TOPOLOGICAL $\sigma$-MODELS IN FOUR DIMENSIONS
AND TRIHOLOMORPHIC MAPS

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Abstract

It is well-known that topological $\sigma$-models in two dimensions constitute a path-integral approach to the study of holomorphic maps from a Riemann surface $\Sigma$ to an almost complex manifold $\mathcal{K}$, the most interesting case being that where $\mathcal{K}$ is a Kähler manifold. We show that, in the same way, topological $\sigma$-models in four dimensions introduce a path integral approach to the study of triholomorphic maps $q: M \to N$ between a four dimensional Riemannian manifold $M$ and an almost quaternionic manifold $N$. The most interesting cases are those where $M, N$ are hyperKähler or quaternionic Kähler. BRST-cohomology translates into intersection theory in the moduli-space of this new class of instantonic maps, that are named by us hyperinstantons. The definition of triholomorphicity that we propose is expressed by the equation

$$q^* - J_u \circ q^* \circ j_u = 0,$$

where $\{j_u, u = 1, 2, 3\}$ is an almost quaternionic structure on $M$ and $\{J_u, u = 1, 2, 3\}$ is an almost quaternionic structure on $N$. This is a generalization of the Cauchy-Fueter equations. For $M, N$ hyperKähler, this generalization naturally arises by obtaining the topological $\sigma$-model as a twisted version of the N=2 globally supersymmetric $\sigma$-model. We discuss various examples of hyperinstantons, in particular on the torus and the K3 surface. We also analyse the coupling of the topological $\sigma$-model to topological gravity. The classification of triholomorphic maps and the analysis of their moduli-space is a new and fully open mathematical problem that we believe deserves the attention of both mathematicians and physicists.

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1 Introduction

A \( \sigma \)-model is a theory of maps \( \phi : \mathcal{M} \rightarrow \mathcal{N} \) from a Riemannian manifold \( \mathcal{M} \) to a Riemannian manifold \( \mathcal{N} \), described by the action

\[
S = \frac{1}{2} \int_{\mathcal{M}} d^m x \sqrt{g(x)} g^{\mu\nu}(x) \partial_\mu \phi^i(x) \partial_\nu \phi^j(x) h_{ij}(\phi(x)),
\]

where \( m = \dim \mathcal{M} \), \( x \) denote the points of \( \mathcal{M} \), while \( g_{\mu\nu}(x) \) is the metric of \( \mathcal{M} \) and \( h_{ij}(\phi) \) is the metric of \( \mathcal{N} \). In general, \( \mathcal{M} \) is called the world-manifold; it is called the world-sheet when it is two-dimensional. \( \mathcal{N} \) is the target manifold.

The action \( S \) is invariant under those infinitesimal deformations \( \phi + \delta \phi \) of the map \( \phi \) that are isometries of \( \mathcal{N} \). If the \( \mathcal{M} \) metric \( g_{\mu\nu}(x) \) is a given background metric, then we say that gravity is external. One can also consider the case in which gravity is dynamical. In this case \( \mathcal{M} \) can be arbitrarily chosen only as a topological space: its metric \( g_{\mu\nu} \), instead, is to be determined consistently with the map \( \phi \), from the variation of an action that is the sum of the \( \sigma \)-model action (1) plus the Einstein-Hilbert action

\[
- \frac{1}{2} \int_{\mathcal{M}} d^m x \sqrt{g(x)} R(x),
\]

\( R \) being the scalar curvature of \( g_{\mu\nu} \).

A topological \( \sigma \)-model \([1, 2, 3, 4]\) is a theory dealing with the homotopy classes of the maps \( \phi : \mathcal{M} \rightarrow \mathcal{N} \). It is described by an action which is invariant under any continuous deformation \( \phi \rightarrow \phi + \delta \phi \) of the map. This is intrinsically a quantum field theory, since the classical action is either zero or a topological invariant, due to the large symmetry that it is required to possess. Indeed, the functional integral formulation of quantum field theory provides very powerful methods for the study of such a theory of maps. This large symmetry is BRST-quantized \([5, 6]\) in the usual ways as any other gauge symmetry and the gauge is fixed by choosing suitable representatives in the homotopy classes of the maps \( \phi \). These representatives are usually some kind of instantons, because it is in this case that the topological theory turns out to be most interesting. The theory is not independent of the chosen gauge-fixing. Indeed, topological field theories \([1, 2, 3, 4, 5, 6, 7, 8, 9, 10]\) show very clearly that in general two gauge-conditions that are not continuously deformable one into the other give rise to inequivalent quantum field theories. Typically, the gauge-fixing does not fix the gauge completely, namely there is a subset of continuous deformations \( \phi \rightarrow \phi + \delta \phi \) that preserve the gauge condition. The set of maps that satisfy the instantonic equations is called the moduli-space of the theory and in general it is a finite dimensional manifold. The topological field theory is a cohomological theory in this space. Indeed, the functional integral is projected onto an integral over the moduli-space (that is an ordinary integral) and the physical amplitudes are suitable topological invariants of this space.

In the case of external gravity there is an alternative definition of topological field theory, that is a theory characterized by a BRST-exact energy-momentum tensor \( T_{\mu\nu} = \)

This is so, because the lagrangian formulation makes explicit use of the world-metric $g_{\mu\nu}$ although a topological field theory is expected to describe quantities that are independent of any metric. This feature is guaranteed by the very BRST-exactness of $T_{\mu\nu}$. In the case of dynamical gravity, independence of the metric is a meaningless requirement, because the metric is a quantum field. Nevertheless, one can define the concept of topological gravity by saying that it is a theory that quantizes the most general continuous deformation of the metric [4, 3].

We see that topological field theories represent a beautiful joint-venture for physics and mathematics. One conveniently formulates a mathematical problem in the language of physics and moreover the object of the study is of physical interest, since instantons are peculiar solutions to the field equations, namely solutions that give a leading contribution to the functional integral (in the Euclidean region).

Topological $\sigma$-models have been extensively studied in two dimensions ($m = 2$) [1, 2, 3, 4, 11]. The world-sheet $\Sigma$ is a Riemann surface and the target manifold $\mathcal{N}$ is almost complex. The instanton equations are the Cauchy-Riemann equations

$$\partial_\alpha \phi^i - \varepsilon_{\alpha\beta} J^i_{\beta j} \partial_\beta \phi^j = 0,$$

(3)

where $\alpha = 1, 2$ labels the world-sheet coordinates, $\varepsilon$ is the world-sheet complex structure and $J$ is the almost complex structure of $\mathcal{N}$. Witten showed [1] that a convenient starting point is provided by choosing $\mathcal{N}$ to be a Kähler manifold. The theory then describes the holomorphic embeddings $\phi : \Sigma \rightarrow \mathcal{N}$ of Riemann surfaces into Kähler manifolds. If $K$ is the Kähler form of $\mathcal{N}$, one can start from the classical action [2]

$$S_{\text{class}} = \int_{\Sigma} \phi^* K,$$

(4)

where $\phi^* K$ denotes the pull-back of $K$ onto the world-sheet $\Sigma$. The action [1] is clearly invariant under any continuous deformation of the map $\phi$.

One reason why a Kählerian $\mathcal{N}$ is convenient comes once more from physics. Indeed, $\mathcal{N}$=Kähler manifold is the condition for a two-dimensional $\sigma$-model to have an N=2 supersymmetry [12]. In that case, the topological $\sigma$-model arises naturally from the N=2 supersymmetric $\sigma$-model [13], by performing a set of formal manipulations and redefinitions that is called the topological twist [14, 1]. The topological theory that comes from the twist is already gauge-fixed and the natural gauge-fixing is precisely the instantonic condition.

As a matter of fact, the twist was introduced in four dimensions, when Witten formulated topological Yang-Mills theory [14] and showed that it can be obtained by twisting N=2 super Yang-Mills theory. In general, although the starting N=2 theory is defined on a flat world-manifold $\mathcal{M}$ (when supersymmetry is global), yet the final theory can be defined on any $\mathcal{M}$ and it is independent of the choice of the metric on $\mathcal{M}$. Recently [15, 16, 17, 18], we have shown that any N=2 globally but also locally supersymmetric theory in four dimensions can be twisted, if the topological twist is suitably improved. One gets a topological version of the theories, namely topological gravity, topological
Yang-Mills theory and topological σ-model, eventually coupled together. In pure topological gravity [15, 16] or topological gravity coupled to topological Yang-Mills theory [17, 18], the gravitational instantons [19] are the solutions to the condition of selfduality for the spin connection $\omega^{ab}$, namely $\omega^{-ab} = 0$. On the other hand, topological Yang-Mills theory is gauge-fixed by the equations of the usual Yang-Mills instantons. The novelty comes from the twist of the N=2 σ-model [17, 18] describing the self-interaction of hypermultiplets [20, 21]. The new instantons that gauge-fix this theory were called by us hyperinstantons and are the main subject of the present paper. Moreover, the twist of N=2 supergravity coupled to hypermultiplets gives a theory of topological gravity coupled to topological σ-model in which the condition $\omega^{-ab} = 0$ is modified by a contribution due to the hypermultiplets. The hyperinstanton equations, instead, are unmodified.

In this paper we formulate the topological σ-model in four dimensions as a theory of maps $q : M \to N$ from a four dimensional Riemannian manifold $M$ to an almost quaternionic manifold $N$ (if $M$ is Riemannian and four dimensional, then it is also almost quaternionic [22]). Inspired by the results coming from the topological twist [17, 18], we propose the following concept of triholomorphic maps $q : M \to N$. Let $TM$ and $TN$ be the tangent bundles to $M$ and $N$. Let $\{j_u : TM \to TM, u = 1, 2, 3\}$ and $\{J_u : TN \to TN, u = 1, 2, 3\}$ be almost quaternionic structures on $M$ and $N$, respectively, namely triplets of $(1, 1)$-tensors satisfying the quaternionic algebras

$$j_u \circ j_v = -\delta_{uv} \text{id}_{TM} + \varepsilon_{uvz} j_z,$$
$$J_u \circ J_v = -\delta_{uv} \text{id}_{TN} + \varepsilon_{uvz} J_z,$$ (5)

where $\text{id}_{TM}$ and $\text{id}_{TN}$ are the identity maps on $TM$ and $TN$, respectively. By convention, $j_u$ (resp. $J_u$) will be called the almost quaternionic $(1, 1)$-tensors of $M$ (resp. $N$).

Consider the pull-forward $q_* : TM \to TN$ of the map $q : M \to N$ and the following diagram

$$\begin{array}{c}
TM & \xrightarrow{q_*} & TN \\
\uparrow j_u & & \downarrow J_v \\
TM & \xrightarrow{J_u \circ q_* \circ j_u} & TN
\end{array}$$ (6)

We see that $J_v \circ q_* \circ j_u$ acts from $TM$ to $TN$, precisely as $q_*$. Consider the equation

$$q_* - J_u \circ q_* \circ j_u = 0.$$ (7)

The sum over the repeated index $u$ is understood. This is an equation on the map $q : M \to N$ and it is our proposal for the definition of triholomorphic maps from a four dimensional Riemannian manifold $M$ to an almost quaternionic manifold $N$. The definition (4) of triholomorphic maps is not the only possible choice. There is no uniqueness of the relative ordering of the three almost quaternionic $(1, 1)$-tensors of the two manifolds. Moreover, the almost quaternionic $(1, 1)$-tensors are in general only locally defined (1, 1)-tensors [23]. For the moment we suppose that they can be defined globally. We shall soon come back to this point.
defined \[23\], i.e. defined on neighborhoods \(U(\alpha)\) such that on the intersection \(U(\alpha) \cap U(\beta)\) of two neighborhoods the transition functions are \(SO(3)\) matrices \(\Lambda^{uv}\). Consequently, eq. (7) should be substituted by the more general condition

\[ q_* - \Lambda^{uv} J_u \circ q_* \circ j_v = 0, \]  

(8)

where \(\Lambda\) is an \(SO(3)\) matrix that can depend on the point. Then, triholomorphic maps are those maps \(q\) such that there exists a \(\Lambda^{uv}\) such that (8) holds. We postpone the discussion of this ambiguity to later sections, where we shall see that it has a quite simple physical interpretation.

For eq. (7) to be meaningful, it is only required that both \(\mathcal{M}\) and \(\mathcal{N}\) possess almost quaternionic structures. The most interesting cases are when \(\mathcal{M}, \mathcal{N}\) are hyperKähler \[24\] or quaternionic Kähler \[23, 25, 26\] (the latter will be simply called “quaternionic”), namely when the almost quaternionic structures possess more properties.

We are going to show that eq. (7) is a good definition and that it agrees with the concept of triholomorphic maps as formulated in ref.s \[17, 18\] coming from the topological twist (despite the explicit appearance of the metric in the equations that naturally follow from the twist).

In section 2 we formulate the general topological \(\sigma\)-model, where the gauge-fixing is provided by the triholomorphic instanton condition (7). This topological \(\sigma\)-model naturally describes the moduli-space of triholomorphic maps (that we call hyperinstantons, when dealing with physics). In section 3 we make the match with the theories coming from the topological twist. Indeed, we show that when \(\mathcal{M}\) and \(\mathcal{N}\) are both hyperKähler manifolds, the general topological \(\sigma\)-model gauge-fixed by (7) is nothing else but the topological twist of the \(N=2\) globally supersymmetric \(\sigma\)-model (hypermultiplets). On the other hand, when either \(\mathcal{M}\) or \(\mathcal{N}\) is quaternionic (external gravity), the topological \(\sigma\)-model gauge-fixed by (7) is not obtainable as the topological twist of any \(N=2\) globally supersymmetric theory. It is notorious that a quaternionic \(\mathcal{N}\) corresponds to the case of a locally supersymmetric \(\sigma\)-model, so that the case \(\mathcal{N}\) quaternionic requires dynamical gravity in the twisting procedure. This simply shows the well-known fact that the set of possible topological theories is larger than the set of those obtainable from the twist.

The complete details of the topological twist are shown in appendix B, while the general definitions of hyperKähler and quaternionic manifolds can be found in appendix A.

As far as we know, eq.s (7) have not been proposed in the mathematical literature, apart from the case \(\mathcal{M} = \mathcal{N} = \mathbb{R}^4\) where they reduce to the Cauchy-Fueter equations \[24, 28\]. In section 4 we recall the main properties of the solutions to the Cauchy-Fueter equations, to give the reader an idea of what they are. In section 5 we derive the general form of triholomorphic maps in the case \(\mathcal{M} = \mathcal{N} = T_4\). In section 6 we exhibit a class of solutions with \(\mathcal{M} = \mathcal{N} = K3\) (we consider the Fermat surface for simplicity) to convince the reader that the set of solutions to these new equations is non-empty and non-trivial. This class of solutions will be described directly by the properties of the polynomial that defines the Fermat surface, to address the possibility that eq.s (7) have an algebraic counterpart in the case of algebraic varieties. One can also generalize the concept of
“rational curves” of a Kähler manifold by choosing $\mathcal{M} = S^4$ or $\mathbb{C}P^2$ and studying the triholomorphic embeddings of these manifolds into hyperKähler or quaternionic manifolds of any dimension. Interesting embeddings are, of course, also those of $\mathcal{M} = T^4$ or K3.

Physics also provides a topological theory of dynamical gravity coupled to the $\sigma$-model, as mentioned above, in which case the target manifold $\mathcal{N}$ is quaternionic. The coupled equations (something like the “square root” of the coupled Einstein and matter equations) are very difficult to solve, as one can expect. In section 5 we show that the simplest ansätze are not solutions. Nevertheless, the same ansätze are instantons of the general $\sigma$-model presented in the next section (with external gravity), that also contemplates the case of a quaternionic $\mathcal{N}$.

When the world-manifold and the target one are both hyperKähler or quaternionic and four dimensional, we can think of hyperinstantons as maps that go form a gravitational instanton (the world-manifold) to a gravitational instanton (the target) and that are themselves instantons. So, the study of hyperinstantons may be useful for getting insight into gravitational instantons (here intended as manifolds with a self-dual Riemann tensor or a self-dual Weyl tensor). We hope that our work will stimulate research into this subject, because we think that it can be source of insight into the problem of gravitational instantons.

2 Topological $\sigma$-model for triholomorphic maps

In this section we build the topological $\sigma$-model for hyperinstantons, by generalizing the method used by Witten in ref. [1] and further clarified by Baulieu and Singer in ref. [2]. We need Riemannian metrics both on $\mathcal{M}$ and $\mathcal{N}$. We suppose that they are Hermitian with respect to the almost quaternionic $(1,1)$-tensors of the corresponding manifolds. We notice that any four dimensional Riemannian manifold can be naturally endowed with an almost quaternionic structure such that the metric is Hermitian [22].

Introducing indices explicitly, equation (3) takes the form

$$\partial_\mu q^i - (j_u)_\mu^\nu \partial_\nu q^j(J_u)_{ji} = 0,$$

where $\mu = 1, \ldots, 4$ are the world indices and $i = 1, \ldots, 4n$ are the target ones (we set $\dim \mathcal{N} = 4n$). We see that (4) is the natural generalization of the Cauchy-Riemann equations (3). The $16n$ equations (4) are not all independent. Indeed, we expect $4n$ equations. The correct counting is retrieved by observing that the matrix

$$H_\mu^i = \partial_\mu q^i - (j_u)_\mu^\nu \partial_\nu q^j(J_u)_{ji}$$

satisfies identically the duality condition

$$H_\mu^i + \frac{1}{3}(j_u)_\mu^\nu H_\nu^j(J_u)_{ji} = 0.$$

This condition, as the reader can easily verify, reduces the number of equations by a factor four.
The BRST-quantization of the theory is achieved as follows. We introduce topological ghosts $\xi^i$ (ghost number $g = 1$), as well as topological antighosts $\zeta^i$ (ghost number $g = -1$) and Lagrange multipliers $b^i_\mu$ (ghost number $g = 0$) for the gauge-fixing (9). Antighosts and Lagrange multipliers are required to satisfy the same duality condition as the left hand side of (9), namely

$$
\zeta^i + \frac{1}{3} (j_a)_\mu \nu \zeta^i_\nu (J_a)_j^i = 0, \quad b^i_\mu + \frac{1}{3} (j_a)_\mu \nu b^i_\nu (J_a)_j^i = 0.
$$

The BRST operator will be denoted by $s$ and the BRST-variation of $q^i$ will be $sq^i = \xi^i$, so that $s\xi^i = 0$. On the other hand, the BRST-variation of the topological antighost $\zeta^i$ is not simply the Lagrange multiplier $b^i_\mu$, since we have to make sure that the duality condition of $\zeta^i$ is preserved by the BRST-algebra. The correct form of $s\zeta^i$ is $s\zeta^i = b^i_\mu - \Gamma^i_{jk} \xi^j \zeta^k - \frac{1}{4} (j_a)_\mu \nu D_k (J_a)_j^i \xi^k \zeta^j$, where $\Gamma^i_{jk}$ is the Levi-Civita connection on the target manifold $\mathcal{N}$, while $D_k$ is the covariant derivative on $\mathcal{N}$. The BRST-variation of $b^i_\mu$ is obtained by demanding $s^2 b^i_\mu = 0$. One then checks consistency by verifying that $s^2 \zeta^i = 0$ and that the duality condition (12) on $b^i_\mu$ is preserved. The complete BRST algebra is given by

$$
sq^i = \xi^i,
$$
$$
\s\xi^i = 0,
$$
$$
s\zeta^i = \frac{1}{2} R_{j k} \xi^j \zeta^i_\mu \mu - \frac{1}{4} \Gamma^i_{j k} \xi^j b^k_\mu - \frac{1}{4} (j_a)_\mu \nu D_k (J_a)_j^i \xi^k b^j_\nu + \frac{1}{4} (j_a)_\mu \nu D_m D_k (J_a)_j^i \xi^m \xi^j \zeta^\mu - \frac{1}{16} D_k (J_a)_j^i D_l (J_a)_m^j \xi^k \xi^l \zeta^\mu - \frac{1}{16} \xi_{u v z} (j_a)_\mu \nu D_k (J_a)_j^i \xi^m \xi^j \zeta^\mu.
$$

To find a Lagrangian for the theory, we have to choose a gauge fermion $\Psi$ that fixes the gauge according to eq. (9). This is achieved by setting

$$
\Psi = \int_\mathcal{M} d^4 x \sqrt{g} g^{a \mu} h_{ij}^i \xi^i_\mu \left( \partial_{\nu} q^i - \frac{1}{8} b^i_\nu \right).
$$

The action $S = s \Psi$ then turns out to be

$$
S = S_{bosonic} + S_{ghost},
$$

where

$$
S_{bosonic} = \int_\mathcal{M} d^4 x \sqrt{g} g^{a \mu} h_{ij}^i b^i_\mu \left( \partial_{\nu} q^i - \frac{1}{8} b^i_\nu \right),
$$
$$
S_{ghost} = \int_\mathcal{M} d^4 x \sqrt{g} \left( - g^{a \mu} h_{ij}^i \xi^i_\mu \nu D^\nu \xi^j + \frac{1}{16} R_{ijkl} g^{a \mu} \xi^i_\mu \nu D^\nu \xi^j \xi^k \xi^l \right).
$$
The covariant derivative $D_\mu \xi^i$ of $\xi^i$ is defined, as usual, according to

$$D_\mu \xi^i = \partial_\mu \xi^i + \Gamma^i_{jk} \partial_\mu q^j \xi^k.$$  (17)

Keeping into account the self-duality condition (12), the equation of motion of the Lagrange multiplier $b^i_\mu$ is

$$b^i_\mu = \partial_\mu q^i - (J_u)_\mu^\nu \partial_\nu q^i (J_u)_j^i.$$  (18)

By eliminating the Lagrange multiplier from the action (16), one arrives at the following final form of the bosonic action (the ghost action $S_{\text{ghost}}$ is not affected),

$$S_{\text{bosonic}} = \int_M d^4x \left( \frac{1}{2} \sqrt{g} g^{\mu\nu} h_{ij} \partial_\mu q^i \partial_\nu q^j + \frac{1}{2} \sqrt{g} (J_u)^{\mu\nu} (J_u)_ij \partial_\mu q^i \partial_\nu q^j \right).$$

This is the usual $\sigma$-model action (1) plus the term

$$S_T = \frac{1}{2} \int_M d^4x \sqrt{g} (J_u)^{\mu\nu} (J_u)_ij \partial_\mu q^i \partial_\nu q^j.$$  (20)

When $\mathcal{M}$ and $\mathcal{N}$ are both hyperKähler manifolds, $S_T$ is a topological invariant. Indeed, in this case let us introduce the Kähler forms (see appendix A)

$$\Omega_u = (J_u)_i^j h_{jk} dq^j \wedge dq^k,$$

$$\Theta_u = (J_u)_\mu^\nu g_{\nu \rho} dx^\mu \wedge dx^\rho.$$  (21)

$\Theta_u$ are self-dual or anti-self-dual. We choose them to be anti-self-dual ($\ast \Theta_u = -\Theta_u$). Then, (20) can be written as

$$S_T = -\frac{1}{4} \int_M q^* \Omega_u \wedge \Theta_u.$$  (22)

The topological character of $S_T$ is now evident, since both $\Omega_u$ and $\Theta_u$ are closed. The theory with $\mathcal{M}$ and $\mathcal{N}$ hyperKähler is the theory that can be obtained by topological twist (see section 3). In general, the quantum action $S_q$ is the sum of a classical action $S_{\text{class}}$ plus the BRST-variation $s\Psi$ of a gauge-fermion $\Psi$. $S_{\text{class}}$ should be a topological invariant. So far, we have taken $S_{\text{class}} = 0$. In the case when $\mathcal{M}$ and $\mathcal{N}$ are hyperKähler, a good classical action can be $S_{\text{class}} = -S_T$, so that the BRST-quantized action $S = -S_T + s\Psi$ is the usual $\sigma$-model action plus the ghost action. However, this cannot always be achieved, since in general $S_T$ is not a topological invariant (e.g. when either $\mathcal{M}$ or $\mathcal{N}$ are quaternionic).
The equation of the deformations $\delta q^i$ of the hyperinstanton condition (7) is

$$D_\mu \delta q^i - (j_u)_\mu'^\nu \partial_\nu q^i (J_u)_j^i - (j_u)_\mu'^\nu \partial_\nu q^i D_k (J_u)_j^i \delta q^k$$

$$- \Gamma^i_{jk} (\partial_\mu q^j - (j_u)_\mu'^\nu \partial_\nu q^i (J_u)_l^i) \delta q^k = 0,$$  \hspace{1cm} (23)

Consequently, a deformation $q^i + \delta q^i$ of a solution $q^i$ to eq. (7) still satisfies the triholomorphicity condition if and only if

$$D_\mu \delta q^i \equiv D_\mu q^i - (j_u)_\mu'^\nu \partial_\nu q^i (J_u)_j^i - (j_u)_\mu'^\nu \partial_\nu q^i D_k (J_u)_j^i \delta q^k = 0.$$  \hspace{1cm} (24)

This is also the equation of the zero modes of the topological ghosts $\xi^i$, namely their field equation, linearized in the Fermi fields and calculated in the instantonic background. The ghost number anomaly $\Delta g$ is the index of the operator $D_\mu$ and is also called the formal dimension of the moduli-space.

Let us discuss the observables of the theory. Any closed and nonexact form $\Omega$ on the target manifold generates (after pull-back to the world-manifold) descent equations and nontrivial observables, via the BRST-extension of the identity $d \Omega = 0$. Let $\Omega^{\alpha_p}_p$ be a representative of a $p$-cocycle on the target manifold $\mathcal{N}$, $0 \leq p \leq 4n = \dim \mathcal{N}$, $1 \leq \alpha_p \leq b^p(\mathcal{N})$, where $b^p(\mathcal{N})$ denote the Betti numbers of $\mathcal{N}$. Let $\hat{\Omega}^{\alpha_p}_p$ be the BRST-extension of $\Omega^{\alpha_p}_p$, namely

$$\hat{\Omega}^{\alpha_p}_p = \Omega^{\alpha_p}_p + \Omega^{\alpha_p}_{(p-1,1)} + \ldots + \Omega^{\alpha_p}_{(0,p)},$$  \hspace{1cm} (25)

where $\Omega^{\alpha_p}_{(p,0)} = \Omega^{\alpha_p}_p$ and the terms $\Omega^{\alpha_p}_{(p-k,k)}$ are obtained by substituting $k$ differentials $dq^i$ with the topological ghosts $\xi^i = sq^i$. Let $\hat{d}$ be the sum of the exterior derivative operator $d$ plus the BRST operator $s$, $\hat{d} = d + s$. The descent equations $\hat{d}\hat{\Omega}^{\alpha_p}_p = 0$ read

$$d\Omega^{\alpha_p}_{(p-k,k)} + s\Omega^{\alpha_p}_{(p-k+1,k-1)} = 0,$$  \hspace{1cm} (26)

for $k = 0, \ldots p + 1$ and with the convention $\Omega^{\alpha_p}_{(p+1,-1)} = \Omega^{\alpha_p}_{(-1,p+1)} = 0$. Thus we have the following observables

$$\mathcal{O}^{\alpha_p,\beta^k}_{p,k} \equiv \int_{\gamma^k_b} q^*\hat{\Omega}^{\alpha_p}_{(k,p-k)},$$  \hspace{1cm} (27)

where $\gamma^k_b$ is a basis of $k$-cycles on the world-manifold $\mathcal{M}$, $\beta^k = 1, \ldots b_k(\mathcal{M})$. Notice that the property $q^* \circ d = d \circ q^*$ is BRST extended to $q^* \circ \hat{d} = \hat{d} \circ q^*$ and so we also have $q^*s = sq^*$.

The physical amplitudes of the theory are average values of products of observables

$$< \mathcal{O}_1 \cdots \mathcal{O}_l >.$$

If $g_i$, $i = 1, \ldots l$ is the ghost number of $\mathcal{O}_i$, the condition for the amplitude to be possibly nonvanishing is that the sum $\sum_{i=1}^l g_i$ must be equal to the ghost number anomaly $\Delta g$. The physical amplitudes are topological invariants of the moduli-space. They generalize the Donaldson polynomials that one finds in pure topological Yang-Mills theory [14, 22].
A topological theory can be deformed by adding extra terms to the action. We call deformation any BRST-invariant term appearing in the quantum action, in addition to the BRST-exact term $s\Psi$. It must be a BRST-invariant integral of a world-four-form over $\mathcal{M}$. Let $\omega^\beta_k$ denote a basis of world-cocycles of degree $k = 1, \ldots, 4$, $\beta_k = 1, \ldots, b^k(\mathcal{M})$. Then the most general deformation is a linear combination of

$$D_{p,k}^{\alpha p, \beta_k} \equiv \int_\mathcal{M} q^* \Omega_{(1-k,p-4+k)}^{\alpha p} \wedge \omega^\beta_k.$$  \hspace{1cm} (29)

So, the partition function

$$Z = \int d\mu \exp \{-s\Psi\}$$  \hspace{1cm} (30)

can be deformed into

$$Z = \int d\mu \exp \left\{-s\Psi + \sum_{k=0}^{4} \sum_{p=4-k}^{4n} b^k(\mathcal{M}) b^p(\mathcal{N}) \sum_{\beta_k=1}^{b^k(\mathcal{M})} \sum_{\alpha_p=1}^{b^p(\mathcal{N})} s_{\alpha_p, \beta_k}^{p,k} D_{p,k}^{\alpha p, \beta_k} \right\},$$  \hspace{1cm} (31)

where $s_{\alpha_p, \beta_k}^{p,k}$ are the parameters of the deformations. Similarly, the deformed amplitudes are

$$< O_1 \cdots O_k \exp \left\{ \sum_{k=0}^{4} \sum_{p=4-k}^{4n} b^k(\mathcal{M}) b^p(\mathcal{N}) \sum_{\beta_k=1}^{b^k(\mathcal{M})} \sum_{\alpha_p=1}^{b^p(\mathcal{N})} s_{\alpha_p, \beta_k}^{p,k} D_{p,k}^{\alpha p, \beta_k} \right\} >.$$  \hspace{1cm} (32)

If $\mathcal{M}$ is compact, the Poincaré duality theorem says that there exists a world-$(4-k)$-cycle, say $\gamma_{4-k}^\beta_k$, such that

$$D_{p,k}^{\alpha p, \beta_k} = \int_\gamma_{4-k}^\beta_k q^* \Omega_{(4-k,p-4+k)}^{\alpha p} = O_{p,4-k}^{\alpha p, \beta_k}.$$  \hspace{1cm} (33)

The total number $d$ of deformations $D_{p,k}^{\alpha p, \beta_k}$ (which is the same as the total number of observables, when $\mathcal{M}$ is compact) is

$$d = \sum_{q=0}^{4} b^q(\mathcal{M}) \sum_{p=q}^{4n} b^p(\mathcal{N}).$$  \hspace{1cm} (34)

In general, the total number of observables $O_{p,k}^{\alpha p, \beta_k}$ is given by a similar formula, where $b^q(\mathcal{M})$ is replaced by $b_q(\mathcal{M})$.

### 3 Relation with N=2 theories through the topological twist

In this section we discuss the topological field theories that are originated by twisting the N=2 supersymmetric $\sigma$-models (both the case of external and dynamical gravity). We show that in the case of external gravity the twisted topological $\sigma$-model corresponds to the general $\sigma$-model that was formulated in the previous section, provided $\mathcal{M}$ and $\mathcal{N}$...
are both hyperKähler. When either \(\mathcal{M}\) or \(\mathcal{N}\) are quaternionic and gravity is external, we have not an N=2 theory generating the corresponding topological \(\sigma\)-model by twist. Indeed, a quaternionic \(\mathcal{N}\) requires coupling to supergravity in order to exhibit N=2 supersymmetry. The equations that one obtains by twisting N=2 supergravity coupled to hypermultiplets are more general than (7), since they are also required to gauge-fix the world-metric. For a discussion of these equations, see section 7.

Let \(V^a = V^a_\mu dx^\mu\) be a vierbein for the world-manifold \(\mathcal{M}\) (\(g_{\mu\nu} = V^a_\mu V_{\nu a}\)) and \(E^{ak} = E^{ak}_i dq^i\) be a vielbein for the target-manifold \(\mathcal{N}\) (\(h_{ij} = 2E^{ak}_i E^{jk}_i\) in our notation). This way of writing the target vielbein \((k = 1, \ldots, n\) if \(\text{dim} \mathcal{N} = 4n\)) in which the index \(a\) is identified with the indices of the Lorentz group of the world-manifold comes naturally from the twist [17], as we show in detail in appendix B, and can always be done, at least locally.

Let us start by analysing the theory that comes from the N=2 globally supersymmetric \(\sigma\)-model (hypermultiplets). In this case \(\mathcal{N}\) is hyperKähler and \(\mathcal{M}\) should be flat; however, we use a covariantized notation, since we expect the topological theory to be defined on more general world-manifolds. In fact, as anticipated, the topological theory is well-defined for any hyperKähler world-manifold \(\mathcal{M}\). We choose a Lorentz frame for \(\mathcal{M}\) such that \(\omega^{-ab} = 0\). The hyperinstanton equations [17] read

\[
V^{\mu[a} E^{b]k}_i \partial_\mu q^i = 0, \quad V^{\mu}_a E^{ak}_i \partial_\mu q^i = 0.
\]

\([ab]^+\) means antisymmetrization and selfdualization in the indices \(a, b\).

The proof that hyperinstantons minimize the action consists in showing that the total action is a sum of squares of the left hand sides of equations (35) plus a total derivative [17], namely

\[
\int_\mathcal{M} d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu q^r \partial_\nu q^s h_{ij} = 8\lambda \int_\mathcal{M} d^4x \sqrt{-g}([V^{\mu}_a E^{ak}_i \partial_\mu q^i]^2 + 4(V^{\mu[a} E^{b]k}_i \partial_\mu q^i)^2]
+ 16i\lambda \int_\mathcal{M} E^{[ak} \wedge E^{b]k} \wedge V_a \wedge V_b.
\]

\(E^{[ak} \wedge E^{b]k} \wedge V_a \wedge V_b\) is proportional to the contraction of the Pauli matrices \(\sigma_a\) with the Kähler forms \(\Omega_u\) of the target manifold \(\mathcal{N}\) (see formula (122) of appendix A). Consequently, \(E^{[ak} \wedge E^{b]k} \wedge V_a \wedge V_b\) is a linear combination of \(\Omega^u \wedge V[a \wedge V_b]^{-}\). \(\Omega_u\) are closed forms if \(\mathcal{N}\) is hyperKähler. Using this fact one can easily show that the last term in (36) is a topological invariant, provided \(\omega^{-ab} = 0\). This is the reason why \(\mathcal{M}\) cannot be any four dimensional manifold, but it is required to be hyperKähler. We conclude that in case

\[^3\]We retain the Minkowskian notation, namely the notation that comes naturally from the starting N=2 theory. In the formulæ that come directly from the topological twist, the Wick rotation to the Euclidean region will be understood. The parameter \(\lambda\) also comes from the formulation of N=2 supergravity coupled to hypermultiplets, as elaborated in ref. [26]; see also the appendices. We retain it to facilitate the comparison between the various formulæ.

\[^4\]The fact that one has to be cautious when covariantizing the theory obtained by twisting the N=2 supersymmetric \(\sigma\)-model has been recently confirmed by the results of ref. [30]. We recall, on the other hand, that no similar problem arises in the case of topological Yang-Mills theory [14].
of external gravity the action of the topological theory is the sum of the classical action
\[
S_T = 16i\lambda \int_M E^{[ak} \wedge E^{b]} \wedge V_a \wedge V_b,
\]
(37)
plus the squares of the gauge-fixings plus the ghost terms, i.e.
\[
S = S_T + s\Psi.
\]
(38)
We can thus distinguish the following two cases.

i) \(\mathcal{M}\) is not hyperKähler. Then \(S_T\) is not a topological invariant and we are not
guaranteed that the solutions to (35) solve the field equations, so we cannot consider
(35) as good instantonic equations.

ii) \(\mathcal{M}\) is hyperKähler. Then \(S_T\) is a topological invariant and (35) are good instantonic
equations. The twisted theory is a well-defined topological \(\sigma\)-model. We shall prove in
a moment that \(S_T\) corresponds to (22), which is clearly independent of the \(\mathcal{M}\) metric.
In any case, one can always get rid of \(S_T\) by deciding that the quantum action is not
\(S_T + s\Psi\), but simply \(s\Psi\), as in the previous section. The difference, being a topological
invariant, is immaterial from the point of view of the \(N=2\) theory. This change of action
can be also viewed as a deformation of the topological theory of the kind \(D_{a2,b2}^{a2}\).

We now show that eq.s (35) are equivalent to eq.s (7). Let us introduce three matrices
\(I_{ab}^i\) that are antiselfdual in \(ab\) and satisfy the quaternionic algebra
\[
I_u I_v = -\delta_{uv} + \epsilon_{uvw} I_w.
\]
(39)
For future use, we fix an explicit form for these matrices, for example
\[
I_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad I_2 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}, \quad I_3 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.
\]
(40)
Let
\[
A^a_{\mu a} \equiv V^{\mu a} E^{bk} \partial_{\mu} q^i.
\]
(41)
Equations (35) can be written as
\[
A^a_{\mu a} = 0, \quad A^a_{[ab]} = 0.
\]
(42)
These are \(4n\) equations and can be grouped together into
\[
A_k - I_u A_k I_u = 0,
\]
(43)
\(^5\)In ref. [30] an analysis of the breaking of topological symmetry due to the non-BRST-exactness of
\(T_{\mu\nu}\) can be found.
which are indeed $4n$ independent equations, because of a duality condition similar to the one in eq. (33). Then we can write

$$(A_k - I_u A_k I_u)^{ab} = V^{\mu} E_i^{bk} (\partial_\mu q^i - (j_u)_\mu^\nu \partial_\nu q^i (J_u)_j^i),$$

(44)

where $(j_u)_\mu^\nu = I_u^{ab} V^{\mu} V^{\nu}$ and $(J_u)_j^i = (I_u)_a^b E^{ai} E^{bj}$ are the three almost quaternionic $(1,1)$-tensors of $\mathcal{M}$ and $\mathcal{N}$, respectively, (compare with formula (122) for the proof).

Thus we have shown that the hyperinstanton equations (35) are equivalent to the triholomorphicity condition (7). At this point, it is simple to check that (37) corresponds to (22).

In the case when gravity is dynamical, there is a further equation that adds to (35) and defines the gravitational instantons of the theory, namely the condition on the world metric, that turns out to be [17]

$$\omega^{-ab} + \frac{1}{2} I_u^{ab} q^* \omega^u = 0.$$

(45)

Here $\omega^u$, $u = 1, 2, 3$ denote the $Sp(1)$ connection of the quaternionic target manifold $\mathcal{N}$, $\Omega^u$ being the corresponding field strength (see appendix A). Equation (45) implies

$$R^{-ab} = -\frac{1}{2} I_u^{ab} q^* \Omega^u,$$

(46)

which is the generalization of the self-duality condition on the Riemann tensor. If the target manifold is four dimensional, we can also write $\omega^{-ab} = \tilde{\omega}^{-ab}$ or $R^{-ab} = \tilde{R}^{-ab}$, where the tilded forms are the pull-backs of the corresponding target forms. So, in the case of four dimensional target manifold, the first equation of (35) simply states that the anti-self-dual part of the world-manifold spin connection is equal to the pull-back of the anti-self-dual part of the target spin-connection. The analogous statement on the anti-self-dual part of the Riemann tensor will be useful in section 7. Equations (35) can still be rewritten in the form (9), as before.

The total kinetic lagrangian (Einstein lagrangian plus $\sigma$-model kinetic lagrangian) can be written as a sum of squares of the left hand sides of equations (35) and (45) plus a total derivative, namely

$$\mathcal{L}_{kin} = \varepsilon_{abcd} R^{ab} \wedge V^c \wedge V^d - \frac{1}{6} \lambda \varepsilon_{abcd} V^a \wedge V^b \wedge V^c \wedge V^d g^{\mu \nu} h_{ij} \partial_\mu q^i \partial_\nu q^j =$$

$$= 4i \left( \omega^{-ab} + \frac{1}{2} I_u^{ab} q^* \omega^u \right) \wedge \left( \omega_{-ac} + \frac{1}{2} (I_u)_v^{ac} q^* \omega^v \right) \wedge V_b \wedge V^c +$$

$$- \frac{\lambda}{3} \varepsilon_{CDEF} V^c \wedge V^d \wedge V^e \wedge V^f [4 (V^{\mu[a} E^{b]^k}_i \partial_\mu q^i)^2 + (V^\mu E^{ak}_i \partial_\mu q^i)]^2 +$$

$$+ \int_M d\varepsilon_{abcd} \omega^{ab} \wedge V^c \wedge V^d + 2i V^a \wedge dV^a - 2i I_u^{ab} q^* \omega^a \wedge V_a \wedge V_b].$$

(47)

The last term represents the topological action, which is also expressed by

$$S_T = -4i \int_M d \left( \left( \omega^{-ab} + \frac{1}{2} I_u^{ab} q^* \omega^u \right) \wedge V_a \wedge V_b \right).$$

(48)
We see that this expression is zero for any hyperinstanton, due to (45). The coupled action is indeed zero on any solution to the field equations. In this sense, gravitational instantons are not privileged with respect to any solution to the field equations (differently from what happens for Yang-Mills instantons). Only in the topological version of the theory they are privileged, because of the topological gauge-fixing.

The case of external gravity can be considered as a particular case of quaternionic $\mathcal{N}$. To perform the limit from hyperKähler to quaternionic manifold, one substitutes $\Omega_u$ with $\lambda \Omega_u$ everywhere, simplifies the $\lambda$’s in all the formulæ in which it is possible and then lets $\lambda$ go to zero. $\omega^u$ are set to zero ($\Omega^u$ become closed) and so equation (45) reduces to $\omega^{-ab} = 0$ for the world-manifold. The almost quaternionic $(1, 1)$-tensors become (globally defined) covariantly constant complex structures.

Let us discuss the observables of the topological theories under consideration. When gravity is dynamical, in addition to the observables of the topological $\sigma$-model that we have exhibited at the end of the previous section, there are observables due to topological gravity. They are generated as usual by the identities $\hat{d} \text{tr}[\hat{R} \wedge \hat{R}] = 0$ and $\hat{d} \text{tr}[\hat{R} \wedge \hat{R}] = 0$ (in the case when topological Yang-Mills theory is also coupled [17, 18], there are also observables generated by identities like $\hat{d} \text{tr}[\hat{F} \wedge \hat{F}] = 0$, $\hat{d} \text{tr}[\hat{F} \wedge \hat{F} \wedge \hat{F} \wedge \hat{F}] = 0$, and so on (the total number of possibilities being the rank of the gauge group), where $F$ is the field strength of the matter vectors and the trace refers to gauge indices).

The condition for the average value $\langle O_1 \cdots O_k \rangle$ of observables $O_i$ with ghost numbers $g_i$ to be non-vanishing is $\sum_{i=1}^k g_i = \Delta g$, where $\Delta g$ is the ghost number anomaly. In the theories coming from the topological twist, $\Delta g$ can also be viewed as the R-duality anomaly of the starting $N=2$ theory, since it is R-duality that defines the ghost number of the twisted theory [17]. So far, a complete analysis of this anomaly has not appeared in the literature.

Notice that the physical amplitudes of the theory of topological gravity (coupled or not to topological Yang-Mills and topological $\sigma$-models) that one gets by twisting $N=2$ supergravity [15, 16, 17, 18], correspond to well-defined (because topological) amplitudes of a nonrenormalizable theory ($N=2$ supergravity). So, even if quantum gravity does not still exist, there is a subset of physical amplitudes that are well-defined and calculable. This interesting property is not common either to the theories of topological gravity that are formulated “by hand”, i.e. without any topological twist [10], or to any other topological field theory obtained by twist [13] (in general they are twisted versions of renormalizable field theories). That is why we think that our theory of topological gravity deserves particular investigation.

Recalling (31), we have that in the most general case, i.e. topological gravity coupled to topological Yang-Mills theory and to topological sigma model, the partition function can be deformed by adding $D_{\mu \nu \rho \kappa}^{\alpha \beta \gamma \delta}$ (29) or the other deformations that can be built in a similar way from the descent equations generated by the identities $\hat{d} \text{tr}[\hat{R} \wedge \hat{R}] = 0$, $\hat{d} \text{tr}[\hat{R} \wedge \hat{R}] = 0$, $\hat{d} \text{tr}[\hat{F} \wedge \hat{F}] = 0$, $\hat{d} \text{tr}[\hat{F} \wedge \hat{F} \wedge \hat{F} \wedge \hat{F}] = 0$ and so on.
We conclude this section with some remarks about equations (35) and (45). The hyperinstanton equations (35) are invariant under diffeomorphisms of the world-manifold $\mathcal{M}$. However, they break local Lorentz invariance. Moreover, equations (45) are a generalization of the equations of gravitational instantons $\omega^{-ab} = 0$, which also break local Lorentz invariance. The deep reason of the breakdown becomes clearer when considered from the point of view of the topological twist, as described in appendix B and refs. [15, 17, 18]. When gravity is dynamical, the redefinition of the Lorentz group according to the rules of the twist, mixes up the original local Lorentz symmetry with some global $SU(2)_Q \otimes SU(2)_I$. This explains the loss of local invariance under the new Lorentz group. Moreover, there is an arbitrariness which is intrinsic to the definition of the twist, consisting in the choice of the reference frame for $SU(2)_Q \otimes SU(2)_I$ before the twisting identifications. This freedom will be called the relative Lorentz gauge of $\mathcal{N}$ with respect to $\mathcal{M}$.

The hyperinstanton problems coming from the twist can thus be stated as follows.

i) External gravity: given a four dimensional hyperKähler manifold $\mathcal{M}$, with a metric and a Lorentz frame such that $\omega^{-ab} = 0$, and a hyperKähler manifold $\mathcal{N}$, one has to find a map $q : \mathcal{M} \rightarrow \mathcal{N}$ such that there exists a relative Lorentz gauge of $\mathcal{M}$ with respect to $\mathcal{N}$ such that (35) are satisfied;

ii) dynamical gravity: given a four dimensional topological space $\mathcal{M}$ and a quaternionic manifold $\mathcal{N}$, one has to find a metric and a Lorentz frame for $\mathcal{M}$ and a map $q : \mathcal{M} \rightarrow \mathcal{N}$ such that there exist a relative Lorentz gauge of $\mathcal{M}$ with respect to $\mathcal{N}$ such that equations (35) and (45) are satisfied.

However, it can happen that different solutions to the hyperinstanton problem do not contribute to the same topological field theory. It is when two different solutions require different relative Lorentz gauges. Indeed, the relative Lorentz gauge is a fundamental point of the topological theory. It must be fixed once for all when defining the theory, because it enters in the expression of the topological gauge-fixing. Changing the relative Lorentz gauge is equivalent to changing the topological theory. We do not know, so far, whether this is a meaningful change of theory or not. In all the examples discussed in this paper it turns out that this is not an essential change. The properties of the relative Lorentz gauge are well illustrated in the examples of section 6.

In the topological $\sigma$-model that we formulated in section 4, we never introduced Lorentz indices. The concept of relative Lorentz gauge is substituted by the ambiguity $\Lambda^{uv}$ of equations (7)-(8), since there is no preferred way to contract the index $u$ of the world complex structures with the same index of the target ones. Eq. (7) is the simplest choice, but in the following it will be convenient to deal with a more general choice of the kind (8). The relative Lorentz gauge is thus the physical interpretation of the intrinsic ambiguity of the triholomorphicity conditions.

Since it is possible to formulate the topological $\sigma$-model with the method of section 2 at least in the case of external gravity (so far, we do not possess a twist-independent formulation of topological $\sigma$-models coupled to topological gravity), it is evident that the breakdown of local Lorentz invariance is not a meaningful breakdown. Moreover, it is
also clear that the splitting of the Lorentz index of the target vielbein $E^a_i$ into a couple of indices, one of which is identified with the Lorentz indices of the world-manifold, can be simply considered as a convenient intermediate step, but does not put restrictions on the target manifold.

4 Triholomorphic functions

In this section we study the case in which both the world-manifold and the target space are $\mathbb{R}^4 \approx \mathbb{H}$. Moreover, at the end of the section we prove very simple general theorems about the solutions to (35) that generalize analogous theorems about holomorphic functions.

The elements $x \in \mathbb{H}$ are called quaternionic numbers. Reverting to Euclidean signature, equations (53) reduce to

$$\partial_\mu q_\mu = 0, \quad \partial_\mu q_\nu - \partial_\nu q_\mu + \varepsilon_{\mu\nu\rho\sigma} \partial_\rho q_\sigma = 0. \quad (49)$$

At first sight, one is tempted to interpret these equations as the anti-self-duality condition on the field strength of some four-potential $q_\mu$ in the Lorentz gauge. However, $q_\mu$ has nothing to do with a four-potential and the first of Eq. (49) is not a gauge choice, but a true equation. In fact, the correct interpretation of Eq.s (49) is quite different, since these equations are equivalent to the Cauchy-Fueter equations that define holomorphicity in the quaternionic sense and that generalize the Cauchy-Riemann equations. Let $I$, $J$ and $K$ be a representation of the quaternionic algebra,

$$I^2 = J^2 = K^2 = -1, \quad IJ = -JI = K \quad \& \text{cyclic permutations.} \quad (50)$$

Let

$$q = I q_1 + J q_2 + K q_3 + q_4, \quad \bar{\partial} = -I \partial_1 - J \partial_2 - K \partial_3 + \partial_4, \quad (51)$$

where $\partial_\mu = \partial / \partial x^\mu$. Then equations (49) can be written in the form

$$\bar{\partial} q = 0 \quad (52)$$

which are indeed the Cauchy-Fueter equations [27, 28].

A common representation of the complex structures (108) is given by

$$I = -i \sigma_1 \quad J = -i \sigma_2 \quad K = -i \sigma_3 \quad (53)$$

where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. If $e^\mu = (I, J, K, 1)$ and $\bar{e}^\mu = (-I, -J, -K, 1)$, then we have $q = q_\mu e^\mu$ and $\bar{\partial} = \bar{e}^\mu \partial_\mu$.

The Cauchy-Fueter equations have not a unique form, due to the ambiguity noticed in eq. (3). For example, an alternative choice is

$$(-i \partial_1 + j \partial_2 + k \partial_3 + \partial_4) q = 0. \quad (54)$$
It is clear that the identity \( q = x = Ix_1 + Jx_2 + Kx_3 + x_4 \) is a solution of the new equations (54), even if it is not a solution to the old ones (52). It is also clear that essentially no new solution is created by this trick. The set of solutions to (49) and (54) are into one-to-one correspondence. The non-uniqueness of the form of the Cauchy-Fueter equations, namely the lack of a canonical form among the possible ones such as (52), (54) and similar, has a very simple interpretation in our description, as we remarked at the end of the previous section, namely it is the relative Lorentz gauge of the world-manifold with respect to the target one.

The operator \( \bar{\partial} \) can be thought as the Weyl operator in the Euclidean signature. Indeed, the chiral Euclidean representation of the Dirac matrices is

\[
\gamma_i = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

with \( i = 1, 2, 3 \), so that the Dirac operator is

\[
\bar{\partial} = \begin{pmatrix} 0 & \bar{\partial} \\ \partial & 0 \end{pmatrix},
\]

where \( \partial = e^\mu \partial_\mu \). The Weyl equation for right handed Weyl spinors is simply \( \bar{\partial}\psi = 0 \). The Weyl spinor \( \psi \) is a doublet of complex numbers and can be interpreted as a quaternionic number \( q \) via the isomorphism \( H \approx C \oplus JC \approx (R \oplus iR) \oplus J(R \oplus iR) \). In this sense the Cauchy-Fueter equations \( \bar{\partial}q = 0 \) correspond to the Weyl equation \( \bar{\partial}\psi = 0 \).

This remark extends to the case in which the target manifold \( \mathcal{N} \) is \( H \) and the world-manifold \( \mathcal{M} \) is a four dimensional generic hyperKähler manifold (we choose it to satisfy \( \omega^{-ab} = 0 \)). Then the hyperinstanton equations are still equivalent to the Dirac equation \( \mathcal{P}\psi = 0 \), on a right handed Weyl spinor \( \psi = [0, 0, \psi_1, \psi_2], \psi_1, \psi_2 \in C \) that parametrizes \( C \oplus JC \approx H \). Noticing that on such a spinor the covariant derivative is the same as the simple exterior derivative (since \( \omega^{-ab} = 0 \)) and defining

\[
q_1 = -\text{Im } \psi_2, \quad q_2 = \text{Re } \psi_2, \\
q_3 = -\text{Im } \psi_1, \quad q_4 = \text{Re } \psi_1,
\]

we see that \( \mathcal{P}\psi = 0 \) becomes

\[
V^\mu_a \partial_\mu q^a = 0, \quad V^{[a}_\mu \partial_\mu q^{b]} = 0,
\]

as expected. On the other hand, the correspondence between the Dirac equation for right handed Weyl spinors and the hyperinstanton equations is quite natural in the case \( \mathcal{N} \) is flat. Indeed, since the hyperinstanton equations are linear in \( q^i \), they have exactly the same form as the equations of their deformations \( \delta q^i \). Consequently, they also have the same form as the equation of the zero modes \( \zeta^a \) of the topological ghosts. On the other hand, in the twisting procedure, the topological ghosts originate from fermions \( \zeta^t \), and the equation of their zero modes comes from the Dirac equations \( \mathcal{P}\zeta^t = 0 \). Hence, it is no wonder that the Cauchy-Fueter equations correspond to the Dirac equation in the
case $\mathcal{N}$ is flat. To be precise, the Dirac equations $\mathcal{D}\zeta^I = 0$ can be put into the form $\mathcal{D}\psi = 0$, if the four real components of the right handed Weyl spinor $\psi$ are regarded as the four right handed components of two Majorana spinors $\zeta^I$, $I = 1, 2$, via the formula $(\zeta^a)^I = (\bar{\sigma}^a)_I^{\dot{a}}q^a$, $q^a$ being related to $\psi_1$ and $\psi_2$ by (57). We then see that $\mathcal{D}\psi = 0$ is equivalent to $\mathcal{D}\zeta^I = 0$ and thus to the hyperinstanton equations, as claimed. This interpretation of the hyperinstanton equations, however, cannot be extended to the case in which $\mathcal{N}$ is not flat. We thus conclude that the correct interpretation is given by the Cauchy-Fueter equations, since they can be fully generalized to the triholomorphicity condition that we defined in the introduction.

There are three equivalent ways of defining holomorphic functions. One requirement is differentiability in the complex sense, another is analyticity in the complex sense and the third definition is provided by the Cauchy-Riemann equations. The corresponding definitions of triholomorphic functions are not equivalent. As a matter of fact, we know, from the mathematical literature on the subject [27], that the Cauchy-Fueter equations are the best definition of triholomorphic maps. The other possible ways of generalizing the definition of holomorphic maps, are either too restrictive or too general. The requirement of differentiability in the quaternionic sense is too restrictive, because only the linear functions satisfy this condition [27]. The other possibility, that is analyticity in the quaternionic sense, is too general, because it turns out that all real analytic functions in four variables are analytic in the quaternionic sense. The simple argument that proves it is that, if $x = Ix_1 + Jx_2 + Kx_3 + x_4$, then one has

$$
\begin{align*}
x_1 &= \frac{1}{4}(-Ix - xI - KxJ + JxK), \\
x_2 &= \frac{1}{4}(-Jx + KxI - xJ - IxK), \\
x_3 &= \frac{1}{4}(-Kx - JxI + IxJ - xK), \\
x_4 &= \frac{1}{4}(x - IxI - JxJ - KxK).
\end{align*}
$$

(59)

It is clear that any power series in $x_1, x_2, x_3, x_4$, is a power series in $x \in \mathbb{H}$. We thus accept the Cauchy-Fueter equations as the best definition of triholomorphic functions. Triholomorphic functions will be, in general, solutions to the Cauchy-Fueter equations, while hyperinstantons will be those solutions that are defined globally.

Some properties of holomorphic functions are extended to triholomorphic ones, some others are not. We shall briefly recall the main properties to give the reader an idea of what triholomorphic functions are.

The complex conjugate $\bar{x}$ of $x$ is defined to be $\bar{x} = -Ix_1 - Jx_2 - Kx_3 + x_4$. The norm of $x$ is $|x| = \sqrt{x\bar{x}} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$.

Notice that $q(x) = x$ is not a solution to (52), while $q(x) = Ix$, $q(x) = Jx$ and $q(x) = Kx$ are solutions. As a matter of fact, one can easily show that all the linear solutions are of the form

$$
q(x) = Ix\alpha_1 + Jx\alpha_2 + Kx\alpha_3 + \beta,
$$

(60)

where $\alpha_1, \alpha_2, \alpha_3, \beta \in \mathbb{H}$.

The analogue of the Cauchy formula is the Cauchy-Fueter formula [27]

$$
q(x) = \frac{1}{2\pi^2} \int_{\partial \Omega} \frac{x' - x}{|x' - x|^4} \, dx' q(x'),
$$

(61)

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where $\Omega$ is an open set containing $x$ and such that $q$ is triholomorphic in an open set containing $\Omega$. The three-form $Dx$ is defined as \[Dx = dx_1 \wedge dx_2 \wedge dx_3 + idx_4 \wedge dx_2 \wedge dx_3 + jdx_4 \wedge dx_3 \wedge dx_1 + kdx_4 \wedge dx_1 \wedge dx_2.\]

Any triholomorphic function is harmonic \[\Box, 28\], because $\bar{\partial} \partial = \partial \bar{\partial} = \Delta$, \[(62)\]

$\Delta$ being the Laplacian. Hence the maximum modulus principle holds. The harmonic character of triholomorphic functions is a consequence of the general fact that any solution to the hyperinstanton equations is also a solution to the equations of motion.

A theorem about analyticity holds \[27\]. It states that any triholomorphic function can be expanded in power series in the quaternionic numbers (but not the most general power series) with a well-defined convergence radius. For the details and other properties of the Cauchy-Fueter equations, the reader should look at the mathematical literature.

The above definition and the above properties of triholomorphic functions hold for the flat case $\mathbb{H}$. An interesting problem would be to generalize the Cauchy-Fueter formula and the analyticity theorem to the triholomorphic maps that we defined with equation (7) in the introduction.

We conclude this section by mentioning a couple of straightforward properties of the general equations (1) or (35) that are simple generalizations of the corresponding properties of holomorphic functions. First of all, any constant function is trivially triholomorphic. From the point of view of topological $\sigma$-models, the constant map corresponds to mapping the world-manifold into a single point of the target manifold. The moduli space of these hyperinstantons is clearly the target manifold itself and, if the formal dimension is positive (in which case we expect it to be equal to the real dimension), then the physical amplitudes are the intersection forms of the target manifold. The only nonvanishing observables are the local ones. These features are common to topological $\sigma$-models in two dimensions.

Secondly, there exists no triholomorphic function whose image is one-dimensional, i.e. a curve in the target manifold. This corresponds to the property that any holomorphic function is an open mapping (and so its image cannot be a curve in $\mathbb{C}$). Indeed, let $q = q(t)$ be a curve in the target manifold, such that there exists a map $t(x)$ from an open subset of the world-manifold into it. Then we write $\partial_\mu q^i = t_\mu \dot{q}^i$, where the dot denotes the derivative with respect to $t$ and $t_\mu = \partial_\mu t$. Equations (4) give

\[t_\mu \dot{q}^i - (j_\alpha)_\mu^\nu t_\nu \dot{q}^i (J_\alpha)_j^i = 0.\]

These equations and the hermiticity of the $\mathcal{M}$- and $\mathcal{N}$-metrics $g_{\mu\nu}$ and $h_{ij}$, respectively, imply

\[0 = g^{\mu\nu} h_{ij} [t_\mu \dot{q}^i - (j_\alpha)_\mu^\nu t_\nu \dot{q}^k (J_\alpha)_k^i] [t_\nu \dot{q}^j - (j_\sigma)_\nu^\mu t_\sigma \dot{q}^k (J_\sigma)_l^j] = 4t^2 \dot{q}^2,\]

where $t^2 = g^{\mu\nu} t_\mu t_\nu$ and $\dot{q}^2 = h_{ij} \dot{q}^i \dot{q}^j$. Consequently, either $t_\mu = 0$ or $\dot{q}^i = 0$. In both cases $q$ is the constant map. The property that we have now proven will be confirmed explicitly in the case of the torus, where it will be also shown that solutions mapping the
world-manifold into two-, three- or four-dimensional submanifolds of the target manifold can exist.

5 Hyperinstantons on the torus

In this section we consider the case in which the world-manifold and the target manifold are four-tori. We find all the solutions to the hyperinstanton equations. Moreover, at the end of the section we exhibit some examples of hyperinstantons of $S^4$ and $\mathbb{CP}^2$.

The torus $T_4$ is described by four quaternionic numbers $a_i$, $T_4 = \mathbb{H}/\sim$, where, if $x, x'$ denote points of $\mathbb{H}$, the equivalence relation $\sim$ is defined according to

$$x \sim x' \text{ if } \exists n_1, n_2, n_3, n_4 \in \mathbb{Z} \text{ such that } x - x' = n_i a_i.$$  \hfill (65)

We adopt the following convention. The index of $a_i$ enumerates the four quaternionic numbers that describe the torus. When we want to specify the components of $a_i$ as a vector of $\mathbb{R}^4$, we introduce a second index according to $a_i = a_{ij} e_j$, where $(e_1, e_2, e_3, e_4) = (I, J, K, 1)$. Thus $a_{ij}$ denotes the $j$-th component of the $i$-th vector. The torus $T_4$ is described by the matrix $a = (a_{ij})$. In fact $a$ is the period matrix of the torus. We shall not distinguish the indices $\mu$ of the world-manifold from the indices $i$ of the target. Moreover, all indices will be lower indices.

We look for the solutions $q : T_4 \rightarrow T_4'$ to the equation $\bar{\partial} q = 0$. $T_4'$ will be described by quaternionic numbers $b_i$ instead of $a_i$. We shall write $T_4' = \mathbb{H}/\approx$. For $T_4'$ the analogous of the matrix $a = (a_{ij})$ will be denoted by $b = (b_{ij})$.

We must impose $q(x) \approx q(x')$ if $x \sim x'$. This requirement implies that $q$ is linear in $x$, as we now prove. Consider the partial derivatives $\partial_i q_j$ of $q$. We have $\partial_i q_j (x + n_k a_k) = \partial_i q_j (x)$. The partial derivatives, being periodic, can be expanded in Fourier series, and consequently the functions $q_i (x)$ are the sums of linear functions plus Fourier series

$$q_i = \beta_i + x_j \alpha_{ji} + \sum_{n=(n_j) \in \mathbb{Z}^4 - \{0\}} c_i (n) e^{2\pi i x a^{-1} n}. \hfill (66)$$

The reality condition gives $c_i^* (n) = c_i (-n)$, as usual. The partial derivatives

$$\partial_i q_j = \alpha_{ij} + 2\pi i \sum_{n=(n_j) \in \mathbb{Z}^4} (a^{-1})_{ij} n_l c_j (n) e^{2\pi i x a^{-1} n} \hfill (67)$$

must satisfy equation (66). For the coefficients $c_i (n)$ this means, if we put $f_i^{(n)} = (a^{-1})_{ik} n_k$,

$$f_i^{(n)} c_i = 0, \quad f_i^{(n)} c_j^+ = 0. \hfill (68)$$

These are four equations in four unknowns. The solution is trivial since the determinant of the matrix of coefficients is $f_i^{(n)} f_i^{(m)} = n^i (a^{-1})^i a^{-1} n > 0$ for $n \neq 0$. We conclude that $q$ is linear in $x$. The theorem that we have just proved is the four dimensional counterpart of an analogous theorem holding for holomorphic maps between two-tori: the derivative
of the map is not only holomorphic, but also periodic and this forces it to be constant, because there does not exist a nontrivial bounded holomorphic function (in our case the derivative of the map) defined all over \( \mathbb{C} \). Thus the map is forced to be linear.

The topological field theory under consideration is a free theory, namely the lagrangian has the form

\[
L = \partial_\mu q^i \partial_\mu q^i + \bar{\zeta}_I \partial /\zeta_I.
\]

Consequently, the formal dimension of the moduli space is zero. In the partition function the bosonic and fermionic contributions factorize. Due to the zero modes of the fermions (that are independent of the particular hyperinstanton) the fermion factor integrates to zero. We shall only be interested in the bosonic factor of the partition function (which we shall call “partition function”, for simplicity). This function collects in a simple expression a lot of information about hyperinstantons.

Among all the possible maps \( q : T_4 \rightarrow T'_4 \), that can be grouped according to homotopy classes, the topological gauge-fixing (i.e. the condition of triholomorphicity) picks up special representatives (hyperinstantons) in the homotopy classes. The preferred representatives are linear in our theory. We recall from the previous section that the only linear triholomorphic maps are of the form (60). It remains to study what restrictions are to be imposed on the tori in order to have nontrivial solutions. The condition is

\[
\exists \alpha_1, \alpha_2, \alpha_3 \in H \text{ such that } \forall n = (n_i) \in \mathbb{Z}^4 \exists m(n) = (m_i(n)) \in \mathbb{Z}^4 \text{ such that } In_i a_i \alpha_1 + Jn_i a_i \alpha_2 + Kn_i a_i \alpha_3 = m_i(n)b_i.
\]

A necessary and sufficient condition can be obtained by choosing \( n_i = v_i \), where \( v_i \) are the unit vectors of \( \mathbb{R}^4 \) (and so also of \( \mathbb{Z}^4 \)), namely there must exist a matrix of integers \( m_{ij} \) and three quaternionic numbers \( \alpha_1, \alpha_2, \alpha_3 \), such that

\[
Ia_i \alpha_1 + Ja_i \alpha_2 + Ka_i \alpha_3 = m_{ij}b_j.
\]

(70)

There are twelve unknowns (the components of \( \alpha_1, \alpha_2, \alpha_3 \in H \)) and sixteen equations. It follows that condition (70) is nontrivial. Two tori \( T_4 \) and \( T'_4 \) that satisfy a condition like (70) for nontrivial \( \alpha \)'s and \( m \) will be called \textit{commensurable tori} and condition (70) will be called \textit{commensurability condition}.

In the notation \( q_i = \beta_i + x_j \alpha_{ji} \) the commensurability condition reads

\[
a \alpha = mb.
\]

(71)

Solving \( \alpha = a^{-1}mb \), the hyperinstanton condition (63) reads

\[
\text{tr } a^{-1}mb = 0, \quad (a^{-1}mb)_{[ij]} = 0.
\]

(72)

Given \( a_i, b_i \in H \), the solutions (if any) to the commensurability condition are discrete, due to the presence of the matrix of integers \( m \). We conclude that the only moduli of the hyperinstantons of the torus are the translation modes \( \beta \) of equation (60), i.e. the moduli space is \( T'_4 \). It is clear that the space of solutions \( \alpha \) to the condition \( aab^{-1} \in \mathbb{Z}^{4x4} \) is a lattice, so that any linear integer combination of solutions is a solution. Let \( g \) be its dimension \( (g \leq 12) \) and \( \alpha_n, n = 1, \ldots g \) be a set of generators. \( g \) will be called \textit{hyperinstanton dimension}. We can write \( \alpha = m_n \alpha_n, m_n \in \mathbb{Z} \).
The bosonic action, suitably normalized, is equal to

$$S = \frac{1}{2} \text{tr} [\alpha \alpha^t] = \frac{1}{2} \text{tr} [a^{-1} m b b^t (a^{-1})^t] = \frac{1}{2} m_n m_k \text{tr} [\alpha_n \alpha_k^t] = M^t G M,$$

(73)

where $M$ is the vector $(m_n)$ and $G$ is the matrix $(G_{nk})$ with

$$G_{nk} = \frac{1}{2} \text{tr} [\alpha_n \alpha_k^t].$$

(74)

The partition function (bosonic factor) is then a theta function,

$$Z = \Theta(G) = \sum_{M \in \mathbb{Z}^g} e^{-M^t G M}.$$

(75)

Concluding, the hyperinstanton problem associates to a couple of tori, $T_4$ and $T'_4$, a third torus, $T''_4$, the dimension of which (i.e. the hyperinstanton dimension) takes values between zero and twelve. Given a torus, say $T_4$, described by the matrix $a$, one can associate to it the theta function $\Theta(aa^t/2)$, since the matrix $aa^t$ is positive definite (the factor $1/2$ is a convention). Correspondingly, the theta function $\Theta(bb^t/2)$ is associated to $T'_4$. One easily verifies that the theta function associated to $T''_4$ is the partition function $Z$. Indeed, let us expand $\alpha_n$ in a basis of $g$ matrices $E_k$ which are orthonormal with respect to the canonical scalar product $\langle E_k E_l \rangle = \text{tr} [E_k E_l^t]$. Then, if $\alpha_n = \alpha_{nk} E_k$, the torus $T''_4$ is described by the period matrix $\alpha_{nk}$ and the associated theta function is $\Theta(\alpha \alpha^t/2) = Z$.

Let us consider the simple case $a_i = a e_i$ and $b_i = b e_i$ for certain $a, b$ real and positive. The hyperinstantons are then $q_i = \beta_i + x_j \alpha_{ij}$, where

$$\alpha = \frac{b}{a} \left( \begin{array}{cccc} n_1 & m_1 + p_1 & m_2 + p_2 & m_3 + p_3 \\ -m_1 + p_1 & n_2 & -m_3 + p_4 & m_2 + p_5 \\ -m_2 + p_2 & m_3 + p_4 & n_3 & -m_1 + p_6 \\ -m_3 + p_3 & -m_2 + p_5 & m_1 + p_6 & -n_1 - n_2 - n_3 \end{array} \right),$$

(76)

with $n_j, m_j, p_j \in \mathbb{Z}$. As one can see, there is no solution that maps the world-torus into a one dimensional submanifold of the target torus. On the contrary there are solutions whose image are two-, three- or four-dimensional. For example, in the first case, take $n_1 \neq 0$ and all the rest zero; in the second case, take $n_1, n_2 \neq 0$ and all the rest zero; in the third case take $n_1, n_2, n_3 \neq 0$ and all the rest zero.

The partition function (75) is in this case

$$Z(t) = [\Theta(t)]^9 \Theta \left( \frac{t}{2} A \right),$$

(77)

where $t = b^2/a^2$ and $A$ denotes the $3 \times 3$ matrix

$$A = \left( \begin{array}{ccc} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{array} \right).$$

(78)
As mentioned in the introduction, an interesting problem (the generalization of the problem of counting the number of rational curves of a Kähler manifold) is to count the number of triholomorphic embeddings of $S^4$, $\mathbb{CP}^2$, $T_4$ and K3 into hyperKähler or quaternionic manifolds. The partition function that we have just computed gives the answer in the case of triholomorphic embeddings of a torus $T_4$ into a second torus $T'_4$. If we write

$$Z(t) = \sum_{n \in \mathbb{N}} m(n) e^{-nt}, \quad (79)$$

then $m(n)$ represents the number of triholomorphic embeddings such that the topological invariant $\langle 20 \rangle$ or $\langle 37 \rangle$ (which is not the winding number) takes the value $n$.

Let us exhibit another example. It is a case in which the hyperinstanton dimension in neither twelve nor zero. Let $a = \text{diag} \ (1, 1, v, z)$ with $v$ and $z$ irrational and such that there exists no vanishing integer linear combination of $v, z, vz$. Let $b = \zeta \cdot 1$, with $\zeta \in \mathbb{R}$. Then one can show that the most general matrix $m$ is

$$m = \zeta^{-1} \alpha = \begin{pmatrix} n_1 & n_2 & n_3 & n_4 \\ n_2 & -n_1 & n_4 & -n_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (80)$$

where $n_1, n_2, n_3, n_4 \in \mathbb{Z}$, so that the hyperinstanton dimension is $g = 4$ and the partition function is

$$Z(\zeta^2) = [\Theta(\zeta^2)]^4. \quad (81)$$

Choosing $a = \text{diag} \ (1, u, v, z)$ with $u$, $v$ and $z$ irrational and such that there exists no vanishing integer linear combination of $uvz$, $uv$, $uz$, $vz$, we get a case in which the hyperinstanton dimension is zero. The above examples were chosen so as to justify the name commensurability condition that we gave to condition $\langle 70 \rangle$.

We now study the deformations $D_{p,k}^{\alpha \beta}$ [29] of the topological theory. We shall restrict to the case $D_{4-k,k}^{\alpha \beta}$, that is the case when the deformations contain no ghosts, to preserve the factorization between bosonic and fermionic integrations. Then the deformed (bosonic) partition function is of the general form

$$Z(Q^{i}, Q^{jk}, Q^{lmn}, Q^{pqr}) \equiv \sum_{N=(N_i) \in \mathbb{Z}^g} \exp \{Q_i^1 N_i + Q_2^{jk} N_j N_k + Q_3^{lmn} N_l N_m N_n + Q_4^{pqr} N_p N_q N_r N_s \}. \quad (82)$$

Functions of this form will be called hyper-theta-functions. They have the property

$$\begin{aligned}
Z(Q^{i}, Q^{jk}, Q^{lmn}, Q^{pqr}) &= \\
&\exp \{Q_1^1 M_1 + Q_2^{jk} M_j M_k + Q_3^{lmn} M_l M_m M_n + Q_4^{pqr} M_p M_q M_r M_s \} \\
&\cdot Z(Q_1^1 + 2Q_2^{ij} M_j + 3Q_3^{ijk} M_j M_k + 4Q_4^{ijkl} M_j M_k M_l, \\
&Q_2^{jk} + 3Q_3^{kli} M_l + 6Q_4^{klm} M_l M_m, Q_3^{lmn} + 4Q_4^{lmp} M_p, Q_4^{pqr}), \quad (83)
\end{aligned}$$

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∀M ∈ Z^4. This property expresses the fact that under suitable deformations the partition function changes in a very simple way, namely it is multiplied by a factor. Hyper-theta-functions with Q_3 = Q_4 = 0 are the usual theta-functions. Since the winding number is quartic in the integers N_i, we see that the number of triholomorphic embeddings of T_4 into T_4' with a given winding number is encoded into a hyper-theta-function, precisely as the number of triholomorphic embeddings with a given value of the topological invariant (20) or (37) is encoded into a theta-function like (75).

We conclude this section with some remarks about the case M = N = S^4 and the case M = N = CP^2. Let us begin with M = N = S^4. Using stereographic coordinates, the hyperinstanton equations have the form (49) is both the northern and the southern chart. Let x and q be quaternionic numbers referring to the northern charts of M and N, respectively, and x' = x/|x|^2, q' = q/|q|^2 be the southern coordinates. Differently from the case of the torus, the triholomorphic functions are not subject to periodicity conditions, rather to the (nontrivial) condition that they should satisfy the hyperinstanton equations in both charts. For instance, the map q(x) = (r_1 I + r_2 J + r_3 K)xα, where r_1, r_2, r_3 ∈ R and α ∈ H, is a good solution. r_1, r_2, r_3 and α are moduli. Indeed, when x approaches the southern pole of M, then q also approaches the southern pole of N. Changing charts in both cases, we find q'(x') = (r'_1 I + r'_2 J + r'_3 K)x'α', where r'_j = r_j/(r_1^2 + r_2^2 + r_3^2) and α' = α/|α|^2, which is clearly a solution to the southern equations. These solutions have winding number one. We do not know what is the complete set of solutions q : S^4 → S^4. It would be certainly interesting, to be compared with the two-dimensional analogue of the holomorphic functions f : S^2 → S^2. As a matter of fact, S^2 is also CP^1, so that the best four dimensional counterpart could well be the set of triholomorphic maps q : CP^2 → CP^2. As an example, writing the equations in a suitable relative Lorentz gauge, the identity map CP^2 → CP^2 is easily shown to be a solution. Other interesting possibilities are the embeddings of S^4 or CP^2 into the Grassmannian SO(m + 4)/(SO(4) ⊗ SO(m)).

6 Hyperinstantons on the K3 surface

Now we analyze the case in which both the world-manifold and the target manifold are the K3 surface and search for hyperinstantons. We first discuss the hyperinstanton equations in the form (35). We suppose that the world-manifold satisfies ω^{-ab} = 0, i.e. we choose the suitable Lorentz reference frame in which this happens (this is required when writing the hyperinstanton equations in the form (35), that comes from the topological twist).

The first attempt that one tries is to see if the identity map is a hyperinstanton. In order for this to be true, we have to use a trick, because there is an obstruction. Looking back to the first of equations (49), one is lead to infer that the identity map is not a solution, since the equation V _{a}^{µ}E^i_{a}∂_µq^i = 0 of (35), that generalizes the flat equation ∂_µq_µ = 0 of (13), would not be satisfied. However, a suitable trick can make it a solution.

We choose the target vierbein E^a_i(q) equal, in form, to the world-manifold vierbein

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V^a_\mu(x), \text{ apart from a global Lorentz rotation } \Lambda^{ab} (\text{the relative Lorentz gauge of section } 3), \text{ namely we set}

\[ E^a_i(q) = \Lambda^{a}_{ib} V^b_\mu(q) \delta^\mu_i. \] (84)

Equations (35) then become

\[ V^\mu_a(x) \Lambda^{ab} V^b_\rho(q(x)) \delta^\rho_i \partial_\mu q^i(x) = 0, \]
\[ V^{\mu[a}(x) \Lambda^{b]}^+ c V^c_\rho(q(x)) \delta^\rho_i \partial_\mu q^i(x) = 0. \] (85)

It is then clear that the identity map \( q: K^3 \to K^3, q = x \) is a solution, provided the orthogonal matrix \( \Lambda^{ab} \) is traceless and its antisymmetric part is antiselfdual

\[ \Lambda^a_a = 0, \quad \Lambda^{[ab]}^+] = 0. \] (86)

For example, we can choose \( \Lambda^{ab} = I_1^{ab} \), with \( I_1 \) given in equation (40) (in fact any matrix \( I_u \) of (40) is good for this purpose).

Notice that the identity map is always a solution of the equations of motion when the four dimensional target manifold is the same as the world-manifold. The equation of motion

\[ \partial_\mu (\sqrt{|g(x)|} g^\mu\nu(x) \partial_\nu q^i(x)) + \sqrt{|g(x)|} \Gamma^i_{jk}(q(x)) \partial_\mu q^j(x) \partial_\nu q^k(x) g^{\mu\nu}(x) = 0 \] (87)

is surely satisfied by \( q^i(x) = x^\mu \delta^i_\mu \), if \( h_{ij}(q) = g_{\mu\nu}(q) \delta^\mu_i \delta^\nu_j \), as follows from (84), because it reduces to the identity

\[ \partial_\mu (\sqrt{|g|} g^{\mu\nu}) + \sqrt{|g|} \Gamma^{\nu}_{\rho\sigma} g^{\rho\sigma} = 0. \] (88)

With some more computational effort, one can check that any isometry is a solution to the field equations. The natural question is whether any isometry is also solution to the hyperinstanton equations and what classification between isometries is induced by the hyperinstanton equations that is not induced by the field equations.

An isometry is a map \( q: K^3 \to K^3 \) such that

\[ g_{\rho\sigma}(q(x)) \delta^\rho_i \delta^\sigma_j \partial_\mu q^i(x) \partial_\nu q^j(x) = g_{\mu\nu}(x). \] (89)

This means that there must exist an orthogonal matrix \( A^a_b(x) \) such that

\[ V^a_\rho(q(x)) \delta^\rho_i \partial_\mu q^i = A^a_b(x) V^b_\mu(x). \] (90)

In this case, the hyperinstanton equations reduce to

\[ \Lambda^a_b A^b_a = 0, \quad (\Lambda A)^{[ab]}^+ = 0. \] (91)

It is clear that it is sufficient to choose \( \Lambda^a_b = (I_1)^a_c (A^{-1})^c_b \). This shows that for any given isometry there exists a relative Lorentz gauge such that the hyperinstanton equations are solved. However, we must notice that not all these solutions contribute to the same topological field theory. As we noticed at the end of section 3, the relative Lorentz gauge
must be chosen once for all, because $\Lambda^a_b$ enters in the expression (84) of the vierbein of the target manifold. This is the classification between isometries induced by the hyperinstanton equations.

It is simple to see that, if the target K3 manifold is rescaled with respect to the world-manifold, then the above isometries can be turned into rescalings, thus giving a set of solutions that is in one-to-one correspondence with the above set of isometries.

Before going on with the argument, let us reexamine the above reasoning on equations (7). To make the identity map a solution, we use the ambiguity in the formulation of the equations (related to the concept of relative Lorentz gauge, as already pointed out), to put them into the form (8), namely

$$q^* - \Lambda^{uv} J_u \circ q^* \circ j_v = 0,$$

where $\Lambda^{uv}$ is an $SO(3)$ matrix that parametrizes the ambiguity. We choose $\Lambda^{uv} \equiv \Lambda_0^{uv} = \text{diag}(1, -1, -1)$ and, since $\mathcal{M} = \mathcal{N}$, we can set $J_u = j_u$, so that the identity map satisfies (92), since

$$1 - j_1^2 + j_2^2 + j_3^2 = 0.$$  

(93)

It is clear that in general not all isometries satisfy (92) with the above $\Lambda^{uv}_0$. We can derive how many of them do satisfy (92) with the chosen $\Lambda^{uv}_0$, by making use of the results by Alekseevsky [31].

It is well known that K3 possesses no conformal Killing vector, so the set of isometries of K3 is a discrete one.

In fact, K3 possesses lots of isometries\textsuperscript{6} and there is a simple argument for finding them [31]. For simplicity, we shall limit ourselves to consider K3 as the Fermat surface $F$ in $\mathbb{CP}^3$

$$X_1^4 + X_2^4 + X_3^4 + X_4^4 = 0.$$  

(94)

The embedding of K3 in $\mathbb{CP}^3$ induces a Kähler metric $\kappa$. $\kappa$ is not the Calabi-Yau metric, of course, however the Calabi-Yau metric $g$ is defined as the unique Ricci-flat Kähler metric $g$ whose Kähler form is cohomologous to the Kähler form of the metric $\kappa$. Let $G$ be the set of holomorphic transformations of $\mathbb{CP}^3$ that preserve the surface (94). These transformations leave $\kappa$ invariant. Since $g$ is uniquely determined by $\kappa$, $g$ is invariant under $G$. $G$ is thus a set of isometries of the Calabi-Yau metric $g$. One finds [31] $G = S_4 \cdot (\mathbb{Z}_4)^3$, where $S_4$ represents the permutation group of the four homogeneous $\mathbb{CP}^3$-coordinates $X_1, X_2, X_3$ and $X_4$, while $\mathbb{Z}_4$ represents the possibility of multiplying them by the fourth roots of the identity $1, i, -1, -i$. There are only three $\mathbb{Z}_4$'s and not four, because the overall one is immaterial. One can then consider the set $G' = \tau G$ of antiholomorphic transformations of $\mathbb{CP}^3$ that preserve $F$, where $\tau$ denotes the antiholomorphic involution of $F \subset \mathbb{CP}^3$ defined by complex conjugation of the homogeneous coordinates of $\mathbb{CP}^3$. These transformations are also isometries. In ref. [31] it is also proven that any isometry is of one of the two types $G, \tau G$ that we have described. So, the full group of isometries of $g$ turns out to be $G \cup \tau G$, which contains $2^{10} \cdot 3 = 3072$ elements.

\textsuperscript{6}We are grateful to M. Pontecorvo and D.V. Alekseevsky and to S. Cecotti about this point.
Let $E$ be the space of parallel two-forms on the Fermat surface, $E \approx \mathbb{R}^3$. Define the following scalar product on $E$. Given $\alpha, \beta \in E$, let $\alpha \cdot \beta$ be equal to

$$\alpha \cdot \beta = \int_{K^3} \alpha \wedge \beta.$$  \hfill (95)

Alekseevsky shows that an isometry $q$ induces a rotation on $E$. The set of isometries is a discrete group, so it has a natural representation as a finite subgroup of $SO(3)$ acting on $E$. Since the complex structures contracted with the metric span $E$, an isometry has got the same effect on the set of complex structures as on the space $E$. Let $R_{q}^{uv}$ be the rotation induced by $q$ on $E$. We have

$$q_\ast \circ j_u = R_{q}^{uv} j_v \circ q_\ast.$$  \hfill (96)

Then $q$ satisfies (92), if we choose, for example, $\Lambda = \Lambda_0 \cdot (R_q)^{-1}$.

Let $G_0$ be the set of isometries $q$ that belong to the kernel of the representation on $E$ (i.e. such that $R_q = 1$). $G_0$ is determined by Alekseevsky \([1]\) and can be characterized in the following way. Represent an isometry of $G \cup \tau G$ by a $4 \times 4$ complex matrix acting on the vector $(X_1, X_2, X_3, X_4)$ in $\mathbb{C}^4$. The determinant of this matrix can only take the values $1, i, -1, -i$. $G_0$ is the set of isometries such that this determinant is one. It is a normal subgroup of $G \cup \tau G$ and contains $2^7 \cdot 3 = 384$ elements. Moreover, $G \cup \tau G$ acts on $E \approx \mathbb{R}^3$ as the dyhedral group $D_4$. Precisely,

$$\frac{G \cup \tau G}{G_0} \approx D_4.$$  \hfill (97)

Thus, $R_q \in D_4 \subset SO(3)$. We can restrict the matrices $\Lambda$ of eq. (92) to be also in $D_4 \subset SO(3)$, since, for any given isometry $q$ there exists a $\Lambda \in D_4$ such that $q$ solves (92). In some sense, the group $D_4$ “measures” the ambiguity in the condition of triholomorphicity.

Using (96), it is immediate to see that an isometry $q$ solves eq. (92) if and only if $\text{tr } \Lambda R_q = -1$ and $\Lambda R_q$ is symmetric ($\Lambda_0$ is an example of such symmetric $D_4$-matrices with trace $-1$). By inspection in the eight matrices of $D_4$, one checks that there are five such $R_q$’s. This number is independent of $\Lambda \in D_4$, of course, but the set of “good” $R_q$’s does depend on $\Lambda$. In conclusion, due to (97), the total number of hyperinstantonic isometries $q : K^3 \rightarrow K^3$ is $384 \cdot 5 = 1920$, whatever $\Lambda \in D_4$ we choose, but the set of hyperinstantonic isometries depends on the chosen $\Lambda$.

We have thus shown that the hyperinstanton equations induce an interesting structure in the group of isometries, that does not follow from the field equations. Moreover, we have characterized the solutions by simple properties of the Fermat surface. We thus address the possibility that the condition of triholomorphicity has a purely algebraic formulation that applies to the cases of algebraic varieties.

To conclude this section, we prove that isometries are isolated hyperinstantons in the moduli space, namely that equation (24) admits no nontrivial solution. Equation (24) is simplified by the fact that $\mathcal{D}_k(J_u)_j^i = 0$ for K3. Moreover, it must be adapted to our choice of the relative Lorentz gauge, so we substitute it with

$$\mathcal{D}_\mu \xi^i - \Lambda^{uv} (j_u)_\mu^v \mathcal{D}_v \xi^i (J_v)_j^i = 0.$$  \hfill (98)
Let $\xi^\mu$ be defined by $\xi^i = \xi^\mu \partial_\mu q^i$. Using the identity $D_\mu \xi^i = D_\mu \xi^\nu \partial_\nu q^i$ and eq. (96) with $\Lambda = \bar{\Lambda}(R_q)^{-1}$ where $\bar{\Lambda}$ is any symmetric $D_4$-matrix with trace $-1$, any explicit $q$-dependence disappears from (98) and we get

$$D_\mu \xi^\nu - \bar{\Lambda}^{uv}(j_u)_\mu^\rho D_\rho \xi^\sigma (j_v)_\sigma^\nu = 0.$$  

Using a standard argument, we show that the solutions $\xi^\mu$ of the above equation are Killing vectors. Since K3 admits no Killing vector, there are no solutions. In fact, with some standard manipulations such as integration by parts and the use of the self-duality of the Riemann tensor (we are thinking of K3 with the Calabi-Yau metric) and the covariant constancy of the three complex structures $(j_u)_\mu^\nu$, one proves the following identity

$$0 = \int_{K^3} d^4x \sqrt{g} g^{\mu\alpha} g_{\nu\beta} (D_\mu \xi^\nu - \bar{\Lambda}^{uv}(j_u)_\mu^\rho D_\rho \xi^\sigma (j_v)_\sigma^\nu)(D_\alpha \xi^\beta - \bar{\Lambda}^{st}(j_s)_\alpha^\gamma D_\gamma \xi^\zeta (j_t)_\zeta^\beta)$$

$$= 4 \int_{K^3} d^4x \sqrt{g} g^{\mu\alpha} g_{\nu\beta} D_\mu \xi^\nu D_\alpha \xi^\beta. \quad (100)$$

This shows that $\xi^\nu$ must necessarily satisfy $D_\mu \xi^\nu = 0$, so it vanishes. The formal dimension of the moduli-space turns out to be negative, since one easily checks that the topological antighosts do possess zero-modes (the constants).

We notice that the set of hyperinstantons of the K3 surface that we have exhibited may be not the whole set of hyperinstantons. The complete identification of K3 hyperinstantons remains an open problem. A convenient way towards this classification is possibly offered by the analysis of the algebraic counterpart of the hyperinstanton equations.

All the above arguments can be extended, with the obvious modifications, to the cases when K3 is described by other algebraic surfaces than the Fermat surface.

### 7 Hyperinstantons with dynamical gravity

In this section we discuss the case in which gravity is dynamical and we illustrate the difficulty inherent to the problem of solving the coupled hyperinstanton equations (35) and (45), of which no nontrivial solution has so far been found. In particular, we show that the simplest possibilities, namely the identity map between identical four dimensional quaternionic manifolds, cannot be turned into solutions to the hyperinstanton equations. The same maps are good solutions for the topological $\sigma$-model formulated in section 2.

Let the target manifold be any quaternionic $\mathcal{N}$. The trivial solution is when $q(\mathcal{M})$ is a point in $\mathcal{N}$. This means $q =$constant, so that the hyperinstanton equations (35) are satisfied provided the world-manifold $\mathcal{M}$ is hyperKähler ($\mathcal{M} = T_4$ or $\mathcal{M} = K3$ if it is compact). Clearly, the moduli space of these solutions is the entire target manifold $\mathcal{N}$ times the moduli space of the world-manifold $\mathcal{M}$.

From now on, we concentrate on four dimensional $\mathcal{N}$ (in four dimensions, the only quaternionic manifolds are $S^4$ and $\mathbf{CP}^2$, by definition). The first reasonable guess would
be to conjecture that the solutions that we have found in the case of K3 have a counterpart in the present case. In particular one might think that the identity map and the worldmanifold equal to the target manifold (or equal to it up to a rescaling) is a solution to the hyperinstantonic equations. This is not the case. The trick that has been successful in solving the last two equations of (35) is ruled out by the first one.

Before showing this, let us consider a different problem, namely a generic $\sigma$-model coupled to dynamical gravity such that the target manifold is four dimensional. We can write $M = M(N)$ and $q = q(N)$, to mean that $N$ is given, while $M$ and $q$ must be determined (to be precise, $M$ is given as a topological space and its metric is the unknown). We do not restrict $N$ to be a quaternionic manifold, for now. We wonder what are the eigenmanifolds $M(N) = N$, so that the diffeomorphisms $q : N \to N$ are solutions. The Euclidean lagrangian is

$$\frac{1}{2} \sqrt{g}(-R + \lambda g^{\mu\nu} h_{ij}(q) \partial_\mu q^i \partial_\nu q^j).$$  \hfill (101)

The equations of motion are

$$0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - T_{\mu\nu},$$

$$0 = \partial_\mu (\sqrt{g(x)} g^{\mu\nu}(x) \partial_\nu q^i(x)) + \sqrt{g(x)} \Gamma^i_{jk}(q(x)) \partial_\mu q^j(x) \partial_\nu q^k(x) g^{\mu\nu}(x),$$ \quad (102)

where the energy-momentum tensor $T_{\mu\nu}$ turns out to be

$$T_{\mu\nu} = \lambda h_{ij}(q) \partial_\mu q^i \partial_\nu q^j - \frac{1}{2} \lambda g_{\mu\rho} g^{\rho\sigma} h_{ij}(q) \partial_\mu q^i \partial_\sigma q^j.$$ \hfill (103)

Let us consider the identity map $q : N \to N$. Then the first equation of (102) gives

$$R_{\mu\nu} = \lambda g_{\mu\nu},$$ \quad (104)

so that the target manifold is forced to be Einstein, with cosmological constant equal to $\lambda$. On the other hand, the second equation of (102) is surely satisfied, in force of the identity (88) that we have already used in the previous section. Thus we have proved that any Einstein manifold is an eigenmanifold (if the cosmological constant is equal to $\lambda$). In view of the invariance under diffeomorphisms, we can extend the conclusion to any diffeomorphism $q : N \to N$.

Notice that the cosmological constant of the target manifold is forced to be equal to the parameter $\lambda$ that appears in (101). One can find more general solutions if a cosmological term $\lambda_0 \sqrt{g}$ is added to (101), but we are not interested to this case, since it cannot come from the topological twist of N=2 supergravity coupled to matter.

\footnote{It is sufficient to study the identity map. Now diffeomorphisms are a gauge symmetry and they should be gauge-fixed. Thus, under suitable gauge-fixing, only one diffeomorphism matters. We choose it to be the identity.}
and a twist-independent formulation of the topological $\sigma$-model coupled to topological gravity is still missing.

If $\mathcal{M}$ is chosen to be equal to a rescaling of $\mathcal{N}$, then the identity map $q = x$ can be replaced by $q = \xi x$, $\xi$ being the rescaling factor. As before, one easily checks that $q = \xi x$ is a solution only if $\mathcal{N}$ is Einstein and its cosmological constant of $\mathcal{N}$ is equal to $\lambda$.

Now, the theory of N=2 supergravity coupled to hypermultiplets \[26, 21\] gives another relation between the cosmological constant $\Lambda$ of $\mathcal{N}$ and the factor $\lambda$ appearing in the lagrangian \((101)\), namely $\Lambda = 3\lambda$ (in general, $\Lambda = \lambda(n^2 + 2n)$, if $\dim \mathcal{N} = 4n$), so that the above simple ansatz are not even solutions to the field equations of the theory that comes from the twist, and \textit{a fortiori} they are not hyperinstantons. We now prove this fact on the hyperinstanton equations themselves, to illustrate where the trick that was successful for K3 fails.

Let us consider the identity map $q : \mathcal{N} \to \mathcal{N}$. We already know how to satisfy equations \((35)\). It is sufficient to impose \((84)\) (with a $\Lambda^{ab}$ that can now depend on the point) and to choose a relative Lorentz gauge $\Lambda^{ab}$ such that $\Lambda^{aa} = 0$ and $\Lambda^{[ab]} = 0$. Then, the problem is to satisfy \((45)\). With $q^i = \delta^i_\mu x^\mu$ and $E^a_i(q) = \Lambda^a_b V^b_\mu(q) \delta^\mu_\mu$, this equation becomes

$$\omega^{-ab} = \mathcal{L}_\Lambda \omega^{-ab}, \quad (105)$$

where $\mathcal{L}_\Lambda$ denotes the Lorentz transformation performed by the orthogonal matrix $\Lambda^{ab}$. In other words, there must exist a traceless Lorentz transformation such that its antisymmetric part is antiselfdual and to which the antiselfdual part of the spin connection is insensitive. We can easily prove that this cannot be. Let us suppose that eq. \((105)\) is satisfied. Then we would have

$$R^{-ab} = \mathcal{L}_\Lambda R^{-ab}. \quad (106)$$

Expanding $R^{-ab}$ in a basis of anti-self-dual matrices $I_u^{ab}$ like the ones of \((40)\), $R^{-ab} = I_u^{ab} R_u$, we conclude $[\Lambda, I_u] = 0$ for any $u$ such that $R_u \neq 0$. Since all $R_u$ are different from zero, because the target manifold has $SU(2)$ right-holonomy, $\Lambda$ must be proportional to the identity. This is absurd, since the trace of $\Lambda$ should be zero.

In conclusion, a nontrivial example of hyperinstanton with dynamical gravity is so far still missing, due to the difficulty of the coupled equations. So, the eventual solutions are surely very peculiar ones and could exhibit very interesting properties. A much simpler problem is given by the case of quaternionic world and target manifolds in the external gravity regime, as shown in section 2. The maps that we have considered in this section are indeed hyperinstantons of the topological $\sigma$-model of section 2 (see the end of section 5).

8 Conclusions

There are two ways of formulating a topological field theory: the first is by topologically twisting an N=2 supersymmetric theory and the second is by BRST-quantizing a suitable continuous deformation as a gauge-symmetry and imposing “by hand” a suitable
instantonic condition as a gauge-fixing. The second method is more general, since not all
topological field theories can be obtained by the twist procedure. On the other hand, the
topological twist has the advantage that it automatically yields a gauge-fixed topological
field theory. In this paper, we combined the properties of the two methods to get the
most general formulation of topological \(\sigma\)-models in four dimensions. We first took ad-
venture of the topological twist in order to get a hint of the so far unknown instantonic
equations. Secondly, we searched for their most general mathematical interpretation
and we found that they are a condition of triholomorphicity on the map. Finally, we
went back and formulated the most general topological \(\sigma\)-model by BRST-quantizing the
continuous deformations of the map and imposing the triholomorphicity condition as a
gauge-fixing. So, the spirit of our investigation was not the search for a reformulation of
a mathematical problem and of mathematical results in the language of physics. Rather,
we formulated a new mathematical problem inspired by a physical theory.

Indeed, the identification of topological \(\sigma\)-models in four dimensions with the concept
of triholomorphic maps proposes the study of a quite interesting class of mappings on
which very few results are so far known in the mathematical literature, i.e. the triholo-
morphic embeddings of four dimensional Riemannian manifolds into almost quaternionic
manifolds. From the physical point of view, some subtleties that are not obvious in
the two-dimensional case are also revealed by our analysis of topological field theories
in four dimensions. More generally, the problem of coupling topological \(\sigma\)-models to
topological gravity (a problem whose two-dimensional analogue was solved in terms of
classical integrable hierarchies) is shown by our work to be related to an even less studied
mathematical problem, namely that of hyperinstantons consistently coupled to gravita-
tional instantons \[15\]. The difficulties inherent to the solutions of the coupled differential
equations have been illustrated in our work.

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A HyperKähler and quaternionic manifolds

In this appendix we give definitions and properties of a quaternionic manifold \(Q(m)\). We
shall mention the changes that occur for \(Q(m)\) hyperKähler. In any case, the formulæ for
a hyperKähler manifold can be retrieved by formally substituting the symbol \(\Omega_u\) with
\(\lambda\Omega_u\) everywhere, simplifying the \(\lambda\)'s whenever possible and then letting \(\lambda\) go to zero.
Moreover, \(\omega^\mu\) are set to zero.
\[ Q(m) \] is first of all a 4\(m\)-dimensional Riemannian manifold. We denote its metric by
\[
d s^2 = h_{i j}(q) dq^i \otimes dq^j.
\]
(107)

Moreover, \( Q(m) \) possesses an almost quaternionic structure, namely three locally defined \((1,1)\)-tensors \( J^u \), \( u = 1, 2, 3 \), fulfilling the quaternionic algebra
\[
J^u J^v = -\delta^{uv} + \varepsilon^{uvz} J^z.
\]
(108)

The metric \( h_{i j} \) is Hermitian with respect to all the almost quaternionic \((1,1)\)-tensors. \( J_u \) are indeed globally defined and covariantly constant complex structures if \( Q(m) \) is hyperKähler.

We introduce the three forms
\[
\Omega^u = \lambda h_{i k}(J^u)_j^k dq^i \wedge dq^j.
\]
(109)
\( \lambda \) is a real constant that is related to the cosmological constant of \( Q(m) \). Indeed, any quaternionic manifold is an Einstein manifold.

In the hyperKähler case, the forms \( \Omega^u \) are Kähler forms, namely
\[
d\Omega^u = 0.
\]
(110)

If \( Q(m) \) is quaternionic, there exist three one-forms \( \omega^u \) that make an \( SU(2) \) connection, with respect to which the forms \( \Omega^u \) are covariantly closed and such that \( \Omega^u \) is the field strength of this connection, namely
\[
d\Omega^u + \varepsilon_{uwz} \omega^w \wedge \Omega^z = 0,
\]
\[
d\omega^u + \frac{1}{2} \varepsilon_{uwz} \omega^w \wedge \omega^z = \Omega^u.
\]
(111)

The general feature of \( Q(m) \) is that its holonomy group \( Hol(Q(m)) \) is contained in \( SU(2) \otimes Sp(m) \). In the hyperKähler case, the \( SU(2) \) part of the spin connection of \( Q(m) \) is flat, while in the quaternionic case its curvature is proportional to \( \Omega^u \), where \( h_{i j} \) is the metric of \( Q(m) \).

We can introduce a quaternionic vielbein \( U^A_i \) where \( A = 1, 2 \) is an index of \( SU(2) \) and \( I = 1, \ldots 2m \) is an index of \( Sp(m) \). Let us introduce the vielbein one form
\[
U^A_i = U^A_i dq^i.
\]
(112)

We have
\[
h_{i j} = U^A_i U^B_j C_{i j} \varepsilon_{A B},
\]
(113)
where \( C_{i j} \) is the flat \( Sp(m) \) invariant metric, while \( \varepsilon_{A B} \) is, of course, the flat \( Sp(1) \equiv SU(2) \) flat invariant metric.

\(^8\)In our notation, \( Sp(m) \) denotes the symplectic group in \( 2m \) dimensions.
The vielbein $U^{AI}$ is covariantly closed with respect to the $SU(2)$-connection $\omega^u$ and to some $sp(m)$-valued connection $\Delta^{IJ} = \Delta^{JI}$, namely

$$
\nabla U^{AI} \equiv dU^{AI} - \frac{i}{2} \omega^u (\varepsilon^u \varepsilon^{-1})_A^B \wedge U^{BI} + \Delta^{IJ} \wedge U^{AK} \mathcal{C}_{JK} = 0,
$$

(114)

where $(\sigma^x)_A^B$ are the standard Pauli matrices. Furthermore $U^{AI}$ satisfies the reality condition

$$
U_{AI} \equiv (U^{AI})^* = \varepsilon_{AB} \mathcal{C}_{IJ} U^{BJ}.
$$

(115)

Eq. (115) defines the rule to lower the symplectic indices by means of the flat symplectic metrics $\varepsilon_{AB}$ and $\mathcal{C}_{IJ}$.

We also introduce the inverse vielbein $U^i_{AI}$ defined by the equation

$$
U^i_{AI} U^{AI}_j = \delta^i_j.
$$

(116)

Flattening a pair of indices of the Riemann tensor $R_{ijkl}$ we obtain

$$
R_{ij}^{st} U^i_{AI} U^j_{BJ} = \Omega^u_{st} i (\varepsilon^{-1})_A^B \mathcal{C}^{IJ} + R_{IJ}^{st} \varepsilon_{AB},
$$

(117)

where $R_{IJ}^{st}$ is the field strength of the $Sp(m)$ connection

$$
d\Delta^{IJ} + \Delta^{IK} \wedge \Delta^{LJ} \mathcal{C}_{KL} \equiv R_{IJ}^{st} dq^s \wedge dq^t.
$$

(118)

Eq. (117) is the explicit statement that the Levi Civita connection associated with the metric $h$ has a holonomy group contained in $SU(2) \otimes Sp(m)$. Consider now eqs (108) and (109). We easily derive the following relation

$$
h_{ij}^{st} \Omega^u_{st} \Omega^w_{ij} = -\lambda^2 \delta^{wu} h_{ij} + \lambda \varepsilon^{uwz} \Omega^z_{ij}.
$$

(119)

Eq. (119) implies that the intrinsic components of the 2-form $\Omega^u$ yield a representation of the quaternionic algebra. Hence we can set

$$
\Omega^u_{AI, BJ} \equiv \Omega^u_{ij} U^i_{AI} U^j_{BJ} = -i \lambda \mathcal{C}_{IJ} (\sigma^u \varepsilon)_{AB}.
$$

(120)

Alternatively eq. (120) can be rewritten in an intrinsic form as

$$
\Omega^u = i \lambda \mathcal{C}_{IJ} (\sigma^u \varepsilon^{-1})_{AB} U^{AI} \wedge U^{BJ},
$$

(121)

wherefrom we also get

$$
\frac{i}{2} \Omega^u (\sigma^u)_A^B = \lambda U^{AI} \wedge U^{BI}.
$$

(122)
B The topological twist

In this appendix we briefly recall the most general description of hypermultiplets [21, 26], also coupled to N=2 supergravity and we perform the BRST quantization. Then we proceed to define the topological twist and to show that the gauge-fixing equations are those that we expect. The main steps, without details, were given in ref. [17].

To fix the notation, we denote the hypermultiplets by $(q^i, \zeta^I, \bar{\zeta}^I)$. $q^i$ are the coordinates of the $4m$-dimensional target manifold $Q(m)$ ($i = 1, \ldots, 4m$). $\zeta^I$ and $\bar{\zeta}^I$ are the left handed and right handed components of the fermionic superpartners ($I = 1, \ldots, 2m$).

Specifically, $Q(m)$ is a hyperKähler manifold when gravity is external, while it is a quaternionic manifold when gravity is dynamical (i.e. the hypermultiplets are coupled to supergravity). In the first case one should put restrictions on the gravitational background in order to have global supersymmetry. However, we know that the topological theory is meaningful in a more general background. We focus on hypermultiplets coupled to supergravity, since the case of N=2 global supersymmetry can be retrieved as a suitable limit of the N=2 locally supersymmetric theory.

The parameter $\lambda$ of (109) appears in front of the kinetic term in the action of the hypermultiplets [26] [see also formula (128)]. In order to get a physical kinetic term in the case of dynamical gravity, the sign of $\lambda$ should be fixed and the target manifold should be noncompact (when the signature is Minkowskian) [21]. However, we shall not put this restriction, since the complete action is anyway nonpositive definite (due to the Einstein term). Moreover, the topological version of the theory is well defined in itself and the action is zero on any solution to the field equations and a fortiori on any instantonic solution. As we know, any quaternionic manifold is Einstein. Precisely, in the Minkowskian signature, we have $R_{ij} = -(m^2 + 2m)\lambda h_{ij}$.

The “generalized curvatures” are the one forms (112) and the following covariant derivatives of the fermions $\zeta^I$ and $\bar{\zeta}^I$

\[
\nabla \zeta^I = d\zeta^I - \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \zeta^I + \Delta^I J \zeta^J = D\zeta^I + \Delta^I J \zeta^J,
\]

\[
\nabla \bar{\zeta}^I = d\bar{\zeta}^I - \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \bar{\zeta}^I - \Delta^I J \bar{\zeta}^J = D\bar{\zeta}^I - \Delta^I J \bar{\zeta}^J,
\]

(123)

where $\omega^{ab}$ is the world-manifold Lorentz spin connection while $\Delta^I J$ is the $Sp(m)$ connection $\Delta^I J = C_{IK} \Delta^{KJ}$, $\Delta^I J = C_{JK} \Delta^I K$.

The superspace rheonomic parametrizations of the generalized curvatures are easily found [20]:

\[
U^A^I = U^A^I V^a + \epsilon^{AB} C^I J \bar{\psi}_B \zeta^J + \bar{\psi}^A \zeta^I,
\]

\[
\nabla \zeta^I = \nabla^a \zeta^I V^a + iU^B^J \gamma^a \psi^A \epsilon_{AB} \zeta^J,
\]

\[
\nabla \bar{\zeta}^I = \nabla^a \bar{\zeta}^I V^a + iU^A^I \gamma^a \bar{\psi}_A.
\]

(124)

We also report the definitions and the rheonomic parametrizations of the curvatures of N=2 supergravity, which are necessary in the case $Q(m)$ is quaternionic. The definitions
are

\[ R^a = dV^a - \omega^{ab} \wedge V_b - i \bar{\psi}_A \wedge \gamma^a \psi^A = \mathcal{D}V^a - i \bar{\psi}_A \wedge \gamma^a \psi^A, \]
\[ R^{ab} = d\omega^{ab} - \omega^a_c \wedge \omega^{cb}, \]
\[ \rho_A = d\psi_A - \frac{1}{4} \gamma_{ab} \omega^{ab} \wedge \psi_A + \omega_A^B \wedge \psi_B = \mathcal{D}\psi_A + \omega_A^B \wedge \psi_B, \]
\[ \rho^A = d\psi^A - \frac{1}{4} \gamma_{ab} \omega^{ab} \wedge \psi^A + \omega^A_B \wedge \psi_B = \mathcal{D}\psi^A + \omega^A_B \wedge \psi_B, \]
\[ F = dA + \bar{\psi}_A \wedge \psi_B \varepsilon^{AB} + \bar{\psi}^A \wedge \psi^B \varepsilon_{AB}, \]

(125)

where \( \psi_A \) and \( \psi^A \) are the left handed and right handed components of the gravitinos, respectively, while \( \omega_A^B = \frac{i}{2} (\sigma_u^u) A^B \omega^a \) and \( \omega^A_B = \varepsilon^{AL} \omega_L^M \varepsilon_{MB} \).

The rheonomic parametrizations are

\[ A^B_{A|a} = \frac{1}{4} \lambda \delta^B_{\tilde{A}} \zeta_I \gamma_A \zeta_I. \]

(127)

Now we give the kinetic lagrangian of N=2 supergravity coupled to hypermultiplets [29]. In the case \( \mathcal{Q}(m) \) is hyperKähler, the lagrangian is formally the same, upon suppression of the terms involving gravitinos and the Einstein kinetic term and upon setting \( \omega^u \) equal to zero.

\[ \mathcal{L}_{\text{kin}} = \varepsilon_{abcd} R^{ab} \wedge V^c \wedge V^d - 4 (\bar{\psi}^A \wedge \gamma_a \rho_A - \bar{\psi}_A \wedge \gamma^a \rho^A) \wedge V^a - \frac{4}{3} \lambda \varepsilon_{ABCD} U^A_{IJK} (U^{BJ} - \bar{\psi}^B \zeta^J - \varepsilon^{BC} C^{JK} \psi_C \zeta_K) \wedge V_{b} \wedge V_{c} \wedge V_{d} \varepsilon_{abcd} + i \frac{2}{3} \lambda (\bar{\zeta}^I \gamma_a \nabla \zeta^I + \bar{\zeta}_I \gamma_a \nabla \zeta^I) \wedge V_{b} \wedge V_{c} \wedge V_{d} \varepsilon_{abcd} + \frac{1}{6} \lambda \varepsilon_{ABCD} U^A_{IJK} V_{a} \wedge V_{b} \wedge V_{c} \wedge V_{d} \varepsilon_{abcd} + \frac{1}{12} F^{ab} F_{ab} \varepsilon_{cdef} V^c \wedge V^d \wedge V^e \wedge V^f - \varepsilon_{abcd} F^{ab} V^c \wedge V^d \wedge F. \]

(128)

We now perform the BRST quantization of the theory. We follow the general procedure described in refs. [34] [13] [17]. The local symmetries involved are diffeomorphisms, Lorentz
rotations, the gauge symmetry related to the graviphoton $A$ and supersymmetries. The ghosts will be denoted by $c^a$, $c^{ab}$, $c$ and $c_A$, $c^A$, respectively. One finds

$$s q^i = U_A^i (U_a^A c^a + \varepsilon^{AB} C^I c_B \xi^I + \bar{c}^A \xi^I),$$

$$s \xi_a = \frac{1}{4} \gamma_{ab} c^b \wedge \xi_a - \Delta^0_{(1,1)} \xi^a + \nabla_a \xi^a + i U_a^B \gamma^a c^A \varepsilon_{AB} C^J,$$

$$s \xi^I = \frac{1}{4} \gamma_{ab} c^b \wedge \xi^I + \Delta^I_{(1,1)} \xi^j + \nabla_a \xi^a + i U_a^A \gamma^a c^A_{(1)},$$

$$s V^a = - \mathcal{D} e^a + c^{ab} \wedge V_b + i \bar{c} A \gamma^a \psi^A + i \bar{\psi} A \gamma^a c^A,$$

$$s e^a = \varepsilon^{ab} \wedge \psi^b + i \bar{c} A \gamma^a \epsilon^a,$$

$$s \omega^{ab} = - \mathcal{D} e^{ab} + 2 R^{ab} c^c \wedge V^d - \frac{i}{2} (\bar{c} A \wedge c^A + \bar{\psi} A \epsilon^A) (2 \gamma^c \rho^A_{[a} b] c - \gamma^c \rho^A_{a b})$$

$$+ i \varepsilon^{abcd} (\bar{c}^b A \wedge \gamma^d \psi^B + \bar{\psi}^B A \wedge \gamma^d c^B) (A^B_{A[a} - \bar{A}^B_{A[a})$$

$$- \varepsilon^{AB} (\bar{c} A \wedge \psi^B + \bar{\psi} A \wedge c^B) F^{+ab} - \varepsilon_{AB} (\bar{c}^A A \wedge \psi^B + \bar{\psi}^A A \wedge c^B) F^{ab},$$

$$s c^{ab} = - \varepsilon^{ab} + i \varepsilon^{abc} \wedge \rho^B A_{a b} \epsilon^a \wedge \epsilon^b + A^B A_{a b} \gamma^a \epsilon^b,$$

$$s \epsilon^a = - \mathcal{D} \epsilon^a + \frac{1}{4} \gamma_{ab} \epsilon^a \wedge \epsilon^b - \omega^A A_{(1,1)} \wedge \psi_B$$

$$+ 2 \rho^B A_{a b} \epsilon^a \wedge V^b + A^B A_{a b} \gamma^a \epsilon^b \wedge V^b + \bar{\psi}^a B A_{a b} \wedge \epsilon^b,$$

$$s c_A = \frac{1}{4} \gamma_{ab} c^a \wedge C_A - W^A A_{(1,1)} \wedge c_B + \rho A_{a b} \epsilon^a \wedge \epsilon^b + A^B A_{a b} \gamma^a c_B \wedge \epsilon_b$$

$$+ i \varepsilon_{AB} F^{+ab} \gamma^a c_B \wedge \epsilon_b,$$

$$s \psi^A = - \mathcal{D} \psi^A + \frac{1}{4} \gamma_{ab} \epsilon^a \wedge \psi^A - \omega^A A_{(0,1)} \wedge \psi_B$$

$$+ 2 \rho^B A_{a b} \epsilon^a \wedge V^b + \bar{A}^B A_{b} \gamma^a (c^B \wedge V_b + \psi^B \wedge \epsilon_b),$$

$$s c^A = \frac{1}{4} \gamma_{ab} c^a \wedge c^A - \omega^A A_{(0,1)} \wedge c^B + \rho A_{a b} \epsilon^a \wedge \epsilon_b + \bar{A}^A B_{a b} \gamma^a c_B \wedge \epsilon_b$$

$$+ i \varepsilon_{AB} F^{-ab} \gamma^a c_B \wedge \epsilon_b,$$

$$s A = - dc - 2 \bar{c} A \wedge \psi^B A_{(1,1)} - 2 \bar{c} A \wedge \psi^B \wedge \epsilon_B + 2 F_{a b} \epsilon^a \wedge V^b,$$

$$s c = - \bar{c} A \wedge c_B \wedge \epsilon_B + F_{a b} \epsilon^a \wedge \epsilon_B,$$

(129)

where $\Delta^I_{(1,1)}$ and $\omega^A A_{(0,1)}$ are obtained from the one forms $\Delta^J$ and $\omega^A A$ upon substitution of the differential $dq^i$ with the BRST variation $sq^i$, which appears in the first equation of (129).

Now we perform the topological twist of $N=2$ supergravity coupled to hypermultiplets. As mentioned in appendix [A], $\text{Hol}(\mathbb{Q}(m)) \subset SU(2) \otimes Sp(m)$. This $SU(2)$ is the internal
supersymmetry automorphism $SU(2)_I$ [26, 17]. In the twisted version of the theory, $SU(2)_I$ is identified with the right handed part $SU(2)_R$ of the Lorentz group, to give the new $SU(2)_R'$. We recall [17] that one also has to identify an $SU(2)_Q$ for the redefinition of $SU(2)_L$: it is the $SU(2)$ factor in the $SU(2) \otimes SO(m)$ maximal subgroup of $Sp(m)$. Moreover, there must exist a suitable internal $U(1)$, in order to perform the redefinition of the ghost number. This $U(1)_I$ is R-duality [17].

The complete twisting procedure can be divided in three steps. Step A corresponds to the redefinitions of $SU(2)_L$, $SU(2)_R$ and ghost number $U(1)_g$ according to the following scheme

$$SU(2)_L \longrightarrow SU(2)_L' = \text{diag}[SU(2)_L \otimes SU(2)_Q],$$
$$SU(2)_R \longrightarrow SU(2)_R' = \text{diag}[SU(2)_R \otimes SU(2)_I],$$
$$U(1)_g \longrightarrow U(1)_g' = \text{diag}[U(1)_g \otimes U(1)_I],$$
$$c(L, R, I, Q)^g_f \longrightarrow (L \otimes Q, R \otimes I)^{g+c}_f,$$  \hspace{1cm} (130)

where $L$ denotes the representation of $SU(2)_L$, $R$ the representation of $SU(2)_R$, $Q$ is the representation of $SU(2)_Q$, $c$ is the $U(1)_I$ charge, $g$ the ghost number and $f$ the form number. $U(1)_g' = \text{diag}[U(1)_g \otimes U(1)_I]$ is intended to mean that the new ghost number is the sum of the old ghost number plus the R-duality charge. Step B is the correct identification of the topological ghosts (fields with $g + c = 1$ from $g = 0$, $c = 1$) by contraction with a suitable vielbein (in our case the quaternionic vielbein $U^A_i$ for the identification of the topological ghosts $\xi^i$ [17]). Step C is the topological shift, namely the shift by a constant of the $(0, 0)^{0}_0$-field coming by applying step A on the right handed components of the supersymmetry ghosts, $c^A$, namely $(c^A)^A \rightarrow -i/2 e^{-\hat{a}A} + (e^A)^A$ [13, 17], where $e$ is the broker (a zero form with fermionic statistic and ghost number one, with the convention that $e^2$ is set equal to 1 [17]).

The $U(1)_I$ internal symmetry (R-duality) that adds to ghost number to give the ghost number of the topological version of the theory is chirality on the gravitinos, duality on the graviphoton and the opposite of chirality on the superpartners of the quaternionic coordinates, the hyperini $\zeta_I$ and $\zeta^I$, namely [17]

\begin{align*}
\hat{\delta} V^a &= 0, & \hat{\delta} A &= 0, \\
\hat{\delta} \psi_A &= \psi_A, & \hat{\delta} \psi^A &= -\psi^A, \\
\hat{\delta} F^+_a &= 2F^+_a, & \hat{\delta} F^-_a &= -2F^-_a, \\
\hat{\delta} \zeta_I &= -\zeta_I, & \hat{\delta} \zeta^I &= \zeta^I.
\end{align*}  \hspace{1cm} (131)

R-duality extends to the BRST-quantized theory by simply stating that any field has the same R-duality behaviour as its ghost partner.

In general, the R-duality anomaly is the formal dimension of the moduli space, because, after the topological twist, it represents the ghost number anomaly. The R-duality anomaly is not only due to the axial anomaly (R-duality is proportional to chirality on the fermions) but is also due to the dual anomaly of the graviphoton [35]. This anomaly
is related to the difference between the numbers of zero modes of self-dual and anti-self-dual field strengths $F^{ab}$ and not to the zero modes of the vector $A$. So, the problem of the ghost-number anomaly is not so simple as in two dimensions or as in topological Yang-Mills Theory. So far, a complete analysis of this anomaly has not been performed.

The indices $I = 1, \ldots, 2m$ of $Sp(m) \supset SU(2)_Q \otimes SO(m)$ are splitted into a couple of indices, according to $I = (\alpha, k)$, where $\alpha = 1, 2$ is an index of $SU(2)_Q \approx SU(2)_L$ and $k = 1, \ldots, m$ is an index of $SO(m)$. The $Sp(m)$-invariant metric $C_{IJ}$ becomes $\varepsilon_{\alpha \beta} \delta_{kl}$, if $I = (\alpha, k)$ and $J = (\beta, l)$. We can introduce the vielbein $E_{i}^{a k} \equiv \frac{1}{2} \mathcal{U}_{i}^{A k} (\sigma^a)_{\alpha \dot{A}}$. Moreover, we can define the true topological antighosts

$$\zeta_{k}^{+ ab} = - e(\sigma_{ab})_{\alpha}^{\beta} \varepsilon_{\alpha \gamma} (\zeta_{\beta})_{\gamma k},$$

$$\zeta_{k} = - e \varepsilon_{\alpha \beta} (\zeta_{\alpha})_{\beta k},$$

which, under the new Lorentz group transform as $(1, 0)$ and $(0, 0)$ respectively.

The twisted-shifted BRST algebra is, up to nonlinear terms containing ghosts,

$$sV^{a} = \tilde{\psi}^{a} - dc^{a} + \epsilon^{ab} \wedge V_{b},$$

$$se^{a} = C^{a},$$

$$s\psi^{a} = - dC^{a} + \frac{1}{2} F^{+ ab} \wedge V_{b}, \quad s\tilde{\psi}^{ab} = - dC^{ab} + i \frac{1}{2} (\omega^{- ab} + \frac{1}{2} F_{u}^{ab} q^{u} \omega^{u}),$$

$$sC^{ab} = \frac{1}{2} \epsilon^{ab} + \frac{1}{4} I_{u}^{ab} q^{u} \omega_{(0, 1)},$$

$$sA = i \tilde{\psi} - dc, \quad sc = - \frac{1}{2} + i C,$$

$$sq^{i} = - \frac{1}{2} \epsilon^{ij} \tilde{\psi} \leftrightarrow \zeta_{i},$$

$$s\zeta_{+ a}^{i k} = - 2 c V_{a}^{i} \epsilon_{j}^{a k} \partial_{\mu} q^{j},$$

where the formulæ relating $\tilde{\psi}^{a}, \tilde{\psi}^{ab}, \tilde{\psi}$ to $\psi_{A}, \psi^{A}$ are

$$\tilde{\psi}^{a} = \frac{e}{2} (\psi_{A})^{A} (\sigma^{a})_{A}, \quad \tilde{\psi}^{ab} = - e (\sigma^{ab})_{A} (\psi^{A})_{A}, \quad \tilde{\psi} = - e (\psi_{A})^{A} \delta^{A}_{A},$$

and similar formulare $C^{a}, C^{ab}, C$ to $c_{A}, c^{A}$. Moreover, $[ab]^{+}$ means antisymmetrization and selfdualization in the indices $a, b$. Thus we see that both $\zeta_{k}^{+ ab}$ and $\zeta_{k}$ are topological antighosts (otherwise we would not have enough equations to fix the gauge completely).

Let us give the gauge-free algebra of the topological theory in full generality.
where the subscript 0 means that the spin connection $\omega^{ab}$ is the usual one (i.e. in the second order formalism it does not contain the terms quadratic in the fermions that characterize the supergravity spin connection). As we see, the $\sigma$-model sector of the BRST-algebra is trivial.

The complete identification of the gauge-free BRST algebra (135) with the minimal subalgebra of the gauge-fixed BRST algebra (129) is given by (the dots stand for nonlinear terms containing ghosts)

\[
\begin{align*}
\xi^i &= U_{A|I}(U_a^M \epsilon^a + \varepsilon^{AB}C^{IJ}\bar{c}_B \zeta_J + \bar{c}^A \zeta^I) = \tilde{\xi}^i + \cdots, \\
\psi^a &= i\bar{c}_A \gamma^a \psi^A + i\bar{c}_A \gamma^a c^A - A^{ab}\epsilon_b = \tilde{\psi}^a + \cdots, \\
\phi^a &= i\bar{c}_A \gamma^a c^A = C^a + \cdots, \\
\eta^{ab} &= R^{ab}_{\quad cd\,e} \epsilon^c \wedge \epsilon^d - i\bar{c}_A (2\gamma^a \rho^{A|b|c} - \gamma^c \rho^{A|ab}) \wedge \epsilon_c - i\bar{c}^A (2\gamma^a \rho^{B|b|c} - \gamma^c \rho^{B|ab}) \wedge \epsilon_c \\
&\quad + \varepsilon^{abcd} c^A \gamma^d c_B (\bar{A}^B A^c - A^B \bar{A}^c) - \varepsilon^{AB} c^A \wedge c_B F^{-ab} - \varepsilon^{AB} \bar{c}^A \wedge c^B F^{+ab} \\
&= -\frac{1}{2} F^{+ab} + \cdots.
\end{align*}
\]

The total action can be written as the sum of the classical topological action $S_T$ (138) plus the BRST variation of the a gauge fermion $\Psi$, that, up to interaction terms containing ghosts, turns out to be

\[
\begin{align*}
\Psi &= -16i \left[ \frac{B^{ab} - i \left( \omega^{ab} + \frac{1}{2} f^{ab}_u \omega^u + 2i dC^{ab} \right) \wedge \psi_{ac} \wedge V_b \wedge V^c}{2} \\
&\quad + 8i F \wedge \psi_a \wedge V^a + \frac{\lambda}{3} \varepsilon_{cde} \nu^c \wedge V^d \wedge V^e \wedge \nu^f [2\eta^{ab} \epsilon_{ab} \\
&\quad + (4V^{\nu|a} E_i^{b|c} + \Lambda^{abk}) \zeta_{abk} + (\Lambda^k - 2V^k \nu^a E_i^{ak} \partial_\mu q^i) \zeta_k] \right].
\end{align*}
\]

where $B^{ab}$, $\Lambda^{abk}$ and $\Lambda^k$ are Lagrange multipliers ($s\psi^{ab} = B^{ab}$, $s\zeta^{abk} = \Lambda^{abk}$, $s\zeta^k = \Lambda^k$, $sB^{ab} = s\Lambda^{abk} = s\Lambda^k = 0$).

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