GLOBAL BOUNDEDNESS OF SOLUTIONS TO A CHEMOTAXIS-FLUID SYSTEM WITH SINGULAR SENSITIVITY AND LOGISTIC SOURCE

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ABSTRACT. In this paper, we investigate the chemotaxis-fluid system with singular sensitivity and logistic source in bounded convex domain with smooth boundary. We present the global existence of very weak solutions under appropriate regularity assumptions on the initial data. Then, we show that system possesses a global bounded classical solution. Finally, we present a unique globally bounded classical solution for a fluid-free system. In addition, the asymptotic behavior of the solutions is studied, and our results generalize and improve some well-known results in the literature, and partially results are new.

1. Introduction. In this paper, we consider the following chemotaxis-fluid system with singular sensitivity and logistic source:

$$\begin{align*}
n_t + u \cdot \nabla n &= \Delta n - \chi \nabla \cdot (\frac{n}{v} \nabla v) + an - bn^\theta, \quad x \in \Omega, \; t > 0, \\
v_t + u \cdot \nabla v &= \Delta v - v + n, \quad x \in \Omega, \; t > 0, \\
u_t + \kappa (u \cdot \nabla) u &= \Delta u + \nabla P + n \nabla \phi, \quad \nabla \cdot u = 0, \quad x \in \Omega, \; t > 0, \\
\frac{\partial n}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \quad u = 0, \quad x \in \partial \Omega, \; t > 0, \\
n(x, 0) &= n_0(x), \quad v(x, 0) = v_0(x), \quad u(x, 0) = u_0(x), \quad x \in \Omega,
\end{align*}$$

(1.1)

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where $\Omega \subset \mathbb{R}^3$ is a bounded convex domain with smooth boundary $\partial \Omega$ and $\frac{\partial}{\partial \nu}$ denotes the derivative with respect to the outer normal of $\partial \Omega$. $n, v, u, P$ represent the cell density, the chemical attractant concentration, the fluid velocity field and the pressure of the fluid, respectively. $\chi, b, \theta > 0$, $a \geq 0$ are constants. $\kappa \in \{0, 1\}$, initial data $n_0, v_0, u_0$ are known functions satisfying

$$\begin{align*}
    n_0 &\in C^0(\overline{\Omega}) \text{ with } n_0 \geq 0 \text{ and } n_0 \not\equiv 0 \text{ on } \overline{\Omega}, \\
v_0 &\in W^{2,\infty}(\Omega) \text{ with } v_0 > 0 \text{ on } \overline{\Omega}, \\
u_0 &\in D(A') \text{ for some } \epsilon \in (\frac{3}{4}, 1),
\end{align*}$$

(1.2)

where $A$ represents the Stokes operator with domain $D(A') := W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \cap L^2(\Omega)$, where $L^2(\Omega) := \{\varphi \in L^2(\Omega) | \nabla \cdot \varphi = 0\}$. As for the time-independent gravitational potential function $\phi$, we assume for simplicity that

$$\phi \in W^{2,\infty}(\Omega).$$

(1.3)

The system (1.1) consists of two extensions of the classical Keller-Segel model [11]. One is the chemotaxis-(Navier)-Stokes model, the other is the chemotaxis model with singular sensitivity and logistic source. When the system (1.1) without logistic source, Black et al. [5] showed that the system (1.1) possesses a unique global-in-time classical solution in two- and three-dimensional bounded domains if $\chi < \sqrt{\frac{2}{N}}$ ($N = 2, 3$), it is unknown whether the solution is bounded or not. In order to understand the development of the system (1.1), let us mention some previous contributions in this direction.

Processes of directed movement of cells in response to a chemical stimulus, referred to as chemotaxis, play an important role in the interaction of cells with their environment. The pioneering works of the chemotaxis model was introduced by Keller and Segel in [11], describing aggregation of cellular slime mold toward a higher concentration of a chemical signal, which reads

$$\begin{align*}
u_t = \Delta u - \nabla \cdot (u \nabla v), & \quad x \in \Omega, \quad t > 0, \\
v_t = \Delta v + u - v, & \quad x \in \Omega, \quad t > 0,
\end{align*}$$

(1.4)

where $u$ denotes the cell density and $v$ represents the chemical concentration. The mathematical analysis of (1.4) and the variants thereof mainly concentrate on the boundedness and blow-up of the solutions [8, 20, 24]. As the blow-up has not been observed in the real biological process, many mechanisms, such as nonlinear porous medium diffusion, saturation effect, logistic source, may prevent the blow-up of solutions [16, 21, 22, 60, 61]. During the past four decades, classical Keller-Segel model and the variants has attracted extensive attention. And we refer the reader to the survey [1, 7] where a comprehensive information of further examples illustrating the outstanding biological relevance of chemotaxis can be found.

Keller and Segel [12] introduced a phenomenological model of wave-like solution behavior without any type of cell kinetics, a prototypical version of which is given by:

$$\begin{align*}
u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v), & \quad x \in \Omega, \quad t > 0, \\
u_t = \Delta v - \nu v, & \quad x \in \Omega, \quad t > 0,
\end{align*}$$

(1.5)

where $u$ represents the density of bacteria and $v$ denotes the concentration of the nutrient. The second equation models consumption of the signal. In the first equation, the chemotactic sensitivity is determined according to the Weber-Fechner law, which says that the chemotactic sensitivity is proportional to the reciprocal of...
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signal density. Winkler [47] proved that if initial data satisfies appropriate regularity assumptions, the system (1.5) possesses at least one global generalized solution in two dimensional bounded domains. Moreover, he took into account asymptotic behavior of solutions to the system (1.5), and proved that \( v(\cdot, t) \xrightarrow{t \to \infty} 0 \) in \( L^\infty(\Omega) \) and \( v(\cdot, t) \to 0 \) in \( L^p(\Omega) \) as \( t \to \infty \) provided \( \int_\Omega u_0 \leq m, -\int_\Omega \ln(\frac{u_0}{\|u_0\|_{L^\infty(\Omega)}}) \leq M \), where \( m, M \) are positive constants. When \( \Delta u \) is replaced by \( \Delta u^m (m \geq 1) \), Yan et al. [52] showed that if \( m > 1 + \frac{N-2}{2} \) (\( N \geq 2 \)), the corresponding Neumann initial-boundary value problem admits a global generalized solution. When \( v \) does not stand for a nutrient to be consumed but a signalling substance produced by the bacteria themselves, i.e. the second equation is replaced by \( v_t = \Delta v - v + u \), which is given by:

\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (\frac{\mu}{\rho} \nabla v), \quad x \in \Omega, \quad t > 0, \\
v_t &= \Delta v - v + u, \quad x \in \Omega, \quad t > 0.
\end{align*}
\]

Winkler and Yokota [50] proved that the system (1.6) possesses a uniquely determined global classical solution if \( \chi \in (0, \chi_0) \) and \( \chi^2 \leq \delta \), where \( \chi_0 \in (0, \frac{\sqrt{2}}{N}) \), \( \delta > 0 \) are constants. Furthermore, the solution of (1.6) converges to the homogeneous steady state \((\bar{u}_0, \bar{u}_0)\) at an exponential rate with respect to the norm in \( (L^\infty(\Omega))^2 \) as \( t \to \infty \), where \( \bar{u}_0 = \frac{1}{|\Omega|} \int_\Omega u_0 \). Zheng et al. [64] showed that system (1.6) admits a generalised supersolution with appropriate condition, and assumed that \((u, v)\) satisfies additional conditions and that the generalised supersolution is a classical solution. When the system (1.6) has a logistic source \( ru - mu^2 \), Zhao and Zheng [56] proved that the system (1.6) exists globally bounded classical solutions if \( r > \frac{\chi}{2} \) for \( 0 < \chi \leq 2 \), or \( r > \chi - 1 \) for \( \chi > 2 \) in two-dimensional bounded domains. Based on existing results in [55], Zheng et al. [63] showed that the global bounded solution \((u, v)\) exponentially converges to the steady state \((\bar{u}, \bar{u})\). Zhao and Zheng [56] generalized their own work [55] to the higher dimensional case.

The most extensively studied chemotaxis model is probably the classical Keller-Segel system. From the viewpoint of biology, models of types (1.4)-(1.6) ignore the interaction between the cells or chemicals and the surroundings in which they live. Taking this into account, Tuval et al. proposed in [32] for the spatio-temporal evolution in populations of oxytactically moving bacteria that interact with a surrounding fluid through transport and buoyancy. The corresponding model is given by:

\[
\begin{align*}
    n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \chi(n, v) \cdot \nabla v), \quad x \in \Omega, \quad t > 0, \\
v_t + u \cdot \nabla v &= \Delta v - n f(v), \quad x \in \Omega, \quad t > 0, \\
u_t + \kappa (u \cdot \nabla) u &= \Delta u + \nabla P + n \nabla \phi, \quad \nabla \cdot u = 0, \quad x \in \Omega, \quad t > 0.
\end{align*}
\]

When \( \Delta n \) is replaced by \( \Delta n^m \) \((m \geq 1)\), Winkler’s recent analysis [44] revealed that for \( m > \frac{7}{6} \) all reasonably regular initial data, a corresponding no-flux Neumann initial-boundary value problem possesses a globally defined weak solution, which is bounded in three-dimensional bounded convex domains. Tao et al. [27] assured global solvability within the larger range \( m > \frac{9}{8} \), but only in a class of weak solutions locally bounded in \( \Omega \times [0, \infty) \). Based on energy-based arguments and maximal Sobolev regularity theory, Winkler’s result in [48], which allows for the construction of global weak solution to an associated initial-boundary value problem under the milder assumption that \( m > \frac{9}{8} \). Moreover, it is shown that such solution stabilizes to spatially homogeneous state \((\frac{1}{|\Omega|} \int_\Omega n_0, 0, 0)\) in the large time limit.
\((\frac{\chi}{\eta})' > 0, (\frac{\chi}{\eta})'' \leq 0\) and \(\chi \cdot f)' \geq 0\), Winkler \[46\] proved that the system (1.7) possesses a global weak solution in three-dimensional bounded convex domains. Moreover, he asserted that this solution stabilizes to the spatially uniform equilibrium \((\frac{\chi}{\eta})' > 0, (\frac{\chi}{\eta})'' \leq 0\) and \(\chi \cdot f)' \geq 0\). When \((\frac{\chi}{\eta})' > 0, (\frac{\chi}{\eta})'' \leq 0\) and \(\chi \cdot f)' \geq 0\). When \((\frac{\chi}{\eta})' > 0, (\frac{\chi}{\eta})'' \leq 0\) and \(\chi \cdot f)' \geq 0\). Winkler \[43\] showed that the system (1.7) possesses a unique global-in-time classical solution in two-dimensional bounded convex domains. Moreover, he asserted that this solution stabilizes to the spatially uniform equilibrium \((\frac{\chi}{\eta})' > 0, (\frac{\chi}{\eta})'' \leq 0\) and \(\chi \cdot f)' \geq 0\). Winkler \[46\] proved that the system (1.7) has at least one global weak solution in three-dimensional bounded convex domains. When \(\Delta \alpha > \frac{1}{2}\), the system (1.7) has a logistic source \(\kappa n\) and \(\alpha > \frac{1}{2}\), \(S(n) = \frac{n}{1 + n}\), the system (1.7) has a logistic source \(\kappa n \cdot \mu n\). Wang \[38\] proved that the system (1.7) has at least one global weak solution. When \(\Delta n\) is replaced by \(\nabla \cdot (|\nabla n|^{p-2} \nabla n)\), Tao et al. \[31\] proved that if \(p > \frac{32}{15}\) and appropriate structural assumptions on \(f\) and \(\chi\), for all sufficiently smooth initial data \((n_0, v_0, u_0)\), the model (1.7) possesses at least one global weak solution. When the system (1.7) has a logistic source \(\kappa n - \mu n^2\), \(\Delta n\) is replaced by \(\Delta n^m\). Kurima et al. \[13\] proved that for any \(m > 0\), the system (1.7) has a global weak solution. Very recently, when \(\Delta n\) is replaced by \(\nabla \cdot (|\nabla n|^{p-2} \nabla n)\), Tao et al. \[31\] proved that if \(p > \frac{32}{15}\) and appropriate structural assumptions on \(f\) and \(\chi\), for all sufficiently smooth initial data \((n_0, v_0, u_0)\), the model (1.7) possesses at least one global weak solution. When \(\Delta n\) is replaced by \(\Delta n^m\). Kurima et al. \[13\] proved that for any \(m > 0\), the system (1.7) has a global weak solution.

As in the classical Keller-Segel model, where the chemoattractant is produced rather than consumed by bacteria, the relevant Keller-Segel-Navier-Stokes system with rotational effect of the form is given by:

\[
\begin{aligned}
&n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n S(x, n, v) \cdot \nabla v), \quad x \in \Omega, \quad t > 0, \\
&c_t + u \cdot \nabla c = \Delta c - c + n, \quad x \in \Omega, \quad t > 0, \\
&u_t + \kappa (u \cdot \nabla)u = \Delta u + \nabla P + n \nabla \phi, \quad \nabla \cdot u = 0, \quad x \in \Omega, \quad t > 0.
\end{aligned}
\] (1.8)

Compared with (1.7), the mathematical analysis of (1.8) is relatively sparse. When \(S(x, n, v)\) is a tensor-valued sensitivity satisfying some dampening condition, such as \(|S(x, n, v)| \leq C_S(1 + n)^{-\alpha}\), Wang and Xiang \[39\] obtained global existence and boundedness in a Keller-Segel-Stokes system \((\kappa = 0)\) in two-dimensional smoothly bounded domains, to the best of our knowledge, which is the first result on global existence and boundedness in a Keller-Segel-Stokes system with tensor-valued sensitivity. With the same author \[40\], when \(\alpha > \frac{1}{2}\), they also obtained global classical solutions which are uniformly bounded in three-dimensional smoothly bounded domains. Parallel to the case of the corresponding Keller-Segel-Navier-Stokes system, Liu and Wang \[18\] proved that the system (1.8) admits at least one global weak solution with \(\alpha \geq \frac{1}{3}\) in three-dimensional smoothly bounded domains. Wang \[36\] proved the system (1.8) possesses at least one global very weak solution with \(\alpha > \frac{1}{3}\) in three-dimensional smoothly bounded domains. When \(\Delta n\) is replaced by \(\Delta n^m\) \((m \geq 1)\), Zheng \[59\] showed that if \(m > 2\), for all reasonably regular initial data, a corresponding initial boundary value problem possesses a globally weak solution with \(S(x, n, v) \equiv 1\). More recently, when \(S(x, n, v) \equiv 1\), Black \[3\] proved that if \(m > \frac{5}{3}\), the system (1.8) possesses at least one global very weak solution. Moreover, if \(m > \frac{5}{3}\), the system (1.8) admits at least one global weak solution. Black \[4\] improved his previous work \[3\] and showed that if \(m + \alpha > \frac{4}{3}\), system (1.8) possesses at least one global very weak solution. Moreover, if \(m + 2\alpha > \frac{5}{3}\), the system (1.8) admits at least one global weak solution. When the system (1.8) has
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a logistic source \( r n - \mu n^2 \), Tao and Winkler [28] showed that the corresponding initial-boundary problem possesses a global classical solution which is bounded in three-dimensional smoothly bounded domains with \( \kappa = 0 \) under the explicit condition \( \mu \geq 23 \) and suitable regularity assumption on the initial data. Apart from this, they also proved that if \( r = 0 \), then both \( n(\cdot, t) \) and \( v(\cdot, t) \) decay to zero with respect to the norm in \( L^\infty \) as \( t \to \infty \), and that if moreover \( \int_0^\infty \int_\Omega |g|^2 < 0 \), they also showed that \( u(\cdot, t) \to 0 \) in \( L^\infty \) as \( t \to \infty \). In two-dimensional smoothly bounded domains, Tao and Winkler [30] proved that the system (1.8) with \( \kappa = 1 \) possesses a global classical bounded solution when \( \mu > 0 \), and also got the same large time behavior.

Liu et al. [17] showed that under the conditions \( m \geq \frac{1}{3} \) and \( \alpha > \frac{6}{5} - m \), and proper regularity hypotheses on the initial data, the corresponding initial-boundary problem possesses at least one global bounded weak solution for the Keller-Segel-Stokes system with nonlinear diffusion and logistic source in the three-dimensional bounded domains. Jin [9] improved the results in [17], and established the global existence and boundedness of weak solutions for any \( m > 0 \) and \( \alpha > 0 \). When the system controlled by a given external force \( g \), Zheng [58] revealed that \( m > \max\{\frac{6}{5} - \alpha, \frac{1}{3}\} \) and \( \alpha > 0 \), for all reasonably regular initial data, a corresponding no-flux Neumann initial-boundary value problem possesses at least one global weak solution. And as to other related studies of system (1.8), we recommend that readers refer to the literature [51].

To the best of our knowledge, the global very weak solution of system (1.1) remains under-explored. Motivated by the arguments in previous studies [5, 28, 37, 49, 56], we mainly revealed that the system (1.1) admits a global bounded very weak solution and classical solution for different conditions. Finally, we show that fluid-free system possesses a unique global bounded classical solution. In addition, the asymptotic behavior of the solutions is studied. Theorem 1.5 partially generalizes and improves Theorem 1.1 in [6, 57].

Now, we state the main results of this paper.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded convex domain with smooth boundary. \( b > 0, \kappa = 1, \phi \) satisfies (1.3). If one of the following holds:

(i) \( \theta \in (\frac{8}{5}, 2], a, \chi > 0 \) fulfill

\[
a > \begin{cases} \frac{\chi^2}{4}, & 0 < \chi \leq 2, \\ \max\{\frac{\chi^2}{4}(1 - \tilde{p}^2), \chi - 1\}, & \chi > 2. \end{cases} \tag{1.9}
\]

where \( \tilde{p} := \frac{4(\theta - 1)}{8\theta(2 - \theta)\theta^2} \).

(ii) \( \theta > 2 \) and \( a, \chi > 0 \) fulfill

\[
\chi < \sqrt{\min\left\{\frac{2a(1 + a)}{\theta}, \frac{4}{\theta(\theta - 1)(\theta - 2)}\right\}}. \tag{1.10}
\]

Then for any choice of the initial data \((n_0, v_0, u_0)\) satisfying (1.2), the system (1.1) admits a global very weak solution in the sense of Definition 2.3. Moreover, if exist \( \varsigma_1, \varsigma_2 > 0 \) such that \( \frac{a}{2} < \varsigma_1 \) and \( \|n_0\|_{L^2(\Omega)} + \|\nabla v_0\|^2_{L^2(\Omega)} < \varsigma_2 \), then the very weak solution is globally bounded.

**Remark 1.2.** When the fluid equation add to nonzero external force \( f \), assume that \( f \in C^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^3) \), and that moreover \( f \) has the bounded property, the conclusion also holds.
Theorem 1.3. Let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain with smooth boundary. $\phi$ satisfy (1.3). Assume that $\kappa = 0$, $\theta > \frac{15}{7}$ and $a, \chi > 0$ fulfill (1.10). Then for any choice of the initial data $(n_0, v_0, u_0)$ satisfying (1.2), the system (1.1) possesses a global bounded classical solution.

Remark 1.4. Similar to [10], When the first equation of (1.1) add to nonnegative source term $g(x, t)$, $g, \nabla \phi \in C^{\alpha, \frac{2}{3}}(\Omega \times \mathbb{R})$ with $\alpha \in (0, 1)$, $g$ and $\phi$ are time periodic functions with periodicity $T$. Then we conjecture the problem (1.1) admits a time periodic classical solution $(n, v, u)$.

Theorem 1.5. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and assume that the initial data $(n_0, v_0)$ fulfilling (1.2). If one of the following holds:

(i) $\theta \in \left(\frac{8}{5}, 2\right], 0 < \chi < \frac{\sqrt{2}}{3}$, $a > 0$ fulfill

$$a > \begin{cases} \frac{\chi^2}{4}, & 0 < \chi \leq 2, \\ \max\left\{\frac{\chi^2}{4}(1 - \tilde{p}^2), \chi - 1\right\}, & \chi > 2. \end{cases}$$

(ii) $\theta > 2$ and $a > 0$ fulfill

$$\chi < \sqrt{\min\left\{\frac{2}{3}, \frac{2(1 + a)}{\theta \theta (\theta - 1)(\theta - 2)}\right\}}.$$ (1.12)

Then the system (5.1) admits a unique global bounded classical solution.

Remark 1.6. Theorem 1.5 partially generalizes and improves Theorem 1.1 in [6, 57].

Theorem 1.7. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, $a > 0$, $\theta \geq 2$ and $(n, v)$ be the solution of (5.1). Then there exists $b^* > 0$ with the property that if $b > b^*$,

one can find $C > 0$ and $\eta > 0$ such that for all $t > 0$,

$$\|u(\cdot, t) - \left(\frac{a}{b}\right)^{\frac{1}{\theta - 1}}\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \left(\frac{a}{b}\right)^{\frac{1}{\theta - 1}}\|_{L^\infty(\Omega)} \leq Ce^{-\frac{\eta t}{2}}.$$
**Definition 2.1.** Let $T > 0$. A triplet $(n, v, u)$ of functions
\[ n \in L^1(\Omega \times (0, T)), \quad v \in L^1((0, T); W^{1,1}(\Omega)), \quad u \in L^1((0, T); W^{1,1}_0(\Omega; \mathbb{R}^3)), \]
will be called a very weak subsolution to (1.1) in $\Omega \times (0, T)$ if
\[ n^\theta \in L^1(\Omega \times (0, T)), \quad \frac{n}{v} \nabla v \in L^1(\Omega \times (0, T); \mathbb{R}^3) \]
as well as
\[ uv \in L^1(\Omega \times (0, T); \mathbb{R}^N), \quad u \otimes u \in L^1(\Omega \times (0, T); \mathbb{R}^{N \times 3}) \]
and moreover
\[
- \int_0^T \int_\Omega n \psi_t - \int_\Omega n_0 \psi(\cdot, 0) \leq \int_0^T \int_\Omega n \Delta \psi + \chi \int_0^T \int_\Omega \frac{n}{v} \nabla v \cdot \nabla \psi + a \int_0^T \int_\Omega n \psi
\]
\[ + \int_0^T \int_\Omega n(u \cdot \nabla \psi) - b \int_0^T \int_\Omega n^\theta \psi, \quad (2.1) \]
holds for all $\psi \in C^\infty_0(\Omega \times (0, T))$ with $\psi \geq 0$ and $\frac{\partial \psi}{\partial \nu} = 0$ on $\partial \Omega \times (0, T)$, as well as
\[- \int_0^T \int_\Omega v \psi_t - \int_\Omega v_0 \psi(\cdot, 0) = - \int_0^T \int_\Omega \nabla v \cdot \nabla \psi - \int_0^T \int_\Omega v \psi + \int_0^T \int_\Omega n \psi
\]
\[ + \int_0^T \int_\Omega uv \cdot \nabla \psi \quad (2.2) \]
holds for all $\psi \in L^\infty(\Omega \times (0, T)) \cap L^2((0, T); W^{1,2}(\Omega))$ with $\psi_t \in L^2(\Omega \times (0, T))$, and
\[- \int_0^T \int_\Omega u \psi_t - \int_\Omega u_0 \psi(\cdot, 0) = - \int_0^T \int_\Omega \nabla u \cdot \nabla \psi + \kappa \int_0^T \int_\Omega (u \otimes u) \cdot \nabla \psi
\]
\[ + \int_0^T \int_\Omega n \nabla \phi \cdot \psi \quad (2.3) \]
is satisfied for all $\psi \in C^\infty_0(\Omega \times (0, T); \mathbb{R}^3)$ with $\nabla \psi \equiv 0$ in $\Omega \times (0, T)$.

**Definition 2.2.** Let $T > 0$ and $\gamma \in (0, 1)$. A triplet of functions $(n, v, u)$
\[ n \in L^{\gamma+1}(\Omega \times (0, T)), \quad v \in L^1((0, T); W^{1,1}(\Omega)), \quad u \in L^1(\Omega \times (0, T); \mathbb{R}^3) \]
is said to be a weak $\gamma$-entropy supersolution of (1.1) in $\Omega \times (0, T)$ if
\[ n^{\gamma-2} \lvert \nabla n \rvert^2, \quad n^{\gamma} \frac{\lvert \nabla v \rvert}{v^2} \in L^1(\Omega \times (0, T)) \]
and moreover
\[
- \int_0^T \int_\Omega n^{\gamma} \psi_t - \int_\Omega n_0^{\gamma} \psi(\cdot, 0)
\]
\[ \geq \gamma(1 - \gamma) \int_0^T \int_\Omega n^{\gamma-2} \lvert \nabla n \rvert^2 \psi + \int_0^T \int_\Omega n^{\gamma} \Delta \psi + \chi \gamma (\gamma - 1) \int_0^T \int_\Omega n^{\gamma-1} \nabla n \cdot \frac{\nabla v}{v} \psi
\]
\[ + a \gamma \int_0^T \int_\Omega n^{\gamma} \psi + \chi \gamma \int_0^T \int_\Omega n^{\gamma} u \cdot \nabla \psi - b \gamma \int_0^T \int_\Omega n^{\gamma-1+\theta} \psi \quad (2.4) \]
is valid for all $\psi \in C^\infty_0(\Omega \times (0, T))$ with $\psi \geq 0$ and $\frac{\partial \psi}{\partial \nu} = 0$ on $\partial \Omega \times (0, T)$ and (2.2), (2.3) hold.
Definition 2.3. Let $T > 0$. A triplet $(n,v,u)$ of functions is called a very weak solution of (1.1) in $\Omega \times (0,T)$ if it is both a very weak subsolution and a weak $\gamma$-entropy supersolution to (1.1) in $\Omega \times (0,T)$ for some $\gamma \in (0,1)$. A global very weak solution of (1.1) is a triplet $(n,v,u)$ of functions defined in $\Omega \times (0,\infty)$ which is a very weak solution to (1.1) in $\Omega \times (0,T)$ for all $T > 0$.

Remark 2.4. Analogous to the reasoning of Theorem 2.5 in [15], multiplying the inequality $nt+u \cdot \nabla n \geq \Delta n - \chi \nabla \cdot (\frac{n}{\epsilon} \nabla v) + an - bn^\theta$ by $\gamma n^{\gamma - 1} \psi$, the above definition in accordance with classical solution if $(n,v,u)$ is a global very weak solution in the above sense which additionally fulfills $(n,v,u) \in (C^0(\Omega \times [0,\infty)) \cap C^{2,1}(\Omega \times (0,\infty)))^3$, then $(n,v,u)$ already solves (1.1) classically in $\Omega \times (0,\infty)$.

In order to construct such very weak solutions by an approximation procedure, we introduce the following regularized problems

$$
\begin{aligned}
\begin{cases}
  n_{\epsilon} + u_{\epsilon} \cdot \nabla n_{\epsilon} &= \Delta n_{\epsilon} - \chi \nabla \cdot (\frac{n_{\epsilon}}{(1+\epsilon n_{\epsilon})^3} \nabla v_{\epsilon}) + an_{\epsilon} - bn_{\epsilon}^\theta, & x \in \Omega, \ t > 0, \\
  v_{\epsilon} + u_{\epsilon} \cdot \nabla v_{\epsilon} &= \Delta v_{\epsilon} - n_{\epsilon} + n_{\epsilon}, & x \in \Omega, \ t > 0, \\
  u_{\epsilon} + \kappa (Y_{\epsilon} u_{\epsilon} \cdot \nabla) u_{\epsilon} &= \Delta u_{\epsilon} + \nabla P_{\epsilon} + n_{\epsilon} \nabla \phi_{\epsilon}, \ n_{\epsilon} = 0, & x \in \Omega, \ t > 0, \\
  \frac{\partial n_{\epsilon}}{\partial \nu} &= 0, \ u_{\epsilon} = 0, & x \in \partial \Omega, \ t > 0, \\
  n_{\epsilon}(x,0) &= n_0(x), \ v_{\epsilon}(x,0) = v_0(x), \ u_{\epsilon}(x,0) = u_0(x), & x \in \Omega,
\end{cases}
\end{aligned}
$$

where $Y_\omega := (1 + \epsilon A)^{-1} \omega$ for all $\omega \in L^1_{\Omega}$. According to the well-established fixed point arguments, the local solvability of (2.5) can be obtained, the proof is similar to [4, 28, 30, 56], so here we omit the proof.

Lemma 2.5. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, $\theta > 1$ and $q > 3$. If initial data $(n_0,v_0,u_0)$ satisfying (1.2), $\phi$ satisfies (1.3). Then for each $\epsilon \in (0,1)$, there exist $T_{\max,\epsilon} \in (0,\infty)$ and a classical solution $(n_{\epsilon},v_{\epsilon},u_{\epsilon},P_{\epsilon})$ such that

$$
\begin{aligned}
\begin{cases}
  n_{\epsilon} \in C^0(\Omega \times [0,T_{\max,\epsilon})) \cap C^{2,1}(\Omega \times (0,T_{\max,\epsilon})), \\
  v_{\epsilon} \in C^0(\Omega \times [0,T_{\max,\epsilon})) \cap C^{2,1}(\Omega \times (0,T_{\max,\epsilon})) \cap L^\infty([0,T_{\max,\epsilon}); W^{1,q}(\Omega)), \\
  u_{\epsilon} \in C^0(\Omega \times [0,T_{\max,\epsilon}); \mathbb{R}^3) \cap C^{2,1}(\Omega \times (0,T_{\max,\epsilon}); \mathbb{R}^3), \\
  P_{\epsilon} \in C^{1,0}(\Omega \times (0,T_{\max,\epsilon})),
\end{cases}
\end{aligned}
$$

where $T_{\max,\epsilon}$ denotes the maximal existence time. Also, the above solution is unique up to addition of spatially constant to the pressure $P_{\epsilon}$. Moreover, we have $n_{\epsilon} > 0$, and $v_{\epsilon} > 0$ in $\Omega \times [0,T_{\max,\epsilon})$ and if $T_{\max,\epsilon} < +\infty$, then

$$
n_{\epsilon}(\cdot,t)||_{L^\infty(\Omega)} + ||v_{\epsilon}(\cdot,t)||_{W^{1,q}(\Omega)} + ||u_{\epsilon}(\cdot,t)||_{L^2(\Omega)} \to \infty \quad (2.7)
$$

as $t \to T_{\max,\epsilon}$, where $\epsilon$ is taken from (1.2).

By the comparison principle with the positivity of $n_{\epsilon}$, we know that $v_{\epsilon} \geq \inf_{x \in \Omega} v_0(x)e^{-t} =: \sigma_0(t)$ for all $(x,t) \in \Omega \times (0,T_{\max,\epsilon})$ and $\epsilon \in (0,1)$. (2.8)

Finally, we will give useful lemmas, which are important to prove the main theorems.

Lemma 2.6. Suppose that $(n_{\epsilon},v_{\epsilon},u_{\epsilon})$ is a classical solution to (2.5). Then for each $\epsilon \in (0,1)$, $\theta > 1$, we have

$$
\int_{\Omega} n_{\epsilon}(\cdot,t) \leq m := \max\left\{ \int_{\Omega} n_0, \left(\frac{a}{b}\right)^{\frac{1}{\theta-1}} \right\} \text{ for all } t \in (0,T_{\max,\epsilon}), \quad (2.9)
$$
\[
\int_{\Omega} v_\varepsilon(\cdot, t) \leq m_1 := \max \left\{ \int_{\Omega} v_0, \ m \right\} \text{ for all } t \in (0, T_{\text{max}, \varepsilon}) \quad (2.10)
\]

and
\[
\int_{0}^{T} \int_{\Omega} n_\varepsilon^\theta(x, s) \, dx \, ds \leq \frac{L_1}{b}(1 + t) \text{ for all } t \in (0, T_{\text{max}, \varepsilon}) \quad (2.11)
\]

where \( L_1 := m \cdot \max\{1, a\} \).

**Proof.** Integrating the first equation in (2.5) over \( \Omega \) and using Hölder inequality, we have
\[
\frac{d}{dt} \int_{\Omega} n_\varepsilon = a \int_{\Omega} n_\varepsilon - b \int_{\Omega} n_\varepsilon^\theta \leq a \int_{\Omega} n_\varepsilon - \frac{b}{|\Omega|} \left( \int_{\Omega} n_\varepsilon \right)^\theta \quad (2.12)
\]

for all \( t \in (0, T_{\text{max}, \varepsilon}) \), which yields (2.9) on an ODE comparison arguments. The estimate (2.11) comes from by integrating the first identity in (2.12) from 0 to \( t \) and (2.9). We integrate the second equation in (2.7) and use (2.9), we obtain
\[
\frac{d}{dt} \int_{\Omega} v_\varepsilon + \int_{\Omega} v_\varepsilon = \int_{\Omega} n_\varepsilon \leq m \text{ for all } t \in (0, T_{\text{max}, \varepsilon}),
\]

from which (2.8) follows by comparison arguments. This completes the proof. \( \square \)

**Lemma 2.7** (Gagliardo-Nirenberg interpolation inequality [23]). Let \( \Omega \in \mathbb{R}^N \) be a bounded domain with smooth boundary. Assume \( p, q \in [1, \infty] \), \( r \in (0, p) \) with \( p < \infty \) for \( q = N \) and \( p \leq \frac{qN}{N-q} \) for \( q < N \). Then, for \( \theta \in (0, 1] \) given by:
\[
-\frac{N}{p} = (1 - \frac{N}{q})\theta - \frac{N}{r}(1 - \theta)
\]

and some \( C > 0 \), we have
\[
\|z\|_{L^p(\Omega)} \leq C_{GN} \|\nabla z\|_{L^q(\Omega)}^\theta \|z\|_{L^r(\Omega)}^{1-\theta} + C_{GN} \|z\|_{L^r(\Omega)}^{1-\theta}
\]

and
\[
\|z\|_{L^p(\Omega)} \leq C_{GN} \|z\|_{W^{1,q}(\Omega)}^\theta \|z\|_{L^r(\Omega)}^{1-\theta}
\]

for any \( z \in W^{1,q}(\Omega) \cap L^r(\Omega) \).

**Lemma 2.8** ([49]). Let \( T > 0 \) and \( g \in C^0([0, T]) \cap C^1(0, T) \) be such that
\[
g'(t) + ay(t) \leq g(t) \text{ for all } t \in (0, T),
\]

where \( g \in L^1_{\text{loc}}(\mathbb{R}) \) has the property that
\[
\frac{1}{\tau} \int_{t}^{t+\tau} g(s) \, ds \leq b \text{ for all } t \in (0, T)
\]

with some \( \tau > 0 \) and \( b > 0 \). Then
\[
g(t) \leq g(0) + \frac{bt}{1 - e^{-at}} \text{ for all } t \in [0, T).
\]

**Lemma 2.9** ([42]). Let \( \Omega \subset \mathbb{R}^N \) (\( N \in \mathbb{N} \)) be a bounded domain with smooth boundary and let \( e^{t\Delta} \) be the Neumann heat semigroup in \( \Omega \). Then there exist constants \( C, \lambda_1 > 0 \) depending only on \( \Omega \) such that
(i) if \( 1 \leq q \leq p \leq \infty \), then
\[
\|e^{t\Delta} \psi\|_{L^p(\Omega)} \leq C(1 + t^{-\frac{N}{2} \left(\frac{1}{q} - \frac{1}{p}\right)}) e^{-\lambda_1 t} \|\psi\|_{L^q(\Omega)}
\]

holds for all \( t > 0 \) and each \( \psi \in L^q(\Omega) \) and \( \int_{\Omega} \psi = 0 \).
(ii) if \( 1 \leq q \leq p \leq \infty \), then
\[
\|\nabla e^{t\Delta} \psi\|_{L^p(\Omega)} \leq C(1 + t^{-\frac{1}{2} - \frac{N}{q} \left(\frac{1}{q} - \frac{1}{p}\right)}) e^{-\lambda_1 t} \|\psi\|_{L^q(\Omega)}
\]
holds for all $t > 0$ and each $\psi \in L^q(\Omega)$.

(iii) if $2 \leq p < \infty$, then
\[ \| \nabla e^{t\Delta} \psi \|_{L^p(\Omega)} \leq C e^{-\lambda_2 t} \| \nabla \psi \|_{L^p(\Omega)} \]
is valid for all $\psi \in W^{1,p}(\Omega)$.

3. The proof of Theorem 1.1. We already know from the Section 2 that $v_\varepsilon$ has a lower bound on time. However, for the sake of proving the global boundedness of very weak solution to (1.1), we should set up a positive uniform-in-time lower bound of $v_\varepsilon$ for all $\varepsilon \in (0,1)$. Inspired by [55, 56, 57], we have the following lemma.

Lemma 3.1. (1) Under the assumptions of Theorem 1.1(i), there exists a constant $\sigma_1 > 0$ such that
\[ v_\varepsilon(x,t) \geq \sigma_1 \text{ for all } t \in (0,T_{\text{max},\varepsilon}) \text{ and } x \in \Omega. \]  \hspace{1cm} (3.1)

(2) Under the assumptions of Theorem 1.1(ii), there exists a constant $\sigma_2 > 0$ such that
\[ v_\varepsilon(x,t) \geq \sigma_2 \text{ for all } t \in (0,T_{\text{max},\varepsilon}) \text{ and } x \in \Omega. \]  \hspace{1cm} (3.2)

Proof. Let $p, q > 0$. By the straightforward calculation we have
\[
\frac{d}{dt} \int_\Omega n_\varepsilon^{-p} v_\varepsilon^{-q} dx = -p \int_\Omega n_\varepsilon^{-p-1} v_\varepsilon^{-q}[\Delta n_\varepsilon - \chi \nabla \cdot \left( \frac{n_\varepsilon}{(1+\varepsilon n_\varepsilon)^3} \nabla v_\varepsilon \right) - u_\varepsilon \cdot \nabla v_\varepsilon] dx \\
- a \int_\Omega n_\varepsilon^{-p} v_\varepsilon^{-q} + b \int_\Omega n_\varepsilon^{-p-1} - q \int_\Omega n_\varepsilon^{-p} v_\varepsilon^{-q-1} (\Delta v_\varepsilon - u_\varepsilon \cdot \nabla v_\varepsilon) \\
+ q \int_\Omega n_\varepsilon^{-p} v_\varepsilon^{-q} - q \int_\Omega n_\varepsilon^{-p} v_\varepsilon^{-q-1} \\
= -p(p+1) \int_\Omega n_\varepsilon^{-p-2} v_\varepsilon^{-q} |\nabla n_\varepsilon|^2 + bp \int_\Omega n_\varepsilon^{-p-1} v_\varepsilon^{-q} \\
+ \left[ \frac{p(p+1)}{(1+\varepsilon n_\varepsilon)^3} - 2pq \right] \int_\Omega n_\varepsilon^{-p-1} v_\varepsilon^{-q-1} \nabla n_\varepsilon \cdot \nabla v_\varepsilon \\
+ \left[ \frac{pq\chi}{(1+\varepsilon n_\varepsilon)^3} - q(q+1) \right] \int_\Omega n_\varepsilon^{-p} v_\varepsilon^{-q-2} |\nabla v_\varepsilon|^2 + (q - ap) \int_\Omega n_\varepsilon^{-p} v_\varepsilon^{-q} \\
+ p \int_\Omega n_\varepsilon^{-p-1} v_\varepsilon^{-q} u_\varepsilon \cdot \nabla n_\varepsilon - q \int_\Omega n_\varepsilon^{-p+1} v_\varepsilon^{-q-1} + q \int_\Omega n_\varepsilon^{-p} v_\varepsilon^{-q-1} u_\varepsilon \cdot \nabla v_\varepsilon \] \hspace{1cm} (3.3)

for all $t \in (0,T_{\text{max},\varepsilon})$. Noticing
\[
- p \int_\Omega n_\varepsilon^{-p-1} v_\varepsilon^{-q} u_\varepsilon \cdot \nabla n_\varepsilon + q \int_\Omega n_\varepsilon^{-p} v_\varepsilon^{-q-1} u_\varepsilon \cdot \nabla v_\varepsilon \\
= - \int_\Omega v_\varepsilon^{-q} u_\varepsilon \cdot \nabla n_\varepsilon - \int_\Omega n_\varepsilon^{-p} v_\varepsilon^{-q} = \int_\Omega n_\varepsilon^{-p} v_\varepsilon^{-q}(\nabla \cdot u_\varepsilon) = 0, \hspace{1cm} (3.4)
\]
together with Young’s inequality, we obtain
\[
\frac{d}{dt} \int_\Omega n_\varepsilon^{-p} v_\varepsilon^{-q} dx \\
\leq \left\{ \frac{p}{4(p+1)} \left[ \frac{(p+1)\chi}{(1+\varepsilon n_\varepsilon)^3} - 2q \right]^2 + \frac{pq\chi}{(1+\varepsilon n_\varepsilon)^3} - q(q+1) \right\} \int_\Omega n_\varepsilon^{-p} v_\varepsilon^{-q-2} |\nabla v_\varepsilon|^2
\]
for all $t \in (0, T_{\text{max}, \varepsilon})$. Let

$$h_1(q; p, \chi, n_\varepsilon) := \frac{p}{4(p+1)} \left[ \frac{(p+1)\chi}{(1+\varepsilon n_\varepsilon)^3} - 2q \right]^2 + \frac{pq\chi}{(1+\varepsilon n_\varepsilon)^3} - q(q+1).$$

Multiplying both sides of above equation by $4(p+1)$ gets that

$$4(p+1)h_1(q; p, \chi, n_\varepsilon) = \frac{p(p+1)^2\chi^2}{(1+\varepsilon n_\varepsilon)^6} + 4pq^2 - 4(p+1)q(q+1) \leq -4q^2 - 4(p+1)q + p(p+1)^2\chi^2.$$  

Since $\Delta_q = 16(p+1)^2(1+p\chi^2) > 0$, we know $h_1(q; p, \chi, n_\varepsilon) < 0$ provided that $q > q_+ = q_+(p) := \frac{p+1}{2}((1+\sqrt{1+p\chi^2}) - 1).$ Combining with (3.5), we have

$$\frac{d}{dt} \int_\Omega n_\varepsilon^{-p}v_\varepsilon^{-q} \leq (q-ap) \int_\Omega n_\varepsilon^{-p}v_\varepsilon^{-q} + dp \int_\Omega n_\varepsilon^{-p+1+\theta}v_\varepsilon^{-q} - q \int_\Omega n_\varepsilon^{-p+1}v_\varepsilon^{-q-1}$$

for all $\varepsilon \in (0, 1)$ and each $t \in (0, T_{\text{max}, \varepsilon})$. We divide the following proof into two cases.

**Case 1:** $\theta \in (1, 2)$ Let $\xi_0 := \frac{1}{2} \inf_{x \in \Omega} v_0(x)$. By Lemma 2.5, there exists $t_0 \in (0, T_{\text{max}, \varepsilon})$ such that $v_\varepsilon(x, t) \geq \xi_0$ for all $x \in \Omega$ and $t \in [0, t_0]$, and $n_\varepsilon(x, t) \geq \xi_0$ for all $x \in \Omega$ with constant $\xi_0 > 0$. So we just have to prove (3.1) for $t \in (t_0, T_{\text{max}, \varepsilon})$. By the Young’s inequality, we have

$$bp \int_\Omega n_\varepsilon^{-p+1+\theta}v_\varepsilon^{-q} \leq q \int_\Omega n_\varepsilon^{-p+1}v_\varepsilon^{-q-1} + \left( \frac{bp}{p+1} \right)^{q+1} \int_\Omega n_\varepsilon^{1-p+(\theta-2)(q+1)}$$

for all $t \in (t_0, T_{\text{max}, \varepsilon})$. We first claim that $1-p+(\theta-2)(q+1) \in (0, 1)$. In reality, due to $p, q > 0$ and $\theta \in (1, 2)$, readily conclude that $1-p+(\theta-2)(q+1) < 1$, we just have to assume that $1-p+(\theta-2)(q+1) > 0$. Denote $g_1(p) := 1-p+(\theta-2)(q+1) = \theta - p - 1 + \frac{p+1}{2}((\theta-2)(\sqrt{1+p\chi^2} - 1), p \in (0, \theta-1).$ Using the fundamental inequality $\sqrt{s+1} < 1 + \frac{s}{2}$ for all $s > 0$, we know that only if $p \in (0, \frac{4(\theta-1)}{4+2(\theta-2)\theta\chi}),$ we have

$$g_1(p) > \theta - p - 1 + \frac{p+1}{4}(\theta-2)p\chi^2 > \theta - p - 1 + \frac{p+1}{4}\theta\chi^2(\theta-2) > 0,$$

so there exists constant $q > q_+$ such that $1-p+(\theta-2)(q+1) > 0$. By the Hölder inequality, Lemma 2.6, (3.6) and (3.7), we obtain

$$\frac{d}{dt} \int_\Omega n_\varepsilon^{-p}v_\varepsilon^{-q} \leq (q-ap) \int_\Omega n_\varepsilon^{-p}v_\varepsilon^{-q} + c_1 b^{q+1}$$

for all $t \in (t_0, T_{\text{max}, \varepsilon})$, (3.8) where $c_1 > 0$ is a constant. Then, let

$$H_1(p) := q_+ - ap = \frac{(p+1)}{2}((\sqrt{1+p\chi^2} - 1) - ap, p > 0.$$

We immediately find that

$$H_1(p) < 0 \iff H_2(p) := \chi^2 p^2 + 2(\chi^2 - 2a - 2a^2)p + \chi^2 - 4a < 0, p > 0.$$
Since $\Delta_\rho = 16a^2(1 + a^2 - \chi^2) > 0$ provided that $a > \max\{0, \chi - 1\}$. Then $H_1(p) < 0$ for $p \in (p_*, p^*)$, where

$$
p_* = \frac{2a^2 + 2a - \chi^2}{\chi^2} - \frac{2a\sqrt{(1+a)^2 - \chi^2}}{\chi^2}, \quad p^* = \frac{2a^2 + 2a - \chi^2}{\chi^2} + \frac{2a\sqrt{(1+a)^2 - \chi^2}}{\chi^2}.
$$

By the relationship between the two roots, we know that $p_* \leq 0 < p^*$ if $a > \frac{\chi^2}{4}$, $\chi > 0$ and $p \in (p_*, p^*)$ if $\chi - 1 < a < \frac{\chi^2}{4}$, $\chi > 0$. Therefore, $0 < p_* < \sqrt{p_* \cdot p^*} \leq \sqrt{\frac{\chi^2 - 4a}{\chi^2}} < \tilde{p}$ provided that $\max\{\chi - 1, \frac{\chi^2}{4}(1 - \tilde{p}^2)\} < a < \frac{\chi^2}{4}$. Taking $p_1 \in (0, \tilde{p})$ and $q_1 > q_+$ such that $1 - p_1 + (\theta - 2)(q_1 + 1) \in (0, 1)$ and $q_1 - ap_1 < 0$ for $a, \chi > 0$ and $\theta \in (1, 2)$ fulfilling (1.9). Denote $\bar{\theta} := \frac{p_1 + q_1}{q_1 + 1} \in (0, p_1)$. Then $\frac{q_1}{p_1 - \bar{\theta}} = 1$, and thus

$$
\int_{\Omega} n_{\varepsilon}^{-\theta} \leq \left( \int_{\Omega} n_{\varepsilon}^{-p_1} v_{\varepsilon}^{q_1} \right)^{\bar{\theta}} \left( \int_{\Omega} v_{\varepsilon} \right)^{1 - \bar{\theta}} \text{ for all } t \in (t_0, T_{\text{max}, \varepsilon}).
$$

(3.10)

Integrating (3.6) from $t_0$ to $t$ gets that

$$
\int_{\Omega} n_{\varepsilon}^{-p_1} v_{\varepsilon}^{q_1} \leq c_2 (1 + b)^q_{\varepsilon} \int_{\Omega} n_{\varepsilon}^{-p_1}(x, t_0)v_{\varepsilon}^{q_1}(x, t_0) + \frac{c_1}{a p_1 - q_1} b_{\varepsilon}^{q_1 + 1}
$$

(3.11)

for all $t \in (0, T_{\text{max}, \varepsilon})$, with some certain positive constant $c_2$. Combining (3.10), (3.11) with (2.10), we obtain

$$
\int_{\Omega} n_{\varepsilon}^{-\theta} \leq c_3 (1 + b)^{\frac{(q_1 + 1)\bar{\theta}}{\bar{\theta}}} \text{ for all } t \in (t_0, T_{\text{max}, \varepsilon}),
$$

where $c_3 > 0$ is a constant. By the H"older inequality, we have

$$
\int_{\Omega} n_{\varepsilon} \geq |\Omega|^{\frac{\bar{\theta}}{\bar{\theta} + 1}} \left( \int_{\Omega} n_{\varepsilon}^{-\theta} \right)^{-\frac{1}{\bar{\theta}}} \geq c_3^{-\frac{1}{\bar{\theta}}} |\Omega|^{\frac{\bar{\theta}}{\bar{\theta} + 1}} (1 + b)^{-\frac{q_1 + 1}{\bar{\theta}}} \quad \forall \theta > 0
$$

(3.12)

for all $t \in (t_0, T_{\text{max}})$. By the pointwise lower bound estimate for the Neumann heat semigroup $\{e^{t\Delta}\}_{t \geq 0}$ in Lemma 2.9, we obtain

$$
v_{\varepsilon}(x, t) = e^{t(\Delta - 1)} v_0 + \int_0^t e^{(t-s)(\Delta - 1)}(n_{\varepsilon}(x, s) - u_{\varepsilon}(x, s) \cdot \nabla v_{\varepsilon}(x, s))ds
$$

$$
\geq e^{t(\Delta - 1)} v_0 + \int_0^t \frac{1}{4\pi(t-s)} e^{-(s-t)(\frac{diam(\Omega)}{4} + \frac{G(t-s)}{4})} \|n_{\varepsilon}(x, s) - u_{\varepsilon}(x, s) \cdot \nabla v_{\varepsilon}(x, s)\|_{L^1(\Omega)}ds
$$

$$
\geq \varrho_0 \int_0^t \frac{1}{4\pi\tau} e^{-(\tau - (\frac{diam(\Omega)}{4} + \frac{G(t-s)}{4}))}d\tau =: \xi_1 > 0
$$

(3.13)

for all $t \in (t_0, T_{\text{max}, \varepsilon})$ and $x \in \Omega$, where $\text{diam} \Omega := \max_{x, y \in \Omega} |x - y|$. Choosing $\sigma_1 := \{\xi_0, \xi_1\}$, we obtain (3.1).

**Case 2 : $\theta = 2$** Let $\xi_2 := \frac{1}{2} \inf_{x \in \Omega} v_0(x)$. By Lemma 2.5, there exists $t_1 \in (0, T_{\text{max}, \varepsilon})$ such that $v_{\varepsilon}(x, t) > \xi_2$ for all $x \in \Omega$ and $t \in [0, t_1]$, and $n_{\varepsilon}(x, t) > \xi_1$ for all $x \in \Omega$ with some certain constant $\xi_1 > 1$. So we only to prove (3.1) for $t \in (t_1, T_{\text{max}, \varepsilon})$. Similar to (3.6), we have

$$
\frac{d}{dt} \int_{\Omega} n_{\varepsilon}^{-p_1} v_{\varepsilon}^{-q} \leq (q - ap_1) \int_{\Omega} n_{\varepsilon}^{-p_1} v_{\varepsilon}^{-q} + bp \int_{\Omega} n_{\varepsilon}^{-p_1 + 1} v_{\varepsilon}^{-q} - q \int_{\Omega} n_{\varepsilon}^{-p_1 + 1} v_{\varepsilon}^{-q - 1}.
$$
for all $t \in (t_1, T_{\text{max}, \varepsilon})$. Choosing $p \in (0, 1)$, by the Young’s inequality, there exists a constant $c_4 > 0$ such that
\[
bp \int_{\Omega} n^{p+1}_\varepsilon v^{q-1}_\varepsilon \leq q \int_{\Omega} n^{p+1}_\varepsilon v^{q-1}_\varepsilon + \left( \frac{bp}{q+1} \right)^{q+1} \int_{\Omega} n^{-p}_\varepsilon
\]
\[
\leq q \int_{\Omega} n^{p+1}_\varepsilon v^{q-1}_\varepsilon + \left( \frac{bp}{q+1} \right)^{q+1} [(1-p)m + p|\Omega|]
\]
\[
= : q \int_{\Omega} n^{p+1}_\varepsilon v^{q-1}_\varepsilon + c_4
\]
for all $t \in (t_1, T_{\text{max}, \varepsilon})$. The following processes is similar to Case 1, to avoid duplication, we omit it.

(2) Since $\theta > 2$, choosing $p \in (0, \theta - 1)$ and $q > 0$, we readily conclude that $1 - p + (\theta - 2)(q + 1) > (\theta - 2)q > 0$. Let
\[
g_2(p) : = 1 - p + (\theta - 2)(q + 1) = \theta - p - 1 + \frac{p+1}{2} (\theta - 2)(\sqrt{1 + \chi^2}) - 1, \quad p \in (0, \theta - 1).
\]
On account of $\chi^2 < m(\frac{p}{\theta - 2 - 1})$, only if $p \in (\frac{4(\theta - 2)}{4 - \theta^2}, \theta - 1)$, we have
\[
g_2(p) < \theta - p - 1 + \frac{(p+1)}{4} (\theta - 2)p\chi^2 < \theta - p - 1 + \frac{p}{4} \theta^2 < 1.
\]
Thus, there exists $q > q_+$ such that $1 - p + (\theta - 2)(q + 1) > (\theta - 2)q \in (0, 1)$. Analogous to (3.1), we have
\[
bp \int_{\Omega} n^{p-1}_\varepsilon v^{q-1}_\varepsilon \leq q \int_{\Omega} n^{p+1}_\varepsilon v^{q-1}_\varepsilon + c_4 b^{q+1}_\varepsilon
\]
with $c_4 > 0$. Due to $0 < \chi < a + 1$, we know that $p_\ast < 1$, where $p_\ast$ is the same as (3.9). Since $\chi^2 < \frac{2a(a+1)}{\theta}$, $\theta > 2$ and $a > 0$, by simple calculation, we have
\[
p_\ast \geq \frac{2a(a+1) - \chi^2}{\chi^2} > \theta - 1 > \frac{4(\theta - 2)}{4 - (\theta - 2)\theta \chi^2},
\]
it is sufficient to assure that $(\frac{4(\theta - 2)}{4 - \theta^2}, \theta - 1) \cap (p_\ast, p_\ast^*) \neq \emptyset$. Choosing $p_2 \in (\frac{4(\theta - 2)}{4 - \theta^2}, \theta^*)$ and $q_2 > q_+$, similar to (3.10)-(3.13), we obtain (3.2) with certain constant $\sigma_2 > 0$. This completes the proof. \hfill \Box

Lemma 3.2. For each $\varepsilon \in (0, 1)$, the solution of (1.1) is global-in-time; that is, we have $T_{\text{max}, \varepsilon} = \infty$ in Lemma 2.5.

Proof. The proof is similar to [3, 36, 37], we only give the sketch here for the sake of completeness. Firstly, we claim that for all $p \in [2, 6]$, there exists $C(p, \varepsilon) > 0$ such that
\[
\int_{\Omega} n^p_\varepsilon(\cdot, t) + \int_{\Omega} v^p_\varepsilon(\cdot, t) \leq C(p, \varepsilon) \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}). \quad (3.14)
\]
Indeed, we multiply the first equation in (2.5) by $pn^{p-1}_\varepsilon$, use Young’s inequality and $\nabla \cdot u_\varepsilon \equiv 0$, we have
\[
\begin{aligned}
\frac{d}{dt} \int_{\Omega} n^p_\varepsilon &= \int_{\Omega} n^{p-1}_\varepsilon [\Delta n_\varepsilon - u_\varepsilon \cdot \nabla n_\varepsilon - \chi \nabla \cdot (\frac{n_\varepsilon}{1 + \varepsilon n_\varepsilon^a} \nabla v_\varepsilon) + an_\varepsilon - bn^\theta_\varepsilon] \\
&= -p(p - 1) \int_{\Omega} n^{p-2}_\varepsilon |\nabla n_\varepsilon|^2 + \frac{\chi p(p - 1)}{1 + \varepsilon n_\varepsilon^a} \int_{\Omega} n_\varepsilon \nabla n_\varepsilon \cdot \nabla v_\varepsilon \\
&\quad + ap \int_{\Omega} n^p_\varepsilon - bp \int_{\Omega} n^{p-1+\theta}_\varepsilon
\end{aligned}
\]
respectively and integrating them over \( \Omega \), we have
\[
\int_{\Omega} |\nabla v|^{2} - \frac{bp}{2} \int_{\Omega} n^{p-1+\theta} + c_{1},
\]
which yields
\[
\int_{\Omega} \frac{\chi^{2} p(p-1)}{4\varepsilon^{2} \sigma^{2}} \int_{\Omega} |\nabla v|^{2} - \frac{bp}{2} \int_{\Omega} n^{p-1+\theta} + c_{1},
\]
where
\[
c_{1} = \left( \frac{2(p-1+\theta)}{bp} \right)^{\frac{\theta-1}{\theta}} \frac{\chi^{1+\theta}}{p-1+\theta} \frac{|\Omega|(\theta-1)}{\sigma} \quad \sigma := \min\{\sigma_{1}, \sigma_{2}\}.
\]
Then, multiplying the second equation in (2.5) by \((v_{\varepsilon} + 1)^{p-1}\), integrating by parts and using Young’s inequality, we obtain
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} (v_{\varepsilon} + 1)^{p} + (p-1) \int_{\Omega} (v_{\varepsilon} + 1)^{p-2} |\nabla v|^{2} + \int_{\Omega} v_{\varepsilon}(v_{\varepsilon} + 1)^{p-1}
\]
\[
\leq \frac{1}{p} \frac{d}{dt} \int_{\Omega} n^{p} + \frac{p-1}{p} \int_{\Omega} (v_{\varepsilon} + 1)^{p},
\]
which yields
\[
\frac{\chi^{2}}{4\varepsilon^{2} \sigma^{2}} \frac{d}{dt} \int_{\Omega} (v_{\varepsilon} + 1)^{p} + \frac{\chi^{2} p(p-1)}{4\varepsilon^{2} \sigma^{2}} \int_{\Omega} |\nabla v|^{2} \leq \frac{\chi^{2} p(p-1)}{4\varepsilon^{2} \sigma^{2}} \int_{\Omega} n^{p} + \frac{\chi^{2} p(p-1)}{4\varepsilon^{2} \sigma^{2}} \int_{\Omega} (v_{\varepsilon} + 1)^{p}
\]
for all \( t \in (0, T_{\text{max},\varepsilon}) \). Combining (3.15) with (3.16), we obtain
\[
\frac{d}{dt} \left\{ \int_{\Omega} n^{p} + \frac{\chi^{2}}{4\varepsilon^{2} \sigma^{2}} \int_{\Omega} (v_{\varepsilon} + 1)^{p} \right\} \leq K \left\{ \int_{\Omega} n^{p} + \frac{\chi^{2}}{4\varepsilon^{2} \sigma^{2}} \int_{\Omega} (v_{\varepsilon} + 1)^{p} \right\}
\]
for all \( t \in (0, T_{\text{max},\varepsilon}) \), where \( K := \max\{\frac{\chi^{2}}{4\varepsilon^{2} \sigma^{2}}, p-1\} \), and thus establishes (3.14) is direct results on integrating above inequality in time. Then, we use the Helmholtz projection \( P \) to both sides of the fourth equation in (2.5) and testing by \( u_{\varepsilon}, A_{u_{\varepsilon}} \), respectively and integrating them over \( \Omega \), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_{\varepsilon}|^{2} + \int_{\Omega} |\nabla u_{\varepsilon}|^{2} = \int_{\Omega} P\left(-Y_{\varepsilon} u_{\varepsilon} \cdot \nabla \right) u_{\varepsilon} + n_{\varepsilon} \nabla \phi_{\varepsilon} u_{\varepsilon}
\]
\[
\leq \| \nabla \phi_{\varepsilon} \|_{L^{\infty}(\Omega)} \| n_{\varepsilon} \|_{L^{\frac{4}{3}}(\Omega)} \| u_{\varepsilon} \|_{L^{4}(\Omega)} - \int_{\Omega} (Y_{\varepsilon} u_{\varepsilon} \cdot \nabla) u_{\varepsilon} u_{\varepsilon}
\]
\[
\leq c_{2}^{2} \| n_{\varepsilon} \|_{L^{2}(\Omega)}^{2} + \frac{1}{2} \| \nabla u_{\varepsilon} \|_{L^{2}(\Omega)}^{2}
\]
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |A_{\varepsilon} u_{\varepsilon}|^{2} + \int_{\Omega} |A_{u_{\varepsilon}}|^{2} = \int_{\Omega} P\left(-Y_{\varepsilon} u_{\varepsilon} \cdot \nabla \right) u_{\varepsilon} + n_{\varepsilon} \nabla \phi_{\varepsilon} u_{\varepsilon}
\]
\[
\leq \frac{1}{2} \int_{\Omega} |A_{\varepsilon} u_{\varepsilon}|^{2} + \int_{\Omega} |(Y_{\varepsilon} u_{\varepsilon} \cdot \nabla) u_{\varepsilon}|^{2} + \| \nabla \phi_{\varepsilon} \|_{L^{\infty}(\Omega)}^{2} \int_{\Omega} n_{\varepsilon}^{2}
\]
for all \( t \in (0, T_{\text{max},\varepsilon}) \), for any \( \varepsilon > 0 \)
\[
\int_{\Omega} (Y_{\varepsilon} u_{\varepsilon} \cdot \nabla) u_{\varepsilon} u_{\varepsilon} = - \int_{\Omega} \nabla \cdot (Y_{\varepsilon} u_{\varepsilon}) |u_{\varepsilon}|^{2} - \frac{1}{2} \int_{\Omega} Y_{\varepsilon} u_{\varepsilon} \cdot \nabla |u_{\varepsilon}|^{2}
\]
\[
= \frac{1}{2} \int_{\Omega} (\nabla \cdot Y_{\varepsilon} u_{\varepsilon}) |u_{\varepsilon}|^{2} = 0
\]
for all \( t \in (0, T_{\text{max},\varepsilon}) \), where we have been used the fact that \( \nabla \cdot u_{\varepsilon} = 0 \). By Lemma 2.8, (2.11) and (3.17), we have \( \int_{\Omega} |u_{\varepsilon}|^{2} \leq c_{2} \) for all \( t \in (0, T_{\text{max},\varepsilon}) \), where \( c_{2} > 0 \) is a constant. Using the fact that \( \int_{\Omega} |A_{\varepsilon} u_{\varepsilon}|^{2} = \int_{\Omega} |\nabla u_{\varepsilon}|^{2} \) [26] (p. 133, Lemma 2.2.1
Combining (3.18) with (3.19), there exists \( c_3 > 0 \) such that
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla u_\varepsilon|^2 + \frac{1}{2} \int_\Omega |\Delta u_\varepsilon|^2 \leq \int_\Omega (Y_{\varepsilon} u_\varepsilon \cdot \nabla) u_\varepsilon^2 + c_3^2 \|n_\varepsilon\|_{L^2(\Omega)}^2 \tag{3.20}
\]
for all \( t \in (0, T_{\text{max}, \varepsilon}) \). Similar to Lemma 3.6 in [30], we obtain
\[
\int_\Omega |\nabla u_\varepsilon|^2 \leq c_4 \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}),
\]
where \( c_4 > 0 \) is a constant. For \( u_\varepsilon \), we have
\[
\|A^\varepsilon u_\varepsilon\|_{L^2(\Omega)}^2 \leq e^{-t} \|A^\varepsilon u_0\|_{L^2(\Omega)}^2 + \int_0^t \|A^\varepsilon e^{-(t-s)}A\|
\]
\[
\leq e^{-t} \|A^\varepsilon u_0\|_{L^2(\Omega)}^2 + \int_0^t e^{-\lambda_1(t-s)}(t-s)^{-\varepsilon} \|n_\varepsilon \nabla \phi_\varepsilon\|_{L^2(\Omega)} ds
\]
\[
+ \kappa \int_0^t e^{-\lambda_1(t-s)}(t-s)^{-\varepsilon} \|\nabla \phi_\varepsilon\|_{L^2(\Omega)} ds \tag{3.22}
\]
\[
\leq e^{-t} \|A^\varepsilon u_0\|_{L^2(\Omega)} + \int_0^t e^{-\lambda_1(t-s)}(t-s)^{-\varepsilon} \|n_\varepsilon \nabla \phi_\varepsilon\|_{L^2(\Omega)} ds
\]
\[
+ \kappa \int_0^t e^{-\lambda_1(t-s)}(t-s)^{-\varepsilon} \|Y_{\varepsilon} u_\varepsilon \|_{L^\infty(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega)} ds
\]
\[
\leq c_3 \quad \text{for any } \varepsilon \in \left(\frac{3}{4}, 1\right),
\]
we have been used the fact that
\[
\|Y_{\varepsilon} u_\varepsilon \|_{L^\infty(\Omega)} \leq \|(I + \varepsilon A)^{-1} u_\varepsilon\|_{L^\infty(\Omega)} \leq c_5 \|u_\varepsilon\|_{L^2(\Omega)} \leq c_6,
\]
where \( c_5, c_6 > 0 \) are constants. By the embedding theorem [14], we have
\[
\|u_\varepsilon\|_{L^\infty(\Omega)} \leq c_7 (\|u_\varepsilon\|_{L^2(\Omega)} + \|A^\varepsilon u_\varepsilon\|_{L^2(\Omega)}) \leq c_8 \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}),
\]
where \( c_7, c_8 > 0 \) are constants. The following proof is similar to [3, 36, 37], so we omit it.

**Lemma 3.3.** Let \( \theta > \frac{8}{9} \). Then there exists \( L_2 > 0 \) such that
\[
\int_0^T \int_\Omega |\nabla v_\varepsilon(x, s)|^2 dx ds \leq L_2(1 + T), \quad T > 0
\]
for all \( \varepsilon \in (0, 1) \).

**Proof.** Multiplying the second equation in (2.5) by \( v_\varepsilon \), integrating by parts, using the facts that \( \nabla \cdot u_\varepsilon = 0 \) and Young’s inequality, we have
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega v_\varepsilon^2 \leq - \int_\Omega |\nabla v_\varepsilon|^2 - \int_\Omega v_\varepsilon^2 + \frac{1}{\theta} \int_\Omega n_\varepsilon^\theta + \frac{\theta - 1}{\theta} \int_\Omega v_\varepsilon^\frac{\theta}{\theta - 1} \quad t > 0.
\]
By the Gagliardo-Nirenberg inequality and Young’s inequality, we obtain
\[
\frac{\theta - 1}{\theta} \int_\Omega v_\varepsilon^\frac{\theta}{\theta - 1} \leq \frac{\theta - 1}{\theta} (C_G \|\nabla v_\varepsilon\|_{L^2(\Omega)} \|v_\varepsilon\|_{L^\theta(\Omega)}^\frac{\theta - 1}{\theta} + C_{GN} \|v_\varepsilon\|_{L^\theta(\Omega)}^\frac{\theta - 1}{\theta}) \leq \frac{1}{2} \int_\Omega |\nabla v_\varepsilon|^2 + c_1 \quad \text{for all } t \in (0, T_{\text{max}}),
\]
\[
\tag{3.26}
\]
where \( \epsilon_1 = \frac{1 - \frac{\theta - 1}{\theta - 1}}{2}, \) and
\[
c_1 := \frac{\theta - 1 - \theta \epsilon_1}{\theta - 1} \left[ \frac{\theta - 1}{\theta - 1} 2^{\frac{1}{\theta + 1}} (C_G \|
abla \varphi^{(1)}\|_{L^p(\Omega)})^{\frac{\theta}{\theta + 1}} \right]^{\frac{\theta - 1}{\theta - 1}} \left[ \frac{\theta - 1}{2}\epsilon_1 \right]^{-\frac{\theta - 1}{\theta - 1}}.
\]

In fact, for \( \theta > \frac{8}{5} \), by some simple calculation, we readily conclude that \( \epsilon_1 \in (0,1) \) and \( \frac{\theta}{\theta - 1} \epsilon_1 < 2 \). Together (3.25) with (3.26), we have
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \nabla \varphi^2 \leq -\frac{1}{2} \int_\Omega |\nabla \varphi|^2 - \int_\Omega \nabla \varphi^2 + \frac{1}{\theta} \int_\Omega n_\varphi^\theta + c_1, \quad t > 0.
\]

Integrating (3.27) from 0 to \( T \) and using (2.11), we immediately obtain (3.24). This completes the proof.

**Lemma 3.4.** Assume that the conditions of Theorem 1.1(i) hold, there exists \( k \in (0, \theta - 1) \) such that
\[
\int_0^T \int_\Omega n_\varphi^{k-2} |\nabla n_\varphi| dx dt \leq L_3 (1 + T), \quad T > 0
\]
for all \( \varphi \in (0,1) \), where \( L_3 > 0 \) is a constant.

**Proof.** Letting \( k \in (0, \theta - 1) \), multiplying the first equation in (2.5) by \( n_\varphi^{k-1} \), integrating by part and using the fact that \( \nabla \cdot u_\varphi = 0 \), we have
\[
\frac{1}{k} \int_\Omega n_\varphi^{k} = \int_\Omega n_\varphi^{k-1} |\nabla n_\varphi| - u_\varphi \cdot \nabla n_\varphi - \chi \nabla \cdot \left( \frac{n_\varphi}{(1 + \varepsilon n_\varphi)^3} \nabla \varphi \right) + a n_\varphi^\theta - bn_\varphi^\theta
\]
\[
= (1-k) \int_\Omega n_\varphi^{k-2} |\nabla n_\varphi| + \chi (k-1) \int_\Omega \frac{n_\varphi}{(1 + \varepsilon n_\varphi)^3} \nabla n_\varphi \cdot \nabla n_\varphi
\]
\[
+ a \int_\Omega n_\varphi^k - b \int_\Omega n_\varphi^{k-1+\theta}.
\]

Using Young’s inequality, we obtain
\[
\chi (k-1) \int_\Omega \frac{n_\varphi}{(1 + \varepsilon n_\varphi)^3} \nabla n_\varphi \cdot \nabla n_\varphi \leq \frac{1-k}{2} \int_\Omega n_\varphi^{k-2} |\nabla n_\varphi|^2 + \frac{(1-k)\chi^2}{2} \int_\Omega \frac{n_\varphi}{(1 + \varepsilon n_\varphi)^3} |\nabla n_\varphi|^2.
\]
Collecting (3.29)-(3.30), we thus infer that
\[
\frac{1-k}{2} \int_\Omega n_\varphi^{k-2} |\nabla n_\varphi|^2 \leq \frac{1}{k} \int_\Omega n_\varphi + \frac{(1-k)\chi^2}{2} \int_\Omega n_\varphi \frac{n_\varphi^k}{(1 + \varepsilon n_\varphi)^3} |\nabla n_\varphi|^2 - a \int_\Omega n_\varphi^k + b \int_\Omega n_\varphi^{k-1+\theta}.
\]
By the Young’s inequality and (3.1), we have
\[
\int_\Omega n_\varphi \frac{n_\varphi^k}{(1 + \varepsilon n_\varphi)^3} |\nabla n_\varphi|^2 \leq \frac{1}{\sigma_1} \int_\Omega n_\varphi |\nabla n_\varphi|^2 \leq c_1 \int_\Omega n_\varphi^\theta + c_1 \int_\Omega |\nabla n_\varphi|^\frac{2\theta}{p-1}
\]
with certain constant \( c_1 = c_1(k) > 0 \). By the Gagliardo-Nirenberg inequality, we obtain
\[
c_1 \int_\Omega |\nabla n_\varphi|^\frac{2\theta}{p-1} \leq c_1 (C_G \|\nabla n_\varphi\|_{L^{\theta}(\Omega)}^\theta + C_G \|\nabla n_\varphi\|_{L^2(\Omega)}^{1-\theta}) \frac{2\theta}{p-1}
\]
\[
\leq c_2 \int_\Omega |\Delta n_\varphi|^\theta + c_2 \quad \text{for all} \quad t \in (0, T_{\text{max}}),
\]
where \( \epsilon_2 = \frac{1 - \frac{\theta - k}{\theta - 1}}{2} \), where \( c_2 = c_2(k) > 0 \) is a constant. Since \( \theta \in (\frac{8}{5}, 2) \), taking \( k \in (0, \theta - \frac{5}{8}) \), \( t \in (0, \theta - \frac{5}{8}) \cap (0, \theta - 1) \), we have \( \frac{2\theta}{p-1} \epsilon_2 < \theta \). Substituting (3.33) into (3.32), there
exists a constant $c_3 > 0$ such that
\[ \int_0^T \frac{1}{2} \int_\Omega \left| \nabla v_\varepsilon \right|^2 \, dx \, dt \leq c_3 \left( 1 + \int_\Omega \left| n_\varepsilon \right|^2 + \int_\Omega |\Delta v_\varepsilon|^2 \right) \quad \text{for all } t \in (0, T_{\max}). \tag{3.34} \]
Combining (3.31)-(3.34) with (2.9), (2.11), integrating it from 0 to $T$ gets that
\[ \frac{1}{k} \int_0^T \int_\Omega n_\varepsilon^{k-2} |\nabla n_\varepsilon|^2 \, dx \, dt \leq \frac{1}{k} \int_\Omega n_\varepsilon^k (\cdot, t) - \frac{1}{k} \int_\Omega n_\varepsilon^k + \frac{(1-k)\chi^2}{2} \int_0^T \int_\Omega |\nabla v_\varepsilon|^2 \, dx \, dt \]
\[ - a \int_0^T \int_\Omega n_\varepsilon^{k-1} \, dx \, dt + b \int_0^T \int_\Omega n_\varepsilon^{k-1+\theta} \, dx \, dt \leq c_4 (1 + T) + c_4 \int_0^T \int_\Omega |\Delta v_\varepsilon|^\theta \, dx \, dt \tag{3.35} \]
with certain constant $c_4 > 0$. Based on the maximal $L^p$-$L^q$ estimates for parabolic equations of Theorem 3.1 in [19], we have
\[ \int_0^T \int_\Omega |\Delta v_\varepsilon|^\theta \, dx \, dt \leq c_5 (1 + \int_0^T \int_\Omega n_\varepsilon^\theta \, dx \, dt + \int_0^T \int_\Omega \left| \nabla v_\varepsilon \right|^\theta \, dx \, dt) \leq c_6 (1 + T), \quad T > 0, \tag{3.36} \]
where $c_5, c_6 > 0$ are constants. Collecting (3.35)-(3.36), we immediately obtain (3.28). This completes the proof. \hfill \Box

**Lemma 3.5.** Assume that the conditions of Theorem 1.1(i) hold, there exists $L_4 > 0$ such that
\[ \int_0^T \left\| \partial_t (1 + n_\varepsilon) \right\|_{L^4(\Omega)}^2 \, dt \leq L_4 (1 + T), \quad T > 0 \tag{3.37} \]
for all $\varepsilon \in (0, 1)$.

**Proof.** Let $k \in (0, \theta - 1)$. Given $T > 0$, due to Sobolev embedding theorem, we see that
\[ \|\varphi\|_{L^\infty((0,T);W^{1,\infty}(\Omega))} \leq c_1 \|\varphi\|_{L^\infty((0,T);W^{4,2}_0(\Omega))} \]
for all $\varphi \in L^\infty((0,T);W^{4,2}_0(\Omega))$ with some certain constant $c_1 > 0$. Without loss of generality, we assume that $\|\varphi\|_{L^\infty((0,T);W^{4,2}_0(\Omega))} \leq 1$. Multiplying the first equation in (2.5) by $(1 + n_\varepsilon)^{k-2} \varphi$, integrating by parts and using Hölder inequality, we have
\[ \frac{2}{k} \frac{d}{dt} \int_\Omega (1 + n_\varepsilon)^{k-2} \varphi = \int_\Omega \left( \frac{1}{2} n_\varepsilon^2 \varphi \Delta n_\varepsilon - u_\varepsilon \cdot \nabla n_\varepsilon - \chi \nabla \cdot \left( \frac{n_\varepsilon}{(1 + \varepsilon n_\varepsilon)\varepsilon} \nabla v_\varepsilon \right) + a n_\varepsilon - b n_\varepsilon^\theta \right) \]
\[ = - \frac{k-2}{k} \int_\Omega \nabla n_\varepsilon \cdot \nabla \varphi \left( \frac{k-2}{k} \chi \int_\Omega \nabla n_\varepsilon \cdot \nabla v_\varepsilon - \frac{k-2}{k} \int_\Omega (1 + n_\varepsilon)^{k-2} \varphi \nabla n_\varepsilon \cdot \nabla v_\varepsilon \right) \]
\[ + \frac{k-2}{k} \chi \int_\Omega (1 + n_\varepsilon)^{k-2} \varphi n_\varepsilon \nabla \varphi \left( \frac{k-2}{k} \chi \int_\Omega \nabla n_\varepsilon \cdot \nabla v_\varepsilon - \frac{k-2}{k} \chi \int_\Omega (1 + n_\varepsilon)^{k-2} \varphi \nabla n_\varepsilon \cdot \nabla v_\varepsilon \right) \]
\[ \leq \frac{2-k}{2} \|\varphi\|_{L^\infty(\Omega)} \cdot \int_\Omega n_\varepsilon^{k-2} |\nabla n_\varepsilon|^2 + \|\nabla \varphi\|_{L^2(\Omega)} \left( \int_\Omega n_\varepsilon^{k-2} |\nabla n_\varepsilon|^2 \right)^{\frac{1}{2}} \]
This completes the proof.

Together with (2.11), there exist some constants 

\[ \text{Moreover, there exists} \]

\[ \text{By the Young’s inequality and integrate it from } 0 \text{ to } T, \text{ we have} \]

\[ \int_0^T \frac{d}{dt} \int_\Omega (1 + \varepsilon)^{\frac{\varepsilon}{2}} v^2 \varphi \, dt \]

\[ \leq c_2 \left( 1 + \int_0^T \int_\Omega n_{\varepsilon}^{k-2}(x,s)v_{\varepsilon} n_{\varepsilon}(x,s) \right)^2 ds + \int_0^T \int_\Omega n_{\varepsilon}^\theta(x,s) dx ds + \int_0^T \int_\Omega |v_{\varepsilon}(x,s)|^\theta dx ds \]

\[ \leq L_4(1 + T), \quad T > 0. \]

Then, we have

\[ \int_0^T \| \partial_t (1 + \varepsilon)^{\frac{\varepsilon}{2}} v (W^{1,2}_0(\Omega)), \| \, dt \leq L_4(1 + T), \quad T > 0. \]

This completes the proof.

\[ \boxdot \]

**Lemma 3.6.** Let \( \theta > 1 \) and \( p \in \left( \frac{6}{5}, \theta \right), p \leq 24 \). Then there exists \( L_5 > 0 \) such that

\[ \int_0^T \| v_{\varepsilon} \|_{L^p(\Omega)} dt + \int_0^T \| v_{\varepsilon} \|_{W^{2, p}(\Omega)} dt \leq L_5(1 + T), \quad T > 0 \text{ for all } \varepsilon \in (0, 1). \]  

(3.39)

Moreover, there exists \( L_6 > 0 \) such that

\[ \int_0^T \left\| \nabla v_{\varepsilon} \right\|_{L^p(\Omega)} dt \leq L_6(1 + T), \quad T > 0 \text{ for all } \varepsilon \in (0, 1). \]  

(3.40)

**Proof.** Using the Hölder inequality with \( p \in \left( \frac{6}{5}, \theta \right) \), we have

\[ \int_\Omega n_{\varepsilon}^p \leq \left( \int_\Omega n_{\varepsilon}^{\frac{p}{\theta - 1}} \right)^{\frac{\theta - 1}{p}} \left( \int_\Omega n_{\varepsilon}^{\theta} \right)^{\frac{p - 1}{\theta}} \leq c_0. \]  

(3.41)

Together with (2.9) with (2.11), there exist some constants \( c_1, c_2 > 0 \) such that

\[ \int_0^T \| n_{\varepsilon} \|_{L^p(\Omega)} ds \leq c_1 \int_0^T \int_\Omega n_{\varepsilon}^\theta(x,s) dx ds \leq c_2(1 + T), \quad T > 0. \]  

(3.42)

Similar to construction methods of Lemmas 3.6 and 4.3 in [49], we obtain

\[ \int_\Omega n_{\varepsilon} \leq c_3 \text{ and } \int_t^{t+1} \int_\Omega \| \nabla v_{\varepsilon}(x,s) \|^2 dx ds \leq c_4 \text{ for all } t > 0. \]  

(3.43)

with some positive constants \( c_3, c_4 \) depend on initial value \( v_0 \). By the Gagliardo-Nirenberg inequality and embedding theorem, there exists a constant \( c_5 > 0 \) such that

\[ \| \nabla v_{\varepsilon} \|_{L^p(\Omega)} \leq C_{GN} \| \Delta v_{\varepsilon} \|_{L^{\frac{2p}{p+1}}(\Omega)}^{\frac{1}{p+1}} \| \nabla v_{\varepsilon} \|_{L^2(\Omega)}^{\frac{p}{p+1}} + C_{GN} \| \nabla v_{\varepsilon} \|_{L^2(\Omega)} \leq c_5 \| \Delta v_{\varepsilon} \|_{L^{\frac{2p}{p+1}}(\Omega)} + c_5, \]

(3.44)
where \( \epsilon_3 = \frac{24}{19}(\frac{5}{6} - \frac{1}{7}) \in (0,1) \) provided that \( p \leq 24 \). Applying maximal \( L^p-L^q \) estimates for parabolic equation of Theorem 3.1 in [19], we have

\[
\int_0^T \|v_{t\epsilon}\|_{L^p(\Omega)}^p \, ds + \int_0^T \|v_{\epsilon}\|_{W^{2,1}_p(\Omega)}^{p-1} \, ds \\
\leq c_0 \int_0^T \left( \|n_\epsilon\|_{L^p(\Omega)}^p + \|u_\epsilon\cdot \nabla v\|_{L^p(\Omega)}^p \right) \\
\leq c_0 \int_0^T \left[ \|n_\epsilon\|_{L^p(\Omega)}^p + c_5 \|u_\epsilon\|_{L^p(\Omega)} \times (\|\Delta v_\epsilon\|_{L^\frac{p}{2}(\Omega)} + 1) \right]^{p-1} \, ds \\
\leq L_5 (1 + T),
\]

here we have been used (3.36), (3.40) is a consequence of (3.39) when combined with Lemma 3.1. This completes the proof. \( \square \)

**Lemma 3.7.** Suppose that the conditions of Theorem 1.1(i) hold, there exists \( L_T > 0 \) such that

\[
\int_0^T \|u_{t\epsilon}(\cdot, t)\|_{(W^{1,2}_{0,\sigma}(\Omega))}^\frac{q}{2} \, dt \leq L_T (1 + T), \quad T > 0 \quad \text{for all} \ \epsilon \in (0,1), \quad (3.44)
\]

where \( W^{1,2}_{0,\sigma}(\Omega) := W^{1,2}_0(\Omega) \cap L^2_\sigma(\Omega) \).

**Proof.** Inspired by [3, 36, 49], we fix an arbitrary \( \psi \in C_0^\infty(\Omega) \) with \( \nabla \cdot \psi \equiv 0 \) in \( \Omega \) and multiply the fourth equation in (2.5) and Hölder’s inequality, we have

\[
\left| \int_{\Omega} u_{\cdot \epsilon}(\cdot, t) \psi \right| = - \int_{\Omega} \nabla u_\epsilon \cdot \nabla \psi + \int_{\Omega} (Y_\epsilon u_\epsilon \otimes u_\epsilon) \cdot \nabla \psi + \int_{\Omega} n_\epsilon \cdot \nabla \phi \cdot \psi \\
\leq \|\nabla u_\epsilon\|_{L^2(\Omega)} \|\nabla \psi\|_{L^2(\Omega)} + \|Y_\epsilon u_\epsilon\|_{L^4(\Omega)} \|u_\epsilon\|_{L^4(\Omega)} \|\nabla \psi\|_{L^2(\Omega)} \| \nabla \phi\|_{L^\infty(\Omega)} \|n_\epsilon\|_{L^\infty(\Omega)} \|\psi\|_{L^{\frac{q}{2}}(\Omega)}
\] (3.45)

on \( (0, \infty) \) for all \( \epsilon \in (0,1) \). Since \( W^{1,2}_{0,\sigma}(\Omega) \hookrightarrow L^{\frac{q}{2}}(\Omega) \), we obtain

\[
\|Y_\epsilon u_\epsilon\|_{L^2(\Omega)} \leq \|\nabla Y_\epsilon u_\epsilon\|_{L^2(\Omega)} = \|A^\frac{1}{2} Y_\epsilon u_\epsilon\|_{L^2(\Omega)}
\]

\[
= \|Y_\epsilon A^\frac{1}{2} u_\epsilon\|_{L^2(\Omega)} \leq \|A^\frac{1}{2} u_\epsilon\|_{L^2(\Omega)} = \|\nabla u_\epsilon\|_{L^2(\Omega)}
\]

for all \( t > 0 \) and all \( \epsilon \in (0,1) \), here we have been used the fact that \( A^\frac{1}{2} \) commutes with \( Y_\epsilon \) and \( Y_\epsilon \) is nonexpansive on \( L^2_\sigma(\Omega) \). Combining (3.45) with (1.3), using Young’s inequality, there exist \( c_1, c_2 > 0 \) such that

\[
\|u_{t\epsilon}(\cdot, t)\|_{(W^{1,2}_{0,\sigma}(\Omega))}^\frac{q}{2} \leq c_1 \left\{ \|\nabla u_\epsilon\|_{L^2(\Omega)}^\frac{q}{2} + \|\nabla u_\epsilon\|_{L^2(\Omega)}^\frac{q}{2} + \|u_\epsilon\|_{L^2(\Omega)} + \|n_\epsilon\|_{L^2(\Omega)} \right\}
\]

\[
\leq c_2 \left\{ \|\nabla u_\epsilon\|_{L^2(\Omega)}^\frac{q}{2} + \|\nabla u_\epsilon\|_{L^2(\Omega)}^\frac{q}{2} + \|u_\epsilon\|_{L^2(\Omega)} + \|n_\epsilon\|_{L^2(\Omega)} + 1 \right\}
\] (3.46)

for all \( t > 0 \) and all \( \epsilon \in (0,1) \). Ulteriorly, by the Gagliardo-Nirenberg inequality, there exists a constant \( c_3 > 0 \) such that

\[
\|u_\epsilon\|_{L^2(\Omega)} \leq c_3 \|\nabla u_\epsilon\|_{L^2(\Omega)}^\frac{q}{2} + c_3 \|u_\epsilon\|_{L^2(\Omega)}^\frac{q}{2}.
\] (3.47)

Substituting (3.47) into (3.46) and integrating with respect to time, we have

\[
\int_0^T \|u_{t\epsilon}(\cdot, t)\|_{(W^{1,2}_{0,\sigma}(\Omega))}^\frac{q}{2} \, dt \\
\leq c_2 \int_0^T \|\nabla u_\epsilon\|_{L^2(\Omega)}^\frac{q}{2} \, dt + c_2 c_3 \int_0^T \|\nabla u_\epsilon\|_{L^2(\Omega)}^\frac{q}{2} \, dt
\]
for all \( t > 0 \) and all \( \varepsilon \in (0, 1) \), by Lemmas 2.6 and 3.2 result in (3.44). This completes the proof. \( \square \)

Now, we are preparing to extract a suitable sequence of number \( \varepsilon \) along with the respective solutions approach a limit in appropriate topologies.

**Lemma 3.8.** Under the assumptions of Theorem 1.1(i), for \( 0 < k \leq \frac{\theta-1}{1-\theta} \) and \( p \in \left( \frac{\theta}{\theta-1}, \theta \right) \), there exist \( n \in L^1_{\text{loc}}(\Omega \times [0, \infty)) \), \( v \in L^1_{\text{loc}}([0, \infty); W^{1,2}(\Omega)) \) and \( u \in L^1_{\text{loc}}([0, \infty); W^{1,2}_{0,\text{loc}}(\Omega)) \) with some \( q > \frac{\theta}{\theta-1} \) such that

\[
\begin{align*}
t^k_n & \rightarrow n^k \text{ in } L^2_{\text{loc}}([0, \infty); W^{1,2}(\Omega)), \\
n \varepsilon & \rightarrow n \text{ in } L^q_{\text{loc}}(\Omega \times [0, \infty)), \\
\varepsilon & \rightarrow n \text{ a.e. in } \Omega \times [0, \infty) \text{ and } L^q_{\text{loc}}(\Omega \times [0, \infty)) \tag{3.48}
\end{align*}
\]

and

\[
\begin{align*}
v \varepsilon & \rightarrow v \text{ a.e. in } \Omega \times [0, \infty) \text{ and } L^q_{\text{loc}}([0, \infty); W^{1,q}(\Omega)), \\
\frac{\nabla v \varepsilon}{v \varepsilon} & \rightarrow \frac{\nabla v}{v} \text{ in } L^q_{\text{loc}}(\Omega \times (0, \infty)), \\
\frac{1}{v \varepsilon} & \rightarrow \frac{1}{v} \text{ in } L^\infty_{\text{loc}}(\Omega \times (0, \infty)) \tag{3.50}
\end{align*}
\]

as well as

\[
\begin{align*}
u \varepsilon & \rightarrow u \text{ in } L^2_{\text{loc}}(\Omega \times [0, \infty)) \text{ and a.e. in } \Omega \times [0, \infty), \\
u \varepsilon(\cdot, t) & \rightarrow u(\cdot, t) \text{ in } L^2(\Omega) \text{ for a.e. } t > 0, \\
\nabla u \varepsilon & \rightarrow \nabla u \text{ in } L^2_{\text{loc}}(\Omega \times [0, \infty)), \\
Y \varepsilon \varepsilon & \rightarrow u \text{ in } L^2_{\text{loc}}(\Omega \times [0, \infty)) \tag{3.57}
\end{align*}
\]

as \( \varepsilon = \varepsilon_j \searrow 0 \).

**Proof.** The verdicts (3.48), (3.49) and (3.53) are direct results from (3.28), (2.11) and (3.1). Since \( W^{1,2}(\Omega) \hookrightarrow \hookrightarrow L^2(\Omega) \), by the Aubin-Lions lemma with (3.28) and (3.37), we have \( (1+n_\varepsilon)^\frac{1}{2} \rightarrow (1+n)^\frac{1}{2} \) in \( L^2(\Omega \times (0, T)) \), and \( n_\varepsilon \rightarrow n \) a.e. in \( \Omega \times (0, T) \), as \( \varepsilon = \varepsilon_j \searrow 0 \). This combine with (3.49) yields (3.50) by a similar reason of Lemma 1.4 in [41]. Since \( W^{2,p}(\Omega) \hookrightarrow \hookrightarrow W^{1,r}(\Omega) \) provided that \( r < \frac{3p}{3-p} \), using Aubin-Lions lemma again and (3.39) to obtain that \( v \varepsilon \rightarrow v \) in \( L^{\frac{p(\theta-1)}{p-1}}((0, T), W^{1,r}(\Omega)) \) for \( p \in \left( \frac{\theta}{\theta-1}, \theta \right) \) and \( r < \frac{3p}{3-p} \), and \( v_\varepsilon \rightarrow v \) a.e. in \( \Omega \times (0, \infty) \), as \( \varepsilon = \varepsilon_j \searrow 0 \). Due to \( \theta \in \left( \frac{\theta}{\theta-1} \right) \), by a simple calculation, there exists \( p \in \left( \frac{\theta}{\theta-1}, \theta \right) \) such that \( \frac{p(\theta-1)}{p-1} > \frac{\theta}{\theta-1} \) and \( \frac{3p}{3-p} > \frac{\theta}{\theta-1} \). This conclude (3.51) and (3.52) with \( q \in \left( \frac{\theta}{\theta-1}, \min\{ \frac{p(\theta-1)}{p-1}, \frac{3p}{3-p} \} \right) \). The properties (3.54)-(3.56) can similarly be obtained for Lemma 7.1 in [49]. Finally, depending on arguments as Lemma 4.1 in [46], we use the properties of the Yosida approximation to see that for all \( \zeta \in L^2(\Omega) \), we have \( \|Y \varepsilon \zeta\|_{L^2(\Omega)} \leq \|\zeta\|_{L^2(\Omega)} \) and \( \varepsilon \zeta \rightarrow \zeta \) in \( L^2(\Omega) \) as \( \varepsilon \searrow 0 \). So

\[
\|Y \varepsilon u_\varepsilon \|_{L^2(\Omega)} \leq \|u_\varepsilon - u\|_{L^2(\Omega)} + \|Y \varepsilon u - u\|_{L^2(\Omega)} \leq 0
\]
for a.e. \( t > 0 \) as \( \varepsilon_j \searrow 0 \) as well as
\[
\|Y_{\varepsilon_j}u_{\varepsilon_j} - u\|^2_{L^2(\Omega)} \leq 4 \sup_{\varepsilon \in (0,1)} \|u_{\varepsilon}\|^2_{L^2(\Omega)} \quad \text{for a.e. } t > 0,
\]

from dominated convergence theorem yields (3.57). This completes the proof. \( \square \)

Inspired by [3, 36, 45, 56], choosing a suitable test functions in (2.2), we will derive the following lemma.

**Lemma 3.9.** There exists a null set \( \Lambda \subset (0, \infty) \) such that the functions \( n, v \) and \( u \) obtained in Lemma 3.8 fulfill the inequality
\[
\int_0^T \int_\Omega \frac{\nabla v(x,t)}{v^2(x,t)} dxdt \geq |\Omega|/T - \int_0^T \int_\Omega \frac{n(x,t)}{v(x,t)} dxdt + \int_\Omega \ln v(x,T) - \int_\Omega \ln v_0 \quad (3.58)
\]
for all \( T \in (0, \infty) \setminus \Lambda \).

**Proof.** Owing to \( 0 < \max\{\sigma_1, \sigma_2\} \leq v \in L^1_{loc}(\Omega \times (0, \infty)) \), we readily conclude that
\[
z(t) := \int_\Omega \ln v(x,t) \in L^1_{loc}(0, \infty).\]

Thus, there exists a null set \( \Lambda \subset (0, \infty) \) such that each \( T \in (0, \infty) \setminus \Lambda \) is a Lebesgue point of \( z \). In particular,
\[
\frac{1}{\delta} \int_T^{T+\delta} \int_\Omega \ln v(x,t) dxdt \to \int_\Omega \ln v(x,T) \quad \text{for all } T \in (0, \infty) \setminus \Lambda \text{ as } \delta \searrow 0. \quad (3.59)
\]

Given any \( T \in (0, \infty) \setminus \Lambda \) and \( \delta \in (0, 1) \), we let
\[
\rho_\delta(t) := \begin{cases} 1, & t \in [0, T], \\ \frac{T + \delta - t}{\delta}, & t \in (T, T + \delta), \\ 0, & t \in [T + \delta, \infty); \end{cases}
\]

and defined
\[
\bar{v}(x,t) := \begin{cases} \frac{1}{v(x,t)}, & (x,t) \in \Omega \times (0, \infty), \\ \frac{1}{v_0(x)}, & (x,t) \in (-1, 0]. \end{cases}
\]

Then for \( \delta \in (0, 1) \), \( h \in (0, \delta) \), we introduce
\[
\varphi(x,t) := \rho_\delta(t) \cdot (A_h \bar{v})(x,t), \quad (x,t) \in \Omega \times (0, \infty),
\]
where \( (A_h \bar{v})(x,t) := \frac{1}{h} \int_{t-h}^{t} \bar{v}(x,s) ds \), \( (x,t) \in \Omega \times (0, \infty) \). By Lemmas 3.1(1) and 3.8, we know that \( \frac{1}{\sigma} \in L^\infty_{loc}(\Omega \times (0, \infty)) \cap L^2_{loc}((0, \infty); W^{1,2}(\Omega)) \), it is easy to obtain that \( \varphi \in L^\infty_{loc}(\Omega \times (0, \infty)) \cap L^2_{loc}((0, \infty); W^{1,2}(\Omega)) \), and in addition \( \varphi \) is supported in \( \Omega \times [0, T + \delta] \) and
\[
\varphi_t(x,t) = \rho_\delta(t) \cdot (A_h \bar{v})(x,t) + \rho_\delta(t) \cdot \frac{1}{h} (\bar{v}(x,t) - \bar{v}(x,t - h)), \quad (x,t) \in \Omega \times (0, \infty).
\]

This implies \( \varphi_t(x,t) \in L^2_{loc}(\Omega \times (0, \infty)) \). Thus, we now insert \( \varphi \) into (2.2) to obtain
\[
J(\delta, h) := -\int_0^\infty \int_\Omega \rho_\delta(t) \nabla v(x,t) \cdot \nabla (A_h \bar{v})(x,t) dxdt
\]
\[
= \int_0^\infty \int_\Omega \rho_\delta(t) v(x,t) (A_h \bar{v})(x,t) dxdt - \int_0^\infty \int_\Omega \rho_\delta(t) n(x,t) (A_h \bar{v})(x,t) dxdt
\]
\[
- \int_0^\infty \int_\Omega \rho_\delta(t) \cdot (A_h \bar{v})(x,t) dxdt + \int_0^\infty \int_\Omega \rho_\delta(t) u(x,t) v(x,t) \cdot \nabla (A_h \bar{v})(x,t) dxdt
\]
By a simple calculation, we have

\[ -|\Omega| - \int_0^\infty \int_\Omega \rho_\delta(t)v(x,t) \cdot \frac{1}{h}(\bar{v}(x,t) - \bar{v}(x,t-h))dxdt \]

\[ = J_1(\delta, h) + J_2(\delta, h) + J_3(\delta, h) + J_4(\delta, h) + J_5(\delta, h) + J_6(\delta, h). \tag{3.60} \]

By the definition of \( \rho_\delta(t) \) and \( \bar{v}(x,t) \), we have

\[ \varphi(x,0) = \rho_\delta(0) \cdot \frac{1}{h} \int_{-h}^h \bar{v}(x,s)ds = \frac{1}{v_0(x)}, \quad x \in \Omega. \]

Since \( v_0 \in W^{2,\infty}(\Omega) \), it follows that \( \nabla \bar{v} \in L^2(\Omega \times (-1, T+\delta)) \), we can apply Lemma A.2(a) in [18] to obtain

\[ \nabla(A_h\bar{v}) = A_h \nabla \bar{v} \rightarrow \nabla \tilde{v} = -\frac{\nabla v}{v^2} \quad \text{in} \quad L^2(\Omega \times (0, T+\delta)) \quad \text{as} \quad h \searrow 0, \]

and thus,

\[ J(\delta, h) \rightarrow \int_0^{T+\delta} \int_\Omega \rho_\delta(t)\frac{\nabla v(x,t)^2}{v^2(x,t)}dxdt \quad \text{as} \quad h \searrow 0. \tag{3.61} \]

Similarly, \( \bar{v} \in L^\infty(\Omega \times (-1, T+\delta)) \) complies with Lemma A.2(b) in [18] assures that

\[ A_h\bar{v} \xrightarrow{a.s.} \bar{v} = \frac{1}{v} \quad \text{in} \quad L^\infty(\Omega \times (0, T+\delta)) \quad \text{as} \quad h \searrow 0, \tag{3.62} \]

and thus,

\[ J_1(\delta, h) \rightarrow \int_0^{T+\delta} \int_\Omega \rho_\delta(t)dxdt \quad \text{as} \quad h \searrow 0, \tag{3.63} \]

and

\[ J_2(\delta, h) \rightarrow -\int_0^{T+\delta} \int_\Omega \rho_\delta(t)\frac{n(x,t)}{v(x,t)}dxdt \quad \text{as} \quad h \searrow 0, \tag{3.64} \]

as well as

\[ J_3(\delta, h) \rightarrow -\int_0^{T+\delta} \int_\Omega \rho_\delta(t)dxdt \quad \text{as} \quad h \searrow 0. \tag{3.65} \]

Since \( u \) is solenoidal, integrations by parts show that

\[ J_6(\delta, h) \rightarrow -\int_0^{T+\delta} \int_\Omega \rho_\delta(t)u(x,t)\frac{\nabla v(x,t)}{v(x,t)}dxdt = 0 \quad \text{as} \quad h \searrow 0. \tag{3.66} \]

By a simple calculation, we have

\[ J_5(\delta, h) := -\frac{1}{h} \int_0^{T+\delta} \int_\Omega \rho_\delta(t)dxdt + \frac{1}{h} \int_0^h \int_\Omega \rho_\delta(t)\frac{v(x,t)}{v_0(x)}dxdt \\
+ \frac{1}{h} \int_{T+\delta}^{T+\delta-h} \int_\Omega \rho_\delta(t)\frac{v(x,t)}{v(x,t-h)}dxdt \\
\geq -\frac{1}{h} \int_0^h \int_\Omega \rho_\delta(t)dxdt + \frac{1}{h} \int_0^h \int_\Omega \rho_\delta(t)\frac{v(x,t)}{v_0(x)}dxdt \\
- \frac{1}{h} \int_0^h \int_\Omega \rho_\delta(t)\ln v(x,t)dxdt + \frac{1}{h} \int_{T+\delta-h}^{T+\delta} \int_\Omega \rho_\delta(t)\ln v(x,t)dxdt \\
+ \int_0^{T+\delta-h} \int_\Omega \rho_\delta(t) - \rho_\delta(t + h) \frac{1}{h} \ln v(x,t)dxdt. \tag{3.67} \]

Since \( \rho_\delta(0) = 1 \), we have

\[ -\frac{1}{h} \int_0^h \int_\Omega \rho_\delta(t)dxdt \rightarrow -|\Omega| \quad \text{as} \quad h \searrow 0, \tag{3.68} \]
Therefore, owing to our choice of $T$ (sequence still denoted by $\xi$), Lemma 3.10. Collecting (3.67)-(3.72), we thus infer that

$$\frac{1}{h} \int_0^h \int_\Omega \rho_\delta(t) \frac{v(x,t)}{v_0(x)} dxdt \to |\Omega| \text{ as } h \searrow 0, \tag{3.69}$$

and

$$-\frac{1}{h} \int_0^h \int_\Omega \rho_\delta(t) \ln v(x,t) dxdt \to -\int_\Omega \ln v_0(x) \text{ as } h \searrow 0. \tag{3.70}$$

From the dominated convergence theorem, we obtain

$$\int_0^{T+h} \int_\Omega \rho_\delta(t) v(x,t) \ln v(x,t) dxdt \to \int_0^T \int_\Omega \rho_\delta(t) v(x,t) dxdt \text{ as } h \searrow 0, \tag{3.71}$$

and

$$\frac{1}{h} \int_{T+h}^{T+\delta} \int_\Omega \rho_\delta(t) v(x,t) \ln v(x,t) dxdt \to 0 \text{ as } h \searrow 0. \tag{3.72}$$

Collecting (3.67)-(3.72), we thus infer that

$$\lim \inf_{h \to 0} J_5(\delta, h) \geq -\int_\Omega \ln v_0(x) - \int_T^{T+\delta} \int_\Omega \rho_\delta(t) \ln v(x,t) dxdt. \tag{3.73}$$

Substituting (3.61), (3.63)-(3.66) and (3.73) into (3.60), let $h \searrow 0$, we have

$$\int_0^{T+\delta} \int_\Omega \rho_\delta(t) \frac{|\nabla v(x,t)|^2}{v_0(x,t)^2} dxdt \geq \int_0^{T+\delta} \int_\Omega \rho_\delta(t) dxdt - \int_0^{T+\delta} \int_\Omega \rho_\delta(t) \frac{n(x,t)}{v(x,t)} dxdt + \int_0^{T+\delta} \int_\Omega \rho_\delta(t) dxdt$$

$$+ |\Omega| - \int_\Omega \ln v_0(x) - \int_T^{T+\delta} \int_\Omega \rho_\delta(t) \ln v(x,t) dxdt$$

$$= \int_0^{T+\delta} \int_\Omega \rho_\delta(t) dxdt - \int_0^{T+\delta} \int_\Omega \rho_\delta(t) \frac{n(x,t)}{v(x,t)} dxdt - \int_\Omega \ln v_0(x) + \int_\Omega \ln v(x, T), \tag{3.74}$$

here we have been used the definition of $\rho_\delta(t)$ and the Lebesgue point property of $T$. Using the monotone convergence theorem for the above inequality and letting $\delta \searrow 0$ result in (3.58). This completes the proof.

Inspired by [3, 36, 45, 56], we will derive the following lemma.

**Lemma 3.10.** Let $(\varepsilon_j)_{j \in N}$ be as provided by Lemma 3.8. The there exists a subsequence still denoted by $(\varepsilon_j)_{j \in N}$ such that for each $T > 0$

$$\frac{\nabla v_{\varepsilon_j}}{v_{\varepsilon_j}} \to \frac{\nabla v}{v} \text{ in } L^2(\Omega \times (0, T)) \text{ as } \varepsilon = \varepsilon_j \searrow 0. \tag{3.75}$$

**Proof.** Since $\ln v_{\varepsilon_j} \to \ln v$ in $L^\infty_{loc}(\Omega \times (0, \infty))$, we choose a subsequence again denoted by $(\varepsilon_j)_{j \in N}$ such that

$$\int_\Omega \ln v_{\varepsilon_j}(x, T) \to \int_\Omega \ln v(x, T) \text{ for all } T \in (0, \infty) \setminus \Lambda_1 \text{ as } \varepsilon = \varepsilon_j \searrow 0 \tag{3.76}$$

with a null set $\Lambda_1 \subset (0, \infty)$. Taking $\Lambda \subset (0, \infty)$ as in Lemma 3.9, then, we only need to verify (3.75) for all $T \in (0, \infty) \setminus (\Lambda \cup \Lambda_1)$. Given any such $T$, from (3.50) and (3.53), we have

$$\int_0^T \int_\Omega \frac{n_{\varepsilon_j}(x,t)}{v_{\varepsilon_j}(x,t)} dxdt \to \int_0^T \int_\Omega \frac{n(x,t)}{v(x,t)} dxdt \text{ in } L^1(\Omega \times (0, T)) \text{ as } \varepsilon_j \searrow 0. \tag{3.77}$$

Therefore, owing to our choice of $T$, Lemma 3.9 shows that

$$\int_0^T \int_\Omega \frac{|\nabla v(x,t)|^2}{v^2(x,t)} dxdt$$
\[
\geq |\Omega|T - \int_0^T \frac{n(x,t)}{v(x,t)} dx dt + \int_\Omega \ln v(x,T) - \int_\Omega \ln v_0(x)
\]

\[
= \lim_{\varepsilon \to 0} \left\{ |\Omega|T - \int_0^T \frac{n(x,t)}{v(x,t)} dx dt + \int_\Omega \ln v(x,T) - \int_\Omega \ln v_0(x) \right\}.
\]

Multiplying the second equation in (2.7) by \(1\) gets that

\[
\int_0^T \frac{\nabla v(x,t)}{v^2(x,t)} dx dt = |\Omega|T - \int_0^T \frac{n(x,t)}{v(x,t)} dx dt + \int_\Omega \ln v(x,T) - \int_\Omega \ln v_0(x),
\]

complies with (3.78), which yields

\[
\int_0^T \frac{\nabla v(x,t)}{v^2(x,t)} dx dt \geq \liminf_{\varepsilon \to 0} \int_0^T \frac{\nabla v_\varepsilon(x,t)}{v_\varepsilon^2(x,t)} dx dt.
\]

On the other hand, by lower semicontinuity of the norm in \(L^2(\Omega \times (0,T))\) with respect to weak convergence,

\[
\int_0^T \frac{\nabla v(x,t)}{v^2(x,t)} dx dt \leq \liminf_{\varepsilon \to 0} \int_0^T \frac{\nabla v_\varepsilon(x,t)}{v_\varepsilon^2(x,t)} dx dt,
\]

combined (3.79) with (3.80), we immediately obtain (3.75). This completes the proof.

Finally, we prove the main theorem.

\textbf{The proof of Theorem 1.1 (Existence).} (i) Since the regularity properties of \(n,v,u\) and \(\psi\) have been proved in Lemma 3.8, we will show that \((n,v,u)\) is a very weak subsolution of (1.1) in \(\Omega \times (0,T)\) for all \(T > 0\). We fixed an arbitrary nonnegative \(\psi_1 \in C_0^\infty(\overline{\Omega} \times [0,T])\) with \(\frac{\partial \psi_1}{\partial v} = 0\) on \(\partial \Omega \times (0,T)\), and then multiply the first equation in (2.5) by \(\psi_1\), we have

\[
- \int_0^T \int_\Omega n(x,t)\psi_1(x,t) dx dt - \int_\Omega n_0\psi_1(x,0)
\]

\[
= \chi \int_0^T \int_\Omega \frac{n(x,t)}{1 + \varepsilon n(x,t)} \nabla v_\varepsilon(x,t) \cdot \nabla \psi_1(x,t) dx dt
\]

\[
+ \int_0^T \int_\Omega n(x,t)(u_\varepsilon(x,t) \cdot \nabla \psi_1(x,t)) dx dt + a \int_0^T \int_\Omega n_\varepsilon(x,t)\psi_1(x,t) dx dt
\]

\[
+ \int_0^T \int_\Omega n(x,t) \cdot \Delta \psi_1(x,t) dx dt - b \int_0^T \int_\Omega n_\varepsilon(x,t)\psi_1(x,t) dx dt
\]

From (3.49) and (3.54), we readily conclude that

\[
- \int_0^T \int_\Omega n(x,t)\psi_1(x,t) dx dt \to - \int_0^T \int_\Omega n(x,t)\psi_1(x,t) dx dt,
\]

\[
\int_0^T \int_\Omega n(x,t) \cdot \Delta \psi_1(x,t) dx dt \to \int_0^T \int_\Omega n(x,t) \cdot \Delta \psi_1(x,t) dx dt
\]

\[
= - \int_0^T \int_\Omega \nabla n(x,t) \cdot \nabla \psi_1(x,t) dx dt,
\]

\[
\int_0^T \int_\Omega n(x,t)(u_\varepsilon(x,t) \cdot \nabla \psi_1(x,t)) dx dt \to \int_0^T \int_\Omega n(x,t)(u(x,t) \cdot \nabla \psi_1(x,t)) dx dt,
\]

and

\[
\int_0^T \int_\Omega n_\varepsilon(x,t)\psi_1(x,t) dx dt \to \int_0^T \int_\Omega n(x,t)\psi_1(x,t) dx dt.
\]
Together (3.82)-(3.86) with the Fatou lemma, we have

\[
a \int_0^T \int_\Omega n_\varepsilon(x,t) \psi_1(x,t) dx \, dt \to a \int_0^T \int_\Omega n(x,t) \psi_1(x,t) dx \, dt, \tag{3.85}
\]
as \varepsilon = \varepsilon_j \searrow 0. Meanwhile, by (3.50) and (3.52),

\[
\chi \int_0^T \int_\Omega \frac{n_\varepsilon(x,t)}{(1 + \varepsilon n_\varepsilon(x,t))^4} \nabla v_\varepsilon(x,t) \cdot \nabla \psi_1(x,t) dx \, dt \\
\to \chi \int_0^T \int_\Omega \frac{n(x,t)}{v(x,t)} \nabla v(x,t) \cdot \nabla \psi_1(x,t) dx \, dt \text{ as } \varepsilon = \varepsilon_j \searrow 0. \tag{3.87}
\]

Together (3.82)-(3.86) with the Fatou lemma, we have

\[
b \int_0^T \int_\Omega n^\theta(x,t) \psi_1(x,t) dx \, dt \\
\leq b \liminf_{\varepsilon \to \varepsilon_j} \int_0^T \int_\Omega n_\varepsilon(x,t) \psi_1(x,t) dx \, dt \\
= \int_0^T \int_\Omega n(x,t) \psi_1(x,t) dx \, dt + \int_\Omega n(\cdot,0) - \int_0^T \int_\Omega \nabla n(x,t) \cdot \nabla \psi_1(x,t) dx \, dt \\
+ a \int_0^T \int_\Omega n(x,t) \psi_1(x,t) dx \, dt + \chi \int_0^T \int_\Omega \frac{n(x,t)}{v(x,t)} \nabla v(x,t) \cdot \nabla \psi_1(x,t) dx \, dt \\
+ \int_0^T \int_\Omega n(x,t) (u(x,t) \cdot \nabla \psi_1(x,t)) dx \, dt. \tag{3.88}
\]

Fixed arbitrary \( \psi_2 \in L^\infty(\Omega \times (0,T)) \cap L^2((0,T);W^{1,2}(\Omega)) \) and \( \psi_2t \in L^2(\Omega \times (0,T)) \).

Multiplying the second equation in (2.5) by \( \psi_2 \), we obtain

\[
- \int_0^T \int_\Omega v_\varepsilon(x,t) \psi_2(x,t) dx \, dt - \int_\Omega v_0(x) \psi_2(x,0) \\
= - \int_0^T \int_\Omega \nabla v_\varepsilon(x,t) \cdot \nabla \psi_2(x,t) dx \, dt - \int_0^T \int_\Omega v_\varepsilon(x,t) \psi_2(x,t) dx \, dt \\
+ \int_0^T \int_\Omega n_\varepsilon(x,t) \psi_2(x,t) dx \, dt + \int_0^T \int_\Omega u_\varepsilon(x,t) v_\varepsilon(x,t) \cdot \nabla \psi_2(x,t) dx \, dt. \tag{3.89}
\]

In accordance with (3.49), (3.51) and (3.54), the equality about \( v \) is obtained.

Similarly, testing the third equation of (2.5) by \( \psi_3 \in C_0^\infty(\Omega \times (0,\infty);\mathbb{R}^N) \) with \( \nabla \cdot \psi_3 \equiv 0 \) in \( \Omega \times (0,\infty) \), we have

\[
- \int_0^T \int_\Omega u_\varepsilon(x,t) \psi_3(x,t) dx \, dt - \int_\Omega u_0(x) \psi_3(x,0) \\
= - \int_0^T \int_\Omega \nabla u_\varepsilon(x,t) \cdot \nabla \psi_3(x,t) dx \, dt + \int_0^T \int_\Omega n_\varepsilon(x,t) \nabla \phi(x,t) \cdot \psi_3(x,t) dx \, dt \\
+ \kappa \int_0^T \int_\Omega (Y_\varepsilon u_\varepsilon(x,t) \otimes u_\varepsilon(x,t)) : \nabla \psi_3(x,t) dx \, dt. \tag{3.90}
\]

Relying on the convergence properties (3.50) and (3.54)-(3.57), we get (2.3) by taking \( \varepsilon = \varepsilon_j \searrow 0 \). This shows that \( (n,v,u) \) is a very weak subsolution of (1.1).

Multiplying the first equation in (2.5) by \( kn_\varepsilon^{k-1} \psi_1 \), integrating by part, we have

\[
- \int_0^T \int_\Omega n^k_\varepsilon(x,t) \psi_1(x,t) dx \, dt - \int_\Omega n^k_0(x) \psi_1(x,0) \\
= \int_0^T \int_\Omega n_\varepsilon(x,t) \psi_1(x,t) dx \, dt + \int_\Omega n(\cdot,0) - \int_0^T \int_\Omega \nabla n_\varepsilon(x,t) \cdot \nabla \psi_1(x,t) dx \, dt.
\]
By using (3.49), we have
\begin{align*}
- \int_0^T \int_\Omega n^k_\varepsilon(x,t)\Delta \psi_1(x,t)dxdt + ak \int_0^T \int_\Omega n^k_\varepsilon(x,t) \psi_1(x,t)dxdt \\
+ \chi k(k-1) \int_0^T \int_\Omega n^{k-1}_\varepsilon(x,t) \nabla n_\varepsilon(x,t) \cdot \nabla v_\varepsilon(x,t) \psi_1(x,t)dxdt \\
+ \int_0^T \int_\Omega n^k_\varepsilon(x,t)(u_\varepsilon(x,t) \cdot \nabla \psi_1(x,t))dxdt - bk \int_0^T \int_\Omega n^{k-1+\theta}_\varepsilon(x,t) \psi_1(x,t)dxdt.
\end{align*}
(3.91)

By using (3.49), we have
\begin{align*}
- \int_0^T \int_\Omega n^k_\varepsilon(x,t)\psi_{11}(x,t)dxdt & \to - \int_0^T \int_\Omega n^k(x,t)\psi_{11}(x,t)dxdt, \\
\int_0^T \int_\Omega n^k_\varepsilon(x,t)\Delta \psi_1(x,t)dxdt & \to \int_0^T \int_\Omega n^k(x,t)\Delta \psi_1(x,t)dxdt, \\
ak \int_0^T \int_\Omega n^k_\varepsilon(x,t)\psi_1(x,t)dxdt & \to ak \int_0^T \int_\Omega n^k(x,t)\psi_1(x,t)dxdt, \\
- bk \int_0^T \int_\Omega n^{k-1+\theta}_\varepsilon(x,t) \psi_1(x,t)dxdt & \to - bk \int_0^T \int_\Omega n^{k-1+\theta}(x,t) \psi_1(x,t)dxdt.
\end{align*}
(3.92) (3.93) (3.94) (3.95)

as \( \varepsilon = \varepsilon_j \to 0 \). Together (3.48) with (3.75) to obtain
\begin{align*}
\chi k(k-1) \int_0^T \int_\Omega n^{k-1}_\varepsilon(x,t) \nabla n_\varepsilon(x,t) \cdot \nabla v_\varepsilon(x,t) \psi_1(x,t)dxdt \\
\to \chi k(k-1) \int_0^T \int_\Omega n^{k-1}(x,t) \nabla n(x,t) \cdot \nabla v(x,t) \psi_1(x,t)dxdt
\end{align*}
(3.96)

as \( \varepsilon = \varepsilon_j \to 0 \). By using (3.50) and (3.52), we have
\begin{align*}
\chi k \int_0^T \int_\Omega \frac{n^k_\varepsilon(x,t)}{(1+\varepsilon n_\varepsilon(x,t))^3} \cdot \frac{\nabla v_\varepsilon(x,t)}{v(x,t)} \psi_1(x,t)dxdt \\
\to \chi k \int_0^T \int_\Omega \frac{n^k(x,t)}{v(x,t)} \nabla v(x,t) \cdot \nabla \psi_1(x,t)dxdt
\end{align*}
(3.97)

as \( \varepsilon = \varepsilon_j \to 0 \). In addition, from (3.49) and (3.54) to get
\begin{align*}
\int_0^T \int_\Omega n^k_\varepsilon(x,t)(u_\varepsilon(x,t) \cdot \nabla \psi_1(x,t))dxdt & \to \int_0^T \int_\Omega n^k(x,t)(u(x,t) \cdot \nabla \psi_1(x,t))dxdt.
\end{align*}
(3.98)

as \( \varepsilon = \varepsilon_j \to 0 \). Therefore, combining (3.91) with (3.92)-(3.98) and Fatou lemma, we obtain
\begin{align*}
k(1-k) \int_0^T \int_\Omega n^{k-2}_\varepsilon(x,t) |\nabla n_\varepsilon(x,t)|^2 \psi_1(x,t)dxdt \\
\leq \liminf_{\varepsilon = \varepsilon_j \to 0} k(1-k) \int_0^T \int_\Omega n^{k-2}(x,t) |\nabla n(x,t)|^2 \psi_1(x,t)dxdt \\
= - \int_0^T \int_\Omega n^k(x,t)\psi_{11}(x,t)dxdt - \int_0^T \int_\Omega n^k_0(x)\psi_1(x,0)dxdt - \int_0^T \int_\Omega n^k(x,t)\Delta \psi_1(x,t)dxdt.
\end{align*}
Lemma 3.1. We have

Combining (3.101) with (3.102), we have

3.2 in [16], we obtain

Multiplying the first equation in (2.5) by $\varsigma$, we have

Theorem 1.1. There exist $\sigma^2 \min \{\sigma_1^2, \sigma_2^2\}$, $c_1 = (a + 2)2^{\alpha + 1} b^{-\frac{\alpha + 1}{2}}$. We differentiate the second equation in (2.5) and multiply the second equation in (2.5) by $4v_\varepsilon^3$, similar to Lemma 3.2 in [16], we obtain

Combining (3.101) with (3.102), we have

\[
\begin{align*}
\frac{d}{dt} \left( \int_{\Omega} n_{\varepsilon}^2 + \int_{\Omega} |\nabla v_{\varepsilon}|^4 \right) \\
\leq -4 \left( \int_{\Omega} n_{\varepsilon}^2 + \int_{\Omega} |\nabla v_{\varepsilon}|^4 \right) - \int_{\Omega} |\nabla n_{\varepsilon}|^2 - \int_{\Omega} |\nabla |\nabla v_{\varepsilon}|^2|^2 + 2c_1 m \\
+ c_2 \int_{\Omega} n_{\varepsilon}^3 + c_2 \int_{\Omega} |\nabla v_{\varepsilon}|^6 \leq 14 \int_{\Omega} n_{\varepsilon}^2 |\nabla v_{\varepsilon}|^2 + 14 \int_{\Omega} |v_{\varepsilon}|^2 |\nabla v_{\varepsilon}|^4 .
\end{align*}
\]
where \( c_2 = 14 + \frac{x^2}{\varepsilon} \). We deduce from the embedding theorem, the Gagliardo-Nirenberg inequality and the Young’s inequality that

\[
14 \left( \int_{\Omega} |u_\varepsilon|^6 \right)^{\frac{1}{3}} \left( \int_{\Omega} |\nabla u_\varepsilon|^6 \right)^{\frac{1}{3}} \leq c_3 \| \nabla u_\varepsilon \|_{L^2(\Omega)}^2 + c_5, \\
\leq c_4 \left( \| \nabla \nabla u_\varepsilon \|_{L^3(\Omega)}^2 \| \nabla u_\varepsilon \|_{L^2(\Omega)}^2 + \| \nabla u_\varepsilon \|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \leq \frac{1}{2} \int_{\Omega} |\nabla \nabla u_\varepsilon|^2 + c_5,
\]

where \( c_3, c_4, c_5 > 0 \) are constants. By the Gagliardo-Nirenberg inequality and the Young’s inequality, we have

\[
\int_{\Omega} n_\varepsilon^3 \leq c_6 \left( \| \nabla n_\varepsilon \|_{L^2(\Omega)}^2 \| n_\varepsilon \|_{L^2(\Omega)}^2 + \| n_\varepsilon \|_{L^2(\Omega)}^2 \right) \leq \frac{1}{c_2} \int_{\Omega} |\nabla n_\varepsilon|^2 + c_6 \left( \int_{\Omega} n_\varepsilon^2 \right)^{\frac{3}{2}} + c_7 \left( \int_{\Omega} n_\varepsilon^2 \right)^{\frac{3}{2}},
\]

where \( c_7 = c_2 c_6^4 \). Similar to (3.105), we have

\[
\int_{\Omega} |\nabla u_\varepsilon|^6 \leq \frac{1}{2c_2} \int_{\Omega} |\nabla \nabla u_\varepsilon|^2 + c_6 \left( \int_{\Omega} |\nabla u_\varepsilon|^4 \right)^{\frac{3}{2}} + c_8 \left( \int_{\Omega} |\nabla u_\varepsilon|^4 \right)^{\frac{3}{2}},
\]

where \( c_8 = 16 c_2\varepsilon^4 \). Denote \( y(t) := \int_{\Omega} n_\varepsilon^2 + \int_{\Omega} |\nabla v_\varepsilon|^4 \), from (3.103)-(3.106), we have

\[
y'(t) \leq -4y(t) + c_1 y^{\frac{3}{2}}(t) + c_1 c_8 y^3(t) + 2c_1 m, \quad t > 0,
\]

where \( y(0) = \int_{\Omega} n_0^2 + \int_{\Omega} |\nabla v_0|^4 \). Then, we let

\[
g_3(\eta, m) := -4\eta + c_1 c_8 \eta^{\frac{3}{2}} + c_1 c_8 \eta^3 + 2c_1 m, \quad \eta > 0.
\]

Then there exists \( m_0 > 0 \) such that \( g_3(\eta, m) \) has the unique positive root \( \eta_0 \). It is clear to see that \( y(t) \equiv \eta_0 \) fulfill the ordinary differential equation

\[
y'(t) = g_3(y(t), m_0), \quad t > 0,
\]

where initial condition \( y(0) = \eta_0 \). Then, we let

\[
\zeta_1 := \left( \frac{m_0}{\Omega} \right)^{\theta - 1} \quad \text{and} \quad \zeta_2 := \min \{ \sigma_0, \frac{m_0^2}{|\Omega|} \}
\]

complies with \( \frac{a}{\varepsilon} < \zeta_1 \) and \( \| n_0 \|_{L^2(\Omega)} + \| \nabla v_0 \|^2 \|_{L^2(\Omega)} < \zeta_2 \). Then

\[
\int_{\Omega} n_0 < |\Omega|^\frac{1}{2} \| n_0 \|_{L^2(\Omega)} < m_0,
\]

and thus,

\[
\int_{\Omega} n_\varepsilon \leq m := \max \left\{ \int_{\Omega} n_0, \left( \frac{a}{b} \right)^{\frac{1}{\theta}} |\Omega| \right\} < m_0
\]

for all \( \varepsilon \in (0, 1) \). It is obvious that \( g_3(\eta, m) < g_3(\eta, m_0) \) provided \( m < m_0 \), then the function \( g_3(\eta, m) \) has fitly two positive roots \( \eta_* < \eta_0 < \eta^* \), and thus, \( g_3(\eta, m) < 0 \) if \( \eta \in (\eta_*, \eta_0) \). By the comparison arguments, we conclude that

\[
y(t) = \int_{\Omega} n_\varepsilon^2 + \int_{\Omega} |\nabla v_\varepsilon|^4 \leq \eta_0, \quad t > 0
\]

for all \( \varepsilon \in (0, 1) \). This completes the proof.

Finally, we prove the main theorem.
The proof of Theorem 1.1 (Boundedness). Based on Lemma 3.11, we obtain the global boundedness of solutions to the regularization problem by a similar argument as that in [28, 55] that
\[
\|n_\varepsilon\|_{L^\infty(\Omega)} + \|v_\varepsilon\|_{W^{1,\infty}(\Omega)} + \|u_\varepsilon\|_{L^\infty(\Omega)} \leq C, \quad t > 0
\]
with some \(C > 0\) for all \(\varepsilon \in (0, 1)\). Consequently, we conclude that the very weak solution \((n, v, u)\) is globally bounded in time as well by taking \(\varepsilon = \varepsilon_j \searrow 0\). This completes the proof.

4. The proof of Theorem 1.2. In this section, we consider classical solution to (1.1) with \(\kappa = 0\) and \(\theta > 2\). At the first steps, we give the following estimate is crucial to prove the main theorem.

Lemma 4.1. Let \((n, v, u)\) be the local classical solution of (1.1). Then for \(q > 1\), there exists \(C_1 > 0\) such that
\[
\frac{1}{4} \frac{d}{dt} \int_\Omega n^4 + \frac{1}{q} \frac{d}{dt} \int_\Omega |\nabla v|^{2q} \leq -\frac{1}{4} \int_\Omega n^4 - \frac{1}{q} \int_\Omega |\nabla v|^{2q} - \frac{q-1}{4q^2} \int_\Omega |\nabla |\nabla v|^q|^2 + C_1 \int_\Omega \left( \frac{|\nabla v|}{v} \right)^{\frac{2q(\theta + 3)}{q-2}} + C_1 \int_\Omega |\nabla v|^{(2q-2)\frac{\theta + 3}{q-2}} + C_1
\]
for all \(t \in (0, T_{\text{max}})\).

Proof. Multiplying the first equation in (1.1) by \(n^3\), integrating by part and using Young’s inequality, we have
\[
\frac{1}{4} \frac{d}{dt} \int_\Omega n^4 = \int_\Omega n^3 [\Delta n - u \cdot \nabla n - \chi \nabla \cdot (\frac{n}{v} \nabla v) + an - bn^\theta] \\
= -3 \int_\Omega n^2 |\nabla n|^2 + 3\chi \int_\Omega \frac{n^3}{v} \nabla n \cdot \nabla v + a \int_\Omega n^4 - b \int_\Omega n^{\theta + 3} \\
\leq 3\chi^2 \int_\Omega n^4 \frac{|\nabla v|^2}{v^2} + a \int_\Omega n^4 - b \int_\Omega n^{\theta + 3} \\
\leq -\frac{1}{4} \int_\Omega n^4 + c_1 \int_\Omega \left( \frac{|\nabla v|}{v} \right)^{\frac{2q(\theta + 3)}{q-2}} - \frac{b}{2} \int_\Omega n^{\theta + 3} + c_1,
\]
where \(c_1 > 0\) is a constant. Similar to [17, 18], we obtain
\[
\frac{1}{4} \frac{d}{dt} \int_\Omega |\nabla v|^{2q} + \frac{q-1}{2q^2} \int_\Omega |\nabla |\nabla v|^q|^2 + 2 \int_\Omega |\nabla v|^{2q} \\
\leq (2q + 1) \int_\Omega n^2 |\nabla v|^{2(q-1)} + (4q + 2) \int_\Omega |u|^2 |\nabla v|^{2q} + c_2 \\
\leq b \int_\Omega n^{\theta + 3} + c_3 \int_\Omega |\nabla v|^{2(q-1)} + (4q + 2) \left( \int_\Omega |u|^6 \right)^{\frac{1}{4}} \left( \int_\Omega |\nabla v|^{3q} \right)^{\frac{q}{4}} + c_2,
\]
where \(c_2, c_3 > 0\) are constants. Analogous to Lemma 3.11, we deduce from the embedding theorem, the Gagliardo-Nirenberg inequality and the Young’s inequality that
\[
(4q + 2) \left( \int_\Omega |u|^6 \right)^{\frac{1}{4}} \left( \int_\Omega |\nabla v|^{3q} \right)^{\frac{2}{4}} \leq \frac{q-1}{4q^2} \int_\Omega |\nabla |\nabla v|^q|^2 + c_4
\]
with some \(c_4 > 0\). Choosing \(C_1 := \max\{c_1, c_3, c_2 + c_4\}\), the combination of (4.2)-(4.4), we obtain (4.1). This completes the proof.
Lemma 4.2. Assume that the conditions of Theorem 1.3 hold, there exists \( L_9 > 0 \) such that
\[
\|n(\cdot, t)\|_{L^4(\Omega)} \leq L_9 \quad \text{for all } t \in (0, T_{\text{max}}).
\] (4.5)

Proof. Since \( \theta > 2 \) fulfilling the condition of Theorem 1.1(ii), together Lemma 3.1(2) with (4.1), we have
\[
\frac{1}{4} \frac{d}{dt} \int_\Omega n^{4} + \frac{1}{q} \frac{d}{dt} \int_\Omega |\nabla v|^{2q} \leq -\frac{1}{4} \int_\Omega n^{4} - \frac{1}{q} \int_\Omega |\nabla v|^{2q} - \frac{q-1}{4q^2} \int_\Omega |\nabla |\nabla v|^{q}|^2 + C_1
\]
\[
+ C_2 \sigma_2 \frac{2(\theta+3)}{\theta+1} \int_\Omega |\nabla v|^{2(\theta+3)} + C_1 \int_\Omega |\nabla v|^{(2q-2)(\theta+3)q}
\] (4.6)
for all \( t \in (0, T_{\text{max}}) \). Letting \( \max\{2, \frac{4}{\theta+1} + \frac{1}{3}\} < q < \frac{1}{5}(5\theta + 11) \). By the embedding theorem, the Gagliardo-Nirenberg inequality and the Young’s inequality, there exists \( c_1, C > 0 \) such that
\[
\|\nabla v\|_{L^\frac{2(\theta+3)}{\theta+1}(\Omega)} \leq \left( C \|\nabla |\nabla v|^{q}\|_{L^5(\Omega)} \|\nabla v\|_{L^2(\Omega)}^{\frac{1}{3}} \|\nabla v\|_{L^\frac{2}{2}(\Omega)}^{\frac{1}{3}} + C \|\nabla v\|_{L^\frac{2}{2}(\Omega)}^{1}\right)^{\frac{1}{2}}
\]
\[
\leq c_1 \|\nabla v\|_{L^\frac{2}{2}(\Omega)}^{\frac{2(\theta+3)}{\theta+1}} + c_2 \leq \left( \frac{q-1}{8q^2 C_1} \right)^{\frac{2(\theta+3)}{\theta+1}} \int_\Omega |\nabla v|^{q} + c_2, \quad (4.7)
\]
where
\[
\eta_4 = \frac{3q}{2} \frac{3(q-1)\theta}{2(\theta+3)},
\]
\[
c_2 = \left( \frac{2(\theta+3)}{\theta+1} q(\theta-1)(q-1) \right)^{\frac{1}{q}} - \frac{\eta_4(q-1)}{q(\theta-1)} \frac{q(\theta-1)}{q(\theta+3)} - \frac{\eta_4(q-1)}{q(q+3)} - \frac{\eta_4(q-1)}{q(q+3)}.
\]
Since \( q > \frac{4}{\theta+1} + \frac{1}{3} \), by a simple calculation, we know that \( \eta_4 \in (0, 1) \) and \( \frac{\eta_4(q-1)}{q(q+3)} < 1 \). Similar to (4.7), there exists \( c_3, C > 0 \) such that
\[
\|\nabla v\|_{L^\frac{2(\theta+3)}{\theta+1}(\Omega)} \leq \left( C \|\nabla |\nabla v|^{q}\|_{L^5(\Omega)} \|\nabla v\|_{L^2(\Omega)}^{\frac{1}{3}} \|\nabla v\|_{L^\frac{2}{2}(\Omega)}^{\frac{1}{3}} + C \|\nabla v\|_{L^\frac{2}{2}(\Omega)}^{1}\right)^{\frac{1}{2}}
\]
\[
\leq c_3 \|\nabla v\|_{L^\frac{2}{2}(\Omega)}^{\frac{2q-2(\theta+3)q}{\theta+1}} + c_3 \leq \left( \frac{q-1}{8q^2 C_1} \right)^{\frac{2q-2(\theta+3)q}{\theta+1}} \int_\Omega |\nabla v|^{q} + c_4, \quad (4.8)
\]
where
\[
\eta_5 = \frac{3q}{2} \frac{3(q-1)\theta}{2(\theta+3)},
\]
\[
c_4 = \left( \frac{q-1}{8q^2 C_1} \right)^{\frac{2q-2(\theta+3)q}{\theta+1}} - \frac{\eta_5(q-1)}{q(\theta+3)} - \frac{\eta_5(q-1)}{q(q+3)} - \frac{\eta_5(q-1)}{q(q+3)} \times \frac{(q-1)(\theta+1)q}{(q+1)(\theta+3)q}.
\]
Since \( 2 < q < \frac{1}{5}(5\theta + 11) \), a simple calculation, we readily conclude that \( \eta_5 \in (0, 1) \) and \( \frac{\eta_4(q-1)}{q(q+3)} < 1 \). In accordance with the above analysis, we know that
\[
\frac{q-1}{4q^2} - C_1 \sigma_2 \frac{2(\theta+3)}{\theta+1} \frac{q-1}{8q^2 C_1} - C_1 \frac{q-1}{8q^2 C_1} = 0. \quad (4.9)
\]
The combination of (4.6)-(4.9), we obtain
\[
\frac{1}{4} \frac{d}{dt} \int_\Omega n^{4} + \frac{1}{q} \frac{d}{dt} \int_\Omega |\nabla v|^{2q} \leq -\frac{1}{4} \int_\Omega n^{4} - \frac{1}{q} \int_\Omega |\nabla v|^{2q} + c_5 \quad (4.10)
\]
with $c_5 = c_1 + c_2 + c_4$ for all $t \in (0, T_{\max})$, we obtain the desired result. This completes the proof.

**Lemma 4.3.** Assume that the conditions of Theorem 1.3 hold. Then for all $\epsilon \in (\frac{3}{4}, 1)$ one can find $C(\epsilon) > 0$ such that
\[
\|A' u(\cdot, t)\|_{L^2(\Omega)} \leq C(\epsilon) \quad \text{for all } t \in (0, T_{\max}).
\]  
(4.11)

**Proof.** By the variation-of-constants formula for the projected version of the third equation in (1.1), we have
\[
\|A' u_\varepsilon\|_{L^2(\Omega)} \leq \|A' e^{-tA} u_0\|_{L^2(\Omega)} + \int_0^t \|A' e^{-(t-s)A} P(n_\varepsilon \nabla \phi_\varepsilon)\|_{L^2(\Omega)} ds
\]
\[
\leq \|A' u_0\|_{L^2(\Omega)} + c_1 \int_0^t e^{-\lambda_1 (t-s)} (t-s)^{-\epsilon} \|n_\varepsilon \nabla \phi_\varepsilon\|_{L^2(\Omega)} ds
\]
\[
\leq e^{-t} \|A' u_0\|_{L^2(\Omega)} + c_1 \int_0^t e^{-\lambda_1 (t-s)} (t-s)^{-\epsilon} \|n_\varepsilon \nabla \phi_\varepsilon\|_{L^2(\Omega)} ds
\]
\[
\leq c_2 \quad \text{for any } \epsilon \in (\frac{3}{4}, 1),
\]
here we have been used the Lemma 4.2. This completes the proof.

**Lemma 4.4.** Assume that the conditions of Theorem 1.3 hold. There exists constant $C > 0$ such that
\[
\|n(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}).
\]  
(4.12)

**Proof.** Given $T \in (0, T_{\max})$. Let $M(T) := \sup_{t \in (0, T]} \|n(\cdot, t)\|_{L^\infty(\Omega)}$ and $g := \frac{\nabla u}{\|\nabla u\|} + u$. From Lemma 3.1 and Lemma 4.2 with $q = 2$, there exists constant $c_1 > 0$ such that
\[
\|g(\cdot, t)\|_{L^1(\Omega)} \leq c_1 \quad \text{for all } t \in (0, T_{\max}).
\]  
(4.13)

By the variation-of-constants formula with respect to $n$, for every $t \in (0, T)$, we have
\[
n(\cdot, t) = e^{(t-t_0)\Delta} n(\cdot, t_0) - \int_{t_0}^t e^{(t-s)\Delta} \nabla \cdot (n(\cdot, s) g(\cdot, s)) ds
\]
\[
+ \int_{t_0}^t e^{(t-s)\Delta} (an(\cdot, s) - bn^\theta(\cdot, s)) ds
\]
\[
=: I_1(\cdot, t) + I_2(\cdot, t) + I_3(\cdot, t),
\]  
(4.14)

where $t_0 := (t-1)^+$. By the maximum principle, we obtain
\[
\|I_1(\cdot, t)\|_{L^\infty(\Omega)} \leq \|n_0\|_{L^\infty(\Omega)} \quad \text{if } t \in (0, 1],
\]  
(4.15)

if $t > 1$, from standard $L^p$-$L^q$ estimates for the Neumann heat semigroup, there exists $c_2 > 0$ such that
\[
\|I_1(\cdot, t)\|_{L^\infty(\Omega)} \leq c_2 (t-t_0)^{-\frac{3}{2}} \|n(\cdot, t_0)\|_{L^1(\Omega)} = c_2 \|n(\cdot, t_0)\|_{L^1(\Omega)} \leq c_2 m.
\]  
(4.16)

Since $an - bn^\theta \leq c_3 := a^{\frac{n}{2\tau}} (b \theta)^{-\frac{1}{\tau}} (1 - \frac{t}{2})$ for all $n \geq 0$, by the maximum principle once more, we obtain
\[
I_3(\cdot, t) \leq \int_{t_0}^t e^{(t-s)\Delta} c_3 ds - c_3 (t-t_0) \leq c_3.
\]  
(4.17)
Then, we fix arbitrary \( p \in (3, 4) \), complies with Lemma 2.5 and Hölder inequality, there exists a constant \( c_4 > 0 \) such that
\[
\|I_2(\cdot, t)\|_{L^\infty(H)} \leq c_4 \int_{t_0}^t (t - s)^{-\frac{1}{2} - \frac{3}{4p}} \|n(\cdot, s)g(\cdot, s)\|_{L^p(H)} ds \\
\leq c_4 \int_{t_0}^t (t - s)^{-\frac{1}{2} - \frac{3}{4p}} \|n(\cdot, s)\|_{L^{1 + \eta}(H)} ||g(\cdot, s)||_{L^4(H)} ds \\
\leq c_4 \int_{t_0}^t (t - s)^{-\frac{1}{2} - \frac{3}{4p}} \|n(\cdot, s)\|_{L^{1 + \eta}(H)} ||n(\cdot, s)||^{1-\eta}_{L^4(H)} ||g(\cdot, s)||_{L^4(H)} ds,
\]
where \( \eta := \frac{5p-4}{4p} \in (0, 1) \). Combining (2.11) with (4.13), we immediately find that
\[
\|I_2(\cdot, t)\|_{L^\infty(H)} \leq c_4 c_5 m_1^{-\eta} \int_{t_0}^t \sigma^{-\frac{1}{2} - \frac{3}{4p}} d\sigma \cdot M^\eta(T),
\]
offering to \( p > 3 \), we readily conclude that \( \frac{1}{2} + \frac{3}{2p} < 1 \). Substituting (4.15)-(4.18) into (4.14), there exists \( c_5 > 0 \) such that
\[
\|n(\cdot, t)\|_{L^\infty(H)} \leq c_5 + c_6 M^\eta(T) \quad \text{for all } t \in (0, T),
\]
and thus,
\[
M(T) \leq c_5 + c_6 M^\eta(T) \quad \text{for all } t \in (0, T_{\max}).
\]
Applying Lemma 3.10 in [30], we have
\[
M(T) \leq \max\{2c_5, (2c_5)^{-1}\} \quad \text{for all } t \in (0, T_{\max}).
\]
This completes the proof. 

Finally, we prove the main theorem.

The proof of Theorem 1.3. With the regularity properties from Lemmas 4.3 and 4.4, by the standard parabolic regularity arguments applied to the second equation in (1.1), we readily conclude that \( \|v(\cdot, t)\|_{W^{1, q}(H)} \leq C(q) \) for every \( q > 1 \) and all \( t \in (0, T_{\max}) \), together with Lemmas 2.5, 4.3 and 4.4, we have \( T_{\max} = \infty \). This completes the proof.

5. A special case of fluid-free system with singular sensitivity and logistic source.

5.1. Global boundedness of chemotaxis system with singular sensitivity and logistic source. In this subsection, we will consider the following chemotaxis system with singular sensitivity and logistic source
\[
\begin{aligned}
nt = & \Delta n - \chi \nabla \cdot \left( \frac{n}{v} \nabla v \right) + \alpha n - \beta n^p, \quad x \in \Omega, \quad t > 0, \\
v_t = & \Delta v - v + n, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial n}{\partial \nu} = & \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \quad t > 0, \\
in(x, 0) = & n_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega,
\end{aligned}
\tag{5.1}
\]
where \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary \( \partial \Omega \) and \( \frac{\partial}{\partial \nu} \) denotes the derivative with respect to the outer normal of \( \partial \Omega \). Same as chemotaxis-fluid system, we also obtain a positive uniform-in-time lower bound of \( v \). At the first steps, we state the local solvability of (5.1).
Lemma 5.1. Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary. Assume that the initial data \((n_0, v_0)\) fulfill (1.2). Then for any \(a, b > 0\) and \(\theta > 1\), there exists a maximal existence time \(T_{\text{max}} \in (0, \infty)\) and a uniqueness pair of nonnegative functions

\[
(n, v) \in \left( C^0(\overline{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}})) \right)^2,
\]

which solves (5.1) classically and fulfills \(u, v > 0\) in \(\Omega \times (0, T_{\text{max}})\). Moreover, if \(T_{\text{max}} < +\infty\), then

\[
\|n_v(\cdot, t)\|_{L^\infty(\Omega)} + \|v_v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \to \infty \quad \text{as } t \to T_{\text{max}}.
\]

Lemma 5.2. Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary. Assume that \(\chi < 1\). Let (1.11), (1.12) hold and \((n, v)\) be a solution to (5.1) on \((0, T_{\text{max}})\). Then for all \(p \in (1, \frac{1}{\chi})\) and each \(q \in (\bar{q}_-, \bar{q}_+)\), we can find \(C > 0\) such that

\[
\int_{\Omega} n^p v^{-q} \leq Cb^{-1} \text{ for all } t \in (0, T_{\text{max}}),
\]

where \(C\) is independent of \(b\) and \(\bar{q}_\pm := \frac{p+1}{p}(1 \pm \sqrt{1-p\chi^2})\).

Proof. Similar to (3.5), we have

\[
\frac{d}{dt} \int_{\Omega} n^p v^{-q} \leq \left(\frac{p}{4(p-1)}[(p-1)\chi + 2q]^2 - pq\chi - q(q+1)\right) \int_{\Omega} n^p v^{-q} |\nabla v|^2
\]

\[
+ (q + ap) \int_{\Omega} n^p v^{-q} - bp \int_{\Omega} n^{p-1+\theta} v^{-q} - q \int_{\Omega} n^{p+1} v^{-q-1},
\]

for all \(t \in (0, T_{\text{max}})\). Let

\[
h_2(p; q, \chi) := \frac{p}{4(p-1)}[(p-1)\chi + 2q]^2 - pq\chi - q(q+1).
\]

Similar to Lemma 3.1, we claim that \(h_2(p; q, \chi) < 0\) provided that \(q \in (\bar{q}_-, \bar{q}_+)\), together with (5.5), we obtain

\[
\frac{d}{dt} \int_{\Omega} n^p v^{-q} \leq (q + ap) \int_{\Omega} n^p v^{-q} - bp \int_{\Omega} n^{p-1+\theta} v^{-q} - q \int_{\Omega} n^{p+1} v^{-q-1}
\]

for all \(t \in (0, T_{\text{max}})\). By the Young’s inequality and Lemma 3.1, we have

\[
(q + ap + 1) \int_{\Omega} n^p v^{-q}
\]

\[
\leq bp \int_{\Omega} n^{p-1+\theta} v^{-q} + \frac{1}{p-1+\theta} |b(q+1)|^{-p}(q + ap + 1)^{p-1+\theta} \int_{\Omega} v^{-q}
\]

\[
\leq bp \int_{\Omega} n^{p-1+\theta} v^{-q} + \frac{1}{p-1+\theta} |b(q+1)|^{-p}(q + ap + 1)^{p-1+\theta} \sigma^{-q}|\Omega|,
\]

where \(\sigma := \min\{\sigma_1, \sigma_2\}\), substituting (5.7) into (5.6), we obtain

\[
\frac{d}{dt} \int_{\Omega} n^p v^{-q} + \int_{\Omega} n^p v^{-q} \leq \frac{1}{p - 1 + \theta} |b(q+1)|^{-p}(q + ap + 1)^{p-1+\theta} \sigma^{-q}|\Omega|
\]

for all \(t \in (0, T_{\text{max}})\), integrating (5.8) from 0 to \(t\), we readily conclude that

\[
\int_{\Omega} n^p(\cdot, t) v^{-q}(\cdot, t) \leq e^{-t} \int_{\Omega} n_0^p v_0^{-q}
\]

\[
+ \frac{1}{p - 1 + \theta} |b(q+1)|^{-p}(q + ap + 1)^{p-1+\theta} \sigma^{-q}|\Omega|(1 - e^{-t}).
\]

This completes the proof. \(\square\)
Lemma 5.3. Assume that $0 < \chi < \sqrt{\frac{2}{3}}$. Then there exist $p_0 > \frac{3}{2}$, $q_0 \in (0, \frac{3}{2})$ and $\tilde{C} > 0$ such that
\[
\int_\Omega n^{p_0} v^{-q_0} \leq \tilde{C} b^{-1} \quad \text{for all } t \in (0, T_{\text{max}}),
\]
where $\tilde{C}$ is independent of $b$.

Proof. Since $0 < \chi < \sqrt{\frac{2}{3}}$, it is clear to see that $\frac{1}{\chi} > \frac{3}{2}$. Thus, we can choose $p := p_0$ such that $p \in (1, \frac{1}{\chi})$ and $q \in (\tilde{q}_-, \tilde{q}_+) \subset (0, \frac{3}{2})$, where $\tilde{q}_-, \tilde{q}_+$ are same as Lemma 5.2. By Lemma 5.2 results in (5.9). This completes the proof.

Finally, we prove the main theorem.

The proof of Theorem 1.5. In accordance with Lemma 5.3, we select $p_0 > \frac{3}{2}$ and $q_0 \in (0, \frac{3}{2})$ such that
\[
\int_\Omega n^{p_0} v^{-q_0} \leq c_1 \quad \text{for all } t \in (0, T_{\text{max}}),
\]
with certain constant $c_1 > 0$. Since $q_0 < \frac{3}{2}$ and $p_0 > \frac{3}{2}$, we can choose $r \in (\frac{3}{2}, p_0)$ such that $r < \frac{3(p_0 - q_0)}{3 - 2q_0}$. Using H"{o}lder inequality, we have
\[
\|n\|_{L^r(\Omega)} \leq \left(\int_\Omega n^{p_0} v^{-q_0}\right)^\frac{1}{r} \left(\int_\Omega v^{p_0 - r} \right)^{\frac{r}{p_0 - r}} \leq c_1^\frac{1}{r} \left(\int_\Omega v^{p_0 - r} \right)^{\frac{r}{p_0 - r}} \tag{5.11}
\]
for all $t \in (0, T_{\text{max}})$. For $v$, by Lemmata 2.6 and 2.9, we obtain
\[
\|v(t, \cdot)\|_{L^\frac{p_0}{q_0}(\Omega)} \leq \frac{1}{\Omega} \int_\Omega n, c_2, c_3 \geq 0 \text{ are constants, since } r < \frac{3(p_0 - q_0)}{3 - 2q_0}, \text{ by a simple calculation, we have } \frac{3}{2}\left(\frac{1}{r} - \frac{p_0 - r}{p_0 - q_0}\right) < 1. \text{ Therefore,}
\]
\[
\sup_{s \in (0, T_{\text{max}})} \|v(t, \cdot)\|_{L^\frac{p_0}{q_0}(\Omega)} \leq c_3 \left(1 + \sup_{s \in (0, T_{\text{max}})} \|n(\cdot, s)\|_{L^r(\Omega)}\right). \tag{5.13}
\]
Together with (5.11), we have
\[
\sup_{s \in (0, T_{\text{max}})} \|n(\cdot, s)\|_{L^r(\Omega)} \leq c_4 \left(1 + \left(\sup_{s \in (0, T_{\text{max}})} \|n(\cdot, s)\|_{L^r(\Omega)}\right)\frac{p_0}{r_0}\right) \tag{5.14}
\]
with some positive constant $c_4$. Since $\frac{p_0}{r_0} < 1$, by Lemma 3.10 in [30], we have
\[
\sup_{s \in (0, T_{\text{max}})} \|n(\cdot, s)\|_{L^r(\Omega)} \leq \tilde{m} := \max\{2c_4, (2c_4)\frac{p_0 - q_0}{p_0}\}. \tag{5.15}
\]
Combining (2.9) with (5.15), there exists \( \bar{r} > \frac{3}{2} \) such that

\[
\sup_{s \in (0, T_{\text{max}})} \| n(\cdot, s) \|_{L^p(\Omega)} \leq \tilde{m}_0,
\]

where \( \tilde{m}_0 > 0 \) is a constant. According to heat semigroup theory, for all \( k \in [1, \frac{3r}{3-r}] \), there exist some constants \( c_5, c_6 > 0 \) such that

\[
\| \nabla v(\cdot, t) \|_{L^k(\Omega)} \\
\leq \| \nabla e^{(t-A)\omega} \|_{L^k(\Omega)} + \int_0^t \| \nabla e^{(t-s)(A-1)} n(\cdot, s) \|_{L^k(\Omega)} ds \\
\leq c_5 \| \nabla v_0 \|_{L^\infty(\Omega)} + c_5 \int_0^t (1 + (t-s)^{-\frac{1}{2} - \frac{3}{2}(1 + \frac{1}{r})}) e^{-\lambda_1(t-s)} \| n(\cdot, s) \|_{L^p(\Omega)} ds \\
\leq c_5 \| \nabla v_0 \|_{L^\infty(\Omega)} + c_6 \tilde{m}_0 \int_0^t (1 + (t-s)^{-\frac{1}{2} - \frac{3}{2}(1 + \frac{1}{r})}) e^{-\lambda_1(t-s)} ds \\
\leq c_6 \text{ for all } t \in (0, T_{\text{max}}),
\]

(5.17)

Here we have been used Lemma 2.9 and the fact that \( \frac{1}{2} + \frac{3}{2}(1 + \frac{1}{r}) \leq 1 \). Therefore, there exists a positive constant \( c_7 > 0 \) such that

\[
\int_\Omega |\nabla v|^k \leq c_7 \text{ for all } t \in (0, T_{\text{max}}) \text{ and } k \in [1, \frac{3r}{3-r}).
\]

(5.18)

Then, given \( T \in (0, T_{\text{max}}) \), denote \( M(T) := \sup_{t \in [0, T]} \| n(\cdot, t) \|_{L^\infty(\Omega)} \). Since \( \bar{r} > \frac{3}{2} \), we immediately find that \( \frac{3r}{3-r} > \bar{r} \), there exists \( k \in (3, \frac{3r}{3-r}) \) such that (5.18) holds.

For \( n \), according to variation-of-constants formula, for each \( t \in (t_0, T) \),

\[
n(t) = e^{(t-t_0)\Delta} n(\cdot, t_0) - \int_{t_0}^t e^{(t-s)\Delta} \nabla \cdot \left( \frac{n(\cdot, s)}{v(\cdot, s)} \nabla v(\cdot, s) \right) ds \\
+ \int_{t_0}^t e^{(t-s)\Delta} (an(\cdot, s) - bn^\theta(\cdot, s)) ds,
\]

(5.19)

where \( t_0 := (t-1)_+ \). If \( t \in (0, 1] \), by means of the maximum principle, we have

\[
\| e^{(t-t_0)\Delta} n(\cdot, t_0) \|_{L^\infty(\Omega)} \leq \| n_0 \|_{L^\infty(\Omega)},
\]

(5.20)

while if \( t > 1 \), in accordance with Lemma 2.9 and (2.9), there exists a positive constant \( c_8, c_9 > 0 \) such that

\[
\| e^{(t-t_0)\Delta} n(\cdot, t_0) \|_{L^\infty(\Omega)} \leq c_8 (t-t_0)^{-\frac{2}{2}} \| n(\cdot, t_0) \|_{L^p(\Omega)} \leq c_9.
\]

(5.21)

Arbitrary given \( p \in (3, K) \), there exist some constants \( c_{10}, c_{11} > 0 \) such that

\[
\chi \int_{t_0}^t e^{(t-s)\Delta} \nabla \cdot \left( \frac{n(\cdot, s)}{v(\cdot, s)} \nabla v(\cdot, s) \right) ds \\
\leq c_{10} \int_{t_0}^t (1 + (t-s)^{-\frac{1}{2} - \frac{3}{2}e}) e^{-\lambda_1(t-s)} \| n(\cdot, s) \cdot \nabla v(\cdot, s) \|_{L^p(\Omega)} ds \\
\leq c_{10} \int_{t_0}^t (1 + (t-s)^{-\frac{1}{2} - \frac{3}{2}e}) e^{-\lambda_1(t-s)} \| n(\cdot, s) \|_{L^p(\Omega)}^{\frac{p}{2}} \| \nabla v(\cdot, s) \|_{L^\infty(\Omega)} ds \\
\leq c_{10} \int_{t_0}^t (1 + (t-s)^{-\frac{1}{2} - \frac{3}{2}e}) e^{-\lambda_1(t-s)} \| n(\cdot, s) \|_{L^\infty(\Omega)}^{\frac{p}{2}} \| \nabla v(\cdot, s) \|_{L^\infty(\Omega)} ds \\
\leq c_{11} M^{\alpha_k}(T) \text{ for all } t \in (0, T),
\]

(5.22)
where \( \iota_6 = \frac{p k - k + \nu}{p k} \in (0, 1) \), here we have been used the fact that \(-\frac{1}{2} - \frac{3}{2p} > -1\). Similarly,

\[
\int_{t_0}^{t} e^{(t-s)\Delta} (an(\cdot, s) - bn^\theta(\cdot, s)) \leq \int_{t_0}^{t} \sup_{\Omega} (an - bn^\theta)_+ \, ds \leq a^\theta \tau (b^\theta)^{-\frac{1}{\theta}} (1 - \frac{1}{\theta}), \tag{5.23}
\]

where \( \Lambda_+ \) denotes the positive part of \( \Lambda \). Combining with (5.20)-(5.23), there exists a positive constant \( c_{12} \) such that

\[
M(T) \leq c_{12} + c_{12} M^{\iota_6}(T) \text{ for all } T \in (0, T_{\text{max}}). \tag{5.24}
\]

Using Lemma 3.10 in \([30]\) again, we have

\[
\|n(\cdot, t)\|_{L^\infty(\Omega)} \leq c_{13} \tag{5.25}
\]

with some constant \( c_{13} > 0 \). By the Sobolev embedding theorem and parabolic regularity arguments, we readily conclude that

\[
\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq c_{14} \tag{5.26}
\]

with some constant \( c_{14} > 0 \). Combining with Lemma 5.1, we know that \((n, v)\) is global bounded classical solution. The proof of uniqueness similar to our recent work Lemma 2.2 in \([25]\), so we omit it. This completes the proof.

5.2. Large time behavior for the chemotaxis system with singular sensitivity and logistic source. In this subsection, motivated by our recent work \([23, 24]\), we deal with the large time behavior for a chemotaxis system with singular sensitivity and logistic source. For simplicity, we denote

\[
U := \vartheta^{-1} n, \quad V := v - \vartheta, \quad \vartheta := (\frac{a}{b})^{\frac{1}{\theta} - \frac{1}{\theta}}.
\]

Then, we rewriting the system (5.1) as:

\[
\begin{cases}
U_t = \Delta U - \chi \nabla \cdot (\frac{U}{v} \nabla V) + a U (1 - U^\theta - 1), & x \in \Omega, \ t > 0, \\
V_t = \Delta V - V + (\frac{a}{b})^{\frac{1}{\theta} - \frac{1}{\theta}} (U - 1), & x \in \Omega, \ t > 0, \\
\frac{\partial U}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\
U(x, 0) = (\frac{b}{a})^{\frac{1}{\theta} - \frac{1}{\theta}} n_0(x), \ V(x, 0) = v_0(x) - (\frac{a}{b})^{\frac{1}{\theta} - \frac{1}{\theta}}, & x \in \Omega.
\end{cases} \tag{5.27}
\]

From the proof of Lemma 3.1, it immediately derives that: there exists a constant \( l_0 > 0 \) independent of \( b \) such that

\[
\frac{1}{q^2} \leq l_0 \text{ for all } (x, t) \in \Omega \times (0, \infty), \tag{5.28}
\]

where \( l_0 = \frac{1}{\sigma_0}, \quad \sigma_0 = \min \{\sigma_1, \sigma_2\} \).

At the first steps, we state the main theorem.

**Lemma 5.4.** Let \( a > 0 \). Assume that

\[
b > \max \{1, (\frac{\chi^2 l_0}{8})^{-\frac{1}{\theta} - \frac{1}{\theta}} a^{\frac{\theta}{2}}\},
\]

the global solution of (5.1) has the property that there exists constant \( \eta_0 > 0 \) such that the nonnegative functions \( E \) and \( F \) defined by

\[
E(t) := \int_{\Omega} (U - 1 - \ln U) + \frac{K}{2} \int_{\Omega} V^2
\]


\[ F(t) := \int_{\Omega} (U - 1)^2 + \frac{K}{2} \int_{\Omega} V^2 \]
satisfies
\[ E'(t) \leq -\eta_0 F(t), \]
where
\[ \eta_0 := \min\{a - \frac{K}{2} \left( \frac{a}{b} \right)^{\frac{2}{\theta - 1}}, K - \frac{\chi^2 l_0}{4}\} \]
and \( K > 0 \) is a constant which fulfills that
\[ \frac{\chi^2 l_0}{4} < K < 2b^{\frac{2}{\theta - 1}} a^{\frac{\theta - 3}{\theta - 1}} \]
and \( l_0 \) is the same as (5.28).

**Proof.** The nonnegative of \( E(t) \) is similar to Lemma 3.3 in our recent work [23]. In accordance with strong maximal principle and \( U_0 > 0, U \) is positive in \( \Omega \times (0, \infty) \). Multiplying the first equation of (5.27) by \( 1 - \frac{1}{U} \) and using Young’s inequality, we have
\[
\frac{d}{dt} \int_{\Omega} (U - 1 - \ln U) = -\int_{\Omega} \frac{\vert \nabla U \vert^2}{U^2} + \int_{\Omega} \frac{\nabla U \cdot \nabla V}{U v} - a \int_{\Omega} (U - 1)(U^{\theta - 1} - 1)
\leq \frac{\chi^2}{4} \int_{\Omega} \frac{\vert \nabla V \vert^2}{v^2} - a \int_{\Omega} (U - 1)^2
\leq \frac{\chi^2 l_0}{4} \int_{\Omega} \vert \nabla V \vert^2 - a \int_{\Omega} (U - 1)^2. \tag{5.29}
\]
Then, we multiply the second and third equations of (5.27) by \( V \) and using Young’s inequality, which yields
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} V^2 \leq -\int_{\Omega} \vert \nabla V \vert^2 + \frac{1}{2} \varrho^2 \int_{\Omega} \vert U - 1 \vert^2 - \frac{1}{2} \int_{\Omega} V^2 \tag{5.30}
\]
The linear combination of (5.29) and (5.30), we conclude that
\[ E'(t) \leq -(a - K \frac{\varrho^2}{2}) \int_{\Omega} (U - 1)^2 - (K - \frac{\chi^2 l_0}{4}) \int_{\Omega} \vert \nabla V \vert^2 + \frac{K}{2} \int_{\Omega} V^2. \tag{5.31}
\]
Choosing
\[ \eta_0 := \min\{a - \frac{K}{2} \left( \frac{a}{b} \right)^{\frac{2}{\theta - 1}}, K - \frac{\chi^2 l_0}{4}\}, \]
since
\[ \frac{\chi^2 l_0}{4} < K < 2b^{\frac{2}{\theta - 1}} a^{\frac{\theta - 3}{\theta - 1}}, \]
by a simple calculation, we readily conclude that \( \eta_0 > 0 \), and thus,
\[ E'(t) \leq -\eta_0 F(t). \]
This completes the proof. \( \square \)

Finally, we prove the main theorem.

*The proof of Theorem 1.7.* From Lemma 5.4, it immediately derives that \( E(t) \) is nonnegative, and \( E'(t) \leq -\eta_0 F(t) \), we have
\[
\int_{t_0}^{t} F(t) \leq \frac{1}{\eta_0} (E(t_0) - E(t)) \leq \frac{1}{\eta_0} E(t_0) \quad \text{for all } t > t_0,
\]
which implies
\[ \int_{t_0}^t \int_\Omega (U - 1)^2 + \frac{K}{2} \int_{t_0}^t \int_\Omega V^2 < \infty. \]

By the standard parabolic regularity for parabolic equations [14], there exist $\delta \in (0, 1)$ and $C > 0$ such that $\| (u, v) \|_{C^{2+s, 1+s, 2/5}[\Omega, [t, t+1])] \leq C$ for all $t \geq 1$. By the similar reason of Lemma 3.10 in [29], we obtain
\[ \| U - 1 \|_{L^\infty(\Omega)} \to 0, \quad \| V \|_{L^\infty(\Omega)} \to 0 \]
as $t \to \infty$. Then there exists $t_0 > 0$ such that for all $t > t_0$, $\| U - 1 \|_{L^\infty(\Omega)} \leq \frac{1}{2}$, and similar to the proof of Theorem 3.3 in [23], by the L’Hospital rule, we obtain
\[ \lim_{U \to 1} \frac{U - 1 - \ln U}{(U - 1)^2} = \frac{1}{2}, \]
which implies
\[ \frac{1}{3} \left\{ \int_\Omega (U - 1)^2 + \frac{K}{2} \int_\Omega V^2 \right\} \leq E(t) \leq F(t) \]for all $t > t_0$. Therefore,
\[ E'(t) \leq -\eta_0 F(t) \leq -\eta_0 E(t), \]
which yields
\[ E(t) \leq E(t_0) e^{-\eta_0 (t-t_0)}. \]Substituting (5.33) into (5.32), we have
\[ \frac{1}{3} \int_{t_0}^t \left\{ \int_\Omega (U - 1)^2 + \frac{K}{2} \int_\Omega V^2 \right\} \leq E(t_0) e^{-\eta_0 (t-t_0)}, \]
and thus, there exists a constant $C > 0$ such that for all $t > t_0$,
\[ \| U(\cdot, t) - 1 \|_{L^2(\Omega)} \leq C e^{-\frac{\eta_0 t}{2}}, \quad \| V(\cdot, t) \|_{L^2(\Omega)} \leq C e^{-\frac{\eta_0 t}{2}}. \]
Moreover, from Lemma 5.2 in [62], there exists a constant $c_1 > 0$ such that
\[ \| U(\cdot, t) - 1 \|_{W^{1,\infty}(\Omega)} \leq c_1 e^{-\frac{\eta_0 t}{4}}, \quad \| V(\cdot, t) \|_{W^{1,\infty}(\Omega)} \leq c_1 e^{-\frac{\eta_0 t}{4}}. \]
Finally, by the Gagliardo-Nirenberg inequality, there exist some constants $c_2, c_3, c_4 > 0$ such that
\[ \| U(\cdot, t) - 1 \|_{L^\infty(\Omega)} \leq c_2 \| U(\cdot, t) - 1 \|_{W^{1,\infty}(\Omega)} \| U(\cdot, t) - 1 \|_{L^2(\Omega)} + c_2 \| U(\cdot, t) - 1 \|_{L^2(\Omega)} \]
\[ \leq c_3 \| U(\cdot, t) - 1 \|_{L^2(\Omega)} \leq c_4 e^{-\frac{\eta_0 t}{4}} \]
for all $t > 0$. Similarly, we have
\[ \| V(\cdot, t) \|_{L^\infty(\Omega)} \leq c_5 e^{-\frac{\eta_0 t}{4}} \]
with certain constant $c_5 > 0$. This completes the proof. \[ \square \]

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