On Modules over a G–set

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Abstract

Let $R$ be a commutative ring with unity, $M$ a module over $R$ and let $S$ be a $G$–set for a finite group $G$. We define a set $MS$ to be the set of elements expressed as the formal finite sum of the form $\sum_{s \in S} m_s s$ where $m_s \in M$. The set $MS$ is a module over the group ring $RG$ under the addition and the scalar multiplication similar to the $RG$–module $MG$ defined by Kosan, Lee and Zhou in [9]. With this notion, we not only generalize but also unify the theories of both of the group algebra and the group module, and we also establish some significant properties of $(MS)_{RG}$. In particular, we describe a method for decomposing a given $RG$–module $MS$ as a direct sum of $RG$–submodules. Furthermore, we prove the semisimplicity problem of $(MS)_{RG}$ with regard to the properties of $M_R$, $S$ and $G$.

1 Introduction

Throughout this paper, $G$ is a finite group with identity element $e$, $R$ is a commutative ring with unity 1, $M$ is an $R$–module, $RG$ is the group ring, $H \leq G$ denotes that $H$ is a subgroup of $G$ and $S$ is a $G$–set with a group action of $G$ on $S$. If $N$ is an $R$–submodule of $M$, it is denoted by $N_R \leq M_R$.

$MS$ denote the set of all formal expression of the form $\sum_{s \in S} m_s s$ where $m_s \in M$ and $m_s = 0$ for almost every $s$. For elements $\mu = \sum_{s \in S} m_s s$, $\eta = \sum_{s \in S} n_s s \in MS$, by writing $\mu = \eta$ we mean $m_s = n_s$ for all $s \in S$.

We define the sum in $MS$ componentwise

$$\mu + \eta = \sum_{s \in S} (m_s + n_s)s$$

It is clear that $MS$ is an $R$–module with the sum defined above and the scalar product of $\sum_{s \in S} m_s s$ by $r \in R$ that is $\sum_{s \in S} (rm_s)s$.

For $\rho = \sum_{g \in G} r_g g \in RG$, the scalar product of $\sum_{s \in S} m_s s$ by $\rho$ is

$$\rho \mu = \sum_{s \in S} r_g m_s (sg), \quad sg = s' \in S,$$

$$= \sum_{s' \in S} m_{s'} s' \in MS$$

It is easy to check that $MS$ is a left module over $RG$, and also as an $R$–module, it is denoted by $(MS)_{RG}$ and $(MS)_R$, respectively. The $RG$–module $MS$ is called $G$–set module of $S$ by
$M$ over $RG$. It is clear that $MS$ is also a $G$–set. If $S$ is a $G$–set and $H$ is a subgroup of $G$, then $S$ is also an $H$–set and $MS$ is an $RH$–module. In addition, if $S$ is a $G$–set and a group, and $M = R$, then it is easy to verify that $RS$ is a group algebra. On the other hand, if a group acts on itself by multiplication then naturally we have $(MS)_{RG} = (MG)_{RG}$. Since there is a bijective correspondence between the set of actions of $G$ on a set $S$ and the set of homomorphisms from $G$ to $\Sigma_S$ ($\Sigma_S$ is the group of permutations on $S$), the $G$–set modules is a large class of $RG$–modules and we would say that $(MS)_{RG}$ introduced in [9] considering the group acting itself by multiplication is a first example of the $G$–set modules. That is why the notion of the $RG$–module $MS$ presents a generalization of the structure and discussions of $RG$–module $MG$ and some principal module-theoretic questions arise out of the structure of $(MS)_{RG}$. Therefore, this new concept generalizes not only the group algebra but also the group module, and also unifies the theory of these two concepts.

The purpose of this paper is to introduce the concept of the $RG$–module $MS$, and show the close connection between the properties of $(MS)_{RG}$, $M_R$, $S$ and $G$. The semisimplicity of $(MS)_{RG}$ with regard to the properties of $M_R$, $S$ and $G$ and the decomposition of $(MS)_{RG}$ into $RG$–submodules will occupy a significant portion of this paper. In Section 1, we present some examples and some properties of $(MS)_{RG}$ to show that an $R$–module can be extended to $RG$–modules in various ways via the change of the $G$–set and the group ring. In Section 2, we give our first major result about the decomposition of a given $RG$–module $MS$ as a direct sum of $RG$–submodules. In Section 3, in order to go further into the structure of $(MS)_{RG}$, we first require $\varepsilon_{MS}$ that is an extension of the usual augmentation map $\varepsilon_R$ and the kernel of $\varepsilon_{MS}$ denoted by $\triangle_G(MS)$. Then we give the condition for when $\triangle_G(MS)$ is an $RG$–submodule of $(MS)_{RG}$. Finally, we are interested in the semisimplicity of $(MS)_{RG}$ according to the properties of $M_R$, $S$ and $G$.

We start to set out the idea of $G$–set modules in more detail by considering some examples of $G$–set modules and establishing some properties of $(MS)_{RG}$. The following examples for $(MS)_{RG}$ show how useful the notion of $G$–set module for extension of an $R$–module $M$ to an $RG$–module. They also point the relations among $G$–set $S$, $RG$–module $MS$, $G$ and $H$ where $H \leq G$. Example 1.1 shows that for different group actions on different $G$–sets of the same finite group we get different extensions of an $R$–module $M$ to an $RG$–module. Moreover, we see that these are also $RH$–modules unsurprisingly in Example 1.2.

**Example 1.1** Let $M$ be an $R$–module, $G = D_6 = \langle a, b : a^3 = b^2 = e, b^{-1}ab = a^{-1} \rangle$ and $r = \sum g \in D_6 r_g g = r_1 e + r_2 a + r_3 a^2 + r_4 b + r_5 ba + r_6 ba^2 \in RD_6$.

1. Let $S = G$ and let the group act itself by multiplication. Then $MS = MG$ is an $RG$–module.

2. Let $S = \{D_6, C_3, C_2, Id\}$ and let $G$ act on its set of subgroups $C_3 = \langle a : a^3 = e \rangle \leq D_6$, $C_2 = \langle b : b^2 = e \rangle \leq D_6$, $Id = \{e\} \leq D_6$ by $g * H = gHg^{-1}$ for $H \leq G, g \in G$. Then
Example 1.2 Let $M$ be an $R$–module, $G = D_6 = \langle a : a^3 = b^2 = e, b^{-1}ab = a^{-1} \rangle$, $H = C_3 = \langle a : a^3 = e \rangle \leq D_6$ and $k = \sum_{g \in D_6} g = k_1 e + k_2 a + k_3 a^2 \in RC_3$.

1. Let $S = G$ and let the group act itself by multiplication. Then $MS = MG$ is an $RH$–module.

2. Let $S = \{D_6, C_3, C_2, Id\}$ with the group action defined in Example 1.1 (2). For $\mu = \sum_{s \in S} m_s s = m_{1d} Id + m_{C_2} C_2 + m_{C_3} C_3 + m_{D_6} D_6 \in MS$, we get

$$k \mu = (k_1 m_1 + k_2 m_2 + k_3 m_1) Id + (k_1 m_{C_2} + k_2 m_{C_2} + k_3 m_{C_2}) C_2$$
$$+ (k_1 m_{C_3} + k_2 m_{C_3} + k_3 m_{C_3}) C_3 + (k_1 m_{D_6} + k_2 m_{D_6} + k_3 m_{D_6}) D_6.$$

3. Let $S = \{K_1 = \{e, b\}, K_2 = \{a, ba\}, K_3 = \{a^2, ba^2\}\}$ with the group action defined in Example 1.1 (3). For $\mu = \sum_{s \in S} m_s s = m_{K_1} K_1 + m_{K_2} K_2 + m_{K_3} K_3 \in MS$, we get

$$k \mu = (k_1 m_{K_1} + k_2 m_{K_2} + k_3 m_{K_3}) K_1 + (k_2 m_{K_1} + k_1 m_{K_2} + k_3 m_{K_3}) K_2$$
$$+ (k_3 m_{K_1} + k_2 m_{K_2} + k_1 m_{K_3}) K_3.$$
Lemma 1.3 Let $N_1$, $N_2$ be $R$–submodules of $M$. Then $N_1S + N_2S = MS$ if and only if $N_1 + N_2 = M$.

Proof Let $N_1S + N_2S = NS$. Take $m \in M$ and so $ms \in MS$ for any $s \in S$. We write $ms = \sum_{s_i \in S} n_{s_i} s_i + \sum_{s_j \in S} n_{s_j} s_j$ for $\sum n_{s_i} s_i \in N_1S$ and $\sum n_{s_j} s_j \in N_2S$ where $n_{s_i} \in N_1$, $n_{s_j} \in N_2S$. So, there exists $i, j$ such that $m = m_{s_i} + m_{s_j}$.

Let $N_1 + N_2 = M$ and $\mu = \sum_{s \in S} m_s s \in MS$. For all $s \in S$, we can write $m_s = n_s + n'_s$ where $n_s \in N_1$, $n'_s \in N_2$. Hence, $\mu = \sum_{s \in S} n_s s + \sum_{s \in S} n'_s s$, and so $N_1S + N_2S = NS$. □

Lemma 1.4 Let $N_1$, $N_2$ be $R$–submodules of $M$. Then $N_1S \cap N_2S = 0$ if and only if $N_1 \cap N_2 = 0$.

Proof Let $N_1S + N_2S = 0$. Take $n \in N_1 \cap N_2$, and so $ns \in N_1S \cap N_2S$. So, $n = 0$ since $ns = 0$.

Conversely, let $N_1 \cap N_2 = 0$. Take $\eta = \sum_{s \in S} n_s s \in N_1S \cap N_2S$. So $n_s \in N_1 \cap N_2$ and $n_s = 0$ for all $s \in S$. Hence, $N_1S \cap N_2S = 0$. □

From [2] we recall that if $G$ is a finite group, $S$ and $T$ are $G$–sets, then $\varphi : S \rightarrow T$ is said to be a $G$–set homomorphism if $\varphi(gs) = g\varphi(s)$ for any $g \in G$, $s \in S$. If $\varphi$ is bijective, then $\varphi$ is a $G$–set isomorphism. Then we say that $S$ and $T$ are isomorphic $G$–sets, and we write $S \simeq T$.

For $s \in S$, $Gs = \{gs : g \in G\}$ is the orbit of $s$. It is easy to see that $Gs$ is also a $G$–set under the action induced from that on $S$. In addition, a subset $S'$ of $S$ is a $G$–set under the action induced from $S$ if and only if $S'$ is a union of orbits.

Proposition 1.5 Let $M$ be an $R$–module, $N$ an $R$–submodule of $M$, $G$ a finite group, $S$ a $G$–set. Then $\frac{MS}{NS} \simeq \frac{(M)}{N}S$.

Proof We know that $NS$ is an $RG$–submodule of $MS$. Define a map $\theta$ such that
\[
\theta : MS \rightarrow \frac{(M)}{N}S, \quad \mu = \sum_{s \in S} m_s s \mapsto \theta(\mu) = \sum_{s \in S} (m_s + N)s
\]
\[
\theta(g\mu) = \theta(g \sum_{s \in S} m_s s) = g\theta(\mu)
\]
So, $\theta$ is a $G$–set homomorphism. It is clear that $\theta$ is a $G$–set epimorphism. Furthermore, $\theta$ is an $RG$–epimorphism and we get ker $\theta = NS$. □

Lemma 1.6 Any proper subset of an orbit $Gs$ of $s \in S$ is not a $G$–set under the action induced from $S$.

Proof Suppose that a proper subset $T$ of an orbit $Gs$ of $s \in S$ is a $G$–set. Then there exist $sg \in G$, $sg \not\in T$ for some $g \in G$. Take an element $sh$ in $T$, $h \in G$, and so
\[
(gh^{-1})(hs) = g(h^{-1}(hs)) = gh^{-1}(hs) \not\in T.
\]
Lemma 1.7 Let \( N \) be an \( R \)-submodule of an \( R \)-module \( M \), \( S \) a \( G \)-set. Let \( I \) denote the index of disjoint orbits of \( S \), \( J \) a subset of \( I \) and \( S' = \bigcup_{j \in J} Gs_j \) and let \( Gs_i \) be an orbit \( Gs_i \) of \( s_i \in S \) for \( i \in I \). Then we have the following results:

1. \( NGs_i \) is an \( RG \)-submodule of \( MS \) for \( s_i \in S \). Moreover, \( NGs_i \) is a minimal \( RG \)-submodule of \( MS \) containing \( N \) under the action induced from that on \( S \).

2. \( NS' = N(\bigcup_{j \in J} Gs_j) = \bigcup_{j \in J} (NGs_j) \).

3. \( NS' \) is an \( RG \)-submodule of \( MS \).

Proof

1. It is clear that \( NGs_i \subseteq MS \). Let \( \eta = \sum_{g \in G} n_g gs_i \in NGs_i \), \( r \in R \), \( h \in G \). Then we have \( r\eta = \sum_{g \in G} n_g g's_i \in NGs_i \) and \( h\eta = h(\sum_{g \in G} n_g gs_i) =\sum_{g \in G} n_g hgs_i = \sum_{h\eta = g \in G} n_g g's_i \in NGs_i \). Hence, \( NGs_i \) is an \( RG \)-submodule of \( MS \). Assume that there is an \( RG \)-submodule \( N_1 \) of \( MS \) such that \( N_R \leq (N_1)_{RG} \leq (NGs_i)_{RG} \). Take an element \( n \in N \), and so \( nh s_i \in N_1 \) for some \( h \in G \) since \( (N_1)_{RG} \leq (NGs_i)_{RG} \). Then \( h^{-1}(nh s_i) = (nes_i) = ns_i \in N_1 \) and \( g(ns_i) = ngs_i \in N_1 \) for all \( g \in G \). This means that \( N_1 = NGs_i \).

2. Clear by the definition of \( MS \).

Lemma 1.8 Let \( L \) be an \( RG \)-submodule of \( MS \), a fixed \( s \in S \). Then,

1. \( L_s = \{ x \in M \mid \text{there is } y \in L \text{ such that } y = xs + k, k \in MS \} \) is an \( R \)-submodule of \( M \).

2. \( S_L = \{ s \in S \mid \text{there is } x \in M, \text{and also } k \in L \text{ such that } y = xs + k \in L \} \) is a \( G \)-set in \( S \) under the action induced from that on \( S \).

Proof

1. It is obvious that \( L_s \) is in \( M \). Let \( x_1, x_2 \in L_s \) and \( r \in R \). Then, there is \( y_1 = x_1 s + k_1, y_2 = x_2 s + k_2 \in L \) and \( y_1 + y_2 = (x_1 + x_2)s + k_1 + k_2 \in L \) where \( x_1 + x_2 \in MS \). Furthermore, \( ry_1 = rx_1 s + rk_1 \in L \), and so \( rx_1 \in L_s \).

2. Let \( s \in S' \) and \( g, h \in G \). Then \( \exists x \in M, \exists k \in L \) such that \( y = xs + k \in L \) and

\[
xs + k = y = ey = e(xs + k) = xes + ek = xes + k
\]
So, $s = es$. Since $s$ is also an element of $S$, we have

$$(hg)y = (hg)(xs + k) = (hg)xs + (hg)k.$$  

Hence, we get $(hg)s = h(gs)$. □

Lemma 1.9 Let $M$ be an $R$-module and $S$ a $G$-set. Let $I$ denote the index of disjoint orbits of $S$ such that $S = \bigcup_{i \in I} GS_{i}$ and let $GS_{i}$ be an orbit of $s_{i} \in S$ for $i \in I$. If $NGS_{i}$ is a simple $RG$-submodule of $MS$, then $N$ is a simple $R$-submodule of $M$ and $G$ is a finite group whose order is invertible in $\text{End}_{R}(M)$ $(|G|^{-1} \in \text{End}_{R}(M))$.

Proof Assume that there is an $R$-submodule $L$ of $M$ such that $L \leq N \leq M$. Then $(LGS_{i})_{RG} \leq (NGS_{i})_{RG}$, and by Lemma 1.9 this is a contradiction. So, $N$ is a simple $R$-submodule of $M$. □

Theorem 1.10 Let $L$ be a simple $RG$-submodule of $MS$. Then there is a unique simple $R$-submodule $N$ of $M$ and a unique orbit $Gs$ such that $L = NGs$.

Proof For some $s \in S$, by Lemma 1.8 $L_{s}$ is a non-zero $R$-module. And so, $L_{s}Gs \neq 0$ is an $RG$-submodule of $L$. Since $L$ is simple $RG$-submodule, we have $L_{s}Gs = L$. Then, by Lemma 1.9 $L_{s}$ is a simple $R$-submodule of $M$.

Take an element $s' \in S$ such that $L_{s'}$ is non-zero $R$-submodule of $M$. Hence, $L_{s'}Gs' = L = L_{s}Gs$. Take an element $x \in L_{s'}Gs'$. And so, we write

$$x = \sum_{i=1}^{n} l_{i}g_{i}s' = \sum_{i=1}^{n} k_{i}g_{i}s$$

where $l_{i} \in L_{s'}$, $k_{i} \in L$, $g_{i} \in G$ and $n = |G|$. Then, there exists $g_{j} \in G$ such that $g_{j}s = g_{j}s'$, and $s = g_{1}^{-1}g_{j}s'$. So, we get $Gs = Gs'$. That is why we can write

$$Gs = S_{L} = \{ s \in S \mid \text{there is } x \in M, \text{ and also } k \in L \text{ such that } y = xs + k \in L \}.$$  

Moreover, $N = L_{s} = L_{s'}$ is unique by the definition of $MS$. □

On the other hand, the following example shows that the converse of the theorem does not hold.

Example 1.11 Let $R = \mathbb{Z}_{3}$, $M = \mathbb{Z}_{3}$, $G = C_{2} = \langle a : a^{2} = e \rangle$ and $RG = \mathbb{Z}_{3}C_{2}$. If $S = G$ and $G$ acts on itself by group multiplication then $MS = \mathbb{Z}_{3}C_{2}$ where $\mathbb{Z}_{3}C_{2}$ is semisimple $RG$-module since $|G| \leq \infty$ and characteristic of $R$ does not divide $|G|$ by Maschke’s Theorem. Since $\mathbb{Z}_{3}C_{2}$ is semisimple there is a unique decomposition of $\mathbb{Z}_{3}C_{2}$ by Artin-Wedderburn Theorem. Then, $\mathbb{Z}_{3}C_{2} \simeq \mathbb{Z}_{3} \oplus \mathbb{Z}_{3}$ as $R$-module since $|C_{2}| = 2$. Here, $\mathbb{Z}_{3}$ is a simple $R$-submodule of $\mathbb{Z}_{3}C_{2}$. Moreover, by [11] we have $\mathbb{Z}_{3}C_{2} \simeq \mathbb{Z}_{3}C_{2}(\frac{1+a}{2}) \oplus \mathbb{Z}_{3}C_{2}(\frac{1-a}{2})$ as $RG$-module where $\mathbb{Z}_{3}C_{2}(\frac{1+a}{2})$ and $\mathbb{Z}_{3}C_{2}(\frac{1-a}{2})$ are simple $RG$-submodules of $\mathbb{Z}_{3}C_{2}$. Let $N = \mathbb{Z}_{3}$ that is a simple $R$-submodule of $M$. However, $NGS = \mathbb{Z}_{3}C_{2}$ is not simple $RG$-module.
Lemma 1.12 Let \{M_i : i \in I\} be a family of right \(R\)-modules, \(G\) a finite group and \(S\) a \(G\)-set. Then
\[
\left( \bigoplus_{i \in I} M_i \right) S = \left( \bigoplus_{i \in I} M_i S \right)_{RG}
\]

Proof Consider the following map
\[
\left( \bigoplus_{i \in I} M_i \right) S \to \bigoplus_{i \in I} M_i S, \quad \sum_{s \in S} (\ldots, m_s^{(i)}, \ldots) S \mapsto \sum_{s \in S} (\ldots, m_s^{(i)} s, \ldots)
\]
that is an isomorphism.

\[\square\]

Theorem 1.13 An \(R\)-module \(M_R\) is projective if and only if \((MS)_{RG}\) is projective.

Proof Assume that \(M_R\) is projective. Then for an index \(I\), \((R)^{(I)} \simeq M \oplus A\) where \(A\) is a right \(R\)-module. So, by Lemma 1.12
\[
((RS)^{(I)})_{RG} \simeq ((R)^{(I)} S)_{RG} \\
\simeq ((M \oplus A)S)_{RG} \\
\simeq (MS)_{RG} \oplus (AS)_{RG}
\]
So, \((MS)_{RG}\) is projective.

Now, assume that \((MS)_{RG}\) is projective. Then \(((RS)^{(I)})_{RG} \simeq (MS)_{RG} \oplus B\) where \(B\) is a right \(RG\)-module for some set \(I\). All this concerning modules are also \(R\)-modules and \(((RS)^{(I)})_{R} \simeq (MS)_R \oplus B_R\). \(((RS)^{(I)})_{R}\) is a free module because \((RS)_R\) is free. Since \((MS)_R\) is direct summand of a free module, it is projective. So, \(M_R\) is projective.

\[\square\]

2 The Decomposition of \((MS)_{RG}\)

The theme of this section is the examination of a \(G\)-set module \((MS)_{RG}\) through the study of a decomposition of it. The decompositions of \(RG\) and \((MG)_{RG}\) obtained from the idempotent defined as \(e_H = \frac{H}{|H|}\), where \(|H|\) is the order of \(H\) and \(\bar{H} = \sum_{h \in H} h\), explained in [11] and [15], respectively. A similar method give a criterion for the decomposition of a \(G\)-set module \((MS)_{RG}\). In addition, \(End_{RG} MS\) denotes all the \(RG\)-endomorphisms of \(MS\).

Lemma 2.1 Let \(M\) be an \(R\)-module and \(H\) a normal subgroup of finite group \(G\). If \(|H|\), the order of \(H\), is invertible in \(R\) then \(\bar{e}_H = \frac{H}{|H|}\) is an idempotent in \(End_{RG}(MS)\). Moreover, \(\bar{e}_H\) is central in \(End_{RG}(MS)\).

Proof Firstly, we will show that \(\bar{e}_H\) is an \(RG\)-homomorphism. We start with proving that \(Hg = gH\) for \(g \in G\). Since for all \(h_i \in H\), there is \(h_i g \in H\) such that \(h_i g = gh_{i g}\), we have that \(\bar{H}g = \sum_{h_i \in H} h_{i g} = \sum_{h_i \in H} gh_{i g} = \bar{g}H\). Therefore, \(\frac{\bar{H}}{|H|}rg = r\frac{\bar{H}}{|H|}\) and we have \(\bar{e}_H(r gm) = r g e_H(m)\) for \(m \in MS, r \in R\) and \(g \in G\). It is also clear that \(\bar{e}_H(m + n) = \bar{e}_H(m) + \bar{e}_H(n)\) for \(m, n \in MS, g \in G\).
Secondly, by using the fact that $\hat{H} \cdot \hat{H} = |H| \cdot \hat{H}$, we get
\[
\bar{e}_H(\bar{e}_H(m)) = \bar{e}_H\left(\frac{\hat{H}}{|H|} m\right) = \bar{e}_H(m)
\]
So, $\bar{e}_H$ is an idempotent.

Finally, we prove that $\bar{e}_H$ is a central idempotent in $\text{End}_{RG}(MS)$. We will show that $\bar{e}_H$ commutes with every element of $\text{End}_{RG}(MS)$. Let $f$ be in $\text{End}_{RG}(MS)$ and so $\hat{H} f(m) = f(\hat{H} m)$ for $m \in MS$. Thus, we have
\[
\bar{e}_H f(m) = \frac{\hat{H}}{|H|} f(m)
= f(\frac{\hat{H}}{|H|} m) = f\bar{e}_H(m).
\]
\[\square\]

For $\mu = \sum_{g \in G} m_g g \in MG$ and $s_i \in S$, we write
\[
\mu s_i = \sum_{g \in G} m_g (gs_i)
= \sum_{gs_i \in S} m_{gs_i} (gs_i) \in MS
\]
Then for $i \in I$ and $\alpha \in M(Gs_i)$, we write $\alpha = \sum_{gs_i \in Gs_i} m_{gs_i} gs_i$. Moreover, we write $\beta = \sum_{i \in I, gs_i \in Gs_i} m_{gs_i} gs_i$ for $\beta = \sum_{s \in S} m_s s \in MS$ since $MS = M(\bigcup_{i \in I} Gs_i)$.

Let $H$ be a normal subgroup of $G$. It is well known that on $G/H$ we have the group action $g(tH) = gtH$ for $g, t \in G$. Consider $g(\sum_{s \in S} m_s (sH)) = (\sum_{s \in S} m_s (gsH))$ for $m_s \in M$.

Let $S' \subset S$ be a $G/H$-set. Then $S' = \bigcup_{j \in J} G/H s'_j$ where $J$ denotes the index of disjoint orbits of $S'$ and $MS' = M(\bigcup_{j \in J} G/H s'_j)$. Then for $\eta = \sum_{s' \in S'} m_{s'} s' \in MS$, we can write $\eta = \sum_{j \in J} \sum_{s' \in G/H s'_j} m_{s'} s'$.

Hence, we have the following result.

Lemma 2.2 Let $M$ be an $R$–module, $G$ a finite group, $H$ a normal subgroup of $G$, $S$ a $G$–set and $S' \subset S$ a $G/H$–set. Then $MS'$ is an $RG$–module with action defined as $g \eta = g(\sum_{j \in J, s' \in G/H s'_j} m_{s'} s') = \sum_{j \in J, s' \in G/H s'_j} m_{s'} (tH s'_j)$ where $\eta = \sum_{j \in J, s' \in G/H s'_j} m_{s'} s' \in MS'$ and $s' = tH s'_j$ for $t \in G$.

Theorem 2.3 Let $H$ be a normal subgroup of $G$, $|H|$ invertible in $R$ and $\bar{e}_H$, defined above, then we have $MS = \bar{e}_H . MS \oplus (1 - \bar{e}_H). MS$ and there exists a $G/H$–set $S' \subset S$ such that $\bar{e}_H . MS \simeq MS'$. More precisely,
\[
\bar{e}_H . MS = \bar{e}_H \left(M(\bigcup_{i \in I} Gs_i)\right) \simeq M(\bigcup_{i \in I} \bar{e}_H Gs_i)
\]
Proof Firstly, we know that \( MG = \tilde{e}_H MG \oplus (1 - \tilde{e}_H) MG \) and \( \tilde{e}_H MG \simeq M(G/H) \) by the theorem in [15]. Since \( \tilde{e}_H \) is a central idempotent by Lemma [2.1], we get \( MS = \tilde{e}_H MS \oplus (1 - \tilde{e}_H) MS \). Now, consider \( \theta : G \rightarrow G \tilde{e}_H \) where \( g \rightarrow g \tilde{e}_H \). This is a group homomorphism since \( \theta (gh) = gh \tilde{e}_H = g h \tilde{e}_H = g \tilde{e}_H h \tilde{e}_H = \theta (g) \theta (h) \). It is clear that \( \theta \) is a group epimorphism.

We have \( \ker \theta = \{ g \in G \mid g \tilde{e}_H = \tilde{e}_H \} = \{ g \in G \mid (g - 1) \tilde{e}_H = 0 \} = H \) since \( (g - 1) \frac{\tilde{e}_H}{H} = 0 \) and \( gH = H \) for \( g \in H \). Moreover, we get \( \frac{G}{\ker \theta} = \frac{G}{H} \simeq \text{Im} \theta = G \tilde{e}_H \). So,

\[
\tilde{e}_H MS = \tilde{e}_H \left( M \left( \bigcup_{i \in I} GS_i \right) \right) = M \left( \bigcup_{i \in I} G \tilde{e}_H S_i \right) \simeq M \left( \bigcup_{i \in I} (G/H) S_i \right)
\]

Since \( gH s_i = gH s_l \) for \( s_i, s_l \in S, i, l \in I \), we get a \( G/H \)-set \( S' \subset S \) where \( \bigcup_{j \in J} (G/H) s_j = S' \subset S \). Hence

\[
\tilde{e}_H MS \simeq M \left( \bigcup_{i \in I} (G/H) S_i \right) = M \left( \bigcup_{j \in J} (G/H) s_j \right) = MS'
\]

So, \( \tilde{e}_H MS \simeq MS' \).

\[\square\]

**Theorem 2.4** Let \( M \) be an \( R \)-module and \( G \) a finite group. For a \( G \)-set \( S = \bigcup_{i \in I} GS_i \) (\( I \) denotes the index of disjoint orbits of \( S \)), \( MS \simeq \bigoplus_{i \in I} MG_i \ker \theta_i \) where \( \theta_i : MG \longrightarrow MG_i \) are \( RG \)-epimorphisms.

Proof Since \( MS_i \cap MS_j = \emptyset \) for \( i \neq j \in I \) where \( S = \bigcup_{i \in I} GS_i \) and \( I \) denotes the index of disjoint orbits of \( S \), we have \( MS = M \left( \bigcup_{i \in I} GS_i \right) = \bigoplus_{i \in I} MG_i \).

Consider

\[
\theta_i : MG \longrightarrow MG_i, \quad \sum_{g \in G} m_g g \mapsto \sum_{g \in G} m_g gs_i
\]

For \( \mu = \sum_{g \in G} m_g g \in MG, r \in R, h \in G \), we have

\[
\theta_i (r \mu) = \theta_i \left( r \left( \sum_{g \in G} m_g g \right) \right) = \theta_i \left( \sum_{g \in G} rm_g g \right) = \sum_{g \in G} rm_g gs_i
\]

\[
= r \left( \sum_{g \in G} m_g gs_i \right) = r \theta_i \left( \sum_{g \in G} m_g g \right) = r \theta_i (\mu).
\]

\[
\theta_i (h \mu) = \theta_i \left( h \left( \sum_{g \in G} m_g g \right) \right) = \theta_i \left( \sum_{g \in G} m_g hg \right) = \sum_{g \in G} m_g hgs_i
\]

\[
= h \left( \sum_{g \in G} m_g gs_i \right) = h \theta_i \left( \sum_{g \in G} m_g g \right) = h \theta_i (\mu).
\]

Hence, \( \theta_i \) is an \( RG \)-homomorphism. It is clear that \( \theta_i \) is an epimorphism. Moreover, \( MG_i \ker \theta_i \simeq \text{Im} \theta_i = MG_i \). Then,

\[
MS = M \left( \bigcup_{i \in I} GS_i \right) = \bigoplus_{i \in I} MG_i \ker \theta_i \simeq \bigoplus_{i \in I} MG_i \ker \theta_i.
\]

\[\square\]
3 Augmentation Map on MS and Semisimple G–set Modules

In the theory of the group ring, the augmentation ideal denoted by $\triangle(RG)$ is the kernel of the usual augmentation map $\varepsilon_R$ such that

$$\varepsilon_R : RG \rightarrow R, \quad \sum_{g \in G} r_g g \mapsto \sum_{g \in G} r_g.$$

The augmentation ideal is always the nontrivial two-sided ideal of the group ring and we have $\triangle(RG) = \left\{ \sum_{g \in G} r_g(g-1) : r_g \in R, g \in G \right\}$. The augmentation ideal $\triangle(RG)$ is of use for studying not only the relationship between the subgroups of $G$ and the ideals of $RG$ but also the decomposition of $RG$ as direct sum of subrings.

In [9], $\varepsilon_R$ is extended to the following homomorphism of $R$–modules

$$\varepsilon_M : MG \rightarrow M, \quad \sum_{g \in G} m_g g \mapsto \sum_{g \in G} m_g.$$

The kernel of $\varepsilon_M$ is denoted by $\triangle(MG)$ and $\triangle(MG) = \left\{ \sum_{g \in G} m_g(g-1) : m_g \in M, g \in G \right\}$.

We devote this section to $\varepsilon_{MS}$ that is an extension of $\varepsilon_M$, and to the kernel of $\varepsilon_{MS}$ denoted by $\triangle_G(MS)$.

Definition 3.1 The map

$$\varepsilon_{MS} : MS \rightarrow M, \quad \sum_{s \in S} m_s s \mapsto \sum_{s \in S} m_s$$

is called augmentation map on MS.

In addition, $\varepsilon_{MS}(m_s s_1) = \varepsilon_{MS}(m_s s_2) = m_s$ for $m_s s_1, m_s s_2 \in MS$ where $m_s \in M, s_1, s_2 \in S$, however $m_s s_1 \neq m_s s_2$. Hence, $\varepsilon_{MS}$ is not one-to-one.

Lemma 3.2 Let $M$ be an $R$–module, $G$ a group and $S$ a $G$–set. Then $\varepsilon_{MS}(r \mu) = \varepsilon(r) \varepsilon_{MS}(\mu)$ for $\mu = \sum_{s \in S} m_s s \in MS, r = \sum_{g \in G} r_g g \in RG$. In particular, $\varepsilon_{MS}$ is an $R$–homomorphism.

Proof Let $\mu = \sum_{s \in S} m_s s \in MS, r = \sum_{g \in G} r_g g \in RG$, then

$$\varepsilon_{MS}(r \mu) = \varepsilon_{MS} \left( \sum_{gs \in S} (r_g m_s)(gs) \right)$$

$$= \varepsilon_{MS} \left( \sum_{s' \in S} m_{s'} s' \right), \quad m_{s'} = r_g m_s, gs = s' \in S,$$

$$= \left( \sum_{g \in G} r_g \right) \left( \sum_{s \in S} m_s \right)$$

$$= \varepsilon(r) \varepsilon_{MS}(\mu).$$
In addition, for $\mu = \sum_{s \in S} m_s s, \eta = \sum_{s \in S} n_s s \in MS, t \in R,$

$$\varepsilon_{MS}(\mu + \eta) = \varepsilon_{MS}\left(\sum_{s \in S} (m_s + n_s) s\right)$$

$$= \sum_{s \in S} m_s + \sum_{s \in S} n_s$$

$$\varepsilon_{MS}(t\mu) = \varepsilon_{MS}\left(\sum_{s \in S} (tm_s) s\right)$$

$$= t \sum_{s \in S} m_s$$

Furthermore,

$$\ker(\varepsilon_{MS}) = \{ \mu = \sum_{s \in S} m_s s \in MS \mid \varepsilon_{MS}(\mu) = \varepsilon_{MS}\left(\sum_{s \in S} m_s s\right) = \sum_{s \in S} m_s = 0 \}.$$ 

It is clear that $\ker(\varepsilon_{MS}) \neq 0$ because for $m_s s_1 + (-m_s s_2) \in MS,$ where $m \in M, s_1 \neq s_2 \in S,$ we have

$$\varepsilon_{MS}(m_s s_1 + (-m_s s_2)) = \varepsilon_{MS}(m_s s_1) + \varepsilon_{MS}(-m_s s_2)$$

$$= 0$$

Thus, $m_s s_1 + (-m_s s_2) \in \text{er}(\varepsilon_{MS}).$ Moreover, we will characterize the elements of the kernel of $\varepsilon_{MS}$ in detail. For this purpose, we define $\triangle_{G,H}(MS) = \{ \sum_{h \in H} (h - 1)\mu_h \mid \mu_h \in MS \}$ where $H$ is a subgroup of finite group $G.$

**Theorem 3.3** Let $M$ be an $R$–module, $H$ a subgroup of $G,$ $|H|$ invertible in $R,$ $S$ a $G$–set and $\bar{e}_H,$ defined in Lemma 27.1. Then, $\triangle_{G,H}(MS)$ is an $RG$–module and $\triangle_{G,H}(MS) = (1 - \bar{e}_H).MS.$

**Proof** $\triangle_{G,H}(MS)$ is obviously an $RG$–module. Now, take any element $\alpha \in \triangle_{G,H}(MS).$ Then we get

$$\alpha = \sum_{h \in H} (h - 1)\mu_h$$

$$= \sum_{h \in H} (h - 1)(\sum_{s \in S} m_s s)$$

$$= \sum_{h \in H} (\sum_{s \in S} m_s (h - 1)s)$$

$$= \sum_{h \in H} (\sum_{s \in S} m_s (hs - s))$$

$$= \sum_{h \in H} (\sum_{s \in S} m_s (hs - 1) - (s - 1)).$$
On the other hand, for any element $\beta \in (1 - \bar{e}_H).MS$

$$\beta = (1 - \bar{e}_H)\eta$$

$$= (1 - \bar{e}_H)(\sum_{s \in S} n_s s)$$

$$= (1 - \frac{\bar{H}}{|H|})(\sum_{s \in S} n_s s)$$

$$= \frac{1}{|H|}((\sum_{h \in H}(h - 1)))(\sum_{s \in S} n_s s)$$

$$= (\sum_{h \in H}(h - 1))(\sum_{s \in S} n'_s)$$

$$= \sum_{h \in H}(h - 1)(\sum_{s \in S} n'_s)$$

$$= \sum_{h \in H}(\sum_{s \in S} n'_s (hs - 1) -(s - 1))$$

where $\eta \in MS$, $n'_s = -\frac{1}{|H|} n_s$. Hence, $\beta \in \Delta_{G,H}(MS)$. Similarly, $\alpha \in MS.(1 - \bar{e}_H)$. □

Furthermore, we write $\Delta_{G,G}(MS) = \Delta_G(MS)$. It is clear that $\text{ker}(\varepsilon_{MS}) = \Delta_G(MS)$ and we have $\text{ker}(\varepsilon_{MS}) = \Delta_G(MS) = (1 - \bar{e}_G).MS$.

Recall that $\Delta_R(G)$ is the augmentation ideal of $RG$ and for a normal subgroup $N$ of $G$, $\Delta_R(G,N)$ denote the kernel of the natural epimorphism $RG \rightarrow R(G/N)$ induced by $G \rightarrow G/N$. Moreover, $\Delta_R(G,N)$ is a two-sided ideal of $RG$ generated by $\Delta_R(N)$.

**Theorem 3.4** If $N$ is a normal subgroup of $G$, then $\Delta_{G,N}(MS) = \Delta_R(N).MS$.

**Proof** We know that $\Delta_R(N) = \{ \sum_{n \in N} r_n (n - 1) \mid r_n \in R \}$ and $\Delta_{G,H}(MS) = \{ \sum_{h \in H}(h - 1)\mu_h \mid \mu_h \in MS \}$. For $\alpha = \sum_{n \in N} r_n (n - 1) \in \Delta_R(N)$, $\mu = \sum_{s \in S} m_s s \in MS$,

$$\alpha \mu = \left( \sum_{n \in N} r_n (n - 1) \right) \left( \sum_{s \in S} m_s s \right)$$

$$= \sum_{n \in N} r_n (n - 1) \left( \sum_{s \in S} m_s s \right)$$

$$= \sum_{n \in N} (n - 1) \left( \sum_{s \in S} (r_n m_s) s \right)$$

$$= \sum_{n \in N} (n - 1)\mu_n$$

where $\mu_n = \sum_{s \in S} (r_n m_s)s \in MS$. □

In examination of the studies in group rings which make use of the theory of group modules (see [4], [9], [13]), the semisimplicity problem of the $G$–set module arises. In [4], the generalized Maschke’s Theorem states that a group ring $RG$ is a semisimple Artinian ring if and only if $R$ is a semisimple Artinian ring, $G$ is finite and $|G|^{-1} \in R$. A module theoretic version of the Maschke’s Theorem is proven in [9]. This version states that for a nonzero $R$–module $M$ and a group $G$, $MG$ is a semisimple module over $RG$ if and only if $M$ is a semisimple module and
Lemma 3.5 Let $M$ be a nonzero $R$-module, $G$ a group, $S$ a $G$-set. If $X \cap \triangle_G(\text{MS}) = 0$ for some nonzero $RG$-submodule $X$ of $(\text{MS})_{RG}$, then $G$ is a finite group.

Proof Firstly, we know that $\triangle_G(\text{MS})$ is an $RG$-submodule of $(\text{MS})_{RG}$. Assume that $G$ is an infinite group. Then for any $0 \neq x = m_1s_1 + \ldots + m_ks_k \in X$ where $s_1, \ldots, s_k \in S$ are distinct and $m_is_i \neq 0$, there is an element $g$ of $G$ such that $s_ig \neq s_j$ for $1 \leq i \leq k$. Hence, $(1-g)x = \sum_{s_i \in S} m_is_i - \sum_{s_i \in S} m_ig_{s_i} \neq 0$, and also $(1-g)x \in Y$. On the other hand, $0 \neq (1-g)x = \sum_{s_i \in S} m_is_i - 1 - \sum_{s_i \in S} m_is_{gs_i} - 1 \in \triangle_G(\text{MS})$. Then, $X \cap \triangle_G(\text{MS}) \neq 0$ and this is a contradiction. □

We recall the following lemma in [10], and also in [9].

Lemma 3.6 [10,9] Let $X \subseteq Y$ be right $RG$-modules and $G$ be a finite group whose order is invertible in $End_R(V)$. If $X$ is a direct summand of $Y$ as $R$-modules, then $X$ is a direct summand of $Y$ as $RG$-modules.

Theorem 3.7 Let $M$ be a nonzero $R$-module, $G$ a group, $S$ a $G$-set. Then, $MS$ is a semisimple module over $RG$ if and only if $M$ is a semisimple $R$-module, $G$ is a finite group whose order is invertible in $End_R(M)$ ($|G|^{-1} \in End_R(M)$).

Proof Assume that $M$ is a semisimple $R$-module, $G$ is a finite group whose order is invertible in $End_R(M)$. Let $Y$ be an $RG$-submodule of $MS$. Firstly, $(MS)_R$ is semisimple, hence $Y_R$ is a direct summand of $(MS)_R$. Moreover, $|G|^{-1} \in End_R(MS)$ since $G$ is finite and $|G|^{-1} \in End_R(M)$. So, $Y_{RG}$ is a direct summand of $(MS)_{RG}$ by Lemma 3.6 that means $(MS)_{RG}$ is semisimple.

Assume that $MS$ is a semisimple module over $RG$. $\triangle_G(\text{MS})$ is an $RG$-submodule of $MS$ and we know that $\triangle_G(\text{MS}) \neq MS$. So, $\triangle_G(\text{MS})$ is a proper direct summand of $(MS)_{RG}$. Hence, $G$ is a finite group by Lemma 3.6.

Let $N$ be an $R$-submodule of $M$. Then, $(NS)_{RG}$ is an $RG$-submodule of $(MS)_{RG}$. $(NS)_{RG}$ is a direct summand of $(MS)_{RG}$ because $(MS)_{RG}$ is semisimple, so there is $\alpha^2 = \alpha \in End_{RG}(MS)$ such that $NS = \alpha(MS)$. Let $\alpha|_M$ be the restriction of $\alpha$. Consider the composition such that $\gamma : M \xrightarrow{\alpha|_M} MS \xrightarrow{\varepsilon_{MS}} M$, and so $\gamma \in End_R(M)$. It is clear that $\gamma(M) \subseteq N$. For any $z \in N$, write $z = \alpha(y)$ where $y \in MG$. Then $\gamma(z) = \varepsilon_{MS}\alpha(y) = \varepsilon_{MS}(y) = z$. Hence, $N = \gamma(M)$, $\gamma(\gamma(z)) = \gamma(z) = z$ and $\gamma^2 = \gamma$ which means that $N$ is a direct summand of $M$. Therefore, $M_R$ is a semisimple $R$-module.

Assume that $|G|^{-1} \notin End_R(M)$. Then there is a prime divisor $p$ of $|G|$ such that $p^{-1} \notin End_R(M)$. We prove that $p : M \rightarrow M$ is not one-to-one. Indeed, if $p : M \rightarrow M$ is one-to-one, then $pM \neq M$ because $p^{-1} \notin End_R(M)$. $M = pM \oplus Z$ for some nonzero $R$-submodule $Z$ of $M$ because $M_R$ is semisimple. Since $pM \cap Z = 0$, we get $pZ = 0$. Thus, $p : M \rightarrow M$ is not one-to-one. So, there exists a nonzero direct summand $N$ of $M_R$ such that $pN = 0$ because $M_R$ is semisimple.
Now consider $N\hat{G}$ that is an $RG$–submodule of $(MS)_{RG}$ and $N\hat{G} \subseteq \triangle_G(NS)$ since $|G|N = 0$. We claim that $\triangle_G(NS)$ is an essential $RG$–submodule of $(NS)_{RG}$. Let $\sum_{s \in S} n_s s \in NS\setminus \triangle_G(NS)$. Then, $0 \neq \sum_{s \in S} n_s s \in N$, and thus $(\sum_{s \in S} n_s s)\hat{G} = (\sum_{s \in S} n_s)\hat{G}$ is a nonzero element of $\triangle_G(NS)$. So $\triangle_G(NS)$ is an essential $RG$–submodule of $(NS)_{RG}$. Since $MS$ is a semisimple module over $RG$ by hypothesis and $(NS)_{RG}$ is submodule of $(MS)_{RG}$, $(NS)_{RG}$ is semisimple $RG$–module. Hence, $NS = \triangle_G(NS)$, and so $0 = \varepsilon_{MS}(\triangle_G(NS)) = \varepsilon_{MS}(NS) = N$. This is a contradiction. So, $|G|^{-1} \in \text{End}_R(MS)$.

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\section*{References}

[1] Anderson, F.W., Fuller, K.R.: Rings and Categories of Modules. Springer-Verlag, New York (1992).

[2] Alperin, J.L., Rowen, B.B.: Groups and Representation. Springer-Verlag, New York (1995).

[3] Auslander, M.: On regular group rings. Proc. Am. Math. Soc. 8, 658–664 (1957).

[4] Connell, I.G.: On the group ring. Canadian J. Math. 15, 650–685 (1963).

[5] Curtis, C.W., Reiner I.: Methods of Representation Theory Vol. 2. Wiley-Interscience, New York (1987).

[6] Curtis, C.W., Reiner I.: Methods of Representation Theory: With Applications to Finite Groups and Orders Vol. 1. Wiley-Interscience, New York (1990).

[7] Karpilovsky G.: Commutative Group Algebras. Marcel Dekker, New York (1983).

[8] Karpilovsky G.: Group and Semigroup Rings. North-Holland, Amsterdam (1986).

[9] Kosan, M. T., Lee T., Zhou Y.: On modules over group rings. Algebras and Representation Theory 17 (1), 87-102 (2014).

[10] Lam, T.Y.: A First Course in Noncommutative Rings, 2nd edn. Grad. Texts Math. 131. Springer, New York (2001)

[11] Milies C. P., Sehgal, S. K.: An Introduction to Group Rings. Kluwer Academic Publishers, Dordrecht, The Netherlands (2002).

[12] Passmann, D.S.: The Algebraic Structure of Group Rings. Dover Publications, Inc., New York (2011).

[13] Passi, I.B.S.: Group Rings and Their Augmentation Ideals. Springer-Verlag, Berlin, Heidelberg (1979).

[14] Uc, M., Ones, O., Alkan, M.: On modules over groups. Filomat 30:4, 1021–1027 (2016).

[15] Uc, M., Alkan, M.: On submodule characterization and decomposition of modules over group rings, AIP, (2016) (in press).