Congruences for Stochastic Automata

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Abstract

Congruences for stochastic automata are defined, the corresponding factor automata are constructed and investigated for automata over analytic spaces. We study the behavior under finite and infinite streams. Congruences consist of multiple parts, it is shown that factoring can be done in multiple steps, guided by these parts.

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1 Introduction

Stochastic automata [1, 4] are the natural generalization to non-deterministic Mealy automata; they take an input while being in an internal state, change their state and return an output. Both the new state and the output are distributed according to the automaton’s transition law. The basic scenario may be finite or infinite, in the infinite case one may deal with countable or uncountable carrier sets for input, outputs, and states, resp. The finite and the countably infinite case is usually dealt with through methods from linear algebra, since matrices with a finite or countable number of entries are manipulated, the uncountable case required methods from measure theory. This is so since the events an automata is assumed to handle are not all possible events, but come from Boolean σ-algebras of events (using all possible events, i.e., defining the probabilities on the respective power sets, will lead to foundational problems).

This kind of automata — without the bells and whistles one finds in later extensions — have been used, e.g., for modelling simple learning processes along a behavioral taxonomy from psychology [14, 11, 4]. In such a scenario, in which the automaton models a learner, the automaton receives inputs from the environment while being in a specific state, it makes a state transition and responds with an output. This happens in a sequential fashion. We are interested in the single-step behavior. The learning situation is characterized by the observation that equivalent inputs may lead to equivalent outputs, and that there may be equivalent states as well; note that the set of states represents an abstraction obtained through a modelling process, hence is not accessible from the outside. For conceptual clarity, and for minimizing the machine at least conceptually, one is interested in these equivalences, i.e., one wants to form equivalence classes and have the transition law respect these classes. This leads to the notion of a congruence, well known in (universal) algebra. But we must not ignore

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a slightly inconvenient fact: while a congruence, say, on a group, relates group elements to each other, an automaton congruence relates pairs of inputs and states to pairs of states and outputs, so we have a slightly heterogeneous situation at hand. One might be reminded of bisimilarity, where sets of two possibly different transition systems are related to each other.

The latter problem is resolved by introducing the notion of friendship for two equivalence relations, comparing their probabilistic behavior in a straightforward manner. This leads to a notion of congruence for automata, which is exploited by relating it to morphisms and their kernels and constructing factor automata.

An automaton works sequentially, so we study the automaton’s behavior for finite and for infinite input sequences. Here we adopt a black box point of view, hiding state changes from the outside world. This is studied first for finite sequences, then we construct a limit which permits us also to specify behavior under an infinite input stream. It turns out that friendship is a surprisingly stable relationship which can be maintained also for infinite streams.

Finally we want to know whether we can form longer chains of reduced automata, and it turns out that this is not possible: factoring a factored automaton yields an automaton which can be obtained through one-step factoring through a suitably modifies congruence. The result also enables us to reduce automata in a step wise fashion along its components.

Most of the material depends heavily on the coalgebraic approach to stochastic relations [12, 7, 5]. The present paper rests on the well-known fact that the very old problem of reducing an automaton may be solved in a more general fashion without much effort with tools from coalgebras [7, 10].

**Notation and all that**

A measurable space $(F, \mathcal{F})$ is a set $F$ together with a Boolean $\sigma$-algebra $\mathcal{F}$ of subsets of $F$. Measurable spaces form a category, taking measurable maps as morphism. A map $f : F \to H$ for the measurable spaces $(F, \mathcal{F})$ and $(H, \mathcal{H})$ is said to be $\mathcal{F}:\mathcal{H}$ measurable iff $f^{-1}[\mathcal{H}] \subseteq \mathcal{F}$, i.e., iff $f^{-1}[Q] \in \mathcal{F}$ holds for every $Q \in \mathcal{H}$; we will omit the $\sigma$-algebras from the notation of maps whenever possible. The Giry functor $\mathcal{G}$ acts as an endofunctor on this category. It assigns to each measurable space $(F, \mathcal{F})$ the set $\mathcal{G}(F, \mathcal{F})$ of all subprobabilities on $F$ equipped with the smallest $\sigma$-algebra rendering the evaluations $\mu \mapsto \mu(Q)$ for all $Q \in \mathcal{F}$ measurable.

To complete the definition of the functor $\mathcal{G}$, map the measurable map $f : F \to H$ to the measurable map $\mathcal{G}(f)$ which assigns each subprobability $\mu$ on $F$ its image $\lambda P.\mu(f^{-1}[P])$ on $\mathcal{H}$.

Assume an equivalence relation $\xi$ on the measurable space $(F, \mathcal{F})$. The map $\eta_\xi : x \mapsto [x]_\xi$ sends an element to its $\xi$-class. Denote as usual the set of $\xi$-classes by $F/\xi$. This set will be furnished with the $\sigma$-algebra $\mathcal{F}/\xi$ which is the final $\sigma$-algebra on $F/\xi$ with respect to $\mathcal{F}$ and $\eta_\xi$, thus $V \in \mathcal{F}/\xi$ iff $\eta_\xi^{-1}[V] \in \mathcal{F}$. We denote the measurable space $(F/\xi, \mathcal{F}/\xi)$ by $(F, \mathcal{F})/\xi$. $1_F$ denotes the identity relation on $F$. 


2 Stochastic Automata

A stochastic relation \( K : (X,A) \Rightarrow (Y,B) \) is a measurable map \( K : X \to \mathcal{G}(Y,B) \), thus \( K(x) \) is a subprobability measure on \( (Y,B) \) for each \( x \in X \), and the map \( x \mapsto K(x)(B) \) is \( A \)-measurable for each \( B \in B \). Actually — but inconsequentially for the present note — a stochastic relation is a Kleisli morphism for the Giry monad, the functorial part of which is the Giry functor \( \mathcal{G} \) [7][13].

Recall that the category of measurable spaces is closed under finite products: \( (X,A) \otimes (Y,B) \) has the Cartesian product \( X \times Y \) as a carrier set and \( \sigma(\{A \times B \mid A \in A, B \in B\}) =: A \otimes B \) as a \( \sigma \)-algebra. Here \( \sigma(\{\ldots\}) \) denotes the smallest \( \sigma \)-algebra on the carrier containing the generator \( \{\ldots\} \). A \( \sigma \)-algebra is countably generated iff it has a countable generator, and it separates points iff given two distinct points there is a measurable set containing exactly one of them. It is well known that countably generated, point separating \( \sigma \)-algebras are precisely the Borel sets for second countable metric spaces [15].

**Definition 2.1** A stochastic automaton \( K = ((X,A), (Y,B), (Z,C), K) \) is a stochastic relation \( K : (X \times Z, A \otimes C) \Rightarrow (Y \times C \otimes B) \).

Thus the new state and the output of \( K \) is a member of the measurable set \( D \in C \otimes B \) with probability \( K(x,z)(D) \) upon input \( x \in X \) in state \( z \in Z \). Because we work in the realm of subprobabilities, mass may get lost, so that we cannot always reckon with \( K(x,z)(Z \times Y) = 1 \). This suggests the possibility that events cannot be accounted for.

The automata may work in different environments, so different input and output spaces have to be taken into account. Morphisms are used for relating automata. Assume that we have another stochastic automaton \( K' = ((X',A'), (Y',B'), (Z',C'), K') \). A morphism \( \hat{f} : K \to K' \) is a triplet \( \hat{f} = (f,g,h) \) of surjective measurable map \( f : Z \to Z' \), \( g : Y \to Y' \) and \( h : Z \to Z' \) rendering this diagram commutative (with, e.g., \( f \times h : (x,z) \mapsto (f(x),h(z)) \)):

\[
\begin{array}{ccc}
X \times Z & \xrightarrow{K} & \mathcal{G}((Z \times Y,C \otimes B)) \\
\downarrow{f \times h} & & \downarrow{\mathcal{G}(h \times g)} \\
X' \times Z' & \xrightarrow{K'} & \mathcal{G}((Z' \times Y',C' \otimes B'))
\end{array}
\]

Thus

\[
K'(f(x),h(z))(E) = (K' \circ (f \times h))(x,z)(E) = \\
\mathcal{G}(h \times g)(K)(x,z)(E) = K(x,z)((h \times g)^{-1}(E))
\]

whenever \( E \in C' \otimes B' \) indicates the operation of automaton \( K' \).

3 Congruences

Before we define congruences for stochastic automata, we need to talk about friendly relations, i.e., relations on different states which behave nevertheless like congruences. To be specific: Given a stochastic relation \( K : (F,F) \Rightarrow (H,H) \) and equivalence relations \( \xi \) and \( \vartheta \) on \( F \).
resp. \( H \), call \( \xi \) friendly to \( \vartheta \) iff there exists a stochastic relation \( K_{\xi, \vartheta} : (F, \mathcal{F})/\xi \Rightarrow (H, \mathcal{H})/\vartheta \) rendering this diagram commutative:

\[
\begin{array}{ccc}
F & \xrightarrow{K} & \mathcal{G}(H, \mathcal{H}) \\
\downarrow{\eta_\xi} & & \downarrow{\mathcal{G}(\eta_\vartheta)} \\
F/\xi & \xrightarrow{K_{\xi, \vartheta}} & \mathcal{G}((H, \mathcal{H})/\vartheta)
\end{array}
\]

(1)

We observe for friendly \( \xi, \vartheta \) that

\[
K_{\xi, \vartheta}([x]_\xi)(T) = \left( \mathcal{G}(\eta_\vartheta) \circ K \right)(x)(T) = K(x)(\eta_\vartheta^{-1}[T]),
\]

so that \( \xi \) and \( \vartheta \) indeed cooperate in a congruential manner.

We will also need the concept of a small equivalence relation, given that equivalence is a very broad notion. It needs to be restricted somewhat for being useful in our context.

Again, assume an equivalence relation \( \xi \) on the measurable space \((F, \mathcal{F})\). Call the set \( Q \in \mathcal{F} \) \( \xi \)-invariant iff \( Q \) is the union of equivalence classes, thus iff \( x \in Q \) and \( x \xi x' \) entails \( x' \in Q \). It is not difficult to see that

\[
[\mathcal{F}, \xi] := \{Q \in \mathcal{F} \mid Q \text{ is } \xi\text{-invariant}\}
\]

(2)

is a \( \sigma \)-algebra, the \( \sigma \)-algebra of \( \xi \)-invariant sets. Observe that \( \eta_\xi[U] \in \mathcal{F}/\xi \) for \( U \in [\mathcal{F}, \xi] \), because \( \eta_\xi^{-1}[\eta_\xi[U]] = U \). Call the equivalence relation \( \xi \) small iff there exists a countable family \( (U_n)_{n \in \mathbb{N}} \subseteq \mathcal{F} \) such that

\[
x \xi x' \text{ iff } \forall n \in \mathbb{N} : x \in U_n \iff x' \in U_n.
\]

\((U_n)_{n \in \mathbb{N}}\) is said to create relation \( \xi \). Then \([\mathcal{F}, \xi] = \sigma([U_n \mid n \in \mathbb{N}])\) is countably generated, so is \( \mathcal{F}/\xi \), which also separates points.

**Example 3.1** Let \( f : (F, \mathcal{F}) \rightarrow (H, \mathcal{H}) \) be measurable, and assume that \( \mathcal{H} \) is countably generated and separates points. Then the kernel relation

\[
\text{ker}(f) := \{(x, x') \mid f(x) = f(x')\}
\]

is small. In fact, let \( (U_n)_{n \in \mathbb{N}} \) be the generator for \( \mathcal{H} \), then we show that \( \{U_n \mid n \in \mathbb{N}\} \) separates points. Take \( y, y' \in H \) such that \( y \in U_n \) iff \( y' \in U_n \) for all \( n \in \mathbb{N} \). Since \( \{U \subseteq H \mid \forall n \in \mathbb{N} : y \in U \iff y' \in U\} \) is a \( \sigma \)-algebra which contains the generator, it contains \( \mathcal{H} \). From this we conclude that \( y = y' \). But this means that \( (f^{-1}[U_n])_{n \in \mathbb{N}} \) creates \( \text{ker}(f) \). —

The following observation helps characterizing friendly equivalence relations.

**Lemma 3.2** Let \( K : (F, \mathcal{F}) \Rightarrow (H, \mathcal{H}) \) be a stochastic relation and assume equivalence relations \( \xi \) and \( \vartheta \) on \( F \) resp. \( H \), are given. Then these conditions are equivalent:

1. \( \xi \) is friendly to \( \vartheta \).
2. \( \mathcal{G}(m_\vartheta) \circ K : (F, [\mathcal{F}, \xi]) \Rightarrow (H, [\mathcal{H}, \vartheta]) \) with \( m_\vartheta : (H, \mathcal{H}) \rightarrow (H, [\mathcal{H}, \vartheta]) \) as the identity.
3. \( \text{ker}(\mathcal{G}(m_\vartheta) \circ K) \supseteq \xi. \)
Proof Abbreviate the map \( G(m_\vartheta) \circ K \) by \( L \), and note that \( G(m_\vartheta) \) restricts measures on \( \mathcal{H} \) to its sub \( \sigma \)-algebra \([\mathcal{H}, \vartheta]\).

1 \( \Rightarrow \) 2 It is clear that \( L : (F, \mathcal{F}) \Rightarrow (H, [\mathcal{H}, \vartheta]) \), because \( G(m_\vartheta) \) acts as restriction to \([\mathcal{H}, \vartheta]\). So it has to be shown that \( x \mapsto L(x)(G) \) is \([\mathcal{F}, \xi]\)-measurable for each \( G \in [\mathcal{H}, \vartheta] \). Let \( G_0 := \eta_\vartheta [G] \in \mathcal{H}/\vartheta \), then \( L(x, G) = L(x, \eta_\vartheta^{-1}[G_0]) = (G(\eta_\vartheta) \circ K)(x)(G_0) \), thus \( L(x)(G) < r \) if \( K_{\xi, \vartheta}([x]\xi)(G_0) < r \), which implies measurability of \( x \mapsto L(x)(G) \).

2 \( \Rightarrow \) 3 The assumption that there exists \( T \in [\mathcal{H}, \vartheta] \) such that \( K(x)(T) < r < K(x')(T) \) for some \( x, x' \) with \( x \xi x \) gives immediately a contradiction.

3 \( \Rightarrow \) 1 Define \( K_{\xi, \vartheta}([x]\xi) := (G(\eta_\vartheta) \circ K)(x) \), then \( K_{\xi, \vartheta} \) is well-defined, satisfies the measurability conditions and renders diagram (1) commutative. \( \dagger \)

This useful characterization permits testing friendship without actually constructing the factors. It extends to bounded, measurable functions:

**Corollary 3.3** Under the assumptions of Lemma 3.2 these statements are equivalent

1. \( \xi \) is friendly to \( \vartheta \).

2. For each bounded and \([\mathcal{H}, \vartheta]\)-measurable \( f : H \to \mathbb{R} \)

\[
x \xi x' \Rightarrow \int_H f \, dK(x) = \int_H f \, dK(x').
\]

Proof The implication 1 \( \Rightarrow \) 2 follows from part 3 in Lemma 3.2 together with the observation that a bounded measurable function is the pointwise limit of a sequence of step functions, and Lebesgue’s Convergence Theorem. The converse implication observes that the indicator function of a measurable set is a bounded measurable function. An application of part 3 in Lemma 3.2 yields the result. \( \dagger \)

An interesting example for friendship is given by kernels of morphisms for stochastic relations. Recall that finality of a measurable map \( f : (F, \mathcal{F}) \to (H, \mathcal{H}) \) may be characterized by the property that \( \mathcal{H} = \{ R \subseteq H \mid \mathcal{F}[R] \subseteq \mathcal{F} \} \). Thus we may conclude from \( f^{-1}(R) \in \mathcal{F} \) that \( R \in \mathcal{H} \), provided \( f \) is final and onto.

**Example 3.4** Let \( K_i : (F_i, \mathcal{F}_i) \Rightarrow (H_i, \mathcal{H}_i) \) be stochastic relations for \( i = 1, 2 \), and assume that \((f, g) : K_1 \to K_2\) is a morphism, which means \( K_2 \circ f = G(g) \circ K_1 \) for the surjective measurable maps \( f : F_1 \to F_2 \) and \( g : H_1 \to H_2 \). We claim that \( \text{terr}(f) \) is friendly to \( \text{terr}(g) \), provided \( g \) is final and onto.

In fact, let \( f(x) = f(x') \), then we have to show that \( K_1(x)(G) = K_1(x')(G) \) for all \( G \in [\mathcal{H}_1, \text{ter}(g)] \). Fix such a set \( G \), then we know that \( G = \eta_{\text{ter}(g)}^{-1} [\eta_{\text{ter}(g)} [G]] \) with \( \eta_{\text{ter}(g)} [G] \in [\mathcal{H}_1, \text{er}(g)] \). Factoring \( g = g_* \circ \eta_{\text{er}(g)} \) with \( g_* : H_1/\text{er}(g) \to H_2 \) measurable, final and injective yields the surjective map \( g_*^{-1} \) between powersets. We find therefore \( H_0 \subseteq H_2 \) with \( g_*^{-1} [H_0] \subseteq [\mathcal{H}_1, \text{er}(g)] \subseteq \mathcal{H}_1 \). Because

\[
g^{-1} [H_0] = \eta_{\text{er}(g)}^{-1} [g_*^{-1} [H_0]] = \eta_{\text{er}(g)}^{-1} [\eta_{\text{er}(g)} [G]] = G \in [\mathcal{H}_1, \text{er}(g)] \subseteq \mathcal{H}_1
\]
we conclude from finality of $g_\bullet$ that $H_0 \in \mathcal{H}_2$, so that

$$K_1(x)(G) = K_1(x)(g^{-1}[H_0]) = (\mathcal{G}(g) \circ K_1)(x)(H_0) = K_2(f(x))(H_0) = K_2(f(x'))(H_0) = K_1(x')(G).$$

This gives the assertion. —

After all these preparations we are in a position to define congruences for stochastic automata.

**Definition 3.5** Let $K = ((X, A), (Y, B), (Z, C), K)$ be a stochastic automaton, then a triplet $c = (\alpha, \beta, \gamma)$ of equivalence relations on $X, Y, Z$ is called a congruence for $K$ iff $\alpha \times \gamma$ is friendly to $\gamma \times \beta$.

A congruence $c$ for stochastic automaton $K$ is characterized by the existence of a stochastic relation

$$K_c : ((X, A) \otimes (Z, C))/\alpha \times \gamma \Rightarrow ((Z, C) \otimes (Y, B))/\gamma \times \beta$$

which renders this diagram commutative:

\[
\begin{array}{ccc}
(X, A) \otimes (Z, C) & \xrightarrow{K} & \mathcal{G}((Z, C) \otimes (Y, B)) \\
\downarrow_{\eta_\alpha \times \gamma} & & \downarrow_{\mathcal{G}(\eta_\gamma \times \beta)} \\
((X, A) \otimes (Z, C))/\alpha \times \gamma & \xrightarrow{K_c} & \mathcal{G}(((Z, C) \otimes (Y, B))/\gamma \times \beta)
\end{array}
\]

This is an immediate consequence:

**Proposition 3.6** In the notation of Definition 3.5 $(\eta_\alpha, \eta_\beta, \eta_\gamma) : K \rightarrow K_c$ is a morphism. ⊣

The classic case of state reduction by a relation $\gamma$ for automaton $K = ((X, A), (Y, B), (Z, C), K)$ is captured through the triplet $s = (1_X, 1_Y, \gamma)$; that $s$ is a congruence for $K$ is characterized through

$$\forall B \in \mathcal{B} : K(x, z)(E \times B) = K(x, z')(E \times B),$$

whenever $z \gamma z'$, and $E \in [C, \gamma]$ is a $\gamma$-invariant measurable subset of $Z$. This is quite close to the intuition of a (state-) congruence for an automaton: equivalent states behave in the same way on measurable sets which cannot separate equivalent states.

On the other hand, one probably wants to leave the states alone and cater only for inputs and outputs. Here one would work with $t = (\alpha, \beta, 1_Z)$, and $t$ is a congruence iff

$$\forall C \in \mathcal{C} : K(x, z)(C \times B) = K(x', z)(C \times B),$$

whenever $x \alpha x'$ and $B \in [B, \beta]$, so the behavior of $K$ on inputs which are identified through $\alpha$ is the same on sets which cannot separate $\beta$-equivalent outputs. Certainly other combinations are possible.

It is noted that the behavior of an automaton is completely characterized by its assigning values to sets of the form $C \times B$. This is so because these sets determine the respective product $\sigma$-algebras uniquely, and their collection is closed under intersections [7, Lemma 1.6.31].
4 Factoring

We will restrict the class of measurable spaces to analytic spaces now, and we will deal only with small equivalence relations.

Recall that an analytic space is the measurable image of a Polish space, i.e., of a second countable, completely metrizable topological space. Analytic spaces are topological spaces in their own right with a countable and point separating base for their topology. As topological spaces they carry the $\sigma$-algebra of Borel sets. For the rest of the paper we will assume that analytic spaces are equipped with just these Borel sets. This will render notation lighter as well, because it will permit us to omit the $\sigma$-algebra for an analytic space from notation. Measurability refers to the Borel sets, unless otherwise noted.

Analytic spaces have a number of desirable technical properties \cite{15, 7}, among them the closure under countable products; we note that $\mathcal{B}(F \times H) = \mathcal{B}(F) \otimes \mathcal{B}(H)$ for analytic spaces $F$ and $H$, $\mathcal{B}(\ldots)$ denoting the Borel sets. Alas, that the product of Borel sets equals the Borel sets is far from being common among topological spaces. In general this requires some additional assumptions. Just to emphasize this property, we have for the analytic spaces $F$ and $H$

$$\mathcal{B}(F \times H) \overset{(*)}{=} \sigma(\{W \mid W \subseteq F \times H \text{ is open}\})$$

$$\overset{(+)\quad}= \sigma(\{U \times V \mid U \in \mathcal{B}(F), V \in \mathcal{B}(H)\})$$

$$= \mathcal{B}(F) \otimes \mathcal{B}(H)$$

Here equation $(*)$ derives from the definition of the Borel sets as the smallest $\sigma$-algebra containing the open sets, and equation $(+)$ derives from the definition of the product $\sigma$-algebra. Analytic spaces are also closed under factoring through small equivalence relations \cite{15, Exercise 5.1.14, 7, Proposition 4.4.22}.

A first witness to usefulness is given by the following observation (cp. \cite{6, Corollary 2.11}).

Lemma 4.1 Assume that $\xi$ and $\zeta$ are small equivalence relations on the analytic spaces $F$ esp. $H$. Then

1. $[\mathcal{B}(F \times H), \xi \times \zeta] = [\mathcal{B}(F), \xi] \otimes [\mathcal{B}(H), \zeta]$.

2. The measurable spaces $(F \times H)/(\xi \times \zeta)$ and $F/\xi \times H/\zeta$ are isomorphic.

Writing down the second assertion in its full beauty means that $(F \times H, \mathcal{B}(F \times H))/(\xi \times \zeta)$ is isomorphic to $(F, \mathcal{B}(F))/\xi \otimes (H, \mathcal{B}(H))/\zeta$.

Proof 1. Assume that $\xi$ and $\zeta$ have the respective generators $(U_n)_{n \in \mathbb{N}}$ and $(V_m)_{m \in \mathbb{N}}$. Since

$$\langle x, y \rangle \quad (\xi \times \zeta) \quad \langle x', y' \rangle \iff \forall n \in \mathbb{N} \forall m \in \mathbb{N} : [x \in U_n \iff x' \in U_n] \land [z \in V_m \iff z' \in V_m]$$

we see that

$$[\mathcal{B}(F \times H), \xi \times \zeta] =$$

$$\sigma(\{U_n \times V_m \mid n, m \in \mathbb{N}\}) = \sigma(\{U_n \mid n \in \mathbb{N}\}) \otimes \sigma(\{V_m \mid m \in \mathbb{N}\}) =$$

$$[\mathcal{B}(F), \xi] \otimes [\mathcal{B}(H), \zeta]$$
2. It is not difficult to see that \([x, y]_{\xi \times \zeta} \mapsto ([x]_{\xi}, [y]_{\zeta})\) is a bijection and measurable. Now look at the inverse \(\ell\). We want to show that \(\ell^{-1}[E] \in (F, \mathcal{B}(F))/\xi \otimes (H, \mathcal{B}(H))/\zeta\) for each \(E \in (F \times H, \mathcal{B}(F \times H))/\xi \times \zeta\). By the observation following (2) it is sufficient to show that 
\(\ell^{-1}[\eta_{\xi \times \zeta}[D]] \in \mathcal{B}(F/\xi) \otimes \mathcal{B}(H/\zeta)\) for every \(D \in [\mathcal{B}(F \times H), \xi \times \zeta]\). The set
\[
D := \{D \in [\mathcal{B}(F \times H), \xi \times \zeta] \mid \ell^{-1}[\eta_{\xi \times \zeta}[D]] \in \mathcal{B}(F/\xi) \otimes \mathcal{B}(H/\zeta)\}
\]
certainly contains all rectangles \(P \times Q\) with \(P \in [\mathcal{B}(F), \xi]\) and \(Q \in [\mathcal{B}(H), \zeta]\) and, because the complement of an invariant set is invariant again, it is closed under complementation. Also, \(D\) is closed under disjoint countable unions. Since the set of rectangles with invariant sides is closed under intersection, Dynkin’s celebrated \(\pi\)-\(\lambda\)-Theorem [7, Theorem 1.6.30] together with part (1) tells us that \(D = [\mathcal{B}(F \times H), \xi \times \zeta]\). \(\dashv\)

This result is not only of structural importance, as we will see in a moment. It will also permit us to use, e.g., \(\langle [x]_{\xi}, [y]_{\zeta}\rangle\) and \([x, y]_{\xi \times \zeta}\) interchangeably, similarly with maps. This will simplify notation somewhat and thus make life a bit easier.

From now on all automata are working over analytic spaces.

A decent morphism generates a congruence via its kernel [3, 8]. The following counterpart to Proposition 3.3 shows that this is also the case with stochastic automata.

**Proposition 4.2** Given the stochastic automata \(K\) and \(K'\) with \((f, g, h) : K \to K'\) an automata morphism. Then \((\text{ker}(f), \text{ker}(g), \text{ker}(h))\) is a congruence for \(K\), provided \(g\) and \(h\) are final. 

**Proof** 1. Write \(K = (X, Y, Z, K)\) and \(K' = (X', Y', Z', K')\). We show first that we can find for \(V \in [\mathcal{B}(Z), \text{ker}(h)]\) a Borel set \(V_0 \in \mathcal{B}(Z')\) such that \(V = h^{-1}[V_0]\), and for \(W \in [\mathcal{B}(Y), \text{ker}(g)]\) another Borel set \(W_0 \in \mathcal{B}(Y')\) with \(W = g^{-1}[W_0]\). This is done exactly as in Example 3.3 using finality of the respective maps.

2. Assume \(f(x) = f(x')\) and \(h(z) = h(z')\), and take \(G \in [\mathcal{B}(Z \times Y), \text{ker}(h) \times \text{ker}(g)]\). We want to show that \(K(x, z)(G) = K(x', z')(G)\) holds. Assume first that \(G = V \times W\) with \(V \in [\mathcal{B}(Z), \text{ker}(h)]\) and \(W \in [\mathcal{B}(Y), \text{ker}(g)]\) and determine \(V_0, W_0\) as above, so that \(G = (h \times g)^{-1}[V_0 \times W_0]\). But now
\[
K(x, z)(G) = K(x, z)((h \times g)^{-1}[V_0 \times W_0]) = K'(f(x), h(z))(V_0 \times W_0) = K'(x', z')(G).
\]
This argument shows that
\[
D := \{G \in [\mathcal{B}(Z \times Y), \text{ker}(h) \times \text{ker}(g)] \mid K(x, z)(G) = K(x', z')(G)\}
\]
contains all rectangles \(V \times W\) with \(V \in [\mathcal{B}(Z), \text{ker}(h)]\) and \(W \in [\mathcal{B}(Y), \text{ker}(g)]\). The set of these rectangles is closed under finite intersections, and \(D\) is closed under complementation as well as under countable disjoint unions. By Dynkin’s \(\pi\)-\(\lambda\)-Theorem \(D\) equals
\([\mathcal{B}(Z), \text{ker}(h)] \otimes [\mathcal{B}(Y), \text{ker}(g)]\), which is equal to \([\mathcal{B}(Z \times Y), \text{ker}(h) \times \text{ker}(g)]\) by the first part of Lemma 4.1.

3. We have shown that \(\text{ker}(f) \times \text{ker}(h)\) is a subset of \(\text{ker}(G(m_{\text{ker}(h) \times \text{ker}(g)} \circ K))\), which establishes the claim by Lemma 3.2. \(\dashv\)

Recall that a map \(f : F \to H\) has an em-factorization \(f = f_\bullet \circ \eta_{\text{ker}(f)}\). If \(f\) is measurable, so are the components (but this does not entail the em-factorization living in the category
of measurable spaces). We obtain a similar decomposition for stochastic automata: Let \( \mathfrak{f} = (f, g, h) : K \to K' \) be a morphism. To express this in a concise manner, put \( \eta_\text{ter}(f) := (\eta_\text{ter}(f), \eta_\text{ter}(g), \eta_\text{ter}(h)) \) and \( \mathfrak{f}_\bullet := (f_\bullet, g_\bullet, h_\bullet) \).

We immediately obtain as a consequence of Proposition 4.2

**Corollary 4.3** In the notation of Proposition 4.2, \( \eta_\text{ter}(f) : K \to K_\text{ter}(f) \) and \( \mathfrak{f}_\bullet : K_\text{ter}(f) \to K' \) are morphisms, and \( \mathfrak{f} = \mathfrak{f}_\bullet \circ \eta_\text{ter}(f) \).

**Proof** The first part follows from Proposition 4.2 together with Proposition 3.6. As for the second part, a somewhat lengthy but straightforward computation shows that

\[
(\mathcal{G} (h_\bullet \times g_\bullet) \circ K_\text{ter}(f))([x]_\text{ter}(f), [z]_\text{ter}(h))(E') = (K' \circ (f_\bullet \times h_\bullet))([x]_\text{ter}(f), [z]_\text{ter}(h))(E')
\]

whenever \( E' \in \mathfrak{B}(Z' \times Y') \). The last equation is obvious. □

5 Sequential Work

A stochastic automaton works sequentially and synchronously: input is fed into it, in each step an output is produced, then a new input is given, a new output is produced, etc. Of course, state changes occur as part of these operations. Formally, suppose the automaton \( K = (X, Y, Z, K) \) is in state \( z \) and receives first \( x_1 \), then \( x_2 \) as the input. Quite apart from the salient state changes, an output of length two is produced, and the probability for the measurable set \( E \subseteq Z \times Y \times Y \) is computed so:

\[
K(x_1, x_2, z)(E) := \int_{Z \times Y} dK(x_1, z)(\langle \langle z'', y_2 \rangle, \langle z'', y_1 y_2 \rangle \in E \rangle)
\]  

(4)

After input \( x_1 \) in state \( z \) the automaton makes a transition to state \( z' \) and gives an output \( y_1 \) with probability \( dK(x_1, z)(\langle \langle z', y_1 \rangle \rangle) \). The new input \( x_2 \) is met in state \( z' \) and produces a new state \( z'' \) as well as an output \( y_2 \) so that \( \langle \langle z'', y_1 y_2 \rangle \rangle \in E \) with probability \( K(x_2, z')(\langle \langle z'', y_2 \rangle, \langle z'', y_1 y_2 \rangle \in E \rangle) \). We have to average over \( z' \) and \( y_1 \). Standard arguments [3] show that we have extended the transition law to a stochastic relation \( K : X^2 \times Z \Rightarrow Z \times Y^2 \) (we could use indices showing the length of the automaton’s work so far, but there is already enough notation around).

Let \( v \in X^n \) be an input word of length \( n \), and assume that we have extended the transition law already to a stochastic relation \( K : X^n \times Z \Rightarrow Z \times Y^n \), all products carrying the corresponding product \( \sigma \)-algebras. Define for the input \( x \in X \), and the state \( z \) for the Borel set \( E \subseteq Z \times Y^{n+1} \)

\[
K(vx, z)(E) := \int_{Z \times Y} dK(v, z)(\langle \langle z'', y \rangle, \langle z'', wy \rangle \in E \rangle)
\]  

(5)

Then it is shown in [3] that \( K : X^{n+1} \times Z \Rightarrow Z \times Y^{n+1} \) is a stochastic relation.

In this way we extend the probabilistic transition law to finite input sequences in a natural manner.

Now assume that \( c = (\alpha, \beta, \gamma) \) is a congruence for \( K \). We will show now that friendship is not lost during the automaton’s sequential work as outlined above. Define for the equivalence
Proposition 5.1

Summarizing, we obtain

\[ (x_1, \ldots, x_n) \overset{\alpha^n}{\leftrightarrow} (x'_1, \ldots, x'_n) \iff x_i \sim x'_i \text{ for } i = 1, \ldots, n, \]

similarly for the other equivalence relations, and for, e.g., \( \alpha^\infty \) when dealing with infinite sequences. We claim that \( \alpha^n \times \gamma \) is friendly to \( \gamma \times \beta^n \) for each \( n \in \mathbb{N} \), so that congruence \( \sim \) induces an infinite sequence of friendships. This will be demonstrated for \( n = 2 \) now, the general case is shown exactly in the same way using induction and eq. (5).

We do these steps:

**Step 1:** The set \( \{ (z, y') \mid (z', y, y') \in E \} \) is a member of \( \mathcal{B}(Z) \otimes \mathcal{B}(Y), \gamma \times \beta \) for each \( E \in [\mathcal{B}(Z) \otimes \mathcal{B}(Y^2), \gamma \times \beta^2] \) and for each \( y \in Y \). It is easy to see that the set in question is a Borel set, and because \( E \) is \( \gamma \times \beta^2 \)-invariant, and \( \beta \) is a reflexive relation, the set is also \( \gamma \times \beta \)-invariant.

**Step 2:** Let \( E \in [\mathcal{B}(Z) \otimes \mathcal{B}(Y^2), \gamma \times \beta^2] \) and fix \( \bar{x} \in X, \bar{y} \in Y \), then the map

\[
(\bar{z}, \bar{y}) \mapsto K(\bar{x}, z)\{(\{z'', \bar{y}\} \mid (\{z'', y, \bar{y}\} \in E)\}
\]

is \([\mathcal{B}(Z) \otimes \mathcal{B}(Y), \gamma \times \beta]\)-measurable. Assume that \( E \) is a measurable rectangle, say, \( E = C_1 \times B_1 \times B_2 \), then \( K(\bar{x}, z)\{(\{z'', \bar{y}\} \mid (\{z'', y, \bar{y}\} \in E)\} = K(\bar{x}, z)(C_1 \times B_2) \cdot I_{B_1}(x) \) with \( I_{B_1} \) the indicator function of the set \( B_1 \). This constitutes certainly a \([\mathcal{B}(Z) \otimes \mathcal{B}(Y), \gamma \times \beta]\)-measurable function by Lemma 3.2. Applying the principle of good sets, Dynkin’s \( \pi-\lambda \)-Theorem shows that the set of all \( E \) for which the claim is true is all of \([\mathcal{B}(Z) \otimes \mathcal{B}(Y^2), \gamma \times \beta^2] \), because the latter \( \sigma \)-algebra is generated by these rectangles, and because of the first part of Lemma 4.1.

Now we are poised to show that \( \alpha^2 \times \gamma \) and \( \gamma \times \beta^2 \) are friends. For this, take \( E \in [\mathcal{B}(Z) \otimes \mathcal{B}(Y^2), \gamma \times \beta^2] \) and assume that \( (x_1 x_2, z) \overset{\alpha^2}{\leftrightarrow} \gamma (\bar{x}_1 \bar{x}_2, \bar{z}) \), then we have according to eq. (4)

\[
K(x_1 x_2, z)(E) = \int_{Z \times Y} K(x_2, z')\{(\{z'', y_2\} \mid (\{z'', y_1 y_2\} \in E)\} \, dK(x_1, z)((z', y_1))
= \int_{Z \times Y} K(x_2, z')\{\ldots\} \, dK(\bar{x}_1, \bar{z})(\{z', y_1\})
\text{(Corollary 3.3 since the integrand is } [\mathcal{B}(Z) \otimes \mathcal{B}(Y), \gamma \times \beta]\text{-measurable})
= \int_{Z \times Y} K(x_2, z')\{\ldots\} \, dK(\bar{x}_1, \bar{z})(\{z', y_1\})
\text{(by Step 1, because } \{\ldots\} \text{ is in } [\mathcal{B}(Z) \otimes \mathcal{B}(Y), \gamma \times \beta])
= K(\bar{x}_1 \bar{x}_2, \bar{z})(E)
\]

Now the claim is established by Lemma 3.2.

Summarizing, we obtain

**Proposition 5.1** Let \( (\alpha, \beta, \gamma) \) be a countably generated congruence for the stochastic automaton \( K \) over analytic spaces. Then \( \alpha^n \times \gamma \) is friendly to \( \gamma \times \beta^n \) for every \( n \in \mathbb{N} \).

In what follows, we will deal with finite or infinite sequences of inputs resp. outputs. Denote as usual for a set \( M \) by \( M^+ \) the set of all finite non-empty words with letters taken from \( M \), \( |v| \) denotes the length of word \( v \in M^+ \). \( M^\infty \) is the set of all infinite sequences, and
$M^{≤∞} := M^+ \cup M^∞$ are all non-empty finite or infinite sequences over $M$. For $τ ∈ M^∞$ the first $n$ letters are denoted by $τ_n$. If $M$ carries a $σ$-algebra $ℳ$, $M^n$ carries for $n \leq ∞$ the $n$-fold product $ℳ^n$, and $M^+$ the coproduct $ℳ^+$ of $(ℳ^n)_{n∈ℕ}$, finally $M^{≤∞}$ has the coproduct of $ℳ^+$ and $ℳ^∞$.

Having thus fixed notation, we turn to automata again. Viewed from the outside, a learning system, or a reactive one, receives an input and responds through an output, the internal states are hidden from the observer. They are usually assumed to follow some initial probability distribution $µ ∈ ℭ(Z)$. So we put

$$K^{|v|}_µ(v)(G) := \int_Z K(v, z)(Z × G) dµ(z)$$

with $v ∈ X^+, G ∈ ℭ(Y^{|v|})$, thus $K^{|v|}_µ(v)$ specifies the probability distribution of outputs of length $n$ given input $v$ with $|v| = n$, provided the initial states are distributed according to $µ$. Note that the state changes after each input are recorded through $K$, but are kept hidden behind a kind of smoke screen (indicated by computing the probability $µ_n(v, z)$, thus not betraying which new state is adopted specifically). Finally, define

$$K^+_µ(v)(G) := K^{|v|}_µ(v)(G \cap Y^{|v|})$$

(with $G ∈ ℭ(Y^+)$) as the black box associated with the stochastic automaton $K$.

We note for later use

**Lemma 5.2** For every $µ ∈ ℭ(Z)$, $K^+_µ : X^+ ⇒ Y^+$; given a countably generated congruence $(α, β, γ)$ on $K$, $α^n$ is a friend to $β^n$ with respect to $K^+_µ$ for each $n ∈ ℕ$.

**Proof** It is shown first that $K^n_µ : X^n ⇒ Y^n$ is a stochastic relation for each $µ ∈ ℭ(Z)$ [7, Example 2.4.8, Exercise 4.14]. Since $X^+$ is the coproduct of the measurable spaces $(X^n)_{n∈ℕ}$, the first assertion follows. For the second one, fix $G ∈ [ℬ(Y^n), β^n]$, and assume $v α^n v'$.

We observe $(v, z) α^n × γ (v', z)$ for all $z \in Z$, so in particular $K^n_µ(v, z)(Z × G) = K^n_µ(v', z)(Z × G)$, because $α^n × γ$ is friendly to $γ × β^n$ by Proposition 5.1 Integrating with respect to $µ ∈ ℭ(Z)$ yields $K^n_µ(v)(G) = K^n_µ(v')(G)$. So the second assertion follows from Lemma 5.2.

In fact, we may educate our black box to work on infinite sequences in such a way that the finite initial parts are respected. To be specific, we claim that we find a stochastic relation $K^∞_µ$ between $X^∞$ and $Y^∞$ such that for the cylinder set $G = G_n × \prod_{m>n} Y$ with $G_n ∈ ℭ(Y^n)$

$$K^∞_µ(τ)(G) = K^n_µ(τ_n)(G_n)$$

holds. Consequently, we intend to find $K^∞_µ : X^∞ ⇒ Y^∞$ with $K^n_µ = G(π^Y_n) ∘ K^∞_µ$ for all $n$, with $π^Y_n : τ ⇒ τ_n$ as the projection of an infinite sequence to its first $n$ letters. Thus we want to close the gap in this diagram

$$X^∞ \xrightarrow{π^n_Y} G(π^Y_n) \xrightarrow{G} \xrightarrow{G} Y^∞$$

Evidently this requires the automaton to be fully probabilistic, i.e., that $(K(x, z)(Z × Y) = 1$ always holds. For measure-theoretic reasons, we need also a topological assumption.
Proposition 5.3 Let $K = (X, Y, Z, K)$ be a stochastic automaton such that $X$ and $Y$ are Polish spaces, and $Z$ is an analytic space. Then there exists for each initial distribution $\mu \in \mathcal{G}(Z)$ with $\mu(Z) = 1$ a uniquely determined stochastic relation $K^\infty_\mu : X^\infty \Rightarrow Y^\infty$ such that $K^\infty_\mu \circ \pi_n = \mathcal{G}(\pi_n) \circ K^\infty_\mu$ for all $n \in \mathbb{N}$, provided $K(x, Z)(Z \times Y) = 1$ for all $x \in X, z \in Z$.

Proof Fix $\mu \in \mathcal{G}(Z)$ with $\mu(Z) = 1$. Define $\pi_{m,n} : y_1 \ldots y_m \mapsto y_1 \ldots y_n$ as the projection $Y^m \Rightarrow Y^n$ for $m > n$, and put for $\tau \in X^\infty$

$$L_k(\tau)(G) := K^k_\mu(\tau_k)(G),$$

whenever $k \in \mathbb{N}$ and $G \in \mathcal{B}(Y^k)$. Then $L_k : X^\infty \Rightarrow Y^k$ with $L_k(\tau)(Y^k) = 1$ for all $\tau$, and

$$L_n(\tau) = \mathcal{G}(\pi_{m,n}) \left( L_m(\tau) \right)$$

holds for $m > n$. Thus $(L_n(\tau))_{n \in \mathbb{N}}$ is a projective system in the sense of $[7, \text{Definition 4.9.18}]$ for every $\tau \in X^\infty$. The assertion now follows from $[7, \text{Corollary 4.9.21}]$, a mild variant of the famous Kolmogorov Consistency Theorem. $\dagger$

Because of its genesis, the stochastic relation $K^\infty_\mu$ might be called the projective limit associated with automaton $K$ and distribution $\mu$.

Our black box works also for infinite sequences of inputs, answering with a uniquely determined distribution on the set of output sequences. The price to pay for this is on one hand the full probabilistic nature of the underlying stochastic relation (given the requirement, this is only too obvious), and on the other hand the assumption of working in Polish spaces. This, however, cannot be relaxed, as $[2] \text{Example 7.7.3}$ shows.

The distribution on infinite output sequences is consistent with its initial pieces. Suppose you stop the input sequence at point $n$, then you obtain the corresponding distribution on the outputs of length $n$. This observation permits us to decorate trees. Call a subset $\mathcal{T}$ of $X^{\leq \infty}$ a tree iff it is prefix free (so if $p \in \mathcal{T}$ and $q$ is a prefix of $p$, then $q = p$). We interpret the elements of $\mathcal{T}$ and their prefixes as paths, and we associate to each path in the tree a probability: If $\vec{x} = x_1 \ldots x_n$ is a path of length $n$, then there is some $p \in \mathcal{T}$ such that $\vec{x}$ is the prefix of length $n$ to $p$. So assign to $\vec{x}$ the distribution

$$T(\vec{x}) := \mathcal{G}(\pi^n_X)(K^{|p|}_\mu)(p)$$

(with $|p| = \infty$, if $p$ is infinitely long). This is well defined: if $\vec{x}$ is the prefix of $q \in \mathcal{T}$ as well, we have by construction $\mathcal{G}(\pi^n_X)(K^{|p|}_\mu)(p) = \mathcal{G}(\pi^n_X)(K^{|q|}_\mu)(q)$. In fact, being the prefix of more than one path may occur in case the tree branches out at some node later on.

Thus $T(v)(G)$ is the probability that the output is a member of $G \in \mathcal{B}(G^{\|v\|})$ after input of the finite sequence $v$ into the tree. If the finite path associated with $v$ ends in the leaf $x$ (so that $v = w x$ for some $w$, and $v$ is not a prefix of another word in $\mathcal{T}$), then the probability that the final output is a member of $G_0 \in \mathcal{B}(Y)$ is just $T(v)(Y^{\|v\| - 1} \times G_0)$.

Friendship is maintained also for infinite sequences. We first show that the extension $\xi^\infty$ of a small equivalence relation $\xi$ on the measurable space $(F, \mathcal{F})$ is small again. Assume that $\xi$ is created by the countable set $\mathcal{U} := \{ U_n \mid n \in \mathbb{N} \} \subseteq \mathcal{F}$, which we may assume to be closed under finite intersections (otherwise take $\{ \bigcap_{i \in S} U_i \mid \emptyset \neq S \subseteq \mathbb{N} \text{ finite} \}$ as a countable creator).
Put
\[ D_{U,ξ} := \{ \prod_{i=1}^{k} U_{n_i} \times \prod_{m>k} F \mid n_i \in \mathbb{N} \text{ for } 1 \leq i \leq k, k \in \mathbb{N} \}, \]
then it is easy to see that this countable set creates \( ξ^∞ \). Note that \( D_{U,ξ} \) is also closed under finite intersections. Now let \( V \) be a countable creator for \( β \). Assume that \( τ^∞ \tau' \), fix \( µ \in \mathcal{G}(Z) \) with \( µ(Z) = 1 \) as before, and let \( G \in D_{V,β} \), then \( G = \text{prod}_{i=1}^{k} V_i \times \prod_{m>k} Y \) for some \( V_1, \ldots, V_k \in V \). Hence \( G_0 := \prod_{i=1}^{k} V_i \in [\mathfrak{B}(Y^k) , β^k] \), and \( τ_k \alpha^k \tau'_k \), so that we have
\[ K^∞_μ(τ)(G) = K^k_μ(τ_k)(G_0) \overset{(1)}{=} K^k_μ(τ'_k)(G_0) = K^∞_μ(τ')(G). \]
Equality \((1)\) is implied by the friendship of \( α^k \) to \( β^k \) (Lemma 5.2). Thus \( K^∞_μ(τ) \) agrees with \( K^∞_μ(τ') \) on \( D_{V,β} \), so these measures agree on \([\mathfrak{B}(Y^∞), β^∞] = σ(D_{V,β})\) by Dynkin’s \( π-λ\)-Theorem (see [7, Lemma 1.6.31]).

We have shown

**Proposition 5.4** Let \( K = (X, Y, Z, K) \) be a stochastic automaton with \( K(x, Z)(Z \times Y) = 1 \) for all \( x \in X, z \in Z \). Assume that \( X \) and \( Y \) are Polish spaces, and \( Z \) is an analytic space and initial distribution \( µ \in \mathcal{G}(Z) \) with \( µ(Z) = 1 \). If \( (α, β, γ) \) is a countably generated congruence on \( K \), then \( α^n \) is friendly to \( β^n \) with respect to \( K^∞_μ \) for every \( n \in \mathbb{N} \cup \{∞\} \).

So friendship turns out to be a surprisingly stable relationship, maintained even through finite and infinite streams.

### 6 Play it again, Sam

Assume that we have factored an automaton, and subsequently we need to factor again the factored automaton. It will turn out that the resulting automaton may be obtained by factoring once the automaton from which we started, albeit with a modified congruence. This result will also enable us to do the reduction iteratively along the multiple components.

Given an equivalence relation \( ξ \) on a set \( F \), and an equivalence relation \( ζ \) on the set \( F/ξ \), define

\[ x \ (ξ * ζ) x' \text{ iff } [x]_ξ \ ξ \ [x']_ζ, \]
hence \( x \) is related to \( x' \) through the new relation \( ξ * ζ \) iff the class \( [x]_ξ \) of \( x \) is related to the class \( [x']_ξ \) through relation \( ζ \). We may think of \( ξ * ζ \) as a lifting operation (visually, a \( ζ \)-class may be seen as a sea in which \( ξ \)-classes swim; the \(*\) operator lifts these classes to the level of the base space). It is clear that \( ξ * ζ \) is countably generated if both \( ξ \) and \( ζ \) are.

Define the bijections
\[
\varphi_{ξ,ζ} : \begin{cases} F/ξ \times ζ & \rightarrow (F/ξ)/ζ \\ [x]_ξ × ζ & \mapsto [x]_ξ \end{cases}
\]
and
\[ ψ_{ξ,ζ} : \begin{cases} (F/ξ)/ζ & \rightarrow F/(ξ * ζ) \\ [x]_ξ & \mapsto [x]_ξ \end{cases} \]
Let \( G \) be a cg congruence on the factor automaton \( K \), and we have to show that \( \alpha \) is \( G \)-invariant. From \( \varphi_{\xi,\zeta} \circ \eta_{\xi,\zeta} = \eta_{\xi} \circ \eta_{\xi} \) we see that \( \varphi_{\xi,\zeta} \circ \eta_{\xi,\zeta} \) is \( \eta_{\xi} \)-invariant, and since \( \eta_{\xi,\zeta} \) is final, we conclude measurability of \( \varphi_{\xi,\zeta} \). Similarly, since the composition of final morphisms is \( \eta_{\xi} \) again, measurability of \( \psi_{\xi,\zeta} \circ \eta_{\xi} \circ \eta_{\xi} \) by (6) implies measurability of \( \psi_{\xi,\zeta} \).

Thus we may and do identify \( \xi \) with \( 1_{F} \circ \xi \), and with \( \xi \circ 1_{F/\xi} \).

A congruence on a factor automaton, each equivalence relation on a factored component space generates individually a new equivalence on the component proper through lifting, as we have seen above. These new equivalences are countably generated, if their components are. Combining all these lifted equivalences will yield a congruence, as will be shown now.

In a slight abuse of terminology, call a congruence countably generated (abbreviated cg) iff all its components are.

**Lemma 6.1** Let \( (F,F) \) be a measurable space, then \( (F,F)/(\xi \star \zeta) \) and \( ((F,F)/\xi)/\zeta \) are isomorphic as measurable spaces. Moreover \( \eta_{\xi}[G] \in [F/\xi,\zeta] \), provided \( G \in [F,\xi \star \zeta] \).

**Proof** 1. We show that the bijections \( \varphi_{\xi,\zeta} \) and \( \psi_{\xi,\zeta} \) from above are measurable. From \( \varphi_{\xi,\zeta} \circ \eta_{\xi,\zeta} = \eta_{\xi} \circ \eta_{\xi} \) we see that \( \varphi_{\xi,\zeta} \circ \eta_{\xi,\zeta} \) is \( \eta_{\xi} \)-invariant, and since \( \eta_{\xi,\zeta} \) is final, we conclude measurability of \( \varphi_{\xi,\zeta} \). Similarly, since the composition of final morphisms is \( \eta_{\xi} \) again, measurability of \( \psi_{\xi,\zeta} \circ \eta_{\xi} \circ \eta_{\xi} \) by (6) implies measurability of \( \psi_{\xi,\zeta} \).

2. For establishing the second part, we have to show that \( \eta_{\xi}[G] \) is \( \zeta \)-invariant, since clearly \( \eta_{\xi}[G] \in F/\xi \) on account of \( G \in F \) being \( \xi \)-invariant. Given \( t \in \eta_{\xi}[G] \) and \( t' \) with \( t \circ \zeta \) \( t' \), we find \( x \in G' \) and \( x' \) with \( t = \langle x \rangle_{\xi} \) and \( t' = \langle x' \rangle_{\xi} \). Hence \( t \circ \zeta \) \( t' \) translates to \( \langle x \rangle_{\xi} \zeta \zeta \zeta \zeta = \langle x' \rangle_{\xi} \zeta \zeta \zeta \zeta \), equivalently \( \langle x \rangle_{\xi,\zeta} = \langle x' \rangle_{\xi,\zeta} \). Since \( x \in G \), and because \( G \) is \( \xi \star \zeta \)-invariant, we find that \( x' \in G \) holds, which means \( t' \in \eta_{\xi}[G] \), so that the latter set is \( \zeta \)-invariant.  \( \dashv \)

Assume our stage is a measurable space, then we obtain

**Proposition 6.2** Let \( \xi = (\alpha,\beta,\gamma) \) be a cg congruence on the stochastic automaton \( K \), and \( \xi' = (\alpha',\beta',\gamma') \) a cg congruence on the factor automaton \( K_{\xi} \). Then \( \xi' \circ \xi' := (\alpha \star \alpha',\beta \star \beta',\gamma \star \gamma') \) is a cg congruence on \( K \).

**Proof** 1. Write \( K = (X,Y,Z,K) \). We know already that \( \xi \circ \xi' \) is countably generated, so we have to show that \( \xi \circ \xi' \) is friendly to \( \gamma \star \gamma' \). \( \star \) binds stronger than \( \times \). This is done through Lemma 6.2.

2. Let \( \langle x, z \rangle \) (\( \alpha \star \alpha' \times \gamma \star \gamma' \) \( \langle x', z' \rangle \), we want to show \( K(x, z)(G) = K(x', z')(G) \) for all \( G \in [B(Z \times Y), \gamma \star \gamma' \times \beta \star \beta'] \). Fix such a set \( G \). Because \( \alpha' \times \gamma' \) is friendly to \( \gamma' \times \beta' \), we know that

\[
K_{\xi}(\langle x \rangle_{\alpha}, \langle z \rangle_{\gamma})(H) = K_{\xi}(\langle x' \rangle_{\alpha}, \langle z' \rangle_{\gamma})(H)
\]
for all $H \in [\mathfrak{B}(Z/\gamma) \otimes \mathfrak{B}(Z/\beta), \gamma' \times \beta']$. From the second part of Lemma 6.1 we see that $\eta_{\gamma \times \beta} [G] \in [\mathfrak{B}(Z/\gamma) \otimes \mathfrak{B}(Y/\beta), \gamma' \times \beta']$. Thus
\[
K(x, z)(G) = K(x, z)(\eta_{\gamma \times \beta}^{-1} [\eta_{\gamma \times \beta} [G]]) \\
= K_\epsilon([x]_\alpha, [z]_\gamma)(\eta_{\gamma \times \beta} [G]) \\
\equiv K_\epsilon([x']_\alpha, [z']_\gamma)(\eta_{\gamma \times \beta} [G]) \\
= K(x', z')(G),
\]
and we are done. ⊣

Factoring twice, each time with a countably generated congruence, has — up to isomorphism — the same effect as factoring once through a suitably constructed congruence. This observation is similar to the Third Isomorphism Theorem in Group Theory [9, Corollary 5.10], which tells us what happens when factoring iteratively through normal subgroups.

**Proposition 6.3** Let $K$ be a stochastic automaton with a countably generated congruence $c$, and $c'$ a countably generated congruence on the factor automaton $K_\epsilon$. The factor automaton of $K$ for the congruence $c * c'$, and the factor automaton of $K_\epsilon$ for the congruence $c'$ are isomorphic.

This could more suggestively and more comprehensively be written as $(K/c)/c' = K/(c*c')$.

**Proof.** We assume that the automaton $K$ is defined over the analytic spaces $X$, $Y$, and $Z$, and that the congruences are $c = (\alpha, \beta, \gamma)$ resp. $c' = (\alpha', \beta', \gamma')$. Denote by $K_1$ the factor automaton of $K$ for the congruence $c * c'$, and by $K_2$ the factor automaton of $K_\epsilon$ for the congruence $c'$. $K$ is assumed to be the transition law for automaton $K$, $K_i$ the one for $K_i$, $i = 1, 2$.

1. The candidates for the isomorphism are the suspects already indicated in the equations (6), specifically
\[
a^\beta : [x]_{\alpha \ast \alpha'} \mapsto [x]_{\alpha \ast \alpha'},
\]
\[
b^\beta : [y]_{\beta \ast \beta'} \mapsto [y]_{\beta \ast \beta'}.
\]

2. We show that this diagram commutes
\[
\begin{array}{ccc}
X/(\alpha \ast \alpha') \times Z/(\gamma \ast \gamma') & \xrightarrow{K_1} & G(Z/(\gamma \ast \gamma') \times y/(\beta \ast \beta')) \\
\xrightarrow{a^\beta \times b^\beta} & & \xrightarrow{G(c^\beta \times b^\beta)} \\
(X/\alpha)/\alpha' \times (Z/\gamma)/\gamma' & \xrightarrow{K_2} & G((Z/\gamma)/\gamma' \times (y/\beta)/\beta')
\end{array}
\]

For this, fix $J \in \mathfrak{B}((Z/\gamma)/\gamma' \times (y/\beta)/\beta')$. An easy manipulation shows that
\[
(\eta_{\gamma \ast \gamma'} \times \eta_{\beta \ast \beta'})^{-1} (c^\beta \times b^\beta)^{-1} [J] = (\eta_\gamma \times \eta_\beta)^{-1} (\eta_{\gamma'} \times \eta_{\beta'})^{-1} [J]. \tag{8}
\]
But now we obtain

\[
K(a^\#([x]_{\alpha*\alpha'}), c^\#([z]_{\gamma*\gamma'}))(J) = K_2(x, z)((\eta_\gamma \times \eta_\beta)^{-1} \left[ (\eta_{\gamma'} \times \eta_{\beta'})^{-1} [J] \right])
\]

Thus the diagram in question commutes indeed. A similar diagram for \((a^\flat, b^\flat, c^\flat)\) is shown to be commutative in exactly the same manner. Because all contributing maps are bijective and measurable by the remarks at the beginning of this section, we have found the desired isomorphisms.

This result indicates that a stepwise reduction is possible. Suppose that we want to first reduce states according to \(\gamma\), and then reduce inputs and outputs through \(\alpha\) resp. \(\beta\). We observe that up to isomorphism

\[
(\alpha, \beta, \gamma) = (1_X, 1_Y, \gamma) \ast (\alpha, \beta, 1_{Z/\gamma}).
\]

Reducing inputs and outputs first and then dealing with states gives rise to a similar isomorphism:

\[
(\alpha, \beta, \gamma) = (\alpha, \beta, 1_Z) \ast (1_{X/\alpha}, 1_{Y/\beta}, \gamma).
\]

7 Conclusion and Discussion

The notion of a congruence for stochastic automata is defined and investigated, the interplay of congruences with the kernels of morphisms is briefly shed light on. The central notion is the friendship of equivalence relations with respect to stochastic relations, which is studied extensively. We investigate also the behavior of an automaton when the input comes from a finite or infinite stream; this permits to have the automaton work on trees with possibly infinite paths. Some topological assumptions had to me made in order to face measure theoretic problems adequately. Finally an isomorphism result is stated which permits the reduction of an automaton in a stepwise fashion.

An extension to these ideas extends equivalence relations to act on subprobabilities. To be specific, let \((F, \mathcal{F})\) be a measurable space, \(\xi\) an equivalence relation on \(F\) with the \(\sigma\)-algebra \([\mathcal{F}, \xi]\) of \(\xi\)-equivalent sets. Define for \(\mu, \nu \in \mathcal{G}(F, \mathcal{F})\)

\[
\mu \xi^\circ \nu \text{ iff } \forall E \in [\mathcal{F}, \xi] : \mu(E) = \nu(E).
\]

This is the randomization of \(\xi\); note that \(x \xi x'\) iff \(\delta_x \xi^\circ \delta_{x'}\) with \(\delta_x\) the point mass on \(x\). Furthermore, extend the stochastic relation \(K : (F, \mathcal{F}) \Rightarrow (H, \mathcal{H})\) to a measurable map \(K^* : \mathcal{G}(F, \mathcal{F}) \to \mathcal{G}(H, \mathcal{H})\) upon setting

\[
K^*(\mu)(E) := \int_F K(x)(E)\mu(dx)
\]

for \(E \in \mathcal{H}\) (remember, stochastic relation \(K\) is really a Kleisli morphism for the Giry monad, \(K^*\) is its Kleisli extension). Call then the equivalence relation \(\xi\) a random friend to the

\[\footnote{The present author does not know whether a random friend is a casual acquaintance, or a friend for life, or something in between.}\]
equivalence relation $\zeta$ iff we have $K^*(\mu) \preceq K^*(\nu)$ provided $\mu \preceq \nu$ with $\mu, \nu \in \mathcal{G}(F,F)$. An equivalent formulation without explicit randomization reads

$$\ker(\mathcal{G}(m_\xi)) \subseteq (K^* \times K^*)^{-1}[\ker(\mathcal{G}(m_\zeta))],$$

where $m$ is defined in Lemma 3.2. Transporting these ideas to automata, one would have to decide whether one wants friendship of the level of, say, $(\alpha \times \gamma)^\circ$, or of $\alpha^\circ \times \gamma^\circ$; the latter one indicates a much tighter pairing than the former one (recall that a finite measure on a product space in not necessarily a product measure).

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