A SPECIAL CHAIN THEOREM FOR THE EMBEDDING DIMENSION (*)

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Abstract. This paper establishes an analogue of the “special chain theorem” for the embedding dimension of polynomial rings, with direct application on the (embedding) codimension given by codim(\(R\)) := embdim(\(R\)) - \(\dim(R)\). Namely, let \(R\) be a Noetherian ring and let \(P\) be a prime ideal of \(R[1, ..., n]\) with trace \(p := P \cap R\). We prove that \(\text{embdim}(R[1, ..., n], p) = \text{embdim}(R, p) + \text{ht}(\mathfrak{p}[p])\) and then \(\text{codim}(R[1, ..., n], p) = \text{codim}(R, p)\). In particular, this recovers a classic result on the transfer of regularity to polynomial rings (initially proved via a combination of Serre’s result on finite global dimension and Hilbert theorem on syzygies). A second application characterizes regularity in general settings of localizations of polynomial rings, including Nagata rings and Serre’s conjecture rings.

1. Introduction

Throughout, all rings are commutative with identity elements and ring homomorphisms are unital. The embedding dimension of a Noetherian local ring \((R, \mathfrak{m})\), denoted by \(\text{embdim}(R)\), is the least number of generators of \(\mathfrak{m}\). The ring \(R\) is regular if its Krull dimension and embedding dimensions coincide. Serre proved that a ring is regular if and only if it has finite global dimension. This allowed to see that regularity is stable under localization and then the definition got generalized as follows: a Noetherian ring is regular if its localizations with respect to all prime ideals are regular. The (embedding) codimension of \(R\) measures the defect of regularity of \(R\) and is given by the formula \(\text{codim}(R) := \text{embdim}(R) - \text{dim}(R)\).

One of the cornerstones of dimension theory of polynomial rings in several variables is the special chain theorem, which essentially asserts that the height of any prime ideal of the polynomial ring can always be realized via a special chain of prime ideals passing by the extension of its contraction over the basic ring; namely, if \(R\) is a Noetherian ring and \(P\) is a prime ideal of \(R[1, ..., n]\) with \(p := P \cap R\), then \(\dim(R[1, ..., n], p) = \dim(R, p) + \text{dim}(\kappa(p)[1, ..., n], p)\).

The aim of this paper is to establish an analogue of this result for the embedding dimension, with direct application on the (embedding) codimension which recovers a classic result on the transfer of regularity to polynomial rings. A second

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application characterizes regularity in general settings of localizations of polynomial rings, including the particular cases of Nagata and Serre’s conjecture rings.

Suitable background on regular rings (and also on the larger class of Cohen-Macaulay rings) is [9, 12, 14, 15]. For a geometric treatment of these concepts, we refer the reader to the excellent book of Eisenbud [10]. Early and recent developments on prime spectra and dimension theory are to be found in [1, 2, 3, 4, 7, 11, 13, 14, 15, 16, 18, 19, 20, 21, 22]. Any unreferenced material is standard, as in [5, 6, 14, 15, 23].

2. Result and applications

The main result of this paper (Theorem 2.1) settles a formula for the embedding dimension for the localizations of polynomial rings over Noetherian rings. As a first application, we get a formula for the (embedding) codimension (Corollary 2.2), which recovers a classic result on the transfer of regularity to polynomial rings; that is, \( R[x_1, \ldots, x_n] \) is regular if and only if so is \( R \). Moreover, Theorem 2.1 leads to investigate the regularity of two famous localizations of polynomial rings in several variables; namely, the Nagata ring \( R(x_1, x_2, \ldots, x_n) \) and the Serre’s conjecture ring \( R(x_1, x_2, \ldots, x_n) \). We show that the regularity of these two constructions is entirely characterized by the regularity of \( R \) (Corollary 2.3).

Recall that one of the cornerstones of dimension theory of polynomial rings in several variables is the special chain theorem, which essentially asserts that the height of any prime ideal \( P \) of \( R[x_1, \ldots, x_n] \) can always be realized via a special chain of prime ideals passing by the extension \( (P \cap R)[x_1, \ldots, x_n] \). This result was first proved by Jaffard in [13] and, later, Brewer, Heinzer, Montgomery and Rutter reformulated it in the following simple way (Theorem 1): Let \( P \) be a prime ideal of \( R[x_1, \ldots, x_n] \) with \( p := P \cap R \). Then

\[
\text{ht}(P) = \text{ht}(p[x_1, \ldots, x_n]) + \text{ht}\left(\frac{P}{p[x_1, \ldots, x_n]}\right).
\]

In a Noetherian setting, this formula becomes:

\[
\dim(R[x_1, \ldots, x_n]_P) = \dim(R_p) + \text{ht}\left(\frac{P}{p[x_1, \ldots, x_n]}\right) = \dim(R_p) + \dim\left(\frac{\kappa_R(p)[x_1, \ldots, x_n]}{p[x_1, \ldots, x_n]}\right)
\]

(1)

where \( \kappa_R(p) \) denotes the residue field of \( R_p \). Notice that the second equality holds on account of the basic fact that

\[
\frac{P}{p[x_1, \ldots, x_n]} \cap \frac{R}{p} = 0.
\]

Here is the main result of this paper which features a “special chain theorem” for the embedding dimension:
Theorem 2.1. Let $R$ be a Noetherian ring and $X_1, \ldots, X_n$ be indeterminates over $R$. Let $P$ be a prime ideal of $R[X_1, \ldots, X_n]$ with $p := P \cap R$. Then:

$$\text{embdim}(R[X_1, \ldots, X_n]_P) = \text{embdim}(R_p) + \text{ht}\left(\frac{P}{p[X_1, \ldots, X_n]}\right)$$

$$= \text{embdim}(R_p) + \text{embdim}\left(\frac{\kappa R(p)[X_1, \ldots, X_n]}{p\kappa R(p)[X_1, \ldots, X_n]}\right)$$

Proof. Throughout, given a ring $A$, an ideal of $A$ and $q$ a prime ideal of $A$, when no confusion is likely, we will denote by $I_q$ the ideal $IA_q$ of the local ring $A_q$. For the proof we use induction on $n$. Assume $n = 1$ and let $P$ be a prime ideal of $R[X]$ with $p := P \cap R$ and $r := \text{embdim}(R_p)$. Then

$$p_p = (a_1, \ldots, a_r)R_p$$

for some $a_1, \ldots, a_r \in p$. We envisage two cases; namely, either $P$ is an extension of $p$ or an upper to $p$. For both cases, we will use induction on $r$.

**Case 1:** $P$ is an extension of $p$ (i.e., $P = pR[X]$). We prove that

$$\text{embdim}(R[X]_p) = r.$$ 

Indeed, we have

$$P_p = pR_p[X]_{pR_p[X]}$$

$$= (a_1, \ldots, a_r)R_p[X]_{pR_p[X]}$$

$$= (a_1, \ldots, a_r)R[X].$$

So, obviously, if $p_p = (0)$, then $P_p = 0$. Next, we may assume $r \geq 1$. One can easily check that the canonical ring homomorphism

$$\varphi : R_p \rightarrow R[X]_p$$

is injective with $\varphi(p_p) \subseteq P_p$. This forces

$$\text{embdim}(R[X]_p) \geq 1.$$ 

Hence, there exists $j \in \{1, \ldots, n\}$, say $j = 1$, such that $a := a_1 \in p$ with $\frac{a}{1} \in P_p \setminus P_p^2$ and, a fortiori, $\frac{a}{1} \in P_p \setminus P_p^2$. By [14] Theorem 159, we get

$$\begin{align*}
\text{embdim}(R[X]_p) &= 1 + \text{embdim}\left(\frac{R}{(a)}[X]_{\frac{p}{a}X}\right) \\
\text{embdim}(R_p) &= 1 + \text{embdim}\left(\frac{R}{(a)}_{\frac{p}{a}}\right)
\end{align*}$$

(2)

Therefore

$$\text{embdim}\left(\frac{R}{(a)}_{\frac{p}{a}}\right) = r - 1$$

and then, by induction on $r$, we obtain

$$\text{embdim}\left(\frac{R}{(a)}[X]_{\frac{p}{a}X}\right) = \text{embdim}\left(\frac{R}{(a)}_{\frac{p}{a}}\right).$$

(3)
A combination of (2) and (3) leads to
\[ \text{embdim}(R[X]_P) = r \]
as desired.

**Case 2**: \( P \) is an upper to \( p \) (i.e., \( P \neq pR[X] \)). We prove that \( \text{embdim}(R[X]_P) = r + 1 \).

Note that \( PR_p[X] \) is also an upper to \( p \) and then there exists a (monic) polynomial \( f \in R[X] \) such that \( \overline{f} \) is irreducible in \( \kappa_R(p)[X] \) and \( PR_p[X] = pR_p[X] + fR_p[X] \). Notice that
\[ pR[X] + fR[X] \subseteq P \]
and we have
\[
\begin{align*}
P_P &= PR_p[X]_{PR_p[X]} \\
    &= (pR_p[X] + fR_p[X])_{PR_p[X]} \\
    &= (p[X] + fR[X])R_p[X]_{PR_p[X]} \\
    &= (p[X] + fR[X])_P \\
    &= p[X]_P + fR[X]_P \\
    &= (a_1, ..., a_r, f)R[X]_P.
\end{align*}
\]

Assume \( r = 0 \). Then \( P \) is an upper to zero with \( P_P = fR[X]_P \). So
\[ \text{embdim}(R[X]_P) \leq 1. \]

Further, by the principal ideal theorem [14, Theorem 152], we have
\[
\text{embdim}(R[X]_P) \geq \dim(R[X]_P) = \text{ht}(P) = 1.
\]
It follows that \( \text{embdim}(R[X]_P) = 1 \), as desired.

Next, assume \( r \geq 1 \). We claim that
\[ pR[X]_P \not\subseteq P^2_P. \]
Deny and suppose that \( pR[X]_P \subseteq P^2_P \). This assumption combined with the fact
\[ P_P = p[X]_P + fR[X]_P \]
yields
\[
\frac{P_P}{P^2_P} = \overline{f}R[X]_P
\]
as modules over \( R[X]_P \). Hence, by [14, Theorem 158], we get
\[ P_P = fR[X]_P. \]

Next, let \( a \in p \). Then, as \( \frac{a}{1} \in P_P = fR[X]_P \), there exist \( g \in R[X] \) and \( s, t \in R[X] \setminus P \) such that \( t(sa - fg) = 0 \). So that \( tfg \in p[X] \), whence \( tg \in p[X] \subset P \) as \( f \notin p[X] \). It follows that \( tsa = tfg \in P^2 \) and thus
\[
\frac{a}{1} \in P^2_P = f^2R[X]_P.
\]
We iterate the same process to get
\[
\frac{a}{1} \in P^n_P = f^nR[X]_P
\]
for each integer \( n \geq 1 \). Since \( R[X]_p \) is a Noetherian local ring, \( \cap P^n_p = (0) \) and thus \( \overline{f} = 0 \) in \( R[X]_p \). By the canonical injective homomorphism

\[ R_p \hookrightarrow R[X]_p \]

we have \( \overline{f} = 0 \) in \( R_p \). Thus

\[ p_p = (0), \]

the desired contradiction.

Consequently,

\[ pR[X]_p = (a_1, \ldots, a_r)R[X]_p \subseteq P^2_p. \]

Hence, there exists \( j \in \{1, \ldots, n\} \), say \( j = 1 \), such that \( a_1 \in P \setminus P^2_p \). Similar arguments as in Case 1 lead to the same two formulas displayed in (2). Therefore

\[ \text{embdim}(R[X]_p) = r + 1, \]

as desired.

Now, assume that \( n \geq 2 \) and, for \( k := 1, \ldots, n \), set

\[ R[k] := R[X_1, \ldots, X_k], \quad p[k] := p[X_1, \ldots, X_k]. \]

Let \( P' := p \cap R[n-1] \). We prove that

\[ \text{embdim}(R[n]_p) = r + \text{ht}\left( \frac{P}{p[n]} \right), \]

Indeed, by virtue of the case \( n = 1 \), we have

\[ \text{embdim}(R[n]_p) = \text{embdim}(R[n-1]_p) + \text{ht}\left( \frac{P}{P'[X_n]} \right). \]

Moreover, by induction hypothesis, we get

\[ \text{embdim}(R[n-1]_p) = r + \text{ht}\left( \frac{P'}{p'[n]} \right). \]

Note that the prime ideals \( \frac{P'}{p'[n]} \) and \( \frac{P}{p[n]} \) both survive in \( \kappa_R(p)[n] \), respectively. Hence, as \( \kappa_R(p)[n] \) is catenarian and \( (R/p)[n-1] \) is Noetherian, we obtain

\[ \text{ht}\left( \frac{P}{p[n]} \right) = \text{ht}\left( \frac{P'}{p'[n]} \right) + \text{ht}\left( \frac{P}{p'[X_n]} \right). \]
Further, the fact that $\kappa_R(p)[X_1,\ldots,X_n]$ is regular yields
\[
\text{ht}\left(\frac{P}{p[X_1,\ldots,X_n]}\right) = \text{embdim}\left(\frac{\kappa_R(p)[X_1,\ldots,X_n]}{p[p[X_1,\ldots,X_n]]}\right).
\] (8)
So a combination of (5), (6), (7), and (8) leads to the conclusion, completing the proof of the theorem. \hfill \Box

As a first application of Theorem 2.1, we get the next corollary on the (embedding) codimension. In particular, it recovers a well-known result on the transfer of regularity to polynomial rings (initially proved via Serre’s result on finite global dimension and Hilbert Theorem on syzygies [17, Theorem 8.37]. See also [14, Theorem 171]).

**Corollary 2.2.** Let $R$ be a Noetherian ring and $X_1,\ldots,X_n$ indeterminates over $R$. Let $P$ be a prime ideal of $R[X_1,\ldots,X_n]$ with $p := P \cap R$. Then:
\[
\text{codim}(R[X_1,\ldots,X_n]|_P) = \text{codim}(R_p).
\]
In particular, $R[X_1,\ldots,X_n]$ is regular if and only if $R$ is regular.

Theorem 2.1 allows us to characterize the regularity for two famous localizations of polynomial rings; namely, Nagata rings and Serre’s conjecture rings. Let $R$ be a ring and $X, X_1,\ldots,X_n$ indeterminates over $R$. Recall that
\[
R(X_1,\ldots,X_n) = S^{-1}R[X_1,\ldots,X_n]
\]
is the Nagata ring, where $S$ is the multiplicative set of $R[X_1,\ldots,X_n]$ consisting of the polynomials whose coefficients generate $R$. Let
\[
R\langle X \rangle := U^{-1}R[X]
\]
where $U$ is the multiplicative set of monic polynomials in $R[X]$, and
\[
R\langle X_1,\ldots,X_n \rangle := R\langle X_1,\ldots,X_{n-1} \rangle(X_n).
\]
Then $R\langle X_1,\ldots,X_n \rangle$ is called the Serre’s conjecture ring and is a localization of $R[X_1,\ldots,X_n]$.

**Corollary 2.3.** Let $R$ be a Noetherian ring and $X_1,\ldots,X_n$ indeterminates over $R$. Let $S$ be a multiplicative subset of $R[X_1,\ldots,X_n]$. Then:
(a) $S^{-1}R[X_1,\ldots,X_n]$ is regular if and only if $R_p$ is regular for each prime ideal $p$ of $R$ such that $p[X_1,\ldots,X_n] \cap S = \emptyset$.
(b) In particular, $R(X_1,\ldots,X_n)$ is regular if and only if $R\langle X_1,\ldots,X_n \rangle$ is regular if and only if $R\langle X_1,\ldots,X_n \rangle$ is regular if and only if $R$ is regular.

**Proof.** (a) Let $Q = S^{-1}P$ be a prime ideal of $S^{-1}R[X_1,\ldots,X_n]$, where $P$ is the inverse image of $Q$ by the canonical homomorphism
\[
R[X_1,\ldots,X_n] \to S^{-1}R[X_1,\ldots,X_n]
\]
and let $p := P \cap R$. Notice that

$$S^{-1}R[X_1, \ldots, X_n]|_Q \cong R[X_1, \ldots, X_n]|_P$$

and

$$Q \quad \cong \quad S^{-1}p[X_1, \ldots, X_n]$$

where $\overline{S}$ denotes the image of $S$ via the natural homomorphism

$$R[X_1, \ldots, X_n] \to R[p[X_1, \ldots, X_n]].$$

Therefore, by (1), we obtain

$$\dim(S^{-1}R[X_1, \ldots, X_n]|_Q) = \dim(R[X_1, \ldots, X_n]|_P) = \dim(R_p) + \text{ht}
\left(\frac{Q}{S^{-1}p[X_1, \ldots, X_n]}\right)$$

and, by Theorem 2.1, we have

$$\text{embdim}(S^{-1}R[X_1, \ldots, X_n]|_Q) = \text{embdim}(R[X_1, \ldots, X_n]|_P) = \text{embdim}(R_p) + \text{ht}
\left(\frac{Q}{S^{-1}p[X_1, \ldots, X_n]}\right).$$

Now, observe that the set $\{Q \cap R \mid Q \text{ is a prime ideal of } S^{-1}R[X_1, \ldots, X_n]\}$ is equal to the set $\{p \mid p \text{ is a prime ideal of } R \text{ such that } p[X_1, \ldots, X_n] \cap S = \emptyset\}$. Therefore, (9) and (10) lead to the conclusion.

(b) Combine (a) with the fact that the extension of any prime ideal of $R$ to $R[X_1, \ldots, X_n]$ does not meet the multiplicative sets related to the rings $R(X_1, \ldots, X_n)$ and $R(X_1, \ldots, X_n)$.

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