Vanishing contact structure problem and convergence of the viscosity solutions

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ABSTRACT
This article is devoted to studying the vanishing contact structure problem which is a generalization of the vanishing discount problem. Let $H^k(x, p, u)$ be a family of Hamiltonians of contact type with parameter $k > 0$ and converge to $G(x, p)$. For the contact type Hamilton-Jacobi equation with respect to $H^k$, we prove that, under mild assumptions, the associated viscosity solution $u^k$ converges to a specific viscosity solution $u^0$ of the vanished contact equation. As applications, we give some convergence results for the nonlinear vanishing discount problem.

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1. Introduction and main results

Let $M$ be a connected and compact smooth manifold without boundary, equipped with a smooth Riemannian metric $g$, the associated Riemannian distance on $M$ will be denoted by $d$. Let $TM$ and $T^*M$ denote the tangent and cotangent bundles respectively. A point of $TM$ will be denoted by $(x, v)$ with $x \in M$ and $v \in T_xM$, and a point of $T^*M$ by $(x, p)$ with $p \in T^*_xM$ is a linear form on the vector space $T_xM$. With a slight abuse of notation, we shall both denote by $| \cdot |_x$ the norm induced by $g$ on the fiber $T_xM$ and also the dual norm on $T^*_xM$. Let $H^\lambda(x, p, u) : T^*M \times \mathbb{R} \to \mathbb{R}$ be a family of Hamiltonians with parameter $\lambda \in (0, 1]$ and $G : T^*M \to \mathbb{R}$ be a Hamiltonian such that $H^\lambda$ uniformly converges, as $\lambda \to 0$, to $G$ on any compact subsets.

Consider the contact type Hamilton-Jacobi equation

$$H^\lambda(x, Du(x), u(x)) = c, \quad x \in M. \tag{1.1}$$

This equation, from the view of physics, naturally arises in contact Hamiltonian mechanics (see for instance [1, 2]). Systematic discussions of the contact transformations and Hamilton-Jacobi equations can be found in [3]. Since $H^\lambda$ uniformly converges to $G$, a natural and important question is the convergence of viscosity solutions of (1.1),
that is, whether or not the whole family \( \{u^\lambda\}_\lambda \) of viscosity solutions uniformly converges, as \( \lambda \to 0 \), to a unique function. By the stability property of viscosity solution, it is known that the limits of \( \{u^\lambda\}_\lambda \) must be viscosity solutions of
\[
G(x, Du(x)) = c, \quad x \in M. \tag{1.2}
\]

An immediate comment to be made is that, as is well known, there is only one special critical value \( c = c(G) \) such that (1.2) admits a viscosity solution. It is thus natural to take the value \( c = c(G) \) in Eqs. (1.1) and (1.2). This leads us to consider the following contact type Hamilton-Jacobi equations
\[
H^\lambda(x, Du(x), u(x)) = c(G), \quad x \in M. \tag{HJ}\lambda
\]
and
\[
G(x, Du(x)) = c(G), \quad x \in M. \tag{HJ}\0
\]

Our main goal in this article is to study the vanishing contact structure problem, which mainly focuses on the asymptotic behavior of the whole family \( \{u^\lambda\}_\lambda \) of solutions of (HJ) as \( \lambda \) goes to zero, and the characterization of the possible limits. This problem is a natural generalization of the vanishing discount problem where \( H^\lambda \) is a linear discounted system \( H^\lambda(x, p, u) = \lambda u + G(x, p) \). The vanishing discount problem is also called ergodic approximation in PDE, which was first studied in a general framework by Lions, Papanicolaou, and Varadhan in [4] and has been widely studied since then. Recently, great progress has been made in the vanishing discount problem under various type of settings and methods, see [5–14], etc., especially the results in [7] where the authors first prove a convergence result under very mild conditions, and characterize the unique limit in terms of Peierls barrier and projected Mather measures from a dynamical viewpoint.

However, the vanishing contact structure problem has not been fully studied yet, as far as we know. The difficulties are that the general \( H^\lambda(x, p, u) \) might not be linear in \( u \) as discounted case, and hence one cannot directly obtain a convenient explicit representation formula for the solution \( u^\lambda \). In recent works [15–17], certain implicit variational principle is applied to give representation formula for the viscosity solutions or weak KAM solutions of (HJ). An alternative approach following Herglotz’ generalized variational principle is also obtained from the Lagrangian formalism [18], which is later used to obtain a vanishing contact structure result on relevant Cauchy problem for evolutionary equations [19]. But, these results are established under \( C^2 \) Tonelli conditions, so they are no longer applicable to the settings in our current article. Therefore, in this article, we will develop some new techniques, under suitable assumptions, to handle the vanishing contact structure problem.

Now we begin to state the main assumptions here. Suppose that

(SH1) \( H^\lambda \in C(T^*M \times \mathbb{R}) \) and \( H^\lambda(x, p, 0) = G(x, p) \) for each \( \lambda \).
(SH2) The map \( p \mapsto G(x, p) \) is convex on \( T_x^*M \), for every \( x \in M \).
(SH3) \( G(x, p) \) is coercive in the fibers, that is, \( \lim_{|p|_\lambda \to +\infty} G(x, p) = +\infty \), uniformly for all \( x \in M \).
(SH4) The map \( u \mapsto H^\lambda(x, p, u) \) is strictly increasing, and for any \( r > 0 \), there exists a constant \( \kappa^r_\lambda > 0 \) such that \( \lim_{\lambda \to 0} \kappa^r_\lambda = 0 \) and
\[ |H^\lambda(x, p, u) - H^\lambda(x, p, 0)| \leq \kappa^\lambda_R |u|, \text{ for all } |u| \leq r. \] (1.3)

(SH5) There exist positive constants \( \delta^\lambda_R \leq K^\lambda_R \) depending on \( R \) and \( \lambda \) such that

\[ \delta^\lambda_R |u| \leq |H^\lambda(x, p, u) - H^\lambda(x, p, 0)| \leq K^\lambda_R |u| \text{ for all } |p| \leq R, |u| \leq R \]

and for a suitably large \( R_0 \),

\[ \lim_{\lambda \to 0} \frac{\delta^\lambda_{R_0}}{K^\lambda_{R_0}} = 1. \] (1.4)

Under hypotheses (SH1) and (SH4), it is clear that \( H^\lambda \) uniformly converges, as \( \lambda \to 0 \), to \( G \) on any compact subsets of \( T^* M \times \mathbb{R} \).

Notice that assumptions (SH4) and (SH5) together imply that \( \delta^\lambda_R \leq \kappa^\lambda_R \) and \( \lim_{\lambda \to 0} \delta^\lambda_{R_0} = \lim_{\lambda \to 0} K^\lambda_{R_0} = 0 \).

Here we give some notes and remarks on our assumptions: throughout this article the Hamiltonians are required to be only continuous, in such a setting, of course, no Hamiltonian dynamics can necessarily be defined in the usual sense. \( H^\lambda(x, p, 0) \equiv G(x, p) \) in assumption (SH1) is necessary because otherwise, the convergence result might not hold, see Example 4.2. The coercivity in assumption (SH3) is used to prove that each subsolution of \((HJ^\lambda)\) should be Lipschitz continuous. The strict monotonicity assumption (SH4), as is known to all, guarantees the uniqueness of Lipschitz continuous viscosity solution of \((HJ^\lambda)\), and the local Lipschitz continuity (1.3) is used to give uniform estimates for the whole family of solutions \( \{u^\lambda\}_{\lambda} \). As for assumption (SH5), the constant \( R_0 \) in (1.4) will be specified in the proof of Theorem 4.3, this hypothesis might seem too strict at first sight, however, in the sequel (see Main Results 2 and 3 below) we will show that besides the well-known discounted system, it is really satisfied by a large class of models. Last but not least, to characterize the limit solution more precisely, we make a crucial use of the convexity assumption (SH2) and the coercivity assumption (SH3) to apply Aubry-Mather theory.

We denote by \( \mathcal{M}(L_G) \) the set of all projected Mather measures with respect to \( L_G \) (see Section 2 for the precise definition), and by \( \mathcal{F}(G) \) the set of all viscosity subsolutions \( w \) of \((HJ_0)\) satisfying

\[ \int_M w(x) d\nu(x) \leq 0, \quad \forall \ \nu \in \mathcal{M}(L_G). \]

Then we address the main results in this article:

**Main Result 1.** Let \( \{H^\lambda\}_{\lambda \in (0, 1]} \) satisfy (SH1)–(SH5). Then equation \((HJ^\lambda)\) has a unique continuous viscosity solution \( u^\lambda \) which is also Lipschitzian, and the convergence

\[ u^0(x) = \lim_{\lambda \to 0} u^\lambda(x) \text{ uniformly for all } x \in M \] (1.5)
holds for some function \( u^0 \in \text{Lip}(M) \). Furthermore, the limit function \( u^0 \) is a viscosity solution of (HJ\(_0\)) and is characterized by formula

\[
\begin{align*}
   u^0(x) &= \sup_{u \in \mathcal{F}(G)} u(x) = \min_{\mu \in \mathcal{M}(L_G)} \int_M h(y, x) d\mu(y),
\end{align*}
\]

where \( h(y, x) \) is the Peierls barrier of the Lagrangian \( L_G \).

It should be noted that unlike [7], we do not use the superlinearity assumption of \( G \) in the fibers, to obtain the representation formula (1.6). In this respect, we remark that the technique for dealing with lower semicontinuous Lagrangians in [5] allows us to conclude (1.6).

Our assumptions (SH1)–(SH5) are true of many models. In the sequel, as applications, we give some generalizations of discounted systems which are nonlinear in \( u \), and naturally satisfy the aforementioned assumptions. Some of the readers might more interested in the following one:

**Application I:** We consider a direct generalization of the discounted equations. Suppose that

(C1) \( H \in C(T^* M) \) is convex and coercive in the fibers.
(C2) \( f(x, u) \in C^1(M \times \mathbb{R}) \) with \( f_u > 0 \) and

\[
\limsup_{|u| \to 0} \frac{|f_u(x, u) - f_u(x, 0)|}{|u|} < +\infty \quad \text{uniformly for } x \in M.
\]

Let \( c = c(G) \) be the critical value of the Hamiltonian \( G(x, p) := f(x, 0) + H(x, p) \).

Now for each \( \lambda > 0 \), we consider the following Hamilton-Jacobi equations:

\[
\begin{align*}
   f(x, \lambda u) + H(x, Du) &= c, \quad x \in M. 
\end{align*}
\]

and

\[
\begin{align*}
   f(x, 0) + H(x, Du) &= c, \quad x \in M. 
\end{align*}
\]

**Main Result 2.** Under the above assumptions (C1)–(C2), Eq. (1.7) has a unique continuous viscosity solution \( u^\lambda \), which is also Lipschitz continuous, and, for some \( u^0 \in \text{Lip}(M) \), which is indeed a viscosity solution of (1.8), the following convergence holds:

\[
\begin{align*}
   u^0(x) &= \lim_{\lambda \to 0} u^\lambda(x) \quad \text{uniformly for all } x \in M.
\end{align*}
\]

Moreover, \( u^0 \) is characterized by

\[
\begin{align*}
   u^0(x) &= \sup_{w \in \mathcal{F}(G)} w(x) = \min_{\mu \in \mathcal{M}(L_G)} \frac{\int_M f_u(y, 0) h(y, x) d\mu(y)}{\int_M f_u(y, 0) d\mu(y)},
\end{align*}
\]

where \( h(y, x) \) is the Peierls barrier of the Lagrangian \( L_G \) and \( \mathcal{F}(G) \) is the set of viscosity subsolutions \( w \) of (1.8) satisfying \( \int_M f_u(y, 0) w(y) d\mu(y) \leq 0 \) for all \( \mu \in \mathcal{M}(L_G) \).

Similar discussions and convergence results of (1.7) have also been obtained in [9, Theorem 2.1] by using the nonlinear adjoint method.
However, compared with their result and proof, our assumptions are milder and, moreover, we deal with the problem by a single application of a comparison theorem (Theorem 3.2 below) combined with now a classical convergence result by Davini et al. [7] (see Theorem 2.6 below).

**Application II:** Now we study a more general class of systems, let $H = H(x, p, u) : T^*M \times \mathbb{R} \to \mathbb{R}$ and $G(x, p) := H(x, p, 0)$ be the Hamiltonian with $c = c(G)$ the critical value of $G$. For each $\lambda > 0$, we consider the equations

$$H(x, Du(x), \lambda u(x)) = c, \quad x \in M. \quad (1.9)$$

and

$$H(x, Du(x), 0) = c, \quad x \in M. \quad (1.10)$$

Then we can prove

**Main Result 3.** Suppose that

a. $H \in C(T^*M \times \mathbb{R})$, the partial derivative $H_u \in C(T^*M)$ with $H_u > 0$. For any $R > 0$, there exists $B_R > 0$ so that

$$|H(x, p, u) - H(x, p, 0)| \leq B_R |u| \quad \text{for all } |u| \leq R$$

and

$$|H_u(x, p, u) - H_u(x, p, 0)| \leq B_R |u| \quad \text{for all } |p| \leq R, |u| \leq R;$$

b. $G(x, p)$ and $\frac{G(x, p) - c}{H_u(x, p, 0)}$ is convex and coercive in the fibers.

Then Eq. (1.9) has a unique continuous viscosity solution $u^\lambda$, which is also Lipschitzian, and, moreover, for some Lipschitz viscosity solution $u^0$ of (1.10),

$$u^0(x) = \lim_{\lambda \to 0} u^\lambda(x) \quad \text{uniformly for all } x \in M.$$

Furthermore, $u^0$ is represented as

$$u^0(x) = \sup_{u \in \mathcal{F}(\tilde{G})} u(x) = \min_{\mu, \in \mathcal{M}(\tilde{G}, L_{\tilde{G}})} \int_M \tilde{h}(y, x) d\mu(y). \quad (1.11)$$

where $\tilde{h}(y, x)$ is the Peierls barrier of the Lagrangian $L_{\tilde{G}}$ associated with the Hamiltonian $\tilde{G}(x, p) = \frac{G(x, p) - c}{H_u(x, p, 0)}$.

The article is organized as follows. Section 2 provides basic terminologies and notations which are necessary for the understanding of our subsequent work, and we collect some necessary results in Aubry-Mather theory and weak KAM theory under a nonsmooth setting and without superlinearity assumption. In Section 3, we establish the existence and uniqueness of viscosity solution for equation (HJ$_\lambda$), based on Perron’s construction and the comparison principle. Section 4 is the main part of the present article which consists of the discussion, statement and proof of Main Result 1. Section 5 provides some applications and gives the proof of Main Results 2 and 3, which strongly rely on the nonlinear analysis of our models. Finally, in Section 6, for the reader’s
convenience, we give the detailed proof for Theorem 2.6 which is crucial in this article. Appendices A and B examine briefly the validity of the variational or optimal control formula for the solutions of the Cauchy problem for Hamilton-Jacobi equations as well as the existence of solutions and the comparison principle. The role of the appendices is to bridge a gap, at least in the literature, in the basic theory of the Hamilton-Jacobi equations with Lagrangians having possibly the value \( +\infty \).

2. Preliminaries

In this section, we provide some useful results from Aubry-Mather theory and weak KAM theory which are necessary for the purpose of this article. Aubry-Mather theory is classical and well known for \( C^2 \) Hamiltonians or Lagrangians satisfying Tonelli conditions, we refer the reader to Mather’s original papers [20, 21] and Mañé’s papers [22, 23]. For a complete introduction to the weak KAM theory under Tonelli settings, we refer the reader to Fathi’s book [24]. An analog of Aubry-Mather theory or weak KAM theory has been developed for contact Hamiltonian, see for example [25–28].

However, in this article our systems are only required to be continuous and are lack of Hamiltonian or Lagrangian dynamics. So we need the generalizations of Aubry-Mather theory and weak KAM theory for non-smooth systems, the main references are [5, 7, 29–32].

Throughout this section, \( M \) is a connected and compact manifold without boundary, and \( H(x, p) : T^*M \to \mathbb{R} \) is assumed to satisfy

(H1) \( H \in C(T^*M) \).

(H2) For every \( x \in M \), the map \( p \mapsto H(x, p) \) is convex in the fiber \( T^*_x M \).

(H3) \( H(x, p) \) is coercive in the fibers, that is, \( \lim_{|p|_x \to +\infty} H(x, p) = +\infty \) uniformly in \( x \in M \).

By the Legendre-Fenchel transformation, we define the Lagrangian associated with \( H \) by

\[
L(x, v) = \sup_{p \in T^*_x M} \{ \langle p, v \rangle_x - H(x, p) \}, \quad \forall (x, v) \in TM,
\]

where \( \langle p, v \rangle_x \) denotes the value of the linear form \( p \in T^*_x M \) evaluated at \( v \in T_x M \).

It is a classical result in convex analysis that if, in addition, \( H \) has superlinear growth in the fibers, then \( L \in C(TM) \). But we do not assume here the superlinearity of \( H \) in the fibers, which results in a possibility of \( L \) taking value \( +\infty \). However, by the definition of \( L \), it is clear that \( L \) is lower semicontinous in \( TM \). Moreover, \( L \) is bounded in a neighborhood of zero section of \( TM \) and superlinear in the fibers. Indeed, we observe that

\[
L(x, v) \geq -H(x, 0), \tag{2.1}
\]

Also, since \( p \mapsto H(x, p) \) is convex and coercive and \( M \) is compact, there exist constants \( \delta > 0 \) and \( C_0 > 0 \) such that \( H(x, p) \geq \delta |p|_x - C_0 \). Hence,

\[
\langle p, v \rangle_x - H(x, p) \leq |p|_x |v|_x - \delta |p|_x + C_0 = (|v|_x - \delta) |p|_x + C_0
\]
and therefore
\[ L(x, v) \leq C_0 \text{ if } |v|_x \leq \delta. \] (2.2)

Furthermore, for any nonzero \( v \in T_xM \) and \( R > 0 \), by definition,
\[ L(x, v) \geq \max_{|p|_x = R} (p, v) - \max_{|p|_x = R} H(x, p) = R|v|_x - \max_{|p|_x = R} H(x, p). \] (2.3)

Since \( R > 0 \) is arbitrary, the Lagrangian \( L \) has a superlinear growth in the fibers. The critical value \( c(H) \) associated with \( H \) is defined by
\[ c(H) = \inf \{ c \in \mathbb{R} : H(x, Du) = c \text{ admits a viscosity subsolution} \}. \] (2.4)

It is well known that the critical value is the unique real number \( c \) such that the equation
\[ H(x, Du) = c, \quad x \in M, \]
admits a global viscosity solution (see [30]). The notion of viscosity solution has been introduced by Crandall and Lions [33]. A function \( u : V \to \mathbb{R} \) is a viscosity subsolution of \( H(x, Du) = c \) on the open subset \( V \subset M \) if, for every \( C^1 \) function \( \phi : V \to \mathbb{R} \), with \( \phi \geq u \) and \( \phi(x_0) = u(x_0) \), we have \( H(x_0, D\phi(x_0)) \leq c \). It is a viscosity supersolution if, for every \( C^1 \) function \( \psi : V \to \mathbb{R} \), with \( \psi \leq u \) and \( \psi(x_0) = u(x_0) \), we have \( H(x_0, D\psi(x_0)) \geq c \). A function \( u : V \to \mathbb{R} \) is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

The following result is classical in viscosity solution theory (see [33, Theorem I.14], [34, Example 1]) and we refer the reader to [35, Theorem 5.2] for the case \( M \) is a compact manifold.

**Proposition 2.1.** Let \( H : T^*M \to \mathbb{R} \) be coercive in the fibers and \( c \in \mathbb{R} \), then any continuous viscosity subsolution of \( H(x, Du) = c \) is Lipschitz continuous. Moreover, the Lipschitz constant is bounded above by \( \kappa_c \) with
\[ \kappa_c = \sup \{ |p|_x : H(x, p) \leq c \}. \]

For every \( t > 0 \) and \( x, y \in M \), let \( \Gamma^t_{x,y} \) denote the set of absolutely continuous curves \( \xi : [0, t] \to M \) with \( \xi(0) = x \) and \( \xi(t) = y \). We define the action function
\[ h_t(x, y) = \inf_{\xi \in \Gamma^t_{x,y}} \int_0^t \left[ L\left( \xi(s), \dot{\xi}(s) \right) + c(H) \right] ds, \]
which might be \( +\infty \) if the distance between \( x \) and \( y \) is large and \( t > 0 \) is small. Then, we define a real-valued function \( h \) on \( M \times M \) by
\[ h(x, y) := \lim_{t \to \infty} \inf h_t(x, y). \]

In the literature, \( h(x, y) \) is called the Peierls barrier. This leads us to define the so-called projected Aubry set \( A \) by
\[ A := \{ x \in M : h(x, x) = 0 \}. \]

Since \( M \) is compact, the projected Aubry set \( A \) is nonempty and closed.

We recall the “semi-distance” \( S : M \times M \to \mathbb{R} \), as introduced in [30] (see also [24, Chapter 8]), given by
$$S(y, x) = \sup \left\{ \psi(x) - \psi(y) : \psi \in S^-(H - c(H)) \right\}.$$ 

Here and henceforth, we denote by $S^-(G)$ the set of all viscosity subsolutions of $G(x, Du) = 0$.

We collect some basic properties of the function $S$:

**Proposition 2.2.**

1. $S$ is finite-valued function and satisfies the triangle inequality: $S(x, y) \leq S(x, z) + S(z, y)$.
2. If $w$ is a viscosity subsolution of $H(x, Du) = c(H)$, then $w(y) - w(x) \leq S(x, y)$.
3. For any $y \in M$, the function $x \mapsto S(y, x)$ is a viscosity subsolution of $H(x, Du) = c(H)$.

The claims (1)–(3) above are easy to check. Indeed, the set $S^-(H - c(H))$ is not empty and, by Proposition 2.1, it is equi-Lipschitz continuous in $M$. Consequently, $S$ is well defined as a real-valued function and Lipschitz continuous in $M \times M$. Property (2) above is a simple consequence of the definition of $S$. In particular, since $z \mapsto S(x, z)$ is a member of $S^-(H - c(H))$, we have $S(x, y) \leq S(x, z) + S(z, y)$. As for property (3), by the stability of viscosity subsolutions under the sup-operation, it follows that $x \mapsto S(y, x)$ is a viscosity subsolution.

The results in the proposition below seem to be new, at least for the Hamiltonian $H$ without a superlinear growth in the fibers. The argument in [5] is easily adapted to our current setting, we will present the proof in Section 6 for the reader’s convenience.

**Proposition 2.3.** $z \in A$ if and only if the function $x \mapsto S(z, x)$ is a viscosity solution of the equation $H(x, Du) = c(H)$. In addition, we have the following representation:

$$h(x, y) = \inf_{z \in A} [S(x, z) + S(z, y)].$$

As a result, we have the following properties for $h$ where item (1) and item (3) are a direct consequence of Propositions 2.2 and 2.3. As for item (2), note that by Proposition 2.3, for any $z \in A, x \mapsto S(z, x)$ is a viscosity solution of $H(x, Du) = c(H)$ in $M$, and we deduce by the stability property of viscosity solutions that $x \mapsto h(y, x) = \inf_{z \in A} [S(y, z) + S(z, x)]$ is also a viscosity solution. Item (4) is exactly [24, Theorem 8.5.5].

**Proposition 2.4.**

1. $h$ is finite valued and Lipschitz continuous.
2. For any $y \in M$, the function $x \mapsto h(y, x)$ is a viscosity solution of $H(x, Du) = c(H)$ and the function $x \mapsto -h(x, y)$ is a viscosity subsolution of $H(x, Du) = c(H)$.
3. If $v$ is a viscosity subsolution of $H(x, Du) = c(H)$, then $v(y) - v(x) \leq h(x, y)$.
4. If $f$ and $g$ are a viscosity subsolution and a viscosity supersolution of $H(x, Du) = c(H)$ and $f \leq g$ on the set $A$, then $f \leq g$ in $M$.

In this article, we need a well-known approximation argument for subsolutions of convex Hamilton-Jacobi equations, this technique is standard and can be found in many places, for instance, in the proof of the assertion (d) of [36, Lemma 2.2]. To be better applied to our problem, we refer the reader to [35, Theorem 10.6].
Proposition 2.5. Let $u : M \rightarrow \mathbb{R}$ be a Lipschitz viscosity subsolution of the equation $H(x, Du) = c(H)$. Then for any number $\epsilon > 0$, we can find a $C^\infty$ function $w : M \rightarrow \mathbb{R}$ such that $\|u - w\|_\infty \leq \epsilon, H(x, Dw) \leq c(H) + \epsilon$ for every $x \in M$, and $\|Dw\|_\infty \leq \text{Lip}(u) + 1$.

Next, we will introduce the notion of Mather measure. For the purpose of this article, we adopt the equivalent definition originate from Mañé [22], see also [37]. Recall that a Borel probability measure $\mu$ on $TM$ is called closed if it satisfies
\[
\int_{TM} \langle D\phi, \nu \rangle_x d\tilde{\mu}(x, \nu) = 0, \quad \text{for all } \phi \in C^1(M).
\]

Let $P$ be the set of closed probability measures on $TM$. The set is nonempty and it has been shown that the critical value $c(H)$ can be obtained by considering a minimizing problem. More precisely, the following relation holds (see [7, Theorem 5.7] in the case of $H$ having a superlinear growth in the fibers, and [5] in the case of $H$ with only coercive condition):
\[
-c(H) = \min_{\mu \in P} \int_{TM} L(x, \nu) d\tilde{\mu}(x, \nu).
\]

Here, we remark that $L$ may take the value $+\infty$ and the identity above, in particular, requires at least that $L < +\infty \tilde{\mu}$-almost everywhere.

For any probability measure $\tilde{\mu}$ on $TM$, we define the corresponding projected measure on $M$ by $\mu := \pi \tilde{\mu}$, where $\pi : TM \rightarrow M$ is the canonical projection, namely $\mu(A) = \tilde{\mu}(\pi^{-1}(A))$ and, equivalently,
\[
\int_M f(x) \, d\mu(x) = \int_{TM} (f \circ \pi)(x, \nu) \, d\tilde{\mu}(x, \nu), \quad \text{for every } f \in C(M).
\]

A Mather measure for the Lagrangian $L$ is a measure $\tilde{\mu} \in P$ satisfying the following minimizing property:
\[
\int_{TM} L(x, \nu) d\tilde{\mu} = -c(H).
\] (2.5)

We denote by $\mathcal{M}(L)$ the set of all Mather measures, and by
\[
\mathcal{M}(L) = \pi \mathcal{M}(L)
\] (2.6)
the set of all projected Mather measures.

Now we end up this section with a remarkable result which is vital in the following sections.

Theorem 2.6 ([5, 7]). Let $H(x, p)$ satisfy (H1)–(H2) and be coercive in the fibers. Then
\[
\lambda u + H(x, Du) = c(H), \quad x \in M
\]
has a unique continuous viscosity solution $u^\lambda$, the family $\{u^\lambda\}$ converges uniformly, as $\lambda \rightarrow 0$, to a single critical solution $u^0$ of $H(x, Du) = c(H)$, and the limit $u^0$ is represented as
\[
u^0(x) = \sup_{u \in \mathcal{F}(H)} u(x) = \min_{\mu \in \mathcal{M}(L)} \int_M h(y, x) d\mu(y),
\] (2.7)
where $F(H)$ denotes the set of all viscosity subsolutions $w$ of $H(x, Du) = c(H)$ satisfying $\int_M w(x) d\nu(x) \leq 0$ for every $\nu \in \mathcal{M}(L)$.

The convergence result in the first part of the theorem above has been obtained in [7] and the representation formula (2.7), under the superlinearity assumption on $H$ in the fibers, is contained in [7]. Formula (2.7) in the general case can be easily obtained by adapting the proof in [5] to our case, or, in other words, by modifying the argument of [7] with some technicalities from [5]. Some of the details of such modifications are explained in Section 6 for the reader’s convenience.

3. Existence of viscosity solutions and some Lipschitz estimates

In this section, we establish the existence and uniqueness of a solution of equation (HJ$_c$), and give some uniform estimates for the whole family of solutions.

We begin with a lemma which describes that every continuous viscosity subsolution is necessarily Lipschitz continuous.

**Lemma 3.1.** Let $H^k$ satisfy (SH1), (SH3) and (SH4), then any continuous viscosity subsolution $u^k$ of (HJ$_c$) is Lipschitz continuous.

**Proof.** Since $u^k$ is bounded on $M$, we set

$$\tilde{H}^k(x, p) := H^k(x, p; -||u^k||_{\infty})$$

it is a classical continuous Hamiltonian which is coercive in the fibers by assumptions (SH3) and (SH4). So $u^k$ is a continuous viscosity subsolution of $\tilde{H}^k(x, Du) = c(G)$ Then by Proposition 2.1, $u^k$ is Lipschitz continuous. \(\square\)

Next, we prove a version of comparison principle. The proof below is easily extended to the case of general semicontinuous sub- and super-solutions, however, the continuous version is enough for the applications in this article.

**Theorem 3.2.** Let $H(x, p, u) : T^*M \times \mathbb{R} \to \mathbb{R}$ be a continuous Hamiltonian which is strictly increasing in $u$ for every $(x, p) \in T^*_x M$. Assume that the equation

$$H(x, Du(x), u(x)) = 0, \quad x \in M,$$

admits a Lipschitz continuous viscosity solution $w$. Then for any continuous viscosity subsolution $f$ and any continuous viscosity supersolution $g$, we have $f \leq g$.

Although the above comparison theorem is well-known, for the reader’s convenience, we present here a proof of it.

**Proof.** We will prove $f \leq g$ by showing $f \leq w$ and $w \leq g$. To see this, we only need to prove $f \leq w$ since the proof of $w \leq g$ is similar.

Let $\hat{x}$ be any maximum point such that $f(\hat{x}) - w(\hat{x}) = \max(f - w)$. We need only to show that

$$f(\hat{x}) - w(\hat{x}) \leq 0.$$  \hspace{1cm} (3.2)

We select a function $\psi \in C^1(M)$ so that $\psi(\hat{x}) = 0$ and $\psi(x) < 0$ for all $x \neq \hat{x}$. Note that the function $f - w + \psi$ has a strict maximum $(f - w)(\hat{x})$ at $\hat{x}$ and that $D\psi(\hat{x}) = 0$. 

Let \( D \subset M \) be a closed domain contained in one of the coordinate charts of \( M \) and \( \hat{x} \in \text{int} D \). Because of the local nature, we can assume, without loss of generality, that \( D = \overline{B}(0,1) \) is a unit ball in \( \mathbb{R}^n \) and \( \hat{x} = 0 \).

Now let \( \alpha > 0 \), consider the function \( \Phi_{\alpha} : D \times D \to \mathbb{R} \) defined by
\[
\Phi_{\alpha}(x,y) := (f + \psi)(x) - w(y) - \alpha|x - y|^2,
\]
and let \((x_0, y_0) \in D \) be a maximum point of \( \Phi_{\alpha} \). From \( \Phi_{\alpha}(x_0, y_0) \geq \Phi_{\alpha}(x, y) \) we get
\[
\alpha|x_0 - y_0|^2 \leq w(x_0) - w(y_0) \leq \text{Lip}(w)|x_0 - y_0|.
\]
From this, we get
\[
\alpha|x_0 - y_0| \leq \text{Lip}(w) \quad \text{and} \quad \lim_{\alpha \to +\infty} (x_0 - y_0) = 0. \tag{3.3}
\]

Hence, by compactness, there exist a sequence \( \{x_j\}_{j \in \mathbb{N}} \) divergent to \(+\infty\) and a point \((\hat{z}, \hat{p}) \in D \times \mathbb{R}^n \) such that
\[
(x_{j_n}, 2\alpha(x_{j_n} - y_{j_n})) \to (\hat{z}, \hat{p}) \quad \text{and} \quad \lim_{j \to +\infty} y_{j_n} = \lim_{j \to +\infty} x_{j_n} = \hat{z}.
\]
Now, since \( \Phi_{\alpha}(x_{j_n}, y_{j_n}) \geq \Phi_{\alpha}(x, x) = (f + \psi - w)(x) \) for all \( x \in D \), we see that
\[
(f + \psi)(x_{j_n}) - w(y_{j_n}) \geq \Phi_{\alpha}(x_{j_n}, y_{j_n}) \geq \max_D (f + \psi - w),
\]
and, moreover, after taking limit for \( \alpha = x_j \),
\[
(f + \psi - w)(\hat{z}) \geq \max_D (f + \psi - w).
\]
This implies that \( \hat{z} = \hat{x} = 0 \) since \( f + \psi - w \) has a strict maximum at \( \hat{x} = 0 \). We may thus assume without loss of generality that \( x_{j_n} \) lies in the interior of \( D \) for every \( j \in \mathbb{N} \).

Since the function \( f(x) - [w(y_{j_n}) + \alpha|x - y_{j_n}|^2 - \psi(x)] \) of \( x \) has a maximum at \( x_{j_n} \) and the function \( w(y) - [(f + \psi)(x_{j_n}) - \alpha|y - x_{j_n}|^2] \) of \( y \) has a minimum at \( y_{j_n} \), by the viscosity property of \( f \) and \( w \), we have
\[
H(x_{j_n}, 2\alpha(x_{j_n} - y_{j_n}) - D\psi(x_{j_n}), f(x_{j_n})) \leq 0 \quad H(y_{j_n}, 2\alpha(x_{j_n} - y_{j_n}), w(y_{j_n})) \geq 0.
\]
Taking limit in \( j \) yields
\[
H(\hat{x}, \hat{p} - D\psi(\hat{x}), f(\hat{x})) \leq 0 \quad \text{and} \quad H(\hat{x}, \hat{p}, w(\hat{x})) \geq 0,
\]
and furthermore, since \( D\psi(\hat{x}) = 0 \),
\[
H(\hat{x}, \hat{p}, f(\hat{x})) \leq 0 \leq H(\hat{x}, \hat{p}, w(\hat{x})).
\]
Now, the strict monotonicity in \( u \) of \( H(x, p, u) \) implies that \( f(\hat{x}) \leq w(\hat{x}) \). Thus, (3.2) holds.

Now we can prove the following result:

**Theorem 3.3.** Let \( H^\lambda \) satisfy (SH1)–(SH4). For each \( \lambda \in (0,1] \), equation \((HJ_\lambda)\) admits a unique continuous viscosity solution \( u^\lambda \). In addition, \( u^\lambda \) is Lipschitzian, and the family \( \{u^\lambda : \lambda \in (0,1]\} \) is equi-bounded and equi-Lipschitzian, namely there exist constants \( C_0 > 0 \) and \( M_0 > 0 \) which are independent of \( \lambda \) such that
\[ \|u^\lambda\|_\infty \leq C_0, \quad \operatorname{Lip}(u^\lambda) \leq M_0. \quad (3.4) \]

**Proof.** Under our assumptions, by the choice of \( c(G) \) and Lemma 2.1, \((HJ_0)\) always admits a Lipschitz viscosity solution, denoted by \( u_0 \). Because \( H^\lambda(x,p,0) = G(x,p) \), by the monotonicity assumption \((SH4)\), \( u^-_0 := u_0 - \|u_0\|_\infty \) and \( u^+_0 := u_0 + \|u_0\|_\infty \) are, respectively, a Lipschitz viscosity subsolution and a Lipschitz viscosity supersolution of \((HJ_\lambda)\). Fix \( \lambda \) and set

\[ u^\lambda(x) := \sup\{ v(x) : u^-_0 \leq v \leq u^+_0, v \text{ is a continuous subsolution of } (HJ_\lambda) \}. \quad (3.5) \]

Notice that if \( v \) is a subsolution of \((HJ_\lambda)\) and \( u^-_0 \leq v \leq u^+_0 \), then

\[ |v(x)| \leq C_0 := 2\|u_0\|_\infty, \quad \text{for all } x \in M. \]

Without loss of generality, we assume \( K_0 := \max_{\lambda} \kappa^2_{\lambda C_0} \) is finite since \( \lim_{\lambda \to 0} \kappa^2_{\lambda C_0} = 0 \), by Assumption \((1.3)\) in \((SH4)\). Now that \( v \) is a subsolution of the equation

\[ G(x,Dv) = a, \]

with \( a = 2K_0C_0 + c(G) \), \( v \) is Lipschitz by Lemma 2.1. In addition, the coercivity of \( G \) implies that for some constant \( N_0 > 0 \),

\[ (x,p) \in T^*_x M, \|p\|_x > N_0 \Rightarrow G(x,p) > a, \]

this shows that, for every \( x \in M \), \( v \) is Lipschitz continuous with Lipschitz bound \( N_0 \) in a neighborhood of \( x \). Since \( M \) is compact and connected, there is a constant \( M_0 \) such that \( \operatorname{Lip}(v) \leq M_0 \).

Thus, by Lemma 3.1, the function \( u^\lambda \) constructed in \((3.5)\) is also Lipschitz continuous with \( \operatorname{Lip}(u^\lambda) \leq M_0 \). Since \( u^\lambda \) is defined by the Perron method (see \([34]\)), it is a viscosity solution of \((HJ_\lambda)\). This proves the existence of continuous viscosity solutions of \((HJ_\lambda)\), and the uniform estimates \((3.4)\).

Finally, since we have shown that \((HJ_\lambda)\) admits a Lipschitz viscosity solution, Theorem 3.2 implies that \((HJ_\lambda)\) has only one continuous viscosity solution, which finishes our proof. \( \square \)

**4. Convergence of the viscosity solutions**

In this section, we investigate the asymptotic behavior of the viscosity solution \( u^\lambda \) of equation \((HJ_\lambda)\), we aim to study whether or not the limit of \( u^\lambda \) exists as \( \lambda \to 0 \), and if the limit exists, what the characterization of this limit is.

We firstly clarify a useful result on the limiting behavior of solutions of \((HJ_\lambda)\), which is a direct consequence of Theorem 3.3, Ascoli-Arzelà theorem, and a well-known stability property of viscosity solutions (see e.g. \([38, 39]\)).

**Proposition 4.1** (Stability). Let \( H^\lambda \) satisfy \((SH1)-(SH4)\) and \( u^\lambda \) be the viscosity solution of \((HJ_\lambda)\), then the family \( \{u^\lambda\}_{\lambda \in [0,1]} \) has a uniformly convergent subsequence. Moreover, for any subsequence \( \{u^{\lambda_n}\}_n \) which converges uniformly to \( u^0 \) as \( \lambda_n \to 0 \), \( u^0 \) is a viscosity solution of equation \((HJ_0)\).
Now it is natural to ask whether the limit $u^0$ is unique. Before that, we give an example where the family of solutions does not converge.

**Example 4.2 (Non-convergence).** Let $G(x, p) \in C(T^*M)$ be coercive and convex in the fibers. For any continuous viscosity solution $w$ of $G(x, Du) = c(G)$, we choose a family $\{w_\lambda\}$ of $C^2$ functions satisfying $||w-w_\lambda|| \leq \lambda$ and consider the Hamiltonians

$$H^\lambda(x, p, u) = \lambda u + G(x, p) - \lambda w_\lambda(x), \quad \lambda > 0.$$ 

Then $w(x) - \lambda$ and $w(x) + \lambda$ are, respectively, a subsolution and a supersolution of the equation $H^\lambda(x, Du(x), u(x)) = c(G)$. Thus by the comparison principle for discounted equations, if $u^k \in C(M)$ is the solution of (HJ$^\lambda$), then

$$w(x) - \lambda \leq u^k(x) \leq w(x) + \lambda.$$ 

As $\lambda$ goes to zero, $u^k$ converges to $w$.

Now let $f$ and $g$ be two distinct viscosity solutions of the equation $G(x, Du) = c(G)$. We select two families $\{f_\lambda\}$ and $\{g_\lambda\}$ of $C^2$ functions satisfying $||f-f_\lambda|| \leq \lambda$ and $||g-g_\lambda|| \leq \lambda$. Consider the following family of Hamiltonian:

$$H^\lambda(x, p, u) = \begin{cases} 
\lambda u + G(x, p) - \lambda f_\lambda(x), & \text{if } \lambda \in \left(\frac{1}{2n+1}, \frac{1}{2n}\right]; \\
\lambda u + G(x, p) - \lambda g_\lambda(x), & \text{if } \lambda \in \left(\frac{1}{2n+2}, \frac{1}{2n+1}\right]. 
\end{cases}$$

It is easy to check that $H^\lambda$ satisfies all requirements in assumptions (SH1)–(SH5) except the hypothesis $H^\lambda(x, p, 0) = G(x, p)$. However, $H^\lambda(x, p, 0)$ converges uniformly to $G(x, p)$. By the same argument as above, one can find that, as $\lambda \to 0$, the limits of $\{u^k\}$ cannot be unique, one is $f(x)$, the other is $g(x)$.

Under assumptions (SH2) and (SH3), we can define the Lagrangian $L_G : TM \to (-\infty, +\infty]$ associated with the Hamiltonian $G$ by

$$L_G(x, v) = \sup_{p \in \Gamma_x^*M} \{ \langle p, v \rangle_x - G(x, p) \}. \quad (4.1)$$

We denote by $\mathfrak{M}(L_G)$ the set of all projected Mather measures, associated with $L_G$, and by $\mathcal{F}(G)$ the set of all viscosity subsolutions $w$ of $G(x, Du) = c(G)$ satisfying

$$\int_M w(x) d\nu(x) \leq 0, \quad \forall \nu \in \mathfrak{M}(L_G).$$

Now we are ready to prove the main theorem:

**Theorem 4.3.** Let $\{H^\lambda\}_{\lambda \in (0,1]}$ be the Hamiltonians satisfying (SH1)–(SH5). Then equation (HJ$^\lambda$) has a unique continuous viscosity solution $u^\lambda$, which is also Lipschitzian, and the family $\{u^\lambda\}_{\lambda > 0}$ converges uniformly, as $\lambda \to 0$, to a single $u^0$ which is a Lipschitz continuous viscosity solution of (HJ$^0$). Furthermore, the limit $u^0$ is characterized by

$$u^0(x) = \sup_{u \in \mathcal{F}(G)} u(x) = \min_{\mu \in \mathfrak{M}(L_G)} \int_M h(y, x) d\mu(y), \quad (4.2)$$

where $h(y, x)$ is the Peierls barrier of the Lagrangian $L_G$. 
Proof. By Theorem 3.3, we can fix a positive constant \( R_0 > 0 \) so that
\[
\max \left\{ ||u^\lambda||_{\infty}, \ \text{Lip}(u^\lambda) \right\} \leq R_0, \quad \text{for all} \ \lambda \in (0, 1].
\]

Now we claim that the functions
\[
u^-_\lambda := u^\lambda - \left( \frac{K^\lambda_{R_0} - \delta^\lambda_{R_0}}{K^\lambda_{R_0}} \right) R_0, \quad \nu^+_\lambda := u^\lambda + \left( \frac{K^\lambda_{R_0} - \delta^\lambda_{R_0}}{K^\lambda_{R_0}} \right) R_0
\]
are, respectively, a subsolution and a supersolution of the discounted equation
\[
K^\lambda_{R_0} u + G(x, Du) = c(G), \quad x \in M.
\]

Indeed, this can be easily deduced from assumption (SH5), that is, for any \( |p|_x \leq R_0 \) and \( |u| \leq R_0 \),
\[
K^\lambda_{R_0} u - \left( \frac{K^\lambda_{R_0} - \delta^\lambda_{R_0}}{K^\lambda_{R_0}} \right) R_0 \leq H^\lambda(x, p, u) - G(x, p) \leq K^\lambda_{R_0} u + \left( \frac{K^\lambda_{R_0} - \delta^\lambda_{R_0}}{K^\lambda_{R_0}} \right) R_0.
\]

On the other hand, under our assumptions, we know from Theorem 2.6 that (4.3) admits a unique continuous viscosity solution, denoted by \( w^\lambda \). By the comparison principle (Theorem 3.2) for the discounted Eq. (4.3), we obtain
\[
u^-_\lambda \leq w^\lambda \leq \nu^+_\lambda.
\]

Condition (SH5) implies that
\[
\lim_{\lambda \to 0} \frac{K^\lambda_{R_0} - \delta^\lambda_{R_0}}{K^\lambda_{R_0}} = 0,
\]
this yields \( \lim_{\lambda \to 0} (u^\lambda - w^\lambda) = 0 \). As \( K^\lambda_{R_0} \to 0 \), by Theorem 2.6, \( w^\lambda \) converges uniformly to a unique \( u^0 \), which is a viscosity solution of (HJ0), and formula (4.2) is valid. The proof is complete. \( \square \)

An argument completely different from the above applies to obtain a convergence result when the constant functions are viscosity subsolutions of (HJ0), and then we could remove assumption (SH5) and obtain a simpler representation formula for the limit \( u^0 \).

Theorem 4.4. Let \( \{H^\lambda\}_{\lambda} \) satisfy (SH1)–(SH4) and assume that the constant functions are viscosity subsolutions of (HJ0). Then equation (HJ\_\lambda) has a unique continuous viscosity solution \( u^\lambda \), which is also Lipschitzian, the collection \( \{u^\lambda\} \) converges uniformly, as \( \lambda \to 0 \), to a Lipschitz continuous viscosity solution \( u^0 \) of (HJ0), and
\[
u_0(x) = \min_{y \in A} h(y, x),
\]
where \( h(y, x) \) and \( A \) are, respectively, the Peierls barrier and the Aubry set of \( LG \).

Proof. The existence and uniqueness of a viscosity solution follow from Theorem 3.3. Now, we set
ux(\cdot) := \min_{y \in A} h(y, x).

Since constant functions are viscosity subsolutions of \((HJ_0)\), we see by (3) of Proposition 2.4 that \(h(y,x) \geq 0\) for all \(x, y\). Hence, in view of (2) of Proposition 2.4, we see immediately that \(u \geq 0\) in \(M\) and it is a viscosity solution of \((HJ_0)\). Moreover, it is easy to see that \(u\) and \(0\) are, respectively, a supersolution and a subsolution of \((HJ_\lambda)\) because of the monotonicity of \(u \mapsto H^\lambda(x,p,u)\) and the identity \(H^\lambda(x,p,0) = G(x,p)\). Thus, by the comparison principle,

\[0 \leq u^\lambda \leq \hat{u}.
\]

In particular, since \(\hat{u}\) equals to zero on the Aubry set \(A\), this implies

\[u^\lambda(x) = 0, \quad \forall x \in A.
\]

(4.5)

To finish our proof, it suffices, by Proposition 4.1, to prove that any converging subsequence has the same limit \(\hat{u}\). Indeed, if \(u^0\) is the limit of a convergent sequence \(\{u^\lambda_n\}\), then, by (4.5), \(u^0(x) = 0\) for all \(x \in A\). Furthermore, the assertion (4) of Proposition 2.4, a well-known property of Aubry sets, ensures that \(u^0 = \hat{u}\). \(\square\)

5. Applications: Nonlinear vanishing discount problem

In order to verify that our assumptions in Section 4 are reasonable and rational, we deal with some nonlinear discounted systems and demonstrate the convergence results in this section.

5.1. Application I

Firstly, we consider a simple example which is a direct generalization of the discounted equations. Suppose that

(C1) \(H \in C(T^*M)\) is convex and coercive in the fibers.
(C2) \(f(x,u) \in C^1(M \times \mathbb{R})\) with \(f_u > 0\) and

\[
\limsup_{|u| \to 0} \frac{|f_u(x,u) - f_u(x,0)|}{|u|} < +\infty, \quad \text{uniformly for } x \in M.
\]

(5.1)

Let \(c = c(G)\) be the critical value of the Hamiltonian

\[G(x,p) := f(x,0) + H(x,p).
\]

Now for each \(\lambda \in (0,1]\), we consider the following equation:

\[f(x,\lambda u) + H(x,Du) = c, \quad x \in M.
\]

(5.2)

and

\[f(x,0) + H(x,Du) = c, \quad x \in M.
\]

(5.3)
Theorem 5.1. Under the above assumptions (C1)–(C2), equation (5.2) has a unique continuous viscosity solution \( u^\lambda \), which is also Lipschitz continuous, and the convergence

\[
u^0(x) = \lim_{\lambda \to 0} u^\lambda(x), \quad \text{uniformly for all } x \in M.
\]

holds, where the limit \( u^0 \) is a Lipschitz viscosity solution of (5.3).

The limit \( u^0 \) is characterized by

\[
u^0(x) = \sup_{u \in \mathcal{F}(G)} u(x) = \min_{\mu_\ast \in \mathcal{M}(L_\mu)} \int_M \bar{h}(y, x) d\mu_\ast(y), \quad \text{(5.4)}
\]

where \( \bar{h} \) is the Peierls barrier associated with the new Hamiltonian \( \bar{G}(x, p) = \frac{G(x, p) - c}{f_u(x, 0)} \).

**Proof.** We apply Theorem 4.3 to prove the theorem. For each \( \lambda > 0 \), set \( H^\lambda(x, Du, u) = f(x, \lambda u) + H(x, p) \). Since \( f_u(x, 0) > 0 \), one can easily observe that \( u^\lambda \) is a viscosity solution of (5.2) if and only if \( u^\lambda \) is a viscosity solution of

\[
\frac{H^\lambda(x, Du) - c}{f_u(x, 0)} = 0.
\]

Similarly, \( u \) is a viscosity solution of (5.3) if and only if \( u \) is a viscosity solution of the equation

\[
\frac{G(x, Du) - c}{f_u(x, 0)} = 0.
\]

This leads us to consider a new family of continuous Hamiltonians

\[
\bar{H}^\lambda(x, p, u) = \frac{H^\lambda(x, p, u) - c}{f_u(x, 0)}.
\]

and the corresponding

\[
\bar{G}(x, p) = \frac{G(x, p) - c}{f_u(x, 0)}.
\]

The partial derivative of \( \bar{H}^\lambda \) with respect to \( u \) is

\[
\bar{H}_u^\lambda = \frac{H_u^\lambda}{f_u(x, 0)} = \frac{\lambda f_u(x, \lambda u)}{f_u(x, 0)} > 0.
\]

Thus one can easily find that \( \bar{H}^\lambda \) satisfies all assumptions (SH1)–(SH4).

It only remains to check assumption (SH5). For every \( R > 0 \), let

\[
\delta^\lambda_R := \min_{(x, p) \in T^M \mid |u| < R} \bar{H}_u^\lambda(x, p, u), \quad K^\lambda_R := \max_{(x, p) \in T^M \mid |u| < R} \bar{H}_u^\lambda(x, p, u).
\]

It suffices to prove that

\[
\lim_{\lambda \to 0} \frac{\delta^\lambda_R}{K^\lambda_R} = 1, \quad \text{for every } R > 0.
\]

Indeed, from (5.1) we know that there exist \( \epsilon > 0 \) and \( B > 0 \) such that

\[
|f_u(x, u) - f_u(x, 0)| \leq B|u|, \quad \text{for all } x \in M, |u| \leq \epsilon,
\]
then as long as \( \lambda \) is small enough so that \( \lambda R < \varepsilon \), we have
\[
\lambda \left(1 - \frac{\lambda RB}{f_u(x,0)}\right) \leq \tilde{H}_u^\lambda(x,p,u) = \frac{\lambda f_u(x,\lambda u)}{f_u(x,0)} \leq \lambda \left(1 + \frac{\lambda RB}{f_u(x,0)}\right), \quad \forall |u| \leq R.
\]

Hence, if we denote \( a = \min_x f_u(x,0) \), then for \( \lambda < 1 \),
\[
\delta^\lambda_R \geq \lambda \left(1 - \frac{\lambda RB}{a}\right), \quad K^\lambda_R \leq \lambda \left(1 + \frac{\lambda RB}{a}\right).
\]
this leads to
\[
1 = \liminf_{\lambda \to 0} \frac{1 - \frac{\lambda RB}{a}}{1 + \frac{\lambda RB}{a}} \leq \liminf_{\lambda \to 0} \frac{\delta^\lambda_R}{K^\lambda_R} \leq \limsup_{\lambda \to 0} \frac{\delta^\lambda_R}{K^\lambda_R} \leq 1,
\]
which proves assumption (SH5). Thus, by Theorem 4.3, the equation
\[
\tilde{H}^\lambda(x, Du, u) = 0
\]
admits a unique continuous viscosity solution \( u^\lambda \), and \( u^\lambda \) uniformly converges, as \( \lambda \to 0 \), to a unique \( u^0 \), which is a continuous viscosity solution of \( \tilde{G}(x, Du) = 0 \). Theorem 4.3, moreover, guarantees that (5.4) is valid. \( \square \)

Notice that the limit \( u^0 \) in Theorem 5.1 is characterized by a new Hamiltonian \( \tilde{G} \). A natural question is: Can we represent the limit \( u^0 \) in terms of the original Hamiltonian \( G \)? To answer this question, we analyze the dynamical links between \( G \) and \( \tilde{G} \).

**Proposition 5.2.** Let \( H \in C(T^*M) \) be convex and coercive in the fibers with critical value \( c \), and \( f \in C^1(M) \) with \( f > 0 \). Set
\[
\tilde{H}(x,p) := \frac{H(x,p) - c}{f(x)}, \quad x \in M, p \in T^*_x M.
\]
Let \( L \) and \( \tilde{L} \) be the Lagrangians associated with \( H \) and \( \tilde{H} \), respectively.

1. Let \( \tilde{h} \) and \( h \) be the Peierls barriers of the Lagrangians \( \tilde{L} \) and \( L \), respectively. Then
\[
\tilde{h}(y,x) = h(y,x). \tag{5.9}
\]
In particular, the projected Aubry sets \( A(\tilde{L}) \) and \( A(L) \) are identical.

2. There is a one-to-one correspondence between the Mather measures \( \tilde{\mathcal{M}}(\tilde{L}) \) and \( \mathcal{M}(L) \) associated with \( L \) and \( \tilde{L} \), respectively. More precisely, for any \( \tilde{\mu} \in \tilde{\mathcal{M}}(\tilde{L}) \), if the Borel measure \( \tilde{\mu}_* \) is defined by
\[
\int \psi(x,v) d\tilde{\mu}_* := \frac{1}{\int f(x) d\tilde{\mu}} \int \psi(x, \frac{v}{f(x)}) f(x) d\tilde{\mu}, \quad \forall \psi \in C_c(TM), \tag{5.10}
\]
then \( \tilde{\mu}_* \in \tilde{\mathcal{M}}(\tilde{L}) \). Conversely, for any \( \tilde{\mu}_* \in \tilde{\mathcal{M}}(\tilde{L}) \), if \( \tilde{\mu} \) is defined by
\[
\int \psi(x,v) d\tilde{\mu} := \frac{1}{\int 1/f(x) d\tilde{\mu}_*} \int \psi(x,f(x)v)/f(x) d\tilde{\mu}_*, \quad \forall \psi \in C_c(TM). \tag{5.11}
\]
then \( \tilde{\mu} \in \tilde{\mathcal{M}}(\tilde{L}) \).
Proof. It is immediate to see that if \( u \in C(M) \) is a viscosity solution of \( H(x, Du) = c \), then it is also a viscosity solution of \( \hat{H}(x, Du) = 0 \). Since \( c \) is the critical value of \( H \), this means that 0 is the critical value of \( \hat{H} \).

A simple manipulation shows that

\[
\bar{L}(x, v) = \frac{L(x, f(x)v) + c}{f(x)}, \quad x \in M, \quad v \in T_x M.
\]

(1) Now, we define the semi-distances \( S \in C(M \times M) \) and \( \bar{S} \in C(M \times M) \), associated with the Hamiltonians \( H - c \) and \( \bar{H} \), respectively, by

\[
S(y, x) = \sup \left\{ \psi(x) - \psi(y) : \psi \in S^-(H - c) \right\},
\]

\[
\bar{S}(y, x) = \sup \left\{ \psi(x) - \psi(y) : \psi \in S^-(\bar{H}) \right\}.
\]

It is clear that \( S^-(H - c) = S^-(\bar{H}) \), hence \( S = \bar{S} \). Thus, by Proposition 2.3, we see that \( h = \bar{h} \). Furthermore, we have \( \mathcal{A}(\bar{L}) = \{ x \in M : \hat{h}(x, x) = 0 \} = \{ x \in M : h(x, x) = 0 \} = \mathcal{A}(L) \).

(2) If \( \tilde{\mu} \in \mathcal{M}(L) \), then the measure \( \tilde{\mu}_s \), defined by (5.10), is a Borel probability measure. Because \( \tilde{\mu} \) is closed, then \( \tilde{\mu}_s \) is also a closed measure since

\[
\int \langle D\psi, v \rangle_x d\tilde{\mu}_s - \int \frac{\langle D\psi, v \rangle_x d\tilde{\mu}}{f(x) d\tilde{\mu}} = 0, \quad \forall \, \psi \in C^1(M).
\]

As the critical value of \( \bar{L} \) is zero, now it only remains to check that \( \int \bar{L} d\tilde{\mu}_s = 0 \). Indeed,

\[
\int \bar{L}(x, v) d\tilde{\mu}_s = \int \frac{L(x, v) + cd\tilde{\mu}}{f(x) d\tilde{\mu}} = 0,
\]

which yields \( \tilde{\mu}_s \in \mathcal{M}(L) \). Similarly, one can prove that if \( \tilde{\mu}_s \in \mathcal{M}(\bar{L}) \), then \( \tilde{\mu} \in \mathcal{M}(L) \) with \( \tilde{\mu} \) defined as (5.11). Thus there is a one-to-one correspondence between \( \mathcal{M}(L) \) and \( \mathcal{M}(\bar{L}) \).

**Theorem 5.3.** Under the same assumptions as in Theorem 5.1, the limit \( u^0 \) in (5.4) is also characterized by

\[
u^0(x) = \sup_{w \in \mathcal{F}(G)} w(x) = \min_{\mu \in \mathcal{M}(L)} \frac{\int_M f_u(y, 0) h(y, x) d\mu(y)}{\int_M f_u(y, 0) d\mu(y)},
\]

where \( \mathcal{F}(G) \) is the set of all viscosity subsolutions \( w \) of (5.3) satisfying

\[
\int_M f_u(y, 0) w(y) d\mu(y) \leq 0, \quad \forall \, \mu \in \mathcal{M}(L).
\]

**Proof.** If \( L_G \) is the Lagrangian associated with \( G = f(x, 0) + H(x, p) \), then the Lagrangian associated with \( \tilde{G}(x, p) = \frac{G(x, p) - c}{f_u(x, 0)} \) is

\[
\tilde{L}_G(x, v) = \frac{L_G(x, f_u(x, 0) v) + c}{f_u(x, 0)}.
\]
By Proposition 5.2 and (5.4) in Theorem 5.1, we easily get

$$\min_{\mu, \in \mathfrak{N}(G)} \int_M h(y, x) d\mu(y) = \min_{\mu, \in \mathfrak{N}(G)} \int_M h(y, x) d\mu(y)$$

$$= \min_{\mu, \in \mathfrak{N}(G)} \frac{\int_M f_u(y, 0) h(y, x) d\mu(y)}{\int_M f_u(y, 0) d\mu(y)}.$$

On the other hand, $w$ is a viscosity subsolution of (5.3) if and only if $w$ is a viscosity subsolution of $G(x, Du) = 0$. Thus, if $w \in \mathcal{F}(G)$, that is, $\int_M w(y) d\mu(y) \leq 0$ for all $\mu, \in \mathfrak{N}(L_G)$, then by Proposition 5.2 again,

$$\int_M f_u(y, 0) w(y) d\mu(y) \leq 0, \quad \forall \mu, \in \mathfrak{N}(L_G).$$

Equivalently, we have that

$$\int_M f_u(y, 0) w(y) d\mu(y) \leq 0, \quad \forall \mu, \in \mathfrak{N}(L_G),$$

since $\int_M f_u(y, 0) d\mu(y) > 0$. This leads to

$$u^0 = \sup_{w \in \mathcal{F}(G)} w(x) = \sup_{w \in \mathcal{F}(G)} w(x).$$

5.2. Application II

Now we study a more general class of systems, let $H = H(x, p, u)$ and $G(x, p) := H(x, p, 0)$ be the Hamiltonians satisfying

(D1) $H \in C(T'M \times \mathbb{R})$, the partial derivative $H_u \in C(T'M)$ with $H_u > 0$. For any $R > 0$, there exists a constant $B_R > 0$ so that

$$|H(x, p, u) - H(x, p, 0)| \leq B_R |u|, \quad \text{for all } |u| \leq R. \quad (5.12)$$

and

$$H_u(x, p, u) - H_u(x, p, 0) \leq B_R |u|, \quad \text{for all } |p| \leq R, |u| \leq R. \quad (5.13)$$

(D2) $G(x, p)$ and $\frac{G(x, p) - c}{H_u(x, p, 0)}$ is convex and coercive in the fibers.

Let $c = c(G)$ be the critical value of $G$, for each $\lambda > 0$, we consider the equations

$$H(x, Du, \lambda u) = c, \quad x \in M. \quad (5.14)$$

and

$$H(x, Du, 0) = c, \quad x \in M. \quad (5.15)$$
For example, we can take
\[ H(x, p, u) = u + \frac{1}{2} \cos^2 u \cdot \sin p + \frac{p^2}{2} + V(x) \]
where \((x, p) \in \mathbb{T}^n \times \mathbb{R}^n\) and \(\delta \ll 1\). Notice that \(\frac{1}{2} \leq H_u \leq \frac{3}{2}\) and \(H_u(x, p, 0) = 1\).

**Theorem 5.4.** Under assumptions (D1)–(D2) above, equation (5.14) has a unique continuous viscosity solution \(u^\lambda\), which is also Lipschitz continuous, and the convergence
\[ u^0(x) = \lim_{\lambda \to 0} u^\lambda(x), \quad \text{uniformly for all } x \in M, \]
holds. Moreover, the limit function \(u_0\) is a Lipschitz viscosity solution of (5.15) and it is characterized by
\[ u^0(x) = \sup_{u \in \mathcal{C}(\mathcal{D}\mathcal{L})} u(x) = \min_{\mu \in M} \int_M h(y, x) d \mu(y), \quad (5.16) \]
where \(h(y, x)\) is the Peierls barrier of the Lagrangian \(L_c\) associated with the Hamiltonian \(G(x, p) = \frac{G(x, p) - c}{H_u(x, p, 0)}\).

**Proof.** Set \(H^\lambda(x, p, u) = H(x, p, \lambda u)\). It follows from hypotheses (D1)–(D2) that \(H^\lambda\) satisfies assumptions (SH1)–(SH4). Hence, by Theorem 3.3, Eq. (5.14) has a unique continuous viscosity solution, and the whole family \(\{u^\lambda\}_{\lambda}\) is equi-bounded and equi-Lipschitz, that is, there exists a constant \(R_0 > 0\) so that
\[ \max\left\{ \|u^\lambda\|_\infty, \text{ Lip}(u^\lambda) \right\} \leq R_0, \quad \text{for all } \lambda \in (0, 1]. \]

Since \(H_u \geq 0\), one can easily observe that \(u^\lambda\) is a viscosity solution of (5.14) if and only if \(u^\lambda\) is a viscosity solution of
\[ \frac{H(x, Du, \lambda u) - c}{H_u(x, Du, 0)} = 0. \quad (5.17) \]

Similarly, \(u\) is a viscosity solution of Equation (5.15) if and only if \(u\) is a viscosity solution of
\[ \frac{H(x, Du, 0) - c}{H_u(x, Du, 0)} = 0. \quad (5.18) \]

This leads us to introduce a family of new Hamiltonians
\[ \tilde{H}^\lambda(x, p, u) = \frac{H(x, p, \lambda u) - c}{H_u(x, p, 0)}. \]
and the corresponding Lagrangians
\[ \tilde{G}(x, p) = \frac{G(x, p) - c}{H_u(x, p, 0)}. \]

The partial derivative of \(\tilde{H}^\lambda\) with respect to \(u\) is
\[ \tilde{H}_u^\lambda(x, p, u) = \frac{\lambda H_u(x, p, \lambda u)}{H_u(x, p, 0)} > 0. \]
For every number $R > 0$, let
\[
\delta^\lambda_R := \min_{x \in M} \tilde{H}_u^\lambda(x, p, u), \quad K^\lambda_R := \max_{x \in M} \tilde{H}_u^\lambda(x, p, u).
\]

Now we claim that $\delta^\lambda_R$ satisfy assumption (SH5). It suffices to prove that
\[
\lim_{\lambda \to 0} \delta^\lambda_R = 1, \quad \text{for every } R > 0.
\]

Indeed, for all $|p|, |u| \leq R$, by (5.13) we have
\[
\lambda \left(1 - \frac{\lambda R B_R}{H_u(x, p, 0)} \right) \leq \tilde{H}_u^\lambda(x, p, u) = \frac{\lambda H_u(x, p, \lambda u)}{H_u(x, p, 0)} \leq \lambda \left(1 + \frac{\lambda R B_R}{H_u(x, p, 0)} \right).
\]

Thus if we denote $a := \min_{|p| \leq R} |H_u(x, p, 0)| > 0$, then
\[
\delta^\lambda_R \geq \lambda \left(1 - \frac{\lambda R B_R}{a} \right), \quad K^\lambda_R \leq \lambda \left(1 + \frac{\lambda R B_R}{a} \right).
\]

This leads to
\[
1 = \liminf_{\lambda \to 0} \frac{\lambda R B_R}{a} \leq \liminf_{\lambda \to 0} \delta^\lambda_R \leq \limsup_{\lambda \to 0} \delta^\lambda_R \leq 1.
\]

So (5.19) is valid, which implies assumption (SH5).

On the other hand, by the same argument as in the proof of Theorem 4.3, we know that
\[
\bar{u}^- := u^\lambda - \frac{\lambda R B_R}{K^\lambda_R} R_0, \quad \bar{u}^+ := u^\lambda + \frac{\lambda R B_R}{K^\lambda_R} R_0
\]
are, respectively, a subsolution and a supersolution of the discounted equation
\[
K^\lambda_R u + \bar{G}(x, Du) = 0, \quad \forall \ x \in M.
\]

So by Theorem 2.6, Eq. (5.20) admits a unique continuous viscosity solution, denoted by $w^\lambda$. By the comparison principle for Eq. (5.20), we obtain
\[
\bar{u}^\lambda \leq w^\lambda \leq \bar{u}^\lambda.
\]

Condition (SH5) implies that
\[
\lim_{\lambda \to 0} \frac{\lambda R B_R}{K^\lambda_R} = 0,
\]
this yields $\lim_{\lambda \to 0} (\bar{u}^\lambda - \bar{u}^\lambda) = 0$. As $K^\lambda_R \to 0$, by Theorem 2.6, $w^\lambda$ converges uniformly to a unique $u^0$, which is a viscosity solution of $\bar{G}(x, Du) = 0$, and the function $u^0$ is characterized by (5.16). This finishes our proof.

\[\Box\]

6. Peierls barriers and Theorem 2.6

In this section, we mainly focus on the proof of Theorem 2.6 and Proposition 2.3. As in Section 2, we always assume in this section that $M$ is a connected and compact
manifold without boundary and $H(x, p) : T^*M \to \mathbb{R}$ is a given Hamiltonian that satisfies (H1)–(H3). We consider the discounted problem

$$\dot{u} + H(x, Du) = c(H) \text{ in } M,$$

and the limit problem

$$H(x, Du) = c(H) \text{ in } M.$$ 

It might be worth recalling (see (2.1), (2.2), and (2.3)) that, under hypotheses (H1)–(H3), the formula

$$L(x, v) = \sup_{p \in T^*_x M} [\langle p, v \rangle - H(x, p)]$$

defines an extended real-valued, lower semicontinuous function in $T^*_x M$ which is convex and superlinear in each fiber $T_x M$ and bounded in a neighborhood of the zero section of $T^*_x M$.

We give some details of the proof of the representation formula (2.7) in Theorem 2.6 in the generality of hypotheses (H1)–(H3). The argument is a modification of that in [7] as suggested or indicated in [5]. Recall again that the convergence assertion of Theorem 2.6 has been established in [7] under the hypotheses (H1)–(H3).

The following proposition asserts that one of the representation formula (2.7) is valid.

**Proposition 6.1.** Let $u^0 \in \text{Lip}(M)$ be the uniform limit of the family $\{u^x\}_{x > 0}$ of the viscosity solutions $u^x \in \text{Lip}(M)$ of $(DP_\lambda)$, then

$$u^0(x) = \max \left\{ w(x) : w \in S^-(H - c(H)), \int_M w(x) \, d\mu(x) \leq 0 \text{ for all } \mu \in \mathcal{M}(L) \right\}. \quad (6.2)$$

**Proof.** We show first that

$$\int_M u^0(x) \, d\mu(x) \leq 0 \text{ for all } \mu \in \mathcal{M}(L). \quad (6.3)$$

Indeed, by approximation (Proposition 2.5), for each $\lambda > 0$ and $\varepsilon > 0$, there exists $u^x_\varepsilon \in C^1(M)$ such that

$$H(x, Du^x_\varepsilon) \leq c(H) - \lambda u^x_\varepsilon + \varepsilon \text{ for all } x \in M,$$

which leads to

$$\langle Du^x_\varepsilon, v \rangle \leq c(H) + L(x, v) - \lambda u^x_\varepsilon + \varepsilon \text{ for all } (x, v) \in TM,$$

Integration by $\tilde{\mu} \in \mathcal{M}(L)$ yields

$$0 = \int_{TM} \langle Du^x_\varepsilon, v \rangle \, d\tilde{\mu}(x, v) \leq \int_{TM} \left( c(H) + L - \lambda u^x_\varepsilon + \varepsilon \right) \, d\tilde{\mu}(x, v)$$

$$= -\lambda \int_M u^x_\varepsilon(x) \, d\mu(x) + \varepsilon,$$

where $\mu := \pi \tilde{\mu}$, which shows, in the limit as $\varepsilon \to 0$ and $\lambda \to 0$, that (6.3) is valid.

Now the only thing we need to show is that, for any $w \in S^-(H - c(H))$ with $\int_M w \, d\mu(x) \leq 0$ for all $\mu \in \mathcal{M}(L)$, $w \leq u^0$ in $M$.

Indeed, according to Theorem 3.3, we have a constant $R > 0$ such that $w$ and the functions $u^x_\varepsilon$ are Lipschitz continuous with Lipschitz bound $R$. Choose any sequence $\{\delta_j\}_{j \in \mathbb{N}}$ of positive numbers converging to zero and define $H_{\delta_j}^x$ by
where \( B_{x,R} \) denotes the ball in \( T_x^* M \) of radius \( R \) with the center at the origin, and \( \text{dist}(p, B_{x,R}) \) denotes the distance in \( T_x^* M \) between \( p \) and \( B_{x,R} \). Let \( L^j_R \) be the Lagrangian of \( H^j_R \). Since \( H^j_R \) is superlinear in the fiber \( T_x^* M \) for every \( x \in M \), the Lagrangians \( L^j_R \) are finite valued and continuous in \( TM \) and have superlinear growth in each fiber \( T_x M \). Furthermore, \( u^\lambda \), with \( \lambda \geq 0 \), is a viscosity solution of \( \lambda u^\lambda + H^j_R(x, Du^\lambda) = c(H) \) in \( M \) and \( w \) is a viscosity subsolution of \( H^j_R(x, Dw) = c(H) \) in \( M \).

By [7, Proposition 3.5], for each \( z \in M \), \( \lambda > 0 \) and \( j \in \mathbb{N} \), there exists \( \tilde{\mu}^{z,\lambda,j} \in \mathcal{P}(TM) \) such that

\[
\lambda u^\lambda(z) = \int_{TM} \left[ L^j_R(x, v) + c(H) \right] d\tilde{\mu}^{z,\lambda,j}(x, v), \tag{6.4}
\]

and, moreover, as a direct consequence of [7, (3.5)],

\[
\lambda \psi(z) = \int_{TM} \left[ (Du \psi)_x + \lambda \psi(x) \right] d\tilde{\mu}^{z,\lambda,j}(x, v) \text{ for all } \psi \in C^1(M). \tag{6.5}
\]

Note that \( u := w \) is a viscosity subsolution of \( \lambda u + H^j_R(x, Du) = c(H) + \lambda w \) in \( M \), and, by approximation (Proposition 2.5), we may choose \( w_\varepsilon \) for each \( \varepsilon > 0 \) so that \( ||w_\varepsilon - w||_{\infty} \leq \varepsilon \) and

\[
\lambda w_\varepsilon + (Dw_\varepsilon)_x \leq c(H) + L^j_R(x, v) + \lambda w + \varepsilon \text{ in } TM.
\]

Integration with respect to \( \tilde{\mu}^{z,\lambda,j} \), combined with (6.5) and (6.4), yields

\[
\lambda w_\varepsilon(z) = \int_{TM} \left[ (Dv_\varepsilon)_x + \lambda w_\varepsilon \right] d\tilde{\mu}^{z,\lambda,j}(x, v) \\
\leq \int_{TM} \left[ c(H) + L^j_R + \lambda w + \varepsilon \right] d\tilde{\mu}^{z,\lambda,j}(x, v) = \lambda u^\lambda(z) + \lambda \int_M w(x) \ d\mu^{z,\lambda,j}(x) + \varepsilon,
\]

where \( \mu^{z,\lambda,j} := \pi \tilde{\mu}^{z,\lambda,j} \). Hence, by sending \( \varepsilon \to 0 \) and dividing by \( \lambda \), we get

\[
w(z) \leq u^\lambda(z) + \int_M w(x) \ d\mu^{z,\lambda,j}(x). \tag{6.6}
\]

Noting that \( L^j_R \leq L^{j+1}_R \) and \( \lim_{j \to \infty} L^j_R = L \), we find that for all \( j, k \in \mathbb{N} \),

\[
\int_{TM} \left[ c(H) + L^j_R \right] d\tilde{\mu}^{z,\lambda,j+k}(x, v) \leq \int_{TM} \left[ c(H) + L^{j+k}_R \right] d\tilde{\mu}^{z,\lambda,j+k}(x, v) = \lambda u^\lambda(z). \tag{6.7}
\]

Since every Lagrangian \( L^j_R \) has superlinear growth in the fibers, (6.7) shows that the collection of probability measures \( \{\tilde{\mu}^{z,\lambda,j}\}_{j \in \mathbb{N}} \) is tight and has a weakly convergent subsequence (in the sense of measures), which we denote still by the same symbol, to a probability measure \( \tilde{\mu}^{z,\lambda} \in \mathcal{P}(TM) \). From (6.7), we get

\[
\int_{TM} \left[ c(H) + L^j_R \right] d\tilde{\mu}^{z,\lambda}(x, v) \leq \lambda u^\lambda(z) \text{ for all } j \in \mathbb{N},
\]

and then by the monotone convergence theorem that

\[
\int_{TM} \left[ c(H) + L \right] d\tilde{\mu}^{z,\lambda}(x, v) \leq \lambda u^\lambda(z).
\]
This shows that the family \( \{\mu^\lambda\}_\lambda > 0 \) is tight, and we can choose a sequence \( \{\mu^\lambda\}_{k \in \mathbb{N}} \) converging weakly in the sense of measures to a \( \tilde{\mu}^z \in \mathcal{P}(TM) \). By the lower semicontinuity of \( L \) and the fact that \( \lim_{\lambda \to 0} u^\lambda(x) = u^0(x) \) uniformly, we deduce that

\[
\int_{TM} [c(H) + L] \, d\tilde{\mu}^z(x, v) \leq 0. \tag{6.8}
\]

Also, it is easily seen from (6.5) and (6.6) that

\[
\int_{TM} \langle D\psi, v \rangle_x \, d\tilde{\mu}^z(x, v) = 0 \text{ for } \psi \in C^1(M), \tag{6.9}
\]

and

\[
w(z) \leq u^0(z) + \int_M w(x) d\mu^z(x), \tag{6.10}
\]

where \( \mu^z := \pi \tilde{\mu}^z \).

The identity (6.9) means that \( \tilde{\mu}^z \) is a closed probability measure on \( TM \) and, this together with (6.8) ensures that \( \tilde{\mu}^z \) is a Mather measure for \( L \), that is, \( \tilde{\mu}^z \in \mathcal{M}(L) \). Moreover, it follows that

\[
\int_{TM} [c(H) + L] \, d\tilde{\mu}^z(x, v) = 0.
\]

By the choice of \( w \), we have \( \int_M w(x) \, d\mu^z(x) \leq 0 \). Thus, we conclude from (6.10) that \( w(z) \leq u^0(z) \), where \( z \in M \) is arbitrary, and that formula (6.2) holds.

The previous and next propositions together validate Theorem 2.6 concerning the representation of the limit \( u^0 \).

**Proposition 6.2.** Assume (H1)–(H3). Let \( u^\lambda \in \text{Lip}(M) \) be a unique solution of (DP\( _{\lambda} \)) for \( \lambda > 0 \) and \( u^0 \in \text{Lip}(M) \) be the uniform limit of \( \{u^\lambda\} \) as \( \lambda \to 0 \). Then

\[
u^0(x) = \min \left\{ \int_M h(y, x) \, d\mu(y) : \mu \in \mathcal{M}(L) \right\} \text{ for } x \in M. \tag{6.11}
\]

It should be remarked that Proposition 6.2 above guarantees the existence of Mather measures for \( L \); otherwise the right side of (6.11) would equal \( +\infty \). The following proof parallels that of [7, Theorem 4.3], with basis on the formula \( h(y, x) = \min_{z \in A} [S(z, x) + S(y, z)] \) in Proposition 2.3.

**Proof.** We write \( w(x) \) for the right side of (6.11). First, we show that \( u^0 \leq w \). Indeed, by (3) of Proposition 2.4,

\[
u^0(x) - u^0(y) \leq h(y, x) \text{ for all } x, y \in M.
\]

Integration in \( y \) with respect to \( \mu \in \mathcal{M}(L) \), together with Proposition 6.1, yields

\[
u^0(x) \leq \int_M u^0(y) \, d\mu(y) + \int_M h(y, x) \, d\mu(y) \leq \int_M h(y, x) \, d\mu(y).
\]

This assures that \( u^0 \leq w \).

Next, by (2) of Proposition 2.4, the function \( y \mapsto -h(y, x) + w(x) \) is a viscosity subsolution of (DP\( _0 \)). Integrating this function with respect to \( \mu \in \mathcal{M}(L) \),
\[ \int_{M} \left[ -h(y, x) + w(x) \right] d\mu(y) = -\int_{M} h(y, x) \ d\mu(y) + w(x) \leq 0 \text{ for any } x \in M. \]

The characterization of \( u^0 \) in Proposition 6.1 guarantees that
\[ u^0(y) \geq -h(y, x) + w(x) \text{ for all } x, y \in M, \quad (6.12) \]
this yields \( u^0(z) \geq w(z) \) for all \( z \in A \). Since \( u^0 \) is a viscosity solution and \( w \in S^\prime (H-c(H)) \), it follows from (4) of Proposition 2.4 that \( w \leq u^0 \). Thus, \( u^0 = w \) in \( M \). \( \square \)

Now, we start to prove Proposition 2.3, which is a critical issue in this section.

The following proposition is fundamental to connect the functions \( h_0 \) and the solutions of the Hamilton-Jacobi equation \( \partial_t u + H(x, Du) = c(H) \), a proof of which is given in the Appendix.

**Proposition 6.3.** Let \( u_0 \in \text{Lip}(M) \), and set
\[ U(x, t) = \inf_{y \in M} \left[ u_0(y) + h_t(y, x) \right] \text{ for } (x, t) \in M \times (0, \infty). \quad (6.13) \]
Then,
1. \( \lim_{t \to 0^+} U(x, t) = u_0(x) \) uniformly for \( x \in M \),
2. The function \( U \) is bounded and Lipschitz continuous in \( M \times (0, \infty) \),
3. The function \( U \) is a viscosity solution of
\[ \partial_t u + H(x, Du) = c(H) \text{ in } M \times (0, \infty). \quad (6.14) \]

**Proposition 6.4.** Let \( 0 < T \leq \infty \) and \( v, w : M \times [0, T) \to \mathbb{R} \) be an upper semicontinuous viscosity subsolution and a lower semicontinuous viscosity supersolution of (6.14) in \( M \times (0, T) \). Assume that \( v(x, 0) \leq w(x, 0) \) for \( x \in M \). Then, \( v \leq w \) in \( M \times [0, T) \).

Semicontinuous viscosity sub- and super-solutions are needed in our proof of Proposition 6.3 in the Appendix. By definition, an upper semicontinuous function \( v : \ V \to \mathbb{R} \), where \( V \) is an open subset of \( M \times (0, \infty) \), is a viscosity subsolution of \( \partial_t v + H(x, Dv) = c(H) \) in \( V \) if, for any \( \varphi \in C^1(V) \) and \( (\bar{x}, \bar{t}) \in V \) such that \( (v-\varphi)(x, t) \leq (v-\varphi)(\bar{x}, \bar{t}) \) for \( (x, t) \in V \), we have \( \partial_t \varphi(\bar{x}, \bar{t}) + H(\bar{x}, D\varphi(\bar{x}, \bar{t})) \leq c(H) \).

Similarly, a lower semicontinuous function \( v : \ V \to \mathbb{R} \), where \( V \) is an open subset of \( M \times (0, \infty) \), is a viscosity subsolution of \( \partial_t v + H(x, Dv) = c(H) \) in \( V \) if, for any \( \varphi \in C^1(V) \) and \( (\bar{x}, \bar{t}) \in V \) such that \( (v-\varphi)(x, t) \geq (v-\varphi)(\bar{x}, \bar{t}) \) for \( (x, t) \in V \), we have \( \partial_t \varphi(\bar{x}, \bar{t}) + H(\bar{x}, D\varphi(\bar{x}, \bar{t})) \geq c(H) \).

The proof of Proposition 6.4 is similar to that of Theorem 3.2 once one knows the existence of a Lipschitz continuous, viscosity solution of (6.14) for each Lipschitz initial data, as stated in the next proposition. We give main ideas of the proof of Propositions 6.4 and 6.5 in Appendix B.

**Proposition 6.5.** For any \( u_0 \in \text{Lip}(M) \), there exists a viscosity solution \( u \in \text{Lip}(M \times [0, \infty)) \) of (6.14) satisfying the initial condition \( u(\cdot, 0) = u_0 \).

The next lemma is a simple adaptation of Proposition 2.5 and it is left to the reader to prove it.
Lemma 6.6. Let \( u \in \text{Lip}(M \times [0, \infty)) \) be a viscosity solution of (6.14). Then, for each \( \varepsilon > 0 \), there exists \( u_{\varepsilon} \in C^0(M \times [0, \infty)) \) such that
\[
\partial_t u_{\varepsilon} + H(x, Du_{\varepsilon}) \leq c(H) + \varepsilon \quad \text{in } M \times (0, \infty) \quad \text{and} \quad \|u - u_{\varepsilon}\|_{\infty} < \varepsilon.
\]

We need the dynamic programing principle, stated as

Lemma 6.7. Let \( \tau > 0, \quad \sigma > 0 \) and set \( t = \tau + \sigma \). Then
\[
h_t(x, y) = \inf_{z \in M} [h_\tau(x, z) + h_\sigma(z, y)].
\]

Remark that both sides of the formula in the lemma above can be \( +\infty \). This lemma can be proved easily by the definition of \( h_t \).

Proposition 6.8. For any \( x, y \in M \), we have
\[
S(x, y) = \inf_{t > 0} h_t(x, y).
\]

A standard proof of the proposition above is the one similar to that of Proposition 6.3 presented in the Appendix below and based on the dynamic programing principle (Lemma 6.7). The following proof is more dependent on Proposition 6.3.

Proof. Fix any \( y \in M \) and set
\[
u(x, t) = \inf_{z \in M} [S(y, z) + h_t(z, x)].
\]

According to Proposition 6.3, the function \( u \in \text{Lip}(M \times (0, \infty)) \) is a viscosity solution of
\[
\partial_t u + H(x, Du) = c(H) \quad \text{in } M \times (0, \infty) \quad \text{(6.15)}
\]
and satisfies
\[
\lim_{t \to 0} u(x, t) = S(y, x) \quad \text{uniformly for } x \in M. \quad \text{(6.16)}
\]

Since the function \( (x, t) \mapsto S(y, x) \) is also a viscosity subsolution of (6.15) with (6.16), it follows from Proposition 6.4 that \( S(y, x) \leq u(x, t) \), which implies that
\[
S(y, x) \leq h_t(y, x) \quad \text{for } t > 0.
\]

If we set \( h^-(x, y) = \inf_{t > 0} h_t(x, y) \), the inequality above reads
\[
S(y, x) \leq h^-(y, x) \quad \text{for } x, y \in M. \quad \text{(6.17)}
\]

Let \( \delta \) and \( C_0 \) be the constants in (2.2). We choose a constant \( r > 0 \) so that for any \( x, y \in M \), if \( d(x, y) < r \), then there is a geodesic curve \( \gamma \) with speed \( \delta \) connecting \( x \) and \( y \) and
\[
h_{d(x, y)/\delta}(x, y) \leq \int_0^{d(x, y)/\delta} L(\dot{\gamma}(s), \dot{\gamma}(s)) + c(H) \, ds \leq (C_0 + c(H))d(x, y)/\delta.
\]
Accordingly, since $M$ is compact and connected, for some constant $C > 0$, we have
\[ h^-(x, y) \leq Cd(x, y) \text{ for all } x, y \in M. \]

We may assume that $C$ is large enough so that $S$ is Lipschitz continuous with Lipschitz bound $C$. By (6.17), we have
\[ h^-(x, y) \geq S(x, y) \geq -Cd(x, y). \]

It follows from Lemma 6.7 that
\[ h^-(x, y) \leq h^-(x, z) + h^-(z, y) \text{ for all } x, y, z \in M. \]

These show that $h^-(x, x) = 0$ for $x \in M$ and $h^- \in \text{Lip}(M \times M)$. By Lemma 6.7,
\[ \inf_{s > 0} h_{t+s}(y, x) = \inf_{s > 0} \inf_{z \in M} \left[ h_i(y, z) + h_t(z, x) \right] = \inf_{z \in M} \left[ h^-(y, z) + h_t(z, x) \right]. \tag{6.18} \]

Hence, fixing $y \in M$ and setting
\[ v(x, t) = \inf_{z \in M} \left[ h^-(y, z) + h_t(z, x) \right], \]
we observe by Proposition 6.3 that $v$ is a viscosity solution of (6.15) and satisfies
\[ \lim_{t \to 0} v(x, t) = h^-(y, x) \text{ uniformly for } x \in M. \]

By (6.18), we have $v(x, t) = \inf_{s > 0} h_{t+s}(y, x)$, which shows that the function $t \mapsto v(x, t)$ is nondecreasing. Hence we see that, for any $t > 0$, $x \mapsto v(x, t)$ is a viscosity subsolution of $H(x, Du) = c(H)$ in $M$ and, since $h^-(y, x) = \lim_{t \to 0} v(x, t)$ uniformly, the function $x \mapsto h^-(y, x)$ is a viscosity subsolution of $H(x, Du) = c(H)$ in $M$. By the definition of $S$, we have
\[ h^-(y, x) = h^-(y, x) - h^-(y, y) \leq S(y, x). \]

This and (6.17) yield that $h^-(y, x) = S(y, x)$. \hfill \Box

We set temporarily
\[ A_S = \{ z \in M : x \mapsto S(z, x) \text{ is a viscosity solution of } H(x, Du) = c(H) \text{ in } M \}. \]

Then we have an equivalent description for the projected Aubry set.

**Theorem 6.9.** The sets $A_S = A$.

**Proof.** Fix any $z \in A_S$, namely the function $x \mapsto S(z, x)$ is a viscosity solution of $H(x, Du) = c(H)$ in $M$. As in the proof of Proposition 6.8, we set
\[ u(x, t) = \inf_{y \in M} \left[ S(z, y) + h_t(y, x) \right] \text{ for } (x, t) \in M \times (0, \infty). \tag{6.19} \]

Observe that the functions $u(x, t)$ and $w(x, t) := S(z, x)$ are both viscosity solutions of (6.14) with the initial condition $\lim_{t \to 0} u(x, t) = \lim_{t \to 0} w(x, t) = S(z, x)$ uniformly for $x \in M$. Proposition 6.4 then guarantees that $u = w$. That means
\[ \inf_{y \in M} \left[ S(z, y) + h_t(y, x) \right] = S(z, x). \]

This combined with Propositions 6.8 and 6.7 reveals
\[ S(z, x) = \inf_{y \in M} \left[ h_t(y, x) + \inf_{s > 0} h_s(z, y) \right] \geq \inf_{s > 0} h_{t+s}(z, x), \]
and evaluated at \( x = z \),
\[
0 = \inf_{s > 0} h_{t+s}(z, z) \quad \text{and} \quad h(z, z) = \lim_{t \to +\infty} \inf_{s > 0} h_{t+s}(z, z) = 0.
\]

Thus, we see that \( z \in \mathcal{A} \).

Now, let \( z \in \mathcal{A} \), and define \( u : M \times (0, \infty) \to \mathbb{R} \) by (6.19) as before. By Proposition 6.8, \( u(x, t) \geq \inf_{y \in M} [S(z, y) + S(y, x)] \geq S(z, x) \) for \( (x, t) \in M \times (0, \infty) \). Next, we observe that \( \lim_{t \to 0} u(x, t) = S(z, x) \) uniformly for \( x \in M \), and
\[
u(x, t) = \inf_{s > 0} h_{t+s}(z, x),
\]
the last of which implies that the function \( t \mapsto u(x, t) \) is nondecreasing. Hence,
\[
0 = h(z, z) = \lim_{t \to -\infty} u(z, t) \geq u(z, t) \geq \lim_{t \to 0} u(z, t) = S(z, z) = 0, \quad t > 0
\]
This yields
\[
u(z, t) = 0 \quad \text{for all} \quad t > 0.
\]
The monotonicity implies as well that, for each \( t > 0 \), \( v(x) := u(x, t) \) is a viscosity subsolution of \( H(x, Dv) = c(H) \) in \( M \). By the definition of \( S \), we have \( u(x, t) = v(x) - v(z) \leq S(z, x) \) for all \( x \in M \) and \( t > 0 \), so \( u(x, t) = S(z, x) \) for \( (x, t) \in M \times (0, \infty) \).
This shows that \( (x, t) \mapsto S(z, x) \) is a viscosity solution of (6.14), which implies that \( w : x \mapsto S(z, x) \) is a viscosity solution of \( H(x, Dw) = c(H) \) in \( M \). Hence, \( z \in \mathcal{A}_S \), finishing the proof.

Theorem 6.9 has proven the first part of Proposition 2.3, while the second part is as follows:

**Theorem 6.10.** For any \( x, y \in M \),
\[
h(x, y) = \inf_{z \in \mathcal{A}} [S(x, z) + S(z, y)].
\]

We divide the proof of Theorem 6.10 into proving the following three lemmas.

**Lemma 6.11.** For any \( x, y \in M \),
\[
h(x, y) \leq \inf_{z \in \mathcal{A}} [S(x, z) + S(z, y)].
\]

**Proof.** Let \( \tau, \sigma, \theta \in (0, \infty) \) and set \( t = \tau + \sigma + \theta \). By Lemma 6.7, we have
\[
h_t(x, y) = \inf_{z_1, z_2 \in M} [h_t(x, z_1) + h_\sigma(z_1, z_2) + h_\theta(z_2, y)]
\]
\[
\leq h_t(x, z) + h_\sigma(z, z) + h_\theta(z, y) \quad \text{for any} \quad z \in \mathcal{A}.
\]
Noting that \( h(z, z) = 0 \) for \( z \in \mathcal{A} \), we take the liminf of the both sides as \( \sigma \to \infty \), to obtain
\[
h(x, y) \leq h_t(x, z) + h_\theta(z, y) \quad \text{for all} \quad z \in \mathcal{A}.
\]
This and Proposition 6.8 yield
\[
h(x, y) \leq S(x, z) + S(z, y) \quad \text{for} \quad z \in \mathcal{A},
\]
which completes the proof.
The following lemma is a basic observation in weak KAM theory. For the proof, see [24, Proposition 8.5.3], where a more refined version of the lemma is discussed. See also [31, Lemma 8.4] for details on the proof.

**Lemma 6.12.** Let $K \subset M$ be a compact set such that $K \cap A = \emptyset$. Then there exists a function $\psi \in \text{Lip}(M)$ and $f \in C(M)$ such that $H(x, D\psi) \leq c(H) + f$ a.e. in $M$ and $\max_K f < 0$.

**Lemma 6.13.** We have

$$h(x,y) \geq \inf_{z \in A} [S(x,z) + S(z,y)].$$

**Proof.** Let $\varepsilon \in (0, 1)$ and set $U = \{x \in M : \text{dist}(x,A) < \varepsilon\}$, where $\text{dist}(x,A)$ denotes the distance of a point $x$ and the set $A$ induced by the metric $d$. By Lemma 6.12, there exist functions $\psi \in \text{Lip}(M)$, $f \in C(M)$ such that $H(x, D\psi) \leq c(H) + f$ a.e. in $M$ and $\max_{M \setminus U} f < 0$. Fix $\delta > 0$ so that $\max_{M \setminus U} f < -2\delta$. By approximation of $\psi$, we may select $\psi_{\delta} \in C^1(M)$ such that $H(x, D\psi_{\delta}) \leq c(H) + f + \delta$ in $M$, so that $H(x, D\psi_{\delta}) \leq c(H) - \delta$ in $M \setminus U$.

Let $C > 0$ be a constant such that $|S(x,y)| \leq C$ for all $x,y \in M$ and $|\psi_{\delta}(x)| \leq C$ for all $x \in M$. By Proposition 6.8 and Lemma 6.11, we have

$$h(x,y) \leq 2C \text{ for all } x,y \in M.$$

Let $T > 0$ be such that

$$h_T(x,y) < h(x,y) + \varepsilon,$$

and select $\gamma \in \Gamma_{x,y}^T$ so that

$$\int_0^T [L(\gamma(s), \dot{\gamma}(s)) + c(H)] \ ds < h(x,y) + \varepsilon.$$ 

Observe then that if $\gamma(s) \in M \setminus U$ for all $s \in [0, T]$, then

$$\psi_{\delta}(x) - \psi_{\delta}(y) = \int_0^T \langle D\psi_{\delta}(\gamma(s)), \dot{\gamma}(s) \rangle_{\gamma(s)} \ ds$$

$$\leq \int_0^T [L(\gamma(s), \dot{\gamma}(s)) + H(\gamma(s), D\psi_{\delta}(\gamma(s)))] \ ds$$

$$\leq \int_0^T [L(\gamma(s), \dot{\gamma}(s)) + c(H) - \delta] \ ds < h(x,y) + \varepsilon - \delta T,$$

and hence,

$$\delta T < 4C + 1.$$

Conversely, if $\delta T \geq 4C + 1$, then such curves $\gamma$ as above must intersect the set $U$.

By the argument above, where $\varepsilon \in (0, 1)$ is arbitrarily chosen, we deduce that there exist sequences $\{t_j\}_{j \in \mathbb{N}}$ and $\{\varepsilon_j\}_{j \in \mathbb{N}}$ of positive numbers and, for each $j \in \mathbb{N}$, a curve $\gamma_j \in \Gamma_{x,y}^{t_j}$ such that $\lim_{j \to \infty} \varepsilon_j = 0, \lim_{j \to \infty} t_j = +\infty$, and
\[
\lim_{{j \to \infty}} \int_0^t L(\gamma_j(s), \dot{\gamma}_j(s)) \, ds = h(x, y) \quad \text{and} \quad \text{dist}(\gamma_j([0, t]), A) < \varepsilon_j,
\]

where \(\text{dist}(A, B)\) denotes the distance of two sets \(A, B\) induced by the metric \(d\). We choose \(\tau_j \in (0, t_j)\) and \(z_j \in A\) so that \(d(\gamma_j(\tau_j), z_j) < \varepsilon_j\) and compute by using Lemma 6.8 that

\[
\int_0^{\tau_j} L(\gamma_j(s), \dot{\gamma}_j(s)) \, ds = \int_0^{\tau_j} L(\gamma_j(s), \dot{\gamma}_j(s)) \, ds + \int_{\tau_j}^t L(\gamma_j(s), \dot{\gamma}_j(s)) \, ds \\
\geq S(x, \gamma_j(\tau_j)) + S(\gamma_j(\tau_j), y) \\
\geq S(x, z_j) + S(z_j, y) - S(\gamma_j(\tau_j), z_j) - S(z_j, \gamma_j(\tau_j)) \\
\geq \inf_{z \in A} [S(x, z) + S(z, y)] + O(\varepsilon_j)
\]
as \(j \to \infty\). Thus, we have \(h(x, y) \geq \inf_{z \in A} [S(x, z) + S(z, y)]\) and finish the proof. \(\square\)

**Proof of Theorem 6.10.** We only need to combine Lemmas 6.11 and 6.13, to finish the proof. \(\square\)

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In this Appendix, we give a proof of Proposition 6.3, which follows mostly that of [32, Theorem 5.1].

We need to use a version of [32, Lemma 5.5] stated as follows:

**Lemma A.1.** Let $\varphi \in C^1(M \times [0, \infty)), (\bar{x}, \bar{t}) \in M \times (0, \infty)$, and $\varepsilon \in (0, 1)$. Then, there exists an absolutely continuous curve $\gamma : [0, \bar{t}] \to M$ such that $\gamma(\bar{t}) = \bar{x}$ and

$$L(\gamma(s), \dot{\gamma}(s)) + H(\gamma(s), D\varphi(\gamma(s), s)) \leq \varepsilon + \langle D\varphi(\gamma(s), s), \dot{\gamma}(s) \rangle$$

for a.e. $s \in [0, \bar{t}]$.

Furthermore, there exists a constant $C > 0$, depending only on $\|D\varphi\|_{C(M \times [0, \bar{t}])}$ and $H$, such that $|\dot{\gamma}(s)|_{\gamma(s)} < C$ for a.e. $s \in [0, \bar{t}]$.

The proof of [32, Lemmas 5.5 and 5.6] can be easily modified when one works in a local chart, with the Euclidean inner product replaced by those given by the Riemannian metric of $M$ and with interpretation of the curve $\gamma$ above as the map $s \mapsto (\gamma(t-s), -\gamma(t-s), 0)$ being a member of $\text{SP}(x)$, where $\text{SP}(x)$ is defined as the collection of solutions of the Skorokhod problem (see [32]). We leave to the reader to check the validity of the lemma above.

**Proof of Proposition 6.3.** Fix any $u_0 \in \text{Lip}(M)$ and let $U$ and $V$ be the function we are given by (6.13) and the function $u$ in Proposition 6.5, respectively.

We first prove that

$$V(x, t) \leq U(x, t) \text{ for } (x, t) \in M \times (0, \infty).$$  \hspace{1cm} (A.1)

For this, fix any $\varepsilon > 0$ and, in view of Lemma 6.6, choose $V_\varepsilon \in C^1(M \times [0, \infty))$ such that

$$\partial_t V_\varepsilon + H(x, DV_\varepsilon) \leq c(H) + \varepsilon \text{ in } M \times (0, \infty) \quad \text{ and } \quad V_\varepsilon - \varepsilon \leq V \leq V_\varepsilon + \varepsilon \text{ in } M \times [0, \infty).$$
Let $t > 0$, $x, y \in M$, and $\gamma \in \Gamma^t_{x,x}$. Compute that
\[
V_\varepsilon(x, t) - V_\varepsilon(y, 0) = V_\varepsilon(\gamma(t), t) - V_\varepsilon(\gamma(0), 0) = \int_0^t \left[ (DV_\varepsilon(\gamma(s), \gamma(s)), \dot{\gamma}(s)) + \partial V_\varepsilon(\gamma(s), s) \right] ds \\
\leq 2 \int_0^t \left[ H(\gamma(s), DV_\varepsilon(\gamma(s))) + L(\gamma(s), \dot{\gamma}(s)) + \partial V_\varepsilon(\gamma(s), s) \right] ds \\
\leq 2t + \int_0^t \left[ L(\gamma(s), \dot{\gamma}(s)) + c(H) \right] ds,
\]
and hence
\[
V(x, t) \leq u_0(y) + 2t + 2 \int_0^t \left[ L(\gamma(s), \dot{\gamma}(s)) + c(H) \right] ds.
\]
As $\gamma \in \Gamma^t_{x,x}$ and $\varepsilon > 0$ are arbitrary, we get from the above
\[
V(x, t) \leq u_0(y) + h_t(y, x),
\]
which implies furthermore that inequality (A.1) holds.

Next, we show that there is a constant $C_0 > 0$ such that
\[
U(x, t) \leq u_0(x) + C_0 t \quad \text{for} \quad (x, t) \in M \times (0, \infty).
\]
To see this, we choose $C_0 > 0$ so that
\[
c(H) - \min_{\Gamma^t_M} H(x, p) \leq C_0,
\]
and observe by the convex duality that
\[
L(x, 0) + c(H) = - \min_{p \in \Gamma^t_M} H(x, p) + c(H) \leq C_0 \quad \text{for all} \quad x \in M,
\]
and, for any $(x, t) \in M \times (0, \infty)$, the curve $\gamma_x(s) \equiv x$ belongs to $\Gamma^t_{x,x}$ and
\[
h_t(x, x) \leq \int_0^t \left[ L(\gamma_x(s), 0) + c(H) \right] ds \leq C_0 t,
\]
which assures that
\[
U(x, t) \leq h_t(x, x) + u_0(x) \leq u_0(x) + C_0 t,
\]
which is exactly inequality (A.2).

To show the reverse inequality to (A.1), we only need to show that the upper semicontinuous envelope $U^*$ is a viscosity subsolution of (6.14). Note that, by definition, we have
\[
U^*(x, t) = \lim_{r \to 0^+} \sup \left\{ U(y, s) : (y, s) \in B_r(x, t) \right\},
\]
where $B_r(x, t)$ denotes the ball of radius $r$ with center at $(x, t)$ with respect to the distance induced in the Riemannian manifold $M \times \mathbb{R}$. By (A.1) and (A.2), we get
\[
V(x, t) \leq U^*(x, t) \leq u_0(x) + C_0 t \quad \text{for} \quad (x, t) \in M \times [0, \infty),
\]
which, in particular, shows that $V(x, 0) = U^*(x, 0) = u_0(x)$ for $x \in M$. Once these observations are done, by Proposition 6.4, we get $U^* \leq V$ in $M \times [0, \infty)$, which shows that $U^* = V$ in $M \times (0, \infty)$, and the proof ends.

Now, the only thing we need to show is that, the upper semicontinuous envelope $U^*$ is a viscosity subsolution of (6.14). Let $\phi \in C^1(M \times [0, \infty))$ and assume that $U^* - \phi$ has a strict maximum at $(\bar{x}, \bar{t}) \in Q := M \times (0, \infty)$ and $(U^* - \phi)(\bar{x}, \bar{t}) = 0$. 


We need to prove the inequality
\begin{equation}
\partial_t \varphi(x, t) + H(\dot{x}, D\varphi(x, t)) \leq c(H).
\end{equation}

We argue by contradiction, and thus suppose
\begin{equation}
\partial_t \varphi(x, t) + H(\dot{x}, D\varphi(x, t)) > c(H).
\end{equation}

By continuity, we may choose constants \(r > 0\) with \(\dot{t} - r > 0\), and \(\varepsilon \in (0, 1)\) so that.
\begin{equation}
\partial_t \varphi(x, t) + H(x, D\varphi(x, t)) > c(H) + \varepsilon \quad \text{for all } (x, t) \in B_r(\dot{x}) \times [\dot{t} - r, \dot{t} + r],
\end{equation}
where \(B_r(x)\) denotes the geodesic ball of radius \(r\) and center \(x\).

Next, we apply Lemma A.1, in order to find an appropriate curve \(\gamma\). First of all, let \(C > 0\) be the constant from Lemma A.1 depending only on \(H\) and \(\|D\varphi\|_{C(M \times [0, \dot{t} + r])}\).

We choose \(\rho \in (0, r)\) so that \(4Cr \leq r\). Since the maximum value of \(U^\ast - \varphi\) is zero and it is a strict maximum, we set
\[\delta : = - \max_{\partial B_{\rho} \times [\dot{t} - \rho, \dot{t} + \rho]} (U^\ast - \varphi) > 0.\]

We may select a point \((x_0, t_0) \in B_{\rho/2}(\dot{x}) \times [\dot{t} - \rho/2, \dot{t} + \rho/2]\) so that
\[(U - \varphi)(x_0, t_0) > -\delta.\]

We invoke Lemma A.1, to obtain \(\gamma\) with \(\gamma(t_0) = x_0\) such that for a.e. \(s \in [0, t_0]\),
\begin{equation}
\begin{cases}
H(\gamma(s), D\varphi(\gamma(s), s)) + L(\gamma(s), \dot{\gamma}(s)) < \varepsilon + \langle D\varphi(\gamma(s), s), \dot{\gamma}(s) \rangle_{\gamma(s)}, \\
|\dot{\gamma}(s)|_{\gamma(s)} \leq C.
\end{cases}
\end{equation}

Now, by our choice of \((x_0, t_0)\), setting \(\sigma = \dot{t} - \rho\), we have
\[d(\gamma(s), x_0) \leq \int_{t_0}^{t_0} |\dot{\gamma}(\tau)|_{\gamma(\tau)} d\tau \leq C(t_0 - s) < 2Cr \leq \frac{r}{2}\] for \(s \in [\sigma, t_0]\),
which shows that \(\gamma(s) \in B_r(\dot{x})\) for all \(s \in [\sigma, t_0]\), and also,
\[(U - \varphi)(x_0, t_0) > \max_{\partial B(\dot{x}) \times [\sigma, \dot{t} + \rho]} (U^\ast - \varphi) \geq (U - \varphi)(\gamma(\sigma), \sigma).\]

Hence,
\[U(x_0, t_0) - U(\gamma(\sigma), \sigma) > \varphi(x_0, t_0) - \varphi(\gamma(\sigma), \sigma) = \int_{\sigma}^{t_0} \langle D\varphi(\gamma(s), s), \dot{\gamma}(s) \rangle_{\gamma(s)} + \partial_t \varphi(\gamma(s), s) \rangle ds,\]
and, moreover, using (A.5) and (A.4),
\[U(x_0, t_0) - U(\gamma(\sigma), \sigma) \geq \int_{\sigma}^{t_0} [-\varepsilon + L(\gamma(s), \dot{\gamma}(s)) + H(\gamma(s), D\varphi(\gamma(s), s)) + \partial_t \varphi(\gamma(s), s)] ds \geq h_{t_0 - \sigma}(\gamma(\sigma), x_0).\]

Thus, we obtain
\[U(x_0, t_0) > \inf_{y \in M} [U(y, \sigma) + h_{t_0 - \sigma}(y, x_0)],\]
which is a contradiction in view of Lemma 6.7. This completes the proof. \(\square\)
Appendix B

Here we give some ideas of how to prove Propositions 6.4 and 6.5.

Outline of proof of Proposition 6.5. For $R > 0$, we define the function $\Theta_R : \mathbb{R} \rightarrow \mathbb{R}$ by $\Theta_R(r) = \min\{r, R\}$. Fix $R > 0$, set $H_R = \Theta_R \circ (H - c(H))$, and observe that $H_R$ is bounded and uniformly continuous in $T^*M$. To establish the existence of a viscosity solution $u \in \text{Lip}(M \times [0, \infty))$ of (6.14), we consider

$$\partial_t u + H_R(x, Du) = 0 \text{ in } M \times (0, \infty).$$

(B.1)

Given a function $u_0 \in \text{Lip}(M)$, it is important to obtain a viscosity solution $u^R$ of (B.1) satisfying $u(\cdot, 0) = u_0$ whose Lipschitz constant is relatively small compared to $R$, so that $u^R$ is a viscosity solution of (6.14) as well.

The existence and uniqueness (including comparison) of a viscosity solution of (B.1), with initial condition $u(\cdot, 0) = u_0$, is a consequence of the classical theory of viscosity solutions. Let $u^R$ be such a solution and we seek for an estimate on the Lipschitz bound for $u^R$ that is independent of $R$. It is easy to select a constant $C_0 > 0$ so that

$$|H(x, p) - c(H)| < C_0 \text{ for all } x \in M, p \in B_\kappa,$$

where $\kappa = \text{Lip}(u_0)$ and $B_\kappa$ denotes the ball $\subset T^*_0 M$ of radius $\kappa$ with center at the origin. It is then easy to see that the functions $\frac{x}{\kappa(t)} \rightarrow u_0(x) - C_0 t$ and $\frac{x}{\kappa(t)} \rightarrow u_0(x) + C_0 t$ are viscosity sub- and super-solutions of (B.1), respectively. By the comparison principle, we get

$$u_0(x) - C_0 t \leq u^R(x, t) \leq u_0(x) + C_0 t.$$

(B.3)

Fix any $\tau > 0$ and consider the function $v : (x, t) \mapsto u^R(x, t + \tau)$ in $M \times [0, \infty)$, which is a viscosity solution of (B.1). By (B.3), we have $|v(x, 0) - u^R(x, 0)| \leq C_0 \tau$ for $x \in M$. By the comparison principle applied to the functions $u^R(x, t + \tau)$ and $u^R(x, t) \pm C_\tau$, we see that $|v(x, t) - u^R(x, t)| \leq C_\tau$ for all $(x, t) \in M \times [0, \infty)$ and, hence, the function $u^R$ is Lipschitz continuous in $t$, with Lipschitz bound less or equal to $C_0$, where $C_0$ is independent of choice of $R$.

Further, for each $t > 0, x \mapsto u^R(x, t)$ is a viscosity subsolution of $H_R(x, Du) = C_0$ in $M$. If $R > C_0$, then, by the definition of $\Theta_R$, this implies that $x \mapsto u^R(x, t)$ is a viscosity subsolution of $H(x, Du) = c(H) + C_0$ in $M$ for any $t > 0$, which, together with the coercivity assumption (H3), yields a Lipschitz bound, independent of $R$, of $u^R(x, t)$ as functions of $x$, uniform in $t$. Moreover, we deduce that, if $R > C_0$, then $u^R$ is a viscosity solution of (6.14) as well.

To conclude, we select $C_0 > 0$ so that (B.2) is satisfied, fix an $R > C_0$, and solve (B.1). Then the solution $u^R$ is a viscosity solution of (6.14) and, moreover, it is Lipschitz continuous in $M \times [0, \infty)$.

Outline of proof of Proposition 6.4. We need only to consider the case when $T < \infty$. We follow the proof of Theorem 3.2 with minor modifications. It is enough to show that $v \leq w$ in $M \times (0, T)$ for all $\varepsilon > 0$. Thus, we may assume by adding a positive constant to $w$ that $v(x, 0) < w(x, 0)$ for $x \in M$. It is then possible to select a function $u_0 \in \text{Lip}(M)$ so that $v(x, 0) \leq u_0(x) \leq w(x, 0)$ for $x \in M$. By Proposition 6.5 there exists a viscosity solution $u \in \text{Lip}(M \times [0, \infty))$ of (6.14) satisfying the initial condition $u(\cdot, 0) = u_0$. We need to prove that $v \leq u \leq w$ in $M \times (0, T)$.

The next step is to show that $v \leq u$ in $M \times (0, T)$. The argument for proving the inequality $u \leq w$ is similar and we skip it here. We argue by contradiction and thus suppose that $\sup_{M \times [0, T]} (v - u) > 0$. We choose an $S \in (0, T)$, so close to $T$, that $\sup_{M \times [0, S]} (v - u) > 0$. Note that $v$ is bounded above in $M \times [0, S]$ since $v$ is real-valued and upper semicontinuous in $M \times [0, T]$. For $\delta > 0$ we consider the function $\Psi_\delta : (x, t) \mapsto v(x, t) - u(x, t) - \delta(T - t)^{-1}$, which attains a maximum at a point $(x_\delta, t_\delta) \in M \times [0, S]$. Observe that, if $\delta > 0$ is small enough, then the maximum value of $\Psi_\delta$ is positive, and that the function $v_\delta : (x, t) \mapsto v(x, t) - \delta(T - t)^{-1}$ is a viscosity subsolution of $\partial_t v_\delta + H(x, Du v_\delta) = c(H) - \delta T^{-2}$ in $M \times (0, S)$. We fix such a small $\delta$ in what follows. Note that, since $v(x, 0) \leq u(x, 0)$, we have $t_\delta > 0$. We fix a $\varphi \in C^1(M \times [0, S])$ so
that \( \psi(x_0, t_0) = 0 \), \( \partial \psi(x_0, t_0) = \partial_x \psi(x_0, t_0) = 0 \), and \( \psi(x, t) > 0 \) for all \( (x, t) \neq (x_0, t_0) \). The function \( \Psi - \phi \) achieves a strict maximum at \((x_0, t_0)\).

We are now ready to apply the argument of doubling variables. Passing to local coordinates around \( x_0 \), we may assume that \( x_0 \in D \) for some open subset \( D \) of \( \mathbb{R}^n \) such that \( D \subseteq M \). We choose \( \rho > 0 \) so that \( \{x_0 - \rho, t_0 + \rho\} \cap D = D \). For \( \alpha > 0 \) we consider the function

\[
\Phi_\alpha(x, t, y, s) = (v_\alpha - \phi)(x, t) - u(y, s) - \alpha \left( |x-y|^2 + (t-s)^2 \right)
\]

in \( \bar{D} \times [t_0-\rho, t_0+\rho] \times \bar{D} \times [t_0-\rho, t_0+\rho] \). Since \( \Phi_\alpha \) is upper semicontinuous, \( \Phi_\alpha \) achieves a maximum at a point \((x_\alpha, t_\alpha, y_\alpha, s_\alpha)\). Since \( u \) is Lipschitz continuous, the inequality \( \Phi_\alpha(x_\alpha, t_\alpha, y_\alpha, s_\alpha) \geq \Phi_\alpha(x_\alpha, t_\alpha, x_\alpha, t_\alpha) \) yields

\[
\alpha \left( |x_\alpha - y_\alpha|^2 + (t_\alpha - s_\alpha)^2 \right) \leq \operatorname{Lip}(u) \left( |x_\alpha - y_\alpha|^2 + |t_\alpha - s_\alpha|^2 \right)^{1/2},
\]

which implies

\[
\alpha \left( |x_\alpha - y_\alpha|^2 + |t_\alpha - s_\alpha|^2 \right)^{1/2} \leq \operatorname{Lip}(u).
\]

This shows that the collections \( \{x_\alpha(x_\alpha - y_\alpha)\}_{\alpha > 0} \subset \mathbb{R}^n \) and \( \{x_\alpha(t_\alpha - s_\alpha)\}_{\alpha > 0} \subset \mathbb{R} \) are bounded, and, in particular, for some sequence \( \alpha_j \rightarrow +\infty \), the sequence \( \{(x_{\alpha_j}, t_{\alpha_j}, y_{\alpha_j} - y_{\alpha_j}), (t_{\alpha_j} - s_{\alpha_j})\}_{j \in \mathbb{N}} \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \) is convergent. Set

\[
x_0 = \lim_{j \rightarrow \infty} x_{\alpha_j}, \quad t_0 = \lim_{j \rightarrow \infty} t_{\alpha_j}, \quad p_0 = \lim_{j \rightarrow \infty} 2x_{\alpha_j}(x_{\alpha_j} - y_{\alpha_j}), \quad q_0 = \lim_{j \rightarrow \infty} 2x_{\alpha_j}(t_{\alpha_j} - s_{\alpha_j}),
\]

and note that

\[
\lim_{j \rightarrow \infty} y_{\alpha_j} = x_0 \quad \text{and} \quad \lim_{j \rightarrow \infty} s_{\alpha_j} = t_0.
\]

From the inequality

\[
(\Psi - \phi)(x_0, t_0) \leq \max_{(x,t) \in \bar{D} \times [t_0-\rho, t_0+\rho]} \Phi_\alpha(x, t, x, t) = \Phi_\alpha(x_\alpha, t_\alpha, y_\alpha, s_\alpha),
\]

using the upper semicontinuity of \( v_\alpha \), we obtain

\[
(\Psi - \phi)(x_0, t_0) \leq (v_\alpha - \phi)(x_0, t_0) - u(x_0, t_0) = (\Psi - \phi)(x_0, t_0),
\]

which ensures, since \((x_0, t_0)\) is a strict maximum point of \( \Psi - \phi \), that \( (x_0, t_0) = (x_0, t_0) \). Thus, for sufficiently large \( j \), we have \( x_{\alpha_j}, y_{\alpha_j} \in D \) and \( t_{\alpha_j}, s_{\alpha_j} \in (t_0 - \rho, t_0 + \rho) \). For such \( j \), by the viscosity properties of \( v_\alpha \) and \( u \), we have

\[
2x_{\alpha_j}(t_{\alpha_j} - s_{\alpha_j}) + H(x_{\alpha_j}, \partial_x \phi(x_{\alpha_j}, t_{\alpha_j}) + 2x_{\alpha_j}(x_{\alpha_j} - y_{\alpha_j})) \leq c(H) - \delta T^{-2},
\]

and

\[
2x_{\alpha_j}(t_{\alpha_j} - s_{\alpha_j}) + H(y_{\alpha_j}, 2x_{\alpha_j}(x_{\alpha_j} - y_{\alpha_j})) \geq c(H).
\]

Moreover, in the limit as \( j \rightarrow \infty \), we obtain

\[
q_0 + H(x_0, p_0) \leq c(H) - \delta T^{-2} \quad \text{and} \quad q_0 + H(x_0, p_0) \geq c(H).
\]

These yield a contradiction. \( \square \)