An approximation to the Lamb shift and hyperfine splitting as nonlinear effective Coulomb-like interactions in the Dirac equation

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The Dirac equation for the Coulomb problem is restated by incorporating a nonlinear effective interaction into the Dirac Hamiltonian: one keeps the $1/r$ dependence for the Coulomb field, but the coupling constant is modified by a factor depending on the $n$ (principal quantum number) power of the mean value of the Hamiltonian. In this simple context we study the Lamb shift and the hyperfine splitting of the $s$-levels of hydrogenic atoms. We discuss to what extent the corresponding calculations fit the energy splittings to the appropriate order in the fine structure constant.

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I. INTRODUCTORY BACKGROUND

In 1947, Lamb and Retherford [1–3] observed that the $2s_{1/2}$ and the $2p_{1/2}$ energy levels of the hydrogen atom were split: the $2p_{1/2}$ energy level was depressed more than 1000 MHz below the $2s_{1/2}$ energy level. The original theory of a Dirac electron in a classical Coulomb potential predicted that the energy levels of the hydrogen atom should depend only on the principal quantum number $n$ and the total spin $j$, so these two levels should be degenerate.

The calculation of the Lamb-shift is rather intricate, because one is dealing with the hydrogen atom ($Z = 1$) as a bound-state problem, and also because we must sum over all radiative corrections to the electron interacting with a Coulomb potential that modify

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the naïve $\Psi^\dagger A_0 \Psi$ vertex. These corrections include the vertex correction, the anomalous magnetic moment, the self-energy of the electron, the vacuum polarization graph, and even infrared divergences.

The original nonrelativistic bound-state calculation of Bethe [4], which ignored many of these subtle higher-order corrections, could account for about 1000 MHz of the Lamb-shift, but only a fully relativistic quantum treatment could calculate the rest of the difference. To begin the discussion, we first see that the vacuum polarization term can be attached to the photon line, changing the photon propagator. This, of course, translates into a shift in the effective coupling of an electron to the Coulomb potential [3]. We know from ordinary nonrelativistic quantum mechanics that, by taking matrix elements of this modified potential between Coulomb wave functions, we can calculate the first-order correction to the energy levels of the one-electron atom due to the vacuum polarization graph.

This discussion can be generalized to include the other corrections to the calculation of the Lamb-shift. The method is the same: corrections to the vertex function $\Psi^\dagger \gamma_\mu \Psi$, take the zeroth component, and then take the low-energy limit. If we add the various contributions to the vertex correction, we find the well-known formula:

$$\Delta E_{\text{Lamb}}^{(n\kappa)} = \frac{4mc^2}{3\pi n^2\alpha (Z\alpha)^4} \left( L_{n\kappa} + \frac{19}{30} - 2 \ln (Z\alpha) \right) \delta_{\kappa,-1} + \frac{3}{8\kappa (2|\kappa| - 1)} (1 - \delta_{\kappa,-1})$$

(1)

to order $\alpha (Z\alpha)^4 \ln (Z\alpha)$, where $m$ is the mass of the electron and $\alpha = e^2/\hbar c$ is the fine structure constant. The term

$$L_{n\kappa} \equiv \frac{n^3}{2m^2c^2 (Z\alpha)^4} \sum_{n',\kappa'} \left| \left< \Psi_{n',\kappa'}^{\text{NR}} | \hat{p} | \Psi_{n\kappa}^{\text{NR}} \right> \right|^2 \times \left( E_{n',\kappa'} - E_{n\kappa} \right)$$

(2)

$$\times \left[ 2 \ln (Z\alpha) - \ln \left( \frac{mc^2/2}{E_{n',\kappa'} - E_{n\kappa}} \right) \right] \times \left( E_{n',\kappa'} - E_{n\kappa} \right) \times \left[ 2 \ln (Z\alpha) - \ln \left( \frac{mc^2/2}{E_{n',\kappa'} - E_{n\kappa}} \right) \right]$$

is also known as the Bethe logarithm which has to be evaluated numerically. In the above $\kappa = \pm (j + 1/2)$ for $l = j \pm 1/2$, and $\Psi_{n\kappa}^{\text{NR}}$ is a nonrelativistic atomic state.

The vertex correction, for $Z = 1$ for example, gives us a value of 1010 MHz. The anomalous magnetic moment of the electron contributes 68 MHz. The vacuum polarization
graph contributes \(-27.1\) MHz. Adding these corrections together, we find, to the lowest
loop level, the Lamb-shift to within 6 MHz accuracy.

Since then, higher-order corrections have been calculated, so that the difference between
experiment and theory has been reduced to 0.01 MHz. Theoretically, the \(2s_{1/2}\) level is
above the \(2p_{1/2}\) energy level by 1057.864 MHz. The experimental result is 1057.862 within
0.02 MHz. This is an excellent indicator of the basic correctness of QED [6–8].

Another important effect, not contained in the energy levels of the Dirac-Coulomb prob-
lem, arises from the interaction between the magnetic moment of the nucleus and the mag-
netic moment of the electron. In the case of the hydrogen atom, for instance, when we
combine the electron spin with the proton spin, the net result is \(S = 1\) (triplet) or \(S = 0\)
(singlet), where \(S\) is the quantum number corresponding to the total spin. Since the magnetic
interaction is dependent on the relative orientation of the two magnetic dipole moments, each
level of the hydrogen atom characterized by \(njl\), is further split into two sublevels corre-
sponding to the two possible values of \(S\) even in the absence of any external magnetic field.
This is known as the hyperfine splitting [8–11].

Using the nonrelativistic wave function \(\Psi_{\text{NR}}^{(\text{Hyp})}\), we obtain the energy shift

\[
\Delta E_{n\kappa}^{(\text{Hyp})} = \frac{1}{3} g_p \left( \frac{m}{M_p} \right) \frac{mc^2}{n^3} (\delta_{S,1} - 3\delta_{S,0}) \delta_{\kappa,-1} \alpha^4,
\]

where \(g_p = 2(1 + \kappa_p) = 5.58568\) is the \(g\)-factor of the proton. Note that the order of
magnitude of this splitting is the fine-structure splitting multiplied by \(m/M_p\). For the \(2s_{1/2}\)-
state, the above energy difference corresponds to a radio microwave of 1420 MHz. It is an
accurately measured quantity: 1420.40575180 within at least \(3 \times 10^{-8}\) MHz [13].

There are other corrections to the Dirac formula. First, we must take into account the
motion of the nucleus since the mass of the nucleus is not infinite. A major part of this
correction can be taken care of if we use the reduced mass in place of \(m\). Second, the finite
size of the nucleus, especially for the \(s\) states which are sensitive to small deviations from
Coulomb’s law at close distances: in the interesting case of the \(2s\) state of the hydrogen
atom, however, we can estimate the energy shift due to this effect to be only 0.1 MHz.

The utility of Dirac theory in atomic physics is not limited to light hydrogen-like atoms.
For heavy atoms where \(Z\alpha\) is not very small compared with unity, the relativistic effects
must be taken into account even for understanding the qualitative features of the energy
levels. Although we cannot, in practice, study one-electron ions of heavy atoms, it is actually possible to check the quantitative predictions of Dirac theory by looking at the energy levels of the innermost electrons of high Z atoms which can be inferred experimentally. Similar studies have been carried out with muonic atoms [14].

This paper concerns a simple, and restricted, approach to the study of the splitting structure of the Dirac-Coulomb energy levels, given in Eqs. (1) and (3), in the context of relativistic quantum mechanics. This procedure circumvents second quantization on both the electromagnetic and the electron fields. However, it does not pretend to be an alternative way of reproducing the corresponding detailed and precise modern calculations by QED. In fact, for the case of the Lamb-shift, we do in fact make use of second quantization information (the self-energy of the electron, the vertex correction, the anomalous magnetic moment, and the vacuum polarization graphs to order $\alpha (Z\alpha)^4mc^2$) to define the basic coupling constant. In Sec. II, we present the general approach. It considers the introduction of a nonlinear effective interaction into the Dirac Hamiltonian. Although the radiative processes involved in the Lamb-shift will somehow be hidden in the corresponding effective Hamiltonian, it is illustrative to view them from a different perspective. The case of the hyperfine structure is treated in the same framework, though it does not involve radiative corrections to the first level, i.e., when only relativistic corrections are taken into account.

Higher order corrections to the Lamb-shift to order $\alpha (Z\alpha)^5mc^2$ and to the hyperfine-splitting are considered in Sec. III. Finally, Sec. IV contains our conclusions and some open questions.

II. EFFECTIVE NONLINEAR COULOMB-LIKE INTERACTION

In this paper we want to show that the splittings (1) and (3) can be derived from the Dirac-Coulomb problem by incorporating a nonlinear effective interaction into the Dirac Hamiltonian. To this end, we shall assume that the interaction does not modify the $\sim (1/q)$ ($q \equiv |q|$) law in the Coulomb-like gauge (central) field. This radial behavior is one of the few cases where the Dirac wave equation can be solved analytically. The problem to study is then the following
\[
H \Psi(q, t) = \left(-i \hbar \alpha \cdot \nabla_q + \beta m c^2 + e A_0^{(\text{eff})} (q) \right) \Psi(q, t) = i \hbar \frac{\partial}{\partial t} \Psi(q, t),
\]
\[\text{in the standard Dirac representation [15]. Here the normalization condition is}\]
\[
\int_{\mathbb{R}^3} \Psi^\dagger(q, t) \Psi(q, t) \, d^3 q = 1,
\]
\[\text{and}\]
\[
A_0^{(\text{eff})} (q) \equiv \hat{g}_\nu \frac{Z |e|}{q}
\]
is an effective potential, where \( \hat{g}_\nu \) is a diagonal constant matrix: It is taken not to be \( q \)-dependent since we shall not consider corrections in the vicinity of the nucleus. The dimensionless elements of this matrix depend on the quantum numbers represented by \( \nu \), labeling the stationary states \( \Psi_{E_\nu}(q, t) = \exp(-i E_\nu t / \hbar) \Psi_{E_\nu}(q) \), whose concrete structure will be specified, in each case, below.

Before solving the eigenvalue problem associated with (4), we recall that the operators \( \hat{K} \equiv \beta (\Sigma \hat{L} + \hbar) \), \( \hat{J} \equiv \hat{L} + \frac{\hbar}{2} \Sigma \), with \( \hat{L} = q \times \hat{p} \) the orbital angular momentum operator, are constants of motion: \( [H, \hat{K}] = 0, [H, \hat{J}] = 0 \). Following a standard procedure [15], the stationary states of energy \( E \) can be written as
\[
\Psi_{E_\nu}(q, t) = \left( \psi_a(q, t), \psi_b(q, t) \right) = \left( \psi_a(q) \mathcal{Y}_{jjsl_a}(\hat{q}), i \psi_b(q) \mathcal{Y}_{jjsl_b}(\hat{q}) \right) \exp(-i E_\nu t / \hbar),
\]
where \( \mathcal{Y}_{jjsl} \) are the normalized total angular momentum functions, with
\[
\hat{L}^2 \mathcal{Y}_{jjsl} = \hbar^2 l(l + 1) \mathcal{Y}_{jjsl}, \quad \hat{J}^2 \mathcal{Y}_{jjsl} = \hbar^2 j(j + 1) \mathcal{Y}_{jjsl}, \quad \hat{K} \mathcal{Y}_{jjsl} = \hbar \kappa \mathcal{Y}_{jjsl},
\]
where \( l_a = j \pm 1/2, l_b = j \mp 1/2 \) when \( \kappa = \pm (j + 1/2) \). Let us write \( \hat{g}_\nu \) in the general form
\[
\hat{g}_\nu \equiv \frac{1}{2} g_{\nu a} (I + \beta) + \frac{1}{2} g_{\nu b} (I - \beta).
\]
where \( \nu_{a,b} \) are labels defined in Eq.(21). Thus the Dirac equation is equivalent to the set of first-order (nonlinear) differential equations
\[
c_\sigma \hat{p} \psi_b(q) = \left( E_\nu - mc^2 + g_{\nu a} \frac{Ze^2}{q} \right) \psi_a(q), \quad c_\sigma \hat{p} \psi_a(q) = \left( E_\nu + mc^2 + g_{\nu b} \frac{Ze^2}{q} \right) \psi_b(q).
\]
From (8), (9) and (11) we find that

\[
\left( \frac{d}{dr} R_B(r) - \frac{\kappa}{r} R_B(r) \right) = \left( \sqrt{\frac{M_2}{M_1} - g_{\nu_a} \frac{Z\alpha}{r}} \right) R_A(r),
\]

(12)

\[
\left( \frac{d}{dr} R_A(r) + \frac{\kappa}{r} R_A(r) \right) = \left( \sqrt{\frac{M_1}{M_2} + g_{\nu_b} \frac{Z\alpha}{r}} \right) R_B(r),
\]

where \( R_{A,B}(r) \equiv r\psi_{a,b}(r) \) and

\[
r = \sqrt{M_1 M_2 q}, \quad M_1 = \frac{mc^2 + E_\nu}{\hbar c}, \quad M_2 = \frac{mc^2 - E_\nu}{\hbar c}, \quad Z\alpha = Z e^2 \hbar c.
\]

(13)

Next we look for solutions in the form of series

\[
R_A(r) = \exp(-r)s \sum_{\mu=0} a_\mu r^\mu, \quad R_B(r) = \exp(-r)s \sum_{\mu=0} b_\mu r^\mu.
\]

(14)

Thus from (12) and (14) we get

\[
\begin{align*}
(s - \kappa) a_0 - g_{\nu_b} Z\alpha b_0 &= 0, \\
g_{\nu_a} Z\alpha a_0 + (s + \kappa) b_0 &= 0,
\end{align*}
\]

for \( \mu = 0 \),

(15)

and

\[
\begin{align*}
(s + \mu + \kappa) a_\mu - a_{\mu-1} - g_{\nu_b} Z\alpha b_\mu - \sqrt{M_1/M_2} b_{\mu-1} &= 0, \\
(s + \mu - \kappa) b_\mu - b_{\mu-1} + g_{\nu_a} Z\alpha a_\mu - \sqrt{M_2/M_1} a_{\mu-1} &= 0,
\end{align*}
\]

for \( \mu > 0 \).

(16)

Given that \( a_0, b_0 \neq 0 \), from (13) we obtain

\[
s = \pm \sqrt{\left( \kappa^2 - (Z\alpha)^2 g_{\nu_a} g_{\nu_b} \right)} \geq \pm \sqrt{\left( \kappa^2 - (Z\alpha)^2 \right)} > -\frac{1}{2}.
\]

(17)

The negative sign must be excluded since it would make the functions \( R_{A,B} \) singular at the origin. Choosing \( \mu = n' + 1 \) and \( a_{n'+1} = b_{n'+1} = 0 \), in order to terminate the series, we have that \( b_{n'} = -a_{n'} \sqrt{M_2/M_1} \). Then from (16) we get

\[
2 (s + n') \sqrt{M_1 M_2} = Z\alpha (g_{\nu_a} M_1 - g_{\nu_b} M_2).
\]

(18)

Finally, from (17) and (18) we obtain the energy eigenvalues from

\[
2 (s + n - |\kappa|) \sqrt{(mc^2)^2 - E_{n\kappa}^2} = Z\alpha g_{\nu_a} \left( mc^2 + E_{n\kappa} \right) - Z\alpha g_{\nu_b} \left( mc^2 - E_{n\kappa} \right),
\]

(19)

where

6
\[ n \equiv n' + |\kappa| = n' + j + \frac{1}{2} \]  

(20)

is the principal quantum number. Note that for the point nucleus there exist bound solutions (for \( \kappa = -1 \)) only up to \( Z \simeq 1/\alpha \sqrt{g_{n\alpha} g_{n\beta}} \simeq 1/\alpha \).

Given the fact that \( <1/q>_{\Psi^{(NR)}} \sim 1/n^2 \), we expect that \( g_{\nu_a} - 1 \propto 1/n \), to be able to meet the factor \( 1/n^3 \) in both (1) and (3). Thus, in the examples that follow, we shall choose a particular form for \( g_{\nu_a} \) and \( g_{\nu_b} \):

\[
g_{\nu_a} \equiv g_{n\kappa l_a} \equiv 1 - \lambda_{n\kappa l_a} (\alpha, Z\alpha) \left( 1 - \left( \langle \hat{H}_D \rangle_{\Psi^{(Dirac)}_{n\kappa}} \right)^n \right),
\]

\[
g_{\nu_b} \equiv g_{n\kappa l_b} \equiv 1 - \lambda_{n\kappa l_b} (\alpha, Z\alpha) \left( 1 - \left( \langle \hat{H}_D \rangle_{\Psi^{(Dirac)}_{n\kappa}} \right)^n \right),
\]

(21)

where \( \hat{H}_D \equiv H_D/mc^2 \). The factors \( \lambda_{n\kappa l}(\alpha, Z\alpha) \) may depend on the original Dirac-Coulomb potential quantum numbers \( \nu_{a,b} \equiv n\kappa l_{a,b} \), the binding (powers of \( Z\alpha \)), and radiative corrections (powers of \( \alpha \)). The term containing the expectation value of \( \hat{H}_D \) has the required property, namely

\[
\left( 1 - \left( \langle \hat{H}_D \rangle_{\Psi^{(Dirac)}_{n\kappa}} \right)^n \right) = \frac{1}{2} \frac{1}{n} (Z\alpha)^2 + O \left( (Z\alpha)^4 \right) \propto \frac{1}{n}.
\]

(22)

Thus the factors \( g_{n\kappa l} \) includes a nonrelativistic coupling and a radiative term: \( \lambda_{n\kappa l}(\alpha, Z\alpha) \) (see Eq. (21)), times a purely quantum relativistic factor: \( \left( 1 - \left( \langle \hat{H}_D \rangle_{\Psi^{(Dirac)}_{n\kappa}} \right)^n \right) \).

Now from (13) and (21) we finally get

\[
2 \left( s + n - |\kappa| \right) \sqrt{(mc^2)^2 - E_{n\kappa}^2} = 2E_{n\kappa} Z\alpha g_{n\kappa l_a} + O \left( mc^2 (Z\alpha)^3 (g_{n\kappa l_{a,b}} - 1) \right).
\]

(23)

On the right-hand side of (23) we are neglecting a term proportional to \( (Z\alpha)^3 (g_{n\kappa l_{a,b}} - 1) \) which is, for instance, of order \( \alpha (Z\alpha)^5 \) for the case of the Lamb-shift.

A. The Lamb-shift

As a first instance, we shall consider an approximation to the Lamb-shift. To this end notice that \( \langle H_D \rangle_{\Psi^{(Dirac)}_{n\kappa}} \equiv E_{n\kappa} (\hat{g} = I) \), with \( H_D \equiv H(\hat{g} = I) \) the Dirac-Coulomb Hamiltonian, where the eigenstates \( \Psi^{(Dirac)}_{n\kappa} \) satisfy the normalization condition

\[
\int_{\mathbb{R}^3} \Psi^{(Dirac)}_{n\kappa}(\mathbf{q}) \Psi^{(Dirac)}_{n\kappa}(\mathbf{q}) \, d^3q = 1.
\]

(24)
Let
\[
\lambda_{\text{nel}}(\alpha, Z\alpha) \equiv \frac{8}{3\pi} \alpha \left( \left( L_{nl} + \frac{19}{30} - 2 \ln (Z\alpha) \right) \delta_{l,0} + \frac{3}{8} \frac{1}{\kappa (2 |\kappa| - 1)} (1 - \delta_{l,0}) \right) \,.
\] (25)

The effective gauge potential to be considered is of the form
\[
A_0^{(\text{eff})}(q) = \hat{g}_{\text{nel}}^{(\text{Lamb})} \frac{Z |e|}{q},
\] (26)

where
\[
\hat{g}_{\text{nel}}^{(\text{Lamb})} = g_{\text{nel}}^{(\text{Lamb})}(\alpha, Z\alpha) = 1 - \lambda_{\text{nel}}(\alpha, Z\alpha) \left( 1 - \left( \langle \hat{H}_D > \psi_{\text{Dirac}}^{(\text{Dirac})} \rangle \right)^n \right).
\] (27)

Thus \(\hat{g}_{\text{nel}}^{(\text{Lamb})}\) diminishes the Coulomb binding \(-Z\alpha/q\), and as a consequence, the \(s\) levels are pushed higher. From (19) and (27) we find
\[
(s + n - |\kappa|) \sqrt{(mc^2)^2 - \left( E_{nl}^{(\text{Lamb})} \right)^2} = Z\alpha g_{\text{nel}}^{(\text{Lamb})} E_{nl}^{(\text{Lamb})} + O (mc^2 \alpha (Z\alpha)^5). \] (28)

Expanding this equation in powers of \(\alpha\), we find the spectrum
\[
\left( E_{nl}^{(\text{Lamb})} - mc^2 \right) / mc^2 = -\frac{1}{2} \frac{n}{n^2} (Z\alpha)^2 + \left( \frac{3}{8} \frac{1}{n^4} - \frac{1}{2} \frac{1}{n^3} \frac{1}{|\kappa|} \right) (Z\alpha)^4 \] (29)

\[
+ \frac{4}{3\pi} \frac{1}{n^3} \left( \left( L_{nl} + \frac{19}{30} - 2 \ln (Z\alpha) \right) \delta_{\kappa,-1} + \frac{3}{8} \frac{1}{\kappa (2 |\kappa| - 1)} (1 - \delta_{\kappa,-1}) \right) \alpha (Z\alpha)^4 + O (\alpha (Z\alpha)^5) .
\]

We observe that the last term in (29) corresponds to the Lamb-shift splitting to the Dirac levels to order \(\alpha (Z\alpha)^4\) \cite{8}. The Coulomb potential for the hydrogen atom \((Z = 1)\) and the corresponding Coulomb-like radiative corrections for \(n = 4, 8, 12\), are shown in fig. 1.

**B. The hyperfine splitting**

The second instance regards the *hyperfine splitting* in the energy levels of the hydrogen atom \((Z = 1)\). In hydrogenic atoms, the interaction of the magnetic moment of the orbital electron with the magnetic moment of the nucleus leads to a splitting of the *fine structure* levels with fixed orbital angular momentum \(l\) \((\kappa = -1)\) into the *hyperfine structure* levels. In this case we choose
\[
g_{\text{nel}}^{(\text{Hyp})} = 1 - \lambda_{\text{Hyp}} \left( 1 - \left( \langle \hat{H}_D > \psi_{\text{Dirac}}^{(\text{Dirac})} \rangle \right)^n \right) , \] (30)
where

\[ \lambda_{kl} \equiv \frac{2}{3} g_p \left( \frac{m}{M_p} \right) (\delta_{S,1} - 3\delta_{S,0}) \delta_{l,0} , \tag{31} \]

with \( g_p = 2(1 + \kappa_p) \), \( \kappa_p = 1.79284 \), the \( g \)-factor of the neutron. Thus from (13) and (30) we get

\[ 2 (s + n - |\kappa|) \sqrt{(mc^2)^2 - (E_{n\kappa}^{(Hyp)})^2} = 2Zag_{n\kappa l_a}^{(Hyp)} E_{n\kappa}^{(Hyp)} + O \left( mc^2 (Z\alpha)^3 \right) \tag{32} \]

from which we get

\[ \frac{(E_{n\kappa}^{(Hyp)} - mc^2)}{mc^2} = -\frac{1}{2} \frac{\alpha^2}{n^2} + \left( \frac{3}{8} \frac{1}{n^4} - \frac{1}{2} \frac{1}{n^3 |\kappa|} \right) \alpha^4 \]

\[ + \frac{1}{3} \frac{1}{n^2} g_p \left( \frac{m}{M_p} \right) (\delta_{S,1} - 3\delta_{S,0}) \delta_{\kappa,-1}\alpha^4 \]

\[ + O (\alpha^6) . \tag{33} \]

This is again the correct spectrum for the relativistic Dirac levels to order \( O (\alpha^4) \) [13].

Notice that the first term in both (29) and (33) gives the energy spectrum of the bound states in the non-relativistic approximation. The second terms are the corresponding leading corrections to the Balmer formula: the fine splitting. These expressions are a consequence of the modification \( Z\alpha \rightarrow \hat{g}_{n\kappa}Z\alpha \) introduced by the nonlinear factor \( \left( \langle \hat{H} | \psi_{n\kappa}^{(Dirac)} \rangle \right)^n \) (together with \( \lambda_{n\kappa}(\alpha, Z\alpha) \) in the case of the Lamb-shift), which depend on the solution of the original (\( \hat{g}_{n\kappa} = I \)) eigenvalue problem itself.

Finally, an interesting exercise to consider regards the eigenvalue problem containing both corrections: the Lamb-shift and the hyperfine splitting for the case of the hydrogen atom \( (Z = 1) \). This is easily done by defining

\[ \hat{g}_{n\kappa} \equiv \hat{g}_{n\kappa}^{(Lamb)} + \hat{g}_{n\kappa}^{(Hyp)} - I . \tag{34} \]

Replacing \( \hat{g}_{n\kappa} \) in (23) and expanding in powers of \( \alpha \) yields

\[ \frac{(E_{n\kappa} - mc^2)}{mc^2} = -\frac{1}{2} \frac{\alpha^2}{n^2} + \left( \frac{3}{8} \frac{1}{n^4} - \frac{1}{2} \frac{1}{n^3 |\kappa|} \right) \alpha^4 + \frac{1}{3} g_p \left( \frac{m}{M_p} \right) \frac{1}{n^3} (\delta_{S,1} - 3\delta_{S,0}) \delta_{\kappa,-1}\alpha^4 \]

\[ + \frac{4}{3\pi} \frac{1}{n^3} \left( \left( L_{n\kappa} + \frac{19}{30} - 2 \ln (\alpha) \right) \delta_{\kappa,-1} + \frac{3}{8} \frac{1}{\kappa (2 |\kappa| - 1)} (1 - \delta_{\kappa,-1}) \right) \alpha^5 \]

\[ + O (\alpha^6) , \tag{35} \]

which is the spectrum of the hydrogen atom to order \( \alpha^5 \). Notice that any series expansion of \( \ln (\alpha) \) rapidly overcomes the relativistic (Dirac) corrections of \( E_{n\kappa} \) beginning from order \( \alpha^5 \).
III. HIGHER ORDER CORRECTIONS

To achieve a still better quantitative agreement, several contributions of higher order must be included \[8\], namely corrections to order \( mc^2 \alpha (Z\alpha)^5 \), \( mc^2 \alpha (Z\alpha)^6 \) and \( mc^2 \alpha^2 (Z\alpha)^4 \). Here we only want to examine the \( mc^2 \alpha (Z\alpha)^5 \) term (the 2nd order binding) as the next correction to the Lamb-shift: \( 2s_{1/2} - 2p_{1/2} \) \[29\] for the hydrogen atom (see fig. 2). To this end we can approximately fit the corresponding QED theoretical value by making the replacement

\[
\lambda_{nl}(\alpha, Z\alpha) \rightarrow (1 + Z\alpha) \lambda_{nl}(\alpha, Z\alpha), \tag{36}
\]

in the \( \lambda_{nl} \) factors contained in \( \text{(27)} \). For \( Z = 1 \), the correction calculated with \( \text{(19)}, \text{(27)} \) and \( \text{(36)} \), and the QED value \[8\], both to order \( \alpha (Z\alpha)^5 \), are respectively:

\[
\delta E \simeq 7.663 \text{ MHz} , \quad (\delta E)_{\text{QED}} \simeq 7.243 \text{ MHz} , \tag{37}
\]

which corresponds to a \( \sim 5\% \) of discrepancy. These calculations have been performed with the value \( \alpha \simeq 1/137.036 \simeq 7.29735 \times 10^{-3} \) for the fine structure constant.

On the other hand, in fig. 2 we observe that there is good agreement between the present approach and the QED numerical values for \( Z = 1, 2, \ldots, 10 \). In fact, the Lamb-shift for \( Z = 1, n = 2 \), is given by

\[
\Delta E_{n=2}^{(\text{Lamb})} = 1,046.54 \text{ MHz} , \quad \left( \Delta E_{n=2}^{(\text{Lamb})} \right)_{\text{QED}} = 1,046.45 \text{ MHz} , \tag{38}
\]

where

\[
\Delta E_{n=2}^{(\text{Lamb})} \equiv E^{(\text{Lamb})}(2s_{1/2}) - E^{(\text{Lamb})}(2p_{1/2}), \tag{39}
\]

(see the red line in fig. 2); while for \( Z = 10, n = 2 \), we get

\[
\Delta E_{n=2}^{(\text{Lamb})} = 4,469.82 \times 10^3 \text{ MHz} , \quad \left( \Delta E_{n=2}^{(\text{Lamb})} \right)_{\text{QED}} = 4,860.51 \times 10^3 \text{ MHz} , \tag{40}
\]

with a \( \sim 8\% \) of discrepancy.

For the hyperfine structure we make

\[
\lambda_{nl}^{(\text{Hyp})} \rightarrow \lambda_{nl}^{(\text{Hyp})}(\alpha, Z\alpha) = (1 + \delta_{\text{Breit}} + \delta(\alpha, Z\alpha)) \lambda_{nl}^{(\text{Hyp})} , \tag{41}
\]
where $\delta_{\text{Breit}} = (3/2)(Z\alpha)^2$, and $\delta (\alpha, Z\alpha)$ contains higher order corrections: radiative, binding, finite mass and structure of the proton [10,12]. Higher (radiative and binding) order corrections to the Lamb-shift seem to be meaningless and difficult to reproduce by using this simple approach.

IV. OUTLOOK

In this article we have introduced a nonlinear effective Coulomb-like gauge potential (electromagnetic in nature) into the Hamiltonian of a Dirac particle to describe the Lamb-shift and the hyperfine structure of the energy spectrum. In all the calculations we have kept track of the corresponding orders in both the fine structure constant $\alpha$ and the combination $Z\alpha$ introduced by the $\hat{g}_{n\kappa}$ matrix factor. In Eq. (19) there still is room for further research. For instance, we can study systems (apart from the example of the Lamb-shift) where $\hat{g}_\nu(\alpha)$ is of order $\zeta > 0$ ($\zeta$ some integer number) in $\alpha$ and $Z\alpha$. In addition to this, the prospect of defining an iterative recursion procedure from (19) and (27) in order to improve the (eventually convergent) coupling constant $Z\alpha$ should be studied.

Much work needs to be done to give a precise characterization of the various physical models included in Eq.(19). Particularly, there still is lacking a thorough explanation of the structure of the $\hat{g}_\nu(\alpha)$ factors contained in the different effective Hamiltonians that we have considered.

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Figure captions

Fig. 1. Schematic plots of the Coulomb potential per unit charge $A_0(q)/|e|$ for the hydrogen atom and the corresponding Coulomb-like (radiative) $|\lambda_{n0\alpha}(\alpha, Z\alpha)| A_0(q)/|e|$ ($\kappa = -1, l_a = 0$) corrections ($Rc$) for $n = 4, 8, 12$.

Fig. 2. The lamb-shift $2s_{1/2} - 2p_{1/2}$ for $Z = 1, ..., 40$. The calculations with the present approach to order $\alpha (Z\alpha)^4$ (green color) are improved with the corrections to order $\alpha (Z\alpha)^5$ (red color). The continuous lines are drawn in order to visualize the corresponding patterns. There is good agreement for values of $Z$ in the range 1, ..., 10 with those of QED (see Refs. [8,16]) with a relative accuracy between 0.01%. ($Z = 1$) and 8% ($Z = 10$).
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