A characterization of vertex operator algebras $V_{Z\alpha}^+$: I

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Abstract

A characterization of vertex operator algebra $V_{Z\alpha}^+$ with $(\alpha, \alpha)/2$ not being a perfect square is given in terms of dimensions of homogeneous subspaces of small weights. This result contributes to the classification of rational vertex operator algebras of central charge 1.

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1 Introduction

Classification of rational vertex operator algebras is definitely one of the most important problems in the theory of vertex operator algebras and is crucial in classification of rational conformal field theory. Although there is a lot of progress in the field, one still has very limited knowledge on the structure theory for vertex operator algebras. There are two different directions in classification currently. One direction is the classification of holomorphic vertex operator algebras which have the simplest representation theory (see [S], [DM2], [LS]). Another direction is the classification of rational vertex operator algebras with small central charges. It is established in [DZ] that if the central charges are less than one, the vertex operator algebra is an extension of the vertex operator algebra associated to the discrete series for the Virasoro algebra. But such extensions have not been constructed and classified except for some special cases. In the operator algebra setting, classification of local conformal nets with $c < 1$ has been completed in [KL]. It is natural to consider classification of rational vertex operator algebras of central charge 1 next. This has been achieved at the character level in the physical literature under the assumption that the character of each irreducible module is a modular function over a congruence subgroup of the modular group [K]. The classification of conformal nets of central charge 1 has been given in [X] with some extra assumption.

It is conjectured that there are three classes of rational vertex operator algebras of central charge 1: (a) vertex operator algebras $V_L$ associated with positive definite lattices $L$ of

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rank one, (b) orbifold vertex operator algebras $V_L^+$ under the automorphism of $V_L$ induced from the $-1$ isometry of $L$, (c) $V_{Z\alpha}^+$ where $(\beta, \beta) = 2$ and $G$ is the finite subgroup of $SO(3)$ isomorphic to $A_4, S_4, A_5$. The lattice vertex operator algebra $V_L$ for any positive definite even lattice $L$ has been characterized in [DM1]. This paper gives a characterization of $V_L^+$ for $L = Z\alpha$ with $(\alpha, \alpha)/2$ not being a perfect square. A complete characterization of $V_L^+$ for any rank one positive definite even lattice $L$ will be given in a subsequent paper. The characterization of $V_{Z\alpha}^+$ with $(\alpha, \alpha) = 4$ has been obtained previously in [ZD] and [DJ]. We should mention that the conjecture also requires the effective central charge $\tilde{c} = 1$ (see [ZD], [DJ]).

There are three major steps in the characterization of $V_L^+$. Let $V$ be a rational vertex operator algebra of central charge 1 satisfying certain conditions. The first step is to show that the Virasoro vector $\omega$ and a weight 4 primary vector $J$ for the Virasoro algebra generate a subalgebra isomorphic to $M(1)^+$ which is the fixed points of the Heisenberg vertex operator algebra of rank one under the $-1$ automorphism. The main idea is to use the fusion rules for the Virasoro vertex operator algebra $L(1, 0)$ (see [M], [DJ]) and a result on the $W$-algebra. The second step is to show that $V$ is a completely reducible $M(1)^+$-module by using the fusion rules for the vertex operator algebras $L(1, 0)$ and $M(1)^+$. The last step is to establish that the subalgebra $M(1)^+$ and a primary vector $F \not\in M(1)^+$ of minimal weight generate a subalgebra isomorphic to $V_{Z\alpha}^+$ with $(\alpha, \alpha) = 2k$ where $k$ is the weight of $F$. The argument depends heavily on the decomposition of $V_{Z\alpha}^+$ as an $M(1)^+$-module and the fusion rules.

We now explain why we only consider the characterization of $V_{Z\alpha}^+$ with $(\alpha, \alpha)/2$ not being a perfect square in the present paper. In this case, $V_{Z\alpha}^+$ contains a primary vector whose weight is not a perfect square. On the other hand, if $(\alpha, \alpha)/2$ is a perfect square, the weight of any primary vector in $V_{Z\alpha}^+$ is a perfect square. One assumption in the paper is that there is a primary vector whose weight is not a perfect square. This assumption is crucial in producing a vertex operator subalgebra isomorphic to $M(1)^+$. Without this assumption, it is much more difficult to obtain a subalgebra inside $V$ isomorphic to $M(1)^+$ as the fusion rules among irreducible $L(1, 0)$-modules $L(1, n^2)$ are more complicated (see Theorem 2.6).

As in [DJ] there is an assumption that $V$ is a sum of highest weight modules for the Virasoro algebra in the paper. Although we believe that this assumption is not necessary but we do not know how to obtain this assumption from the others. On the other hand, we do not need to assume that the effective central charge $\tilde{c}$ is equal to the central charge $c$ in this paper as $V_2$ is assumed to be one dimensional.

This paper is organized as follows. In Section 2 we recall some basic and important results on vertex operator algebras $L(1, 0)$, $M(1)^+$ and $V_L^+$ including the fusion rules. Section 3 is about the $W_3$ algebra. We show that the simple vertex operator algebra associated to $W_3$ of central charge 1 is not a completely reducible module for the Virasoro algebra. This result will be used in later sections to deal with the case that the dimension of the weight 3 subspace of the vertex operator algebra is greater than 2. In Section 4 we demonstrate that the vertex operator algebra $V$ contains a subalgebra isomorphic to $M(1)^+$. In Section 5, we first prove that $V$ is a completely reducible $M(1)^+$-module.
Then we give the main result of this paper. That is, a vertex operator algebra satisfying certain conditions is isomorphic to $V^+_L$ for some rank one lattice $L$.

## 2 Preliminaries

Let $V = (V, Y, 1, \omega)$ be a vertex operator algebra [E], [FLM]. We assume the authors are familiar with various notions of $V$-modules and the definition of rational vertex operator algebras (cf. [FLM], [Z], [DLM]). We briefly review the vertex operator algebras associated to the highest weight representations for the Virasoro algebra and the rank one rational vertex operator algebra $V^+_L$.

Here are some basic facts about the highest weight modules for the Virasoro algebra Vir. Let $c, h \in \mathbb{C}$ and $V(c, h)$ be the highest weight module for the Virasoro algebra Vir with central charge $c$ and highest weight $h$. Let $\hat{V}(c, 0) = V(c, 0)/U(Vir)L_{-1}v$ where $v$ is a highest weight vector with highest weight 0 and denote the irreducible quotient of $V(c, h)$ by $L(c, h)$. We have (see [KR], [FZ]):

**Proposition 2.1.** Let $c$ be a complex number.

1. $\hat{V}(c, 0)$ is a vertex operator algebra and $L(c, 0)$ is a simple vertex operator algebra.
2. For any $h \in \mathbb{C}$, $V(c, h)$ is a module for $\hat{V}(c, 0)$.
3. $V(c, h) = L(c, h)$, $\hat{V}(c, 0) = L(c, 0)$, for $c > 1$ and $h > 0$.
4. $V(1, h) = L(1, h)$ if and only if $h \neq \frac{m^2}{4}$ for all $m \in \mathbb{Z}$. In case $h = m^2$ for a nonnegative integer $m$, the unique maximal submodule of $V(1, m^2)$ is generated by a highest weight vector with highest weight $(m + 1)^2$ and is isomorphic to $V(1, (m + 1)^2)$.

We need to review the vertex operator algebras $M(1)^+, V^+_L$ and related results [A1], [A2], [AD], [ADL], [DN1], [DN2], [DN3], [DJL], [FLM].

Let $L = \mathbb{Z}\alpha$ be a positive definite even lattice of rank one. That is, $(\alpha, \alpha) = 2k$ for some positive integer $k$. Set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend $(\cdot, \cdot)$ to a $\mathbb{C}$-bilinear form on $\mathfrak{h}$. Let $\mathfrak{h} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h} \oplus CK$ be the affine Lie algebra associated to the abelian Lie algebra $\mathfrak{h}$ so that

$$[\alpha(m), \alpha(n)] = 2km\delta_{m,-n}K \text{ and } [K, \mathfrak{h}] = 0$$

for any $m, n \in \mathbb{Z}$ where $\alpha(m) = \alpha \otimes t^m$. Then $\mathfrak{h}^{\geq 0} = \mathbb{C}[t] \otimes \mathfrak{h} \oplus CK$ is a commutative subalgebra. For any $\lambda \in \mathfrak{h}$, we define a one dimensional $\mathfrak{h}^{\geq 0}$-module $\mathbb{C}e^\lambda$ such that $\alpha(m) \cdot e^\lambda = (\lambda, \alpha)\delta_{m,0}e^\lambda$ and $K \cdot e^\lambda = e^\lambda$ for $m \geq 0$. We denote by

$$M(1, \lambda) = U(\mathfrak{h}) \otimes_{U(\mathfrak{h}^{\geq 0})} \mathbb{C}e^\lambda \cong S(t^{-1}\mathbb{C}[t^{-1}])$$

the $\mathfrak{h}$-module induced from $\mathfrak{h}^{\geq 0}$-module $\mathbb{C}e^\lambda$. Set

$$M(1) = M(1, 0).$$

Then there exists a linear map $Y : M(1) \to \operatorname{End} M(1)[[z, z^{-1}]]$ such that $(M(1), Y, 1, \omega)$ carries a simple vertex operator algebra structure and $M(1, \lambda)$ becomes an irreducible
operator algebra associated to $L$ isomorphic to one of the following modules:

Any irreducible module for the vertex operator algebra $V_L$ is given by

$$V_L = M(1) \otimes \mathbb{C}[L].$$

The dual lattice $L^\circ$ of $L$ is

$$L^\circ = \{ \lambda \in \mathfrak{h} \mid (\alpha, \lambda) \in \mathbb{Z} \} = \frac{1}{2k}L.$$

Then $L^\circ = \bigcup_{i=-k+1}^k (L + \lambda_i)$ is the coset decomposition with $\lambda_i = \frac{i}{2k} \alpha$. In particular, $\lambda_0 = 0$. Set $\mathbb{C}[L + \lambda_i] = \bigoplus_{\beta \in L} \mathbb{C} e^{\beta + \lambda_i}$. Then each $\mathbb{C}[L + \lambda_i]$ is an $L$-submodule in an obvious way. Set $V_{L+\lambda_i} = M(1) \otimes \mathbb{C}[L + \lambda_i]$. Then $V_L$ is a rational vertex operator algebra and $V_{L+\lambda_i}$ for $i = -k + 1, \ldots, k$ are the irreducible modules for $V_L$ (see [FLM], [F], [D1], [DLM]).

Define a linear isomorphism $\theta : V_{L+\lambda_i} \rightarrow V_{L-\lambda_i}$ for $i \in \{-k + 1, \ldots, k\}$ by

$$\theta(\alpha(-n_1)\alpha(-n_2)\cdots\alpha(-n_k) \otimes e^{\beta + \lambda_i}) = (-1)^k \alpha(-n_1)\alpha(-n_2)\cdots\alpha(-n_k) \otimes e^{-\beta - \lambda_i}$$

where $n_j > 0$ and $\beta \in L$. Then $\theta$ defines a linear isomorphism from $V_{L^\circ} = M(1) \otimes \mathbb{C}[L^\circ]$ to itself such that

$$\theta(Y(u, z)v) = Y(\theta u, \theta v)$$

for $u \in V_L$ and $v \in V_{L^\circ}$. In particular, $\theta$ is an automorphism of $V_L$ which induces an automorphism of $M(1)$.

For any $\theta$-stable subspace $U$ of $V_{L^\circ}$, let $U^\pm$ be the $\pm 1$-eigenspace of $U$ for $\theta$. Then $V_L^+$ is a simple vertex operator algebra.

Also recall the $\theta$-twisted Heisenberg algebra $\mathfrak{h}[-1]$ and its irreducible module $M(1)(\theta)$ from [FLM]. Let $\chi_s$ be a character of $L/2L$ such that $\chi_s(\alpha) = (-1)^s$ for $s = 0, 1$ and $T_{\chi_s} = \mathbb{C}$ the irreducible $L/2L$-module with character $\chi_s$. It is well known that $V^{T_{\chi_s}}_L = M(1)^{}(\theta) \otimes T_{\chi_s}$ is an irreducible $\theta$-twisted $V_L$-module (see [FLM], [D2]). We define actions of $\theta$ on $M(1)(\theta)$ and $V^{T_{\chi_s}}_L$ by

$$\theta(\alpha(-n_1)\alpha(-n_2)\cdots\alpha(-n_k)) = (-1)^k \alpha(-n_1)\alpha(-n_2)\cdots\alpha(-n_k)$$

$$\theta(\alpha(-n_1)\alpha(-n_2)\cdots\alpha(-n_k) \otimes t) = (-1)^k \alpha(-n_1)\alpha(-n_2)\cdots\alpha(-n_k) \otimes t$$

for $n_j \in \frac{1}{2} + \mathbb{Z}_+$ and $t \in T_{\chi_s}$. We denote the $\pm 1$-eigenspaces of $M(1)(\theta)$ and $V^{T_{\chi_s}}_L$ under $\theta$ by $M(1)(\theta)^\pm$ and $(V^{T_{\chi_s}}_L)^\pm$ respectively. We have the following results:

**Theorem 2.2.** Any irreducible module for the vertex operator algebra $M(1)^+ \,$ is isomorphic to one of the following modules:

$$M(1)^+, M(1)^-, M(1, \lambda) \cong M(1, -\lambda) \quad (0 \neq \lambda \in \mathfrak{h}), M(1)(\theta)^+, M(1)(\theta)^-.$$
Theorem 2.3. Any irreducible $V^+_L$-module is isomorphic to one of the following modules:

$$V^\pm, V_{\lambda+L}(i \neq k), V^\pm_{\lambda+L}, (V^T_{\lambda^*})^\pm.$$ 

Theorem 2.4. $V^+_L$ is rational.

We remark that the classification of irreducible modules for arbitrary $M(1)^+$ and $V^+_L$ are obtained in [DN1]-[DN3] and [AD]. The rationality of $V^+_L$ is established in [A2] for rank one lattice and [DJL] in general.

We next turn our attention to the fusion rules of vertex operator algebras. Let $V$ be a vertex operator algebra, and $W^i (i = 1, 2, 3)$ be ordinary $V$-modules. We denote by $I_V(W^3 W^1 W^2)$ the vector space of all intertwining operators of type $W^3 W^2 W^1$. For a $V$-module $W$, let $W'$ denote the graded dual of $W$. Then $W'$ is also a $V$-module [FHL]. It is well known that fusion rules have the following symmetry (see [FHL]).

Proposition 2.5. Let $W^i (i = 1, 2, 3)$ be $V$-modules. Then

$$\dim I_V\left(\begin{array}{c} W^3 \\ W^1 W^2 \end{array}\right) = \dim I_V\left(\begin{array}{c} W^3 \\ W^2 W^1 \end{array}\right), \quad \dim I_V\left(\begin{array}{c} W^3 \\ W^1 W^2 \end{array}\right) = \dim I_V\left(\begin{array}{c} (W^2)' \\ W^1 (W^3)' \end{array}\right).$$

The following two results were obtained in [M] and [DJ].

Theorem 2.6. (1) We have

$$\dim I_{L(1,0)}\left(\begin{array}{c} L(1, k^2) \\ L(1, m^2) L(1, n^2) \end{array}\right) = 1, \quad k \in \mathbb{Z}_+, \ |n - m| \leq k \leq n + m,$$

$$\dim I_{L(1,0)}\left(\begin{array}{c} L(1, k^2) \\ L(1, m^2) L(1, n^2) \end{array}\right) = 0, \quad k \in \mathbb{Z}_+, \ k < |n - m| \text{ or } k > n + m,$$

where $n, m \in \mathbb{Z}_+$.  

(2) For $n \in \mathbb{Z}_+$ such that $n \neq p^2$, for all $p \in \mathbb{Z}_+$, we have

$$\dim I_{L(1,0)}\left(\begin{array}{c} L(1, n) \\ L(1, m^2) L(1, n) \end{array}\right) = 1,$$

$$\dim I_{L(1,0)}\left(\begin{array}{c} L(1, k) \\ L(1, m^2) L(1, n) \end{array}\right) = 0,$$

for $k \in \mathbb{Z}_+$ such that $k \neq n$.

(3) Let $U$ be a highest weight module for the Virasoro algebra generated by the highest weight vector $u^{(r)}$ such that

$$L(0)u^{(r)} = r^2 u^{(r)}, \quad L(k)u^{(r)} = 0, \quad k \in \mathbb{Z}_+ \setminus \{0\}.$$

Let $m, n \in \mathbb{Z}_+ \setminus \{0\}$ be such that $m \neq n$ and $m, n$ are not perfect squares. Then

$$I_{L(1,0)}\left(\begin{array}{c} U \\ L(1, m) L(1, n) \end{array}\right) = 0.$$
We also need the following result from [1] later on.

**Theorem 2.7.** Let $M, N$ and $T$ be irreducible $M(1)^+$-modules. If $M = M(1, \lambda)$ such that $\lambda \neq 0$, then

$$\dim I_{M(1)^+}(T_{MN}) = 0 \text{ or } 1$$

and

$$\dim I_{M(1)^+}(T_{MN}) = 1$$

if and only if $(N, T)$ is one of the following pairs:

$$(M(1)^{\pm}, M(1, \mu))(\lambda^2 = \mu^2), \ (M(1, \mu), M(1, \nu)), \ (\nu^2 = (\lambda \pm \mu)^2),$$

$$(M(1)(\theta)^{\pm}, M(1)(\theta)^{\pm}), \ (M(1)(\theta)^{\pm}, M(1)(\theta)^{\mp}).$$

### 3 $W_3$ algebra

In this section, we recall $W_3$ algebra, the associated vertex operator algebra and its irreducible quotient (see [BMP], [W] and references therein). We will also give some new results on $W_3$ with the central charge $C = 1$.

Let $W_3$ be the associative algebra generated by $L_m, W_m, m \in \mathbb{Z}$ subject to the following relations:

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} C,$$

$$[L_m, W_n] = (2m - n)W_{m+n},$$

$$[W_m, W_n] = (m - n)[\frac{1}{15}(m + n + 2)(m + n + 3) - \frac{1}{6}(m + 2)(n + 2)]L_{m+n}$$

$$+ \frac{16}{22 + 5C}(m - n)\Lambda_{m+n} + \frac{m(m^2 - 1)(m^2 - 4)}{360} \delta_{m+n,0} C,$$

(3.2)

where

$$\Lambda_m = \sum_{k \geq 2} L_{-k}L_{m+k} + \sum_{k \geq -1} L_{m-k}L_k - \frac{3}{10}(m + 2)(m + 3)L_m,$$

and $C$ is a non-zero central element. Denote by $W_{3,\pm}$ and $W_{3,0}$ the subalgebras of $W_3$ generated by $\{L_m, W_m|m \geq 0\}$ and $\{L_0, W_0, C\}$, respectively. Then $W_{3,0} = W_{3,+} + W_{3,0}$ is also a subalgebra. For $c, \lambda, \mu \in \mathbb{C}$, let $C_{\lambda,\mu} = \mathbb{C}1_{\lambda,\mu}$ be the one dimensional module of $W_{3,0}$ such that

$$L_01_{\lambda,\mu} = \lambda 1_{\lambda,\mu}, \ W_01_{\lambda,\mu} = \mu 1_{\lambda,\mu},$$

$$C \cdot 1_{\lambda,\mu} = c1_{\lambda,\mu}, \ L_m1_{\lambda,\mu} = W_m1_{\lambda,\mu} = 0$$

for $m > 0$. Denote by $\mathcal{M}(c, \lambda, \mu)$ the induced module:

$$\mathcal{M}(c, \lambda, \mu) = W_3 \otimes_{W_{3,0}} C_{\lambda,\mu}.$$
It is well known that \( \mathcal{M}(c, \lambda, \mu) \) has a unique irreducible quotient which is denoted by \( \mathcal{L}(c, \lambda, \mu) \). It is easy to see that

\[
CL_{-1}1 + CW_{-1}1 + CW_{-2}1
\]

is an invariant subspace of \( \mathcal{M}(c, 0, 0) \) under the action of \( W_{3,+} \). Let \( \mathcal{J} \) be the submodule of \( \mathcal{M}(c, 0, 0) \) generated by the three vectors \( L_{-1}1, W_{-1}1 \) and \( W_{-2}1 \). Denote by \( \mathcal{M}(c, 0, 0) \) the quotient \( \mathcal{M}(c, 0, 0)/\mathcal{J} \). Let

\[
L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \quad W(z) = \sum_{n \in \mathbb{Z}} W_n z^{-n-3}.
\]

Then \( \mathcal{M}(c, 0, 0) \) is a vertex operator algebra generated by \( \omega = L_{-2}1 \) and \( w = W_{-3}1 \) such that \( Y(\omega, z) = L(z) \) and \( Y(w, z) = W(z) \). The irreducible quotient \( \mathcal{L}(c, 0, 0) \) is the associated simple vertex operator algebra. The following lemma is clear (see [BMP], [W]).

**Lemma 3.1.** \( \hat{\mathcal{M}}(c, 0, 0) \) has a linear basis:

\[
L_{-m_1}L_{-m_2} \cdots L_{-m_s}W_{-n_1}W_{-n_2} \cdots W_{-n_t}1,
\]

where \( s, t \geq 0, m_1 \geq m_2 \geq \cdots \geq m_s \geq 2, n_1 \geq n_2 \geq \cdots \geq n_t \geq 3 \).

In the following discussion, we assume that the central charge \( c = 1 \). Note that the vertex operator subalgebra \( \langle \omega \rangle \) is isomorphic to the irreducible module \( L(1, 0) \) for the Virasoro algebra. In the following we will identify \( L(1, h) \) with its isomorphic image in \( \hat{\mathcal{M}}(1, 0, 0) \) if there is no confusion arising.

For convenience, we call a non-zero element \( v \) in \( \hat{\mathcal{M}}(1, 0, 0) \) a primary vector if \( L_mv = 0 \), for all \( m \geq 1 \). From the definition of \( \hat{\mathcal{M}}(1, 0, 0) \), it is obvious that there is no primary element of weight 4 or 5, and

\[
\dim \hat{\mathcal{M}}(1, 0, 0)_r = r - 1, \quad 3 \leq r \leq 5.
\]

By Lemma 3.1, \( \hat{\mathcal{M}}(1, 0, 0)_6 \) has a basis:

\[
W_{-3}W_{-3}1, \; W_{-6}1, \; L_{-2}W_{-4}1, \; L_{-3}W_{-3}1, \; L_{-6}1, \; L_{-4}L_{-2}1, \; L_{-3}^21, \; L_{-2}^31.
\]

The \( L(1,0) \)-submodule of \( \mathcal{M}(1,0,0) \) generated by \( w = W_{-3}1 \) is isomorphic to \( L(1,3) \). Note that

\[
\dim L(1,0)_6 = 4, \quad \dim L(1,3) \cap \hat{\mathcal{M}}(1,0,0)_6 = 3,
\]

then there is a non-zero element of weight 6 in \( \hat{\mathcal{M}}(1,0,0) \) which is not in \( L(1,0) \oplus L(1,3) \). Since \( \dim(\hat{\mathcal{M}}(1,0,0)_4 + \mathcal{M}(1,0,0)_5) = 7 < 8 \), it follows that there is a primary element \( u^{(6)} \) of weight 6 in \( \hat{\mathcal{M}}(1,0,0) \). Assume that

\[
u^{(6)} = a_1W_{-3}w + a_2L_{-3}w + a_3L_{-2}L_{-1}w + a_4L_{-1}^3w + a_5v,
\]

for some \( v \in L(1,0) \) and \( a_i \in \mathbb{C}, 1 \leq i \leq 5 \). From the commutation relation \( 3.2 \) we see that

\[
L_iW_{-3}w \in L(1,0), \quad i \geq 1.
\]

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Note that
\[ L_i(a_2L_{-3}w + a_3L_{-2}L_{-1}w + a_4L_{-1}^3w) \in L(1, 3) \]
for \( i \geq 1 \). This implies that \( a_i = 0 \) for \( i = 2, 3, 4 \). Consequently, \( a_1 \neq 0 \) and
\[ W_{-3}w \in L(1, 0) \oplus L(1, 6). \quad (3.3) \]

The \( L(1, 0) \)-submodule generated by \( u^{(6)} \) is isomorphic to \( L(1, 6) \). By computing the dimensions of \( L(1, 0)_7, L(1, 3)_7, L(1, 6)_7 \) and \( \hat{M}(1, 0, 0)_7 \) we immediately see that there is no primary vector in \( \hat{M}(1, 0, 0)_7 \). Note that \( \hat{M}(1, 0, 0)_8 \) has a basis:
\[ \{W_{-5}w, W_{-4}W_{-4}1, L_{-2}W_{-3}w, W_{-8}1, L_{-2}W_{-6}1, L_{-3}W_{-5}1, \\
L_{-4}W_{-4}1, L_{-2}W_{-4}1, L_{-5}w, L_{-3}L_{-2}w\} \cup S_8, \]
and the weight 8 subspace of \( L(1, 6) \oplus L(1, 3) \oplus L(1, 0) \) has a basis:
\[ \{L_{-2}W_{-3}w, L_{-1}^2W_{-3}w, L_{-5}w, L_{-4}L_{-1}w, L_{-3}L_{-2}w, \\
L_{-3}L_{-1}^2w, L_{-2}L_{-1}w, L_{-2}L_{-1}^2w, L_{-1}^5w\} \cup S_8, \]
where \( S_8 \) is a basis of \( L(1, 0)_8 \). It follows that there exists a non-zero element in \( \hat{M}(1, 0, 0)_8 \) which is not in the \( L(1, 0) \)-submodule generated by \( u^{(6)}, w \) and 1.

We are now in a position to state the main result of this section.

**Theorem 3.2.** The simple vertex operator algebra \( L(1, 0) \) is not completely reducible as an \( L(1, 0) \)-module.

**Proof:** By the above discussion, we have
\[ \dim \hat{M}(1, 0, 0)_8 - \dim (L(1, 6) \oplus L(1, 3) \oplus L(1, 0))_8 = 1. \]
Then there exists \( u^{(8)} \in \hat{M}(1, 0, 0)_8 \) such that \( u^{(8)} \) is either a primary element or the irreducible quotient of \( L(1, 0) \)-module generated by \( u^{(8)}, u^{(6)}, w \) and \( \omega \) modulo the submodule \( L(1, 6) \oplus L(1, 3) \oplus L(1, 0) \) is isomorphic to \( L(1, 8) \). It is easy to check that \( u^{(8)}, u^{(6)} \) and \( w \) do not lie in the maximal submodule \( J \).

Now let
\[
\begin{align*}
  u^{(9)} = & 3312738W_{-3}W_{-3}w - 214776L_{-6}w - 4379460L_{-5}L_{-1}w - 10304064L_{-4}L_{-2}w \\
  & - 7494075L_{-2}^2w + 2682708L_{-4}L_{-1}^2w + 8127252L_{-3}L_{-2}L_{-1}w + 424032L_{-2}^2L_{-1}w \\
  & - 2068431L_{-3}L_{-1}^2w - 2594664L_{-2}^2L_{-1}w + 350889L_{-2}L_{-1}^2w + 159578L_{-1}^6w
\end{align*}
\]
A direct calculation yields that
\[ L(n)u^{(9)} = 0, \quad n \geq 1. \]
Furthermore, we have

\[
W_1 W_{-3} W_{-3} w = \frac{146}{3} L_{-2} W_{-3} w + \frac{10070}{27} L_{-8} 1 + \frac{49664}{729} L_{-5} L_{-3} 1 + \frac{20480}{729} L_{-3} L_{-2} 1 \\
+ \frac{7232}{729} L_{-4}^2 1 + \frac{28672}{729} L_{-4} L_{-2}^2 1 + \frac{95872}{729} L_{-6} L_{-2} 1 + \frac{310}{9} W_{-5} w,
\]

\[
W_1 L_{-6} w = 13 W_{-5} w + 2 L_{-6} L_{-2} 1, W_1 L_{-5} L_{-1} w = 11 W_{-4} W_{-4} 1 + 5 L_{-5} L_{-3} 1,
\]

\[
W_1 L_{-4} L_{-2} w = 9 L_{-2} W_{-3} w + 9 W_{-5} w - \frac{5}{9} L_{-4}^2 1 + \frac{214}{27} L_{-4} L_{-2}^2 1,
\]

\[
W_1 L_{-3}^2 w = 28 W_{-5} w - \frac{56}{27} L_{-8} 1 - \frac{28}{27} L_{-5} L_{-3} 1 + \frac{502}{27} L_{-3}^2 L_{-2} 1,
\]

\[
W_1 L_{-4} L_{-1}^2 w = -10 L_{-8} 1 + 18 W_{-5} w + \frac{128}{3} L_{-6} L_{-2} 1 + \frac{128}{3} L_{-5} L_{-3} 1 \\
+ \frac{110}{3} L_{-4}^2 1 + \frac{64}{9} L_{-4} L_{-2}^2 1,
\]

\[
W_1 L_{-3} L_{-2} L_{-1} w = 14 W_{-4} W_{-4} 1 + \frac{704}{27} L_{-8} 1 + \frac{280}{9} L_{-6} L_{-2} 1 \\
+ \frac{448}{27} L_{-4} L_{-2}^2 1 + \frac{688}{27} L_{-5} L_{-3} 1 + \frac{839}{27} L_{-3}^2 L_{-2} 1,
\]

\[
W_1 L_{-3}^3 w = 45 L_{-2} W_{-3} w + 15 W_{-5} w - \frac{40}{3} L_{-8} 1 \\
- \frac{20}{3} L_{-6} L_{-2} 1 - \frac{5}{3} L_{-4} L_{-2}^2 1 + \frac{178}{9} L_{-2}^4 1,
\]

\[
W_1 L_{-3} L_{-1}^3 w = \frac{1064}{9} L_{-8} 1 + \frac{1792}{9} L_{-6} L_{-2} 1 + \frac{2520}{9} L_{-5} L_{-3} 1 \\
+ \frac{896}{9} L_{-4}^2 1 + \frac{448}{9} L_{-3} L_{-2} 1,
\]

\[
W_1 L_{-2}^2 L_{-1}^2 w = 30 W_{-5} w + \frac{8462}{27} L_{-8} 1 + \frac{3152}{9} L_{-6} L_{-2} 1 + \frac{4480}{27} L_{-5} L_{-3} 1 + \frac{320}{9} L_{-4}^2 1 \\
+ \frac{2974}{27} L_{-4} L_{-2}^2 1 + \frac{1280}{27} L_{-3} L_{-2} 1 + \frac{64}{9} L_{-2}^4 1,
\]

\[
W_1 L_{-2} L_{-1}^4 w = \frac{25144}{9} L_{-8} 1 + \frac{5280}{3} L_{-6} L_{-2} 1 + \frac{9728}{9} L_{-5} L_{-3} 1 \\
+ \frac{1280}{3} L_{-4}^2 1 + \frac{2048}{9} L_{-4} L_{-2}^2 1 + \frac{1024}{9} L_{-3}^2 L_{-2} 1,
\]

\[
W_1 L_{-1}^6 w = 26800 L_{-8} 1 + \frac{25600}{3} L_{-6} L_{-2} 1 + \frac{25600}{3} L_{-5} L_{-3} 1 + \frac{12800}{3} L_{-4}^2 1.
\]

Note that the monomial $L_{-2}^4 1$ appears only in $W_1 L_{-2}^3 w$ and $W_1 L_{-2}^2 L_{-1}^2 w$. Then it is easy to check that

\[
W_1 u^{(9)} \neq 0.
\]
So \( u^{(9)} \) does not lie in \( J \). This implies that \( u^{(9)} \) is a primary element of \( L(1,0,0) \). If \( L(1,0,0) \) is completely reducible as an \( L(1,0) \)-module, then it follows from (3.3) and the expression of \( u^{(9)} \) that the space of intertwining operators of type

\[
\begin{pmatrix}
L(1,9) \\
L(1,3) \\
L(1,6)
\end{pmatrix}
\]

is non-zero, which is a contradiction with Theorem 2.6. Thus \( L(1,0,0) \) is not completely reducible as an \( L(1,0) \)-module. \( \Box \)

4 Existence of \( M(1)^+ \) in \( V \)

From now on, we always assume that \( V \) is a simple rational vertex operator algebra of central charge 1 satisfying the following conditions:

1. \( V = \bigoplus_{n=0}^{\infty} V_n, \ V_0 = \mathbb{C}, \ V_1 = 0, \ \dim V_2 = 1 \);
2. \( \dim V_3 \geq 2, \) or \( \dim V_3 = 1 \) and \( \dim V_4 \geq 3, \)
3. \( V \) is a sum of highest weight modules of \( L(1,0) \).

Remark 4.1. Let \( V \) be a rational, \( C_2 \)-cofinite CFT type vertex operator algebras with \( c = \tilde{c} = 1 \). It is shown in [DM1] that if \( V_1 \neq 0 \) then \( V \) is isomorphic to \( V_L \) for a rank one lattice \( L \). It is established in [ZD] and [DJ] that if \( V_1 = 0 \) and \( \dim V_2 > 1 \) then \( V \) is isomorphic to \( V_{Z\alpha}^+ \) with \( (\alpha, \alpha) = 4 \). So for the purpose of the classification we need only to consider the case that \( V_1 = 0 \) and \( \dim V_2 = 1 \). For the characterization of the rational vertex operator algebra \( V_{Z\alpha}^+ \), it is very natural to have the assumption (2). We expect the assumption (3) is true for any rational vertex operator algebra with \( c = \tilde{c} = 1 \), but it seems difficult to prove it right now.

We have the following lemma from [DJ].

Lemma 4.2. \( V \) is a completely reducible module for the Virasoro vertex operator algebra \( L(1,0) \).

It is obvious that \( V \) carries a non-degenerate symmetric bilinear form \((\cdot, \cdot)\) such that \((1,1) = 1\) ([FHL], [L1]). We will prove in this section that \( V \) contains a vertex operator subalgebra isomorphic to \( M(1)^+ \). Let \( X^1 \) and \( X^2 \) be two subsets of \( V \). Set

\[ X^1 \cdot X^2 = \text{span}\{x_n y | x \in X^1, y \in X^2, n \in \mathbb{Z}\}. \]

Recall from Section 3 that a non-zero element \( v \) in \( V \) is called a primary vector if \( L_n v = 0 \) for all \( n \geq 1 \). We first have the following lemma.

Lemma 4.3. Let \( u^1, u^2 \in V \) be two primary elements. Let \( U^1 \) and \( U^2 \) be the two \( L(1,0) \)-submodules of \( V \) generated by \( u^1 \) and \( u^2 \) respectively. Then

\[ U^1 \cdot U^2 = \text{span}\{L(-m_1) \cdots L(-m_s) u^1_n u^2 | s \geq 0, m_1, \ldots, m_s \in \mathbb{Z}_+, n \in \mathbb{Z}\}. \]
Proof: Let \( x = L(-m_1) \cdots L(-m_s)u^1 \) and \( u = x_n L(-n_1) \cdots L(-n_k)u^2 \). Using the formula
\[
(a \cdot b)_m = \sum_{j \geq 0} (-1)^j {l \choose j} a_{t-j} b_{m+j} - \sum_{j \geq 0} (-1)^{l+j} {l \choose j} b_{m+l-j} a_j
\]
(which holds for any vertex operator algebra and any vectors \( a, b \) in the vertex operator algebra) and the commutator formula
\[
[L(p), u_i^j] = (-p - q - 1 + (p + 1)wt)u_{i+q}^j
\]
for \( l, p, q \in \mathbb{Z} \) and \( i = 1, 2 \) we see that

\[
u = x_n(L(-n_1) \cdots L(-n_k)u^2)
= \sum_{p_1, \ldots, p_s, p \in \mathbb{Z}} a_{p_1, \ldots, p_s, p} L(p_1) \cdots L(p_s) u_1^1 L(-n_1) \cdots L(-n_k) u^2
\]

lies in
\[
\text{span}\{L(-m_1) \cdots L(-m_s)u^1_n u^2 | s \geq 0, m_1, \ldots, m_s \in \mathbb{Z}_+, n \in \mathbb{Z}\},
\]
where \( a_{p_1, \ldots, p_s, p} \in \mathbb{C} \). The proof is complete. \( \square \)

Lemma 4.4. \( V \) contains a primary element of weight 4.

Proof: If \( \dim V_3 = 1 \), then the lemma follows from the assumption (2) and Lemma 4.2 as \( \dim L(1,0) = 2 \). If \( \dim V_3 \geq 2 \), then there exists at least one primary vector of weight 3 in \( V \). Let \( F \) be a primary element of weight 3 such that \( (F,F) = 2 \). Set
\[
u = F_1 F + \frac{2}{3}L(-4)1 - \frac{64}{9}L(-2)^2 1.
\]

Then it is easy to check that \( L_n u = 0 \), for all \( n \geq 1 \).

If \( \nu \neq 0 \), then the lemma holds. If \( u = 0 \), denote \( W_n = \frac{1}{\sqrt{6}} F_{n+2} \), for \( n \in \mathbb{Z} \). Then it is easy to check that \( L_m, W_n \) for \( m, n \in \mathbb{Z} \) satisfy relations (3.1) and (3.2) with \( C = 1 \). Then the vertex operator subalgebra of \( V \) generated by \( F \) and \( \omega \) is a quotient of \( \mathcal{M}(1,0,0) \). By Theorem [3.2], the vertex operator subalgebra of \( V \) generated by \( F \) and \( \omega \) is not a completely reducible \( L(1,0) \)-module, which contradicts Lemma 4.2. \( \square \)

Now let \( J \) be a primary element of weight 4. We may assume that
\[
(J,J) = 54, \tag{4.1}
\]
So \( J_7 J = 54I \).

Since \( V \) is rational, it follows that \( V \) contains infinitely many primary vectors. In this paper we will deal with the case that \( V \) contains a primary vector whose weight is not a perfect square.

Let \( k \) be the smallest positive integer such that \( k \) is not a perfect square and \( V \) contains a primary vector of weight \( k \). Then \( k \geq 3 \). Let \( U \) be the subalgebra of \( V \) generated by
\( \omega \) and \( J \). We will prove that \( U \) is isomorphic to \( M(1)^+ \). Let \( V^{(4)} \) be the irreducible \( L(1,0) \)-submodule of \( U \) generated by \( J \). Then \( V^{(4)} \cong L(1,4) \). By Theorem 2.6 for \( n \in \mathbb{Z} \),

\[
\dim I_{L(1,0)} \left( \begin{array}{c}
L(1,n) \\
V^{(4)} \\
L(1,k)
\end{array} \right) \neq 0
\]

if and only if \( n = k \). Let

\[
A_k = \{ v \in V_k | L_n v = 0, \ n \geq 1 \}.
\]

For any \( 0 \neq x \in A_k \), the \( L(1,0) \)-submodule \( U(Vir)x \) of \( V \) generated by \( x \) is isomorphic to \( L(1,k) \) and \( \text{span}\{u_n v | u \in V^{(4)}, v \in U(Vir)x, n \in \mathbb{Z}\} \) is an irreducible \( L(1,0) \)-module isomorphic to \( L(1,k) \) again. In particular, \( J x \in A_k \). Let \( F \) be an eigenvector of \( J \) in \( A_k \).

Let \( \sigma_1 : M^{(4)} \rightarrow V^{(4)} \) and \( \sigma_2 : M^{(k)} \rightarrow V^{(k)} \) be the intertwining operators of type \( L(1,0)M^{(4)} \rightarrow V^{(4)} \) and \( M^{(k)} \rightarrow V^{(k)} \) such that

\[
\sigma_1(J^1) = J, \quad \sigma_2(E) = F
\]

and

\[
(u^1, v^1) = (\sigma_1(u^1), \sigma_1(v^1))
\]

for \( u^1, v^1 \in M^{(4)} \).

We identify the Virasoro vertex operator algebra \( L(1,0) \) in \( V \) and \( V^+_L \). Let

\[
\mathcal{I}^0(u^1, z)v^1 = \mathcal{P}^0 \circ Y(u^1, z)v^1
\]

for \( u^1, v^1 \in M^{(4)} \) be the intertwining operator of type

\[
\left( \begin{array}{c}
L(1,0) \\
M^{(4)} \\
M^{(4)}
\end{array} \right)
\]

and

\[
\mathcal{I}^0(\sigma_1(u^1), z)\sigma_1(v^1) = \mathcal{Q}^0 \circ Y(\sigma_1(u^1), z)\sigma_1(v^1)
\]

for \( u^1, v^1 \in M^{(4)} \) be the intertwining operator of type

\[
\left( \begin{array}{c}
L(1,0) \\
V^{(4)} \\
V^{(4)}
\end{array} \right)
\]
where $\mathcal{P}^0$ and $Q^0$ are the projections of $V_L^+$ and $V$ to $L(1,0)$ respectively. By (4.1)-(4.2) and Theorem 2.6 we have

$$\mathcal{I}^0(u^1, z)v^1 = \mathcal{I}^0(\sigma_1(u^1), z)\sigma_1(v^1),$$

(4.4)

for $u^1, v^1 \in M^{(4)}$. Furthermore, we have the following lemma.

**Lemma 4.5.** Replacing $J$ by $-J$ if necessary, we have

$$\sigma_2(Y(u^1, z)v^2) = Y(\sigma_1(u^1), z)\sigma_2(v^2),$$

(4.5)

for $u^1 \in M^{(4)}, v^2 \in M^{(k)}$.

**Proof:** Since $M^{(4)} \cong V^{(4)} \cong L(1,4)$ and $M^{(k)} \cong V^{(k)} \cong L(1,k)$, we may identify $M^{(4)}$ with $V^{(4)}$ through $\sigma_1$ and $M^{(k)}$ with $V^{(k)}$ through $\sigma_2$. So both $Y(u^1, z)v^2$ and $Y(\sigma_1(u^1), z)\sigma_2(v^2)$ for $u^1 \in M^{(4)}, v^2 \in M^{(k)}$ are intertwining operators of type

$$
\begin{pmatrix}
L(1,k) \\
L(1,4) \\
L(1,k)
\end{pmatrix}.
$$

By Theorem 2.6 and (4.3), we have

$$\sigma_2(Y(u^1, z)v^2) = \varepsilon Y(\sigma_1(u^1), z)\sigma_2(v^2),$$

for some $\varepsilon \in \mathbb{C}$.

By the Jacobi identity, we have

$$(J_7^1J^1)_{-1}E = \sum_{i=0}^{\infty} (-1)^i \begin{pmatrix} 7 \\ i \end{pmatrix} (J_{7-i}^1J_{1+i}^1 + J_{6-i}^1J_i^1)E,$$

$$(J_7J)_{-1}F = \sum_{i=0}^{\infty} (-1)^i \begin{pmatrix} 7 \\ i \end{pmatrix} (J_{7-i}J_{1+i} + J_{6-i}J_i)F.$$

By (4.2), we have $\sigma_2((J_7^1J^1)_{-1}E) = 54F = (J_7J)_{-1}F$. On the other hand,

$$\sigma_2((J_7^1J^1)_{-1}E) = \varepsilon^2 (J_{7-i}J_{1+i} + J_{6-i}J_i)F$$

This forces $\varepsilon = \pm 1$. If $\varepsilon = 1$ we are done. Otherwise, we may replace $J$ by $-J$ and have the desired result.

**Lemma 4.6.** If there exists a primary element $v$ of weight $4$ in $V$ such that $v_3F \in \mathbb{C}F$, then $v \in \mathbb{C}J$. 

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**Proof:** Let $X$ be the irreducible $L(1,0)$-submodule of $V$ generated by $v$. Then we also have

$$X \cdot V^{(k)} = V^{(k)}$$

and $Y(w^2, z)|_{V^{(k)}}$ for $w^2 \in X$ is an intertwining operator of type $\left( V^{(k)} \atop X V^{(k)} \right)$ from the assumption.

Notice that $V^{(4)}$ is isomorphic to $X$ as $L(1,0)$-modules. Let $\varphi$ be an isomorphism from $V^{(4)}$ to $X$. Then $Y(\varphi(w^1), z)|_{V^{(k)}}$ for $w^1 \in V^{(4)}$ is an intertwining operator of type $\left( V^{(k)} \atop V^{(4)} V^{(k)} \right)$. By Theorem 2.6 there exists some nonzero $c \in \mathbb{C}$ such that

$$Y(w^1 - c\varphi(w^1), z)|_{V^{(k)}} = 0,$$

for all $w^1 \in V^{(4)}$. By Proposition 11.9 of [DL], we see that $w^1 - c\varphi(w^1) = 0$ for any $w^1 \in V^{(4)}$. This implies that $\varphi(w^1) \in V^{(4)}$ and $V^{(4)} = X$. The lemma is proved. □

By Theorem 2.6 and the fact that $V_1 = 0$, we may assume that

$$J_3 J = x + y,$$

where $x \in L(1,0)$, $y$ is either zero or a primary element of weight 4. Since

$$(J_3 J)_3 F \in V^{(k)},$$

it follows that $y_3 F \in V^{(k)}$. Then by Lemma 4.6 $y \in V^{(4)}$. Thus

$$J_3 J \in L(1,0) \oplus V^{(4)}.$$  \hspace{1cm} (4.6)

Now let $P^4$ and $Q^4$ be the projections of $V^+_L$ and $V$ to $M^{(4)}$ and $V^{(4)}$ respectively. Let $I^4(u^1, z)v^1 = P^4 \circ Y(u^1, z)v^1$ for $u^1, v^1 \in M^{(4)}$ be the intertwining operator of type

$$\left( \begin{array}{c} M^{(4)} \\ M^{(4)} \end{array} \right),$$

and $I^4(\sigma_1(u^1), z)\sigma_1(v^1) = Q^4 \circ Y(\sigma_1(u^1), z)\sigma_1(v^1)$ for $u^1, v^1 \in M^{(4)}$ be the intertwining operator of type

$$\left( \begin{array}{c} V^{(4)} \\ V^{(4)} \end{array} \right).$$

We have the following lemma.

**Lemma 4.7.** For $u^1, v^1 \in M^{(4)}$, we have

$$\sigma_1 I^4(u^1, z)v^1 = I^4(\sigma_1(u^1), z)\sigma_1(v^1).$$
Proof: By Theorem 2.6 we have
\[
\sigma_1\mathcal{I}^I(u^I, z)v^I = c\mathcal{I}^I(\sigma_1(u^I), z)\sigma_1(v^I),
\] (4.7)
for some \( c \in \mathbb{C} \). Using Lemma 4.5 we see that for \( n \in \mathbb{Z} \),
\[
\sum_{i=0}^{\infty} (-1)^i \binom{3}{i} (J_{3-i}J_{n+i} + J_{3+n-i}J_1)F = \sigma_2 \sum_{i=0}^{\infty} (-1)^i \binom{3}{i} (J_{3-i}J_{n+i} + J_{3+n-i}J_1)E).
\]
From the Jacobi identity we know that
\[
(J_3J)_n = \sum_{i=0}^{\infty} (-1)^i \binom{3}{i} (J_{3-i}J_{n+i} + J_{3+n-i}J_1).
\]
So
\[
(J_3J)_n F = \sigma_2 ((J_3J)^1)_n E).
\]
That is,
\[
((\sigma_1 J^1)_3 (\sigma_1 J^1))_n (\sigma_2 E) = \sigma_2 ((J_3 J^1)^1)_n E).
\]
From the discussion above we may assume that
\[
J_3J = x + y, \quad J_3^1 J^1 = x^1 + y^1,
\]
where \( x, x^1 \in L(1, 0), \ y^1 \in M^{(4)}, \ y \in V^{(4)} \). Then
\[
(x + y)_n F = \sigma_2 ((x^1 + y^1)_n E) = (\sigma_1 (x^1 + y^1))_n \sigma_2 (E) = (\sigma_1 (x^1 + y^1))_n F
\]
for all \( n \). Thus \( \sigma_1 (x^1 + y^1) = x + y, \ x = \sigma_1 (x^1) \) and \( y = \sigma_1 (y^1) \). On the other hand, by (4.7), we have \( \sigma_1 (y^1) = cy \). This forces \( c = 1 \).

By the skew-symmetry, we have
\[
Y(J, z)J = e^{L(-1)z} Y(J, -z)J.
\]
It follows that
\[
J_{-2}J = -J_{-2}J + \sum_{j=1}^{9} (-1)^{j+1} \frac{1}{j!} L(-1)^j J_{-2+j}J.
\]
This together with Theorem 2.6 and Lemma 4.3 deduces that
\[
V^{(4)} \cdot V^{(4)} \subseteq L(1, 0) \oplus L(1, 4) \oplus L(1, 4^2).
\]
Actually we have

Lemma 4.8. \( V^{(4)} \cdot V^{(4)} \cong L(1, 0) \oplus L(1, 4) \oplus L(1, 4^2) \).
Proof: Again by Lemma 4.5 we see that for $n \in \mathbb{Z}$,
\[
\sum_{i=0}^{\infty} (-1)^i \left( -\frac{9}{i} \right) (J_{-9-i}J_{n+i} + J_{-9+n-i}J_i)F = \sigma_2(\sum_{i=0}^{\infty} (-1)^i \left( -\frac{9}{i} \right) (J_{-9-i}J_{n+i} + J_{-9+n-i}J_i)E).
\]
It follows from the Jacobi identity that
\[
(J_{-9}J)_nF = \sigma_2((J_{-9}J)_nE).
\]
That is,
\[
((\sigma_1J^1)_{-9}(\sigma_1J^1))_n(\sigma_2E) = \sigma_2((J_{-9}J^1)_nE).
\]
Let $M^{(4^2)}$ be the irreducible $L(1,0)$-submodule of $M(1)^+$ isomorphic to $L(1,16)$. Let
\[
J_{-9}J^1 = x^0 + x^4 + x^{4^2}, \quad J_{-9}J = y^0 + y^4 + y^{4^2},
\]
where $x^0, y^0 \in L(1,0)$, $x^4 \in M^{(4)}$, $y^4 \in V^{(4)}$, $x^{4^2} \in M^{(4^2)}$ and $y^{4^2} \in V$ are primary vectors of weight 16. Then
\[
(y^0 + y^4 + y^{4^2})_nF = \sigma_2((x^0 + x^4 + x^{4^2})_nE).
\]
By (4.4), we have
\[
x^0 = y^0.
\]
By Lemma 4.5 and Lemma 4.7, we have
\[
\sigma_2(x^4_nE) = (\sigma_1 x^4)_nF = y^4_nF.
\]
So
\[
y^4_nF = \sigma_2(x^{4^2}_nE).
\]
Since $M^{(4)} \cdot M^{(4)} \cong L(1,0) \oplus L(1,4) \oplus L(1,16)$, we conclude that $x^{4^2}_nE \neq 0$. This can also be verified directly. As a result, $y_n^{4^2}F \neq 0$. The lemma follows.

Denote by $V^{(4^2)}$ the irreducible $L(1,0)$-submodule of $V$ generated by $y^{4^2}$ as in the proof of Lemma 4.8. Let $\mathcal{P}^{4^2}$ and $\mathcal{Q}^{4^2}$ be the projections of $V^+_L$ and $V$ to $M^{(4^2)}$ and $V^{(4^2)}$ respectively. Let $\mathcal{I}^{4^2}(u^1, z)v^1 = \mathcal{P}^{4^2}Y(u^1, z)v^1$ for $u^1, v^1 \in M^{(4)}$ be the intertwining operator of type
\[
\begin{pmatrix}
M^{(4^2)} \\
M^{(4)} \\
M^{(4)}
\end{pmatrix},
\]
and $\mathcal{I}^{4^2}(\sigma_1(u^1), z)\sigma_1(v^1) = \mathcal{Q}^{4^2}Y(\sigma_1(u^1), z)\sigma_1(v^1)$ for $u^1, v^1 \in M^{(4)}$ be the intertwining operator of type
\[
\begin{pmatrix}
V^{(4^2)} \\
V^{(4)} \\
V^{(4)}
\end{pmatrix}.
\]
By the proof of Lemma 4.8, the $L(1,0)$-module isomorphism $\sigma_1: M^{(4)} \rightarrow V^{(4)}$ can be extended to the $L(1,0)$-module isomorphism $\sigma_1$:
\[
L(1,0) \oplus M^{(4)} \oplus M^{(4^2)} \rightarrow L(1,0) \oplus V^{(4)} \oplus V^{(4^2)}
\]
such that

\[ \sigma_1(Y(u^1, z)v^1) = Y(\sigma_1(u^1), z)\sigma_1(v^1), \]

\[ \sigma_2(Y(u, z)w) = Y(\sigma_1(u), z)\sigma_2(w), \]

for \( u^1, v^1 \in M^{(1)}, u \in L(1, 0) \oplus M^{(4)} \oplus M^{(4)}, w \in M^{(k)}. \)

It is well known that \( M^{(1)} = \oplus_{i \geq 0} L(1, (2i)^2) \) as modules for the Virasoro algebra. We denote the \( L(1, 0) \)-submodule \( L(1, r^2) \) of \( M^{(1)} \) by \( M^{(r^2)}. \) Repeating the above procedure we deduce that \( U \) contains \( L(1, 0) \)-submodules \( V^{(r^2)} \) isomorphic to \( L(1, r^2) \) for even \( r. \) Then studying \( M^{(r^2)} \), \( M^{(s^2)} \) and \( V^{(r^2)} \cdot V^{(s^2)} \), for \( (r, s) \) such that \( r, s \in 2\mathbb{Z}^+, r \geq 2, s \geq 4 \) and \( r \leq s \), respectively (from small numbers to large ones), we conclude that

\[ U = L(1, 0) \bigoplus_{r=1}^{\infty} L(1, (2r)^2), \]

and there exists an \( L(1, 0) \)-module isomorphism \( \sigma_1 \) from \( M^{(1)} \) to \( U \) such that

\[ \sigma_1(Y(u^1, z)v^1) = Y(\sigma_1(u^1), z)\sigma_1(v^1), \]

\[ \sigma_2(Y(u^1, z)w) = Y(\sigma_1(u^1), z)\sigma_2(w), \]

for \( u^1, v^1 \in M^{(1)} \), \( w \in M^{(k)}. \) Then we prove that

**Theorem 4.9.** \( M^{(1)} \) is isomorphic to the vertex operator subalgebra \( U \) generated by \( \omega \) and \( J. \)

**Lemma 4.10.** In the decomposition of \( V \) into direct sum of irreducible \( L(1, 0) \)-modules, the multiplicity of \( L(1, (2r)^2) \) is 1, and the multiplicity of \( L(1, (2r+1)^2) \) is zero, for all \( r \in \mathbb{Z}, r \geq 1. \)

**Proof:** Suppose the lemma is false. Let \( r \) be the smallest positive integer such that there is a primary vector with weight \( r^2 \) satisfying \( u \notin U \cong M^{(1)}. \) Then \( r \geq 2. \) Let \( X \) be the \( U \)-submodule of \( V \) generated by \( \sum_{0 \leq s < r^2} V_s. \) Then \( X = U \oplus X^1 \) where \( X^1 \) is the \( U \)-submodule generated by the primary vectors whose weights are less than \( r^2 \) and are not perfect square. Since \( U \) is a simple vertex operator algebra, the restriction of the bilinear form \( \langle , \rangle \) to \( U \) is nondegenerate. Clearly, the restriction of the bilinear form to \( X^1 \) is also nondegenerate and \( (U, X^1) = 0. \) As a result, the restriction of the bilinear form to \( X \) is nondegenerate.

Let \( X^\perp \) be the orthogonal complement of \( X \) in \( V. \) Using the invariant property of the bilinear form we see that \( X^\perp \) is also a \( U \)-module. Then \( X^\perp = \sum_{s \geq r^2} X_s^\perp \) with \( X_{r^2}^\perp \neq 0 \) and each non-zero vector in \( X_{r^2}^\perp \) is a primary vector. Note that \( J_t u = 0 \) for \( t > 3. \) Since \( J_3 \) preserves \( X_{r^2}^\perp, \) there exists \( u \in X_{r^2}^\perp \) satisfying

\[ J_3 u = cu \]

for some \( 0 \neq c \in \mathbb{C}. \) Let \( Z \) be the \( L(1, 0) \)-submodule generated by \( u. \) Then \( Z \) is isomorphic to \( L(1, r^2). \) By Theorem 2.7 we see that \( V^{(4)} \cdot Z \subset Z \oplus (t \geq (r+1)^2) X_{r^2}^\perp. \) In particular, \( J_i u \) for \( i = 0, \ldots, 3 \) is linear combination of \( u, L(-1)u, L_{-2}u, L_{-2}^2u \) and \( L_{-3}u, L_{-2}L_{-1}u, L_{-1}^3u. \)
Let
\[ A_k = \{ v \in V_k | L(n)v = 0, n \geq 1 \}. \]
By Theorem \[2.6\], \( A_k \) is invariant under the actions of \( J_3 \) and \( u_{r^2-1} \), and \( u_i|_{A_k} = 0 \), for \( i \geq r^2 \). Note that
\[ J_3u_{r^2-1} - u_{r^2-1}J_3 = \sum_{j=0}^{3} (J_ju)_{r^2+2-j}. \]
It follows from the proof of Lemma \[4.3\] that for \( j \geq 0 \), \( (J_ju)_{r^2+2-j}|_{A_k} \in \mathbb{C}u_{r^2-1}|_{A_k} \). We deduce that on \( A_k \),
\[ [J_3, u_{r^2-1}] \in \mathbb{C}u_{r^2-1}. \]
So the Lie subalgebra of \( gl(A_k, \mathbb{C}) \) generated by \( J_3 \) and \( u_{r^2-1} \) is solvable. Since \( A_k \) is finite-dimensional, by the well-known Lie theorem, there exists \( 0 \neq F \in A_k \) such that
\[ J_3F, u_{r^2-1}F \in \mathbb{C}F. \] (4.14)
Let \( N \) be the \( L(1, 0) \)-submodule of \( V \) generated by \( F \). Then \( V(4) \cdot N = N \) and \( Z \cdot N = N \). In particular, \( J_3F \neq 0 \), \( u_{r^2-1}F \neq 0 \).
Let \( W \) be the \( M(1)^+ \)-submodule of \( V \) generated by \( u \). Then \( W \) has lowest weight \( r^2 \). Although it is not clear that \( W \) is an irreducible \( M(1)^+ \)-module, \( W \) has a unique irreducible quotient isomorphic to the irreducible module \( M(1, \sqrt{2r}) \) (see [DN1] for the notation). Since \( W \) is a completely reducible \( L(1, 0) \)-module, it follows from [DG] that \( W \) contains an \( L(1, 0) \)-submodule \( T \) isomorphic to \( L(1, s^2) \) for some even \( s \geq r \). Assume that \( w \) is a nonzero primary vector of \( T \). Then \( w \) is not an element of \( U = M(1)^+ \).
By (4.14), we have
\[ M(1)^+ \cdot N = N, \ W \cdot N = N. \]
Let \( v \in M(1)^+ \) be a primary vector of weight \( s^2 \) and \( T^1 \) the irreducible \( L(1, 0) \)-submodule of \( V \) generated by \( v \). Then both \( T^1 \) and \( T \) are isomorphic to \( L(1, s^2) \). By the above discussion,
\[ T^1 \cdot N = N, \ T \cdot N = N. \]
Similar to the proof of Lemma \[4.6\] we deduce that \( v = cw \), for some nonzero \( c \in \mathbb{C} \), contradicting the fact that \( w \notin M(1)^+ \), \( v \in M(1)^+ \). So the lemma holds.

5 Characterization of \( V_{\mathbb{Z}, \alpha}^+ \)

Let \( V \) be the vertex operator algebra as in Section 4. In this section, we prove the main result of the paper. That is, \( V \) is isomorphic to \( V_{\mathbb{Z}, \alpha}^+ \) with \((\alpha, \alpha) = 2k\).

Let \( n \) be a positive integer. Set \( X^1 = e^\beta \), \( X^2 = e^{-\beta} \) where \( (\beta, \beta) = 2n \). Let \( W^1 \) and \( W^2 \) be the irreducible \( M(1)^+ \)-submodules of \( V_{\mathbb{Z}, \beta} \) generated by \( X^1 \) and \( X^2 \) respectively. Then
\[ (X^1, X^1) = (X^2, X^2) = 0, \ (X^1, X^2) = 1 \] (5.1)
and

\[ W^2 \cdot W^2 = W^3, \quad W^1 \cdot W^3 = W^2, \]

where \( W^3 \) is the irreducible \( M(1)^+ \)-module generated by \( X^3 = e^{-2\beta} \). Using the vertex operator algebra structure of \( V_{\mathbb{Z}^2} \), we see that \( Y(u^2, z)v^2 \) for \( u^2, v^2 \in W^2 \) and \( Y(u^1, z)v^3 \) for \( u^1 \in W^1, v^3 \in W^3 \) are the intertwining operators of type

\[
\begin{pmatrix}
W^3 \\
W^2 \\
W^2
\end{pmatrix},
\]

and

\[
\begin{pmatrix}
W^2 \\
W^1 \\
W^3
\end{pmatrix}
\]

for the vertex operator algebra \( M(1)^+ \). In particular, we have

\[ X^2_{-2n-1}X^2 = X^3, \quad X^1_{4n-1}X^3 = X^2, \quad X^2_iX^2 = 0, \quad i \geq -2n. \]

Using the explicit definition of \( Y(X^1, z) \), Proposition 4.5.8 of [LL] and the fact that \( X^2_mX^1 = 0 \) for \( m \geq 2n \), we have

\[
(X^1_{2n-2}X^2_0)X^2 = 2nX^2 \neq 0.
\]

For \( n \in \mathbb{Z}_+ \), denote

\[ A_n = \{ v \in V_n | L(m)v = 0, \ m \geq 1 \}. \]

Then \( \dim A_n < \infty \). Let \( S \) be the set of all the positive integers \( n \) such that \( n \geq k, n \) is not a perfect square and \( \dim A_n \neq 0 \). We prove in Section 4 that

\[
V = M(1)^+ \bigoplus \left( \bigoplus_{n \in S} (\dim A_n)L(1, n) \right),
\]

where we identify the vertex subalgebra \( U \) generated by \( J \) and \( \omega \) with \( M(1)^+ \). Recall that the restriction of the bilinear form to \( M(1)^+ \) is non-degenerate and the orthogonal complement \( M \) of \( M(1)^+ \) in \( V \) is also an \( M(1)^+ \)-module such that \( M \cap M(1)^+ = 0 \). Then clearly, as an \( L(1, 0) \)-module,

\[ M = \bigoplus_{n \in S} (\dim A_n)L(1, n). \]

For \( n \in S \), by Theorem 2.6,

\[ J_3A_n = A_n, \quad J_mA_n = 0, \quad \text{for } m \geq 4. \]

The following lemma is obvious.
Lemma 5.1. Let $n \in S$ and $u \in A_n$ be such that $J_3u \in Cu$, then the irreducible $L(1,0)$-module $L(1,n)$ generated by $u$ is an irreducible $M(1)^+$-module.

Note that the bilinear form on $V$ restricted to $A_n$ is still non-degenerate. We decompose $A_n$ into direct sum of indecomposable $\mathbb{C}[J_3]$-modules: $A_n = \bigoplus_{i=1}^{s} X_i$. Since the form $(\cdot, \cdot)$ on $A_n$ is nondegenerate and $(J_3u, v) = (u, J_3v)$ for $u, v \in A_n$ we may assume that
\[(X_i, X_j) = 0 \quad (5.4)\]
if $i \neq j$.

Lemma 5.2. $J_3$ is semisimple on $A_n$.

Proof: It is equivalent to show that each $X_i$ has dimension 1. Suppose that $J_3$ is not semisimple on $A_n$. We may assume that $\dim X_1 > 1$. Let\[
\{x^1, \cdots, x^r\}
\]
be a basis of $X_1$ such that
\[(J_3 - \lambda_n \text{id})x^j = x^{j-1} \quad (5.5)\]
for $j = 1, 2, \cdots, r$ where $x^0 = 0$ and $\lambda_n = 4n^2 - n$ (cf. [DNI]). Then
\[(J_3x^j, x^1) = (\lambda_n x^j + x^{j-1}, x^1),\]
\[(x^j, J_3x^1) = (x^j, \lambda_n x^1).\]
Since $(J_3x^j, x^1) = (x^j, J_3x^1)$, it follows that
\[(x^{j-1}, x^1) = 0, \quad j = 2, \cdots, r. \quad (5.6)\]
Using the non-degeneracy of $(\cdot, \cdot)$, we may choose $x^r$ such that
\[(x^r, x^1) = 1, \quad (x^r, x^r) = 0. \quad (5.7)\]
In fact, we may replace $x^r$ by $x^r - \frac{1}{2}(x^r, x^r)x^1$ if $(x^r, x^r) \neq 0$. Let $M^j$ be the irreducible $L(1,0)$-module generated by $x^j$. Then $M^1$ is an irreducible $M(1)^+$-module. Using the invariant property of the bilinear form and (5.6) we see that $(M^1, M^1) = 0$.

Claim: $P = M^1 \cdot M^1 = L(1,4n)$ is an irreducible $M(1)^+$-module.

Note that
\[M^1 \cdot M^1 = \text{span}\{u_l v | u, v \in M^1, l \in \mathbb{Z}\}\]
is an $M(1)^+$-module as $M^1$ is an $M(1)^+$-module. For any $u, v \in M^1, w \in M(1)^+$ we have
\[(Y(u, z)v, w) = (v, Y(e^{zL(1)}(-z)^{(L(0)} v, z^{-1})w) = 0\]
as the coefficients of $z^n$ in $Y(e^{zL(1)}(-z)^{(L(0)} v, z^{-1})w$ lie in $M^1$. This implies that
\[(P, M(1)^+) = 0.\]
and $P$ is an $M(1)^+$-submodule of $M$. In particular, $P$ is a direct sum of irreducible $L(1,0)$-modules whose lowest weights are not perfect squares. Let $p \in \mathbb{Z}$ such that

$$x_p^1 x^1 \neq 0, \quad x_m^1 = 0, \quad m > p.$$  

By Lemma 4.3 we see that $x_p^1 x^1$ is the unique highest weight vector up to a constant for the Virasoro algebra with the highest weight $2n - p - 1$. As a result, $x_p^1 x^1$ generates a highest weight module for the Virasoro algebra and isomorphic to $L(1, 2n - p - 1)$ which is also an irreducible $M(1)^+$-module. Thus by Theorem 2.6 we have $M(1)^+$-module decomposition $P = L(1, 2n - p - 1) \oplus N$ where $N$ is the sum of highest weight modules for the Virasoro algebra whose highest weights are greater than $2n - p - 1$. Using the fact that $(x^1, x^1) = 0$ and Theorem 2.7 we see that $2n - p - 1 = 4n$.

Let $N_m$ be the sum of irreducible highest weight modules for the Virasoro algebra in $N$ isomorphic to $L(1, m)$ for $m > 4n$. Then $N_m$ is also an $M(1)^+$-module. If $N_m \neq 0$ for some $m > 4n$, then $N_m$ has a finite composition series as an $M(1)^+$-module. In particular, $N_m$ has a maximal $M(1)^+$-submodule such that the quotient is isomorphic to $M(1, \sqrt{2m})$. This yields an intertwining operator of type

$$\left( \begin{array}{c} M(1, \sqrt{2m}) \\ M(1, \sqrt{2n}) M(1, \sqrt{2n}) \end{array} \right)$$

for $m \neq 4n$. This is a contradiction by Theorem 2.7. This implies that $P = L(1, 4n)$ is an irreducible $M(1)^+$-module.

Notice that $x_i^1 x^1 = 0$ for $i \geq 0$ and $V_1 = 0$. We have

$$(x_{2n-2}^1 x^1)_0 x^1 = \sum_{i=0}^{2n-2} (-1)^{i+1} \binom{2n-2}{i} x_{2n-2-i} x_i^1 x^1$$

$$= \sum_{i=0}^{2n-2} \sum_{j=0}^{2n-i} (-1)^{i+j+s+1} \binom{2n-2}{i} \binom{2n}{j} \binom{2n}{s} x_{i+j+s} x_{2n-2-i-j-s} x^1$$

$$= 0.$$  

On the other hand, as $L(1,0)$-modules, $M^r \cong W^1$, $M^1 \cong W^2$, $P \cong W^3$ by module isomorphisms

$$\tau_1 : M^r \to W^1, \quad x^r \mapsto X^1;$$

$$\tau_2 : M^1 \to W^2, \quad x^1 \mapsto X^2;$$

$$\tau_3 : P \to W^3, \quad x_{-2n-1} x^1 \mapsto X^3.$$  

Clearly, $M^r \cdot P$ is a direct sum of irreducible $L(1,0)$-modules. It is easy to see from [FZ] and [L2] that

$$\dim_{L(1,0)} \begin{pmatrix} W^2 \\ W^3 \\ W^1 \end{pmatrix} = 1.$$  

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It follows from (5.2) that
\[(x_{2n-2}^r x_{1}^1)_{0} x_1^1 \neq 0\]

unless \(\mathcal{P}(Y(u,z) w) = 0\), for \(u \in M^r\) and \(w \in P\), where \(\mathcal{P}(Y(u,z) w)\) is a projection of \(Y(u,z) w\) to \(M^1\). So
\[\mathcal{P}(Y(u,z) w) = 0,\]

for \(u \in M^r\) and \(w \in P\). Then we have
\[
(x_{2n-1}^r x_{1}^1)_{-1} x_1^1
= \sum_{i=0}^{2n-1} (-1)^i \binom{2n-1}{i} x_{2n-2-i}^r x_i^1
= \sum_{i=0}^{2n-1} \sum_{j=0}^{2n-i} \sum_{s=0}^{2n} (-1)^{i+j+s} \binom{2n-1}{i} \binom{-2n}{j} \binom{2n}{s} x_{i+j+s}^r x_{2n-2-i-j-s}^1
= 0.
\]

But by (5.7), we have
\[(x_{2n-1}^r x_{1}^1)_{-1} x_1^1 = x_1^1 \neq 0,\]
a contradiction. The proof is complete. \(\square\)

By Lemmas 5.1 and 5.2, we immediately have

**Lemma 5.3.** As an \(M(1)^+\)-module, \(V\) is completely reducible.

It follows from Lemma 5.2 that there is a primary vector \(F \in V_k\) satisfying \(J_3 F = (4k^2 - k) F\) and \((F,F) = 2\). Let \(V^0\) be the vertex operator subalgebra of \(V\) generated by \(\omega, J\) and \(F\).

**Lemma 5.4.** \(V^0\) is linearly spanned by

\[L(-m_1) \cdots L(-m_s) J_{-n_1} \cdots J_{-n_t} 1, \quad L(-m_1) \cdots L(-m_s) J_{-n_1} \cdots J_{-n_t} F_{-p_1} \cdots F_{-p_l} F\]

where \(m_1 \geq m_2 \geq \cdots m_s \geq 1, n_1 \geq n_2 \geq \cdots n_t \geq 1, \text{ and } p_1 \geq p_2 \geq \cdots p_l \geq 1.\)

**Proof:** Clearly, \(V^0\) is spanned by

\[L(m_1) \cdots L(m_s) J_{n_1} \cdots J_{n_t} F_{p_1} \cdots F_{p_l} F\]

where \(m_i, n_j, p_l \in \mathbb{Z}\).

Recall that \(M(1)^+ = U\) is generated by \(\omega\) and \(J\). Furthermore, \(J_i J \in L(1,0) + V^{(4)}\) and \(J_i F \in V^{(k)}\) for \(i \geq 0\) where \(V^{(4)}, V^{(k)}\) are the irreducible \(L(1,0)\)-modules generated by \(J\) and \(F\), respectively. Using Theorem 2.7 and Lemma 5.3, we see that \(F_i F \in U\) for \(i \geq 0\). The Lemma then follows from the commutator relations

\[[J_m, J_n] = \sum_{i=0}^{\infty} \binom{m}{i} (J_i J)_{m+n-i},\]
\[ [F_m, F_n] = \sum_{i=0}^{\infty} \binom{m}{i} (F_i F)_{m+n-i} \]

and

\[ [J_m, F_n] = \sum_{i=0}^{\infty} \binom{m}{i} (J_i F)_{m+n-i} \]

immediately.

For \( r, s \in \mathbb{Z}_+ \) such that \( r \neq s \), let \( X, Y \) be irreducible \( M(1)^+ \)-submodules isomorphic to \( L(1, r^2 k) \) and \( L(1, s^2 k) \), respectively. It follows from Theorem 2.7 and Lemma 5.3 that \( X \cdot Y \) is an \( M(1)^+ \)-submodule of \( V \) isomorphic to either \( L(1, (r-s)^2 k) \), \( L(1, (r+s)^2 k) \) or \( L(1, (r-s)^2 k) \oplus L(1, (r+s)^2 k) \). Using Lemma 5.4 and Theorem 2.7 gives

\[ V^0 = \left( \bigoplus_{m=0}^{\infty} L(1, (2m)^2) \bigoplus \bigoplus_{r=1}^{\infty} c_r L(1, r^2 k) \right), \tag{5.8} \]

where \( c_r \geq 0 \).

**Lemma 5.5.** \( c_r \leq 1 \).

**Proof:** We prove the result by induction on \( r \). By Lemma 5.4, \( F \) is the only one linearly independent primary vector of weight \( k \) in \( V^0 \). That is, \( c_1 = 1 \). Recall that \( V^{(k)} \) is the irreducible \( M(1)^+ \)-module generated by \( F \) and \( V^{(k)} = L(1, k) \). By Lemma 4.3, we have

\[ V^{(k)} \cdot V^{(k)} = \text{span}\{L(-m_1) \cdots L(-m_s)F_n F|m_1, \ldots, m_s \in \mathbb{Z}_+, n \in \mathbb{Z}\}. \]

Notice that by Theorem 2.7 and Lemma 5.3, \( F_i F \in U \) for \( i \geq -2k \). So \( F_m F_i F \in V^{(k)} \) for \( m \in \mathbb{Z}, i \geq -2k \). From Lemma 5.4, we see that any \( u \in V_{4k}^0 \) can be written as

\[ u = x + aF_{-2k-1}F \]

for some \( x \in U + V^{(k)}, a \in \mathbb{C} \). Let

\[ v^1 = x^1 + a_1 F_{-2k-1} F, \quad v^2 = x^2 + a_2 F_{-2k-1} F \]

be two primary vectors of weight \( 4k \) where \( x^1, x^2 \in U \bigoplus V^{(k)} \), \( a_1, a_2 \in \mathbb{C} \). Since there is no primary vector of weight \( 4k \) in \( U + V^{(k)} \), \( a_1, a_2 \) are nonzero. Note that

\[ a_2 v^1 - a_1 v^2 = a_2 x^1 - a_1 x^2 \in U \bigoplus V^{(k)} \]

is either zero or a primary element of weight \( 4k \). This implies that \( a_2^2 v^1 - a_1 v^2 = a_2^2 x^1 - a_1 x^2 = 0 \). That is, there exists at most one linearly independent primary element of weight \( 4k \).

By Theorem 2.7 and Lemma 5.3 for \( p_1 \in \mathbb{Z}_+ \),

\[ F_{-p_1} F = u^1 + \sum_{i=1}^{r_1^{(1)}} c_i^{(1)} L(-m_i^{(1)}) \cdots L(-m_{i_{s_1}}^{(1)}) F_{-2k-1} F, \]

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for some \( u^1 \in U, c_i^{(1)} \in \mathbb{C}, m_{i1}^{(1)} \geq m_{i2}^{(1)} \cdots \geq m_{is_1}^{(1)} \geq 1 \). If there exists no primary element of weight \( 4k \), then
\[
V^0 = U \bigoplus V^{(k)}.
\]
The lemma holds.

Otherwise, let \( v^{(4k)} \) be the primary element of weight \( 4k \) (unique up to a scalar). Then \( v^{(4k)} \) generates an irreducible \( U \)-module \( V^{(4k)} \) which is isomorphic to \( L(1, 4k) \) as \( L(1, 0) \)-modules.

Note that
\[
F_m F_{-2k-1} F \in U + V^{(k)} + V^{(4k)}
\]
for \( m \geq -4k \). Then by Lemma 5.3, any element in \( V^0 \) of weight \( 9k \) is a linear combination of an element in \( U + V^{(k)} + V^{(4k)} \) and \( F_{-4k-1} F_{-2k-1} F \). Now let \( v^{(9k)} \) be a primary element of weight \( 9k \), then we may assume that
\[
v^{(9k)} = u + a F_{-4k-1} F_{-2k-1} F
\]
for some \( u \in U \bigoplus V^{(k)} \bigoplus V^{(4k)} \) and \( 0 \neq a \in \mathbb{C} \). As above we can prove that there exists at most one linearly independent primary element of weight \( 9k \) and for \( p_1, p_2 \in \mathbb{Z}_+ \)
\[
F_{-p_2} F_{-p_1} F = \sum_{i=1}^{t_i^{(2)}} d_i^{(2)} L(-n_{i1}^{(2)}) \cdots L(-n_{it_i^{(2)}2}^{(2)}) F
\]
\[
+ \sum_{i=1}^{t_i^{(2)}} c_i^{(2)} F_{-4k-1} F_{-2k-1} F
\]
for some \( c_i^{(2)}, d_i^{(2)} \in \mathbb{C}, m_{i1}^{(2)} \geq m_{i2}^{(2)} \cdots \geq m_{is_i}^{(2)} \geq 1, n_{i1}^{(2)} \geq n_{i2}^{(2)} \cdots \geq n_{it_i^{(2)}2}^{(2)} \geq 1 \).

Now assume that there exists only one linearly independent primary vector \( v^{(m^2k)} \) in \( V^0 \) of weight \( m^2k \), for some \( m \geq 3 \), and for \( p_{m-1}, \ldots, p_1 \in \mathbb{Z}_+ \),
\[
F_{-p_{m-1}} \cdots F_{-p_2} F_{-p_1} F
\]
\[
= u + \sum_{i=1}^{m-1} \sum_{j=1}^{k_i} c_{ij} L(-m_{ij}^{(i)}) \cdots L(-m_{j_{k_i}}^{(i)}) F_{-2ik_1} \cdots F_{-4k-1} F_{-2k-1} F,
\]
where \( u \in U \bigoplus V^{(k)}, c_{ij} \in \mathbb{C}, m_{j1}^{(i)} \geq m_{j2}^{(i)} \geq \cdots \geq m_{j_{k_i}}^{(i)} \geq 1 \). Then by Theorem 2.6, \( J_{3v^{(m^2k)}} \subset \mathbb{C}v^{(m^2k)} \). So \( v^{(m^2k)} \) generates an irreducible \( U \)-module \( V^{(m^2k)} \) which is isomorphic to \( L(1, m^2k) \) as \( L(1, 0) \)-modules. By Theorem 2.7
\[
F_n F_{-2(m-1)k-1} \cdots F_{-4k-1} F_{-2k-1} F \in U \bigoplus_{i=1}^{m} ( \bigoplus V^{(i^2k)} ) \text{ for } n \geq -2mk.
\]
So any homogeneous element in \( V^0 \) of weight \( (m+1)^2k \) is a linear combination of an element in \( U \bigoplus ( \bigoplus_{i=1}^{m} V^{(i^2k)} ) \) and \( F_{-2mk-1} \cdots F_{-4k-1} F_{-2k-1} F \).

Let
\[
v^1 = x^1 + a_1 F_{-2mk-1} \cdots F_{-4k-1} F_{-2k-1} F, \quad v^2 = x^2 + a_2 F_{-2mk-1} \cdots F_{-4k-1} F_{-2k-1} F
\]

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be two primary vectors of weight \((m+1)^2k\), where \(x^1, x^2 \in U \bigoplus (\bigoplus_{i=1}^{m} V^{(i^2k)})\), \(a_1, a_2 \in \mathbb{C}\). If \(a_1 = a_2 = 0\), then \(x^1 = x^2 = 0\). So we may assume that \(a_1 \neq 0, a_2 \neq 0\). Note that

\[
a_2v^1 - a_1v^2 = a_2x^1 - a_1x^2 \in U \bigoplus \left( \bigoplus_{i=1}^{m} V^{(i^2k)} \right)
\]

is either zero or a primary vector of weight \((m+1)^2k\). This implies that \(a_2v^1 - a_1v^2 = a_2x^1 - a_1x^2 = 0\). This proves that there exists at most one linearly independent primary vector of weight \((m+1)^2k\).

\[
\square
\]

**Remark 5.6.** We actually also prove in Lemma 5.5 that \(V^0\) is linearly spanned by

\[
L(-m_1) \cdots L(-m_s)J_{-n_1} \cdots J_{-n_t} 1, \quad L(-m_1) \cdots L(-m_s)F,
\]

\[
L(-m_1) \cdots L(-m_s)F_{2mk-1} \cdots F_{-2k-1} F;
\]

where \(m_1 \geq m_2 \geq \cdots m_s \geq 1, m \geq 1, s, t \geq 0\).

**Lemma 5.7.** Let \(V_k\) be the vertex operator algebra associated to the positive definite even lattice \(L = \mathbb{Z}\alpha\) such that \((\alpha, \alpha) = 2k\). Then \(V^0 \cong V^+_L\).

**Proof:** For \(m \in \mathbb{Z}^+_+\), set

\[
E^m = e^{m\alpha} + e^{-m\alpha}, \quad E = E^1
\]

and denote \(N^{(m^2k)} \cong M(1, m\sqrt{2k}) \cong L(1, m^2k)\) the irreducible \(M(1)^+\)-module generated by \(E^m\). Then

\[
V^+_L = M(1)^+ \bigoplus \left( \bigoplus_{m \in \mathbb{Z}^+_+} N^{(m^2k)} \right).
\]

It is known that \((E^m, E^m) = 2\) and \(J_3E^m = (4m^4k^2 - m^2k)E^m\).

By (5.8) and Lemma 5.5 we may assume that

\[
V^0 = M(1)^+ \bigoplus \left( \bigoplus_{m \in \mathbb{Z}^+_+} c_m V^{(m^2k)} \right),
\]

where \(c_m \leq 1, V^{(m^2k)} \cong M(1, m\sqrt{2k})\). Since \((E, E) = (F, F) = 2\) we may assume that

\[
E_{-2k-1}E = u + aE^2,
\]

\[
F_{-2k-1}F = u + v,
\]

where \(u \in M(1)^+, 0 \neq a \in \mathbb{C}\) (see Lemma 5.5) and \(v \in V\) is either zero or a primary vector of weight \(4k\). Note that

\[
J_3E = (4k^2 - k)E, \quad J_3F = (4k^2 - k)F,
\]

\[
(E_{-2k-1}E, E_{-2k-1}E) = (E, \sum_{j=0}^{4k-1} \binom{4k-1}{j} (E_jE)_{2k-2-j} E),
\]

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\[(F_{-2k-1}F, F_{-2k-1}F) = (F, \sum_{j=0}^{4k-1} \binom{4k-1}{j} (F_j F)_{2k-2-j} F).\]

Since \(E_j E = F_j F \in M(1)^+\) for \(j \geq 0\), it follows that
\[(E_{-2k-1}E, E_{-2k-1}E) = (F_{-2k-1}F, F_{-2k-1}F).\]

So
\[(u + aE^2, u + aE^2) = (u + v, u + v) = (u, u) + 2a^2 = (u, u) + (v, v).\]

This proves that \(v \neq 0\). So we have \(v = bF^2\) where \(0 \neq b \in \mathbb{C}\) and \(F^2 \in V_{4k}^0\) is a primary vector such that \((F^2, F^2) = 2\). It follows that \(c_2 = 1\). If \(a = -b\), we replace \(F^2\) by \(-F^2\). So we may assume that \(a = b\).

Let \(\sigma\) be an \(M(1)^+\)-module isomorphism from
\[P^{(2)} = M(1)^+ \bigoplus N^{(k)} \bigoplus N^{(4k)} \quad \text{to} \quad Q^{(2)} = M(1)^+ \bigoplus V^{(k)} \bigoplus V^{(4k)}\]
defined by
\[\sigma(E^i) = F^i, \sigma(u) = u, \ i = 1, 2\]
for \(u \in M(1)^+\) where \(F^1 = F\). For \(u, v \in M(1)^+ \bigoplus N^{(k)}\) and \(n \in \mathbb{Z}\), we have from Theorem 2.7 that
\[\sigma(u_n v) = (\sigma u)_n (\sigma v).\]

Now assume that \(c_i = 1\), for \(i = 1, 2, \cdots, r\) and there exist irreducible \(M(1)^+\)-modules \(V^{(i^2)}\) generated by non-zero elements \(F^i \in A_{i^2}, i = 1, 2, \cdots, r\) such that \(F^1 = F\), \((F_i, F_i) = 2\) and the \(\sigma\) can be extended to an \(M(1)^+\)-module isomorphism from
\[P^{(r)} = M(1)^+ \bigoplus \bigoplus_{i=1}^{r} N^{(i^2 k)} \quad \text{to} \quad Q^{(r)} = M(1)^+ \bigoplus \bigoplus_{i=1}^{r} V^{(i^2 k)}\]
satisfying
\[\sigma(E^i) = F^i, \ i = 1, 2, \cdots, r\]
and for \(u, v \in P^{(r)}, n \in \mathbb{Z}\) with \(u_n v \in P^{(r)}\),
\[\sigma(u_n v) = (\sigma u)_n (\sigma v).\]

Note that \(E^1_{-2r^2,k-1} E^r \in P^{(r)} + N^{(r^2+1)}\), \(F^1_{-2r^2,k-1} F^r \in Q^{(r)}\) or \(Q^{(r)} + V^{(r^2+1)^2 k}\) is an irreducible \(M(1)^+\)-module generated by a primary vector \(F^{r+1}\) of weight \(k(r + 1)^2\). By inductive assumption, we may assume that
\[E^1_{-2r^2,k-1} E^r = x + a_1 E^{r+1}, \ F^1_{-2r^2,k-1} F^r = \sigma(x) + b_1 F^{r+1},\]
where \(x \in P^{(r)}, 0 \neq a_1 \in \mathbb{C}, b_1 \in \mathbb{C}\). We have
\[(E^1_{-2r^2,k-1} E^r, E^1_{-2r^2,k-1} E^r) = (E^r, E^1_{-2r^2,k-1} E^1_{-2r^2,k-1} E^r) \]
\[= (E^r, \sum_{j=0}^{\infty} \binom{2r k + 2k - 1}{j} (E^1_j E^1)^{2k-2-j} E^r) + (E^r, E^1_{-2r^2,k-1} E^1_{-2r^2,k-1} E^r)\]

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\[
(F_{-2r-1}^{1}, F_{-2r-1}^{1} F^{r}) = (F^{r}, F_{2r+2k-1}^{1} F_{-2r-1}^{1} F^{r})
\]
\[
= (F^{r}, \sum_{j=0}^{\infty} (2r_{j} + 2k - 1) (F^{1}_{j} F^{1}) (2k - j) F^{r}) + (F^{r}, F_{-2r-1}^{1} F_{2r+2k-1}^{1} F^{r})
\]

Notice that \(E_{j}^{1} E^{1} = F_{j}^{1} F^{1}\), for \(j \geq 0\) and

\[
E_{2r+2k-1}^{1} F^{r} \in N^{(r-1)^{2k}}, \quad F_{2r+2k-1}^{1} F^{r} \in V^{(r-1)^{2k}}
\]

Then by the inductive assumption, we have

\[
(E_{-2r-1}^{1} E^{r}, E_{-2r-1}^{1} E^{r}) = (F_{-2r-1}^{1} F^{r}, F_{-2r-1}^{1} F^{r}).
\]

This implies that

\[
a_{i}^{2} = b_{i}^{2} \neq 0.
\]

Thus \(c_{r+1} \neq 0\). Replacing \(F^{r+1}\) by \(-F^{r+1}\) if necessary we may assume that \(a_{1} = b_{1}\).

We extend the \(M(1)^{+}\)-module \(\sigma\) to an \(M(1)^{+}\)-module isomorphism from

\[
P^{(r+1)} = M(1)^{+} \bigoplus_{i=1}^{r+1} N^{(i^{2}k)} \to Q^{(r+1)} = M(1)^{+} \bigoplus_{i=1}^{r+1} V^{(i^{2}k)}
\]

satisfying

\[
\sigma E^{i} = F^{i}, \quad i = 1, 2, \ldots, r + 1
\]

and for \(u \in N^{(1)}, v \in P^{(r)} \) and \(n \in \mathbb{Z}\),

\[
\sigma(u, v) = (\sigma u)_{n}(\sigma v).
\]  (5.9)

Now consider \(E_{-4r+4k-1}^{2} F^{r-1} \in P^{(r)} + N^{(r+1)^{2k}}\). From the above discussion we know that

\[
E^{2} = u + a E_{-2k-1}^{1} E^{1}, \quad F^{2} = u + a F_{-2k-1}^{1} F^{1},
\]

where \(u \in M(1)^{+}, 0 \neq a \in \mathbb{C}\). Note that for any \(n \in \mathbb{Z}\)

\[
E_{n}^{2} = u_{n} + \sum_{i,j \in \mathbb{Z}} a_{i,j} E_{i} E_{j}, \quad F_{n}^{2} = u_{n} + \sum_{i,j \in \mathbb{Z}} a_{i,j} F_{i} F_{j}
\]

for some \(a_{i,j} \in \mathbb{C}\). Combining this with (5.9) yields that for \(u \in N^{(4k)}, v \in N^{(r-1)^{2k}}\) and \(n \in \mathbb{Z}\),

\[
\sigma(u, v) = (\sigma u)_{n}(\sigma v).
\]  (5.10)

Continuing in this way, we show that for \(u, v \in P^{(r+1)}, n \in \mathbb{Z}\) with \(u_{n} v \in P^{(r+1)}\),

\[
\sigma(u, v) = (\sigma u)_{n}(\sigma v).
\]

As a result,

\[
V^{0} = M(1)^{+} \bigoplus_{m \in \mathbb{Z}_{+}} V^{(m^{2}k)},
\]

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where $V^{(m^2k)}$ is an irreducible $M(1)^+$-module generated by non-zero element $F^m \in A_{m^2k}$ such that $(F^m, F^m) = 2$. Moreover, there exists a linear map $\sigma$ from $V^+_L$ to $V$ such that for $u, v \in V^+_L$, $n \in \mathbb{Z}$,

$$\sigma(u_n v) = (\sigma u)_n (\sigma v).$$

Thus $V^0$ is isomorphic to $V^+_L$.

Here is the main theorem of this paper:

**Theorem 5.8.** Let $V$ be as before. Then

$$V \cong V^+_L.$$  

**Proof:** By Lemma 5.7

$$V^0 \cong V^+_L.$$  

Then $V$ is an extension of $V^0$ and $V$ is a direct sum of irreducible $V^0$-modules. By the representation theory of $V^+_L$, $V^+_L$ has only two irreducible modules with integral lowest weights 0 and 1 [DN2]. The assumption that $V_1 = 0$ and $\text{dim } V_0 = 1$ forces $V = V^+_L$. □

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