THE INTEGRABLE OPEN XXZ CHAIN
WITH BROKEN Z\textsubscript{2} SYMMETRY

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The hamiltonian of an asymmetric diffusion process with injection and ejection of particles at the ends of a chain of finite length is known to be relevant to that of the spin-$\frac{1}{2}$ XXZ chain with integrable boundary terms. However, the inclusion of boundary sources and sinks of particles breaks the arrow-reversal symmetry necessary for solution via the usual Bethe Ansatz approach. Developments in solving the model in the absence of arrow-reversal symmetry are discussed.

1 Introduction

The spin-$\frac{1}{2}$ XXZ Heisenberg chain is the canonical example of an integrable quantum hamiltonian. The integrability is assured by the triangle or factorisation equations,

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v).$$

(1)

Among other interpretations, this relation is often described pictorially in terms of the scattering of three particles. This notion of integrability has been extended to include the presence of a boundary with a similar interpretation in terms of particles reflecting from a wall. The reflection equation is

$$R_{12}(u-v)K_{1}(u)R_{21}(u+v)K_{2}(v) = K_{2}(v)R_{12}(u+v)K_{1}(u)R_{21}(u-v).$$

(2)

Our interest here lies in the fact that the above $R$ and $K$ matrices define the Boltzmann weights for exactly solvable two-dimensional lattice models. In particular, the reflection equation follows as the condition for two Sklyanin double-row transfer matrices (Fig. 1(a)) to commute. The boundary vertex weights can be simply written in terms of the $K$-matrix elements.

Consider the concrete example of the six-vertex model and the related XXZ spin chain, for which the $R$ and $K$ matrices are given by

$$R = \begin{pmatrix} a & \sin u & \sin \lambda \\ \sin \lambda & \sin u & a \\ \sin u & \sin \lambda & a \end{pmatrix}, \quad K = \begin{pmatrix} k \sin(\xi + u) & \mu \sin(2u) \\ \nu \sin(2u) & k \sin(\xi - u) \end{pmatrix}$$

(3)
where $a = \sin(\lambda - u)$ with $k, \xi, \mu, \nu$ free parameters. This $K$-matrix was found only quite recently.

The vertex weights along the right boundary in Fig. 1(b) follow as

$$
\begin{align*}
\langle \uparrow \rangle &= k \sin(\xi + \frac{1}{2}u), \\
\langle \downarrow \rangle &= \mu \sin(u), \\
\langle \downarrow \rangle &= k \sin(\xi - \frac{1}{2}u), \\
\langle \uparrow \rangle &= \nu \sin(u).
\end{align*}
$$

(4)

In general there is a set of parameters $k_{\pm}, \xi_{\pm}, \mu_{\pm}, \nu_{\pm}$ for each boundary. These correspond to the two free ends of the related spin chain, with hamiltonian

$$
H = \sum_{j=1}^{N-1} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right) + \left( p_- \sigma_1^z + p_+ \sigma_N^z + c_- \sigma_1^z + c_+ \sigma_N^z + d_- \sigma_1^z + d_+ \sigma_N^z \right),
$$

(5)

where $\sigma = (\sigma^x, \sigma^y, \sigma^z)$ are the Pauli matrices, with $\sigma^\pm = \frac{1}{\sqrt{2}}(\sigma^z \pm i \sigma^y)$. The parameters $p_{\pm}$ (related to $\xi_{\pm}$) control the strength of the $z$-component of the surface fields. The parameters $c_{\pm}$ and $d_{\pm}$ (related to $\nu_{\pm}$ and $\mu_{\pm}$) are the terms responsible for breaking the familiar “up-down” $Z_2$ symmetry.

Despite some intensive effort, the six-vertex model with boundary weights defined by the above $K$-matrix, and equivalently the $XXZ$ chain with the above boundary terms, still defy an exact solution, by which I mean that both the transfer matrix and the hamiltonian remain to be diagonalised. The major obstacle is readily apparent – the boundary terms arising from the non-diagonal elements of the $K$-matrix break the $Z_2$ symmetry by the introduction of sources and sinks of arrows or particles. The total spin along a row of vertical bonds in the vertex model is no longer a conserved quantity. Such
a good quantum number is essential input into the Bethe Ansatz method of solution.

2 Special cases

2.1 Solution for \( \mu = \nu = 0 \).

Consider first the spin chain, which has been solved for the corresponding choice of \( c_\pm = d_\pm = 0 \).

\[
H_1(\Delta, p_-, p_+) = \sum_{j=1}^{N-1} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right) + p_- \sigma_1^z + p_+ \sigma_N^z \tag{6}
\]

The eigenvalues are given by

\[
E = (N-1)\Delta + 4 \sum_{j=1}^{n} (\cos k_j - \Delta) \tag{7}
\]

with the Bethe equations

\[
e^{i2(L-1)k_j} = \frac{f_{k_j}(p_-, \Delta) f_{-k_j}(p_+, \Delta)}{f_{k_j}(p_-, \Delta) f_{k_j}(p_+, \Delta)} \prod_{l=1}^{n} \frac{S(k_l, k_j)}{S(k_l, -k_j)} \tag{8}
\]

where \( f_k(a, b) = a - b + e^{ik} \) and the prime denotes \( l \neq j \). Also \( S(p, q) = s(p, q)/s(q, p) \), where \( s(p, q) = 1 - 2\Delta e^{i\lambda} + e^{i(p+q)} \). This hamiltonian has \( U_q[su(2)] \)-symmetry when \( \Delta = -\cos \lambda \) with \( p_- = p_+ = \pm i \sin \lambda \), known also as the ‘Potts case’.

The six-vertex model on the particular open lattice shown in Fig. 1(b) has been considered by a number of authors. For this lattice there is a diagonal-to-diagonal transfer matrix as indicated. It was solved by means of the co-ordinate Bethe Ansatz for boundary weights corresponding to \( \mu = \nu = 0 \). The Sklyanin double-row transfer matrix (Fig. 1(a)) was also diagonalised for \( \mu = \nu = 0 \) via the algebraic Bethe Ansatz. It is worth noting that a similar double-row transfer matrix was diagonalised earlier by Baxter, who used the Bethe Ansatz solution to obtain the surface free energy. Later it was shown how to pass from the double-row transfer matrix to the diagonal-to-diagonal transfer matrix by means of a special choice of alternating inhomogeneities. The more general double-row transfer matrix was also diagonalised. A key point is that the diagonal-to-diagonal transfer matrix does not commute – rather the underlying integrability lies in the commutation of the Sklyanin transfer matrix with alternating inhomogeneities. A number of other models have also
been solved on the lattice in Fig. 1(b) with diagonal $K$-matrices. Solutions of the various quantum invariant spin chains are also known.

I shall not reproduce the solution of the six-vertex model with diagonal $K$-matrices here. However, it is worth noting that the parametrisation of the diagonal boundary weights in (6), given via the solution of the reflection equation, is particularly convenient when substituted into the corresponding weights $d$ and $e$ of Owczarek and Baxter.

2.2 The case $\mu = 0$ or $\nu = 0$.

Given the impasse on solving the general problem it came as quite a surprise when a solution was reported for the hamiltonian

$$H_2(\alpha, \beta, \gamma, \delta) = \sum_{j=1}^{N-1} \left( \sigma^z_j \sigma^z_{j+1} + \sigma^y_j \sigma^y_{j+1} + \sigma^x_j \sigma^x_{j+1} \right) + H_s$$

The surface term $H_s$ following from the stochastic dynamics of symmetric hopping of particles in one dimension, with particle injection and ejection at the boundaries, is

$$H_s = (\alpha + \gamma)(\sigma^+_1 - 1) + (\gamma - \alpha)(i \sigma^y_1 - \sigma^+_1) + (\beta + \delta)(\sigma^+_L - 1) + (\beta - \delta)(i \sigma^y_L - \sigma^+_L)$$

$$= 2\alpha \sigma^-_1 + 2\gamma \sigma^+_1 - (\alpha + \gamma) - (\gamma - \alpha)\sigma^+_1 + 2\beta \sigma^+_L + 2\delta \sigma^-_L - (\beta + \delta) - (\beta - \delta)\sigma^+_L$$

This is clearly a special case of the general hamiltonian (8). Bethe equations were obtained similar in form to those given in (8) with $\Delta = 1$. However, checking the operator algebra in Ref. reveals that the solution is precisely that given in (8), with the identification $\Delta = 1$, $p_- = \alpha + \gamma$, $p_+ = \beta + \delta$.

Moreover, numerical diagonalisation reveals the equivalence

$$H_1(1, \alpha + \gamma, \beta + \delta) = H_2(\alpha, \beta, \gamma, \delta)$$

This is a consequence of the eigenvalues of

$$H_1(\Delta, p_-, p_+) + d_- \sigma^+_1 + d_+ \sigma^+_L$$

being independent of the variables $d_{\pm}$ (and similarly for $\sigma^+_1$ and $\sigma^+_L$).

Similar considerations apply to the six-vertex model. Certain combinations of sources and sinks at the boundary do not change the eigenspectrum. In particular, a boundary sink of arbitrary weight may be included on both edges, or equivalently a boundary source on both edges. Such behaviour is consistent with inversion relations in which the off-diagonal terms involve the prefactors $\mu_-\nu_-$ and $\mu_+\nu_+$. 4
3 Concluding remarks

Obtaining the solution with the general $K$-matrix remains an outstanding problem. It may still be that the solution can be obtained via the pair propagation through a vertex technique used for the six-vertex model with antiperiodic boundary conditions, where the $Z_2$ symmetry is also broken.\[15\]

The implications of the special solutions with sources and sinks at the boundary to the diffusion problems remain to be fully explored. These include the isotropic hamiltonian $[1]$ of relevance to symmetric hopping with particle injection and ejection at the boundary;\[12\] (see also the vertex model in Ref.\[15\]). It is clear that the corresponding anisotropic $XXZ$ chain enjoys a similar property, however its precise meaning in terms of the asymmetric hopping of particles needs to be clarified.

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References

1. I. Cherednik, *Theor. Math. Phys.* 61, 977 (1984).
2. E.K. Sklyanin, *J. Phys.* A 21, 2375 (1988).
3. L. Mezincescu and R.I. Nepomechie, *J. Phys.* A 24, L17 (1991).
4. C.M. Yung and M.T. Batchelor, *Nucl. Phys.* B 435, 430 (1995).
5. H.J. de Vega and A. González Ruiz, *J. Phys.* A 26, L519 (1993).
6. F.C. Alcaraz, M.N. Barber, M.T. Batchelor, R.J. Baxter and G.R.W. Quispel, *J. Phys.* A 20, 6397 (1987).
7. V. Pasquier and H. Saleur, *Nucl. Phys.* B 330, 523 (1990).
8. A.L. Owczarek and R.J. Baxter, *J. Phys.* A 22, 1141 (1989).
9. R.J. Baxter, unpublished notes (1973).
10. C. Destri and H.J. de Vega, *Nucl. Phys.* B 374, 692 (1992).
11. R.B. Stinchcombe and G.M. Schütz, *Europhys. Lett.* 29, 663 (1995).
12. R.B. Stinchcombe and G.M. Schütz, *Phys. Rev. Lett.* 75, 140 (1995).
13. F.C. Alcaraz and M.T. Batchelor, unpublished.
14. Y.K. Zhou, *Nucl. Phys.* B 458, 504 (1996).
15. M.T. Batchelor, R.J. Baxter, M.J. O’Rourke and C.M. Yung, *J. Phys.* A 28, 2759 (1995); C.M. Yung and M.T. Batchelor, *Nucl. Phys.* B 446, 461 (1995).
16. A. Honecker and I. Peschel, *J. Stat. Phys.* 88, 319 (1997).