AREA BOUNDS FOR FREE BOUNDARY MINIMAL SURFACES IN A GEODESIC BALL IN THE SPHERE

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Abstract

We extend to higher dimensions earlier sharp bounds for the area of two dimensional free boundary minimal surfaces contained in a geodesic ball of the round sphere. This follows work of Brendle and Fraser-Schoen in the euclidean case.

1. Introduction

A problem of recent interest in geometric analysis is to identify sharp area bounds for free boundary minimal surfaces. Fraser-Schoen proved [10] Theorem 5.4] any free boundary \( \Sigma^2 \subset B^n \), where \( B^n \) is a unit \( n \)-dimensional euclidean ball, has area at least \( \pi \); equality holds precisely when \( \Sigma \) is congruent to a disk. Following a question of Guth, Schoen conjectured the analogous sharp bound \( |\Sigma^k| \geq |B^k| \) for free boundary \( \Sigma^k \subset B^n \) of any dimension. This was later proved by Brendle [3]. In [11], the authors proved analogous bounds for free boundary \( \Sigma^2 \) in certain positively curved geodesic balls, including any such ball contained in a hemisphere of the round \( S^n \). In this article we extend results of [11] to higher dimensions.

**Theorem 1.1.** Let \( \Sigma^k \subset B^n_R \) be a free boundary minimal surface, where \( B^n_R \subset S^n \) is a geodesic ball with radius \( R \leq \pi/2 \), and \( k = 4 \) or \( k = 6 \). Then \( |\Sigma| \geq |B^k_R| \), where \( |B^k_R| \) is the volume of a \( k \)-dimensional geodesic ball of radius \( R \). If equality holds, then \( \Sigma \) coincides with some such ball.

Applying the proof of Theorem 1.1 to a sequence of balls \( B^n_R \) as above with radii going to zero in combination with a rescaling argument recovers in the limit (see [11] for a precise statement) the euclidean bounds in [3], giving another proof of those results in the dimensions above.

A corollary of the euclidean area bounds mentioned above is that free boundary submanifolds of a euclidean ball satisfy the sharp isoperimetric inequality \( |\partial \Sigma^k|^k/|\Sigma^k|^{k-1} \geq |\partial B^k|^k/|B^k|^{k-1} \). The class of minimal submanifolds \( \mathbb{R}^n \) for which this sharp isoperimetric inequality is known to hold is relatively small and includes also absolutely area minimizing submanifolds [1] and two-dimensional minimal surfaces with radially connected boundary [5]. It would be interesting to know whether the submanifolds considered in Theorem 1.1 satisfy the sharp spherical isoperimetric inequality. In dimension 2, Choe-Gulliver [8] Remark 1] have asked more generally whether every minimal surface \( \Sigma^2 \) contained in a hemisphere of \( S^n \) (with no conditions on the boundary) satisfies the sharp \( S^2 \)-isoperimetric inequality \( 4\pi|\Sigma| \leq |\partial \Sigma|^2 + |\Sigma|^2 \).

A properly immersed submanifold \( \Sigma^k \subset \Omega^n \) in a domain of a Riemannian manifold is a free boundary minimal submanifold if \( \Sigma \) is minimal, \( \partial \Sigma \subset \partial \Omega \), and \( \Sigma \) intersects \( \partial \Omega \) orthogonally. Such submanifolds are volume-critical among all deformations which preserve the condition \( \partial \Sigma \subset \partial \Omega \). Free boundary minimal submanifolds have been widely studied in the last decade, and many fundamental questions regarding their existence and uniqueness remain unanswered.
The proof of Theorem 1.1 is motivated by Brendle’s ingenious approach in [3]. There Brendle applies the divergence theorem to a vector field $W$ with the following properties:

(i). $W$ is defined on $B^n \setminus \{y\}$ and has a prescribed singularity at $y \in \partial B^n$.
(ii). $W$ is tangent to $\partial B^n$ along $\partial B^n \setminus \{y\}$.
(iii). $\text{div}_B W \leq 1$ for any submanifold $\Sigma^k \subset B^n$.

In the euclidean setting of [3], $W$ is a sum of a radial field with divergence bounded above by 1 centered at 0 and a singular field with nonpositive divergence centered at $y$. When the dimension $k$ of the submanifold is greater than two, $W$ contains an integral term manufactured to cancel an unfavorable term arising from the dominant singular part.

Unfortunately, the analogous field – even in dimension two – in the setting of Theorem 1.1 no longer satisfies (iii). It turns out however that a judiciously chosen convex combination of fields – each of which has divergence bounded above by 1 – can be arranged which satisfies (i)-(iii). The singular part is governed by a vector field $Z$ of the form

$$Z = \Psi_y + \int_R^\pi h(s)\Psi_{\gamma(s)} \, ds,$$

where $\Psi_y$ is a dominant term with a singularity at $y$, and the singular integral term (integrated along the geodesic segment $\gamma$ connecting $y$ and the antipode of $B_R$’s center) is manufactured as in the euclidean case to ensure that $W$ is tangent to the geodesic sphere $\partial B_R$ along $\partial B_R \setminus \{y\}$. Idiosyncratic aspects of the formula for the volume $|B_R^k|$ of a $k$-dimensional geodesic ball of radius $R$ in $S^n$ make this term fundamentally more complicated than its counterpart in [3], which has several consequences.

One such aspect is a structural difference between expressions for $|B_R^k|$ when $k$ is even and when $k$ is odd. Because of this, we are presently able to propose a scheme to adequately construct $Z$ only for even $k$ (see [3.11] and [4.8]). A similar dichotomy is present in formulae related to other PDE, for example in the solution of the wave equation on $\mathbb{R}^n$ [9] Theorems 2.4.2, 2.4.3] and in formulas for the heat kernel on hyperbolic space $\mathbb{H}^n$ [12] and on the sphere $S^n$ [15].

Another consequence is that it is rather trivial (see [24]) to prove the sharp bound of Theorem 1.1 in the special case when $B_R^2$ is a hemisphere – one may actually take $W = \Psi_y$ and $h$ identically zero – but more challenging to understand the state of affairs for general $R$ and $k$.

Indeed, when $k$ is an even integer $2j$, $h$ is determined by the solution of an initial value problem associated with a $(j-1) \times (j-1)$ first order linear system of differential equations (see [3.10]). Even for small $j$, the associated $h$ is quite involved – when $j = 2$, for example,

$$h(s) = \xi \left\{ (1 + 2 \csc^2 R) \sin^3 s + (\cos^2 s - \cos R \cos s - \frac{1}{3} \sin^2 R) \sin s \left( \frac{\cot(s/2)}{\cot(R/2)} \right)^{\cos R} \right\},$$

where $\xi := (3 \cos R \csc^2 R)/(1 + 3 \sin^2 R)$. By contrast, the appropriate euclidean analogue of $h$ [14] equation (1) in dimension $k \geq 3$ is simply $s^{k-3}$. A key step in our method of proof is to verify that $h$ is nonnegative. We are able to confirm this for $j = 2$ and $j = 3$ and thus prove Theorem 1.1 in dimensions $k = 4$ and $k = 6$. When $k = 8$ and for certain values of $R$, numerical computations indicate that $h$ is not strictly nonnegative and the method appears to break down.

The calibration vector field strategy in the spirit of [3] appears to be quite flexible and has been used recently by Brendle-Hung [4] to prove a sharp lower bound for the area of a minimal submanifold $\Sigma^k \subset B^n$ passing through a prescribed point $y \in B^n$ (see also [16]).
The approach here is also closely related to work of Choe [6] and Choe-Gulliver [7, 8] on isoperimetric inequalities for domains on minimal surfaces. While in that setting the geometric inequalities are favorable in a negative curvature background, in the present context positive ambient curvature is essential (see [2.6] to the proof of Theorem 1.1). Similar interactions between curvature and geometric inequalities lead to a generalization of the classical monotonicity formula for minimal submanifolds of hyperbolic space $\mathbb{H}^n$ [2], whereas no monotonicity formula is known for minimal submanifolds of the sphere (see however [13] Lemma 2.1 for a weaker result).

2. Notation and auxiliary results

Let $(\mathbb{S}^n, g_\mathbb{S})$ denote the unit $n$-sphere equipped with the round metric. Given $p \in M$, we write $d_p$ for the geodesic distance function from $p$ and define a closed geodesic ball about $p$ by

$$B^n_R(p) := \{ q \in M : d_p(q) \leq \delta \}.$$  

Given $p \in \mathbb{S}^n$, recall that the punctured round unit sphere $\mathbb{S}^n \setminus \{-p\}$ is isometric to

$$\mathbb{S}^{n-1} \times [0, \pi), \quad g = dr \otimes dr + w(r)^2 g_\mathbb{S},$$  

where $r := d_p$ and $w(r) := \sin r$. Let $|B^k_R|$ be the area of any geodesic $k$-ball with radius $R$.

Throughout, we fix $R \in (0, \pi/2]$ and a geodesic ball $B^n_R(p)$, which we shall refer to in abbreviated fashion as $B_R$. Let $\Sigma^k \subset B_R$ be a minimal surface. Let $\nabla$ be the covariant derivative on $\mathbb{S}^n$ and $\nabla^\Sigma$, $\text{div}^\Sigma$, and $\Delta_\Sigma$ respectively be the covariant derivative, divergence, and Laplacian operators on $\Sigma$. It is convenient to define $\nabla^\Sigma r^\perp := \nabla r - \nabla^\Sigma r$; note that $|\nabla^\Sigma r^\perp|^2 = 1 - |\nabla^\Sigma r|^2$.

**Definition 2.2.** Define a function $I_k \in C^\infty ([0, \pi])$ by

$$I_k(r) = \int_0^r w^{k-1}(s) \, ds.$$  

*When the context is clear, we may omit the subscript $k$.***

**Remark 2.3.** Note that

$$|B^k_r| = \int_{B^k_r} dV = \int_0^r \int_{S^{k-1}} w^{k-1}(s) d\omega ds = \omega_{k-1} I_k(r),$$  

where $\omega_{k-1} := \int_{S^{k-1}} d\omega$ is the euclidean area of the unit $(k-1)$-sphere.

Theorem 1.1 follows from the following general argument which shifts the difficulty of the problem to the construction of a vector field with certain properties.

**Proposition 2.4.** Suppose for each $y \in \partial B_R$, there exists a vector field $W$ on $B_R \setminus \{y\}$ satisfying:

(i). $\text{as } d_y \searrow 0, W = -2I(R)d^{-k}_y \nabla d_y + o(d^{-k}_y).$

(ii). $W$ is tangent to $\partial B_R$ along $\partial B_R \setminus \{y\}$.

(iii). $\text{div}^\Sigma W \leq 1$ for any minimal surface $\Sigma^k \subset B_R$, with equality only if $\nabla^\Sigma d_y = \nabla d_y$ on $\Sigma$.

Then the conclusion of Theorem 1.1 holds.

**Proof.** Fix $y \in \partial \Sigma$ and $W$ as above. From the divergence theorem, the minimality of $\Sigma$, and (iii),

$$|\Sigma \setminus B_\varepsilon(y)| \geq \int_{\Sigma \setminus B_\varepsilon(y)} \text{div}^\Sigma W = \int_{\partial \Sigma \setminus B_\varepsilon(y)} \langle W, \eta \rangle + \int_{\Sigma \cap \partial B_\varepsilon(y)} \langle W, \eta \rangle.$$
By the free boundary condition, \( \eta = \nabla r \) on \( \partial \Sigma \); using (ii) and letting \( \varepsilon \searrow 0 \), we find
\[
|\Sigma| \geq \lim_{\varepsilon \searrow 0} \int_{\Sigma \cap \partial B_{\varepsilon}(y)} \langle W, \eta \rangle.
\]
On \( \Sigma \cap \partial B_{\varepsilon}(y) \), the free boundary condition implies \( \eta = -\nabla d_p + o(1) \); in combination with (i) this implies \( \langle W, \eta \rangle = 2I(R)\varepsilon^{1-k} + o(\varepsilon^{1-k}) \) on \( \Sigma \cap \partial B_{\varepsilon}(y) \). The free boundary condition also implies
\[
|\Sigma \cap \partial B_{\varepsilon}(y)| = \omega_{k-1}I(R)\varepsilon^{k-1} + o(\varepsilon^{k-1}).
\]
Taking \( \varepsilon \searrow 0 \), we conclude from the preceding that \( |\Sigma| \geq \omega_{k-1}I(R) = |B_{R_k}| \), where the last equality follows from Remark 2.3.

In the case of equality, (iii) implies that the integral curves in \( \Sigma \) of \( \nabla r \) are about \( \partial B \) of radius \( k \) about \( p \).

**Definition 2.5.** Define a function \( \varphi \) on \( [0, \pi) \) by \( \varphi(t) = I(t)w^{1-k}(t) \), where we define \( \varphi(0) = 0 \).

Given \( p \in \mathbb{S}^n \), define vector fields \( \Phi_p \) and \( \Psi_p \) on respectively \( \mathbb{S}^n \setminus \{p\} \) and \( \mathbb{S}^n \setminus \{p\} \) by
\[
\Phi_p = (\varphi \circ d_p) \nabla d_p, \quad \Psi_p = \Phi_p.
\]

**Lemma 2.6.** Given \( p \in \mathbb{S}^n \), the following hold.

(i) \( \text{div}_\Sigma \Phi_p \leq 1 \) and \( \text{div}_\Sigma \Psi_p \leq 1 \).

(ii) \( \Psi_p = (\psi \circ d_p) \nabla d_p \), where \( \psi(r) := \frac{I(r) - I(\pi)}{\sin^{k/(k-1)} r} \). If \( k = 2j \) is even, moreover
\[
\psi(r) = \sum_{i=1}^{j} \frac{-a_i \sin r}{(1 - \cos r)^2}, \quad a_1 := \frac{1}{2j - 1}, \quad a_{i+1} := \frac{2(j - i)}{2j - (i + 1)} a_i, \quad i = 1, \ldots, j - 1.
\]

**Proof.** In this proof, denote \( r = d_p \). Take coordinates for \( \mathbb{S}^n \setminus \{p\} \) as in (2.1). As in the proof of [6] Lemma 3], \( \nabla^2 r = (w'/w)(g - dr \otimes dr) \). It follows that \( \Delta_{\Sigma} r = (w'/w)(k - |\nabla_{\Sigma} r|^2) \). Then
\[
\text{div}_\Sigma \Phi_p = \varphi' \left| \nabla_{\Sigma} r \right|^2 + \varphi \Delta_{\Sigma} r
\]
\[
= \varphi' \left| \nabla_{\Sigma} r \right|^2 + \varphi \frac{w'}{w}(k - |\nabla_{\Sigma} r|^2)
\]
\[
= \varphi' + (k - 1)\varphi \frac{w'}{w} + \left( \varphi \frac{w'}{w} - \varphi' \right) |\nabla_{\Sigma} r|^2
\]
\[
= 1 + w^{-k} \left( kIw' - w' \right) |\nabla_{\Sigma} r|^2
\]
\[
= 1 + w^{-k} \left( k \int_0^r (Iw')' dt - w' \right) |\nabla_{\Sigma} r|^2
\]
\[
= 1 + kw^{-k} \left( \int_0^r Iw'' dt \right) |\nabla_{\Sigma} r|^2.
\]
(i) follows from this and Definition 2.5

For (ii), denote \( r = d_p \) and compute
\[
\Psi_p = -\varphi(\pi - r) \nabla r = \frac{-I(\pi - r)}{\sin^k(\pi - r)} \sin r \nabla r.
\]
The first equality follows by using that \( I(r) + I(\pi - r) = I(\pi) \). Next, note that the general solution to (2.7)
\[
(s/r) y' + k(\cos r) y = -1
\]
is \( y(r) = (-I_k(r) + C) \csc^k r \). By the change of variable \( t(r) = \cos r \), (2.1) is equivalent to
\[
(t^2 - 1) \frac{du}{dt} + ktu = -1.
\]
Assume now \( k = 2j \) is even. Define 
\[
    u(t) = \sum_{i=1}^{j} a_i (1 - t)^{-i}
\]
where the \( a_i \) are as in the statement of the lemma and compute
\[
    (t^2 - 1) \frac{du}{dt} + 2jt u = \sum_{i=1}^{j} \frac{i(t^2 - 1)a_i}{(1 - t)^{i+1}} + \sum_{i=1}^{j} \frac{2ja_i}{(1 - t)^i} \\
    = \sum_{i=1}^{j} \frac{-i(1 + t)a_i}{(1 - t)^i} + \sum_{i=1}^{j} \frac{2ja_i}{(1 - t)^i} - \sum_{i=1}^{j} \frac{2ja_i}{(1 - t)^{i-1}} \\
    = \sum_{i=1}^{j} \frac{-2ia_i}{(1 - t)^i} + \sum_{i=1}^{j} \frac{ia_i}{(1 - t)^{i-1}} + \frac{2ja_i}{(1 - t)^i} + \sum_{i=1}^{j} \frac{2j(a_i - a_{i+1})}{(1 - t)^i} - 2ja_1 \\
    = \sum_{i=1}^{j-1} \frac{-2ia_i}{(1 - t)^i} + \sum_{i=1}^{j-1} \frac{(i + 1)a_{i+1}}{(1 - t)^i} + a_1 + \sum_{i=1}^{j-1} \frac{2j(a_i - a_{i+1})}{(1 - t)^i} - 2ja_1 \\
    = -1,
\]
where the last step uses the definition of the coefficients \( a_i \). Since both \( u \) and \( I_k(r) \csc^k r \) are nonsingular at \( r = 0 \), the conclusion follows. \( \square \)

**Remark 2.8.** When \( j = 2, a_1 = a_2 = 1/3 \); when \( j = 3, a_1 = a_2 = 1/5 \) and \( a_3 = 2/15 \).

Now fix \( p \in S^n \) and \( x, y \in \partial B_R \), where \( B_R := B_R(p) \). Denote \( r := d_y(x) = d_x(y) \).

**Remark 2.9.** In the special case where \( B_R^n \) is a hemisphere, the area bound \( |\Sigma|^k \geq |B_R^k| \) is rather trivial – one simply defines \( W = \Psi_y \) and verifies that \( 2.3 \)(i)-(iii) hold: (i) follows from \( 2.5 \)(ii), (ii) follows from the definition of \( \Psi_y \) and using that \( \langle \nabla d_y, \nabla d_p \rangle \big|_{\partial B_R} = 0 \) since \( B_R^n \) is a hemisphere, and (iii) follows from \( 2.7 \)(i). Note that the preceding holds for all \( k \) and not just the values asserted in Theorem 1.1. The bound in this special case can also be anticipated from the following geometric heuristic. Because \( B_R \) is a hemisphere and \( \Sigma \) meets \( \partial B_R \) orthogonally, the union of \( \Sigma \) and its reflection across \( \partial B_R \) is a closed minimal submanifold \( \hat{\Sigma} \) in \( S^n \). Applying the euclidean monotonicity formula to the cone over \( \hat{\Sigma} \) implies \( \hat{\Sigma} \geq |S^k| \) and hence \( |\Sigma|^k \geq |B_R^k| \) after dividing by two.

By Remark 2.9, we may henceforth assume \( R \in (0, \pi/2) \). The more general definition of \( W \) (see 1.1 and 3.1) reduces to \( \Psi_y \) when \( R = \pi/2 \).

**Lemma 2.10.** \( \tan R \langle \Psi_y, \nabla d_p \rangle \big|_x = -\sum_{i=0}^{j-1} a_{i+1} (1 - \cos r)^{-i} \).

**Proof.** Combine 2.6(iii) and the spherical law of cosines at vertex \( x \) in the geodesic triangle \( p x y \), namely \( \sin R \sin r \langle \nabla d_y, \nabla d_p \rangle \big|_x = \cos R(1 - \cos r) \). \( \square \)

**Remark 2.11.** The vector field \( W \) used to prove the two-dimensional version of Theorem 1.1. Its definition of \( \Psi_y \) and \( \Psi_y \). In dimension \( k = 2j, j > 1 \), \( \langle \Psi_y, \nabla d_p \rangle \big|_x \) depends on \( r \) by Lemma 2.10 and consequently no such linear combination satisfies 2.3(iii). For this reason the definition of \( W \) here is more involved.

**Notation 2.12.** Let \( \gamma : [R, \pi] \to S^n \) be the minimizing geodesic from \( y \) to \( -p \) parametrized by arc length. When there is no risk of confusion, we often write \( d_x \) in place of \( d_x \circ \gamma \).

We now outline the proof of Theorem 1.1. Our general aim is to construct from \( \Psi_y \) a vector field \( W \) satisfying Proposition 2.4(i)-(iii). It turns out that the undesirable inner product property of Lemma 2.10 can be ameliorated by adding to \( \Psi_y \) a term of the form \( f_R^\pi h(s) \Psi_{\gamma(s)} ds \) so that the
resulting vector field \( Z \) has constant inner product with \( \nabla d_p \) along \( \partial B^n_{R} \setminus \{y\} \). In 4.1 we define \( W \) to be an appropriate linear combination of \( \Phi_p \) and \( Z \) so that 2.4.(ii) holds. 2.4.(i) then follows in a straightforward way after observing (Lemma 2.14 below) that \( \int_{R}^{\pi} h(s) \Psi_{\gamma(s)} \, ds \) is less singular than \( \Psi_y \) at \( y \). On the other hand, verification of 2.4.(iii) requires a detailed understanding of the function \( h(s) \) which occupies most of the rest of the paper. In particular, our argument requires us to prove that \( h \) is nonnegative so we can conclude from 2.6.(i) that \( \text{div}_{S} Z \leq 1 + \int_{R}^{\pi} h(s) \, ds \).

Lemma 2.13 (The spherical law of cosines). The following hold.

(i). \( \sin R \sin (d_{x} \circ \gamma) \langle \nabla d_{\gamma(s)}, \nabla d_{p} \rangle \big|_{x} = \cos s - \cos R \cos (d_{x} \circ \gamma) \).
(ii). \( \sin s \sin (d_{x} \circ \gamma) \langle \nabla d_{x}, \nabla d_{p} \rangle \big|_{\gamma(s)} = \cos R - \cos s \cos (d_{x} \circ \gamma) \).
(iii). \( \sin s \frac{d}{ds} (1 - \cos (d_{x} \circ \gamma)) = \cos R - \cos s + \cos s (1 - \cos (d_{x} \circ \gamma)) \).

Proof. (i)-(ii) are the spherical law of cosines, applied to vertices \( x \) and \( \gamma(s) \) of the geodesic triangle \( px\gamma(s) \). (iii) is just a reformulation of (ii). \( \square \)

Lemma 2.14. Suppose \( u \) is a bounded integrable function on \([R, \pi]\). Then

\[
\int_{R}^{\pi} u(s) \Psi_{\gamma(s)} \, ds = o \left( d_{y}^{1-k} \right) \quad \text{as} \quad d_{y} \searrow 0.
\]

Proof. In this proof, denote \( r = d_{x}(y) \). Using 2.6(ii), estimate

\[
\left| \int_{R}^{\pi} u(s) \Psi_{\gamma(s)} \, ds \right| \leq C \int_{R}^{\pi} \frac{1}{(d_{x} \circ \gamma)^{k-1}} \, ds.
\]

There exists a constant \( c > 0 \) such that along \( \gamma \), \( d_{x} \circ \gamma > c(r + s - R) \). Therefore,

\[
\left| r^{k-1} \int_{R}^{\pi} h(s) \Psi_{\gamma(s)} \, ds \right| \leq C \frac{1}{c^{k-1}} \int_{R}^{\pi} \left( \frac{r}{r + (s - R)} \right)^{k-1} \, ds.
\]

The result now follows from the dominated convergence theorem. \( \square \)

3. Constructing \( Z \)

The 4 dimensional case.

Definition 3.1. Define a vector field \( Z \) on \( B_{R} \setminus \{y\} \) by

\[
Z = \Psi_y + \int_{R}^{\pi} h(s) \Psi_{\gamma(s)} \, ds, \quad h(s) := \frac{\cos R}{\sin^2 R} \sin s.
\]

Lemma 3.2. \( -3 \tan R \langle Z, \nabla d_p \rangle |_{\partial B_{R}} = 1 + \int_{R}^{\pi} h(s) \, ds \).
Proof. Fix \( x \in \partial B_R \). Using (2.13) and (2.6(ii)),

\[-3 \sin R \int_{\partial B_R} \langle \Psi_{\gamma(s)}, \nabla d_p \rangle \sin s ds = -3 \sin R \int_{\partial B_R} \psi(d_x) \langle \nabla d_{\gamma(s)}, \nabla d_p \rangle \sin s ds\]

\[
= \int_{\partial B_R} \frac{1}{1 - \cos d_x} + \frac{1}{1 - \cos d_x} \) \sin s ds \\
= \int_{\partial B_R} \cos s - \cos R \left( \frac{\cos s}{1 - \cos d_x} + \cos R \right) \sin s ds \\
= \int_{\partial B_R} \sin^2 s \frac{d(1 - \cos d_x)}{1 - \cos d_x} + 2 \cos s \sin s \frac{1 - \cos d_x}{1 - \cos d_x} \sin R \sin s ds \\
= \int_{\partial B_R} \frac{d}{ds} \left( \frac{\sin^2 s}{1 - \cos d_x} \right) + \cos R \sin s ds \\
= -3 \sin R \frac{\sin^2 R}{1 - \cos R} + \cos R \int_{\partial B_R} \sin s ds,
\]

where the fourth equality uses the following rearrangement of (2.13(iii)):

\[\cos s - \cos R = -\sin s \frac{d}{ds} (1 - \cos d_x) + \cos s (1 - \cos d_x).\]

By Lemma (2.10)

\[-3 \tan R \langle \Psi_y, \nabla d_p \rangle \mid_x = \frac{1}{1 - \cos R} + 1.\]

Combining these calculations with Definition 3.1 finishes the proof. \( \square \)

The 6 dimensional case. Before defining \( Z \), we need to derive a system of first order linear equations which specifies \( h(s) \) when supplied with appropriate initial values.

**Lemma 3.3.** The equation

\[-15 \sin R \langle \Psi_{\gamma(s)}, \nabla d_p \rangle \mid_x h(s) - 3 \cos R h(s) = \frac{d}{ds} \sum_{i=1}^2 f_i(s) \sin^4 s \]

is equivalent to the conditions that \( h(s) = f_2(s) \sin^3 s \) and \( f := (f_1, f_2) \) solves the system

\[Af = f', \quad A := \frac{1}{\sin s} \left[ \begin{array}{cc} -3 \cos s & 3 \cos s \\ \cos R - \cos s & \cos s - \cos R \end{array} \right].\]

**Proof.** Using Lemma (2.6(ii) and (2.13)

\[-15 \sin R \langle \Psi_{\gamma(s)}, \nabla d_p \rangle \mid_x = -15 \sin R \psi(d_x) \langle \nabla d_{\gamma(s)}, \nabla d_p \rangle \mid_x\]

\[
= \sum_{i=1}^3 \frac{b_i}{(1 - \cos d_x)^i} (\cos s - \cos R + \cos R(1 - \cos d_x)) \\
= \sum_{i=1}^3 \frac{b_i (\cos s - \cos R)}{(1 - \cos d_x)^i} + \sum_{i=1}^3 \frac{b_i \cos R}{(1 - \cos d_x)^{i-1}} \\
= \frac{2(\cos s - \cos R)}{(1 - \cos d_x)^3} + \frac{3 \cos s - \cos R}{(1 - \cos d_x)^2} + \frac{3 \cos s}{1 - \cos d_x} + 3 \cos R,
\]

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The general even dimensional case. Assume $\Sigma$ has even dimension. By inspection, the vector of constant functions $(1, 2, \ldots)$ is equivalent to a first order system $A \mathbf{s} = \mathbf{f}'$ where $\mathbf{f} = (f_1, \ldots, f_{j-1})$. The system (3.4) follows from multiplying (3.5) by $h(s)$ and matching coefficients above.

**Definition 3.6.** Define a smooth vector field $Z$ on $B_R \setminus \{y\}$ by

$$Z = \Psi_y + \int_R^s h(s)\Psi_{\gamma(s)} ds,$$

where $h(s) := f_2(s)\sin^3 s$ and $\mathbf{f} := (f_1, f_2)$ is the solution of (3.4) satisfying $f(R)\sin^4(R) = \cos R(3, 2)$.

**Lemma 3.7.** $-5\tan R \langle Z, \nabla d_p \rangle|_{\partial B_R} = 1 + \int_R^\pi h(s) ds$.

**Proof.** By Lemma 2.10

$$-15\tan R \langle \Psi_y, \nabla d_p \rangle|_x = \frac{2}{(1 - \cos r)^2} + \frac{3}{1 - \cos r} + 3.$$n

Then Lemma 3.3 implies

$$-15\sin R \int_R^\pi h(s) \langle \Psi_{\gamma(s)}, \nabla d_p \rangle|_x ds - 3\cos R \int_R^\pi h(s) ds = \int_R^\pi \frac{d}{ds} \left( \sum_{j=1}^2 f_j(s)\sin^4 s \right) ds$$

$$-3\cos R \int_R^\pi h(s) ds = -\cos R \left( \frac{2}{(1 - \cos r)^2} + \frac{3}{1 - \cos r} \right),$$

where the second equality uses that $\lim_{s \to \pi} f_i(s)\sin^4 s = 0$, which follows either from standard ODE theory or the explicit formulae in Proposition A.1 and the last step follows from Definition 3.6. Combining these calculations proves the lemma.

**Lemma 3.8.** $h$ is nonnegative on $[R, \pi]$.

**Proof.** It suffices to prove that $f_2$ is increasing on $(R, \pi)$. From (3.4), $f_2$ satisfies the equation

$$\frac{d}{ds} f_2^2 = (\cos R - \cos s)(f_1 - f_2).$$

By inspection, the vector of constant functions $(1, 1)$ solves (3.4). By the initial condition in 3.6 $f_1(R) > f_2(R)$, so $\mathbf{f}$ is not a constant multiple of $(1, 1)$. Hence, by uniqueness of ODE solutions, $f_1 - f_2 > 0$ on $(R, \pi)$. Since $\cos R - \cos s > 0$ on $(R, \pi)$, it follows from (3.9) that $f_2' > 0$ on $(R, \pi)$. □

The general even dimensional case. Assume $\Sigma$ has even dimension $k = 2j$, where $j$ is at least 4.

**Lemma 3.10.** The equation

$$-\sin R \langle \Psi_{\gamma(s)}, \nabla d_p \rangle|_x h(s) - \frac{\cos R}{k - 1} h(s) = \frac{d}{ds} \sum_{i=1}^{j-1} \frac{a_{i+1} f_i(s)\sin^{k-2} s}{(1 - \cos d_x)^3},$$

is equivalent to a first order system $A \mathbf{f} = \mathbf{f}'$, where $\mathbf{f} = (f_1, \ldots, f_{j-1})$. 8
Proof. One one hand, using Lemma 2.6(ii) and 2.13,

\[-\sin R(\Psi_{\gamma(s)}, \nabla d_p) = -\sin R\psi(d_x)\langle \nabla d_{\gamma(s)}, \nabla d_p \rangle|_x\]

\[= \sum_{i=1}^{j} \frac{a_i}{(1 - \cos d_x)^i} (\cos s - \cos R + \cos R(1 - \cos d_x)) \]

\[= \sum_{i=1}^{j} \frac{a_i(\cos s - \cos R)}{(1 - \cos d_x)^i} + \sum_{i=1}^{j} \frac{a_i \cos R}{(1 - \cos d_x)^i-1} \]

\[= a_j(\cos s - \cos R) \frac{1}{(1 - \cos d_x)^j} + \sum_{i=1}^{j} \frac{a_i \cos s + (a_{i+1} - a_i) \cos R}{(1 - \cos d_x)^i} + a_1 \cos R, \]

where the final step uses that \(a_{i+1} - a_i = \frac{1-i}{2j-i+1} a_i = (1-i) c_i a_i\) from the recursion formula in Lemma 2.6(ii).

On the other hand, using 2.13(iii), compute

\[\frac{d}{ds} \sum_{i=1}^{j-1} \frac{a_{i+1} f_i \sin^{k-2} s}{(1 - \cos d_x)^i} = \sum_{i=1}^{j-1} \frac{a_{i+1}(f_i \sin^{k-2} s)'}{(1 - \cos d_x)^i} - \sum_{i=1}^{j-1} \frac{ia_{i+1} f_i \sin^{k-3} s(\cos R - \cos s + \cos s(1 - \cos d_x))}{(1 - \cos d_x)^{i+1}} \]

\[= \frac{(j-1)a_j f_{j-1} \sin^{k-3} s(\cos R - \cos R)}{(1 - \cos d_x)^j} + \sum_{i=1}^{j-1} \frac{a_{i+1}(f_i \sin^{k-2} s)'}{(1 - \cos d_x)^i} - \sum_{i=1}^{j-1} \frac{(i-1)a_i f_{i-1} \sin^{k-3} s(\cos R - \cos s)}{(1 - \cos d_x)^i} \]

\[= \frac{(j-1)a_j f_{j-1} \sin^{k-3} s(\cos R - \cos R)}{(1 - \cos d_x)^j} + \sum_{i=1}^{j-1} \frac{\sin^{k-3} s a_{i+1}(f'_i \sin s + (k-2-i)f_i \cos s) - a_{i}(i-1) f_{i-1} (\cos R - \cos s)}{(1 - \cos d_x)^i} \]

Matching coefficients on the terms over \((1 - \cos d_x)^j\) implies \(h(s) = (j-1)f_{j-1} \sin^{k-3} s\). Using this and matching coefficients in the other terms, we find for \(i = 1, \ldots, j-1\)

\(a_i (\cos s + (1-i)c_i \cos R) (j-1)f_{j-1} = a_{i+1} (f'_i \sin s + (k-2-i)f_i \cos s) - a_{i}(i-1) f_{i-1} (\cos R - \cos s)\).

Solving each such equation for \(f'_i\) establishes the system \(A f = f'\) and completes the proof. \(\square\)

**Definition 3.11.** Define a vector field \(Z\) on \(B^n_R \setminus \{y\}\) by

\[Z = \Psi_y + \int_0^\pi h(s)\Psi_{\gamma(s)} ds,\]

where \(h(s) = (j-1)f_{j-1}(s) \sin^{k-3} s\) and \(f\) is the solution of the system in 3.7.10 satisfying \(f(R) \sin^{k-2}(R) = \cos R(1, \ldots, 1)\).

4. Proof of Theorem 1.1

Assume now \(k = 2j\) and \(j > 1\), and let \(Z\) be defined as in 3.1 3.6 and 3.11.
Definition 4.1.

\[ W = \cos R \sin^{k-2}(R) \phi_p + \frac{2I(R)}{I(\pi)} Z. \]

The following calculus identity (recall the notation of Definition 2.2) will be useful:

\[ (4.2) \quad I_k(R) = -\frac{1}{k-2} \cos R \sin^{k-2}(R) + \frac{k-2}{k-1} I_{k-2}(R). \]

**Remark 4.3.** It will be useful to rewrite \( W \) using (4.2) as follows:

\[ W = \left( 1 - \frac{k-1}{k-2} \frac{I(R)}{I_{k-2}(R)} \right) \phi_p + \frac{2I(R)}{I(\pi)} Z. \]

**Example 4.4.** When \( k = 4 \) and \( k = 6 \), calculations using 2.2 show that \( W \) satisfies

\[ W = \cos R \cos^2(R/2) \phi_p + \frac{3}{2} I(R) Z, \quad (k = 4) \]

\[ W = \frac{3 \cos^4(R/2) \cos R}{2 + \cos R} \phi_p + \frac{15}{8} I(R) Z, \quad (k = 6). \]

**Lemma 4.5 (Constraint for \( h \)).** Let \( Z \) and \( h \) be as in Definition 3.1, 3.6, or 3.11.

(i). \( 1 + \int_R h(s) ds = \frac{k-1}{k-2} \frac{I(\pi)}{I_{k-2}(\pi)} = \frac{I_{k-2}(\pi/2)}{I_{k-2}(R)}. \)

(ii). \( \frac{2I(R)}{I(\pi)} \langle Z, \nabla d_p \rangle_{\partial B_R} = -\frac{\cot R}{k-2} \frac{I(\pi)}{I(R)} \langle Z, \nabla d_p \rangle_{\partial B_R}. \)

**Proof.** In this proof, denote \( C = 1 + \int_R h(s) ds. \) Let \( \Sigma \) be a geodesic \( k \)-ball about \( p \). As in the proof of 3.10 (see also 3.2 and 3.7),

\[ -\tan R \langle Z, \nabla d_p \rangle_{\partial B_R} = \frac{C}{k-1} \quad \text{and} \quad \text{div}_Z Z = C. \]

Using the divergence theorem, we have for small \( \varepsilon > 0 \)

\[ C |z \setminus B_\varepsilon(y)| = \int_{\Sigma \setminus B_\varepsilon(y)} \text{div}_Z Z = \int_{\Sigma \setminus B_\varepsilon(y)} \langle Z, \eta \rangle + \int_{\Sigma \cap B_\varepsilon(y)} \langle Z, \eta \rangle. \]

Letting \( \varepsilon \rightarrow 0 \), we find (recall Remark 2.3)

\[ \omega C I(R) = -\frac{\omega C}{k-1} \cos R \sin^{k-2}(R) + \lim_{\varepsilon \rightarrow 0} \int_{\Sigma \cap B_\varepsilon(y)} \langle Z, \eta \rangle. \]

Arguing as in the proof of Proposition 2.4 and using (4.2) to simplify, we find

\[ C \frac{k-2}{k-1} I_{k-2}(R) = \frac{I(\pi)}{2} \]

and the conclusion follows after simplifying and using (4.2). \( \square \)

**Lemma 4.6.** When \( k = 4 \) and \( k = 6 \), \( W \) satisfies Proposition 2.4 (i)-(iii).

**Proof.** It follows from 2.6 (ii) that as \( r := d_p \searrow 0 \),

\[ \Psi_r = -I(\pi) r^{1-k} \nabla r + o(r^{1-k}). \]

By Lemma 2.14 the integral term in \( Z \) contributes a singularity of order \( o(r^{1-k}) \) as \( r \searrow 0 \). (i) follows from combining these facts with the definition of \( W \). For (ii), compute using Definition 2.3 and Lemma 4.5 (ii)

\[ \langle W, \nabla d_p \rangle_{\partial B_R} = \frac{\cos R \sin^{k-2}(R)}{(k-2) I_{k-2}(R)} \langle \phi_p, \nabla d_p \rangle_{\partial B_R} + \frac{2I(R)}{I(\pi)} \langle Z, \nabla d_p \rangle_{\partial B_R} = 0. \]
For (iii), calculate using 2.6(i) and 4.5(ii) and Remark 4.3
\[
\div_{\Sigma} W = \left(1 - \frac{k - 1}{k - 2} \frac{I(R)}{I_{k-2}(R)}\right) \div_{\Sigma} \Phi_p + \frac{2I(R)}{I(\pi)} \div_{\Sigma} Z
\]
\[
\leq 1 - \frac{k - 1}{k - 2} \frac{I(R)}{I_{k-2}(R)} + \frac{k - 1}{k - 2} \frac{I(R)}{I_{k-2}(R)}
\]
\[
= 1,
\]
where before applying Lemma 4.5(ii) we have used that \(\div_{\Sigma} Z \leq 1 + \int_R^\pi h(s) \, ds\), which uses that \(h \geq 0\) (via Definition 3.11 when \(k = 4\) and Lemma 3.8 when \(k = 6\)) in conjunction with Lemma 2.6(ii). \(\square\)

REMARK 4.7. To prove the generalization of Theorem 1.1 in dimension \(k = 2j\), \(j \geq 4\) using the method above, it would suffice to prove that \(h\) (recall Definition 3.11) is nonnegative on \([R, \pi]\). When \(k = 8\), numerical calculations suggest that \(h\) is not strictly nonnegative for certain values of \(R\) and the method appears to break down.

REMARK 4.8. When \(k = 3\), calculations using Definition 2.6 and Lemma 2.6(ii) show that
\[
\psi(r) = \frac{r - \sin r \cos r - \pi}{2 \sin^2 r} \quad \text{and} \quad \tan R \langle \Psi_y, \nabla d_p \rangle|_x = \frac{r - \sin r \cos r - \pi}{2(1 + \cos r) \sin r}.
\]
These expressions should be contrasted with their even dimensional counterparts, respectively the formula in the second part of 2.6(ii) and the statement of Lemma 2.10. When \(k = 2j\) is even, \(h\) is defined as in 3.1, 3.6 and 3.11 so that the \(r\) dependent terms in \(\langle \Psi_y, \nabla d_p \rangle|_x\) are cancelled after adding \(\int_R^\pi h(s) \Psi_{\gamma(s)} ds, \nabla d_p\). It would be interesting to know to define \(h\) when \(k\) is odd to produce the analogous cancellation.

Appendix A.

The system (3.4) can be solved explicitly, and we sketch the details below for completeness.

**Proposition A.1.** A matrix of solutions for (3.4) is
\[
\phi(s) = \begin{bmatrix}
1 & (1 - \cos R \cos s + \cos^2 s) \csc^2 s \left(\cot \frac{s}{2}\right)^{\cos R} \\
1 & (\cos^2 s - \cos R \cos s - \frac{1}{3} \sin^2 R) \csc^2 s \left(\cot \frac{s}{2}\right)^{\cos R}
\end{bmatrix}.
\]
Moreover, \(h\) (recall Definition 3.6) satisfies
\[
h(s) = \mathcal{C} \begin{bmatrix}
(1 + 2 \csc^2 R) \sin^3 s + (\cos^2 s - \cos R \cos s - \frac{1}{3} \sin^2 R) \sin s \left(\frac{\cot(s/2)}{\cot(R/2)}\right)^{\cos R} \\
\end{bmatrix},
\]
where \(\mathcal{C} := (3 \cos R \csc^2 R)/(1 + 3 \sin^2 R)\).

**Proof.** Observe that \(f = (1, 1)\) solves (3.4), and let \(\phi = ((1, 1), (f_1, f_2))\) be a matrix of solutions, where \(f_1\) and \(f_2\) are to be determined. Liouville’s formula implies
\[
det(\phi(s)) = f_2 - f_1 = C \exp \left(\int \text{tr} A(s) \, ds\right) = C \csc^2 s \left(\cot \frac{s}{2}\right)^{\cos R}.
\]
Solving for \(f_2\) in (A.2) and substituting into the first item of (3.4) implies
\[
f_1' = 3C \cos s \csc^3 s \left(\cot \frac{s}{2}\right)^{\cos R},
\]
\[
\text{det}(\phi(s)) = f_2 - f_1 = C \exp \left(\int \text{tr} A(s) \, ds\right) = C \csc^2 s \left(\cot \frac{s}{2}\right)^{\cos R}.
\]
Solving for \(f_2\) in (A.2) and substituting into the first item of (3.4) implies
\[
f_1' = 3C \cos s \csc^3 s \left(\cot \frac{s}{2}\right)^{\cos R},
\]
\[
\text{det}(\phi(s)) = f_2 - f_1 = C \exp \left(\int \text{tr} A(s) \, ds\right) = C \csc^2 s \left(\cot \frac{s}{2}\right)^{\cos R}.
\]
Solving for \(f_2\) in (A.2) and substituting into the first item of (3.4) implies
\[
f_1' = 3C \cos s \csc^3 s \left(\cot \frac{s}{2}\right)^{\cos R},
\]
\[
\text{det}(\phi(s)) = f_2 - f_1 = C \exp \left(\int \text{tr} A(s) \, ds\right) = C \csc^2 s \left(\cot \frac{s}{2}\right)^{\cos R}.
\]
which after integrating gives a solution \( f_1 \) of the form

\[
f_1 = \frac{-3C}{3 + \sin^2 R} \csc^2 s \left( 1 - \cos R \cos s + \cos^2 s \right) \left( \cot \frac{s}{2} \right)^\cos R.
\]

Taking \( C = -1 - \frac{1}{3} \sin^2 R \) and substituting back into \((A.2)\) solves for \( f_2 \) and completes the proof of the formula for \( \Phi \). Next, define \( f = (f_1, f_2) \) by

\[
(A.3) \quad f = \mathcal{L} \Phi \cdot \left[ 1 + 2 \csc^2 R \right] \left[ \tan \frac{R}{2} \right]^\cos R,
\]

where \( \mathcal{L} \) is as in the statement of the proposition. A straightforward but omitted calculation shows that this solution of \((3.4)\) satisfies the initial values in Definition \(3.6\). \( \square \)

**Remark A.4.** The integral of \( h \) may be computed directly, using that

\[
\int \left( \cos^2 s - \cos R \cos s - \frac{1}{3} \sin^2 R \right) \sin s \left( \cot \frac{s}{2} \right)^\cos R ds = -\frac{1}{3} (\cos R - \cos s) \sin^2 s \left( \cot \frac{s}{2} \right)^\cos R.
\]

Finally, by taking \( R \searrow 0 \) in combination with a rescaling argument, we show below that the proof of Theorem \(1.1\) recovers the Euclidean area bounds in \([3, \text{Theorem 4}]\) in dimensions \( k = 2, 4 \) and \( 6 \).

**Definition A.5.** Given \( R \in (0, \pi) \), define \( R : T_p\mathbb{S}^n \to T_p\mathbb{S}^n \) by \( Rv = Rv \), a magnified metric \( \tilde{g} = \tilde{g}[R] \) on \( B_R \subset \mathbb{S}^n \) and a metric \( g_R \) on \( B_1(0) \subset T_p\mathbb{S}^n \) by

\[
\tilde{g} = R^{-2} g, \quad g_R := (\exp_p \circ R)^* \tilde{g}.
\]

Denote by \( \hat{\nabla} \) and \( \hat{d} \) the Levi-Civita connection and the distance function induced by the metric \( \tilde{g} \).

By Definition \( A.5 \), \( \exp_p \circ R : (B_1(0), g_R) \to (B_R(p), \tilde{g}) \) is an isometry which we use to identify the two spaces. Note that as \( R \searrow 0 \), \( g_R \) converges smoothly to the euclidean metric \( g|_p \). Using the identification above, we shall abuse notation by referring to \( y \) both as a point on \( \partial B_1(0) \) as well as a point on \( \partial B_R(p) \subset \mathbb{S}^n \).

**Proposition A.6** (Euclidean asymptotics). As \( R \searrow 0 \), the vector fields \( W \) in \([11, \text{Definition 2.11}]\) when \( k = 2 \) and in \([4, \text{Definition 2.7}]\) when \( k = 4, 6 \) converge smoothly to fields \( W_0 \) on the euclidean ball \( (B := B_1(0), g|_p) \) (using the notation above) satisfying

(i). \( W_0 = -\frac{2}{k} \hat{d}_y \nabla \hat{d}_y \) as \( \hat{d}_y \searrow 0 \).

(ii). \( W_0 \) is tangent to \( \partial B \) along \( \partial B \setminus \{y\} \).

(iii). \( \text{div}_2 W_0 \leq 1 \) for any minimal surface \( \Sigma^k \subset B \), with equality only if \( \nabla \Sigma d_p = \nabla d_p \) on \( \Sigma \).

With these conditions, an appropriately modified version of Proposition \( 2.4 \) implies the area bounds in the euclidean setting (see the proof of \([3, \text{Theorem 4}]\)).

In the proof we show slightly more: the limit \( W_0 \) is the field \( W \) defined in \([3]\), up to a factor \( 2/k \).

**Proof.** Let \( q \in B_p(R) \) and denote \( \hat{r} = \hat{d}_q \). Note that \( \hat{\nabla} \hat{r} \) is a unit vector with respect to the \( \tilde{g} \) metric. By straightforward expansions using the definitions (recall \(2.7\)) we have

\[
\Phi_q = \frac{\hat{r}}{k} \hat{\nabla} \hat{r} + O(R^2)
\]

\[
2 \frac{I(R)}{I(\pi)} \Psi_q = \frac{2}{k} \left( -\hat{r}^{1-k} + O(R) \right) \hat{\nabla} \hat{r}.
\]
Taking a limit as \( R \downarrow 0 \), on the limit euclidean ball \( (B_1(0), g|_p) \), \( \Phi_q \) and \( 2 \frac{I(R)}{I(\pi)} \Psi_q \) converge to

\[
\frac{x}{k} \quad \text{and} \quad -\frac{2}{k} \frac{x}{|x|^k},
\]

where here \( x \) is the position vector field on \( B_1(0) \).

We first discuss the \( k = 2 \) case. The field \( W \) from [11, Definition 2.11] used to prove the two-dimensional area bound is \( W = (\cos R) \Phi_p + (1 - \cos R) \Psi_y \). Noting that in this case \( 2 \frac{I(R)}{I(\pi)} \) is equal to \( 1 - \cos R \), it follows from (A.7) and (A.8) by taking \( R \downarrow 0 \) that on the limit euclidean ball \( (B_1(0), g|_p) \), \( W \) converges to (in the notation of [3])

\[
\frac{x}{2} \frac{x - y}{|x - y|^2},
\]

which is the vector field used by Brendle in dimension 2. (i)-(iii) can be checked either by passing to the limit from items (i)-(iii) of Proposition 2.4 or by direct calculation from the limit formula above.

In dimensions \( k = 4 \) and 6, the integral terms require some care. Straightforward but tedious calculations using the explicit formulas for \( h(s) \) in (3.1) and (A.1) when \( k = 4 \) and 6 show that as \( R \downarrow 0 \),

\[
2 \frac{I(R)}{I(\pi)} \int_0^\pi h(s) \Psi_\gamma(s) \, ds \quad \text{converges to}
\]

\[
\frac{k - 2}{k} \int_1^\infty u^{k-3} \frac{x - uy}{|x - uy|^k} \, du,
\]

(cf. also [14, Equation (1)]) and the discussion thereafter) which after the change of variable \( u(t) = 1/t \) is equal to

\[
\frac{k - 2}{k} \int_0^1 \frac{tx - y}{|tx - y|^k} \, dt.
\]

Using [3, (A.8)] and the preceding, it follows that the limit \( W_0 \) is

\[
W_0 = \frac{x}{k} \frac{2}{k} \frac{x - y}{|x - y|^k} - \frac{k - 2}{k} \int_0^1 \frac{tx - y}{|tx - y|^k} \, dt,
\]

which is up to a factor of \( k/2 \) the vector field defined in [3].

\[ \square \]

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