AN APPLICATION OF λ-METHOD
ON INEQUALITIES OF SHAFER-FINK’S TYPE

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Abstract. In this article λ-method of Mitrinović-Vasić [1] is applied to improve
the upper bound for the arc sin function of L. Zhu [4].

1. Inequalities of Shafer-Fink’s type

D. S. Mitrinović in [1] considered the lower bound of the arc sin function, which
belongs to R. E. Shafer. Namely, the following statement is true.

Theorem 1.1 For 0 ≤ x ≤ 1 the following inequalities are true:

\[
\frac{3x}{2 + \sqrt{1 - x^2}} \leq \frac{6(\sqrt{1 + x} - \sqrt{1 - x})}{4 + \sqrt{1 + x} + \sqrt{1 - x}} \leq \arcsin x.
\]

(1)

A. M. Fink proved the following statement in [2].

Theorem 1.2 For 0 ≤ x ≤ 1 the following inequalities are true:

\[
\frac{3x}{2 + \sqrt{1 - x^2}} \leq \arcsin x \leq \frac{\pi x}{2 + \sqrt{1 - x^2}}.
\]

(2)

B. J. Malešević proved the following statement in [3].

Theorem 1.3 For 0 ≤ x ≤ 1 the following inequalities are true:

\[
\frac{3x}{2 + \sqrt{1 - x^2}} \leq \arcsin x \leq \frac{\pi - 2x}{\pi - 2 + \sqrt{1 - x^2}} \leq \frac{\pi x}{2 + \sqrt{1 - x^2}}.
\]

(3)

The main result of the article [3] can be formulated with the next statement.

Proposition 1.4 In the family of the functions:

\[
f_b(x) = \frac{(b + 1)x}{b + \sqrt{1 - x^2}} \quad (0 \leq x \leq 1),
\]

according to the parameter b > 0, the function \( f_2(x) \) is the greatest lower bound of
the \( \arcsin x \) function and the function \( f_{2/(\pi - 2)}(x) \) is the least upper bound of the
\( \arcsin x \) function.
L. Zhu proved the following statement in [4].

**Theorem 1.5** For \( x \in [0, 1] \) the following inequalities are true:

\[
\frac{3x}{2 + \sqrt{1 - x^2}} \leq \frac{6(\sqrt{1 + x} - \sqrt{1 - x})}{4 + \sqrt{1 + x} + \sqrt{1 - x}} \leq \arcsin x
\]

(5)

\[
\leq \frac{\pi(\sqrt{2} + \frac{1}{2})(\sqrt{1 + x} - \sqrt{1 - x})}{4 + \sqrt{1 + x} + \sqrt{1 - x}} \leq \frac{\pi x}{2 + \sqrt{1 - x^2}}
\]

In this article we further improve the upper bound of the \( \arcsin \) function. Namely, in the next section we will give proof of the following theorem:

**Theorem 1.6** For \( x \in [0, 1] \) the following inequalities are true:

\[
\frac{3x}{2 + \sqrt{1 - x^2}} \leq \frac{6(\sqrt{1 + x} - \sqrt{1 - x})}{4 + \sqrt{1 + x} + \sqrt{1 - x}} \leq \arcsin x
\]

(6)

\[
\leq \frac{\pi(\sqrt{2} - \sqrt{2})}{\pi - 2\sqrt{2}}(\sqrt{1 + x} - \sqrt{1 - x})
\]

\[
\leq \frac{\sqrt{2}(4 - \pi)}{\pi - 2\sqrt{2}} + \sqrt{1 + x} + \sqrt{1 - x}
\]

\[
\leq \frac{\pi(\sqrt{2} + \frac{1}{2})(\sqrt{1 + x} - \sqrt{1 - x})}{4 + \sqrt{1 + x} + \sqrt{1 - x}} \leq \frac{\pi x}{2 + \sqrt{1 - x^2}}
\]

**Remark 1.7** Using numerical method from [5] we have the following conclusions:

1º. For values \( x \in (0, 0.387 266 274 \ldots) \) the following inequality is true:

(7) \( \arcsin x < \frac{\pi}{2 + \sqrt{1 - x^2}} < \frac{\pi(\sqrt{2} + \frac{1}{2})(\sqrt{1 + x} - \sqrt{1 - x})}{4 + \sqrt{1 + x} + \sqrt{1 - x}} \),

and for values \( x \in (0.387 266 274 \ldots, 1) \) the following inequality is true:

(8) \( \arcsin x < \frac{\pi(\sqrt{2} + \frac{1}{2})(\sqrt{1 + x} - \sqrt{1 - x})}{4 + \sqrt{1 + x} + \sqrt{1 - x}} < \frac{\pi}{2 + \sqrt{1 - x^2}} \).

Numerically determined constant \( c = 0.387 266 274 \ldots \) is the unique number where the previous bounds have the same values over \((0, 1)\).

2º. For values \( x \in (0, 1) \) the following inequality is true:

(9) \( \arcsin x < \frac{\pi(\sqrt{2} - \sqrt{2})}{\pi - 2\sqrt{2}}(\sqrt{1 + x} - \sqrt{1 - x}) \)

\[
\leq \frac{\sqrt{2}(4 - \pi)}{\pi - 2\sqrt{2}} + \sqrt{1 + x} + \sqrt{1 - x} \leq \frac{\pi}{2 + \sqrt{1 - x^2}}.
\]
2. The main results

In this article, using \( \lambda \)-method of Mitrinović-Vasić we give an analogous statement to Proposition \[1.4\]. Let us notice that from inequality given by L. Zhu \[4\]:

\[
\frac{6(\sqrt{1+x} - \sqrt{1-x})}{4 + \sqrt{1+x} + \sqrt{1-x}} \leq \text{arcsin} \, x \leq \frac{\pi (\sqrt{2} + \frac{1}{2}) (\sqrt{1+x} - \sqrt{1-x})}{4 + \sqrt{1+x} + \sqrt{1-x}},
\]

for \( x \in [0, 1] \), we can conclude that the function \( \varphi(x) = \text{arcsin} \, x \) has a lower bound and upper bound in the family of the functions:

\[
\Phi_{\alpha,\beta}(x) = \alpha(\sqrt{1+x} - \sqrt{1-x}) + \frac{1}{\beta + \sqrt{1+x} + \sqrt{1-x}} (0 \leq x \leq 1),
\]

for some values of parameters \( \alpha, \beta > 0 \). Next for \( x = 0 \) it is true that \( \Phi_{\alpha,\beta}(0) = 0 \), for \( \alpha, \beta > 0 \). On the other hand, for values \( x \in (0, 1] \) it is true:

\[
\Phi_{\alpha_1,\beta_1}(x) > \Phi_{\alpha_2,\beta_2}(x) \iff \alpha_1 \beta_2 - \alpha_2 \beta_1 > (\alpha_2 - \alpha_1)(\sqrt{1+x} + \sqrt{1-x}),
\]

for \( \alpha_{1,2}, \beta_{1,2} > 0 \). Let us apply \( \lambda \)-method of Mitrinović-Vasić on the considered two-parameters family \( \Phi_{\alpha,\beta}(x) \) in order to determine the bounds of the function \( \varphi(x) \) under the following conditions:

\[
\Phi_{\alpha,\beta}(0) = \varphi(0) \quad \text{and} \quad \frac{d}{dx}\Phi_{\alpha,\beta}(0) = \frac{d}{dx}\varphi(0).
\]

It follows that \( \alpha = \beta + 2 \). In that way we get one-parameter subfamily:

\[
f_{\beta}(x) = \Phi_{\beta+2,\beta}(x) = \frac{(\beta + 2)(\sqrt{1+x} - \sqrt{1-x})}{\beta + \sqrt{1+x} + \sqrt{1-x}} (0 \leq x \leq 1),
\]

according to the parameter \( \beta > 0 \). For that family the condition \[13\] is true:

\[
f_{\beta}(0) = \varphi(0) \quad \text{and} \quad \frac{d}{dx}f_{\beta}(0) = \frac{d}{dx}\varphi(0).
\]

Additionally, we have:

\[
\frac{d^2}{dx^2}f_{\beta}(0) = \frac{d^2}{dx^2}\varphi(0) \quad \text{and} \quad \frac{d^3}{dx^3}f_{\beta}(0) = \frac{d^3}{dx^3}\varphi(0) + \frac{4 - \beta}{4(2+\beta)}
\]

and

\[
\frac{d^4}{dx^4}f_{\beta}(0) = \frac{d^4}{dx^4}\varphi(0) \quad \text{and} \quad \frac{d^5}{dx^5}f_{\beta}(0) = \frac{d^5}{dx^5}\varphi(0) + \frac{3(128 + 18\beta - 13\beta^2)}{16(2+\beta)^2}.
\]

Let us notice that for the family of the functions \( f_{\beta}(x) \), on the basis of \[12\], for values \( x \in (0, 1] \) the following equivalence is true:

\[
f_{\beta_1}(x) > f_{\beta_2}(x) \iff \beta_1 < \beta_2,
\]
for $\beta_{1,2} > 0$. Let us emphasize that there is a better upper bound $f_{b_1}(x)$ than upper bound $\Phi_{\pi(\sqrt{x+1/2}),4}(x)$ of the function $\varphi(x)$ over $(0,1]$. It is true that the parameter $\beta = b_1$ fulfils:

(19) \[ f_{b_1}(1) = \varphi(1) = \frac{\pi}{2} \]

hence:

(20) \[ b_1 = \frac{\sqrt{2}(4 - \pi)}{\pi - 2\sqrt{2}} = 3.876452527 \ldots < 4. \]

Let us prove that the function $f_{b_1}(x)$ is the upper bound of the function $\varphi(x)$ over $[0,1]$. Let us define the function:

(21) \[ h(x) = f_{b_1}(x) - \varphi(x) \]

for $0 \leq x \leq 1$. For the function $h(x)$ we introduce two substitutions $x = \cos t$ ($t \in [0, \frac{\pi}{2}]$) and $t = 4 \arctg u$ ($u \in [0, \frac{\pi}{8}]$) respectively, and we get a new function:

(22) \[ w(u) = h(\cos(4 \arctg u)) = \frac{\sqrt{2}(b_1 + 2)(u^2 + 2u - 1)}{(\sqrt{2} - b_1)u^2 - 2\sqrt{2}u - b_1 - \sqrt{2}} - \frac{\pi}{2} + 4 \arctg u \]

for $0 \leq u \leq \tan \frac{\pi}{8} = \sqrt{2} - 1$. Then:

(23) \[
\frac{d}{du} w(u) = \left( \left( 4b_1^2 + 2\sqrt{2}b_1^2 - 8b_1 - 4\sqrt{2}b_1 - 8 \right) u^4 + \left( -4\sqrt{2}b_1^2 + 8\sqrt{2}b_1 - 32 \right) u^3 + \right.
\left. \left( 8b_1^2 - 16b_1 - 16 \right) u^2 + \left( -4\sqrt{2}b_1^2 + 8\sqrt{2}b_1 + 32 \right) u + \right.
\left. \left( -4\sqrt{2}b_1^2 - 8b_1 + 4\sqrt{2}b_1 - 8 \right) \right) / \left( \left( u^2 + 1 \right) \left( b_1 u^2 - 2u^2 + 2\sqrt{2}u + b_1 + \sqrt{2} \right) \right). \]

All solutions of the equation $\frac{d}{du} w(u) = 0$ are determined by terms:

(24) \[
\begin{align*}
  u_{1,4} & = \frac{2\sqrt{2} \mp \sqrt{-b_1^4 + 4b_1^4 + 4b_1^2 - 16b_1}}{b_1^2 - 2b_1 + 2\sqrt{2} - 4}, \\
  u_{2,3} & = \sqrt{2} - 1;
\end{align*}
\]

or by numerical values: $u_1 = 0.0869 \ldots$, $u_{2,3} = 0.4142 \ldots$, $u_4 = 0.8400 \ldots$. The function $w(u)$ has local maximum at the point $u_1$ and $w(0) = w(\sqrt{2} - 1) = 0$. Hence $w(u) \geq 0$ for $u \in [0, \sqrt{2} - 1]$. Therefore the function:

(25) \[
\begin{align*}
  f_{b_1}(x) = & \frac{\pi(2 - \sqrt{2})}{\pi - 2\sqrt{2}} \left( \sqrt{1 + x} - \sqrt{1 - x} \right) \\
   & \frac{\sqrt{2}(4 - \pi)}{\pi - 2\sqrt{2}} + \sqrt{1 + x} + \sqrt{1 - x}.
\end{align*}
\]
An application of Lambda-method on inequalities of Shafer-Fink’s type

is the upper bound of \( \varphi(x) \) over \([0, 1]\). Let us notice that, for values \( x \in (0, 1] \), on the basis (12), the following inequalities are true:

\[
\varphi(x) < f_{b_1}(x) = \Phi_{b_1+2,b_1}(x) < \Phi_{\pi(\sqrt{2}+1/2),4}(x).
\]

Let us prove that the function \( f_{b_1}(x) \) is the least upper bound of the function \( \varphi(x) \) from the family (14). The following implication is true:

\[
b_1 < b = \Rightarrow f_{b_1}(1) < f_b(1) = \varphi(1) = \frac{\pi}{2}.
\]

Hence for \( b > b_1 \) the function \( f_b(x) \) is not the upper bound for the function \( \varphi(x) \) over \([0, 1]\). According to the previous consideration we can conclude that the function \( f_{b_1}(x) \) is the least upper bound of the function \( \varphi(x) \) over \([0, 1]\).

The lower bound of the function \( f_4(x) \) of the function \( \varphi(x) \) over \([0, 1]\), which belongs to R. E. Shafer, according to formulas (15) - (17), has at \( x = 0 \) the root of the fifth order. Let us prove that the function \( f_4(x) \) is the greatest lower bound of the function \( \varphi(x) \) from the family (14). For fixed \( b \in (b_1, 4) \) let us define the function:

\[
g(x) = \begin{cases} 
\alpha & : x = 0, \\
\frac{f_b(x) - \varphi(x)}{x^3} & : x \in (0, 1]; 
\end{cases}
\]

with the constant:

\[
\alpha = \frac{d^3 f_b(0) - d^3 \varphi(0)}{6} = \frac{4 - b}{24(2 + b)} > 0.
\]

The function \( g(x) \) is continuous over \([0, 1]\) and the following is true:

\[
g(0) > 0 \quad \text{and} \quad g(1) < 0.
\]

Therefore we can conclude that there is \( c_b \in (0, 1) \) such that \( g(c_b) = 0 \). Let us notice that \( g(0) > 0 \) and \( g(c_b) = 0 \). Then, there is some point \( \xi_b \in (0, c_b) \) such that \( g(\xi_b) > 0 \) \((g \in C[0, c_b])\). This is sufficient for conclusion that, for each \( b \in (b_1, 4) \), the function \( f_b(x) \) is not the lower bound of the function \( \varphi(x) \) over \([0, 1]\). According to the previous consideration we can conclude that the function \( f_4(x) \) is the greatest lower bound of the function \( \varphi(x) \) over \([0, 1]\).

On the basis of the previous consideration the following statement is true.

**Proposition 2.1** In the family of the functions:

\[
f_b(x) = \Phi_{b+2,b}(x) = \frac{(b + 2)(\sqrt{1 + x} - \sqrt{1 - x})}{b + \sqrt{1 + x} + \sqrt{1 - x}} \quad (0 \leq x \leq 1),
\]

according to the parameter \( b > 0 \), the function \( f_4(x) \) is the greatest lower bound of the arc sin \( x \) function and the function \( f_{\sqrt{\pi(4-\pi)/(\pi-2\sqrt{\pi})}}(x) \) is the least upper bound of the arc sin \( x \) function.
Remark 2.2 Let us emphasize that Theorem 1.6 has been recently considered in [6] and [7]. In the article [7] a simple proof of Theorem 1.6 based on "L'Hospital rule for monotonicity" is obtained.

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