Robustness of the S-deformation method for black hole stability analysis

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Abstract

The S-deformation method is a useful way to show the linear mode stability of a black hole when the perturbed field equation takes the form of the Schrödinger equation. While previous works where many explicit examples are studied suggest that this method works well, general discussion is not given yet explicitly. In this paper, we show the existence of a regular S-deformation when a black hole spacetime is stable under some reasonable assumptions. This S-deformation is constructed from a solution of a differential equation. We also show that the boundary condition for the differential equation which corresponds to a regular S-deformation has a one-parameter degree of freedom with a finite range. This is the reason why any fine-tune technique is not needed to find S-deformation numerically.

Keywords: black hole, stability analysis, S-deformation

(Some figures may appear in colour only in the online journal)

1. Introduction

When the black hole spacetime is highly symmetric, the linear gravitational perturbation equation usually takes the form of the two-dimensional wave equation \[1–13\]

\[
\left[ -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - V(x) \right] \tilde{\Phi} = 0.
\]  

Using the Fourier transformation with respect to the time coordinate, \(\tilde{\Phi}(t, x) = e^{-i\omega t}\Phi(x)\), this equation takes the form of the Schrödinger equation.
\[
\left[ \frac{d^2}{dx^2} + V \right] \Phi = \omega^2 \Phi =: E \Phi. \tag{2}
\]

It is known that the non-existence of a bound state with \( E < 0 \) implies the non-existence of an exponentially growing mode in time, i.e. the linear mode stability of the black hole.

For a continuous function \( S \), from equation (2), we can show

\[
- \left[ \Phi^* \frac{d\Phi}{dx} + S|\Phi|^2 \right] \bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \left[ \left( \frac{d\Phi}{dx} + S\Phi \right)^2 + \left( V + \frac{dS}{dx} - S^2 \right) |\Phi|^2 \right] = E \int_{-\infty}^{\infty} |\Phi|^2, \tag{3}
\]

where \( \Phi^* \) is the complex conjugate of \( \Phi \). We consider the boundary condition such that the boundary term vanishes. We can regard that the potential is deformed as

\[
\tilde{V} := V + \frac{dS}{dx} - S^2, \tag{4}
\]

which is called the \( S \)-deformation. If \( \tilde{V} \) is non-negative everywhere by choosing an appropriate function \( S \), we can say \( E \geq 0 \) for any \( \Phi \), i.e. the non-existence of a negative energy bound state. This method was introduced in [5–7] and used for the stability analysis in many cases, e.g. in [8–15].

In the previous work [16], a simple method to find a regular \( S \) numerically by solving an equation \( \tilde{V} = 0 \), i.e.

\[
V + \frac{dS}{dx} - S^2 = 0, \tag{5}
\]

was discussed, when it is hard to find it analytically. In some examples, the existence of a regular solution of equation (5) was used to show that the spacetime is stable [16], but it is not known whether a regular solution always exists or not when the spacetime is stable. In fact, equation (5) is satisfied by

\[
S = -\frac{1}{\Phi_{E=0}} \frac{d\Phi_{E=0}}{dx}, \tag{6}
\]

with a solution of the Schrödinger equation with zero energy \( \Phi_{E=0} \). Thus, the existence of \( \Phi_{E=0} \) which does not have zero anywhere implies the existence of a regular solution of equation (5). In this paper, we show the existence of such a wave function when the spacetime is stable (under some assumptions). This result is an almost immediate consequence of the Sturm–Liouville theory, but we think that this fact is not well recognized in the community of the black hole stability and the proof does not seem trivial when the domain of the potential \( V \) is \(-\infty < x < \infty\). We also discuss the robustness of this method, i.e. we can find a regular \( S \)-deformation numerically without fine-tuning.

2. \( S \)-deformation in a finite box

The discussion in this section is almost trivial from the usual Sturm–Liouville theory, but we think that pedagogically it is still worth giving an explicit proof. We focus on the domain where the potential is bounded and piecewise continuous. First, we introduce a proposition (e.g. see [17, 18].)

\(^5\)Note that this \( \Phi_{E=0} \) is not necessary square integrable, and usually it corresponds to a growing mode.
Proposition 1. Let us consider two solutions $\Phi_1$ and $\Phi_2$ of the Schrödinger equation (2) for energies $E_1$ and $E_2$, with $E_1 < E_2$, respectively. If $\Phi_1$ has two consecutive zeros at $x = x_L$ and $x = x_R$, $\Phi_2$ has at least one zero between $x_L$ and $x_R$.

**Proof.** From the Schrödinger equation (2) for energies $E_1$ and $E_2$, we have

\[ (\Phi'_1 \Phi_2 - \Phi'_2 \Phi_1)' = (E_2 - E_1) \Phi_1 \Phi_2, \]

where a prime denotes the derivative with respect to $x$. We can assume $\Phi_1 > 0$ for $x_L < x < x_R$ without loss of generality. Then, $\Phi'_1 |_{x_L} > 0$ and $\Phi'_2 |_{x_R} < 0$ are satisfied. Integrating this equation, we have

\[ \Phi'_1 |_{x_L} \Phi_2 |_{x_R} - \Phi'_2 |_{x_L} \Phi_2 |_{x_R} = (E_2 - E_1) \int_{x_L}^{x_R} dx \Phi_1 \Phi_2. \]

If $\Phi_2$ does not have zero in $x_L < x < x_R$, we have $\Phi_2 \geq 0$ for $x_L \leq x \leq x_R$. In that case, while the left-hand side of equation (8) is not positive, the right-hand side of equation (8) is positive. This is a contradiction. Thus, $\Phi_2$ has a zero in $x_L < x < x_R$.

From this proposition, we can immediately show the well known result: the $n$th excited state has $n$ nodes. Also, we can show the following lemma.

**Lemma 1.** If there exists a solution for $E = E_1$ which has two consecutive zeros at $x = x_L$ and $x = x_R$, there also exists a solution for $E = E_0 (< E_1)$ which does not have zero in $x_L \leq x \leq x_R$.

**Proof.** Solving the Schrödinger equation for $E_0$ from $x = x_L$ with the boundary condition $\Phi |_{x_L} = 0$, we obtain a solution $\Phi_0$. From the above proposition, $\Phi_0$ does not have zero for $x_L < x < x_R$. We denote $a = \Phi_0 |_{x_L} (> 0), b = \Phi'_0 |_{x_L}$. Solving the Schrödinger equation for $E_0$ from $x = x_R$ with the boundary condition $\Phi |_{x_R} = a$ and $\Phi'_0 |_{x_R} = b (< b)$, we obtain a solution $\tilde{\Phi}_0$. Integrating the Wronskian conservation equation $(\Phi'_0 \tilde{\Phi}_0 - \tilde{\Phi}'_0 \Phi_0)' = 0$ from a point $y (x_R > y > x_L)$ to $x_R$, we have

\[ -\Phi'_0 |_{y} \tilde{\Phi}_0 |_{y} + \tilde{\Phi}'_0 |_{y} \Phi_0 |_{y} = -a(b - \tilde{b})(< 0). \]

If $\tilde{\Phi}_0 |_{y} = 0$ and $\Phi_0$ is positive in $y < x < x_R$, $\tilde{\Phi}'_0 |_{y}$ takes positive value. Thus, the above equation implies that $\tilde{\Phi}_0 |_{y}$ becomes negative. However, this contradicts with that $\Phi_0 \geq 0$ for $x_L \leq x < x_R$. Thus, $\Phi_0$ is positive for $x_L \leq x \leq x_R$ and this is the desired solution.

If the energy of the ground state is larger than zero, from lemma 1, we can construct a solution of the Schrödinger equation with zero energy which does not have zero anywhere when the potential is in a finite box, i.e. $V$ is bounded and piecewise continuous in $-L < x < L$ with a positive constant $L$, but $V = \infty$ in $|x| > L$. Using such a solution $\Phi_{E=0}$, we can construct a regular $S$-deformation as

\[ S = -\frac{1}{\Phi_{E=0}} \frac{d\Phi_{E=0}}{dx}. \]

Thus, we have the following:

*If $\Phi'_1 |_{x_L} = 0$ or $\Phi'_2 |_{x_R} = 0$, $\Phi_1$ becomes a trivial solution, i.e. $\Phi_1 = 0$ everywhere, since both $\Phi_1$ and its derivative are zero at $x = x_L$ or $x = x_R$. However, this cannot happen because $\Phi_1 > 0$ for $x_L < x < x_R$. 
Proposition 2. If the energy of the ground state is larger than zero and the potential is in a finite box, there exists a regular S-deformation.

Since this proposition holds for arbitrary value of \( L \), we can expect that there also exists a regular S-deformation for a stable spacetime even when \( L \to \infty \). In the next section, we discuss the case with \( L \to \infty \).

3. Infinite size box

We consider the case in ‘an infinite size box’, i.e. the domain of our interest is \(-\infty < x < \infty\). Even in this case, if there exists a bound state with \( E > 0 \), we can easily show that the non-existence of a bound state with negative energy implies the existence of a regular S-deformation by a similar discussion in lemma 1. In fact, this is a well known result (e.g. see [19]). This condition is not usually satisfied in a black hole perturbation problem since \( V \to 0 \) at the horizon. Hereafter, we do not assume the existence of a bound state with \( E > 0 \).

In this section, we assume the following conditions:

(i) The potential \( V \) is bounded and piecewise continuous in \(-\infty < x < \infty\). At \( x \to \pm \infty \), \( V \) takes non-negative constants \( V_{\pm} \), respectively.

(ii) For \( E < 0 \), the Schrödinger equation has solutions \( \Phi^L_E(x) \) and \( \Phi^R_E(x) \) whose leading behaviors are \( \Phi^L_E \simeq e^{\sqrt{V-E}x+o(x)} \) at \( x \to -\infty \) and \( \Phi^R_E \simeq e^{\sqrt{V-E}x+o(x)} \) at \( x \to \infty \), respectively.

(iii) For \( E = 0 \), the Schrödinger equation has solutions \( \Phi^L_0(x) \) and \( \Phi^R_0(x) \) which asymptote to a non-negative constant at \( x \to -\infty \) and \( x \to \infty \), respectively.

Hereafter, we omit to write \( o(x) \) since this does not affects the following discussions.

3.1. Existence of regular S for stable spacetime

First, we show the following three lemmas:

Lemma 2. For an energy \( E_1 < 0 \), if there exists \( x_1 \) such that \( \Phi^L_{E_1} |_{x_1} = 0 \) and \( \Phi^L_{E_1} > 0 \) for \(-\infty < x < x_1 \), there exists a bound state with negative energy.

Proof. For brevity, we denote \( \Phi^L_{E_1} \) as \( \Phi_E \) in this proof. Integrating the equation \( (\Phi'_E + \Phi_E \Phi_{E_1}) = (E - E_1) \Phi_{E_1} \Phi_E \) with \( E < E_1 \) from \( -\infty \) to a point \( \gamma \), we have

\[
(\Phi'_E \Phi_E - \Phi'_E \Phi_{E_1}) |_{\gamma} = (E - E_1) \int_{-\infty}^{\gamma} dx \Phi_{E_1} \Phi_E.
\]

(11)

Let us assume \( \Phi_{E_1}|_{x_2} = 0 \) and \( \Phi_E > 0 \) for \(-\infty < x < x_2 \), then \( \Phi'_E |_{x_2} \) is negative. If \( x_2 \leq x_1 \), the above equation with \( y = x_2 \) becomes

\[
-\Phi'_E |_{x_2} \Phi_{E_1} |_{x_2} = (E - E_1) \int_{-\infty}^{x_2} dx \Phi_{E_1} \Phi_{E}.
\]

(12)

Now the integrand is non-negative, and the integral is finite because \( \Phi_E \simeq e^{\sqrt{V-E}x} \) at \( x \to -\infty \). Since the right hand side is negative and \( \Phi'_E |_{x_2} \) is negative, \( \Phi_{E_1} |_{x_2} \) should be negative. However, this contradicts with the fact that \( \Phi_{E_1} \geq 0 \) for \(-\infty < x \leq x_2 \leq x_1 \). Thus, \( x_2 \) should satisfies \( x_1 < x_2 \) if it exists.
If we consider \( E \leq V_{\text{min}} \), where \( V_{\text{min}} \) is the minimum value of \( V \), since \( \Phi_E^0/\Phi_E = (V - E) \geq (V_{\text{min}} - E) \geq 0 \), once \( \Phi_E \) and \( \Phi_E^0 \) take positive value at some point, \( \Phi_E \) remains to be positive as \( x \) increases. Since the boundary condition is \( \Phi_E \simeq e^{\sqrt{V_+ - E} x} \) at \( x \to -\infty \), we find \( \Phi_E > 0 \) everywhere for \( E \leq V_{\text{min}} \).

Since the position of zero \( x_2 \) is a continuous function of \( E(\leq E_1) \) if it exists (see appendix A) and it can take all values in \( x_1 < x_2 < \infty \) by changing \( E \), we can say that there exists \( E_0 \) \((V_{\text{min}} < E_0 < E_1)\) such that \( x_2 \to \infty \) for \( E = E_0 \).

The asymptotic behavior near \( x \to \infty \) is \( \Phi_E \simeq c_1 e^{-\sqrt{V_+ - E} x} + c_2 e^{\sqrt{V_+ - E} x} \). If \( 0 < 1 - E/E_0 \ll 1 \), \( x_2 \) is in the asymptotic region. So, from the condition \( \Phi_E|_{x_2} = 0 \), we obtain \( \Phi_E \simeq c_1(e^{-\sqrt{V_+ - E} x} - e^{\sqrt{V_+ - E} (x - 2c_2)}) \) when \( 0 < 1 - E/E_0 \ll 1 \). Thus, in the limit \( E \to E_0 + 0 \), i.e. \( x_2 \to \infty \), the second term vanishes, and then \( \Phi_E|_{E_0} \simeq c_1 e^{-\sqrt{V_+ - E_0} x} \to 0 \) at \( x \to \infty \). Thus, the solution \( \Phi_{E_0} \) is the ground state.

\[ \text{Lemma 3.} \quad \text{If the Schrödinger equation for zero energy has a solution which has two consecutive zeros at } x = x_0 \text{ and } x = x_1, \text{ there exists a bound state with negative energy.} \]

\[ \text{Proof.} \quad \text{We denote the solution of the Schrödinger equation for zero energy which has two consecutive zeros at } x = x_0 \text{ and } x = x_1 \text{ as } \Phi_0 \text{ in this proof. Let us consider to solve the Schrödinger equation for } E \leq 0 \text{ with the boundary condition } \Phi|_{x_0} = 0 \text{ and } \Phi'|_{x_1} = \Phi|_{x_1}. \]

If we change the value of \( E \) from zero, the position of zero also changes from \( x = x_0 \). From the similar discussion in lemma 2, for some energy \( E < 0 \), the solution becomes decaying mode at \( x \to -\infty \). This implies the existence of a bound state with negative energy from lemma 2.

\[ \text{Lemma 4.} \quad \text{If } S\text{-deformation function that satisfies equation (5) is continuous for } -\infty < x < \ell \text{ with a constant } \ell, S \text{ asymptotes to } V_- \text{ at } x \to -\infty. \]

\[ \text{Proof.} \quad \text{Since } S \text{ satisfies equation (5) and the potential } V \text{ is bounded above and below, if } S \text{ is divergent in the asymptotic regions, the only possibility is that } dS/dx \geq S^2 \text{ and then } S \geq 1/(c_2 - x). \]

However, it does not happen since \( 1/(c_2 - x) \) can be divergent only for finite \( c_2 \). Thus, \( S \) is also bounded above and below.

If \( S \) oscillates and does not have a limit in \( x \to -\infty \), \( S \) has infinite number of local maximum/minimum at \( x = \alpha_n < \ell \) \((n = 1, 2, \cdots)\) with \( \alpha_{n+1} < \alpha_n \) and \( \lim_{n \to \infty} \alpha_n \to -\infty \). When \( S \) is smooth, since \( dS/dx = 0 \) at each \( x = \alpha_n \), we have

\[ V|_{\alpha_n} = S^2|_{\alpha_n}. \tag{14} \]

\[ \text{Note that } \Phi_E \text{ is not necessary a continuous function of } E, \text{ but we can say the position of zero } x_2 \text{ is a continuous function of } E. \]

\[ S \text{ can be non-smooth at } x = \alpha_n \text{ when } V \text{ is not continuous there. In that case, } S \text{ is still continuous but the two limits } \lim_{x \to \alpha_n^+} S'|_{x=\alpha_n^+} \text{ and } \lim_{x \to \alpha_n^-} S'|_{x=\alpha_n^-} \text{ can be different values. Since } x = \alpha_n \text{ is a local maximum/minimum of } S, \lim_{x \to \alpha_n}(S'|_{x=\alpha_n^+} S'|_{x=\alpha_n^-}) \text{ takes a negative value, and there exists positive constants } c_n, d_n \text{ such that } \lim_{x \to \alpha_n}(c_n S'|_{x=\alpha_n^+} + d_n S'|_{x=\alpha_n^-}) = 0. \]

Thus, the equation

\[ S|_{\alpha_n} = \lim_{n \to \infty} \frac{c_n V|_{\alpha_n^+} + d_n V|_{\alpha_n^-}}{c_n + d_n} \tag{13} \]

holds, and in the limit \( n \to \infty \), the right hand side becomes \( V_- \), but the left hand side does not have a limit point. This is a contradiction, and then \( S \) should asymptote to a constant at \( x \to -\infty \).
This implies that $V_{\alpha_n} \neq V_{\alpha_{n+1}}$ for an arbitrary large $n$, which contradicts the fact that $V$ asymptotes to $V_-$ at $x \to -\infty$. So, $S$ should asymptote to a constant at $x \to -\infty$.

Since $S$ asymptotes to a constant at $x \to -\infty$, from equation (5),

$$S = \int dx(-S^2 + V),$$

(15)

we can see that $S^2 \to V_-$ so that the integral is finite.

We can show the existence of a regular $S$-deformation for a stable black hole as follows.

**Proposition 3.** If bound states with negative energy do not exist, there exists a regular $S$-deformation function that satisfies equation (5).

**Proof.** We assume that $\Phi_{0 \to}^L$ has zero at $x = x_1$. If $\Phi_{0 \to}^L$ has another zero except at $x \to \pm \infty$, there exists a bound state with negative energy from lemma 3. So, we only need to consider the case $\Phi_{0 \to}^L > 0$ in $-\infty < x < x_1$. We denote $\Phi_{0 \to}^L$ by $\phi_1$. From lemma 4, $(\phi_1' / \phi_1)^2 \to V_-$ at $x \to -\infty$. Since $\phi_1$ is not a growing mode at $x \to -\infty$, we can say

$$\frac{\phi_1'}{\phi_1} \to \sqrt{V_-},$$

(16)

at $x \to -\infty$.

Let us consider to solve the Schrödinger equation for zero energy with the boundary condition $\Phi = 0$ at $x = x_2 > x_1$. We denote the solution by $\phi_2$. If $\phi_2$ has zero in $x_1 \leq x < x_2$, this implies the existence of a bound state with negative energy from lemma 3. So, we can consider $\phi_2 > 0$ in $x_1 \leq x < x_2$. From $(\phi_1 \phi_2' - \phi_1' \phi_2)^2 = 0$, we obtain a relation

$$\phi_1 \phi_2' - \phi_1' \phi_2 = \epsilon,$$

(17)

with a positive constant $\epsilon$ by evaluating equation (17) at $x = x_1$, where we used the relations $\phi_1 |_{x_1} = 0, \phi_1' |_{x_1} < 0, \phi_2 |_{x_1} > 0$. Equation (17) can be written in the form

$$\frac{\phi_2'}{\phi_2} = \frac{\phi_1'}{\phi_1} + \frac{\epsilon}{\phi_1 \phi_2}.$$  

(18)

If $\phi_2 > 0$ in $-\infty < x < x_1$, from equations (16), (18) and lemma 4, we have

$$\frac{\epsilon}{\phi_1 \phi_2} \to -\sqrt{V_-} \pm \sqrt{V_-} \leq 0$$  

(19)

at $x \to -\infty$. This is possible only when $\phi_2 \to 0$ at $x \to -\infty$ since $\phi_1$ is not a growing mode. However, setting $\phi_2 = \phi_1 Z$, $Z$ satisfies

$$Z' = \frac{\epsilon}{\phi_1} \geq 0,$$

(20)

then $Z$ and $\phi_2$ cannot asymptotes to $+\infty$ at $x \to -\infty$. Thus, $\phi_2$ should have a zero somewhere in $-\infty < x < x_1$. This implies the existence of a bound state with negative energy from lemma 3.

Thus, $\phi_1$ does not have a zero except at $x \to -\infty$ if bound states with negative energy do not exist. Then
becomes a regular solution of equation (5) in $-\infty < x < \infty$, since $S$ is continuous and it is bounded above and below from lemma 4.

3.2. Robustness of S-deformation method

We can construct the general regular S-deformation $S$ that satisfies equation (5), from $\Phi_{0}^{L^{-}}$ and $\Phi_{0}^{R^{-}}$, as follows:

**Corollary 1.** If there exists no bound state with $E \leq 0$, the general non-trivial solution of the Schrödinger equation for zero energy which does not have a node is given by

$$\Psi = c_{L}\Phi_{0}^{L^{-}} + c_{R}\Phi_{0}^{R^{-}}$$

with $c_{L}c_{R} \geq 0$, $(c_{L}^{2} + c_{R}^{2} \neq 0)$. Then, the general regular S-deformation such that $V$ vanishes is given by $S = -\Psi^{-1}\Psi'$.

**Proof.** From proposition 3, $\Phi_{0}^{L^{-}}$ and $\Phi_{0}^{R^{-}}$ do not have a node, then $\Phi_{0}^{L^{-}} > 0$ and $\Phi_{0}^{R^{-}} > 0$. Since it is clear that $\Psi$ does not have a node, we only need to show that the other solutions have a node. The other solutions are given by $\tilde{\Psi} = a_{1}\Phi_{0}^{L^{-}} + a_{2}\Phi_{0}^{R^{-}}$ with $a_{1} > 0, a_{2} < 0$ (or $a_{1} < 0, a_{2} > 0$). Since $\tilde{\Psi} \to a_{2}\Phi_{0}^{R^{-}} < 0$ at $x \to -\infty$ and $\tilde{\Psi} \to a_{1}\Phi_{0}^{L^{-}} > 0$ at $x \to \infty$, $\tilde{\Psi}$ should have a node at some finite $x$.

Corollary 1 implies that the general regular S-deformation that satisfies equation (5) has a one-parameter degree of freedom. Moreover, we can show the following:

**Proposition 4.** If there exists no bound state with $E \leq 0$, defining $S_{L} = -\left(\Phi_{0}^{L^{-}}\right)'/\Phi_{0}^{L^{-}}, S_{R} = -\left(\Phi_{0}^{R^{-}}\right)'/\Phi_{0}^{R^{-}}$, the general regular S-deformation such that $V$ vanishes can only take values between $S_{L}$ and $S_{R}$ at any point, and it can take all values between $S_{L}$ and $S_{R}$ at any point.

**Proof.** From corollary 1, the general regular S-deformation that satisfies equation (5) can be written in the form

$$S = S_{L} \sin^{2} \Theta + S_{R} \cos^{2} \Theta,$$

where $\sin \Theta, \cos \Theta$ are defined by

$$\sin \Theta = \sqrt{\frac{c_{L}\Phi_{0}^{L^{-}}}{c_{L}\Phi_{0}^{L^{-}} + c_{R}\Phi_{0}^{R^{-}}}}, \quad \cos \Theta = \sqrt{\frac{c_{R}\Phi_{0}^{R^{-}}}{c_{L}\Phi_{0}^{L^{-}} + c_{R}\Phi_{0}^{R^{-}}}}.$$

with $0 \leq \Theta \leq \pi/2$.

At any point $x = a$, the functions $\Phi_{0}^{L^{-}}, \Phi_{0}^{R^{-}}$ are positive definite, and hence $\Theta$ can take all values in the range $0 \leq \Theta \leq \pi/2$ by changing $c_{L}$ and $c_{R}$ with $c_{L}c_{R} \geq 0, (c_{L}^{2} + c_{R}^{2} \neq 0)$. Since $S_{R} > S_{L}$ holds as shown in appendix B, the relations

$$S = -(S_{R} - S_{L}) \sin^{2} \Theta + S_{R} \leq S_{R},$$

$$S = S_{L} + (S_{R} - S_{L}) \cos^{2} \Theta \geq S_{L},$$




Proposition 4 implies that all initial conditions that give regular $S$-deformations by solving equation (5) correspond to values between $S_L$ and $S_R$ at any point (see figure 1). Thus, if we slightly change the initial condition in solving equation (5), the solution still corresponds to a regular $S$-deformation. This is the reason why we could obtain a regular $S$-deformation by solving equation (5) numerically without any fine-tuning in [16]. Typically, the potential $V$ becomes negative only near the horizon. In this case, if the spacetime is stable, $S = 0$ at a point in the region far from the horizon becomes an appropriate initial condition to obtain a regular solution of equation (5) because $S_L$ is negative and $S_R$ is positive there like an example shown in figure 1.

### 3.3. On the existence of zero mode

Even if there exists a regular $S$-deformation such that $\tilde{V}$ vanishes, there might still exist a zero mode, i.e. $E = 0$ or $\omega = 0$ mode\(^9\). We sometimes call $\omega = 0$ mode as a ‘marginally stable mode’ because this is not an exponentially growing mode in time but a static perturbation. However, since $r\Phi$ becomes also a solution of the wave equation (1), this implies the existence of a linearly growing mode in time. So, it is important to discuss whether there exists $\omega = 0$ mode or not.

If there exists a zero mode $\Phi = \Phi_0$, from equation (3), $S$ should satisfy

$$\frac{d\Phi_0}{dx} + S\Phi_0 = 0,$$

which implies that there exists only a single $S$ if there exists a zero mode. Thus, we can say the following:

\(^9\)If the zero mode asymptotes to a non-zero constant at either of $x \to \pm\infty$, this is not normalizable. However, this mode might be physically acceptable if $x \to \infty$ or $-\infty$ corresponds to the event horizon or cosmological horizon which locates at the finite proper distance. We note that this happens only when $V \to 0$ at $x \to \infty$ or $x \to -\infty$ and the boundary term in equation (3) still vanishes since $S \to 0$ in this case.
Proposition 5. If there exist two different regular functions $S = S_1, S_2$ such that $\tilde{V}$ vanishes, the zero mode does not exist.

This is a merit for the $S$-deformation method. If we wish to show the non-existence of a zero mode by solving the Schrödinger equation directly, we need to solve it from the decaying boundary condition, which is not always easy.

4. Summary

In this paper, we showed the existence of a regular $S$-deformation function for a stable spacetime under some reasonable assumptions. This implies that the $S$-deformation method can be used for showing the linear mode stability when the spacetime is stable, and that the conjecture in the previous work [16] is correct. We also showed that the general regular $S$-deformation such that $\tilde{V}$ vanishes contains a one-parameter degree of freedom. This is the reason why we can find a regular $S$-deformation numerically without any fine-tuning [16].

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Appendix A. Continuity of the position of zero as a function of $E$

Even if $\Phi_E^L$ is not a continuous function of $E$, we can show the position of the first zero of $\Phi_E^L$ is a continuous function of $E$ if it exists. In this section, we denote $\Phi_E^L$ as $\Phi_E$ for simplicity. When $\Phi_E$ has a zero, we denote the position by $\rho_E$, i.e. $\Phi_E|_{\rho_E} = 0$ and $\Phi_E > 0$ for $-\infty < x < \rho_E$. Let us assume that $\rho_{E_0}$ exists for an energy $E_0$ from the same discussion in lemma 2, we can say $\rho_E > \rho_{E_0}$ if it exists.

For $y > \rho_{E_0}$, we consider to solve the Schrödinger equation for $E \leq E_0$ with the boundary condition $\Phi|_y = 0, \Phi|'_y = -1$. We denote this solution by $\phi_E$ and $\Phi_E$ is a continuous function of $E$ because the boundary condition is given at the finite point $y$. Integrating the Wronskian conservation equation $(\phi_E' \Phi_E - \Phi_E' \phi_E)' = 0$ from $x = -\infty$, with sufficiently large $L > 0$, to $x = \rho_{E_0}$, we obtain $\Phi_E|_{\rho_{E_0}} \phi_E|_{\rho_{E_0}} \approx 2b_1 b_2 \sqrt{V_0 - E_0}$ where we used $\Phi_{E_0} \approx b_1 e^{\sqrt{V_0 - E_0} x}$ and $\phi_{E_0} \approx 2b_2 e^{-\sqrt{V_0 - E_0} x}$.

Since $b_1 > 0$ and $\Phi_E|_{\rho_{E_0}} < 0$ from the assumption, $b_2 \phi_{E_0}|_{\rho_{E_0}}$

10 Precisely speaking, $\phi_{E_0} \approx b_1 e^{\sqrt{V_0 - E_0} x - f_2}$ and $\phi_{E_0} \approx b_2 e^{-\sqrt{V_0 - E_0} x + f_3}$, with $f_2 = o(1)$, near $x \to -\infty$. From the similar discussion in the proof of lemma 4, we can say that $-\Phi_{E_0}/\phi_{E_0}$ and $-\phi_{E_0}/\Phi_{E_0}$ should be constants $-\sqrt{V_0 - E_0}$ and $\sqrt{V_0 - E_0}$ at $x \to -\infty$, respectively. Thus, $f_2 = o(1)$ near $x \to -\infty$.

From the condition that $\phi_{E_0} \Phi_{E_0} - \Phi_{E_0} \phi_{E_0}$ is constant, $f_2 + f_3 = o(1)$ should hold near $x \to -\infty$. Hence, $\phi_{E_0} \Phi_{E_0} - \Phi_{E_0} \phi_{E_0} \approx -b_1 b_2 e^{\sqrt{V_0 - E_0} x} + f_2 - f_3 \to -b_1 b_2 \sqrt{V_0 - E_0}$ at $x \to -\infty$, and we can see that $o(1)$ terms do not affect the discussion.
should be negative. Thus, \( \phi_{E_0} \) has another zero in \( x < \rho_{E_0} (\leq y) \), and there exists \( z \) such that \( \phi_{E_0} |_{z} = 0 \) and \( \phi_{E_0} > 0 \) for \( z < x < y \). Since \( \phi_E \) is a continuous function of \( E \), \( z \) is also a continuous function of \( E \) if it exists. Also, for an energy \( E < V_{\text{min}} \), where \( V_{\text{min}} \) is the minimum of \( V \), \( \phi_E \) does not have a zero except at \( x = y \). Thus, we can say that there exists \( E_y \) (\( < E_0 \)) such that \( z \to -\infty \) for \( E = E_y \).

The asymptotic behavior near \( x \to -\infty \) is \( \phi_E \simeq c_1 e^{-\sqrt{V_0-E_x}x} + c_2 e^{\sqrt{V_0-E_x}x} \). If \( 0 < 1 - E/E_y \ll 1 \), \( z \) is in the asymptotic region. So, from the condition \( \phi_E |_{z} = 0 \), we obtain \( \phi_E \simeq c_2 e^{-\sqrt{V_0-E}x} + e^{\sqrt{V_0-E}x} \) when \( 0 < 1 - E/E_0 \ll 1 \). Therefore, in the limit \( E \to E_y + 0 \), i.e. \( z \to -\infty \), the first term vanishes, and then \( \phi_E \simeq c_1 e^{V_0-E_0}x \to 0 \) at \( x \to -\infty \). Thus, \( \phi_E = c \Phi_{E_0} \) with a constant \( c \). This shows the existence of \( \rho_E \) with \( E > E_y > \rho_{E_0} \).

Also, since \( \Phi_E \) does not have a zero except at \( x \to -\infty \) for \( E < V_{\text{min}} \), there should exist an energy \( E_{\infty} \) such that \( \rho_E \to \infty \) when \( E \to E_{\infty} + 0 \). Hence, \( \rho_E \) is defined for \( E_{\infty} < E \leq E_0 \). For any \( E_1 \) and \( E_2 \) with \( E_0 \geq E_1 > E_2 > E_{\infty} \), there exist \( \rho_{E_1} \) and \( \rho_{E_2} \) with \( \rho_{E_1} < \rho_{E_2} \). Thus, \( \rho_E \) is a continuous function of \( E \) and it monotonically increases as \( E \) decreases.

### Appendix B. \( S_R > S_L \)

Defining \( S_L = -\left( \Phi_{0}^{L-} \right)' / \Phi_{0}^{L-} \), \( S_R = -\left( \Phi_{0}^{R-} \right)' / \Phi_{0}^{R-} \), we can show \( S_R > S_L \) if there exists no bound state with \( E \leq 0 \). Moreover, we can say stronger statement that \( S_L \) is the minimum function near \( x \to -\infty \) in the solutions of equation (5). Let us denote the general solution of the Schrödinger equation for \( E = 0 \) by \( \Phi = z \Psi_{0}^{L-} \), then \( z \) satisfies

\[
z'' + 2 \left( \Psi_{0}^{L-} \right)^{-1} \left( \frac{\Psi_{0}^{L-}}{z} \right)' = 0. \tag{B.1}
\]

Integrating this equation, \( S = -\Phi'/\Phi = S_L - z'/z \) becomes

\[
S = S_L + \left( \Psi_{0}^{L-} \right)^{-2} \left( c_1 + \int_{x}^{\infty} \frac{dx}{\left( \Phi_{0}^{L-} \right)^{2}} \right)^{-1}. \tag{B.2}
\]

Note that \( |c_1| \to \infty \) corresponds to the case \( S = S_L \). Since \( \Phi_{0}^{L-} \) is a positive constant or zero at \( x \to -\infty \), the function \( \int_{x}^{\infty} \frac{dx}{\left( \Phi_{0}^{L-} \right)^{2}} \) is divergent at \( x \to -\infty \). Thus,

\[
S \simeq S_L + \left( \Psi_{0}^{L-} \right)^{-2} \left( \int_{x}^{\infty} \frac{dx}{\left( \Phi_{0}^{L-} \right)^{2}} \right)^{-1}, \tag{B.3}
\]

unless \( |c_1| \to \infty \), and the second term in the right hand side is positive. This implies that all \( S \) which satisfies equation (5) is larger than \( S_L \) near \( x \to -\infty \) except for the case \( S = S_L \).

If there exits no bound state with \( E \leq 0 \), both \( S_L \) and \( S_R \) are regular everywhere. Also, \( S_L \) cannot coincide with \( S_R \) at a finite point because they are solutions of equation (5) with different boundary conditions. Thus, the relation \( S_R > S_L \) holds everywhere in this case.

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