$r$-extension of Dunkl operator in one variable and Bessel functions of vector index

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Abstract

In this work we present an operator $D_\mu$ constructed with the help of the cyclic group set of the $r^{th}$ roots of unity. This operator constitute an $r$-extension of the Dunkl operator in one variable because when $r = 2$ it reduces to the classical one and admits as eigenfunctions the Bessel functions of vector index early deeply studied by Klyuchantsev. This paper is argued by specific examples and contains some interesting results which are the prelude of harmonic analysis related to this operator.

Keywords : Dunkl operator, Bessel functions, Fourier transform, transmutation operator.

2000 AMS Mathematics Subject Classification—Primary 33D15, 47A05.

1 Introduction

At the beginning of the last decade of the twentieth century C. Dunkl [3, 4] in a series of articles using reflection groups introduced a differential-difference operator now commonly called Dunkl operator and became a great center of interest and inspiration in many areas of pure and applied mathematics. This operator has generated a rich harmonic analysis developed by several authors, and involves a combination of Bessel functions of index $\alpha$ as eigenfunctions. So exploiting specific properties of these well-known special functions great analysis and an armada of applications was born.

Having knowledge of the progress of this topic in many scientific areas, we are always asked and highly intrigued by his extension in higher $r$-order which involves Bessel functions of index vector $j_\mu$ which are eigenfunctions of $\Delta_\mu, \quad \mu = (\alpha_1, ... , \alpha_{r-1})$ a differential operator of order $r$. These last functions are one of the generalized Bessel functions mentioned by Watson in his venerable book [13] and greatly studied with applications by many authors (see [1, 7, 8, 9, 10]) and the references therein . The cyclic group $C_r$ plays a central role in the definition of our $r$-extension particulary for define the $r$-even and the $r$-odd functions of order $l = 1, ... r - 1$ and leads to decompose the space of functions in direct sum of invariant subspaces with appropriate projectors $T_k, \quad k = 1, ... r - 1$ useful to construct $D_\mu$ the $r$-extension of the Dunkl operator having as fundamental property $D_\mu^r = \Delta_\mu$ on a particular subspace $F_k$. In addition to the construction process of the operator $D_\mu$, we give a particular interest in many cases especially

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in all the paragraphs discussed namely representation integral, associated Riemann-Liouville transform, transmutation and r-extension Dunkl transform. Recently, someone tells us that our operator $D_\mu$ can be included in the class of operators presented by Dunkl and Opdam [5]. We give at the end of this work our commentary and the link between the two buildings and we are grateful to our informant. Nerveless, in both cases no reliable harmonic analysis concerning these operators is made in addition in our case explicit eigenfunctions are obtained expressed via Bessel functions with index vector.

## 2 The operator $D_\mu$

Throughout this paper $r$ is an integer greater than 1, $\omega = e^{\frac{2\pi}{r}}$ and we put

$$C_r = \{1, \omega, \omega^2, ...\omega^{r-1}\}$$

the cyclic group of order $r$. Let $F$ be the space of complex valued functions on which we consider the following actions

$$s_k g(x) = \omega^k g(\omega x), \quad k = 0, 1, 2, ....$$

Putting $F_k$ the subspace of $F$ invariant by $s_k$; namely

$$g \in F_k \Leftrightarrow s_k g = g.$$  \hspace{1cm} (1)

Now we introduce the collection of the projector operators defined by the relations

$$T_k = \frac{1}{r} \sum_{n=0}^{r-1} s_k^n; \quad k = 0, 1, 2, ...$$  \hspace{1cm} (2)

which are slightly different from those introduced in [11].

We recall that in some mathematical literature [9, 10], we often say the function $T_0 g$ the $r$-even part of $g$ and the functions $T_k g, \quad r = 1, 2, ...r - 1$ the $r$-odd of order $k$ of $g$.

Taking account of the fact that $s_k^r = id$, one can easily show that the following properties hold:

1. The operators $T_k$ and $s_k, \quad k = 0, 1, 2...$ commute in the sense

$$T_k s_k = s_k T_k$$

2. The subspace $F_k$ can be also characterized as:

$$g \in F_k \Leftrightarrow T_k g = g.$$  \hspace{1cm} (1)

3. We have

$$k \neq l \Leftrightarrow T_k T_l = 0.$$  \hspace{1cm} (2)

Starting of the fact that $T_k$ are projectors namely $T_k^2 = T_k$ one can see easily that

$$F = F_1 \oplus ... \oplus F_{r-1}.$$  

For more clarity and taking account of their importance and their interference in the demonstrations we summarize here the useful properties:

\[ \]
1. The derivative operator \( \frac{d}{dx} \) maps the space \( F_k \) into \( F_{k+1} \).

2. The multiplication operator by \( \frac{1}{x} \) satisfies
\[
\frac{1}{x} s_k = s_{k+1} \frac{1}{x}
\]
and it maps the space \( F_k \) into \( F_{k+1} \). Moreover
\[
\frac{1}{x} T_k = T_{k+1} \frac{1}{x}
\]

3. For a real, let \( L_a \) be the operator:
\[
L_a(f) = x^{-a} \frac{d}{dx} (x^a f) = f' + \frac{a}{x} f. \tag{3}
\]
We have \( x^{-b} L_a x^b = L_{a+b} \) and \( L_a \) maps \( F_k \) into \( F_{k+1} \).

**Definition 1** Let \( \mu = (\alpha_0, \alpha_1, \ldots, \alpha_{r-1}) \) be a vector of \( \mathbb{R}^r \). We define the Bessel operator of order \( r \) associated to the index vector \( \mu \) by
\[
\Delta_\mu = L_{a_{r-1}} \circ \ldots \circ L_{a_0}
\]
where we have put
\[
a_k = r\alpha_k + k \quad k = 0, 1, \ldots, r - 1, \tag{4}
\]
and \( L_a \) is the operator given by (3).

**Definition 2** The \( r \)-extension of Dunkl operator is defined by
\[
D_\mu = \frac{d}{dx} + \frac{1}{x} \sum_{k=0}^{r-1} a_k T_k
\]
where the coefficients \( a_k \) and the operators \( T_k \) are given respectively by (4) and (2).

Before any thing let us justify the appellation by the following proposition

**Proposition 1** For \( f \in F_k, \quad k = 0, \ldots, (r - 1) \), we have
\[
D_\mu^r(f) = \Delta_\mu(f).
\]
**Proof.** This result is first consequence of the fact that \( D_\mu \) maps the \( F_k \) in \( F_{k+1} \) because the operators \( \frac{d}{dx} \) and \( T_k \) have the same properties. Since if \( g \in F_k \) then \( D_\mu g = L_{a_k} g \) we deduce then
\[
D_\mu g = L_{a_k} g \in F_{k+1},
\]
hence
\[
D_\mu^2 g = L_{a_k+1} L_{a_k} g \in F_{k+2}.
\]
By induction and the fact that \( F_{k+r} = F_k \) we find
\[
\text{if } g \in F_k \Rightarrow D_\mu g = \Delta_\mu g.
\]
Having defined the operators $\Delta_\mu$ and $D_\mu$, it quite natural to seek their eigenfunctions. For this we introduce the Bessel functions of vector index

$$\mu = (\alpha_0, \alpha_1, ..., \alpha_{r-1}) \in \mathbb{R}^r \setminus \mathbb{Z}_r$$

by

$$j_\mu(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(\alpha_0 + 1)_n(\alpha_1 + 1)_n... (\alpha_{r-1} + 1)_n} x^{nr},$$

where $(\beta)_n = \frac{\Gamma(\beta + 1)}{\Gamma(\beta)}$.

The knowledgeable reader should note that this function differs from that studied in [10] by the number of components of the index vector.

The above series is entire and taking account of

$$\Delta_\mu x^{rn} = r^r (\alpha_0 + n)(\alpha_1 + n)...(\alpha_{r-1} + n)x^{(n-1)r},$$

one can state:

**Proposition 2** For a complex $\lambda$ we have

$$\Delta_\mu j_\mu(\lambda x) = -\lambda^r j_\mu(\lambda x);$$

with $j_\mu(0) = 1$.

Now we put

$$\theta = e^{i\frac{\pi}{r}}$$

and we consider the $r$-extension function which call it also $r$-Dunkl kernel

$$E_\mu(x) = j_\mu(x) + \frac{1}{\theta} D_\mu j_\mu(x) + ... + \frac{1}{\theta^{r-2}} D_\mu^{r-2} j_\mu(x) + \frac{1}{\theta^{r-1}} D_\mu^{r-1} j_\mu(x).$$

(5)

**Proposition 3** For complex $\lambda$, the function $x \mapsto E_\mu(\lambda x)$ is an eigenfunction for the generalized Dunkl operator $D_\mu$ with $\theta \lambda$ as eigenvalue:

$$D_\mu E_\mu(\lambda x) = \theta \lambda E_\mu(\lambda x).$$

**Proof.** Indeed, to be convinced it suffices to make the following computations:

$$D_\mu E_\mu = D_\mu j_\mu + \frac{1}{\theta} D_\mu^2 j_\mu + ... + \frac{1}{\theta^{r-2}} D_\mu^{r-2} j_\mu + \frac{1}{\theta^{r-1}} D_\mu^{r-1} j_\mu$$

$$= D_\mu j_\mu + \frac{1}{\theta} D_\mu^2 j_\mu + ... + \frac{1}{\theta^{r-2}} D_\mu^{r-2} j_\mu + \frac{\theta^r}{\theta^{r-1}} j_\mu$$

$$= \theta \left[ j_\mu + \frac{1}{\theta} D_\mu j_\mu + ... + \frac{1}{\theta^{r-1}} D_\mu^{r-1} j_\mu \right]$$

$$= \theta E_\mu.$$
so the result follows.

In the reminder we must compute the action of the $r$-extension of Dunkl operator $D_\mu$ on the Bessel functions of vector index $j_\mu$.

This is can be deduced from the fact that $D_\mu x^{nr} = L_\alpha x^{nr} = r(\alpha_0 + n)x^{nr-1}$, we obtain

$$D_\mu j_\mu(x) = r \sum_{n=0}^{\infty} (-1)^n \frac{1}{(\alpha_0 + 1)_n \cdots (\alpha_{r-1} + 1)_n} (\alpha_0 + n)x^{nr-1}.$$ 

Then we distinguish two cases:

**Case 1 :** $\alpha_0 \neq 0$.

We have

$$D_\mu j_\mu(x) = \frac{r}{x} \alpha_0 j_{\mu - 1},$$

remark that we have adopt the convention : for $\mu = (\alpha_0, \alpha_1, \ldots, \alpha_{r-1})$ we put

$$\mu - 1 = (\alpha_0 - 1, \alpha_1, \ldots, \alpha_{r-1}).$$

**Case 2 :** $\alpha_0 = 0$.

With a slice change of computation we have

$$D_\mu j_\mu(x) = - \frac{1}{(\alpha_1 + 1) \cdots (\alpha_{r-1} + 1)} \left( \frac{x}{r} \right)^{r-1} j_{\mu + 1}.$$ 

We have also adopt the convention : for $\mu = (\alpha_0, \alpha_1, \ldots, \alpha_{r-1})$ we put

$$\mu + 1 = (\alpha_0, \alpha_1 + 1, \ldots, \alpha_{r-1} + 1).$$

As mentioned in abstract we present here three explicit examples which illustrate the operators $\Delta_\mu$ and $D_\mu$.

**Example 1 :** $r = 2, \omega = -1, \theta = i, \mu = (0, \alpha)$.

Then

$$a_0 = 0, a_1 = 2\alpha + 1.$$ 

So that

$$\Delta_\mu = L_{2\alpha + 1} L_0 = \frac{d^2}{dx^2} + \frac{2\alpha + 1}{x} \frac{d}{dx},$$

which is exactly the well known Bessel operator having as eigenfunction the normalized Bessel function

$$j_\alpha(x) = j_\mu(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(\alpha + 1)_n} \frac{x^{2n}}{2^n}.$$
Now since
\[ T_1g(x) = \frac{g(x) + s_1g(x)}{2} = \frac{g(x) - g(-x)}{2} \]
The operator \( D_\mu \) is the classical Dunkl operator in one variable:
\[ D_\mu = D_\alpha = \frac{d}{dx} + \frac{\alpha + 1}{x}T_1, \]
and using the conventional notation introduced before namely
\[ \mu + 1 = (0, \alpha + 1) \]
and the fact that \( \theta = i \), lead to show that the eigenfunctions \( E_\mu \) coincide with the classical one.

**Example 2** : \( r = 3, \omega = e^{i\pi/3}, \theta = e^{i\pi/3}, \mu = (0, \alpha - \frac{1}{3}, -\frac{2}{3}) \).

Taking account of the relation (41) between \( a_k \) and \( \alpha_k \) we deduce that
\[ a_0 = 0, \ a_1 = 3\alpha, \ a_2 = 0. \]
So
\[ \Delta_\mu = L_0L_{3\alpha}L_0 = \frac{d^3}{dx^3} - 3\alpha \frac{d^2}{x \cdot dx^2} + \frac{3\alpha d}{x^2 \cdot dx}. \]
The previous operator was greatly studied in [9]. Its eigenfunction is given by
\[ j_\mu(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n! (\alpha + \frac{2}{3}) n (\frac{1}{3}) n 3^{3n}} x^{3n}. \]
The correspondent Dunkl operator is:
\[ D_\mu = \frac{d}{dx} + \frac{3\alpha}{x}T_1 \]
with
\[ T_1g(x) = \frac{g(x) + \omega g(\omega x) + \omega^2 g(\omega^2 x)}{3}, \]
we can deduce
\[ D_\mu j_\mu(x) = \frac{d}{dx} j_\mu(x) = -\frac{1}{(\alpha + \frac{2}{3})(\frac{1}{3})} \left( \frac{x}{3} \right)^2 j_{\mu+1}(x), \]
with \( \mu + 1 = (0, \alpha + \frac{2}{3}, \frac{1}{3}) \) and then
\[ j_{\mu+1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(\alpha + \frac{2}{3}) n (\frac{1}{3}) n 3^{3n}} x^{3n}. \]
From the fact that \( D_\mu j_\mu \in F_1 \), we obtain
\[ D_\mu^2 j_\mu(x) = \frac{d}{dx} D_\mu j_\mu(x) + \frac{3\alpha}{x} T_1 D_\mu j_\mu(x). \]
Direct computations give
\[ D_\mu^2 j_\mu(x) = \frac{x^4}{4(3\alpha + 2)(3\alpha + 5)} j_{\mu+2}(x) - x j_{\mu+1}; \]
with \( \mu + 2 = (0, \mu + \frac{5}{3}, \frac{1}{3}) \) and

\[
j_{\mu+2}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n! (\alpha + \frac{2}{3})^n (\frac{2}{3})^n} x^{3n}.
\]

Finally taking account of the above results we state that the eigenfunction of the correspondent Dunkl operator is then

\[
E_\mu(x) = j_\mu(x) + e^{-\frac{ix}{3}} D_\mu j_\mu(x) + e^{-\frac{2ix}{3}} D^2_\mu j_\mu(x)
\]

\[
E_\mu(x) = j_\mu(x) - \left[ \frac{e^{-\frac{ix}{3}}}{3^2 (\alpha + \frac{2}{3})^2 (\frac{1}{3})} \right] j_{\mu+1}(x) + \frac{e^{-\frac{2ix}{3}}}{4(3\alpha + 2)(3\alpha + 5)} x^4 j_{\mu+2}(x).
\]

**Remark 1 :** It is easy to see that the following commutation holds:

\[ L_\alpha T_k = T_{k+1} L_\alpha \]

and as \( T_{k+r} = T_k \) this leads that the operators \( \Delta_\mu \) and \( T_k \) commute in the sense

\[ \Delta_\mu T_k = T_k \Delta_\mu. \]

Note that if \( g \) is a function such that \( \Delta_\mu g = -g \) and from the unique decomposition :

\[ g = T_0 g + \ldots + T_{r-1} g, \]

one can interpret the component \( T_k g \) as the unique solution of the previous equation restraint to the subspace \( F_k \).

**Example 3 :** \( \mu = (0, -\frac{1}{r}, \ldots, -\frac{r-1}{r}) \), \( \theta = e^{\frac{ir\pi}{2}} \).

In this situation as \( \alpha_k = -\frac{k}{r} \) we have \( a_k = 0 \), hence \( \Delta_\mu = (\frac{d}{dx})^r \) and \( D_\mu = \frac{d}{dx}. \) It is clear that the function \( e^{\theta x} = e^{\theta x} \) satisfies the equation

\[ \Delta_\mu e^{\theta x} = -e^{\theta x} \]

The components \( T_k e^{\theta x}, k = 0, 1, \ldots, r-1 \) are called the r-trigonometric functions [9]. We have in particular

\[
\cos_r(x) = T_0 e^{\theta x} = 1 + \sum_{k=0}^{r-1} e^{\theta \omega^k x} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(1)(\frac{1}{r} + 1)n\ldots(-\frac{1}{r} + 1)n} x^{nr}
\]

This last function is the unique eigenfunction of \( \Delta_\mu = (\frac{d}{dx})^r \) with take the value 1 at \( x = 0 \). We notice that the Dunkl kernel can be written

\[
E_\mu(x) = \cos_r(x) + \frac{1}{\theta} D_\mu \cos_r(x) + \ldots + \frac{1}{\theta^{r-1}} D^{r-1}_\mu \cos_r(x)
\]

\[ = (T_0 + \ldots + T_{r-1}) e^{\theta x} = e^{\theta x}. \]
3 Integral representations

In this section we attempt to show that the functions $j_\mu$ and $E_\mu$ have some useful integral representations. For the first function the reader can found tis integral representation already shown by klyuchantsev [10] and which is recalled in the proof of Theorem 1.

The following lemma is basic and it is a consequence of properties of Euler functions.

**Lemma 1** We have

$$r \int_0^1 (1 - u^r)^{y-1} u^{rx-1} du = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)}$$

provided the integral converges.

We deduce then the following identities

$$r \int_0^1 u^{nr}(1 - u^r)^{\alpha_i + \frac{i}{r} - 1} u^{r-(i+1)} du = \frac{\Gamma(\alpha_i + \frac{i}{r} + 1)}{\Gamma(\alpha_i + 1 + n)}.$$

So we have

$$r \int_0^1 \frac{\Gamma(\alpha_i + 1)}{\Gamma(\alpha_i + \frac{i}{r} + 1)} \Gamma(-\frac{i}{r} + 1) \int_0^1 u^{nr}(1 - u^r)^{\alpha_i + \frac{i}{r} - 1} u^{r-(i+1)} du = \frac{1}{(\alpha_i + 1)_n}.$$

and in general case

$$\int_0^1 \ldots \int_0^1 (xu_0 \ldots u_{r-1})^{nr} w_0(u_0) \ldots w_{r-1}(u_{r-1}) du_0 \ldots du_{r-1}$$

$$= (-1)^n \frac{1}{(\alpha_0 + 1)_n \ldots (\alpha_{r-1} + 1)_n} x^{nr}.$$

Let

$$w_\mu(u) = \prod_{i=0}^{r-1} (1 - u_i^r)^{\alpha_i + \frac{i}{r} - 1} u_i^{-r(i+1)}$$

(6)

and

$$c_\mu = \prod_{i=0}^{r-1} r \frac{\Gamma(\alpha_i + 1)}{\Gamma(\alpha_i + \frac{i}{r} + 1)}$$

(7)

and using the function $\cos_r$ presented in Example 3, we have
**Theorem 1** The Bessel function of vector index possess the following integral representation

\[ j_\mu(x) = c_\mu \int_{[0,1]^r} \cos r(xu_r) w_\mu(u) du \]

where \( c_\mu \) and \( w_\mu \) are given respectively by (7) and (6).

Note that in this representation we can remove the components \( u_i \) associated with indices \( i \) such that \( a_i = 0 \). In this case the Mehler representation takes the following form

\[ j_\mu(x) = c'_\mu \int_{[0,1]^r} \cos r(xu_r) w'_\mu(u) du \]

with \( r' \) is the number of index \( i \) such that \( a_i \neq 0 \) and \( \mu' \) contains only the associate \( \alpha_i \).

**Theorem 2** the \( r \)-Dunkl kernel possess the following integral representation

\[ E_\mu(x) = c_\mu \int_{[0,1]^r} \left( T_0 + \sum_{k=1}^{r-1} \frac{1}{\theta_k} T_k L_{a_k-1} \ldots L_{a_0}\right) e_\theta(xu_r) w_\mu(u) du. \]

**Proof.** This is a consequence of definition of \( E_\mu(x) \) and the identity

\[
\frac{1}{\theta_k} D^k_\mu j_\mu(x) = c_\mu \int_{[0,1]^r} \frac{1}{\theta_k} D^k_\mu T_0 e_\theta(xu_r) w_\mu(u) du
\]

\[
= c_\mu \int_{[0,1]^r} \frac{1}{\theta_k} L_{a_k-1} \ldots L_{a_0} T_0 e_\theta(xu_r) w_\mu(u) du
\]

\[
= c_\mu \int_{[0,1]^r} \frac{1}{\theta_k} T_k L_{a_k-1} \ldots L_{a_0} e_\theta(xu_r) w_\mu(u) du,
\]

which prove the result. \( \blacksquare \)

**Example 4:** \( r = 2, w = -1, \theta = i, \mu = (0, \alpha) \)

Since \( a_0 = 0 \) then we can remove the index \( i = 0 \) in the representation of the function \( j_\mu \) which gives

\[ j_\mu(x) = j_\alpha(x) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + \frac{1}{2})} \int_0^1 \cos (xu) (1 - u^2)^{\alpha - \frac{1}{2}} du. \]

On the other hand

\[ E_\mu(x) = E_\alpha(x) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + \frac{1}{2})} \int_0^1 \left[ T_0 e^{ixu} + \frac{1}{i} T_1 \frac{d}{dx} e^{ixu} \right] (1 - u^2)^{\alpha - \frac{1}{2}} du \]

\[ = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + \frac{1}{2})} \int_0^1 \left[ T_0 e^{ixu} + u T_1 e^{ixu} \right] (1 - u^2)^{\alpha - \frac{1}{2}} du \]

\[ = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + \frac{1}{2})} \int_0^1 \left[ T_0 e^{ixu} + T_1 \frac{1}{x} (xu) e^{ixu} \right] (1 - u^2)^{\alpha - \frac{1}{2}} du \]
To find the classical form we can write

\[
E_\alpha(x) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + \frac{1}{2})} \int_0^1 \left[ \frac{e^{ixu} + e^{-ixu}}{2} + u \frac{e^{ixu} - e^{-ixu}}{2} \right] (1 - u^2)^{\alpha - \frac{1}{2}} du
\]

\[
= \frac{1}{\sqrt{\pi}} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + \frac{1}{2})} \int_{-1}^1 e^{ixu} (1 + u)(1 - u^2)^{\alpha - \frac{1}{2}} du.
\]

**Example 5**: \( r = 3, w = e^{i\frac{2\pi}{3}} = j, \theta = e^{i\frac{\pi}{3}}, \mu = (0, v - \frac{1}{3}, -\frac{2}{3}) \).

We have

\[ j_\mu(x) = j_\nu(x) = 3 \frac{\Gamma(\nu + \frac{2}{3})}{\Gamma(\nu) \Gamma(\frac{2}{3})} \int_0^1 \cos (xu) (1 - u^3)^{\nu - 1} du. \]

Then

\[ E_\nu(x) = 3 \frac{\Gamma(\nu + \frac{2}{3})}{\Gamma(\nu) \Gamma(\frac{2}{3})} \int_0^1 \left[ T_0 e^{\theta xu} + \frac{1}{\theta} T_1 \frac{d}{dx} e^{\theta xu} + \frac{1}{\theta^2} T_2 \left( \frac{d}{dx} + \frac{3v}{x} \right) \frac{d}{dx} e^{\theta xu} \right] (1 - u^3)^{\nu - 1} du \]

which can be written in the following form

\[ E_\nu(x) = 3 \frac{\Gamma(\nu + \frac{2}{3})}{\Gamma(\nu) \Gamma(\frac{2}{3})} \times \]

\[ \int_0^1 \left[ T_0 \frac{1}{x} e^{\theta xu} + T_1 \frac{1}{x^2} (xu)^2 e^{\theta xu} + T_2 \frac{1}{x^3} (xu)^3 e^{\theta xu} + \frac{3v}{\theta} T_2 \frac{1}{x^3} (xu)^2 e^{\theta xu} \right] (1 - u^3)^{\nu - 1} du. \]

### 4 Riemann–Liouville transform

Considering the \( r \)-Riemann Liouville operators of the form

\[ R_\alpha g(x) = \int_0^1 g(xt)(1 - t^\alpha)^{\alpha - 1} dt. \]

The integral representation in Theorem I of the Bessel function of vector index can be rewritten as follows

\[ j_\mu(x) = c_\mu \int_{[0,1]^r} \cos_r (xu_r) w_\mu(u) du = c_\mu \prod_{i=0}^{r-1} \left( \frac{1}{x^{r-(i+1)}} R_{\alpha_i + \frac{i}{r} - 1} x^{r-(i+1)} \right) \cos_r(x). \quad (8) \]

Now we study the inverse of \( R_\alpha \). We begin by state:

**Theorem 3** For \( k \) integer and \( 0 < \alpha < 1 \) we have

\[
R_{\alpha + \frac{k}{r}}^{-1} g(x) = \frac{r^2}{\Gamma(k + 1) \Gamma(\alpha) \Gamma(1 - \alpha)} x^{r-1} \left( \frac{1}{r x^{r-1}} \frac{d}{dx} \right)^{k+1} \int_0^x g(u) (x^r - u^r)^{-\alpha} u^{(k+\alpha)r} du.
\]

**Proof.** The operator \( R_\alpha \) can be take the form

\[ R_\alpha g(x) = \frac{1}{x^{1+(\alpha-1)}} \int_0^2 g(u) [(x^r - u^r)^{\alpha-1} du \]



and for $0 < \alpha < 1$ admits as inverse:

$$R^{-1}_\alpha g(t) = \frac{r}{\Gamma(\alpha) \Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t g(x) \left[ t^r - x^r \right]^{-\alpha} x^{\alpha r} \, dx$$

which is shown as follows

$$R^{-1}_\alpha R_\alpha g(t) = \frac{r}{\Gamma(\alpha) \Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t R_\alpha g(x) \left[ t^r - x^r \right]^{-\alpha} x^{\alpha r} \, dx$$

$$\quad = \frac{r}{\Gamma(\alpha) \Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t \left[ \frac{1}{x^{1+\alpha r}(\alpha-1)} \int_0^x g(u) [x^r - u^r]^{\alpha-1} \, du \right] \left[ t^r - x^r \right]^{-\alpha} x^{\alpha r} \, dx$$

$$\quad = \frac{r}{\Gamma(\alpha) \Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t \left[ \int_0^t [x^r - u^r]^{\alpha-1} [t^r - x^r]^{-\alpha} x^{\alpha r} \, dx \right] g(u) \, du$$

$$\quad = \frac{1}{\Gamma(\alpha) \Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t \left[ \int_0^t [y - u^r]^{\alpha-1} [t^r - y]^{-\alpha} \, dy \right] g(u) \, du$$

$$\quad = \frac{d}{dt} \int_0^t g(u) \, du = g(t).$$

Therefore

$$R^{-1}_\alpha = \frac{r}{\Gamma(\alpha) \Gamma(1 - \alpha)} \frac{d}{dx} x^{1-\alpha r} R_{1-\alpha x^{\alpha r}}.$$

For an integer $k$ we have the following relation

$$\frac{r}{k!} x^{r-1} \left( \frac{1}{r x^{r-1}} \frac{d}{dx} \right)^{k+1} \int_0^x g(u) (x^r - u^r)^k \, du = g(x)$$

then

$$R^{-1}_{k+1} = \frac{r}{k!} x^{r-1} \left( \frac{1}{r x^{r-1}} \frac{d}{dx} \right)^{k+1} x^{1+kr}.$$

On the other hand

$$R_{k+\alpha} g(x) = \frac{1}{x^{1+(k+\alpha-1)r}} \int_0^x (x^r - u^r)^{k+\alpha-1} \, du$$

$$\quad = \frac{1}{x^{1+(k+\alpha-1)r}} \int_0^x (x^r - u^r)^k (x^r - u^r)^{\alpha-1} \, du$$

$$\quad = \frac{1}{x^{kr}} x^{\alpha r} \frac{1}{x^{1+(\alpha-1)r}} \int_0^x \left[ \frac{d}{du} \int_0^u (x^r - s^r)^k \, ds \right] (x^r - u^r)^{\alpha-1} \, du$$

then

$$R_{k+\alpha} = \frac{1}{x^{kr}} R_\alpha \frac{d}{dx} x^{1+rk} R_{k+1}.$$

So we have

$$R^{-1}_{k+\alpha} = R^{-1}_{k+1} \frac{1}{x^{1+rk}} \left( \frac{d}{dx} \right)^{-1} R^{-1}_\alpha x^{kr}$$

$$R^{-1}_{k+\alpha} = \frac{r^2}{\Gamma(k+1) \Gamma(\alpha) \Gamma(1 - \alpha)} x^{r-1} \left( \frac{1}{r x^{r-1}} \frac{d}{dx} \right)^{k+1} x^{1-\alpha r} R_{1-\alpha x^{(k+\alpha)r}}.$$

The result is then established.
5 Hilbertian structure

We equipped the space $F$ of complex valued functions by the hermitian scalar product given by:

$$\langle f, g \rangle_a = \int_0^\infty \left[ \sum_{m=0}^{r-1} f(w^m t)\overline{g(w^m t)} \right] t^a dt$$

(9)

where $a$ is a suitable positive real number.

We need to list some properties of the resulting hermitian structure. We begin by showing that the projectors $T_i$ given by (2) are then symmetric. Indeed

$$\langle f, T_i g \rangle_a = \int_0^\infty \left[ \sum_{m=0}^{r-1} r^{-1} \sum_{k=0}^{r-1} w^{ik} f(w^m t)\overline{g(w^{m+k} t)} \right] t^a dt$$

$$= \frac{1}{r} \int_0^\infty \left[ \sum_{m=0}^{r-1} r^{-1} \sum_{k=0}^{r-1} w^{-ik} f(w^{m-k} t)\overline{g(w^{m'} t)} \right] t^a dt$$

$$= \frac{1}{r} \int_0^\infty \left[ \sum_{m'=0}^{r-1} r^{-1} \sum_{k'=0}^{r-1} w^{ik'} f(w^{m'+k'} t)\overline{g(w^{m'} t)} \right] t^a dt$$

$$= \langle T_i f, g \rangle_a.$$

As a direct consequence we notice that if $i \neq j$ and $f \in F_i, g \in F_j$ then we have

$$\langle f, g \rangle = \langle T_i f, T_j g \rangle = (T_j T_i f, g) = 0.$$

One can also verify the following identities

$$\langle f, \frac{1}{x} g \rangle_a = \langle \frac{1}{x} f, g \rangle_a$$

and

$$\langle f, x g \rangle_a = \langle \overline{x} f, g \rangle_a.$$

Hence we have:

**Proposition 4** Let $f$ and $g$ be two complex valued functions such as

$$\lim_{x \to 0, \infty} \left[ f(x)\overline{g(x)} x^a \right]$$

is finite.

Then

$$\langle \frac{d}{dx} f, g \rangle_a = - \langle f, \left( \frac{d}{dx} + \frac{a}{x} \right) g \rangle_a.$$
Proof. We have
\[ \left\langle \frac{d}{dx} f, g \right\rangle_a = \sum_{m=0}^{r-1} \int_0^\infty \frac{df}{dx}(w^m x) \overline{g(w^m x)} x^a dx \]
\[ = \sum_{m=0}^{r-1} \int_0^\infty \frac{1}{w^m} \frac{d}{dx} f(w^m x) \overline{g(w^m x)} x^a dx \]
\[ = \left( \sum_{m=0}^{r-1} \frac{1}{w^m} \right) \left\{ \lim_{x \to \infty} \left[ f(x) \overline{g(x)} x^a \right] - \lim_{x \to 0} \left[ f(x) \overline{g(x)} x^a \right] \right\} \]
\[ - \sum_{m=0}^{r-1} \int_0^\infty f(w^m x) \left[ \frac{dg}{dx}(w^m x) + \frac{a}{w^m} g(w^m x) \right] x^a dx \]
\[ = - \left\langle f, \left( \frac{d}{dx} + \frac{a}{x} \right) g \right\rangle_a . \]
This is true because we have
\[ \sum_{m=0}^{r-1} \frac{1}{w^m} = 0. \]

Now we are able to determine the adjoint of the \( r \)-extension of the Riemann Liouville operator and those related to the \( r \)-extension of Dunkl operator:

**Proposition 5** For \( k \) integer and \( 0 < \alpha < 1 \) the adjoint of the Riemann–Liouville operator is given by
\[ R^*_\alpha g(u) = \int_1^\infty g(ut) \left[ t^r - 1 \right]^{\alpha-1} u^{\alpha-1-r(\alpha-1)} dt \]
and
\[ R^*_{k+\alpha} g(\lambda) = (-1)^{k+1} \frac{r^{1-k}}{\Gamma(k+1)\Gamma(\alpha)\Gamma(1-\alpha)} \lambda^{(k+1+\alpha)r-1} \]
\[ \times \int_1^\infty \left( \frac{d}{dx} \frac{1}{x^{r-1}} + \frac{a}{x^r} \right)^{k+1} g(\lambda x) (x^r - 1) x^{\alpha-1-2r(\alpha-2)} dx. \]

Proof. In fact we have
\[ \left\langle R_\alpha f, g \right\rangle_a = \sum_{m=0}^{r-1} \int_0^\infty R_\alpha f(w^m x) \overline{g(w^m x)} x^a dx \]
\[ = \sum_{m=0}^{r-1} \int_0^\infty \left[ \int_0^x f(w^m u) [x^r - u^r]^{\alpha-1} du \right] \overline{g(w^m x)} x^{a-1-r(\alpha-1)} dx \]
\[ = \sum_{m=0}^{r-1} \int_0^\infty f(w^m u) \left[ \int_u^\infty \overline{g(w^m x)} [x^r - u^r]^{\alpha-1} x^{a-1-r(\alpha-1)} dx \right] du \]
\[ = \sum_{m=0}^{r-1} \int_0^\infty f(w^m u) R^*_\alpha g(w^m u) du \]
\[ = \left\langle f, R^*_\alpha g \right\rangle_a. \]
Therefore
\[ R^*_\alpha g(u) = u^{-a} \int_0^\infty g(x) [x^r - u^r]^{\alpha - 1} x^{a - r(\alpha - 1)} \, dx \]
\[ = \int_1^\infty g(u t) [t^r - 1]^{\alpha - 1} t^{a - r(\alpha - 1)} \, dt. \]

Since we have
\[ R^{-1}_{k + \alpha} = \frac{r^2}{\Gamma(k + 1) \Gamma(\alpha) \Gamma(1 - \alpha)} x^{r-1} \left( \frac{1}{r x^{r-1}} \frac{d}{dx} \right)^{k+1} x^{1-\alpha} R_{1-\alpha} x^{(k+\alpha)r} \]
then
\[ (R^{-1}_{k+\alpha})^* = R^*_{k+\alpha} = (-1)^{k+1} \frac{r^{1-k}}{\Gamma(k + 1) \Gamma(\alpha) \Gamma(1 - \alpha)} x^{(k+\alpha)r} R^*_{1-\alpha} x^{1-\alpha} \left( \left( \frac{d}{dx} + \frac{a}{x} \right) \frac{1}{x^{r-1}} \right)^{k+1} x^{r-1} \]
which leads to the result. \[\square\]

**Proposition 6** The corresponding adjoint of the \( r \)-extension Dunkl operator namely

\[ D_\mu = \frac{d}{dx} + \frac{1}{x} \sum_{k=0}^{r-1} a_k T_k \]

is given by

\[ D^*_\mu = - \left( \frac{d}{dx} + \frac{1}{x} \sum_{k=0}^{r-1} (a - a_k) T_{k+1} \right), \]

where \( a \) is the real taking place in the definition of the inner product (9).

**Proof.** We performs the following calculation

\[ \left\langle \left( \frac{d}{dx} + \frac{1}{x} \sum_{k=0}^{r-1} a_k T_k \right) f, g \right\rangle_a = \left\langle \frac{d}{dx} f, g \right\rangle_a + \left\langle \frac{1}{x} \sum_{k=0}^{r-1} a_k T_k f, g \right\rangle_a \]
\[ = - \left\langle f, \frac{d}{dx} g \right\rangle_a - \left\langle f, \frac{a}{x} g \right\rangle_a + \left\langle f, \sum_{k=0}^{r-1} a_k T_k \frac{1}{x} g \right\rangle_a \]
\[ = - \left\langle f, \frac{d}{dx} g \right\rangle_a - \left\langle f, \frac{1}{x} \sum_{k=0}^{r-1} a T_{k+1} g \right\rangle_a + \left\langle f, \frac{1}{x} \sum_{k=0}^{r-1} a_k T_{k+1} g \right\rangle_a \]
\[ = - \left\langle f, \left( \frac{d}{dx} + \frac{1}{x} \sum_{k=0}^{r-1} (a - a_k) T_{k+1} \right) g \right\rangle_a. \]

This proves the result. \[\square\]

**Example 6 :** \( r = 2, w = -1, \theta = i, \mu = (0, \alpha) \)

We choose \( a = 2\alpha + 1 \) then we get

\[ \langle f, g \rangle = \int_0^\infty [f(t)g(t) + f(-t)g(-t)] t^{2\alpha + 1} \, dt = \int_{-\infty}^\infty f(t)g(t) |t|^{2\alpha + 1} \, dt. \]
On the other hand the Dunkl operator is given by

\[ D_\alpha = \frac{d}{dx} + \frac{2\alpha + 1}{x} T_1 \]

which implies

\[ D_\alpha^* = -\left( \frac{d}{dx} + \frac{2\alpha + 1}{x} T_1 \right) = -D_\alpha. \]

**Example 7:** \( r = 3, w = e^{i\frac{2\pi}{3}} = j, \theta = e^{i\frac{\pi}{3}}, \mu = (0, v - \frac{1}{3}, -\frac{2}{3}) \)

We choose \( a = 3v \). The Dunkl operator is given by

\[ D_v = \frac{d}{dx} + \frac{3v}{x} T_1 \]

which implies

\[ D_v^* = -\left( \frac{d}{dx} + \frac{3v}{x} T_1 \right) = -D_v. \]

We note that in general we have \( D_{\mu}^* \neq -D_{\mu} \); the equality depends of a suitable choice of the real \( a \).

### 6 Transmutation operator \( V_\mu \)

An interesting topics is to seek an operator \( V_\mu \) (see [6, 12] for the classical one case \( r = 2 \)) which transforms \( e^{\theta x} \) into \( E_\mu(x) \). To make this section self containing we recall some properties shown early .

\[ R_\alpha T_i = T_i R_\alpha, \quad T_1 \frac{1}{x} = \frac{1}{x} T_{i-1}, \quad T_i x = x T_{i+1}, \quad T_{i+r} = T_i, \quad T_i^2 = T_i, \quad T_i T_j = 0 \text{ if } i \neq j \]

**Theorem 4** The transmutation kernel \( V_\mu \) has the following form

\[ V_\mu = c_\mu T_0 \prod_{i=0}^{r-1} \left( \frac{1}{x^{r-(i+1)}} R_{\alpha_i + \frac{1}{r} - 1} x^{r-(i+1)} \right) \]

\[ + c_\mu \sum_{k=1}^{r-1} \sum_{j=0}^{k} \frac{P_j}{\theta^j} T_k \prod_{i=0}^{r-1} \left( \frac{1}{x^{r-(i+1)}} R_{\alpha_i + \frac{1}{r} - 1} x^{r-(i+1)} \right) x^{k-j}. \]  

(10)

**Proof.** We start with the representation integral (8) and since we can write

\[ L_{a_{k-1}} \ldots L_{a_0} = \sum_{j=0}^{k} \frac{1}{a_j} \left( \frac{d}{dx} \right)^{k-j}. \]
The constants $P_j$ will be explained later. So we have

$$
\left( \sum_{k=0}^{r-1} \frac{1}{\theta^k} T_k L_{a_{k+1}} \ldots L_{a_0} \right) e_\theta (x u_r) = \sum_{k=0}^{r-1} \frac{1}{\theta^k} T_k \left( \sum_{j=0}^{k} P_j \frac{1}{x^j} \frac{d}{dx}^{k-j} e_\theta (x u_r) \right)

= \sum_{k=0}^{r-1} T_k \left( \sum_{j=0}^{k} P_j \frac{1}{\theta^j x^j} u_r^{k-j} \right) e_\theta (x u_r).
$$

Hence

$$
E_\mu (x) = c_\mu \int_{[0,1]^r} \left( T_0 + \sum_{k=0}^{r-1} \frac{1}{\theta^k} T_k L_{a_{k+1}} \ldots L_{a_0} \right) e_\theta (x u_r) w_\mu (u) du

= c_\mu \int_{[0,1]^r} \left( T_0 + \sum_{k=1}^{r-1} T_k \left( \sum_{j=0}^{k} P_j \frac{1}{\theta^j x^j} u_r^{k-j} \right) \right) e_\theta (x u_r) w_\mu (u) du.
$$

The transmutation operator $V_\mu$ is written as

$$
V_\mu g (x) = c_\mu \int_{[0,1]^r} \left( T_0 + \sum_{k=1}^{r-1} T_k \left( \sum_{j=0}^{k} P_j \frac{1}{\theta^j x^j} u_r^{k-j} \right) \right) g (x u_r) w_\mu (u) du

= c_\mu \left[ T_0 \int_{[0,1]^r} g (x u_r) w_\mu (u) du + \sum_{k=1}^{r-1} \sum_{j=0}^{k} T_k \frac{P_j}{\theta^j x^j} \int_{[0,1]^r} g (x u_r) w_\mu (u) u_r^{k-j} du \right].
$$

Note that

$$
\int_{[0,1]^r} g (x u_r) w_\mu (u) du = \prod_{i=0}^{r-1} \left( \frac{1}{x^{r-(i+1)} R_{\alpha_i + \frac{1}{r} - 1}} x^{r-(i+1)} \right) g(x),
$$

then

$$
\int_{[0,1]^r} g (x u_r) w_\mu (u) u_r^{k-j} du = \frac{1}{x^{k-j}} \int_{[0,1]^r} g (x u_r) (x u_r)^{k-j} w_\mu (u) du

= \frac{1}{x^{k-j}} \prod_{i=0}^{r-1} \left( \frac{1}{x^{r-(i+1)} R_{\alpha_i + \frac{1}{r} - 1}} x^{r-(i+1)} \right) x^{k-j} g(x).
$$

Finally

$$
V_\mu = c_\mu T_0 \prod_{i=0}^{r-1} \left( \frac{1}{x^{r-(i+1)} R_{\alpha_i + \frac{1}{r} - 1}} x^{r-(i+1)} \right)

+ c_\mu \sum_{k=1}^{r-1} \sum_{j=0}^{k} \frac{P_j}{\theta^j} T_k \frac{1}{x^k} \prod_{i=0}^{r-1} \left( \frac{1}{x^{r-(i+1)} R_{\alpha_i + \frac{1}{r} - 1}} x^{r-(i+1)} \right) x^{k-j}.
$$

In this representation we can remove the components associated with indices $i$ such that $a_i = 0$. To explicate the constants $P_j$ we recall that

$$
L_a = x^{-a} \frac{d}{dx} x^a = \frac{d}{dx} + \frac{a}{x}.
$$
We check that
\[
\frac{1}{x^k} \prod_{i=0}^{k-1} \left( x \frac{d}{dx} + a_i + i \right) = L_{a_{k-1}} \cdots L_{a_0}.
\]

The use of the modified identity proven by Klushantsev [10]
\[
\frac{1}{x^k} \prod_{j=0}^{k-1} \left( x \frac{d}{dx} + a_j + j \right) = \sum_{j=0}^{k} P_j \frac{1}{x^j} \left( \frac{d}{dx} \right)^{k-j},
\]
where
\[
P_{k-s} = \frac{1}{s!} \sum_{j=0}^{s} (-1)^{s-j} C_s^j \prod_{i=0}^{k-1} (a_i + i + j)
\]
leads to the result.

\[\blacksquare\]

Remark 2 : Thanks to relation (10) of the Theorem 4, we can compute the inverse of the operator \(V_\mu\) but being given the complicity of writing we just give \(V_\mu^{-1}\) in the case of the following example.

Example 8 : \(r = 2, w = -1, \theta = i, \mu = (0, \alpha)\)

We have
\[
R_{\alpha + \frac{1}{2}} g(x) = \int_0^1 g(xu)(1 - u^2)^{\alpha - \frac{1}{2}} du,
\]
then
\[
c_\mu = c_\alpha = 2 \frac{\Gamma (\alpha + 1)}{\Gamma (\alpha + 1/2) \Gamma (1/2)}.
\]

The operator \(V_\mu = V_\alpha\) is then
\[
V_\alpha = c_\alpha \left[ T_0 R_{\alpha + \frac{1}{2}} + T_1 \frac{1}{x} R_{\alpha + \frac{1}{2}} x \right]
\]
and we obtain
\[
V_\alpha^{-1} = \frac{1}{c_\alpha} \left[ R_{\alpha + \frac{1}{2}}^{-1} T_0 + \frac{1}{x} R_{\alpha + \frac{1}{2}}^{-1} x T_1 \right].
\]

That can be justified as follows
\[
V_\alpha^{-1} V_\alpha = R_{\alpha + \frac{1}{2}}^{-1} T_0 R_{\alpha + \frac{1}{2}} + \frac{1}{x} R_{\alpha + \frac{1}{2}}^{-1} x T_1 \frac{1}{x} R_{\alpha + \frac{1}{2}} x
\]
\[
= R_{\alpha + \frac{1}{2}}^{-1} R_{\alpha + \frac{1}{2}} T_0 + \frac{1}{x} R_{\alpha + \frac{1}{2}}^{-1} x T_0 R_{\alpha + \frac{1}{2}} x
\]
\[
= T_0 + \frac{1}{x} R_{\alpha + \frac{1}{2}}^{-1} R_{\alpha + \frac{1}{2}} T_0 x
\]
\[
= T_0 + T_1 = id.
\]

On the other hand the adjoint of \(V_\alpha\) take the form:
\[
V_\alpha^* = c_\alpha \left[ R_{\alpha + \frac{1}{2}}^* T_0^* + x R_{\alpha + \frac{1}{2}}^* \frac{1}{x} T_1^* \right]
\]
\[
= c_\alpha \left[ R_{\alpha + \frac{1}{2}}^* T_0 + x R_{\alpha + \frac{1}{2}}^* \frac{1}{x} T_1 \right]
\]
where
\[ R_{\alpha+\frac{1}{2}}^* g(u) = \int_1^\infty g(ut) \left[ t^2 - 1 \right]^{\frac{\alpha}{2}} t dt, \]
and
\[ V_{\alpha}^{*-1} = \frac{1}{c_\alpha} \left[ T_0 R_{\alpha+\frac{1}{2}}^* + T_1 x R_{\alpha+\frac{1}{2}}^{*-1} \right]. \]

**Example 9:** \( r = 3, w = e^{\frac{2\pi}{3}} = j, \theta = e^{\frac{\pi}{3}}, \mu = (0, v - \frac{4}{3}, -\frac{2}{3}) \)

We have
\[ R_v g(x) = \int_0^1 g(xu) (1 - u^3)^v du \]
\[ c_\mu = c_v = 3 \frac{\Gamma (v + \frac{2}{3})}{\Gamma (v) \Gamma (\frac{4}{3})}. \]

The operator \( V_\mu = V_v \) is given by
\[ V_v = c_v \left[ T_0 \frac{1}{x} R_v x + T_1 \frac{1}{x^2} R_v x^2 + T_2 \frac{1}{x^3} R_v x^3 + \frac{3v}{\theta} T_2 \frac{1}{x^3} R_v x^2 \right] \]
then its inverse is given by
\[ V_v^{-1} = \frac{1}{c_v} \left[ \frac{1}{x} R_v^{-1} x T_0 + \frac{1}{x^2} R_v^{-1} x^2 T_1 + \frac{1}{x^3} R_v^{-1} x^3 T_2 - \frac{3v}{\theta} \frac{1}{x^3} R_v^{-1} x^2 T_1 \right]. \]

Since
\[ V_v^{-1} V_v = 1 \frac{1}{x} R_v^{-1} x T_0 + \frac{1}{x^2} R_v^{-1} x^2 T_1 + \frac{1}{x^3} R_v^{-1} x^3 T_2 = T_0 + T_1 + T_2 = id \]
On the other hand
\[ V_v^* = c_v \left[ \pi R_v^* \frac{1}{x} T_0 + \pi^2 R_v^* \frac{1}{x^2} T_1 + \pi^3 R_v^* \frac{1}{x^3} T_2 + \frac{3v}{\theta} \pi^2 R_v^* \frac{1}{x^3} T_2 \right] \]
where
\[ R_v^* g(u) = \int_1^\infty g(ut) \left[ t^3 - 1 \right]^{v-1} t^2 dt. \]

**7 The operators** \( D_\mu \) **and** \( \frac{d}{dx} \)

In this section, we tackle the crucial subject concerning the research of functional spaces on which the following transmutation relation is valid
\[ D_\mu V_\mu = V_\mu \frac{d}{dx} \]

The first idea that comes to mind is to verify that
\[ D_\mu V_\mu x^n = V_\mu \frac{d}{dx} x^n, \quad \forall n \in \mathbb{N} \] (11)
and when this last fact is true then the transmutation act on the space of entire function.

\[ D_\mu V_\mu g(x) = V_\mu \frac{d}{dx} g(x), \quad g(x) = \sum_{n=0}^{\infty} a_n x^n. \]

In fact we can permute each of the following operators

\[ \frac{d}{dx}, R_\alpha, T_k, x^k, \frac{1}{x^k} \]

with the infinite sum \( \sum_{n=0}^{\infty} \).

We will give two examples and we will constat that in the first, formula (11) is true but for the second it is false.

**Example 10**: \( r = 2, w = -1, \theta = i, \mu = (0, \alpha) \)

We will prove that formula (11) is true in this case. For this we use the following result

\[ R_\alpha x^n = \left( \int_0^1 (1 - u^r)^{\alpha-1} u^n du \right) x^n = \frac{1}{r} \Gamma \left( \frac{n+\alpha}{r} \right) \Gamma \left( \frac{\alpha}{r} \right) x^n = l_n^\alpha x^n, \]

then

\[
\begin{align*}
D_\alpha V_\alpha x^{2n} &= c_\alpha \left( \frac{d}{dx} + \frac{2\alpha + 1}{x} T_1 \right) \left( T_0 R_\alpha + \frac{1}{x} T_1 \right) x^{2n} \\
&= c_\alpha \left( \frac{d}{dx} + \frac{2\alpha + 1}{x} T_1 \right) l_n^{\alpha+\frac{1}{2}} x^{2n} = c_\alpha l_n^{\alpha+\frac{1}{2}} x^{2n-1}
\end{align*}
\]

\[
\begin{align*}
V_\alpha \frac{d}{dx} x^{2n} &= c_\alpha \left( T_0 R_\alpha + \frac{1}{x} T_1 \right) x^{2n-1} \\
&= c_\alpha l_n^{\alpha+\frac{1}{2}} x^{2n-1}.
\end{align*}
\]

On the other hand

\[
\begin{align*}
D_\alpha V_\alpha x^{2n+1} &= c_\alpha \left( \frac{d}{dx} + \frac{2\alpha + 1}{x} T_1 \right) \left( T_0 R_\alpha + \frac{1}{x} T_1 \right) x^{2n+1} \\
&= c_\alpha \left( \frac{d}{dx} + \frac{2\alpha + 1}{x} T_1 \right) l_n^{\alpha+\frac{1}{2}} x^{2n+1} = c_\alpha l_n^{\alpha+\frac{1}{2}} [(2n + 1) + (2\alpha + 1)] x^{2n},
\end{align*}
\]

and

\[
\begin{align*}
V_\alpha \frac{d}{dx} x^{2n+1} &= c_\alpha \left( T_0 R_\alpha + \frac{1}{x} T_1 \right) x^{2n} \\
&= c_\alpha l_n^{\alpha+\frac{1}{2}} x^{2n}.
\end{align*}
\]

To show equality we use the identity
\[ \ell_{2n}^{\alpha+\frac{1}{2}}(2n+1) = \ell_{2n+2}^{\alpha+\frac{1}{2}}[(2n+1) + (2\alpha + 1)]. \]

**Example 11**: \( r = 3, w = e^{i\frac{2\pi}{3}} = j, \theta = e^{i\frac{\pi}{3}}, \mu = (0, v - \frac{1}{3}, -\frac{2}{3}) \)

The transmutation operator is given by

\[
V_v = c_v \left[ T_0 \frac{1}{x} R_v x + T_1 \frac{1}{x^2} R_v x^2 + T_2 \frac{1}{x^3} R_v x^3 + \frac{3v}{\theta} T_2 \frac{1}{x^3} R_v x^2 \right].
\]

The 3-extension of Dunkl operator takes the following form

\[
D_v = \frac{d}{dx} + \frac{3v}{x} T_1
\]

We check easily that

\[
\left( \frac{d}{dx} + \frac{3v}{x} T_1 \right) V_v x^{3n} \neq V_v \frac{d}{dx} x^{3n}.
\]

So the spaces of entire function seems not suitable for transmutation for all \( r \) except for the case \( r = 2 \).

In the following statement we show that the transmutation is true over the following suitable functional space.

**Theorem 5** Let \( g \) be a continuously differentiable function on an interval \([-\frac{T}{2}, \frac{T}{2}]\) such that

\[
\sum_{n=-\infty}^{\infty} |c_n(g)| e^{\pi|n\text{Im}(w^k)|} < \infty, \quad \forall k = 0 \ldots r - 1
\]

(12)

then we have

\[
D_\mu V_\mu g(x) = V_\mu \frac{d}{dx} g(x).
\]

**Proof.** Since we have

\[
D_\mu V_\mu e^{\theta \mu x} = V_\mu \frac{d}{dx} e^{\theta \mu x}, \quad \forall \mu \in \mathbb{C} \Rightarrow D_\mu V_\mu e^{i\lambda x} = V_\mu \frac{d}{dx} e^{i\lambda x}, \quad \forall \lambda \in \mathbb{C}.
\]

Let \( g \) be a continuously differentiable function on an interval \([-\frac{T}{2}, \frac{T}{2}]\) then we have

\[
g(x) = \sum_{n=-\infty}^{\infty} c_n(g) e^{\frac{2\pi n}{T} x}, \quad \forall t \in \left[-\frac{T}{2}, \frac{T}{2}\right].
\]

The coefficients \( c_n(g) \) so called the Fourier coefficients of \( g \), defined by the formula

\[
c_n(g) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} g(t) e^{\frac{2\pi n}{T} t} dt.
\]

We have

\[
\sum_{n=-\infty}^{\infty} |c_n(g)| < \infty.
\]
The action of the operator \( T_k \) at the function \( g \) shows in the Fourier series a term of the form

\[
e^{\frac{2\pi i}{T} nw_k t}, \quad w = e^{\frac{2\pi i}{T}}, \quad 0 \leq k \leq r - 1.
\]

As

\[
|e^{\frac{2\pi i}{T} nw_k t}| \leq e^{\pi|n\operatorname{Im}(w_k)|}.
\]

So if we impose the condition of normal convergence

\[
\sum_{n=-\infty}^{\infty} |c_n(g)| e^{\pi|n\operatorname{Im}(w_k)|} < \infty, \quad \forall k = 0 \ldots r - 1
\]

we say that

\[
D_{\mu}V_{\mu}g(x) = V_{\mu}\frac{d}{dx}g(x).
\]

Now we understand why this relationship is verified for \( x^n \) in the case \( r = 2 \) because \( w = -1 \) and then \( \operatorname{Im}(w_k) = 0 \).

\section{r-extension of Dunkl transform}

Before anything let us introduce the integral transform of Laplace type given for \( \theta = i\frac{T}{r} \) by:

\[
\mathcal{L}_\theta g(\lambda) = \int_0^{\infty} e^{\theta t\lambda} g(t)dt.
\]

\textbf{Proposition 7} The inversion formula of \( \mathcal{L}_\theta \) is given by

\[
\mathcal{L}_\theta^{-1} g(x) = \frac{1}{2\pi i \theta} \lim_{T \to \infty} \int_{-\tau - i \theta T}^{-c\theta + i \theta T} e^{-\theta xs} g(s)ds
\]

which is valid for any function of exponential type \( \alpha < c \).

\textbf{Proof.} To prove this formula, given a function \( g \) of exponential type \( \alpha < c \). Then there exists \( M > 0 \) such that

\[
|g(t)| \leq M e^{\alpha t}, \quad \forall t \in \mathbb{R}.
\]

If \( s = -c\theta + iy\theta \) where \( c > \alpha \) and \( y > 0 \) we get

\[
\mathcal{L}_\theta g(s) = \int_0^{\infty} e^{\theta t(-c\theta + iy\theta)} g(t)dt = \int_0^{\infty} e^{iyt} e^{-ct} g(t)dt.
\]

Therefore

\[
|\mathcal{L}_\theta g(s)| \leq \int_0^{\infty} e^{-ct} |g(t)| dt \leq \int_0^{\infty} e^{(\alpha-c)t} dt < \infty.
\]

So we have

\[
\mathcal{L}_\theta^{-1}\mathcal{L}_\theta g(x) = \mathcal{L}_\theta^{-1}\mathcal{L}_\theta g(x) = e^{cx} \left[ \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T}^{T} e^{-iyx} \left( \int_0^{\infty} e^{iyt} e^{-ct} g(t)dt \right) dy \right] = e^{cx} e^{-cx} g(x) = g(x), \quad \forall x \in \mathbb{R}.
\]
Definition 3 For $a > 0$, we define the $r$-extension of the Dunkl transform associated with the vector $\mu = (\alpha_0, \alpha_1, \ldots, \alpha_{r-1})$ as follows

$$F_\mu g(\lambda) = \langle g, E_\mu(\lambda x) \rangle_a = \int_0^\infty \left[ \sum_{m=0}^{r-1} g(w^m t) E_\mu(w^m \lambda t) \right] t^a dt$$

where $E_\mu$ denote the $r$-Dunkl kernel [5].

Taking account of the fact that $E_\mu(\lambda x) = V_\mu e^{\theta \lambda x}$ then we can write

$$F_\mu g(\lambda) = \langle g, E_\mu(\lambda x) \rangle_a = \langle g, V_\mu e^{\theta \lambda x} \rangle_a.$$ 

The integral transform associated with $\mu = (0, -\frac{1}{r}, \ldots, -\frac{1}{r})$ is given by

$$F_r g(\lambda) = \langle g, e^{\theta \lambda x} \rangle_0.$$ 

This operator coincide with the Laplace transform and Fourier transform respectively for $r = 1$ and $r = 2$.

We deduce that

$$F_\mu g(\lambda) = \langle V_\mu^* g, e^{\theta \lambda x} \rangle_a = \langle |x|^a V_\mu^* g, e^{\theta \lambda x} \rangle_0.$$ 

Therefore

$$F_\mu = F_r |x|^a V_\mu^*.$$ 

Proposition 8 Let $g$ be a function of exponential type belongs in $F_{r-k}$ the subspace defined by (7) then we have

$$F_{\mu}^{-1} g(\lambda) = \frac{1}{r} V_\mu^{s-1} |x|^{-a} L_\theta^{-1} g(\lambda).$$

Proof. We write the transformation $F_r$ as follows

$$F_r g(\lambda) = \langle g, e^{\theta \lambda x} \rangle_0 = \int_0^\infty \left( \sum_{m=0}^{r-1} g(w^m t) e^{w^m \theta t \lambda} \right) dt$$

then

$$F_r T_k g(\lambda) = \int_0^\infty \left( \sum_{m=0}^{r-1} T_k g(w^m t) e^{w^m \theta t \lambda} \right) dt = \int_0^\infty \left( \sum_{m=0}^{r-1} w^{(r-k)m} e^{w^m \theta t \lambda} \right) T_k g(t) dt = r T_{r-k} L_\theta T_k g(\lambda).$$

Furthermore we have

$$F_\mu T_k = F_r |x|^a V_\mu^* T_k = F_r T_k |x|^a V_\mu^* = r T_{r-k} L_\theta T_k |x|^a V_\mu^* = r T_{r-k} L_\theta |x|^a V_\mu^* T_k,$$
then
\[ F_\mu = \sum_{k=0}^{r-1} F_\mu T_k = r \sum_{k=0}^{r-1} T_{r-k} \mathcal{L}_\theta |x|^a V^*_\mu T_k, \]

which implies
\[ T_m F_\mu = r T_m \sum_{k=0}^{r-1} T_{r-k} \mathcal{L}_\theta |x|^a V^*_\mu T_k = r T_m \mathcal{L}_\theta |x|^a V^*_\mu T_{r-m}. \]

Notice that
\[ F_\mu : F_k \to F_{r-k}. \]

In the space \( F_k \) we have the following equality
\[ F_\mu = r T_{r-k} \mathcal{L}_\theta |x|^a V^*_\mu. \]

This leads to the result.

\[ \blacksquare \]

**Proposition 9** If \( D^*_\mu = -D_\mu \) then we have
\[ F_\mu D_\mu g(\lambda) = -\theta \lambda F_\mu g(\lambda). \]

**Proof.** In fact
\[
F_\mu D^*_\mu g(\lambda) = \langle D^*_\mu g, E_\mu(\lambda x) \rangle_a = \langle D^*_\mu g, V_\mu e^{\theta \lambda x} \rangle_a
\]
\[ = \langle g, D_\mu V_\mu e^{\theta \lambda x} \rangle_a = \langle g, V_\mu \frac{d}{dx} e^{\theta \lambda x} \rangle_a
\]
\[ = \theta \lambda \langle g, E_\mu(\lambda x) \rangle_a = \theta \lambda F_\mu g(\lambda). \]

In the case \( D^*_\mu = -D_\mu \) we obtain the result.

Note that the function \( x \mapsto e^{\theta \lambda x} \) is a continuously differentiable function which satisfies [12].

\[ \blacksquare \]

**Epilogue**

We just built an \( r \)-extension of Dunkl operator focusing on examples. This approach is very positive and encourages researchers to determine adequate harmonic analysis and especially look for applications.

In forthcoming papers we will study in great detail the associated heat and wave equations.

**Appendix**

The one-dimensional specialization of the Dunkl-Opdam operators defined in [5, p.20] is a particular case of this introduced in our paper. We begin by recalling that for a fixed \( r = 1, 2, \ldots \) the complex reflection group \( W \) of type \( G(r,1,N) \) is generated by the \( N \times N \) permutation matrices with the nonzero entries being powers of \( \omega = e^{2i\pi/r} \), an \( r \)th root of unity, and by the complex reflection \( \tau_i \) defined by
\[ x\tau_i = \left(x_1, \ldots, \omega^i x_i, \ldots \right), \quad 1 \leq i \leq N \]
An element \( w \) of the groups \( W \) acting on a complex valued function \( f \) as follows
\[
wf(x) = f(wx).
\]

Given a list of complex numbers \( \kappa = (\kappa_0, \ldots, \kappa_{r-1}) \). The Dunkl-Opdam operators for complex reflection groups \( W \) defined by
\[
T_i(\kappa) = \frac{\partial}{\partial x_i} + \kappa_0 \sum_{j \neq i} \sum_{s=0}^{r-1} \frac{1 - \tau^{-s}_i (i,j) \tau^s_j}{x_i - \omega^s x_j} + \sum_{t=1}^{r-1} \kappa_t \sum_{s=0}^{r-1} \frac{\omega^{-st} \tau^{-s}_i}{x_i},
\]
where \((i,j)\) is a transposition
\[
x(i,j) = (x_1, \ldots, x_j, \ldots, x_i, \ldots).
\]

The one-dimensional specialization of this operators is then
\[
T(\kappa) = \frac{d}{dx} + \frac{1}{x} \sum_{t=1}^{r-1} \kappa_t \left( \sum_{s=0}^{r-1} \frac{1}{\omega^{-st}} \tau^s \right)
\]
\[
= \frac{d}{dx} + \frac{1}{x} \sum_{s=0}^{r-1} \left( \sum_{t=1}^{r-1} \kappa_t \omega^{-st} \right) \tau^s,
\]
where \( \tau \) is a complex reflection given by
\[
\tau f(x) = f(\omega x).
\]

The operators introduced in our paper can be written in the following form
\[
D_\mu = \frac{d}{dx} + \frac{1}{x} \sum_{t=0}^{r-1} a_t T_t
\]
\[
= \frac{d}{dx} + \frac{1}{x} \sum_{t=0}^{r-1} a_t \left( \frac{1}{r} \sum_{s=0}^{r-1} s_t^s \right)
\]
\[
= \frac{d}{dx} + \frac{1}{x} \sum_{s=0}^{r-1} \left( \frac{1}{r} \sum_{t=0}^{r-1} a_t \omega^{st} \right) \tau^s.
\]

If we have
\[
\frac{1}{r} \sum_{t=0}^{r-1} a_t \omega^{st} = \sum_{t=1}^{r-1} \kappa_t \omega^{-st}, \quad s = 0 \ldots r - 1
\]
then we obtain
\[
D_\mu = T(\kappa).
\]

Hence, to reduce an operator \( D_\mu \) to a fixed operator \( T(\kappa) \) it is necessary to solve a linear system of \( r \) indeterminate \( (a_0, \ldots, a_{r-1}) \) and \( r \) equations \( (s = 0, \ldots, r - 1) \). There’s one and unique solution because
\[
(\omega^{st})_{0 \leq t,s \leq n} \in GL_n(\mathbb{C})
\]

Conversely, if we want to reduce an operator \( T(\kappa) \) to a fixed operator \( D_\mu \), then it is necessary to solve a linear system of \( r - 1 \) indeterminate \( (\kappa_1, \ldots, \kappa_{r-1}) \) and \( r \) equations \( (s = 0, \ldots, r - 1) \). In general there’s no solution. This prove that the operator introduced in our paper is a generalization of the Dunkl-Opdam operators in one dimension.
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