JACOBI–TRUDI FORMULA FOR REFINED DUAL STABLE GROTHENDIECK POLYNOMIALS

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Abstract. In 2007 Lam and Pylyavskyy found a combinatorial formula for the dual stable Grothendieck polynomials, which are the dual basis of the stable Grothendieck polynomials with respect to the Hall inner product. In 2016 Galashin, Grinberg, and Liu introduced refined dual stable Grothendieck polynomials by putting additional sequence of parameters in the combinatorial formula of Lam and Pylyavskyy. Grinberg conjectured a Jacobi–Trudi type formula for refined dual stable Grothendieck polynomials. In this paper this conjecture is proved by using bijections of Lam and Pylyavskyy.

1. Introduction

In 1982 Lascoux and Schützenberger [11] introduced Grothendieck polynomials, which are representatives of the structure sheaves of the Schubert varieties in a flag variety. Fomin and Kirillov [4] studied Grothendieck polynomials combinatorially and introduced stable Grothendieck polynomials, which are stable limits of Grothendieck polynomials. Buch [3] found a combinatorial formula for stable Grothendieck polynomials using set-valued tableaux. Lam and Pylyavskyy [10] first studied dual stable Grothendieck polynomials \( g_\lambda(x) \), which are the dual basis of the stable Grothendieck polynomials under the Hall inner product. They also found a combinatorial formula for \( g_\lambda(x) \) in terms of reverse plane partitions. Their formula gives a combinatorial way to expand \( g_\lambda(x) \) in terms of Schur functions \( s_\mu(x) \).

Refined dual stable Grothendieck polynomials \( g_\lambda/\mu(x) \) are inhomogeneous symmetric functions in variables \( x = (x_1, x_2, \ldots) \). Galashin, Grinberg, and Liu [7] introduced refined dual stable Grothendieck polynomials \( \tilde{g}_{\lambda/\mu}(x;t) \) by putting an additional sequence \( t = (t_1, t_2, \ldots) \) of parameters in the combinatorial formula of Lam and Pylyavskyy. They showed that \( \tilde{g}_{\lambda/\mu}(x;t) \) is also symmetric in \( x \). Refined dual stable Grothendieck polynomials generalize both dual stable Grothendieck polynomials and Schur function: if \( t_i = 1 \) for all \( i \geq 1 \), then \( \tilde{g}_{\lambda/\mu}(x;t) = g_{\lambda/\mu}(x) \), and if \( t_i = 0 \) for all \( i \geq 1 \), then \( \tilde{g}_{\lambda/\mu}(x;t) = s_{\lambda/\mu}(x) \). Galashin [5] found a Littlewood–Richardson rule to expand \( \tilde{g}_{\lambda/\mu}(x;t) \) in terms of Schur functions. Yeliuszizov [15] further studied (dual) stable Grothendieck polynomials and showed the following Jacobi–Trudi type formula for \( \tilde{g}_\lambda(x;t) \) originally conjectured by Darij Grinberg:

\[
\tilde{g}_\lambda(x;t) = \det \left( e_{\lambda'_i - i + j}(x_1, x_2, \ldots, t_1, t_2, \ldots, t_{\lambda'_i - 1}) \right)_{1 \leq i, j \leq n},
\]

where \( e_k(z_1, z_2, \ldots) = \sum_{i_1 < i_2 < \cdots < i_k} z_{i_1} z_{i_2} \cdots z_{i_k} \) is the \( k \)th elementary symmetric function and we define \( e_0(z_1, z_2, \ldots) = 1 \) and \( e_k(z_1, z_2, \ldots) = 0 \) for \( k < 0 \). We also note that the Jacobi–Trudi formula (1.1) for the case \( t_i = 1 \) for all \( i \geq 1 \) was first studied and conjectured by Bozgan [2] in 2011.

The main result of this paper is the following Jacobi–Trudi formula for the refined dual stable Grothendieck polynomial \( \tilde{g}_{\lambda/\mu}(x;t) \), which was also conjectured by Darij Grinberg [9, slide 72] in 2015. See Section 2 for the precise definitions.

Theorem 1.1. Let \( \lambda \) and \( \mu \) be partitions with \( \ell(\lambda') \leq n \). Then

\[
\tilde{g}_{\lambda/\mu}(x;t) = \det \left( e_{\lambda'_i - \mu'_j - i + j}(x_1, x_2, \ldots, t_{\mu'_j + 1}, t_{\mu'_j + 2}, \ldots, t_{\lambda'_i - 1}) \right)_{1 \leq i, j \leq n},
\]

where, if \( \mu'_j + 1 > \lambda'_i - 1 \), the \( (i, j) \) entry is defined to be \( e_{\lambda'_i - \mu'_j - i + j}(x_1, x_2, \ldots) \).
Note that if \( t_i = 0 \) for all \( i \geq 1 \), then Theorem 1.1 reduces to the classical (dual) Jacobi–Trudi formula for the Schur function \( s_{\lambda/\mu}(x) \). Theorem 1.1 gives another proof of the fact that \( \tilde{g}_{\lambda/\mu}(x;t) \) is symmetric in the variables \( x \).

There is a standard combinatorial method to prove a Jacobi–Trudi type formula using the Lindström–Gessel–Viennot lemma \([8,13]\). First interpret the determinant as a signed sum of \( n \)-paths, i.e., sequences \( (p_1, \ldots, p_n) \) of \( n \) paths in a certain lattice. If there are intersections among the \( n \) paths, choose an intersection in a controlled way and exchange the “tails” of the two paths through this intersection. This will give a sign-reversing involution on the total \( n \)-paths leaving only the non-intersecting \( n \)-paths as fixed points. Then one interprets the non-intersecting \( n \)-paths as the desired tableaux by a simple bijection.

However, the Jacobi–Trudi formula in Theorem 1.1 cannot be proved in this way. Because of the restriction of a path depending on the initial point, the usual method of exchanging tails is not applicable. In this paper we prove Theorem 1.1 by finding a sign-reversing involution on certain \( n \)-paths using two maps introduced by Lam and Pylyavskyy \([10]\) as intermediate steps in their bijection between reverse plane partitions and pairs of semistandard Young tableaux and so-called elegant tableaux.

The remainder of this paper is organized as follows. In Section 2 we give necessary definitions and notation, and show that the determinant in Theorem 1.1 is a generating function for “semi-noncrossing” \( n \)-paths. In Section 3 we define vertical tableaux and give a bijection between them and \( n \)-paths. In Section 4 we define RSE-tableaux and review two bijections \( \phi^- \) and \( \phi^+ \) on RSE-tableaux due to Lam and Pylyavskyy. In Section 5 we extend the definition of RSE-tableaux to skew shapes and study properties of the maps \( \phi^- \) and \( \phi^+ \) on skew RSE-tableaux. In Section 6 we give a sign-reversing involution on semi-noncrossing \( n \)-paths using these maps and complete the proof of Theorem 1.1. In Section 7 we give a concrete example of the sign-reversing involution defined in Section 6.

We note that Amanov and Yeliussizov \([1]\) proved Theorem 1.1 independently about the same time this paper was written. Their proof uses a sign-reversing involution on 3-dimensional lattice paths. It would be interesting to see whether there is a connection between their sign-reversing involution and ours.

2. Definitions and notation

In this section we give basic definitions and notation which will be used throughout this paper.

A **partition** \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \) is a weakly decreasing sequence of positive integers. Each \( \lambda_i \) is called a part of \( \lambda \). The **length** \( \ell(\lambda) \) of \( \lambda \) is the number of parts. Sometimes we will append some zeros at the end of \( \lambda \) so that for example \( (4, 3, 1) \) and \( (4, 3, 1, 0, 0) \) are considered as the same partition, and \( \lambda_i = 0 \) whenever \( i > \ell(\lambda) \).

The **Young diagram** of \( \lambda \) is defined to be the set \( \{(i,j) \in \mathbb{Z}^2 : 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\} \). From now on we will identify \( \lambda \) with its Young diagram. The Young diagram of \( \lambda \) will be visualized by placing a unit square, called a **cell**, in the \( ith \) row and \( jth \) column for each \( (i,j) \in \lambda \). The **conjugate** \( \lambda' \) of \( \lambda \) is defined to be the partition given by \( \lambda' = \{(i,j) : (j,i) \in \lambda\} \), see Figure 1.

For two partitions \( \lambda \) and \( \mu \), we write \( \mu \subseteq \lambda \) if \( \mu_i \leq \lambda_i \) for all \( i \geq 1 \). In this case the **skew shape** \( \lambda/\mu \) is defined to be the set-theoretic difference \( \lambda - \mu \) of their Young diagrams.
where $c$ is just a map define col $\leq R$ therefore the weight of $\lambda/\mu$ skew shape obtained from $\lambda/\mu$ strictly increasing in each column. The set of SSYTs of shape $\lambda/\mu$ strictly increasing in each row and column, i.e., $R(i,j) \leq R(i,j+1)$ and $R(i,j) \leq R(i+1,j)$ whenever these values are defined. The set of RPPs of shape $\lambda/\mu$ is denoted by RPP($\lambda/\mu$). For $R \in$ RPP($\lambda/\mu$), the weight of $R$ is defined by

\[ \text{wt}(R) = \prod_{i \geq 1} a_i(R) \frac{b_i(R)}{t_i}, \]

where $a_i(R)$ is the number of columns containing an $i$ and $b_i(R)$ is the number of cells $(i,j)$ such that $R(i,j) = R(i+1,j)$. For example if $R$ is the RPP in Figure 3, then \( \text{wt}(R) = x_1^3 x_2^2 t_1 t_2^2 t_4. \)

A reverse plane partition (RPP) of shape $\lambda/\mu$ is a tableau $R : \lambda/\mu \to \mathbb{Z}^+$ such that the entries weakly increase in each row and column. An elegant tableau $E$ of a certain skew shape $\lambda/\mu$ such that $1 \leq E(i,j) \leq i - 1$ for all $(i,j) \in \lambda/\mu$. See Figure 3. Elegant tableaux were first defined by Lenart [12, Theorem 2.7] and further studied by Lam and Pylyavskyy [10].

Note that if $R \in$ SSYT($\lambda/\mu$) $\subset$ RPP($\lambda/\mu$), then $R$ has no repeated entries in each column and therefore the weight of $R$ defined in (2.1) is given by

\[ \text{wt}(R) = x_R = x_1^{c_1(T)} x_2^{c_2(T)} \cdots, \]

where $c_i(T)$ is the number of $i$’s in $R$. For example, if $R$ is the SSYT in Figure 3 then \( \text{wt}(R) = x_1^2 x_2^2 x_4^4 x_5. \)
Let \( x = (x_1, x_2, \ldots) \) and \( t = (t_1, t_2, \ldots) \) be sequences of variables. The refined dual stable Grothendieck polynomial \( \tilde{g}_{\lambda/\mu}(x; t) \) is defined by

\[
\tilde{g}_{\lambda/\mu}(x; t) = \sum_{R \in \mathcal{RPP}(\lambda/\mu)} \text{wt}(R).
\]

Now we recall the main result, Theorem \[\ref{main_theorem}\] if \( \lambda \) and \( \mu \) are partitions with \( \ell(\lambda') \leq n \),

\[
\tilde{g}_{\lambda/\mu}(x; t) = \det \left( e_{\lambda'_i - \mu'_j - i + j}(x_1, x_2, \ldots, t_{\mu'_j+1}, t_{\mu'_j+2}, \ldots, t_{\lambda'_i-1}) \right)_{1 \leq i, j \leq n}.
\]

Note that if \( \mu \not\subseteq \lambda \), by definition, \( \tilde{g}_{\lambda/\mu}(x; t) = 0 \). It is easy to see that in this case the above determinant also vanishes because if \( \mu \not\subseteq \lambda \), then \( \lambda'_i < \mu'_j \) for some \( 1 \leq r \leq n \), which implies that the \((i, j)\) entry is zero for all \( r + 1 \leq i \leq n \) and \( 1 \leq j \leq r \). Moreover, it is also easy to see that if \( \ell(\lambda') = m < n \), then the above determinant is equal to its principal minor consisting of the first \( m \) rows and columns. Therefore it is sufficient to show Theorem \[\ref{main_theorem}\] for the case \( \mu \subseteq \lambda \) and \( \ell(\lambda') = n \).

From now on we always assume that \( \mu \subseteq \lambda \) and \( \ell(\lambda') = n \).

Let \( \omega \) be the smallest infinite ordinal number and let

\[
\mathbb{N} = \{0, 1, 2, \ldots\},
\]

\[
\mathbb{N}_\omega = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, 2\omega\},
\]

\[
G = \mathbb{N} \times \mathbb{N}_\omega,
\]

where the numbers are ordered as usual by

\[
0 < 1 < 2 < \cdots < \omega < \omega + 1 < \omega + 2 < \cdots < 2\omega.
\]

We will also write \( \omega + i \) as \( i^* \).

A path from \((a, 0)\) to \((b, 2\omega)\) is a pair \((s_1, s_2)\) of infinite sequences \( s_j = ((u^{(1)}_{i,j}, v^{(1)}_{i,j}), (u^{(2)}_{i,j}, v^{(2)}_{i,j}), \ldots) \), \( j = 1, 2 \), of points in \( G \) satisfying the following conditions:

- The steps \((u^{(1)}_{i,j}, v^{(1)}_{i,j}) - (u^{(1)}_{i,j}, v^{(1)}_{i,j})\), for \( i \geq 0 \) and \( j = 1, 2 \), consist of up steps \((0, 1)\) and diagonal steps \((1, 1)\).
- \((u^{(2)}_{0,0}, v^{(2)}_{0,0}) = (a, 0), \) \( v^{(2)}_{0,0} = \omega \) and

\[
\lim_{m \to \infty} u^{(1)}_{m,0} = u^{(2)}_{m,0}, \quad \lim_{m \to \infty} u^{(2)}_{m,0} = b.
\]

The weight of a path \( p \) is defined to be

\[
\text{wt}(p) = \prod_{i \geq 1} a_i(p)^{a_i(t)} b_i(t)^{b_i(t)}
\]

where \( a_i(p) \) (resp. \( b_i(p) \)) is the set of diagonal steps of \( p \) ending at height \( i \) (resp. \( \omega + i \)). See Figure \[\ref{fig:paths}\].

Suppose that \( \lambda \) and \( \mu \) are partitions with \( \mu \subseteq \lambda \) and \( \ell(\lambda') \leq n \). Denote by \( \mathcal{L}_{\lambda/\mu}(i, j) \) the set of all paths from \((\mu'_i + n - i, 0)\) to \((\lambda'_j + n - j, 2\omega)\) in which there is no diagonal step between the lines \( y = \omega \) and \( y = \omega + \mu'_i \) and no diagonal step above the line \( y = \omega + \lambda'_j - 1 \). See Figure \[\ref{fig:paths}\] for a typical example of a path in \( \mathcal{L}_{\lambda/\mu}(i, j) \). It is clear from the construction that

\[
e_{\lambda'_i - \mu'_j - i + j}(x_1, x_2, \ldots, t_{\mu'_j+1}, t_{\mu'_j+2}, \ldots, t_{\lambda'_i-1}) = \sum_{p \in \mathcal{L}_{\lambda/\mu}(i, j)} \text{wt}(p).
\]

An \( n \)-path is an \( n \)-tuple of paths. Denote by \( \mathfrak{S}_n \) the set of permutations on \( \{1, 2, \ldots, n\} \). For a permutation \( \pi \in \mathfrak{S}_n \), let \( \mathcal{L}_{\lambda/\mu}(\pi) \) denote the set of \( n \)-paths \( p = (p_1, \ldots, p_n) \) such that \( p_i \in \mathcal{L}_{\lambda/\mu}(i, \pi(i)) \) for all \( 1 \leq i \leq n \). Define

\[
\mathcal{L}_{\lambda/\mu} = \bigcup_{\pi \in \mathfrak{S}_n} \mathcal{L}_{\lambda/\mu}(\pi).
\]

The type of an \( n \)-path \( p \in \mathcal{L}_{\lambda/\mu} \), denoted \( \text{type}(p) \), is the permutation \( \pi \) for which \( p \in \mathcal{L}_{\lambda/\mu}(\pi) \). Note that \( \text{type}(p) \) is uniquely determined because the starting points \((\mu'_i + n - i, 0)\) and the ending points \((\lambda'_j + n - j, 2\omega)\) are all distinct. The weight of \( p = (p_1, \ldots, p_n) \in \mathcal{L}_{\lambda/\mu} \) is defined by

\[
\text{wt}(p) = \text{sign}(\text{type}(p)) \prod_{i=1}^n \text{wt}(p_i).
\]
The following notation will be used throughout this paper. See Figure 5 for an illustration.

**Notation 2.1.** Fix partitions $\lambda$ and $\mu$ with $\mu \subseteq \lambda$, $\ell(\lambda) = \ell$, and $\ell(\lambda') = \lambda_1 = n$. Define $d_1 > d_2 > \cdots > d_r$ to be the distinct integers in $\{\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}\}$ and let $d_0 = n$ and $d_{r+1} = 0$. For $1 \leq i \leq r + 1$,

- $m_i$ denotes the multiplicity of the part $d_i$ in $\mu$, where the multiplicity of $d_{r+1} = 0$ in $\mu$ is defined to be $\ell(\lambda) - \ell(\mu)$,
- $M_i = m_1 + \cdots + m_i$, and
- $D_i = \{d_i + 1, d_i + 2, \ldots, d_{i-1}\}$.

Note that for $1 \leq i, j \leq n$, we have $\mu_i' = \mu_j'$ if and only if $i, j \in D_k$ for some $1 \leq k \leq r$. 

**Figure 4.** The left diagram is a path from $(2, 0)$ to $(7, 2\omega)$ with weight $x_2x_5t_1t_4t_5$. The height of the ending point of each diagonal step is shown. The right diagram illustrates a typical path in $L_{\lambda/\mu}(i, j)$, which cannot have diagonal steps in the gray areas.

**Figure 5.** An illustration of Notation 2.1 for a given $\lambda/\mu$. Every letter is an integer except $D_i$'s, which are sets of column indices.
Figure 6. Let \( \lambda = (2, 2, 2, 1) \) and \( \mu = (1) \) so that \( \lambda'_1 = 4, \lambda'_2 = 3 \) and \( \mu'_1 = 1 \).

The left diagram shows a 2-path \((p_1, p_2) \in \mathcal{L}_{\lambda/\mu}\). If we switch the tails of \( p_1 \) and \( p_2 \) after their unique intersection, we obtain the resulting 2-path \((p'_1, p'_2) \) as shown on the right. The gray area shows (part of) the restriction on the path whose starting point is \( A_1 = (\mu_1 + 2 - 1, 0) = (2, 0) \). Since \( p'_1 \) has a diagonal step in the gray area, \((p'_1, p'_2) \notin \mathcal{L}_{\lambda/\mu}\).

Suppose \( p = (p_1, \ldots, p_n) \in \mathcal{L}_{\lambda/\mu} \). We say that \( p \) is noncrossing if \( p_i \) and \( p_j \) have no common points for all \( i \neq j \), and that \( p \) is semi-noncrossing if \( p_i \) and \( p_j \) have no common points whenever \( i \) and \( j \) are distinct elements in \( D_k \) for some \( 1 \leq k \leq r \). Denote by \( \mathcal{L}^{SNC}_{\lambda/\mu} \) (resp. \( \mathcal{L}^{NC}_{\lambda/\mu} \)) the set of noncrossing (resp. semi-noncrossing) paths in \( \mathcal{L}_{\lambda/\mu} \). Note that if \( \mu = \emptyset \), then \( \mathcal{L}^{NC}_{\lambda/\mu} = \mathcal{L}^{SNC}_{\lambda/\mu} \).

By expanding the determinant in Theorem 1.1 using (2.2) we have

\[
\det \left( e_{\lambda'_i - \mu'_j - i + j} (x_1, x_2, \ldots, t_{\mu'_j + 2}, \ldots, t_{\lambda'_i - 1}) \right)_{1 \leq i, j \leq n} = \sum_{p \in \mathcal{L}_{\lambda/\mu}} \wt(p).
\]

The standard method of the Lindström–Gessel–Viennot lemma [8, 13] interprets a determinant as a weighted sum of noncrossing \( n \)-paths via a sign-reversing involution which exchanges “tails” of intersecting paths. Roughly speaking, in order for this to work “local” changes of the steps in an \( n \)-path must be allowed. Such “local” changes are not allowed for an \( n \)-path \( p = (p_1, \ldots, p_n) \) in \( \mathcal{L}_{\lambda/\mu} \), because each path \( p_i \) has the “global” restriction that there are no diagonal steps between the lines \( y = \omega \) and \( y = \omega + \mu'_i \). For example, see Figure 6.

However, it is possible to cancel all \( n \)-paths except for the semi-noncrossing \( n \)-paths.

**Proposition 2.2.** Let \( \lambda \) and \( \mu \) be partitions with \( \mu \subseteq \lambda \), \( \ell(\lambda) = \ell \), and \( \ell(\lambda') = n \). Then

\[
\det \left( e_{\lambda'_i - \mu'_j - i + j} (x_1, x_2, \ldots, t_{\mu'_j + 2}, \ldots, t_{\lambda'_i - 1}) \right)_{1 \leq i, j \leq n} = \sum_{p \in \mathcal{L}^{SNC}_{\lambda/\mu}} \wt(p).
\]

**Proof.** By (2.3) it is sufficient to show that

\[
\sum_{p \in \mathcal{L}_{\lambda/\mu}} \wt(p) = \sum_{p \in \mathcal{L}^{SNC}_{\lambda/\mu}} \wt(p).
\]

We will cancel all paths in \( \mathcal{L}_{\lambda/\mu} \setminus \mathcal{L}^{SNC}_{\lambda/\mu} \) using the standard method of switching tails of two paths. More precisely, suppose \( p = (p_1, \ldots, p_n) \in \mathcal{L}_{\lambda/\mu} \setminus \mathcal{L}^{SNC}_{\lambda/\mu} \). Then we can find the smallest integer \( k \) such that \( p_i \) and \( p_j \) have common points for some \( i \neq j \) in \( D_k \). Choose such \( i \) and \( j \) so that \( (i, j) \) is the smallest in the lexicographic order. Let \((a, b)\) be the last intersection of \( p_i \) and \( p_j \). Let \( p'_i \) and \( p'_j \) be the paths obtained from \( p_i \) and \( p_j \) respectively by exchanging the subpaths after \((a, b)\).

If \( \text{type}(p) = \pi \), then \( p_i \in \mathcal{L}_{\lambda/\mu}(i, \pi(i)) \) and \( p_j \in \mathcal{L}_{\lambda/\mu}(j, \pi(j)) \). Since \( i, j \in D_k \), we have \( \mu'_i = \mu'_j \). Therefore neither \( p_i \) nor \( p_j \) has diagonal steps between heights \( \omega \) and \( \omega + \mu'_i = \omega + \mu'_j \), which ensures that \( p'_i \notin \mathcal{L}_{\lambda/\mu}(i, \pi(j)) \) and \( p'_j \notin \mathcal{L}_{\lambda/\mu}(j, \pi(i)) \). Let \( p' \) be the \( n \)-path obtained from \( p \) by replacing \( p_i \) and \( p_j \) by \( p'_i \) and \( p'_j \) respectively. Then \( p \in \mathcal{L}_{\lambda/\mu} \setminus \mathcal{L}^{SNC}_{\lambda/\mu} \) and \( \text{type}(p') = \pi(i, j) \), where
1 2 3 4 5 6 7 8 9

Figure 7. The vertical diagram $V(\alpha)$ on the left and the vertical diagram $\text{col}_{\geq 4}(V(\lambda))$ on the right for the composition $\alpha = (0, 0, 4, 3, 5, 0, 2, 1, 1)$. For visibility the column indices are written above the diagrams.

1 2 3 4 5 6 7 8 9

Figure 8. The diagram $V(\beta) / V(\alpha)$ for $\alpha = (1, 3, 1, 2, 2, 0)$ and $\beta = (1, 6, 5, 7, 5, 5)$ is shown with the white cells.

$(i, j)$ is the transposition. Therefore $\text{wt}(p) = - \text{wt}(p')$. It is easily seen that this argument shows that $\sum_{p \in L(\alpha)/L(\mu)} \text{wt}(p) = 0$, hence (2.4). □

3. Vertical tableaux and $n$-paths

In this section we introduce a notion of vertical tableaux and give a simple bijection between them and certain $n$-paths.

A composition is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of nonnegative integers. The vertical diagram of a composition $\alpha$ is defined by

$$V(\alpha) = \{(i, j) \in \mathbb{Z}^2 : 1 \leq j \leq n, 1 \leq i \leq \alpha_j\}.$$  

Similarly to Young diagrams each element $(i, j)$ in the vertical diagram is represented by a cell in row $i$ and column $j$. Since $\lambda = V(\lambda')$ as subsets of $\mathbb{Z}^2$, we will also consider the Young diagram of $\lambda$ as a vertical diagram. The notation used for Young diagrams is naturally extended to vertical diagrams. For example, for a vertical diagram $V$, define $\text{col}_{\geq k}(V) = \{(i, j) \in V : j \geq k\}$, and for two vertical diagrams $V_1$ and $V_2$ with $V_1 \subseteq V_2$, define $V_2 / V_1$ to be the set-theoretic difference $V_2 - V_1$. We say that $V_1$ and $V_2$ are the inner shape and the outer shape of $V_2 / V_1$, respectively. See Figures 7 and 8.

For vertical diagrams $V_1$ and $V_2$ with $V_1 \subseteq V_2$, a vertical tableau of shape $V_2 / V_1$ is a filling of $V_2 / V_1$ with numbers in $\{1 < 2 < \cdots < 1^* < 2^* < \cdots\}$ such that the entries are strictly increasing in each column. See the right diagram in Figure 9 for an example of a vertical tableau. Let $\text{VT}(V_2 / V_1)$ denote the set of vertical tableaux of shape $V_2 / V_1$.

**Definition 3.1.** [The map Tab sending $n$-paths to vertical tableaux] Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be compositions with $V(\alpha) \subseteq V(\beta)$. Define $\mathcal{L}(\alpha, \beta)$ to be the set of $n$-paths $p = (p_1, \ldots, p_n)$, where $p_i$ is a path from $(\alpha_i + n - i, 0)$ to $(\beta_i + n - i, 2\omega)$.

For $p = (p_1, \ldots, p_n) \in \mathcal{L}(\alpha, \beta)$, define $\text{Tab}(p)$ to be the vertical tableau $T \in \text{VT}(V(\beta) / V(\alpha))$ constructed as follows. For each diagonal step of $p_i$, if its ending point is $(a, b)$, fill the $(a - n + i - 1, i)$-entry of $T$ with $b$.  

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
\end{array}
\]
Proposition 3.2. Suppose that \( p \in L_{\lambda/\mu} \) satisfies \( \text{Tab}(p) \in \text{VT}(\pi(\lambda)/\mu) \). Let type\( (p) = \sigma \). Then \( \sigma(\lambda) = \pi(\lambda) \) or \( \sigma(\lambda) = \pi(\lambda) \), which implies \( \sigma(\lambda) = \pi(\lambda) \), or equivalently,

\[
(\lambda'_{\pi_1} - \pi_1 + 1, \ldots, \lambda'_{\pi_n} - \pi_n + n) = (\lambda'_{\pi_1} - \sigma_1 + 1, \ldots, \lambda'_{\pi_n} - \sigma_n + n).
\]

By subtracting \( i \) from the \( i \)th component we also have

\[
(\lambda'_{\pi_1} - \pi_1, \ldots, \lambda'_{\pi_n} - \pi_n) = (\lambda'_{\sigma_1} - \sigma_1, \ldots, \lambda'_{\sigma_n} - \sigma_n).
\]

See Figure 9 for an example of the map \( \text{Tab} \) in Definition 3.1. The following proposition is straightforward to verify.

**Proposition 3.2.** Following the notation in Definition 3.1, the map \( \text{Tab} \) is a bijection from \( L(\alpha, \beta) \) to \( \text{VT}(\beta/\alpha) \). Moreover, if \( \text{Tab}(p) = T \), then for every positive integer \( h \) the total number of diagonal steps in \( p \) ending at height \( h \) (resp. \( \omega + h \)) is equal to the number of times \( h^* \) appears in \( T \).

For a partition \( \lambda \) with \( \ell(\lambda') = n \) and a permutation \( \pi \in \mathcal{S}_n \), we define \( \pi(\lambda) \) to be the vertical diagram given by

\[
\pi(\lambda) = \{(i, j) \in \mathbb{Z}^2 : 1 \leq j \leq n, 1 \leq i \leq \lambda'_{\pi_j} - \pi_j + j\}.
\]

Note that if \( \pi \) is the identity permutation then \( \pi(\lambda) \) is the Young diagram of \( \lambda \). One may worry about the situation that \( \lambda'_{\pi_j} - \pi_j + j < 0 \) in the definition of \( \pi(\lambda) \). Since we will only consider \( \pi(\lambda) \) when \( \text{VT}(\pi(\lambda)/\mu) \) is nonempty (or equivalently, when \( \mu \subseteq \pi(\lambda) \)) this will never occur, see the paragraph after the proof of Lemma 3.3.

The following lemma shows that the type of \( p \in L_{\lambda/\mu} \) is encoded in the outer shape of the vertical tableau \( \text{Tab}(p) \) while the inner shape of \( \text{Tab}(p) \) is always \( \mu \). See Figure 18 for an example.

**Lemma 3.3.** For \( p \in L_{\lambda/\mu} \), we have \( \text{type}(p) = \pi \) if and only if \( \text{Tab}(p) \in \text{VT}(\pi(\lambda)/\mu) \).

**Proof.** Suppose that \( p = (p_1, \ldots, p_n) \in L_{\lambda/\mu} \) has \( \text{type}(p) = \pi \). Let \( \alpha \) and \( \beta \) be the compositions given by \( \alpha_i = \mu'_i \) and \( \beta_i = \lambda'_{\pi_i} - \pi_i + i \). Then \( V(\beta)/V(\alpha) = \pi(\lambda)/\mu \). Since \( p_i \) is a path from \((\mu'_i + n - i, 0) = (\alpha_i + n - i, 0)\) to \((\lambda'_{\pi_i} + n - \pi_i, 2\omega) = (\beta_i + n - i, 2\omega)\), we have \( p \in \text{Tab}(\beta/\alpha) \). By Proposition 3.2, \( \text{Tab}(p) \in \text{VT}(V(\beta)/V(\alpha)) = \text{VT}(\pi(\lambda)/\mu) \).

Conversely, suppose that \( p \in L_{\lambda/\mu} \) satisfies \( \text{Tab}(p) \in \text{VT}(\pi(\lambda)/\mu) \). Let \( \text{type}(p) = \sigma \). Then by what we just proved, we obtain \( \text{Tab}(p) \in \text{VT}(\pi(\lambda)/\mu) \), which implies \( \sigma(\lambda) = \pi(\lambda) \), or equivalently,

\[
(\lambda'_{\sigma_1} - \pi_1 + 1, \ldots, \lambda'_{\pi_n} - \pi_n + n) = (\lambda'_{\sigma_1} - \sigma_1 + 1, \ldots, \lambda'_{\pi_n} - \sigma_n + n).
\]

By subtracting \( i \) from the \( i \)th component we also have

\[
(\lambda'_{\pi_1} - \pi_1, \ldots, \lambda'_{\pi_n} - \pi_n) = (\lambda'_{\sigma_1} - \sigma_1, \ldots, \lambda'_{\sigma_n} - \sigma_n).
\]
Both sequences in the above equation are rearrangements of \((\lambda'_1 - 1, \ldots, \lambda'_n - n)\), which is a strictly decreasing sequence. Since there are no repeated entries in this sequence, the rearrangements must be identical and we obtain \(\pi = \sigma\). Hence \(\text{type}(p) = \pi\) and the proof is completed. \(\square\)

Note that in the proof of the above lemma if \(\text{type}(p) = \pi\), we must have \(\lambda'_1 + n - \pi_i \geq \mu'_i + n - i\), or, equivalently, \(\lambda'_i - \pi_i + i \geq \mu'_i \geq 0\). Hence in this case we always have \(\lambda'_i - \pi_i + i \geq 0\) in the definition of \(\pi(\lambda)\).

4. RSE-tableaux and bijections of Lam and Pylyavskyy

In this section we define RSE-tableaux and two maps \(\phi_-\) and \(\phi_+\) on these objects. The notion of RSE-tableaux was introduced implicitly by Lam and Pylyavskyy [10, Proof of Theorem 9.8] in their bijection between RPPs and pairs of SSYTs and elegant tableaux. The maps \(\phi_-\) and \(\phi_+\) described in this sections are intermediate steps in their bijection. We assume reader’s familiarity with the RSK algorithm and its basic properties. See [5, Section 1.1] or [13, Section 7.11] for a standard reference. In particular, we will use Row Bumping Lemma and Proposition in [5, Section 1.1].

Definition 4.1. An RSE-tableau of shape \(\lambda\) of level \(k\) is a pair \(T = (R, E)\) satisfying the following conditions:

- \(R \in \text{RPP}(\nu)\) with row \(k\) of \(R\) marked,
- \(\text{row}_{\geq k}(R)\) is an SSYT,
- \(E\) is an elegant tableau of shape \(\lambda/\nu\), and
- \(\nu\) is a partition with \(\nu \subseteq \lambda\) and \(\text{row}_{\leq k}(\nu) = \text{row}_{\leq k}(\lambda)\).

The set of RSE-tableaux of shape \(\lambda\) and level \(k\) is denoted by \(\text{RSE}_k(\lambda)\).

We will represent an RSE-tableau \(T = (R, E)\) as the tableau obtained by drawing both \(R\) and \(E\) in which every entry \(i\) in \(E\) is written as \(i^*\). See Figure 10 for an example of an RSE-tableau.

Note that if \(T = (R, E) \in \text{RSE}_1(\lambda)\), then both \(R\) and \(E\) are SSYTs. Thus \(T\) can be considered as an SSYT whose entries are from \(\{1, 2, \ldots, 1^*, 2^*, \ldots\}\). Using this observation the following proposition is easy to verify.

Proposition 4.2. The map \(\text{Tab}\) is a weight-preserving bijection between \(\mathcal{L}^{\text{NC}}_{\lambda/\emptyset}\) and \(\text{RSE}_1(\lambda)\).

If \(T = (R, E) \in \text{RSE}_k(\lambda)\), then \(E = \emptyset\) and \(R\) is an RPP of shape \(\lambda\) with no extra conditions. Hence, we will identify \(\text{RSE}_k(\lambda)\) with \(\text{RPP}(\lambda)\).

The weight of \(T = (R, E) \in \text{RSE}_k(\lambda)\) is defined by

\[
\text{wt}(T) = \text{wt}(R)t_E,
\]

where \(t_E = t_1^{c_1(E)}t_2^{c_2(E)}\cdots\) and \(c_i(E)\) is the number of \(i\)'s in \(E\). For example, if \(T = (R, E)\) is the RSE-tableau in Figure 10, then \(\text{wt}(R) = x_1^4x_2^4x_3^3x_4^2x_5t_2t_3^2t_4t_5\), \(t_E = t_3t_4^2t_5\), and \(\text{wt}(T) = \text{wt}(R)t_E = x_1^4x_2^4x_3^3x_4^2x_5t_2t_3^2t_4t_5\).

Figure 10. An RPP \(R\) of shape \(\nu\) on the left with row 3 marked, an elegant tableau \(E\) of shape \(\lambda/\nu\) in the middle, and the RSE-tableau \(T = (R, E)\) of level 3 and shape \(\lambda\) on the right, where \(\lambda = (5, 5, 4, 4, 2)\) and \(\nu = (5, 5, 2, 2)\). Note that \(\text{row}_{\geq 3}(R)\) is an SSYT.
We now describe two maps $\phi_-$ and $\phi_+$ on RSE-tableaux, where the level of an RSE-tableau is decreased by $\phi_-$ and increased by $\phi_+$. These maps are due to Lam and Pylyavskyy [10] who used them as intermediate steps in their bijection between RPP($\lambda$) and RSE$_{1}(\lambda)$. See Figures 11 and 12 for illustrations of these maps.

**Definition 4.3.** [The level-decreasing map $\phi_- : \text{RSE}_{k+1}(\lambda) \to \text{RSE}_k(\lambda)$] Let $\lambda$ be a partition with $\ell(\lambda) = \ell$ and let $T = (R, E) \in \text{RSE}_{k+1}(\lambda)$ with $1 \leq k \leq \ell - 1$. Then $\phi_-(T)$ is defined as follows.

**Step 1:** For $1 \leq j \leq \lambda_k$, the entry $R(k, j)$ is novel if $j > \lambda_{k+1}$ or $R(k, j) \neq R(k+1, j)$. Let $a_1 \leq a_2 \leq \cdots \leq a_r$ be the novel entries. Let $R'$ be the tableau obtained from $R$ by removing row $k$ and shifting row$_{\geq k+1}(R')$ up by one (so that row$_{\geq k}(R') = \text{row}_{\geq k+1}(R))$. Then $H := \text{sh}(R)/\text{sh}(R')$ is a horizontal strip.

**Step 2:** Update $R'$ by inserting $a_1, a_2, \ldots, a_r$ in this order into row$_{\geq k}(R')$ using the RSK algorithm. By the property of the RSK algorithm, if $a_i$ was the novel entry in column $j$, then $a_i$ bumps the $(k, j)$-entry of row$_{\geq k}(R')$ (in case it exists) or $a_i$ is simply placed at position $(k, j)$ (in case row$_{\geq k}(R')$ has no $(k, j)$-entry). Therefore row $k$ of $R'$ becomes the original row $k$ of $R$ and the newly created cells of $R'$ lie in the horizontal strip $H$. Let $E'$ be the union of $E$ and the remaining empty cells in $H$, which we fill with $k$'s. Finally, mark row $k$ of $R'$ as the level and define $\phi_-(T) = (R', E')$.

**Definition 4.4.** [The level-increasing map $\phi_+ : \text{RSE}_k(\lambda) \to \text{RSE}_{k+1}(\lambda)$] Let $\lambda$ be a partition with $\ell(\lambda) = \ell$ and let $T = (R, E) \in \text{RSE}_k(\lambda)$ with $1 \leq k \leq \ell - 1$. Then $\phi_+(T)$ is defined as follows.

**Step 1:** Let $c_1 < c_2 < \cdots < c_r$ be the column indices $j$ such that column $j$ of $E$ does not contain $k$. Let $E'$ be the tableau obtained from $E$ by removing the cells containing $k$. For $i = r, r-1, \ldots, 1$ in this order, apply the reverse RSK algorithm to row$_{\geq k}(R)$ starting from the last cell of column $c_i$ and denote the resulting tableau by $R_1$. Let $a_i$ and $b_i$ be the integers such that the reverse RSK algorithm bumps $a_i$ at position $(k, b_i)$ at the end.

**Step 2:** Let $R'$ be the tableau obtained from $R$ by replacing row$_{\geq k}(R)$ by $R_1$. Shift row$_{\geq k}(R')$ down by one so that row $k$ of $R'$ is now empty. For each $1 \leq j \leq \lambda_k$, if $j = b_i$ for some $i$, then let $R'(k, j) = a_i$, and otherwise let $R'(k, j)$ equal $R'(k+1, j)$. Finally, mark row $k + 1$ of $R'$ as the level and define $\phi_+(T) = (R', E')$.

The following proposition is shown in [10] Proof of Theorem 9.8.

**Proposition 4.5.** Let $\lambda$ be a partition with $\ell(\lambda) = \ell$. For $1 \leq k \leq \ell - 1$, the maps $\phi_+ : \text{RSE}_k(\lambda) \to \text{RSE}_{k+1}(\lambda)$ and $\phi_- : \text{RSE}_{k+1}(\lambda) \to \text{RSE}_k(\lambda)$ are weight-preserving bijections and they are mutual inverses.
As a corollary to Proposition 4.7, we obtain that the map \( \phi_+^{\ell-1} : \text{RSE}_\ell(\lambda) \rightarrow \text{RSE}_1(\lambda) \) is a weight-preserving bijection. Since we can identify \( \text{RSE}_\ell(\lambda) \) with \( \text{RPP}(\ell) \), it follows that

\[
\bar{g}_\lambda(x; t) = \sum_{R \in \text{RPP}(\ell)} \text{wt}(R) = \sum_{T \in \text{RSE}_\ell(\lambda)} \text{wt}(T) = \sum_{T \in \text{RSE}_1(\lambda)} \text{wt}(T).
\]

Note that since \( L_{\lambda/\mu}^{\text{NC}} = L_{\lambda/\mu}^{\text{SNC}} \) for \( \mu = \emptyset \), Proposition 2.2 shows

\[
\det (e_{\lambda_i' - i+j}(x_1, x_2, \ldots, t_1, t_2, \ldots, t_{\lambda_i'-1}))_{1 \leq i, j \leq n} = \sum_{p \in L_{\lambda}^{\text{NC}}} \text{wt}(p).
\]

By Proposition 4.7

\[
\sum_{p \in L_{\lambda}^{\text{NC}}} \text{wt}(p) = \sum_{T \in \text{RSE}_1(\lambda/\mu)} \text{wt}(T).
\]

Combining the above three equations we obtain (1.1). This proof is essentially the same as Yeliussizov’s [15, §10.1].

5. Skew \( \text{RSE}\)-tableaux

In this section we extend the definition of \( \text{RSE}\)-tableaux to skew shapes and study properties of the maps \( \phi_- \) and \( \phi_+ \) on them. We first need to extend the definition of elegant tableaux.

Let \( \mu \) be a partition. A \( \mu \)-elegant tableau is an SSYT \( E \) of a certain skew shape \( \lambda/\nu \) with \( \mu \subseteq \nu \) such that \( \mu_j' + 1 \leq E(i, j) \leq i - 1 \) for all \( (i, j) \in \lambda/\nu \). Note that the \( \emptyset \)-elegant tableaux are the usual elegant tableaux.

Definition 5.1. A (skew) \( \text{RSE}\)-tableau of shape \( \lambda/\mu \) of level \( k \) is a pair \( T = (R, E) \) satisfying the following conditions:

- \( R \in \text{RPP}(\nu/\mu) \) with row \( k \) of \( R \) marked,
- \( E \) is a \( \mu \)-elegant tableau of shape \( \lambda/\nu \),
- \( \nu \) is a partition with \( \mu \subseteq \nu \subseteq \lambda \) and \( \text{row}_{\leq k}(\nu) = \text{row}_{\leq k}(\lambda) \), and
- \( \text{row}_{\geq k}(R) \) is an SSYT.

The set of \( \text{RSE}\)-tableaux of shape \( \lambda/\mu \) and level \( k \) is denoted by \( \text{RSE}_k(\lambda/\mu) \).

The weight of \( T = (R, E) \in \text{RSE}_k(\lambda/\mu) \) is defined in the same way by

\[
\text{wt}(T) = \text{wt}(R)t_E.
\]
For example, if $T$ is the skew RSE-tableau in Figure 13, then $\text{wt}(R) = x_1^4 x_2^3 x_3^2 t_1^3 t_2^2 t_3 t_4$, and $\text{wt}(T) = x_1^4 x_2^3 x_3^2 t_1^3 t_2^2 t_4$.

We also define $\text{RSE}_k(\lambda)$ to be the set of RSE-tableaux $(R, E)$ of shape $\lambda$ and level $k$, where the entries of $R$ are taken from

\[ \{ T < \overline{T} < \cdots < 1 < 2 < \cdots \} , \]

while the entries of $E$ are still positive integers.

For $T = (R, E) \in \text{RSE}_k(\lambda/\mu)$, let $\overline{T}$ be the RSE-tableau $(\overline{R}, E) \in \text{RSE}_k(\lambda)$, where $\overline{T}$ is obtained from $R$ by filling the cells in row $i$ of $\mu$ with $\overline{t}_i$s for each $1 \leq i \leq \ell(\mu)$. We will identify $T$ with $\overline{T}$ so that $\text{RSE}_k(\lambda/\mu) \subseteq \text{RSE}_k(\lambda)$. By replacing each entry $i$ in $E$ by $i^*$ and putting $\overline{R}$ and $E$ together we will also consider $T = \overline{T} = (\overline{R}, E) \in \text{RSE}_k(\lambda/\mu)$ as a tableau of shape $\lambda$ whose entries are taken from

\[ \{ T < \overline{T} < \cdots < 1 < 2 < \cdots < 1^* < 2^* \cdots \} . \]

We call the elements in $\overline{T}$ the extended integers. See Figure 13 for an example of this correspondence. Sometimes we will also consider $T \in \text{RSE}_k(\lambda/\mu)$ as an RPP of shape $\lambda$ whose entries are extended integers. We call $\overline{t}$ a negative entry and $i^*$ an $\omega$-entry.

The following proposition is immediate from the definition of $\text{RSE}_k(\lambda/\mu)$.

**Proposition 5.2.** Let $T \in \text{RSE}_k(\lambda)$. Then $T \in \text{RSE}_k(\lambda/\mu)$ if and only if the following conditions hold:

1. The cells containing a negative entry are exactly those in $\mu$.
2. If $(i, j) \in \mu$, then $T(i, j) = \overline{t}_i$.
3. If $T(i, j) = a^*$, then $\mu_j^i + 1 \leq a \leq \lambda_j^i - 1$.

Note that if $T = (R, E) \in \text{RSE}_1(\lambda/\mu)$ then we can regard $T$ as an SSYT of shape $\lambda/\mu$ whose entries are from $\{1, 2, \ldots, 1^*, 2^* \cdots\}$. Using this observation, similarly to Proposition 4.2, the following proposition is easy to verify.

**Proposition 5.3.** The map $\text{Tab}$ is a weight-preserving bijection between $\mathcal{L}^\text{NC}_{\lambda/\mu}$ and $\text{RSE}_1(\lambda/\mu)$.

If $T = (R, E) \in \text{RSE}_k(\lambda/\mu)$, then by definition of an RSE-tableau, we must have $E = \emptyset$ and $R$ can be any RPP of shape $\lambda/\mu$ whose entries are positive integers. Hence, we will identify $\text{RSE}_k(\lambda/\mu)$ with $\text{RPP}(\lambda/\mu)$.

Using the ordering given by (5.1), the same maps $\phi_-$ and $\phi_+$ are applied to $\text{RSE}_k(\lambda)$. By the identification $\text{RSE}_k(\lambda/\mu) \subseteq \text{RSE}_k(\lambda)$, these maps $\phi_-$ and $\phi_+$ are also applied to $\text{RSE}_k(\lambda/\mu)$. See Figure 14 for an example.

The following definitions will be used frequently for the rest of this paper. See Figure 15 for an example.

**Definition 5.4.** Let $T_1$ and $T_2$ be RPPs whose entries are extended integers. Define $T_1 \triangleright_k T_2$ to be the tableau obtained by concatenating $T_1$ and $T_2$, i.e., $\text{col}_{\leq k}(T_1 \triangleright_k T_2) = T_1$ and $\text{col}_{\geq k+1}(T_1 \triangleright_k T_2) = T_2$, where $k$ is the number of columns in $T_1$. Define $T_1 \leq T_2$ if $T_1 \triangleright_k T_2$ is also an RPP (with extended integers).
Lemma 5.5. Let \( T \in \text{RSE}_k(\lambda/\mu) \). Then \( \phi_- (T) \in \text{RSE}_k(\lambda/\mu) \) and, for all \( 1 \leq s \leq \mu_k \),

\[
\phi_- (T) = \text{col}_{\leq s}(T) \cup \phi_- (\text{col}_{\geq s+1}(T)).
\]

Proof. Let \( T = (R, E) \). Since \( T \in \text{RSE}_k(\lambda/\mu) \subseteq \text{RSE}_{k+1}(\lambda) \), we have \( \phi_- (T) \in \text{RSE}(\lambda) \). In order to show \( \phi_- (T) \in \text{RSE}_k(\lambda/\mu) \), we must show that \( \phi_- (T) \) satisfies the three conditions in Proposition 5.2. To this end we first prove the following claim, which is equivalent to (5.3).

Claim: If \( 1 \leq s \leq \mu_k \), then \( \text{col}_{\leq s}(\phi_- (T)) = \text{col}_{\leq s}(T) \) and \( \text{col}_{\geq s+1}(\phi_- (T)) = \phi_- (\text{col}_{\geq s+1}(T)) \).

The first \( s \) entries of row \( k \) in \( T \) are all \( k \) and every entry in row \( k+1 \) is either \( k+1 \) or a positive integer. Thus the first \( s \) entries are novel entries. In the definition of \( \phi_- \), we delete row \( k \) and insert the novel entries into row \( k+1 \) of \( T \) (after shifting it up by one). Since \( k \) is smaller than every entry in row \( k+1 \), each of the first \( s \) insertion paths is a straight vertical path. Since insertion paths never intersect, the first \( s \) columns are not changed after the insertion of the first \( s \) rows. This shows the first identity of the claim. The fact that the insertion paths starting from columns of index greater than \( \mu_k \) never enter columns of index at least \( \mu_k \) also implies the second identity of the claim.

We now show that \( \phi_- (T) \) satisfies the three conditions in Proposition 5.2. For the first two conditions it is enough to show that the restrictions of \( T \) and \( \phi_- (T) \) to \( \mu \) are equal because \( \phi_- \)...
preserves the total number of negative entries. Since $\mu$ is contained in row$_{\leq k}(\lambda) \cup \text{col}_{\leq \mu_k}(\lambda)$, this follows from the special case col$_{\leq \mu_k}(\phi_-(T)) = \text{col}_{\leq \mu_k}(T)$ of the claim and the fact row$_{\leq k}(\phi_-(T)) = \text{row}_{\leq k}(T)$. For the third condition note that in the construction of $\phi_-(T) = (R', E')$ from $T = (R, E)$, $E'$ is obtained from $E$ by adding some $k^*$’s. Suppose $\phi_-(T)(i, j) = a^*$. If $a \neq k$, then we must have $T(i, j) = a^*$. Since $T \in \text{RSE}_{k+1}(\lambda/\mu)$, in this case $\mu'_j + 1 \leq a \leq \lambda'_j - 1$. If $a = k$, we must have $j \geq \mu_k + 1$ since col$_{\leq \mu_k}(\phi_-(T)) = \text{col}_{\leq \mu_k}(T)$ and $T$ has no $k^*$. But $j \geq \mu_k + 1$ implies that $\mu'_j < k$, and $\phi_-(T) \in \text{RSE}_{k}(\lambda/\mu)$ implies $k \leq \lambda'_j - 1$. Hence the third condition also holds and the proof is completed. \hfill \Box

**Lemma 5.6.** Let $T \in \text{RSE}_{k}(\lambda/\mu)$. Then the following are equivalent:

1. $\phi_+(T) \in \text{RSE}_{k+1}(\lambda/\mu)$,
2. $T \in \phi_-(\text{RSE}_{k+1}(\lambda/\mu))$,
3. $\phi_+(T) = \text{col}_{\leq s}(T) \cup \phi_+(\text{col}_{\geq s+1}(T))$, for all $1 \leq s \leq \mu_k$, and
4. $\text{col}_{\leq \mu_k}(T) \leq \phi_+(\text{col}_{\geq \mu_k+1}(T))$.

**Proof.** We will prove the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Let $T' = \phi_+(T) \in \text{RSE}_{k+1}(\lambda/\mu)$. Then $T = \phi_-(T') \in \phi_-(\text{RSE}_{k+1}(\lambda/\mu))$.

(2) $\Rightarrow$ (3): Suppose $T = \phi_-(T')$ for some $T' \in \text{RSE}_{k+1}(\lambda/\mu)$. Then $T' = \phi_+(T)$. We need to show that for $1 \leq s \leq \mu_k$,

$$\text{col}_{\leq s}(T') = \text{col}_{\leq s}(T),$$

$$\text{col}_{\geq s+1}(T') = \phi_+(\text{col}_{\geq s+1}(T)).$$

By Lemma [5.3] $T = \phi_-(T') = \text{col}_{\leq s}(T') \cup \phi_-(\text{col}_{\geq s+1}(T'))$. This shows that $\text{col}_{\leq s}(T) = \text{col}_{\leq s}(T')$, which is the first equality, and $\text{col}_{\geq s+1}(T) = \phi_-(\text{col}_{\geq s+1}(T'))$, which is equivalent to the second equality after applying $\phi_+$.

(3) $\Rightarrow$ (4): The fact that $\phi_+(T) = \text{col}_{\leq \mu_k}(T) \cup \phi_+(\text{col}_{\geq \mu_k+1}(T))$ is an RPP (with extended integers as entries) shows that $\text{col}_{\leq \mu_k}(T) \leq \phi_+(\text{col}_{\geq \mu_k+1}(T))$.

(4) $\Rightarrow$ (1): Suppose that $T = (R, E) \in \text{RSE}_{k}(\lambda/\mu)$ satisfies $\text{col}_{\leq \mu_k}(T) \leq \phi_+(\text{col}_{\geq \mu_k+1}(T))$. To show $\phi_+(T) \in \text{RSE}_{k+1}(\lambda/\mu)$ we must show that $\phi_+(T)$ satisfies the three conditions in Proposition 5.2. Since the restriction of $\phi_+(T)$ to the $\omega$-entries is exactly the same as that of $T$ with $k^*$ deleted and $T \in \text{RSE}_{k+1}(\lambda/\mu)$ satisfies the third condition, so does $\phi_+(T)$. For the first two conditions, it is enough to show that the restrictions of $T$ and $\phi_+(T)$ to $\mu$ are the same. Since $\mu$ is contained in row$_{\leq k}(\lambda) \cup \text{col}_{\geq \mu_k+1}(\lambda)$ and row$_{\leq k}(\phi_+(T)) = \text{row}_{\leq k}(T)$, it suffices to prove the following equality:

$$\text{col}_{\leq \mu_k}(\phi_+(T)) = \text{col}_{\leq \mu_k}(T).$$

To show [5.4] we investigate the construction of $\phi_+(T)$ in Definition [4.3]. Let $c_1 < c_2 < \cdots < c_r$ be the column indices $j$ such that column $j$ of $E$ does not contain $k$. Since $(R, E) \in \text{RSE}_{k}(\lambda/\mu)$, $E$ is a $\mu$-elegant tableau. Thus for every cell $(i, j)$ of $E$ with $1 \leq j \leq \mu_k$ we have $E(i, j) \geq \mu'_j + 1 \geq k + 1$. This shows that $E$ has no entries equal to $k$ in the first $\mu_k$ columns, i.e., $c_j = j$ for all $1 \leq j \leq \mu_k$.

Recall that in the definition of $\phi_+(T)$ we apply the reverse RSK algorithm to row$_{\geq k}(R)$ starting from the last cell of column $c_j$ for $j = r, r - 1, \ldots, 1$ in this order. We denote by $P_j$ each inverse bumping path.

**Claim:** $P_{\mu_k}, P_{\mu_k-1}, \ldots, P_1$ are straight vertical paths.

By the construction of $\phi_+(T)$ the claim implies [5.4]. Hence it suffices to prove the claim. To this end let $Q$ be the tableau obtained from row$_{\geq k}(R)$ by applying the reverse RSK algorithm to the last cell of column $c_j$ for $j = r, r - 1, \ldots, 1$. Note that the same process is applied to row$_{\geq k}(\text{col}_{\geq \mu_k+1}(T))$ when we compute $(R_1, E_1) = \phi_+(\text{col}_{\geq \mu_k+1}(T))$. Since row$_{\geq k+1}(R_1)$ has been shifted down in Step 2 of the definition of $\phi_+$, we have

$$\text{row}_{\geq k+1}(R_1) = \text{row}_{\geq k}(Q).$$

Suppose that $R_1(k, \mu_k + 1) < R_1(k + 1, \mu_k + 1)$. This means that the entry $R_1(k, \mu_k + 1)$ was bumped during the applications of the reverse RSK algorithm. Since column $\mu_k + 1$ is the leftmost nonempty column of col$_{\geq \mu_k+1}(T)$, the reverse bumping path that pushed this cell must be
a straight vertical path and we must have \( c_{\mu_k+1} = \mu_k + 1 \). Since each path \( P_j \), for \( 1 \leq j \leq \mu_k + 1 \),
starts from column \( j \) and \( P_{\mu_k+1} \) is a straight vertical path, by the non-intersecting property of the
reverse bumping paths, the claim follows.

Suppose now that \( R_1(k, \mu_k + 1) = R_1(k + 1, \mu_k + 1) \). By the same reasoning as in the
previous paragraph, it is sufficient to show that \( P_{\mu_k} \) is a straight vertical path. By the assumption
\( \text{col}_{\leq \mu_k}(T) \leq \phi_+ (\text{col}_{\geq \mu_k+1}(T)) \), we have \( \text{col}_{\leq \mu_k} (R) \leq R_1 \). This together with (5.7) implies that
\( R(i, \mu_k) \leq R_1(i, \mu_k + 1) = Q(i-1, \mu_k + 1) \) for all \( i \geq k + 1 \). These inequalities and the assumption
\( R_1(k, \mu_k + 1) = R_1(k + 1, \mu_k + 1) \) ensure that \( P_{\mu_k} \) is a straight vertical path. This completes the
proof of the claim.

The following two lemmas will be useful in the next section.

**Lemma 5.7.** Suppose \( T \in \text{RSE}_b(\text{col}_{\geq c+1}(V/\mu)) \), where \( b, c \) are nonnegative integers, \( V \) is a
vertical diagram, and \( \mu \) is a partition with \( \mu \subseteq V \) such that \( \text{col}_{\geq c+1}(V/\mu) \) is a skew shape and
\( 1 \leq c \leq \mu_b \). Then for all \( 1 \leq d \leq b - 1 \) and \( 1 \leq j \leq \mu_{b-1} \), we have

\[
\begin{align*}
\phi^d (T) & \in \text{RSE}_b-d(\text{col}_{\geq c+1}(V/\mu)), \\
\phi^d (T) & = \text{col}_{\leq j}(T) \cup \phi^d (\text{col}_{\geq j+1}(T)).
\end{align*}
\]

In other words (5.7) means that applying \( \phi^d \) to \( T \) is the same thing as applying \( \phi^d \) only to the
columns of index greater than \( j \) while keeping \( \text{col}_{\leq j}(T) \) unmodified.

**Proof.** Since \( \text{col}_{\geq c+1}(V/\mu) \) is a skew shape, applying Lemma 5.5 repeatedly gives (5.3).

We prove (5.7) by induction on \( d \). If \( d = 1 \), it is just Lemma 5.5. Suppose that (5.7) is true for \( 1 \leq d \leq b - 2 \) and consider the \( d + 1 \) case. Using Lemma 5.5 with (5.6) and the inequality
\( j \leq \mu_{b-1} \leq \mu_{b-d-1} \), we have

\[
\phi_-(\phi^d (T)) = \text{col}_{\leq j}(\phi^d (T)) \cup \phi_-(\text{col}_{\geq j+1}(\phi^d (T))).
\]

Note that the induction hypothesis (5.7) for the \( d \) case is equivalent to

\[
\text{col}_{\leq j}(\phi^d (T)) = \text{col}_{\leq j}(T), \quad \text{col}_{\geq j+1}(\phi^d (T)) = \phi^d (\text{col}_{\geq j+1}(T)).
\]

Hence (5.8) can be written as

\[
\phi^{d+1}_-(T) = \text{col}_{\leq j}(T) \cup \phi_-(\phi^d (\text{col}_{\geq j+1}(T))) = \text{col}_{\leq j}(T) \cup \phi^{d+1}_-(\text{col}_{\geq j+1}(T)),
\]

which is (5.7) for the \( d + 1 \) case. This completes the proof.

**Lemma 5.8.** Suppose \( T \in \phi^-_a(\text{RSE}_b(\text{col}_{\geq c+1}(V/\mu))) \), where \( a, b, c \) are nonnegative integers, \( V \) is a
vertical diagram, and \( \mu \) is a partition with \( \mu \subseteq V \) such that \( \text{col}_{\geq c+1}(V/\mu) \) is a skew shape and
\( 1 \leq c \leq \mu_b \). Then for all \( 1 \leq d \leq a \) and \( 1 \leq j \leq \mu_{b-a+d-1} \), we have

\[
\begin{align*}
\phi^d_a (T) & \in \text{RSE}_b-a+d(\text{col}_{\geq c+1}(V/\mu)), \\
\phi^d_a (T) & = \text{col}_{\leq j}(T) \cup \phi^d (\text{col}_{\geq j+1}(T)).
\end{align*}
\]

In other words (5.10) means that applying \( \phi^d_a \) to \( T \) is the same thing as applying \( \phi^d_a \) only to the
columns of index greater than \( j \) while keeping \( \text{col}_{\leq j}(T) \) unmodified.

**Proof.** Since \( \phi_- \) and \( \phi_+ \) are inverses of each other, the assumption \( T \in \phi^-_a(\text{RSE}_b(\text{col}_{\geq c+1}(V/\mu))) \)
together with Lemma 5.7 implies that for all \( 1 \leq d \leq a \),

\[
\phi^d_+(T) \in \phi_-^{d-d}(\text{RSE}_b(\text{col}_{\geq c+1}(V/\mu))) \subseteq \text{RSE}_b-a+d(\text{col}_{\geq c+1}(V/\mu)),
\]

which shows (5.9). For the second statement, let \( T' = \phi^d_+(T) \). Then (5.10) is equivalent to

\[
\text{col}_{\leq j}(T') = \text{col}_{\leq j}(T), \quad \text{col}_{\geq j+1}(T') = \phi^d (\text{col}_{\geq j+1}(T)).
\]

Since \( T' = \phi^d_+(T) \in \text{RSE}_b-a+d(\text{col}_{\geq c+1}(V/\mu)) \), by Lemma 5.7

\[
T = \phi^d_-(T') = \phi^d_a (T').
\]

This shows that

\[
\text{col}_{\leq j}(T) = \text{col}_{\leq j}(T'), \quad \text{col}_{\geq j+1}(T) = \phi^d (\text{col}_{\geq j+1}(T')).
\]
By taking $\phi^d_+$ in each side of the second equation we obtain \([5.11]\), completing the proof. \qed

Recall that at the end of the previous section we showed that $\phi_{\ell-1}^{\ell-1} : RSE_\ell(\lambda) \rightarrow RSE_1(\lambda)$ is a weight-preserving bijection and

$$
\tilde{g}_\lambda(x; t) = \sum_{R \in RPP(\lambda)} \text{wt}(R) = \sum_{T \in RSE_\ell(\lambda)} \text{wt}(T) = \sum_{T \in RSE_1(\lambda)} \text{wt}(T).
$$

For the skew shape case the map $\phi_- : RSE_k(\lambda/\mu) \rightarrow RSE_{k-1}(\lambda/\mu)$ is not a bijection but just an injection.

**Proposition 5.9.** Let $\lambda$ and $\mu$ be partitions with $\mu \subseteq \lambda$ and $\ell(\lambda) = \ell$. Then, for $2 \leq k \leq \ell$, the map $\phi_- : RSE_k(\lambda/\mu) \rightarrow RSE_{k-1}(\lambda/\mu)$ is a weight-preserving injection. In other words, $\phi_- : RSE_k(\lambda/\mu) \rightarrow \phi_- (RSE_k(\lambda/\mu))$ is a weight-preserving bijection.

**Proof.** This follows from Proposition 4.5 and Lemma 5.5. \qed

The fact that $\phi_- : RSE_k(\lambda/\mu) \rightarrow RSE_{k-1}(\lambda/\mu)$ is an injection can still be used to give a different expression for $\tilde{g}_\lambda/\mu(x; t)$.

**Proposition 5.10.** Let $\lambda$ and $\mu$ be partitions with $\mu \subseteq \lambda$ and $\ell(\lambda) = \ell$. Then

$$
\tilde{g}_\lambda/\mu(x; t) = \sum_{T \in \phi_-^{\ell-1}(RSE_\ell(\lambda/\mu))} \text{wt}(T).
$$

**Proof.** By the identification of $RPP(\lambda/\mu)$ and $RSE_\ell(\lambda/\mu)$,

$$
\tilde{g}_\lambda/\mu(x; t) = \sum_{R \in RPP(\lambda/\mu)} \text{wt}(R) = \sum_{T \in RSE_\ell(\lambda/\mu)} \text{wt}(T).
$$

Applying Proposition 5.9 repeatedly we obtain that $\phi_{\ell-1}^{\ell-1} : RSE_\ell(\lambda/\mu) \rightarrow \phi_{\ell-1}^{\ell-1}(RSE_\ell(\lambda/\mu))$ is a weight-preserving bijection. Therefore

$$
\sum_{T \in RSE_\ell(\lambda/\mu)} \text{wt}(T) = \sum_{T \in \phi_-^{\ell-1}(RSE_\ell(\lambda/\mu))} \text{wt}(T),
$$

and the proof follows. \qed

By Propositions 2.2 and 5.10 in order to prove Theorem 1.1 it is sufficient to show the following proposition whose proof will be given in the next section.

**Proposition 5.11.** Let $\lambda$ and $\mu$ be partitions with $\mu \subseteq \lambda$, $\ell(\lambda) = \ell$, and $\ell(\lambda') = n$. Then

$$
\sum_{p \in L^{\text{SNC}}_{\lambda/\mu}} \text{wt}(p) = \sum_{T \in \phi_-^{n-1}(RSE_\ell(\lambda/\mu))} \text{wt}(T).
$$

6. **Sign-reversing involution**

In this section we define a sign-reversing involution on $L^{\text{SNC}}_{\lambda/\mu}$ to prove Proposition 5.11. Recall Notation 2.1. For any tableau $Q$ denote by $\text{col}_{D_k}(Q)$ the part of $Q$ consisting of column $j$ for all $j \in D_k$. The following proposition is an immediate consequence of Proposition 5.3.

**Proposition 6.1.** Let $p = (p_1, \ldots, p_n) \in L_{\lambda/\mu}$ and $T = \text{Tab}(p)$. Then $p \in L^{\text{SNC}}_{\lambda/\mu}$ if and only if each $\text{col}_{D_k}(T)$ (with row 1 marked) is an RSE-tableau of level 1 (and of some skew shape).

We now define the sign-reversing involution $\Phi$ on $L^{\text{SNC}}_{\lambda/\mu}$. See Section 7 for a concrete example of the map $\Phi$.

**Definition 6.2** (The sign-reversing involution $\Phi$ on $L^{\text{SNC}}_{\lambda/\mu}$). Let $p \in L^{\text{SNC}}_{\lambda/\mu}$. Then $\Phi(p)$ is defined as follows. Here we use the letters defined in Notation 2.1.

**Step 1:** Suppose $T = \text{Tab}(p) \in \text{VT}(\pi(\lambda/\mu))$ and write $T = T_{r+1} \sqcup \cdots \sqcup T_2 \sqcup T_1$, where each $T_i = \text{col}_{D_i} T$ is considered as an RSE-tableau of level 1.
Figure 17. The construction of $\widetilde{T}_{k+1}$ and $\widetilde{U}_k$. The tableau $\widetilde{T}_{k+1} \sqcup \widetilde{U}_k$ is obtained from $T_{k+1} \sqcup \phi^m(U_k)$ by replacing Tab(q) by Tab(q').

Step 2: Let $U_1 = T_1$. For $i = 1, 2, \ldots, r$, if $U_i$ has been defined and $T_{i+1} \leq \phi^m(U_i)$, define $U_{i+1}$ to be the RSE-tableau $T_{i+1} \sqcup \phi^m(U_i)$ with level $M_i + 1$.

Step 3: If $U_{i+1}$ is defined, set $\Phi(p) = p$. Otherwise, let $k$ be the smallest integer such that $T_{k+1} \not\leq \phi^m(U_k)$. In order to define $\Phi(p)$ we proceed as follows.

Step 3-1: Let $\gamma = (\gamma_1, \ldots, \gamma_s)$ be the partition defined by

$$
\gamma_i = \begin{cases}
\lambda_i, & \text{if } 1 \leq i \leq M_k, \\
\min(\lambda_i, d_k), & \text{if } M_k + 1 < i \leq \ell.
\end{cases}
$$

Considering row $\geq M_k + 1(T_{k+1} \sqcup \phi^m(U_k))$ as an element in $\operatorname{VT}(\pi(\lambda)/\gamma)$, let $q = (q_1, \ldots, q_n)$ be the $n$-path such that

$$
\text{Tab}(q) = \text{row} \geq M_k + 1(T_{k+1} \sqcup \phi^m(U_k)).
$$

Step 3-2: Let $s$ be the largest integer such that column $s$ of row $\geq M_k + 1(\phi^m(U_k))$ is nonempty. Choose the intersection point $(a, b)$ of $q_i$ and $q_j$ for $d_k + 1 < i, j \leq s$ in such a way that $(b, a)$ is the largest in the lexicographic order. Let $q_i'$ and $q_j'$ be the paths obtained from $q_i$ and $q_j$, respectively, by exchanging their subpaths after the intersection $(a, b)$. Define $q'$ to be the $n$-path $q$ in which $q_i$ and $q_j$ are replaced by $q_i'$ and $q_j'$, respectively.

Step 3-3: Note that row $D_k(T_\gamma(q)) = \text{row} \geq M_k + 1(T_{k+1})$ and row $D_k(T_\gamma(q')) = \text{row} \geq M_k + 1(\phi^m(U_k))$. Let $\widetilde{T}_{k+1}$ be the RSE-tableau of level 1 obtained from $T_{k+1}$ by replacing row $D_k(T_\gamma(q))$ by row $D_k(T_\gamma(q'))$. Let $\widetilde{U}_k$ be the RSE-tableau of level $M_k + 1$ obtained from $\phi^m(U_k)$ by replacing row $\geq d_k + 1(T_\gamma(q'))$ by row $\geq d_k + 1(T_\gamma(q'))$. See Figure 17 for an illustration of the construction of $\widetilde{T}_{k+1}$ and $\widetilde{U}_k$.

Let $T' = T_r \sqcup \cdots \sqcup T_{k+2} \sqcup \widetilde{T}_{k+1} \sqcup \phi^m(\widetilde{U}_k)$.

Finally, define $\Phi(p)$ to be the $n$-path $p'$ satisfying $\text{Tab}(p') = T'$.

The main theorem in this section is as follows.

**Theorem 6.3.** The map $\Phi$ is a sign-reversing involution on $\mathcal{S}^\text{SNC}_{\lambda/\mu}$ whose fixed point set is

$$
\left\{ p \in \mathcal{S}^\text{SNC}_{\lambda/\mu} : \text{Tab}(p) \in \phi^{e-1}(\text{RSE}_c(\lambda/\mu)) \right\}.
$$

Note that Theorem 6.3 immediately implies Proposition 5.11 and hence completes the proof of Theorem 1.4. The rest of this section is devoted to proving Theorem 6.3. We will constantly use the notation in Definition 6.2.

We first show that $\Phi$ is a well-defined map on $\mathcal{S}^\text{SNC}_{\lambda/\mu}$. The only thing that needs to be checked is Step 3-3 in the construction of $\Phi(p)$. More precisely we must check the three assertions in the following lemma. One can see that these three assertions imply $\Phi(p) = p' \in \mathcal{S}^\text{SNC}_{\lambda/\mu}$ as follows. By
the second assertion, we obtain that $\phi_{M_k}^-(U_k)$ is an RSE-tableau of level 1. This together with the first assertion implies that $\text{col}_{D_k}(T')$ is an RSE-tableau of level 1 for all $1 \leq k \leq r + 1$. Then Propositions 3.2, 6.1 and the third assertion imply that $p' \in L_{\lambda/\mu}^{\text{SNC}}$.

**Lemma 6.4.** Let $p \in L_{\lambda/\mu}^{\text{SNC}}$ with type$(p) = \pi$. Suppose that $U_{r+1}$ is not defined in the construction of $\Phi(p)$. Then

1. $\bar{T}_{k+1}$ is an RSE-tableau of level 1,
2. $\bar{U}_k$ is an RSE-tableau of level $M_k + 1$, and
3. $T' \in \text{VT}(\pi'(\lambda)/\mu)$, where $\pi' = \pi(i, j)$ and $(i, j)$ is the transposition exchanging $i$ and $j$.

**Proof.** Recall that $q' = (q'_1, \ldots, q'_n)$ and $q = (q_1, \ldots, q_n)$ differ only by the $i$th and $j$th paths, where $q'_i$ and $q'_j$ are obtained from $q_i$ and $q_j$ by exchanging the subpaths after the intersection $(a, b)$. Suppose $i < j$ so that $i \in D_k$ and $j \geq d_k + 1$. The choice of the intersection point $(a, b)$ in Step 3-2 guarantees that both $\{q'_l: d_k + 1 \leq l \leq d_k\}$ and $\{q'_l: d_k + 1 \leq l \leq s\}$ are nonintersecting. This implies that $\text{col}_{D_k}(\text{Tab}(q'))$ and $\text{col}_{d_k+1}(\text{Tab}(q'))$ are SSYTs (whose entries are extended integers). Moreover, since $i < j$, the initial point $(\mu'_j + n - j, 0)$ of $q_j$ is to the left of the initial point $(\mu'_i + n - i, 0)$ of $q_i$. Therefore the intersection $(a, b)$ of $q_i$ and $q_j$ must occur after the first diagonal step of $q_j$. This shows that row $M_k + 1$ of $\text{col}_{d_k+1}(\text{Tab}(q'))$ is the same as that of $\text{col}_{d_k+1}(\text{Tab}(q'))$.

Note that

$$\text{row}_{\leq M_k}(\bar{T}_{k+1}) = \text{row}_{\leq M_k}(T_{k+1}),$$

$$\text{row}_{\leq M_k}(\bar{U}_k) = \text{row}_{\leq M_k}(\phi_{M_k}^+(U_k)),$$

$$\text{row}_{\geq M_k+1}(\bar{T}_{k+1}) = \text{col}_{D_k}(\text{Tab}(q')),$$

$$\text{row}_{\geq M_k+1}(\bar{U}_k) = \text{col}_{d_k+1}(\text{Tab}(q')),$$

where everything in the right-hand side is an SSYT (whose entries are extended integers) except $\text{row}_{\leq M_k}(\phi_{M_k}^+(U_k))$, which is an RPP. Thus checking the first and second assertions reduces to checking the following:

1. Rows $M_k$ and $M_k + 1$ of $\bar{T}_{k+1}$ form an SSYT.
2. Rows $M_k$ and $M_k + 1$ of $\bar{U}_k$ form an RPP.

The first statement is true because row $M_k$ of $\bar{T}_{k+1}$ is empty (or equivalently filled with negative entries $\overline{\mathbf{M}}_k$). The second statement is also true because rows $M_k$ and $M_k + 1$ of $\bar{U}_k$ are identical with those of $\text{col}_{d_k+1}(\phi_{M_k}^+(U_k))$ by the last sentence of the previous paragraph.

For the final assertion recall that $\text{Tab}(p) \in \text{VT}(\pi(\lambda)/\mu)$ and $\text{Tab}(q) \in \text{VT}(\pi(\lambda)/\gamma)$. By Lemma 6.3 type$(p) = \text{type}(q) = \pi$. Since $q'$ is obtained from $q$ by changing the tails of $q_i$ and $q_j$, we have $\text{type}(q') = \pi'$. Hence, by Lemma 6.3 $\text{Tab}(q') \in \text{VT}(\pi'(\lambda)/\gamma)$. Since $T'$ has the same inner shape as $T$ and the same outer shape as $\text{Tab}(q')$, the third assertion follows. □

The following lemma gives a more direct way of computing $U_i$.

**Lemma 6.5.** Using the notation in Definition 6.2 for $1 \leq i \leq r + 1$, if $U_i$ is defined, then

$$\phi_{M_i}^{-1}(U_i) = \text{col}_{d_i+1}(T),$$

or equivalently,

$$U_i = \phi_{M_i}^{-1}(\text{col}_{d_i+1}(T)),$$

where $M_0 = 0$.

**Proof.** We proceed by induction on $i$. It is true for the case $i = 1$, which is $U_1 = T_1$. Assume true for the case $i \geq 1$ and consider the case $i + 1$. Since $U_{i+1} \in \text{RSE}_{M_i+1}(\text{col}_{d_i+1}(\pi(\lambda)/\mu))$ and $\mu_{M_i} = d_i$, by Lemma 6.3 $\phi_{M_i}^-(U_{i+1}) \in \text{RSE}_{M_{i+1}+1}(\text{col}_{d_{i+1}+1}(\pi(\lambda)/\mu))$ and

$$\phi_{M_i}^-(U_{i+1}) = \text{col}_{d_i}(U_{i+1}) \cup \phi_{M_i}^-(\text{col}_{d_{i+1}+1}(U_{i+1})).$$

Using the construction of $U_{i+1} = T_{i+1} \cup \phi_{M_i}^-(U_i)$ the above equation can be rewritten as

$$\phi_{M_i}^-(U_{i+1}) = T_{i+1} \cup \phi_{M_i}^-(\phi_{M_i}^-(U_i)) = T_{i+1} \cup U_i.$$
which implies
\( \col_{≤d_s}(\phi^m_+(U_{i+1})) = T_{i+1}, \ \col_{≥d_s+1}(\phi^m_+(U_{i+1})) = U_i. \)

Since \( \phi^m_-(U_{i+1}) \) is an element of \( \RSE_{M_{i-1}+1}(\col_{≥d_s+1}(λ/μ)) \) and \( d_i = μ_{M_i} < μ_{M_{i-1}} \), by using Lemma 5.8 and (6.1) we obtain
\( \phi^m_{M_{i-1}}(U_{i+1}) = \col_{≤d_s}(\phi^m_-(U_{i+1})) \cup \phi^m_{M_{i-1}}(\col_{≥d_s+1}(\phi^m_-(U_{i+1}))) = T_{i+1} \cup \phi^m_{M_{i-1}}(U_i). \)

By the induction hypothesis, \( \phi^m_{M_{i-1}}(U_i) = \col_{≥d_s+1}(T) \). Hence the above equation can be rewritten as
\( \phi^m_{M_{i}}(U_{i+1}) = T_{i+1} \cup \col_{≥d_s+1}(T) = \col_{≥d_s+1+1}(T), \)
which is the desired statement for the \( i + 1 \) case. This completes the proof. \( \square \)

The following lemma gives an equivalent condition for \( U_i \) to be defined.

**Lemma 6.6.** Using the notation in Definition 6.2 for \( 1 ≤ i ≤ r + 1 \), \( U_i \) is defined if and only if
\( \col_{≥d_s+1}(T) ∈ \phi^m_{M_{i-1}}(\RSE_{M_{i-1}+1}(\col_{≥d_s+1}(ρ))). \)

where \( ρ = sh(T) = π(λ)/μ. \)

**Proof.** Suppose that \( U_i \) is defined. Then (6.2) follows immediately from Lemma 6.5 since \( U_i ∈ \RSE_{M_{i-1}+1}(\col_{≥d_s+1}(ρ)). \)

Conversely suppose that (6.2) holds. We will prove by induction that \( U_s \) is defined for all \( 1 ≤ s ≤ i \). The case \( s = 1 \) is trivial. Assume that \( U_s \) is defined and \( 1 ≤ s ≤ i - 1 \). Then by Lemma 6.5
\( U_s = φ^m_{M_{i-1}}(\col_{≥d_s+1}(T)). \)

Recall that \( U_{s+1} \) is defined if \( T_{s+1} ≤ φ^m_+(U_s) \). To show this let \( Q ∈ \col_{≥d_s+1}(T) \). Since \( Q ∈ φ^m_{M_{i-1}}(\RSE_{M_{i-1}+1}(\col_{≥d_s+1}(ρ))) \), by Lemma 5.8 with \( a = M_{i-1}, b = M_{i-1} + 1, d = M_{i-1}, \) and \( j = μ_{b-a-d} = μ_{M_{i-1}} = d_s \), we have
\( φ^m_{M_{i-1}}(Q) = \col_{≤d_s}(Q) \cup φ^m_{M_{i-1}}(\col_{≥d_s+1}(Q)) ∈ \RSE_{M_{i-1}+1}(\col_{≥c_s+1}(ρ)). \)

This implies that \( \col_{≤d_s}(Q) ≤ φ^m_+(\col_{≥d_s+1}(Q)) \). Since the leftmost columns of \( \col_{≤d_s}(Q) \) and \( T_{s+1} \) coincide, we also have
\( T_{s+1} ≤ φ^m_+(\col_{≥d_s+1}(Q)). \)

On the other hand, by (6.3),
\( φ^m_+(\col_{≥d_s+1}(Q)) = φ^m_+(\col_{≥d_s+1}(T)) = φ^m_+(U_s). \)

The above two equations show that \( T_{s+1} ≤ φ^m_+(U_s) \) and hence \( U_{s+1} \) is defined. Therefore by induction \( U_i \) is also defined, which completes the proof. \( \square \)

The following lemma shows that \( Φ \) has the desired fixed points.

**Lemma 6.7.** We have \( Φ(ρ) = ρ \) if and only if \( T = \text{Tab}(p) ∈ φ^m_{r-1}(\RSE_{r}(λ/μ)). \)

**Proof.** Suppose \( Φ(p) = p \). Then \( U_{r+1} \) is defined and therefore by Lemma 6.5
\( T = \col_{≥d_r+1}(T) ∈ φ^m_r(\RSE_{M_r+1}(λ/μ)). \)

Thus \( T = φ^m_r(Q) \) for some \( Q ∈ \RSE_{M_r+1}(λ/μ) \). To obtain \( T ∈ φ^m_{r-1}(\RSE_{r}(λ/μ)) \) it is enough to show that \( Q ∈ φ^m_{r-1}(\RSE_{r}(λ/μ)) \) because it would imply that there is \( Q′ ∈ \RSE_{r}(λ/μ) \) satisfying
\( T = φ^m_r(Q) = φ^m_r(φ^m_{r-1}(Q′)) = φ^m_{r-1}(Q′). \)

Since \( Q = φ^m_{r-1}(Q′) \) if and only if \( φ^m_{r-1}(Q′) = Q′ \), the condition \( Q ∈ φ^m_{r-1}(\RSE_{r}(λ/μ)) \) is equivalent to \( φ^m_{r-1}(Q) ∈ \RSE_{r}(λ/μ) \). We will show by induction that
\( φ^m_+(Q) ∈ \RSE_{M_r+1+i}(λ/μ), \quad 0 ≤ i ≤ m_{r+1} - 1, \)
which is true for \( i = 0 \) by assumption. Let \( 0 ≤ i ≤ m_{k+1} - 2 \) and suppose (6.6) is true for \( i \). Now we apply Lemma 5.6 with \( T = φ^m_+(Q) \) and \( k = M_r + i + 1. \) Since \( μ_{M_r+1+i} = 0 \), the fourth
condition of the lemma trivially holds. Hence the first condition of the lemma also holds and we obtain \( \phi_+(\phi_i^+(Q)) \in \text{RSE}_{M_{r+1}+1}(\lambda/\mu) \), which is exactly \( (6.6) \) with \( i+1 \). By induction \( (6.6) \) is true for all \( 0 \leq i \leq m_{r+1} - 1 \), and in particular we obtain \( \phi_M^{m_{r+1}-1}(Q) \in \text{RSE}_{\ell}(\lambda/\mu) \) as desired.

Conversely, suppose that \( T \in \phi_{\ell-1}(\text{RSE}_{\ell}(\lambda/\mu)) \). To show \( \Phi(p) = p \), we must show that \( U_{r+1} \) is defined, which is by Lemma \( (6.5) \) equivalent to \( (6.5) \). By the assumption there is \( Q \in \text{RSE}_{\ell}(\lambda/\mu) \) with \( T = \phi_{\ell-1}(Q) \). Let \( Q' = \phi_{m_{r+1}-1}(Q) \in \text{RSE}_{M_{r+1}}(\lambda/\mu) \), which shows \( (6.5) \).

The following lemma shows that \( \Phi \) is indeed an involution.

**Lemma 6.8.** If \( \Phi(p) = p' \) and \( p \neq p' \), then \( \Phi(p') = p \).

**Proof.** In this proof we will use the notation \( X(p) \) to indicate the object \( X \), for example \( X = U_i \) or \( X = \tilde{U}_k \), in Definition \( (6.2) \) when we apply \( \Phi \) to \( p \).

Let \( \rho = \text{sh}(T(p')) \). Then \( \tilde{U}_k(p) \in \text{RSE}_{M_{k+1}}(\text{col}_{d_{k+1}}(\rho)) \), and by Lemma \( (6.5) \) we have \( \phi^m_{\rho_k}(\tilde{U}_k(p)) \in \text{RSE}_{M_{k+1}}(\text{col}_{d_{k+1}}(\rho)) \). Since

\[
\text{col}_{d_{k+1}}(T(p')) = \phi^{M_{k+1}}_{\rho_k}(\tilde{U}_k(p)) = \phi^{M_{k+1}}_{\rho_k}(\phi^{M_{k+1}}_{\rho_k}(\tilde{U}_k(p))) \in \phi^{M_{k+1}}_{\rho_k}(\text{RSE}_{M_{k+1}}(\text{col}_{d_{k+1}}(\rho))),
\]

by Lemma \( (6.6) \) \( \tilde{U}_k(p') \) is defined. Then by Lemma \( (6.5) \)

\[
U_k(p') = \phi^{M_{k+1}}_{\rho_k}(\text{col}_{d_{k+1}}(T(p'))) = \phi^{M_{k+1}}_{\rho_k}(\phi^{M_{k+1}}_{\rho_k}(\tilde{U}_k(p))) = \phi^{M_{k+1}}_{\rho_k}(\tilde{U}_k(p)),
\]

or equivalently,

\[
\tilde{U}_k(p) = \phi^{M_{k+1}}_{\rho_k}(U_k(p')).
\]

By the construction we have \( \tilde{T}_{k+1}(p) \leq \tilde{U}_k(p) \). Since \( T_{k+1}(p') = \tilde{T}_{k+1}(p) \), we obtain

\[
T_{k+1}(p') \leq \phi^{M_{k+1}}(U_k(p')).
\]

Therefore in the construction of \( \Phi(p') \), \( U_1(p'), U_2(p'), \ldots, U_k(p') \) are defined but not \( U_{k+1}(p') \). Observe that in Step 3-1 we compute

\[
\text{Tab}(q(p')) = \text{row}_{M_k+1}(T_{k+1}(p') \cup \phi^{M_k}_{\rho_k}(U_k(p))) = \text{row}_{M_k+1}(\tilde{T}_{k+1}(p) \cup \tilde{U}_k(p)) = \text{Tab}(q(p)).
\]

Since the map in Step 3-2 sending \( q \) to \( q' \) is easily seen to be an involution, we obtain that \( \text{Tab}(q'(p')) = \text{Tab}(q(p)) \) and therefore

\[
\tilde{T}_{k+1}(p') = T_{k+1}(p), \quad \tilde{U}_k(p') = \phi^{M_k}_{\rho_k}(U_k(p)).
\]

Then

\[
T'(p') = T_r(p') \cup \cdots \cup T_{k+1}(p) \cup \phi^{M_k}_{\rho_k}(U_k(p'))
\]

\[
= T_r(p) \cup \cdots \cup T_{k+1}(p) \cup T_{k+1}(p) \cup \phi^{M_{k+1}}_{\rho_k}(U_k(p)).
\]

By Lemma \( (6.5) \) \( \phi^{M_{k+1}}_{\rho_k}(U_k(p)) = \text{col}_{d_{k+1}}(T(p)) \), and we obtain \( T'(p') = T(p) \). This means that \( \Phi(p') = p \) as desired.

Now we can prove Theorem \( (6.7) \) easily. By Lemmas \( (6.7) \) and \( (6.8) \) \( \Phi \) is an involution on \( \mathcal{L}^{\Sigma_{\text{NC}}} \) with the desired fixed point set. Suppose that \( \Phi(p) = p' \) and \( p \neq p' \). Lemma \( (6.3) \) and the third assertion of Lemma \( (6.4) \) show that \( \text{sign}(\text{type}(p')) = -\text{sign}(\text{type}(p)) \). Note that \( \text{wt}(p') = \text{sign}(\text{type}(p)) \text{wt}(T) \) and \( \text{wt}(p') = \text{sign}(\text{type}(p')) \text{wt}(T') \). By the construction of \( \Phi \), we have \( \text{wt}(T) = \text{wt}(T') \), and hence \( \text{wt}(p) = -\text{wt}(p') \), which completes the proof.
In this section we give a concrete example of the involution $\Phi$ applied to $p$ which is not a fixed point.

Let $n = 6$, $\lambda = (6, 6, 5, 5, 5, 5, 5, 4)$ and $\mu = (5, 3, 3, 1, 1, 1, 1)$. Then $\lambda' = (9, 9, 9, 8, 2)$ and $\mu' = (6, 3, 3, 1, 1)$. Let $A_i = (\mu_i' + n - i, 0)$ and $B_i = (\lambda_i' + n - i, 2\omega)$. Consider the $n$-path $p \in \mathcal{L}_{\lambda/\mu}^{\text{SNC}}$ in Figure 18 Note that $\text{type}(p) = \pi = (3, 2, 4, 1, 5, 6)$. We construct $\Phi(p) = p'$ as follows.

In Step 1, we find $T = \text{Tab}(p)$ and express $T = T_4 \sqcup T_3 \sqcup T_2 \sqcup T_1$ as shown in Figure 18 Then $T \in \text{VT}(\pi(\lambda)/\mu)$.

In Step 2, we find $U_1$ and $\phi_+^{m_1}(U_1) = \phi_+(U_1)$ as in Figure 19 and $U_2 = T_2 \sqcup \phi_+^{m_2}(U_2)$ as in Figure 20. Observe that $T_3$, which is columns 2 and 3 in the right diagram of Figure 20 does not satisfy $T_3 \sqcup \phi_+^{m_2}(U_2)$. Therefore $U_3$ is not define and Step 2 is finished.

In Step 3, $k = 2$ is the smallest integer such that $T_{k+1} \not\leq \phi_+^{m_k}(U_k)$.

In Step 3-1, we have $s = 5$ and $\gamma = (6, 6, 5, 1, 1, 1, 1)$. The vertical tableau row $\geq \mu_4 + 1(T_k \cup \phi_+^{m_k}(U_{k-1})) \in \text{VT}(\pi(\lambda)/\gamma)$ and its corresponding $n$-path $q$ are shown in Figure 21.

In Step 3-2, there are 7 intersections among $\{q_i : d_{k+1} + 1 \leq i \leq s\} = \{q_2, q_3, q_4, q_5\}$ and the intersection $(a, b) = (8, 8)$ of $q_i = q_3$ and $q_i = q_5$ is chosen. Then $q'$ is obtained from $q$ by exchanging the subpaths of $q_3$ and $q_5$ after $(8, 8)$. See Figure 22 for $q'$ and its corresponding vertical tableau.

In Step 3-3, $T_{k+1} \sqcup \bar{U}_k = \bar{T}_3 \sqcup \bar{U}_2$ is obtained from $T_{k+1} \cup \phi_+^{m_k}(U_k) = T_3 \cup \phi_+^{2}(U_2)$ by replacing the part equal to $\text{Tab}(q)$ by $\text{Tab}(q')$, see Figure 23. We then compute $\phi_+^{M_k}(\bar{U}_k) = \phi_+^{1}(\bar{U}_2)$ as in Figure 24. Finally, the tableau $T' = T_{k+1} \sqcup \cdots \sqcup T_{k+2} \sqcup \bar{T}_k \sqcup \phi_+^{M_k}(U_k) = T_4 \sqcup \bar{T}_3 \sqcup \phi_+^{1}(\bar{U}_2)$ and the corresponding $n$-path $p'$ are obtained as in Figure 25.
Figure 19. $U_1 = T_1$ is the last column in the left diagram and $\phi_+^{m_1}(U_1) = \phi_+(U_1)$ is the last column in the right diagram. The level of each RSE-tableau is marked by a star.

Figure 20. $U_2 = T_2 \sqcup \phi_+^{m_1}(U_1)$ is the last three columns in the left diagram, $\phi_+(U_2)$ is the last three columns in the middle diagram, and $\phi_+^{m_2}(U_2) = \phi_+^2(U_2)$ is the last three columns in the right diagram.

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Figure 21. The vertical tableau row $\geq M_{\lambda+1}(T_3 \sqcup \phi^{m_2}_{\lambda}(U_2)) \in \text{VT}(\pi(\lambda)/\gamma)$ and the corresponding $n$-path $q$. The chosen intersection $(a, b)$ is circled.

Figure 22. The $n$-path $q'$ and the corresponding vertical tableau $\text{Tab}(q') \in \text{VT}(\pi'(/\lambda)/\gamma)$, where $\pi' = \pi(4, 5)$.

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Figure 23. The left diagram is $T_{k+1} \sqcup \phi_+^m(U_k) = T_3 \sqcup \phi_2^2(U_2)$, where the part equal to $\text{Tab}(q)$ is colored yellow. The right diagram is $\tilde{T}_{k+1} \sqcup \tilde{U}_k = \tilde{T}_3 \sqcup \tilde{U}_2$, where the part equal to $\text{Tab}(q')$ is colored yellow.

Figure 24. The RSE-tableaux $\tilde{U}_2 = \tilde{U}_2, \phi_-(\tilde{U}_2), \phi_2^2(\tilde{U}_2)$ and $\phi_3^2(\tilde{U}_2) = \phi_M(\tilde{U}_k)$ from left to right. The level of each RSE-tableau is marked by a star.
Figure 25. The tableau $\mathcal{T}' = T_{r+1} \sqcup \cdots \sqcup T_{k+2} \sqcup \mathcal{T}_{k+1} \sqcup \mathcal{U}_k^M (\tilde{U}_k) = T_4 \sqcup \mathcal{T}_3 \sqcup \mathcal{U}_2^3 (\tilde{U}_2)$ and the corresponding $n$-path $p'$. 

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