EFFICIENT STEADY-STATE SIMULATION OF HIGH-DIMENSIONAL
STOCHASTIC NETWORKS

JOSE BLANCHET, XINYUN CHEN, PETER GLYNN AND NIAN SI

ABSTRACT. We propose and study an asymptotically optimal Monte Carlo estimator for steady-state expectations of a $d$-dimensional reflected Brownian motion. Our estimator is asymptotically optimal in the sense that it requires $\tilde{O}(d)$ (up to logarithmic factors in $d$) i.i.d. Gaussian random variables in order to output an estimate with a controlled error. Our construction is based on the analysis of a suitable multi-level Monte Carlo strategy which, we believe, can be applied widely. This is the first algorithm with linear complexity (under suitable regularity conditions) for steady-state estimation of RBM as the dimension increases.

1. Introduction

The complexity of supply chains and communication networks have resulted in ever increasing stochastic networks. On the other hand, the steady-state analysis of these systems is of significant interest because operators often focus on performance analysis or control of long term average rewards/costs per unit of time. These reasons motivate our focus in this paper, namely, the study of efficient Monte Carlo methods for steady-state analysis of high-dimensional stochastic networks.

We consider a family of multidimensional reflected Brownian motion (RBM) living in the positive orthant. Under natural uniformity conditions, as the dimension increases, we propose a steady-state simulation estimator which is optimal in the sense of requiring almost a linear number of i.i.d. Gaussian random variables to output an estimate that is close to the steady-state expectation of the underlying RBM. We will provide an explicit description of the assumptions that we impose in Section 2. It suffices to say at the moment that these conditions correspond basically to uniform stability and uniformly bounded variances.

As far as we understand, this paper provides the first class of optimal steady-state estimators for a reasonably general class of stochastic networks as the dimension increases.

RBM can be used to approximate the workload process of a wide range of stochastic networks in heavy traffic. In addition, RBM can be succinctly parameterized in terms of means, variances, and the routing architecture of the network. These properties make it an ideal vehicle for the study of Monte Carlo estimators for stochastic networks indexed by a set of parameters growing in the number of dimensions. Precisely, RBM is a parsimonious, yet powerful, stylized model capturing the features that make steady-state analysis of stochastic networks challenging. In particular, direct computation of the steady-state distribution for RBM is a very challenging problem, even in low dimensions.
There is no closed-form expression in general and even numerical methods are difficult to apply. These difficulties arise from the fact that RBM is defined in terms of a system of constrained stochastic differential equations known as the Skorokhod problem which involves delicate local-time-like dynamics.

Our analysis builds on recent work by [1] and [2]. The first results showing a polynomial rate of convergence to steady-state for a high-dimensional RBM are given in [2]. The proof technique used in [2] involves the following three ingredients: a) the use of a coupling between a steady-state version of the RBM and one starting from a given initial condition driven by the same Brownian motion; b) the application of results from [12] which leads to a contraction factor as the product of certain random matrices when hitting the constrain boundaries at a certain epochs; c) a Lyapunov bound which estimates the return times of the contraction epochs - basically the return time to the constrain boundaries. By combining a)-c), [2] provided an estimate of the form $O\left(d^4 \log^2(d)\right)$ for the relaxation time (measured in terms of the Wasserstein distance) between an RBM starting from the origin and its steady-state distribution. The work of [1] introduced a weighted Lyapunov function (i.e. modifying step c)), greatly improving these estimates and obtaining a relaxation time of $O\left(\log^2(d)\right)$. This suggests simulating the RBM of interest for a time of $O\left(\log^2(d)\right)$ to control the size of the initial transient bias.

In addition to dealing with the initial transient bias, numerical simulation also involves discretization bias. In particular, discretizing a one dimensional Brownian motion with a grid of size $\varepsilon$ induces an error of $\tilde{O}\left(\varepsilon^{1/2}\right)$ in uniform norm on compact intervals. (The tilde notation here means that we are ignoring logarithmic factors in $\log(1/\varepsilon)$). Because RBM is a Lipschitz function of Brownian motion (the Lipschitz constant depends on $d$), we could combine all the above bias analysis for a $d$-dimensional RBM using the triangle inequality, resulting in an error bound of $\tilde{O}\left(d\varepsilon^{1/2}\right)$ on any given compact interval. Consequently, a back-of-the-envelop calculation suggests that direct simulation, even using the sharp analysis in [1], yields a complexity of $\varepsilon = \tilde{O}\left(d^{-2}\right)$ in order to guarantee a controlled error. In turn, this yields that the overall number of Gaussian random variables simulated to obtain an estimate with controlled error is of $\tilde{O}\left(d^3\right)$, which is superlinear in $d$. Our analysis in this paper, in contrast, shows that the estimation can actually be done in complexity $\tilde{O}\left(d\right)$ measured by the number of i.i.d. standard Gaussian random variables simulated.

Moreover, because the results in [1] build from the elements a)-c), the contraction estimate may be difficult to translate to other situations of interest, for example, in the analysis of other types of stochastic networks or general high-dimensional processes whose steady-state distribution may be of interest (e.g. Markov chain Monte Carlo models).

This takes us to the main contributions of this paper, which are summarized as follows.

I) First, we theoretically show that our simulation estimator approximates the steady-state distribution of the underlying RBM in $\tilde{O}\left(d\right)$ time (i.e. almost linear time), measured in terms of i.i.d. Gaussian random variables generated.
II) Second, we provide an alternative method to deriving the contraction estimates for the initial transient bias, which is based on the derivative of the underlying RBM with respect to the initial condition. The intuition is that the rate of convergence to stationarity is dictated by how fast the process ‘forgets’ its initial condition, i.e. how fast the derivative with respect to the initial condition converges to zero. This approach, although analyzed explicitly only in the context of RBM in this paper, is, we believe, applicable to many other settings.

A key idea behind our first contribution is that for numerical simulation of a \(d\)-dimension RBM on finite time intervals, we analyze the contribution of discretization bias of the \(d\) Brownian motions altogether instead of separately, in order to obtain a finer bound on the simulation bias.

A crucial aspect in the development of II) is the use of derivative estimates of RBM with respect to the initial condition, using tools developed in [13]. These derivatives, as it turns out, can be computed as the product of random matrices precisely arising in item b) mentioned earlier in the analysis of [2]. This is both reassuring and convenient because we can simply take advantage of the analysis both in [2] and [1]. However, studying the derivative process with respect to the initial condition is a type of strategy that can be applied in a wide range of settings of interest. So, we believe that the strategy deployed in this paper can be used as a blueprint for the development of efficient Monte Carlo methods for high-dimensional steady-state analysis in many other settings. These developments will be studied in future research.

Our estimators are built using the multilevel Monte Carlo (MLMC) method (see [5]) in conjunction with the key idea discussed earlier and also the contraction property mentioned in II). For a review of multilevel Monte Carlo the reader is referred to [6]. The MLMC method and its randomized variant, which can be used to remove bias under certain conditions (see [14]), have been investigated both in the discretization of stochastic differential equations and, more recently, also in the context of steady-state expectations, see [7] and also [8].

As in [7], we are concerned both with the error in the numerical discretization of the underlying SDE and the time horizon contraction property. We both use a synchronous coupling, which, in our case, is motivated by the analysis in [2]. A key difference, however, is that our goal is to study the complexity of the method as the dimension \(d\) increases to infinity and showing that our estimator has essentially linear complexity in the dimension, as measured by the total number of generated random seeds. Indeed, we believe that this is also a key difference between our work and virtually every work to the date which uses multilevel Monte Carlo methods or steady-state Monte Carlo estimation in generic stochastic networks.

The rest of the paper is organized as follows. In Section 2 we review the definition of RBM and discuss the uniformity conditions which we use to test the asymptotic optimality of our algorithm. The simulation algorithm is given in Section 3 together with the main result of this paper, Theorem 1. A numerical experiment that validates the theoretical performance of the algorithm, tested in the setting of networks of increasing size, is given in Section 4. Finally, the proof of Theorem 1 is given in Section 5.
2. Model and Assumptions

2.1. Skorokhod Problem and RBM. A multidimensional reflected Brownian motion (RBM) can be defined as the solution to a Skorokhod problem with Brownian input. In particular, let \( X(\cdot) \) be a multi-dimensional Brownian motion with drift vector \( \mu \), covariance matrix \( \Sigma := CC^T \), and initial value \( X(0) = 0 \). Let \( Q \) be a substochastic matrix, i.e. \( Q \geq 0 \) and all its row sums \( \leq 1 \), and define \( R = (I - Q)^T \). We assume \( R \) is an \( M \)-matrix, i.e.

\[
R^{-1} \text{ exists and it has non-negative entries.} \tag{1}
\]

The seminal paper [9] shows that the following Skorokhod problem (2) is well posed (i.e. it has a unique solution) in the case where the input \( X(\cdot) \) is continuous and \( R \) is an \( M \)-matrix.

**Skorokhod Problem:** Given a process \( X(\cdot) \) and a matrix \( R \), we say that the pair \((Y, L)\) solves the associated Skorokhod problem if

\[
0 \leq Y(t) = Y(0) + X(t) + RL(t), \quad L(0) = 0 \tag{2}
\]

where the \( i \)-th entry of \( L(\cdot) \) is non-decreasing and \( \int_0^t Y_i(s) dL_i(s) = 0 \).

When the input process \( X \) is a multi-dimensional Brownian motion with parameter \((\mu, \Sigma)\), we call the process \( Y(\cdot) \) solved from (2) a \((\mu, \Sigma, R)\)-RBM.

**Remark 1.** From the perspective that RBM \( Y(\cdot) \) is an approximation to the workload process of a stochastic network, the assumption that \( R \) is an \( M \)-matrix is equivalent to \( Q^n \to 0 \), i.e. the network is open in the sense that all jobs will eventually leave the network.

For general Skorokhod problems, under the \( M \)-condition and some mild conditions on \( X(\cdot) \), the assumption that

\[
R^{-1}EX(1) = R^{-1}\mu < 0 \tag{3}
\]

implies that \( Y(t) \Rightarrow Y(\infty) \) as \( t \to \infty \), where \( Y(\infty) \) is a random variable with the (unique) stationary distribution of \( Y(\cdot) \). (We use \( \Rightarrow \) to denote weak convergence.) In particular, according to [10], condition (3) is necessary and sufficient for stability of the \((\mu, \Sigma, R)\)-RBM (i.e. a unique stationary distribution exists) under the \( M \)-condition (1) (see also [12] which studies necessary and sufficient conditions for more general types of input processes).

2.2. Assumptions. The goal of our simulation algorithm is to estimate the steady-state expectation of certain function value of a multi-dimension RBM. In particular, let \((\mu, \Sigma, R)\) be the parameters of the RBM and \( f(\cdot) \) be the function to be evaluated. To study the complexity of the algorithm as the number of dimension grows, we shall consider a family of \((\mu, \Sigma, R)\)-RBMs under certain uniformity assumptions for arbitrary dimension \( d \), as in [2]. Implicitly, \( R, \mu, \) and \( \Sigma \) are indexed by their dimension. Now we state the uniformity conditions imposed throughout the paper.
A1) Uniform contraction: We let \( R = I - Q^T \), where \( Q \) is substochastic and assume that there exists \( \beta_0 \in (0, 1) \) and \( \kappa_0 \in (0, \infty) \) independent of \( d \) such that
\[
\|1^TQ^n\|_{\infty} \leq \kappa_0(1 - \beta_0)^n. \tag{4}
\]
Under (4) we observe that
\[
\|R^{-1}\|_{\infty} \leq b_1 := \frac{\kappa_0}{\beta_0} < \infty.
\]

A2) Uniform stability: We write \( X(t) = \mu t + CB(t) \), where \( B(t) = (B_1(t), \ldots, B_d(t))^T \) and the \( B_i(\cdot) \)'s are standard Brownian motions, and the matrix \( C \) satisfies \( \Sigma = CC^T \). We assume that there exists \( \delta_0 > 0 \) independent of \( d \) such that
\[
R^{-1}\mu < -\delta_01.
\]

A3) Uniform marginal variability: Define \( \sigma^2_i = \Sigma_{i,i} \) (i.e. the variance of the \( i \)-th coordinate of \( X \)). We assume that there exists \( b_0 \in (0, \infty) \), independent of \( d \geq 1 \), such that
\[
b_0^{-1} \leq \sigma^2_i \leq b_0.
\]

A4) Lipschitz functions: Throughout the rest of the paper, we assume that the function \( f: \mathbb{R}^d_{++} \rightarrow \mathbb{R} \) for which we shall estimate \( E[f(Y(\infty))] \) is Lipschitz continuous in \( l_\infty \) norm, i.e. there exists a constant \( L > 0 \) such that
\[
|f(y) - f(y')| \leq L\|y - y'\|_{\infty}, \text{ for all } y, y' \in \mathbb{R}^d.
\]

Remark 2. A detailed discussion on the Assumptions A1) to A3) is given in Section 2.2 of \([2]\). Assumption A4) holds if \( f \) is chosen to quantify the performance of a finite number of servers in the network, or when the performance measure of the system is scaled by \( d \), for instance, the average workload in the servers.

3. Two-Parameter Multilevel Monte Carlo Algorithm

Any simulation estimator for stationary expectations of RBM is bound to contain two types of sources of bias. The first one is the discretization error, due to the fact that we can only simulate discrete approximation of continuous Brownian paths. The second source of bias is the initial transient bias or non-stationary error, due to the fact that we can only simulate the RBM during a finite time horizon. We call our simulation method a two-parameter multilevel Monte Carlo (MLMC) algorithm because when constructing the MLMC estimator, we use two parameters \( \gamma \in (0, 1) \) and \( T > 0 \) to control the discretization and non-stationary errors, respectively.

As in the classic MLMC algorithm (\([2]\)), the precision of the MLMC estimator is controlled by the total number of levels \( L \). Besides, we need to specify the initial state \( y_0 \) to simulate the RBM paths. Given the parameter set \((\gamma, T, L, y_0)\), plus the parameters \((\mu, \Sigma, R)\) for the RBM, and the function \( f \) to evaluate, we now describe how to construct the two-parameter MLMC estimator for \( E[f(Y(\infty))] \) and will summarize the whole procedure at the end of this section.
Let \( B(t) = (B_1(t), ..., B_d(t))^T \in \mathbb{R}^d \) be a standard Brownian motion with drift 0 and covariance matrix \( I \). Given parameter \( \gamma \in (0, 1) \), we denote \( D_m = \{0, \gamma^m, 2\gamma^m, ...\} \) for any integer \( m \geq 0 \). For every \( t \geq 0 \), we define \( t_m^+ = \inf\{r \in \mathbb{D}_m : r > t\} \) and \( t_m^- = \sup\{r \in \mathbb{D}_m : r \leq t\} \). Note that, following the definition, \( t_m^- = t \) for \( t \in \mathbb{D}_m \). Define a discretization of the standard Brownian motion of level \( m \) as \( B^m(t) = (B^m_1(t), ..., B^m_d(t))^T \) such that

\[
B^m_i(t) = B_i(t_m^-) + (t - t_m^-) \frac{B_i(t_m^+) - B_i(t_m^-)}{t_m^+ - t_m^-}, \text{ for all } t \geq 0 \text{ and } i = 1, 2, ..., d.
\]

It is easy to see that \( B^m(\cdot) \) is continuous and piecewise linear, and \( B^m(t) = B(t) \) for all \( t \in \mathbb{D}_m \). The corresponding discretization of the Brownian motion \( X(\cdot) \) driving the RBM is defined as

\[
X^m(t) = \mu t + CB^m(t).
\]

For any \( 0 \leq s \leq t < \infty \), we write \( X_{s,t} \) (resp. \( X_{s,t}^m \)) to denote the increment of \( X(\cdot) \) over \([s, t]\), i.e., \( X_{s,t} = \{X(s + u) - X(s) : 0 \leq u \leq t - s\} \) (resp. \( X_{s,t}^m = \{X^m(s + u) - X^m(s) : 0 \leq u \leq t - s\} \)). We use \( Y(t - s; y, X_{s,t}) \) (resp. \( Y^m(t - s; y, X_{s,t}^m) \)) to denote the value of RBM driven by \( X_{s,t} \) at time point \( t - s \) given initial value \( Y(0) = y \) (resp. \( Y^m(0) = y \)). Following this notation, we have

\[
Y(t + s; y, X_{0:s+t}) = Y(t; Y(s; y, X_{0:s}), X_{s:s+t}), \quad Y^m(t + s; y, X_{0:s+t}) = Y^m(t; Y^m(s; y, X_{0:s}), X_{s:s+t}).
\]

To construct the multi-level estimator, we introduce an integer-valued random variable \( M \in \{0, 1, 2, ..., L - 1\} \), where \( L \) is the total number of levels. The random variable \( M \) is independent of the process \( X(\cdot) \) and follows probability distribution

\[
P(M = m) = p(m) = \gamma^m (1 - \gamma)/(1 - \gamma^L) = K(\gamma) \gamma^m, \text{ for } 0 \leq m < L.
\]

Now, we give the formal definition of the two-parameter MLMC estimator \( Z \) for \( E[f(Y(\infty))] \) with input parameter set \( (\gamma, T, L, y_0) \) as

\[
Z = \frac{1}{p(M)} \left( f\left( Y^{M+1}(M; Y^M, X_M, X_{T:(M+1)T}) \right) - f\left( Y^M(M; y_0, X_T, X_{T:(M+1)T}) \right) \right) + f(y_0).
\]

To see that \( Z \) is indeed a good estimator for \( E[f(Y(\infty))] \), we compute

\[
E[Z] = E[E[Z|M]] = \sum_{m=0}^{L-1} \left( E\left[ f\left( Y^{m+1}(mT; Y^m, X_{0:mT}, X_{T:(m+1)T}) \right) \right] - E[f(Y^m(mT; y_0, X_{0:mT}))] + f(y_0) \right) = E[f(Y^L(LT; y_0, X_{0:LT}))].
\]
The last equality holds because $Y^m(m^T; y_0, X_{0;m^T}) = y_0$ for $m = 0$. Consequently, we can split the estimation bias into two parts:

$$E[f(Y^L(TL; y_0, X_{0;LT}))] - E[f(Y(\infty))]$$

$$= (E[f(Y^L(TL; y_0, X_{0;LT}))] - E[f(Y(TL; y_0, X_{0;LT}))]) + (E[f(Y(TL; y_0, X_{0;LT}))] - E[f(Y(\infty))])$$

$$= \text{Discretization Error} + \text{Non-stationarity Error.}$$

Intuitively, as $L \to \infty$, the two errors will both go to 0, and as a consequence, we can obtain accurate estimate of $E[f(Y(\infty))]$ by taking $L$ large enough. In Section 5.1 and 5.2, we shall provide theoretical upper bounds for those two errors in terms of $L$, and also analyze their dependence on the number of dimension $d$. Then, we apply these theoretical error bounds to control the mean square error (MSE) of the simulation estimator, and obtain the main complexity analysis result for our simulation algorithm in Section 5.3.

The above description of the two-parameter multilevel Monte Carlo method is summarized in Algorithm 1. The main result of the paper as follows. We show that, under proper choice of algorithm hyperparameters, the computational budget for Algorithm 1 to obtain estimator of a fixed accuracy level is almost linear in the dimension $d$. The proof relies on a sequence of analysis on the dimension dependence of the discretization and non-stationary error in the simulation procedures, and will be given in Section 5.

**Theorem 1.** Suppose $Y$ (indexed by the number of dimensions $d$) is a sequence of RBM satisfying Assumption 1-4. Then, the total expected cost, in terms of the number of random seeds, for the 2-dimensional MLMC Algorithm 1 to produce an estimator of $E[f(Y(\infty))]$ with mean square error (MSE) $\varepsilon^2$ is

$$O \left( \varepsilon^{-2}d \log(d)^3(\log(\log(d)) + \log(1/\varepsilon))^3 \right).$$

4. Numerical Experiments

We test the theoretical performance guarantee (i.e. Theorem 1) of our algorithm using the so-called symmetric RBMs. In this case, the true value of $E[Y_1(\infty)]$ is known with closed-form expression so that we can check the dimension dependence of the simulation MSE and complexity. To do this, we consider a sequence of symmetric RBMs of different dimensions from 5 up to 200. In detail, for each $d \in \{5, 6, ..., 200\}$, the covariance matrix takes the form

$$\Sigma = \begin{bmatrix}
1 & \rho_\sigma & \cdots & \rho_\sigma \\
\rho_\sigma & 1 & \cdots & \rho_\sigma \\
\vdots & \vdots & \ddots & \vdots \\
\rho_\sigma & \cdots & \rho_\sigma & 1
\end{bmatrix}.$$
Algorithm 1 Two-Parameter Multilevel Monte Carlo for RBM

Input:
The parameters of the RBM: \((\mu, \Sigma, R)\);
The function to evaluate: \(f : \mathbb{R}_+^d \to \mathbb{R}\);
The target error level \(\epsilon\);

Output:
An estimator for \(E[Y(\infty)]\), \(\bar{Z}\);

Hyperparameter Setting:
Step size: \(1 > \gamma > 0\);
Path length: \(T = O(\log(d))^2\);
Number of levels: \(L = \lceil \log(\log(d)) + 2 \log(1/\epsilon) + k_1 \rceil / \log(1/\gamma), \) for a numerical constant \(k_1\);
Initial value: \(y_0 = 0\);
Simulation rounds: \(N = \lceil K(\gamma)^{-1}(1 - \gamma)^{-L} \rceil\), where \(K(\gamma) = (1 - \gamma) / (1 - \gamma^L)\);

Algorithm procedure:
1: for \(i = 1\) to \(N\) do
2: Generate \(M\) with \(P(M = m) = p(m) = K(\gamma)^m\);
3: Simulate a discrete Brownian path \(B_{M+1}(t)\) with step size \(\gamma^{M+1}\) on \([0, (M + 1)T]\);
4: Compute \(B_M(t)\) as a discrete Brownian path such that \(B_M(t) = B_{M+1}(t)\) for all \(t \in D_M\);
5: Compute \(X^M(t) = \mu t + CB^M(t)\) and \(X^{M+1}(t) = \mu t + CB^{M+1}(t)\);
6: Compute
\[
Z_i = \frac{1}{p(M)} \left( f(Y^{M+1}((M + 1)T, y_0, X_{0:(M+1)T})) - f(X^{M+1}(MT, y_0, X_{T:(M+1)T})) \right);
\]
return \(\bar{Z} = f(y_0) + \frac{1}{N} \sum_{i=1}^{N} Z_i\).

and the reflection matrix takes the form
\[
R = \begin{bmatrix}
1 & -r & \ldots & -r \\
-r & 1 & \ldots & -r \\
\vdots & 1 & \ddots & \vdots \\
-r & \ldots & -r & 1
\end{bmatrix}.
\]

To be consistent with Assumptions A1) to A3), we pick
\[
\rho_\sigma = \frac{1 - \beta}{d - 1} \quad \text{and} \quad r = \frac{1 - \beta}{d - 1},
\]
for given \(0 < \beta < 1\). According to \([1]\), the steady-state expectation of workload in each station equals to
\[
E[Y_1(\infty)] = \frac{1 - (d - 2)r + (d - 1)r\rho_\sigma}{2(1 + r)} = \frac{\beta}{2}.
\]
For \(\beta = 0.8\), the true value of \(E[Y_1(\infty)] = 0.4\).
In the first group of numerical experiments, we compare the algorithm performance for different choices of parameter \( \gamma \in \{0.01, 0.05, 0.1\} \) at target error level \( \varepsilon = 0.01 \). The other parameters are as follows: \( T = \log(d)^2/2 \), \( L = \lceil (\log(\log(d)) + 2 \log(1/\varepsilon) - 2)/\log(1/\gamma) \rceil \), and \( N = \lceil K(\gamma)^{-1}\gamma^{-L}L \rceil \). Figure 1 shows the estimated mean and total complexity across dimensions from \( d = 5 \) to \( d = 200 \) for different choices of \( \gamma \). It shows that most of the absolute error fluctuates around 0.01 and the total complexity grows approximately linear in the number of dimension for all three values of \( \gamma \). The simulation error is not sensitive to the choice of \( \gamma \). Besides, the complexity is best when \( \gamma = 0.05 \), as indicated by our theoretic analysis (Lemma 7).

In our second group of numerical experiments, we aim to show that our choice of the parameters is optimal in the sense that the precision level of the algorithm is stable across different number of dimensions. In particular, we estimate the mean square error (MSE) of the estimators for \( \gamma = 0.05 \) and target error level \( \varepsilon = 0.05 \) with the other parameters remain the same. For each dimension range from \( \{10, 20, 30, \ldots, 200\} \), we generate 250 estimators to estimate the MSE as well as the 95% confidence band of the MSE and the results are reported in Figure 2. We see the MSE is stable around \( 5 \times 10^{-4} \) across different dimensions, which is smaller than the target level \( \varepsilon^2 = 0.0025 \).

5. Proof of Theorem 1

In this section, we develop theoretic MSE bounds of the 2-parameter MLMC estimator in terms of the number of dimensions \( d \) using the hyperparameters \((\gamma, T, L, y_0)\) specified in the algorithm 1. As in (7), the estimation bias of the 2-parameter MLMC estimator \( Z \) can be split into two parts corresponding to the discretization error and non-stationary error. The sketch of the proof is as follows:

1. In Section 5.1, we derive an upper bounds for the discretization error in Lemma 4 which is based on the discretization error for Brownian motion (Lemma 2) and an explicit upper bound for the Lipschitz constant of Skorokhod mapping (Lemma 3).
2. In Section 5.2, we provide a bound for the non-stationary error in Lemma 6 by analyzing the derivative of RBM with respect to its initial value (Lemma 5).
3. Finally, in Section 5.3, we derive a theoretic upper bound for the algorithm complexity based using the error bounds.

5.1. Discretization Error Bounds. To bound the discretization error, we first bound the discretization error of multi-dimensional Brownian motion.

Lemma 1. Suppose \( Z_1, Z_2, \ldots, Z_n \) are Gaussian variables (not necessarily independent) with mean 0 and variance 1. Then, we have \( E[\max_{1 \leq i \leq n} Z_i^2] \leq 4(\log n + 1/2 \log(2)) \).
Figure 1: Simulation results for symmetric RBMs.
Figure 2: Mean square error of the estimators. The shade represents 95% confidence band of the MSE.

**Proof of Lemma 1.** For $\lambda \in (0, 1/2)$, we have

\[
E \left[ \max_{1 \leq i \leq n} Z_i^2 \right] = \frac{1}{\lambda} E \left[ \log \left( \exp \left( \lambda \max_{1 \leq i \leq n} Z_i^2 \right) \right) \right] \\
\leq \frac{1}{\lambda} \log E \left[ \exp \left( \lambda \max_{1 \leq i \leq n} Z_i^2 \right) \right] \\
\leq \frac{1}{\lambda} \log E \left[ \sum_{i=1}^{n} \exp \left( \lambda Z_i^2 \right) \right] \\
= \frac{1}{\lambda} (\log n - 1/2 \log(1 - 2\lambda)).
\]

We can pick $\lambda = 1/4$ and then

\[
E \left[ \max_{1 \leq i \leq n} Z_i^2 \right] \leq 4 \left( \log n + 1/2 \log(2) \right).
\]

\[\square\]

**Lemma 2.** For $0 < \gamma < 1$ and $m \geq 1$, let $X^m(\cdot)$ be a discretized $d$-dimension Brownian path with step size $\gamma^m$. Then, there exists a positive constant $C_0$, such that for any $d \geq 1, m \geq 1, t > 0$,

\[
E \left[ \max_{1 \leq i \leq d} \max_{0 \leq s \leq t} (X_i^m(s) - X_i(s))^2 \right] \leq C_0 \gamma^m \left( \log(t) + \log(d) + m \log(1/\gamma) \right).
\]
Proof of Lemma 2. Let \( \tilde{X}(t) = X(t) - \mu t \) and \( \tilde{X}^m(t) = X^m(t) - \mu t \). Note that
\[
\max_{1 \leq i \leq d} \max_{0 \leq s \leq t} (X^m_i(s) - X_i(s))^2
\]
\[
\leq \max_{1 \leq i \leq d} \max_{0 \leq s \leq \gamma m [t/\gamma m]} (X^m_i(s) - X_i(s))^2
\]
\[
= \max_{1 \leq i \leq d} \max_{0 \leq k \leq [t/\gamma m]} \max_{0 \leq s \leq \gamma m} \left( \tilde{X}_i (\gamma^m k + s) - \tilde{X}_i^m (\gamma^m k + s) \right)^2.
\]

For \( 0 \leq s < \gamma^m \) and \( 0 \leq k \leq [t/\gamma^m] \), we have
\[
\left( \tilde{X}_i (\gamma^m k + s) - \tilde{X}_i^m (\gamma^m k + s) \right)^2
\]
\[
\leq \max \left\{ \left( \tilde{X}_i (\gamma^m k + s) - \tilde{X}_i (\gamma^m k) \right)^2, \left( \tilde{X}_i (\gamma^m k + \gamma^m) - \tilde{X}_i (\gamma^m k + s) \right)^2 \right\}
\]
\[
\leq \left( \tilde{X}_i (\gamma^m k + s) - \tilde{X}_i (\gamma^m k) \right)^2 + \left( \tilde{X}_i (\gamma^m k + \gamma^m) - \tilde{X}_i (\gamma^m k + s) \right)^2.
\]

By time-reversibility of the Brownian process, we have
\[
\left( \tilde{X}_i (\gamma^m k + s) - \tilde{X}_i (\gamma^m k) \right)^2 \overset{d}{=} \left( \tilde{X}_i (\gamma^m k + \gamma^m) - \tilde{X}_i (\gamma^m k + (\gamma^m - s)) \right)^2.
\]

Then, according to the increment independence of Brownian motion, we have
\[
\max_{1 \leq i \leq d} \max_{0 \leq k \leq [t/\gamma^m]} \max_{0 \leq s \leq \gamma^m} \left( \tilde{X}_i (\gamma^m k + s) - \tilde{X}_i (\gamma^m k) \right)^2
\]
\[
\overset{d}{=} \gamma^m \max_{1 \leq i \leq d} \max_{0 \leq k \leq [t/\gamma^m]} \max_{0 \leq s \leq \gamma^m} \left( \tilde{X}_i^{(k)} (s) \right)^2,
\]
where \( \tilde{X}^{(0)}, \tilde{X}^{(1)} \ldots \) are i.i.d. copies of \( \tilde{X} \) and \( \tilde{X}^{(k)} = \{ \tilde{X}_1^{(k)}, \tilde{X}_2^{(k)}, \ldots, \tilde{X}_d^{(k)} \} \). Recall that (e.g. [11], page 346)
\[
\max_{0 \leq s \leq 1} \left( \tilde{X}_i^{(k)} (s) \right)^2 \overset{d}{=} \left( \tilde{X}_i^{(k)} (1) \right)^2.
\]

Then, by Lemma 4 we have for \( d > 1, \)
\[
E \left[ \max_{1 \leq i \leq d} \max_{0 \leq s \leq \gamma m [t/\gamma m]} (X^m_i(s) - X_i(s))^2 \right]
\]
\[
\leq 2 \gamma^m E \left[ \max_{1 \leq i \leq d} \max_{0 \leq k \leq [t/\gamma^m]} \left( \tilde{X}_i^{(k)} (s) \right)^2 \right]
\]
\[
\leq 2h_0 \gamma^m (4 \log(d [t/\gamma^m]) + 2 \log(2))
\]
\[
\leq C_0 \gamma^m (\log(t) + \log(d) + m \log(1/\gamma)).
\]

\( \Box \)

The following Lemma 3 shows that the Skorokhod mapping, from \( X \) to \( Y \), is Lipschitz continuous and provides an uniform upper bound for the Lipschitz constant. As a result, the discretization error \( \sup_{0 \leq s \leq T} \| Y(s) - Y^m(s) \|_\infty \) can be uniformly bounded.
Lemma 3. Suppose $Y(t)$ and $Y'(t) \in \mathbb{R}^d$ are the solutions to two Skorokhod problems \cite{2} with the same reflection matrix $R$ satisfying Assumption A1), and input processes $X(t)$ and $X'(t)$ respectively for $t \in [0,T]$. Then,

$$|Y(T) - Y'(T)| \leq 2R \sup_{0 \leq s \leq T} |X(s) - X'(s)|.$$ 

As a direct consequence, under Assumptions A1) to A3), we have

$$\|Y(T) - Y'(T)\|_\infty \leq \frac{2K_0}{\beta} \sup_{0 \leq s \leq T} \|X(s) - X'(s)\|_\infty.$$ 

Proof of Lemma 3. The proof uses the fixed-point representation of the Skorokhod mapping as constructed in the proof of Theorem 1 in \cite{9}. In detail, we first need to do a transform on the space $\mathbb{R}^d$ with respect to a diagonal matrix $\Theta$ with positive diagonal elements, which depends only on $R$, such that $(\Theta Y, \Theta L) \ ((\Theta Y', \Theta L'))$ is the solution to a new Skorokhod problem of the form \cite{2} with input process $\Theta X$ ($\Theta X'$) and reflection matrix $R^* = I - (\Theta^{-1}Q\Theta)^T$. (Note that in our notation, all vectors are column vectors while in \cite{9}, they are treated as row vectors.)

Let $Q^* = \Theta^{-1}Q\Theta$. Then, according to \cite{9}, the amount of reflection $\Theta L$ and $\Theta L'$ solve the following fixed point problem:

$$\Theta L(t) = \sup_{0 \leq s \leq t} \left( Q^T \Theta L(s) - \Theta X \right)^+ \quad \text{and} \quad \Theta L'(t) = \sup_{0 \leq s \leq t} \left( Q^T \Theta L'(s) - \Theta X' \right)^+ \quad \text{for all } 0 \leq t \leq T.$$ 

Here the supreme is taken coordinate by coordinate. Since the elements of $Q^T$ are non-negative, we have

$$\Theta(L(t) - L'(t)) \leq Q^T \Theta \sup_{0 \leq s \leq t} |L(s) - L'(s)| + \sup_{0 \leq s \leq t} \Theta |X(s) - X'(s)|.$$ 

The inequality here also holds coordinate by coordinate. As $\Theta$ is a diagonal matrix with positive diagonal elements, we have

$$(L(t) - L'(t)) \leq \Theta^{-1}Q^T \Theta \sup_{0 \leq s \leq t} |L(s) - L'(s)| + \sup_{0 \leq s \leq t} |X(s) - X'(s)|$$

$$= Q^T \sup_{0 \leq s \leq t} |L(s) - L'(s)| + \sup_{0 \leq s \leq t} |X(s) - X'(s)|.$$ 

As a result,

$$\sup_{0 \leq s \leq T} |L(s) - L'(s)| \leq Q^T \sup_{0 \leq s \leq T} |L(s) - L'(s)| + \sup_{0 \leq s \leq T} |X(s) - X'(s)|.$$ 

Since $(I - Q^T)^{-1} = R^{-1}$ has non-negative elements, we have

$$\sup_{0 \leq s \leq T} |L(s) - L'(s)| \leq R^{-1} \sup_{0 \leq s \leq T} |X(s) - X'(s)|.$$ 

In the end, we have

$$\sup_{0 \leq s \leq T} |Y(s) - Y'(s)| \leq \sup_{0 \leq s \leq T} |X(s) - X'(s)| + |R| \sup_{0 \leq s \leq T} |L(s) - L'(s)|$$

$$\leq \sup_{0 \leq s \leq T} |X(s) - X'(s)| + |R|R^{-1} \sup_{0 \leq s \leq T} |X(s) - X'(s)|.$$
Let us denote $R^{-1}$ by $S$, then $S_{ij} \geq 0$ for all $1 \leq i, j \leq d$. Based on the fact that $R_{ii} = 1$, $R_{ij} \leq 0$ for all $1 \leq i \neq j \leq d$ and $\sum_k R_{ik} S_{ki} = 1$ for all $1 \leq i \leq d$, we have

$$(|R|S)_{ii} = \sum_{k=1}^{d} |R_{ik}|S_{ki} = R_{ii}S_{ii} - \sum_{k \neq i} R_{ik}S_{ki} = R_{ii}S_{ii} + (-1 + R_{i}S_{ii}) = 2S_{ii} - 1.$$  

Note that $2S_{ii} - 1 > 0$ as all diagonal elements of $R^{-1} \geq 1$. Similarly, as $\sum_k R_{ik} S_{kj} = 0$ for all $1 \leq i \neq j \leq d$, we have

$$(|R|S)_{ij} = \sum_{k=1}^{d} |R_{ik}|S_{kj} = R_{ii}S_{ij} - \sum_{k \neq i} R_{ik}S_{kj} = R_{ii}S_{ij} + R_{i}S_{ij} = 2S_{ij}.$$  

Therefore, $|R|R^{-1} = 2R^{-1} - I$ where $I$ is the identity matrix of dimension $d$, and we conclude

$$\sup_{0 \leq s \leq T} |Y(s) - Y'(s)| \leq \sup_{0 \leq s \leq T} |X(s) - X'(s)| + |R|R^{-1} \sup_{0 \leq s \leq T} |X(s) - X'(s)|$$

$$= 2R^{-1} \sup_{0 \leq s \leq T} |X(s) - X'(s)|.$$  

Recall Assumption A1, $\|R^{-1}1\|_{\infty} \leq \kappa_0/\beta_0$, the desired result follows. \hfill \Box

Given Lemma 2 and Lemma 3 we now are ready to provide a theoretic upper bound for the discretization error.

**Lemma 4.** For fixed $\gamma \in (0, 1)$, $t > 0$ and the number of dimensions $d$, there exists a positive constant $C_1$ such that

$$E[\|Y^m(t) - Y(t)\|_{\infty}^2] \leq C_1 \gamma^m (\log(t) + \log(d) + m \log(1/\gamma)).$$  

**Proof of Lemma 4.** By Lemma 3 we have

$$E[\|Y^m(t) - Y(t)\|_{\infty}^2] \leq \frac{4\kappa_0^2}{\beta^2} E \left[ \sup_{0 \leq s \leq t} \|X^m(s) - X(s)\|_{\infty}^2 \right]$$

$$= \frac{4\kappa_0^2}{\beta^2} E \left[ \max_{1 \leq i \leq d} \max_{0 \leq s \leq t} |X^m_i(s) - X_i(s)|^2 \right] \leq C_1 \gamma^m (\log(t) + \log(d) + m \log(1/\gamma))$$,

the last inequality follows from Lemma 2 with $C_1 = C_0 \cdot \frac{4\kappa_0^2}{\beta^2}$. \hfill \Box

### 5.2. Non-stationary Error Bound.

Convergence rate to stationarity of RBM has been analyzed in [11] and [12] based on the synchronous coupling technique. Here we provide an alternative method based on the derivative of RBM with respect to the initial condition. Intuitively, the non-stationary error should have the same order to this derivative, as it reflects the impact of the initial condition on the RBM.

To do this, we first introduce the directional derivative of RBM as defined in [13]. For every continuous input $X_{0:t}$ and any initial condition $y$, [13] shows that there exists a matrix valued process $D(t; y, X_{0:t})$ such that

$$D(t; y, X_{0:t}) \cdot h = \lim_{\varepsilon \to 0} \frac{Y(t; y+\varepsilon h, X_{0:t}) - Y(t; y, X_{0:t})}{\varepsilon}, \forall \, h \in \mathbb{R}^d.$$
We first show that the derivative matrix is bounded by the product of a series of matrices. Following the notations introduced in Section 3 of [2], for the RBM $Y(\cdot; y, X)$ starting from position $y$ at time 0, define a series of stopping times:

$$
\eta^k(y) = \inf\{t > \eta^{k-1}(y) + 1 : Y_i(t; y) = 0\}, \\
\eta^k(y) = \sup\{\eta^k(y) : 1 \leq i \leq d\},
$$

(8)

and set-valued functions

$$
\Gamma_i(t, y) = \{\eta^k : \eta^k \leq t\}, \quad \text{and} \quad \Gamma(t, y) = \bigcup_{i=1}^d \Gamma_i(t, y).
$$

For any time point $t \geq 0$, define

$$
C(t) = \{1 \leq i \leq d : Y_i(t) = 0\} \quad \text{and} \quad \bar{C}(t) = \{1 \leq j \leq d : j \notin C(t)\}
$$

For any subset $S$ of $\{1, 2, ..., d\}$, define the $d \times d$ matrix $\Lambda(S)$ as

$$
\Lambda_{i,j}(S) = P_i(\tau(S) < \tau(\{0\}), W(\tau(S)) = j) \quad \text{for} \quad i, j \in \{1, ..., d\}.
$$

The following result provides an explicit bound for the derivative matrix in terms of the product of $\Lambda$ matrices.

**Lemma 5.** The derivative process

$$
\mathcal{D}(t; y, X_{0:t}) \leq R^{-1} \prod_{s \in \Gamma(t, y)} \Lambda^T(\bar{C}(s)).
$$

**Remark 3.** Under the uniformity assumptions, $\|R^{-1}1\| \leq b_1$, as a result, for any $1 \leq i, j \leq d$,

$$
\mathcal{D}_{ij}(t; y, Z_{0:t}) \leq b_1 \prod_{s \in \Gamma(t, y)} \|\Lambda^T(\bar{C}(s))\|_\infty.
$$

**Proof of Lemma 5.** For the simplicity of notation, we shall write $\mathcal{D}(t; y, X_{0:t}) = \mathcal{D}(t)$ and define $\gamma(t) = R^{-1} \mathcal{D}(t) - I$, i.e., $\gamma(t)$ is the derivative of $L(t)$ with respect to the initial value $y$ (see [13]).

According to Theorem 1.1 of [12], the process $R^{-1}(Y(t, y_1, X_{0:t}) - Y(t, y_2, X_{0:t}))$ is non-increasing in $t$, for any $y_1 \geq y_2$. As a direct consequence, we can conclude that $\gamma(t)$ is non-increasing in $t$ component by component.

Suppose $\Gamma(t, y) = \{\tau_1, \tau_2, ...\}$ with $\tau_1 < \tau_2 < ...$ in order. We define

$$
D_n = \prod_{k \leq n} \Lambda^T(\bar{C}(\tau_k)), \quad \text{and} \quad \gamma_n = R^{-1}(D_n - I).
$$

In particular, $D_0 = I$ and $\gamma_0 = 0$. We shall prove by induction that for any $\tau_n < t \leq \tau_{n+1}$,

$$
\gamma(t) \leq \gamma_n \quad \text{and hence} \quad R^{-1} \mathcal{D}(t) \leq R^{-1} \prod_{k \leq N(t)} \Lambda^T(\bar{C}(\tau_k)).
$$

(9)

First, when $t < \tau_1$, by definition $\gamma(t) = \gamma_0 = 0$. Now suppose (9) holds for all $n \leq m - 1$ and we consider a fixed time $\tau_m < t \leq \tau_{m+1}$. According to [13], the derivative processes...
\[ \gamma(t) \] is the unique solution to the following system of equations:
\[ \gamma_{ij}(t) = \sup_{s \in \Phi^i(t)} \left[ -I_{ij} + (P\gamma(s))_{ij} \right], \]
where \( \Phi^i(t) = \{ s \leq t : L_i(s) = L_i(t) \} \) and \( P = I - R \geq 0 \). For any \( i \in \mathcal{C}(\tau_k) \), \( L_i(s) > L_i(\tau_k) \) with probability 1. By the fact that \( \gamma(t) \) is non-increasing in \( t \) and \( P \geq 0 \), we have
\[ \gamma_{ij}(t) \leq -I_{ij} + (P\gamma(\tau_m))_{ij} \leq -I_{ij} + (P\gamma_{m-1})_{ij}, \]
where the last inequality holds by the induction assumption. For any \( i \in \mathcal{C}(\tau_m) \), we have \( \gamma_{ij}(t) \leq \gamma_{ij}(\tau_m) \leq \gamma_{m-1,ij} \). Suppose \( \bar{\gamma} \) is the solution to the systems of linear equations:
\[ \bar{\gamma}_{ij} = \begin{cases} -I_{ij} + (P\gamma_{m-1})_{ij} & \text{if } i \in \mathcal{C}(\tau_m), \\ \gamma_{m-1,ij} & \text{if } i \in \bar{\mathcal{C}}(\tau_m). \end{cases} \]
Then, \( \gamma(t) \leq \bar{\gamma} \) component by component. For the simplicity of notation, we write \( \mathcal{C} = \mathcal{C}(\tau_m) \). Then, \( \bar{\gamma}_{ij} \) can be solved explicitly as
\[ \bar{\gamma}_{i\bar{c}} = \gamma_{m-1,\bar{c}}; \bar{\gamma}_c = R^{-1}_{cc}(-I_c + P_{cc}\gamma_{m-1,\bar{c}}). \]

More precisely, we write
\[ \bar{\gamma} = \begin{pmatrix} -R^{-1}Ic \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & R^{-1}P_{cc} \\ 0 & I_{cc} \end{pmatrix} \gamma_{m-1}. \]

One can check that
\[ \Lambda^T_m \triangleq \Lambda^T(\bar{\mathcal{C}}(\tau_m)) = I + R \begin{pmatrix} -R^{-1}Ic \\ 0 \end{pmatrix}, \]
and
\[ R^{-1}\Lambda^T_m R = R^{-1} \left[ I + R \begin{pmatrix} -R^{-1}Ic \\ 0 \end{pmatrix} \right] R = I + \begin{pmatrix} -R^{-1}Icc & 0 \\ 0 & R_{cc} \end{pmatrix} \begin{pmatrix} R_{cc} & R_{cc} \\ R_{cc} & R_{cc} \end{pmatrix} \begin{pmatrix} -I_{cc} \\ 0 \end{pmatrix} - R^{-1}R_{cc}I_{cc} = \begin{pmatrix} 0 & R^{-1}P_{cc} \\ 0 & I_{cc} \end{pmatrix}. \]

Therefore, we have
\[ \bar{\gamma} = R^{-1}(\Lambda^T_m - I) + R^{-1}\Lambda^T_m R_{\gamma_{m-1}} = R^{-1}(\Lambda^T_m - I) + R^{-1}\Lambda^T_m R \cdot R^{-1}(\bigcap_{k \leq m-1} \Lambda^T_k - I) = R^{-1}(\bigcap_{k \leq m} \Lambda^T_k - I) = \gamma_m. \]

As a result, \( \mathbf{D}(t) \) holds by induction and we have
\[ R^{-1}\mathbf{D}(t) \leq R^{-1} \prod_{k \leq N(t)} \Lambda^T(\bar{\mathcal{C}}(\tau_k)). \]

Since all the components of \( R^{-1} \) are nonnegative and all its diagonal entries are greater or equal to 1, we can conclude that, component by component
\[ \mathbf{D}(t) \leq R^{-1} \prod_{k \leq N(t)} \Lambda^T(\bar{\mathcal{C}}(\tau_k)). \]
Now, we are ready to derive the upper bound for non-stationary error.

**Lemma 6.** There exists constants $C_2$ and $\xi_1 > 0$ such that
\[
E[\|Y(t; Y(\infty), X_{0:t}) - Y(t; 0, X_{0:t})\|^2] \leq C_2 d^\delta \exp \left(-\xi_1 \frac{t}{\log(d)} \right).
\]

**Proof of Lemma 6.** By the definition of directional derivative of RBM, for any $y \in \mathbb{R}^d$,
\[
Y(t; y, X_{0:t}) - Y(t; 0, X_{0:t}) = \left( \int_0^1 \mathcal{D}(t; u \cdot y, X_{0:t}) \, du \right) y.
\]

Then, following Lemma 5 for $j = 1, 2, ..., d$,
\[
|Y_j(t; y, X_{0:t}) - Y_j(t; 0, X_{0:t})| \leq \sum_{i=1}^d b_i y_i \int_0^1 \| \prod_{s \in \Gamma(t, u, y)} \Lambda^T(C(s)) \| du.
\]

Therefore,
\[
\|Y(t; y, X_{0:t}) - Y(t; 0, X_{0:t})\| \leq b_1 \int_0^1 \| \prod_{s \in \Gamma(t, u, y)} \Lambda^T(C(s)) \| du \cdot \|y\|.
\]

Let’s denote $\| \prod_{s \in \Gamma(t, u, y)} \Lambda^T(C(s)) \| = \Theta(u)$. Then we have
\[
\|Y(t; y, X_{0:t}) - Y(t; 0, X_{0:t})\| \leq b_1^2 \|y\|^2 \left( \int_0^1 \Theta(u) \, du \right)^2 \leq b_1^2 \|y\|^2 \int_0^1 \Theta(u) \, du.
\]

The last equality holds as $\Theta(u) \leq 1$ for all $0 \leq u \leq 1$.

The rest of proof follows the same argument as in [1]. By Lemma 2 and Lemma 3 of [2], all $0 \leq u \leq 1$,
\[
\Theta(u) \leq \|Q^{N(t, u, y)} 1\|_\infty,
\]

where $N(t, y)$ is a random positive integer equals basically to the number of stopping times $\eta_k$ (defined by (8)), observed by time $t$, (see also [2] for the details). Then, we have
\[
E[\|Y(t; Y(\infty), X_{0:t}) - Y(t; 0, X_{0:t})\|_\infty^2] \leq b_1^2 E[\|Y(\infty)\|_\infty^2] \|Q^{N(t, Y(\infty))} 1\|_\infty^2]
\]

\[
\leq b_1^2 E[\|Y(\infty)\|_\infty^2]^{1/2} E[\|Q^{N(t, Y(\infty))} 1\|_\infty^{1/2}] \leq b_1^2 \|y\|^2 \int_0^1 \Theta(u) \, du.
\]

The proof of Theorem 1 of [1] (page 20) shows that, under Assumptions A1) to A3),
\[
E[\|Y(\infty)\|_\infty^{1/2}] \leq \frac{4b_1^2}{\delta_0^2} d^\delta,
\]

\[
E[\|Q^{N(t, Y(\infty))} 1\|_\infty^{1/2}] \leq C_0 d \left( \exp \left(-\xi_1 \frac{t}{\log(d)} \right) \right).
\]

Therefore, let $C_2 = \frac{4b_1^2}{\delta_0^2} C_0$, we get
\[
E[\|Y(t; Y(\infty), X_{0:t}) - Y(t; 0, X_{0:t})\|_\infty^2] \leq C_2 d^\delta \exp \left(-\xi_1 \frac{t}{\log(d)} \right).
\]
5.3. **Complexity Analysis.** Given the error bounds Lemma 2 and Lemma 6, we are ready to show that, for the two-parameter multilevel Monte Carlo Algorithm 1, the computational budget to obtain estimator of a fixed accuracy level is almost linear in dimension $d$.

**Proof of Theorem**. Recall that for a given sequence of RBMs and the $f$ function to evaluate, the algorithm has five input parameters $(\gamma, T, L, y_0, N)$. In the following analysis, we choose $y_0 = 0$.

For fixed $d$ and $\varepsilon$, the mean square error of the estimator $\tilde{Z}$ can be expressed as

$$E[(\tilde{Z} - E[f(Y(\infty))])^2]$$

$$= Var[Z] + (E[Z] - E[f(Y(\infty))])^2$$

$$\leq \frac{1}{N} E[(Z - f(y_0))^2] + (E[Z] - E[f(Y(\infty))])^2$$

$$= \frac{1}{N} \sum_{m=0}^{L-1} p(m)^{-1} E \left[ \left( f(Y^{m+1}((m+1)T; y_0, X_{0:(m+1)T}^T)) - f(Y^m(mT; y_0, X_{T:(m+1)T}^T)) \right)^2 \right]$$

$$+ (E[Z] - E[f(Y(\infty))])^2$$

$$\triangleq \frac{1}{N} \sum_{m=0}^{L-1} K(\gamma)^{-1}\gamma^{-m}V_m + Bias^2. \quad (10)$$

We first analyze the variance terms $V_m$ for each $m = 0, 1, ..., L - 1$. Following Assumption A4,

$$f(Y^{m+1}((m+1)T; y_0, X_{0:(m+1)T}^T)) - f(Y^m(mT; y_0, X_{T:(m+1)T}^T)) \leq \mathcal{L} \| Y^{m+1}((m+1)T; y_0, X_{0:(m+1)T}^T) - Y^m(mT; y_0, X_{T:(m+1)T}^T) \|_{\infty},$$

and

$$\| Y^{m+1}((m+1)T; y_0, X_{0:(m+1)T}^T) - Y^m(mT; y_0, X_{T:(m+1)T}^T) \|_{\infty} \leq \| Y^{m+1}((m+1)T; y_0, X_{0:(m+1)T}^T) - Y((m+1)T; y_0, X_{0:(m+1)T}^T) \|_{\infty}$$

$$+ \| Y^m(mT; y_0, X_{T:(m+1)T}^T) - Y((m+1)T; y_0, X_{T:(m+1)T}^T) \|_{\infty}$$

$$+ \| Y((m+1)T; y_0, X_{0:(m+1)T}^T) - Y((m+1)T; y_0, X_{T:(m+1)T}^T) \|_{\infty}$$

Following Lemma 3, we have

$$E[\| Y^{m+1}((m+1)T; y_0, X_{0:(m+1)T}^T) - Y((m+1)T; y_0, X_{0:(m+1)T}^T) \|_{\infty}^2] \leq C_1 \gamma^{-m+1} (\log((m+1)T) + \log(d) + (m+1)\log(1/\gamma)),$$

and

$$E[\| Y^m(mT; y_0, X_{T:(m+1)T}^T) - Y((m+1)T; y_0, X_{T:(m+1)T}^T) \|_{\infty}^2] \leq C_1 \gamma^{-m} (\log(mT) + \log(d) + m\log(1/\gamma)).$$
Following Lemma 3, we have
\[ E \left[ \left\| Y \left( (m + 1)T; y_0, X_{0:(m+1)T} \right) - Y \left( mT; y_0, X_{0:mT} \right) \right\|^2_\infty \right] = E \left[ \left\| Y \left( mT; Y(T; y_0, X_{0:T}), X_{0:mT} \right) - Y \left( mT; y_0, X_{0:mT} \right) \right\|^2_\infty \right] \leq C_2 \cdot d^3 \exp \left( -\xi_1 \frac{mT}{\log d} \right). \]

Therefore, recall that \((a + b + c)^2 \leq 3(a^2 + b^2 + c^2)\), we have
\[ V_m \leq 3C^2 \left( 2C_1 \gamma^m \log((m + 1)T) + \log(d) + (m + 1) \log(1/\gamma) \right) + C_2 \cdot d^3 \exp \left( -\xi_1 \frac{mT}{\log d} \right). \]

Let \( T = [(3 \log(d)^2 + \log(1/\gamma) \log(d)) / \xi_1] \) and \( C_3 = 3C^2(2C_1 + C_2) \). We have
\[ V_m \leq 3C^2 \left( 2C_1 \gamma^m \log((m + 1)T) + \log(d) + (m + 1) \log(1/\gamma) \right) + C_2 \gamma^m \]
\[ \leq C_3 \gamma^m \log((m + 1)T) + \log(d) + (m + 1) \log(1/\gamma)). \]

Therefore, the total variance of our estimator is
\[ V_{total} = \frac{1}{N} \sum_{m=0}^{L-1} K(\gamma)^{-1} \gamma^{-m} V_m \]
\[ \leq \frac{1}{N} K(\gamma)^{-1} \sum_{m=0}^{L-1} C_3 \log((m + 1)T) + \log(d) + (m + 1) \log(1/\gamma)) \]
\[ \leq \frac{1}{N} C_3 K(\gamma)^{-1} L \log(LT) + \log(d) + L \log(1/\gamma)) \leq \varepsilon^2 / 2. \]

Now we turn to the term of bias in (10). Following Assumption A4), we have
\[ Bias^2 = (E[Z] - E[f(Y(\infty))])^2 = (E[f(Y^L(TL; y_0, X_{0:LT}))] - f(Y(\infty)))^2 \]
\[ \leq 2 \left( (E[f(Y^L(TL; y_0, X_{0:LT}))] - f(Y(TL; y_0, X_{0:LT})))^2 + (E[f(Y(TL; y_0, X_{0:LT}))] - f(Y(\infty)))^2 \right) \]
\[ \leq 2 \left( E[\left\| Y^L(TL; y_0, X_{0:LT}) - Y(TL; y_0, X_{0:LT}) \right\|^2_\infty] + E[\left\| Y(TL; y_0, X_{0:LT}) - Y(\infty) \right\|^2_\infty] \right). \]

Following Lemma 3, we have
\[ E[\left\| Y^L(TL; y_0, X_{0:LT}) - Y(TL; y_0, X_{0:LT}) \right\|^2_\infty] \leq C_1 \left( \gamma^L \log(LT) + \log(d) + L \log(1/\gamma) \right). \]

Following Lemma 3, we have
\[ E[\left\| Y(TL; y_0, X_{0:LT}) - Y(\infty) \right\|^2_\infty] \leq C_2 \cdot d^3 \exp \left( -\xi_1 \frac{LT}{\log(d)} \right) \leq C_2 \cdot \gamma^L, \]
for \( T = [(3 \log(d)^2 + \log(1/\gamma) \log(d)) / \xi_1] \).

Therefore,
\[ Bias^2 \leq C_3 \left( \gamma^L \log(LT) + \log(d) + L \log(1/\gamma) \right) \leq \varepsilon^2 / 2, \]
for $T = [(3 \log(d)^2 + \log(1/\gamma) \log(d)) / \xi_1]$ and $L = [(\log(\log(d)) + 2 \log(1/\varepsilon) + k_1) / \log(1/\gamma)]$, where $k_1$ is a numerical constant.

To equalize the variance and bias of our estimator, we enforce

$$C_3(\gamma^L(\log(LT) + \log(d) + L \log(1/\gamma))) = \frac{1}{N} C_3 K(\gamma)^{-1} L(\log(LT) + \log(d) + L \log(1/\gamma)).$$

So, $N = K(\gamma)^{-1} L / \gamma^L = O(\varepsilon^{-2} K(\gamma)^{-1} L \log(d)).$

Note that the complexity, in terms of expected random seeds used, to simulate one sample of $Z$, should be

$$C = \sum_{m=0}^{L-1} p(m) \gamma^{-(m+1)} T(m+1)d = \frac{1}{2} K(\gamma) \gamma^{-1} dTL(L+1).$$

Then, the total complexity to compute $\bar{Z}$ by $N$ rounds of simulation, with our choice of $(\gamma, T, L, N)$, is

$$N \times C = O(\varepsilon^{-2} K(\gamma)^{-1} L \log(d)) \times \left(\frac{1}{2} K(\gamma) \gamma^{-1} dTL(L+1)\right)$$

$$= O(\varepsilon^{-2}dT \log(d)L^3) = O(\varepsilon^{-2}d \log(d)^3(\log(\log(d)) + \log(1/\varepsilon))^3). \quad (11)$$

**Lemma 7.** The optimal $\gamma^* = 0.05$.

**Proof of Lemma 7.** According to (11), we have the dependence of the total complexity on $\gamma$ is approximately $\gamma^{-1} (\log(1/\gamma))^{-3}$. We shall optimize $\gamma$ to obtain the optimal complexity. Therefore, the optimal $\gamma$ is

$$\gamma^* = \arg \min_{0 < \gamma < 1} \gamma^{-1} (\log(1/\gamma))^{-3} = 0.05.$$

**□**

6. **Conclusion**

We have presented and analyzed a Monte Carlo strategy which provides asymptotically optimal estimators for steady-state expectations of high-dimensional RBM. We believe that the strategy that we present can be applied to more general networks. A key idea is to consider the so-called asynchronous expectations in combination of multilevel Monte Carlo. While this idea is not new (see, for example, [8]), the analysis, which is based on the rate of decay to zero of the product of sub-stochastic random matrices is, we believe, applicable to other settings. In particular, the sensitivity to the initial condition in every stochastic flow naturally yields to the study of product of random matrices and the analysis of the so-called top-Lyapunov exponent. In this paper, we are able to use implicit estimates for this product from [1] and [2]. This, we expect, will provide a blueprint that can be used in other settings, as we expect to report in future research.
Acknowledgement: J. Blanchet gratefully acknowledges NSF grants No. 1915967, 1820942, 1838576. X. Chen gratefully acknowledges NSFC grants No. 91646206 and 11901493.

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Routine to Solve Skorokhod Problem in Algorithm 1

In Step 6 of Algorithm 2, once the piece-wise linear approximation is obtained for the underlying Brownian motion, we obtain the solution to the Skorokhod problem by solving, at each time-step, a static linear complementarity problem (see, for example [3]). Since $R$ is an $M$-matrix, we here provide a simple yet numerical stable algorithm to solve the linear complementarity problem in Algorithm 2.

**Algorithm 2** Algorithm for the Linear Complementarity Problem

- **Input:**
  - The reflection matrix: $R$;
  - The initial vector: $x$;
- **Output:**
  - The solution of the linear complementarity problem: $y \geq 0$, where $y = x + RL$ for $L \geq 0$.

1. Set $\epsilon = 10^{-8}$;
2. $y = x$;
3. while Exists $y_i < -\epsilon$ do
4. Compute the set $B = \{i : y_i < \epsilon\}$;
5. Compute $L_B = -R^{-1}_{B,B}x_B$;
6. Compute $y = x + R_{i:B} \times L_B$;
return $y$.

Lower Bound on Constant $\xi_1$

We also provide a lower bound for the constant $\xi_1$, which is not given explicitly in either [1] or [2]. The lower bound is computed based on a worst-case analysis in [1]. We believe that it is far from tight, as shown in the numerical experiments in Section 4. We provide this, nevertheless, for completeness.

**Lemma 8.** The constant $\xi_1$ satisfies

$$\xi_1 \geq D_1 \left(\frac{\log(2)}{\log(1 - \beta_0)} + 1\right)^{-1} \left(2 + \frac{\kappa_0^2 \beta_0}{\beta_0^2 \delta_0^2}\right)^{-1}$$

with $D_1 = 1/557065$.

**Proof of Lemma** Our $\xi_1$ is equivalent to $E_2$ as defined in Theorem 3 in [1], i.e.

$$E_2 = D_1 \left(\frac{\log(2)}{\log(1 - \beta_0)} + 1\right)^{-1} \left(2 + \frac{\kappa_0^2 \beta_0}{\beta_0^2 \delta_0^2}\right)^{-1},$$

with $D_1 = \delta'/128$, $\delta' = (64C_1)^{-1}$ and $C_1 = C_0 = A_0 = 68$ according to Lemma 7 and Lemma 8 in [1].