Long-distance asymptotics of spin-spin correlation functions for the XXZ spin chain

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Abstract

We study asymptotic expansions of spin-spin correlation functions for the XXZ Heisenberg chain in the critical regime. We use the fact that the long-distance effects can be described by the Gaussian conformal field theory. Comparing exact results for form factors in the XYZ and sine-Gordon models, we determine correlation amplitudes for the leading and main sub-leading terms in the asymptotic expansions of spin-spin correlation functions. We also study the isotropic (XXX) limit of these expansions.
1 Introduction

In the domain of two-dimensional integrable models, it is, in general, still a challenging problem to compute correlation functions in the form of compact and manageable expressions. For lattice systems, a few methods of computation have been developed [1–4]: in particular, it is possible in some cases to obtain exact integral representations of correlation functions. However it is still quite difficult to analyse those expressions, and especially to extract their long-distance asymptotic behavior.

On the other hand, at the critical point, when the gap in the spectrum of the lattice Hamiltonian $H_{\text{latt}}$ vanishes, the correlation length becomes infinite in units of the lattice spacing. As a result, the leading long-distance effects can be described by Conformal Field Theory (CFT) with the Hamiltonian $H_{\text{CFT}}$ [5]. In general, it is expected that the critical lattice Hamiltonian admits an asymptotic power series expansion involving an infinite set of local scaling fields (see e.g. [6, 7]):

$$H_{\text{latt}} \sim H_{\text{CFT}} + \lambda_1 \varepsilon^{d_1-2} \int dx O_1 + \lambda_2 \varepsilon^{d_2-2} \int dx O_2 + \ldots .$$  \hspace{1cm} (1.1)

Here $d_k$ denotes the scaling dimension of the field $O_k$, and the explicit dependence on the lattice spacing $\varepsilon$ is used to show the relative smallness of various terms. All fields occurring in (1.1) are irrelevant and the corresponding exponents of $\varepsilon$ are positive. Such an expansion also involves a set of non-universal coupling constants $\lambda_k$ depending on the microscopic properties of the model, and whose values rely on the chosen normalization for $O_k$.

The asymptotic expansion (1.1) is a powerful tool to study the low energy spectrum of lattice theories. At the same time, in order to analyse lattice correlation functions, one should also know how local lattice operators $O_{\text{latt}}$ are represented in terms of scaling fields. Just as the Hamiltonian density, they can be expressed as formal series in powers of the lattice spacing $\varepsilon$,

$$O_{\text{latt}} \sim C_{m_1} \varepsilon^{d_{m_1}} O_{m_1} + C_{m_2} \varepsilon^{d_{m_2}} O_{m_2} + \ldots ,$$  \hspace{1cm} (1.2)

where again the non-universal constants $C_m$ depend on the normalization of the scaling fields.

The knowledge of the explicit form of (1.1) and (1.2) enables one to obtain the long-distance asymptotic expansion of lattice correlation functions in powers of lattice distances. For example, in the case of a vacuum correlator, the leading term is given by the CFT correlation function of the operator occurring in (1.2) at the lowest order in $\varepsilon$, whereas subleading asymptotics come from both higher order terms in (1.2) and from the perturbative corrections to the CFT vacuum toward the lattice ground state. In this expansion, the exact values of the exponents follow from the scaling dimensions of the CFT fields, while the corresponding amplitudes come from the values of the constants appearing in (1.1) and (1.2). Thus, to investigate quantitatively the long-distance behavior of lattice correlation functions, one needs to determine (for a fixed suitable normalization of the scaling fields) exact analytical expressions for the non-universal constants $\lambda_k$ and $C_m$.

The aim of this article is to study expansions of the type (1.2) in the case of the XXZ spin 1/2 Heisenberg chain. In particular, we calculate the first constants $C_m$ occurring in the expansion (1.2) of local spin operators.
The article is organized as follows.

In Section 2, we recall the definition of the XXZ spin chain (see e.g. [8] for details), and discuss its continuous limit, the Gaussian CFT [13–11]. We review in particular how, from the analysis of the global symmetries of the model, one can derive selection rules for the expansions (1.2) of the lattice operators in terms of the scaling fields of the Gaussian model. This enables one to predict the structure of the long-distance asymptotic expansions of the spin-spin correlation functions.

The problem of computing the constants occurring in the expansions (1.2) of spin operators is the subject of Section 3. There we explain how, moving slightly away from criticality, and comparing exact results obtained for the XYZ and sine-Gordon models, one can quantitatively connect lattice spin operators to scaling fields. This gives us access to the correlation amplitudes of the spin-spin correlation functions. Our predictions are gathered in Section 4, where they are compared to existing numerical data.

In Section 5, we study these spin-spin correlation functions in the isotropic (XXX) limit by means of the exact Renormalization Group (RG) approach. This section contains an erratum of Section 5 of [12].

Finally, we conclude this article with several remarks.

## 2 XXZ spin chain in the continuous limit

### 2.1 Preliminaries [8,11]

To illustrate the problem of the determination of correlation amplitudes, we consider an example, the XXZ Heisenberg chain of spins 1/2. It is defined by the Hamiltonian

\[
H_{XXZ} = -\frac{J}{2} \sum_{l=-\infty}^{\infty} \left\{ \sigma_l^x \sigma_{l+1}^x + \sigma_l^y \sigma_{l+1}^y + \Delta \sigma_l^z \sigma_{l+1}^z \right\},
\]

where the spin operators \( \sigma_l^a \) \((a = x, y, z)\) denote the conventional Pauli matrices associated with the \( l \)-th site of the infinite lattice, and \( \Delta \) is an anisotropy parameter. As \( H_{XXZ}(-J, -\Delta) \) can be obtained from \( H_{XXZ}(J, \Delta) \) by a unitary transformation, we choose in the following the coupling constant \( J \) to be positive.

The nature of the spectrum of the infinite chain depends on the value of the anisotropy parameter \( \Delta \). The study we present here concerns the critical regime of the chain, which corresponds to the domain \(-1 \leq \Delta < 1\). We shall use the parameterizations

\[
\Delta = \cos(\pi \eta) \quad (0 < \eta \leq 1),
\]

and

\[
J = \frac{1 - \eta}{\varepsilon \sin(\pi \eta)} \quad (\varepsilon > 0).
\]

To describe leading long-distance (low-energy) effects, it is useful to consider a continuous limit of the lattice model. This limit can be obtained from the representation of (2.1) in terms of lattice fermions,

\[
H_{XXZ} = -J \sum_l \left\{ \psi_l^\dagger \psi_{l+1} + \psi_{l+1}^\dagger \psi_l + \Delta \left( 1 - 2 \psi_l^\dagger \psi_l \right) \left( 1 - 2 \psi_{l+1}^\dagger \psi_{l+1} \right) \right\},
\]

\[
(2.4)
\]
where the fermionic operators are related to the spin operators through the Jordan-Wigner transformation,
\[
\sigma_l^z = 1 - 2 \psi_l^\dagger \psi_l, \quad \psi_l^\dagger = \prod_{j<l} \sigma_j^z \cdot \sigma_l^-, \quad \psi_l = \prod_{j<l} \sigma_j^z \cdot \sigma_l^+, \tag{2.5}
\]
and \(\sigma_l^\pm = (\sigma_l^x \pm i \sigma_l^y)/2\). With the parameterization (2.3) of the constant \(J\), the spin-1/2 spin wave dispersion relation has the form [13, 14]
\[
E(k) = -\cos(k) \varepsilon. \tag{2.6}
\]
The ground state of the chain has all levels filled with \(|k| < \pi/2\). Linearizing the dispersion relation in the vicinity of the Fermi points \(k_F = \pm \pi/2\), one can take the continuous limit of the model in which the lattice operators \(\psi_l\) are replaced by two fields, \(\psi_R(x)\) and \(\psi_L(x)\), varying slowly on the lattice scale:
\[
\psi_l \propto e^{i \pi l/2} \psi_R(x) + e^{-i \pi l/2} \psi_L(x). \tag{2.7}
\]
Here \(x = l \varepsilon\), and thus the parameter \(\varepsilon\) can be interpreted as a lattice spacing. The continuous Fermi fields are governed by the Hamiltonian
\[
H_{\text{Thirring}} = \int \frac{dx}{2\pi} \left\{ -i \psi_R^\dagger \partial_x \psi_R + i \psi_L^\dagger \partial_x \psi_L + g \psi_R^\dagger \psi_R \psi_L^\dagger \psi_L \right\}. \tag{2.8}
\]

A precise relation between the anisotropy parameter \(\Delta\) and the four-fermion coupling constant \(g\) depends on the choice of the regularization procedure for the continuous Hamiltonian (2.8) and is not essential for our purposes. What is important is that the quantum field theory model (2.8) (which is known as the Thirring model [15,16]) is conformally invariant and equivalent to the Gaussian CFT (see the preprint collection [17] for a historical review of bosonization).

### 2.2 Exponential fields in the Gaussian model

The Gaussian model is defined in terms of one scalar field \(\varphi\), which satisfies the D’Alembert equation\footnote{The value of the spin-wave velocity \(v\) is determined from the slope of the dispersion relation (2.6) at the Fermi points and from the identification of \(x\) with \(l \varepsilon\).}
\[
(\partial_t^2 - v^2 \partial_x^2) \varphi = 0, \quad \text{with} \quad v = 1, \tag{2.9}
\]
and the boundary condition
\[
\partial_t \varphi(x, t)|_{x \to \pm \infty} = \partial_x \varphi(x, t)|_{x \to \pm \infty} = 0. \tag{2.10}
\]
Assuming that the equal-time canonical commutation relations are imposed on the field,
\[
[\varphi(x), \partial_t \varphi(x')] = 8\pi i \delta(x - x'),
\]
\[ 
one can write the Hamiltonian of the model in the form
\[ H_{\text{Gauss}} = \int \frac{dx}{2\pi} \left( T_R + T_L \right) + \text{const}, \] (2.11)

where
\[ T_R(x) = \frac{1}{16} (\partial_x \varphi - \partial_t \varphi)^2, \quad T_L(x) = \frac{1}{16} (\partial_x \varphi + \partial_t \varphi)^2. \] (2.12)

As is usual in CFT \cite{18}, it is convenient to set a class of conformal primary fields among all scaling fields. In the case of the Gaussian model, these conformal primaries include right and left currents,
\[ (\partial_x - \partial_t) \varphi, \quad (\partial_x + \partial_t) \varphi, \] (2.13)
along with exponential fields\cite{10}. With the boundary condition (2.10), the latter can be defined as
\[ O_{s,n}(x,t) = \Lambda^{d_{s,n}} \lim_{\varepsilon \to +0} \exp \left\{ \frac{in}{4\sqrt{\eta}} \int_{-\infty}^{x} dx' \partial_t \varphi(x',t) \right\} \exp \left\{ \frac{is \sqrt{\eta}}{2} \varphi(x+\varepsilon,t) \right\}, \] (2.14)

where \( s, n \) are integers and
\[ d_{s,n} = \frac{s^2 \eta}{2} + \frac{n^2}{8\eta}. \] (2.15)

The regularization parameter \( \Lambda \), which has the dimension \([\text{length}]^{-1}\), is introduced in the definition (2.14) in order to provide a multiplicative renormalization of the fields. Notice that \( O_{s,n} \) obey the simple Hermiticity relation
\[ O_{s,n}^\dagger = O_{-s,-n}. \] (2.16)

To completely define these exponential fields, we should also specify some condition which fixes their multiplicative normalization. By a proper choice of \( \Lambda \) in (2.14), one can impose the following form of the causal Green’s functions in the Euclidean domain \( x^2 - t^2 > 0 \):
\[ \langle T O_{s,n}(x,t) O_{s,n}^\dagger(0,0) \rangle = \left( \frac{t-x}{t+x} \right)^{ \frac{s^2 \eta}{2} } (x^2 - t^2)^{-d_{s,n}}. \] (2.17)

We will later refer to Eq. (2.17) as the “CFT normalization condition”.

2.3 Global symmetries

To draw a precise link between the Gaussian CFT and the XXZ spin chain, it is important to examine and identify their global symmetries.

Let us first consider the Gaussian model. The Hamiltonian (2.11) is manifestly invariant under the \( U(1) \) rotations,
\[ U_\alpha \varphi U_\alpha^{-1} = \varphi + \frac{2\alpha}{\sqrt{\eta}}, \quad U_\alpha O_{s,n} U_\alpha^{-1} = e^{isa} O_{s,n}, \] (2.18)

\[ ^2 \text{Here we are not considering the orbifold (Ashkin-Teller) sector of the Gaussian model.} \]
where the operator $U_\alpha$ can be written in the form

$$U_\alpha = \exp \left\{ \frac{i\alpha}{i\pi \sqrt{\eta}} \int_{-\infty}^{\infty} dx \partial_t \varphi \right\}.$$  \hspace{1cm} (2.19)

The CFT model is also invariant under the parity transformation $P: \varphi(x, t) \rightarrow \varphi(-x, t)$, the time reversal $T: \varphi(x, t) \rightarrow \varphi(x, -t)$, and the reflection $C: \varphi \rightarrow -\varphi$. Using the definition (2.14), one can show that these transformations act on the exponential fields as

$$P O_{s,n}(x,t) P = e^{i\alpha s \pi} U_\pi O_{s,-n}(-x,t),$$ \hspace{1cm} (2.20)

$$T O_{s,n}(x,t) T = O_{-s,n}(x,-t),$$ \hspace{1cm} (2.21)

$$C O_{s,n} C = O_{-s,-n}(x,t).$$ \hspace{1cm} (2.22)

Note that the reflection $C$ can be naturally considered as a charge conjugation in the theory.

As usual, $C$ and $P$ are intrinsic automorphisms of the operator algebra, contrary to the anti-unitary transformation $T$ which acts on $c$-numbers as follows,

$$T (c\text{-number})^* = (c\text{-number}).$$

Let us now identify the above transformations with the global symmetries of the spin chain. First of all, it is natural to define the action of the $U(1)$ rotation on the lattice as follows,

$$U_\alpha \sigma_i^\pm U_\alpha^{-1} = e^{\pm i\alpha} \sigma_i^\pm, \quad U_\alpha \sigma_i^z U_\alpha^{-1} = \sigma_i^z,$$ \hspace{1cm} (2.23)

where

$$U_\alpha = e^{i\alpha S_z} \quad \text{with} \quad S_z = \frac{1}{2} \sum_j \sigma_j^z.$$ \hspace{1cm} (2.24)

Such an identification has an important consequence. Indeed, as the two rotations of angle $2\pi$ and $-2\pi$ are indistinguishable trivial transformations in the lattice theory, we should set

$$U_{2\pi} = U_{-2\pi} = \nu \mathbb{I} \quad \text{with} \quad \nu = \pm 1$$ \hspace{1cm} (2.25)

in the corresponding Gaussian model. The condition (2.25), along with (2.18), implies that $\varphi$ has to be treated as a compactified field:

$$\varphi \equiv \varphi + \frac{4\pi}{\sqrt{\eta}} \mathbb{Z}.$$

According to (2.24), the sign factor $\nu = +1$ in Eq. (2.25) occurs for states with an integer eigenvalue of the operator $S_z$. The corresponding linear subspace of the whole Hilbert space can be constructed from the thermodynamic limit of finite chains with an even number of sites, and will therefore be referred to as the “even sector”. Similarly, the condition $\nu = -1$ defines another linear subspace which will be called the “odd sector”.

The actions of charge conjugation and time reversal can be naturally identified, in the lattice theory, with the following transformations,

$$C \sigma_i^\pm C = \sigma_i^\mp, \quad C \sigma_i^z C = -\sigma_i^z,$$ \hspace{1cm} (2.26)

$$T \sigma_i^\pm T = \sigma_i^\mp, \quad T \sigma_i^z T = -\sigma_i^z.$$ \hspace{1cm} (2.27)
Let us recall here that the time reversal is an anti-unitary transformation, so that $C$ and $T$ correspond to different symmetries of the XXZ chain, even though they act identically on the spin operators.

As for the parity transformation, its action in the lattice model depends on the choice of the sector specified by the sign factor $\nu$ in (2.25). Indeed, $\nu = +1$ implies that the considered infinite chain is defined as the thermodynamic limit of finite lattices with an even number of sites. Such finite lattices clearly do not possess any invariant site with respect to the parity transformation. Therefore,

$$P \sigma^a_1 P = \sigma^a_{-1} \quad \text{for} \quad \nu = +1,$$

whereas the “naive” action of $P$ is valid only in the odd sector:

$$P \sigma^a_1 P = \sigma^a_{-l} \quad \text{for} \quad \nu = -1.$$  

### 2.4 Selection rules

The global symmetries and the knowledge of their action on lattice and continuous operators provide selection rules for the set of scaling fields which can occur in expansions (1.1) and (1.2). Here we examine what are these selection rules for the conformal primary fields in the expansions of the spin operators.

First, since the whole operator content of the Gaussian model is given by the primary fields described above and by their conformal descendants, it is easy to see that $O_{\pm 1,0}$ are the local fields with the lowest scaling dimension which may occur in the expansions of $\sigma^{\pm}_{0}$:

$$\sigma^{\pm}_{0} \sim C_0 \varepsilon^{d_{1,0}} \mathcal{O}_{\pm 1,0}(0) + \ldots. \quad (2.30)$$

Furthermore, assuming that the conjugation in the lattice theory is defined in such a way that $\sigma^a_{\pm}$ ($a = x, y, z$) are Hermitian operators, we conclude from (2.16) that the constant $C_0$ in (2.30) is real.

Let us now consider the expansion of the lattice operator $\sigma^z_{0}$. Due to the $U(1)$ invariance, it can contain only the primary fields $(\partial_x \pm \partial_t)\varphi$ and $\mathcal{O}_{0,n}$, $n \in \mathbb{Z}$, along with their conformal descendants. Using definition (2.14), it is easy to show that the fields $\mathcal{O}_{0,n}$ with odd $n$ are not mutually local with respect to $\mathcal{O}_{\pm 1,0}$. As the latter are the leading terms in the expansions (2.30) of $\sigma^z_{0}$, it would contradict the mutual locality of the lattice operators $\sigma^+_0$ and $\sigma^-_0$ if there were any $\mathcal{O}_{0,2m+1}$, $m \in \mathbb{Z}$, in the series for $\sigma^z_{0}$. From the $C$, $T$ invariances and the Hermiticity of $\sigma^z_{0}$, one can moreover predict that the primary fields are allowed to appear only as linear combinations of $\partial_t \varphi$ and $i(\mathcal{O}_{0,2m} - \mathcal{O}_{0,-2m})$ with real coefficients. Finally, let us consider the parity transformation. According to equations (2.20) and (2.23),

$$P \mathcal{O}_{s,2m}(x,t) P = \nu^m \mathcal{O}_{s,-2m}(-x,t), \quad (2.31)$$

and thus one has to examine even and odd sectors separately. In the odd sector ($\nu = -1$), the parity (2.23) prohibits the presence of the fields $\mathcal{O}_{0,4m} - \mathcal{O}_{0,-4m}$ with $m \in \mathbb{Z}$. Therefore the expansion has to be of the form,

$$\sigma^z_{0} \sim \varepsilon C^z_0 \partial_t \varphi(0) + \frac{1}{2i} \sum_{m=1}^{+\infty} C^z_m \varepsilon^{d_{0,4m-2}} (\mathcal{O}_{0,4m-2} - \mathcal{O}_{0,-4m+2})(0) + \text{descendants}. \quad (2.32)$$
But, since (2.32) is a local operator expansion, it cannot depend on the choice of the sector in the Hilbert space. Thus, (2.32) should be valid in the even sector as well. Owing to this and to Eqs. (2.28), (2.31), we can determine the action of the lattice translation,

$$K\sigma_l^o K^{-1} = \sigma_{l+1}^o,$$

(2.33)
on the primary fields $O_{0,4m-2}$:

$$K O_{0,4m-2}(x) K^{-1} = -O_{0,4m-2}(x + \varepsilon).$$

(2.34)

With this relation, the expansion (2.32) can be written in the more general form,

$$\sigma_l^z \sim \frac{\varepsilon}{2\pi\sqrt{\eta}} \partial_x \varphi(x) + \frac{(-1)^l}{2l} \sum_{m=1}^{+\infty} C_m^z \varepsilon^{d_{0,4m-2}} (O_{0,4m-2} - O_{0,-4m+2})(x) + \text{descendants},$$

(2.35)

where $x = l \varepsilon$. Notice that the exact value of the coefficient $C_0^z$ in (2.35) is determined from the comparison of Eqs. (2.19) and (2.24).

Using the same line of arguments, one can extend the expansion (2.30),

$$\sigma_l^\pm \sim \frac{1}{2} \sum_{m=0}^{+\infty} (-1)^m C_m \varepsilon^{d_{1,2m}} (O_{\pm1,2m} + O_{\pm1,-2m})(x) + \text{descendants},$$

(2.36)

and determine the action of the lattice translation $K$ on $O_{\pm1,2m}$:

$$K O_{\pm1,2m}(x) K^{-1} = (-1)^m O_{\pm1,2m}(x + \varepsilon).$$

(2.37)

Since all exponential fields $O_{\pm s,2m}(x)$ with integers $s$ and $m$ can be obtained by means of operator product expansions of the fields $O_{\pm1,2m}(x)$ and $O_{0,4m-2}(x)$, we deduce from (2.34) and (2.37) that

$$K O_{s,2m}(x) K^{-1} = (-1)^m O_{s,2m}(x + \varepsilon).$$

(2.38)

Notice that, in the process of bosonization, the fermionic fields $\psi_R$ and $\psi_L$ (2.7) are identified with the exponential fields $O_{1,-1}$ and $O_{1,1}$ respectively. Hence equation (2.38) is indeed consistent with (2.7).

It is not difficult to extend our symmetry analysis to derive the selection rules for expansion (1.2) of any local lattice operator. This procedure can, in particular, be applied to the case of the Hamiltonian density. As a result, the following form of the low energy effective Hamiltonian for the XXZ spin chain is suggested:

$$H_{XXZ} \sim H_{\text{Gauss}} + \int \frac{dx}{4\pi} \left\{ \sum_{m=1}^{\infty} \lambda_m \varepsilon^{d_{0,4m-2}} (O_{0,4m} + O_{0,-4m})(x) + \text{descendants} \right\}.$$

(2.39)

As was mentioned in Introduction, it is necessary to choose the normalization of the scaling fields to give a precise meaning to the couplings in (2.39) as well as to the real
constants $C_m$, $C_n$, in expansions (2.35) and (2.36). In this paper we adapt the CFT normalization (2.17). With this normalization condition, the first coupling constant $\lambda_1$ in (2.39) has been obtained in [12], together with the leading contributions of descendant fields (see also Ref. [19] for a qualitative analysis of the descendant field contributions):

$$H_{XXZ} \sim H_{Gauss} + \lambda_1 \varepsilon^{2/\eta-2} \int \frac{dx}{4\pi} \left( O_{0,4} + O_{0,-4} \right)(x)$$

$$-\varepsilon^2 \int \frac{dx}{2\pi} \left\{ \lambda_+ T_R T_L(x) + \lambda_- (T_R^2 + T_L^2)(x) \right\} + \ldots , \quad (2.40)$$

where

$$\lambda_1 = -\frac{4 \Gamma(1/\eta)}{\Gamma(1 - 1/\eta)} \left[ \frac{\Gamma(1 + \frac{\eta}{2 - 2\eta})}{2\sqrt{\pi} \Gamma(1 + \frac{1}{2 - 2\eta})} \right]^{2/\eta-2} , \quad (2.41)$$

$$\lambda_+ = \frac{1}{2\pi} \tan \left( \frac{\pi}{2 - 2\eta} \right) , \quad (2.42)$$

$$\lambda_- = \frac{\eta}{12\pi} \frac{\Gamma\left(\frac{3}{2 - 2\eta}\right)}{\Gamma\left(\frac{2}{2 - 2\eta}\right)} \frac{\Gamma^3(\frac{\eta}{2 - 2\eta})}{\Gamma^3(\frac{1}{2 - 2\eta})} , \quad (2.43)$$

2.5 Vacuum spin-spin correlation functions

At this stage, without the knowledge of the constants occurring in expansions (2.35) and (2.36), it is already possible to predict the exact values of the exponents for the vacuum spin-spin correlation functions. They follow immediately from the scaling dimensions of the fields occurring in (2.35), (2.36) and (2.40).

It may be worth recalling at this point, that the vacuum sector of the infinite XXZ chain is infinitely degenerate. In general, the boundary conditions imposed on the finite critical chain do not preserve all the global symmetries discussed above, and they may be spontaneously broken at the thermodynamic limit. Here we consider only translational invariant vacuums with unbroken parity:

$$\mathbb{P} | \text{vac} \rangle = \mathbb{K} | \text{vac} \rangle = | \text{vac} \rangle . \quad (2.44)$$

To fulfill these requirements, we shall treat the infinite XXZ chain as the thermodynamic limit of finite chains subject to periodic boundary conditions. Then, different vacuum states can be distinguished by means of the operator $S_z$ (2.24):

$$S_z | s \rangle = s | s \rangle , \quad 2s \in \mathbb{Z} . \quad (2.45)$$

Also, by a proper choice of phase factors, one can always set up the conditions

$$\mathbb{C} | s \rangle = | -s \rangle \quad \text{and} \quad \mathbb{T} | s \rangle = | -s \rangle . \quad (2.46)$$

In the continuous limit, the vacuum $| s \rangle$ flows toward the conformal primary state with right and left conformal dimensions equal to $d_{s,0}/2$, where $d_{s,n}$ is given by (2.15) [20–22]. This

\[^{3}\text{It has been done for the first time in Ref. [4].}\]
implies in particular that, for the XXZ chain with a finite number of sites $N \gg 1$, the difference of vacuum energies corresponding to the states $|s\rangle$ and $|0\rangle$ is $2\pi d_{s,0}/N + o(N^{-1})$.

From our previous analysis, one can now predict the following asymptotic expansions for the time-ordered correlation functions:

$$\langle T \sigma_{l+j}^x(t) \sigma_j^x(0) \rangle \sim \frac{A}{(l_+ l_-)^{\eta}} \left\{ 1 - \frac{B}{(l_+ l_-)^{\eta/2}} + O(l^{-2}\log l, l^{8-8/\eta}) \right\}$$

$$- \frac{(-1)^l \tilde{A}}{(l_+ l_-)^{\eta/2+1/\eta}} \left\{ \frac{1}{2} \left( l_+ l_+ - l_- l_- \right) + \frac{\tilde{B}}{(l_+ l_-)^{\eta/2-1}} + O(l^{-2}\log l, l^{4-4/\eta}) \right\} + \ldots ,$$

(2.47)

$$\langle T \sigma_{l+j}^z(t) \sigma_j^z(0) \rangle \sim -\frac{1}{\pi^{2\eta} l_+ l_-} \left\{ \frac{1}{2} \left( \frac{l_+}{l_-} + \frac{l_-}{l_+} \right) \right\}$$

$$+ \frac{\tilde{B}_z}{(l_+ l_-)^{\eta/2-1/2}} \left( 1 + \frac{2 - \eta}{4(1 - \eta)} \left( \frac{l_+}{l_-} + \frac{l_-}{l_+} \right) \right) + O(l^{-2}\log l, l^{8-8/\eta}) \right\}$$

$$+ \frac{(-1)^l A_z}{(l_+ l_-)^{\eta/2}} \left\{ 1 - \frac{B_z}{(l_+ l_-)^{\eta/2-1}} + O(l^{-2}\log l, l^{4-4/\eta}) \right\} + \ldots ,$$

(2.48)

where

$$l_\pm = l \pm \frac{t}{\varepsilon} \gg 1 .$$

(2.49)

The coefficients in the asymptotic expansions (2.47) and (2.48) do not really depend on the choice of the vacuum state $|s\rangle$, thus any finite $s \in \mathbb{Z}/2$ can be chosen for the averaging $\langle \ldots \rangle \equiv \langle s | \ldots | s \rangle$.

In Eqs. (2.47), (2.48), the correlation amplitudes $A$, $\tilde{A}$ and $A_z$ are simply related to the first constants occurring in the expansions (2.35) and (2.36):

$$A = 2(C_0)^2 , \quad \tilde{A} = (C_1)^2 , \quad A_z = \frac{1}{2} (C_z^2)^2 .$$

(2.50)

They will be computed in the next section. At the same time, the constants $B$, $\tilde{B}$, $B_z$ and $\tilde{B}_z$ appearing in (2.47) and (2.48) can be determined by methods of conformal perturbation theory based on the effective Hamiltonian (2.40). In particular, the constant $B$ was obtained in Ref. [12] from second order perturbative calculations,

$$B = \frac{\lambda_1^2}{16} \left\{ \frac{2\pi^2}{\sin^2(2\pi/\eta)} - \frac{\eta^2}{(1 - \eta)(2 - \eta)} - \psi'(1/\eta) - \psi'(3/2 - 1/\eta) \right\} ,$$

(2.51)

where $\psi'(z) = \partial_z^2 \log \Gamma(z)$ and $\lambda_1$ is given by (2.41). The constant $\tilde{B}_z$ can be computed similarly:

$$\tilde{B}_z = \frac{\lambda_1^2}{4} \frac{\eta}{(\eta - 2)^2} .$$

(2.52)
On the contrary, the determination of $\tilde{B}$ and $B_z$ do not require calculations beyond the first order. They read explicitly,

$$B_z = -\lambda_1 2^{\frac{4}{\eta} - 5} \frac{\Gamma\left(\frac{4}{\eta} - \frac{1}{2}\right) \Gamma\left(1 - \frac{1}{\eta}\right)}{\Gamma\left(\frac{2}{\eta} - \frac{1}{2}\right) \Gamma\left(\frac{1}{\eta}\right)} ,$$

(2.53)

$$\tilde{B} = (1 - \eta)^2 B_z .$$

(2.54)

### 3 Calculation of correlation amplitudes

The purpose of this section is to explain how the local operators $\sigma^\pm_l$ and $\sigma^z_l$ can be quantitatively related to the scaling fields (2.13) and (2.14), that is, how one can compute explicit analytic expressions for the constants $C_m$ and $\tilde{C}_m$ occurring in (2.35) and (2.36) for the fixed CFT normalization (2.17). We concentrate here on the first terms of the expansions (2.35), (2.36), which provide the main asymptotic behavior of the spin-spin correlation functions (2.47) and (2.48):

$$\sigma^\pm_l \sim C_0 \varepsilon^\frac{\eta}{2} O_{\pm,1,0}(x) + (-1)^l \frac{C_1}{2} \varepsilon^{\frac{\eta}{2} + \frac{1}{\eta}} (O_{\pm,1,2} + O_{\pm,-2})(x) + \ldots ,$$

(3.1)

$$\sigma^z_l \sim \frac{\varepsilon}{2\pi\sqrt{\eta}} \partial_t \varphi(x) + (-1)^l \frac{C_z}{2} \varepsilon^{\frac{1}{2\eta}} (O_{0,2} - O_{0,-2})(x) + \ldots .$$

(3.2)

Note that in (3.2) each of the two terms is either leading or sub-leading according to the value of $\eta$, i.e. of the anisotropy parameter $\Delta$, whereas in (3.1) the second term is always subleading. Nevertheless, this term gives rise to noticeable numerical corrections to the leading asymptotic behavior of correlation functions and has already been studied numerically (see Section 4).

#### 3.1 Even sector of the infinite XYZ chain

To find the quantitative relation between the spin operators and the local scaling fields, it is useful to move slightly away from criticality and, instead of (2.1), to consider the XYZ chain,

$$H_{\text{XYZ}} = -\frac{1}{2} \sum_{l=-\infty}^{\infty} \left\{ J_x \sigma^x_l \sigma^x_{l+1} + J_y \sigma^y_l \sigma^y_{l+1} + J_z \sigma^z_l \sigma^z_{l+1} \right\} .$$

(3.3)

Without loss of generality we assume here that $J_x > J_y \geq |J_z|$. The XYZ deformation has a remarkable feature: it preserves the integrability of the original theory \[8,24\]. Nowadays, the structure of the Hilbert space of the infinite XYZ chain is well understood. We recall here some basic facts that will be useful for our analysis.

In the case of the XYZ spin chain, the global symmetry group discussed in the previous section is explicitly broken to the subgroup generated by the lattice translation $K$, the $C$, $P$, $T$ transformations and the rotation (2.23) with $\alpha = \pi$: $U_{\pi}$. In this section, we concentrate on the even sector of the XYZ spin chain (defined as the thermodynamic limit of finite chains with an even number of sites), which implies the condition

$$U^2_{\pi} = 1 .$$

(3.4)
Let us denote by $H_s$ the eigenspace of the operator $S_z$ corresponding to a given eigenvalue $s \in \mathbb{Z}$. Then, the Hamiltonian (3.3) acts as,

$$H_{\text{XYZ}} : H_s \to H_{s-2} \oplus H_s \oplus H_{s+2},$$

and therefore the infinite degeneracy in the vacuum sector of the XXZ chain is reduced to the two states

$$|s\rangle_{\text{XYZ}} \in \bigoplus_{k=-\infty}^{\infty} H_{s+2k} \quad (s = 0, 1)$$

satisfying the condition

$$U_\pi |s\rangle_{\text{XYZ}} = e^{is\pi} |s\rangle_{\text{XYZ}}. \quad (3.5)$$

This requirement, along with the conventional normalization of vacuum states, $\langle \text{vac} | \text{vac} \rangle = 1$, defines $|s\rangle_{\text{XYZ}}$ up to an overall complex phase factor. Since the time reversal transformation acts on states as the complex conjugation, one can eliminate (up to sign) the ambiguity of such a phase by imposing the condition

$$T|s\rangle_{\text{XYZ}} = |s\rangle_{\text{XYZ}}. \quad (3.6)$$

Moreover, since the charge conjugation matrix (2.26) can be identified with $\prod_l \sigma_i^x$, one has

$$C|\text{vac}\rangle^{(j)} = |\text{vac}\rangle^{(1-j)}. \quad (3.7)$$

When the couplings $J_y$ and $J_z$ are non-vanishing, the pure ferromagnetic states are no longer stationary states, but it is still possible to introduce two vacuums $|\text{vac}\rangle^{(j)}$ satisfying (3.6) and (3.7) through the relation

$$|s\rangle_{\text{XYZ}} = \frac{1}{\sqrt{2}} \left\{ |\text{vac}\rangle^{(0)} + (-1)^s |\text{vac}\rangle^{(1)} \right\} \quad (s = 0, 1). \quad (3.8)$$

The Hilbert space of the XYZ chain contains two linear subspaces $\mathcal{V}^{(j)}$ ($j = 0, 1$) associated with the vacuums $|\text{vac}\rangle^{(j)}$. In the spectrum of the model, there exist kink-like “massive” excitations, $B_+$ and $B_-$, such that the corresponding Zamolodchikov-Faddeev operators intertwine these subspaces $\mathcal{V}^{(j)}$:

$$B_\pm : \mathcal{V}^{(j)} \to \mathcal{V}^{(1-j)}.$$

One can therefore generate two sets of asymptotic states in the form

$$B_{\sigma_2 m}(k_{2m}) \ldots B_{\sigma_1}(k_1) |\text{vac}\rangle^{(j)} \in \mathcal{V}^{(j)} \quad (m = 0, 1, 2 \ldots) \quad (3.9)$$

$^4$ The corresponding energies $E_N^{(s)}$ for a chain with a finite number of sites $N \gg 1$ are asymptotically degenerate in the sense that $E_N^{(1)} - E_N^{(0)} = O(e^{-\text{const} N})$ (see e.g. Ref. [8]).
where the Zamolodchikov-Faddeev operators $\mathbf{B}_\sigma(k) (\sigma = \pm)$ depend on a quasi-momentum $k$:

$$
\Xi \mathbf{B}_\pm(k) \Xi^{-1} = e^{ik} \mathbf{B}_\pm(k), \quad [H_{XYZ}, \mathbf{B}_\pm(k)] = \mathcal{E}(k) \mathbf{B}_\pm(k).
$$

(3.10)

The dispersion relation $\mathcal{E} = \mathcal{E}(k)$ of the fundamental excitations was calculated in work \[13\].

It is recalled in Appendix A (see Eqs. (A.8), (A.9)). The operators $\mathbf{B}_\pm$ satisfy also the conditions:

$$
\mathbb{C} \mathbf{B}_\pm^{(1-j,j)} \mathbb{C} = \mp (-1)^j \mathbf{B}_\pm^{(1-j,j)}, \quad \mathbb{U}_\pi \mathbf{B}_\pm^{(1-j,j)} \mathbb{U}_\pi = \pm (-1)^j \mathbf{B}_\pm^{(j,1-j)}.
$$

(3.11)

In Eqs. (3.11), $\mathbf{B}_\pm^{(1-j,j)}$ denotes the restriction of the operator $\mathbf{B}_\pm$ when it acts on the subspace $\mathcal{V}^{(j)}$.

Any local lattice operator $\mathcal{O}_{\text{latt}}$ (i.e. any operator which can be written as a local combination of $\sigma^\alpha_i$) leaves the subspaces $\mathcal{V}^{(j)}$ invariant:

$$
\mathcal{O}_{\text{latt}} : \mathcal{V}^{(j)} \rightarrow \mathcal{V}^{(j)}.
$$

Furthermore, the algebra of local operators $\mathcal{A}_{\text{loc}}$ acts invariantly on the component of the subspace $\mathcal{V}^{(j)}$ generated by the states (3.9). For $J_z \leq 0$, the kinks $\mathbf{B}_\pm$ do not produce bound states, and the two sets of asymptotic states (3.9) obeying the condition

$$
-\pi/2 \leq k_1 < k_2 \ldots < k_{2m} < \pi/2
$$

form complete in-bases in these unitary equivalent spaces of representation of $\mathcal{A}_{\text{loc}}$. At first glance, to construct explicitly the representations of $\mathcal{A}_{\text{loc}}$, one should put at one’s disposal the whole collection of in-basis matrix elements for an arbitrary local operator $\mathcal{O}_{\text{latt}} \in \mathcal{A}_{\text{loc}}$. As a matter of fact, using the so-called crossing symmetry (see e.g. 26, 27), one can express all possible matrix elements in terms of those of the form

$$
\langle \text{vac} | \mathcal{O}_{\text{latt}} | \mathbf{B}_{\sigma_2m}(k_{2m}) \ldots \mathbf{B}_{\sigma_1}(k_1) | \text{vac} \rangle = \langle \text{vac} | \mathbf{B}_{\sigma_1}(k_1) \ldots \mathbf{B}_{\sigma_2m}(k_{2m}) | \text{vac} \rangle.
$$

(3.12)

Such matrix elements are known as form factors.

Currently, there exists a formal procedure which allows one to express form factors of local operators in terms of multiple integrals [26, 27]. Unfortunately, it is difficult to apply in practice, even in the case of the local spin operators $\sigma^\alpha_i$ themselves and for form factors involving only a small number of excitations. For our purposes, we merely need the explicit form of Vacuum Expectation Values (VEV) and of two-particle form factors. From the relations (2.44), (3.6) and (3.7), it follows immediately that the VEVs of $\sigma^\alpha_i$ and $\sigma^\beta_j$ vanish, whereas

$$
\langle \sigma^\alpha_j \rangle = (-1)^j F,
$$

(3.13)

where $F$ depends on the two ratios $J_y/J_x$ and $J_z/J_x$. This VEV was found in work \[28\] (see Appendix A, Eq. (A.7)). As for the two-particle form factors, the $\mathbb{Z}_2$-symmetry generated by $\mathbb{U}_\pi$ (3.11) enables one to predict their general form:

$$
\langle \sigma^\alpha_0 | \mathbf{B}_\pm(k_1) \mathbf{B}_\pm(k_2) \rangle_{\text{in}} = (-1)^j F^\alpha_1(k_1, k_2) \pm F^\alpha_2(k_1, k_2),
$$

(3.14)

$$
\langle \sigma^\beta_0 | \mathbf{B}_\pm(k_1) \mathbf{B}_\pm(k_2) \rangle_{\text{in}} = F^\beta_1(k_1, k_2) \pm (-1)^j F^\beta_2(k_1, k_2),
$$

(3.15)

$$
\langle \sigma^\delta_0 | \mathbf{B}_\pm(k_1) \mathbf{B}_\pm(k_2) \rangle_{\text{in}} = (-1)^j F^\delta_1(k_1, k_2) \pm F^\delta_2(k_1, k_2),
$$

(3.16)
where $F_{a1,2}$ are some functions of the quasi-momentums $k_1$ and $k_2$. The invariance with respect to charge conjugation $C$ (3.11) dictates that all other two-particle form factors vanish. Form factor (3.14) was calculated in Ref. [27]. It is possible to extend the result of that work and calculate form factors (3.15) and (3.16) as well. We collect the explicit expressions of $F$ and $F_{a1,2}$ in Appendix A.

To conclude this subsection, let us note that, in the current treatment of the infinite XYZ model as the thermodynamic limit of finite periodical chains with an even number of sites, we cannot construct excited states containing an odd number of the elementary excitations

$$B_{\sigma_{2m+1}}(k_{2m+1}) \ldots B_{\sigma_1}(k_1) | \text{vac} \rangle^{(1-j)} \in \mathcal{V}^{(j)} \quad (m = 0, 1 \ldots) .$$

(3.17)

Such states are deduced from finite chains with boundary conditions breaking the translation invariance. In particular, the linear subspace of $\mathcal{V}^{(j)}$ spanned by the states (3.17) can be constructed from the thermodynamic limit of chains with an odd number of sites $N$ and subject to the so called twist boundary condition:

$$\sigma_{-N/2}^2 = -\sigma_{N/2}^2, \quad \sigma_{-N/2}^z = \sigma_{N/2}^z .$$

The in-asymptotic states (3.9), (3.17) form complete bases in $\mathcal{V}^{(j)}$ for $J_z \leq 0$.

### 3.2 XYZ spin chain in the scaling limit

For $J_x = J_y$, the gap in the spectrum of the XYZ chain vanishes, and its correlation length [29]

$$R_c \simeq \frac{1}{4} \left[ \frac{8 (J_x^2 - J_z^2)}{J_x (J_x - J_y)} \right]^{1/2} \left( \frac{\eta}{\pi} \arccos(J_z/J_x) \right)$$

(3.18)

becomes infinite. In the limit $J_x \rightarrow J_y$, the correlation functions at large lattice separation ($\sim R_c$) assume a certain scaling form which can be described by quantum field theory. If $(J_x - J_y)/J_x \ll 1$, it is natural to treat the XYZ model as the perturbation of the XXZ chain by the lattice operator

$$\sigma_l^x \sigma_{l+1}^x - \sigma_l^y \sigma_{l+1}^y .$$

The leading term in the expansion (1.2) for this operator is given by the relevant field $\mathcal{O}_{2,0} + \mathcal{O}_{-2,0}$ of scaling dimension $2\eta$. Therefore, the scaling limit of the XYZ chain is described by the sine-Gordon quantum field theory [29],

$$H_{sg} = H_{\text{Gauss}} - \mu \int dx \left( \mathcal{O}_{2,0} + \mathcal{O}_{-2,0} \right)(x) .$$

(3.19)

Up to a numerical factor, which was obtained in Ref. [30], the coupling constant $\mu$ coincides with the quantity $M^{2-2\eta}$, where the combination

$$M = (\varepsilon R_c)^{-1}$$

(3.20)

---

5 This fact follows from the result of work [22].
can be naturally identified with the soliton mass in the sine-Gordon model \((3.19)\).

The sine-Gordon theory admits a class of local soliton-creating operators characterized by two integers \(s, n \in \mathbb{Z}\), where \(n\) gives the topological charge and \(sn/2\) represents the Lorentz spin of the field. These operators can also be expressed in a form similar to \((2.14)\), in which \(\varphi\) denotes the sine-Gordon field instead of the Gaussian field obeying the simple D’Alembert equation \((2.9)\) (see e.g. Ref. [31] for details). They moreover coincide with the Gaussian fields \((2.14)\) in the conformal limit \(\mu \to 0\). Hence, with some abuse of notation, we will denote such soliton-creating operators in the sine-Gordon model by the same symbol \(\mathcal{O}_{s,n}\).

To proceed further, one needs to draw a link between local lattice operators in the XYZ chain and local fields in the sine-Gordon model. Let us note at this point that local expansions of the type \((1.2)\) are based on dimensional analysis and do not necessarily imply the criticality of the original lattice system. Similar expansions are expected to be applicable to describe the near-critical behavior of lattice systems. Usually, the term with the smallest scaling dimension in \((1.2)\) governs the universal scaling behavior of lattice correlators, whereas the next terms produce non-universal corrections. In particular, relations \((3.1), (3.2)\) obtained for the XXZ spin chain can be used to study the leading scaling behavior and first non-universal corrections of the XYZ correlation functions and form factors. In the XYZ case, the continuous fields which appear in \((3.1), (3.2)\) should be understood as operators in the sine-Gordon model rather than their conformal limits. The numerical constants \(C_0, C_1\) and \(C_z\) remain, of course, the same as for the critical XXZ chain.

Let us now discuss the relation between the Hilbert spaces of the XYZ and sine-Gordon models. In general, the theory \((3.19)\) admits a discrete symmetry \(\varphi \to \varphi + 2\pi j/\sqrt{\eta} \ (j \in \mathbb{Z})\), which is generated by the operator \(U_\pi\) defined similarly to \((2.19)\). For \(0 < \eta \leq 1\), the above symmetry is spontaneously broken, so that the theory has an infinite number of ground states \(|0_j\rangle\ (j \in \mathbb{Z})\) characterized by the corresponding VEVs of the field \(\varphi\):

\[
\frac{\langle 0_j | \varphi(x) | 0_j \rangle}{\langle 0_j | 0_j \rangle} = \frac{2\pi j}{\sqrt{\eta}}.
\]

(3.21)

The sine-Gordon model which governs the scaling behavior of the even sector of the XYZ chain is subject to the additional constraint \(U_\pi^2 = 1\). This equation implies in particular that the field \(\varphi\) is compactified, \(\varphi \equiv \varphi + 4\pi / \sqrt{\eta}\), and that, unlike the uncompactified case, there exist only two non-equivalent vacuum states \(|0_j\rangle\) with \(j = 0, 1\). These states are naturally identified with the scaling limit of the two XYZ vacuums \(|\text{vac}\rangle\).

To describe the scaling limit of XYZ excited states, one has to relate the Zamolodchikov-Faddeev operators of the lattice and continuous theories. Let us recall here that the sine-Gordon model admits a global continuous \(U(1)\) symmetry generated by the operator

\[
V_\alpha = e^{i\alpha Q}, \quad \text{where} \quad Q = \frac{\sqrt{\eta}}{2\pi} \int_{-\infty}^{\infty} dx \partial_x \varphi,
\]

(3.22)

which acts on the exponential fields as follows,

\[
V_\alpha \mathcal{O}_{s,n} V_\alpha^{-1} = e^{i\alpha} \mathcal{O}_{s,n}.
\]

\[\text{In Ref. [31], the soliton-creating operators were denoted as } \mathcal{O}_{a}, \text{ where } a = s\sqrt{\eta}/2. \text{ The parameter } \beta \text{ in [31] coincides with } \sqrt{\eta}. \text{ Note that there, the quantity } 2a/\beta \text{ was not assumed to be integer.}\]
Notice that the Gaussian CFT also possesses such a global symmetry, contrary to the XXZ and XYZ lattice models. Nevertheless, the form of the expansions (2.35), (2.36) and (2.39) suggests that the lattice models are invariant with respect to the transformation \( V_\pi \) which acts trivially on all local lattice fields. Such symmetry manifests itself in the existence of two subspaces \( V^{(j)} \) \((j = 0, 1)\) which can be treated as eigenspaces of the operator \( V_\pi \):

\[
V_\pi V^{(j)} = (-1)^j V^{(j)} .
\]

The fundamental sine-Gordon kink-like excitations, the soliton \( A_- \) and the antisoliton \( A_+ \), carry, respectively, negative and positive units of the topological charge \( Q \) (3.22):

\[
V_\alpha A_\pm(\theta) V_\alpha^{-1} = e^{\pm i \alpha} A_\pm(\theta) ,
\]

where the argument \( \theta \) denotes kink rapidity. The relation with the Zamolodchikov-Faddeev operators of the XYZ model was established in Ref. [32]. In our notations it can be summarized as follows: the operators \( A_\pm(k) \) defined in the lattice model as

\[
A_{\pm}^{(1,0)} = i \sqrt{2} \left( \pm B_{\pm}^{(1,0)} + B_{\pm}^{(1,0)} \right), \quad A_{\pm}^{(0,1)} = i \sqrt{2} \left( - B_{\pm}^{(0,1)} \pm B_{\pm}^{(0,1)} \right),
\]

(3.24) turn out, in the scaling limit, to be the Zamolodchikov-Faddeev operators \( A_\pm(\theta) \) (3.23) of the sine-Gordon model. Here, as well as in Eq. (3.11), we denote the restriction of the operators \( B_\pm \) and \( A_\pm \) acting on the subspace \( V^{(j)} \) as \( B_{\pm}^{(1-j,j)} \) and \( A_{\pm}^{(1-j,j)} \). Again, with some abuse of notation, we use the same symbol \( A_\pm \) for the XYZ operators and for their scaling limits. Notice that the quasi-momentum \( k \) of the low-lying fundamental excitation becomes the usual particle momentum in the scaling limit:

\[
\lim_{\varepsilon \to 0} \frac{k}{\varepsilon} = M \sinh(\theta) \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{E(k)}{\varepsilon} = M \cosh(\theta),
\]

where \( M \) is the soliton mass (3.20).

### 3.3 Scaling behavior of the form factors

We are now in a position to calculate the constants appearing in expansions (3.1), (3.2). To illustrate the procedure, let us consider first the scaling behavior of the lattice VEV (3.13). From relation (3.1) we deduce that

\[
F \sim 2 C_0 \varepsilon^{\eta/2} \langle O_{1,0} \rangle + \ldots .
\]

(3.25)

Here \( \langle \ldots \rangle \) means an averaging with respect to the vacuum (3.24) with \( j = 0 \). To write the equation (3.25), we also use the fact that the VEV of the fields \( O_{1,0} \) and \( O_{-1,0} \) are equal by virtue of charge conjugation symmetry. The VEV of the operator \( O_{s,0} \) was found in [33]. We denote it as

\[
\langle O_{s,0} \rangle = \sqrt{Z_{s,0}} .
\]

On the other hand, the function \( F \) is given by the Baxter-Kelland formula [28] (see Eq. (A.7)). Its leading scaling behavior reads

\[
F \sim \frac{1}{1 - \eta} (4 R c)^{-\eta/2} + \ldots .
\]

(3.26)
Comparing Eqs. (3.25) and (3.26), and using the relation $R_c^{-1} = M\varepsilon$, one can deduce the constant $C_0$:

$$C_0 = \frac{1}{2(1 - \eta) \sqrt{Z_{1,0}}} \left( \frac{M}{4} \right)^{\frac{3}{4}}. \quad (3.27)$$

A similar strategy can be applied to calculate the constants $C_1$ and $C_1^\tau$ in (3.1), (3.2). The relations (3.24) allows one to express the leading scaling behavior of the functions that appear in Eqs. (3.14)-(3.16) through the form factors of the sine-Gordon fields:

$$F_1^x \pm i F_1^y \sim 2 C_0 \varepsilon^{\frac{3}{2}} \langle O_{\pm 1,0}(0) | A_+(\theta_1)A_-(\theta_2) \rangle_{\text{in}} + \ldots,$$

$$F_2^x \pm i F_2^y \sim -C_1 \varepsilon^{\frac{3}{2} + \frac{1}{2\eta}} \langle O_{\pm 1,2}(0) | A_-(\theta_1)A_-(\theta_2) \rangle_{\text{in}} + \ldots,$$

and

$$F_1^z \sim \frac{\varepsilon}{2\pi \sqrt{\eta}} \langle \partial_t \varphi(0) | A_+(\theta_1)A_-(\theta_2) \rangle_{\text{in}} + \ldots,$$

$$F_2^z \sim \frac{i}{2} C_1^\tau \varepsilon^{\frac{1}{2\eta}} \langle O_{0,2}(0) | A_-(\theta_1)A_-(\theta_2) \rangle_{\text{in}} + \ldots.$$  \quad (3.28)

We use here an abbreviated notation similar to (3.12), except that the index specifying the vacuum states is omitted since it is always assumed to be $j = 0$. The two-particle form factors of the topologically neutral operators $O_{\pm 1,0}$ and $\partial_t \varphi$ have been known for a long time (see e.g. [25]). They read explicitly,

$$\langle O_{\pm 1,0}(0) | A_+(\theta_1)A_-(\theta_2) \rangle_{\text{in}} = \sqrt{Z_{1,0}} \frac{G(\theta_1 - \theta_2)}{\xi G(-i\pi)} \frac{2i\varepsilon^{\alpha}(\theta_1 - \theta_2 + i\pi)/(2\xi)}{\sinh((\theta_1 - \theta_2 + i\pi)/(\xi))}, \quad (3.32)$$

$$\langle \partial_t \varphi(0) | A_+(\theta_1)A_-(\theta_2) \rangle_{\text{in}} = \frac{\sqrt{G(-i\pi)}}{\xi G(-i\pi)} \frac{i\pi M}{\cosh((\theta_1 - \theta_2 + i\pi)/(2\xi))}, \quad (3.33)$$

where $\xi = \frac{n}{1 - \eta}$ and the function $G(\theta)$ is a so-called minimal form factor. The form factors of the topologically charged operators $O_{s,n}$ have been proposed in [31]. In particular, for $n = 2$ one has:

$$\langle O_{s,2}(0) | A_-(\theta_1)A_-(\theta_2) \rangle_{\text{in}} = \sqrt{Z_{s,2}} \varepsilon^{\frac{s_2}{2}} \varepsilon^{\frac{s_1 + s_2}{2}} G(\theta_1 - \theta_2). \quad (3.34)$$

The explicit formulae for the minimal form factor $G(\theta)$ and the field-strength renormalization $Z_{s,n}$ are recalled in Appendix B.

We are now able to compare (3.28)-(3.33) with expansions of the exact lattice form factors (3.13)-(3.16) given in Appendix A, and to relate the values of the constants $C_0$, $C_1$ and $C_1^\tau$ to the constants $Z_{s,n}$:

$$C_1 = \frac{4}{\eta G(-i\pi) \sqrt{Z_{1,2}}} \left( \frac{M}{4} \right)^{\frac{3}{4} + \frac{1}{2\eta}}, \quad (3.35)$$

$$C_1^\tau = \frac{8}{\eta G(-i\pi) \sqrt{Z_{0,2}}} \left( \frac{M}{4} \right)^{\frac{1}{2\eta}}. \quad (3.36)$$
4 Correlation amplitudes. Comparison with numerical results

With the explicit expression (B.3) for the normalization constants $Z_{1,0}$, $Z_{1,2}$ and $Z_{0,2}$, the relations (3.27), (3.35) and (3.36) lead to the following formulae for the correlation amplitudes $A$, $\tilde{A}$ and $A_z$ (2.50):

$$A = \frac{1}{2(1-\eta)^2} \left[ \frac{\Gamma(\frac{\eta}{2-2\eta})}{2\sqrt{\pi}\Gamma(\frac{1}{2-2\eta})} \right]^\eta \exp \left\{ - \int_0^\infty \frac{dt}{t} \left( \frac{\sinh(\eta t)}{\sinh((1-\eta)t)} - \eta e^{-2t} \right) \right\},$$

(4.1)

$$\tilde{A} = \frac{2}{\eta(1-\eta)} \left[ \frac{\Gamma(\frac{\eta}{2-2\eta})}{2\sqrt{\pi}\Gamma(\frac{1}{2-2\eta})} \right]^{\eta+\frac{1}{2}} \exp \left\{ - \int_0^\infty \frac{dt}{t} \left( \frac{\cosh(2\eta t)e^{-2t} - 1}{2\sinh(\eta t)\sinh(t)} + \frac{1}{\sinh(\eta t)} - \frac{\eta^2 + 1}{\eta} e^{-2t} \right) \right\},$$

(4.2)

and

$$A_z = \frac{8}{\pi^2} \left[ \frac{\Gamma(\frac{\eta}{2-2\eta})}{2\sqrt{\pi}\Gamma(\frac{1}{2-2\eta})} \right]^\frac{1}{\eta} \exp \left\{ \int_0^\infty \frac{dt}{t} \left( \frac{\sinh((2\eta - 1)t)}{\sinh(t)\sinh((1-\eta)t)} - \frac{2\eta - 1}{\eta} e^{-2t} \right) \right\}.$$  

(4.3)

Note that these amplitudes obey the simple relation:

$$\frac{\tilde{A}}{AA_z} = \pi \frac{\Gamma^2(1+\frac{\eta}{2-2\eta})}{4 \Gamma^2(\frac{\eta}{2}+\frac{\eta}{2-2\eta})}. \quad (4.4)$$

The correlation amplitude $A$ was already obtained in [33]. Our computations also confirm the conjecture from [34] concerning the amplitude $A_z$.

In [35], numerical values of the spin-spin equal-time correlation functions have been obtained for an open chain of 200 sites by the density-matrix renormalization-group technique [36]. In Table 1, we compare, for different values of the anisotropy parameter $\Delta$, the numerical values that follow from (4.2), (4.3) with those estimated in [35] from the fitting of the numerical data. The corresponding plots are represented in Figure 1.

5 Spin-spin correlation functions in the isotropic limit

5.1 Marginal perturbations of the Wess-Zumino-Witten model

As long as the parameter $\eta$ is not too close to unity, the first terms of the asymptotic expansions (2.47), (2.48) provide a good approximation to the spin-spin correlation functions even for moderate space separations $l$. However, these expansions cannot be directly applied in the isotropic limit $\eta \to 1$. Indeed, in this limit, the operator $\mathcal{O}_{0,4} + \mathcal{O}_{0,-4}$ in the effective
# Table 1: Correlation amplitudes.

| $\Delta$ | $\tilde{A}^{NUM}/4$ | $\tilde{A}/4$ | $A^{NUM}_z/4$ | $A_z/4$ |
|----------|---------------------|--------------|----------------|--------|
| 0.7      | 0.004720            | 0.008(1)     | 0.00893        |        |
| 0.6      | 0.0048(14)          | 0.006643     | 0.01314        |        |
| 0.5      | 0.0076(9)           | 0.008656     | 0.01795        |        |
| 0.4      | 0.0099(7)           | 0.010696     | 0.02332        |        |
| 0.3      | 0.0122(4)           | 0.012717     | 0.02924        |        |
| 0.2      | 0.0144(2)           | 0.014687     | 0.03574        |        |
| 0.1      | 0.0164(2)           | 0.016583     | 0.04285        |        |
| 0.0      | 0.0182(2)           | 0.018368     | 0.05066        |        |
| -0.1     | 0.01995(7)          | 0.020082     | 0.05929        |        |
| -0.2     | 0.02154(5)          | 0.021657     | 0.06891        |        |
| -0.3     | 0.02296(4)          | 0.023098     | 0.07978        |        |
| -0.4     | 0.02420(4)          | 0.024392     | 0.09231        |        |
| -0.5     | 0.02525(6)          | 0.025522     | 0.10713        |        |
| -0.6     | 0.0261(2)           | 0.026464     | 0.12539        |        |
| -0.7     | 0.0267(3)           | 0.027182     | 0.14930        |        |
| -0.8     | 0.0271(7)           | 0.027608     | 0.18414        |        |
| -0.9     | 0.027(2)            | 0.027570     | 0.24844        |        |

The suitable way to explore the $\eta \to 1$ limit is based on the low-energy effective theory defined as a perturbation of the Gaussian model with $\eta = 1$. As is well known (see e.g. [17]), the Gaussian model coincides in this case with the SU(2) level one Wess-Zumino-Witten (WZW) theory. In particular, the WZW holomorphic currents are identified with the following primary operators of the Gaussian CFT,

\[
J^z_R = \frac{1}{4} (\partial_t - \partial_x) \varphi, \quad J^{\pm}_R = \mathcal{O}_{1,\pm 2}, \quad (5.1)
\]

\[
J^z_L = \frac{1}{4} (\partial_t + \partial_x) \varphi, \quad J^{\pm}_L = \mathcal{O}_{\pm 1,\pm 2}, \quad (5.2)
\]

whereas the matrix of the fundamental WZW field is bosonized as

\[
\begin{pmatrix}
\mathcal{O}_{0,2} & i \mathcal{O}_{-1,0} \\
i \mathcal{O}_{1,0} & \mathcal{O}_{0,-2}
\end{pmatrix}.
\]

The low-energy effective Hamiltonian can be expressed as a marginal current-current perturbation of the WZW Hamiltonian \[37\],

\[
H_{XXZ} = H_{WZW} + \int \frac{dx}{2\pi} \left\{ g_+ J^+_R J^+_L + \frac{g}{2} \left( J^+_R J^-_L + J^-_R J^+_L \right) + \cdots \right\}. \quad (5.3)
\]
In this expression, the coupling constants $g_\parallel$ and $g_\perp$ should be understood as “running” ones, i.e. depending on the renormalization scale $r$ which has the dimension of length. The corresponding Renormalization Group (RG) flow is known as the Kosterlitz-Thouless flow. For our purpose, we need only to consider the domain

$$|g_\perp| \leq g_\parallel,$$  \hspace{1cm} (5.4)

in which all RG trajectories flow toward the line $g_\perp = 0$ of the infrared-stable fixed points associated with the Gaussian CFT. These trajectories are characterized by the limiting values of the running coupling $g_\parallel$,

$$\epsilon = \frac{1}{2} \lim_{r \to +\infty} g_\parallel(r),$$  \hspace{1cm} (5.5)

and the parameter $\epsilon$ is simply related with the parameter $\eta$ of the Gaussian model,

$$\epsilon = 1 - \eta.$$  \hspace{1cm} (5.6)

The RG flow of the running coupling constants is defined by a system of ordinary differential equations,

$$r \frac{dg_\parallel}{dr} = - \frac{g_\parallel^2}{f_\parallel(g_\parallel, g_\perp)}, \hspace{1cm} r \frac{dg_\perp}{dr} = - \frac{g_\parallel g_\perp}{f_\perp(g_\parallel, g_\perp)}.$$  \hspace{1cm} (5.7)

Perturbatively, the functions $f_\parallel, f_\perp(g_\parallel, g_\perp) = 1 + O(g)$ admit loop expansions as power series in $g_\parallel$ and $g_\perp$, and their precise form depends on the choice of a renormalization scheme. We
use here the scheme introduced by Al.B. Zamolodchikov \[30, 38\], who showed that, under a suitable diffeomorphism in \(g\|\) and \(g\perp\), the functions \(f\|\) and \(f\perp\) can be taken equal to each other and to the quantity
\[f\| = f\perp = 1 - \frac{g\|}{2}.	ag{5.8}\]

With this particular choice of the \(\beta\)-function, it is possible to integrate the RG flow equations exactly. To do this, let us first note that the system of differential equations (5.7), (5.8) admits a first integral, the numerical value of which is determined by means of the condition (5.5):
\[g\|^2 - g\perp^2 = (2\epsilon)^2.	ag{5.9}\]

Then the equations (5.7) are solved as
\[g\| = 2\epsilon \frac{1 + q}{1 - q}, \quad g\perp = 4\epsilon \frac{\sqrt{q}}{1 - q},\tag{5.10}\]

where \(q = q(r)\) is the solution of
\[q^{\frac{1}{2} - \frac{1}{2}} (1 - q) = \epsilon \left(\frac{r_0}{r}\right)^2.\tag{5.11}\]

As well as \(\epsilon\), the dimensional parameter \(r_0\) is a RG invariant. It is of the same order as the lattice spacing \(\epsilon\), and is supposed to have a regular loop expansion of the form
\[\frac{\epsilon}{r_0} = \exp \left( c_0 + c_1 \epsilon + c_2 \epsilon^2 + \ldots \right).\tag{5.12}\]

It should be noted that the even coefficients \(c_0, c_2, \ldots\) in (5.12) are essentially ambiguous and can be chosen as one wants. A variation of these coefficients corresponds to a smooth redefinition of the coupling constants which does not affect the \(\beta\)-function. On the contrary, the odd constants \(c_{2k+1}\) are unambiguous and precisely specified, once the form of the RG equations is fixed. It is possible to show \[30, 38\] that the odd constants vanish in Zamolodchikov’s scheme: \(c_{2k+1} = 0\ (k = 0, 1, \ldots)\). Therefore, once the coefficients \(c_{2k}\) in (5.12) are chosen, the renormalization scheme is completely specified.

As already mentioned, the perturbation by the marginal operators produces logarithmic corrections to the scale-invariant correlation functions. Hence, the conformal normalization condition imposed on the field \(O_{s,n}\) turns out to be singular for \(\epsilon = 0\), and we should define renormalized fields \(O_{s,n}^{(\text{ren})}\), which are rescaled version of the “bare” exponential operators:
\[O_{s,n}^{(\text{ren})}(x, t; r) = Z_{s,n}^{-\frac{1}{2}}(r) \ O_{s,n}(x, t).\tag{5.13}\]

Notice that, in writing (5.13), we assume that there is no resonance mixing of the operator \(O_{s,n}\) with other fields, so that it is renormalized as a singlet. In particular, one can easily check that this is indeed the case when
\[|n| < 2 + 2 |s| .\tag{5.14}\]
The renormalized fields \( \psi_{s,n} \) are no longer singular at \( \epsilon = 0 \), but depend on the auxiliary RG scale \( r \). To specify them completely, we have to impose some non-singular normalization condition. The conventional condition, which is usually imposed on Green’s function for a space-like interval \( t^2 - x^2 < 0 \), is

\[
\langle T \mathcal{O}_{s,n}^{\text{ren}}(x,t; r) \mathcal{O}_{s,n}^{\text{ren}}(0,0; r) \rangle \bigg|_{\sqrt{x^2-t^2}=r} = \left( \frac{t-x}{t+x} \right)^{2n} \tag{5.15}
\]

Eqs. (5.13) and (5.15) imply that the correlators of the “bare” exponential fields should take the form:

\[
\langle T \mathcal{O}_{s,n}(x,t) \mathcal{O}_{s,n}^\dagger(0,0) \rangle = \left( \frac{t-x}{t+x} \right)^{2n} \mathcal{Z}_{s,n} \left( \sqrt{x^2-t^2} \right). \tag{5.16}
\]

Then, using (3.1), (3.2), one can express the spin-spin correlation functions through the renormalization factors \( \mathcal{Z}_{s,n} \). For example, for the time-ordered correlation function of \( \sigma^x \), one has:

\[
\langle T \sigma^x_{i+j}(t) \sigma^x_j(0) \rangle \sim A \varepsilon^{2d_{i,j}} \mathcal{Z}_{1,0}(\varepsilon \sqrt{l_t^+ - l_t^-}) - (-1)^j \tilde{A} \varepsilon^{2d_{1,2}} \mathcal{Z}_{1,2}(\varepsilon \sqrt{l_t^+ - l_t^-}) \left\{ \frac{1}{2} \left( \frac{l_t^+ + l_t^-}{l_t^+ - l_t^-} \right) - \mathcal{R}(\varepsilon \sqrt{l_t^+ - l_t^-}) \right\} + \ldots. \tag{5.17}
\]

Here we use notations (2.49), and the function \( \mathcal{R} \) is related to the following causal Green’s function as,

\[
\langle T \mathcal{O}_{1,2}(x,t) \mathcal{O}_{-1,2}(0,0) \rangle = \mathcal{Z}_{1,2}(r) \mathcal{R}(r) \bigg|_{r=\sqrt{x^2-t^2}}. \tag{5.18}
\]

The first terms of the perturbative expansion for the scalar factor \( \mathcal{Z}_{s,n}(r) \) in (5.13) can be deduced from the results of work \[39\]:

\[
\mathcal{Z}_{s,n}(r) = \tilde{Z}_{s,n} \left( \frac{\varepsilon}{r} \right)^{\frac{n^2}{2} + s^2(1+\epsilon^2)} \left( g_{\perp}^2 \right)^{\frac{n^2}{2} - s^2(1-\epsilon^2)} \times e^{u_1 g_{\parallel} + u_2 g_{\perp}^3} \left( 1 + g_{\perp}^2 (v_1 - v_2 g_{\parallel}) + \mathcal{O}(g^4) \right), \tag{5.19}
\]

where

\[
\tilde{Z}_{s,n} = \varepsilon^{-2d_{s,n}} \left( 2^{1-\epsilon} \sqrt{e^{-c_0(1-\epsilon^3)+\ldots}} \right)^{2s^2-2d_{s,n}} e^{-2\epsilon u_1 - (2\epsilon^3) u_2 + \ldots},
\]

and \( d_{s,n} \) is given by (2.15) with \( \eta = 1 - \epsilon \). The coefficients \( u_1, u_2, v_1 \) and \( v_2 \) in these equations are listed in Appendix C. Notice that to derive (5.19) one should assume that the field \( \mathcal{O}_{s,n} \) is mutually local with respect to the density of the effective Hamiltonian (2.40). This assumption implies that \( s \in \mathbb{Z} \), but does not impose any restriction on \( n \) in addition to (5.14).

As follows from the Callan-Symanzik equation, the function \( \mathcal{R} \) in (5.18) admits a perturbative expansion in terms of the running coupling constants. Explicitly, one can obtain

\[
\mathcal{R} = -\frac{g_{\perp}^4}{4} \left\{ g_{\parallel} + \left( c - \frac{1}{2} \right) g_{\parallel}^2 + c g_{\perp}^2 + \mathcal{O}(g^3) \right\}. \tag{5.20}
\]
The constant $c$ appearing in Eq. (5.20) is related to $c_0$ from (5.12) as

$$c_0 = c + \gamma E + \frac{1}{2} \ln(2\pi),$$

(5.21)

where $\gamma E = 0.5772\ldots$ is the Euler constant. Combining relation (5.17) with (5.19) and (5.20), one can deduce the RG improved expansion of the $\sigma^x$ lattice correlator which is applicable for $\epsilon \ll 1$.

We can similarly derive an expansion for the correlation function of $\sigma^z$. The relation

$$\frac{1}{2\pi\sqrt{\eta}} \partial_t \varphi = \frac{2}{i\pi} \partial_x \partial_n O_{0,n} \bigg|_{n=0},$$

(5.22)

which follows from the definition (2.14), is useful to perform this computation. Note also that the operators

$$O_+ = \frac{O_{0,2} + O_{0,-2}}{\sqrt{2}}, \quad O_- = \frac{O_{0,2} - O_{0,-2}}{\sqrt{2}i},$$

renormalize as singlets:

$$O_{\pm}^{(\text{ren})}(x, t; r) = Z_{\pm}^{-\frac{1}{2}}(r) O_{\pm}(x, t).$$

(5.23)

Indeed, since

$$\mathbb{C} O_\pm \mathbb{C} = \pm O_\pm,$$

invariance with respect to the charge conjugation prevents resonance mixing of $O_+$ and $O_-$. Now, using equations (5.22), (5.23) and (3.2), one obtains

$$\langle T \sigma^z_{i+j}(t) \sigma^z_j(0) \rangle \sim -\frac{2}{\pi^2} (\partial_+ + \partial_-)^2 \partial_n^2 Z_{0,n}(\varepsilon \sqrt{l_+ l_-}) \bigg|_{n=0}$$

+ $(-1)^l A_\pm \varepsilon^{2l_{0,2}} Z_{\pm}(\varepsilon \sqrt{l_+ l_-}) + \ldots$.  

(5.24)

We collect in Appendix D the RG improved expansions of the different-time two-point correlation functions (5.17) and (5.24).

5.2 Equal-time correlation functions for the XXX spin chain. Comparison with numerical results

Using expansions (D.1), (D.2), it is easy to perform the isotropic limit. Setting $\epsilon = 0$ and $g_\perp = g_\parallel = g$, one obtains the following large $l$ expansion for the equal-time spin-spin correlation functions:

$$\langle \sigma^z_{i+j} \sigma^z_j \rangle = (-1)^l \langle \sigma^z_{i+j} \sigma^z_j \rangle$$

$$\sim \sqrt{\frac{2}{\pi^3 l}} \frac{1}{\sqrt{g}} \left\{ 1 + \left( \frac{3}{8} - \frac{c}{2} \right) g + \left( \frac{5}{128} - \frac{c}{16} - \frac{c^2}{8} \right) g^2 \right.$$  

$$+ \left( \frac{21}{1024} + \frac{7c}{256} - \frac{7c^2}{64} - \frac{c^3}{16} + \frac{13\zeta(3)}{32} \right) g^3 + O(g^4) \right\}$$

$$- \left( -1 \right)^l \frac{1}{\pi^2 l^2} \left\{ 1 + \frac{g}{2} + \left( c + \frac{3}{4} \right) \frac{g^2}{2} + \frac{c(c+2)}{2} g^3 + O(g^4) \right\} + \ldots. \tag{5.25}$$

The coefficient $\sqrt{2/\pi^3}$ in Eq. (5.24) was originally obtained in Ref. [40].
Here $g = g(l)$ is a solution of the equation
\[ \sqrt{g} e^{\frac{1}{g}} = 2\sqrt{2\pi} e^{\gamma E + c l}, \]
which corresponds to the limit $\epsilon \to 0$ of Eqs. (5.10) and (5.11). Let us stress that, if the perturbation series in (5.25) can be summed, then the correlation function should not depend on the auxiliary parameter $c$ (5.12), (5.21). This, however, is not true if we truncate the perturbative series at some finite order. Thus, when fitting the numerical data with (5.25), we may treat $c$ as an optimization parameter, allowing us to minimize the remainder of the series, or at least to develop some feeling concerning the effects of this remainder.

The correlation function (5.25) has been computed numerically in [43] for $1 \leq l \leq 30$. The authors used the density-matrix algorithm [36] to study the large-distance decay of the correlation function for XXX spin chains with $14 \leq N \leq 70$ sites. To extract the values of the correlation function in the infinite chain case, they adopted the phenomenological scaling relation of Kaplan et al. [41] (see also [42]). The relative error of the interpolation procedure was estimated to be of order 1% for the largest $l$ values. In Table 2, we compare those numerical data to the results obtained from (5.25) in the cases $c = -1$ and $c = -2$. The corresponding plots (numerical data against RG result for $c = -1$) are given in Figure 2. It appears that the numerical data are consistent with our prediction within the given errors.

| $l$ | NUM | RG ($c = -1$) | RG ($c = -2$) | $l$ | NUM | RG ($c = -1$) | RG ($c = -2$) |
|-----|-----|--------------|--------------|-----|-----|--------------|--------------|
| 1   | 0.1477 | 0.1616 | *** | 16  | 0.1628 | 0.1624 | 0.1630 |
| 2   | 0.1214 | 0.1213 | 0.1918 | 17  | 0.1676 | 0.1666 | 0.1671 |
| 3   | 0.1510 | 0.1509 | 0.1583 | 18  | 0.1646 | 0.1642 | 0.1647 |
| 4   | 0.1384 | 0.1381 | 0.1424 | 19  | 0.1689 | 0.1679 | 0.1683 |
| 5   | 0.1541 | 0.1541 | 0.1566 | 20  | 0.1661 | 0.1657 | 0.1661 |
| 6   | 0.1463 | 0.1462 | 0.1482 | 21  | 0.1700 | 0.1690 | 0.1694 |
| 7   | 0.1567 | 0.1571 | 0.1586 | 22  | 0.1674 | 0.1670 | 0.1674 |
| 8   | 0.1513 | 0.1514 | 0.1527 | 23  | 0.1712 | 0.1700 | 0.1704 |
| 9   | 0.1596 | 0.1596 | 0.1607 | 24  | 0.1687 | 0.1682 | 0.1686 |
| 10  | 0.1550 | 0.1552 | 0.1561 | 25  | 0.1723 | 0.1710 | 0.1713 |
| 11  | 0.1620 | 0.1618 | 0.1626 | 26  | 0.1699 | 0.1693 | 0.1697 |
| 12  | 0.1581 | 0.1581 | 0.1588 | 27  | 0.1734 | 0.1719 | 0.1722 |
| 13  | 0.1641 | 0.1636 | 0.1643 | 28  | 0.1710 | 0.1703 | 0.1706 |
| 14  | 0.1606 | 0.1604 | 0.1611 | 29  | 0.1746 | 0.1727 | 0.1730 |
| 15  | 0.1659 | 0.1652 | 0.1658 | 30  | 0.1722 | 0.1712 | 0.1716 |

Table 2: Numerical values of the correlation function $\frac{1}{4} \langle \sigma^x_{l+j} \sigma^x_j \rangle$ of the XXX spin chain according to the distance $l$. The column “NUM” has been obtained in [36], whereas the columns “RG ($c = -1$)” and “RG ($c = -2$)” follow from (5.25) with the corresponding values of the free parameter $c$. 
Figure 2: The correlation function $W(l) = \frac{1}{4} \langle \sigma^x_{l+j} \sigma^x_j \rangle$ of the XXX spin chain according to the distance $l$. The dots represent the numerical data obtained in [13], and the continuous line connects odd and even terms obtained from Eq. (5.25) with $c = -1$.

5.3 Erratum of [12]

The spin-spin correlation function in the limit $\epsilon \to 0$ was previously studied in Section 5 of Ref. [12]. The analysis performed in that paper, along with the numerical results obtained in [13] (see Fig. 2 of [12]), strongly suggested the existence of an additional staggered term of the form

$$\propto (-1)^l \frac{1}{j^{l+1}}$$

in the large distance asymptotic expansion of the correlation function $\langle \sigma^x_{l+j} \sigma^x_j \rangle$. It was argued in [12] that such a term occurs because of the presence of a correction of the type $(-1)^l \partial_x O_{x=1,0}$ in expansions (3.1). However, the RG computation from [12] appears to be erroneous. Indeed, Eqs. (5.6) and (5.7) from [12] have to be replaced respectively by our equations (D.7) and (5.25). Therefore, contrary to what was claimed in [12], the numerical data are consistent (within the numerical errors) with (3.1).

6 Conclusion and further remarks

The purpose of this article is the quantitative study of the long-distance behavior of spin-spin correlation functions for the XXZ Heisenberg chain in the critical regime. Our main result here is the determination of analytical expressions for the correlation amplitudes involved in the corresponding asymptotic expansions. To obtain these values, we considered quantum field theory which describes the scaling limit of the lattice model, and compared, in this
limit, the respective normalizations of the lattice operators and of the corresponding local fields. This comparison was achieved by considering known exact matrix elements (form factors).

We would like to conclude the article with the following remarks.

- The method used in this work can be applied to higher order terms in expansion (2.35) or (2.36). For example, we were able to compute (up to sign factors) all constants $C_m$ from (2.36) for odd integers $m = 2p + 1$:

\[
(C_{2p+1})^2 = \frac{2}{\eta(1-\eta)} \left[ \frac{\Gamma\left(\frac{\eta}{2-2\eta}\right)}{2\sqrt{\pi} \Gamma\left(\frac{1}{2-2\eta}\right)} \right]^{\eta+\frac{(2p+1)^2}{\eta}} \prod_{j=1}^{p} \left\{ \sin^2\left(\frac{2\pi j}{\eta}\right) \cot^2\left(\frac{\pi(2j-1)}{2-2\eta}\right) \right\} 
\times \exp \left\{ - \int_0^\infty dt \left( \frac{\cosh(2\eta t)e^{-2(2p+1)t} - 1}{2\sinh(\eta t)\sinh((1-\eta)t)} \right) \right. 
\left. + \frac{2p + 1}{\sinh(\eta t)} - \left( \eta + \frac{(2p+1)^2}{\eta} \right) e^{-2t} \right\} \right). \tag{6.1}
\]

- One can also study expansions of other local lattice operators in terms of the scaling fields. For example, we have calculated the constant $C_0^{(s)}$ in the leading term of the expansion of the lattice operators

\[
\sigma_i^+ \sigma_{i+1}^+ \ldots \sigma_{i+s-1}^+ \sim C_0^{(s)} \epsilon \frac{2p}{\eta} \mathcal{O}_{\pm,s,0} + \ldots, \tag{6.2}
\]

for which we obtained $(p = 0, 1 \ldots)$:

\[
C_0^{(2p)} = \left[ \frac{\Gamma\left(\frac{\eta}{2-2\eta}\right)}{2\sqrt{\pi} \Gamma\left(\frac{1}{2-2\eta}\right)} \right]^{2\eta p^2} \frac{1}{\pi^{2p(1-\eta)^{2p^{2p}-p}}} \prod_{j=1}^{p} \frac{\Gamma^2\left(\frac{1}{2} + \frac{\eta(2j-1)}{2-2\eta}\right) \Gamma^2(\eta/2j - 1))}{\Gamma^2\left(\frac{1}{2} + \frac{\eta(2j-1)}{2-2\eta}\right) \Gamma^2(\eta/2j - 1))}, \tag{6.3}
\]

\[
C_0^{(2p+1)} = \left[ \frac{\Gamma\left(\frac{\eta}{2-2\eta}\right)}{2\sqrt{\pi} \Gamma\left(\frac{1}{2-2\eta}\right)} \right]^{2\eta(p+\frac{1}{2})^2} \frac{1}{2\pi^{2p(1-\eta)^{2p^{2p+p+1}}} \prod_{j=1}^{p} \frac{\Gamma^2\left(\frac{1}{2} + \frac{\eta(2j-1)}{1-\eta}\right) \Gamma^2(2\eta j))}{\Gamma^2\left(\frac{1}{2} + \frac{\eta(2j-1)}{1-\eta}\right) \Gamma^2(2\eta j))} 
\times \exp \left\{ - \int_0^\infty dt \left( \frac{\sinh(\eta t)}{2\sinh(t)\cosh((1-\eta)t)} - \frac{\eta}{2} e^{-2t} \right) \right\}. \tag{6.4}
\]

- Eventually, one can wonder if it is possible to confirm our predictions from existing integral representations of lattice correlators. Up to now, although explicit expressions for the equal-time spin-spin correlation functions at finite lattice distances are known \[14\] \[17\], their long-distance behavior was studied only for the so-called “free fermion point”, $\Delta = 0$. In this case the XXZ spin chain can be mapped onto two non-interacting critical Ising models and the long-distance asymptotics are readily derived from results of works \[13\] \[17\].

- The approach of this work is actually quite general for lattice solvable models at criticality: from a knowledge of particular form factors of a lattice theory and of its quantum field theory counterpart at the scaling limit, it is possible to predict the amplitudes which govern the large distance behavior of lattice correlation functions. It would indeed be interesting to obtain effective results for other critical exactly solvable lattice models.
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A Two-particle form factors in the XYZ model

In this appendix, we collect explicit expressions of two-particle form factors of local spin operators in the XYZ model. Following Baxter \[8, 24\], we use the parameterization of the coupling constants \( J_x > J_y > |J_z| \) of the Hamiltonian \([3, 3]\) in terms of \( 0 < \eta < 1 \) and of the elliptic nome \( 0 < p < 1 \):

\[
J_x = \frac{1 - \eta}{\pi \varepsilon} \left( \frac{\vartheta_4(\eta) \vartheta_4'(0)}{\vartheta_4(0) \vartheta_1(\eta)} + \frac{\vartheta_1(\eta) \vartheta_4'(0)}{\vartheta_4(0) \vartheta_4(\eta)} \right), \tag{A.1}
\]

\[
J_y = \frac{1 - \eta}{\pi \varepsilon} \left( \frac{\vartheta_4(\eta) \vartheta_4'(0)}{\vartheta_4(0) \vartheta_1(\eta)} - \frac{\vartheta_1(\eta) \vartheta_4'(0)}{\vartheta_4(0) \vartheta_4(\eta)} \right), \tag{A.2}
\]

\[
J_z = \frac{1 - \eta}{\pi \varepsilon} \left( \frac{\vartheta_1'(\eta)}{\vartheta_1(\eta)} - \frac{\vartheta_4'(\eta)}{\vartheta_4(\eta)} \right). \tag{A.3}
\]

Here \( \vartheta_i(u) \equiv \vartheta_i(u,p) \) denote the elliptic theta-functions

\[
\vartheta_1(u, p) = 2p^{1/4} \sin(\pi u) \prod_{n=1}^{\infty} (1 - p^{2n})(1 - 2p^{2n} \cos(2\pi u) + p^{4n}), \tag{A.4}
\]

\[
\vartheta_4(u, p) = \prod_{n=1}^{\infty} (1 - p^{2n})(1 - 2p^{2n-1} \cos(2\pi u) + p^{2(2n-1)}), \tag{A.5}
\]

and the prime in Eqs. (A.1)-(A.3) means a derivative: \( \vartheta_i' = \partial_u \vartheta_i \). We shall also use the other conventional theta-functions

\[
\vartheta_2(u) = \vartheta_1(u + 1/2), \quad \vartheta_3(u) = \vartheta_4(u + 1/2),
\]

and the notation,

\[
\xi = \frac{\eta}{1 - \eta}. \tag{A.6}
\]

With this parameterization, the VEV of \( \sigma^x \) \([3, 13]\) is given by the Baxter-Kelland formula \([28]\):

\[
F = (1 + \xi) p^{\xi} \prod_{n=1}^{\infty} \left( \frac{1 - p^{n(1+\xi)}}{1 - p^{(n-\frac{1}{2})(1+\xi)}} \frac{1 - p^{n-\frac{1}{2}}}{1 - p^{n}} \right)^2. \tag{A.7}
\]
In order to describe the two-particle form factors, one needs to know the explicit form of the dispersion relation (3.10). For this purpose, it is convenient to parameterize the quasi-momentum \( k \) by means of the so-called rapidity variable \( \theta \):

\[
e^{ik(\theta)} = \frac{\partial_4(\frac{\theta}{2\pi} - \frac{1}{4}, p^{\frac{1+i}{2}})}{\partial_4(\frac{\theta}{2\pi} + \frac{1}{4}, p^{\frac{1-i}{2}})}.
\]

As a function of \( \theta \), the excitation energy explicitly reads [13],

\[
E(\theta) = \frac{\partial k(\theta)}{\partial \theta}.
\]

Equations (A.8) and (A.9) define the dispersion relation \( E = E(k) \) in parametric form.

The two-particle form factors of the spin operators can be computed by means of the q-vertex operator approach, for which progress has been made recently in the XYZ case [26,27]. In [27], the two-particle form factors of the \( \sigma^x \) operator were obtained:

\[
F_1^x = \frac{F_0 \tilde{G}(\theta_1 - \theta_2, p) \partial_4(0, p^\frac{1}{2})}{\partial_4(\frac{\theta_1 + \theta_2}{2\pi} - \frac{1}{4}, p^{\frac{1+i}{2}}) \partial_4(\frac{\theta_1 - \theta_2 + i\pi}{2\pi}, p^{\frac{1+i}{2}})},
\]

\[
F_2^x = \frac{F_0 \tilde{G}(\theta_1 - \theta_2, p) \partial_4(0, p^\frac{1}{2})}{\partial_4(\frac{\theta_1 + \theta_2}{2\pi} - \frac{1}{4}, p^{\frac{1+i}{2}}) \partial_4(\frac{\theta_1 - \theta_2 - i\pi}{2\pi}, p^{\frac{1+i}{2}})}.
\]

For \(-2\pi < \Im n(\theta) < 0\) the meromorphic function \( \tilde{G} \) reads

\[
\tilde{G}(\theta, p) = e^{\frac{\delta(1+i\pi)}{8\pi \xi} (\theta + i\pi)^2} \exp \left\{ \sum_{n=1}^{\infty} \frac{\sin^2(\delta n(\theta + i\pi)/2) \sinh(\pi\delta n) \sinh(\pi\delta n/2) \cosh(\pi\delta n/2)}{n \sinh(\pi\delta n) \sinh(\pi\delta n/2) \cosh(\pi\delta n/2)} \right\},
\]

and it is defined through an analytic continuation outside this domain. The parameter \( \delta \) in (A.12) is related to the elliptic nome as \( p = e^{-\pi\xi(\theta+i\pi)} \), and the constant \( F_0 \) in (A.10), (A.11) is given by

\[
F_0 = \frac{1 + \xi}{\pi \xi} \frac{\theta_1(0, p^{\frac{1+i}{2}}) \theta_1(0, p^{\frac{1-i}{2}})}{\theta_1(0, p^{\frac{i}{2}})}.
\]

Notice that, in writing the form factors, we always assume the conventional normalization of vacuum states, \( \langle \text{vac} | \text{vac} \rangle = 1 \), and of in-asymptotic states:

\[
in\langle B_{\sigma_n^x}(k'_n) \ldots B_{\sigma_1^x}(k'_1) | B_{\sigma_n}(k_n) \ldots B_{\sigma_1}(k_1) \rangle_{\text{in}} = (2\pi)^n \prod_{j=1}^{n} \delta_{\sigma_j \sigma'_j} \delta(\theta_j - \theta'_j),
\]

where \( k_j = k(\theta_j) \) and \( k'_j = k(\theta'_j) \).

\(^9\)Note that the regime considered in [27] is the so-called principal one \((-J_x > J_y \geq |J_z|)\). To apply the results obtained there to the case with \( J_y > J_z \geq |J_x| \), one has to replace \( \sigma^x, \sigma^y \) and \( \sigma^z \) from [27] respectively by \( \sigma^y, (-1)^0 \sigma^x \) and \( (-1)^1 \sigma^z \), which corresponds to a similarity transformation of the Hamiltonian (3.3).
Using the method proposed in [27], one can also compute the two-particle form factors of the other spin fields, $\sigma^y$ and $\sigma^z$:

\[
F_1^y = - \frac{F_0 \tilde{G}(\theta_1 - \theta_2, p) \partial_3(0, p^\mp)}{\partial_4(\frac{\theta_1}{2\pi} - \frac{1}{4}, p^\mp) \partial_4(\frac{\theta_2}{2\pi} - \frac{1}{4}, p^\mp)} \frac{\partial_3(\frac{\theta_1 + \theta_2}{2\pi}, p^{\pm \mp})}{\partial_3(\frac{\theta_1 - \theta_2 + i\pi}{2\pi \xi}, p^{\pm \mp})}, \tag{A.14}
\]

\[
F_2^y = - \frac{F_0 \tilde{G}(\theta_1 - \theta_2, p) \partial_3(0, p^\mp)}{\partial_4(\frac{\theta_1}{2\pi} - \frac{1}{4}, p^\mp) \partial_4(\frac{\theta_2}{2\pi} - \frac{1}{4}, p^\mp)} \frac{\partial_3(\frac{\theta_1 + \theta_2}{2\pi}, p^{\pm \mp})}{\partial_3(\frac{\theta_1 - \theta_2 + i\pi}{2\pi \xi}, p^{\pm \mp})}. \tag{A.15}
\]

and

\[
F_1^z = i \frac{F_0 \tilde{G}(\theta_1 - \theta_2, p) \partial_2(0, p^\mp)}{\partial_4(\frac{\theta_1}{2\pi} - \frac{1}{4}, p^\mp) \partial_4(\frac{\theta_2}{2\pi} - \frac{1}{4}, p^\mp)} \frac{\partial_2(\frac{\theta_1 + \theta_2}{2\pi}, p^{\pm \mp})}{\partial_2(\frac{\theta_1 - \theta_2 + i\pi}{2\pi \xi}, p^{\pm \mp})}, \tag{A.16}
\]

\[
F_2^z = i \frac{F_0 \tilde{G}(\theta_1 - \theta_2, p) \partial_2(0, p^\mp)}{\partial_4(\frac{\theta_1}{2\pi} - \frac{1}{4}, p^\mp) \partial_4(\frac{\theta_2}{2\pi} - \frac{1}{4}, p^\mp)} \frac{\partial_3(\frac{\theta_1 + \theta_2}{2\pi}, p^{\pm \mp})}{\partial_3(\frac{\theta_1 - \theta_2 + i\pi}{2\pi \xi}, p^{\pm \mp})}. \tag{A.17}
\]

To compare the lattice and the sine-Gordon two-particle form factors, one should take the limit $p \to 0$. Notice that

\[
p \simeq (4 R_c)^{-\frac{4}{\ell + 1}},
\]

where the correlation length is defined as in (3.18). The following relation between the function $\tilde{G}$ (A.12) and the sine-Gordon minimal form factor (B.2):

\[
\lim_{p \to 0} \tilde{G}(\theta, p) = \frac{G(\theta)}{G(-i\pi)},
\]

is useful to proceed with this limit.

**B  Form factors of topologically charged operators in the sine-Gordon model**

In this appendix, we recall the expressions obtained in [31] concerning the form factors of the topologically charged (or soliton-creating) operators $O_{s,n}$ in the sine-Gordon model. The simplest non-vanishing form factor of $O_{s,n}$ is given by the formula:

\[
\langle O_{s,n}(0) \mid A_-(\theta_1) \cdots A_-(\theta_n) \rangle_{in} = \sqrt{Z_{s,n}} \ e^{\frac{4\pi s}{4}} \prod_{m=1}^{n} e^{\frac{s\theta_m}{2}} \prod_{m<j} G(\theta_m - \theta_j). \tag{B.1}
\]

Here, the minimal form factor $G$ has a form:

\[
G(\theta) = i C_1 \sinh(\theta/2) \exp \left\{ \int_{0}^{\infty} \frac{dt}{t} \frac{\sinh^2(t - i\theta/\pi)}{\sinh(2t) \cosh(t) \sinh(\xi t)} \right\}. \tag{B.2}
\]
The explicit expression of the normalization constant \( Z_{s,n} \), which has been conjectured in [31], is the following:

\[
Z_{s,n} = \left( \frac{C_2}{2C_1^2} \right)^{\frac{n^2}{4}} \left( \frac{\xi C_2}{16} \right)^{\frac{\pi^2}{4}} \left[ \frac{\sqrt{\pi} M \Gamma \left( \frac{3}{2} + \frac{\xi}{2} \right)}{\Gamma \left( \frac{3}{2} \right)} \right]^{2d_{s,n}} \\
\times \exp \left\{ \int_0^\infty dt \left[ \frac{\cosh(2\xi st) e^{-(1+\xi)nt} - 1}{2 \sinh(\xi t) \sinh((1+\xi)t) \cosh(t)} + \frac{n}{2 \sinh(t\xi)} - 2d_{s,n} e^{-2t} \right] \right\}. \tag{B.3}
\]

In the previous formulae we use the notations,

\[
C_1 \equiv G(-i\pi) = \exp \left\{ -\int_0^\infty dt \sinh^2(t/2) \sinh((\xi-1)t) \right\}, \tag{B.4}
\]

\[
C_2 = \exp \left\{ 4 \int_0^\infty dt \frac{\sinh^2(t/2) \sinh((\xi-1)t)}{\sinh(2t) \sinh(\xi t)} \right\}, \tag{B.5}
\]

and \( \xi \) is given by (A.6).

## C Numerical coefficients for equation (5.19)

We collect in this appendix the explicit expressions of the coefficients \( u_1, u_2, v_1 \) and \( v_2 \) which occur in the expansion (5.19).

\[
u_1 = \frac{n^2 - 4s^2}{16} \left( T_s \left( \frac{n}{2} \right) - \frac{3}{2} \right) + \frac{s(s-1)}{4},
\]

\[
u_2 = \frac{(n^2 - 4s^2)(n^2 + 4s^2 - 8)}{3072} T_s'' \left( \frac{n}{2} \right) + \frac{n(n^2 - 4)}{192} T_s' \left( \frac{n}{2} \right) + \frac{3n^2 - 4}{192} T_s \left( \frac{n}{2} \right) - \frac{s(s+2)}{192} - \frac{11n^2}{768} + \frac{c}{24} + \frac{c_2(n^2 - 4s^2)}{32}, \tag{C.1}
\]

where

\[
T_s(z) = \psi(z+s) + \psi(-z+s) + 2\gamma_E + 2c,
\]

\[
T_s'(z) = \partial_z T_s(z), \quad T_s''(z) = \partial_z^2 T_s(z),
\]
and \( \psi(z) = \partial_z \log \Gamma(z) \). The constants \( c \) and \( c_2 \) are the same as in Eqs. (5.12), (5.21). The coefficients \( v_1 \) and (using the expressions \( u_{1,2} \) from (C.1)) \( v_2 \) are

\[
v_1 = \frac{n(n^2 - 4s^2)}{128} T'_s \left( \frac{n}{2} \right) + \frac{n^2 - 4s^2}{64} T_s \left( \frac{n}{2} \right) - \frac{3n^2 + 4s(2 - 5s)}{64} T_s \left( \frac{n}{2} \right) + \frac{7n^2 + 4s(10 - 17s)}{128} + \frac{u_1}{2},
\]

\[
v_2 = \frac{(n^2 - 4s^2)(8 - 3n^2)}{3072} T''_s \left( \frac{n}{2} \right) - \left( \frac{n(n^2 - 4s^2)}{128} T_s \left( \frac{n}{2} \right) + \frac{n(n^2 + 4s^2 - 8s - 8)}{256} \right) T'_s \left( \frac{n}{2} \right) - \frac{n^2 - 4s^2}{192} T'_s \left( \frac{n}{2} \right) - \frac{n^2 + 4s(s - 2)}{128} T'_s \left( \frac{n}{2} \right) - \frac{n^2 - 8(s^2 - s + 1)}{128} T_s \left( \frac{n}{2} \right) - \frac{n^2 - 4s^2}{128} (2c_2 - 14 \zeta(3) - 3) - \frac{s(s - 4)}{64} + \frac{u_1}{8} + \frac{v_1}{2} + \frac{3u_2 - c}{8}.
\]

(C.2)

### D Spin-spin correlation functions for \( \epsilon \ll 1 \)

In this appendix, we give RG improved expansions for the different-time spin-spin correlation functions which were discussed in Section 5:

\[
\langle T \sigma_i^z(t) \sigma_j^z(0) \rangle \sim \sqrt{\frac{2}{\pi^3}} e^{-\left(\epsilon_+ + \frac{1}{4} \log(8\pi) + \frac{1}{2}\right) \epsilon^2} \times e^{\eta_1 g_1 + \eta_2 g_2} \left\{ 1 + g_1^2 (v_1^2 - v_2^2 g_1) + O(g^4) \right\} - \frac{(-1)^t}{\pi^2} \left( g_1^2 \right)^{\frac{3}{2}} e^{(\epsilon_+ + \frac{1}{4} \log(8\pi)) \epsilon^2} \times \exp \left\{ \frac{g_1}{2} + \frac{c}{2} g_1^2 + \left( \frac{1}{96} + \frac{c}{8} \right) g_1^3 - \frac{1}{32} - \frac{c}{8} - \frac{c^2}{2} \right\} g_1 g_2^2 + O(g^4) \right\} \times \left[ \frac{1}{2} \left( \frac{l_+}{l_-} + \frac{l_-}{l_+} \right) + \frac{g_4}{4} \left( g_4 + \left( c - \frac{1}{2} \right) g_1^2 + c g_2^2 + O(g^3) \right) \right] + \cdots \quad (D.1)
\]

and

\[
\langle T \sigma_i^z(t) \sigma_j^z(0) \rangle \sim \sqrt{\frac{8}{\pi^3}} \left( -1 \right)^t \sqrt{l_+ l_-} e^{\left( \frac{1}{4} + c \right) \epsilon^2} \times e^{\eta_1 g_1 + \eta_2 g_2 + \sqrt{2}(v_1^2 - v_2^2 g_1)} \left( 1 - g_1 \left( w_1 - w_2 g_1 + w_3 g_1^2 + w_4 g_2^2 + O(g^4) \right) \right) - \frac{1}{\pi^2} \left( l_+ l_- \left( \frac{1}{2} \right) \right) \left[ \frac{1}{2} \left( \frac{l_+}{l_-} + \frac{l_-}{l_+} \right) \left( 1 + \left( c - \frac{1}{2} \right) g_1^2 \right) + \left( 2c^2 + c - \frac{1}{4} \right) \frac{g_1^2}{4} + O(g^4) \right] + \frac{g_1^2}{4} \left( 1 + \left( 2c - \frac{1}{2} \right) g_1^2 + O(g^2) \right) \right] + \cdots \quad (D.2)
\]
In these expressions, the constants are given by

\[
\begin{align*}
  u_1^x &= \frac{3}{8} - \frac{c}{2}, \\
  v_1^x &= -\frac{1}{32} + \frac{c}{8} - \frac{c^2}{4}, \\
  w_1 &= c,
\end{align*}
\]

\[
\begin{align*}
  u_2^x &= -\frac{1}{64} - \frac{\zeta(3)}{48} - \frac{c^2}{8}, \\
  v_2^x &= -\frac{5}{128} - \frac{41}{96} \zeta(3) + \frac{c^3}{6} - \frac{c^2}{8}, \\
  w_2 &= \frac{1}{8} - \frac{c(1+c)}{2}, \\
  u_3^x &= \frac{3}{8} + \frac{c}{2}, \\
  v_3^x &= -\frac{5}{32} + \frac{5c}{8} + \frac{3c^2}{4}, \\
  w_3 &= -\frac{1}{16} - \frac{7}{12} \zeta(3) + \frac{c}{8} + \frac{c^2}{2} + \frac{c^3}{6} + \frac{c^2}{4}, \\
  u_4^x &= -\frac{1}{32} - \frac{13}{24} \zeta(3) - \frac{c}{8} + \frac{c^2}{4} - \frac{c^3}{6} - \frac{c^2}{4}, \\
  v_4 &= \frac{3}{8} + \frac{c(1+c)}{2}, \\
  u_5^x &= \frac{3}{8} + \frac{c}{2}, \\
  v_5^x &= -\frac{5}{32} + \frac{5c}{8} + \frac{3c^2}{4}, \\
  w_5 &= \frac{1}{64} + \frac{\zeta(3)}{48} + \frac{c}{8} + \frac{c^2}{4}, \\
  w_6 &= \frac{1}{64} + \frac{\zeta(3)}{48} + \frac{c}{8} + \frac{c^2}{4}, \\
  w_7 &= \frac{1}{64} + \frac{\zeta(3)}{48} + \frac{c}{8} + \frac{c^2}{4}, \\
  w_8 &= \frac{1}{64} + \frac{\zeta(3)}{48} + \frac{c}{8} + \frac{c^2}{4},
\end{align*}
\]

and

\[
\begin{align*}
  w_1 &= c, \\
  w_2 &= \frac{1}{8} - \frac{c(1+c)}{2}, \\
  u_3^x &= \frac{3}{8} + \frac{c}{2}, \\
  v_3^x &= -\frac{5}{32} + \frac{5c}{8} + \frac{3c^2}{4}, \\
  w_3 &= -\frac{1}{16} - \frac{7}{12} \zeta(3) + \frac{c}{8} + \frac{c^2}{2} + \frac{c^3}{6} + \frac{c^2}{4}, \\
  u_4^x &= -\frac{1}{32} - \frac{13}{24} \zeta(3) - \frac{c}{8} + \frac{c^2}{4} - \frac{c^3}{6} - \frac{c^2}{4}, \\
  v_4 &= \frac{3}{8} + \frac{c(1+c)}{2}, \\
  u_5^x &= \frac{3}{8} + \frac{c}{2}, \\
  v_5^x &= -\frac{5}{32} + \frac{5c}{8} + \frac{3c^2}{4}, \\
  w_5 &= \frac{1}{64} + \frac{\zeta(3)}{48} + \frac{c}{8} + \frac{c^2}{4}, \\
  w_6 &= \frac{1}{64} + \frac{\zeta(3)}{48} + \frac{c}{8} + \frac{c^2}{4}, \\
  w_7 &= \frac{1}{64} + \frac{\zeta(3)}{48} + \frac{c}{8} + \frac{c^2}{4}, \\
  w_8 &= \frac{1}{64} + \frac{\zeta(3)}{48} + \frac{c}{8} + \frac{c^2}{4}.
\end{align*}
\]

The running couplings \(g_\parallel, g_\perp\) in (D.1), (D.2) are defined by equations (5.10) and (5.11) where

\[
\frac{r}{r_0} = \sqrt{2\pi l_+ l_-} e^{\gamma_E + c + c^2 + \ldots} \quad \text{and} \quad l_\pm = l \pm \frac{t}{\varepsilon}.
\]

Setting \(l_+ = l_- = l\) (equal time) and \(g_\perp = g_\parallel = g, \epsilon = 0\) (isotropic limit) in (D.1), (D.2), one obtains (5.25).

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