Towards precision determination of the top quark mass from $M_{b\ell}$ distribution in semi-leptonic decays

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Abstract

We explore a possibility of optimization of the method of determination of the top quark mass from $M_{b\ell}$ distribution in semi-leptonic decays $t \rightarrow b \ell \nu$ at LHC and a future linear collider (LC). We discover that the systematic and statistical errors of $M_t$ determination can be diminished if considering the high moments over the distribution. In the case of LHC this allows one to reduce more than in twice the errors, and in the case of LC to approach to the precision expected at studying the threshold scan of the total cross-section $e^+e^- \rightarrow t\bar{t}$.

1 Introduction

The precision determination of the top quark mass is one of the major research problems at colliders of next generation [1, 2, 3, 4, 5]. Being a fundamental parameter of the Standard Model (SM), the top quark mass is tightly constrained by quantum level calculations with other fundamental parameters. This enables one to test the SM and/or to select the probable scenario of its extension on the basis of an independent $M_t$ measurement.

A considerable progress in this direction is expected at Tevatron and LHC where the accuracy of the $M_t$ determination is anticipated of about 1-2 GeV [1]. At LHC in view of the copious production of the top quarks, for the increase of the accuracy the decrease of systematic errors is crucial. An analysis of Ref. [1] shows that the method most promising from the point of view of optimization of the errors is based on the investigation of a distribution over the invariant mass of the observable products of semi-leptonic decays $t \rightarrow bW \rightarrow b\ell\nu$; more precisely of the isolated lepton $\ell$ and the $\mu^+\mu^-$ pair indicating a $J/\psi$ meson produced from the decay of the $b$ quark [6]. In this channel one can obtain experimentally very clean final states. Correspondingly, the systematic error of the $M_t$ measurement can be made low. The evaluation made by Monte-Carlo (MC) modelling gives 0.6-0.8 GeV at the statistical error of about 1 GeV for 4 years of LHC operation [6]. This result is recognized as the best one among others obtained by various methods [1].

In the case of a future linear collider (LC) [2, 3, 4, 5] the most promising method for precision $M_t$ determination is based on the investigation of the threshold scan of
the total cross-section $e^+e^- \rightarrow t\bar{t}$. In this region the form and the height of the cross-section are very sensitive to the mass of the top quark. This gives an opportunity to determine $M_t$ with very high accuracy. A serious difficulty in this approach is a precise theoretical calculation of the behavior of the cross-section in the vicinity of the threshold, which becomes additionally complicated because of the resonant effects due to the strong $t-\bar{t}$ interaction. A major progress in the calculations was made by way of the summation of QCD contributions via solving Lippmann-Schwinger equation for the Green function describing the $t\bar{t}$ production [7]. At the present moment the theoretical error of the top mass determined by this method is estimated at 100-200 MeV [8, 9], with the experimental error of about 20 MeV [10].

Alternate methods of the $M_t$ determination are based on the reconstruction of events of the decays of the top quarks. In the basic features they are common at LC and at the hadron colliders, but at LC the precision is anticipated better. Thus, for example, the systematic error of $M_t$ determination by direct reconstruction of $t\bar{t}$ events in $e^+e^-$ collisions at $\sqrt{s} = 500$ GeV is expected [11] at 340 and 250 MeV in hadronic and semi-leptonic channels, respectively, with statistical errors of about 100 MeV for 1-2 years of LC [12]. Since far above the threshold one can expect very high precision of the necessary theoretical calculations, the resultant errors should be close to that expected at studying the threshold scan of the cross-section. This promising anticipation again excites a question about the precision of the top mass determination by the method of Ref. [6], but this time in the LC case. Actually this method in the LC case has been discussed as a preliminary in [13] (see also review [3]), but the errors have not been determined. So the prospect of this method at LC is still not practically known.

In this article we clear up this question. In contrast to Ref. [6], however, we consider the full reconstructed jet of the $b$ quark instead of $J/\psi$ or $\mu^+\mu^-$ pair only. Such an approach has been considered in [13], and partially in [14]. We follow it by keeping in mind that the $M_{b\ell}$ distribution in any case does emerge in a certain stage of the analysis. So from very beginning the analysis can be made in terms of the data converted to the form of $M_{b\ell}$ distribution. (Of course, the systematic errors that arise in the course of the converting of the data must be taken into account.) An obvious advantage of this approach is a possibility to consider the data in a uniform fashion in both cases, LHC and LC. Moreover, this allow us in a simple way to explore a possibility of optimization of the algorithm of the extraction of the top mass from the data. The elaboration of the latter problem actually is the major purpose of the present article.

In the next section we detail the statement of the problem. In sections 3 and 4 we discuss a model for the calculation of the errors. The parameters of the model are fixed in section 5 and in the same place the quantitative outcomes are determined. In section 6 we discuss the theoretical uncertainty, and in section 7 we discuss the results.
2 Statement of the problem

We consider the processes

\[ e^+ e^- (q\bar{q}, gg) \to t\bar{t} \to bWbW \to b\ell \nu \ bq_1q_2 \to \{b\text{-jet} + \ell\} + \{3 \text{ jets}\}, \]  
(1)

with the $b$-jet, isolated lepton $\ell = \{e, \mu\}$, and invisible in the final states neutrino coming from one $t$ quark, and the remaining three jets coming from another $t$ quark. In the experiment the above mentioned states are registered, and measured is a distribution

\[ F(q) = \frac{1}{\sigma} \frac{d\sigma}{dq}. \]  
(2)

Here $\sigma$ is the cross-section of the process (1), $q \equiv m_{b\ell}$ is the reconstructed invariant mass of the system $\{b\text{-jet} + \ell\}$.

We simulate the results of the experiment under the following suppositions. First we suppose that there is a satisfactory method for extracting signal from the data. Actually this means the existence of a satisfactory model for the background processes that survive after setting of kinematic cuts. Further, we describe the signal in the Born approximation, identifying the $b$-jet with the $b$ quark. Finally, on the basis of the results of Ref. [6] we disregard the effects of finite width of the top quarks. The latter assumption means that $\sigma^{-1} d\sigma/dq$ is equal to $\Gamma_{t\rightarrow b\ell\nu} d\Gamma_{t\rightarrow b\ell\nu}/dq$, where $\Gamma_{t\rightarrow b\ell\nu}$ is a partial width of the decay $t \rightarrow b\ell\nu$. (Thus the distribution $F$ becomes process-independent.)

Direct calculation gives the following formula for the distribution of the partial width:

\[ \frac{d\Gamma_{b\ell\nu}}{dq^2} = \frac{3G_F|V_{tb}|^2 \Gamma_{W\rightarrow\ell\nu} M_W}{4\sqrt{2} \pi^2 M_t^3} \times \left\{ q^2 - \Lambda^2 - M_W^2 + \left( \frac{\Lambda^2 - M_W^2}{2} - q^2 \right) \ln \left( \frac{(\Lambda^2 - q^2)^2 + M_W^2 \Gamma_W^2}{M_W^4 + M_W^2 \Gamma_W^2} \right) \right. 
\]  
\[ + \left. \left( \frac{\Lambda^2 - q^2)(q^2 + M_W^2) + M_W^2 \Gamma_W^2}{M_W \Gamma_W} \right] \left[ \arctan \left( \frac{\Lambda^2 - q^2}{M_W \Gamma_W} \right) + \arctan \left( \frac{M_W}{\sqrt{\Gamma_W}} \right) \right] \right\}. \]  
(3)

Here $\Lambda^2 = M_t^2 - M_W^2$, $\Gamma_W$ is the total and $\Gamma_{W\rightarrow\ell\nu}$ is the partial width of the $W$ boson, and we neglect the masses of the lepton $\ell$ and the $b$ quark. In this approximation $\Gamma_{W\rightarrow\ell\nu} = 2/9 \Gamma_W$ and $q$ ranges between 0 and $M_t$. Fig. 1 shows the distribution $F(q)$ defined by formula (3) at $M_t = 170, 175, 180$ GeV. From Fig. 1 a dependence of $F(q)$ on $M_t$ is obvious. So, by comparing the experimental distribution with a set of theoretical curves one can determine, in principle, the experimental value of $M_t$.

In a practical respect, however, it is convenient to compare integrated parameters of the distributions. For instance, in Ref. [6] the $M_t$ was extracted from the mean value (position of the maximum) of the Gaussian distribution approximating the

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1 The set of the cuts and the background processes in the LHC case have been discussed in Ref. [3]. In the LC case that has been done in Refs. [11, 12]. At this stage we do not take manifestly into account the kinematic cuts but do that at deriving the quantitative outcomes.

2 The influence of the mass of the $b$ quark is noticeable at very small $q$, but this region is inessential when considering the moments over the distribution.
measured distribution. Refs. [13, 14] determined $M_t$ by the first moment $\langle q \rangle$ over the distribution. In the present article we consider a method of $M_t$ determination by the higher moments

$$\langle q^n \rangle = \int_0^M dq \; q^n F(q).$$ (4)

Here $M$ is a fixed quantity close to $M_t$, see below for details. In fact this method means the matching of the experimental distribution $q^n F(q)$ with the corresponding theoretical distribution which depends on the parameter $M_t$.

As we will see below, the insertion of $q^n$ factor will significantly increase the precision of the $M_t$ determination. Eventually this can be checked by a quantitative analysis. Nevertheless some hints on this can be a priori seen. Really, with increasing $n$ the moment $\langle q^n \rangle$ becomes in rising measure dependent on the behavior of $F(q)$ in a region located between the position of its maximum and a large-$q$ tail where $F(q)$ almost vanishes. (More precisely, by the tail we mean a range $\Lambda < q < M_t$, where in the limit $\Gamma_W = 0$ the distribution identically vanishes by reason of kinematics.) Further, in the mentioned region the behavior of $F(q)$ in the greatest measure is sensitive to the value of $M_t$, which is seen from Fig. 1. As a result, with increasing $n$ the sensitivity of $\langle q^n \rangle$ with respect to $M_t$ is increasing. That is why one can expect the increasing of the precision of $M_t$ extracted from the higher moments.

Now let us dwell on the details of definition (4). The point of the discussion is the upper limit in the integral. We set it $M$ instead of conventional $M_t$, meaning the upper bound of the region allowed by kinematics, because $M_t$ also is a parameter which is subject to determination. In order to avoid an inconvenience, we put in the place of the upper limit a certain predetermined value $M$ fixed close to $M_t$. Simultaneously we adjust the normalization of the distribution $F(q)$ so that to provide the equality $\langle 1 \rangle = 1$. The moments $\langle q^n \rangle$ at $n \geq 1$ after this redefinition practically do not change (at not too large $n$) in view of almost vanishing $F$ in the tail at large $q$.

So, we define the experimentally measured value of the top quark mass as a solution to the equation

$$\langle q^n \rangle = \langle q^n \rangle_{\text{exp}}.$$ (5)

Here in the r.h.s. the moment is determined (at a given $M$) on the basis of the experimental data, and that in the l.h.s. on the basis of the theoretical distribution which depends on the parameter $M_t$. Let at a given $n$ a solution to the equation (5) be $M_t = M_t(n)$. Then the error of the solution can be determined as

$$\Delta M_t(n) = \Delta \langle q^n \rangle_{\text{exp}} \left/ \frac{d\langle q^n \rangle}{dM_t} \right|_{M_t=M_t(n)}.$$ (6)

Our aim is to estimate $\Delta M_t(n)$ and find an optimal value of $n$ which would minimize $\Delta M_t(n)$. Since by virtue of (3) the derivative $d\langle q^n \rangle/dM_t$ is known, the problem is reduced to the determination of the statistical and systematic errors, the components of the experimental error $\Delta \langle q^n \rangle_{\text{exp}}$. 

4
3 Statistical errors

We determine the statistical errors of the moments on the supposition that the data averaged over ensemble are described by $F(q) = \Gamma_{bl}^{-1} d\Gamma_{blV}/dq$ with $\Gamma_{blV}$ determined by formula (3) at $M_t = 175$ GeV.

Let $\delta q_i$ be the size of a bin, within which $i$-th element of the distribution is measured, and let $\overline{N}_i$ be the number of events counted in this bin on the average. Then

$$F(q_i)\delta q_i = \overline{N}_i / \overline{N}. \tag{7}$$

Here $\overline{N}$ is the total number of events counted in all bins on the average. Further we do not distinguish between $\overline{N}$ and $N = \sum_i N_i$, the total number of events counted in all bins in the given experiment. The experimentally measured $n$-th moment is

$$\langle q^n \rangle_{\text{exp}} = \sum_i q^n_i \frac{N_i}{N}. \tag{8}$$

By virtue of (7) the averaged experimental moment $\overline{\langle q^n \rangle}_{\text{exp}}$ is found by formula (4).

Since $N_i$ is distributed by Poisson law with parameter $\overline{N}_i$, the variance of $\langle q^n \rangle_{\text{exp}}$ is

$$D\langle q^n \rangle_{\text{exp}} = \sum_i q^{2n}_i \frac{N_i}{N^2} \equiv \frac{1}{N}\langle q^{2n} \rangle. \tag{9}$$

Formula (9) implies the following estimation for the statistical error:

$$\Delta_{\text{stat}}\langle q^n \rangle_{\text{exp}} = \sqrt{\frac{1}{N}\langle q^{2n} \rangle}. \tag{10}$$

To give an idea of the behavior of $\Delta_{\text{stat}}\langle q^n \rangle_{\text{exp}}$, we present in Fig. 2 by a continuous curve the ratio $\Delta_{\text{stat}}\langle q^n \rangle_{\text{exp}} / \langle q^n \rangle_{\text{exp}}$ calculated at $N = 4000$ (corresponds to LHC case, see Sect. 5). It is seen from the figure that with increasing $n$ the ratio is growing. This is explained by the shift (to the right) of the position of maximum of $q^n F(q)$ from the position of maximum of $F(q)$, where the statistics is largest. As a result a statistical reliability of $\langle q^n \rangle_{\text{exp}}$ comes down. Another important property of the ratio is the change of the mode of the growth beginning with $n \approx 15$. This is explained by the emergence of a noticeable contribution from the large-$q$ tail in $q^n F(q)$. The latter property is illustrated by the set of the curves represented by Fig. 3.

In fact the emergence of a noticeable contribution from the tail is an undesirable effect since on the tail the uncertainty from the background is comparable with the signal process. In order to avoid this difficulty one can correct the definition of the moments by introducing a cutoff in the integral in (4). The position of the cutoff should be determined so that to isolate the second (unphysical) peak in the tail of $q^n F(q)$ but simultaneously to keep as much as possible a statistical significance of the sample of events. It is clear that the optimal cutoff should be placed in the neighborhood of a local minimum between the two peaks of $q^n F(q)$ (if the second peak appears). From Fig. 3 it is seen that at $n \approx 40$ the sought-for point is distant by about two half-widths to the right of the position of the maximum of $q^n F(q)$. So a simplified
algorithm for the cutoff may be determined by setting \( \Lambda_n = \min\{q_{n,\text{extr}} + 2\Gamma_{n,\text{right}}, M\} \), where \( q_{n,\text{extr}} \) is the position of the maximum of \( q^n F(q) \) and \( \Gamma_{n,\text{right}} \) is the half-width from the right. Thus we come to the following definition of the effective moments:

\[
\langle q^n \rangle_{\text{eff}} = \frac{\int_0^{\Lambda_n} dq q^n F(q)}{\int_0^{\Lambda_n} dq F(q)}.
\]

In the experimentally determined effective moments the cutoff must be the same. Ultimately \( \Delta_{\text{stat}} \langle q^n \rangle_{\text{exp}} \) is defined by formula (10) with \( \langle q^{2n} \rangle_{\text{eff}} \) replaced by \( \langle q^{2n} \rangle_{\text{eff}} \) but with introducing the cutoff \( \Lambda_n \) instead of \( \Lambda_{2n} \). The latter anomalous prescription follows immediately from the derivation of formula (10).

The behavior of \( \Delta_{\text{stat}} \langle q^n \rangle_{\text{eff}} \) is shown by the dashed curve in Fig. 2. It is seen from the figure that the transition to the effective moments implies no noticeable modification up to \( n \approx 15 \), while at the larger \( n \) the growth of the ratio becomes stabilized. A similar behavior is observed in the basic formalism (without the transition to the effective moments) in the limit \( \Gamma_W \to 0 \), when the large-\( q \) tail identically vanishes.

### 4 Systematic errors

Proceeding to the systematic errors it is necessary at first to clarify a reason of their origin. For this purpose we use the analysis of Ref. [6] of the errors of the \( M_{b\ell} \) distribution simulated with the PYTHIA and/or HERVIG event generators. By the main sources of the systematic errors Ref. [6] found the uncertainties in the \( b \) quark fragmentation (including the final state radiation) and the uncertainties in the background processes. It is clear that the same sources should be the main ones at solving the inverse problem, the determination of the \( M_{b\ell} \) distribution from \( M_{b\ell} \) distribution which is considered virtually as the data. In the LC case we expect the same pattern of the origin of systematic errors.

On this basis we consider at first the error resulting from the uncertainty in the \( b \) quark fragmentation. For brevity we call it by the error of the type I. At the level of \( M_{b\ell} \) distribution it appears as the uncertainty in the number of the bin within which the number of events, \( N_i \), is measured. In the continuous case this error becomes the uncertainty \( \Delta q \) in the determining of \( q \) variable.

Suppose that \( \Delta q \) is sufficiently small. Then, neglecting the nonlinear effects, we have

\[
\Delta^{\text{sys}}_{\text{I}} \langle q^n \rangle_{\exp} = \int_0^M dq \left[ q^n F(q) \right]' \Delta q.
\]

Here the prime means the derivative with respect to \( q \). The systematic error I of the effective moment \( \langle q^n \rangle_{\text{eff}} \) is estimated similarly, with replacing the upper bound \( M \) by \( \Lambda_n \) and, then, dividing the result by the normalization factor as in formula (11).

The determination of \( \Delta q \) we carry out with the aid of the following reasoning. First we note that the invariant mass \( q^2 \) actually is the doubled scalar product of
4-momenta of the $b$ quark and the lepton $\ell$. So in the laboratory frame it can be represented as $q^2 = E_b K$, where $E_b$ is the energy of the $b$ quark, and $K$ is a factor proportional to the energy of the lepton $\ell$. (Additionally $K$ includes a dependence on angular variables which, however, is relatively weak.) Further, by the calculating of the differential we get $\Delta q = \frac{1}{2} \left( \Delta E_b / E_b + \Delta K / K \right) q$, where $\Delta E_b$ and $\Delta K$ are the corresponding errors. A more precise estimation is determined by the sum in the quadratures. Thus we come to a linear dependence with a certain coefficient $r$,

$$\Delta q = rq, \quad r = \frac{1}{2} \left[ \left( \frac{\Delta E_b}{E_b} \right)^2 + \left( \frac{\Delta K}{K} \right)^2 \right]^{1/2}. \quad (13)$$

The systematic error arising after subtraction of the background we name by the error of type II. It appears in the absolute value of the distribution function. So it should be described as an additive contribution $\delta F$ to function $F$. Correspondingly, we get the following formula for the error II of the moments:

$$\Delta^{sys}_{II}\langle q^n \rangle_{exp} = \int_0^M dq \ q^n \delta F(q). \quad (14)$$

The error II of the effective moments $\langle q^n \rangle_{eff}$ is defined by a similar formula to within modifications listed below $[12]$. It is reasonable to determine $\delta F(q)$ on supposition that it vanishes at the boundaries of the phase space and when passing from small $q$ to large $q$ it only once changes the sign. The simplest form of a function satisfying to these requirements is a polynomial of degree three,

$$\delta F = h q \ (q - M/2) \ (q - M). \quad (15)$$

Parameter $h$ in $[15]$ describes the amplitude of the error and it is subject to further determination.

5 Numerical results

We assign the following values for the parameters having a global meaning:

$$M_W = 80.4 \text{ GeV}, \ \Gamma_W = 2.1 \text{ GeV}, \ M_t = M = 175 \text{ GeV}. \quad (16)$$

The remaining parameters $N$, $r$, and $h$ depend on the conditions of the consideration. Recall that $N$ means the volume of the representative sample of events, parameter $r$ characterizes the error in the invariant mass of the $b\ell$-system, and $h$ describes the error arising after the subtraction of the background processes.

With reference to LHC case, the parameters $N$, $r$, $h$ we fixe on the basis of the results of Ref. [6]. Since in that work the $M_{3\ell}$ distribution was determined at $N = 4000$ (with kinematic cuts and for 4 years of LHC), in our investigation we set this value for $N$, as well. Parameters $r$ and $h$ we fix based on the properties of $M_{3\ell}$ distribution and the direct results derived in Ref. [6] from these properties. First we use the estimation $\Delta^{sys}(M_{3\ell}) = +0.3/-0.4 \text{ GeV}$ and the derived from it
Table 1: Statistical, systematic I and II, and the systematic summed in quadrature errors presented in GeV in the LHC case. The last column represents the sum of the statistical and the systematic errors. In the brackets we show the results calculated by the method of the effective moments (if they are different with the results calculated by the basic method).

| n  | \(\Delta^{\text{stat}}M_{t(n)}\) | \(\Delta^{\text{sys} \ I}M_{t(n)}\) | \(\Delta^{\text{sys} \ II}M_{t(n)}\) | \(\Delta^{\text{sys}}M_{t(n)}\) | \(\Delta M_{t(n)}\) |
|----|---------------------------------|----------------|----------------|----------------|----------------|
| 1  | 2.07                           | 0.62           | 0.30           | 0.69           | 2.18           |
| 5  | 0.62                           | 0.13           | 0.20 (0.17)    | 0.24 (0.21)    | 0.66           |
| 10 | 0.45                           | 0.07 (0.06)    | 0.20 (0.13)    | 0.21 (0.14)    | 0.50 (0.48)    |
| 15 | 0.41 (0.40)                    | 0.05 (0.03)    | 0.24 (0.12)    | 0.24 (0.12)    | 0.48 (0.42)    |
| 20 | 0.42 (0.39)                    | 0.03 (0.00)    | 0.32 (0.11)    | 0.32 (0.11)    | 0.52 (0.40)    |
| 30 | 0.59 (0.38)                    | 0.02 (0.03)    | 0.63 (0.11)    | 0.63 (0.11)    | 0.86 (0.39)    |

the result \(\Delta M_t = +0.6/−0.8\) GeV. Considering in the framework of our investigation the latter quantity as the uncertainty of the input parameter \(M_t\), we get by direct calculation \(\Delta^{\text{sys}}\langle q \rangle_{\exp} = +0.47/−0.62\) GeV. By comparing this with \(\Delta^{\text{sys}}\langle M_{3\phi/4l} \rangle\) we obtain an energy scale factor of 1.6, which describes the spreading of the \(M_{3\phi/4l}\) distribution when converting it to the \(M_{b\ell}\) distribution. Using the mentioned factor, from \(\Delta^{\text{sys} \ II}\langle M_{3\phi/4l} \rangle_{\exp} \lesssim 0.15\) GeV [6] we further derive an estimation \(\Delta^{\text{sys} \ II}\langle q \rangle_{\exp} \lesssim 0.24\) GeV. From this result and (14), (15) we get \(h \simeq 1.7 \times 10^{-10}\) GeV\(^{-4}\). (Hereinafter we take the upper bounds as the estimations.)

Knowing \(\Delta^{\text{sys}}\langle q \rangle_{\exp}\) and \(\Delta^{\text{sys} \ II}\langle q \rangle_{\exp}\), we immediately get \(\Delta^{\text{sys}}\langle q \rangle_{\exp} \simeq +0.41/−0.57\). From here and formula (12) there follows \(r \simeq 0.004-0.006\). Further we use the average value \(r = 0.005\). It is worth noticing that the same estimation for \(r\) follows from formula (13) when taking into consideration the 1%-precision of the determination of the energy of \(b\) jets expected at LHC [1], and additionally neglecting \(\Delta K/K\) as compared to \(\Delta E_b/E_b\).

Now as we know \(N, r, h\), we can calculate \(\Delta^{\text{stat}}\langle q^n \rangle_{\exp}\) and \(\Delta^{\text{sys} \ I, II}\langle q^n \rangle_{\exp}\) at any \(n\). Then we calculate \(\Delta^{\text{stat}}M_{t(n)}\) and \(\Delta^{\text{sys} \ I, II}M_{t(n)}\). The dependence on \(n\) of these errors is shown by the solid lines on Figs. 4–6. The dashed lines show the errors obtained by the method of the effective moments. (The break of slope in the dashed line in Fig. 5 is explained by the change of the sign in the integral in formula (12) appearing after introducing the cutoff \(\Lambda_n\).) In Table 1 we present the numerical results at some \(n\). In the same place we show the summed in the quadrature errors \(\Delta^{\text{sys}}M_{t(n)}\) and \(\Delta M_{t(n)}\). It should be noted that at \(n = 1\) the systematic errors in Table 1 practically coincide with those in [6]. The reason is that we have fixed the parameters of the model actually by matching the errors of the first moments.

In the LC case, unfortunately, there are no published results that could allow us in a similar way to fix the parameters of the model. Therefore we make use mainly of indirect methods. Parameter \(N\) we fix by the following reasoning. First we note that \(\sigma(e^+e^- \to t\bar{t}) \approx 0.6\) pb at \(\sqrt{s} = 500\) GeV. So at the integrated luminosity of 300 fb\(^{-1}\), corresponding to 1-2 years of running, approximately 180 000 \(t\bar{t}\) pairs must be generated. Since the branching of the process [1] is near of 30\%, only 54 000 events of \(t\bar{t}\) are related to our investigation. The efficiency of their detection we estimate as
follows. Suppose that at LC the efficiency of the detecting of $W$-pairs decaying in a semi-leptonic channel will be the same as at LEP2, i.e. $\sim 80\%$ \cite{15}. In addition, following Ref. \cite{3}, we suppose that the $b$-jet tagging efficiency at LC will be about 80%. In summary this gives an acceptance of 50% which implies $N = 27000$.

Parameter $r$ we fix based on the systematic error of $M_t$ obtained in \cite{11} in the approach of the direct reconstruction of $t\bar{t}$ events in the semi-leptonic channel. Additionally we use the note that in the kinematic range near the upper $M_{b\ell}$ endpoint the neutrino practically does not contribute to the total invariant mass of the decay products of the top quark. Therefore the determination of the $M_t$ in the mentioned range practically is the same that the determination of the $M_{b\ell}$ invariant mass. Ref. \cite{11} obtained $\Delta_{\text{sys}} M_t = 250$ MeV. So we set $\Delta_{\text{sys}} M_{b\ell} = 250$ MeV. Assuming that this is the error of the type I, we equate $\Delta q$ to this value. Finally, by setting $\Delta q = rq$, $q \simeq M_t$ we get $r = 0.0014$.

Parameter $h$ we fix by proceeding to the note of Ref. \cite{6} about the decreasing of the systematic error II of the average $\langle M_{J/\psi\ell} \rangle$ below of 0.1 GeV at the increasing of statistics up to $N \sim 10^4$. From this by using the above method we get a rough estimate $h \simeq 1.1 \times 10^{-10}$ GeV$^{-4}$.

Knowing $N$, $r$, $h$, we find $\Delta_{\text{stat}} \langle q^n \rangle_{\exp}$, $\Delta_{\text{sys}} M_{t(1)}$ and $\Delta_{\text{sys}} M_{t(1)}$. Since in our model the pattern of the dependence on $n$ is the common one in the LC and LHC cases, the difference between these cases appears in the scales of the errors only. This allows us to present the results on Figs. \ref{fig:1} \ref{fig:2} with adding new scales. The numerical results are presented in Table \ref{table:2}. It is interesting to note that $\Delta_{\text{sys}} M_{t(1)}$ turns out smaller than $\Delta_{\text{sys}} M_t$ obtained in Ref. \cite{11} in the framework of the direct reconstruction of events. Nevertheless this does not mean an inconsistency. Really, we equate $\Delta_{\text{sys}} M_t$ of \cite{11} to $\Delta q$ at $q \simeq M_t$, but the dominant contributions to $\Delta_{\text{sys}} \langle q \rangle_{\exp}$ are formed at strictly smaller $q$ than $M_t$, which is obvious from formulas \((12)\) and \((13)\). This effect diminishes $\Delta_{\text{sys}} M_{t(1)}$ compared to $\Delta_{\text{sys}} M_t$ of \cite{11}.

### Table 2: The same that in Table \ref{table:1} in the LC case.

| n  | $\Delta_{\text{stat}} M_{t(n)}$ | $\Delta_{\text{sys}} M_{t(n)}$ | $\Delta_{\text{sys}} M_{t(n)}$ | $\Delta_{\text{sys}} M_{t(n)}$ | $\Delta_{\text{sys}} M_{t(n)}$ |
|----|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| 1  | 0.80                          | 0.17                          | 0.19                          | 0.26                          | 0.84                          |
| 5  | 0.24                          | 0.04                          | 0.13 (0.11)                   | 0.13 (0.12)                   | 0.27                          |
| 10 | 0.17                          | 0.02                          | 0.13 (0.09)                   | 0.13 (0.09)                   | 0.27 (0.20)                   |
| 15 | 0.16                          | 0.01                          | 0.15 (0.08)                   | 0.15 (0.08)                   | 0.22 (0.17)                   |
| 20 | 0.16 (0.15)                   | 0.01 (0.00)                   | 0.21 (0.07)                   | 0.21 (0.07)                   | 0.26 (0.16)                   |
| 30 | 0.23 (0.15)                   | 0.01                          | 0.41 (0.07)                   | 0.41 (0.07)                   | 0.46 (0.16)                   |

6 \textbf{Theoretical uncertainty}

The analysis of the previous sections shows that the experimental accuracy of the $M_t$ determination can be considerably improved by the transiting to the high degrees of the moments. So, for the achieving of the eventual high accuracy a theoretical
uncertainty becomes more and more crucial. In this connection it is important to understand whether the theoretical error in the $M_t$ determination can be made smaller than the experimental error at the high degrees of the moments. If it will be possible then the theoretical error will not spoil the expected accuracy. Below we discuss this question in somewhat qualitative manner since the possibility of the solution first of all is important.

Let us begin with the note that the origin of the theoretical uncertainty in the $M_t$ determination is connected with the uncertainty of the calculation of the theoretical moment $\langle q^n \rangle$ in equation (5). Further, at the determining of $\Delta M_{t(n)}$ the corresponding error $\Delta^{\text{th}} \langle q^n \rangle$ is to be added (in quadratures) to $\Delta \langle q^n \rangle_{\text{exp}}$ in formula (6). So the problem is reduced to the question about a possibility of carrying out the calculations so precisely that to keep the error $\Delta^{\text{th}} \langle q^n \rangle$ smaller than $\Delta \langle q^n \rangle_{\text{exp}}$.

In practice it is convenient to compare relative errors like $\Delta \langle q^n \rangle / \langle q^n \rangle$ instead of the proper errors $\Delta \langle q^n \rangle$. Fortunately the experimental relative error $\Delta \langle q^n \rangle_{\text{exp}} / \langle q^n \rangle_{\text{exp}}$ is growing at the transiting to the high degrees of the moments; its behavior is similar to that represented in Fig. 2. In particular, at transiting from $n = 1$ to $n = 15$ the $\Delta \langle q^n \rangle_{\text{exp}} / \langle q^n \rangle_{\text{exp}}$ increases from 1.8% to 5.2%(4.5%) in the LHC case, and from 0.7% to 2.4%(1.9%) in the LC case. So with increasing $n$ the requirement for the theoretical relative error $\Delta^{\text{th}} \langle q^n \rangle / \langle q^n \rangle$ is weakening.

Generally a theoretical error arises from a parametric uncertainty and an intrinsic uncertainty of the itself calculation. The parametric uncertainty originates mainly from the parameters that are worse known. In the given case they are the widths of the $W$ boson and of the top quark. The analysis of Ref. [6] shows that the uncertainties in these parameters practically do not affect the first moment. Moreover, even the switching-off of the widths gives a negligible effect. In the 15-th moment, the varying of $\Gamma_W$ within the experimental error $\Delta \Gamma_W = 0.04$ GeV results in $\Delta \langle q^{15} \rangle / \langle q^{15} \rangle = 0.09\%(0.06\%)$, which is insignificant, as well. Unfortunately we cannot estimate the variance of the moments with varying the width of the top quark, since from the very beginning we use for the top quarks the narrow width approximation. Nevertheless by basing on the results of Ref. [6] we expect a negligible variance of the moments in this case too. This is corroborated by the lack of reasons leading to appreciably greater sensitivity of the moments with respect to the width of the top quarks than to the width of the $W$ boson.

It should be mentioned, however, that the complete switching-off of the widths can vary noticeably the high-degree moments. Thus, the setting $\Gamma_W = 0$ implies a shift of $\langle q^{15} \rangle$ on 4.6%(3.2%) which can be compared with the above estimations for the experimental relative errors. This means that the calculation of the high-degree moments must be carried out with the taking into consideration of the realistic values of the widths. The latter requirement, of course, is unnecessary for the estimation of the errors only, the case of the investigation of the present article.

Now let us consider the errors of the itself calculation. First we note that all the processes in (1) go far above the thresholds of the production of unstable particles, the $W$ bosons and the top quarks. Therefore their production and decay can be described by the standard methods [16], namely with the Dyson resummation in the leading order calculation and in the pole approximation at calculating the perturbation-theory.
corrections. Thus, the problem is reduced to the estimation of the order of the perturbation theory, which is necessary for satisfying the required precision of the calculation.

Further we note that the corrections to the moments are to be calculated by way of calculating the corrections to the distribution \( F(q) \). For kinematic reasons the behavior of the latter corrections in the basic features should follow the behavior of the distribution. Namely, \( \Delta^{\text{th}}F(q) \) must vanish at the ends of the kinematic region because of the vanishing phase volume. Furthermore, \( \Delta^{\text{th}}F(q) \) must almost vanish on the tail at large \( q \) due to the smallness of the width of the \( W \) boson. (Recall that in the limit \( \Gamma_W = 0 \) the distribution is completely suppressed on the tail by reason of kinematics.) So \( \Delta^{\text{th}}F(q) \) must be precisely known mainly in the middle of the kinematic region but not near its ends including the tail. At transiting to the high-degree moments this condition is maintained. Moreover, in some sense it becomes even more strong. Really, at low \( q \) the contributions to the moments are additionally suppressed by the factor \( q^n \). At large \( q \), in the case of the effective moments, the contributions are completely suppressed by the cutoff \( \Lambda_n \). In addition, the larger \( n \) the more the distant between the cutoff and the \( M_t \), the right boundary of the actual range of kinematic variable (since \( \Lambda_n \to \Lambda \) from the right as \( n \to \infty \)). In particular, \( \Lambda_1 = 171 \text{ GeV} \) but \( \Lambda_{15} = 160 \text{ GeV} \) which is distant form \( M_t \) by 15 GeV. The mentioned feature is valuable for our consideration as the cut-off of the ends of the region of the kinematic variable implies a suppression of large logarithms that can arise near the ends at calculating the perturbation-theory corrections. In the final analysis this allows us to use the naive counting method for estimating the corrections to the moments.

With this in mind we farther use a rather rough approach which is based on a comparison between the corrections to the moments and to the width of the top quark. (Notice that the width actually is the zero moment accurate to the normalization.) The key reason of the approach is the observation that the integrals for the moments and the width, and for the corrections to the moments and the width, accumulate the contributions mainly from the middle region of the kinematic variable. So, supposing that in this region the correction to the distribution \( \Delta^{\text{th}}F(q) \) varies weakly in the units of \( F(q) \), one can expect that the corrections to the moments and to the width should be close to each other in the relative units. By the closeness we admit here a factor of order of several units. It is worth mentioning that even in the case of the experimental errors, which depend strongly on the shape of the distribution, the relative errors at \( n = 1 \) and \( n = 15 \) differ from each other by a factor of 2.5–3.5 only.

As we know, the electroweak one-loop correction to the top quark width amounts approximately 2%. The QCD one-loop correction is near of 10%, while the two-loop one is near of 2%. (See [11] and the references therein.) The comparison of these values

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3As variants, one can exploit the method of an effective field theory for calculating the resonant processes [17] or the modified perturbation theory based on distribution theory [18, 19].

4It should be emphasized that we discuss here the corrections to the \( t \to b \) transition but not to the \( b \)-quark fragmentation, including the perturbative fragmentation. The latter process is described by the convolution of the cross-section with the fragmentation function, and this operation is to be fulfilled in the framework of MC event generators.
with the above estimations for the experimental relative errors demonstrates that the one-loop electroweak and two-loop QCD corrections are enough to remain within the required limits (in both cases, LHC and LC). It should be noted that the mentioned corrections to the distribution can be certainly calculated since the corrections to the width have been calculated. Finally we note also that only the direct calculations of these corrections can explicitly solve the problem of the theoretical errors to the moments.

The mentioned calculation, however, would not yet entirely close the problem of the theoretical error of the $M_t$ determination because of the problem of nonperturbative nature caused by a renormalon contribution. Below for the completeness we only briefly consider this problem as its solution is known, at least in a conceptual respect. The problem actually is connected with the kind of the mass which is to be determined through an experimental measurement. In fact there are different masses, but only a Lagrangian mass is ultimately valuable since only the Lagrangian mass can be constrained with other fundamental parameters of the theory. The important representatives of the Lagrangian mass are the pole and $\overline{\text{MS}}$ masses. The directly measurable one is the pole mass, which is determined by kinematics. Correspondingly, the currently used algorithms of extracting $M_t$ from the data are turned to the pole mass. However, because of the renormalon contribution the pole mass determination faces an extra uncertainty of order of $O(\Lambda_{QCD})$ [20]. Numerically it can amount hundreds of MeVs.

The above mentioned difficulty can be bypassed in the framework of the following algorithm (below we state one of its possible variants) [20]: First, all theoretical calculations are to be fulfilled in the terms of the pole mass. Then the value of the pole mass is to be determined from the matching with data. Remember, at this stage the result includes the renormalon contribution. Further, by means of the well-known formula relating the pole mass with the $\overline{\text{MS}}$ mass (see Ref. [1], for example), the $\overline{\text{MS}}$ mass is determined. At this step the result gets again the renormalon contribution but, as is declared, it cancels the previous one. (So the inaccuracy in the relation between the pole and $\overline{\text{MS}}$ masses is charged to the pole mass.) Direct calculations in certain examples [8, 9] demonstrate effectiveness of the above algorithm.

So, the problem is initially stated as though for the pole mass determination, but at the final stage the $\overline{\text{MS}}$ mass is determined. This allows one to avoid a theoretical systematic uncertainty of order of $O(\Lambda_{QCD})$ caused by the renormalon contribution. Returning to our outcomes, we see that the theoretical error of the top mass determination can be really made smaller than the experimental error.

7 Discussion

The major result of this article is the detection of the effect of decreasing of statistical and systematic errors of the top quark mass measured from $M_{b\ell}$ distribution, when applying the technique of the moments and proceeding to the high degrees of the moments. The optimal value of the degree minimizing the errors is found near $n = 15$.

For the determining of the errors we have attracted a simple enough model. Its
parameters in the LHC case have been fixed on the basis of the results obtained earlier \[6\] by the MC modelling method. As applied to LC the parameters have been fixed mainly by indirect methods. Knowing the parameters and the dependence on the degree \(n\) of the moments, we have estimated the errors at varied \(n\) and have found the optimal value of \(n\), minimizing the errors. The optimal value \(n = 15\) is clearly visible in the framework of the basic method of calculating the moments. The applying of the technique of the effective moments diminishes the errors at \(n = 15\) by 10-20\%, but at the further increasing of \(n\) the results practically do not vary (see Figs. 4 and 6 and Tables 1-2).

At the optimal value \(n = 15\) the total error \(\Delta M_t\) is found close to 500 MeV in the LHC case, and close to 200 MeV in the LC case. In the LHC case the above accuracy more than in twice exceeds the accuracy obtained by the other methods \[1\] including the original method of Ref. \[6\]. In the LC case the estimated accuracy of the \(M_t\) determination is close to that expected at scanning the \(t\bar{t}\) production threshold \[8, 9\].

In conclusion it should be mentioned, once again, that at the intermediate stage of the analysis we have introduced simplifications allowing to minimize calculations. However at the final stage all estimations have been made on the basis of realistic values of the parameters. This peculiarity should not reduce the legitimacy of the detected behavior of the errors and, moreover, of their rounded estimations. Nevertheless the quantitative outcomes could be improved by further calculations based on the direct applying of a proper MC event generator.

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Figure 1: Distribution $F(q) = \Gamma^{-1} \frac{d\Gamma}{dq}$, $q \equiv M_{b\ell}$, at $M_t = 170$, 175, and 180 GeV.

Figure 2: The ratio $\Delta^{\text{stat}} \langle q^n \rangle_{\exp}/\langle q^n \rangle_{\exp}$ depending on $n$ ($M_t = 175$ GeV, $N = 4000$). The continuous curve represents the results described by formula (10). The dashed curve represents the results obtained by the method of the effective moments.

Figure 3: The shape of the function $q^n F(q)$ at $M_t = 175$ GeV, $n = 1, 5, 15, 40$ (in arbitrary normalization).

Figure 4: The statistical error $\Delta^{\text{stat}} M_{t(n)}$ depending on $n$. The dashed curve represents the results obtained by the method of the effective moments. The left and right vertical axes scale in GeV the results for LHC and LC cases, respectively.

Figure 5: The same that on Fig. 4 for $\Delta^{\text{sys 1}} M_{t(n)}$.

Figure 6: The same that on Fig. 4 for $\Delta^{\text{sys 2}} M_{t(n)}$. 

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