Electromagneto-mechanical fields of giant magnetostrictive/piezoelectric laminates under concentrated load

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This paper presents the nonlinear electromagneto-mechanical behavior of magnetostrictive/piezoelectric laminates under three-point bending both numerically and experimentally. The laminates are fabricated using thin Terfenol-D and PZT layers. The three-point bending test was conducted on the Terfenol-D/PZT laminates, and the displacement, induced magnetic field and induced voltage due to mechanical loads were measured. Three-dimensional finite element analysis was also carried out, and the electromagneto-mechanical fields in the laminates were predicted by introducing a second-order magnetoelastic constant for Terfenol-D. Comparison was then made between simulation and experiment.

Keywords: Magnetostrictive/piezoelectric mechanics; finite element method; material testing; electronic laminates; detection and response characteristics; smart materials and structures

1. Introduction

Magnetostrictive materials have been studied due to their great potential as sensor, actuator and switch elements in a wide variety of applications that can benefit from their remote operation, high energy density and short response time [1]. Among magnetostrictive materials, Terbium Dysprosium Iron alloy (Terfenol-D) is the most attractive one because of its high saturation magnetostrain (1600 ppm), good coupling coefficient (as high as 60%), and high Young’s modulus [2]. On the other hand, piezoelectric materials, especially lead zirconate titanate (PZT), are widely used in smart structures due to their high bandwidth, high output force, compact size, and high power density. Magnetostrictive (ME) composites require magnetostrictive and piezoelectric materials, with a strong coupling between them, and many applications of these composites such as magnetic field sensing devices [3,4], coil-less transformers, tunable microwave and read/write devices [5] are currently under investigation.

In general, the implementation of any methodology for monitoring and control requires some form of measurement [6]. These measurements are usually mechanical motions (e.g. strain, acceleration) and can include other optical, magnetic, electrical, or pressure sensing measurements. Accordingly, a wide range of sensing technologies has been used in the research and practice of monitoring and control. For the magnetostrictive materials,
however, prediction of sensor or monitoring performance is complicated under bending. Recently, Datta et al. [7] carried out a four-point bending test of magnetostrictive Galfenol sensors under different magnetic bias fields, and discussed the induced strain and the magnetic induction due to bending load.

In this work, we study the electromagneto-mechanical behavior of Terfenol-D/PZT laminates under bending in a combined numerical and experimental approach. The displacement, induced magnetic field and induced voltage of the Terfenol-D/PZT laminates due to mechanical loads were measured by using a three-point bending test. The electromagneto-mechanical fields in the laminates were also calculated through nonlinear finite element analysis. Results produced by the analysis were then compared with experimental values.

2. Experimental procedures

We consider a two-layered Terfenol-D/PZT laminate as shown in Figure 1. Terfenol-D (Etrema Products, Inc., USA) of length \( l_m = 15 \) mm, width \( w_m = 5 \) mm and thickness \( h_m = 1, 3 \) mm and PZT C-91 (Fuji Ceramics, Co. Ltd., Japan) with \( l_p = 15 \) mm, \( w_p = 5 \) mm and \( h_p = 1 \) mm were used to make Terfenol-D/PZT laminates by epoxy bonding (EP-34B; Kyowa Electronic Instruments Co. Ltd., Japan). Subscripts \( m \) and \( p \) refer to Terfenol-D and PZT layers, respectively. A rectangular Cartesian coordinate system \( O-xyz \) is employed, and the origin of the coordinate system is located at the center of the bottom left side of the Terfenol-D layer. The easy axis of the magnetization of the Terfenol-D layer is in the \( z \)-direction, while the polarization of the PZT layer is in the \( x \)-direction.
The Terfenol-D/PZT laminate is loaded in three-point bending, as shown in Figure 2. The span between the bottom two supports is $S = 13$ mm, and a load ($P$) controlled mode is employed to apply with load rate $0.02$ Ns$^{-1}$. First, the load versus deflection curve of the Terfenol-D/PZT laminate is evaluated with a micro-testing machine. Next, the average induced magnetic field over the total area on the $z = 15$ mm plane of the Terfenol-D layer is measured using a Tesla meter (see Figure 3a). The Hall probe is touched on the edge of the Terfenol-D layer, and this setup allows a precision of induced magnetic field measurements of $\pm 0.01$ mT. The induced voltage is then measured using an oscilloscope (see Figure 3b).

3. Analysis

3.1. Basic equations

The basic equations for magnetostrictive and piezoelectric materials are outlined here. Consider the rectangular Cartesian coordinate system $O-x_1x_2x_3$. The equilibrium equations are given by

\begin{align}
\sigma_{ji,j} &= 0 \\
B_{i,j} &= 0 \\
D_{i,j} &= 0
\end{align}

Figure 3. Experimental setup for measuring (a) the induced magnetic field and (b) the induced voltage.
where $\sigma_{ij}$ is the stress tensor, $B_i$ is the magnetic induction vector, $D_i$ is the electric displacement vector, a comma followed by an index denotes partial differentiation with respect to the space coordinate $x_i$, and the summation convention for repeated tensor indices is applied. The constitutive laws are given as follows:

$$\varepsilon_{ij} = s^H_{ijkl}\sigma_{kl} + d'_{kij}H_k$$

(4)

$$B_i = d'_{kij}\sigma_{kl} + \mu_{ij}H_k$$

(5)

for the magnetostrictive material, and

$$\varepsilon_{ij} = s^E_{ijkl}\sigma_{kl} + d_{kij}E_k$$

(6)

$$D_i = d_{kij}\sigma_{kl} + \varepsilon^T_{ij}E_k$$

(7)

for the piezoelectric material. Here, $\varepsilon_{ij}$ is the strain tensor, $H_i$ is the magnetic field intensity vector, $E_i$ is the electric field intensity vector, $s^H_{ijkl}, d'_{kij}, \mu_{ij}$ are the constant magnetic field elastic compliance, magnetoelastic constant and magnetic permittivity of the magnetostrictive material, and $s^E_{ijkl}, d_{kij}, \varepsilon^T_{ij}$ are the constant electric field elastic compliance, direct piezoelectric constant and dielectric permittivity of the piezoelectric material. In the case of a magnetostrictive/piezoelectric laminate under bending, the constitutive laws are naturally expressed in strain (magnetic induction, electric displacement)–stress form. For each material, the magnetic and electric fields are not coupled. Valid symmetry conditions for the material constants are

$$s^H_{ijkl} = s^H_{ijkl} = s^H_{klij} = s^H_{klij}, \quad d'_{kij} = d'_{kij}, \quad \mu_{ij} = \mu_{ji}$$

(8)

$$s^E_{ijkl} = s^E_{ijkl} = s^E_{klij} = s^E_{klij}, \quad d_{kij} = d_{kij}, \quad \varepsilon^T_{ij} = \varepsilon^T_{ji}.$$  

(9)

The relation between the strain tensor and the displacement vector $u_i$ is given by

$$\varepsilon_{ij} = \frac{1}{2}(u_{ij} + u_{ji}).$$

(10)

The magnetic and electric field intensities are written as

$$H_i = \varphi_{,i}$$

(11)

$$E_i = -\phi_{,i}$$

(12)

where $\varphi$ and $\phi$ are the magnetic and electric potentials, respectively.

3.2. Model

A three-dimensional finite element model is created in order to calculate the deflection, stresses, induced magnetic field and induced voltage for the laminates. Figure 4 shows the finite element mesh and boundary conditions. Let the coordinate axes $y = x_2$ and $z = x_3$ be chosen such that they coincide with the interface plane of the laminate and let the $x = x_1$
Figure 4. Finite element model of the Terfenol-D/PZT laminate: (a) top view and (b) side view.

axis be perpendicular to this plane. The geometry of the finite element model is defined by the geometry of the specimen. The easy axis of the magnetization of Terfenol-D and the poling axis of PZT are in the \( z \) - and \( x \) -directions, respectively. The constitutive relations for the Terfenol-D and PZT layers are given in Appendix A.

As we know, nonlinearity of magnetostriction versus magnetic field curves arises from the rotation of magnetic domains [8]. Magnetic domain switching gives rise to the changes of the magnetoelastic constants, and the constants \( d'_{15}, d'_{31} \) and \( d'_{33} \) for the Terfenol-D layer under a uniform magnetic field of magnetic induction \( B_z = B_0 \) are

\[
\begin{align*}
   d'_{15} &= d''_{15} \\
   d'_{31} &= d''_{31} + m_{31} H_z \\
   d'_{33} &= d''_{33} + m_{33} H_z
\end{align*}
\]

where \( d''_{15}, d''_{31}, d''_{33} \) are the piezo-magnetic constants, and \( m_{31} \) and \( m_{33} \) are the second-order magnetoelastic constants. When the length of Terfenol-D is much longer than the other two sizes (width and thickness) and a magnetic field is along the length direction (easy axis), the longitudinal (33) magnetostrictive deformation mode is dominant. So it is assumed that only the constant \( d'_{33} \) varies with magnetic field \( H_z \), and the constant \( m_{31} \) equals zero. The constant \( m_{33} \) can predict well the nonlinearity, without complex computation and more parameters [9].

More realistic material models are much more complicated, see e.g. [10], but the present formulation is able to capture the nonlinear effect, which is a remarkable feature of this model. Coupled-field solid elements in ANSYS were used in the analysis. A mechanical load \( P \) was produced by the application of a uniformly distributed load per unit width at \( z = l_m/2 (x = -h_m, -w_m/2 \leq y \leq w_m/2) \). The average induced magnetic field \( B_{in} \) in the
Table 1. Material properties of Terfenol-D.

|                  | Elastic compliance ($\times 10^{-12} m^2/N$) | Piezo-magnetic constant ($\times 10^{-9} m/A$) | Magnetic permittivity ($\times 10^{-6} H/m$) |
|------------------|-----------------------------------------------|-----------------------------------------------|---------------------------------------------|
|                  | $s_{11}^H$ | $s_{33}^H$ | $s_{44}^H$ | $s_{12}^H$ | $d_{31}^m$ | $d_{33}^m$ | $d_{15}^m$ | $\mu_{11}$ | $\mu_{33}$ |
| Terfenol-D       | 17.9       | 17.9       | 26.3       | -5.88      | -5.88      | -5.3       | 11         | 28         | 6.29       | 6.29       |

Table 2. Material properties of C-91.

|                  | Elastic compliance ($\times 10^{-12} m^2/N$) | Direct piezoelectric constant ($\times 10^{-12} m/V$) | Dielectric permittivity ($\times 10^{-10} C/Vm$) |
|------------------|-----------------------------------------------|------------------------------------------------------|-----------------------------------------------|
|                  | $s_{11}^E$ | $s_{33}^E$ | $s_{44}^E$ | $s_{12}^E$ | $s_{13}^E$ | $d_{31}$ | $d_{33}$ | $d_{15}$ | $\varepsilon_{11}^T$ | $\varepsilon_{33}^T$ |
| C-91             | 17.1       | 18.6       | 41.4       | -6.3       | -7.3       | -340     | 645       | 836       | 395         | 490         |

$z$-direction at the side surface (at the $z = l_m$ plane) and the induced voltage $V_{in}$ at the interface (at the $x = 0$ mm plane) between the Terfenol-D and PZT layers were calculated. $B_{in}$ is given by

$$B_{in} = \frac{1}{A} \int_A B_z(x, y, l_m) dA$$

(14)

where the integration is over the surface area, $A = w_m h_m$, of the Terfenol-D layer. The model was meshed using an eight-node element. A finite element formulation is presented in Appendix B. In total, 12 000, 21 600 elements and 12 810, 24 339 nodes were used for $h_m = 1, 3$ mm, respectively. It should be noted that before carrying out the simulations, a mesh sensitivity study was performed to ensure that the mesh was fine enough. The finite element computations were provided by modifying the program with routines developed by our previous work [11]. Material properties of Terfenol-D [12,13] and C-91 [14] are listed in Tables 1 and 2, and the constants $m_{33}$ of the Terfenol-D layer with $h_m = 1, 3$ mm are $5.0 \times 10^{-12}, 3.3 \times 10^{-12} m^2/A^2$, respectively [15].

4. Results and discussion

Figure 5 shows the load-point displacement $w_{max}$ versus applied load $P$ for the Terfenol-D/PZT laminate with $h_m = 1$ mm. Also shown is the average induced magnetic field $B_{in}$. The lines and plots denote the results of nonlinear finite element analysis (FEA) and test, respectively. As the applied load increases, both the load-point displacement and the average induced magnetic field increase. The comparison between the FEA and test is reasonable. Similar phenomena for the load-point displacement and induced magnetic field for $h_m = 3$ mm are also observed in Figure 6. Only one datum is plotted due to the accuracy limit of the Tesla meter. As the thickness of the Terfenol-D layers increases, both the load-point displacement and the average induced magnetic field decrease. Figure 7 shows the induced voltage $V_{in}$ versus applied load $P$ for the Terfenol-D/PZT laminates with $h_m = 1$,
3 mm, obtained from the FEA and test. As the applied load increases, the induced voltage increases. The induced voltage also increases as the thickness of the Terfenol-D layers decreases. It can be seen that the trend is sufficiently similar between analysis and test. The measured induced voltage for $h_m = 1$ mm is smaller than the simulated one, due to uncertainty of the assembly alignment. The results show that the induced voltage is capable of monitoring for small loading conditions. On the other hand, it is noted that the induced magnetic field is capable of monitoring when a large load is applied (see Figures 5 and 6).

The variations of normal stress $\sigma_{zz}$ along the thickness direction are calculated at the center ($y = 0$ mm and $z = 7.5$ mm) for the Terfenol-D/PZT laminates and the results are shown in Figure 8. All calculations were done at a load-point displacement of $w_{\text{max}} = 1 \mu\text{m}$. The applied loads at $w_{\text{max}} = 1 \mu\text{m}$ are about $P = 4.9, 28$ N for $h_m = 1, 3$ mm and the corresponding average induced magnetic fields are about $B_{\text{in}} = 0.048, 0.041$ mT for $h_m = 1, 3$ mm, respectively. There is a stress gap at the interface between the Terfenol-D and PZT layers for $h_m = 1$ mm. At $h_m = 3$ mm, high normal stress is found for the same displacement. Figure 9 shows the variations of shear stress $\sigma_{xz}$ along the length direction at
Figure 7. Induced voltage versus applied load for the Terfenol-D/PZT laminates.

Figure 8. Normal stress distribution along the thickness direction at $y = 0$ mm and $z = 7.5$ mm for the Terfenol-D/PZT laminates.

$x = y = 0$ mm for the Terfenol-D/PZT laminates at $w_{\text{max}} = 1$ µm. Owing to symmetry, only a half of the specimen was presented. High shear stress is observed near the center for $h_m = 1$ mm. On the other hand, for $h_m = 3$ mm, the shear stress exhibits a peak at $z \approx 13$ mm. This is because the shear stress at the interface is affected by the load-point for $h_m = 1$ mm, while the shear stress at the interface is affected by the support for $h_m = 3$ mm. Low shear stress is noted for small Terfenol-D layer thickness at the same displacement. It seems that the loading condition and the presence of supports create the most suitable stress conditions for the initiation of cracking. This situation may be applicable to the other magnetostrictive/piezoelectric laminates with different size and shape. The results for the evaluation of stresses may help magnetostrictive/piezoelectric laminates designers to estimate the fracture risk and to optimize in-service loading conditions.

Figure 10 plots the normal stress $\sigma_{zz}$ versus applied load $P$ at a chosen point ($x = 1$ mm, $y = 0$ mm and $z = 7.5$ mm here) of Terfenol-D/PZT laminates with $h_m = 1, 3$ mm, obtained from the FEA. Also shown is the calculated average induced magnetic field $B_{\text{in}}$ for the Terfenol-D/PZT laminates with $h_m = 1, 3$ mm. As the applied load increases, the normal stress and average induced magnetic field increase linearly and nonlinearly, respectively.
Figure 9. Shear stress distribution along the length direction at $x = 0\,\text{mm}$ and $y = 0\,\text{mm}$ for the Terfenol-D/PZT laminates.

Figure 10. Normal stress (at $x = 1\,\text{mm}$, $y = 0\,\text{mm}$, $z = 7.5\,\text{mm}$) and the average induced magnetic field versus applied load for the Terfenol-D/PZT laminates.

The results show that the induced magnetic field is capable of self-monitoring of internal stresses.

5. Conclusions
A numerical and experimental investigation of magnetostrictive/piezoelectric laminates under three-point bending was conducted. The displacement, induced magnetic field and induced voltage are predicted using finite element simulations, and comparison with the measured data shows that current predictions are reasonable. It was found that a smaller magnetostrictive layer thickness gives smaller internal stresses at the same displacement. Also, the induced magnetic field and voltage depended on the magnetostrictive layer thickness and were correlated with the load-point displacement and internal stresses. This study may be useful in designing advanced magnetostrictive/piezoelectric laminates with monitoring capabilities.

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Appendix A

For Terfenol-D, the constitutive relations can be written in the following form:

\[
\begin{align*}
\{\epsilon_{xx} & \epsilon_{yy} \epsilon_{zz} \epsilon_{yz} \epsilon_{zx} \epsilon_{xy}\} = \\
\{\frac{s_{11}H}{d_{31}} \frac{s_{12}H}{d_{31}} \frac{s_{13}H}{d_{31}} 0 0 0 \} & + \\
0 0 0 \frac{s_{44}H}{d_{15}/2} 0 0 & + \\
0 0 0 0 \frac{s_{44}H}{d_{15}/2} \frac{s_{66}H}{d_{15}/2} & + \\
\{\sigma_{xx} & \sigma_{yy} \sigma_{zz} \sigma_{yz} \sigma_{zx} \sigma_{xy}\} \left[ \begin{array}{c}
H_x \\
H_y \\
H_z
\end{array} \right]
\end{align*}
\]

(A.1)

\[
\begin{align*}
\{B_x & B_y B_z\} = \\
\{0 0 0 0 d_{15}' & 0 \} & + \\
0 0 0 d_{15}' 0 0 & + \\
d_{31}' d_{31}' d_{33}' 0 0 & + \\
\{\sigma_{xx} & \sigma_{yy} \sigma_{zz} \sigma_{yz} \sigma_{zx} \sigma_{xy}\} \left[ \begin{array}{c}
\mu_{11} 0 0 0 \\
0 \mu_{11} 0 0 & + \\
0 0 \mu_{33} & + \\
\end{array} \right] \left[ \begin{array}{c}
H_x \\
H_y \\
H_z
\end{array} \right]
\end{align*}
\]

(A.2)
where

\[ s_{11}^H = s_{1111}^H = s_{2222}^H, \quad s_{12}^H = s_{1122}^H, \quad s_{13}^H = s_{1133}^H = s_{2233}^H, \quad s_{33}^H = s_{3333}^H \]

\[ s_{44}^H = 4s_{2323}^H = 4s_{3131}^H, \quad s_{66}^H = 4s_{1212}^H = 2(s_{11}^H - s_{12}^H) \]  

(A.3)

\[ d_{15}' = 2d_{3131}' = 2d_{223}' = d_{31}' = d_{322}' = d_{33}' = d_{333}' \]  

(A.4)

The constitutive relations for PZT (hexagonal crystal of class 6mm) are

\[
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\varepsilon_{yz} \\
\varepsilon_{zx} \\
\varepsilon_{xy}
\end{bmatrix} =
\begin{bmatrix}
s_{E11} & s_{E12} & s_{E13} & 0 & 0 & 0 \\
s_{E12} & s_{E22} & s_{E23} & 0 & 0 & 0 \\
s_{E13} & s_{E23} & s_{E33} & 0 & 0 & 0 \\
0 & 0 & 0 & s_{E66}/2 & 0 & 0 \\
0 & 0 & 0 & 0 & s_{E44}/2 & 0 \\
0 & 0 & 0 & 0 & 0 & s_{E44}/2
\end{bmatrix}
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\sigma_{yz} \\
\sigma_{zx} \\
\sigma_{xy}
\end{bmatrix}
+ \begin{bmatrix}
d_{33} \\
d_{31} \\
d_{31} \\
d_{15}/2 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
E_x \\
E_y \\
E_z
\end{bmatrix}
\]  

(A.5)

\[
\begin{bmatrix}
D_x \\
D_y \\
D_z
\end{bmatrix} =
\begin{bmatrix}
d_{33} & d_{31} & d_{31} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_{15}/2 \\
0 & 0 & 0 & 0 & 0 & d_{111}
\end{bmatrix}
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\sigma_{yz} \\
\sigma_{zx} \\
\sigma_{xy}
\end{bmatrix}
+ \begin{bmatrix}
e_{33}^T \\
e_{11}^T \\
e_{11}^T \\
e_{11}^T \\
e_{11}^T \\
e_{11}^T
\end{bmatrix}
\begin{bmatrix}
E_x \\
E_y \\
E_z
\end{bmatrix}
\]  

(A.6)

where

\[ s_{E11} = s_{E22} = s_{E33}, \quad s_{E12} = s_{E23} = s_{E13} = s_{E1122}, \quad s_{E33} = s_{E1133}, \quad s_{E66} = 2s_{E11} - s_{E12} \]

(A.7)

\[ d_{15} = 2d_{3131} = 2d_{223} = d_{31} = d_{3122} = d_{333} = d_{111} \]  

(A.8)

**Appendix B**

The displacements, and electric and magnetic potentials are

\[
\begin{bmatrix}
u_x \\
u_y \\
u_z
\end{bmatrix} =
\begin{bmatrix}
N_1 & 0 & 0 & 0 & N_8 & 0 & 0 \\
0 & N_1 & 0 & \ldots & 0 & N_8 & 0 \\
0 & 0 & N_1 & \ldots & 0 & 0 & N_8
\end{bmatrix}
\begin{bmatrix}
u
\end{bmatrix}
\]

(B.1)

\[
\phi = (N_1N_2 \ldots N_8)v
\]

(B.2)

\[
\psi = (N_1N_2 \ldots N_8)w
\]

(B.3)

where \{\nu\} is the vector of nodal displacements, \{v\} and \{w\} are the vectors of nodal electric and magnetic potentials, and

\[
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix} =
\begin{bmatrix}
u_1 & u_2 & u_3 & \ldots & u_8 & u_{13} & u_{14} & \ldots & u_{18} & u_{23} & u_{24} & \ldots & u_{28} & u_{33} & \ldots & u_{38} & u_{48} & \ldots & u_{48}
\end{bmatrix}^T
\]

(B.4)

\[
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix} =
\begin{bmatrix}
\phi_1 & \phi_2 & \ldots & \phi_8
\end{bmatrix}^T
\]

(B.5)

\[
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix} =
\begin{bmatrix}
\psi_1 & \psi_2 & \ldots & \psi_8
\end{bmatrix}^T
\]

(B.6)
In Equations (B.1)–(B.3), \( N_i \) for \( i = 1, \ldots, 8 \) are the shape functions for node \( i \). As usual a topological transformation from the physical volume of element \((x, y, z)\) to the natural volume of element \((s, t, r)\) is accomplished, \( s, t, r \) being the local, non-dimensional nodal coordinates with origin at the center of the element, and the shape functions are

\[
\begin{align*}
N_1 &= \frac{1}{8}(1-s)(1-t)(1-r), \quad N_2 = \frac{1}{8}(1+s)(1-t)(1-r) \\
N_3 &= \frac{1}{8}(1+s)(1+t)(1-r), \quad N_4 = \frac{1}{8}(1-s)(1+t)(1-r) \\
N_5 &= \frac{1}{8}(1-s)(1-t)(1+r), \quad N_6 = \frac{1}{8}(1+s)(1-t)(1+r) \\
N_7 &= \frac{1}{8}(1+s)(1+t)(1+r), \quad N_8 = \frac{1}{8}(1-s)(1+t)(1+r)
\end{align*}
\]

The coupled finite element matrix equations are

\[
\begin{bmatrix}
[ K_{cp} ] & [ K_\varepsilon ] \\
[ K_{ec} ] & [ K_\varepsilon ]^T
\end{bmatrix}
\begin{bmatrix}
\{ u \} \\
\{ v \}
\end{bmatrix} = 0 \quad (B.8)
\]

\[
\begin{bmatrix}
[ K_{cm} ] & [ K_\mu ] \\
[ K_q ] & [ K_\mu ]^T
\end{bmatrix}
\begin{bmatrix}
\{ u \} \\
\{ w \}
\end{bmatrix} = 0. \quad (B.9)
\]

The various stiffness matrices in Equations (B.8) and (B.9) are defined as shown below:

\[
[K_{cp}] = \int_V [B_u]^T [\varepsilon] [B_u] dV \quad (B.10)
\]

\[
[K_\varepsilon] = - \int_V [B_r]^T [\varepsilon] [B_r] dV \quad (B.11)
\]

\[
[K_\varepsilon] = \int_V [B_u]^T [\varepsilon] [B_r] dV \quad (B.12)
\]

\[
[K_{cm}] = \int_V [B_u]^T [\varepsilon^m] [B_u] dV \quad (B.13)
\]

\[
[K_\mu] = - \int_V [B_u]^T [\mu] [B_u] dV \quad (B.14)
\]

\[
[K_q] = \int_V [B_u]^T [q] [B_u] dV \quad (B.15)
\]
where $V$ is the volume of element, and

$$\varepsilon = \begin{bmatrix} \varepsilon_{T_{33}} & 0 & 0 \\ 0 & \varepsilon_{T_{11}} & 0 \\ 0 & 0 & \varepsilon_{T_{11}} \end{bmatrix} - \begin{bmatrix} d_{33} & d_{31} & d_{31} & 0 & 0 & 0 & d_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 & d_{15} \end{bmatrix}$$

(B.16)

$$\varepsilon = \begin{bmatrix} \varepsilon_{E_{33}} & \varepsilon_{E_{13}} & \varepsilon_{E_{13}} & 0 & 0 & 0 \\ \varepsilon_{E_{13}} & \varepsilon_{E_{12}} & \varepsilon_{E_{12}} & 0 & 0 & 0 \\ \varepsilon_{E_{13}} & \varepsilon_{E_{11}} & \varepsilon_{E_{11}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_{E_{66}/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_{E_{44}/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon_{E_{44}/2} \end{bmatrix} - \begin{bmatrix} d_{33} & 0 & 0 \\ 0 & d_{31} & 0 \\ 0 & d_{31} & 0 \\ 0 & 0 & d_{15}/2 \\ 0 & 0 & d_{15}/2 \\ 0 & 0 & d_{15}/2 \end{bmatrix}$$

(B.17)

$$\varepsilon = \begin{bmatrix} d_{33} & d_{31} & d_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{15} \\ 0 & 0 & 0 & 0 & d_{15} & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{E_{33}} & \varepsilon_{E_{13}} & \varepsilon_{E_{13}} & 0 & 0 & 0 \\ \varepsilon_{E_{13}} & \varepsilon_{E_{12}} & \varepsilon_{E_{12}} & 0 & 0 & 0 \\ \varepsilon_{E_{13}} & \varepsilon_{E_{11}} & \varepsilon_{E_{11}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_{E_{66}/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_{E_{44}/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon_{E_{44}/2} \end{bmatrix}^{-1}$$

(B.18)

$$\varepsilon = \begin{bmatrix} s_{H_{11}} & s_{H_{12}} & s_{H_{13}} & 0 & 0 & 0 \\ s_{H_{12}} & s_{H_{11}} & s_{H_{13}} & 0 & 0 & 0 \\ s_{H_{13}} & s_{H_{11}} & s_{H_{13}} & 0 & 0 & 0 \\ s_{H_{13}} & s_{H_{13}} & s_{H_{33}} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{H_{44}/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{H_{44}/2} & 0 \end{bmatrix} - \begin{bmatrix} d_{33} & d_{31} & d_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & d_{31} & d_{31} & 0 & 0 & 0 \end{bmatrix}$$

(B.19)
\[
\begin{bmatrix}
  s_{11}^H & s_{12}^H & s_{13}^H & 0 & 0 & 0 \\
  s_{12}^H & s_{11}^H & s_{13}^H & 0 & 0 & 0 \\
  s_{13}^H & s_{12}^H & s_{33}^H & 0 & 0 & 0 \\
  0 & 0 & 0 & s_{44}^H/2 & 0 & 0 \\
  0 & 0 & 0 & 0 & s_{44}^H/2 & 0 \\
  0 & 0 & 0 & 0 & 0 & s_{66}^H/2
\end{bmatrix}
\times
\begin{bmatrix}
  0 & 0 & d_{31}^e \\
  0 & 0 & d_{31}^e \\
  0 & 0 & d_{33}^e \\
  0 & d_{15}^e/2 & 0 \\
  d_{15}^e/2 & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix}^{-1} (B.20)
\]

\[
[q] =
\begin{bmatrix}
  0 & 0 & 0 & d_{15}^e & 0 \\
  0 & 0 & 0 & d_{15}^e & 0 \\
  d_{31}^e & d_{31}^e & d_{33}^e & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  s_{11}^H & s_{12}^H & s_{13}^H & 0 & 0 & 0 \\
  s_{12}^H & s_{11}^H & s_{13}^H & 0 & 0 & 0 \\
  s_{13}^H & s_{12}^H & s_{33}^H & 0 & 0 & 0 \\
  0 & 0 & 0 & s_{44}^H/2 & 0 & 0 \\
  0 & 0 & 0 & 0 & s_{44}^H/2 & 0 \\
  0 & 0 & 0 & 0 & 0 & s_{66}^H/2
\end{bmatrix}^{-1} (B.21)
\]

\[
[B_u] =
\begin{bmatrix}
  \frac{\partial}{\partial x} & 0 & 0 \\
  0 & \frac{\partial}{\partial y} & 0 \\
  0 & 0 & \frac{\partial}{\partial z} \\
  \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\
  0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\
  \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}
\end{bmatrix}
\begin{bmatrix}
  N_1 & 0 & 0 & N_8 & 0 & 0 \\
  0 & N_1 & 0 & \ldots & 0 & N_8 \\
  0 & 0 & N_1 & \ldots & 0 & N_8
\end{bmatrix} (B.22)
\]

\[
[B_v] = [B_w] = \begin{bmatrix}
  \frac{\partial}{\partial x} \\
  \frac{\partial}{\partial y} \\
  \frac{\partial}{\partial z}
\end{bmatrix}
(N_1N_2\ldots N_8). (B.23)
\]