The Iterated Prisoner’s Dilemma on a Cycle

Martin Dyer and Velumailum Mohanaraj
School of Computing, University of Leeds
Leeds LS2 9AS, United Kingdom

Abstract

Pavlov, a well-known strategy in game theory, has been shown to have some advantages in the Iterated Prisoner’s Dilemma (IPD) game. However, this strategy can be exploited by inveterate defectors. We modify this strategy to mitigate the exploitation. We call the resulting strategy Rational Pavlov. This has a parameter \( p \) which measures the “degree of forgiveness” of the players. We study the evolution of cooperation in the IPD game, when \( n \) players are arranged in a cycle, and all play this strategy. We examine the effect of varying \( p \) on the convergence rate and prove that the convergence rate is fast, \( O(n \log n) \) time, for high values of \( p \). We also prove that the convergence rate is exponentially slow in \( n \) for small enough \( p \). Our analysis leaves a gap in the range of \( p \), but simulations suggest that there is, in fact, a sharp phase transition.

Keywords: Games on graphs, prisoner’s dilemma, convergence, Pavlov strategy

1 Introduction

1.1 Overview

The Prisoner’s Dilemma (PD) is one of the most famous strategic games in game theory (see, for example, [13]). This game is widely used as a prototype for the study of the evolution of cooperation among selfish agents. It has attracted a large amount of interest from researchers in diverse fields, due to the fact that it represents a very common strategic situation that needs to be understood. Many real-life problems can be modelled by the Prisoner’s Dilemma [3].

In the standard form of PD, the payoff obtained when both prisoners cooperate with each other is denoted by \( R \), the reward for mutual cooperation. The payoff gained when both defect is denoted by \( P \), the punishment for mutual defection. Finally, \( T \) (the temptation to defect) is earned by the informer and \( S \) (the sucker’s payoff) is earned by the other when one defects and the other cooperates. These four outcomes are shown in Figure 1 in a matrix form. In this game, the payoff of an action does not depend on the player. Hence the game is said to be symmetric.

| Row Player | Cooperate | Defect |
|------------|-----------|--------|
| Cooperate  | \( R = 3 \) | \( S = 0 \) |
| Defect     | \( T = 5 \) | \( P = 1 \) |

Figure 1: The payoff matrix for the Prisoner’s Dilemma with Axelrod’s numerical example. The game is symmetric, therefore only the payoffs to the row player are shown.

The interesting feature of the PD game is the property that the four payoffs satisfy: \( T > R > P > S \). Hence, in one shot game, it is always best to defect. Thus, self-interest of the players
leads to the payoff $P$ which is worse than the $R$ that both players would get by cooperating, hence the dilemma. In the Iterated Prisoner’s Dilemma (IPD), the same players meet again with a high probability, thus getting an opportunity to punish each other for any previous non-cooperative moves. Understandably, the fear of retaliation here is likely to encourage cooperation. This was studied in [1], which stimulated work in this area. Another constraint $2R > S + T$ is usually added to the standard form of the IPD [11]. If this constraint is not present, players could benefit from receiving $S$ and $T$ on alternate rounds rather than $R$ on every round through continuous cooperation.

A great deal of research has been done to find out an ideal strategy for the IPD. A strategy helps players decide whether to cooperate or defect in the current round. A simple strategy called *tit-for-tat* (TFT) surprisingly won Axelrod’s seminal computer tournaments [1]. TFT cooperates on the first round, and copies what the opponent has done on the previous round thereafter. However, this strategy has two main problems: firstly, it is not *evolutionarily stable* [2, 7]; and secondly, any mistakes by the agents or any noise in the responses may cause a misinterpretation leading to irrecoverable retaliation sequences. (Informally, a strategy is evolutionary stable if a population of players adopting the strategy can not be overrun by any mutant strategy.)

Another well-known strategy is called *Pavlov*. The Pavlov, an exemplar of the *win-stay lose-shift* strategy, works as follows. On each iteration of the game, if a Pavlov player’s payoff is one of the two smaller payoffs, i.e. $P$ or $S$, then he switches his action in the next round of the game, otherwise he keeps the same action. It is claimed in [7, 10, 16] that the Pavlov performs better than TFT. This is due to its ability to recover from noise and errors and its capability to exploit unconditional cooperators (All-C). However Pavlov has two main issues. Firstly, Pavlov is deterministic, thus it cannot represent uncertainties present in the real world, such as the stochastic nature of biological interactions [15]. Secondly, it fares poorly against all-time defectors (All-D). This is because, when played against All-D, Pavlov is punished for defecting, so switches to cooperation, just to be punished even more. This is repeated forever, and consequently Pavlov collects the sucker’s payoff ($S$) on alternate rounds.

A family of stochastic Pavlovian strategies $\mathcal{P}(k, \ell)$, for a fixed $\ell$ and $0 < k < \ell$, has also been studied and hailed as a near-ideal strategy for the IPD in [10]. $\mathcal{P}(k, \ell)$ cooperates with probability $k/\ell$. At the end of each round of the game, $k$ is increased if the player gains $T$ or $R$, and decreased otherwise. The advantages of these strategies are: they are adaptive and naturally stochastic. The disadvantages are: they take exponential time in $\ell$ for learning to cooperate and are exploitable by All-D. It is worth to mention that $\mathcal{P}(1, 1)$ is equivalent to the Pavlov strategy described above.

Before we move on, let us represent the Pavlov strategy as a (deterministic) Markov chain. Suppose two agents play the IPD using Pavlov. This can be modelled as a Markov chain having four states, each representing a possible combination of the strategies of the agents. We denote these states by $++$, $+-$, $-+$ and $--$. (Here + stands for *cooperation* and − stands for *defection.*) Thus, $++$, for example, represents the scenario where both agents cooperate. The transition diagram for this process is shown in Figure 2(a).

### 1.2 Rational Pavlov strategies on IPD

It is now clear that the main weakness of the Pavlov is that it can be exploited by All-D. Thus we suggest an enhancement to this strategy. We modify it to add randomness. This, we think, makes the resultant strategy more rational and robust. The details of the modification are given below.

A Pavlov player cooperates in the current round if both he and his opponent cooperated or defected at their previous play. Thus, a transition from $--$ to $++$ happens in a single repetition with certainty. We will modify this in two ways, so that the transition from $--$ to $++$ will only happen with some probability less than 1. More precisely, the modifications introduced to the $--$ to $++$ transition are: if both players defected, i.e. in state $--$, in the previous play, then

1. each player decides independently whether to cooperate in the current round with probability $p$. The transition diagram of the strategy obtained after this modification is shown
in Figure 2(b). As we believe that this modification adds some rationality to Pavlov we call this strategy Rational Pavlov (RP).

2. both cooperate in the current round with probability $p$. The transition diagram of this strategy is shown in Figure 2(c). This is a simplified version of the RP, hence the name Simplified Rational Pavlov (SRP). Even with the absence of communication, players deciding together with probability $p$ can also be justified using the superrationality principle [6]. Thus, SRP might also be expanded as Super Rational Pavlov.

It is noteworthy that both RP and SRP are equivalent when $p = 1$ or $p = 0$. And, both RP and SRP reduce to the original Pavlov strategy when $p = 1$.

![Transition diagrams of the original and the modified Pavlovian strategies.](image)

**Figure 2:** Transition diagrams of the original and the modified Pavlovian strategies. Here, “−” represents cooperation and “+” represents defection. The transition probabilities are shown on the edges.

### 1.3 Previous work

Although our work appears to be the first to formally define strategies like RP and SRP, there is some evidence in the literature that support our intuition behind the proposed improvements. Firstly, the results from experiments with humans in [20] overwhelmingly support this. The results show that humans use a Pavlov-like strategy that is smarter than the classic Pavlov strategy when dealing with All-D. This Pavlov-like strategy cooperates after −− state with probability less than 1, like RP and SRP do. Not surprisingly, the players using this strategy were more successful than the others in the experiments. Furthermore, a similar modification has been suggested as a possible improvement of the Pavlov in [10]. Finally, a strategy similar to RP and SRP has proved to be the winner in computer simulations as well [5].

Apart from reigning supreme in evolutionary game theory, the Pavlov has been studied in distributed Artificial Intelligence as a learning model. Shoham and Tennenholtz [19] introduced the notion of co-learning where agents try to adapt to their environment by adapting to one another’s behaviour. In the same paper, they also defined a simple co-learning update rule, namely *Highest Cumulative Reward* (HCR). This rule states that an agent should adapt to the action that resulted in favourable feedback in the latest $\mu$ iterations, where $\mu$ is the memory size. The HCR update rule ensures that cooperation emerges at the end in the IPD game. This update rule with $\mu = 0$, which is one of the most efficient memory sizes [9], is precisely the Pavlov strategy.

Shoham and Tennenholtz [19] studied the evolution of cooperation for the HCR update rule in unstructured population and concluded that it is an impractical model for the evolution of cooperation. This conclusion is not surprising, as it is now well known that, in an unstructured population, natural selection favours defection over cooperation [17]. Hence, there is a growing interest in studying the evolution of cooperation when the topology for interactions is not
complete (see, for example, [18, 17]). Thus we consider the players to be arranged as the vertices of a graph, and they can interact only along the edges of the graph. Kittock [9] studied the effects of an interaction graph on the emergence of cooperation under the dynamics in which, at every step, two adjacent players are selected uniformly at random to play the IPD game using the Pavlov strategy. The paper [9] presented the results from an empirical study which shows that the time needed for the emergence of cooperation in the IPD game is polynomial on cycles and exponential on complete graphs.

Most of the work we have mentioned above is empirical. However the need for rigorous results has rightly been emphasised. (See, for example, [9, 11, 19].) The reason for the lack of rigorous analysis of games on graphs is that it is complicated due to the vast number of patterns that can be generated [14]. While the empirical results do give some insights into the evolution, some of the results are far too complicated to be understood without theoretical backing. The results obtained through rigorous analysis are often more revealing and contribute to a clearer understanding of the problem. Hence, in this paper, we analyse the behaviour of RP rigorously. More precisely, we establish the conditions for fast convergence, and determine the rate of convergence to cooperation when all players play RP. These measures are central to understanding the emergence of cooperation among selfish agents [2].

On the theoretical side, Dyer et al. [4] studied the two cases examined in [9], using rigorous analysis. Mossel and Roch [12] did a similar study for some expander graphs and bounded degree trees and showed that the convergence is slow in both settings for the Pavlov strategy. Istrate et al. [8] investigated the robustness of these convergence results under adversarial scheduling in which an adversary selects which players update at every step. Their results show that if an adversary can specify two players for the update, the game might never converge. Along this line of work, we carry out a rigorous analysis of RP in this paper. In particular, we attempt to find the range of \( p \) that favours fast evolution of cooperation and the range of \( p \) that makes the evolution of cooperation exponentially slow, when the IPD is played on the cycle using RP. (Here, we consider speed of convergence as a function of the number of players, \( n \).) All our results are complemented by simulation results. Our choice of graph, the cycle, is an extreme case, where every player has only two neighbours. Game dynamics have previously been analysed for the cycle [4, 9, 17]. Our results show some interesting results, for instance, we show that the emergence of cooperation is exponentially slow for small values of \( p \). Thus a high degree of forgiveness seems necessary for cooperation to emerge. Perhaps, our most important message is that a Rational Pavlov player can reduce the risk of being exploited without compromising the emergence of cooperation.

We have analysed SRP as well. The analysis is quite similar to that of RP. Therefore, we do not present it in this paper. Instead, we make some remarks on the final results under relevant sections.

1.4 Preliminaries

Much of the notation and terminology used in this paper is adopted from [4]. We consider \( n \) players arranged as the vertices of a cycle graph \( G = (V, E) \), where \( V = \{0, \ldots, n-1\} \) and \( E = \{(i, i+1) : 0 \leq i \leq n-1\} \). Hence, vertex \( i \) can interact only with the vertices \( i-1 \) and \( i+1 \). Here and throughout the paper, addition and subtraction on vertices is performed modulo \( n \).

The agent at the vertex \( i \) (\( 0 \leq i < n \)) has a state \( S_i \in \{-1, 1\} \), where \(-1\) represents defection and \(1\) represents cooperation. We will denote the cooperator-states, or 1’s, also as +’s (pluses), and the defector-states, or \(-1\)’s, as −’s (minuses). Each edge of the graph has a state which is determined by the states of its end vertices. Thus, an edge of the graph might be in any of four states, \(-\), \(-+\), \(+−\), \(+\), as shown in the state transition diagrams in Figure 2.

In this study, the game is played in the following way. At each stage, an edge of the cycle is selected uniformly at random. The agents connected by this edge play the game using RP and update their strategies accordingly. In this process, emergence of cooperation means reaching the state where everyone cooperates, in other words, reaching the state \( S^* \) with \( S_i = 1 \) for all
\(i \in V\). The state \(S^*\) is the unique absorbing state of this process.

We will use the following terminology. Let \(S \in \{-1, 1\}^V\) be given. A plus-run (resp. minus-run) in \(S\) is an interval \([i, j]\) where \(0 \leq i, j < n\), such that \(S_k = 1\) (resp. \(-1\)) for \(i \leq k \leq j\) and \(S_{i-1} = -1\) (resp. \(1\)). (It is possible to have \(j \leq i\), since we are working modulo \(n\).) Clearly all runs are disjoint. The length of a minus-run \(R_d\), denoted by \(\ell(R_d)\), equals the number of minuses in the run. We will refer to a minus-run of length \(\ell\) as an \(\ell_d\)-run where the subscript “d” stands for defectors. We use similar variables for a plus-run with subscript “c”, which stands for cooperators.

We now give some definitions for minus-runs, which are equally applicable to plus-runs if the signs are changed, and the subscript \(c\) is used. A 1\(_d\)-run is also called a singleton minus, and a 2\(_d\)-run is also called a pair of minuses. There are two outer rim edges associated with a minus-run \(R_d = [i, j]\), namely \(\{i - 1, i\}\) and \(\{j, j + 1\}\). The all-minuses configuration is not a run as we have defined it, since it has no bordering pluses, we will nevertheless refer to it as the \(n_d\)-run.

Finally, the parameter of both RP and SRP will be denoted by \(p\), but the context should always make the meaning clear. The following theorems summarise our results.

**Theorem 1.** Suppose \(n\) players, arranged as the vertices of a cycle, play the IPD game using Rational Pavlov (RP) with parameter \(p \geq 0.870\). Then, there is a constant \(\omega > 0\) such that, the probability that the all-cooperate state is not reached in time \(\omega n \log(n\varepsilon)\) is at most \(\varepsilon\), for any \(\varepsilon > 0\).

**Theorem 2.** Suppose \(n\) players, arranged as the vertices of a cycle, play the IPD game using Rational Pavlov (RP) with parameter \(p\). Suppose all players play defect when the game is started. Then there exists a constant \(p_1 > 0\) such that, for all \(p \leq p_1\), it takes time exponential in \(n\) for the all-cooperate state to be reached, except for probability exponentially small in \(n\).

**Theorem 3.** Suppose \(n\) players, arranged as the vertices of a cycle, play the IPD game using Rational Pavlov (RP) with \(p = 0\). Provided there is at least one defector on the cycle at the beginning of the game, the game converges to defection in time \(T_n\) where \(T_n\) lies within the range \(\left[\frac{n(n-1)}{2} \pm O(n^{3/2} \log n)\right]\) with high probability.

**Remark 1.** In this paper, an event \(Y_n\) which depends on the size of the graph \(n\) is said to happen with high probability, or in short w.h.p., to mean that \(\Pr(Y_n) \to 1\) as \(n \to \infty\).

The outline of the rest of this paper is as follows: In Section 2, we derive the conditions for fast convergence to cooperation when RP is used on the cycle. In Section 3, we prove that convergence to cooperation is slow for small values of \(p\). Section 4 concentrates on a special case where defection emerges fast on the cycle. Experimental results are presented in Section 5. Finally, Section 6 presents our concluding remarks.

## 2 Fast convergence on the cycle

The convergence rate of the IPD has been analysed in [4] by finding a nonnegative integer-valued potential function \(\xi : \{-1, 1\}^V \to \mathbb{R}\) such that \(\xi(S) = 0\) when \(S = S^*\) and \(\xi(S) > 0\) otherwise. Then, Dyer et al. [4] proved that the expectation of the function \(\xi\), which measures the distance from the absorbing state \(S^*\) to any given state \(S\), decreases with non-null probability till the
absorbing state is reached. We use a similar approach here, but with a simpler potential function \( \phi(S) \). This function is defined as

\[
\phi(S) = \sum_{\ell=1}^{n} w_\ell r_\ell, \quad \text{where} \quad r_\ell \text{ is the number of } \ell\text{-runs, and} \quad w_\ell > 0 \text{ is the weight of an } \ell\text{-run.} \tag{1}
\]

Note that \( \phi(S) = 0 \) when \( S = S^* \) and \( \phi(S) > 0 \) otherwise. In this section, we prove Theorem \( \square \) which shows that the emergence of cooperation is fast when using RP in the IPD for high values of \( p \). This is done by studying the changes in the total weight of the minus-runs. Hence, in this section, a run means a run of minuses unless otherwise stated.

2.1 Analysis

We first consider the minus-runs that are separated from their adjacent runs by at least two pluses. When two minus-runs are separated by a singleton plus, choosing the outer rim edges of the singleton plus causes the two runs to merge together. This case therefore needs some special consideration and is addressed at the end of this section.

We need to show that the expectation of \( \phi \) decreases after every iteration of the game. This requirement can be modelled by having a constraint that the expected total weight of the runs created by hitting an overlapping edge of an \( \ell\text{-run} \) \((\ell = 1, 2, \ldots, n)\), denoted by \( \mathbb{E}[s_\ell] \), is strictly less than the original weight \( w_\ell \). We will now consider runs of different lengths in turn, and find the corresponding constraint.

A 1\( \ell\)-run. For a 1\( \ell\)-run, there are only two edges which overlap this run. Choosing either of these edges will produce a 2\( \ell\)-run. Therefore, the 1\( \ell\)-run can be handled by adding the following constraint to the formulation.

\[
\mathbb{E}[s_1] = \frac{1}{n} \left(2w_2 + (n-2)w_1\right) \leq (1 - \delta)w_1,
\]

for small \( \delta > 0 \). Let \( \delta = \omega/n \). Thus we obtain

\[
2w_2 - (2 - \omega)w_1 \leq 0. \tag{2}
\]

An \( \ell\)-run, where \( 2 \leq \ell \leq n - 1 \). There are \( \ell + 1 \) edges which overlap this run. Two of them are outer rim edges, and selecting either for the update causes the run to grow in length by 1. All other \( \ell - 1 \) overlapping edges are in state \(--\). Let us number these edges 1, 2, \ldots, \ell - 1. According to the strategy RP, if the edge \( i \in \{1, 2, \ldots, \ell - 1\} \) is chosen for the play, this edge will become ++ with probability \( p^2 \), producing a \((i-1, \ell - i - 1)\)-split. Similarly, this edge might go to the state -- or -- with a probability of \( p(1 - p) \), resulting in a \((i-1, \ell - i)\)-split or a \((i, \ell - i - 1)\)-split respectively. The edge might also remain in the same state with probability \((1 - p)^2\). Finally, there is a chance of not hitting any of the overlapping edges of the run, leaving the \( \ell\)-run intact. We can now compute the expected new weight of the run after one step of the game, by combining these cases. Hence we have

\[
\mathbb{E}[s_\ell] = \frac{1}{n} \left(2w_{\ell+1} + p^2 \sum_{i=1}^{\ell-1}(w_{i-1} + w_{\ell-1}) + p(1 - p) \sum_{i=1}^{\ell-1}(w_{i-1} + w_{\ell-i}) +
\right.
\]

\[
\left. p(1 - p) \sum_{i=1}^{\ell-1}(w_i + w_{\ell-i-1}) + (1 - p)^2(\ell - 1)w_\ell + \frac{n - (\ell + 1)}{n}w_\ell \right) \leq (1 - \delta)w_\ell.
\]

This inequality can be simplified to

\[
2w_{\ell+1} + p^2 \sum_{i=1}^{\ell-1}(w_{i-1} + w_{\ell-1}) + p(1 - p) \sum_{i=1}^{\ell-1}(w_{i-1} + w_{\ell-i}) +
\]

\[
p(1 - p) \sum_{i=1}^{\ell-1}(w_i + w_{\ell-i-1}) + (1 - p)^2(\ell - 1)w_\ell \leq (\ell + 1 - \omega)w_\ell.
\]
Hence, we have

\[
2w_{\ell+1} + 2p^2 \sum_{i=0}^{\ell-2} w_i + 2p(1-p) \left( \sum_{i=0}^{\ell-2} w_i + \sum_{i=1}^{\ell-1} w_i \right) + (1-p)^2(\ell-1)w_\ell \leq (\ell + 1 - \omega)w_\ell .
\]

Thus,

\[
2w_{\ell+1} + 2p^2 \sum_{i=0}^{\ell-2} w_i + 2p(1-p) \left( 2\sum_{i=0}^{\ell-2} w_i - w_0 + w_{\ell-1} \right) + (1-p)^2(\ell-1)w_\ell \leq (\ell + 1 - \omega)w_\ell .
\]

Let \( w_0 = 0 \). Then, for \( 2 \leq \ell \leq n - 1 \), we have

\[
2w_{\ell+1} + 2p(2-p) \sum_{i=0}^{\ell-2} w_i + 2p(1-p)w_{\ell-1} + (\ell(p^2 - 2p) - (p^2 - 2p + 2) + \omega)w_\ell \leq 0 . \tag{3}
\]

The \( n_4 \)-run. For the \( n_4 \)-run, choosing any edge will cause the run to decrease in length by 2 with the probability \( p^2 \), to decrease in length by 1 with probability \( 2p(1 - p) \), and to remain the same with probability \( (1 - p)^2 \). Thus we obtain

\[
\frac{1}{n}(p^2 w_{n-2} + 2p(1-p)w_{n-1} + (1-p)^2 w_n)n \leq (1-\delta)w_n .
\]

Simplifying this inequality yields

\[
p^2 w_{n-2} + 2p(1-p)w_{n-1} + (p^2 - 2p + \delta)w_n \leq 0 . \tag{4}
\]

Finally, consider the case where two adjacent runs are separated by a singleton plus. Suppose the lengths of these runs are \( (\ell_1 - 1) \) and \( \ell_2 \). If we delete the singleton plus which separates them, a run of length \( \ell_1 + \ell_2 \) is created. Let us count this as two runs of length \( \ell_1 \) and \( \ell_2 \). In other words, we calculate the resulting weight as \( w_{\ell_1} + w_{\ell_2} \) whereas the true weight is \( w_{\ell_1 + \ell_2} \). We need to know that this underestimates the true cost. This can be done by adding the inequalities

\[
w_{\ell_1} + w_{\ell_2} \geq w_{\ell_1 + \ell_2} . \tag{5}
\]

### 2.2 Determining the weights

We now show that we can find appropriate values for the weights \( w_\ell \) satisfying inequalities (2) to (5). This will imply that the expectation of the total weight of the runs in a cycle decreases in expectation after every iteration of the game, leading to a fast (polynomial) convergence rate. We also determine a range of \( p \) favouring fast convergence.

Solving for \( w_{\ell+1} \) in inequality (3) gives the recurrence

\[
\hat{w}_{\ell+1} = -p(2-p) \sum_{i=0}^{\ell-2} \hat{w}_i - p(1-p)\hat{w}_{\ell-1} - \frac{1}{2}(\ell(p^2 - 2p) - (p^2 - 2p + 2) + \delta n)\hat{w}_\ell , \tag{6}
\]

for \( 2 \leq \ell \leq n - 1 \). And, from (2), we have

\[
\hat{w}_2 = (1-\frac{1}{2}\omega)\hat{w}_1 . \tag{7}
\]

Define \( g(\ell) \) by

\[
g(\ell) = \frac{\hat{w}_\ell}{\ell} .
\]
Lemma 4. There exists \( p_0 \leq 0.870 \) such that \( g(\ell) \) has a minimum when \( \ell \geq p_0 \).

Proof. To prove the lemma, we determine polynomial functions of \( p \) satisfying the first few terms of the recurrences \( \ell \), with seeds \( \hat{w}_0 = 0 \) and \( \hat{w}_1 = 1 \). We use these to find inequalities which determine the range of \( p \) such that \( g(\ell) \) has a minimum. We then solve these numerically. In fact, we use the decrease in \( g(\ell) \) at a given \( \ell \), which we denote by \( h(\ell) \). That is,

\[
h(\ell) = g(\ell + 1) - g(\ell) .
\]

Thus, if \( g(\ell) \) has its first local minimum at \( \ell = \ell_0 \), \( h(\ell) \) will be negative for \( \ell = 1, 2, \ldots, \ell_0 - 1 \) and positive at \( \ell = \ell_0 \). For simplicity, we assume that \( \omega = 0 \) in the calculations that follow. Now, solving \( \ell \) and \( \ell \) for \( h(\ell) \), with \( \hat{w}_0 = 0 \) and \( \hat{w}_1 = 1 \), we obtain:

1. \( h(1) = -\frac{1}{2} \). Hence, \( h(1) < 0 \) for all \( 0 \leq p \leq 1 \).
2. \( h(2) = -\frac{1}{6} + \frac{1}{2}p^2 \). Hence, \( h(2) \leq 0 \) for all \( 0 \leq p \leq 1 \).
3. \( h(3) = -\frac{1}{12} - \frac{1}{3}p + \frac{5}{12}p^2 - \frac{3}{4}p^4 + \frac{1}{4}p^3 \). Hence, \( h(3) \leq 0 \) for all \( 0 \leq p \leq 1 \).
4. \( h(4) = -\frac{1}{20} - \frac{1}{4}p^2 - \frac{1}{3}p^2 + \frac{1}{12}p^4 + \frac{1}{6}p^3 - \frac{1}{8}p^5 + \frac{1}{24}p^6 \). Hence, \( h(4) \leq 0 \) for \( 0 \leq p \leq 0.897 \).

Continuing in this way, as \( \ell \) goes from 5 to 10, the range of \( p \) for which \( h(\ell) \leq 0 \) becomes gradually smaller:

- \( h(5) \leq 0 \) for \( 0 \leq p \leq 0.877 \).
Lemma 6. \( h(6) \leq 0 \) for \( 0 \leq p \leq 0.871 \).
\( h(7) \leq 0 \) for \( 0 \leq p \leq 0.870 \).
\( h(8) \leq 0 \) for \( 0 \leq p \leq 0.869 \).

Here the upper bounds for \( p \) are rounded to three decimal places. Note that \( h(8) \) is positive if \( p \geq 0.870 \). Therefore, if \( p \geq 0.870 \), \( h(\ell) \) is negative for \( 1 \leq \ell \leq 7 \) and positive for \( \ell = 8 \). Thus \( g(\ell) \) decreases up to \( \ell = 8 \) and increases at \( \ell = 9 \). Hence, by definition, \( \ell_0 \leq 8 \) when \( p \geq 0.870 \), and the lemma is proved.

Next we prove two properties of the function \( w_\ell \), which will be used later in the proof.

**Lemma 5.** \( w_\ell \) is a non-decreasing sequence.

*Proof.* Recall that \( w_\ell = \hat{w}_\ell \) for \( \ell \leq \ell_0 \). Furthermore, from Lemma 4, we know that \( \ell_0 \leq 8 \) for \( p \geq p_0 \). Hence, we first prove that \( w_\ell \) is increasing up to \( \ell_0 = 8 \), by proving that \( \hat{w}_\ell \) is increasing as \( \ell \) goes from 1 to 8. This is done by solving the recurrences (6) and (7), with seeds \( \hat{w}_0 = 0 \) and \( \hat{w}_1 = 1 \). Here also, for simplicity, we assume that \( \omega = 0 \). Let \( f(\ell) \) be defined by

\[
f(\ell) = \hat{w}_{\ell+1} - \hat{w}_\ell .
\]

Then it suffices to show that \( f(\ell) \geq 0 \) for \( \ell = 0, 1, \ldots, 7 \). But we have

1. \( f(0) = 1 \). Hence, \( f(0) \geq 0 \) for all \( 0 \leq p \leq 1 \).
2. \( f(1) = 0 \). Hence, \( f(1) \geq 0 \) for all \( 0 \leq p \leq 1 \).
3. \( f(2) = \frac{1}{2}p^2 \). Hence, \( f(2) \geq 0 \) for all \( 0 \leq p \leq 1 \).
4. \( f(3) = -p + p^2 + p^3 - \frac{1}{2}p^4 \). Hence, \( f(3) \geq 0 \) for \( 0.689 \leq p \leq 1 \).
5. \( f(4) = -2p - \frac{3}{4}p^2 + \frac{1}{4}p^3 + \frac{5}{4}p^4 + \frac{3}{4}p^5 - 3p^6 \). Hence, \( f(4) \geq 0 \) for \( 0.805 \leq p \leq 1 \).

 Likewise, we obtain

- \( f(5) \geq 0 \) for \( 0.850 \leq p \leq 1 \).
- \( f(6) \geq 0 \) for \( 0.865 \leq p \leq 1 \).
- \( f(7) \geq 0 \) for \( 0.869 \leq p \leq 1 \).

All these together show that \( \hat{w}_\ell \) is increasing in the range \( \ell = 1, 2, \ldots, \ell_0 \) for \( 0.869 \leq p \leq 1 \). The lemma then follows from the definition that \( w_\ell \) is increasing when \( \ell \geq \ell_0 \).

**Lemma 6.** \( \frac{w_\ell}{\ell} \) is a non-increasing sequence.

*Proof.* From the definition of \( \ell_0 \), \( \frac{w_\ell}{\ell} \) is decreasing for \( 1 \leq \ell \leq \ell_0 \). Moreover, \( w_\ell = \hat{w}_\ell \) for \( \ell \leq \ell_0 \). Therefore \( \frac{w_\ell}{\ell} \) also decreases for \( 1 \leq \ell \leq \ell_0 \). When \( \ell > \ell_0 \), we have \( \frac{w_\ell}{\ell} = \alpha \) which is obviously non-increasing.

The following lemmas show that the inequalities (2) to (5) are satisfied by the proposed weights.

**Lemma 7.** The weights \( w_\ell \) defined in (3), with seeds \( \hat{w}_0 = 0 \) and \( \hat{w}_1 = 1 \), satisfy inequalities (2) and (3).

*Proof.* Let \( k \geq \ell_0 \). Then, from (3), we get

\[
w_{k+1} \leq -p(2 - p) \sum_{i=0}^{k-2} w_i - p(1-p)w_{k-1} - \frac{1}{2} (k(p^2 - 2p) - (p^2 - 2p + 2) + \omega)w_k .
\]

Let

\[
c_i = w_i - i\alpha .
\]
Then, substituting (8) and (10) into (9), we obtain
\[
(k + 1)\alpha \leq -p(2 - p) \sum_{i=0}^{k-2} (i\alpha + c_i) - p(1 - p)((k - 1)\alpha + c_{k-1})
- \frac{1}{2}(k(p^2 - 2p) - (p^2 - 2p + 2) + \omega)k\alpha.
\]
Simplifying yields
\[
k \geq \frac{\alpha + \alpha p + p(2 - p) \sum_{i=0}^{k-2} c_i + p(1 - p)c_{k-1}}{\alpha(p - \frac{1}{\omega})}.
\]
If \( k = \ell_1 > \ell_0 \), we get
\[
\ell_1 \geq \frac{\alpha + \alpha p + p(2 - p) \sum_{i=0}^{\ell_1 - 2} c_i + p(1 - p)c_{\ell_1-1}}{\alpha(p - \frac{1}{\omega})}.
\]
But, from (8) and (10), we know that \( c_k = 0 \) for \( k \geq \ell_0 \). Since \( \ell_1 > \ell_0 \), we have \( c_{\ell_1-1} = 0 \) and \( \sum_{i=0}^{\ell_1 - 2} c_i = \sum_{i=0}^{\ell_0 - 1} c_i \). Hence
\[
\ell_1 \geq \frac{\alpha + \alpha p + p(2 - p) \sum_{i=0}^{\ell_0 - 1} c_i}{\alpha(p - \frac{1}{\omega})} = \ell_1^* \text{ (say)}.
\]
Thus, inequality (3) holds for \( \ell \geq \ell_1^* \). We also know that (3) holds when \( \ell \leq \ell_0 \) by (8). Therefore, showing \( \ell_1^* - \ell_0 \leq 1 \) will mean that (3) is true for all values of \( \ell \). To do this, we determine a lower bound on \( \ell_0 \) by substituting \( k = \ell_0 \) into (11). Thus
\[
\ell_0 \geq \frac{\alpha + \alpha p + p(2 - p) \sum_{i=0}^{\ell_0 - 2} c_i + p(1 - p)c_{\ell_0-1}}{\alpha(p - \frac{1}{\omega})} = \ell_0^* \text{ (say)}.
\]
Hence we have
\[
\ell_1^* - \ell_0 \leq \ell_1^* - \ell_0 = \frac{\alpha p(2 - p)c_{\ell_0-1} - p(1 - p)c_{\ell_0-1}}{\alpha(p - \frac{1}{\omega})}
\approx \frac{c_{\ell_0-1}}{\alpha}, \quad \text{since } \frac{\alpha}{p} \to 1 \text{ when } \omega \to 0.
\]
Substituting (10) into the above inequality yields
\[
\ell_1^* - \ell_0 \leq \frac{w_{\ell_0-1} - \alpha(\ell_0 - 1)}{\alpha}
= 1 - \frac{w_{\ell_0} - w_{\ell_0-1}}{\alpha}
\leq 1, \quad \text{since } w_{\ell_0} > w_{\ell_0-1} \text{ from Lemma 5}
\]
Finally, it can easily be verified that (2) holds with \( w_1 = \hat{w}_1 = 1 \) and \( w_2 = \hat{w}_2 = 1 - \omega/2 \), completing the proof. \( \square \)

Lemma 8. The weights \( w \) defined in (8), with seeds \( \hat{w}_0 = 0 \) and \( \hat{w}_1 = 1 \), satisfy inequality (4).

Proof. Assume \( \ell_0 \leq 8 \) and \( n \geq 10 \), so \( w_\ell = \ell\alpha \) for \( \ell = n - 2, n - 1, n \). Then we require
\[
p^2(n - 2)\alpha + 2p(1 - p)(n - 1)\alpha + \left( p^2 - 2p + \frac{\omega}{n} \right) n\alpha \leq 0,
\]
which simplifies to
\[
\omega \leq 2p,
\]
so is satisfied for small enough \( \omega \). \( \square \)
Lemma 9. The weights $w_\ell$ defined in (5), with seeds $\hat{w}_0 = 0$ and $\hat{w}_1 = 1$, satisfy inequality (5).

Proof. $w_{\ell_1} + w_{\ell_2} \geq \ell_1 \frac{w_{\ell_1}}{\ell_1} + \ell_2 \frac{w_{\ell_2}}{\ell_2}$.

But, from Lemma 3 $w_\ell$ is a decreasing sequence. Thus,

$$w_{\ell_1} + w_{\ell_2} \geq \ell_1 \frac{w_{\ell_1 + \ell_2}}{\ell_1 + \ell_2} + \ell_2 \frac{w_{\ell_1 + \ell_2}}{\ell_1 + \ell_2} = w_{\ell_1 + \ell_2},$$

proving the lemma. □

Proof of Theorem 1. Consider an $\ell_d$-run where $1 \leq \ell \leq n$. Denote by $E[s_\ell]$ the expected weight of the resulting runs. Then, we have

$$E[s_\ell] \leq (1 - \delta)w_\ell,$$

for $p \geq p_0$, since the weights (5) satisfy the constraints (2) to (5) by Lemma 7, Lemma 8, and Lemma 9. Furthermore, Lemma 4 showed that the weights in RP can be defined this way for $p \geq 0.870$.

Now, suppose the initial state of the cycle is $S_0$. Let $S_\ell$ be the resulting state of the cycle after $t$ steps. Then total weight after one step of the game is therefore

$$E[\phi(S_1) \mid S_0] = \sum_\ell E[s_\ell] r_\ell$$

$$\leq \sum_\ell (1 - \delta)w_\ell r_\ell$$

$$= (1 - \delta)\phi(S_0).$$

Thus, by total expectation, we have

$$E[\phi(S_1)] \leq (1 - \delta)E[\phi(S_0)].$$

We have $\phi(S) \leq n$. To see this, note that for an $\ell_d$-run, $\hat{w}_1 = 1 \geq w_\ell/\ell = w_\ell/\ell$ when $1 \leq \ell \leq \ell_0$. Thus $\alpha \ell \leq w_\ell \leq \ell$ when $1 \leq \ell \leq \ell_0$. In particular, this implies $\alpha \leq 1$. When $\ell \geq \ell_0$, we have $w_\ell = \alpha \ell \leq \ell$, implying $w_\ell \leq \ell$ for all $1 \leq \ell \leq n$. Summing this over all runs in $S$, we have $\phi(S) \leq n$. Since $\delta = \omega/n$, we have

$$E[\phi(S_1)] \leq \left(1 - \frac{\omega}{n}\right)E[\phi(S_0)].$$

Applying this for $t$ steps, we obtain

$$E[\phi(S_t)] \leq \left(1 - \frac{\omega}{n}\right)^t E[\phi(S_0)] \leq \left(1 - \frac{\omega}{n}\right)^t \leq e^{-\omega t} n \leq \epsilon,$$

when

$$t > \frac{n}{\omega \log \left(\frac{n}{\epsilon}\right)}.$$ 

We also know that, for any $S \neq S^*$, $\phi(S) \geq w_1 \geq 1$ by Lemma 3. Thus, using Markov’s inequality, we obtain

$$\Pr[\phi(S_t) \neq 0] = \Pr[\phi(S_t) \geq 1] \leq E[\phi(S_t)] \leq \epsilon,$$

and the theorem is proved. □

Remark 2. The above requires satisfying (2) to (5). These are all linear inequalities. Therefore, we can solve them by linear programming. Initially, we solved the problem this way, obtaining the same results as above.

Remark 3. The problem for SRP can be formulated and solved in the same way as described in Section 2 for RP. We did this and found that the convergence to cooperation is fast when $p \geq 0.699$. So, the range of $p$ for which the convergence is fast is bigger for SRP than for RP. This is somewhat expected because, for a given $p$, SRP is more forgiving than RP.
3 Slow convergence on the cycle

In Section 2, we proved that the IPD game converges to cooperation fast for high values of $p$. It raises an interesting research question: how fast or slow is the convergence when $p$ is small? In this section, we answer this question by proving Theorem 2 which shows that the convergence to cooperation takes time exponential in $n$ for small enough $p$. The idea of the proof is to show that it takes exponential time for a plus-run of length $\Omega(n)$ to be formed. (This is done by analysing plus-runs on the cycle. Therefore, in this section, a run refers to a run of “pluses” unless otherwise.) It obviously follows that it takes exponential time for the all-cooperate state to be reached.

3.1 Problem formulation

Let $R_i^\ell(t)$ denote the event that a run of $\ell$ pluses (an $\ell$-run) starts at position $i$ at time $t$, i.e. $S_k = 1$ for $i \leq k \leq i + \ell - 1$ and $S_{i-1} = S_{i+\ell} = -1$. By the symmetry of the cycle, and the initial configuration, $\Pr(R_i^\ell)$ will be the same for all $i$. Let $\delta_j(t)$ denote the event that $S_j$ is a minus at time $t$. i.e. $\delta_j(t) = \{S_j = -1\}$. We will write $R_i^\ell(t)$ and $\delta_j(t)$ simply as $R_i^\ell$ and $\delta_j$, respectively, to ease the notation. Then we will define $P_i^\ell (\ell = 0, 1, \ldots, n-1)$ to be

$$P_i^\ell = \Pr(R_i^\ell \mid \delta_{i-1}) \quad (i = 0, 1, \ldots, n).$$

The conditioning on $\delta_{i-1}$ means that the probability $P_i^\ell$ is an upper bound on $\Pr(R_i^\ell)$. This follows since, if $S_{i-1} = +1$, a plus-run cannot start at $i$. Recall that $\ell = 0$ means the length of the plus-run is 0. Hence, in particular, $P_i^0$ is an upper bound on the probability that there are minuses at positions $i - 1$ and $i$. An advantage of this approximation is that the $P_i^\ell$ are a probability distribution for $\ell = 1, 2, \ldots, n$, whereas the quantities $\Pr(R_i^\ell)$ do not sum to 1 in general.

Later in the proof, we will need to calculate an upper bound on the probability that two plus runs are separated by two minuses. That is, we need to calculate an upper bound on the joint probability $\Pr(R_i^\ell \land R_m^j)$ where $i = j + m + 2$. But, we have

$$\Pr(R_i^\ell \land R_m^j) = \Pr(R_i^\ell \mid R_m^j) \Pr(R_m^j) \ldots \text{(13)}$$

We will use the fact that, conditional on $\delta_r$, the $S_q$ for $q > r$ and the $S_k$ for $k < r$ are independent, if the vertices $k$ and $q$ belong to different plus-runs and there is at least one more plus-run on the cycle. Under this condition, changes to the $S_q$ occur independently from those to the $S_k$, since all steps are independent and affect only two adjacent vertices. The structure of the cycle means that changes to the $S_q$ can only be percolated to the $S_k$ through the vertex $i$, on which we have conditioned. Thus, given $\delta_{i-1}$, the $R_i^\ell$ is conditionally independent of the $R_m^j$, provided there is at least another plus-run on the cycle. The assumption of having at least three plus-runs holds initially because the game is started with all-minuses, which means there are $n$ 0-runs of pluses. Moreover, we will then show that it takes exponential time for a plus-run of length $n/4$ to be formed. To summarise, we may assume

$$\Pr(R_i^\ell \mid R_m^j) = \Pr(R_i^\ell \mid \delta_{i-1}) \ldots \text{(14)}$$

Therefore, from (13) and (14), we have

$$\Pr(R_i^\ell \land R_m^j) = \Pr(R_i^\ell \mid \delta_{i-1}) \Pr(R_m^j) \leq \Pr(R_i^\ell \mid \delta_{i-1}) \Pr(R_m^j \mid \delta_{j-1}) = P_i^\ell P_m^j \ldots \text{Note that this inequality and the argument are also applicable when one or both runs are of length 0 and separated by one minus, i.e. for } \Pr(R_i^\ell \land R_{m+1}^j) \text{ and } \Pr(R_i^\ell \land R_{n+2}^j). \text{ We use this below without referring further to the details.}$$

We do not explicitly determine a $p_0$ for which slow convergence occurs. Though this is possible in principle with our methods, the simpler approach we have chosen already leads to very cumbersome calculations. Our approach, therefore, is to regard $p$ as small, and use the $O$
and $o$ notation to indicate the order of approximations. Thus there will be some small enough constant $p_1$, for which our results hold, but we cannot estimate it. In order that the $O$ etc. notation can be applied to both $p$ and $n$ without confusion, we will assume that $n > e^{1/p}$.

We will first consider short runs. For simplicity, we will leave the investigation of a $0_c$-run to the end of this section, and start with $1_c$-runs.

A $1_c$-run. Let $R_c$ be a $1_c$-run at position $i$, i.e. $R_c = [i, i]$. Choosing either of its outer rim edges causes $R_c$ to be deleted. On the other hand, $R_c$ is created from a $2_c$-run at position $i - 1$ if the edge $\{i - 2, i - 1\}$ is selected and from a $2_c$-run at position $i$ if the edge $\{i + 1, i + 2\}$ is selected. In addition, $R_c$ is created from three consecutive minuses at positions $(i - 1), i,$ and $(i + 1)$ with probability $p(1 - p)$ if either $\{i - 1, i\}$ or $\{i, i + 1\}$ is selected. The probability of finding three consecutive minuses is at most $P^t_0$. Combining all this information, we obtain

$$P^{t+1}_1 = P^t_1 + \frac{1}{n} \left(-2P^t_1 + 2P^t_2 + 2p(1-p)P^t_0\right). \tag{15}$$

Note that the coefficient of $P^t_1$ on the right hand side of (15) is positive if $n \geq 2$ and other two variables $P^t_0$ and $P^t_2$ also have positive coefficients. Hence, using the upper bounds of these three variables yield an upper bound for $P^{t+1}_1$ as required. Also, as we only need an upper bound, we have ignored the cases where both $i - 2$ and $i - 1$ are minuses. In that case, choosing the edge $\{i - 2, i - 1\}$ causes the $1_c$-run to increase in length by 2 with probability $p^2$ and by 1 with probability $p(1 - p)$, effectively deleting $R_c$. We will perform similar approximations for the other runs investigated below, without mentioning these details further.

The equation (15) is a difference equation with time step 1. Let us rescale so that the new time step is $1/n$. The difference equation corresponding to the new step size is then as follows.

$$\frac{P^{t+\frac{1}{n}}_1}{\frac{1}{n}} = P^t_1 + \frac{1}{n} \left(-2P^t_1 + 2P^t_2 + 2p(1-p)P^t_0\right). \tag{16}$$

Let $\tau = \frac{t}{n}$ and $h = \frac{1}{n}$. Then the equation (16) can be written as

$$\frac{P^{\tau+\frac{h}{n}}_1 - P^\tau_1}{\frac{h}{n}} = -2P^\tau_1 + 2P^\tau_2 + 2p(1-p)P^\tau_0. \tag{17}$$

Now, the difference equation (17) can be approximated by the following differential equation, with error up to $O(h) = O(1/n) = O(e^{-1/p})$, say, on the right hand side.

$$\frac{dP^\tau_1}{d\tau} = -2P^\tau_1 + 2P^\tau_2 + 2p(1-p)P^\tau_0. \tag{18}$$

A $2_c$-run. Let $R_c$ be a $2_c$-run starting at position $i$, hence $R_c = [i, i + 1]$. Similarly to a $1_c$-run, choosing either of the two outer rim edges of $R_c$ causes the run to decrease in length by 1, reducing the number of $2_c$-runs on the cycle by 1. $R_c$ is created by choosing the outer rim edge $\{i - 2, i - 1\}$ of a $3_c$-run at $(i - 1)$. Similarly, $R_c$ is created by choosing the outer rim edge $\{i + 2, i + 3\}$ of a $3_c$-run at $i$. In addition, $R_c$ can be created from a singleton plus adjacent to a pair of minuses. This happens if the edge connecting the pair of minuses is selected and only the minus next to the singleton plus becomes a plus. The probability for this, given that the corresponding edge has been selected, is $p(1 - p)$. Finally, $R_c$ is created with probability $p^2$ by selecting the middle edge of four consecutive minuses. The probability of having four consecutive minuses at location $i$ is at most $P^t_0^{2}$. Therefore we get

$$P^{t+1}_2 = P^t_2 + \frac{1}{n} \left(-2P^t_2 + 2P^t_3 + 2p(1-p)P^t_0 P^t_1 + p^2 P^t_0^{2}\right). \tag{19}$$

As before, rescaling and approximating, we obtain

$$\frac{dP^\tau_2}{d\tau} = -2P^\tau_2 + 2P^\tau_3 + 2p(1-p)P^\tau_0 P^\tau_1 + p^2 P^\tau_0^{2}. \tag{20}$$
An $\ell_c$-run, where $\ell \geq 3$. Suppose $R_c = [i, j]$ is an $\ell_c$-run for some $\ell \geq 3$. Selecting either of the two outer rim edges causes $R_c$ to decrease in length by 1. On the other hand, an $(\ell + 1)_c$-run starting at position $(i - 1)$ is turned into an $\ell_c$-run starting at position $i$ if the edge $(i - 2, i - 1)$ is chosen; and, an $(\ell + 1)_c$-run starting at position $i$ becomes an $\ell_c$-run starting at the same position $i$ if the edge $(j + 1, j + 2)$ is chosen. Also, if there is a 0-run at $i$ and an $\ell_c$-run at $i + 1$, choosing the edge $(i - 1, i)$ will create an $\ell_c$-run starting at location $i$ with probability $p(1 - p)$. We will get the same result if these two runs are in the reverse order: $(\ell - 1)_c$-run at $i$ and a 0-run at $j$. If there is an $(\ell - 2)_c$-run starting at $(i + 2)$ and there are minuses at $i - 1$, $i$, and $i + 1$, then choosing the edge $(i, i + 1)$ produces an $\ell$-run at $i$ with probability $p^3$. Similarly if there is an $(\ell - 2)_c$-run at position $i$ and there are minuses at positions $j - 1$, $j$, and $j + 1$, the run increases in length by 2 with probability $p^2$ if the edge $(j - 1, j)$ is selected. Finally a $k_c$-run and an $(\ell - 2 - k)_c$-run, $1 \leq k \leq \ell - 3$, at positions $i$ and $(i + k + 2)$ respectively merge with probability $p^2$, introducing an $\ell_c$-run, if the edge between the runs, namely $(i + k, i + k + 1)$, is selected. Thus we have

$$P^{t+1}_\ell = P^t_\ell + \frac{1}{n}\left(-2P^t_\ell + 2P^t_{\ell+1} + 2p(1-p)P^{t+1}_{\ell-1}P^t_0 + 2p^2P^{t}_{\ell-2}P^t_0 + p^2\sum_{k=1}^{\ell-3}P^t_kP^{t+1}_{\ell-2-k}\right).$$

This could be written as

$$P^{t+1}_\ell = P^t_\ell + \frac{1}{n}\left(-2P^t_\ell + 2P^t_{\ell+1} + 2p(1-p)P^{t+1}_{\ell-1}P^t_0 + p^2\sum_{k=0}^{\ell-2}P^t_kP^{t+1}_{\ell-2-k}\right).$$

(21)

Here also, we have used the fact that the probability of finding three consecutive minuses is at most $P^t_0$. Observe that, in this form, the difference equation (19) is equivalent to (21) when $\ell = 2$. Therefore, we can use (21) for $\ell = 2$ also.

A 0$_c$-run. Finally, consider a run of length zero, i.e. a 0$_c$-run. Recall that we have defined $P^t_0$ and $P^t_1$ to be upper bounds on the probability of finding a 0$_c$-run and 1$_c$-run respectively, at position $i$ at time $t$. Now let $\bar{P}^t_0$ and $\bar{P}^t_1$ denote the exact values of these probabilities respectively, i.e. $\bar{P}^t_0 = \text{Pr}(R^t_0)$ and $\bar{P}^t_1 = \text{Pr}(R^t_1)$. We can now examine the dynamics of a 0$_c$-run. A 0$_c$-run at position $i$ means there are minuses at positions $(i - 1)$ and $i$. Then, $i + 1$ can be a minus or a plus. It is not difficult to verify that, if it is a minus then the 0$_c$-run might be deleted with probability $(3p - p^2)/n$, and if it is a plus, the 0$_c$-run might be deleted with probability $(2p - p^2)/n$. Also note that probability of finding each of these configurations is at most $\bar{P}^t_0$. On the creation side, a 0$_c$-run at $i$ could be created from a 1$_c$-run at $i$ with probability $2/n$ and from any longer plus-runs at position $i$ with probability $1/n$. By definition, the probability of finding a 1$_c$-run at $i$ is $\bar{P}^t_1$. It then follows that the probability of finding a plus-run of length greater than 2 at position $i$ is $(1 - \bar{P}^t_0 - \bar{P}^t_1)$. Hence we obtain

$$P^{t+1}_0 = \bar{P}^t_0 + \frac{1}{n}\left(-\bar{P}^t_0(3p - p^2) - \bar{P}^t_1(2p - p^2) + 2\bar{P}^t_1(1 - \bar{P}^t_0 - \bar{P}^t_1)\right).$$

Thus

$$P^{t+1}_0 = \bar{P}^t_0\left(1 - \frac{1 + 5p - 2p^2}{n}\right) + \frac{1}{n}(1 + \bar{P}^t_1).$$

(22)

Note that $P^{t+1}_0$ in (22) is an upper bound. Furthermore, the coefficient of $\bar{P}^t_0$ is positive when $n \geq 1 + 5p - 2p^2$, and the coefficient of $\bar{P}^t_1$ is also positive. We can therefore replace $P^t_0$ and $P^t_1$ with their upper bounds $\bar{P}^t_0$ and $\bar{P}^t_1$ respectively, obtaining

$$P^{t+1}_0 = \bar{P}^t_0 + \frac{1}{n}\left(-(1 + 5p - 2p^2)\bar{P}^t_0 + \bar{P}^t_1 + 1\right).$$

(23)

Hence we get

$$\frac{d\bar{P}^t_0}{dt} = -(1 + 5p - 2p^2)\bar{P}^t_0 + \bar{P}^t_1 + 1.$$
3.2 The analysis

In the previous section we modelled the game dynamics by a set of differential equations. We first solve the ones corresponding to the runs of length shorter than 3.

**Lemma 10.** If the game is started in the all-minuses configuration, the solution to the system of differential equations (15), (20) and (24) is given by

\[
\begin{bmatrix}
P^*_0 \\
P^*_1 \\
P^*_2 \\
\end{bmatrix} = \begin{bmatrix}
1 + (-4 + 3e^{-\tau} + e^{-2\tau})p + (\frac{7}{2} + 5e^{-2\tau} - 9e^{-\tau})p + 2e^{-2\tau} \tau^2 - 31e^{-\tau} + \frac{32}{5}e^{-2\tau} \\
(1 - e^{-2\tau})p + (-\frac{3}{2} + 6e^{-\tau} - \frac{3}{2}e^{-2\tau} - e^{-2\tau} \tau - 2e^{-2\tau} \tau^2)p^2 + o(p^2) \\
(\frac{3}{2} - \frac{3}{2}e^{-2\tau} - 2e^{-2\tau} \tau)p^3 + o(p^2)
\end{bmatrix}.
\]

*Proof.* Note that the differential equations (18) and (24) are linear, while (20) is nonlinear. Fortunately, we can approximately linearise (20) using some knowledge of the system.

We approximate the solutions with error terms \(o(p^2)\). Then, assuming \(P^*_0 = 1 + O(p)\), \(P^*_1 = O(p)\) and \(P^*_3 = o(p^2)\) linearises (20). The linearised version is given by

\[
\frac{dP^*_2}{d\tau} = 2p(1 - p)P^*_1 - 2P^*_2 + p^2 + o(p^2).
\]

Hence, for short runs, we have the following nonhomogeneous linear system of first order differential equations.

\[
\begin{align*}
\frac{dP^*_0}{d\tau} &= -(1 + 5p - 2p^2)P^*_0 + P^*_1 + 1, \\
\frac{dP^*_1}{d\tau} &= 2p(1 - p)P^*_0 - 2P^*_1 + 2P^*_2, \\
\frac{dP^*_2}{d\tau} &= 2p(1 - p)P^*_1 - 2P^*_2 + p^2 + o(p^2).
\end{align*}
\]

In matrix form, the system can be written as

\[
\frac{d}{d\tau} \begin{bmatrix} P^*_0 \\ P^*_1 \\ P^*_2 \end{bmatrix} = \begin{bmatrix}
-1 + 5p - 2p^2 & 1 & 0 \\
2p(1 - p) & -2 & 2 \\
0 & 2p(1 - p) & -2
\end{bmatrix} \begin{bmatrix} P^*_0 \\ P^*_1 \\ P^*_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} p^2 + o(p^2) \end{bmatrix}.
\]

Let us denote this system by

\[
P' = AP + F.
\]

(25)

Since the game is started with the all-minuses configuration, we have the initial condition

\[
P(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
\]

Thus we have an initial value problem which we solve by the method of decoupling. We first find the eigenvalues and eigenvectors of \(A\). The characteristic polynomial of \(A\) is

\[
\lambda^3 + (5 + 5p - 2p^2)\lambda^2 + (8 + 14p - 2p^2)\lambda + 4 + 12p - 20p^2 + 28p^3 - 8p^4.
\]

An analysis of this cubic polynomial shows that all three roots are real, different, and negative for small \(p\). The eigenvalues of \(A\) are

\[
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{bmatrix} = \begin{bmatrix}
-1 - 3p + 14p^2 + o(p^2) \\
-2 - 2\sqrt{p} - p + \frac{11}{4}p^{3/2} - 6p^2 + o(p^2) \\
-2 + 2\sqrt{p} - p - \frac{11}{4}p^{3/2} - 6p^2 + o(p^2)
\end{bmatrix}.
\]

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Now, the eigenvector of $A$ corresponding to eigenvalue $\lambda_1$ is
\[
e_1 = \begin{bmatrix} \frac{1}{2}p^{-2} - 2p^{-1} + 6 + O(p) \\ \frac{1}{2}p^{-1} - 1 + 6p + O(p^2) \\ 1 \end{bmatrix} = \frac{1}{4p^2} \begin{bmatrix} 1 - 8p + 24p^2 + O(p^3) \\ 2p - 4p^2 + O(p^3) \\ 4p^2 \end{bmatrix}.
\]

The eigenvector corresponding to eigenvalue $\lambda_2$ is
\[
e_2 = \begin{bmatrix} p^{-1/2} - \frac{3}{2} + \frac{53}{8}p^{1/2} - 13p + \frac{4167}{128}p^{3/2} - \frac{101}{2}p^2 + \frac{40525}{1024}p^{5/2} + O(p^3) \\ -p^{-1/2} - \frac{1}{2} + \frac{3}{8}p^{1/2} - \frac{7}{2}p + \frac{1001}{128}p^{3/2} - \frac{43}{2}p^2 + \frac{45627}{1024}p^{5/2} + O(p^3) \\ \frac{1}{\sqrt{p}} \left(1 + \frac{1}{2}\sqrt{p} + \frac{53}{8}p - 13p^{3/2} + \frac{4167}{128}p^2 - \frac{101}{2}p^5/2 + O(p^3)\right) \end{bmatrix} \cdot \frac{1}{\sqrt{p}} \left(-1 - \frac{1}{2}\sqrt{p} - \frac{53}{8}p - 13p^{3/2} - \frac{4167}{128}p^2 - \frac{101}{2}p^5/2 + O(p^3)\right) \sqrt{p}.
\]

Finally, the eigenvector corresponding to eigenvalue $\lambda_3$ is
\[
e_3 = \begin{bmatrix} -\frac{1}{\sqrt{p}} - \frac{1}{2} - \frac{53}{8}\sqrt{p} - 13p - \frac{4167}{128}p^{3/2} - \frac{101}{2}p^2 - \frac{40525}{1024}p^{5/2} + O(p^3) \\ -1 - \frac{1}{2}\sqrt{p} - \frac{53}{8}p - 13p^{3/2} - \frac{4167}{128}p^2 - \frac{101}{2}p^5/2 + O(p^3) \end{bmatrix} \cdot \frac{1}{\sqrt{p}} \left(-1 - \frac{1}{2}\sqrt{p} - \frac{53}{8}p - 13p^{3/2} - \frac{4167}{128}p^2 - \frac{101}{2}p^5/2 + O(p^3)\right) \sqrt{p}.
\]

We now form the matrix $T$ whose columns are constant multiples of the eigenvectors of $A$. That is
\[
T = \begin{bmatrix} 4p^2e_1 & \sqrt{p}e_2 & \sqrt{p}e_3 \end{bmatrix}.
\]

Since all three eigenvalues are different, the eigenvectors $e_1, e_2,$ and $e_3$ are linearly independent. Hence the matrix $T$ is non-singular and $T^{-1}$ exists. Let us calculate the determinant of $T$ to confirm that the approximated $T$ is non-singular.
\[
\det T = -2\sqrt{p} + \frac{51}{4}p^{3/2} - \frac{4663}{64}p^{5/2} \neq 0.
\]

Now, we calculate the inverse of the matrix $T$. Since the determinant of $T$ is $O(p^{1/2})$ and $T$ is accurate up to $O(p^{5/2})$, $T^{-1}$ will be correct up to $O(p^2)$.
\[
T^{-1} = \begin{bmatrix} 1 + 6p - 6p^2 + O(p^2) \\ 1 + 13p + 79p^2 + O(p^2) \\ 2 + 32p + 226p^2 + O(p^2) \end{bmatrix} \begin{bmatrix} 1 + 6p - 6p^2 + O(p^2) \\ 1 + 13p + 79p^2 + O(p^2) \\ 2 + 32p + 226p^2 + O(p^2) \end{bmatrix}^{-1} \begin{bmatrix} 1 + 6p - 6p^2 + O(p^2) \\ 1 + 13p + 79p^2 + O(p^2) \\ 2 + 32p + 226p^2 + O(p^2) \end{bmatrix}.
\]

Ignoring the error terms, we can verify that $T^{-1}AT$ is a diagonal matrix whose diagonal elements are the eigenvalues of $A$, converging with the theory. That is
\[
T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.
\]

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Let $P = TY$. Then we have a new system of differential equations given by

$$Y' = DY + G,$$

with initial condition $Y(0) = T^{-1}P(0)$ where $D = T^{-1}AT$ and $G = T^{-1}F$. Hence

$$Y(0) = T^{-1}P(0) = \begin{bmatrix} 1 + 6p - 6p^2 + o(p^2) \\ p - 2p^{3/2} + \frac{43}{3}p^2 + o(p^2) \\ -p - 2p^{3/2} - \frac{41}{8}p^2 + o(p^2) \end{bmatrix}. $$

We also know that

$$F = \begin{bmatrix} 1 \\ 0 \\ p^2 + o(p^2) \end{bmatrix}. $$

Thus

$$G = T^{-1}F = \begin{bmatrix} 1 + 6p - 4p^2 + o(p^2) \\ p - \frac{3}{2}p^{3/2} + \frac{41}{8}p^2 + o(p^2) \\ -p - \frac{3}{2}p^{3/2} - \frac{41}{8}p^2 + o(p^2) \end{bmatrix}. $$

Now, solving the three decoupled differential equations (20) yields

$$Y = \begin{bmatrix} \frac{1}{3} + (3e^{-\tau})p + (1 - 7e^{-\tau} + 9e^{-2\tau})p^2 + o(p^2) \\ (\frac{1}{2} - \frac{1}{2}e^{-2\tau})p + (1 - e^{-2\tau} - \frac{1}{4}e^{-2\tau})p^{3/2} + (e^{-2\tau} - e^{-2\tau} + \frac{1}{4}e^{-2\tau} + \frac{1}{3}e^{-2\tau} + \frac{1}{3}e^{-2\tau})p^2 + o(p^2) \end{bmatrix}. $$

Finally, the solution for (25) can be computed using $P = TY$. What is remaining to be shown is that the three assumptions used in the proof are valid. They are: $P_3^\tau = o(p^2)$, $P_1^\tau = O(p)$ and $P_0^\tau = 1 - O(p)$. The assumption on $P_3^\tau$ is validated in Lemma 12. Let us consider the other two here. The final solution confirms that our assumptions are valid at any time $\tau$ if they were valid initially. Clearly the assumptions hold initially as, at time $\tau = 0$, we have $P_0^0 = 1$ and $P_1^0 = 0$. Hence, the final solution holds for any $\tau$ and the proof is complete. \hfill \Box

We will use generating functions to solve the recurrence (21). Let the function $F(x,t)$ be defined by

$$F(x,t) = \sum_{\ell=0}^{\infty} P_{\ell}\,x^\ell. $$

Now, multiplying (21) by $x^{\ell+1}$ and summing over all $\ell \geq 2$, we obtain

$$\sum_{\ell=2}^{\infty} P_{\ell+1} x^{\ell+1} = \sum_{\ell=2}^{\infty} P_{\ell} x^{\ell+1} + \frac{1}{n} \left( -2 \sum_{\ell=2}^{\infty} P_{\ell} x^{\ell+1} + 2 \sum_{\ell=2}^{\infty} P_{\ell+1} x^{\ell+1} \\ + 2p(1-p) \sum_{\ell=2}^{\infty} P_{\ell-1} P_{0} x^{\ell+1} + p^2 \sum_{\ell=2}^{\infty} x^{\ell+1} \sum_{k=0}^{\ell/2} P_k x^{\ell-k} \right). $$

The indices of (27) can be adjusted to get

$$x \sum_{i=2}^{\infty} P_{i} x^i = x \sum_{i=2}^{\infty} P_{i} x^i + \frac{1}{n} \left( -2x \sum_{i=2}^{\infty} P_{i} x^i + 2 \sum_{i=3}^{\infty} P_{i} x^i \\ + 2p(1-p) x^2 \sum_{i=2}^{\infty} P_{i} x^i + p^2 \sum_{i=0}^{\infty} x^{i+3} \sum_{k=0}^{i/2} P_k x^{i-k} \right). $$

(28)
Note that the last term in (28) can be thought of as relating the sequence \(P_i^t\) to its own convolution, thus can be replaced by their product, obtaining

\[ x \sum_{i=2}^{\infty} \sum_{i=2}^{\infty} P_i^t x^i + \frac{1}{n} \left( -2x \sum_{i=2}^{\infty} P_i^t x^i + 2 \sum_{i=3}^{\infty} P_i^t x^i \right) + 2p(1-p)x^2 P_0^t \sum_{i=1}^{\infty} P_i^t x^i + p^2x^3 \left( \sum_{n=0}^{\infty} P_n^t x^n \right) \left( \sum_{n=0}^{\infty} P_n^t x^n \right) . \]

Hence

\[ x(F(x, t+1) - P_t^x - P_1^t x) = x \left( F(x, t) - P_0^t - P_1^t x \right) + \frac{1}{n} \left( -2x \left( F(x, t) - P_0^t - P_1^t x \right) + 2(F(x, t) - P_0^t - P_1^t x - P_2^t x^2) + 2p(1-p)x^2 P_0^t (F(x, t) - P_0^t) + p^2x^3 F(x, t)^2 \right) . \]

This can be rearranged to get

\[ x \frac{F(x, t+1) - F(x, t)}{n} - x \frac{(P_t^{x+1} - P_t^x) - x^2 (P_1^{x+1} - P_1^x)}{n} = -2x \left( F(x, t) - P_0^t - P_1^t x \right) \]

\[ + 2(F(x, t) - P_0^t - P_1^t x - P_2^t x^2) + 2p(1-p)x^2 P_0^t (F(x, t) - P_0^t) + p^2x^3 F(x, t)^2 . \]

Substituting (15) and (23) into the above equation yields

\[ x \frac{F(x, t+1) - F(x, t)}{n} = p^2x^3 F(x, t)^2 + 2F(x, t) (1 - x + x^2 P_0 t (1-p)) \]

\[ - 2x^2 P_0^2 (1-p) + P_0 t (1-5p+2p^2) + 2x^2 p (1-p) - 2 - P_1^t x + x . \]

Now, let \( y(\tau) = F(x, t) \) where \( \tau = \frac{t}{n} \) as defined before. Then, approximating and rescaling, we get

\[ x \frac{dy(\tau)}{d\tau} = p^2 x^3 y(\tau)^2 + 2p (1 - x + x^2 P_0^t p (1-p)) - 2x^2 P_0^2 p (1-p) \]

\[ + P_0^t (x(1-5p+2p^2) + 2x^2 p (1-p) - 2) - P_1^t x + x . \]

The following lemma proves that \( y(\tau) \) has a radius of convergence greater than 1.

**Lemma 11.** The generating function \( y(\tau) \) is bounded above and converges for some \( x > 1 \).

**Proof.** Without loss of generality, let \( x = 1 + p^3 \). Substituting this value into the differential equation (30) gives

\[ \frac{dy(\tau)}{d\tau} = p^2 y(\tau)^2 + 2 y(\tau) P_0^t p (1-p) - 2 P_0^2 p (1-p) \cdot P_0^t (1+3p) - P_1^t + 1 + o(p^2) . \quad (31) \]

Differential Equation (31) is nonlinear. But, we can linearise this by assuming \( y(\tau) = 1 + O(p) \). This assumption will be validated later. Under this assumption, the nonlinear term

\[ p^2 y(\tau)^2 = p^2 + o(p^2) . \]

Then, substituting the solutions for \( P_0^t \) and \( P_1^t \) from Lemma 10 into (31) and simplifying, we get the following linear differential equation.

\[ \frac{dy(\tau)}{d\tau} - (2p - (10 - 2e^{-2\tau} - 6e^{-\tau})p^2) y(\tau) = (-2 - 3e^{-\tau}) p \]

\[ + (16 - 17e^{-2\tau} + 4e^{-\tau} + 9e^{-\tau} - 4e^{-2\tau} \tau) p^2 + o(p^2) . \quad (32) \]

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This is a first order linear differential equation which could be solved by the method of integrating factor. The integrating factor is
\[ \mu(\tau) = e^{\int -2\tau + (10 - 2e^{-2\tau} - 6e^{-\tau}) d\tau} = e^{-2\tau p + (10\tau + e^{-2\tau} + 6e^{-\tau})p^2}. \]

Using Taylor approximations, we can approximate the integrating factor and its inverse to get
\[ \mu(\tau) = 1 - 2\tau p + (10\tau + 2\tau^2 + 6e^{-\tau} + e^{-2\tau})p^2 + o(p^2), \quad (33) \]
and
\[ \frac{1}{\mu(\tau)} = 1 + 2\tau p - (10\tau - 2\tau^2 + 6e^{-\tau} + e^{-2\tau})p^2 + o(p^2). \quad (34) \]

On multiplying (32) by \( \mu(\tau) \), we obtain
\[ \frac{d(y(\tau)\mu(\tau))}{d\tau} = \mu(\tau)((-2 - 3e^{-\tau})p + (16 - 17e^{-2\tau} + 4e^{-\tau} + 9e^{-\tau} - 4e^{-2\tau})p^2 + o(p^2)) . \]

By using the approximation of the integrating factor in (33), the above equation could be simplified to
\[ \frac{d(y(\tau)\mu(\tau))}{d\tau} = (-3e^{-\tau} - 2)p + (16 + 4\tau + 15e^{-\tau} - 4\tau e^{-2\tau} - 17e^{-2\tau} + 4e^{-\tau})p^2 + o(p^2). \]

Now let \( u \) be defined by
\[ u = (3e^{-\tau} + 2) - (16 + 4\tau + 15e^{-\tau} - 4\tau e^{-2\tau} - 17e^{-2\tau} + 4e^{-\tau})p, \]
such that
\[ \frac{d(y(\tau)\mu(\tau))}{d\tau} = -pu + o(p^2). \]

Let us now find the integral of \( u \) which we will need later.
\[ \int_0^\tau u \, d\tau = (-3e^{-\tau} + 2\tau) - (16\tau + 2\tau^2 - 15\tau e^{-\tau} - 19e^{-\tau} + 2\tau e^{-2\tau} + \frac{19}{2} e^{-2\tau})p . \]

Now, suppose \( u = \Omega(1) \). Then we have
\[ \frac{d(y(\tau)\mu(\tau))}{d\tau} = -(1 + o(p))pu . \]

Integrating both sides of this equation, we get
\[ y(\tau)\mu(\tau) = -(1 + o(p))\left((-3e^{-\tau} + 2\tau)p - (16\tau + 2\tau^2 - 15\tau e^{-\tau} - 19e^{-\tau} + 2\tau e^{-2\tau} + \frac{19}{2} e^{-2\tau})p^2\right) + C, \quad (35) \]
where \( C \) is an arbitrary constant. We can determine the value of \( C \) using the initial condition \( y(0) = 1 \). Thus we have
\[ 1 + 7p^2 + o(p^2) = 3p - \frac{19}{2} p^2 + C . \]

Hence, the initial condition will be satisfied if \( C = 1 - 3p + \frac{33}{2} p^2 + o(p^2) \). Substituting this value and (34) into (35) and simplifying using Taylor approximations, we get the following solution.
\[ y(\tau) = 1 - 3(1 - e^{-\tau})p + (2e^{-2\tau} + \frac{17}{2} e^{-2\tau} - 9e^{-\tau} - 25e^{-\tau} + \frac{33}{2})p^2 + o(p^2). \]
We can therefore conclude that while \( u = \Omega(1) \), \( y(\tau) \) cannot deviate much from the above solution. The solution is bounded above as required. It is easily verified that this solution agrees with our assumption that \( y(\tau) = 1 + O(p) \). Since our assumption is valid initially, i.e. \( y(\tau = 0) = 1 \), the solution is valid for all \( \tau \). The lemma is proved. \( \square \)

We have just proved that the generating function \( F(x, t) \) converges when \( x = 1 + p^3 \). Before looking at the subsequent results, let us validate an assumption made in Lemma 10 that \( P_3 = o(p^2) \).

**Lemma 12.** The assumption that \( P_3 = o(p^2) \) is valid. In fact, we have \( P_\ell = o(p^2) \) for all \( \ell \geq 3 \).

**Proof.** From Lemma 10 we have

\[
P_0 + P_1 + P_2 = 1 - 3(1 - e^{-\tau})p + (2e^{-2\tau} + \frac{17}{2}e^{-2\tau} - 9e^{-\tau} - 25e^{-\tau} + \frac{33}{2})p^2 + o(p^2). \tag{36}
\]

Now let \( g(\tau) = F(1, t) \). Then, from (29), we obtain

\[
\frac{dg(\tau)}{d\tau} = p^2 g(\tau)^2 + 2g(\tau)P_0p(1 - p) - 2pP_0^2(1 - p) - P_0(1 + 3p) - P_1 + 1. \tag{37}
\]

Note that, by definition, \( g(\tau) \) is equal to the sum of the probability bounds \( P_\ell \). Now comparing the equations (37) and (31) reveals that both \( g(\tau) \) and \( y(\tau) \) are identical except some error terms in \( o(p^2) \). It is then readily verified that the solution for \( g(\tau) \) will be identical to \( y(\tau) \). Hence, from Lemma 11, we have

\[
g(\tau) = 1 - 3(1 - e^{-\tau})p + (2e^{-2\tau} + \frac{17}{2}e^{-2\tau} - 9e^{-\tau} - 25e^{-\tau} + \frac{33}{2})p^2 + o(p^2). \tag{38}
\]

Notice that both (36) and (38) have the functions of the same order on the right hand side. Hence, the additional terms that are missing on the left hand side in (36) must be of the order \( o(p^2) \). That is,

\[
\sum_{\ell \geq 3} P_\ell = o(p^2),
\]

proving the Lemma. \( \square \)

In Lemma 11 we proved that the generating function \( F(x, t) \) converges when \( x = 1 + p^3 \). Hence, if \( \ell \) is sufficiently large, the following holds

\[
P_\ell x^\ell < 1, \text{ i.e. } P_\ell < \frac{1}{(1 + p^3)^\ell}.
\]

Otherwise, there is an infinite sequence with \( P_\ell x^\ell \geq \frac{1}{x^\ell} \), which contributes an infinite amount to the sum, contradicting the lemma. Thus, for some constant \( \gamma > 0 \), we have

\[
P_\ell < \frac{\gamma}{(1 + p^3)^\ell}, \tag{39}
\]

for all \( \ell \). Using this result, the following lemma proves that it takes exponential time before a plus-run of length \( \Omega(n) \) can be formed on the cycle.

**Lemma 13.** The following statement fails with probability exponentially small in \( n \): if the game is started with all minuses on the cycle, it would take exponential time before a plus-run of length \( n/4 \) or longer can be created.
Proof. By definition, probability that a run of length \( n/4 \) starts at position \( i \) at a given time \( \tau \) is at most \( P_{\tau}^{n/4} \). As the game is started with a symmetrical configuration (i.e. all-minuses) the result at any time will be symmetrical too. Hence the probability that such a run exists at any position on the cycle’s \( n \) positions at a given time \( \tau \) is equal to \( nP_{\tau}^{n/4} \). Finally, the probability of finding such a run at any position on the cycle at any time within \( T \) steps is at most

\[
TnP_{\tau}^{n/4}.
\]

It has already been shown in (39) that, when \( \ell \) is sufficiently large,

\[
P_{\ell} < \frac{\gamma}{(1+p^3)^{\ell}}.
\]

Hence the probability that a run of length \( n/4 \) is created in \( T \) steps is at most

\[
TnP_{\tau}^{n/4} \leq Tn\frac{\gamma}{(1+p^3)^{n/4}}.
\]

This probability is exponentially small whenever \( T \) is polynomially bounded. In other words, \( T \) has to be exponentially large before a run of length \( n/4 \) can appear on the cycle. Clearly, longer runs require even longer time, proving the lemma.

Remark 4. As mentioned earlier, the discretisation error is \( O(e^{-1/p}) \). However, the analysis above has error terms \( o(p^2) \). Thus, for small enough \( p \), the former is insignificant.

Proof of Theorem 4: For the game to converge to all-cooperation, at some point in time, there must be a plus-run of length \( n/4 \) or longer. The result then follows from Lemma 13.

As the error in the analysis is \( o(p^2) \), the value for \( p \) should be small enough so that \( o(p^2) \) terms can be ignored. This completes the proof.

Remark 5. SRP also shows behaviour similar to RP for small enough \( p \). That is, we can prove that there exists a small enough \( p \) for which it takes exponential time for the evolution of cooperation for SRP. The same approach as the one used for RP can be used here. We performed the analysis in this way and found that it is predictably much simpler.

4 Emergence of defection

The case where \( p = 0 \) is easy to analyse because \( p = 0 \) implies that there is no randomness in the strategies. As mentioned before, both RP and SRP are equivalent when \( p = 0 \), thus the same analysis applies to both strategies. The transition diagram of the resultant strategy is shown in Figure 4. Clearly, the process converges to all-minuses state if there are any minuses on the cycle at the beginning of the game. Theorem 3 computes the time it takes for this.

Proof of Theorem 3: It is easy to check that it will take the longest to reach the absorbing state (all-minuses state) if there is only one minus, i.e. a singleton, on the
cycle at the beginning of the game. Therefore, we can use this setting as the initial configuration for the worst case analysis.

Note that at each step of the game, the probability of spreading minus to a neighbour is $2/n$. Let $T_i$ denote the number of steps it takes to go from $i$-minuses to $(i+1)$-minuses on the cycle. Thus we have

$$Pr(T_i = t) = \left(1 - \frac{2}{n}\right)^{(t-1)} \frac{2}{n}.$$ Clearly, $T_i$ has geometric distribution with probability of success $2/n$. Therefore $E[T_i] = n/2$. Hence we get

$$E[T] = \sum_{i=1}^{n-1} E[T_i] = \sum_{i=1}^{n-1} \frac{n}{2} = \frac{n(n-1)}{2}.$$

Let us now get a bound on the probability of getting large deviations from the mean $E[T]$. Let $X^t$ denote the event that the number of minuses was not increased in the first $t$ trials. Then,

$$Pr(X^t) = \left(1 - \frac{2}{n}\right)^t \leq e^{-\frac{2t}{n}}.$$ If $t = \beta n \log n/2$, then $Pr(X^t) \leq e^{-\frac{\beta n \log n}{2n}} = \frac{1}{n^\beta}$. But we know from the definition of $X^t$ that

$$Pr\left(T_i > \frac{\beta n \log n}{2}\right) \leq Pr(X^t) \leq \frac{1}{n^\beta}.$$ Thus, deviations of size $\frac{\beta n \log n}{2}$ are unlikely. In other words, $T_i$ lies within the range $\left[0, \frac{\beta n \log n}{2}\right]$ with high probability. Now, define a set of random variables $Y_i$ such that $Y_i = \frac{2T_i}{\beta n \log n}$. Then, $Y_i \in [0, 1]$ with high probability. Also, we have

$$E[Y] = \frac{2E[T]}{\beta n \log n} = \frac{n-1}{\beta \log n}.$$ As $Y_1, Y_2, \ldots, Y_n$ are independent random variables taking values in $[0,1]$, we can apply Chernoff bound to get

$$Pr(Y \notin [(1 \pm \varepsilon)E[Y]]) \leq 2e^{-\frac{\varepsilon^2 n-1}{n \log n}}.$$
If \( \varepsilon = \frac{3\beta \log n}{\sqrt{n-1}} \), the following holds.

\[
\Pr \left( Y \notin \left[ \left(1 \pm \varepsilon \right) \mathbb{E}[Y] \right] \right) \leq 2e^{-3\beta \log n} = \frac{2}{n^{3\beta}}.
\]

It follows immediately that \( T \) lies within the range \( \left[ \left(1 \pm \varepsilon \right) \mathbb{E}[T] \right] \) with high probability. Thus we can conclude that \( T \in \left[ \frac{n(n-1)}{2} \pm O(n^{3} \log n) \right] \) with high probability. \( \square \)

5 Experimental results

Theorem 1 proves that cooperation emerges fast when \( p \) is high, and Theorem 2 shows that cooperation emerges exponentially slowly when \( p \) is small enough. As it is not clear what happens for \( p \) between these two ranges, we carried out an empirical study. The results of this study are presented in this section.

5.1 Simulation model

The experimental results presented in this paper were obtained from a computer program which we developed to simulate the IPD game played by agents arranged as the vertices of a cycle. This program takes the length of the cycle and a value for \( p \) as the input parameters and plays the game until cooperation emerges or the number of iteration reaches a predefined maximum, whichever happens first. The maximum number of iteration attempted is \( 43 \times 10^6 \). At each step of the game, an edge is chosen uniformly at random and the game is played by the associated agents based on RP. Experiments were performed in a homogeneous setting where all players on the cycle adopt the same strategy. In our experiments, the game was started with all players playing defect.

When all agents on the cycle play RP, the time taken for reaching cooperation was measured in terms of the number of steps required and plotted against the values of \( p \) in Figure 5(a). For the cases where the all-cooperate state was not reached in \( 43 \times 10^6 \) steps, the number of cooperators were counted before abandoning the game and plotted against \( p \) in Figure 5(b). Each data point in the graphs represents an average value of 100 repetitions.

5.2 Observations

Figure 5(a) suggests that the absorption time decreases as \( p \) increases, which is to be expected from the definition of the strategy. The results also support our theoretical results that cooperation emerges quite fast for high \( p \), and takes a very long time for low \( p \). However there is a large gap between the minimum value of \( p \) that we proved to give fast convergence and the lowest \( p \) having relatively faster convergence. To be more precise, Figure 5(a) shows that the absorption time increases rapidly when \( p \) is in the region \( 0.5 - 0.6 \). In other words, the convergence is relatively much faster when \( p \) is greater than 0.6. Theorem 1 however, rigorously proves the fast convergence for RP only when \( p \geq 0.870 \).

For small values of \( p \), the emergence of cooperation took so long that we could not reliably measure the time. This substantiates our theoretical result that it takes exponential time for cooperation to emerge for small values of \( p \). Interestingly, in Figure 5(b), the proportion of the cooperators is seemingly about \( p \). This can be explained intuitively as follows. When the game starts, all agents are defectors. Thereafter, every
one of them decides to cooperate with probability \( p \). These are exactly the ones we will see for smaller \( p \), since their decision to cooperate will not lead to others cooperating.

In summary, the absorption time is exponentially large when \( p \) is in the region \( 0 - 0.5 \). This drops considerably in the region \( 0.5 - 0.6 \) and is relatively small when \( p \) is greater than 0.6. The results suggest that there is a sharp “phase transition” in the region \( 0.5 - 0.6 \).

Remark 6. We carried out simulations for SRP as well, but the results are not included in this paper. The results obtained are quite similar to the results presented above for RP. The main difference is that the apparent phase transition happens when \( p \) is in the range \( 0.3 - 0.4 \) for SRP whereas it happens in the range \( 0.5 - 0.6 \) for RP.

6 Conclusions and open problems

We have proposed randomised improvements to the Pavlov strategy for the multiplayer Iterated Prisoner’s Dilemma game. This gives two new strategies called RP (Rational Pavlov) and SRP (Simplified Rational Pavlov) with a parameter \( p \). We have studied the rate of convergence of these strategies both rigorously and experimentally when used on the cycle for playing the IPD. We have presented a complete analysis for RP and briefly remarked upon similar results we obtained for SRP.

Since a rational player would choose to minimise risk without affecting long term return, a player playing RP or SRP should choose the lowest possible \( p \) that guarantees fast convergence to cooperation. Our results provide evidence (both theoretical and empirical) that players can safely choose \( p = 0.870 \) for RP and \( p = 0.699 \) for SRP, and still achieve fast cooperation. We have also shown that cooperation emerges exponentially slow when \( p \) is small enough and defection emerges (fast) when \( p = 0 \), for both strategies. It is not clear what happens for intermediate \( p \). Simulation results suggest that there is a sharp phase transition in this range.

It remains as an open question whether the phase transition can be proved rigorously. Two other interesting open questions are: whether this process can be analysed on graphs other than cycles, and whether there are graphs with average degree greater than 2 where fast convergence to cooperation for RP and SRP occurs for any \( p \).
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