Pascal’s triangle and word bases
for blob algebra ideals

P P Martin

Mathematics Department, City University,
Northampton Square, London EC1V 0HB, UK.

1 Introduction

The Temperley–Lieb algebra [20] plays a role in many different branches of Mathematics and Physics (see for example [1, 7, 12, 13, 10]). The blob algebra [16] is a generalisation which preserves this versatility (see for example [15, 16, 17, 18, 14, 4, 19, 6, 5, 9]).

In particular, the blob algebra is used to study (affine) Hecke algebra representation theory [18, 9], and boundary integrable lattice models [6, 5]. In these settings it is useful to have a rather explicit understanding of the relationship between the well-known ‘diagram’ bases for ‘standard’ blob modules [16] and bases of words in abstract generators. This connection is sketched, in principle, both in [16] and (using [8]) in [3]. The present paper is intended simply to provide an explicit self-contained version (which can thus be put to direct practical use in the study of further generalisations of the Temperley–Lieb algebra not amenable to the methods of [3], that we shall report on elsewhere). The main results are Definition 4 (a map from walks on the Pascal triangle to ‘reduced’ words in the blob algebra); Proposition 13 and Proposition 17 (which together show that these words aggregate naturally into bases for certain blob algebra modules, including the regular module).

The blob algebra $b_n$ is usually defined in terms of a certain basis of diagrams (16, and see later), from which it derives its name. This blob algebra is isomorphic to an algebra defined by a presentation. There is a map which takes each generator in the presentation to a diagram, which, it is fairly easy to see, provides a surjective algebra homomorphism. The isomorphism follows from the vanishing of the kernel of this homomorphism, which is somewhat harder to exhibit. A direct way to see it, however, is from a co-enumeration of the diagram basis with a certain set of words, which is then shown to be spanning. (The image of any spanning set under a surjective map is again spanning, and hence, if of order no greater than the rank of the diagram algebra, independent. Then the original set is independent, and hence a basis. We have a surjective homomorphism between free modules of equal rank, and hence an isomorphism.) This co-enumeration uses the natural tower structure $b_i \subseteq b_{i+1}$, and follows from the bra-ket formalism introduced in [16], via a generalisation of the corresponding proof of isomorphism between the
diagram and presentation forms [13 §6.4-6.5] for the Temperley–Lieb algebra. In this way one gets bases not only for the algebra, but also for Specht modules (cf. [11]), and various double sided ideals, in a way which helps exhibit the structure of the algebra. Martin and Saleur [16] eschewed details of this argument in the interest of brevity, providing an explicit enumeration only on the diagram side. On the presentational side a brief summary can be found in [3] (using Graham [8] which, motivated by the study of Hecke algebras, extends the ideas of [13 §6.5] to an axiomatic framework broad enough to include the blob algebra). Here we provide an explicit self-contained version on the presentation side, paralleling the use in [16] of an ‘enriched’ Pascal triangle.

1.1 The algebra \( b_n \) by presentation

For \( K \) a ring, \( x \) an invertible element in \( K \), \( q = x^2 \), and \( \gamma, \delta_e \in K \), define \( b^K_n \) to be the unital \( K \)–algebra with generator set \( U^e = \{e, U_1, \ldots, U_{n-1}\} \) and relations

\[
\begin{align*}
U_i U_i &= (q + q^{-1}) U_i \\
U_i U_{i \pm 1} U_i &= U_i \\
U_i U_j &= U_j U_i \\& \quad (|i - j| \neq 1) \\
U_1 e U_1 &= \gamma U_1 \\
e e &= \delta_e e \\
U_i e &= e U_i \\& \quad (i \neq 1).
\end{align*}
\]

It will be evident that \( e \) can be rescaled to change \( \gamma \) and \( \delta_e \) by the same factor. Thus, if we require that \( \delta_e \) is invertible, then we might as well replace it by 1. (This brings us to the original two–parameter definition of the algebra.) We take \( b^K_0 = K\{1\} \), and to be clear, note that \( b^K_1 = K\{1, e\} \).

Note from the form of the relations that we have

**Proposition 1** Let \( (b^K_n)^{op} \) be the opposite algebra of \( b^K_n \). Then there is an algebra isomorphism

\[
b^K_n \cong (b^K_n)^{op}
\]

which fixes the generators. \( \Box \)

For \( k \) a field which is a \( K \)–algebra define \( k \)–algebra \( b_n = k \otimes_K b^K_n \).

**Proposition 2** For \( n = 2 \)

\[
b^K_2 = K\{1, e, U_1, e U_1, U_1 e, e U_1 e\}
\]

(with the given set independent).

**Proof:** It will be evident that this set of words is spanning. (It is left as an exercise to show independence.) \( \Box \)
Proposition 3  Set \( U_0 = e \). Then for \( n \geq 1 \)
\[
  b_n^K = b_{n-1}^K + b_{n-1}^K U_{n-1} b_{n-1}^K.
\]  \( (7) \)

Proof: By induction on \( n \). The case \( n = 1 \) of (7) is clear, and the case \( n = 2 \) follows from proposition \( \[2\] \) Now suppose true at level \( n - 1 \), and consider \( n > 2 \). Trivially
\[
  b_n^K = b_{n-1}^K + b_{n-1}^K U_{n-1} b_{n-1}^K + b_{n-1}^K U_{n-1} b_{n-1}^K U_{n-1} b_{n-1}^K + \ldots
\]
but by assumption
\[
  U_{n-1} b_{n-1}^K U_{n-1} = U_{n-1} b_{n-1}^K U_{n-1} + U_{n-1} b_{n-1}^K U_{n-1} b_{n-2} U_{n-2} + \ldots \quad (8)
\]
\[ \square \]

Proposition 4
\[
  U_1 b_2^K U_1 = ([2] K + \gamma K) U_1 b_0^K
\]  \( (9) \)
and for \( n \geq 3 \)
\[
  U_{n-1} b_n^K U_{n-1} = U_{n-1} b_n^K
\]  \( (10) \)

Proof: \( (9) \) follows from proposition \( \[2\] \) and the defining relations. By proposition \( \[3\] \)
\[
  U_{n-1} b_n^K U_{n-1} = U_{n-1} b_{n-1}^K U_{n-1} + U_{n-1} b_{n-1}^K U_{n-1} b_{n-1}^K U_{n-1} = U_{n-1} b_{n-2}^K \quad (8)
\]  \( \square \)

In \([16]\) Proposition 2] an explicit enumeration of the diagram basis of each blob diagram algebra is given. Indeed this basis is put in explicit bijection with the set of pairs of walks to the same location in level \( n \) of Pascal’s triangle. Here we do the same thing for bases for each algebra \( b_n \) as defined above. We proceed by constructing bases for a series of ideals, the last of which is \( b_n \) itself.

## 2 Preliminaries on words in \( U^e \)

### 2.1 Preliminary construction of ideals

Let \( U = \{U_1, U_2, \ldots, U_{n-1}\} \), so \( U^e = \{e\} \cup U \). For \( S \) a set let \( \mathcal{P}(S) \) denote its power set. For \( m \geq 0, m \equiv n \, (\text{mod. } 2) \) define
\[
  E_m(n) = U_1 U_3 \ldots U_{n-m-1} \quad E'_m(n) = U_{n-1} U_{n-3} \ldots U_{m+1}
\]
(take \( E_n(n) = E'_n(n) = 1 \)). When \( n \) is fixed we may simply write \( E_m \) for \( E_m(n) \), but note that
\[
  E_{m+1}(n+1) = E_m(n). \quad (11)
\]

For \( m > 0, m \equiv n \, (\text{mod. } 2) \) define
\[
  E_{m+}(n) = E_{m+} = E_m(n) e U_2 U_4 \ldots U_{n-m} E_m(n).
\]
**Definition 1** Define double sided ideals in $b_n$ by

$$I_m = b_n E_m(n) b_n \quad \quad I_m^+ = b_n E_{m+}(n) b_n.$$ 

Thus $I_n = b_n$. It follows from proposition 4 that

**Proposition 5** For $m > 0$

$$E_m'(n) b_n E_m(n) = E_m'(n) b_m$$

(for $m = 0$ a similar result holds, but one must take care with the ring, as in proposition 2 — at least one of [2], $\gamma$ must be invertible).

**Definition 2** Define $U^2 \subseteq P(U)$ as the maximal subset such that $U_i, U_j \in V \in U^2$ implies $i - j \not\in \{\pm 1\}$ (i.e. the set of commutative subsets of $U$).

**Proposition 6** Let $W \in U^2$. Then

$$b_n \left( \prod_{w \in W} w \right) b_n = I_{n-2|W|}$$

$$I_m \subset I_{m+2}$$

$$I_m^+ \subset I_m$$

$$\gamma I_m \subset I_{m+2}$$

The proofs are elementary applications of relations (2) (and, in the last case (4), which provides the only means to reduce out the ‘last’ $e$ in any word).

Accordingly, for $i \equiv n \pmod{2}, i \geq 0$, define quotient algebras

$$b_n^i = b_n/I_i \quad \quad b_n^{-i} = b_n/(I_i^+ \cup I_{i-2})$$

Note that, for $n \geq 2$, $b_n^{n-2}$ has basis $\{1, e\}$ (consider iterating proposition 3 for example). Similarly, for $n - 2 > 0$, $E_{n-2}'(n)b_n^{n-4}E_{n-2}'(n)$ has basis $\{U_{n-1}, eU_{n-1}\} = \{1, e\} E_{n-2}'(n)$ (as it were). Indeed, provided that $n - 2r > 0$, $E_{n-2r}'(n)b_n^{n-2r-2}E_{n-2r}'(n)$ has basis $\{1, e\} E_{n-2r}'(n)$.

For $n$ even, define left $b_n$-module $\Delta_0(n) = b_n E_0(n)$. For $n$ odd, define left $b_n$-module $\Delta_1(n) = b_n E_1(n)$, and $\Delta_i(n) = b_n E_i(n)$ mod. $\Delta_{i+2}(n)$. Define left $b_n$-module $\Delta_{i+2}(n)$ as the restriction of the $b_n^{-i}$-module $b_n^{-i}E_{i+2}(n)$. Define left $b_n$-module $\Delta_{-(i+2)}(n)$ as the restriction of the $b_n^i$-module $b_n^i E_{(i+2)+}(n)$.

### 2.2 Notations and identities

Define

$$U_{i,j} = U_i U_{i-1} \ldots U_j \quad \quad (i \geq j)$$

$$U_{i,k,j} = U_i U_{i-2} \ldots U_j \quad \quad (i - j \in 2\mathbb{N})$$

(and if the argument condition is violated we will take any such product to evaluate to 1). NB the following elementary identities

$$E_0(i) U_{i+1 \backslash 1} = E_0(i + 2) \quad (12)$$

4
2.3 Word reduction

Note from (1-6) that every relation which shortens a word introduces a scalar factor from $K$ (but that this factor may be 1). We call a word algebra reduced if it cannot be expressed as a product of a scalar in $K \setminus \{1\}$ times another word. (Thus the $K$-span of algebra reduced words is the whole algebra $[2]$.) For example, $U_1 U_2 U_1$ is algebra reduced.

**Proposition 7** (a) Word $w \in b_{n-1}^K$ is algebra reduced iff $wU_1 \in b_{n+1}^K$ is algebra reduced.

(b) Word $w \in b_{n-1}^K$ is algebra reduced iff $wU_1 wU_2 \ldots U_1 \in b_n^K$ is algebra reduced.

(c) A word of form $wU_{n-2} \in b_{n-1}^K$ is algebra reduced iff $wU_{n-2} U_{n-2} U_{n-2} \in b_n^K$ is algebra reduced.

**Proof:** The first claim follows from the commutation of $w$ with $U_n$ in $b_{n+1}^K$ (noting that any word which is algebra reduced in $b_{n+1}^K$ but expressible in $b_{n-1}^K$ is algebra reduced in $b_{n-1}^K$). The second then follows since $(wU_n)(U_{n-1}U_{n-2} \ldots U_1)U_2 U_3 \ldots U_n = wU_n$. For the third note that a word of the form $wU_{n-2} \in b_{n-1}^K$ is algebra reduced if and only if $wU_{n-2} eU_{n-2} \in b_{n}^K$ is algebra reduced. This in turn is algebra reduced if and only if $wU_{n-2} eU_{n-2} U_{n-2} \in b_{n}^K$ is algebra reduced, since $U_n(wU_{n-2} eU_{n-2} U_{n-2})U_{n-2} eU_{n-3} = wU_{n-2} eU_{n-3}$, whereupon we can use (a) again (NB, the last identity is verified in $b_{n+1}^K$). □

3 The Pascal triangle

We now associate certain elements of $b_n$ to descending paths of length $n$ on the Pascal triangle.
3.1 Paths on the Pascal triangle

Label vertices (positions) on the Pascal triangle by pairs of numbers giving level (row) and weight (column):

```
level  0   1   1
    1   1   1
    2   1   2   1
    ...
```

Label edges by vertex pairs: 

\(((n, m), (n + 1, m \pm 1))\).

For \(m \in \mathbb{Z} \setminus \{0\}\), \(i \in \mathbb{N}\), define

\[
m \uparrow i = \begin{cases} 
m + i & \text{in case } m > 0 \\
m - i & \text{otherwise.} \end{cases}
\]

Define \(m \downarrow i = m \uparrow -i\) similarly (except \(m \downarrow i = 0\) in case \(i > |m|\)).

**Definition 3** Let \(S_n\) denote the set of walks from level 0 to level \(n\) (any weight) on the Pascal triangle; and \(S_{n,m}\) the subset to weight \(m\).

It will be evident that \(|S_n| = 2^n\). For \(p \in S_n\) we will write \(p_i\) for the \(i^{th}\) edge of \(p\).

There are various ways of specifying a particular \(p \in S_n\). In particular, let \(\sigma(p) = (\sigma(p)_0, \sigma(p)_1, \ldots)\) be the encoding of \(p \in S_n\) as a sequence of weights. For example, \((0, 1, 0)\). The edges \(p_i\) are then just the sequence of adjacent pairs from this sequence. More robustly, we may specify a walk, or part of a walk, as a sequence of (level,weight) pairs. (Our example becomes \(((0, 0), (1, 1), (2, 0))\).) Then each edge is a pair of such pairs.

3.2 Words and paths

Associate words in the generators \(U^e (b_\infty)\) to edges on the Pascal triangle as:

\[
\begin{align*}
w(((n, |m|), (n + 1, |m| + 1))) &= 1 \\
w(((n, -|m|), (n + 1, -|m| - 1))) &= 1 \\
w(((n, m), (n + 1, m \downarrow 1))) &= U_n U_{n-1} \ldots U_1 \\
w(((n, 0), (n + 1, +1))) &= e U_n b_2 U_{n-1} b_1 \\
w(((n, 0), (n + 1, -1))) &= 1
\end{align*}
\]

Each descending path \(p\) on the Pascal triangle may be described by the sequence \(p_1, p_2, \ldots\) of edges passed through.
Definition 4 For any \( n \) define

\[ w : S_n \to \langle U^e \rangle \in b^K_n \]

as follows. The word \( w(p) \) associated to path \( p \) is the product of words associated to this sequence of edges by \( E \), written from left to right: \( w(p) = w(p_1)w(p_2) \)....

The first several such are given in table 1.

Note that if we write \( w(p) = x \in b_n \) we mean the identity in \( b_n \), not necessarily as words.

Definition 5 The set of words \( w(p) \) associated to paths \( p \in S_{n,m} \) (i.e. starting at vertex \((0,0)\) and terminating at vertex \((n,m)\)), is denoted \( S_{(n,m)} \).

Proposition 8

\[ S_{(n,m)} \subset b^K_n E_{|m|}(n). \]  

(23)

For \( m > 0 \)

\[ S_{(n,m)} \subset b^K_n E_m(n) \]  

(24)

For \( m \leq 0 \), \( E_m(n) \in S_{(n,m)} \); for \( m > 0 \), \( E_m(n) \in S_{(n,m)} \).

Proof: (Of (23)): Suppose true at level \( n - 1 \) (the base case is trivial). For \( m = 0 \) we have

\[ S_{(n,0)} = \{xU_{n-1\downarrow}|x \in S_{(n-1,1)}\} \cup \{xU_{n-1\downarrow}|x \in S_{(n-1,-1)}\} \]

But \( S_{(n-1,1)} \subset b^K_{n-1} E_1(n - 1) \) by the inductive hypothesis, and \( E_1(n - 1)U_{n-1\downarrow} = E_0(n - 2)U_{n-1\downarrow} = E_0(n) \) so the second subset lies in \( b^K_n E_0(n) \) (and similarly for the \( S_{(n-1,1)} \) part). For \( m \neq 0, 1 \) we have

\[ S_{(n,m)} = S_{(n-1,m\downarrow)} \cup \{xU_{n-1\downarrow}|x \in S_{(n-1,m\uparrow)}\} \]

The first subset obeys (23) by equation (11) and the inductive hypothesis, the second by equation (12). For \( m = 1 \) (24) use (13) similarly.

(Of content claim): In case \( m = 0 \) note that \( w(0 \ -\ 1\ 0 \ -\ 1\ \ldots\ 0) = E_0(n) \) by (12). In case \( m = 1 \) note then that \( w(0 \ -\ 1\ 0 \ -\ 1\ \ldots\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) = E_{1+}(n) \) by (21). All the other cases follow by observing that every point on the Pascal triangle can be reached by a path of the form \( 0 \ -\ 1\ 0 \ -\ 1\ \ldots\ 0\ -\ 1\ -\ 2\ \ldots\ -m \) or \( 0 \ -\ 1\ 0 \ -\ 1\ \ldots\ 0\ 12\ldots m \), for which the last \( m \) factors in \( w(p) \) do not change the word. This \( w(p) \) is thus the required word (noting (11)). \( \square \)

It follows that we may replace (20) by

\[ w(((n,m),(n+1,m \downarrow 1))) = U_{n\downarrow |m\downarrow 1|} \]

changing the word \( w(p) \) only by an algebra equivalence. We will use the two forms interchangeably in what follows, unless the length of words is an issue (in which case we will take the latter form, this being never longer than the former).

It then follows from Proposition 7 that
Figure 1: Words associated to edges, and hence paths, on the Pascal triangle. Unlabelled edges have \( w(\text{edge}) = 1 \).
Proposition 9  For all $n, m$: $S_{(n,m)}$ is a set of algebra reduced words; $S_{(n,m)}$ can be expressed in the form

\[
S_{(n,m)} = \{ wE_{-m}(n) \mid w \in s(n,m) \} \quad (m \leq 0)
\]
\[
S_{(n,m)} = \{ wE_{m+}(n) \mid w \in s(n,m) \} \quad (m > 0)
\]

where in each case $s(n,m)$ is some set of algebra reduced words. \(\square\)

4  Ideals, modules and bases

Consider the action of $U_i$ ($i \in \{1, 2, ..., n-1\}$) from the left on each element $w(p)$ of $S_{(n,m)}$. For this purpose the most significant part of the path $p$ is the neighbourhood of the $i^{th}$ vertex, as we will see. This part may be specified by expanding $p$ in the form

\[
p = p'(i - 1, l)(i, m)(i + 1, n)p'' = p'(l, m, n)p''
\]

for some weight triple $(l, m, n)$. (This is an abuse of notation, since weight $(i - 1, l)$ is also a part of the final edge in subpath $p'$, but it is still useful.) In this sense we may write $w(p) = w(p')w((l, m, n))w(p'')$ with $w(p') \in b_{i-1}$ (and hence commuting with $U_i$). For example

\[
w(p'(l, l \downarrow 1, l)p'') = w(p') U_{i-1\backslash1} w(p'')
\]

and hence

Proposition 10  For $|l| > 1$

\[
U_i w(p'(l, l \downarrow 1, l)p'') = w(p'(l, l \uparrow 1, l)p'')
\]

\(\square\)

It will be convenient to picture the difference between the two walks here as follows:

Let us generalise the notation of the triple $(l, m, n)$ (whose middle element is the weight at the $i^{th}$ vertex) to any sequence of weights of consecutive vertices (NB, the position of the $i^{th}$ vertex must be indicated in some way). For example

Proposition 11  For $l \in \mathbb{N}$

\[
U_i w(p'(0(-1 -2)^l -1 \quad 0) \quad 1)p'' = w(p'(0(1 2)^l 1 2 1)p'')
\]

9
Proof: Note that $i$ is even. Put $j = i - 2l - 2$, then simply applying (18-22) we get
\begin{align*}
w(p'(0(-1 -2)^l -1 0 1)p'') \\
= w(p') \ U_{j\setminus 1}U_{j+2\setminus 1} \ldots U_{j+2l\setminus 1}U_{j+2l+1\setminus 1}eU_2U_4 \ldots U_{j+2l+2}U_1U_3 \ldots U_{j+2l+1}w(p'').
\end{align*}
Thus
\begin{align*}
U_iw(p'(0(-1 -2)^l -1 0 1)p'') \\
= U_iw(p') \ U_{j\setminus 1}U_{j+2\setminus 1} \ldots U_{j+2l\setminus 1}U_{j+2l+1\setminus 1}eU_2U_4 \ldots U_{j+2l+2}U_1U_3 \ldots U_{j+2l+1}w(p'')
\end{align*}
But using equation (16) repeatedly
\begin{align*}
U_{j\setminus 1}U_{j+2\setminus 1} \ldots U_{j+2l\setminus 1}U_{j+2l+1\setminus 1}eU_2U_4 \ldots U_{j+2l+2}U_1U_3 \ldots U_{j+2l+1}
\end{align*}
while
\begin{align*}
w(p'(0(1 2)^l 1 2 1)p'') =
\end{align*}
\begin{align*}
w(p')(eU_2U_4 \ldots U_{j-2}U_1U_3 \ldots U_{j-3})U_{j\setminus 1}U_{j+2\setminus 1} \ldots U_{j+2l\setminus 1}U_{j+2l+2\setminus 1}w(p'')
\end{align*}
and the result follows from repeated application of equation (17). \qed

**Proposition 12** For $l \in \mathbb{N}$
\begin{align*}
U_iw(p'(0\hat{1}^{(i)th}(23)^l 2 1)p'') = w(p'(0 -1(0 -1)^l 0 1)p'')
\end{align*}

Proof: The $l = 0$ case follows from proposition 11 and the others by a simple iteration:
If $p = p'(01232\ldots)$
\begin{align*}
U_iw(p) = w(p')(U_i(eU_2 \ldots U_{i-1}U_1 \ldots U_{i-2})U_{i+2}U_{i+1}U_{i-1} \cdot U_1w(p''))
\end{align*}
\[ = (U_{i+2}w(p')(U_i)(eU_2 \ldots U_{i-1}U_1 \ldots U_{i-2})U_{i+1}U_iw(p'')) \]
\[ = (U_{i+2})w(p')(U_i)(eU_2 \ldots U_{i-1}U_{i+1}U_1 \ldots U_{i-2}U_i)w(p'') = (U_{i+2})w(p'(0 -1 012...)) \]
and so on. \( \square \)

**Proposition 13** If \( m = 0,1 \) then \( S_{(n,m)} \) spans \( b_nS_{(n,m)} \);

if \( m = -1 \) then \( S_{(n,m)} \) spans a left subideal mod. \( b_nS_{(n,1)} \).

if \( m \geq 2 \) then \( S_{(n,m)} \) spans a left subideal mod. \( b_nE_{m|2}(n)b_n \);

and if \( m < -1 \) then \( S_{(n,m)} \) spans a left subideal mod. \( b_nS_{(n,-m)} \cup b_nE_{-(m|2)}(n)b_n \).

**Proof:** It is relatively straightforward to check that the action of \( e \) on \( S_{(n,m)} \) stays within the indicated span. Thus we concentrate on the action of \( U_i \). Again consider the action of \( U_i \) on \( w(p) \), and characterise the path in the neighbourhood of \( i \) by the triple \( (l,m,n) \).

First suppose that one or both of the edges touching the \( i^{th} \) vertex touches \( m = 0 \). There are various cases:

- \((0, \pm 1, 0)\):
  \[ p = p'(i - 1, 0)(i, \pm 1)(i + 1, 0)p'' \]
  (NB, \( i \) odd) then \( w(p) = w(p')w((i - 1, 0)(i, \pm 1))U_i \ldots U_1w(p'' \) so \( U_iw(p) = [2]w(p) \).

- \((0, 1, 2)\): If \( p'' \) never turns over then \( w(p'') = 1 \) and \( U_iw(p) \in b_nE_{\pm(m|2)}(n) \) by proposition 6. If \( p'' \) never touches \( m = 1 \) again then we may use equation (25) to equate to a case in which it never turns over, and again \( U_iw(p) \in b_nE_{\pm(m|2)}(n) \) by proposition 6. Otherwise we may use proposition 12 (in combination with proposition 10).

Cases \((0, -1, -2), (2, 1, 0)\) and \((-2, -1, 0)\) are simpler, but with a similar strategy.

- In case \((-1, 0, -1) \) (\( i \) even) then \( w(p) = w(p')U_{i-1}w(p'') \) so \( U_iw(p) = w(p')w((i + 1, +2, +1))w(t(p'')) \)
  where \( t \) reflects that part of its argument up to the first \( m = 0 \) in \( m = 0 \). If \( p \) ends at this first zero or beyond we are done. If \( p \) ends before touching zero again then we changed the sign of the end point of the walk (to positive) and we are done by the quotient.

In case \((-1, 0, +1) \) (\( i \) even) every case is covered by some combination of propositions 10 and 11.

If neither of the edges touching the \( i^{th} \) vertex touches \( m = 0 \), then:
\begin{itemize}
\item If 
\[ p = p'(i-1,m)(i,m \uparrow 1)(i+1,m)p'' \]
then \( w(p) = w(p')U_1 \ldots U_1w(p'') \) so 
\[ U_1w(p) = [2]w(p). \]

If \( p = p'(i-1,m)(i,m \downarrow 1)(i+1,m)p'' \) then \( w(p) = w(p')U_{i-1} \ldots U_1w(p'') \) so 
\[ U_iw(p) = w(p'(i-1,m)(i,m \uparrow 1)(i+1,m)p''). \]

If \( p = p'(i-1,m \downarrow 2)(i,m \downarrow 1)(i+1,m)p'' \) then \( w(p) = w(p')w(p'') \). If \( p'' \) never turns over then \( w(p'') = 1 \) and \( U_iw(p) \in b_1 \prod_j U_j \) by proposition 6. If \( p'' \) turns over at \( j > i \) then \( w(p'') = U_jU_{j-1} \ldots U_{i+1}U_i \ldots U_1w(p'') \) so 
\[ U_iw(p) = (U_jU_{j-1} \ldots U_{i+2})w(p')U_iU_{i-1} \ldots U_1w(p'') \] 
and 
\[ w(p')U_iU_{i-1} \ldots U_1w(p'') = w(p'(i-1,m \downarrow 2)(i,m \downarrow 1)(i+1,m \downarrow 2)) \ldots . \]

We may partially order the set of walks which are identical except in some interval where neither crosses \( m = 0 \) (and hence the sign of \( m \) never changes) by \( p \geq p' \) if \( |m| \geq |m'| \) throughout this interval. We have converted the action of \( U_i \) on \( w(p) \) in the case above to an action on a lower walk. We will return to this case shortly.

If \( p = p'(i-1,m \uparrow 2)(i,m \uparrow 1)(i+1,m)p'' \) then \( w(p) = w(p')U_{i-1} \ldots U_iU_i \ldots U_1w(p'') \) so \( U_iw(p) \) may be reduced to the action of a string of \( U_j \)'s (depending on \( p' \)) on a lower walk, in a manner analogous to the case above.

We have converted the action of \( U_i \) on \( w(p) \) in each of the two cases above to an action on a lower walk, thus we may apply an induction with one of the \( m \) touching zero cases as base.
\end{itemize}

\[ 1 \]

**Definition 6** For \( a \in b_n \) let \( a^\circ \) denote the image under the opposite isomorphism. Let \( S^2(\mathbb{n},m) \) denote the set of words \( \{a(E_m+1)b^\circ \mid a,b \in s(n,m)\} \) (here if \( m \leq 0 \) then \( m+ \) means \( -m \)).

For example, reading from figure \[ 1 \] we have
\[ S_{(3,1)} = \{U_1eU_2U_1,e(U_1eU_2U_1),U_2(U_1eU_2U_1)\} \]
so
\[ S^2_{(3,1)} = \{ U_1eU_2U_1, \quad eU_1eU_2U_1, \quad eU_2U_1, \]
\[ U_1eU_2, \quad eU_1eU_2, \quad eU_2 \} \]

By Proposition 8 every \( S_{(n,m)} \) contains an element invariant under the opposite isomorphism \( (U_1eU_2U_1 \text{ in our example}) \) so, noting that the above argument works analogously for right ideals, from the definition and the last proposition we have
Proposition 14 $S_{(n,m)}^2$ spans the double sided ideal it generates, modulo the double sided ideals generated by all $S_{(n,m')}^2$ with $|m'| < |m|$ and $m' = -m$ if $m < 0$. \(\square\)

Since $S_{(n,-n)}^2 = \{1\}$ we have that

$$S_{(n)}^2 := \bigcup_m S_{(n,m)}^2$$

(26)

spans $b_n$.

Finally, in the next section, we show linear independence.

5 Modules, bases and diagrams

A blob diagram is a Temperley–Lieb diagram in which any line which may be deformed isotopically to touch the western edge of the frame may be decorated with a ‘blob’. The set of such diagrams with $n$ vertices on each of the northern and southern edge is denoted $B_n$. Two diagrams $d_1, d_2 \in B_n$ are ‘concatenated’ by a juxtaposition which identifies each southern vertex of $d_1$ with a northern vertex of $d_2$ in the natural way. The blob (diagram) algebra, here denoted $b'_n$, is defined in [16]. In short, $b'_n$ has basis $B_n$, with composition on this basis defined as follows. The concatenation of $a, b \in B_n$ gives another diagram, except that this may contain some extra features. We interpret this pseudodiagram in $b'_n$ as follows. Firstly each internalised vertex is ignored. Each undecorated loop is removed and interpreted as a scalar factor $\delta$; each occurrence of a second blob on a given line is removed and interpreted (in our implementation) as a scalar factor $\delta e$; each decorated loop is then removed and interpreted as a scalar factor $\gamma$. (Note that after these removals the diagram will again lie in $B_n$.)

The map $\phi : b_n \rightarrow b'_n$ is given by

$U_i \mapsto$

\[\begin{array}{c}
\includegraphics{blob_diagram1}
\end{array}\]

$e \mapsto$

\[\begin{array}{c}
\includegraphics{blob_diagram2}
\end{array}\]

Proposition 15 The map $\phi : b_n \rightarrow b'_n$ is a surjective algebra homomorphism.

Proof: That this is an algebra homomorphism follows from a straightforward check of the relations. For surjectivity, compare the image of $U_{i+1}$ with the usual rule for constructing half–diagrams using the Pascal triangle [16]. (Alternatively, just note that the images of the generators generate $b'_n$.) \(\square\)

Since the degree of $S_{(n)}^2$ coincides with the rank of $b'_n$, the surjectivity of the map $b_n \rightarrow b'_n$ implies

Proposition 16 For any $n$, $\phi$ defines an isomorphism $b_n \cong b'_n$. \(\square\)
And hence

**Proposition 17** Every spanning set constructed in propositions 13 and 14, and (26), is a basis for the corresponding module. □

**Acknowledgement.** I thank R J Marsh for useful comments, and RJM and A E Parker for encouraging me to make these notes available.

**Appendix**

A.1 A compendium of related combinatorial facts

Consider the ‘diamond’ grid of side length \(n\) (that is, the square grid with \(n+1\times n+1\) vertices, oriented at 45°). Let \(T_n\) denote the set of right-stepping walks from the set of vertices on the centre vertical of this grid to the rightmost vertex (in bijection with the set of walks from leftmost to rightmost which are symmetric about the centre vertical).

There is an obvious bijection between \(T_n\) and \(S_n\) got by rotating through 90°. We wish to construct a different bijection. For \(p \in S_n\), parse the edge sequence \(\sigma(p)\) to a sequence \(\pi(p)\) of elements from the set \(\{N, S\}\) as follows: reading \(\sigma(p)\) from left to right,

- if \(|\sigma(p)_i| < |\sigma(p)_{i-1}|\) then \(\pi(p)_i = S\);
- if \((\sigma(p)_{i-1}, \sigma(p)_i) \neq (0, 1)\) and \(|\sigma(p)_i| > |\sigma(p)_{i-1}|\) then \(\pi(p)_i = N\);
- if \((\sigma(p)_{i-1}, \sigma(p)_i) = (0, 1)\) then \(\pi(p)_i = S\).

For example \(\pi((0, 1, 0)) = (S, S)\). Each sequence \(\pi(p)\) encodes a walk in \(T_n\) by regarding \(N\) as a northeast step and \(S\) as a southeast step, with the starting point determined by the requirement that the finishing point is fixed for all walks. Some examples are shown in figures 2 and 3.

Let ‘heights’ on the diamond grid be measured from the lowest vertex (height 0). Define a poset \((T_n, \geq)\) by \(t \geq t'\) if, reading from left to right, at each point the height of \(t\) is \(\geq\) the height of \(t'\). (We will induce a poset \((S_n, \geq)\) from this, via \(\pi^{-1}\).)

**Proposition 18** \(\pi\) is a bijection. \(\pi(S_{n,m})\) is the subset of \(T_n\) containing those walks whose lowest point is at height \(n - |m| + \frac{m+|m|}{2m}\).

The proof is elementary.

Define a map \(W : T_n \rightarrow b_n\) as follows. Draw the walk \(t\) together with the maximal walk with lowest point \(n - |m|\) \((m\) taken from \(t = \pi(p)\) as above). In each small diamond in the envelope so created write the generator corresponding to that position (writing \(es\) in the centre ‘half–diamonds’). Now read off \(W(t)\) from this picture from left to right, top to bottom. The walks from the example in figure 2 are shown arranged in the partial order (top to bottom) in figure 4. The lower envelopes and indices for the relevant \(U_i\)s have also been drawn (with \(U_0 = e\)). (A bigger example, with \(n = 6\), is given in figure 5.)
Figure 2: $\pi : S_{4,0} \leftrightarrow T_4$.

Figure 3: $\pi : S_{4,\pm 2} \leftrightarrow T_4$. 
Figure 4: $\pi(S_{4,0})$ arranged in poset order. For the topmost case $w(p) = W(\pi(p)) = eU_1eU_2U_1U_3$. 

16
Figure 5: $\pi(S_{6,0})$ arranged in poset order. For the topmost case $w(p) = W(\pi(p)) = eU_1eU_2U_1U_3eU_2U_4U_1U_3U_5$. 
Proposition 19 \( W(\pi(p)) = w(p) \)

The proof is elementary.

A.2 Word basis: variant form

The variant form of the word set on the Pascal triangle (from the end of section 3.2) is shown in figure 6.

References

[1] R J Baxter, *Exactly solved models in statistical mechanics*, Academic Press, New York, 1982.

[2] G M Bergman, *The diamond lemma for ring theory*, Adv. Math. 29 (1978), 178–218.

[3] A G Cox, J J Graham, and P P Martin, *The blob algebra in positive characteristic*, J Algebra 266 (2003), 584–635.

[4] J de Gier, *Loops, matchings and alternating-sign matrices*, 14th International Conference on Formal Power Series and Algebraic Combinatorics (Melbourne 2002), math.CO/0211285 (2002).

[5] J de Gier and P Pyatov, *Bethe ansatz for the Temperley-Lieb loop model with open boundaries*, hep-th/0312235 (2003).

[6] A Doikou and P P Martin, *Hecke algebraic approach to the reflection equation for spin chains*, J Phys A 36 (2003), 2203–2225, hep-th/0206076.

[7] F M Goodman, P de la Harpe, and V F R Jones, *Coxeter graphs and towers of algebras*, Math Sci Research Inst Publications 14, Springer–Verlag, Berlin, 1989.

[8] J J Graham, *Modular representations of Hecke algebras and related algebras*, Ph.D. thesis, Mathematics, University of Sydney, 1995.

[9] J. J. Graham and G. I. Lehrer, *Diagram algebras, Hecke algebras and decomposition numbers at roots of unity*, Annales Scientifiques de l’École Normale Supérieure 36 (2003), no. 4, 479–524.

[10] M Henkel, *Conformal invariance and critical phenomena*, Texts and monographs in Physics, Springer, 1999.

[11] G D James and A Kerber, *The representation theory of the symmetric group*, Addison-Wesley, London, 1981.

[12] L H Kauffman, *Knots and physics*, World Scientific, Singapore, 1991.
Figure 6: Words associated to edges, and hence paths, on the Pascal triangle. Unlabelled edges have $w(\text{edge}) = 1$. 
[13] P P Martin, *Potts models and related problems in statistical mechanics*, World Scientific, Singapore, 1991.

[14] P P Martin and S Ryom-Hansen, *Virtual algebraic Lie theory: Tilting modules and Ringel duals for blob algebras*, Proc LMS 89 (2004), 655–675, (math.RT/0210063).

[15] P P Martin and H Saleur, *On an algebraic approach to higher dimensional statistical mechanics*, Commun. Math. Phys. 158 (1993), 155–190.

[16] ——, *The blob algebra and the periodic Temperley–Lieb algebra*, Lett. Math. Phys. 30 (1994), 189–206, (hep-th/9302094).

[17] P P Martin and D Woodcock, *On the structure of the blob algebra*, J Algebra 225 (2000), 957–988.

[18] ——, *Generalized blob algebras and alcove geometry*, LMS J Comput Math 6 (2003), 249–296, (math.RT/0205263).

[19] A Nichols, V Rittenberg, and J de Gier, *One-boundary Temperley–Lieb algebras in the XXZ and loop models*, J Stat (2005, to appear), cond-mat/0411512.

[20] H N V Temperley and E H Lieb, *Relations between percolation and colouring problems and other graph theoretical problems associated with regular planar lattices: some exact results for the percolation problem*, Proceedings of the Royal Society A 322 (1971), 251–280.