LITTLESTONE AND VC-DIMENSION OF FAMILIES OF ZERO SETS

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Abstract. We prove that, for any $d$ linearly independent functions from some set into a $d$-dimensional vector space over any field, the family of zero sets of all non-trivial linear combination of these functions has VC-dimension and Littlestone dimension $d - 1$. Additionally, we characterize when such families are maximal of VC-dimension $d - 1$ and give a sufficient condition for when they are maximal of Littlestone dimension $d - 1$.

1. Introduction

Complexity, in the sense of machine learning theory, of the sets of positivity of linear combinations of real-valued functions is fairly well understood. If $f : \mathbb{R}^k \to \mathbb{R}$ is a function, the set of positivity of $f$ is the set $\{ \mathbf{x} \in \mathbb{R}^k : f(\mathbf{x}) > 0 \}$. If $\{f_1, \ldots, f_d\}$ is a set of linearly independent functions from $\mathbb{R}^k$ to $\mathbb{R}$, then the set system of sets of positivity of all the linear combinations of $\{f_1, \ldots, f_d\}$ has the VC dimension $d$ (see [4] or [5, Theorem A]). If $X$ is a finite subset of the domain of the functions, by Sauer–Shelah lemma, the size of the induced set system on $X$ is bounded above by $\sum_{i=0}^{d} \binom{|X|}{i}$ (we abbreviate this sum as $\binom{|X|}{\leq d}$).

There are known sufficient conditions on functions guaranteeing that for suitable finite sets $X$, the induced set systems have maximal size $\binom{|X|}{\leq d}$. Floyd shows in [6] that a condition on linear dimension of the family of functions relative to the set $X$ and a condition on the number of zeros are sufficient for maximality. Johnson provides analytic sufficient conditions in [7] for maximality of the set system. It is worth pointing out that set systems given by the sets of positivity generally have infinite Littlestone dimension.

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We study set systems of zero-sets of non-trivial linear combinations of functions from a set to a field. One expects to see a lower combinatorial complexity of this set system, and indeed we show that both Littlestone dimension and VC-dimension of zero-sets of linear combinations of \(d\) linearly independent functions is \(d - 1\). Somewhat surprisingly, the system of zero-sets can be maximal (we formally define the notion in Definition 2.8). We obtain a characterization of maximality of the set system.

Bhaskar has studied a shatter function related to Littlestone dimension (called “thicket dimension” in [1]). Bhaskar showed that the size of the set system that well-labels the leaves of the tree of depth \(n\) (this is a natural notion for the shatter function for Littlestone dimension) is bounded above by \((^n_d)\), where \(d\) is the Littlestone dimension. It is natural to ask for examples of maximal families for this shatter function.

In this paper, we establish the following results: Let \(\{f_1, \ldots, f_d\}\) be a set of functions from a set \(X\) to a vector space \(F^d\), where \(F\) is a field and \(d\) is a positive integer, and let \(\mathcal{C}\) be the set of all zero-sets of non-trivial linear combinations of \(f_1, \ldots, f_d\).

**Theorem 3.10**: If \(\{f_1, \ldots, f_d\}\) is linearly independent (in the vector space of all functions from \(X\) to \(F\)), then \(\mathcal{C}\) has VC-dimension and Littlestone dimension \(d - 1\).

**Theorem 5.6**: \(\mathcal{C}\) is maximum of VC-dimension \(d - 1\) if and only if the image of \((f_1, \ldots, f_d)\) is not contained in the union of finitely many proper subspaces of \(F^d\).

**Corollary 4.5**: If the image of \((f_1, \ldots, f_d)\) is not contained in the union of finitely many proper subspaces of \(F^d\), then \(\mathcal{C}\) is maximum of Littlestone dimension \(d - 1\).

2. Preliminaries

For this paper, let \(\mathbb{N}\) denote the set of natural numbers, which for us is the set of all non-negative integers (including 0). Let \(\mathbb{Z}^+\) denote the set of positive integers (i.e., \(\mathbb{Z}^+ = \mathbb{N} \setminus \{0\}\)).

For \(n, k \in \mathbb{N}\) with \(k \leq n\), let

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{and} \quad \binom{n}{\leq k} = \sum_{i=0}^{k} \binom{n}{i}.
\]

For all sets \(X\) and \(Y\), let \(X^Y\) denote the set of all functions from \(X\) to \(Y\). For any set \(X\) and \(f : X \to \mathbb{R}\), let \(\text{supp}(f) = \{a \in X : f(a) \neq 0\}\). For all sets \(X\) and all \(n \in \mathbb{N}\), let \(\binom{X}{n}\) denote the set of all subsets of \(X\).
of size $n$. Similarly, let

$$\left( X \leq n \right) = \bigcup_{i \leq n} \binom{X}{i} \quad \text{and} \quad \left( X < n \right) = \bigcup_{i < n} \binom{X}{i}.$$ 

For all $n \in \mathbb{N}$, let

$$[n] = \{ k \in \mathbb{N} : 0 \leq k < n \}.$$ 

(Note that $[0] = \emptyset$.) Then, we have that, for all $0 \leq k \leq n$,

$$\left| \binom{[n]}{k} \right| = \binom{n}{k} \quad \text{and} \quad \left| \binom{[n]}{\leq k} \right| = \binom{n}{\leq k}.$$ 

**Definition 2.1.** Let $X$ be a set, $C \subseteq \mathcal{P}(X)$, and $d \in \mathbb{N}$. A $(X, C)$-labeled tree of depth $d$ is a function $T$ from $\bigcup_{k \leq d} [k][2]$ to $X \cup C$ such that

1. for all $0 \leq k < d$ and $\sigma \in [k][2]$, $T(\sigma) \in X$; and
2. for all $\tau \in [d][2]$, $T(\tau) \in C$

For $0 \leq k \leq d$ and $\sigma \in [k][2]$, we call $(\sigma, T(\sigma))$ a node if $k \neq d$ and a leaf if $k = d$. A leaf $(\tau, T(\tau))$ is well-labeled if, for all $0 \leq k < d$,

$$T(\tau|\{k\}) \in T(\tau) \iff \tau(k) = 1.$$ 

We say that $T$ is well-labeled if all leaves of $T$ are well-labeled. We say that $T$ of depth $d$ is level-balanced if, for all $k \leq d$ and for all $\sigma_0, \sigma_1 \in [k][2]$, $T(\sigma_0) = T(\sigma_1)$.

**Definition 2.2.** Let $X$ be a set and $C \subseteq \mathcal{P}(X)$.

1. The Littlestone dimension of $C$, denoted $\text{Ldim}(C)$, is the largest natural number $d$ such that there exists a well-labeled $(X, C)$-labeled tree of depth $d$. If there exists a well-labeled $(X, C)$-labeled tree of depth $d$ for all natural numbers $d$, we say $C$ has infinite Littlestone dimension, denoted $\text{Ldim}(C) = \infty$. If there exists no well-labeled $(X, C)$-labeled tree of depth 0, we say $\text{Ldim}(C) = -\infty$.

2. The Littlestone shatter function of $C$, denoted $\rho_C$, is the function from $\mathbb{N}$ to $\mathbb{N}$ given by, for all $n \in \mathbb{N}$, the maximum number of well-labeled leaves of a $(X, C)$-labeled tree $T$ of depth $n$.

3. The Littlestone density of $C$, denoted $\text{Lden}(C)$, is the infimum over all positive $\ell \in \mathbb{R}$ such that there exists $K \in \mathbb{R}$ such that, for all $n \geq 1$, $\rho_C(n) \leq Kn^\ell$.

4. If we restrict our attention to only level-balanced trees $T$ in the above definitions, we define instead the VC-dimension of $C$, denoted $\text{VCdim}(C)$, the VC-shatter function of $C$, denoted $\pi_C$, and the VC-density of $C$, denoted $\text{VCden}(C)$, respectively.
Remark 2.3. For any set $X$ and $\mathcal{C} \subseteq \mathcal{P}(X)$, we clearly have

1. $\text{VCdim} (\mathcal{C}) \leq \text{Ldim} (\mathcal{C})$,
2. $\pi_{\mathcal{C}}(n) \leq \rho_{\mathcal{C}}(n)$ for all $n \in \mathbb{N}$, and
3. $\text{VCden} (\mathcal{C}) \leq \text{Lden} (\mathcal{C})$.

The above definitions for the VC-dimension and the VC-shatter function are equivalent to the usual definitions given in terms of shattering. In particular, Lemma 2.5 below can be found, for example, in [3].

Definition 2.4. Let $X$ be a set, $Y \subseteq X$, and $\mathcal{C} \subseteq \mathcal{P}(X)$. Let

$$\mathcal{C}|_Y = \{ A \cap Y : A \in \mathcal{C} \},$$

we call $\mathcal{C}|_Y$ an induced set system on $Y$. We say that $\mathcal{C}$ shatters $Y$ if $\mathcal{C}|_Y = \mathcal{P}(Y)$.

Lemma 2.5. Let $X$ be a set and $\mathcal{C} \subseteq \mathcal{P}(X)$.

1. the VC-dimension of $\mathcal{C}$ is the largest $d \in \mathbb{N}$ such that there exists $Y \in \binom{X}{d}$ with $|\mathcal{C}|_Y| = 2^d$.
2. the VC-shatter function of $\mathcal{C}$ is given by, for all $n \in \mathbb{N}$,

$$\pi_{\mathcal{C}}(n) = \max \left\{ |\mathcal{C}|_Y| : Y \in \binom{X}{n} \right\}.$$

Lemma 2.6 (Sauer–Shelah Lemma). For any set $X$ and $\mathcal{C} \subseteq \mathcal{P}(X)$, if $\mathcal{C}$ has VC-dimension $d$, then, for all $n \geq d$,

$$\pi_{\mathcal{C}}(n) \leq \binom{n}{\leq d}.$$

An analogous lemma holds in the Littlestone case.

Lemma 2.7 ([1]). For any set $X$ and $\mathcal{C} \subseteq \mathcal{P}(X)$, if $\mathcal{C}$ has Littlestone dimension $d$, then, for all $n \geq d$,

$$\rho_{\mathcal{C}}(n) \leq \binom{n}{\leq d}.$$

This motivates the following two definitions:

Definition 2.8. Let $X$ be a set and $\mathcal{C} \subseteq \mathcal{P}(X)$.

1. For $d \in \mathbb{N}$, we say $\mathcal{C}$ is maximal of VC-dimension $d$ if, for all $n \geq d$,

$$\pi_{\mathcal{C}}(n) = \binom{n}{\leq d}.$$

2. For $d \in \mathbb{N}$, we say $\mathcal{C}$ is maximal of Littlestone dimension $d$ if, for all $n \geq d$,

$$\rho_{\mathcal{C}}(n) = \binom{n}{\leq d}.$$
Remark 2.9. For any set $X$ and $\mathcal{C} \subseteq \mathcal{P}(X)$, by the Sauer-Shelah Lemma,
$$\text{VCden}(\mathcal{C}) \leq \text{VCdim}(\mathcal{C}).$$

Similarly, by Lemma 2.7
$$\text{Lden}(\mathcal{C}) \leq \text{Ldim}(\mathcal{C}).$$

Moreover, if $\mathcal{C}$ is maximal of VC-dimension $d$, then
$$\text{VCden}(\mathcal{C}) = \text{VCdim}(\mathcal{C}) = d$$
and, if $\mathcal{C}$ is maximal of Littlestone dimension $d$, then
$$\text{Lden}(\mathcal{C}) = \text{Ldim}(\mathcal{C}) = d.$$

If $\mathcal{C}$ is maximal of VC-dimension $d$ and $\text{Ldim}(\mathcal{C}) = d$, then $\mathcal{C}$ is maximal of Littlestone dimension $d$ as well. This is because, in this case, for all $n \geq d$,
$$\left( \begin{array}{c} n \\ \leq d \end{array} \right) = \pi_{\mathcal{C}}(n) \leq \rho_{\mathcal{C}}(n) \leq \left( \begin{array}{c} n \\ \leq d \end{array} \right).$$

3. VC-dimension and Littlestone dimension of zero sets.

Let $X$ be a set and $F$ a field. The family of functions from $X$ to $F$ has a natural structure of a vector space over $F$ with the pointwise addition operation. The notion of linear independence below is taken in the sense of that vector space. In particular, if $X = F = \mathbb{F}_3$, then the set of functions $\{x, x^3\}$ is linearly dependent, because $x$ and $x^3$ define the same function from $\mathbb{F}_3$ to $\mathbb{F}_3$.

Definition 3.1. Let $F$ be a field, $d \in \mathbb{Z}^+$, and $\overline{a}, \overline{b} \in F^d$. Let $\overline{a} \cdot \overline{b}$ denote the usual dot product of $a$ and $b$. That is,
$$(a_0, \ldots, a_{d-1}) \cdot (b_0, \ldots, b_{d-1}) = \sum_{i=0}^{d-1} a_i b_i$$
for all $a_0, \ldots, a_{d-1}, b_0, \ldots, b_{d-1} \in F$. We say that $\overline{a}, \overline{b} \in F^d$ are orthogonal if $\overline{a} \cdot \overline{b} = 0$.

We note that the term orthogonal is used as a short-hand; the vector space $F^d$ with the dot product may not be an inner product space.

If $\overline{f} = (f_0, \ldots, f_{d-1})$ is a tuple of functions from $X$ to $F$ and $\overline{a} = (a_0, \ldots, a_{d-1})$ is a tuple of scalars from $F$, it is convenient to write the linear combination $a_0 f_0 + \cdots + a_{d-1} f_{d-1}$ as the dot product $\overline{a} \cdot \overline{f}$. 

Definition 3.2. Let $X$ be a set, $F$ a field, $d \in \mathbb{Z}^+$, $\overline{\pi} \in F^d$, and $\overline{f}: X \to F^d$. Define the zero set of $\overline{f}$ and $\overline{a}$ by
\[ Z_{\overline{f}, \overline{a}} = \{ c \in X : \overline{a} \cdot \overline{f}(c) = 0 \}. \]
Define
\[ C_{\overline{f}} = \{ Z_{\overline{f}, \overline{a}} : \overline{a} \in F^d \setminus \{ \overline{0} \} \}. \]

Example 3.3. If $X = \mathbb{R}^2$, $F = \mathbb{R}$, $d = 6$, and
\[ \overline{f}(x, y) = (x^2, xy, y^2, x, y, 1), \]
then $C_{\overline{f}}$ is the set of all conic sections in $\mathbb{R}^2$.

Lemma 3.4. Let $X$ be a set, $F$ a field, $d \in \mathbb{Z}^+$, and $\overline{f}: X \to F^d$. The following are equivalent:

1. \{ $f_0, f_1, \ldots, f_{d-1}$ \} is a linearly independent subset of the $F$-vector space $X^F$;
2. for all $\overline{a} \in F^d \setminus \{ \overline{0} \}$, there exists $c \in X$ such that $\overline{a} \cdot \overline{f}(c) \neq 0$; and
3. $\overline{f}(X)$ is not contained in a proper subspace of $F^d$.

Proof. (1) $\Rightarrow$ (2): If \{ $f_0, f_1, \ldots, f_{d-1}$ \} is a linearly independent set, then, for all $\overline{a} \in F^d$, $\sum_{k<d} a_k f_k$ is identically zero if and only if $\overline{a} = \overline{0}$. Therefore, for any non-zero $\overline{a} \in F^d$, $\sum_{k<d} a_k f_k$ is not identically zero, so there exists $c \in X$ such that $\overline{a} \cdot \overline{f}(c) \neq 0$.

(2) $\Rightarrow$ (3): Suppose that (3) fails. So $\overline{f}(X)$ is contained in a proper subspace $V$ of $F^d$. Take $\overline{a} \in F^d$ non-zero and orthogonal to $V$. Then, for all $c \in X$, $\overline{a} \cdot \overline{f}(c) = 0$. Hence, (2) fails.

(3) $\Rightarrow$ (1): Suppose that (1) fails. So, there exists $\overline{a} \in F^d \setminus \{ \overline{0} \}$ such that $\sum_{k<d} a_k f_k$ is identically zero. Therefore, $\overline{f}(X)$ is contained in the subspace of $F^d$ orthogonal to $\overline{a}$. \( \square \)

Definition 3.5. Let $X$ be a set, $F$ a field, $d \in \mathbb{Z}^+$, and $\overline{f}: X \to F^d$. If $\overline{f}$ satisfies any of the conditions of Lemma 3.4, we say that $\overline{f}$ is linearly independent.

In Proposition 3.6 below, we show that the Littlestone dimension of $C_{\overline{f}}$ is bounded above by $d - 1$. This is proven by contradiction: from a well-labeled $(X, C_{\overline{f}})$-labeled tree of depth $d$, we create a $d \times d$ matrix that has both full rank and less than full rank.

Proposition 3.6. Let $X$ be a set, $F$ a field, $d \in \mathbb{Z}^+$, and $\overline{f}: X \to F^d$. Then, $\text{Ldim}(C_{\overline{f}}) < d$. 
Proof. For each natural number $k$, let $1_k$ denote the function from $[k]$ to $\mathbb{N}$ that is constantly 1. Assume that $T$ is a well-labeled $(X, \mathcal{C}_f)$-labeled tree of depth $d$. First, we will show that the set

$$V = \{ \overline{f}(T(1_k)) : k \in [d] \} \subseteq F^d$$

is linearly dependent.

Since $T(1_d) \in \mathcal{C}_f$, there exists $\overline{a} \in F^d \setminus \{\overline{0}\}$ such that $T(1_d) = Z_{\overline{a}}$. On the other hand, since $T$ is well-labeled, $T(1_k) \in T(1_d)$ for all $k < d$. Therefore, for all $k < d$,

$$\overline{a} \cdot \overline{f}(T(1_k)) = 0.$$

Therefore, we have

$$\begin{bmatrix}
    f_0(T(1_0)) & \cdots & f_{d-1}(T(1_0)) \\
    \vdots & \ddots & \vdots \\
    f_0(T(1_{d-1})) & \cdots & f_{d-1}(T(1_{d-1}))
\end{bmatrix}
\begin{bmatrix}
    a_0 \\
    \vdots \\
    a_{d-1}
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    \vdots \\
    0
\end{bmatrix}.$$

Since $\overline{a} \neq \overline{0}$, we conclude that $V$ is linearly dependent.

Since $V$ is linearly dependent, there exists $0 \leq k < d$ and $\overline{b} \in F^k$ such that

$$\overline{f}(T(1_k)) = \sum_{j=0}^{k-1} b_j \overline{f}(T(1_j)).$$

Consider $\tau : [d] \to [2]$ given by

$$\tau(j) = \begin{cases} 
1 & \text{if } j < k \\
0 & \text{if } k \leq j < d
\end{cases}.$$

Then, for all $j < k$, $T(1_j) \in T(\tau)$ and $T(1_k) \notin T(\tau)$. Choose $\overline{a} \in F^d \setminus \{\overline{0}\}$ such that $T(\tau) = Z_{\overline{a}}$. Then, for all $j < k$,

$$\overline{a} \cdot \overline{f}(T(1_j)) = 0 \text{ and } \overline{a} \cdot \overline{f}(T(1_k)) \neq 0.$$

However,

$$\overline{a} \cdot \overline{f}(T(1_k)) = \overline{a} \cdot \left( \sum_{j=0}^{k-1} b_j \overline{f}(T(1_j)) \right) = \sum_{j=0}^{k-1} b_j (\overline{a} \cdot \overline{f}(T(1_j))) = 0.$$

This is a contradiction.

Therefore, no such well-labeled tree $T$ exists. That is, $\text{Ldim}(\mathcal{C}_f) < d$. \qed
Next, in Theorem 3.10 below we establish that, if $f$ is linearly independent, then the VC-dimension and Littlestone dimension of $C_f$ are both exactly $d - 1$. This is done through Lemma 3.7, where we create a new tuple of functions in the span of $f$ that acts as an indicator function on a subset of $X$ of size $d$. Then, in Proposition 3.8, we use this auxiliary tuple of functions to produce a large subset of $X$ that is shattered by $C_f$.

**Lemma 3.7.** Let $X$ be a set, $F$ a field, $d \in \mathbb{Z}^+$, and $f : X \to F^d$ linearly independent. Then, there exists $\mathbf{c} \in X^d$ and $g : X \to F^d$ such that $g_k \in \text{span}\{f_0, \ldots, f_{d-1}\}$ for all $k < d$ and, for all $i, j < d$,

$$g_j(c_i) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}.$$

**Proof.** By induction on $d$. If $d = 1$, choose any $c_0 \in X$ where $f_0(c_0) \neq 0$ and set $g_0(x) = (f_0(c_0))^{-1}f_0(x)$.

Fix $d > 1$ and suppose that $f : X \to F^d$ is linearly independent. By the induction hypothesis, there exists $\mathbf{c} \in X^{d-1}$ and $g' : X \to F^{d-1}$ such that $g'_k \in \text{span}\{f_0, \ldots, f_{d-2}\}$ for all $k < d - 1$ and, for all $i, j < d - 1$,

$$g'_j(c_i) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}.$$

Define

$$g'_{d-1}(x) = f_{d-1}(x) - \sum_{i=0}^{d-2} f_{d-1}(c_i) g'_i(x).$$

Then, $g'_{d-1}(c_i) = 0$ for all $i < d - 1$. On the other hand, since $g'_{d-1}$ is a non-trivial linear combination of $\{f_0, \ldots, f_{d-1}\}$, which is linearly independent, $g'_{d-1}$ is non-zero. Thus, there exists $c_{d-1} \in X$ such that $g'_{d-1}(c_{d-1}) \neq 0$. Finally, set

$$g_{d-1}(x) = (g'_{d-1}(c_{d-1}))^{-1}g'_{d-1}(x).$$

and set

$$g_i(x) = g'_i(x) - g'_i(c_{d-1})g_{d-1}(x)$$

for $i < d - 1$. Check that this gives the desired conclusion. \qed

**Proposition 3.8.** Let $X$ be a set, $F$ a field, $d \in \mathbb{Z}^+$, and $f : X \to F^d$ linearly independent. Then, $\text{VCdim}(C_f) \geq d - 1$. 
Proof. By Lemma 3.7 there exists $\mathbf{c} \in X^d$ and $\mathbf{f} : X \rightarrow F^d$ such that $g_k \in \text{span}\{f_0, \ldots, f_{d-1}\}$ for all $k < d$ and, for all $i, j < d$,

$$g_j(c_i) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}.$$ 

For any non-empty $S \subseteq [d]$, let

$$g_S(x) = \sum_{i \in S} g_i(x).$$

Then, for any non-empty $S \subseteq [d]$ and any $i < d$, note that

$$g_S(c_i) = 0 \iff i \notin S.$$ 

On the other hand, since $g_k \in \text{span}\{f_0, \ldots, f_{d-1}\}$ and $g_k$ is not identically zero, for each non-empty $S \subseteq [d]$, there exists $\mathbf{a}_S \in F^d \setminus \{0\}$ such that $g_S = \mathbf{a}_S \cdot \mathbf{f}$. Therefore, for any non-empty $S \subseteq [d]$ and any $i < d$,

$$c_i \in \text{Z}_{f,a_S} \iff i \notin S.$$ 

Thus, $C_{\mathbf{f}}$ shatters $\{c_0, \ldots, c_{d-2}\}$. So the VC-dimension of $C_{\mathbf{f}}$ is at least $d - 1$. \[\square\]

Remark 3.9. We see from the above proof that, if $\mathbf{f} : X \rightarrow F^d$ is linearly independent, then $\pi_{C_{\mathbf{f}}}(d) \geq 2^d - 1$. Using the next theorem together with the Sauer–Shelah Lemma, we conclude that $\pi_{C_{\mathbf{f}}}(d) = 2^d - 1$.

Theorem 3.10. Let $X$ be a set, $F$ a field, $d \in \mathbb{Z}^+$, and $f : X \rightarrow F^d$ linearly independent. Then,

$$\text{VCdim}(C_{\mathbf{f}}) = \text{Ldim}(C_{\mathbf{f}}) = d - 1.$$ 

Proof. By Proposition 3.6, Remark 2.3, and Proposition 3.8 we obtain

$$d - 1 \leq \text{VCdim}(C_{\mathbf{f}}) \leq \text{Ldim}(C_{\mathbf{f}}) < d.$$ 

The conclusion follows. \[\square\]

4. VC-density and Littlestone density of the system of zero sets.

Now that we have established the VC-dimension and Littlestone dimension of $C_{\mathbf{f}}$, we turn our attention to analyzing the VC-density and Littlestone density of $C_{\mathbf{f}}$.

Proposition 4.1. Let $X$ be a set, $F$ a field, $d \in \mathbb{Z}^+$, and $\mathbf{f} : X \rightarrow F^d$. Suppose that there exist 1-dimensional subspaces $V_i \subseteq F^d$ for $i < k$ and suppose that $\mathbf{f}(X) \subseteq \bigcup_{i<k} V_i$. Then, $|C_{\mathbf{f}}| \leq 2^k$. In particular, $C_{\mathbf{f}}$ has VC-density 0 and Littlestone density 0.
Proof. Without loss of generality, we may assume that \( V_i \neq V_j \) for all \( i \neq j \). Therefore, since \( V_i \) and \( V_j \) are 1-dimensional subspaces, \( V_i \cap V_j = \{0\} \). For each \( i < k \), let
\[
S_i = \overline{f}^{-1}(V_i \setminus \{0\})
\]
and let \( S_k = \overline{f}^{-1}(\{0\}) \). For each \( i < k \), consider \( C_f|_{S_i} \). By assumption, for all \( c \in S_i \), \( \overline{f}(c) \in V_i \setminus \{0\} \). Thus, for all \( \overline{\alpha} \in F^d \) and \( c \in S_i \), \( \overline{\alpha}.\overline{f}(c) = 0 \) if and only if \( \overline{\alpha} \) is orthogonal to \( V_i \). Since this is independent of the choice of \( c \in S_i \), we see that \( Z_{\overline{f},\alpha} \cap S_i = \emptyset \) or \( Z_{\overline{f},\alpha} \cap S_i = S_i \). Therefore,
\[
C_f|_{S_i} = \{\emptyset, S_i\}.
\]
Similarly, one can check that \( C_f|_{S_k} = \{S_k\} \). Therefore,
\[
C_f \subseteq \left\{ \bigcup_{i \in I} S_i \cup S_k : I \subseteq [k] \right\}.
\]
Thus, \( |C_f| \leq 2^k \). \( \square \)

Remark 4.2. Let \( X \) be a set, \( F \) a field, \( d \in \mathbb{Z}^+ \), and \( \overline{f} : X \to F^d \). If \( F \) is a finite field, then \( C_f \) is finite (since \( F^d \setminus \{0\} \) is finite). Therefore, \( C_f \) has VC-density 0 and Littlestone density 0.

In Proposition 4.3 below, we establish that, if \( \overline{f}(X) \) is not contained in a finite union of proper subspaces of \( F^d \), then \( C_f \) is maximal of VC-dimension \( d - 1 \). This is done by establishing an infinite sequence of elements from \( X \) that have \( d \)-wise linearly independent images under \( \overline{f} \). Then, we use linear independence to select every subset of this sequence of size at most \( d - 1 \) with some \( Z_{\overline{f},\alpha} \), establishing the maximality of the VC-dimension.

**Proposition 4.3.** Let \( X \) be a set, \( F \) a field, \( d \in \mathbb{N} \) with \( d \geq 2 \), and \( \overline{f} : X \to F^d \). If \( \overline{f}(X) \) is not contained in a finite union of proper subspaces of \( F^d \), then \( C_f \) is maximal of VC-dimension \( d - 1 \). In particular, \( \text{VCden}(C_f) = d - 1 \).

Proof. First, create a sequence \( (c_i)_{i \in \mathbb{N}} \) with \( c_i \in X \) for all \( i \in \mathbb{N} \) such that, for all \( I \in \binom{\mathbb{N}}{\leq d} \),
\[
\{\overline{f}(c_i) : i \in I\} \text{ is linearly independent}.
\]
We do this recursively as follows:
First, choose \( c_0 \in X \) such that \( \overline{f}(c_0) \neq \overline{0} \) (which can be done, since \( \overline{f}(X) \not\subseteq \{0\} \)). Now, assume that \( c_0, \ldots, c_{n-1} \) have been constructed so
that, for all $I \in \binom{[n]}{\leq d}$, (I) holds. Consider the set

$$U = \bigcup_{I \in \binom{[n]}{\leq d}} \text{span} \{ \mathbf{f}(c_i) : i \in I \}.$$ 

This is a union of finitely many proper subspaces of $F^d$, so, by the assumption, there exists $c_n \in X$ such that $\mathbf{f}(c_n) \notin U$. It is easy to check that, for all $I \in \binom{[n+1]}{\leq d}$, (I) holds. This concludes our construction.

For all $I \in \binom{[n]}{d-1}$, consider the homogeneous system

$$\begin{bmatrix}
  f_0(c_{i_0}) & \cdots & f_{d-1}(c_{i_0}) \\
  \vdots & \ddots & \vdots \\
  f_0(c_{i_{d-2}}) & \cdots & f_{d-1}(c_{i_{d-2}})
\end{bmatrix} \begin{bmatrix}
  x_0 \\
  \vdots \\
  x_{d-1}
\end{bmatrix} = \begin{bmatrix}
  0 \\
  \vdots \\
  0
\end{bmatrix},$$

where $i_0 < \cdots < i_{d-2}$ enumerate $I$. Since the matrix has rank $d-1$, this has a non-trivial solution $\mathbf{b}_I \in F^d$. That is, for all $i \in I$, $\mathbf{b}_I \cdot \mathbf{f}(c_i) = 0$. On the other hand, for any $j \in \mathbb{N} \setminus I$, $\{ \mathbf{f}(c_i) : i \in I \cup \{ j \} \}$ is linearly independent, so we have that

$$\begin{bmatrix}
  f_0(c_{i_0}) & \cdots & f_{d-1}(c_{i_0}) \\
  \vdots & \ddots & \vdots \\
  f_0(c_{i_{d-2}}) & \cdots & f_{d-1}(c_{i_{d-2}})
\end{bmatrix} \begin{bmatrix}
  \mathbf{b}_I \\
  \vdots \\
  \mathbf{b}_I
\end{bmatrix} \neq \begin{bmatrix}
  0 \\
  \vdots \\
  0
\end{bmatrix}. $$

(This holds since $\mathbf{b}_I \neq 0$ and this matrix has rank $d$.) Therefore, we see that $\mathbf{b}_I \cdot \mathbf{f}(c_j) \neq 0$. In other words, we have constructed, for each $I \in \binom{[n]}{d-1}$, $\mathbf{b}_I \in F^d$ such that, for all $i \in \mathbb{N}$,

$$\mathbf{b}_I \cdot \mathbf{f}(c_i) = 0 \iff i \in I.$$ 

In other words,

$$c_i \in Z_{\mathbf{f}, \mathbf{b}_I} \iff i \in I.$$ 

For any $n \geq d$, for any $I \in \binom{[n]}{\leq d}$, set $I' = I \cup \{ n, n+1, \ldots, n+(d-|I|-2) \}$ and we have, for all $i \in [n]$,

$$c_i \in Z_{\mathbf{f}, \mathbf{b}_{I'}} \iff i \in I.$$ 

Thus,

$$|\mathcal{C}_{\mathbf{f}} \{ c_i : i \in [n] \} | \geq \binom{n}{<d}.$$ 

Thus, since $\text{VCdim}(\mathcal{C}_{\mathbf{f}}) = d - 1$, $\mathcal{C}_{\mathbf{f}}$ is maximal of VC-dimension $d - 1$. \qed
Unfortunately, we do not necessarily get an upper bound on the VC-density of $C_f$, even if $f(X)$ is contained in a union of proper subspaces of $F^d$.

Example 4.4. Fix $d \in \mathbb{N}$ with $d \geq 3$ and $F$ an infinite field. For each $i < d$, let $\tau_i \in F^d$ denote the $i$th standard basis vector. For each $i < d - 1$, let
\[
L_i = \text{span}\{\tau_0, \tau_{i+1}\}.
\]
Let $X = \bigcup_{i=0}^{d-2} L_i$ and let $\overline{f} : X \to F^d$ be the identity embedding. Then, VCden($C_f$) = $d - 1$, even though $f(X)$ is contained in the union of 2-dimensional subspaces of $F^d$.

Proof. Since $F$ is infinite, there exists $g : \mathbb{N} \to F \setminus \{0\}$ an injective function. For each $i < d - 1$ and $j \in \mathbb{N}$, define
\[
\overline{c}_{i,j} = \tau_0 + g(j)\tau_{i+1}.
\]
For each $j_0, \ldots, j_{d-2} \in \mathbb{N}$, let
\[
\overline{b}_{j_0, \ldots, j_{d-2}} = \left( \prod_{k<d-1} g(j_k) \right) \tau_0 + \sum_{\ell<d-1} \left( - \prod_{k<d-1, k \neq \ell} g(j_k) \right) \tau_{\ell+1}.
\]
Then, for all $i < d - 1$, $j \in \mathbb{N}$, and $j_0, \ldots, j_{d-2} \in \mathbb{N}$,
\[
\overline{b}_{j_0, \ldots, j_{d-2}} \cdot \overline{f}(c_{i,j}) = \overline{b}_{j_0, \ldots, j_{d-2}} \cdot \overline{c}_{i,j} = \prod_{k<d-1} g(j_k) - g(j) \prod_{k<d-1, k \neq i} g(j_k) = (g(j_i) - g(j)) \prod_{k<d-1, k \neq i} g(j_k).
\]
Therefore, $\overline{c}_{i,j} \in ZF_{\overline{b}_{j_0, \ldots, j_{d-2}}}$ if and only if $g(j) = g(j_i)$ if and only if $j = j_i$. Thus, for any $n \in \mathbb{N}$,
\[
|C_f|_{\{c_{i,j} : i < d-1, j < n\}} \geq n^{d-1}.
\]
Thus, VCden($C_f$) $\geq d - 1$. However, by Theorem 3.10 and Remark 2.9 we conclude that VCden($C_f$) = $d - 1$.

Corollary 4.5. Let $X$ be a set, $F$ a field, $d \in \mathbb{N}$ with $d \geq 2$, and $\overline{f} : X \to F^d$. If $\overline{f}(X)$ is not contained in a finite union of proper subspaces of $F^d$, then $C_f$ is maximal of Littlestone dimension $d - 1$. In particular, Lden($C_f$) = $d - 1$.

Proof. By Proposition 4.3 $C_f$ is maximal of VC dimension $d - 1$. By Theorem 3.10, $C_f$ has Littlestone dimension $d - 1$. By Remark 2.9 $C_f$ is maximal of Littlestone dimension $d - 1$.

□
5. Characterization of maximality of systems of zero sets.

In this section, we prove the converse to Proposition 4.3.

**Definition 5.1.** Let $F$ be a field, $d \in \mathbb{Z}^+$, $S \subseteq F^d$ finite, and $C \subseteq \mathcal{P}(S)$. We say that $C$ is span injective if

1. for all $A \in C$, $\text{span}(A) \neq F^d$, and
2. for all $A, B \in C$, if $\text{span}(A) = \text{span}(B)$, then $A = B$.

In other words, $C$ is span injective if span is an injective function from $C$ to the set of all proper subsets of $F^d$.

**Lemma 5.2.** Let $X$ be a set, $F$ a field, $d \in \mathbb{Z}^+$, and $f : X \rightarrow F^d$. Then, the set

$$\{\bar{f}(Z_{\bar{a}}) : \bar{a} \in F^d \setminus \{\bar{0}\}\}$$

is span injective.

**Proof.** For all $\bar{a} \in F^d \setminus \{\bar{0}\}$, for all $c \in Z_{\bar{a}}$, $\bar{a} \cdot \bar{f}(c) = 0$, so every element of $\bar{f}(Z_{\bar{a}})$ is orthogonal to $\bar{a}$. Hence, $\text{span}(\bar{f}(Z_{\bar{a}})) \neq F^d$. Suppose that $\bar{a}, \bar{b} \in F^d \setminus \{\bar{0}\}$ and $\text{span}(\bar{f}(Z_{\bar{a}})) = \text{span}(\bar{f}(Z_{\bar{b}}))$. Then, as $\bar{a}$ is orthogonal to $\bar{f}(Z_{\bar{a}})$, it is orthogonal to $\bar{f}(Z_{\bar{b}})$, hence it is orthogonal to $\text{span}(\bar{f}(Z_{\bar{b}}))$, so $\bar{a}$ is orthogonal to $\bar{f}(Z_{\bar{b}})$. Thus, for any $c \in Z_{\bar{a}}$, $\bar{a} \cdot \bar{f}(c) = 0$, so $c \in Z_{\bar{a}}$. By symmetry, we obtain that $Z_{\bar{a}} = Z_{\bar{b}}$. \qed

**Lemma 5.3.** Let $F$ be a field, $d \in \mathbb{Z}^+$, $S \subseteq F^d$ finite, and $C \subseteq \mathcal{P}(S)$. If $C$ is span injective, then there exists $C' \subseteq \mathcal{P}(S)$ such that

1. $\left|C'\right| = \left|C\right|$,
2. $C'$ is span injective,
3. the elements of $C'$ are linearly independent, and
4. $C' \subseteq \left({S \choose <d}\right)$.

**Proof.** For each $A \in C$, let $I_A \subseteq A$ be minimal such that $\text{span}(I_A) = \text{span}(A)$ and let

$$C' = \{I_A : A \in C\}.$$  

Clearly $C'$ is span injective. Fix $A \in C$. Since $I_A$ is chosen to be minimal, $I_A$ is linearly independent. Since $C$ is span injective, $\text{span}(I_A) = \text{span}(A) \neq F^d$, so $|I_A| < d$. Thus,

$$C' \subseteq \left({S \choose <d}\right).$$

Fix $A, B \in C$ and suppose that $I_A = I_B$. Then,

$$\text{span}(A) = \text{span}(I_A) = \text{span}(I_B) = \text{span}(B).$$
Since \( C \) is span injective, \( A = B \). Therefore, \( A \mapsto I_A \) is a bijection from \( C \) to \( C' \), so \( |C| = |C'| \).

**Lemma 5.4.** Let \( F \) be a field, \( d \in \mathbb{Z}^+ \), \( S \subseteq F^d \) finite, and \( C \in \mathcal{P}(S) \). If \( |S| > k(d-1) \), \( S \) is contained in the union of \( k \) proper subsets of \( F^d \), and \( C \) is span injective, then

\[
|C| < \binom{|S|}{<d}.
\]

**Proof.** Let \( C' \) be given as in Lemma 5.3. Since \( |C| = |C'| \) and \( C' \subseteq \binom{S}{<d} \), it suffices to show that there exists a \((d-1)\)-element subset of \( S \) not in \( C' \).

Since \( S \) is contained in the union of \( k \) proper subsets of \( F^d \) and \( |S| > k(d-1) \), by the pigeonhole principle, there exists a proper subspace \( L \) of \( F^d \) that contains at least \( d \) elements of \( S \). Suppose that there exists \( A, B \in \binom{S \cap L}{d-1} \) such that \( A, B \in C' \). Since \( A \) and \( B \) are linearly independent of cardinality \( d-1 \) and \( L \) has dimension less than \( d \), \( \text{span}(A) = \text{span}(B) = L \). Since \( C' \) is span injective, \( A = B \). That is, only one element of \( \binom{S \cap L}{d-1} \) belongs to \( C' \).

Therefore,

\[
|C| = |C'| < \binom{|S|}{<d}.
\]

We are ready to establish the converse to Proposition 4.3.

**Proposition 5.5.** Let \( X \) be a set, \( F \) a field, \( d \in \mathbb{N} \) with \( d \geq 2 \), and \( \overline{f} : X \to F^d \). If \( \overline{f}(X) \) is contained in a finite union of proper subspaces of \( F^d \), then \( C_\overline{f} \) is not maximal of VC-dimension \( d - 1 \).

**Proof.** Let \( n = k(d-1) + 1 \) and fix some \( X_0 \in \binom{X}{n} \). Consider \( \overline{f}|_{X_0} : X_0 \to F^d \) and, for each \( \overline{\sigma} \in F^d \setminus \{0\} \), the corresponding \( Z_{\overline{f}|_{X_0} \overline{\sigma}} \in \mathcal{P}(X_0) \). By Lemma 5.2 the set

\[
C = \{ \overline{f}|_{X_0}(Z_{\overline{f}|_{X_0} \overline{\sigma}}) : \overline{\sigma} \in F^d \setminus \{0\} \}
\]

is span injective. By Lemma 5.4,

\[
|C| < \binom{n}{<d}.
\]

It is clear that, for all \( \overline{\sigma} \in F^d \setminus \{0\} \),

\[
Z_{\overline{f} \overline{\sigma}} \cap X_0 = Z_{\overline{f}|_{X_0} \overline{\sigma}}.
\]
Thus, $C_f|_{X_0} = C_f|_{X_0}$. Therefore,

$$|C_f|_{X_0} = |C_f|_{X_0} \leq |C| < \binom{n}{d}.$$  

Thus, $C_f$ is not maximal of VC-dimension $d - 1$. □

We thus obtain the following result.

**Theorem 5.6.** Let $X$ be a set, $F$ a field, $d \in \mathbb{N}$ with $d \geq 2$, and $f : X \to F^d$. Then, the following are equivalent:

1. $C_f$ is maximal of VC-dimension $d - 1$;
2. $f(X)$ is not contained in a finite union of proper subspaces of $F^d$.

**Proof.** (1) $\Rightarrow$ (2) follows from the contrapositive of Proposition [5.5] and (2) $\Rightarrow$ (1) follows from Proposition [4.3]. □

It turns out that the conditions in Theorem 5.6 are not equivalent to $C_f$ being maximal Littlestone dimension $d - 1$.

**Example 5.7.** Consider the function $\overline{f} : X \to F^d$ in Example 4.4. That is, $F$ is an infinite field, $d \geq 3$,

$$X = \bigcup_{i=0}^{d-2} \text{span}\{e_0, e_{i+1}\},$$

and $\overline{f} : X \to F^d$ is the identity embedding. Then, $C_f$ is maximal of Littlestone dimension $d - 1$, even though $\overline{f}(X)$ is contained in a union of 2-dimensional subspaces of $F^d$.

**Proof.** Define $\overline{c}_{i,j}$ for $i \in [d - 1]$ and $j \in \mathbb{N}$ and define $\overline{b}_{j_0, \ldots, j_{d-2}}$ for $j_0, \ldots, j_{d-2} \in \mathbb{N}$ as in Example [4.4]. In particular, $\overline{c}_{i,j} \in X$ for all $i \in [d - 1]$ and $j \in \mathbb{N}$ and

$$\overline{c}_{i,j} \in Z_{\overline{b}_{j_0, \ldots, j_{d-2}}} \iff j = j_i$$

for all $i \in [d - 1], j, j_0, \ldots, j_{d-2} \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and we define a $(X, C_f)$-labeled tree $T$ of depth $n$ with $\binom{n}{<d}$ well-labeled leaves, showing that $C_f$ is maximal of Littlestone dimension $d - 1$.

For each $k \in [n]$ and $\sigma \in [k][2]$, let $i = |\text{supp}(\sigma)|$ and let

$$T(\sigma) = \begin{cases} \overline{c}_{i,k} & \text{if } i < d - 1 \\ \overline{c}_{0,n} & \text{otherwise} \end{cases}$$
For $\tau \in [n][2]$, let $j_0 < j_1 < \cdots < j_{\ell-1}$ enumerate the set $\text{supp}(\sigma)$. If $\ell < d - 1$, let $j_\ell = j_{\ell+1} = \cdots = j_{d-2} = n$. Then, let

$$T(\tau) = Z_{\overline{f}_{j_0, \ldots, j_{d-2}}}.$$

We claim that, for each $\tau \in [n][2]$, if $|\text{supp}(\tau)| < d$, then $(\tau, T(\tau))$ is well-labeled. This exhibits $\binom{n}{d}$ well-labeled leaves of $T$.

$$\begin{array}{c}
\overline{c}_0,0 \\
\overline{c}_0,1 \\
\overline{c}_1,0,1 \\
\overline{c}_1,4 \\
\overline{c}_{1,5} \\
Z_{\overline{f}_{j_0, \ldots, j_{d-2}}} \\
\end{array}$$

Fix $\tau \in [n][2]$ with $|\text{supp}(\tau)| < d$. Say $j_0 < \cdots < j_{\ell-1}$ enumerates $\text{supp}(\tau)$. Clearly $\ell \leq d - 1$. Set $j_\ell = j_{\ell+1} = \cdots = j_{d-2} = n$ and $j_{d-1} = -1$. Fix $k \in [n]$. If $j_{i-1} < k \leq j_i$ for some $i \in [d-1]$, then

$$|\text{supp}(\tau|_{[k]})| = |\{j_0, \ldots, j_{i-1}\}| = i.$$ 

Therefore, $T(\tau|_{[k]}) = \overline{c}_{i,k}$. Moreover,

$$T(\tau) = Z_{\overline{f}_{j_0, \ldots, j_{d-2}}}.$$

By construction, $T(\tau|_{[k]}) \in T(\tau)$ if and only if $k = j_i$ if and only if $\tau(k) = 1$. Finally, if $j_{d-2} < k < n$, then $|\text{supp}(\tau|_{[k]})| = d - 1$, so $T(\tau|_{[k]}) = \overline{c}_{0,n}$. Thus, since $j_0 < n$, $T(\tau|_{[k]}) \notin T(\tau)$. Moreover, $\tau(k) = 0$.

In any case, we see that, for all $k \in [n]$, $T(\tau|_{[k]}) \in T(\tau)$ if and only if $\tau(k) = 1$. Therefore, we conclude that $(\tau, T(\tau))$ is well-labeled. \(\square\)

6. Conclusion

We have determined the VC-dimension and Littlestone dimension of $C_{\overline{f}}$, as well as given a characterization for when $C_{\overline{f}}$ is maximal in VC-dimension. We can apply these results, for example, to the set of conic sections in $\mathbb{R}^2$ via Example 3.3.

Example 6.1. Let $X = \mathbb{R}^2$, $F = \mathbb{R}$, $d = 6$, and

$$\overline{f}(x, y) = (x^2, xy, y^2, x, y, 1).$$
As noted above, $C_f$ is the set of all conic sections in $\mathbb{R}^2$. For example, we have in the image of $f$,
\[
\begin{align*}
    f(0,0) &= (0,0,0,0,0,1), \\
    f(1,0) &= (1,0,0,1,0,1), \\
    f(0,1) &= (0,0,1,0,1,1), \\
    f(1,1) &= (1,1,1,1,1,1), \\
    f(2,1) &= (4,2,1,2,1,1), \\
    f(1,2) &= (1,2,4,1,2,1).
\end{align*}
\]

It is easy to check that these vectors form a basis for $\mathbb{R}^6$. By Lemma 3.4, $f$ is linearly independent. Thus, by Theorem 3.10, $C_f$ has VC-dimension and Littlestone dimension 5. Moreover, $f(\mathbb{R}^2)$ is not contained in a finite union of proper subspaces of $\mathbb{R}^6$.

Towards a contradiction, suppose that $f(\mathbb{R}^2)$ is contained in finitely many proper subspaces of $\mathbb{R}^6$, say $L_0, \ldots, L_{m-1}$. For each $i < m$, suppose that $\pi_i \in \mathbb{R}^6$ is orthogonal to $L_i$. For each $\overline{b} \in \mathbb{R}^2$, $f(\overline{b}) \in f(\mathbb{R}^2)$, hence there exists $i < m$ such that $f(\overline{b}) \in L_i$. That is, $\pi_i \cdot f(\overline{b}) = 0$, so $\overline{b} \in Z_{f,\pi_i}$. Thus,
\[
\mathbb{R}^2 \subseteq \bigcup_{i<m} Z_{f,\pi_i}.
\]
However, each $Z_{f,\pi_i}$ is a conic section. This is a contradiction, since we cannot cover $\mathbb{R}^2$ with finitely many conic sections.

Therefore, by Theorem 5.6, $C_f$ is maximal of VC-dimension 5. By Corollary 4.5, $C_f$ is maximal of Littlestone dimension 5. In particular, the set of conic sections in $\mathbb{R}^2$ has VC-density 5 and Littlestone density 5.

The results of this paper can be applied to more general situations. For example, consider the set of axes-aligned ellipses. We can use Theorem 3.10 to compute the VC-dimension and Littlestone dimension of this class.

**Example 6.2.** Let $C$ be the set of axes-aligned ellipses in $\mathbb{R}^2$. Formally, let
\[
E_{a,b,c,d} = \{(x,y) \in \mathbb{R}^2 : a(x-b)^2 + c(y-d)^2 = 1\}
\]
and let $C$ be the class of all $E_{a,b,c,d}$ where $a, b, c, d \in \mathbb{R}$ and $a, c > 0$. Note that $C$ is a subclass of the class $C_f$, where $f : \mathbb{R}^2 \to \mathbb{R}^5$ given by
\[
f(x,y) = (x^2, y^2, x, y, 1).
\]
It is easy to show that \( \overline{f} \) is linearly independent. By Theorem 3.10, \( \mathcal{C}_{\overline{f}} \) has VC-dimension and Littlestone dimension 4. Moreover, it is not hard to show that \( \mathcal{C} \) has VC-dimension at least 4 (for example, the set \( \{(1,0),(0,1),(-1,0),(0,-1)\} \) is shattered by \( \mathcal{C} \)). Therefore,

\[
4 \leq \text{VCdim}(\mathcal{C}) \leq \text{Ldim}(\mathcal{C}) \leq \text{Ldim}(\mathcal{C}_{\overline{f}}) = 4.
\]

Thus, \( \mathcal{C} \) has VC-dimension and Littlestone dimension 4. On the other hand, although we can show that \( \mathcal{C}_{\overline{f}} \) is maximal of VC-dimension and Littlestone dimension 4, it is more difficult to apply Theorem 5.6 on the class of axes-aligned ellipses.

In general, it may be interesting to examine what happens when we restrict the parameters of \( \mathcal{C}_{\overline{f}} \). Under what conditions does the VC-dimension or Littlestone dimension drop below \( d - 1 \)? What can be said about the maximality of VC-dimension or Littlestone dimension in this case?

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