A renormalized equation for the three-body system with short-range interactions

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Abstract

We study the three-body system with short-range interactions characterized by an unnaturally large two-body scattering length. We show that the off-shell scattering amplitude is cutoff independent up to power corrections. This allows us to derive an exact renormalization group equation for the three-body force. We also obtain a renormalized equation for the off-shell scattering amplitude. This equation is invariant under discrete scale transformations. The periodicity of the spectrum of bound states originally observed by Efimov is a consequence of this symmetry. The functional dependence of the three-body scattering length on the two-body scattering length can be obtained analytically using the asymptotic solution to the integral equation. An analogous formula for the three-body recombination coefficient is also obtained.

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Recently, there has been renewed interest in the three-body system with short-range interactions [1–9]. From the effective field theory (EFT) perspective, an understanding of this system is an important ingredient for a successful description of few- and many-body systems in nuclear and atomic physics [10–13]. Despite its simplicity, this system exhibits many interesting features [14–16]. For example, when the two-body scattering length $a_2$ approaches infinity, the three-body system exhibits an infinite number of shallow bound states [15]. The equations describing three particles interacting via strong short-range two-body forces have been known for a long time [17]. While these equations are well behaved in some channels, they do not have a unique solution for spinless bosons or three nucleons with total spin $S = 1/2$ [18,19].

The renormalization of the EFT for the three-body system with short-range interactions has been discussed in detail in Ref. [3]. Here we give a brief summary of the results. For nonrelativistic particles interacting via short-range forces, the Lagrangian consists of a nonrelativistic kinetic term and an infinite number of contact interactions with an increasing number of derivatives. For systems in which the two-body scattering length $a_2$ is much larger than the characteristic range of the interactions, the leading two-body contact interaction with no derivatives needs to be treated nonperturbatively [20–22]. For three-body systems, the leading order Feynman diagrams can be summed using an integral equation which is identical to that of Ref. [17] if the three-body force is of natural size and subleading. For three bosons or for the spin-1/2 state of three nucleons, this integral equation exhibits strong dependence on the cutoff used to regulate the theory. From the viewpoint of effective field theory, this is surprising as the Feynman diagrams that the integral equation is designed to sum are individually finite. Their sum, however, does not converge and this is the origin of the cutoff dependence in the integral equation.

The cutoff dependence of the three-body integral equation is properly interpreted via renormalization theory. In this case, the sensitivity to the cutoff indicates that a three-body contact interaction, which one would regard as subleading on the basis of naive dimensional analysis, is in fact leading order. In Ref. [3], it was shown that the cutoff dependence in the three-body equation can be properly renormalized with the inclusion of this three-body force. The three-body force exhibits a very unusual renormalization group flow: it is characterized by a limit cycle. The relevance of a single three-body operator provides a compelling explanation for the existence of the Phillips line [16]. The effective theory has enjoyed successful phenomenological applications in neutron-deuteron scattering [1,2,4–6], the scattering of $^4$He molecules [3], and three-body recombination of atoms in Bose-Einstein condensates [7].

In this paper, we will show that once the three-body force is included, the cutoff dependence of off-shell three-body amplitudes vanishes as the cutoff is taken to infinity. This fact allows us to write down renormalized equations in which the cutoff is completely removed. These renormalized equations exhibit invariance under discrete scale transformations. This invariance is exact at leading order in the effective theory and can be used to constrain the functional form of three-body observables. The spacing of energy levels of low-lying three-body bound states is a direct consequence of this symmetry. The invariance of the three-body observables under the discrete scale transformations and the spacing of the energy levels has been derived previously in a very different manner by Efimov [15,23]. We also show how asymptotic solutions to the equations can be used to determine analytically
the dependence of the three-body scattering length and the three-body recombination rate on the two-body scattering length. The formula for the three-body scattering length has also been derived in earlier work by Efimov [15,23], while the behavior of the three-body recombination rate has been extracted from fits to numerical solutions of the equations [7].

We begin with the integral equation for the elastic scattering of a particle and a bound state of the other two particles:

\[ [F(p; k)]^{-1}K(k, p) = M(k, p; k; \Lambda) + \frac{2}{\pi} \int_0^\Lambda dq M(q, p; k; \Lambda)\mathcal{P}\left(\frac{q^2}{q^2 - k^2}\right)K(k, q), \]  

where

\[ F(p; k) = \frac{8}{3} \left( \frac{1}{a_2} + \sqrt{\frac{3p^2}{4} - mE_k} \right), \]

\[ M(q, p; k; \Lambda) = \frac{1}{2pq} \ln \left| \frac{q^2 + qp + p^2 - mE_k}{q^2 - qp + p^2 - mE_k} \right| + \frac{H(\Lambda)}{\Lambda^2}. \]

Equation (1) has been derived in Ref. [3]. Here \( a_2 \) is the two-body scattering length, and the total energy is \( mE_k = 3k^2/4 - 1/a_2^2 \). \( K(k, p) \) describes the S-wave scattering of a particle and a bound state with momentum \( \pm k \) into a state with momentum \( \pm p \). This is an off-shell quantity except at \( p = k \), where it is related to the scattering phase shift, \( k \cot \delta = 1/K(k, k) \). The three-body scattering length is simply \( a_3 = -K(0, 0) \). The integral equation is regulated by the ultraviolet cutoff \( \Lambda \) and \( H(\Lambda)/\Lambda^2 \) is the contribution from the three-body force. \( H(\Lambda) \) depends on the three-body parameter, \( \Lambda_* \), and evolves in such a way as to render the solution insensitive to \( \Lambda \).

We concentrate now on threshold scattering. Setting \( k = 0 \) and defining \( K(0, p) \equiv a(p) \), we obtain the equation:

\[ [F(p; 0)]^{-1}a(p) = \frac{1}{p^2 + 1/a_2^2} + \frac{H(\Lambda)}{\Lambda^2} + \frac{2}{\pi} \int_0^\Lambda dq M(q, p; 0; \Lambda) a(q). \]  

It is clear from the numerical solutions of Eq. (2) in Ref. [3] that \( a(p) \) is independent of \( \Lambda \) up to power suppressed corrections. This fact is crucial for the derivation of the renormalized equation. On-shell quantities such \( K(k, k) \) are necessarily cutoff independent if the theory is properly renormalized. However, \( a(p) \) is an off-shell quantity for \( p \neq 0 \), so it is not obvious that \( a(p) \) should be cutoff independent as well. To see why this is the case we will derive a renormalization group equation (RGE) for \( a(p) \). We begin by noting that it is possible to derive an equation in which the three-body force is eliminated by taking the difference of Eq. (2) with \( p \neq 0 \) and the same equation with \( p = 0 \),

\[ [F(p; 0)]^{-1}a(p) - \frac{3a_2}{16}a(0) = \frac{1}{p^2 + 1/a_2^2} - a_2 \]

\[ + \frac{2}{\pi} \int_0^\Lambda dq \left[ \frac{1}{2pq} \ln \left| \frac{q^2 + qp + p^2 + 1/a_2^2}{q^2 - qp + p^2 + 1/a_2^2} \right| - \frac{1}{q^2 + 1/a_2^2} \right] a(q). \]

Acting on Eq. (3) with \( \Lambda \, d/d\Lambda \), and using the fact that \( a(0) \) must be cutoff independent, we derive the following RGE for \( a(p) \):

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The function is cutoff independent up to corrections of \( O(\Lambda) \), \( \Lambda \gg Q = p, 1/\alpha_2 \). In the limit of large \( \Lambda \), Eq. (4) becomes a homogeneous integral equation:

\[
[F(p; 0)]^{-1} \Lambda \frac{d}{d\Lambda} a(p) = \frac{2}{\pi} \int_0^\Lambda dq \left[ \frac{1}{2pq} \ln \left| \frac{q^2 + qp + p^2 + 1/\alpha_2^2}{q^2 - qp + p^2 + 1/\alpha_2^2} \right| - \frac{1}{q^2 + 1/\alpha_2^2} \right] \Lambda \frac{d}{d\Lambda} a(q),
\]

which is trivially solved by

\[
\Lambda \frac{d}{d\Lambda} a(p) = 0.
\]  

Thus Eq. (5) provides an explanation for the cutoff independence of \( a(p) \). It is conceivable that Eq. (5) has other solutions, however, the numerical calculations of Ref. [3](cf. Fig. 6) show that Eq. (6) is the correct solution up to corrections that vanish as \( \Lambda \) goes to infinity. We have not yet obtained an analytic proof of Eq. (6).

It is possible to repeat this analysis for the off-shell amplitude \( K(k, p) \). After eliminating \( H(\Lambda) \) from Eq. (1), acting with \( \Lambda d/d\Lambda \), and dropping power suppressed terms, we find

\[
[F(p; k)]^{-1} \Lambda \frac{d}{d\Lambda} K(k, p) = [F(p; 0)]^{-1} \Lambda \frac{d}{d\Lambda} K(k, 0) + 2 \int_0^\Lambda dq \left[ \frac{1}{2pq} \ln \left| \frac{q^2 + qp + p^2 - mE_k}{q^2 - qp + p^2 - mE_k} \right| - \frac{1}{q^2 - mE_k} \right] \Lambda \frac{d}{d\Lambda} K(k, q).
\]

Note that \( K(k, p) \) is not symmetric so \( K(k, 0) \neq a(k) \). Eq. (7) is solved by

\[
\Lambda \frac{d}{d\Lambda} K(k, p) = 0.
\]  

In Fig. 1, we have plotted \( K(0.8/\alpha_2, p) \) for four different values of the cutoff and \( \alpha_2 \lambda_s = 16 \). The function is cutoff independent up to corrections of \( O(1/\Lambda \alpha_2) \). For the lowest value of the cutoff, \( \lambda_2 = 20 \), these corrections can be seen in Fig. 1. For the larger values of \( \Lambda \), they are negligible.

We can now use the cutoff independence of \( a(p) \) to derive the RGE for the three-body force. Setting \( p = 0 \) in Eq. (2) gives the following equation for the three-body scattering length \( a_3 \):

\[
\frac{3\alpha_2}{16} a_3 = a_2^2 + \frac{H(\Lambda)}{\Lambda^2} + \int_0^\Lambda dq \left( \frac{1}{q^2 + 1/\alpha_2^2} + \frac{H(\Lambda)}{\Lambda^2} \right) a(q),
\]

where we have used \( a(0) = -a_3 \). All cutoff dependence in this expression is explicit. Taking derivatives with respect to \( \Lambda \) we obtain the exact RGE

\[
\Lambda \frac{d}{d\Lambda} \left( \frac{H(\Lambda)}{\Lambda^2} \right) = - \frac{2\alpha_2}{\pi} \left( 1 + \frac{2}{\pi} \int_0^\Lambda dq a(q) \right)^{-1} \left( \frac{1}{\Lambda^2 + 1/\alpha_2^2} + \frac{H(\Lambda)}{\Lambda^2} \right) a(\Lambda).
\]
From a study of the asymptotics of Eq. (2) (cf. Refs. [18,3]), we know the behavior of \( a(p) \) for large \( p \):

\[
a(p) = a_2 C(a_2 \Lambda_*) \cos(s_0 \ln(p/\Lambda_*)),
\]

where \( s_0 \approx 1.006 \) and \( \Lambda_* \) is the three-body force parameter. \( C \) is an unknown dimensionless function of \( a_2 \Lambda_* \) and expected to be of \( O(1) \). We can substitute this asymptotic solution for \( a(p) \) into Eq. (10) in order to obtain an approximate RGE for \( H(\Lambda) \). If we neglect terms of \( O(1/(a_2 \Lambda)) \) we find

\[
\Lambda \frac{d}{d\Lambda} \left( \frac{H(\Lambda)}{\Lambda^2} \right) = \frac{- (1 + s_0^2) \cos(s_0 \ln(\Lambda/\Lambda_*) + s_0 \sin(s_0 \ln(\Lambda/\Lambda_*)))}{\cos(s_0 \ln(\Lambda/\Lambda_*) + s_0 \sin(s_0 \ln(\Lambda/\Lambda_*)) + \arctan(1/s_0)) \left( \frac{1}{\Lambda^2} + \frac{H(\Lambda)}{\Lambda^2} \right)},
\]

which is solved by

\[
H(\Lambda) = - \frac{\sin(s_0 \ln(\Lambda/\Lambda_*) - \arctan(1/s_0))}{\sin(s_0 \ln(\Lambda/\Lambda_*) + \arctan(1/s_0))}.
\]

This solution for \( H(\Lambda) \) was previously obtained in Ref. [3] by requiring the low-energy solution \( a(p \sim 1/a_2) \) to be invariant under finite changes of the cutoff. Equations (12, 13) receive corrections that scale as \( 1/(a_2 \Lambda) \), which could in principle be computed with the help of Eq. (10). In practice, Eq. (13) is an excellent approximation to the exact evolution of \( H(\Lambda) \) (cf. Fig. 8 of Ref. [3]).

We are now in a position to write renormalized equations for \( K(k, p) \) and \( a(p) \). Cutoff independence of \( K(k, p) \) implies that we are free to choose the cutoff to be whatever we like. Up to corrections of \( O(1/(a_2 \Lambda)) \), \( H(\Lambda) \) vanishes if \( \Lambda = \Lambda_n \), where
\[ \Lambda_n = \Lambda_* \exp \left[ \frac{1}{s_0} \left( n\pi + \arctan \left( \frac{1}{s_0} \right) \right) \right] \approx \Lambda_* \exp \left( \left( n + \frac{1}{4} \right) \pi \right). \]  

Setting \( \Lambda = \Lambda_n \) in Eqs. (1,2) results in

\[ [F(p; 0)]^{-1} K(k, p) = \frac{1}{2pk} \ln \left| \frac{k^2 + kp + p^2 - mE}{k^2 - kp + p^2 - mE} \right| \]

\[ + \frac{2}{\pi} \int_0^{\Lambda_n(\Lambda_*)} dq \frac{1}{2pq} \ln \left| \frac{q^2 + qp + p^2 - mE}{q^2 - qp + p^2 - mE} \right| \theta \left( \frac{q^2}{p^2 + 1/a_2^2} \right) K(k, q), \]

and

\[ [F(p; 0)]^{-1} a(p) = \frac{1}{p^2 + 1/a_2^2} + \frac{2}{\pi} \int_0^{\Lambda_n(\Lambda_*)} dq \frac{1}{2pq} \ln \left| \frac{q^2 + qp + p^2 + 1/a_2^2}{q^2 - qp + p^2 + 1/a_2^2} \right| a(q). \]

These are interesting equations because all dependence on \( \Lambda \) has been removed in favor of the physical parameter characterizing the three-body force, \( \Lambda_* \), which appears in the upper limit of the integral. These equations receive corrections which fall off as powers of \( \Lambda \). The leading corrections in \( 1/\Lambda \) come from the running of \( H(\Lambda) \) and scale as \( 1/(a_2 \Lambda) \). They are suppressed in the limit \( a_2 \to \infty \) and can be made negligible by choosing \( n \) sufficiently large. Note that we are not taking the cutoff to infinity in a continuous manner, rather in a series of discrete steps for which \( H(\Lambda) = 0 \).

The renormalized equations have an exact symmetry that is a consequence of the freedom to choose an arbitrary integer value for \( n \) in Eq. (14). This corresponds to rescaling \( \Lambda_* \) by a factor of \( \exp(n\pi/s_0) \) and holding all other dimensionful quantities fixed. Note that this symmetry is trivially satisfied in the two-body system. As a consequence, physical observables should not change if the three-body force parameter is rescaled by \( \exp(n\pi/s_0) \).

This symmetry can also be seen in the unrenormalized equation. The amplitude \( a(p) \) is a function of three variables: \( a(p; a_2, \Lambda_*) \). This function has a remnant of scale invariance due to the periodicity of \( H(\Lambda) \). If we perform the scale transformations:

\[ p \to \alpha p, \]

\[ 1/a_2 \to \alpha/a_2, \]

\[ \Lambda \to \alpha \Lambda, \]

\[ a(p; a_2, \Lambda_*) \to \alpha^{-1} a(\alpha p; \alpha^{-1} a_2, \Lambda_*) \]

in Eq. (2) where the cutoff is still present, we find

\[ [F(p; 0)]^{-1} a(\alpha p; \alpha^{-1} a_2, \Lambda_*) = \frac{1}{p^2 + 1/a_2^2} + \frac{H(\alpha \Lambda)}{\Lambda^2} + \frac{2}{\pi} \int_0^{\Lambda} dq \frac{M(q, p; 0; \alpha \Lambda)}{\theta(\frac{q^2}{p^2 + 1/a_2^2})} a(\alpha q; \alpha^{-1} a_2, \Lambda_*). \]

Note that all dimensionful quantities except for \( \Lambda_* \) are rescaled in the transformation in Eq. (17). Up to corrections of \( O(1/(a_2 \Lambda)) \) (which can be made arbitrarily small by choosing an appropriate cutoff \( \Lambda \)), the three-body force runs according to Eq. (13) and has a limit cycle with period \( \exp(n\pi/s_0) \). As a consequence, if \( a(p; a_2, \Lambda_*) \) is a solution, then \( a(\alpha p, \alpha^{-1} a_2, \Lambda_*) \) is a solution as well for all \( \alpha = \exp(n\pi/s_0) \).
This symmetry leads immediately to the unique spectrum of three-body bound states originally discovered by Efimov [15]. The equation for the three-body bound state can easily be obtained from Eq. (2). Since the bound state solutions correspond to standing waves, we drop the inhomogeneous term and replace $E_k$ with $-B_3$. It is sufficient to consider the equation in the limit $k = 0$, and we have

$$\tilde{a}(p) = \frac{16}{3\pi} \left( \frac{1}{a_2} + \sqrt{\frac{3p^2}{4} + m B_3} \right) \int_0^\Lambda dq \left[ \frac{1}{2pq} \ln \left| \frac{q^2 + qp + p^2 + m B_3}{q^2 - qp + p^2 + m B_3} \right| + \frac{H(\Lambda)}{\Lambda^2} \right] \tilde{a}(q),$$

(19)

where $\tilde{a}(p)$ is related to the bound state wave function. The binding energies are then given by those values of $B_3$ for which Eq. (19) has nontrivial solutions. Equation (19) has the same symmetry as Eq. (2) if we transform $B_3 \to e^{2B_3}$. Consequently, if Eq. (19) has a solution for one value of $B_3 \sim 1/(ma_2^2)$, the symmetry then implies the existence of a spectrum of bound states with binding energies [15]

$$B_3^n = d(s_0 \ln(a_2 \Lambda_*)) \frac{\exp(2n\pi/s_0)}{ma_2^n},$$

(20)

where $n > 0$ is an integer. The prefactor $d(s_0 \ln(a_2 \Lambda_*))$ is a periodic function of order one and has been calculated numerically in Ref. [3]. Because a stable three-body bound state cannot have a binding energy smaller than the two-body binding energy $B_2 = 1/(ma_2^2)$, $n$ has to be larger than zero. Furthermore, due to the presence of the momentum cutoff $\Lambda$ the maximal binding energy is of order $\Lambda^2/m$. Inverting Eq. (20), we then find for the total number of bound states [15]

$$N = \frac{s_0}{\pi} \ln(\Lambda a_2).$$

(21)

If we increase the cutoff, the maximal binding energy increases as well [14]. The shallow bound states, however, are kept at constant binding energy through the running of $H$ with the cutoff $\Lambda$ [3]. Corrections to Eq. (20) from irrelevant operators have been studied in Ref. [9].

The discrete scale symmetry plays an important role in constraining the functional dependence of other three-body observables. We will consider the three-body scattering length $a_3$, the three-body effective range parameter $r_3$, and the recombination coefficient $\alpha$. In the recombination process, three atoms start at rest, then two of the atoms form a bound state, and the third atom balances energy and momentum. The recombination rate for three atoms in a gas of density $n$ is $\nu = \alpha n^3$, where $\alpha$ is the matrix element squared for the transition to take place. The integral equation that determines $\alpha$ is very similar to that of $a(p)$ and will be given below. The quantities $a_3$, $r_3$ and $\alpha$ depend only on $a_2$ and $\Lambda_*$ and the requirement that observables be invariant under $\Lambda_* \to \exp(\pi/s_0)\Lambda_*$ highly restricts their functional form:

$$a_3(a_2, \Lambda_*) = a_2 f(s_0 \ln(a_2 \Lambda_*)),
$$

$$r_3(a_2, \Lambda_*) = a_2 g(s_0 \ln(a_2 \Lambda_*)),
$$

$$\alpha(a_2, \Lambda_*) = \frac{a_2^4}{m} h(s_0 \ln(a_2 \Lambda_*)),$$

where $f(x)$, $g(x)$ and $h(x)$ are all periodic functions: $f(x) = f(x + \pi)$. The dependence on factors of $a_2$ and $m$ is determined by dimensional analysis.
In the following, we will investigate the observables in Eq. (22) in more detail. It turns out that the functional form of \(a_3\) and \(\alpha\) can be obtained from the asymptotic solution. We first turn to the three-body recombination rate \(\alpha\). For three atoms approximately at rest in the initial state, the final momentum for the bound state is \(p_f = 2/(\sqrt{3} a_2)\). In Ref. [7], it was shown that \(\alpha = |b(p_f)|^2\), where \(b(p)\) is the solution of

\[
b(p) = \frac{32\pi \sqrt{2}}{\sqrt{m} \sqrt{3}} \left( \frac{1}{p^2} + \frac{H(\Lambda)}{\Lambda^2} \right) + \frac{2}{\pi} \int_0^\Lambda dq \frac{q^2 b(q)}{-1/a_2 + \sqrt{3}q/2 - \imath \epsilon} \ln \left| \frac{p + q + p^2}{q^2 - p q + p^2} \right| + \frac{H(\Lambda)}{\Lambda^2}.
\]  

It is straightforward to show that \(p b(p)\) satisfies the same asymptotic equation as \(a(p)\) from Eqs. (2, 16). Consequently, the ultraviolet behavior of \(a(p)\) and \(p b(p)\) is the same and Eq. (23) is renormalized by the same three-body force [7]. Furthermore, \(b(p)\) is independent of \(\Lambda\) up to power corrections, and we can choose the cutoff as in Eq. (14) to obtain a renormalized equation.

The asymptotic solution for \(b(p)\) is expected to be valid only for \(p \gg 1/a_2\). However, in the numerical calculations of Ref. [3] it was observed that the asymptotic solution works surprisingly well down to momenta of order \(1/a_2\). In this case we should be able to use the asymptotic solution to get the functional form of \(\alpha\). The solution to the asymptotic equation for \(b(p)\) has the form:

\[
b(p) \sim \frac{a_2}{\sqrt{m}} \frac{\cos(s_0 \ln(p/\Lambda_\ast))}{p},
\]  

where the factors of \(a_2\) and \(m\) are determined from dimensional analysis. Note that the asymptotic equation does not determine the normalization of the solution. Evaluating the formula for \(\alpha\), we obtain

\[
\alpha = |b(p_f)|^2 \propto \frac{a_4^4}{m} \cos^2[s_0 \ln(a_2 \Lambda_\ast) - 0.1].
\]  

The recombination coefficient was computed numerically in Ref. [7]. The numerical solution can be fit with the following expression:

\[
\alpha \approx \frac{a_4^4}{m} 68 \cos^2[s_0 \ln(a_2 \Lambda_\ast) + 1.7].
\]  

Equation (26) is invariant under \(\Lambda_\ast \rightarrow \exp(\pi/s_0)\Lambda_\ast\) as expected. The asymptotic solution gives the correct functional form but the normalization is not predicted. It does not predict the correct phase in Eq. (26) because the phase of the full solution at \(p \sim 1/a_2\) is not equal to the asymptotic phase.

Now we turn to the three-body scattering length which is given by \(a_3 = -a(0)\). For \(p \ll 1/a_2\), the asymptotic form is not a good approximation anymore. In fact, it does not even have a well defined limit as \(p \rightarrow 0\). The only extra piece of information we need is the existence of momenta for which the value of \(a(p)\) is independent of \(\Lambda_\ast\). These meeting points were first observed in Ref. [3] (cf. Fig. 5). The first of these points occurs at \(p_0 = 1.1/a_2\),
In Ref. [24], it was conjectured that nonrelativistic systems interacting through short-range
is critical for understanding implications of nonrelativistic scale and conformal symmetry.

Eq. (1). In Fig. 2, we have plotted appearing in Eq. (28) but do not change the functional form.

The effective range has not been derived. We have extracted coefficient 

$$c_9 = \sqrt{a_2}$$

These values of \(\Lambda_\star\) is [3]

$$a(p) \sim a_2 C' \frac{\cos(s_0 \ln(p/L_\star))}{\cos(s_0 \ln(p_0/L_\star))}$$  \(\text{(27)}\)

where \(C'\) is a constant that is independent of \(a_2\Lambda_\star\).

We now insert this form into Eq. (16), set \(p = 0\), and keep only the terms that dominate in the limit \(a_2 \to \infty\). We split the integral over \(q\) into an integral from 0 to \(p_0\) and an integral from \(p_0\) to \(\Lambda_n\). For large \(a_2\) the first integral is parametrically suppressed and can be neglected. The second integral can be performed by expanding the kernel in a formal power series,

$$\frac{a_3}{a_2} = \frac{16}{3} + a_2 C' \frac{32}{3\pi} \int_{p_0}^{\Lambda_n} \frac{dq}{q} \sum_{n=1}^{\infty} (a_2 q)^{-2n} (-1)^{n+1} \frac{\cos(s_0 \ln(q/L_\star))}{\cos(s_0 \ln(p_0/L_\star))} .$$

$$= \frac{16}{3} - \frac{32}{3\pi} C' \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (p_0 a_2)^{-2n}}{s_0^2 + (2n-1)^2} [2n-1 + s_0 \tan(s_0 \ln(a_2 L_\star) - 0.1)] .$$  \(\text{(28)}\)

In Eq. (28), we have kept only the leading terms for large \(a_2\). The three-body scattering length obtained from a fit to numerical solutions in Ref. [7] is

$$a_3 \approx a_2 (1.4 - 1.8 \tan[s_0 (a_2 L_\star) + 3.2]) .$$  \(\text{(29)}\)

Again this is invariant under \(\Lambda_\star \to \exp(\pi/s_0)\). Efimov gave an argument for this functional form in Ref. [23]. The formula obtained using Eq. (27) has the same functional form as Eq. (29) but does not predict the numbers in Eq. (29). One can also consider the effect of corrections to the asymptotic equation of the form

$$\frac{\cos(s_0 \ln(p/L_\star))}{\cos(s_0 \ln(p_0/L_\star))} \frac{c_n}{(p a_2)^n} .$$  \(\text{(30)}\)

These are subleading in the limit of large \(p\) but become important for \(p \sim 1/a_2\). The coefficient \(c_1\) was computed in Ref. [18]. Corrections of this form change the coefficients appearing in Eq. (28) but do not change the functional form.

It would be interesting to know the functional form of \(r_3\). An analytic expression for the effective range has not been derived. We have extracted \(r_3\) from numerical solutions to Eq. (1). In Fig. 2, we have plotted \(r_3/a_2\) as a function of \(\Lambda_\star a_2\) for \(\Lambda_\star a_2\) between 4.1 and 92.9. These values of \(\Lambda_\star a_2\) differ by 22.7 \(\approx \exp(\pi/s_0)\), so this interval corresponds to one period of the limit cycle, after which \(r_3\) returns to its original value. In order to compute the functional form of \(r_3\), it is necessary to know the \(k\) dependence of the amplitude \(K(k, q)\).

The asymptotic solution, however, is independent of \(k\).

Knowing how three-body observables depend on the underlying parameters \(a_2\) and \(\Lambda_\star\) is critical for understanding implications of nonrelativistic scale and conformal symmetry. In Ref. [24], it was conjectured that nonrelativistic systems interacting through short-range
forces could be invariant under nonrelativistic scale and conformal transformations in the limit where \( a_2 \) goes to infinity and the two-body effective range parameters are set to zero. This is interesting as it would give an example of a strongly interacting system that is nevertheless scale and conformally invariant. The many-body problem in this limit has been suggested as a model for neutron matter \([25]\) and could be relevant for gases of trapped atoms. In Ref. \([24]\), it was shown that on- and off-shell two-body Green’s functions respect the nonrelativistic scale and conformal Ward identities. Ward identities were also derived for three-body scattering. Defining the limit \( a_2 \to \infty \) for the three-body system is subtle because of the oscillations in Eq. (22). When we take \( a_2 \) to infinity continuously, \( a_3 \) will oscillate rapidly between \(-\infty\) and \(+\infty\). While this limit is appropriate for atomic systems near Feshbach resonances \([7]\), other limits may be necessary to find scale and conformally invariant theories. One sensible limiting procedure for this purpose would be to take \( a_2 \to \infty \) in discrete steps

\[
\lim_{a_2 \to \infty} \equiv \lim_{n \to \infty} (a_2 = \exp(n\pi/s_0)a_2^0),
\]

where \( n \) is a sequence of integers and \( a_2^0 \) is an arbitrarily chosen starting point for the sequence of discrete transformations. An alternative limiting procedure would be to take \( a_2 \to \infty \) continuously while taking \( \Lambda_* \to 0 \) in such a way that \( a_2\Lambda_* \) remains constant. When the \( a_2 \to \infty \) limit is defined in either of these two ways, observables of the form in Eq. (22) live at one point in the limit cycle. \( a_3 \) does not oscillate when the limit \( n \to \infty \) is taken but tends to either zero or \( \pm\infty \), depending on the initial choice \( a_2^0 \). Obviously obtaining zero requires some fine tuning of \( a_2^0 \) (or equivalently, \( \Lambda_* \)). \( r_3 \) will always tend to \(+\infty\) because it is positive for all values of \( \Lambda_* \) in its limit cycle, as is seen in Fig. 2. Higher order terms in the effective range expansion will tend to either zero or \( \pm\infty \). Because these higher order terms can also diverge as \( a_2 \to \infty \), one needs to understand the behavior of the entire function

**FIG. 2.** Three-body effective range \( r_3 \) as a function of \( \Lambda\).
$K(k; k)$ as $a_2 \to \infty$. It would be interesting to determine this behavior and check if the three-body amplitude obeys the scale and conformal Ward identities derived in Ref. [24].

One very useful property of the renormalized equations is that it is possible to demonstrate analytically that the theory is being renormalized properly rather than having to check this numerically with each calculation. The existence of renormalized equations will facilitate future numerical work, since there are no delicate cancellations between two different terms in the kernel. We emphasize that we are not eliminating the three-body force but using the renormalization group to simplify the form of the equation and make the dependence on the three-body force parameter, $\Lambda_*$, explicit. In some early treatments of the three-body problem [26], the integral equation was solved with a finite cutoff that was tuned to fit observed data and then treated as a universal parameter. Though this procedure may seem somewhat ad hoc, Eq. (16) shows that this is in fact a rigorous procedure.

For many applications it will be important to understand how to explicitly renormalize higher orders in the EFT. For low-energy neutron-deuteron scattering it is clear that effective range corrections are crucial for obtaining accurate agreement with data [1,2,4–6]. Being able to renormalize equations analytically at higher orders will facilitate these increasingly complex calculations. It would be advantageous to understand the renormalization group evolution of higher dimension three-body operators to see if they evolve according to similar limit cycles and to understand at what order these enter into calculations.

A renormalized equation for two-nucleon systems in theories with explicit pions would also be of great practical value. Pion exchange gives rise to a $1/r^3$ potential which must be treated nonperturbatively. It is not obvious that the resummed pion graphs can be renormalized by a single local operator. If this is possible, one would expect this operator to exhibit a limit cycle similar to the one observed in the three-body system, and one would like to calculate the evolution of this operator analytically. Investigations along these lines have been carried out in a position space treatment using a nonrelativistic Schrödinger equation regulated with square well potentials [27].

Interesting applications also include coupling photons and weak currents to few nucleon systems. In these applications one would like to employ gauge invariant regulators such as dimensional regularization. This is also important for theories with pions where one would like to preserve chiral symmetry explicitly. In this context, it would be useful to have renormalized integral equations which are regulated using dimensional regularization.

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