A NOTE ON AN OPEN PROBLEM

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Abstract. The function $\Gamma\left(\frac{x+1}{x+\beta}\right)$ is logarithmically completely monotonic on $(0, \infty)$ for $\alpha \geq 1$ and $0 \leq \beta \leq 1$, and is logarithmically completely monotonic in $(-1, 0)$ for $0 < \alpha \leq \frac{2\beta}{1+2\beta}$ and $\beta > 1$. This gives an answer to an open problem proposed by Feng Qi.

1. Introduction

The classical gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt \quad (x > 0)$$

(1)

is one of the most important functions in analysis and its applications. The history and development of this function are described in detail [2]. The psi or digamma function $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed [6, p.16] as

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} \, dt,$$  

(2)

$$\psi^{(k)}(x) = (-1)^{k+1} \int_0^\infty \frac{t^k e^{-xt}}{1 - e^{-t}} \, dt$$

(3)

for $x > 0$ and $k = 1, 2, \ldots$, where $\gamma = 0.57721566490153286\ldots$ is the Euler-Mascheroni constant.

We recall that a function $f : (0, \infty) \rightarrow \mathbb{R}$ is said to be completely monotonic if $f$ has derivatives of all orders and

$$(-1)^n f^{(n)}(x) \geq 0$$

(4)

for $x > 0$ and $n = 0, 1, 2, \ldots$. If $f$ is nonconstant and completely monotonic, then the inequality (4) is strict, see [3]. Let $\mathcal{C}$ denote the set of completely monotonic functions.

A function $f$ is said to be logarithmically completely monotonic on $(0, \infty)$ if $f$ is positive and, for all $n \in \mathbb{N}$,

$$0 \leq (-1)^n [\log f(x)]^{(n)} < \infty,$$

(5)

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see [1, 7]. If inequality (5) is strict for all \( x \in (0, \infty) \) and for all \( n \geq 1 \), then \( f \) is said to be strictly logarithmically completely monotonic. Let \( \mathcal{L} \) on \((0,\infty)\) stand for the set of logarithmically completely monotonic functions.

The notion that logarithmically completely monotonic function was posed explicitly in [8] and published formally in [7] and a much useful and meaningful relation \( \mathcal{L} \subset \mathcal{C} \) between the completely monotonic functions and the logarithmically completely monotonic functions was proved in [7, 8].

In [5], H. Minc and L. Sathre proved that, if \( n \) is a positive integer and \( \phi(n) = (n!)^\frac{1}{n} \), then
\[
1 < \frac{\phi(n + 1)}{\phi(n)} < \frac{n + 1}{n},
\]
which can be rearranged as
\[
[\Gamma(n + 1)]^{1/n} < [\Gamma(n + 2)]^{1/(n+1)}
\]
and
\[
\frac{1}{n} \frac{[\Gamma(n + 1)]^{1/n}}{n} > \frac{1}{n+1} \frac{[\Gamma(n + 2)]^{1/(n+1)}}{n+1},
\]
since \( \Gamma(n + 1) = n! \).

In [4], the following monotonicity results for the Gamma function were established. The function \( [\Gamma(1 + \frac{1}{x})]^x \) decreases with \( x > 0 \) and \( x \left[ \Gamma(1 + \frac{1}{x}) \right]^x \) increases with \( x > 0 \), which recover the inequalities in (6) which refer to integer values of \( n \). These are equivalent to the function \( [\Gamma(1 + x)]^{1/x} \) being increasing and \( \frac{[\Gamma(1 + x)]^{1/x}}{x} \) being decreasing on \((0,\infty)\), respectively. In addition, it was proved that the function \( x^{1-y} \left[ \Gamma(1 + \frac{1}{x}) \right]^x \) decreases for \( 0 < x < 1 \), which is equivalent to \( \frac{\Gamma(1 + x + 1)^{1/x}}{x^{1-y}} \) being increasing on \((1,\infty)\).

In [9], Qi and Chen showed that the function \( \sqrt[1/x]{\frac{\Gamma(x+1)}{x+1}} \) is strictly decreasing and strictly logarithmically convex in \((0,\infty)\), and the function \( \sqrt[1/x]{\frac{\Gamma(x+1)}{\sqrt{x+1}}} \) is strictly increasing and strictly logarithmically concave in \((0,\infty)\). Using the monotonicity of above functions, Qi and Chen presented the following double inequality
\[
\frac{x+1}{y+1} < \frac{\Gamma(x+1)]^{1/x}}{\Gamma(y+1)]^{1/y}} < \sqrt[1/y]{\frac{x+1}{y+1}}
\]
for \( 0 < x < y \), see Corollary 1 of [9].

In [8], Qi and Guo proposed an open problem

**Open Problem 1.** Find conditions about \( \alpha \) and \( \beta \) such that the ratio
\[
F(x) = \frac{[\Gamma(x+1)]^{1/x}}{(x+\beta)^\alpha}
\]
is completely (absolutely, regularly) monotonic (convex) with \( x > -1 \).
In this paper, we give an answer to this problem and establish new inequalities.

**Theorem 1.** The function $F(x)$ defined by (7) is strictly logarithmically completely monotonic in $(0, \infty)$ for $\alpha \geq 1$ and $0 \leq \beta \leq 1$. Moreover, the function $F(x)$ is strictly completely monotonic in $(0, \infty)$ for $\alpha \geq 1$ and $0 \leq \beta \leq 1$.

**Proof.** Taking the logarithm of $F(x)$ defined by (7),

$$
\log F(x) = \frac{\log \Gamma(x + 1)}{x} - \alpha \log(x + \beta) \triangleq g(x) - \alpha \log(x + \beta).
$$

Using Leibnitz’ rule

$$
[u(x)v(x)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)}(x)v^{(n-k)}(x),
$$

we have

$$
g^{(n)}(x) = \frac{1}{x^{n+1}} \sum_{k=0}^{n} \frac{(-1)^{n-k} n! x^k \psi^{(k-1)}(x+1)}{k!} \triangleq \frac{h_n(x)}{x^{n+1}}.
$$

Using the representations

$$
\frac{(n-1)!}{(x+1)^n} = \int_{0}^{\infty} t^{n-1} e^{-(x+1)t} dt, x > 0, n = 1, 2, \ldots,
$$

and (3), we conclude

$$
v'_{\alpha, \beta}(x) = (-1)^n x^n \psi^{(n)}(x + 1) + \frac{n! x^n \alpha \beta}{(x + \beta)^{n+1}} + \frac{(n-1)! x^n \alpha}{(x + \beta)^n}
$$

$$
\triangleq x^n \int_{0}^{\infty} \phi(t) \frac{t^{n-1} e^{-(x+\beta)t}}{e^t - 1} dt
$$

where $\psi^{(-1)}(x+1) = \log \Gamma(x+1)$ and $\psi^{(0)}(x+1) = \psi(x+1)$.
where

\[ \phi(t) = \alpha \beta t(e^t - 1) - te^{\beta t} + \alpha (e^t - 1) = (\alpha - 1) + \sum_{m=2}^{\infty} \frac{[\alpha + m\beta(\alpha - \beta(m-2))]}{m!}. \]

If \( \alpha \geq 1 \) and \( 0 \leq \beta \leq 1 \), then \( \phi(t) > 0 \) and \( v'_{\alpha,\beta}(x) > 0 \). Hence, \( v_{\alpha,\beta}(x) > v_{\alpha,\beta}(0) = 0 \) and \((-1)^n \log F(x)(n) > 0\), and thus, the function \( F(x) \) is strictly logarithmically completely monotonic. The proof of Theorem 1 is complete.

**Corollary 1.** For \( \alpha \geq 1 \) and \( 0 \leq \beta \leq 1 \),

\[ \frac{\Gamma(x + 1)^{\frac{1}{x}}}{\Gamma(y + 1)^{\frac{1}{y}}} > \left( \frac{x + \beta}{y + \beta} \right)^{\alpha}, \tag{14} \]

in which \( 0 < x < y \).

**Theorem 2.** The function \( F(x) \) defined by (7) is strictly logarithmically completely monotonic in \((-1, 0)\) for \( 0 < \alpha \leq \frac{2\beta}{1 + 2\beta} \) and \( \beta > 1 \). Moreover, the function \( F(x) \) is strictly completely monotonic in \((-1, 0)\) for \( 0 < \alpha \leq \frac{2\beta}{1 + 2\beta} \) and \( \beta > 1 \).

**Proof.** By (13),

\[ \phi(t) = \alpha \beta t(e^t - 1) - te^{\beta t} + \alpha (e^t - 1) \]
\[ \phi(0) = 0 \]
\[ \phi'(t) = e^t(\alpha + \alpha \beta + \alpha \beta t) - \alpha \beta - e^{\beta t}(1 + \beta t) \]
\[ \phi''(0) = \alpha - 1 \]
\[ \phi''(t) = e^t \left[ \alpha + 2\alpha \beta + \alpha \beta t - \beta e^{(\beta - 1)t}(2 + \beta t) \right] \]
\[ \triangleq e^t u(t) \]

\[ u(0) = \alpha + 2\alpha \beta - 2\beta \]
\[ u'(t) = \alpha \beta - \beta(\beta - 1)e^{(\beta - 1)t}(2 + \beta t) - \beta^2 e^{(\beta - 1)t} \]
\[ u'(0) = -3\beta^2 + \alpha \beta + 2\beta \]
\[ u''(t) = e^{(\beta - 1)t} \left[ -\beta^2(\beta - 1)^2 t - 2\beta(\beta - 1)(2\beta - 1) \right] \]

If \( 0 < \alpha \leq \frac{2\beta}{1 + 2\beta} \) and \( \beta > 1 \), then \( u''(t) < 0 \) and \( u'(t) \) is strictly decreasing. So \( u'(t) < u'(0) = 0 \) and \( u(t) \) is strictly decreasing. Hence, \( u(t) < u(0) = 0 \) and \( \phi''(t) < 0 \). Since \( 0 < \alpha \leq \frac{2\beta}{1 + 2\beta} \), we have \( \phi'(t) < \phi'(0) < 0 \). So we conclude that \( \phi(t) < \phi(0) = 0 \).
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The proof of Theorem 2 is complete.

Motivated by the open problem, we established a new function

\[ G(x) = \frac{[\Gamma(x + \alpha)]^{\frac{1}{\beta}}}{(x + \beta)^\gamma} \]  \tag{16}

in which \(\alpha, \beta, \gamma\) are nonnegative. Our Theorem 3 consider its logarithmically completely monotonicity.

Theorem 3. The function \(G(x)\) defined by (16) is strictly logarithmically completely monotonic in \((0, \infty)\) for \(\alpha \in (0, 1) \cup [2, \infty), \alpha - 1 \leq \beta \leq \alpha\) and \(\gamma \geq \max\left\{\frac{1}{\beta}, 1\right\}\). Moreover, the function \(G(x)\) is strictly completely monotonic in \((0, \infty)\) for \(\alpha \in (0, 1) \cup [2, \infty), \alpha - 1 \leq \beta \leq \alpha\) and \(\gamma \geq \max\left\{\frac{1}{\beta}, 1\right\}\).

Proof. Using (9), we obtain

\[
\begin{align*}
(\log G(x))^{(n)} &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{x}\right)^{(n-k)} \left[\log \Gamma(x + \alpha)\right]^{(k)} - \frac{(-1)^{n-1} \gamma(n-1)!}{(x + \beta)^n} \\
&= \left(\frac{1}{x}\right)^{(n)} \log \Gamma(x + \alpha) + \sum_{k=1}^{n} \binom{n}{k} \left(\frac{1}{x}\right)^{(n-k)} \psi^{(k-1)}(x + \alpha) + \frac{(-1)^n \gamma(n-1)!}{(x + \beta)^n} \\
&= \frac{(-1)^n n!}{x^{n+1}} \log \Gamma(x + \alpha) + \sum_{k=1}^{n} \frac{n! (-1)^{n-k}}{k! x^{n-k+1}} \psi^{(k-1)}(x + \alpha) + \frac{(-1)^n \gamma(n-1)!}{(x + \beta)^n} \\
&\triangleq (-1)^n \frac{1}{x^{n+1}} \delta(x),
\end{align*}
\]

and

\[
\delta'(x) = x^n \left( (-1)^n \psi^{(n)}(x + \alpha) + \frac{n! \beta \gamma}{(x + \beta)^{n+1}} + \frac{(n-1)! \gamma}{(x + \beta)^n} \right).
\]
Using (3) and (12) for \( x > 0 \) and \( n \in \mathbb{N} \), we conclude

\[
\frac{1}{x^n} \delta'(x) = (-1)^n \psi^{(n)}(x + \alpha) + \frac{n! \beta \gamma}{(x + \beta)^{n+1}} + \frac{(n-1)! \gamma}{(x + \beta)^n} \\
= \int_0^\infty \left[ \gamma(e^t - 1) + \beta \gamma t(e^t - 1) - t e^{(\beta - \alpha + 1)t} \right] \frac{t^{n-1} e^{-(x+\beta)t}}{e^t - 1} \, dt \\
\triangleq \int_0^\infty u(t) \frac{t^{n-1} e^{-(x+\beta)t}}{e^t - 1} \, dt,
\]

where

\[
u(t) = \beta \gamma t(e^t - 1) - t e^{(\beta - \alpha + 1)t} + \gamma(e^t - 1) \\
= (\gamma - 1) t + \sum_{m=2}^{\infty} \{ \gamma + m [ \beta \gamma - (\beta - \alpha + 1)^{m-1}] \} \frac{t^m}{m!}.
\]

If \( \alpha - 1 \leq \beta \leq \alpha \) and \( \gamma \geq \max \{ \frac{1}{\beta}, 1 \} \), then \( u(t) > 0 \) and \( \delta'(x) > 0 \). Notice that \( \Gamma(\alpha) \geq 1 \) for \( \alpha \in (0, 1] \cup [2, \infty) \). Hence, \( \delta(x) > \delta(0) = n! \log \Gamma(\alpha) \geq 0 \) and \( (-1)^n \left( \log G(x) \right)^{(n)} > 0 \) in \((0, \infty)\), and thus, the function \( G(x) \) is strictly logarithmically completely monotonic. The proof of Theorem 3 is complete. \( \square \)

**Corollary 3.** For \( \alpha \in (0, 1] \cup [2, \infty) \), \( \alpha - 1 \leq \beta \leq \alpha \) and \( \gamma \geq \max \{ \frac{1}{\beta}, 1 \} \),

\[
\frac{\Gamma(x + \alpha)\frac{x}{y + \beta}}{\Gamma(y + \alpha)\frac{x}{y + \beta}} > \left( \frac{x + \beta}{y + \beta} \right)^\gamma,
\]

in which \( 0 < x < y \).

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