Variety and Volatility in Financial Markets

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We study the price dynamics of stocks traded in a financial market by considering the statistical properties both of a single time series and of an ensemble of stocks traded simultaneously. We use the $n$ stocks traded in the New York Stock Exchange to form a statistical ensemble of daily stock returns. For each trading day of our database, we study the ensemble return distribution. We find that a typical ensemble return distribution exists in most of the trading days with the exception of crash and rally days and of the days subsequent to these extreme events. We analyze each ensemble return distribution by extracting its first two central moments. We observe that these moments are fluctuating in time and are stochastic processes themselves. We characterize the statistical properties of ensemble return distribution central moments by investigating their probability density functions and temporal correlation properties. In general, time-averaged and portfolio-averaged price returns have different statistical properties. We infer from these differences information about the relative strength of correlation between stocks and between different trading days. Lastly, we compare our empirical results with those predicted by the single-index model and we conclude that this simple model is unable to explain the statistical properties of the second moment of the ensemble return distribution.

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I. INTRODUCTION

In recent years physicists started to interact with economists to concur to the modeling of financial markets as model complex systems [1]. This triggered the interest of a group of physicists into the analysis and modeling of price dynamics in financial markets performed by using paradigms and tools of statistical and theoretical physics [2]. One target of these researches is to implement a stochastic model of price dynamics in financial markets which reproduces the statistical properties observed in the time evolution of stock prices. In the last few years physicists interested in financial analysis have performed several empirical researches investigating the statistical properties of stock price and volatility time series of a single stock (or of an index) at different temporal horizons [3,4]. Such a kind of analysis does not take into account any interaction of the considered financial stock with other stocks traded simultaneously in the same market. It is known that the synchronous price returns time series of different stocks are pair correlated [5] and several researches has been performed also by physicists in order to extract information from the correlation properties [6,7]. A precise characterization of collective movements in a financial market is of key importance in understanding the market dynamics and in controlling the risk associated to a portfolio of stocks. The present study contributes to the understanding of collective behavior of a portfolio of stocks in normal and extreme days of market activity.

Specifically, we address the question: Is the complexity of a financial market essentially limited to the statistical behavior of each financial time series or rather a complexity of the overall market exists? To answer this question, we present the results of an empirical analysis performed adopting the following point of view. We investigate the price returns of an ensemble of $n$ stocks simultaneously traded in a financial market at a given day. With this approach we quantify what we call the variety of a financial market at a given trading day. The variety provides statistical information about the amount of different behavior observed in stock return in a given ensemble of stocks at a given trading time horizon (in the present case, one trading day). We observe that the distribution of variety is sensitive to the composition of the portfolio investigated (especially to the capitalization of the considered stocks).

The return distribution shows a typical shape for most of the trading days. However, the typical behavior is not observed during crash and rally days. The shape and parameters characterizing the ensemble return distribution are relatively stable during normal phases of the market activity while become time dependent in the periods subsequent to crashes. The variety is characterized by a long-range correlated memory showing that no typical time scale can be expected after a rally or a crash for the expected relaxation to a “normal” market phase. Moreover a simple model such as the single-index model is not able to reproduce the statistical properties empirically observed.

The paper is organized as follow. In Section II we illustrate our database and the ensemble of stocks considered.
Sect. III is devoted to the investigation of the statistical properties of the time evolution of each single stock. In Section IV, we discuss the statistical properties of ensemble return distribution. Specifically we consider the behavior of the central lowest moments, their distribution and correlation, a comparison of time and portfolio average, and the role of the size and homogeneity of the investigated portfolio. In Section V we compare the statistical properties observed in a real financial market with the prediction of the single-index model. In Section VI we present a discussion of the obtained results.

II. DATABASE AND INVESTIGATED VARIABLES

The investigated market is the New York Stock Exchange (NYSE) during the 12-year period from January 1987 to December 1998 which corresponds to 3032 trading days. We consider the ensemble of all stocks traded in the NYSE. The number of stocks traded in the NYSE is increasing in the investigated period and it ranges from 1128 at the beginning of 1987 to 2788 at the end of 1998. The total number of data records exceeds 6 millions.

The variable investigated in our analysis is the daily price return, which is defined as

$$R_i(t) = \frac{Y_{i}(t+1) - Y_{i}(t)}{Y_{i}(t)},$$

where $Y_{i}(t)$ is the closure price of $i$-th stock at day $t$ ($t = 1, 2, ...$). For each trading day $t$, we consider $n$ returns, where $n$ is depending on the total number of stocks traded in the NYSE at the selected day $t$. In our study we use a “market time”. With this choice, we consider only the trading days and we remove the weekends and the holidays from the calendar time.

A database of more than 6 millions records unavoidably contains some errors. A direct control of a so large database is not realistic. For this reason, to avoid spurious results we filter the data by not considering price returns which are in absolute values greater than 50%.

The companies traded in the NYSE are quite different the one from the other. Differences among the companies are observed both with respect to the sector of their economic interests and with respect to their size. One measure of the size of a company is its capitalization. The capitalization of a stock is the stock price times the number of outstanding shares. In this study, we discuss the role of the different capitalization in the price dynamics.

III. SINGLE STOCK PROPERTIES

The distribution of returns with different time horizons of a single stock or index has been studied by several authors [2–4].

The stocks traded in a financial market have different capitalization. An important point is whether the differences in capitalization are reflected in the statistical properties of the price returns of the stocks. To answer this question we investigate the distribution of daily returns of 2188 stocks traded in the NYSE at an arbitrarily chosen day that we select as June 10th, 1996.

We compare the statistical properties of daily price return distribution of each stock as a function of its capitalization. We order the 2188 stocks in decreasing order according to their capitalization at June 10th, 1996. Our ordering procedure gives to the most capitalized stock (the General Electric Co., GE) the rank $i = 1$, to the second one (the Coca Cola Company) the rank $i = 2$, and so on. An analysis of the return probability density function (pdf) for the 2188 stocks shows that the distributions are different. This is due in general to: (i) different scale and (ii) different shape of the return pdfs. In order to eliminate one source of difference we analyze the pdf of the normalized returns $(R_i(t) - \mu_i)/\sigma_i$ ($i = 1, 2, ..., 2188$), where $\mu_i$ and $\sigma_i$ are the first two central moments of the time series $R_i(t)$ defined as

$$\mu_i = \frac{1}{T_i} \sum_{t=1}^{T_i} R_i(t),$$

$$\sigma_i = \left(\frac{1}{T_i} \sum_{t=1}^{T_i} (R_i(t) - \mu_i)^2\right)^{1/2},$$

where $T_i$ is the number of trading days of the stock $i$ during the investigated period. The quantity $\mu_i$ gives a measure of the overall performance of stock $i$ in the period. The standard deviation $\sigma_i$ is called historical volatility in the financial literature and quantifies the risk associated with the $i$-th stock. This quantity is of primary importance in risk management and in option pricing.

The pdf of normalized daily returns of all the stocks ordered by capitalization is shown in Fig. 1. The central part of the distribution of the most capitalized stocks has
FIG. 2. Each circle represents the $h$ parameter defined in Eq. (4) of the daily return distribution of a stock as a function of its capitalization. The dashed line is the value $\sqrt{2/\pi} \approx 0.80$ which is the lower bound for $h_G$ expected for a Gaussian distribution of daily return. Values of $h$ smaller than $h_G$ indicate a leptokurtic distribution of returns. The parameter $h$ slowly increases by increasing the capitalization.

The typical estimation of the degree of leptokurtosis of a pdf is done by considering its kurtosis. The evaluation of the kurtosis of the pdf is in general difficult for small set of records because the fourth moment and all the moments higher than the second are extremely sensible to the highest absolute returns. This implies that the kurtosis calculated from a relatively small set of records is dominated by the highest absolute returns rather than by the shape of the pdf and therefore it is not a good statistical estimation. To avoid this problem, we quantify the distance between the empirically calculated pdf of daily returns of $i-$th stock and the Gaussian distribution by considering the quantity

$$h = \frac{<|x|>}{{\sqrt{<x^2>} - <x>^2}}.$$ (4)

The quantity $h$ is nondimensional and depends on the first two moments. For the Gaussian distribution

$$P_G(x) = \frac{1}{\sqrt{2\pi\sigma_G^2}} \exp\left(-\frac{(x - \mu_G)^2}{2\sigma_G^2}\right),$$ (5)

the parameter $h$ is equal to

$$h_G = \sqrt{\frac{2}{\pi}} \left(\exp\left(-\frac{\mu_G^2}{2\sigma_G^2}\right) + \sqrt{\frac{\pi}{2\sigma_G}} \text{Erf}\left(\frac{\mu_G}{\sqrt{2\sigma_G}}\right)\right).$$ (6)

The parameter $h_G$ is a function of the ratio $\mu_G/\sigma_G$ ranging from the lower bound $\sqrt{2/\pi}$ when $\mu_G/\sigma_G = 0$ to infinity.

FIG. 3. Surface plot of the logarithm of the ensemble return distribution for the 12-year investigated period from January 1987 to December 1998. From the Figure is clearly recognizable the 1987 crash (trading day index equal to 200) and the high volatility two-year period 1997-1998 (trading day index from 2500 to 3032).

For a leptokurtic pdf, as for example a Laplace distribution or a Student’s t-distribution with finite variance, $h$ is always smaller than $h_G$. The distance of $h$ from $h_G$ is able to quantify the degree of leptokurtosis of the considered pdf. Figure 2 shows the parameter $h$ for the stocks traded in the NYSE as a function of their capitalization. In the figure, we show also the lower bound of $h_G$ for comparison. The empirically calculated parameter $h$ is systematically smaller than $h_G$. The mean value $<h>$ of the overall market is $<h>=0.67$ and its standard deviation is $\sigma_h=0.06$. Hence this result suggests that as a first approximation one can assume that the large majority of stocks are characterized by a roughly similar pdf. However we wish to point out that this conclusion is only valid as a first approximation because a trend of $h$ is clearly detected in Fig. 2. Specifically $h$ increases as the capitalization increases. Therefore the less capitalized stocks have a more leptokurtic daily return pdf than the more capitalized ones.

The second moment of return distribution has been found finite in recent research $[1,14]$. In order to verify the convergence of the pdf towards a Gaussian pdf at large temporal horizons, we evaluate the $h$ parameter for weekly $<h_w>$ and monthly $<h_m>$ return pdfs. We obtain from our analysis $<h_w>=0.70$ and $<h_m>=0.74$. These results show that the values of $h$ moves towards $h_G = \sqrt{2/\pi} \approx 0.80$ when the time horizon of returns is increased, supporting the conclusion of finite second moment.

IV. ENSEMBLE RETURN DISTRIBUTION

In the previous section we focused on statistical properties of time evolution of price returns for each single stock traded in the NYSE. In this section we perform a synchronous analysis on the return of all the stock traded in the NYSE. To this aim we extract the $n$ returns of the $n$ stocks for each trading day $t$. 

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The distribution of these returns $P_t(R)$ provides information about the kind of activity occurring in the market at the selected trading day $t$.

Figure 3 shows the logarithm of the pdf as a function of the return and of the trading day. In this figure we show the interval of daily returns from $-25\%$ to $25\%$. The central part of the distribution is roughly triangular in a logarithmic scale and this shape and its scale are conserved for long time periods. Sometimes the shape and scale of the ensemble return pdf changes abruptly either in the presence of large average positive returns or large average negative returns. Figure 4 shows the same data of Fig. 3 in a contour plot. The contour lines describe equiprobability regions. In order to point out the properties of the central part of the distribution, in Fig. 4 we plot only the returns which are less than $15\%$ in absolute value. Only a few points of the contour lines fall behind this limit during the 1987 and 1998 crises. In Fig. 4 there are long time periods in which the central part of the distribution maintains its shape and the equiprobability contour lines are approximately parallel one to each other. As an example, one can consider the three-year period 1993-1995.

![Contour plots of logarithm of ensemble return distribution](image)

**FIG. 4.** Contour plot of the logarithm of the ensemble return distribution for the 12-year investigated period from January 1987 to December 1998 (same data as in Fig. 3). The contour plot is obtained for equidistant intervals of the logarithmic probability density. The brightest area of the contour plot corresponds to the most probable value.

![Ratio between h parameter](image)

**FIG. 5.** Ratio between $h$ parameter defined in Eq. (4) of the ensemble return distribution and the value of $h_G$ expected by a Gaussian distribution and defined by Eq. (6) for each trading day. The ratio $h/h_G$ is systematically smaller than one, indicating that the ensemble return distribution is leptokurtic for each trading day.

On the other hand there are time periods in which the shape of the distribution changes drastically. In general these periods corresponds to financial turmoil in the market. For example a dramatic change of the shape and of the scale of the pdf is observed in Fig. 4 during and after the 19 Oct. 1987 crash, at the beginning of 1991 and at the end of 1998. A systematic analysis of the change of the shape and scale of the ensemble return distribution during extreme events of the market has been discussed elsewhere.

One key aspect of the ensemble return distribution concerns its shape during the normal periods of activity of the market. Is the distribution approximately Gaussian or systematic deviation from a Gaussian shape are quantitatively observed? We already cited that a direct inspection of Fig. 3 suggests that the central part of the empirical return distribution is roughly Laplacian (triangular in a logarithmic scale) and not Gaussian. To make this analysis more quantitative, we show in Fig. 5 the ratio between the value of $h$ determined for each trading day from the ensemble return distribution and the quantities $h_G$ calculated by determining the mean and the standard deviation of $P_t(R)$ and hypothesizing a Gaussian shape by using Eq. (6). The ratio $h/h_G$ is systematically smaller than one and this implies that the Gaussian hypothesis for the shape of the distribution is not verified by the empirical analysis. In other words the Gaussian distribution is not a good approximation both for the central part and for the tails of the distribution and the deviation from the Gaussian behavior is systematically observed for all the trading days of the 12 years time period analyzed in our study.

In summary the ensemble return distribution well char-
characterizes the market activity. It has a typical shape and scale during long periods of “normal” activity of the

market characterized by moderately low average daily return. During extreme events the shape and scale are dramatically changed in a systematic way. Specifically during crises the ensemble return distribution becomes negatively skewed whereas during rallies a positive skewness is observed [15]. Figure 4 clearly shows that extreme events (such as for example October 87 crash) triggers an “aftershock” period, in the ensemble return pdf, that can last for a period of time of several months.

A. Central moments

In order to characterize more quantitatively the ensemble return distribution at day $t$, we extract the first two central moments at each of the 3032 trading days. Specifically, we consider the average and the standard deviation defined as

$$\mu(t) = \frac{1}{n_t} \sum_{i=1}^{n_t} R_i(t),$$  \hfill (7)

$$\sigma(t) = \sqrt{\frac{1}{n_t} \left( \sum_{i=1}^{n_t} (R_i(t) - \mu(t))^2 \right)},$$  \hfill (8)

where $n_t$ indicates the number of stocks traded at day $t$.

The mean of price returns $\mu(t)$ quantifies the general trend of the market at day $t$. The standard deviation $\sigma(t)$ gives a measure of the width of the ensemble return distribution. We call this quantity *variety* of the ensemble because it gives a measure of the variety of behavior observed in a financial market at a given day. A large value of $\sigma(t)$ indicates that different companies are characterized by rather different returns at day $t$. In fact in days of high variety some companies perform great gains whereas others have great losses. The mean and the standard deviation of price returns are not constant and fluctuate in time. We study the temporal series of $\mu(t)$ and $\sigma(t)$ in order to characterize the temporal evolution of the ensemble return distribution quantitatively. We investigate these fluctuating parameters by investigating their time correlation properties and their pdfs.

B. Probability distributions of the central moments

The empirical pdf of the mean $\mu(t)$ for the 3032 trading days investigated is shown in Fig. 6. The central part of this distribution is non-Gaussian and is roughly described by a Laplace distribution.

The mean $\mu(t)$ is proportional to the sum of $n$ random variables $R_i(t)$ ($i = 1, 2, ..., n$). The Central Limit Theorem prescribes that the sum of $n$ independent random variables with finite variance converges to a Gaussian pdf. By assuming a finite value for the volatility of stocks, the observation that the pdf of the mean return $\mu(t)$ is non-Gaussian can be therefore attributed to the presence of correlation between the stocks.

Figure 7 shows the pdf of the variety $\sigma(t)$. The central part of this distribution is approximated by a lognormal distribution. A deviation from the lognormal behavior is observed in the tail of higher values of variety. This deviation is depending on the size of the portfolio and will be discussed in subsection IV E.
C. Correlations in the central moments

Another important statistical property of $\mu(t)$ and $\sigma(t)$ concerns their correlation properties. For the considered portfolio, we calculate the autocorrelation function of a variable $x(t)$ which is defined as

$$R(\tau) \equiv \frac{< x(t)x(t+\tau) > - < x(t) > < x(t+\tau) >}{< x(t)^2 > - < x(t)^2 >}.$$  \hspace{1cm} (9)

In agreement with previous results, we find that the mean $\mu(t)$ is approximately delta correlated, whereas the autocorrelation function of $\sigma(t)$ is long-range correlated. The empirical autocorrelation function of $\sigma(t)$ is well approximated by a power-law function $R(\tau) \propto \tau^{-\delta}$. By performing a best fit with a maximum time lag of 50 trading days, we determine the exponent $\delta = 0.230 \pm 0.006$. This result indicates that the variety $\sigma(t)$ has a long-time memory in the market. We recall that the historical volatility is characterized by long time memory of the same nature.

Another way to investigate the long-range correlation is to determine the power spectrum of the investigated variable. We evaluate the power spectrum of $\sigma(t)$ and we perform a best fit of the power spectrum with a functional form of the kind

$$S(f) \propto \frac{1}{f^\eta}.$$  \hspace{1cm} (10)

Our best fit for the power spectrum of $\sigma(t)$ gives for the exponent $\eta \approx 1.1$. This result confirms that the variety $\sigma(t)$ is a long-range correlated random variable.

D. Time and portfolio average

Figure 6 shows two curves. In fact in Fig. 6 we also show the pdf of the mean $\mu_i$. The quantity $\mu_i$ (see Eq. (2)) is the mean return of stock $i$ averaged over the investigated time interval. The pdf of $\mu_i$ is non-Gaussian and it is much more peaked than the pdf of $\mu(t)$. Hence the statistical behavior observed by investigating a large portfolio in a market day is not representative of the statistical behavior observed by investigating the time evolution of single stocks.

This comparison can be performed also for the second moment of the distributions. In Fig. 7 we compare the pdf of the volatility $\sigma_i$ and the pdf of the variety $\sigma(t)$. Also in this case, the statistical properties of $\sigma_i$ and $\sigma(t)$ are different. Specifically, the pdf of $\sigma(t)$ is more peaked than the pdf of $\sigma_i$.

In order to understand the different behavior of the time-averaged and the portfolio-averaged quantities, for the sake of simplicity, we consider a portfolio composed by $N$ stocks which are traded in a period of $T$ trading days. We first study the properties of the two means, $\mu_i$ and $\mu(t)$. It is straightforward to verify that

$$< \mu_i >_i = < \mu(t) >_i \equiv \mu,$$  \hspace{1cm} (11)

where $< .. >_i$ indicates temporal average and $< .. >_t$ indicates ensemble average. The variances of $\mu_i$ and $\mu(t)$ are in general different. We obtain for the variance of $\mu(t)$ the expression

$$\text{Var}[\mu(t)]_t = \frac{1}{T} \sum_{t=1}^{T} (\mu(t) - \mu)^2 = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij}^2,$$  \hspace{1cm} (12)

where $\sigma_{ij}^2$ is the return covariance between stock $i$ and $j$ defined as

$$\sigma_{ij}^2 = < R_i(t) R_j(t') >_t - < R_i(t) >_t < R_j(t') >_t.$$  \hspace{1cm} (13)

The width of the pdf of $\mu(t)$ (shown in Fig. 6) is the square root of $\text{Var}[\mu(t)]_t$. Equations (12) and (13) indicate that this quantity depends both on the ensemble averaged square volatility (terms with $i = j$ in Eq. (12)) and on the mean of the synchronous cross-covariances between pairs of stocks (terms with $i \neq j$ in Eq. (12)).

With similar methods we show that the variance of $\mu_i$ can be written as

$$\text{Var}[\mu_i]_i = \frac{1}{N} \sum_{i=1}^{N} (\mu_i - \mu)^2 = \frac{1}{T^2} \sum_{t=1}^{T} \sum_{t'=1}^{T} \sigma_{tt'}^2,$$  \hspace{1cm} (14)

where we define the return covariance between trading day $t$ and $t'$ as

$$\sigma_{tt'}^2 = < R_i(t) R_i(t') >_i - < R_i(t) >_i < R_i(t') >_i.$$  \hspace{1cm} (15)

This quantity gives an estimate of the correlation present in the whole portfolio at trading day $t$ and $t'$. The double sum in Eq. (14) can be split in a term depending on the average square variety $(t = t')$ and in a term depending on the correlation between different trading days $(t \neq t')$.

We verify that the average square variance and volatility satisfy the sum rule

$$\text{Var}[\mu_i]_i + < \sigma_i^2 >_i = \text{Var}[\mu(t)]_t + < \sigma^2(t) >_t.$$  \hspace{1cm} (16)

Combining Eqs. (12), (14) and (16) we show that

$$\frac{T-1}{T} < \sigma^2(t) >_t + \frac{2}{N^2} \sum_{j=1}^{N} \sum_{i<j} \sigma_{ij}^2 =$$

$$= \frac{N-1}{N} < \sigma_i^2 >_i + \frac{2}{T^2} \sum_{t=1}^{T} \sum_{t'=1}^{T} \sigma_{tt'}^2.$$  \hspace{1cm} (17)

Since $N, T >> 1$, we approximate $(N-1)/N \approx (T-1)/T \approx 1$ and Eq. (17) becomes

$$< \sigma_i^2 >_i - < \sigma^2(t) >_t \leq < \sigma_{ij}^2 >_{i \neq j} - < \sigma_{tt'}^2 >_{t \neq t'},$$  \hspace{1cm} (18)

or equivalently

$$\text{Var}[\mu(t)]_t - \text{Var}[\mu_i]_i \leq < \sigma_{ij}^2 >_{i \neq j} - < \sigma_{tt'}^2 >_{t \neq t'}. $$  \hspace{1cm} (19)
Figure 6 shows that $\text{Var}[\mu(t)]_2 > \text{Var}[\mu_i]_i$. This empirical observation together with the last relation tell us that the synchronous cross-correlations between the stocks are on average stronger than the single stock correlation present in the whole portfolio at two different trading day. This result is consistent with previous observations that synchronous returns of different stocks are significantly cross-correlated [5–9], whereas single price returns are poorly autocorrelated in time. This conclusion is also verified by our empirical observation that $<\sigma^2_i >_i < \sigma^2(t) >_t$.

E. Portfolio size

One key aspect of the previous results concerns the degree of generality of the observed stylized facts. In other words, are the empirical properties of the variety depending on the considered portfolio? In Section II we have shown that all the stocks are not equivalent with respect to their statistical properties (see the spread of points observed in Fig. 2). In fact a trend is observed in the degree of non-Gaussian shape of the return distribution as a function of the stock capitalization.

To test the degree of sensitivity of our results to the average capitalization of the selected portfolio, we repeat the analysis presented in subsection III.B for three other portfolios of stocks traded in the NYSE. Specifically we investigate: (a) the set of 30 stocks used to compute the Dow Jones Industrial Average index; (b) the set of stocks traded in the NYSE and used to compute the Standard & Poor’s 100 index; and (c) the set of stocks traded in the NYSE and used to compute the Standard & Poor’s 500 index. The results obtained for all the stocks traded in the NYSE are also considered for reference. The four sets are different with respect to two aspects. They differ for the number of stocks present in the set and for the average capitalization of the considered stocks. The empirical pdfs of $\mu(t)$ for the four considered sets are roughly the same. An evident different behavior is observed for the variety. In Fig. 8 we show the pdf of the variety of the considered portfolios of stocks. Specifically panels (a), (b), (c) and (d) of Fig. 8 are the results obtained for the Dow Jones 30, Standard & Poor’s 100, Standard & Poor’s 500 and NYSE sets of stocks, respectively. By moving from the smallest to the largest portfolio of stocks two effects take place. The pdf of the variety becomes progressively sharper and deviates more from a lognormal profile. The fact that the pdf of the variety becomes progressively sharper is probably due to the fact the number of elements in the considered set increases whereas we interpret the progressive deviation from the lognormal profile as a direct manifestation of the progressive increases of the degree of inhomogeneity of the portfolio of stocks.

In summary the presence of inhomogeneity in capitalization in the portfolio of stocks affects the statistical properties of the variety of the portfolio. This fact should be kept in mind when results about the variety such as results about other statistical properties included return distribution are obtained by considering the statistical properties of a set of inhomogeneous stocks.

V. SINGLE-INDEX MODEL

In this section we compare the results of our empirical analysis obtained for the NYSE portfolio of stocks with the results obtained by modeling the stock price dynamics with the single-index model. The single-index model [5,6] is a basic model of price dynamics in financial markets. It assumes that the returns of all stocks are controlled by one factor, usually called the “market”. In this model, for any stock $i$ we have

$$R_i(t) = \alpha_i + \beta_i R_M(t) + \epsilon_i(t), \quad (20)$$

where $R_i(t)$ and $R_M(t)$ are the return of the stock $i$ and of the “market” at day $t$, respectively, $\alpha_i$ and $\beta_i$ are two real parameters and $\epsilon_i(t)$ is a zero mean noise term characterized by a variance equal to $\sigma^2_i$. The noise terms of different stocks are assumed to be uncorrelated, $<\epsilon_i(t)\epsilon_j(t) >_t = 0$ for $i \neq j$. Moreover the covariance between $R_M(t)$ and $\epsilon_i(t)$ is set to zero for any $i$.

Each stock is correlated with the market and the presence of such a correlation induces a correlation between any pair of stocks. It is customary to adopt a broad-based stock index for the market $R_M(t)$. Our choice for the “market” time series is the Standard and Poor’s 500 index. The best estimation of the model parameters $\alpha_i$, $\beta_i$ and $\sigma^2_i$ is done with the ordinary least squares method [5]. In order to compare our empirical results with those
FIG. 9. (a) Time series of the mean of the ensemble return distribution $\mu(t)$. (b) Time series of the mean of the ensemble return distribution for the surrogate data generated according to the single-index model. (c) Time series of the variety $\sigma(t)$ of the ensemble return distribution. (d) Time series of the variety of the ensemble return distribution for the surrogate data generated according to the single-index model.

predicted by the single-index model we build up an artificial market according to Eq. (20). To this end we first evaluate the model parameters for all the stocks traded in the NYSE and then we generate a set of $n$ of surrogate time series according to Eq. (20). To make the simulation as realistic as possible, in the generation of our surrogate data set we use as “market” time series the true time series of the Standard and Poor’s 500 index.

We evaluate the central moments $\mu(t)$ and $\sigma(t)$ defined in Eqs (7-8) for the surrogate data. In Fig. 9(a) we show the time series of $\mu(t)$ of the real data and in Fig. 9(b) we show the same quantity for the surrogate market data generated according to the single-index model. The agreement between the two time series is pretty high and therefore the single-index model describes quite well the mean returns of the market at time $t$ provided that the behavior of the “market” $R_M(t)$ is known. This result is also confirmed by Fig. 10 where the pdf of $\mu(t)$ for real and surrogate data are shown. Also the time correlation properties of surrogate $\mu(t)$ are pretty similar to the real ones. In fact, a fast decaying autocorrelation function of $\mu(t)$ is observed in surrogate data. A good agreement is also observed when one investigates the statistical properties of $\mu_i$ and $\sigma_i$. The single-index model approximates quite well the empirical distribution of $\mu_i$ and $\sigma_i$.

A different behavior is observed for the variety $\sigma(t)$. Figure 9(c) and 9(d) show the time series of $\sigma(t)$ for real and surrogate data, respectively. The real time series of the variety is non stationary and shows several bursts of activity. On the contrary the surrogate time series is quite stationary with the exception of the 1987 crash. Figure 11 shows the pdfs of $\sigma(t)$ for real and surrogate data. The model fails in describing the distribution of $\sigma(t)$.

FIG. 10. Comparison of the probability density function of the mean $\mu(t)$ of the ensemble return distribution obtained from real (diamond) with the one obtained from surrogate data generated according to the single-index model (continuous line).

FIG. 11. Comparison of the probability density function of the variety $\sigma(t)$ obtained from real (diamond) with the one obtained from surrogate data generated according to the single-index model (continuous line).

In summary, the single-index model gives a good approximation of the statistical behavior of $\mu(t)$, $\mu_i$ and $\sigma_i$ whereas it describes poorly the statistical behavior of the variety of a portfolio of stocks traded in a financial market. This conclusion is also supported by the observation that the autocorrelation function of the surrogate variety decays in 2 – 3 trading days to the value 0.1 and the power spectrum is very similar to a white noise spectrum, whereas long-range correlation is observed in real data.

A more refined analysis shows that the artificial ensemble return distribution is systematically less leptokurtic
than the real one. Moreover, in Ref. [15] we show that the single-index model is unable to predict the change in the symmetry properties of the ensemble return distribution in crash and rally days. The differences observed between the behavior of real data and the behavior of surrogate data suggest that the correlations among the stocks can be explained by the single-index model only for “normal” periods in first approximation whereas the model miss completely to reproduce the correlation behavior during extreme events.

VI. CONCLUSIONS

The present study shows that one needs to consider not only the statistical properties characterizing the time evolution of price for each stock traded but also the synchronous collective behavior of the portfolio considered to reveal the overall complexity of a financial market. We show that such a collective behavior of a portfolio of stock is efficiently monitored by the variability of the ensemble return distribution. This variable is directly observable for each portfolio and presents interesting statistical properties. It is non-Gaussian distributed and long-range correlated. The detailed statistical properties depend on the considered portfolio of stocks. We verify that for a portfolio of stocks characterized by comparable capitalization the distribution of the variety is approximately lognormal. Deviation from the lognormal behavior are observed for less homogeneous (in capitalization) portfolios.

The shape of the distribution and the long-term memory of the variety are not reproduced by considering surrogate data simulated by using a single-index model with a realistic time series for the “market”. This implies that the complexity detected by the performed empirical analysis cannot be modeled with a similar simple stock price model. The correlations present in the market are more complex than the ones hypothesized by the single-index model.

The correct modeling of the statistical properties of the variety can be then used as a benchmark for stock price models more sophisticated than the single-index model.

The ensemble return distribution shows a qualitatively and quantitatively different behavior in “normal” and extreme trading days. The variety of a portfolio is then able to detect quite clearly shocks and aftershocks occurring in the market. Hence, it is a promising direct observable able to measure how much a portfolio is under pressure and how distant is from the typical market activity in a specific trading day. A theoretical challenge is to relate this empirical ensemble observation directly with the correlations active between pairs of stocks of a correlation.

In summary, we believe that the overall complexity of a financial market can be detected and modeled only by considering simultaneously – (i) the statistical properties of the time evolution of stock prices of the considered portfolio and (ii) the statics and dynamics of the correlations existing between stocks.

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[1] P. W. Anderson, K. J. Arrow and D. Pines Editors, The Economy as an Evolving Complex System (Addison-Wesley, Redwood City) 1988.
[2] For a collection of papers see for example: J. Kertesz and I. Kend, eds. Econophysics: Proceedings of the Budapest Workshop, Kluwer Academic Press (Dordrecht, in press). R. N. Mantegna, ed., Proceedings of the International Workshop on Econophysics and Statistical Finance, Physica A 269, Issue 1 (1999). J.-P. Bouchaud, ed., Proceedings of the EPS conference on Applications of Physics in Financial Analysis, Int. J. Theor. and Appl. Finance (In press).
[3] R. N. Mantegna and H. E. Stanley, An Introduction to Econophysics: Correlations and Complexity in Finance, (Cambridge Univ. Press, 2000).
[4] J.-P. Bouchaud and M. Potters, Theory of financial risk, (Cambridge Univ. Press, In press).
[5] E. J. Elton and M. J. Gruber Modern Portfolio Theory and Investment Analysis, (J. Wiley & Sons, New York, 1995).
[6] J. Y. Campbell, A. W. Lo, A. C. MacKinlay The Econometrics of Financial Markets, Princeton University Press (Princeton, 1997).
[7] R. N. Mantegna, Eur. Phys. J. B 11, 193 (1999).
[8] L. Laloux, P. Cizeau, J.-P. Bouchaud and M. Potters, Phys. Rev. Lett. 83, 1467 (1999).
[9] V. Plerou, P. Gopikrishnan, B. Rosenow, L. A. N. Amaral and H. E. Stanley, Phys. Rev. Lett. 83, 1471 (1999).
[10] F. Lillo and R. N. Mantegna, Statistical Properties of Statistical Ensembles of Stock Returns, International Journal of Theoretical and Applied Finance, (In press); cond-mat/9909302.
[11] V. Akgiray, and G. G. Booth, Journal of Business & Economic Statistics 6, 51 (1988).
[12] R. N. Mantegna and H. E. Stanley, Nature 376 (1995) 46.
[13] T. Lux, Applied Financial Economics 6 (1996) 463.
[14] P. Gopikrishnan, M. Meyer, L. A. N. Amaral, and H. E. Stanley, Eur. Phys. J. B 3 (1998) 139.
[15] F. Lillo and R. N. Mantegna, Eur. Phys. J. B 15, 603 (2000).
[16] M. M. Dacorogna, U. A. Miller, R. J. Nagler, R. B. Olsen,
and O. V. Pictet, *Journal of International Money and Finance*, 12, 413 (1993).

[17] Y. Liu, P. Gopikrishnan, P. Cizeau, M. Meyer, C.-K. Peng, and H. E. Stanley, *Phys. Rev E* 60, 1390 (1999)

[18] M. Pasquini and M. Serva, e-print, cond-mat/9810232 e-print, cond-mat/9903334