A completely monotonic function involving the gamma and trigamma functions

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Abstract

In this paper the author provides necessary and sufficient conditions on $a$ for the function

$$
\frac{1}{2} \ln(2\pi) - x + (x - \frac{1}{2}) \ln x - \ln \Gamma(x) + \frac{1}{12} \psi'(x + a)
$$

and its negative to be completely monotonic on $(0, \infty)$, where $a \geq 0$ is a real number, $\Gamma(x)$ is the classical gamma function, and $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function. As applications, some known results and new inequalities are derived.

Keywords: Completely monotonic function; gamma function; inequality; logarithmically completely monotonic function; trigamma function.

1. Introduction

It is well known that the classical Euler gamma function is defined by

$$
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt
$$

for $x > 0$, that the logarithmic derivative of $\Gamma(x)$ is called the psi or digamma function and denoted by

$$
\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}
$$

for $x > 0$, that the derivatives $\psi'(x)$ and $\psi''(x)$ for $x > 0$ are respectively called the trigamma and tetragamma functions, and that the derivatives $\psi^{(i)}(x)$ for $i \in \mathbb{N}$ and $x > 0$ are called polygamma functions.

We recall from Mitrinović et al. (1993) and Widder (1946) that a function $f(x)$ is said to be completely monotonic on an interval $I$, if it has derivatives of all orders on $I$ and satisfies

$$
0 \leq (-1)^n f^{(n)}(x) < \infty \quad (1)
$$

for $x \in I$ and all integers $n \geq 0$. If $f(x)$ is non-constant, then the inequality (1) is strict (Dubourdieu, 1939). The class of completely monotonic functions may be characterized by the celebrated Bernstein-Widder Theorem (Widder, 1946), which reads that a necessary and sufficient condition that $f(x)$ should be completely monotonic in $0 \leq x < \infty$ is that

$$
f(x) = \int_0^\infty e^{-t} d\alpha(t),
$$

where $\alpha(t)$ is bounded and non-decreasing and the integral converges for $0 \leq x < \infty$.

For $x \in (0, \infty)$ and $a \geq 0$, let

$$
F_a(x) = \ln \Gamma(x) - (x - \frac{1}{2}) \ln x - \frac{1}{12} \psi'(x + a).
$$

Merkle (1998) proved that the function $F_a(x)$ is strictly concave and the function $F_a(x)$ for $a \geq \frac{1}{2}$ is strictly convex on $(0, \infty)$. This was surveyed and reviewed in Qi (2010).

In recent years, some new results on the complete monotonicity of functions involving the gamma and polygamma functions have been obtained (Guo & Qi, 2012a; Guo & Qi, 2012b; Guo & Qi, 2013a; Guo & Qi, 2013b; Guo et al., 2012; Li et al., 2013; Lü et al., 2011; Qi & Berg, 2013; Qi et al., 2013a; Qi et al., 2013b; Qi et al., 2012; Srivastava et al., 2012; Zhao et al., 2011; Zhao et al., 2012b), for example.
The aims of this paper are to generalize the convexity of the function $F_a(x)$ and to derive known results and some new inequalities.

2. Complete monotonicity

The first aim of this paper is to generalize the convexity of $F_a(x)$ to complete monotonicity, which may be stated as Theorem 1 below.

Theorem 1 For $x \in (0, \infty)$ and $a \geq 0$, let

$$f_a(x) = \frac{1}{2} \ln(2\pi) - x + (x - \frac{1}{2}) \ln(x - \ln \Gamma(x) + \frac{1}{12} \psi'(x + a)).$$

Then the functions $f_a(x)$ and $-f_a(x)$ for $a \geq \frac{1}{2}$ are completely monotonic on $(0, \infty)$.

Proof. Using recursion formulas $\Gamma(x + 1) = x\Gamma(x)$ and

$$\psi'(x + 1) - \psi'(x) = -\frac{1}{x^2}$$

for $x > 0$, an easy calculation yields

$$f_a(x) - f_a(x + 1) = 1 + \frac{1}{2} \ln \left( \frac{x}{x + 1} \right) + \frac{1}{12} \left[ \psi'(x + a) - \psi'(x + a + 1) \right]$$

$$\leq 1 + \frac{1}{2} \ln \left( \frac{x}{x + 1} \right) + \frac{1}{12} \ln \left( \frac{x}{x + 1} \right).$$

Utilizing formulas

$$\Gamma(z) = k^z \int_0^\infty e^{-zt} \Gamma(t) \, dt$$

and

$$\ln \left( \frac{b}{a} \right) = \int_0^1 \frac{e^{-au} - e^{-bu}}{u} \, du$$

for $Re(z) > 0$, $Re(k) > 0$, $a > 0$ and $b > 0$, (Abramowitz & Stegun, 1972), gives

$$[f_a(x) - f_a(x + 1)]' = \int_0^\infty \left[ \frac{1}{2} e^{-t} + \frac{1}{2} \frac{t^2 e^{-t}}{2} \psi'(x + a) \right] \, dt$$

$$+ \frac{e^{-t} - 1}{t} e^{-a} \, dt$$

$$= \int_0^\infty \psi'(t) e^{-a} \, dt.$$  \hspace{1cm} (2)

It is easy to see that

$$\phi_0(t) = -\frac{(t^3 - 6t + 12)e^t - 6(t + 2)}{12te^t}$$

and

$$\phi_{i/2}(t) = \frac{6(t - 2)e^t - t^i e^{it/2} + 6(t + 2)}{12te^t}$$

$$= \frac{1}{12e^t} \sum_{i=5}^\infty \frac{(i - 2)(3\cdot 2^{i-2} - i^2 + i)}{i!} t^{i-1} > 0$$

on $(0, \infty)$, where the inequality $3\cdot 2^{i-2} - i^2 + i > 0$ for $i \geq 5$ may be verified by induction. As a result, the function

$$[f_a(x + 1) - f_a(x)]' = f_a'(x + 1) - f_a'(x)$$

and

$$[f_{i/2}(x) - f_{i/2}(x + 1)]' = f_{i/2}'(x) - f_{i/2}'(x + 1)$$

are completely monotonic on $(0, \infty)$, that is,

$$(-1)^k [f_a(x + 1) - f_a(x)]^{(k)} \geq 0$$

and

$$(-1)^k [f_{i/2}(x) - f_{i/2}(x + 1)]^{(k)} \geq 0$$

for $k \geq 0$. By induction, we have

$$(-1)^k f_0^{(k+1)}(x) \leq (-1)^k f_0^{(k+1)}(x + 1)$$

$$\leq (-1)^k f_0^{(k+1)}(x + 2)$$

$$\leq (-1)^k f_0^{(k+1)}(x + 3) \leq \cdots$$

$$\leq (-1)^k \lim_{m \to \infty} f_0^{(k+1)}(x + m)$$

and

$$(-1)^k f_{i/2}^{(k+1)}(x) \geq (-1)^k f_{i/2}^{(k+1)}(x + 1)$$

$$\geq (-1)^k f_{i/2}^{(k+1)}(x + 2)$$

$$\geq (-1)^k f_{i/2}^{(k+1)}(x + 3) \geq \cdots$$

$$\geq (-1)^k \lim_{m \to \infty} f_{i/2}^{(k+1)}(x + m)$$

for $k \geq 0$.

It is not difficult to obtain
\[ f_a(x) = \frac{\psi''(x) + a}{12} - \psi(x) + \ln x - \frac{1}{2x} \]

and

\[ f_a^{(i)}(x) = \frac{\psi^{(i+1)}(x) + a}{12} - \psi^{(i)}(x) \]

\[ = \frac{(-1)^i(i-2)!}{x^{i-1}} + \frac{(-1)^i(i-1)!}{2x^i}, \quad i \geq 2. \]

In the light of the double inequalities

\[ \ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x} \]

and

\[ \frac{(i-1)!}{x^i} + \frac{i!}{2x^{i+1}} < |\psi^{(i)}(x)| < \frac{(i-1)!}{x^i} + \frac{i!}{x^{i+1}} \]

for \( x > 0 \) and \( i \in \mathbb{N} \), (Guo & Qi, 2011b; Guo & Qi, 2010c; Qi & Guo, 2010a, and Qi et al. (2010), we immediately derive

\[ \lim_{x \to \infty} f_a(x) = 0 \quad \text{and} \quad \lim_{x \to 0^+} f_a^{(i)}(x) = 0 \]

for \( i \geq 2 \) and \( a \geq 0 \). Combining this with (3) and (4), we deduce

\[ (-1)^k f_a^{(k+1)}(x) \leq 0 \quad \text{and} \quad (-1)^k f_a^{(k+1)}(x) \geq 0 \]

for \( k \geq 0 \) on \( (0, \infty) \).

From the formula

\[ \ln \Gamma(z) = (z - \frac{1}{2}) \ln z - z + \frac{1}{2} \ln(2\pi) + 2 \int_0^{\pi} \frac{\arctan(t/z)}{e^{2\pi t} - 1} \, dt \]  
(7)

for \( \Re(z) > 0 \), (Abramowitz & Stegun, 1972), and the double inequality (5) for \( i = 1 \), we easily obtain

\[ \lim_{x \to \infty} f_a(x) = 0 \]  
(8)

for \( a \geq 0 \). Inequalities in (6) imply that the functions \(-f_a(x)\) and \(f_{1/2}(x)\) are increasing on \( (0, \infty) \). Hence, we have

\[ f_a(x) > 0 \quad \text{and} \quad f_{1/2}(x) < 0 \]  
(9)

on \( (0, \infty) \).

From (6) and (9), we conclude that the functions \( f_a(x) \) and \(-f_{1/2}(x)\) are completely monotonic on \( (0, \infty) \).

It is clear that

\[ -f_a(x) = -f_{1/2}(x) + \frac{1}{12}[\psi'(x + \frac{1}{2}) - \psi'(x + a)]. \]

From the facts that the trigamma function

\[ \psi'(x) = \int_0^\infty \frac{t}{1 - e^{-t} e^{-x}} \, dt \]

for \( x > 0 \), (Abramowitz & Stegun, 1972), is completely monotonic on \( (0, \infty) \), that the difference \( f(x) - f(x + \alpha) \) for any given real number \( \alpha > 0 \) of any completely monotonic function \( f(x) \) on \( (0, \infty) \) is also completely monotonic on \( (0, \infty) \), and that the sum of finitely many completely monotonic functions on an interval \( I \) is still completely monotonic on \( I \), it readily follows that the function \(-f_a(x)\) for \( a > \frac{1}{2} \) is also completely monotonic on \( (0, \infty) \). The proof of Theorem 1 is complete.

3. Necessary and sufficient conditions

The second aim of this paper is to answer a natural question: Find the best constants \( \alpha \geq 0 \) and \( \beta \leq \frac{1}{2} \) such that \( f_a(x) \) and \(-f_a(x)\) are both completely monotonic on \( (0, \infty) \).

**Theorem 2.** The function \( f_a(x) \) is completely monotonic on \( (0, \infty) \) if and only if \( \alpha = 0 \), and so is the function \(-f_a(x)\) if and only if \( \beta \geq \frac{1}{2} \).

**Proof.** The first proof. The conclusion that the function \( \phi(t) \) defined in (2) is positive or negative on \( (0, \infty) \) is equivalent to

\[ a \geq -\frac{1}{t} \ln \frac{12}{t^2} \left[ \frac{e^{x} + 1}{2} + \frac{e^{x} - 1}{t} \right] \]

\[ \Delta -\phi(t) = -\frac{1}{t} \ln \phi(t), \quad t > 0. \]

By the L’Hôspital rule, we have

\[ \lim_{t \to 0^+} \phi(t) = \lim_{t \to 0^+} \frac{t(e^{x} + 1)(e^{x} - 1)}{t^2} = \frac{e^{x} - 1}{t} = 1 \]

and \( \lim_{t \to \infty} \phi(t) = 0 \). Hence, the function \( \phi(t) \) can be represented as

\[ \phi(t) = \frac{\ln \phi(t) - \ln \phi(0)}{t} = \frac{1}{t} \int_0^t \phi(u) \, du \]

\[ = \frac{1}{t} \int_0^t 2(u-3)e^u + u^2 + 4u + 6 + 4u + 6 \, du \]

\[ = -\frac{1}{t} \int_0^t \phi(u) \, du \]

for \( t > 0 \). Since
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\[-\{u^3[(u-2)e^u + u + 2]\} \varphi_2(u)\]
\[= 2(u^2 - 6u + 6)e^{2u} + (u^4 + 8u^2 - 24)e^u + 2(u^3 + 6u + 6)\]
\[\geq \sum_{i=8}^{\infty} \frac{2^{-i}(i^2 - 13i + 24) + i^4}{i!} \leq \sum_{i=8}^{\infty} \frac{-6i^3 + 19i^2 - 14i - 24}{i!} > 0\]

for \(u > 0\), where

\[2i^{-i}(i^2 - 13i + 24) + i^4 - 6i^3 + 19i^2 - 14i - 24 > i(i^2 - 13i + 24) + i^4 - 6i^3 + 19i^2 - 14i - 24 = (i - 5)i^2 + 6i^2 + 2(5i - 12) > 0\]

for \(i \geq 8\), the function \(\varphi_2(u)\) is strictly decreasing on \((0, \infty)\), and, by Qi et al. (1999) and Qi & Zhang (1999), the arithmetic mean

\[-\varphi(t) = \frac{1}{t} \int_0^t \varphi_2(u)du\]

is strictly decreasing, and \(\varphi(x)\) is strictly increasing, on \((0, \infty)\). From the L'Hôpital rule and limits

\[\lim_{u \to \infty} \varphi_2(u) = \frac{1}{2} \quad \text{and} \quad \lim_{u \to \infty} \varphi_2(u) = 0,\]

we obtain

\[\lim_{t \to 0^+} \varphi(t) = \frac{1}{2} \quad \text{and} \quad \lim_{t \to \infty} \varphi(t) = 0.\]

As a result, from (10), it follows that

1. when \(a = 0\), the function \([f_a(x+1) - f_a(x)]\) is completely monotonic;
2. when \(a \geq \frac{1}{2}\), the function \([f_a(x) - f_a(x+1)]\) is completely monotonic.

Along with the corresponding argument as in the proof of Theorem 1, we obtain that the sufficient condition for \(f_a(x)\) or \(-f_a(x)\) to be completely monotonic on \((0, \infty)\) is \(a = 0\) or \(a \geq \frac{1}{2}\) respectively.

Conversely, if \(-f_a(x)\) is completely monotonic on \((0, \infty)\), then \(f_a(x)\) is increasing and negative on \((0, \infty)\), so

\[x^2 f_a(x) < 0\]

on \((0, \infty)\). From the double inequality

\[\frac{1}{2x^2} \leq \frac{1}{2} - \psi'(x+1) < \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5}\]

it is easy to see that

\[\lim_{x \to \infty} \{x^2[\psi'(x) - \frac{1}{x}]\} = \frac{1}{2}.\]

Using the asymptotic formula

\[\ln \Gamma(z) \sim (z - \frac{1}{2}) \ln z - z + \frac{1}{2} \ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \ldots\]

as \(z \to \infty\) in \(|\arg z| < \pi\), see [Abramowitz & Stegun, 1972], gives

\[\lim_{x \to \infty} \{x^2[\frac{1}{2} \ln(2\pi) - x + (x - \frac{1}{2}) \ln x - \ln \Gamma(x) + \frac{1}{12(x+a)}]\} = \lim_{x \to \infty} \{x^2[\frac{1}{12(x+a)} - \frac{1}{12x} + O(\frac{1}{x^2})]\} = \frac{a}{12}.\]

In virtue of (11) and the above limit, we obtain

\[x^2 f_a(x) = x^2[\frac{1}{2} \ln(2\pi) - x + (x - \frac{1}{2}) \ln x - \ln \Gamma(x) + \frac{1}{12(x+a)} + \frac{x^2}{12} \psi'(x+a) - \frac{1}{x+a}]\]

\[\to \frac{a}{12} + \frac{1}{12x} - \frac{1}{2}\]

as \(x\) tends to \(\infty\). So the necessary condition for \(-f_a(x)\) to be completely monotonic on \((0, \infty)\) is \(a \geq \frac{1}{2}\).

If \(f_a(x)\) for \(a > 0\) is completely monotonic on \((0, \infty)\), then \(-f_a(x)\) should be decreasing and positive on \((0, \infty)\), but utilizing (7) leads to

\[\lim_{x \to \infty} f_a(x) = \lim_{x \to \infty} \frac{1}{2} \ln(2\pi) - x + (x - \frac{1}{2}) \ln x - \ln \Gamma(x) + \frac{1}{12 \psi'(a)}\]
which leads to a contradiction. So the necessary condition for $f_0(x)$ to be completely monotonic on $(0,\infty)$ is $a=0$. The first proof of Theorem 2 is complete.

Proof. The second proof. The famous Binet’s first formula of $\ln \Gamma(x)$ for $x > 0$ is given by

$$\ln \Gamma(x) = \left(1 - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \theta(x),$$

where

$$\theta(x) = \int_0^x \frac{1}{e^t - 1} \frac{1}{t} e^{-t} dt$$

for $x > 0$ is called the remainder of Binet’s first formula for the logarithm of the gamma function $\Gamma(x)$ (Magnus et al. 1966; Qi & Guo, 2010c). Combining this with the integral representation

$$\psi^{(k)}(x) = (-1)^{k-1} \int_0^x t^k \frac{e^{-t}}{1-e^{-t}} dt$$

for $x > 0$ and $k \in \mathbb{N}$, (Abramowitz & Stegun, 1972), yields

$$f_a(x) = \frac{1}{12} \psi'(x + a) - \theta(x)$$

$$= \int_0^x \left[ -ie^{(a-1)i\pi} \left( \frac{1}{e^{t} - 1} - \frac{1}{t} \right) \right] e^{-it} dt. \quad (12)$$

It is not difficult to see that the positivity and negativity are equivalent to

$$a \leq -\frac{1}{t} \ln \left[ \frac{12(e^t-1)}{t^2 e^t} \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) \right] = -\phi(t),$$

where $\phi(t)$ is defined by (10). The rest proof is the same as in the first proof of Theorem 2. The second proof of Theorem 2 is complete.

4. Remarks

In this section, we list more results in the form of remarks.

Remark 1. From proofs of Theorem 1 and Theorem 2, we can abstract a general and much useful conclusion below.

Theorem 3 A function $f(x)$ defined on an infinite interval $I$ tending to $\infty$ is completely monotonic if and only if

1. there exist positive numbers $\alpha_i$ such that the differences $(-1)^i [f(x) - f(x + \alpha_i)]^{(i)}$ are nonnegative for all integers $i \geq 0$ on $I$;
2. the limits $\lim_{x \to \infty} [(-1)^i f^{(i)}(x)] = a_i \geq 0$ exist for all integers $i \geq 0$.

This is essentially a generalization of Guo et al. (2006); Guo & Qi (2011a); Guo & Qi (2010b); Guo & Qi (2010c); Qi (2007); Qi & Guo (2009) and Qi et al. (2010) which was also implicitly applied in Guo et al. (2010) and Qi & Guo (2010a), for example.

Remark 2. Because $f_a(x) = \frac{1}{2} \ln(2\pi) - x - F_a(x)$ and $f_a'(x) = -F_a'(x)$ on $(0,\infty)$, the concavity of $F_a(x)$ and the convexity of $F_a(x)$ for $a \geq \frac{1}{2}$ obtained in Merkle (1998) can be concluded readily from the complete monotonicity of $f_a(x)$ and $-f_a(x)$ on $(0,\infty)$ established in Theorems 1 and 2.

Remark 3. We also recall from Atanassov & Tsoukrovski (1988); Qi & Guo (2004); Qi et al. (2006); Qi et al. (2004) that a function $f(x)$ is said to be logarithmically completely monotonic on an interval $I$ if it has derivatives of all orders on $I$ and its logarithm $\ln(f(x))$ satisfies $0 \leq (-1)^k [\ln(f(x))]^{(k)} < \infty$ for all integers $k \geq 1$ on $I$. It was proved once again in Berg (2004); Guo & Qi (2010a); Qi & Chen (2004); Qi & Guo (2004); Qi et al. (2006) that logarithmically completely monotonic functions on an interval $I$ must be completely monotonic on $I$, but not conversely. For more information on the history and properties of logarithmically completely monotonic functions, please refer to Atanassov & Tsoukrovski (1988); Berg (2004); Guo & Qi (2010a); Qi (2010); Qi et al. (2010) and closely related references therein.

For $a \geq 0$ and $x > 0$, let

$$g_a(x) = -\ln \Gamma(x) + (x - \frac{1}{2}) \ln x - x + \frac{1}{12} \psi'(x + a).$$

It is obvious that

$$f_a(x) = \frac{1}{2} \ln(2\pi) + g_a(x)$$

on $(0,\infty)$ for $a \geq 0$, with the limit (8). It is not difficult to see that Theorem 1 in Alzer (1993) may be reworded as follows: For $0 < s < 1$ the function $\exp[g_a(x + s) - g_a(x + 1)]$ is logarithmically completely monotonic on $(0,\infty)$ if and only if $a \geq \frac{1}{2}$, and so is the function $\exp[g_a(x + 1) - g_a(x + s)]$ if and only if $a = 0$. This was reviewed in Qi (2010).

In virtue of complete monotonicity of $f_a(x)$ and
Remark 1, it follows that the difference
\[ f_a(x+s) - f_a(x+t) = g_a(x+s) - g_a(x+t) \]
for \( t > s \) and \( a \geq 0 \) is completely monotonic with respect to \( x \in (-s, \infty) \) if and only if \( a = 0 \), and so is its negative if and only if \( a \geq \frac{1}{2} \). Therefore, by the second item of Theorem 5 in Qi & Guo (2010b), it follows that the function \[ \exp\left[f_a(x+s) - f_a(x+t)\right] \text{ for } t > s \text{ and } a \geq 0 \]
is logarithmically completely monotonic with respect to \( x \) on \((-s, \infty)\) if and only if \( a \geq \frac{1}{2} \), and so is the function \[ \exp[f_a(x+t) - f_a(x+s)] \text{ if and only if } a = 0. \]
In other words, the function
\[
\frac{\Gamma(x+s)}{\Gamma(x+t)} \frac{(x+t)^{s+t-1/2}}{(x+s)^{s+t-1/2}} \exp[s-t] \left[ \frac{\psi'(x+t+\alpha) - \psi'(x+s+\alpha)}{12} \right]
\]
for \( s < t \) and \( \alpha \geq 0 \) is logarithmically completely monotonic with respect to \( x \in (-s, \infty) \) if and only if \( \alpha \geq \frac{1}{2} \), and so is the reciprocal of (13) if and only if \( \alpha = 0 \).

The monotonicity of (13) and its reciprocal implies that the double inequality
\[
\exp[t-s + \frac{\psi'(x+s+\beta) - \psi'(x+t+\beta)}{12}] \leq \frac{\Gamma(x+s)}{\Gamma(x+t)} \frac{(x+t)^{s+t-1/2}}{(x+s)^{s+t-1/2}} \leq \exp[t-s + \frac{\psi'(x+s+\alpha) - \psi'(x+t+\alpha)}{12}]
\]
for \( \alpha \geq \beta \geq 0 \), \( s < t \) and \( x \in (-s, \infty) \) is valid if and only if \( \beta = 0 \) and \( \alpha \geq \frac{1}{2} \).

5. Conclusion

In conclusion, from complete monotonicity in Theorem 2, Theorem 1 and Corollary 1 in Alzer (1993) together with Theorem 3 and its corollary in Li et al. (2006) may be deduced and extended straightforwardly. This means that Theorem 2 is stronger not only than Alzer (1993) but also than Li et al. (2006).

Remark 4. As a consequence of Theorem 2, the following double inequality is easily obtained: for \( x > 0 \), the double inequality
\[
\sqrt{2\pi} x^{x^{1/2}} \exp[\frac{\psi'(x+\beta) - x}{12}] < \frac{\Gamma(x)}{12} < \sqrt{2\pi} x^{x^{1/2}} \exp[\frac{\psi'(x+\alpha) - x}{12}]
\]
is valid if and only if \( \alpha = 0 \) and \( \beta \geq \frac{1}{2} \).

For more inequalities for bounding the gamma function \( \Gamma(x) \), please refer to Guo & Qi (2010c); Guo et al. (2008); Zhao et al. (2012a) and closely related references therein.

Remark 5. The equation (12) in the second proof of Theorem 2 tells us the integral representations of the completely monotonic functions \( f_a(x) \) and \(-f_a(x)\) for \( a \geq \frac{1}{2} \).

Remark 6. For the history, background, motivations, and recent developments of this topic, please refer to the survey and expository papers (Qi (2010); Qi (2014); Qi & Luo (2012); Qi & Luo (2013)) and plenty of references therein.

Remark 7. This paper is a revised version of the preprint (Qi, 2013).

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1.2.3

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خلاصة

تقدم في هذا البحث شروط ضرورية و كافية على الثابت $\alpha$ حتى تكون الدالة

$$\frac{1}{2} \ln(2\pi) - x + (x - \frac{1}{2}) \ln x - \ln \Gamma(x) + \frac{1}{12} \psi'(x + a)$$

ويكون سالبًا أيضًا تام الرابطة على $(0, \infty)$ حيث أن $0 \leq a$ هو عدد حقيقي، $\Gamma(x)$ هي دالة غاما الكلاسيكية و $\psi(x)$ هي دالة داي غاما. و كتطبيق لما وجدناه، نستخرج بعض النتائج المعروفة، و بعض المبانيات الجديدة.