ON FOURIER INTEGRAL OPERATORS WITH HÖLDER-CONTINUOUS PHASE

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Abstract. We study continuity properties in Lebesgue spaces for a class of Fourier integral operators arising in the study of the Boltzmann equation. The phase has a Hölder-type singularity at the origin. We prove boundedness in $L^1$ with a precise loss of decay depending on the Hölder exponent, and we show by counterexamples that a loss occurs even in the case of smooth phases. The results can be seen as a quantitative version of the Beurling-Helson theorem for changes of variables with a Hölder singularity at the origin. The continuity in $L^2$ is studied as well by providing sufficient conditions and relevant counterexamples. The proofs rely on techniques from Time-frequency Analysis.

1. Introduction

In the study of the Boltzmann equation one faces with integral operators (Boltzmann collision operators)

$$Af(x) = \int_{\mathbb{R}^d} K(x, y)f(y)dy, \quad f \in \mathcal{S}(\mathbb{R}^d),$$

with (collision) kernels

$$K(x, y) = \int_{\mathbb{R}^d} \Phi(u)e^{-2\pi i(\beta(|u|)u \cdot y - u \cdot x)}du,$$

and is interested in estimates of the type

$$\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} |K(x, y)| \, dx < \infty,$$

[AL17] (related references are provided by [AL10, BD98, MM06]). The estimate [3] would imply the corresponding operator $A$ is bounded on $L^1(\mathbb{R}^d)$. The function $\Phi(u)$ has a good decay at infinity but could be not smooth at the origin $u = 0$. A typical example is given by radial functions

$$\Phi(u) = \frac{|u|}{(1 + |u|^2)^m}$$

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with large real $m$.

The phase $\beta(r)$ is real-valued and smooth on $(0, +\infty)$ but could have a Hölder type singularity at the origin. As an oversimplified model the reader can consider the case

$$\beta(r) = a + br^\gamma, \quad 0 < r \leq 1,$$

for some $a, b \in \mathbb{R}, \gamma \in (0, 1)$. As $r \to +\infty$, $\beta(r)$ is assumed to approach a constant.

As basic case suppose $\beta(r) = a$, $r > 0$, is a constant function. Rapid granular flows are described by the Boltzmann equation and $\beta(r) = a$ corresponds to the case of inelastic interactions with constant restitution coefficient. Indeed, the loss of mechanical energy due to collisions is characterized by the restitution coefficient $\beta$ which quantifies the loss of relative normal velocity of a pair of colliding particles after the collision with respect to the impact velocity. Now, when $\beta(r) = a$ is constant,

$$K(x, y) = \hat{\Phi}(ay - x)$$

and the estimate (3) holds if and only if $\Phi \in \mathcal{F}L^1(\mathbb{R}^d)$, i.e. $\Phi$ has Fourier transform in $L^1(\mathbb{R}^d)$. The major part of the research, at the physical as well as the mathematical levels, has been devoted to this particular case of a constant restitution coefficient. However, as described in [BP04, AL10], a more relevant description of granular gases should involve a variable restitution coefficient $\beta(r)$.

In the model case above $\beta(r)$ approaches a constant both as $r \to 0^+$ and $r \to +\infty$ and is smooth in between, so that one could conjecture that the same estimate holds in that case. Now, this is not the case, even for smooth phases: we will prove in Proposition 3.1 that, in dimension $d = 1$, if $\tilde{\varphi}(u) := \beta(|u|)u$ is any nonlinear smooth diffeomorphism $\mathbb{R} \to \mathbb{R}$ with $\tilde{\varphi}(u) = u$ (hence $\beta(|u|) = 1$) for $|u| \geq 1$, and $\Phi \in C_0^\infty(\mathbb{R})$, $\Phi \equiv 1$ on $[-1, 1]$, then the weighted estimate

$$\int_{\mathbb{R}^d} |K(x, y)| dx \lesssim (1 + |y|)^s$$

does not hold for $s < 1/2$.

This looks surprising at first glance, but it can be regarded as a manifestation of the Beurling-Helson phenomenon [BH53, CNR10, LO94, Oko09, RSTT11], which roughly speaking states that the change-of-variable operator $f \mapsto f \circ \psi$ is not bounded on $\mathcal{F}L^1(\mathbb{R}^d)$ except in the case $\psi : \mathbb{R}^d \to \mathbb{R}^d$ is an affine mapping. Indeed the operator $A$ in (1) with kernel $K(x, y)$ in (2) can be written as

$$Af = \mathcal{F}^{-1}\Phi \ast \mathcal{F}^{-1}(\mathcal{F}f \circ \tilde{\varphi}), \quad \text{with } \tilde{\varphi}(u) := \beta(|u|)u.$$
Suppose $\beta(r)$ as in (5) for $0 < r \leq 1$, with $\gamma \in (-1, 1]$ and assume $\beta$ has at most linear growth as $r \to +\infty$. Let $\Phi$ be as in (1), with $m > (d+1)/2$. Then (6) holds with $s = d/((\gamma + 1)$.

As expected, the growth in (6) is therefore the weakest one when $\gamma = 1$, being $s = d/2$ in that case. The same growth occurs for smooth phases, as the following result shows.

Suppose that $\tilde{\varphi}(u) := \beta(|u|)u$ extends to a smooth function in $\mathbb{R}^d$, with at most quadratic growth at infinity. Let $\Phi$ be as in (4), with $m > (d+1)/2$. Then (6) holds with $s = d/2$.

By the above mentioned counterexample, this estimate is sharp, at least in dimension $d = 1$.

Notice that the estimate (6) implies a continuity property for the corresponding operator between weighted $L^1$ spaces, precisely $L^1_{v_s} \to L^1$, where $v_s(x) = (1 + |x|)^s$.

A natural question is therefore whether similar continuity estimates hold without a loss of decay at least in $L^2(\mathbb{R}^d)$, under the above assumptions. We will show in Proposition 4.1 below that, again, this is not the case. Sufficient conditions are instead given in 4.1 below. Here is a simplified version of Theorem 4.1 (and subsequent remark).

Suppose $\beta(r)$ as in (5) for $0 < r \leq 1$, with $\gamma > 0$. Let $\Phi \in C^{\infty}(\mathbb{R}^d)$ supported in $|u| \leq 1$. Then, if $a(a + (\gamma + 1)b) > 0$ the operator $A$ in (1), (2) is bounded in $L^2(\mathbb{R}^d)$.

Actually the results below are stated for $\beta$ and $\Phi$ in classes of functions with minimal regularity and are inspired by the models above. It turns out that, in all the results it is sufficient to take $\Phi$ in the so-called Segal algebra $M^1(\mathbb{R}^d)$ [Fei81b, Fei89, Fei06]. Roughly speaking, a function $\Phi \in L^\infty(\mathbb{R}^d)$ belongs to $M^1(\mathbb{R}^d)$ if locally has the regularity of a function in $\mathcal{F}L^1(\mathbb{R}^d)$ (in particular is continuous) and globally it decays as a function in $L^1(\mathbb{R}^d)$, but no differentiability conditions are required. We have $M^1(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^d) \cap \mathcal{F}L^1(\mathbb{R}^d)$. To compare this space with the usual Sobolev spaces we observe that $W^{k,1}(\mathbb{R}^d) \subset M^1(\mathbb{R}^d)$ for $k \geq d + 1$, but functions in $M^1(\mathbb{R}^d)$ do not need to have any derivatives. For examples, the functions $\Phi$ in (4) are in $M^1(\mathbb{R}^d)$ if $m > (d+1)/2$ (see Example 4.3 below). It is important to observe that the very weak assumption $\Phi \in M^1(\mathbb{R}^d)$ prevents us to use classical tools such as stationary phase estimates; instead we use techniques and function spaces from Time-frequency Analysis, which will be recalled in the next Section. Recently, such function spaces and more generally Time-frequency Analysis have been successfully applied in the study of partial differential equations with rough data by a large number of authors, see, e.g., [CN085, RWZ16, STW11, WH07] and references therein. We also refer to the papers [CNR10, CR14] and the references therein for the problem of the continuity in $L^p(\mathbb{R}^d)$, $1 < p < \infty$, and from the
Hardy space to \( L^1(\mathbb{R}^d) \), of general Fourier integral operators of Hörmander’s type (i.e. arising in the study of hyperbolic equations).

In short the paper is organized as follows.

In Section 2 we briefly recall the definitions of modulation and Wiener amalgam spaces and exhibit the main properties and preliminary results we need in the sequel.

In Section 3 we study the \( L^1 \)-continuity for the integral operators in (i) having phases with Hölder-type singularity at the origin. The boundedness is attained at the cost of a loss of decay. Such a loss is unavoidable, as testified by an example sequel.

In Section 4 we study the \( L^2 \)-continuity properties of \( A \) in (ii). Under the same assumptions of the \( L^1 \)-boundedness results we provide a counterexample even in this framework (cf. Proposition 3.11). We then show conditions on the phase of the operators which guarantee \( L^2 \)-boundedness without loss of decay.

2. Preliminaries

**Notation.** We define \(|x|^2 = x \cdot x\), for \( x \in \mathbb{R}^d \), where \( x \cdot y = xy \) is the scalar product on \( \mathbb{R}^d \). The space of smooth functions with compact support is denoted by \( C_0^\infty(\mathbb{R}^d) \), the Schwartz class is \( \mathcal{S}(\mathbb{R}^d) \), the space of tempered distributions \( \mathcal{S}'(\mathbb{R}^d) \). The Fourier transform is normalized to be \( \hat{f}(u) = \mathcal{F}f(u) = \int f(t)e^{-2\pi i ut}dt \). Translation and modulation operators (time and frequency shifts) are defined, respectively, by

\[
T_x f(t) = f(t-x) \quad \text{and} \quad M_u f(t) = e^{2\pi i ut}f(t).
\]

We have the formulas \((T_x f)' = M_{-x} \hat{f}, (M_u f)' = T_u \hat{f}, \) and \( M_u T_x = e^{2\pi i xu}T_x M_u \). The notation \( A \lesssim B \) means \( A \leq cB \) for a suitable constant \( c > 0 \), whereas \( A \simeq B \) means \( c^{-1}A \leq B \leq cA \), for some \( c \geq 1 \). The symbol \( B_1 \hookrightarrow B_2 \) denotes the continuous embedding of the linear space \( B_1 \) into \( B_2 \).

2.1. Wiener amalgam spaces \( \text{[Fei83, Fei81a, Fei90, FSS85, FZ98]} \). Let \( g \in C_0^\infty(\mathbb{R}^d) \) be a test function that satisfies \( \|g\|_{L^2} = 1 \). We will refer to \( g \) as a window function. For \( 1 \leq p \leq \infty \), recall the \( \mathcal{F}L^p \) spaces, defined by

\[
\mathcal{F}L^p(\mathbb{R}^d) = \{ f \in \mathcal{S}'(\mathbb{R}^d) : \exists h \in L^p(\mathbb{R}^d), \hat{h} = f \};
\]

they are Banach spaces equipped with the norm

\[
\|f\|_{\mathcal{F}L^p} = \|h\|_{L^p}, \quad \text{with} \quad \hat{h} = f.
\]

Let \( B \) one of the following Banach spaces: \( L^p, \mathcal{F}L^p, 1 \leq p \leq \infty \). For any given function \( f \) which is locally in \( B \) (i.e. \( gf \in B \), \( \forall g \in C_0^\infty(\mathbb{R}^d) \)), we set \( f_B(x) = \|fT_x g\|_B \). The Wiener amalgam space \( W(B, L^p)(\mathbb{R}^d) \) with local component \( B \) and global component \( L^p(\mathbb{R}^d), 1 \leq p \leq \infty \), is defined as the space of all functions \( f \) locally in \( B \) such that \( f_B \in L^p(\mathbb{R}^d) \). Endowed with the norm \( \|f\|_{W(B, L^p)} = \|f_B\|_{L^p} \)
\[ \|f_B\|_{L^p(\mathbb{R}^d)}, \ W(B, L^p) \] is a Banach space. Moreover, different choices of \( g \in C_0^\infty(\mathbb{R}^d) \) generate the same space and yield equivalent norms.

If \( B = \mathcal{F}L^1 \), the Fourier algebra, the space of admissible windows for the Wiener amalgam spaces \( W(\mathcal{F}L^1, L^p)(\mathbb{R}^d) \) can be enlarged to the so-called Feichtinger algebra \( M^1(\mathbb{R}^d) = W(\mathcal{F}L^1, L^1)(\mathbb{R}^d) \), which is also a modulation space, as shown below. Recall that the Schwartz class \( S(\mathbb{R}^d) \) is dense in \( W(\mathcal{F}L^1, L^1)(\mathbb{R}^d) \).

2.2. Modulation spaces [Grö01]. Let \( g \in S(\mathbb{R}^d) \) be a non-zero window function. The short-time Fourier transform (STFT) \( V_g f \) of a function/tempered distribution \( f \) with respect to the window \( g \) is defined by

\[
V_g f(z, u) = \int e^{-2\pi i uy} f(y) g(y - z) \, dy,
\]
i.e., the Fourier transform \( \mathcal{F} \) applied to \( fT_z g \).

For \( 1 \leq p, q \leq \infty \), the modulation space \( M^{p,q}(\mathbb{R}^d) \) is defined as the space of measurable functions \( f \) on \( \mathbb{R}^d \) such that the norm

\[
\|f\|_{M^{p,q}} = \|\|V_g f(\cdot, u)\|_{L^p}\|_{L^q_u}
\]
is finite. Among the properties of modulation spaces, we record that \( M^{2,2}(\mathbb{R}^d) = L^2(\mathbb{R}^d), M^{p_1,q_1}(\mathbb{R}^d) \hookrightarrow M^{p_2,q_2}(\mathbb{R}^d) \), if \( p_1 \leq p_2 \) and \( q_1 \leq q_2 \). If \( p, q < \infty \), then \( (M^{p,q}(\mathbb{R}^d))' = M^{p',q'}(\mathbb{R}^d) \).

For comparison, notice that the norm in the Wiener amalgam spaces \( W(\mathcal{F}L^p, L^q)(\mathbb{R}^d) \) reads

\[
\|f\|_{W(\mathcal{F}L^p, L^q)} = \|\|V_g f(z, \cdot)\|_{L^p_z}\|_{L^q}.
\]
The relationship between modulation and Wiener amalgam spaces is expressed by the following result.

**Proposition 2.1.** For \( 1 \leq p, q \leq \infty \), the Fourier transform establishes an isomorphism \( \mathcal{F} : M^{p,q}(\mathbb{R}^d) \rightarrow W(\mathcal{F}L^p, L^q)(\mathbb{R}^d) \).

Consequently, convolution properties of modulation spaces can be translated into point-wise multiplication properties of Wiener amalgam spaces, as shown below.

**Proposition 2.2.** For every \( 1 \leq p, q \leq \infty \) we have

\[
\|fu\|_{W(\mathcal{F}L^p, L^q)} \lesssim \|f\|_{W(\mathcal{F}L^1, L^{\infty})} \|u\|_{W(\mathcal{F}L^p, L^q)}.
\]
If \( p = q \), we have

\[
\|fu\|_{M^p} \lesssim \|f\|_{W(\mathcal{F}L^1, L^{\infty})} \|u\|_{M^p}.
\]

**Proof.** From Proposition 2.1 the estimate to prove is equivalent to

\[
\|\hat{f} \ast \hat{u}\|_{M^{p,q}} \lesssim \|\hat{f}\|_{M^{1,\infty}} \|\hat{u}\|_{M^{p,q}},
\]
but this a special case of [CG03, Proposition 2.4].
Modulation and Wiener amalgam spaces are invariant with respect to modulation and translation operators. Namely, from [Grö01, Theorem 11.3.5] we infer

**Proposition 2.1.** For \(1 \leq p, q \leq \infty\), \(M^{p,q}(\mathbb{R}^d)\) is invariant under time-frequency shifts, with

\[
\|T_x M_u f\|_{M^{p,q}} \asymp \|f\|_{M^{p,q}}.
\] (10)

The result is known for Wiener amalgam spaces as well.

**Proposition 2.2.** For \(1 \leq p, q \leq \infty\), \(W(F\mathcal{L}^p, L^q)(\mathbb{R}^d)\) is invariant under time-frequency shifts, with

\[
\|T_x M_u f\|_{W(F\mathcal{L}^p, L^q)} \asymp \|f\|_{W(F\mathcal{L}^p, L^q)}.
\] (11)

Indeed, both these results are a consequence of the covariance property of STFT, namely

\[
|V_g(T_x M_u f)(y, \omega)| = |V_g f(y - x, \omega - u)|, \quad x, y, u, \omega \in \mathbb{R}^d,
\]

which follows from direct inspection.

### 2.3. Preliminary results

In the sequel we shall list issues preparing for our later argumentation. To study the properties of our phase function, we shall relay on the following results.

**Lemma 2.3.** ([MNR+09, Lemma 3.2]) Let \(\epsilon > 0\). Suppose \(\mu\) is a real-valued function of class \(C^{[d/2]+1}\) on \(\mathbb{R}^d \setminus \{0\}\) satisfying

\[
|\partial^\alpha \mu(u)| \leq C_\alpha |u|^{[-|\alpha|} \quad \text{for } |\alpha| \leq [d/2] + 1.
\]

Then \(\mathcal{F}^{-1}[\eta e^{i\mu}] \in L^1(\mathbb{R}^d)\) for each \(\eta \in \mathcal{S}(\mathbb{R}^d)\) with compact support. The norm of \(\eta e^{i\mu}\) in \(F\mathcal{L}^1(\mathbb{R}^d)\) is indeed controlled by a constant depending only on \(d\), \(\eta\) and the constants \(C_\alpha\) in (12).

**Lemma 2.4.** ([BGOR07, Theorem 5]) For \(d \geq 1\), let \(l = [d/2] + 1\). Assume that \(\mu\) is \(2l\)-times continuously differentiable function on \(\mathbb{R}^d\) and \(\|\partial^\alpha \mu\|_{L^\infty} \leq C_\alpha\), for \(2 \leq |\alpha| \leq 2l\), and some constants \(C_\alpha\). Then \(e^{i\mu} \in W(F\mathcal{L}^1, L^\infty)(\mathbb{R}^d)\).

The norm of \(e^{i\mu}\) in \(W(F\mathcal{L}^1, L^\infty)(\mathbb{R}^d)\) is indeed controlled by a constant depending only on \(d\), and the above constants \(C_\alpha\).

Dilation properties for Wiener amalgam spaces will play a key role in the proof of our main results.

**Lemma 2.5.** ([CN08a, Corollary 3.2]) Let \(1 \leq p, q \leq \infty\) and \(\lambda \geq 1\). Then, for every \(f \in W(F\mathcal{L}^p, L^q)(\mathbb{R}^d)\),

\[
\|f_\lambda\|_{W(F\mathcal{L}^p, L^q)} \lesssim \lambda^{d/p - d/q} \|f\|_{W(F\mathcal{L}^p, L^q)},
\]

where \(f_\lambda(x) = f(\lambda x)\).
Lemma 2.6. ([CNR15, Proposition 2.5]) Let $h \in C^\infty(\mathbb{R}^d \setminus \{0\})$ be positively homogeneous of degree $r > 0$, i.e., $h(\lambda x) = \lambda^r h(x)$ for $x \neq 0$, $\lambda > 0$. Consider $\chi \in C^\infty_0(\mathbb{R}^d)$ and set $f = h\chi$. Then, for $\psi \in \mathcal{S}(\mathbb{R}^d)$, there exists a constant $C > 0$ such that
\[ |V_\psi f(x, u)| \leq C(1 + |u|)^{-r-d}, \text{ for every } x, u \in \mathbb{R}^d. \]

We recall the definition, for $k \in \mathbb{N}$, of the Sobolev space
\[ W^{k,1}(\mathbb{R}^d) = \{ u \in L^1(\mathbb{R}^d) : \partial^\alpha u \in L^1(\mathbb{R}^d), \forall |\alpha| \leq k \} \]
with the obvious norm. Sharp inclusion relations between Bessel potential Sobolev spaces and modulation spaces were proved in [BLB06, Oko04, KS11, ST07, Tof04]. Here we are interested in an easier embedding of the Sobolev space $W^{k,1}(\mathbb{R}^d)$. We remark that for $p = 1$ (and $p = \infty$) Sobolev and Bessel potential Sobolev spaces are different (cf. [Ste70, pag. 160]).

Lemma 2.7. We have
\[ W^{k,1}(\mathbb{R}^d) \hookrightarrow M^1(\mathbb{R}^d) \text{ for } k \geq d + 1. \]

Proof. First of all we observe that the following embedding holds:
\[ W^{k,1}(\mathbb{R}^d) \hookrightarrow \mathcal{F}L^1(\mathbb{R}^d), \quad k \geq d + 1. \]
This follows by an integration by parts, namely, for $f \in \mathcal{S}(\mathbb{R}^d)$ and every $\alpha$,
\[ |\mathcal{F}f(u)| \leq (2\pi)^{-|\alpha|}|u^\alpha|^{-1}\|\partial^\alpha f\|_{L^1(\mathbb{R}^d)}. \]
Taking the minimum with respect to $|\alpha| \leq d+1$ and using the fact that $\min_{|\alpha| \leq d+1} |u^\alpha|^{-1}$ is in $L^1(\mathbb{R}^d)$ (see e.g. [Gro01, page 321]) we obtain (13).

We can then write
\[ \|\mathcal{F}(fT_xg)\|_{L^1} \lesssim \sum_{|\alpha| \leq d+1} \|\partial^\alpha(fT_xg)\|_{L^1}. \]
Using Leibniz’ rule and integrating with respect to $x \in \mathbb{R}^d$ gives
\[ \|f\|_{M^1} \lesssim \|f\|_{W^{k,1}(\mathbb{R}^d)}, \quad k \geq d + 1. \]

In order to exhibit the counterexample anticipated in the introduction we will make use of a result proved in [CNR10, Proposition 6.1] which can be stated as follows.

Proposition 2.3. Let $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ be any nonlinear smooth diffeomorphism satisfying
\[ \tilde{\varphi}(u) = u, \quad \text{for } |u| \geq 1, \]
and let $\Phi \in C^\infty_0(\mathbb{R}^d)$, $\Phi \equiv 1$ on $[-1,1]^d$. 

For $2 \leq p \leq \infty$, $m < d(1/2 - 1/p)$, the so-called type I FIO $T_{1,\varphi,\sigma}$, defined as

$$T_{1,\varphi,\sigma} f(x) = \int e^{2\pi i \varphi(x,u)} \sigma(x,u) \hat{f}(u) \, du,$$

having phase $\varphi(x,u) = \sum_{k=1}^{d} \tilde{\varphi}(u_k) x_k$, and symbol $\sigma(x,u) = \langle x \rangle^m \Phi(u)$, does not extend to a bounded operator on $L^p(\mathbb{R}^d)$.

3. Continuity in $L^1$ with loss of decay

We consider an integral operator $A$ formally defined in (1) and with kernel $K(x,y)$ in (2). We assume

$$\Phi \in M^1(\mathbb{R}^d), \quad \beta : (0, \infty) \to \mathbb{R}. $$

Then, the kernel $K$ is well-defined for every $x, y \in \mathbb{R}^d$. Indeed, since $M^1(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^d)$, the integral in (2) is absolutely convergent. Inserting the kernel expression (2) in the operator $A$, defined in (1), and using the absolute convergence of the integrals we can apply Fubini’s Theorem and infer

$$A f(x) = \int_{\mathbb{R}^{2d}} e^{-2\pi i [\beta(|u|)u \cdot y - x \cdot u]} \Phi(u) f(y) \, dy \, du. $$

That is, the operator $A$ can be written in the form of a Fourier integral operator of type II. We recall that a FIO of type II with phase $\varphi$ and symbol $\sigma$ has the general form

$$T_{II,\varphi,\sigma} f(x) = \int_{\mathbb{R}^{2d}} e^{-2\pi i (\varphi(y,u) - x \cdot u)} \sigma(y,u) f(y) \, dy \, du$$

hence

$$A = T_{II,\varphi,\sigma}, \quad \text{with} \quad \varphi(y,u) = \beta(|u|)u \cdot y \quad \text{and} \quad \sigma(y,u) = \Phi(u).$$

FIO’s of type II are the formal adjoints of FIO’s of type I, defined in (15). Namely,

$$(T_{I,\varphi,\sigma})^* = T_{II,\varphi,\sigma}. $$

In general we do not expect that the integral operator $A$ in (1) with kernel $K$ in (2) is continuous on $L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, $p \neq 2$. Indeed, we expect a loss of decay, as witnessed by the following example.

**Proposition 3.1.** In dimension $d = 1$, for any $1 \leq p \leq 2$, consider the weight function

$$v_m(y) = (1 + |y|)^m, \quad y \in \mathbb{R},$$

with $m \in \mathbb{R}$ such that

$$m < \frac{1}{p} - \frac{1}{2}. $$

Let $\beta \in C^\infty((0, +\infty))$ such that
\begin{equation}
\phi(u) = \beta(|u|)u
\end{equation}
extends to a nonlinear smooth diffeomorphism $\mathbb{R} \to \mathbb{R}$ satisfying
\begin{equation}
\bar{\phi}(u) = u, \quad |u| \geq 1.
\end{equation}
(hence, $\beta(|u|) = 1$, for $|u| > 1$). Let $\Phi \in C_0^\infty(\mathbb{R})$, $\Phi(u) = 1$ for $|u| \leq 1$.

Then the operator $A$ in (17) does not extend to a bounded operator from $L^p_{\nu_m}(\mathbb{R})$ to $L^p(\mathbb{R})$.

Proof. Step 1: Rephrasing the thesis. Since $v_m(y) = (1 + |y|)^m$ is a weight equivalent to $w_m(y) := \langle y \rangle^m = (1 + y^2)^{m/2}$, we can work with $w_m$ in place of $v_m$. Since $A$ can be written as a type II Fourier integral operator, this amounts to considering the continuity from $L^p_{w_m}(\mathbb{R})$ to $L^p(\mathbb{R})$ of the operator $T_{II,\phi,\sigma}$ in (18) with $\phi(y, u) = \bar{\phi}(u)y$ and symbol $\sigma(y, u) = \Phi(u)$, with $\bar{\phi}$ and $\Phi$ as in the statement.

Step 2: From type II FIOs to type I FIOs. By duality, the continuity of $T_{II,\phi,\sigma}$ is equivalent to the boundedness of the adjoint $(T_{II,\phi,\sigma})^* = T_{I,\phi,\sigma}$, from $L^p(\mathbb{R})$ to $L^p_{w_m}(\mathbb{R})$, for $2 \leq p \leq \infty$.

Step 3: Results for type I FIOs. The continuity of $T_{I,\phi,\sigma}$ from $L^p(\mathbb{R})$ to $L^p_{w_m}(\mathbb{R})$ is equivalent to the boundedness of the operator $w_mT_{I,\phi,\sigma}$ on $L^p(\mathbb{R})$. Now, observe that
\begin{align*}
\langle x \rangle^{-m}T_{I,\phi,\sigma}f(x) &= \langle x \rangle^{-m} \int e^{2\pi i \phi(x, u)} \sigma(x, u) \hat{f}(u) \, du \\
&= \int e^{2\pi i \phi(x, u)} \langle x \rangle^{-m} \sigma(x, u) \hat{f}(u) \, du \\
&= \int e^{2\pi i \phi(x, u)} \bar{\sigma}(x, u) \hat{f}(u) \, du := T_{I,\phi,\bar{\sigma}}
\end{align*}
with $\bar{\sigma}(x, u) = \langle x \rangle^{-m} \sigma(x, u) = \langle x \rangle^{-m} \Phi(u)$.

Now the type I FIO $T_{I,\phi,\bar{\sigma}}$ is not bounded on $L^p$, $2 \leq p \leq \infty$, by Proposition 2.3.

Continuity in weighted $L^1$ spaces, i.e. with a loss of decay, for the operator $A$ in (17) can be proved by a Schur-type estimate for the kernel $K$. The following result addresses such estimates and Corollary 3.1 the corresponding continuity result.

Theorem 3.1. Consider functions $\Phi \in M^1(\mathbb{R})$ and $\beta : (0, +\infty) \to \mathbb{R}$. Moreover, assume that for some exponent $\gamma \in (-1, 1]$, with $l = \lceil d/2 \rceil + 1$,
\begin{equation}
|\partial^\alpha \beta(|u|)u| \leq C_\alpha |u|^{|\gamma|+1-|\alpha|}, \quad \text{for } 0 \neq |u| \leq 1, \ |\alpha| \leq l,
\end{equation}
where $C_\alpha > 0$, and
\begin{equation}
|\partial^\alpha \beta(|u|)u| \leq C_\alpha', \quad \text{for } |u| \geq 1, \ 2 \leq |\alpha| \leq 2l,
\end{equation}
where $C_\alpha'$ is a constant depending on $\alpha$. Then the operator $A$ in (17) does not extend to a bounded operator from $L^p_{\nu_m}(\mathbb{R})$ to $L^p(\mathbb{R})$.
with $C' > 0$. Then the integral kernel in (2) satisfies

$$
\int_{\mathbb{R}^d} |K(x, y)| dx \leq C(1 + |y|)^{d/(\gamma+1)},
$$

for a suitable constant $C > 0$ independent of $y$.

**Proof.** To infer the estimate in (26), we write

$$
\|K(\cdot, y)\|_{L^1} = \|F^{-1}(\Phi e^{-2\pi i \varphi(y, \cdot)})\|_{L^1} = \|\Phi e^{-2\pi i \varphi(y, \cdot)}\|_{\mathcal{F}L^1},
$$

where the phase is $\varphi(y, u) = \beta(|u|)u \cdot y$.

We are going to show that

$$
e^{-2\pi i \varphi(y, \cdot)} \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d),
$$

with

$$
\|e^{-2\pi i \varphi(y, \cdot)}\|_{W(\mathcal{F}L^1, L^\infty)} \leq C(1 + |y|)^{d/(\gamma+1)};
$$

then the algebra property of $W(\mathcal{F}L^1, L^\infty)$ for $p = 1$ in (9) yields

$$
\|\Phi e^{-2\pi i \varphi(y, \cdot)}\|_{\mathcal{F}L^1} \lesssim \|\Phi e^{-2\pi i \varphi(y, \cdot)}\|_{M^1} \lesssim \|e^{-2\pi i \varphi(y, \cdot)}\|_{W(\mathcal{F}L^1, L^\infty)} \|\Phi\|_{M^1}
$$

and this gives the claim (26).

We study the cases $|y| \leq 1$ and $|y| \geq 1$ separately. The more difficult point is $|y| \geq 1$, which is proved as follows. Using the dilation properties of $W(\mathcal{F}L^1, L^\infty)$ in Lemma 2.5, the estimate (27) follows from

$$
e^{-2\pi i \varphi(y, |y|^{1/(\gamma+1)}u)} \in W(\mathcal{F}L^1, L^\infty),
$$

uniformly with respect to $y$. Indeed,

$$
\|e^{-2\pi i \beta(|u|)u \cdot y}\|_{W(\mathcal{F}L^1, L^\infty)} \lesssim |y|^{d/(\gamma+1)} \left\| e^{-2\pi i \beta \left(\frac{|u| y}{|y|^{1/(\gamma+1)}}\right)} \frac{u}{|y|^{1/(\gamma+1)}} y \right\|_{W(\mathcal{F}L^1, L^\infty)}

\lesssim (1 + |y|)^{d/(\gamma+1)} \left\| e^{-2\pi i \beta \left(\frac{|u| y}{|y|^{1/(\gamma+1)}}\right)} \frac{u}{|y|^{1/(\gamma+1)}} y \right\|_{W(\mathcal{F}L^1, L^\infty)}.
$$

The key idea is to use Lemmas 2.3 and 2.4 with $\mu(u)$ being the phase $\varphi(y, u) = \beta(|u|)u \cdot y$ now rescaled by $|y|^{-1/(\gamma+1)}$: set

$$
B_y(u) := 2\pi \beta \left(\frac{|u|}{|y|^{1/(\gamma+1)}}\right) \frac{u}{|y|^{1/(\gamma+1)}} \cdot y.
$$

We consider a test function $\chi \in C^\infty_0(\mathbb{R}^d)$, such that $\chi(u) = 1$ for $|u| \leq 4/3$ and $\chi(u) = 0$ when $|u| \geq 5/3$ and write

$$
e^{-iB_y(u)} = e^{-iB_y(u)} \chi(u) + e^{-iB_y(u)}(1 - \chi(u)).$$
We first show that
\[ e^{-iB_y} \chi \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d), \]
uniformly with respect to \( y \), that is, since \( \chi \) is compactly supported,
\[ e^{-iB_y} \chi \in \mathcal{F}L^1(\mathbb{R}^d). \]  
Using Lemma 2.3, it is enough to verify that the phase \( B_y \) satisfies the estimates
\[ |\partial^\alpha B_y(u)| \leq C_\alpha |u|^\gamma + 1 - |\alpha| \]
for \( |u| \leq 2 \) (which is a neighborhood of the support of \( \chi \)) and \( \gamma \leq [d/2] + 1 \).
Observe that the estimates in the assumption (24) actually hold for, say, \( |u| \leq 2 \), because (25) implies (24) for \( 1 \leq |u| \leq 2 \). Hence we have
\[ |\partial^\alpha B_y(u)| \leq 2 \pi C'_\alpha \frac{|u|}{|y|^{\gamma + 1/2}} |y| = 2 \pi C_\alpha |y|^\gamma + 1 - |\alpha| \]
for \( |u| \leq 2 |y|^{1/(\gamma + 1)} \), hence for \( |u| \leq 2 \). Since \( \gamma + 1 > 0 \), by Lemma 2.3 we have (30).
We now use Lemma 2.4 to show that
\[ e^{-iB_y}(1 - \chi) \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d). \]
It is sufficient to verify that the phase satisfies
\[ |\partial^\alpha B_y(u)| \leq C_\alpha \]
for \( |u| \geq 1 \) and \( 2 \leq |\alpha| \leq 2l \), with \( l = [d/2] + 1 \). If \( 1 \leq |u| \leq |y|^{1/(\gamma + 1)} \) the estimate (33) follows by (32), because \( \gamma \leq 1 \). On the other hand, if \( |u| \geq |y|^{1/(\gamma + 1)} \) we use the hypothesis (25):
\[ |\partial^\alpha B_y(u)| \leq \frac{2 \pi C'_\alpha}{|y|^{\gamma + 1/2}} |y| = \frac{2 \pi C''_\alpha}{|y|^{(\gamma + 1)/(\gamma + 1)}} \leq C''_\alpha \]
for \( 2 \leq |\alpha| \leq 2l \), because \( \gamma \leq 1 \). Then, Lemma 2.4 gives the claim.

In the previous result the weakest growth is reached when \( \gamma = 1 \), the exponent in (26) in that case being \( d/2 \). That growth is the same obtained even for smooth phase, as proved in the following result, and cannot be further reduced, as shown in Proposition 3.1.

**Corollary 3.2.** Consider functions \( \Phi \in M^1(\mathbb{R}^d) \) and \( \beta : (0, +\infty) \to \mathbb{R} \). Moreover, setting \( l = [d/2] + 1 \), assume that the function \( \beta(|u|) \) extends to a \( \mathcal{C}^{2l} \) function on \( \mathbb{R}^d \) and satisfies
\[ |\partial^\alpha \beta(|u|)u| \leq C_\alpha, \quad \text{for } u \in \mathbb{R}^d \text{ and } 2 \leq |\alpha| \leq 2l. \]
Then, the integral kernel in \((2)\) satisfies

\[
\int_{\mathbb{R}^d} |K(x,y)| dx \leq C(1 + |y|)^\frac{d}{4}.
\]

Proof. The proof uses the same arguments as in Theorem 3.1. We split into the cases \(|y| \geq 1\) and \(|y| < 1\). We study first \(|y| \geq 1\) and prove that

\[
e^{-2\pi i \beta \left( \frac{|u|}{|y|^{1/2}} \right) \frac{u}{|y|^{1/2}} \cdot y} \in W(\mathcal{F} L^1, L^\infty)(\mathbb{R}^d)
\]

uniformly with respect to \(y\). Using Lemma 2.4, we are reduced to verify that the rescaled phase

\[
B_y(u) := 2\pi \beta \left( \frac{|u|}{|y|^{1/2}} \right) \frac{u}{|y|^{1/2}} \cdot y
\]

satisfies the estimate

\[
|\partial^\alpha B_y(u)| \leq C_\alpha, \quad \forall u \in \mathbb{R}^d
\]

and \(2 \leq |\alpha| \leq 2l\). By the hypothesis \((34)\),

\[
|\partial^\alpha B_y(u)| \leq C_\alpha \frac{2\pi}{|y|^{1/2} \cdot \alpha} \cdot \frac{1}{|y|^{1/2} \cdot \alpha} = C_\alpha'\frac{1}{|y|^{|(\alpha|-2)/2}} \leq C_\alpha''
\]

since \(|\alpha| \geq 2\) and \(|y| \geq 1\); this gives \((36)\).

Then, by Lemma 2.5 we have

\[
\left\| e^{-2\pi i \beta (|u|) u \cdot y} \right\|_{W(\mathcal{F} L^1, L^\infty)} \lesssim (1 + |y|)^{d/2} \left\| e^{-2\pi i \beta \left( \frac{|u|}{|y|^{1/2}} \right) \frac{u}{|y|^{1/2}} \cdot y} \right\|_{W(\mathcal{F} L^1, L^\infty)}.
\]

As \(\Phi \in M^1(\mathbb{R}^d)\), we have that \(\Phi(u)e^{-i\beta (|u|) u \cdot y} \in M^1(\mathbb{R}^d)\). Moreover, we can write

\[
K(x,y) = \mathcal{F}^{-1} \left[ \Phi(u)e^{-i\beta (|u|) u \cdot y} \right](x).
\]

Hence,

\[
\int_{\mathbb{R}^d} |K(x,y)| dx = \left\| \mathcal{F}^{-1} \left[ \Phi(u)e^{-i\beta (|u|) u \cdot y} \right] \right\|_{L^1} \lesssim \left\| \mathcal{F}^{-1} \left[ \Phi(u)e^{-i\beta (|u|) u \cdot y} \right] \right\|_{M^1}
\]

\[
\lesssim \left\| \Phi(u)e^{-i\beta (|u|) u \cdot y} \right\|_{M^1} \lesssim \left\| \Phi \right\|_{M^1} \left\| e^{-2\pi i \beta (|u|) u \cdot y} \right\|_{W(\mathcal{F} L^1, L^\infty)}
\]

\[
\leq C(1 + |y|)^\frac{d}{2},
\]

for some \(C > 0\).

The case \(|y| < 1\) is attained with the same pattern above, without the dilation factor \(|y|^{-\frac{d}{4}}\). \qed
Consider functions \( \Phi \in M^1(\mathbb{R}^d) \) and \( \beta : (0, +\infty) \to \mathbb{R} \). Assume that for some \( \gamma \in (-1, 1) \) and \( a \in \mathbb{R} \),

\[
\tilde{\beta} := \beta - a
\]
satisfies, with \( l = \lfloor d/2 \rfloor + 1 \),

\[
\left| \partial^{\alpha} \tilde{\beta}(|u|)u \right| \leq C_{\alpha} |u|^{|\gamma+1-|\alpha||}, \quad \text{for } |u| \leq 1, |\alpha| \leq l,
\]

for \( C_{\alpha} > 0 \), and

\[
\left| \partial^{\alpha} \tilde{\beta}(|u|)u \right| \leq C'_{\alpha}, \quad \text{for } |u| \geq 1, 2 \leq |\alpha| \leq 2l,
\]

with \( C'_{\alpha} > 0 \). Then the integral kernel in (2) satisfies

\[
\int_{\mathbb{R}^d} |K(x, y)|dx \leq C(1 + |y|)^{d/(\gamma+1)},
\]

for a suitable constant \( C > 0 \), independent of the variable \( y \).

**Proof.** By the proof of Theorem 3.1 we know that \( e^{-2\pi i \tilde{\beta}(|u|)u}y \in W(FL^1, L^\infty)(\mathbb{R}^d) \) with

\[
\left\| e^{-2\pi i \tilde{\beta}(|u|)u}y \right\|_{W(FL^1, L^\infty)} \lesssim (1 + |y|)^{d/(\gamma+1)}.
\]

By (37),

\[
e^{-2\pi i \tilde{\beta}(|u|)u}y = e^{-2\pi i au} \cdot e^{-2\pi i \tilde{\beta}(|u|)u}y = M_{-ay} e^{-2\pi i \tilde{\beta}(|u|)u}y.
\]

Using the invariance property of \( W(FL^1, L^\infty)(\mathbb{R}^d) \) with respect to time-frequency shifts in (11):

\[
\left\| e^{-2\pi i \tilde{\beta}(|u|)u}y \right\|_{W(FL^1, L^\infty)} = \left\| M_{-ay} e^{-2\pi i \tilde{\beta}(|u|)u}y \right\|_{W(FL^1, L^\infty)}
\]

\[
\lesssim \left\| e^{-2\pi i \tilde{\beta}(|u|)u}y \right\|_{W(FL^1, L^\infty)}
\]

\[
\leq C(1 + |y|)^{d/(\gamma+1)}.
\]

This concludes the proof.

We end up this section by using the previous results for the integral kernel \( K(x, y) \) to attain the \( L^1 \)-boundedness for the corresponding operator \( A \). The cost is a loss of decay, as explained below.

**Corollary 3.4.** Assume the hypotheses of Corollary 3.3 and consider the weight function

\[
v(y) = (1 + |y|)^{d/(\gamma+1)}.
\]

Then the integral operator \( A \) in (17) with kernel \( K \) in (2) is bounded from \( L^1_v(\mathbb{R}^d) \) into \( L^1(\mathbb{R}^d) \).
Proposition 4.1. Let \( f \in L^1(\mathbb{R}^d) \); using Fubini’s Theorem and the estimate in (40),

\[
\|Af(x)\|_{L^1} = \int_{\mathbb{R}^d} |Af(x)| \, dx = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} K(x, y) f(y) \, dy \right| \, dx \\
\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |K(x, y)||f(y)| \, dy \, dx = \int_{\mathbb{R}^d} |f(y)| \left( \int_{\mathbb{R}^d} |K(x, y)| \, dx \right) \, dy \\
\leq \int_{\mathbb{R}^d} |f(y)| C(1 + |y|)^{d/(\gamma+1)} \, dy = C\|f\|_{L^1}^2,
\]

as desired.

4. Continuity in \( L^2 \)

A natural question is whether the assumptions of Corollary 3.3 that give continuity of the operator \( A \) on \( L^1(\mathbb{R}^d) \) with a loss of decay, guarantee at least continuity of \( A \) on \( L^2(\mathbb{R}^d) \) without any loss. The answer is negative even in dimension \( d = 1 \), as shown by the following result.

Proposition 4.1. Let \( d = 1 \). There exists an operator \( A \) as in (17), with \( \beta \) and \( \Phi \) satisfying the assumptions of Corollary 3.3, that is not bounded on \( L^2(\mathbb{R}^d) \).

Proof. Consider a function \( h(r) \) such that \( \Phi(u) = h(|u|) \in C_c^\infty(\mathbb{R}) \), and \( h(0) \neq 0 \). For \( \gamma \in (0, 1) \), set \( \beta(u) = u^\gamma \). Finally, take \( \chi \in C_c^\infty(\mathbb{R}) \) such that \( \chi(u) = 1 \) when \( u \in \text{supp} \Phi \), and consider the function \( \beta(u) = \chi(u) \beta(u) \). Consider the operator \( A \) with integral kernel

\[
K(x, y) = \int_\mathbb{R} h(|u|) e^{-2\pi i (ux + \tilde{\beta}(|u|) uy)} \, du = \int_\mathbb{R} h(|u|) e^{-2\pi i (ux + \beta(|u|) uy)} \, du.
\]

We now show that \( A \) is not bounded on \( L^2(\mathbb{R}^d) \).

For \( f \in \mathcal{S}(\mathbb{R}) \),

\[
Af(x) = \int_\mathbb{R} K(x, y) f(y) \, dy = \int_\mathbb{R} \int_\mathbb{R} h(|u|) e^{-2\pi i (ux + \beta(|u|) uy)} f(y) \, dy \, du \\
= \int_\mathbb{R} h(|u|) e^{-2\pi i u \cdot x} \left( \int_\mathbb{R} f(y) e^{-2\pi i \beta(|u|) u \cdot y} \, dy \right) \, du \\
= \int_\mathbb{R} h(|u|) e^{-2\pi i u \cdot x} \hat{f}(\beta(|u|) u) \, du = \mathcal{F} \left[ h(|u|) \hat{f}(\beta(|u|) u) \right] (x).
\]

Then, by Parseval’s Theorem,

\[
\|Af\|_2^2 = \left\| h(|u|) \hat{f}(\beta(|u|) u) \right\|_2^2 = \int_\mathbb{R} |h(|u|)|^2 |\hat{f}(\beta(|u|) u)|^2 \, dx.
\]
We perform the change of variable
\[
\tilde{u} = \beta(|u|)u = |u|^\gamma u = \begin{cases} 
  u^{\gamma+1}, & u \geq 0, \\
  -|u|^{\gamma+1}, & u < 0,
\end{cases}
\]
so that
\[
u = \begin{cases} 
  \tilde{u}^{1+\gamma}, & \tilde{u} \geq 0, \\
  -(\tilde{u})^{1+\gamma}, & \tilde{u} < 0.
\end{cases}
\]
and \(du = \frac{1}{1+\gamma}|\tilde{u}|^{1+\gamma-1}d\tilde{u}\). In this way, we obtain
\[
\|Af\|_2^2 = \int_{\mathbb{R}} |h(|u|)|^2|\hat{f}(\beta(|u|)u)|^2du = \frac{1}{1+\gamma} \int_{\mathbb{R}} |\tilde{u}|^{1+\gamma-1}|h(|\tilde{u}|^{1+\gamma})|^2|\hat{\tilde{f}}(\tilde{u})|^2d\tilde{u}.
\]
Now, the last expression is controlled by \(C\|f\|_{L^2}^2\), for a suitable constant \(C > 0\) and for every \(f \in S(\mathbb{R})\), if and only if
\[
|h(|\tilde{u}|^{1+\gamma})| \in L^\infty(\mathbb{R}),
\]
(notice that \(h(|\tilde{u}|^{1+\gamma})\) has compact support) and this fails since \(-\gamma/(1+\gamma) < 0\) and \(|h(|u|)| \geq \delta > 0\) in a neighborhood of 0.

We now look for suitable assumptions on the functions \(\Phi\) and \(\beta\) which guarantee \(L^2\)-continuity of the operator \(A\). A successful choice is shown below.

**Theorem 4.1.** Consider \(\Phi \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)\). Let \(\beta: (0, \infty) \to \mathbb{R}\) satisfy the following assumptions:
(i) \(\beta \in C^1((0, \infty))\);
(ii) There exists \(\delta > 0\) such that \(\beta(r) \geq \delta\), for all \(r > 0\);
(iii) There exist \(B_1, B_2 > 0\), such that
\[(41) \quad B_1 \leq \frac{d}{dr}(\beta(r)r) \leq B_2, \quad \forall r > 0.
\]

Then the integral operator \(A\) with kernel \(K\) in (2) is bounded on \(L^2(\mathbb{R}^d)\).

**Proof.** We first observe that, since \(\Phi \in L^1(\mathbb{R}^d)\), the integral defining the kernel \(K(x, y)\) is absolutely convergent and \(K\) is well-defined. Let \(f \in S(\mathbb{R}^d)\), using Fubini’s Theorem we can write
\[
Af(x) = \int_{\mathbb{R}^d} K(x, y)f(y)dy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Phi(u)e^{-2\pi i(\beta(|u|)u\cdot y-x)}f(y)dudy
\]
\[
= \int_{\mathbb{R}^d} \Phi(u)e^{2\pi iu\cdot x} \left( \int_{\mathbb{R}^d} f(y)e^{-2\pi i\beta(|u|)u\cdot y}dy \right) du
\]
\[
= \int_{\mathbb{R}^d} \Phi(u)e^{2\pi iu\cdot x} \hat{f}(\beta(|u|)u)du = \mathcal{F}^{-1} \left[ \Phi(u)\hat{f}(\beta(|u|)u) \right](x).
\]
Then, by Parseval’s Theorem,

$$\|Af\|_2^2 = \|\Phi(u)\hat{f}(|u|)u\|_2^2 = \int_{\mathbb{R}^d} |\Phi(u)|^2 |\hat{f}(|u|)u|^2 du.$$ 

Changing to polar coordinates $u = r\theta$, with $r > 0$ and $\theta \in S^{d-1}$, we have $du = r^{d-1}drd\theta$ and

$$\|Af\|_2^2 = \int_0^\infty \int_{S^{d-1}} |\Phi(r\theta)|^2 |\hat{f}(r\theta(r\theta))|^2 r^{d-1} d\theta dr.$$ 

Observe that the function $\varphi(r) := \beta(r)r$ is strictly increasing by assumption (iii). Performing the change of variable $\tilde{r} := \varphi^{-1}(r)$ is strictly increasing.

Further, by assumption (ii),

$$\frac{1}{\delta} \geq \frac{1}{\beta(r)} = \frac{r}{\beta(r)r} = \frac{\varphi^{-1}(\tilde{r})}{\tilde{r}}.$$ 

Then we can write,

$$\|Af\|_2^2 \leq \int_0^\infty \int_{S^{d-1}} |\Phi(\varphi^{-1}(\tilde{r})\theta)|^2 |\hat{f}(\tilde{r}\theta)|^2 (\varphi^{-1}(\tilde{r}))^{-1} d\theta d\tilde{r} \leq \sup_{r,\theta} \left\{ |\Phi(\varphi^{-1}(\tilde{r})\theta)|^2 \left( \frac{\varphi^{-1}(\tilde{r})}{\tilde{r}} \right)^{-1} \frac{d}{d\tilde{r}} (\varphi^{-1}(\tilde{r})) \right\} \int_0^\infty \int_{S^{d-1}} |\hat{f}(\tilde{r}\theta)|^2 (\tilde{r})^{-1} d\theta d\tilde{r}.$$

This gives $\|Af\|_2 \leq C\|f\|_2$, for every $f \in \mathcal{S}(\mathbb{R}^d)$. By density argument we obtain the claim for every $f \in L^2(\mathbb{R}^d)$. 

\begin{remark}

The previous proof still works if we change the function $\beta$ with $-\beta$. Hence, under the assumptions of Theorem 4.1 with assumptions (ii) and (iii) replaced by:

(ii)' There exists $\delta < 0$ such that $\beta(r) \leq \delta$, for all $r > 0$;

\end{remark}
There exist $B_1, B_2 < 0$, such that

$$B_1 \leq \frac{d}{dr}(\beta(r)r) \leq B_2, \quad \forall r > 0;$$

the integral operator $A$ with kernel $K$ in (2) is bounded on $L^2(\mathbb{R}^d)$.

We now exhibit a class of examples of functions $\Phi \in M^1(\mathbb{R}^d)$, hence fulfilling the assumptions of Theorem 3.1 and related corollaries, as well as those of Theorem 4.1, which are of special interest in the study of Boltzmann equation, cf. [AL10].

**Example 4.3.** Consider the function

$$\Phi(u) = \frac{|u|}{(1 + |u|^2)^m}, \quad \text{for } m > \frac{d+1}{2}. \quad (42)$$

Then $\Phi \in M^1(\mathbb{R}^d)$. (Observe that $\Phi(u) = h(|u|))$.

**Proof.** We consider a function $\chi \in C^\infty_0(\mathbb{R}^d)$, such that $\chi(u) = 1$ when $|u| \leq 1/2$ and $\chi(u) = 0$ when $|u| \geq 1$. We write

$$\Phi(u) = \Phi(u)\chi(u) + \Phi(u)(1 - \chi(u))$$

and show that

$$\Phi(u)\chi(u) \in M^1(\mathbb{R}^d) \quad (43)$$

and

$$\Phi(u)(1 - \chi(u)) \in M^1(\mathbb{R}^d). \quad (44)$$

To prove (43), we choose another cut-off function $\tilde{\chi} \in C^\infty_0(\mathbb{R}^d)$ such that

$$\tilde{\chi}(u) = 1 \quad \text{for } u \in \text{supp } \chi;$$

then $\tilde{\chi} \cdot \chi = \chi$. Consider now the function $h(u) = |u|$, which is in $C^\infty(\mathbb{R}^d \setminus \{0\})$ and positively homogeneous of degree 1 and set $f = h\chi$. Lemma 2.6 gives, for $\psi \in S(\mathbb{R}^d)$,

$$|V_\psi f(x, \xi)| \leq C(1 + |\xi|)^{-(d+1)}$$

hence, by (8),

$$\|f\|_{W(\mathcal{F} L^1, L^\infty)} = \|V_\psi f(x, \cdot)\|_{L^1} \|L^\infty < \infty,$$

that is $|u|\chi \in W(\mathcal{F} L^1, L^\infty)(\mathbb{R}^d)$. Since

$$\frac{\tilde{\chi}(u)}{(1 + |u|^2)^m} \in S(\mathbb{R}^d) \subseteq M^1(\mathbb{R}^d).$$

we can write

$$\Phi(u)\chi(u) = |u|\chi(u) \cdot \frac{\tilde{\chi}(u)}{(1 + |u|^2)^m} \in M^1(\mathbb{R}^d),$$

by Proposition 2.2.
Finally, to show (44), we observe that \( \Phi(u)(1 - \chi(u)) = 0 \) for \( |u| \leq 1/2 \), hence the singularity at the origin is removed and \( \Phi(u)(1 - \chi(u)) \in W^{k,1}(\mathbb{R}^d) \) for all \( k \in \mathbb{N} \), provided that \( 2m - 1 > d \). We then choose \( k > d \) and apply the inclusion relations between the Potential Sobolev space \( W^{k,1}(\mathbb{R}^d) \) and the Feichtinger’s algebra \( M^1(\mathbb{R}^d) \) in Lemma 2.7 which gives (44).

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