Oriented straight lines and twistor correspondence

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Abstract

The tangent bundle to the \( n \)-dimensional sphere is the space of oriented lines in \( \mathbb{R}^{n+1} \). We characterise the smooth sections of \( TS^n \to S^n \) which correspond to points in \( \mathbb{R}^{n+1} \) as gradients of eigenfunctions of the Laplacian on \( S^n \) with eigenvalue \( n \). The special case of \( n = 6 \) and its connection with almost complex geometry is discussed.

1 Oriented lines in \( \mathbb{R}^{n+1} \)

Oriented geodesics in \( \mathbb{R}^{n+1} \) are straight lines. They can be parametrised by choosing a unit vector \( u \) giving a direction, and taking the position vector \( v \) of the point on the geodesic nearest to the chosen origin. A pair of vectors \( (u, v) \) corresponds to the oriented line \( v + tu \), where \( t \in \mathbb{R} \). The space of oriented geodesics is then given by

\[ T = \{ (u, v) \in S^n \times \mathbb{R}^{n+1}, \ u \cdot v = 0 \}. \tag{1.1} \]

For each fixed \( u \) this space restricts to a tangent plane to a unit \( n \)-sphere, and so \( T \) is just the tangent bundle \( TS^n \). We shall call \( T \) the twistor space. There exists a fix-point-free map \( \tau : T \to T \), such that \( \tau^2 = 1 \), obtained by reversing the orientation of each geodesic, i.e. \( \tau(u, v) = (-u, v) \).

Let \( p \) be a point in \( \mathbb{R}^{n+1} \) with a position vector \( p \). The oriented lines through \( p \) are parametrised by the unit \( n \)-sphere in \( T_p \mathbb{R}^{n+1} \), and therefore each \( p \) corresponds to a section \( L_p : S^n \to TS^n \) given by

\[ u \to (u, s(u)), \quad \text{where} \quad s(u) = p - (p \cdot u)u. \tag{1.2} \]

Note that these sections are preserved by \( \tau \). Each section vanishes at two points, where \( \pm p \) is normal to the sphere.
1.1 Laplace sections

The Euclidean group $E(n+1)$ acts on $\mathbb{R}^{n+1}$ and on $TS^n$, and $E(n+1)/so(n+1) = \mathbb{R}^{n+1}$, so the preferred sections are orbits of $so(n+1)$. The $(n+1)$-dimensional space of preferred sections of $T$ corresponding to points in $\mathbb{R}^{n+1}$ can be characterised as the eigenspace of the Laplacian on the n-sphere with eigenvalue $n$, with the vector fields being the gradients for the eigenfunctions.

**Definition 1.1** The gradients of eigenfunctions of the Laplacian on $S^n$ with eigenvalue $n$ are called the Laplace sections of $TS^n$.

**Theorem 1.2** There is a one-to-one correspondence between (Fig. 1)

$$\mathbb{R}^{n+1} \longleftrightarrow TS^n$$

Points $\longleftrightarrow$ Laplace sections

Oriented lines $\longleftrightarrow$ Points.

**Proof.** To complete the proof we need to show that all Laplace sections are of the form (1.2) in some coordinates. To see it consider a unit sphere $S^n$ isometrically immersed in $\mathbb{R}^{n+1}$, and identify a point of $S^n$ with a unit position vector $u$. Let $h$ be the Riemannian metric on $S^n$ induced by the Euclidean inner product on $\mathbb{R}^{n+1}$, and let $X, Y \in T_uS^n$.

Then

$$\nabla'_{X}Y = \nabla X Y + h(X,Y)u$$

where $\nabla'$ is the flat connection on $\mathbb{R}^{n+1}$, and $\nabla$ is the induced connection on the sphere. If $F: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$, then

$$\triangle_{\mathbb{R}^{n+1}}(F) = -r^{-n} \frac{\partial}{\partial r} \left( r^n \frac{\partial F}{\partial r} \right) + r^{-2} \triangle_{S^n}(F|_{S^n(r)}),$$

(1.3)

where $\triangle_{\mathbb{R}^{n+1}} = -\nabla' \cdot \nabla'$ is the Laplacian on $\mathbb{R}^{n+1}$, $\triangle_{S^n}$ is the Laplacian on the unit $n$–sphere, and $F|_{S^n(r)}$ is the restriction of $F$ to an $n$–sphere of radius $r$.

For any constant vector $p \in \mathbb{R}^{n+1}$ consider a function $\chi(u) = u \cdot p$ on $S^n$. We verify that

$$\nabla'(\chi) = -r^{-1}(p - (u \cdot p)u), \quad \triangle_{\mathbb{R}^{n+1}}(\chi) = \frac{n}{r^2} \chi,$$

(1.4)
Restricting the Laplacian to the unit sphere with \( r = 1 \) we deduce that
\[
\triangle_{S^n}(\chi) = n\chi.
\] (1.5)

In particular each coordinate function in \( \mathbb{R}^{n+1} \) regarded as a function on \( S^n \) is an eigenfunction of \( \triangle_{S^n} \) with an eigenvalue \( n \).

The space of solutions to (1.5) is \( n + 1 \) dimensional and the bijection between linear functions on \( \mathbb{R}^{n+1} \) and solutions to (1.5) can be established as follows: We have already verified that restrictions of linear functions from \( \mathbb{R}^{n+1} \) to \( S^n \) satisfy (1.5). Conversely, let \( \chi : S^n \rightarrow \mathbb{R} \) satisfy (1.5). Using the representation (1.3) we deduce that \( r\chi \) is a harmonic function homogeneous of degree one on \( \mathbb{R}^{n+1} \). Let \( x_i \) be local coordinates on \( \mathbb{R}^{n+1} \) with \( |x| = r \). Therefore for each \( i = 1, ..., n + 1, \) \( \partial(r\chi)/\partial x_i \) is harmonic and homogeneous of degree 0, and so it descends to a harmonic function on \( S^n \). There are no such functions apart from the constants, so we deduce that \( r\chi = x \cdot p \), thus establishing the bijection\(^1\).

\[\square\]

This argument can be extended to show that the space of homogeneous harmonic polynomials on \( \mathbb{R}^{n+1} \) of degree \( k > 1 \), when restricted to \( S^n \) constitute the eigenspace of \( \triangle_{S^n} \) with eigenvalue \( k(k+n-1) \). The multiplicity of this eigenvalue is (consult [1] for details)
\[
\binom{n+k}{k} - \binom{n+k-2}{k-2}.
\]

Let us list the properties of the Laplace sections which follow from Theorem 1.2

• Laplace sections are invariant under a map \( \tau : TS^n \rightarrow TS^n \) given by reversing orientations of lines in \( \mathbb{R}^{n+1} \).

• Each non-zero Laplace section vanishes at exactly two points on \( S^n \). Two distinct non-zero Laplace sections \( L_p \) and \( L_q \) intersect at two points in \( TS^n \). These points correspond to two oriented lines joining \( p, q \in \mathbb{R}^{n+1} \)
\[
\begin{align*}
u &= \pm \frac{p - q}{|p - q|}, \quad v = \frac{p \cdot q - |q|^2}{|p - q|} p + \frac{p \cdot q - |p|^2}{|p - q|} q.
\end{align*}
\]

Three (or more) Laplace sections generically don’t meet.

To make the whole construction independent on the choice of the origin in \( \mathbb{R}^{n+1} \), we should regard the twistor space as an affine vector bundle over \( S^n \) with no preferred zero section.

The twistor space \( T \) can also be obtained by factoring the correspondence space \( S^n \times \mathbb{R}^{n+1} \) by the action \( (u,v) \rightarrow (u,tu + v) \) for \( t \in \mathbb{R} \). This action is generated by the geodesic flow \( X \), and leads to a double fibration
\[
\begin{array}{ccc}
S^n \times \mathbb{R}^{n+1} & \rightarrow & T \\
p_2 \searrow & & \swarrow p_1 \\
\mathbb{R}^{n+1} & & \\
\end{array}
\]
given by
\[
p_2(u,v) = v, \quad p_1(u,v) = (u,v - (v \cdot u)u).
\]

Let us look at some special cases: (here \( \nabla = \partial/\partial u \))

\(^1\)Another (equivalent) characterisation of the preferred sections (1.2) is a direct consequence of (1.4). Consider the infinitesimal generators \( s \) of non-homothetic conformal transformations, such that \( s = \nabla \chi \). The equation \( \mathcal{L}_s \chi = 2\chi h \) will then imply that \( \chi \) satisfies (1.5).
• For $n = 1$ the unit circle $S^1$ is parametrised by $\phi \in [0, 2\pi]$, $p = (x_1, x_2)$, and

$$s(u) \cdot \nabla = \text{Re} \left( (x_1 + ix_2) \exp(i\phi) \frac{d}{d\phi} \right).$$

• For $n = 2$ one easily verifies

$$s(u) \cdot \nabla = \text{Re} \left( \left((x_1 + ix_2) + 2\lambda x_3 - \lambda^2(x_1 - ix_2)\right) \frac{d}{d\lambda} \right),$$

where $\lambda = (u_1 + iu_2)/(1 - u_3)$ is a holomorphic coordinate on $\mathbb{CP}^1 = S^2$, and $p = (x_1, x_2, x_3)$. The Laplace sections are in this case holomorphic sections of $T\mathbb{CP}^1$ preserved by $\tau$. This is the original twistor correspondence established by Hitchin [6] in his construction of magnetic monopoles, and recently used in [5] in a study of generalised surfaces in $\mathbb{R}^3$.

A much older application goes back to Whittaker [10]. We shall explain it in a modern language of Hitchin: Given an element of $f \in H^1(T\mathbb{CP}^1, \mathcal{O}(-2))$ restrict it to a Laplace section. The general harmonic function on $\mathbb{R}^3$ is then given by

$$V(x_1, x_2, x_3) = \oint_{\Gamma} f(\lambda, (x_1 + ix_2) + 2\lambda x_3 - \lambda^2(x_1 - ix_2)) d\lambda,$$

where $\Gamma \subset L_p \cong \mathbb{CP}^1$ is a real closed contour.

A different integral transform (the X-ray transform introduced by John [7]) can be used to construct solutions to ultra-hyperbolic wave equation on the twistor space. This takes a smooth function on $\mathbb{RP}^3$ (a compactification of $\mathbb{R}^3$) and integrates it over an oriented geodesic. The resulting function is defined on the Grassmannian $\text{Gr}_2(\mathbb{R}^4)$ of two-planes in $\mathbb{R}^4$ and satisfies the wave equation for a flat metric in $(++-\cdots)$ signature.

## 2 Almost complex structure and $TS^6$

The Riemannian connection $\nabla$ on $S^n$ can be used to define an almost complex structure on $TS^n$ for any $n$. Let $T(TS^n) = V \oplus H$ be the splitting of the tangent space to $TS^n$ into vertical and horizontal components. Define $J_D : TS^n \to TS^n$ by

$$J_D(X_H) = X_V, \quad J_D(X_V) = -X_H,$$

where $X_V$ and $X_H$ are the vertical and horizontal parts of a vector on $TS^n$. This structure was studied by Dombrowski [3] who showed that the torsion of $J_D$ does not vanish unless both the torsion and the curvature of $\nabla$ are zero. This almost complex structure has nothing to do with the Laplace sections defined in Def. 1.1. From now on we shall restrict to the case $n = 6$ where another (inequivalent) almost complex structure can be defined on $T$. The basic facts about the cross products on $\mathbb{R}^7$ will be recalled, and used to show that the Laplace sections are almost complex.

### 2.1 Cross product in $\mathbb{R}^7$ and the group $G_2$

Let $(x_1, ..., x_7)$ be coordinates on $\mathbb{R}^7$, and let $dx_{ijk}$ be a shorthand notation for $dx_i \wedge dx_j \wedge dx_k$. Following Bryant [2] we define the exceptional group $G_2$ as

$$G_2 = \{ \rho \in GL(7, \mathbb{R}) | \rho^*(\phi) = \phi \},$$
where
\[ \phi = dx_{123} + dx_1 \wedge (dx_{45} + dx_{67}) + dx_2 \wedge (dx_{46} - dx_{57}) - dx_3 \wedge (dx_{47} + dx_{56}). \]

It is a compact, connected, and simply connected Lie group of dimension 14. It also preserves the Euclidean metric
\[ g = dx_1^2 + \ldots + dx_7^2, \]
the orientation
\[ dx_{1234567}, \]
and the four-form
\[ *\phi = dx_{4567} + dx_2 \wedge (dx_{45} + dx_{67}) - dx_3 \wedge (dx_{46} - dx_{57}) - dx_1 \wedge (dx_{47} + dx_{56}). \]

The group \( G_2 \) acts transitively on a unit sphere \( S^6 \subset \mathbb{R}^7 \) with a stabiliser \( SU(3) \).

A cross product \( \times : \mathbb{R}^7 \times \mathbb{R}^7 \rightarrow \mathbb{R}^7 \) can be defined by
\[ g(X \times Y, Z) = \phi(X, Y, Z). \]

This cross product has the same properties as the one induced by the octonion multiplication, which leads to a more standard definition of \( G_2 \) as the group of automorphisms of the octonions. The induced cross product satisfies the identities analogous to those in three-dimensions
\[ g(X \times Y, X \times Y) = g(X, X)g(Y, Y) - g(X, Y)^2, \quad X \times (X \times Y) = g(X, Y)X - g(X, X)Y. \]  

\[ (2.6) \]

### 2.2 Pseudoholomorphic sections of \( TS^6 \)

Consider a curve \( \gamma(s, t) \) of oriented lines in \( \mathbb{R}^7 \) parametrised by \( s \in \mathbb{R} \), and given by
\[ \gamma(s, t) = v(s) + tu(s). \]

A \( u \)-orthogonal projection of tangent vector \( t\dot{u} + \dot{v} \) gives rise to a normal Jacobi field
\[ V = (\dot{v} - (\dot{v} . u)u + t\dot{u})|_{s=0} = (\dot{u}, \dot{v} - (\dot{v} . u)u), \quad \text{where} \quad \dot{\cdot} = \frac{\partial}{\partial s}. \]  

All vectors tangent to a space of oriented geodesics are of this form.

Let us define a map \( \tilde{J} : TT \rightarrow TT \) by
\[ V \rightarrow \tilde{J}(V) = u \times V, \quad \text{where} \quad V \in T_{(u,v)}\mathbb{T}. \]  

From the properties (2.6) of cross-product \( \times \) in \( \mathbb{R}^7 \) it follows that \( \tilde{J} \) is an almost complex structure. Indeed,
\[ \tilde{J}^2(V) = u \times (u \times V) = (u.V)u - (u.u)V = -V. \]

Note that \( \tau(\tilde{J}) = -\tilde{J} \).

This almost complex structure is related to a standard almost complex structure \( J \) on \( S^6 \) defined by \( J(v) = u \times v \). To see this consider the restriction of the Euclidean scalar product from \( \mathbb{R}^7 \) to \( S^6 \). This gives the unique nearly Kähler metric \( h \) on \( S^6 \) compatible with \( J \) in a sense that
\[ h(X, Y) = h(JX, JY), \quad \nabla_X J(X) = 0, \quad \forall X, Y \in TS^6, \]
where $\nabla$ is the Levi–Civita connection of $h$. Let

$$T(TS^6) = V \oplus H$$

be the splitting of the tangent space to $TS^6$ into vertical and horizontal components with respect to $\nabla$. The almost complex structure on $TS^6$ defined by taking the standard almost complex structure $J$ on each factor $H$ and $V$ coincides with the almost–complex structure (2.8), because the splitting (2.7) coincides with the splitting $T(TS^6)$ induced by $\nabla$ (which is a projection of splitting given by restricting $\nabla'$ to a tangent space). In particular $\tilde{J}$ is not integrable, since $J$ isn’t.

Let $\rho : S^6 \to S^6$ be an element of $G_2$, and let $v \in T_uS^6$. Then

$$\rho_*(J(v)) = \rho(u) \times \rho_*(v) = \tilde{J}(\rho_*(v)) \in T_{\rho(u)}S^6.$$ 

and we deduce that the Laplace sections $L_p$ of $T \to S^6$ which correspond to points in $\mathbb{R}^7$ are $G_2$–invariant in a sense that

$$\rho_*(L_p(u)) = L_{\rho(p)}(\rho(u)).$$

Now we want to argue that the Laplace sections are also almost complex in the sense that

$$\tilde{J} \circ (L_p)_* = (L_p)_* \circ J.$$ 

This follows directly from the geometrical construction because $u$ is a unit normal to a sphere of geodesics $L_p$ through $p$, and the cross product preserves the almost complex structure on $S^6$ (the almost complex structure on the space of lines is a rotation in $\mathbb{R}^7$ through 90 degrees about the direction of the line which preserves the tangent spaces of $L_p$).

It can also be seen by applying $\tilde{J}$ to (2.7) and performing a direct calculation. This leads to an overdetermined system of equations for $L : S^6 \to TS^6$, $L(u) = (u^j, L^j(u))$

$$\left( \phi_{ijm}u^j\Sigma_{pk} + \phi_{kjp}u^j\Sigma_{ml} \right) \frac{\partial L^m}{\partial u^p} = 0,$$

where $\Sigma_{ij} = \delta_{ij} - u_iu_j$. These equations are satisfied by the Laplace sections.

### 3 Other twistor correspondences

In this final section we shall mention two other generalisations of the Hitchin correspondence. The first one (due to Study [9] for $n = 2$) is more than hundred years old. The second one (due to Murray [8]) gives a way of solving the Laplace equation.

**Study’s correspondence.** The correspondence between oriented lines in $\mathbb{R}^{n+1}$ and points in $TS^n$ can be re-expressed in terms of the dual numbers of the form

$$a + \tau b$$

where $a, b \in \mathbb{R}$, and $\tau^2 = 0$. Let $\mathbb{D}$ denote the space of the dual numbers. Any oriented line in $\mathbb{R}^{n+1}$ can be represented by a vector in $\mathbb{D}^{n+1}$

$$A = u + \tau v$$
which is of unit length with respect to an Euclidean norm in $\mathbb{D}^{n+1}$ induced from $\mathbb{R}^{n+1}$. This gives an analogue of Study’s result [9]: There is a one to one correspondence between oriented lines in $\mathbb{R}^{n+1}$ and points on the dual unit sphere in $\mathbb{D}^{n+1}$. Comparing this with (1.1), we see that the dual unit sphere in $\mathbb{D}^{n+1}$ is equivalent to $TS^n$ with an additional structure (that of dual numbers) selected on the fibres.

Let $\theta$ and $\rho$ be the angle and the distance between two oriented lines represented by $A$ and $B$. Define a dual angle by

$$\Theta = \theta + \tau \rho.$$ 

Using a formal definition

$$\cos \Theta = 1 - \frac{1}{2!} \Theta^2 + \frac{1}{4!} \Theta^4 + ... = \cos \theta - \tau \sin \theta,$$

one can verify an attractive looking formula

$$A \cdot B = \cos \Theta,$$

and deduce that group of Euclidean motions in $\mathbb{R}^{n+1}$ is equivalent to $O(n+1, \mathbb{D})$.

**Murray’s correspondence.** Let $[z_0, z_1, ..., z_n]$ be homogeneous coordinates on $\mathbb{C}P^n$, and let $f = z_0^2 + z_1^2 + ... + z_n^2$ define a section of $O(2) \rightarrow \mathbb{C}P^n$. This section vanishes on a hyper-quadric

$$X = \{f = 0, [z] \in \mathbb{C}P^n\} \subset \mathbb{C}P^n.$$ 

Murray [8] defines a twistor space $Z$ to be a restriction of the total space of $O(1) \rightarrow \mathbb{C}P^n$ to $X$. This leads to a double fibration

$$X \times \mathbb{R}^{n+1} \xrightarrow{m_2} \mathbb{R}^{n+1} \xleftarrow{m_1} Z.$$ 

The canonical bundle of $K_X$ of $X$ in $Z$ is $O(-n+1)$.

**Theorem 3.1 (Murray [8])** Let $\Delta_{\mathbb{R}^{n+1}}$ be the Laplacian on $\mathbb{R}^{n+1}$. There exists an isomorphism

$$T : H^{n-1}(Z, K_X) \rightarrow Ker(\Delta_{\mathbb{R}^{n+1}})$$

given by

$$T(\omega)(z) = \int_{X_z} \omega,$$

where $(\omega)$ is a $K_X$-valued $(0,n)$ form on $Z$ pulled back to $X \times \mathbb{R}^{n+1}$.

The twistor spaces $T$ and $Z$ have the same dimensions, but the connection between Theorem 1.2 and the Murray correspondence is not clear.

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