Newton Law in Brane-World Scenario with 4d Induced Gravity:
Singular Quantum Mechanical Approach

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Abstract

From the viewpoint of the singular quantum mechanics the effect of the energy-dependent coupling constant for \( \delta \)-function potential is examined. The energy-dependence of the coupling constant naturally generates the time-derivative in the boundary condition of the Euclidean propagator. This is explicitly confirmed by making use of the simple 1\( d \) model. The result is applied to the linearized gravity fluctuation equation for the brane-world scenario with 4\( d \) induced gravity. Our approach generates 5\( d \) Newton potential at a certain intermediate range of distance between two test massive sources. For other range of distance 4\( d \) Newton potential is recovered.

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The original brane-world scenarios are designed to solve the gauge-hierarchy problem, one of the longstanding puzzle in physics, by introducing large extra dimensions [1–3] or warped extra dimension [4]. Especially, Randall-Sundrum(RS) scenario may also provide a mechanism for the localization of the gravity on the single positive-tension brane [5]. As a result much attention is paid to understand the higher-order corrections of Newton potential on the brane by employing the various brane-world pictures. Furthermore, these activities may motivate the experimental investigation for the short range gravitational effect [6].

For the case of $AdS_5$ bulk with a flat brane as RS scenario the computation of Newton law carried out in original RS paper [5] is improved by examining the brane-bending effect [7,8] or by computing the one-loop corrections to the gravitational propagator [9]. These improvements introduce a mutiplication factor in the subleading term of Newton potential contributed from Kaluza-Klein excitations. In addition, the linearized gravity fluctuation equation is also treated from the viewpoint of the singular quantum mechanics(SQM) [10,11]. In SQM it is well-known that when the potential is too singular, Hamiltonian loses its self-adjoint property, and thus the conservation of the probability met a serious problem. In order to overcome this difficulty we should maintain the self-adjoint property of Hamiltonian by extending its domain of definition appropriately, which is refered to a self-adjoint extension [12,13]. It is known that this mathematical approach is effectively identical with the physically-oriented coupling constant renormalization scheme [14,15]. The method has been applied to the gravitation fluctuation equations of RS single brane and two brane scenarios for the compromise of the gravitational localization with a small cosmological constant [10,11]. It also generates the logarithmic correction in the short range of Newton potential [16].

Newton law with a different setup is also examined. The gravitational potential for the flat brane in $dS_5$ bulk and for the curved $dS$ brane in $dS_5$ or $AdS_5$ bulk are examined [17]. For the case of the flat brane in $dS_5$ bulk the sign of the subleading correction is changed to be negative. The physical implication of this sign change is discussed in Ref. [17] in the context of $dS$/CFT correspondence [18].
Another type of the scenario which attracts an attention recently is a brane-world with a 4d induced Einstein term where the brane has its own gravity term *ab initio*. For the case of the flat bulk in this picture the gravitational potential becomes 4d type $1/r$ at the short range, *i.e.* $r << \lambda/2$ and 5d type $1/r^2$ at the long range, *i.e.* $r >> \lambda/2$, where $\lambda$ is a ratio of 4d Planck scale with that of 5d: $\lambda \equiv M_4^2/M_5^3$ [19]. This fact can be used for explaining the acceleration of the universe [20]. Newton law with 4d induced gravity in the $AdS_5$ background is also examined [21,22]. In this case there is an intermediate range of distance where 5d gravitational potential plays a dominant role. At other ranges 4d gravitational potential is recovered.

The most remarkable feature of the linearized fluctuation equation for the case of $AdS_5$ bulk with the induced Einstein term is the fact that the coupling constant of the $\delta$-function potential is dependent on energy(or mass) as following:

$$\hat{H}\hat{\psi}(z) = E\hat{\psi}(z) \quad (1)$$

where

$$\hat{H} = \hat{H}_{RS} - \lambda E\delta(z) \quad (2)$$

$$\hat{H}_{RS} = -\frac{1}{2} \frac{\partial^2}{\partial z^2} + \frac{15k^2}{8(k|z| + 1)^2} - \frac{3}{2}k\delta(z)$$

and $\lambda \equiv M_4^2/M_5^3$. If we regard Eq.(1) as a Schrödinger equation, we should rely on the SQM with an energy-dependent coupling constant of $\delta$-function potential. Upon our knowledge the effect of the energy-dependent coupling constant was not fully examined in the context of SQM. The purpose of this letter is to analyze Eq.(1) from the viewpoint of SQM.

To understand the effect of the energy-dependent coupling constant in quantum mechanics let us consider the Schrödinger equation $H\phi = E\phi$, where

$$H = H_V(\vec{p}, \vec{r}) + \hat{v}(E)\delta(\vec{r}) \quad (3)$$

$$H_V(\vec{p}, \vec{r}) = \frac{\vec{p}^2}{2} + V(\vec{r})$$

Let $G[\vec{r}_1, \vec{r}_2; t]$ and $G_V[\vec{r}_1, \vec{r}_2; t]$ be the Euclidean propagators for $H$ and $H_V$ respectively. Then, it is well-known that $G[\vec{r}_1, \vec{r}_2; t]$ satisfies the following integral equation
\[
G[r_1, r_2; t] = G_V[r_1, r_2; t] - \hat{v}(E) \int_0^t ds G_V[r_1, 0; t-s] G[0, r_2; s].
\] (4)

If we take a Laplace transform

\[
\mathcal{L}[f] \equiv \hat{f}(E) \equiv \int_0^t dt f(t) e^{-Et}
\] (5)
in Eq.(4), one can easily derive

\[
\hat{G}[r_1, r_2; E] = \hat{G}_V[r_1, r_2; E] - \hat{v}(E) \hat{G}_V[r_1, 0; E] G[0, r_2; E].
\] (6)

Solving Eq.(6) one can express the fixed-energy amplitude \(\hat{G}[r_1, r_2; E]\) from \(\hat{G}_V[r_1, r_2; E]\);

\[
\hat{G}[r_1, r_2; E] = \hat{G}_V[r_1, r_2; E] - \frac{\hat{G}_V[r_1, 0; E] \hat{G}_V[0, r_2; E]}{1 - \hat{v}(E) \hat{G}_V[0, 0; E]}. \tag{7}
\]

As an example let us consider a simple 1d free particle case for \(H_V\). Then, the fixed-energy amplitude for \(H_V\) is simply

\[
\hat{G}_V[x, y; E] \equiv \hat{G}_F[x, y; E] = e^{-\sqrt{2E}|x-y|}/\sqrt{2E}
\] (8)

where the subscript ‘F’ stands for the free particle. Thus, Eq.(7) can be re-written as

\[
\hat{G}[x, y; E] = e^{-\sqrt{2E}|x-y|}/\sqrt{2E} - \frac{\hat{v}(E)}{\sqrt{2E} \sqrt{2E + \hat{v}(E)}} e^{-\sqrt{2E}(|x|+|y|)}. \tag{9}
\]

If \(\hat{v}(E)\) is independent of \(E\), one can easily take an inverse Laplace transform to Eq.(9) using the formulae

\[
\begin{align*}
\mathcal{L}^{-1} \left( \frac{1}{\sqrt{E}} e^{-\alpha t/2E} \right) &= \pi^{-\frac{1}{2}} t^{-\frac{1}{2}} e^{-\alpha t/4} \\
\mathcal{L}^{-1} \left( (\sqrt{E} + \beta)^{-1} e^{-\alpha t/2E} \right) &= \pi^{-\frac{1}{2}} t^{-\frac{1}{2}} e^{-\alpha^2 t/4} - \beta e^{\alpha \beta + \beta^2 t} \text{Erfc} \left( \frac{1}{2} \alpha t^{-\frac{1}{2}} + \beta t^{\frac{1}{2}} \right)
\end{align*}
\] (10)

where \(\text{Erfc}(z)\) is an usual error function:

\[
\text{Erfc}(z) \equiv \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} dt. \tag{11}
\]

After some manipulations one can show the Euclidean propagator in this case is simply

\[
G[x, y; t] = G_0[x, y; t] - \hat{v} \int_0^t dz e^{-\hat{v}z} G_0[|x|, -|y|-|z|; t]
\] (12)
where $G_0[x, y; t]$ is a propagator for a 1d free-particle system;

$$G_0[x, y; t] = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}. \quad (13)$$

One can show that $G[x, y; t]$ and $\hat{G}[x, y; E]$ satisfy the usual boundary condition for the $\delta$-function potential;

$$\frac{\partial G}{\partial x}[0^+, y; t] - \frac{\partial G}{\partial x}[0^-, y; t] = 2\hat{v}G[0, y; t] \quad (14)$$

$$\frac{\partial \hat{G}}{\partial x}[0^+, y; E] - \frac{\partial \hat{G}}{\partial x}[0^-, y; E] = 2\hat{v}\hat{G}[0, y; E].$$

Next, let us consider $\hat{v}(E) = \alpha E$ case. Then, Eq.(9) shows that the fixed-energy amplitude $\hat{G}[x, y; E]$ becomes

$$\hat{G}[x, y; E] = e^{-\frac{\sqrt{2E|x-y|}}{\sqrt{E}}} - \frac{1}{\sqrt{2}} \frac{1}{\sqrt{E + \frac{\sqrt{2E}}{\alpha}}} e^{-\frac{\sqrt{2E(|x|+|y|)}}{\alpha}}. \quad (15)$$

From Eq.(15) one can show $\hat{G}[x, y; E]$ satisfies the following boundary condition at $x = 0$;

$$\frac{\partial \hat{G}}{\partial x}[0^+, y; E] - \frac{\partial \hat{G}}{\partial x}[0^-, y; E] = 2\hat{v}(E)\hat{G}[0, y; E]. \quad (16)$$

Eq.(16) enables us to derive a boundary condition for the corresponding Euclidean propagator $G[x, y; t]$ to be satisfied at $x = 0$. Taking an inverse Laplace transform in Eq.(16) leads to

$$\frac{\partial G}{\partial x}[0^+, y; t] - \frac{\partial G}{\partial x}[0^-, y; t] = 2 \int_0^t ds v(t - s) G[0, y; s] \quad (17)$$

where $v(t) = L^{-1}[\hat{v}(E)]$. Since $v(t)$ becomes

$$v(t) = \alpha \lim_{\epsilon \to 0^+} \delta'(t - \epsilon) \quad (18)$$

for $\hat{v}(E) = \alpha E$, Eq.(17) reduces to

$$\frac{\partial G}{\partial x}[0^+, y; t] - \frac{\partial G}{\partial x}[0^-, y; t] = 2\alpha \frac{\partial}{\partial t} G[0, y; t]. \quad (19)$$

Thus the energy-dependence of the coupling constant yields an time-derivative at the boundary condition of the propagator. This may be understood from the usual energy-time uncertainty principle. The explicit form of the Euclidean propagator in this case can be derived by taking an inverse Laplace transform to Eq.(15);
\[ G[x, y; t] = G_0[x, y; t] - \frac{2}{\alpha} \int_0^\infty dz e^{-\frac{z}{\alpha}} G_0[x, -|y| - |z|; t] \]  

where \( G_0[x, y; t] \) is given in Eq.(13). It is straightforward to show that \( G[x, y; t] \) really satisfies the boundary condition (19).

Now, let us go back to the gravitational fluctuation equation (1). Firstly, let us comment briefly how Eq.(1) is derived. The 5\( d \) Einstein equation we consider is

\[ \left( \tilde{R}_{MN} - \frac{1}{2} G_{MN} \tilde{R} \right) + \frac{\Lambda}{M_5^2} G_{MN} + \lambda \left[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right] \delta_5^{\mu\nu} \delta(y) = 0 \]  

where \( \tilde{R} \) and \( R \) are 5\( d \) and 4\( d \) curvature scalar respectively. Of course \( \Lambda \) and \( v_b \) are 5\( d \) cosmological constant and brane tension. In fact, Eq.(21) can be derived by varying the Einstein-Hilbert action

\[ S = \int d^2 y \sqrt{-G} \left[ \frac{M_5^2}{2} \tilde{R} - \Lambda + \left( \frac{M_4^2}{2} R - v_b \right) \delta(y) \right]. \]  

The solution of Eq.(21) we have interest is same with that of the usual RS scenario

\[ ds^2 = e^{-2k|y|} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \]  

with the fine-tuning conditions \( \Lambda = -6M_5^3 k^2 \) and \( v_b = 6kM_5^3 \). The coincidence of the classical solution with that of the usual RS scenario is in fact evident because the 3-brane in Eq.(23) is flat and hence the 4\( d \) induced gravity does not play any role. However, this induced gravity yields an important effect when we consider the linearized gravitational fluctuation \( h_{\mu\nu}(x, y) \) defined as

\[ ds^2 = (e^{-2k|y|} \eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu + dy^2. \]  

Inserting Eq.(24) into the Einstein equation (21) with redefinition \( h_{\mu\nu} = e^{-k|y|/2} \tilde{\psi}(y) e^{ipx}, \)

\[ z = \epsilon(y)(e^{k|y|} - 1)/k, \]  

and \( E = -p^2/2, \) one can derive Eq.(1) straightforwardly. Of course we should use RS gauge \( h_{\mu,\nu} = h_{\mu} = 0 \) in the course of calculation.

The fixed-energy amplitude \( \tilde{G}_{RS}[a, b; E] \) for \( \tilde{H}_{RS} \) is examined in detail in Ref. [10,11]. In fact, \( \tilde{G}_{RS}[a, b; E] \) is crucially dependent on the boundary condition at \( x = R \equiv 1/k, \) where
\[ x = z + R, \text{ which is parametrized by } \xi \text{ in Ref. [10,11]. Here, we choose only } \xi = 1/2 \text{ case which means the Dirichlet and the Neumann boundary conditions are included with equal weight. In this case the fixed-energy amplitude } \hat{G}_{RS}[a, b; E] \text{ on the brane simply reduces to } \hat{G}_{RS}[a, b; E] = \Delta_0^{RS} + \Delta_K^{RS} \] 

where

\[
\Delta_0^{RS} = \frac{1}{RE} \quad \Delta_K^{RS} = \frac{1}{\sqrt{2E}} K_0(\sqrt{2ER}) K_1(\sqrt{2ER})
\]

and \(K_\nu(z)\) is an usual modified Bessel function. Of course, \(\Delta_0^{RS}\) and \(\Delta_K^{RS}\) represent the zero mode and Kaluza-Klein excitations of the RS scenario respectively.

The general formula Eq.(7) enables us to derive a fixed-energy amplitude \(\hat{G}[a, b; E]\) for Hamiltonian \(\hat{H}\) defined in Eq.(2) as following:

\[
\hat{G}[a, b; E] = \hat{G}_{RS}[a, b; E] + \frac{\hat{G}_{RS}[a, R; E] \hat{G}_{RS}[R, b; E]}{\frac{1}{\lambda E} - \hat{G}_{RS}[R, R; E]}. \tag{26}
\]

Thus, the fixed-energy amplitude at the location of the brane for the system involving the 4d induced gravity is

\[
\hat{G}[R, R; E] = (\Delta_0^{RS} + \Delta_K^{RS}) + \frac{(\Delta_0^{RS} + \Delta_K^{RS})^2}{\frac{1}{\lambda E} - (\Delta_0^{RS} + \Delta_K^{RS})} = \frac{K_2(mR)}{m 2K_1(mR) - \lambda K_2(mR)} \tag{27}
\]

where \(m \equiv \sqrt{2E}\). The fixed-energy we derived in Eq.(27) is proportional to the momentum-dependent Green function \(\hat{G}_R(p, y = 0)\) which is expressed at Eq.(2.10) of Ref. [21]. If we adjust our conventions with those of Ref. [21], the relation between them is simply \(\hat{G}[R, R; E] = M_0^2 \hat{G}_R(p = m, y = 0)\). This simple relation makes us to analyze Newton law of the gravitation localized on the brane.

Newton potential localized on the brane is generally obtained from the space-time dependent Green function as following [7,19]

\[
V(\vec{x}) = \int dt G_R(t, \vec{x}, y = 0; 0, \vec{0}, 0) \tag{28}
\]

where the subscript ‘R’ stands for the retarded Green function. Using a Fourier transform of \(G_R\)

\[
G_R(t, \vec{x}, y; 0, \vec{0}, 0) \equiv \int \frac{d^4p}{(2\pi)^4} e^{ipt} \hat{G}_R(p, y) \tag{29}
\]
one can show the potential $V(\vec{x})$ in Eq.(28) reduces to

$$V(r) = \frac{1}{2\pi^2 r} \int_0^\infty dpp \sin pr \tilde{G}_R(p, y = 0) = \frac{1}{2\pi^2 M_3^2 r} \int_{m_0}^\infty dm \sin mr \tilde{G}[R, R; E].$$  \hspace{1cm} (30)$$

Since the continuum states start from the asymptotic value of the quantum-mechanical potential, the singular $\delta$-function potential can not generate any mass gap. Thus we can conclude $m_0 = 0$ in Eq.(30). The factor $\sin mr$ in Eq.(30) is crucial to extract an information on the long-range and short-range behaviors of the gravitational potential $V(r)$. To show this more explicitly we re-express $V(r)$ as following

$$V(r) = \frac{1}{\pi^2 M_3^2 Rr} \int_0^\infty du \sin \left( \frac{ru}{R} \right) \frac{K_2(u)}{2K_1(u) - \frac{1}{R} u K_2(u)}. \hspace{1cm} (31)$$

If $r >> R$, the high oscillation of $\sin(\rho u/R)$ results in a negligible contribution to the integral from large $u$. Thus, one can approximate $K_1(u)$ and $K_2(u)$ as

$$K_1(u) \sim \frac{1}{u} + \frac{u}{2} \ln u, \quad K_2(u) \sim \frac{1}{u^2} - \frac{1}{2}. \hspace{1cm} (32)$$

Then it is straightforward to show that the potential becomes

$$V(r) \sim \frac{1}{2\pi M_3^2 R} \left[ 1 - \frac{2}{\pi} \lim_{\epsilon \to 0^+} \int_0^\infty du e^{-\epsilon u} \sin \left( \frac{r}{R} u \right) \ln u \right] \hspace{1cm} (33)$$

where the infinitesimal parameter $\epsilon$ is introduced for the regularization. Performing the integration in Eq.(33) makes $V(r)$ to be

$$V(r) \sim \frac{1}{2\pi M_3^2 R} \left[ 1 - \frac{1}{2 - \frac{1}{R}} \right] \left( 1 + \frac{1}{2 - \frac{1}{R}} \frac{R^2}{r^2} \right). \hspace{1cm} (34)$$

Thus the potential exhibits the $4d$ behavior at the long-range.

If $r << R$, the large $u$ region mainly contributes to the integral of Eq.(31). Thus, one can use the asymptotic expansion of $K_1(u)$ and $K_2(u);

$$K_1(u) \sim \sqrt{\frac{\pi}{2u}} e^{-u} \left( 1 + \frac{3}{8u} \right), \quad K_2(u) \sim \sqrt{\frac{\pi}{2u}} e^{-u} \left( 1 + \frac{15}{8u} \right). \hspace{1cm} (35)$$

Then it is straightforward to compute $V(r)$ whose final approximate expression is

$$V(r) \sim -\frac{1}{\pi^2 M_3^2 \lambda r} \left[ \frac{\pi}{2} - \frac{2R}{\lambda} - \frac{15}{8} \left\{ \frac{\pi}{2} - f \left( \frac{r}{R} \left( \frac{2R}{\lambda} - \frac{15}{8} \right) \right) \right\} \right] \hspace{1cm} (36)$$
where \( f(z) \equiv ci(z) \sin z - si(z) \cos z \) and, \( ci(z) \) and \( si(z) \) are usual sine and cosine integral functions. If \( \lambda << R \), \( V(r) \) simply reduce to

\[
V(r) \sim -\frac{1}{\pi^2 M_0^3 \lambda^2 r} f\left(\frac{2r}{\lambda}\right). 
\]

Thus if \( \lambda << r \), one can use the asymptotic expansion of \( f(z) \) [23]

\[
\lim_{z \rightarrow \infty} f(z) \sim \frac{1}{z} \left(1 - \frac{2}{z^2}\right) 
\]

which results in

\[
V(r) \sim -\frac{1}{2\pi^2 M_0^3 \lambda^2 r^2} \left(1 - \frac{\lambda^2}{2r^2}\right). 
\]

Therefore, in the region \( \lambda << r << R \), \( V(r) \) displays the 5d gravitational behavior. If \( r << \lambda << R \), Eq.(37) shows \( V(r) \) reduces to

\[
V(r) \sim -\frac{1}{2\pi M_0^3 \lambda r} \left(1 + \frac{4r}{\pi \lambda} \ln \frac{2r}{\lambda}\right). 
\]

Thus at this region \( V(r) \) recovers the 4d behavior.

The exact computation of Newton potential in the full range of \( r \) with arbitrary \( \lambda \) seems to be interesting. It may need an highly nontrivial numerical method because our computation requires a regularization. We hope to return to this issue in the near future. Another point we should stress is that our result comes from \( \hat{G}_{RS}[R, R; E] = \Delta_{0}^{RS} + \Delta_{KK}^{RS} \). In Ref. [10,11], however, \( \hat{G}_{RS}[a, b; E] \) is crucially dependent on the boundary conditions, which is parametrized by \( \xi \). It is interesting to study an effect of the induced gravity when the general boundary conditions are employed. From the viewpoint of SQM Eq.(7) is a formal solution for the fixed-energy amplitude because \( \hat{G}_{V}[\vec{r}, \vec{0}; E] \) is in general ill-defined at the higher dimensional theory. Thus one should adopt an appropriate regularization scheme for obtaining the finite result [14,15]. It seems to be interesting to study further the effect of the energy-dependent coupling constant from the aspect of pure SQM.

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REFERENCES

[1] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B 429 (1998) 263 [hep-ph/9803315].

[2] L. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B 436 (1998) 257 [hep-ph/9804398].

[3] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Rev. D 59 (1999) 086004 [hep-ph/9807344].

[4] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370 [hep-ph/9905221].

[5] L. Randall and R. Sundrum, Phys. Rev. Lett. B 83 (1999) 4690 [hep-th/9906064].

[6] C. D. Hoyle, U. Schmidt, B. R. Heckel, E. G. Adelberger, J. H. Gundlach, D. J. Kapner and H. E. Swanson, Phys. Rev. Lett. 86 (2001) 1418 [hep-ph/0011014].

[7] J. Garriga and T. Tanaka, Phys. Rev. Lett. 84 (2000) 2778 [hep-th/9911055].

[8] S. B. Giddings, E. Katz, and L. Randall, JHEP 0003 (2000) 023 [hep-th/0002091].

[9] M. J. Duff and J. T. Liu, Phys. Rev. Lett. 85 (2000) 2052 [hep-th/0003237].

[10] D. K. Park and S. Tamaryan, Phys. Lett. B532 (2002) 305 [hep-th/0108068].

[11] D. K. Park and H. S. Kim, Nucl. Phys. B 636 (2002) 179 [hep-th/0204122].

[12] A. Z. Capri, Nonrelativistic Quantum Mechanics (Benjamin/Cummings, Menlo Park, CA, 1985).

[13] M. Reed and B. Simon, Methods of Modern Mathematical Physics (Academic, New York, 1975).

[14] R. Jackiw, in M. A. Bég Memorial Volume, edited by A. Ali and P. Hoodbhoy (World Scientific, Singapore, 1991).

[15] D. K. Park, J. Math. Phys. 36 (1995) 5453 [hep-th/9405020].
[16] D. K. Park and S. Tamaryan, Phys. Lett. B 554 (2003) 92 [hep-th/0212023].

[17] S. Nojiri and S. D. Odintsov, Phys. Lett. B 548 (2002) 215 [hep-th/0209066].

[18] A. Strominger, JHEP 0110 (2001) 034 [hep-th/0106113].

[19] G. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B 485 (2000) 208 [hep-th/0005016].

[20] V. A. Rubakov, hep-th/0303125.

[21] E. Kiritsis, N. Tetradis and T. N. Tomaras, JHEP 0203 (2002) 019 [hep-th/0202037].

[22] M. Ito, Phys. Lett. B 554 (2003) 180 [hep-th/0211268].

[23] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (Dover, New York, NY, 1972).