Optimized $\delta$ - Expansion and Triviality or Non - Triviality of Field Theories

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Abstract: We use a very simple version of the optimized (linear) $\delta$ - expansion by scaling the free part of the Lagrangian with a variational parameter. This method is well suited to calculate the renormalized coupling constant in terms of the free one and the cutoff. One never has to calculate any new Feynman graphs but simply can modify existing results from the literature. We find that $\Phi^4_4$ -theory as well as QED are free in the limit where the cutoff goes to infinity. In contrast to this, the structure of Yang-Mills theories enforces a special choice of the Lagrangian of the $\delta$ - expansion. Together with the change in the sign of the $\beta$ - function, this leads to a different behavior and allows Yang-Mills theory to become non trivial.

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1 Introduction

The “optimized $\delta$-expansion” (for the older literature see Stevenson [1] and references therein), also called “linear $\delta$-expansion”, is a powerful method which combines the merits of perturbation theory with those of variational approaches. The underlying idea is simple. Generically, the Lagrangian is split into a free and an interacting part in such a way that an arbitrary parameter $\lambda$ (or more) is artificially introduced. The interacting part is multiplied by a factor $\delta$ which serves as expansion parameter and is put equal to one at the end. The exact solution should be independent of the parameter $\lambda$ while any approximate solution will depend on it. The idea, often called “Stevenson’s principle of minimal sensitivity” [2], is that the approximate solution should depend as little upon the parameter as possible. This means that $\lambda$ should be chosen such that the quantity to be calculated has an extremum. In this way the result becomes non-perturbative because $\lambda$ becomes a non-linear function of the coupling constant. In every order of perturbation theory the parameter has to be calculated again and usually goes to infinity with growing order.

The method has been applied with great success to simple cases like the zero dimensional and one dimensional (quantum mechanics) anharmonic oscillator [3], where rigorous proofs for the (rapid) convergence exist. There are also interesting applications to the calculation of the effective potential and the question of spontaneous symmetry breaking [4] - [7]. In this context the method is usually called “Gaussian effective potential” (GEP), or, in higher orders, “post Gaussian effective potential” (PGEP). We also mention applications in lattice theory [8].

In [4] - [7] the mass parameter of the free Lagrangian was treated as variational parameter. In our approach we will fix it at the physical mass. Instead we scale the free part of the Lagrangian with a factor $\zeta$. That’s all! Due to the “benevolent paradox” [9] of the linear $\delta$-expansion we don’t need to calculate any Feynman graphs. We simply can use existing calculations and modify the results. We find that this method is not only quite simple but also very useful in order to calculate the renormalized coupling constant in terms of the bare one and the cutoff. All our results are already obtained in one-loop order.

For $\Phi^4_4$ - theory, treated in sect. 2, and QED in sect. 3, one easily finds that these theories are free, i.e. that the renormalized coupling constant goes to zero when the cutoff goes to infinity, irrespective of the behavior of the bare coupling. In the case of $\Phi^4_4$ - theory one can explicitly show that the conclusion holds in any (even) order of perturbation theory in $\delta$. In QED one can use a realistic cutoff and give an upper bound for the fine structure constant. Yang-Mills theories are treated in sect. 4. Here two important changes happen. Firstly, the change of the sign of the $\beta$-function is crucial. Secondly, gauge invariance enforces special conditions on the splitting of the Lagrangian. Both together implies that the theory can become non-trivial. In all cases generalization to higher orders is straightforward in principle.

Our methods are not rigorous but they give transparent analytical expressions in a simple way and lead to an understanding of the relevant features of the various theories.
2 \( \Phi^4 \) - Theory

We split the Lagrangian

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{m_0^2}{2} \Phi^2 - \frac{g_0}{4!} \Phi^4
\]

(2.1)

by introducing an artificial parameter \( \zeta \) which scales the free Lagrangian. The expansion parameter is called \( \delta \) as usual. The parameter \( M \) will not be treated as a variational parameter but will be fixed at the physical mass. This is very convenient and will lead to the results in a simple way.

\[
\mathcal{L} = \mathcal{L}_0 + \delta \mathcal{L}_I
\]

(2.2 a)

with

\[
\mathcal{L}_0 = \frac{1}{\zeta} \left[ \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{M^2}{2} \Phi^2 \right],
\]

(2.2 b)

\[
\mathcal{L}_I = (1 - 1/\zeta) \left[ \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{M^2}{2} \Phi^2 \right] - \frac{m_0^2 - M^2}{2} \Phi^2 - \frac{g_0}{4!} \Phi^4.
\]

(2.2 c)

For \( \delta = 1 \) the original Lagrangian is recovered.

The Feynman rules are directly read off from (2.2). The essential modification is that the propagator acquires a factor \( \zeta \) and that we obtain insertions containing the free Lagrangian. They will be denoted by a thick dot in order to distinguish them from the mass insertions which, as usual, are denoted by a cross:

**Propagator:**

\[
\frac{i\zeta}{p^2 - M^2 + i\epsilon}
\]

(2.3 a)

**Free Lagrangian insertion:**

\[
i\delta(1 - 1/\zeta)(p^2 - M^2)
\]

(2.3 b)

**Mass insertion:**

\[-i\delta(m_0^2 - M^2)
\]

(2.3 c)

**Vertex:**

\[-i\delta g_0.
\]

(2.3 d)

It is easy to see how the insertions of the free Lagrangian work. If we combine an insertion with one adjacent propagator, the \( p^2 - M^2 \) cancels and one is left with a factor \( \delta(1 - \zeta) \). Summing up the geometrical series consisting of the bare propagator together with 1, 2, \( \cdots \), \( n \) insertions and putting \( \delta = 1 \) gives

\[
\frac{i\zeta}{p^2 - M^2 + i\epsilon} \sum_{\nu=0}^{n} (1 - \zeta)\nu = \frac{i}{p^2 - M^2 + i\epsilon} [1 - (1 - \zeta)^{n+1}].
\]

(2.4)

Clearly the extremum is at the “natural” value \( \zeta = 1 \). For \( 0 < \zeta < 2 \) the series converges and the limit is independent of \( \zeta \) as it should be. In a theory with interactions there are, of course, additional contributions which will shift the extremum away from 1.
Calculations with the Lagrangian (2.2) are easily performed by just modifying the expressions of usual perturbation theory. Although our essential results are already obtained at the one loop level we present the general procedure. Let $\Gamma_{2N}(p_i)$ be the connected one particle irreducible Green function with $2N$ external legs, the propagators for the external legs are not included. In usual perturbation theory with respect to the bare coupling constant $g_0$, one has an expansion of the form

$$\Gamma_{2N}(p_i) = \sum_{V=N-1}^{\infty} g_0^V \Gamma^{(V)}_{2N}(p_i).$$

In our approach the Green function will depend on $\zeta$ and $\delta$ in any finite order, therefore we denote it by $\Gamma_{2N}(p_i, \zeta, \delta)$. If we expand it with respect to $\delta$, every internal line gets an extra factor $\zeta$. There are $I = 2V - N$ internal lines in the graphs with $V$ vertices which contribute to $\Gamma^{(V)}_{2N}(p_i)$. Furthermore, we have to take into account the insertions of the free Lagrangian which give a factor $\delta(1 - \zeta)$ compared to the graph without the insertion. The number of possibilities to place $J$ insertions on $I$ internal lines (compare the well known analogous problem of Bose statistics to put $J$ indistinguishable particles into $I$ boxes) is

$$\binom{I + J - 1}{J} = \binom{2V - N + J - 1}{J}.$$ 

(2.6)

For the expansion of $\Gamma_{2N}(p_i, \zeta, \delta)$ with respect to $\delta$ up to order $n$ we therefore obtain, if we substitute $V = \nu - J$

$$\Gamma_{2N}(p_i, \zeta, \delta) = \sum_{\nu=N-1}^{n} \delta^{\nu} \sum_{J=0}^{J_{\text{max}}} \binom{2\nu - N - 1 - J}{J} \zeta^{2\nu - N - 2J}(1 - \zeta)^J g_0^{\nu - J} \Gamma^{(\nu - J)}_{2N}(p_i),$$

(2.7)

with $J_{\text{max}} = \text{Max}\{\nu - 1 - [N/2], 0\}$. Consider next the two point function $G_2(p^2, \zeta, \delta)$ of all one particle irreducible contributions to the propagator, which will be needed to calculate the wave function renormalization $Z_\Phi(\zeta, \delta)$. The graphs which contribute to $G_2(p^2, \zeta, \delta)$ up to order $\delta^2$ are shown in fig. 1. In general one has the expansion

$$G_2(p^2, \zeta, \delta) = \frac{i\zeta}{p^2 - M^2 + i\epsilon} \left( 1 + \delta (1 - \zeta) + \frac{\zeta \Sigma(p^2, \zeta, \delta)}{p^2 - M^2 + i\epsilon} \right)$$

(2.8. a)

with

$$\Sigma(p^2, \zeta, \delta) \equiv \Gamma_2(p^2, \zeta, \delta) = \sum_{\nu=1}^{n} \delta^{\nu} \sum_{J=0}^{J_{\text{max}}} \binom{2\nu - 2 - J}{J} \zeta^{2\nu - 1 - 2J}(1 - \zeta)^J g_0^{\nu - J} \Gamma^{(\nu - J)}_{2N}(p_i).$$

(2.8. b)

The propagator $D(p^2, \zeta, \delta)$ is obtained by summing up the geometrical series of all one particle irreducible contribution contained in $G(p^2, \zeta, \delta)$. Putting

$$\Sigma(p^2, \zeta, \delta) = (p^2 - M^2) \Sigma'(M^2, \zeta, \delta) + \sigma(p^2, \zeta, \delta)$$

(2.9)

one finds
\[ D(p^2, \zeta, \delta) = \frac{i\zeta}{(p^2 - M^2 + i\epsilon)(1 - \delta(1 - \zeta) - \zeta \Sigma'(M^2, \zeta, \delta)) - \zeta \sigma(p^2, \zeta, \delta)} = \frac{iZ_\Phi(\zeta, \delta)}{p^2 - M^2 + i\epsilon - \Sigma_{\text{ren}}(p^2, \zeta, \delta)}, \]

\[(2.10)\]

with

\[ Z_\Phi(\zeta, \delta) = \frac{\zeta}{1 - \delta(1 - \zeta) - \zeta \Sigma'(M^2, \zeta, \delta)} \]

and \( \Sigma_{\text{ren}}(p^2, \zeta, \delta) = Z_\Phi(\zeta, \delta)\sigma(p^2, \zeta, \delta). \)  \((2.11)\)

Besides the propagator and the wave function renormalization constant we need the vertex function \( \Gamma(p_i, \zeta, \delta) \) (normalized such that the expansion starts with 1). This, in turn, determines the vertex renormalization constant \( Z_V(\zeta, \delta) \) through

\[ \Gamma(p_i, \zeta, \delta) \to 1/Z_V(\zeta, \delta). \]  \((2.12)\)

Since we are only interested in the behavior for cutoff to infinity, the special choice of the momenta for the renormalization prescription of \( \Gamma \) is unessential.

In ordinary perturbation theory with respect to the bare coupling constant \( g_0 \), one has the formal expansions

\[ Z_\Phi = 1 + \sum_{\nu = 1}^{\infty} g_0^\nu Z_\Phi^{(\nu)}, \quad \Gamma(p_i) = 1 + \sum_{\nu = 1}^{\infty} g_0^\nu \Gamma^{(\nu)}(p_i), \quad 1/Z_V = 1 + \sum_{\nu = 1}^{\infty} g_0^\nu \bar{\Gamma}^{(\nu)}, \]  \((2.13)\)

where \( \Gamma^{(\nu)} \) are the coefficients \( \Gamma^{(\nu)}(p_i) \) taken at the external momenta where the renormalization prescription is imposed.

The corresponding expansion for \( Z_\Phi(\zeta, \delta) \) with respect to \( \delta \) is obtained from (2.11), (2.9), (2.8). The quadratically divergent tadpole in fig. 1 is independent of the external momentum and only contributes to the mass renormalization. If \( M \) is chosen as the physical mass, it is canceled by the mass counterterm. Therefore we may omit all tadpole contributions here and in the following. From the graphs in fig. 1, together with the foregoing considerations we get

\[ Z_\Phi(\zeta, \delta) = \zeta \left\{ 1 + \delta[1 - \zeta] + \delta^2[(1 - \zeta)^2 + g_0^2 \zeta^4 Z_\Phi^{(2)}] + O(\delta^3) \right\}. \]  \((2.14)\)

In general, a term \( \delta g_0 \zeta^2 Z_\Phi^{(1)} \) would also show up in the curly bracket of (2.14) which, however, vanishes in \( \Phi^4 \) - theory.

For the vertex functions the graphs of fig. 2 contribute. This results in

\[ 1/Z_V(\zeta, \delta) = 1 + \delta g_0 \zeta^2 \bar{\Gamma}^{(1)} + \delta^2[2g_0 \zeta^2 (1 - \zeta) \bar{\Gamma}^{(1)} + g_0^2 \zeta^4 \bar{\Gamma}^{(2)}] + O(\delta^3). \]  \((2.15)\)

The relation between bare and renormalized coupling constant is:

\[ g = \delta g_0 Z_\Phi^2(\zeta, \delta)/Z_V(\zeta, \delta). \]  \((2.16)\)
We need this relation only up to order $\delta^2$ at the moment:

$$g = \delta g_0 \zeta^2 \left\{ 1 + \delta [2(1 - \zeta) - 3g_0\zeta^2C/2] + O(\delta^2) \right\}.$$  \hspace{1cm} (2.17)

The constant $C$ is defined by

$$\bar{\Gamma}^{(1)} = -3C/2, \quad \text{with} \quad C \to \frac{b}{3} \ln \frac{\Lambda^2}{M^2} = \frac{1}{(4\pi)^2} \ln \frac{\Lambda^2}{M^2} \quad \text{for} \quad \Lambda \to \infty,$$  \hspace{1cm} (2.18)

and $b$ is the first coefficient in the usual expansion of the $\beta$-function, $\beta(g) = bg^2 + \cdots$. Here and in the following we use the results of well known one loop calculations which may be found e.g. in [10].

Before going on we have to clarify an important conceptional point. In quantum mechanics or any other theory without infinities one would prescribe $g_0$ and then calculate $g$ as a power series in $\delta$. In the usual treatment of quantum field theory, however, one proceeds the other way round. One fixes $g$ at its physical value and calculates $g_0$ as a power series with coefficients that are divergent in the limit $\Lambda \to \infty$. We stress that the latter method is not applicable here! The reason is that the expansion of $g$ starts with a term proportional to $\delta$. If we invert (2.17), we obtain

$$\delta g_0 \zeta^2 = \{1 - 2\delta(1 - \zeta) + 3g_0\zeta^2C/2 + \text{rest} \}.$$  \hspace{1cm} (2.19)

But now the rest contains an infinity of terms of order $\delta^0$, because $\delta^\nu g_0^{\nu} \zeta^{2\nu} = g + \cdots$ starts with order 1, not with order $\delta^\nu$. Therefore it is not allowed to truncate the series. If we do it nevertheless, this means that we return to naive perturbation theory in $g$. In the latter case one easily finds that the renormalized vertex function $\Gamma_{\text{ren}}(p_i) = Z_V \Gamma(p_i)$ as well as other renormalized quantities like the self energy, become independent of $\zeta$. This also happens in higher orders. The renormalization procedure is powerful enough to remove our manipulations with the Lagrangian!

We will see now, however, that the linear $\delta$-expansion in our special formulation is extremely useful and simple in order to obtain information about the renormalized coupling constant in terms of the bare coupling and the cutoff. We return to (2.17), truncate after the order $\delta^2$, and put $\delta = 1$, thus ending up with

$$g = g_0 \{3\zeta^2 - 2\zeta^3 - 3g_0\zeta^4/2 \}.$$  \hspace{1cm} (2.20)

The equation for the extremum in $\zeta$ (dropping the unacceptable solution $\zeta = 0$) reads

$$1 - \zeta = g_0 C \zeta^2.$$  \hspace{1cm} (2.21)

Eliminating $g_0 C$ from this, (2.20) becomes

$$g = \frac{1}{2C}(1 - \zeta)(3 - \zeta).$$  \hspace{1cm} (2.22)

Let us first assume, as usual, that $g_0 > 0$. Then (2.21) has just one positive solution for $\zeta$. The solution lies between 0 and 1, in this interval $(1 - \zeta)(3 - \zeta) \leq 3$ and thus
\[ g \leq \frac{3}{2C} \rightarrow \frac{3(4\pi)^2}{2 \ln(\Lambda^2/M^2)}. \]  

(2.23)

Therefore, in the limit \( \Lambda \rightarrow \infty \) the renormalized coupling constant \( g \) will necessarily converge to zero, irrespective whether \( g_0 \) becomes constant, goes to zero, or diverges.

In the literature there are suggestions for possible non-trivial and stable \( \Phi^4 \) theories, the “precarious” theory [11], [1], [5], [6] (\( g_0 < 0 \) infinitesimal, \( g < 0 \) finite) and the “autonomous” theory [12], [5], [7] (\( g_0 > 0 \) infinitesimal, \( g > 0 \) finite, infinite wave function renormalization). As we have just seen, an autonomous theory cannot arise in our approach. Let us look for the possibility of a precarious theory by putting \( g_0 \equiv -\gamma < 0 \). In this case (2.21) has two solutions with \( \zeta > 1 \). A finite value of \( g \) in the limit \( C \rightarrow \infty \) can now be obtained if \( \gamma C \rightarrow 0, \zeta \rightarrow 1/\gamma C \rightarrow \infty \). Therefore the negative bare coupling constant must become infinitesimally small. For \( g \) one then would get \( g \rightarrow 1/2\gamma^2C^3 \) which is positive. Depending on how fast \( \gamma \) vanishes in the limit \( \Lambda \rightarrow \infty \), this may converge to 0, \( \infty \), or to a finite value. We consider this possibility, however, as unacceptable. According to the general philosophy of the principle of minimal sensitivity the second extremum at \( \zeta \approx 1 + \gamma C \) should be preferred because the second derivative is (drastically!) smaller. (This would also, and even stronger, be the case if we would have chosen \( 1/\zeta \) as variation variable.) This second solution would lead to a negative \( g \approx -\gamma \rightarrow 0 \). Therefore these exotic possibilities do not show up in our approach.

It is instructive to look at the perturbative features and the analyticity properties contained in (2.20), (2.21). One may expand the solution (2.21) for the extremum

\[ \zeta = \left( \sqrt{1 + 4g_0C} - 1 \right) / 2g_0C = 1 - g_0C + 2(g_0C)^2 - 5(g_0C)^3 + 14(g_0C)^4 + \cdots \]  

(2.24)

and introduce this into (2.20) to obtain

\[ g = g_0 \left\{ 1 - 3g_0C/2 + 3(g_0C)^2 - 7(g_0C)^3 + 18(g_0C)^4 + \cdots \right\}. \]  

(2.25)

The linear term in \( g_0C \) coincides with that of naive perturbation theory as it should, because we expand at the extremum.

Obviously (2.24) has a branch cut at \( g_0C = -1/4 \), corresponding to \( \zeta = 2 \), which determines the radius of convergence of (2.25). We expect that in higher orders of the optimized \( \delta \)-expansion the branch cut approaches zero as it happens in simple models. So the method shows how the non-analytic behavior at \( g_0 = 0 \) and the divergence of ordinary perturbation theory for any \( g_0 \) arises.

It is surprising that all the previous conclusions can formally be extended to arbitrary orders \( n \) of the optimized \( \delta \)-expansion. Instead of (2.17) one obtains a more complicated formula of similar structure, which again has a factor \( \delta g_0\zeta^2 \) in front. What we need is information about the behavior of the various contributions in the limit \( \Lambda \rightarrow \infty \) and about the presence of an extremum for positive \( \zeta \).

If we expand the factors \( Z^2_\phi(\zeta, \delta) \) and \( 1/Z_V(\zeta, \delta) \) in (2.16) with respect to \( \delta \) we obtain series involving the coefficients \( Z^{(\nu)}_\phi \) and \( \bar{\Gamma}^{(\nu)} \) in (2.13), modified by the changes performed in the Lagrangian. Because we have chosen \( M \) as the physical mass, all quadratically
divergent tadpole contributions cancel and only the logarithmic divergences survive in
the coefficients. For $\Lambda \to \infty$ one has

$$Z^{(\nu)}_\Phi \sim z_\nu (\ln \frac{\Lambda^2}{M^2})^{\nu-1} \quad \text{(for } \nu \geq 2), \quad (2.26)$$

$$\bar{\Gamma}^{(\nu)} \sim (-1)^\nu C_\nu (\ln \frac{\Lambda^2}{M^2})^\nu \quad \text{with } C_\nu > 0. \quad (2.27)$$

The important point is that we know the signs of the coefficients $C_\nu$. These signs are
easily obtained from inspecting the various factors $\pm i$ stemming from the propagators,
vertices, and rotations to euclidean space in the integration variables, for a graph con-
tributing to $\bar{\Gamma}^{(\nu)}$. The remaining integrand is then positive definite. A specific cutoff
prescription for higher order graphs is e.g. the replacement of the euclidean propagator
by $\exp[-(P^2 + M^2)/\Lambda^2]/(P^2 + M^2)$. There is no simple statement concerning the signs
of the coefficients $z_\nu$, but fortunately these signs are unimportant. In our approach we
have the same modifications as in (2.7): Every propagator gets an additional factor $\zeta$,
this has the consequence that $Z^{(\nu)}_\Phi$ as well as $\bar{\Gamma}^{(\nu)}$ always appear together with a factor
$\delta^\nu \zeta^{2\nu}$. The insertions of the free Lagrangian give factors of $\delta(1 - \zeta)$ compared to the
 corresponding graph without the insertion.

The clue for the existence of an extremum is the sign of the term with the highest
power of $\zeta$ in the expansion of $g$ in (2.16). From the previous remarks it is clear that the highest power of $\zeta$ contains no insertions. Furthermore, the coefficients of $\bar{\Gamma}^{(\nu)}$ have one
power of $\ln(\Lambda^2/M^2)$ more than those of $Z^{(\nu)}_\Phi$. To find the leading term in the coefficient
for $\Lambda \to \infty$, we therefore have to take the lowest order of $Z^2_\Phi(\zeta, \delta)$ and the highest order
of $1/Z_V(\zeta, \delta)$. Therefore the term with the highest power of $\zeta$ in the expansion of (2.16)
reads

$$\delta g_0 \zeta^n (\ln(\Lambda^2/M^2))^{n-1}. \quad (2.28)$$

It has a definite sign and is negative for $g_0 > 0$ and even order $n$. This term with the
highest power of $\zeta$ is also the one with the highest power of $\ln(\Lambda^2/M^2)$.

The term with the lowest power of $\zeta$ is, correspondingly, obtained if we take the
lowest order in $Z^2_\Phi(\zeta, \delta)$ and $1/Z_V(\zeta, \delta)$, and only consider the insertions. This gives

$$1/Z_V(\zeta, \delta) \to 1 \quad \text{and} \quad Z^2_\Phi(\zeta, \delta) \to \frac{\zeta^2}{[1 - \delta(1 - \zeta)]^2} \to \frac{\zeta^2}{[1 - \delta]^2} \to \zeta^2 \sum_{\nu=0}^{n-1} (\nu + 1)\delta^\nu. \quad (2.29)$$

Obviously this factor is positive. Together with the previous result this guarantees that
for even $n$ there will always be at least one maximum at positive $\zeta$, i.e. the principle of
minimal sensitivity is applicable.

It is now easy to see what happens in the limit $\Lambda \to \infty$. In the case that $g_0 \ln(\Lambda^2/M^2)$
stays finite (or goes to zero) there will be a maximum at some finite $\zeta$. In this case $g_0$
necessarily goes to zero, therefore the extra factor $g_0$ in front as in (2.17) will imply that
$g$ vanishes for $\Lambda \to \infty$. Let us next assume that $g_0 \ln(\Lambda^2/M^2)$ diverges for $\Lambda \to \infty$. Then
$\zeta$ at the maximum has to go to zero, because otherwise the highest order term would
dominate all the other ones. Therefore the expression may be simplified considerably.
In any term we only need to consider the highest power of \( \ln(\Lambda^2/M^2) \) and the lowest power of \( \zeta \). Insertions therefore only give powers of \( \delta \). We end up with an expression of the form

\[
g = \delta g_0 \zeta^2 \left\{ \sum_{\nu=0}^{n-1} (\nu + 1)\delta^\nu + \sum_{\nu=1}^{n-1} \delta^\nu \sum_{j=0}^{\nu-1} S_{\nu,j} [\zeta^2 g_0 \ln(\Lambda^2/M^2)]^{\nu-j} \right\}.
\]

(2.30)

The \( S_{\nu,j} \) are uninteresting numerical coefficients, the important point is that we know the highest one: \( S_{n-1,0} = (-1)^{n-1} C_{n-1} < 0 \) for \( n \) even.

The rest is trivial. Putting \( \zeta^2 g_0 \ln(\Lambda^2/M^2) = x \) and setting \( \delta = 1 \), (2.30) becomes

\[
g = \frac{x}{\ln(\Lambda^2/M^2)} \left\{ \frac{n(n + 1)}{2} + \sum_{\nu=1}^{n-1} \sum_{j=0}^{\nu-1} S_{\nu,j} x^{\nu-j} \right\}.
\]

(2.31)

For even \( n \) there is a maximum for some finite \( x \), if this is chosen one finds that \( g \to 0 \) for \( \Lambda \to \infty \), i.e. the theory is trivial.

The case of a precarious theory with \( g_0 < 0 \) can be excluded in the same way if one chooses \( n \) odd.

We cannot make a general statement whether there is an extremum for \( g_0 > 0 \) and \( n \) odd or vice versa; for \( n = 3 \) there is none in the limit \( g_0 \ln(\Lambda^2/M^2) \to \infty \). This does not matter at all because we always may choose a convenient subset of values of \( n \). It is a well known feature of the optimized \( \delta \) - expansion, which shows up already in the completely understood toy model of a “zero dimensional \( \Phi^4 \) - partition function” \[3\], that one has to restrict to even or odd \( n \), respectively.

Of course we don’t claim that the previous considerations provide a proof that \( \Phi^4 \) - theory is free. (See \[3\] for this topic.) However, they certainly give some new and alternative insight into the problem and add further evidence for the triviality of this theory.
The situation is very similar to that in $\Phi^4$-theory, therefore we can concentrate on the modifications. For simplicity we work in the Feynman gauge. The free Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial - m_0)\psi - \frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - e_0 \bar{\psi} A \psi$$

is split in the following way:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I(\delta), \quad \text{with}$$

$$\mathcal{L}_0 = \frac{1}{\zeta_2} \bar{\psi}(i\partial - M)\psi - \frac{1}{2\zeta_3} \partial_\mu A_\nu \partial^\mu A^\nu,$$

$$\mathcal{L}_I(\delta) = \delta^2 [(1 - \frac{1}{\zeta_2}) \bar{\psi}(i\partial - M)\psi - \frac{1}{2} (1 - \frac{1}{\zeta_3}) \partial_\mu A_\nu \partial^\mu A^\nu + (M - m_0)\bar{\psi}\psi]$$

$$- \delta e_0 \bar{\psi} A \psi.$$

Some comments are appropriate here.

We have now introduced two scaling parameters, $\zeta_2$ for the free electron Lagrangian, and $\zeta_3$ for the free photon Lagrangian. $M$ is chosen as the physical mass of the electron. The expansion parameter is again $\delta$ which is put equal to 1 at the end. Only for $\delta = 1$ the Lagrangians have to coincide. This freedom was used to choose different powers of $\delta$ in the various terms of $\mathcal{L}_I(\delta)$, namely $\delta$ for the electron-photon vertex but $\delta^2$ for the free Lagrangian insertions and the mass insertions. The reason for doing this is that in the familiar perturbative treatment of QED the contributions to the mass counter term $\delta m$ as well as to the renormalization constants $Z_1, Z_2, Z_3$ always arise in connection with loop graphs, which in lowest order are proportional to $\alpha_0 = e_0^2/4\pi$. The Lagrangian (3.2) has the corresponding structure, it is not only invariant under the transformation $e_0 \rightarrow -e_0$, $A_\mu \rightarrow -A_\mu$, but also under $\delta \rightarrow -\delta$, $A_\mu \rightarrow -A_\mu$. This guarantees that at the end expansions go with $\alpha_0$, not with $e_0$ itself.

The Feynman rules which are derived from (3.2) read:

**Electron propagator:**

$$\frac{i\zeta_2}{\hat{p} - M + i\epsilon}$$

(3.3. a)

**Photon propagator:**

$$- \frac{i\zeta_3 g_{\mu\nu}}{k^2 + i\epsilon}$$

(3.3. b)

**Electron free Lagrangian insertion:**

$$i\delta^2 (1 - 1/\zeta_2)(\hat{p} - M)$$

(3.3. c)

**Photon free Lagrangian insertion:**

$$- i\delta^2 (1 - 1/\zeta_3) k^2 g_{\mu\nu}$$

(3.3. d)
Electron mass insertion: \[ i\delta^2(M - m_0) \] \hspace{0.5cm} (3.3. e)

Electron photon vertex: \[ -i\delta\epsilon_0\gamma^\mu. \] \hspace{0.5cm} (3.3. f)

As in the previous section, Green functions can be calculated by using the expansion coefficients of ordinary perturbation theory. Because it is more convenient to write the expansion in \( \alpha_0 \) instead of \( \epsilon_0 \) we introduce the following notation. Let \( 2N_e \) denote the number of external electron lines, \( 2N_\gamma + \sigma \) with \( \sigma = 0 \) or \( 1 \) the external photon lines, and \( 2V + \sigma \) the vertices. The usual perturbation expansion of the vertex function then reads

\[
\Gamma_{2N_e,2N_\gamma+\sigma}(p_\ell) = \epsilon_0^\sigma \sum_{V=N_e+N_\gamma-1}^\infty \alpha_0^V \Gamma^{(V)}_{2N_e,2N_\gamma+\sigma}(p_\ell). \quad (3.4)
\]

The graphs which contribute to \( \Gamma^{(V)}_{2N_e,2N_\gamma+\sigma} \) have \( I_e = 2V + \sigma - N_e \) internal electron lines and \( I_\gamma = V - N_\gamma \) internal photon lines. Considering the additional factors in the propagators and the insertions in analogy to (2.7), we obtain for the expansion up to order \( 2n + \sigma \)

\[
\Gamma_{2N_e,2N_\gamma+\sigma}(p_\ell, \zeta_2, \zeta_3, \delta) = \]

\[
(\delta\epsilon_0)^\sigma \sum_{\nu=N_e+N_\gamma-1}^n \frac{\delta^{2\nu}}{J_e,J_\gamma=0} \left( \frac{2\nu + \sigma - N_e - 1 - J_e - 2J_\gamma}{J_e} \right) \left( \frac{\nu - N_\gamma - 1 - J_\gamma}{J_\gamma} \right) \times \]

\[
\times \zeta_2^{2\nu - \sigma - 2J_e - 2J_\gamma} (1 - \zeta_2)^{J_e} \zeta_3^{\nu - N_\gamma - J_e - J_\gamma} (1 - \zeta_3)^{J_\gamma} \alpha_0^{\nu - J_e - J_\gamma} \Gamma^{(\nu - J_e - J_\gamma)}_{2N_e,2N_\gamma+\sigma}(p_\ell). \quad (3.5)
\]

We keep as close to the usual notation as possible. The vertex renormalization, electron wave function renormalization, and photon renormalization constants are now functions of \( \zeta_2, \zeta_3, \delta \) and are denoted by \( Z_k(\zeta_2, \zeta_3, \delta) \). For \( \zeta_2 = \zeta_3 = \delta = 1 \) they go over into \( Z_1, \ Z_2, \ Z_3 \). The formal expansion of the latter in usual perturbation theory reads

\[
Z_k = 1 + \sum_{\nu=1}^\infty Z_k^{(\nu)} \alpha_0^\nu \quad \text{for} \quad k = 1, 2, 3. \quad (3.6)
\]

The Ward identity guarantees that \( Z_1^{(\nu)} = Z_2^{(\nu)} \) in any order.

The quantities \( Z_k(\zeta_2, \zeta_3, \delta) \) can be expanded into series in \( \delta \). The calculation of order \( \delta^2 \) is completely parallel to the one in the last section, therefore we just give the result:

\[
Z_1(\zeta_2, \zeta_3, \delta) = 1 + \delta^2 \zeta_2^2 \zeta_3 \alpha_0 Z_1^{(1)}, \quad (3.7)
\]

\[
Z_2(\zeta_2, \zeta_3, \delta) = \zeta_2 [1 + \delta^2 (1 - \zeta_2 + \zeta_2^2 \zeta_3 \alpha_0 Z_2^{(1)})], \quad (3.8)
\]

\[
Z_3(\zeta_2, \zeta_3, \delta) = \zeta_3 [1 + \delta^2 (1 - \zeta_3 + \zeta_2^2 \zeta_3 \alpha_0 Z_3^{(1)})]. \quad (3.9)
\]

Clearly \( Z_1(\zeta_2, \zeta_3, \delta) \neq Z_2(\zeta_2, \zeta_3, \delta) \), i.e. the Ward identity does not hold in its usual form! The reason for this is easily traced back to the extra contribution of the free
electron Lagrangian insertion in fig. 3 which contributes to the self energy and therefore to \( Z_2(\zeta_2, \zeta_3, \delta) \), but does not appear in the vertex function and in \( Z_1(\zeta_2, \zeta_3, \delta) \). It is, however, easy to derive the modified Ward identity from the familiar one. The result is

\[
\frac{\zeta_2}{Z_2(\zeta_2, \zeta_3, \delta)} + \delta^2 (1 - \zeta_2) = \frac{1}{Z_1(\zeta_2, \zeta_3, \delta)}. \tag{3.10}
\]

The renormalized fine structure constant \( \alpha \) is obtained from the relation

\[
\alpha = \delta^2 \alpha_0 Z_2^2(\zeta_2, \zeta_3, \delta) Z_3(\zeta_2, \zeta_3, \delta)/Z_1^2(\zeta_2, \zeta_3, \delta). \tag{3.11}
\]

In order \( \delta^4 \) this gives

\[
\alpha = \delta^2 \alpha_0 \zeta_2^2 \{ 1 + \delta^2 [2(1 - \zeta_2) + 1 - \zeta_3 + \zeta_2^2 \zeta_3 \alpha_0 Z_3^{(1)}] \}. \tag{3.12}
\]

Originally, instead of \( Z_3^{(1)} \) one had \( Z_3^{(1)} + 2 Z_2^{(1)} - 2 Z_1^{(1)} \), the last two terms cancel due to the Ward identity. This cancellation also happens in higher orders. Therefore, though the Ward identity is modified in our approach, the “old” Ward identity for the coefficients, \( Z_1^{(\nu)} = Z_2^{(\nu)} \), still does its job.

We have to determine the extremum of (3.12) for \( \delta = 1 \) with respect to the two variables \( \zeta_2, \zeta_3 \). One easily sees that

\[
\frac{\partial \alpha}{\partial \zeta_2} = 2 \frac{\partial \alpha}{\partial \zeta_3} \quad \text{for} \quad \zeta_2 = \zeta_3. \tag{3.13}
\]

Thus there is an extremum with \( \zeta_2 = \zeta_3 \). This is in fact a necessary condition. Therefore, from now on we shall take

\[
\zeta_2 = \zeta_3 \equiv \zeta. \tag{3.14}
\]

For \( \delta = 1 \) the relation (3.12) then simplifies to

\[
\alpha = \alpha_0 \left\{ 4 \zeta^3 - 3 \zeta^4 - \alpha_0 C \zeta^6 \right\}. \tag{3.15}
\]

Here we have defined

\[
C = -Z_3^{(1)} \rightarrow b \ln \frac{\Lambda^2}{M^2} = \frac{1}{3 \pi} \ln \frac{\Lambda^2}{M^2} \quad \text{for} \quad \Lambda \rightarrow \infty, \tag{3.16}
\]

with \( b \) again being the first coefficient in the expansion of the \( \beta \)-function, \( \beta(\epsilon) = b \epsilon \alpha + \cdots \). The structure of (3.15) is very similar to (2.20) and can be discussed along the same lines. The maximum is at

\[
1 - \zeta = \alpha_0 C \zeta^3/2. \tag{3.17}
\]

This equation has just one real solution \( \zeta \) and the solution lies in the interval between 0 and 1. (The possibility of a precarious theory does not arise here from the beginning, because \( \alpha_0 \) is necessarily positive). Eliminating \( \alpha_0 C \) one obtains

\[
\alpha = \frac{2}{C} (1 - \zeta)(2 - \zeta) \leq \frac{4}{C} \quad \text{for} \quad 0 \leq \zeta \leq 1. \tag{3.18}
\]
So we find that $\alpha \rightarrow 0$ for $\Lambda \rightarrow \infty$, irrespective of the behavior of $\alpha_0$. For the triviality of QED see [4].

As before, one can expand the solution of (3.17),

$$\zeta = 1 - \alpha_0 C/2 + 3(\alpha_0 C)^2/4 - 3(\alpha_0 C)^3/2 + 55(\alpha_0 C)^4/16 + \cdots,$$

(3.19)

and introduce it into (3.15):

$$\alpha = \alpha_0 \{1 - \alpha_0 C + 3(\alpha_0 C)^2/2 - 11(\alpha_0 C)^3/4 + 91(\alpha_0 C)^4/16 + \cdots\}.$$  \hspace{1cm} (3.20)

The branching point can be found by simultaneously requiring the vanishing of the first and second derivative of (3.15). It is located at $\alpha_0 C = -8/27$, $\zeta = 3/2$.

It is tempting to look into the consequences of the inequality (3.18) if one takes it seriously. To do this we have to extend the theory by including all elementary charged fermions and the charged $W$-bosons (assuming, of course, that there exist no further elementary charged particles). We use the same factor $\zeta$ for all particles in generalization of (3.14). In the order in which we work, the various contributions to the vacuum polarization simply add up. We obtain again the relations (3.15), (3.18) but now

$$C = - \sum_f Q_f^2 Z_{3,f}^{(1)} - Z_{3,W}^{(1)}.$$  \hspace{1cm} (3.21)

For the fermionic contributions one has, as in (3.16)

$$-Z_{3,f}^{(1)} \sim \frac{1}{3\pi} \ln \frac{\Lambda^2}{M_f^2},$$

(3.22)

while the $W$-boson contributes with opposite sign (see e.g. [5]).

$$-Z_{W,3}^{(1)} \sim -\frac{3}{4\pi} \ln \frac{\Lambda^2}{M_W^2}.$$  \hspace{1cm} (3.23)

For a numerical estimate we use the physical masses for leptons and $W$, while for the quarks (to be counted three times each for color) we take current masses as given in [6], supplemented by the value for the top quark. To be definite we give the values used, though they are not essential.

$$M_u = 7.6, M_d = 13.3, M_s = 260, M_c = 1270, M_b = 4250, M_t = 176000 \text{MeV}.$$  \hspace{1cm} (3.24)

For $\Lambda$ we may either choose the unification scale $\Lambda_U = 10^{15}$ GeV, or the Planck mass, $\Lambda_P = 10^{19}$ GeV, depending on where we expect that the theory becomes modified and a natural cutoff is provided. We then find $C_U = 46.35$ and $C_P = 57.59$ which leads to

$$\alpha \leq 0.086 \quad \text{or} \quad \alpha \leq 0.069,$$  \hspace{1cm} (3.25)

respectively. Because $\alpha$ is a monotonically increasing function of $\alpha_0$ these bounds are reached for $\alpha_0 \rightarrow \infty$. The value $\alpha \approx 1/137$ would be obtained for $\alpha_0 \approx 1/98$ and $\alpha_0 \approx 1/90$ respectively. Clearly these considerations are highly speculative at this stage.
4 Yang - Mills Theory

The results of the last two sections, \( \Phi^4 \)-theory and QED, clearly depended on the sign of the first non-trivial correction to the renormalized coupling constant. This, in turn, is directly connected to the positive sign of the \( \beta \)-function for small coupling. Yang-Mills theories are asymptotically free, the \( \beta \)-function is negative and the sign of the first non-trivial correction changes. Naively one would expect that we again obtain an equation like (3.15) in QED, but with the sign of the term with \( C \) reversed. This would imply that there is no extremum in the limit \( \alpha_0 C \to \infty \). So the method would simply avoid to run into the previous conclusions by refusing to produce an extremum.

Actually, what happens is much more subtle. The structure of non-abelian gauge theories will enforce a special choice of the Lagrangian of the \( \delta \)-expansion, in order that all the usual compensations of the theory still take place. This in turn will reduce the power of \( \zeta \) in the term with \( \alpha_0 C \) so that there will be an extremum now. For a suitable behavior of \( \alpha_0 C \) the position of this extremum moves to infinity when \( C \to \infty \). So, a finite renormalized coupling constant arises.

The usual YM - Lagrangian (the following considerations hold for any non-abelian gauge group, for simplicity we use the language of QCD) has the form

\[
\mathcal{L} = \mathcal{L}_0 + g_0 \mathcal{L}_3 + g_0^2 \mathcal{L}_4 + g_0 \mathcal{L}_{Gh} + g_0 \mathcal{L}_F + \mathcal{L}_{GF}.
\]

(4.1)

Here \( \mathcal{L}_0 \) denotes the free part of \( \mathcal{L} \), \( \mathcal{L}_3 \) and \( \mathcal{L}_4 \) denote the three-gluon and four-gluon couplings, \( \mathcal{L}_{Gh} \) the ghost Lagrangian, \( \mathcal{L}_F \) the fermion-gluon coupling, and \( \mathcal{L}_{GF} \) the gauge fixing term. The explicit expressions are well known and there is no need to repeat them here. We use again Feynman gauge for simplicity. Remembering the result for QED in the last section we introduce a common scaling parameter \( \zeta \) for all fields from the beginning. An essential modification is necessary now in order not to destroy the compensations between various graphs of the same order in \( g_0 \). Consider, as illustration, the second order contributions of fig. 4 to the gluon propagator. The three graphs in the second line have two internal lines and would get a factor \( \zeta^2 \), the graph in the third line which contains the four gluon vertex has only one internal gluon line and would get a factor \( \zeta \) only. This would, of course, be a disaster.

Fortunately this apparent difficulty can be simply overcome by an appropriate choice of the Lagrangian of the \( \delta \)-expansion. We introduce three functions \( w_0(\zeta, \delta) \), \( w_3(\zeta, \delta) \), and \( w_4(\zeta, \delta) \) to be specified later, which will be multiplied with the free part and the three- and four-gluon couplings respectively. Our Lagrangian reads

\[
\mathcal{L} = \frac{1}{\zeta} \mathcal{L}_0 + \mathcal{L}_I \text{ with}
\]

(4.2. a)

\[
\mathcal{L}_I = \delta^2 w_0(\zeta, \delta)(1 - \frac{1}{\zeta}) \mathcal{L}_0 + \delta w_3(\zeta, \delta) g_0[\mathcal{L}_3 + \mathcal{L}_{Gh} + \mathcal{L}_F] + \delta^2 w_4(\zeta, \delta) g_0^2 \mathcal{L}_4 + \mathcal{L}_{GF}.
\]

(4.2. b)

The requirement that the original Lagrangian is recovered for \( \delta = 1 \) gives the conditions

\[
w_k(\zeta, 1) = 1,
\]

(4.3)
but we cannot simply put all $w_k$ identical to one for the reasons just mentioned; we have to impose the conditions imposed by gauge invariance. Consider the vertex function $\Gamma_N(p_i)$ with $N$ external gluon lines. In ordinary perturbation theory it has the expansion

$$\Gamma_N(p_i) = \sum_V g_0^V \Gamma_N^{(V)}(p_i). \quad (4.4)$$

The number $I$ of internal gluon lines is now not fixed by $N$ and $V$, because $\Gamma_N^{(V)}(p_i)$ contains graphs with different numbers $V_3$ of three gluon and $V_4$ of four gluon vertices. (We forget ghosts and fermions for the moment for simplicity). Obviously $V = V_3 + 2V_4$. We may write

$$\Gamma_N^{(V)}(p_i) = \sum_I \Gamma_N^{(V,I)}(p_i). \quad (4.5)$$

The numbers of vertices can be expressed in terms of $N$ and $I$:

$$V_3 = N - 2V + 2I, \quad V_4 = 3V/2 - N/2 - I. \quad (4.6)$$

The expansion for the vertex function in the $\delta$-expansion, using the Lagrangian (4.2) reads

$$\Gamma_N(p_i, \zeta, \delta) = \sum_V \sum_I [\delta^3 w_3 g_0]^{V_3} [\delta^2 w_4 g_0^2]^{V_4} \zeta^I \Gamma_N^{(V,I)}(p_i) \sum_{J=0}^{\infty} \left( \binom{I + J - 1}{J} \right) \delta^2 w_0(1 - \zeta)^J. \quad (4.7)$$

This should be obvious from comparison with the analogous expansions in the previous sections. The sum over $J$ which represents the insertions, can be performed using $\sum_J \left( \binom{I + J - 1}{J} \right) x^J = (1 - x)^{-I}$, and $V_3$ and $V_4$ eliminated from (4.6). This results in

$$\Gamma_N(p_i, \zeta, \delta) = \sum_V \delta^V g_0 [w_3]^{N-2V} [w_4]^{3V/2-2N/2} \zeta \sum_I \frac{\zeta w_3^2}{w_4[1 - \delta^2 w_0(1 - \zeta)]^I} \Gamma_N^{(V,I)}(p_i). \quad (4.8)$$

We must insist that the compensations between various graphs of the same order in $g_0$ have to take place also in the $\delta$-expansion. This will be the case if and only if all the contributions $\Gamma_N^{(V,I)}$ will get a factor which depends on the order $V$ only, but not on the number $I$ of internal gluon lines. Therefore the curly bracket in (4.8) must be equal to one. This gives a relation between the $w_k$:

$$\zeta w_3^2 = w_4(1 - \delta^2 w_0(1 - \zeta)). \quad (4.9)$$

For $\delta = 1$, where the $w_k$ are equal to one, this relation is fulfilled.

It is easily seen that the inclusion of ghosts and fermions does not alter the previous considerations because the latter always appear in three particle vertices.

Using (4.9), the sum over $I$ in (4.8) simplifies to $\Gamma_N^{(V)}(p_i)$. The insertions have, of course, only apparently disappeared, they are now taken into account through the functions $w_3$ and $w_4$ and their behavior dictated by (4.9).
There is some freedom in solving (4.9). If we insist on a reasonable and simple solution, however, the result becomes essentially unique. We want all $w_{k}$ to be polynomials in $\delta$, in order to be sure that we don’t introduce any singularities into the Lagrangian (4.2). In order that we can take the square root of $w_{3}^{2}(\zeta, \delta)$ in (4.9) we have to choose $w_{4}(\zeta, \delta) = [1 - \delta^{2}w_{0}(\zeta, \delta)(1 - \zeta)]/\zeta$ where the factor $1/\zeta$ is implied by (4.3). Finally we may simply choose $w_{0}(\delta, \zeta) = 1$. So we arrive at

$$w_{0}(\zeta, \delta) = 1, \quad w_{3}(\zeta, \delta) = w_{4}(\zeta, \delta) = [1 - \delta^{2}(1 - \zeta)]/\zeta.$$  

(4.10)

The calculation of the renormalized coupling constant $\alpha(\mu^{2})$ at the renormalization scale $\mu$ in order $\delta^{2}$ is now straightforward along the lines of the two preceding sections. The only difference is the consideration of the additional factors (4.10) in the appropriate order. It is the factor of $1/\zeta$ there which implies essential changes in the powers of $\zeta$. The result reads

$$\alpha(\mu^{2}) = \delta^{2} \alpha_{0} \zeta \left\{ 1 + \delta^{2}[(1 - \zeta) + \zeta \alpha_{0} C] \right\}.$$  

(4.11)

We now have

$$C \to 4\pi b \ln \frac{\Lambda^{2}}{\mu^{2}},$$  

(4.12)

with $b$ the coefficient in the expansion $\beta(g) = -bg^{3} + \cdots$ of the $\beta$-function. (Strictly speaking, we should better use dimensional regularization, $d = 4 - 2\epsilon$, now, and later retranslate $1/\epsilon \to \ln(\Lambda^{2}/\mu^{2})$.) For $\delta = 1$ (4.11) becomes

$$\alpha(\mu^{2}) = \alpha_{0} \{2\zeta - \zeta^{2} + \alpha_{0} C \zeta^{2}\}.$$  

(4.13)

The extremum is at $\zeta = 1/(1 - \alpha_{0} C)$. Obviously we now have $\zeta > 1$ as long as $\alpha_{0} C < 1$. Introducing this value for $\zeta$ gives the simple relation

$$\alpha(\mu^{2}) = \frac{\alpha_{0}}{1 - \alpha_{0} C}.$$  

(4.14)

Now, if for $C \to \infty$ the bare coupling $\alpha_{0}$ goes to zero such that $1 - \alpha_{0} C \to \epsilon \to 0^{+}$ one has $\alpha \to 1/C\epsilon$ which will be finite if $\epsilon$ is proportional to $1/C$.

To be more explicit we have to note that $\alpha_{0} \sim 1/C$ must not depend upon the arbitrary renormalization scale $\mu$. Therefore we have to introduce a dimensional parameter $\Lambda_{QCD}$ and take $\alpha_{0} \sim 1/[4\pi b \ln(\Lambda^{2}/\Lambda_{QCD}^{2})]$. This leads to $\epsilon = \ln(\mu^{2}/\Lambda_{QCD}^{2})/\ln(\Lambda^{2}/\Lambda_{QCD}^{2})$ and

$$\alpha(\mu^{2}) \to \frac{1}{C\epsilon} \to \frac{1}{4\pi b \ln(\mu^{2}/\Lambda_{QCD}^{2})}.$$  

(4.15)

as it should. In a higher-order calculation it should be possible to relate $\Lambda_{QCD}$ to the cutoff $\Lambda$.

If there were so many fermions that the theory would no longer be asymptotically free, $b$ would be negative and the theory would become trivial as in the examples of the previous sections.
5 Conclusions

The ansatz of scaling the whole free Lagrangian, including the kinetic term, appears both simple and powerful. In this paper we restricted to one loop order and to a calculation of the renormalized coupling constant. Generalizations to higher orders and/or to the calculation of the whole effective potential appear straightforward in principle, without any new conceptional problems.

A characteristic of the method is, that one has to consider the bare constant $g_0$ as given and the renormalized coupling constant $g$ as a function of the expansion parameter $\delta$. One cannot do it the other way round, because then an expansion with respect to $\delta$ becomes impossible. Therefore it will need some further considerations before the method can be applied to other problems in field theory.

Finally, one should frankly admit that all these methods of going beyond perturbation theory have a “distinctly alchemical flavor” as phrased by Duncan and Jones. The splitting of the Lagrangian into $L_0$ and $L_I$ is widely ambiguous, sometimes it may be advantageous or even mandatory to use a more complicated dependence of $L_I$ upon $\delta$, and finally every quantity to be calculated (even a function, say a power or a logarithm, of the original quantity) leads to a different position of the extremum. Nevertheless, the inherent ambiguities of the method can be overcome by using physical principles and simplicity arguments and lead to results which go far beyond naive perturbation theory.

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Figure Captions

Fig. 1: One particle irreducible contributions to the two point function in $\Phi^4$ - theory up to order $\delta^2$. Here and in the following, the thick dot denotes the insertion of the free Lagrangian.

Fig. 2: Contributions to the vertex function in $\Phi^4$ - theory up to order $\delta^2$.

Fig. 3: One particle irreducible contributions to the electron two point function, photon two point function, and vertex function in QED up to order $\delta^2$.

Fig. 4: Contributions to the two point function in Yang-Mills theory.
Fig. 1
Fig. 2
Test

Fig. 4