Equiangular tight frames and unistochastic matrices

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Abstract
We demonstrate that a complex equiangular tight frame composed of $N$ vectors in dimension $d$, denoted ETF $(d, N)$, exists if and only if a certain bistochastic matrix, univocally determined by $N$ and $d$, belongs to a special class of unistochastic matrices. This connection allows us to find new complex ETFs in infinitely many dimensions and to derive a method to introduce non-trivial free parameters in ETFs. We present an explicit six-parametric family of complex ETF(6,16), which defines a family of symmetric POVMs. Minimal and maximal possible average entanglement of the vectors within this qubit–qutrit family are described. Furthermore, we propose an efficient numerical procedure to compute the unitary matrix underlying a unistochastic matrix, which we apply to find all existing classes of complex ETFs containing up to 20 vectors.

Keywords: equiangular tight frames, unistochastic matrices, SIC POVM

(Some figures may appear in colour only in the online journal)

1. Introduction

Equiangular tight frames (ETFs) represent regular geometrical structures in Hilbert spaces. ETFs composed of $N$ normalized vectors \{$\phi_1, \ldots, \phi_N$\} minimize the highest possible coherence
max_{j \neq k} |⟨ϕ_j|ϕ_k⟩|. These frames, also known as Grassmannian frames, are related to optimal solutions of practical problems in communications, coding theory, sparse approximation and compressed sensing [1–5]. In particular, ETFs provide an error-correcting code which is maximally robust against two erasures [6]. ETFs are also closely related to strongly regular graphs [7], difference sets [8, 9], Clifford groups [10] and Steiner systems [11]. Other applications have appeared in quantum information theory [12]. A special class of ETFs consisting of \(d^2\) vectors in \(\mathbb{C}^d\) is known in quantum mechanics as a symmetric informationally complete positive operator valued measure (SIC-POVM) [13]. SIC-POVMs are optimal measurements in the sense that the redundancy of information provided by any pair of outcomes of a positive operator valued measure is minimized. In particular, rank-one SIC-POVM allows one to to tomographically reconstruct any quantum state. Note that while full rank SIC-POVMs are known to exist in every dimension [14], the existence of rank-one SIC-POVMs is a hard open problem.

Over the years, many important results have been derived for real and complex ETFs. A summary of the current state of the art can be found in a recent work of Fickus and Mixon [15]. However, the theory is still far from being complete. To illustrate this fact, let us mention that the full classification of complex ETFs is only known in dimensions \(d = 2\) and \(d = 3\) [16]. The aim of the present work is to shed new light on ETFs by establishing a connection with the unistochastic matrices theory. The paper is organised as follows: in section 2 we review the basic properties of equiangular tight frames. Section 3 is devoted to establishing the connection between ETFs and unistochastic matrices. We study some fundamental properties and present new inequivalent classes of complex ETFs in infinitely many dimensions. In section 4 we present a method to introduce free parameters in a given real or complex ETF. In section 5 we study entanglement properties of ETF. In section 6 we define an iterative procedure to find the unitary matrix underlying a unistochastic matrix. In section 7 we summarize the results and pose some open questions. In appendix A, we explicitly derive a six-parametric family of complex ETFs stemming from the free parameters in a given real or complex ETF. In section 7 we summarize the results and pose some open questions. In appendix B contains all the proofs of our propositions.

2. Equiangular tight frames

A complex ETF\((d, N)\) is a set of \(N\) vectors \(\{ϕ_k\}_{k=0,\ldots,N-1}\) in \(\mathbb{C}^d\) such that they are:

(i) Normalized: \(∥ϕ_j∥ = 1\) for every \(j = 0,\ldots,N - 1\).

(ii) Equiangular: \(|⟨ϕ_j,ϕ_l⟩| = \frac{1}{d}\) for every \(j \neq l = 0,\ldots,N - 1\) and a fixed \(α > 0\).

(iii) Tight: \(\frac{d}{N} \sum_{j=0}^{N-1} ⟨ϕ_j,ϕ⟩ϕ_j = ϕ\) for every \(ϕ ∈ \mathbb{C}^d\).

If the set of vectors forming the ETF belongs to \(\mathbb{R}^d\), then we have a real ETF. It is easy to prove that the parameter \(α > 0\) in property (ii), which is called inverse coherence, satisfies \(α = \sqrt{\frac{d(N-1)}{N-d}}\) (Welch bound). In quantum mechanics, the third property (tightness) is equivalent to the fact that the rank-one projectors \(Π_j = ⟨ϕ_j,·⟩ϕ_j\) \(j = 0,\ldots,N - 1\) satisfy \(\frac{d}{N} \sum_{j=0}^{N-1} Π_j = Ι\), where \(Ι\) denotes the identity matrix, i.e. the projectors form a POVM [17].

The ETF problem can be equivalently posed in terms of Gram matrices \(G_{jk} = ⟨ϕ_j,ϕ_k⟩\). A Gram matrix \(G ∈ \mathbb{C}^{N×N}\) is associated with an ETF\((d, N)\) if the following properties hold:

\[
G_{jj} = 1 \quad \text{for every} \ j = 0,\ldots,N - 1; \tag{1a}
\]

\[
|G_{jl}| = \frac{1}{α} \quad \text{for every} \ j \neq l = 0,\ldots,N - 1 \text{ and a fixed} \ α > 0; \tag{1b}
\]

\[
σ(G) ∈ \{0,N/d\}, \tag{1c}
\]
where $\sigma(G)$ is the spectrum of $G$. It is easy to prove that the eigenvalues $N/d$ and 0 of $G$ have multiplicities $d$ and $N - d$, respectively, where $N > d$ is assumed. When property (1c) is replaced by the weaker property $\text{Rank}(G) = d$, we have a set of $N$ equiangular lines in dimension $d$ [18]. We recall that the vectors forming an ETF can be explicitly found from considering the Cholesky decomposition of the Gram matrix $G$, which can be efficiently implemented (see [19, p 52]).

3. ETFs and unistochastic matrices

A bistochastic matrix $B$ is a square matrix of size $N$ having non-negative real entries such that

$$\sum_{i=0}^{N-1} B_{ik} = 1 \quad \text{and} \quad \sum_{j=0}^{N-1} B_{ij} = 1,$$

for every $k = 0, \ldots, N - 1$. The matrix $B$ is called unistochastic if there exists a unitary matrix $U$ such that $B_{ij} = |U_{ij}|^2$, for every $i, j = 0, \ldots, N - 1$. If $U$ is a real orthogonal matrix then $B$ is called orthostochastic. The problem of characterizing the full set of unistochastic matrices of order $N$ is simple for $N = 2$. Its solution is known also for $N = 3$ [20]. The problem for $N = 4$ is still open. A detailed explanation on unistochastic matrices and their applications can be found, e.g., in [21].

From now on we will restrict our attention to a particular type of unistochastic matrix, denoted $B_N(\theta)$. Those are unistochastic matrices for which there exists a unitary hermitian matrix $U_N(\theta)$ having a real and non-negative constant diagonal such that

$$B_N(\theta)_{ij} = |U_N(\theta)_{ij}|^2 = \begin{cases} \cos^2(\theta) & \text{if } i = j; \\ \frac{1}{N-1} \sin^2(\theta) & \text{if } i \neq j, \end{cases}$$

where $\theta \in [0, \pi/2]$ (see figure 1). We are now in a position to establish a connection between ETFs and unistochastic matrices $B_N(\theta)$.

**Proposition 1.** A complex ETF$(d, N)$ exists if and only if an unistochastic matrix $B_N(\theta)$ exists for $d = N \sin^2(\theta/2)$, $\theta \in [0, \pi/2]$. In particular, the ETF$(d, N)$ is real if and only if $B_N(\theta)$ is orthostochastic.

Proofs of the propositions can be found in appendix B. Unitary matrices having prescribed moduli of their entries have been previously studied [22–25]. However, those approaches are only marginally related to our problem because we also require hermiticity of the underlying unitary matrix and uniformity of its main diagonal. On the other hand, there already exists a characterization of (real) orthogonal matrices with a constant diagonal [26]. Our proposition 1 implies a straightforward relation of those matrices to the existence of real ETFs. Examining the results of [26], we noticed that every ETF associated with the orthogonal matrices found in [26] was independently rediscovered in the theory of ETFs.

Hermitian complex Hadamard matrices have a remarkable property, namely $U_N(\theta_{N_1}) \otimes U_N(\theta_{N_2}) = U_{N_1N_2}(\theta_{N_1N_2})$, where $\otimes$ denotes the Kronecker product. This property allows us to derive a wide range of inequivalent equiangular tight frames:

**Proposition 2.** Let $\sqrt{N} = p_1^{r_1} \times \cdots \times p_a^{r_a}$ be the prime power decomposition of $\sqrt{N}$, where $N$ is a square and $p_1, \ldots, p_a$ are distinct prime numbers. Then there exists

$$N = \mathcal{P}(r_1) \mathcal{P}(r_2) \cdots \mathcal{P}(r_a),$$

where $\mathcal{P}(r_i)$ is the prime power decomposition of $r_i$. The proof of this proposition can be found in appendix B.
inequivalent ETF
\[
\left( \frac{N - \sqrt{N}}{2}, N \right),
\]
where \( P(r) \) denotes the number of unrestricted partitions of the integer number \( r \).

Our approach based on unistochastic matrices enables us to address another interesting problem in ETF theory, namely, the possibility of having the off-diagonal entries in the Gram matrix proportional to the roots of unity. In proposition 3 below, we show that for a given value of \( N \) only certain \( m \)th roots of unity are allowed.

Proposition 3. Consider an ETF with a Gram matrix such that its off-diagonal entries, when normalized to have an absolute value 1, are \( m \)th roots of unity. Let \( m = p_1^{a_1} \cdots p_r^{a_r} \) be the decomposition of \( m \) in prime power factors. Then there exist numbers \( x_j \in \mathbb{N} \cup \{0\} \) such that
\[
k = \frac{N - 2d}{2} \sqrt{\frac{N - 1}{d(N - d)}}.
\]

Using relation \( d = N \sin^2(\theta/2) \) derived in proposition 1, one can express \( \cot(\theta) \) in terms of \( N \) and \( d \). Hence we obtain
\[
k = \frac{N - 2d}{2} \sqrt{\frac{N - 1}{d(N - d)}}.
\]

Proposition 3 implies in particular that if \( 2k \) is not an integer number, then the normalized off-diagonal entries of the Gram matrix cannot be a root of unity. Note that for a real ETF the number \( k \) is always an integer [27].

In order to illustrate the result, let us consider \( N = 64 \) and the fourth roots of unity. According to proposition 3, for \( m = 2^4 \) we have \( 2k + 64 - 2 = 2x_1 \). The only ETFs compatible with this equation and equation (5) are ETF(32,64), ETF(28,64) and ETF(8,64). Let us note that ETF(8,64) is a special kind of SIC-POVM known as Hoggar lines [28]. In general, the following restriction holds for SIC-POVMs.

Corollary 1. A SIC-POVM in dimension \( d \) admits a Gram matrix with off-diagonal entries proportional to roots of unity if \( d = 2 \) or \( d + 1 \) is a square.
The proof is straightforward from the fact that $2k$ has to be an integer (see proposition 3) combined with equation (5). Concerning the existence of SIC-POVMs having a Gram matrix composed of roots of unity up to dimension $d = 100$, we have:

- $d = 2, 3, 8$: the existence of SIC-POVMs confirmed with fourth, sixth and fourth roots of unity in the off-diagonal entries of the Gram matrix, respectively;
- $d = 15, 24, 35, 48$: SIC-POVMs exist, but the possibility of having roots of unity in the off-diagonal entries is open;
- $d = 63, 80, 99$: these are special cases of $r^2 - 1$ for $r$ being integer. SIC-POVMs in dimensions $d = r^2 - 1$ are known numerically for all $d$ up to 195, and in 323 [29].

4. Free parameters for ETFs

In this section, we propose a method to construct a family of complex ETFs by introducing free parameters in a given ETF in dimension $d = (N - \sqrt{N})/2$. The presence of free parameters allows, for example, an optimization of the geometrical structure according to a given application. For instance, within a one-parametric family of SIC-POVMs existing for one-qutrit systems, a particular element maximizes the amount of classical information that can be extracted by a quantum measurement, i.e. the informational power [30].

The construction is based on the following idea, introduced in [31]: two vectors $v, w \in \mathbb{C}^N$ are called equivalent to a real (ER) pair if $v \circ w^* \in \mathbb{R}^N$, where the circle and asterisk denote the Hadamard (entrywise) product and complex conjugation, respectively. Let $C_A$ and $C_B$ be two columns of a complex Hadamard matrix that form an ER pair. Note that the entries of the vector $C_A \circ C_B^*$ are $\pm 1$. We introduce a parameter $\alpha \in [0, 2\pi)$ in vectors $C_A(\alpha)$ and $C_B(\alpha)$ as follows:

- If $(C_A \circ C_B^*)_j = -1$, we set $(C_A(\alpha))_j = e^{i\alpha}(C_A)_j$ and $(C_B(\alpha))_j = e^{i\alpha}(C_B)_j$;
- If $(C_A \circ C_B^*)_j = 1$, we set $(C_A(\alpha))_j = (C_A)_j$ and $(C_B(\alpha))_j = (C_B)_j$.

Columns $C_A(\alpha)$ and $C_B(\alpha)$ are orthogonal for any $\alpha \in [0, 2\pi)$ and they belong to the plane defined by $C_A(0)$ and $C_B(0)$, i.e. they are orthogonal to the rest of the columns of the Hadamard matrix. Therefore, if we replace columns $C_A$ and $C_B$ in the original Hadamard matrix with $C_A(\alpha)$ and $C_B(\alpha)$, respectively, then we obtain a family of complex Hadamard matrices parametrized by $\alpha$. Note that real pairs of columns form trivial ER pairs, thus they always allow us to introduce free parameters.

In order to illustrate the procedure, let us consider the Hadamard matrix

$$H = \begin{pmatrix}
-1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1
\end{pmatrix}. \quad (6)$$

Let $C_A = (1, 1, -1, 1)^T$ and $C_B = (1, 1, 1, -1)^T$ be the third and fourth column of $H$, respectively. Since $C_A \circ C_B^* = (1, 1, -1, -1)^T$, we set $C_A(\alpha) = (1, 1, -e^{i\alpha}, e^{i\alpha})^T$ and $C_B(\alpha) = (1, 1, e^{i\alpha}, -e^{i\alpha})^T$. Then

$$H(\alpha) = \begin{pmatrix}
-1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -e^{i\alpha} & e^{i\alpha} \\
1 & 1 & e^{i\alpha} & -e^{i\alpha}
\end{pmatrix} \quad (7)$$

is a one-parametric family of complex Hadamard matrices.
As we have mentioned in section 3, real and complex Hadamard matrices of size $N$ are associated with real and complex ETF\((N - \sqrt{N})/2, N\), respectively. The technique of introducing free parameters in a Hadamard matrix can be thus used for constructing parametric families of ETF\((N - \sqrt{N})/2, N\)—see figure 2. It is important to remark that our method has to be applied to both columns and rows in order to obtain a hermitian matrix. That is, the ER pair of columns \(\{C_A(\alpha), C_B(\alpha)\}\) has to be complemented with the associated ER pair of rows \(\{R_A(-\alpha), R_B(-\alpha)\}\), where the parameter \(\alpha\) is reflected due to the requirement of hermiticity.

Let us add some clarifying remarks:

- In the special case \(N = 4\), a hermitian family of complex Hadamard matrices cannot be constructed from a real symmetric Hadamard matrix with a constant diagonal because introducing rows \(\{R_A(-\alpha), R_B(-\alpha)\}\) in addition to columns \(\{C_A(\alpha), C_B(\alpha)\}\) in (7) results in the matrix \(H(0)\). Equivalently, a family of ETF\((3,4)\) does not exist.

- Hermitian families exist for every real symmetric Hadamard matrix having a constant diagonal \(1\) of even square size \(N > 4\), as ER pairs always exist for such cases. The first hermitian family stemming from a real Hadamard occurs for \(N = 16\). The six-parametric family is presented in appendix A. It is worth noting that this method allows us to introduce up to \(N/2 - 1\) free parameters, but some of them can be dependent on others.

- We can also introduce \(N/2 - 1\) free parameters for hermitian Fourier matrices when \(N\) is even \([31]\). For other hermitian complex Hadamard matrices of even square size \(N\) the method can be applied when ER pairs exist.

Now we are ready to formulate the main result of this section, which is based on the considerations above.

**Proposition 4.** There exists complex ETF\((N - \sqrt{N})/2, N\) admitting the introduction of \(N/2 - 1\) linearly-independent free parameters for every even square value of \(N > 4\). For real ETF\((N - \sqrt{N})/2, N\) the result holds if and only if a real symmetric Hadamard matrix exists.

The proof follows from \([31,\text{theorem 3.1}]\). For odd values of \(N\), free parameters cannot be introduced by using our method. This is so because ER pairs do not exist if \(N\) is odd. Given that the ER pairs method is the most general method to introduce free parameters in a pair of columns, we conclude that the rotation of two vectors of any ETF\((N - \sqrt{N})/2, N\) cannot generate a family of ETFs for any odd \(N\).

### 5. Entanglement in equiangular tight frames

Equiangular tight frames have important applications in quantum mechanics as they represent symmetric POVM quantum measurements. Indeed, a connection existing between entanglement and designs has been recently studied \([32]\). It is therefore interesting to study entanglement of the symmetric vectors forming a POVM, which could be important for some theoretical and experimental implementation purposes. In this section we discuss the average entanglement properties of both SIC-POVM and the six-parametric family of ETF\((6,16)\), defined in appendix A.

Before proceeding, let us recall the notion of the purity of a quantum state \(\rho\), which is given by \(P = \text{Tr}(\rho^2)\). It quantifies how close a quantum state is to the set of pure states, i.e., the set of rank one-projectors: if \(\rho\) is defined in dimension \(d\), then \(1/d \leq P \leq 1\), where \(P = 1/d\) holds for the maximally mixed state \(\rho = \mathbb{I}/d\) and \(P = 1\) for any pure state \(\rho = \langle \phi, \phi \rangle\). For example, the entire set of quantum states for one-qubit systems satisfying \(P = 1\) determines the surface of the Bloch sphere (Poincaré sphere) \([17]\), whereas the center of the sphere represents the maximally mixed state.
An ETF ($d_1$, $d_2$), i.e. SIC-POVM in dimension $d$, defined for a bipartite system $d = d_A \times d_B$ has a fixed average purity of reductions [33]:

$$\frac{1}{d_A d_B} \sum_{j=0}^{d_A d_B - 1} \text{Tr}(\rho_{\alpha_j}^2) = (d_A + d_B)/(d_A d_B + 1).$$  (8)

Here, $\rho_{\alpha_j} = \text{Tr}_B(\langle \psi_j, \cdot \rangle \psi_j)$ is the reduction to the first subsystem $A$ and $\psi_j$ is the $j$th element of the SIC-POVM. The symbol $\text{Tr}_B$ means partial trace over the second subsystem $B$. Equation (8) also holds for the average purity of reductions to the subsystem $B$, as the bipartite states forming the POVM are pure. The key property required to derive (8) is the fact that SIC-POVMs are 2-designs. Other classes of ETFs are only 1-design and, therefore, it is expected that similar results do not hold.

As a novel contribution, here we show that the average purity of reductions for the six-parametric family of ETF(6,16), presented in appendix A, depends on the parameters, and we present approximate lower and upper bounds. This family of symmetric POVMs is defined for a qubit–qutrit system ($d = 6 = d_A \times d_B = 2 \times 3$). The average purity for the qubit subsystem (A) seems to lie in the range

$$0.576737 \lesssim \frac{1}{16} \sum_{j=1}^{16} \text{Tr}(\rho_{\alpha_j}^2) \lesssim 0.804885,$$  (9)

where the approximate lower (LB) and upper (UB) bounds are attained for parameters

$$\tilde{\alpha}_{\text{LB}} = \{0.0970, 0.0957, 0.4536, 0.7275, 0.7287, 0.2258\},$$  (10)

and

$$\tilde{\alpha}_{\text{UB}} = \{2.2222, 2.2233, 3.1401, 0.4173, 2.9043, 2.6317\},$$  (11)

Figure 2. Introduction of a free parameter $\alpha$ in a pair of vectors of an ETF. A complex cone is generated by vectors $v_1(\alpha)$ and $v_2(\alpha)$ in such a way that the $N$ vectors \{$v_1(\alpha), v_2(\alpha), v_3, \ldots, v_N$\} define a complex ETF for every $\alpha \in [0, \pi]$. Note that $v_2(\pi) = v_1(0)$ and $v_1(\pi) = v_2(0)$.
respectively. The approximate bounds given in (9) were obtained by considering numerical optimizations in Mathematica. An analytic derivation of the bounds seems to be a difficult problem. As an interesting consequence of (9), we noted that the 16 vectors forming the ETF(6,16) can be neither fully separable nor maximally entangled because the average purity of reductions can attain neither the value 1 nor the value 1/2, respectively.

In general, for \( N \) vectors defining an ETF in dimension \( 2d \ll N \) it is reasonable to expect weak restrictions on the average purity of reductions. Indeed, vectors forming an orthonormal basis can be either fully separable (e.g. tensor product basis in dimension \( d = 2^N \) for an \( N \) qubit system) or maximally entangled (e.g. generalized Bell basis in dimension \( d = d' \times d' \) for a two-qudit system having \( d' \) internal levels each).

6. Algorithm to find the underlying unitary matrix

Let us present an iterative algorithm that efficiently computes the underlying unitary matrix \( U \) existing behind a given unistochastic matrix \( B \). The algorithm works as follows:

1. (Seed): start from a random matrix \( A^{(0)} \) of size \( N \).
2. (Bistochasticity): \( A^{(k)}_{ij} \rightarrow A^{(k+1)}_{ij} = \frac{A^{(k)}_{ij}}{|A^{(k)}_{ij}|} \sqrt{B_{ij}} \).
3. (Unistochasticity): apply the Schmidt orthogonalization to columns of \( A^{(k+1)} \).

The iteration of steps 2 and 3 generates a sequence of matrices that converges to a unitary matrix \( U \) such that \( B_{ij} = |U_{ij}|^2 \), for a suitable choice of the seed in step 1. Indeed, unistochastic matrices are attractive fixed points of the composed non-linear operators \( T = T_2 T_3 \), where \( T_2 \) and \( T_3 \) are the non-linear operators associated with the mappings defined in steps 2 and 3, respectively. A solution is found when the seed belongs to the basin of attraction of a unistochastic matrix \( B \).

The additional steps:

2’. (Hermiticity): \( A \rightarrow (A + A^\dagger)/2 \),
2”’. (Constant diagonal): \( A_{ij} \rightarrow \cos(\theta) \), where \( d = N \sin^2(\theta/2) \),

allow us to generate a unitary hermitian matrix \( U_N(\theta) \) having a constant diagonal and, consequently, the unistochastic matrix \( B_N(\theta) \), where \( d = N \sin^2(\theta/2) \). In this way, we determine a complex ETF(\( d \), \( N \)). A similar procedure can be applied to generate a real ETF. In a case where the imposed matrix \( B \) is bistochastic but not unistochastic, the algorithm exhibits oscillations without converging (e.g. for \( d = 3 \) and \( N = 8 \)).

We have implemented the above-described procedure for matrices of size \( N = 4, \ldots, 22 \). Up to \( N = 19 \) we were able to reproduce all known classes of complex ETFs [15] by considering \( 10^3 \) seeds. This number of seeds was no more sufficient for \( N = 20 \), where \( 10^4 \) seeds were required to find ETF(10, 20). We found oscillations in any other case, which strongly suggests that all classes of ETF existing up to \( N = 20 \) vectors are known.

For \( N = 22 \) we focused our attention on the existence of complex ETF(11, 22), which is an open case. Such an ETF would have associated with a hermitian complex conference matrix \( U_{22}(\pi/2) \). Our simulations using \( 10^6 \) random seeds did not lead to a single successful convergence of the iterative procedure, which suggests that a hermitian complex conference matrix of size 22 does not exist. Note that the existence of a real ETF(11, 22) is excluded by the nonexistence of a symmetric conference matrix of size 22, which follows from the fact that 22 is not a sum of two integer squares [34].
7. Conclusions

We have introduced a one-to-one connection between complex ETFs and a special kind of unistochastic matrix $B_N(\theta)$, defined in (3). The connection was established in proposition 1. As a direct consequence, we have found new classes of complex ETFs (see proposition 2). Additionally, we presented new integrality restrictions for equiangular tight frames whose Gram matrices have off-diagonal entries proportional to roots of unity (see proposition 3). In particular, the only possible SIC-POVMs of such a kind may only exist in dimension $d = 2$ or when $d + 1$ is a square number (see corollary 1). We also proposed a method to introduce non-trivial free parameters in real and complex ETFs (see proposition 4). To illustrate our method, we explicitly derived a six-parametric family of complex ETFs stemming from the real ETF(6, 16) (see appendix A). Moreover, we studied the average purity of reductions ($P$) for this family, which defines a symmetric POVM for a qubit–qutrit quantum system. As consequence, we found that $P$ can be neither maximal ($P = 1$ for separable states) nor minimal ($P = 1/2$ for maximally entangled states). Lower and upper bounds for $P$ were derived (see section 6).

Furthermore, we presented an efficient algorithm to find unistochastic matrices, which is simple to implement in a computer language. With a refinement of the algorithm, we can also compute the unitary hermitian matrices lying behind unistochastic matrices $B_N(\theta)$, which allows us to find an ETF($d, N$) for $d = N\sin^2(\theta/2)$. By using this procedure, we calculated all the parameters ($d, N$) for which a complex ETF($d, N$) exists, up to $N = 20$. The results are consistent with known classes described in the literature [15]. Additionally, we exhaustively studied the existence of the complex ETF(11, 22), which seems non-existent. A confirmation of this result would imply that an hermitian complex conference matrix of size 22 does not exist.

We conclude the paper with two important open questions: (i) Find integrality restrictions for a general complex ETF, and (ii) solve the Fickus conjecture [39]: consider the three numbers $d, N - 1$ and $N - d$. If a complex ETF($d, N$) exists then one of these three numbers divides the product of the other two. Our numerical simulations up to $N = 20$, as well as all solutions presented in the most complete catalog of ETFs [15], are consistent with the conjecture.

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Appendix A. ETF(6,16) having six free parameters

In order to illustrate the method introduced in section 4, let us present an explicit six-parametric family of hermitian unitary matrices $U(\vec{\alpha})$ which determines the existence of a family of complex ETF(6, 16). The result is given in (A.1) below. The parametric family $U(\vec{\alpha})$ is constructed from the real symmetric Hadamard matrix $U(\vec{0}) = H^{\otimes 4}$, where $\otimes$ denotes the Kronecker product. We used the following ER pairs of columns and rows, $\{2, 3\}; \{4, 13\}; \{5, 8\}; \{6, 7\}; \{9, 12\}; \{10, 11\}; \{14, 15\}$ which are associated with parameters $\alpha_1$ to $\alpha_7$, respectively. Here, only six parameters are linearly independent and, therefore, $\alpha_7$ can be set to zero without loss of generality.

The explicit form of the six-parametric family composed of 16 vectors can be found by considering the Cholesky decomposition of the Gram matrix $G = 4/3(\mathbb{I} - U_{16}(\vec{\alpha}))$. 
Appendix B. Proof of propositions

**Proposition 1.** A complex ETF(d, N) exists if and only if a unistochastic matrix \(B(\theta)\) exists for \(d = N \sin^2(\theta/2)\). In particular, the ETF(d, N) is real if and only if \(B(\theta)\) is orthostochastic.

**Proof.** Suppose a complex ETF(d, N) exists, having associated with a Gram matrix \(G\). Therefore, \(U = I - \frac{d}{2}G\) is a unitary hermitian matrix that implies the existence of the unistochastic matrix \(B(\theta)\) defined in (3). Furthermore, by using the fact that \(U_{ii} = 1 - 2d/N\) and (3), we have \(N \cos(\theta) = N - 2d\), which is equivalent to \(d = N \sin^2(\theta/2)\). Reciprocally, suppose that there exists a unistochastic matrix \(B(\theta)\) with an underlying hermitian unitary matrix \(U_N(\theta)\). By checking properties \((1a)-(1c)\), let us prove that \(G = \frac{N}{2p}(I - U_N(\theta))\) for \(d = N \sin^2(\theta/2)\) is a Gram matrix of an ETF(d, N). As for property \((1a)\), we have \(G_{ii} = \frac{N}{2p}(1 - \cos(\theta)) = \frac{N}{2p} \sin^2(\theta/2) = 1\), where we used the fact that \(U_{ii} = \cos(\theta)\). Similarly, we prove \((1b)\), i.e. \(|G_{ij}|^2 = \frac{N - d}{4p^2} \sin^2(\theta/2)\) for \(i \neq j\) (Welch bound). As for \((1c)\), since \(U_N(\theta)\) is hermitian with a constant diagonal, \(U_N(\theta)\) has eigenvalues 1 and \(-1\) and \(\text{Tr}(G) = N\); hence \(G\) has eigenvalues \(\lambda_0 = 0\) and eigenvalues \(\lambda_1 = \frac{N}{2p}\) with degeneracies \(N - d\) and \(d\), respectively. To sum up, \(G\) is a Gram matrix of a complex ETF(d, N). In particular, real matrices \(G\) associated with real ETF(d, N) are related one-to-one with orthostochastic matrices \(B_N(\theta)\), where \(U_N(\theta)\) is an orthogonal symmetric matrix.

**Proposition 2.** Let \(\sqrt{N} = p_1^r \times \cdots \times p_d^s\) be the prime power decomposition of \(\sqrt{N}\), where \(N\) is a square and \(p_1, \ldots, p_d\) are distinct prime numbers. Then there exist at least
\[N = \mathcal{P}(r_1)\mathcal{P}(r_2)\cdots\mathcal{P}(r_s),\] (B.1)
inequivalent ETF((N – \(\sqrt{N}\))/2, N), where \(\mathcal{P}(r)\) denotes the number of unrestricted partitions of the integer number \(r\).

**Proof.** Let \(\sqrt{N} = p_1^i \times \cdots \times p_d^s\) be the prime power decomposition of \(\sqrt{N}\). Given that hermitian Fourier matrices exist in every square dimension (see [35, 36]) we can construct matrices \(U(\theta)\) with \(\cos(\theta) = 1/\sqrt{N}\) by considering the tensor product of hermitian Fourier matrices of sizes \(p_1^{2i_1}, p_2^{2i_2}, \ldots, p_d^{2i_d}\) for every \(s_1 = 1, \ldots, r_1\). In particular, all possible products of matrices of size \(p_1^{2i_1}\) generate \(\mathcal{P}(r_1)\) different matrices of size \(p_1^{2i_1}\). Therefore, there are \(N = \mathcal{P}(r_1)\mathcal{P}(r_2)\cdots\mathcal{P}(r_s)\) different ways to construct a matrix \(U(\theta)\) of size \(N = p_1^{2i_1} \times \cdots \times p_d^{2i_d}\). Furthermore, all of these matrices are inequivalent in the sense that we cannot transform one into the other by permuting rows/columns and multiplying rows/columns by unimodular complex numbers. This is so because the set of invariants under equivalence known as the Haagerup set [37]:
\[\Lambda(U) = \{U_{uv}U_{uv}^*: s, t, u, v = 0, \ldots, N - 1\},\] (B.2)
is different for all the constructed matrices \(U(\theta)\). Therefore, the ETFs obtained from them are inequivalent. 

To exemplify unrestricted partitions, note that \(4 = 4 + 0 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1\), so \(\mathcal{P}(4) = 5\). Proposition 2 can be extended to a construction of other ETF((N – \(\sqrt{N}\))/2, N) by considering some existing real symmetric Hadamard matrices having a constant diagonal. For example, for \(N = 144\) the real case \(4 \times 36\) is not considered in proposition 2. Also, there are further complex cases such as \(4 \times 36\), generated by considering
the tensor product of the symmetric Hadamard matrix of size 4 and the hermitian Fourier matrix of size 36, and so on for higher dimensions.

**Proposition 3.** Consider an ETF with a Gram matrix such that its off-diagonal entries, when normalized to have absolute value 1, are mth roots of unity. Let \( m = p_1^{a_1} \cdots p_r^{a_r} \) be the decomposition of \( m \) in prime power factors. Then there exist numbers \( x_j \in \mathbb{N} \cup \{0\} \) such that \( 2k + N - 2 = x_1p_1 + \cdots + x_rp_r \), where \( k = \cot(\theta) \sqrt{N - 1} \).

**Proof.** As we have seen in the proof of proposition 1, the off-diagonal entries of the Gram matrix coincide with the off-diagonal entries of the associated matrix \( U(\theta) \). We multiply the columns of \( U(\theta) \) so that the first two rows have the form

\[
\{ \cos(\theta), 1, \ldots, 1 \} \quad \text{and} \quad \{ 1, \cos(\theta), e^{i\alpha_1} \sin(\theta)/\sqrt{N - 1}, \ldots, e^{i\alpha_{N-2}} \sin(\theta)/\sqrt{N - 1} \}. \tag{B.3}
\]

Since \( U(\theta) \) is unitary, the inner product between these two rows must be 0, i.e.

\[
2k + e^{i\alpha_1} + \cdots + e^{i\alpha_{N-2}} = 0, \quad \text{where} \quad k = \cot(\theta) \sqrt{N - 1}. \tag{B.4}
\]

It is known that a sum of \( N' \) terms of the form \( e^{i\alpha_1} + \cdots + e^{i\alpha_{N'}} \), composed of \( m \)th roots of unity with \( m = p_1^{a_1} \cdots p_r^{a_r} \), can vanish only if there exist numbers \( x_j \in \mathbb{N}_0 \) such that \( N' = x_1p_1 + \cdots + x_rp_r \), where \( p_1, \ldots, p_r \) are prime numbers and \( a_1, \ldots, a_r \in \mathbb{N} \) [38]. Since \( 2k = 1 + 1 + \cdots + 1 \), equation \( \text{(B.4)} \) can be understood as a vanishing sum of \( N' = 2k + N - 2 \) unimodular numbers. Therefore, a necessary condition for the existence of a Gram matrix composed of \( m \)th roots of unity with \( m = p_1^{a_1} \cdots p_r^{a_r} \) is the existence of numbers \( x_j \in \mathbb{N}_0 \) such that \( 2k + N - 2 = x_1p_1 + \cdots + x_rp_r \). \( \square \)

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