Lipschitz Regularity for Elliptic Equations with Random Coefficients

SCOTT N. ARMSTRONG & JEAN-CHRISTOPHE MOURRAT

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Abstract

We develop a higher regularity theory for general quasilinear elliptic equations and systems in divergence form with random coefficients. The main result is a large-scale $L^\infty$-type estimate for the gradient of a solution. The estimate is proved with optimal stochastic integrability under a one-parameter family of mixing assumptions, allowing for very weak mixing with non-integrable correlations to very strong mixing (for example finite range of dependence). We also prove a quenched $L^2$ estimate for the error in homogenization of Dirichlet problems. The approach is based on subadditive arguments which rely on a variational formulation of general quasilinear divergence-form equations.

1. Introduction

1.1. Informal Summary of the Main Results

We are interested in the stochastic homogenization of divergence-form elliptic equations and systems which take the general quasilinear form

$$-\nabla \cdot (a(\nabla u(x), x)) = 0 \quad \text{in } U \subseteq \mathbb{R}^d. \quad (1.1)$$

The precise assumptions are stated in detail below, but let us mention here that the map $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is uniformly monotone and Lipschitz in its first argument. In addition, $x \mapsto a(\cdot, x)$ is a stationary random field satisfying a quantitative ergodic assumption.

In this paper, we prove a “quenched” Lipschitz estimate for solutions of (1.1) which is optimal in terms of the stochastic integrability of the random variables appearing in the estimates. While the results are new for linear equations and even under strongly mixing conditions (see however the discussion of [3,24] below), they hold for the most general nonlinear divergence-form elliptic operators and...
under quite weak mixing conditions. We are primarily motivated by the fact that strong gradient bounds play a central role in the quantitative theory of stochastic homogenization for elliptic equations. This is because the magnitude of the gradient of a solution controls how sensitively it depends on the coefficients—and it turns out that good estimates of the latter, combined with appropriate concentration inequalities, yield optimal quantitative bounds in homogenization. This was shown for linear equations by Gloria and Otto [25–27] and Gloria, Neukamm and Otto [22,23]. The work in this paper is the crucial step to developing an optimal quantitative theory of stochastic homogenization for general quasilinear equations and systems in divergence form, as well as for improving the stochastic integrability of the current linear theory, as we will show in a future work.

To give a better idea of the gradient estimate we prove, we recall that a solution \( u \) of the linear equation
\[
-\nabla \cdot (A(x) \nabla u) = \nabla \cdot f(x) \quad \text{in } B_1,
\]
under the assumption that the coefficient matrix \( A(x) \) and vector field \( f(x) \) are Hölder continuous, satisfies the following pointwise bound:
\[
|\nabla u(0)|^2 \leq C \left( 1 + \int_{B_1} |\nabla u(x)|^2 \, dx \right),
\]
where the constant \( C \) depends in particular on the regularity of the coefficients. Of course, the assumption of the Hölder continuity of \( A \) is crucial, as the best regularity for general equations with rapidly oscillating coefficients is \( W^{1,2+\varepsilon} \) (Meyers’ estimate, for Sobolev regularity) and \( C^{0,\varepsilon} \) (De Giorgi-Nash-Moser, for Hölder regularity) for some \( \varepsilon > 0 \) depending on the ellipticity. Therefore the estimate (1.2) is not scale invariant, and in particular at very large scales (that is, with the ball \( B_1 \) replaced by \( B_R \) with radius \( R \gg 1 \)) the estimate is false, in general, even if \( A \) is smooth.

The primary purpose of this paper is to show that, nevertheless, an equation with random coefficients has better regularity than a general equation, and we can in fact prove that, due to statistical effects, an estimate like (1.2) holds on large scales. The main result, stated precisely in Theorem 1.1 below, states roughly that, under appropriate mixing conditions on the coefficients, for any \( R \gg 1 \), a solution \( u \) of (1.1) in a domain \( U \supseteq B_R \) satisfies
\[
\int_{B_r} |\nabla u(x)|^2 \, dx \leq C \left( 1 + \int_{B_R} |\nabla u(x)|^2 \, dx \right) \quad \text{for every } r \in \left[ X, \frac{1}{2} R \right].
\]
(1.3)

The main difference between (1.2) and (1.3) is that the latter estimate holds only for balls with radii larger than a random “minimal radius” \( X \), while the former holds in every ball (hence pointwise). The random variable \( X \) is almost surely finite—but not bounded—and thus the central task is to estimate the probability that \( X \) is very large. That is, we would like to specify which of the stochastic moments of \( X \) are bounded under various quantitative “mixing” assumptions on the coefficients. In this paper, we prove (1.3), with optimal stochastic integrability estimates on \( X \), under a continuum of mixing conditions on the coefficients, ranging from relatively weak mixing (allowing for non-integrable correlations) to very strong assumptions.