PRODUCT GENERALIZED LOCAL MORREY SPACES AND COMMUTATORS OF MULTI-SUBLINEAR OPERATORS GENERATED BY MULTILINEAR CALDERÓN-ZYGMUND OPERATORS AND LOCAL CAMPANATO FUNCTIONS

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Abstract. The aim of this paper is to get the boundedness of the commutators of multi-sublinear operators generated by local campanato functions and multilinear Calderón-Zygmund operators on the product generalized local Morrey spaces.

1. Introduction and main results

Because of the need for study of the local behavior of solutions of second order elliptic partial differential equations (PDEs) and together with the now well-studied Sobolev Spaces, constitute a formidable three parameter family of spaces useful for proving regularity results for solutions to various PDEs, especially for non-linear elliptic systems, in 1938, Morrey [17] introduced the classical Morrey spaces $L^p,\lambda$ which naturally are generalizations of the classical Lebesgue spaces.

We will say that a function $f \in L^p,\lambda = L^p,\lambda(R^n)$ if

\[ \sup_{x \in R^n, r > 0} \left[ r^{-\lambda} \int_{B(x, r)} |f(y)|^p \, dy \right]^{1/p} < \infty. \]  

(1.1)

Here, $1 < p < \infty$ and $0 < \lambda < n$ and the quantity of (1.1) is the $(p, \lambda)$-Morrey norm, denoted by $\|f\|_{L^p,\lambda}$. We also refer to [1] for the latest research on the theory of Morrey spaces associated with Harmonic Analysis. In recent years, more and more researches focus on function spaces based on Morrey spaces to fill in some gaps in the theory of Morrey type spaces (see, for example, [9, 10, 11, 12, 13, 18]). Moreover, these spaces are useful in harmonic analysis and PDEs. But, this topic exceeds the scope of this paper. Thus, we omit the details here.

First of all, we recall some explanations and notations used in the paper.

Recall that the concept of the generalized local (central) Morrey space $LM^{x_0}_{p,\varphi}$ has been introduced in [2] and studied in [9, 10].

Definition 1. (Generalized local (central) Morrey space) Let $\varphi(x, r)$ be a positive measurable function on $R^n \times (0, \infty)$ and $1 \leq p < \infty$. For any fixed $x_0 \in R^n$ we denote by $LM^{x_0}_{p,\varphi}$ the generalized local Morrey space, the space...

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of all functions \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \) with finite quasinorm
\[
\|f\|_{LM_p(x_0)} = \sup_{r>0} \varphi(x_0, r)^{-1} |B(x_0, r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x_0, r))} < \infty.
\]

According to this definition, we recover the local Morrey space \( LL_{p,\lambda}^{(x_0)} \) under the choice \( \varphi(x_0, r) = r^{\frac{\lambda-n}{p}} \):
\[
LL_{p,\lambda}^{(x_0)} = LM_{p,\varphi}^{(x_0)} |_{\varphi(x_0, r) = r^{\frac{\lambda-n}{p}}}.
\]

For the properties and applications of generalized local (central) Morrey spaces \( LM_{p,\varphi} \), see also [2, 9, 10].

On the other hand, in 1976, Coifman et al. [4] introduced the commutator \( T_b \) generated by the Calderón-Zygmund operator \( T \) and a locally integrable function \( b \) as follows:
\[
(1.2) \quad T_b f(x) = [b, T]f(x) \equiv b(x)T f(x) - T (b f)(x) = \int_{\mathbb{R}^n} K(x, y) [b(x) - b(y)] f(y) dy,
\]
with the kernel \( K \) satisfying the following size condition:
\[
K(x, y) \leq C |x - y|^{-n}, \quad x \neq y,
\]
and some smoothness assumption. A celebrated result is that \( T \) is bounded operator on \( L^p \) space, where \( 1 < p < \infty \). Sometimes, the commutator defined by (1.2) is also called the commutator in Coifman et al.’s sense, which has its root in the complex analysis and harmonic analysis (see [4]). The main result from [4] states that, if and only if \( b \in \text{BMO} \) (bounded mean oscillation space), \( T_b \) is a bounded operator on \( L^p (\mathbb{R}^n) \), \( 1 < p < \infty \). It is worth noting that for a constant \( C \), if \( T \) is linear we have,
\[
[b + C, T] f = (b + C) T f - T ((b + C) f) = bT f + C T f - T (b f) - C T f = [b, T] f.
\]
This leads one to intuitively look to spaces for which we identify functions which differ by constants, and so it is no surprise that \( b \in \text{BMO} \) or \( LC_{q,\lambda}^{(x_0)} (\mathbb{R}^n) \) (local Campanato space) has had the most historical significance. Also, the definition and some properties of local Campanato space \( LC_{q,\lambda}^{(x_0)} (\mathbb{R}^n) \) that we need in the proof of commutators are as follows.

**Definition 2.** [2, 9] Let \( 1 \leq q < \infty \) and \( 0 \leq \lambda < \frac{1}{n} \). A local Campanato function \( b \in L^q_{\text{loc}} (\mathbb{R}^n) \) is said to belong to the \( LC_{q,\lambda}^{(x_0)} (\mathbb{R}^n) \), if
\[
(1.3) \quad \|b\|_{LC_{q,\lambda}^{(x_0)}} = \sup_{r>0} \left( \frac{1}{|B(x_0, r)|^{1+\lambda q}} \int_{B(x_0, r)} \|b(y) - b_{B(x_0, r)}\|_q^q dy \right)^{\frac{1}{q}} < \infty,
\]
where
\[
b_{B(x_0, r)} = \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} b(y) dy.
\]
Define
\[ \text{LC}_{q,\lambda}^{(x_0)}(\mathbb{R}^n) = \left\{ b \in L^q_{\text{loc}}(\mathbb{R}^n) : \|b\|_{\text{LC}_{q,\lambda}^{(x_0)}} < \infty \right\}. \]

**Remark 1.** If two functions which differ by a constant are regarded as a function in the space \( \text{LC}_{q,\lambda}^{(x_0)}(\mathbb{R}^n) \), then \( \text{LC}_{q,\lambda}^{(x_0)}(\mathbb{R}^n) \) becomes a Banach space. The space \( \text{LC}_{q,\lambda}^{(x_0)}(\mathbb{R}^n) \) when \( \lambda = 0 \) is just the \( \text{LC}_q^{(x_0)}(\mathbb{R}^n) \).Apparently, (1.3) is equivalent to the following condition:
\[
\sup_{r > 0} \inf_{c \in C} \left( \frac{1}{|B(x_0, r)|^{1+q}} \int_{B(x_0, r)} |b(y) - c|^q \, dy \right)^{\frac{1}{q}} < \infty.
\]

Also, in [10], Lu and Yang have introduced the central BMO space \( \text{CBMO}_q(\mathbb{R}^n) = \text{LC}_{q,0}^{(0)}(\mathbb{R}^n) \). Note that \( \text{BMO}(\mathbb{R}^n) \subset \bigcap_{q > 1} \text{LC}_q^{(x_0)}(\mathbb{R}^n), 1 \leq q < \infty \). Moreover, one can imagine that the behavior of \( \text{LC}_{q,\lambda}^{(x_0)}(\mathbb{R}^n) \) may be quite different from that of \( \text{BMO}(\mathbb{R}^n) \), since there is no analogy of the famous John-Nirenberg inequality of \( \text{BMO}(\mathbb{R}^n) \) for the space \( \text{LC}_{q,\lambda}^{(x_0)}(\mathbb{R}^n) \).

**Lemma 1.** [2, 9] Let \( b \) be a local Campanato function in \( \text{LC}_{q,\lambda}^{(x_0)}(\mathbb{R}^n), 1 \leq q < \infty, 0 \leq \lambda < \frac{1}{n} \) and \( r_1, r_2 > 0 \). Then
\[
\left( \frac{1}{|B(x_0, r_1)|^{1+q}} \int_{B(x_0, r_1)} |b(y) - b_{B(x_0, r_2)}|^q \, dy \right)^{\frac{1}{q}} \leq C \left( 1 + \left| \ln \frac{r_1}{r_2} \right| \right) \|b\|_{\text{LC}_{q,\lambda}^{(x_0)}},
\]
where \( C > 0 \) is independent of \( b, r_1 \) and \( r_2 \).

From this inequality (1.4), we have
\[
|b_{B(x_0, r_1)} - b_{B(x_0, r_2)}| \leq C \left( 1 + \ln \frac{r_1}{r_2} \right) |B(x_0, r_1)|^\lambda \|b\|_{\text{LC}_{q,\lambda}^{(x_0)}},
\]
and it is easy to see that
\[
\|b - (b)_{B}\|_{L_q(B)} \leq C \left( 1 + \ln \frac{r_1}{r_2} \right) r_i^{q+\lambda} \|b\|_{\text{LC}_{q,\lambda}^{(x_0)}}.
\]

**Remark 2.** Let \( x_0 \in \mathbb{R}^n, 1 < p_i, q_i < \infty, \) for \( i = 1, \ldots, m \) such that \( \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m} \) and \( \vec{b} \in \text{LC}_{q_i,\lambda_i}^{(x_i)}(\mathbb{R}^n) \) for \( 0 \leq \lambda_i < \frac{1}{n}, i = 1, \ldots, m \). Then, from Lemma (7), it is easy to see that
\[
\|b_i - (b_i)_{B}\|_{L_{q_i}(B)} \leq C r_i^{q_i+\lambda_i} \|b_i\|_{\text{LC}_{q_i,\lambda_i}^{(x_i)}},
\]
and
\[
\|b_i - (b_i)_{B}\|_{L_{q_i}(2B)} \leq \|b_i - (b_i)_{2B}\|_{L_{q_i}(2B)} + \|(b_i)_{B} - (b_i)_{2B}\|_{L_{q_i}(2B)} \lesssim r_i^{q_i+\lambda_i} \|b_i\|_{\text{LC}_{q_i,\lambda_i}^{(x_i)}}.
\]

for \( i = 1, 2 \).

On the other hand, multilinear Calderón-Zygmund theory is a natural generalization of the linear case. The initial work on the class of multilinear Calderón-Zygmund operators has been done by Coifman and Meyer in [3]. Moreover, the
study of multilinear singular integrals has motivated not only as the generalization of the theory of linear ones but also their natural appearance in analysis. It has received increasing attention and much development in recent years, such as the study of the bilinear Hilbert transform by Lacey and Thiele [14, 15] and the systematic treatment of multilinear Calderón-Zygmund operators by Grafakos-Torres [6, 7, 8] and Grafakos-Kalton [5]. Meanwhile, the commutators generated by the multilinear singular integral and BMO functions of Lipschitz functions also attract much attention, since the commutator is more singular than the singular integral operator itself.

Let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space of points \( x = (x_1, \ldots, x_n) \) with norm \( |x| = \left( \sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} \) and \((\mathbb{R}^n)^m = \mathbb{R}^n \times \ldots \times \mathbb{R}^n\) be the \( m \)-fold product spaces \((m \in \mathbb{N})\). For \( x \in \mathbb{R}^n \) and \( r > 0 \), we denote by \( B(x, r) \) the open ball centered at \( x \) of radius \( r \), and by \( B^C(x, r) \) denote its complement and \( |B(x, r)| \) is the Lebesgue measure of the ball \( B(x, r) \) and \( |B(x, r)| = v_n r^n \), where \( v_n = |B(0, 1)| \). Throughout this paper, we denote by \( \vec{y} = (y_1, \ldots, y_m) \), \( d\vec{y} = dy_1 \ldots dy_m \), and by \( \vec{f} \) the \( m \)-tuple \((f_1, \ldots, f_m)\), \( m, n \) the nonnegative integers with \( n \geq 2 \), \( m \geq 1 \).

Suppose that \( T^{(m)} \) represents a multilinear or a multi-sublinear operator, which satisfies that for any \( m \in \mathbb{N} \) and \( \vec{f} = (f_1, \ldots, f_m) \), suppose each \( f_i \) \((i = 1, \ldots, m)\) is integrable on \( \mathbb{R}^n \) with compact support and \( x \notin \bigcap_{i=1}^{m} \text{supp} f_i \),

\[(1.8) \quad \left| T^{(m)} (\vec{f}) (x) \right| \leq c_0 \int_{(\mathbb{R}^n)^m} \frac{1}{| (x - y_1, \ldots, x - y_m) |^{mn}} \left\{ \prod_{i=1}^{m} |f_i (y_i)| \right\} \, d\vec{y}, \]

where \( c_0 \) is independent of \( \vec{f} \) and \( x \).

We point out that the condition (1.8) in the case of \( m = 1 \) was first introduced by Soria and Weiss in [14]. The condition (1.8) is satisfied by many interesting operators in harmonic analysis, such as the \( m \)-linear Calderón-Zygmund operators, \( m \)-sublinear Carleson’s maximal operator, \( m \)-sublinear Hardy–Littlewood maximal operator, C. Fefferman’s singular multipliers, R. Fefferman’s \( m \)-linear singular integrals, Ricci–Stein’s \( m \)-linear oscillatory singular integrals, the \( m \)-linear Bochner–Riesz means and so on (see [2, 9, 10, 19] for details).

We are going to be working on \( \mathbb{R}^n \). Let’s begin with the recalling of the multilinear Calderón-Zygmund operator \( \mathcal{T}^{(m)} (m \in \mathbb{N}) \). Let \( \vec{f} \in L^1_{\text{loc}}(\mathbb{R}^n) \times \ldots \times L^1_{\text{loc}}(\mathbb{R}^n) \). The \( m \)(multi)-linear Calderón-Zygmund operator \( \mathcal{T}^{(m)} \) is defined by

\[(1.9) \quad \left| T^{(m)} (\vec{f}) (x) \right| = T^{(m)} (f_1, \ldots, f_m) (x) = \int_{(\mathbb{R}^n)^m} K (x, y_1, \ldots, y_m) \left\{ \prod_{i=1}^{m} f_i (y_i) \right\} \, dy_1 \cdots dy_m, \]

for test vector \( \vec{f} = (f_1, \ldots, f_m) \) and \( x \notin \bigcap_{i=1}^{m} \text{supp} f_i \), where \( K \) is an \( m \)-Calderón-Zygmund kernel which is a locally integrable function defined away from the diagonal \( y_0 = y_1 = \cdots = y_m \) on \((\mathbb{R}^n)^{m+1}\) satisfying the following size estimate:
for some $C > 0$ and some smoothness estimates, see [5]-[6] for details.

At the same time, Grafakos and Torres [6] have proved that the multilinear Calderón-Zygmund operator is bounded on the product of Lebesgue spaces.

**Theorem 1.** [6, 8] Let $\mathcal{T}^{(m)}$ is a $m$-linear Calderón-Zygmund operator. Then, for any numbers $1 \leq p_1, \ldots, p_m < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\mathcal{T}^{(m)}$ can be extended to a bounded operator from $L_{p_1} \times \cdots \times L_{p_m}$ into $L_p$, and bounded from $L_1 \times \cdots \times L_1$ into $L_{\frac{p}{p-1}}$.

Recently, Xu [20] has established the boundedness on the product of Lebesgue space for the commutators generated by $m$-linear Calderón-Zygmund singular integrals and RBMO functions with nonhomogeneous. Inspired by [6], [8], [20], we will introduce the commutators $\mathcal{T}^{(m)}_{b}$ generated by $m$-linear Calderón-Zygmund operators $\mathcal{T}^{(m)}$ and local Campanato functions $b = (b_1, \ldots, b_m)$

$$\mathcal{T}^{(m)}_{b} \left( \mathcal{F} \right)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) \left[ \prod_{i=1}^{m} \left[ b_i(x) - b_i(y_i) \right] f_i(y_i) \right] d\mathcal{F}(y),$$

where $K(x, y_1, \ldots, y_m)$ is a $m$-linear Calderón-Zygmund kernel, $b_i \in LC^{(x_i)}_{\lambda_i, \lambda}(\mathbb{R}^n)$ (local Campanato spaces) for $0 \leq \lambda_i < \frac{1}{n}$, $i = 1, \ldots, m$. We would like to point out that $\mathcal{T}^{(m)}_b$ is the special case of $\mathcal{T}^{(m)}_b$ with taking $m = 1$.

Closely related to the above results, in this paper in the case of $b_i \in LC^{(x_i)}_{\lambda_i, \lambda}(\mathbb{R}^n)$ for $0 \leq \lambda_i < \frac{1}{n}$, $i = 1, \ldots, m$, we find the sufficient conditions on $(\varphi_1, \ldots, \varphi_m, \varphi)$ which ensures the boundedness of the commutator operators $\mathcal{T}^{(m)}_{b}$ from $LM^{(x_1)}_{\varphi_1} \times \cdots \times LM^{(x_m)}_{\varphi_m}$ to $LM^{(x_o)}_{\varphi}$, where $1 < p_1, q_i < \infty$, for $i = 1, \ldots, m$ such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m}$. In fact, in this paper the results of [2] and [6] (by taking $\Omega \equiv 1$ there) will be generalized to the multilinear case; we omit the details here.

**Remark 3.** Our results in this paper remain true for the inhomogeneous versions of local Campanato spaces $LC^{(x_o)}_{\lambda, \lambda}(\mathbb{R}^n)$ for $0 \leq \lambda < \frac{1}{n}$ and generalized local Morrey spaces $LM^{(x_o)}_{\varphi}$.

We now make some conventions. Throughout this paper, we use the symbol $A \lesssim B$ to denote that there exists a positive constant $C$ such that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, we then write $A \approx B$ and say that $A$ and $B$ are equivalent. For a fixed $p \in [1, \infty)$, $p'$ denotes the dual or conjugate exponent of $p$, namely, $p' = \frac{p}{p-1}$ and we use the convention $1' = \infty$ and $\infty' = 1$.

Our main results can be formulated as follows.

**Theorem 2.** Let $x_0 \in \mathbb{R}^n$, $1 < p_1, q_i < \infty$, for $i = 1, \ldots, m$ such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m}$ and $b \in LC^{(x_o)}_{\lambda, \lambda}(\mathbb{R}^n)$ for $0 \leq \lambda_i < \frac{1}{n}$, $i = 1, \ldots, m$. Let also, $T^{(m)}$ ($m \in \mathbb{N}$) be a multilinear operator satisfying condition (LS), bounded
Corollary 2. Let \( \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} \) and \( \vec{b} \in BMO^m(\mathbb{R}^n) \) for \( i = 1, \ldots, m \). Let also, \( T^{(m)}(m \in \mathbb{N}) \) be a multilinear operator satisfying condition \((L8)\), bounded from \( L_{p_1} \times \cdots \times L_{p_m} \) into \( L_p \) for \( p_i > 1(i = 1, \ldots, m) \). Moreover, we have for \( p_i > 1(i = 1, \ldots, m) \)

\[
\| T^{(m)}_{\vec{b}}(\vec{f}) \|_{L_p(B(x_0, r))} \lesssim \prod_{i=1}^m \| b_i \|_{\mathcal{L}C^{(\xi_0)}_{\lambda_i}} \prod_{i=1}^m \| f_i \|_{\mathcal{L}C^{(\xi_0)}_{\lambda_i}}
\]

for the \( m \)-sublinear commutator of the \( m \)-sublinear maximal operator

\[
M^{(m)}_{\vec{b}}(\vec{f})(x) = \sup_{t > 0} \frac{1}{|B(x, t)|} \int_{B(x, t)} \prod_{i=1}^m |b_i(x) - b_i(y_i)| |f_i(y_i)| \, dy
\]

from Theorem 3 we get the following new results.

Corollary 1. Let \( x_0 \in \mathbb{R}^n, 1 < p_i, q_i < \infty, \) for \( i = 1, \ldots, m \) such that \( \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{q_1} + \cdots + \frac{1}{q_m} \) and \( \vec{b} \in L^{1+\ln t}_r(B(x_0, t)) \) for \( 0 \leq \lambda_i < \frac{1}{n}, i = 1, \ldots, m \) and \( (\varphi_1, \ldots, \varphi_m, \varphi) \) satisfies condition \((L8)\). Then, the operators \( M^{(m)}_{\vec{b}} \) and \( T^{(m)}_b \) \( (m \in \mathbb{N}) \) are bounded from product space \( L^{p_1}_{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}_{p_m}(\mathbb{R}^n) \) to \( L^{p}_{p}(\mathbb{R}^n) \) for \( p_i > 1(i = 1, \ldots, m) \).

Remark 4. Note that, in the case of \( m = 1 \) Theorem 3 and Corollary 1 have been proved in [20].

Corollary 2. Let \( 1 < p_i, q_i < \infty, \) for \( i = 1, \ldots, m \) such that \( \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_n} \) and \( \vec{b} \in BMO^m(\mathbb{R}^n) \) for \( i = 1, \ldots, m \). Let also, \( T^{(m)}(m \in \mathbb{N}) \) be a multilinear operator satisfying condition \((L8)\), bounded from \( L_{p_1} \times \cdots \times L_{p_m} \) into \( L_p \) for \( p_i > 1(i = 1, \ldots, m) \).
1 (i = 1, . . . , m). If functions $\varphi, \varphi_i : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$, (i = 1, . . . , m) and $(\varphi_1, \ldots, \varphi_m, \varphi)$ satisfies the condition

$$\int_r^{\infty} \left(1 + \ln \frac{1}{r}\right)^m \frac{\text{essinf}_{t<r<\infty} \prod_{i=1}^m \varphi_i(x, \tau) T_r^\tau}{n \sum_{i=1}^n \frac{1}{p_i} + 1} \, dt \leq C \varphi(x, r),$$

where $C$ does not depend on $r$.

Then the operator $T(t)^{(m)}_b$ (m $\in \mathbb{N}$) is bounded from product space $M_{p_1, \varphi_1} \times \cdots \times M_{p_m, \varphi_m}$ to $M_{p, \varphi}$ for $p_i > 1$ (i = 1, . . . , m). Moreover, we have for $p_i > 1$ (i = 1, . . . , m)

$$\left\| T(t)^{(m)}_b \left( \overline{f} \right) \right\|_{M_{p, \varphi}} \lesssim \prod_{i=1}^m \left\| \overline{b} \right\|_{BMO} \prod_{i=1}^m \| f_i \|_{M_{p_i, \varphi_i}} .$$

**Corollary 3.** Let $1 < p_i, q_i < \infty$, for $i = 1, \ldots, m$ such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\overline{b} \in BMO^m (\mathbb{R}^n)$ for $i = 1, \ldots, m$ and also $(\varphi_1, \ldots, \varphi_m, \varphi)$ satisfies condition L. Then, the operators $M_{p_i}^{(m)}$ and $\overline{T(t)}_b^{(m)}$ (m $\in \mathbb{N}$) are bounded from product space $M_{p_1, \varphi_1} \times \cdots \times M_{p_m, \varphi_m}$ to $M_{p, \varphi}$ for $p_i > 1$ (i = 1, . . . , m).

2. **Proofs of the main results**

2.1. **Proof of Theorem 2**

Proof. In order to simplify the proof, we consider only the situation when $m = 2$. Actually, a similar procedure works for all $m \in \mathbb{N}$. Thus, without loss of generality, it is suffice to show that the conclusion holds for $T_{(b_1, b_2)}^{(2)} (\overline{f}) = T_{(b_1, b_2)}^{(2)} (f_1, f_2)$.

We just consider the case $p_i, q_i > 1$ for $i = 1, 2$. For any $x_0 \in \mathbb{R}^n$, set $B = B (x_0, r)$ for the ball centered at $x_0$ and of radius $r$ and $2B = B (x_0, 2r)$. Thus, we have the following decomposition,

$$T_{(b_1, b_2)}^{(2)} (f_1, f_2) (x) = \left( (b_1 (x) - \{b_1\}_B) \left[ (b_2 (x) - \{b_2\}_B) \right] T_{(b_1, b_2)}^{(2)} (f_1, f_2) (x) - (b_1 (x) - \{b_1\}_B) \left[ (b_2 (x) - \{b_2\}_B) \right] T_{(b_1, b_2)}^{(2)} (f_1, f_2) (x) \right)$$

$$- [(b_1 (x) - \{b_1\}_B)] T_{(b_1, b_2)}^{(2)} [(b_2 (x) - \{b_2\}_B)] f_1, f_2 (x)$$

$$- [(b_2 (x) - \{b_2\}_B)] T_{(b_1, b_2)}^{(2)} [(b_1 (x) - \{b_1\}_B)] f_1, f_2 (x) + T_{(b_1, b_2)}^{(2)} [(b_1 (x) - \{b_1\}_B)] f_1, f_2 (x)$$

$$\equiv H_1 (x) + H_2 (x) + H_3 (x) + H_4 (x) .$$

Thus,

$$\left\| T_{(b_1, b_2)}^{(2)} (f_1, f_2) \right\|_{L_p (B(x_0, r))} = \left( \int_B \left( \int_B \left| T_{(b_1, b_2)}^{(2)} (f_1, f_2) (x) \right|^p dx \right) \right) \frac{1}{p} \leq \sum_{i=1}^4 \left( \int_B \left| H_i (x) \right|^p dx \right) \frac{1}{p} = \sum_{i=1}^4 G_i .$$

One observes that the estimate of $G_2$ is analogous to that of $G_3$. Thus, we will only estimate $G_1$, $G_2$ and $G_4$.

Indeed, we also decompose $f_i$ as $f_i (y_i) = f_i (y_i) \chi_{2B} + f_i (y_i) \chi_{(2B)'}$ for $i = 1, 2$. And, we write $f_1 = f_1^0 + f_1^\infty$ and $f_2 = f_2^0 + f_2^\infty$, where $f_i^0 = f_i \chi_{2B}$, $f_i^\infty = f_i \chi_{(2B)'}$, for $i = 1, 2.$
For $G_1 = \left\| \left( f_1^0, f_2^0 \right) \right\|_{L_p(B(x_0, r))}$, we decompose it into four parts as follows:

$$G_1 \leq \left\| \left( b_1 - \left\{ b_1 \right\}_B \right) \left[ \left( b_2 - \left\{ b_2 \right\}_B \right) T^2 \left( f_1^0, f_2^0 \right) \right] \right\|_{L_p(B(x_0, r))}$$

Hence, we get

$$\leq \left\| \left[ \left( b_1 - \left\{ b_1 \right\}_B \right) \right] \left[ \left( b_2 - \left\{ b_2 \right\}_B \right) T^2 \right] \left( f_1^0, f_2^0 \right) \right\|_{L_p(B(x_0, r))}$$

$$+ \left\| \left[ \left( b_2 - \left\{ b_2 \right\}_B \right) T^2 \left[ \left( b_1 - \left\{ b_1 \right\}_B \right) f_1^0, f_2^0 \right] \right] \right\|_{L_p(B(x_0, r))}$$

$$+ \left\| \left[ \left( b_1 - \left\{ b_1 \right\}_B \right) T^2 \left[ \left( b_2 - \left\{ b_2 \right\}_B \right) f_1^0, f_2^0 \right] \right] \right\|_{L_p(B(x_0, r))}$$

$$f_1^0, f_2^0 \right\|_{L_p(B(x_0, r))}$$

$$+ \left\| \left[ \left( b_1 - \left\{ b_1 \right\}_B \right) f_1^0, f_2^0 \right] \right\|_{L_p(B(x_0, r))}$$

$$G_{11} + G_{12} + G_{13} + G_{14}.$$
of $T^{(2)}$ from $L_{r_1} \times L_{r_2}$ into $L_p$ (see Theorem 1), Hölder’s inequality and (1.7), we obtain

$$G_{14} \lesssim \| (b_1 - \{b_1\}_B) f_1^0 \|_{L_{r_1}(R^n)} \| (b_2 - \{b_2\}_B) f_2^0 \|_{L_{r_2}(R^n)} \lesssim \| (b_1 - \{b_1\}_B) \|_{L_{r_1}(2B)} \| (b_2 - \{b_2\}_B) \|_{L_{r_2}(2B)} \| f_1 \|_{L_{p_1}(2B)} \| f_2 \|_{L_{p_2}(2B)}$$

$$\lesssim \frac{2}{r} \prod_{i=1}^2 b \| f_i \|_{L_{p_i}(B(x_0,t))} \frac{\int_{r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)}{t^n \left( \left( \frac{1}{q_1} + \frac{1}{q_2} \right) - (\lambda_1 + \lambda_2) \right) + 1}.$$ 

Combining all the estimates of $G_{11}, G_{12}, G_{13}, G_{14}$: there is

$$G_1 = \left\| T_{(b_1,b_2)}^{(2)} (f_1^0, f_2^0) \right\|_{L_p(B(x_0,r))} \lesssim \prod_{i=1}^2 b \| f_i \|_{L_{p_i}(B(x_0,t))} \frac{\int_{r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)}{t^n \left( \left( \frac{1}{q_1} + \frac{1}{q_2} \right) - (\lambda_1 + \lambda_2) \right) + 1}.$$ 

(ii) For $G_2 = \left\| T_{(b_1,b_2)}^{(2)} (f_1^0, f_2^\infty) \right\|_{L_p(B(x_0,r))}$, we also write

$$G_2 \lesssim \left\| (b_1 - \{b_1\}_B) \| (b_2 - \{b_2\}_B) \right\| \left( f_1^0, f_2^\infty \right) \left\| T_{(b_1,b_2)}^{(2)} (f_1^0, f_2^\infty) \right\|_{L_p(B(x_0,r))}$$

$$\lesssim \left\| (b_1 - \{b_1\}_B) \| (b_2 - \{b_2\}_B) \right\| \left( f_1^0, f_2^\infty \right) \left\| T_{(b_1,b_2)}^{(2)} (f_1^0, f_2^\infty) \right\|_{L_p(B(x_0,r))}$$

$$\lesssim \left\| (b_1 - \{b_1\}_B) \right\|_{L_{r_1}(2B)} \left\| (b_2 - \{b_2\}_B) \right\|_{L_{r_2}(2B)} \| f_1 \|_{L_{p_1}(2B)} \| f_2 \|_{L_{p_2}(2B)}$$

$$\lesssim \prod_{i=1}^2 b \| f_i \|_{L_{p_i}(B(x_0,t))} \frac{\int_{r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)}{t^n \left( \left( \frac{1}{q_1} + \frac{1}{q_2} \right) - (\lambda_1 + \lambda_2) \right) + 1}.$$ 

Let $1 < \frac{1}{p}, \frac{1}{q} < \infty$, such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then, using Hölder’s inequality we have

$$G_{21} = \left\| (b_1 - \{b_1\}_B) \| (b_2 - \{b_2\}_B) \right\| \left( f_1^0, f_2^\infty \right) \left\| T_{(b_1,b_2)}^{(2)} (f_1^0, f_2^\infty) \right\|_{L_p(B(x_0,r))}$$

$$\lesssim \left\| (b_1 - \{b_1\}_B) \| (b_2 - \{b_2\}_B) \right\| \left( f_1^0, f_2^\infty \right) \left\| T_{(b_1,b_2)}^{(2)} (f_1^0, f_2^\infty) \right\|_{L_p(B(x_0,r))}$$

$$\lesssim \prod_{i=1}^2 b \| f_i \|_{L_{p_i}(B(x_0,t))} \frac{\int_{r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)}{t^n \left( \left( \frac{1}{q_1} + \frac{1}{q_2} \right) - (\lambda_1 + \lambda_2) \right) + 1}.$$ 

where in the second inequality we have used the following fact:
It is clear that \(|(x_0 - y_1, x_0 - y_2)^{2n} \geq |x_0 - y_2|^{2n}\). By the condition (1.8) with \(m = 2\) and Hölder’s inequality, we have

\[
\left| T^{(2)}(f_1^0, f_2^\infty)(x) \right| \lesssim \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f_1^0(y_1)| |f_2^\infty(y_2)|}{|x - y_1, x - y_2|^{2n}} \, dy_1 \, dy_2
\]

\[
\lesssim \int_{2B} |f_1(y_1)| \, dy_1 \int_{(2B)^c} \frac{|f_2(y_2)|}{|x_0 - y_2|^{2n}} \, dy_2
\]

\[
\approx \int_{2B} |f_1(y_1)| \, dy_1 \int_{(2B)^c} |f_2(y_2)| \int_{|x_0 - y_2|}^{\infty} \frac{dt}{t^{2n+1}} \, dy_2
\]

\[
\lesssim \left\| f_1 \right\|_{L_{p_1}(2B)} |2B|^{1 - \frac{n}{p_1}} \int_{2r}^{\infty} \left\| f_2 \right\|_{L_{p_2}(B(x_0, t))} |B(x_0, t)|^{1 - \frac{n}{p_2}} \frac{dt}{t^{2n+1}}
\]

\[
\lesssim \prod_{i=1}^{2} \left\| f_i \right\|_{L_{p_i}(B(x_0, r))} \frac{dt}{t^{\frac{n}{p_1} + 1}}
\]

where \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\). Thus, the inequality

\[
\left\| T^{(2)}(f_1^0, f_2^\infty) \right\|_{L_{p}(B(x_0, r))} \lesssim \prod_{i=1}^{2} \left\| f_i \right\|_{L_{p_i}(B(x_0, r))} \frac{dt}{t^{\frac{n}{p_1} + 1}}
\]

is valid.

On the other hand, for the estimates used in \(G_{22}, G_{23}\), we have to prove the below inequality:

(2.2)

\[
\left| T^{(2)}\left[f_1^0, \left< b_2, \cdot \right> - \left< b_2, B \right> \right] f_2^\infty(x) \right| \lesssim \left\| b_2 \right\|_{LC_{y_1}^{(y_2)}} \int_{2r}^{\infty} \left( 1 + \frac{t}{r} \right)^{\frac{2}{p_2}} \prod_{i=1}^{2} \left\| f_i \right\|_{L_{p_i}(B(x_0, t))} \frac{dt}{t^{n\left(\frac{1}{p_1} + \frac{1}{p_2} - \lambda_2\right) + 1}}
\]

Indeed, it is clear that \(|(x_0 - y_1, x_0 - y_2)^{2n} \geq |x_0 - y_2|^{2n}\). Moreover, using the conditions (1.9) and (1.8) with \(m = 2\), we have

\[
\left| T^{(2)}\left[f_1^0, \left< b_2, \cdot \right> - \left< b_2, B \right> \right] f_2^\infty(x) \right|
\]

\[
\lesssim \int_{2B} |f_1(y_1)| \, dy_1 \int_{(2B)^c} \frac{|b_2(y_2) - \left< b_2, B \right> |f_2(y_2)|}{|x_0 - y_2|^{2n}} \, dy_2.
\]

It’s obvious that

(2.3)

\[
\int_{2B} |f_1(y_1)| \, dy_1 \lesssim \left\| f_1 \right\|_{L_{p_1}(2B)} |2B|^{1 - \frac{n}{p_1}},
\]
and using Hölder’s inequality and by (1.7)
\[
\int_{(2B)^c} \frac{|b_2(y_2) - \{b_2\}_B|}{|x_0 - y_2|^{2n}} |f_2(y_2)| \, dy_2
\]
\[
\lesssim \int_{(2B)^c} \frac{|b_2(y_2) - \{b_2\}_B|}{|x_0 - y_2|^{2n}} \left( \int_{|x_0 - y_2|^{2n+1}}^\infty \frac{dt}{t} \right) \, dy_2
\]
\[
\lesssim \int_{2r}^\infty \left( B(x_0, t) \right)^{-\frac{n}{p} + \lambda_2} \|f_2\|_{L_{p_2}(B(x_0, t))} \left( B(x_0, t) \right)^{-\frac{n}{p} + \lambda_2} \frac{dt}{t^{2n+1}}
\]
\[
\lesssim \|b_2\|_{L_{C_{\rho_2, \lambda_2}}^{(\rho_0)}} \int_{2r}^\infty \left( 1 + \ln \frac{t}{r} \right) \left( B(x_0, t) \right)^{-\frac{n}{p} + \lambda_2} \|f_2\|_{L_{p_2}(B(x_0, t))} \left( B(x_0, t) \right)^{-\frac{n}{p} + \lambda_2} \frac{dt}{t^{2n+1}}
\]
\[
(2.4) \quad \lesssim \|b_2\|_{L_{C_{\rho_2, \lambda_2}}^{(\rho_0)}} \int_{2r}^\infty \left( 1 + \ln \frac{t}{r} \right) \left( B(x_0, t) \right)^{-\frac{n}{p} + \lambda_2} \|f_2\|_{L_{p_2}(B(x_0, t))} \left( B(x_0, t) \right)^{-\frac{n}{p} + \lambda_2} \frac{dt}{t^{n(\frac{n}{p} + \lambda_2) - \frac{1}{2}}}.
\]
Hence, by (2.3) and (2.4), it follows that:
\[
|T^{(2)} \left[ f_1^0, (b_2(\cdot) - \{b_2\}_B) f_2^\infty \right] (x) |
\]
\[
\lesssim \|b_2\|_{L_{C_{\rho_2, \lambda_2}}^{(\rho_0)}} \|f_1\|_{L_{p_1}(2B)} \|2B\|^{-1} \int_{2r}^\infty \left( 1 + \ln \frac{t}{r} \right) \|f_2\|_{L_{p_2}(B(x_0, t))} \left( B(x_0, t) \right)^{-\frac{n}{p} + \lambda_2} \frac{dt}{t^{n(\frac{n}{p} + \lambda_2) - \frac{1}{2}}}
\]
\[
\lesssim \|b_2\|_{L_{C_{\rho_2, \lambda_2}}^{(\rho_0)}} \int_{2r}^\infty \left( 1 + \ln \frac{t}{r} \right) \left( B(x_0, t) \right)^{-\frac{n}{p} + \lambda_2} \|f_2\|_{L_{p_2}(B(x_0, t))} \left( B(x_0, t) \right)^{-\frac{n}{p} + \lambda_2} \frac{dt}{t^{n(\frac{n}{p} + \lambda_2) - \frac{1}{2}}}.
\]
This completes the proof of inequality (2.2).

Thus, let \( 1 < \tau < \infty \), such that \( \frac{n}{p} = \frac{1}{\tau_1} + \frac{1}{\tau_2} \). Then, to estimate \( G_{22} \), similar to the estimates for \( G_{11} \), using Hölder’s inequality and from (2.3), we get
\[
G_{22} = \left\| \left[ (b_1 - \{b_1\}_B) \right] T^{(2)} [f_1^0, (b_2 - \{b_2\}_B) f_2^\infty] \right\|_{L_r(B(x_0, r))}
\]
\[
\lesssim \left\| (b_1 - \{b_1\}_B) \right\|_{L_{\rho_1}(B)} \left\| T^{(2)} [f_1^0, (b_2 - \{b_2\}_B) f_2^\infty] \right\|_{L_{\tau}(B)}
\]
\[
\lesssim \prod_{i=1}^2 \|b_i\|_{L_{C_{\rho_i, \lambda_i}}^{(\rho_0)}} \left( B(x_0, t) \right)^{-\frac{n}{p} + \lambda_i} \int_{2r}^\infty \left( 1 + \ln \frac{t}{r} \right) \left( B(x_0, t) \right)^{-\frac{n}{p} + \lambda_i} \frac{dt}{t^{n(\frac{n}{p} + \lambda_i) - \frac{1}{2}}}
\]
\[
\lesssim \prod_{i=1}^2 \|b_i\|_{L_{C_{\rho_i, \lambda_i}}^{(\rho_0)}} \int_{2r}^\infty \left( 1 + \ln \frac{t}{r} \right) \left( B(x_0, t) \right)^{-\frac{n}{p} + \lambda_i} \frac{dt}{t^{n(\frac{n}{p} + \lambda_i) - \frac{1}{2}}}
\]
Similarly, $G_{23}$ has the same estimate above, here we omit the details, thus the inequality

$$G_{23} = \left\| [b - \{b\}_{B}] T(x, r) [f_1^0, f_2^\infty] \right\|_{L_p(B(x_0, r))}$$

is valid.

Therefore, by (2.5) we deduce that

$$G_{23} = \left\| [b - \{b\}_{B}] T(x, r) [f_1^0, f_2^\infty] \right\|_{L_p(B(x_0, r))}$$

$$\approx \sum_{i=1}^2 \left\| \overrightarrow{b} \right\|_{LC_{\lambda_i}} \int_{2r}^\infty \left( 1 + \frac{t}{r} \right) \prod_{i=1}^2 \left\| f_i \right\|_{L_p(B(x_0, t))} \frac{dt}{t^\left( \frac{1}{\lambda_i} + \frac{1}{\lambda_2} \right) - (\lambda_1 + \lambda_2)} + 1$$

is valid.

Now we turn to estimate $G_{24}$. Similar to (2.3), we have to prove the following estimate for $G_{24}$:

$$\left| T(x, r) [f_1^0, f_2^\infty] \right|$$

(2.5) \begin{align*}
\lesssim \prod_{i=1}^2 \left\| \overrightarrow{b} \right\|_{LC_{\lambda_i}} \int_{2r}^\infty \left( 1 + \frac{t}{r} \right) \prod_{i=1}^2 \left\| f_i \right\|_{L_p(B(x_0, t))} \frac{dt}{t^\left( \frac{1}{\lambda_i} + \frac{1}{\lambda_2} \right) - (\lambda_1 + \lambda_2)} + 1
\end{align*}

Firstly, using the condition (1.8) with $m = 2$, we have

$$\left| T(x, r) [f_1^0, f_2^\infty] \right|$$

\begin{align*}
\lesssim \prod_{i=1}^2 \left\| \overrightarrow{b} \right\|_{LC_{\lambda_i}} \int_{2r}^\infty \left( 1 + \frac{t}{r} \right) \prod_{i=1}^2 \left\| f_i \right\|_{L_p(B(x_0, t))} \frac{dt}{t^\left( \frac{1}{\lambda_i} + \frac{1}{\lambda_2} \right) - (\lambda_1 + \lambda_2)} + 1
\end{align*}

It’s obvious that

$$G_{24} = \left\| T(x, r) [f_1^0, f_2^\infty] \right\|_{L_p(B(x_0, r))}$$

Then, by (2.4) and (2.6) we get (2.5). This completes the proof of inequality (2.5).

Therefore, by (2.5) we deduce that

$$G_{21} = \left\| T(x, r) [f_1^0, f_2^\infty] \right\|_{L_p(B(x_0, r))}$$

$$\approx \sum_{i=1}^2 \left\| \overrightarrow{b} \right\|_{LC_{\lambda_i}} \int_{2r}^\infty \left( 1 + \frac{t}{r} \right) \prod_{i=1}^2 \left\| f_i \right\|_{L_p(B(x_0, t))} \frac{dt}{t^\left( \frac{1}{\lambda_i} + \frac{1}{\lambda_2} \right) - (\lambda_1 + \lambda_2)} + 1$$

Considering estimates $G_{21}, G_{22}, G_{23}, G_{24}$ together, we get the desired conclusion

$$G_2 = \left\| T(x, r) [f_1^0, f_2^\infty] \right\|_{L_p(B(x_0, r))}$$

$$\approx \prod_{i=1}^2 \left\| \overrightarrow{b} \right\|_{LC_{\lambda_i}} \int_{2r}^\infty \left( 1 + \frac{t}{r} \right) \prod_{i=1}^2 \left\| f_i \right\|_{L_p(B(x_0, t))}$$

$$\times \frac{dt}{t^\left( \frac{1}{\lambda_i} + \frac{1}{\lambda_2} \right) - (\lambda_1 + \lambda_2)} + 1$$

Similar to $G_2$, we can also get the estimates for $F_3$,

$$G_3 = \left\| T(x, r) [f_1^0, f_2^\infty] \right\|_{L_p(B(x_0, r))}$$

$$\approx \prod_{i=1}^2 \left\| \overrightarrow{b} \right\|_{LC_{\lambda_i}} \int_{2r}^\infty \left( 1 + \frac{t}{r} \right) \prod_{i=1}^2 \left\| f_i \right\|_{L_p(B(x_0, t))}$$

$$\times \frac{dt}{t^\left( \frac{1}{\lambda_i} + \frac{1}{\lambda_2} \right) - (\lambda_1 + \lambda_2)} + 1.$$
Finally, for $G_4 = \| T^{(2)}_{(b_1, b_2)} (f_1^\infty, f_2^\infty) \|_{L_p(B(x_0, r))}$, we write

$$G_4 \lesssim \left\| [(b_1 - \{b_1\}) B] [(b_2 - \{b_2\}) B] T^{(2)} (f_1^\infty, f_2^\infty) \right\|_{L_p(B(x_0, r))}$$

$$+ \left\| [(b_1 - \{b_1\}) B] T^{(2)} [(b_1 - \{b_1\}) B, f_2^\infty] \right\|_{L_p(B(x_0, r))}$$

$$+ \left\| [(b_2 - \{b_2\}) B] T^{(2)} [(b_1 - \{b_1\}) B, f_2^\infty] \right\|_{L_p(B(x_0, r))}$$

$$+ \left\| T^{(2)} [(b_1 - \{b_1\}) B] (f_1^\infty, (b_2 - \{b_2\}) B, f_2^\infty) \right\|_{L_p(B(x_0, r))}$$

$$\equiv G_{41} + G_{42} + G_{43} + G_{44}.$$

Now, let us estimate $G_{41}$, $G_{42}$, $G_{43}$, $G_{44}$, respectively.

For the term $G_{41}$, let $1 < \tau < \infty$, such that $\frac{1}{p} = \left( \frac{1}{q_1} + \frac{1}{q_2} \right) + \frac{1}{\tau}$, $\frac{1}{\tau} = \frac{1}{p_1} + \frac{1}{p_2}$.

Then, by Hölder’s inequality we get

$$G_{41} = \left\| [(b_1 - \{b_1\}) B] [(b_2 - \{b_2\}) B] T^{(2)} (f_1^\infty, f_2^\infty) \right\|_{L_p(B(x_0, r))}$$

$$\lesssim \| (b_1 - \{b_1\}) B \|_{L_{q_1}(B)} \| (b_2 - \{b_2\}) B \|_{L_{q_2}(B)} \left\| T^{(2)} (f_1^\infty, f_2^\infty) \right\|_{L_r(B)}$$

$$\lesssim \prod_{i=1}^2 \| b_i \|_{L_{\nu_i}(\nu_i, \lambda_i)} \| B^{(\lambda_1 + \lambda_2) + \left( \frac{\lambda_1}{\nu_1} + \frac{\lambda_2}{\nu_2} \right)} \|_{L_r(B)}$$

$$\lesssim \prod_{i=1}^2 \| b_i \|_{L_{\nu_i}(\nu_i, \lambda_i)} r^\frac{\lambda_1}{\nu_1} \int_2^{\infty} \left( 1 + \ln \frac{t}{r} \right)^2 \prod_{i=1}^2 \| f_i \|_{L_p(B(x_0, t))} dt \frac{dt}{t^{\frac{\lambda_1}{\nu_1} + 1}}$$

where in the second inequality we have used the following fact:
Noting that \(|(x_0 - y_1, x_0 - y_2)|^{2n} \geq |x_0 - y_1|^n |x_0 - y_2|^n\). Using the condition (1.8), with \(m = 2\) and by Hölder’s inequality, we get

\[
\left| T^{(2)}_{\alpha}(f_1^\infty, f_2^\infty)(x) \right| \lesssim \int \int_{\mathbb{R}^n} \frac{|f_1(y_1)\chi(2B)^c|}{|(x_0 - y_1, x_0 - y_2)|^{2n}} dy_1 dy_2 \lesssim \int \int_{(2B)^c} \frac{|f_1(y_1)||f_2(y_2)|}{|x_0 - y_1|^n |x_0 - y_2|^n} dy_1 dy_2
\]

\[
\lesssim \sum_{j=1}^{\infty} \prod_{i=1}^{2} \int_{B(x_0, 2^{j+1}r) \setminus B(x_0, 2^j r)} \frac{|f_i(y_i)|}{|x_0 - y_i|^n} dy_i \lesssim \sum_{j=1}^{\infty} \prod_{i=1}^{2} (2^j r)^{-n} \int_{B(x_0, 2^{j+1}r)} |f_i(y_i)| dy_i \lesssim \sum_{j=1}^{\infty} (2^j r)^{-2n} \prod_{i=1}^{2} \|f_i\|_{L_p(B(x_0, 2^{j+1}r))} \|B(x_0, 2^{j+1}r)|^{1-\frac{1}{p_i}}
\]

\[
\lesssim \sum_{j=1}^{\infty} \int_{2^{j+1}r} \prod_{i=1}^{2} \|f_i\|_{L_p(B(x_0, t))} |B(x_0, t)|^{1-\frac{1}{p_i}} \frac{dt}{t^{2n+1}}
\]

\[
\lesssim \int \prod_{i=1}^{2} \|f_i\|_{L_p(B(x_0, t))} |B(x_0, t)|^{2\left(-\frac{1}{p_1} + \frac{1}{p_2}\right)} \frac{dt}{t^{2n+1}} \lesssim \int \|f_1\|_{L_p(B(x_0, t))} \|f_2\|_{L_{p_2}(B(x_0, t))} \frac{dt}{t^{2n+1}}
\]

where \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\). Thus, for \(p_1, p_2 \in [1, \infty)\) the inequality

\[
\left\| T^{(2)}(f_1^\infty, f_2^\infty) \right\|_{L_p(B(x_0, r))} \lesssim r^{\frac{n}{p}} \int \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0, t))} \frac{dt}{t^{2n+1}}
\]

is valid.

For the terms \(G_{42}, G_{43}\), similar to the estimates used for (2.2), we have to prove the following inequality:

(2.7)

\[
\left| T^{(2)}(f_1^\infty, (b_2 - b_2 B) f_2^\infty)(x) \right| \lesssim \|b_2\|_{L_C^{(s_1,s_2)}} \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^{2\left(-\frac{1}{s_1} + \frac{1}{s_2}\right)} \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0, t))} \frac{dt}{t^{n\left(-\frac{1}{s_1} + \frac{1}{s_2}\right)-1}}
\]

Indeed, noting that \(|(x_0 - y_1, x_0 - y_2)|^{2n} \geq |x_0 - y_1|^n |x_0 - y_2|^n\). Recalling the estimates used for \(G_{22}, G_{23}, G_{24}\) and also using the condition (1.8) with \(m = 2\), we
have

\[
\left| T^{(2)} \left[ f_1^\infty, (b_2 - \{b_2\}_B) f_2^\infty \right](x) \right|
\]

\[
\lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|b_2(y_2) - \{b_2\}_B| |f_1(y_1)| \chi_{(2B)^c} |f_2(y_2)| \chi_{(2B)^c}}{|(x_0 - y_1, x_0 - y_2)|^{2n}} dy_1 dy_2
\]

\[
\lesssim \int_{(2B)^c(2B)^c} \int_{B(x_0, 2^j r)} \frac{|f_1(y_1)| |f_2(y_2)|}{|x_0 - y_1|^{n}} dy_1 \int_{B(x_0, 2^j r)} \frac{|b_2(y_2) - \{b_2\}_B| |f_2(y_2)|}{|x_0 - y_2|^{n}} dy_2
\]

It’s obvious that

\[
(2.8) \quad \int_{B(x_0, 2^j r)} |f_1(y_1)| dy_1 \lesssim \|f_1\|_{L_{p_1}(B(x_0, 2^{j+1} r))} \|B(x_0, 2^{j+1} r)\|^{1 - \frac{1}{p_1}}
\]

and using Hölder’s inequality and by (1.7)

\[
\int_{B(x_0, 2^{j+1} r)} |b_2(y_2) - \{b_2\}_B| |f_2(y_2)| dy_2
\]

\[
\lesssim \|b_2(y_2) - \{b_2\}_B\|_{L_{q_2}(B(x_0, 2^{j+1} r))} \|f_2\|_{L_{p_2}(B(x_0, 2^{j+1} r))} \|B(x_0, 2^{j+1} r)\|^{1 - \frac{1}{q_2} + \frac{1}{p_2}}
\]

\[
+ \|\{b_2\}_B - \{b_2\}_B\|_{L_{q_2}(B(x_0, 2^{j+1} r))} \|f_2\|_{L_{p_2}(B(x_0, 2^{j+1} r))} \|B(x_0, 2^{j+1} r)\|^{1 - \frac{1}{q_2}}
\]

\[
\lesssim \|b_2\|_{L_C(x_0)} \|B(x_0, 2^{j+1} r)\|^{\frac{1}{q_2} + \lambda_2} \|f_2\|_{L_{p_2}(B(x_0, 2^{j+1} r))} \|B(x_0, 2^{j+1} r)\|^{1 - \frac{1}{q_2} + \frac{1}{p_2}}
\]

\[
+ \|b_2\|_{L_C(x_0)} \left(1 + \ln \frac{2^{j+1} r}{r}\right) \|B(x_0, 2^{j+1} r)\|^{\lambda_2} \|f_2\|_{L_{p_2}(B(x_0, 2^{j+1} r))} \|B(x_0, 2^{j+1} r)\|^{1 - \frac{1}{q_2} + \frac{1}{p_2}}
\]

(2.9)

\[
\lesssim \|b_2\|_{L_C(x_0)} \int_{2r}^{\infty} \left(1 + \ln \frac{2^{j+1} r}{r}\right)^2 \|B(x_0, 2^{j+1} r)\|^{\lambda_2 - \frac{1}{q_2} + \frac{1}{p_2}} \|f_2\|_{L_{p_2}(B(x_0, 2^{j+1} r))}.
\]
Hence, by (2.8) and (2.9), it follows that:

\[
\left| T^{(2)} \left[ f_1^\infty, (b_2 - \{b_2\}_B) f_2^\infty \right] (x) \right| \\
\lesssim \sum_{j=1}^\infty (2^j r)^{-2n} \int_{B(x_0,2^j r)} |f_1(y_1)| \, dy_1 \int_{B(x_0,2^j r+1)} |b_2(y_2) - \{b_2\}_B| \, dy_2 \\
\lesssim \|b_2\|_{L^p_C(\gamma_0^2,\lambda_2^2)} \sum_{i=1}^{\infty} (2^j r)^{2n} \left( 1 + \ln \frac{2^{j+1} r}{r} \right)^2 |B(x_0,2^{j+1} r)| \lambda_2^2 \left( \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2} \right)^{2} + \frac{2}{2} \|f_i\|_{L^p_{\gamma_i}(B(x_0,2^{j+1} r))} \\
\lesssim \|b_2\|_{L^p_C(\gamma_0^2,\lambda_2^2)} \sum_{j=1}^{\infty} (2^j r)^{2n-1} \left( 1 + \ln \frac{2^{j+1} r}{r} \right)^2 |B(x_0,2^{j+1} r)| \lambda_2^2 \left( \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2} \right)^{2} \\
\times \prod_{i=1}^2 \|f_i\|_{L^p_{\gamma_i}(B(x_0,2^{j+1} r))} \, dt \\
\lesssim \|b_2\|_{L^p_C(\gamma_0^2,\lambda_2^2)} \sum_{j=1}^{\infty} \left( 1 + \ln \frac{2^{j+1} r}{r} \right)^2 |B(x_0,2^{j+1} r)| \lambda_2^2 \left( \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2} \right)^{2} \prod_{i=1}^2 \|f_i\|_{L^p_{\gamma_i}(B(x_0,2^{j+1} r))} \, dt \\
\lesssim \|b_2\|_{L^p_C(\gamma_0^2,\lambda_2^2)} \prod_{i=1}^2 \|f_i\|_{L^p_{\gamma_i}(B(x_0,t))} \frac{dt}{t^{n \left( \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2} \right) + 1}}.
\]

This completes the proof of inequality (2.7).

Now we turn to estimate $G_{42}$. Let $1 < \tau < \infty$, such that $\frac{1}{p} = \frac{1}{q_0} + \frac{1}{\tau}$. Then, by Hölder’s inequality and (2.7), we obtain

\[
G_{42} = \left\| \left( b_1 - \{b_1\}_B \right) T^{(2)} \left[ f_1^\infty, (b_2 - \{b_2\}_B) f_2^\infty \right] \right\|_{L^p(B(x_0,r))} \\
\lesssim \|b_1 - \{b_1\}_B\|_{L_{\gamma_0}^{\tau}(B)} \left\| T^{(2)} \left[ f_1^\infty, (b_2 - \{b_2\}_B) f_2^\infty \right] \right\|_{L^{\tau}(B)} \\
\lesssim \prod_{i=1}^2 \|b_i\|_{L^p_{\gamma_i}(\gamma_0^2,\lambda_1^2)} \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^2 \prod_{i=1}^2 \|f_i\|_{L^p_{\gamma_i}(B(x_0,t))} \frac{dt}{t^{n \left( \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2} \right) + 1}}.
\]

Similarly, $G_{43}$ has the same estimate above, here we omit the details, thus the inequality

\[
G_{43} = \left\| \left( b_2 - \{b_2\}_B \right) T^{(2)} \left[ (b_1 - \{b_1\}_B) f_1^\infty, f_2^\infty \right] \right\|_{L^p(B(x_0,r))} \\
\lesssim \prod_{i=1}^2 \|b_i\|_{L^p_{\gamma_i}(\gamma_0^2,\lambda_1^2)} \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^2 \prod_{i=1}^2 \|f_i\|_{L^p_{\gamma_i}(B(x_0,t))} \frac{dt}{t^{n \left( \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2} \right) + 1}}.
\]

is valid.
By the estimates of $G(2.10)$, to prove Theorem 3, we will use the following relationship between the essential supremum and essential infimum:

$$\left( \text{essinf}_{x \in E} f(x) \right)^{-1} = \text{esssup}_{x \in E} \frac{1}{f(x)}.$$

Finally, to estimate $G_{44}$, similar to the estimate of (2.7), we have

$$\left| T^{(2)} [(b_1 - \{b_1\}_B) f_1^\infty, (b_2 - \{b_2\}_B) f_2^\infty] (x) \right|$$

$$\lesssim \sum_{j=1}^{\infty} (2^j r)^{-2n} \left[ \int_{B(x_0, 2^{j+1} r)} |b_1(y) - \{b_1\}_B| |f_1(y)| \, dy_1 \right] \left[ \int_{B(x_0, 2^{j+1} r)} |b_2(y) - \{b_2\}_B| |f_2(y)| \, dy_2 \right]$$

$$\lesssim \prod_{i=1}^{2} \| b \|_{L^C_{q_i, \lambda_i}} \frac{\int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^2 \prod_{i=1}^{2} \| f_i \|_{L^p(B(x_0, t))} \, dt}{t^n \left( \frac{1}{p_1} + \frac{1}{p_2} \right) - (\lambda_1 + \lambda_2) + 1}.$$

Thus, we have

$$G_{44} = \left\| T^{(2)} [(b_1 - \{b_1\}_B) f_1^\infty, (b_2 - \{b_2\}_B) f_2^\infty] \right\|_{L^p(B(x_0, r))}$$

$$\lesssim \prod_{i=1}^{2} \| b \|_{L^C_{q_i, \lambda_i}} \frac{\int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^2 \prod_{i=1}^{2} \| f_i \|_{L^p(B(x_0, t))} \, dt}{t^n \left( \frac{1}{p_1} + \frac{1}{p_2} \right) - (\lambda_1 + \lambda_2) + 1}.$$

By the estimates of $G_{4j}$ above, where $j = 1, 2, 3, 4$. We know that

$$G_4 = \left\| T^{(2)} (f_1^\infty, f_2^\infty) \right\|_{L^p(B(x_0, r))} \lesssim \prod_{i=1}^{2} \| b \|_{L^C_{q_i, \lambda_i}} \frac{\int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^2 \prod_{i=1}^{2} \| f_i \|_{L^p(B(x_0, t))} \, dt}{t^n \left( \frac{1}{p_1} + \frac{1}{p_2} \right) - (\lambda_1 + \lambda_2) + 1}.$$

Recalling (2.11), and combining all the estimates for $G_1, G_2, G_3, G_4$, we get

$$\left\| T^{(2)} (b_1, b_2) (f_1, f_2) \right\|_{L^p(B(x_0, r))} \lesssim \prod_{i=1}^{2} \| b \|_{L^C_{q_i, \lambda_i}} \frac{\int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^2 \prod_{i=1}^{2} \| f_i \|_{L^p(B(x_0, t))} \, dt}{t^n \left( \frac{1}{p_1} + \frac{1}{p_2} \right) - (\lambda_1 + \lambda_2) + 1}.$$

Therefore, Theorem $[\text{3}]$ is completely proved. \hfill $\square$

### 2.2. Proof of Theorem $[\text{3}]$

**Proof.** To prove Theorem $[\text{3}]$, we will use the following relationship between essential supremum and essential infimum

(2.10) \[ \left( \text{essinf}_{x \in E} f(x) \right)^{-1} = \text{esssup}_{x \in E} \frac{1}{f(x)}, \]
where \( f \) is any real-valued nonnegative function and measurable on \( E \) (see [21], page 143). Indeed, since \( \mathcal{F} \in LM_{P_{1}, \varphi}^{\{x_0\}} \times \cdots \times LM_{P_{m}, \varphi}^{\{x_0\}} \) by (2.10) and the non-decreasing, with respect to \( t \), of the norm \( \prod_{i=1}^{m} \| f_i \|_{L_{p_{i}}(B(x_0,t))} \), we get

\[
\prod_{i=1}^{m} \| f_i \|_{L_{p_{i}}(B(x_0,t))} \leq \sup_{0 < t < \tau < \infty} \prod_{i=1}^{m} \varphi_i(x_0,\tau) \tau^{\frac{m}{p_{i}}} \prod_{i=1}^{m} \| f_i \|_{L_{p_{i}}(B(x_0,t))} \leq \sup_{0 < t < \infty} \prod_{i=1}^{m} \varphi_i(x_0,\tau) \tau^{\frac{m}{p_{i}}} \prod_{i=1}^{m} \| f_i \|_{L_{P_{i}}(x_0)}.
\]

(2.11)

For \( 1 < p_1, \ldots, p_m < \infty \), since \( (\varphi_1, \ldots, \varphi_m, \varphi) \) satisfies (1.11) and by (2.11), we have

\[
\int_{r}^{\infty} \left( 1 + \ln \frac{t}{r} \right) \prod_{i=1}^{m} \| f_i \|_{L_{p_{i}}(B(x_0,t))} dt \leq \prod_{i=1}^{m} \| f_i \|_{LM_{P_{i}, \varphi_i}^{\{x_0\}}} \varphi(x_0,r).
\]

(2.12)

Then by (1.10) and (2.12), we get

\[
\left\| T_{B}^{(m)} \left( \mathcal{F} \right) \right\|_{LM_{P_{i}, \varphi}^{\{x_0\}}} = \sup_{r > 0} \varphi(x_0,r)^{-1} |B(x_0,r)|^{-\frac{1}{p}} \left\| T_{B}^{(m)} \left( \mathcal{F} \right) \right\|_{L_{p}(B(x_0,r))} \\
\leq m \prod_{i=1}^{m} \left\| \mathcal{F} \right\|_{LC_{\varphi_i}^{\{x_0\}}} \sup_{r > 0} \varphi(x_0,r)^{-1} \int_{r}^{\infty} \left( 1 + \ln \frac{t}{r} \right) \prod_{i=1}^{m} \| f_i \|_{L_{p_{i}}(B(x_0,t))} dt \leq m \prod_{i=1}^{m} \left\| \mathcal{F} \right\|_{LC_{\varphi_i}^{\{x_0\}}} \prod_{i=1}^{m} \| f_i \|_{LM_{P_{i}, \varphi_i}^{\{x_0\}}}.
\]

Thus we obtain (1.12). Hence the proof is completed. \( \square \)
References

[1] D.R. Adams, Morrey spaces. Lecture Notes in Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, Cham, 2015.

[2] A.S. Balakishiyev, V.S. Guliyev, F. Gurbuz and A. Serbetci, Sublinear operators with rough kernel generated by Calderón-Zygmund operators and their commutators on generalized local Morrey spaces. J. Inequal. Appl. 2015, 2015:61. doi:10.1186/s13660-015-0582-y.

[3] R.R. Coifman, Y. Meyer, On commutators of singular integrals and bilinear singular integrals, Trans. Amer. Math. Soc., 212 (1975), 315-331.

[4] R.R. Coifman, R. Rochberg, G. Weiss, Factorization theorems for Hardy spaces in several variables, Ann. of Math., 103 (3) (1976), 611-635.

[5] L. Grafakos, N. Kalton, Multilinear Calderón-Zygmund operators on Hardy spaces, Collect. Math. 52 (2001), 169-179.

[6] L. Grafakos, R.H. Torres, Multilinear Calderón-Zygmund theory, Adv. Math., 165 (2002), 124-164.

[7] L. Grafakos, R.H. Torres, Maximal operator and weighted norm inequalities for multilinear singular integrals, Indiana Univ. Math. J., 51 (2002), 1261-1276.

[8] L. Grafakos, R.H. Torres, On multilinear singular integrals of Calderón-Zygmund type, Publ. Mat., 46 (2002), 57-91.

[9] F. Gurbuz, Boundedness of some potential type sublinear operators and their commutators with rough kernels on generalized local Morrey spaces [Ph.D. thesis], Ankara University, Ankara, Turkey, 2015.

[10] F. Gurbuz, Sublinear operators with a rough kernel generated by fractional integrals and local Campanato space estimates for commutators with rough kernel on generalized local Morrey spaces, Int. J. Appl. Math. & Stat., 2016, in press.

[11] F. Gurbuz, Weighted Morrey and Weighted fractional Sobolev-Morrey Spaces estimates for a large class of pseudo-differential operators with smooth symbols, J. Pseudo-Differ. Oper. Appl., 7 (4) (2016), 595-607. doi:10.1007/s11868-016-0158-8.

[12] F. Gurbuz, Some estimates for generalized commutators of rough fractional maximal and integral operators on generalized weighted Morrey spaces, Canad. Math. Bull., 60 (1) (2017), 131-145.

[13] F. Gurbuz, Sublinear operators with rough kernel generated by Calderón-Zygmund operators and their commutators on generalized Morrey spaces, Math. Notes, 2016, in press.

[14] M. Lacey, C. Thiele, $L^p$ estimates on the bilinear Hilbert transform for $2 < p < \infty$, Ann. Math., 146 (1997), 693-724.

[15] M. Lacey, C. Thiele, On Calderón’s conjecture, Ann. Math., 149 (1999), 475-496.

[16] S.Z. Lu and D.C. Yang, The central BMO spaces and Littlewood-Paley operators, Approx. Theory Appl. (N.S.), 11 (1995), 72-94.

[17] C.B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc., 43 (1938), 126-166.

[18] D.K. Palagachev, L.G. Sofova, Singular integral operators, Morrey spaces and fine regularity of solutions to PDE’s, Potential Anal., 20 (2004), 237-263.

[19] F. Soria, G. Weiss, A remark on singular integrals and power weights, Indiana Univ. Math. J., 43 (1994) 187-204.

[20] J. Xu, Boundedness in Lebesgue spaces for commutators of multilinear singular integrals and $RBMO$ functions with non-doubling measures, Sci. China (Series A), 50 (2007), 361-376.

[21] R.L. Wheeden and A. Zygmund, Measure and Integral: An Introduction to Real Analysis, vol. 43 of Pure and Applied Mathematics, Marcel Dekker, New York, NY, USA, 1977.

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