The Extension of the Physical and Stochastic Problems to Space-Time-Fractional Differential Equations

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Abstract. The fractional calculus gains wide applications nowadays in all fields. The implementation of the fractional differential operators on the partial differential equations make it more reality. The space-time-fractional differential equations mathematically model physical, biological, medical, etc., and their solutions explain the real life problems more than the classical partial differential equations. Some new published papers on this field made many treatments and approximations to the fractional differential operators making them loose their physical and mathematical meanings. In this paper, I answer the question: why do we need the fractional operators?. I give brief notes on some important fractional differential operators and their Grünewald-Letnikov schemes. I implement the Caputo time fractional operator and the Riesz-Feller operator on some physical and stochastic problems. I give some numerical results to some physical models to show the efficiency of the Grünewald-Letnikov scheme and its shifted formulae.

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1. introduction
The fractional calculus has gained so much attention at the last fifty years. Actually, hundreds of papers appear on fractional calculus and its applications every year. Before going on the important definitions of the fractional operators, I answer the question being why do we need the fractional differential operators?.

The random motion, the erratic motion, of a small particle immersed in a fluid is called the Brownian motion and is mathematically modelled by the diffusion equation

\[
\frac{\partial u(x,t)}{\partial t} = D \frac{\partial^2 u(x,t)}{\partial x^2}, \quad u(x,0) = \delta(x), \quad 0 < \alpha \leq 2.
\] (1.1)

Albert Einstein deduced the value of the diffusion constant \(D\) as

\[
D = \frac{RT}{6N_A \pi \eta r} = \frac{k_B T}{6\pi \eta r},
\]
where \( R \) is the gas constant, \( N_A \) is the Avogadro number, \( T \) is the temperature, \( \eta \) is the liquid’s velocity, and \( r \) is the radius of the Brownian particle, see [1] and [2].

One needs the space–fractional operators to mathematically model many physical, biological, chemical, medical, etc., phenomena that move through fractal media commonly exhibits large deviations from the Brownian motion and do not require finite velocity. The extension to Lévy stable motion is a straight forward generalization due to the common properties of Lévy stable motion and Brownian motion, but the Lévy flights differ from the regular Brownian motion by the occurrence of extremely long jumps whose length is distributed according to the Lévy long tail \( \sim |x|^{-1-\alpha}, \ 0 < \alpha < 2 \), see [3], [4], [5], [6] and [7]. In this paper, I give a brief review to the Brownian motion, Langevin equation, and the Lévy distribution. This extension requires the replacement of \( \frac{\partial^2 u(x,t)}{\partial x^2} \) to the symmetric Riesz-Feller operator, namely \( D_x^\alpha \). The space-fractional diffusion equation reads
\[
\frac{\partial u(x,t)}{\partial t} = D_x^\alpha u(x,t), \quad u(x,0) = \delta(x), \quad 0 < \alpha \leq 2,
\] (1.2)

The continuous time random walk (ctrw) associated with the space-fractional diffusion equation is described as Lévy flights, see in Fig[1] to find out the difference between the Brownian jump and the Lévy long jump. The transport of particles under the earth surface is a common phenomenon. For example, the transport of solute and contaminant particles in surface and subsurface water flows, the motion of soil particles and associated soil particles, and the transport of sediment particles and sediment-born substances in turbulent flow. These phenomena require a large number of data and are mathematically modelled by the Fokker-Planck equation
\[
\frac{\partial u(x,t)}{\partial t} = \frac{1}{2} \frac{\partial^2 a(x,t)u(x,t)}{\partial x^2} - \frac{\partial b(x,t)u(x,t)}{\partial x} = L_{fp}u(x,t), \quad u(x,0) = \delta(x).
\] (1.3)

This equation represents a stochastic process under the natural assumptions \( u(x,t) \to 0 \) and \( x^n u(x,t) \to 0 \) as \( |x| \to \infty \), besides \( \int_{-\infty}^{\infty} u(x,t)dx = 1 \). By supposing that the functions \( a(x,t) \) and \( b(x,t) \) are power functions of \( x \) only or constants, equation (1.3) could mathematically modelling many physical, biological, chemical, medical and etc. phenomena. Equation (1.2) and equation (1.3) represent a Markov process with exponential awaiting time. For more details about the theory of the Brownian motion, the passage limit to diffusion processes, and the Fokker Planck equations, see [8], [9], [10], [11], [12], [13], [14], [15], and [16].

Not only the diffusion equations are associated with the Fokker-Planck operator and its special cases. The movement of the potential and current in an electric transmission line (Cable equation), namely
\[
\frac{\partial^2 I(x,t)}{\partial t^2} + kc^2 \frac{\partial I(x,t)}{\partial t} = c^2 \frac{\partial^2 I(x,t)}{\partial x^2} + bc^2 I(x,t) = L_{fp}I(x,t),
\] (1.4)

with the resistance \( R \), inductance \( L \), capacitance \( c \) and leakage conductance \( G \), the resistance \( k \), and the function \( I(x,t) \) to represent the electric transmitted current. If \( I(x,t) \to 0 \), and \( x^n I(x,t) \to 0 \) as \( |x| \to \infty \), then it could be specified as stochastic processes. This wave equation with attenuation (damping term) represents also the movement of population of individuals as a famous biological problem. Equation(1.3) and equation(1.4) should be extended to space-fractional differential equations if the motion of the solutes or the propagation of the wave are in fractal medium. For more physical applications of this equation and its special forms, see [17], [18] and [19].

Second, one needs the time–fractional operators when the velocity variations are heavy tailed with power law. For examples, many physical, biological, medical, chemical processes exhibit
a power law with a non integer frequency of order \( t^{-\beta} \), where \( 0 < \beta < 1 \) or \( 1 < \beta < 2 \). Such processes have a memory. Only the Caputo-time fractional operator reflects this memory. The Caputo-time-fractional operator is defined as

\[
D^\beta_0 t f(t) = \begin{cases} 
\frac{1}{\Gamma(m-\beta)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\beta+1-m}} d\tau & \text{for } m-1 < \beta < m, \\
\frac{d^m}{dt^m} f(t) & \text{for } \beta = m, 
\end{cases}
\]

where

\[
K_\beta(t,\tau) = \frac{(t-\tau)^{\beta+1-m}}{\Gamma(m-\beta)},
\]
is its kernel and is called the memory function. This kernel enables reflects the memory effects of many physical, biological and etc. processes. The Caputo fractional derivative \( D^\beta_0 t \) can also be defined through its image in the Laplace transform domain, which is

\[
\mathcal{L}\{D^\beta_0 t f(t); s\} = s^\beta \mathcal{L}\{f(t); s\} - s^{\beta-1} f(0) - \hat{f}(0) s^{\beta-2} - \cdots - f^{(m-1)}(0) s^{\beta-m}, \quad s > 0.
\]

The Caputo fractional operator is dependent on the initial condition \( f(0) \), therefore it is suitable to be natural generalization of the first order time derivative.

This paper is organized as follows: section 1 is devoted to the introduction. Section 2 is to review some important fractional differential operators. The Grunwald-Letnikov scheme of the famous fractional differential operators and their algorithms are given in section 3. Section 4 is devoted to the simulation of the stochastic Ehrenfest model that is mathematically formulated by Fokker-Planck equation. The simulation of the multi-term wave equation (Cable equation) is given in section 5. Section 6 is to introduce the effects of the memory on the hereditary process in human-beings. The numerical results and the interpretations of the discussed models are given in section 7.

2. important fractional differential operators

A growing number of articles and books have appeared in the last 50 years (see for example: [3], [4], [21], [22], and [23]). For a very general theory, see [24]. There is no unique definition for the fractional integral operator or for the fractional differential operator. Many versions are applied to functions defined on a half-axis or on the whole real line. The fractional integral usually represents the convolution with a power function, while the fractional derivative is usually defined as the left-inverse to the fractional integration operator and directly depends on the fractional integral operators.

First we consider the Riemann-Liouville fractional integral operator, denoted by \( J^\beta \), and defined by

\[
J^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\beta}} d\tau, \quad t > 0, \quad \beta > 0.
\]

For completeness, set \( J^0 f(t) = f(t) \). This means \( J^0 \) is its identity operator. The Riemann-Liouville integral satisfies also the semi-group property

\[
J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t) = J^{\alpha+\beta} f(t), \quad \alpha, \beta > 0.
\]

This can directly be seen by applying the definition and interchanging the order of the integration. An interesting example is the integral of the power function \( t^\nu, \nu > -1 \)

\[
J^\beta t^\nu = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu+\beta)} t^{\nu+\beta}.
\]
From the definition (2.1), one can deduce that the operator \( J \) is a natural generalization of the integration of an integer order, which is according to the \((n\text{-fold})\) iterated integration can be shown to be
\[
J^n f(t) = \frac{1}{n!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau, t > 0.
\] (2.2)

From this equation, one can deduce that
\[
D^n J^n = I, \text{ but } J^n D^n \neq I,
\]
where \( n \in \mathbb{N} \) and \( I \) is an identity operator. Actually
\[
J^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k.
\]

Assuming now \( m - 1 < \beta \leq m \in \mathbb{N} \), and as a consequence of the last discussion, we expect to have \( D^\beta J^\beta g(t) = g(t) \). Formally one obtains
\[
g(t) = D^\beta f(t),
\]
as a solution of the Abelian integral equation
\[
J^\beta g(t) = f(t), t > 0, \beta > 0,
\]
if \( f(t) \) is a sufficiently smooth function. Setting
\[
J^m g(t) = J^{m-\beta} J^\beta g(t) = J^{m-\beta} f(t),
\]
letting \( \phi(t) = J^{m-\beta} f(t) \), and assuming all \( \phi^{(k)}(0) = 0 \), for \( k = 0, 1, \ldots, m - 1 \), then we can set
\[
D^m \phi(t) = D^m J^{m-\beta} f(t).
\]

Now by using this equation and the definition of the identity operator \( I \), we can set
\[
g(t) = D^m J^m g(t) = D^m J^{m-\beta} f(t).
\]

This means
\[
D^\beta f(t) = D^m J^{m-\beta} f(t).
\] (2.3)

Therefore, the Riemann-Liouville fractional derivative, being the most used fractional derivative operator, for a sufficiently smooth function \( f(t) \) in an interval \([0, \infty)\), is defined as (see for example:[26], [23], and [27])
\[
(D^\beta f)(t) := \begin{cases} 
\frac{1}{\Gamma(m-\beta)} \frac{d^m}{dt^m} \int_0^t \frac{f(\tau)}{(t-\tau)^{\beta+1-m}} d\tau, & m - 1 < \beta < m, \\
\frac{d^m}{dt^m} f(t), & m = \beta.
\end{cases}
\] (2.4)

For the important case \( 0 < \beta < 1 \), we have \( m = 1 \), and the widely used relation
\[
D^\beta f(t) = D^1 J^{1-\beta} f(t).
\]
From the definitions (2.1) and (2.4), one can define the Riemann-Liouville fractional derivative $D^\beta$ as the left inverse of $J^\beta$, $\beta \geq 0$. Therefore, some authors prefer to write the Riemann-Liouville fractional integral operator $J^\beta$ as $D^{-\beta}$ (e.g. [22], [26] and [4]). For completeness, we have $D^0 = I$. The Riemann-Liouville fractional derivative of the power function $t^\mu$, gives

$$D^\beta t^\mu = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + 1 - \beta)} t^{\mu - \beta}, \beta \geq 0.$$ 

This means the Riemann-Liouville fractional derivative of a non-zero constant is different from zero. In fact

$$D^\beta 1 = D^\beta t^0 = \frac{\Gamma(1)}{\Gamma(1 - \beta)} t^{-\beta} \text{ if } 0 < \beta < 1.$$ 

The alternative fractional derivative operator is the Caputo fractional derivative of order $\beta > 0$ (see [23]). It can be defined by interchanging the operators in the R. H. S. of equation (2.3). This gives

$$D^*_\beta f(t) = J^{m-\beta} D^m f(t), \quad m - 1 < \beta \leq m,$$

and more precisely

$$D^*_\beta f(t) = \begin{cases} \frac{1}{\Gamma(m-\beta)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\beta+1-m}} d\tau & \text{for } m - 1 < \beta < m, \\ \frac{d^m}{dt^m} f(t) & \text{for } \beta = m, \end{cases}$$

where

$$K_\beta(t-\tau) = \frac{(t-\tau)^{\beta+1-m}}{\Gamma(m-\beta)},$$

is its kernel and is called the memory function. This kernel enables reflects the memory effects of many physical, biological and etc. processes. The Caputo fractional derivative $D^*_\beta$ can also be defined through its image in the Laplace transform domain, which is

$$\mathcal{L}\{D^*_\beta f(t); s\} = s^\beta \tilde{f}(s) - s^{\beta-1} f(0) - \tilde{f}(0) s^{\beta-2} - \cdots - f^{(m-1)}(0) s^{\beta-m}, \quad s > 0.$$ 

The definition (2.5) is more restrictive than (2.4) because it requires the existence of $f^{(m)}(t)$.

The relation between the Riemann-Liouville fractional derivative and integral operators and the Caputo fractional derivative of order, in the special case $0 < \beta \leq 1$. Since we have

$$D^*_\beta f(t) = J^{1-\beta} D f(t) = D^*_\beta (f(t) - f(0^+)),$$ 

and

$$D^\beta (f(t) - f(0)) = D J^{1-\beta} (f(t) - f(0)) = D^\beta f(t) - \frac{f(0)}{\Gamma(1 - \beta)} t^{-\beta},$$

one can deduce that

$$D^*_\beta f(t) = D^\beta (f(t) - f(0)) \quad 0 < \beta \leq 1,$$

which in the Laplace domain reads

$$\mathcal{L}\{D^*_\beta f(t); s\} = s^\beta \tilde{f}(s) - s^{\beta-1} f(0) s > 0.$$
Equation (2.8) represents the relation between the Riemann-Liouville and the Caputo fractional derivative, and the dependence on the initial conditions. This equation is important for solving the fractional differential equations.

Recently Caputo-Fabrizio extended the Caputo time fractional operator to have an exponential kernel and called the new derivative the Caputo-Fabrizio derivative of order \( \beta \) that reads

\[
0D_t^{\text{CF}} f(t) = \begin{cases} 
\frac{(2-\beta)M(\beta)}{\Gamma(1-\beta)} \left\{ \frac{d}{dt} \int_0^t f(\tau) K_\beta(t-\tau) d\tau \right\} & \text{for } m - 1 < \beta < m , \\
\frac{d^m}{dt^m} f(t) & \text{for } \beta = m , 
\end{cases} 
\tag{2.9}
\]

where

\[ K_\beta(t - \tau) = \exp\left\{ -\frac{\beta}{1 - \beta} (t - \tau) \right\} . \]

This improved Caputo-Fabrizio fractional operator has been used successfully to prove the existence of solutions of some integro-differential equations, see [28] and [29]. Baleanu et al [30] theoretically studied the fractional differential equations of the Rubella disease model. Another modification to Caputo-fractional differential operator is done by Atanagana and Baleanu [31], namely

\[
0D_t^{\text{ABC}} f(t) = \begin{cases} 
\frac{B[\beta]}{\Gamma(1-\beta)} \left\{ \int_a^t f(\tau) K_\beta(t-\tau) d\tau \right\} & \text{for } m - 1 < \beta < m , \\
\frac{d^m}{dt^m} f(t) & \text{for } \beta = m , 
\end{cases} 
\tag{2.10}
\]

where

\[ K_\beta(t - \tau) = E_\beta\left( t - \tau \right) \beta \]

This operator has been used successfully to mathematically model some life problems. Veeresha et al [25] used this Atanagana-Baleanu fractional operator to give analytic solution to the Fisher-Kolmogorov. The \( E_\beta(t) \) is the Mittag-Leffler function of order \( \beta \). The general form of the infinite series of the Mittag-Leffler function of order \( \alpha \) and \( \beta \) is defined as

\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + k\alpha)} , \quad \alpha \in \mathbb{R} , \ \beta \in \mathbb{R} , \ z \in \mathbb{C} . \tag{2.11}
\]

The simulation of \( E_\beta(t) \) in the long run is simulated in figure[2]. The simulation shows that it asymptotically behaves as the fast convergent exponential function \( e^{-t} \) as \( t \gg 1 \). The figure shows also that \( E_\beta(t) \) converges faster to zero as \( \beta << 1 \), see [32] and the references therein for more details about the importance of the Mittag-Leffler function on the theory and applications of the fractional calculus. In figure[2], I plot the Mittag-Leffler function for different values of \( \beta \).

3. the Grünwald-Letnikov scheme

In this section, I discuss the concepts of the Grünwald-Letnikov scheme. Before beginning, one must begin with introducing the discretization of the independent variables \((x, t)\) by defining the the grids \((x_j, t_n)\), where \( x_j = jh, j = 0, 1, 2, ..., r, \) \( t_n = n\tau, n = 0, 1, 2, ..., T, \) \( r \) and \( T \) are integer numbers with \( h = \frac{1}{r} \). Then introduce the column vector

\[
y^{(n)} = \{ y_j^{(n)} \} = \{ y_0^{(n)}, y_1^{(n)}, \ldots, y_{r-1}^{(n)}, y_r^{(n)} \} , \tag{3.1}
\]

as an approximation to \( u(x_j, t_n) \), where \( u(x, t) \) is the solution of the discussed partial differential equations. Now, I begin by introducing the difference operators \( \Delta, \Delta^2, \Delta^3, \ldots \), being defined as

\[
\Delta y_j^{(n)} = y_{j+1}^{(n)} - y_j^{(n)} , \tag{3.2}
\]
\[ \Delta^2 y_j^{(n)} = \Delta(\Delta y_j^{(n)}) = \Delta(y_{j+1}^{(n)} - y_j^{(n)}) = y_{j+2}^{(n)} - 2y_{j+1}^{(n)} + y_j^{(n)}, \quad (3.3) \]
\[ \Delta^3 y_j^{(n)} = \Delta^2(y_{j+1}^{(n)} - y_j^{(n)}) = y_{j+3}^{(n)} - 3y_{j+2}^{(n)} + 3y_{j+1}^{(n)} - y_j^{(n)}. \quad (3.4) \]

More generally the \( n \)th difference operator is defined as
\[ \Delta^n y_j^{(n)} = \Delta^{n-1}(y_{j+1}^{(n)} - y_j^{(n)}) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} y_j^{n+1-k}. \quad (3.5) \]

In what follows, we consider the finite difference scheme for a function \( f(t) \) which is differentiable up to an integer order \( n \in \mathbb{N} \), or up to a fractional order \( \beta \in \mathbb{R}^+ \). The backward finite difference operator of an integer order \( n \in \mathbb{N} \), is defined as
\[ (\Delta^n f)(t) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t - k\tau). \quad (3.6) \]

If the function is differentiable up to an integer order \( n \in \mathbb{N} \), then we have
\[ f^{(n)}(t) = \lim_{\tau \to 0} \frac{(\Delta^n f)(t)}{\tau^n}. \quad (3.6) \]

By generalizing formula (3.6) to the fractional order \( \beta > 0 \) instead of \( n \) and taking the limit of the upper limit of the summation to \( \infty \), one gets the definition of the Grünwald-Letnikov fractional derivative of order \( \beta > 0 \). The fractional finite difference operator of a positive order \( \beta \) is defined for a smooth function \( f(t) \) as
\[ (\Delta^\beta f)(t) = \sum_{k=0}^{\infty} (-1)^k \binom{\beta}{k} f(t - k\tau), \quad (3.7) \]

and is defined on the whole line, see [26], as
\[ f^{(\beta)}(t) = \lim_{\tau \to 0^+} \frac{(\Delta^\beta f)(t)}{\tau^\beta}. \quad (3.8) \]

One can note for \( \beta > 0 \), that the series \( \sum_{k=0}^{\infty} |\binom{\beta}{k}| \) converges absolutely and uniformly. It is also clear that
\[ \sum_{k=0}^{n} \binom{n}{k} = 2^n. \]

\( f^{(\beta)}(t) \) is called the Grünwald-Letnikov fractional derivative of order \( \beta > 0 \) for a function \( f(t) \). It can also be defined on a half line \( t \geq 0 \) by the finite difference operator
\[ (\Delta^\beta f)(t) = \sum_{k=0}^{t-a} (-1)^k \binom{\beta}{k} f(t - k\tau), \quad (3.9) \]

where \( t > a \). Then we have
\[ f^{(\beta)}(t) = \lim_{\tau \to 0^+} \frac{1}{\tau^\beta} \sum_{k=0}^{t-a} (-1)^k \binom{\beta}{k} f(t - k\tau), \quad (3.10) \]
where $\tau = \frac{b-a}{n} \to 0$ (see [22] and [26]). This operator is modified by Gorenflo [33] in order to show that the error committed by approximating $(D^\beta f)(t)$ by $\tau^{-\beta}(\Delta^\beta f)(t)$ possesses an asymptotic expansion in integer powers of the step length $\tau$ (as $\tau \to 0$). For more information see [33].

We express now the Riesz-Feller operator $D^\alpha_0$ as the inverse of the suitable integral operator (Riesz potential) $I^\alpha_0$ whose symbol (Fourier transformation) is $(|\kappa|^{-\alpha})$. We write

$$D^\alpha_0 = -I^{-\alpha}_0, \quad 0 < \alpha \leq 2, \quad \alpha \neq 1,$$

(3.11)

where $I^{-\alpha}_0$ is the inverse of the symmetric Riesz potential (see Samko & Marichev [26])

$$I^\alpha_0 \Phi(x) = c_-(\alpha) I^\alpha_+ \Phi(x) + c_+(\alpha) I^\alpha_- \Phi(x),$$

(3.12)

where

$$c_-(\alpha) = c_+(\alpha) = 1/(2 \cos \frac{\alpha \pi}{2}) \quad \text{iff} \quad \alpha \neq 1.$$

(3.13)

Therefore, the symmetric Riesz potential can be written as

$$I^\alpha_0 \Phi(x) = \frac{1}{2 \cos \frac{\alpha \pi}{2}} (I^\alpha_+ \Phi(x) + I^\alpha_- \Phi(x)), \quad 0 < \alpha \leq 2, \quad \alpha \neq 1,$$

(3.14)

where $I^\alpha_{\pm}$ denotes the Riemann-Liouville fractional integral operators, by some people called Weyl integrals that are defined as

$$I^\alpha_+ \Phi(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x - \xi)^{\alpha-1} \Phi(\xi) d\xi,$$

$$I^\alpha_- \Phi(x) = \frac{1}{\Gamma(\alpha)} \int_{\infty}^{x} (\xi - x)^{\alpha-1} \Phi(\xi) d\xi.$$

(3.15)

The symmetric Riesz potential is then defined as

$$I^\alpha_0 \Phi(x) = \frac{1}{2 \Gamma(\alpha) \cos(\alpha \pi/2)} \int_{-\infty}^{\infty} |x - \xi|^{\alpha-1} \Phi(\xi) d\xi.$$

(3.16)

The Riesz potential operator is well defined if its index is located in the range $(0, 1)$ and we have the semi group property,

$$I^\alpha_0 I^\beta_0 = I^{\alpha+\beta}_0, \quad 0 < \alpha < 1, \quad 0 < \beta < 1.$$

Now, we can extend these definitions according to Samko [26] to introduce the inverse Riesz potential operator in the whole range $0 < \alpha \leq 2$ as

$$D^\alpha_0 = \frac{-1}{2 \cos(\alpha \pi/2)} [I^{\alpha-\alpha}_+ + I^{-\alpha}_-], \quad 0 < \alpha \leq 2, \quad \alpha \neq 1,$$

(3.17)

where $I^{-\alpha}_{\pm}$ are the inverse of the operators $I^\alpha_{\pm}$. This means the operators $I^{-\alpha}_{\pm}$ are obtained from the definitions of $I^\alpha_{\pm}$ equation (3.15) by change the sign of $\alpha$. Off course, appropriate assumptions are required for the functions to which these operators are applied. In the Fourier
domain, $I^0_0$ is represented by the symbol $-|\kappa|^{\alpha}$, and we see that $D^0_0$ is also meaning full as a peseudo-differential operator if $\alpha = 1$.

The discrete scheme of the Riesz fractional operator $D^\alpha_0$ by a suitable finite difference scheme is given for $0 < \alpha < 1$, $1 < \alpha < 2$, and $\alpha = 1$. To do so, one can approximate the inverse operators $I^\pm_\pm$ by the Gr"{u}nwald-Letnikov scheme (see Oldham & Spanier [3], Ross & Miller [4], and recently Gorenflo & Mainardi [34]).

\begin{equation}
I^\pm_\pm = \lim_{h \to 0} I^\pm_h ,
\end{equation}

where $I^\pm_h$ denotes the approximating Gr"{u}nwald-Letnikov scheme which reads

(a) $0 < \alpha < 1$
\begin{equation}
I^\pm_\pm \phi(x) = \frac{1}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \phi(x_j \mp kh) ,
\end{equation}

(b) $1 < \alpha \leq 2$
\begin{equation}
I^\pm_\pm \phi(x) = \frac{1}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \phi(x_j \mp (k-1)h) .
\end{equation}

So far, the discretization of the Riesz potential operator is as follows
\begin{equation}
D^\alpha_0 y_j(t_n) = \frac{-1}{2\cos \frac{\alpha \pi}{2}} \left( I^{-\alpha}_h + I^{-\alpha}_h \right) y_j(t_n) ,
\end{equation}

where $0 < \alpha \leq 2$ and $\alpha \neq 1$.

We must distinguish the discretization of $I^{-\alpha}_h$ with respect to the value of $\alpha$:
\begin{equation}
I^{-\alpha}_h y_j(t_n) = \frac{1}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} y_{j \mp k} ,
\end{equation}

$0 < \alpha \leq 1$,
\begin{equation}
I^{-\alpha}_h y_j(t_n) = \frac{1}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} y_{j \pm 1 \mp k} ,
\end{equation}

$1 < \alpha \leq 2$.

(c) $\alpha = 1$ This case is related to the Cauchy distribution and one cannot use the Gr"{u}nwald-Letnikov scheme for discretizing $D^1_0$ because the dominator $c_{\pm} \to 0$ at equation (3.13) is undefined for $\alpha = 1$. Instead of Gr"{u}nwald-Letnikov scheme, I use the discretization introduced in [36] and is successfully numerically applied by Abdel-Rehim [35]. In these references, Gorenflo and Mainardi deduced the discretization of $D^1_0$ from the Cauchy density $p_1(x, 0) = \frac{1}{\pi} \frac{1}{1+x^2}$.

They replaced the factor $(-1)^k \binom{\alpha}{k}, k \in \mathbb{Z}$, by $\frac{2}{k}$ as $k = 0$, and by $\frac{1}{\pi k (k+1)}$ as $k \in [-R, R], k \neq -1$.

The following known identity is very helpful in this case
\begin{equation}
\sum_{k=0}^{\infty} \frac{1}{k(k+1)} < \infty ,
\end{equation}

In the following sections, I give some applications and their simulations.
4. the space-time-fractional diffusion with central linear drift

In this section, I discuss a famous stochastic problem being mathematically modelled by partial differential equation. The modified Ehrenfest model is described by Vincze [15]. Vincze considered $N$ balls, numbered from 1 to $N$, $K$ of them in an urn $U_1$, $N-K$ in an urn $U_2$. In an urn $U_0$ there are $N+s$ slips of papers ($s \geq 0$) each of them having probability $(N+s)^{-1}$ of being randomly drawn. $N$ of the slips are numbered from 1 to $N$, the other $s$ slips are not numbered. We repeat indefinitely the following experiment.

Draw a slip from the urn $U_0$, and if it carries a number, move the ball which has the same number from the urn ($U_1$ or $U_2$) in which it is lying to the other urn ($U_2$ or $U_1$). If the slip is not numbered, one has to leave the ball in its urn. Then we put the slip back into the urn $U_0$. If the states are reconded as the number of balls in the urn $U_1$ after $n$-steps. The transition probabilities among the states are defined as

$$p_{k,j} = P(x_{n+1}^{(s)} = k | x_n^{(s)} = k) = \frac{k}{s+N}, \quad k = 0, 1, \ldots, N,$$

$$p_{k-1,j} = P(x_{n+1}^{(s)} = k-1 | x_n^{(s)} = k) = \frac{k}{s+N}, \quad k = 1, 2, \ldots, N,$$

$$p_{k+1,j} = P(x_{n+1}^{(s)} = k+1 | x_n^{(s)} = k) = \frac{N-k}{s+N}, \quad k = 0, 1, 2, \ldots, N-1,$$

with

$$p_{k,k} + p_{k,k-1} + p_{k,k+1} = 1 \quad \forall k = 0, 1, 2, \ldots, N, \quad p_{k,k+j} = 0 \quad \forall j \geq 1.$$

This model is mathematically modelled by the classical diffusion equation with central linear drift, namely

$$\frac{\partial u(x,t)}{\partial t} = a \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial}{\partial x} (F(x)u(x,t)) \quad (4.1),$$

where $a > 0$ is the diffusion constant and it is applied to the boundary conditions and the initial condition

$$u(\infty, t) = u(-\infty, t) = 0, \quad u(x, 0) = \delta(x) \quad (4.2).$$

As $s = 0$, is called the Ehrenfest model or the urn model and is mathematically modelled by the free diffusion equation. To have a stochastic process, the external force $F(x)$ should be replace by attractive force in the form $-bx^n$, where $b > 0$ is called the drift term. Equation (4.1) is a special form of the Fokker-Planck equation (1.3). On other words, the diffusion of molecules, smokes on air, pollution on water in which no force acts rather than the attractive force between molecules or any particles is modelled by equation(1.1). If equation(1.1) satisfies the boundary conditions (4.2), then it is called the free diffusion. The simulation of the free diffusion is simulated at Fig[3] for $\alpha = 2, \quad \beta = 1$. Here, I choose the external force is $F(x) = -bx$, to represent an attractive linear force, where $b \geq 0$ is the drift constant. This equation has a huge applications on stochastic processes, physics, biology, chemistry, fluid dynamics and many other fields. This equation can be modified to space-time-fractional diffusion with central linear drift to mathematically models the diffusion processes along all the $x$–axis and the memory effects on the diffusion processes, see [37] and [6].

$$D^\beta_s u(x,t) = a D^\alpha_0 u(x,t) + b \frac{\partial}{\partial x} (xu(x,t)) \quad (4.3)$$
where $D^\beta_*$ is the predefined Caputo derivative. Without lose of generality, let the constants $a = b = 1$, discretize this model by the predefined finite difference methods and the Grünwald-Letnikov scheme, and finally solve for $y_j^{n+1}$, to get

$$y_j^{(n+1)} = \sum_{k=0}^{n} (-1)^k \left( \frac{\beta_j}{k} \right) y_j^{(0)} + \sum_{k=1}^{n} (-1)^{k+1} \left( \frac{\beta_j}{k} \right) y_j^{(n+1-k)} + y_j^{(n)} \left[ \mu + \frac{\mu h^2}{2} (j+1) \right] - 2\mu y_j^{(n)} + y_j^{(n)} \left[ \mu - \frac{\mu h^2}{2} (j-1) \right], \quad (4.4)$$

where $\mu = \frac{\omega_{\beta\alpha}}{2\pi}$ is called the scaling relation. The scheme (4.4) is stable only if $\mu$ satisfies the condition

$$0 < \mu \leq \frac{\beta}{2}. \quad (4.5)$$

This model is simulated in Figure[4-6] for $\beta = 0.5$ and $\beta = 0.75$, respectively. The figures show that the diffusion is very slow because of the presence of the external attractive linear force. The difference scheme for equation (4.3) for $1 < \alpha < 2$ reads

$$y_j^{(n+1)} = \sum_{k=0}^{n} (-1)^k \left( \frac{\beta_j}{k} \right) y_j^{(0)} + \sum_{k=2}^{n} (-1)^{k+1} \left( \frac{\beta_j}{k} \right) y_j^{(n+1-k)} +$$

$$\left( \beta - \frac{a\mu}{\cos \frac{\alpha\pi}{2}} \right) y_j^{(n)} + a\mu \left( 1 + \frac{j+1}{R} + \frac{1}{2\cos \alpha \pi^2} \left( 1 \right) \right) y_j^{n-1} +$$

$$a\mu \left( 1 + \frac{j-1}{R} + \frac{1}{2\cos \alpha \pi^2} \left( 1 \right) \right) y_j^{n-1} + \frac{a\mu}{2\cos \alpha \pi^2} \sum_{k\geq 2} (-1)^k \left( \frac{\alpha}{k} \right) \left( y_j^{n-k} + y_j^{n+k} \right), \quad (4.6)$$

with the scaling relation $\mu = \frac{\alpha^\beta}{2\pi}$. This scheme is convergent if

$$0 < \mu < \frac{\beta \cos \frac{\alpha\pi}{2}}{a}. \quad (5.1)$$

This case is simulated at Fig[6]. The simulation shows that the fractional derivatives make the diffusion slower than the classical Brownian case.

5. the space-time-fractional multi-term wave equation

In this section, I discuss the approximate solution of equation(1.4). Equation (1.4) besides its physical implementations it could also mathematically model the propagation of the over diagnostic ultrasound waves through complex biological vascular networks such as the tumor tissue is studied. The formal general form of equation (1.4) is written as

$$\frac{\partial^2 u(x,t)}{\partial t^2} + k \frac{\partial u(x,t)}{\partial t} = \frac{1}{2} \frac{\partial^2 a(x,t)u(x,t)}{\partial x^2} - \frac{\partial b(x,t)u(x,t)}{\partial x} = L_{fp} u(x,t), \quad u(x,0) = \delta(x). \quad (5.1)$$

It could also be called the classical damped-forced wave equation. Here $u(x,t)$ represents the pressure amplitude, $a > 0$ is the general positive constant and $0 < k < 1$ is the attenuation coefficient. $b(x)$ represents the external force which supplies energy to the ultrasound wave. In addition to the boundary condition (4.2), one needs another initial condition to solve (5.1), namely $u(x,0) = g(t)$. This equation could be extended to be a space-time fractional multi term equation because evidence shows that the over diagnostic wave propagates through complex media with power law of non integer order $t^{-\nu}$, $1 < \nu < 2$. Evidence shows also that the
vascular morphology of the tumor is non smooth in a complex media. That means it is a fractal media. The wave propagates through this fractal media exhibits with extremely long jumps whose length is distributed according to the Lévy long tail $|x|^{-1-\alpha}$, $0 < \alpha < 2$. The space-time fractional multi term equation, with $a(x,t) = 1$ and $b(x,t) = -bx = -x$ as $b = 1$, reads

$$D_t^\beta u(x,t) + k \frac{\partial u(x,t)}{\partial t} = D_x^\alpha u(x,t) + \frac{\partial x u(x,t)}{\partial x}, \quad (5.2)$$

where $1 < \beta \leq 2$ and $0 < \alpha < 2$. Joining the Grünwald-Letnikov of the Caputo and Riesz operators together with the common finite difference rules and beginning by $1 < \alpha < 2$, $1 < \beta < 2$, see for more details [38], one gets after minor mathematical manipulating

$$y_j^{(n+1)} = \frac{b_n}{1+k\tau} y_j^{(0)} + \frac{1}{1+k\tau} \sum_{m=2}^{n} c_m y_j^{(n+1-m)} + \left(\frac{\beta + k\tau}{1+k\tau} + \frac{a\mu}{\cos \frac{\alpha \pi}{2}(1+k\tau)} \right) y_j^{(n)} - \frac{1}{1+k\tau} \left(\frac{b\tau j - 1}{2} + \frac{a\mu}{2 \cos \frac{\alpha \pi}{2}} \left(1 + \frac{\alpha}{2}\right) \right) y_{j-1}^{(n)} + \frac{1}{1+k\tau} \left(\frac{b\tau j + 1}{2} - \frac{a\mu}{2 \cos \frac{\alpha \pi}{2}} \left(1 + \frac{\alpha}{2}\right) \right) y_{j+1}^{(n)} - \frac{1}{1+k\tau} \frac{a\mu}{2 \cos \frac{\alpha \pi}{2}} \sum_{s=1}^n (-1)^s \left(\frac{\alpha}{s}\right) (y_{j+1-s}^{(n)} + y_{j-1+s}^{(n)}). \quad (5.3)$$

The scaling relation is defined as $\mu = \tau^\beta / h^\alpha$. This scheme is stable if and only if the coefficient of $y_j^{(n)}$ is positive, specifically

$$\frac{\beta + k\tau}{1+k\tau} + \frac{a\mu}{\cos \frac{\alpha \pi}{2}(1+k\tau)} \left(1 + \frac{\alpha}{2}\right) > 0.$$

Similarly, one can write the schemes of the cases $0 < \alpha < 1$ and $\alpha = 1$. The time evolution of the difference scheme (5.3) is simulated at Figures[7-9].

6. The hereditary process in human beings

Another application to the need of the time fractional differential equation is to study the hereditary of the human beings. The DNA is the history books of human beings. It carries the genetics that are responsible on all the physical characteristics of any individual. The characteristics of any human as for examples: his hair colour, his eyes colours, his skin colour are inherited from his parents or his grandparents or grand grandparents and so on. Is he tall or short or is he fat or thin?. The answer depends also partially on his family and about his way of habits. Some diseases are also inherited from the family, and some families have long history with specific diseases. Those families are advised to marry individuals from individuals being outside their own families. All these examples are mathematically modelled by the dependence on the memory and only the Caputo time fractional differential operator and his kernel reflects this memory. Abdel-Rehim and et al [39] recently published a simulation of the time-fractional genetic diffusion. The time fractional diffusion genetic equation with selection and mutation reads

$$D_t^\beta u(x,t) = \frac{1}{4N_e(t)} \frac{\partial^2}{\partial x^2} \{x(1-x)u(x,t)\} - s \frac{\partial}{\partial x} \{x(1-x)u(x,t)\}, \quad 0 < \beta \leq 1, \quad (6.1)$$
where $0 < x < 1$. Here $u(x, t)$ represents the conditional probability of finding a specific gene in the generation $t$ with frequency $x$, with $0 < x < 1$. $N_e(t)$ is the number of individuals and $s$ is the rate of selection, where $[N_e, s] \leq 1$. This equation satisfies that $u(x, t) = 0$ and $x^n u(x, t) \to 0$ as $x \to \infty$. Actually $x$ represents the frequency of finding the gene at the generation $t$ and $x \in (0, 1)$. Also $\int_0^1 u(x, t)dx = 1$ as conditional probability function. The mutation rate is 0.5. This equation according to [40] who studied only its classical case, i.e. the case with the first order time partial differential equation, represents the random mating with selection and mutation. Discretizing (6.1) and solve for $y_j^{(n+1)}$ to get

$$y_j^{(n+1)} = \sum_{k=0}^{n} (-1)^k \binom{\beta}{k} y_j^0 + \sum_{m=2}^{n} (-1)^{k+1} \binom{\beta}{k} y_j^{n+1-m} + \left( \frac{2\mu j h(1-jh)}{4N_e} \right) y_j^n$$

where $h = \frac{1}{R}$ and $0 < t \leq T$. On the next section, I give some simulation of this model with various values of $\beta$. The condition needed to preserve the non-negativity in equation (6.2) is given by the scaling relation

$$0 < \mu = \frac{\tau^\beta}{h^2} \leq 800\beta$$

I simulate the drift genetic diffusion for $\beta = 1$ and $\beta < 1$ at Figure[10], to show that the time-fractional genetic drift maintains the genes in families longer than the classical case. For more details about this important model, see [41] and the references therein. The authors [41] proves that using the time-fractional operator in this genetic diffusion equation and henceforth the dependence of the memory on the hereditary process makes makes the results more accurate and there is no loose of any gene within the same big family for long long time (many generations).

7. numerical results

In this section, I give the simulation of the discussed models.

Figure[1] is devoted to the simulation of the discrete random walk of Covid-19. Left shows the simulation of the Gaussian case which is typically Brownian motion. Right is to simulate the discrete random walk of the space fractional differential equation that mathematically models the spreading of Covid-19. The figure shows that the diffusion is typically Lévy flights.

Figure[2] represents the comparison between the Mittag-Leffler function behaviours corresponding to different values of $\beta$ and different values of $t$. The Mittag-Leffler converges faster than $e^{-at}$, with $c > 0$.

Figure [3] represents the diffusion equation (4.1) as $\alpha = 2$, $\beta = 1$. Figure[4] represents the diffusion equation (4.1) as $\alpha = 2$, $\beta = 0.5$. Figure[5] represents the diffusion equation (4.1) as $\alpha = 2$, $\beta = 0.75$. Figure[6] represents the diffusion equation (4.1) as $\alpha = 1.5$, $\beta = 0.75$. These figures shows that the diffusion of the classical case is much faster than the fractional cases and both diffuse very slowly diffuse comparing with other models.

Figures[7-9] represents the simulation of the time-fractional forced wave equation with fractional damping term for $\alpha = 2, \beta_1 = 0.6, \beta_2 = 1.8$ and for different values of $t$. The simulation shows how the wave evolves with the time. The wave grows up and moves from the boundaries and reaches its stationary form near the origin.

Figure[10] compares the classical approximate solution of the classical (Markov) genetic drift equation till the 50th generation with the approximate solution of the time–fractional (Non
Markov) genetic drift equation for $\beta = 0.75$. Here we used $N_e = 100$ and the selection rate $s = 0.02$ and mutation rate with 0.5 and the scaling relation $\mu = 250$ and finally $0 < x < 1$, see for more details [39]. The simulation shows that the classical Markov case diffuses faster than the fractional case.

Finally, I can conclude my results in the following words. I have given three models to cover the stochastic processes, the physical, and finally biological models. In all these models, the extension from the first and second order derivative with respect to $t$ and $x$, to fractional orders was natural. These time-fractional and space-time fractional equations represent real life problems and gives many answers to history questions.

Figure 1. left gauss diffusion equation with $\alpha = 2$ and right the space-fractional diffusion equation with $\alpha = 1.5$

Figure 2. left Mittag-Leffler for $t : 0 \to 1$, right $t : 0 \to 6$ with tstep=0.01
Figure 3. The simulation of equation (1.3) for $\alpha = 2$, $\beta = 1.0$, $a=1$ and $F(x) = -x$

Figure 4. The simulation of equation (4.1) for $\alpha = 2$ and $\beta = 0.5$

Figure 5. The simulation of equation (4.1) for $\alpha = 2$ and $\beta = 0.75$
Figure 6. The simulation of equation (4.1) for $\alpha = 1.5$ and $\beta = 0.75$

Figure 7. Time-fractional forced wave with fractional damping term for $\beta_1 = 0.6, \beta = 2.$, left $t = 1$ right $t = 34$

Figure 8. Time-fractional forced wave with fractional damping term for $\beta_1 = 0.6, \beta = 2.$, left $t = 46$, $t = 57$
Figure 9. Time-fractional forced wave with fractional damping term for $\beta_1 = 0.6, \beta = 2.$, left $t = 58$ and $t = 59$

Figure 10. left: The classical approximate solution of the classical (Markov) genetic drift equation till the 50th generation. right: The approximate solution of the time–fractional (Non Markov) genetic drift equation for $\beta = 0.75$. 
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