IDEMPOTENT (ASYMPTOTIC) MATHEMATICS
AND THE REPRESENTATION THEORY

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1. Introduction. Idempotent Mathematics is based on replacing the usual arithmetic operations by a new set of basic operations (e.g., such as maximum or minimum), that is on replacing numerical fields by idempotent semirings and semifields. Typical (and the most common) examples are given by the so-called (max, +) algebra $\mathbb{R}_{\text{max}}$ and (min, +) algebra $\mathbb{R}_{\text{min}}$. Let $\mathbb{R}$ be the field of real numbers. Then $\mathbb{R}_{\text{max}} = \mathbb{R} \cup \{ -\infty \}$ with operations $x \oplus y = \max\{x, y\}$ and $x \odot y = x + y$. Similarly $\mathbb{R}_{\text{min}} = \mathbb{R} \cup \{ +\infty \}$ with the operations $\oplus = \min$, $\odot = +$. The new addition $\oplus$ is idempotent, i.e., $x \oplus x = x$ for all elements $x$. Idempotent Mathematics can be treated as a result of a dequantization of the traditional mathematics over numerical fields as the Planck constant $\hbar$ tends to zero taking pure imaginary values. Some problems that are nonlinear in the traditional sense turn out to be linear over a suitable idempotent semiring (idempotent superposition principle [1]). For example, the Hamilton-Jacobi equation (which is an idempotent version of the Schrödinger equation) is linear over $\mathbb{R}_{\text{min}}$ and $\mathbb{R}_{\text{max}}$.

The basic paradigm is expressed in terms of an idempotent correspondence principle [2]. This principle is similar to the well-known correspondence principle of N. Bohr in quantum theory (and closely related to it). Actually, there exists a heuristic correspondence between important, interesting and useful constructions and results of the traditional mathematics over fields and analogous constructions and results over idempotent semirings and semifields (i.e., semirings and semifields with idempotent addition).

A systematic and consistent application of the idempotent correspondence principle leads to a variety of results, often quite unexpected. As a result, in parallel with the traditional mathematics over rings, its “shadow”, the Idempotent Mathematics, appears. This “shadow” stands approximately in the same relation to the traditional mathematics as classical physics to quantum theory. In many respects Idempotent Mathematics is simpler than the traditional one. However, the transition from traditional concepts and results to their idempotent analogs is often nontrivial.

There is an idempotent version of the theory of linear representations of groups. We shall present some basic concepts and results of the idempotent representation theory. In the framework of this theory the well-known Legendre transform can be treated as an $\mathbb{R}_{\text{max}}$-version of the traditional Fourier transform (this observation

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is due to V.P. Maslov). We shall discuss some unexpected theorems of the Engel type.

In this paper we present the so-called algebraic approach to Idempotent Mathematics: basic notions and results are ‘simulated’ in pure algebraic terms. Historical surveys and the corresponding references can be found in [2–6].

2. Semirings, semifields, and idempotent dequantization. Consider a set $S$ equipped with two algebraic operations: addition $\oplus$ and multiplication $\odot$. It is a semiring if the following conditions are satisfied:

- the addition $\oplus$ and the multiplication $\odot$ are associative;
- the addition $\oplus$ is commutative;
- the multiplication $\odot$ is distributive with respect to the addition $\oplus$: $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$ and $(x \odot y) \odot z = (x \odot z) \oplus (y \odot z)$ for all $x, y, z \in S$.

A unity of a semiring $S$ is an element $1 \in S$ such that $1 \odot x = x \odot 1 = x$ for all $x \in S$. A zero of a semiring $S$ is an element $0 \in S$ such that $0 \neq 1$ and $0 \odot x = x$, $0 \odot x = x \odot 0 = 0$ for all $x \in S$. A semiring $S$ is called an idempotent semiring if $x \odot x = x$ for all $x \in S$. A semiring $S$ with neutral elements $0$ and $1$ is called a semifield if every nonzero element of $S$ is invertible.

Let $R$ be the field of real numbers and $R_+$ the semiring of all nonnegative real numbers (with respect to the usual addition and multiplication). The change of variables $x \mapsto u = h \ln x$, $h > 0$, defines a map $\Phi_h: R \to S = R \cup \{\infty\}$. Let the addition and multiplication operations be mapped from $R$ to $S$ by $\Phi_h$, i.e., let $u \oplus_h v = h \ln(\exp(u/h) + \exp(v/h))$, $u \odot v = u + v$, $0 = -\infty = \Phi_h(0)$, $1 = 0 = \Phi_h(1)$. It can easily be checked that $u \oplus_h v \to \max\{u, v\}$ as $h \to 0$ and $S$ forms a semiring with respect to addition $u \oplus v = \max\{u, v\}$ and multiplication $u \odot v = u + v$ with zero $0 = -\infty$ and unit $1 = 0$. Denote this semiring by $R_{\max}$; it is idempotent, i.e., $u \oplus u = u$ for all its elements. The semiring $R_{\max}$ is actually a semifield. The analogy with quantization is obvious; the parameter $h$ plays the rôle of the Planck constant, so $R_+$ (or $R$) can be viewed as a “quantum object” and $R_{\max}$ as the result of its “dequantization”. A similar procedure gives the semiring $R_{\min} = R \cup \{+\infty\}$ with the operations $\oplus = \min$, $\odot = +$; in this case $0 = +\infty$, $1 = 0$. The semirings $R_{\max}$ and $R_{\min}$ are isomorphic. Connections with physics and imaginary values of the Planck constant are discussed below. The idempotent semiring $R \cup \{-\infty\} \cup \{+\infty\}$ with the operations $\oplus = \max$, $\odot = \min$ can be obtained as a result of a “second dequantization” of $R$ (or $R_+$). Dozens of interesting examples of nonisomorphic idempotent semirings may be cited as well as a number of standard methods of deriving new semirings from these (see, e.g., [2–6] and below).

Idempotent dequantization is related to logarithmic transformations that were used in the classical papers of E. Schrödinger [7] and E. Hopf [8]. The subsequent progress of E. Hopf’s ideas has culminated in the well-known vanishing viscosity method (the method of viscosity solutions), see, e.g., [9].

3. Idempotent Analysis. Let $S$ be an arbitrary semiring with idempotent addition $\oplus$ (which is always assumed to be commutative), multiplication $\odot$, zero $0$, and unit $1$. The set $S$ is supplied with the standard partial order $\preceq$: by definition, $a \preceq b$ if and only if $a \oplus b = b$. Thus all elements of $S$ are positive: $0 \preceq a$ for all $a \in S$. Due to the existence of this order, Idempotent Analysis is closely related to lattice theory, the theory of vector lattices, and the theory of ordered spaces.
Moreover, this partial order allows to model a number of basic notions and results of Idempotent Analysis at the purely algebraic level; in this paper we develop this line of reasoning systematically. Let us notice that the standard partial order can be defined for an arbitrary commutative semigroup with idempotent addition.

Calculus deals mainly with functions whose values are numbers. The idempotent analog of a numerical function is a map $X \to S$, where $X$ is an arbitrary set and $S$ is an idempotent semiring. Functions with values in $S$ can be added, multiplied by each other, and multiplied by elements of $S$.

The idempotent analog of a linear functional space is a set of $S$-valued functions that is closed under addition of functions and multiplication of functions by elements of $S$, or an $S$-semimodule. Consider, e.g., the $S$-semimodule $\mathcal{B}(X, S)$ of functions $X \to S$ that are bounded in the sense of the standard order on $S$.

If $S = \mathbb{R}_{\text{max}}$, then the idempotent analog of integration is defined by the formula

\begin{equation}
I(\varphi) = \int_{X}^{\oplus} \varphi(x) \, dx = \sup_{x \in X} \varphi(x),
\end{equation}

where $\varphi \in \mathcal{B}(X, S)$. Indeed, a Riemann sum of the form $\sum_{i} \varphi(x_{i}) \cdot \sigma_{i}$ corresponds to the expression $\bigoplus_{i} \varphi(x_{i}) \odot \sigma_{i} = \max\{\varphi(x_{i}) + \sigma_{i}\}$, which tends to the right-hand side of (1) as $\sigma_{i} \to 0$. Of course, this is a purely heuristic argument.

Formula (1) defines the idempotent integral not only for functions taking values in $\mathbb{R}_{\text{max}}$, but also in the general case when any of bounded (from above) subsets of $S$ has the least upper bound.

An idempotent measure on $X$ is defined by $m_{\psi}(Y) = \sup_{x \in Y} \psi(x)$, where $\psi \in \mathcal{B}(X, S)$. The integral with respect to this measure is defined by

\begin{equation}
I_{\psi}(\varphi) = \int_{X}^{\oplus} \varphi(x) \, dm_{\psi} = \int_{X}^{\oplus} \varphi(x) \odot \psi(x) \, dx = \sup_{x \in X} (\varphi(x) \odot \psi(x)).
\end{equation}

Obviously, if $S = \mathbb{R}_{\text{min}}$, then the standard order $\preceq$ is opposite to the conventional order $\leq$, so in this case equation (2) assumes the form

\begin{equation}
\int_{X}^{\oplus} \varphi(x) \, dm_{\psi} = \int_{X}^{\oplus} \varphi(x) \odot \psi(x) \, dx = \inf_{x \in X} (\varphi(x) \odot \psi(x)),
\end{equation}

where $\inf$ is understood in the sense of the conventional order $\leq$.

The functionals $I(\varphi)$ and $I_{\psi}(\varphi)$ are linear over $S$; their values correspond to limits of Lebesgue (or Riemann) sums. The formula for $I_{\psi}(\varphi)$ defines the idempotent scalar product of the functions $\psi$ and $\varphi$. Various idempotent functional spaces and an idempotent version of the theory of distributions can be constructed on the basis of the idempotent integration, see, e.g., [1, 3–6, 10]. The analogy of idempotent and probability measures leads to spectacular parallels between optimization theory and probability theory. For example, the Chapman–Kolmogorov equation corresponds to the Bellman equation (see, e.g., the survey of Del Moral [11] and [5]). Many other idempotent analogs may be cited (in particular, for basic constructions and theorems of functional analysis [4]).
4. The superposition principle and linear problems. Basic equations of quantum theory are linear (the superposition principle). The Hamilton–Jacobi equation, the basic equation of classical mechanics, is nonlinear in the conventional sense. However it is linear over the semiring $\mathbb{R}_{\text{min}}$. Also, different versions of the Bellman equation, the basic equation of optimization theory, are linear over suitable idempotent semirings (V. P. Maslov’s idempotent superposition principle), see [1, 3, 6, 10]. For instance, the finite-dimensional stationary Bellman equation can be written in the form $X = H \odot X \oplus F$, where $X, H, F$ are matrices with coefficients in an idempotent semiring $S$ and the unknown matrix $X$ is determined by $H$ and $F$ [12]. In particular, standard problems of dynamic programming and the well-known shortest path problem correspond to the cases $S = \mathbb{R}_{\text{max}}$ and $S = \mathbb{R}_{\text{min}}$, respectively. In [12], it was shown that main optimization algorithms for finite graphs correspond to standard methods for solving systems of linear equations of this type (i.e., over semirings). Specifically, Bellman’s shortest path algorithm corresponds to a version of Jacobi’s algorithm, Ford’s algorithm corresponds to the Gauss–Seidel iterative scheme, etc.

Linearity of the Hamilton–Jacobi equation over $\mathbb{R}_{\text{min}}$ (and $\mathbb{R}_{\text{max}}$) is closely related to the (conventional) linearity of the Schrödinger equation. Consider a classical dynamical system specified by the Hamiltonian

\[
H = H(p, x) = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + V(x),
\]

where $x = (x_1, \ldots, x_N)$ are generalized coordinates, $p = (p_1, \ldots, p_N)$ are generalized momenta, $m_i$ are generalized masses, and $V(x)$ is the potential. In this case the Lagrangian $L(x, \dot{x}, t)$ has the form

\[
L(x, \dot{x}, t) = \sum_{i=1}^{N} m_i \frac{\dot{x}_i^2}{2} - V(x),
\]

where $\dot{x} = (\dot{x}_1, \ldots, \dot{x}_N)$, $\dot{x}_i = dx_i/dt$. The value function $S(x, t)$ of the action functional has the form

\[
S(x, t) = \int_{t_0}^{t} L(x(t), \dot{x}(t), t) \, dt,
\]

where the integration is performed along a trajectory of the system. The classical equations of motion are derived as the stationarity conditions for the action functional (the Hamilton principle, or the least action principle).

The action functional can be considered as a function taking the set of curves (trajectories) to the set of real numbers. Assume that its range lies in the semiring $\mathbb{R}_{\text{min}}$. In this case the minimum of the action functional can be viewed as the idempotent integral of this function over the set of trajectories or the idempotent analog of the Feynman path integral. Thus the least action principle can be considered as the idempotent version of the well-known Feynman approach to quantum mechanics (which is presented, e.g., in [13]); here, one should remember that the exponential function involved in the Feynman integral is monotone on the real axis. The representation of a solution to the Schrödinger equation in terms of
the Feynman integral corresponds to the Lax–Oleĭnik formula for a solution to the Hamilton–Jacobi equation.

Since \( \partial S/\partial x_i = p_i \), \( \partial S/\partial t = -H(p, x) \), the following Hamilton–Jacobi equation holds:

\[
\frac{\partial S}{\partial t} + H \left( \frac{\partial S}{\partial x_i}, x_i \right) = 0.
\]

Quantization (see, e.g., [13]) leads to the Schrödinger equation

\[
-\hbar i \frac{\partial \psi}{\partial t} = \hat{H} \psi = H(\hat{p}_i, \hat{x}_i) \psi,
\]

where \( \psi = \psi(x, t) \) is the wave function, i.e., a time-dependent element of the Hilbert space \( L^2(\mathbb{R}^N) \), and \( \hat{H} \) is the energy operator obtained by substitution of the momentum operators \( \hat{p}_i = \hbar \frac{\partial}{\partial x_i} \) and the coordinate operators \( \hat{x}_i: \psi \mapsto x_i \psi \) for the variables \( p_i \) and \( x_i \) in the Hamiltonian function, respectively. This equation is linear in the conventional sense (the quantum superposition principle). The standard procedure of limit transition from the Schrödinger equation to the Hamilton–Jacobi equation is to use the following ansatz for the wave function: \( \psi(x, t) = a(x, t) e^{i S(x, t)/\hbar} \), and to keep only the leading order as \( \hbar \to 0 \) (the ‘semiclassical’ limit).

Instead of doing this, we switch to imaginary values of the Planck constant \( \hbar \) by the substitution \( h = i \hbar \), assuming \( h > 0 \). Thus the Schrödinger equation (1.10) turns to an analog of the heat equation:

\[
\hbar \frac{\partial u}{\partial t} = H \left( -\hbar \frac{\partial}{\partial x_i}, \hat{x}_i \right) u,
\]

where the real-valued function \( u \) corresponds to the wave function \( \psi \). A similar idea (the switch to imaginary time) is used in the Euclidean quantum field theory (see, e.g., [14]); let us remember that time and energy are dual quantities.

Linearity of equation (10) implies linearity of equation (11). Thus if \( u_1 \) and \( u_2 \) are solutions of (11), then so is their linear combination

\[
u = \lambda_1 u_1 + \lambda_2 u_2.
\]

Let \( S = -\hbar \ln u \) or \( u = e^{-S/\hbar} \) as in Section 2 above. It can easily be checked that equation (11) thus turns to

\[
\frac{\partial S}{\partial t} = V(x) + \sum_{i=1}^{N} \frac{1}{2m_i} \left( \frac{\partial S}{\partial x_i} \right)^2 - \hbar \sum_{i=1}^{n} \frac{1}{2m_i} \frac{\partial^2 S}{\partial x_i^2}.
\]

This equation is nonlinear in the conventional sense. However, if \( S_1 \) and \( S_2 \) are its solutions, then so is the function

\[
S = \lambda_1 \odot S_1 \oplus h \odot S_2
\]

obtained from (12) by means of our substitution \( S = -\hbar \ln u \). Here the generalized multiplication \( \odot \) coincides with the ordinary addition and the generalized addition \( \oplus_h \) is the image of the conventional addition under the above change of variables.
As $h \to 0$, we obtain the operations of the idempotent semiring $R_{\min}$, i.e., $\oplus = \min$ and $\odot = +$, and equation (13) turns to the Hamilton–Jacobi equation (9), since the third term in the right-hand side of equation (13) vanishes.

Thus it is natural to consider the limit function $S = \lambda_1 \odot S_1 \oplus \lambda_2 \odot S_2$ as a solution of the Hamilton–Jacobi equation and to expect that this equation can be treated as linear over $R_{\min}$. This argument (clearly, a heuristic one) can be extended to equations of a more general form. For a rigorous treatment of (semiring) linearity for these equations see [3, 6] and also [1]. Notice that if $h$ is changed to $-h$, then the resulting Hamilton–Jacobi equation is linear over $R_{\max}$.

The idempotent superposition principle indicates that there exist important problems that are linear over idempotent semirings.

5. Convolution and the Fourier–Legendre transform. Let $G$ be a group. Then the space $B(X, R_{\max})$ of all bounded functions $G \to R_{\max}$ (see above) is an idempotent semiring with respect to the following analog $\star$ of the usual convolution:

$$ (\varphi(x) \star \psi)(g) = \int_G \varphi(x) \odot \psi(x^{-1} \cdot g) \, dx = \sup_{x \in G} (\varphi(x) + \psi(x^{-1} \cdot g)). $$

(15)

Of course, it is possible to consider other “function spaces” (and other basic semirings instead of $R_{\max}$). In [3] “group semirings” of this type are referred to as convolution semirings.

Let $G = \mathbb{R}^n$, where $\mathbb{R}^n$ is considered as a topological group with respect to the vector addition. The conventional Fourier–Laplace transform is defined as

$$ \varphi(x) \mapsto \tilde{\varphi}(\xi) = \int_G e^{i\xi \cdot x} \varphi(x) \, dx, $$

(16)

where $e^{i\xi \cdot x}$ is a character of the group $G$, i.e., a solution of the following functional equation:

$$ f(x + y) = f(x)f(y). $$

The idempotent analog of this equation is

$$ f(x + y) = f(x) \odot f(y) = f(x) + f(y), $$

so “continuous idempotent characters” are linear functionals of the form $x \mapsto \xi \cdot x = \xi_1 x_1 + \cdots + \xi_n x_n$. As a result, the transform in (16) assumes the form

$$ \varphi(x) \mapsto \tilde{\varphi}(\xi) = \int_G \xi \cdot x \odot \varphi(x) \, dx = \sup_{x \in G} (\xi \cdot x + \varphi(x)). $$

(17)

The transform in (17) is nothing but the Legendre transform (up to some notation) [10]; transforms of this kind establish the correspondence between the Lagrangian and the Hamiltonian formulations of classical mechanics.

Of course, this construction can be generalized to different classes of groups and semirings. Transformations of this type convert the generalized convolution $\star$ to the pointwise (generalized) multiplication and possess analogs of some important properties of the usual Fourier transform. For the case of semirings of Pareto sets the corresponding version of the Fourier transform reduces the multicriterial optimization problem to a family of singlecriterial problems [15].
The examples discussed in this sections can be treated as fragments of an idempotent version of the representation theory. In particular, “idempotent” representations of groups can be examined as representations of the corresponding convolution semirings (i.e. idempotent group semirings) in semimodules. To present nontrivial examples from the idempotent version of the representation theory we need some preliminary material.

6. Idempotent semimodules and linear spaces. Recall that an idempotent semigroup is an arbitrary commutative (additive) semigroup with idempotent addition. It can be treated as an ordered set with the following partial order: $x \leq y$ if and only if $x \oplus y = y$. It is easy to see that this order is well-defined and $x \oplus y = \sup\{x, y\}$. For an arbitrary subset $X$ of an idempotent semigroup, we put $\oplus X = \sup(X)$ and $\wedge X = \inf(X)$ if the corresponding right-hand sides exist. An idempotent semigroup is called $b$-complete (or boundedly complete) if any of its subsets bounded from above (including the empty subset) has the least upper bound. In particular, any $b$-complete idempotent semigroup contains zero (denoted by $0$), which coincides with $\oplus \emptyset$, where $\emptyset$ is the empty set. A homomorphism of $b$-complete idempotent semigroups is called a $b$-homomorphism if $g(\oplus X) = \oplus g(X)$ for any subset $X$ bounded from above.

An idempotent semifield is called $b$-complete if it is $b$-complete as an idempotent semigroup. In any $b$-complete semifield, the generalized distributive laws

\begin{equation}
(18) \quad a \circ (\oplus X) = \oplus(a \circ X), \quad a \circ (\wedge X) = \wedge(a \circ X)
\end{equation}

are valid; here $a$ is an element of the semifield and $X$ is a nonempty bounded subset. It is easy to see that $\mathbb{R}_{\text{max}}$ is a $b$-complete semifield.

An *idempotent semimodule* over an idempotent semiring $K$ is an idempotent semigroup $V$ endowed with a multiplication $\circ$ by elements of $K$ such that, for any $a, b \in K$ and $x, y \in V$, the usual laws

\begin{align*}
(19) \quad &a \circ (b \circ x) = (a \circ b) \circ x, \\
(20) \quad & (a \oplus b) \circ x = a \circ x \oplus b \circ x, \\
(21) \quad &a \circ (x \oplus y) = a \circ x \oplus a \circ y, \\
(22) \quad &0 \circ x = 0
\end{align*}

are valid. An idempotent semimodule over an idempotent semifield is called an *idempotent space*. An idempotent $b$-complete space $V$ over a $b$-complete semifield $K$ is called an *idempotent $b$-space* if, for any nonempty bounded subset $Q \subset K$ and any $x \in V$, the relations

\begin{equation}
(23) \quad (\oplus Q) \circ x = \oplus(Q \circ x), \quad (\wedge Q) \circ x = \wedge(Q \circ x)
\end{equation}

hold. A homomorphism $g : V \rightarrow W$ of $b$-spaces is called a $b$-homomorphism, or a $b$-linear operator (mapping), if $g(\oplus X) = \oplus g(X)$ for any bounded subset $X \subset V$. More general definitions (for spaces which may not be $b$-complete) can be found in [4]. Homomorphisms taking values in $K$ (treated as a semimodule over itself) are called linear functionals. A subset of an idempotent space is called a *subspace* if it is closed with respect to addition and multiplication by coefficients. A subspace in a $b$-space is called a *$b$-closed subspace* if it is closed with respect to summation over
arbitrary bounded (in V) subsets. This subspace has a natural structure of b-space; it is also a b-subspace in V in the sense of [4].

For an arbitrary set X and an idempotent space V over a semifield K, we use B(X, V) to denote the semimodule of all bounded mappings from X into V with pointwise operations. If V is an idempotent b-space, then B(X, V) is a b-space. A mapping f from a topological space X into an ordered set V is called upper semicontinuous if, for any b ∈ V, the set \( \{ x \in X | f(x) \geq b \} \) is closed in X, see [4]. In the case where V is the set of real numbers, this definition coincides with the usual definition of upper semicontinuity of a real function. The set of all bounded upper semicontinuous mappings from X to V is denoted by USC(X, V). If V is an idempotent b-space, then USC(X, V) is also a b-space with respect to the operations \( f \oplus g = \sup \{ f, g \} \) and \( (k \circ f)(x) = k \circ f(x) \).

7. Archimedean spaces [16]. In what follows, unless otherwise specified, the symbol K stands for a b-complete idempotent semifield and all idempotent spaces are over K.

A subset M of idempotent b-space V is called wo-closed if \( \land X \in M \) and \( \oplus X \in M \) for any linearly ordered subset \( X \subseteq M \) in V. A nondecreasing mapping \( f : V \to W \) of b-spaces is called wo-continuous if \( f(\oplus X) = \oplus f(X) \) and \( f(\land X) = \land f(X) \) for any bounded linearly ordered subset \( X \subseteq V \). Note that an arbitrary isomorphism of ordered sets is wo-continuous. It can be shown that the notions of wo-closedness and wo-continuity coincide with the closedness and continuity with respect to some T1 topology defined in an intrinsic way in terms of the order.

**Proposition 1.** Suppose that V is an idempotent b-space and W is a wo-closed subsemigroup of V. Then \( \oplus X \in W \) for any subset \( X \subseteq W \) bounded in V. In particular, each wo-closed subspace is a b-closed subspace.

An element \( x \) of an idempotent space V is called Archimedean if, for any \( y \in V \), there exists a coefficient \( \lambda \in K \) such that \( \lambda \odot x \geq y \). For an Archimedean element \( x \in V \), the formula \( x^*(y) = \land \{ k \in K | k \odot x \geq y \} \) defines a mapping \( x^* : V \to K \). If V is an idempotent b-space, then \( x^* \) is a b-linear functional and \( x^*(y) \odot x \geq y \) for any \( y \in V \) [6]. We say that an Archimedean element \( x \in V \) is wo-continuous if the functional \( x^* \) is wo-continuous, and that an idempotent b-space V is Archimedean if V contains a wo-continuous Archimedean element.

**Proposition 2.** If X is a compact topological space, then USC(X, K) is an Archimedean space and the function e identically equal to 1 is a wo-continuous Archimedean element.

Note that \( e^*(f) = \sup \{ f(x) | x \in X \} \).

**Theorem 1.** Any wo-closed subspace of an Archimedean space is an Archimedean space. Any linearly ordered (with respect to the inclusion) family of nonzero wo-closed subspaces of an Archimedean space V has a nonzero intersection.

Let V be a b-space. A subset \( W \subseteq V \) is called a \( \land \)-subspace if it is closed with respect to multiplication by scalars and taking greatest lower bounds of nonempty subsets. By this definition, any such W is a boundedly complete lattice with respect to the order inherited from V. Therefore, any \( \land \)-subspace \( W \subseteq V \) can be treated as a semimodule with respect to the inherited multiplication by scalars and the operations \( x \oplus_W y = \sup \{ x, y \} \), where sup is over W. In what follows, all \( \land \)-subspaces are considered as semimodules with respect to these operations. The
definitions immediately imply that any ∧-subspace of a b-space is a b-space. It is
easy to show that USC(X, V) is a ∧-subspace in B(X, V) for any b-space V and
any topological space X.

**Proposition 3.** If V is an Archimedean b-space and x ∈ V is a wo-continuous
Archimedean element, then any ∧-subspace W of V containing x is an Archimedean
b-space.

An arbitrary semiring K is called algebraically closed (or radicable, see, e.g. [5])
if for any element x ∈ K and any positive integer number n there exists an element
y ∈ K such that y^n = x. It is easy to show that R_max is a b-complete algebraically
closed semifield.

**Theorem 2.** An idempotent b-space V over an algebraically closed b-complete
semifield K is Archimedean if and only if there exists a space of the form USC(X, K),
where X is a compact topological space, such that V is isomorphic to its ∧-subspace
containing constants.

8. **Representations of groups in Archimedean spaces.** Suppose that V is an
Archimedean idempotent b-space over an algebraically closed b-complete semifield
(e.g., over R_max). By End(V) denote the set of all b-linear operators V → V. This
set is an idempotent semigroup with respect to the pointwise sum and it is a
b-space over K with respect to the standard multiplication by coefficients from K.
The usual multiplication (composition) of maps turns End(V) into an idempotent
semiring (and a b-complete semialgebra over K).

Let G be an abstract group. A linear representation π : G → End(V) of G in
an Archimedean space V is a homomorphism from G to the group of all invertible
elements in End(V) (with respect to the composition of operators). The representa-
tion π is (topologically) irreducible if the space V has no nontrivial wo-closed
π(G)-invariant subspaces.

**Theorem 3.** Every linear representation of a group G in an Archimedean idempo-
tent space V has a nontrivial irreducible subrepresentation in a wo-closed subspace
of V.

**Theorem 4.** Let π be a linear representation of a group G in an Archimedean
idempotent space V and for a nonzero element x ∈ V the orbit π(G)x is bounded.
Set a = ⊕(π(G)x). Then π(g)a = a for each g ∈ G.

We shall say that a representation π of G in V has a (nonzero) joint eigenvector
a ∈ V if π(g)a = λ(g)a for all g ∈ G, where λ(g) ∈ K.

**Corollary 1.** Every linear representation of a finite group in an Archimedean
idempotent space has a joint eigenvector with a unique eigenvalue 1.

**Corollary 2.** Every upper semicontinuous linear representation of a compact group
in an Archimedean idempotent space has a joint eigenvector with a unique eigen-
value 1.

9. **An Engel type theorem for representations of nilpotent groups.** Let
G be an abstract group. For elements a, b ∈ G we set [a, b] = a^{-1}b^{-1}ab; for
subsets X and Y in G we denote by [X, Y] a subgroup in G generated by the set
\{[x, y]|x ∈ X, y ∈ Y\}; we set Γ_i(G) = [G, Γ_{i-1}(G)], i = 1, 2, 3, . . . .
Recall that an abstract group G is nilpotent if and only if there exists a positive
integer number $n$ such that $\Gamma_n(G) = \{e\}$, where $e$ is the neutral element (identity) of $G$.

**Theorem 5.** Every linear representation of a nilpotent abstract group in an Archimedean idempotent space over an algebraically closed $b$-complete semifield (e.g., over $\mathbb{R}_{\text{max}}$) has a joint eigenvector.

**Corollary 3.** Every collection of commuting invertible $b$-linear operators in an Archimedean idempotent space has a joint eigenvector.

**Corollary 4.** Every invertible $b$-linear operator in an arbitrary Archimedean idempotent space over an algebraically closed $b$-complete semifield has an eigenvector.

**Remark.** There is no idempotent version of the well known Lie theorem for representations of abstract solvable groups in idempotent spaces. Moreover, there exists an irreducible linear representation of a solvable group in the idempotent space $V = \mathbb{R}_{\text{max}} \times \mathbb{R}_{\text{max}}$ over $\mathbb{R}_{\text{max}}$.

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