THE COHOMOLOGY OF $C_2$-EQUIVARIANT $A(1)$ AND THE HOMOTOPY OF $ko_{C_2}$

B. J. GUILLOU, M. A. HILL, D. C. ISAKSEN, AND D. C. RAVENEL

Abstract. We compute the cohomology of the subalgebra $A_{C_2}(1)$ of the $C_2$-equivariant Steenrod algebra $A_{C_2}$. This serves as the input to the $C_2$-equivariant Adams spectral sequence converging to the $RO(C_2)$-graded homotopy groups of an equivariant spectrum $ko_{C_2}$. Our approach is to use simpler $C$-motivic and $R$-motivic calculations as stepping stones.

1. Introduction

The $RO(G)$-graded homotopy groups are among the most fundamental invariants of the stable $G$-equivariant homotopy category. This article is a first step towards systematic application of the equivariant Adams spectral sequence to calculate these groups.

Araki and Iriye [AI] [Ir] computed much information about the $C_2$-equivariant stable homotopy groups using EHP-style techniques in the spirit of Toda [T]. Our approach is entirely independent from theirs.

We work only with the two-element group $C_2$ because it is the most elementary non-trivial case. In order to compute $C_2$-equivariant stable homotopy groups of the $C_2$-equivariant sphere spectrum, one needs to work with the full $C_2$-equivariant Steenrod algebra $A_{C_2}$ for the constant Mackey functor $F_2$. In this article, we tackle a computationally simpler situation by working over the subalgebra $A_{C_2}(1)$. This means that we are computing the $C_2$-equivariant stable homotopy groups not of the sphere but of the $C_2$-equivariant analogue of connective real $K$-theory $ko$. We will explicitly construct this $C_2$-equivariant spectrum in Section 10.

Our calculational program is carried out for $A_{C_2}(1)$ in this article as a warmup for the full Steenrod algebra $A$ to be studied in future work. Roughly speaking, $A$ contains Steenrod squaring operations $Sq^i$ with the expected properties, and $A_{C_2}(1)$ is the subalgebra generated by $Sq^1$ and $Sq^2$. A key point is that our program works just as well for $A_{C_2}$ as for $A_{C_2}(1)$, except that the details are even more complicated.

Our strategy is to build up to the complexity of the $C_2$-equivariant situation by first studying the $C$-motivic and $R$-motivic situations. The $C$-motivic cohomology of a point is equal to $F_2[\tau]$ [V1]. The $C$-motivic Steenrod algebra $A_C$ is very similar to the classical Steenrod algebra, but there are some small complications related to $\tau$. In particular, these complications allow the element $h_1$ in the cohomology of $A_C$ to be non-nilpotent. In the cohomology of $A_C(1)$, the non-nilpotence of $h_1$ is essentially the only difference to the classical case.
The \( \mathbb{R} \)-motivic cohomology of a point is equal to \( \mathbb{F}_2[\tau, \rho] \) \([V1]\). Now an additional complication enters because \( \text{Sq}^1(\tau) = \rho \). The computation of the cohomology of the \( \mathbb{R} \)-motivic Steenrod algebra \( A^{\mathbb{R}} \) becomes more difficult because the cohomology of a point is a non-trivial \( A^{\mathbb{R}} \)-module. In addition, the \( \mathbb{R} \)-motivic Steenrod algebra \( A^{\mathbb{R}} \) has additional complications associated with terms involving higher powers of \( \rho \) \([V2]\).

A natural way to avoid this problem is to filter by powers of \( \rho \). In the associated graded object, \( \text{Sq}^1(\tau) \) becomes zero. Therefore, the \( \rho \)-Bockstein spectral sequence starts from the cohomology of \( A^{\mathbb{C}} \) and converges to the cohomology of \( A^{\mathbb{R}} \).

This \( \rho \)-Bockstein spectral sequence has lots of differentials and hidden extensions. Nevertheless, a complete calculation for \( A^{\mathbb{R}}(1) \) is reasonable. A key point is to first carry out the \( \rho \)-inverted calculation. This turns out to be much simpler. With a priori knowledge of the \( \rho \)-inverted calculation in hand, there is just one possible pattern of \( \rho \)-Bockstein differentials.

Relying on our experience from the \( \mathbb{R} \)-motivic situation, we are now ready to tackle the \( C_2 \)-equivariant situation. The \( C_2 \)-equivariant cohomology of a point contains \( \mathbb{F}_2[\tau, \rho] \), but there is an additional “negative cone” that is infinitely divisible by both \( \tau \) and \( \rho \) \([HK]\). Except for the complications in the cohomology of a point, the \( C_2 \)-equivariant Steenrod algebra \( A^{C_2} \) is no more complicated than the \( \mathbb{R} \)-motivic one \([HK]\).

Again, a \( \rho \)-Bockstein spectral sequence allows us to compute the cohomology of \( A^{C_2}(1) \). Because of infinite \( \tau \)-divisibility, the starting point of the spectral sequence is more complicated than just the cohomology of \( A^{C_2}(1) \). Once identified, this issue presents only a minor difficulty.

The \( \rho \)-inverted calculation determines the part of the cohomology of \( A^{C_2}(1) \) that supports infinitely many \( \rho \) multiplications. Dually, it is also helpful to determine in advance the part of the cohomology of \( A^{C_2}(1) \) that is infinitely \( \rho \)-divisible, i.e., the inverse limit of an infinite tower of \( \rho \)-multiplications. We anticipate that this approach via infinitely \( \rho \)-divisible classes will be essential in the more complicated calculation over the full Steenrod algebra \( A^{C_2} \), to be studied in future work.

As for the \( \mathbb{R} \)-motivic case, the \( \rho \)-Bockstein spectral sequence is manageable, even though it does have lots of differentials and hidden extensions.

All of these calculations lead to a thorough understanding of the cohomology of \( A^{C_2}(1) \). The charts in Section 12 display the calculation graphically.

The next step is to consider the \( C_2 \)-equivariant Adams spectral sequence. For degree reasons, there are no Adams differentials. The same simple situation occurs in the classical, \( \mathbb{C} \)-motivic, and \( \mathbb{R} \)-motivic cases.

However, it turns out that there are many hidden extensions to be analyzed. The presence of so many hidden extensions suggests that the Adams filtration may not be optimal for equivariant purposes. Unfortunately, we do not have an alternative to propose.

The final description of the homotopy groups is complicated. We refer to Section 11 and the charts in Section 12 for details.

1.1. Organization. In Section 2, we provide the basic algebraic input to our calculation by thoroughly describing the \( C_2 \)-equivariant cohomology of a point and the \( C_2 \)-equivariant Steenrod algebra \( A^{C_2} \). In Section 3, we set up the \( \rho \)-Bockstein spectral sequence, which is our main tool for computing the cohomology of \( A^{C_2}(1) \).
In Sections 4 and 5, we carry out the \(\rho\)-inverted and the infinitely \(\rho\)-divisible calculations. In Section 6, we carry out the \(\mathbb{R}\)-motivic \(\rho\)-Bockstein spectral sequence, as a warmup for the \(C_2\)-equivariant \(\rho\)-Bockstein spectral sequence in Section 7. Section 8 provides some information about Massey products in the \(C_2\)-equivariant cohomology of \(\mathcal{A}(1)\), which is used in Section 9 to determine multiplicative structure that is hidden by the \(\rho\)-Bockstein spectral sequence. Section 10 gives the construction of the \(C_2\)-equivariant spectrum whose homotopy groups are computed by the cohomology of \(\mathcal{A}^{C_2}(1)\), and Section 11 analyzes multiplicative structure in these homotopy groups that is hidden by the Adams spectral sequence. Finally, Section 12 includes a series of charts that graphically describe our calculation.

1.2. Notation. We employ notation as follows:

1. \(M^\text{cl}_2 = \mathbb{F}_2[\tau]\) is the motivic cohomology of \(\mathbb{C}\) with \(\mathbb{F}_2\) coefficients, where \(\tau\) has bidegree \((0, 1)\).

2. \(M^\text{cl}_2 = \mathbb{F}_2[\tau, \rho]\) is the motivic cohomology of \(\mathbb{R}\) with \(\mathbb{F}_2\) coefficients, where \(\tau\) and \(\rho\) have bidegrees \((0, 1)\) and \((1, 1)\), respectively.

3. \(M^{C_2}_2\) is the bigraded equivariant cohomology of a point with coefficients in the constant Mackey functor \(\mathbb{F}_2\). See Section 2.1 for a description of this algebra.

4. \(NC\) is the “negative cone” part of \(M^{C_2}_2\). See Section 2.1 for a precise description.

5. \(H^*_{C_2}(X)\) is the \(C_2\)-equivariant cohomology of \(X\), with coefficients in the constant Mackey functor \(\mathbb{F}_2\).

6. \(\mathcal{A}^{cl}, \mathcal{A}^c, \mathcal{A}^{\mathbb{R}}, \text{ and } \mathcal{A}^{C_2}\) are the classical, \(C\)-motivic, \(\mathbb{R}\)-motivic, and \(C_2\)-equivariant \(\mathbb{R}\)=Bockstein spectral sequences.

7. \(\mathcal{A}^{cl}(n), \mathcal{A}^c(n), \mathcal{A}^{\mathbb{R}}(n), \text{ and } \mathcal{A}^{C_2}(n)\) are the classical, \(C\)-motivic, \(\mathbb{R}\)-motivic, and \(C_2\)-equivariant subalgebras generated by \(Sq^1, Sq^2, Sq^4, \ldots, Sq^{2^n}\).

8. \(\mathcal{E}^{C_2}(1)\) is the subalgebra of \(\mathcal{A}^{C_2}\) generated by \(Q_0 = Sq^1\) and \(Q_1 = Sq^1Sq^2 + Sq^2Sq^1\).

9. \(\text{Ext}_{cl}\) is the bigraded ring \(\text{Ext}_{\mathcal{A}^{cl}}(\mathbb{F}_2, \mathbb{F}_2)\), i.e., the cohomology of \(\mathcal{A}^{cl}\).

10. \(\text{Ext}_{C}\) is the trigraded ring \(\text{Ext}_{\mathcal{A}^c}(M^C_2, M^C_2)\), i.e., the cohomology of \(\mathcal{A}^{C}\).

11. \(\text{Ext}_R\) is the trigraded ring \(\text{Ext}_{\mathcal{A}^{\mathbb{R}}}(M^{\mathbb{R}}_2, M^{\mathbb{R}}_2)\), i.e., the cohomology of \(\mathcal{A}^{\mathbb{R}}\).

12. \(\text{Ext}_{C_2}\) is the trigraded ring \(\text{Ext}_{\mathcal{A}^{C_2}}(M^{C_2}_2, M^{C_2}_2)\), i.e., the cohomology of \(\mathcal{A}^{C_2}\).

13. \(\text{Ext}_{NC}\) is the \(\mathcal{A}^{cl}\)-module \(\text{Ext}_{\mathcal{A}^{cl}}(NC, M^C_2)\).

14. \(\text{Ext}_{cl}(n)\) is the bigraded ring \(\text{Ext}_{\mathcal{A}^{cl}(n)}(\mathbb{F}_2, \mathbb{F}_2)\), i.e., the cohomology of \(\mathcal{A}^{cl}(n)\).

15. \(\text{Ext}_{C}(n)\) is the trigraded ring \(\text{Ext}_{\mathcal{A}^c(n)}(M^C_2, M^C_2)\), i.e., the cohomology of \(\mathcal{A}^{C}(n)\).

16. \(\text{Ext}_{\mathbb{R}}(n)\) is the trigraded ring \(\text{Ext}_{\mathcal{A}^{\mathbb{R}}(n)}(M^{\mathbb{R}}_2, M^{\mathbb{R}}_2)\), i.e., the cohomology of \(\mathcal{A}^{\mathbb{R}}(n)\).

17. \(\text{Ext}_{C_2}(n)\) is the trigraded ring \(\text{Ext}_{\mathcal{A}^{C_2}(n)}(M^{C_2}_2, M^{C_2}_2)\), i.e., the cohomology of \(\mathcal{A}^{C_2}(n)\).

18. \(\text{Ext}_{NC}(n)\) is the \(\mathcal{A}^{cl}(n)\)-module \(\text{Ext}_{\mathcal{A}^{cl}(n)}(NC, M^C_2)\).

19. \(E^+\) is the \(\rho\)-Bockstein spectral sequence

\[\text{Ext}_{C}(1)[\rho] \Rightarrow \text{Ext}_{\mathbb{R}}(1)\]

20. \(E^-\) is the \(\rho\)-Bockstein spectral sequence that converges to \(\text{Ext}_{NC}(1)\).
(21) $\frac{F_2[x]}{x^n}$ is the infinitely $x$-divisible module $\text{colim}_n F_2[x]/x^n$, consisting of elements of the form $\frac{1}{x^n}$ for $k \geq 1$. See Remark 2.1.

(22) $koC_2$ is a $C_2$-equivariant spectrum such that $H_{*,*}^*(koC_2) \cong A^{C_2}/A^{C_2}(1)$. See Section 10.

(23) $\pi_{*,*}(X)$ are the bigraded $C_2$-equivariant stable homotopy groups of $X$, appropriately completed so that the equivariant Adams spectral sequence converges.

(24) $\Pi_n(X)$ is the Milnor-Witt $n$-stem $\bigoplus_{p} \pi_{p+n,p}$.

We use grading conventions that are common in motivic homotopy theory but less common in equivariant homotopy theory. In equivariant homotopy theory, $RO(C_2) \cong \mathbb{Z}[\sigma]/\sigma^2 - 1$ is the real representation ring of $C_2$, where $\sigma$ is the 1-dimensional sign representation. The main points of translation are:

1. Equivariant degree $p + q\sigma$ corresponds to motivic degree $(p + q, q)$, where $p + q$ is the total degree and $q$ is the weight.
2. Equivariant $u$ corresponds to motivic $\tau$.
3. Equivariant $a$ corresponds to motivic $\rho$.

We grade Ext groups in the form $(s, f, w)$, where $s$ is the stem, i.e., the total degree minus the homological degree; $f$ is the Adams filtration, i.e., the homological degree; and $w$ is the weight. We will also refer to the Milnor-Witt degree, which equals $s - w$.

2. Ext groups

2.1. The equivariant cohomology of a point. The purpose of this section is to carefully describe the structure of the equivariant cohomology ring $M_{C_2}^{C_2}$ of a point from a perspective that will be useful for our calculations. This section is a reinterpretation of results from [HK, Proposition 6.2].

Additively, $M_{C_2}^{C_2}$ equals

1. $F_2$ in degree $(s, w)$ if $s \geq 0$ and $w \geq s$.
2. $F_2$ in degree $(s, w)$ if $s \leq 0$ and $w \leq s - 2$.
3. 0 otherwise.

This additive structure is represented by the dots in Figure 1. The non-zero element in degree $(0, 1)$ is called $\tau$, and the non-zero element in degree $(1, 1)$ is called $\rho$. We remind the reader that we are here employing cohomological grading. Thus the class $\rho$ has degree $(-1, -1)$ when considered as an element of the homology ring $\pi_{*,*}H_{C_2}^F$.

The “positive cone” refers to the part of $M_{C_2}^{C_2}$ in degrees $(s, w)$ with $w \geq 0$. The positive cone is isomorphic to the $\mathbb{R}$-motivic cohomology ring $M_{C_2}^{R}$ of a point. Multiplicatively, the positive cone is just a polynomial ring on two variables $\rho$ and $\tau$.

The “negative cone” $NC$ refers to the part of $M_{C_2}^{C_2}$ in degrees $(s, w)$ with $w \leq -2$. Multiplicatively, the product of any two elements of $NC$ is zero, so $M_{C_2}^{C_2}$ is a square-zero extension of $M_{C_2}^{R}$. Also, multiplications by $\rho$ and $\tau$ are non-zero in $NC$ whenever they make sense. Thus, the elements of $NC$ are infinitely divisible by both $\rho$ and $\tau$. 
We use the notation $\theta^s \rho^k \tau^j$ for the non-zero element in degree $(-j+1, -j-k)$. This is consistent with the multiplicative properties described in the previous paragraph. So $\tau \cdot \theta^s \rho^k \tau^j$ equals $\theta^s \rho^k \tau^{j+1}$ when $k \geq 2$, and $\rho \cdot \theta^s \rho^k \tau^j$ equals $\theta^{s+1} \rho^k \tau^j$ when $j \geq 2$.

The symbol $\theta$, which does not correspond to an actual element of $M_{C_2}^2$, has degree $(1, 0)$. The $F_2[\tau]$-module structure on $M_{C_2}^2$ is essential for later calculations, since we will filter by powers of $\rho$. Therefore, we explore further the $F_2[\tau]$-module structure on $NC$.

**Remark 2.1.** Recall that $F_2[\tau]/\tau^\infty$ is the $F_2[\tau]$-module colim $F_2[\tau]/\tau^k$, which consists entirely of elements that are divisible by $\tau$. We write $\Phi_{2[\tau]}(x)$ for the infinitely divisible $F_2[\tau]$-module consisting of elements of the form $\frac{\Phi_{2[\tau]}}{\tau^k}$ for $k \geq 1$. Note that $x$ itself is not an element of $\Phi_{2[\tau]}(x)$. The idea is that $x$ represents the infinitely many relations $\tau^k \cdot \frac{\Phi_{2[\tau]}}{\tau^k} = 0$ that define $\Phi_{2[\tau]}(x)$.

With this notation in place, $M_{C_2}^2$ is equal to

$$M_{R_2}^R \oplus NC = M_{R_2}^R \oplus \bigoplus_{s \geq 1} \frac{F_2[\tau]}{\tau^\infty} \left\{ \frac{\theta^s}{\rho^s} \right\}$$

as an $F_2[\tau]$-module.

### 2.2. The equivariant Steenrod algebra

As a Hopf algebroid, the equivariant dual Steenrod algebra can be described [R, Proposition 6.10(2)] as

$$A_{C_2}^C \cong M_{C_2}^C \otimes_{M_{C_2}^R} A_{C_2}^R.$$
Recall [V2] that
\[ \mathcal{A}_R^s \cong \mathbb{M}_2^R[\tau_0, \tau_1, \ldots, \xi_0, \xi_1, \ldots]/(\xi_0 = 1, \tau_i^2 = \rho \tau_{i+1} + \tau \xi_{i+1} + \rho \tau_0 \xi_{i+1}), \]
with \( \eta_R(\rho) = \rho \) and \( \eta_R(\tau) = \tau + \rho \tau_0 \). The formula for the right unit \( \eta_R \) on the negative cone given in [HK, Theorem 6.41] appears in our notation as
\[
\eta_R \left( \frac{\theta}{\rho^j \tau^k} \right) = \frac{\theta}{\rho^j \tau^k} \left[ \sum_{l \geq 0} \left( \frac{\rho \tau_0}{\tau} \right)^l \right]^k.
\]
Note that the sum is finite because \( \frac{\rho \tau_0}{\tau} \cdot \rho^n = 0 \) if \( n \geq j \).

We have quotient Hopf algebroids
\[ \mathcal{A}_R^s(n) := \mathbb{M}_2^R[\tau_0, \ldots, \tau_n, \xi_1, \ldots, \xi_n]/(\xi_n = 1, \tau_i^2 = \rho \tau_{i+1} + \tau \xi_{i+1} + \rho \tau_0 \xi_{i+1}), \]
and their equivariant analogues
\[ \mathcal{A}_{R_2}^s(n) := \mathcal{M}_2^{C_2} \otimes_{\mathcal{M}_2} \mathcal{A}_s(n), \quad \mathcal{E}_{R_2}^s(n) := \mathcal{M}_2^{C_2} \otimes_{\mathcal{M}_R} \mathbb{E}_s^R(n) \]
Their duals define subalgebras \( \mathcal{A}_{C_2}^s(n) \subseteq \mathcal{A}_{C_2}^s \) and \( \mathcal{E}_{C_2}^s(n) \subseteq \mathcal{E}_{C_2}^s \).

The relationship between the equivariant and \( \mathbb{R} \)-motivic Steenrod algebras leads to an analogous relationship between Ext groups. For convenience, we write \( \text{Ext}_{C_2} \) and \( \text{Ext}_R \) for \( \text{Ext}_{C_2}(\mathcal{M}_2^{C_2}, \mathcal{M}_2^{C_2}) \) and \( \text{Ext}_R(\mathcal{M}_2^R, \mathcal{M}_2^R) \) respectively. We also write \( \text{Ext}_{NC} \) for \( \text{Ext}_{\mathbb{A}^s}(NC, \mathcal{M}_2^R) \).

**Proposition 2.2.** The algebra \( \text{Ext}_{\mathcal{A}_{C_2}} \) is a square-zero extension of \( \text{Ext}_{\mathcal{A}_s} \). In particular,

\[ \text{Ext}_{\mathcal{A}_{C_2}} \cong \text{Ext}_{\mathcal{A}_s} \oplus \text{Ext}_{NC} \]
as \( \text{Ext}_{\mathcal{A}_s} \)-modules.

**Proof.** The extension (2.2) allows us to express the cobar complex as:
\[
\text{coB}^*(\mathcal{M}_2^{C_2}, \mathcal{A}_s^{C_1}) = \mathcal{M}_2^{C_2} \otimes_{\mathcal{M}_2} (\mathcal{A}_s^{C_1})^s \cong \mathcal{M}_2^{C_2} \otimes_{\mathcal{M}_2} (\mathcal{M}_2^{C_2} \otimes_{\mathcal{M}_2} \mathcal{A}_s^R)^s \cong \mathcal{M}_2^{C_2} \otimes_{\mathcal{M}_2} (\mathcal{A}_s^R)^s.
\]
The splitting (2.1) is a splitting as \( \mathcal{A}_s^R \)-comodules, so that there results a splitting
\[
\text{coB}^*(\mathcal{M}_2^R, \mathcal{A}_s^{C_2}) \oplus \text{coB}^*(NC, \mathcal{A}_s^{C_2})
\]
of the cobar complex. This splitting is square-zero, in the sense that the product of any two elements in the second factor is equal to zero. This observation follows from the fact that the product of any two elements of \( NC \) is zero.

In \( \text{Ext}_{\mathcal{A}_{C_2}} \), this yields
\[ \text{Ext}_{\mathcal{A}_{C_2}} \cong \text{Ext}_{\mathcal{A}_s} \oplus \text{Ext}_{NC} \]
The square-zero property of \( \text{Ext}_{\mathcal{A}_{C_2}} \) comes from the square-zero property of the splitting of the cobar complex.

Thus from the point of view of \( \mathbb{R} \)-motivic homotopy theory, the cohomology of the negative cone is the only new feature in \( \text{Ext}_{\mathcal{A}_{C_2}} \). The same decomposition holds when computing Ext over the subalgebras \( \mathcal{A}_{C_2}^s(n) \) of \( \mathcal{A}_{C_2}^s \).
3. The $\rho$-Bockstein spectral sequence

Our tool for computing $\mathbb{R}$-motivic or $C_2$-equivariant Ext is the $\rho$-Bockstein spectral sequence [H, DI2]. The $\rho$-Bockstein spectral sequence arises by filtering the cobar complex by powers of $\rho$. More precisely, we can define an $A^\mathbb{R}$-module filtration on $M_{C_2}^\mathbb{C}$, where $F_p(M_{C_2}^\mathbb{C})$ is the part of $M_{C_2}^\mathbb{C}$ concentrated in degrees $(s, w)$ with $s \geq p$. Dualizing, we get a filtration of comodules over the dual Steenrod algebra, which induces a filtration on the cobar complex that computes $\text{Ext}_{C_2}$.

Recall that the $C_2$-motivic cohomology of a point is $M_{C_2}^\mathbb{C} = F_2[\tau]$, and the $C_2$-motivic Steenrod algebra is $A^\mathbb{C} = A^\mathbb{R}/[V_1] [V_2]$. For convenience, we write $\text{Ext}_{C}$ for $\text{Ext}_{A^\mathbb{C}}(M_{C_2}^\mathbb{C}, M_{C_2}^\mathbb{C})$.

**Proposition 3.1.** There is a $\rho$-Bockstein spectral sequence $E_1 = \text{Ext}_{gr, A_{C_2}}(gr, \rho M_{C_2}^\mathbb{C}, gr, \rho M_{C_2}^\mathbb{C}) \Rightarrow \text{Ext}_{C_2}$.

Under the splitting of Proposition 2.2, this decomposes as

$E_1^+ = \text{Ext}_C[\rho] \Rightarrow \text{Ext}_R$

and

$E_1^- \Rightarrow \text{Ext}_{NC}$,

where $E_1^-$ belongs to a split short exact sequence

$$\bigoplus_{s \geq 1} \frac{M_{C_2}^\mathbb{C}}{\tau^\infty} \left\{ \frac{\theta}{\rho^s} \right\} \otimes_{M_{C_2}^\mathbb{C}} \text{Ext} \rightarrow E_1^- \rightarrow \bigoplus_{s \geq 1} \text{Tor}_{M_{C_2}^\mathbb{C}} \left( \frac{M_{C_2}^\mathbb{C}}{\tau^\infty} \left\{ \frac{\theta}{\rho^s} \right\}, \text{Ext} \right).$$

**Remark 3.2.** Beware that the short exact sequence for $E_1^-$ does not split canonically.

**Remark 3.3.** The same spectral sequences occur in the same form when $A_{C_2}, A^\mathbb{R},$ and $A^\mathbb{C}$ are replaced by $A_{C_2}(1), A^\mathbb{R}(1),$ and $A^\mathbb{C}(1)$.

**Proof.** See [H, Proposition 2.3] (or [DI2, Section 3]) for the description of $E_1^+$. For $E_1^-$, the associated graded of $NC$ is

$$\text{gr}_p NC \cong \bigoplus_{s \geq 1} \frac{M_{C_2}^\mathbb{C}}{\tau^\infty} \left\{ \frac{\theta}{\rho^s} \right\},$$

as described in Section 2.1. It follows that the Bockstein spectral sequence begins with

$$E_0 \cong \bigoplus_{s \geq 1} \frac{M_{C_2}^\mathbb{C}}{\tau^\infty} \left\{ \frac{\theta}{\rho^s} \right\} \otimes_{M_{C_2}^\mathbb{C}} \text{coB}(M_{C_2}^\mathbb{C}, A^\mathbb{C}).$$

The ring $M_{C_2}^\mathbb{C} \cong F_2[\tau]$ is a graded principal ideal domain (in fact, it is a graded local ring with maximal ideal generated by $\tau$). Therefore, the Kunneth split exact sequence gives

$$\left( \bigoplus_{s \geq 1} \frac{M_{C_2}^\mathbb{C}}{\tau^\infty} \left\{ \frac{\theta}{\rho^s} \right\} \right) \otimes_{M_{C_2}^\mathbb{C}} \text{Ext} \rightarrow E_1^- \rightarrow \text{Tor}_{M_{C_2}^\mathbb{C}} \left( \bigoplus_{s \geq 1} \frac{M_{C_2}^\mathbb{C}}{\tau^\infty} \left\{ \frac{\theta}{\rho^s} \right\}, \text{Ext} \right).$$

The first and third terms of the short exact sequence may be rewritten as in the statement of the proposition because the direct sum in each case is a splitting of $M_{C_2}^\mathbb{C}$-modules.

We write:
(1) \( \text{Ext}_C(n) \) for \( \text{Ext}_{A^c(n)}(\mathbb{M}_x^C, \mathbb{M}_y^C) \).
(2) \( \text{Ext}_R(n) \) for \( \text{Ext}_{A^p(n)}(\mathbb{M}_x^R, \mathbb{M}_y^R) \).
(3) \( \text{Ext}_{NC}(n) \) for \( \text{Ext}_{A^p(n)}(\mathbb{NC}, \mathbb{M}_y^R) \).

We shall completely analyze the spectral sequence

\[ E_1^+ = \text{Ext}_C(1)[\rho] \Rightarrow \text{Ext}_R(1) \]

in Section 6. While non-trivial, this part of our calculation is comparatively straightforward.

On the other hand, analysis of the spectral sequence \( E_1^- \Rightarrow \text{Ext}_{NC}(1) \) requires significantly more work. The first step is to compute \( E_1^- \) more explicitly. In particular, we must describe the Tor groups that arise.

**Lemma 3.4.** \( \text{Tor}_{\mathbb{M}_z^C} \left( \frac{\mathbb{M}_x^C}{\tau}, \mathbb{M}_y^C \right) \) equals \( \frac{\mathbb{M}_y^C}{\tau} \), concentrated in homological degree zero.

**Proof.** This is a standard fact about the vanishing of higher Tor groups for free modules.

**Lemma 3.5.** \( \text{Tor}_{\mathbb{M}_z^C} \left( \frac{\mathbb{M}_x^C}{\tau}, \mathbb{M}_y^C \right) \) equals \( \frac{\mathbb{M}_y^C}{\tau} \), concentrated in homological degree one.

**Proof.** This follows from direct computation, using the resolution

\[
\frac{\mathbb{M}_x^C}{\tau} \leftarrow \mathbb{M}_y^C \leftarrow \mathbb{M}_z^C \leftarrow 0.
\]

After tensoring with \( \frac{\mathbb{M}_y^C}{\tau} \), this gives the map

\[
\frac{\mathbb{M}_y^C}{\tau} \leftarrow \mathbb{M}_z^C \leftarrow \mathbb{M}_x^C \leftarrow 0.
\]

that takes \( \frac{\mu}{\tau} \) to \( \frac{\nu}{\tau} \) if \( a > k \), and takes \( \frac{\mu}{\tau} \) to zero if \( a \leq k \). This map is onto, and its kernel is isomorphic to \( \mathbb{M}_z^C/\tau^k \).

**Remark 3.6.** Lemmas 3.4 and 3.5 give a practical method for identifying \( E_1^- \) in Proposition 3.1. Copies of \( \mathbb{M}_z^C \) in \( \text{Ext}_C(1) \) lead to copies of the negative cone in \( E_1^- \). On the other hand, copies of \( \mathbb{M}_z^C/\tau \), such as the submodule generated by \( h_1 \), lead to copies of \( \mathbb{M}_z^C/\tau \) in \( E_1^- \) that are infinitely divisible by \( \rho \). These copies of \( \mathbb{M}_z^C/\tau \) occur with a degree shift because they arise from \( \text{Tor}^1 \).

4. \( \rho \)-inverted \( \text{Ext}_R(1) \)

As a first step towards computing \( \text{Ext}_{C^2}(1) \), we will invert \( \rho \) in the \( \mathbb{R} \)-motivic setting and study \( \text{Ext}_R(1)[\rho^{-1}] \). This gives partial information about \( \text{Ext}_R(1) \) and also about \( \text{Ext}_{C^2}(1) \). Afterwards, it remains to compute \( \rho^k \) torsion, including infinitely \( \rho \)-divisible elements.

We write \( A^c \) for the classical Steenrod algebra. For convenience, we write \( \text{Ext}_{c^1} \) and \( \text{Ext}_{c^1}(n) \) for \( \text{Ext}_{A^c}(\mathbb{F}_2, \mathbb{F}_2) \) and \( \text{Ext}_{A^c(n)}(\mathbb{F}_2, \mathbb{F}_2) \) respectively.

**Proposition 4.1.** There is an injection \( \text{Ext}_{c^1}(n - 1)[\rho^{\pm 1}] \hookrightarrow \text{Ext}_R(n)[\rho^{-1}] \) such that:

1. The map is highly structured, i.e., preserves products, Massey products, and algebraic squaring operations.
2. The element \( h_i \) of \( \text{Ext}_{c^1}(n - 1) \) corresponds to \( h_{i+1} \) of \( \text{Ext}_R(n) \).
3. The map induces an isomorphism

\[
\text{Ext}_R(n)[\rho^{-1}] \cong \text{Ext}_{c^1}(n - 1)[\rho^{\pm 1}] \otimes \mathbb{F}_2[\tau_{2^{n+1}}].
\]
(4) An element in \( \text{Ext}_3(n-1) \) of degree \((s,f)\) corresponds to an element in \( \text{Ext}_2(n) \) of degree \((2s+f,f,s+f)\).

**Proof.** The proof is similar to the proof of Theorem 4.1 of [DI2]. Since localization is exact, we may compute the cohomology of the Hopf algebroid \((M_2^2[\rho^{-1}],\mathcal{A}^R(n+1))\) to obtain \( \text{Ext}_2(n+1)[\rho^{-1}] \). After inverting \( \rho \), we have

\[
\tau_{k+1} = \rho^{-1}\tau_k^2 + \rho^{-1}\tau\xi_{k+1} + \tau_0\xi_{k+1},
\]

and it follows that

\[
\mathcal{A}^R(n)[\rho^{-1}] \cong M_2^2[\rho^{-1}][\tau_0,\xi_1, \ldots, \xi_n]/(\tau_0^{2^n+1},\xi_1^2, \ldots, \xi_n^2).
\]

This splits as

\[
(M_2^2[\rho^{-1}],\mathcal{A}(n)[\rho^{-1}]) \cong (M_2^2[\rho^{-1}],\mathcal{A}'(n)) \otimes_{\mathbb{F}_2}(\mathbb{F}_2,\mathcal{A}''(n)),
\]

where

\[
\mathcal{A}'(n) = M_2^2[\rho^{-1}]/\tau_0^{2^n+1}
\]

and

\[
\mathcal{A}''(n) = \mathbb{F}_2[\xi_1, \ldots, \xi_n]/(\xi_1^{2^n}, \ldots, \xi_n^2).
\]

The Hopf algebra \((\mathbb{F}_2,\mathcal{A}''(n))\) is isomorphic to the classical Hopf algebra \((\mathbb{F}_2,\mathcal{A}(n-1))\) with altered degrees, so its cohomology is \( \text{Ext}_3(n-1) \).

For the Hopf algebroid \((M_2^2[\rho^{-1}],\mathcal{A}'(n))\), we have an isomorphism

\[
(M_2^2[\rho^{-1}],\mathcal{A}'(n)) \cong \mathbb{F}_2[\rho^{\pm 1}] \otimes_{\mathbb{F}_2}(\mathbb{F}_2[\tau],\mathbb{F}_2[\tau][x]/x^{2^n+1})
\]

with

\[
\eta_L(\tau) = \tau, \quad \eta_R(\tau) = \tau + x.
\]

An argument like that of [DI2, Lemma 4.2] shows that the cohomology of this Hopf algebroid is \( \mathbb{F}_2[\tau^{2^n+1}] \). □

**Corollary 4.2.**

\[
\text{Ext}_R(1)[\rho^{-1}] \cong \mathbb{F}_2[\rho^{\pm 1},\tau^4,h_1].
\]

**Proof.** This follows immediately from Proposition 4.1, given that \( \text{Ext}_c(0) \cong \mathbb{F}_2[h_0] \). □

**Remark 4.3.** Corollary 4.2 implies that the products \( \tau^4 \cdot h^k_1 \) are non-zero in \( \text{Ext}_R(1) \). But \( \tau^4 h^k_1 = 0 \) in \( \text{Ext}_c(1) \) when \( k \geq 3 \), so the products \( \tau^4 \cdot h^k_1 \) are hidden in the \( \rho \)-Bockstein spectral sequence for \( k \geq 3 \). We will sort this out in detail in Section 6.

5. **Infinitely \( \rho \)-divisible elements of \( \text{Ext}_{C_2}(1) \)**

Having computed the effect of inverting \( \rho \) in Section 4, we now consider the dual question and study infinitely \( \rho \)-divisible elements. This gives additional partial information about \( \text{Ext}_{C_2}(1) \). Afterwards, it remains only to compute the \( \rho^k \) torsion classes that are not infinitely \( \rho \)-divisible.

In fact, this section is not strictly necessary to carry out the computation of \( \text{Ext}_{C_2}(1) \). Nevertheless, the infinitely \( \rho \)-divisible calculation works out rather nicely and provides some useful insight into the main computation. We also anticipate that this approach via infinitely \( \rho \)-divisible classes will be essential in the much more complicated calculation of \( \text{Ext}_{C_2} \), to be studied in further work.
For a $\mathbb{F}_2[\rho]$-module $M$, the $\rho$-colocalization, or $\rho$-cellularization, is the limit $\lim_\rho M$ of the inverse system

$$\cdots \xrightarrow{\rho} M \xrightarrow{\rho} M.$$ 

While $\rho$-localization detects $\rho$-torsion-free elements, the $\rho$-colocalization detects infinitely $\rho$-divisible elements.

An alternative description is given by the isomorphism

$$\lim_\rho M \cong \text{Hom}_{\mathbb{F}_2[\rho]}(\mathbb{F}_2[\rho^{\pm 1}], M)$$

because $\mathbb{F}_2[\rho^{\pm 1}]$ is isomorphic to $\text{colim}_\rho \mathbb{F}_2[\rho]$. It follows that $\lim_\rho M$ is an $\mathbb{F}_2[\rho^{\pm 1}]$-module, and the functor $M \mapsto \lim_\rho M$ is right adjoint to the restriction

$$\text{Mod}_{\mathbb{F}_2[\rho^{\pm 1}]} \to \text{Mod}_{\mathbb{F}_2[\rho]}.$$

**Lemma 5.1.**

1. Let $M$ be a cyclic $\mathbb{F}_2[\rho]$-module $\mathbb{F}_2[\rho]$ or $\mathbb{F}_2[\rho]/\rho^k$. Then $\lim_\rho M$ is zero.
2. Let $M$ be the infinitely divisible $\mathbb{F}_2[\rho]$-module $\mathbb{F}_2[\rho]/\rho^\infty$. Then $\lim_\rho M$ is isomorphic to $\mathbb{F}_2[\rho^{\pm 1}]$.

**Proof.** If $M$ is cyclic, then no nonzero element is infinitely $\rho$-divisible, which implies the first statement. For the case $M = \mathbb{F}_2[\rho]/\rho^\infty$, a (homogeneous) element of the limit is either of the form

$$\left(\frac{1}{\rho^k}, \frac{1}{\rho^k+1}, \ldots\right)$$

or of the form

$$\left(0, \ldots, 0, 1, \frac{1}{\rho^k}, \frac{1}{\rho^k+1}, \ldots\right).$$

For $k \geq 0$, the isomorphism $\mathbb{F}_2[\rho^{\pm 1}] \to \lim_\rho M$ sends $\rho^k$ to the tuple $(0, \ldots, 0, 1, \frac{1}{\rho^k}, \ldots)$ having $k - 1$ zeroes and sends $\frac{1}{\rho^k}$ to $(\frac{1}{\rho^k}, \frac{1}{\rho^{k+1}}, \ldots)$. $\blacksquare$

We will now compute the $\rho$-colocalization of $\text{Ext}_{\mathcal{C}_2}(1)$.

**Proposition 5.2.**

$$\lim_\rho \text{Ext}_{\mathcal{C}_2}(1) \cong \bigoplus_{k \geq 1} \mathbb{F}_2[\rho^{\pm 1}, h_1] \left\{\frac{\theta}{\tau^{4k}}\right\} \cong \mathbb{F}_2[\rho^{\pm 1}, h_1] \otimes_{\mathbb{F}_2[\tau^4]} \{\theta\}.$$ 

Recall that $\theta$ itself is not an element of $\lim_\rho \text{Ext}_{\mathcal{C}_2}(1)$, as described in Remark 2.1. The main point of Proposition 5.2 is that the elements $\frac{\theta}{\rho^{4k}} h_1$ are infinitely $\rho$-divisible classes in $\text{Ext}_{\mathcal{C}_2}(1)$, and there are no other infinitely $\rho$-divisible families in $\text{Ext}_{\mathcal{C}_2}(1)$.

**Proof.** Since the cobar complex $\text{coB}^*([\mathcal{M}_2^C], A_{C_2}(1))$ is finite-dimensional in each tridegree, the inverse systems

$$\cdots \xrightarrow{\rho} \text{coB}^*([\mathcal{M}_2^C], A_{C_2}(1)) \xrightarrow{\rho} \text{coB}^*([\mathcal{M}_2^C], A_{C_2}(1))$$

and

$$\cdots \xrightarrow{\rho} \text{Ext}_{\mathcal{C}_2}(1) \xrightarrow{\rho} \text{Ext}_{\mathcal{C}_2}(1)$$

satisfy the Mittag-Leffler condition, so that ([W, Theorem 3.5.8])

$$\lim_\rho \text{Ext}_{\mathcal{C}_2}(1) \cong H^* \left(\lim_\rho \text{coB}^*([\mathcal{M}_2^C], A_{C_2}(1))\right).$$
Now we compute

$$\lim_{\rho} \mathrm{coB}^s\left(\mathbb{M}_2^{C_2}, A^{C_2}(1)\right) = \lim_{\rho} \left(\mathbb{M}_2^{C_2} \otimes_{\mathbb{M}_2^{C_2}} A^{C_2}(1)^{\otimes s}\right)$$

$$\cong \lim_{\rho} \left(\mathbb{M}_2^{C_2} \otimes_{\mathbb{M}_2^{C_2}} A^R(1)^{\otimes s}\right).$$

The splitting $\mathbb{M}_2^{C_2} = \mathbb{M}_2^R \otimes NC$ yields a splitting

$$\left(\mathbb{M}_2^R \otimes_{\mathbb{M}_2^R} A^R(1)^{\otimes s}\right) \oplus \left(\mathbb{M}_2^R \otimes_{\mathbb{M}_2^R} A^R(1)^{\otimes s}\right)$$

of $\mathbb{M}_2^{C_2} \otimes_{\mathbb{M}_2^{C_2}} A^R(1)^{\otimes s}$ as an $F_2[\rho]$-module. The first piece of the splitting contributes nothing to the $\rho$-colocalization by Lemma 5.1(1) because $\mathbb{M}_2^R$ is free as an $F_2[\rho]$-module.

On the other hand, the $F_2[\rho]$-module $NC$ is a direct sum of copies of $F_2[\rho]/\rho^\infty$. By Lemma 5.1(2), we have that $\lim_{\rho} \left(\mathbb{M}_2^R \otimes_{\mathbb{M}_2^R} A^R(1)^{\otimes s}\right)$ is isomorphic to

$$\left(\frac{\mathbb{M}_2^R[\rho^{-1}]}{\tau^\infty}\right) \otimes_{\mathbb{M}_2^R} A^R(1)^{\otimes s}.$$ 

Now the argument of Proposition 4.1 provides a splitting

$$\mathrm{coB}^s_{\mathbb{M}_2^R} \left(\frac{\mathbb{M}_2^R[\rho^{-1}]}{\tau^\infty}\right), A^R(1) \simeq$$

$$\mathrm{coB}^s_{F_2[\tau]} \left(\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\right) \left(\frac{\mathbb{F}_2[\tau, x]}{x^4}\right) \left[\rho^x\right] \otimes_{F_2} \mathrm{coB}^s_{F_2} \left(\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\right),$$

where $x = \rho\tau_0$. The cohomology of the second factor is $\mathbb{F}_2[h_1]$.

It remains to show that the cohomology of

$$\mathrm{coB}^s_{F_2[\tau]} \left(\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\right)$$

is equal to $\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{\theta\}$. As in Formula (2.3), the comodule structure on $\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{\theta\}$ is given by

$$\eta_R \left(\frac{\theta \tau^k}{\tau^x}\right) = \theta \frac{\tau^k}{\tau^x} \left(1 + \frac{x}{\tau} + \frac{x^2}{\tau^2} + \frac{x^3}{\tau^3}\right)^k.$$ 

Now we filter $\mathrm{coB}^s_{F_2[\tau]} \left(\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{\theta\}, \frac{\mathbb{F}_2[\tau, x]}{x^4}\right)$ by powers of $x$. We then have

$$E_1 \cong \frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{\theta\} \otimes_{F_2} \mathbb{F}_2[v_0, v_1],$$

where $v_0 = [x]$ and $v_1 = [x^2]$. The differential $d_1 \left(\frac{\theta \tau^k}{\tau^x}\right) = \frac{\theta \tau^k}{\tau^x} v_0$ gives

$$E_2 \cong \frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{\theta\} \otimes_{F_2} \mathbb{F}_2[v_1].$$

Finally, the differential $d_2 \left(\frac{\theta \tau^k}{\tau^x}\right) = \frac{\theta \tau^k}{\tau^x} v_1$ gives

$$E_3 = E_\infty \cong \frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{\theta\}.$$

\[\blacksquare\]
Our next step in working towards the calculation of $\text{Ext}_{C_2}(1)$ is to describe the simpler $\mathbb{R}$-motivic $\text{Ext}_{\mathbb{R}}(1)$. The reader is encouraged to consult the charts on pages $36$–$38$ to follow along with the calculations described in this section. This calculation was originally carried out in [H]. We include the details of the $\mathbb{R}$-motivic $\rho$-Bockstein spectral sequence, but we take the approach of [DI2], rather than [H], in establishing $\rho$-Bockstein differentials. The point is that there is only one pattern of differentials that is consistent with the $\rho$-inverted calculation of Corollary 4.2. This observation avoids much technical work with Massey products that would otherwise be required to establish relations that then imply differentials.

For $\mathcal{A}^R(1)$, the $\mathbb{R}$-motivic $\rho$-Bockstein spectral sequence takes the form

$$\text{Ext}_{C_2}(1)[\rho] \Rightarrow \text{Ext}_{\mathbb{R}}(1),$$

where $\text{Ext}_{C_2}(1) \cong M^C_2[h_0, h_1, a, b]/h_0 h_1, \tau h_1^2, h_0 a, a^2 + h_0^2 b]$.

**Proposition 6.1.** In the $\mathbb{R}$-motivic $\rho$-Bockstein spectral sequence, we have differentials

- (1) $d_1(\tau) = \rho h_0$.
- (2) $d_2(\tau^2) = \rho^2 \tau h_1$.
- (3) $d_3(\tau^3 h_1^2) = \rho^3 a$.

All other differentials on multiplicative generators are zero, and $E_4$ equals $E_\infty$.

**Proof.** By Corollary 4.2, the infinite $\rho$-towers that survive the $\rho$-Bockstein spectral sequence occur in the Milnor-Witt $4k$-stem. All other infinite $\rho$-towers are either truncated by a differential or support a differential.

For example, the permanent cycle $h_0$ must be $\rho$-torsion in $\text{Ext}_{\mathbb{R}}(1)$, which forces the Bockstein differential

$$d_1(\tau) = \rho h_0.$$ 

Next, the $\rho$-tower on $\tau h_1$ cannot survive, and the only possibility is that there is a differential

$$d_2(\tau^2) = \rho^2 \tau h_1.$$ 

The $\rho$-tower on $\tau^3 h_1^2$ cannot survive, and we conclude that it must support a differential

$$d_3(\tau^3 h_1^2) = \rho^3 a.$$ 

There is no room for further differentials, so $E_4 = E_\infty$. 

Proposition 6.1 leads to an explicit description of the $\mathbb{R}$-motivic $\rho$-Bockstein $E_\infty$-page. However, there are hidden multiplications in passing from $E_\infty$ to $\text{Ext}_{\mathbb{R}}(1)$.

**Theorem 6.2.** $\text{Ext}_{\mathcal{A}^R(1)}$ is the $\mathbb{F}_2$-algebra on generators given in Table 1 with relations given in Table 2.

The horizontal lines in Table 2 group the relations into families. The first family describes the $\rho^k$-torsion. The remaining families are associated to the classical products $h_0^2$, $h_0 h_1$, $h_1^2$, $h_0 a$, $h_1 a$, and $a^2 + h_0^2 b$ respectively.

**Proof.** The family of $\rho^k$-torsion relations follows from the $\rho$-Bockstein differentials of Proposition 6.1.

Many relations follow immediately from the $\rho$-Bockstein $E_\infty$-page because there are no possible additional terms.
Table 1. Generators for $\text{Ext}_R(1)$

| $mw$ | $(s, f, w)$     | generator  |
|------|----------------|------------|
| 0    | $(-1, 0, -1)$  | $\rho$     |
| 0    | $(0, 1, 0)$    | $h_0$      |
| 0    | $(1, 1, 1)$    | $h_1$      |
| 1    | $(1, 1, 0)$    | $\tau h_1$|
| 2    | $(0, 1, -2)$   | $\tau^2 h_0$|
| 2    | $(4, 3, 2)$    | $a$        |
| 4    | $(4, 3, 0)$    | $\tau^2 a$|
| 4    | $(8, 4, 4)$    | $b$        |
| 4    | $(0, 0, -4)$   | $\tau^4$   |

Table 2. Relations for $\text{Ext}_R(1)$

| $mw$ | $(s, f, w)$     | relation                                          |
|------|----------------|---------------------------------------------------|
| 0    | $(-1, 1, -1)$  | $\rho h_0$                                       |
| 2    | $(-1, 1, -3)$  | $\rho \cdot \tau^2 h_0$                         |
| 1    | $(-1, 1, -2)$  | $\rho^2 \cdot \tau h_1$                        |
| 2    | $(1, 3, -1)$   | $\rho^3 a$                                       |
| 4    | $(0, 2, -4)$   | $(\tau^4 h_0)^2 + \tau^4 h_0^3$                 |
| 0    | $(1, 2, 1)$    | $h_0 h_1$                                        |
| 1    | $(1, 2, 0)$    | $h_0 \cdot \tau h_1 + \rho h_1 \cdot \tau h_1$|
| 2    | $(1, 2, -1)$   | $\tau^2 h_0 \cdot h_1 + \rho(\tau h_1)^2$     |
| 3    | $(1, 2, -2)$   | $\tau^2 h_0 \cdot \tau h_1$                    |
| 1    | $(3, 3, 2)$    | $h_1^2 \cdot \tau h_1$                         |
| 2    | $(3, 3, 1)$    | $h_1 (\tau h_1)^2 + \rho a$                     |
| 3    | $(3, 3, 0)$    | $(\tau h_1)^3$                                |
| 4    | $(3, 3, -1)$   | $\tau^4 \cdot h_1^3 + \rho \cdot \tau^2 a$   |
| 4    | $(4, 4, 0)$    | $\tau^2 h_0 \cdot a + h_0 \cdot \tau^2 a$     |
| 6    | $(4, 4, -2)$   | $\tau^2 h_0 \cdot \tau^2 a + \tau^4 h_0 a$   |
| 2    | $(5, 4, 3)$    | $h_1 a$                                          |
| 3    | $(5, 4, 2)$    | $\tau h_1 \cdot a$                             |
| 4    | $(5, 4, 1)$    | $h_1 \cdot \tau^2 a + \rho^3 b$                 |
| 5    | $(5, 4, 0)$    | $\tau h_1 \cdot \tau^2 a$                      |
| 4    | $(8, 6, 4)$    | $a^2 + h_0^2 b$                                  |
| 6    | $(8, 6, 2)$    | $a \cdot \tau^2 a + \tau^2 h_0 \cdot h_0 b$   |
| 8    | $(8, 6, 0)$    | $(\tau^2 a)^2 + \tau^4 h_0^2 b + \rho^4 \tau^4 h_1^2 b$ |

Corollary 4.2 implies that $\tau^4 \cdot h_1^3$, is non-zero in $\text{Ext}_R(1)$. It follows that there must be a hidden relation

$$\tau^4 \cdot h_1^3 = \rho \cdot \tau^2 a.$$ 

Similarly, there is a hidden relation

$$h_1 \cdot \tau^2 a = \rho^3 b$$

because $\tau^4 \cdot h_1^3$ is non-zero in $\text{Ext}_R(1)$. This last relation then gives rise to the extra term $\rho^2 \tau^4 h_1^2 b$ in the relation for $(\tau^2 a)^2 + \tau^4 h_0^2 b$. 
Shuffling relations for Massey products imply the remaining three relations, namely
\[ h_0 \cdot \tau h_1 = h_0(h_1, h_0, \rho) = \langle h_0, h_1, h_0 \rangle \rho = \rho h_1 \cdot \tau h_1, \]
\[ \tau^2 h_0 \cdot h_1 = \langle \rho \tau h_1, \rho, h_0 \rangle h_1 = \rho \tau h_1(\rho, h_0, h_1) = \rho(\tau h_1)^2, \]
and
\[ p a = \rho(h_0, h_1, \tau h_1 \cdot h_1) = \langle \rho, h_0, h_1 \rangle \tau h_1 \cdot h_1 = h_1(\tau h_1)^2. \]
See Table 5 in Section 8 for more details on these Massey products, whose indeterminacies are all zero.

7. BOCKSTEIN DIFFERENTIALS IN THE NEGATIVE CONE

We finally come to the key step in our calculation of \( \Ext_{\mathbb{C}}^n(1) \). We are now ready to analyze the \( \rho \)-Bockstein differentials associated to the negative cone, i.e., to the spectral sequence \( E^- \) of Proposition 3.1. We already analyzed the spectral sequence \( E^+ \) in Section 6.

7.1. The structure of \( E^-_1 \). First, we need some additional information about the algebraic structure of \( E^-_1 \). Since \( E_1 = E^+_1 \oplus E^-_1 \) is defined in terms of \( \Ext \) groups, it is a ring and has higher structure in the form of Massey products. The subobject \( E^-_1 \) is a module over \( E^+_1 \), and it possesses Massey products of the form \( \langle x_1, \ldots, x_n, y \rangle \), where \( x_1, \ldots, x_n \) belong to \( E^+_1 \) and \( y \) belongs to \( E^-_1 \).

**Definition 7.1.** Suppose that \( x \) is a non-zero element of \( \Ext_{\mathbb{C}}^n(1) \) such that \( \tau x \) is zero. According to Remark 3.6, for each \( s \geq 1 \), the element \( x \) gives rise to a copy of \( \mathbb{M}_2^{\mathbb{C}}/\tau \) in \( \Tor_{\mathbb{M}_2^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \Ext_{\mathbb{C}}^n(1)) \left\{ \varrho \right\} \) that is infinitely divisible by \( \rho \). In particular, it gives a non-zero element of the Tor group. Let \( \varrho_{\rho} x \) be any lift to \( E^-_1 \) of this non-zero element.

**Remark 7.2.** There is indeterminacy in the choice of \( \varrho_{\rho} x \) which arises from the first term of the short exact sequence for \( E^-_1 \) in Proposition 3.1.

**Lemma 7.3.** The element \( \varrho_{\rho} x \) of \( E^-_1 \) is contained in the Massey product \( \langle x, \tau, \varrho \rangle \).

**Proof.** If \( d(u) = \tau \cdot x \) in the cobar complex for \( \Ext_{\mathbb{C}}^n(1) \), then \( \varrho_{\rho} u \) is a cycle, since \( \tau \varrho_{\rho} = 0 \). This cycle \( \varrho_{\rho} u \) represents both the Massey product as well as \( \varrho_{\rho} x \).

**Remark 7.4.** The most important example is the element \( \varrho_{\rho} h_1^3 \), which is defined because \( \tau h_1^3 \) equals zero in \( \Ext_{\mathbb{C}}^n(1) \). Another possible name for \( \varrho_{\rho} h_1^3 \) is \( \varrho_{\rho} v_1^2 \), since \( v_1^2 \) is the element of the May spectral sequence that creates the relation \( \tau h_1^3 \).

**Remark 7.5.** Beware that the Massey product description for \( \varrho_{\rho} x \) holds in \( E^-_1 \), not in \( \Ext_{\mathbb{C}}^n(1) \). In fact, we have already seen in Section 6 that \( \tau \) is not a permanent cycle in the \( \rho \)-Bockstein spectral sequence.

Nevertheless, minor variations on these Massey products may exist in \( \Ext_{\mathbb{C}}^n(1) \). For example, \( \langle h_1^2, \tau h_1, \varrho_{\rho} \rangle \) equals \( \varrho_{\rho} h_1^3 \).

We can now deduce a specific computational property of \( E^-_1 \) that we will need later.

**Lemma 7.6.** In \( E^-_1 \), there is a relation \( h_0 \cdot \varrho_{\rho} h_1^3 = \varrho_{\rho} a \).
Proof. Use Lemma 7.3 and the Massey product shuffle

\[ h_0 \cdot \frac{Q}{\rho} h_3^1 = h_0 \left\langle h_3^1, \tau, \frac{\theta}{\rho^2} \right\rangle = \left\langle h_0, h_3^1, \frac{\theta}{\rho^2} \right\rangle = \frac{\theta}{\rho^2} a. \]

Table 3 gives multiplicative generators for the Bockstein \( E_1 \)-page. The elements above the horizontal line are multiplicative generators for \( E_1^+ \). The elements below the horizontal generate \( E_1^- \) in the following sense. Every element of \( E_1^- \) can be formed by starting with one of the these listed elements, multiplying by elements of \( E_1^+ \), and then dividing by \( \rho \). The elements in Table 6 are not multiplicative generators for \( \text{Ext}_{C_2}(1) \) in the usual sense, because we allow for division by \( \rho \). The point of this notational approach is that the elements of \( E_1^- \) and of \( \text{Ext}_{NC} \) are most easily understood as families of \( \rho \)-divisible elements.

**Table 3. Generators for the Bockstein \( E_1 \)-page**

| \( mw \) | \( (s, f, w) \) | element |
|-------|----------------|---------|
| 0     | \((-1, 0, -1)\) | \( \rho \) |
| 0     | \((0, 1, 0)\)   | \( h_0 \) |
| 0     | \((1, 1, 1)\)   | \( h_1 \) |
| 1     | \((0, 0, -1)\)  | \( \tau \) |
| 2     | \((4, 3, 2)\)   | \( a \) |
| 4     | \((8, 4, 4)\)   | \( b \) |
| \(-k - 1\) | \((0, 0, k + 1)\) | \( \frac{\rho}{h_1^3} \) |

7.2. \( \rho \)-Bockstein differentials in \( E^- \). Our next goal is to analyze the \( \rho \)-Bockstein differentials in \( E^- \). We will rely heavily on the \( \rho \)-Bockstein differentials for \( E^+ \) established in Section 6, using that \( E^- \) is an \( E^+ \)-module.

As an \( E_1^+ \)-module, \( E_1^- \) is generated by the elements \( \frac{a}{\rho^2} \) and \( \frac{Q}{\rho} h_3^1 \). This arises from the observation that the \( \tau \) torsion in \( \text{Ext}_{C_2}(1) \) is generated as an \( \text{Ext}_{C_2}(1) \)-module by \( h_3^1 \).

Proposition 7.7 gives the values of the \( \rho \)-Bockstein \( d_1 \) differential on these generators of \( E_1^- \). All other \( d_j \) differentials can then be deduced from the Leibniz rule and the \( E_1^+ \)-module structure.

All of the differentials in \( E^- \) are infinitely divisible by \( \rho \), in the following sense. When we claim that \( d_r(x) = y \), we also have differentials \( d_r \left( \frac{a}{\rho^j} \right) = \frac{a}{\rho^j} \) for all \( j \geq 0 \). For example, in Proposition 7.7, the formula \( d_1 \left( \frac{a}{\rho^2} \right) = \frac{a}{\rho^2} h_0 \) implies that

\[ d_1 \left( \frac{a}{\rho^{2j+2}} \right) = \frac{a}{\rho^{2j+2}} h_0 \text{ for all } j \geq 1. \]

**Proposition 7.7.** For all \( k \geq 0 \),

1. \( d_1 \left( \frac{a}{\rho^{2k+2}} \right) = \frac{a}{\rho^{2k+2}} h_0. \)
2. \( d_1 \left( \frac{Q}{\rho} h_3^1 \right) = \frac{Q}{\rho} a. \)

These differentials are infinitely divisible by \( \rho \).
Proof. We give three proofs for the first formula. First, it follows from $\text{Sq}^1 \left( \frac{\theta}{\rho^2 \tau + \tau^2} \right) = \frac{\theta}{\rho^2 \tau + \tau^2}$, using the relationship between $d_1$ and the left and right units of the Hopf algebroid. Second, we have

$$0 = d_1 \left( \tau^{2k+1} \frac{\theta}{\rho^2 \tau + \tau^2} \right) = \tau^{2k+1} d_1 \left( \frac{\theta}{\rho^2 \tau + \tau^2} \right) + \frac{\theta}{\rho^2 \tau + \tau^2} \rho^2 \tau^{2k} h_0$$

$$= \tau^{2k+1} d_1 \left( \frac{\theta}{\rho^2 \tau + \tau^2} \right) + \frac{\theta}{\rho^2 \tau + \tau^2} h_0.$$

Third, we can use Proposition 5.2 to conclude that the infinitely $\rho$-divisible elements $\frac{\theta}{\rho^2 \tau + \tau^2}$ cannot survive the $\rho$-Bockstein spectral sequence. The only possibility is that they support a $d_1$ differential.

For the second formula, use the first formula to determine that $d_1 \left( \frac{\theta}{\rho^2 \tau + \tau^2} a \right) = \frac{\theta}{\rho^2 \tau + \tau^2} h_0 a$. Then use the relation of Lemma 7.6. Alternatively, this differential is also forced by Proposition 5.2.

It is now straightforward to compute $E_2^-$, since the $\rho$-Bockstein $d_1$ differential is completely known. The charts in Section 12 depict $E_2$ graphically.

Next, Proposition 7.8 gives a $\rho$-Bockstein $d_2$ differential in $E_2^-$. This is the essential calculation, in the sense that the $d_2$ differential is zero on all other $E_2^+$-module generators of $E_2^-$. 

**Proposition 7.8.** $d_2 \left( \frac{\theta}{\rho^2 \tau + \tau^2} \right) = \frac{\theta}{\rho^2 \tau + \tau^2} h_1$ for all $k \geq 0$. This differential is infinitely divisible by $\rho$.

**Proof.** As for Proposition 7.7, we give three proofs. First, $\text{Sq}^2 \left( \frac{\theta}{\rho^2 \tau + \tau^2} \right) = \frac{\theta}{\rho^2 \tau + \tau^2}$. Second, we have

$$0 = d_2 \left( \tau^{4k+2} \frac{\theta}{\rho^2 \tau + \tau^2} \right) = \tau^{4k+2} d_2 \left( \frac{\theta}{\rho^2 \tau + \tau^2} \right) + \rho^2 \tau^{4k+1} \frac{\theta}{\rho^2 \tau + \tau^2} h_1$$

$$= \tau^{4k+2} d_2 \left( \frac{\theta}{\rho^2 \tau + \tau^2} \right) + \frac{\theta}{\rho^2 \tau + \tau^2} h_1.$$

Third, use Proposition 5.2 to conclude that the infinitely $\rho$-divisible elements $\frac{\theta}{\rho^2 \tau + \tau^2}$ cannot survive the $\rho$-Bockstein spectral sequence. The only possibility is that they support a $d_2$ differential.

At this point, the behavior of $E^-$ becomes qualitatively different than $E^+$. For $E^+$, there are $d_3$ differentials, and then the $E_3^+$-page equals the $E_\infty^+$. For $E^-$, it turns out that the $d_r$ differential is non-zero for infinitely many values of $r$. This does not present a convergence problem, because there are only finitely many non-zero differentials in any given degree. One consequence is that the orders of the $\rho$-torsion in $\text{Ext}_{C_2}(1)$ are unbounded. In other words, for every $s$, there exists an element $x$ of such that $\rho^s x$ is non-zero but $\rho^{s+t} x$ is zero for some $t > 0$. This is fundamentally different than $\text{Ext}_{\mathbb{Z}}(1)$, where $\rho^3 x$ is zero if $x$ is not $\rho$-torsion free.

Proposition 7.9 makes explicit these higher differentials.

**Proposition 7.9.** For all $k \geq 1$,

1. $d_{4k} \left( \frac{Q}{\rho^2 \tau + \tau^2} h_1^{4k} \right) = \frac{\theta}{\rho^2 \tau + \tau^2} b^k$.
2. $d_{4k+1} \left( \frac{Q}{\rho^2 \tau + \tau^2} h_1^{4k+3} \right) = \frac{\theta}{\rho^2 \tau + \tau^2} a b^k$. 
These differentials are infinitely divisible by \(\rho\).

**Proof.** The first formula relies on the relation \(\tau^4 h_1^4 = \rho^b \) in \(\text{Ext}_C(1)\). We know that \(\frac{\theta}{\rho^4} \) and \(b\) are permanent cycles. On the other hand, in \(\text{Ext}_{C_2}(1)\) we have

\[
\frac{\theta}{\rho^4} \theta^k C_{k+1} b^k = \tau^4 \frac{\theta}{\rho^4} \theta^k C_{k+1} b^k = \tau^4 \frac{\theta}{\rho^4} \theta^k C_{k+1} b^k-1.
\]

Thus \(\frac{\theta}{\rho^4} b^k\) is \(h_1\)-divisible, which implies that it must be zero in \(\text{Ext}_{C_2}(1)\), as there is no surviving class in the appropriate degree to support the \(h_1\)-multiplication. The only Bockstein differential that could hit \(\frac{\theta}{\rho^4} b^k\) is the claimed one.

For the second formula, the classes \(\frac{\theta}{\rho^4} \theta^k C_{k+1} a b^k\) are permanent cycles, yet

\[
\frac{\theta}{\rho^4} \theta^k C_{k+1} a b^k = \tau^4 \frac{\theta}{\rho^4} \theta^k C_{k+1} a b^k = \tau^4 \frac{\theta}{\rho^4} \theta^k C_{k+1} a b^k-1
\]

in \(\text{Ext}_{C_2}(1)\). But \(h_1 a = 0\), so \(\frac{\theta}{\rho^4} \theta^k C_{k+1} a b^k\) must be zero in \(\text{Ext}_{C_2}(1)\), forcing the claimed differential.

Alternatively, one can use Proposition 5.2 to obtain both differentials. ■

Table 4 summarizes the Bockstein differentials that we computed in Sections 6 and 7.2. The differentials below the horizontal line occur in \(E^-\), while the differentials below the horizontal line occur in \(E^-\) and are infinitely divisible by \(\rho\).

| \(mw\) | \((s, f, w)\) | \(\text{element}\) | \(r\) | \(d_r\) | proof |
|-------|---------------|----------------|------|--------|-------|
| 1     | \((0, 0, -1)\) | \(\tau\)       | 1    | \(\rho h_0\) | Prop. 6.1 |
| 2     | \((0, 0, -2)\) | \(\tau^2\)     | 2    | \(\rho^2 \tau h_1\) | Prop. 6.1 |
| 3     | \((2, 2, -1)\) | \(\tau^3 h_1^2\) | 3    | \(\rho^3 a\) | Prop. 6.1 |
| \(-2k-2\) | \((1, 0, 2k+3)\) | \(\frac{\theta}{\rho^4} \theta^k C_{k+1} a b^k\) | 1     | \(\frac{\theta}{\rho^4} \theta^k C_{k+1} a b^k\) | Prop. 7.7 |
| 0     | \((5, 2, 5)\) | \(\rho^3 h_1\) | 1     | \(\rho^3 a\) | Prop. 7.7 |
| \(-4k-3\) | \((2, 0, 4k+5)\) | \(\frac{\theta}{\rho^4} \theta^k C_{k+1} a b^k\) | 2     | \(\frac{\theta}{\rho^4} \theta^k C_{k+1} a b^k\) | Prop. 7.8 |
| 0     | \((8k+1, 4k-1, 8k+1)\) | \(\frac{\theta}{\rho^4} \theta^k C_{k+1} a b^k\) | 4k    | \(\frac{\theta}{\rho^4} \theta^k C_{k+1} a b^k\) | Prop. 7.9 |
| 0     | \((8k+5, 4k+2, 8k+5)\) | \(\frac{\theta}{\rho^4} \theta^k C_{k+1} a b^k\) | 4k+1  | \(\frac{\theta}{\rho^4} \theta^k C_{k+1} a b^k\) | Prop. 7.9 |

The \(\rho\)-Bockstein differentials of Sections 6 and 7 allow us to completely compute the \(E^-\)-page of the \(\rho\)-Bockstein spectral sequence for \(\text{Ext}_{C_2}(1)\).

### 8. Some Massey Products

The final step in the computation of \(\text{Ext}_{C_2}(1)\) is to determine multiplicative extensions that are hidden in the \(\rho\)-Bockstein \(E^-\)-page. In order to do this, we will need some Massey products in \(\text{Ext}_{C_2}(1)\). Table 5 summarizes the information that we will need.

**Theorem 8.1.** Some Massey products in \(\text{Ext}_{C_2}(1)\) are given in Table 5. All have zero indeterminacy, except for the last two.
Table 5. Some Massey products in $\text{Ext}_{C_2}(1)$

| $mw$ $(s, f, w)$ | $\langle h_0, h_1, h_0 \rangle$ | $\tau h_1$ | classical |
|-------------------|----------------------------------|------------|-----------|
| 1 $(2, 2, 1)$     | $\langle h_0, h_1, h_0 \rangle$ | $\tau h_1$ | classical |
| 1 $(1, 1, 0)$     | $\langle \rho, h_0, h_1 \rangle$ | $\tau h_1$ | $d_3(\tau) = \rho h_0$ |
| 2 $(4, 3, 2)$     | $\langle \tau h_1 \cdot h_1, h_1, h_0 \rangle$ | $\rho \cdot \theta$ | classical |
| 2 $(0, 1, -2)$    | $\langle \rho \tau h_1, h, h_0 \rangle$ | $\tau^2 h_0$ | $d_2(\tau^2) = \rho^2 \tau h_1$ |
| 4 $(8, 5, 4)$     | $\langle a, h_1, \tau h_1^2 \rangle$ | $h_0 b$ | classical |

Proof. For some Massey products in Table 5, a $\rho$-Bockstein differential is displayed in the last column. In these cases, May’s Convergence Theorem [MI] [I, Chapter 2.2] applies, and the Massey product can be computed with the given differential. Roughly speaking, May’s Convergence Theorem says that Massey products in $\text{Ext}_{C_2}(1)$ can be computed with any $\rho$-Bockstein differential. Beware that May’s Convergence Theorem requires technical hypotheses involving “crossing differentials” that are not always satisfied. Failure to check these conditions can lead to mistaken calculations.

The proofs for other Massey products in Table 5 are described as “classical”. In these cases, the Massey product already occurs in $\text{Ext}_{cl}$. ■

Remark 8.2. The eight Massey products in the middle Section of Table 5 are only the first examples of infinite families that are $\tau^4$-periodic. For example, $\langle \tau^2 h_0, \rho, \frac{\theta}{\rho \tau^{4k+3}} \rangle$ equals $\frac{\theta}{\rho \tau^{4k+3}}$ for all $k \geq 0$, and $\langle \rho, \frac{\theta}{\rho \tau^{4k+3}}, \tau h_1 \rangle$ equals $\frac{\theta}{\rho \tau^{4k+3}}$ for all $k \geq 0$. 

9. Hidden extensions

We now determine multiplicative extensions that are hidden in the $\rho$-Bockstein $E_\infty$-page. We have already determined some of these hidden extensions in Section 6. In this section, we establish additional hidden relations on elements associated
with the negative cone. We have not attempted a completely exhaustive analysis of the ring structure of Ext_{C_2}(1).

Recall that Ext_{C_2}(1) is a square-zero extension of Ext_{\mathbb{G}}(1). This eliminates many possible hidden extensions. For example, \((Q/\theta h_1)^2\) is zero in Ext_{C_2}(1).

**Proposition 9.1.** For all \(k \geq 0\),
\begin{enumerate}
\item \(h_0 \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+3} = \theta \frac{\theta}{\rho^{4k+1}} a b^k h_0 b^{k+1}\).
\item \(a \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+3} = \frac{\theta}{\rho^{4k+1}} h_0 b^{k+1}\).
\end{enumerate}

Proof.
\begin{enumerate}
\item \(h_0 \langle \rho, \frac{\theta}{\rho^{4k+2}} a b^k \rangle = \langle h_0, \rho, \frac{\theta}{\rho^{4k+2}} \rangle a b^k\).
\item Using part (1), we have that
\[h_0 a \cdot \frac{Q}{\rho^{4k+1} h_1^{4k+3}} = a \cdot \frac{\theta}{\rho^{4k+1}} a b^k h_0^{2} b^{k+1}\]
which is non-zero. Therefore, \(a \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+3}\) must also be non-zero, and there is just one possibility.
\end{enumerate}

**Proposition 9.2.** For all \(k \geq 1\),
\begin{enumerate}
\item \(\tau^2 a \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+3} = \frac{\theta}{\rho^{4k+1}} h_0 b^{k+1} + \frac{Q}{\rho^{4k+3}} h_1^{4k+2} b\).
\item \(\tau^4 \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+3} = \frac{Q}{\rho^{4k+3}} h_1^{4k-1} b\).
\item \(\tau^2 h_0 \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+3} = \frac{\theta}{\rho^{4k+1}} a b^k\).
\end{enumerate}

Proof.
\begin{enumerate}
\item Using Proposition 9.1(1), we have that
\[h_0 \cdot \tau^2 a \cdot \frac{Q}{\rho^{4k+1} h_1^{4k+3}} = \tau^2 a \cdot \frac{\theta}{\rho^{4k+1}} a b^k = \frac{\theta}{\rho^{4k+1}} h_0^{2} b^{k+1}\]
which is non-zero. This shows that \(\tau^2 a \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+3}\) is either \(\frac{\theta}{\rho^{4k+1}} h_0 b^{k+1}\) or \(\frac{\theta}{\rho^{4k+1}} h_0 b^{k+1} + \frac{Q}{\rho^{4k+3}} h_1^{4k+2} b\).
On the other hand,
\[h_1 \cdot \tau^2 a \cdot \frac{Q}{\rho^{4k+1} h_1^{4k+3}} = \tau^2 b \cdot \frac{Q}{\rho^{4k+1} h_1^{4k+3}} = \frac{Q}{\rho^{4k+2}} h_1^{4k+3} b\]
Therefore, \(\tau^2 a \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+3}\) must equal \(\frac{\theta}{\rho^{4k+1}} h_0 b^{k+1} + \frac{Q}{\rho^{4k+3}} h_1^{4k+2} b\).
\item Using Proposition 9.1(1), we have that
\[h_0 \cdot \tau^4 \cdot \frac{Q}{\rho^{4k+1} h_1^{4k+3}} = \tau^4 a \cdot \frac{\theta}{\rho^{4k+1}} a b^k = \frac{\theta}{\rho^{4k+1}} a b^k\]
which is non-zero. This shows that \(\tau^4 \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+3}\) is also non-zero, and there is just one possible value.
\item \(\tau^2 h_0 \langle \rho, \frac{\theta}{\rho^{4k+1}}, b^{k+1} \rangle = \langle \tau^2 h_0, \rho, \frac{\theta}{\rho^{4k+1}} \rangle b^{k+1}\).
\end{enumerate}
Proposition 9.3. For all $k \geq 0$,

1. $h_0 \cdot \frac{Q}{\rho^{4k+4}} h_1^{4k+4} = \theta \frac{\theta}{\rho^{4k+3}} b^{k+1}$.
2. $\tau h_1 \cdot \frac{Q}{\rho^{4k+4}} h_1^{4k+4} = \theta \frac{\theta}{\rho^{4k+3}} b^{k+1}$.
3. $\tau^2 h_0 \cdot \frac{Q}{\rho^{4k+4}} h_1^{4k+4} = \theta \frac{\theta}{\rho^{4k+3}} b^{k+1}$.
4. $a \cdot \frac{Q}{\rho^{4k+4}} h_1^{4k+4} = \theta \frac{\theta}{\rho^{4k+3}} h_1^2 b^{k+1}$.
5. $\tau^2 a \cdot \frac{Q}{\rho^{4k+4}} h_1^{4k+4} = \frac{Q}{\rho^{4k+3}} h_1^{4k+3} b$.

Proof.

1. $h_0 \left\langle \rho, \frac{\theta}{\rho^{4k}}, b^k \right\rangle = \left\langle h_0, \rho, \frac{\theta}{\rho^{4k}} \right\rangle b^k$.

2. $\rho \tau h_1 \left\langle \rho, \frac{\theta}{\rho^{4k+1}}, b^{k+1} \right\rangle = \left\langle \rho \tau h_1, \rho, \frac{\theta}{\rho^{4k+1}} \right\rangle b^{k+1}$.

3. $\tau^2 h_0 \left\langle \rho, \frac{\theta}{\rho^{4k+1}}, b^{k+1} \right\rangle = \left\langle \tau^2 h_0, \rho, \frac{\theta}{\rho^{4k+1}} \right\rangle b^{k+1}$.

4. Using part (1), we have that

   $h_0 a \cdot \frac{Q}{\rho^{4k+4}} h_1^{4k+4} = a \cdot \frac{\theta}{\rho^{4k+3}} b^{k+1} = \frac{\theta}{\rho^{4k+3}} a b^{k+1}$,

   which is non-zero. Therefore, $a \cdot \frac{Q}{\rho^{4k+4}} h_1^{4k+4}$ must also be non-zero, and

   there is just one possibility.

5. Using part (1), we have that

   $h_0 a \cdot \frac{Q}{\rho^{4k+4}} h_1^{4k+4} = a \cdot \frac{\theta}{\rho^{4k+3}} b^{k+1} = \frac{\theta}{\rho^{4k+3}} a b^{k+1}$,

   which is non-zero. This shows that $\tau^2 a \cdot \frac{Q}{\rho^{4k+4}} h_1^{4k+4}$ is also non-zero, and

   there is just one possible value.

Proposition 9.4. For all $k \geq 0$,

1. $h_0 \cdot \frac{\theta}{\rho^{4k+1}} h_1^2 = \theta \frac{\theta}{\rho^{4k+2}} a$.
2. $a \cdot \frac{\theta}{\rho^{4k+1}} h_1^2 = \frac{\theta}{\rho^{4k+3}} h_0 b$.
3. $\tau^2 a \cdot \frac{\theta}{\rho^{4k+1}} h_1^2 = \theta \frac{\theta}{\rho^{4k+3}} h_0 b$.

Proof.

1. $\left\langle \rho^{4k+3}, h_1, \tau h_1 \cdot h_1 \right\rangle h_0 = \frac{\theta}{\rho^{4k+3}} \left(h_1, \tau h_1 \cdot h_1, h_0\right)$.

2. Using part (1), we have that

   $h_0 a \cdot \frac{\theta}{\rho^{4k+1}} h_1^2 = a \cdot \frac{\theta}{\rho^{4k+1}} a$,

   which equals $\frac{\theta}{\rho^{4k+1}} h_1^2 b$ modulo a possible error term involving higher powers of $\rho$. Using that $h_1 a = 0$, we conclude that the error term is zero.
Using part (1), we have that
\[ h_0 \cdot \tau^2 a \cdot \frac{\theta}{\rho^4 \tau^{4k+1}} a \cdot \frac{\theta}{\rho^4 \tau^{4k+3}} a = \frac{\theta}{\rho^4 \tau^{4k+1}} h_1^2 b, \]
which is non-zero. This shows that \( \tau^2 a \cdot \frac{\theta}{\rho^4 \tau^{4k+1}} h_1^2 \) is also non-zero, and there is just one possible value.

\[ \square \]

Proposition 9.5. For all \( k \geq 0 \),

(1) \[ h_0 \cdot \frac{\theta}{\rho^4 \tau^{4k+1}} h_1 = \frac{\theta}{\rho^4 \tau^{4k+2}} h_1^2. \]

(2) \[ h_0 \cdot \frac{\theta}{\rho^4 \tau^{4k+2}} = \frac{\theta}{\rho^4 \tau^{4k+1}} h_1. \]

Proof. All of these extensions follow from Massey product shuffles:

(1) \[ h_0 \left( h_1, h_0, \frac{\theta}{\rho^4 \tau^{4k+2}} \right) = \left( h_0, h_1, h_0 \right) \frac{\theta}{\rho^4 \tau^{4k+2}}. \]

(2) \[ h_0 \left( \rho, \frac{\theta}{\rho^4 \tau^{4k+4}}, \tau h_1 \right) = \left( h_0, \rho, \frac{\theta}{\rho^4 \tau^{4k+4}} \right) \tau h_1. \]

\[ \square \]

Proposition 9.6. For all \( k \geq 0 \),

(1) \[ h_1 \cdot \frac{\theta}{\rho^4 \tau^{4k+1}} h_1^2 = \frac{\theta}{\rho^4 \tau^{4k+3}} a. \]

(2) \[ h_1 \cdot \frac{\theta}{\rho^4 \tau^{4k+3}} a = \frac{\theta}{\rho^4 \tau^{4k+6}} b. \]

Proof.

(1) \[ \tau h_1 \cdot h_1 \left( h_1, h_0, \frac{\theta}{\rho^4 \tau^{4k+6}} \right) = \left( \tau h_1 \cdot h_1, h_0 \right) \frac{\theta}{\rho^4 \tau^{4k+6}}. \]

Alternatively, this \( h_1 \) extension is forced by Proposition 5.1.

(2) We have
\[ h_1 \cdot \frac{\theta}{\rho^4 \tau^{4k+6}} a = \frac{\theta}{\rho^4 \tau^{4k+8}} h_1 \cdot \tau^2 a = \frac{\theta}{\rho^4 \tau^{4k+8}} \rho^3 b = \frac{\theta}{\rho^4 \tau^{4k+8}} a. \]

where the second equality follows from Table 2.

\[ \square \]

9.1. Ext\(_{C_2}\)(1). The charts in Section 12 depict Ext\(_{C_2}\)(1) graphically. Table 6 gives generators for Ext\(_{C_2}\)(1). The elements above the horizontal line are multiplicative generators for Ext\(_R\)(1). The elements below the horizontal generate Ext\(_{NC}\) in the following sense. Every element of Ext\(_{NC}\) can be formed by starting with one of these listed elements, multiplying by elements of Ext\(_R\)(1), and then dividing by \( \rho \).

The elements in Table 6 are not multiplicative for Ext\(_{C_2}\)(1) in the usual sense, because we allow for division by \( \rho \). For example, \( \frac{\theta}{\rho} h_1^2 \) is indecomposable in the usual sense, yet it does not appear in Table 6 because \( \rho \cdot \frac{\theta}{\rho} h_1^2 = \frac{\theta}{\rho^2} h_1^2 \) is decomposable.

The point of this notational approach is that the elements of Ext\(_{NC}\) are most easily understood as families of \( \rho \)-divisible elements.
Table 6. Generators for $\text{Ext}_{C_2}(1)$

| $mw$ | $(s, f, w)$ | element |
|------|-------------|---------|
| 0    | $(-1, 0, -1)$ | $\rho$  |
| 0    | $(0, 1, 0)$  | $h_0$   |
| 0    | $(1, 1, 1)$  | $h_1$   |
| 1    | $(1, 1, 0)$  | $\tau h_1$ |
| 2    | $(0, 1, -2)$ | $\tau^2 h_0$ |
| 2    | $(4, 3, 2)$  | $a$     |
| 4    | $(0, 0, -4)$ | $\tau^4$ |
| 4    | $(4, 3, 0)$  | $\tau^2 a$ |
| 4    | $(8, 4, 4)$  | $b$     |

Let $\text{Sp}$ denote the category of spectra, and let $\text{Sp}^{C_2}$ denote the category of “genuine” $C_2$-spectra \[M2,\] Chapter XII, obtained from the category of $C_2$-spaces by inverting the regular representation $(C, z \mapsto \overline{z})$. There are restriction and fixed-point functors $\iota^* : \text{Ho}(\text{Sp}^{C_2}) \to \text{Ho}(\text{Sp})$, $(-)^{C_2} : \text{Ho}(\text{Sp}^{C_2}) \to \text{Ho}(\text{Sp})$

which detect the homotopy theory of $C_2$-spectra, meaning that a map $f$ in $\text{Ho}(\text{Sp}^{C_2})$ is an equivalence if and only if $\iota^*(f)$ and $f^{C_2}$ are equivalences in $\text{Ho}(\text{Sp})$. Moreover, a sequence $X \to Y \to Z$ is a cofiber sequence in $\text{Ho}(\text{Sp}^{C_2})$ if and only if applying both functors $\iota^*$ and $(-)^{C_2}$ yield cofiber sequences. Both statements follow from the fact \[SS,\] Example 3.4(i) that the pair of $C_2$-spectra $\{\Sigma_{x}^\infty S^0, \Sigma_{x}^\infty C_2 + \}$ give a compact generating set for $\text{Ho}(\text{Sp}^{C_2})$. Beware that we are discussing actual fixed-point spectra here, not geometric fixed-point spectra.

Recall (cf. \[L,\] Proposition 3.3) that for a $C_2$-spectrum $X$, the equivariant connective cover $X(0) \xrightarrow{q} X$ is a $C_2$-spectrum such that:

1. $q(q)$ is the connective cover of the underlying spectrum $X$, and
2. $q^{C_2}$ is the connective cover of $X^{C_2}$.

Recall that $KOC_2$ is the $C_2$-spectrum representing the $K$-theory of $C_2$-equivariant real vector bundles \[M2,\] Ch. XIV.

**Definition 10.1.** Let $koC_2$ be the equivariant connective cover $KOC_2(0)$ of $KOC_2$.

We also have a description from the point of view of equivariant infinite loop space theory.

**Theorem 10.2.** \[Me,\] Theorem 7.1] $koC_2 \simeq K_{C_2}(R)$, where $R$ is considered as a topological ring with trivial $C_2$-action.

The underlying spectrum of $koC_2$ is $ko$, and its fixed-point spectrum $(koC_2)^{C_2}$ is $ko \vee ko$. The latter claim is a specialization of the statement that, if $X$ is any space equipped with a trivial $G$-action, then $KOG(X)$ is isomorphic to $RO(G) \otimes KO(X)$ \[M2,\] Section XIV.2. Alternatively, from the point of view of algebraic $K$-theory,
we have $\mathbb{K}_C(R)^{C_2} \simeq \mathbb{K}(R[C_2])$ [Me, Theorem 1.2], and $R[C_2] \cong R \times R$. It follows that

$$koC_2 \cong \mathbb{K}_C(R)^{C_2} \simeq \mathbb{K}(R) \times \mathbb{K}(R) \cong ko \vee ko.$$  

We are working towards a description of the $C_2$-equivariant cohomology of $koC_2$ as the quotient $\mathcal{A}^{C_2}/\mathcal{A}^{C_2}(1)$. This will allow us to express the $E_2$-page of the Adams spectral sequence for $koC_2$ in terms of the cohomology of $\mathcal{A}^{C_2}(1)$. The main step will be to establish the cofiber sequence of Proposition 10.12. In preparation, we first prove some auxiliary results.

**Definition 10.3.** Let $\rho$ be the element of $\pi_{-1,-1}$ determined by the inclusion $S^{0,0} \hookrightarrow S^{1,1}$ of fixed points.

Note that the element $\rho \in \pi_{-1,-1}$ induces multiplication by $\rho$ in cohomology under the Hurewicz homomorphism.

Recall that the real $C_2$-representation ring $RO(C_2)$ is a rank two free abelian group. Generators are given by the trivial one-dimensional representation $1$ and the sign representation $\sigma$. Let $A(C_2)$ denote the Burnside ring of $C_2$, defined as the Grothendieck group associated to the monoid of finite $C_2$-sets. This is also a rank two free abelian group, with generators the trivial one-point $C_2$-set and the free $C_2$-set $C_2$. As a ring, $A(C_2)$ is isomorphic to $\mathbb{Z}[[C_2]/(C_2^2 - 2C_2)]$.

The linearization map $A(C_2) \rightarrow RO(C_2)$ sending a $C_2$-set to the induced permutation representation is an isomorphism, sending the free orbit $C_2$ to the regular representation $1 \oplus \sigma$. Recall that the Euler characteristic moreover gives an isomorphism from $A(C_2)$ to $\pi_0(S^{0,0})$ [Se].

**Lemma 10.4.** The $C_2$-fixed points spectrum of $\Sigma^{1,1}koC_2$ is equivalent to $ko$.

**Proof.** Recall the cofiber sequence $C_2 + \xrightarrow{\pi} S^{0,0} \xrightarrow{\rho} S^{1,1}$ of $C_2$-spaces. This yields a cofiber sequence $C_2 + \wedge koC_2 \xrightarrow{\pi} koC_2 \xrightarrow{\rho} \Sigma^{1,1}koC_2$ of equivariant spectra. Passing to fixed point spectra gives the cofiber sequence

$$ko \xrightarrow{\pi C_2} ko \vee ko \xrightarrow{\rho C_2} (\Sigma^{1,1}koC_2)^{C_2}.$$  

The map $\pi C_2$ is induced by the split inclusion $\mathbb{Z} \rightarrow RO(C_2)$ that takes $1$ to the regular representation $pC_2$, and this induces a splitting of the cofiber sequence. Therefore, $(\Sigma^{1,1}koC_2)^{C_2}$ is equivalent to $ko$. \hfill \blacksquare

Recall that $k\mathbb{R}$ denotes the equivariant connective cover $K\mathbb{R}(0)$ of Atiyah’s $K$-theory ‘with reality’ spectrum $K\mathbb{R}$ [A]. The latter theory classifies complex vector bundles equipped with a conjugate-linear action of $C_2$. The underlying spectrum of $k\mathbb{R}$ is $ku$, and its fixed-point spectrum is $ko$.

**Theorem 10.5.** [Me, Theorem 7.2] $k\mathbb{R} \cong \mathbb{K}_C(\mathbb{C})$, where $\mathbb{C}$ is considered as a topological ring with $C_2$-action given by complex conjugation.

**Definition 10.6.** The $C_2$-equivariant Hopf map $\eta$ is

$$\mathbb{C}^2 - \{0\} \rightarrow \mathbb{C}P^1 : (x, y) \mapsto [x : y],$$  

where both source and target are given the complex conjugation action.

As $\mathbb{C} \cong R[C_2]$, the punctured representation $\mathbb{C}^2 - \{0\}$ is homotopy equivalent to $S^{3,2}$, and $\mathbb{C}P^1$ is homeomorphic to $S^{2,1}$. It follows that $\eta$ gives rise to a stable homotopy class in $\pi_{1,1}$. 


**Remark 10.7.** The element $\eta$ only defines a specific element of $\pi_{1,1}$ after choosing isomorphisms $S^2 - \{0\} \cong S^{3,2}$ and $\mathbb{CP}^1 \cong S^{2,1}$ in the homotopy category. We follow the choices of [DI, Example 2.12]. By Proposition C.5 of [DI], with these choices, the induced map $\eta^{C_2}: S^1 \to S^1$ on fixed points is a map of degree $-2$.

**Lemma 10.8.** The element $\eta\rho$ in $\pi_{0,0}$ corresponds to the element $C_2 - 2$ of $A(C_2)$.

**Proof.** In $\pi_{0,0}$, we have $(\eta\rho)^2 = -2\eta\rho$ ([DI, Mo]). The non-zero solutions to $x^2 = -2x$ in $A(C_2)$ are $x = -2$, $x = C_2 - 2$, and $x = -C_2$. The only such solution which restricts to zero at the trivial subgroup is $x = C_2 - 2$. 

**Lemma 10.9.** The induced map $\eta^{C_2}: (\Sigma^{1,1}k_{O_{C_2}})^{C_2} \to (k_{O_{C_2}})^{C_2}$ is equivalent to $ko \xrightarrow{(-1,1)} ko \vee ko$.

**Proof.** To determine the fixed map $\eta^{C_2}$, we use that a map $X \to Y$ of $C_2$-spectra induces a commutative diagram

$$
\begin{array}{ccc}
X^{C_2} & \xrightarrow{\eta^{C_2}} & Y^{C_2} \\
\downarrow & & \downarrow \\
X^e & \xrightarrow{\eta^e} & Y^e,
\end{array}
$$

in which the vertical maps are the inclusions of fixed points. In the case of $\eta$ on $k_{O_{C_2}}$, this gives the diagram

$$
\begin{array}{ccc}
ko \cong (\Sigma^{1,1}k_{O_{C_2}})^{C_2} & \xrightarrow{\eta^{C_2}} & ko \vee ko \cong (k_{O_{C_2}})^{C_2} \\
\downarrow & & \downarrow \\
\Sigma^{1}ko & \xrightarrow{\iota^*\eta} & ko,
\end{array}
$$

where $\nabla$ is the fold map. This shows that $\eta^{C_2}$ factors through the fiber of $\nabla$, so that $\eta^{C_2}$ must be of the form $(k,-k)$ for some integer $k$. On the other hand, we have the commutative diagram

$$
\begin{array}{ccc}
ko \otimes RO(C_2) & \xrightarrow{\cong} & ko \\
\downarrow & & \downarrow \\
(k_{O_{C_2}})^{C_2} & \xrightarrow{\eta^{C_2}} & (k_{O_{C_2}})^{C_2} \\
\downarrow & & \downarrow \\
(S^{0,0})^{C_2} & \xrightarrow{\eta^{C_2}} & (S^{0,0})^{C_2}
\end{array}
\quad \quad
\begin{array}{ccc}
ko \otimes RO(C_2) & \xrightarrow{\cong} & ko \\
\downarrow & & \downarrow \\
(k_{O_{C_2}})^{C_2} & \xrightarrow{\eta^{C_2}} & (k_{O_{C_2}})^{C_2} \\
\downarrow & & \downarrow \\
(S^{0,0})^{C_2} & \xrightarrow{\eta^{C_2}} & (S^{0,0})^{C_2}
\end{array}
$$

According to Lemma 10.8, on the sphere $\eta\rho$ induces multiplication by $(C_2 - 2)$ under the isomorphism $\pi_{0,0} \cong A(C_2)$. The outer vertical compositions induce the linearization isomorphism $A(C_2) \cong RO(C_2)$ on $\pi_0$. It follows that the top row induces multiplication by $(\sigma - 1)$ on homotopy. We conclude that $\eta^{C_2}$ is $(-1,1)$. 

**Definition 10.10.** The complexification map $KOC_2 \xrightarrow{\simeq} k\mathbb{R}$ assigns to an equivariant real bundle $E \to X$ the associated bundle $\mathbb{C} \otimes_k E \to X$, where $C_2$ acts on $\mathbb{C}$ via complex conjugation. We denote by $k_{O_{C_2}} \xrightarrow{\simeq} k\mathbb{R}$ the associated map on connective covers.
Remark 10.11. Alternatively, from the point of view algebraic \( K \)-theory, the complexification map can be described as \( \mathbb{K}_{C_2}(\iota) \), where \( \mathbb{R} \to \mathbb{C} \) is the inclusion of \( C_2 \)-equivariant topological rings.

Proposition 10.12. The Hopf map \( \eta \) induces a cofiber sequence

\[
\Sigma^{1,1} koC_2 \overset{\eta}{\to} koC_2 \overset{c}{\to} ku.
\]

Proof. On underlying spectra, this is the classical cofiber sequence

\[
\Sigma ko \overset{\eta}{\to} ko \overset{c}{\to} ku.
\]

On fixed points, according to Lemma 10.4 the sequence (10.1) induces a sequence

\[
ko \overset{\eta C_2}{\to} ko \vee ko \overset{c C_2}{\to} ko.
\]

By Lemma 10.9, the map \( \eta C_2 \) is of the form \((-1, 1)\). For any real \( C_2 \)-representation \( V \), the construction \( \mathbb{C} \otimes \mathbb{R} V \) only depends on the dimension of \( V \), which implies that \( c C_2 \) is the fold map. So the fixed points sequence is also a cofiber sequence. ■

Remark 10.13. From the point of view of spectral Mackey functors [GM, Ba], the cofiber sequence (10.1) is the cofiber sequence of Mackey functors

\[
\begin{array}{cccccc}
ko & \overset{(1, -1)}{\to} & ko \vee ko & \overset{c}{\to} & ko \\
\Sigma^{1,1} ko & \overset{\eta}{\to} & ko & \overset{c}{\to} & ku \\
& & \text{sign} & \text{triv} & \text{conj}
\end{array}
\]

where \( ku \overset{r}{\to} ko \) considers a rank \( n \) complex bundle as a rank \( 2n \) real bundle.

Remark 10.14. It is also possible to build \( koC_2 \) using the “Tate diagram” approach. See, for example, [G] for a nice description of this approach. According to this approach, one specifies a \( C_2 \)-spectrum \( X \) by giving three pieces of data:

1. an underlying spectrum \( X^e \) with \( C_2 \)-action,
2. a geometric fixed points spectrum \( X^F C_2 \), and
3. a map \( X^F C_2 \to (X^e)^T C_2 \) from the geometric fixed points to the Tate construction.

In our case, we would take: (1) the spectrum \( ko \) with trivial \( C_2 \)-action, and (2) a wedge \( \bigvee_{k \geq 0} \Sigma^{4k} H\hat{Z}2 \). The Tate construction \( ko^{C_2} \) was computed by Davis-Mahowald to be \( \bigvee_{n \in \mathbb{Z}} \Sigma^{4n} H\hat{Z}2 \) [DM2, Theorem 1.4]. For the interpretation of the Davis-Mahowald calculation in terms of the Tate construction, see [M2, Section XXI.3]. We may then take the map (3) to be the connective cover.

Theorem 10.15. The \( C_2 \)-equivariant cohomology of \( koC_2 \), as a module over \( A^{C_2} \), is

\[
H^*_C(koC_2; \mathbb{F}_2) \cong A^{C_2} / A^{C_2}(1).
\]

Proof. According to [R, Corollary 6.19], we have \( H^*_C(k\mathbb{R}) \cong A^{C_2} / E^{C_2}(1) \), where \( E^{C_2}(1) \) is the \( \mathbb{M}^{C_2}_2 \)-subalgebra of \( A^{C_2} \) generated by \( Q_0 \) and \( Q_1 \).
Since $\eta$ induces the trivial map on equivariant cohomology, the sequence \((10.1)\) induces a short exact sequence
\[
0 \longrightarrow H_{C_2}^{s-2,s-1}(k\mathcal{O}_{C_2}) \xrightarrow{i} \mathcal{A}_{C_2}^{s}/E^{C_2}(1) \xrightarrow{j} H_{C_2}^{s}(k\mathcal{O}_{C_2}) \longrightarrow 0
\]
of $\mathcal{A}_{C_2}$-modules.

The cofiber $C\eta$ is a 2-cell complex that supports a $\text{Sq}^2$ in cohomology. It follows that the composition
\[
k\mathbb{R} \simeq k\mathcal{O}_{C_2} \wedge C(\eta) \longrightarrow \Sigma^{2-1}k\mathcal{O}_{C_2} \hookrightarrow \Sigma^{2-1}k\mathcal{O}_{C_2} \wedge C(\eta)
\]
induces the map
\[
\mathcal{A}_{C_2}^{s}/E^{C_2}(1) \xrightarrow{i} \mathcal{A}_{C_2}^{s}/E^{C_2}(1) : 1 \mapsto \text{Sq}^2.
\]
In particular, the composition $\mathcal{A}_{C_2} \rightarrow \mathcal{A}_{C_2}^{s}/E^{C_2}(1) \xrightarrow{j} H_{C_2}^{s}(k\mathcal{O}_{C_2})$ factors through $\mathcal{A}_{C_2}^{s}/E^{C_2}(1)$. Given the right $E^{C_2}(1)$-module decomposition $\mathcal{A}_{C_2}^{s}(1) \cong E^{C_2}(1) \oplus \Sigma^{2,1}E^{C_2}(1)$, it follows that the sequence \((10.2)\) sits in a diagram
\[
\begin{array}{cccccc}
0 & \longrightarrow & H_{C_2}^{2,s-1}(k\mathcal{O}_{C_2}) & \longrightarrow & \mathcal{A}_{C_2}^{s}/E^{C_2}(1) & \longrightarrow & H_{C_2}^{s}(k\mathcal{O}_{C_2}) & \longrightarrow & 0 \\
\uparrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma^{2-1}\mathcal{A}_{C_2}^{s}/\mathcal{A}_{C_2}^{s}(1) & \longrightarrow & \mathcal{A}_{C_2}^{s}/E^{C_2}(1) & \longrightarrow & \mathcal{A}_{C_2}^{s}/\mathcal{A}_{C_2}^{s}(1) & \longrightarrow & 0.
\end{array}
\]
The outer two maps agree up to suspension, so they are both isomorphisms. ■

**Corollary 10.16.** The $E_2$-page of the Adams spectral sequence for $k\mathcal{O}_{C_2}$ is
\[
E_2 \cong \text{Ext}_{\mathcal{A}_{C_2}}(H_{C_2}^{s}(k\mathcal{O}_{C_2}), M_{C_2}^{s}) \cong \text{Ext}_{C_2}(1).
\]

**Proof.** This is a standard change of rings isomorphism, using that $H_{C_2}^{s}(k\mathcal{O}_{C_2})$ is isomorphic to $\mathcal{A}_{C_2}^{s}/\mathcal{A}_{C_2}^{s}(1)$. ■

### 11. The Homotopy Ring

In this section, we will describe the bigraded homotopy ring $\pi_{s,t}(k\mathcal{O}_{C_2})$ of $k\mathcal{O}_{C_2}$. We are implicitly completing the homotopy groups appropriately so that the Adams spectral sequence converges.

It turns out that the Adams spectral sequence collapses, so that $\text{Ext}_{C_2}(1)$ is an associated graded object of $\pi_{s,t}(k\mathcal{O}_{C_2})$. Nevertheless, the Adams spectral sequence hides much of the multiplicative structure.

Recall that the Milnor-Witt stem of $X$ is defined (cf. [DI2]) as the direct sum
\[
\Pi_n(X) \cong \bigoplus_i \pi_{n+i,i}(X).
\]

**Proposition 11.1.** There are no differentials in the Adams spectral sequence for $k\mathcal{O}_{C_2}$.

**Proof.** This follows by inspection of the $E_2$-page, shown in the charts in Section 12. Adams differentials decrease the Milnor-Witt stem by 1. Every class in Milnor-Witt stem congruent to 3 modulo 4 is infinitely $\rho$-divisible. As there are no infinitely $\rho$-divisible classes in Milnor-Witt stem congruent to 2 modulo 4, it follows that there are no differentials supported in the Milnor-Witt $(4k + 3)$-stem.

Every class in Milnor-Witt stem $4k$ supports an infinite tower of either $h_0$-multiples or $h_1$-multiples, while there are no such towers in Milnor-Witt stem.
Table 7. Notation for $\pi_{*,*}(koC_2)$

| mw     | (s, w)   | element detected by | defining relation |
|--------|----------|---------------------|-------------------|
| 0      | (−1, −1) | $\rho$             | $\rho$            |
| 0      | (1, 1)   | $\eta$             | $h_1$            |
| 0      | (4, 4)   | $\alpha$           | $Q^3 \eta_1$     |
| 0      | (0, 0)   | $\omega$           | $h_0$            |
| 0      | (0, −4)  | $\tau^4$           | $\tau^4$         |
| 0      | (8, 8)   | $\beta$            | $Q^4 \eta_1$     |
| 2      | (0, −2)  | $\tau^2 \omega$    | $\tau^2 h_0$    |
| −2     | (0, 2)   | $\frac{\omega}{\rho}$ | $\frac{\omega}{\rho}$ |
| −4     | (0, 4)   | $\frac{\omega}{\rho^2}$ | $\frac{\omega}{\rho^2}$ |
| −k − 1 | (0, k + 1)| $\frac{\omega}{\rho^k}$ | $\frac{\omega}{\rho^k}$ |
| 1      | (1, 0)   | $\tau \eta$        | $\tau h_1$       |
| 2      | (4, 2)   | $\tau^2 \alpha$    | $a$              |

4k + 1. It follows that there cannot be any differentials emanating from the (4k + 1)-Milnor-Witt-stem. Finally, direct inspection shows there cannot be any differentials starting in the Milnor-Witt (4k + 2) or 4k-stems. ■

The structure of the Milnor-Witt n-stem $\Pi_n(koC_2)$ of course depends on $n$. The description of these Milnor-Witt stems naturally breaks into cases, depending on the value of $n \pmod{4}$.

Table 7 summarizes the notation that we will use for specific elements of $\pi_{*,*}(koC_2)$. The definition of each element is discussed in detail in the following sections.

11.1. The Milnor-Witt 0-stem. Our first task is to describe the Milnor-Witt 0-stem $\Pi_0(koC_2)$. The other Milnor-Witt stems are modules over $\Pi_0(koC_2)$, and we will use this module structure heavily in order to understand them.

Proposition 11.2. Let $X$ be a $C_2$-equivariant spectrum, and let $\alpha$ belong to $\pi_{n,k}(X)$. The element $\alpha$ is divisible by $\rho$ if and only if its underlying class $\rho^* \eta^* \alpha$ in $\pi_{n}(\eta^* X)$ is zero.

Proof. The $C_2$-equivariant cofiber sequence

$$C_2 + \longrightarrow S^{0,0} \overset{\rho}{\longrightarrow} S^{1,1}$$

induces a long exact sequence

$$\cdots \longrightarrow \pi_{n+1,k+1}(X) \overset{\rho}{\longrightarrow} \pi_{n,k}(X) \overset{\rho^*}{\longrightarrow} \pi_{n}(\eta^* X) \longrightarrow \pi_{n+2,k+1}(X) \overset{\rho}{\longrightarrow} \cdots .$$

Proposition 11.3. There is a hidden $\rho$ extension from $Q^3 h_1$ to $h_3$ in the Adams spectral sequence.

Proof. Recall that $\eta^3$ is zero in $\pi_3(ko)$. Proposition 11.2 implies that $\eta^3$ in $\pi_{3,3}(koC_2)$ is divisible by $\rho$. The only possibility is that there is a hidden extension from $Q^3 h_1$ to $h_3$. ■

Proposition 11.4. The element $\eta$ in $\pi_{1,1}(koC_2)$ is detected by $h_1$. ■
Proof. The restriction $\iota^*(\eta)$ of $\eta$ is the classical $\eta$, which is detected by the classical element $h_1$. As all other elements of $\text{Ext}_{A^C_{1(1)}}$ in the 1-stem and weight 1 all live in higher filtration, the result follows.

**Definition 11.5.** Let $\alpha$ be an element in $\pi_{4,4}(koC_2)$ detected by $\frac{Q}{\rho} h_1^3$ such that $\rho \alpha = \eta^3$.

Proposition 11.3 guarantees that such an element $\alpha$ exists.

There are many elements of $\pi_{4,4}$ detected by $\frac{Q}{\rho} h_1^3$ because of the presence of elements in higher Adams filtration. The condition $\rho \alpha = \eta^3$ narrows the possibilities, but still does not determine a unique element because of the elements $\frac{Q}{\rho} h_0^k a$ in higher Adams filtration. For our purposes, this remaining choice makes no difference.

**Definition 11.6.** Let $\omega$ be the element $\eta \rho + 2$ of $\pi_{0,0}(koC_2)$.

As for $\rho$ and $\eta$, the element $\omega$ comes from the homotopy groups of the equivariant sphere spectrum. Strictly speaking, there is no need for the notation $\omega$ since it can be written in terms of other elements. Nevertheless, it is convenient because $\omega$ plays a central role.

Note that $\omega$ is detected by $h_0$, while 2 is detected by $h_0 + \rho h_1$. For this reason, $\omega$, rather than 2, plays the role of the zeroth Hopf map in the equivariant (and $\mathbb{R}$-motivic) context. Also note that $\omega$ equals $1 - \epsilon$, where $\epsilon$ is the twist $S^1,1 \rightarrow S^1,1 \wedge S^1,1$.

**Proposition 11.7.** The homotopy class $\eta^5$ is divisible by 2.

*Proof.* The relation $\omega \eta = 0$ was established by Morel [Mo] in the $\mathbb{R}$-motivic stable stems, and the equivariant stems agree with the $\mathbb{R}$-motivic ones in the relevant degrees [DI3, Theorem 4.1]. (See also [DI] for a geometric argument for this relation given in the motivic context. This geometric argument works just as well equivariantly.)

Using the defining relation for $\alpha$, it follows that

$$-2 \eta \alpha = \rho \eta^3 \alpha = \eta^5.$$ 

Proposition 11.7 was already known to be true in the homotopy of the $C_2$-equivariant sphere spectrum [B]. The divisibility of the elements $\eta^k$ is very much related to work of Landweber [La].

**Definition 11.8.** Let $\tau^4$ be an element of $\pi_{0,-4}(koC_2)$ that is detected by $\tau^4$.

The element $\tau^4$ is not uniquely determined because of elements in higher Adams filtration. For our purposes, we may choose an arbitrary such element.

**Proposition 11.9.**

1. There is a hidden $\tau^4$ extension from $\frac{Q}{\rho} h_1^3$ to $\tau^2 a$.
2. There is a hidden $\tau^4$ extension from $\frac{Q}{\rho} h_1^3$ to $b$.

*Proof.*

1. The product $\rho \alpha \cdot \tau^4$ equals $\tau^4 \cdot \eta^3$, which is detected by $\tau^4 \cdot h_1^3$. This last expression equals $\rho \cdot \tau^2 a$ in Ext.
Let \( b \) detect \( \pi \) Adams filtration, because there is no 2-torsion in \( \beta \) isfying 4 \( \tau \) is an isomorphism. By considering the effect of multiplication by \( [\tau] \) is also an isomorphism. Thus it suffices to show that \((\tau^4)^2 \alpha^2 \) is 4-divisible in \( \pi_{8,0}(k\text{o}_C_2) \). But \((\tau^2)^2 \cdot \alpha^2 \) is detected by \((\tau^2)^2 \) by Proposition 11.9 (1), which equals \((h_0 + \rho h_1)^2 \tau^4 b \) in Ext. Finally, observe that \( h_0 + \rho h_1 \) detects 2.

**Lemma 11.10.** The class \( \alpha^2 \) in \( \pi_{8,0}(k\text{o}_C_2) \) is divisible by 4.

**Proof.** By Proposition 11.9, the multiplication map

\[
\tau^4 : \pi_{8,0}(k\text{o}_C_2) \xrightarrow{\cong} \pi_{8,4}(k\text{o}_C_2)
\]

is an isomorphism. By considering the effect of multiplication by \( \tau^4 \) in Ext, we see that

\[
\tau^4 : \pi_{8,4}(k\text{o}_C_2) \xrightarrow{\cong} \pi_{8,0}(k\text{o}_C_2)
\]

is also an isomorphism. Thus it suffices to show that \((\tau^4)^2 \alpha^2 \) is 4-divisible in \( \pi_{8,0}(k\text{o}_C_2) \). But \((\tau^2)^2 \cdot \alpha^2 \) is detected by \((\tau^2)^2 \) by Proposition 11.9 (1), which equals \((h_0 + \rho h_1)^2 \tau^4 b \) in Ext. Finally, observe that \( h_0 + \rho h_1 \) detects 2.

**Definition 11.11.** Let \( \beta \) be the element of \( \pi_{8,8}(k\text{o}_C_2) \) detected by \( \frac{Q}{\rho^3} h_1^4 \) and satisfying \( 4\beta = \alpha^2 \).

Note that \( \beta \) is uniquely determined, even though there are elements of higher Adams filtration, because there is no 2-torsion in \( \pi_{8,8}(k\text{o}_C_2) \).

**Proposition 11.12.** \( \rho^3 \beta = \eta \alpha \).

**Proof.** The defining relation for \( \beta \) implies that \( 4\rho^3 \beta \) equals \( \rho^3 \alpha^2 \), which equals \( \rho \beta \alpha^2 \) by the defining relation for \( \alpha \). Using the relation \((\eta \rho + 2) \eta = 0 \), this element equals \( 4\beta \eta \). Finally, there is no 2-torsion in \( \pi_{5,5}(k\text{o}_C_2) \).

**Proposition 11.13.** The (2-completed) Milnor-Witt 0-stem of \( k\text{o}_C_2 \) is

\[
H_0(k\text{o}_C_2) \cong \mathbb{Z}_2[\eta, \rho, \alpha, \beta]/(\rho(\eta \rho + 2), \eta(\eta \rho + 2), \rho \alpha - \eta \beta, \rho^3 \beta - \eta \alpha, \alpha^2 - 4\beta),
\]

where the generators have degrees \((1, 1)\), \((-1, -1)\), \((4, 4)\), and \((8, 8)\) respectively. These homotopy classes are detected by \( h_1 \), \( \rho \), \( \frac{Q}{\rho^3} h_1^4 \), and \( \frac{Q}{\rho^4} h_1^4 \) in the Adams spectral sequence.

**Proof.** The relations \( \rho(\eta \rho + 2) \) and \( \eta(\eta \rho + 2) \) are already true in the sphere [Mo][DI]. The third and fifth relations are part of the definitions of \( \alpha \) and \( \beta \), while the fourth relation is Proposition 11.12.

It remains to show that \( \beta^k \) is detected by \( \frac{Q}{\rho^3} h_1^{4k+4} \) and that \( \alpha \beta^k \) is detected by \( \frac{Q}{\rho^k} h_1^{4k+4} \).

We assume for induction on \( k \) that \( \beta^k \) is detected by \( \frac{Q}{\rho^k} h_1^{4k} \). We have the relation \( h_0 \cdot \frac{Q}{\rho^3} h_1^4 = \frac{\theta}{\rho^3 + \rho \alpha} b^k \) in Ext, so \( \omega \beta^k \) is detected by \( \frac{\theta}{\rho^3 + \rho \alpha} b^k \) in Ext. Now \( b \) detects \( \tau^4 \) by Proposition 11.9 (2), so \( \omega \beta^k + 1 \) is detected by \( \frac{\theta}{\rho^3 + \rho \alpha} b^{k+1} \). Finally, \( \frac{\theta}{\rho^3 + \rho \alpha} b^{k+1} \) equals \( \tau^4 \cdot \frac{\theta}{\rho^3 + \rho \alpha} b^{k+1} \) in Ext, which equals \( \tau^4 \cdot h_0 \cdot \frac{Q}{\rho^4} h_1^{4k+4} \).

We have now shown that \( \tau^4 \cdot h_0 \cdot \frac{Q}{\rho^4} h_1^{4k+4} \) detects \( \tau^4 \cdot \omega \beta^k + 1 \). It follows that \( \frac{Q}{\rho^3} h_1^{4k+4} \) detects \( \beta^k + 1 \).

A similar argument handles the case of \( \alpha \beta^k \).
11.2. \( \tau^4 \)-periodicity. Before analyzing the other Milnor-Witt stems of \( ko_{C_2} \), we will explore a piece of the global structure involving the element \( \tau^4 \) of \( \pi_{-4}(ko_{C_2}) \).

**Proposition 11.14.** There are hidden \( \tau^4 \) extensions:

1. from \( \frac{\rho}{\tau^2} \cdot a \) to \( \tau^2 h_0 \).
2. from \( \frac{\rho}{\tau^2} h_1^2 \) to \( a \).
3. from \( \frac{\rho}{\tau^2} \) to \( h_0 \).
4. from \( \frac{\rho}{\tau^2} \) to \( \tau h_1 \).

**Proof.**

1. Recall that \( \frac{\rho}{\tau^2} \cdot a \) equals \( h_0 \cdot \frac{\rho}{\tau^2} h_1^3 \) in Ext, so the hidden \( \tau^4 \) extension on \( \frac{\rho}{\tau^2} h_1^3 \) from Proposition 11.9 (1) implies that there is a hidden \( \tau^4 \) extension from \( \frac{\rho}{\tau^2} \cdot a \) to \( \tau^2 h_0 a \). It follows that there is a hidden \( \tau^4 \) extension from \( \frac{\rho}{\tau^2} \) to \( \tau^2 h_0 \).

2. Using that \( h_2^2 \cdot \tau^2 h_0 \) equals \( \rho^2 a \) in Ext, part (1) implies that there is a hidden \( \tau^4 \) extension from \( \frac{\rho}{\tau^2} h_2^2 \) to \( \rho^2 a \).

3. Recall that \( \frac{\rho}{\tau^2} \cdot b \) equals \( h_0 \cdot \frac{\rho}{\tau^2} h_1^4 \) in Ext, so the hidden \( \tau^4 \) extension on \( \frac{\rho}{\tau^2} h_1^4 \) from Proposition 11.9 (2) implies that there is a hidden \( \tau^4 \) extension from \( \frac{\rho}{\tau^2} \cdot b \) to \( h_0 b \). It follows that there is a hidden \( \tau^4 \) extension from \( \frac{\rho}{\tau^2} \) to \( h_0 \).

4. Using that \( \rho a \) equals \( h_1 (\tau h_1)^2 \) in Ext, part (2) implies that there is a hidden \( \tau^4 \) extension from \( \frac{\rho}{\tau^2} h_1^2 \) to \( h_1 (\tau h_1)^2 \). Now \( \frac{\rho}{\tau^2} h_1^2 \) equals \( \frac{\rho}{\tau^2} h_1 \cdot \tau h_1 \), so there is also a hidden \( \tau^4 \) extension on \( \frac{\rho}{\tau^2} \).

The homotopy of \( ko_{C_2} \) is nearly \( \tau^4 \)-periodic, in the following sense.

**Theorem 11.15.** Multiplication by \( \tau^4 \) gives a homomorphism on Milnor-Witt stems

\[ \Pi_n(ko_{C_2}) \rightarrow \Pi_{n+4}(ko_{C_2}) \]

which is

1. injective if \( n = -4 \).
2. surjective (and zero) if \( n = -5 \).
3. bijective in all other cases.

**Proof.**

1. This is already true in Ext, except in the 0-stem. But the 0-stem is handled by Proposition 11.14(3).

2. There is nothing to prove here, given that \( \Pi_{-1}(ko_{C_2}) = 0 \).

3. We give arguments depending on the residue of \( n \) modulo 4.

   - \( n \equiv 0 \pmod{4} \): If \( n < -4 \), this is already true in Ext. For \( n \geq 0 \), this follows from the relation \( \rho a = \eta^3 \) and the hidden \( \tau^4 \) extensions on \( \alpha \) and \( \beta \) given in Proposition 11.9.
   - \( n \equiv 1 \pmod{4} \): For \( n < -3 \), this is already true in Ext. For \( n \geq -3 \), this follows from Proposition 11.14(4).
   - \( n \equiv 2 \pmod{4} \): For \( n < -2 \), this is already true in Ext. For \( n \geq -2 \), this follows from Proposition 11.14(1) and (2).
   - \( n \equiv 3 \pmod{4} \): This is already true in Ext.
11.3. The Milnor-Witt $n$-stem with $n \equiv 0 \pmod{4}$. Theorem 11.15 indicates that $\tau^4$ multiplications are useful in describing the structure of the homotopy groups of $kOC_2$. Therefore, our next task is to build on our understanding of $\Pi_0(kOC_2)$ and to describe the subring $\bigoplus_{k \in \mathbb{Z}} \Pi_{4k}(kOC_2)$ of $\pi_* kOC_2$.

The Ext charts indicate that the behavior of these groups differs for $k \geq 0$ and for $k < 0$.

**Proposition 11.16.** $\bigoplus_{k \geq 0} \Pi_{4k}(kOC_2)$ is isomorphic to $\Pi_0(kOC_2)[\tau^4]$.

**Proof.** This follows immediately from Theorem 11.15. ■

**Definition 11.17.** Define $\tau^2 \omega$ to be an element in $\pi_{0,-2}(kOC_2)$ that is detected by $\tau^2 h_0$ such that $(\tau^2 \omega)^2 = 2\omega \cdot \tau^4$.

An equivalent way to specify a choice of $\tau^2 \omega$ is to require that the underlying map $i^*(\tau^2 \omega)$ equals 2 in $\pi_0(k\omega)$.

**Definition 11.18.** For $k \geq 1$, let $\frac{\Theta}{\rho \tau^4}$ be an element of $\pi_{0,k+1}$ detected by $\frac{\theta}{\rho \tau^4}$ such that:

1. $\tau^4 \cdot \frac{\theta}{\rho \tau^4} = \tau^2 \omega$.
2. $\tau^4 \cdot \frac{\Theta}{\rho \tau^4} = \omega$.
3. $\tau^4 \cdot \frac{\Theta}{\rho \tau^4} = \frac{\Theta}{\rho \tau^4}$ when $k \geq 5$.

The elements $\frac{\Theta}{\rho \tau^4}$ are uniquely determined by the stated conditions. Proposition 11.14 (1) and (3) allow us to choose $\frac{\Theta}{\rho \tau^4}$ and $\frac{\Theta}{\rho \tau^4}$ with the desired properties.

**Proposition 11.19.** As a $\pi_0(kOC_2)[\tau^4]$-module, $\bigoplus_{k \in \mathbb{Z}} \Pi_{4k}(kOC_2)$ is isomorphic to the $\pi_0(kOC_2)[\tau^4]$-module generated by 1 and the elements $\frac{\Theta}{\rho \tau^4}$, subject to the relations

1. $\tau^4 \cdot \frac{\Theta}{\rho \tau^4} = \frac{\Theta}{\rho \tau^4}$.
2. $\rho \cdot \frac{\Theta}{\rho \tau^4} = 0$.
3. $\eta \cdot \frac{\Theta}{\rho \tau^4} = 0$.
4. $\tau^4 \cdot \frac{\Theta}{\rho \tau^4} = \omega$.

**Proof.** This follows by inspection of the Ext charts, together with the defining relations for $\frac{\Theta}{\rho \tau^4}$. ■

11.4. The Milnor-Witt $n$-stem with $n \equiv 1 \pmod{4}$.

**Definition 11.20.** Denote by $\tau \eta$ an element of $\pi_{1,0}(kOC_2)$ that is detected by $\tau h_1$.

Note that $\tau \eta$ is not uniquely determined because of elements in higher Adams filtration, but the choice makes no practical difference. One way to specify a choice of $\tau \eta$ is to use the composition

$$S^{1,0} \to S^{0,0} \to kOC_2,$$

where the first map is the image of the classical Hopf map $\eta : S^1 \to S^0$, and the second map is the unit.

**Proposition 11.21.** As a $\Pi_0(kOC_2)[\tau^4]$-module, there is an isomorphism

$$\bigoplus_{k \in \mathbb{Z}} \Pi_{1+4k}(kOC_2) \cong \{\Pi_0(kOC_2)[(\tau^4)^{\pm 1}]/(2, \rho^2, \eta^2, \omega)\} \{\tau \eta\}. $$
Proof. This follows from inspection of the Ext charts, together with Theorem 11.15. ■

11.5. The Milnor-Witt $n$-stem with $n \equiv 2 \pmod{4}$. Recall from Definition 11.17 that $\tau^2 \omega$ is an element of $\pi_{n,2}(kOC_2)$ that is detected by $\tau^2 h_0$.

Lemma 11.22. The product $\alpha \cdot \tau^2 \omega$ in $\pi_{4,2}(kOC_2)$ is detected by $h_0 \alpha$.

Proof. The product $\tau^4 \cdot \alpha \cdot \tau^2 \omega$ is detected by $\tau^4 h_0 \alpha$ by Proposition 11.9 (1). ■

Definition 11.23. Define $\tau^2 \alpha$ to be an element of $\pi_{4,2}(kOC_2)$ that is detected by $\alpha$ such that $2 \cdot \tau^2 \alpha$ equals $\alpha \cdot \tau^2 \omega$.

Proposition 11.24. As a $\Pi_0(kOC_2)[\tau^4]$-module, $\bigoplus_{k \in \mathbb{Z}} \Pi_{2+4k}(kOC_2)$ is isomorphic to the free $\Pi_0(kOC_2)[(\tau^4)^{\pm 1}]$-module generated by $\tau^2 \omega$, $(\tau \eta)^2$, and $\tau^2 \alpha$, subject to the relations

1. $\rho \cdot \tau^2 \omega = 0$.
2. $\alpha \cdot \tau^2 \omega = 2 \cdot \tau^2 \alpha$.
3. $\rho(\tau \eta)^2 = \eta \cdot \tau^2 \omega$.
4. $2(\tau \eta)^2 = 0$.
5. $\eta(\tau \eta)^2 = \rho \cdot \tau^2 \alpha$.
6. $\alpha(\tau \eta)^2 = 0$.
7. $\eta \cdot \tau^2 \alpha = 0$.
8. $\alpha \cdot \tau^2 \alpha = 2 \beta \cdot \tau^2 \omega$.

Proof. Except for the last relation, this follows from inspection of the Ext charts, together with Theorem 11.15.

For the last relation, use that $2 \alpha \cdot \tau^2 \alpha$ equals $\tau^2 \omega \cdot \alpha^2$ by the definition of $\tau^2 \alpha$, and that $\tau^2 \omega \cdot \alpha^2$ equals $4 \beta \cdot \tau^2 \omega$ by the defining relation for $\beta$. ■

11.6. The Milnor-Witt $n$-stem with $n \equiv 3 \pmod{4}$. The structure of $\bigoplus_{k \in \mathbb{Z}} \Pi_{4k+3}(kOC_2)$ is qualitatively different than the other cases because it contains elements that are infinitely divisible by $\rho$. The Ext charts show that $\bigoplus_{k \in \mathbb{Z}} \Pi_{4k+3}(kOC_2)$ is concentrated in the range $k \leq -2$.

The elements $\frac{\Theta}{\rho \tau^l}$ are infinitely divisible by both $\rho$ and $\tau^4$. We write $\frac{\Theta}{\rho^{l+1} \tau^m}$ for an element such that $\rho^l \cdot \frac{\Theta}{\rho^{l+1} \tau^m}$ equals $\frac{\Theta}{\rho^{l+1} \tau^{m+1}}$.

By inspection of the Ext charts, we see that $\bigoplus_{k \leq 0} \Pi_{4k-5}(kOC_2)$ is generated as an abelian group by the elements $\frac{\Theta}{\rho^l \tau^m}$. The $\Pi_0(kOC_2)[\tau^4]$-module structure on $\bigoplus_{k \leq 0} \Pi_{4k-5}(kOC_2)$ is then governed by the orders of these elements, together with the relations

$$\alpha \cdot \frac{\Theta}{\rho \tau^l} = -8 \frac{\Theta}{\rho^5 \tau^{4k}}$$

and

$$\beta \cdot \frac{\Theta}{\rho \tau^l} = 16 \frac{\Theta}{\rho^7 \tau^{4k}}.$$ 

The first relation follows from the calculation

$$\alpha \cdot \frac{\Theta}{\rho \tau^l} = \rho \alpha \cdot \frac{\Theta}{\rho^2 \tau^l} = \eta^3 \cdot \frac{\Theta}{\rho^2 \tau^l} = (\eta \rho)^3 \cdot \frac{\Theta}{\rho^3 \tau^l} = (-2)^3 \cdot \frac{\Theta}{\rho^3 \tau^{4k}} = -8 \frac{\Theta}{\rho^5 \tau^{4k}}.$$ 

The second relation follows from a similar argument, using that $\rho^3 \beta = \eta \alpha$.

Proposition 11.25. The order of $\frac{\Theta}{\rho^{l+1} \tau^m}$ is $2^{(j+1)}$, where $\varphi(j)$ is the number of positive integers $0 < i \leq j$ such that $i \equiv 0, 1, 2$ or 4 (mod 8).
Proof. This is follows from inspection of the Ext charts. □

Remark 11.26. Proposition 11.25 is an independent verification of a well-known calculation. We follow the argument given in [D, Appendix B].

Let $\mathbb{R}^{q,q}$ be the antipodal $C_2$-representation on $\mathbb{R}^q$. Consider the cofiber sequence
\[
S(q,q) \to D(q,q) \to S^{q,q},
\]
where $S(q,q) \subset D(q,q) \subset \mathbb{R}^{q,q}$ are the unit sphere and unit disk respectively. Since $D(q,q)$ is equivariantly contractible, this gives the exact sequence
\[
\pi_{m,0}(koC_2) \to \pi_{m+q,q}(koC_2) \to ko\mathbb{C}_2^{-m-1,0}(S(q,q)) \to \pi_{m+1,0}(koC_2).
\]
If $m \leq -2$, the outer groups vanish. Moreover, $C_2$ acts freely on $S(q,q)$, and the orbit space is $S(q,q)/C_2 \cong \mathbb{R}^{q-1}$. It follows ([M2, Section XIV.1]) that $\pi_{m,-1}(1)$ is cyclic of order $\varphi(j)$. The proof of the third relation is similar. Use the definitions of $\tau^2\alpha \cdot \tau^2\omega$ and $\tau^2\alpha$ to see that
\[
2 \cdot \tau^2\omega \cdot \tau^2\alpha = (\tau^2\omega)^2\alpha = 2\tau^4 \cdot \omega\alpha.
\]
The group $\pi_{4,0}(koC_2)$ has no 2-torsion, so it follows that $\tau^2\omega \cdot \tau^2\alpha$ equals $\tau^4 \cdot \omega\alpha$.

The proof of the third relation is similar. Use the definitions of $\tau^2\alpha$ and $\beta$ and part (2) to see that
\[
2(\tau^2\alpha)^2 = \tau^2\omega \cdot \tau^2\alpha \cdot \alpha = \tau^4 \cdot \omega\alpha^2 = 4\tau^4 \cdot \omega\beta.
\]
The group $\pi_{8,4}(koC_2)$ has no 2-torsion. □

12. Charts

12.1. Bockstein $E^+$ and $\text{Ext}_{\mathcal{A}^8(1)}$. The charts on pages 36–38 depict the Bockstein $E^+$ spectral sequence that converges to $\text{Ext}_{\mathcal{A}^8(1)}$. The details of this calculation are described in Section 6.

The $E^+_2$-page is too complicated to present conveniently in one chart, so this page is separated into two parts by Milnor-Witt stem modulo 2. Similarly, the $E^+_3$-page is separated into four parts by Milnor-Witt stem modulo 4. The $E^+_4$-page in Milnor-Witt stems 0 or 1 modulo 4 is not shown, since it is identical to the $E^+_1$-page in those Milnor-Witt stems. The $E^+_3$-page in Milnor-Witt stems 3 modulo 4 is not shown because it is zero.

Here is a key for reading the Bockstein charts:

(1) Gray dots and green dots indicate groups as displayed on the charts.
12.2. Bockstein $E^{-}$ and $\text{Ext}_{NC}$. The charts on pages 39–44 depict the Bockstein $E^{-}$ spectral sequence that converges to $\text{Ext}_{NC}$. The details of this calculation are described in Section 7.

The $E^{-}_{2}$-page is too complicated to present conveniently in one chart, so this page is separated into two parts by Milnor-Witt stem modulo 2. Similarly, the $E^{-}_{3}$-page is separated into four parts by Milnor-Witt stem modulo 4. The $E^{-}_{4}$-page in Milnor-Witt stems 0 or 3 modulo 4 is not shown, since it is identical to the $E^{-}_{4}$-page in those Milnor-Witt stems. The $E^{-}_{5}$-page and $E^{-}_{6}$-page in Milnor-Witt stems 1 or 2 modulo 4 is not shown, since it is identical to the $E^{-}_{4}$-page in those Milnor-Witt stems.

Here is a key for reading the Bockstein charts:

1. Gray dots indicate groups as displayed on the charts.
2. Horizontal lines indicate multiplications by $\rho$.
3. Vertical lines indicate multiplications by $h_{0}$.
4. Diagonal lines indicate multiplications by $h_{1}$.
5. Horizontal rightward arrows indicate infinite sequences of divisions by $\rho$, i.e., infinitely $\rho$-divisible elements.
6. Vertical arrows indicate infinite sequences of multiplications by $h_{0}$.
7. Diagonal arrows indicate infinite sequences of multiplications by $h_{1}$.

(2) Horizontal lines indicate multiplications by $\rho$.
(3) Vertical lines indicate multiplications by $h_{0}$.
(4) Diagonal lines indicate multiplications by $h_{1}$.
(5) Horizontal arrows indicate infinite sequences of multiplications by $\rho$.
(6) Vertical arrows indicate infinite sequences of multiplications by $h_{0}$.
(7) Diagonal arrows indicate infinite sequences of multiplications by $h_{1}$.

Here is a key for the charts of $\text{Ext}_{A^{u}(1)}$:

1. Gray dots indicate copies of $\mathbb{F}_{2}[\tau^{4}]$ that arise from a copy of $\mathbb{F}_{2}[\tau^{4}]$ in the $E_{\infty}^{-}$-page.
2. Green dots indicate copies of $\mathbb{F}_{2}[\tau^{4}]$ that arise from a copy of $\mathbb{F}_{2}$ and a copy of $\mathbb{F}_{2}[\tau^{4}]$ in the $E_{\infty}^{+}+$-page, connected by a $\tau^{4}$ extension that is hidden in the Bockstein spectral sequence. For example, the green dot at (3, 3) arises from a hidden $\tau^{4}$ extension from $h_{3}^{1}$ to $\rho \cdot \tau^{2}a$.
3. Blue dots indicate copies of $\mathbb{F}_{2}[\tau^{4}]$ that arise from two copies of $\mathbb{F}_{2}$ and one copy of $\mathbb{F}_{2}[\tau^{4}]$ in the $E_{\infty}^{+}$-page, connected by $\tau^{4}$ extensions that are hidden in the Bockstein spectral sequence. For example, the blue dot at (7, 7) arises from hidden $\tau^{4}$ extensions from $h_{7}^{1}$ to $\rho^{4}h_{3}^{1}b$, and from $\rho^{4}h_{3}^{1}b$ to $\rho^{5} \cdot \tau^{2}a \cdot b$.
4. Horizontal lines indicate multiplications by $\rho$.
5. Vertical lines indicate multiplications by $h_{0}$.
6. Diagonal lines indicate multiplications by $h_{1}$.
7. Dashed lines indicate extensions that are hidden in the Bockstein spectral sequence.
8. Orange horizontal lines indicate $\rho$ multiplications that equal $\tau^{4}$ times a generator. For example, $\rho \cdot \tau^{2}a$ equals $\tau^{4} \cdot h_{1}^{3}$.
9. Horizontal arrows indicate infinite sequences of multiplications by $\rho$.
10. Vertical arrows indicate infinite sequences of multiplications by $h_{0}$.
11. Diagonal arrows indicate infinite sequences of multiplications by $h_{1}$.
The structure of \( \text{Ext}_{NC} \) is too complicated to present conveniently in one chart, so it is separated into parts by Milnor-Witt stem modulo 4. Unfortunately, the part in positive Milnor-Witt stems 0 modulo 4 alone is still too complicated to present conveniently in one chart. Instead, we display \( \text{Ext}_{C_2} \), including both \( \text{Ext}_{A^p(1)} \) and \( \text{Ext}_{NC} \), for the Milnor-Witt 0-stem and the Milnor-Witt 4-stem.

Here is a key for the charts of \( \text{Ext}_{NC} \):

1. Gray dots indicate copies of \( \mathbb{F}_2[\tau^4]/\tau^{\infty} \).
2. Horizontal lines indicate multiplications by \( \rho \).
3. Vertical lines indicate multiplications by \( h_0 \).
4. Diagonal lines indicate multiplications by \( h_1 \).
5. Dashed lines indicate extensions that are hidden in the Bockstein spectral sequence.
6. Dashed lines of slope \(-1\) indicate \( \rho \) extensions that are hidden in the Adams spectral sequence.
7. Horizontal rightward arrows indicate infinite sequences of divisions by \( \rho \), i.e., infinitely \( \rho \)-divisible elements.
8. Vertical arrows indicate infinite sequences of multiplications by \( h_0 \).
9. Diagonal arrows indicate infinite sequences of multiplications by \( h_1 \).

12.3. **Milnor-Witt stems.** The diagram on page 45 depicts the Milnor-Witt stems in families as described in Section 11.

The top figure represents the Milnor-Witt 4\( k \)-stem, where \( k \geq 0 \). The middle three figures represent the \( \tau^4 \)-periodic classes, as in Theorem 11.15. The bottom figure represents the Milnor-Witt stem \( \Pi_n \), where \( n \equiv 3 \pmod{4} \) and \( n \leq -5 \).

Here is a key for reading the Milnor-Witt charts:

1. Black dots indicate copies of \( \mathbb{F}_2 \).
2. Hollow circles indicate copies of \( \mathbb{Z}_2^2 \).
3. Circled numbers indicate cyclic groups of given order. For instance, the 1-stem of \( \Pi_{-5} \) is \( \mathbb{Z}/4 \).
4. Blue lines indicate multiplications by \( \eta \).
5. Red lines indicate multiplications by \( \rho \).
6. Curved green lines denote multiplications by \( \alpha \).
7. Lines labelled with numbers indicate that a multiplication equals a multiple of an additive generator. For example, \( \alpha \cdot \eta^4 \) equals \( 4\eta\rho\beta \) in \( \Pi_0 \).

For clarity, some \( \alpha \) multiplications are not shown in the first and last diagrams. For example, the \( \alpha \) multiplication on \( \eta \) is not shown in the first diagram.
Bockstein $E_2$-page

$E_2[τ]

E_2

Bockstein $E_2$-page, $m_2w \equiv 0 \pmod{2}$

Bockstein $E_2$-page, $m_2w \equiv 1 \pmod{2}$
THE COHOMOLOGY OF $C_2$-EQUIVARIANT $A(1)$ AND THE HOMOTOPY OF $ko_{C_2}$
Bockstein $E_2^+ = E_2^+$. PAGE, $mw \equiv 2 \pmod{4}$

$\phi_2[\tau^4]$ $\tau^2 h_0 b$

$\tau^2 h_1 b$

$\tau h_0$

Ext $A_{(1)}$. $mw \equiv 0 \pmod{4}$

$\phi_2[\tau^4]$ $\tau^2 h_0 b$

hidden $\tau^4$ extension

two hidden $\tau^3$ extensions

Ext $A_{(1)}$. $mw \equiv 1 \pmod{4}$

$\phi_2[\tau^4]$ $\tau h_1 b$

$\tau h_1$

Ext $A_{(1)}$. $mw \equiv 2 \pmod{4}$

$\phi_2[\tau^4]$ $\tau^2 h_0 b$

$\tau^2 h_1 b$

$\tau h_0$
THE COHOMOLOGY OF $C_2$-EQUIVARIANT $A(1)$ AND THE HOMOTOPY OF $ko$
THE COHOMOLOGY OF $C_2$-EQUIVARIANT $A(1)$ AND THE HOMOTOPY OF $ko_{C_2}$
Bockstein $E^l_0$-page, $m\omega \equiv 0 \pmod{4}$

$\mathcal{E}_2 [\tau^1] / \tau^\infty$

$\mathcal{E}_2 [\tau^2] / \tau^\infty$

$\mathcal{E}_2 [\tau^3] / \tau^\infty$

$\mathcal{E}_2 [\tau^4] / \tau^\infty$

Bockstein $E^l_0$-page, $m\omega \equiv 3 \pmod{4}$

$\mathcal{E}_2 [\tau^1] / \tau^\infty$

$\mathcal{E}_2 [\tau^2] / \tau^\infty$

$\mathcal{E}_2 [\tau^3] / \tau^\infty$

$\mathcal{E}_2 [\tau^4] / \tau^\infty$
THE COHOMOLOGY OF $\mathbb{C}_2$-EQUIVARIANT $A(1)$ AND THE HOMOTOPY OF $ko_{C_2}$
Milnor-Witt modules

\[ \Pi_{4k} \]

-2 -1 0 1 2 3 4 5 6 7 8 9 10 11 12 13

\[ \Pi_{-4-4k} \]

\[ \Pi_{1\pm 4k} \]

\[ \Pi_{2\pm 4k} \]

\[ \Pi_{-5-4k} \]
References

[AI] S. Araki, and K. Iriye, Equivariant stable homotopy groups of spheres with involutions I. Osaka J. Math. 19 (1982), 1–55.

[A] M. F. Atiyah, K-theory and reality. Quart. J. Math. Oxford Ser. (2) 17 (1966), 367–386.

[Ba] C. Barwick. Spectral Mackey functors and equivariant algebraic K-theory (I). Adv. Math. 304 (2017), 646–727.

[B] G. E. Bredon, Equivariant homotopy, Proc. Conf. on Transformation Groups (New Orleans, 1967), Springer, 1968, 281–292.

[BG] R. R. Bruner and J. P. C. Greenlees, Connective real K-Theory of finite groups. Math. Surveys and Monographs 169, Amer. Math. Soc., 2010.

[DM] D. Davis and M. Mahowald, The Euler class for connective ko-theory and an application to immersions of quaternionic projective space. Indiana Univ. Math. J. 28 (1979), 1025–1034.

[DM2] D. Davis and M. Mahowald. The spectrum (P ∧ bo)−∞. Math. Proc. Cambridge Philos. Soc. 96 (1984), no. 1, 85–93.

[D] D. Dugger, An Atiyah-Hirzebruch spectral sequence for KR-theory. K-Theory 35 (2005), 213–256.

[DI] D. Dugger and D. C. Isaksen, Motivic Hopf elements and relations. New York J. Math. 19 (2013), 823–871.

[DI2] D. Dugger and D. C. Isaksen, Low dimensional Milnor-Witt stems over R, Ann. K-Theory 2 (2017), 175–210.

[DI3] D. Dugger and D. C. Isaksen, Z/2-equivariant and R-motivic stable stems, Proc. Amer. Math. Soc. to appear.

[G] J. P. C. Greenlees, Four approaches to cohomology theories with reality. Preprint, 2017. Available as arXiv preprint arXiv:1705.09305.

[GM] D. Guillou and J. P. May. Models of G-spectra as presheaves of spectra. Available as arXiv preprint arXiv:1110.3571.

[H] M. A. Hill, Ezt and the motivic Steenrod algebra over R. J. Pure Appl. Algebra 215 (2011), 715–727.

[HK] P. Hu and I. Kriz, Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence. Topology 40 (2001), 317–399.

[Ie] K. Iriye, Equivariant stable homotopy groups of spheres with involutions II. Osaka J. Math. 19 (1982), 733–743.

[I] D. Isaksen, Stable Stems, Mem. Amer. Math. Soc., to appear.

[La] P. S. Landweber, On equivariant maps between spheres with involutions, Ann. of Math. 89 (1969), 125–137.

[Li] L. G. Lewis, Jr., Change of universe functors in equivariant stable homotopy theory. Fund. Math. 148 (1995) 117–158.

[MJ] J. P. May, Matric Massey products. J. Algebra 12 (1969) 533–568.

[M2] J. P. May et al, Equivariant homotopy and cohomology theory. CBMS Regional Conference Series in Mathematics 91, Amer. Math. Soc., 1996.

[Me] M. Merling, Equivariant algebraic K-theory of G-rings. Math. Z. 285 (2017), no. 3-4, 1205–1248.

[Mo] F. Morel, On the motivic π0 of the sphere spectrum. Axiomatic, enriched and motivic homotopy theory, NATO Sci. Ser. II Math. Phys. Chem., vol. 131, Kluwer Acad. Publ., 2004.

[R] N. Ricka. Subalgebras of the Z/2-equivariant Steenrod Algebra. Homology Homotopy Appl. 17 (2015), 281–305.

[SS] S. Schwede and B. Shipley. Stable model categories are categories of modules. Topology 42 (2003), 103–153.

[Se] G. Segal. Equivariant stable homotopy theory. Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, pp. 59–63, 1971.

[T] H. Toda, Composition methods in homotopy groups of spheres. Annals of Mathematics Studies 49, Princeton University Press, 1962.

[V1] V. Voevodsky, Motivic cohomology with Z/2-coefficients. Publ. Math. Inst. Hautes Études Sci. 98 (2003), 59–104.
THE COHOMOLOGY OF $C_2$-EQUIVARIANT $\mathcal{A}(1)$ AND THE HOMOTOPIY OF $koC_2$

[V2] V. Voevodsky. Reduced power operations in motivic cohomology. Publ. Math. Inst. Hautes Études Sci. 98 (2003), 1–57.

[W] C. A. Weibel, An Introduction to Homological Algebra. Cambridge Studies in Advanced Mathematics 38, Cambridge University Press, 1994.

Department of Mathematics, The University of Kentucky, Lexington, KY
E-mail address: bertguillou@uky.edu

Department of Mathematics, University of California, Los Angeles, Los Angeles, CA
E-mail address: mikehill@math.ucla.edu

Department of Mathematics, Wayne State University, Detroit, MI
E-mail address: isaksen@wayne.edu

Department of Mathematics, University of Rochester, Rochester, NY
E-mail address: doug@math.rochester.edu