A refinement of a classic theorem on continued fractions

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Abstract

We express the set of transformations occurring in two different continued fraction algorithms as subspaces of PGL(2, \(\mathbb{Z}\)) defined by certain simple linear inequalities. As a consequence, we improve a Hurwitz’s classic theorem on continued fractions giving, for \(\gamma \in \text{PGL}(2, \mathbb{Z})\), a bound depending only on \(\gamma\) for the index of the term from which the continued fractions of two irrational numbers related by \(\gamma\) start being identical.

Different algorithms of Diophantine approximation give different expansions of a real number \(x\) in continued fraction. In the first section we express the set of transformations appearing in the algorithm of classic positive continued fraction for an irrational number \(x\) as a subspace of PGL(2, \(\mathbb{Z}\)) defined by linear inequalities on the action on \(\infty\) and \(x\). In the second section we give a similar description for the transformations appearing in a different algorithm of continued fraction playing an important role in [B]. This proves a special case of a more general reduction conjecture formulated by Zagier.

In the third section, we improve Hurwitz’s classic theorem which says that the classic continued fractions of two irrational numbers are the same after a finite number of steps if and only if the numbers are PGL(2, \(\mathbb{Z}\))-equivalent. In Hurwitz’s theorem the index from which the continued fractions start being identical depends on the irrational numbers themselves and hence it is not bounded. We give, for \(\gamma \in \text{PGL}(2, \mathbb{Z})\), a bound depending only on \(\gamma\) for the index of the term from which the continued fractions of two irrational numbers on the orbit of \(\gamma\) coincide.

1 Classic continued fraction

We denote by \(\Gamma\) the group PGL(2, \(\mathbb{Z}\)) and by \(\varepsilon\) and \(T\) the transformations that correspond to the inversion and the translation for the usual action \(\begin{pmatrix} a & b \\ c & d \end{pmatrix} x := \frac{ax + b}{cx + d}\)
on the projective line:
\[ \varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \]

The positive continued fraction of a real number \( x \),
\[ x = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \ddots}}, \quad (n_0 \in \mathbb{Z}, \quad n_i \geq 1 \forall i \geq 1), \]
also denoted by \( x = [n_0, n_1, \ldots] \), is given by the algorithm
\[ x_0 = x, \quad n_i = [x_i], \quad x_{i+1} = \frac{1}{x_i - n_i} = \varepsilon T^{-n_i}(x_i) \quad (i \geq 0). \quad (1.1) \]
Clearly each \( x_i \) is the image of \( x \) by a matrix \( \gamma_i = \gamma_{i,x} \in \Gamma \), given explicitly by
\[ \gamma_0 = \gamma_{0,x} := \text{Id}, \quad \gamma_i = \gamma_{i,x} := \begin{pmatrix} 0 & 1 \\ 1 & -n_{i-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & -n_0 \end{pmatrix} \quad (i \geq 1) \quad (1.2) \]
and recursively by
\[ \gamma_0 = \text{Id}, \quad \gamma_{i+1} = \varepsilon T^{-n_i} \gamma_i \quad (i \geq 0). \quad (1.3) \]
A key idea is to replace the sequence \((\gamma_1, \gamma_2, \gamma_3, \ldots)\) of elements in \( \Gamma \) by the unordered set \( \Gamma(x) = \{\gamma_1, \gamma_2, \gamma_3, \ldots\} \subset \Gamma \).

The \( i \)-th convergent of \( x \) is denoted by \( \frac{p_i}{q_i} = [n_0, \ldots, n_i] \). The integers \( p_i \) and \( q_i \) satisfy the recurrence
\[ p_{-2} = 0, \quad p_{-1} = 1, \quad p_i = n_i p_{i-1} + p_{i-2} \quad (i \geq 0), \]
\[ q_{-2} = 1, \quad q_{-1} = 0, \quad q_i = n_i q_{i-1} + q_{i-2} \quad (i \geq 0), \]
the equation
\[ p_{i+1}q_i - p_i q_{i+1} = (-1)^i \quad (1.4) \]
and the inequalities
\begin{enumerate}
\item \( q_i \geq q_{i-1} \geq 0 \) for all \( i \geq 0 \) and \( q_i > q_{i-1} > 0 \) for all \( i \geq 2 \),
\item \( |p_i| \geq |p_{i-1}| \) for all \( i \geq 2 \) and \( |p_i| > |p_{i-1}| \) for all \( i \geq 3 \),
\item \( q_i \geq \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^i \) for all \( i \geq 0 \),
\item \( |p_i| \geq \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^i \) for all \( i \geq 0 \).
\end{enumerate}
It is well known that any rational number \( p/q \) satisfying
\[
\left| \frac{p}{q} - x \right| < \frac{1}{2q^2}
\]
is a convergent of \( x \).

The numbers \( \delta_i \) (\( i \geq -1 \)) defined by
\[
\delta_i = (-1)^i (p_{i-1} - q_{i-1}x)
\]
satisfy the recurrence
\[
\delta_{-1} = x, \quad \delta_0 = 1, \quad \delta_{i+1} = -n_i \delta_i + \delta_{i-1} \quad \text{with} \quad n_i = \left[ \frac{\delta_{i-1}}{\delta_i} \right]
\]
and the inequalities \( 1 = \delta_0 > \delta_1 > \ldots > 0 \). If \( x \) is rational, then \( x_i = p_i/q_i \) for some \( i \) and the recurrence stops with \( \delta_{i+1} = 0 \); if \( x \) is irrational, the \( \delta_i \) are all positive and converge to 0 with exponential rapidity. With these notations, one has
\[
\gamma_i^{-1} = \begin{pmatrix} p_{i-1} & p_{i-2} \\ q_{i-1} & q_{i-2} \end{pmatrix}, \quad \gamma_i \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \delta_{i-1} \\ \delta_i \end{pmatrix}.
\]

The algorithm (1.1) is a reduction algorithm in the sense that the norms of the vectors \( \gamma_i \begin{pmatrix} x \\ 1 \end{pmatrix} \) decrease to zero.

**Lemma 1.1** If \( r/t \) and \( s/u \) are two rational numbers such that \( t, u > 0, \ r/t \leq x \leq s/u \) and \( ru - st = \pm 1 \), then \( r/t \) or \( s/u \) is a convergent of \( x \).

**Proof.** Let \( x \) be an element in \( [r/t, s/u] \). If one of the inequalities
\[
\left| \frac{r}{t} - x \right| < \frac{1}{2t^2} \quad \text{or} \quad \left| \frac{s}{u} - x \right| < \frac{1}{2u^2}
\]
is satisfied, then \( r/t \) or \( s/u \) is a convergent of \( x \). If not, then
\[
\frac{1}{tu} = \left| \frac{r}{t} - \frac{s}{u} \right| = \left| \frac{r}{t} - x \right| + \left| \frac{s}{u} - x \right| \geq \frac{1}{2t^2} + \frac{1}{2u^2},
\]
which can only happen if \( t = u = 1 \). Then \( s = r + 1 \), so
\[
\begin{align*}
\frac{s}{u} &= x = \frac{p_0}{q_0} \quad \text{or} \quad \frac{r}{t} = \lfloor x \rfloor = \frac{p_0}{q_0}.
\end{align*}
\]

\[\square\]
Theorem 1.2 For all \( x \in \mathbb{R} \) irrational, the set \( \Gamma(x) \) equals \( W - (W_1 \cup W_2) \), where
\[
W = \{ \gamma \in \Gamma \mid -1 \leq \gamma(\infty) \leq 0, \gamma(x) > 1 \}
\]
and
\[
W_1 = \{ \gamma \in W \mid \gamma(\infty) = 0, \det(\gamma) = 1 \}, \quad W_2 = \{ \gamma \in W \mid \gamma(\infty) = -1, \det(\gamma) = -1 \}.
\]
The sets \( W_1 \) and \( W_2 \) have respectively exactly one and at most one element.

Proof. One easily checks that
\[
W_1 = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & -n_0 \end{pmatrix} \right\}, \quad W_2 = \left\{ \begin{pmatrix} -1 & 1 + n_0 \\ 1 & -n_0 \end{pmatrix} \right\} \quad \text{if } n_1 \geq 2,
\]
\[
\left\{ \begin{pmatrix} \delta_{i-1} & \gamma_i \\ 1 & -n_0 \end{pmatrix} \right\} \quad \text{if } n_1 = 1.
\]
This proves the second statement. It is also easy to see that \( \Gamma(x) \subseteq W - (W_1 \cup W_2) \).
Indeed \( \gamma_i(x) = \frac{\delta_{i-1}}{\gamma_i} > 1 \) and \( \gamma_i(\infty) = -\frac{q_i - 2}{q_i - 1} \) for \( i \geq 1 \) with inequalities (1) imply \( \gamma_i \in W - (W_1 \cup W_2) \).

We therefore have to show that \( W \subseteq \Gamma(x) \cup W_1 \cup W_2 \). Let \( \gamma \in \Gamma \) satisfy
\[
-1 \leq \gamma(\infty) < 0, \quad \gamma(x) > 1.
\]
Denote
\[
\gamma^{-1} = \begin{pmatrix} r & s \\ t & u \end{pmatrix}, \quad \gamma = \begin{pmatrix} u & -s \\ -t & r \end{pmatrix}.
\]
The conditions \( \gamma(\infty) < 0 \) and \( \gamma(x) > 0 \) imply that \( u \) and \( t \) have the same sign, as well as \( ux - s \) and \( -tx + r \), so \( x \in \left[ \frac{r}{t}, \frac{s}{u} \right] \) \((\left[ \frac{r}{t}, \frac{s}{u} \right] \) denotes \( [\frac{r}{t}, \frac{s}{u}] \) if \( \frac{r}{t} \leq \frac{s}{u} \) or \( [\frac{s}{u}, \frac{r}{t}] \) otherwise). By Lemma 1.1 \( \frac{r}{t} \) or \( \frac{s}{u} \) is a convergent of \( x \).

If \( \frac{r}{t} \) is a convergent of \( x \), then \( \gamma \) and \( \gamma^{-1} \) are
\[
\gamma^{-1} = \begin{pmatrix} p_i & p_i - 1 + kp_i \\ q_i & q_i - 1 + kq_i \end{pmatrix}, \quad \gamma = \begin{pmatrix} q_i - 1 + kq_i & -p_i - 1 - kp_i \\ -q_i & p_i \end{pmatrix}\tag{1.7}
\]
or
\[
\gamma^{-1} = \begin{pmatrix} p_i & -p_i - 1 - kp_i \\ q_i & q_i - 1 - kq_i \end{pmatrix}, \quad \gamma = \begin{pmatrix} -q_i - 1 - kq_i & p_i - 1 + kp_i \\ -q_i & p_i \end{pmatrix}\tag{1.8}
\]
with \( k \in \mathbb{Z} \) and \( i \geq 0 \). In the case \( \frac{r}{t} \), \( -1 \leq \gamma(\infty) = -\frac{q_i + 1}{q_i} \) if and only if \( k = 0 \) and \( i \geq 1 \), or \( k = 1 \) and \( i = 0 \), because \( q_i+1 \leq q_i \). Moreover \( \gamma(x) = \frac{\delta_i}{\delta_{i+1}} - k > 1 \) if and only if \( k \leq n_{i+1} - 1 \) because \( \left\lfloor \frac{\delta_i}{\delta_{i+1}} \right\rfloor = n_{i+1} \). Hence
\(-1 \leq \gamma(\infty) < 0\) and \(\gamma(x) > 1\) if and only if \(k = 0\) and \(i \geq 1\) or \(n_i \geq 2, k = 1\) and \(i = 0\). In the first case

\[
\gamma = \begin{pmatrix} q_{i-1} & -p_{i-1} \\ -q_i & p_i \end{pmatrix} = \gamma_{i+1} \quad (i \geq 1);
\]

in the second case

\[
\gamma = \begin{pmatrix} -1 & 1 + n_0 \\ 1 & -n_0 \end{pmatrix} \in W_2.
\]

In the case (1.8), if \(-1 \leq \gamma(\infty) = \frac{q_i - 1}{q_i} + k < 0\), then \(k < 0\) and \(\gamma(x) = -\frac{\delta_i}{\delta_{i+1}} + k < 0\), so \(\gamma \not\in W\).

If \(\frac{s}{u}\) is a convergent of \(x\), then \(\gamma\) and \(\gamma^{-1}\) are:

\[
\gamma^{-1} = \begin{pmatrix} p_{i-1} + kp_i & p_i \\ q_{i-1} + kq_i & q_i \end{pmatrix}, \quad \gamma = \begin{pmatrix} q_i & -p_i \\ -q_{i-1} - kp_i & p_{i-1} + kp_i \end{pmatrix} \quad (1.9)
\]

or

\[
\gamma^{-1} = \begin{pmatrix} -p_{i-1} - kp_i & p_i \\ -q_{i-1} - kp_i & q_i \end{pmatrix}, \quad \gamma = \begin{pmatrix} q_i & -p_i \\ q_{i-1} + kq_i & -p_{i-1} - kp_i \end{pmatrix} \quad (1.10)
\]

with \(k \in \mathbb{Z}\) and \(i \geq 0\). In the case (1.9), \(\gamma(x) = \frac{\delta_{i+1}}{\delta_i - k\delta_{i+1}} > 1\) if and only if \(k = n_{i+1}\) because \(\frac{\delta_i}{\delta_{i+1}} = n_{i+1}\). If \(k = n_{i+1}\), then \(-1 \leq \gamma(\infty) = -\frac{q_i}{q_{i+1}} < 0\). So

\[
\gamma = \begin{pmatrix} q_i & -p_i \\ -q_{i+1} & p_{i+1} \end{pmatrix} = \gamma_{i+2} \quad (i \geq 0).
\]

In the case (1.10), if \(-1 \leq \gamma(\infty) = \frac{q_i}{q_{i+1} + kq_i} < 0\), then \(k < 0\) and \(\gamma(x) = \frac{\delta_{i+1}}{-\delta_i + k\delta_{i+1}} < 0\), so \(\gamma \not\in W\).

Let \(\gamma = \begin{pmatrix} 0 & 1 \\ 1 & u \end{pmatrix}\) be the unique element of \(W\) that satisfies \(\gamma(\infty) = 0\) and \(\det(\gamma) = -1\). We have \(1 < \gamma(x) = \frac{1}{x + u}\) if and only if \(0 < x + u < 1\), namely \(u = -[x]\), and \(\gamma = \gamma_1 \in \Gamma(x)\).

\[\square\]

**Remark 1.3** For \(x \in \mathbb{Q}\), Theorem 1.2 is still true if the value \(\infty\) is allowed for \(\gamma(x)\), with \(\gamma \in W\).
2 The slow continued fraction

We consider the slower version of the classic algorithm of reduction (1.1)

\[
x_0 = x, \quad x_{i+1} = \begin{cases} 
    x_i + 1 = T(x_i) & \text{if } x_i \leq 0, \\
    \frac{1}{x_i} - 1 = T^{-1} \varepsilon(x_i) & \text{if } 0 < x_i \leq 1, \\
    x_i - 1 = T^{-1}(x_i) & \text{if } x_i > 1.
\end{cases}
\] (2.1)

With this algorithm the expansion of \(x\) in continued fraction is

\[
x = \pm 1 \pm \cdots \pm 1 + \frac{1}{1 + \cdots + \frac{1}{1 + \cdots}}
\]

where \(n_0, n_1, n_2, \ldots\), are given in (1.1) and each \(\pm\) equals the sign of \(|n_0|\).

Each \(x_i\) is the image of \(x\) by a matrix \(\gamma'_i = \gamma'_{i,x} \in \Gamma\) given recursively by

\[
\gamma'_0 = \text{Id}, \quad \gamma'_{i+1} = \begin{cases} 
    T \gamma'_i & \text{if } x_i \leq 0, \\
    T^{-1} \varepsilon \gamma'_i & \text{if } 0 < x_i \leq 1, \\
    T^{-1} \gamma'_i & \text{if } x_i > 1.
\end{cases}
\] (2.2)

In a similar way as in the previous section, we will replace the sequence \(\gamma'_1, \gamma'_2, \gamma'_3, \ldots\) of elements in \(\Gamma\) by the unordered set \(\Gamma(x)' = \{\gamma'_1, \gamma'_2, \gamma'_3, \ldots\} \subset \Gamma\), given explicitly by

\[
\Gamma(x)' = \left\{ \left( \begin{array}{cc}
    q_{i-2} + kq_{i-1} & -p_{i-2} - kp_{i-1} \\
    -q_{i-1} & p_{i-1}
  \end{array} \right) \right\}, \quad 1 \leq k \leq n_i \right\}_{i \geq 1}.
\] (2.3)

**Theorem 2.1** For all \(x \in \mathbb{R}\) irrational, the set \(\Gamma(x)\) equals \(W' - W'_1\), where

\[
W' = \{ \gamma \in \Gamma \mid \gamma(\infty) \leq -1, \gamma(x) > 0 \}
\]

and

\[
W'_1 = \{ \gamma \in W' \mid \gamma(\infty) = -1, \det(\gamma) = 1 \}.
\]

The set \(W'_1\) has exactly one element.

**Proof.** One easily proves the second statement checking that

\[
W'_1 = \left\{ \left( \begin{array}{cc}
    1 & -n_0 \\
    -1 & n_0 + 1
  \end{array} \right) \right\}.
\]
It is also easy to see that \( \Gamma(x) \subseteq W' - W'_1 \). Indeed, \( \gamma'_i(x) = \frac{\delta_{i-1}}{\delta_i} - k > 0 \) because \( \left[ \frac{\delta_{i-1}}{\delta_i} \right] = n_i \) and \( \gamma'_i(\infty) = -\frac{q_{i-2}}{q_{i-1}} - k \) and inequalities (1) imply \( \gamma'_i \in W' - W'_1 \).

We therefore have to see that \( W' \subseteq \Gamma(x) \cup W'_1 \). Let \( \gamma \in \Gamma \) satisfy

\[
\gamma(\infty) \leq -1, \quad \gamma(x) > 0.
\]

As for Theorem 1.2, the conditions \( \gamma(\infty) < 0 \) and \( \gamma(x) > 0 \) imply that \( \gamma^{-1}(\infty) \) or \( \gamma^{-1}(0) \) is a convergent of \( x \). Hence \( \gamma \) is equal to

\[
\begin{align*}
(q_{i-1} + kq_i & \quad -p_{i-1} - kp_i) \quad \text{or} \quad (-q_{i-1} - kq_i & \quad p_{i-1} + kp_i), \\
q_{i-1} - kq_i & \quad -p_i \quad \text{or} \quad (-q_{i-1} - kq_i & \quad p_{i-1} - kp_i),
\end{align*}
\]

with \( k \in \mathbb{Z} \) and \( i \geq 0 \).

In the first case we have, when \( n_i \geq 2 \), \( \gamma(\infty) = -\frac{q_{i-1}}{q_i} - k \leq -1 \) if and only if \( k \geq 1 \); when \( n_i = 1 \), \( \gamma(\infty) \leq -1 \) if and only if \( k \geq 0 \) and \( k \neq 0 \) for \( i \neq 1 \), because \( q_{i-1} \leq q_i \). Moreover, \( \gamma(x) = \frac{\delta_i}{\delta_{i+1}} - k > 0 \) if and only if \( k \leq n_{i+1} \) because

\[
\left[ \frac{\delta_i}{\delta_{i+1}} \right] = n_{i+1}.
\]

So \( \gamma(\infty) \leq -1 \) and \( \gamma(x) > 0 \) if and only if

\[
\gamma = \begin{pmatrix} (q_{i-1} + kq_i & -p_{i-1} - kp_i) \\ q_i & -p_i \end{pmatrix} (1 \leq k \leq n_{i+1})
\]
or, when \( n_i = 1 \),

\[
\gamma = \begin{pmatrix} 1 & -n_0 \\ -1 & n_0 + 1 \end{pmatrix} \in W'_1.
\]

In the second case, if \( \gamma(\infty) = \frac{q_{i-1}}{q_i} + k \leq -1 \), then \( k < 0 \) and \( \gamma(x) = -\frac{\delta_i}{\delta_{i+1}} + k < 0 \), so \( \gamma \notin W' \).

In the third case we have \( \gamma(\infty) = -\frac{q_i}{q_{i-1} + kq_i} \leq -1 \) if and only if \( 0 < \frac{q_{i-1}}{q_i} + k \leq 1 \), if and only if \( k = 0 \) and \( i \geq 1 \), or \( k = 1 \) and \( i = 0 \). Moreover, \( \gamma(x) = \frac{\delta_{i+1}}{\delta_i - k\delta_{i+1}} > 0 \) if and only if \( k \leq n_{i+1} \). So \( \gamma(\infty) \leq -1 \) and \( \gamma(x) > 0 \) if and only if

\[
\gamma = \begin{pmatrix} q_i & -p_i \\ -q_{i-1} & p_{i-1} \end{pmatrix} (i \geq 1),
\]
or

\[
\gamma = \begin{pmatrix} 1 & -n_0 \\ -1 & 1 + n_0 \end{pmatrix} \in W'_1.
\]
In the last case, if $\gamma(\infty) = \frac{q_i}{q_{i-1} + kq_i} \leq -1$, then $k < 0$ and $\gamma(x) = \frac{\delta_{i+1}}{-\delta_i + k\delta_{i+1}} < 0$, so $\gamma \notin W'$. □

**Remark 2.2** Theorem 2.1 proves a special case of a more general reduction conjecture formulated by Zagier. For a Fuchsian group $G$, consider a partition of $\mathbb{P}^1(\mathbb{R})$ into a finite set of intervals $I_\alpha$, where $\alpha$ is a generator of $G$, together with a map $\rho : \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$ such that $\rho|_{I_\alpha} = \alpha$. The map $\rho$ can be viewed as the map $\bar{\rho} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $(0, 0) \mapsto (0, 0)$ and $(x, y) \neq (0, 0) \mapsto \alpha \left( \begin{array}{c} x \\ y \end{array} \right)$ if $(x : y) \in I_\alpha$. Such an algorithm is a reduction algorithm if $\|\bar{\rho}(x, y)\| \leq \|(x, y)\|$ for all $(x, y) \in \mathbb{R}^2$ with equality only if $(x, y) = (0, 0)$ or $\alpha$ is parabolic and $(x : y) \in \partial I_\alpha$. Zagier conjectured that, if $\Delta$ is the diagonal $x = y$ of $\mathbb{R}^2$, the map $\bar{\rho}$ possesses a global attractor set $D = \cap_{r=0}^\infty \bar{\rho}^r(\mathbb{R}^2 - \Delta)$ with a finite rectangular structure, on which $\bar{\rho}$ is essentially bijective and such that every point $(x, y)$ of $\mathbb{R}^2 - \Delta$ is mapped to $D$ after finitely many iterations of $\bar{\rho}$. Katok and Ugarcovici ([KU]) proved it for the special two-parameter family of maps $\rho_{a,b}$ (suggested by Zagier) defined by the transformations of $\text{SL}(2, \mathbb{Z})$

$$
\rho_{a,b}(x) = \begin{cases} 
  x + 1 & \text{if } x < a \\
  -\frac{1}{x} & \text{if } a \leq x < b \\
  x - 1 & \text{if } x \geq b.
\end{cases}
$$

### 3 Main theorem

For the classic continued fraction defined in Section 1, the following theorem is well known:

**Theorem 3.1** (Hurwitz) Two irrational numbers $x = [n_0, n_1, \ldots]$ and $y = [m_0, m_1, \ldots]$ are $\Gamma$-equivalent if and only if there exists $s, t \geq 0$ such that $n_{s+i} = m_{t+i}$ for all $i \geq 0$.

The proof of this classic theorem can be found in [HW]. As we said in the introduction, if we consider $x$ and $y$ belonging to a fixed irrational orbit of $\Gamma$, the indexes $s$ and $t$ in Theorem 3.1 are not bounded. In the next theorem, we show that there is a bound for $s$ and $t$ in terms of the matrix in $\Gamma$ relating $x$ and $y$.

**Definition 3.2** For $\alpha = \frac{A}{B} \in \mathbb{Q}$ with $B > 0$ and $(A, B) = 1$, we define

$$
M(\alpha) := \frac{\log(\sqrt{5}\min(|A|, B))}{\log((1 + \sqrt{5})/2)} + 2r + 3,
$$

where $r$ is the number of convergents of $\alpha$. 
Theorem 3.3 Let $\gamma$ be an element of $\Gamma$. There exists $N = N(\gamma)$ such that for all $x \in \mathbb{R} - \mathbb{Q}$, if we write $x = [n_0, n_1, \ldots]$ and $\gamma(x) = [m_0, m_1, \ldots]$, then there exists $0 \leq s, t \leq N$ with $n_{s+i} = m_{t+i}$ for all $i \geq 0$. We can take $N = 3$ if $\gamma(\infty) = \infty$ and $N = \max(M(\gamma(\infty)), M(\gamma^{-1}(\infty)))$ otherwise.

Proof. Denote by $(x_i)_{i \geq 0}$ and $(x'_i)_{i \geq 0}$ the respective series defined in (1.1) for $x$ and $\gamma(x)$. Because $x_s = \gamma_s(x)$ and $x'_t = \gamma_t(\gamma(x))$, if there exists $s, t \geq 0$ such that $\gamma_{s,x} = \gamma_{t,\gamma(x)}$, then $x_s = x'_t$. In this case, $n_s = m_t$ and so $\gamma_{s+1,x} = \gamma_{t+1,\gamma(x)}$ (by (1.3)). Again $x_{s+1} = x'_{t+1}$ and $n_{s+1} = m_{t+1}$. By induction we would have $\gamma_{s+i,x} = \gamma_{t+i,\gamma(x)}$ and $n_{s+i} = m_{t+i}$ for all $i \geq 0$.

To prove that such $s, t$ smaller than $N$ exist, we may bound by $N$ the cardinal number of the sets $\{\gamma_i, x \notin \Gamma(\gamma(x))\}_{i \geq 0}$ and $\{\gamma_i, x \notin \Gamma(\gamma^{-1})\}_{i \geq 0}$.

If $\gamma(\infty) = \infty$, then $\gamma = \begin{pmatrix} \pm 1 & b \\ 0 & 1 \end{pmatrix}$ with $b \in \mathbb{Z}$. One easily sees from Theorem 1.2 that

$$\Gamma(x + b)T^b = \Gamma(x), \quad \gamma_{i,x} \in \Gamma(-x) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ if } \gamma_{i,x}(\infty) \notin \{0, 1\},$$

which is the case for all $i \geq 3$. So

$$|\{\gamma_{i,x} \notin \Gamma(\gamma(x))\}_{i \geq 0}| < 3, \quad |\{\gamma_{i,\gamma(x)} \notin \Gamma(\gamma^{-1})\}_{i \geq 0}| < 3.$$

Suppose $\gamma(\infty) \neq \infty$; we will prove the inequality

$$|\{\gamma_{i,x} \gamma^{-1} \notin \Gamma(\gamma(x))\}_{i \geq 0}| \leq M(\gamma^{-1}(\infty)).$$

Applying this inequality to $\gamma^{-1}$ instead of $\gamma$ and $\gamma(x)$ instead of $x$, we will obtain

$$|\{\gamma_{i,\gamma(x)} \gamma \notin \Gamma(x)\}_{i \geq 0}| \leq M(\gamma(\infty))$$

and thus the theorem will be proved.

Suppose there exists $i \geq 0$ such that $\gamma_{i,x} \gamma^{-1} \notin \Gamma(\gamma(x))$. We put $\gamma^{-1}(\infty) = \frac{p}{q}$ with $(p, q) = 1$. The description of $\Gamma(\gamma(x))$ given by Theorem 1.2 and the fact that $\gamma_{i,x} \gamma^{-1}(\gamma(x)) = \gamma_{i,x}(x) > 1$ reduce the supposition above to one of the following conditions:

(i) $\gamma_{i,x} \left(\frac{p}{q}\right) \geq 0$,

(ii) $\gamma_{i,x} \left(\frac{p}{q}\right) \leq -1$.

For the case (i) we have

$$\gamma_{i,x} \left(\frac{p}{q}\right) = \frac{q_i - 2p - p_i - 2q}{-q_i - p + p_i - 1} \geq 0.$$
This implies \( \frac{p}{q} \in \left[ \frac{p_{i-2}}{q_{i-2}}, \frac{p_{i-1}}{q_{i-1}} \right] \) and hence (Lemma 1.1) \( \frac{p_{i-2}}{q_{i-2}} \) or \( \frac{p_{i-1}}{q_{i-1}} \) is a convergent of \( \frac{p}{q} \).

For the case (ii) we have

\[
\gamma_{i,x} \left( \frac{p}{q} \right) = \frac{q_{i-2}p - p_{i-2}q}{-q_{i-1}p + p_{i-1}q} \leq -1.
\]

This implies \( \frac{p}{q} \in \left[ \frac{p_{i-1} - p_{i-2}}{q_{i-1} - q_{i-2}}, \frac{p_{i-1}}{q_{i-1}} \right] \) and hence (Lemma 1.1) \( \frac{p_{i-1} - p_{i-2}}{q_{i-1} - q_{i-2}} \) or \( \frac{p_{i-1}}{q_{i-1}} \) is a convergent of \( \frac{p}{q} \).

If \( \frac{p_{i-1} - p_{i-2}}{q_{i-1} - q_{i-2}} \) is a convergent of \( \frac{p}{q} \), then we deduce from inequalities (1) and (2) and the recurrence satisfied by the convergents that

\[
\begin{align*}
|p_{i-3}| & \leq |(n_{i-1} - 1)p_{i-2} + p_{i-3}| = |p_{i-1} - p_{i-2}| \leq |p| \quad (i \geq 4) \\
q_{i-3} & \leq (n_{i-1} - 1)q_{i-2} + q_{i-3} = q_{i-1} - q_{i-2} \leq q \quad (i \geq 2).
\end{align*}
\]

We deduce from inequalities (3) and (4)

\[
i \leq \frac{\log(\sqrt{5}|p|)}{\log((1 + \sqrt{5})/2)} + 3, \quad i \leq \frac{\log(\sqrt{5}q)}{\log((1 + \sqrt{5})/2)} + 3 \quad (i \geq 4).
\]

Finally we obtain

\[
|\{\gamma_{i,x} \gamma^{-1} \notin \Gamma(\gamma(x))\}|_{i \geq 0} \leq M(\gamma^{-1}(\infty)).
\]

\(\square\)

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