Constraints on Brane-Localized Gravity

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Abstract

In this paper we explore some general aspects of the embeddings associated
with brane-localized gravity. In particular we show that the consistency of
such embeddings can require (or impose) very specific relations between all the
involved bulk and brane matter source parameters. We specifically explore the
embeddings of 3-branes with non-zero spatial 3-curvature $k$ into 5-dimensional
spacetime bulks, and show that for such embeddings, a 5-dimensional bulk
cosmological constant is not able to produce the exponential suppression of
the geometry thought necessary to localize gravity to the brane.

I. GENERAL INTRODUCTORY REMARKS

It has been suggested recently [1–4] that it is possible for our 4-dimensional universe to
be a 3-brane§ embedded in some higher dimensional bulk spacetime whose spacelike extra
dimensions need not in fact be as minuscule as their string theory Planck length expectation.
And while the original motivation of these studies was an attempt to solve the hierarchy
problem, nonetheless the potential existence of any large such extra dimension is a matter of
great interest in and of itself.¶ Moreover, Randall and Sundrum [3,4] were able to show that
the embedding of a flat Minkowski 3-brane into a 5-dimensional anti de Sitter spacetime
($AdS_5$) would then explicitly localize gravity to the 4-dimensional world, thereby releasing
the extra higher dimension from needing to be tiny. Now while this is a very nice property of
the $AdS_5$ embedding, it is important to see just how generic it in fact is, and to what extent

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§The dimensionality of a brane is defined by the number of its spatial dimensions just like that
of a sheet of material.

¶For a recent compilation of some of the rapidly growing literature in this field see e.g. [5].
the embedding into a 5-dimensional spacetime of any given 4-dimensional set of matter fields in any given 4-dimensional geometry[2] would in fact then result in a gravity that actually was localized to the 4-dimensional world. Moreover, it is equally necessary to see whether any given such 4-space configuration of matter fields can even be consistently embedded into a higher dimensional space at all. And indeed, in one sense the whole localization issue is initially somewhat puzzling, since gravitating material always produces a gravitational field in the empty space around it, and it is straightforward to produce source configurations in which the associated gravitational potentials can actually grow at distances far from such sources. While an immediately obvious example of such a source might be a uniform density, zero curvature sheet of non-relativistic static gravitating material, a source which produces a constant, non-declining, Newtonian gravitational force away from the sheet when the sheet is immersed in an otherwise flat, empty background, such a non-relativistic gravitational field is just a coordinate artifact, being equivalent to a uniform acceleration in flat space. Thus, as we will show below, an embedding of such a sheet into a bulk with a non-zero 5-dimensional cosmological constant $\Lambda_5$ does in fact produce a gravity which is localized to the sheet, though as we shall also see, the covariantizing of such a sheet (to then produce a true gravitational field) proves to be instructive, with the consistency of its embedding (even into a source free bulk) being found to only be achievable for very specific brane equations of state. Motivated by this analysis, we shall then extend our study to the case where the static sheet or brane is endowed with a non-zero spatial 3-curvature $k$ (a configuration whose embedding into a source free bulk leads to a gravitational force which even grows with distance, one which is not a coordinate artifact), to then find in this case that the bulk $\Lambda_5$ is found incapable of producing exponentially suppressed localization of the geometry to the brane.

To begin our analysis it is instructive to recall some general properties of $AdS_5$ spacetime itself. As well as being constructible as a constant surface in a flat 6-dimensional space, the $AdS_5$ metric can also be given by the convenient form

$$ds^2 = (R^2/z^2)(dz^2 - dt^2 + d\tilde{x}^2)$$

(1)

where $R$ is the radius of curvature. To see that this metric is in fact an $AdS_5$ metric, we note that since this metric is conformal to flat, we can explicitly determine its associated curvature by conformally transforming the flat $\eta_{\mu\nu}(x)$ metric according to $\eta_{\mu\nu}(x) \rightarrow \Omega^2(z)\eta_{\mu\nu}(x) = g_{\mu\nu}(x)$ where $\Omega(z) = R/z$. Under such a transformation the initially zero 5-dimensional Ricci tensor is found to transform to

3In discussing embeddings it is important to recognize that the geometry of a given bulk can be modified by the introduction into the bulk spacetime of a brane with its own specific symmetry, and with the brane setting up its own gravitational field in the bulk, it is important to ascertain whether a given $AdS_5$ bulk geometry remains so after the embedding of the brane. To illustrate the point we recall that the familiar Schwarzschild de Sitter metric $ds^2 = -B(r)dt^2 + dr^2/B(r) + r^2d\Omega$ (where $B(r) = 1 - 2MG/r - kr^2$) is a metric for which the geometry exterior to the source is not maximally 4-symmetric ($R_{2323} = -kr^4(1 + 2MG/kr^3)\sin^2\theta \neq -k(g_{22}g_{33} - g_{23}^2)$), with it only becoming so asymptotically far from the source. In the presence of the mass source then the geometry exterior to the source is not the $dS_4$ geometry it would have been had the source not been there.
Moreover, since the Weyl tensor vanishes in geometries which are conformal to flat, the Riemann tensor associated with the metric of Eq. (1) is determinable from its associated Ricci tensor $R_{\mu\nu} = 4g_{\mu\nu}/R^2$ alone, with it then immediately being found to take the form $R_{\lambda\rho\sigma\nu} = -(g_{\sigma\rho}g_{\lambda\nu} - g_{\sigma\nu}g_{\lambda\rho})/R^2$. We thus recognize the spacetime associated with the metric of Eq. (1) to be that of a 5-space with constant negative curvature $K = -1/R^2$, viz. $AdS_5$. As the above analysis shows, we could construct metrics of the form $ds^2 = (R^2/x^2)(dz^2 - dt^2 + dx^2)$ where $x$ is any one of the four spacelike coordinates and still have an $AdS_5$ spacetime. However, because of the signature change in the flat d’Alambertian operator $\partial_\mu \partial^\mu$, the metric $ds^2 = (R^2/t^2)(dz^2 - dt^2 + dx^2)$ would have constant positive curvature $K = +1/R^2$ and thus be a de Sitter rather than an anti de Sitter space. For negative curvature spaces then the multiplying overall factor in the metric of Eq. (1) must only be associated with one of the spacelike coordinates. Since for such coordinates the transformation $z = Re^{y/R}$ allows us to rewrite the metric in the form

$$ds^2 = dy^2 - e^{-2y/R}(dt^2 - dx^2)$$

we see that in $AdS$ spaces the spatial exponential $e^{-2y/R}$ factor acts just like its temporal analog $e^{-2t/R}$ in de Sitter spacetimes. Such exponential behavior in $AdS$ spaces is thus the spatial analog of inflation, with the $e^{-2y/R}$ factor leading to rapid suppression as we go out in $y$ away from the 4-dimensional space associated with the metric $dt^2 - dx^2$. Now since the multiplying factor in the metric of Eq. (1) is quadratic in $z$ (and uniquely so for $AdS$ spaces\footnote{Multiplying a flat space metric by any conformal factor will always lead to a new metric which is conformal to flat. However, it is only the choice $\Omega^2(z) = R^2/z^2$ which leads to a spacetime with the same maximal number of Killing vectors as the original flat spacetime itself.}), we can make transformations of either of the forms $z = Re^{y/R}$ and $z = Re^{-y/R}$ on the metric of Eq. (1). Thus we can make the transformation $z = Re^{y/R}$ in the $y > 0$ region and the transformation $z = Re^{-y/R}$ in the $y < 0$ region to then give us $e^{-2|y|/R}$ exponential suppression for every value of $y$, positive or negative. However, now the two regions in $y$ will be two separate patches of $AdS_5$ with there then necessarily being a discontinuity at $y = 0$ where the two patches meet. It is thus at just such a discontinuity that our 4-dimensional universe can be located with our universe then being a 3-brane embedded in a higher dimensional bulk space containing two separate geometrical patches; with gravity potentially then being localized to the brane through the $e^{-2|y|/R}$ suppression\footnote{The dependence of the geometry away from the brane on $|y|$ rather than on $y$ itself is characteristic of the requirement made in all brane localized gravity studies (this one included) that there is to be a $y \rightarrow -y$ symmetry of the metric around the $y = 0$ brane.} It is thus to the implications of the embedding of such brane universes, and to the dynamical interplay of the bulk and the brane entailed by the very fact of such embeddings (even when the bulk itself contains no matter fields at all), to which we now turn, first in a source free empty background and then in one with a bulk $\Lambda_5$ (viz. a bulk that would be $AdS_5$ in the absence of the brane).
II. EMBEDDING A BRANE IN AN EMPTY BULK

For the embedding of a homogeneous, isotropic standard 4-dimensional Robertson-Walker universe with spatial 3-curvature $k$ into an arbitrary (and thus not necessarily $AdS_5$) 5-dimensional bulk space, the most general allowed maximally $r, \theta, \phi$ 3-symmetric metric takes the generic form

$$ds^2 = -n^2(y, t)dt^2 + a^2(y, t)[dr^2/(1 - kr^2) + r^2d\Omega] + b^2(y, t)dy^2 + 2c(y, t)dtdy,$$

(4)

up to arbitrary coordinate transformations involving $y$ and $t$. Recognizing the pure $y, t$ sector of this metric to be the most general 2-dimensional metric in a $y, t$ space, we can thus make a coordinate transformation in this space to remove the $y, t$ cross-term, and in the illustrative static limit which we study here can then reabsorb $b(y)$ into a redefinition of $y$ to thus yield

$$ds^2 = -n^2(y)dt^2 + a^2(y)[dr^2/(1 - kr^2) + r^2d\Omega] + dy^2$$

(5)

a metric whose embedding and localization of gravity aspects we now study. While in brane-localized studies it is desired to recover standard gravity only in the 4-dimensional world, it is conventional to assume that the full 5-space gravity is given simply by the 5-dimensional Einstein equations (rather than by some more complicated set of 5-dimensional equations), viz.

$$G_{AB} = R_{AB} - g_{AB}R^C_C/2 = -\kappa_5^2[T_{AB} + T_{\mu\nu}\delta^A_\mu\delta^B_\nu\delta(y)]$$

(6)

where $T_{AB}$ ($A, B = 0, 1, 2, 3, 5$) is due to sources in the bulk and $T_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$) is due to sources on the $y = 0$ brane. For the symmetry of Eq. (5) both of these energy-momentum tensors are given as perfect fluids, viz.

$$T^A_B = diag(-\rho_B, P_B, P_B, P_B, P_T), \quad T^\mu_\nu = diag(-\rho_0, p_b, p_b, p_b)$$

(7)

($B$ denotes bulk and $b$ denotes brane). For the metric of Eq. (5) it is straightforward to write the 5-dimensional Einstein equations, with the resulting expressions simplifying to (the prime denotes differentiation with respect to $y$)

$$G_{00} = 3e^2f''/2f^2 - 3e^2k/f^2 = -\kappa_5^2e^2[\rho_B + \rho_0\delta(y)]/f$$

(8)

$$G_{ij} = [-f''/2 - f e''/e + k]\gamma_{ij} = -\kappa_5^2f[p_B + p_0\delta(y)]\gamma_{ij}$$

(9)

$$G_{55} = -3f'e'/2fe + 3k/f = -\kappa_5^2P_T$$

(10)

when the identification $a(y) = f^{1/2}(y)$, $n(y) = e(y)/f^{1/2}(y)$ is made [here $\gamma_{ij} = diag(1/(1 - kr^2), r^2, r^2\sin^2\theta)$].

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6While we concentrate here on embedding and localization issues, the brane theory associated with Eq. (4) has also been studied as a cosmology in and of itself, see e.g. [1, 3].
While we shall discuss the implications of these equations in various situations below, we note immediately that when the bulk $T_{AB}$ is set to zero and when the spatial $i, j$ coordinates are restricted to a flat 2-dimensional $(x, y)$ plane and $y$ is replaced by the usual spatial $z$, Eq. (4) then describes a uniform, infinite, flat 2-dimensional sheet of static matter embedded in (what otherwise would have been) ordinary empty spacetime, a system whose Newtonian limit is known to correspond to a gravitational potential which grows linearly with $z$ and a gravitational force $F(z)$ per unit mass which is independent of $z$. Moreover, in such a case, the Newtonian gravitational force points toward the $z = 0$ sheet no matter which side we consider, with Gauss’ Law yielding $F(z = 0^+) - F(z = 0^-) = 4\pi G \sigma$ for a sheet of surface matter density $\sigma$, with the gravitational potential $\phi = 2\pi G \sigma |z|$ thus being discontinuous across the surface. As such the relation $F(z = 0^+) - F(z = 0^-) = 4\pi G \sigma$ is a non-relativistic analog of the fully covariant relativistic Israel junction conditions [9] (see e.g. [10,11] for some recent derivations)

\[ K_{\mu\nu}(y = 0^+) - K_{\mu\nu}(y = 0^-) = -\kappa_5^2(T_{\mu\nu} - q_{\mu\nu}T^\alpha_\alpha/3) \]

across a discontinuous surface with normal $n^A$ where $q_{AB} = g_{AB} - n_A n_B \equiv q_{\mu\nu}$ is the induced metric on the surface and $K_{\mu\nu} = q^\alpha_{\mu} q^\beta_{\nu} n_{\alpha\beta\alpha}$ is its extrinsic curvature. We shall thus expect to see an analog of this Newtonian gravity discontinuity in the treatment of the relativistic case associated with Eq. (5), something to which we now turn.

For the simplest case first of a $k = 0$ spatially flat Robertson-Walker geometry embedded in a source free bulk, on taking the metric coefficient $a(y)$ to be a function of $|y| = y[\theta(y) - \theta(-y)]$ (where $|y| = \theta(y) - \theta(-y)$, $|y|^2 = 1$, $|y|^n = 2\delta(y)$), integration of Eq. (8) is then found to yield

\[ a^2(y) = \alpha(1 - \kappa_5^2 \rho_b |y|/3) \]

where $\alpha$ is an arbitrary constant which can be absorbed in a redefinition of the spatial $x_i$ coordinates. With Eq. (10) then obliging $e(y) = e(|y|)$ to be a constant, we can then set $n(y) = 1/a(y)$, with Eq. (9) then recovering the solution of Eq. (12) provided

\[ p_b = -\rho_b/3. \]

As a check on this solution, we note that the Israel junction conditions which follow from Eq. (11) in our particular case, viz. [5]

\[ (a'(y = 0^+) - a'(y = 0^-))/a(y = 0) = -\kappa_5^2 \rho_b/3, \]

\[ (n'(y = 0^+) - n'(y = 0^-))/n(y = 0) = \kappa_5^2(3p_b + 2\rho_b)/3, \]

7With brane studies always requiring the gravitational potential to only be a function of $|z|$, we see that such a requirement nicely dovetails with the constraints of Gauss’ Law.
are indeed satisfied by our obtained discontinuity.\footnote{With $n^2(y) \to 1 + \kappa_5^2\rho_b|y|/3$ in the weak gravity limit, we see that we nicely recover the linear potential characteristic of a Newtonian gravity sheet (though, as will be clarified below, one actually not with the standard weak gravity coefficient), with gravity not at all being localized to the brane, and with the strong gravity limit even possessing a singularity at $|y| = 3/\kappa_5^2\rho_b$.}

Nonetheless, even in that case the gravity independent input, and the gravity which it produces is then determined as output.

While we thus see that we can obtain the anticipated non-localized solution, we find that it is only obtainable for a very particular equation of state, one with a negative pressure.\footnote{To see that this is in fact a generic effect we note that for a flat 2-brane embedded in an empty 4 space, viz. one described by the metric $ds^2 = -n^2(z)dt^2 + a^2(z)(dx^2 + dy^2) + dz^2$, the 4-dimensional Einstein equations $G_{\mu\nu} = -\kappa_4^2T_{\mu\nu}$ take the form $G^0_0 = -(2aa'' + a'^2)/a^2 = \kappa_4^2\rho_b\delta(z)$, $G^x_x = -(a''n + a'n' + an'')/an = -\kappa_4^2\rho_b\delta(z)$, $G^z_z = -a'(a'n + 2an')/a^2n = 0$, with solution $a(z) = 1/n^2(z) = (1 - 3\kappa_4^2\rho_b|z|/8)^{2/3}$, constraint $p_b = -\rho_b/4$, Israel junction conditions of the form $K_{\mu\nu}(y = 0^+) - K_{\mu\nu}(y = 0^-) = -\kappa_4^2(T_{\mu\nu} - q_{\mu\nu}T_{\alpha\beta}/2)$ (viz. $(a'(y = 0^+) - a'(y = 0^-))/a(y = 0) = -\kappa_4^2\rho_b/2$, $(n'(y = 0^+) - n'(y = 0^-))/n(y = 0) = \kappa_4^2(2p_b + \rho_b)/2$), and a Riemann tensor which is again of order $\kappa_4^2$. In passing we note also that our result confirms an old result of Vilenkin\footnote{In such a case the harmonic phonon mode fluctuations in the lattice will still have positive pressure, it is just that they have nothing to do with the mechanism which put the atoms onto the lattice sites in the first place by minimizing the free energy. Rather they are only a perturbation around such a minimum, a minimum to which gravity however is sensitive.} that no static solution is possible for 2-branes with equation of state $p_b = -\rho_b$. However, we now see that a static solution is possible when $p_b = -\rho_b/4$.} 

As regards first the conventional pure 4-dimensional situation with no embedding into an ordinary 4-dimensional world and the one that it appears to be playing in the embedded case. As regards first the conventional pure 4-dimensional situation with no embedding into a fifth dimension, we note that there the fluid equation of state is usually taken as a fixed, gravity independent input, and the gravity which it produces is then determined as output. Nonetheless, even in that case the $p_b/\rho_b$ ratio need not necessarily be positive. Thus even while the energy density and pressure of a high temperature ideal gas due to the kinematic motions of the gas particles are both positive, if the gas is cooled into a solid phase it then undergoes a phase transition, a long range order effect such as condensation into an ordered crystal lattice, an effect which can be associated with the negative pressure characteristic of vacuum breaking,\footnote{8With all of the metric coefficients being functions of $|y|$, the junction conditions require the terms linear in $|y|$ to be first order in $\kappa_5^2$, viz. $a(|y|) = \alpha(1 - \kappa_5^2\rho_b|y|/6) + O(|y|^2)$, $n(|y|) = \beta(1 + \kappa_5^2\rho_b|y|/6) + O(|y|^2)$. The junction conditions thus generate a contribution to the bulk Riemann tensor (see Eq. (41) below) which only begins in order $\kappa_5^4$, something which is to be expected since the order $\kappa_5^2$ constant Newtonian gravitational acceleration associated with a metric with $n(y) = \beta(1 + \kappa_5^2\rho_b|y|/6)$ is removable by a coordinate transformation, with coordinate independent gravitational effects thus only beginning in order $\kappa_5^4$, with true gravity thus needing the $O(\kappa_5^3)$ terms of both of the $n(y)$ and $a(y)$ metric coefficients to be non-zero.} with non-gravitational physics thus being capable of leading to fluids...
with negative pressure. Thus, if the fluid equation of state is to be taken as a fixed, gravity independent input in the embedded gravity case as well, we would have to conclude that unless the fluid actually possesses the needed equation of state, then no (static) embedding would in fact be possible.

However, in the brane-embedded case, it turns out that gravity can potentially play a different role, one in which it could be instrumental in actually fixing the fluid equation of state in the first place, with the equation of state then being output to the problem rather than input. In particular, the key difference between the embedded and non-embedded cases is that in the embedded case the matter sources are assumed to be confined to the brane, with no brane matter contribution to $T_{55}$ being permitted. As a consequence, the $G_{55}$ component of the empty bulk Einstein tensor has to vanish, a quite non-trivial requirement which has dynamical implications not present in a non-brane-embedded situation where the fluid is otherwise free to flow in all available spatial directions. Such a vanishing is then a constraint imposed by the geometry, and even when there are no explicit bulk matter fields to apply stresses on the brane, nonetheless there is still a non-vanishing Riemann tensor in the bulk to thus enable the bulk gravitational field to provide such stresses instead, with the bulk curvature and the brane pressure then potentially being able to dynamically adjust to each other to thereby fix the pressure on the brane and yield the brane equation of state as output. Thus rather than a given input fluid equation of state imposing an output...

11The $p_b = -\rho_b/3$ equation of state required above, for instance, could be associated with an isotropic network of cosmic strings, with such a negative pressure fluid potentially leading to the cosmic acceleration (see e.g. [13] for a recent review) associated with quintessence models [14].

12In their original papers Randall and Sundrum noted the need to have a fixed relation between bulk and brane cosmological constants in models in which $n(y)$ was initially set equal to $a(y)$. We now see that restrictions on the structure of the energy-momentum tensor are of much broader generality, in principle involving the energy densities and pressures of all brane matter sources, restrictions which are intrinsic to all embeddings (even ones into source-free bulks) and not just only to those associated with $\text{AdS}_5$.

13I.e. the very presence of the brane modifies the geometry in the source-free bulk, to thus prevent it from being the flat one that it would have been in the absence of the brane.

14Thus while the Israel junction condition \( (n'(y = 0^+) - n'(y = 0^-))/n(y = 0) = \kappa_4^2(2p_b + \rho_b)/2 \) associated with the embedding of a flat 2-brane in an empty 4 space would actually yield the standard weak gravity Gauss’ Law \( (n'(y = 0^+) - n'(y = 0^-))/n(y = 0) = \kappa_4^2\rho_b/2 \) if we were to ignore the pressure $p_b$ on the brane (the usual weak gravity assumption), we see that the consistency of the embedding requires a very different brane pressure, one of the same order of magnitude as $\rho_b$, to thus yield to a weak ($\kappa_4^2$ small) gravity limit whose potential has a different normalization than that associated with a standard pressureless weak Newtonian gravity sheet. (Since the standard Newtonian potential of a sheet is just a coordinate artifact, there is no reason for it to have to correspond to the non-relativistic limit of the true gravity associated with a fully covariantized uniform sheet.)
geometry on gravity, gravity instead could impose an output equation of state on the fluid. The notion that the presence of an extra dimension might have dynamical implications and that higher dimensional gravity might play a role in stabilizing lower dimensional systems is certainly a very interesting one which requires further study. To illustrate its capability it is instructive to recall Einstein’s attempt to construct a static 4-dimensional model of the universe. As is well-known, in his attempt to do so Einstein introduced a cosmological constant and was then able to find a non-trivial static universe solution provided the spatial 3-curvature $k$ of the universe was taken to be positive. In such a situation the ordinary 4-dimensional Einstein equations take the form $G^0_0 = 3k = \kappa^2 \rho_b$, $G^i_j = k \delta^i_j = -\kappa^2 p_b \delta^i_j$ to precisely impose the selfsame $p_b = -\rho_b/3$ equation of state which we obtained above while fixing $k = \kappa^2 \rho_b/3$. As we now see, in the 5-dimensional $k = 0$ model discussed above, the binding role played by positive $k$ in the 4-dimensional world is instead provided by the embedding, with the constraint condition $G_{55} = 0$ then providing a dynamics not otherwise present in the 4-dimensional system itself.

In fact this phenomenon is actually a quite general one. Specifically if the Einstein equations are assumed to hold in the bulk, it turns out \cite{19,20} that the induced gravitational equations on the brane actually deviate from the standard 4-dimensional Einstein equations, with the additional terms that are found being explicit consequences of the embedding, viz. they do not represent new matter sources in the 4 space, but rather they arise though the constraints associated with the very existence of the embedding. In particular the authors of \cite{19} noted that since the difference between the 4-dimensional Riemann tensor $(4) R^\alpha^\beta^\gamma^\delta$ of a general 4-surface and the 5-dimensional Riemann tensor $R^A_{BCD}$ of some general 5-bulk into which it is embedded can be completely characterized by a function quadratic in the extrinsic curvature tensor $K_{\mu\nu}$ of the 4-surface according to the Gauss embedding formula

\begin{equation}
(4) R^\alpha^\beta^\gamma^\delta = R^A_{BCD} q_A^\alpha q_B^\beta q_C^\gamma q_D^\delta - K^\alpha^\gamma K^\beta^\delta + K^\alpha^\delta K^\beta^\gamma,
\end{equation}

use of the bulk Einstein equations, the Israel junction conditions at the surface of the brane and the assumption of a $y \to -y$ symmetry around the $y = 0$ brane then enable us to express the 4-dimensional Einstein tensor in terms of quantities on the brane which must necessarily be quadratic in the energy-momentum tensor of the brane. In particular, for generic metrics of the form $ds^2 = dy^2 + q_{\mu\nu} dx^\mu dx^\nu$ and a brane energy-momentum tensor of the specific

\begin{itemize}
  \item[15] Einstein himself satisfied the relation $p_b = -\rho_b/3$ by taking $\rho_b = \rho_m + \lambda$, $p_b = -\lambda$ where $\rho_m$ is the energy density of ordinary matter. In this solution then ordinary matter was taken to have no pressure and the cosmological constant was tuned to be given as $\lambda = \rho_m/2$. Thus already in this now quite ancient model we see the need for constraints (either input or output) on the components of the 4-dimensional matter energy-momentum tensor. In passing we note that current observations \cite{15–17} almost a century later are apparently requiring a similar such fine tuning between 4-dimensional matter and vacuum energy densities (for some remedies to this perplexing problem see e.g. \cite{18}).
  \item[16] Negative pressure solutions to the cosmic acceleration problem could thus arise as a 4-dimensional reflection of a higher dimensional embedding.
\end{itemize}
the authors of \[19\] found that the 4-dimensional Einstein tensor on the brane is given by

\[(4) \quad G_{\mu\nu} = \Lambda_4 q_{\mu\nu} - 8\pi G_N \tau_{\mu\nu} - \kappa_5^2 \pi_{\mu\nu} - \bar{E}_{\mu\nu}\]  

where

\[G_N = \lambda \kappa_5^2 / 48\pi, \quad \Lambda_4 = \kappa_5^2 (\Lambda_5 + \kappa_5^2 \lambda^2 / 6) / 2, \quad E_{\mu\nu} = C^{A}_{BCD} n_{A\nu} q^{B}_{\mu} q^{D}_{\nu}\]  

and where \(\bar{E} = [E(y = 0^+) + E(y = 0^-)] / 2\) is the mean value of \(E_{\mu\nu}\) at the brane. Thus, even in the event that gravity gets to be localized to the brane, we see in general that on the brane we would expect gravity to depart from that given by just the standard 4-dimensional Einstein equations associated with a non-embedded 4-dimensional world. This is because measurements within a 4-dimensional world embedded in a higher dimensional bulk would in principle be able to reveal the presence of the higher dimensional space even if the gravitational field is localized to the 4-dimensional world.\[17\]

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\[17\]This particular form was chosen so that one of the quadratic terms would then yield a term linear in \(\tau_{\mu\nu}\).

\[18\]For a perfect fluid \(\tau_{\mu\nu} = (\rho_m + p_m) U_{\mu} U_{\nu} + p_m q_{\mu\nu}\) for instance, the additional \(\pi_{\mu\nu}\) tensor takes the form

\[\pi_{\mu\nu} = -\tau_{\mu\alpha} \tau_{\nu}^{\alpha} / 4 + \tau_{\alpha}^{\alpha} \tau_{\mu\nu} / 12 + q_{\mu\nu} \tau_{\alpha\beta} \tau^{\alpha\beta} / 8 - q_{\mu\nu} (\tau_{\alpha}^{\alpha})^2 / 24,\]  

and where \(\bar{E} = [E(y = 0^+) + E(y = 0^-)] / 2\) is the mean value of \(E_{\mu\nu}\) at the brane. Thus, even in the event that gravity gets to be localized to the brane, we see in general that on the brane we would expect gravity to depart from that given by just the standard 4-dimensional Einstein equations associated with a non-embedded 4-dimensional world.\[20\] This is because measurements within a 4-dimensional world embedded in a higher dimensional bulk would in principle be able to reveal the presence of the higher dimensional space even if the gravitational field is localized to the 4-dimensional world.\[19\]

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\[19\]Noting the special role played by the brane cosmological constant \(\lambda\) in establishing the Newton constant term in Eq. \[18\], we see that the effective Newton constant \(G_N\) would vary (possibly even in sign as well as magnitude) in different epochs separated by phase transitions, with early universe cosmology then potentially no longer being controlled by the Newton constant measured in a low energy Cavendish experiment. It is thus of interest to note that it is precisely an epoch dependence to both the sign and magnitude of the effective gravitational coupling constant which has recently been identified \[18\] as a possible solution to the cosmological constant problem.

\[20\]In passing, we also note a subtlety in applying Eq. \[18\] to the Schwarzschild problem. Specifically, even though the 4-dimensional \(R_{\mu\nu} = 0\) vacuum Schwarzschild solution can \[21\] explicitly be embedded into a 5-space with a bulk cosmological constant according to the localizing

\[ds^2 = e^{-2|y|} \left[ \frac{dr^2}{(1 - 2MG/r) + r^2d\Omega} - (1 - 2MG/r)d\tau^2 \right] + dy^2,\]  

a case where every single term in Eq. \[18\] is found to vanish, nonetheless, the bulk is not \(AdS_5\) in this case. Specifically, Eq. \[18\] only involves the projections of the Weyl tensor along the normal direction to the brane and
To study further implications of such embeddings we turn now to our second soluble model, namely a static 3-brane with non-zero \( k \) embedded in an empty bulk. In the event of non-zero \( k \) the most general empty bulk solution to Eq. (8) is directly given as

\[
a^2(y) = \alpha (1 - \kappa^2_5 \rho_b |y|/3 + k|y|^2/\alpha),
\]

with Eq. (11) then leading to

\[
n^2(y) = (\kappa^2_5 \rho_b \alpha/3 + 2k|y|)^2/\alpha (1 - \kappa^2_5 \rho_b |y|/3 + k|y|^2/\alpha),
\]

and with Eq. (11) then entailing the equation of state

\[
p_b = -\rho_b/3 - 12k/\alpha \kappa^4_5 \rho_b.
\]

Thus we see that the consistency of the embedding (cf. the non-trivial vanishing of \( G_{55} \)) again imposes constraints on the brane equation of state, with the metric away from the brane now growing quadratically with distance. It is thus to the issue of whether or not there is to be a quenching of this metric when a bulk cosmological constant is introduced to which we now turn.

### III. EMBEDDING A BRANE IN A NON-EMPTY BULK

In the event of there being a bulk cosmological constant \(-\rho_B = p_B = P_T = -\Lambda_5\) the structure of the solutions to the 5-dimensional Einstein equations will depend on whether there is a spatial curvature \( k \) on the brane. Thus on setting \( k = 0 \) first and taking \( a(y) \) to be a function only of \(|y|\) just as before, Eq. (8) is then found to lead to

\[
\frac{3}{2} \frac{d^2 f(|y|)}{d|y|^2} = -\kappa^2_5 \Lambda_5 f(|y|)
\]

\[
[3 \frac{df(|y|)}{d|y|} + \kappa^2_5 \rho_b f(|y|)] \delta(y) = 0.
\]

these components indeed do vanish in the bulk in the solution of [21]. However, explicit calculation shows that the other components of the Weyl tensor (viz. the ones not involved in Eq. (18)) do not in fact vanish (cf. \( C_{0101} = 2MGe^{-2|y|}/r^3 \)), so that (just like in our earlier discussion of the 4-dimensional Schwarzschild de Sitter metric) we find that in the presence of Schwarzschild on the brane the geometry cannot be the maximally symmetric \( AdS_5 \) in the bulk (though it does become so asymptotically far from the brane). Now in their paper the authors of [19] showed that when there is a spatially inhomogeneous matter distribution on the brane, Eq. (18) then prevents the exterior bulk geometry from being pure \( AdS_5 \). Since setting \( R_{\mu\nu} = 0 \) on the brane actually requires a delta function singularity at \( r = 0 \) (if the \( 1/r \) term in \( g_{00} \) is to have a non-zero coefficient), Schwarzschild on the brane actually entails an inhomogeneous source on the brane, to thus yield a non \( AdS_5 \) bulk.
According to Eq. (24) (viz. a pure bulk Einstein equation which would hold even in the absence of any brane at $y = 0$) and its counterpart which comes from Eq. (9), the most general allowed metric coefficient in the $k = 0$ case is then found to have an unbounded exponential dependence

$$a^2(y) = f(y) = \alpha e^{\nu|y|} + \beta e^{-\nu|y|}, \quad n(y) f^{1/2}(y) = e(y) = \alpha e^{\nu|y|} - \beta e^{-\nu|y|},$$

(26)
on distance if $\Lambda_5$ is negative (viz. anti de Sitter), with the square of the exponent $\nu$ being given by

$$\nu^2 = -2\kappa_5^2\Lambda_5/3.$$  

(27)

Thus no matter what sign we take for $\nu$, we see that in and of itself a bulk cosmological constant does not automatically lead to exponential suppression away from the brane in this case - as we recall from Eq. (1), the $AdS_5$ metric is quadratic in $R/z$ - with the most general solution to Eq. (24) in fact necessarily being unbounded, with having a $\Lambda_5 \neq 0$ bulk in and of itself thus not being sufficient to guarantee brane-localization of gravity. Thus we see that while the bulk might have been a pure $AdS_5$ bulk with only a single exponential in its metric in the absence of the brane, the introduction of the lower symmetry brane then lowers the symmetry in the bulk with the unbounded exponential no longer automatically being excluded.

In order to try to remove this undesired growing exponential anyway, we note that the insertion of Eq. (26) into the brane discontinuity formulas associated with Eq. (8) and its Eq. (9) counterpart yields

$$3\nu(\alpha - \beta) = - (\alpha + \beta)\kappa_5^2\rho_b,$$

(28)

$$6\nu(\alpha + \beta) = (\alpha - \beta)\kappa_5^2(\rho_b + 3p_b),$$

(29)

relations whose solubility requires the quantity $\rho_b(\rho_b + 3p_b)$ to expressly be negative. Thus if in addition to the equation of state (once again we see that embeddings entail constraints on both the bulk and brane matter fields)

$$\Lambda_5 - \kappa_5^2(\rho_b + 3p_b)/12 = 0$$

(30)

which then follows, we additionally now impose the condition

$$p_b = -\rho_b$$  

(31)

In the original Randall-Sundrum study the brane geometry was taken to maximally 4-symmetric (viz. Minkowski), to thus obliged the metric coefficients $a^2(y)$ and $n^2(y)$ to be equal to each other, and thereby only allow solutions with a single exponential. (Whether this single exponential itself would then be converging or diverging depends on the sign of $\Lambda_b$.) However, once the brane geometry is lowered to maximally 3-symmetric, two exponential terms with opposite sign exponents are then allowed.
we will then explicitly force the coefficient $\beta$ to vanish and reduce the metric to just one exponential term. Thus on dropping the $\beta$ dependent term in Eq. (24) (a point we shall return to below), and retaining only the $\alpha$ dependent one, we then find that the discontinuity condition at the brane then fixes the sign of $\nu$ according

$$\nu = -\kappa_5^2 \rho_b / 3,$$

(32)

with the geometry now being localized to the brane according to

$$a^2(y) = \alpha e^{\nu |y|}, \quad n^2(y) = \alpha e^{\nu |y|},$$

(33)

when $\rho_b$ is positive (for $\rho_b$ negative no localization would be obtained), with Eq. (30) then yielding the compatibility condition

$$\Lambda_5 + \kappa_5^2 \rho_b^2 / 6 = 0.$$  

(34)

Thus when $\beta$ is set to zero the exponential dependence associated with Eq. (33) nicely quenches the linear metric dependence found in Eq. (12) in the empty bulk case just as desired, while precisely imposing on the brane the equation of state associated with a cosmological constant $\lambda$, with the condition $-\Lambda_5 = \kappa_5^2 \rho_b^2 / 6 \equiv \kappa_5^2 \lambda^2 / 6$ then entailing the vanishing of the net effective brane cosmological constant $\Lambda_4$ of the general Eq. (19) just as found in the original Randall Sundrum study.

Returning now to the more general $\beta$ dependent case given in Eq. (26), we see that it is its more general equation of state $\Lambda_5 - \kappa_5^2 \rho_b (\rho_b + 3 p_b) / 12 = 0$, rather than the restricted one of Eq. (34), which actually matches on continuously to the $p_b = -\rho_b / 3$ equation of state obtained earlier as Eq. (13) in the $\Lambda_5 = 0$ case. The reason for this is due to the limiting process needed to extract the term linear in $|y|$ as needed for Eq. (12) from a function only containing exponentials, with it being only a linear combination of two exponentials with appropriately chosen singular coefficients ($\sim (1 \pm 1/\nu)$) which can generate a non-vanishing linear term when the exponent $\nu = (-2\kappa_5^2 \Lambda_5 / 3)^{1/2}$ is allowed to go to zero. Hence, by not retaining the $\beta$ dependent term, we have then taken a $\Lambda_5$ dependent metric, viz. that of Eq. (33), whose $\Lambda_5 \to 0$ limit does not generate any term linear in $|y|$. It is thus our assumed $\beta$ vanishing $|y| = \infty$ boundary condition which takes care of the Eq. (12) empty bulk linear term, with the parameter $\beta$ itself actually only vanishing when the very specific equation of state $p_b = -\rho_b$ is imposed, to thus then lead to a geometry which is exponentially suppressed as we go away from the brane. As we shall now show, however, in the presence of a non-zero spatial curvature, i.e. in the presence of a non-trivial topology on the brane, even picking a brane field configuration which retains only the exponentially damped $\alpha$ dependent term will not in fact prove sufficient to suppress the geometry away from the brane.

In the $k \neq 0$ non-empty bulk case, Eqs. (26), (28) and (29) are found to be replaced by

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22Equation (19) thus explains why this condition is in fact quadratic in $\lambda$.

23Since we should in principle be able to consider brane field configurations other than this particular one, we see that in general some configurations lead to suppression and some do not.
\[ f(y) = \alpha e^{\nu|y|} + \beta e^{-\nu|y|} - 2k/\nu^2, \quad e(y) = \alpha e^{\nu|y|} - \beta e^{-\nu|y|}, \quad (35) \]

\[ 3\nu(\alpha - \beta) = -(\alpha + \beta - 2k/\nu^2)\kappa_5^2\rho_b, \quad (36) \]

\[ 6\nu(\alpha + \beta) = (\alpha - \beta)\kappa_5^2(\rho_b + 3p_b), \quad (37) \]

where again \( \nu^2 = -2\kappa_5^2\Lambda_5/3 \). If we now drop the \( \beta e^{\nu|y|} \) type term by hand by requiring the matter fields to obey \( \kappa_5^2(3p_b + \rho_b)^2 + 24\Lambda_5 = 0 \), the solution reduces to

\[ a^2(y) = \alpha e^{\nu|y|} + 3k/\kappa_5^2\Lambda_5, \quad n^2(y) = \alpha e^{\nu|y|}/(1 + 3ke^{-\nu|y|}/\alpha\kappa_5^2\Lambda_5), \quad (38) \]

where

\[ \nu = (-2\kappa_5^2\Lambda_5/3)^{1/2} = -\kappa_5^2\rho_b(1 + 3k/\alpha\kappa_5^2\Lambda_5)/3, \quad p_b = -\rho_b(1 + 2k/\alpha\kappa_5^2\Lambda_5), \quad (39) \]

with the net brane cosmological constant \( \Lambda_4 \) vanishing this time if the brane matter energy density and brane cosmological constant (defined via \( \rho_b = \rho_m + \lambda \)) are fine-tuned according to

\[ \lambda = -\rho_m(1 + \alpha\kappa_5^2\Lambda_5/3k), \quad (40) \]

so that the brane matter pressure defined via \( p_b = p_m - \lambda \) is then related to the brane matter energy density according to \( p_m = -\rho_m/3 \). As we thus see, even though the \( AdS_5 \) exponential damping factor does make an appearance in the \( k \neq 0 \) case, nonetheless we find that \( a^2(y) \) is not asymptotically suppressed far away from the brane. Rather \( a^2(y) \) tends to the non-vanishing value \( 3k/\kappa_5^2\Lambda_5 \) if \( k \) is negative (even as \( n^2(y) \) is then being suppressed), while if \( k \) is positive \( n(y) \) actually becomes singular. This lack of suppression is also evidenced in the Riemann tensor, with its \( R^{12}_{12} \) component for instance, viz.

\[ R^{12}_{12} = f'^2/4f^2 - k/f, \quad (41) \]

tending to \(-\kappa_5^2\Lambda_5/3\) at large \( y \) (viz. twice the pure \( AdS_5 \) value), with the bulk embedding thus never being able to counteract the effect of the spatial curvature of the brane, no matter how judicious a choice of matter fields we may make. A similar situation is also found for the Weyl tensor where

\[ C^{12}_{12} = [-2eff'' + 3ef'^2 - 2efk - 3ef'f' + 2f^2e'']/12ef^2 \quad (42) \]

\[ ^{24}\text{For metrics of the form given in Eq. (3) all non-vanishing components of the Weyl tensor are kinematically proportional to } C^{12}_{12}. \]
asymptotes to the non-vanishing value $-\kappa_5^2 \Lambda_5/6$. Thus, unlike pure $AdS_5$, the metric associated with Eq. (38) is not conformal to flat $^{24}$ with the bulk cosmological constant $\Lambda_5$ term not being able to completely quench the quadratic growth previously found for $a^2(y)$ in Eq. (21) in the $k \neq 0$, $\Lambda_5 = 0$ case. $^7$ Thus, to conclude, we see that while embedding in a higher dimensional $\Lambda_5 < 0$ bulk might lead to a brane-localized geometry in certain specific cases, it would appear from study of our somewhat idealized static cosmological model that such embeddings may not always lead to exponential suppression in general, with localization thus needing to be checked on a case by case basis.

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$^{25}$As Eq. (12) also shows, even when $e(y) = f(y) = 1$, $C_{12}^{12}$ is equal to $-k/6$ and still does not vanish. Thus even while a constant curvature 3 space embedded in an otherwise flat 4 space (viz. standard 4-dimensional Robertson-Walker) produces a 4-dimensional metric which is conformal to flat, the same is not true of the same 3 space embedded in an otherwise flat 5 space, something a bulk cosmological constant is simply unable to alter. With only the $k = 0$ Robertson-Walker geometry thus being localizable by an $AdS_5$ embedding, it would be of interest to see whether geometries with non-zero $k$ but with a negligibly small current era value of $\Omega_k(t) = -k^2/\dot{R}^2(t)$ could still admit of an effective brane-localized current era gravity.

$^{26}$It is possible to force the bulk Weyl tensor to vanish, to then force the bulk to actually be $AdS_5$. However, this requires the reinstatement of the divergent $\beta$ type term, with the coefficients in the metric of Eq. (35) having to then obey $\alpha \beta = k^2/\nu^4 \neq 0$, and with the fields then having to be related according to $\kappa_5^2 \rho_b (2\rho_b + 3p_b) - 6\Lambda_5 = 0$. Thus even in this case there is still no localization of gravity. Further details of this particular case are presented separately in $^{24}$.

$^{27}$While, as had already been noted above, the bulk geometry would not be pure $AdS_5$ in the event that the brane matter source was spatially inhomogeneous, we also see that even when the brane distribution is homogeneous, the bulk geometry may still not be pure $AdS_5$. 

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