The method of non-local transformations: Applications to blow-up problems

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Abstract. The method for numerical integration of Cauchy problems for ODEs with blow-up solutions is described. It is based on introducing a new non-local variable that reduces a single nth-order ODE to a system of first-order coupled ODEs. This method leads to problems whose solutions are presented in parametric form and do not have blowing-up singular points; therefore the standard fixed-step numerical methods can be applied. The efficiency of the proposed method is illustrated with two test problems. It is shown that the first Painlevé equation with suitable initial conditions have non-monotonic blow-up solutions.

Keywords: nonlinear differential equations, non-local transformations, blow-up problems, numerical solutions, Painlevé equations

1. Introduction

We will consider Cauchy problems for ODEs, whose solutions tend to infinity at some finite value of the independent variable $t = t_*$, where $t_*$ does not appear explicitly in the differential equation under consideration and it is not known in advance. Similar solutions exist on a bounded interval and are called blow-up solutions. This raises the important question for practice: how to determine the position of a singular point $t_*$ and the solution in its neighborhood by numerical methods.

In the general case, monotonic blow-up solutions, having a power singularity, can be represented in a neighborhood of the singular point $t_*$ in the form

$$x \simeq A(t_* - t)^{-\beta}, \quad \beta > 0,$$

(1)

where it is assumed that $t_0 \leq t < t_*$. For such solutions we have $\lim_{t \to t_*} |x| = \infty$ and $\lim_{t \to t_*} |x'| = \infty$. Differentiating (1), we obtain the derivatives near the singular point

$$x'_t \simeq A\beta(t_* - t)^{-\beta-1}, \quad x''_t \simeq A\beta(\beta + 1)(t_* - t)^{-\beta-2}.$$

(2)
The formulas (1)–(2) remain valid for non-monotonic blow-up solutions if there is a neighborhood on the left of the singular point \( t_1 \leq t < t_*, \) where \( t_0 \leq t_1 \), in which the solution is monotonic.

The direct application of the standard fixed-step numerical methods to blow-up problems leads to certain difficulties because the range of variation of the independent variable is unknown in advance [1]. One of the basic ideas of numerical integration of blow-up problems consists in the application of an appropriate transformation at the initial stage, which leads to the equivalent problem whose solutions have no singularities at a priori unknown point. In particular, for such problems with monotonic solutions, the hodograph transformation can be used [2]. Some special methods of numerical integration of blow-up problems are described, for example, in [3–7]. This paper generalizes the results of [9–11], in which to numerical integration of these problems we used the method of non-local transformations.

2. Blow-up Cauchy problems for second-order ODEs

2.1. Solution method based on the introduction of a non-local variable

The Cauchy problem for a second-order differential equation has the form

\[
\begin{align*}
x''(t) &= f(t, x, x', t > t_0); \\
x(t_0) &= x_0, \quad x'(t_0) = x_1.
\end{align*}
\]

We assume that the problem (3)–(4) has a non-monotonic blow-up solution.

We introduce a new non-local variable \( \xi \) by means of the first-order ODE and the initial condition:

\[
\xi' = g(t, x, x', \xi), \quad \xi(t_0) = 0.
\]

Here \( g = g(t, x, x', \xi) \) is the regularizing function that can be varied.

We represent the second-order equation (3) in the form of an equivalent system of two equations of the first order

\[
\begin{align*}
x'_t &= y, \\
y'_t &= f(t, x, y).
\end{align*}
\]

By using (5), we pass in (6) and (4) from \( t \) to a new independent variable \( \xi \). As a result, the Cauchy problem (3)–(4) is transformed to the following problem for the system of three equations:

\[
\begin{align*}
t'_\xi &= \frac{1}{g(t, x, y, \xi)}, \\
x'_\xi &= \frac{y}{g(t, x, y, \xi)}, \\
y'_\xi &= \frac{f(t, x, y)}{g(t, x, y, \xi)} (\xi > 0); \\
t(0) &= t_0, \quad x(0) = x_0, \quad y(0) = x_1.
\end{align*}
\]

Choosing an appropriate function \( g = g(t, x, y, \xi) \) in (7), we can obtain the Cauchy problem whose solution will not have blow-up singularities and which can be integrated by applying the standard fixed-step numerical methods [13–15].

In the particular case, when \( g = g(t, x, x'_t) \), the function \( \xi \) by integrating the equation (5), can be expressed in terms of the solution of the Cauchy problem (3)–(4) in the form of an integral

\[
\xi = \int_{t_0}^{t} g(t, x, y) \, dt, \quad x = x(t), \quad y = y(t).
\]

For numerical solution of blow-up problems it is reasonable to apply the regularizing functions of the form

\[
g = G(|y|, |f|),
\]
where \( f = f(t, x, y) \) is the right-hand side of the equation (3) and \( y = x'_t \). We impose the following conditions on the function \( G = G(u, v) \):

\[
G > 0; \quad G_u \geq 0, \quad G_v \geq 0, \quad G \rightarrow \infty \quad \text{as} \quad u + v \rightarrow \infty,
\]

where \( u \geq 0, v \geq 0 \). In this case the non-local variable (8) will be monotonically increasing and unbounded with increasing \( t \).

Further, in the main, we will consider a five-parameter class of the regularizing functions of the form

\[
g = (1 + k_1|y|^{p_1} + k_2|f|^{p_2})^q,
\]

where \( k_1 \geq 0, k_2 \geq 0 \) \((k_1 + k_2 \neq 0)\), \( p_1 > 0, p_2 > 0 \), and \( q > 0 \). The particular case \( g = \sqrt{1 + y^2 + f^2} \lim_{t \rightarrow t^*} I = \infty \), \( I = \int_{t_0}^{t} g(t, x, y) dt \).

The convergence or divergence of the integral \( I \) is determined by the convergence or divergence of the integral \( I_2 \).

**Remark 1.** Non-local transformations of a special form were used in [16, 17] for finding exact solutions and linearizing some ordinary differential equations of the second order.

### 2.2. Conditions that the regularizing functions must satisfy. Examples of regularizing functions

Let \( g = g(t, x, x'_t) \). Then the formula (8) is valid. Since the transformed problem (7) should not have blow-up singularities, it is necessary that its solution \( x = x(t), y = y(t) \) satisfies the following condition:

\[
\lim_{t \rightarrow t^*} I = \infty, \quad I = \int_{t_0}^{t} g(t, x, y) dt.
\]

We have:

\[
I = I_1 + I_2, \quad I_1 = \int_{t_0}^{t_1} g(t, x, y) dt, \quad I_2 = \int_{t_1}^{t} g(t, x, y) dt,
\]

where \( t_1 > 0 \) is a point sufficiently close to the blow-up point \( t^* \) \((t_1 < t^*)\). The convergence or divergence of the integral \( I \) is determined by the convergence or divergence of the integral \( I_2 \).

1°. **Regularizing functions of the first type.** First, we consider regularizing functions of the form \( g = g(|y|) > 0 \), which in addition to the normalization condition \( g(0) = 1 \) satisfy the asymptotic condition of power growth for large \( |y| \):

\[
g(|y|) \rightarrow C|y|^{\alpha} \quad \text{as} \quad |y| \rightarrow \infty \quad \text{with} \quad \alpha > 0 \quad (C > 0).
\]

We use the asymptotics (2) and (13) to analyze the convergence or divergence of the integral \( I_2 \) in (12). We have

\[
I_2 \approx |A\beta|^{\alpha} C \int_{t_1}^{t} (t^* - \tau)^{-\alpha(\beta+1)} d\tau.
\]

Hence it follows that the integral \( I_2 \) diverges if \( \alpha(\beta+1) \geq 1 \) that is equivalent to the condition

\[
\alpha \geq \frac{1}{\beta + 1}.
\]

For the most common singularity of the solution that has a first-order pole, which corresponds to the value \( \beta = 1 \), we should choose \( \alpha \geq \frac{1}{2} \). Since \( \beta > 0 \), then \( \alpha = 1 \) is suitable for any blow-up solution of the power (and logarithmic) type.
The asymptotics as $|y| \to \infty$ of regularizing functions of the form
\[ g = (1 + k|y|^p)^q \quad (k > 0, \ p > 0, \ q > 0) \] (16)
is determined by the value $\alpha = pq$ in (13). For the function (16) with $p = 2$ and $q = 1/2$ we have $\alpha = 1$ and the inequality (15) holds for any positive $\beta$. For the function (16) with $p = 1$ and $q = 1/2$ we have $\alpha = 1/2$. In this case, the inequality (15) holds for a first-order pole, which is determined by the value $\beta = 1$, and also for all $\beta \geq 1$ (that is, for integer poles of any order).

2°. Regularizing functions of the second type. We now consider regularizing functions of the form $g = g(|z|) > 0$, where $z = x''_t = f(t, x, y)$, which in addition to the normalization condition $g(0) = 1$ satisfy the asymptotic condition of power growth for large $|z|$: \[ g(|z|) \to C|z|^\alpha \quad \text{as} \quad |z| \to \infty \quad \text{with} \quad \alpha > 0 \quad (C > 0). \] (17)

We use the asymptotics (2) and (17) to analyze the convergence or divergence of the integral $I_2$ in (12). We have
\[ I_2 \simeq |A\beta(\beta + 1)|^{\alpha}C \int_{t_1}^{t} (t_* - \tau)^{-\alpha(\beta+2)} d\tau. \] (18)
It follows that the integral $I_2$ diverges if $\alpha(\beta + 2) \geq 1$, or
\[ \alpha \geq \frac{1}{\beta + 2}. \] (19)
For the most common singularity of the solution that has a first-order pole, which corresponds to the value $\beta = 1$, we should choose $\alpha \geq 1/3$. Since $\beta > 0$, then $\alpha \geq 1/2$ is suitable for any blow-up solution of the power (and logarithmic) type.

The asymptotics as $|z| \to \infty$ of regularizing functions of the form
\[ g = (1 + k|z|^p)^q \quad (k > 0, \ p > 0, \ q > 0) \] (20)
is determined by the value $\alpha = pq$ in (17). For the function (20) with $p = 1$ and $q = 1/2$ or $p = 1/2$ and $q = 1$ we have $\alpha = 1/2$ and the inequality (19) holds for any positive $\beta$. For the function (20) with $p = 1$ and $q = 1/3$ or $p = 1/3$ and $q = 1$ we have $\alpha = 1/3$. In this case, the inequality (19) holds for a first-order pole, which is determined by the value $\beta = 1$, and also for all $\beta \geq 1$ (that is, for integer poles of any order).

Remark 2. A comparison of the exact solutions of a number of test problems and the corresponding numerical solutions obtained by the method of non-local transformations shows that the most efficient regularizing functions are those that have the least admissible value of $\alpha$, the exponent in the asymptotics (13) and (17), and are determined by the equality sign in (15) and (19).

3°. Regularizing functions of the mixed type. In a similar way, we can define the domain of divergence of regularizing functions of the mixed type $g = g(|y|, |z|)$, starting from the asymptotics of this function as $|y| + |z| \to \infty$. In particular, the function $g = (1 + |y| + |z|)^{1/3}$ can be used if a solution has a singularity in the form of a pole of any integer order ($\beta = 1, 2, \ldots$).

Example 1. Consider a Cauchy problem for the first Painlevé equation [12]
\[ x''_t = 6x^2 + t \quad (t > 0); \quad x(0) = a, \quad x'_t(0) = b. \] (21)
For $a > 0$ and $b < 0$, the problem (21) has non-monotonic blow-up solutions.
Figure 1. Numerical solutions of the problem (21) for the first Painlevé equation with $a = 1$, regularizing function $g = (1 + |y| + |f|)^{1/3}$, and $h = 0.01$.

Numerical solution of the problem (21) for the first Painlevé equation with $a = 1$ and the three values of the parameter $b$ ($b = 0$, $b = -40$, $b = -80$) which is obtained by integrating the transformed system (7) with $f = 6x^2 + t$ and $g = (1 + |y| + |f|)^{1/3}$ by the Runge–Kutta method of the fourth-order approximation for the fixed stepsize $h = 0.01$ are shown by circles in Fig. 1. For $b = -40$ and $b = -80$, the solutions of the first Painlevé equation have a non-monotonic character and exist in the finite regions $0 \leq t < t_* = 1.0577704$ and $0 \leq t < t_* = 0.8323860$, respectively. Reducing the stepsize by one-half, the maximum module of difference between the numerical solutions (with $h = 0.01$ and $h = 0.005$) for problem (21) for $x \leq 100$ with $b = -40$ is equal to 0.0000033 and for this problem for $x \leq 100$ with $b = -80$ is equal to 0.0000044.

Remark 3. It can be shown that the Cauchy problem for the second Painlevé equation

$$x_{tt}'' = 2x^3 + tx + c \quad (t > 0); \quad x(0) = a, \quad x_t'(0) = b,$$

for $a > 0$ and $b < 0$, can have non-monotonic blow-up solutions.

Remark 4. Several blow-up test problems and their non-monotonic exact solutions are described in [11].

3. Blow-up Cauchy problems for higher-order ODEs

3.1. General description of the method of non-local transformations

We consider the Cauchy problem for the $n$th-order differential equation of the general form

$$x^{(n)}_t = f(t, x, x_t', \ldots, x_t^{(n-1)}) \quad (t > t_0); \quad x(t_0) = x_0, \quad x_t'(t_0) = x_t^{(1)}(t_0), \quad \ldots, \quad x_t^{(n-1)}(t_0) = x_t^{(n-1)}(t_0),$$

(22)

where $x_t^{(k)} = d^k x/dt^k$ ($k = 3, \ldots, n$).

As before, we introduce a new non-local independent variable $\xi$ by means of the first-order differential equation (with respect to $\xi$) and the initial condition:

$$\xi_t = g(t, x, x_t', \ldots, x_t^{(n-1)}, \xi), \quad \xi(t_0) = 0.$$
Here \( g = g(t, x, x_1', \ldots, x_t^{(n-1)}, \xi) \) is a regularizing function that can be varied.

Now we represent the Cauchy problem for one \( n \)th-order equation (22) in the form of an equivalent problem for a system consisting of \( n \) equations of the first order (of special form):

\[
\begin{align*}
(x_1')_t &= x_2, & (x_2')_t &= x_3, & \ldots, & (x_{n-1}')_t &= x_n, & (x_n')_t &= f(t, x_1, x_2, \ldots, x_n); \\
x_1(t_0) &= x_0, & x_2(t_0) &= x_0^{(1)}, & \ldots, & x_n(t_0) &= x_0^{(n-1)},
\end{align*}
\]

(24)

where \( x_1 = x \). Then, by using (23), we pass from \( t \) to a new independent variable \( \xi \) in (24).

As a result, the Cauchy problem for one equation of the \( n \)th order (22) is transformed to the following problem for a system consisting of \( (n + 1) \)st equation of the first order:

\[
\begin{align*}
t' &= \frac{1}{g}, & x_1' &= \frac{x_2}{g}, & x_2' &= \frac{x_3}{g}, & \ldots, & x_{n-1}' &= \frac{x_n}{g}, & x_n' &= \frac{f}{g}; \\
t(0) &= t_0, & x_1(0) &= x_0, & x_2(0) &= x_0^{(1)}, & \ldots, & x_n(0) &= x_0^{(n-1)},
\end{align*}
\]

(25)

where \( f = f(t, x_1, x_2, \ldots, x_n) \), \( g = g(t, x_1, x_2, \ldots, x_n, \xi) \), and the prime denotes the derivative with respect to \( \xi \).

In the particular case, when \( g = g(t, x, x_1', \ldots, x_t^{(n-1)}) \), the function \( \xi \) by integrating the equation (23), can be expressed in terms of the solution of the Cauchy problem (23) in the form of an integral

\[
\xi = \int_{t_0}^t g(t, x_1, x_2, \ldots, x_n) \, dt, \quad x_k = x_k(t) \equiv x_t^{(k)}.
\]

(26)

3.2. Examples of regularizing functions for \( n \)th-order equations

From (1), we obtain an approximate formula for the derivative of an arbitrary order in a neighborhood of the singular point \( t_\star \):

\[
x_t^{(n)} \simeq A_n (t_\star - t)^{-\beta-n}, \quad A_n = A\beta(\beta + 1) \cdots (\beta + n - 1).
\]

(27)

We consider regularizing functions of the form \( g = g(|f|) > 0 \), where \( f \) is the right-hand side of the equation (22). Suppose, in addition to the normalization condition \( g(0) = 1 \), that the function \( g \) satisfies the asymptotic condition of power growth for large \(|f|\):

\[
g(|f|) \to C|f|^\alpha \quad \text{as} \quad |f| \to \infty \quad \text{with} \quad \alpha > 0 \quad (C > 0).
\]

(28)

We use the asymptotics (27) and (28) to analyze the convergence or divergence of the integral \( I_2 \) in (12). Arguing as in Section 2.2, we can show that the integral \( I_2 \) diverges if the following condition is satisfied:

\[
\alpha \geq \frac{1}{\beta + n}.
\]

(29)

Since \( \beta > 0 \), then \( \alpha = 1/n \) is suitable for any blow-up power-type solution. For the most common singularity of the solution that has a first-order pole, which corresponds to the value \( \beta = 1 \), we can choose \( \alpha = 1/(n + 1) \) (this value of \( \alpha \) is also suitable for a pole of any integer order).

The asymptotics as \(|f| \to \infty\) of regularizing functions of the form

\[
g = (1 + k|f|^p)^q \quad (k > 0, \ p > 0, \ q > 0)
\]

(30)
is determined by the value \( \alpha = pq \) in (28). For the function (30) with \( p = 1 \) and \( q = 1/n \) or \( p = 2 \) and \( q = 1/(2n) \) we have \( \alpha = 1/n \) and the inequality (29) holds for any positive \( \beta \). For the function (30) with \( p = 1 \) and \( q = 1/(n+1) \) or \( p = 1/(n+1) \) and \( q = 1 \) we have \( \alpha = 1/(n+1) \). In this case the inequality (29) holds for a first-order pole, which is determined by the value \( \beta = 1 \), and also for all \( \beta \geq 1 \) (that is, for integer poles of any order).

Below, in more detail, we describe the procedure for applying the method for third-order ODEs.

3.3. Equations of the third order. A test problems and its numerical integration

We consider the Cauchy problem for the third-order differential equation

\[
x'''_{ttt} = f(t, x, x', x''_{ttt}) \quad (t > 0); \quad x(0) = x_0, \quad x'_t(0) = x_1, \quad x''_{ttt}(0) = x_2,
\]

which is equivalent to the problem for a system consisting of the three first-order equations:

\[
x'_t = y, \quad y'_t = z, \quad z'_t = f(t, x, y, z) \quad (t > 0); \quad x(0) = x_0, \quad y(0) = x_1, \quad z(0) = x_2.
\]

Introducing a non-local variable \( \xi \) by means of the differential equation

\[
\xi'_t = g(t, x, x'_t, x''_{ttt}, \xi), \quad \xi(0) = 0,
\]

we transform the system (32) to the following form:

\[
t'_\xi = \frac{1}{g}, \quad x'_\xi = \frac{y}{g}, \quad y'_\xi = \frac{z}{g}, \quad z'_\xi = \frac{f}{g} \quad (\xi > 0); \quad t(0) = 0, \quad x(0) = x_0, \quad y(0) = x_1, \quad z(0) = x_2,
\]

where \( f = f(t, x, y, z) \) and \( g = g(t, x, y, z, \xi) \).

Let us consider various possibilities for choosing the function \( g \) in the system (33).

1°. We can take \( g = y/x \). In this case, the system (33) is simplified, since the second equation is directly integrated, and taking into account the second initial condition, we obtain \( x = x_0 e^\xi \).

2°. We can take \( g = z/y \). In this case, the system (33) is simplified, since the third equation is directly integrated, and we obtain \( y = x_1 e^\xi \). Taking into account the relations (32), we also have \( \xi = \ln(x'_t/x_1) \).

3°. Also, we can take \( g = f/z \). In this case, the system (33) is also simplified, since the fourth equation is directly integrated, and we obtain \( z = x_2 e^\xi \).

4°. We can take \( g = (1 + k_1 |y|^{p_1} + k_2 |z|^{p_2} + k_3 |f|^{p_3})^q \) for \( k_m \geq 0, \quad p_m > 0, \quad \) and \( q > 0 \). The case \( k_m = 1, \quad p_m = 2, \) and \( q = 1/2 \) corresponds to the method of the arc-length transformation [3].

The transformations corresponding to the first three Items, 1°, 2°, and 3°, will be called the special exp-type transformations, they lead to the solutions, in which the variable \( t \) tends exponentially rapidly to a blow-up point \( t_* \). These transformations can not be applied if for non-monotonic solutions (since the function \( g \) function can vanish in such cases). For non-monotonic solutions one can use the last function from Item 4° (for example, with \( p_m = 1, \quad q = 1/2 \) or \( p_m = 1, \quad q = 1/3 \)).
Example 2. We consider in more detail the test Cauchy problem of the form

\[ x'''(t) = 6x^4(t > 0); \quad x(0) = x'(0) = 1, \quad x''(0) = 2. \] (34)

The exact solution of this problem is determined by the formula

\[ x(t) = \frac{1}{1-t}. \]

It has a power-type singularity (a first-order pole) at the point \( t_s = 1 \) and does not exist for \( t > t_s \).

In Table 1, a comparison of the efficiency of the numerical integration methods, based on various nonlocal transformations of the form (5) is presented by using the example of the test blow-up problem for the third-order ODE (34). The comparison is based on the number of grid points needed to make calculations with the same maximum error, \( \% = 0.01 \).

| Transformation             | Function \( g \)                                                                 | Max. interval \( \xi_{\text{max}} \) | Stepsize \( h \) | Grid points \( N \) |
|----------------------------|---------------------------------------------------------------------------------|--------------------------------------|----------------|-------------------|
| Arc-length, Item 4°        | \( g = (1+y^2+z^2+f^2)^{1/2} \)                                               | 1996500.000                         | 0.3630         | 5500000           |
| Hodograph                 | \( g = y \)                                                                     | 98.987                               | 0.0790         | 1253              |
| Special exp-type, Item 3° | \( g = f/z \)                                                                  | 13.800                               | 0.1150         | 120               |
| Nonlocal, Item 4°         | \( g = (1+|f|)^{1/4} \)                                                        | 7.222                                | 0.0675         | 107               |
| Special exp-type, Item 1° | \( g = y/x \)                                                                   | 4.590                                | 0.0450         | 102               |
| Nonlocal, Item 4°         | \( g = (1+|y|+|z|)^{1/3} \)                                                     | 6.030                                | 0.0603         | 100               |
| Special exp-type, Item 2° | \( g = z/y \)                                                                   | 9.212                                | 0.0940         | 98                |

Table 1. Various types of analytical transformations applied for numerical integration of the problem (34) with a given accuracy (percent error is 0.01 for \( 1 \leq x \leq 100 \)) and their basic parameters (maximum interval, stepsize, grid points number).

It can be seen that the arc-length transformation is the least effective, since the use of this transformation is associated with a large number of grid points. In particular, when using the last four transformations, you need 52401–56122 times less of a number of grid points. The hodograph transformation has an intermediate (moderate) efficiency.

4. Brief Conclusions

The method for numerical integration of blow-up problems for ODEs \( x^{(n)}_i = f(t, x, x'_1, \ldots, x'^{(n-1)}_i) \), based on the introduction of a new non-local independent variable \( \xi \), which is related to the original variables \( t \) and \( x \) by the equation \( \xi'_t = g(t, x, x'_1, \ldots, x'^{(n-1)}_i, \xi) \), is described. With a suitable choice of the regularizing function \( g \), the proposed method leads to problems whose solutions are presented in parametric form and do not have blowing-up singular points; therefore the transformed problems allow the application of the standard fixed-step numerical methods. The numerical integration of test problems has shown high efficiency of methods based on non-local transformations of a special type (and the practical inapplicability of the arc-length transformation for numerical solving blow-up problems with ODEs of high order). It is shown that the first Painlevé equation with suitable initial conditions have non-monotonic blow-up solutions.

It is important to note that the method of non-local transformations can be useful for numerical integration of other problems with large solution gradients (for example, in problems with solutions of boundary-layer type).

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