The Bethe-Salpeter Equation and the Low Energy Theorems for $\pi N$ Scattering

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Abstract. The Bethe-Salpeter (BS) amplitude for $\pi N$ scattering is evaluated at the off mass shell points corresponding to the Low Energy Theorems (LET) based on PCAC and current algebra. The results suggest a way of maintaining constructing between BS equation and LET.

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1 Introduction

The formulations of $\pi N$ scattering can be divided into two approaches. On the one hand, we have the Effective Field Theory (EFT) approach where the emphasis is on preserving the symmetries of QCD. This is achieved by the expansion for the amplitude in powers of the pion mass or external momenta divided by a typical QCD cutoff of $\approx 1$ GeV. For the $\pi N$ system the commonly used EFT is Chiral Perturbation Theory (ChPT) which results in an off-mass-shell amplitude that is consistent with the Low Energy Theorems (LET). These LET are based on current algebra and PCAC. On the other hand, we have the more traditional approach of using the two-body scattering equation (e.g., the Bethe-Salpeter equation) in which the potential is based on $s$-, $t$-, and $u$-channel pole diagrams derived from a chirally invariant Lagrangian. In this case unitarity is the main uniting feature which allows the examination of $\pi N$ scattering at higher energies.

2 The Bethe-Salpeter Amplitude

With the advent of solutions to the Bethe-Salpeter (BS) equation for the off-mass-shell $\pi N$ amplitude, it is now possible to compare the results of the two approaches, with the LET. Here we will present solutions to the BS equation based on a potential derived from the chirally invariant Lagrangian.

\begin{equation}
\mathcal{L}_{\text{int}} = \frac{g_{\pi NN}}{2m_N} \bar{\psi}_N \gamma_5 \gamma^\mu \tau \cdot \partial_\mu \pi \psi_N
\end{equation}

The tree level diagrams that contribute to the potential include the $s$- and $u$-channel diagrams with nucleon and $\Delta$ poles, and $t$-channel diagrams with $\rho$ and $\sigma$ poles. To solve the four-dimensional BS equation, we have introduced form factors for each of the vertices in the Lagrangian. Here we considered two classes of form factors

\begin{equation}
f_{\alpha\beta\gamma}(p_\alpha^2, p_\beta^2, p_\gamma^2) = f_{\alpha}(p_\alpha^2) f_{\beta}(p_\beta^2) f_{\gamma}(p_\gamma^2) \quad \text{(Type I)}
\end{equation}

and

\begin{equation}
f_{\alpha\beta\gamma}(p_\alpha^2, p_\beta^2, p_\gamma^2) = f_{\pi}(p_\pi^2) \quad \text{(Type II)}
\end{equation}

where

\begin{equation}
f_{\alpha}(p_\alpha^2) = \left( \frac{A_\alpha^2 - m_\alpha^2}{A_\alpha^2 - p_\alpha^2} \right)^{n_\alpha}.
\end{equation}

In the above, $m_\alpha$ and $A_\alpha$ are the mass and cut off associated with the hadron $\alpha$. The parameters in the Lagrangian are adjusted to fit the $s$- and $p$-wave scattering data up to pion laboratory energy of 300 MeV. In Fig. 1 we present a fit to the data for the form factors type I and II (n=4). Similar fits are achieved for the form factors II (n=2) and II (n=10). Here, the form factors determine the off mass shell behaviour of the BS amplitudes. In this way we can vary the off mass shell amplitudes when comparing the results of the BS equations with those based on current algebra and PCAC.
3 The Low Energy Theorems

The LET theorems for $\pi N$ scattering are based on current algebra and PCAC. The latter is implemented by defining the pion field, $\pi^a$, in terms of the derivative of the axial vector current, $i.e. \partial^a A_\mu^a = f_\pi m_\pi^2 \pi^a$. This allows us to write the $\pi N$ amplitude in terms of the commutation relation of the currents. In this way we can use current algebra to determine the $\pi N$ amplitude in the soft pion limit.

The $\pi N$ amplitude with off shell pion can be written as

$$ T_{ab}^{\pi N} = \bar{u}(p') \left\{ T^{(+)} \delta_{ab} + \frac{i}{2m} [\tau_a, \tau_b] T^{(-)} \right\} u(p), \quad (5) $$

where $p$ ($p'$) is the initial (final) on-shell nucleon momentum, and

$$ T^{(\pm)} = D^{(\pm)} + \frac{i}{2m} \sigma^{\mu\nu} q_\mu q_\nu B^{(\pm)} . \quad (6) $$

Here the $(+)$, $(-)$ refer to isospin even and odd components of the amplitude, and $q$ ($q'$) are the off mass shell pion initial (final) momentum. The amplitudes $D$ and $B$ with the pion off mass shell are a function of $\nu, \nu_B, q^2$ and $q'^2$, $i.e. D^{(\pm)}(\nu, \nu_B, q^2, q'^2)$ with $\nu = \frac{1}{m}(s-u)$ and $\nu_B$ being the value of $\nu$ at the $s$ channel nucleon pole, and $s$ and $u$ being the standard Mandelstam variables. Current algebra and PCAC can impose constraints on these off mass shell amplitudes. In particular, we can write the isospin even amplitude with the nucleon pole subtracted $i.e. \tilde{D}^{(+)}(\nu, \nu_B, q^2, q'^2)$ at three off shell points. The amplitude at Weinberg (W)\textsuperscript{3}, Adler (A)\textsuperscript{5} and the Cheng-Dashen (CD)\textsuperscript{6} points are:

$$ \tilde{D}^{+}(0, 0, 0, 0) = -\frac{\sigma_{\pi N}(0)}{f_\pi^2} \quad (7) $$

$$ \tilde{D}^{+}(0, 0, m_\pi^2, 0) = 0 = \tilde{D}^{+}(0, 0, 0, m_\pi^2) \quad (8) $$

$$ \tilde{D}^{+}(0, 0, m_\pi^2, m_\pi^2) = \frac{\sigma_{\pi N}(0)}{f_\pi^2} + \mathcal{O}(m_\pi^4) + \cdots , \quad (9) $$

respectively. Here we observe that the amplitude at the Weinberg and Cheng-Dashen points are opposite in sign, while that at the Adler point is zero. Since the sigma term $\sigma_{\pi N}(0)$ is a measure of chiral symmetry breaking $i.e.$

$$ \sigma_{\pi N}(0) = \frac{1}{2} \sum_{a=1}^{3} \langle N(p) | [Q^a_{\pi N}, H] | N(p) \rangle , \quad (10) $$

in the absence of any mechanism for chiral symmetry breaking, the $\pi N$ amplitude is zero at all three points.

4 Results

To examine the variation in the off mass shell BS amplitude when comparing with the results from the LET,
we have considered four possible form factors for our potential. In Table 1 we have the parameters that give the optimum fit to the data up to pion energy of 300 MeV for the form factor types I(n=1), II(n=2), and II(n=4) and II(n=10). Also included in the table are the equivalent cut off mass for a monopole $A_R^{(n)}$, and the difference in the form factor at the pion pole and at $q^2 = 0$, i.e. $\Delta_\pi = 1 - f^{(n)}_+ (0)$. Here, $R$ refers to the fact that these quantities are calculated for the renormalised form factor. From the table we observe that the dressing of the nucleon and $\Delta$ is substantially more for type I form factors than is the case for type II form factors. At the same time the type I form factors give a value for $\Delta_\pi$ that is closer to the commonly accepted value of 3% from the Goldberger-Treiman relation.

Table 1. The coupling constants and masses for the optimum fit to the data for different choices for the form factors. All coupling constants are $g^2/4\pi$.

|          | I(n=1) | II(n=2) | II(n=4) | II(n=10) |
|----------|--------|---------|---------|---------|
| $g^{(0)}_{\pi NN}$ | 1.80   | 4.23    | 4.68    | 5.98    |
| $f^{(0)}_{\pi NN}$ | 0.37   | 0.17    | 0.20    | 0.196   |
| $x_\Delta$ | -0.11  | -0.13   | -0.24   | -0.18   |
| $g_{\rho NN}g_{\rho NN}$ | 2.88   | 2.67    | 2.63    | 2.80    |
| $\kappa_{\rho}$ | 2.66   | 2.18    | 2.03    | 2.15    |
| $g_{\rho NN}g_{\rho NN}$ | -0.41  | 0.86    | 0.39    | 0.48    |
| $m_N^{(0)}$ | 1.34   | 1.18    | 1.14    | 1.11    |
| $m_\Delta^{(0)}$ | 2.305  | 1.495   | 1.492   | 1.435   |
| $m_\sigma$ | 0.65   | 0.88    | 0.62    | 0.64    |
| $A_R^{(n)}$ | 1.22   | 0.874   | 0.868   | 0.822   |
| $\Delta_\pi$ | 1.3%   | 2.47%   | 2.51%   | 2.79%   |

In Table 2 we present the off shell amplitude resulting from the solution of the BS equations at the Adler (A), Weinberg (W), and the Cheng-Dashen(CD) points for different choices for our form factor. Also included are the values for the $\pi N$ $\sigma$-term $\sigma_{\pi N}(0)$ and the isospin even scattering length $a^\pi_\sigma$. Here we observe that:

Table 2. The BS amplitude at the Adler(A), Weinberg(W) and Cheng-Dashen(CD) points in units of $m_{\pi}^{-1}$ for different form factors. Also included are the $\sigma$-term $\sigma_{\pi N}(0)$ and the isospin even $S$-wave scattering length.

| Model | A     | W     | CD    | $\sigma_{\pi N}(0)$ | $a^\pi_\sigma$ |
|-------|-------|-------|-------|---------------------|----------------|
| I     | 0.366 | 0.355 | 0.379 | 23.8 -0.025         | -0.025         |
| II n=2 | 0.0949 | 0.102 | 0.106 | 6.64 -0.05          | -0.05          |
| II n=4 | 0.0411 | 0.0462 | 0.0494 | 3.10 -0.048        | -0.048         |
| II n=10 | 0.0180 | 0.0218 | 0.0259 | 1.62 -0.049        | -0.049         |

1. The off mass shell amplitude is sensitive to the choice of cut off form factor, and in particular, model I gives a larger $\sigma$-term than model II.

2. The amplitude at the three off mass shell points are approximately the same. This is in contrast to the fact that the amplitude at the Weinberg and Cheng-Dashen points are equal and opposite in sign.

3. Finally, the amplitude at the Adler point is not zero. To understand the difference between the amplitude resulting from the solution of the BS equation and the LET, we first examined the Born amplitude for the $\pi N$ potentials considered. Here we found that the Born amplitude at all three points is zero, consistent with the requirement of chiral symmetry conservation. (The $\sigma$-term is a measure of chiral symmetry breaking). This suggests that the higher terms in the multiple scattering series give all the contribution to the amplitude at the three off mass shell points. To examine the source of this chiral symmetry breaking contribution, we have examined the contribution of each term in the potential to the amplitude at the Adler, Weinberg and Cheng-Dashen points. This exercise revealed that the source of chiral symmetry breaking is the higher order multiple scattering $t$-channel $\rho$ exchange. In particular, it is the $\rho t \pi$ term in the Lagrangian that gives rise to the symmetry breaking. Since this $\rho t \pi$ Lagrangian is of the form $g_{\rho NN} \rho_{\mu} \cdot (\partial^\rho \pi \times \pi)$, then the only time this term in the Lagrangian contributes to chiral symmetry breaking is when the external pion is represented by $\pi$ rather than $\partial^\rho \pi$. It is possible to restore chiral symmetry to the amplitude by introducing a symmetric form for the coupling of the $\rho$ to the $\pi$, e.g. $(\partial^\rho \rho_{\mu} - \partial_{\rho} \rho_{\mu}) \cdot (\partial^\rho \pi \times \partial^\rho \pi)^2$, which is equivalent on the mass shell to the form employed in the present calculation. The resultant amplitude would then satisfy chiral symmetry. Chiral symmetry breaking could then be introduced in a controlled form via a non-derivative $\sigma \pi \pi$ coupling.

5 Conclusion

In the above analysis we have established that the Bethe-Salpeter amplitude for a Lagrangian that satisfies chiral symmetry is inconsistent with the Low Energy Theorems of current algebra. The source of the disagreement is the mode of chiral symmetry breaking via the higher order multiple scattering of $t$-channel $\rho$ exchange. This could be overcome with the introduction of a symmetric $\rho t \pi$ Lagrangian, and a mode of chiral symmetry breaking that is consistent with the LET.

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