Weihrauch degrees of finding equilibria in sequential games

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Various operations related to infinite sequential games are classified in the Weihrauch lattice.

1 Overview

We continue the investigation of the Weihrauch degrees of operations mapping games to their equilibria started in [29]. There, finding pure and mixed Nash equilibria in two-player games with finitely many actions in strategic form were classified. In the present work, we consider infinite sequential games played by any countable number of players. The best-known example of such games are Gale-Stewart games [15], which are two-player win/loose games. As such, Borel determinacy [25] in principle falls into the scope of this research, although we will be concerned only with determinacy for much smaller pointclasses.

One motivation for this line of inquiry is the general stance that solution concepts in game theory can only be convincing if the players are capable of (at least jointly) computing them taken e.g. in [31]. Even if we allow for some degree of hypercomputation, or are, e.g., willing to tacitly replace actually attaining a solution concept by some process (slowly) converging to it, we still have to reject solution concepts with too high a Weihrauch degree.

The results for determinacy of specific pointclasses that we provide are a refinement of results obtained in reverse mathematics by Nemoto, MedSalem and Tanaka [28]; the first is also a uniformization of a result by Cenzer and Remmel [12]. For some represented pointclass \( \Gamma \), let \( \text{Det}_\Gamma : \Gamma \to \{0,1\}^N \) be the map taking a \( \Gamma \)-subset \( A \) of Cantor space to a (suitably encoded) Nash equilibrium in the sequential two-player game with alternating moves where the first player wins if the induced play is in \( A \), and the second player wins otherwise. Let \( \mathcal{A} \) be the closed subsets of Cantor space, and \( \mathcal{D} := \{ U \setminus U' \mid U, U' \in \mathcal{A} \} \). Some of our results are:

**Theorem.** \( \text{Det}_\mathcal{A} \equiv_W C_{\{0,1\}^N} \) and \( \text{Det}_\mathcal{D} \equiv_W C_{\{0,1\}^N} \star \lim \).

We have two remarks. One, by combining the preceding theorem with the main result of [9], we find that \( \text{Det}_\mathcal{D} \) is equivalent to the Bolzano-Weierstrass-Theorem. This may be a bit unexpected in particular seeing that \( C_{\{0,1\}^N} \star \lim \) is not (yet) known to contain a plethora of mathematical theorems (unlike, e.g., \( C_{\{0,1\}^N} \)). Two, we already need to use a limit operator in order to move up one level of the difference hierarchy – rather than being able to move up one level in the Borel hierarchy as one may have expected naively. Thus, this observation may complement Harvey Friedman’s famous result [14] that proving Borel determinacy requires repeated use of the axiom of replacement.

Another group of results is based on inspecting the various results extending Borel determinacy to more general classes of games (and solution concepts) in [22 23 24]. If we instantiate
these generic results with specific determinacy version as above, we can prove for some of them that they are actually optimal w.r.t. Weihrauch reducibility. We shall state two such classifications explicitly.

Consider two-player sequential games with finitely many outcomes, where each player has some acyclic preference relation over the outcomes. For any upper set of outcomes w.r.t. some player’s preference let the corresponding set of plays be open or closed. Let \( \text{NE}_{O∪A} \) be the operation taking such a game (suitably encoded) and producing a Nash equilibrium. Then:

**Theorem.** \( \text{NE}_{O∪A} \equiv_w C_{\{0,1\}^n} \times \text{LPO}^* \)

Next, we restrict the aforementioned class of games to abstract zerosum games, that is, games where the preferences of one player are the inverse of the preferences of the other player. Those games will have subgame-perfect equilibria, and we let \( \text{SPE}_{O∪A} \) be the operation mapping such games to a subgame-perfect equilibrium.

**Theorem.** \( \text{SPE}_{O∪A} \equiv_w \lim \)

Various further classifications are obtained, and adhere to the scheme that algebraic combinations of very common Weihrauch degrees appear, which is already exhibited by our examples above.

## 2 Fundamentals

### 2.1 Background on represented spaces

We briefly recall some fundamental concepts on represented spaces following [32], to which the reader shall also be referred for a more detailed presentation. The concept behind represented spaces essentially goes back to Weihrauch and Kreitz [21], the name may have first been used by Brattka [3]. A *represented space* is a pair \( X = (X, \delta_X) \) of a set \( X \) and a partial surjection \( \delta_X : \subseteq \mathbb{N}^N \rightarrow X \). A function between represented spaces is a function between the underlying sets. For \( f : X \rightarrow Y \) and \( F : \subseteq \mathbb{N}^N \rightarrow \mathbb{N}^N \), we call \( F \) a realizer of \( f \) (notation \( F \vdash f \)), iff \( \delta_Y(F(p)) = f(\delta_X(p)) \) for all \( p \in \text{dom}(f\delta_X) \), i.e. if the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{N}^N & \xrightarrow{f} & \mathbb{N}^N \\
\downarrow \delta_X & & \downarrow \delta_Y \\
X & \xrightarrow{f} & Y
\end{array}
\]

A map between represented spaces is called computable (continuous), iff it has a computable (continuous) realizer. Similarly, we call a point \( x \in X \) computable, iff there is some computable \( p \in \mathbb{N}^N \) with \( \delta_X(p) = x \). A priori, the notion of a continuous map between represented spaces and a continuous map between topological spaces are distinct and should not be confused!

Given two represented spaces \( X, Y \) we obtain a third represented space \( C(X,Y) \) of functions from \( X \) to \( Y \) by letting \( 0^n 1p \) be a \( [\delta_X \rightarrow \delta_Y] \)-name for \( f \), if the \( n \)-th Turing machine equipped with the oracle \( p \) computes a realizer for \( f \). As a consequence of the UTM theorem, \( C(\dash,\dash) \) is the exponential in the category of continuous maps between represented spaces, and the evaluation map is even computable (as are the other canonical maps, e.g. currying).

This function space constructor, together with two represented spaces \(, \mathbb{N} = (\mathbb{N},\delta_N) \) and \( S = (\{\bot,\top\},\delta_S) \), allows us to obtain a model of Escardó’s synthetic topology [13]. The
representation are given by $\delta_n(0^n1^n) = n$, $\delta_S(0^n) = \bot$ and $\delta(p) = \top$ for $p \neq 0^n$. It is straightforward to verify that the computability notion for the represented space $\mathbb{N}$ coincides with classical computability over the natural numbers. The Sierpiński space $\mathbb{S}$ in turn allows us to formalize semi-decidability. The computable functions $f : \mathbb{N} \to \mathbb{S}$ are exactly those where $f^{-1}(\{\top\})$ is recursively enumerable (and thus $f^{-1}(\{\bot\})$ co-recursively enumerable).

In general, for any represented space $\mathbb{X}$ we obtain two spaces of subsets of $\mathbb{X}$: the space of open sets $\mathbb{O}(\mathbb{X})$ by identifying $f \in \mathbb{C}(\mathbb{X}, \mathbb{S})$ with $f^{-1}(\{\top\})$, and the space of closed sets $\mathbb{A}(\mathbb{X})$ by identifying $f \in \mathbb{C}(\mathbb{X}, \mathbb{S})$ with $f^{-1}(\{\bot\})$. The properties of the spaces of open and closed sets, namely computability of the usual operations, follow from a few particular computable functions on Sierpiński space $\mathbb{S}$ and the fundamental function space properties.

We require further classes of sets (often called pointclasses in this context) as represented spaces. A general approach to this is found in synthetic descriptive set theory suggested in [33]. Here, we shall just directly provide representations that suffice. Some definitions essentially already appeared in [5].

**Definition 1.** Given a represented pointclass $\Gamma$, we represent $\{A^C \mid A \in \Gamma\}$ by reinterpreting a name for $A$ as a name for $A^C$. Furthermore, we represent $\bigcup_{i \in \mathbb{N}} A_i \mid \forall i \in \mathbb{N} A_i \in \Gamma$ by identifying suitable sequences in $\prod_{i \in \mathbb{N}} \Gamma$. Given represented pointclasses $\Gamma_1, \Gamma_2$, we represent $\{A \setminus B \mid A \in \Gamma_1, B \in \Gamma_2\}$ by identifying the suitable pairs in $\Gamma_1 \times \Gamma_2$.

Note that only the first of the three constructions preserves admissibility of the representations.

We will make use of the jump of a represented space. This is based on the limit operator $\lim : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ defined via $\lim(p)(n) = \lim_{i \to \infty} p((n, i))$. We then define $(X, \delta_X)' = (X, \delta_X \circ \lim)$. This is iterated along $X^{(0)} := X$ and $X^{(n+1)} := (X^{(n)})'$. In [9], the jump was extended to multivalued functions via $(f : \mathbb{X} \to \mathbb{Y})^{[n]} := (f : \mathbb{X}^{[n]} \to \mathbb{Y})$.

### 2.2 Background on infinite sequential games

Our presentation of the required background on infinite sequential games is modelled closely on the corresponding section in [21], and on [22].

It is convenient to introduce the abstract notion of a game first, before adding additional structure later on. A game is a tuple $\langle A, (S_a)_{a \in A}, (\prec_a)_{a \in A} \rangle$ consisting of a non-empty set $A$ of agents or players, for each agent $a \in A$ a non-empty set $S_a$ of strategies, and for each agent $a \in A$ a preference relation $\prec_a \subseteq (\prod_{a \in A} S_a) \times (\prod_{a \in A} S_a)$. The generic setting suffices to introduce the notion of a Nash equilibrium: A strategy profile $\sigma \in (\prod_{a \in A} S_a)$ is called a Nash equilibrium, if for any agent $a \in A$ and any strategy $s_a \in S_a$ we find $- (\sigma \prec_a \sigma_{a \to s_a})$, where $\sigma_{a \to s_a}$ is defined by $\sigma_{a \to s_a}(b) = \sigma(b)$ for $b \in A \setminus \{a\}$ and $\sigma_{a \to s_a}(a) = s_a$. In words, no agent prefers over a Nash equilibrium some other situation that only differs in her choice of strategy.

**Definition 2 (Infinite sequential game, cf. [22] Definition 1.1)).** An infinite sequential game is an object $\langle A, C, d, O, v, (\prec_a)_{a \in A} \rangle$ complying with the following.

1. $A$ is a non-empty set (of agents).
2. $C$ is a non-empty set (of choices).
3. $d : C^* \to A$ (assigns a decision maker to each stage of the game).
4. $O$ is a non-empty set (of possible outcomes of the game).
5. $\nu : C^\omega \to O$ (uses outcomes to value the infinite sequences of choices).
6. Each $\prec_a$ is a binary relation over $O$ (modelling the preference of agent $a$).

The intuition behind the definition is that agents take turns to make a choice. Whose turn it is depends on the past choices via the function $d$ (which often will be just alternating play). Over time, the agents thus jointly generate some infinite sequence, which is mapped by $\nu$ to the outcome of the game. Note that using a single set of action $C$ for each step just simplifies the notation, a generalization to varying action sets is straightforward. In the present paper, typically $A$ and $O$ will be countable, and $C$ even finite.

The infinite sequential games are linked to abstract games as follows: the agents remain the agents and the strategies of agent $a$ are the functions $s_a : d^{-1}(\{a\}) \to C$. We can then safely regard a strategy profile as a function $\sigma : C^* \to C$ whose induced play is defined below, where for an infinite sequence $p \in C^\omega$ we let $p_n$ be its $n$-th value, and $p_{\leq n} = p_{<n+1} \in C^*$ be its finite prefix of length $n$.

**Definition 3** (Induced play and outcome, cf. [22, Definition 1.3]). Let $s : C^* \to C$ be a strategy profile. The play $p = p^\lambda(s) \in C^\omega$ induced by $s$ starting at $\lambda \in C^*$ is defined inductively through its prefixes: $p_n = \lambda_n$ for $n \leq |\lambda|$ and $p_n := s(p_{<n})$ for $n > |\lambda|$. Also, $\nu \circ p_\lambda(s)$ is the outcome induced by $s$ starting at $\lambda$. The play (resp. outcome) induced by $s$ is the play (resp. outcome) induced by $s$ starting at $\varepsilon$.

In the usual way to regard an infinite sequential game as a special abstract game, an agent prefers a strategy profile $\sigma$ to $\sigma'$, iff he prefers the outcome induced by $\sigma$ to the outcome induced by $\sigma'$. And indeed we shall call a strategy profile of an infinite sequential game a Nash equilibrium, iff it is a Nash equilibrium with these preferences. In a certain notation overload, we will in particular use the same symbols for the preferences over strategy profiles and the preferences over outcomes.

We also consider a refinement of the concept of a Nash equilibrium, namely subgame-perfect Nash equilibria. Intuitively, a strategy profile is subgame-perfect, if it still forms an equilibrium if the game were started in an arbitrary position.

### 2.3 Background on Weihrauch reducibility

**Definition 4** (Weihrauch reducibility). Let $f, g$ be multi-valued functions on represented spaces. Then $f$ is said to be Weihrauch reducible to $g$, in symbols $f \leq_W g$, if there are computable functions $K, H : \equiv N \to N^N$ such that $K(\text{id}, GH) \vdash f$ for all $G \vdash g$.

The relation $\leq_W$ is reflexive and transitive. We use $\equiv_W$ to denote equivalence regarding $\leq_W$, and by $<_W$ we denote strict reducibility. By $W$ we refer to the partially ordered set of equivalence classes. As shown in [30, 8], $W$ is a distributive lattice, and also the usual product operation on multivalued function induces an operation $\times$ on $W$. The algebraic structure on $W$ has been investigated in further detail in [18, 11].

There are two relevant unary operations defined on $W$, both happen to be closure operators. The operation $^*$ was introduced in [30, 29] by setting $f^0 := \text{id}_N$, $f^{n+1} := f \times f^n$ and then $f^*(n, x) := f^n(x)$. It corresponds to making any finite number of parallel uses of $f$ available. Similarly, the operation $\hat{\cdot}$ from [8, 7] makes countably many parallel uses available by $\hat{f}(x_0, x_1, x_2, \ldots) := (f(x_0), f(x_1), f(x_2), \ldots)$. 


We will make use of an operation $\star$ defined on $\mathcal{W}$ that captures aspects of function decomposition. Following \cite{Grz96} \cite{Sip97}, let $f \star g := \max_{W} \{ f_0 \circ g_0 \mid f \equiv_W f_0 \land g \equiv_W g_0 \}$. We understand that the quantification is running over all suitable functions $f_0, g_0$ with matching types for the function decomposition. It is not obvious that this maximum always exists, this is shown in \cite{Sip97} using an explicit construction for $f \star g$. Like function composition, $\star$ is associative but generally not commutative. We use $\star$ to introduce iterated composition via setting $f^{(0)} := \text{id}_{\mathbb{N}}$ and $f^{(n+1)} = f^{(n)} \star f$.

All computable multivalued functions with a computable point in their domain are Weihrauch equivalent, this degree is denoted by 1.

An important source for examples of Weihrauch degrees relevant in order to classify theorems are the closed choice principles studied in e.g. \cite{Brattka04} \cite{Brattka05}:

**Definition 5.** Given a represented space $X$, the associated closed choice principle $C_X$ is the partial multivalued function $C_X : \subseteq A(X) \Rightarrow X$ mapping a non-empty closed set to an arbitrary point in it.

For any uncountable compact metric space $X$ we find that $C_X \equiv_W C_{[0,1]}$. For well-behaved spaces, using closed choice iteratively does not increase its power, in particular $C_{\mathbb{N}} \star C_{\mathbb{N}} \equiv_W C_{\mathbb{N}}$ and $C_{[0,1]} \star C_{[0,1]} \equiv_W C_{[0,1]}$. Likewise, it was shown that $C_{\mathbb{R}} \equiv_W C_{\mathbb{R}} \star C_{\mathbb{R}} \equiv_W C_{\mathbb{N}} \times C_{[0,1]} \equiv_W C_N \star C_{[0,1]} \equiv_W C_{[0,1]}$ for any $n > 0$. Closed choice for $[0,1]$ and $\{0,1\}^N$ is incomparable. Furthermore, $C_{[0,1]} \equiv_W C_{[0,1]} \equiv_W C_{[0,1]}$. The degree $C_{[0,1]}$ is closely linked to WKL in reverse mathematics, while $C_N$ is Weihrauch-complete for functions computable with finitely many mindchanges.

Another typical degree is LPO, which has important representatives such as $\rightarrow : \mathbb{S} \rightarrow \mathbb{S}$, the characteristic function of $0^\mathbb{N}$, the characteristic function of 0 in $\mathbb{R}$, $\neq : \{0,1\}^N \times \{0,1\}^N \rightarrow \{0,1\}$ and IsEmpty : $A(\{0,1\}^N) \rightarrow \{0,1\}$.

Furthermore, we require the degree obtained from the limit operator $\lim : \subseteq \mathbb{N}^N \rightarrow \mathbb{N}^N$. This degree was studied by von Stein \cite{Stein84}, Mylatz \cite{Mylatz03}, and Brattka \cite{Brattka99} \cite{Brattka04}, with the latter noting in \cite{Brattka04} that it is closely connected to the Borel hierarchy. Hoyrup, Rojas and Weihrauch have shown that limit is equivalent the Radon-Nikodym derivative in \cite{Hoyrup05}. It also appears in the context of model of hypercomputation as shown by Ziegler \cite{Ziegler98} \cite{Ziegler99}, and captures precisely the additional computational power certain solutions to general relativity could provide beyond computability \cite{Simpson11}. It is related to the examples above via $C_N \times C_{[0,1]} \triangleleft_W \lim \equiv_W \lim \times \lim <_W \lim \star \lim$, and LPO $<_W LPO^* <_W C_N <_W \lim \equiv_W \lim$. The (strict) hierarchy $(\lim^{(n)})_{n \in \mathbb{N}}$ plays a very similar role in the Weihrauch degrees as the iterated Halting problems fill in the Turing degrees. We always find $f^{[n]} \leq_W f \star \lim^{(n)}$, and for many natural functions $f$, equivalence holds, since $(f \times \text{id})^{[n]} \equiv_W f \star \lim^{(n)}$.

### 2.4 Defining the problems of interest

Let $\Gamma$ be a represented pointclass over $\{0,1\}^N$. In a straightforward fashion, we can obtain a representation of the infinite sequential games with countably many agents, countably many outcomes, $C = \{0,1\}$ where the valuation function $v : \{0,1\}^N \rightarrow O$ is $\Gamma$-measurable. The representation encodes the number of agents and outcomes available, for each outcome the $\Gamma$-set of plays resulting in it, the map $d$ (as a look-up table), and the relations $\prec_a$ as look-up tables. We always assume that the inverse of any preference relation is well-founded (this guarantees
that equilibria exist). Using a canonic isomorphism \(\{0,1\}^* \cong \mathbb{N}\), we will pretend that the space of strategy profiles in such a game is \(\{0,1\}^\mathbb{N}\).

We now consider the following multivalued functions:

1. \(\text{Det}_\Gamma\) takes a two player win-loose game as input, where the first player has a winning set in \(\Gamma\). Valid outputs are the Nash equilibria, i.e. the pairs of strategies where one strategy is a winning strategy.

2. \(\text{Win}_\Gamma\) has the same inputs as \(\text{Det}_\Gamma\), and decides which player has a winning strategy.

3. \(\text{FindWS}_\Gamma\) is the restriction of \(\text{Det}_\Gamma\) to games where the first player has a winning strategy.

4. \(\text{NE}_\Gamma\) takes a game with countably many players and finitely many outcomes as input, where each finite combination of outcome comes from a \(\Gamma\)-set. The valid outputs are the Nash equilibria.

5. \(\text{NE}_\Gamma^{2p}\) is the restriction of \(\text{NE}_\Gamma\) to games with only two players.

6. \(\text{NE}_\Gamma^{\omega_0}\) takes a game with countably many players and countable many outcomes as input, where each finite combination of outcome comes from a \(\Gamma\)-set. The valid outputs are the Nash equilibria.

7. \(\text{SPE}_\Gamma\) takes as input a two-player game with finitely many outcomes and abstract-zero sum preferences, i.e. \(\prec_a = \prec_b^{-1}\). Valid outputs are the subgame perfect equilibria.

Some trivial reducibilities between these problems are: \(\text{FindWS}_\Gamma \leq^W \text{Det}_\Gamma \leq^W \text{NE}_\Gamma^{2p}\), \(\text{Det}_\Gamma \leq^W \text{SPE}_\Gamma\), and \(\text{NE}_\Gamma^{2p} \leq^W \text{NE}_\Gamma \leq^W \text{NE}_\Gamma^{\omega_0}\).

We always assume that \(\Gamma\) is closed under rescaling and finite intersection with clopens.

3 The computational content of some determinacy principles

We begin by classifying the simplest non-computable games, namely games where the first player wants to reach some closed set. This classification essentially is a uniform version of a result by Cenzer and Remmel [12].

**Theorem 6.** \(\text{FindWS}_A \equiv^W \text{Det}_A \equiv^W C_{\{0,1\}^\mathbb{N}}\).

**Proof.** \(C_{\{0,1\}^\mathbb{N}} \leq^W \text{FindWS}_A\) Given a closed subset \(A \in A(\{0,1\}^\mathbb{N})\), we can easily obtain the game where only player 1 moves, and player 1 wins iff the induced play falls in \(A\). If \(A\) is non-empty, then player 1 has a winning strategy: Play any infinite sequence in \(A\).

\(\text{FindWS}_A \leq^W \text{Det}_A\) Trivial.

\(\text{Det}_A \leq^W C_{\{0,1\}^\mathbb{N}}\) Given the open winning set of player 2, we can modify the game tree by ending the game once we know for sure that player 2 will win. Now the set of strategy profiles where either player 1 wins and player 2 cannot win, or player 2 wins and player 1 cannot prolong the game, is a closed set effectively obtainable from the game. Moreover, it is non-empty, and any such strategy profile is a Nash equilibrium.

**Proposition 7.** \(\text{Win}_A \equiv^W \text{LPO}\)

**Proof.** This follows by combining the constructions from the preceding theorem with the fact that \(\text{IsEmpty} : A(\{0,1\}^\mathbb{N}) \to \{0,1\}\) is equivalent to LPO.
We can use the results for $A$ as the base case for classifying the strength of determinacy for the difference hierarchy. For some pointclass $\Gamma$, let $\mathcal{D}_\Gamma := \{U \setminus X, (U \setminus X)^C \mid U \in \mathcal{O}, X \in \Gamma\}$.

**Lemma 8** \([\text{[2]}]\). $\text{Det}_{\mathcal{D}_\Gamma} \leq_{W} C_{\{0,1\}^n} \star \text{Det}_{\Gamma \cup \mathcal{F}} \times \text{Win}_{\Gamma \cup \mathcal{F}}$ and $\text{Win}_{\mathcal{D}_\Gamma} \leq_{W} LPO \star s \times \text{Win}_{\Gamma \cup \mathcal{F}}$.

**Proof.** We consider the winning set of the form $U \setminus X$ (w.l.o.g. let this be the winning set of the first player), and assume that $U$ is given as $U = \bigcup_{n \in \mathbb{N}} v_n \{0,1\}^n$ for some prefix-free sequence $(v_n)$ of positions in the game. Each of the countably many subgames rooted in a $v_n$ is a game with winning sets from $\Gamma \cup \mathcal{F}$, so we may use $\text{Det}_{\Gamma \cup \mathcal{F}} \times \text{Win}_{\Gamma \cup \mathcal{F}}$ to find out who wins the subgame, and a Nash equilibrium for each such game. Let $v_{n_i}$ be the subsequence of the $v_n$ where the first player wins.

Now consider the game where the first player’s winning set is the open set $U' = \bigcup_{i \in \mathbb{N}} v_{n_i} \{0,1\}^n$. The first player is winning this derived game, if he is winning the original game. Thus, by Proposition \([\text{[2]}]\) the claim $\text{Win}_{\mathcal{D}_\Gamma} \leq_{W} LPO \star s \times \text{Win}_{\Gamma \cup \mathcal{F}}$. By Theorem \([\text{[6]}]\) we can use $C_{\{0,1\}^n}$ to find a Nash equilibrium of this game. Then we combine this Nash equilibrium with those of the subgames to get a Nash equilibrium of the entire game, which is the claim $\text{Det}_{\mathcal{D}_\Gamma} \leq_{W} C_{\{0,1\}^n} \star \text{Det}_{\Gamma \cup \mathcal{F}} \times \text{Win}_{\Gamma \cup \mathcal{F}}$. \(\square\)

Let $\mathcal{D}_n$ be the $n$-th level of the difference hierarchy, i.e. $\mathcal{D}_0 = \mathcal{O} \cup A$ and $\mathcal{D}_{n+1} = \mathcal{D}_n \cup A$.

Let $\Sigma^n_0 - \text{LLPO} \subseteq \{0,1\}^n \times \{0,1\}^n \Rightarrow \{0,1\}$ be defined via $i \in \Sigma^0_0 - \text{LLPO}(p_0, p_1)$ iff $\forall k_0 \exists k_1 \ldots \exists k_n p_i(\langle k_0, \ldots, k_n \rangle) = 1$ (where $\exists = \forall$ if $n$ is odd and $\exists = \exists$ otherwise). Let $\Sigma^n_0 - \text{LEM} : \{0,1\}^n \Rightarrow \{0,1\}$ be defined via $\Sigma^n_0 - \text{LEM}(p) = 1$ iff $\forall k_0 \exists k_1 \ldots \exists k_n p(\langle k_0, \ldots, k_n \rangle) = 1$ and $\Sigma^n_0 - \text{LEM}(p) = 0$ otherwise. (cf. \([\text{[1]} \; \text{[9]} \; \text{[17]}]\)) Then:

**Proposition 9.** $\Sigma^n_{n+1} - \text{LLPO} \equiv_{W} \text{LLPO}^{[n]}$ and $\Sigma^n_{n+1} - \text{LEM} \equiv_{W} \text{LPO}^{[n]}$

**Lemma 10.** $\Sigma^n_{n+1} - \text{LLPO} \leq_{W} \text{Det}_{\mathcal{D}_n}$ and $\Sigma^n_{n+1} - \text{LEM} \leq_{W} \text{Win}_{\mathcal{D}_n}$.

**Proof.** We will first describe the construction for $\Sigma^n_{n+1} - \text{LEM} \leq_{W} \text{Win}_{\mathcal{D}_n}$, which will then be reused for the remaining claim. The game structure will only depend on the parameter $n$, but not on the actual input for $\Sigma^n_0 - \text{LEM}$. This input acts only on the winning set.

The game works as follows: The second player may pick some $k_1 \in \mathbb{N}$, or refuse to play. If the second player picks a number, then the first player may pick $k_2 \in \mathbb{N}$ or refuse to play. This alternating choice continues until $k_{n-1}$ has been chosen, or a player refuses to pick. A player refusing to pick a number loosens. If all numbers are picked, the winner depends on the input $p$ to $\Sigma^n_0 - \text{LEM}$ as follows: If $n$ is even and $\exists k_n \forall k_1 \exists k_2 \ldots \exists k_n p(\langle k_1, \ldots, k_n \rangle) = 1$, then player 1 wins. If $n$ is odd, and $\exists k_n \forall k_1 \exists k_2 \ldots \exists k_{n-1} \exists k_n p(\langle k_1, \ldots, k_n \rangle) = 0$, then player 2 wins. Note that this always describes an open component $U$ picked of the winning set of the respective player.

Furthermore, note that the set of plays $U_j$ where a value for $k_j$ was chosen is always an open set. Now the condition that the second player refused to pick first is $U_j \cup (U_2 \cap U_3) \cup (U_4 \cap U_5) \cup \ldots$. This makes for a winning set in $\mathcal{D}_n$, as required. If player 1 has a winning strategy in the game, the answer to $\Sigma^n_0 - \text{LEM}(p)$ is 1, if player 2 wins, it is 0.

The game for the reduction $\Sigma^n_{n+1} - \text{LLPO} \leq_{W} \text{Det}_{\mathcal{D}_n}$ adds two layers above the game discussed before. First, player 2 picks an index $j \in \mathbb{N}$ of one of the input pairs $\langle p^j_0, p^j_1 \rangle, \langle p^j_2, p^j_3 \rangle, \ldots$ of $\Sigma^n_0 - \text{LLPO}$, and looses the game if he refuses to pick. Then player 1 picks $i \in \{0,1\}$, and they play the game above on $p^j_i$. The extra layers do not impact the complexity of the winning.

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1This is a generalization of a proof idea from \([\text{[28]}]\) by NEMOTO, MedSALEM and TANAKA.
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set, in particular since the first two natural numbers are both to be chosen by player 2. The map \( j \mapsto i \) extractable from player 1’s winning strategy is a valid solution to \( \Sigma^0_n - \text{LLPO} \).

**Theorem 11.** \( \text{Det}_n \equiv^W \mathcal{C}_{\{0,1\}^n}^{[n]} \) and \( \text{Win}_n \equiv^W \text{LPO}^{[n]} \).

**Proof.** Note that \( \text{LPO}^{[n]} \equiv^W \text{LLPO}^{[n]} \) and \( \text{LPO}^{[n]} \equiv^W \text{lim}^{(n+1)} \). One direction of the equivalences is provided by Lemma 10 (while taking into consideration Proposition 9). The other direction is shown by induction. The base case is provided by Theorem 6 and Proposition 7. The induction step uses Lemma 8.

Before ending this section, we shall make explicit a feature of the constructions used above:

**Proposition 12.** Consider a pointclass \( \Gamma \) that is closed under the operation \((A_n)_{n \in \mathbb{N}} \mapsto (\{1^{\mathbb{N}}\} \cup \bigcup_{n \in \mathbb{N}} 0^n 1 A_n)\). Then:

\[
\text{FindWS}_\Gamma \equiv^W \text{FindWS}_\Gamma
\]

**Proof.** Given a sequence of games, we construct a single combined game as follows: The second player picks a natural number, and loses the game if he fails to do so. If a number is chosen, the players proceed to the game with the corresponding index. If the first player has winning strategies in all input games, he can win the combined game by playing their combination.

4 The complexity of equilibrium transfer

In [22, 23, 24], various results were provided that transfer Borel determinacy (or, somewhat more general, determinacy for some pointclass), to prove the existence of Nash equilibria (and sometimes even subgame-perfect equilibria) in multi-player multi-outcome infinite sequential games. In this section, we shall inspect those constructions and extract Weihrauch reductions from them.

In [23], the first author gave a very general construction that allows to extend determinacy of win/lose games to the existence of Nash equilibria for two player games of the same type. The computational content is the following:

**Theorem 13 (Equilibrium transfer).** \( \text{NE}_p^2 \leq^W \text{Det}_p^2 \ast \text{Win}_p^2 \) and \( \text{NE}_p^2 \leq^W \text{Det}_p^p \times \text{Win}_p^p \)

**Proof.** For any upper set of outcomes (for either players preferences), we construct the win/lose derived game where that player wins, iff he enforces such an outcome, and loses otherwise. There are finitely many such games, so we can use \( \text{Win}_p^\ast \) to decide which are won and which are lost. As shown in [23], there will be a combination of upper sets of outcomes for each player, such that if both players enforce their upper set, this forms a Nash equilibrium. If can either use \( \text{Det}_p^\ast \) twice after having learned which outcomes we need to enforce, or use \( \text{Det}_p^\ast \times \text{Win}_p^\ast \) to compute Nash equilibria for all derived games in parallel, and then simply select the suitable strategies.

Techniques suitable for multiplayer sequential games where then introduced in [22], again by the first author. The computational content differs somewhat depending on whether there are finitely many different outcomes, or countably many.

**Theorem 14.** \( \text{NE}_p \leq^W \text{Win}_p \times \text{Det}_p \), \( \text{NE}^\omega \leq^W (\lim \ast \text{Win}_p) \times \text{Det}_p \)
Proof. Let us consider a game with finitely many players and linear preference relations. (Considering countably many players would be possible, but it would reduce to the finite case since there are only finitely many possible preference relations; considering acyclic preference relations would be possible, but it would reduce to the linear case by linear extension.) For each node of the game and each upper interval of the preference of the player owning the node (that is, for countably many cases) let us do two things in parallel: on the one hand, invoke Win \( \Gamma \) and ask who is winning the win/lose subgame rooted at the node where all the opponents of the node owner team up and try to yield an outcome outside of the given preference-upper interval; on the other hand, invoke Det \( \Gamma \) and obtain a Nash equilibrium for that same game. We claim that this information suffices to computably build a Nash equilibrium for the original game along the proof from [22]: indeed, the best guarantee [22, Definition 2.5] of a player, which is the smallest preference-upper interval that the player can enforce, can be computed since we already know who wins the relevant derived games thanks to Win \( \Gamma \); the existential witness from [22, Lemma 2.6] is a Nash equilibrium that has been already computed by Det \( \Gamma \); which is enough to deepen the guarantee as in [22, Definition 2.7] and build the strategy profile \( \sigma \) [22, Lemma 2.8]; and finally, the threats that [22, Theorem 2.9] attaches along the play of \( \sigma \) are given by the Nash equilibria that have been already computed by Det \( \Gamma \).

Let us now consider the similar case with countably many players and outcomes where the inverses of the preferences are well-ordered. At each node, the associate games are now countably many, so knowing who wins each of them no longer suffices to compute the best guarantee of the node owner. To this purpose we use one lim operator per node, and the rest follows as in the finite case above, so NE\( \Gamma^\omega \) \( \leq_W (\hat{\lim} \ast \text{Win} \Gamma) \times \hat{\text{Det}} \Gamma \), and the claim follows since \( \lim \equiv_W \hat{\lim} \).

A further improvement on the techniques in [22] where provided by the authors in [24]. These techniques in particular suffice to prove the existence of subgame-perfect equilibria in abstract zero-sum games (this implies two players).

**Theorem 15.** SPE\( \Gamma \) \( \leq_W \text{Win} \Gamma \ast \hat{\text{Det}} \Gamma \)

**Proof.** Like in the proof of Theorem 14 let us ask at every node who wins the derived games and ask for a corresponding Nash equilibrium. These two pieces information together provide the existential witness needed in the third condition of [24, Lemma 16], so by the recursive construction in the proof of this lemma, its conclusion follows computably. Let us invoke this conclusion once for each of the two players and combine the obtained strategies into a strategy profile, which is a subgame-perfect equilibrium by determinacy assumptions. (See [24, Lemma 17] for the details.)

5 Deciding the winner and finding Nash equilibria

The results in Section 3 show that for many concrete examples of \( \Gamma \), the problem Det \( \Gamma \) is inherently multivalued, i.e. not equivalent to any functions between admissible spaces. On the other hand, the upper bounds provided in Section 4 all include Win \( \Gamma \), which is of course single-valued. In the current section, we will explore some converse reductions, from deciding the winner to finding Nash equilibria. This generally requires some (rather tame) requirements on the pointclasses involved.
Lemma 16. Let $\Gamma$ be obtained by $\Gamma_1$ by first closing under finite union, rescaling and union with clopens; and then adding complements. Then:

$$\text{Win}_i^\ast \leq_W \text{NE}_T^{2p}$$

Proof. We first informally describe the construction. Given $n$ win/lose games, the first player starts by announcing which of these games she believes she can win. Then the second player can choose one of the listed games to play. If the first player did not claim any winnable games, the game ends and the outcome is 0. If the first player claimed to be able to win $k$ out of $n$ games, then the outcomes of the games subsequently chosen by the second player are scaled up to $k, -k$. Thus, the first player has every reason to list precisely those games she can actually win: If she would not list a game she could win, she trades payoff $k - 1$ for payoff $k$. If she lists a game she cannot win, the second player will subsequently choose and win it, and then the first player is punished by $-k$.

The following depicts the construction in case of two input games:

```
G_0 is played with payoffs 1; -1
  / \
G_1 is played with payoffs 2; -2
```

It remains to argue that the resulting game actually is a valid input to $\text{NE}_T^{2p}$. We need to show that any upper set of outcomes is associated with a set of plays belonging to $\Gamma$. For $k > 0$, the upper set is a finite union of rescaled sets from $\Gamma_1$, which by assumption is a member of $\Gamma$. For $k = 0$, to the former we add the clopen set of plays resulting in 0. For $k < 0$, we add additional clopens for those subgames played with stakes $< |k|$.

Lemma 17. Let $\Gamma$ be closed under taking unions with $\Gamma_1$ and $\overline{\Gamma_1}$. Then:

$$\text{NE}_T^{2p} \times \text{FindWS}_{\Gamma_1} \equiv_W \text{NE}_T^{2p}$$

Proof. As clearly $1 \leq_W \text{FindWS}_{\Gamma_1}$, we only need to show $\text{NE}_T^{2p} \times \text{FindWS}_{\Gamma} \leq_W \text{NE}_T^{2p}$. Let $G_0$ be the input game to $\text{NE}_T^{2p}$ on the left, and $G_1$ be the input game to $\text{FindWS}_{\Gamma}$. If $\Omega$ is the (twice ordered) set of outcomes used in $G_0$, then we use the outcome set $\Omega \cup \{0, 1\}$ for the newly constructed game, and extend the preferences by $0 \prec_a o \prec_a 1$ and $1 \prec_b o \prec_b 0$ for each $o \in \Omega$. The game tree looks as follows: The second player can choose whether to play in $G_1$ for outcomes 0 (second player wins) and 1 (second player loose), or to play in $G_0$ for the original outcomes:

```
G_0 \rightarrow b \rightarrow G_1 \rightarrow 0; 1
```

In any Nash equilibrium of this game the second player is choosing to play in $G_0$, both players play a Nash equilibrium inside $G_0$, and the first player is using a winning strategy inside $G_1$. Thus, all desired information can be recovered.
In order for the constructed game to be valid for $\text{NE}^2p$, we need that any upper set from $\Gamma$ is closed under union with a winning set of player 1.

**Corollary 18.** Let $\Gamma_1$ and $\Gamma$ simultaneously satisfy the criteria of the two preceding lemmata. Then:

$$\text{FindWS}^*_\Gamma_1 \times \text{Win}^*_\Gamma_1 \leq_W \text{NE}^2p$$

Unfortunately, the restrictions on $\Gamma$, $\Gamma_1$ in place in Lemma 17 (and subsequently Corollary 18) are too strong for the application we have in mind. The result of Corollary 18 can be obtained with weaker conditions though:

**Lemma 19.** Let $\Gamma$ be obtained by $\Gamma_1$ by first closing under finite union, rescaling and union with clopens; and then adding complements. Then:

$$\text{FindWS}^*_\Gamma_1 \times \text{Win}^*_\Gamma_1 \leq_W \text{NE}^2p$$

**Proof.** The reduction directly combines the constructions in Lemma 16 and Lemma 17. We only need to argue that the weaker condition on $\Gamma_1$ and $\Gamma$ suffices to have the valuation $\Gamma$-measurable in the resulting game. For this, note that the same reasoning as in Lemma 16 applies, with the addition of a $\Gamma_1$ set above the other outcomes, and a $\Gamma_1$ set below. □

### 6 General games with concrete pointclasses

The general constructions put together with the classifications for specific pointclasses allow us to obtain some concrete Weihrauch degrees. First, we shall see that moving from a win/lose game with closed and open outcomes to a two player game with several outcomes just complicates the operation of finding Nash equilibria by finitely many uses of LPO in parallel:

**Theorem 20.** $\text{NE}^2p_{\mathcal{O} \cup \mathcal{A}} \equiv_W C_{\{0,1\}^N} \times \text{LPO}^*$

**Proof.** For the reduction $\text{NE}^2p_{\mathcal{O} \cup \mathcal{A}} \leq_W C_{\{0,1\}^N} \times \text{LPO}^*$, instantiate Theorem 13 with the results from Theorem 6 and Proposition 7. For the other direction, note $\text{FindWS}_{\mathcal{A}} \equiv_W \text{FindWS}_{\mathcal{O} \cup \mathcal{A}} \equiv_W C_{\{0,1\}^N}$ as in Theorem 6 and that $\Gamma_1 := \mathcal{A}$ and $\Gamma := \mathcal{O} \cup \mathcal{A}$ satisfy the requirements of Lemma 19 which then provides the desired result. □

If one wishes to have subgame-perfect equilibria instead of mere Nash equilibria, then countably many uses of LPO become necessary, and the problem becomes equivalent to lim:

**Theorem 21.** $\text{SPE}_{\mathcal{O} \cup \mathcal{A}} \equiv_W \text{lim}$

**Proof.** For $\text{SPE}_{\mathcal{O} \cup \mathcal{A}} \leq_W \text{lim}$, instantiate Theorem 15 with the results from Theorem 6 and Proposition 7, note that $\text{LPO} \equiv_W \text{lim}$ and use the uniform low basis theorem [6], which provides $\text{lim}^* C_{\{0,1\}^N} \equiv_W \text{lim}$. For the other direction we shall prove $\text{lim} \leq_W \text{SPE}_{\mathcal{O} \cup \mathcal{A}}$. We shall construct a single player game with three outcomes 0, $\frac{1}{2}$, 1. The player can pick a natural number $n \in \mathbb{N}$, or refuse to pick. After picking $n$, he can play left or play right. Playing right always provides a payoff of $\frac{1}{2}$. If the $n$-th input to $\text{LPO}$ is not $0^N$, then going left after picking $n$ gives 0, if the $n$-th input is $0^N$, then the player gets 1 instead. If the player refuses to pick a number, he gets 1, too. Then
the plays yielding 0 are an open set, those yielding $\frac{1}{2}$ are an open set, too, and those yielding 1 are a closed set.

In a subgame perfect equilibrium, the (hypothetical) choice of the player after $n$ was picked has to be left, if the $n$-th input to $\hat{LPO}$ is not $0^N$, and right if it was $0^N$. Thus, we can read off the correct answer from the subgame perfect equilibrium.

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