In this comment, we address a number of erroneous discussions and conclusions presented in a recent preprint by the HALQCD collaboration, arXiv:1703.07210 [1]. In particular, we show that lattice QCD determinations of bound states at quark masses corresponding to a pion mass of \( m_\pi = 806 \text{ MeV} \) are robust, and that the extracted phases shifts for these systems pass all of the “sanity checks” introduced in arXiv:1703.07210 [1].

In the last decade, significant progress has been made in the study of multi-hadron systems using lattice QCD, with the first calculations of multi-baryon bound states and their electroweak properties and decays having been performed [2–28]. It is imperative that the methods used in these calculations be robust; investigations such as those of the HALQCD collaboration in Ref. [1] are vital provided they are carried out correctly. However, as we show in detail, many of the conclusions reached in Ref. [1] (henceforth referred to as HAL), that cast doubt on the validity of multi-baryon calculations, are incorrect. Since we have recently refined one of the analyses that is criticized in HAL, we focus our attention on the conclusions drawn regarding this case in particular, see Ref. [29].

The central point addressed by HAL is whether there exist bound states in the \( ^1S_0 \) and \( ^3S_1 \) two-nucleon channels at heavy quark masses. Three independent groups have analysed lattice QCD calculations at quark masses corresponding to a heavy pion mass of \( \sim 800 \text{ MeV} \) (one set of calculations used quenched QCD) and found that there are bound states in these channels. Each of these groups has concluded this by extracting energies from two-point correlation functions (with the quantum numbers of interest) at two or more lattice volumes and demonstrating, through extrapolations based on the finite-volume formalism of Lüscher [30, 31], that these energies correspond to an infinite-volume state that is below the two-particle threshold and is hence a bound state. Each group has used different technical approaches, and all are in reasonable agreement given the uncertainties that are reported. The HALQCD collaboration has also investigated these two-particle channels using a method (also based on the work of Lüscher [30, 31]) that involves constructing Bethe-Salpeter wavefunctions, but do not find evidence for bound states in these
channels [4] [5] [8] [13] [17]. We note, however, that the HALQCD method introduces unquantified systematic effects as discussed in, e.g., Refs. [32] [33] and the nuclear physics overview talks in recent proceedings of the International Symposium on Lattice Field Theory [35] [36]. Here, we focus our criticisms of HAL on several specific points.

1. Misinterpretation of energies and source independence

Figure 2 of HAL contains a compilation of results for the ground states of the $^1S_0$ and $^3S_1$ two-nucleon channels. Unfortunately the figure includes a second state from Ref. [22] that the authors of Ref. [22] explicitly indicate is not the ground state, and reporting it as such is a significant error on which many of the invalid arguments of HAL are based. There is a small scatter in the remaining results that is due to statistical fluctuations, discretisation artifacts and exponentially-small residual finite-volume effects, but, taken as a whole, there is no inconsistency in these results. In addition, a further recent study of axial-current matrix elements using a different set of interpolators [26] [28] (denoted in Fig. 1 by NPLQCD17) also finds a consistent negatively-shifted energy on the $32^3 \times 48$ ensemble used in this comparison. Figure 2 in HAL also fails to include the energies extracted in Ref. [13] on the largest volume, which dominate the extraction of the binding energy. Without the results from this large volume, the confidence in the binding energy in Ref. [13] would be significantly diminished. It is therefore vital that this information be included in any discussion of these results. Figure 1 below shows a (corrected) summary of the energy levels extracted for the ground states of the $^1S_0$ and $^3S_1$ two-nucleon systems in different volumes that are published in the literature at this particular quark mass. No significant interpolator dependence is observed, as is indicated by simple fits to the reported results for each volume, with all these fits having acceptable values of $\chi^2$ per degree of freedom. Figure 13 of HAL is also erroneously described as indicating that scattering state results are not source independent. The results show three energy levels where different interpolating operators are consistent within one standard deviation, and one energy level that differs at two standard deviations. This indicates broad agreement within the reported uncertainties and, contrary to statements in HAL, does not provide a sound statistical basis for a claim of inconsistency.

In summary, comparison of results from the different interpolators in Refs. [13] [14] [22] [28] shows that both bound and scattering-state energy levels are source-independent within reported uncertainties. This is contrary to the claims in HAL.

2. Volume scaling of energies

The authors of HAL claim that the single-exponential behaviour found in our work, Refs. [13] [14], and in that of Ref. [22], is a “mirage” arising from the cancellation of two or more scattering eigenstates contributing to the correlation functions with opposite signs (see Ref. [38] for elaborations on possible “mirage” plateaus). This interpretation of the negatively-shifted states in these works is exceedingly unlikely, however, as such cancellation would need to occur in an almost identical way for multiple different volumes. For each of the different analyses of the

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1 In the ΛΛ channel, the HALQCD approach does indicate a bound state, but the binding energy is found to be significantly different from that determined by extrapolating finite-volume energy levels [13].
2 Whether the quoted value for the second energy in Ref. [22] is a true estimate of an excited-state energy is a question for future discussion. However for the ground states, all results unambiguously agree.
3 The scattering states are loosely used here to denote states in a finite volume that correspond to the continuum states of infinite volume.
FIG. 1: Binding energies of the $^3S_1$ and $^1S_0$ ground states at $m_\pi = 806$ MeV found in the literature: NPLQCD13 [13], Berkowitz16 [22], and NPLQCD17 [26–28] ($d = 0$ and $d = 2$ refer to the magnitude of the centre-of-mass momentum used in the calculations in units of $2\pi/L$). The three regions in each panel correspond to three different volumes: $L = 24, 32$, and $48$ from left to right. Uncertainties listed in the original references are combined in quadrature. The horizontal lines and shaded bands represent the central value and one standard deviation bands from uncorrelated fits, respectively.

806 MeV ensembles in Fig. [ ] (NPLQCD2013 $d = 0$, NPLQCD2013 $d = 2$ and Berkowitz2016 $d = 0$), identical sources and sinks were used in each of three volumes (two volumes in the case of Berkowitz2016). Scattering-state eigenenergies necessarily change significantly with volume, having power-law dependence as dictated by the Lüscher quantisation condition. While it is possible that, in a given volume, a correlator for a particular source-sink interpolator combination could exhibit a cancellation between contributions of two scattering states that produces an energy level below threshold, it is very unlikely that the cancellation would persist in different volumes as the scattering-state eigenenergies change significantly with volume. As shown in Fig. [ ], for example, the volume-independent interpolators used in Ref. [13, 14] produce energy levels in the three different volumes that are statistically indistinguishable, and even the approach to single-exponential behaviour does not depend on volume. The figure shows the effective masses of the smeared-point correlation functions, but the same features are seen in all other source-sink interpolator combinations that are studied. This rules out the possibility that the negatively-shifted signals are caused by cancellations between scattering states. The largest volume used in our works [13, 14] makes this an extremely robust statement as the spatial volumes from which we draw these conclusions vary by a factor of eight.

3. Consistency of Effective Range Expansion (“HAL Sanity Check (i)”)

If the effective range expansion (ERE) is a valid parametrization of the scattering amplitude at low energies, the analyticity of the amplitude as a function of the centre-of-mass energy implies that the ERE obtained from states with positively-shifted energies ($k^*^2 > 0$, where $k^*$ is the
centre-of-mass interaction momentum) must be consistent with that obtained from states with negatively-shifted energies \(k^{*2} < 0\). Although HAL finds that the NPLQCD results pass this test, we demonstrate how robust the results in Refs. \[13, 14\] are in this regard through the plots presented in Fig. 3. This figure shows fits to the ERE using both ground states \((n = 1)\) and first excited states \((n = 2)\) (color-shaded bands). These are overlaid on ERE fits using only the ground states (hashed bands). The two sets of bands are fully consistent with each other, proving that this check is unambiguously passed. The same feature is seen for three-parameter ERE fits, with significantly larger uncertainty bands (see also Ref. \[29\]).

The difference in the size of uncertainties in the phase shift between the fits with and without the \(n = 2\) data shows that conclusions about the behaviour and/or validity of the ERE for datasets only near the bound-state pole are likely subject to significant uncertainties. We note that scattering parameters extracted in the region near \(k^{*2} = 0\) from a linear ERE will in general differ from those determined in the vicinity of a bound-state pole due to higher order terms in the ERE. Indeed, it is known that in nature, the ERE of the \(3S_1\) phase shift around \(k^{*2} = 0\) and around the deuteron pole are different (albeit slightly) \[39\].

4. **Residue of the S-matrix at the bound-state pole (“HAL Sanity check (iii)”)**

The sign of the residue of the S-matrix at the bound-state pole is fixed. This requirement leads to the following condition on \(k^{*} \cot \delta\):

\[
\frac{d}{dk^{*2}} \left( k^{*} \cot \delta + \sqrt{-k^{*2}} \right) \bigg|_{k^{*2} = -\kappa(\infty)^2} < 0,
\]

\( \kappa(\infty)^2 \)

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\(4\) Our analysis of two-nucleon correlation functions generated from these ensembles of gauge-field configurations has been recently refined in a comprehensive re-analysis \[29\], including results at additional kinematic points. This new analysis has been used in obtaining the results shown in Figs. 3 and 4. All of the energies extracted from the three lattice volumes, and the binding energies and ERE parameters subsequently obtained, are in agreement with our previous results; i.e., the differences in the mean values of the results from the previous and the new analyses are within one standard deviation as defined by the (statistical and systematic) uncertainties of the results combined in quadrature \[13, 14\].
FIG. 3: $k^*\cot \delta$ vs. the square of the centre-of-mass momentum of two baryons, $k^{*2}$, along with the bands representing fits to two-parameter EREs obtained from i) only the ground states ($n = 1$) and ii) from both the ground states ($n = 1$) and the first excited states ($n = 2$). The plots show the consistency of the ERE between negative and positive $k^{*2}$ regions in both the $1S_0$ and $3S_1$ channels. These results are from our recent re-analysis of these ensembles [29], and are consistent with the initial analysis [13, 14], with the mean values in agreement within one standard deviation as defined by the combined (statistical and systematic) uncertainties of each result. Quantities are expressed in lattice units (l.u.).

where $\kappa(\infty)$ is the infinite-volume binding momentum. As is seen from Fig. 4 which displays the results of the 2017 refined analysis [29] of the correlation functions analyzed in Refs. [13, 14], the slope of the two-parameter ERE fit to the $k^*\cot \delta$ function (colored regions) is less than the slope of $-\sqrt{-k^{*2}}$ (grey regions) at the corresponding bound-state pole in both channels. In the $1S_0$ channel the difference is at the $1\sigma$ level, while the difference is more than $3\sigma$ in the coupled $3S_1-3D_1$ channels. The uncertainty in the tangent line to the $-\sqrt{-k^{*2}}$ function at $k^{*2} = -\kappa(\infty)^2$ arises from the uncertainty in the values of $\kappa(\infty)$ (see also Ref. [29]). A similar conclusion can be drawn from three-parameter ERE fits. For the sake of clarity, the two-parameter ERE fits to the results of only the 2013 analysis of the same correlation functions are shown in Fig. 5 and are seen to be consistent with the criterion in Eq. (1) as well.

For comparison, in Fig. 6 we show the results of two-parameter ERE fits obtained in the 2013 analysis [13, 14] and in the 2017 refined analysis of the same correlation functions [29]. Both analyses of these channels yield results that are consistent with each other and with the criterion in Eq. (1) within the uncertainties of the calculations, thus passing check (iii).
Given the discussion above, the NPLQCD results presented in the “NPL2013” row of Table IV of the published version of HAL [1], reproduced below,

| Data          | Source independence | Sanity check (i) | Sanity check (ii) | Sanity check (iii) | Source independence | Sanity check (i) | Sanity check (ii) | Sanity check (iii) |
|---------------|---------------------|------------------|-------------------|-------------------|---------------------|------------------|-------------------|-------------------|
| NPL2013 [28,29] | No                  | *                | *                 | No                | No                  | *                | *                 | ?                 |

should be replaced by (with reference numbers changed to relate to the present bibliography)

| Data          | Source independence | Sanity check (i) | Sanity check (ii) | Sanity check (i) | Sanity check (ii) | Sanity check (iii) | Sanity check (i) | Sanity check (ii) | Sanity check (iii) |
|---------------|---------------------|------------------|-------------------|------------------|------------------|-------------------|------------------|------------------|-------------------|
| NPL2013 [13,14] | Yes                 | Passed           | Passed            | Passed           | Yes              | Passed            | Passed           | Passed           |
| NPL2017 [29] of NPL2013 | Yes             | Passed           | Passed            | Passed           | Yes              | Passed            | Passed           | Passed           |
where we have taken the liberty of changing the notation (in their published version) used to indicate passing a “sanity check” in HAL from a “*” entry to “Passed”. We are currently revisiting the other NPLQCD analyses discussed in HAL. Ref. [40] refutes the HAL criticisms of source-dependence leveled at the works of the PACS-CS collaboration [7]. Ref. [37] provides a summary of the evidence for the validity of ground-state identifications in two-nucleon systems. With the robust conclusion of the existence of bound states reached by independent groups, and argued in this Comment, the systematic uncertainties of the potential method used by the HALQCD
collaboration requires further investigation to better understand the origin of its failure to identify two-nucleon bound states.

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