A New Stochastic Strategy for the Minority Game

G. Reents, R. Metzler, and W. Kinzel
Institut für Theoretische Physik, Universität Würzburg, Am Hubland, D-97074 Würzburg, Germany

July 21, 2000

We present a variant of the Minority Game in which players who were successful in the previous timestep stay with their decision, while the losers change their decision with a probability \( p \). Analytical results for different regimes of \( p \) and the number of players \( N \) are given and connections to existing models are discussed. It is shown that for \( p \propto 1/N \) the average loss \( \sigma^2 \) is of the order of 1 and does not increase with \( N \) as for other known strategies.

PACS numbers: 02.50.-r, 02.50.Le, 05.40.-a, 87.23.Ge

I. INTRODUCTION

Game theory describes situations in which players must make decisions, i.e. choose between different alternatives, and receive payoffs according to their and the other players’ choices. The question how players decide on a strategy, i.e. how they find out what to do if they do not possess full information on the strategies of the other players, was addressed in Ref. [1]. There it was suggested that each player has a number of models that prescribe an action for a given state of the player’s world, for example, for a given game history. The model that has proven most successful so far is actually used by the player.

This approach was applied in the Minority Game introduced and studied in [2–4]. The rules for this game and its variations are as follows:

- There is an odd number \( N \) of players.
- At each time step \( t \) each player \( i \) makes a decision \( \sigma_i(t) \in \{+1, -1\} \), the majority is determined, \( S(t) = \text{sign} \left( \sum_{i=1}^{N} \sigma_i(t) \right) \), and those players who are in the minority, \( \sigma_i(t) = -S(t) \), win, the others lose.
- A measure of global loss is

\[
\sigma^2 = \left\langle \left( \sum_{i=1}^{N} \sigma_i(t) \right)^2 \right\rangle_t .
\]

Random guessing leads to \( \sigma^2 = N \).
- The only information accessible to players is the history of the majority \( (S(t-M), \ldots, S(t)) \). In many cases, the history can be replaced by a random sequence without essentially affecting the results [5].
- Accordingly, no contracts between players are allowed.

In the original Minority Game, each player has a small number of randomly picked decision tables that prescribe an action for each possible history. Those tables receive points according to how well they have predicted the best action in the course of the game, and the best table is used to actually make the decision.

Other publications studied variants in which the agents used neural networks to make their decisions [6], or in which each agent has a probability that determines whether he chooses the action that was successful in the last step or its opposite [7].

II. THE MODEL

In this paper, we introduce a very simple prescription for the agents that still is a reasonable way of behaving in the absence of detailed information. It is in some ways related to Johnson’s model [8–10], but different in decisive details. The model is this:

- If an agent \( i \) is successful in a given turn, he will make the same decision the next turn: \( \sigma_i(t+1) = \sigma_i(t) \). After all, there’s no reason to change anything.
- Otherwise, the agent will change his output with a probability \( p \): \( \text{prob} (\sigma_i(t+1) = -\sigma_i(t)) = p \). The agent is reluctant to give up his position, but eventually, something must change.

This is evidently a stochastic one-step process and can be handled well with the tools for Markov processes. We therefore introduce variables to describe an ensemble of games.

Instead of using the whole set \( \{\sigma_i(t)\}_{i=1}^{N} \) of time dependent random variables we consider the stochastic process

\[
K(t) = \frac{1}{2} \sum_i \sigma_i(t) .
\]

The possible values \( k \) that \( K(t) \) can take are half-integer and run from \( -N/2 \) to \( N/2 \) in steps of 1. Then, the probabilities

\[
\pi_k(t) = \text{prob} (K(t) = k)
\]
together with the transition probabilities
\[ W_{k\ell} = \text{prob}(K(t + 1) = k \mid K(t) = \ell) \]
are the basic quantities to describe the system. To shorten notation we consider the probabilities \( \pi_k(t) \) as components of the state vector \( \pi(t) = (\pi_{-N/2}(t), \ldots, \pi_{N/2}(t))^T \). The number of players in the majority at time \( t \) is \( N/2 + |K(t)| \). Since the individual players perform independent Bernoulli trials, the transition probability \( W_{k\ell} = \mathcal{W}(\ell \to k) \) from a state with \( K(t) = \ell \) to \( K(t+1) = k \) is given by the binomial distribution
\[
W_{k\ell} = \binom{N}{\ell} p^{\ell-k} (1-p)^{N-k} \quad \text{for } \ell > 0,
\]
\[
W_{k\ell} = \binom{N}{k} p^{k-\ell} (1-p)^{N-\ell} \quad \text{for } \ell < 0.
\]
It is understood that \( \binom{N}{m/|t|} = 0 \) for \( m < 0 \).

This stochastic process may be considered a random walk in one dimension, where steps of arbitrary size with probability \( \binom{N}{m} \) are allowed only in the direction of the origin.

Given the initial state \( \pi(0) \), the state \( \pi(t) \) is updated at each time step by multiplying it by the transition matrix \( \mathcal{W} \):
\[ \pi(t+1) = \mathcal{W} \pi(t). \]

The mathematical theory dealing with this kind of problems is that of Markov chains with stationary transition probabilities \( \mathcal{W} \). Since \( \mathcal{W}^2 > 0 \), the chain is irreducible as well as ergodic \( \mathcal{W} \), which implies that irrespective of the initial distribution the state \( \pi(t) \) converges for \( t \to \infty \) to a unique stationary state \( \pi(\infty) \equiv \pi^* \). In view of Eq. (\ref{eigenvalue_equation}) \( \pi^* \) corresponds to an eigenvector of \( \mathcal{W} \) with eigenvalue 1:
\[ \mathcal{W} \pi^* = \pi^* \quad \text{and} \quad \sum_k \pi_k^* = 1. \]

The properties of this eigenvector, which by the stated normalization condition becomes unique, are our main interest.

The problem can be simplified by exploiting the symmetry \( \mathcal{W}_{-k,-\ell} = \mathcal{W}_{k\ell} \), which implies the symmetry \( \pi_k^* = \pi_k^\dagger \) of the stationary state. Reformulating the eigenvalue problem for the independent components of \( \pi^* \), the eigenvector can be calculated numerically up to \( N \approx 1200 \) in reasonable time with standard linear algebra packages.

### III. Solution for Small Probabilities

A closer look reveals that as \( N \to \infty \), there are two scaling regimes for \( \sigma^2 \), depending on how \( p \) depends on \( N \). We will first consider \( p = x/(N/2) \), where \( x \) is constant and much smaller than \( N \). As \( N \) is increased, the number of players that switch sides every turn stays constant to first order: since the majority is approximately \( N/2 \), on the average \( x \) agents will change their opinion.

In this case the matrix elements \( W_{k\ell} \) can be approximated by Poisson probabilities \( \mathcal{W} \):
\[
W_{k\ell} \to W_{k\ell}^P = e^{-x} \frac{x^{\ell-k}}{(\ell-k)!} \quad \text{for } \ell > 0,
\]
\[
W_{k\ell} \to W_{k\ell}^P = e^{-x} \frac{x^{k-\ell}}{(k-\ell)!} \quad \text{for } \ell < 0,
\]
where, again, \( 1/m! \) for negative \( m \) has to be interpreted as zero. In the limit \( N \to \infty \) we are thus looking for an infinite component vector \( \pi^* \) satisfying the eigenvalue equation together with the proper normalization:
\[ \mathcal{W}^P \pi^* = \pi^* \quad \text{and} \quad \sum_k \pi_k^* = 1. \]

Making use of (\ref{eigenvalue_equation}) and (\ref{eigenvalue_equation}) we were able to derive equations for the moments of the stationary distribution:
\[
\langle |k| - \frac{1}{2} \rangle = \frac{x}{2},
\]
\[
\langle (|k| - \frac{1}{2})(|k| - \frac{3}{2}) \rangle = \frac{x^2}{3},
\]
\[
\langle (|k| - \frac{1}{2})(|k| - \frac{3}{2})(|k| - \frac{5}{2}) \rangle = \frac{x^3}{4},
\]
etc.

These in turn determine the characteristic function of \( \pi_k^* \), and a Fourier transform finally leads to
\[
\pi_k^* \equiv \frac{1}{2(|k| - \frac{1}{2})!} \sum_{j=0}^\infty (-1)^j \frac{x^j}{j!(j + |k| + \frac{1}{2})}. \]

It has been proven that (\ref{eigenvalue_equation}) indeed satisfies the eigenvalue equation (\ref{eigenvalue_equation}). Note that \( \pi_k^* \) can be expressed by the incomplete gamma function:
\[
\pi_k^* = \frac{\gamma(\frac{1}{2} + \frac{x}{2})}{2 x (|k| - \frac{1}{2})!}. \]

A comparison with numerically determined eigenvectors of the matrix (\ref{eigenvalue_equation}) for \( N = 801 \) gives excellent agreement, as seen in Fig. 1. The distribution is roughly flat for \( |k| > x \) and much smaller than \( N \), with a turning point near \( |k| = x \) and falls off exponentially with \( k \) for larger values of \( |k| \). From (\ref{eigenvalue_equation}), the variance \( \sigma^2 = \langle (2k)^2 \rangle \) can be calculated:
\[
\sigma^2 = 1 + 4x + 4 \frac{x^2}{3}. \]

For small \( x \), this approaches the optimal value \( \sigma^2 = 1 \) that occurs if the majority is always as narrow as possible, but even for larger \( x \), \( \sigma^2 \) does not increase with \( N \).
Numerical calculations show that the eigenvector \( \pi^s(\kappa) \) takes the shape of two Gaussian peaks centered at symmetrical distances \( \pm \kappa_0 \) from the origin (see Fig. 3).

The physical interpretation is that the majority switches from one side to the other in every time step. Since approximately \( (\kappa_0 + 1/2)pN \) agents switch sides every turn and the distance between the two peaks amounts to a number of \( 2\kappa_0N \) agents, we get \( \kappa_0 = p/(4 - 2p) \).

This reasoning can be made more precise, and also the width of the peaks for large but finite \( N \) can be calculated by the following argument: The well known normal approximation for the binomial coefficients in (3) leads to

\[
W(\kappa, \lambda) = N W_{k\ell} \approx \frac{1}{\sqrt{2\pi s(\lambda)}} \exp \left[ -\frac{1}{2} \frac{(\kappa - f(\lambda))^2}{s^2(\lambda)} \right],
\]

where \( f(\lambda) = (1-p)\lambda - \text{sign}(\lambda)\frac{p}{2} \) and \( s^2(\lambda) = \frac{p(1-p)}{N} (\frac{\ell}{N} + |\lambda|) \). (13)

A double gaussian of the form

\[
\pi^s(\kappa) = \frac{1}{2} \frac{1}{\sqrt{2\pi b}} \left[ \exp \left( \frac{(\kappa + \kappa_0)^2}{2b^2} \right) + \exp \left( \frac{(\kappa - \kappa_0)^2}{2b^2} \right) \right]
\]

is transformed by the integral kernel (13) into a double peak of the same type if in the integral equation we approximate the variance \( s^2(\lambda) \) of (13) by \( s^2(\pm\kappa_0) \) and if the assumption \( b^2 \ll \kappa_0^2 \) is justified. It means that the peaks are well separated and that the integral can be extended from \(-\infty \) to \( \infty \). By requiring \( \pi^s(\kappa) \) from (14) to satisfy the eigenvalue equation (12) we get

\[
\kappa_0 = \frac{p}{2(2-p)} \quad \text{and} \quad b^2 = \frac{1 - p}{(2-p)^2N}. \quad (15)
\]

The result for \( \kappa_0 \) confirms the simple argument given above, whereas the term for \( b^2 \) is slightly surprising: it does not depend on \( p \) in the leading order, i.e. it is not simply the number of players who switch sides. Eq. (15) also allows to check whether the assumptions made for its derivation are true for a given \( p \) and \( N \). For example, for \( p = x/N, \kappa_0^2/b^2 \to 0 \) for \( N \to \infty \) according to (15), so one cannot expect the formation of double peaks in this limit. The crossover from single-peak to double-peak distribution occurs for \( p \propto 1/\sqrt{N} \).

It is now easy to integrate over the probability distribution to get an expression for \( \sigma^2 \):

\[
\sigma^2 = \frac{N}{(2-p)^2} (Np^2 + 4(1-p)). \quad (16)
\]

This holds well if the condition \( \kappa_0 \gg b \) is fulfilled, i.e. for sufficiently large \( p \) and \( N \), as seen in Fig. 3.

IV. SOLUTION FOR LARGE PROBABILITIES

The other scaling regime assumes that \( p \) is of order one and \( pN \gg 1 \). To handle this regime, we will use a rescaled (continuous) coordinate \( \kappa = k/N = \sum \sigma_i/(2N) \), the range of which is \(-1/2 \leq \kappa \leq 1/2 \). Multiplied by \( N \), the stationary state \( \pi^s_k \) for large \( N \) turns into a probability density function \( \pi^s(\kappa) \), and the matrix \( W_{k\ell} \) becomes an integral kernel \( W(\kappa, \lambda) \), hence (3) is transformed into an integral equation:

\[
\pi^s(\kappa) = \int W(\kappa, \lambda) \pi^s(\lambda) d\lambda \quad \text{and} \quad \int \pi^s(\kappa) d\kappa = 1. \quad (12)
\]

FIG. 1. Stationary solution \( \pi^s_k \) for \( p = 2x/N \). The numerical solution for \( N = 801 \) (symbols) is in good agreement with the analytical solution for \( N \to \infty \).

FIG. 2. Stationary solution \( \pi^s(\kappa) \) for \( p = 0.4 \). With increasing \( N \), the width of the peaks becomes narrower.
FIG. 3. $\sigma^2$ for several values of $p$ and $N$, compared to predictions by Eq. (16).

V. CONCLUDING REMARKS

The presented strategy can be related to the decision tables of Challet and Zhang’s Minority Game as follows: if every player keeps only one decision table with entries for all possible histories and changes the entries individually with a probability of $p$ if he loses in a given time step, the mean result will be exactly the same as for the presented one-step memory. A similar argument was given for Johnson’s variant in [12]. The memory size, which determines the number of entries in the tables, is completely irrelevant for the average loss of each player, but does influence the time series of minority decisions generated by the system.

In summary, we have found an analytic solution of a stochastic strategy for the Minority Game. Although this strategy is very simple, it yields an average loss of order one even in the limit of infinitely many agents. Questions that will be discussed in future publications include the dynamics and relaxation time of the system, interactions with players using other strategies and individual probabilities for each player.

VI. ACKNOWLEDGEMENT

R. M. and W. K. acknowledge financial support by the German-Israeli Foundation. We would like to thank Christian Horn, Andreas Engel and Ido Kanter for helpful discussions.