Around the normal derivative lemma

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Abstract

This survey provides a description of the history and the state of the art of one of the most important fields in the qualitative theory of elliptic partial differential equations including the strong maximum principle, the boundary point principle (the normal derivative lemma) and related topics.

Keywords: Strong maximum principle, boundary point principle, normal derivative lemma, Hopf–Oleinik lemma, Harnack inequality, Aleksandrov–Bakelman maximum principle.

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**PDMI RAS and St. Petersburg State University
1 Introduction

The qualitative theory of partial differential equations has been intensively developed over the past century. Among the most important tools for studying solutions of elliptic and parabolic equations are, in particular, the normal derivative lemma (also known as the Hopf–Oleinik lemma or the boundary point principle) and the strong maximum principle. They play a key role in proving uniqueness theorems for boundary value problems. They are also used in studying the symmetry properties of solutions, the behavior of solutions in unbounded domains (Phragmén–Lindelöf type theorems), and in other applications.

The first results in this area can be traced back to the works of C.F. Gauss, who proved the strong maximum principle for harmonic functions in 1840, in Section 21 of the famous paper [140], see also [139], [117]. In modern notation, the Gauss statement reads as follows:

*Let u be a non-constant harmonic function in a domain \( \Omega \subset \mathbb{R}^3 \), that is \( \Delta u = 0 \) in \( \Omega \). Then the function u attains neither maximum nor minimum in the interior points of \( \Omega \).*

In what follows, by strong maximum principle for a second-order linear elliptic operator \( \mathbb{L} \) we mean the following assertion:
The strong maximum principle. Let $u$ be a super-elliptic function in a domain $\Omega \subset \mathbb{R}^n$, that is $Lu \geq 0$ in $\Omega$. If $u$ attains its minimum at an interior point of the domain then $u \equiv \text{const}$ and $Lu \equiv 0$.

We also recall the formulation of the weak maximum principle:

The weak maximum principle. Let $u$ be a super-elliptic function in a bounded domain $\Omega \subset \mathbb{R}^n$. If $u$ is non-negative on the boundary of $\Omega$ then $u$ is non-negative in $\Omega$.

The boundary version of the strong maximum principle is the so-called normal derivative lemma, first formulated by S. Zaremba in 1910 [357] for harmonic functions in a (three-dimensional, bounded) domain satisfying the interior ball condition.

The normal derivative lemma. Let $u$ be a non-constant super-elliptic function in a domain $\Omega \subset \mathbb{R}^n$. If $u$ attains its minimum at a boundary point $x^0 \in \partial \Omega$ then the following inequality holds:

$$\liminf_{\varepsilon \to +0} \frac{u(x^0 + \varepsilon n) - u(x^0)}{\varepsilon} > 0,$$

(1.1)

where $n$ is the interior normal vector to the boundary at the point $x^0$.

In particular, if the function $u$ has a derivative with respect to the direction $n$ at the point $x^0$ then $\partial_n u(x^0) > 0$.

It is important to note that the strong maximum principle is a property of the operator $L$ whereas the normal derivative lemma also depends on the behavior of $\partial \Omega$ in a neighborhood of $x^0$.

Closely adjacent to the main subjects of the survey is the Harnack inequality, which can be regarded as a quantitative version of the strong maximum principle. It was first proved by C.G.A. Harnack in 1887 [152, § 19] for the

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1We assume that the principal coefficients of the operator $L$ form a non-positive matrix.

2Notice that Zaremba used this lemma to prove the uniqueness theorem for a mixed problem (the boundary of the domain is split into two parts, one of which is subject to the Dirichlet condition and the other to the Neumann condition). Nowadays, it is called the Zaremba problem, although Zaremba himself in [357] points out that it was posed to him by W. Wirtinger.

3The mathematician Carl Gustav Axel Harnack had a twin brother Carl Gustav Adolf von Harnack, historian and theologian, founding president of the Kaiser Wilhelm Gesellschaft (now the Max Planck Society for Scientific Research). The highest award of the Max Planck Society bears his name.
harmonic functions on the plane. The classical formulation of this inequality is as follows:

**The Harnack inequality.** Let $\mathcal{L}$ be an elliptic operator in a domain $\Omega$. If $u$ is a non-negative solution of the equation $\mathcal{L}u = 0$ in $\Omega$ then in any bounded subdomain $\Omega'$ such that $\overline{\Omega'} \subset \Omega$, we have the inequality

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u,$$

(1.2)

where $C$ is a constant independent of $u$.

**Remark 1.1.** It is clear from a compactness argument that it suffices to prove (1.2) in the case where $\Omega$ and $\Omega'$ are concentric balls. In this case, it is important for applications that the constant $C$ does not depend on the radii of the balls (but only on their ratio) or, in the worst case, remains bounded as the radii tend to zero with a fixed ratio.

Some a priori estimates of solutions, in particular, the Aleksandrov–Bakelman maximum principle, can also be considered as a quantitative version of the strong maximum principle. On the other hand, it has become clear relatively recently that the boundary gradient a priori estimate for solutions is a statement dual to the normal derivative lemma.

The discussed topic is almost boundless, so in this paper we focus on the elliptic case only. The main part of the article is split into three sections. Section 2 discusses the properties of classical and strong (sub/super)solutions of non-divergence type equations, whereas Section 3 concerns the properties of weak (sub/super)solutions of divergence type equations. Finally, Section 4 is a “patchwork” of various generalizations and applications. Here we do not claim to be complete, and the choice of topics reflects the personal interests of the authors.

Various aspects of the topic under discussion are reflected in monographs and survey papers [297], [338], [195], [73], [136], [305], [271], [182], [306], [35]. In this paper, we have used some information from these sources, as well as from our articles [42], [43], but we tried to cover the history of the mentioned issues as deeply as possible.

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Basic notation

\( x = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n) \) are points in \( \mathbb{R}^n, \ n \geq 2 \).
\(|x|, |x'|\) are the Euclidean norms in the corresponding spaces.
\( \mathbb{R}_+ = [0, +\infty) \) denotes the closed half-axis.
\( \Omega \) is a domain (i.e., a connected open set) in \( \mathbb{R}^n \) with a boundary \( \partial\Omega \).
\( \Omega \) is assumed to be bounded, unless otherwise (as in §1.2) specified.
\( \overline{\Omega} \) denotes the closure of \( \Omega \).
\( |\Omega| \) is the Lebesgue measure of \( \Omega \).
\( \text{diam}(\Omega) \) is the diameter of \( \Omega \).
\( d(x) = \text{dist}(x, \partial\Omega) \) is the distance from the point \( x \) to \( \partial\Omega \).
\( B^n_r(x^0) = \{ x \in \mathbb{R}^n \mid |x - x^0| < r \} \) is the open ball in \( \mathbb{R}^n \) with center \( x^0 \) and radius \( r \); \( B^n_r = B^n_r(0) \). If the dimension of the Euclidean space is clear from the context, we will simply write \( B_r(x^0) \) and \( B_r \).
\( Q_{r,h} = B^n_{r-1} \times (0, h) \).

The indices \( i \) and \( j \) run from 1 to \( n \). \( D_i \) denotes the operator of (weak) differentiation with respect to \( x_i \). We adopt the standard convention regarding summation with respect to repeated indices.

For a function \( f \) we set \( f_\pm = \max\{\pm f, 0\} \) and

\[ \int_{\Omega} f \, dx = \frac{1}{|\Omega|} \int_{\overline{\Omega}} f \, dx. \]

We use the letters \( C \) and \( N \) (with or without indices) to denote various positive constants. To indicate that, say, \( C \) depends on some parameters, we list them in parentheses \( C(\ldots) \).

Function spaces and classes of domains

\( C^k(\overline{\Omega}) \) is the space of functions defined on \( \overline{\Omega} \) and having continuous derivatives up to the order \( k \ (k \geq 0) \). For brevity we write \( C \) instead of \( C^0 \).
We denote by $L_p(\Omega)$, $W^k_p(\Omega)$, and $\tilde{W}^k_p(\Omega)$ the standard Lebesgue and Sobolev spaces, see, e.g., [344, § 4.2.1]; $\| \cdot \|_{p, \Omega}$ stands for the norm in $L_p(\Omega)$. Further, we write $f \in L_{p, \text{loc}}(\Omega)$ if $f \in L_p(\Omega')$ for arbitrary subdomain $\Omega'$ such that $\overline{\Omega'} \subset \Omega$. In a similar way we understand $f \in W^k_{p, \text{loc}}(\Omega)$.

$L_{p,q}(\Omega)$ is the Lorentz space, see, e.g., [344, § 1.18.6].

We say that $\sigma : [0, 1] \to \mathbb{R}_+$ is a function of the $D$ class, if

- $\sigma$ is continuous, increasing, and $\sigma(0) = 0$;
- $\sigma(\tau)/\tau$ is decreasing and integrable.

**Remark 1.2.** Notice that the monotonicity assumption for $\sigma(\tau)/\tau$ is not restrictive. Indeed, for an increasing function $\sigma : [0, 1] \to \mathbb{R}_+$ such that $\sigma(0) = 0$ and $\sigma(\tau)/\tau$ is integrable, we define

$$\tilde{\sigma}(t) = t \sup_{\tau \in [t, 1]} \frac{\sigma(\tau)}{\tau}, \quad t \in (0, 1).$$

Obviously, $\tilde{\sigma}(t)/t$ decreases on $[0, 1]$ and $\sigma(t) \leq \tilde{\sigma}(t)$ on $(0, 1]$ (the latter inequality allows us to put $\tilde{\sigma}$ instead of $\sigma$ in all estimates). Further, the set of points where $\sigma(t) < \tilde{\sigma}(t)$ is at most a countable union of the intervals $(t_j, t_{j+1})$. Evidently, $\tilde{\sigma}$ is increasing on each of these intervals and therefore it is increasing on $[0, 1]$.

Now we consider the integral

$$\int_0^1 \frac{\tilde{\sigma}(\tau)}{\tau} d\tau = \int_{\{\tilde{\sigma} = \sigma\}} \frac{\sigma(\tau)}{\tau} d\tau + \sum_j \int_{t_{j-1}}^{t_j} \frac{\tilde{\sigma}(\tau)}{\tau} d\tau.$$

However, on $(t_j, t_{j+1})$ we have

$$\frac{\tilde{\sigma}(t)}{t} \equiv \frac{\sigma(t_{j+1})}{t_{j+1}} = \frac{\sigma(t_{j+1})}{t_{j+1}},$$

whence, taking into account the monotonicity of $\sigma$ we arrive at

$$\int_0^1 \frac{\tilde{\sigma}(\tau)}{\tau} d\tau = \int_{\{\tilde{\sigma} = \sigma\}} \frac{\sigma(\tau)}{\tau} d\tau + \sum_j \left(\sigma(t_{j+1}) - \sigma(t_j)\right) < \infty.$$

Thus, $\tilde{\sigma} \in \mathcal{D}$. 


Remark 1.3. Without loss of generality, we can also assume that $\sigma$ is continuously differentiable on $(0; 1]$. Indeed, for any $\sigma \in \mathcal{D}$ we can define

$$\hat{\sigma}(r) := 2 \int_{r/2}^{r} \frac{\sigma(\tau)}{\tau} d\tau = 2 \int_{1/2}^{1} \frac{\sigma(r\tau)}{\tau} d\tau, \quad r \in (0; 1]. \quad (1.3)$$

By the monotonicity of the functions $\sigma$ and $\frac{\sigma(\tau)}{\tau}$, we conclude from the second equality in (1.3) that $\hat{\sigma}$ also increases whereas $\frac{\hat{\sigma}(r)}{r}$ decreases on $(0; 1]$. Further, the first equality in (1.3) easily implies that $\hat{\sigma} \in \mathcal{C}^1(0; 1]$, and the following inequalities hold:

$$\sigma(r) \leq \hat{\sigma}(r) \leq 2\sigma(r/2), \quad r \in (0; 1]. \quad (1.4)$$

The second inequality in (1.4) provides $\hat{\sigma} \in \mathcal{D}$. Finally, the first inequality in (1.4) allows us to put $\hat{\sigma}$ instead of $\sigma$ in all estimates.

We say that a function $\zeta : \Omega \to \mathbb{R}$ satisfies:

- the Hölder condition with exponent $\alpha \in (0, 1]$, if
  $$|\zeta(x) - \zeta(y)| \leq C|x - y|^{\alpha} \quad \text{for all} \quad x, y \in \Omega;$$

- the Dini condition, if there is a function $\sigma \in \mathcal{D}$ such that
  $$|\zeta(x) - \zeta(y)| \leq \sigma(|x - y|) \quad \text{for all} \quad x, y \in \Omega.$$

Further, $\mathcal{C}^{k,\alpha}(\Omega)$ and $\mathcal{C}^{k,\mathcal{D}}(\Omega)$ for $k \geq 0$ are the spaces of functions which have derivatives of order $k$ satisfying the Hölder condition with exponent $\alpha \in (0, 1]$ (respectively, the Dini condition). Functions in $\mathcal{C}^{0,1}(\Omega)$ are called Lipschitz.

We say that a domain $\Omega \subset \mathbb{R}^n$ belongs to the class $\mathcal{C}^k$ or is $\mathcal{C}^k$-smooth for some $k \geq 0$, if there is an $r > 0$ such that for every point $x^0 \in \partial\Omega$ the set $B_r(x^0) \cap \partial\Omega$ (in an appropriate Cartesian coordinate system) is the graph of a function $x_n = f(x')$, $f \in \mathcal{C}^k(G)$ (here $G$ is a domain in $\mathbb{R}^{n-1}$). In a similar way we define domains of classes $\mathcal{C}^{k,\alpha}$ and $\mathcal{C}^{k,\mathcal{D}}$.

\[\text{The set } B_r(x^0) \cap \Omega \text{ lies on one side of the graph.}\]
Domains of $\mathcal{C}^{0,1}$ class are called strongly Lipschitz.

Recall that the **interior ball condition** means that one can touch any point of the boundary $\partial \Omega$ with a ball of fixed radius lying in $\Omega$.

In a similar way, denote by $\mathcal{T}(\phi, h)$ (here $\phi : [0, +\infty) \mapsto [0, +\infty)$ is a convex function, $\phi(0) = 0$, and $h > 0$) the domain (body)

$$\mathcal{T}(\phi, h) = \{ x \in \mathbb{R}^n | \phi(|x'|) < x_n < h \}.$$

Assume that one can touch any point of the boundary $x^0 \in \partial \Omega$ with a body congruent to $\mathcal{T}(\phi, h)$ with vertex at the point $x^0$, and this body lies in $\Omega$. Suppose also that $\phi, h$ do not depend on $x^0$. Then we say that $\Omega$ satisfies

- **the interior $C^{1,\alpha}$-paraboloid condition**, $\alpha \in (0, 1]$, if $\phi(s) =Cs^{1+\alpha}$ (for $\alpha = 1$ this condition coincides with the interior ball condition);

- **the interior $C^{1,D}$-paraboloid condition** if $\phi'(0+) = 0$, and $\phi'$ satisfies the Dini condition;

- **the interior cone condition** if $\phi(s) = Cs$.

In a similar way we define conditions of **exterior ball**, **exterior paraboloid** and **exterior cone**.

It is easy to see that any domain of $\mathcal{C}^{1,1}$ class satisfies the interior and exterior ball conditions. Moreover, these conditions together are equivalent to the $\mathcal{C}^{1,1}$-smoothness of the domain, see, e.g., [273, Lemma 2]. In a similar way, the $\mathcal{C}^{1,\alpha}$-smooth domains are exactly domains satisfying the interior and exterior $\mathcal{C}^{1,\alpha}$-paraboloid conditions; the $\mathcal{C}^{1,D}$-smooth domains are exactly domains satisfying the interior and exterior $\mathcal{C}^{1,D}$-paraboloid conditions; and strongly Lipschitz domains satisfy the interior and exterior cone conditions.

## 2 Non-divergence type operators

In this Section, we consider operators with the following structure:

$$\mathcal{L} \equiv -a^{ij}(x)D_iD_j + b^i(x)D_i. \quad (2.1)$$

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6Without a priori assumption “the boundary is locally the graph of a function” these equivalences were proved in [35].

7Here, contrary to the assertion made in [35], there is no longer any equivalence. A counterexample is a Lipschitz, but not strongly Lipschitz, domain composed of two “bricks”, see, e.g., [257, p.39].
We introduce the notation $A = (a_{ij})$, $b = (b_i)$. If $b \equiv 0$ then we write $\mathcal{L}_0$ instead of $\mathcal{L}$.

The matrix of principal coefficients $A$ is symmetric and satisfies either the \textbf{degenerate ellipticity condition}
\begin{equation}
    a^{ij}(x)\xi_i\xi_j \geq 0 \quad \text{for all } \xi \in \mathbb{R}^n,
\end{equation}

or the \textbf{uniform ellipticity condition}
\begin{equation}
    \nu|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \nu^{-1}|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n
\end{equation}

(here $\nu \in (0, 1]$ is the so-called \textit{ellipticity constant}).

In Subsections 2.1–2.2 we assume that the condition (2.2) or (2.3) is satisfied for all $x \in \Omega$. Starting from Subsection 2.3, it is assumed that the entries of the matrix $A$ are measurable functions, and the condition (2.2) or (2.3) holds for almost all $x \in \Omega$.

\textbf{Remark 2.1.} \textit{For operators in the more general form $\mathcal{L} + c(x)$, both the strong maximum principle and the normal derivative lemma obviously do not hold in the formulation given in the Introduction. Indeed, the first eigenfunction of the Dirichlet problem for the Laplacian is a counterexample even for the weak maximum principle. In this case, one usually imposes a condition on the sign of the coefficient $c(x)$ in a neighborhood of the minimum point. We provide two pairs of simple assertions.}

1. Assume that the strong maximum principle holds for the operator $\mathcal{L}$.
   \begin{enumerate}
   \item Let $c \geq 0$, $c \not\equiv 0$. If $\mathcal{L}u + cu \geq 0$ in $\Omega$ then $u$ cannot attain its \textbf{negative} minimum in $\Omega$.
   \item Let $c \leq 0$, $c \not\equiv 0$. If $\mathcal{L}u + cu \geq 0$ in $\Omega$ then $u$ cannot attain its \textbf{non-negative} minimum in $\Omega$ unless $u \equiv 0$.
   \end{enumerate}

2. Assume that the normal derivative lemma holds for the operator $\mathcal{L}$ in the domain $\Omega$.
   \begin{enumerate}
   \item Let $\mathcal{L}u + cu \geq 0$ in $\Omega$, $c \geq 0$, $c \not\equiv 0$. If $u$ attains its \textbf{negative} minimum at a point $x^0 \in \partial \Omega$ then the inequality $\partial_n u(x^0) > 0$ holds.
   \item Let $\mathcal{L}u + cu \geq 0$ in $\Omega$, $c \leq 0$, $c \not\equiv 0$. If $u$ attains its \textbf{non-negative} minimum at a point $x^0 \in \partial \Omega$ then the inequality $\partial_n u(x^0) > 0$ holds unless $u \equiv 0$.
   \end{enumerate}
All four assertions follow from the fact that the inequality $Lu + cu \geq 0$ implies $Lu \geq 0$ in some neighborhood of the minimum point.

### 2.1 Classical results: from Gauss and Neumann to Hopf and Oleinik

Recall that the strong maximum principle for harmonic functions in a three-dimensional domain was obtained by C.F. Gauss [140] on the basis of his mean value theorem. Since this theorem is valid for harmonic functions in $\mathbb{R}^n$ for any $n$, the Gauss proof is obviously valid in any dimension and, moreover, can be easily extended to superharmonic functions.

Proof of the strong maximum principle for uniformly elliptic operators of the more general form $\mathcal{L} + c(x)$ with $C^2$-smooth coefficients (in the form given in the item 1(a) of Remark 2.1) was given:

- in 1892, for $c(x) > 0$ in the two-dimensional case [290];
- in 1894, for $c(x) > 0$ in the multidimensional case [260];
- in 1905, for $c(x) \geq 0$ in the two-dimensional case [294], see also [232].

The most important step was taken in 1927 by E. Hopf [158], see also [160]. Although in this paper the validity of the strong maximum principle is established for uniformly elliptic operators of the form (2.1) with continuous coefficients, actually the Hopf proof runs without changes for operators with bounded coefficients.

Another important observation was made in [158] for operators of the form $\mathcal{L} + c(x)$. In addition to the obvious assertion of item 1(a) of Remark 2.1, Hopf showed that if $Lu + cu \geq 0$ in $\Omega$, then without any conditions on the sign of the coefficient $c(x)$, the function $u$ cannot attain a zero minimum in $\Omega$ unless $u \equiv 0$.

As mentioned in the Introduction, the normal derivative lemma was first established for harmonic functions by S. Zaremba [357] under the interior

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8 An extensive survey of mean value theorems for various classes of functions is contained in [210], see also [276].

9 A similar idea is contained in [295], but the strong maximum principle is not established in this paper.

10 In 1954, A.D. Aleksandrov [10] gave another (purely geometric) proof of this statement.
ball condition on the boundary of a three-dimensional domain. The Zaremba proof uses only the weak maximum principle and the Green’s function of the Dirichlet problem for the Laplacian in a ball. So it is valid in any dimension, and also runs for superharmonic functions.

It should be noted that for the Laplace operator there is an alternative (and equivalent) formulation of the normal derivative lemma:

Let $G$ be the Green’s function of the Dirichlet problem for the Laplacian in $\Omega$. Then the inequality $\partial_n G(x, x^0) > 0$ holds for $x \in \Omega$ and $x^0 \in \partial \Omega$.

This assertion was proved by C. Neumann [277] back in 1888 for a two-dimensional $C^2$-smooth convex domain. Later it was generalized:

- in 1901, for a two-dimensional $C^2$-smooth domain, strictly star-shaped with respect to a point [196];
- in 1909, for a general two-dimensional $C^2$-smooth domain [231];
- in 1912, for a two-dimensional $C^{1,\alpha}$-smooth domain, $\alpha \in (0, 1)$ [185];
- in 1918, for a three-dimensional $C^{1,1}$-smooth [11] domain [233], see also [234].

For the operator $-\Delta + b^i(x)D_i + c(x)$ with $c(x) \geq 0$ in a two-dimensional $C^{2,\alpha}$-smooth domain, $\alpha \in (0, 1)$, this statement was established in 1924 [235]. Later, however, almost all the results we know were formulated in the form of the conventional normal derivative lemma [12].

In 1931, it was first noted [74] (for the operator $-\Delta + c(x)$ with $c(x) \geq 0$ in a two-dimensional $C^2$-smooth domain) that the normal derivative lemma is actually true for a derivative along any strictly interior direction $\ell$ (i.e., along a direction that forms an acute angle with the interior normal).

In 1932, G. Giraud [146, Ch. V] proved the normal derivative lemma for uniformly elliptic operators $L + c(x)$, $c(x) \geq 0$, with coefficients of class $C^{0,\alpha}$,

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11In [233] and [234], the author claims the statement for a domain of class $C^{1,\alpha}$, $\alpha \in (0, 1)$. However, the proof relies on the following fact: for any point $x^0 \in \partial \Omega$ one can choose a point $x \in \Omega$ so that $x^0$ is the boundary point closest to $x$. This fact is not true for domains of class $C^{1,\alpha}$ with $\alpha < 1$.

12Probably, this is due to the fact that for operators with variable principal coefficients, the proof of the alternative formulation is essentially more difficult, and in the general case of measurable principal coefficients, the Green’s function is not defined.

13Instead of the normal $\mathbf{n}$, Giraud uses the conormal $n^\mathcal{L}$ with coordinates $n_i^\mathcal{L} = a^{ij} \mathbf{n}_j$. This gives an equivalent statement. It is essential that he also considers the case $u(x^0) = 0$ without conditions on the sign of $c(x)$; cf. item 2(a) of Remark [24].
\( \alpha \in (0, 1) \), in an \( n \)-dimensional domain of class \( C^{1,1} \). In \[147\] this result was extended to the case where the lower-order coefficients can have singularities on a set \( \mathcal{M} \), that is the union of a finite number of \( C^{1,\alpha} \)-smooth, codimension 1 manifolds, and

\[
|b^i(x)|, |c(x)| \leq C \cdot \text{dist}^{-1}(x, \mathcal{M}), \quad \gamma \in (0, 1).
\]

In 1937, for the first time, the condition on the boundary was significantly weakened: in the paper \[184\], the normal derivative lemma was proved for the Laplacian in a (three-dimensional) domain satisfying the interior \( C^{1,\alpha} \)-paraboloid condition.\[14\]

Finally, a key step was taken by E. Hopf \[159\] and O.A. Oleinik \[282\], who simultaneously and independently proved the normal derivative lemma for uniformly elliptic operators with continuous coefficients under the interior ball condition on the boundary of a domain. The proofs in \[282\] and \[159\] are based on the same idea and, like in \[158\], run without changes for operators with bounded coefficients.\[15\]

Now we give a complete proof of the classical results \[158\] and \[159, 282\].

**Theorem 2.1.** A. Let \( \mathcal{L} \) be an operator of the form (2.1), let the functions \( a^{ij}, b^i \) and \( c \) be bounded in \( \Omega \), and let the assumption (2.3) hold. Suppose that \( u \in C^2(\Omega) \), and \( \mathcal{L}u + cu \geq 0 \) in \( \Omega \).

A1. The function \( u \) cannot attain its zero minimum in \( \Omega \) unless \( u \equiv 0 \).

A2. If \( c \geq 0 \) then \( u \) can attain no negative minimum in \( \Omega \) unless \( u \equiv \text{const} \) and \( c \equiv 0 \).

A3. If \( c \leq 0 \) then \( u \) can attain no positive minimum in \( \Omega \) unless \( u \equiv \text{const} \) and \( c \equiv 0 \).

B. In addition, let the domain \( \Omega \) satisfy the interior ball condition, and let the function \( u \neq \text{const} \) be continuous in \( \overline{\Omega} \). Denote by \( x^0 \) the point \( \partial \Omega \).

\[14\] In some sources (for example, \[171\] and \[35\]), it is stated that a similar condition on the domain was considered already by Giraud. Indeed, in \[146\] and \[147\], some theorems are proved for domains of class \( C^{1,\alpha} \), but the normal derivative lemma requires \( \alpha = 1 \).

\[15\] Hopf considers operators of the form (2.1), Oleinik deals with operators \( \mathcal{L} + c(x) \) under the condition \( c(x) \geq 0, u(x^0) \leq 0 \). Moreover, in \[282\], instead of the normal, she takes an arbitrary direction that forms an acute angle with \( \mathbf{n} \).
at which \(u\) attains its minimum. Then the inequality (1.1) holds true under any of the following conditions:

- **B1.** \(u(x^0) = 0\);
- **B2.** \(u(x^0) < 0 \text{ and } c \geq 0\);
- **B3.** \(u(x^0) > 0 \text{ and } c \leq 0\).

Moreover, in (1.1) one can replace the normal \(n\) by any strictly interior direction \(\ell\).

**Proof.**

1. We begin with the case \(c \equiv 0\). First of all, we establish the weak maximum principle for the operator \(L\) in a domain \(\pi\) with sufficiently small diameter \(d\).

   Assume the contrary: let \(Lu \geq 0\) in \(\pi\), and let \(u|_{\partial\pi} \geq 0\), but for some \(x^0 \in \pi\) we have \(u(x^0) = -A < 0\). Consider the function
   \[
   u^\varepsilon(x) = u(x) - \varepsilon|x - x^0|^2.
   \]

   Obviously, for all sufficiently small \(\varepsilon\) we have
   \[
   u^\varepsilon|_{\partial\pi} \geq -\varepsilon d^2 > -A = u^\varepsilon(x^0).
   \]

   Therefore \(u^\varepsilon\) attains its minimum at some point \(x^1 \in \pi\). At this point we have \(Du^\varepsilon(x^1) = 0\), and the matrix \(D^2u^\varepsilon(x^1)\) is non-negative definite. Thus \(Lu^\varepsilon(x^1) \leq 0\).

   However, the assumption \(Lu \geq 0\) implies
   \[
   Lu^\varepsilon \geq 2\varepsilon (a^{ij} \delta_{ij} - b^i(x - x^0_i)) \geq 2\varepsilon(n\nu - d \sup |b(x)|) > 0 \quad \text{in } \pi,
   \]

   provided \(d < d_0 := \frac{n\nu}{\sup |b(x)|}\). This contradiction proves the statement.

2. Now we prove the strong maximum principle for the operator \(L\). Assume the contrary: let \(Lu \geq 0\) in \(\Omega\), and let \(u \not\equiv \text{const}\), but let the set
   \[
   M = \{x \in \Omega \mid u(x) = \inf_{\overline{\Omega}} u\}
   \]

   be not empty. The complement \(\Omega \setminus M\) is open, and therefore there is a ball lying in it whose boundary contains a point in \(M\). We place the origin at the

\[16\] The assertion B1, without the sign condition for \(c(x)\), was apparently first highlighted in [112], see also [330].
center of this ball and denote by $r$ the radius of the ball, and by $x^0$ a point in $\partial B_r \cap M$. Without loss of generality, we can assume that $r < \frac{d_0}{2}$.

In the annulus $\pi = B_r \setminus \overline{B_{\frac{r}{2}}}$, we consider the barrier function\footnote{Hopf and Oleinik used different barrier functions. Apparently the function was first introduced for this purpose in [221], see also [222] and [223, Ch. 1].}

$$v_s(x) = |x|^{-s} - r^{-s}. \tag{2.5}$$

We estimate $\mathcal{L}v_s$ taking into account the ellipticity condition (2.3):

$$D_i v_s(x) = -sx_i|x|^{-s-2}; \quad D_i D_j v_s(x) = s(s + 2)x_i x_j |x|^{-s-4} - s \delta_{ij} |x|^{-s-2};$$

$$\mathcal{L}v_s(x) = |x|^{-s-2} \cdot \left( - s(s + 2) a^{ij} \frac{x_i x_j}{|x|} + sa^{ij} \delta_{ij} - sb^i x_i \right)$$

$$\leq s|x|^{-s-2} \cdot \left[ - (s + 2) \nu + n \nu^{-1} + r \sup_{\Omega} |b(x)| \right].$$

We choose $s$ so large that the expression in the square brackets is negative. Then for any $\epsilon > 0$ the function $w^\varepsilon = u - \inf_{\Omega} u - \varepsilon v_s$ satisfies the inequality $\mathcal{L}w^\varepsilon \geq 0$ in $\pi$.

Further, $\partial \pi = \partial B_r \cup \partial B_{\frac{r}{2}}$. Obviously, $w^\varepsilon|_{\partial B_r} \geq 0$. Since $B_r \subset \Omega \setminus M$ by construction, the function $u - \inf_{\Omega} u$ is bounded away from zero on the set $\partial B_{\frac{r}{2}}$, and for sufficiently small $\varepsilon > 0$ we have $w^\varepsilon|_{\partial B_{\frac{r}{2}}} \geq 0$. Therefore, we can apply the weak maximum principle to $w^\varepsilon$ in $\pi$, that gives $w^\varepsilon \geq 0$ in $\pi$.

However, $w^\varepsilon(x^0) = 0$. Therefore, for any vector $\ell$ directed into $\pi$ we have $\partial_\ell w^\varepsilon(x^0) \geq 0$, that is

$$\partial_\ell u(x^0) \geq \varepsilon \partial_\ell v_s(x^0) > 0.$$ 

This gives a contradiction, since at the minimum point $Du(x^0) = 0$, and the statement follows.

3. Next, we prove the normal derivative lemma for $\mathcal{L}$. By assumption, one can choose a ball of radius $r$ touching $\partial \Omega$ at the point $x^0$. We place the origin at the center of this ball. According to the strong maximum principle, $u > u(x^0)$ in $B_r$. Further, verbatim repetition of part 2 of the proof gives the inequality (1.1), where $n$ can be replaced by $\ell$.

4. Finally, we drop the assumption $c \equiv 0$. Statements $A2$, $A3$, $B2$ and $B3$ follow immediately from Remark 2.1.
To prove $A1$ и $B1$, we represent $u$ in the form $u = \psi v$, with $\psi > 0$ and $v \geq 0$ in $\Omega$. Direct computation gives

$$0 \leq \frac{Lu + cu}{\psi} = \tilde{L}v := -a^{ij}D_iD_jv + \tilde{b}^i D_i v + \tilde{c}v,$$  \hfill (2.6)

where

$$\tilde{b}^i = b^i - \frac{2a^{ij}D_j\psi}{\psi}, \quad \tilde{c} = \frac{L\psi + c\psi}{\psi}.$$  

Now we put $\psi(x) = \exp(\lambda x_1)$. Then

$$L\psi + c\psi = \psi(-a^{11}\lambda^2 + b^1\lambda + c) \leq \psi\left[-\nu\lambda^2 + \sup_{\Omega} b^1(x)\lambda + \sup_{\Omega} c(x)\right].$$

We choose $\lambda$ so large that the expression in the square brackets is negative. Then for the operator $\tilde{L}$ defined in (2.6), the statements of items 1(b) and 2(b) in Remark 2.1 hold true. In particular, $v$ cannot vanish inside the domain, which gives $A1$. Since $u(x^0) = 0$ implies $Du(x^0) = \psi(x^0)Du(x^0)$, item 2(b) for $v$ provides $B1$ for $u$.

2.2 Generalizing of classical results and refining the conditions on $\partial\Omega$

After the basic results of [159], [282], through the efforts of many authors, the topic was developed in several directions:

1. extension of the class of differential operators, that is, weakening the requirements for the principal and lower-order coefficients;

2. extension of the class of domains, that is, reduction of requirements on the boundary (for the normal derivative lemma);

3. refinement of the applicability limits for the corresponding statements by constructing various counterexamples.

We begin the description of the results with the article of C. Pucci [298]–[299], in which the normal derivative lemma was established in the domain $\Omega = B_r$ for a wider class of operators than in [159], [282]. Namely, the ellipticity condition is allowed to degenerate in the directions tangent to $\partial\Omega$, and the lower-order coefficients satisfy the conditions

$$|b^i(x)| \leq \frac{\sigma(d(x))}{d(x)}, \quad 0 \leq c(x) \leq \frac{\sigma(d(x))}{d^2(x)}, \quad \sigma \in D.$$  \hfill (2.7)
The Pucci proof is based on the barrier function
\[
v(x) = \int_0^d(x) \int_0^{\tau} \frac{\sigma(t)}{t} \, dt \, d\tau + \kappa d(x)
\]
with an appropriate choice of the constant \(\kappa\). This function and its variations were used later in many papers.

If the ellipticity condition degenerates even more, then the strong maximum principle in its classical form does not hold. A.D. Aleksandrov in a series of papers [13], [16], [17], [19], [20] gave for such operators a description of the zero set structure of a non-negative function \(u\), satisfying the inequality \(L u + c u \geq 0\) in \(\Omega\).

In the papers [352]−[353] by R. Výborný the normal derivative lemma was proved for the operator \(L + c(x)\) in a \(C^{1,\mathcal{D}}\)-smooth domain. The conditions imposed on the coefficients of the operator were the same as in [298].

Unfortunately, the results of [352]−[353] are not widely known.

In [354], sharp estimates were obtained for the derivatives of the Green’s function of the Dirichlet Laplacian in a \(C^{1,\mathcal{D}}\)-smooth domain. In particular, the normal derivative lemma was proved in the Neumann form (the normal derivative of the Green’s function on \(\partial \Omega\) is positive). Also there was given a counterexample showing that the condition \(C^{1,\mathcal{D}}\) on the boundary cannot be relaxed to \(C^{1}\). Namely, if \(\phi'\) does not satisfy the Dini condition at zero, then the relation \(\partial_n G(x, 0) = 0\) holds in the paraboloid \(\Sigma(\phi, h)\).

The note [351] was published at the same time as [354]. Subtle asymptotics of harmonic functions in a neighborhood of non-smooth boundary points were derived in this note. As a corollary, the following statement was proved. Let a function \(u\) be harmonic in the paraboloid \(\Sigma(\phi, h)\), and let \(u\) attain its minimum at the vertex \(x^0 = 0\). Then the necessary and sufficient

\[^{18}\text{This problem is also discussed in [283] Ch. III and in [308]−[309]; some operators with unbounded coefficients are considered in [307]; see also [128].}\]

\[^{19}\text{More precisely, Výborný assumes that there exists a function } \rho \in C^2(\Omega) \cap C^1(\overline{\Omega}) \text{ such that } \rho(x) = 0 \text{ and } D^2 \rho = 0 \text{ on } \partial \Omega, \rho > 0 \text{ and } |D^2 \rho(x)| \leq \frac{\sigma(\rho(x))}{\rho(x)} \text{ in } \Omega, \text{ where } \sigma \in \mathcal{D}. \text{ The (local) existence of such a function for a domain of } C^{1,\mathcal{D}} \text{ class was proved in [236].}\]

\[^{20}\text{Výborný proves the assertion of item 2(a) of Remark 2.1 in this case the upper bound for the coefficient } c(x) \text{ in (2.7) is redundant.}\]

\[^{21}\text{Under more restrictive conditions on the domain, some of these estimates were established earlier in [123] and [337].}\]
condition for the relation \( \partial \ell u(0) > 0 \) (for any strictly interior direction \( \ell \)) is the Dini condition for the function \( \phi' \) at zero. This statement is equivalent to that obtained in [354].

The behavior of solutions to the equation \( Lu = 0 \) in a neighborhood of the point \( x^0 \in \partial \Omega \) under the assumption \( b^i(x) = o(|x - x^0|^{-1}) \) was studied in [280], [281], and [256] in the cases where \( \partial \Omega \) satisfies, respectively, the interior/exterior cone condition at the point \( x^0 \).

A large series of papers generalizing the normal derivative lemma is due to B.N. Himchenko and L.I. Kamynin.

In the article [154] (see also [156]), the normal derivative lemma for the Laplacian was established for domains satisfying the interior \( C^{1,D} \)-paraboloid condition. Further, in this paper (see also [155]), the normal derivative estimate on \( \partial \Omega \) was obtained for solutions of the problem

\[
-\Delta u = f \quad \text{in} \quad \Omega, \quad u|_{\partial \Omega} = 0,
\]

provided that \( \Omega \) satisfies the exterior \( C^{1,D} \)-paraboloid condition, while the right-hand side is subject to \( |f(x)| \leq C d^{\gamma-1}(x), \gamma \in (0, 1) \). Finally, [154] gives examples showing that the conditions on the boundary cannot be noticeably improved (these examples repeat in essence the corresponding counterexamples from [354] and [351]).

In the paper [157], the results of [154] were extended to uniformly elliptic operators of the form \( L + c(x) \) with bounded coefficients \( b^i(x) \). The normal derivative lemma is stated there (for any strictly interior direction) “under the assumption that the maximum principle holds” (apparently this means \( c(x) \geq 0 \), and the gradient estimate on \( \partial \Omega \) for the solution of the problem

\[
Lu + cu = f \quad \text{in} \quad \Omega, \quad u|_{\partial \Omega} = g
\]

is stated under the assumptions

\[
|c(x)|, |f(x)| \leq C d^{\gamma-1}(x), \quad \gamma \in (0, 1); \quad g \in C^{1,D}(\partial \Omega).
\]

In the article [171] (see also [170]), the normal derivative lemma is extended to elliptic-parabolic operators

\[
-a^{ij}(x,y)D_{x_i} D_{x_j} - \tilde{a}^{kl}(x,y)D_{y_k} D_{y_l} + b^i(x,y)D_{x_i} + \tilde{b}^k(x,y)D_{y_k} + c(x,y),
\]

\(22\)Here (apparently, for the first time in the literature) one can see the duality of the gradient estimate for the solution on \( \partial \Omega \) and the normal derivative lemma.
with bounded coefficients under the following conditions: the matrix $A$ satisfies the uniform ellipticity condition, the matrix $\tilde{A}$ is non-negative definite, $c(x) \geq 0$, and the domain $\Omega$ satisfies the interior $C^{1,\sigma}$-paraboloid condition, and the paraboloid axis is not perpendicular to the plane $y = 0$.

The In [173] and [174] (see also [172]), the results of [157] are generalized to the class of weakly degenerate operators whose principal coefficients satisfy conditions similar to [298], [352] (lower-order coefficients are bounded)\textsuperscript{23}.

Finally, in the series of papers [175]–[181] some subtle generalizations of the results from [13]–[20] are given.

A very interesting “weakened” form of the normal derivative lemma was established by N.S. Nadirashvili [266] (see also [265]) in a domain $\Omega$ satisfying the interior cone condition. Namely, let $\mathcal{L}$ be a uniformly elliptic operator of the form (2.1), and $c(x) \geq 0$. Suppose that a non-constant function $u$ satisfying the condition $\mathcal{L}u + cu \geq 0$ attains its non-positive minimum at the point $x^0 \in \partial \Omega$. Then \textit{in any neighborhood of $x^0$ there is a point} $x^* \in \partial \Omega$ such that for any strictly interior direction $\ell$ the inequality

$$\liminf_{\varepsilon \to +0} \frac{u(x^* + \varepsilon \ell) - u(x^*)}{\varepsilon} > 0.$$}

holds true. In [169] this result was generalized to a certain class of domains with outer “peaks” and to weakly degenerate (in the spirit of [173]) non-divergence type operators.

In the article [236] by G. Lieberman, the important notion of \textbf{regularized distance}\textsuperscript{24} was introduced. In particular, it was shown that in any domain $\Omega$ of class $C^1$ there exists a function $\rho \in C^2(\mathbb{R}^n \setminus \partial \Omega) \cap C^1(\mathbb{R}^n)$ for which the estimates

$$C^{-1}d(x) \leq \pm \rho(x) \leq C d(x);$$

$$|D\rho(x) - D\rho(y)| \leq C \sigma(|x - y|);$$

$$|D^2\rho(x)| \leq C \frac{\sigma(|\rho(x)|)}{|\rho(x)|}$$

hold true (the + and − signs are related to the points $x \in \overline{\Omega}$ and $x \in \mathbb{R}^n \setminus \Omega$, respectively). Here $\sigma$ stands for the common modulus of continuity for the derivatives of the functions describing $\partial \Omega$ in local coordinates.

\textsuperscript{23}Further development of this topic can be found, for example, in [98].

\textsuperscript{24}In particular cases, this construction has been used earlier, see, e.g., [278], [352], [353].
As a corollary, the normal derivative lemma in a domain of class $C^{1,D}$ was obtained in [236] under conditions on the coefficients (both principal and lower-order) close to [298], [352]. Later, in [237], the gradient estimates on $\partial \Omega$ for solutions to the Dirichlet problem were established in a domain of class $C^{1,D}$ with boundary data $g \in C^{1,D}(\partial \Omega)$. Also the boundary smoothness of the solution was analyzed in [237] in the case where $Dg \in C(\partial \Omega)$ does not satisfy the Dini condition.

Finally, we mention the monumental work [35]. In this paper, the assumptions on the coefficients providing the validity of the normal derivative lemma and the strong maximum principle are somewhat weakened compared to the works listed earlier, although it is much more difficult to verify these conditions. Also in [35] some new counterexamples are given, showing the sharpness of the introduced conditions.

### 2.3 The Aleksandrov–Bakelman maximum principle

This subsection is devoted to one of the most beautiful geometric ideas in the theory of PDEs, the maximum principle of A.D. Aleksandrov and I.Ya. Bakelman. This name is given to a priori maximum estimates for solutions of non-divergence type equations. These estimates have a huge number of applications and, in particular, play a key role in proving the strong maximum principle and the normal derivative lemma for equations with unbounded lower-order coefficients in Lebesgue spaces.

The first estimates of this type were published in the papers [18] and [52]. An estimate for solutions of the Dirichlet problem in the general case was obtained in [22]. In this work the sharpness of the obtained estimates was proved as well [24]. In 1963 Aleksandrov gave in Italy a series of lectures about his method. These lectures were published in Rome [30].

To prove the Aleksandrov–Bakelman estimate, we need some definitions.

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25The history of this result is complicated. The article [52] was published later than the short communication [18] but was submitted somewhat earlier. In [53, §28.1], it is written: “The first version of these maximum principles was obtained by Bakelman [51], [54] in 1959”. In fact, these papers do not yet contain the estimates under consideration, although the idea of studying normal images for estimating solutions was developed earlier by Aleksandrov in [12] as well as by Bakelman in [49], [51]. On the other hand, the survey [271] does not describe the importance of [52] correctly.

26The results of [22] were later rediscovered in [300], [302]. In this regard, the name “Aleksandrov–Bakelman–Pucci (ABP) maximum principle” often occurs in literature.
Let a function $u$ be continuous in $\Omega$, and let $u|_{\partial\Omega} < 0$. We denote by $\tilde{\Omega} = \text{conv}(\Omega)$ the convex hull of $\Omega$. In what follows, we assume that the function $u_+$ is extended by zero to $\tilde{\Omega} \setminus \Omega$.

The convex hull of $u_+$ is the minimal upward convex function that majorizes $u_+$ in $\tilde{\Omega}$. We denote this function by $z$. It is obvious that $z|_{\partial\tilde{\Omega}} = 0$, and the subgraph of the function $z$ is a convex set (the convex hull of the subgraph of $u_+$). It can also be shown (see [273]) that if $\Omega$ is a $C^{1,1}$-smooth domain and $u \in C^{1,1}(\Omega)$, then $z \in C^{1,1}(\tilde{\Omega})$. We also introduce the so-called contact set

$$Z = \{ x \in \Omega \mid z(x) = u(x) \}.$$ 

Now we define the (in general, multi-valued) normal mapping (hodograph mapping) $\Phi : \tilde{\Omega} \to \mathbb{R}^n$ generated by the function $z$. This mapping assigns to any point $x^0 \in \tilde{\Omega}$ all possible vectors $p \in \mathbb{R}^n$ such that the graph of the function $\pi(x) = p \cdot (x - x^0) + z(x^0)$ is the supporting plane to the subgraph of the function $z$ at the point $x^0$. Obviously, if $z \in C^{1}(\tilde{\Omega})$, then the mapping $\Phi$ is single-valued in $\tilde{\Omega}$ (but not in its closure!) and is given by the formula $\Phi(x) = Dz(x)$.

First, we consider an operator $L_0$ with measurable coefficients.

**Lemma 2.1.** Let $\Omega$ be a $C^{1,1}$ class domain, let $u \in C^{1,1}(\Omega)$ and $u|_{\partial\Omega} < 0$. Suppose that the uniform ellipticity condition (2.3) is satisfied. Then the inequality

$$\int_{\Phi(\tilde{\Omega})} g(p) \, dp \leq \frac{1}{n^2} \int_{\tilde{\Omega}} g(Du) \cdot \frac{(L_0 u)^n}{\det(A)} \, dx$$

holds true for any non-negative function $g$.

**Proof.** Note that under the assumptions of Lemma the mapping $\Phi$ satisfies the Lipschitz condition. Changing variables in the integral we obtain

$$\int_{\Phi(\tilde{\Omega})} g(p) \, dp = \int_{\tilde{\Omega}} g(Dz) | \det(D^2z)| \, dx = \int_{\tilde{\Omega}} g(Dz) \det(-D^2z) \, dx$$

(the latter equality follows from the fact that $-D^2z$ is a non-negative definite matrix).

---

27Note that this statement is false if the condition $u|_{\partial\Omega} < 0$ is relaxed to $u|_{\partial\Omega} \leq 0$. 

20
If \( x \notin \mathcal{Z} \) then we apply the Caratheodory theorem, see, e.g., [311, § 17], to claim that \((x, z(x))\) is an interior point of a simplex\(^{28}\) that completely belongs to the graph of \( z \). Therefore, the second derivative of \( z \) in some direction vanishes. But since \( D^2 z(x) \) is non-positive definite matrix, this direction is a principal one, and thus \( \det(-D^2 z(x)) = 0 \).

Otherwise, if \( x \in \mathcal{Z} \) then the tangency condition at the point \( x \) gives

\[
Dz(x) = Du(x); \quad -D^2 z(x) \leq -D^2 u(x)
\]

(the second relation is understood in the sense of quadratic forms and holds for almost all \( x \)). Therefore, (2.9) implies

\[
\int_{\Phi(\overline{\Omega})} g(p) \, dp \leq \int_{\mathcal{Z}} g(Du) \det(-D^2 u) \, dx.
\]

Further, since \( A \) and \(-D^2 u\) are non-negative definite matrices on the set \( \mathcal{Z} \), the matrix \(-A \cdot D^2 u\) has non-negative eigenvalues. By the inequality of arithmetic and geometric means, we have (here and below, \( \text{Tr} \) is the matrix trace)

\[
\det(-D^2 u) = \frac{\det(-A \cdot D^2 u)}{\det(A)} \leq \frac{1}{n^n} \cdot \frac{(\text{Tr}(-A \cdot D^2 u))^n}{\det(A)} = \frac{1}{n^n} \cdot \frac{(L_0 u)_n^n}{\det(A)},
\]

and (2.8) follows.

**Remark 2.2.** Since the inequalities \( u > 0 \) \( L_0 u \geq 0 \) hold on the set \( \mathcal{Z} \), one often uses the more convenient estimate

\[
\int_{\Phi(\overline{\Omega})} g(p) \, dp \leq \frac{1}{n^n} \int_{\{u>0\}} g(Du) \cdot \frac{(L_0 u)_n^n}{\det(A)} \, dx \quad (2.10)
\]

instead of \((2.8)\).

**Theorem 2.2.** Let the condition \((2.2)\) hold, and let \( \text{Tr}(A) > 0 \) almost everywhere in \( \Omega \). Then any function \( u \in W^2_{n,\text{loc}}(\Omega) \) such that\(^{29}\) \( u \big|_{\partial \Omega} \leq 0 \) satisfies the estimate

\[
(\max_{\overline{\Omega}} u_+)^n \leq \frac{\text{diam}^n(\Omega)}{n^n |B_1|} \int_{\mathcal{Z}} \frac{(L_0 u)_n^n}{\det(A)} \, dx \quad (2.11)
\]

(here and below, we set \( 0/0 = 0 \) if such uncertainty arises).

---

\(^{28}\)In this case the dimension of the simplex can be any number from 1 to \( n \).

\(^{29}\)This means that for any \( \varepsilon > 0 \) the inequality \( u - \varepsilon < 0 \) holds in some neighborhood of \( \partial \Omega \).
Proof. Let us first assume that the matrix \( A \), the function \( u \), and the domain \( \Omega \) satisfy the conditions of Lemma 2.1. It suffices to consider the case when 
\[
M = \max_{\Omega} u = \max_{\Omega} z > 0.
\]
We set 
\[
d = \text{diam}(\Omega) = \text{diam}(\tilde{\Omega})
\]
and claim that the set \( \Phi(\tilde{\Omega}) \) contains the ball \( B_{M/d} \). Indeed, let \( p \in B_{M/d} \). Consider the graph of the function 
\[
\pi(x) = p \cdot x + h.
\]
By choosing an appropriate \( h \), we can ensure that this is a supporting plane to the subgraph of the function \( z \) at some point \( x^0 \), and write 
\[
\pi(x) = p \cdot (x - x^0) + z(x^0).
\]
If \( x^0 \in \partial \tilde{\Omega} \) then \( z(x^0) = 0 \), and the maximum point of \( z \) satisfies 
\[
M = z(x) \leq p \cdot (x - x^0) \leq |p| \cdot d < M,
\]
which is a contradiction. Therefore, \( x^0 \in \tilde{\Omega} \), whence 
\[
p = Dz(x^0) = \Phi(x^0) \in \Phi(\tilde{\Omega}),
\]
and the claim follows.

We use the estimate (2.8) with \( g \equiv 1 \) and obtain 
\[
|B_1| \cdot \left( \frac{M}{d} \right)^n = |B_{M/d}| \leq |\Phi(\tilde{\Omega})| \leq \frac{1}{n^n} \int_\Omega (L_0 u)^n \frac{1}{\det(A)} dx,
\]
which immediately implies (2.11).

Consider now the general case. The integrand in (2.11) does not change if the matrix \( A \) is multiplied by a positive function. Therefore, without loss of generality, we can assume that \( \text{Tr}(A) \equiv 1 \). Let us take the function 
\[
u^\varepsilon = u - \varepsilon \]
and approximate \( \Omega \) from the inside by domains with smooth boundaries. Further, since the estimate (2.11) keeps under a passage to the limit in \( W_0^2 \), we can assume that \( u^\varepsilon \) is a smooth function. We apply the estimate (2.11) to the function \( u^\varepsilon \) and the uniformly elliptic operator \( L_0 - \nu \Delta \). Then we can push \( \nu \to 0 \) and then \( \varepsilon \to 0 \). 

Theorem 2.3. Let \( \mathcal{L} \) be an operator of the form (2.7), let the assumption (2.2) hold, and let \( \text{Tr}(A) > 0 \) almost everywhere in \( \Omega \). Assume that 
\[
\mathfrak{h} \equiv \frac{|b|}{\det \frac{1}{2}(A)} \in L_\infty(\Omega)
\]
(2.12)
Then the estimate
\[
\max_{\Omega} u_+ \leq N\left(n, \|h\|_{n, \{u>0\}}\right) \cdot \text{diam}(\Omega) \left\| \frac{(Lu)_+}{\det \pi(A)} \right\|_{n, \{u>0\}} \tag{2.13}
\]
holds true for any function \(u\) satisfying the assumptions of Theorem 2.2.

**Proof.** We can assume that the matrix \(A\), the function \(u\), and the domain \(\Omega\) satisfy the conditions of Lemma 2.1. The general case is obtained from this particular one analogously to the second part of the proof of Theorem 2.2.

Let \(g = g(|p|)\). Taking into account the inclusion \(B_{M/d} \subset \Phi(\tilde{\Omega})\), we obtain from (2.10)
\[
n |B_1| \cdot \int_0^{M/d} g(\rho) \rho^{n-1} d\rho \leq \frac{1}{n^n} \int_{\{u>0\}} g(|Du|) \cdot \frac{(Lu - b_i D_i u)^n}{\det(A)} dx. \tag{2.14}
\]

We introduce the notation
\[
F = \left\| \frac{(Lu)_+}{\det \pi(A)} \right\|_{n, \{u>0\}} + \varepsilon, \quad \varepsilon > 0.
\]

Then the quotient in the right-hand side of (2.14) can be estimated by the Hölder inequality:
\[
\frac{(Lu - b_i D_i u)^n}{\det(A)} \leq \left( F^{\frac{n}{n-1}} + |Du|^{\frac{n}{n-1}} \right)^{n-1} \cdot \left( \frac{(Lu)^n}{\det(A) F^n} + h^n \right).
\]

Now we put \(g(\rho) = (F^n + \rho^n)^{-1}\). Then (2.14) implies
\[
n |B_1| \int_0^{M/d} \frac{\rho^{n-1}}{F^n + \rho^n} d\rho \leq \frac{1}{n^n} \int_{\{u>0\}} \left( F^{\frac{n}{n-1}} + |Du|^{\frac{n}{n-1}} \right)^{n-1} \cdot \left( \frac{(Lu)^n}{\det(A) F^n} + h^n \right) dx.
\]

By the elementary inequality \((x + y)^{n-1} \leq 2^{n-2}(x^{n-1} + y^{n-1})\) we deduce
\[
\ln \left( 1 + \frac{M^n}{n^n |B_1|} \right) \leq \frac{2^{n-2}}{n^n |B_1|} \left( 1 + \|h\|^n_{n, \{u>0\}} \right),
\]
and therefore,
\[
M \leq d \cdot F \left( \exp \left( \frac{2^{n-2}}{n^n |B_1|} \left( 1 + \|h\|^n_{n, \{u>0\}} \right) \right) - 1 \right)^{\frac{1}{n}}.
\]

Pushing \(\varepsilon \to 0\) in the expression for \(F\), we arrive at (2.13). \(\square\)
Remark 2.3. If the uniform ellipticity condition \((2.3)\) is fulfilled then, in view of Remark 2.2, \((2.13)\) implies the simpler estimate:

\[
\max_{\Omega} u_+ \leq N \left( n, \tfrac{|b|_{n, \{u>0\}}}{\nu} \right) \cdot \frac{\text{diam}(\Omega)}{\nu} \cdot \| (\mathcal{L}u)_+ \|_{n, \{u>0\}}.
\]  \((2.15)\)

Remark 2.4. For uniformly elliptic operators of the form \((2.1)\), the following Hopf’s maximum estimate is well known (see, e.g., [145, Theorem 3.7]):

\[
\max_{\Omega} u_+ \leq C \left( \text{diam}(\Omega), \tfrac{|b|_{\infty, \{u>0\}}}{\nu} \right) \cdot \| (\mathcal{L}u)_+ \|_{\infty, \{u>0\}}.
\]

Here, the maximum of the solution is estimated in terms of the \(L_{\infty}\)-norm of the right-hand side, which turns out to be insufficient in some applications.

On the other hand, coercive estimates in \(L_r\) ([145, Theorem 9.13]) together with the Sobolev embedding theorem imply

\[
\max_{\Omega} u_+ \leq C \cdot \| (\mathcal{L}_0 u)_+ \|_{r, \Omega}, \quad r > n/2.
\]  \((2.16)\)

However, in this estimate the constant \(C\) depends on the continuity moduli of the coefficients \(a^{ij}\). Therefore, for example, for quasilinear equations, where the coefficients \(a^{ij}\) depend on the solution \(u\) itself and on its derivatives, the estimate \((2.16)\) is of little use.

The Aleksandrov–Bakelman estimate differs in that it requires neither the continuity of the principal coefficients nor the boundedness of the lower-order coefficients and the right-hand side of the equation.

In connection with Theorem 2.3, we mention the so-called Bony type maximum principle.

Let \(\mathcal{L}\) be an operator of the form \((2.1)\), and let the assumption \((2.2)\) be satisfied. If a function \(u\) attains its minimum at the point \(x^0 \in \Omega\), then the inequality \(\liminf_{x \to x^0} \mathcal{L}u \leq 0\) holds true.

This statement was proved for operators with bounded coefficients in [71] (for \(u \in W^2_q(\Omega)\) with any \(q > n\)) and in [243] (for \(u \in W^2_n(\Omega)\)).\(^{30}\) We prove its variant for operators with unbounded lower-order coefficients.

\(^{30}\)In [243], a stronger property was proved:

\[
\liminf_{x \to x^0} |Du| = 0; \quad \liminf_{x \to x^0} D^2u \geq 0
\]

(the second relation is understood in the sense of quadratic forms). However, for operators with unbounded coefficients, the relation \((2.17)\) does not follow directly from this.
Corollary 2.1. Assume that the coefficients of the operator $L$ satisfy the conditions of Theorem 2.3. If a function $u \in W_{n,\text{loc}}^2(\Omega)$ attains its minimum at the point $x^0 \in \Omega$, then
\[
\operatorname{ess lim inf}_{x \to x^0} \frac{L u}{\operatorname{Tr}(A)} \leq 0. \quad (2.17)
\]

Proof. As in Theorem 2.2, we can assume without loss of generality that $\operatorname{Tr}(A) \equiv 1$. Let us place the origin at the point $x^0$.

We proceed by contradiction. Suppose that in some neighborhood of the origin the inequality $L u \geq \delta > 0$ holds almost everywhere. Consider the function
\[
w^\varepsilon(x) = \varepsilon \left(1 - \frac{|x|^2}{r^2}\right) - u(x) + u(0)
\]
in the ball $B_r$. Then $w^\varepsilon(0) = \varepsilon$, and we have $w^\varepsilon|_{\partial B_r} \leq 0$ for sufficiently small $r$. Applying the estimate (2.15) to $w^\varepsilon$ in $B_r$ we obtain
\[
\varepsilon \leq N(n, \|h\|_{n,B_r}) \cdot 2r \cdot \left\| \frac{(L w^\varepsilon)_+}{\det \frac{1}{\pi}(A)} \right\|_{n,B_r}.
\]
Since
\[
L w^\varepsilon = \frac{2\varepsilon}{r^2} \left(\operatorname{Tr}(A) + b^i x_i\right) - L u \leq \frac{2\varepsilon}{r^2} \left(1 + r|b|\right) - \delta,
\]
we have for $\varepsilon < \frac{\delta r^2}{4}$
\[
\varepsilon \leq N \left\| \frac{(4\varepsilon|b| - r\delta)_+}{\det \frac{1}{\pi}(A)} \right\|_{n,B_r} \leq 4\varepsilon N \left\| \left(h - \frac{r\delta}{4\varepsilon}\right)_+ \right\|_{n,B_r} = o(\varepsilon) \quad \text{as} \quad \varepsilon \to 0
\]
(here the inequality (*) follows from the relation $\det \frac{1}{\pi}(A) \leq \operatorname{Tr}(A) = 1$).

This contradiction proves (2.17). \qed

A.D. Aleksandrov repeatedly developed and improved the results of [22]. In [25], pointwise estimates of solutions to the Dirichlet problems are obtained in terms of the distance to the boundary of the domain; in [23] these estimates are extended to a wider class of equations. The paper [26] is devoted to proving the attainability of the obtained estimates, while a short note [24] shows that in the general case it is impossible to relax the requirements on the right-hand side of the equation. Finally, in [28], pointwise
estimates for the solution in terms of fine characteristics of the domain $\Omega$ are obtained. Based on this result, a gradient estimate for the solution on $\partial\Omega$ is obtained for some special cases (a summary of these results is given in [27], [29]).

In the mid-1970s, N.V. Krylov ([201]–[203]) first obtained an Aleksandrov–Bakelman type estimate for parabolic operators. After that, the study of elliptic and parabolic problems proceeded almost in parallel, but the discussion of the results for nonstationary equations is beyond the scope of our survey.

Later, the techniques based on the normal image was applied to other boundary value problems. For the oblique derivative problem where a non-tangential directional derivative is given on the boundary, a local Aleksandrov type maximum estimate was established in [267] for bounded coefficients $b^i$ and in [268] for the general case (see also [97] and [238]).

For the Venttsel problem where a second order operator in tangential variables

$$L' \equiv -\alpha^{ij}(x)d_id_j + \beta^i(x)D_i, \quad d_i \equiv D_i - n_in_kD_k, \quad \beta^i(x)n_i \leq 0,$$

is given on the boundary, the corresponding estimates were obtained in [248], [249] in the non-degenerate case (the operator $L'$ is uniformly elliptic with respect to tangential variables) and in the degenerate case (the second-order terms in the boundary operator can vanish on a set of positive measure but the vector field $(\beta^i)$ is non-tangential to $\partial\Omega$). Later these estimates were generalized to the case of operators $L$ and $L'$ with unbounded lower-order coefficients [39], [40]. In the paper [41], local Aleksandrov type estimates were established for solutions to the so-called two-phase Venttsel problem. In all these cases, these estimates served as a “launching pad” for obtaining a series of a priori estimates required to prove existence theorems for solutions of quasilinear and fully nonlinear boundary value problems.

Another direction in the development of Aleksandrov’s ideas is the transfer of the maximum estimates to equations with lower-order coefficients and right-hand sides from other functional classes. The papers [38], [238], and [269] dealt with various classes of operators with “composite” coefficients. The article [270] is devoted to an Aleksandrov type estimate in terms of the norms of the right-hand side in the weighted Lebesgue spaces. Each of these results led to a corresponding extension of the class of nonlinear equations for which one can prove the solvability of the basic boundary value problems.
L. Caffarelli [81] established the Aleksandrov–Bakelman estimate for the so-called viscosity solutions of elliptic equations. Later this idea was actively applied to various classes of nonlinear equations (see, e.g., [82], [83], Ch.3], [103], [166], [45], [90]).

A further group of papers is devoted to weakening the conditions on the right-hand side of the equation for certain classes of operators. In 1984, E. Fabes and D. Stroock [126] obtained the estimate (2.16) for operators with measurable principal coefficients under the assumption $r > r_0$, where $r_0 < n$ is an exponent depending on the ellipticity constant of the operator (see also [138]). In [186] and [239] this estimate was established for the oblique derivative problem. On the other hand, C. Pucci [301] introduced the concept of maximal and minimal operators. Using this concept he established a lower bound for the values of $r_0$ allowing such an estimate (in this connection see [303] and references therein). Necessary and sufficient conditions for the fulfillment of (2.16) are obtained only in the two-dimensional case [47]. In some papers (see [194] and references therein), the results of [126] have been extended to viscosity solutions of nonlinear equations.

In [208], a series of maximum estimates for the solution was established in terms of the $L_m$-norm of the right-hand side (here $m \in (n/2, n]$ is an integer) under the assumption that the matrix of principal coefficients belongs to some special convex cone in the space of matrices (for almost all $x \in \Omega$). Among recent advances in this direction, we mention the paper [349] by N.S. Trudinger. Undoubtedly, these studies are still far from complete.

We should also quote the paper [76], which studies the dependence of the maximum estimate on the domain characteristics. In particular, the author managed to obtain an estimate in terms of $|\Omega|^4$ instead of the domain diameter (notice that for convex domains this was already done in [22]).

We also mention the paper [207], in which a discrete analogue of the Aleksandrov–Bakelman estimate for difference operators was obtained.

2.4 Results for operators with coefficients $b^i(x)$ in Lebesgue spaces

The simplest consequence of the Aleksandrov–Bakelman estimate is the weak maximum principle for functions $u \in W^2_n(\Omega)$ and operators of the form (2.1) with $b^i \in L_n(\Omega)$. Moreover, as pointed out already in [22], this estimate allows us to consider the operator $\mathcal{L} + c(x)$ with coefficient $c(x)$ of “bad sign”.
Corollary 2.2. Assume that the coefficients of the operator $L$ satisfy the assumptions of Theorem 2.3. Then there exists $\delta > 0$ depending only on $n$, $\text{diam}(\Omega)$ and $\|h\|_{n,\Omega}$ (the function $h$ is introduced in (2.12)) such that if

$$h \equiv \frac{c_+}{\det^{\frac{1}{n}}(A)} \in L_n(\Omega), \quad \|h\|_{n,\Omega} < \delta$$

(recall that we set $0/0 = 0$ if such uncertainty arises), then the weak maximum principle holds for the operator $L + c(x)$ and functions $u \in W^{2,n}_{\text{loc}}(\Omega)$.

Proof. We proceed by contradiction. Suppose that $Lu + cu \geq 0$ in $\Omega$ and $u \geq 0$ on $\partial \Omega$, but $\min_{\Omega} u = -A < 0$. Apply the estimate (2.13) to the function $u^\varepsilon = -u - \varepsilon$. Since the inequality $Lu^\varepsilon = -Lu \leq cu \leq Ac_- \cdot A$ holds on the set $\{u^\varepsilon > 0\}$, we obtain

$$(A - \varepsilon)^+ \leq N(n, \|h\|_{n,\Omega}) \cdot \text{diam}(\Omega) \cdot \|h\|_{n,\Omega} \cdot A.$$ 

This is impossible if $N(n, \|h\|_{n,\Omega}) \cdot \text{diam}(\Omega) \cdot \|h\|_{n,\Omega} < 1$ and $\varepsilon > 0$ is small enough. 

It is easy to see that now the proof of Theorem 2.1 runs without changes for the so-called strong supersolutions: $u \in W^2_n(\Omega)$, and $Lu + cu \geq 0$ almost everywhere in $\Omega$ (the coefficients of the operator $L$ are assumed to be measurable and bounded). However, in order to consider lower-order coefficients in Lebesgue spaces, new ideas were needed.

Note that one cannot reduce the assumption $b^i \in L_n(\Omega)$ to $b^i \in L_p(\Omega)$ with $p < n$. Indeed, the function $u(x) = |x|^2$ satisfies the equation

$$-\Delta u + \frac{n x_i}{|x|^2} D_i u = 0 \quad \text{in} \quad B_1,$$

but does not satisfy the maximum principle; the coefficients $b^i(x) = \frac{n x_i}{|x|^2}$ belong to the space $L_p(B_1)$ with any $p < n$ and even to the weak-$L_n$ space (the Lorentz space $L_{n,\infty}(B_1)$) but do not belong to $L_n(B_1)$.

The strong maximum principle for operators with $b^i \in L_n(\Omega)$ was established in [21]. We prove the simplest version of this result.\footnote{In [21], operators of the form $\mathcal{L} + c(x)$ are considered under the condition $c(x) \leq \frac{h(x)}{|x-x_0|^2}$, where $x_0$ is a (zero) minimum point of the function $u$, and $h \in L_n(\Omega)$. In addition, restrictions on the coefficients in this work can depend on the direction.}
**Theorem 2.4.** Let $\mathcal{L}$ be an operator of the form (2.1), let the condition (2.3) be satisfied, and let $b^i \in L_{n,\text{loc}}(\Omega)$. Assume that $u \in W^2_{n,\text{loc}}(\Omega)$, and $\mathcal{L}u \geq 0$ almost everywhere in $\Omega$. If $u$ attains its minimum at an interior point of the domain, then $u \equiv \text{const}$ and $\mathcal{L}u \equiv 0$.

**Proof.** Suppose that $u \not\equiv \text{const}$, but the set (2.4) is not empty. As in the proof of Theorem 2.1, one can see that there exists a ball in the set $\Omega \setminus M$ whose boundary contains a point $x^0 \in M$. We denote the radius of the ball by $\frac{r}{2}$ and assume, without loss of generality, that $B_r \subset \Omega$. Denote $\pi = B_r \setminus B_{\frac{r}{4}}$ and consider the barrier function (2.5) in $\pi$.

We have

$$\mathcal{L}v_s(x) \leq s|x|^{-s-2} \cdot \left( - (s + 2)\nu + n\nu^{-1} + r|b(x)| \right).$$

In contrast to Theorem 2.1, here we cannot achieve the inequality $\mathcal{L}v_s \leq 0$. However, choosing $s = n\nu^{-2}$, we obtain

$$\mathcal{L}v_s(x) \leq sr|x|^{-s-2}|b(x)| \leq 4^{s+2}sr^{-s-1}|b(x)| \quad \text{in } \pi.$$

By construction, the inequality $u(x) - u(x^0) > 0$ holds on $\partial B_{\frac{r}{4}}$. Therefore, for sufficiently small $\varepsilon > 0$, the function $w^\varepsilon(x) = \varepsilon v_s(x) - u(x) + u(x^0)$ is non-positive on both parts of the boundary of $\pi$.

Application of the estimate (2.15) to $w^\varepsilon$ in $\pi$ gives

$$w^\varepsilon(x) \leq C(n, \nu, \|b\|_{n,\pi}) \cdot r \cdot \varepsilon \cdot (\mathcal{L}v_s(x))_+ \|v_s\|_{n,\pi} \leq C(n, \nu, s, \|b\|_{n,\pi}) \cdot \varepsilon r^{-s} \|b\|_{n,\pi},$$

and therefore,

$$u(x) - u(x^0) \geq \varepsilon \left( |x|^{-s} - r^{-s} - C(n, \nu, s, \|b\|_{n,\pi}) \|b\|_{n,\pi} r^{-s} \right). \quad (2.18)$$

By the Lebesgue theorem, for any $\delta > 0$ one can choose $r$ so small that $\|b\|_{n,\pi} \leq \delta$. Then the inequality (2.18) taken at the point $x^0$ becomes

$$0 \geq \varepsilon r^{-s} \left( 2^s - 1 - C(n, \nu, s, \delta) \delta \right).$$

The latter is impossible if $\delta$ is small enough. \hfill $\square$

As a corollary, the following statement was proved in [21].

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32 This result is also given here in a simplified version.

29
Corollary 2.3. Let the operator $L$ and the function $u$ satisfy the assumptions of Theorem 2.4. Let the domain $\Omega$ satisfy the interior ball condition in a neighborhood $U$ of a point $x^0 \in \partial \Omega$. Suppose that
\[ u|_{\partial \Omega \cap U} = \inf_{\Omega} u; \quad Du|_{\partial \Omega \cap U} = 0. \tag{2.19} \]
Then $u \equiv \text{const}$ in $\Omega$.

Proof. Extending the function $u$ by a constant outside $\Omega$ in a neighborhood of the point $x^0$, we fall into the conditions of Theorem 2.4. \qed

It is easy to see that Corollary 2.3 is much weaker than the normal derivative lemma, since the condition (2.19) must be satisfied on $\partial \Omega \cap U$, i.e. on a piece of the boundary. However, in contrast to the case of bounded lower-order coefficients, where the proofs of the strong maximum principle and of the normal derivative lemma are almost the same, the normal derivative lemma fails under the conditions of Theorem 2.4! We provide the corresponding counterexample (see [274], [319], [272]).

Let $u(x) = x_n \cdot \ln^\alpha(|x|^{-1})$ in the half-ball $B_r^+ = B_r \cap \{x_n > 0\}$. Then it is easily seen that $u \in W^{2,n}_r(B_r^+)$ for $r \leq \frac{1}{2}$ and $\alpha < \frac{n-1}{n}$. Further, direct calculation shows that $u$ satisfies the equation
\[-\Delta u + b^a(x)D_n u = 0 \quad \text{with} \quad |b^a| \leq \frac{C(\alpha)}{|x| \ln(|x|^{-1})} \in L^\infty(B_r^+).\]
Finally, $u > 0$ in $B_r^+$, and $u$ attains its minimum at the boundary point $0$. However, for $\alpha < 0$ we evidently have $D_n u(0) = 0$.

Remark 2.5. Notice that the weakened form of the normal derivative lemma (see [266]) holds true in this example. We conjecture that such a statement is fulfilled for a general uniformly elliptic operator $L$ with $b^i \in L^\infty(\Omega)$, but as far as we know, this question remains open.

Remark 2.6. The above example also shows that the condition $b^i \in L^\infty(\Omega)$ is insufficient for the gradient estimates on $\partial \Omega$ of the solution to the Dirichlet problem. Indeed, for $\alpha > 0$ we have $D_n u(0) = +\infty$.

The article by O.A. Ladyzhenskaya and N.N. Ural'tseva [214] is of primary importance (a brief report was published three years earlier in [212]). There, for the first time, an iterative method for estimating the solution in
a neighborhood of the boundary was applied. In the simplest case, it is as follows.

Assume that a function $u$ is defined in the cylinder $Q_{1,1}$, satisfies the equation $L u = f$ and the boundary condition $u|_{x_n=0} = 0$. Let us introduce a sequence of cylinders $Q_{r_k,h_k}$, where $r_k = 2^{-k}$ and $h_k$ is a suitably chosen sequence such that $h_k = o(r_k)$ as $k \to \infty$. Denote

$$M_k = \sup_{Q_{r_k,h_k}} u(x)/h_k$$

and apply the Aleksandrov–Bakelman estimate to the difference

$$u(x) - M_k h_k \cdot \mathbf{v}\left(\frac{x'}{r_k}, \frac{x_n}{h_k}\right),$$

where $\mathbf{v}$ is a certain special barrier function. The resulting estimate, taken at the points $x \in Q_{r_{k+1},h_{k+1}}$, gives a recurrence relation between $M_{k+1}$ and $M_k$. Iterating it, we obtain $\limsup_k M_k < \infty$, which gives an upper bound for $D_n u(0)$ in terms of $\sup u$ and some integral norm of the right-hand side.

In [214], this scheme was applied to the equation $L u = f$ with a uniformly elliptic operator $L$ under the following assumptions:

$$u \in W^2_n(\Omega), \quad b^i \in L_q(\Omega), \quad f_+ \in L_q(\Omega), \quad q > n, \quad (2.20)$$

in a domain in one of the following two classes:

1) convex domains;
2) $W^2_q$-smooth domains.

As we already mentioned in §2.3, the paper [38] established an Aleksandrov–Bakelman type estimate in $\Omega \subset Q_{R,R}$ for operators of the form (2.11) with “composite” lower-order coefficients $b^i = b^i_{(1)} + b^i_{(2)}$ provided

$$b^i_{(1)} \in L_n(\Omega), \quad |b^i_{(2)}(x)| \leq C x_n^{-1}, \quad \gamma \in (0, 1). \quad (2.21)$$

Based on this result, a bound for $\esssup \partial_n u$ on $\partial \Omega$ in a $W^2_q$-smooth domain, $q > n$, was established in [38] subject to the conditions

$$b^i = b^i_{(1)} + b^i_{(2)}; \quad b^i_{(1)} \in L_q(\Omega), \quad |b^i_{(2)}(x)| \leq C x_n^{-1},$$

$$L u = f^{(1)} + f^{(2)}; \quad f^{(1)}_+ \in L_q(\Omega), \quad f^{(2)}_+ (x) \leq C x_n^{-1}, \quad \gamma \in (0, 1).$$

This means that any point $x^0 \in \partial \Omega$ has a neighborhood $U$ such that there is a $W^2_q$-diffeomorphism mapping the set $U \cap \Omega$ onto $Q_{1,1}$, and the norms of direct and inverse diffeomorphisms are estimated uniformly with respect to $x^0$. This assumption ensures that the conditions (2.20) are invariant under local flattening of the boundary.
M.V. Safonov \cite{318} (see also \cite{321}) developed a new approach based on the boundary Harnack inequality (see § 4.3). By this approach, he established in a unified way

1. the normal derivative lemma under the condition $L_0 u \geq 0$, in a domain satisfying the interior $C^{1,\mathcal{D}}$-paraboloid condition\textsuperscript{34};

2. an upper bound for $\partial_n u(0)$ under the conditions $L_0 u \leq 0$, $u|_{\partial\Omega \cap B_r} = 0$, in a domain satisfying the exterior $C^{1,\mathcal{D}}$-paraboloid condition\textsuperscript{34}.

In \cite{319}, the (slightly improved) iterative method by Ladyzhenskaya–Uraltseva was applied\textsuperscript{35} to derive the normal derivative lemma in the domain $\Omega = \mathcal{Q}_{R,R}$ under the conditions

$$u \in W^2_{n,\text{loc}}(\Omega) \cap C(\overline{\Omega}), \quad \min_{\overline{\Omega}} u = u(0); \quad b^i \in L_n(\Omega), \quad b^n \in L_q(\Omega), \quad q > n.$$ 

Thus, it turns out that, in comparison with $b^i \in L_n(\Omega)$, it suffices to strengthen the assumption only on the normal component of the vector $b$.

In \cite{272}, both the normal derivative lemma and the gradient estimate of the solution to the Dirichlet problem on the boundary of the domain are obtained under the currently sharpest conditions. Moreover, the duality of these statements is explicitly demonstrated. The result is achieved by a combination of the Ladyzhenskaya–Uraltseva–Safonov technique and the Aleksandrov–Bakelman type estimate \cite{238}, where the assumption on $b^{i}_{(2)}$ from (2.21) is refined to $|b^{i}_{(2)}(x)| \leq \frac{\sigma(x_n)}{x_n}, \sigma \in \mathcal{D}$.

We give the formulation of this result.

**Theorem 2.5.** Let $\mathcal{L}$ be a uniformly elliptic operator of the form (2.1) in $\Omega = \mathcal{Q}_{R,R}$. Let $b^i = b^i_{(1)} + b^i_{(2)}$, and let the following conditions be satisfied:

$$b^i_{(1)} \in L_n(\Omega), \quad \|b^i_{(1)}\|_{n,Q,r} \leq \sigma(r) \quad \text{for} \quad r \leq R; \quad |b^i_{(2)}(x)| \leq \frac{\sigma(x_n)}{x_n}; \quad \sigma \in \mathcal{D}.$$ 

Suppose that $u \in W^2_{n,\text{loc}}(\Omega) \cap C(\overline{\Omega})$.

\textsuperscript{34}In \cite{318}, the function $\phi$ defining an interior or exterior paraboloid satisfies the assumption $\int_0^\tau \tau^{-2}\phi(\tau) \, d\tau < \infty$. This assumption is formally more general than the standard $C^{1,\mathcal{D}}$-condition, but Lemma 2.4 in \cite{272} shows that the obtained requirement on the domain is in essence equivalent to the usual one.

\textsuperscript{35}It was noted in \cite{319} that the normal derivative lemma under assumption $b^i \in L_q(\Omega)$, $q > n$, was in fact obtained already in \cite{214} Lemma 4.4. This fact remained unnoticed for more than 20 years!
1. If $u > 0$ in $Q_{R,R}$, $u(0) = 0$, and $\mathcal{L}u \geq 0$, then

$$\inf_{0 < x_n < R} \frac{u(0,x_n)}{x_n} > 0.$$ 

2. If $u|_{x_n=0} \leq 0$, $u(0) = 0$, and $\mathcal{L}u = f^{(1)} + f^{(2)}$, where

$$\|f^{(1)}_+\|_{n,Q,r,r} \leq \sigma(r) \text{ for } r \leq R; \quad f^{(2)}_+(x) \leq \frac{\sigma(x_n)}{x_n},$$

then

$$\sup_{0 < x_n < R} \frac{u(0,x_n)}{x_n} \leq C,$$

where $C < \infty$ is determined by known quantities.

It is important to note that the occurrence of term $b^{(2)}_{(2)}$ allows us to perform a coordinate transformation using the regularized distance in a neighborhood of an insufficiently smooth boundary. Thus, Theorem 2.5 implies corresponding assertions in domains satisfying the interior/exterior $C^{1,D}$-paraboloid condition.

In [42], a new counterexample was constructed. It shows the sharpness of the interior $C^{1,D}$-paraboloid condition for the normal derivative lemma. We present its formulation in the simplest case.

**Theorem 2.6.** Let $\Omega$ be locally convex in a neighborhood of the origin, that is,

$$\Omega \cap B_R = \{x \in \mathbb{R}^n \mid F(x') < x_n < \sqrt{R^2 - |x'|^2}\}$$

for some $R > 0$. Here $F$ is a convex function, $F \geq 0$, and $F(0) = 0$.

Further, suppose that $u \in W^{2, \text{loc}}(\Omega) \cap C(\overline{\Omega})$ is a solution of the equation $\mathcal{L}_0u = 0$ with a uniformly elliptic operator $\mathcal{L}_0$, and $u|_{\partial\Omega \cap B_R} = 0$.

If the function

$$\delta(r) = \sup_{|x'| \leq r} \frac{F(x')}{|x'|}$$

does not satisfy the Dini condition at zero then $\lim_{\varepsilon \to 0} \frac{u(\varepsilon x_n)}{\varepsilon} = 0$.

\[36\text{Cf. [161], where the existence of } D_n u(0) \text{ is proved for viscosity solutions of the equation } \mathcal{L}_0 u = f.\]
Note that if $\delta(r)$ satisfies the Dini condition at zero then $\Omega$ satisfies the interior $C^{1,\mathcal{D}}$-paraboloid condition at the origin. Thus, for \textit{locally convex} domains, the Dini condition at zero for the function $\delta(r)$ is necessary and sufficient for the validity of the normal derivative lemma.

We emphasize that all previous counterexamples of this type ([354], [351], [157] and [318]) require the absence of the Dini condition for the function $\inf_{|x'| \leq r} \frac{F(x')}{|x'|}$. Roughly speaking, in those counterexamples the Dini condition must fail in all directions, whereas in Theorem 2.6 it is enough to violate it in one direction.

For general domains, a more subtle counterexample was constructed in [321]. However, it is too complicated to describe it here.

2.5 The Harnack inequality

As already mentioned in the Introduction, the Harnack inequality, which can be considered as a quantitative version of the strong maximum principle, was first proved by C.G.A. Harnack [152] for harmonic functions on the plane. Since Harnack’s proof is based on the Poisson formula, it is obviously valid in any dimension. Harnack’s formulation is included into most textbooks:

$\begin{equation}
\text{If } u \geq 0 \text{ is a harmonic function in } B_R \subset \mathbb{R}^n \text{ then }
\frac{u(0) (R - |x|) R^{n-2}}{(R + |x|)^{n-1}} \leq u(x) \leq \frac{u(0) (R + |x|) R^{n-2}}{(R - |x|)^{n-1}}.
\end{equation}$

For $\Omega = B_R$ and $\Omega' = B_{\theta R}$, $\theta < 1$, this immediately implies (1.2) with $C = (1 + 1/\theta)^n$.

In this Section we assume that the uniform ellipticity condition (2.3) is satisfied.

L. Lichtenstein in [232] proved the inequality (1.2) for operators of general form $\mathcal{L} + c(x)$, $c \geq 0$, with $C^2$-smooth coefficients (as in [152], in the two-dimensional case).

J. Serrin [327] established the Harnack inequality in the two-dimensional case for operators $\mathcal{L} + c(x)$, $c \geq 0$, with \textit{bounded} coefficients. This result was obtained simultaneously and independently in [63]. For the case $n \geq 3$, Serrin also proved (1.2) under the condition \footnote{More precisely, the principal coefficients of the operator must satisfy the Dini condition in some neighborhood of $\partial \Omega$.} $a^{ij} \in C^{0,\mathcal{D}}(\Omega)$.
An important improvement was made by E.M. Landis [220] (see also [223, Ch. 1]). Using the growth lemma proposed by himself, he proved the Harnack inequality in arbitrary dimension for the operator $\mathcal{L}_0$ with bounded coefficients under the additional assumption that the eigenvalues of the matrix $\mathcal{A}$ have sufficiently small dispersion.\footnote{Conditions of this form were first introduced in \cite{99}, which is why Landis calls \cite{223} the Cordes type condition.} Namely, the following relations are assumed to hold (after multiplying the matrix $\mathcal{A}$ by a suitable positive function):

$$\text{Tr}(\mathcal{A}) \equiv 1, \quad \nu > \frac{1}{n + 2}$$ \hspace{1cm} (2.23)

(Obviously, the inequality $\nu \leq \frac{1}{n}$ always holds, and equality is possible only for the Laplace operator).

Notice that all the above results were obtained for classical solutions $u \in C^2(\Omega)$.

Finally, the key step belongs to N.V. Krylov and M.V. Safonov [206, 317] (see also [205]). Combining the Landis method with Aleksandrov–Bakelman estimates (in the elliptic case) and Krylov estimates [201]–[203] (in the parabolic case), they managed to obtain the inequality \cite{12} for strong solutions of elliptic [317] and parabolic [206] equations with general operators $\mathcal{L} + c(x)$, $c \geq 0$ (with bounded coefficients), without assuming that the matrix $\mathcal{A}$ is continuous or that the dispersion of its eigenvalues is small.\footnote{Note that if $c \equiv 0$, then the Harnack inequality easily implies an a priori estimate for the Hölder norm of a solution. Extending this estimate (also obtained in \cite{206, 317}) to quasilinear equations, O.A. Ladyzhenskaya and N.N. Uraltseva further established the solvability of the Dirichlet problem for non-divergence quasilinear equations under natural structure conditions only (see the survey \cite{213}). Subsequently, this result was extended to other boundary value problems for quasilinear and fully non-linear equations.}

For operators $\mathcal{L}$ with $b^i \in L^\infty(\Omega)$, the Harnack inequality was proved in [319] (see also [268]). The papers [131] and [320] demonstrate a unified approach to proving the Harnack inequality for divergence and non-divergence type operators. At the same time, [131] showed\footnote{See also \cite{93} in this connection.} that for operators of mixed (divergence-non-divergence) form

$$-D_i(a^{ij}(x)D_j) - \tilde{a}^{ij}(x)D_iD_j$$

(matrices of the principal coefficients $\mathcal{A}$ and $\tilde{\mathcal{A}}$ satisfy the uniform ellipticity condition) the Harnack inequality can fail even for $n = 1$. 

\textsuperscript{38} Conditions of this form were first introduced in \cite{99}, which is why Landis calls \cite{223} the Cordes type condition.

\textsuperscript{39} Note that if $c \equiv 0$, then the Harnack inequality easily implies an a priori estimate for the Hölder norm of a solution. Extending this estimate (also obtained in \cite{206, 317}) to quasilinear equations, O.A. Ladyzhenskaya and N.N. Uraltseva further established the solvability of the Dirichlet problem for non-divergence quasilinear equations under natural structure conditions only (see the survey \cite{213}). Subsequently, this result was extended to other boundary value problems for quasilinear and fully non-linear equations.

\textsuperscript{40} See also \cite{93} in this connection.
We also mention the papers [8] and [322], where the Harnack inequality and the Hölder continuity of solutions were considered in the “abstract” context of metric and quasi-metric spaces.

3 Divergence type operators

In this Section, we consider operators with the following structure:
\[ \mathcal{L} \equiv -D_i(a^{ij}(x)D_j) + b^i(x)D_i \]  
(3.1)

(in the case \( b \equiv 0 \), we write \( \mathcal{L}_0 \) instead of \( \mathcal{L} \) and operators of more general form)
\[ \hat{\mathcal{L}} \equiv -D_i(a^{ij}(x)D_j + d^i) + b^i(x)D_i + c(x). \]  
(3.2)

The matrix of principal coefficients \( A \) is symmetric and satisfies the ellipticity condition
\[ \nu(x)|\xi|^2 \leq a^{ij}(x)\xi_i \xi_j \leq V(x)|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \]  
(3.3)
or the uniform ellipticity condition \( \text{[223]} \) for almost all \( x \in \Omega \). In (3.3), the functions \( \nu(x) \) and \( V(x) \) are positive and finite \( \text{[41]} \) almost everywhere in \( \Omega \).

The solution of the equation \( \hat{\mathcal{L}}u = 0 \) is understood here as a weak solution, i.e. a function \( u \in W^{1,2}_{1,\text{loc}}(\Omega) \) such that the integral identity
\[ \langle \hat{\mathcal{L}}u, \eta \rangle := \int_{\Omega} (a^{ij}D_juD_i\eta + b^iD_iu \eta + d^iD_i\eta + cu\eta) \, dx = 0 \]
is satisfied for arbitrary test function \( \eta \in C_0^\infty(\Omega) \). Respectively, a weak supersolution \( (\hat{\mathcal{L}}u \geq 0) \) is a function \( u \in W^{1,2}_{1,\text{loc}}(\Omega) \) such that
\[ \int_{\Omega} (a^{ij}D_juD_i\eta + b^iD_iu \eta + d^iD_i\eta + cu\eta) \, dx \geq 0 \]  
(3.4)
for arbitrary non-negative test function \( \eta \in C_0^\infty(\Omega) \). A weak subsolution \( (\hat{\mathcal{L}}u \leq 0) \) is defined in a similar way.

Let us prove the weak maximum principle for the operator \( \hat{\mathcal{L}} \) under the simplest restrictions on the coefficients.

\[ \text{We emphasize that, in contrast to operators of non-divergence type, the properties of the operator } \mathcal{L} \text{ are not preserved when multiplied by an arbitrary positive function. Therefore, the behavior of the functions } \nu(x) \text{ and } V(x) \text{ should be considered separately.} \]
Theorem 3.1. Let \( n \geq 3 \). Suppose that \( \hat{L} \) is an operator of the form (3.2) in a domain \( \Omega \subset \mathbb{R}^n \), the condition (2.3) is fulfilled, \( b^i, d^i \in L_n(\Omega), \ c \in L_{\frac{n}{2}}(\Omega), \) and the function \( u \equiv 1 \) is a weak supersolution of the equation \( \hat{L}u = 0 \) in \( \Omega \). Let \( u \in W^{1,2}_{2,\text{loc}}(\Omega), \hat{L}u \geq 0 \) in \( \Omega \), and \( u \geq 0 \) on \( \partial \Omega \). Then \( u \geq 0 \) in \( \Omega \).

Proof. 1. First of all, note that the bilinear form \( \langle \hat{L}u, \eta \rangle \) is continuous on \( W^{1,2}_{2,\text{loc}}(\Omega) \times \overset{\circ}{W}^{1,2}_{2}(\Omega) \) if \( \Omega' \subset \Omega \). Indeed, applying the Hölder and Sobolev inequalities, we obtain

\[
|\langle \hat{L}u, \eta \rangle| \leq \nu^{-1} \|Du\|_{2,\Omega} \|D\eta\|_{2,\Omega} + \|b\|_{n,\Omega} \|Du\|_{2,\Omega} \|\eta\|_{2^*,\Omega} + \|b\|_{n,\Omega} \|Du\|_{2,\Omega} \|\eta\|_{2^*,\Omega} + \|c\|_{\frac{n}{2},\Omega} \|u\|_{2,\Omega} \|\eta\|_{2^*,\Omega} \leq C (\|Du\|_{2,\Omega} + \|u\|_{2,\Omega}) \|D\eta\|_{2,\Omega},
\]

(here and below \( 2^* = \frac{2n}{n-2} \) is the critical Sobolev exponent). Therefore, in the definition of a weak solution (sub/supersolution), one can take any test functions \( \eta \in W^{1,2}_{2}(\Omega) \) with compact support.

2. Assume the converse. Let \( \inf_{\Omega} u = -A < 0 \) (the case \( A = \infty \) is not excluded). Then for any \( 0 < k < A \) the function \( \eta = (u + k)^- \in \overset{\circ}{W}^{1,2}_{2}(\Omega) \) is non-negative and compactly supported in \( \Omega \), and therefore the inequality (3.4) holds true. Since \( D(u + k)^- = -Du \cdot \chi_{\{u < -k\}} \), this gives

\[
\int_{\{u < k\}} a^{ij} D_j u D_i \eta dx \leq \int_{\{u < -k\}} (b^i D_i u \eta + d^i u D_i \eta + cu \eta) dx = \int_{\{u < -k\}} (b^i - d^i) D_i u \eta dx + \int_{\{u < -k\}} (d^i D_i (u \eta) + c(u \eta)) dx.
\]

The latter term here is non-positive, since \( u \equiv 1 \) is a weak supersolution. Using (2.3) in the left-hand side and the Hölder and Sobolev inequalities in the right-hand side, we arrive at

\[
\nu \|Du\|_{2,\{u < -k\}}^2 \leq (\|b\|_{n,\{u < -k\}} + \|d\|_{n,\{u < -k\}}) \|Du\|_{2,\{u < -k\}}^2.
\]

\[
(3.5)
\]

\[\text{Similarly to the footnote 29, this means that for any } \varepsilon > 0 \text{ the inequality } u + \varepsilon > 0 \text{ holds in some neighborhood of } \partial \Omega.\]
If $A = \infty$ then the first factor in the right-hand side tends to zero as $k \to \infty$, which gives a contradiction.

If $A < \infty$ then $Du = 0$ almost everywhere on the set $\{u = -A\}$, and we can rewrite (3.5) as follows: $\nu \leq \|b\|_{n, \mathcal{A}_k} + \|d\|_{n, \mathcal{A}_k}$, where

$$\mathcal{A}_k = \{x \in \Omega \mid -A < u(x) < -k, Du(x) \neq 0\}.$$  

Evidently, $|\mathcal{A}_k| \to 0$ as $k \to A$. Therefore, $\|b\|_{n, \mathcal{A}_k} + \|d\|_{n, \mathcal{A}_k} \to 0$, and we again have a contradiction. \hfill \Box

**Remark 3.1.** In a recent paper [188], the weak maximum principle is proved in the so-called John domain for functions $u \in W^{1,2}_2(\Omega)$ under the following assumptions:

- $\hat{L}u \geq 0$ in $\Omega$;
- the conormal derivative condition $(a^{ij}D_j u + d^i u)n_i \geq 0$ is satisfied instead of $u \geq 0$ on $\partial \Omega$ (this means that the inequality (3.4) holds for all non-negative functions $\eta \in W^{1,2}_2(\Omega)$).

### 3.1 The Harnack inequality and the strong maximum principle

In contrast to non-divergence type operators, almost all results on the strong maximum principle for divergence type operators were obtained as a consequence of the corresponding Harnack inequalities. In this regard, we present the history of these results in parallel.

The works of W. Littman [244], [245] stand somewhat apart. They deal with operators

$$\mathcal{L}^* \equiv -D_i D_j a^{ij}(x) - D_i b^i(x), \quad \text{(3.6)}$$

formally adjoint to operators of the form (2.1). A weak supersolution to the equation $\mathcal{L}^*u + cu = 0$ is a function $u \in L^1_{1,loc}(\Omega)$ such that for any

|                      | Strong max. principle | The Harnack inequality |
|----------------------|-----------------------|------------------------|
| The Laplace operator | 1839 [139], [117]    | 1887 [152]             |
| Operators with smooth coeff. | 1892 [290]    | 1912 [232]             |
| Operators with discontinuous coeff. | 1927 [168]   | 1955 [271], [63]      |

\[38\] Compare the years of the first obtained results in the table:
non-negative test function $\eta \in C^\infty_0(\Omega)$ the inequality
\[
\langle \mathcal{L}^* u + cu, \eta \rangle := \int_{\Omega} u(\mathcal{L}\eta + c\eta) \, dx \geq 0
\]
holds true. In [244] the coefficients of the operator were assumed smooth, while in [245] the conditions were substantially weakened. Let us formulate the latter result.

Let $\mathcal{L}$ be an operator of the form (2.1). Suppose that $a^{ij}, b^i,$ and $c$ belong to $C^{0,\alpha}(\Omega)$, $\alpha \in (0,1)$, and the assumption (2.3) is satisfied. Let $u$ be a weak supersolution to the equation $\mathcal{L}^* u + cu = 0$ in $\Omega$. Then

1. $u$ cannot attain zero minimum in $\Omega$ unless $u \equiv 0$.

2. If $u \equiv 1$ is a weak supersolution to the equation $\mathcal{L}^* u + cu = 0$ in $\Omega$ then $u$ cannot attain negative minimum in $\Omega$ unless $u \equiv \text{const}$ (in this case $u$ is a weak solution).

3. If $-u$ is a weak supersolution to the equation $\mathcal{L}^* u + cu = 0$ in $\Omega$ then $u$ cannot attain positive minimum in $\Omega$ unless $u \equiv \text{const}$ (in this case $u$ is a weak solution).

Further developments of these results for operators of the form (3.6) can be found in the papers [120], [121], [253], [115] (see also the references therein).

Let us return to divergence type equations. The Harnack inequality for a uniformly elliptic operator $\mathcal{L}_0$ with measurable coefficients was first proved by J. Moser [259]. In the paper by G. Stampacchia [339] this result was generalized to operators of the form (3.2) under the conditions
\[
b^i \in L_n(\Omega), \quad d^i \in L_q(\Omega), \quad c \in L_{q/2}(\Omega), \quad q > n. \tag{3.7}
\]
A similar result can be extracted from the paper [328] devoted to quasilinear equations.

As a corollary, the strong maximum principle is proved in [339] in two versions:

\footnote{In this case, it means that $-D_iD_j(a^{ij}) - D_i(b^i) + c \geq 0$ in the sense of distributions.}

\footnote{As shown in [114] (see also [115] and [226]), the inequality (1.2) can also be obtained from E. De Giorgi’s proof [107] of the Hölder continuity of weak solutions to the equation $\mathcal{L}_0 u = 0$.}

\footnote{See also [96] and [153].}
1. for the operator $\mathcal{L}$ provided $\operatorname{ess inf}_\Omega u = 0$;

2. for the operator $\mathcal{L}$.

We give a somewhat simplified proof of the second assertion. This proof is based on Moser’s idea \[258\], but does not use the Harnack inequality.

**Theorem 3.2.** Let $\mathcal{L}$ be a uniformly elliptic operator of the form \[(3.1)\] in the domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, and let $b^i \in L^p_\text{loc}(\Omega)$. Suppose that $u \in W^{1,2}_\text{loc}(\Omega)$ and $\mathcal{L}u \geq 0$ in $\Omega$. If $u$ attains its minimum at a point $x^0 \in \Omega$, then $u \equiv \text{const}$.

**Proof.**

1. Similarly to Step 1 of the proof of Theorem 3.1, in the definition of a weak solution (sub/supersolution) one can take any compactly supported test function $\eta \in \mathcal{W}^{1,2}(\Omega)$.

2. Now let $v$ be a weak subsolution, i.e. $\mathcal{L}v \leq 0$ in $\Omega$. We substitute the test function $\eta = \varphi'(v) \cdot \varsigma$ into the inequality $\langle \mathcal{L}v, \eta \rangle \leq 0$. Here $\varsigma$ is a non-negative Lipschitz function supported in $B_{2R} \subset \Omega$, and $\varphi$ is a convex Lipschitz function on $\mathbb{R}$ that vanishes on the negative semiaxis. This gives

$$\int_{B_{2R} \cap \{u > 0\}} \left( a^{ij} D_j V D_i \varsigma + \frac{\varphi''(v)}{\varphi'(v)} a^{ij} D_j V D_i \varsigma + b^i D_i V \varsigma \right) dx \leq 0, \quad (3.8)$$

where $V = \varphi(v) \in W^{1,2}_\text{loc}(\Omega)$. In particular, since the second term in (3.8) is non-negative, $V$ is also a weak subsolution.

We set $\varphi(\tau) = \tau^p$, $p > 1$, and $\varsigma = V \varsigma^2$, where $\varsigma$ is a smooth cut-off function in $B_{2R}$. We arrive at

$$\int_{B_{2R}} 2^{p-1} a^{ij} D_j V D_i V \varsigma^2 \ dx \leq - \int_{B_{2R}} \left( 2 a^{ij} D_j V V D_i \varsigma + b^i D_i V V \varsigma^2 \right) dx. \quad (3.9)$$

We estimate the left-hand side in (3.9) from below using (2.3), and the right-hand side from above by the Hölder and Sobolev inequalities:

$$\nu \|DV \varsigma\|_{2, B_{2R}}^2 \leq 2 \nu^{-1} \|DV \varsigma\|_{2, B_{2R}} \|VD \varsigma\|_{2, B_{2R}} + \|b\|_{n, B_{2R}} \|DV \varsigma\|_{2, B_{2R}} \|V \varsigma\|_{2^*, B_{2R}} \leq N(n) \|b\|_{n, B_{2R}} \|DV \varsigma\|_{2, B_{2R}}^2 + C \|DV \varsigma\|_{2, B_{2R}} \|VD \varsigma\|_{2, B_{2R}}^2.$$
By the Lebesgue theorem, we have $N(n)\|b\|_{n,B_{2R_\ast}} \leq \frac{\nu}{2}$ for sufficiently small $R_\ast$. For $R \leq R_\ast$ this implies
\[
\|DV \zeta\|_{2,B_{2R}} \leq C(n, \nu, \|b\|_{n,\Omega}) \cdot \|VD\zeta\|_{2,B_{2R}}.
\] (3.10)

We put in (3.10) $R_k = R(1 + 2^{-k})$, $k \in \mathbb{N} \cup \{0\}$, and take $\zeta = \zeta_k$ such that
\[
\zeta_k \equiv 1 \text{ in } B_{R_k+1}, \quad \zeta_k \equiv 0 \text{ outside } B_{R_k}, \quad |D\zeta_k| \leq \frac{2^{k+2}}{R}.
\]

We obtain
\[
\|DV \zeta_k\|_{2,B_{R_k}} \leq \frac{C(n, \nu, \|b\|_{n,\Omega}) \cdot 2^k \|V\|_{2,B_{R_k}}}{R}.
\] (3.11)

Now for $p = p_k \equiv (2^*/2)^k$ we deduce from the Sobolev inequality and (3.11) that
\[
\left( \int_{B_{R_k+1}} v_+^{2p_k+1} \, dx \right)^{\frac{1}{2p_k+1}} \leq \left( N(n) \int_{B_{R_k}} (V\zeta_k)^{2^*} \, dx \right)^{\frac{1}{2^*}} \leq \left( 4^k C \int_{B_{R_k}} V^2 \, dx \right)^{\frac{1}{p_k}} = \left( 4^k C \int_{B_{R_k}} v_+^{2p_k} \, dx \right)^{\frac{1}{p_k}},
\] (3.12)

where $C$ depends only on $n, \nu, \|b\|_{n,\Omega}$.

Iterating (3.12), we conclude that any (weak) subsolution $v$ admits the estimate
\[
\text{ess sup}_{B_R} v_+ \leq C(n, \nu, \|b\|_{n,\Omega}) \cdot \left( \int_{B_{2R}} v_+^2 \, dx \right)^{\frac{1}{2}}, \quad R \leq R_\ast.
\] (3.13)

3. Let us now turn to the proof of the Theorem. Without loss of generality, we can suppose that $\text{ess inf}_{\Omega} u = 0$.

Assume that the statement is wrong. Then there is an interior point $x^0 \in \Omega$ such that $\text{ess lim inf}_{x \to x^0} u = 0$, but for some $k > 0$, $\delta > 0$ and $R \leq R_\ast$ the inequality
\[
\left| \{u \geq k\} \cap B_R(x^0) \right| \geq \delta \cdot |B_R| \quad (3.14)
\]
holds. Without loss of generality, we assume that $\overline{B_{2R}}(x^0) \subset \Omega$. We place the origin at the point $x^0$ and introduce the function $v_\varepsilon(x) = 1 - \frac{u}{k} - \varepsilon$, $\varepsilon > 0$. Obviously $v_\varepsilon$ is a subsolution.
We apply the inequality (3.8) with \( V = \varphi(v^\varepsilon) \equiv (\ln \frac{1}{1 - v^\varepsilon})_+ \) (this is allowed since \( v^\varepsilon < 1 \)) and \( \varsigma = \zeta^2 \), where \( \zeta \) is a smooth cut-off function equal to 1 in \( B_R \). Taking into account that \( \frac{\varphi'}{\varphi^\varepsilon} \equiv 1 \) and using (2.3) together with the Hölder and Sobolev inequalities, we obtain

\[
\nu \| DV \zeta \|_{2, B_{2R}}^2 \leq \int_{B_{2R}} \left( 2a^{ij} D_j V \zeta D_i \zeta + b_i^j D_i V \zeta^2 \right) dx \\
\leq C(n, \nu, \| b \|_{n, \Omega}) \cdot \| DV \zeta \|_{2, B_{2R}} \| D \zeta \|_{2, B_{2R}},
\]

whence

\[
\| DV \zeta \|_{2, B_{2R}} \leq C(n, \nu, \| b \|_{n, \Omega}) R^{\frac{n}{2} - 1}.
\] (3.15)

Now we observe that \( V \) vanishes on the set \( \{ u \geq k \} \cap B_R \), and \( \zeta \equiv 1 \) on this set. It follows from the proof of Lemma 5.1 in [211, Chapter II] that this implies

\[
\left| \{ u \geq k \} \cap B_R \right| \cdot V(x) \zeta(x) \leq \left( \frac{4R}{n} \right)^n \int_{B_{2R}} \frac{|DV(y)| \zeta(y)}{|y - x|^{n-1}} dy.
\]

We take in both parts the norm in \( L^2^* \) and estimate the right-hand side by the Hardy–Littlewood–Sobolev inequality (see, e.g., [344, Theorem 1.18.9/3]). Taking into account (3.14) and (3.15) we obtain

\[
\| V \zeta \|_{2^*, B_{2R}} \leq \frac{C(n)}{\delta} \| DV \zeta \|_{2, B_{2R}} \leq C(n, \nu, \delta, \| b \|_{n, \Omega}) R^{\frac{n}{2} - 1},
\]

and therefore,

\[
\left( \int_{B_R} V^2 dx \right)^{\frac{1}{2}} \leq C(n) R^{1 - \frac{n}{2}} \| V \zeta \|_{2^*, B_{2R}} \leq C(n, \nu, \delta, \| b \|_{n, \Omega}).
\]

Finally, since \( V \) is a subsolution, we can apply the estimate (3.13). This gives \( \text{ess sup}_{B_{R/2}} V_+ \leq C \), which is equivalent to

\[
\text{ess inf}_{B_{R/2}} u \geq k(\exp(-C) - \varepsilon).
\]

Since the constant \( C \) does not depend on \( \varepsilon \), we get a contradiction with the assumption \( \text{ess lim inf}_{x \to 0} u = 0 \). \( \square \)
Remark 3.2. The last term in (3.9) can be estimated as follows:

$$\int_{B_{2R}} |b^i D_i V \zeta^2| \, dx \leq \|b\|_{L_{n,\infty}(B_{2R})} \|DV \zeta\|_{2,B_{2R}} \|V \zeta\|_{L_{2^*,2}(B_{2R})}$$

(recall that $L_{p,q}$ is the Lorentz space). Then one can use the strengthened Sobolev embedding theorem $W_1^1(\Omega) \hookrightarrow L_{2^*,2}(\Omega)$ (see [292]). This implies that the assumption $b^i \in L_n(\Omega)$ can be weakened to $b^i \in L_{n,q}(B_{2R})$ with any $q < \infty$. The counterexample in the beginning of § 2.4 shows that one cannot put in general $q = \infty$. However, if the norm $\|b\|_{L_{n,\infty}(\Omega)}$ is sufficiently small then the proof runs without changes.

The Harnack inequality also holds true under the same assumptions (the proof of Theorem 2.5' in [275] can be transferred completely to this case).

Remark 3.3. In the two-dimensional case, the statement of Theorem 3.2 (and even Theorem 3.1) is false; here is a corresponding counterexample from the paper [132].

For $n = 2$ we set $u(x) = \ln^{-1}(|x|^{-1})$. Obviously, for $r \leq \frac{1}{2}$ the function $u \in W_2^1(B_r)$ is a weak solution of the equation

$$-\Delta u + b^i(x)D_i u = 0 \quad \text{with} \quad b^i(x) = \frac{2x_i}{|x|^2 \ln(|x|^{-1})} \in L_2(B_r).$$

However, $u$ attains its minimum at the origin.

Thus, for $n = 2$ the condition on $b^i$ must be strengthened. For example, one can estimate the last term in (3.9) as follows (cf. [44, Theorem 3.1]):

$$\int_{B_{2R}} |b^i D_i V \zeta^2| \, dx \leq \|b\|_{L_{\Phi_1}(B_{2R})} \|DV \zeta\|_{2,B_{2R}} \|V \zeta\|_{L_{\Phi_2}(B_{2R})},$$

where $L_{\Phi}$ stands for the Orlicz space with the $N$-function $\Phi$ (see, e.g., [66, Section 10]),

$$\Phi_1(t) = t^2 \ln(1 + t), \quad \Phi_2(t) = \exp(t^2) - 1.$$

Using the Yudovich–Pohozhaev embedding theorem $W_1^1(\Omega) \hookrightarrow L_{\Phi_2}(\Omega)$ (see, e.g., [66, Subsection 10.6]), we obtain the strong maximum principle under

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49 This fact is not noted in [339] and [66].
the assumption $b \ln^{1/2} (1 + |b|) \in L^2(\Omega)$ which was introduced in [275]. Under the same assumption, the Harnack inequality also holds (see [275, Theorem 2.5′]). The above example shows that the power $1/2$ of the logarithm cannot be reduced.

Since the second half of the 1960s the number of papers on the Harnack inequality for divergence type equations (even linear ones) has grown rapidly. We will focus on three important directions in the development of this topic.

1. Non-uniformly elliptic operators. In several papers, operators with the ellipticity condition (3.3) were studied under various assumptions about the functions $\nu(x)$ and $\mathcal{V}(x)$.

N.S. Trudinger [346] proved the Harnack inequality for operators $\mathcal{L}_0$ under the assumption

$$
\nu^{-1} \in L^q(\Omega), \quad \nu^{-1}\mathcal{V}^2 \in L^r(\Omega), \quad \frac{1}{q} + \frac{1}{r} < \frac{2}{n}.
$$

In [347], operators of more general form (3.2) were considered under a weaker condition

$$
\nu^{-1} \in L^q(\Omega), \quad \mathcal{V} \in L^r(\Omega), \quad \frac{1}{q} + \frac{1}{r} < \frac{2}{n};
$$

(3.16)

the lower-order coefficients were assumed to satisfy some weighted summability conditions determined by the matrix $A$.

Under these conditions, the Harnack inequality was established in [347], as well as the strong maximum principle in the following form:

Let $u$ be a weak supersolution of the equation $\mathcal{L}u = 0$ in $\Omega$. If $u \equiv 1$ is also a supersolution, then $u$ cannot attain its negative minimum in $\Omega$ unless $u \equiv \text{const}$ (in this case $u$ is a weak solution).

For operators of the simplest form $\mathcal{L}_0$, the restriction on exponents in (3.16) was weakened to $\frac{1}{q} + \frac{1}{r} < \frac{2}{n-1}$ in the recent paper [60]. On the other hand, an example in the paper [137] shows that for $n \geq 4$ and $\frac{1}{q} + \frac{1}{r} > \frac{2}{n-1}$ the equation $\mathcal{L}_0 u = 0$ in $B_R$ can have a weak solution unbounded in $B^{2}_R$. The question of the validity of the Harnack inequality in the borderline case $\frac{1}{q} + \frac{1}{r} = \frac{2}{n-1}$ is still open.

50 In the case of a uniformly elliptic operator, these conditions are close to the Stampacchia conditions (3.7).
In [125], operators $L_0$ were considered under the following conditions:

1. there exists $N \geq 1$ such that $V(x) \leq N \cdot \nu(x)$ for almost all $x \in \Omega$;
2. $\nu$ belongs to the Muckenhoupt class $A_2$, i.e.

$$\sup_{x \in \mathbb{R}^n, r > 0} \left( \int_{B_r(x)} \nu(y) \, dy \cdot \int_{B_r(x)} \nu^{-1}(y) \, dy \right) < \infty. \quad (3.17)$$

Under these conditions, the Harnack inequality and the strong maximum principle are proved in [125]. In addition, a counterexample showing that weakening the condition $\nu \in A_2$ to $\nu \in \bigcup_{p>2} A_p$ does not ensure the fulfillment of the Harnack inequality is given there.

In [106], the results of [125] were generalized to operators of the form (3.2). In this case, the lower-order coefficients satisfy the following conditions:

$$\frac{b^i}{\nu} \in L_m(\Omega), \quad \frac{d^i}{\nu} \in L_q, \quad \frac{c}{\nu} \in L_{\frac{q}{2}}, \quad q > m, \quad (3.18)$$

(here $m$ is the exponent called in [106] “intrinsic dimension” generated by the behavior of the weight $\nu$; for uniformly elliptic operators we have $m = n$, and these conditions become (3.7)).

We also mention the papers [89] and [95], where the Harnack inequality was established for the operator $L_0$ under “abstract” conditions on the functions $\nu(x)$ and $V(x)$. Namely, certain weighted Sobolev and Poincaré inequalities should be satisfied.

2. Lower-order coefficients from the Kato classes. Notice that the Lebesgue spaces (as well as the Lorentz and Orlicz spaces) are rearrangement invariant: the norm of a function $f$ in these spaces is determined only by the behavior of the measure of the set $\{x \in \Omega \mid |f(x)| > N\}$ as $N \to \infty$. A more subtle description of the coefficients singularities can be given in terms of the Kato classes.

Recall that a function $f \in L_1(\Omega)$ belongs to the class $\mathcal{K}_{n,\beta}$, $\beta \in (0, n)$, if

$$\omega_{\beta}(r) := \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} \frac{|f(y)|}{|x - y|^{n-\beta}} \, dy \to 0 \quad \text{as} \quad r \to 0. \quad (3.19)$$

51 These conditions appeared earlier in the paper [118], devoted to quasilinear equations, but there additional restrictions (3.16) were imposed on the functions $\nu(x)$ and $V(x)$.
52 This counterexample does not violate the strong maximum principle.
As usual, \( f \in K_{n,\beta,\text{loc}} \) means that \( f\chi_{\Omega'} \in K_{n,\beta} \) for arbitrary subdomain \( \Omega' \) such that \( \overline{\Omega'} \subset \Omega \).

The functionals \( \omega_{\beta}(r) \) and the spaces defined by them were introduced by M. Schechter in [323], [324, Ch. 5, §1; Ch. 7, §7] and studied in detail in [325]. For further development of the subject and references see [360].

All the results of this subsection refer to the case \( n \geq 3 \).

In the paper [9], the Harnack inequality was established for the operator \(-\Delta + c(x)\) under the assumption \( c \in K_{n,2} \). In [94] this result was extended to uniformly elliptic operators of the form \( \mathcal{L}_0 + c(x) \) under the same condition.

In the paper [209], the Harnack inequality was proved for uniformly elliptic operators of a more general form \( \mathcal{L} + c(x) \) under the assumption

\[
(b^i)^2, c \in K_{n,2,\text{loc}}.
\]

Finally, the paper [356] combines the two directions described above. Namely, the Harnack inequality is proved for the non-uniformly elliptic operators of the form (3.22). The functions \( \nu(x) \) and \( \mathcal{V}(x) \) in (3.3) satisfy the assumptions \( \mathcal{V}(x) \leq N \cdot \nu(x) \) and (3.17), while the functions \( (b^i)^2 \), \( (d^i)^2 \) and \( c \) belong to the weighted analogue of the Kato class \( K_{n,2} \) with an additional constraint: the corresponding analog of the function \( \omega_2 \) from (3.19) admits the estimate \( O(r^{\gamma}) \) for some \( \gamma > 0 \) as \( r \to 0 \).

The assumption (3.20) in the general case is very close to optimal. Variations of (3.20) are possible if some additional conditions are imposed on the matrix \( A \).

The paper [358] considers a uniformly elliptic operator of the form (3.1) with \( a^{ij} \in C^{0,\alpha}(\Omega), \alpha \in (0,1) \). This restriction allowed to prove the Harnack inequality under the assumption \( b^i \in K_{n,1} \).

Note that the Hölder condition on the principal coefficients in [358] is redundant: using the estimates of the Green’s function and its derivatives from [150], the same result can be obtained for \( a^{ij} \in C^{0,\mathcal{D}}(\Omega) \).

\[53\] For particular values of \( \beta \), condition (3.19) was used in [340] and [183]. In this regard, \( K_{n,\beta} \) are usually called the Kato or Kato–Stummel classes, which is a typical example of Arnold’s principle [46]. Some generalizations of the \( K_{n,\beta} \) classes can be found, for instance, in [119].

\[54\] See also [100] and [332] in this connection.

\[55\] In an earlier paper [101], the operator \(-\Delta + b^i(x)D_i\) was considered under stronger restrictions \( (b^i)^2 \in K_{n,2,\text{loc}} \) and \( b^i \in K_{n,1,\text{loc}} \).

\[56\] We assume that this constraint is of a technical nature, but as far as we know, this question remains open.
In the recent paper [199], a case in a sense intermediate has been considered. The principal coefficients of the uniformly elliptic operator \( L \) in this paper belong to the Sarason space \( VMO(\Omega) \). This means that \( \omega^{ij}(\rho) \to 0 \) as \( \rho \to 0 \), where

\[
\omega^{ij}(\rho) := \sup_{x \in \Omega} \sup_{r \leq \rho} \int_{\Omega \cap B_r(x)} |a^{ij}(y) - \int_{\Omega \cap B_r(x)} a^{ij}(z) \, dz| \, dy.
\]  

(3.21)

In this case, the condition \( |b^i|^{\beta} \in \mathcal{K}_{n,\beta}, \beta > 1 \) is imposed on the lower-order coefficients with the additional restriction \( \sup_{x \in \Omega} \int_{\Omega \cap B_r(x) \setminus B_{r/2}(x)} \frac{|b^i(y)|^{\beta}}{|x - y|^{n-\beta}} \, dy \leq \sigma^{\beta}(r), \ \sigma \in \mathcal{D}. \)

(3.22)

For such operators the strong maximum principle is proved in [199]. Note that the Harnack inequality can also be proved under these assumptions. Whether it is possible to remove or at least relax the restriction (3.22) is still unclear.

3. Operators with \( \text{div}(b) \leq 0 \). When studying hydrodynamic problems, one often encounters (see, for example, [359], [91], [92], [193]) the operators \( -\Delta + b^i(x)D_i \) (or, more generally, operators of the form (3.1)) with the additional structure condition \( D_i(b^i) = 0 \) or \( D_i(b^i) \leq 0 \) understood in the sense of distributions. Recall that this means, respectively,

\[
\int_{\Omega} b^i D_i \eta \, dx = 0 \quad \text{for all} \quad \eta \in \mathcal{C}^\infty_0(\Omega)
\]

or

\[
\int_{\Omega} b^i D_i \eta \, dx \geq 0 \quad \text{for all non-negative} \quad \eta \in \mathcal{C}^\infty_0(\Omega).
\]

This condition allows to significantly weaken the regularity assumptions for the coefficients \( b^i \).

In the paper [326], the Harnack inequality was established for the operator \( -\Delta + b^i(x)D_i \) with \( D_i(b^i) = 0 \) under the assumption \( b^i \in BMO^{-1}(\Omega) \). It

\[57\] In the case of \( n = 2 \), also studied in [199], the condition (3.22) is somewhat modified.
means that \( b^i = D_j(B^{ij}) \) in the sense of distributions, where \( B^{ij} \in BMO(\Omega) \), i.e. functions \( \omega^{ij}(\rho) \) defined in (3.21) (with \( B^{ij} \) instead of \( a^{ij} \)) are bounded\(^{58}\).

If this is true, the relation \( D_i(b^i) = 0 \) is ensured by the additional condition \( B^{ij}(x) = -B^{ji}(x) \) for almost all \( x \in \Omega \).

The paper \(^{275}\) studied uniformly elliptic operators of the form (3.1) with \( D_i(b^i) \leq 0 \). The requirements on lower-order coefficients were described in terms of the Morrey spaces.

Recall that the space \( \mathcal{M}_p^{\alpha}(\Omega), 1 \leq p < \infty, \alpha \in (0, n) \), consists of functions \( f \in L^p(\Omega) \) for which

\[ \|f\|_{\mathcal{M}_p^{\alpha}(\Omega)} := \sup_{B_r(x) \subset \Omega} r^{-\alpha} \|f\|_{L^p(B_r(x))} < \infty. \]

In particular, in \(^{275}\) the Harnack inequality was proved under the assumption\(^{59}\) \( b^i \in \mathcal{M}_q^{\alpha q^{-1}}(\Omega), \frac{n}{2} < q < n \). N.D. Filonov constructed an extremely subtle counterexample (\[^{132}\] Theorem 1.6) showing that even under the assumption \( D_i(b^i) = 0 \) the exponent \( \alpha = \frac{n}{q} - 1 \) cannot be reduced.

The strong maximum principle was established in \(^{275}\) for Lipschitz supersolutions\(^{61}\) under the assumption \( b^i \in L_q(\Omega), q > \frac{n}{2} \). However, using the approximation (\[^{134}\] Theorem 3.1) one can obtain the following partial generalization of this result:

Let \( \Omega \subset \mathbb{R}^n, n \geq 3 \). Suppose that the function \( u \in W^{1,2}_{2,loc}(\Omega) \) is a weak solution of the equation \(-\Delta u + b^i(x)D_iu = 0 \) in \( \Omega \), and

\[ D_i(b^i) = 0; \quad b^i \in L_q(\Omega); \quad q > \frac{n}{2} \text{ for } n \geq 4; \quad q = 2 \text{ for } n = 3. \]

If \( u \) attains its minimum at a point \( x^0 \in \Omega \) then \( u \equiv \text{const} \).

On the other hand, the following counterexample was constructed in \(^{134}\).

Let \( n \geq 4 \), and let \( u(x) = \ln^{-1}(|x'|^{-1}) \). Then \( u \in W^{1,2}_r(\mathbb{B}_r) \) for \( r \leq \frac{1}{3} \). Further, direct calculation shows that \( u \) is a weak solution of the equation

\[ -\Delta u + b^i(x)D_iu = 0 \text{ with}^{61} \]

\(^{58}\)Obviously, \( L^p(\Omega) \subset BMO^{-1}(\Omega) \) due to the embedding \( W^{1,2}_1(\Omega) \hookrightarrow BMO(\Omega) \).

\(^{59}\)Obviously, \( L^p(\Omega) \subset \mathcal{M}_q^{\alpha q^{-1}}(\Omega) \) by the Hölder inequality.

\(^{60}\)For weak supersolutions the requirements on \( b^i \) in \(^{275}\) are somewhat stronger.

\(^{61}\)There is a typo in \(^{134}\) in the formula for \( b^u \).
It is easy to see that $D_i(b^i) = 0$, and $b^i \in L_q(B_r)$ for all $q < \frac{n-1}{2}$. However, the strong maximum principle does not hold.

In the recent paper [133] (see also [187]), a vector field $b \in L^{n-1}_2(B_r)$ with $D_i(b^i) = 0$ is constructed, for which the equation $-\Delta u + b^i(x)D_i u = 0$ has a weak solution unbounded in $B_r$. This can also be considered as violation of the strong maximum principle. The question of the validity of the strong maximum principle for $\frac{n-1}{2} < q \leq \frac{n}{2}$ under the assumption $D_i(b^i) = 0$ is open.

3.2 The normal derivative lemma

The history of the normal derivative lemma for weak (super)solutions of the equation $Lu = 0$ is rather short. The first result here was obtained by R. Finn and D. Gilbarg in 1957, see [135]. They considered uniformly elliptic operators of the form (3.1) with $a^{ij} \in C^{0,\alpha}(\Omega)$ and $b^i \in C(\Omega)$ in a two-dimensional $C^{1,\alpha}$-smooth domain, $\alpha \in (0,1)$.

Only in 2015 this result was generalized to the $n$-dimensional case [316]; the boundary of the domain in this paper was assumed to be smooth. In [197] the normal derivative lemma was proved for all $n \geq 3$ under the same conditions on $a^{ij}$ and $\partial \Omega$ as in [135], and for $b^i \in L^q(\Omega)$, $q > n$.

Back in 1959, a counterexample showing that the requirement for principal coefficients cannot be relaxed to $a^{ij} \in C(\Omega)$ was constructed in [144]. Here we give a more general example (see [272]).

Let $\Omega$ be a domain in $\mathbb{R}^n$ such that $\Omega \cap \{x_n < h\} = \mathcal{I}(\phi, h)$ and $\phi \in C^1$, but $\phi'$ does not satisfy the Dini condition at zero. As mentioned in §2.2 it is shown in [351] that the normal derivative lemma for the Laplace operator does not hold in such a domain.

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62 Also in [316] some papers with incorrect use of the normal derivative lemma for weak solutions were listed.

63 It is given in various forms in [145, Ch. 3], [306, Ch.2].
Now we flatten the boundary in a neighborhood of the origin. This gives us an operator $\mathcal{L}_0$ with \textit{continuous} principal coefficients for which the normal derivative lemma fails in a \textit{smooth} domain.

This example shows that the natural condition on the principal coefficients of the operator is the Dini condition. In this regard, we note the work of V.A. Kozlov and V.G. Maz'ya [198]. In this paper, for the operator $\mathcal{L}_0$ a more subtle condition on the coefficients $a^{ij}$ is obtained, which provides the gradient estimate for the solution at points of the (smooth) boundary $\partial \Omega$. Apparently, from the asymptotics of the solution obtained in [198], one can also deduce a condition for the fulfillment of the normal derivative lemma, which is more precise than the Dini condition.

To demonstrate the main idea we prove the normal derivative lemma for the simplest operator $\mathcal{L}_0$ with $a^{ij} \in C^0, D(\Omega)$ under minimal assumptions on the boundary of the domain.

\textbf{Theorem 3.3.} Let the domain $\Omega \subset \mathbb{R}^n$ satisfy the interior $C^{1,D}$-paraboloid condition. Suppose that the coefficients of the operator $\mathcal{L}_0$ satisfy the conditions (2.3) and $a^{ij} \in C^0, D(\Omega)$. Let $u \not\equiv \text{const}$ be a weak supersolution of the equation $\mathcal{L}_0 u = 0$ in $\Omega$.

If $u$ is continuous in $\overline{\Omega}$ and attains its minimum at $x^0 \in \partial \Omega$, then for any strictly interior direction $\ell$ the inequality

$$\liminf_{\varepsilon \to +0} \frac{u(x^0 + \varepsilon \ell) - u(x^0)}{\varepsilon} > 0.$$ 

holds true.

\textbf{Proof.} Without loss of generality, we assume that $x^0 = 0$ and $\Omega = \Sigma(\phi, h)$, with $\phi \in C^{1,D}$. Further, the restrictions on $a^{ij}$ are preserved under coordinate transformations of the class $C^{1,D}$. Therefore, we can flatten $\partial \Omega$ in a neighborhood of $x^0$ and assume that $B_R \cap \{x_n > 0\} \subset \Omega$ for some $R > 0$.

\textsuperscript{64}B. Sirakov informed us in private communication that he has proved the normal derivative lemma provided that $a^{ij}$ satisfy the \textbf{mean-Dini condition}, that is, $\omega^{ij} \in D$ in (3.21). This assumption is stronger than $a^{ij} \in C(\Omega)$ but weaker that $a^{ij} \in C^{0,D}(\Omega)$. See in this connection [115], where $C^1$-estimate up to the boundary was proved for solutions to the equations under the same assumption.

\textsuperscript{65}Obviously, it suffices to fulfill this condition only in some neighborhood of $\partial \Omega$. Apparently, this condition can be kept only on $\partial \Omega$, see [198].
For \(0 < r < R/2\) consider the point \(x^r = (0, \ldots, 0, r)\) and the annulus \(\pi = B_r(x^r) \setminus \overline{B_{\frac{r}{2}}}(x^r) \subset \Omega \).

The assumption \(a^{ij} \in C^{0,\varphi}(\Omega)\) gives
\[
|a^{ij}(x) - a^{ij}(y)| \leq \sigma(|x - y|), \quad x, y \in \pi, \quad \sigma \in \mathcal{D}.
\] (3.23)

Let \(x^*\) be an arbitrary point in \(\pi\). Following [135], we define the barrier function \(\mathcal{V}\) and the auxiliary function \(\Psi_{x^*}\) as solutions to the following boundary value problems:
\[
\begin{aligned}
\mathcal{L}_0 \mathcal{V} &= 0 \quad \text{in } \pi, \\
\mathcal{V} &= 1 \quad \text{on } \partial B_{\frac{r}{2}}(x^r), \\
\mathcal{V} &= 0 \quad \text{on } \partial B_{r}(x^r),
\end{aligned}
\]
\[
\begin{aligned}
\mathcal{L}_0^{x^*} \Psi_{x^*} &= 0 \quad \text{in } \pi, \\
\Psi_{x^*} &= 1 \quad \text{on } \partial B_{\frac{r}{2}}(x^r), \\
\Psi_{x^*} &= 0 \quad \text{on } \partial B_{r}(x^r),
\end{aligned}
\]
where \(\mathcal{L}_0^{x^*}\) is the operator with constant coefficients
\[
\mathcal{L}_0^{x^*} \Psi_{x^*} := -D_i(a^{ij}(x^*) D_j \Psi_{x^*}).
\]

It is well known that \(\Psi_{x^*} \in C^\infty(\pi)\). Further, the existence of a (unique) weak solution \(\mathcal{V}\) follows from the general linear theory. Moreover, Lemma 3.2 in [150] shows that \(\mathcal{V} \in C^1(\pi)\), and for \(y \in \pi\) the following estimate holds:
\[
|D \mathcal{V}(y)| \leq \frac{N_1(n, \nu, \sigma)}{r}.
\] (3.24)

We set \(w = \mathcal{V} - \Psi_{x^*}\) and notice that \(w = 0\) on \(\partial \pi\). Therefore, \(w\) admits the representation via the Green’s function \(G_{x^*}\) of the operator \(\mathcal{L}_0^{x^*}\) in \(\pi\):
\[
w(x) = \int_\pi G_{x^*}(x, y) \mathcal{L}_0^{x^*} w(y) \, dy \overset{(*)}{=} \int_\pi G_{x^*}(x, y) \left( \mathcal{L}_0^{x^*} \mathcal{V}(y) - \mathcal{L}_0 \mathcal{V}(y) \right) \, dy,
\]
(the equality \((*)\) follows from relation \(\mathcal{L}_0^{x^*} \Psi_{x^*} = \mathcal{L}_0 \mathcal{V} = 0\)).

Integrating by parts we have
\[
w(x) = \int_\pi D_y G_{x^*}(x, y) \left( a^{ij}(x^*) - a^{ij}(y) \right) D_j \mathcal{V}(y) \, dy.
\] (3.25)

Differentiating both parts of the equality \((3.25)\) with respect to \(x_k\), we obtain
\[
D_k w(x^*) = \int_\pi D_{x_k} D_y G_{x^*}(x^*, y) \left( a^{ij}(x^*) - a^{ij}(y) \right) D_j \mathcal{V}(y) \, dy,
\] (3.26)
\[\quad k = 1, \ldots, n.\]
The derivatives of the Green’s function $G_{x^*}(x, y)$ can be estimated as follows (see, e.g., [150, Theorem 3.3]):

\[ |D_x D_y G_{x^*}(x, y)| \leq \frac{N_2(n, \nu)}{|x - y|^n}, \quad x, y \in \overline{\pi}. \quad (3.27) \]

The substitution of (3.24), (3.27) and (3.23) into (3.26) gives

\[ |Dw(x^*)| \leq \frac{N_1 N_2}{r} \int_{B_{2r}(x^*)} \frac{\sigma(|x^* - y|)}{|x^* - y|^n} dy, \]

and we arrive at

\[ |D\mathfrak{Q}(x^*) - D\Psi_{x^*}(x^*)| \leq \frac{N_3(n, \nu, \sigma)}{r} \int_0^{2r} \frac{\sigma(\tau)}{\tau} d\tau, \quad x^* \in \overline{\pi}. \quad (3.28) \]

Since the normal derivative lemma holds for operators with constant coefficients, we obtain for any strictly interior direction $\ell$

\[ \partial_\ell \Psi_0(0) \geq \frac{N_4(n, \nu, \ell)}{r} > 0. \]

By (3.28), for sufficiently small $r > 0$ we have

\[ \partial_\ell \mathfrak{M}(0) \geq \partial_\ell \Psi_0(0) - |D\mathfrak{M}(0) - D\Psi_0(0)| \geq \frac{N_4}{r} - \frac{N_3}{r} \int_0^{2r} \frac{\sigma(\tau)}{\tau} d\tau \geq \frac{N_4}{2r}. \]

We fix such $r$. Since $u \neq const$, the strong maximum principle yields $u - u(0) > 0$ on $\partial B_{2r}(x^*)$. Therefore, for sufficiently small $\varepsilon > 0$

\[ \mathfrak{L}_0(u - u(0) - \varepsilon \mathfrak{M}) \geq 0 \quad \text{in } \pi; \quad u - u(0) - \varepsilon \mathfrak{M} \geq 0 \quad \text{on } \partial \pi. \]

Now the weak maximum principle gives $u - u(0) \geq \varepsilon \mathfrak{M}$ in $\pi$. Since at the origin this inequality becomes equality, we have

\[ \liminf_{\varepsilon \to +0} \frac{u(\varepsilon \ell) - u(0)}{\varepsilon} \geq \varepsilon \partial_\ell \mathfrak{M}(0), \]

and the statement follows. \qed
Now we formulate a more general result established in [43]. The conditions on the lower-order coefficients for the validity of the normal derivative lemma obtained in this paper are the most precise at the moment.

**Theorem 3.4.** Let the domain $\Omega \subset \mathbb{R}^n$ and the principal coefficients of the operator $L$ satisfy the conditions of Theorem 3.3. Let us also assume that

$$\sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} \frac{|b(y)|}{|x-y|^{n-1}} \cdot \frac{d(y)}{d(y) + |x-y|} \, dy \to 0 \quad \text{as} \quad r \to 0.$$  \tag{3.29}

Let $u \in W^1_2(\Omega)$ be a weak supersolution of the equation $Lu = 0$, let $u \not \equiv \text{const}$ in $\Omega$, and let $b \cdot D_i u \in L^1(\Omega)$. Then the conclusion of Theorem 3.3 holds true.

**Remark 3.4.** In any subdomain of $\Omega'$ such that $\overline{\Omega'} \subset \Omega$, the condition (3.29) coincides with $b \in K_{n,1}$, cf. (3.22). Therefore, (3.29) implies $b \in K_{n,1,\text{loc}}$. On the other hand, it is shown in [43] that the assumptions on $b$ imposed in Theorem 2.5 imply (3.29).

**Remark 3.5.** The normal derivative lemma for divergence type operators is directly related to the properties of the Green’s functions for these operators.

The Green’s function for a uniformly elliptic operator $L_0$ with measurable coefficients was first constructed in the seminal paper [246]. Among other results of this work, we notice the estimate

$$C^{-1} \frac{1}{|x-y|^{n-2}} \leq G(x, y) \leq \frac{C}{|x-y|^{n-2}},$$

which holds for the Green’s function in the whole $\mathbb{R}^n$, $n \geq 3$ (here $C$ depends only on $n$ and $\nu$).

A very important role belongs also to the article [150], where, among other results, the following estimates were proved for the Green’s function of a uniformly elliptic operator $L_0$ with coefficients satisfying the Dini condition, in the domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, satisfying the exterior ball condition:

$$|D_x G(x, y)| \leq \frac{C}{|x-y|^{n-1}} \cdot \frac{d(x)}{d(x) + |x-y|} \cdot \frac{d(y)}{d(y) + |x-y|};$$

$$|D_x D_y G(x, y)| \leq \frac{C}{|x-y|^{n}}.$$  

66 Later this estimate was extended to more general operators of the form (3.1). Recent results in this area, as well as a historical survey, can be found in [33].
(the constant $C$ depends on $n, \nu$, on the function $\sigma$ in the Dini condition for the coefficients and on the domain $\Omega$).

Thus, the assumption (3.29), roughly speaking, means that the function $|b(y)| \cdot |D_x G(x, y)|$ is integrable uniformly with respect to $x$.

4 Some generalizations and applications

As already mentioned in the Introduction, in this Section we provide a brief presentation of some subjects that either generalize the main assertions of our survey or directly rely on them.

4.1 Symmetry of solutions to nonlinear boundary value problems

We start with the celebrated moving plane method. It was first applied by A.D. Aleksandrov [15] to the problem of characterizing a sphere by the property of constancy of its mean curvature (or some other functions of the principal curvatures)\footnote{The problem statement and the history of the problem are given in [11]; see also [14]. For generalizations of this result see, e.g., [227], [229].}. The method was later rediscovered by J. Serrin [330] when solving the following overdetermined problem in the unknown $C^2$-smooth domain:

$$-\Delta u = 1 \text{ in } \Omega, \quad u|_{\partial \Omega} = 0, \quad \partial_n u|_{\partial \Omega} = \text{const.}$$

It is shown in [330] that such a problem is solvable only if $\Omega$ is a ball.

The method owes its popularity to the article [142], which considered the problem

$$-\Delta u = f(u) \text{ in } B_R, \quad u|_{\partial B_R} = 0 \quad (4.1)$$

and its generalizations. Let us formulate the basic result of this work.

**Theorem 4.1.** Let $f \in C^1_\text{loc}(\mathbb{R}_+)$, and let $u \in C^2(\overline{B_R})$ be a positive in $B_R$ solution to the problem (4.1). Then $u = u(r)$ (the function $u$ is radially symmetric) and $u'(r) < 0$ for $0 < r < R$.

Let us sketch the proof of Theorem 4.1. Obviously, it suffices to show that $u$ is an even function of $x_n$ and $D_n u(x) < 0$ for $x_n > 0$.\footnote{67}
For $0 < \lambda < R$, denote by $\Sigma_\lambda$ the segment cut off from the ball by the plane $\Pi_\lambda = \{x \mid x_n = \lambda\}$. For $x \in \Sigma_\lambda$ we denote by $\widehat{x}_\lambda = (x', 2\lambda - x_n)$ the point symmetric to $x$ with respect to $\Pi_\lambda$.

Consider the function $v_\lambda(x) = u(\widehat{x}_\lambda) - u(x)$ in $\Sigma_\lambda$. It satisfies the equation

$$-\Delta v_\lambda + c(x)v_\lambda = 0; \quad c(x) = \frac{f(u(\widehat{x}_\lambda)) - f(u(x))}{u(x) - u(\widehat{x}_\lambda)} \in L_\infty(\Sigma_\lambda).$$

For $\lambda$ sufficiently close to 1, the function $v_\lambda$ is positive in $\Sigma_\lambda$ (the graph of the “reflected” function lies above the original one) and attains \textbf{zero} minimum at $\Pi_\lambda$. By the normal derivative lemma (item B1 of Theorem 2.1), we have $\partial_n v_\lambda(x) = 2D_nu(x) < 0$ on $\Pi_\lambda$.

Therefore, one can slightly reduce $\lambda$ (shift the plane $\Pi_\lambda$ to the center of the ball) such that the inequality $v_\lambda > 0$ in $\Sigma_\lambda$ will still be satisfied.

Denote by $\lambda_0$ the greatest lower boundary of those $\lambda$ for which $v_\lambda > 0$ in $\Sigma_\lambda$. If we assume that $\lambda_0 > 0$ then $v_{\lambda_0} > 0$ on the “circular” part of $\partial \Sigma_{\lambda_0}$.

By the strong maximum principle (item A1 of Theorem 2.1), $v_{\lambda_0} > 0$ in $\Sigma_{\lambda_0}$. But then we can repeat the previous argument and obtain that the plane $\Pi_{\lambda_0}$ can be shifted a little more towards the center, which is impossible. Thus, $\lambda_0 = 0$, and $v_0 \equiv 0$, i.e. $u(x', -x_n) \equiv u(x)$. The theorem is proved.

As pointed out in \cite{142}, if $f(0) \geq 0$ then the a priori positivity of $u$ can be replaced by the assumption $u \geq 0$, $u \not\equiv 0$. It is also obvious that the condition $f \in C^1_{\text{loc}}(\mathbb{R}_+)$ can be replaced by the local Lipschitz condition. The following example given in \cite{142} shows that the Hölder condition on $f$ is, in general, not sufficient.

Let $p > 2$, and let $u(x) = (1 - |x - x^0|^2)^p_+$. Direct calculation shows that $u$ is a solution to the problem \cite{142} for $R > |x^0| + 1$ if we put

$$f(u) = 2p(n - 2 + 2p)u^{1 - \frac{2}{p}} - 4p(p - 1)u^{1 - \frac{2}{p}} \in C^0_{\text{loc}}(\mathbb{R}_+).$$

The Hölder exponent can be made arbitrarily close to 1 by choosing $p$, but the assertion of the theorem does not hold.

---

\textsuperscript{68}Recall that the sign of the coefficient $c(x)$ is not important here. Note also that if $f(0) < 0$, then $D_nu$ can vanish at $x \in \Pi_\lambda \cap \partial B_R$, but it is shown in \cite{142} that in this case $D_nD_nu(x) > 0$; this is sufficient for the subsequent argument.

\textsuperscript{69}There is a typo in \cite{142} in this formula.

\textsuperscript{70}Nevertheless, if $f > 0$ then the Lipschitz condition on $f$ can be weakened, see, e.g., \cite{242} and \cite{149}.  

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The article [142] (as well as [143], where equations of the form (4.1) were considered in the whole space) gave rise to a huge number of improvements and generalizations. Among them, we lay emphasis on the paper by H. Berestycki and L. Nirenberg [61]. There, by using the Aleksandrov–Bakelman maximum principle, the results of [142] are extended to strong solutions for a rather wide class of uniformly elliptic nonlinear equations. Applications of the moving plane method to the $p$-Laplacian type degenerate operators can be found in [103], [104], [122], [285], see also references therein.

A number of papers use the moving sphere method, that is a combination of the moving plane method with conformal transformations (see, e.g., [230] and [289]). We also mention the paper [254], where discrete analogues of the results in [142] were obtained.

Other applications of the strong maximum principle and the normal derivative lemma to the proof of symmetry properties in geometric problems can be found, for example, in [5], [329], [291], [343], [284], [261] (see also [338]). The applications of the Aleksandrov–Bakelman maximum principle and its variants to the study of symmetry properties for solutions to nonlinear boundary value problems and to the proof of isoperimetric inequalities are discussed in [77], [79], [80], see also the survey [78].

### 4.2 Phragmén–Lindelöf type theorems

The Phragmén–Lindelöf principle in its original formulation [293] describes the behavior at infinity of a function analytic in an unbounded domain.

For solutions of uniformly elliptic (non-divergence type) equations of general form, such theorems were first proved by E.M. Landis [218]–[219] (brief reports were previously published in [215]–[217]). The principal coefficients of the operator in [219] satisfy the Dini condition, and the behavior of the domain at infinity is described in terms of measure.

More exact Phragmén–Lindelöf type theorems can be obtained if the domains are described in terms of capacity. The first results of this kind were established in [251], [69] for divergence type equations with measurable principal coefficients and in [68], [69] for non-divergence type equations under the Hölder condition on the principal coefficients.

Finally, the crucial step was taken by E.M. Landis [221] (see also [223, Ch. 1]), who, using the concept of $s$-capacity introduced by himself, proved
the Phragmén–Lindelöf type theorems for non-divergence type equations with measurable principal coefficients.

We present, for instance, one of the results from [223] Ch. 1, § 6.

**Theorem 4.2.** Let \( \Omega \) be an unbounded domain lying inside the infinite layer \( \Omega \subset \{ x \in \mathbb{R}^n \mid |x_n| < h \} \).

Let an operator \( L_0 \) satisfy the condition (2.3), and let \( u \in C^2(\Omega) \) be a classical subsolution\footnote{Using the Aleksandrov–Bakelman maximum principle, this result can be adapted for strong subsolutions \( u \in W^{2}_{n,\text{loc}}(\Omega) \).} of the equation \( L_0 u = 0 \) satisfying the condition \( u|_{\partial \Omega} \leq 0 \).

If \( u(x) > 0 \) at some point \( x \in \Omega \), then

\[
\liminf_{R \to \infty} \frac{\max_{|x|=R} u(x)}{\exp \left( \frac{C}{\rho} R \right)} > 0,
\]

where the constant \( C > 0 \) depends only on \( n \) and \( \nu \).

We also note the work of V.G. Maz’ya [252], where related questions were studied for quasilinear \( p \)-Laplacian type operators.

In the case where the non-tangential derivative is given on a part of \( \partial \Omega \), the Phragmén–Lindelöf type theorems were proved in [224], [163], [164] for divergence type equations and in [85] (see also [165]) for non-divergence type equations. Note that in the last two papers, a weakened form of the normal derivative lemma [266] was used.

The above results are linked to the **Landis conjecture** that is the problem of the fastest possible rate of convergence to zero for a nontrivial solution to a uniformly elliptic equation in the domain \( \Omega = \mathbb{R}^n \setminus B_R \). It was first formulated in the survey [195] for the equation

\[
- \Delta u + c(x)u = 0 \quad (4.2)
\]

with \( c \in L_{\infty}(\Omega) \) (in this case, the expected answer is exponential decay: if \( |u(x)| = O(\exp(-N|x|)) \) as \( |x| \to \infty \) for any \( N > 0 \), then \( u \equiv 0 \)). This problem has not been completely solved even for the simplest equation (4.2). Recent results in this area, as well as a historical survey, can be found in [335] (see also [247]).
4.3 Boundary Harnack inequality

If the normal derivative lemma does not hold, the following statement can be considered as its weaker version:

**Boundary Harnack inequality.** Let \( 0 \in \Omega \) and let \( L \) be an elliptic operator in \( \Omega \). If \( u_1 \) and \( u_2 \) are positive solutions of the equation \( Lu = 0 \) in \( \Omega \) satisfying the condition \( u_1|_{\partial \Omega \cap B_R} = u_2|_{\partial \Omega \cap B_R} = 0 \), then the inequality

\[
C^{-1} \frac{u_1(0)}{u_2(0)} \leq \frac{u_1(x)}{u_2(x)} \leq C \frac{u_1(0)}{u_2(0)} \tag{4.3}
\]

holds true in the subdomain \( \Omega \cap B_{R/2} \), where \( C \) is a constant independent on \( u_1 \) and \( u_2 \).

**Remark 4.1.** If, for example, \( \Omega \) is a \( C^{1,D} \)-smooth domain, and \( L \) is a uniformly elliptic operator of the form \( (2.1) \) with bounded coefficients, then (4.3) easily follows from the normal derivative lemma, the gradient estimate for solutions on \( \partial \Omega \), and the usual Harnack inequality.

**Remark 4.2.** In the important particular case of flat boundary \( x_n = 0 \) and an operator \( L_0 \), where \( u_2(x) = x_n \) can be taken, the boundary Harnack inequality was first obtained by N.V. Krylov [204] in order to obtain boundary estimates in \( C^{2,\alpha} \) for solutions of nonlinear equations.

To describe the results of this subsection, we need new classes of domains:

- Nontangentially accessible domains (NTA domains);
- Uniform domains;
- Domains satisfying the \( \lambda \)-John condition, \( \lambda \geq 1 \); when \( \lambda = 1 \) just say “John domains”;
- Twisted Hölder domains (THD domains), with clarification “of order \( \alpha \in (0, 1] \)” (THD-\( \alpha \)) if necessary.

The exact definitions of these classes can be found in the corresponding works listed in Table 1. For the reader’s convenience, we present only the relations between them (see, for example, [189]):

\[
C^{0,1} \subset \text{NTA} \subset \text{Uniform} \subset \text{John} = \text{THD-1};
\]

\[
C^{0,\alpha} \subset \frac{1}{\alpha}\text{-John (\(\triangle\))} \equiv \text{THD-\(\alpha\)}.
\]

\[72\text{The relation (\(\triangle\)) is not stated explicitly in [189] but follows from Remark 2.5 in this paper.}\]
In Table 1 it is assumed by default that the principal coefficients of the operators are measurable and satisfy the condition (2.3).

| Operator | $C^{0,1}$ | NTA | Unif. | John | $C^{0,\alpha}$ | THD$^{73}$ |
|----------|-----------|-----|-------|------|----------------|-----------|
| $-\Delta$ | 102$^{74}$ | 167 | 6     |      |                |           |
| $\mathcal{L} + c(x)$ $^{75}$ | 84 | 48$^{76}$ | 56 |      |                |           |
| $\mathfrak{L}_0$ | 124$^{78}$ |      |      |      |                |           |
| $-\Delta + b^i(x) D_i$ $^{77}$ | 101 |      |      |      |                |           |
| $\mathcal{L}_0 + c(x), c \in \mathcal{K}_{n,2}$ | 100$^{79}$ |      |      |      |                |           |
| $\hat{\mathfrak{L}}$ | 100$^{80}$ |      |      |      |                |           |
| $\mathcal{L}, b^i \in L_\infty(\Omega)$ | 191 | 191 | 189 |      |                |           |
| $\mathcal{L}, b^i \in L_n(\Omega)$ | 319 |      |      |      |                |           |

Table 1: Boundary Harnack inequality in various classes of domains

In the recent papers $^{108}$, $^{109}$, a unified approach to the proof of the boundary Harnack inequality for divergence and non-divergence types operators is demonstrated $^{82}$.

A variation of the boundary Harnack inequality for supersolutions and “almost supersolutions” of the equation $\mathfrak{L}u + cu = 0$ with bounded coefficients was obtained in $^{34}$.

$^{73}$Results are obtained for $\alpha > \frac{1}{2}$; counterexamples are constructed in $^{56}$ for $\alpha < \frac{1}{2}$ and in $^{190}$ for $\alpha = \frac{1}{2}$.

$^{74}$See also $^{355}$.

$^{75}$The coefficients satisfy the Hölder condition.

$^{76}$See also $^{130}$.

$^{77}$For the assumptions on the coefficients $b^i$, see Footnote $^{55}$.

$^{78}$See also $^{59}$; a somewhat more general condition on the domain is considered in $^{17}$.

$^{79}$See Footnote $^{54}$.

$^{80}$The principal coefficients satisfy the assumptions $^{33}$, $\mathcal{V}(x) \leq N \cdot \nu(x)$ and $^{31}$, and the lower-order ones are subject to condition $^{31}$.

$^{81}$The result is obtained for $\alpha > \frac{1}{2}$, while for $\alpha < \frac{1}{2}$ a counterexample is constructed. If $\partial \Omega$ additionally satisfies the condition (A) introduced by O.A. Ladyzhenskaya and N.N. Uraltseva (see, for instance, $^{213}$), then the result is obtained for all $\alpha > 0$.

$^{82}$Earlier, similar ideas appeared in the works of M.V. Safonov, see $^{131}$, $^{319}$, $^{189}$.

$^{83}$See in this connection $^{341}$, where an estimate is established for a superharmonic function satisfying the zero Dirichlet condition in a two-dimensional domain with corners in terms of the first eigenfunction of the Dirichlet Laplacian.
The boundary Harnack inequality is linked to results similar to the weak Harnack inequality for the quotient $u(x)/d(x)$ (see [334] and references therein). Let us give, for example, one of the results of [334].

**Theorem 4.3.** Let $u$ be a non-negative weak supersolution of the equation $\mathcal{L}u = f$ in a $C^{1,1}$-smooth domain. Assume that the condition (2.3) is satisfied as well as the following assumptions:

$$a^{ij} \in W^{1,q}_q(\Omega), \quad b^i \in L^{q}(\Omega), \quad f_ - \in L^{q}(\Omega); \quad q > n.$$ 

Then

$$\left( \int_{\Omega} \left( \frac{u(x)}{d(x)} \right)^s dx \right)^{\frac{1}{s}} \leq C \left( \inf_{x \in \Omega} \frac{u(x)}{d(x)} + \|f_ -\|_{q,\Omega} \right)$$

for any $s < 1$. The constant $C$ depends on $n$, $\nu$, $s$, $q$, on the norms of coefficients $a^{ij}$ and $b^i$ in corresponding spaces, on $\text{diam}(\Omega)$, and on the properties of $\partial \Omega$.

The harmonic function $x_n \cdot |x|^{-n}$ in the half-ball $B_r^+ = B_r \cap \{x_n > 0\}$ shows that the constraint $s < 1$ is sharp.

We also mention some papers (see, e.g., [241], [240], [54]) where the boundary Harnack inequality was obtained in the “abstract” context of metric spaces.

### 4.4 Other results for linear operators

In the papers [37], [75], a generalized strong maximum principle is established for the operators $-\Delta + c(x)$ with $c \in L_1(\Omega)$; solutions are understood in the sense of measures. For further results in this direction see [63], [286], [296].

It is well known that the validity of the weak maximum principle for a second-order elliptic operator is equivalent to the positivity of the first eigenvalue for the corresponding Dirichlet problem. In the paper [52], a generalized first eigenvalue is defined for uniformly elliptic operators $\mathcal{L} + c(x)$.

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84It is important that no condition is imposed on the behavior of $u$ near $\partial \Omega$. 

60
with bounded coefficients in an arbitrary bounded domain\(^85\) (the supremum is taken over \(\phi \in W^{2,\text{loc}}_{n,\text{loc}}(\Omega), \phi > 0 \text{ in } \Omega\))

\[
\lambda_1 = \sup_{\phi} \inf_{x \in \Omega} \frac{\mathcal{L}\phi(x) + c(x)\phi(x)}{\phi(x)}.
\]

(4.4)

It is shown in \([62]\) that the weak maximum principle (as well as the “improved” weak maximum principle introduced in this article) for the operator \(\mathcal{L} + c(x)\) is equivalent to the inequality \(\lambda_1 > 0\).

In recent decades the study of partial differential equations on complicated structures has become very popular. In a number of papers (see, e.g., \([148], [111], [58], [250]\) and references therein), conditions for the validity of the strong maximum principle, the Harnack inequality, the normal derivative lemma, and the boundary Harnack inequality were studied for subelliptic operators, including sub-Laplacians on homogeneous Carnot groups.

In the papers \([141], [287], [288]\) the strong maximum principle and the normal derivative lemma were considered for the simplest elliptic operators on stratified sets, which are cell complexes with some special properties\(^86\).

### 4.5 Nonlinear operators

Even the simplest keyword search shows that in recent years the number of articles on the topic of the survey, concerning nonlinear operators, can been estimated at dozens per year. Therefore, this subsection has an obviously dotted character without even a minimum completeness.

The Harnack inequality for divergence form quasilinear operators was first proved in \([328]\) and then for wider classes of operators in \([345]\) and \([348]\). These works are now classics. We also note the paper \([114]\), where the Harnack inequality was established for \textit{quasi-minimizers} of variational problems.

In the paper \([116]\), the normal derivative lemma from \([352], [353]\) was generalized to the quasilinear case.

\(^85\)For operators with smooth coefficients in smooth domains, this formula actually gives the first eigenvalue; in this case, the supremum can be taken over smooth functions \(\phi\) positive in \(\Omega\). For the Laplacian, the formula \([144]\) was apparently first highlighted in \([55]\). Then it was generalized to various classes of operators (see \([62]\) and references therein).

\(^86\)The simplest examples of such operators are the operators of the Venttsel problem and the two-phase Venttsel problem.
For the operators similar to $p$-Laplacian
\[ \Delta_p u \equiv D_i(|Du|^{p-2}D_i u), \quad p > 1 \]  
the normal derivative lemma was first proved in \[342\]. Among the recent generalizations of this result, we mention the papers \[255\] and \[87\].

In the papers \[88\] and \[64\], sharp conditions for the strong maximum principle and the normal derivative lemma were obtained for the minimizers of the functional
\[ J[u] = \int_\Omega f(Du) \, dx. \]

J.L. Vázquez \[350\] proved the strong maximum principle for the equation
\[ -\Delta_p u + f(u) = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^n, \quad n \geq 2. \]  

**Theorem 4.4.** Let $f\in C(\mathbb{R}^+)$ be a nondecreasing function, and let $f(0) = 0$. Then a necessary and sufficient condition providing that arbitrary (nonzero) nonnegative supersolution of the equation (4.6) does not vanish in $\Omega$ is the relation
\[ \int_0^\delta \frac{dt}{(F(t))^{\frac{1}{p}}} = \infty, \quad \text{where} \quad F(t) = \int_0^t f(s) \, ds. \]  

For generalizations of this result to wider classes of quasilinear operators see \[304\], \[129\], \[335\]. In the paper \[168\], the corresponding Harnack inequality is established (for $p = 2$):

**Theorem 4.5.** Suppose that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing function. Let $u \in W^{1,2}_0(B_R)$ be a solution to the equation $L_0 u + f(u) = 0$, where $L_0$ is a (divergence type) uniformly elliptic operator with measurable coefficients. Denote $M = \sup_{B_R} u$ and $m = \inf_{B_R} u$. Then \[87\]
\[ \int_m^M \frac{dt}{(F(t))^{\frac{1}{p}} + t} \leq C, \]
where $F$ is defined in (4.7), and the constant $C$ depends only on $n$ and $\nu$ (in particular, it does not depend on $f$!).

\[87\] Notice that for $f \equiv 0$ this relation becomes the classical Harnack inequality.
The boundary Harnack inequality for operators of \( p \)-Laplacian type in a \( C^2 \)-smooth domain was established in [67]. Subsequently, it was proved for the wider classes of domains discussed in § 4.3 (see [279] and references therein). The boundary Harnack inequality for the maximal and minimal Pucci operators was proved in [333] (see also [72]).

Nowadays, popular objects of research are also \( p(x) \)-Laplacians, i.e. operators of the form (4.5), where the exponent \( p \) is a function of the \( x \) variables. The Harnack inequality for such operators was first proved in [31] (for recent generalizations see, e.g., [32] and [336]). In [4], the boundary Harnack inequality was established in a \( C^{1,1} \)-smooth domain.

4.6 Nonlocal operators

In recent decades, interest in the study of nonlocal ( integro-differential) operators has increased significantly. Among them, fractional Laplacians show up. The simplest of these (and historically the first), the fractional Laplacian in \( \mathbb{R}^n \) of order \( s \), is defined using the Fourier transform:

\[
(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}u)(\xi)), \quad s > 0;
\]

for \( s \in (0, 1) \) this operator can be defined via a hypersingular integral:

\[
((-\Delta)^s u)(x) = C_{n,s} \cdot \text{PV.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy, \quad C_{n,s} = \frac{s2^{2s}\Gamma\left(\frac{n}{2} + s\right)}{\pi^{n/2}\Gamma(1-s)}.
\]

M. Riesz [310] (see also [225, Ch. IV, § 5]) proved a direct analog of the Harnack inequality (2.22) for \((-\Delta)^s \) with \( s \in (0, 1) \):

Let \( u \geq 0 \) in \( \mathbb{R}^n \), and let \((-\Delta)^s u = 0 \) in \( B_R \). Then, for \( x \in B_R \) we have

\[
\frac{(R - |x|)^s R^{n-2s}}{(R + |x|)^{n-s}} \leq u(x) \leq \frac{(R + |x|)^s R^{n-2s}}{(R - |x|)^{n-s}}.
\]

In contrast to the case of the whole space, the fractional Laplacians in the domain \( \Omega \subset \mathbb{R}^n \) certainly depend on the boundary conditions (there are

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88To define accurately this and similar operators, as well as the notion of a weak ( sub/super)solution to the corresponding equations, it would be necessary to introduce the Sobolev–Slobodetskii spaces ([344, Ch. 2–4]; see also [124]). We will not do this, pitying the reader.
fractional Dirichlet Laplacians, Neumann Laplacians, etc.). Moreover, even for a fixed type of boundary conditions, there are several essentially different definitions of fractional Laplacians: restricted, spectral, and some others. Note that to compare restricted and spectral Dirichlet Laplacian, in [262], [263] the classical normal derivative lemma for weakly degenerate operators (see [173], [35]) was used.

Proofs of the strong maximum principle for various fractional Laplacians of order \( s \in (0, 1) \) in \( \Omega \) can be found in [331], [86], [162]; in the paper [264], a unified approach was proposed for a large family of fractional Laplacians and more general nonlocal operators. On the other hand, it is shown in [2], [3] that even the weak maximum principle does not hold for the restricted fractional Dirichlet Laplacian with \( s > 1 \) in a domain of general form.

The boundary Harnack inequality for the operator \((−\Delta)^s\), \( s \in (0, 1) \), in a Lipschitz domain was proved in [70]. Due to the non-locality of the operator, its formulation differs from the standard one (see §4.3):

Let \( 0 \in \Omega \). If \( u_1 \) and \( u_2 \) are non-negative functions in \( \mathbb{R}^n \), continuous in the ball \( B_R \), satisfying the equation \((−\Delta)^s u = 0 \) in \( \Omega \cap B_R \) and the condition \( u_1|_{B_R \setminus \Omega} = u_2|_{B_R \setminus \Omega} = 0 \), then the inequality \((4.3)\) holds true in the subdomain \( \Omega \cap B_{R/2} \) with constant \( C \) depending only on \( n, s, \Omega \) and \( R \).

Later this result was extended to arbitrary domains \( \Omega \) and to a wide class of integro-differential operators (see [315] and references therein).

In [314] was constructed a barrier which is sufficient to prove an analogue of the normal derivative lemma in the following form:

Let \( \Omega \) be a \( C^{1,1} \)-smooth domain, and let \( s \in (0, 1) \). Suppose that \( u \) is a weak supersolution to the equation \((−\Delta)^s u = 0 \) in \( \Omega \), and \( u = 0 \) in \( \mathbb{R}^n \setminus \Omega \). If \( u \not\equiv 0 \) then

\[
\inf_{x \in \Omega} \frac{u(x)}{d^s(x)} > 0.
\]

Further generalizations of this result can be found, for instance, in [312], [313]. For operators of fractional \( p \)-Laplacian type, a similar assertion was proved in [110].

\(^{89}\)Note that in \( \mathbb{R}^n \) as well as in the ball \( \Omega = B_R \) the strong maximum principle holds for any \( s > 0 \), which is also shown in [2].

\(^{90}\)If \( u_2(0) = 0 \) then \( u_2 \equiv 0 \). Therefore, we can assume that \( u_2(0) > 0 \).
For the spectral fractional Dirichlet Laplacian, instead of (4.8), the inequality 
\[ \inf_{x \in \Omega} \frac{d(x)}{u(x)} > 0 \]
holds under the same assumptions (see Theorem 1.2 in [192], where more general functions of the Laplace operator with Dirichlet conditions are also considered). An analog of the normal derivative lemma for the regional fractional Laplacian with \( s \in (\frac{1}{2}, 1) \) was obtained in a recent preprint [1].

In [151], a generalization of the Aleksandrov–Bakelman maximum principle for non-local analogs of the maximal and minimal Pucci operators is obtained.

An application of the moving plane method to problems with fractional Laplacians can be found in [127] and in the papers cited there.

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