QUIVER VARIETIES AND SELF-INJECTIVE ALGEBRAS

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ABSTRACT. In this paper, we introduce a new class of quiver varieties, recovering as special cases the cyclic and classical Nakajima quiver varieties. We show that the geometry of the new quiver varieties is closely linked to the representation theory of a suitable finitely-generated algebra $\tilde{P}$, which is self-injective if $Q$ is of ADE Dynkin type. The affine quiver variety, defined as GIT quotient, is shown to be isomorphic to representations of a finitely-generated algebra $\tilde{S}$. We construct a functor $\Psi$ establishing a bijection between the strata of the affine quiver variety and the isomorphism classes of objects in $\text{proj} \tilde{P}$. Under $\Psi$, the degeneration order between strata corresponds to Jensen–Su–Zimmermann’s degeneration order on $\text{proj} \tilde{P}$. Furthermore, we use $\Psi$ to realize the fibres of Nakajima’s projection map $\pi$ as quiver Grassmannians of $\tilde{P}$. The algebra $\tilde{P}$ specializes to the preprojective algebra if we consider classical quiver varieties. Finally, we use our results to construct desingularisations of quiver Grassmannians of modules of self-injective algebras of finite type.

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1. Introduction

Quiver varieties associated with a finite and acyclic quiver $Q$ were introduced by Nakajima in [37]. Since then they have been of great importance in Nakajima’s geometric study of Kac Moody algebras and their representations [37], [38]. Furthermore, Nakajima quiver varieties yield important examples of symplectic hyper-Kähler manifolds appearing in various fields of representation theory.

The graded quiver varieties, defined as fixed point sets of ordinary quiver varieties in [39], found an interesting application in constructing a monoidal categorification of cluster algebras (see [40], [41], [30]) in the sense of Hernandez and Leclerc [20] and [21]. Cyclic quiver varieties have recently been used by Qin to realize a geometric construction of quantum groups [43]. Motivated by the construction of monoidal categorifications of Cluster algebras, we studied in [28] properties of graded quiver varieties.

In this work we introduce generalized quiver varieties and investigate many of their properties. Classical and cyclic quiver varieties are both recovered as special cases of our generalized quiver varieties. This allows us to achieve two goals. First, we can prove results for generalized quiver varieties that are new even for classical and cyclic quiver varieties in a uniform way. Second, we extend to generalized quiver varieties many results that were known for classical and cyclic quiver varieties. Furthermore, we show that generalized quiver varieties give the right framework to construct desingularisation maps for quiver Grassmannians of self-injective algebras as explained below.

To the choice of dimension vectors $w$ and $v$ for the set of vertices of $Q$, Nakajima associates the smooth quiver variety $\mathcal{M}(v, w)$ and the affine quiver variety $\mathcal{M}_0(w)$. Two features of quiver varieties play an important role in their application to Kac Moody algebras and cluster algebras:

(1) There is a proper and projective map $\pi : \mathcal{M}(v, w) \to \mathcal{M}_0(w)$. 
2. If \( Q \) is of Dynkin type, i.e. a quiver whose underlying graph is an ADE Dynkin diagram, then \( \pi \) induces a stratification

\[
\mathcal{M}_0(w) = \bigsqcup_v \mathcal{M}_0^{\text{reg}}(v, w)
\]

into finitely many non-empty smooth and locally closed strata.

We will show that generalized quiver varieties do satisfy analogous properties. Furthermore, we will show that their geometry is governed by an associated self-injective algebra. This greatly generalizes work of \([28]\) and extends results of \([49]\). In the case of classical Nakajima quiver varieties, the self-injective algebra describing the geometry of quiver varieties is the preprojective algebra \( \mathcal{P}_Q \), associated to the quiver \( Q \). Preprojective algebras were introduced in \([14]\) and play an important role in Lusztig’s study of canonical bases \([32]\), \([33]\).

1.1. Generalized Nakajima categories and quiver varieties. To treat the case of classical and cyclic quiver varieties simultaneously, we introduce a new class of Nakajima quiver varieties, the generalized quiver varieties. Namely, we consider configurations of the regular (graded) Nakajima categories \( \mathcal{R}_C^{gr} \), which are certain quotient categories of the Nakajima category \( \mathcal{R}_{C}^{gr} \), and were introduced in \([28]\). Taking the orbit category with respect to an isomorphism \( F \) of \( \mathcal{R}_C^{gr} \) yields the generalized regular Nakajima category \( \mathcal{\tilde{R}} \). In analogy to the classical case, we can define generalized quiver varieties associated to \( \mathcal{\tilde{R}} \) using geometric invariant theory and obtain two varieties \( \mathcal{M}(v, w) \) and \( \mathcal{M}_0(v, w) \) satisfying the properties (1) and (2). The cyclic and classical quiver varieties appear as special case of this construction.

Associated to the regular Nakajima category \( \mathcal{\tilde{R}} \), we consider a certain full subcategory \( \mathcal{\tilde{S}} \) of \( \mathcal{\tilde{R}} \), the singular Nakajima category. We show that the quotient category \( \mathcal{\tilde{P}} \cong \mathcal{\tilde{R}}/\mathcal{\tilde{S}} \) describes the geometry of these generalized quiver varieties, that is their stratification into strata \( \mathcal{M}_0^{\text{reg}}(v, w) \), the degeneration order between the strata and the fibres of \( \pi \). In fact, if we consider the special case of classical quiver varieties, \( \mathcal{\tilde{P}} \) is isomorphic to the preprojective algebra \( \mathcal{P}_Q \). We show that if \( Q \) is a Dynkin quiver the category \( \mathcal{\tilde{P}} \) has similar properties than the preprojective algebra: it is self-injective, finite-dimensional and its category of projective modules is triangulated. The triangulated structure of projective \( \mathcal{\tilde{P}} \)-modules will be of crucial to establish an equivalence between the degeneration order on strata and a degeneration order on triangulated categories defined by \([23]\).

1.2. Affine quiver varieties as moduli spaces. It was shown in \([28]\) and \([31]\) that graded affine quiver varieties can be realized as spaces of representations of the (graded) singular Nakajima category \( \mathcal{S}^{gr} \). As shown in Theorem \([5,5]\), an analogous result holds for affine quiver varieties associated with quivers \( Q \) whose diagram is of Dynkin type. As algebraic variety, the
affine quiver variety $\mathcal{M}_0(w)$ is isomorphic to $\text{rep}(w, \tilde{S})$, the space of representations with dimension vector $w$ of the singular Nakajima category $\tilde{S}$. If $Q$ is not of Dynkin type, $\text{rep}(w, \tilde{S})$ is isomorphic to the union the smooth spaces $\mathcal{M}_0^{\text{reg}}(v, w)$.

Motivated by this result, we give a more explicit description of $\tilde{S}$. If $Q$ is of Dynkin type, then $\tilde{S}$ is given as an algebra by $\mathbb{C}Q_{\tilde{S}}/I$, where $Q_{\tilde{S}}$ is a finite quiver and $I$ is an admissible ideal of the path algebra of $Q_{\tilde{S}}$. We determine the quiver $Q_{\tilde{S}}$ and the number of minimal relations of paths between two vertices of $Q_{\tilde{S}}$ in terms of dimensions of morphism spaces in $\tilde{P}$ (see Proposition 3.16). Hence the self-injective algebra $\tilde{P}$ plays a key role in the description of the affine quiver variety.

1.3. Stratification of affine quiver varieties. The main part of this work consists in the construction of a so-called stratification functor $\Psi$ from the category of finite-dimensional $\tilde{S}$–modules to the category $\text{inj}^{\text{nil}}\tilde{P}$ of finitely-cogenerated injective $\tilde{P}$–modules that have nilpotent socle. The functor $\Psi$ is called a stratification functor as it satisfies the following properties:

- two points $M$ and $M'$ belonging to the regular part $\bigcup_v \mathcal{M}^{\text{reg}}_0(v, w) \cong \text{rep}(w, \tilde{S})$ of $\mathcal{M}_0(w)$ map into the same stratum if and only if their images under $\Psi$ are isomorphic (Theorem 4.7 and 4.8).
- If $Q$ is of Dynkin type, $\text{inj}^{\text{nil}}\tilde{P}$ equals $\text{proj}\tilde{P}$, the category of finitely-generated projective $\tilde{P}$–modules which has a triangulated structure. In this case, we show that $\Psi$ is a $\delta$–functor and behaves well with respect to the degeneration order on strata: the degeneration order in $\mathcal{M}_0(w)$ corresponds under the stratification functor to the degeneration order in the sense of [23] applied to the triangulated category $\tilde{P}$.
- Finally, we use $\Psi$ to describe the fibers of Nakajima’s map $\pi$ as quiver Grassmannians of modules of $\tilde{P}$ extending thereby results of Savage–Tingley [49] (see Theorems 4.13 and 4.14).

1.4. Frobenius models of orbit categories of the derived category. Let $Q$ be of Dynkin type and let us consider a category $\mathcal{P}$ given as an orbit category of $D_Q$, the bounded derived category of finite-dimensional representations of $Q$, by an isomorphism $F$ satisfying the assumptions in [27]. Let $\mathcal{R}$ be the regular Nakajima category defined in Section 2. This category depends on the choice of an admissible configuration $C$, which is a subset of the vertices of the repetition quiver $\mathbb{Z}Q$ and on an isomorphism $F'$ of the mesh category $k(\mathbb{Z}Q)$ satisfying certain conditions. Via Happel’s Theorem (see [22]), the functor $F'$ extends to an isomorphisms of $D_Q$. If $F \cong F'$ as isomorphisms of $D_Q$, then we show that $\text{proj}\mathcal{R}$, the category of finitely generated projective $\mathcal{R}$–modules, is a Frobenius model for $\tilde{P}$.
Conversely, we show that all Frobenius models of $\tilde{P}$ which are standard are indeed equivalent to $\text{proj} \tilde{\mathcal{R}}$ for some pair $F'$ and $C$ such that $F' \cong F$ seen as isomorphisms of $\mathcal{D}_Q$.

1.5. **Desingularization of quiver Grassmanians.** In the last section, we apply our results to quiver Grassmannians for modules of self-injective algebras of finite type. We show that self-injective algebras can be realized as singular Nakajima categories $\tilde{S}$ and use techniques developed in [29] to construct desingularisation maps for quiver Grassmannians of modules of self-injective algebras of finite type. The domain of the desingularisation maps are smooth quiver Grassmannians of $\tilde{\mathcal{R}}$–modules (see Theorem 6.4).

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2. **Generalized Nakajima categories**

2.1. **Notations.** For later use, we introduce the following notations. Let $k$ be a field and $\text{Mod}k$ be the category of $k$-vector spaces. Recall that a $k$-category is a category whose morphism spaces are endowed with a $k$-vector space structure such that the composition is bilinear. Let $\mathcal{C}$ be a $k$-category and let $\text{Mod}(\mathcal{C})$ be the category of right $\mathcal{C}$–modules, i.e. $k$-linear functors $\mathcal{C}^{\text{op}} \rightarrow \text{Mod}(k)$. For each object $x$ of $\mathcal{C}$, we obtain a free module $x^\wedge = x^\wedge_\mathcal{C} = \mathcal{C}(?,x) : \mathcal{C}^{\text{op}} \rightarrow \text{Mod}k$ and a cofree module $x^\vee = x^\vee_\mathcal{C} = D(\mathcal{C}(x,?)) : \mathcal{C}^{\text{op}} \rightarrow \text{Mod}k$.

Here, we write $\mathcal{C}(u,v)$ for the space of morphisms $\text{Hom}_\mathcal{C}(u,v)$ and $D$ for the duality over the ground field $k$. Recall that for each object $x$ of $\mathcal{C}$ and each $\mathcal{C}$–module $M$, we have canonical isomorphisms

\begin{equation}
\text{Hom}(x^\wedge,M) = M(x) \quad \text{and} \quad \text{Hom}(M,x^\vee) = D(M(x)).
\end{equation}

In particular, the module $x^\wedge$ is projective and $x^\vee$ is injective. We will denote $\text{proj} \mathcal{C}$ the full subcategory of $\mathcal{C}$–modules with objects the finite direct sums of objects $x^\wedge$ and dually, we denote $\text{inj} \mathcal{C}$ the full subcategory of $\mathcal{C}$–modules with objects the finite direct sums of objects $x^\vee$.

Furthermore, we denote throughout the paper by $\mathcal{C}_0$ the set of objects of $\mathcal{C}$ and mean by a dimension vector of $\mathcal{C}$ a function $w : \mathcal{C}_0 \rightarrow \mathbb{N}$ with finite support. We define $\text{rep}(\mathcal{C},w)$ to be the space of $\mathcal{C}$–modules $M$ such that $M(u) = k^{w(u)}$ for each object $u$ in $\mathcal{C}_0$.

Finally, we call a $\mathcal{C}$–module $M$ pointwise finite-dimensional if $M(x)$ is finite-dimensional for each object $x$ of $\mathcal{C}$. For an object $x$, we denote by $S_x$
the simple module given by the functor that sends the object \( x \) to \( k \) and every other object to zero.

2.2. Nakajima Categories and Happel’s Theorem. In this section we define the singular and regular Nakajima categories \( R \) and \( S \) both in the graded and ungraded setting. The regular Nakajima category is a mesh category which we can associate to any acyclic finite quiver and \( S \) is a full subcategory of \( R \). The graded Nakajima categories have also been defined in [28], but for the convenience of the reader, we give a short definition here. In [28], we denoted the graded Nakajima categories by \( R \) and \( S \), here we will denote them by \( R^{gr} \) and \( S^{gr} \) to distinguish them from the ungraded Nakajima categories.

Let \( Q \) be a finite acyclic quiver with set of vertices \( Q_0 \) and set of arrows \( Q_1 \). We will assume throughout the article that \( Q \) is connected. This allows a simplified exposition of results as we will have to differentiate between the case when \( Q \) is of Dynkin type, that is the unoriented underlying graph of \( Q \) is a Dynkin diagram, or not. The framed quiver \( Q^f \) is obtained from \( Q \) by adding, for each vertex \( i \), a new vertex \( i' \) and a new arrow \( i \to i' \). For example, if \( Q \) is the quiver \( 1 \to 2 \), the framed quiver is

\[
\begin{array}{c}
\cdot \cdot \cdot \\
\tau(x) \xrightarrow[\sigma(x)]{} x \xrightarrow[\sigma^{-1}(x)]{} \tau^{-1}(x) \\
\end{array}
\]

Let \( ZQ^f \) be the repetition quiver of \( Q^f \) (see [44]). We refer to the vertices \((i',p), i \in Q_0, p \in \mathbb{Z}\) as the frozen vertices of \( ZQ^f \). We define the automorphism \( \tau \) of \( ZQ \) to be the shift by one unit to the left, so that we have in particular \( \tau(i,p) = (i,p-1) \) for all vertices \((i,p) \in Q_0 \times \mathbb{Z} \). Similarly, we define \( \sigma \) by \((i,p) \mapsto (i',p-1) \) and \((i',p) \mapsto (i,p) \) for all \( i \in Q_0, p \in \mathbb{Z} \).

For example, if \( Q \) is the Dynkin diagram \( A_2 \), the repetition \( ZQ^f \) is the quiver

\[
\begin{array}{c}
\cdot \cdot \cdot \\
\tau(x) \xrightarrow[\sigma(x)]{} x \xrightarrow[\sigma^{-1}(x)]{} \tau^{-1}(x) \\
\end{array}
\]

Finally we denote by \( \bar{\alpha} \) the unique arrow to \( \alpha : y \to x \) that runs from \( \tau x \to y \). As in [11] and [44], we denote by \( k(ZQ) \) the mesh category of \( ZQ \), that is the objects are given by the vertices of \( ZQ \) and the morphisms are \( k \)-linear combinations of paths modulo the ideal spanned by the mesh relations

\[
R_x : \sum_{\alpha: y \to x} \bar{\alpha} \alpha,
\]

where the sum runs through all arrows of \( ZQ \) starting in \( \tau x \).
By Happel’s Proposition 4.6 of [18] and Theorem 5.6 of [19], there is a fully faithful embedding

\[ H : k(ZQ) \hookrightarrow \text{ind} \mathcal{D}_Q \]

where \( \text{ind} \mathcal{D}_Q \) denotes the category of indecomposable complexes in the bounded derived category of \( \text{mod} \, kQ \). The functor \( H \) is an equivalence if and only if \( Q \) is of Dynkin type.

The action of \( \tau \) on \( ZQ \) corresponds to the action of the Auslander-Reiten translation on \( \mathcal{D}_Q \) if \( Q \) is a Dynkin quiver. We will therefore use also the notation \( \tau \) for the Auslander-Reiten translation.

The regular Nakajima category \( \mathcal{R}^{gr} \) has objects the vertices of \( ZQ^f \) and the morphism space from \( a \) to \( b \) is the space of all \( k \)-linear combinations of paths from \( a \) to \( b \) modulo the subspace spanned by all elements \( uR_xv \), where \( u \) and \( v \) are path and \( x \) is a non-frozen vertex.

Note that \( \mathcal{R}^{gr} \) is not equivalent to the mesh category \( k(ZQ^f) \), as we do not impose mesh relations in frozen objects.

The singular graded Nakajima category \( \mathcal{S}^{gr} \) is the full subcategory of \( \mathcal{R}^{gr} \) whose objects are the frozen vertices.

We define the Nakajima category in the ungraded setting using the double quiver.

The double quiver \( \overline{Q} \) is obtained from \( Q \) by adding, for each arrow \( \alpha \), a new arrow \( \overline{\alpha} \) with inverted orientation. For example, if \( Q \) is the quiver \( 1 \rightarrow 2 \), the doubled quiver is

\[
\begin{array}{c}
1 & \overline{1} & \rightarrow & 2 & \overline{2} \\
\end{array}
\]

The regular Nakajima category \( \mathcal{R} \) is the category with objects the vertices of \( \overline{Q}^f \) and whose morphism space from \( a \) to \( b \) is the space of all \( k \)-linear combinations of paths from \( a \) to \( b \) modulo the subspace spanned by all elements \( ur_xv \), where \( u \) and \( v \) are paths and

\[ r_x = \sum_{\beta: y \rightarrow x} \overline{\beta} \beta - \sum_{\alpha: x \rightarrow y} \overline{\alpha} \alpha \]

is the mesh relation associated with a vertex \( x \) of \( Q \). Here the sum runs over all arrows of \( Q^f \). We call the objects corresponding to the \( i' \) for \( i \in Q_0 \) the frozen vertices. For example, if \( Q \) is the quiver \( 1 \rightarrow 2 \), the category \( \mathcal{R} \) has as objects the vertices of the quiver

\[
\begin{array}{c}
\begin{array}{c}
2 & \overline{a} & \rightarrow & 2' \\
\downarrow_{\overline{\gamma}} & & & \downarrow_{\overline{\gamma'}} \\
1 & \overline{\beta} & \rightarrow & 1' \\
\end{array}
\end{array}
\]

and the mesh relations are given by

\[ \alpha \overline{\alpha} - \overline{\gamma} \gamma = 0 \text{ and } \beta \overline{\beta} + \gamma \overline{\gamma} = 0. \]
The singular (ungraded) Nakajima category $S$ is the full subcategory of $R$ whose objects are the frozen vertices.

2.3. Configurations and orbit categories of Nakajima categories. Let $C$ be a subset of the set of vertices of the repetition quiver $\mathbb{Z}Q$. We denote $R^\text{gr}_C$ the quotient of $R^\text{gr}$ by the ideal generated by the identities of the frozen vertices not belonging to $\sigma(C)$ and by $S^\text{gr}_C$ the full subcategory of $R^\text{gr}_C$ formed by the objects corresponding to the vertices in $\sigma(C)$. Furthermore, we denote by $\mathbb{Z}Q_C$ the quiver obtained from $\mathbb{Z}Q$ by adding for all $x \in C$ a vertex $\sigma(x)$ and arrows $\tau(x) \to \sigma(x)$ and $\sigma(x) \to x$.

Let $F$ be a $k$-linear isomorphism on $k(\mathbb{Z}Q)$. We make the following assumption on $C$ and $F$.

**Assumption 2.4.** For each vertex $x$ of $\mathbb{Z}Q$, the sequences

$$(2.4.1) \quad 0 \to R^\text{gr}_C(?, x) \to \bigoplus_{x \to y} R^\text{gr}_C(?, y) \quad \text{and} \quad 0 \to R^\text{gr}_C(x, ?) \to \bigoplus_{y \to x} R^\text{gr}_C(y, ?)$$

are exact, where the sums range over all arrows of $\mathbb{Z}Q$ whose source (respectively, target) is $x$.

Furthermore, we have that $F(C) \subseteq C$ and $F^n \neq 1$ for all $n \in \mathbb{Z}$.

We call all $C$ satisfying the above assumption admissible configuration and $(C, F)$ an admissible pair. For example if $C$ is the set of all vertices of $\mathbb{Z}Q$, then $C$ is admissible. Another class of examples is given in [31].

Note that $F$ commutes with $\tau$ and extends uniquely to $R^\text{gr}_C$ by setting $F\sigma(c) := \sigma(F(c))$ for all $c \in C$. We will denote by $\tilde{R}$ the orbit category of $R^\text{gr}_C/F$ and by $\tilde{S}$ its full subcategory with objects $\sigma(C)$. Note that the quotient $R^\text{gr}_C/\langle S^\text{gr}_C \rangle \cong k(\mathbb{Z}Q)$ does not depend on $C$ and is equivalent to the image of the Happel functor $H$. Furthermore $F$ lifts to an isomorphism on the quotient. It is easy to see that the orbit category of $R^\text{gr}_C/\langle S^\text{gr}_C \rangle$ by $F$ is naturally equivalent to the quotient $\tilde{P} := \tilde{R}/\langle \tilde{S} \rangle$.

In the case that $F = \tau$ and that $C$ is the set of all vertices of $\mathbb{Z}Q$, our assumptions are satisfied and we recover $\tilde{R} = \mathcal{R}$, $\tilde{S} = \mathcal{S}$ as introduced in the previous section. We see that $\tilde{P}$ is equivalent to the preprojective algebra $\mathcal{P}_Q$ which carries a triangulated structure if $Q$ is of Dynkin type. This holds in greater generality as shows the next lemma.

**Lemma 2.5.** Suppose that $Q$ is of Dynkin type. Under the assumption of 2.4, $F$ lifts to a triangulated functor of $\mathcal{D}_Q$, the category $\tilde{P} \cong \mathcal{D}_Q/F$ is triangulated, $\tilde{P}$ is Hom-finite, has up to isomorphism only finitely many objects and proj $\tilde{P} = \text{inj} \tilde{P}$.

**Proof.** By definition, $F$ sends Auslander-Reiten sequences to Auslander-Reiten sequence and therefore commutes with $\tau$. By Serre duality, we conclude that

$$\text{Ext}^1(X, Y) \cong D \text{Hom}(\tau X, Y) \cong D \text{Hom}(\tau F(X), F(Y)) \cong \text{Ext}^1(FX, FY),$$
hence $F(kQ)$ is a tilting object in $\mathcal{D}_Q$ and induces an auto equivalence of $\mathcal{D}_Q$. As $F$ acts without torsion, we have that $F^n = \tau^k$ for some $k, n \in \mathbb{Z} - 0$ and as $Q$ is of Dynkin type, we also know that there are $s, l \in \mathbb{Z} - 0$ with $\tau^s \cong \Sigma^l$. Hence $F^m \cong \Sigma^h$ for some non-zero integers $m$ and $h$. From this equality, it is easy to see that all conditions of [27] are satisfied and as a consequence $\bar{\mathcal{P}}$ is naturally a triangulated category. Furthermore, we have that $F^n = \tau^k$ for some $k, n \in \mathbb{Z}$. Hence $F$ has only finitely many orbits and therefore $\bar{\mathcal{P}}$ has only finitely many objects up to isomorphism. As $F$ does not have a fixed point on ind $\mathcal{D}_Q$, and all projective $\mathcal{D}_Q$–modules are finite-dimensional, we know that $\mathcal{D}_Q(x, F^i(y))$ vanishes for all but finitely many $i \in \mathbb{Z}$. Hence $\bar{\mathcal{P}}$ is Hom-finite. As $F$ is a triangulated functor on $\mathcal{D}_Q$, $F$ commutes with the Serre functor $S$. We find that for all $x \in \text{ind} \mathcal{D}_Q$ the following isomorphism is given

$$x^\vee_{\bar{\mathcal{P}}} = D\bar{\mathcal{P}}(?, x) \cong D\bigoplus_{i \in \mathbb{Z}} \mathcal{D}_Q(?, F^i x) \cong \bigoplus_{i \in \mathbb{Z}} D\mathcal{D}_Q(?, F^i x) \cong \mathcal{D}_Q(S^{-1}F^i x, ?) \cong \bar{\mathcal{P}}(S^{-1}x, ?) = (S^{-1}x)_{F^i}.$$ 

Note that as $\mathcal{D}_Q \to \bar{\mathcal{P}}$ is a triangulated functor, the shift functor $\Sigma$ induces an auto equivalence on $\bar{\mathcal{P}}$ which we will also denote by $\Sigma$. Furthermore, as $F^m \cong \tau^k$ and $\Sigma^2$ is also equivalent to a power of $\tau$, we find that $\bar{\mathcal{P}}$ is fractional Calabi-Yau, that is $S^k \cong \Sigma^k$ for some $k \in \mathbb{Z}$.

Finally, we denote by $\bar{Q}$ the quiver $\mathbb{Z}Q_C/F$ with vertices the $F$–orbits on the set of objects of $\mathcal{R}^{gr}_C$ and the number of arrows $x \to y$ between two fix representatives $x$ and $y$ of $F$–orbits is given by the numbers of arrows from $x \to F^i y$ for all $i \in \mathbb{Z}$ in the quiver $\mathbb{Z}Q_C$. By our condition on $F$, it is easy to see that $\bar{Q}$ is a finite quiver and the canonical map $\mathbb{Z}Q_C \to \bar{Q}$ is a Galois covering. For example the quiver $\bar{Q}_{f}$ of the previous section identifies with $\bar{Q}$ for the choice of $F = \tau$ and $C = (\mathbb{Z}Q)_0$. In the sequel, we will identify the orbit categories $\bar{\mathcal{R}}, \bar{\mathcal{S}}$ and $\bar{\mathcal{P}}$ with their equivalent skeleta categories, in which we identify all objects lying in the same $F$–orbit. Note also, that seen as an algebra, $\bar{\mathcal{R}}$ is equivalent to a path algebra of $\bar{Q}$.

**Example 2.6.** We consider the example, where $Q = A_2$ and $C = (\mathbb{Z}Q)_0$. Let us consider the triangulated functor $F = \Sigma \tau$ acting on $\mathcal{D}_Q$ and by Happel’s theorem on the mesh category $k(\mathbb{Z}Q)$. The action of $F$ can be extended to $\mathcal{R}$ by setting $F(\sigma(x)) = \sigma(F(x))$ for all $x \in \mathcal{R} - \mathcal{S}$ which we can naturally identify with objects of $k(\mathbb{Z}Q)$. Clearly $F$ satisfies all conditions in [24] and $\bar{\mathcal{R}} := \mathcal{R}/F$ is given by the path category to the quiver $\bar{Q}$ obtained by
identifying the two vertices labelled by $S_1$ and $P_2$ respectively

$$
P_2 \rightarrow \sigma(\Sigma S_2) \rightarrow \Sigma S_1 \rightarrow \sigma(S_1) \rightarrow S_1
$$

$$
S_1 \rightarrow \sigma(S_2) \rightarrow S_2 \rightarrow \sigma(\Sigma P_2) \rightarrow \Sigma P_2 \rightarrow \sigma(P_2) \rightarrow P_2
$$

The relations in $\tilde{R}$ are given by the mesh relations in the non-frozen objects corresponding to the vertices $S_1, P_2, S_2, \Sigma S_1, \Sigma P_2$. The labelling of the vertices corresponds to the position of the indecomposable objects in the Auslander-Reiten quiver of the derived category of $\mathcal{D}_Q$, that is $S_1$ and $S_2$ are the stalk complexes of the simple $kQ$–modules to the vertices 1 and 2 and $P_2$ is the projective $kQ$–module to the vertex 2 seen as a stalk complex.

Furthermore $\mathcal{D}_Q/F$ is the Cluster category associated to $Q$ introduced in [4] which is triangulated by [27].

2.7. Kan extensions and Stability. In [29], we introduced in more generality the notion of intermediate extensions and stability. We recall them here adapted to the setup of this paper. We call an $\tilde{R}$–module $M$ stable if $\text{Hom}_{\tilde{R}}(S, M) = 0$ vanishes for all modules $S$ supported only in non-frozen vertices. Equivalently, $M$ does not contain any non zero submodule supported only on non frozen vertices. We call $M$ costable if we have $\text{Hom}_{\tilde{R}}(M, S) = 0$ for each module $S$ supported only in non-frozen vertices. Equivalently, $M$ does not contain any non zero quotient supported only on non frozen vertices. A module is bistable, if it is both stable and costable.

As the restriction functor

$$
\text{res} : \text{Mod} \tilde{R} \rightarrow \text{Mod} \tilde{S}
$$

is a localization functor in the sense of [10], it admits a right and a left adjoint which we denote $K_R$ and $K_L$ respectively: the left and right Kan extension cf. [35]. We define the intermediate extension

$$
K_{LR} : \text{Mod} \tilde{S} \rightarrow \text{Mod} \tilde{R}
$$

as the image of the canonical map $K_R \rightarrow K_L$ (see [29] for general properties). To distinguish the graded from the non graded case, we will denote the Kan extensions by $K^g_R$, $K^g_L$ and $K^g_{LR}$ in the graded setting.

In the next Lemma, we summarize some basic properties of Kan extensions used in the sequel of this article. They are proven in Lemma 2.2 of [29].

**Lemma 2.8.** Let $M$ be a $\tilde{R}$–module, then the following holds.

1. The adjunction morphism $M \rightarrow K_R \text{res} M$ is injective if and only if $M$ is stable.
2. Dually, $M$ is costable if the adjunction morphism $K_L \text{res} M \rightarrow M$ is surjective.
3. $M$ is bistable if and only if $K_{LR} \text{res} M \cong M$. 
The adjunction morphism $M \to K_R \text{res} M$ is invertible if and only if $M$ is stable and $\text{Ext}^1(N, M)$ vanishes for all $N$ which lie in the kernel of res.

Dually, $K_L \text{res} M \to M$ is invertible if and only if $M$ is costable and $\text{Ext}^1(M, N)$ vanishes for all $N$ which lie in the kernel of res.

Note that, as a consequence of (3), the intermediate extension $K_{LR}$ establishes an equivalence between $\text{mod} \tilde{S}$ and the full subcategory of $\text{mod} \tilde{R}$ with objects the bistable modules.

In this article we construct a functor from representations of the singular Nakajima category $\tilde{S}$ to $\tilde{P}$, establishing a bijection between the strata of the affine quiver variety and the objects of $\tilde{P}$ (see Theorem 4.7).

To this end, we define for every $\tilde{S}$–module $M$ the modules $CK(M)$ and $KK(M)$ given by

$$CK(M) = \text{ker}(K_L(M) \to K_{LR}(M))$$
$$KK(M) = \text{cok}(K_{LR}(M) \to K_R(M)).$$

Note that both $CK$ and $KK$ are supported only in $\tilde{R}_0 - \tilde{S}_0$. Now we have an obvious isomorphisms $\tilde{R}/\langle \tilde{S} \rangle \to \tilde{P}$. Therefore, we may view $CK(M)$ and $KK(M)$ as $\tilde{P}$–modules. In the graded setting, we have an isomorphism $R^{gr}/\langle S^{gr} \rangle \to D_{Q}$ and the respective functors in the graded setting which we denote here by $CK^{gr}$ and $KK^{gr}$ can be seen as functors from $\text{Mod} S^{gr}$ to $\text{Mod} D_Q$.

2.9. **Preprojective algebras.** Let us denote by $P_Q$ the preprojective algebra associated with $Q$ and by $D_Q$ the bounded derived category of finite-dimension $kQ$–modules. The preprojective algebra is the path algebra $kQ/I$ where $I$ is the ideal generated by the mesh relations $r_x$ for all $x \in Q_0$. Recall that

$$r_x = \sum_{\beta: y \to x} \beta \beta - \sum_{\alpha: x \to y} \alpha \alpha$$

is the mesh relation associated with a vertex $x$ of $Q$, where the sum runs over all arrows of $Q$.

It will also be convenient to view $P_Q$ as the mesh category associated to $Q$, that is the objects of $P_Q$ are the vertices of $Q$ and the morphisms are the $k$–linear combinations of paths in $Q$ modulo the ideal generated by the mesh relations $r_x$. From this point of view $\text{proj} P_Q$ and $P_Q$ are equivalent categories.

If $Q$ is of Dynkin type, the isomorphism class of $P_Q$ does not depend on the orientation of $Q$, that is quivers with underlying graph the same Dynkin diagram give rise to isomorphic categories $P_Q$. We will use the following well-known properties of preprojective algebras. The first statement can be found in [47].

**Theorem 2.10.** The following holds.
The algebra $\mathcal{P}_Q$ is finite-dimensional if and only if $Q$ is of Dynkin type.

If $Q$ is of Dynkin type, $\text{mod}\, \mathcal{P}_Q$ is self-injective and otherwise $\text{mod}\, \mathcal{P}_Q$ is of global dimension two.

Furthermore we know by [27, 7.3] that $\text{proj}\, \mathcal{P}_Q = \text{inj}\, \mathcal{P}_Q$ is equivalent to the orbit category $\mathcal{D}_Q/\tau$ if $Q$ is a Dynkin quiver. Hence, the shift $\Sigma$ in the derived category induces an auto equivalence on $\text{proj}\, \mathcal{P}_Q$ which we will also denote by $\Sigma$. As $\Sigma^2$ is isomorphic to a power of $\tau$ on $\mathcal{D}_Q$, it acts as the identity on $\text{proj}\, \mathcal{P}_Q$. Furthermore $\Sigma$ is the Serre functor of $\text{proj}\, \mathcal{P}_Q$ and therefore $\text{proj}\, \mathcal{P}_Q$ is a one Calabi-Yau triangulated category. Note that $\Sigma$ seen as an automorphism of $\text{proj}\, \mathcal{P}_Q$ is given the Nakayama automorphism $\nu := D(\mathcal{P}_Q) \otimes -$. As the category $\text{proj}\, \mathcal{P}_Q$ is equivalent to the mesh category $\mathcal{P}_Q$, the functor $\Sigma$ can be seen as an auto equivalence of $\mathcal{P}_Q$.

3. Generalized Nakajima quiver varieties

Nakajima introduces in [39] two types of quiver varieties, the affine quiver variety $M_0$ and the smooth quiver variety $M$. Both quiver varieties give rise to a graded version: the graded affine quiver variety and the graded quiver variety, which we denote in this paper by $M^g_0$ and $M^g$.

We define generalized quiver varieties as geometric invariant quotients of representations of the regular Nakajima category $\tilde{\mathcal{R}}$. To obtain the definition of the graded quiver variety it suffices to replace $\tilde{\mathcal{R}}$ by $\mathcal{R}^g$. We refer to [28] for more details. As a special case of generalized quiver varieties, we obtain the classical quiver varieties for the choice $\tilde{\mathcal{R}} = \mathcal{R}$ and $\mathcal{S} = S$ and the cyclic quiver varieties for the choice of $F = \tau^n$ for some $n \in \mathbb{N}$.

We fix two dimension vector $v : \mathcal{R}_0 - \mathcal{S}_0 \to \mathbb{N}$ and $w : \mathcal{S}_0 \to \mathbb{N}$ and set $\text{St}(v, w)$ to be the subset of $\text{rep}(v, w, \tilde{\mathcal{R}})$ consisting of all $\tilde{\mathcal{R}}$–modules with dimension vector $(v, w)$ that are stable in the sense of [2.7]. Let $G_v$ be the product of the groups $\text{GL}(k^{v(x)})$, where $x$ runs through the non frozen vertices. By base change in the spaces $k^{v(x)}$, the group $G_v$ acts freely on the set $\text{St}(v, w)$. The generalized quiver variety $M(v, w)$ is the quotient $\text{St}(v, w)/G_v$.

For the next statement, we refer to Nakajima’s work [39, 40] for the case where $Q$ is Dynkin or bipartite and to Qin [41, 42] and Kimura-Qin [30] for the extension to the case of an arbitrary acyclic quiver $Q$.

Theorem 3.1. The $G_v$–action is free on $\text{St}(v, w)$ and the quasi-projective variety $M(v, w)$ is smooth.

Proof. Suppose that $gM = M$ for some $g \in G_v$ and $M \in \text{St}(v, w)$. Then $\text{im}(1 - g)M$ is a submodule of $M$ which has support only in objects $\mathcal{R}_0 - \mathcal{S}_0$. This is a contradiction to the stability of $M$. As the action of $G_v$ is free, it
remains to show that $Sl(v, w)$ is smooth. We consider the map
\[ \nu : rep(v, w, \tilde{Q}) \rightarrow \bigoplus_{x \in \tilde{R}_0} \text{Hom}(C^{v(x)}, C^{v(\tau x)}), M \mapsto \bigoplus_{x \in \tilde{R}_0 - \tilde{S}_0} \sum_{\alpha:y \rightarrow x} M(\tilde{\alpha})M(\alpha) \]
where the sum runs through all arrows $\alpha : y \rightarrow x$ in $\tilde{Q}$ and $\tilde{\alpha}$ denotes the unique arrow $\tau x \rightarrow y$ in $\tilde{Q}$ corresponding to $\alpha$. The set $\nu^{-1}(0)$ can be identified with $rep(v, w, \tilde{R})$. Furthermore the tangent space at a point $M \in rep(v, w, \tilde{R})$ corresponds to the space of all $N \in rep(v, w, Q)$ satisfying that
\[ d\nu_M(N) = \bigoplus_{x \in \tilde{R}_0} \sum_{\alpha:y \rightarrow x} N(\tilde{\alpha})M(\alpha) + M(\tilde{\alpha})N(\alpha) \]
vanishes. We show that in every mesh to $x \in \tilde{R}_0 - \tilde{S}_0$, and every $f \in \text{Hom}(C^{v(x)}, C^{v(\tau x)})$ there is a representation $N \in rep(v, w, Q)$ with image $f$. Clearly, if $M_x : M(x) \rightarrow M(\sigma(x))$ is injective, this is possible by choosing $N_{\sigma(x)} : C^{w(\sigma(x))} \rightarrow C^{v(\tau x)}$ such that $N_{\sigma(x)}M_x = f$ and all other linear maps of $N$ to be zero. Then $d\nu_M(N) = f$. Assume now that $\text{ker} i \neq 0$ and $s \in \text{ker} i$. Then, by the stability of $M$, there is a vertex $z \in \tilde{R}_0 - \tilde{S}_0$ such that the image of $s$ along a path from $z$ to $x$ in $\tilde{Q}$ does not lie in the kernel of $M_z : C^{v(z)} \rightarrow C^{w(\sigma(z))}$. Let us assume that $z = x_n, \ldots, x_0 = x$ are the vertices along a minimal path with arrows $\alpha_i : x_{i+1} \rightarrow x_i$ and that $f$ is a projection of $s$ onto $s' \in C^{v(z)}$. Then, we can choose inductively along the path for every arrow of the path linear maps $N(\tilde{\alpha}_i)$ such that we get components $N(\tilde{\alpha}_0)M(\alpha_0) = f$ and $M(\tau \alpha_{i-1})N(\tilde{\alpha}_{i-1}) + N(\tilde{\alpha}_i)M(\alpha_i) = 0$ for all $1 \leq i \leq n - 1$. In the last mesh, we choose $N_{\sigma(z)}$ such that $N_{\sigma(z)}M_x + M(\tau \alpha_{n-1})N(\tilde{\alpha}_{n-1}) = 0$. All other linear maps defining $N$ are set to zero. By the definition of $N$, we have $d\nu_M(N) = f$ and $d\nu_M$ is therefore surjective. Hence the tangent spaces at stable points are of the same dimension. We conclude that $Sl(v, w)$ and $M(v, w)$ are smooth.

The affine quiver variety $M_0(w)$ is constructed as follows: we consider for a fixed $w$ the affine varieties given by the geometric quotients $M_0(v, w) := rep(v, w, \tilde{R})//G_v$. For all $v' \leq v$, where the order is component wise, we have an inclusion by extension by zero, that is by adding semi-simple nilpotent representations with dimension vector $v - v'$$rep(v', w, \tilde{R})//G_{v'} \hookrightarrow rep(v, w, \tilde{R})//G_v$.

The affine quiver variety $M_0(w)$ is defined as the colimit of the quotients $rep(v, w, \tilde{R})//G_v$ over all $v$ along the inclusions.

**Theorem 3.2.** The set $M(v, w)$ canonically becomes a smooth quasi-projective variety and the projection map
\[ \pi : \sqcup_v M(v, w) \rightarrow M_0(w) \]
taking the $G_v$–orbit of a stable $\tilde{R}$–module $M$ to the unique closed $G_v$–orbit in the closure of $G_vM$, is proper.
We denote by $\mathcal{M}^\text{reg}(v, w) \subset \mathcal{M}(v, w)$ the open set consisting of the union of closed $G_v$–orbits of stable representations. Furthermore, we denote $\mathcal{M}_0^\text{reg}(v, w) := \pi(\mathcal{M}^\text{reg}(v, w))$ the open possibly empty subset of $\mathcal{M}_0(v, w)$. Note that it follows from the definition that $\pi$ is one to one when restricted to $\mathcal{M}_0^\text{reg}(v, w)$.

3.3. Affine quiver varieties as $\tilde{S}$–modules. Based on [31], we have proven in [28] that the graded affine quiver variety $\mathcal{M}_0^g(r)(w)$ is equivalent to $\text{rep}(\mathcal{S}^g, w)$ as an algebraic variety. In this section, we show that this result remains true in the ungraded setting if $Q$ is a Dynkin quiver.

First, we proceed by describing the closed points of $\mathcal{M}_0(w)$ more concretely. By abuse of language, we say that a $G_v$–stable subset of $\text{rep}(v, w, \tilde{R})$ contains a module, if it contains the orbit corresponding to the module.

Lemma 3.4.

- The closed $G_v$–orbits in $\text{rep}(v, w, \tilde{R})$ are represented by $L \oplus N \in \text{rep}(v, w, \tilde{R})$, where $L$ is a bistable module and $N$ is a semi-simple module such that $\text{res}(N)$ vanishes.
- A stable $\tilde{R}$–module $M$ belongs to $\mathcal{M}^\text{reg}(v, w)$ if and only if it is bistable.

Proof. (1) Given an exact sequence

\[ 0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0 \]

of finite-dimensional $\tilde{R}$–modules with $\text{res}(N) = 0$ (resp. $\text{res}(L) = 0$), the closure of the $G_v$–orbit of $M$ contains $N_{ss} \oplus L$ (resp. $L_{ss} \oplus N$) where $N_{ss}$ is the semi-simple module with the same composition series than $N$.

Let $M$ be a module whose $G_v$–orbit is closed. Then the module $K_{LR}(\text{res } M)$ is contained in $\text{im}(\varepsilon) \subset K_R(\text{res } M)$, where $\varepsilon$ denotes the adjunction morphism $M \to K_R(\text{res } M)$. Let $i$ denote the inclusion $K_{LR}(\text{res } M) \subset \text{im}(\varepsilon)$. Now by the first step, the closure of the orbit of $M$ contains $\text{im}(\varepsilon) \oplus (\ker(\varepsilon))_{ss}$ and the closure of the orbit of $\text{im}(\varepsilon)$ contains $K_{LR}(\text{res } M) \oplus (\cok(i))_{ss}$. Thus the $G_v$–orbit of $M$, which equals its closure, contains the object

\[ K_{LR}(\text{res } M) \oplus (\cok(i))_{ss} \oplus (\ker(\varepsilon))_{ss}. \]

Conversely, given a point $L \oplus N$ as in the statement of the Lemma. As $N$ is semi-simple, every element in the closure of the orbit of $L \oplus N$ contains $N$ as direct summand. By GIT theory, we know that the closure of the orbit of $L \oplus N$ contains a closed orbit $G_vX$. By the first part $X \cong K_{LR} \text{res } X \oplus Z \oplus N$, where $\text{res } Z$ vanishes. As the restriction functor is $G_v$–invariant and algebraic, it takes constant value on the orbit closure. Therefore $\text{res } L \cong \text{res } X$ and as $L$ is bistable, we have $L \cong K_{LR}(\text{res } L) \cong K_{LR}(\text{res } X)$. Furthermore, by dimension reasons, $Z$ vanishes. Hence the orbit of $L \oplus N$ and the orbit of $X$ coincide. This finishes the proof.

(2) Suppose that $M$ is stable and belongs to $\mathcal{M}^\text{reg}(v, w)$, then by the first part $M \cong K_{LR}(\text{res } M) \oplus N$, where $\text{res } N$ vanishes. As $M$ is stable, it forces
$N = 0$. Hence $M$ is bistable by Lemma 2.8 and conversely, by the first part, all bistable modules give rise to closed $G_v$-orbits.

As in the proof of the previous Lemma, we will consider the algebraic map

$$\text{res} : \mathcal{M}_0(w) \to \text{rep}(w, \bar{S}), \ G_v M \mapsto \text{res} M,$$

induced by the restriction map $\text{res}$. The map is well-defined as $\text{res}$ is $G_v$-invariant. Furthermore it is a surjective map, as for every $X \in \text{rep}(w, \bar{S})$, the $G_v$-orbit of $K_{LR} X$ is a closed point of $\mathcal{M}_0(w)$ satisfying $\text{res} K_{LR} X = X$. An equivalent statement to the next Theorem can also be found in [38, 3.27].

Remarkably, in the graded setting, the varieties $\mathcal{M}_{\text{gr}}_0(w)$ and $\text{rep}(w, S_{\text{gr}})$ are always isomorphic for any choice of $Q$. This is proven in [28] and [31]. It relies on the fact that all simple $S_{\text{gr}}$-modules are nilpotent. This result remains only true if $Q$ is of Dynkin type. Nevertheless, we have a nice description of $\text{rep}(w, \bar{S})$ as unions of the strata $\mathcal{M}_{\text{reg}}_0(v, w)$.

We have that $\mathcal{M}_0(v, w)$ embeds into $\sqcup \text{rep}(w, \bar{S}) \times \text{rep}(v, \bar{P}_Q)/G_v$. Applying the colimit over $v'$ yields an embedding of $\mathcal{M}(w)$ into $\text{rep}(w, \bar{S}) \times \text{colim} \text{rep}(v, \bar{P}_Q)/G_v$. Clearly this embedding is a subjection.

**Theorem 3.5.** The restriction

$$\text{res} : \bigsqcup_v \mathcal{M}_{\text{reg}}^e(v, w) \to \text{rep}(w, \bar{S})$$

induces an isomorphism of algebraic varieties. If $Q$ is of Dynkin type, the affine quiver variety $\mathcal{M}_0(w)$ is equivalent to $\text{rep}(w, \bar{S})$ and is stratified by the images of the non empty ones among the open subsets $\mathcal{M}_{\text{reg}}^e(v, w)$.

**Proof.** The functor $K_{LR}$ establishes a bijection between bistable modules and $\text{rep}(w, \bar{S})$. As by Lemma 3.4 all objects in the open sets $\mathcal{M}_{\text{reg}}^e(v, w)$ consist exactly of the bistable modules we have established the first part.

The fibres of $\text{res}$ over any point correspond to $G_v$-orbits of semi-simple $P_Q$-modules, Now the variety $\text{rep}(v, \bar{P})/G_v$ consists of the orbits of semi-simple $\bar{P}$-modules with dimension vector $v$. If $Q$ is a Dynkin quiver, all simple $\bar{P}$-modules are nilpotent and therefore $\text{rep}(v, \bar{P})/G_v$ is just a point and $\text{res}$ is an isomorphism of algebraic varieties.

An analogous result holds in the graded setting without restriction on the quiver $Q$. Note that on the other hand, if $Q$ is not of Dynkin type there are finite-dimensional simple non-nilpotent $P_Q$-modules by Theorem 1.2 of [7]. Hence the restriction functor $\text{res}$ has non-trivial fibres and therefore the affine variety $\mathcal{M}_0(w)$ is never isomorphic to $\text{rep}(w, \bar{S})$ for any choice of dimension vector $w$.

For later use we record the next corollary which is an easy consequence of the previous results in this section.

**Corollary 3.6.** Every point $G_v L \in \mathcal{M}_0(v, w)$ corresponds uniquely to a pair $(L_1, L_2)$ where $L_1 = \text{res} L \in \text{rep}(w, \bar{S})$ and $L_2$ a representative of
the isomorphism class of a semi-simple \( \tilde{P} \)-module. With this identification the map \( \pi : \mathcal{M}(v, w) \to \mathcal{M}_0(w) \) is given by \( G_v N \mapsto (\text{res} N, N_2) \) where \( N_2 \) is the semi-simple module with the same composition series than \( \text{Coker}(K_{LR}(\text{res} N) \to N) \).

3.7. From graded to ungraded quiver varieties. In Nakajima’s original definition, the graded and cyclic quiver varieties are defined as a fixed point set of the quiver varieties with respect to a \( \mathbb{C}^* \)-action (see [39]). Hence they are naturally closed subvarieties of the original quiver varieties. We used a different approach in this paper: we started by introducing the graded quiver varieties as moduli spaces of graded Nakajima categories and introduced quiver varieties as moduli spaces of orbit categories of the graded Nakajima categories. In this section, we will relate the two different approaches.

The graded Nakajima categories and the ungraded Nakajima categories are related as follows: There is a canonical equivalence

\[
\mathcal{R}^\text{gr}/\tau \cong \mathcal{R}
\]

which restricts to an equivalence

\[
\mathcal{S}^\text{gr}/\tau \cong \mathcal{S}.
\]

We denote in more generality \( p : \mathcal{R}_C^\text{gr} \to \tilde{R} \) the induced functor,

\[
p_* : \text{Mod} \mathcal{R}_C^\text{gr} \to \text{Mod} \tilde{R}
\]

the associated pushforward functor and

\[
p^* : \text{Mod} \tilde{R} \to \text{Mod} \mathcal{R}_C^\text{gr}
\]

the pullback functor. More concretely, they are given by

\[
p_* M(x) = \bigoplus_{n \in \mathbb{Z}} M(F^n x) \quad \text{for all } M \in \text{Mod} \mathcal{R}_C^\text{gr}
\]

and

\[
p^*(N)(x) = N(p(x)) \quad \text{for all } N \in \text{Mod} \tilde{R}.
\]

The same functors exists at the level of \( \mathcal{S}_C^\text{gr} \) and \( \tilde{S} \)-modules. As \( p \) commutes with the restriction functor, we will also denote the pushforward and pullback functors by \( p_* : \text{Mod} \mathcal{S}_C^\text{gr} \to \text{Mod} \mathcal{S} \) and \( p^* : \text{Mod} \mathcal{S} \to \text{Mod} \mathcal{S}_C^\text{gr} \) respectively. It is easy to see that the pushforward functor induces an embedding of graded quiver varieties into classical quiver varieties. To obtain the embedding of cyclic quiver varieties into classical quiver varieties, one considers the pushforward functor to the canonical functor \( \mathcal{R}_C^\text{gr}/\tau^n \to \mathcal{R} \cong \mathcal{R}_C^\text{gr}/\tau \).

We call an \( \mathcal{R}_C^\text{gr} \)-module respectively a \( \mathcal{S}_C^\text{gr} \)-module \( M \) left bounded if there is an \( n \in \mathbb{Z} \) such that \( M(m, i) = 0 \) for all \( m \leq n \) and all objects \( i \) of \( \mathcal{R}_C^\text{gr} \) and \( \mathcal{S}_C^\text{gr} \) respectively. Right bounded modules are defined analogously and a bounded module is left and right bounded.
Lemma 3.8. The functors $(p_*, p^*)$ are adjoint functors. If $A$ is a bounded $R_C^{gr}$–module respectively $S_C^{gr}$–module and $B$ a $\bar{R}$–module respectively $\bar{S}$–module, we also have a natural isomorphism \[ \text{Hom}(B, p_* A) \cong \text{Hom}(p^* B, A). \]

By [28] all indecomposable projective modules of $R_C^{gr}$ and $S_C^{gr}$ are of the form $x^\wedge$ and point wise finite dimensional and right bounded for any object $x$ in $R_C^{gr}$ or $S_C^{gr}$. Analogously, all indecomposable injective modules are of the form $x^\vee$ and point wise finite dimensional and left bounded for objects $x$ of $R_C^{gr}$ or $S_C^{gr}$ respectively.

Both the pullback and pushforward functors are exact. Hence $p_*$ maps finitely generated projective modules of the graded Nakajima categories to finitely generated projective modules of the ungraded Nakajima categories.

Remark 3.9. Note that in general $p_*$ does not map finitely cogenerated injective modules to finitely cogenerated injective modules. This can be seen as follows: as $x^\vee$ is left bounded and point wise finite dimensional, its image under $p_*$ is locally nilpotent and has $S_x$ in its socle. But by Theorem 1.2 of [7] there is a non-nilpotent simple $R$–module $S$ and a simple nilpotent $\bar{R}$–module $S_x$ such that $\text{Ext}_R^1(S_x, S)$ does not vanish. It follows that the injective hull of $S_x$ is not locally nilpotent, hence it is not isomorphic to $p_*(x^\vee)$.

In the next Lemma, we investigate the relationship between the Kan extensions in the graded and non-graded case.

Lemma 3.10. Let $M$ be an $S_C^{gr}$–module, then $p_* K_L^{gr} M \cong K_{LP_*} M$ and there is a canonical inclusion \[ i : p_* K_L^{gr} M \hookrightarrow K_{RP_*} M \]
whose restriction to $S$ is the identity. If $Q$ is of Dynkin type and $M$ finite-dimensional, then $i$ is an isomorphism.

Proof. To prove the first isomorphism, we use the adjointness relations
\[
\text{Hom}(p_* K_L^{gr} M, L) \cong \text{Hom}(K_L^{gr} M, p^* L) \cong \text{Hom}(M, \text{res} p^* L)
\]
for any $L \in \text{Mod} R$. Hence by the uniqueness of the left adjoint we find $p_* K_L^{gr} M \cong K_{LP_*} M$. Finally, we have
\[
\text{Hom}(S, p_* K_L^{gr} M) \hookrightarrow \text{Hom}(p^* S, K_R^{gr} M) = 0,
\]
and hence we have that $p_* K_R^{gr} M$ is stable and as $\text{res} p_* K_R^{gr} M \cong M$ there is a canonical injection $p_* K_R^{gr} M \rightarrow K_{RP_*} M$ which restricts to the identity on $S$. If $Q$ is Dynkin and $M$ is finite-dimensional, then $K_R^{gr} M$ is bounded and we obtain
\[
\text{Hom}(L, p_* K_R^{gr} M) \cong \text{Hom}(p^* L, K_R^{gr} M) \cong \text{Hom}(p^* \text{res} L, M)
\]
for any $L \in \text{Mod} R$. Hence by the uniqueness of the left adjoint we find $p_* K_R^{gr} M \cong K_{LP_*} M$. Finally, we have
\[
\text{Hom}(S, p_* K_R^{gr} M) \hookrightarrow \text{Hom}(p^* S, K_R^{gr} M) = 0,
\]
and hence we have that $p_* K_R^{gr} M$ is stable and as $\text{res} p_* K_R^{gr} M \cong M$ there is a canonical injection $p_* K_R^{gr} M \rightarrow K_{RP_*} M$ which restricts to the identity on $S$. If $Q$ is Dynkin and $M$ is finite-dimensional, then $K_R^{gr} M$ is bounded and we obtain
\[
\text{Hom}(L, p_* K_R^{gr} M) \cong \text{Hom}(p^* L, K_R^{gr} M) \cong \text{Hom}(p^* \text{res} L, M)
\]
for any $L \in \text{Mod}\ R$. Hence by the uniqueness of the right adjoint we find $p_*K_R^{gr}M \cong K_{RP_*}M$.

It follows now from the first two parts that $K_{LRP_*}M$ is the image of the canonical map $p_*K_L^{gr}M \to p_*K_R^{gr}M$ and by the exactness of $p_*$, the image is $p_*K_{LR}^M$.

We obtain the following easy consequence.

Lemma 3.11. For $M \in \text{mod}\ S$, we have

$$K_{LRP_*}M \cong p_*K_{LR}^{gr}M \text{ and } p_*K^{gr}(M) \cong KK(p_*M).$$

Proof. It follows from the first two parts of the previous Lemma that $K_{LRP_*}M$ is the image of the canonical map $p_*K_L^{gr}M \to p_*K_R^{gr}M$ and by the exactness of $p_*$, the image is $p_*K_{LR}^M$. By Lemma 3.10, the functor $p_*$ commutes with the intermediate extension and the left Kan extension. Hence this statement follows from the exactness of $p_*$. √

3.12. Homological properties of Nakajima categories. First we determine the projective resolutions of nilpotent simple $\tilde{R}$–modules.

Lemma 3.13. Let $x$ be a non-frozen vertex.

a) The nilpotent simple $\tilde{R}$–modules have projective resolutions given by

$$0 \to \tau(x)^\wedge \to \bigoplus_{y \to x} y^\wedge \to x^\wedge \to S_x \to 0$$

where the sum runs over all arrows $y \to x$ in $\tilde{Q}$. If $x \in C$, then the projective resolution of $S_{\sigma x}$ is given by

$$0 \to \tau(x)^\wedge \to \sigma(x)^\wedge \to S_{\sigma(x)} \to 0.$$ 

b) The nilpotent simple $\tilde{R}$–modules have injective resolutions given by

$$0 \to S_x \to x^\vee \to \bigoplus_{x \to y} y^\vee \to \tau^{-1}(x)^\vee \to 0$$

where the sum runs over all arrows $x \to y$ in $\tilde{Q}$. If $x \in C$, then the projective resolution of $S_{\sigma x}$ is given by

$$0 \to S_{\sigma(x)} \to \sigma(x)^\vee \to x^\vee \to 0$$

if $x \in C$.

Proof. We apply $p_*$ to the projective resolution of simple $\mathcal{R}_{gr}$–modules given in [28]. As $p_*$ is an exact functor, we obtain the above exact sequence. Now applying a dual argument to $\tilde{R}^{op}$ yields the injective resolutions. √

Note that in general $x^\vee$ is not the indecomposable injective module with socle $S_x$. This can be seen as follows. By adding mesh relations in the frozen vertices, we obtain a quotient of $\mathcal{R}$ that is equivalent to $\mathcal{P}_{Q^f}$, the preprojective algebra associated to the framed quiver $Q^f$. By Theorem 1.2 of [7] there exists a simple non-nilpotent $\mathcal{P}_{Q^f}$–module. Let $S$ be such a
simple module with nonzero support in \( x \). Then there is a non zero map from \( S \) to \( x^\vee \), which is necessarily an injection.

Next, we determine projective resolutions of simple nilpotent \( S \)-modules. Let \( C \) be either \( \tilde{S} \) or \( \tilde{R} \). Let us denote by

\[
P(x) := \bigoplus_{y \in C} \tilde{P}(y, x)\sigma(y)^{\wedge}
\]

the projective \( C \)-module and by

\[
I(x) := \bigoplus_{y \in C} D\tilde{P}(x, y)\sigma(y)^{\vee}
\]

the injective \( C \)-module which we associate with an object \( x \in \tilde{S}_0 \). The graded analogues of \( C^{gr} \)-modules for \( C^{gr} = R^{gr} \) or \( C^{gr} = S^{gr} \), where we replace \( \tilde{P} \) with \( D_Q \) shall be denoted \( P^{gr}(x) \) and \( I^{gr}(x) \).

**Lemma 3.14.** Let \( x \in C \).

a) Let \( Q \) be of Dynkin type, then the nilpotent simple \( \tilde{S} \)-modules have infinite projective resolutions

\[
\cdots \rightarrow P(\Sigma^2\tau x) \rightarrow P(\Sigma\tau x) \rightarrow P(\tau x) \rightarrow \sigma(x)^{\wedge} \rightarrow S_{\sigma(x)} \rightarrow 0.
\]

b) If \( Q \) is not of Dynkin type, we obtain a projective resolution for the simple nilpotent \( \tilde{S} \)-modules

\[
0 \rightarrow P(\tau x) \rightarrow \sigma(x)^{\wedge} \rightarrow S_{\sigma(x)} \rightarrow 0.
\]

Dualizing these sequence yields injective resolutions.

**Proof.** We apply \( p_* \) to the projective resolution of the simple \( S^{gr} \)-module \( S_{\sigma(x)} \) given in [28] and we use that

\[
p_*(P^{gr}(\Sigma^k x)) = \bigoplus_{y \in D_Q} D_Q(y, \Sigma^k x)p_*\sigma(y)^{\wedge}
\]

and

\[
\bigoplus_{l \in \mathbb{Z}} D_Q(F^l(y), \Sigma^k x) \cong \tilde{P}(y, \Sigma^k x).
\]

Dually, we obtain the injective resolution.

We summarize some consequences of the previous Lemmata.

**Corollary 3.15.**

- The category of finite-dimensional nilpotent \( \tilde{R} \)-modules has global dimension 2.
- If \( Q \) is of Dynkin type, the category of finite-dimensional nilpotent \( \tilde{S} \)-modules has infinite global dimension and it is hereditary if \( Q \) is not of Dynkin type.

Note also that as an algebra, \( \tilde{S} \) is finitely generated if \( Q \) is of Dynkin type. The results on the homological properties of \( \tilde{S} \) allow us to describe \( \tilde{S} \) as an algebra. We describe \( \tilde{S} \) as the path algebra of a finite quiver \( Q_{\tilde{S}} \) such
that $\tilde{S}$ is isomorphic to a path algebra $kQ_{\tilde{S}}$ modulo an admissible ideal $I$ of the path algebra $kQ_S$ cf. [13, Ch. 8] [1, II.3]. We determine the quiver $Q_{\tilde{S}}$ and the number of minimal relations between two vertices.

**Proposition 3.16.** Let $Q$ be of Dynkin type. Then the quiver $Q_{\tilde{S}}$ has vertices $x \in C$ and the number of arrows $x \to y$ is $\dim \tilde{P}(y, \Sigma x)$. The number of minimal relations of paths from $x$ to $y$ is given by the dimension of $\tilde{P}(y, \Sigma^2 x)$.

**Proof.** As all objects in $\tilde{S}$ are pairwise non-isomorphic, the vertices of $Q_{\tilde{S}}$ are in bijection with the objects of $\tilde{S}$, which are via $\sigma$ in bijection with the objects in $C$. Let us hence denote the vertices of $Q_{\tilde{S}}$ by the object in $C$. The number of arrows from $x$ to $y$ is given by the dimension of $\text{Ext}^1_{\tilde{S}}(S(\sigma(x)), S(\sigma(y)))$ and the number of minimal relations of paths between $x$ and $y$ is given by $\text{Ext}^2_{\tilde{S}}(S(\sigma(x)), S(\sigma(y)))$ which by Lemma 3.14 are given by the dimensions of $\tilde{P}(y, \Sigma x)$ and $\tilde{P}(y, \Sigma^2 x)$ respectively.

We determine $\tilde{S}$ in terms of a quiver with relations for two different choices of functors $F$.

**Example 3.17.** Let $Q$ be the $A_2$ quiver $1 \to 2$. We choose $F = \tau$ and hence $\tilde{S} = S$. In this case $Q_{\tilde{S}}$ is given by

```
1
\(\downarrow\)
2
```

The minimal relations defining $S$ are $d^3 - fg = 0$, $e^3 - gf = 0$, $df - fe = 0$ and $eg - gd = 0$. To obtain the minimal relations one computes that $P_Q(x, y)$ is one dimensional for any choice of vertices $x$ and $y$. It follows that there is exactly one minimal relation between paths joining any two vertices $x$ and $y$. As $\tilde{S}$ is an orbit category of $S^{\tau^t}$, the exact minimal relations can also be obtained using the Example of [28] after Lemma 2.7.

**Example 3.18.** Let $F = \Sigma \tau^{-1}$ and $C$ be as in the example 2.6. Then $\tilde{P}$ is the cluster category and $\tilde{S}$ is given as path algebra of the quiver

```
S_1
\(\downarrow b\)
S_2
\(\downarrow a\)
\(\Sigma P_2 \quad a \quad P_2 \)
\(\downarrow \Sigma S_1 \quad a \quad S_1\)
\(\downarrow b\)
```
subject to the relations \( ab = ba, \ a^3 = b^2 \) and \( b^3 = a^2 \).

We can classify simple \( \tilde{\mathcal{R}} \)-modules in terms of simple \( \tilde{\mathcal{P}} \) and simple \( \tilde{\mathcal{S}} \)-modules.

**Lemma 3.19.** The intermediate extension \( \text{KLR} \) establishes a bijection between simple \( \tilde{\mathcal{S}} \)-modules and all simple \( \tilde{\mathcal{R}} \)-modules which do not vanish under restriction to \( \tilde{\mathcal{S}} \).

**Proof.** Let \( S \) be a simple \( \tilde{\mathcal{S}} \)-module. Let \( S' \) be a non-trivial simple submodule \( S' \leq \text{KLR}S \). As \( \text{KLR}S \) is stable, so is \( S' \) and we find that \( 0 \neq \text{res} S' \leq \text{res} \text{KLR}S \cong S \) and hence \( \text{res} S' \cong S \). Furthermore \( S' \) is costable as it is simple and has non-zero support in frozen vertices. As \( \text{KLR}S \) is the unique bistable module up to isomorphism whose restriction is isomorphic to \( S \), we have that \( S' \cong \text{KLR}S \). As every simple \( \mathcal{R} \)-module whose restriction to \( S \) is non-zero is bistable, we have that \( \text{KLR}S \) is the unique simple lift of \( S \). Now let \( L \) be a simple \( \mathcal{R} \)-module. Clearly \( \text{res} L \) vanishes if and only if \( L \) is supported only in non-frozen vertices. Suppose that \( \text{res} L \) does not vanish. Then \( L \) is stable and \( \text{KLR} \text{res} L \) is a submodule of \( L \). Hence \( \text{KLR} \text{res} L \cong L \) is simple. Suppose that \( L' \) is a simple submodule of \( \text{res} L \). Then \( \text{KLR} \text{res} L' \) is also a simple submodule of \( L \) and by the identity

\[ L' \cong \text{res} \text{KLR} \text{res} L' \cong \text{res} L \]

we have that \( L' \cong \text{res} L \).

\( \square \)

4. The Stratification functor of affine quiver varieties

We recall that the moduli space \( \text{rep}(w, \tilde{\mathcal{S}}) \) is stratified into finitely many strata \( \sqcup_v \mathcal{M}_{0,w}^{\text{reg}}(v, w) \). If \( Q \) is of Dynkin type, the affine quiver variety \( \mathcal{M}_0(w) \) and \( \text{rep}(w, \tilde{\mathcal{S}}) \) coincide, while in the non-Dynkin case, \( \text{rep}(w, \tilde{\mathcal{S}}) \) is a closed and proper subvariety of the affine quiver variety. In this section, we will show that there is a functor \( \psi : \text{mod} \tilde{\mathcal{S}} \to \text{inj}^{\text{nil}} \tilde{\mathcal{P}}_Q \) which parametrizes the strata, describes the degeneration order between strata and provides a description of the fibre of the proper map \( \pi : \mathcal{M}(v, w) \to \mathcal{M}_0(w) \).

4.1. The Stratification functor. In analogy with the results of [28], we show that the stratification of the affine quiver variety is parametrized by \( \text{inj}^{\text{nil}} \tilde{\mathcal{P}}_Q \), the category of finitely cogenerated injective \( \tilde{\mathcal{P}}_Q \)-modules with nilpotent socle. In the case that \( Q \) is a Dynkin quiver this category is equivalent to \( \text{proj} \tilde{\mathcal{P}}_Q \).

**Lemma 4.2.** The functors \( \mathcal{KK} \) and \( \mathcal{CK} \) satisfy

\[ \text{Ext}^1(\text{KLR}M, S) = \text{Hom}(\mathcal{KK}(M), S) \text{ and } \text{Ext}^1(S, \text{KLR}M) = \text{Hom}(S, \mathcal{CK}(M)) \]

for all \( \tilde{\mathcal{R}} \)-modules \( S \) which are supported in \( \tilde{\mathcal{R}}_0 - \tilde{\mathcal{S}}_0 \). Furthermore \( \text{Ext}^1(\mathcal{KK}(M), S) \) and \( \text{Ext}^1(S, \mathcal{CK}(M)) \) vanish for all finite-dimensional nilpotent modules \( S \) which are supported in \( \mathcal{R}_0 - \mathcal{S}_0 \).
Proof. We give the proof for the functor $KK$, the proof for $CK$ being dual. By applying $\text{Hom}(-,S)$ to the short exact sequence

$$0 \to KK(M) \to KL M \to KLR(M) \to 0$$

we obtain the exact sequence

$$\text{Hom}(KL M, S) \to \text{Hom}(KK(M), S) \to \text{Ext}^1(KLR(M), S) \to \text{Ext}^2(KLR(M), S).$$

By Lemma 2.8 we have that $\text{Hom}(KL M, S)$ and $\text{Ext}^1(KL M, S)$ vanish, proving the first identity.

We deduce from Lemma 3.13 that $\text{Ext}^2(KLR M, S) \cong \text{Hom}(S, KLR M)$.

The second term vanishes as $KLR M$ is stable, which proves the second identity.

Lemma 4.3. Let $Q$ be of Dynkin type. For any $M \in \text{mod} \tilde{S}$, we have that $KK(M)$ is a finitely generated projective $\tilde{P}$-module and $CK(M)$ is a finitely cogenerated injective $\tilde{P}$-module.

Proof. As $M$ is finitely-generated the same is true for $KL M$ and its submodule $KK(M)$. As $Q$ is of Dynkin type, we have that the algebra $\tilde{P}$ is finite-dimensional. Hence $KK(M)$ is finite-dimensional and finitely presented as $\tilde{P}$-module. Furthermore all simple $\tilde{P}$-modules are nilpotent and hence isomorphic to $S_x$ for some $x \in \tilde{R}_0 - \tilde{S}_0$. As $\text{Ext}^1(KK(M), S_x)$ vanishes by the previous Lemma for all simple modules $S_x$ supported on non-frozen vertices, the module $KK(M)$ is a finitely generated projective $\tilde{P}$-module. The proof that $CK(M)$ is finitely cogenerated injective is dual.

If $Q$ is of Dynkin type, then $\tilde{P}$ carries a triangulated structure, which lifts from the triangulated structure of $D_Q$. Hence the shift functor $\Sigma$ seen as an automorphism of $\tilde{P}$ makes $\tilde{P}$ a triangulated category (see also section 3).

Proposition 4.4. Let $Q$ be of Dynkin type. Then we have $CK = \Sigma KK$.

Proof. Note that the Serre functor on $D_Q$ is given by $\tau \Sigma$ and yields the Serre functor in $\tilde{P}$. The number of direct summands $x^\wedge$ appearing in $KK(M)$ is given by the dimension of $\text{Hom}(KK(M), S_x) \cong \text{Ext}^1(KLR M, S_x)$ and dually the number of direct summands $x^\vee = (\tau \Sigma x)^\wedge$ appearing in $CK(M)$ is given by the dimension of $\text{Hom}(S_x, CK M) \cong \text{Ext}^1(S_x, KLR M)$. Now as the dimension of $\text{Ext}^1(S_x, KLR M)$ and $\text{Ext}^1(KLR M, S_{\tau x})$ are equal, both numbers coincide, which finishes the proof.

The next Lemma shows that the functors $KK$ and $CK$ are essentially surjective if $Q$ is of Dynkin type.

Lemma 4.5. We have $KK(S_{\sigma(x)}) \cong x^\wedge_{\tilde{P}}$ and $CK(S_{\sigma(x)}) \cong x^\vee_{\tilde{P}}$ for all $x \in C$.
Proof. To prove the first identity we use the result of Lemma 3.11. We have shown in [28], that 
\( \text{KK}^{gr}(S_{\sigma(x)}) \) is a projective module of \( \tilde{\mathcal{P}} \) represented by the image of the Happel functor \( H(x) \). Hence we have that \( \text{KK}(S_{\sigma(x)}) = p_{*}(\text{Hom}_{\mathcal{D}}(-, H(x))) = x^\wedge_{\tilde{\mathcal{P}}} \).

To prove the second identity, we observe that by Lemma 2.8 the module \( K_{R}(\sigma(x)^\vee) \) is isomorphic to \( \sigma(x)^\vee \) seen as \( \tilde{\mathcal{R}} \)–module. This follows as \( \sigma(x)^\vee \) is stable and \( \text{Ext}^1(-, \sigma(x)^\vee) \) vanishes. Hence \( K_{R}I(x) \) is isomorphic to \( I(x) \) seen as \( \tilde{\mathcal{R}} \)–module. We consider now the commutative diagram with exact rows and exact columns

\[
\begin{array}{cccccc}
\text{ker } g & \longrightarrow & S_{\sigma(x)} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
K_{R}S_{\sigma(x)} & \longrightarrow & \sigma(x)^\vee & \longrightarrow & I(x) \\
\downarrow & & \downarrow & & \\
x^\vee_{\tilde{\mathcal{P}}} & \longrightarrow & x^\vee & \longrightarrow & I(x) \\
\end{array}
\]

The exactness of the middle row follows from applying \( K_{R} \) to the start of the injective resolution of \( S_{\sigma(x)} \) in Lemma 3.13. To obtain the exactness of the last row we use Theorem 3.7 of [28]. Hence it follows that \( \text{ker } g \) equals \( S_{\sigma(x)} \cong K_{LR}(S_{\sigma(x)}) \) and therefore \( \text{CK}(S_{\sigma(x)}) \cong x^\vee_{\tilde{\mathcal{P}}} \).

We will call \textit{stratification functor} an additive functor \( \Psi : \text{mod} \tilde{\mathcal{S}} \rightarrow \mathcal{A} \) satisfying that any two \( \mathcal{S} \)–modules \( M_1 \in \mathcal{M}^{reg}(v, w) \) and \( M_2 \in \mathcal{M}^{reg}(v', w) \) lie in the same stratum if and only if their images under \( \Psi \) are isomorphic.

For a vector \( v : \tilde{\mathcal{R}}_{0} - \tilde{\mathcal{S}}_{0} \rightarrow \mathbb{Z} \), we define \( C_{q}v : \tilde{\mathcal{R}}_{0} - \tilde{\mathcal{S}}_{0} \rightarrow \mathbb{Z} \) by

\[
(C_{q}v)(x) = v(x) - \left( \sum_{y \rightarrow x} v(y) \right) + v(\tau(x)) , \quad x \in \tilde{\mathcal{R}}_{0} - \tilde{\mathcal{S}}_{0} ,
\]

where the sum ranges over all arrows \( y \rightarrow x \) of \( \tilde{Q} \) and \( y \in \tilde{\mathcal{R}}_{0} - \tilde{\mathcal{S}}_{0} \). The index \( q \) reminds us that \( C_{q} \) is a ‘quantum Cartan matrix’, cf. section 3.1 of [40]. Note that in the case that \( Q \) is of Dynkin type and \( \mathcal{P} \) isomorphic to the preprojective algebra \( \mathcal{P}_{Q} \), the map \( C_{q} \) induces an injective map. Indeed in this case, the map \( C_{q} \) corresponds to the Cartan matrix associated with the diagram of \( Q \).

We will also denote by \( w_{\sigma} : \tilde{\mathcal{R}}_{0} - \tilde{\mathcal{S}}_{0} \rightarrow \mathbb{N} \) the dimension vector which sends \( x \) to \( w(\sigma(x)) \) if \( x \in C \) and vanishes otherwise. Recall that a \( \delta \)–functor is a functor from an exact category to a triangulated category, which sends short exact sequences to distinguished triangles up to equivalence, cf. e. g. [26].

**Proposition 4.6.** Let \( Q \) be of Dynkin type. The functors \( \text{KK} \) and \( \text{CK} \) are \( \delta \)–functors

\[
\text{mod} \tilde{\mathcal{S}} \rightarrow \text{proj} \tilde{\mathcal{P}} .
\]
Let $K_{LR}M$ have dimension vector $(v, w)$, then the multiplicity of $z^\alpha_P$ as direct summand of $KK(M)$ is given by $(w\sigma - C_qv)(z)$.

Proof. Let $M \in \text{mod} \, S$. We have already shown in Lemma 4.3 that $KK(M)$ is a finitely generated projective $P$–module and that $KK(S_{\sigma(x)}) \cong x^\alpha_P$ in Lemma 4.5. Hence it remains to show that the isomorphism class of $KK(M)$ depends only on $\dim K_{LR}M$. The multiplicity of $x^\alpha_P$ as a direct summand of $KK(M)$ is given by $\dim \text{Hom}(KK(M), S_x) = \dim \text{Ext}^1(K_{LR}M, S_x)$. Now $\text{Ext}^1(K_{LR}M, S_x)$ is given by the cohomology of the right and left exact complex

$$0 \to K_{LR}M(\tau x) \to K_{LR}M(y) \to K_{LR}M(x) \to 0$$

which equals $w\sigma - C_qv$ and hence depends uniquely on the dimension vector of $K_{LR}M$.

Let us now show that $CK$ is a $\delta$–functor. Let $0 \to M \to N \to L \to 0$ be an exact sequence in $\text{mod} \, S$. Then we obtain the following commutative diagram

As $K_R(N)/K_R(M)$ is stable, it embeds into $K_R(L)$.

Furthermore, as $K_{LR}N/K_{LR}M$ is costable, the image of $f$ in $K_R(L)$ is given by $K_{LR}L$. Hence, we have that $\ker(f) \cong \ker(K_{LR}N/K_{LR}M \to K_{LR}L)$ and as $K_{LR}N/K_{LR}M$ is costable and restricts under res to $L$, the canonical map $K_L(L) \to K_{LR}L$ factors through $K_{LR}N/K_{LR}M \to K_{LR}L$. Hence, we obtain the following commutative diagram

 Applying the snake Lemma, we conclude that $KK(L) \cong \Sigma^{-1} CK(L)$ maps onto $\ker(f) \cong KK(L)/S$. We consider next the diagram

As $K_{LR}(N)/K_{LR}(M)$ is stable, it embeds into $K_{LR}(L)$.

Furthermore, as $K_{LR}N/K_{LR}M$ is costable, the image of $f$ in $K_{LR}(L)$ is given by $K_{LR}L$. Hence, we have that $\ker(f) \cong \ker(K_{LR}N/K_{LR}M \to K_{LR}L)$ and as $K_{LR}N/K_{LR}M$ is costable and restricts under res to $L$, the canonical map $K_L(L) \to K_{LR}L$ factors through $K_{LR}N/K_{LR}M \to K_{LR}L$. Hence, we obtain the following commutative diagram

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Furthermore, as $K_{LR}N/K_{LR}M$ is costable, the image of $f$ in $K_R(L)$ is given by $K_{LR}L$. Hence, we have that $\ker(f) \cong \ker(K_{LR}N/K_{LR}M \to K_{LR}L)$ and as $K_{LR}N/K_{LR}M$ is costable and restricts under res to $L$, the canonical map $K_L(L) \to K_{LR}L$ factors through $K_{LR}N/K_{LR}M \to K_{LR}L$. Hence, we obtain the following commutative diagram

Applying the snake Lemma, we conclude that $KK(L) \cong \Sigma^{-1} CK(L)$ maps onto $\ker(f) \cong KK(L)/S$. We consider next the diagram

As $K_R(N)/K_R(M)$ is stable, it embeds into $K_R(L)$. 

Furthermore, as $K_{LR}N/K_{LR}M$ is costable, the image of $f$ in $K_R(L)$ is given by $K_{LR}L$. Hence, we have that $\ker(f) \cong \ker(K_{LR}N/K_{LR}M \to K_{LR}L)$ and as $K_{LR}N/K_{LR}M$ is costable and restricts under res to $L$, the canonical map $K_L(L) \to K_{LR}L$ factors through $K_{LR}N/K_{LR}M \to K_{LR}L$. Hence, we obtain the following commutative diagram

Applying the snake Lemma, we conclude that $KK(L) \cong \Sigma^{-1} CK(L)$ maps onto $\ker(f) \cong KK(L)/S$. We consider next the diagram

As $K_R(N)/K_R(M)$ is stable, it embeds into $K_R(L)$.
As $K_{LR}M$ is costable, there is no map from $\ker(K_{LR}N \to K_{LR}N/K_{LR}M)$ to $S$. So by the snake Lemma the map $KK(N) \cong \Sigma^{-1}CK(L)$ maps surjectively onto $S$.

We conclude that

$$\Sigma^{-1}CK(N) \to \Sigma^{-1}CK(L) \to CK(M) \to CK(N) \to CK(L)$$

is an exact sequence in $\text{proj}\, \tilde{P}_Q$ and as an exact sequence,

$$\Sigma^{-1}CK(L) \to CK(M) \to CK(N) \to CK(L)$$

is therefore isomorphic to a distinguished triangle in $\text{proj}\, \tilde{P}$. Hence $CK$ and $KK$ give rise to a $\delta$-functor.

Note that, as $CK$ is a $\delta$-functor, we are able to recover the image of $CK$ on $\tilde{S}$–modules from the image $CK$ takes on simple $\tilde{S}$–modules. In the case of classical quiver varieties, the map $C_q$ is invertible. Hence we obtain a stronger statement.

**Theorem 4.7.** Let $Q$ be of Dynkin type. The functors $KK$ and $CK$ are $\delta$–functors

$$\text{mod}\, S \to \text{proj}\, P_Q$$

satisfying that any two modules $M_1$ and $M_2$ belonging to $M_0(w)$ lie in the same stratum if and only if their image under $CK$ respectively $KK$ are isomorphic.

**Proof.** Let $M \in \text{mod}\, S$. We recall that the dimension vector of $K_{LR}M$ determines the stratum of $M$:

We have that $M \in M_0(v, w)$ if and only if $\dim K_{LR}M = (v, w)$ by Lemma 2.8. We have shown in the previous proposition that the dimension vector of $K_{LR}M$ determines the image of $KK(M)$ and $CK(M)$ up to isomorphism.

So suppose that $M_1, M_2 \in M_0(w)$ and lie in the strata $M_0^{\text{reg}}(v_1, w)$ and $M_0^{\text{reg}}(v_2, w)$ respectively. If $KK(M_1) \cong KK(M_2)$ then $w\sigma - C_qv_1 = w\sigma - C_qv_2$, and as $C_q$ is invertible, we find $v_1 = v_2$. As by Proposition 4.4 we have $CK = \Sigma KK$, the analogous statement also holds for $CK$.

In the non-Dynkin case, we have a similar statement, relating the strata to the isomorphism classes of the objects in $\text{inj}^{\text{nil}}(\tilde{P})$, which denotes the category of finitely cogenerated injective modules with nilpotent socle. Let us denote by $I_x$ the injective hull of $S_x$ seen as a $\tilde{P}$–module.

**Theorem 4.8.** There is a functor

$$\Psi : \text{rep}(w, \tilde{S}) \to \text{inj}^{\text{nil}}(\tilde{P}), \quad S_{\sigma(x)} \mapsto I_x,$$

such that two objects appearing in the same strata are isomorphic under $\Psi$.

**Proof.** Let $I_M$ be the injective hull of $CK(M)$. As

$$\text{Hom}(S_x, I_M) \cong \text{Hom}(S_x, CK(M)) \cong \text{Ext}^1(S_x, K_{LR}M)$$
is finite dimensional, we know that $I_x$ appears only finitely many times as a factor of $I_M$. We define $\psi(M)$ to be the maximal direct factor of $I_M$ with nilpotent socle. As $K_{LR}S_{\sigma(x)} = S_{\sigma(x)}$, we have that $\dim \Ext^1(S_z, K_{LR}S_{\sigma(x)}) = \delta_{z,x}$ and hence $\psi(S_{\sigma(x)}) = I_x$. Proving that this defines a functor parametrizing the strata of $\mod\tilde{S}$ is analogous to the Dynkin case.

Note that in the case when $Q$ is not a Dynkin quiver, the functors $CK$ and $\psi$ do not coincide: we have that $CK(S_{\sigma(x)}) = x^+_P$ which is not isomorphic to $\Psi(S_{\sigma(x)}) = I_x$. 

4.9. Degeneration order on Strata. Let us assume in this section that $Q$ is of Dynkin type and that the map $C_q$ associated with $\tilde{P}$ is invertible. The correspondence between the finitely generated projective $\tilde{P}$–modules and the strata of the affine quiver variety allows us to identify the degeneration order of strata with a degeneration order on objects of the triangulated category $\tilde{P}$.

We recall first the degeneration order on strata of the classical quiver variety $\rep(w, \mathcal{S}) \cong M_0(w)$: we define $M_0^{\text{reg}}(v, w) \leq M_0^{\text{reg}}(v', w)$ if and only if $M_0^{\text{reg}}(v, w) \subset M_0^{\text{reg}}(v', w)$. This is shown to be the case if and only if $v(x) \leq v'(x)$ for all $x \in R_0 - S_0$ by [39, 4.1.3.14].

Analogously, we define a partial order on the strata of $\rep(w, \tilde{S})$. For two points $M, M' \in \rep(w, \tilde{S})$, we set $M \leq M'$ if and only if $v(x) \leq v'(x)$ for all $x \in \tilde{R}_0 - \tilde{S}_0$, where $(v, w)$ and $(v', w)$ are the dimension vectors of $K_{LR}(M)$ and $K_{LR}(M')$ respectively. This clearly defines a partial order on the strata $M_0^{\text{reg}}(v, w)$ of $M_0(w)$, which we also call the degeneration order.

In [23], Jensen, Su and Zimmermann define a partial order on objects of a triangulated category $\mathcal{T}$ satisfying the following conditions:

- $\mathcal{T}$ is $\Hom$-finite and idempotent morphisms are split;
- For all $X, Y \in \mathcal{T}$ there is an $n \in \mathbb{Z} - \{0\}$ such that $\Hom(X, \Sigma^n Y)$ vanishes.

The first condition assures transitivity and the second condition allows to conclude that the preorder is anti-symmetric. Given two objects $X$ and $Y$ of $\mathcal{T}$, we set in convention with our notation, $X \leq Y$ if and only if there is an object $Z$ in $\mathcal{T}$ and a distinguished $X \to Y \oplus Z \to Y \to \Sigma X$.

We show that under the stratification functor $KK$, the degeneration order on strata corresponds to the order on objects of $\tilde{P}$. Note that in the case that $\mathcal{T}$ is $\tilde{P}$ condition one is satisfied by Lemma 2.5, but the second condition is not satisfied. Hence the order of [23] is only a preorder. But it will follow from the next Theorem that the preorder defines in fact a partial order on the objects of $\tilde{P}$. We will identify $\text{proj} \tilde{P}$ with $\tilde{P}$ via the Yoneda embedding.
Theorem 4.10. Under the stratification functor the degeneration order among the strata of $M_{0}(w)$ corresponds to the degeneration order of the triangulated category $\text{proj} \bar{P}$.

Proof. Let $M$ and $M'$ be two elements in $M_{0}(w)$ belonging to the strata $M_{0}^{reg}(v, w)$ and $M_{0}^{reg}(v', w)$ respectively. Let us assume that $KK(M) \leq KK(M')$ in the degeneration order of $[23]$. We will show $v(x) \leq v'(x)$ for all $x \in R_{0} - S_{0}$.

We suppose there is a triangle

\begin{equation}
\tag{4.10.1}
KK(M) \to KK(M') \oplus Z \to Z \to \Sigma KK(M)
\end{equation}

for some $Z \in \text{proj} \bar{P}$.

By Lemma 4.5 we can find semi-simple $\bar{S}$-modules $Y$ and $L$ such that $KK(M) \cong KK(Y)$ and $KK(L) = Z$.

Hence $\dim K_{LR}Y = (0, w - C_{q}v\sigma^{-1})$ and $\dim K_{LR}L = (0, w_{L})$ for the dimension vector $w_{L} = \dim L$ of $\bar{S}$. Note that $KK$ induces an isomorphism

$$\text{Ext}^{1}_{\bar{S}}(S_{\sigma(x)}, S_{\sigma(y)}) \cong \bar{P}(x, \Sigma y).$$

Hence we can lift the triangle 4.10.1 to an exact sequence

$$0 \to Y \to N \to L \to 0$$

for some $N \in \text{mod} \bar{S}$.

Applying the functor $K_{LR}$ to the short exact sequence and using the left and right exactness of $K_{LR}$ yields the inequality

$$\dim K_{LR}Y + \dim K_{LR}L \leq \dim K_{LR}N.$$ 

Let $\dim K_{LR}N = (v_{N}, w_{N})$, then we find $w_{N} = w - C_{q}v\sigma^{-1} + w_{L}$ and $v_{N} \geq 0$.

As $KK(N) \cong KK(L) \oplus KK(M')$, we have that

$$w_{N} - C_{q}v_{N}\sigma^{-1} = (w_{L} + w) - C_{q}v'\sigma^{-1}.$$ 

Hence it follows that $C_{q}(v' - v_{N}) = C_{q}v$ implying $v' - v_{N} = v$ by our assumption on $C_{q}$.

For the converse claim, we refer to [28, 3.18]. The proof is analogous if we replace the split Grothendieck group of $D_{Q}$ by the split Grothendieck group of $\text{proj} \bar{P}$.

Clearly, the degeneration order is anti-symmetric. Hence the same holds for the preorder $\bar{P}$.

4.11. Fibers of $\pi$ as quiver Grassmannians. In this section, we give a description of the fibers of the map $\pi$ in terms of quiver Grassmannians. This generalizes work of Savage and Tingley [49]. The methods used here are entirely different though. The result in the graded case was given in [28, 4.19].
For a dimension vector $u$ of $Q_0$ and a module $M \in \text{mod } \overline{P}$, we denote $\text{Gr}_u(M)$ the quiver Grassmannian, that is the projective variety consisting of submodules $N \subset M$ such that $\dim N = u$.

**Proposition 4.12.** Let $M \in \text{rep}(w, \overline{S})$ with $\dim K_{LR}(M) = (v_0, w)$. Then the fibre $\text{res}^{-1}(M)$ of $\text{res} : M(v, w) \rightarrow \text{rep}(w, \overline{S})$ is isomorphic in the complex analytic topology to the Grassmannian

$$\text{Gr}_{v-v_0}(CK(M)).$$

**Proof.** An $\overline{R}$–module $N \in \text{rep}(v, w, \overline{R})$ lies in the fibre of $M$ if and only if

- $N$ is stable,
- $\text{res} N \cong M$.

Suppose $N$ lies in $\text{res}^{-1}(M)$. Then we have that $K_{LR} \text{res} N \cong K_{LR} M$ and as $N$ is stable, the adjunction morphism $K_{LR} M \rightarrow N$ is an injection whose cokernel is a submodule of $CK(\text{res} N) \cong CK(M)$ with dimension vector $v - v_0$. Conversely, every submodule $X$ of $CK(M)$ with dimension vector $v - v_0$ gives rise via pullback to an $\overline{R}$–module $N$ appearing in the exact sequence

$$0 \rightarrow K_{LR}(M) \rightarrow N \rightarrow X \rightarrow 0.$$  

As $\text{res} X$ vanishes, we obtain $\text{res} N \cong \text{res} K_{LR} M \cong M$ and $N$ is stable as it is by construction a submodule of $K_R M$.

If $Q$ is of Dynkin type, then the restriction induces an isomorphism $\text{rep}(w, \overline{S}) \cong M_0(w)$ of algebraic varieties. Hence Nakajima’s projective map $\pi : M(v, w) \rightarrow M_0(w)$ corresponds to the restriction map $M(v, w) \rightarrow \text{rep}(w, \overline{S})$. Furthermore every point $M \in M_0(w)$ belongs to a unique stratum. In this case $M \in M_0(v_0, w)$ if and only if $K_{LR}(\text{res} M)$ has dimension vector $(v_0, w)$. Therefore the next result is immediate.

**Theorem 4.13.** Suppose that $Q$ is of Dynkin type and $M \in M_0^{\text{reg}}(v_0, w)$, then $\pi^{-1}(M)$ is isomorphic in the complex analytic topology to the Grassmannian

$$\text{Gr}_{v-v_0}(CK(M)).$$

To treat the case that $Q$ is not of Dynkin type, we introduce certain closed subvarieties of quiver Grassmannians. For a semi-simple $\overline{P}$–module $L$ of dimension vector $u$, we denote $\text{Gr}_u^L(M)$ the closed sub variety of $\text{Gr}_u(M)$ containing all submodules of $M$ with dimension vector $u$, that in addition have the same composition series than $L$. Finally, we denote $\text{Gr}_u^{\text{nil}}(M)$ the closed sub variety of nilpotent submodules of $M$ having dimension vector $u$.

**Theorem 4.14.** Let $L$ be a point of $M_0(w)$ given by $L_1 \oplus L_2$, where $L_1 \in M_0^{\text{reg}}(v_0, w) \subset \text{rep}(w, \overline{S})$ and $L_2$ is a semi-simple $\overline{P}$–module (see Corollary 3.6). Then $\pi^{-1}(L)$ is isomorphic to $\text{Gr}_{v-L_2-v_0}(CK(L_1))$ in the complex-analytic topology.
Proof. Being in the fibre of $L$ is equivalent to the following three conditions on $N \in \text{rep}(v, w, \tilde{R})$:

- $\text{res} \ N \cong L_1$;
- $N$ is stable;
- $M := \text{Coker}(K_{LR}(\text{res} \ N) \to N)$ has the same composition series than $L_2$.

Let $N \in \pi^{-1}(L)$. Note that, as $N$ is stable, the adjunction morphism $K_{LR}(\text{res} \ N) \to N$ is injective. Also the first point is equivalent to the existence of isomorphisms

$$K_{LR} \text{res} \ N \cong K_{LR} \text{res} \ L \cong K_{LR}(L_1).$$

Hence there is a commutative diagram where the columns are injections

$$\begin{array}{ccc}
K_{LR}(L_1) & \to & N \\
| & | & | \\
| & | & | \\
K_{LR}(L_1) & \to & K_{R}(L_1) \\
| & | & | \\
| & | & | \\
\text{CK}(L_1).
\end{array}$$

Conversely every submodule $M$ of $\text{CK}(L_1)$ with composition series equal to $L_2$ gives rise via pullback to an $\tilde{R}$–module $N$ appearing as the middle term of the top sequence in the commutative diagram. Clearly, $N$ satisfies by construction the third condition and $N$ is stable as it is a submodule of the stable module $K_{R}(L_1)$. Finally, $N$ satisfies $\text{res} \ N = \text{res} \ L$ as the restriction functor is exact and vanishes on $M$. Hence there is a bijection between the submodules of $\text{CK}(L_1)$ of dimension vector $v - v_0$ that have the same composition series than $L_2$ and fibre of $\pi$ over $L$.

In the case, that $L$ belongs to a stratum $\mathcal{M}^{\text{reg}}_0(v_0, w)$, we have an alternative description.

Lemma 4.15. Suppose $L \in \mathcal{M}^{\text{reg}}_0(v_0, w)$, then $\pi^{-1}(L)$ is isomorphic to $\text{Gr}^{\text{nil}}_{v - v_0}(\psi(\text{res} \ L))$ in the complex-analytic topology.

Proof. Suppose now that $L \in \mathcal{M}^{\text{reg}}_0(v_0, w)$. This implies that $L_2$ is semi-simple and nilpotent. Let $Y \subseteq I$ be a finite-dimensional nilpotent submodule of the injective hull $i : \text{CK}(\text{res} \ L) \hookrightarrow I$. We can assume without loss of generality that $Y$ is indecomposable. Then $Y \cap \text{CK}(\text{res} \ L)$ seen as a submodule of $I$ is non-empty. Let us consider the commutative diagram

$$\begin{array}{ccc}
\text{CK}(\text{res} \ L) & \to & I \\
\uparrow & & \downarrow \\
Y \cap \text{CK}(\text{res} \ L) & \to & Y/\text{CK}(\text{res} \ L)
\end{array}$$

The up going arrows are all injections. We know that $\text{Ext}^1(-, \text{CK}(\text{res} \ L))$ vanishes on all nilpotent finite dimensional $\mathcal{P}$–modules. Therefore, applying $\text{Hom}(Y/\text{Y} \cap \text{CK}(\text{res} \ L), -)$ to the top sequence shows that, as $Y$ is...
nilpotent, the morphism $f$ factors through $I$ via an injective map. But as the image of $Y/Y \cap CK(\mathrm{res} L)$ in $I$ does not intersect with $CK(\mathrm{res} L)$, the image of $Y/Y \cap CK(\mathrm{res} L)$ lies in $Y$. As $Y$ is indecomposable, this implies that $Y/Y \cap CK(\mathrm{res} L)$ vanishes. As a consequence, the image $Y \to I$ lies in $CK(\mathrm{res} L)$. Hence all nilpotent finite-dimensional submodules of the injective hull of $CK(\mathrm{res} L)$ are also submodules of $CK(\mathrm{res} L)$ and $Gr^\text{nil}_{\psi^{-1}}(CK(\mathrm{res} L)) \cong Gr^\text{nil}_{\psi^{-1}}(I)$. Now $Gr^\text{nil}_{\psi^{-1}}(I) \cong Gr^\text{nil}_{\psi^{-1}}(\psi(\mathrm{res} L))$ as $\psi(\mathrm{res} L)$ is by definition the factor of the injective hull of $CK(\mathrm{res} L)$ which has nilpotent socle.

$$\sqrt{}$$

These results allow us to give an alternative description of the Langragian quiver variety in terms of quiver Grassmannians. In analogy with the classical definition of the Langragian Nakajima quiver variety we define the generalized Lagrangian quiver variety $L(v, w)$ to be the sub variety of $M(v, w)$ given by the $G_v$–orbits of stable representations $X \in rep(v, w, \bar{R})$ that satisfy:

- the module $X$ is nilpotent;
- the linear maps $X_\alpha : k_v^{w(\sigma i)} \to k_v^{w(\tau i)}$ associated with the arrows $\alpha : \tau(i) \to \sigma(i)$ vanish for all $i \in C$.

We show next that the Langragian quiver variety consists of all stable and nilpotent modules whose restriction to $\tilde{S}$ is the semi-simple nilpotent module with dimension vector $w$. Let us denote by $s_d$ the semi-simple nilpotent module associated to the dimension vector $d$.

**Lemma 4.16.** A point $X \in L(v, w)$ lies in the Langragian Nakajima quiver variety if and only if $X$ is nilpotent and its restriction to $\tilde{S}$ is semi-simple. Hence $L(v, w)$ is given by the fibre $\pi^{-1}(s_{(w,v)})$.

**Proof.** The first implication is immediate. To obtain the converse inclusion, it remains to show that $X_\alpha$ vanishes. Assume there is an $X_\alpha$ that does not vanish. As $X$ is nilpotent, the same holds for $\mathrm{res} X$. As $\mathrm{res} X$ is semi-simple by assumption, this forces $\mathrm{res} X$ to be isomorphic to $s_w$. As a consequence, the submodule generated by $\mathrm{im} X_\alpha$ is non-zero and has only support in non-frozen vertices. This is a contradiction to the stability condition, so we have proven the first part. Let $X_1 \oplus X_2$ be the image of $X$ under $\pi$, where $X_1 = \mathrm{res} X \in rep(w, \tilde{S})$ and $X_2$ is the semi-simple $\tilde{P}$–module with the composition series of the cokernel of $K_{LR}(\mathrm{res} X) \to X$. Then $X_1 \cong s_w$ and hence $K_{LR}(X_1) = K_{LR}(\mathrm{res} X) = K_{LR}(s_w) = s_w$. Now as $X$ is nilpotent, $X_2$ is also nilpotent and we conclude that $X_2 \cong s_v$.  

$$\sqrt{}$$

We recover a result of Savage and Tingley [49, 4.4]. Let us denote $I_w \in \text{inj} \tilde{P}$ the injective hull of $s_w$.

**Corollary 4.17.** The generalized Lagrangian quiver variety $L(v, w)$ is isomorphic in the complex-analytic topology, to the quiver Grassmannian $Gr_v(I_w)$ if $Q$ is of Dynkin type;
Gr-nil(v) if Q is not of Dynkin type.

Proof. If Q is of Dynkin type, we have that res : M(v, w) → rep(w, tilde(S)) corresponds to the desingularisation map π. Furthermore, L(v, w) is the fibre over s_w of this map. We know by Lemma 4.5 that CK(S_x) is given by x_P which is the injective hull of S_x if Q is a Dynkin quiver. Hence applying Theorem 4.13 finishes the proof. If Q is not of Dynkin type, the Langragian quiver variety L(v, w) is given by the fibre under π of the semi-simple tilde(R)-module s_w ⊕ s_v, and hence by Lemma 4.15, we have

L(v, w) ∼= Gr-nil(v)(I_w)

using that ψ(s_w) ∼= I_w.

As shown in [49] the variety Gr-nil(v)(I_w) is equivalent as algebraic variety to Gr-nil(q_w), where q_w denotes the injective hull of s_w in the category of locally nilpotent tilde(P)-modules.

5. Frobenius models and Nakajima categories

We assume in this section that Q is an orientation of a Dynkin diagram. Recall that tilde(P) is a triangulated category. In this section, by a Frobenius category, we mean a k-linear, Krull–Schmidt category E endowed with the structure of an exact category for which it is Frobenius. Then the stable category E obtained as a quotient of E by all morphisms factoring through projective-injective objects of E is naturally a triangulated category. A Frobenius model for tilde(P) is a Frobenius category E together with a triangle equivalence tilde(P) ∼→ E.

5.1. Gorenstein projective modules of tilde(S). Some results of this section where also announced by [36], we refer to this paper for more details. Recall that, for a k-category C, a C–module M is Gorenstein projective [9] if there is an acyclic complex

P : ... → P_1 → P_0 → P_{-1} → ...

of finitely generated projective modules such that M is isomorphic to the cokernel of P_1 → P_0 and that the complex Hom(P, P') is still acyclic for each finitely generated projective C–module P'. We denote the category of Gorenstein projective modules by gpr(C). By Proposition 5.1 of [2] it follows easily that gpr(C) is a Frobenius exact category and that the subcategory of projective–injective objects is the subcategory of finitely generated projective C–modules.

We have shown in Theorem 5.18 of [28] that the stable category of the Frobenius category gpr(S^{op}) is triangle equivalent to the bounded derived category

D_Q ∼= gpr(S^{op}_C).

As in section , we denote by F both the exact automorphism on Mod tilde(R)^{op}_C and its restriction to Mod S^{op}_C induced by the functor F.
As the pushforward functors
\[ p_* : \text{gpr}(S\text{gr}_C) \to \text{gpr}(\tilde{S}) \quad \text{and} \quad p_* : \text{proj}\,R\text{gr}_C \to \text{proj}\,\tilde{R} \]
are invariant under \( F_* \), we obtain functors
\[ \text{gpr}(S\text{gr}_C)/F_* \to \text{gpr}(\tilde{S}) \]
and
\[ \text{proj}(R\text{gr}_C)/F_* \to \text{proj}\,\tilde{R}. \]
sending the finitely generated projective \( S\text{gr}_C \)-modules to finitely generated projective \( \tilde{S} \)-modules. These functors satisfy the following properties.

**Lemma 5.2.** The functor \( \text{proj}\,R\text{gr}_C/F_* \to \text{proj}\,\tilde{R} \) is an equivalence and \( \text{gpr}(S\text{gr}_C)/F_* \hookrightarrow \text{gpr}(\tilde{S}) \) is fully faithful.

**Proof.** Let now \( R\text{gr}_C \) be either \( S\text{gr}_C \) or \( R\text{gr}_C \) and respectively \( C \) be either \( \tilde{S} \) or \( \tilde{R} \). Let \( x^\wedge \) and \( y^\wedge \) be two projective \( C\text{gr}_C \)-modules associated with \( x, y \in C_{0}^{gr} \).

Then
\[
\text{Hom}_{\text{gpr}\,C\text{gr}_C/(F_*)}(x^\wedge, y^\wedge) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\text{gpr}\,C\text{gr}_C}(x^\wedge, F_i(y)^\wedge)
= \bigoplus_{i \in \mathbb{Z}} C^{gr}(x, F_i(y)) \cong C(x, y) \cong \text{Hom}_{C}(x^\wedge, y^\wedge),
\]
where we identify the vertices \( x \) and \( y \) with their images under \( p \). Hence the first functor is fully faithful. It is an equivalence as every indecomposable projective \( \tilde{R} \)-module lifts to a projective indecomposable \( R\text{gr}_C \)-modules. This proves the first part.

Now we show that the functor is fully faithful when restricted to finitely presented modules. Let \( M \) and \( N \) be finitely presented and let
\[
P_1^M \xrightarrow{f} P_2^M \to M \to 0 \quad \text{and} \quad P_1^N \xrightarrow{f} P_2^N \to N \to 0
\]
be presentations of \( M \) and \( N \) respectively. Applying \( p_* \) to this presentation yields a presentation of \( p_*M \) in \( \tilde{S} \). We have
\[
\text{Hom}_{\text{gpr}\,C\text{gr}_C/(F_*)}(M, N) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{S^{gr}_C}(M, F_i(N))
= \bigoplus_{i \in \mathbb{Z}} \ker \text{Hom}_{S^{gr}_C}(P_1^M, F_i(N)) \to \text{Hom}_{S^{gr}_C}(P_2^M, F_i(N)).
\]
Now \( F_i^*(N) \) is also finitely-presented by
\[
F_i^*(P_2^N) \xrightarrow{g} F_i^*(P_1^N) \to F_i^*(N) \to 0.
\]
Hence
\[
\text{Hom}_{S^{gr}_C}(P_2^M, F_i(N)) \cong \text{Coker} \left( \text{Hom}_{S^{gr}_C}(P_2^M, F_i(P_2^N)) \to \text{Hom}_{S^{gr}_C}(P_2^M, F_i(P_1^N)) \right).
\]
for all \( i \in \mathbb{Z} \) and \( j \in \{1, 2\} \). Furthermore as
\[
\bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\text{gr}(\mathcal{S}_C)}(P^M_j, F^*_i(P^N_1)) \cong \text{Hom}_{\mathcal{S}}(P^M_j, P^N_1),
\]
we obtain that \( p_* \) induces a fully faithful functor. Clearly, all Gorenstein projective modules are finitely presented and are mapped to Gorenstein projective \( \mathcal{S} \)-modules. This proves the second statement.

We show next that \( \text{proj} \mathcal{R} \) is a Frobenius model for \( \mathcal{P} \).

**Theorem 5.3.** The category \( \text{gpr}(\mathcal{S}^{gr})/F_* \) is equivalent to \( \text{proj} \mathcal{R} \) and an exact subcategory of \( \text{gpr}(\mathcal{S}) \). The stable category of \( \text{proj} \mathcal{R} \) is equivalent to \( \mathcal{P} \).

**Proof.** We refer to [36] for the fact that \( \text{gpr}(\mathcal{S}^{gr})/F_* \) is an exact category. In Theorem 5.23 of [28] we have shown that the map \( \text{proj} \mathcal{R}^{gr}_C \rightarrow \text{gpr}(\mathcal{S}^{gr}_C), x^\Lambda \mapsto \text{res} x^\Lambda \) induces an isomorphism of exact categories. This implies that \( \text{gpr}(\mathcal{S}^{gr}_C)/F_* \cong \text{proj} \mathcal{R}^{gr}_C/F_* \). Furthermore, we have shown in the Lemma 5.2 that \( \text{proj} \mathcal{R}^{gr}_C/F_* \) is equivalent to \( \text{proj} \mathcal{R} \). Hence the category \( \text{gpr}(\mathcal{S}^{gr}_C)/F_* \) is equivalent to \( \text{proj} \mathcal{R} \). Furthermore, the fully faithful embedding of \( \text{proj} \mathcal{R} \) into \( \text{gpr}(\mathcal{S}) \) is exact which proves the first part. Clearly the stable category \( \text{proj} \mathcal{R} \) is equivalent to \( \mathcal{P} \). This finishes the proof.

Combining the results of this section, yields the following.

**Corollary 5.4.** There is a fully faithful embedding
\[
\mathcal{P} \rightarrow \text{gpr}(\mathcal{S}), x^\Lambda \mapsto \text{res} x^\Lambda
\]
for all \( x \in \mathcal{R}_0 - \mathcal{S}_0 \).

**Proof.** By the previous results, we have that \( \text{res} : \text{proj} \mathcal{R} \rightarrow \text{gpr}(\mathcal{S}), x^\Lambda \mapsto \text{res} x^\Lambda \) yields a fully faithful functor between two Frobenius categories, such that the indecomposable projective-injective objects in \( \text{proj} \mathcal{R} \) are mapped to the projective-injective objects in \( \text{gpr}(\mathcal{S}) \) given by \( x^\Lambda \) for all \( x \in \mathcal{S}_0 \). Hence this functor lifts to a functor on the stable categories. As \( \text{proj} \mathcal{R} \cong \mathcal{P} \), we obtain the above statement.

5.5. **Frobenius models of self-injective algebras.** In this section we classify all Frobenius models of \( \mathcal{P} \) that are standard (cf. section 2.3, page 63 of [48]) in the following sense:

(P1) For each indecomposable non projective object \( X \) of \( \mathcal{E} \), there is an almost split sequence starting and an almost split sequence ending at \( X \).

(P2) The category of indecomposables is equivalent to the mesh category of its Auslander-Reiten quiver.

(P3) Every projective-injective indecomposable object appears in exactly one mesh of the Auslander-Reiten quiver.
Recall from 2.3 that \(\widetilde{\mathcal{P}}\) is equivalent to the orbit category \(k(\mathbb{Z}Q)/F\) where we view \(F\) as an automorphism on the mesh category \(k(\mathbb{Z}Q)\). Similarly, \(\widetilde{\mathcal{R}}\) is defined by the choice of an automorphism \(F\) of \(k(\mathbb{Z}Q)\) and an admissible configuration \(C \subset (\mathbb{Z}Q)_0\).

Let us fix an automorphism \(F\) of \(k(\mathbb{Z}Q)\) with \(\widetilde{\mathcal{P}} \cong k(\mathbb{Z}Q)/F\).

**Theorem 5.6.** There is a bijection between the admissible \(F\)-stable configurations \(C \subset \mathbb{Z}Q_0\) and the equivalence classes of Frobenius models \(E\) of \(\mathcal{P}\) which are standard in the above sense by mapping \(C\) to the Nakajima category \(\text{proj} \widetilde{\mathcal{R}}\) where \(\widetilde{\mathcal{R}}\) is defined by the datum \((C, F)\).

**Proof.** Let \(\widetilde{\mathcal{R}}\) be defined by the admissible pair \((C, F)\). By the Yoneda embedding \(\widetilde{\mathcal{R}} \to \text{proj} \widetilde{\mathcal{R}}, x \mapsto x^\wedge\), it is clear that \(\text{proj} \widetilde{\mathcal{R}}\) is equivalent to the mesh category \(\widetilde{\mathcal{R}}\). Hence \(\text{proj} \widetilde{\mathcal{R}}\) is a Krull-Schmidt category whose Auslander-Reiten quiver is \(\widetilde{\mathcal{Q}} \cong \mathbb{Z}Q_{C}/F\). Now, by Theorem 5.3, the category \(\text{proj} \widetilde{\mathcal{R}}\) is exact with projective-injective objects \(\sigma(x)^\wedge\) for all \(x \in C\) and we have a natural equivalence of triangulated categories between the stable category of \(\text{proj} \widetilde{\mathcal{R}}\) and \(\widetilde{\mathcal{P}}\) by (P1)–(P3). Hence \(\text{proj} \widetilde{\mathcal{R}}\) is a Frobenius model of \(\widetilde{\mathcal{P}}\).

Conversely, if \(E\) is a Frobenius model of \(\mathcal{P}\) satisfying (P1)–(P3), then the corresponding set \(C_E\) of objects in \(\mathcal{P}\) such that \(\sigma(c)\) corresponds to the projective-injective objects in \(E\) lifts to an \(F\)-invariant subset \(C \subset \mathbb{Z}Q_0\). Clearly, this configuration gives rise to a category \(\widetilde{\mathcal{R}}\) which is equivalent to \(E\) by (P1)–(P3). It remains to show that \(C\) is admissible. Applying \(\text{Hom}(-, -)\) to the almost split sequences of \(E\) starting in \(x\) yields the exact sequences

\[
0 \to \widetilde{\mathcal{R}}(?, x) \to \bigoplus_{x \to y} \widetilde{\mathcal{R}}(?, y)
\]

and

\[
0 \to \widetilde{\mathcal{R}}(x, ?) \to \bigoplus_{y \to x} \widetilde{\mathcal{R}}(y, ?)
\]

where the arrows run between representatives of \(F\)-orbits in \(\widetilde{Q}\). By the definition of \(\widetilde{\mathcal{R}}\) as an orbit category, this yields exact sequences in \(\mathcal{R}_C^{gr}\) as in 2.4. Hence \(C\) is an admissible configuration.

It follows by 28 that \(\text{proj} \mathcal{R}_C^{gr} \cong \text{gpr}(\mathcal{S}_C^{gr})\) is a Frobenius model of \(\mathcal{D}_Q\) and hence we know that all Frobenius models of \(\mathcal{P}\) are orbit categories of Frobenius models of \(\mathcal{D}_Q\).

6. **Desingularization of Quiver Grassmannians**

We can apply our results to desingularize quiver Grassmannians of modules of self-injective algebras of finite representation type. We refer to the survey 51 for an overview on self-injective algebras. We denote by \(\tilde{B}\) the repetitive algebra associated with a finite-dimensional algebra \(B\), see 22 and 13. Furthermore, we denote by \(\Gamma(A)\) the mesh category associated to the Auslander-Reiten quiver of an algebra \(A\).
Proposition 6.1. Let $A$ be a self-injective algebra of finite representation type. Then there is an admissible pair $(C, F)$ such that $\Gamma(A)$ is isomorphic to $\tilde{R}$ and $A$ is Morita equivalent to $\tilde{S}$.

Proof. By [46], every self-injective algebra of finite type over a field of characteristic $\neq 2$ is necessarily standard. Furthermore, if $A$ is self-injective, standard and of finite representation type, then there is by [45] [52] an algebra $B$ which is tilted of Dynkin type $Q$ such that $A \cong \hat{B}/G$, where $G$ is generated by an autoequivalence $F$ which satisfies 2.4. By [12] (see also [51] Section 3.2) the mesh category $\Gamma(A)$ is given by the quotient $\Gamma(\hat{B})/G$ and the canonical pushforward functor induces a Galois cover $\Gamma(\hat{B}) \to \Gamma(A)$. By Theorem of [29] the mesh category $\Gamma(\hat{B})$ is equivalent to $\text{Rgr}_C$ of the Nakajima category $\text{Rgr}$ and the vertices $\sigma(x)$ with $x \in C$ correspond to the position of the projective-injective $\hat{B}$–modules in the Auslander-Reiten quiver. Furthermore $\hat{B}$ is standard and therefore equivalent to $\text{Sgr}_C$, the subcategory of $\text{Rgr}_C$ generated by the projective-injective objects $\sigma(x)$ with $x \in C$. Hence $A$ is Morita equivalent to $\tilde{S}$ and the mesh category of $\Gamma(A)$ is $\tilde{R}$.

Let $A$, $B$ and $F$ be as in Proposition 6.1. As shown in the Proposition 6.1 the algebra $A$ is Morita equivalent to $\tilde{S}$ which is the full subcategory of $\tilde{R}$ generated by the objects $\sigma(x)$ with $x \in C$. Furthermore, as $A$ is standard, we have the equivalences $\text{mod} A \cong \Gamma(A) \cong \tilde{R}$. Hence the intermediate extension $K_{LR} : \text{mod} \tilde{S} \to \text{mod} \tilde{R}$ can be seen as a functor

$$K_{LR} : \text{mod} A \to \text{mod} \text{mod} A.$$ 

Theorem 6.2. Let $A$ be a self-injective algebra of finite representation type and $M \in \text{mod} A$. Then the projective variety $\text{Gr}_d(K_{LR}M)$ is smooth. Furthermore there are finitely many dimension vectors $d_1, \ldots, d_n$ such that the restriction induces a map

$$\bigsqcup_i \text{Gr}_{d_i}(K_{LR}M) \to \text{Gr}_e(M), \ L \mapsto \text{res} L$$

which is proper and surjective.

Proof. By Theorem 2.7 of [29], the image under the intermediate extension $K_{LR}$ of every $M \in \text{mod} A$ is rigid and has projective dimension one. As $\Gamma(A)$ is an Auslander Algebra, it has global dimension at most two and using Proposition 7.1 of [5] we have that $\text{Gr}_d(K_{LR}M)$ is smooth and equidimensional. Now the restriction induces a map $\text{Gr}_d(K_{LR}M) \to \text{Gr}_e(M)$ for any dimensionvector $d$ of $\Gamma(A)$ and $e$ the restriction of $d$ to objects $\sigma(x)$ for $x \in C$. This map is proper as its domain is projective. Furthermore as $K_{LR}$ is left exact, every submodule $N \subset M$ gives rise to a submodule $K_{LR}N \subset K_{LR}M$. Hence there are finitely many dimensionvectors $d_1 \ldots d_r$ such that

$$\bigsqcup_i \text{Gr}_{d_i}(K_{LR}M) \to \text{Gr}_e(M)$$

is surjective. √
We define $\text{Gr}_{d}^{bs}(K_{LR}M)$ to be the closure of the open subset
\[ \{ L \in \text{Gr}_{d}(K_{LR}M) \mid L \text{ is bistable} \}. \]
It follows that $\text{Gr}_{d}^{bs}(K_{LR}M)$ is smooth. Following [5], let us denote by $C(N)$ the irreducible subvariety of $\text{Gr}_{e}(M)$ containing all submodules isomorphic to an $A$–module $N$. As $A$ is of finite representation type, all irreducible components of $\text{Gr}_{e}(M)$ are of the form $C(N)$. Let $C(N_{i})$ for $i = 1, \ldots, n$ denote the irreducible components of $\text{Gr}_{e}(M)$ for some representatives $N_{i} \in \text{Gr}_{e}(M)$ and let $d_{i}$ be the dimension vector of $K_{LR}(N_{i})$. We denote $V(M) := \{ d_{1}, \ldots, d_{n} \}$ the set of dimension vectors.

**Lemma 6.3.** The restriction $\pi^{gr} : \text{Gr}_{d}^{bs}(K_{LR}M) \rightarrow \text{Gr}_{e}(M)$ maps birationally onto all components $C(N_{i})$ with $\dim K_{LR}(N_{i}) = d_{i}$.

**Proof.** Clearly, to every $N \in \text{Gr}_{e}(M)$ which is isomorphic to some $N_{i}$ as in the Lemma, we obtain an element $K_{LR}N \in \text{Gr}_{d}^{bs}(K_{LR}M)$ such that $\pi^{Gr}(K_{LR}N)$ maps to $N$. Furthermore all bistable modules form an open subset of $\text{Gr}_{d}^{bs}(K_{LR}M)$ and are of the form $K_{LR}L$ for some $L \in \text{mod} A$. Hence the open subset of bistable modules in $\text{Gr}_{d}^{bs}(K_{LR}M)$ is mapped bijectively to the open subsets of modules isomorphic to some $N_{i}$ as in the Lemma. Therefore $\pi^{gr}$ is surjective and birational. \[ \sqrt{\text{ }} \]

Recall that a desingularisation map between algebraic varieties is a proper, surjective and birational map with smooth domain. We conclude with the following result.

**Theorem 6.4.** The map
\[ \pi^{gr} : \bigsqcup_{d \in V} \text{Gr}_{d}^{bs}(K_{LR}M) \rightarrow \text{Gr}_{e}(M) \]
is a desingularisation map.

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