.$$ 

(2.40)
In the presence of a prescribed external field, the energy-momentum vector and, consequently, the Hamiltonian $\hat{H}(n)$ can depend explicitly on $\tau$. Combining Eq. (2.39) with the adjoint equation for the bra-vector, one finds that the statistical operator $\rho(n, \tau)$ for a mixed quantum ensemble obeys the equation

$$\frac{\partial \rho(n, \tau)}{\partial \tau} - i \left[ \rho(n, \tau), \hat{H}(n) \right] = 0 ,$$

(2.41)

which is analogous to the non-relativistic von Neumann equation.

In order to solve Eq. (2.41), some boundary condition have to be imposed on the statistical operator. The standard boundary condition in kinetic theory is Bogoliubov’s boundary condition of weakening of initial correlations which implies the uncoupling of all correlation functions to one-particle density matrices in the distant past, i.e., for $\tau \to -\infty$. In the scheme developed by Zubarev (see, e.g., [14]), such boundary conditions can be included by using instead of Eq. (2.41) the equation with an infinitesimally small source term

$$\frac{\partial \rho(n, \tau)}{\partial \tau} - i \left[ \rho(n, \tau), \hat{H}(n) \right] = -\eta \left\{ \rho(n, \tau) - \rho_{\text{rel}}(n, \tau) \right\} ,$$

(2.42)

where $\eta \to +0$ after the calculation of averages. Here $\rho_{\text{rel}}(n, \tau)$ is the so-called relevant statistical operator which describes a Gibbs state for some given non-equilibrium state variables. In QED kinetics these variables are the Wigner function and the photon density matrix. In general we will call these non-equilibrium state variables the relevant operators $\hat{B}_\mu^\ell$. In the next section we will search for solutions of Eq. (2.42) up to terms linear in $\hat{B}_\mu^\ell$.

### 3 Linear Response Theory

Using the results of the last section, it is possible to construct a kinetic theory on the hyperplanes, which was extensively discussed in [1].

Here we show how the non-equilibrium statistical operator for the case of small deviations from the equilibrium distribution can be constructed. This can be done by solving Eq. (2.42) with a appropriate choice for the relevant distribution. This treatment, known as generalized linear response, was successfully applied in different studies [13].

Further for simplicity we consider the plasma in the adiabatic approximation, where the dynamics of the positively charged component of the plasma is frozen.
3.1 Fluctuations near equilibrium

For a given relevant distribution $\varrho_{\text{rel}}$, a formal solution of the Zubarev equation (2.42) is given by

$$
\varrho(n, \tau) = \eta \int_{-\infty}^{\tau} d\tau' e^{-\varepsilon(\tau-\tau')} U(\tau, \tau') \varrho_{\text{rel}}(n, \tau') U^\dagger(\tau, \tau'),
$$

where the evolution operator can be written as the ordered exponent

$$
U(\tau, \tau') = T_\tau \exp \left\{ -i \int_{\tau'}^\tau \hat{H}(n) d\tau \right\}.
$$

After partial integration, the expression (3.1) becomes

$$
\varrho(n, \tau) = \varrho_{\text{rel}}(n, \tau) + \Delta \varrho(n, \tau),
$$

$$
\Delta \varrho(n, \tau) = -\int_{-\infty}^{\tau} d\tau' e^{-\eta(\tau-\tau')}
\times U(\tau, \tau') \left\{ \frac{\partial \varrho_{\text{rel}}(n, \tau')}{\partial \tau'} - i \left[ \varrho_{\text{rel}}(n, \tau'), \hat{H}(n) \right] \right\} U^\dagger(\tau, \tau').
$$

The Hamiltonian $\hat{H}$, as defined in the Eqs. (2.29) – (2.33), is rewritten in the adiabatic approximation in decomposed form according to

$$
\hat{H}(n) = \hat{H}_s(n) + \hat{H}_{\text{ext}}(n),
$$

$$
\hat{H}_s(n) = \hat{H}_D(n) \bigg|_{c=e} + \hat{H}_{EM}(n) + \hat{H}_{\text{rad}}(n) \bigg|_{c=e} + \hat{H}_{\text{int}}(n).
$$

In (3.5) – (3.6) the system part of the Hamiltonian $\hat{H}_s$ is decomposed into the kinetic parts $\hat{H}_D$ and $H_{EM}$ and some part describing the collisions $\hat{H}_{\text{int}}$ and $\hat{H}_{\text{rad}}$, which will be treated within perturbation theory. The different parts of the Hamiltonian are taken from Eqs. (2.29) – (2.33)

$$
\hat{H}_D(n) = \int d\sigma \frac{\hat{\psi}_e}{\sigma_n} \left( -\frac{i}{2} \gamma_\perp(n) \hat{\nabla}_\mu + m_e \right) \hat{\psi}_e,
$$
\[ \hat{H}_{EM}(n) = \int d\sigma \left( \frac{1}{4} \hat{F}_{\perp \mu \nu} \hat{F}^{\mu \nu} - \frac{1}{2} \hat{\Pi}_{\perp \mu} \hat{\Pi}^{\mu} \right), \tag{3.8} \]

\[ \hat{H}_{rad}(n) = -e \int d\sigma \hat{j}_{\perp \mu}(x_\perp) \hat{A}^{\mu}_\perp(x_\perp), \tag{3.9} \]

\[ \hat{H}_{int}(n) = -e \int d\sigma \left( \hat{j}_{\parallel \mu}(x_\perp) A^{\text{ion}}_{\parallel \mu}(x_\perp) + \hat{\sigma}_{\perp \mu}(x_\perp) A^{\text{ion}}_{\parallel \mu}(x_\perp) \right), \tag{3.10} \]

\[ \hat{H}_{\tau \text{ext}}(n) = -e \int d\sigma \hat{j}_{\mu}(x_\perp) A^{\mu}_{\text{ext}}(\tau, x_\perp). \tag{3.11} \]

In the following we will drop the index “e” and the electron spinors are denoted by \( \hat{\psi} \) and \( \hat{\bar{\psi}} \).

The radiation term, Eq. (3.9), is important for the description of photon emission from the plasma or for photon scattering in the plasma. As an application we will focus to the absorption of a classical electro-magnetic wave by a relativistic plasma in the next section, where the radiation term can be neglected.

In what follows, the non-equilibrium state of the system must be specified. We use Zubarev’s method [14] of a non-equilibrium statistical ensemble in linear response. The so called relevant statistical operator \( \varrho_{\text{rel}}(n, \tau) \) describes a generalized Gibbs distribution, which characterizes the initial non-equilibrium state of our system. In the linear response regime we will only consider small fluctuations from the equilibrium. The relevant statistical operator can be written in the form

\[ \varrho_{\text{rel}}(n, \tau) = Z_{\text{rel}}^{-1}(\beta, \nu, \phi_{\mu}^{\ell}; \tau) \exp \left\{ -\beta \left[ n^{\mu} \hat{P}_{\mu} - \nu \hat{Q} \right. \right. \]

\[ \left. \left. - \int d\sigma \sum_{\ell} \phi_{\mu}^{\ell}(x_\perp; \tau) \hat{B}_{\ell}^{\mu}(x_\perp) \right] \right\}, \tag{3.12} \]

where the relevant observables \( \hat{B}_{\ell}^{\mu}(x_\perp) \) define the non-equilibrium state and will be treated as small quantities. It should be mentioned, that the \( \tau \)-dependence is carried completely by the set of Lagrange multipliers \( \phi_{\mu}^{\ell}(x_\perp; \tau) \). The first two terms in the exponential of Eq. (3.12) describe the generalization of the Gibbsian distribution \( \varrho_0(n) \) of the grand canonical ensemble

\[ \varrho_{0}(n) = Z_{0}^{-1}(\beta, \nu; n) \exp \left\{ -\beta \left[ n^{\mu} \hat{P}_{\mu} - \nu \hat{Q} \right] \right\}, \tag{3.13} \]

where
\[ n^\mu \hat{P}_\mu = \hat{H}_s, \quad \hat{Q} = n_\mu j^\mu. \]  

(3.14)

The equilibrium statistical operator does only depend on the additive integrals of motion \( \hat{P}_\mu, \hat{Q} \).

In the following we demonstrate how correlation functions can be derived in linear response within the hyperplane formalism. This means, that all expressions will be approximated to first order in quantities, describing the deviation from the equilibrium.

Expressing the relevant part of the statistical operator Eq. (3.12) in linear response, as demonstrated in Appendix B, we find in Fourier representation

\[ \varrho_{\text{rel}}(n, \tau) = \varrho_0(n) + \beta e^{-ik_\parallel \tau} \int_0^1 dz \sum_\ell \phi_\mu^\ell(k) \hat{B}_{\ell \parallel}^\mu(k_\perp, iz\beta) \varrho_0(n) . \]  

(3.15)

A similar calculation leads to the irrelevant part of the statistical operator in linear response as (see Appendix B for details)

\[ \Delta \varrho(n, \tau) = \beta e^{-ik_\parallel \tau} \int_0^\infty d\tilde{\tau} e^{ik_\parallel(\tilde{\tau}+i\eta)} \int_0^1 dz \left\{ \sum_\ell \left[ \hat{B}_{\ell \parallel}^\mu(-\tilde{\tau} + iz\beta) - ik_\parallel \hat{B}_{\ell \parallel}^{i\mu}(-\tilde{\tau} + iz\beta) \right] \phi_\mu^\ell(k) \\
+ \varrho_0(n) \right\} \varrho_0(n) . \]  

(3.16)

By definition, the mean values of the relevant observables are some prescribed functions of space and time and are obtained by averaging with the relevant statistical operator only. The self-consistency relations, by which the Lagrange multipliers are determined, can therefore be expressed as

\[ \text{Tr} \left\{ \hat{B}_\ell^{\mu}(x_\perp) \Delta \varrho(n, \tau) \right\} = 0 . \]  

(3.17)

Multiplying Eq. (3.16) by \( \hat{B}_\ell^{\nu} \), taking the trace and rearranging the indices we find the response equation

\[ - \left( \hat{B}_\ell^{\nu} \hat{J}_\nu \right)_{k_\parallel + i\eta} A_\text{ext}^{\nu} = \sum_{\ell'} \left( \hat{B}_\ell^{\nu} \left( \hat{B}_{\ell'}^{\mu} + ik_{\parallel} \hat{B}_{\ell'}^{\mu} \right) \right)_{k_\parallel + i\eta} \phi_\mu^{\ell'} . \]  

(3.18)

In Eq (3.18) the correlation functions \( \langle \hat{A} ; \hat{B} \rangle \) and its Laplace transforms \( \langle \hat{A} ; \hat{B} \rangle_\eta \) are defined according to

14
\[
\langle \hat{A}; \hat{B} \rangle = \int_0^1 dz \, \text{Tr} \left\{ \hat{A}( -i \beta z ) \hat{B}^\dagger \hat{g}_0 \right\} = \int_0^1 dz \, \text{Tr} \left\{ \hat{A} \hat{B}^\dagger( i \beta z ) \hat{g}_0 \right\}, \tag{3.19}
\]

\[
\langle \hat{A}; \hat{B} \rangle = \int_0^\infty d\tau \, e^{-i\beta\tau} \left( \hat{A}(\tau) ; \hat{B} \right) = \int_0^\infty d\tau \, e^{-i\beta\tau} \left( \hat{A}; \hat{B}( -\tau ) \right). \tag{3.20}
\]

Making use of Eqs. (3.15) and (3.16) local observables can be expressed in linear response, in particular the induced current \( j_{\text{ind}}^\mu = \delta \langle \hat{j}_\mu \rangle \), is written as

\[
\delta \langle \hat{O}_\mu \rangle_{k^\parallel} = \delta \langle \hat{O}_\mu \rangle_{k^\parallel} \exp\{ -ik_{\parallel}\tau \}
\]

\[
\delta \langle \hat{j}_\mu \rangle_{k^\parallel} = \beta \sum_{\ell} \left\{ \langle \hat{j}_\mu ; \hat{B}_\nu^\ell \rangle - \langle \hat{j}_\mu ; \hat{B}_\nu^\ell \rangle_{k^\parallel + i\eta} + i k_{\parallel} \langle \hat{j}_\mu ; \hat{B}_\nu^\ell \rangle_{k^\parallel + i\eta} \right\} \phi_\nu^\ell
\]

\[
- \beta \langle \hat{j}_\mu ; \hat{j}_\nu \rangle_{k^\parallel + i\eta} A_{\text{ext}}^\nu . \tag{3.21}
\]

Eq. (3.21) describes the response of the system due to weak pertubations \( A_{\text{ext}}^\mu \). The Lagrange multipliers can be eliminated from Eq. (3.21) by making use the self-consistency relation (3.18).

If the induced current can be represented by a linear combination of the relevant operators \( \hat{j}_\mu = \sum_{\ell} a_\ell \hat{B}_\nu^\ell \), Eq. (3.21) is simplified as

\[
\delta \langle \hat{j}_\mu \rangle_{k^\parallel} = \beta \sum_{\ell\ell'} a_\ell \langle \hat{B}_\mu^\ell ; \hat{B}_\nu^{\ell'} \rangle \phi_\nu^{\ell'} = \beta \sum_{\ell} \langle \hat{j}_\mu ; \hat{B}_\nu^\ell \rangle \phi_\nu^\ell , \tag{3.22}
\]

where the remaining terms in Eq. (3.21) are canceled due to the self-consistency relation (3.18). The Lagrange multipliers can be eliminated in Eq. (3.22) by multiplying Eq. (3.18) by \( a_\ell \) and summing over \( \ell \). Further partial integration of correlation functions

\[
iz \langle \hat{A}; \hat{B} \rangle_z + \langle \hat{A}; \hat{B} \rangle_z = \langle \hat{A}; \hat{B} \rangle_z = -\langle \hat{A}; \hat{B} \rangle_z \tag{3.23}
\]

can be applied and finally we obtain

\[
- \sum_{\ell} a_\ell \langle \hat{B}_\mu^\ell ; \hat{j}_\nu \rangle_{k^\parallel + i\eta} A_{\text{ext}}^\nu = \sum_{\ell\ell'} a_\ell \langle \hat{B}_\mu^\ell ; \hat{B}_\nu^{\ell'} \rangle \phi_\nu^{\ell'} . \tag{3.24}
\]

Now Eq. (3.24) can be plugged into Eq. (3.22) and we have the induced current expressed in terms of the external field within the linear response approximation.
\[ \delta \langle \hat{j}_\mu \rangle_{k||} = -\beta \langle \hat{j}_\mu ; \hat{j}_\nu \rangle_{k||+i\eta} A^\nu_{\text{ext}} \]

\[ = -\beta \{ i k || \langle \hat{j}_\mu ; \hat{j}_\nu \rangle_{k||+i\eta} + (\hat{j}_\mu ; \hat{j}_\nu) \} A^\nu_{\text{ext}}. \]

(3.25)

In the second line again partial integration [Eq. (3.23)] was applied.

The susceptibility tensor \( \chi_{\mu\nu} \), describing the response of the system to an external perturbation, is defined as

\[ j_{\mu}^{\text{ind}}(k) = \chi_{\mu\nu}(k) A^\nu_{\text{ext}}(k) \]

(3.27)

and can be read from the equations (3.25) and (3.26)

\[ \chi_{\mu\nu} = -\beta \langle \hat{j}_\mu ; \hat{j}_\nu \rangle_{k||+i\eta} = -\beta (\hat{j}_\mu ; \hat{j}_\nu) - i\beta k || \langle \hat{j}_\mu ; \hat{j}_\nu \rangle_{k||+i\eta}. \]

(3.28)

Eq. (3.28) gives the susceptibility tensor in terms of the current-current or the current-force correlation function. The major task to proceed is to evaluate these correlation functions within certain approximations. We will show how perturbation theory can be applied. The simplest evaluation of the correlation function is given in the RPA approximation, which will be considered in the next section. As a next step we demonstrate how collisions can be included.

Since we are working in the adiabatic approximation, we chose a reference frame in which the the ions are at rest. This means that it is most convenient to use the instant frame formulation \( (n_\mu = (1, 0, 0, 0)) \).

### 3.2 The RPA-result for the correlation function

The susceptibility tensor (3.28) can be most easily calculated in the RPAapproximation. In this approximation the force operator \( \hat{j}_\nu \) in Eq. (3.28) is calculated with the Dirac part in the Hamiltonian only.

It is convenient to perform the calculation using the plane wave expansion for the field operators \( \hat{\psi} \) and \( \hat{\bar{\psi}} \) according to

\[ \hat{\psi}(x) = \int \frac{d^3p}{(2\pi\hbar)^{3/2}} \sqrt{\frac{m}{E_p}} \sum_s \left( \hat{b}_{ps} u(p, s) e^{-i p \cdot x} + \hat{d}_{ps}^\dagger v(p, s) e^{i p \cdot x} \right), \]

(3.29)

\[ \hat{\bar{\psi}}(x) = \int \frac{d^3p'}{(2\pi\hbar)^{3/2}} \sqrt{\frac{m}{E_{p'}}} \sum_{s'} \left( \hat{d}_{p's'}^\dagger \bar{u}(p', s') e^{-i p' \cdot x} + \hat{b}_{p's'} \bar{v}(p', s') e^{i p' \cdot x} \right). \]

(3.30)
We use the four-dimensional scalar product and the mass on-shell condition

\[ x \cdot p = p_0 \cdot t - x \cdot p, \quad p_0 = E_p = +\sqrt{\mathbf{p}^2 + m^2}. \] (3.31)

The operators \( \hat{b}, \hat{b}^\dagger, \hat{d} \) and \( \hat{d}^\dagger \) are the electron creation and annihilation operators and the corresponding antiparticle creation and annihilation operators. The current operator can be expressed in terms of these operators

\[
\hat{j}_\mu(k) = e \int d^3x :\bar{\psi}(x)(\gamma_\mu)\psi(x) : e^{-ik \cdot x}
\]

\[
= e \int \frac{d^3p}{(2\pi\hbar)^3} \sqrt{\frac{m}{E_p}} \sum_{ss'} \left\{ \sqrt{\frac{m}{E_{p+\hbar k}}}:\hat{\bar{\psi}}_{p+\hbar k, s'} \hat{b}_{p, s'}^\dagger \hat{v}(p+\hbar k, s') \gamma_\mu \psi(p, s) e^{\mp i(E_p-E_{p+\hbar k})t} + \sqrt{\frac{m}{E_{p-\hbar k}}}:\hat{\bar{\psi}}_{p-\hbar k, s'} \hat{\bar{\psi}}_{p, s'} e^{\mp i(E_p-E_{p-\hbar k})t} \right. \\
+ \sqrt{\frac{m}{E_{p+\hbar k}}}:\hat{\bar{\psi}}_{p-\hbar k, s'} \hat{b}_{p, s'} e^{\mp i(E_p+E_{p-\hbar k})t} \\
+ \sqrt{\frac{m}{E_{p-\hbar k}}}:\hat{\bar{\psi}}_{p+\hbar k, s'} \hat{\bar{\psi}}_{p, s'} e^{\mp i(E_p+E_{p+\hbar k})t} \right\}. \] (3.32)

The result (3.32) was obtained after integration over \( x \) and \( p' \).

In a similar way the Dirac Hamiltonian is written in the plane-wave representation

\[
\hat{H}_D = \int d^3p \sum_s E_p(\hat{d}_{ps}^\dagger \hat{d}_{ps} + \hat{\bar{b}}_{ps}^\dagger \hat{b}_{ps}), \] (3.33)

where all products of operators are taken in normal order.

From the Heisenberg equation of motion and the current operator (3.32) the force operator is found

\[
\dot{\hat{j}}_\mu(k) = -\frac{ie}{\hbar} \int \frac{d^3p}{(2\pi\hbar)^3} \sqrt{\frac{m}{E_p}} \sum_{ss'} \left\{ \sqrt{\frac{m}{E_{p+\hbar k}}}:\hat{\bar{\psi}}_{p+\hbar k, s'} \hat{v}(p+\hbar k, s') \gamma_\mu \psi(p, s) e^{\mp i(E_p-E_{p+\hbar k})t} \right. \\
+ \left[ E_p - E_{p+\hbar k} \right] e^{\mp i(E_p-E_{p+\hbar k})t} \right\}.
\]
\begin{equation}
\begin{aligned}
+ & \sqrt{\frac{m}{E_{p-hk}}} \hat{b}_{p-hk,s} \hat{b}_{p,s} \left( \bar{u}(p - h k, s') \gamma_{\mu} u(p, s) \right) \\
& \left[ E_{p} - E_{p-hk} \right] \exp \left[ -\frac{i}{\hbar} (E_{p} - E_{p-hk}) t \right] \\
+ & \sqrt{\frac{m}{E_{p-hk}}} \hat{d}_{-p+hk,s'} \hat{b}_{p,s} \left( \bar{v}(-p + h k, s') \gamma_{\mu} u(p, s) \right) \\
& \left[ E_{p} + E_{p-hk} \right] \exp \left[ -\frac{i}{\hbar} (E_{p} + E_{p-hk}) t \right] \\
+ & \sqrt{\frac{m}{E_{p+hk}}} \hat{d}_{p-hk,s} \hat{b}_{-p-hk,s'} \left( \bar{u}(-p - h k, s') \gamma_{\mu} v(p, s) \right) \\
& \left[ E_{p} + E_{p+hk} \right] \exp \left[ \frac{i}{\hbar} (E_{p} + E_{p+hk}) t \right].
\end{aligned}
\end{equation}

Eq. (3.32) and (3.34) can now be used to calculate the correlation function \( \langle \hat{j}_{\mu}(k) : \hat{j}_{\nu}(k) \rangle_{\omega + i \eta} \). The calculation is shown in Appendix C. Together with the Eqs. (C.4) – (C.7) as well as Eq. (3.28) we have the 4-dimensional RPA susceptibility tensor in the form

\begin{equation}
\chi_{\mu\nu}(k, \omega) = \frac{e^2}{4} \int \frac{d^3p}{(2\pi \hbar)^3} \frac{1}{E_{p}} \left\{ \frac{-1}{E_{p+hk}} \frac{\tilde{f}(E_{p}) - \tilde{f}(E_{p+hk})}{\hbar \omega + E_{p} - E_{p+hk} + i\eta} \text{tr}_{D} \left\{ \gamma_{\mu} [\phi - m] \gamma_{\nu} [\phi + \hbar k - m] \right\} \\
+ \frac{1}{E_{p-hk}} \frac{f(E_{p}) - f(E_{p-hk})}{\hbar \omega - E_{p} - E_{p-hk} + i\eta} \text{tr}_{D} \left\{ \gamma_{\mu} [\phi + m] \gamma_{\nu} [\phi - \hbar k + m] \right\} \\
- \frac{1}{E_{p-hk}} \frac{1 - f(E_{p}) - \tilde{f}(E_{p-hk})}{\hbar \omega - E_{p} - E_{p-hk} + i\eta} \text{tr}_{D} \left\{ \gamma_{\mu} [\phi + m] \gamma_{\nu} [-\phi + \hbar k - m] \right\} \\
+ \frac{1}{E_{p+hk}} \frac{1 - \tilde{f}(E_{p}) - f(E_{p+hk})}{\hbar \omega + E_{p} + E_{p+hk} + i\eta} \text{tr}_{D} \left\{ \gamma_{\mu} [\phi - m] \gamma_{\nu} [-\phi - \hbar k + m] \right\} \right\}. \tag{3.35}
\end{equation}

The fermion and anti-fermion distribution functions, \( f \) and \( \tilde{f} \) respectively, are defined by Eq. (C.3). The equation (3.35) can be written in a more compact form, if we notice, that the susceptibility tensor can be decomposed as

\begin{equation}
\begin{aligned}
\chi_{00} &= -\frac{k_{i}}{k_{0}} \chi_{i0} = -\frac{k_{i}}{k_{0}} \chi_{0i} = -\frac{k_{i}}{k_{0}} \chi_{,i} = -\frac{k_{i}}{k_{0}} \chi_{\ell}, \\
\chi_{ij} &= \frac{k_{i}k_{j}}{k^{2}} \chi_{\ell} + \left( \delta_{ij} - \frac{k_{i}k_{j}}{k^{2}} \right) \chi_{\ell}, \\
\chi_{\ell} &= \frac{k_{i}k_{j}}{k^{2}} \chi_{ij}, \quad \chi_{\ell} = \frac{1}{2} \left[ \delta_{ij} - \frac{k_{j}k_{i}}{k^{2}} \right] \chi_{ij}.
\end{aligned}
\end{equation}

This means that the longitudinal and transverse component completely dete-
mine the susceptibility tensor. This decomposition can be shown, making use of the current conservation \( \omega \dot{j}_0 - \mathbf{k} \cdot \mathbf{j} = 0 \).

In order to give an expression for the \( \chi^\ell \) and \( \chi^t \) we calculate the Dirac traces of the spatial part \( \chi^{ij} \). For instance the first trace in Eq. (3.35) yields

\[
\text{tr}_D \{ \gamma_i [\dot{\gamma} + \mathbf{m}] \gamma_j [\dot{\gamma} + \mathbf{k}] - \frac{\mathbf{k} \cdot \hat{\mathbf{j}}}{\mathbf{k}^2} \}
= 4 \left\{ \dot{\gamma}_j (\dot{\gamma} + \mathbf{k}) \right\}_j + \delta_{ij} \left[ E_p E_{p+hk} - \mathbf{p}(\mathbf{p} + \mathbf{k}) - m^2 \right].
\]

(3.38)

Calculating the remaining traces accordingly and projecting the longitudinal and transverse part [see Eq. (3.37)] as well as shifting \( \mathbf{p} \rightarrow -\mathbf{p} \) in the first and fourth term in Eq. (3.35), we find

\[
\chi^{\ell,t}(k, \omega) = e^2 \int \frac{d^3 p}{(2\pi \hbar)^3} \left\{ \Lambda^{\ell,t}_- \left[ \frac{f(E_p) - f(E_{p-hk})}{\hbar \omega - E_p + E_{p-hk} + i\eta} - \frac{\bar{f}(E_p) - \bar{f}(E_{p-hk})}{\hbar \omega - E_p - E_{p-hk} + i\eta} \right] + \Lambda^{\ell,t}_+ \left[ \frac{1-f(E_p) - f(E_{p-hk})}{\hbar \omega + E_p + E_{p-hk} + i\eta} - \frac{1-f(E_p) - \bar{f}(E_{p-hk})}{\hbar \omega - E_p - E_{p-hk} + i\eta} \right] \right\},
\]

(3.39)

with the longitudinal and transverse projectors \( \Lambda^{\ell,t}_\pm \)

\[
\Lambda^{\ell}_\pm = 1 \pm \frac{E_p^2 + \hbar \mathbf{p} \cdot \mathbf{k} - 2 (\mathbf{p} \cdot \mathbf{k})^2}{E_p E_{p-hk}},
\]

(3.40)

\[
\Lambda^{t}_\pm = 1 \pm \frac{m^2 - \hbar \mathbf{p} \cdot \mathbf{k} + (\mathbf{p} \cdot \mathbf{k})^2}{E_p E_{p-hk}}.
\]

(3.41)

Observing, that \( \Lambda^{\ell,t}_\pm \) is invariant under the shift \( \mathbf{p} \rightarrow -\mathbf{p} + \hbar \mathbf{k} \), we can perform this shift in all terms containing \( f(E_{p-hk}) \) and \( \bar{f}(E_{p-hk}) \) in Eq. (3.39). We finally have the result for the longitudinal and transverse RPA susceptibility tensor

\[
\chi^{\ell,t}(k, \omega) = e^2 \int \frac{d^3 p}{(2\pi \hbar)^3} F(E_p) \left\{ \frac{E_p - E_{p-hk}}{(E_p - E_{p-hk})^2 - \omega^2 - i\eta} \Lambda^{\ell,t}_- \right. \\
+ \frac{E_p + E_{p-hk}}{(E_p + E_{p-hk})^2 - \omega^2 - i\eta} \Lambda^{\ell,t}_+ \left\} + \chi^{\ell,t}_{\text{vac}},
\]

(3.42)

\[
F(E_p) = 2 \left( f(E_p) + \bar{f}(E_p) \right),
\]

(3.43)
\[
\chi_{\nu t}^{\ell,t}(\mathbf{k},\omega) = -2e^2 \int \frac{d^3p}{(2\pi\hbar)^3} \frac{E_p + E_{p-hk}}{(E_p + E_{p-hk})^2 - \omega^2 - i\eta} \Lambda_{\nu t}^{\ell,t}.
\] (3.44)

It should be noted that the result (3.42) corresponds to familiar expressions, for example published by Tsytovich [24], given in the form of the dielectric tensor.

The major difficulties in the theory arise, by the inclusion of collisions, which will be the main issue of the next section.

3.3 Inclusion of collisions

As in the case of the RPA approximation, we restrict ourselves to the instant frame description, where the ions stay at rest \(n_\mu = (1,0,0,0)\). This also implies that we only have to consider the Coulomb interaction term of the electrons with the ions, which will be treated perturbatively. In the case of an arbitrary reference frame, where the ions are moving, we have \(n^\mu = (n^0, \mathbf{n}) = (\gamma, \mathbf{v}\gamma)\), where \(\gamma\) is the relativistic factor \(\gamma^{-1} = \sqrt{1 - v^2}\).

Collisions are included into the response function Eq. (3.28) most conveniently, by making use of partial integration and writing \(\chi_{\mu\nu}\) in terms of correlation functions including the force operator \(\dot{j}_\nu\) in the denominator. We can find the matrix equation

\[
\langle \dot{j}_\mu ; \dot{j}_\nu \rangle = -\frac{M_{\alpha\beta}}{|M_{\alpha\beta}|} \langle \dot{j}_\alpha ; \dot{j}_\nu \rangle + \frac{\langle \dot{j}_\alpha ; \dot{j}_\beta \rangle}{|\langle \dot{j}_\alpha ; \dot{j}_\beta \rangle|}.
\] (3.45)

\[
M_{\mu\nu} = -i\omega\langle \dot{j}_\mu ; \dot{j}_\nu \rangle + \langle \dot{j}_\mu ; \dot{j}_\nu \rangle + \frac{\langle \dot{j}_\alpha ; \dot{j}_\nu \rangle \langle \dot{j}_\alpha ; \dot{j}_\beta \rangle}{\langle \dot{j}_\alpha ; \dot{j}_\beta \rangle}.
\] (3.46)

In Eq. (3.46) the force-force correlation function appears, which is well suited for a perturbative treatment. Different processes can be found in each order of a perturbative expansion. In QED they are well studied and can be represented by Feynman diagrams. In the Figs. (1) and (2) we have shown the second order diagrams for the bremsstrahlung and inverse bremsstrahlung process. These processes are well known and expressions were given by QED S-matrix
calculations [15] as well as by a non-relativistic treatment [16]. We will draw our attention to the inverse bremsstrahlung within our formalism in the next section.

4 Inverse Bremsstrahlung

As a second order process we consider in this section the absorption of electromagnetic waves in a relativistic plasma. For illustration we derive the corresponding absorption coefficient in the long wavelength limit.

4.1 The absorption coefficient

Before calculating the inverse bremsstrahlung in second order, we need to give some relations, which allow to extract the absorption coefficient $\alpha$ out of the 4-dimensional susceptibility tensor.

In Eq. (3.36) and (3.37) a general decomposition of the susceptibility tensor into its longitudinal and transverse part was shown. The transverse component $\chi^t$ can also be projected out of $\chi_{\mu \nu}$ by the two orthogonal transverse polarization vectors $\epsilon_\mu(k,1)$ or $\epsilon_\mu(k,2)$

$$\chi^t(k) = \epsilon^\mu(k,i) \chi_{\mu \nu}(k) \epsilon^\nu(k,i) \quad i = 1,2$$

While the susceptibility tensor describes the response of the plasma due to an external field, the dielectric tensor $\epsilon_{\mu \nu}$ connects the internal and external fields with each other. The longitudinal and transverse part of the dielectric tensor is related to the susceptibility according to [25]

$$\chi^\ell(k,\omega) = \frac{k^2 \epsilon^\ell(k,\omega) - 1}{e^2 \epsilon(k,\omega)}.$$
\[ \chi^t(k, \omega) = \frac{k^2}{e^2} \left( 1 - \frac{k^2}{\omega^2} \right) \frac{\varepsilon^t(k, \omega) - 1}{\varepsilon^t(k, \omega) - k^2/\omega^2} \].

(4.49)

In the long wavelength limit, which will be taken here, it is seen from Eqs. (4.48) and (4.49) that the longitudinal and transverse part of the dielectric tensor and the susceptibility tensor coincide. The long wavelength approximation is well justified in the optical regime. The absorption coefficient is related to the damping of electro-magnetic waves, which is determined by the imaginary part of the dispersion equation. In the long wavelength limit we have

\[ \alpha(\omega) = \frac{\omega}{n(\omega)} \lim_{k \to 0} \text{Im} \varepsilon^t(\omega, k) \],

(4.50)

where in the following the index of refraction is approximated by \( n(\omega) \approx 1 \). From Eq. (4.49) we can find the relation

\[ \lim_{k \to 0} \text{Im} \varepsilon^t(\omega) = \lim_{k \to 0} \frac{k^2}{e^2} \frac{\text{Im} \varepsilon^t}{k^4 + \left[ \text{Re} \chi^t \right]^2 + \left[ \text{Im} \chi^t \right]^2 - 2\frac{k^2}{\omega^2} \text{Re} \chi^t}, \]

with a \( k^4 \)-term dropped due to the limes \( k \to 0 \). It can be shown, that non-trivial solution for the transverse susceptibility, making use of Eq. (3.28) as well as Eqs. (3.45) and (3.46), can be written as

\[ \chi^t(k, \omega) = -\beta \left( \hat{j}_\perp \, \hat{j}_\perp \right) - \frac{i\beta \omega \left( \hat{j}_\perp \, \hat{j}_\perp \right) \left( \hat{j}_\perp \, \hat{j}_\perp \right)}{-i\omega \left( \hat{j}_\perp \, \hat{j}_\perp \right) + \left( \hat{j}_\perp \, \hat{j}_\perp \right)} \]

(4.52)

In these expressions we use the transverse current \( \hat{j}_\perp \equiv \epsilon^\mu \hat{j}_\mu \). Contributions from the determinant, containing \( \left( \hat{j}_\alpha \, \hat{j}_\nu \right) \) are neglected, since these terms are of higher order. In order to derive Eq. (4.52), the current conservation, leading to the composition (3.36) and (3.37) was used. Treating the collisions, i.e. \( \left( \hat{j}_\perp \, \hat{j}_\perp \right) \) in Eq. (4.52) as small contributions, we can write to first order in the force-force correlation function

\[ \chi^t(k, \omega) \approx -\frac{i\beta}{\omega} \left( \hat{j}_\perp \, \hat{j}_\perp \right). \]

(4.53)

Finally with Eq. (4.50), (4.51) and (4.53) we obtain the absorption coefficient containing collisions in first order

\[ \alpha(\omega) = -\beta \lim_{k \to 0} \frac{e^2}{k^2} \text{Re} \left( \hat{j}_\perp \, \hat{j}_\perp \right)_{\omega+i\eta}. \]

(4.54)
The force-force correlation function in Eq. (4.54) contains both, the emission
and the absorption of photons. The evaluation of the correlation functions (see
next section) shows that the absorption is described by

\[ \alpha(\omega) = -\beta \lim_{k \to 0} \frac{e^2}{k^2} \text{Re} \left\langle \dot{j}_{\perp}(-k) ; \dot{j}_{\perp}(-k) \right\rangle_{-\omega + i\eta} , \]  

(4.55)

with \( \omega > 0, k > 0 \). The emission of photons, which will not be considered here
further, is related to the absorption by the interchange \( \omega \to -\omega \) and \( k \to -k \).

The relation, given in this section can straightforwardly generalized to arbi-
trary hyperplanes. Since we work in the adiabatic approximation all expres-
sions were given in the instant frame, which is of course most convenient
here.

4.2 Evaluation of the correlation function

Evaluating the force-force correlation function in Eq. (4.55) we will follow the
same way presented in the last section. The perturbative part in the Hamil-
tonian, i.e. the Coulomb interaction with the ions, is written as

\[
\hat{H}_{\text{ion}} = e \int d^3x \dot{j}_0(x) : A^0_\text{ion}(x) 
= e \int d^3p_1 \int d^3q \frac{1}{(2\pi\hbar)^3} A^0_\text{ion}(q) \sqrt{\frac{m}{E_{p_1}}} \sum_{r,r'} \left\{ \right.
\sqrt{\frac{m}{E_{p_1 - q}}} \hat{d}_{p_1 - q,r} \hat{d}^\dagger_{p_1,r} : \left( \bar{u}(p_1 - q,r') \gamma_0 u(p_1,r) \right) e^{i(E_{p_1} - E_{p_1 - q})t} 
+ \sqrt{\frac{m}{E_{p_1 + q}}} \hat{b}_{p_1 + q,r} \hat{b}^\dagger_{p_1,r} : \left( \bar{u}(p_1 + q,r') \gamma_0 u(p_1,r) \right) e^{-i(E_{p_1} - E_{p_1 + q})t} 
+ \sqrt{\frac{m}{E_{-p_1 - q}}} \hat{d}_{-p_1 - q,r} \hat{d}^\dagger_{-p_1,r} : \left( \bar{u}(-p_1 - q,r') \gamma_0 u(p_1,r) \right) e^{-i(E_{p_1} + E_{p_1 - q})t} 
+ \sqrt{\frac{m}{E_{-p_1 + q}}} \hat{b}_{-p_1 + q,r} \hat{b}^\dagger_{-p_1,r} : \left( \bar{u}(-p_1 + q,r') \gamma_0 u(p_1,r) \right) e^{i(E_{p_1} + E_{p_1 - q})t} \left. \right\} .
\]

(4.56)

In the “rotating wave approximation” (RWA) the last two terms in Eq. (4.56)
are neglected, since they describe rapid oscillating processes with frequencies
\( \omega \approx 2m/\hbar \). We calculate \( \dot{j}_\mu = -\frac{i}{\hbar} [\hat{j}_\mu, \hat{H}_{\text{ion}}] \) using Eqs. (3.32) and (4.56) in
the RWA-approximation. After some algebra, which is given in Appendix D, we
can find the expression.
\[ j^\mu(-k) = -\frac{iem}{2\hbar} \int \frac{d^3p_i}{(2\pi\hbar)^3/2} \frac{d^3p_f}{(2\pi\hbar)^3/2} \sqrt{\frac{1}{E_{p_i}E_{p_f}}} A^0_{\text{ion}}(q)e^{i(E_{p_f} - E_{p_i})t} \]

\[ \sum_{s_i,s_f} \left\{ N^\mu_{\mu}(p_i,s_i) \delta_{p_j,s} \hat{d}_{p_i,s} + N^\mu_{\mu}(p_i,s_i) \hat{b}_{p_j,s}^\dagger \right\}, \] (4.57)

\[ N^\mu_{\mu} = \bar{v}(p_i,s_i) \text{tr}_D \left[ \gamma_0 \gamma_\mu - \gamma_0 \frac{E_{p_i} - m}{E_{p_i} - \hbar k} \right] v(p_f,s_f), \] (4.58)

\[ N^\mu_{\mu} = \bar{u}(p_f,s_f) \text{tr}_D \left[ \gamma_0 \gamma_\mu - \gamma_0 \frac{E_{p_f} - m}{E_{p_f} - \hbar k} \right] u(p_i,s_i). \] (4.59)

It is now straightforward to calculate the force-force correlation function, following the steps in the RPA-approximation (see Appendix C). As seen from Eq. (4.55) we need the expression for the real part of the transverse force-force correlation. With the definitions Eqs. (3.19) and (3.20) we find after performing the contractions using the Wick Theorem [Eq. (C.2)] and integrating as well as summing over the variables appearing in the delta-functions, we obtain

\[ \text{Re}\langle j^\mu, j^\nu \rangle_{-\omega+i\eta} = -\frac{e^2 m^2}{4\hbar^2\beta} \text{Re} \int_0^\infty dt e^{i(-\omega+i\eta)t} \int \frac{d^3p_i}{(2\pi\hbar)^3} \frac{d^3p_f}{(2\pi\hbar)^3} \]

\[ \frac{1}{E_{p_f}E_{p_i}} (A^0_{\text{ion}}(q))^2 e^{i(E_{p_f} - E_{p_i})(t-i\hbar\omega)} \]

\[ \sum_{s_i,s_f} \left\{ \bar{f}(E_{p_f}) \left[ 1 - \bar{f}(E_{p_f}) \right] |e^\mu_{\mu}N^\mu_{\mu}|^2 + f(E_{p_f}) \left[ 1 - f(E_{p_f}) \right] |e^\mu_{\mu}N^\mu_{\mu}|^2 \right\} \] (4.60)

The integration over \( \bar{t} \) and \( \tilde{t} \) can be performed, and we obtain, with the Dirac identity \( \lim_{\eta \to +0} (x \pm i\eta)^{-1} = P(1/x) \mp i\pi\delta(x) \)

\[ \text{Re}\langle j^\mu, j^\nu \rangle_{-\omega+i\eta} = -\frac{e^2 m^2 \pi}{4\hbar^2\omega\beta} \int \frac{d^3p_i}{(2\pi\hbar)^3} \frac{d^3p_f}{(2\pi\hbar)^3} \]

\[ \frac{\delta(E_{p_f} - E_{p_i} - \hbar\omega)}{E_{p_f}E_{p_i}} (A^0_{\text{ion}}(q))^2 \left( e^{\beta(E_{p_f} - E_{p_i})} - 1 \right) \]

\[ \sum_{s_i,s_f} \left\{ \bar{f}(E_{p_f}) \left[ 1 - \bar{f}(E_{p_f}) \right] |e^\mu_{\mu}N^\mu_{\mu}|^2 + f(E_{p_f}) \left[ 1 - f(E_{p_f}) \right] |e^\mu_{\mu}N^\mu_{\mu}|^2 \right\} \] (4.61)

Using the first two relations of Eq. (C.7) as well as Eq. (4.55) and writing \( E_i \equiv E_{p_i} \) and \( E_f \equiv E_{p_f} \)

\[ \text{Re}\langle j^\mu, j^\nu \rangle_{-\omega+i\eta} = \frac{e^2 m^2 \pi}{4\hbar^2\omega\beta} \int \frac{d^3p_i}{(2\pi\hbar)^3} \frac{d^3p_f}{(2\pi\hbar)^3} \frac{\delta(E_f - E_i - \hbar\omega)}{E_{p_f}E_{p_i}} (A^0_{\text{ion}}(q))^2 \]
\[ \sum_{s_i, s_f} \left\{ |\epsilon_i^\mu N_{\mu}^{(p)}|^2 \left[ \bar{f}(E_f) - \bar{f}(E_i) \right] + |\epsilon_i^\mu N_{\mu}^{(e)}|^2 \left[ f(E_f) - f(E_i) \right] \right\} . \quad (4.62) \]

Finally with Eq. (4.55) we find for the absorption coefficient

\[ \alpha(\omega) = \lim_{k \to 0} \frac{e^4 m^2 \pi}{4\hbar^2 \omega k^2} \int \frac{d^3 p_i}{(2\pi \hbar)^3} \frac{d^3 p_f}{(2\pi \hbar)^3} \frac{\delta(E_f - E_i - \hbar \omega)}{E_{p_f} E_{p_i}} (A_{\text{ion}}^0(q))^2 \]

\[ \sum_{s_i, s_f} \left\{ |\epsilon_i^\mu N_{\mu}^{(p)}|^2 \left[ \bar{f}(E_f) - \bar{f}(E_i) \right] + |\epsilon_i^\mu N_{\mu}^{(e)}|^2 \left[ f(E_f) - f(E_i) \right] \right\} . \quad (4.63) \]

It should be emphasized, that the transition matrices \( N_{\mu}^{(p)} \) and \( N_{\mu}^{(e)} \) carry a \( k \)-dependence (see Eqs. (4.58) and (4.59)) leading to a finite absorption coefficient in Eq. (4.63). As already pointed out the polarization vector \( \epsilon_i^\mu \) for a linear polarized wave is one of the two transverse modes, labeled here by \( i = 1, 2 \).

Within the approximation applied here, we observe from Eq. (4.63), that we have an electron as well as a positron contribution, responsible for the absorption of the external wave. In first order perturbation theory these two terms are not coupled to each other. Writing the result in terms of the transition matrices \( N_{\mu}^{(p)} \) and \( N_{\mu}^{(e)} \), it can further be observed, that our result corresponds to the well known Bethe-Heitler \[15\] formula, if treating the absorption of the wave by a single incoming electron with energy \( E_i \) and an outgoing electron of energy \( E_f \).

In \[16\] non-relativistic results for the absorption coefficient in different approximations are derived. The expression given in the paper in terms of the transition matrix correspond to the non-relativistic result of Eq. (4.63) if the matrix element is evaluated in Born approximation. The final result is written for the complex collision frequency \( \nu \), which is related to the absorption coefficient \( \alpha \) by \( \alpha(\omega) = \omega_{pl}^2/\omega^2 \text{Re} \nu(\omega) \), with \( \omega_{pl} \) the electron plasma frequency and the index of refraction \( n(\omega) = 1 \).

5 Discussion and Outlook

We demonstrated in this work, how the hyperplane formalism can be used for a manifest covariant density matrix formulation of relativistic plasmas. The one-time description allows to formulate an initial value problem, which can lead to considerable simplifications in short-time pump-and probe experiments. A covariant scheme is developed in the hyperplane formalism, where Heisenberg operators are defined on spacelike hyperplanes in Minkowski space. In particular, the construction of the quantum Hamiltonian starting from the
classical QED Lagrangian, making use of the canonical quantization scheme is shown.

From the Liouville von Neumann equation an initial value problem in the hyperplane formalism is formulated. The approach used in this work is a generalization of Zubarev’s method of the relevant statistical operator. For the case of moderate fields we applied the linear response approximation, where the statistical operator is expanded near its equilibrium solution. From the self-consistency relation, which determines the Lagrange multipliers of the generalized Gibbsian ensemble, we obtain the response equation, defining the susceptibility tensor on the hyperplane in terms of current-current correlation functions. The relativistic susceptibility tensor, which displays the response of the fermion current to an external electro-magnetic wave, is calculated in the RPA approximation. The result agrees with familiar results, published for instance by Tsytovich [24].

Further it was demonstrated how to include collisions into the formalism in a systematic way, making use of perturbation theory. For that reason the current-current correlation function is expressed by force-force correlation functions by partial integration. The force-force correlation functions contain collisions since the force operators are calculated from the von Neummann equation with the interaction part in the Hamiltonian. The advantage of the representation of the current-current correlation function, is the appearance of the force-force correlation function in denominator, which is convenient for a perturbative expansion.

As an illustration, we derived an expression for the absorption coefficient of inverse bremsstrahlung in first order of the force-force correlation function (which corresponds to second order in the interaction). We made use of the adiabatic approximation, where the dynamic of the positively charged plasma component is frozen as well as the Born approximation in calculating the correlation functions. The result given here is a generalization of the Bethe-Heitler formula for the case the absorption in a electron-positron plasma. The interaction is described by the Coulomb interaction.

Extensions to the approximations assumed here in this work can be done in different ways. It is possible to consider higher orders in the expansion of the current-current correlation function in terms of the force-force correlation function. This will allow to describe higher order processes, like for instance pair production. Further interactions can be taken into account, like the electron-electron interaction, or the radiation part of the Hamiltonian, which implies to couple photons into the plasma.

A general scheme to derive kinetic equations in the hyperplane formalism is explained in [1], valid also for the case of strong external fields. However,
processes beyond the RPA approximation are hard to be calculated, since coupled kinetic equations for the fermions and photons are to be solved.

Appendix A

Commutation relations on hyperplanes

The constraint equations for the canonical variables $A_\perp^\mu$ and $\Pi_\perp^\mu$ on the hyperplane $\sigma_{n,\tau}$ is written in the form $\chi_N(x_\perp) = 0$, where

$$
\begin{align*}
\chi_1(x_\perp) &= \nabla_\mu A_\perp^\mu(x_\perp), & \chi_2(x_\perp) &= \nabla_\mu \Pi_\perp^\mu(x_\perp), \\
\chi_3(x_\perp) &= n_\mu A_\perp^\mu(x_\perp), & \chi_4(x_\perp) &= n_\mu \Pi_\perp^\mu(x_\perp).
\end{align*}
$$

(A.1)

For any functionals $\Phi_1$ and $\Phi_2$ of the field variables $A_\perp$ and $\Pi_\perp$, we define the Poisson bracket

$$
[\Phi_1, \Phi_2]_P \equiv \int d\sigma \left\{ \frac{\delta \Phi_1}{\delta A_\perp^\mu(x_\perp)} \frac{\delta \Phi_2}{\delta \Pi_\perp^\mu(x_\perp)} - \frac{\delta \Phi_2}{\delta A_\perp^\mu(x_\perp)} \frac{\delta \Phi_1}{\delta \Pi_\perp^\mu(x_\perp)} \right\},
$$

(A.2)

where the constraints are ignored in calculating the functional derivatives. Applying this formula to the canonical variables we obtain

$$
[A_\perp^\mu(x_\perp), \Pi_\perp^\nu(x_\perp')]_P = \delta^\mu_\nu \delta^3(x_\perp - x_\perp')
$$

(A.3)

with the three-dimensional delta function (2.17). All other Poisson brackets for the canonical variables are equal to zero. In the Dirac terminology, functions (A.1) correspond to second class constraints since the matrix

$$
C_{NN'}(x_\perp, x_\perp') = [\chi_N(x_\perp), \chi_{N'}(x_\perp')]_P
$$

(A.4)

is non-singular. A straightforward calculation of the Poisson brackets shows that the non-zero elements of $C$ are

$$
\begin{align*}
C_{12}(x_\perp, x_\perp') &= -C_{21}(x_\perp, x_\perp') = -\nabla_\mu \nabla^\mu \delta^3(x_\perp - x_\perp'), \\
C_{34}(x_\perp, x_\perp') &= -C_{43}(x_\perp, x_\perp') = \delta^3(x_\perp - x_\perp').
\end{align*}
$$

(A.5)
According to the general quantization scheme \[20,21\], commutation relations for canonical operators are defined by the Dirac brackets for classical canonical variables. In our case the Dirac brackets are written as

\[
[\Phi_1, \Phi_2]_D = [\Phi_1, \Phi_2]_P - \int d\sigma \int d\sigma' \left[ [\Phi_1, \chi_N(x_\perp)]_P C^{-1}_{NN'}(x_\perp, x'_\perp) \left[ \chi_N'(x'_\perp), \Phi_2 \right]_P \right] (A.6)
\]

(summation over repeated indices). The inverse matrix, \(C^{-1}_{NN'}(x_\perp, x'_\perp)\), satisfies the equation

\[
\int d\sigma'' C_{NN''}(x_\perp, x''_\perp) C^{-1}_{NN'}(x'_\perp, x''_\perp) = \delta_{NN'} \delta^3(x_\perp - x'_\perp). \quad (A.7)
\]

Since the matrix elements (A.5) of \(C\) depend on the difference \(x_\perp - x'_\perp\), Eq. (A.7) can be solved for \(C^{-1}\) using a Fourier transform on \(\sigma_{n,\tau}\), which is defined for any function \(f(x)\) as

\[
\tilde{f}(\tau, p_\perp) = \int d^4x e^{ip_\perp \cdot x_\perp} \delta(x \cdot n - \tau) f(x). \tag{A.8}
\]

The inverse transform is

\[
f(x) \equiv f(\tau, x_\perp) = \int \frac{d^4p}{(2\pi)^3} e^{-ip \cdot x_\perp} \delta(p \cdot n) \tilde{f}(\tau, p_\perp). \tag{A.9}
\]

If we perform the Fourier transformation in Eq. (A.7), we find by inserting (A.5) that the non-zero elements of \(C^{-1}\) are

\[
C^{-1}_{12}(x_\perp, x'_\perp) = -C^{-1}_{21}(x_\perp, x'_\perp) = -\int \frac{d^4p}{(2\pi)^3} e^{-ip \cdot (x - x')} \delta(p \cdot n) \frac{1}{p^2_\perp},
\]

\[
C^{-1}_{34}(x_\perp, x'_\perp) = -C^{-1}_{43}(x_\perp, x'_\perp) = -\delta^3(x_\perp - x'_\perp). \tag{A.10}
\]

Now the Dirac brackets (A.6) for the canonical variables are easily calculated and we obtain

\[
[A^\mu_\perp(x_\perp), \Pi^\nu_\perp(x'_\perp)]_D = \epsilon^{\mu
u}(x_\perp - x'_\perp), \quad (A.11)
\]

\[
[A^\mu_\perp(x_\perp), A^\nu_\perp(x'_\perp)]_D = [\Pi^\mu_\perp(x_\perp), \Pi^\nu_\perp(x'_\perp)]_D = 0, \quad (A.12)
\]

where the functions \(\epsilon^{\mu\nu}(x_\perp - x'_\perp)\) are given by Eq. (2.25). According to the general quantization rules, the commutation relations for canonical operators
correspond to $i[... ]_D$. Thus, in the hyperplane formalism, the commutation relations for the operators of EM field are given by (2.23) and (2.24). Obviously these relations are valid in the Schrödinger and Heisenberg pictures.

The anti-commutation relations on hyperplanes

To find the anticommutation relations for the fermion operators on the hyperplane $\sigma_{n,\tau}$, it is sufficient to consider a free Dirac field. Our starting point is the standard quantization scheme in the frame where $x^\mu = (t, r)$ and $n^\mu = (1, 0, 0, 0)$ (see, e.g., [18]). In that case the field operators $\hat{\psi}_a$ and $\hat{\bar{\psi}}_a$ can be written in terms of creation and annihilation operators according to

\[
\hat{\psi}_a(x) = \int \frac{d^4p}{(2\pi)^{3/2}} \frac{\delta(p^0 - \epsilon(p))}{\sqrt{2\epsilon(p)}} \sum_{s=\pm 1} \left[ \hat{b}_s(p) u_{as}(p) e^{-ip\cdot x} + \hat{d}_s(p) v_{as}(p) e^{ip\cdot x} \right],
\]

\[
\hat{\bar{\psi}}_a(x) = \int \frac{d^4p}{(2\pi)^{3/2}} \frac{\delta(p^0 - \epsilon(p))}{\sqrt{2\epsilon(p)}} \sum_{s=\pm 1} \left[ \hat{d}_s(p) \bar{v}_{as}(p) e^{-ip\cdot x} + \hat{\bar{b}}_s(p) \bar{u}_{as}(p) e^{ip\cdot x} \right],
\]

where $\epsilon(p) = \sqrt{p^2 + m^2}$ is the free fermion dispersion relation. Constructing the expression $\{\hat{\psi}_a(x), \hat{\bar{\psi}}_{a'}(x')\}$ for two arbitrary space-time points and recalling the anticommutation relations

\[
\{\hat{b}_s(p), \hat{d}_{s'}(p')\} = \{\hat{d}_s(p), \hat{\bar{b}}_{s'}(p')\} = \delta_{ss'}\delta^3(p - p'),
\]  

(A.13)

as well as polarization sums

\[
\sum_{s=\pm 1} u_{as}(p) \bar{u}_{a's}(p) = \left[ \gamma^\mu p_\mu + m \right]_{aa'}, \quad \sum_{s=\pm 1} v_{as}(p) \bar{v}_{a's}(p) = \left[ \gamma^\mu p_\mu - m \right]_{aa'},
\]

we arrive at

\[
\{\hat{\psi}_a(x), \hat{\bar{\psi}}_{a'}(x')\} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\epsilon(p)} \left\{ \left[ \gamma^\mu p_\mu + m \right]_{aa'} e^{-ip\cdot(x-x')} + \left[ \gamma^\mu p_\mu - m \right]_{aa'} e^{ip\cdot(x-x')} \right\},
\]  

(A.14)

where $p^0 = \sqrt{p^2 + m^2}$. Using

\[
\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\epsilon(p)} = \int \frac{d^4p}{(2\pi)^4} \delta(p^2 - m^2) \bigg|_{p^0 > 0},
\]  

(A.15)

Eq. (A.14) can be rewritten in a Lorentz invariant form
\[
\left\{ \hat{\psi}_a(x), \hat{\psi}_a'(x') \right\} = \int \frac{d^4p}{(2\pi)^3} \left\{ \left[ \gamma^\mu p_\mu + m \right]_{aa'} e^{-ip(x-x')} \delta(p^2 - m^2) \right\}_{p^\rho > 0} \\
+ \left[ \gamma^\mu p_\mu - m \right]_{aa'} e^{ip(x-x')} \delta(p^2 - m^2) \right\}_{p^\rho > 0}.
\] (A.16)

The anticommutation relation on the hyperplane \( \sigma_{n,\tau} \) is now obtained by setting \( x = n\tau + x_\perp \) and \( x' = n\tau + x'_\perp \). In calculating the integrals, it is convenient to use the decomposition \( p^\mu = n^\mu p^\parallel + p^\mu_\perp, (p^\parallel > 0) \). Then we get

\[
\left\{ \hat{\psi}_a(\tau, x_\perp), \hat{\psi}_a'(\tau, x'_\perp) \right\} = \int \frac{d^4p}{(2\pi)^3} \frac{\delta(p^\parallel - \epsilon(p_\perp))}{2\epsilon(p_\perp)} \\
\times \left\{ \left[ \gamma^\parallel p^\parallel + \gamma^\mu_\perp p^\mu_\perp + m \right]_{aa'} e^{-ip_\perp(x_\perp - x'_\perp)} \\
+ \left[ \gamma^\parallel p^\parallel + \gamma^\mu_\perp p^\mu_\perp - m \right]_{aa'} e^{ip_\perp(x_\perp - x'_\perp)} \right\} \] (A.17)

with the dispersion relation on the hyperplane

\[
\epsilon(p_\perp) = \sqrt{-p_{\perp\mu}p^\mu_{\perp} + m^2}. \] (A.18)

Finally, changing the variable \( p_\perp \rightarrow -p_\perp \) in the second integral in Eq. (A.17), we obtain the anticommutation relation (2.26). The relations (2.27) can be derived by the same procedure.

Appendix B

The relevant statistical operator in linear response

In order to rewrite Eq. (3.12), we make use of the operator identity

\[
e^{\hat{C}_1 + \hat{C}_2} = \left( 1 + \int_0^1 dz \ e^{z(\hat{C}_1 + \hat{C}_2)} \hat{C}_2 e^{-z\hat{C}_1} \right) e^{\hat{C}_1} \] (B.1)

and obtain

\[
\varrho_{\text{rel}}(n, \tau) = \left( 1 + \beta \int_0^1 dz \left\{ e^{-z\beta[H_s - vQ]} \right\} \right)
\]
\[
\times \left( \int \frac{d\sigma}{\sigma_n} \sum_{\ell} \phi_\mu^\ell(x_\perp, \tau) \hat{B}_\ell^\mu(x) e^{z\beta[\hat{H}_s - \nu Q]} \right) \varrho_0(n).
\]

The Lagrange multipliers \( \phi_\mu^\ell \) are already first order contributions for the deviation from the equilibrium. Therefore in zeroth order, we can express the Schrödinger operators \( \hat{B}_\ell^\mu(x) \) by some \( \tau \)-dependent Heisenberg operators according to

\[
e^{-z\beta \hat{H}_s} \hat{B}_\ell^\mu(x_\perp, \tau_0) e^{z\beta \hat{H}_s} = U_0^\dagger (iz\beta, \tau_0) \hat{B}_\ell^\mu(x_\perp, \tau_0) U_0 (iz\beta, \tau_0) = \hat{B}_\ell^\mu(x_\perp, iz\beta),
\]

where we assumed, that at \( \tau_0 \) Schrödinger and Heisenberg operators coincide. The free evolution operator \( U_0 \) is given by

\[
U_0(\tau, \tau_0; n) = e^{-i\hat{H}_s(n)(\tau - \tau_0)}.
\]

Making use of Eq. (B.3) in Eq. (B.2) we find the relevant statistical operator

\[
\varrho_{\text{rel}}(n, \tau) = \left( 1 + \beta \sum_{\ell} \int \frac{d\sigma}{\sigma_n} \phi_\mu^\ell(x) \int_0^1 dz \hat{B}_\ell^\mu(x_\perp, iz\beta) \right) \varrho_0(n)
\]

and finally Eq. (3.15) is obtained after Fourier transformation.

**The irrelevant statistical operator in linear response**

In order to express \( \Delta \varrho(n, \tau) \) [see Eq. (3.4)] in linear response we first consider the commutator

\[
[\varrho_{\text{rel}}(n, \tau), \hat{H}^\tau(n)] = [\varrho_{\text{rel}}(n, \tau), \hat{H}_s(n) + \hat{H}_\text{ext}^\tau(n)]
\]

and note, that the zero order part \( [\varrho_0(n), \hat{H}_s(n)] = 0 \) vanishes by definition. There are two first order terms

\[
[\varrho_0(n), \hat{H}_\text{ext}^\tau(n)] \quad \text{and} \quad [\varrho_{\text{rel}}^{(1)}(n, \tau), \hat{H}_s(n)],
\]

where \( \varrho_{\text{rel}}^{(1)} \) is the first order contribution of \( \varrho_{\text{rel}} \). Making use of the Kubo-identity

31
\[
[\hat{C}_2, e^{\hat{C}_1}] = \int_0^1 dz \, e^{z\hat{C}_1} [\hat{C}_2, \hat{C}_1] e^{-z\hat{C}_1} e^{\hat{C}_1}, \quad (B.8)
\]

we find

\[
[\varrho_0(n), \hat{H}^\tau_{\text{ext}}(n)] = \beta \int d\sigma \, A^\mu_{\text{ext}}(x) \int_0^1 dz \, e^{-z\beta(\hat{H}_s - \nu\hat{Q})} \times \left[ \hat{j}_\mu, (\hat{H}_s - \nu\hat{Q}) \right] e^{z\beta(\hat{H}_s - \nu\hat{Q})} \varrho_0(n). \quad (B.9)
\]

Since \( [\hat{j}_\mu, \hat{Q}] = n^\alpha [\hat{j}_\mu, \hat{j}_\alpha] = 0 \) and also \( [\hat{H}_s, \hat{Q}] = 0 \) (\( \hat{Q} \) is an integral of motion) we can write

\[
[\varrho_0(n), \hat{H}^\tau_{\text{ext}}(n)] = \beta \int d\sigma \, A^\mu_{\text{ext}}(x) \int_0^1 dz \, e^{-z\beta\hat{H}_s} \left[ \hat{j}_\mu, \hat{H}_s \right] e^{z\beta\hat{H}_s} \varrho_0(n). \quad (B.10)
\]

Since the external field \( A^\mu_{\text{ext}}(x) \) appears in (B.10) the commutator \( [\hat{j}_\mu, \hat{H}_s] \) is only treated in zero order, and we can write

\[
\partial_\tau \hat{j}_\mu = -i [\hat{j}_\mu, \hat{H}] = -i \left[ \hat{j}_\mu, \hat{H}_s \right]. \quad (B.11)
\]

Making use of Eq. (B.3), we find

\[
[\varrho_0(n), \hat{H}^\tau_{\text{ext}}(n)] = i\beta \int d\sigma \, A^\mu_{\text{ext}}(x) \int_0^1 dz \, \hat{j}_\mu(x, i\beta z) \varrho_0(n), \quad (B.12)
\]

\[
= i\beta e^{-ik_\parallel \tau} \int_0^1 dz \, A^\mu_{\text{ext}}(k) \hat{j}_\mu(k_z, i\beta z) \varrho_0(n), \quad (B.13)
\]

with the last expression written in Fourier space.

The second commutator in Eq. (B.7) can be written in linear response by making use of Eq. (B.8)

\[
[\varrho_{\text{rel}}^{(1)}(n, \tau), \hat{H}_s(n)] = i\beta \sum_\ell \int d\sigma \, \phi_\ell(x) \int_0^1 dz \, \hat{B}_\ell^\mu(x, i\beta) \varrho_0(n). \quad (B.14)
\]

In Fourier representation Eq. (B.14) reads
\[ [\hat{g}^{(1)}_{\text{rel}}(n, \tau), \hat{H}_s(n)] = i\beta \sum_{\ell} \phi_{\mu}(k) \int_0^1 dz \hat{B}^{\mu}_\ell(k_\perp, iz\beta) \varrho_0(n). \] (B.15)

The derivative term in Eq. (3.4) is calculated using Eq. (3.15)

\[ \partial_{\tau} \hat{g}_{\text{rel}}(n, \tau) = -i\beta k_\parallel e^{-ik_\parallel \tau} \int_0^1 dz \sum_{\ell} \phi_{\mu}(k) \hat{B}^{\mu}_\ell(k_\perp, iz\beta) \varrho_0(n). \] (B.16)

Constructing the expression for \( \Delta \varrho(n, \tau) \) in the linear approximation [Eq. (3.4)], the evolution operators \( U \) are to be taken in zeroth order and yield a \( \tau \)-translation. From the Eqns. (3.4), (B.13), (B.14), (B.16) one finds

\[ \Delta \varrho(n, \tau) = -\beta \int_{-\infty}^\tau d\tau' e^{-\eta(\tau-\tau')} e^{-ik_\parallel \tau'} \int_0^1 dz \left\{ \sum_{\ell} \left[ \hat{B}^{\mu}_\ell(\tau' - \tau + iz\beta) - i k_\parallel \hat{B}^{\mu}_\ell(\tau' - \tau + iz\beta) \right] \phi_{\mu}(k) \right. \\
+ A_{\text{ext}}^\mu(k) \hat{j}_\mu(\tau' - \tau + iz\beta) \left. \right\} \varrho_0(n). \] (B.17)

Eq. (B.17) can be rewritten using the transformation \( \tilde{\tau} = \tau - \tau' \) and Eq. (3.16) is obtained.

Appendix C

The RPA susceptibility tensor

We evaluate the current-force correlation function in the RPA approximation with Eq. (3.32), (3.34) as well as \( \hat{j}_\nu(k) = \hat{j}_\nu(-k) \) and the short-hand notation

\[ \langle \hat{j}_\mu(k) : \hat{j}_\nu(k) \rangle_{\omega + i\eta} \equiv \langle \hat{j}_\mu : \hat{j}_\nu \rangle_{\omega + i\eta} \]

\[ \langle \hat{j}_\mu : \hat{j}_\nu \rangle_{\omega + i\eta}^k \equiv \frac{1}{\beta} \int_0^\infty d\tilde{\tau} e^{i(\omega + i\eta)\tilde{\tau}} \int_0^\beta d\tilde{t} \text{Tr} \left\{ \hat{j}_\mu(k, \tilde{\tau} - i\tilde{t}) \hat{j}_\nu(-k) \varrho_0 \right\} \]

\[ = -\frac{i e^2 m^2}{\hbar^2} \int_0^\infty d\tilde{\tau} e^{i(\omega + i\eta)\tilde{\tau}} \int_0^\beta d\tilde{t} \int d^3p d^3p' \sqrt{\frac{1}{E_p}} \sqrt{\frac{1}{E_{p'}}} \sum_{s' r r'} \left\{ \right. \\
\left. \right. \\
\left. \right. \\
\left. \right. \]
Further we introduced the Fermi-distribution functions $f$ and antiparticles respectively

$$D \text{ and } \bar{f} \text{ for particles and antiparticles respectively} .$$

$$f(E_p) = \frac{1}{e^{\beta(E_p - \mu)} + 1}, \quad \bar{f}(E_p) = \frac{1}{e^{\beta(E_p + \mu)} + 1} \quad \text{(C.3)}$$

It should be noted that contractions inside the normal order do not contribute and that crossing contraction lines (the third term in Eq. (C.1)) give an extra
minus sign. Eq. (C.1) can now be written in terms of Fermi functions if the integration over \( p' \) and the summation over \( r \) and \( r' \) is performed

\[
\langle \hat{j}_\mu \rangle_{\bar{\omega} + i\eta}^k = -\frac{i e^2 m^2}{\hbar \beta} \int_0^\infty d\bar{t} \int_0^{\beta} d\bar{t} \int \frac{d^3 p}{(2\pi \hbar)^3} \frac{1}{E_p} \sum_{ss'} \left\{ 
\right.
\]

\[
(-1) \frac{1}{E_{p+hk}} \bar{f}(E_p) \left[ 1 - \bar{f}(E_{p+hk}) \right] [E_{p+hk} - E_p] e^{\frac{i}{\hbar}(E_{p+hk}) (\bar{t} - i\hbar)} \]

\[
\text{tr}_D \left\{ \left( \bar{v}(p + \hbar k, s') \gamma_\mu v(p, s) \right) \left( \bar{v}(p, s) \gamma_\nu u(p + \hbar k, s') \right) \right\}
\]

\[
+ \frac{1}{E_{p-hk}} \left[ 1 - \bar{f}(E_{p-hk}) \right] \left[ 1 - f(E_p) \right] \left[ E_p + E_{p-hk} \right] e^{-\frac{i}{\hbar}(E_p + E_{p-hk}) (\bar{t} - i\hbar)} \]

\[
\text{tr}_D \left\{ \left( \bar{u}(p - \hbar k, s') \gamma_\mu u(p, s) \right) \left( \bar{u}(p, s) \gamma_\nu v(p - \hbar k, s') \right) \right\}
\]

\[
- \frac{1}{E_{p+hk}} \bar{f}(E_p) \left[ E_p + E_{p+hk} \right] e^{\frac{i}{\hbar}(E_p + E_{p+hk}) (\bar{t} - i\hbar)} \]

\[
\text{tr}_D \left\{ \left( \bar{u}(-p - \hbar k, s') \gamma_\mu v(p, s) \right) \left( \bar{v}(p, s) \gamma_\nu u(-p - \hbar k, s') \right) \right\} \right\} . \tag{C.4}
\]

Finally the spin summation rules

\[
\sum_s u(p, s) \otimes \bar{u}(p, s) = \frac{1}{2m} (\bar{p} + m) \tag{C.5}
\]

\[
\sum_s v(p, s) \otimes \bar{v}(p, s) = \frac{1}{2m} (\bar{p} - m) \tag{C.6}
\]

and the integrations over \( \bar{t} \) and \( \bar{t} \) can be performed in Eq. (C.4). With the relations

\[
\bar{f}(E_p) \left( 1 - \bar{f}(E_{p+hk}) \right) \left( e^{+\beta(E_p - E_{p+hk})} - 1 \right) = -\left( \bar{f}(E_p) - \bar{f}(E_{p+hk}) \right)
\]

\[
f(E_{p-hk}) \left( 1 - f(E_p) \right) \left( e^{-\beta(E_p - E_{p-hk})} - 1 \right) = f(E_p) - f(E_{p-hk})
\]

\[
\left( 1 - f(E_p) \right) \left( 1 - \bar{f}(E_{p-hk}) \right) \left( e^{-\beta(E_p - E_{p-hk})} - 1 \right) = -1 + f(E_p) + \bar{f}(E_{p-hk})
\]

\[
\bar{f}(E_p) f(E_{p+hk}) \left( e^{+\beta(E_p - E_{p+hk})} - 1 \right) = 1 - \bar{f}(E_p) - f(E_{p+hk}) \tag{C.7}
\]

the equation (3.35) is obtained.
Appendix D

Calculation of \( \dot{\hat{j}}_\mu(k) \)

From the Eqs. (3.32) and (4.56) we find

\[
\dot{\hat{j}}_\mu(-k) = -\frac{ie}{\hbar} \int \frac{d^3q}{(2\pi\hbar)^3} d^3p d^3p_1 \sqrt{\frac{m}{E_p}} \sqrt{\frac{m}{E_{p_1}}} A^0_{\text{ion}}(q) \sum_{ss'rr'} \left\{ \\
\sqrt{\frac{m}{E_{p-hk}}} \sqrt{\frac{m}{E_{p_1-q}}} \left[ \hat{d}_{p-hk,s} \hat{d}_{p_1,r}^\dagger \right. \right. \\
\times \left( \bar{v}(p + \hbar k, s') \gamma_\mu v(p, s) \right) \left( \bar{u}(p_1 - q, r') \gamma_0 v(p_1, r) \right) \\
+ \sqrt{\frac{m}{E_{p+hk}}} \sqrt{\frac{m}{E_{p_1+q}}} \left[ \hat{d}_{p+hk,s} \hat{d}_{p_1,r}^\dagger \right. \\
\times \left( \bar{v}(-p + \hbar k, s') \gamma_\mu u(p, s) \right) \left( \bar{u}(p_1 + q, r') \gamma_0 u(p_1, r) \right) \\
+ \sqrt{\frac{m}{E_{p-hk}}} \sqrt{\frac{m}{E_{p_1-q}}} \left[ \hat{d}_{p-hk,s} \hat{d}_{p_1,r}^\dagger \right. \\
\times \left( \bar{v}(-p + \hbar k, s') \gamma_\mu u(p, s) \right) \left( \bar{u}(p_1 + q, r') \gamma_0 u(p_1, r) \right) \\
+ \sqrt{\frac{m}{E_{p+hk}}} \sqrt{\frac{m}{E_{p_1+q}}} \left[ \hat{d}_{p+hk,s} \hat{d}_{p_1,r}^\dagger \right. \\
\times \left( \bar{v}(-p + \hbar k, s') \gamma_\mu v(p, s) \right) \left( \bar{u}(p_1 - q, r') \gamma_0 v(p_1, r) \right) \\
+ \sqrt{\frac{m}{E_{p-hk}}} \sqrt{\frac{m}{E_{p_1-q}}} \left[ \hat{d}_{p-hk,s} \hat{d}_{p_1,r}^\dagger \right. \\
\times \left( \bar{v}(-p + \hbar k, s') \gamma_\mu u(p, s) \right) \left( \bar{u}(p_1 - q, r') \gamma_0 u(p_1, r) \right) \\
+ \sqrt{\frac{m}{E_{p+hk}}} \sqrt{\frac{m}{E_{p_1+q}}} \left[ \hat{d}_{p+hk,s} \hat{d}_{p_1,r}^\dagger \right. \\
\times \left( \bar{v}(-p + \hbar k, s') \gamma_\mu v(p, s) \right) \left( \bar{u}(p_1 - q, r') \gamma_0 v(p_1, r) \right) \\
\right. \\
\left. \times \left( \bar{u}(p_1 + q, r') \gamma_0 u(p_1, r) \right) \left( \bar{u}(p_1 - q, r') \gamma_0 u(p_1, r) \right) \right\}.
\]

After calculating the commutators in Eq. (D.1), the integration over \( p_1 \) as well the summation over the primed variable can be performed. Finally, using the spin summation Eq. (C.5) and (C.6), we find the following equation for the force operator (only the non-vanishing terms in the RWA-approximation are kept)

\[
\dot{\hat{j}}_\mu(-k) = -\frac{iem}{2\hbar} \int \frac{d^3q}{(2\pi\hbar)^3} d^3p A^0_{\text{ion}}(q) \sum_{ss'} \left\{ \\
\right.
\]

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\[
\sqrt{\frac{1}{E_p}} \sqrt{\frac{1}{E_{p-q-hk}}} e^{\frac{\imath}{\hbar} (E_p - E_{p-q-hk}) t} \times (\bar{v}(p - q - \hbar k, s') \gamma_0 \frac{\slashed{p} - \hbar k}{E_{p-hk}} \gamma_{\mu} v(p, s)) \hat{d}_{p,s}^\dagger \hat{d}_{p-q-hk,s'} \\
- \sqrt{\frac{1}{E_{p+q}}} \sqrt{\frac{1}{E_{p-hk}}} e^{\frac{\imath}{\hbar} (E_{p+q} - E_{p-hk}) t} \times (\bar{v}(p - \hbar k, s') \gamma_{\mu} \frac{\slashed{p} - m}{E_p} \gamma_0 v(p + q, s)) \hat{d}_{p+q,s}^\dagger \hat{d}_{p-hk,s'} \\
+ \sqrt{\frac{1}{E_{p-q}}} \sqrt{\frac{1}{E_{p+hk}}} e^{\frac{\imath}{\hbar} (E_{p+hk} - E_{p-q}) t} \times (\bar{u}(p + \hbar k, s') \gamma_{\mu} \frac{\slashed{p} + m}{E_p} \gamma_0 u(p - q, s)) \hat{b}_{p+hk,s'}^\dagger \hat{b}_{p-q,s} \\
- \sqrt{\frac{1}{E_p}} \sqrt{\frac{1}{E_{p+q+hk}}} e^{-\frac{\imath}{\hbar} (E_{p+q+hk}) t} \times (\bar{u}(p + q + \hbar k, s') \gamma_0 \frac{\slashed{p} + \hbar k + m}{E_{p+hk}} \gamma_{\mu} u(p, s)) \hat{b}_{p+q+hk,s}^\dagger \hat{b}_{p,s} \tag{D.2}
\]

We perform the shift \(p \to p + \hbar k\) in the first term and \(p \to p - \hbar k\) in the last term of Eq. (D.2) and introduce the initial and final momentum variables, \(p_i\) and \(p_f\) according to

\[
q = p_f - p_i - \hbar k , \quad p_i = p - \hbar k , \quad p_f = p + q , \\
p_i = p - q , \quad p_f = p + \hbar k \tag{D.3}
\]

Using these redefinitions we can obtain Eqs. (4.57) – (4.59) in terms of the transition matrices \(N^{(e)}_{\mu}\) and \(N^{(p)}_{\mu}\) for electrons and positrons respectively.

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