On the $\mathcal{U}_q[sl(2)]$ Temperley–Lieb reflection matrices

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Abstract. This work concerns the boundary integrability of the spin-$s\mathcal{U}_q[sl(2)]$ Temperley–Lieb model. A systematic computation method is used to construct the solutions of the boundary Yang–Baxter equations. For $s$ half-integer, a general $2s(s + 1) + 3/2$ free parameter solution is presented. It turns that for $s$ integer, the general solution has $2s(s + 1) + 1$ free parameters. Moreover, some particular solutions are discussed.

Keywords: algebraic structures of integrable models, integrable spin chains (vertex models), solvable lattice models

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1. Introduction

The search for integrable models through solutions of the Yang–Baxter equation [1]–[3]

\[ R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v) \] (1.1)

has been performed by the quantum group approach in [4], where the problem is reduced
to a linear one. Indeed, the \( R \) matrices corresponding to vector representations of all
non-exceptional affine Lie algebras have been determined in this way by Jimbo [5].

A similar approach is desirable for finding solutions of the boundary Yang–Baxter
equation [6, 7] where the boundary weights follow from \( K \) matrices which satisfy a pair of
equations, namely the reflection equation

\[ R_{12}(u - v)K_1^-(u)R_{12}^{t_2}(u + v)K_2^-(v) = K_2^-(v)R_{12}(u + v)K_1^-(u)R_{12}^{t_2}(u - v) \] (1.2)

and the dual reflection equation

\[ R_{12}(-u + v)(K_1^+)^{t_1}(u)M_1^{-1}R_{12}^{t_2}(-u - v - 2\rho)M_1(K_2^+)^{t_2}(v) \]

\[ = (K_2^+)^{t_2}(v)M_1R_{12}(-u - v - 2\rho)M_1^{-1}(K_1^+)^{t_1}(u)R_{12}^{t_2}(u + v). \] (1.3)

In this case duality supplies a relation between \( K^- \) and \( K^+ \) [8]

\[ K^+(u) = K^-(u - \rho)^t M, \quad M = V^t V. \] (1.4)

Here \( t \) denotes transposition and \( t_i \) denotes transposition in the \( i \)th space. \( V \) is the crossing
matrix and \( \rho \) the crossing parameter, both being specific to each model [9].

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With this goal in mind, the study of boundary quantum groups was initiated in [10]. These boundary quantum groups have been used to determine the $A_{1}^{(1)}$ reflection matrices for arbitrary spin [11], and the $A_{2}^{(2)}$ and some $A_{n}^{(1)}$ reflection matrices were derived again in [12]. Reflection solutions from $R$-matrices corresponding to vector representations of Yangians and super-Yangians were presented in [13]. However, as observed by Nepomechie [12], an independent systematic method of constructing the boundary quantum group generators is not yet available. In contrast to the bulk case [5], one cannot exploit boundary affine Toda field theory, since appropriate classical integrable boundary conditions are not yet known [14]. Therefore, it is still an open question whether it is possible to find all solutions of the reflection equations by using quantum group generators.

Independently, there has been an increasing amount of effort towards the understanding of two-dimensional integrable theories with boundaries via solutions of the reflection equation (1.2). In field theory, attention is focused on the boundary $S$ matrix [15,16]. In statistical mechanics, the emphasis has been laid on deriving all solutions of (1.2) because different $K$-matrices lead to different universality classes of surface critical behavior [17] and allow the calculation of various surface critical phenomena, both at and away from criticality [18].

Although being a hard problem, the direct computation has been used to derive the solutions of the boundary Yang–Baxter equation (1.2) for given $R$. For instance, we mention the solutions with the $R$ matrix based in non-exceptional Lie algebras [19,20] and superalgebras [21,22]. The regular $K$-matrices for the exceptional $U_{q}[G_{2}]$ vertex model were obtained in [23]. Many diagonal solutions for face and vertex models associated with affine Lie algebras were presented in [18]. For A–D–E interaction-round face (IRF) models, diagonal and some non-diagonal solutions were presented in [24]. Reflection matrices for Andrews–Baxter–Forrester models in the RSOS/SOS representation were presented in [25]. Apart from these $c$-number solutions of the reflection equations there must also exist non-trivial solutions that include the boundary degree of freedom, as were derived for the sine–Gordon theory in [26].

Here we will again touch upon this issue in order to include the Temperley–Lieb lattice models [27] arising from the quantum group $U_{q}[sl(2)]$ [28].

We have organized this paper as follows. In section 2 the model is presented, in section 3 we choose the reflection equations and their solutions. In section 4 some reduced solutions are derived, while section 5 is reserved for the conclusion.

2. The model

The Temperley–Lieb algebra is very useful in the study of two-dimensional lattice statistical mechanics. It provided an algebraic framework for constructing and analyzing different types of integrable lattice models, such as the $Q$-state Potts model, IRF model, $O(n)$ loop model, six vertex model, etc [29].

From the representation of the Temperley–Lieb algebra [28], one can build solvable vertex models with the $R$ operator defined by

$$R(u) = \frac{\sinh(\eta - u)}{\sinh\eta} I + \sqrt{Q} \frac{\sinh u}{\sinh\eta} P_0$$ (2.1)
where $I$ is the identity operator and $P_0$ a suitable projector. Here $u$ is the spectral parameter and the anisotropic parameter $\eta$ is chosen so that

$$2 \cosh \eta = \sqrt{Q}. \quad (2.2)$$

For the spin-$sU_q[sl(2)]$ model, $I$ is the $(2s + 1)^2$ by $(2s + 1)^2$ unity matrix and $\sqrt{Q}P_0 = U$ is the spin zero operator

$$\langle i, j | U | k, l \rangle = (-1)^{i+k} q^{i+k} \delta_{i+k,0} \delta_{k+l,0} \quad \{i, j, k, l\} = -s, -s + 1, \ldots, s \quad (2.3)$$

and $\sqrt{Q} = [2s + 1] = q^{2s} + q^{2(s-1)} + \cdots + q^{-2s}$.

The Hamiltonian limit

$$R(u) = I + u(\alpha^{-1} \mathcal{H} + \beta \mathcal{I}) \quad (2.4)$$

with $\alpha = \sinh \eta$, $\beta = -\coth \eta$ leads to the quantum spin chains

$$\mathcal{H} = \sum_{k=1}^{N-1} U_{k,k+1} + bt \quad (2.5)$$

where, instead of periodic boundary condition, we are taking into account the existence of integrable boundary terms $bt$ [7], derived from the $K^-$ and $K^+$ matrices presented in the next sections.

We also have to consider the permuted operator $R = PR$ that is regular, satisfying $PT$-symmetry, unitarity and crossing symmetry,

$$R_{12}^t(0) = P, \quad R_{12}^{t_1t_2}(u) = PR_{12}(u)P = R_{21}(u),$$

$$R_{12}(u)R_{12}^{t_1t_2}(-u) = x_1(u)x_1(-u)I, \quad R_{21}(u) = (-1)^{2s}(V \otimes 1)R_{12}^{t_2}(-u - \rho)(V \otimes 1)^{-1} \quad (2.6)$$

where $\rho = -\eta$ is the crossing parameter and $V$ is the crossing matrix, specified by

$$V_{i,j} = (-1)^{i-1} q^{s+1-i} \delta_{i,2s+2-j} \quad (2.7)$$

and $P$ is the permutation matrix $\langle i, j | P | k, l \rangle = \delta_{i,l} \delta_{j,k}$.

### 3. The reflection matrices

The reflection equation (1.2) where $K^-_1 = K^- \otimes I$, $K^-_2 = I \otimes K^-$, $R_{12} = R$ and $R_{12}^{t_1t_2} = PRP$, $I$ and $K^-$ are $(2s+1)$ by $(2s+1)$ matrices satisfying the normal condition $K^-(0) = I$. Substituting

$$K^-(u) = \sum_{i,j=1}^{2s+1} k_{i,j}(u)E_{i,j} \quad (3.1)$$

where $(E_{i,j})_{k,l} = \delta_{i,k} \delta_{j,l}$ are the Weyl matrices and $R(u) = P[x_1(u)I + x_2(u)U]$ with

$$x_1(u) = \frac{\sinh(\eta - u)}{\sinh \eta}, \quad x_2(u) = \frac{\sinh u}{\sinh \eta} \quad (3.2)$$

into (1.2), we will have $(2s+1)^4$ functional equations for the $k_{i,j}$ elements, many of them not independent equations. In order to solve these functional equations, we shall

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Finally, we can use the equation (1) and where $k$ we are thus left with several equations involving two diagonal elements and $M$. Here we note that trace of the matrix $\beta - \rho v$, we get algebraic equations involving the single variable $u$ and $(2s + 1)^2$ parameters

$$\beta_{i,j} = \frac{d k_{i,j}(v)}{dv} \bigg|_{v=0}, \quad i, j = 1, 2, \ldots, 2s + 1. \quad (3.3)$$

Analyzing the reflection equations one can see that they possess a special structure. Several equations exist involving only two non-diagonal elements. They can be solved by the relations

$$k_{i,j}(u) = \frac{\beta_{i,j}}{\beta_{1,2s+1}} k_{1,2s+1}(u) \quad (i \neq j = \{1, 2, \ldots, 2s + 1\}). \quad (3.4)$$

We are thus left with several equations involving two diagonal elements and $k_{1,2s+1}(u)$. Such equations are solved by the relations

$$k_{i,i} = k_{1,1}(u) + (\beta_{i,i} - \beta_{1,1}) \frac{k_{1,2s+1}(u)}{\beta_{1,2s+1}} \quad (i = 2, 3, \ldots, 2s + 1). \quad (3.5)$$

Finally, we can use the equation $(1, 2s + 1)$ in order to find the element $k_{1,1}(u)$:

$$k_{1,1}(u) = \frac{k_{1,2s+1}(u)}{\beta_{1,2s+1}} \left\{ \frac{x_1(u)x_2'(x)}{x_2(u)} - \frac{1}{2} \frac{x_1(u)}{\beta_{1,2s+1}} \right. \left[ \sum_{j=2}^{2s} \beta_{1,j} \beta_{j,2s+1} - \frac{1}{2} x_2(u) \sum_{j=2}^{2s} (\beta_{j,j} - \beta_{1,1}) q^{2j-2s-2} - \frac{1}{2} x_1(u) + q^{2s} x_2(u) \right] \right\} \quad (3.6)$$

where $x_i'(u) = dx_i(u)/du$. After these steps, we can write all matrix elements in terms of $k_{1,2s+1}(u)$.

Now, substituting these expressions into the remaining equations $(i, j)$, we are left with several constraint equations involving the $\beta_{i,j}$ parameters.

First, we consider the blocks of four equations [30]

$$B[j, 2s] = \{(j, 2s), (2s, j), (j'', (2s)''), (2s)'', j'')\} \quad (3.7)$$

where $i'' = (2s + 1)^2 + 1 - i$, in order to fix the 2s diagonal parameters $\beta_{i,j}$, $j = 2, 3, \ldots, 2s$ and $\beta_{2s+1,2s+1}$ (the parameter $\beta_{1,1}$ is fixed by the normal condition).

Subsequently, we are embroiled with too large a task, which consists of finding some parameters $\beta_{i,j}$ in terms of the free parameters. For the spin-$s U_q[sl(2)]$ Temperley–Lieb model, the number of free parameters is so great that we need excessive computer resources in order to express the fixed parameters in terms of them.

The dual equation (1.3) is solved by the $K^+$ matrices via the isomorphism (1.4) with $\rho = -\eta$ and the matrix $M$ specified by

$$M_{i,j} = q^{-2(s+1-i)} \delta_{i,j}. \quad (3.8)$$

Here we note that trace of the matrix $M$ is equal to $2 \cosh \eta$.

Now, we explicitly show these computations for the first cases. Firstly, let us make the choice

$$k_{1,2s+1}(u) = \frac{\beta_{1,2s+1} x_2(u)x_2'(x) \cosh \eta + x_1(u)}{x_1(u)x_2'(x) - x_1'(u)x_2(u)} = \frac{1}{2} \beta_{1,2s+1} \sinh(2u) \quad (3.9)$$

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in order to simplify our presentation. From the general solution (3.4) to (3.6) one can see
$k_{1,2s+1}(u)$ as an arbitrary function satisfying the normal condition $K^-(0) = I$. Therefore,
the choice (3.9) does not imply any restriction as compared to the general case.

3.1. The $\mathcal{U}_q[sl(2)]$ Temperley–Lieb $K$ matrix for $s = \frac{1}{2}$

For the case $s = \frac{1}{2}$ we have the well-known three free parameter solution [15]:

$$
K^-(u) = \begin{pmatrix}
\frac{1}{2} \beta_{11}(u) & \frac{1}{2} \beta_{12} \sinh(2u) & \frac{1}{2} \beta_{13} \sinh(2u) \\
\frac{1}{2} \beta_{21} \sinh(2u) & \frac{1}{2} \beta_{11}(u) + \frac{1}{2} (\beta_{22} - \beta_{11}) \sinh(2u) & \frac{1}{2} \beta_{23} \sinh(2u) \\
\frac{1}{2} \beta_{31} \sinh(2u) & \frac{1}{2} \beta_{32} \sinh(2u) & \frac{1}{2} \beta_{11}(u) + \frac{1}{2} (\beta_{33} - \beta_{11}) \sinh(2u)
\end{pmatrix}.
$$

(3.10)

Using (3.9), the expression (3.6) has a simplified form

$$
k_{1,1}(u) = 1 - \frac{1}{2}(\beta_{2,2} - \beta_{1,1})[x_1(u) + qx_2(u)]x_2(u) \sinh \eta
$$

(3.11)

where $\beta_{1,2}$, $\beta_{2,1}$ and $\beta_{2,2}$ are the free parameters and $2 \cosh \eta = q + q^{-1}$.

3.2. The $\mathcal{U}_q[sl(2)]$ Temperley–Lieb $K$ matrix for $s = 1$

For the biquadratic model [31, 32], it follows from (3.4) and (3.5) that

$$
K^{-}(u) = \begin{pmatrix}
k_{1,1}(u) & \frac{1}{2} \beta_{12} \sinh(2u) & \frac{1}{2} \beta_{13} \sinh(2u) \\
\frac{1}{2} \beta_{21} \sinh(2u) & \frac{1}{2} \beta_{11}(u) + \frac{1}{2} (\beta_{22} - \beta_{11}) \sinh(2u) & \frac{1}{2} \beta_{23} \sinh(2u) \\
\frac{1}{2} \beta_{31} \sinh(2u) & \frac{1}{2} \beta_{32} \sinh(2u) & \frac{1}{2} \beta_{11}(u) + \frac{1}{2} (\beta_{33} - \beta_{11}) \sinh(2u)
\end{pmatrix}
$$

(3.12)

where $k_{1,1}(u)$ is given by (3.6),

$$
k_{1,1}(u) = 1 - \frac{1}{2} \left\{ (\beta_{3,3} - \beta_{1,1}) [x_1(u) + q^2x_2(u)] + \frac{\beta_{1,2} \beta_{2,3}}{\beta_{1,3}}x_1(u) \\
+ (\beta_{2,2} - \beta_{1,1})x_2(u) \right\} x_2(u) \sinh \eta.
$$

(3.13)

The diagonal parameters are fixed by the constraint equations (3.7)

$$
\beta_{2,2} = \frac{\beta_{1,1}}{\beta_{1,3}} + \frac{\beta_{1,2} \beta_{2,3}}{\beta_{1,3}} - \frac{\beta_{21} \beta_{13}}{\beta_{23}}, \quad \beta_{3,3} = \beta_{1,1} + \frac{\beta_{1,3} \beta_{3,2}}{\beta_{1,2}} - \frac{\beta_{21} \beta_{13}}{\beta_{2,3}},
$$

(3.14)

and $\beta_{1,1}$ is fixed by the normal condition. Moreover, all the remaining constraint equations are solved by the relation

$$
\beta_{3,1} = \beta_{3,2} \beta_{2,1} \frac{\beta_{1,3}}{\beta_{1,2} \beta_{2,3}}.
$$

(3.15)

And we have obtained a five free parameter solution. Here $2 \cosh \eta = q^{-2} + 1 + q^2$ and we
made the choice $\beta_{1,2}$, $\beta_{1,3}$, $\beta_{2,1}$, $\beta_{2,3}$ and $\beta_{3,2}$ for the free parameters.
3.3. The $\mathcal{U}_q[sl(2)]$ Temperley–Lieb $K$ matrix for $s = \frac{3}{2}$

For $s = \frac{3}{2}$, we have from (3.4) to (3.6) the following non-diagonal entries

$$k_{i,j}(u) = \frac{1}{2} \beta_{i,j} \sinh(2u), \quad (i \neq j = 1-4)$$

(3.16)

and the diagonal one

$$k_{i,i}(u) = k_{1,1}(u) + \frac{1}{2} (\beta_{i,i} - \beta_{1,1}) \sinh(2u), \quad (i = 2-4)$$

(3.17)

with

$$k_{1,1}(u) = 1 - \frac{1}{2} \left\{ (\beta_{2,4} - \beta_{1,1}) [x_1(u) + q^3 x_2(u)] + \frac{\beta_{1,2} \beta_{2,4} + \beta_{1,3} \beta_{3,4}}{\beta_{1,4}} x_1(u) \right.$$ 

$$+ \left[ (\beta_{2,2} - \beta_{1,1}) q^{-1} + (\beta_{3,3} - \beta_{1,1}) q \right] x_2(u) \right\} x_2(u) \sinh \eta.$$  

(3.18)

From the block $B[2,3]$ (3.7), we choose to fix the following diagonal parameters:

$$\beta_{2,2} = \beta_{1,1} + \frac{\beta_{1,2} \beta_{2,4} + \beta_{1,3} \beta_{3,4}}{\beta_{1,4}} - \frac{\beta_{2,1} \beta_{1,4} + \beta_{2,3} \beta_{3,4}}{\beta_{2,4}},$$

$$\beta_{3,3} = \beta_{1,1} + \frac{\beta_{1,2} \beta_{2,4} + \beta_{1,3} \beta_{3,4}}{\beta_{1,4}} - \frac{\beta_{3,1} \beta_{1,4} + \beta_{3,2} \beta_{2,4}}{\beta_{3,4}},$$

$$\beta_{4,4} = \beta_{1,1} + \frac{\beta_{1,2} \beta_{2,4} + \beta_{1,3} \beta_{3,4}}{\beta_{1,3}} - \frac{\beta_{3,1} \beta_{1,4} + \beta_{3,2} \beta_{2,4}}{\beta_{3,4}}.$$  

(3.19)

All the remaining constraint equations are solved by the choice

$$\beta_{4,1} = \frac{\beta_{2,4} \beta_{2,1} + \beta_{1,3} \beta_{3,1}}{\beta_{1,2} \beta_{2,4} + \beta_{1,3} \beta_{3,4}} \beta_{1,4}, \quad \beta_{3,2} = \frac{-\beta_{3,1} \beta_{1,2} + \beta_{3,4} \beta_{4,2}}{\beta_{1,2} \beta_{2,4} + \beta_{1,3} \beta_{3,4}} \beta_{1,4},$$

$$\beta_{2,3} = -\frac{\beta_{2,1} \beta_{1,3} + \beta_{2,4} \beta_{3,4}}{\beta_{1,2} \beta_{2,4} + \beta_{1,3} \beta_{3,4}} \beta_{1,4}.$$  

(3.20)

It means that we have found a $K^-$ matrix with nine free parameters.

Note that from now on that we will be using functions of the type

$$\Psi_{i,j} = \frac{1}{\beta_{i,j}} \sum_{k \neq i \neq j}^4 \beta_{i,k} \beta_{k,j}$$  

(3.21)

so that (3.19) and (3.20) can be written as

$$\beta_{2,2} = \beta_{1,1} + \Psi_{1,4} - \Psi_{2,4}, \quad \beta_{3,3} = \beta_{1,1} + \Psi_{1,4} - \Psi_{3,4},$$

$$\beta_{4,4} = \beta_{1,1} + \Psi_{1,3} - \Psi_{3,4}$$  

(3.22)

and

$$\Psi_{4,1} = \Psi_{1,4}, \quad \Psi_{3,2} = -\Psi_{1,4}, \quad \Psi_{2,3} = -\Psi_{1,4}.$$  

(3.23)

respectively.

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3.4. The $U_q[sl(2)]$ Temperley–Lieb $K$ matrix for $s = 2$

For $s = 2$, the matrix elements are

$$k_{i,j}(u) = \frac{1}{2} \beta_{i,j} \sinh(2u), \quad (i \neq j = 1, \ldots, 5) \quad (3.24)$$

and

$$k_{i,i}(u) = k_{1,1}(u) + \frac{1}{2} (\beta_{i,i} - \beta_{1,1}) \sinh(2u), \quad (i = 2, \ldots, 5). \quad (3.25)$$

where

$$k_{1,1}(u) = 1 - \frac{1}{2} \left\{ \left( \beta_{5,5} - \beta_{1,1} \right) \left[ x_1(u) + q^4 x_2(u) \right] + \frac{\beta_{1,2} \beta_{2,5} + \beta_{1,3} \beta_{3,5} + \beta_{1,4} \beta_{4,5}}{\beta_{1,5}} x_1(u) 
+ \left[ (\beta_{2,2} - \beta_{1,1}) q^{-2} + (\beta_{3,3} - \beta_{1,1}) + (\beta_{4,4} - \beta_{1,1}) q^2 \right] x_2(u) \right\} x_2(u) \sinh \eta. \quad (3.26)$$

Substituting these expressions into the reflection equations belonging to the blocks (3.7), i.e., $B[2, 5]$, $B[3, 5]$ and $B[4, 5]$, we can find the diagonal parameters

$$\beta_{2,2} = \beta_{1,1} + \Psi_{1,5} - \Psi_{2,5}, \quad \beta_{3,3} = \beta_{1,1} + \Psi_{1,5} - \Psi_{3,5}$$
$$\beta_{4,4} = \beta_{1,1} + \Psi_{1,5} - \Psi_{4,5}, \quad \beta_{5,5} = \beta_{1,1} + \Psi_{1,4} - \Psi_{4,5} \quad (3.27)$$

where we have defined the twenty functions

$$\Psi_{i,j} = \frac{1}{\beta_{i,j}} \sum_{k \neq i, j} \beta_{i,k} \beta_{k,j}. \quad (3.28)$$

From the blocks $B[i, k]$ ($k \geq i = 1–4$), we can see that all constraint equations are rewritten by ten symmetric relations

$$\Psi_{j,i} = \Psi_{i,j} \quad (j > i) \quad (3.29)$$

and the five relations

$$\Psi_{2,4} = \Psi_{2,3} + \Psi_{1,4} - \Psi_{1,3}, \quad \Psi_{2,5} = \Psi_{2,3} + \Psi_{1,5} - \Psi_{1,3},$$
$$\Psi_{3,4} = \Psi_{2,3} + \Psi_{1,4} - \Psi_{1,2}, \quad \Psi_{3,5} = \Psi_{2,3} + \Psi_{1,5} - \Psi_{1,2},$$
$$\Psi_{4,5} = \Psi_{2,3} + \Psi_{1,4} + \Psi_{1,5} - \Psi_{1,2} - \Psi_{1,3}. \quad (3.30)$$

Now, all we need is to look for the constraint equations belonging to the blocks $B[5, k]$. There are seven remaining equations, but only four are independent

$$\Theta_{2,2} = \Theta_{1,1} - (\Psi_{2,3} - \Psi_{1,3}) \Psi_{1,2} \quad \Theta_{3,3} = \Theta_{1,1} - (\Psi_{2,3} - \Psi_{1,2}) \Psi_{1,3}$$
$$\Theta_{4,4} = \Theta_{1,1} - (\Psi_{2,3} + \Psi_{1,4} - \Psi_{1,2} - \Psi_{1,3}) \Psi_{1,4}$$
$$\Theta_{5,5} = \Theta_{1,1} - (\Psi_{2,3} + \Psi_{1,5} - \Psi_{1,2} - \Psi_{1,3}) \Psi_{1,5} \quad (3.31)$$

where we have defined five new functions of the type

$$\Theta_{i,i} = \sum_{k \neq i} \beta_{i,k} \beta_{k,i}, \quad i = 1, \ldots, 5. \quad (3.32)$$

Finally, all constraint equations are substituted by these relations.

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Substituting (3.30) into the ten symmetric relations (3.29) we still have to solve them in order to write the \( K^- \) solution explicitly. A way to do this is to solve simultaneously these ten recursive relations and take into account the solution with the smallest numbers of fixed parameters. Following this procedure we have to find a unique solution with seven fixed parameters, for instance, \( \beta_{1,2}, \beta_{1,3}, \beta_{2,4}, \beta_{2,5}, \beta_{3,4}, \beta_{3,5}, \text{ and } \beta_{4,5} \). Its expressions in terms of the free parameters are not written here because they are too large and cumbersome. These seven parameters plus the four diagonal parameters (3.27) and the normal condition, give us a 5 by 5 reflection \( K^- \) matrix solution with 13 free parameters!

### 3.5. The \( \mathcal{U}_q[sl(2)] \) Temperley–Lieb \( K \)-matrix for \( s > 2 \)

The main difficulty is to solve the constraint equations, for which the solution will be to find some of the parameters in the solution given by (3.4)–(3.6). The existence of many free parameters makes this a very hard task. The constraint equations are large and became greatest after algebraic manipulation and our computer resources are insufficient to work with too large algebraic expressions. In its original form, it is a very formidable problem.

In order to proceed with this task we have introduced two new objects instead of working directly with the \( \beta_{i,j} \) parameters

\[
\Psi_{i,j} = \frac{1}{\beta_{i,j}} \sum_{k \neq i,j} \beta_{i,k} \beta_{k,j} \quad \text{and} \quad \Theta_{i,i} = \sum_{k \neq i} \beta_{i,k} \beta_{k,i}.
\]  

(3.33)

After we rewrite the constraint equations in terms of \( \Psi_{i,j} \) and \( \Theta_{i,i} \), we can easily solve the blocks (3.7) in order to find \( 2s \) diagonal parameters,

\[
\beta_{i,i} = \beta_{1,1} + \Psi_{1,2s+1} - \Psi_{1,2s+1}, \quad i = 2, 3, \ldots, 2s
\]

\[
\beta_{2s+1,2s+1} = \beta_{1,1} + \Psi_{1,2s} - \Psi_{2s,2s+1}.
\]

(3.34)

All equations from the block \( B[1,k] \) to the block \( B[2s,k] \) are now substituted by \( s(2s + 1) \) symmetric relations

\[
\Psi_{j,i} = \Psi_{i,j}, \quad j > i
\]  

(3.35)

and \( 4(s - 1) \) relations of the type

\[
\Psi_{2,j} = \Psi_{2,3} + \Psi_{1,j} - \Psi_{1,3}, \quad j = 4, \ldots, 2s + 1,
\]

\[
\Psi_{3,j} = \Psi_{2,3} + \Psi_{1,j} - \Psi_{1,2}, \quad j = 4, \ldots, 2s + 1.
\]

(3.36)

The remaining equations contained in the block \( B[2s+1,k] \) are rewritten by \( 2(s-1)(s-\frac{3}{2}) \) relations involving the \( \Psi_{i,j} \)

\[
\Psi_{i,j} = \Psi_{1,i} + \Psi_{1,j} - \Psi_{1,2} - \Psi_{1,3}, \quad i = 4, \ldots, 2s, \quad j = i + 1, \ldots, 2s + 1
\]  

(3.37)

and \( 4s - 1 \) relations involving the diagonal \( \beta_{k,k} \) parameters, \( \Psi_{1,2s+1} \) and the \( \Theta_{j,j} \),

\[
\Theta_{j,j} = \Theta_{2s+1,2s+1} + (\beta_{2s+1,2s+1} - \beta_{j,j})(\beta_{j,j} - \beta_{1,1} - \Psi_{1,2s+1}), \quad j = 2, 3, \ldots, 2s,
\]

\[
\Theta_{j,j'} = \Theta_{1,1} + (\beta_{1,1} - \beta_{j,j'})(\beta_{j,j'} - \beta_{2s+1,2s+1} - \Psi_{1,2s+1}), \quad j = 2, 3, \ldots, 2s,
\]

\[
\Theta_{2s+1,2s+1} = \Theta_{1,1} - (\beta_{1,1} - \beta_{2s+1,2s+1})\Psi_{1,2s+1}.
\]

(3.38)

where \( j' = 2s + 2 - j \).

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From (3.35) to (3.38) one can count $4s(s + 1) - 2$ constraint equations but, after we have substituted the relations (3.36) and (3.37) into (3.35), we only need to look for these $s(2s + 1)$ symmetric relations.

From these relations we have fixed $2s^2 - 1$ parameters when $s$ is an integer and $2s^2 - 3/2$ parameters when $s$ is a semi-integer.

Again, we recall that the last step requires large computer resources and the final expressions for the $\beta_{i,j}$ parameters are too large and cumbersome to be written in any way. It means that we have found the normal reflection matrices for the $\mathcal{U}_q[sl(2)]$ Temperley–Lieb model. These matrices are solutions of (1.2) with $2s(s + 1) + 1$ free parameters if $s = 1, 2, \ldots$ and $2s(s + 1) + 3/2$ free parameters if $s = 1/2, 3/2, \ldots$.

4. Reduced solutions

Now, for particular choices of the free parameters, we can derive several subclasses of solutions from the general solution presented in section 3. However, using any reduced procedure we can use some of the possible reductions. For instance, it is simpler to solve the reflection equations directly by looking for diagonal solutions instead of deriving them from the general one. In the following, we present some particular solutions which can be obtained from the general solution. Let us start with the diagonal solutions.

4.1. Diagonal solutions

Taking into account only the diagonal $K^-$ matrices, the reflection equations are solved when we find all matrix elements $k_{j,j}(u)$, $j = 2, \ldots, 2s + 1$ as functions of $k_{1,1}(u)$, provided that the diagonal parameters $\beta_{i,j}$ satisfy $s(2s - 1)$ constraint equations of the type

$$(\beta_{2s+1,2s+1} - \beta_{i,i})(\beta_{2s+1,2s+1} - \beta_{j,j})(\beta_{j,j} - \beta_{i,i}) = 0 \quad (i \neq j \neq 2s + 1).$$

From (4.1) we can find diagonal $K^-$ matrix solutions with only two type of entries. Let us normalize one of them to be equal to one such that the other entry is given by

$$k_{p,p}(u) = \frac{\beta_{p,p}x_2(u)[\Delta_1x_2(u) + x_1(u)] + 2[x_1(u)x'_2(u) - x'_1(u)x_2(u)]}{\beta_{p,p}x_2(u)[\Delta_1x_2(u) + x_1(u)] - 2[x_1(u)x'_2(u) - x'_1(u)x_2(u)]}$$

where $\Delta_1 + \Delta_k = 2\cosh \eta$.

Identifying the diagonal positions with the powers of $q$, $(1, 2, \ldots, 2s, 2s + 1) \equiv (q^{-2s}, q^{-2s+2}, \ldots, q^{2s-2}, q^{2s})$ one can see that $\Delta_1$ is the sum of the powers of $q$ corresponding to the positions of the entries 1 and $\Delta_k$ is the sum of the power of $q$ corresponding to the positions of the entries $k_{p,p}(u)$.

Denoting the diagonal solutions by $\mathbb{K}_{a_1}^{(r)}$, where $a_1 = (a_1, a_2, \ldots, a_{2s+1})$ with $a_i = 0$ if $k_{i,i}(u) = 1$ or $a_i = 1$ if $k_{i,i}(u) = k_{p,p}(u)$, $r$ is the number of the entries $k_{p,p}(u)$ distributed on diagonal positions and $p$ is the first position with the entry different from unity. Thus, we have counted

$$N = \sum_{r=1}^{2s} \frac{(2s + 1)!}{r!(2s + 1 - r)!}. \quad (4.3)$$

for the number of diagonal $K^-$ matrix solutions with one free parameter. Again, the $K^+$ solutions are obtained by the isomorphism (1.4). Let us work explicitly with this
description. For the case $s = \frac{1}{2}$ one can find two diagonal solutions:

$$K^{[1]}_{(1,0)} = \begin{pmatrix} k_{1,1}(u) & 0 \\ 0 & 1 \end{pmatrix}, \quad K^{[1]}_{(0,1)} = \begin{pmatrix} 1 & 0 \\ 0 & k_{2,2}(u) \end{pmatrix}$$  \hspace{1cm} (4.4)

where

$$k_{1,1}(u) = \frac{\beta_{1,1}x_2(u)[q x_2(u) + x_1(u)] + 2[x_1(u)x'_2(u) - x'_1(u)x_2(u)]}{\beta_{1,1}x_2(u)[q^{-1}x_2(u) + x_1(u)] - 2[x_1(u)x'_2(u) - x'_1(u)x_2(u)]}$$

$$k_{2,2}(u) = \frac{\beta_{2,2}x_2(u)[q^{-1}x_2(u) + x_1(u)] - 2[x_1(u)x'_2(u) - x'_1(u)x_2(u)]}{\beta_{2,2}x_2(u)[q x_2(u) + x_1(u)] + 2[x_1(u)x'_2(u) - x'_1(u)x_2(u)]}$$  \hspace{1cm} (4.5)

with $2 \cosh \eta = q^{-1} + q$. Of course, they are equivalent by the exchange $q \leftrightarrow q^{-1}$.

Similarly, for $s = 1$, one can find six diagonal solutions, half of them with one entry different from unity

$$K^{[1]}_{(1,0,0)} = \begin{pmatrix} k_{1,1}(u) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad K^{[1]}_{(0,1,0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k_{2,2}(u) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$K^{[1]}_{(0,0,1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k_{3,3}(u) \end{pmatrix}$$  \hspace{1cm} (4.6)

with

$$k_{1,1}(u) = \frac{\beta_{1,1}x_2(u)[(1 + q^2)x_2(u) + x_1(u)] + 2[x_1(u)x'_2(u) - x'_1(u)x_2(u)]}{\beta_{1,1}x_2(u)[q^{-2}x_2(u) + x_1(u)] - 2[x_1(u)x'_2(u) - x'_1(u)x_2(u)]}$$

$$k_{2,2}(u) = \frac{\beta_{2,2}x_2(u)[q^{-2} + q^2)x_2(u) + x_1(u)] + 2[x_1(u)x'_2(u) - x'_1(u)x_2(u)]}{\beta_{2,2}x_2(u)[q x_2(u) + x_1(u)] - 2[x_1(u)x'_2(u) - x'_1(u)x_2(u)]}$$

$$k_{3,3}(u) = \frac{\beta_{3,3}x_2(u)[q^{-2} + q^2)x_2(u) + x_1(u)] + 2[x_1(u)x'_2(u) - x'_1(u)x_2(u)]}{\beta_{3,3}x_2(u)[q^{-2}x_2(u) + x_1(u)] - 2[x_1(u)x'_2(u) - x'_1(u)x_2(u)]}$$  \hspace{1cm} (4.7)

and three further diagonal solutions with two equal entries different from unity

$$K^{[2]}_{(1,1,0)} = \begin{pmatrix} k_{1,1}(u) & 0 & 0 \\ 0 & k_{1,1}(u) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad K^{[2]}_{(1,0,1)} = \begin{pmatrix} k_{1,1}(u) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k_{1,1}(u) \end{pmatrix},$$

$$K^{[2]}_{(0,1,1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k_{2,2}(u) & 0 \\ 0 & 0 & k_{2,2}(u) \end{pmatrix}$$  \hspace{1cm} (4.8)

where

$$k_{1,1}(u) = \frac{\beta_{1,1}x_2(u)[q x_2(u) + x_1(u)] + 2[x_1(u)x'_2(u) - x'_1(u)x_2(u)]}{\beta_{1,1}x_2(u)[q^{-2}x_2(u) + x_1(u)] - 2[x_1(u)x'_2(u) - x'_1(u)x_2(u)]},$$

$$k_{1,1}(u) = \frac{\beta_{1,1}x_2(u)[(1 + q^2)x_2(u) + x_1(u)] + 2[x_1(u)x'_2(u) - x'_1(u)x_2(u)]}{\beta_{1,1}x_2(u)[q^{-2} + q^2)x_2(u) + x_1(u)] + 2[x_1(u)x'_2(u) - x'_1(u)x_2(u)]},$$

$$k_{2,2}(u) = \frac{\beta_{2,2}x_2(u)[q^{-2}x_2(u) + x_1(u)] - 2[x_1(u)x'_2(u) - x'_1(u)x_2(u)]}{\beta_{2,2}x_2(u)[q x_2(u) + x_1(u)] - 2[x_1(u)x'_2(u) - x'_1(u)x_2(u)]}$$  \hspace{1cm} (4.9)

Here we notice again that the difference between the entries (4.9) comes from the partitions of $2 \cosh \eta$ such that $\Delta_1 + \Delta_k = q^{-2} + 1 + q^2$ and the equivalence due to the symmetry $q \leftrightarrow q^{-1}$. It follows similarly for the other values of $s$. 

\begin{verbatim}
doi:10.1088/1742-5468/2011/01/P01009
\end{verbatim}
4.2. $Z_{2s+1}K$ matrix solution

Now we recall the center of $\mathcal{U}_q[sl(2)]$ in order to take into account the $K^-$ matrix solution in terms of the generators of the $Z_{2s+1}$-symmetry. From the general solution it is straightforward to get the following reduced solution

$$K^-(u) = k_{1,1}(u)I + \sum_{k=1}^{2s} k_{1,2s+1}(u)\omega^kZ_k$$

(4.10)

where $I$ is the unity matrix, $\omega = \exp(2i\pi/(2s+1))$ and $Z_k$ are the $Z_{2s+1}$ matrices

$$(Z_k)_{i,j} = \delta_{i,j+k} \mod(2s+1), \quad k = 1, \ldots, 2s$$

(4.11)

and

$$k_{1,1}(u) = \frac{k_{1,2s+1}(u)}{\beta_{1,2s+1}[x_2(u)\cosh\eta + x_1(u)]} \left\{ \frac{x_1(u)x_2'(x) - x_1'(u)x_2(u)}{x_2(u)} - \frac{(2s-1)\omega^{2s}}{2}x_1(u) \right\}.$$  

(4.12)

Here $\beta_{1,2s+1}$ is the free parameter and $k_{1,2s+1}(u)$ is an arbitrary function.

4.3. Spin zero $K$ matrix solution

Let us start again with the $2s + 1$ by $2s + 1$ Temperley–Lieb $R$ matrix (2.1). However, looking for the $K^-$ matrix solutions with the form

$$K^-(u) = f_1(u)I' + f_2(u)U'$$

(4.13)

where $f_1(u)$ and $f_2(u)$ are functions to be determined in order that $U'$ is the spin zero projector (2.3) for the spin $s'$ and $I'$ being the corresponding identity matrix.

When $(2s'+1)^2 = 2s + 1$, we have found that (4.13) is a solution of the reflection equation (1.2) provided that

$$f_1(u) = \frac{f_2(u)}{\beta[x_2(u)\cosh\eta + x_1(u)]} \left\{ \frac{x_1(u)x_2'(x) - x_1'(u)x_2(u)}{x_2(u)} - \frac{\beta}{2} \sum_{j=-s'}^{s'} [q^{2j(2s'+1)}x_2(u) + q^{2j}x_1(u)] \right\}.$$  

(4.14)

where $\beta = (df_2(u)/du)|_{u=0}$ is the free parameter and $f_2(u)$ is an arbitrary function.

5. Conclusion

In this work we have presented the general solutions of the reflection equation for the $\mathcal{U}_q[sl(2)]$ Temperley–Lieb vertex model. Our findings can be summarized into two classes of general solutions depending on whether the spin $s$ is an integer or semi-integer. The large number of free parameters, $2s(s+1) + 1$ for $s$ integer and $2s(s+1) + 3/2$ for $s$ semi-integer, is an important property of this model which follows due to the explicit factorization of the constraint $\delta(2\cosh\eta - [2s + 1])$ in the reflection equation.

These results pave the way to construct, solve and study the physical properties of the underlying quantum spin chains with open boundaries, generalizing the previous efforts.

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made for the case of periodic boundary conditions [33, 34]. Although we do not know the algebraic Bethe ansatz for \( s \neq \frac{1}{2} \), even for the periodic cases, we expect that the coordinate Bethe ansatz solution of the Temperley–Lieb models constructed from diagonal solutions presented here can be obtained by adapting the results of [35], and the algebraic-functional method presented in [36] may be a possibility to treat the non-diagonal cases. We expect the results presented here to motivate further developments on the subject of integrable open boundaries for the Temperley–Lieb vertex models based on other \( q \)-deformed Lie algebras [28] and superalgebras [37].

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