On the Picard number of Fano 3-folds with terminal singularities

To memory of Boris Moishezon

By

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Introduction

Here we continue investigations started in [N6], [N7]. Algebraic varieties we consider are defined over field $\mathbb{C}$ of complex numbers.

In this paper, we get a final result on estimating the Picard number $\rho = \dim N_1(X)$ of a Fano 3-fold $X$ with terminal $\mathbb{Q}$-factorial singularities if $X$ does not have small extremal rays and its Mori polyhedron does not have faces with Kodaira dimension 1 or 2. One can consider this class as a generalization of the class of Fano 3-folds with Picard number 1. There are many non-singular Fano 3-folds satisfying this condition and with Picard number 2 (see [Mo-Mu] and also [Ma]). We also think that studying the Picard number of this class may be important for studying Fano 3-folds with Picard number 1, too (see Corollary 2 below).

Let $X$ be a Fano 3-fold with $\mathbb{Q}$-factorial terminal singularities. Let $R$ be an extremal ray of the Mori polyhedron $\overline{NE}(X)$ of $X$. We say that $R$ has the type (I) (respectively (II)) if curves of $R$ fill an irreducible divisor $D(R)$ of $X$ and the contraction of the ray $R$ contracts the divisor $D(R)$ to a point (respectively to a curve). An extremal ray $R$ is called small if curves of this ray fill a curve on $X$.

A pair $\{R_1, R_2\}$ of extremal rays has the type $\mathcal{B}_2$ if extremal rays $R_1, R_2$ are different, both have the type (II), and have the same divisor $D(R_1) = D(R_2)$.

We recall that a face $\gamma$ of Mori polyhedron $\overline{NE}(X)$ defines a contraction $f: \gamma \rightarrow X'$ (see [Ka1] and [Sh]) such that $f(C)$ is a point for an irreducible curve $C$ if and only if $C$ belongs to $\gamma$. The $\dim X'$ is called the Kodaira dimension of the $\gamma$. A set $\delta$ of extremal rays is called extremal if it is contained in a face of Mori polyhedron.

Basic Theorem. Let $X$ be a Fano 3-fold with terminal $\mathbb{Q}$-factorial sing-
ularities. Assume that $X$ does not have a small extremal ray, and Mori polyhedron $\overline{NE}(X)$ does not have a face of Kodaira dimension 1 or 2.

Then the following statements for the $X$ hold:

1. The $X$ does not have a pair of extremal rays of the type $B_2$ and Mori polyhedron $\overline{NE}(X)$ is simplicial;
2. The $X$ does not have more than one extremal ray of the type (I);
3. If $S$ is an extremal set of $k$ extremal rays of $X$, then the $S$ has one of the types: $A_1 \Pi (k - 1) C_1$, $D_2 \Pi (k - 2) C_1$, $E_2 \Pi (k - 2) C_1$, $kC_1$ (we use notation of Theorem 2.3.3).
4. We have the inequality for the Picard number of the $X$:
\[ \rho (X) = \dim N_1 (X) \leq 7. \]

Proof. See Theorem 2.5.8.

It follows from (4):

**Corollary 1.** Let $X$ be a Fano 3-fold with terminal $\mathbb{Q}$-factorial singularities and $\rho (X) > 7$. Then $X$ has either a small extremal ray or a face of Kodaira dimension 1 or 2 for Mori polyhedron.

We mention that non-singular Fano 3-folds do not have a small extremal ray (by Mori [Mo1]), and their maximal Picard number is equal to 10 according to their classification by Mori and Mukai [Mo–Mu]. Thus, all these statements already work for non-singular Fano 3-folds.

From the statement (2) of the Theorem, we also get the following application of Basic Theorem to geometry of Fano 3-folds.

Let us consider a Fano 3-fold $X$ and its blow-up $X_p$ at different non-singular points $|x_1, ..., x_p|$ of $X$. We say that this is a Fano blow-up if $X_p$ is Fano. We have the following very simple

**Proposition.** Let $X$ be a Fano 3-fold with terminal $\mathbb{Q}$-factorial singularities and without small extremal rays. Let $X_p$ be a Fano blow-up of $X$. Then for any small extremal ray $S$ on $X_p$, the $S$ has a non-empty intersection with one of exceptional divisors $E_1, ..., E_p$ of this blow up and does not belong to any of them. The divisors $E_1, ..., E_p$ define $p$ extremal rays of the type (I) on $X_p$.

Proof. See Proposition 2.2.14.

It is known that a contraction of a face of Kodaira dimension 1 or 2 of $\overline{NE}(X)$ of a Fano 3-fold $X$ has a general fiber which is a rational surface or curve respectively, because this contraction has relatively negative canonical class. See [Ka1], [Sh]. It is also known that a small extremal ray is rational [Mo2].

Then, using Basic Theorem and Proposition, we can divide Fano 3-folds of Basic Theorem on the following 3 classes:

**Corollary 2.** Let $X$ be a Fano 3-fold with terminal $\mathbb{Q}$-factorial singular-
for X (by rational curves over T g E which contain Fano 3-folds with Generalizations of results here one can find in a preprint professors support to these my studies.

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Preliminary variant of this paper was published as a preprint [N8]. Generalizations of results here one can find in a preprint [N9].
CHAPTER I. Diagram Method

Here we give the simplest variant of the diagram method for multi-dimensional algebraic varieties. We shall use this method in the next chapter. This part also contains some corrections and generalizations of the corresponding parts of our papers [N6] and [N7].

Let \( X \) be a projective algebraic variety with \( \mathbb{Q} \)-factorial singularities over an algebraically closed field. Let \( \dim X \geq 2 \). Let \( N_1(X) \) be the \( \mathbb{R} \)-linear space generated by the numerical equivalence classes of all algebraic curves on \( X \), and let \( N^1(X) \) be the \( \mathbb{R} \)-linear space generated by the numerical equivalence classes of all Cartier (or Weil) divisors on \( X \). Linear spaces \( N_1(X) \) and \( N^1(X) \) are dual to one another by the intersection pairing. Let \( NE(X) \) be a convex cone in \( N_1(X) \) generated by all effective curves on \( X \). Let \( NE(X) \) be the closure of the cone \( NE(X) \) in \( N_1(X) \). It is called Mori cone (or polyhedron) of \( X \). A non-zero element \( x \in N^1(X) \) is called nef if \( x \cdot NE(X) \geq 0 \). Let \( NEF(X) \) be the set of all nef elements of \( X \) and the zero. It is the convex cone in \( N^1(X) \) dual to Mori cone \( NE(X) \). A ray \( R \subset NE(X) \) with origin 0 is called extremal if from \( C_1 \in \overline{NE}(X) \), \( C_2 \in \overline{NE}(X) \) and \( C_1 + C_2 \in R \) it follows that \( C_1 \in R \) and \( C_2 \notin R \).

We consider a condition (i) for a set \( \mathcal{E} \) of extremal rays on \( X \).

(i) If \( R \in \mathcal{E} \), then all curves \( C \in \mathcal{E} \) fill out an irreducible divisor \( D(R) \) on \( X \).

In this case, an oriented graph \( G(\mathcal{E}) \) corresponds to \( \mathcal{E} \) in the following way: Two different rays \( R_1 \) and \( R_2 \) are joined by an arrow \( R_1 R_2 \) from \( R_1 \) to \( R_2 \) if \( R_1 \cdot D(R_2) > 0 \). Here and in what follows, for an extremal ray \( R \) and a divisor \( D \) we write \( R \cdot D > 0 \) if \( r \cdot D > 0 \) for \( r \in R \) and \( r \neq 0 \). (The same convention is applied for the symbols \( \leq \), \( \geq \) and \( < \).)

A set \( \mathcal{E} \) of extremal rays is called external if it is contained in a face of \( \overline{NE}(X) \). Equivalently, there exists a nef element \( H \in N^1(X) \) such that \( \mathcal{E} \cdot H = 0 \). Evidently, a subset of an extremal set is extremal, too.

We consider the following condition (ii) for extremal sets \( \mathcal{E} \) of extremal rays.

(ii) An extremal set \( \mathcal{E} = \{ R_1, ..., R_n \} \) satisfies the condition (i), and for any real numbers \( m_1 \geq 0, ..., m_n \geq 0 \) which are not all equal to 0, there exists a ray \( R_j \in \mathcal{E} \) such that \( R_j \cdot (m_1D(R_1) + m_2D(R_2) + \cdots + m_nD(R_n)) < 0 \). In particular, the effective divisor \( m_1D(R_1) + m_2D(R_2) + \cdots + m_nD(R_n) \) is not nef.

A set \( \mathcal{L} \) of extremal rays is called \( E \)-set (extremal in a different sense) if the \( \mathcal{L} \) is not extremal but every proper subset of \( \mathcal{L} \) is extremal. Thus, \( \mathcal{L} \) is a minimal non-extremal set of extremal rays. Evidently, an \( E \)-set \( \mathcal{L} \) contains at least two elements.

We consider the following condition (iii) for \( E \)-sets \( \mathcal{L} \).
(iii) Any proper subset of an $E$-set $\mathcal{L} = |Q_1, \ldots, Q_m|$ satisfies the condition (ii), and there exists a non-zero effective nef divisor $D(\mathcal{L}) = a_1D(Q_1) + a_2D(Q_2) + \cdots + a_mD(Q_m)$.

The following statement is very important.

**Lemma 1.1.** An $E$-set $\mathcal{L}$ satisfying the condition (iii) is connected in the following sense: For any decomposition $\mathcal{L} = \mathcal{L}_1|\cdots|\mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are non-empty, there exists an arrow $Q_1Q_2$ such that $Q_1 \in \mathcal{L}_1$ and $Q_2 \in \mathcal{L}_2$.

If $\mathcal{L}$ and $\mathcal{M}$ are two different $E$-sets satisfying the condition (iii), then there exists an arrow $LM$ where $L \in \mathcal{L}$ and $M \in \mathcal{M}$.

**Proof.** Let $\mathcal{L} = |Q_1, \ldots, Q_m|$. By (iii), there exists a nef divisor $D(\mathcal{L}) = a_1D(Q_1) + a_2D(Q_2) + \cdots + a_mD(Q_m)$. If one of the coefficients $a_1, \ldots, a_m$ is equal to zero, we get a contradiction to the conditions (ii) and (iii). It follows that all the coefficients $a_1, \ldots, a_m$ are positive. Let $\mathcal{L} = \mathcal{L}_1|\cdots|\mathcal{L}_k$ where $\mathcal{L}_1 = |Q_1, \ldots, Q_k|$, and $\mathcal{L}_2 = |Q_{k+1}, \ldots, Q_m|$. The divisors $D_1 = a_1D(Q_1) + \cdots + a_kD(Q_k)$ and $D_2 = a_{k+1}D(Q_{k+1}) + \cdots + a_mD(Q_m)$ are non-zero. By (ii), there exists a ray $Q_s$, $1 \leq i \leq k$, such that $Q_i \cdot D_i < 0$. On the other hand, $Q_s \cdot (D_1 + D_2) \geq 0$. It follows that there exists $j$, $k+1 \leq j \leq m$, such that $Q_s \cdot D(Q_j) > 0$. It means that $Q_sQ_j$ is an arrow.

Let us prove the second statement. By the condition (iii), for every ray $R \in \mathcal{L}$, we have the inequality $R \cdot D(\mathcal{M}) \geq 0$. If $R \cdot D(\mathcal{M}) = 0$ for any $R \in \mathcal{L}$, then the set $\mathcal{L}$ is extremal, and we get the contradiction. It follows that there exists a ray $R \in \mathcal{L}$ such that $R \cdot D(\mathcal{M}) > 0$. It follows our assertion.

Let $\text{NEF}(X) = \overline{\text{NE}}(X) \ast \subset N^1(X)$ be the cone of nef elements of $X$ and $\mathcal{M}(X) = \text{NEF}(X)/\mathbb{R}^+$ its projectivization. We use usual relations of orthogonality between subsets of $\mathcal{M}(X)$ and $\overline{\text{NE}}(X)$. So, for $U \subset \mathcal{M}(X)$ and $V \subset \overline{\text{NE}}(X)$ we write $U \perp V$ if $x \cdot y = 0$ for any $\mathbb{R}^+x \in U$ and any $y \in V$. Thus, for $U \subset \mathcal{M}(X)$, $V \subset \overline{\text{NE}}(X)$ we denote

$$U^\perp = \{y \in \overline{\text{NE}}(X) \mid U \perp y\}, \quad V^\perp = \{x \in \mathcal{M}(X) \mid x \perp V\}.$$

A subset $\gamma \subset \mathcal{M}(X)$ is called a face of $\mathcal{M}(X)$ if there exists a non-zero element $r \in \overline{\text{NE}}(X)$ such that $\gamma = r^\perp$.

A convex set is called a closed polyhedron if it is a convex hull of a finite set of points. A convex closed polyhedron is called simplicial if all its proper faces are simplexes. A convex closed polyhedron is called simple (equivalently, it has simplicial angles) if it is dual to a simplicial one. In other words, any its face of codimension $k$ is contained exactly in $k$ faces of $\gamma$ of the highest dimension. Similar names we use for convex cones and cones over polyhedra. For example, a convex cone is called simplex, simplicial and simple if it is a cone over a simplex, simplicial or simple polyhedron respectively.

We need some relative notions of the notions above.

We say that $\mathcal{M}(X)$ is a closed polyhedron in its face $\gamma \subset \mathcal{M}(X)$ if $\gamma$ is a
closed polyhedron and $M(X)$ is a closed polyhedron in a neighbourhood $T$ of $\gamma$. Thus, there should exist a closed polyhedron $M'$ such that $M' \cap T = M(X) \cap T$.

We will use the following notation. Let $R(X)$ be the set of all extremal rays of $X$. For a face $\gamma \subseteq M(X)$,

$$R(\gamma) = \{R \in R(X) | \exists R^+H \subseteq \gamma; R \cdot H = 0\}$$

and

$$R(\gamma^{\perp}) = \{R \in R(X) | \gamma \perp R\}.$$ Let us assume that $M(X)$ is a closed polyhedron in its face $\gamma$. Then sets $R(\gamma_1)$ and $R(\gamma_1^{\perp})$ are finite for any face $\gamma_1 \subseteq \gamma$. Evidently, the face $\gamma$ is simple if

$$(1) \quad \# R(\gamma_1^{\perp}) - \# R(\gamma^{\perp}) = \text{codim} \gamma_1$$

for any face $\gamma_1$ of $\gamma$. Then we say that the polyhedron $M(X)$ is simple in its face $\gamma$. Evidently, this condition is equivalent to the condition:

$$(2) \quad \dim \mathcal{E} - \dim [R(\gamma_1^{\perp})] = \# \mathcal{E} - \# R(\gamma^{\perp})$$

for any extremal set $\mathcal{E}$ such that $R(\gamma^{\perp}) \subseteq \mathcal{E}$. Here $[\cdot]$ denotes a linear hull. (In [N6], we required a more strong condition for a polyhedron $M(X)$ to be simple in its face $\gamma$: $\# R(\gamma_1^{\perp}) = \dim M(X) - \dim \gamma_1$ for any face $\gamma_1$ of $\gamma$.)

Let $A, B$ be two vertices of an oriented graph $G$. The distance $d(A, B)$ in $G$ is a length (the number of links) of a shortest oriented path of the graph $G$ from $A$ to $B$. The distance is $+\infty$ if this path does not exist. The diameter diam $G$ of an oriented graph $G$ is the maximum distance between ordered pairs of its vertices. By the Lemma 1.1, the diameter of an $E$-set is a finite number if this set satisfies the condition (iii).

Theorem 1.2 below is an analog for algebraic varieties of arbitrary dimension of the Lemma 3.4 of [N2] and the Lemma 1.4.1 of [N5], which were devoted to surfaces.

**Theorem 1.2.** Let $X$ be a projective algebraic variety with $Q$-factorial singularities and $\dim X \geq 2$. Let us suppose that $M(X)$ is closed and simple in its face $\gamma$. Assume that the set $R(\gamma)$ satisfies the condition (i) above. Assume that there are some constants $d, C_1, C_2$ such that the conditions (a) and (b) below hold:

(a) For any $E$-set $L \subseteq R(\gamma)$ such that $L$ contains at least two elements which don't belong to $R(\gamma^{\perp})$ and for any proper subset $L' \subseteq L$ the set $R(\gamma^{\perp}) \cup L'$ is extremal, the condition (iii) is valid and

$$\text{diam} G(L) \leq d.$$ 

(b) For any extremal subset $\mathcal{E}$ such that $R(\gamma^{\perp}) \subseteq \mathcal{E} \subseteq R(\gamma)$, we have: the $\mathcal{E}$ satisfies the condition (i) and for the distance in the oriented graph $G(\mathcal{E})

$$\# \{(R_1, R_2) \in (\mathcal{E} - R(\gamma^{\perp})) \times (\mathcal{E} - R(\gamma^{\perp})) | 1 \leq \rho(R_1, R_2) \leq d\} \leq C_1 \# (\mathcal{E} - R(\gamma^{\perp}));$$

and

$$\# \{(R_1, R_2) \in (\mathcal{E} - R(\gamma^{\perp})) \times (\mathcal{E} - R(\gamma^{\perp})) | d + 1 \leq \rho(R_1, R_2) \leq 2d + 1\} \leq C_2 \# (\mathcal{E} - R(\gamma^{\perp})).$$
Then $\dim \gamma < (16/3)C_1 + 4C_2 + 6$.

Proof. We use the following Lemma 1.3 which was proved in [N1]. The lemma was used in [N1] to get a bound ($\leq 9$) of the dimension of a hyperbolic (Lobachevsky) space admitting an action of an arithmetic reflection group with a field of definition of the degree $> N$. Here $N$ is some absolute constant.

Lemma 1.3. Let $M$ be a convex closed simple polyhedron of a dimension $n$, and $A_{n}^{i,k}$ the average number of $i$-dimensional faces of $k$-dimensional faces of $M$. Then for $n \geq 2k - 1$

$$A_{n}^{i,k} < \left( \frac{n - i}{n - k} \right) \cdot \left( \binom{[n/2]}{i} + \binom{n - [n/2]}{i} \right) \cdot \left( \binom{[n/2]}{k} + \binom{n - [n/2]}{k} \right) .$$

In particular, if $n \geq 3$

$$A_{n}^{0,2} < \begin{cases} 
\frac{4(n-1)}{n-2} & \text{if } n \text{ is even,} \\
\frac{4n}{n-1} & \text{if } n \text{ is odd.}
\end{cases}$$

Proof. See [N1]. We mention that the right side of the inequality of the Lemma 1.3 decreases and tends to the number $2^{k-1} \binom{k}{i}$ of $i$-dimensional faces of $k$-dimensional cube if $n$ increases.

From the estimate of $A_{n}^{0,2}$ of the Lemma, it follows the following analog of Vinberg's Lemma from [V]. Vinberg's Lemma was used by him to obtain an estimate ($\dim < 30$) for the dimension of a hyperbolic space admitting an action of a discrete reflection group with a bounded fundamental polyhedron.

By definition, an angle of a polyhedron $T$ is an angle of a 2-dimensional face of $T$. Thus, the angle is defined by a vertex $A$ of $T$, a plane containing $A$ and a 2-dimensional face $\gamma_2$ of $T$, and two rays with the beginning at $A$ which contain two corresponding sides of the $\gamma_2$. To define an oriented angle of $T$, one should in addition put in order two rays of the angle.

Lemma 1.4. Let $M$ be a convex simple polyhedron of a dimension $n$. Let $C$ and $D$ are some numbers. Suppose that oriented angles (2-dimensional, plane) of $M$ are supplied with weights and the following conditions (1) and (2) hold:

1. The sum of weights of all oriented angles at any vertex of $M$ is not greater than $Cn + D$.

2. The sum of weights of all oriented angles of any 2-dimensional face of $M$ is at least $5 - k$ where $k$ is the number of vertices of the 2-dimensional face.

Then

$$n < 8C + 5 + \begin{cases} 
1 + 8D/n & \text{if } n \text{ is even,} \\
(8C + 8D) / (n-1) & \text{if } n \text{ is odd,}
\end{cases}$$
In particular, for $C \geq 0$ and $D=0$, we have
\[ n < 8C + 6. \]

Proof. We correspond to a non-oriented plane angle of $M$ a weight which is equal to the sum of weights of two corresponding oriented angles. Evidently, the conditions of the Lemma hold for the weights of non-oriented angles too if we forget about the word "oriented". Then we obtain Vinberg's lemma from [V] which we formulate a little bit more precisely here. Since the proof is simple, we give the proof here.

Let $\sum$ be the sum of weights of all (non-oriented) angles of the polyhedron $M$. Let $\alpha_0$ be the number of vertices of $M$ and $\alpha_2$ the number of 2-dimensional faces of $M$. Since $M$ is simple,
\[ \alpha_0 \frac{n(n-1)}{2} = \alpha_2 A_n^{0,2}. \]

From this equality and conditions of the Lemma, we get inequalities
\[ (Cn+D) \alpha_0 \geq \sum \geq \sum \alpha_{2,k}(5-k) = 5\alpha_2 - \alpha_2 A_n^{0,2} = \alpha_2 (5 - A_n^{0,2}) = \alpha_0 (n(n-1)/2) (5/A_n^{0,2} - 1). \]

Here $\alpha_{2,k}$ is the number of 2-dimensional faces with $k$ vertices of $M$. Thus, from this inequality and Lemma 1.3, we get
\[ Cn+D \geq \frac{(n(n-1)/2)}{5/A_n^{0,2} - 1} \left\{ \begin{array}{ll} \frac{n(n-6)}{8} & \text{if } n \text{ is even,} \\ \frac{(n-1)(n-5)}{8} & \text{if } n \text{ is odd.} \end{array} \right. \]

From these calculations, Lemma 1.4 follows.

The proof of Theorem 1.2. (Compare with [V].) Let $\angle$ be an oriented angle of $\gamma$. Let $R(\angle) \subset R(\gamma)$ be the set of all extremal rays of $M(X)$ which are orthogonal to the vertex of $\angle$. Since $M(X)$ is simple in $\gamma$, the set $R(\angle)$ is a disjoint union
\[ R(\angle) = R(\angle^+) \cup |R_1(\angle)| \cup |R_2(\angle)| \]
where $R(\angle^+)$ contains all rays orthogonal to the plane of the angle $\angle$, the rays $R_1(\angle)$ and $R_2(\angle)$ are orthogonal to the first and second side of the oriented angle $\angle$, respectively. Evidently, the set $R(\angle)$ and the ordered pair of rays $(R_1(\angle), R_2(\angle))$ define the oriented angle $\angle$ uniquely. We define the weight $\sigma(\angle)$ by the formula:
\[ \sigma(\angle) = \begin{cases} 2/3, & \text{if } 1 \leq \rho(R_1(\angle), R_2(\angle)) \leq d, \\ 1/2, & \text{if } d+1 \leq \rho(R_1(\angle), R_2(\angle)) \leq 2d+1, \\ 0, & \text{if } 2d+2 \leq \rho(R_1(\angle), R_2(\angle)). \end{cases} \]

Here we take the distance in the graph $G(R(\angle))$. Let us prove conditions of the Lemma 1.4 with the constants $C = (2/3)C_1 + C_2/2$ and $D=0$.

The condition (1) follows from the condition (b) of the theorem. We remark that rays $R_1(\angle)$, $R_2(\angle)$ do not belong to the set $R(\gamma^+)$.

Let us prove the condition (2).

Let $\gamma_3$ be a 2-dimensional triangle face (triangle) of $\gamma$. The set $R(\gamma_3)$ of
all extremal rays orthogonal to points of $\gamma_3$ is the union of the set $R(\gamma_3^+)$ of extremal rays, which are orthogonal to the plane of the triangle $\gamma_3$, and rays $R_1, R_2, R_3$, which are orthogonal to the sides of the triangle $\gamma_3$. Union of the set $R(\gamma_3^+)$ with any two rays of $R_1, R_2, R_3$ is extremal, since it is orthogonal to a vertex of $\gamma_3$. On the other hand, the set $R(\gamma_3) = R(\gamma_3^+) \cup |R_1, R_2, R_3|$ is not extremal, since it is not orthogonal to a point of $M(X)$. Indeed, the set of all points of $M(X)$, which are orthogonal to the set $R(\gamma_3^+) \cup |R_2, R_3|$, $R(\gamma_3^+) \cup |R_1, R_3|$, or $R(\gamma_3^+) \cup |R_1, R_2|$ is the vertex $A_1, A_2, A_3$ respectively of the triangle $\gamma_3$, and the intersection of these sets of vertices is empty. Thus, there exists an $E$-set $\mathcal{L} \subset R(\gamma_3)$, which contains the set of rays $|R_1, R_2, R_3|$. By the condition (a), the graph $G(\mathcal{L})$ contains a shortest oriented path $s$ of the length $\leq d$ which connects the rays $R_1, R_3$. If this path does not contain the ray $R_2$, then the oriented angle of $\gamma_3$ defined by the set $R(\gamma_3^+) \cup |R_1, R_3|$ and the pair $(R_1, R_3)$ has the weight $2/3$. If this path contains the ray $R_2$, then the oriented angle of $\gamma_3$ defined by the set $R(\gamma_3^+) \cup |R_1, R_4|$ and the pair $(R_1, R_4)$ has the weight $2/3$. Thus, we proved that the side $A_2A_3$ of the triangle $\gamma_3$ defines an oriented angle of the triangle with the weight $2/3$ and the first side $A_3A_4$ of the oriented angle. The triangle has three sides. It follows the condition (2) of the Lemma 1.4 for the triangle.

Let $\gamma_4$ be a 2-dimensional quadrangle face (quadrangle) of $\gamma$. In this case,

$$R(\gamma_4) = R(\gamma_4^+) \cup |R_1, R_2, R_3, R_4|$$

where $R(\gamma_4^+)$ is the set of all extremal rays which are orthogonal to the plane of the quadrangle and the rays $R_1, R_2, R_3, R_4$ are orthogonal to the consecutive sides of the quadrangle. As above, one can see that the sets $R(\gamma_4^+) \cup |R_1, R_3|$, $R(\gamma_4^+) \cup |R_2, R_4|$ are not extremal, but the sets $R(\gamma_4^+) \cup |R_1, R_2|$, $R(\gamma_4^+) \cup |R_2, R_3|$, $R(\gamma_4^+) \cup |R_3, R_4|$ and $R(\gamma_4^+) \cup |R_4, R_1|$ are extremal. It follows that there are $E$-sets $\mathcal{L}, \mathcal{N}$ such that $|R_1, R_3| \subset \mathcal{L} \subset R(\gamma_4^+) \cup |R_1, R_3|$ and $|R_2, R_4| \subset \mathcal{N} \subset R(\gamma_4^+) \cup |R_2, R_4|$. By Lemma 1.1, there exist rays $R \in \mathcal{L}$ and $Q \in \mathcal{N}$ such that $RQ$ is an arrow. By the condition (a) of the theorem, one of the rays $R_1, R_3$ is joined by an oriented path $s_1$ of the length $\leq d$ with the ray $R$ and this path does not contain another ray from $R_1, R_3$ (here $R$ is the terminal of the path $s_1$). We can suppose that this ray is $R_1$ (otherwise, one should replace the ray $R_1$ by the ray $R_3$). As above, we can suppose that the ray $Q$ is connected by the oriented path $s_2$ of the length $\leq d$ with the ray $R_2$ and this path does not contain the ray $R_4$. The path $s_1 RQ s_2$ is an oriented path of the length $\leq 2d + 1$ in the oriented graph $G(\mathcal{R}(\gamma_4^+) \cup |R_1, R_3|)$. It follows that the oriented angle of the quadrangle $\gamma_4$, such that consecutive sides of this angle are orthogonal to the rays $R_1$ and $R_2$ respectively, has the weight $\geq 1/2$. Thus, we proved that for a pair of opposite sides of $\gamma_4$ there exists an oriented angle with weight $\geq 1/2$ such that the first side of this oriented angle is one of this
opposite sides of the quadrangle. A quadrangle has two pairs of opposite sides. It follows that the sum of weights of oriented angles of $\gamma_4$ is $\geq 1$. It proves the condition (2) of the Lemma 1.4 and the theorem.

In the sequel, we apply Theorem 1.2 to 3-folds.

CHAPTER II. Threefolds

1. Contractible extremal rays

We consider normal projective 3-folds $X$ with $\mathbb{Q}$-factorial singularities. Let $R$ be an extremal ray of Mori polyhedron $\overline{NE}(X)$ of $X$. A morphism $f: X \to Y$ onto a normal projective variety $Y$ is called the contraction of the ray $R$ if for an irreducible curve $C$ of $X$ the image $f(C)$ is a point if and only if $C \in R$. The contraction $f$ is defined by a linear system $H$ on $X$ ($H$ give rise to a nef element of $N^1(X)$, which we also denote by $H$). It follows that an irreducible curve $C$ is contracted if and only if $C \cdot H = 0$. We assume that the contraction $f$ has properties: $f_*\mathcal{O}_X = \mathcal{O}_Y$ and the sequence

$$ (1.1) \quad 0 \to \mathcal{O}_R \to N_1(X) \to N_1(Y) \to 0 $$

is exact where the arrow $N_1(X) \to N_1(Y)$ is $f_*$. An extremal ray $R$ is called contractible if there exists its contraction $f$ with these properties.

The number $\kappa(R) = \dim Y$ is called Kodaira dimension of the contractible extremal ray $R$.

A face $\gamma$ of $\overline{NE}(X)$ is called contractible if there exists a morphism $f: X \to Y$ onto a normal projective variety $Y$ such that $f_*\gamma = 0$, $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $f$ contracts curves lying in $\gamma$ only. The $\kappa(\gamma) = \dim Y$ is called Kodaira dimension of $\gamma$.

Let $H$ be a general nef element orthogonal to a face $\gamma$ of Mori polyhedron. Numerical Kodaira dimension of $\gamma$ is defined by the formula

$$ \kappa_{\text{num}}(\gamma) = \begin{cases} 3, & \text{if } H^3 > 0; \\ 2, & \text{if } H^3 = 0 \text{ and } H^2 \neq 0; \\ 1, & \text{if } H^2 = 0. \end{cases} $$

It is obvious that for a contractible face $\gamma$ we have $\kappa_{\text{num}}(\gamma) \geq \kappa(\gamma)$. In particular, $\kappa_{\text{num}}(\gamma) = \kappa(\gamma)$ for a contractible face $\gamma$ of Kodaira dimension $\kappa(\gamma) = 3$.

2. Paris of extremal rays of Kodaira dimension three lying in contractible faces of $\overline{NE}(X)$ of Kodaira dimension three

We assume further that $X$ is a projective normal threefold with $\mathbb{Q}$-factorial singularities.

**Lemma 2.2.1.** Let $R$ be a contractible extremal ray of Kodaira dimension
3 and \( f : X \to Y \) its contraction.

Then there are three possibilities:

(I) All curves \( C \subseteq R \) fill an irreducible Weil divisor \( D(R) \), the contraction \( f \) contracts \( D(R) \) to a point and \( R \cdot D(R) < 0 \).

(II) All curves \( C \subseteq R \) fill an irreducible Weil divisor \( D(R) \), the contraction \( f \) contracts \( D(R) \) to an irreducible curve and \( R \cdot D(R) < 0 \).

(III) (small extremal ray) All curves \( C \subseteq R \) give a finite set of irreducible curves and the contraction \( f \) contracts these curves to points.

Proof. Assume that some curves of \( R \) fill an irreducible divisor \( D \). Then \( R \cdot D < 0 \) (this inequality follows from the Proposition 2.2.6 below). Suppose that \( C \subseteq R \) and \( D \) does not contain \( C \). It follows that \( R \cdot D \geq 0 \). We get a contradiction. It follows the lemma.

According to Lemma 2.2.1, we say that an extremal ray \( R \) has the type (I), (II) or (III) (small) if it is contractible of Kodaira dimension 3 and the statement (I), (II) or (III) respectively holds.

Lemma 2.2.2. Let \( R_1 \) and \( R_2 \) are two different extremal rays of the type (I). Then divisors \( D(R_1) \) and \( D(R_2) \) do not intersect one another.

Proof. Otherwise, \( D(R_1) \) and \( D(R_2) \) have a common curve and the rays \( R_1 \) and \( R_2 \) are not different.

For an irreducible Weil divisor \( D \) on \( X \) let

\[
\overline{NE}(X, D) = (\text{image} \overline{NE}(D)) \subset \overline{NE}(X).
\]

Lemma 2.2.3. Let \( R \) be an extremal ray of the type (II), and \( f \) its contraction. Then \( \overline{NE}(X, D(R)) = R + R^* S \), where \( R^* f_* S = R^* (f(D)) \).

Proof. This follows at once from the exact sequence (1.1).

Lemma 2.2.4. Let \( R_1 \) and \( R_2 \) are two different extremal rays of the type (II) such that the divisors \( D(R_1), D(R_2) \) coincide. Then for \( D = D(R_1) = D(R_2) \) we have: \( \overline{NE}(X, D) = R_1 + R_2 \). In particular, do not exist three different extremal rays of the type (II) such that their divisors coincide one another.

Proof. This follows from the Lemma 2.2.3.

Lemma 2.2.5. Let \( R \) be an extremal ray of the type (II) and \( f \) its contraction. Then there does not exist more than one extremal ray \( Q \) of the type (I) such that \( D(Q) \cap D(R) \) is not empty. If \( Q \) is this ray, then \( D(Q) \cap D(R) \) is a curve and any irreducible component of this curve is not contained in fibers of \( f \).

Proof. The last assertion is obvious. Let us prove the first one. Suppose that \( Q_1 \) and \( Q_2 \) are two different extremal rays of the type (I) such that \( D(Q_1) \cap D(R) \) and \( D(Q_2) \cap D(R) \) are not empty. Then the plane angle \( \overline{NE}(X, D(R)) \) (see the Lemma 2.2.3) contains three different extremal rays: \( Q_1, Q_2 \).
and $R$. It is impossible.

The following key proposition is very important.

**Proposition 2.2.6.** Let $X$ be a projective 3-fold with $\mathbb{Q}$-factorial singularities, $D_1, \ldots, D_m$ irreducible divisors on $X$ and $f: X \to Y$ a surjective morphism such that $\dim X = \dim Y$ and $\dim f(D_i) < \dim D_i$. Let $y \in f(D_1) \cap \ldots \cap f(D_m)$. Then there are $a_1 > 0, \ldots, a_m > 0$ and an open $U, y \in U \subseteq f(D_1) \cup \ldots \cup f(D_m)$, such that

$$C \cdot (a_1D_1 + \ldots + a_mD_m) < 0$$

if a curve $C \subseteq D_1 \cup \ldots \cup D_m$ belongs to a non-trivial algebraic family of curves on $D_1 \cup \ldots \cup D_m$ and $f(C) = \text{point} \in U$.

**Proof.** The proof is the same as the well-known case of surfaces (but, for surfaces, it is not necessary to suppose that $C$ belongs to a nontrivial algebraic family). Let $H$ be an irreducible ample divisor on $X$ and $H' = f_*H$. Since $\dim f(D_i) < \dim D_i$, it follows that $f(D_1) \cup \ldots \cup f(D_m) \subset H'$. Let $\phi$ be a non-zero rational function on $Y$ which is regular in a neighbourhood $U$ of $y$ on $Y$ and is equal to zero on the divisor $H$. In the open set $f^{-1}(U)$ the divisor $(f^*\phi)$ can be written in a form

$$(f^*\phi) = \sum_{i=1}^m a_iD_i + \sum_{j=1}^n b_jZ_j,$$

where all $a_i > 0$ and all $b_j > 0$. Here every divisor $Z_j$ is different from any divisor $D_i$. We have

$$0 = C \cdot \sum_{i=1}^m a_iD_i + C \cdot \sum_{j=1}^n b_jZ_j.$$

Here $C \cdot (\sum_{j=1}^n b_jZ_j) > 0$ since $C$ belongs to a nontrivial algebraic family of curves on a surface $D_1 \cup \ldots \cup D_m$ and one of the $Z_j$ is the hyperplane section $H$.

**Lemma 2.2.7.** Let $R_1, R_2$ are two extremal rays of the type (II), divisors $D(R_1), D(R_2)$ are different and $D(R_1) \cap D(R_2) \neq \emptyset$. Assume that $R_1, R_2$ belong to a contractible face of $\overline{NE}(X)$ of Kodaira dimension 3. Let $0 \notin F_1 \in R_1$ and $0 \notin F_2 \in R_2$. Then

$$\langle F_1 \cdot D(R_2) \rangle \langle F_2 \cdot D(R_1) \rangle < \langle F_1 \cdot D(R_1) \rangle \langle F_2 \cdot D(R_2) \rangle.$$

**Proof.** Let $f$ be the contraction of a face of Kodaira dimension 3, which contains both rays $R_1, R_2$. By Proposition 2.2.6, there are $a_1 > 0, a_2 > 0$ such that

$$a_1 \langle F_1 \cdot D(R_1) \rangle + a_2 \langle F_1 \cdot D(R_2) \rangle < 0 \quad \text{and} \quad a_1 \langle F_2 \cdot D(R_1) \rangle + a_2 \langle F_2 \cdot D(R_2) \rangle < 0$$

or

$$-a_1 \langle F_1 \cdot D(R_1) \rangle + a_2 \langle F_1 \cdot D(R_2) \rangle \quad \text{and} \quad -a_2 \langle F_2 \cdot D(R_2) \rangle + a_1 \langle F_2 \cdot D(R_1) \rangle$$

where $F_1 \cdot D(R_1) < 0$, $F_2 \cdot D(R_2) < 0$ and $F_1 \cdot D(R_1) > 0$, $F_2 \cdot D(R_1) > 0$. Multiplying inequalities above, we obtain the lemma.
3. A classification of extremal sets of extremal rays which contain extremal rays of the type (I) and simple extremal rays of the type (II)

As above, we assume that $X$ is a projective normal 3-fold with $\mathbb{Q}$-factorial singularities.

**Definition 2.3.1.** An extremal ray $R$ of the type (II) is called simple if

$$R \cdot (D(R) + D) \geq 0$$

for any irreducible divisor $D$ such that $R \cdot D > 0$.

The following proposition gives a simple sufficient condition for an extremal ray to be simple.

**Proposition 2.3.2.** Let $R$ be an extremal ray of the type (II) and $f: X \to Y$ the contraction of $R$. Suppose that the curve $f(D(R))$ is not contained in the set of singularities of $Y$. Then

1. the ray $R$ is simple;
2. if $X$ has only isolated singularities, then a general element $C$ of the ray $R$ and the divisor $D(R)$ is non-singular along $C$. If additionally $R \cdot K_X < 0$, then $C \cdot D(R) = C \cdot K_X = -1$.
3. In particular, both statements (1) and (2) are true if $X$ has terminal singularities and $R \cdot K_X < 0$.

**Proof.** Let $D$ be an irreducible divisor on $X$ such that $R \cdot D > 0$. Since $R \cdot D(R) < 0$, the divisor $D$ is different from $D(R)$ and the intersection $D \cap D(R)$ is a curve which does not belong to $R$. Then $D' = f_*(D)$ is an irreducible divisor on $Y$ and $f'(D(R))$ is a curve on $D'$. Let $y \in f'(D(R))$ be a non-singular point of $Y$. Then the divisor $D'$ is defined by some local equation $\phi$ in a neighbourhood $U$ of $y$. Evidently, in the open set $f^{-1}(U)$ we can write

$$(f_\ast \phi) = D + m(D(R))$$

where the integer $m \geq 1$. Let a curve $C \subseteq R$ and $f(C) = y \in U \cap f(D(R))$. Then $0 = C \cdot (D + m(D(R))) = C \cdot (D + D(R)) + C \cdot (m - 1)(D(R))$. Since $m \geq 1$ and $C \cdot D(R) < 0$, it follows that $C \cdot (D + D(R)) \geq 0$.

Let us prove (2). Let us consider a linear system $|H|$ of hyperplane sections on $Y$ and the corresponding linear systems on resolutions of singularities of $Y$ and $X$. Let us apply Bertini’s theorem (see, for example, [Ha, ch. III, Corollary 10.9 and the Exercise 11.3]) to these linear systems. Singularities of $X$ and $Y$ are isolated. Then by Bertini theorem, for a general element $H$ of $|H|$ we obtain that (a) $H$ and $H' = f^{-1}(H)$ are irreducible and non-singular; (b) $H$ intersects $\Gamma$ transversely in non-singular points of $\Gamma$. Let us consider the corresponding birational morphism $f' = f|H': H' \to H$ of the non-singular irreducible surfaces. It is a composition of blowing ups at non-singular points. Thus, fibers of $f'$ over $H \cap \Gamma$ are trees of non-singular rational curves. The
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exceptional curve of the first of these blowing ups is identified with the fiber of the projectivization of the normal bundle $\mathbf{P}(\mathcal{N}_{r'/r})$. Thus, we obtain a rational map over the curve $\Gamma$

$$\phi: \mathbf{P}(\mathcal{N}_{r'/r}) \to D(R)$$
of the irreducible surfaces. Evidently, it is an injection at general points of $\mathbf{P}(\mathcal{N}_{r'/r})$. It follows that $\phi$ is a birational isomorphism of the surfaces. Since $\phi$ is a birational map over the curve $\Gamma$, it follows that the general fibers of this maps are birationally isomorphic. It follows that a general fiber of $f'$
is $C = \mathbf{P}^1$. Since $C$ is non-singular and is an intersection of the non-singular surface $H'$ with the surface $D(R)$, and since $X$ has only isolated singularities, it follows that $D(R)$ is non-singular along the general curve $C$.

The $X$ and $D(R)$ are non-singular along $C = \mathbf{P}^1$ and the curve $C$ is non-singular. Then the canonical class $K_C = (K_X + D(R))|C$ where both divisors $K_X$ and $D(R)$ are Cartier divisors on $X$ along $C$. It follows that $-2 = \deg K_C = K_X \cdot C + D(R) \cdot C$, where the both numbers $K_X \cdot C$ and $D(R) \cdot C$ are negative integers. Then $D(R) \cdot C = K_X \cdot C = -1$.

If $X$ has terminal singularities and $R \cdot K_X < 0$, then $Y$ has terminal singularities too (see, for example, [Ka1]). Moreover, 3-dimensional terminal singularities are isolated. From (1), (2), the last statement of the Proposition follows.

In connection with Proposition 2.3.2, see also [Mo2, 1.3 and 2.3.2] and [I, Lemma 1].

Let $R_1, R_2$ are two extremal rays of the type (I) or (II). They are joined if $D(R_1) \cap D(R_2) \neq 0$. It defines connected components of a set of extremal rays of the type (I) or (II).

We recall (see Chapter I) that a set $\mathcal{E}$ of extremal rays is called extremal if it is contained in a face of $\overline{NE}(X)$. We say that $\mathcal{E}$ is extremal of Kodaira dimension 3 if it is contained in a face of numerical Kodaira dimension 3 of $\overline{NE}(X)$.

We prove the following classification result.

**Theorem 2.3.3.** Let $\mathcal{E} = \{R_1, R_2, ..., R_n\}$ be an extremal set of extremal rays of the type (I) or (II). Suppose that every extremal ray of $\mathcal{E}$ of the type (II)
is simple. Assume that $\mathcal{E}$ is contained in a contractible face with Kodaira dimension 3 of $\overline{NE}(X)$. (Thus, $\mathcal{E}$ is extremal of Kodaira dimension 3.) Then every connected component of $\mathcal{E}$ has a type $\mathcal{A}_1$, $\mathcal{B}_2$, $\mathcal{C}_m$ or $\mathcal{D}_2$ below (see figure 1).

(A_1) One extremal ray of the type (I).

(B_2) Two different extremal rays $S_1, S_2$ of the type (II) such that their divisors $D(S_1) = D(S_2)$ coincidied.

(C_m) $m \geq 1$ extremal rays $S_1, S_2, ..., S_m$ of the type (II) such that their divisors $D(S_2), D(S_3), ..., D(S_m)$ do not intersect one another, and $S_1 \cdot D(S_i) = 0$ and $S_i \cdot D(S_i) > 0$ for $i = 2, ..., m$.

(D_2) Two extremal rays $S_1, S_2$, where $S_1$ is of the type (II) and $S_2$ of the
type (I), \( S_1 \cdot D(S_2) > 0 \) and \( S_2 \cdot D(S_1) > 0 \). Either \( S_1 \cdot (b_1D(S_1) + b_2D(S_2)) < 0 \) or \( S_2 \cdot (b_1D(S_1) + b_2D(S_2)) < 0 \) for any \( b_1, b_2 \) such that \( b_1 \geq 0, b_2 \geq 0 \) and one of \( b_1, b_2 \) is not zero.

The following inverse statement is true: If \( \mathcal{E} = \{ R_1, R_2, ..., R_n \} \) is a connected set of extremal rays of the type (I) or (II) and \( \mathcal{E} \) has a type \( \mathcal{A}_1, \mathcal{B}_2, \mathcal{C}_m \) or \( \mathcal{D}_2 \) above, then \( \mathcal{E} \) generates a simplex face \( R_1 + \cdots + R_n \) of the dimension \( n \) and numerical Kodaira dimension 3 of \( NE(X) \). In particular, extremal rays of the set \( \mathcal{E} \) are linearly independent.

Proof. Let us prove the first statement. We can suppose that \( \mathcal{E} \) is connected. We have to prove that \( \mathcal{E} \) has the type \( \mathcal{A}_1, \mathcal{B}_2, \mathcal{C}_m \) or \( \mathcal{D}_2 \). If \( n = 1 \), this is obvious.

Let \( n = 2 \). From Lemma 2.2.2, it follows that one of the rays \( R_1, R_2 \) has the type (II). Let \( R_1 \) have the type (II) and \( R_2 \) the type (I). Since \( D(R_1) \cap D(R_2) \neq \emptyset \), evidently \( R_2 \cdot D(R_1) > 0 \). If \( R_1 \cdot D(R_2) = 0 \), then the curve \( D(R_1) \cap D(R_2) \) belongs to the ray \( R_1 \). It follows that the rays \( R_1 \) and \( R_2 \) contain the same curve. We get a contradiction. Thus, \( R_1 \cdot D(R_2) > 0 \). The rays \( R_1, R_2 \) belong to a contractible face of Kodaira dimension 3 of Mori polyhedron. Let \( f \) be a contraction of this face. By the Lemma 2.2.3, \( f \) contracts the divisors \( D(R_1), D(R_2) \) to the same point. By Proposition 2.2.6, there exist positive \( a_1, a_2 \) such that \( R_1 \cdot (a_1D(R_1) + a_2D(R_2)) < 0 \) and \( R_2 \cdot (a_1D(R_1) + a_2D(R_2)) < 0 \). Now suppose that for some \( b_1 > 0 \) and \( b_2 > 0 \) the inequalities \( R_1 \cdot (b_1D(R_1) + b_2D(R_2)) \geq 0 \) and \( R_2 \cdot (b_1D(R_1) + b_2D(R_2)) \geq 0 \) hold. There exists \( \lambda > 0 \) such that \( \lambda b_1 \leq a_1, \lambda b_2 \leq a_2 \) and one of these inequalities is an equality. For example, let \( \lambda b_1 = a_1 \). Then

\[
R_1 \cdot (a_1D(R_1) + a_2D(R_2)) = R_1 \cdot \lambda(b_1D(R_1) + b_2D(R_2)) + R_1 \cdot (a_2 - \lambda b_2)D(R_2) \geq 0.
\]

We get a contradiction. It proves that in this case \( \mathcal{E} \) has the type \( \mathcal{D}_2 \).

Let \( n = 3 \). Every proper subset of \( \mathcal{E} \) has connected components of types

![Figure 1](image-url)
Using Lemmas 2.2.2–2.2.5, one can see very easily that either $\mathfrak{S}$ has the type $\mathfrak{C}_3$ or we have the following case: The rays $R_1, R_2, R_3$ have the type (II), every two elements subset of $\mathfrak{S}$ has the type 2 and we can find a numeration such that $R_1 \cdot D(R_2) > 0, R_2 \cdot D(R_3) > 0, R_3 \cdot D(R_1) > 0$. Let $f$ be a contraction of the face $\gamma$. By Lemma 2.2.3, $f$ contracts the divisors $D(R_1), D(R_2), D(R_3)$ to a one point. By Proposition 2.2.6, there are positive $a_1, a_2, a_3$ such that
$$R_i \cdot (a_1D(R_1) + a_2D(R_2) + a_3D(R_3)) < 0$$
for $i=1, 2, 3$. On the other hand, from simplicity of the rays $R_1, R_2, R_3$, it follows that
$$R_i \cdot (D(R_1) + D(R_2) + D(R_3)) \geq 0.$$ Let $a_1 = \min \{a_1, a_2, a_3\}$. From the last inequality,
$$R_1 \cdot (a_1D(R_1) + a_2D(R_2) + a_3D(R_3)) =$$
$$= R_1 \cdot (D(R_1) + D(R_2) + D(R_3)) + R_1 ((a_2 - a_1)D(R_2) + (a_3 - a_1)D(R_3)) \geq 0.$$ We get a contradiction with the inequality above.

Let $n > 3$. We have proven that every two or three elements subset of $\mathfrak{S}$ has connected components of types $\mathfrak{W}_1, \mathfrak{Z}_2$. It follows very easily that $\mathfrak{S}$ has the type $\mathfrak{G}_n$ (we suppose that $\mathfrak{S}$ is connected).

Let us prove the inverse statement. For the type $\mathfrak{A}_1$ this is obvious.

Let $\mathfrak{S}$ have the type $\mathfrak{B}_2$. Since the rays $S_1, S_2$ are extremal of Kodaira dimension 3, there are $\text{nef}$ elements $H_1, H_2$ such that $H_1 \cdot S_1 = H_2 \cdot S_2 = 0, H_1^2 > 0, H_2^2 > 0$. Let $0 \neq C_1 \subseteq S_1$ and $0 \neq C_2 \subseteq S_2$. Let $D$ be a divisor of the rays $S_1$ and $S_2$. Let us consider a map
$$H_1, H_2 \rightarrow H = (-D \cdot C_2) (H_2 \cdot C_1) H_1 +$$
$$+ (-D \cdot C_1) (H_1 \cdot C_2) H_2 + (H_2 \cdot C_1) (H_1 \cdot C_2) D.$$ For a fixed $H_1$, we get a linear map $H_2 \rightarrow H$ of the set of $\text{nef}$ elements $H_2$ orthogonal to $S_2$ into the set of $\text{nef}$ elements $H$ orthogonal to $S_1$ and $S_2$. This map has a one dimensional kernel generated by $(-D \cdot C_2) H_1 + (H_1 \cdot C_2) D$. It follows that $S_1 + S_2$ is a 2-dimensional face of $\overline{\text{NE}}(X)$.

For a general $\text{nef}$ element $H = a_1 H_1 + a_2 H_2 + bD$ orthogonal to this face, where $a_1, a_2, b > 0$, we have $H^3 = (a_1H_1 + a_2H_2 + bD)^2 \cdot (a_1H_1 + a_2H_2 + bD) \geq (a_1H_1 + a_2H_2 + bD)^2 \cdot (a_1H_1 + a_2H_2 + bD) \cdot (a_1H_1 + a_2H_2 + bD) \cdot (a_1H_1 + a_2H_2 + bD) \geq (a_1H_1 + a_2H_2)^2 > 0$, since $a_1H_1 + a_2H_2 + bD$ and $a_1H_1 + a_2H_2$ are $\text{nef}$. It follows that the face $S_1 + S_2$ is of the numerical Kodaira dimension 3.

Let $\mathfrak{S}$ have the type $\mathfrak{C}_m$. Let $H$ be a $\text{nef}$ element orthogonal to the ray $S_1$. Let $0 \neq C_1 \subseteq S_i$. Let us consider a map
$$H \rightarrow H' = H + \sum_{i=2}^{m} (- (H \cdot C_i) / (C_i \cdot D(S_i))) D(S_i).$$ It is a linear map of the set of $\text{nef}$ elements $H$ orthogonal to $S_1$ into the set of $\text{nef}$ elements $H'$ orthogonal to the rays $S_1, S_2, ..., S_m$. The kernel of the map
has the dimension \( m - 1 \). It follows that the rays \( S_1, S_2, ..., S_m \) belong to face of \( \overline{NE}(X) \) of a dimension \( \leq m \). On the other hand, multiplying the divisors \( D(S_1), ..., D(S_m) \) by rays \( S_1, ..., S_m \), one can see very easily that the rays \( S_1, ..., S_m \) are linearly independent. Thus, they generate an \( m \)-dimensional face of \( \overline{NE}(X) \). Let us show that this face is \( S_1 + S_2 + \cdots + S_m \). To prove this, we show that every \( m - 1 \) subset of \( \mathfrak{E} \) is contained in a face of \( \overline{NE}(X) \) of a dimension \( \leq m - 1 \).

If this subset contains the ray \( S_1 \), this subset has the type \( c_{m-1} \). By induction, we can suppose that this subset belongs to a face of \( \overline{NE}(X) \) of dimension \( m - 1 \). Let us consider the subset \( |S_2, S_3, ..., S_m| \). Let \( H \) be an ample element of \( X \). For the element \( H \), the map \((3.2)\) gives an element \( H' \) which is orthogonal to the rays \( S_2, ..., S_m \), but is not orthogonal to the ray \( S_1 \). It follows that the set \( |S_2, ..., S_m| \) belongs to a face of the Mori polyhedron of the dimension \( < m \). Like the above, one can see that for a general \( H \) orthogonal to \( S_1 \), the element \( H' \) has \( \langle H' \rangle^2 \geq H'^3 > 0 \).

Let \( \mathfrak{E} \) have the type \( \mathfrak{D}_2 \). Let \( H \) be a \( \text{nef} \) element orthogonal to the ray \( S_2 \). Let \( 0 \neq C_i \in S_i \). Let us consider a map

\[
(3.3) \quad H \rightarrow H' = H + \frac{(H \cdot C_1)}{(D(S_2) \cdot C_1)} \left( \left( (-D(S_2) \cdot C_2) D(S_1) + (D(S_1) \cdot C_2) D(S_2) \right) \right).
\]

Evidently, \( C_2 \cdot \left( (-D(S_2) \cdot C_2) D(S_1) + (D(S_1) \cdot C_2) D(S_2) \right) = 0 \). From this equality and the inequality of the definition of the system \( \mathfrak{D}_2 \), it follows that \( C_1 \cdot \left( (-D(S_2) \cdot C_2) D(S_1) + (D(S_1) \cdot C_2) D(S_2) \right) < 0 \). Thus, the denominator of the formula \((3.3)\) is positive. Then \((3.3)\) is a linear map of the set of \( \text{nef} \) elements \( H \) orthogonal to the ray \( S_2 \) into the set of \( \text{nef} \) elements \( H' \) orthogonal to the rays \( S_1, S_2 \). Evidently, the map has a one dimensional kernel. Thus, the rays \( S_1 \) and \( S_2 \) generate a two dimensional face \( S_1 + S_2 \) of Mori polyhedron. As above, for a general element \( H \) orthogonal to \( S_2 \) we have \( \langle H' \rangle^3 \geq H'^3 > 0 \).

**Corollary 2.3.4.** Let \( \mathfrak{E} = |R_1, R_2, ..., R_n| \) be an extremal set of extremal rays of the type \((I)\) or \((II)\) and every extremal ray of \( \mathfrak{E} \) of the type \((II)\) is simple. Assume that \( \mathfrak{E} \) is contained in a contractible face with Kodaira dimension 3 of the \( \overline{NE}(X) \). Let \( m_1 \geq 0, m_2 \geq 0, ..., m_n \geq 0 \) and at least one of \( m_1, ..., m_n \) is positive.

Then there exists \( i, 1 \leq i \leq n \), such that

\[
R_i \cdot (m_1 D(R_1) + \cdots + m_n D(R_n)) < 0.
\]

Thus, the condition \((ii)\) of Chapter I is valid.

**Proof.** It is sufficient to prove this statement for the connected \( \mathfrak{E} \). For every type \( \mathfrak{A}_1, \mathfrak{B}_2, \mathfrak{C}_m \) and \( \mathfrak{D}_2 \) of the Theorem 2.3.3, one can prove it very easily.

Unfortunately, in general, the inverse statement of the Theorem 2.3.3 holds only for connected extremal sets \( \mathfrak{E} \). We will give two cases where it is true for a non-connected \( \mathfrak{E} \).
Definition 2.3.5. A threefold $X$ is called strongly projective (respectively very strongly projective) if the following statement holds: a set $|Q_1, ..., Q_n|$ of extremal rays of the type $(I)$ is extremal of Kodaira dimension 3 (respectively generates the simplex face $Q_1 + \cdots + Q_n$ of $NE(X)$ of dimension $n$ and Kodaira dimension 3) if its divisors $D(Q_1), ..., D(Q_n)$ do not intersect one another.

Theorem 2.3.6. Let $B = |R_1, R_2, ..., R_n|$ be a set of extremal rays of the type $(I)$ or $(II)$ such that every connected component of $B$ has the type $\mathcal{A}_1, \mathcal{B}_2, \mathcal{C}_m$ or $\mathcal{D}_2$. Then:

1. $B$ is extremal of numerical Kodaira dimension 3 if and only if the same is true for any subset of $B$ containing only extremal rays of the type $(II)$ whose divisors do not intersect one another. In particular, it holds if $X$ is strongly projective.

2. $B$ generates a simplex face $R_1 + \cdots + R_n$ with numerical Kodaira dimension 3 of the Mori polyhedron if and only if the same is true for any subset of $B$ containing only extremal rays of the type $(II)$ whose divisors do not intersect one another. In particular, it is true if $X$ is very strongly projective.

Proof. Let us prove (1). Only the inverse statement is non-trivial. We prove it by induction on $n$. For $n = 1$, the statement is obviously true.

Assume that some connected component of $B$ has the type $\mathcal{A}_1$. Suppose that this component contains the ray $R_1$. By our induction hypothesis, there exists a nef element $H$ such that $H^2 > 0$ and $H \cdot R_1 = 0$ if $i > 1$. Then there exists $k \geq 0$, such that $H' = H + kD(R_1)$ is nef and $H' \cdot B = 0$. As above, one can prove that $(H')^3 \geq H^3 > 0$.

Assume that some connected component of $B$ has the type $\mathcal{B}_2$. Suppose that this component contains the rays $R_1, R_2$ and $D(R_1) = D(R_2) = D$. Then, by induction, there are nef elements $H_1$ and $H_2$ such that $H_1^2 > 0, H_2^3 > 0$ and $H_1 \cdot |R_1, R_3, ..., R_n| = 0, H_2 \cdot |R_2, R_3, ..., R_n| = 0$. As for the proof of the inverse statement of the Theorem 2.3.3 in the case $\mathcal{B}_2$, there are $k_1 \geq 0, k_2 \geq 0, k_3 \geq 0$ such that the element $H = k_1H_1 + k_2H_2 + k_3D$ is nef, $H \cdot B = 0$ and $H^3 > 0$.

Assume that some connected component of $B$ has the type $\mathcal{C}_m, m > 1$. We use the notation of Theorem 2.3.3 for this connected component. Let this be $|S_1, S_2, ..., S_m|$. By induction, there exists a nef element $H$ such that $H$ is orthogonal to $B - |S_2, ..., S_m|$ and $H^2 > 0$. As for the proof of the inverse statement of the Theorem 2.3.3 in the case $\mathcal{C}_m$, there are $k_2 \geq 0, ..., k_m \geq 0$ such that $H' = H + k_2D(S_2) + \cdots + k_mD(S_m)$ is nef, $H' \cdot B = 0$ and $(H')^3 \geq H^3 > 0$.

Assume that some connected component of $B$ has the type $\mathcal{D}_2$. We use the notation of Theorem 2.3.3 for this connected component. Let this be $|S_1, S_2|$. By induction, there exists a nef element $H$ such that $H^3 > 0$ and $H$ is orthogonal to $B - |S_1|$. As for Theorem 2.3.3, there are $k_1 \geq 0, k_2 \geq 0$ such that $H' = H + k_1D(S_1) + k_2D(S_2)$ is nef, $H' \cdot B = 0$ and $(H')^3 \geq H^3 > 0$.

If every connected component of $B$ has the type $\mathcal{C}_1$, then the statement
Let us prove (2). Only the inverse statement is non-trivial. We prove it by induction on \( n \). For \( n=1 \) the statement is true. It is sufficient to prove that \( \mathcal{E} \) is contained in a face of a dimension \( \leq n \) of Mori polyhedron because, by our induction hypothesis, any its \( n-1 \) elements subset generates a simplex face of the dimension \( n-1 \) of Mori polyhedron.

Assume that some connected component of \( \mathcal{E} \) has the type \( \mathcal{A}_1 \). Suppose that the ray \( R_1 \) belongs to this component and \( 0 \neq C_1 \in R_1 \). Let us consider the map

\[
H \to H' = H' + \left( \left( H \cdot C_1 \right) / \left( -D(R_1) \cdot C_1 \right) \right) D(R_1)
\]

of the set of nef elements \( H \) orthogonal to the set \( |R_2, \ldots, R_n| \) into the set of nef elements \( H' \) orthogonal to the \( \mathcal{E} \). It is the linear map with one dimensional kernel. Since, by the induction, the set \( |R_2, \ldots, R_n| \) is contained in a face of Mori polyhedron of the dimension \( n-1 \), it follows that \( \mathcal{E} \) is contained in a face of the dimension \( n \).

If \( \mathcal{E} \) has a connected component of the type \( \mathcal{B}_2, \mathcal{C}_m, m > 1 \), or \( \mathcal{D}_2 \), the proof is the same if one uses the maps (3.1), (3.2) and (3.3) above.

If all connected components of \( \mathcal{E} \) have the type \( \mathcal{C}_1 \), the statement holds by the condition.

Remark 2.3.7. Like the statement (1) of Theorem 2.3.6, one can prove that a set \( \mathcal{E} \) of extremal rays with connected components of the type \( \mathcal{A}_1, \mathcal{B}_2, \mathcal{C}_m \) or \( \mathcal{D}_2 \) is extremal if and only if the same is true for any subset of \( \mathcal{E} \) containing only extremal rays of the type (II) whose divisors do not intersect one another.

The next proposition is simple but important. To simplify the notation, we say that for a fixed \( a_1, \ldots, a_n \), we have a linear dependence condition

\[
a_1R_1 + \cdots + a_nR_n = 0
\]

between extremal rays \( R_1, \ldots, R_n \) if there exist non-zero \( C_i \in R_i \) such that

\[
a_1C_1 + \cdots + a_nC_n = 0.
\]

Proposition 2.3.8. Assume that a set \( \mathcal{E} = |R_1, R_2, \ldots, R_m| \) of extremal rays has connected components of the type \( \mathcal{A}_1, \mathcal{B}_2, \mathcal{C}_m \) or \( \mathcal{D}_2 \) and there exists a linear dependence condition \( a_1R_1 + a_2R_2 + \cdots + a_mR_m = 0 \) with all \( a_i \neq 0 \). Then all connected components of \( \mathcal{E} \) have the type \( \mathcal{B}_2 \). Let these components be \( \mathcal{B}^1, \ldots, \mathcal{B}^t \). Then \( t \geq 2 \), and we can choose a numeration such that \( \mathcal{B}^i = |R_{i1}, R_{i2}| \) and the linear dependence has a form

\[
a_{i1}R_{i1} + a_{i2}R_{i2} + \cdots + a_{i1}R_{i1} = a_{i2}R_{i2} + a_{i3}R_{i3} + \cdots + a_{i2}R_{i2}.
\]

where all \( a_{ij} > 0 \).

Proof. Let us multiply the divisors \( D(R_1), \ldots, D(R_m) \) by the equality \( a_1R_1 + a_2R_2 + \cdots + a_mR_m = 0 \). Then we get that \( a_k = 0 \) if the ray \( R_k \) belongs to a connected component of the type \( \mathcal{A}_1, \mathcal{C}_m \) or \( \mathcal{D}_2 \). Thus, all connected components of \( \mathcal{E} \) have the type \( \mathcal{B}_2 \). Let these components be

\[
\mathcal{B}^1 = |R_{11}, R_{12}|, \mathcal{B}^2 = |R_{21}, R_{22}|, \ldots, \mathcal{B}^t = |R_{t1}, R_{t2}|.
\]
Obviously, \( t \geq 2 \), and we can rewrite the linear dependence as
\[
a_{11}R_{11} + a_{12}R_{12} + a_{21}R_{21} + a_{22}R_{22} + \cdots + a_{11}R_{11} + a_{12}R_{12} = 0,
\]
where all \( a_{ij} \neq 0 \). Multiplying all divisors \( D(R_{ij}) \) by this equation and using inequalities \( R_{ij} \cdot D(R_{ij}) < 0 \), we get the last statement of the proposition.

4. A classification of E-sets of extremal rays of type (I) or (II)

As in the above, we suppose that \( X \) is a projective normal 3-fold with \( \mathbb{Q} \)-factorial singularities.

We recall that a set \( \mathcal{L} \) of extremal rays is called an E-set if it is not extremal but any proper subset of \( \mathcal{L} \) is extremal (it is contained in a face of \( NE(X) \)). Thus, an E-set is a minimal non-extremal set of extremal rays.

**Theorem 2.4.1.** Let \( \mathcal{L} \) be an E-set of extremal rays of the type (I) or (II). Suppose that every ray of the type (II) of \( \mathcal{L} \) is simple and every proper subset of \( \mathcal{L} \) is contained in a contractible face of Kodaira dimension 3 of Mori polyhedron. Then we have one of the following cases:

(a) \( \mathcal{L} \) is connected and \( \mathcal{L} = |R_1, R_2, R_3| \), where any \( R_i \) has the type (II) and each of 2-element subsets \( |R_1, R_2|, |R_2, R_3|, |R_3, R_1| \) of \( \mathcal{L} \) has the type \( \mathcal{C}_2 \). Here \( R_1 \cdot D(R_2) > 0, R_2 \cdot D(R_3) > 0, R_3 \cdot D(R_1) > 0 \) but \( R_2 \cdot D(R_1) = R_3 \cdot D(R_2) = R_1 \cdot D(R_3) = 0 \). The divisor \( D(\mathcal{L}) = D(R_1) + D(R_2) + D(R_3) \) is nef.

(b) \( \mathcal{L} \) is connected and \( \mathcal{L} = |R_1, R_2| \), where at least one of the rays \( R_1, R_2 \) has the type (II). There are positive \( m_1, m_2 \) such that \( R \cdot (m_1 D(R_1) + m_2 D(R_2)) \geq 0 \) for any extremal ray \( R \) of the type (I) or simple extremal ray of type (II) on \( X \). If the divisor \( m_1 D(R_1) + m_2 D(R_2) \) is not nef, both the extremal rays \( R_1, R_2 \) have the type (II).

(c) \( \mathcal{L} \) is connected and \( \mathcal{L} = |R_1, R_2| \), where both \( R_1 \) and \( R_2 \) have the type (II) and there exists a simple extremal ray \( S_1 \) of the type (II) such that the rays \( R_1, S_1 \) define the extremal set of the type \( \mathcal{B}_2 \) (it means that \( S_1 \neq R_1 \) but the divisors \( D(S_1) = D(R_1) \)) and the rays \( S_1, R_2 \) define the extremal set of the type \( \mathcal{C}_2 \), where \( S_1 \cdot D(R_2) = 0 \) but \( R_2 \cdot D(S_1) > 0 \). Here there do not exist positive \( m_1, m_2 \) such that the divisor \( m_1 D(R_1) + m_2 D(R_2) \) is nef, since evidently \( S_1 \cdot (m_1 D(R_1) + m_2 D(R_2)) \) \( < 0 \). See figure 2 below.

(d) \( \mathcal{L} = |R_1, \ldots, R_k| \) where \( k \geq 2 \), all rays \( R_1, \ldots, R_k \) have the type (II) and the divisors \( D(R_1), \ldots, D(R_k) \) do not intersect one another. Any proper subset of \( \mathcal{L} \) is contained in a contractible face of Kodaira dimension 3 of Mori polyhedron but \( \mathcal{L} \) is not contained in a face of Mori polyhedron.

![Figure 2](image-url)
Proof. Let \( \mathcal{L} = |R_1, \ldots, R_n| \) be an \( E \)-set of extremal rays satisfying the conditions of the theorem. Let us consider two cases.

The case 1. Let \( \mathcal{L} \) is not connected. Then every connected component of \( \mathcal{L} \) is extremal and, by Theorem 2.3.3, it has the type \( \mathcal{A}_1, \mathcal{B}_2, \mathcal{C}_m \) or \( \mathcal{D}_2 \). If some of these components does not have the type \( \mathcal{C}_1 \), then, by the statement (1) of Theorem 2.3.6, \( \mathcal{L} \) is extremal and we get a contradiction. Thus, we get the case (d) of the theorem.

The case 2. Let \( \mathcal{L} = |R_1, \ldots, R_n| \) is connected. Let \( n \geq 4 \). By Theorem 2.3.3, any proper subset of \( \mathcal{L} \) has connected components of the type \( \mathcal{A}_1, \mathcal{B}_2, \mathcal{C}_m \) or \( \mathcal{D}_2 \). Like for the proof of Theorem 2.3.3, it follows that \( \mathcal{L} \) has the type \( \mathcal{C}_n \). By Theorem 2.3.3, then \( \mathcal{L} \) is extremal. We get a contradiction.

Let \( n = 3 \). Then, like for the proof of Theorem 2.3.3, we get that \( \mathcal{L} \) has the type (a).

Let \( n = 2 \) and \( \mathcal{L} = |R_1, R_2| \). If both rays \( R_1, R_2 \) have the type (i), then, by Lemma 2.2.2, \( \mathcal{L} \) is not connected and we get a contradiction.

Let \( R_1 \) has the type (I) and \( R_2 \) has the type (II). Since the set \( \mathcal{L} \) is not extremal, by Theorem 2.3.3, there are positive \( m_1, m_2 \) such that \( R_1 \cdot (m_1D(R_1) + m_2D(R_2)) \geq 0 \) and \( R_2 \cdot (m_1D(R_1) + m_2D(R_2)) \geq 0 \). By Lemma 2.2.3, it follows that \( C \cdot (m_1D(R_1) + m_2D(R_2)) \geq 0 \) if the curve \( C \) is contained in the \( D(R_1) \cup D(R_2) \). If \( C \) is not contained in \( D(R_1) \cup D(R_2) \), then obviously \( C \cdot (m_1D(R_1) + m_2D(R_2)) \geq 0 \). It follows, that the divisor \( m_1D(R_1) + m_2D(R_2) \) is nef. Thus, we get the case (b).

Let both rays \( R_1, R_2 \) have the type (II). If \( D(R_1) = D(R_2) \), then we get an extremal set \( \{R_1, R_2\} \) by Theorem 2.3.3. Thus, the divisors \( D(R_1) \) and \( D(R_2) \) are different. By Lemma 2.2.1, the curve \( D(R_1) \cap D(R_2) \) does not have an irreducible component which belongs to both rays \( R_1 \) and \( R_2 \). Since rays \( R_1, R_2 \) are simple, it follows that \( R_1 \cdot (D(R_1) + D(R_2)) \geq 0 \) and \( R_2 \cdot (D(R_1) + D(R_2)) \geq 0 \). Let \( R \) be an extremal ray of type (I) or simple extremal ray of the type (II). If the divisor \( D(R) \) does not coincide with the divisor \( D(R_1) \) or \( D(R_2) \), then obviously \( R \cdot (D(R_1) + D(R_2)) \geq 0 \). Thus, if there does not exist an extremal ray \( R \) which has the same divisor as the ray \( R_1 \) or \( R_2 \), we get the case (b).

Assume that \( D(R) = D(R_1) \). Then, by Lemma 2.2.5, the ray \( R \) has the type (II), too. If \( R \cdot D(R_2) = 0 \), we get the case (c) of the theorem where \( S_1 = R \). If \( R \cdot D(R_2) > 0 \), then \( R \cdot (D(R_1) + D(R_2)) \geq 0 \) since the ray \( R \) is simple. Then we get the case (b) of the theorem.

5. An application of the diagram method to Fano 3-folds with terminal singularities

We restrict ourselves to considering Fano 3-folds with \( \mathbb{Q} \)-factorial terminal singularities, but it is possible to formulate and prove corresponding results for a negative part of Mori cone of 3-dimensional variety with \( \mathbb{Q} \)-factorial terminal singularities like in [N7].
We recall that an algebraic 3-fold $X$ over $\mathbb{C}$ with $\mathbb{Q}$-factorial singularities is called Fano if the anticanonical class $-K_X$ is ample. By results of Kawamata [Ka1] and Shokurov [Sh], any face of $NE(X)$ is contractible and $NE(X)$ is generated by a finite set of extremal rays if $X$ is a Fano 3-fold with terminal $\mathbb{Q}$-factorial singularities.

5.1. Preliminary Results. We need the following

Lemma 2.5.1. Let $X$ be a Fano 3-fold with $\mathbb{Q}$-factorial terminal singularities. Let $\mathcal{E} = \{R_1, ..., R_n\}$ be a set of $n$ extremal rays of the type (II) and with disjoint divisors $D(R_1), ..., D(R_n)$ on $X$. (Thus, $\mathcal{E}$ has the type $n\mathbb{C}_1$).

If we suppose that the set $\mathcal{E}$ is not extremal, then there exists a small extremal ray $S$ and $i, 1 \leq i \leq n$, such that $S \cdot (-K_X + D(R_i)) < 0$ and $S \cdot D(R_i) = 0$ if $j \neq i$.

It follows that any curve of the ray $S$ belongs to the divisor $D(R_i)$.

Proof. By Proposition 2.3.2, the divisor $H = -K_X + D(R_1) + \cdots + D(R_n)$ is orthogonal to $\mathcal{E}$. Besides, $H$ is nef and $H^3 > 0$ if there does not exist a small extremal ray $S$ with the property above. Then, $\mathcal{E}$ is extremal of Kodaira dimension 3.

Definition 2.5.2. A set $|R, S|$ of extremal rays has the type $\mathbb{C}_2$ if the ray $R$ has type (II), the extremal ray $S$ is small and $S \cdot D(R) < 0$. (See Figure 3.)

Thus, by Lemma 2.5.1, the set $R_1, ..., R_n, S$ of extremal rays contains a subset of the type $\mathbb{C}_2$.

By Proposition 2.3.2, any extremal ray of $X$ of the type (II) is simple, and by results of Sections 3 and 4 we get a classification of extremal sets and $E$-sets of extremal rays of the type (I) and (II) on $X$.

We have the following general theorem.

\begin{figure}[h]
\centering
\includegraphics{figure3.png}
\caption{Figure 3.}
\end{figure}

Theorem 2.5.3. Let $X$ be a Fano 3-fold with $\mathbb{Q}$-factorial terminal singularities. Let $\alpha$ be a face of $\overline{NE}(X)$. Then we have the following possibilities:

1. There exists a small extremal ray $S$ such that $\alpha + S$ is contained in a face of $\overline{NE}(X)$ of Kodaira dimension 3.

2. There are extremal rays $R_1, R_2$ of the type (II) and small extremal ray $S$ such that $\alpha + R_1$ and $\alpha + R_2$ are contained in faces of $\overline{NE}(X)$ of Kodaira dimension
3, the ray $R_2$ does not belong to $\alpha$, and one of the sets $|R_1, S|$ or $|R_2, S|$ has the type $G_2$.

(3) The face $\alpha$ is contained in a face of $\text{NE}(X)$ of Kodaira dimension 1 or 2.

(4) There exists an $E$-set $L = |R_1, R_2|$ such that $R_1 \nsubseteq \alpha$, $R_2 \nsubseteq \alpha$, but $\alpha + R_1$ and $\alpha + R_1$ are contained in faces of $\text{NE}(X)$ of Kodaira dimension 3. The $L$ satisfies the condition (c) of Theorem 2.4.1: Thus, both extremal rays $R_1, R_2$ have the type (II) $R_1 \cdot D(R_2) > 0$ and $R_2 \cdot D(R_1) > 0$ and there exists an extremal ray $R'_1$ of the type (II) such that $D(R_1) = D(R'_1)$ and $R'_1 \cdot D(R_2) = 0$.

(5) There are extremal rays $R_1, \ldots, R_n$ of the type (II) such that any of them does not belong to $\alpha$, $\alpha + R_1 + \cdots + R_n$ is contained in face of $\text{NE}(X)$ of Kodaira dimension 3 and

$$\dim \alpha + R_1 + \cdots + R_n < \dim \alpha + n.$$ 

(6) $\dim N_1(X) - \dim \alpha \leq 12$.

**Proof.** Let us consider the face $\gamma = \alpha^+$ of $M(X)$ and apply Theorem 1.2 to this face $\gamma$. We have $\dim \gamma = \dim N_1(X) - 1 - \dim \alpha$.

Assume that $\alpha$ does not satisfy the conditions (1), (3) and (5). Then $R(\gamma)$ contains extremal rays of the type (I) or (II) only and $M(X)$ is closed and simple in the face $\gamma$. By Proposition 2.3.2 and Theorem 2.3.3, any extremal subset $S$ of $R(\gamma)$ has connected components of the types $A_1, B_2, C_n$ or $D_2$. By Corollary 2.3.4, the condition (ii) is valid for extremal subsets of $R(\gamma)$. Let $L \subseteq R(\gamma)$ be a $E$-set. Assume that at least two elements $R_1, R_2 \in L$ don't belong to $R(\gamma^1)$ and for any proper subset $L' \subseteq L$ we have that $L' \cup R(\gamma^1)$ is extremal. Let us apply Theorem 2.4.1 to $L$.

Assume that $L$ has the type (d). By Lemma 2.5.1, one of extremal rays $R_1$ of $L$ together with some small extremal ray $S$ define a set of the type $G_2$. Since $|R_1| \subseteq L$ is a proper subset of $L$, the $R(\gamma^1) \cup |R_1|$ is extremal. Or $\alpha + R_1$ is contained in a face of $\text{NE}(X)$. Since $L$ has at least 2 elements which do not belong to $R(\gamma^1)$, there exists another extremal ray $R_2$ of $L$ which does not belong to $R(\gamma^1)$. Like the above, $\alpha + R_2$ is contained in a face of $\text{NE}(X)$ of Kodaira dimension 3. By definition of the case (d), both extremal rays $R_1, R_2$ have the type (II). Thus, we get the case (2) of the theorem.

Assume that $L$ has the type (c). Then we get the case (4) of the theorem.

Assume that $L = |R_1, R_2|$ has the type (b). Suppose that the divisor $m_1D(R_1) + m_2D(R_2)$ is not nef (see the case (b) of Theorem 2.4.1). Then there exists a small extremal ray $S$ such that $S \cdot (m_1D(R_1) + m_2D(R_2)) < 0$. It follows that one of the sets $|R_1, S|$ or $|R_2, S|$ has the type $G_2$. Thus, we get the case (2).

Assume that $L = |R_1, R_2, R_3|$ has the type (a). Then the divisor $D(R_1) + D(R_2) + D(R_3)$ is nef.

Thus, if we additionally exclude the cases (2) and (4), then all conditions
of Theorem 1.2 are satisfied. By Theorems 2.4.1 and 2.3.3, we can take \( d = 2, C_1 = 1 \) and \( C_2 = 0 \). (See Figure 4 for graphs \( G(\mathcal{B}) \) corresponding to extremal sets \( \mathcal{B} \) of the types \( \mathcal{A}_1, \mathcal{B}_2, \mathcal{C}_m \) and \( \mathcal{D}_2 \).) Thus, by Theorem 1.2, \( \dim \gamma < 34/3 \). It follows that \( \dim N_1(X) = \dim \alpha < 12 \).

![Figure 4](image)

**5.2. General properties of configurations of extremal rays of the type \( \mathcal{B}_2 \).** Let \( |R_{11}, R_{12}| \) be a set of extremal rays of the type \( \mathcal{B}_2 \). By Theorem 2.3.3, they define a 2-dimensional face \( R_{11} + R_{12} \) of \( \overline{\text{NE}}(X) \). Let \( |R_{21}, R_{22}| \) be another set of extremal rays of the type \( \mathcal{B}_2 \). Since two different 2-dimensional faces of \( \overline{\text{NE}}(X) \) may have only a common extremal ray, the divisors \( D(R_{11}) = D(R_{12}) \) and \( D(R_{21}) = D(R_{22}) \) don't have a common point. There exists the maximal set \( |R_{11}, R_{12}|, |R_{21}, R_{22}|, ..., |R_{h1}, R_{h2}| \) of pairs of extremal rays of the type \( \mathcal{B}_2 \).

**Lemma 2.5.4.** Any \( t \) pairs \( |R_{11}, R_{12}|, |R_{21}, R_{22}|, ..., |R_{t1}, R_{t2}| \) of extremal rays of the type \( \mathcal{B}_2 \) generate a face

\[
\sum_{i=1}^{t} \sum_{j=1}^{2} R_{ij} \subset \overline{\text{NE}}(X) \subset N_1(X)
\]

of the Kodaira dimension 3 of \( \overline{\text{NE}}(X) \).

**Proof.** This face is orthogonal to the nef divisor \( H = -K_X + D(R_{11}) + \cdots + D(R_{t1}) \) with \( H^3 \geq (-K_X)^3 > 0 \).

**Lemma 2.5.5.** Under the above notation, there exists a changing of order of pairs of extremal rays \( R_{11}, R_{12} \) such that \( R_{11} + \cdots + R_{11} \) is a simplex face of \( \overline{\text{NE}}(X) \).

**Proof.** For \( t = 1 \), it is obvious. Let us suppose that \( \theta = R_{11} + \cdots + R_{(t-1)1} \) is a simplex face of the face

\[
\alpha_{t-1} = \sum_{i=1}^{t-1} \sum_{j=1}^{2} R_{ij}.
\]

The face

\[
\alpha_t = \sum_{i=1}^{t} \sum_{j=1}^{2} R_{ij}
\]
has $\alpha_{t-1}$ as its face and does not coincide with the face $\alpha_t$. It follows that there exists a face $\beta$ of $\alpha$ of the dimension $t$ such that $\beta \subset \alpha_{t-1}$ but $\theta \subset \beta$ is a face of $\beta$. It follows that all extremal rays of $\beta$ are the extremal rays $R_{11}, \ldots, R_{(t-1)1}$ and some of extremal rays $R_{11}, R_{12}$. Assume that both extremal rays $R_{11}, R_{12}$ belong to $\beta$. Then the extremal rays $R_{11}, \ldots, R_{(t-1)1}, R_{11}, R_{12}$ are linearly dependent, since $\dim \beta = t$. By Proposition 2.3.8, it is impossible. Thus, only one of extremal rays $R_{11}, R_{12}$ belongs to the face $\beta$. Suppose that this is $R_{11}$.

Then $\beta = R_{11} + \cdots + R_{(t-1)1} + R_{t1}$, will be the face we were looking for.

We divide the maximal set $\{R_{11}, R_{12}\}$, $\{R_{21}, R_{22}\}$, ..., $\{R_{n1}, R_{n2}\}$ of pairs of extremal rays of the type $B_2$ into two parts:

$\{R_{11}, R_{12}\}$, $\{R_{21}, R_{22}\}$, ..., $\{R_{m1}, R_{m2}\}$

and

$\{R_{(m+1)1}, R_{(m+1)2}\}$, $\{R_{(m+2)1}, R_{(m+2)2}\}$, ..., $\{R_{(m+k)1}, R_{(m+k)2}\}$

where $n = m + k$. By definition, here the extremal rays $R_{11}, R_{12}$ belong to the first part if and only if they are linearly independent of other extremal rays from the set $\{R_{11}, R_{12}\}$, $\{R_{21}, R_{22}\}$, ..., $\{R_{n1}, R_{n2}\}$. Thus, extremal rays $R_{11}, R_{12}$ belong to the second part if they are linearly dependent of other extremal rays from the set $\{R_{11}, R_{12}\}$, $\{R_{21}, R_{22}\}$, ..., $\{R_{n1}, R_{n2}\}$.

**Lemma 2.5.6.** Let $S$ be an extremal ray of the type (II) such that $|R_{11}, S|$ define a configuration (c) of the Theorem 2.4.1. Thus: $R_{11} \cdot D(S) > 0$, $S \cdot D(R_{11}) > 0$ and $R_{12} \cdot D(S) = 0$. Then the extremal ray $R_{11}$, $R_{12}$ are linearly independent from all other extremal rays in $\{R_{11}, R_{12}\}$, $\{R_{21}, R_{22}\}$, ..., $\{R_{n1}, R_{n2}\}$. Thus, $1 \leq i \leq m$. There does not exist a configuration of this type with the ray $R_{12}$. Thus, there does not exist an extremal ray $S'$ of the type (II) such that $R_{12} \cdot D(S') > 0$, $S' \cdot D(R_{12}) > 0$ and $R_{11} \cdot D(S') = 0$.

**Proof.** The $R_{11} + R_{12}$ and $R_{12} + S$ are 2-dimensional faces of $NE(X)$ with intersection by the extremal ray $R_{12}$. It follows that any curve of $D(S)$ belongs to the face $R_{12} + S$ (by Lemma 2.2.3). It follows that the divisor $D(S)$ has no common point with the divisor $D(R_{11})$ for any other pair $R_{11}, R_{12}$ for $j \neq i$. Multiplying $D(S)$ by a linear relation of extremal rays $R_{12}, R_{12}$ with other extremal rays $\{R_{11}, R_{12}\}$, $\{R_{21}, R_{22}\}$, ..., $\{R_{n1}, R_{n2}\}$ and using Proposition 2.3.8, we get that this linear relation does not exist.

Let us suppose that there exists an extremal ray $S'$ (see formulation of the lemma). Then $R_{11} + S'$ is another 2-dimensional face of $NE(X)$. Evidently, divisors $D(S)$ and $D(S')$ have a non-empty intersection. Thus, faces $R_{12} + S$ and $R_{11} + S'$ have a common ray. But it is possible only if $S = S'$. Thus, we get a contradiction, because $R_{11} \cdot D(S) > 0$ but $R_{11} \cdot D(S') = 0$.

Using this Lemma 2.5.6, we can subdivide the first set

$\{R_{11}, R_{12}\}$, $\{R_{21}, R_{22}\}$, ..., $\{R_{m1}, R_{m2}\}$

into sets.
Viacheslav V. Nikulin

\[ |R_{11}, R_{12}|, |R_{21}, R_{22}|, ..., |R_{m1}, R_{m2}| \]

and

\[ |R_{(m1+1)1}, R_{(m1+1)2}|, ..., |R_{(m1+m2)1}, R_{(m1+m2)2}| \]

where \( m_1 + m_2 = m \). Here \( R_{11}, R_{12} \) belong to the first part if and only if there exists an extremal ray \( S \) such that \( R_{11}, S \) satisfy the condition of Lemma 2.5.6. By Lemma 2.5.6, the order between extremal rays \( R_{11} \) and \( R_{12} \) is then canonical.

Let us consider the second set

\[ |R_{(m+1)1}, R_{(m+1)2}|, |R_{(m+2)1}, R_{(m+2)2}|, ..., |R_{(m+k)1}, R_{(m+k)2}| \]

We introduce an invariant

\[ \delta = \dim \sum_{i=m+1}^{m+k} \sum_{j=1}^{2} R_{ij} - k \]

of \( X \). Evidently, either \( k = \delta = 0 \) or \( k \geq 2 \) and \( 1 \leq \delta < k \). Thus,

\[ \dim \sum_{i=m+1}^{m+k} \sum_{j=1}^{2} R_{ij} = k + \delta \]

Let

\[ \rho_0(X) = \dim N_1(X) - \dim \sum_{i=1}^{n=m+k} \sum_{j=1}^{2} R_{ij} \]

Then

\[ \rho(X) = \dim N_1(X) = \rho_0(X) + 2m + k + \delta \]

The invariants: \( \rho_0(X) \), \( n \), \( m \), \( k \), \( \delta \), \( m_1 \), \( m_2 \) are important invariants of a Fano 3-fold \( X \).

The following lemma will be very useful:

**Lemma 2.5.7.** Let \( X \) be a Fano 3-fold with \( \mathbb{Q} \)-factorial terminal singularities. Let \( \mathcal{S} \) be the set of all extremal rays of a proper face \( [\mathcal{S}] \) of \( NE(X) \).

Let

\[ |R_{11}, R_{12}| \cup ... \cup |R_{t1}, R_{t2}| \]

be a set of different pairs of extremal rays of the type \( \mathcal{B}_2 \). Assume that \( R \cdot D(R_{11}) = 0 \) for any \( R \in \mathcal{S} \) and any \( i, 1 \leq i \leq t \). Then there are extremal rays \( Q_1, ..., Q_r \) such that the following statements hold:

(a) \( r \leq t \);

(b) For any \( i, 1 \leq i \leq r \), there exists \( j, 1 \leq j \leq t \), such that \( Q_i \cdot D(R_{1j}) > 0 \) (in particular, \( Q_i \) is different from extremal rays of pairs of extremal rays \( |R_{u1}, R_{u2}| \) of the type \( \mathcal{B}_2 \));

(c) For any \( j, 1 \leq j \leq r \), there exists an extremal ray \( Q_i \), \( 1 \leq i \leq r \), such that \( Q_i \cdot D(R_{1j}) > 0 \);

(d) The set \( \mathcal{S} \cup |Q_1, ..., Q_r| \) is extremal, and extremal rays \( |Q_1, ..., Q_r| \) are linearly independent.
Proof. If \( t=0 \), we can take \( r=0 \). Thus, we assume that \( t \geq 1 \).

Since \( R_{ii} \cdot D(R_{ii}) < 0, 1 \leq i \leq t, 1 \leq j \leq 2 \), the set \( \mathcal{E} \) does not contain the rays \( R_{ii} \). Let \( H \) be a general nef element orthogonal to \( \mathcal{E} \). Since \( t \geq 1 \), there exists \( a>0 \) such that \( H' = H + aD(R_{11}) \) is nef and \( H' \) is orthogonal to \( \mathcal{E} \) and one of the rays \( R_{11}, R_{12} \). Let this ray be \( R_{11} \). Then the set \( \mathcal{E} \cup \{R_{11}\} \) is extremal and is contained in a (proper) face of \( \overline{NE}(X) \). It follows, \( \dim[\mathcal{E}] < \dim[\mathcal{E} \subset \{R_{11}\}] < \dim\overline{NE}(X) \), and \( \dim[\mathcal{E}] < \dim\overline{NE}(X) - 1 \). Let us consider a linear subspace \( V(\mathcal{E}) \subset N_1(X) \) generated by all extremal rays \( \mathcal{E} \). By our condition, \( V(\mathcal{E}) \) is a linear envelope of the face \( \mathcal{E} \) of \( \overline{NE}(X) \).

Let us consider the factorization map \( \pi: N_1(X) \to N_1(X)/V(\mathcal{E}) \). Since the cone \( \overline{NE}(X) \) is polyhedral, the cone \( \pi(\overline{NE}(X)) \) is generated by images of extremal rays \( T \) such that the set \( \mathcal{E} \cup |T| \) is contained in a face \( \mathcal{E} \subset |T| \) of \( \overline{NE}(X) \) of the dimension \( \dim[\mathcal{E}] + 1 \). In particular, since \( \dim[\mathcal{E}] < \dim N_1(X) - 1 \), the face \( \mathcal{E} \cup |T| \) is proper, and the set \( \mathcal{E} \cup |T| \) is extremal.

There exists a curve \( C \) on \( X \) such that \( C \cdot D(R_{11}) = 0 \). This curve \( C \) (as any element \( x \in \overline{NE}(X) \)) is a linear combination of extremal rays \( T \) with non-negative coefficients and extremal rays from \( \mathcal{E} \) with real coefficients. We have \( R \cdot D(R_{11}) = 0 \) for any extremal ray \( R \in \mathcal{E} \). Thus, there exists an extremal ray \( T \) above such that \( T \cdot D(R_{11}) > 0 \). It follows that \( T \) is different from extremal rays of pairs of the type \( 2 \). We take \( Q_1 = T \). By our construction, the set \( \mathcal{E} \cup |Q_1| \) is extremal. If \( Q_1 \cdot D(R_{11}) > 0 \) for any \( j \) such that \( 1 \leq j \leq t \), then \( r = 1 \), and the set \( |Q_1| \) gives the set we were looking for. Otherwise, there exists a minimal \( j \) such that \( 2 \leq j \leq t \) and \( Q_1 \cdot D(R_{11}) = 0 \). Then we replace \( \mathcal{E} \) by the set \( \mathcal{E}_1 \) of all extremal rays in the face \( \mathcal{E} \cup |Q_1| \) of the dimension \( \dim[\mathcal{E}_1] = \dim[\mathcal{E}] + 1 \), and the set

\[
|R_{11}, R_{12}| \cup \ldots \cup |R_{11}, R_{12}|
\]

by

\[
|R_{11}, R_{12}| 1 \leq j \leq t, Q_1 \cdot D(R_{11}) = 0 |
\]

and repeat this procedure.

5.3. Basic Theorems We want to prove the following basic theorem.

Basic Theorem 2.5.8. Let \( X \) be a Fano 3-fold with terminal \( \mathbb{Q} \)-factorial singularities. Assume that \( X \) does not have a small extremal ray, and Mori polyhedron \( \overline{NE}(X) \) does not have a face of Kodaira dimension 1 or 2.

Then we have the following for the \( X \):

(1) The \( X \) does not have a pair of extremal rays of the type \( \mathcal{B}_2 \) (thus, in notation above, the invariant \( n = 0 \)) and Mori polyhedron \( \overline{NE}(X) \) is simplicial.

(2) The \( X \) does not have more than one extremal ray of the type \( 1 \).

(3) If \( \mathcal{E} \) is an extremal set of \( k \) extremal rays of \( X \), then the \( \mathcal{E} \) has one of the types: \( A_4 \Pi (k - 1) \mathcal{C}_1, B_2 \Pi (k - 2) \mathcal{C}_1, C_2 \Pi (k - 2) \mathcal{C}_1, k \mathcal{C}_1 \) (we use notation of
Theorem 2.3.3).

(4) We have the inequality for the Picard number of $X$

$$\rho(X) = \dim N_1(X) \leq 7.$$ 

Proof. We use notations introduced in the Section 5.2. We divide the proof into several steps.

Let us consider extremal rays

$$\mathcal{E}_0 = |R_{11}, R_{12}| \cup |R_{21}, R_{22}| \cup \ldots \cup |R_{n1}, R_{n2}|.$$ 

Let

$$\mathcal{E}^{\text{ind}}_0 = |R_{11}, R_{12}| \cup |R_{21}, R_{22}| \cup \ldots \cup |R_{m1}, R_{m2}|,$$

and

$$\mathcal{E}^{\text{sep}}_0 = |R_{(m+1)1}, R_{(m+1)2}| \cup |R_{(m+2)1}, R_{(m+2)2}| \cup \ldots \cup |R_{n1}, R_{n2}|.$$ 

By Lemma 2.5.4, the set $\mathcal{E}_0$ is extremal. Let $\mathcal{E}$ be a maximal extremal set of extremal rays which contains $\mathcal{E}_0$. Let $\mathcal{E}_1 = \mathcal{E} - \mathcal{E}_0$. By Proposition 2.3.8, $\mathcal{E}_1 = \rho(X) - 1 - \dim [\mathcal{E}_0]$. By Theorem 2.3.3, for $S \subset \mathcal{E}_1$, the divisor $D(S)$ has no a common point with divisors $D(R_{i1}), 1 \leq i \leq n$.

Lemma 2.5.9. Assume that $X$ satisfies the conditions of Theorem 2.5.8. Let $Q$ be an extremal ray such that $Q$ is different from extremal rays $R_{i1}, 1 \leq i \leq n$, $1 \leq j \leq 2$, and the set $\mathcal{E}_1 \cup |Q|$ is extremal. Then the $Q$ has the type (II) and there exists exactly one $i$ such that $1 \leq i \leq n$ and $Q \cdot D(R_{i1}) > 0$ and $D(Q) \cap D(R_{i1}) = \emptyset$ if $j \neq i$.

Proof. Assume that $Q$ has the type (I). Then the divisor $D(Q)$ has no common point with the divisors $D(R_{i1}), 1 \leq i \leq n$. By Theorems 2.3.3, 2.3.6 and Lemma 2.5.1, the set $|Q| \cup \mathcal{E}_1 \cup \mathcal{E}_0$ is extremal. We then get a contradiction with the condition that $\mathcal{E}_1 \cup \mathcal{E}_0$ is a maximal extremal set. Thus, the extremal ray $Q$ has the type (II).

If $D(Q)$ has no common point with the divisors $D(R_{i1}), 1 \leq i \leq n$, we get a contradiction by the same way. Thus, there exists $i$ such that $1 \leq i \leq n$ and $D(Q) \cap D(R_{i1}) \neq \emptyset$. Let us consider a projectivization $PNE(X)$. By Lemma 2.2.2, $PNE(X, D(Q))$ is an interval with two ends. Its first end is the vertex $PQ$ and its second end is a point of the edge $P(R_{11} + R_{12})$ of the convex polyhedron $PNE(X)$. Thus, the $i$ is defined by the extremal ray $Q$. Evidently, $Q \cdot D(R_{i1}) > 0$.

Lemma 2.5.10. With the conditions of Lemma 2.5.9 above, assume that $m + 1 \leq i \leq n$. Then there exists exactly one extremal ray $Q = Q_i$ with the conditions of Lemma 2.5.9: thus, the set $\mathcal{E}_1 \cup |Q_i|$ is extremal and $Q_i \cdot D(R_{i1}) > 0$, and $D(Q_i) \cap D(R_{j1}) = \emptyset$ if $j \neq i$.

Proof. The

$$\beta = \sum_{S \in \mathcal{E}_1} S + \sum_{R \in \mathcal{E}_0} R.$$
is a face of $\overline{NE}(X)$ of highest dimension $\rho(X) - 1$, and
\[ \beta_i = \sum_{x \in \Delta_i} S + \sum_{x \in \Lambda_i - \{0\}} R \]
is a face $\beta_i \subset \beta \subset \overline{NE}(X)$ of dimension $\rho(X) - 2$ and of the codimension one in $\beta$ (Here we use that $m + 1 \leq i \leq m + k$). It follows that there exists exactly one face $\beta'$ of $\overline{NE}(X)$ such that $\beta'$ contains $\beta_i$, $\dim \beta' = \rho(X) - 1$, and $\beta' \neq \beta$. By Theorems 2.3.3 and 2.3.6, and Lemma 2.5.9, $\beta' = \beta_i + Q_i$ where $Q_i$ is an extremal ray such that the set $S_i \cup \{Q_i\} \cup (S_0 - \{R_{11}, R_{12}\})$ is extremal, and the ray $Q_i$ has the properties of Lemma 2.5.10. It follows that the $Q_i$ is unique and does exist.

**Lemma 2.5.11.** Under the above notation, the set $S_i \cup S_0^{ind} \cup \{Q_{m+1}, ..., Q_n\}$ is extremal.

**Proof.** By Theorems 2.3.3, 2.3.6, Proposition 2.3.8 and Lemma 2.5.1, the set $S = S_i \cup S_0^{ind}$ is extremal and generates a face of $\overline{NE}(X)$. We apply Lemma 2.5.7 to this $S$ and $S_0^{dep}$. By Lemma 2.5.7, there are extremal rays $Q_{m+1}', ..., Q_{m+r}'$ such that the set $S_i \cup S_0^{ind} \cup \{Q_{m+1}', ..., Q_{m+r}'\}$ is extremal and for any $i, m+1 \leq i \leq m+r$, there exists $j, m+1 \leq j \leq n$, such that $Q_j \cdot D(R_{11}) > 0$. Moreover, for any $j, m+1 \leq j \leq n$, there exists an extremal ray $Q_i, m+1 \leq i \leq m+r$, such that $Q_i \cdot D(R_{11}) > 0$. By Lemmas 2.5.9 and 2.5.10, $r = k$ and $S_i \cup S_0^{ind} \cup \{Q_{m+1}', ..., Q_{m+r}'\} = S_i \cup S_0^{ind} \cup \{Q_{m+1}, ..., Q_n\}$.

**Lemma 2.5.12.** The set $S_0^{dep}$ is empty.

**Proof.** By Lemmas 2.5.9, 2.5.10 and 2.5.11, the set of extremal rays $U = S_i \cup S_0^{ind} \cup \{Q_{m+1}, ..., Q_n\}$ is a maximal extremal set which contains $S_i \cup S_0^{ind}$ and does not contain extremal rays from $S_0^{dep}$. Assume that $k = n - m \neq 0$. Then $k \geq 2$ and $\dim U = \rho_0(X) - 1 + 2m + k$. But the dimension of a face of $\overline{NE}(X)$ of highest dimension is equal to $\rho(X) - 1 = \rho_0(X) - 1 + 2m + k + \delta$ where $\delta \geq 1$. Thus, the extremal set $U$ is not maximal, and there exists another extremal ray $S$ such that $U \cup \{S\}$ is extremal. By definition of $U$, the $S \in S_0^{dep}$. Let $S = R_{11}$ where $m+1 \leq i \leq n$. Since $Q_i \cdot D(R_{11}) > 0$, by Theorem 2.3.3, the extremal set $\{Q_i, R_{11}\}$ has the type $\mathfrak{D}_2$. Thus, $R_{11} \cdot D(Q_i) = 0$. By definition of the set $S_0^{dep}$, there exists a linear dependence $\sum_{i=m+1}^n a_{ii} R_{11} + a_{12} R_{12} = 0$ where $a_{ii} \neq 0$ and $a_{12} \neq 0$. Multiplying $D(Q_i)$ by the equality above, we get $a_{12} = 0$. Thus, we get a contradiction. (Compare with Lemma 2.5.6.)

**Lemma 2.5.13** The set $S_0^{ind}$ is empty.
Proof. Since $E^\text{dep}_0 = \emptyset$, the set $U = E_1 \cup E^\text{ind}_0 = E_1 \cup |R_1, R_1| \cup \ldots \cup |R_m, R_m|$ is a maximal extremal set. It follows that $U$ generates a simplex face of $\overline{NE} (X)$ of codimension 1. Thus, $U_1 = E_1 \cup E^\text{ind}_0 - |R_m| = E_1 \cup |R_1, R_1| \cup \ldots \cup |R_{(m-1)}, R_{(m-1)}| \cup |R_m|$ generates a simplex face of $\overline{NE} (X)$ of codimension 2. It follows that there exists an extremal ray $Q_m$ such that $U_1 = E_1 \cup |R_1, R_1| \cup \ldots \cup |R_{(m-1)}, R_{(m-1)}| \cup |R_m|$ generates a simplex face of $\overline{NE} (X)$ of codimension 1, and $Q_m$ is different from $R_m$. By Lemma 2.5.9, $Q_m \cdot D (R_m) > 0$. Thus, by Theorem 2.3.3, $Q_m, R_m$ is an extremal set of the type 2 where $R_m \cdot D (Q_m) = 0$.

Similarly, we can find an extremal ray $Q_1$ such that the set $|Q_1, R_m|$ is extremal of the type 2 where $R_m \cdot D (Q_1) = 0$. Then we get a contradiction to Lemma 2.5.6. Thus, $m = 0$, and the set $E^\text{ind}_0 = \emptyset$.

Thus, we proved that $X$ does not have a pair of extremal rays of the type $\mathcal{S}_2$. By Theorem 2.3.3 and Proposition 2.3.8, the Mori polyhedron $\overline{NE} (X)$ is then simplicial. Thus, we have proven the statement (1).

Now let us prove (2): $X$ does not have more than one extremal ray of the type (I).

By Lemma 2.2.2, divisors of different extremal rays of the type (I) do not have a common point. By Theorem 2.3.6, any set of extremal rays of the type (I) generates a simplex face of $\overline{NE} (X)$ of Kodaira dimension 3. It follows that the set of extremal rays of the type (I) is finite. Let $|R_1, ..., R_s|$ be the whole set of extremal rays of the type (I) on $X$. We should prove that $s \leq 1$.

Let $\mathcal{E}$ be a maximal extremal set of extremal rays on $X$ containing the set $|R_1, ..., R_s|$ and such that each connected component of $\mathcal{E}$ contains one of extremal rays $R_1, ..., R_s$ (see the definition of connected components before Theorem 2.3.3). By Theorem 2.3.3, then $\mathcal{E}$ has exactly $s$ connected components $T_1, ..., T_s$ such that $T_i$ contains the extremal ray $R_i$. The $T_i$ has either the type $\mathcal{A}_1$ (thus, $T_i = |R_i|$) or $\mathcal{D}_2$ (thus, $T_i$ contains two extremal rays: the $R_i$ and another extremal ray which has the type (II)). Evidently, the maximal $\mathcal{E}$ does exist.

By [Ka] and [Sh], any face of $\overline{NE} (X)$ is contractible, and by our conditions, it has Kodaira dimension 3. By Proposition 2.2.6, for any $1 \leq i \leq s$, there exists an effective divisor $D(T_i)$ which is a linear combination of divisors of rays from $T_i$ with positive coefficients and $R \cdot D(T_i) < 0$ for any $R \in T_i$. Since $T_i$ has the type $\mathcal{A}_1$ or $\mathcal{D}_2$, one can see easily by Lemma 2.2.3, that the same it true for each curve of divisors of rays of $T_i$ because this curve belongs to the sum of extremal rays of $T_i$ with positive coefficients.

Using the divisors $D(T_i)$, similarly to Lemma 2.5.7, we can find extremal rays...
Picard number of Fano 3-folds

$|Q_1, ..., Q_r|$ with properties:

(a) $r \leq s$;

(b) For any $i, 1 \leq i \leq r$, there exists $j, 1 \leq j \leq t$, such that $Q_i \cdot D(T_j) > 0$ (in particular, $Q_i$ is different from extremal rays of $\mathcal{E}$ and does not have the type (I));

(c) For any $j, 1 \leq j \leq s$, there exists an extremal ray $Q_i, 1 \leq i \leq r$, such that $Q_i \cdot D(T_j) > 0$ (in particular, $Q_i$ is different from extremal rays of $\mathcal{E}$ and does not have the type (I));

(d) The set $|Q_1, ..., Q_r|$ of extremal rays is extremal.

By our conditions, all extremal rays on $X$ are divisorial. Thus, by (b), the extremal rays $Q_1, ..., Q_r$ have the type (II).

Let us take the ray $Q_1$, and let $Q_1 \cdot D(T_i) > 0$. By Theorem 2.3.3, the set $T_i$ generates a simplex face $\gamma_i$ of $NE(X)$. We have mentioned above that each curve of divisors of rays from $T_i$ belongs to this face. It follows that $NE(X, D(Q_1))$ is a 2-dimensional angle bounded by the ray $Q_1$ and a ray from the face $\gamma_i$ since the divisor $D(Q_1)$ evidently has a common curve with one of divisors $D(R), R \in T_i$. Since any two sets of $T_1, ..., T_s$ do not have a common extremal ray, the faces $\gamma_1, ..., \gamma_s$ do not have a common ray (not necessarily extremal). It follows that the angle $NE(X, D(Q_1))$ does not have a common ray with the face $\gamma_k$ for $k \neq j$. Thus, the divisor $D(Q_1)$ does not have a common point with divisors of rays $T_k$. It follows that $r=s$ and we can choose an order $Q_1, ..., Q_s$ such that $Q_i \cdot D(T_i) > 0$ but $D(Q_i)$ do not have a common point with divisors of extremal rays $T_j$ if $j \neq i$.

Let us fix $i, 1 \leq i \leq s$. By our construction, the set $\mathcal{E} \cup |Q_i|$ has connected components

$T_1, ..., T_{i-1}, T_i \cup |Q_i|, T_{i+1}, T_s$.

By definition of $\mathcal{E}$, then the $\mathcal{E} \cup |Q_i|$ is not extremal. Thus, it contains an $E$-set (minimal non-extremal) $\mathcal{L}_i$ which contains $Q_i$. By Theorem 2.4.1 and Lemma 1.1, the $\mathcal{L}_i$ is connected. Thus, $|Q_i| \subseteq \mathcal{L}_i \subseteq T_i \cup |Q_i|$. Let us consider the sets $\mathcal{L}_1, ..., \mathcal{L}_s$. By Lemma 1.1, the $\mathcal{L}_i, \mathcal{L}_j$ are joint by arrows. By our construction, it follows that $Q_i, Q_j$ are joint by arrows $Q_i Q_j$ and $Q_j Q_i$ for any $1 \leq i < j \leq s$. By Theorem 2.3.3, for the extremal set $|Q_1, ..., Q_s|$ of extremal rays of the type (II), this is possible only if $s \leq 1$. This proves the statement (2).

To prove (3) we use the following.

**Statement.** The contraction of a ray $R$ of the type (II) on $X$ gives a Fano 3-fold $X'$ with terminal $Q$-factorial singularities and without small extremal rays and without faces of Kodaira dimension 1 or 2 for $NE(X')$. Extremal sets $\mathcal{E}'$ on $X'$ are in one to one correspondence with extremal sets $\mathcal{E}$ on $X$ which contain the ray $R$.

**Proof.** Let $\sigma: X \rightarrow X'$ be a contraction of $R$. The $X'$ has terminal $Q$-factorial singularities by [Ka1] and [Sh]. We have, $K_X = \sigma^*(K_{X'}) +$
$dD(R)$. Multiplying this equality by $R$ and using Proposition 2.3.2, we get that $d=1$. By the statement (1), it follows that $\sigma^*(-K_{X'}) = -K_X + D(R)$ is nef and only contracts the extremal ray $R$. Then $-K_{X'}$ is ample on $X'$ and $X'$ is a Fano 3-fold with terminal $\mathbb{Q}$-factorial singularities. Faces of $\overline{NE}(X')$ are in one to one correspondence with faces of $\overline{NE}(X)$ which contain the $R$. Contractions of faces of $\overline{NE}(X')$ are dominated by these of the corresponding faces of $\overline{NE}(X)$. This proves the last statement.

Let $\mathcal{E} = \{R_1, \ldots, R_k\}$ be an extremal set on $X$. By Theorem 2.3.3, it has connected components of the type $\mathcal{A}_1$, $\mathcal{B}_2$, $\mathcal{C}_m$ or $\mathcal{D}_2$. Moreover, by (1) and (2), it does not have a connected component of the type $\mathcal{B}_2$ and does not have more than one connected component of the type $\mathcal{A}_1$. By Statement above, the same should be true for the extremal set $\mathcal{E}'$ which one gets by the contraction of any extremal ray $R_i$ of the type (II) of $\mathcal{E}$. This shows the statement (3).

Now we prove (4): $\rho(X) \leq 7$.

First, we show how to prove $\rho(X) \leq 8$ applying Theorem 1.2 to the face $\gamma = \mathcal{M}(X)$ of $\dim \mathcal{M}(X) = m = \rho(X) - 1$. By the statement (1) of Theorem 2.5.8 and Theorems 2.3.3 and 2.4.1, the $\mathcal{M}(X)$ is simple and all conditions of Theorem 1.2 are valid for some constants $d$, $C_1$, $C_2$. By Theorem 2.4.1, we can take $d=2$. By the proof of Theorem 1.2, we should find the constants $C_1$, and $C_2$ for maximal extremal sets $\mathcal{E}$ only (only this sets we really use). Thus, $\# \mathcal{E} = m$. By the statement (3), then the constants $C_1 \leq 2/m$ and $C_2 = 0$. Thus, we get $m < (16/3)2/m + 6$. Then, $m = \rho(X) - 1 \leq 7$, and $\rho(X) \leq 8$.

To prove the better inequality $\rho(X) \leq 7$, we should analyze the proof of Theorem 1.2 for our case more carefully. We will show that the conditions of Lemma 1.4 hold for the $\mathcal{M}(X)$ with the constants $C=0$ and $D=2/3$. By Lemma 1.4, we then get the inequality $\rho(X) \leq 7$ we want to prove.

Like for the proof of Theorem 1.2, we introduce a weight of an oriented angle, but using a new formula: $\sigma(\angle) = 2/3$ if $\rho(R_1(\angle), R_2(\angle)) = 1$, and $\sigma(\angle) = 0$ otherwise.

By (3) of Theorem 2.5.8, the condition (1) of Lemma 1.4 holds with constants $C=0$ and $D=2/3$.

Let us prove the condition (2) of Lemma 1.4. For $k=3$ (triangle) it is true since an $E$-set which has at least 3 elements has the type (a) of Theorem 2.4.1 (see the proof of Theorem 1.2). Thus, the triangle has at least three oriented angles with the weight 2/3. For $k = 4$ (quadrangle), we proved (when we were proving Theorem 1.2) that one can find at least two oriented angles of the quadrangle such that any of them has finite $\rho(R_1(\angle), R_2(\angle))$. By (3) of Theorem 2.5.8, then $\rho(R_1(\angle), R_2(\angle)) = 1$. Thus, the quadrangle has at least two oriented angles of the weight 2/3. This finishes the proof of Theorem 2.5.8.

Now, we give an application of (2) of Theorem 2.5.8 to the geometry of Fano 3-folds.
Let us consider a Fano 3-fold $X$ and blow-ups $X_p$ at different non-singular points $\{x_1, \ldots, x_p\}$ of $X$. We say that this is a Fano blow-up if $X_p$ is Fano. We have the following very simple

**Proposition 2.5.14.** Let $X$ be a Fano 3-fold with terminal $\mathbb{Q}$-factorial singularities and without small extremal rays. Let $X_p$ be a Fano blow up of $X$. Then for any small extremal ray $S$ on $X_p$, the $S$ has a non-empty intersection with one of exceptional divisors $E_1, \ldots, E_p$ of this blow up and does not belong to any of them. Moreover, the exceptional divisors $E_1, \ldots, E_p$ define $p$ extremal rays $Q_1, \ldots, Q_p$ of the type $(I)$ on $X_p$ such that $E_i = D(Q_i)$.

**Proof.** The last statement is clear. Let $S$ be a small extremal ray on $X_p$ which does not intersect divisors $E_1, \ldots, E_p$. Let $H$ be a general nef element orthogonal to $S$. Let $l_1, \ldots, l_n$ be lines which generate extremal rays $Q_1, \ldots, Q_p$. Then the divisor $H' = H + (l_1 \cdot H)E_1 + \cdots + (l_p \cdot H)E_p$ is a nef divisor on $X_p$ orthogonal to all extremal rays $Q_1, \ldots, Q_p$, $S$, and $(H')^3 > H^3 > 0$. This proves that the extremal rays $Q_1, \ldots, Q_p, S$ generate a face of $\text{NE}(X_p)$ of Kodaira dimension 3. Then, by the contraction of the extremal rays $Q_1, \ldots, Q_p$, the image of $S$ gives a small extremal ray on $X$. This gives a contradiction.

It is known that a contraction of a face of Kodaira dimension 1 or 2 of $\text{NE}(Y)$ of a Fano 3-fold $Y$ has a general fiber which is rational surface or curve respectively, because this contraction has relatively negative canonical class. See [Ka1], [Sh]. It is known that a small extremal ray is rational [Mo2].

Then, using the statement (2) of Theorem 2.5.8 and Proposition 2.5.14, we can divide Fano 3-folds of Theorem 2.5.8 into the following 3 classes:

**Corollary 2.5.15.** Let $X$ be a Fano 3-fold with terminal $\mathbb{Q}$-factorial singularities and without small extremal rays, and without faces of Kodaira dimension 1 or 2 for Mori polyhedron. Let $\varepsilon$ be the number of extremal rays of the type $(I)$ on $X$ (by Theorem 2.5.8, the $\varepsilon \leq 1$).

Then there exists $p, 1 \leq p \leq 2 - \varepsilon$, such that $X$ belongs to one of classes (A), (B) or (C) below:

(A) There exists a Fano blow-up $X_p$ of $X$ with a face of Kodaira dimension 1 or 2. Thus, birationally, $X$ is a fibration of rational surfaces over a curve or of rational curves over a surface.

(B) There exist Fano blow-ups $X_p$ of $X$ for general $p$ points on $X$ such that for all these blow-ups the $X_p$ has a small extremal ray $S$. Then images of curves of $S$ on $X$ give a system of rational curves on $X$ which cover a Zariski open subset of $X$.

(C) There do not exist Fano blow-ups $X_p$ of $X$ for general $p$ points.

We remark that for Fano 3-folds with Picard number 1 the $\varepsilon = 0$. Thus, $1 \leq p \leq 2$. 

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We mention that that statements (3) and (4) of Theorem 2.5.8 give similar information for blow ups of $X$ along curves. Of course, it is more difficult to formulate these statements.

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Picard number of Fano 3-folds

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