Modularity in Orbifold Theory
for Vertex Operator Superalgebras

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Abstract

This paper is about the orbifold theory for vertex operator superalgebras. Given a vertex operator superalgebra $V$ and a finite automorphism group $G$ of $V$, we show that the trace functions associated to the twisted sectors are holomorphic in the upper half plane for any commuting pairs in $G$ under the $C_2$-cofinite condition. We also establish that these functions afford a representation of the full modular group if $V$ is $C_2$-cofinite and $g$-rational for any $g \in G$. 2000MSC:17B69

1 Introduction

Modular invariance of trace functions is fundamental in the theory of vertex operator algebras and conformal field theory (see for example [B2], [DVVV], [DLM3], [FLM3], [H], [Hu], [M2], [MS], [V], [Z]). Many important results such as the existence of twisted sectors in orbifold theory, the Lie algebra structure of weight one subspace of a good vertex operator algebra, and classification of holomorphic vertex operator algebras with small central charges can be obtained by studying the modular invariance property of 1-point functions on torus (see [DM1], [DM2], [Mo], [S]). The modular invariance of trace functions associated to modules for an arbitrary rational vertex operator algebra was first established under the $C_2$-cofinite condition in [Z]. This work was extended in [DLM3] to include a finite automorphism group of the vertex operator algebra. There were further generalizations in [M1], [M2], [Y].

Partially motivated by the generalized moonshine conjecture [N] and orbifold conformal field theory [DVVV], the trace functions associated to twisted modules for a vertex operator algebra $V$ and a finite automorphism group $G$ was studied in [DLM3]. (The trace functions associated to the twisted sectors are exactly the 1-point partition functions on torus for the orbifold theory.) Under some finiteness conditions on a rational vertex operator algebra $V$, among other things, the precise number of inequivalent, simple $g$-twisted $V$-modules was determined and the modular invariance (in a suitable sense) of the 1-point functions on torus associated to any commuting pairs in $G$ was established. If $V$ is holomorphic, the existence and the uniqueness of the twisted sector for each $g \in G$ was obtained. The important case in which $V = V^2$ is the moonshine vertex operator algebra [FLM3] and $G$ is the monster simple group [G] plays a special role in a proof.

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of several parts of the generalized moonshine conjecture [DLM3] except the genus zero property. Also see [T1],[T2] and [T3] on the generalized moonshine conjecture.

In this paper we present a general theory of modular invariance of trace functions associated to twisted sectors for a vertex operator superalgebra $V$ and a finite automorphism group $G$. Note that $V = V_0 \oplus V_1$ has a canonical central automorphism $\sigma$ arising from the superspace structure of $V$ such that $\sigma|_{V_i} = (-1)^i$ for $i = 0, 1$. The $\sigma$ is crucial in formulating the main results in this paper. The involution $\sigma$ is well-known in the literature and is expressed as $(-1)^{\alpha_0}$ or $(-1)^F$ where $\alpha_0$ is the charge operator (e.g. Chapter 5 of [K]) and $F$ is the “Fermion number” (e.g. [GSW], [P]). Let $\tilde{G}$ be the group generated by $G$ and $\sigma$. We now explain the main results in details.

Let $g \in G$ have order $T$ and $\sigma g$ have order $T'$. Then a simple $\sigma g$-twisted module $M = (M,Y_M)$ has a grading of the shape

$$M = \bigoplus_{n=0}^{\infty} M_{\lambda+n/T}$$

with $M_\lambda \neq 0$ for some complex number $\lambda$ which is called the conformal weight of $M$ (see Section 4 and [DZ]). One basic result in [DZ] is that if $V$ is $g$-rational there are only finitely many simple $\sigma g$-twisted $V$-modules up to isomorphism. As in [DLM3], for any $h \in \tilde{G}$ we can define a $h\sigma hg^{-1}$-twisted module $h \circ M$ so that $h \circ M = M$ as vector spaces and $Y_{h \circ M}(v,z) = Y_M(h^{-1}v,z)$ (see Section 6). $M$ is called $h$-stable if $M$ and $h \circ M$ are isomorphic. As a result the stabilizer of $M$ in $\tilde{G}$ acts projectively on $M$, denoting by $\phi$.

Let $v \in V$ and $M$ be $\sigma h$-stable. We set

$$T_M(v,(g,h),q) = q^{\lambda - v/24} \sum_{n=0}^{\infty} \text{tr}_{M_{\lambda+n/T}}(o(v)\phi(\sigma h))q^{n/T}$$

where $Y_M(v,z) = \sum_{m \in \frac{1}{T} \mathbb{Z}} v(m)z^{-m-1}$ and $o(v) = v(wtv-1)$ induces a linear map on each homogeneous subspace of $M$. We extend the notation $T_M(v,(g,h),q)$ to all $v \in V$ linearly.

As in [Z], there is another vertex operator superalgebra structure on $V$ (see Section 3). If the original vertex operator superalgebra is defined over sphere, the second vertex operator superalgebra can be regarded as defined over a torus. We will denote the corresponding weight by $\text{wt}[v]$. Here are our main results on the modular invariance.

**Theorem 1** Suppose that $V$ is a $C_2$-cofinite vertex operator superalgebra and $G$ a finite group of automorphisms of $V$.

(i) Let $g, h \in G$ and $M$ be a simple, $\sigma h$-stable, $\sigma g$-twisted $V$-module. Then the trace function $T_M(v,(g,h),q)$ converges to a holomorphic function in the upper half plane $\mathfrak{h}$ where $q = e^{2\pi i \tau}$ and $\tau \in \mathfrak{h}$.

(ii) Suppose in addition that $V$ is $x$-rational for each $x \in \tilde{G}$. Let $v \in V$ satisfy $\text{wt}[v] = k$. Then the space of (holomorphic) functions in $\mathfrak{h}$ spanned by the trace functions $T_M(v,(g,h),\tau)$ for all choices of $g,h$ in $G$ and $\sigma h$-stable $M$ is a (finite-dimensional) $SL(2,\mathbb{Z})$-module such that

$$T_M|\gamma(v,(g,h),\tau) = (c\tau + d)^{-k}T_M(v,(g,h),\gamma\tau),$$
where $\gamma \in SL(2, \mathbb{Z})$ acts on $\mathfrak{h}$ as usual.

More precisely, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ then we have an equality

$$T_M(v, (g, h), \frac{at+b}{ct+d}) = (ct+d)^k \sum_W \gamma_{M,W} T_W(v, (g^a h^c, g^b h^d), \tau),$$

where $W$ ranges over the $\sigma g^a h^c$-twisted sectors which are $g^b h^d$ and $\sigma$-stable. The constants $\gamma_{M,W}$ depend only on $M, W$ and $\gamma$ only.

**Theorem 2** Let $V$ be a $C_2$-cofinite and $\sigma$-rational vertex operator superalgebra. Let $\{M^1, ..., M^n\}$ be the inequivalent $\sigma$-twisted $V$-modules which are $\sigma$-stable. Then for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2, \mathbb{Z})$ there are constants $\gamma_{ij}$ for $1 \leq i, j \leq n$ such that for any $v \in V$ with $\text{wt}[v] = k$,

$$T_{M^i}(v, (1, 1), \frac{at+b}{ct+d}) = (ct+d)^k \sum_{j=1}^n \gamma_{ij} T_{M^j}(v, (1, 1), \tau).$$

It is important to point out that Theorem 2 is an analogue of the main theorem in [Z]. In fact, if $V_1 = 0$ then $V$ is a vertex operator algebra and $\sigma = id_V$. In this case, Theorem 2 reduces to the main theorem in [Z]. In particular, the space of characters of irreducible $V$-modules is invariant under the action of the modular group. But if $V_1 \neq 0$ the space of characters of irreducible $V$-modules is no longer invariant under the action of the modular group. However, the space of $\sigma$-stable traces on $\sigma$-twisted sectors is modular invariant. So the automorphism $\sigma$ plays a fundamental role in the study of modular invariance for vertex operator superalgebra and its orbifold theory. This will be illustrated in Section 8 with examples in Section 10.

There is an assumption in both Theorems 1 and 2 on twisted modules. That is, the twisted modules are required to be $\sigma$-stable. There are examples in which irreducible $\sigma$-twisted modules are not $\sigma$-stable. So these modules which are not $\sigma$-stable are excluded in the discussion of modular invariance. To include these modules, one can use the Theta group $\Gamma_0 = \Gamma(2) \cup \Gamma(2) S$ instead of the full modular group $SL(2, \mathbb{Z})$. Here $\Gamma(2)$ is the congruence subgroup of $SL(2, \mathbb{Z})$ consisting of elements which are congruent to the identity modulo 2 and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In fact if $G = 1$ this has been done in [H]. The involution $\sigma$ does not play any role in this approach.

Again let $G$ be a finite automorphism group of $V$ and $g \in G$. Let $M$ be a simple $g$-twisted $V$-module. Let $G_M$ be the stabilizer of $M$ in $G$. Then $M$ is $h$-stable for any $h \in G_M$ and $G_M$ acts projectively on $M$. As before we use $\phi$ to denote the projective representation. For $h \in G_M$ and $v \in V$ set

$$F_M(v, (g, h), q) = \text{tr}_M o(\phi(v)\phi(h)) q^{L(0) - c/24}.$$

The analogue of Theorem 3 is the following which extends the modular invariance result in [H] to the orbifold theory.
Theorem 3  Suppose that $V$ is a $C_2$-cofinite vertex operator superalgebra and $G$ a finite group of automorphisms of $V$.

(i) Let $g, h \in G$ and $M$ be a simple, $h$-stable $g$-twisted $V$-module. Then the trace function $F_M(v, (g, h), q)$ converges to a holomorphic function in the upper half plane $\mathfrak{h}$.

(ii) Suppose in addition that $V$ is $x$-rational for each $x \in G$: Let $v \in V$ satisfy $\text{wt}[v] = k$. Then the space of (holomorphic) functions in $\mathfrak{h}$ spanned by the trace functions $F_M(v, (g, h), \tau)$ for all choices of $g, h$ in $G$ and $h$-stable $M$ is a (finite-dimensional) $\Gamma_\theta$-module. That is, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta$ then we have an equality

$$F_M(v, (g, h), \frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k \sum_W \gamma_{M, W} F_W(v, (g^a h^c, g^b h^d), \tau),$$

where $W$ ranges over the $g^a h^c$-twisted sectors which are $g^b h^d$-stable. The constants $\gamma_{M, W}$ depend only on $M, W$ and $\gamma$ only.

This paper is a super version of the theories developed in [DLM3]. The main ideas and the broad outline of our proof follow from those in [DLM3], [Z] and [H]. Two important results on $g$-rationality and the associative algebras $A_g(V)$ in [DZ] (also see Section 4) also play basic roles in this paper. If $V$ is $g$-rational, then $A_g(V)$ is semisimple and the category of $V$-modules and the category of finite dimensional $A_g(V)$-modules are equivalent. Furthermore there are finitely many irreducible $g$-twisted modules. These results enable us to prove that the space of 1-point $(g, h)$ functions is finite dimensional. In fact, from the proof of the main results in this paper one can see that we only need the assumptions that $V$ is $C_2$-cofinite and $A_g(V)$ is a semisimple associative algebra. Although there are a lot of expectations on the relation among rationality, $C_2$-cofiniteness, the semisimplicity of $A_g(V)$, we cannot remove either the $C_2$-cofiniteness condition or the $A_g(V)$ semisimplicity condition.

This paper is organized as follows. In Section 2 we review several families of modular functions and elliptic functions, and their transformation laws following [DLM3]. Section 3 is about the definition of vertex operator superalgebra and vertex operator superalgebra on torus. In Section 4 we define weak, admissible and ordinary $g$-twisted modules for a vertex operator superalgebra $V$ and a finite order automorphism $g$. We also present the main results concerning the associative algebra $A_g(V)$ and $g$-rationality. We define the space of abstract 1-point $(g, h)$-functions in Section 5 and establish that the elements in the full modular group transform the space of 1-point $(g, h)$ functions to another space of 1-point functions. We also discuss the shapes of $q$-expansions of the 1-point functions and prove that these functions satisfy differential equations with regular singularity. This leads us to the definition of formal 1-point $(g, h)$ functions. Sections 6 and 8 are the key sections in this paper. We show in Section 6 that the $\sigma h$-trace on $\sigma g$-twisted modules produce 1-point $(g, h)$-functions. In fact, the inequivalent simple $\sigma g$-twisted modules give linearly independent 1 point functions. In Section 8 we show that if $V$ is $\sigma g$-rational then the space of 1-point $(g, h)$ functions has a basis consisting of the $\sigma h$-trace on the inequivalent simple $\sigma g$-twisted modules. We also give several important corollaries in
Section 8 concerning the rationality of the central charges and the conformal weights of simple $g$-twisted modules. In Section 9 we present the modular invariance result on $\Gamma_\theta$. We give an existence result on the twisted sectors in Section 7 and discuss several examples in Section 10 to illustrate the main theorems.

2 P-functions and Q-functions

In this section, we review some properties of the $P$–functions and $Q$–functions following [DLM3]. These functions will be used extensively in later sections.

Recall that the modular group $\Gamma = SL(2, \mathbb{Z})$ acts on the upper half plane $\mathfrak{h} = \{ z \in \mathbb{C} | \text{Im} z > 0 \}$ via Möbius transformations

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \tau = \frac{a\tau + b}{c\tau + d}. \quad \text{(2.1)}$$

$\Gamma$ also acts on the right of $S^1 \times S^1$ via

$$(\mu, \lambda) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = (\mu^a\lambda^c, \mu^b\lambda^d). \quad \text{(2.2)}$$

Let $\mu = e^{2\pi i j/M}$ and $\lambda = e^{2\pi i l/N}$ for integers $j, l, M, N$ with $M, N > 0$. For each integer $k = 1, 2, \cdots$ and each $(\mu, \lambda)$ we define a function $P_k$ on $\mathbb{C} \times \mathfrak{h}$ as follows

$$P_k(\mu, \lambda, z, q_x) = P_k(\mu, \lambda, z, \tau) = \frac{1}{(k-1)!} \sum_{n \in \frac{1}{\lambda} + \mathfrak{h}} n^{k-1} q_x^n \quad \text{(2.3)}$$

In the above equation, $q_x = e^{2\pi i x}$ and $\sum'$ means omit the term $n = 0$ when $(\mu, \lambda) = (1, 1)$. Then $P_k$ converges uniformly and absolutely on compact subsets of the region $|q_x| < |q_z| < 1$. We have the following transformation properties (see Theorem 4.2 of [DLM3]).

**Theorem 2.1** Suppose that $(\mu, \lambda) \neq (1, 1)$. Then the following equalities hold:

$$P_k(\mu, \lambda, \frac{z}{c\tau + d}, \gamma \tau) = (c\tau + d)^k P_k((\mu, \lambda)\gamma, z, \tau).$$

for all $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma$.

In order to define $Q$–functions we need to recall the Bernoulli polynomials $B_r(x) \in \mathbb{Q}[x]$ which are determined by

$$te^{tx} \frac{t^r}{(e^t - 1)} = \sum_{r=0}^{\infty} B_r(x) t^r \quad \text{for } t \to 0.$$ 

For example, $B_0(x) = 1, B_1(x) = x - \frac{1}{2}, B_2(x) = x^2 - x + \frac{1}{6}$. 

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For \((\mu, \lambda) = (e^{2\pi i j/M}, e^{2\pi i l/N})\) and \((\mu, \lambda) \neq (1, 1)\), when \(k \geq 1\) and \(k \in \mathbb{Z}\), we define

\[
Q_k(\mu, \lambda, q) = Q_k(\mu, \lambda, \tau)
\]

\[
= \frac{1}{(k-1)!} \sum_{n \geq 0} \frac{\lambda(n + j/M)^{k-1} q^{n+j/M}}{1 - \lambda q^{n+j/M}} + \frac{(-1)^k}{(k-1)!} \sum_{n \geq 1} \frac{\lambda^{-1}(n - j/M)^{k-1} q^{-n-j/M}}{1 - \lambda^{-1} q^{-n-j/M}} - \frac{B_k(j/M)}{k!}.
\]

(2.4)

Here \((n + j/M)^{k-1} = 1\) if \(n = 0, j = 0\) and \(k = 1\). Similarly, \((n - j/M)^{k-1} = 1\) if \(n = 1, j = M\) and \(k = 1\). We also define

\[
Q_0(\mu, \lambda, \tau) = -1.
\]

(2.5)

Here are the transformation properties of \(Q\)-functions (see Theorem 4.6 of [DLM3]).

**Theorem 2.2** If \(k \geq 0\) then \(Q_k(\mu, \lambda, \tau)\) is a holomorphic modular form of weight \(k\). If \(\gamma = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \Gamma\) it satisfies

\[
Q_k(\mu, \lambda, \gamma \tau) = (ct + d)^k Q_k((\mu, \lambda) \gamma, \tau).
\]

For the further discussion we also define \(\bar{P}\)-functions. Again we take \(\mu = e^{2\pi i j/M}\) and \(\lambda = e^{2\pi i l/N}\). Set for \(k \geq 1\),

\[
\bar{P}_k(\mu, \lambda, z, q) = \bar{P}_k(\mu, \lambda, z; \tau) = \frac{1}{(k-1)!} \sum_{n \in j/M + \mathbb{Z}}' \frac{n^{k-1} z^n}{1 - \lambda q^n}
\]

(2.6)

with \(\bar{P}_0 = 0\).

Then we have (see Proposition 4.9 of [DLM3]):

**Proposition 2.3** If \(m \in \mathbb{Z}\), \(\mu = e^{2\pi i j/M}, k \geq 0\) and \((\mu, \lambda) \neq (1, 1)\), then

\[
Q_k(\mu, \lambda, \tau) + \frac{1}{k!} B_k(1 - m + j/M)
\]

\[
= \operatorname{Res}_z \left( \sum_{s \geq 0} \frac{z^s}{z^{s+1}} \frac{m-j/M}{z^{m+j/M}} \bar{P}_k(\mu, \lambda, \frac{z_1}{z}, \tau) \right)
\]

\[
+ \operatorname{Res}_z \left( \lambda \sum_{s \geq 0} \frac{z^s}{z^{s+1}} \frac{m-j/M}{z^{m+j/M}} \bar{P}_k(\mu, \lambda, \frac{z_1 q}{z}, \tau) \right).
\]

We also recall the Eisenstein series of weight \(k\) for even \(k \geq 3\):

\[
G_k(\tau) = \sum_{m_1, m_2 \in \mathbb{Z}}' \frac{1}{(m_1 \tau + m_2)^k}
\]

\[
= (2\pi i)^k \left( \frac{-B_k(0)}{k!} + \frac{2}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \right)
\]

(2.8)
where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. For this range of values of $k$, $G_k(\tau)$ is a modular form on $SL(2, \mathbb{Z})$ of weight $k$. We will use of the normalized Eisenstein series
\[
E_k(\tau) = \frac{1}{(2\pi i)^k} G_k(\tau) = \frac{-B_k(0)}{k!} + \frac{2}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n
\]  
for even $k \geq 2$. Note that $E_2(\tau)$ is a quasi-modular form (cf. [EZ]).

3 Vertex operator superalgebras

In this section we recall the definition of vertex operator superalgebra (cf. [B1], [DL], [FLM3]). We also study another vertex operator superalgebra structure on torus following [Z] to give the modularity result in later sections.

Let $V = V_0 \oplus V_1$ be any $\mathbb{Z}_2$-graded vector space. The element in $V_0$ (resp. $V_1$) is called even (resp. odd). For any $v \in V_i$ with $i = 0, 1$ we define $\tilde{v} = i$. For convenience, we let $\epsilon_{\mu, v} = (-1)^{\mu}$ and $\epsilon_v = (-1)^{\tilde{v}}$.

**Definition 3.1** A vertex operator superalgebra (VOSA) is a quadruple $(V, Y, \mathbf{1}, \omega)$, where

$V = \bigoplus_{n \in \frac{1}{2} \mathbb{Z}} V_n = V_0 \oplus V_1$

with $V_0 = \sum_{n \in \mathbb{Z}} V_n$ and $V_1 = \sum_{n \in \frac{1}{2} \mathbb{Z}^+} V_n$ satisfying $\dim V_n < \infty$ for all $n$ and $V_m = 0$ if $m$ is sufficiently small, $\mathbf{1} \in V_0$ is the vacuum of $V$, $\omega \in V_2$ is called the conformal vector of $V$, and $Y$ is a linear map

$V \rightarrow (\text{End} V)[[z, z^{-1}]]$,  
$v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v(n) z^{-n-1} \quad (v(n) \in \text{End} V)$.

satisfying the following axioms for $u, v \in V$:

(i) For any $u, v \in V$, $u(n)v = 0$ for sufficiently large $n$,

(ii) $Y(\mathbf{1}, z) = Id_V$,

(iii) $Y(v, z)\mathbf{1} = v + \sum_{n \geq 2} v(-n) \mathbf{1} z^{n-1}$,

(iv) The component operators of $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$ satisfy the Virasoro algebra relation with central charge $c \in \mathbb{C}$:

$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c$, \hspace{1cm} (3.1)

and

$L(0)|_{V_n} = n, \ n \in \frac{1}{2} \mathbb{Z}$ \hspace{1cm} (3.2)

$\frac{d}{dz} Y(v, z) = Y(L(-1)v, z)$, \hspace{1cm} (3.3)
(v) The Jacobi identity for $\mathbb{Z}_2$-homogeneous $u, v \in V$ holds,

$$
\begin{align*}
 z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(u, z_1) Y(v, z_2) - \epsilon_{u,v} z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y(v, z_2) Y(u, z_1) \\
= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y(Y(u, z_0) v, z_2)
\end{align*}
\tag{3.4}
$$

where $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ and $(z_i - z_j)^n$ is expanded as a formal power series in $z_j$. Throughout the paper, $z_0, z_1, z_2, \text{etc.}$ are independent commuting formal variables.

As in [Z], we also define a vertex operator superalgebra on a torus associated to $V$.

Set

$$
\tilde{\omega} = \omega - \frac{c_2}{24}
$$

and

$$
Y[v, z] = Y(v, e^z - 1) e^{z \omega} = \sum_{n \in \mathbb{Z}} v[n] z^{-n-1}
\tag{3.5}
$$

for homogeneous $v \in V$. Following the proof of Theorem 4.21 of [Z] we have

**Theorem 3.2** $(V, Y[\cdot], 1, \tilde{\omega})$ is a vertex operator superalgebra.

As in [Z] and [DLM3] we have

$$
v[m] = \text{Res}_z Y(v, z)(\log(1 + z))^m (1 + z)^{\omega - 1}.
\tag{3.6}
$$

For integers $i, m$ with $i, m \geq 0$, define numbers $c(p, i, m)$ by

$$
\left( \begin{array}{c} p - 1 + z \\ i \end{array} \right) = \sum_{m=0}^{i} c(p, i, m) z^m.
\tag{3.7}
$$

Then if $m \geq 0$

$$
v[m] = m! \sum_{i=m}^{\infty} c(\omega, i, m) v(i).
\tag{3.8}
$$

In particular,

$$
v[0] = \sum_{i=0}^{\infty} \left( \begin{array}{c} \omega - 1 \\ i \end{array} \right) v(i).
\tag{3.9}
$$

Write

$$
Y[\tilde{\omega}, z] = \sum_{n \in \mathbb{Z}} L[n] z^{-n-2}.
\tag{3.10}
$$

One can verify that

$$
L[-1] = L(-1) + L(0)
\tag{3.11}
$$

$$
L[0] = L(0) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)} L(n).
\tag{3.12}
$$

We denote the eigenspace of $L[0]$ with eigenvalue $n \in \mathbb{Z}$ by $V[n]$. If $v \in V[n]$ we write $\text{wt}[v] = n$. 

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Definition 3.3 An automorphism $g$ of vertex operator superalgebra $V$ is a linear automorphism of $V$ preserving $\mathbf{1}$ and $\omega$ such that the actions of $g$ and $Y(v, z)$ on $V$ are compatible in the sense that
\[
gY(v, z)g^{-1} = Y(gv, z)
\]
for $v \in V$.

Note that an automorphism preserves each homogeneous space $V_n$ and thus each $V_i$. There is a special automorphism $\sigma$ of $V$ with $\sigma | V_i = (-1)^i$ associated to the superspace structure of $V$. That is, $\sigma v = \epsilon_i v$ for homogeneous $v$. Then $\sigma$ is a central element in the full automorphism group $\text{Aut}(V)$. We have already mentioned in the introduction that the involution $\sigma$ is also denoted by $(-1)^F$ in conformal field theory where $F$ is the “Fermion number” (cf. [GSW], [P]). As we will see later, $\sigma$ plays a fundamental role in formulating the modular invariance of trace functions.

4 Twisted Modules for VOSA

The twisted modules are the main objects in this section. In the case of vertex operator algebras, the twisted modules are studied in [D], [DLM2], [FFR], [FLM2], [FLM3], [L]. We will follow [DLM2] and [DZ] in this section. In particular, we will introduce the notion of weak, admissible and ordinary twisted modules. We also present the associative algebra $A_g(V)$ associated to a vertex operator algebra $V$ and a finite order automorphism $g$ and related consequences.

Let $(V, \mathbf{1}, \omega, Y)$ be a VOSA and $g$ an automorphism of $V$ of finite order $T$. Let $T'$ be the order of $g\sigma$. Then we have the following eigenspace decompositions:
\[
V = \bigoplus_{r \in \mathbb{Z}/T} V^r
\]
where $V^r = \{v \in V | g\sigma v = e^{-2\pi i r/T} v\}$ and $V^r = \{v \in V | gv = e^{-2\pi i r/T} v\}$. We now can define various notions of $g$-twisted modules (cf. [DZ]).

Definition 4.1 A weak $g$-twisted $V$-module is a $\mathbb{C}$-linear space $M$ equipped with a linear map $V \to (\text{End}M)[[z^{1/T}, z^{-1/T}]]$, $v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Q}} v(n) z^{-n-1}$ satisfying:

1. For $v \in V, w \in M, v(m)w = 0$ if $m >> 0$,
2. $Y_M(1, z) = \text{Id}_M$,
3. For $v \in V^r$, and $0 \leq r \leq T - 1$,
   \[
   Y_M(v, z) = \sum_{n \in r/T + z} v(n) z^{-n-1}
   \]
4. The twisted Jacobi identity holds for $u \in V^r$ and $v \in V$:
   \[
   z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(u, z_1) Y_M(v, z_2) - \delta \left( \frac{z_2 - z_1}{z_0} \right) Y_M(v, z_2) Y_M(u, z_1)
   = \frac{1}{z_2} \left( \frac{z_1 - z_0}{z_2} \right)^{-r/T} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M(Y(u, z_0) v, z_2).
   \]
Note that (4.3) is equivalent to the following:

\[
\text{Res}_{z_1} Y_M(u, z_1) Y_M(v, z_2) z_1^{r/T} (z_1 - z_2)^{m_1} z_2^n \\
- \epsilon_{uv} \text{Res}_{z_1} Y_M(v, z_2) Y_M(u, z_1) z_1^{r/T} (z_1 - z_2)^{m_2} z_2^n \\
= \text{Res}_{z_1 - z_2} Y_M(Y(u, z_1 - z_2)v, z_2) \epsilon_{z_2, z_1 - z_2} z_1^{r/T} (z_1 - z_2)^{m_1} z_2^n \\
(4.4)
\]

for \( m \in \mathbb{Z}, n \in \mathbb{C} \), where \( \epsilon_{z_2, z_1 - z_2} z_1^{r/T} = \sum_{m \geq 0} \left( \binom{r/T}{m} z_2^{r/T - m} (z_1 - z_2)^m \right) \).

Set

\[
Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} \omega(n) z^{-n - 2}.
\]

Then \( Y_M(L(-1)v, z) = \frac{d}{dz} Y_M(v, z) \) for \( v \in V \) and the operators \( L(n) \) satisfy the Virasoro algebra relation with central charge \( c \) (cf. [DLM1], [DZ]).

**Definition 4.2** An admissible \( g \)-twisted \( V \)-module is a weak \( g \)-twisted \( V \)-module \( M \) which carries a \( \frac{1}{T} \mathbb{Z} \)-grading

\[
M = \bigoplus_{0 \leq n \leq \frac{1}{T} \mathbb{Z}} M(n)
\]

such that

\[
v(m) M(n) \subseteq M(n + wtv - m - 1)
\]

for homogeneous \( v \in V \). We may and do assume that \( M(0) \neq 0 \) if \( M \neq 0 \).

Note that an admissible \( V \)-module \( M \) is \( \frac{1}{T} \mathbb{Z} \)-graded instead of \( \frac{1}{T} \mathbb{Z} \)-graded as \( V \) is \( \frac{1}{2} \mathbb{Z} \)-graded.

**Definition 4.3** An (ordinary) \( g \)-twisted \( V \)-module \( M \) is a \( \mathbb{C} \)-graded weak \( g \)-twisted \( V \)-module

\[
M = \prod_{\lambda \in \mathbb{C}} M_{\lambda}
\]

where \( M_{\lambda} = \{ w \in M | L(0)w = \lambda w \} \) such that \( \dim M_{\lambda} \) is finite and for fixed \( \lambda \), \( M_{\frac{1}{2} + \lambda} = 0 \) for all small enough integers \( n \).

It \( g = 1 \) we have the notion of weak, admissible ordinary \( V \)-modules. It is not hard to prove that any ordinary \( g \)-twisted \( V \)-module is admissible.

If \( M \) is a simple \( g \)-twisted \( V \)-module, then

\[
M = \bigoplus_{n=0}^{\infty} M_{\lambda + n/T}
\]

for some \( \lambda \in \mathbb{C} \) such that \( M_{\lambda} \neq 0 \) (cf. [Z]). We define \( \lambda \) as the conformal weight of \( M \).

**Definition 4.4** (1) A VOSA \( V \) is called rational for an automorphism \( g \) of finite order if the category of admissible modules is semisimple.

(2) \( V \) is called \( g \)-regular if any weak \( g \)-twisted \( V \)-module is a direct sum of irreducible ordinary \( g \)-twisted \( V \)-modules.
It is clear that a $g$-regular vertex operator superalgebra is $g$-rational.
The following theorem is proved in [DZ].

**Theorem 4.5** Let $V$ be a VOSA and $g$ an automorphism of $V$ of finite order. Assume that $V$ is $g$-rational. Then

1. There are only finitely many irreducible admissible $g$-twisted $V$-modules up to isomorphism.
2. Each irreducible admissible $g$-twisted $V$-module is ordinary.

Now we review the $A_g(V)$ theory [DZ]. Let $V^{rs}$ be as in (4.1). For $u \in V^{rs}$ and $v \in V$ we define

$$u \circ_g v = \text{Res}_z (1+z)^{wt u-1+\delta r + \frac{r}{2}} Y(u, z)v$$
$$u \ast_g v = \begin{cases} 
\text{Res}_z (Y(u, z)\frac{(1+z)^{wt u}}{z} v) & \text{if } r = 0 \\
0 & \text{if } r > 0.
\end{cases}$$

where $\delta_r = 1$ if $r = 0$ and $\delta_r = 0$ if $r \neq 0$. Extend $\circ_g$ and $\ast_g$ to bilinear products on $V$. We let $O_g(V)$ be the linear span of all $u \circ_g v$. Then the quotient $A_g(V) = V/O_g(V)$ is an associative algebra with respect to $\ast_g$.

Let $M$ be a weak $\sigma g$-twisted $V$-module. Define

$$\Omega(M) = \{ w \in M | u(wtu - 1 + n)w = 0, u \in V, n > 0 \}.$$ 

For homogeneous $u \in V$ we set

$$o(u) = u(wtu - 1)$$

We have the following Theorem (see [DZ]):

**Theorem 4.6** Let $V$ be a VOSA together with an automorphism $g$ of finite order, and $M$ a weak $g$-twisted $V$-module. Then

1. $\Omega(M)$ is an $A_g(V)$-module such that $v + O_g(V)$ acts as $o(v)$.
2. If $M = \sum_{n \geq 0} M(n/T')$ is an admissible $g$-twisted $V$-module with $M(0) \neq 0$, then $M(0) \subset \Omega(M)$ is an $A_g(V)$-submodule. Moreover, $M$ is irreducible if and only if $M(0) = \Omega(M)$ and $M(0)$ is a simple $A_g(V)$-module.
3. The map $M \to M(0)$ gives a 1-1 correspondence between the irreducible admissible $g$-twisted $V$-modules and simple $A_g(V)$-modules.
4. If $V$ is $g$-rational then $A_g(V)$ is a finite dimensional semisimple associative algebra.

Another important concept is the $C_2$-cofinite condition (cf. [Z]). Set

$$C_2(V) = \{ u(-2)v | u, v \in V \}.$$ 

$V$ is called $C_2$-cofinite if $C_2(V)$ has finite codimension.

It is proved in [Li] and [ABD] that a vertex operator algebra $V$ is regular if and only if $V$ is $C_2$-cofinite and rational. Using the the same proof we have
Theorem 4.7 Assume that \( V \) is \( C_2 \)-cofinite. Then \( V \) is \( g \)-rational if and only if \( V \) is \( g \)-regular.

Regularity of certain vertex operator superalgebras have also been studied recently in [A].

Here is another consequence of the \( C_2 \)-cofiniteness.

Proposition 4.8 Suppose that \( V \) is \( C_2 \)-cofinite and let \( g \) be an automorphism of \( V \) of finite order. Then

1. The algebra \( A_g(V) \) has finite dimension.
2. If \( A_g(V) \neq 0 \) then \( V \) has a simple \( g \)-twisted \( V \)-module.

The proof of (1) is similar to that of Proposition 3.6 in [DLM3] and (2) follows from Theorem 4.6 (3).

5 1-point functions on the torus

We first introduce some notation which will be used in this section. \( V \) again is a VOSA and \( g, h \in \text{Aut}(V) \) such that \( gh = hg \). Let \( o(g) = T, o(h) = T_1 \) be finite.

Let \( A \) be the subgroup of \( \text{Aut}(V) \) generated by \( g \) and \( h \); and \( N = \text{lcm}(T, T_1) \) be the exponent of \( A \). Let \( \Gamma(T, T_1) \) be the subgroup of matrices \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) in \( SL(2, \mathbb{Z}) \) satisfying \( a \equiv d \equiv 1 \pmod{N} \), \( b \equiv 0 \pmod{T} \), \( c \equiv 0 \pmod{T_1} \), and \( M(T, T_1) \) be the ring of holomorphic modular forms on \( \Gamma(T, T_1) \) with natural gradation \( M(T, T_1) = \oplus_{k \geq 0} M_k(T, T_1) \), where \( M_k(T, T_1) \) is the space of forms of weight \( k \).

From the Lemma 5.1 in [DLM3], \( M(T, T_1) \) is a Noetherian ring which contains each \( E_{2k}(\tau), k \geq 2 \), and each \( Q_k(\mu, \lambda, \tau), k \geq 0 \) for \( \mu, \lambda \) a \( T \)-th., resp. \( T_1 \)-th. root of unity.

Let \( V(T, T_1) = M(T, T_1) \otimes_C V \). For \( v \in V \) with \( gv = \mu^{-1}v, hv = \lambda^{-1}v \), we can define a \( M(T, T_1) \)-submodule of \( V(T, T_1) \), say \( O(g, h) \) which consists of the following elements:

\[
v[0]w, w \in V, (\mu, \lambda) = (1, 1) \quad (5.1)
\]
\[
v[-2]w + \sum_{k=2}^{\infty} (2k - 1) E_{2k}(\tau) \otimes v[2k - 2]w, (\mu, \lambda) = (1, 1) \quad (5.2)
\]
\[
v, (\varepsilon_v, \mu, \lambda) \neq (1, 1, 1) \quad (5.3)
\]
\[
\sum_{k=0}^{\infty} Q_k(\mu, \lambda, \tau) \otimes v[k - 1]w, (\mu, \lambda) \neq (1, 1). \quad (5.4)
\]

We have the following Lemmas as in [DLM3].

Lemma 5.1 If \( V \) is \( C_2 \)-cofinite, then \( V(T, T_1)/O(g, h) \) is a finitely-generated \( M(T, T_1) \)-module.
Lemma 5.2 If $V$ is $C_2$-cofinite, then for $v \in V$, there is $m \in \mathbb{N}$ and elements $r_i(\tau) \in M(T, T_1)$, $0 \leq i \leq m - 1$, such that

$$L[-2]^m v + \sum_{i=0}^{m-1} r_i(\tau) \otimes L[-2]^i v \in O(g, h).$$

(5.5)

Definition 5.3 The space of $(g, h)$ 1-point functions $C(g, h)$ is a $\mathbb{C}$-linear space consisting of functions

$$S : V(T, T_1) \times \mathfrak{h} \rightarrow \mathbb{C}$$

satisfying the following conditions:

1. $S(v, \tau)$ is holomorphic in $\tau$ for $v \in V(T, T_1)$.
2. $S(v, \tau)$ is $\mathbb{C}$ linear in $v$ and for $f \in M(T, T_1)$, $v \in V$,

$$S(f \otimes v, \tau) = f(\tau) S(v, \tau)$$

3. $S(v, \tau) = 0$ if $v \in O(g, h)$.
4. If $v \in V$ with $\sigma v = gv = hv = v$, then

$$S(L[-2]v, \tau) = \partial S(v, \tau) + \sum_{l=2}^{\infty} E_{2l}(\tau) S(L[2l - 2]v, \tau).$$

(5.6)

Here $\partial S$ is the operator which is linear in $v$ and satisfies

$$\partial S(v, \tau) = \partial_k S(v, \tau) = \frac{1}{2\pi i} \frac{d}{d\tau} S(v, \tau) + k E_{2}(\tau) S(v, \tau)$$

(5.7)

for $v \in V[\kappa]$.

The 1-point functions satisfy the following modular transformation property.

Theorem 5.4 For $S \in C(g, h)$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we define

$$S|\gamma(v, \tau) = S|_{\kappa} \gamma(v, \tau) = (c\tau + d)^{-k} S(v, \gamma \tau)$$

(5.8)

for $v \in V[\kappa]$, and extend linearly. Then $S|\gamma \in C((g, h)\gamma)$.

This theorem is the ‘super’ analogue of Theorem 5.4 in [DLM3] and the proof is the same.

Now we study the $q$-expansion of $S(v, \tau)$ for $S \in C(g, h)$ and $v \in V$. As argued in [DLM3], the Frobenius-Fuchs theory of differential equations with regular singular points tells us that $S(v, \tau)$ may be expressed in the following form: for some $p \geq 0$,

$$S(v, \tau) = \sum_{i=0}^{p} (\log q_\tau)^i S_i(v, \tau)$$

(5.9)
where
\[ S_i(v, \tau) = \sum_{j=1}^{b(i)} q^{\lambda_{i,j}} S_{i,j}(v, \tau) \]  \hspace{1cm} (5.10)
\[ S_{i,j}(v, \tau) = \sum_{n=0}^{\infty} a_{i,j,n}(v) q^{n/T} \]  \hspace{1cm} (5.11)
are holomorphic on the upper half-plane, and
\[ \lambda_{i,j_1} \neq \lambda_{i,j_2} \pmod{\frac{1}{T} \mathbb{Z}} \]  \hspace{1cm} (5.12)
for \( j_1 \neq j_2 \) where \( q_{\pm} = e^{2\pi i \tau/T} \) and \( \lambda_{i,j} \in \mathbb{C} \). Moreover, \( p \) is bounded independently of \( v \).

It is important to point out that the 1-point functions on the torus can be defined formally regarding \( q \) as a formal variable. That is, we identify elements of \( M(T, T_1) \) with their Fourier expansions at \( \frac{1}{T} \), which lie in the ring of formal power series \( \mathbb{C}[[q_{\pm}]] \). Similarly, the functions \( E_{2k}(\tau) \), \( k \geq 1 \), are considered to lie in \( \mathbb{C}[[q]] \). The operator \( \frac{1}{2\pi i} \frac{d}{d{\tau}} \) acts on \( \mathbb{C}[[q_{\pm}]] \) via the identification \( \frac{1}{2\pi i} \frac{d}{d{\tau}} = q_{\pm} \frac{d}{dq} \).

Then a formal \((g, h)\) 1-point function is a map
\[ S : V(T, T_1) \to P \]
where \( P \) is the space of formal power series of the form
\[ q^\lambda \sum_{n=0}^{\infty} a_n q^{n/T} \]  \hspace{1cm} (5.13)
for some \( \lambda \in \mathbb{C} \), and it satisfies the formal analogues of (2)-(4) of Definition 5.3. Then as in [DLM3], each \( S(v, q) \) converges to a holomorphic function and we have

**Theorem 5.5** Assume that \( V \) is \( C_2 \)-cofinite. Then any formal \((g, h)\) 1-point function \( S \) defines an element of \( \mathcal{C}(g, h) \), also denoted by \( S \), via the identification
\[ S(v, \tau) = S(v, q), q = q_\tau = e^{2\pi i \tau}. \]  \hspace{1cm} (5.14)

### 6 Trace functions

The goal in this section is to construct \((g, h)\) 1-point functions which are essentially the graded \( h\sigma\)-trace functions on \( g\sigma\)-twisted \( V \)-modules.

Let \( M = (M, Y_M) \) be a \( g\sigma\)-twisted \( V \)-module and \( k \in \text{Aut}(V) \). We can define a \( k(\sigma)k^{-1} \)-twisted \( V \)-module \((k \circ M, Y_{k \circ M})\) so that \( k \circ M = M \) as vector spaces and
\[ Y_{k \circ M}(v, z) = Y_M(k^{-1}v, z). \]

\( M \) is called \( k \)-stable if \( k \circ M \) and \( M \) are isomorphic.
Recall from Section 4 that if $M$ is a simple $g\sigma$-twisted module then there exists a complex number $\lambda$ such that

$$M = \bigoplus_{n=0}^{\infty} M_{\lambda + \frac{n}{T}}$$

(see (4.8)) where $T$ is the order of $g$. In particular, if $g = \sigma$ then

$$M = \bigoplus_{n=0}^{\infty} M_{\lambda + \frac{n}{2}}.$$  

Let $h \in \text{Aut}(V)$ as before and assume that $h\sigma \circ M$ and $M$ are isomorphic. Then $g, h$ commute and there is a linear map $\phi(h\sigma) : M \to M$ such that

$$\phi(h\sigma)Y_M(v, z)\phi(h\sigma)^{-1} = Y_M((\sigma h)v, z)$$

for all $v \in V$.

**Lemma 6.1** If $M$ is a simple $g\sigma$-twisted $V$-module then $M$ is $g$-stable. In particular, any simple untwisted $V$-module $M$ is $\sigma$-stable.

**Proof:** We define a map $\phi(g)$ on $M$ such that $\phi(g)|_{M_{\lambda+n/T}} = e^{2\pi i n/T}$. Let $o(g\sigma) = T'$ and $v \in V^r$, $v \in V^s$ (see (4.1)-(4.2)). It is easy to verify that $wtv - \frac{T}{T'}$ and $\frac{n}{T}$ are congruent modulo $\mathbb{Z}$. We immediately have $\phi(g)Y_M(v, z)\phi(g)^{-1} = Y_M((\sigma h)v, z)$ for all $v \in V$. That is, $M$ and $g \circ M$ are isomorphic. □

We remark that this result is different from a similar result in the case of vertex operator algebra where a simple $g$-twisted module $M$ and $g \circ M$ are always isomorphic. But this is not true in the super case.

Now we introduce the function $T$ which is linear in $v \in V$, and define for homogeneous $v \in V$ as follows:

$$T(v) = T_M(v, (g, h), q) = z^{wtv}\text{tr}_M Y_M(v, z)\phi(h\sigma)q^{L(0)-\frac{c}{24}}$$

(6.4)

Here $c$ is the central charge of $V$. If $M = V$ and $h = 1$, $T(1)$ is exactly the supertrace defined in the literature (cf. [AMV], [GSW] and [P]). Note that for $m \in \frac{1}{T}\mathbb{Z}, v(m)$ maps $M_{\mu}$ to $M_{\mu+wtv-m-1}$. So we can write $T(v)$ as follows:

$$T(v) = q^{L(0)-\frac{c}{24}} \sum_{n=0}^{\infty} \text{tr}_{M_{\lambda+n/2}} o(v)\phi(h\sigma)q^{n} = \text{tr}_M o(v)\phi(h\sigma)q^{L(0)-\frac{c}{24}}$$

(6.5)

Here is the main result in this section.

**Theorem 6.2** Let $V$ be $C_2$-cofinite and $g, h \in \text{Aut}(V)$ have finite orders. Let $M$ be a simple $g\sigma$-twisted $V$-module such that $M$ is $h$ and $\sigma$-stable. Then $T \in \mathbb{C}(g, h)$.

By Theorem 5.5 it is enough to prove that $T(v)$ defines a formal $(g, h)$ 1-point function. Clearly, $T(v)$ has the shape of (5.13). So we need to prove that $T(v)$ satisfies (2)-(4) in definition 5.3. But (2) is clear, we are going to prove (3) and (4).

Fix $v \in V$ such that $gv = \mu^{-1}v$, $hv = \lambda^{-1}v$ and $v_1 \in V$. Let $M$ be as in Theorem 6.2.
Lemma 6.3 \( T(v) = 0 \) for \( (\varepsilon, \mu, \lambda) \neq (1, 1, 1) \).

**Proof:** Let \( k \in \{g, h, \sigma\} \). Since \( k \circ M \) and \( M \) are isomorphic (see Lemma 6.1), then there is a linear isomorphism \( \phi(k) : M \rightarrow M \) such that \( \phi(k)Y_M(v, z)\phi(k)^{-1} = Y_M(kv, z) \) for all \( v \in V \). Normalizing \( \phi(k) \) we can assume that \( \phi(k)^t = 1 \) where \( t \) is the order of \( k \). So we can decompose \( M \) into a direct sum of eigenspaces. Since \( g, h, \sigma \) commute with each other, \( \phi(h\sigma) \) preserves each eigenspace. Since \( v \) is an eigenvector for \( k \) with eigenvalue different from 1, \( o(v) \) moves one eigenspace of \( \phi(k) \) to a different eigenspace. As a result, the trace of \( o(v)\phi(h\sigma) \) on any homogeneous subspace \( M_{\lambda+\varepsilon/\mu} \) is zero. The proof is complete. \( \square \)

**Lemma 6.4** \( T(v[0]v_1) = 0 \).

**Proof:** Consider

\[
\text{tr}_M[v(wtv - 1), Y_M(v_1, z)]\phi(\sigma h)q^{L(0) - c/24} = \text{tr}_M v(wtv - 1)Y_M(v_1, z)\phi(\sigma h)q^{L(0) - c/24} - \varepsilon_{v,v_1}\text{tr}_M Y_M(v_1, z)v(wtv - 1)\phi(\sigma h)q^{L(0) - c/24} = \text{tr}_M v(wtv - 1)Y_M(v_1, z)\phi(\sigma h)q^{L(0) - c/24}v(wtv - 1) = (1 - \varepsilon_{v,v_1})\text{tr}_M v(wtv - 1)Y_M(v_1, z)\phi(\sigma h)q^{L(0) - c/24}.
\]

On the other hand

\[
\text{tr}_M[v(wtv - 1), Y_M(v_1, z)]\phi(\sigma h)q^{L(0) - c/24} = \text{tr}_M \sum_{i=0}^{\infty} (wtv - 1)^i z^{wtv-1-i}Y_M(v(i)v_1, z)\phi(\sigma h)q^{L(0) - c/24} = z^{-wtv}T(v[0]v_1)
\]

Thus if \( v \in V_0 \), or both \( v \) and \( v_1 \) are in \( V_1 \), then \( \varepsilon_{v,v_1}\varepsilon_v = 1 \) and \( T(v[0]v_1) = 0 \). If \( v \in V_1 \), and \( v_1 \in V_0 \), then \( v[0]v_1 \in V_1 \). By Lemma 6.3, we also have \( T(v[0]v_1) = 0 \). \( \square \)

**Lemma 6.5**

\[
\sum_{k=0}^{\infty} Q_k(\mu, \lambda, \tau) \otimes T(v[k - 1]v_1) = 0
\]

for \( (\mu, \lambda) \neq (1, 1) \).

In order to prove Lemma 6.5, we first define 2-point correlation functions. These are multi-linear functions \( T(v_1, v_2) \) defined for \( v_1, v_2 \in V \) homogeneous via

\[
T(v_1, v_2) = T((v_1, z_1), (v_2, z_2), (g, h), q) = z_1^{-wtv_1}z_2^{-wtv_2}\text{tr}_M Y_M(v_1, z_1)Y_M(v_2, z_2)\phi(\sigma h)q^{L(0) - c/24}. \tag{6.6}
\]

We need the following sublemma.
Sublemma 6.6 Let \( v, v_1 \in V \) be homogeneous with \( gv = \mu^{-1}v \), \( hv = \lambda^{-1}v \) and \((\mu, \lambda) \neq (1, 1)\). Then

\[
T(v, v_1) = \sum_{k=1}^{\infty} \bar{P}_k(\mu, \lambda, \frac{z_1}{z}, q)T(v[k-1]v_1) \tag{6.7}
\]

\[
T(v_1, v) = \epsilon_v \lambda \sum_{k=1}^{\infty} \bar{P}_k(\mu, \lambda, \frac{z_1}{z} q, q)T(v[k-1]v_1) \tag{6.8}
\]

where \( \bar{P}_k \) is as in (2.6).

**Proof:** First we have the following identities:

\[
(1 - \epsilon_{v,v_1} \epsilon_v \lambda q^k) \text{tr}_M v(wtv - 1 + k)Y_M(v_1, z_1)\phi(\sigma h)q^{L(0)}
= \sum_{i=0}^{\infty} (wtv - 1 + k) z_1^{i} \text{tr}_M Y_M(v(i)v_1, z_1)\phi(\sigma h)q^{L(0)} \tag{6.9}
\]

\[
(1 - \epsilon_{v,v_1} \epsilon_v \lambda q^k) \text{tr}_M Y_M(v_1, z_1)v(wtv - 1 + k)\phi(\sigma h)q^{L(0)}
= \epsilon_v \lambda q^k \sum_{i=0}^{\infty} (wtv - 1 + k) z_1^{i} \text{tr}_M Y_M(v(i)v_1, z_1)\phi(\sigma h)q^{L(0)}. \tag{6.10}
\]

The proofs of these identities are similar to that given in Lemma 8.5 of [DLM3] by noting the following

\[
\text{tr}_M v(wtv - 1 + k)Y_M(v_1, z_1)\phi(\sigma h)q^{L(0)} = \text{tr}_M [v(wtv - 1 + k), Y_M(v_1, z_1)]\phi(\sigma h)q^{L(0)}
+ \epsilon_{v,v_1} \text{tr}_M Y_M(v_1, z_1)v(wtv - 1 + k)\phi(\sigma h)q^{L(0)}, \tag{6.11}
\]

\[
v(wtv - 1 + k)\phi(\sigma h) = \epsilon_v \lambda \phi(\sigma h)v(wtv - 1 + k),
\]

and

\[
\text{tr}_M Y_M(v_1, z_1)v(wtv - 1 + k)\phi(\sigma h)q^{L(0)} = \epsilon_v \lambda q^k \text{tr}_M Y_M(v_1, z_1)\phi(\sigma h)q^{L(0)}. \tag{6.12}
\]

Use identities (6.9), (6.10) and the same proof of Theorem 8.4 of [DLM3] to obtain:

\[
T(v, v_1) = \sum_{k=1}^{\infty} \bar{P}_k(\mu, \epsilon_{v,v_1} \epsilon_v \lambda, \frac{z_1}{z}, q)T(v[k-1]v_1)
\]

\[
T(v_1, v) = \epsilon_v \lambda \sum_{k=1}^{\infty} \bar{P}_k(\mu, \epsilon_{v,v_1} \epsilon_v \lambda, \frac{z_1}{z} q, q)T(v[k-1]v_1). \]

Again if \( \epsilon_{v,v_1} \epsilon_v \neq 1 \) then \( T(v[k-1]v_1) = 0 \) by Lemma 6.3. So in any case we have

\[
T(v, v_1) = \sum_{k=1}^{\infty} \bar{P}_k(\mu, \lambda, \frac{z_1}{z}, q)T(v[k-1]v_1)
\]

\[
T(v_1, v) = \epsilon_v \lambda \sum_{k=1}^{\infty} \bar{P}_k(\mu, \lambda, \frac{z_1}{z} q, q)T(v[k-1]v_1),
\]

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as required. □

We are now in a position to prove Lemma 6.5. Recall the following equality about Bernoulli polynomials from [DLM3], Lemma 8.6

$$\sum_{k=0}^{\infty} \frac{1}{k!} B_k(1 - wtv + \frac{r}{T} + \frac{1}{2} \delta_{v,1})v[k - 1] = \sum_{i=0}^{\infty} \left( \frac{r}{T} + \frac{1}{2} \delta_{v,1} \right) v(i - 1). \quad (6.13)$$

Let $m = wtv - \frac{1}{2} \delta_{v,1} \in \mathbb{Z}$ in Proposition 2.3. Using (6.13) we have

$$\sum_{k=0}^{\infty} Q_k(\mu, \lambda, q) T(v[k - 1]v_1)$$

$$= \sum_{k=0}^{\infty} \left( \frac{r}{T} + \frac{1}{2} \delta_{v,1} \right) z^{m-T\delta_{v,1}} z^{-m+T+\frac{1}{2} \delta_{v,1}} P_k(\mu, \lambda, \frac{z}{v}, q) T(v[k - 1]v_1)$$

$$= \sum_{k=0}^{\infty} \left( \frac{r}{T} + \frac{1}{2} \delta_{v,1} \right) z^{m-T\delta_{v,1}} z^{-m+T+\frac{1}{2} \delta_{v,1}} P_k(\mu, \lambda, \frac{z}{v}, q) T(v[k - 1]v_1)$$

On the other hand, by (4.4),

$$\sum_{i=0}^{\infty} \left( \frac{r}{T} + \frac{1}{2} \delta_{v,1} \right) T(v(i - 1)v_1)$$

$$= \sum_{i=0}^{\infty} \left( \frac{r}{T} + \frac{1}{2} \delta_{v,1} \right) z^{wtv+wtv_1-i} \text{tr} Y_M(v(i - 1)v_1, z_1) \phi(\sigma h) q^{L(0)-c/24}$$

$$= \sum_{i=0}^{\infty} \left( \frac{r}{T} + \frac{1}{2} \delta_{v,1} \right) z^{wtv+wtv_1-i} \text{Res}_{z_1} (z-z_1)^{-1} z^{wtv+wtv_1} \text{tr} Y_M(Y(v, z-z_1)v_1, z_1) \phi(\sigma h) q^{L(0)-c/24}$$

$$= \text{Res}_{z-z_1} \text{tr} Y_M(Y(v, z-z_1)v_1, z_1) \phi(\sigma h) q^{L(0)-c/24}$$

$$= \text{Res}_{z-z_1} \text{tr} Y_M(Y(v, z-z_1)v_1, z_1) \phi(\sigma h) q^{L(0)-c/24}$$

$$= \text{Res}_{z-z_1} \text{tr} Y_M(Y(v, z-z_1)v_1, z_1) \phi(\sigma h) q^{L(0)-c/24}$$

which by Sublemma 6.6 is equal to

$$\sum_{k=1}^{\infty} \text{Res}_{z-z_1} (z-z_1)^{-1} \left( \frac{z}{z_1} \right)^{wtv-\frac{r}{T} - \frac{1}{2} \delta_{v,1}} P_k(\mu, \lambda, \frac{z}{z_1}, q) T(v[k - 1]v_1)$$

$$- \epsilon_v \epsilon_{v_1} \sum_{k=1}^{\infty} \text{Res}_{z-z_1} (z-z_1)^{-1} \left( \frac{z}{z_1} \right)^{wtv-\frac{r}{T} - \frac{1}{2} \delta_{v,1}} P_k(\mu, \lambda, \frac{z}{z_1}, q) T(v[k - 1]v_1).$$

Again, if $v \in V_0$ or both $v, v_1 \in V_1$ then $\epsilon_v \epsilon_{v_1} \epsilon_v = 1$ and the result follows. If $v \in V_1$ and $v_1 \in V_0$, the result follows from Lemma 6.3. □
Lemma 6.7  If $(\mu, \lambda) = (1, 1)$, then

$$T(v[-2]v_1) + \sum_{k=2}^{\infty} (2k-1)E_{2k}(\tau)T(v[2k-2]v_1) = 0.$$ 

Sublemma 6.8  Let $v, v_1 \in V$ be the same as in Lemma 6.7. Then

$$T(v, v_1) = \text{tr}_M o(v) o(v_1) \phi(\sigma h) q^{L(0)} - \frac{\varepsilon}{\pi} + \sum_{k=1}^{\infty} \tilde{P}_k(1, 1, \frac{z_1}{z}, q) T(v[k-1]v_1)$$ (6.14) 

$$T(v_1, v) = \text{tr}_M o(v_1) o(v) \phi(\sigma h) q^{L(0)} - \frac{\varepsilon}{\pi} + \sum_{k=1}^{\infty} \tilde{P}_k(1, 1, \frac{z_1}{z}, q) T(v[k-1]v_1)$$ (6.15) 

where $\tilde{P}_k$ is defined in (2.6).

Proof:  We only prove (6.14) and (6.15) can be proved similarly. By (6.9), (3.7) and (3.8) we have

$$T(v, v_1) = z^{\text{wte}z_{1}^{\text{wte}}} \text{tr} Y_M(v, z) Y_M(v_1, z_1) \phi(\sigma h) q^{L(0)} - \frac{\varepsilon}{\pi}$$

$$= z^{\text{wte}z_{1}^{\text{wte}}} \text{tr} \sum_{k \in \mathbb{Z}} z^{-k-\text{wte}v} (\text{wte}v - 1 + k) Y_M(v_1, z_1) \phi(\sigma h) q^{L(0)} - \frac{\varepsilon}{\pi}$$

$$= z_{1}^{\text{wte}v_1} \text{tr}_M o(v) Y_M(v_1, z_1) \phi(\sigma h) q^{L(0)} - \frac{\varepsilon}{\pi}$$

$$+ \sum_{k \in \mathbb{Z}_{-\{0\}}} \frac{z^{-k}}{1 - \varepsilon_{v,v_1} \varepsilon_q q^k} \sum_{i=0}^{\infty} \left( \frac{\text{wte}v_1 - 1 + k}{i} \right) z^{wte_i+k-i} \text{tr} Y_M(\text{v}i, v_1, z_1) \phi(\sigma h) q^{L(0)} - \frac{\varepsilon}{\pi}$$

$$= z_{1}^{\text{wte}v_1} \text{tr}_M o(v) Y_M(v_1, z_1) \phi(\sigma h) q^{L(0)} - \frac{\varepsilon}{\pi}$$

$$+ \sum_{k \in \mathbb{Z}_{-\{0\}}} \frac{z^{-k}}{1 - \varepsilon_{v,v_1} \varepsilon_q q^k} \sum_{i=0}^{\infty} \sum_{m=0}^{i} c(\text{wte}v_1, i, m) k^m T(v(i)v_1)$$

$$= z_{1}^{\text{wte}v_1} \text{tr}_M o(v) Y_M(v_1, z_1) \phi(\sigma h) q^{L(0)} - \frac{\varepsilon}{\pi}$$

$$+ \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \tilde{P}_{m+1}(1, \varepsilon_{v,v_1} \varepsilon_q, \frac{z_1}{z}, q) m! c(\text{wte}v_1, i, m) T(v(i)v_1)$$

$$= \text{tr}_M o(v_1) \phi(\sigma h) q^{L(0)} - \frac{\varepsilon}{\pi} + \sum_{m=0}^{\infty} \tilde{P}_{m+1}(1, \varepsilon_{v,v_1} \varepsilon_q, \frac{z_1}{z}, q) T(v[m]v_1)$$

$$= \text{tr}_M o(v_1) \phi(\sigma h) q^{L(0)} - \frac{\varepsilon}{\pi} + \sum_{k=1}^{\infty} \tilde{P}_k(1, \varepsilon_{v,v_1} \varepsilon_q, \frac{z_1}{z}, q) T(v[k-1]v_1)$$

Using Lemma 6.3 and discussing the values of $\varepsilon_{v,v_1} \varepsilon_q$ give the result. □

Sublemma 6.9  We have

$$T(v[-1]v_1) = \text{tr}(v) o(v_1) \phi(\sigma h) q^{L(0)} - \frac{\varepsilon}{\pi} + \sum_{m=1}^{\infty} E_{2m}(\tau) T(v[2m-1]v_1).$$
Proof: If $\varepsilon_{v_1}\varepsilon_v \neq 1$ then both sides of the equation are zero by Lemma 6.3. So we now assume that $\varepsilon_{v_1}\varepsilon_v = 1$. Write $v[-1]v_1 = \sum_{i \geq -1} c_i v(i)v_1$ with $c_{-1} = 1$. Then

\[
T(v[-1]v_1) = \sum_{i \geq -1} c_i z_1^{w_{v-1} + \text{wt}v - i - 1} \text{tr} Y_M(v(i)v_1, z_1) \phi(\sigma h)q^{L(0)} - \frac{\pi}{2}
\]

\[
= \sum_{i \geq -1} c_i z_1^{w_{v-1} + \text{wt}v - i} \text{Res}_{z_0} z_0^i \text{tr} Y_M(Y(v, z_0)v_1, z_1) \phi(\sigma h)q^{L(0)} - \frac{\pi}{2}
\]

\[
= \sum_{i \geq -1} c_i z_1^{w_{v-1} + \text{wt}v - i} \text{Res}_{z_0} z_0^i \text{Res}_z \left( \frac{z_1 + z_0}{z} \right)^{\frac{i}{2} \delta_{\text{v}, 1}} \text{tr} X \phi(\sigma h)q^{L(0)} - \frac{\pi}{2}
\]

where

\[
X = z_0^{-1} \delta \left( \frac{z - z_1}{z_0} \right) Y_M(v, z)Y_M(v_1, z_1) - \varepsilon_{v_1} z_0^{-1} \delta \left( \frac{z_1 - z}{-z_0} \right) Y_M(v_1, z_1)Y_M(v, z).
\]

Thus

\[
T(v[-1]v_1) = \sum_{i \geq -1} c_i z_1^{w_{v-1} - i} \text{Res}_z \sum_{j \geq 0} \left( \frac{1}{2} \delta_{\text{v}, 1} \right) z \frac{1}{2} \delta_{\text{v}, 1} - j - j \delta_{\text{v}, 1} \text{wt}v .
\]

\[
\{(z - z_1)^{i+j} T(v, v_1) - \varepsilon_{v_1} (-z_1 + z)^{i+j} T(v_1, v)\}.
\]

Now replace $T(v, v_1)$ and $T(v_1, v)$ with (6.14) and (6.15) respectively and note that

\[
\text{tr}_M o(v_1) o(v) \phi(\sigma h)q^{L(0)} - \frac{\pi}{2} = \varepsilon_v \text{tr}_M o(v) o(v_1) \phi(\sigma h)q^{L(0)} - \frac{\pi}{2}.
\]

Then $T(v[-1]v_1)$ can be written as

\[
T(v[-1]v_1) = (a) + (b)
\]

where

\[
(a) = \text{Res}_{z_1} \frac{1}{2} \delta_{\text{v}, 1} + \text{wt}v z^{-\frac{1}{2} \delta_{\text{v}, 1} + \text{wt}v} \{(z - z_1)^{-1} + (z_1 - z)^{-1}\} \text{tr} o(v) o(v_1) \phi(\sigma h)q^{L(0)} - \frac{\pi}{2}.
\]

\[
= \text{Res}_{z_1} \frac{1}{2} \delta_{\text{v}, 1} + \text{wt}v z^{-\frac{1}{2} \delta_{\text{v}, 1} + \text{wt}v} z_1^{-1} \delta \left( \frac{z}{z_1} \right) \text{tr} o(v) o(v_1) \phi(\sigma h)q^{L(0)} - \frac{\pi}{2}
\]

\[
= \text{tr} o(v) o(v_1) \phi(\sigma h)q^{L(0)} - \frac{\pi}{2}
\]

(we have used the fact that $\frac{1}{2} \delta_{\text{v}, 1} + \text{wt}v$ is always an integer here) and

\[
(b) = \sum_{k=1}^{\infty} \sum_{i \geq -1} c_i \text{Res}_z \sum_{j \geq 0} \left( \frac{1}{2} \delta_{\text{v}, 1} \right) z_1^{\text{wt}v + \frac{1}{2} \delta_{\text{v}, 1} - i - j - \frac{1}{2} \delta_{\text{v}, 1} - \text{wt}v}.
\]

\[
\{(z - z_1)^{i+j} P_k(1, 1, \frac{z_1}{z}, q) - (-z_1 + z)^{i+j} P_k(1, 1, \frac{z_1 q}{z}, q)\} T(v[k - 1]v_1).
\]

From the proof of Proposition 4.3.3 of [Z] we see that

\[
(b) = \sum_{k=1}^{\infty} E_{2k}(q) T(v[2k - 1]v_1).
\]

This completes the proof. □
Remark 6.10 Here is a simplified proof of Sublemma 6.9 suggested by the referee. In fact, if we replace \( z \) and \( z_1 \) in (6.14) by \( e^{2\pi iz} \) and \( e^{2\pi iz_1} \) respectively, Sublemma 6.9 follows by considering the formal expansion in \( z - z_1 \) and comparing the coefficients of \( (z - z_1)^0 \) on both sides of (6.14).

Lemma 6.7 now follows from Sublemma 6.9 (see the proof of Proposition 4.3.6 of [Z]). We also need the following Lemma whose proof is contained in [Z] by using Sublemma 6.9.

Lemma 6.11 If \( v \in V \) satisfies \( \sigma v = gv = hv = v \), i.e. \((\epsilon_v, \mu, \lambda) = (1, 1, 1)\), then

\[
S(L[-2]v, \tau) = \partial S(v, \tau) + \sum_{l=2}^{\infty} E_{2l}(\tau) S(L[2l-2]v, \tau)
\]

where \( \partial S(v, \tau) \) is defined in (5.7).

Theorem 6.2 follows from Lemmas 6.3, 6.4, 6.5, 6.7 and 6.11.

Here is another important theorem whose proof is similar to that of Theorem 8.7 of [DLM3].

Theorem 6.12 Let \( M^1, M^2, \ldots \) be inequivalent simple \( \sigma g \)-twisted \( V \)-modules such that \( h \sigma \circ M^i, \sigma \circ M^i \) and \( M^i \) are isomorphic. Let \( T_1, T_2, \ldots \) be the corresponding trace functions (6.4). Then \( T_1, T_2, \ldots \) are linearly independent elements of \( C(g, h) \).

7 Existence of twisted modules

Although we have the definition of a twisted module, we do not know if there is an irreducible \( g \)-twisted \( V \)-module for a finite order automorphism \( g \) of \( V \). The main result in this section gives a positive answer to the question.

Let \( g, h \in \text{Aut}(V) \) commute and have finite orders.

Lemma 7.1 Let \( v \in V \) satisfy \( gv = \mu^{-1}v, hv = \lambda^{-1}v \). Then the following hold:

1. The constant term of \( \sum_{k=0}^{\infty} Q_k(\mu, \lambda, q)v[k-1]w \) is equal to \(-v \circ \sigma g w\) if \( \mu \neq 1 \).
2. The constant term of \( \sum_{k=0}^{\infty} Q_k(\mu, \lambda, q)(L[-1]v)[k-1]w \) is equal to \(-v \circ \sigma g w\) if \( \mu = 1, \lambda \neq 1 \).
3. The constant term of \( v[-2]w + \sum_{k=2}^{\infty} (2k-1)E_{2k}(\sigma)qE_{2k}v[2k-2]w \) is \( v \circ \sigma g w\) if \( \mu = \lambda = 1 \).

Proof: The proof is the same as that of Lemma 9.2 in [DLM3].

Theorem 7.2 Suppose that \( V \) is a simple, \( C_2 \)-cofinite VOSA and that \( g \in \text{Aut}V \) has finite order. Then \( V \) has at least one simple \( g \)-twisted module.

Proof: Since \( g \) is arbitrary, it is enough to prove that there exists a simple \( g \sigma \)-twisted-module. By Proposition 4.8, it suffices to prove that \( A_{\sigma g}(V) \neq 0 \).
First we show that if $C(g; h) \neq 0$ for some finite order $h \in \text{Aut}(V)$ which commutes with $g$ then $A_{g\sigma}(V)$ is nonzero. As in [DLM3] we take nonzero $S \in C(g, h)$. Then there exists a positive integer $p$ such that (5.9)-(5.12) hold for all $v \in V$ with $S_p \neq 0$. We define a partial order on $C$ as follows: $\lambda \geq \mu$ if $\lambda - \mu \in \mathbb{Z}$ for some nonnegative integer $n \geq 0$. We can arrange the notation so that $\lambda_{p,1}$ is minimal among all $\lambda_{p,j}$ with respect to the partial order and $a_{p,1,0}(v) \neq 0$ for some $v \in V$. Then

$$S_{p,1}(v, \tau) = \alpha(v) + \sum_{n=1}^{\infty} a_{p,1,n}(v)q^{n/T}. \quad (7.1)$$

defines a function $\alpha : V \to \mathbb{C}$ which is not identically zero. Because $S(v, \tau)$ is linear in $v$, $\alpha$ is a linear function on $V$. Furthermore, $\alpha$ vanishes on $O_{g\sigma}(V)$ (see the proof of Lemma 9.3 of [DLM3]). Thus $A_{g\sigma}(V)$ is nonzero.

Since $V$ is a simple $V$-module by the assumption, $T(v) = \text{tr}_{V\sigma}(v)q^{L(0)-c/24}$ defines a nonzero element in $C(\sigma, g)$ by Theorems 6.2 and 6.12. (Since Aut$(V)$ acts on $V$ already we do not need $\phi$ here.) Since $
abla \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ induces a linear isomorphism between $C(\sigma, g)$ and $C(g, \sigma)$ by Theorem 5.4. In particular, $C(g, \sigma)$ is nonzero. The proof is complete. \qed

8. The main theorems

Theorems 1 and 2 are proved in this section. In fact, Theorem 6.12 together with Theorem 8.1 gives Theorem 1, and Theorem 2 follows from Theorem 8.8.

Recall Theorem 6.12. In the case $V$ is $g\sigma$-rational, we can strengthen the results obtained in 6.12.

**Theorem 8.1** Suppose that $V$ is $\sigma g$-rational and $C_2$-cofinite. Let $M^1, ..., M^m$ be all of the inequivalent, simple $\sigma g$-twisted $V$-modules such that $M^i$ is $h$ and $\sigma$-stable. Let $T_1, ..., T_m$ be the corresponding trace functions defined by (6.4). Then $T_1, ..., T_m$ form a basis of $C(g, h)$.

**Remark 8.2** In the proof of Theorem 8.1, we only use the consequence of $\sigma g$-rationality of $V$ that $A_{g\sigma}(V)$ is a finite dimensional semisimple algebra instead of $\sigma g$-rationality of $V$ itself. So we can replace the $\sigma g$-rationality assumption by the semi-simplicity of $A_{g\sigma}(V)$.

We begin with an arbitrary function $S \in C(g, h)$. We have already seen that $S$ can be represented as

$$S(v, \tau) = \sum_{i=0}^{p} (\log q^i)^i S_i(v, \tau) \quad (8.1)$$

for fixed $p$ and all $v \in V$, with each $S_i$ satisfying (5.10)-(5.12). Then Theorem 8.1 follows from the following propositions

**Proposition 8.3** (1) Each $S_i$ is a linear combination of the functions $T_1, ..., T_m$.

(2) $S_i = 0$ if $i > 0$.  

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Using the proof of Proposition 10.5 in [DLM3] we can show that (2) follows from (1). So we only need to prove (1).

We assume the setting in Section 7. In particular, $S_p$ and each $S_{p,j}$ in (5.10) are nonzero, and $\alpha : V \to \mathbb{C}$ vanishes on $O_g(V)$ which induces a linear function

$$\alpha : A_{g\sigma}(V) \to \mathbb{C}.$$

**Lemma 8.4** Suppose that $u, v \in V$ and satisfy $hu = \rho u, hv = \lambda v, \rho, \lambda \in \mathbb{C}$. Then

$$\alpha(u *_{g\sigma} v) = \epsilon_{u,v} \rho \delta_{\rho,1} \alpha(v *_{g\sigma} u). \quad (8.2)$$

**Proof:** Let $gu = \xi u$ and $gv = \nu v$ for scalars $\xi, \nu$. If $\xi$ or $\nu$ is not equal to 1 then $u$ (resp. $v$) lies in $O_{g\sigma}(V)$ by Lemma 3.1 of [DZ]. This implies that $u *_{g\sigma} v$ and $v *_{g\sigma} u$ are zero by Theorem 3.3 of [DZ].

We now assume $gu = u, gv = v$. It is clear that $u *_{g} v$ is an eigenvector for $h$ with eigenvalue $\rho \lambda$. If $\rho \lambda \neq 1$ then $u *_{g} v$ and $v *_{g} u$ lie in $O(g, h)$ by (5.3). Then $S(u *_{g\sigma} v) = S(v *_{g\sigma} u) = 0$ by (C3). This leads to both sides of (8.2) being 0. So finally we can assume that $\rho \lambda = 1$.

Recall the decomposition (4.1). Then $u, v \in V^0$. By Lemma 3.3 of [DZ],

$$u *_{g\sigma} v - \epsilon_{u,v} v *_{g\sigma} u \equiv \text{Res}_z Y(u, z)(1 + z)^{w tu - 1} \pmod{O(V^0)}.$$  

By (3.9)

$$u *_{g\sigma} v - \epsilon_{u,v} v *_{g\sigma} u \equiv \sum_{i=0}^{\infty} \left(\frac{w tu - 1}{i}\right) u(i) v \equiv u[0]v \pmod{O(V^0)}. \quad (8.3)$$

Since $O(V^0) \subset O_{g\sigma}(V)$, $u[0]v \in O(g, h)$ if $\rho = 1$ by (5.1). Thus $\alpha(u *_{g\sigma} v - \epsilon_{u,v} v *_{g\sigma} u) = 0$ if $\rho = 1$.

We now deal with the case $\rho \neq 1$. From the proof of Lemma 9.2 of [DLM3], we see that the constant term of

$$\sum_{k=0}^{\infty} Q_k(1, \rho^{-1}, q) u[k - 1]v$$

is

$$-u *_{g\sigma} v + \frac{1}{1 - \rho^{-1}} u[0]v.$$

The vanishing condition of $S$ on $\sum_{k=0}^{\infty} Q_k(1, \rho^{-1}, q) u[k - 1]v \in O(g, h)$ yields

$$\alpha(u *_{g} v) = \frac{1}{1 - \rho^{-1}} \alpha(u[0]v).$$

Using (8.3) gives the desired result. \(\Box\)

**Lemma 8.5** Suppose that $u, v \in V$ with $h\sigma u = \rho' u, h\sigma v = \lambda' v, \rho', \lambda' \in \mathbb{C}$. Then

$$\alpha(u *_{g\sigma} v) = \rho' \delta_{\rho',\lambda'} \alpha(v *_{g\sigma} u). \quad (8.4)$$
exists a representation of the modular group. That is, for any inequivalent, simple $T = V \to \text{the number of } T = V$.

Theorem 8.7

(1) If the group $\langle g, h \rangle$ generated by $g$ and $h$ is cyclic with generator $k$. Then the dimension of $C(g, h)$ is equal to the number of inequivalent, $\sigma$-stable, $\sigma$-twisted simple $V$-modules.

(2) The number of inequivalent, $\sigma$-stable simple $g$-twisted $V$-modules is at most equal to the number of $g$ and $\sigma$-stable, simple $V$-modules, with equality if $V$ is $g$-rational.

The proof is the same as that of Theorem 10.2 in [DLM3] by using Lemma 6.1 and Theorem 8.1.

As in [DLM3] we have important corollaries.

Theorem 8.6 Suppose that $V$ is rational and $C_2$-cofinite.

(1) If the group $\langle g, h \rangle$ generated by $g$ and $h$ is cyclic with generator $k$. Then the dimension of $C(g, h)$ is equal to the number of inequivalent, $\sigma$-stable, $\sigma$-twisted simple $V$-modules.

(2) The number of inequivalent, $\sigma$-stable simple $g$-twisted $V$-modules is at most equal to the number of $g$ and $\sigma$-stable, simple $V$-modules, with equality if $V$ is $g$-rational.

The proof is the same as that of Theorem 10.2 in [DLM3] by using Lemma 6.1 and Theorem 8.1.

As in the theory of vertex operator algebra, a simple vertex operator superalgebra $V$ is called holomorphic in case $V$ is rational and if $V$ is the unique simple $V$-module.

Theorem 8.7 Suppose that $V$ is holomorphic and is $C_2$-cofinite. For each automorphism $g$ of $V$ of finite order, there is a unique $\sigma$-stable simple $g$-twisted $V$-module $V(g)$. Moreover if $\langle g, h \rangle$ is cyclic then $C(g, h)$ is spanned by $T_{V(g)}(v, g, h, q)$.

It is proved in [Z] that if a vertex operator algebra $V$ is rational and $C_2$-cofinite, then the space spanned by $T_M(v, q) = \text{tr}_M o(v) q^{L(0) - c/24}$ with $M$ running through the inequivalent simple $V$-modules admits a representation of the modular group $\Gamma$. But this is not true anymore in the present situation. In fact, if $V$ is holomorphic such that $V_1 \neq 0$, then the character $\text{tr}_V q^{L(0) - c/24}$ of $V$ is not an eigenvector for the matrix $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Here is the corresponding result for the vertex operator superalgebra, which is an immediate consequence of Theorems 6.2 and 8.1.

Theorem 8.8 Let $V$ be $\sigma$-rational and $C_2$-cofinite VOSA. Let $M^1, \ldots, M^m$ be all of the inequivalent, simple $\sigma$-twisted $V$-modules such that $M^i$ is $\sigma$-stable. Then the space spanned by $T_i(v, \tau) = T_i(v, (1, 1), \tau) = \text{tr}_{M^i} o(v) \phi(\sigma) q^{L(0) - c/24}$ admits a representation of the modular group. That is, for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ there exists a $m \times m$ invertible complex matrix $(\gamma_{ij})$ such that $T_i(v, \frac{a\tau + b}{c\tau + d}) = (c\tau + d)^n \sum_{j=1}^m \gamma_{ij} T_j(v, \tau)$

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for all \( v \in V[n] \). Moreover, the matrix \((\gamma_{ij})\) is independent of \( v \).

Finally we discuss the rationality of the central charge \( c \) and the conformal weights of a \( g \)-twisted module under certain conditions. Let \( M \) be a simple \( g \)-twisted \( V \)-module. Recall from (4.8) that \( M \) is direct sum of \( L(0) \)-eigenspaces

\[
M = \bigoplus_{n=0}^{\infty} M_{\lambda+n/T'}
\]

for some \( \lambda \in \mathbb{C} \) such that \( M_\lambda \neq 0 \). The \( \lambda \) is called the conformal weight of \( M \).

**Theorem 8.9** Assume that \( V \) is \( C_2 \)-cofinite and \( g \in \text{Aut} V \) has finite order. Suppose that \( V \) is \( \sigma^i g^j \)-rational for all integers \( i, j \). Then each \( \sigma \)-stable simple \( \sigma^i g^j \)-twisted \( V \)-module has rational conformal weight, and the central charge \( c \) of \( V \) is rational.

The proof of this theorem is the same as that of Theorem 11.2 in [DLM3] (also see [AM]) using Lemma 6.1.

### 9 Modular invariance over \( \Gamma_\theta \)

In order to get the modular invariance over the full modular group \( \Gamma \) we only considered the \( \sigma \)-stable twisted modules in previous sections. In this section we remove the \( \sigma \)-stable assumption. But we have a modular invariance over the \( \Gamma_{\theta} \) instead of the full modular group \( \Gamma \) as in [H]. Since the proofs of most results in this section are very similar to those given in the previous sections and [DLM3] we present the results without proofs.

Recall that \( G \) is a finite automorphism group of \( V \). Let \( g \in G \) and \( M \) a simple \( g \)-twisted \( V \)-module. Also recall \( M \circ h \) for \( h \in G \). Let \( G_M = \{ h \in G | M \circ h \cong M \} \) be the stabilizer of \( M \). Then \( M \) is a projective module for \( G_M \), denoting by \( \phi \) the projective action. For \( h \in G_M \) and \( v \in V \) set

\[
F_M(v, (g, h), q) = \text{tr}_{M \circ (h)}(v) \phi(h) q^{L(0)} - c/24.
\]

**Theorem 9.1** Let \( V \) be a rational and \( C_2 \)-cofinite vertex operator superalgebra.

1. If \( g \) is an automorphism of \( V \) of finite order then the number of inequivalent, \( g \)-stable, simple \( g \)-twisted \( V \)-modules is at most equal to the number of \( g \)-stable simple untwisted \( V \)-modules. If \( V \) is \( g \)-rational, the number of inequivalent, \( g \)-stable simple \( g \)-twisted \( V \)-modules is precisely the number of \( g \)-stable simple untwisted \( V \)-modules.

2. Let \( g \in \text{Aut} V \) have finite order. Suppose that \( V \) is \( g^i \)-rational for all integers \( i \). Then each \( g \)-stable simple \( g^i \)-twisted \( V \)-module has rational conformal weight, and the central charge \( c \) of \( V \) is rational.

3. Let \( G \) be a finite automorphism group of \( V \). Let \( g \in G \) and \( M \) a simple \( g \)-twisted \( V \)-module and \( h \in G_M \). Then the trace function \( F_M(v, (g, h), q) \) converges to a holomorphic function in the upper half plane \( \mathfrak{h} \).

4. Assume that \( V \) is \( x \)-rational for each \( x \in G \). Let \( v \in V \) satisfy \( \text{wt}[v] = k \). Then the space of (holomorphic) functions in \( \mathfrak{h} \) spanned by the trace functions \( F_M(v, (g, h), \tau) \)
for all choices of $g, h$ in $G$ and $h$-stable $M$ is a (finite-dimensional) $\Gamma_\theta$-module. That is, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta$ then we have an equality

$$F_M(v, (g, h), \frac{a \tau + b}{c \tau + d}) = (c \tau + d)^k \sum_W \gamma_{M,W} F_W(v, (g^a h^c, h^d), \tau),$$

where $W$ ranges over the irreducible $g^a h^c$-twisted modules which are $g^b h^d$-stable. The constants $\gamma_{M,W}$ depend only on $M, W$ and $\gamma$ only.

The statements (3) and (4) are exactly the same as in [DLM3] for vertex operator algebras except that we use the group $\Gamma_\theta$ here instead of $\Gamma$ in [DLM3]. In the case $G = 1$, Parts (3) and (4) have been obtained previously in [H]. Theorem 3 is a corollary of Theorem 9.1.

10 Examples

In this section we will use the examples in [DZ] (cf. Theorem 7.1, Propositions 7.2 and 7.4 of [DZ]) to show the modular invariance. Most results in this section are well known in the literature (cf. [AMV], [GSW] and [P]), but we present them here to explain our main theorems. We are working in the setting of Section 7 of [DZ]. In particular, $H$ is a vector space of dimension $l$ which we assume to be even, $V(H, \mathbb{Z} + \frac{1}{2})$ is a VOSA with central charge $c = \frac{l}{2}$ and $V(H, \mathbb{Z})$ is the unique irreducible $\sigma$-twisted module. We are going to study the trace functions $T(1, (\sigma^i, \sigma^j), q)$ which is the graded $\sigma^{i+1}$-trace on the irreducible $\sigma^{i+1}$-twisted module for $V(H, \mathbb{Z} + \frac{1}{2})$ for $i, j = 0, 1$.

We first recall the important modular form

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n).$$

The trace functions in our examples can be expressed as rational functions in $\eta(\tau)$. Let $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ be the standard generators of $SL(2, \mathbb{Z})$. Then

$$\eta\left(\frac{-1}{\tau}\right) = (-i \tau)^{\frac{1}{2}} \eta(\tau)$$

and

$$\eta(\tau + 1) = e^{\frac{\pi i}{12}} \eta(\tau).$$

Note that the conformal weight of $V(H, \mathbb{Z})$ is $\lambda = \frac{l}{16}$. It is easy to compute that

$$T(1, (1, 1), \tau) = \text{tr}_{V(H, \mathbb{Z})} q^{L(0) - \frac{c}{24}} = q^{\frac{l}{24}} \prod_{m=1}^{l/2} (1 + (-1)) \prod_{n > 0} (1 - q^n)^l = 0$$

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The expression of these trace functions in terms of the Jacobi theta functions are well known in conformal field theory (cf. [AMV] and [P]). The modular transformation law then gives the modular transformation property of functions $T(1, (\sigma^i, \sigma^j), \tau)$ for $i, j = 0, 1$:

$$
T(1, (\sigma^i, \sigma^j), \frac{a\tau + b}{c\tau + d}) = \gamma_{i,j} T(1, (\sigma^{ai+bi}, \sigma^{bj+di}), \tau)
$$

for some constant $\gamma_{i,j} \in \mathbb{C}$ where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$. The result matches what Theorems 5.4 and 8.1 assert.

Here is another example. We still take $V$ to be $V(H, \frac{1}{2} + z)$ with $l = \dim H$ being even. Let $g$ be an isometry of $H$ such that $g(h_1) = -h_2$, $g(h_2) = -h_1$, $g(h_1^*) = -h_2^*$, $g(h_2^*) = -h_1^*$. And $g(h_i) = -h_i$, $g(h_i^*) = -h_i^*$ for $i = 3, \ldots, l/2$ where $\{h_1, h_2, h_i, h_i^*\}$ is a basis of $H$ such that $(h_i, h_j) = (h_i^*, h_j^*) = 0$ and $(h_i, h_i^*) = 0$. Then $g\sigma$ interchanges $h_1$ and $h_2$, $h_1^*$ and $h_2^*$, and leaves other $h_i, h_i^*$ invariant. Then we have

$$
H = H^{0*} \bigoplus H^{1*}, \quad H = H^0 \bigoplus H^1
$$

where $g\sigma|_{H^{0*}} = (-1)^i$ and $g|_{H^{1*}} = (-1)^i$. It is easy to see that

$$
H^{0*} = \mathbb{C}(h_1 + h_2) \bigoplus \mathbb{C}(h_1^* + h_2^*) \bigoplus \bigoplus_{j=3}^{l-2} (\mathbb{C}h_j + \mathbb{C}h_j^*),
$$

and $H^0 = H^{1*}, H^1 = H^{0*}$. Note that $g$ is lifted to an automorphism of the vertex operator superalgebra $\overline{V}(H, Z + \frac{1}{2})$ in an obvious way.

By proposition 7.4 in [DZ],

$$
M = \wedge[(h_1 - h_2)(-n), (h_1^* - h_2^*)(-m), h(-m - \frac{1}{2})|n, m \in \mathbb{Z}, n > 0, m \geq 0, h \in H^1]
$$

is the unique irreducible $g\sigma$-twisted $V(H, Z + \frac{1}{2})$-module whose conformal weight is $\frac{1}{8}$. Thus

$$
T(1, (g, \sigma), \tau) = tr_M q^{L(0) - \frac{1}{8}} = \sum_{n \in 1/2 \mathbb{Z}} (\dim M_{n+1/8}) q^n^{-\frac{1}{8}}
$$

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\[
\begin{align*}
= 2 \left( \prod_{0 < n \in \mathbb{Z}} (1 + q^n) \right)^2 \left( \prod_{0 \leq n \in \mathbb{Z}} (1 + q^{n + \frac{1}{2}}) \right)^{l-2} q^{\frac{1}{2} - \frac{i\pi}{4}} \\
= 2q^{\frac{1}{4} - \frac{i\pi}{4}} \eta(2\tau) \eta(\tau)^4 [q^{\frac{1}{2}} \eta(\tau) \eta(2\tau)]^{l-2} q^{-\frac{1}{4} - \frac{i\pi}{4}} \\
= 2 \frac{\eta(\tau)^{2l-6}}{\eta(2\tau)^{l-4} \eta(\frac{\tau}{2})^{l-2}}.
\end{align*}
\]

Then

\[
T(1, (g, g\sigma), \tau) = 2 \frac{\eta(-\frac{1}{2})^{2l-6}}{\eta(-\frac{2}{2} \tau)^{l-4} \eta(-\frac{1}{2} \tau)^{l-2}}
\]

\[
= 2 \frac{(-i\tau)^{2l-6} \eta(\tau)^{2l-6}}{(-i\tau/2)^{l-4} \eta(\frac{\tau}{2})^{l-4} (-i2\tau)^{l-2} \eta(2\tau)^{l-2}}
\]

\[
= 2 \frac{\mu \eta(\frac{\tau}{2})^{l-4} \eta(2\tau)^{l-2}}{\eta(\frac{\tau}{2})^{l-4} \eta(2\tau)^{l-2}}.
\]

where \(\mu\) is a root of unity. This implies that

\[
T(1, (g, g\sigma), S\tau) = 2\mu T(1, (g, g\sigma)S, \tau) = 2\mu T(1, (g, g), \tau).
\]
Using the fact that
\[ \eta(\frac{\tau + 1}{2}) = \frac{\eta(\tau)^2}{\eta(\frac{\tau}{2})\eta(2\tau)} \]
we have
\[
T(1, (g, \sigma), T\tau) = T(1, (g, \sigma), \tau + 1) \\
= 2\frac{\eta(\tau + 1)^{2l-6}}{\eta(2(\tau + 1))^{l-4}\eta(\frac{\tau + 1}{2})^{l-2}} \\
= 2\nu\frac{\eta(\tau)^{2l-6}\eta(\frac{\tau}{2})^{l-2}\eta(2\tau)^{l-2}}{\eta(2\tau)^{l-4}} \\
= 2\nu\frac{\eta(2\tau)^2\eta(\frac{\tau}{2})^{l-2}}{\eta(\tau)^l}
\]
for some root of unity \( \nu \). Since \((g, \sigma)T = (g, g\sigma)\), we see that
\[
T(1, (g, \sigma), T\tau) = \nu T(1, (g, g\sigma), \tau).
\]
This again verifies Theorems 5.4 and 8.1 in this special case. The reader could determine
\( T(v, (h, k), \tau) \) for other vector \( v \in V \) in these two cases, but the computation will be more complicated and difficult.

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