MINIMAL SYMPLECTICATALASES OF HERMITIAN SYMMETRIC SPACES

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ABSTRACT. In this paper we compute the minimal number of Darboux chart needed to cover a Hermitian symmetric space of compact type $M$ in terms of the degree of its Borel-Weil embedding in $\mathbb{C}P^N$. The proof is based on the recent work of Y. B. Rudyak and F. Schlenk [18] and on the symplectic geometry tool developed by the first author in collaboration with A. Loi and F. Zuddas [12]. As application we compute this number for a large class of Hermitian symmetric spaces of compact type.

1. Introduction and statements of the main results

Consider the open ball of radius $r$,

$$B^{2n}(r) = \{(x, y) \in \mathbb{R}^{2n} \mid \sum_{j=1}^{n} x_j^2 + y_j^2 < r^2\}$$

in the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$, where $\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$. In [18] Y. B. Rudyak and F. Schlenk introduced the invariant $S_B(M, \omega)$ for a closed symplectic manifold $(M, \omega)$ defined by:

$$S_B(M, \omega) := \min\{k \mid M = B_1 \cup \cdots \cup B_k\},$$

where $B_j$ is the image of a Darboux chart $\varphi(B^{2n}(r_j)) \subset M$. This is the minimal number of symplectic charts needed to cover $(M, \omega)$. The problem of estimating this number is closely related to two other problems, namely computing the Gromov width $c_G(M, \omega)$ and the Lusternik-Schnirelmann category $\text{cat}(M)$ of $M$. While the latter can be often computed or estimated very well, computing the former is an open and delicate matter. The Gromov width of a $2n$-dimensional symplectic manifold $(M, \omega)$, introduced in [19], is defined as

$$c_G(M, \omega) = \sup \{\pi r^2 \mid \exists \varphi : (B^{2n}(r), \omega_0) \to (M, \omega)\}$$

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where $\varphi$ is a symplectic embedding. By Darboux’s theorem $c_G(M, \omega)$ is a positive number or $\infty$. Computations and estimates of the Gromov width for various examples can be found in [1, 2, 3, 4, 6, 7, 9, 10, 12, 13, 14, 15, 16, 17, 19, 21].

We adopt the following notation (as in [12]):

**Notation:** From now on we shall use the shortening HSSCT to denote a Hermitian symmetric space of compact type. Further, throughout the paper we shall denote by $\omega_{FS}(B) \in \{-\pi, \pi\}$ when $B$ is a generator of $H_2(M, \mathbb{Z})$, and by $A$ the generator for which $\omega_{FS}(A) = \pi$.

The following theorem and its two corollaries are the main results of this paper.

**Theorem 1.** Let $(M, \omega_{FS})$ be a $2n$-dimensional HSSCT and let $f : M \hookrightarrow \mathbb{C}P^N$ be the Borel-Weil embedding of $M$. Then

(i) If $\deg(f) \geq 2n$, then $S_B(M, \omega_{FS}) = \deg(f) + 1$

(ii) If $\deg(f) < 2n$, then $n + 1 \leq S_B(M, \omega_{FS}) \leq 2n + 1$,

where the Borel-Weil embedding is a full Kähler embedding of $(M, \omega_{FS})$ in $(\mathbb{C}P^N, \omega_{FS})$ (see [5, Section 2.4]). Notice that when $M$ is the complex Grassmannian then $f$ is the Plücker embedding. The proof is based on the results obtained by Y. B. Rudyak and F. Schlenk in [18] about minimal atlas for compact symplectic manifolds together with the explicit computation of the Gromov width given by the first author in collaboration with A. Loi and F. Zuddas in [12] and the properties of the symplectic duality map introduced by A. J. Di Scala and A. Loi in [5] which, in particular, give us a symplectic embedding of the noncompact dual $(\Omega, \omega_0)$ of $(M, \omega_{FS})$ into $(M, \omega_{FS})$.

Using the explicit computation of the volume of a classical domain $(\Omega, \omega_0)$ given by L. K. Hua in [3], we are able to prove the following corollary, which extends the computation of $S_B$ for the Grassmannians given in [18] to any classical irreducible HSSCT.

**Corollary 2.** Let $(M, \omega_{FS})$ be an irreducible HSSCT of dimension $2n$. If $M$ is of type I, II or III with rank$(M) \geq 2$ and $n$ is sufficiently large, then

$$S_B(M, \omega_{FS}) = \deg(f) + 1$$

where $f : M \hookrightarrow \mathbb{C}P^N$ is the Borel-Weil embedding of $M$. Otherwise, we have

$$n + 1 \leq S_B(M, \omega_{FS}) \leq 2n + 1.$$
In the rank one case (i.e. $M = \mathbb{C}P^n$), the degree of the associated Borel-Weil embedding is $\deg(f) = 1$ and [18, Corollary 5.8] says us that $S_B(\mathbb{C}P^n, \omega_{FS}) = n + 1$.

The next corollary, which represent the last result of the paper, is a straightforward consequence of Theorem 1.

**Corollary 3.** Let $(M_1 \times M_2, \omega_{FS})$ be a product of HSSCT of dimension $2n$. If $M_1 \times M_2$ is different from $\mathbb{C}P^1 \times \mathbb{C}P^n$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$, then

$$S_B(M_1 \times M_2, \omega_{FS}) = \deg(f) + 1,$$

where $f : M_1 \times M_2 \hookrightarrow \mathbb{C}P^N$ is the Borel-Weil embedding. Otherwise, we have

$$n + 1 \leq S_B(M, \omega_{FS}) \leq 2n + 1.$$

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## 2. The proofs of Theorem 1, Corollary 2 and Corollary 3

Consider the following lower bound for $S_B(M, \omega)$ given by

$$\Gamma(M, \omega) := \left\lfloor \frac{Vol(M, \omega)n!}{c_G(M, \omega)^n} \right\rfloor + 1,$$

where $\lfloor x \rfloor$ denote the maximal integer smaller than or equal to $x$. The following theorem summarizes the results about minimal atlases obtained in [18] that we need in the proof of Theorem 1.

**Theorem A** (Rudyak–Schlenk [18]). Let $(M, \omega)$ be a compact connected $2n$-dimensional symplectic manifold.

i) If $\Gamma(M, \omega) \geq 2n + 1$, then $S_B(M, \omega) = \Gamma(M, \omega)$.

ii) If $\Gamma(M, \omega) < 2n + 1$ then $n + 1 \leq S_B(M, \omega) \leq 2n + 1$.

### 2.1. Proof of Theorem 1

The proof follows from Theorem A once one observes that the volume of any $n$-dimensional projective variety $X$, with holomorphic embedding $f : X \hookrightarrow \mathbb{C}P^N$, is given by

$$Vol(X, \omega_{FS}) = \deg(f)Vol(\mathbb{C}P^n, \omega_{FS}),$$

(1)

$Vol(\mathbb{C}P^n) = \frac{\pi^n}{n!}$ and that the Gromov width of any HSSNCT (see [12]) is given by $c_G(M, \omega_{FS}) = \pi$. 

2.2. **Proof of Corollary 2** Consider \((\Omega, \omega_0)\), the noncompact dual of \((M, \omega_{FS})\). In [5, Theorem 1.1] it is proved the existence of a global symplectomorphism
\[
\Phi : (\Omega, \omega_0) \rightarrow (M \setminus \text{Cut}_p(M), \omega_{FS})
\]
where \(\text{Cut}_p(M)\) is the cut locus of \((M, \omega_{FS})\) with respect to a fixed point \(p \in M\) (see also [11]). Thus \(\text{Vol}(M, \omega_{FS}) = \text{Vol}(\Omega, \omega_0)\). On the other hand the explicit expression of the volume \(\text{Vol}(\Omega, \omega_0)\) can be found in L. K. Hua [8] and by (1) we are able to write the expression of \(\text{deg}(f)\) associated to any classical HSSCT, as follows.

Let \(I_{k,s}\) be a HSSCT of type I, namely the Grassmannian of \(k\)-planes in \(\mathbb{C}^s\). Notice that the dimension is \(2n = 2(s-k)k\) and that \(\text{rank}(I_{k,s}) = k\). We have that
\[
\text{deg}(f_{k,s}) = \frac{\text{Vol}(I_{k,s}, \omega_{FS})}{\text{Vol}(\mathbb{C}P^{(s-k)k}, \omega_{FS})} = \frac{1! 2! \ldots (s-k-1)! 2! \ldots (k-1)! ((s-k)k)!}{1! 2! \ldots (s-1)!}.
\]
In order to apply Theorem 1 we study when \(\text{deg}(f_{k,s}) \geq 2(s-k)k\), or equivalently when
\[
\frac{\text{deg}(f_{k,s})}{(s-k)k} \geq 2.
\]
One can see that the previous inequality is satisfied for \((k, s) = (2, 7)\) and for \((k, s) = (k, 2k)\) when \(k \geq 3\).

Therefore Corollary 2 is proved, when \(M = I_{k,s}\), once observed that for \(k \geq 2\) we have
\[
\frac{\text{deg}(f_{k,s+1})}{(s+1-k)k} \geq \frac{\text{deg}(f_{k,s})}{(s-k)k}.
\]

When \(M\) is of the second or third type the proof follows the same arguments. The degree of the Borel-Weil embedding \(f_{II}\) and \(f_{III}\) associated to an irreducible HSSCT of the second and the third type are given by:

- Let \(II_s\) be an irreducible HSSCT of the second type. The dimension and the rank are given by \(n = \frac{(s-1)s}{2}\) and \(\text{rank}(II_s) = \left\lceil \frac{s}{2} \right\rceil\), \(s \geq 5\). We have,
\[
\text{deg}(f_{II}) = \frac{s(s+1)}{2} \frac{2! \ldots (2s-2)!}{s! (s+1)! (s+2)! \ldots (2s-1)!}.
\]

- Let \(III_s\) be an irreducible HSSCT of the third type. The dimension and the rank are given by \(n = \frac{(s+1)s}{2}\) and \(\text{rank}(III_s) = s\), \(s \geq 2\). We have,
\[
\text{deg}(f_{III}) = \frac{s(s-1)}{2} \frac{2! \ldots (2s-4)!}{(s-1)! s! \ldots (2s-3)!}.
\]

Finally, since the degree of the Borel-Weil embedding of an irreducible HSSCT of the fourth type (namely the quadric) is always 2 and \(n \geq 3\), the conclusion of the proof of Corollary 2 follows by (ii) of Theorem 1.
2.3. Proof of Corollary 3. Let $\omega^1_{FS}$ and $\omega^2_{FS}$ be the Fubini-Study forms associated to $M_1$ and $M_2$. Since the associated volume form satisfies (with abuse of notation) $v_{\omega_{FS}} = v_{\omega^1_{FS}} \wedge v_{\omega^2_{FS}}$, we have $\text{Vol}(M_1 \times M_2) = \text{Vol}(M_1)\text{Vol}(M_2)$. By (1) we get:

$$\text{deg}(f) = \frac{(n_1 + n_2)!}{n_1!n_2!} \text{deg}(f_1)\text{deg}(f_2),$$

where $n_j$ is the dimension of $M_j$, $j = 1, 2$ and $f_1$ and $f_2$ are the Borel-Weil embedding of $M_1 \times M_2$, $M_1$ and $M_2$. In order to apply (i) of Theorem 1, we have to check when

$$\text{deg}(f_1)\text{deg}(f_2) \left(\frac{(n_1 + n_2 - 1)!}{n_1!n_2!}\right) \geq 2. \tag{2}$$

First notice that when $\text{deg}(f_1) \geq 2$ or $\text{deg}(f_2) \geq 2$, since $\frac{(n_1 + n_2 - 1)!}{n_1!n_2!} \geq 1$, the inequality (2) is satisfied. Finally, when $\text{deg}(f_1) = \text{deg}(f_2) = 1$ is easy to see that (2) is satisfies if and only if $n_1 \geq 3$ and $n_2 \geq 2$ or $n_1 \geq 2$ and $n_2 \geq 3$. The proof is complete.

Remark 4. When $M = \mathbb{C}P^1 \times \mathbb{C}P^n, \mathbb{C}P^2 \times \mathbb{C}P^2$ we are not able to compute $S_B(M, \omega_{FS})$. Even for the simple case of $\mathbb{C}P^1 \times \mathbb{C}P^1$ we know (private communication with F. Schlenk) that one can construct a covering of 4 symplectic balls but we still do not know if this number can be reduced to 3.

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