Quantum localization and cantori in chaotic billiards

Giulio Casati
*International Center for the Study of Dynamical Systems, via Lucini, 3, I–22100 Como, ITALY,*
*Istituto Nazionale di Fisica della Materia and INFN, Unità di Milano*

Tomaž Prosen
*Physics Department, Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, 1111 Ljubljana, Slovenia*
(March 24, 2022)

We study the quantum behaviour of the stadium billiard. We discuss how the interplay between quantum localization and the rich structure of the classical phase space influences the quantum dynamics. The analysis of this model leads to new insight in the understanding of quantum properties of classically chaotic systems.

PACS number: 05.45.+b

The study of quantum mechanics of complex systems at the light of the chaotic behaviour of the corresponding classical systems has greatly improved our understanding of quantum motion \[1\]. For example, the possibility put forward long ago \[2\] that Random Matrix Theory may be a convenient tool to describe spectral properties of classically chaotic systems rests now on more solid grounds \[3\]. However, in spite of the progress in recent years and the growing interest in the so-called “quantum chaos” we are still far from a satisfactory understanding, as a great variety and, to some extent, unexpected rich behaviour of quantum motion continues to emerge \[4\] for which a satisfactory explanation is required. For example, the phenomenon of quantum dynamical localization discovered 20 years ago in systems under external perturbations \[5\] and now experimentally confirmed \[6\], mainly rests on numerical computations and on qualitative considerations. Only few mathematical results exist and it is not clear whether existing semiclassical theories can account for this important feature. Even less understood is the mechanism of dynamical localization in conservative systems \[7\]. Billiards are very convenient models to study quantum localization and cantori \[8\]. They can be, to some extent, studied analytically and in laboratory experiments \[9–11\]. Also they may be relevant for technological applications such as the design of novel microlasers or other optical devices \[12\].

Recently localization has been shown to take place in the stadium billiard \[13\] and on other similar models \[14, 15\]. It is associated to the fact that for small perturbations of the circle, the angular momentum undergoes a classical diffusive process and quantum effects may lead to suppression of this diffusive excitation.

In this paper we study the localization phenomenon and the structure of eigenfunctions (EF) as one moves from the perturbative regime to the ergodic, delocalized regime. The rich variety of classical phase space determines a quite complicated quantum structure. Indeed the classical motion in the stadium billiard can be described by a discontinuous map of the saw-tooth type. This map is known to have cantori \[16\] which may act as barriers to quantum motion \[17\]. This effect has been discussed in \[18\] and recently confirmed in numerical computations on the saw-tooth map on the cylinder \[19\].

In the following we discuss how the combined presence of the cantori structure and of quantum dynamical localization acts on EFs until the regime of quantum ergodic behaviour is reached. We consider the motion of a free point particle of unit mass and velocity \(\vec{v}\) (energy \(E = v^2/2\)) bouncing elastically inside a stadium-shaped well: two semicircles of radius 1 connected by two straight line segments of length \(2\epsilon\). The classical motion, for arbitrary small \(\epsilon\), is ergodic, mixing and exponentially unstable with Lyapunov exponent \(\Lambda \sim \epsilon^{1/2}\). It can be approximated (up to \(O(\epsilon)\)) by the discontinuous stadium-map \[13\]

\[
L_{n+1} = L_n - 2\epsilon \sin \theta_n \text{sgn}(\cos \theta_n) \sqrt{1 - L_n^2}, \theta_{n+1} = \theta_n + \pi - 2\arcsin L_{n+1}
\]

for the rescaled angular momentum \(L = l/\sqrt{2E}\) where \(l = \vec{r} \land \vec{v}\), and the polar angle \(\theta\) (identical to arc-length for small \(\epsilon\)). From rigorous results on the sawtooth-map \[20\] and from the stadium dynamics it can be shown that the angular momentum (for small \(\epsilon\)) undergoes a normal diffusive process with diffusion rate

\[
D = \langle (l_n - l_0)^2 \rangle / n |_{n \geq 1} \approx 2e^{5/2}(2E - l^2)
\]  \(1\)

(Notice that diffusion rate \(D\) depends on the local value of angular momentum). The power 5/2 in eq. \(1\) is due to the existence of *cantori* which form strong obstacles to phase space transport. In ref \[15\] the phenomenon of quantum localization has been shown to take place in the stadium billiard leading to strong deviations from RMT predictions. However the dependence of the localization length on system parameters is not known and in partic-
ular we do not know if, and to what extent, the presence of cantori will influence the quantum dynamics. Indeed it has been conjectured \[22\] that cantori act as perfect barriers for quantum motion provided the flux through cantori is smaller than a Planck's cell \( F < 2\pi \hbar \). On the basis of results on the sawtooth-map \[20\] we can estimate the flux \( F \) — phase space area transported through cantori per iteration (bounce with the boundary) — which is here independent of the winding number of the resonance, and, for small \( \epsilon \), it is given by \( F \approx (2\epsilon)^{1/2} \epsilon^{3/2} \) which leads to the cantori border

\[
\epsilon_c = k^{-2/3},
\]

where \( k = \sqrt{2E} \) is the wavenumber. For \( \epsilon < \epsilon_c \), \( (x := k\epsilon^{3/2} < 1) \), cantori act as perfect barriers, and the quantum system looks as if classically integrable. It is therefore expected that the localization length \( \ell \) of eigenstates in angular momentum variable \( l \) must be of the order of the size of cantori. This size, in rescaled angular momentum variables, averaged over all the resonances, can be estimated from the exact results on sawtooth map \[23\], namely \( \bar{\ell} = \epsilon \ell_c \), where \( \epsilon \approx 12 \) is a numerical constant. \( (c = 10 \text{ for } \epsilon = 0.05, \text{ and } c = 15 \text{ for } \epsilon = 0.005) \). The fact that \( \epsilon \) slowly increases with decreasing \( \epsilon \) is due to the presence of the cantoruses along the separatrix of 2:1 resonance (around \( L = 0 \)) which has a larger size, \( \bar{\ell}(2, 1) \approx \sqrt{7} \). In Fig.\ref{fig:1} we show the classical structure of cantori (\( \epsilon = 0.003 \)) in phase space around the largest 2:1 resonance and the associated quantum eigenstate. In this regime the (average) rescaled localization length of eigenstates \( \sigma = \ell/\ell_{\text{max}} = \epsilon/k \) is indeed found to be equal to the (average) size of cantori (see Fig.\ref{fig:2} Fig.\ref{fig:3}), \( \sigma = \bar{\ell} \).

\[
\sigma \approx D/k \approx \alpha \epsilon^{5/2} = \alpha \epsilon x.
\]

where \( \alpha \approx 1.7 \) is a numerical constant. However, to the above expression, we need to add the average size \( \bar{\ell} \) of cantori. Therefore, for \( x > 1 \), the actual expression for the localization length will be given by

\[
\sigma = \bar{\ell} + (1 - \bar{\ell}) \alpha \epsilon (x - 1)
\]

which takes into account also the fact that we need to rescale the total size of angular momentum space, and that for \( x = 1 \), \( \sigma = \bar{\ell} \). Eigenstates become delocalized (ergodic) when \( \sigma = 1 \), giving the ergodicity border \( \epsilon = \epsilon_c \),

\[
\epsilon_c \approx (\alpha k)^{-2/5}
\]

which agrees with the results of \[23\]. The cantori border can actually be observed if it is below the ergodic border and above the perturbative border \( \epsilon_p \). The perturbative border \( \epsilon_p \) is given by the condition that \( \epsilon \) should be large enough, \( \epsilon > \epsilon_p \), to couple two neighboring eigenvalues of angular momentum, which is equivalent to the intuitive condition of comparing the deformation \( \epsilon \) and the De Broglie wavelength, which leads to

\[
\epsilon_p = k^{-1}.
\]

Therefore, for sufficiently large \( k \), we have, \( \epsilon_p < \epsilon_c < \epsilon_c \). In this situation it is natural to expect that cantori will influence the localization process and we may have here a nice possibility to study the effect of cantori in quantum mechanics.

In order to check the above predictions we numerically computed quantum eigenfunctions \( \Psi_k(\vec{r}) \) of the stadium billiard (solutions of the Schrödinger equation \((\nabla^2 + k^2)\Psi_k = 0 \) where \( \hbar = 1 \)) by expanding them in terms of circular waves (here we consider only odd-odd states) \( \Psi_k(\vec{r}) = \sum_{s=1}^M a_s J_{2s} (kr) \sin(2s\theta) \). Eigenvalues \( k = k_n \) and the associated coefficients \( a_n \) have been computed very efficiently \[14\] by minimizing a special quadratic form defined along the boundary of the billiard \[23\]. The coefficient \( a_n \) is proportional to the probability amplitude of finding angular momentum equal to \( l = 2s \), \( p_k(l) = |\langle l | 2s|\Psi_k \rangle|^2 = |a_s|^2 \int_0^{2\pi} d\theta |J_{2s}(kr)|^2 \approx |a_n|^2 \sqrt{1 - l^2/k^2} \).

![Figure 1](image1)

**FIG. 1.** Time evolution of a single classical orbit, followed up to 20,000 bounces, for the classical billiard with \( \epsilon = 0.003 \). The orbit is initially started in the middle of the largest ‘island’ ( \( L = 0, \theta = \pi/4 - 0.0016 \) (a)). Angular momentum probability distribution \( p_k(l) \) of the corresponding eigenstate \( \Psi_k \) with \( \epsilon = 0.003 \) and eigenvalue \( k = 5999.8166 \). As it is seen the state is uniformly distributed over the cantoruses in the main island (b).

For \( \epsilon > \epsilon_c \), \( (x = k\epsilon^{3/2} > 1) \), when the flux trough turn-
Quantum localization in the stadium is not exponential like for kicked rotor or smooth diffusive billiards, but rather of power-law type. More precisely, the tails of eigenfunctions have been found to decay, on average, as \( p(l) \sim |l - \langle|l|\rangle|^{-4} \). Therefore, in order to characterize the localization length of quantum eigenfunctions \( \Psi_k \), we choose the 99% probability localization length \( \sigma \) rather than the more common inverse participation ratio, information entropy or uncertainty of an eigenstate. These distributions are strongly sparse and ex-}

In Fig.4 we show the probability distributions of a typical eigenstate \( \Psi_k \). In order to illustrate the general structure we show in Fig.4 the probability distributions of a typical circle state in terms of eigenstates of the stadium (one row of the matrix) is (on average) equal to the localization length \( \ell \). It is important to note that unlike for Wigner band random matrices, the matrix \( c_{n,m}^0 \) (ordered, as usual, with increasing quantum wavenumbers \( n_k, k_{n,m}^0 \)), has a symmetric appearance. It has a band structure with the bandwidth \( b \sim k = \delta E \), independent of \( \epsilon \) for \( \epsilon < \epsilon_p \), where \( \delta E \) is the width of the energy shell. However, below the ergodicity border, the matrix \( c_{n,m}^0 \) is uniformly sparse, both in horizontal and vertical directions. The effective number of nonzero elements in each row (or column) is (on average) equal to the localization length \( \ell = \sigma k \). In order to illustrate the general structure we show in Fig.4 the probability distributions of a typical circle state in terms of eigenstates of the stadium (one column of the matrix \( |c_{n,m}^0|^2 \)) and a typical eigenstate of the stadium in terms of circle states (one row of the matrix). These distributions are strongly sparse and extended over the whole energy shell of \( \delta E \sim b = k \) states. In Fig.4e we show, for the same stadium eigenstate, the probability distribution \( p(l) \) in angular momentum: as expected it is strongly localized and non-sparse. The above structure is found in the regime of dynamical localization and in the regime of cantori localization where the quantum system behaves as if classically integrable. Furthermore, a similar structure is found also for the nearby-circular but chaotic rough billiards with smooth boundaries introduced in [14] (above the ‘Breit-Wigner regime’), where the situation is even more complicated due to KAM structure of classical phase space.

As the parameter \( \epsilon \) is increased up to the ergodicity border \( \epsilon_c \), sparsity decreases and the quantum angular momentum distribution \( p(l) \) approaches (apart from fluctuations) the classical steady-state microcanonical distribution, \( p_c(l) \propto 2E - T^2 \). Notice that the scaling parameter \( \sigma \) also controls the deviations from RMT predictions of the statistical properties of eigenvalues and eigenfunctions.

---

**FIG. 2.** Rescaled localization length \( \sigma \) versus the scaling variable \( x = \epsilon^{3/2} k \) for five values of \( \epsilon \) (60 < \( k \) < 12,000). Each point is obtained by averaging over a large number \( \nu \) of consecutive eigenstates (\( \nu = 100 \) for small \( k \) and \( \nu = 1,000 \) for large \( k \)). The numerical data clearly show the cantori border \( x = 1 \). In the cantori region \( x \) is constant as expected, while for \( x > 1 \) the numerical data agree with the theoretical prediction \( \delta \) (dotted curves). For large \( x \), the value of \( \sigma \) approaches the maximal ergodic value \( \sigma = 1 \).

**FIG. 3.** Rescaled localization length \( \sigma \) versus \( \epsilon \) for two different values of \( k \). The full lines give the classical estimate for the average width of cantori. It is seen that below the cantori border \( x = 1 \) (\( \epsilon = 0.003 \) for \( k = 6,000 \) and \( \epsilon = 0.0063 \) for \( k = 2,000 \)) \( \sigma \) is proportional to \( \epsilon \) and independent on \( k \). Above the border \( x = 1 \) instead, the numerical data follow the theoretical estimate \( \delta \) (dotted curves).

In this paper we have discussed a dynamical model, the stadium billiard, for which the classical motion is completely chaotic (without any island of stability) for any value of the control parameter \( \epsilon \) and we have shown that the quantum dynamics instead, exhibits a rich structure and different regimes of motion as a function of \( \epsilon \) and energy \( E \). It has been shown that the presence of cantori in...
classical phase space may have strong effect on the quantum dynamics and leads to a new border which is different from the perturbative and the ergodic border. In the regime of quantum cantori (where the phase space flux through cantori is less than one quantum) the rescaled localization length $\sigma = l/k$ does not depend on energy or wavenumber $k = \sqrt{2\varepsilon}$. However, above the cantori border, quantum dynamical localization takes place and the localization length $l$ is found to be proportional to the rate $D$ of classical diffusion in angular momentum. The mechanism of localization is strongly connected to the sparsity of EFs when expanded on the basis of (unperturbed) circle states (and vice versa). We suggest that the above features are typical of the quantum dynamics of classically chaotic conservative systems.

Discussions with R. Prange, R.S. MacKay, J. Keating, G. Tanner and I. Dana are gratefully acknowledged. T.P. acknowledges financial support from the Ministry of Science and Technology of R. Slovenia.

![FIG. 4.](image)

**FIG. 4.** The structure of high-lying eigenstates of the stadium $k_s \approx 2000$ for $\varepsilon = 0.01$ in terms of circle states and vice versa. In (a) we show the probability distribution of a typical circle state ($l = 2s = 848, k_{sm} = 1999.99349$) versus the wavenumber (energy) of the stadium eigenstates (a column of the matrix $|c_{sm}|^2$ with fixed $s, m$), in (b) we show the probability distribution of a typical eigenstate of the stadium ($k_s = 1999.91397$) versus the wavenumber of circle states (a row of the matrix $|c_{sm}|^2$ with fixed $n$). In (c) we show the same state as in (b) but in angular momentum quantum number $l$ (the sum over radial quantum number $\sum n |c_{sm}|^2$). Notice that the rescaled localization length here is $\sigma_{lk} = 0.10$ and, correspondingly, only a fraction $\sigma_{kn}$ of the total number of $\sim k$ states inside the energy shell is actually significantly excited (a,b). The dotted curve in (c) gives the ergodic distribution $p_\mu(l) = (8\pi/k^2)\sqrt{k^2 - l^2}$.

[1] G. Casati and B.V. Chirikov, eds., *Quantum Chaos: Between Order and Disorder*, Cambridge University Press (1994); Physica D86, 220 (1995).
[2] G. Casati, I. Guarneri and F. Valz-Gris, Lett. Nuovo Cimento 28, 279 (1980); E. McDonald and A.N. Kaufman, Phys. Rev. Lett. 42, 1189 (1979).
[3] A.V. Andreev, O. Agam, B.D. Simons, and B.L. Altshuler, Phys. Rev. Lett. 76, 3947 (1996).
[4] O. Bohigas in *Proceedings of the 1989 Les Houches Summer School on “Chaos and Quantum Physics”,* ed. by M. J. Giannoni, A. Voros, and J. Zinn-Justin (Elsevier Science Publisher B.V., North-Holland, Amsterdam 1991), p. 89.
[5] G. Casati, G. Maspero and D.L. Shepelyansky: quantum strange attractor preprint.
[6] G. Casati, B.V. Chirikov, J. Ford and F.M. Izrailev, Lecture Notes in Physics 39, 334 (1979).
[7] F.L. Moore, J.C. Robinson, C.F. Bharucha, B. Sundaram, and M.G. Raizen, Phys. Rev. Lett. 75, 4598 (1995).
[8] G. Casati, B.V. Chirikov, I. Guarneri and F.M. Izrailev, Phys. Rev. E 48, R1613 (1993); Physics Letters A223 (1996) 430.
[9] R. Artuso, G. Casati, and I. Guarneri, Phys. Rev. E 55, 6384, (1997).
[10] R. Artuso, G. Casati and I. Guarneri, J. Stat. Phys. 83, 145, (1996).
[11] H. Alt, H.-D. Graf, R. Hofferbert, C. Ramgachar, H. Rehfeld, A. Richter, P. Schardt, and A. Wirzba, Phys. Rev. E 54, 2303, (1996).
[12] J. Stein and H.-J. Stockmann, Phys. Rev. Lett. 68, 2867, (1992).
[13] A. Kudrolli, V. Kidambi, and S. Sridhar, Phys. Rev. Lett. 75, 822 (1995).
[14] J. U. Nockel and A.D. Stone, Nature (London) 385, 45 (1997).
[15] F. Borgonovi, G. Casati and B. Li, Phys. Rev. Lett. 77, 4744 (1996).
[16] K. Frahm and D. Shepelyansky, Phys Rev. Lett. 78, 1440 (1997); ibid. 79, 1833 (1997).
[17] G. Casati and T. Prosen, preprint, cond-mat/9704098.
[18] F. Borgonovi, preprint, chao-dyn/9801032.
[19] G. Casati and T. Prosen, preprint, submit.to Physica D 54.
[20] Q. Chen, I. Dana, J. D. Meiss, N. W. Murray, and L. C. Percival, Physica D 46, 217 (1990); see also I. Dana, N. W. Murray, and I. C. Percival, Phys. Rev. Lett. 62, 233 (1989).
[21] T. Geisel, G. Radons, and J. Rubner, Phys. Rev. Lett. 57, 2883 (1986).
[22] R.S. MacKay and J.D. Meiss, Phys. Rev. A37 (1988) 4702.
[23] Average the expression for $h(m, n)$, pp225, weighted by phase-space areas of resonances (eq.33) of [20].
[24] E. Vergini and M. Saraceno, Phys. Rev. E 52, 2204 (1995).
[25] This integrable-like behaviour maybe at the root of the analytical results in: R.E. Prange and R. Nair, “Quasiclassical surface of section perturbation theory”, preprint.