NEW PROPOSAL OF NUMERICAL MODELLING OF BOSE-EINSTEIN CORRELATIONS: BOSE-EINSTEIN CORRELATIONS FROM "WITHIN"

Oleg V. Utyuzh (utyuzh@fuw.edu.pl)
The Andrzej Sołtan Institute for Nuclear Studies; Hoża 69; 00-681 Warsaw, Poland
Grzegorz Wilk (wilk@fuw.edu.pl)
The Andrzej Sołtan Institute for Nuclear Studies; Hoża 69; 00-681 Warsaw, Poland
Zbigniew Włodarczyk (wlod@pu.kielce.pl)
The Institute of Physics, Świętokrzyska Academy, Świętokrzyska 15; 25-406 Kielce, Poland

Abstract. We describe an attempt to numerically model Bose-Einstein correlations (BEC) from "within", i.e., by using them as the most fundamental ingredient of a Monte Carlo event generator (MC) rather than considering them as a kind of (more or less important, depending on the actual situation) "afterburner", which inevitably changes the original physical content of the MC code used to model multiparticle production process.

Key words: Bose-Einstein correlations; Statistical models; Fluctuations

1. Introduction

In all multiparticle production processes one observes specific correlations caused by quantum statistics satisfied by the produced secondaries. Because majority of them are boson (pions, kaons,...) subjected to Bose-Einstein statistics, one usually observes enhancements in the yields of pairs of identical boson produced with small relative momenta. They are called Bose-Einstein correlations [1, 2, 3, 4]. The corresponding depletion observed for fermionic identical particles (like nucleons or lambdas) satisfying Fermi-Dirac statistics is also observed but we shall not discuss it here. There is already extraordinary vast literature (see, for example, [2, 3, 4], and references therein), to which we refer, in what concerns measurements, formulations and interpretation of BEC as well as expectations they cause for our understanding of spatio-temporal details of the multiparticle production processes (or hadronization processes). Here we shall concentrate only on one

* Invited talk delivered by G.Wilk at the International Conference NEW TRENDS IN HIGH-ENERGY PHYSICS (experiment, phenomenology, theory), Yalta, Crimea, Ukraine, September 10-17, 2005.

© 2018 Springer. Printed in the Netherlands.
particular field, namely on the numerical modelling of BEC and we shall propose new approach to this problem, which we call \textit{BEC from within} [5].

The need for numerical modelling of the BEC phenomenon arises because the only effective way to investigate multiparticle production processes is by using one out of many of Monte Carlo (MC) numerical codes developed so far, each based on different physical picture of hadronization process [6]. However, such codes do not contain BEC because they are based on classical probabilistic schemes whereas BEC is of the purely quantum mechanical origin. Modelling BEC is therefore equivalent to modelling quantum mechanical part of hadronization process and by definition can only be some, better or worse, approximation. Actually, so far there is only one attempt in the literature of proper modelling of BEC using them as input of the numerical code. Namely in [7], using information theory approach based on the maximalization of the information (Shannon) entropy, identical particles were allocated in separated cells in the (longitudinal in this case) momentum space. In this way one is at the same time obtaining proper multiplicity distributions and bunching of particles in momentum space mimicking bunching of bosons in energy states, i.e., effect of the Bose-Einstein statistics. The method we shall propose here develops this idea further. However, before continuing along these lines let us first mention other attempts of introducing effects of BEC, which are widely used nowadays and let us point out why they cannot be regarded as satisfactory.

2. Numerical modelling of BEC

Among methods of numerical modelling of BEC using, in one or another way, some known numerical MC codes one can distinguish two approaches. One, represented by [8] (working with plane waves) and [9] (working with wave packets) starts with some distribution of momenta of particles in selected event and then, using a kind of Metropolis rejection method, changes step-by-step their momenta until multiparticle distribution containing effect of BEC is reached. This is now regarded as a true event. The drawback of this method is that it is extremely time consuming and therefore can only be used to explain some fine details of how BEC works, not for comparison with real data.

The other approaches are tacitly assuming that, on the whole, BEC constitutes only a small effect and it is therefore justify to add it in some way to the already known outputs of the MC event generators widely used to model results of high energy collisions in the form of the so called \textit{afterburner} [10]. There are two types of such afterburners.

– The first one modifies accordingly energy-momenta of identical secondaries, $p \rightarrow p + dp$, where $dp$ are selected from some function $f(dp)$ chosen in such way as to reproduce the observed correlation pattern [11]. The advantage of
this approach is that it applies to a single event and does not change multiplicity of secondaries produced in this event. The obvious drawback is that it spoils energy-momentum conservation so afterwards one has to correct for it in some way.

− The second type of afterburners preserves the energy-momentum conservation but it changes multiplicity distribution pattern of given MC code and works only for the whole ensemble of events produced by MC. In it one weights each event depending on how big BEC effect it shows (because of inevitably fluctuations they can be events which already show quite substantial BEC effect - those are multiplied by bigger weight - together with events which show no BEC - they are multiplied by small weight). In practice it leads to distortion of physics because ”suitable” events are counted many times more than the MC code used allows [12, 13]. Again, the weight function is some phenomenological input chosen in such way as to reproduce the observed BEC pattern of the whole ensemble of events considered.

However, even - as it is usually assumed - if the above afterburners do not change in a substantial way the numerical output of MC generators, which they are using, they surely do change their physical basis (i.e., the number and the type of the initial physical assumptions forming a basis of such MC code). This change, neglected in the assumption made above, has never been investigated in detail. We shall now demonstrate how dramatic this change can be by using as example simple MC cascade code (CAS) [14] proposed by us some time ago and endow it with some specific afterburner. The physical basis of CAS is very simple. It starts with decay of the original mass \(M\) into two objects of mass \(M_1\) and \(M_2\) such that \(M_{i=1,2} = k_{i=1,2} \cdot M\) where \(k_{i=1,2} < 1\) are parameters responsible for development of the cascade \((k_1 + k_2 < 1)\). The masses \(M_{i=1,2}\) fly then apart and after some time \(t_{i=1,2}\) they decay (in their rest frame) again into two lighter masses according to the above scheme but now with different values of the new decay factors \(k\). This branching process continues until the original mass \(M\) finally dissipates into a number of masses \(M_{i=1,...,2^N}\) equal to masses of the lightest particles \((N\) is the number of cascade generations). Values of ”life-times” \(t\) and decay factors \(k\) are chosen from some assumed distribution. The charges in the vertices change in the simple possible way: \((0) \rightarrow (+) + (-),\) \((+) \rightarrow (0) + (+)\) and \((-) \rightarrow (0) + (-)\). (see left panel of Fig. 1). This code produces both the energy-momentum distributions of produced particles and spatial distributions of their production points (which show features of the truncated Lévy distributions). The example of the correlation function it leads to is shown in Fig. 3.

CAS being classical scheme produces no BEC effect by itself. However, as we have demonstrated in [15, 16] the effect of BEC can be introduced by special afterburner applicable to every separate event, but, contrary to afterburners mentioned before, working without changing the number of secondaries \(N_{+/0/-}\) and without
Figure 1. Example of charge flows in MC code using simple cascade model for hadronization [14]: left panel - no effect of BEC observed; right panel - after applying afterburner described in [15] (based on new assignment of charges to the produced particles) one has BEC present at the cost of appearance of multicharged vertices.

Figure 2. Charge reassignment method of generating effect of BEC in a single event without changing its energy-momentum and spatio-temporal characteristics [16].

Figure 3. Examples of BEC patterns obtained for $M = 10$, 40 and 100 GeV for constant weights $P = 0.75$ (stars) and $P = 0.5$ (full symbols) and for the weight given by gaussian weights build on the information on momenta and positions of particles considered [16] (open symbols). Upper panels are for CAS, lower for simple statistical model (see [16] for details).
changing the energy-momentum and spatio-temporal structure of this event. It is based on the observation that effect of BEC can be visualized classically as phenomenon originating due to correlations of some specific fluctuations present in such stochastic systems as blob of the hadronizing matter [17, 8, 7]. It means that in phase space there occur bunches of identical particles and BEC arises as correlation of fluctuations effect. To get such effect in our case it is enough to forget about the initial charge assignment in the event under consideration (but keeping in mind the recorded multiplicities $N_{+/0/-}$ and both energy-momenta and spatio-temporal positions of particles emerging from NC code in event under consideration) and look for bunches of identical particles, both in energy-momentum and space-time. The procedure of selection of bunches of particles of the same charge, is therefore the crucial point of such algorithm [15, 16], cf. Fig. 2. It generally consists in selecting particles located nearby in the phase space, forming a kind of cell, and endowing them with the same charge. In what follows we shall call such cell elementary emitting cell, EEC. In this way we explore natural fluctuations of spatio-temporal and energy-momentum characteristic of produced particles resulting from CAS. Referring to [15, 16] for details let us only notice here that this methods works surprisingly well and is very effective. Unfortunately it was not yet developed to fully fledged MC code available for the common use but it allows to follow changes made in the original CAS by imposing on it requirements or reproducing also BEC patter. This is shown in Fig. 1 where CAS without BEC (left panel) is compared with CAS with BEC imposed. Whereas $N_{+/0/-}$ and positions of all secondaries in phase space remain the same the charge flow pattern changes considerably (see the right panel of Fig. 1). BEC in this case enforces occurrence of multicharged vertices, not present in the original CAS.

To summarize, the change in the original CAS required to observe BEC pattern amounts to introduction of bunching of particles of the same charge (in the form of EEC’s). However, when done directly in the CAS it would lead to great problems with the proper ending of cascade (without producing spurious multi-charged hadronic states, not observed in the nature) and from this point of view, once we know what is the physics behind BEC, the proposed afterburner algorithm occurs as a viable numerical short cut solution to this problem. No such knowledge is, however, provided for other afterburners used nowadays (although the idea of bunching origin of BEC can be spotted in the literature, cf., for example [18]). The problem, which is clearly visible in the CAS model, is not at all straightforward in other approaches. However, at least in the string-type models of hadronization [13], one can imagine that it could proceed through the formation of charged (instead of neutral) color dipoles, i.e., by allowing formation of multi(like)charged systems of opposite signs out of vacuum when breaking the string. Because only a tiny fraction of such processes seems to be enough in getting BEC in the case of CAS model, it would probably be quite acceptable modification. It is worth to mention at this point that there is also another possibility in such models, namely
when strings are nearby in the phase space one can imagine that production of given charge with one string enhances emission of the same charge from the string nearby - in this case one would have a kind of *stimulated emission* discussed already in [19, 20].

3. **BEC from ”within”**

The above observations, especially notion of EEC’s, will be the cornerstone of our new proposition. Let us remind that the idea of bunching of particles as quantum statistical (QS) effect is not new and has been used in the phenomenology of multiparticle production already long time ago [21, 22, 23, 24, 25]. In connection with BEC it was mentioned for the first time in [19, 20] and later it formed a cornerstone of the so called *clan model* of multiparticle distributions $P(n)$ leading in natural way to their negative binomial (NB) form observed in experiment [26]. It was introduced in the realm of BEC again in [27], where the notion of EEC has been introduced for the first time, and in [7].

![Diagram](image)

**Figure 4.** Schematic view of our algorithm, which leads to bunches of particles (*clans*). Whereas in [26] these clans could consist of any particles distributed logarithmically in our case they consist of particles of the same charge and (almost) the same energy and are distributed geometrically to comply with their bosonic character. We are therefore led to *Quantum Clan Model* [5].

Because our motivation concerning viewing of BEC from ”within” comes basically from the work [7] let us, for completeness, outline shortly its basic
Figure 5. Upper panels: distribution of cells and particles in a given cell. Lower-left panel: the corresponding summary $P(n)$ which is convolution of both $P(n_{\text{cell}})$ and $P(n_p)$. Lower-right panel: examples of the corresponding correlation functions $C_2(Q)$. Two sets of parameters were used. Data are from [29].

points. It was the first MC code which intended to provide as output distribution of particles containing already, among its physical assumptions, the effect of BEC with no need for any afterburner whatsoever. It deals with the problem of how to distribute in phase space (actually in longitudinal phase space given by rapidity variable) in a least biased way a given number of bosonic secondaries, $\langle n \rangle = \langle n^{(+)} \rangle + \langle n^{(-)} \rangle + \langle n^{(0)} \rangle$, $\langle n^{(+)} \rangle = \langle n^{(-)} \rangle = \langle n^{(0)} \rangle$. Using information theory approach (cf., [28]) their single particle rapidity distribution was obtained in the form of grand partition function with temperature $T$ and with chemical potential $\mu$. To obtain effect of BEC the rapidity space was divided into a number of cells of size $\delta y$ (which was fitted parameter) each. Two very important observations are made there: (i) - whereas the very fact of existence of cells in rapidity space was
enough to obtain reasonably good multiparticle distributions, \( P(n) \), (actually, in the NB-like form) and (ii) their size, \( \delta y \), was crucial for obtaining the characteristic form of the 2–body BEC function \( C_2(Q = |p_i - p_j|) \) (peaked and greater than unity at \( Q = 0 \) and then decreasing in a characteristic way towards \( C_2 = 1 \) for large values of \( Q \)) out of which one usually deduces the spatio-temporal characteristics of the hadronization source [1] (see [7] for more details). The message delivered was obvious: to get correlation function \( C_2 \)

\[
C_2(Q = |p_1 - p_2|) = \frac{d\sigma(p_1, p_2)}{d\sigma(p_1) \cdot d\sigma(p_2)}
\]  

(1)

peaked and greater than unity at \( Q = 0 \) and then decreasing in a characteristic way towards \( C_2 = 1 \) for large values of \( Q \), one must have particles located in cells in phase space which are of nonzero size. It means therefore that from \( C_2 \) one gets not the size of the hadronizing source, as it is frequently said, but only the size of the emitting cell, \( R \sim 1/Q \) [30] (in [7] it is \( R \sim 1/\delta y \)).

It is worth to mention at this point that, as has been demonstrated in [31] in the quantum field theoretical formulation of BEC, the requirement to get nonzero width od \( C_2(Q) \) function corresponds directly to the necessity of replacing delta functions of the type \( \delta(Q) \) in some commutator relations by a well defined, peaked at \( |Q| \to 0 \) functions \( f(Q) \) introducing in this way same dimensional scale to be obtained from the fits to data. This fact was known even before but without any phenomenological consequences [32].

Let us now proceed to our proposition of looking on the problem of BEC. As already mentioned, work [7] has been our inspiration but we would like to allow for dynamically defined EEC’s and to assure energy-momentum conservation. This was only approximate in [7] and rapidity cells there were fixed in size and were consecutively filled with previously preselected number of particles. But already from our previously described afterburner we have learned that EEC’s can be of different sizes (in fact, they even can overlap in phase-space) [16]. What counts most is the fact that distribution of particles in each EEC, \( P(n) \), must follow geometrical distribution in order to get the characteristic energy spectra for bosonic particles,

\[
< n(E) >= [\exp{(E - \mu) / T} - 1]^{-1}.
\]  

(2)

To obtain such effect we proceed in the following way.

1. Using some (assumed) function \( f(E) \) we select a particle of energy \( E_1^{(1)} \) and charge \( Q_1^{(1)} \). The actual form of \( f(E) \) should reflect our \textit{a priori} knowledge of the particular collision process under consideration. In what follows we shall assume that \( f(E) = \exp{(-E/T)} \), with \( T \) being parameter (playing in our example the role of "temperature").

2. Treating this particle as seed of the first EEC we add to it, with probability \( P(E) = P_0 \cdot \exp{(-E/T)} \), until the first failure, other particles of the same
Figure 6. Example of results obtained for $C_2(Q_{mn})$ (left column) and corresponding $C_2(Q_{x,y,z})$ (right column, for parameters used here all $C_2(Q_{x,y,z})$ are the same). Calculations were performed assuming spherical source $\rho(r)$ of radius $R = 1$ fm (spontaneous decay was assumed, therefore there is no time dependence) and spherically symmetric distribution of $p_{x,y,z}$ components of momenta of secondaries $p$. Energies were selected from $f(E) \sim \exp(-E/T)$ distribution. The changes investigated are - from top to bottom: different energies of hadronizing sources, different values of parameter $P_0$ and different spreads $\sigma$ of the energy in EEC.
charge $Q^{(1)}$. Such procedure results in desired geometrical (or Bose-Einstein) distribution of particles in the cell, i.e., in

$$\langle n_p \rangle = P(E)/([1 + P(E)],$$

(3)

accounting for their bosonic character. Together with exponential factor in probability $P(E)$, it assures that occupancy number of state with given energy will eventually follow characteristic bosonic form as given by eq. (2) (here $P_0$ is another parameter playing the role of ”chemical potential” $\mu = T \cdot \ln P_0$). As result $C_2(Q) > 1$ but only at one point, namely for $Q = 0$.

(3) Because process of emission of particles in a given EEC has finite duration, the resultant energies must be spread around the energy of the particle defining given EEC by some amount $\sigma$ (which is another free parameter). It automatically leads to the experimentally observed widths of $C_2(Q)$.

Points (1)-(3) are repeated until all energy is used. Every event is then corrected for energy-momentum conservation caused by the selection procedure adopted and condition $N^{(+) + N^{(-)} = N^{(-)}$ is imposed as well.

As result in each event we get a number of EEC with particles of the same charge and (almost) the same energy, i.e., picture closely resembling classical clans of [26] (with no effects of statistics imposed, see Fig. 4). Our clans (i.e., EECs containing identical bosonic particles subjected to quantum statistics and therefore named quantum clans) are distributed in the same way as the particles forming the seeds for EEC, i.e., according to Poisson distribution. With particles in each clan distributed according to geometrical distribution they lead therefore to the overall distribution being of the so called Pòlya-Aeppli type [33]. This distribution strongly resembles the Negative Binomial distributions obtained in the classical clan model [26] where particles in each clans were assumed to follow logarithmic distribution instead (with differences occurring for small multiplicities [34]). The first preliminary results presented in Fig. 5 are quite encouraging (especially when one remembers that so far effects of resonances and all kind of final state interactions to which $C_2$ is sensitive were neglected here).

The main outcome so far is strong suggestion that EEC’s are among the possible explanations of the BEC effect, in which case BEC provide us mainly with their characteristics, not with the characteristics of the whole hadronizing source. So far our method applies only to one-dimensional example of $C_2(Q)$. This is because process of formation of EEC’s proposed here gives us energies of all particles in a given cell and therefore also $p_i = |\vec{p}_i|$, but says us nothing about their angular distributions (i.e., about their components $p_{ix, iy, iz}$). To extend it to three dimensional case of $C_2(Q_{x,y,z})$ one has to somehow build cells also in $(p_x, p_y, p_z)$ components of the momenta of particles forming EEC. This can be done in many ways. Here we present as example approach using pairwise symmetrization of
the wave functions of every $i > 1$ particle in a given EEC with the first particle ($i = 1$) defining it. As result one gets for each such pair (in the plane wave approximation [1] and assuming instantaneous hadronization) the known 
$1 + \cos \left[ (\vec{p}_1 - \vec{p}_i) \cdot (\vec{r}_1 - \vec{r}_i) \right]$ term which connects the spatial extension of the hadronizing source $\rho(r)$ with its momentum space characteristics obtained before. In this way the assumed shape of $\rho(r)$ translates into the respective cells in momentum space. Preliminary results of this procedure are presented in Fig. 6.

4. Summary

We propose new numerical method of accounting for BEC phenomenon from the very beginning of the modelling process. Once the expected energy spectrum $f(E)$ of produced particles is chosen one constructs EEC’s in energy. We regard this method as very promising but we are aware of the fact that our proposition is still far from being complete. To start with one should allow for time depending emission by including $\delta E \cdot \delta t$ term in the $\cos(\ldots)$ above. The other is the problem of Coulomb and other final state interactions. Their inclusion is possible by using some distorted wave function instead of the plane waves used here. Finally, so far only two particle symmetrization effects have been accounted for: in a given EEC all particles are symmetrized with the particle number 1 being its seed, they are not symmetrized between themselves. To account for this one would have to add other terms in addition to the $\cos(\ldots)$ used above - this, however, would result in dramatic increase of the calculational time.

We shall close with remark that there are also attempts in the literature to model numerically BE condensation effect [35, 36] (or to use notion of BE condensation in other branches of science as well [37, 38]) using ideas of bunching of some quantities in the respective phase spaces.

Acknowledgements

GW is grateful for the support and warm hospitality extended to him by organizers of the International Conference NEW TRENDS IN HIGH-ENERGY PHYSICS (experiment, phenomenology, theory), Yalta, Crimea, Ukraine, September 10-17, 2005, conference. Partial support of the Polish State Committee for Scientific Research (KBN) (grant 621/E-78/SPUB/CERN/P-03/DZ4/99 (GW)) is acknowledged.

References

1 R.M. Weiner, Introduction to Bose-Einstein Correlations and Subatomic Interferometry, J.Wiley, 1999.
2. T. Csörgő, in *Particle Production Spanning MeV and TeV Energies*, eds. W. Kittel et al., NATO Science Series C, Vol. 554, Kluwer Acad. Pub. (2000), p. 203
3. W. Kittel, *Acta Phys. Polon. B* 32, 3927 (2001).
4. G. Alexander, *Rep. Prog. Phys.* 66 (2003) 481.
5. O. V. Utyuzh, G. Wilk, Z. Włodarczyk, *Quantum Clan Model description of Bose Einstein Correlations*, hep-ph/0503046, to be published in *Acta Phys. Hung. A - Heavy Ion Phys.* (2005).
6. K. J. Eskola, *Nucl. Phys. A* 698, 78 (2002).
7. T. Osada, M. Maruyama and F. Takagi, *Phys. Rev. D* 59, 014024 (1999).
8. W. Zajc, *Phys. Rev. D* 35, 3396 (1987).
9. H. Merlitz and D. Pelte, *Z. Phys. A* 357, 175 (1997).
10. K. Geiger, J. Ellis, U. Heinz and U.A. Wiedemann, *Phys. Rev. D* 61, 054002 (2000).
11. L. Lönnblad and T. Sjöstrand, *Eur. Phys. J. C* 2, 165 (1998).
12. K. Fiałkowski, R. Wit and J. Wosiek, *Phys. Rev. D* 58, 094013 (1998).
13. B. Andersson, *Acta Phys. Polon. B* 29, 1885 (1998).
14. O. V. Utyuzh, G. Wilk and Z. Włodarczyk, *Phys. Rev. D* 61, 034007 (1999).
15. O. V. Utyuzh, G. Wilk and Z. Włodarczyk, *Phys. Lett. B* 522, 273 (2001).
16. O. V. Utyuzh, G. Wilk and Z. Włodarczyk, *Acta Phys. Polon. B* 33, 2681 (2002).
17. J.W. Goodman, *Statistical Optics*, John Wiley & Sons, 1985.
18. B. Buschbeck and H. C. Eggers, *Nucl. Phys. B (Proc. Suppl.)* 92, 235 (2001).
19. E. E. Purcell, *Nature* 178, 1449 (1956).
20. A. Giovannini and H. B. Nielsen, *Stimulated emission effect on multiplicity distribution* in: *Proc. of the IV Int. Symp. on Multip. Hadrodynamics*, Pavia 1973, Eds. F. Duimio, A. Giovannini and S. Ratti, p. 538.
21. W.J. Knox, *Phys. Rev. D* 10, 65 (1974).
22. E.H. De Groot and H. Satz, *Nucl. Phys. B* 130, 257 (1977).
23. J. Kripfganz, *Acta Phys. Polon. B* 8, 945 (1977).
24. A.M. Cooper, O. Miyamura, A. Suzuki and K. Takahashi, *Phys. Lett. B* 87, 393 (1979).
25. F. Takagi, *Prog. Theor. Phys. Suppl.* 120, 201 (1995).
26. A. Giovannini and L. Van Hove, *Z. Phys. C* 30, 391 (1986).
27. M. Biyajima, N. Suzuki, G. Wilk and Z. Włodarczyk, *Phys. Lett. B* 386, 297 (1996).
28. F.S. Navarra, O.V. Utyuzh, G. Wilk and Z. Włodarczyk, *Phys. Rev. D* 67, 114002 (2003).
29. P. Abreu et al. (DELPHI Collab.), *Phys. Lett. B* 286, 201 (1992).
30. W.A. Zajc, *A pedestrian's guide to interferometry*, in "Particle Production in Highly Excited Matter", eds. H.H.Guthrod and J.Rafelski, Plenum Press, New York 1993, p. 435.
31. G. A. Kozlov, O. V. Utyuzh and G. Wilk, *Phys. Rev. C* 68, 024901 (2003).
32. K. Zalewski, *Lecture Notes in Physics* 539, 291 (2000).
33. J. Finkelstein, *Phys. Rev. D* 37, 2446 (1988).
34. Ding-wei Huang, *Phys. Rev. D* 58, 017501 (1998).
35. R. Kutner and M. Regulski, *Comp. Phys. Com.* 121-122, 586 (1999).
36. R. Kutner, K. W. Kehr, W. Renz and R. Przenioslo, *J.Phys. A* 28, 923 (1995).
37. A. E. Ezhov and A. Yu. Khrennikov, *Phys. Rev. E* 71, 016138 (2005).
38. K. Stalinas, *Bose-Einstein condensation in classical systems* cond-mat/0001347.