Moduli space intersection duality between Regge surfaces and 2D dynamical triangulations

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Abstract

Deformation theory for 2-dimensional dynamical triangulations with $N_0$ vertices is discussed by exploiting the geometry of the moduli space of Euclidean polygons. Such an analysis provides an explicit connection among Regge surfaces, dynamical triangulations theory and the Witten-Kontsevich model. In particular we show that a natural set of Regge measures and a triangulation counting of relevance for dynamical triangulations are directly connected with intersection theory over the (compactified) moduli space $\mathfrak{M}_{g,N_0}$ of genus $g$ Riemann surfaces with $N_0$ punctures. The Regge measures in question provide volumes of the open strata of $\mathfrak{M}_{g,N_0}$. From the physical point of view, the arguments presented here offer evidence that quantum Regge calculus and dynamical triangulations are related by a form of topological S-duality.

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1 Introduction

The successful analysis of two-dimensional quantum gravity can be partly traced back to the quantum geometry of piecewise-linear surfaces. We discuss in this paper a full-fledged geometrical interplay among two of the main characters of such a theory: dynamical triangulations, and Regge surfaces. We trace a logical path from Riemann moduli theory to triangulations of surfaces that puts clearly to the fore the deep interconnections between dynamical triangulations and Regge calculus, a connection that goes far beyond the few algorithmic elements of piecewise-linear geometry that are common to both theories. We show explicitly that Dynamical triangulations and Regge surfaces with $N_0$ vertices are, in a mathematically well-defined sense, dual under a natural pairing in the (compactified) moduli space of genus $g$ Riemann surfaces with $N_0$ punctures $\mathcal{M}_{g,N_0}$, a duality related to the Witten-Kontsevich intersection numbers $\langle \tau_{\delta_1} \ldots \tau_{\delta_{N_0}} \rangle_1$.[1] Dynamical triangulations have a natural interpretation in terms of open strata of $\mathcal{M}_{g,N_0}$, i.e., to a dynamical triangulation we can associate a generic region of the moduli space. This is not surprising since is just a restatement in the language of dynamical triangulations of the well-known combinatorial parametrization of $\mathcal{M}_{g,N_0}$ in terms of ribbon graph theory (i.e., of graphs which can be drawn on surfaces), exploited by Kontsevich. In our framework what is new is the role of Regge triangulations. They naturally appear upon deforming dynamically triangulated surfaces, and can be related to the geometry of the moduli space of (similarity classes of) polygons in the Euclidean plane. Such a moduli space carries a 2-form that induces a natural volume on the set of Regge surfaces. The structure of the resulting Regge measure is surprisingly elementary, yet it is deeply connected with the geometry of $\mathcal{M}_{g,N_0}$ and provides a natural mean of computing the volume of the open strata in $\mathcal{M}_{g,N_0}$ labelled by the given dynamical triangulations. The actual computation of such a Regge measure is directly related to the volume of a simplex in a Euclidean space whose dimensions are given in terms of the number of vertices (punctures) $N_0$ and of the genus $g$ of the triangulated surface. We leave to a subsequent paper the difficult technical aspects involved in the explicit computation of such a volume. Perhaps what it is more important to stress here is the fact that our analysis provides evidence that two-dimensional quantum Regge calculus and dynamical triangulations theory are related by a form of topological S-duality. Such a duality comes about by observing that the set of Regge-like measures mentioned above and the entropy of dynamical triangulations play against each other, à la Peierls, so as to generate the Witten-Kontsevich intersection numbers.

Coming to the structure of the paper, we tried, as far as possible, to provide a complete and self-contained explanation to the connection between dynamical triangulations and Regge surfaces. Many facts that we present here in detail are often well-known in disguised form to the specialist in moduli theory, but the relation with triangulations is so subtle that we have preferred to be quite explicit. The path we follow is partly related to the geometrical setup of the
Witten-Kontsevich model and the combinatorial parametrization of $\mathcal{M}_g, N_0$ in terms of ribbon graphs theory. However, the different emphasis we put here on Regge calculus gives to our formulation a different flavor which leads to a simple understanding of the interplay between Regge surfaces, dynamical triangulations, and moduli theory.

2 Singular Euclidean structures

Let $T$ denote a $n$-dimensional simplicial complex with underlying polyhedron $|T|$ and $f$-vector $(N_0(T), N_1(T), \ldots, N_n(T))$, where $N_i(T) \in \mathbb{N}$ is the number of $i$-dimensional sub-simplices $\sigma^i$ of $T$. If we consider the (first) barycentric subdivision of $T$, then the closed stars, in such a subdivision, of the vertices of the original triangulation $T$ form a collection of $n$-cells characterizing the polytope $P$ barycentrically dual to $T$. A Regge triangulation of a $n$-dimensional PL manifold $M$, with boundary $\partial M$, is a homeomorphism $|T| \rightarrow M$ where each face of $T$ is geometrically realized by a rectilinear simplex of variable edge-lengths $l(\sigma^i(k))$ of the appropriate dimension. A dynamical triangulation $|T|_{t=a} \rightarrow M$ is a particular case of a Regge PL-manifold realized by rectilinear and equilateral simplices of edge-length $l(\sigma^i(k)) = a$. Similarly, a rectilinear presentation $|P_{T_a}| \rightarrow M$ of the dual cell complex $P$ (with edge-lengths $L = L(l)$) characterizes the Regge polytope (and its rigid equilateral specialization $|P_{T_a}| \rightarrow M$) barycentrically dual to $|T| \rightarrow M$. The metric structure of a Regge triangulation is locally Euclidean everywhere except at the $(n-2)$-simplices $\sigma^{n-2}$, (the bones), where the sum of the dihedral angles, $\theta(\sigma^n)$, of the incident $\sigma^n$'s is in excess (negative curvature) or in defect (positive curvature) with respect to the $2\pi$ flatness constraint. The corresponding deficit angle $r$ is defined by $r = 2\pi - \sum_{\sigma^n} \theta(\sigma^n)$, where the summation is extended to all $n$-dimensional simplices incident on the given bone $\sigma^{n-2}$. If $K_{T}^{n-2}$ denotes the $(n-2)$-skeleton of $|T| \rightarrow M$, then $M \setminus K_{T}^{n-2}$ is a flat Riemannian manifold, and any point in the interior of an $r$-simplex $\sigma^r$ has a neighborhood homeomorphic to $B^r \times C(lk(\sigma^r))$, where $B^r$ denotes the ball in $\mathbb{R}^n$ and $C(lk(\sigma^r))$ is the cone over the link $lk(\sigma^r)$, (the product $lk(\sigma^r) \times [0,1]$ with $lk(\sigma^r) \times \{1\}$ identified to a point), (recall that if we denote by $st(\sigma)$, (the star of $\sigma$), the union of all simplices of which $\sigma$ is a face, then $lk(\sigma^r)$ is the union of all faces $\sigma^f$ of the simplices in $st(\sigma)$ such that $\sigma^f \cap \sigma = \emptyset$). For dynamical triangulations, the deficit angles are generated by the string of integers, the curvature assignments, $\{q(k)\}_{k=1}^{N_n-2} \in \mathbb{N}^{N_n-2(T)}$, viz.,

\begin{equation}
\begin{align*}
    r(i) &= 2\pi - q(i) \arccos(1/n), \quad \sigma^{n-2}(i) \in M \setminus \partial M \\
    &= \pi - q(i) \arccos(1/n), \quad \sigma^{n-2}(i) \in \partial M
\end{align*}
\end{equation}

where
singual Euclidean structure of
write down the singular (conformal) Euclidean metric locally characte
ger the

\[ q(i) : \{\sigma^{n-2}(k)\}_{k=1}^{N_{n-2}(T)} \to \mathbb{N}^+ \]
\[ \sigma^{n-2}(i) \mapsto q(i) \doteq \#\{\sigma^n(h) \perp \sigma^{n-2}(i)\} \]

(2)

provides the numbers of top-dimensional simplices incident on the \(N_{n-2}\) distinct bones. Since each top-dimensional simplex has \(\frac{1}{2}n(n+1)\) bones \(\sigma^{n-2}\), the set of integers \(\{q(k) \geq 3\}_{k=1}^{N_{n-2}}\) is constrained by

\[ \sum_{k}^{N_{n-2}} q(k) = \frac{1}{2}n(n+1)N_n = b(n, n-2)N_{n-2}, \]

(3)

where \(b(n, n-2)\) is the average value of the curvature assignments \(\{q(k)\}_{k=1}^{N_{n-2}}\).

As recalled, a Regge triangulation \(|T_i| \to M\) defines on the PL manifold \(M\) a polyhedral metric with conical singularities associated with the bones \(\{\sigma^{n-2}(i)\}_{i=1}^{N_{n-2}(T)}\) of the triangulation, but which is otherwise flat and smooth everywhere else. Such a metric has important special features, in particular it induces on the PL manifold \(M\) a geometrical structure which turns out to be a particular case of the theory of Singular Euclidean Structure (in the sense of M. Troyanov [2], and W. Thurston [3]). Since each bone \(\sigma^{n-2}(k)\) is \((n-2)\)-dimensional, a point in the interior of \(\sigma^{n-2}(k)\) has a neighborhood in \(|T_k| \to M\) which is homeomorphic to \(B^{n-2}(k) \times B^2(k)\), where \(B^m(k)\) is an \(m\)-dimensional topological ball. Explicitly, the disc \(B^2(k) \simeq C|\ln(\sigma^{n-2}(k))|\) is the cone over the link of the bone. On such a disc we can introduce a locally uniformizing complex coordinate \(\zeta_k \in \mathbb{C}\) in terms of which we can explicitly write down the singular (conformal) Euclidean metric locally characterizing the singular Euclidean structure of \(|T_i| \to M\), viz.,

\[ ds^2(k) = e^{2u} |\zeta_k - \zeta_k(\sigma^{n-2}(k))|^{-2(\frac{\sigma^{n-2}(k)}{2\pi})} |d\zeta_k|^2, \]

(4)

where \(r(k)\) is given by (1), and \(u : B^2 \to \mathbb{R}\) is a continuous function \((C^2\) on \(B^2 - \{\sigma^{n-2}(k)\}\)). Up to the presence of the conformal factor \(e^{2u}\), we immediately recognize in such an expression the metric of a Euclidean cone of total angle \(\theta(k) = r(k) - 2\pi\). Notationally, the metric (4) and its singular structure can be naturally summarized in a formal linear combination of the bones \(\{\sigma^{n-2}(k)\}\) with coefficients given by the corresponding deficit angles (normalized to \(2\pi\), viz., in the real divisor [2]

\[ \text{Div}(T) \doteq \sum_{k=1}^{N_{n-2}(T)} \left( -\frac{r(k)}{2\pi} \right) \sigma^{n-2}(k) = \sum_{k=1}^{N_{n-2}(T\setminus \partial T)} \left( \frac{\theta(k)}{2\pi} - 1 \right) \sigma^{n-2}(k) \]

(5)

\[ + \sum_{h=1}^{N_{n-2}(\partial T)} \left( \frac{\theta(h)}{2\pi} - \frac{1}{2} \right) \sigma^{n-2}(h) \]
supported on the set of bones \( \{ \sigma^{n-2}(i) \}_{i=1}^{N_{n-2}(T)} \). Note that the degree of such a divisor, defined by

\[
|\text{Div}(T)| \doteq \sum_{k=1}^{N_{n-2}(T \setminus \partial T)} \left( \frac{\theta(k)}{2\pi} - 1 \right) + \sum_{h=1}^{N_{n-2}(\partial T)} \left( \frac{\theta(h)}{2\pi} - \frac{1}{2} \right) \quad (6)
\]

is, for dynamical triangulations, a rewriting of the combinatorial constraint (3); in such a sense, the pair \((|T|=a, \text{Div}(T))\), or shortly, \( (T_a, \text{Div}(T)) \), encodes the datum of the triangulation \(|T|=a| \rightarrow M\) and of a corresponding set of curvature assignments \( \{ q(k) \} \) on the labelled bones \( \{ \sigma^{n-2}(i) \}_{i=1}^{N_{n-2}(T)} \).

We conclude this introductory section by recalling a few topological remarks explicitly pertaining to 2-dimensional dynamically triangulated surfaces \(|T|=a| \rightarrow M\). Since, for \( n=2 \), the average incidence of \(|T|=a| \rightarrow M\) is provided by

\[
b(n, n-2)|_{n=2} = 6 \left[ 1 - \frac{\chi(M)}{N_0(T)} \right], \quad (7)
\]

where \( \chi(M) \) denotes the Euler-Poincaré characteristic of the surface, we get \( |\text{Div}(T)|_{|n=2|} = -\chi(M) \). Thus, the real divisor \( |\text{Div}(T)|_{|n=2|} \) characterizes the Euler class of the pair \((T_a, \text{Div}(T))\) and yields for a corresponding Gauss-Bonnet formula. Explicitly, the Euler number associated with \((T_a, \text{Div}(T))\) is defined, [2], by

\[
e(T_a, \text{Div}(T)) \doteq \chi(M) + |\text{Div}(T)|. \quad (8)
\]

and the Gauss-Bonnet formula reads:

**Lemma 1 (Gauss-Bonnet for DT surfaces)** Let \((T_a, \text{Div}(T))\) be a dynamically triangulated surface with divisor

\[
\text{Div}(T) \doteq \sum_{k=1}^{N_0(T)} \left( \frac{\theta(k)}{2\pi} - 1 \right) \sigma^0(k), \quad (9)
\]

associated with the vertex incidences \( \{ \theta(k) \doteq q(k) \frac{\pi}{2\pi} \}_{k=1}^{N_0(T)} \). Let \( ds^2 \) be the conformal metric \( \text{(4)} \) representing the divisor \( \text{Div}(T) \). Then

\[
\frac{1}{2\pi} \int \int_M KdA = e(T_a, \text{Div}(T)), \quad (10)
\]

where \( K \) and \( dA \) respectively are the curvature and the area element corresponding to the metric \( ds^2 \).
Note that such a theorem holds for any singular Riemann surface \( \Sigma \) described by a divisor \( \text{Div}(\Sigma) \) and not just for dynamically triangulated surfaces [2]. Since for a dynamical triangulation, we have \( e(T_a, \text{Div}(T)) = 0 \), the Gauss-Bonnet formula implies

\[
\frac{1}{2\pi} \int_M K dA = 0.
\]  

(11)

Thus, a dynamical triangulation \( |T_{l=a}| \rightarrow M \) naturally carries a conformally flat structure. Clearly this is a rather obvious result, (since the metric in \( M - \{ \sigma^0(i) \}_{i=1}^{N_0(T)} \) is flat). However, it admits a not-trivial converse (recently proved by M. Troyanov, but, in a sense, going back to E. Picard) [2], [4]:

Theorem 2 (Troyanov-Picard) Let \((M, \text{Div})\) be a singular Riemann surface with a divisor such that \( e(M, \text{Div}) = 0 \). Then there exists on \( M \) a unique (up to homothety) conformally flat metric representing the divisor \( \text{Div} \).

This result geometrically characterizes dynamical triangulations (and Regge surfaces) as a particular case of the theory of singular Riemann surfaces, and provides the rationale for understanding at a deeper level the connection between triangulations and some aspects of surface theory of relevance to 2D gravity. A well known example of such a connection is afforded by the relation between the dual polygonalization of \( |P_T| \rightarrow M \) and the space of complete punctured Riemann surfaces exploited in the Witten-Kontsevich theory [1]. A further and strictly related example is provided by the moduli space intersection pairing between Regge surfaces and dynamical triangulations which is the main theme of our paper.

3 Regge surfaces and ribbon graphs

In dimension \( n = 2 \), the geometrical realization of the 1-skeleton of \( |P_{T_L}| \rightarrow M \) is a 3-valent graph \( \Gamma = (\{\varrho^a(k)\}, \{e^1(j)\}) \) where the vertex set \( \{\varrho^a(k)\}_{k=1}^{N_2(T)} \) is identified with the barycenters of the triangles \( \{\sigma^a(k)\}_{k=1}^{N_2(T)} \in |T_i| \rightarrow M \), whereas each edge \( e^1(j) \in \{e^1(j)\}_{j=1}^{N_1(T)} \) is generated by two half-edges \( e^1(j)^+ \) and \( e^1(j)^- \) joined through the barycenters \( \{W(h)\}_{h=1}^{N_1(T)} \) of the edges \( \{\sigma^1(h)\} \) belonging to the original triangulation \( |T_i| \rightarrow M \). Thus, if we formally introduce a degree-2 vertex at each middle point \( \{W(h)\}_{h=1}^{N_1(T)} \), the actual graph naturally associated to the 1-skeleton of \( |P_{T_L}| \rightarrow M \) is

\[
\Gamma_{\text{ref}} = \left( \{\varrho^a(k)\} \bigcup \{W(h)\}, \{e^1(j)^+\} \bigcup \{e^1(j)^-\} \right),
\] 

(12)

the so called edge-refinement [5] of \( \Gamma = (\{\varrho^a(k)\}, \{e^1(j)\}) \). The relevance of such a notion stems from the observation that the natural automorphism group
possible metrics on a (trivalent) ribbon graph $\Gamma$ with given edge-set $\ref{\Gamma}$ the structure of a metric ribbon graph. In general, the set of all refinements of the 1-skeleton of the barycentrical dualization, gives rise to it is natural to attach to the oriented boundaries of punctured disks $D$ may glue to the boundary components of the ribbon graph $\Gamma$. Since we want to keep track of the fact that $\Gamma$ is associated with a polytope $\partial M$ of (the edge-refinement of) a graph does not carry any particular structure. It is precisely the introduction of a cyclic ordering on such a set that allows one to characterize a ribbon graph as a graph which can be drawn on (i.e., embedded into) an oriented surface. Conversely, any ribbon graph $\Gamma$ characterizes an oriented surface $M(\Gamma)$ with boundary possessing $\Gamma$ as a spine, (i.e., the inclusion $\Gamma \hookrightarrow M(\Gamma)$ is a homotopy equivalence). In order to construct $M(\Gamma)$, recall that the half-edges incident to a generic vertex $\partial M(\Gamma)$ is identified with the boundary of the ribbon graph $\Gamma$. Since this latter boundary is oriented, we can attach an oriented disk $D^2(j)$ to each boundary component of $\Gamma$. The resulting space $M$ is a compact oriented topological surface whose genus $g(M)$ is given by

$$2 - 2g(M) = |\nu(\Gamma)| - |e(\Gamma)| + |\partial(\Gamma)|,$$

(14)

where $|\nu(\Gamma)|$, $|e(\Gamma)|$, and $|\partial(\Gamma)|$ respectively denote the number of vertices, edges and boundary components of $\Gamma$. Since we want to keep track of the fact that $\Gamma$ is associated with a polytope $|P_{\ref{\Gamma}}| \rightarrow M$ dual to a Regge triangulation, it is natural to attach to the oriented boundaries $\partial M(\Gamma)$ the length structure naturally associated with $|P_{\ref{\Gamma}}| \rightarrow M$. Note that rather than the disks $D^2(j)$, we may glue to the boundary components of the ribbon graph $\Gamma$ a corresponding set of punctured disks $D^2_{\text{punc}}(j)$. The punctures can be profitably identified with the vertices $\{\sigma^0(k)\}$ of the Regge triangulation $|T_l| \rightarrow M$ which, upon barycentrical dualization, gives rise to $|P_{\ref{\Gamma}}| \rightarrow M$. In this way the edge-refinement of the 1-skeleton of $|P_{\ref{\Gamma}}| \rightarrow M$ acquires from the underlying Regge triangulation the structure of a metric ribbon graph. In general, the set of all possible metrics on a (trivalent) ribbon graph $\Gamma$ with given edge-set $e(\Gamma)$ can

$$Aut(P_L) \simeq Aut(\Gamma_{ref}).$$

(13)

The locally uniformizing complex coordinate $\zeta_k \in \mathbb{C}$ in terms of which we can explicitly write down the singular Euclidean metric $\{4\}$ around each vertex $\sigma^0(k) \in |T_l| \rightarrow M$. Such an orientation gives rise to a cyclic ordering on the set of half-edges $\{e^1(j)^{\pm}\}_{j=1}^{N_1(\Gamma)}$ incident on the vertices $\{\sigma^0(k)\}_{k=1}^{N_2(\Gamma)}$. In particular we assume that the half-edges $e^1(j)^{+}$ and $e^1(j + 1)^{-}$ are incident to the vertex $\sigma^0(j)$ in such a way that $e^1(j)^{+}$ precedes $e^1(j + 1)^{-}$ with respect to the given cyclic ordering. According to such remarks, the 1-skeleton of $|P_{\ref{\Gamma}}| \rightarrow M$ is a ribbon (or fat) graph $[5]$, viz., a graph $\Gamma$ together with a cyclic ordering on the set of half-edges incident to each vertex of $\Gamma$. In general, the set of half-edges of (the edge-refinement of) a graph does not carry any particular structure. It is
be characterized (see [5], definition 3.1) as a topological space homeomorphic to $\mathbb{R}^{|e(\Gamma)|}$, $(|e(\Gamma)|$ denoting the number of edges in $e(\Gamma)$), topologized by the standard $\epsilon$-neighborhoods $U_\epsilon \subset \mathbb{R}^{|e(\Gamma)|}$. On such a space there is a natural action of $\text{Aut}(\Gamma)$, the automorphism group of $\Gamma$ defined by the homomorphism $\text{Aut}(\Gamma) \rightarrow \mathfrak{S}_{e(\Gamma)}$ where $\mathfrak{S}_{e(\Gamma)}$ denotes the symmetric group over $|e(\Gamma)|$ elements. Thus, the resulting space $\mathbb{R}^{|e(\Gamma)|}/\text{Aut}(\Gamma)$ is a differentiable orbifold, (the quotient of a manifold by a finite group). Let

$$\text{Aut}_\partial(P_L) \subset \text{Aut}(P_L),$$

(15)

denote the subgroup of ribbon graph automorphisms of the (trivalent) 1-skeleton $\Gamma$ of $|P_T| \rightarrow M$ that preserve the (labeling of the) boundary components of $\Gamma$. Then, the space of 1-skeletons of Regge polytopes $|P_T| \rightarrow M$, with $N_0(T)$ labelled boundary components, on a surface $M$ of genus $g$ can be defined by [5]

$$RGP^\text{met}_{g,N} = \bigsqcup_{\Gamma \in \text{RGB}_{g,N}} \mathbb{R}_+^{|e(\Gamma)|}/\text{Aut}_\partial(P_L),$$

(16)

where the disjoint union is over the subset of all trivalent ribbon graphs (with labelled boundaries) satisfying the topological condition $2 - 2g - N_0(T) < 0$, and which are dual to triangulations. In this connection, it is worth noticing that not all trivalent ribbon graphs are dual to regular triangulations (e.g., degenerate triangulations with pockets, where two triangles are incident on a vertex, give rise to trivalent dual ribbon graphs with loops, i.e. Regge polytopes containing 2-gons). Our analysis can be extended to such degenerate case as well, however the Regge measure (14) we will use eliminates such singular configurations. Also the set $RGP^\text{met}_{g,N}$ has a natural structure of differentiable orbifold, locally modelled on a stratified space constructed from the components $\mathbb{R}_+^{|e(\Gamma)|}/\text{Aut}_\partial(P_L)$ by means of a (Whitehead) expansion and collapse procedure for ribbon graphs, (see [5] theorems 3.3, 3.4, and 3.5), which basically amounts to collapsing edges and coalescing vertices. This subject has been developed to a high degree of sophistication in [5], and we do not pursue it here any further. As a crude argument, suffices it to say that since in Regge calculus we work at fixed connectivity (i.e., the adjacency matrix of the triangulation $|P_T| \rightarrow M$ is fixed a priori), the relevant model space for the set of 1-skeletons of Regge polytopes $|P_T| \rightarrow M$ is the differential orbifold

$$\mathbb{R}_+^{|e(\Gamma)|}/\text{Aut}_\partial(P_L),$$

(17)

namely a rational cell of the most general orbifold (14). Note that the role of the automorphism group $\text{Aut}_\partial(P_L)$ is highly non-trivial, and the orbifold structure of (17) has a deep impact on the topology of the configuration space of all Regge triangulations, and one cannot simply take $\mathbb{R}_+^{|e(\Gamma)|}$ as a local model for the metric.
3.1 Local deformations of Regge surfaces

The edge-refinement of the 1-skeleton of $|P_{T_k}| \to M$ comes to the fore also in discussing the explicit connection between the metric geometry of a Regge surface $|T_l| \to M$ and the metric structure of the corresponding barycentrically dual complex $|P_{T_k}| \to M$. As a rule, this latter aspect is not emphasized in discussing Regge surfaces, (and it is rather trivial for dynamical triangulations owing to the equilateral constraint), but for later use we need to discuss it here in some detail. To this end, let us fix our attention on the generic vertex $\sigma^0(k) \in |T_l| \to M$, and let us path-order the $q(k)$ vertices of the link $lk(\sigma^0(k))$. If we denote such a collection of ordered vertices by $V_\alpha(k)$, $\alpha = 1, ..., q(k)$, then by splitting open the star $st(\sigma^0(k))$ along the edge connecting $\sigma^0(k)$ to $V_1(k)$ we generate a 2-dimensional Euclidean simplicial complex with $q(k)$ triangles

$$\{ \Delta_\alpha(k) = (V_\alpha(k), \sigma^0(k), V_{\alpha+1}(k)) \}_{\alpha=1}^{q(k)}$$

all incident on the common vertex $\sigma^0(k)$ and such that $\Delta_\alpha(k)$ shares with the adjacent triangle $\Delta_{\alpha+1}(k)$ the edge $V_{\alpha+1}(k)$, (with $\alpha + 1 = 1 \mod q(k)$). In each triangle $\Delta_\alpha(k)$ we can introduce a coordinate system centered at $O \equiv \sigma^0(k)$ with unit basis vectors $\vec{e}_1(\alpha)$ and $\vec{e}_2(\alpha)$, respectively directed along the edges $OV_\alpha(k)$ and $OV_{\alpha+1}(k)$. Thus,

$$\overrightarrow{OV_\alpha(k)} = l_\alpha(k) \vec{e}_1(\alpha),$$

$$\overrightarrow{OV_{\alpha+1}(k)} = l_{\alpha+1}(k) \vec{e}_2(\alpha),$$

where $l_\alpha(k)$ and $l_{\alpha+1}(k)$ denote the respective edge-lengths in the Regge triangulation $|T_l| \to M$. Note that

$$\overrightarrow{V_\alpha(k)V_{\alpha+1}(k)} = \overrightarrow{OV_{\alpha+1}(k)} - \overrightarrow{OV_\alpha(k)},$$

$$\vec{e}_1(\alpha) \cdot \vec{e}_2(\alpha) = \cos \theta_{\alpha,\alpha+1}(k),$$

where $\theta_{\alpha,\alpha+1}(k)$ is the (dihedral) angle at $\sigma^0(k)$ between the edges $\overrightarrow{OV_\alpha(k)}$ and $\overrightarrow{OV_{\alpha+1}(k)}$. The quantity

$$2\pi - \sum_{\alpha=1}^{q(k)} \theta_{\alpha,\alpha+1}(k)$$

is the deficit angle of the Regge triangulation $|T_l| \to M$ at the given vertex $\sigma^0(k)$. In such a setting, the barycenter $g^\omega(\alpha)(k)$ of the triangle $\Delta_\alpha(k)$ is given by

$$\overrightarrow{Og^\omega_\alpha(k)} = \frac{1}{3} \left[ \overrightarrow{OV_\alpha(k)} + \overrightarrow{OV_{\alpha+1}(k)} \right],$$

for dynamical triangulations the adjacency matrix varies, and we must use the general orbifold \[\Box\] as a model of all the rational cells comprising the possible metric geometries of the rigid polytopes $|P_{T_k}| \to M$. For dynamical triangulations the adjacency matrix varies, and we must use the general orbifold \[\Box\] as a model of all the rational cells comprising the possible metric geometries of the rigid polytopes $|P_{T_k}| \to M$. For dynamical triangulations the adjacency matrix varies, and we must use the general orbifold \[\Box\] as a model of all the rational cells comprising the possible metric geometries of the rigid polytopes $|P_{T_k}| \to M$.
whereas the vector \( \overrightarrow{g_\alpha(k)W_{\alpha+1}(k)} \) connecting the barycenter with the midpoint \( W_{\alpha+1}(k) \) of the edge \( OV_{\alpha+1}(k) \) is provided by

\[
\overrightarrow{g_\alpha(k)W_{\alpha+1}(k)} = \frac{1}{2} \overrightarrow{OV_{\alpha+1}(k)} - \frac{1}{6} \overrightarrow{OV_\alpha(k)}.
\]  

(23)

In particular, we get

\[
L_\alpha^+(k) = \left| \overrightarrow{g_\alpha(k)W_{\alpha+1}(k)} \right| \left[ \frac{1}{36} l_{\alpha+1}(k) + \frac{1}{9} l_{\alpha+2}(k) - \frac{1}{9} l_{\alpha+1}(k) l_\alpha(k) \cos \theta_{\alpha,\alpha+1}(k) \right]^{rac{1}{2}},
\]

where \( L_\alpha^+(k) \) denotes the length of the half-edge of \( |PT_k| \rightarrow M \) issuing from the vertex \( g_\alpha(k) \) towards the vertex \( W_{\alpha+1}(k) \). Similarly we can define the length \( L_\alpha^-(k) \), associated with the half-edge issuing from \( W_{\alpha+1}(k) \) towards the vertex \( g_\alpha(k) \), viz.,

\[
L_\alpha^-(k) = \left| \overrightarrow{W_{\alpha+1}(k)g_\alpha(k)} \right| \left[ \frac{1}{36} l_{\alpha+1}(k) + \frac{1}{9} l_{\alpha+2}(k) - \frac{1}{9} l_{\alpha+1}(k) l_\alpha(k) \cos \theta_{\alpha+1,\alpha+2}(k) \right]^{rac{1}{2}},
\]

(25)

(note that \( L_\alpha^-(k) \) for \( \alpha = q(k) \) is given by \( |\overrightarrow{g_\alpha(k)W_1(k)}| \)). Finally, one can also consider the half-edge of \( |PT_k| \rightarrow M \) issuing from the midpoint \( W_{\alpha,\alpha+1}(k) \) of the edge \( V_\alpha(k)W_{\alpha+1}(k) \) towards the vertex \( g_\alpha(k) \). Its length is provided by

\[
L_{\alpha,\alpha+1}^-(k) = \left| \overrightarrow{W_{\alpha,\alpha+1}(k)g_\alpha(k)} \right| = \frac{1}{2} \left| \overrightarrow{OV_\alpha(k)} \right|
\]

(26)

\[
= \frac{1}{6} \left[ l_{\alpha+1}^2(k) + l_\alpha^2(k) + 2 l_{\alpha+1}(k) l_\alpha(k) \cos \theta_{\alpha,\alpha+1}(k) \right]^{rac{1}{2}},
\]

where we have exploited the fact that the medians of the triangles \( \Delta_{\alpha+1}(k) \) are divided in the ratio 2 : 1 by the barycenters \( g_\alpha(k) \). From the relation

\[
l_{\alpha,\alpha+1}^2(k) = \left| \overrightarrow{V_\alpha(k)W_{\alpha+1}(k)} \right|^2
\]

(27)

\[
= l_{\alpha+1}^2(k) + l_\alpha^2(k) - 2 l_{\alpha+1}(k) l_\alpha(k) \cos \theta_{\alpha,\alpha+1}(k),
\]

(28)

(and similarly for \( l_{\alpha+1,\alpha+2}^2(k) \)), it follows that we can conveniently rewrite \( L_\alpha^+(k), L_\alpha^-(k), \) and \( L_{\alpha,\alpha+1}^-(k) \) as

\[
36|L_\alpha^+(k)|^2 = 2 l_\alpha^2(k) + 2 l_{\alpha,\alpha+1}(k) - l_{\alpha+1}^2(k),
\]

(29)
\[36[L_{\alpha,\alpha+1}(k)]^2 = 2l_{\alpha}(k) + 2l_{\alpha+1}(k) - l_{\alpha,\alpha+1}(k). \] (30)

Note that the quantities \((L_{\alpha-1})^2\) and \((L_{\alpha}^+)\) are not independent, being related by

\[\sum_{\alpha=1}^{q(k)}[(L_{\alpha-1})^2 - (L_{\alpha}^+)^2] \equiv 0, \] (31)

(again a consequence of the geometry of the medians). Since the edge connecting the barycenter \(\varrho_{\alpha}(k)\) of the triangle \(\Delta_{\alpha}(k)\) with the barycenter \(\varrho_{\alpha+1}(k)\) of the adjacent triangle \(\Delta_{\alpha+1}(k)\) must pass through the midpoint \(W_{\alpha+1}(k)\) of the edge \(OV_{\alpha+1}(k)\), the length of the edge \(\varrho_{\alpha}(k)\varrho_{\alpha+1}(k)\) \(\in |PT| \rightarrow M\) is provided by

\[L_{\alpha}(k) = L_{\alpha-1}(k) + L_{\alpha}^+(k), \] (32)

namely,

\[L_{\alpha}(k) = \frac{1}{3\sqrt{3}} \left[ l_{\alpha}(k) + l_{\alpha,\alpha+1}(k) - \frac{1}{2}l_{\alpha+1}(k) \right] \]
\[+ \frac{1}{3\sqrt{2}} \left[ l_{\alpha+2}(k) + l_{\alpha+1,\alpha+2}(k) - \frac{1}{2}l_{\alpha+1}(k) \right]^\frac{2}{3}, \] (33)

The set of relations (28), (29), and (30), (as the index \(k\) varies over all vertices of \(|T| \rightarrow M\), or over all 2-cells of \(|PTL| \rightarrow M\)), allow a complete transcription between the metric geometry of a Regge surface \(|T| \rightarrow M\) and the geometry of the corresponding dual polytope \(|PTL| \rightarrow M\).

For a 2-dimensional dynamical triangulation \(|T| \rightarrow M\) the above analysis is only apparently trivial, since it allows us to discuss what happens to the triple of half-edges \(L_{\alpha-1}(k), L_{\alpha}^+(k),\) and \(L_{\alpha,\alpha+1}(k)\) when we deform the edge-lengths of the generic equilateral triangle \(\Delta_{\alpha}(k) \in |T| \rightarrow M\). In particular, by differentiating (28), (29), and (30) with respect to the Regge edge-lengths \(\{l_{\alpha}(k), l_{\alpha,\alpha+1}(k), l_{\alpha+1}(k)\}\) we get

\[dL_{\alpha}(k)_{|l=a} = \frac{1}{3\sqrt{3}} \left[ dl_{\alpha}(k) + dl_{\alpha,\alpha+1}(k) - \frac{1}{2}dl_{\alpha+1}(k) \right], \] (34)

\[dL_{\alpha-1}(k)_{|l=a} = \frac{1}{3\sqrt{3}} \left[ dl_{\alpha+1}(k) + dl_{\alpha,\alpha+1}(k) - \frac{1}{2}dl_{\alpha}(k) \right], \] (35)

\[dL_{\alpha,\alpha+1}(k)_{|l=a} = \frac{1}{3\sqrt{3}} \left[ dl_{\alpha}(k) + dl_{\alpha+1}(k) - \frac{1}{2}dl_{\alpha,\alpha+1}(k) \right]. \] (36)
These relations are invertible and show that the information contained in the deformation of the barycentrically dual polytope \(|P_T| \rightarrow M\) is the same as the information contained in the deformation of the dynamical triangulation \(|T| \rightarrow M\). Thus, if we denote by

\[
\mathcal{P} : \{ |T| \rightarrow M \} \rightarrow \{ |P_T| \rightarrow M \}, \quad \{ l_{(j)} \}_{j=1}^{N_1(T)} \rightarrow \{ L_{\pm(j)}^{\pm} \}_{j=1}^{N_1(P)}, \tag{37}
\]

the map which to the set of edge-lengths \(\{ l_{(j)} \}_{j=1}^{N_1(T)}\) of a Regge surface associates the edge-lengths of the corresponding half-edges in the barycentrically dual polytope, then from the invertibility of (34), (35), and (36) we immediately get the following

Lemma 3 The map \(\mathcal{P}\) is a local isomorphism around any given dynamical triangulation \(|T| \rightarrow M\). In other words, there is a one-to-one correspondence between the local deformations of an equilateral polytope \(|P_T| \rightarrow M\) and the Regge triangulations that can be obtained through the local deformations of the edge-lengths of the dynamical triangulation \(|T| \rightarrow M\) associated with \(|P_T| \rightarrow M\).

Later on we shall discuss in greater detail the geometry of such deformations around a given \(|P_T| \rightarrow M\), and show that is strictly related with the Witten-Kontsevich model.

3.2 Quadratic differentials and singular Euclidean structures

In order to provide a better visualization of the correspondence between singular Euclidean structures, dynamical triangulations, Regge surfaces and moduli spaces of punctured Riemann surfaces we need several basic definitions from surface theory. We apologize to the expert reader who may skip a large part of this section.

We start by recalling that a holomorphic quadratic differential, \(\psi\), (a transverse traceless rank two tensor), on a Riemann surface \(M\) is defined, in a locally uniformizing complex coordinate chart \((U, \zeta)\), by a holomorphic function \(\mu : U \rightarrow \mathbb{C}\) such that \(\psi = \mu(\zeta) d\zeta \otimes d\zeta\). A quadratic differential \(\psi\) is said to have order \(m\) at a point \(p \in M\) if, in a neighborhood of \(p\), we can write \(\psi = \mu(\zeta) d\zeta \otimes d\zeta\), with \(\mu(\zeta)\) possessing a zero of order \(m\) at \(p\). Note that, on a surface of genus \(g > 0\), the number of zeros of a non-trivial quadratic differential \(\psi\) is given by \(4g - 4\) (counting multiplicities). For genus \(g > 0\) the complex vector space of quadratic differentials, \(Q(M)\), is non-empty with complex dimension \(\dim \mathbb{C} Q(M) = 3g - 3\), \((\dim \mathbb{C} Q(M) = 1\), for \(g = 1\)). The geometry of \(Q(M)\) is directly related with the characterization of the Teichmüller space of \(M\), \(\mathcal{T}_g(M)\), the space of all conformal structures on \(M\) under the equivalence relation given by pullback by diffeomorphisms isotopic to the identity map \(id : M \rightarrow M\). It is well known that \(\mathcal{T}_g(M)\) is a smooth finite dimensional manifold that can
be identified with a $6g - 6$ ($\mathbb{R}$)-dimensional cell defined by the open unit ball (in a suitable norm) in the space of quadratic differentials. Conversely, the tangent space to $\mathfrak{f}_g(M)$ at a reference quadratic differential $\psi$, is $\mathbb{C}$-anti-linear isomorphic to $Q(M)$, in other words $Q(M)$ can be canonically identified with the cotangent space to $\mathfrak{f}_g(M)$. If $\pi_1(M)$ denotes the fundamental group of the surface $M$, then the quotient of $\mathfrak{f}_g(M)$ by the action of the outer automorphism group of $\pi_1(M)$, i.e.,

$$\mathcal{M}_g = \mathfrak{f}_g(M)/\text{Out}(\pi_1(M))$$

(38)
is the Riemann moduli space parametrizing conformal equivalence classes of Riemann surfaces of genus $g$. If we fix $\lambda$ distinct points $x_1, ..., x_\lambda \in M$, corresponding to which $M$ is punctured, then the corresponding moduli space acquires one extra (complex) dimension for each puncture, i.e.,

$$\dim \mathcal{M}_{g, \lambda} = 3g - 3 + \lambda.$$  

(39)

The Deligne-Mumford stable curve compactification of such a moduli space is obtained by adding to $\mathcal{M}_{g, \lambda}$ a suitable set of singular noded surfaces according to a prescription that makes $\mathcal{M}_{g, \lambda}$ into a compact (orbifold) space $\mathfrak{M}_{g, \lambda}$. The stability condition guarantees that the surface has only a finite automorphism group, moreover an important feature of $\mathfrak{M}_{g, \lambda}$ is that it compactifies $\mathcal{M}_{g, \lambda}$ without allowing the marked points to come together. Roughly speaking, when points on a (smooth) surface approach each other, the surface sprouts off one or more components and the points distribute themselves on these new components. Recall that on $\mathcal{M}_{g, \lambda}$ one naturally introduces a set of cohomology classes of degree two, $c_1(L_i)$, $i = 1, ..., \lambda$, defined by the Chern class of the line bundle $L_i$ over $\mathcal{M}_{g, \lambda}$ whose fiber at $[M, x_1, ..., x_\lambda] \in \mathcal{M}_{g, \lambda}$ is the cotangent space to $M$ at $x_i$. In terms of such cohomology classes one defines $[5],[6]$ intersection numbers $\langle \tau_{\delta_1}...\tau_{\delta_\lambda} \rangle$ over $\mathfrak{M}_{g, \lambda}$ by setting

$$\langle \tau_{\delta_1}...\tau_{\delta_\lambda} \rangle = \int_{\mathfrak{M}_{g, \lambda}} c_1(L_1)^{\delta_1} \wedge ... \wedge c_1(L_\lambda)^{\delta_\lambda},$$

(40)

where the integral is zero unless the sequence of non-negative integers $\{\delta_i\}_{i=1}^\lambda$ is such that $\sum_{k=1}^\lambda \delta_k = \lambda + 3g - 3$. Since $\mathfrak{M}_{g, \lambda}$ is an orbifold, the $\langle \tau_{\delta_1}...\tau_{\delta_\lambda} \rangle$ are positive rational numbers. Their properties as well as the geometry itself of $\mathfrak{M}_{g, \lambda}$ are strictly connected to a combinatorial stratification of $\mathfrak{M}_{g, \lambda}$ in terms of the graphical data describing inequivalent quadratic differentials. The rationale underlying such stratification is the observation that a quadratic differential $\psi$ may be pictured by a transverse measured foliation which induces on $M$ singular Euclidean structures whose moduli can be connected with $\mathfrak{M}_{g, \lambda}$. For a better visualization of such concepts, recall that a measured foliation with singularities $x_1, ..., x_k$, respectively of order $m_1, ..., m_k$, is a one-dimensional foliation of $M - \{x_1, ..., x_k\}$ with $\{m_j+2\}$ pronged singularities at $\{x_j\}$, together with a transverse measure which assigns to each arc transverse to the foliation a nonnegative real number, such that the natural maps between transversals
are measure-preserving. The foliation by horizontal lines in $\mathbb{C}$ with transverse measure $|dy|$, and more generally the horizontal and vertical trajectories of a holomorphic quadratic differentials provide the typical example of measured foliations. Explicitly, away from the discrete set of the zeros of $\psi \in Q(M) - \{0\}$ we can locally choose a canonical conformal coordinate $\zeta$ (unique up to $\zeta \mapsto \pm \zeta + const$) by integrating the holomorphic 1-form $\sqrt{\psi}$, i.e.,

$$\zeta(z) = \int^z \sqrt{\psi},$$

(41)

so that $\psi = d\zeta \otimes d\zeta$. In such a chart, the measured foliation associated with $\psi$ becomes the usual one whose leaves are the horizontal lines $(\{\text{Im}(\zeta) = \text{const}\}, |d\text{Im}(\zeta)|)$. Similarly, the local sets $\{\text{Re}(\zeta) = \text{const}\}$ endowed with the measure $|d\text{Re}(\zeta)|$ piece together to form the vertical measured foliation associated with $\psi$. At the generic zero $\sigma_j$ of $\psi$ of order $m_j$ we get a cyclically ordered set of $m_j + 2$ pronged singularities counting the number of vertical and horizontal leaves emanating from the singularity. The cyclic order of the prongs (or half-edges) at the singularities is determined by the orientation of the Riemann surface $M$.

If the measured foliation generated by a quadratic differential $\psi \in Q(M) - \{0\}$ has closed horizontal leaves (up to a set of measure zero on the surface), then such a $\psi$ decomposes the surface $M$ into the maximal ring domains foliated by the closed leaves, (typically annuli or punctured disks). As is well known, such a property characterizes the Jenkins-Strebel (JS) quadratic differentials. It is important to realize that if we introduce the local parametrization $\zeta(z)$, the measure associated with $|\psi|$ naturally induces on $M$ a structure of singular flat Riemann surface. Explicitly, let $x_1, ..., x_k$ denote the zeros of $\psi$ with respective multiplicities $m_1, ..., m_k$, then the divisor

$$\sum_{j=1}^k \left( \frac{\theta(j)}{2\pi} - 1 \right) x_j,$$

(42)

with $\theta(j) \doteq (m(j) + 2)\pi$, defines on $M$ a singular Euclidean structure with conical singularities $\theta(j)$ supported on the set of $k$ zeros, $x_1, ..., x_k$, of $\psi$. The corresponding metric is provided by

$$ds^2 = |\psi| = \left( \prod_{j=1}^k |\zeta - (\zeta(x_j))|^{m_j} \right) |d\zeta|^2,$$

(43)

which for the differential $dz^2$ is just the Euclidean metric. Note that the divisor (42) has degree given by

$$\sum_{j=1}^k \left( \frac{(m(j) + 2)\pi}{2\pi} - 1 \right) = \sum_{j=1}^k \frac{m(j)}{2} = -\chi(M),$$

(44)

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as expected. The special feature of the singular Euclidean structure associated with a JS quadratic differential \(|\psi|\) is that it can be naturally put in correspondence to the cell decomposition of the surface \(M\) associated with the rigid polytopes \(|P_{T_a}| \to M\). This is a well-known result that can be considered as a particular case of the Troyanov-Picard theorem, and which in the DT framework can be explicitly described according to

**Proposition 4** Let \(|T_{|a|}| \to M\) be a dynamically triangulated surface of genus \(g\), with a given set of ordered curvature assignments \(\{q(i), \sigma^0(i)\}_{i=1}^{N_0(T)}\) over its \(N_0(T)\) labelled vertices. Let \(|P_{T_a}| \to M\) the equilateral polygonalization of the surface \(M\) associated to the dual polytope of \(|T_{|a|}| \to M\). Identify the labelled vertices \(\{\sigma^0(i)\}_{i=1}^{N_0(T)}\) of \(|T_{|a|}| \to M\) with a corresponding set of marked points of \(M\). If

\[
\begin{cases}
    g \geq 0 \\
    N_0(T) \geq 1 \\
    2 - 2g - N_0(T) < 0
\end{cases}
\]

then there is a unique JS quadratic differential \(\psi \in Q(M) - \{0\}\) on \(M\) satisfying the following conditions: (i) \(\psi\) is holomorphic on \(M\) \(\setminus \{\sigma^0(i)\}_{i=1}^{N_0(T)}\). (ii) \(\psi\) has \(N_2(T)\) zeros (of order 1). (iii) \(\psi\) has a double pole at each \(\sigma^0(k) \in \{\sigma^0(i)\}_{i=1}^{N_0(T)}\). (iv) The union of all noncompact horizontal leaves of \(\psi\) generate the 1-skeleton of \(|P_{T_a}| \to M\). (v) Every compact horizontal leaf of \(\psi\) is a simple loop \(L(\sigma^0(k))\) circling around a vertex \(\sigma^0(k) \in \{\sigma^0(i)\}_{i=1}^{N_0(T)}\), and its (constant) length is proportional to the corresponding curvature assignment \(q(k)\), i.e.,

\[
\left(\frac{\sqrt{3}}{3} a\right) q(k) = \oint_{L(\sigma^0(k))} \sqrt{-\psi}.
\]

Note in particular that the simple loop \(L(\sigma^0(k))\) which hits the corresponding set of zeros of \(\psi\) is the boundary of the \(q(k)\)-gonal 2-cell \(|P_{T_a}| \to M\) baricentrically dual to the vertex \(\sigma^0(k) \in |T_{|a|}| \to M\).

**Proof.** This is basically a rephrasing, in the framework of dynamical triangulation theory, of the classical result of Strebel [7], (we again warmly recommend the remarkable paper [5] for a detailed exposition of Strebel theory, ribbon graph theory, moduli space, and all that). Note that, according to \((46)\), under the correspondence

\[
(|P_{T_a}| \to M) \to (M, \psi),
\]

the \(q(k)\)-gonal 2-cell \(|P_{T_a}| \to M\) dual to the vertex \(\sigma^0(k) \in |T_{|a|}| \to M\) goes into an infinite tube whose constant transverse length is \((\sqrt{3} a) q(k)\). In other words, each dual 2-cell of the dynamical triangulation \(|T_{|a|}| \to M\) gives rise to a punctured disk in the corresponding Riemann surface \((M, \psi)\). Thus the given cut-off \(a\) associated with the fixed edge-length of the \(N_2(T)\) triangles of
a dynamical triangulation, is represented in the corresponding Riemann surface \((M, \psi)\) by the presence of \(N_0(T)\) punctured disks foliated by compact horizontal leaves \(\simeq S^1\) each one of constant length proportional to \(a\). In other words, the tubes of constant section \(\propto a\) are the counterpart, in surface theory, of the \(Diff\)-invariant cut-off characterizing dynamical triangulations.

As is well known, from a mathematical point of view the theory of JS quadratic differentials allows the parametrization of the holomorphic structure of a Riemann surface into the combinatorial data of metric ribbon graphs. Such a parametrization defines a bijective mapping (a homeomorphism of orbifolds) between the space of ribbon graphs \(\text{RGB}_{g,N}^{\text{met}}\) and the moduli space \(\mathcal{M}_{g,N}\) of Riemann surfaces \(M\) of genus \(g\) with \(N\) ordered marked points (punctures) [1], [5],

\[
\mathcal{M}_{g,N} \times \mathbb{R}^N_+ \leftrightarrow \text{RGB}_{g,N}^{\text{met}}
\]

where \((L_1, ..., L_N)\) is an ordered \(n\)-tuple of positive real numbers and \(\Gamma\) is a metric ribbon graphs with \(N\) labelled boundary lengths \(\{L_i\}\) defined by the corresponding JS quadratic differential. In particular, we have

**Proposition 5** Let

\[
\mathcal{D}T_{\{q(k)\}_{k=1}^{N_0}} = \{[T_{l=a}] \rightarrow M : q(\sigma^0(k)) = q(k), k = 1, ..., N_0(T)\}.
\]

\(\mathcal{D}T\) denote the set of distinct dynamically triangulated surfaces of genus \(g\), with a given set of ordered curvature assignments \(\{q(i), \sigma^0(i)\}_{i=1}^{N_0(T)}\) over its \(N_0(T)\) labelled vertices. If \(g \geq 2, 2g - 2 = N_0(T) > 0\), then there is an injective mapping

\[
\mathcal{D}T_{\{q(k)\}_{k=1}^{N_0}} \rightarrow \mathcal{M}_{g,N_0} \times \left(\frac{\sqrt{3}}{3}a\right)^{N_0}_+ \quad (50)
\]

which is defined by associating to the JS quadratic differential \(\psi\), defined by the dual polygonalization of \(a [T_{l=a}] \rightarrow M \in \mathcal{D}T_{\{q(k)\}_{k=1}^{N_0}}\), the corresponding punctured Riemann surface \((M/\{\sigma^0(i)\}_{i=1}^{N_0(T)}), \psi)\).

**Proof.** The injectivity of the map is obvious from the unicity of the JS differential associated with the dual polytope corresponding to \([T_{l=a}] \rightarrow M \in \mathcal{D}T_{\{q(k)\}_{k=1}^{N_0}}\]. Surjectivity fails since the metric ribbon graphs associated with triangulations in \(\mathcal{D}T_{\{q(k)\}_{k=1}^{N_0}}\) do not span the whole \(\text{RGB}_{g,N_0}^{\text{met}}\), but only a subset of \(\text{RGB}_{g,N_0}^{\text{met}}\) generated by ribbon graphs whose labelled boundaries have lengths which are integer multiples of \(\frac{\sqrt{3}}{3}a\). ■

In this connection, it is perhaps worthwhile noticing that the image of the map \(\mathcal{D}T\) coincides with the set of algebraic curves defined over the algebraic closure \(\overline{\mathbb{Q}}\) of the field of rational numbers, (i.e., over the set of complex numbers.
which are roots of non-zero polynomials with rational coefficients). This latter result, due to V. A. Voevodskii and G. B. Shabat [8], establishes a remarkable bijection between dynamical triangulations and curves over algebraic number fields. The proof in [8] exploits the characterization of the collection of algebraic curves defined over \( \mathbb{Q} \) provided by Belyi’s theorem (see e.g., [5]), according to which a nonsingular Riemann surface \( M \) has the structure of an algebraic curve defined over \( \mathbb{Q} \) if and only if there is a holomorphic map (a branched covering of \( M \) over the sphere)

\[ f : M \to \mathbb{C}P^1 \]  

that is ramified only at 0, 1 and \( \infty \), (such maps are known as Belyi maps). It is worthwhile remarking that the inverse image of the line segment \([0, 1] \subset \mathbb{C}P^1\) under a Belyi map is a Grothendieck’s \textit{dessin d’enfant}, thus dynamically triangulated surfaces are eventually connected with the theory of the Galois group \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \) action on the branched coverings \( f : M \to \mathbb{C}P^1 \). The correspondence between Belyi maps, \textit{dessin d’enfant} and JS quadratic differentials has been recently analyzed in depth by M. Mulase and M. Penkava [5], an equally inspiring paper is [9] by M. Bauer and C. Itzykson.

3.3 Triangulations and Moduli spaces of polygons

The relation between a dynamical triangulation and a punctured Riemann surface \((M, \psi)\) appears quite rigid since it exploits both the equilateral structure and the datum of the perimeters \((\sqrt{k\alpha})q(k)\) of the polygons \( |P_{T_a}| \to M \). If we vary the edge lengths \( = (\sqrt{k\alpha}) \) of the polygonal loops \( \{ \mathfrak{L}(\sigma^0(i)) \} \) continuously while leaving the given curvature assignments \( \{ q(i), \sigma^0(i) \}_{i=1}^{N_0(T)} \) fixed, how does this affect the complex structure of the Riemann surface \((M, \psi)\)? What is at issue here is how the complex structure changes as we deform \( |T_{t=a}| \to M \) around the generic vertex \( \sigma^0(k) \) by keeping fixed the adjacency but varying the edge-lengths of the \( q(k) \) triangles incident on \( \sigma^0(k) \), i.e., by considering Regge triangulations associated with \( |T_{t=a}| \to M \). The rationale of this question is to ensure that one is looking at persistent rather than accidental features of the relation between dynamical triangulations, Regge triangulations, and Riemann surfaces. According to lemma 3 there is a one to one correspondence between local deformations of \( |T_{t=a}| \to M \) and local deformations of the equilateral polytope \( |P_{T_a}| \to M \). Thus in order to discuss the above issue, it is sufficient to consider the deformational degrees of freedom of the generic equilateral polygon \( p_{eq}(k, a) \in |P_{T_a}| \to M \). Let

\[ \mathcal{P}_q(k) = \{ p(k) : q(k) - \text{gons in } \mathbb{E}^2 \} \]  

be the space of all (not necessarily equilateral) polygons \( p(k) \) with \( q(k) \) labelled vertices \( v_\alpha = X^\alpha(k) + \sqrt{-1}Y^\alpha(k), v_{q(k)+1} = v_1 \), in the Euclidean plane \( \mathbb{E}^2 \) (\( \simeq \mathbb{C} \)). Note that as we circle around \( p(k) \) in the counterclockwise direction, such
vertices are supposed to be ordered up to a cyclic permutation. We shall consider also the space $[\mathcal{P}_{q(k)}]$ of equivalence classes of all polygons $p(k) \in \mathcal{P}_{q(k)}$ with any two polygons identified if there exists an orientation preserving similarity of $\mathbb{E}^2$, which sends vertices of one polygon to vertices of the other one. Let

$$Z^\alpha(k) \doteq v_{\alpha+1} - v_\alpha = (X^{\alpha+1}(k) - X^\alpha(k)) + \sqrt{-1}(Y^{\alpha+1}(k) - Y^\alpha(k)) \in \mathbb{C}$$ (53)

$$L_\alpha(k) = \sqrt{(X^{\alpha+1}(k) - X^\alpha(k))^2 + (Y^{\alpha+1}(k) - Y^\alpha(k))^2},$$

($\alpha = 1, \ldots, q(k)$), respectively denote the $q(k)$ edges of the polygon $p(k)$ and their lengths. According to (33) we can explicitly write the $L_\alpha(k)$ in terms of the edge-lengths $\{l_\alpha(k)\}$ of the Regge triangulation $[T_l] \to M$ which, upon barycentrical dualization, provides $[P_{T_l}] \to M$.

If we allow all possible deformations of $p(k)$ except the collapse to a point, a generic $q(k)$-gon $p(k)$ is described by the $q(k) - 1$ complex coordinates

$$p(k) = (Z^1(k), \ldots, Z^{q(k)-1}(k)) \in \mathbb{C}^{q(k)-1} \setminus \{0\},$$ (54)

(since the closing condition $\sum_{\alpha=1}^{q(k)} Z^\alpha(k) = 0$ for the edges of the polygon $p(k)$ implies that $Z^{q(k)}(k) = -\sum_{\alpha=1}^{q(k)-1} Z^\alpha(k)$). Moreover, any two polygons $p(k)$ and $\tilde{p}(k)$ such that

$$(\tilde{Z}^1(k), \ldots, \tilde{Z}^{q(k)-1}(k)) = \lambda(Z^1(k), \ldots, Z^{q(k)-1}(k)),$$ (55)

for some $\lambda \in \mathbb{C} \setminus \{0\}$ define the same point in $[\mathcal{P}_{q(k)}]$. In other words, the orbit space $[\mathcal{P}_{q(k)}]$ is canonically isomorphic to the complex projective space over the hyperplane $\sum_{\alpha=1}^{q(k)} Z^\alpha(k) = 0 \subset \mathbb{C}^{q(k)}$, namely to $\mathbb{C}P^{q(k)-2}$. From such an identification it follows that the assignment of $(Z^1(k), \ldots, Z^{q(k)-1}(k))$ to the equivalence class of polygons $[p(k)]$ it defines in $\mathbb{C}P^{q(k)-2}$ is nothing but the natural projection $\pi : \mathbb{C}^{q(k)-1} \setminus \{0\} \to \mathbb{C}P^{q(k)-2}$, [10]. In particular a point in the inverse image

$$\pi^{-1}([p(k)]) \simeq \mathbb{C}^* \simeq \mathbb{C} \setminus \{0\},$$ (56)

is the polygon locally described by

$$p(k) = (\frac{Z^1}{\bar{Z}^\nu}, \ldots, \frac{Z^\nu}{\bar{Z}^\nu}, \ldots, \frac{Z^{q(k)-1}}{\bar{Z}^\nu}; Z^\nu)$$ (57)

with $\{\frac{Z^1}{\bar{Z}^\nu}\}, \nu \neq 1$, being the local coordinates of an equivalence class of polygons, $\bar{Z}^\nu$ denoting an omitted coordinate, and $Z^\nu \in \mathbb{C}^*$. Since any polygon $p(k)$ may occur as the the $q(k)$-gon dual to a vertex of a Regge triangulation, we have the following obvious characterization

**Proposition 6** The set of (labelled) $q(k)$-gons barycentrically dual to the $\sigma_0(k)$ vertex of a Regge triangulation is described by the principal $\mathbb{C}^*$-bundle

$$\mathcal{P}_{q(k)} : \mathbb{C}^{q(k)-1} \setminus \{0\} \xrightarrow{\pi} \mathbb{C}P^{q(k)-2}.$$ (58)
Note that we can equivalently describe such a set of polygons by means of the associated canonical line bundle over $\mathbb{CP}^{q(k)-2}$ (the dual Hopf bundle).

From $[\mathcal{P}_q(k)] \simeq \mathbb{CP}^{q(k)-2}$ it also follows that $[\mathcal{P}_q(k)]$ is a compact connected complex manifold that can be equipped with the standard Fubini-Study Kähler form $\omega_{FS}(k)$,

$$\omega_{FS}(k)|_{\mathcal{L}(k)} = \frac{\sqrt{-1}}{2} \left\{ \frac{dZ^\alpha(k)}{Z^\beta(k)Z^\gamma(k)} \frac{Z^\gamma(k)dZ^\alpha(k) \wedge dZ^\beta(k)}{(Z^\mu(k)Z^\tau(k))^2} \right\}. \tag{59}$$

Such remark implies that $[10]$, if $[Z^1(k),...,Z^{q(k)-1}(k)]$ denote the homogeneous coordinates representing the equivalence class of polygons $[p(k)]$, then each polygon representing $[p(k)] \in [\mathcal{P}_q(k)]$ can be continuously deformed to another polygon, representing a distinct equivalence class $[p(k)] \in [\mathcal{P}_q(k)]$, by means of an interpolating curve of polygons

$$\mathbb{R} \ni I \longrightarrow [\mathcal{P}_q(k)] \tag{60}$$

$$t \mapsto [Z^1(k,t),...,Z^{q(k)-1}(k,t)] = [p(k;t)] \in [\mathcal{P}_q(k)],$$

which can be chosen to be a geodesic with respect to the Fubini-Study metric associated with $\omega_{FS}(k)$. Note that $[10]$, since $[\mathcal{P}_q(k)]$ is compact, such a geodesic segment has length bounded above by a constant which does not depend from $[p(k)]$ and $[\hat{p}(k)]$. Since (59) is invariant under (52), there is also a natural action of $\mathbb{R}^+$ on $[\mathcal{P}_q(k)]$ which corresponds to an overall rescaling of the edge-lengths $(L_1(k),...,L_q(k))(k) \in \mathbb{R}^{q(k)}$ by a common factor $\eta = |\lambda| \in \mathbb{R}^+$. Thus, without loss of generality, we can require that the perimeter of the $q(k)$-gon $p(k,t)$ remains fixed (and equal to the perimeter of $p_{\eta q}(k,a)$) while deforming its edges, i.e.,

$$\sum_{\alpha=1}^{q(k)} L_\alpha(k,t) = \frac{(\sqrt{3}/3)q(k)}{3} \forall t \in I \subset \mathbb{R}. \tag{61}$$

Moreover, any such a deformation is defined up to the action of a cyclic permutation $\alpha_1,\ldots,\alpha_{q(k)} = s(1,\ldots,q(k))$, i.e.,

$$(L_1(k),...,L_q(k))(k) \simeq (L_{\alpha_1}(k),...,L_{\alpha_{q(k)}}(k)), \tag{62}$$

where $s$ is an element of the symmetric group $\mathfrak{S}_{q(k)}$ over $q(k)$ elements. This is a convenient moment for discussing more in detail the map

$$\mathfrak{S}_k : \mathcal{P}_q(k) \longrightarrow \mathbb{R}^{q(k)} \tag{63}$$

$$Z^\alpha(k) \longmapsto \mathfrak{S}_k(Z^\alpha(k)) = (L_1(k),...,L_{q(k)}(k))$$

which assigns to each polygon $p(k)$, in the principal bundle $\mathcal{P}_q(k)$, the $q(k)$-tuple of its edge lengths $\{L_\alpha(k)\}$. The image of $\mathfrak{S}_k$ is contained in the domain $D_{q(k) \subset}$
polygons defined by the pre-image characterized by \( p \). The polygon \( P \) is a smooth submanifold of \( q \) varying its first neighborhood of \( p \). The equilateral \( q \)-gon \( p_{eq}(k,a) \) dual to \( \sigma^a(k) \in [T_{l=a}] \to M \) belongs to the set of polygons defined by the pre-image \( 3^{-1}_k((\frac{\sqrt{2}}{3}a),..., (\frac{\sqrt{2}}{3}a)) \subset C^{q(k)-1} - \{0\} \). However, note that \( 3^{-1}_k((\frac{\sqrt{2}}{3}a),..., (\frac{\sqrt{2}}{3}a)) \) may contain also degenerate polygons (all with the same given perimeter) defined by those \( q(k) \)-gons which collapse into a straight line, \( e.g., \) a parallelogram with equilateral sides of length \((\frac{\sqrt{2}}{3}a)\) may smoothly collapse into a segment of length \(2(\frac{\sqrt{2}}{3}a)\). It is worth noticing that [10] the set of degenerate polygons appears in \( P_{q(3)} \) as the fixed points set of the involution

\[
P_{q(k)} \to P_{q(k)}
\]

\[
((\{Z^a(k)\})_{k=1}^{N_0} \to (\{\overline{Z^a(k)}\})_{k=1}^{N_0}
\]

induced by the complex conjugation. An elementary application of the results of M. Kapovich and J. Millson [10] (pp. 138), provides a suitable characterization of the set of polygons in \( 3^{-1}_k(\{L_{a}(k,t)\}) \) in a neighborhood of the equilateral \( q(k)-gon \) \( p_{eq}(k,a) \) with edge-lengths \( \{L_{a}(k,t)\} = \{(\frac{\sqrt{2}}{3}a)\} \).

**Proposition 7** In the principal bundle \( P_{q(k)} \) there exists a neighborhood \( U_k \) of the equilateral \( q(k)-gon \) \( [p_{eq}(k,a)] \) such that

\[
3^{-1}_k(\{L_{a}(k,t)\}) \cap U_k,
\]

is a smooth submanifold of \( P_{q(k)} \) whose tangent space at \( p_{eq}(k,a) \) is characterized by

\[
T \left[ 3^{-1}_k(\{L_{a}(k,t)\}) \cap U_k \right]_{p_{eq}(k,a)} =
\left\{ (\xi^1, ..., \xi^{q(k)-1}) \in C^{q(k)-1} : \Re[Z^a(\xi^a)]_{a=1, ..., q(k)-1} = 0 \right\}.
\]

In other words, \( 3_k : C^{q(k)-1} - \{0\} \to \mathbb{R}^{q(k)} \) is a smooth submersion in a neighborhood of \( p_{eq}(k,a) \), and \( (3^{-1}_k(\{L_{a}(k,t)\}) \cap U_k \) is a smooth submanifold of \( P_{q(k)} \) around \([p_{eq}(k,a)]\). In order to prove such result, let us define a deformed \( q(k)-gon \), in a neighborhood of the equilateral polygon \( p_{eq}(k,a) \), by smoothly varying its first \( q(k)-1 \) sides, while keeping its perimeter fixed, \( i.e., \)

\[
Z^a(t) \doteq Z^a + t\xi^a|_{a=1, ..., q(k)-1},
\]

\[
Z^{q(k)}(t) \doteq - \sum_{\alpha=1}^{q(k)-1} Z^\alpha(t)
\]

\[
\mathbb{R}^{q(k)} \text{ defined by}
\]

\[
\sum_{\alpha=1}^{q(k)} L_{a}(k,t) = (\frac{\sqrt{3}}{3}a)q(k),
\]

\[
0 < L_{a}(k,t) \leq (\frac{\sqrt{3}}{6}a)q(k),
\]

\[
\forall t \in I \subset \mathbb{R}, \text{ (the upper bound on the } L_{a}(k,t) \text{ follows from the triangle inequalities)}.
\]
with $|Z^\alpha| = (\sqrt{3}/3) a$, and $\sum_{\alpha=1}^{q(k)} Z^\alpha(t) = (\sqrt{3}/3) a q(k)$, $\forall \ t \geq 0$. Thus

$$\Im P \subset \mathbb{R}^2$$

and the corresponding tangent mapping is provided by

$$\partial_k (Z^\alpha(t)) = \left(\frac{d}{dt} |Z^\alpha(t)|_{t=\alpha} \right)_{\alpha=1}^{q(k)-1}, \frac{d}{dt} |Z^q(t)|_{t=0}$$

(69)

and the corresponding tangent space provided by

$$D\mathbb{R}_k \cdot \xi^\alpha = \left\{ (\xi^1, \ldots, \xi^q) \in \mathbb{C}^{q(k)-1} : \text{Re} Z^\alpha \xi^\beta |_{\alpha=1, \ldots, q(k)-1} = 0 \right\}$$

(70)

Note that $\text{dim}_{\mathbb{R}} D\mathbb{R}_k \cdot \xi^\alpha = q(k) - 1$ and $\text{dim}_{\mathbb{R}} \text{Im} D\mathbb{R}_k \cdot \xi^\alpha = q(k) - 1$ which indeed sum up to $\text{dim}_{\mathbb{R}} \mathcal{P}(q(k)) = 2q(k) - 2$.

By projecting $\mathbb{R}_k^{-1}(\{L_a(k, t)\})$ into $\mathbb{C}^{q(k)-2}$ by means of $\pi : \mathcal{P}(q(k)) \to \mathcal{P}(q(k)) \simeq \mathbb{C}^{q(k)-1}$ we get the corresponding (connected component of the) moduli space of polygons [10],

$$\mathbb{M} \mathbb{P}(\{L_a(k, t)\}) = \pi (\mathbb{R}_k^{-1}(\{L_a(k, t)\}))$$

(72)

parametrizing the equivalence classes, under similarities, of distinct (possibly degenerate) polygons with the same edge-lengths $\{L_a(k, t)\}$ (and given fixed perimeter). Since in the neighborhood of regular polygons $\mathbb{R}_k^{-1}(\{L_a(k, t)\})$ is a submanifold of $\mathcal{P}(q(k))$, the space $\mathbb{R}_k^{-1}(\{L_a(k, t)\})$ inherits from $\mathcal{P}(q(k))$ the structure of a principal $\mathbb{C}^*$-bundle over the moduli space $\mathbb{M} \mathbb{P}(\{L_a(k, t)\})$, (with a singularity structure, corresponding to degenerate polygons, which is described in details in [10]).

As the edge-lengths $\{L_a(k, t)\}$ vary in a neighborhood $U_k$ of $p_{eq}(k, a)$, we can characterize the corresponding space of polygons $\mathbb{R}_k^{-1}(\{L_a(k, t)\}) \cap U_k$ by means of the 1-form field defined, on the principal bundle $\mathcal{P}(q(k))$, by

$$\psi(k) : = -a (\sqrt{3}/3) q(k) \sum_{\alpha=1}^{q(k)-1} |Z^\alpha| \left( \sum_{\beta=1}^{3} d|Z^\beta| \right)$$

(73)

where the normalization to the fixed perimeter and the minus sign are chosen for later convenience. As long as $\{L_a(k, t)\} \neq 0$, it follows that $\ker \psi(k) =$
\( \ker D\mathfrak{Z}_k \), so that \( \mathfrak{Z}_k^{-1}(\{L_\alpha(k, t)\}) \) are the integral distributions of \( \psi(k) \). Such \( \psi(k) \) is invariant under the involution \( \mathfrak{Z}_k \), whose fixed point sets characterize degenerate polygons, and is well-defined also if some of the \( L_\alpha(k, t) \to 0 \), (simply remove the corresponding term from \( (73) \)). Thus, it is easily checked that \( \psi(k) \) extends to the whole space \( \mathfrak{Z}_k^{-1}(\{L_\alpha(k, t)\}) \), (degenerate and \( L_\alpha(k, t) \to 0 \) polygons included), and provides a connection on \( \mathfrak{Z}_k^{-1}(\{L_\alpha(k, t)\}) \) thought of as a principal \( \mathbb{C}^* \)-fibration (with singularities) over the moduli space of polygons \( \mathfrak{M}\mathfrak{P}(\{L_\alpha(k, t)\}) \). The curvature 2-form \( \varpi(k) \) associated with such a connection can be locally written as

\[
\varpi(k) = d\psi(k) = -\left[a \left( \frac{\sqrt{3}}{3} q(k) \right) \right]^{-2} q(k)^{\alpha-1} \sum_{\alpha=1}^a d|Z^\alpha| \wedge \left( \sum_{\beta=1}^a d|Z^\beta| \right)
\]

It follows that the class \( d\psi(k) \) represents the first Chern character of the line bundle \( L_{\text{pol}}(k) \) associated with the fibration \( \mathfrak{Z}_k^{-1}(\{L_\alpha(k, t)\}) \) \( \mathfrak{M}\mathfrak{P}(\{L_\alpha(k, t)\}) \),

\[
c_1(L_{\text{pol}}(k)) = \|d\psi(k)\| \in H^2(\mathfrak{M}\mathfrak{P}(\{L_\alpha(k, t)\}), \mathbb{C}),
\]

where \( H^2(\mathfrak{M}\mathfrak{P}(\{L_\alpha(k, t)\}), \mathbb{C}) \) denotes the second deRham cohomology group of the polygons moduli space \( \mathfrak{M}\mathfrak{P}(\{L_\alpha(k, t)\}) \). Actually, \( \|d\psi(k)\| \) lies in the image of the inclusion \( H^2(\mathfrak{M}\mathfrak{P}(\{L_\alpha(k, t)\}), \mathbb{R}) \subseteq H^2(\mathfrak{M}\mathfrak{P}(\{L_\alpha(k, t)\}), \mathbb{C}) \). Indeed, \( d\psi(k) \) is the pull-back, \( d\psi(k) = \mathfrak{Z}_k^* \omega(k) \), under the edge-length map \( \mathfrak{Z}_k : P q(k) \to \mathbb{R}^q(k) \), of the \( \mathfrak{S} q(k)^* \) and scale-invariant 2-form field defined on \( \mathbb{R}^q(k) \) by

\[
\omega(k) = \left[ a \left( \frac{\sqrt{3}}{3} q(k) \right) \right]^{-2} \sum_{1 \leq \alpha \leq \beta \leq q(k)-1} d \left( \frac{L_\alpha(k)}{a} \right) \wedge d \left( \frac{L_\beta(k)}{a} \right)
\]

providing the area elements (normalized to the fixed perimeter) associated with the distinct pairs of bivectors \( \frac{\partial}{\partial \alpha} \wedge \frac{\partial}{\partial \beta} \). Since

\[
dL_\alpha|_a = \frac{1}{3\sqrt{3}} \left[ dl_\alpha(k) - dl_{\alpha+1}(k) + dl_{\alpha+2}(k) + dl_{\alpha,\alpha+1}(k) + dl_{\alpha+1,\alpha+2}(k) \right],
\]

we can also write \( (76) \), as

\[
\omega_R(k) = \left[ 3aq(k) \right]^{-2} \sum_{1 \leq \mu \leq 2q(k)} d\mu(k) \wedge dl_\nu(k),
\]

where \( \{l_\nu(k)\}_{\nu=1}^{2q(k)} \) stand for the \( 2q(k) \) (\( \nu \)-relabelled) edge-lengths of the simplicial loop of triangles \( \{\Delta_\nu(k)\} \) which yields for the polygon \( p(k) \) under dualization, (see section 3). Note that for each given vertex \( \sigma^0(k) \in |T_{l=a}| \to M \)
the 2-form (78) allows for a complete description of the infinitesimal deformations of \([T_{t=a}] \to M\) giving rise to a nearby Regge triangulation \([T_L] \to M\). It is also worthwhile recalling that the 2-form \(\omega(k)\) naturally appears in connection with certain universal polygonal bundles associated with the combinatorial parametrization of the moduli space \(\mathcal{M}_{g,\lambda}\) introduced by Kontsevich in [1].

### 3.4 Isoperimetric Regge measures

In order to extend the above analysis to the deformations of the whole dual polytope \([P_{T_L}] \to M\), associated with a dynamical triangulation \([T_{t=a}] \to M\) with given curvature assignments, let us consider the map,

\[
Def\left\{\{p_{eq}(k,a), \delta_k\}_{k=1}^{N_0(T)}\right\} : \prod_{k=1}^{N_0(T)} [P_{q(k)}] \longrightarrow \prod_{k=1}^{N_0(T)} P_{q(k)}
\]

which associates with an ordered equivalence class of equilateral polygons \(\{[p_{eq}(j,a)]\}_{j=1}^{N_0}\) a deformation realized by the polygonal 2-cells of a polytope \([P_{T_L}] \to M\) obtained by (isoperimetrically) varying the edge-lengths of \(\delta_k \geq 0\) sides of the corresponding equilateral \(q(k)\)-gon \(\in [P_{T_L}] \to M\). Clearly, not all possible deformations of the set of polygons \(\{p_{eq}(k,a)\}_{k=1}^{N_0}\) give rise to a polytope \([P_{T_L}] \to M\). Since the number of edges of \([P_{T_L}] \to M\) is \(N_1(T)\), and each varied edge is shared between two adjacent polygons, the total number of varied edges is \(\sum_{k=1}^{N_0(T)} 2\delta_k\), and must necessarily satisfy

\[
\sum_{k=1}^{N_0(T)} 2\delta_k = N_1(T) - N_0(T),
\]

where the \(N_0(T)\) comes from the isoperimetric constraints (64). From Euler relation, we get

\[
\sum_{k=1}^{N_0(T)} \delta_k = N_0(T) + 3g - 3.
\]

Note that \(\sum_{k=1}^{N_0(T)} 2\delta_k\) also provides the maximal dimension of the space of all possible polytopal deformations \(Def\left\{\{p_{eq}(k,a), \delta_k\}_{k=1}^{N_0}\right\}\), i.e.,

\[
\dim \left\{Def\left\{\{p_{eq}(k,a), \delta_k\}_{k=1}^{N_0}\right\}\right\} = 2N_0(T) + 6g - 6,
\]

which, not surprisingly, coincides with the (real) dimension of the moduli space \(\mathcal{M}_{g,N_0}\) of Riemann surfaces of genus \(g\) with \(N_0(T)\) punctures. It is also important to stress that the set of deformation maps \(\{Def\left\{\{p_{eq}(k,a), \delta_k\}_{k=1}^{N_0}\right\}\}\) is defined up to the action of the automorphism group \(Aut_\partial(P_a)\) preserving the
Proposition 8. For any \(|T_{l=a}| \to M \in \mathcal{DT}\{\{q(i)\}_{i=1}^{N_0}\}\) the polytopal deformation space

\[
\mathcal{R}\mathcal{S}[P_{T_a}; \{q(i)\}_{i=1}^{N_0}] = \bigcup_{\{\delta_k\}} \text{Def}\{\{p_{eq}(k,a), \delta_k\}_{k=1}^{N_0}\}
\]

is naturally isomorphic to the space, \(\mathcal{R}\mathcal{S}[T; \{q(i)\}_{i=1}^{N_0}]\), of all Regge triangulations \(\{\left|T_L\right| \to M\}\), which can be obtained by isoperimetrically deforming the equilateral polytopes \(\{|P_{T_a}| \to M\}\) baricentrically dual to the given \(|T_{l=a}| \to M\). Both spaces \(\mathcal{R}\mathcal{S}[P_{T_a}; \{q(i)\}_{i=1}^{N_0}]\) and \(\mathcal{R}\mathcal{S}[T; \{q(i)\}_{i=1}^{N_0}]\) are differentiable orbifolds modelled after \(\frac{\mathbb{R}^{N_1(T)} - N_0(T)}{\text{Aut}_2(P_a)}\), of (maximal) dimension given by

\[
\dim \mathcal{R}\mathcal{S}[P_{T_a}; \{q(i)\}_{i=1}^{N_0}] = \dim \mathcal{R}\mathcal{S}[T; \{q(i)\}_{i=1}^{N_0}] = 2N_0(T) + 6g - 6,
\]

and can be described in terms of the 2-form

\[
\Omega(\mathcal{R}\mathcal{S}[P_{T_a}; \{q(i)\}_{i=1}^{N_0}]) = \sum_{k=1}^{N_0(T)} \left(\frac{\sqrt{3}}{3}\right)^q(k) \varpi(k),
\]

where \(\varpi(k)\) is given by (74).

Note that in describing deformations of dynamical triangulations, one can use indifferently both \(\mathcal{R}\mathcal{S}[P_{T_a}; \{q(i)\}_{i=1}^{N_0}]\) and \(\mathcal{R}\mathcal{S}[T; \{q(i)\}_{i=1}^{N_0}]\). However, form a mathematical point of view, \(\mathcal{R}\mathcal{S}[P_{T_a}; \{q(i)\}_{i=1}^{N_0}]\) is easier to handle since it represents the natural choice for addressing the issue of the Regge measure for the isoperimetric deformations of a given \(|T_{l=a}| \to M\). In this connection, the volume form naturally associated with (82) is

\[
\frac{\Omega(\mathcal{R}\mathcal{S}[P_{T_a}; \{q(i)\}])^{\dim \mathcal{R}\mathcal{S}[P; \{q(i)\}]}^{\dim \mathcal{R}\mathcal{S}[P_{T_a}; \{q(i)\}]}}{\dim \mathcal{R}\mathcal{S}[P_{T_a}; \{q(i)\}]},
\]

(for notational simplicity we omit the superscripts in \(\{q(i)\}_{i=1}^{N_0}\)), it characterizes the set of all local (isoperimetric) \(\{\delta_k\}_{k=1}^{N_0}\) Regge deformations around the given
\( T \in DT[\{q(i)\}_{i=1}^{N_0}] \). Explicitly, we can write (87) as

\[
\frac{1}{(3g-3+N_0(T))!} \left( \sum_{k=1}^{N_0(T)} \left[ \left( \frac{\sqrt{3}}{3} a \right) q(k) \right]^2 \varphi(k) \right)^{3g-3+N_0(T)} = (88)
\]

\[
= \frac{1}{(3g-3+N_0(T))!} \left( \sum_{k=1}^{N_0(T)} \sum_{1\leq \alpha \leq \beta \leq q(k)-1} d(L_\alpha(k)) \wedge d(L_\beta(k)) \right)^{3g-3+N_0(T)}.
\]

If we relabel the edge lengths \( \{ L_\alpha(k) \}_{\alpha=1}^{N_0(T)} \rightarrow \{ L_h \}_{h=1}^{N_1(T)} \) we may wonder about the connection of such a volume form with the Regge-like measure on the set of all local deformations of \( \{ |P_{\eta_{\alpha}}| \rightarrow M \} \), i.e.,

\[
dL_1 \wedge dL_2 \wedge \ldots \wedge dL_{N_1(T)}, \tag{89}
\]

In order to address such a question, let us remark that if \( \eta(k) \) denotes the perimeter of the generic polygon \( p(k) \), then we can write

\[
\eta(k) \doteq \sum_{\alpha=1}^{q(k)} L_\alpha(k) = \sum_j A_{(k)}^j L_j \tag{90}
\]

where \( A_{(k)}^j \) is a \((0, 1)\) indicator matrix with \( A_{(k)}^j = 1 \) if the edge associated with \( L_j \) belongs to \( p(k) \), and 0 otherwise. Thus, isoperimetric deformations, \( i.e., \eta(k) = (\frac{\sqrt{3}}{3} a)q(k) \), of the \( N_1(T) \) edge lengths \( \{ L_j \} \) are necessarily subjected to the \( N_0(T) \) linear constraints

\[
\left\{ \sum_j A_{(k)}^j L_j = (\frac{\sqrt{3}}{3} a)q(k) \right\}_{k=1}^{N_0(T)} \tag{91}
\]

which, (since each edge is shared between two polygons), imply

\[
2 \sum_{j=1}^{N_1(T)} L_j = (\frac{\sqrt{3}}{3} a) \sum_{k=1}^{N_0(T)} q(k) = (\frac{\sqrt{3}}{3} a) b(n, n-2)|_{n=2N_0(T)} = \tag{92}
\]

\[
= 2\sqrt{3}a \left( N_0(T) + 4g - 4 \right),
\]

where \( b(n, n-2)|_{n=2} \) is the average incidence of the dynamical triangulation associated with \( |P_{\eta_{\alpha}}| \rightarrow M \) and where we have used (7). Thus, under the isoperimetric constraints (91), the possible edge-lengths \( \{ L_j \} \) are necessarily restricted to a subspace \( \Delta(P_T, \{ q(k) \}) \) of the \( (N_1(T)-1) \)-dimensional simplex

\[
\Delta_{N_1}(g, N_0) = \left\{ \{ L_j \} \in \mathbb{R}^{N_1(T)} : \sum_{j=1}^{N_1(T)} L_j = \sqrt{3}a \left( N_0(T) + 4g - 4 \right) \right\}. \tag{93}
\]
The subspace $\Delta(P_T, \{q(k)\})$ is the $(N_1 - N_0)$-dimensional subsimplex of defined by

$$\Delta(P_T, \{q(k)\}) \doteq \left\{ \{L_j\} : \sum_{j=1}^{N_1(T)} L_j = \sqrt{3} a \left( N_0(T) + 4g - 4 \right), \sum_j A^j_{(k)} L_j = \left( \frac{\sqrt{3}}{3} a \right) q(k) \right\}. \quad (94)$$

Note that while $\Delta_{N_1}(g, N_0)$ depends only from the fixed parameter $N_0(T)$ and the genus $g$, the incidence structure of the subsimplex $\Delta(P_T, \{q(k)\})$ explicitly depends from the given dynamical triangulation $|T_a| \to M$.

This is a convenient moment for recalling the following remarkable identity proved by Kontsevich (lemma 3.1 and appendix C of [1]),

$$\prod_{k=1}^{N_0(T)} d\eta(k) \wedge \left( \sum_{k=1}^{N_0(T)} \sum_{1 \leq \alpha \leq \beta \leq q(k) - 1} d(L_\alpha(k)) \wedge d(L_\beta(k)) \right)^{3g - 3 + N_0(T)} = 2^{2N_0(T) + 5g - 5}(3g - 3 + N_0(T)!dL_1 \wedge dL_2 \wedge \ldots \wedge dL_{N_1(T)}), \quad (95)$$

according to which we can write

$$\frac{\Omega(\mathcal{R}\mathcal{E}[P_{T_a}; \{q(i)\}])^{\dim \mathcal{R}\mathcal{E}[P_{T_a}; \{q(i)\}]} d\eta(k, t)}{\dim \mathcal{R}\mathcal{E}[P_{T_a}; \{q(i)\}]!} \prod_{k=1}^{N_0(T)} d\eta(k, t) = 2^{2N_0(T) + 5g - 5}dL_1 \wedge dL_2 \wedge \ldots \wedge dL_{N_1(T)}. \quad (96)$$

If we restrict the above volume form to the subsimplex $\Delta(P_T, \{q(k)\})$, then we immediately get the

**Lemma 9** The volume form $[\mathcal{S}]$ is the Regge volume form restricted to the set of Regge polytopes $\mathcal{R}\mathcal{E}[P_{T_a}; \{q(i)\}]_{i=1}^{N_0}$, i.e.,

$$\frac{\Omega(\mathcal{R}\mathcal{E}[P_{T_a}; \{q(i)\}])^{\dim \mathcal{R}\mathcal{E}[P_{T_a}; \{q(i)\}]} d\eta(k, t)}{\dim \mathcal{R}\mathcal{E}[P_{T_a}; \{q(i)\}]!} = 2^{2N_0(T) + 5g - 5}dL_1 \wedge dL_2 \wedge \ldots \wedge dL_{N_1(T)} \{\Delta(P_T; \{q(k)\})\}. \quad (97)$$

Thus $[\mathcal{S}]$, (up to the fixed normalization factor), represents the natural measure for the set of Regge polytopes that can be obtained by isoperimetrically varying $\delta_1 \geq 0$ edges of the $q(1)$-gone dual to $\sigma^0(1) \in |T_{l=a}| \to M$, then $\delta_2 \geq 0$ edges of the $q(2)$-gone dual to $\sigma^0(2) \in |T_{l=a}| \to M$, and so on.

### 3.5 Regge surfaces and 2D dynamical triangulations as moduli space duals

Since the space of polytopal deformations $Def[q(k,a); T_{\{q(i)\}}^{N_0}]$ around any dynamical triangulation $T \in DT[q(i)]^{N_0}$ is an orbifold locally modelled on
\[ \mathbb{R}^{N(T) - N_0(T)}/\text{Aut}_\beta(P_T), \] we can naturally define an orbifold integration of any (sufficiently regular) functional \( f(\{ L_{\alpha}(k); T \}) \) of the edge-lengths of a \( |T_i| \rightarrow M \in \mathfrak{R}[T; \{ q(i) \}_{i=1}^{N_0}] \) according to

\[
\int_{\Delta(P, \{ q(i) \})} f(\{ L_{\alpha}(k); T \}) = \sum_{T \in \mathcal{D}T[\{ q(i) \}_{i=1}^{N_0}] \left| \text{Aut}_\beta(P_T) \right|} \frac{1}{\left| \Delta(P, \{ q(i) \}) \right|} f(\{ L_{\alpha}(k); T \}),
\]

where the summation is over all distinct dynamical triangulations with given curvature assignments weighted by the order \( |\text{Aut}_\beta(P_T)| \) of the automorphisms group of the corresponding dual polytope. In particular, we can integrate over \( \mathcal{D}T[\{ q(i) \}_{i=1}^{N_0}] \) the volume form \( \Omega \). In this way we can define the volume of all metrical fluctuations around the set \( \mathcal{D}T[\{ q(i) \}_{i=1}^{N_0}] \) by means of a duality pairing between Dynamical Triangulations and Regge polytopes,

\[
\langle \langle \mathcal{D}T[\{ q(i) \}] \mid \mathfrak{R}[P_T; \{ q(i) \}] \rangle \rangle = \int_{\mathcal{D}T[\{ q(i) \}]} \frac{\Omega[\mathfrak{R}[P_T; \{ q(i) \}]]}{\dim \mathfrak{R}[P_T; \{ q(i) \}]!}.
\]

Explicitly, by developing the integrand in \( \langle \langle \mathcal{D}T[\{ q(i) \}] \mid \mathfrak{R}[P_T; \{ q(i) \}] \rangle \rangle \), we get

\[
\langle \langle \mathcal{D}T[\{ q(i) \}] \mid \mathfrak{R}[P_T; \{ q(i) \}] \rangle \rangle = \sum_{\delta_i = 2N_0(T)+6g-6}^{N_0(T)} \prod_{i=1}^{N_0(T)} \left[ \frac{\left( \frac{\Delta}{3} \right)^{\delta_i} q(i)}{\delta_i!} \right] \left\{ \sum_{T \in \mathcal{D}T[\{ q(i) \}]} \frac{1}{\left| \text{Aut}_\beta(P_T) \right|} \prod_k \omega(k)^{\delta_k} \right\},
\]

which, according to the remarks of the previous section, can also be written as

\[
\langle \langle \mathcal{D}T[\{ q(i) \}] \mid \mathfrak{R}[P_T; \{ q(i) \}] \rangle \rangle = \sum_{T \in \mathcal{D}T[\{ q(i) \}]} \frac{2^{2N_0(T)+5g-5}}{\left| \text{Aut}_\beta(P_T) \right|} \int_{\Delta(P, \{ q(i) \})} dL_1 \wedge dL_2 \wedge \ldots \wedge dL_{N_1(T)} \left[ \Delta(P, \{ q(i) \}) \right].
\]

The integral

\[
\int_{\Delta(P, \{ q(i) \})} dL_1 \wedge dL_2 \wedge \ldots \wedge dL_{N_1(T)} \left[ \Delta(P, \{ q(i) \}) \right] = \text{Vol} \left[ \Delta(P, \{ q(i) \}) \right]
\]

is the volume of the \( (N_1 - N_0) \)-dimensional simplex \( \Delta(P, \{ q(i) \}) \), and represents the Regge volume of the isoperimetrical metrical fluctuations \( D\mathfrak{mf}[\{ p_{eq}(k, a), \delta_k \}] \)
around the equilateral polytope $|P_{Ta}| \to M$ associated with the given triangulation. Thus, we can write

$$\langle \Delta DT[[q(i)]] \parallel \mathfrak{M} [P_{Tu}; \{q(i)\}] \rangle =$$

$$= \sum_{T \in DT[[q(i)]]} \frac{2^{2N_0(T)+5g-5}}{|Aut_\theta(P_{Tu})|} Vol[\Delta(P_T, \{q(i)\})] =$$

$$= \left[ \sum_{\delta_i; \sum \delta_i = 2N_0(T)+6g-6} \prod_{i=1}^{N_0(T)} \left( \frac{(\sqrt[2]{a}) q(i)}{\delta_i!} \right)^{2\delta_i} \right] \times \left[ \prod_{T \in DT[[q(i)]]} \frac{1}{|Aut_\theta(P_T)|} \int_{\Delta(P_T, \{q(i)\})} \prod_k \omega(k)^{\delta_i(k)} \right].$$

The distinctive feature of the duality pairing [10] is that it provides a connection between the measure of the isoperimetrical Regge fluctuations and the topology of moduli space. In other words, the metrical fluctuations in $DT[[q(i)]]$ (represented by the distinct $|T_{i=a}| \to M$ which comprise such a space) and the Regge metrical fluctuations represented by $Vol[\Delta(P_T, \{q(i)\})]$ are not independent, but are related to the topology of the moduli space of genus $g$ Riemann surfaces with $N_0$ punctures $\overline{\varpi}_{g,N_0}$. This remark is obviously a direct consequence of the fact that our analysis of the interplay between dynamical triangulations and Regge triangulations is strictly connected to Kontsevich's characterization [1] of the intersection numbers $\langle \tau_{\delta_1} ... \tau_{\delta_k} \rangle$ over the moduli space $\overline{\varpi}_{g,\lambda}$ (see [10]). We get

**Proposition 10** If $\langle \tau_{\delta_1} ... \tau_{\delta_{N_0}} \rangle$ denote the Witten-Kontsevich intersection numbers over the (compactified) moduli space of genus $g$ Riemann surfaces with $N_0$ punctures $\overline{\varpi}_{g,N_0}$, then

$$2^{2N_0(T)+5g-5} \sum_{T \in DT[[q(i)]]} \frac{Vol[\Delta(P_T, \{q(i)\})]}{|Aut_\theta(P_{Tu})|} =$$

$$= \sum_{\delta_i; \sum \delta_i = 2N_0(T)+6g-6} \prod_{i=1}^{N_0(T)} \left( \frac{(\sqrt[2]{a}) q(i)}{\delta_i!} \right)^{2\delta_i} \langle \tau_{\delta_1} ... \tau_{\delta_{N_0}} \rangle.$$

This immediately follows from the combinatorial description of $\overline{\varpi}_{g,N_0}$ in terms of ribbon graphs provided by Kontsevich [1], (see also [5]). In particular, if $f : \mathcal{M}_{g,N_0,comb} \to \mathbb{R}^N_{+}$ denotes the projection which associates to a (Jenkins-Strebel) ribbon graphs (i.e., the ribbon graph associated with the Regge polytope $|P_{Ta}| \to M$) the sequence of perimeters of the corresponding $q(k)$-gons, then according to [1] we have

$$\langle \tau_{\delta_1} ... \tau_{\delta_{N_0}} \rangle = \int_{f^{-1}(a(\sqrt[2]{a}) q(k))}^{orb} \prod_{k} \omega(k)^{\delta_k},$$

(105)
where \( \int_{\text{orb}} \) denotes an orbifold integration similar to (98), and where \( \hat{\omega}(k) \) is the pull-back of the 2-form (76) to \( \mathfrak{M}_{g,N_0} \times \mathbb{R}^{N_0} \) under the map which assigns the edge-lengths to the boundary components of Jenkins-Strebel differentials.

It is easily verified that \( \hat{\omega}(k) \) and \( \varpi(k) \) are cohomologous and that by definition of orbifold integration

\[
\int_{\text{orb}} f^{a(\sqrt{3} 3) q(k)} N_0(T) \prod_k \hat{\omega}(k) = \sum_{T \in \mathcal{DT}([q(i)])} \frac{1}{|\text{Aut}_0(P_{T_a})|} \int_{\Delta(P_{T_a})} N_0(T) \prod_k \varpi(k)^{\delta_k},
\]

from which the stated result follows. □

Thus, we get

\[
2^{2N_0(T)+5g-5} \sum_{T \in \mathcal{DT}([q(i)])} \frac{\text{Vol} [\Delta(P_T, \{q(i)\})]}{|\text{Aut}_0(P_{T_a})|} = \frac{(1/3 a^2)^{N_0+3g-3} F_g(q(1), q(2), \ldots)}{\text{Card} \left[ \mathcal{DT} \left[ \{q(i)\} \right] \right]},
\]

where

\[
F_g(q(1), q(2), \ldots) = \sum_{\{\delta_i\}} \prod_{i=1}^{N_0(T)} \frac{q(i)^{2\delta_i}}{\delta_i!} \langle \tau_{\delta_1} \cdots \tau_{\delta_{N_0}} \rangle,
\]

is the exponential generating function of the intersection numbers on \( \mathfrak{M}_{g,N_0} \) [6], [1]. From (107) we get a non-trivial connection between the Euclidean volumes of the simplices \( \Delta(P_T, \{q(i)\}) \) and intersection theory on \( \mathfrak{M}_{g,N_0} \). Since \( \mathfrak{M}_{g,N_0} \) is locally modelled on the combinatorial orbifold \( \mathbb{R}^{N_0(T)-N_0(T)}/\text{Aut}_0(P_T) \), it is natural to interpret the Regge volumes

\[
\frac{\text{Vol} [\Delta(P_T, \{q(i)\})]}{|\text{Aut}_0(P_{T_a})|}
\]

as the the volumes of the open strata in \( \mathfrak{M}_{g,N_0} \), each stratum being labelled by the distinct dynamical triangulations \( T \in \mathcal{DT}([q(i)]) \). Such an interpretation gives to the (seemingly trivial) isoperimetrical Regge measure [17] a deep geometrical meaning which, a priori, is totally unexpected. From a physical point of view the relation (107) also shows that the average volume \( \langle \text{Vol} [\Delta(P_T, \{q(i)\})] \rangle \), (the average being over the discrete set \( \mathcal{DT}([q(i)]) \)), is such that

\[
\langle \text{Vol} [\Delta(P_T, \{q(i)\})] \rangle \text{ Card} \left[ \mathcal{DT} \left[ \{q(i)\}_{i=1}^{N_0} \right] \right] = \frac{(1/3 a^2)^{N_0+3g-3}}{2^{2N_0(T)+5g-5} F_g(q(1), q(2), \ldots)}.
\]

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where $\text{Card} \left[ DT \left[ \{ q(i) \} \right]_{i=1}^{N_0} \right]$ denotes the number of distinct dynamical triangulations with given curvature assignments $\in DT \left[ \{ q(i) \} \right]_{i=1}^{N_0}$. This latter cardinality comprises an important part of the total triangulations counting, i.e.,

$$\text{Card} \left[ DT \right]_{N_0} = \sum_{\{ q(i) \} \}_{i=1}^{N_0} \text{Card} \left[ DT \left[ \{ q(i) \} \right]_{i=1}^{N_0} \right], \quad (111)$$

where the summation extends over all possible curvature assignments for triangulations (with $N_0$ vertices) over a surface of given topology, (e.g., see [11] and references cited therein), and where $DT \left[ N_0 \right]$ denotes the set of distinct dynamical triangulations with $N_0(T)$ marked vertices admitted by a surface of genus $g$. It is well known that, for large $N_0(T)$, we get

$$\text{Card} \left[ DT \right]_{N_0} \sim N_0(T)^{\gamma_g + N_0 - 3} e^{\mu_0 N_0} \left( 1 + O \left( \frac{1}{N_0} \right) \right), \quad (112)$$

where

$$\gamma_g = \frac{5g - 1}{2}, \quad (113)$$

is the genus-$g$ pure gravity critical exponent, and $\mu_0$ is a (non-universal) parameter independent of $g$ and $N_0$. Thus, (110) expresses a topological duality between a relevant part of the natural counting measure for dynamical triangulation theory and the (isoperimetrical) Regge measure one naturally associates with Regge (polytopal) surfaces. It tells us that intersection theory on $\mathbb{M}_{g,N_0}$ (hence topological 2D gravity) comes from a balance between the entropy $\text{Card} \left[ DT \left[ \{ q(i) \} \right]_{i=1}^{N_0} \right]$ and the typical Regge deformation volume $\langle Vol \left( \Delta(P_T, \{ q(i) \}) \right) \rangle$. Roughly speaking, this reflect the fact that the smaller the typical volume of the strata the higher the number of representative points $| T \in a | \to M$ needed in order to combinatorially approximate $\mathbb{M}_{g,N_0}$. If we define

$$A[T; \{ q(j) \}] = \ln Vol \left[ \Delta(P_T, \{ q(i) \}) \right], \quad (114)$$

then (107) implies that

$$\sum_{T \in DT \left[ N_0 \right]} \frac{1}{| \text{Aut}_0 \left( P_T \right) |} e^{A[T; \{ q(j) \}]} = \left( \frac{4g^2}{2^{2N_0(T)+5g-5}} \right) \sum_{\{ q(i) \} \}_{i=1}^{N_0} F_g (q(1), q(2), ...). \quad (115)$$

Thus, the free energy $A[T; \{ q(j) \}]$ associated with the Regge measure plays against the entropic factor $\text{Card} \left[ DT \right]_{N_0}$ so as to give rise to what is basically a topological quantity. From a physical point of view, such a mechanism, *a la Peierls*, indicates that 2D dynamical triangulations and Regge calculus are related by a form of topological S-duality. Clearly the physical relevance of this
latter remark is biased by the fact that we are explicitly considering here an isoperimetric family of Regge measures and its relation with the standard measure [12], [13] it is not clear. (Notice however that the isoperimetric constraint can be removed by integrating over the perimeters \( \{ \eta(k) \} \) still maintaining, according to Kontsevich’s lemma, a Regge measure structure). One may also stress the fact that, from a deformation theory point of view, we get (87) as a unique choice. Perhaps, this unicity might help in shedding light on the controversial issue of the most appropriate choice of a measure in quantum Regge calculus and related models (see, e.g., [14]). Another challenging open problem is to exploit such a duality to compute the scaling behavior of \( \text{Vol} [\Delta(P_T, \{q(i)\})] \) and the associated critical exponents. This would provide an interesting way of estimating volumes of open strata in \( \mathcal{M}_{g,N_0} \).

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