Generalized Lewis-Riesenfeld invariance for dynamical effective mass in jammed granular media under a potential well in non-commutative space

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Abstract. Consideration of the asteroid belt (Kuiper belt) as a jammed-granular media establishes a bridge between condensed matter physics and astrophysics. It opens up an experimental possibility to determine the deformation parameters for noncommutative space-time. Dynamics of the Kuiper belt can be simplified as dynamics of a dynamical effective mass for a jammed granular media under a gravitational well. Alongside, if one considers the space-time to be noncommutative, then an experimental model for the determination of the deformation parameters for noncommutative space-time can be done. The construction of eigenfunctions and invariance for this model is in general a tricky problem. We have utilized the Lewis-Riesenfeld invariant method to determine the invariance for this time-dependent quantum system. In this article, we have shown that a class of generalized time-dependent Lewis-Riesenfeld invariant operators exist for the system with dynamical effective mass in jammed granular media under a potential well in noncommutative space. To keep the discussion fairly general, we have considered both position-position and momentum-momentum noncommutativity. Since, up to a time-dependent phase-factor, the eigenfunctions of the invariant operator will satisfy the time-dependent Schrödinger equation for the time-dependent Hamiltonian of the system, the construction of the invariant operator fairly solve the problem mathematically, the results of which can be utilized to demonstrate an experiment.

Keywords: Lewis-Riesenfeld invariant; jammed-granular media; Kuiper belt; noncommutative geometry
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PACS numbers:
1. Introduction

There is a consensus that the fundamental concept of space-time is mostly compatible with quantum theory in noncommutative space. It has become a Gospel that the physics in Planck-scale will exhibit the noncommutative structure of space [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. However, lack of any direct experimental evidence is the major criticism for the unified quantum theory of gravity. The main difficulty of experimenting with the validity of theory regarding noncommutative space lies mainly in the limitation of the present-day experimental capacity of attainable energy. Because, most theories of quantum gravity appear to predict departures from classical relativity only at energy scales on the order of $10^{19}$ GeV (By way of comparison, the LHC was designed to run at a maximum collision energy of $1.4 \times 10^4$ GeV [17]). Several proposals are there to deal with the scope to overcome the issue of energy limitation and perform passive experiments with the help of interferometry [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24, 25]. To best of knowledge of present authors, although none of them had yet been performed in an actual experiment.

In this article, we wish to propose a new scheme of the experiment to determine the NC-parameters, which has never been reported so far in the literature. We are advocating that a successful experiment within the purview of present-day engineering advancement requires some bridging between condensed matter experiments and cosmology. To do so, we have utilized the fact that the asteroid belt (Kuiper belt) can be considered as jammed-granular media [26, 27, 28, 29, 30]. Since the inner edge of Kuiper-belt (KB) begins at about 30 AU from the Sun and the outer edge of the main part continues outward to nearly 1000 AU from the Sun, the overall toroidal shape of the Kuiper-belt can be approximated as a cylinder with negligible height (a disc). Despite the distorted orbit, almost all the studied elements of KB had shown to have a periodic rotation around the Sun, except for those who ended their life by crash landing as a meteoroid on the planets including the Earth. Therefore, we can consider that KB has an overall average velocity in which it rotates around the Sun. That means, the entire system can be thought of as a rotating cylinder containing loosely packed jammed granular media (JGM). The overall system experiences a huge gravitational pull towards the nearer gas-giant the Neptune or towards the hypothetical planet in the outer region of the Solar System, named Planet Nine. Our proposal is to utilize this structure for the experimental purpose, even for the measurement of noncommutative space parameters. In particular, the present article deals with the proposition of utilizing the KB as a rotating JGM in noncommutative space (NC) and study the deviation from the commutative counterpart. This will enable us to estimate the values of NC-parameters. Our toy model under consideration consists of a particle with dynamical effective mass subject to NC gravitational quantum well [31, 32, 33, 34, 35, 36, 37]. For example, the quantum well represents the gravitational pull of the Neptune or the Planet Nine. Although the exact system is much complicated than that have been mentiond in
the present article, one can consider the prescribed model to develop more realistic model to perform the exact experiment. However, it is always interesting to obtain an exact analytical solution for a simplified model corresponding to a complicated real-life problem. After all, the approximate behavior of a natural system are often guided from a mathematically tractable model. This is the main spirit of the present article. The system under consideration is a simplified time-dependent quantum toy model in NC-space. The Lewis-Riesenfeld phase-space invariant method (LRM)\cite{38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52} is an effective tool to obtain an analytical solution of a time-dependent quantum system. The mechanism LRM is based on the construction of an invariant operator (IO) in phase space corresponding to a time-dependent Hamiltonian $\hat{H}(t)$. Up to a time-dependent phase factor, the eigenstates of IO ($\hat{I}(t)$) are also the eigenstates of $\hat{H}(t)$, although they will not be isospectral in general. Specifically, the eigenvalues of ($\hat{I}(t)$) are time-independent, whereas the same may be time-dependent for $\hat{H}(t)$. Once one has the LR-invariant operator, it is not difficult to obtain the exact quantum states of the system and utilize it to obtain the measurable quantities.

The organization of the article is the following. At first, a brief description of the system under consideration is given. Then a generalized Lewis-Riesenfeld phase-space invariant method is outlined briefly. After that, we have constructed a generalized invariant operator for dynamical effective mass in jammed granular media. Future aspects and experimental viability are mentioned in discussion.

2. System under consideration: Free falling under gravity in noncommutative space

Since we are dealing with the free-falling under gravity in noncommutative space (NCS), without loss of generality we can confine ourselves in two spatial dimensions in NCS, namely $x'$ and $y'$. Let us choose our $x'$ axis in the direction of the attraction due to gravity. The dynamics for the $y'$ direction in NCS remains something like free-particle. We aim to write down the coherent state structure for this system.

Better or worse, we are adopting the usual technique of writing the quantum version of a theory corresponding to a known classical dynamical system with the aid of the Bohr-correspondence principle. One may wonder why most of the computations of quantum theory can not stand on its ground without the help of a corresponding known classical dynamical system. One may even demand the justification of the applicability of the direct promotion of classical theory to its quantum version for NCS. Keeping aside these foundational issues, let us concentrate on the aims and scope of the present article and write down the energy operator (Hamiltonian) in NCS as follows.

$$\hat{H}_{nc} = \frac{\hat{p}_{x'}^2}{2m(t)} + \frac{\hat{p}_{y'}^2}{2m(t)} + m(t)g(t)x'$$ (1)
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Where the time dependent effective mass (TDEM) of the particle and the acceleration due to gravity is denoted by \( m(t) \) and \( g(t) \) respectively. Acceleration due to gravity \( g(t) \) is allowed to be time dependent, so that the model can be utilized even for the large height for which \( g \) no longer can be considered as a constant. Moreover, this model can be utilized for homogeneous time dependent electric field, which is relatively easy to develop for experimental purpose. \( \hat{p}_x' \) and \( \hat{p}_y' \) are the conjugate momentum operators corresponding to \( \hat{x}' \) and \( \hat{y}' \) respectively. We are considering both position-position and momentum-momentum noncommutativity to keep our discussion fairly general. Following commutation relations for variables \( \{x', y', p'_x, p'_y\} \) in NCS are utilized in present article.

\[
\begin{align*}
[x', y'] &= i\theta, \\
[p'_x, p'_y] &= i\eta, \\
[x'_i, p'_j] &= i\hbar_{\text{eff}}\delta_{ij}.
\end{align*}
\] (2)

Where

\[
\hbar_{\text{eff}} = (1 + \zeta)\hbar,
\] (3)

with \( \zeta = \frac{\theta\eta}{4\hbar^2} \).

\( \delta_{ij} \) are Kronecker delta with the properties

\[
\delta_{ij} = \begin{cases} 
1 & \text{for } i = j \\
0 & \text{for } i \neq j
\end{cases}
\] (4)

One can recover the structure of classical commutative space for quantum mechanics by setting the parameters \( \theta \) and \( \eta \) to zero. If one wishes to be confined in only position-position noncommutativity then \( \eta \) has to be set to zero.

Since our usual notion of calculus are mentally and practically settled in commutative space, it will be a wise decision if we could transform the whole problem to some equivalent commutative space structure. This can be done by the following transformation of co-ordinates.

\[
\begin{pmatrix}
x' \\
y' \\
p'_x \\
p'_y
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & -\frac{\theta}{2\hbar} \\
0 & 1 & \frac{\theta}{2\hbar} & 0 \\
0 & \frac{\eta}{2\hbar} & 1 & 0 \\
-\frac{\eta}{2\hbar} & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
x \\
p_y
\end{pmatrix}.
\] (5)

\( \{x, y, p_x, p_y\} \) are usual co-ordinates in classical commutative space in which the commutation relations are given by

\[
\begin{align*}
[x, y] &= [p_x, p_y] = 0, \\
[x_i, p_j] &= i\hbar\delta_{ij}.
\end{align*}
\] (6)

For computational purpose, with the help of (5) we can now write down the equivalent quantum hamiltonian for the system under consideration (1) in terms of classical
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\[ \hat{H}_c = \frac{\hat{p}_x^2}{2m(t)} + \frac{\hat{p}_y^2}{2m(t)} + \frac{\eta^2}{8m(t)\hbar^2} (\hat{x}^2 + \hat{y}^2) + \frac{\eta}{2m(t)\hbar} (\hat{y}\hat{p}_x - \hat{x}\hat{p}_y) + m(t)g(t) \left( \hat{x} - \frac{\theta}{2\hbar}\hat{p}_y \right). \] (7)

For the purpose of computational convenience, we can rewrite 7 as follows.

\[ \hat{H}_c = \hat{H}_0(t) + \hat{V}(t). \] (8)

With

\[ \hat{H}_0(t) = \hbar \omega(t) \left[ \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + i \left( \hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1 \right) + 1 \right]. \] (9)

And

\[ \hat{V}(t) = \frac{g(t)\sqrt{\eta}}{2\omega(t)} \left[ \hat{a}_1^\dagger + \hat{a}_1 - i \frac{\theta \eta}{4\hbar^2} \left( \hat{a}_2^\dagger - \hat{a}_2 \right) \right]. \] (10)

Where the annihilation operators are defined by

\[ \hat{a}_i = \sqrt{\frac{m(t)\omega(t)}{2\hbar}} \left( \hat{x}_i + \frac{i}{m(t)\omega(t)} \hat{p}_{x_i} \right), \quad i = 1, 2. \] (11)

\( i = 1 \) stands for \( x_1 = x \) and \( i = 2 \) stands for \( x_2 = y \). The creation operators \( \hat{a}_i^\dagger \) are the corresponding adjoint operators of \( \hat{a}_i \). The time dependent frequency \( \omega(t) \) is defined by

\[ \omega(t) = \frac{\eta}{2m(t)\hbar}. \] (12)

The commutation relations among the annihilation and creation operators are given by

\[ [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise}, \end{cases} \] (13)

and

\[ [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0. \] (14)

The frequency \( (\nu_0) \) dependent effective mass, \( m(\nu_0) \), of jammed granular materials which occupy a rigid cavity to a specific filling fraction (the remaining volume being air of normal room condition or controlled humidity) is given by

\[ m(\nu_0) = \pi R^2 \frac{\sqrt{\rho K}}{\nu_0 \tan(qL)}. \] (15)

Where \( \rho \) is the density of the medium, \( L \) is the length of the fluid column, \( R \) is the radius of the cup (container), \( K \) is the bulk-modulus of the medium and \( q = \nu_0 \sqrt{\frac{\rho}{K}} \) is the wave vector. It is worth noting that \( q \) and as well as \( K \) can be complex also. However, for complex \( K \) the medium becomes lossy. We have considered loss-less medium in our discussion. Hence, we have considered only real valued of \( K \) and \( q \). The generalization for complex valued \( K \) is straightforward.

Now we propose a model for the time-varying depth of the granular medium. In particular, we are proposing the model for continuously filling the granular material from
the top of the cylindrical cup, which is subject to the rotation with a fix frequency \(\nu_0\). The rate of filling up the cup is small enough such that the length of the material column can be approximated as

\[ L(t) = L_0(1 + \nu t). \tag{16} \]

Where \(L_0\) is the constant length at \(t = 0\) and \(\nu << 1\) is very small constant such that we can neglect any higher power of \(\nu\), other than linear, in our calculation. Under this assumption, we can approximate \(m_0\) as

\[ m(t) = m_0(q_0 + q_1 \nu t). \tag{17} \]

Where

\[ q_0 = \tan(qL_0), \tag{18} \]
\[ q_1 = qL_0 \sec^2(qL_0), \tag{19} \]
\[ m_0 = \frac{1}{\nu_0} \pi R^2 \sqrt{\rho K}. \tag{20} \]

In the next section we have constructed an invariant operator which is utilized to construct the coherent state structure of the system.

3. Construction of Lewis-Riesenfeld invariant operator

We shall utilize the method described in [53]. Let us assume that we know the eigenfunctions of \(\hat{H}_0(t)\) in 9. Then, for any complex time-dependent parameter \(\mu(t)\), the eigenvalue equation of the following operator is known.

\[ \hat{O}(t) = e^{\mu(t)\hat{H}_0(t)}. \tag{21} \]

However, \(21\) is not an invariant operator associated with \(\hat{H}(t)\). Indeed, one can see that

\[ \dot{\hat{O}}(t) = \frac{\partial \hat{O}(t)}{\partial t} + \frac{1}{i\hbar} \left[ \hat{O}(t), \hat{H} \right] = \hat{\Theta}(t). \tag{22} \]

Where, \(\hat{\Theta}(t)\) is some time-dependent operator. Here dot (\(\cdot\)) denotes total derivative with respect to time. We shall use this shorthand notation throughout this article unless otherwise specified.

Now, using the time -dependent Schrödinger equation

\[ \hat{H}(t)|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle, \tag{23} \]

one can note that

\[ i\hbar \frac{\partial}{\partial t} \left[ \hat{O}(t)|\psi(t)\rangle \right] = \left( \hat{H}(t) + i\hbar \hat{\Theta}(t)\hat{O}^{-1}(t) \right) \left[ \hat{O}(t)|\psi(t)\rangle \right]. \tag{24} \]

That means, \(\hat{O}(t)|\psi(t)\rangle\) is the eigen-function of the deformed hamiltonian

\[ \hat{\hat{H}}(t) = \hat{H}(t) + i\hbar \hat{\Theta}(t)\hat{O}^{-1}(t). \tag{25} \]
Let us define a time-dependent unitary transformation $\Lambda(t)$ which relates the eigenfunctions of $\hat{H}(t)$ and $\hat{\tilde{H}}(t)$. One can readily identify that

$$\hat{\tilde{H}}(t) = i\hbar \frac{\partial}{\partial t} \left[ \Lambda(t) \hat{\tilde{O}}(t) | \psi(t) \rangle \right],$$

with

$$\hat{\tilde{H}}(t) = \hat{\Lambda} \hat{H} \hat{\Lambda}^{-1} + i\hbar \left( \Lambda \Theta \hat{O}^{-1} + \frac{\partial \Lambda}{\partial t} \right) \hat{\Lambda}^{-1}.\quad (27)$$

Therefore, $\Lambda(t) \hat{\tilde{O}}(t) | \psi(t) \rangle$ is the eigen-function of $\hat{\tilde{H}}(t)$. Now we impose the restriction on $\hat{\Lambda}(t)$ such that

$$\hat{H}(t) = \hat{\tilde{H}}(t),\quad (28)$$

which immediately leads to

$$\frac{\partial}{\partial t} \hat{\Lambda}(t) + \frac{i}{\hbar} \left[ \hat{\Lambda}(t), \hat{H}(t) \right] = -\hat{\Lambda}(t) \hat{\Theta}(t) \hat{O}^{-1}(t).\quad (29)$$

Comparing (29) with (22), one can conclude that

$$\frac{\partial}{\partial t} \left[ \hat{\Lambda}(t) \hat{\tilde{O}}(t) \right] + \frac{i}{\hbar} \left[ \hat{\Lambda}(t) \hat{\tilde{O}}(t), \hat{H}(t) \right] = 0.\quad (30)$$

Thus, we have obtained a desired time-dependent Lewis-Riesenfeld invariant operator (LRIO)

$$\hat{\mathcal{I}}(t) = \hat{\Lambda}(t) \hat{\tilde{O}}(t) \quad (31)$$

corresponding to the Hamiltonian $\hat{H}(t)$. If we can construct the eigen-functions of $\hat{\mathcal{I}}(t)$, then we shall also have the eigen-functions of $\hat{H}(t)$. However, they will not be isospectral in general. From the discussion so far, it is clear that in order to construct the LRIO, we shall have to proceed according to the following algorithm.

- Split the Hamiltonian in terms of known part $\hat{H}_0(t)$ and unknown part $\hat{V}(t)$ as follows

  $$\hat{H}(t) = \hat{H}_0(t) + \hat{V}(t).\quad (32)$$

- Construct the time-dependent operator as specified in (21), i.e., $\hat{\tilde{O}}(t) = e^{\mu(t)\hat{H}_0(t)}$.

- Construct $\hat{\Theta}(t)$ according to (22), i.e., $\hat{\Theta}(t) = \frac{\partial \hat{\tilde{O}}(t)}{\partial t} + \frac{1}{\hbar} \left[ \hat{\tilde{O}}(t), \hat{H} \right]$.

Since, this step involves a computation of a commutator of two operators one of which is the exponential of an operator, it will be convenient to use Kubo’s identity which are given as follows.

$$[\hat{A}, e^{-\beta \hat{H}}] = -\int_0^\beta e^{-(\beta-u)\hat{H}} [\hat{A}, \hat{H}] e^{-u\hat{H}} du.\quad (33)$$

The integrand can be easily determined by the Baker-Campbell-Hausdorff formula

$$e^\hat{A} \hat{B} e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \ldots.\quad (34)$$
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• Next is the most crucial step: choose an unitary transformation $\hat{\Lambda}(t)$. There is no consensus for the choice of $\hat{\Lambda}(t)$. However, it is convenient to choose $\hat{\Lambda}(t) = e^{\hat{\pi}}$, where $\hat{\pi}$ is some anti-hermitian operator. Sometime, an anti-hermitian operator constructed by a linear combination of the basic constituent operators of $\hat{V}(t)$ serves the purpose. For example, in this present article this scheme has shown to be useful.

• Final step is to construct the invariant operator $\hat{I}(t)$ and determine its eigenfunctions which are the eigen-functions of our desired time dependent hamiltonian $\hat{H}(t)$.

Now for our system we have $\hat{H}_0(t)$ is given by $9$ and $\hat{V}(t)$ is given by $10$. On the way to determine $\hat{\Theta}(t)$ we observe that

$$\hat{C}_0 = [\hat{V}, \hat{H}_0] = g_0 \left( \hat{b} - \hat{b}^\dagger \right). \quad (35)$$

Where

$$\hat{b} = \hat{a}_1 + i\hat{a}_2. \quad (36)$$

$$g_0(t) = \frac{1}{2} \hbar g(t) \sqrt{\eta} \left( 1 + \frac{\theta_\eta}{4\hbar^2} \right). \quad (37)$$

It is worth noting that

$$[\hat{H}_0, \hat{C}_0] = (-2\hbar \omega)^{-1} g_0 \left( \hat{b} + \hat{b}^\dagger \right). \quad (38)$$

$$\left[ \hat{H}_0, \left[ \hat{H}_0, \hat{C}_0 \right] \right] = (-2\hbar \omega)^2 \hat{C}_0. \quad (39)$$

This leads to the use of Baker-Campbell-Hausdorff formula to obtain the following,

$$\hat{G} = e^{\mu \hat{H}_0} \hat{C}_0 e^{-\mu \hat{H}_0} = g_0 \left( e^{-u_1 \hat{b}} - e^{u_1 \hat{b}^\dagger} \right). \quad (40)$$

Where

$$u_1 = 2\hbar \omega u. \quad (41)$$

Now one can apply the Kubo’s identity and write the following.

$$\{\hat{\Theta}, \hat{H} \} = \frac{g_0}{2\hbar \omega} e^{\mu \hat{H}_0} \left[ (1 - e^{2\hbar \omega \mu}) \hat{b} + (1 - e^{-2\hbar \omega \mu}) \hat{b}^\dagger \right]. \quad (42)$$

Another term in $22$ involves partial derivative of $e^{\mu(t)\hat{H}_0}$ with respect to time. To perform this derivative let us examine the system once again according to the what follows. One can rewrite $\hat{a}_i$ and $\hat{a}_i^\dagger$ as the followings.

$$\hat{a}_i = M \left( \hat{x}_i + \frac{i}{2M^2 \hbar} \hat{p}_x \right); \ i = 1, 2. \quad (43)$$

$$\hat{a}_i^\dagger = M \left( \hat{x}_i - \frac{i}{2M^2 \hbar} \hat{p}_x \right); \ i = 1, 2. \quad (44)$$

The time derivative of them are given by

$$\dot{\hat{a}}_i = \frac{\dot{M}}{M} \hat{a}_i^\dagger; \ i = 1, 2. \quad (45)$$

$$\dot{\hat{a}}_i = \frac{\dot{M}}{M} \hat{a}_i; \ i = 1, 2. \quad (46)$$
Where
\[ M = \sqrt{\frac{m\omega}{2\hbar}}. \]  
(47)

And
\[ \hat{H}_0 = \hbar\omega \left( \hat{N}_1 + \hat{N}_2 + i\hat{N}_{12} + 1 \right). \]  
(48)

Where
\[ \hat{N}_i = \hat{a}_i^\dagger \hat{a}_i; \ i = 1, 2. \]  
(49)
\[ \hat{N}_{12} = \hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1. \]  
(50)

One can identify that the following commutation relations hold.
\[ [\hat{N}_i, \hat{N}_{12}] = 0 ; \ i = 1, 2. \]  
(51)
\[ [\hat{N}_i, \hat{N}_j] = 2\frac{M}{M} \hat{A}_{ij}\delta_{ij} ; \ i, j = 1, 2. \]  
(52)
\[ [\hat{N}_{12}, \hat{N}_2] = -[\hat{N}_{12}, \hat{N}_1] = 2\frac{M}{M} (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2 \hat{a}_1). \]  
(53)

Where
\[ \hat{A}_{ij} = \hat{a}_i^\dagger \hat{a}_j^\dagger - \hat{a}_i \hat{a}_j. \]  
(54)

Using 51, right commutation and 53, one can see that
\[ [\mu \hat{H}_0, \frac{\partial}{\partial t} (\mu \hat{H}_0)] = (\hbar\mu)^2 \frac{\omega}{m} \frac{d}{dt} (m\omega)(\hat{A}_{11} + \hat{A}_{22}). \]  
(55)

Since \( m\omega = \frac{\eta}{2\hbar} \) (eq.12) is a constant, we can conclude
\[ [\mu \hat{H}_0, \frac{\partial}{\partial t} (\mu \hat{H}_0)] = 0. \]  
(56)

One can ask the relevance of the steps from 43 to 56. Since, 11 along with 12 clearly indicates that \( \hat{a}_i \)'s and their corresponding adjoints are time independent for constant \( \eta \) and \( \hbar \) and therefore 56 is trivial. However, we would like to emphasize that equations from 43 to 56 are generally true even for time dependent \( \eta \). In particular, although we are confining ourselves for time-independent background, the results outlined here can be directly used for the time-dependent background. Now, 56 enables us to write
\[ \Theta(t) = e^{\mu \hat{H}_0} \frac{\partial}{\partial t} (\mu \hat{H}_0) + \frac{g_0}{2i\hbar^2\omega} e^{\mu \hat{H}_0} [(1 - e^{2\hbar\omega\mu})\hat{b} + (1 - e^{-2\hbar\omega\mu})\hat{b}^\dagger]. \]  
(57)

Since the following relations hold
\[ [\hat{H}_0, \hat{b}] = -2\hbar\omega\hat{b}, \]  
(58)
\[ [\hat{H}_0, \hat{b}^\dagger] = 2\hbar\omega\hat{b}^\dagger, \]  
(59)
we can write the followings.
\[ e^{\mu \hat{H}_0} \hat{b} e^{-\mu \hat{H}_0} = e^{-2\hbar\omega\mu}\hat{b}, \]  
(60)
\[ e^{\mu \hat{H}_0} \hat{b}^\dagger e^{-\mu \hat{H}_0} = e^{2\hbar\omega\mu}\hat{b}^\dagger. \]  
(61)
Therefore, we can write
\[ \hat{\Theta} \hat{O}^{-1} = \frac{\partial}{\partial t} \left( \mu \hat{H}_0 \right) + \beta_1 \hat{b} + \beta_2 \hat{b}^\dagger \].
(62)

Where
\[ \beta_1 = \frac{g_0}{2i\hbar\omega}(e^{-2i\omega\mu} - 1), \]
(63)
\[ \beta_2 = \frac{g_0}{2i\hbar\omega}(e^{2i\omega\mu} - 1). \]
(64)

Now our task is to define the unitary transformation \( \hat{\Lambda}(t) \). Let us define it as the following displacement operator
\[ \hat{\Lambda}(t) = e^{\hat{\Pi}}, \]
(65)
with \( \hat{\Pi} = v_1 \hat{a}_1 - v_1^* \hat{a}_1^\dagger + v_2 \hat{a}_2 - v_2^* \hat{a}_2^\dagger \).
(66)

Here \( \ast \) denotes the complex-conjugate. If we get some consistent time-dependent complex valued functions \( v_i(t) \), \( i = 1, 2 \), then we are done. To calculate \( \hat{\Lambda} \hat{H} \hat{\Lambda}^{-1} \), we observe the followings.

\[ \hat{\pi}_0 = \left[ \hat{\Pi}, \hat{H}_0 \right] = \hbar \omega (v_{12} \hat{a}_1 + iv_{12} \hat{a}_2 + v_{12}^* \hat{a}_1^\dagger - iv_{12}^* \hat{a}_2^\dagger). \]
(67)
\[ \left[ \hat{\Pi}, \hat{H} \right] = \hat{\pi}_0 + \hbar \omega v_0. \]
(68)
\[ \left[ \hat{\Pi}, \left[ \hat{\Pi}, \hat{H}_0 \right] \right] = \left[ \hat{\Pi}, \left[ \hat{\Pi}, \hat{H} \right] \right] = 2\hbar \omega |v_{12}|^2. \]
(69)

Where
\[ v_{12} = v_1 - iv_2. \]
(70)
\[ v_0 = v_1 g_1 + v_1^* g_1 - iv_2 g_1 g_2 + iv_2^* g_1 g_2. \]
(71)
\[ g_1 = \frac{g\sqrt{\eta}}{2\hbar\omega^2}, \quad g_2 = \frac{\theta\eta}{4\hbar^2 \omega}. \]
(72)

It is clear from 69 that all the higher order commutators vanishes identically. Hence
\[ \hat{\Lambda} \hat{H} \hat{\Lambda}^{-1} = \hat{H} + \hat{\pi}_0 + \hbar \omega (v_0 + |v_{12}|^2). \]
(73)

We can further observe the followings.
\[ e^{\hat{\Pi}} be^{-\hat{\Pi}} = \hat{b} + v_{12}^*. \]
(74)
\[ e^{\hat{\Pi}} b^\dagger e^{-\hat{\Pi}} = \hat{b}^\dagger + v_{12}. \]
(75)

Therefore
\[ \hat{\Theta} \hat{O}^{-1} \hat{\Lambda}^{-1} = \frac{1}{\omega} (\hat{\omega}\mu + \omega \hat{\mu})(\hat{H}_0 + \hat{\pi}_0 + \hbar \omega |v_{12}|^2) + (\beta_1 \hat{b} + \beta_2 \hat{b}^\dagger + \beta_1^* v_{12}^* + \beta_2 v_{12}). \]
(76)

Since \( \hat{\Pi} \) and \( \hat{\Pi} \) commutes to each other, we also have
\[ \frac{\partial \hat{\Lambda}}{\partial t} \hat{\Lambda}^{-1} = \dot{\hat{\Pi}}. \]
(77)
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If we use 73, 76 and 77 in 27 and impose the restriction 28, then the coefficients of the linearly independent operators will satisfy the following set of equations.

\[
\frac{d}{dt}(\omega \mu) = 0. \tag{78}
\]
\[
\omega v_{12} + i\beta_1 + i\dot{v}_1 = 0. \tag{79}
\]
\[
\omega v^*_{12} + i\beta_2 - i\dot{v}^*_1 = 0. \tag{80}
\]
\[
i\omega v_{12} - \beta_1 + i\dot{v}_2 = 0. \tag{81}
\]
\[
- i\omega v^*_{12} + \beta_2 - i\dot{v}^*_2 = 0. \tag{82}
\]
\[
\omega(v_0 + |v_{12}|^2) + i(\beta_1 v^*_{12} + \beta_2 v_{12}) = 0. \tag{83}
\]

Taking complex conjugation of 79 and comparing with 80 (or equivalently 81 and 82) indicates that

\[
\beta_2 = -\beta^*_1. \tag{84}
\]

This implies \(\mu(t)\) is purely imaginary. The specific form of \(\mu(t)\) can be obtained by utilizing 78 and 84 as follows.

\[
\mu(t) = \frac{k_0 i}{\omega(t)}, \quad k_0 \in \mathbb{R}. \tag{85}
\]

Using 84 in 79 and 81, we get

\[
v_1(t) = -i v_2(t), \tag{86}
\]
\[
\dot{v}_2 - 2i\omega(t)v_2 + i\beta_1(t) = 0. \tag{87}
\]

Here we have considered the additive integration constant to be zero (since, we can always redefine \(v_1\) otherwise). Using 85 in 63, we can rewrite 87 as follows

\[
\frac{d}{dt} \begin{pmatrix} v_r \\ v_i \end{pmatrix} = \begin{pmatrix} 0 & -2\omega(t) \\ 2\omega(t) & 0 \end{pmatrix} \begin{pmatrix} v_r \\ v_i \end{pmatrix} + \frac{g_0}{\hbar^2 \omega} \begin{pmatrix} \sin \hbar k_0 \\ \cos \hbar k_0 \end{pmatrix} \sin \hbar k_0. \tag{88}
\]

where the real part and imaginary part of the complex valued function \(v_2(t)\) are \(v_r(t)\) and \(v_i(t)\) respectively. In particular

\[
v_2(t) = v_r(t) + iv_i(t), \quad v_r, v_i : \mathbb{R} \rightarrow \mathbb{R}. \tag{89}
\]

However, 83 provides following constraint on \(v_r(t)\) and \(v_i(t)\).

\[
v_i(g_1(1 + g_2)\omega + \frac{g_0}{\hbar^2 \omega}(\cos(2\hbar k_0) - 1)) + v_r \frac{g_0}{\hbar^2 \omega} \sin(2\hbar k_0) + 2\omega(v_r^2 + v_i^2) = 0. \tag{90}
\]

Given the time-dependent mass \(m(t)\), we can solve \(v_2(t)\) with the help of equations 88 and 90. Before going to some special cases we can write the general form of the invariant operator \(\hat{I}(t) = \hat{\Lambda}(t)\hat{O}(t)\) with the help of BakerCampbellHausdorff formula, which reads

\[
\text{If, } e^X e^Y = e^Z, \tag{91}
\]
\[
\text{then, } Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]]). -
\]
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\[
\frac{1}{24}[Y, [X, [X, Y]]] + \frac{1}{120}((Y, [X, [Y, [X, Y]]]) + [X, [Y, [X, [X, Y]]]]) +
\frac{1}{360}([X, [Y, [Y, X]]]) + [Y, [X, [X, Y]]]) -
\frac{1}{720}((Y, [Y, [Y, [X, Y]]]) + [X, [X, [X, Y]]])) + \ldots \quad (92)
\]

In our case, \( \dot{X} = \hat{\Pi} \) and \( Y = \mu(t)\hat{H}_0 \). Hence one can deduce that

\[
[X, Y] = -2i\hbar\omega\mu(v_2b - v_2^*\hat{b}^i),
\]
\[
[X, [X, Y]] = 8\hbar\omega\mu|v_2|^2,
\]
\[
[Y, [X, Y]] = (2\hbar\omega\mu)^2\hat{\Pi},
\]
\[
[Y, [Y, X]] = (2\hbar\omega\mu)^3i(v_2\hat{b} - v_2^*\hat{b}^i),
\]
\[
[Y, [Y, [Y, X]]] = (2\hbar\omega\mu)^4\hat{\Pi},
\]
\[
[X, [Y, [Y, [X, Y]]]] = -4(2\hbar\omega\mu)^3|v_2|^2.
\]

All other commutators vanish identically. Therefore our invariant operator is given by

\[
\hat{Z}(t) = e^{\hat{Z}},
\]

where

\[
\dot{Z} = (1 - \frac{h^2}{3} - \frac{(h\kappa_0)^4}{45})\hat{\Pi} + \frac{k_0}{\omega}i\hat{H}_0 +
\]
\[
\hbar k_0(v_2\hat{b} - v_2^*\hat{b}^i) + \frac{2}{3}(1 + \frac{2h^2}{15}k_0^2)\hbar k_0i|v_2|^2.
\]

Using 17 in 87, one can write

\[
v_2(t) = v_{20}(1 + \nu_{20}t).
\]

Where

\[
v_{20} = \frac{g_0m_0^2(e^{-2h\kappa_0i} - 1)\tan^2(qL_0)}{i\eta^2 - 2h\nu\eta^2m_0L_0\sec^2(qL_0)}.
\]

\[
\nu_{20} = \frac{4q\nu L_0}{\sin(2qL_0)}.
\]

4. conclusion

We have shown that a class of time-dependent Lewis-Riesenfeld invariance exists for gravitational well in noncommutative space. Since, up to a time-dependent phase factor, the eigenstates of the invariant operator will satisfy the time-dependent Schrödinger equation, one can utilize the eigenstates of the invariant operators to construct the expectation values of the observables. The model under consideration deals with the dynamics of a dynamical effective mass which is a common feature of jammed-granular media. If one considers the asteroid belt as a jammed-granular media, it will open up a possibility to utilize this astronomical system as a condensed matter system with dynamical effective mass for an experiment.

We have partially solved the problem under consideration. Further experimental designs are required for the actual performance of an experiment.
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