Restoring the Boundary Condition and the Initial State in the Variational Data Assimilation Problem for the Black Sea Dynamics Model

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Abstract. In the present paper, we formulate an inverse problem of assimilation of sea surface temperature for a sea thermodynamics model aimed at the reconstruction of the heat flux and the initial state. We assume, that the unique function which is obtained by observation data processing is the function of sea surface temperature (SST). Numerical experiments on restoring the surface heat flux and the initial state for the system of the Black Sea primitive equation hydrodynamics model (temperature, salinity, velocity, and sea surface level) with assimilation procedure were carried. The results of the numerical experiments are presented.

1. Introduction

The four-dimensional variational assimilation of observational data (4D-Var) is currently the most universal and promising technology for solving the problems of monitoring and analyzing the state of the environment [1]–[4]. The methods of variational data assimilation are used for state and parameter estimation for geophysical models, based on minimization of the cost functional related to observation data. Joint initial state and parameter estimation using 4D-Var is very important ([5]–[7]).

In the present paper, we formulate an inverse problem of assimilation of sea surface temperature for a sea thermodynamics model aimed at the reconstruction of the heat flux and the initial state. A numerical model of Black Sea hydrothermodynamics developed at the Institute of Numerical Mathematics RAS [8] is the sea circulation model used in the present paper. A numerical algorithm to solve the variational data assimilation problem is considered and numerical analysis of its convergence is carried out. The results of numerical experiments with the Black Sea dynamics model are presented.

2. Variational data assimilation problem for a sea thermodynamics model

The sea thermodynamics problem has the form [9]:

\[ T_t + (\bar{U}, \text{Grad})T - \text{Div}(\bar{a}_T \cdot \text{Grad} T) = f_T \quad \text{in} \quad D \times (t_0, t_1), \]

\[ T = T_0 \quad \text{for} \quad t = t_0 \quad \text{in} \quad D, \]

\[ -v_T \frac{\partial T}{\partial z} = Q \quad \text{on} \quad \Gamma_s \times (t_0, t_1), \]

\[ \frac{\partial T}{\partial n} = 0 \quad \text{on} \quad \Gamma_{w,c} \times (t_0, t_1), \]

(1)
\[ \begin{align*}
\dot{U}_n^{(-)} + \frac{\partial}{\partial t} \frac{Q_T}{Q} &= Q_T \text{ on } \Gamma_{w,op} \times (t_0, t_1), \\
\frac{\partial T}{\partial n} &= 0 \text{ on } \Gamma_H \times (t_0, t_1),
\end{align*} \]

where \( T = T(x, y, z, t) \) is an unknown temperature function, \( t \in (t_0, t_1) \), \((x, y, z) \in D = \Omega \times (0, H), \)
\( \Omega \subset \mathbb{R}^2 \), \( H = H(x, y) \) is the function of the bottom relief, \( Q = Q(x, y, t) \) is the total heat flux, \( \bar{U} = (u, v, w) \), \( a_T = \text{diag}((a_T)_{ii}) \), \( (a_T)_{11} = (a_T)_{22} = \mu_T \), \( (a_T)_{33} = v_T \), \( f_T = f_T(x, y, z, t) \) are given functions. The domain boundary \( \partial D \) consists of four disjoint parts \( \Gamma_S, \Gamma_{w,op}, \Gamma_{w,e}, \Gamma_H \), where \( \Gamma_S = \Omega \) (sea surface), \( \Gamma_H \) is the sea bottom, \( \Gamma_{w,op} \) is the open part of vertical lateral boundary, \( \Gamma_{w,e} \) is the solid part of the vertical lateral boundary, \( \bar{U}_n^{(-)} = (\bar{U}_n - \bar{U}_n)/2 \), and \( \bar{U}_n \) is the normal component of \( \bar{U} \).

Other notations and a detailed description of the model can be found in [10].

Problem (1) is represented in the form:

\[ \begin{align*}
T_t + LT &= F + BQ, \quad t \in (t_0, t_1), \\
T &= T_0, \quad t = t_0.
\end{align*} \]

(2)

Equation (2) is understood in the weak sense:

\[ (T_t, \hat{T}) + (LT, \hat{T}) = (F, \hat{T}) + (BQ, \hat{T}) \quad \forall \hat{T} \in W^2_2(D), \]

(3)

where case \( L, F, B \) are defined by

\[ \begin{align*}
(LT, \hat{T}) &\equiv \int_D (-TD\text{iv}(\bar{U}\hat{T}))dD + \int_{\Gamma_{w,op}} \bar{U}_n^{(+)} \hat{T}d\Gamma + \int_{\Gamma_{w,e}} a_T \text{Grad}(T) \cdot \text{Grad}(\hat{T})dD, \\
F(\hat{T}) &= \int_{\Gamma_{w,op}} Q_T \hat{T}d\Gamma + \int_D f_T \hat{T}dD, \quad (T_t, \hat{T}) = \int_D T_t \hat{T}dD, \quad (BQ, \hat{T}) = \int_\Omega Q_T|_{z=0}d\Omega,
\end{align*} \]

and the functions \( \bar{a}_T, Q_T, f_T, Q \) are such that (3) makes sense. The operator \( L \) was studied by [10,11].

We consider the data assimilation problem for the sea surface temperature (see [10]). Suppose that the functions \( Q \in L_2(\Omega \times (t_0, t_1)) \) and \( T_0 \in L_2(D) \) are unknown in problem (1). Let also \( T_{obs}(x, y, t) \in L_2(\Omega \times (t_0, t_1)) \) represent the function of observation data defined on a subset of \( \Omega \times (t_0, t_1) \) with the indicator function \( m_0 \).

We consider the data assimilation problem in the form: find \( T_0 \) and \( Q \) such that

\[ \begin{align*}
\begin{cases}
T_t + LT &= F + BQ \quad \text{in } D \times (t_0, t_1), \\
T &= T_0, \quad t = t_0, \\
J(T_0, Q) &= \inf_{w,b} J(w, v),
\end{cases}
\end{align*} \]

(4)

where

\[ J(T_0, Q) = \frac{\alpha}{2} \int_{t_0}^{t_1} \int_\Omega |Q - Q^{(0)}|^2d\Omega dt + \frac{\beta}{2} \int_D |T_0 - T^{(0)}|^2dD \\
+ \frac{1}{2} \int_{t_0}^{t_1} \int_\Omega m_0 |T|_{z=0} - T_{obs}|^2d\Omega dt,
\]

(5)

and \( Q^{(0)} = Q^{(0)}(x, y, t), T^{(0)} = T^{(0)}(x, y, z) \) are given functions, \( \alpha, \beta = \text{const} > 0 \).

For \( \alpha, \beta > 0 \) this optimal control problem has a unique solution. The necessary optimality condition \( \text{grad} J = 0 \) reduces the problem (4)-(5) to the optimality system [2]:

\[ \text{...} \]
\[
T_t + LT = \mathcal{F} + BQ \quad \text{in} \quad D \times (t_0, t_1),
\]
\[
T = T_0, \quad t = t_0,
\]
\[
-(T^*)_t + L^*T^* = Bm_0(T_{obs} - T) \quad \text{in} \quad D \times (t_0, t_1),
\]
\[
T^* = 0, \quad t = t_1,
\]
\[
\alpha(Q - Q^{(0)}) - T^* = 0 \quad \text{on} \quad \Omega \times (t_0, t_1),
\]
\[
\beta(T_0 - T^{(0)}) - T^*|_{t=0} = 0 \quad \text{in} \quad D,
\]

where \(L^*\) is the operator adjoint to \(L\).

To solve the optimality system we consider the iterative process in the form:
\[
T_t^k + LT^k = \mathcal{F} + BQ^k \quad \text{in} \quad D \times (t_0, t_1),
\]
\[
T^k = T_0^k, \quad t = t_0,
\]
\[
-(T^{*k})_t + L^*T^{*k} = Bm_0(T_{obs} - T^k) \quad \text{in} \quad D \times (t_0, t_1),
\]
\[
T^{*k} = 0, \quad t = t_1,
\]
\[
Q^{k+1} = Q^k - \tau_k(\alpha(Q^k - Q^{(0)}) - T^{*k}) \quad \text{on} \quad \Omega \times (t_0, t_1),
\]
\[
T_0^{k+1} = T_0^k - \tau_k(\beta(T^k_0 - T^{(0)}) - T^{*k}|_{t=0}) \quad \text{in} \quad D,
\]

where \(\tau_k\) are iterative parameters. The choice of \(\tau_k\) is discussed in [12,13] in general form.

3. Numerical experiments
The numerical results presented were obtained for the inverse problem of recovering the heat flux \(Q\) and the initial state \(T_0\) via the variational assimilation of sea surface temperature for the Black Sea. As a direct model, we used the numerical sea circulation model developed at the Marchuk Institute of Numerical Mathematics, Russian Academy of Sciences (INM RAS).

The simulation was performed in the Black and Azov seas region. The parameters of the region and its geographical coordinates can be described as follows. We used a \(286 \times 159 \times 40\) grid (latitude \(\times\) longitude \(\times\) depth) with the first point having the coordinates \(27^\circ475^\prime\) E and \(40^\circ93^\prime\) N. The mesh sizes in the \(x\) and \(y\) directions were constants equal to 0.04 and 0.05 degrees, respectively. The time step was \(\Delta t = 5\) min.

The Black Sea surface temperature data (averaged over a day) were used for \(T_{obs}\), the data for 2019 (here \(t_0\) is 17th of June) were obtained from a satellite (portal http://marine.copernicus.eu) and interpolated onto a uniform model grid [14]. For \(Q^{(0)}\) we used the mean climatic flux obtained from the NCEP (National Center for Environmental Prediction) reanalysis.

First, the model was run without the data assimilation procedure. Figure 1 shows the resulting sea surface temperature obtained on day 4. The sea surface temperature observations \(T_{obs}\) on day 4 are displayed in Fig. 2. It can be seen that there are some differences between Figs. 1 and 2.

Next, the model was run with the sea surface temperature assimilation procedure over the period of 3 days (June 2008) and after this run 1 day in a prediction mode. Figure 3 shows the sea surface temperature on day 4 obtained with the parameter \(\alpha = 10^{-5}\) and the parameter \(\beta = 10^{-5}\) in the assimilation period. A comparison of Figs. 2 and 3 reveals that the data \(T_{obs}\) are fairly well assimilated by the model, while the model without assimilation gives a somewhat different pattern (see Fig. 1).
Figure 1. Observation data. °C

Figure 2. Calculation without data assimilation. °C

Figure 3. Calculation with data assimilation. °C
Figure 4. Difference of mean values of SST between model calculation without assimilation and observation data, °C

Figure 5. Difference of mean values of SST between model calculation with assimilation and observation data, °C

The difference between SST obtained by the model without assimilation and the observation data is shown in Figure 4. The difference of mean values of SST between model calculation with assimilation and observation data is presented in Figure 5. This figure demonstrates successful assimilation of incomplete observation data $T_{\text{obs}}$ by the model, which decreases the model solution error obtained without the assimilation procedure. The improvement is up to 2 degrees in some regions of the Black Sea.

4. Conclusions
This work continues the research of the authors on the numerical solution of problems of hydrothermodynamics of the sea using variational data assimilation. In this work, for the first time for a model with splitting, two controls are considered. In the paper we present and study a numerical algorithm for solving the variational data assimilation problem for the sea surface temperature in order to reconstruct simultaneously the heat fluxes on the surface and the initial state using the three-dimensional sea hydrothermodynamics model developed in Marchuk institute of numerical mathematics RAS.
A data assimilation block has been developed, based on the iterative process (10)-(13). This block is incorporated into the global 3D model of the Black Sea.

The results of calculations show good efficiency of the model with assimilation procedures. For small regularization parameters $\alpha = 10^{-5}$ and $\beta = 10^{-5}$ the SST obtained as a result of the assimilation algorithm in the experiment with real atmospheric forcing coincides with observation data. However, the heat flux used in the model without assimilation does not always correspond to the flux obtained as a result of assimilation.

Good convergence of the iterative procedures used for assimilation was demonstrated. It required no more than 5–10 iterations to find the flux $Q$. Therefore, the numerical experiments demonstrate convergence of the considered algorithms and the efficiency of the model assimilation block.

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