Prudent case-based prediction when experience is lacking*

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Abstract

An inexperienced predictor is asked to qualitatively rank eventualities according to their plausibility, given past cases. Inexperience means that, resampling past cases (with replacement) fails to generate a suitably diverse set of rankings. (4-diversity requires that each of the 4! strict rankings of four eventualities arises for some sample.) Along with other essential consistency requirements, 4-diversity yields a matrix representation that may be viewed as an empirical likelihood function (Gilboa and Schmeidler, 2003). We impose 2-diversity and derive a similar representation: provided the predictor is prudent enough to ensure that the arrival of novel cases will not force her into being dogmatic, intransitive or into revising her existing rankings. We build on this to establish a formal tradeoff between inexperience and the cognitive or computational cost of more abstract resampling.

From the past, the present acts prudently, lest it spoil future action.

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1 Model

1.1 Framework

Where possible, we adopt the notation and interpretations of [GS]. The first primitive of our model is the nonempty set $X$ of conceivable eventualities of the present prediction problem. For instance, for a search engine, an eventuality $x \in X$ might be “page such-and-such is the desired webpage”. Recall that search engines present an ordered list of plausible webpages, with the most plausible appearing at the top, followed by the second most plausible, and so on. The forecaster’s present prediction problem is to specify a plausibility ranking on $X$.

Let $\text{Rel}$ denote the set of binary (plausibility) relations on $X$.

Current memory The forecaster is equipped with her current memory $C^*$. We assume that $C^*$ is the union of a (possibly empty) finite set of past cases $D^*$ and a variable or free case $\mathcal{f}$. The cases in $D^*$ collectively represent the forecaster’s relevant observations or experience. Formally, each $c \in D^*$ is a constant (of arity zero). Recall that the arity of a variable, function, operation or relation is the number of arguments it takes. For example, the union operation on sets has arity two and is well-defined independently of any specific domain and range.

Our first and most fundamental modification of the primitives of [GS] is the inclusion of $\mathcal{f}$ in the current memory $C^*$. We model $\mathcal{f}$ as a variable (of positive arity) with unspecified domain and range, to reflect the fact that the forecaster has no experience of it. We include $\mathcal{f}$ in the current memory $C^*$ of the forecaster in order to capture the fact that she is aware of her present prediction problem. This also allows us to model the case where the forecaster has no (relevant) data.
Plausibility given the data  Like [GS], we assume the forecaster possesses a well-defined plausibility relation \( \preceq_D \) that belongs to \( \text{Rel} \), for each nonempty subsample \( D \subseteq D^* \). In contrast, \( \preceq_f \) is indeterminate and not, therefore, a member of \( \text{Rel} \). It is however a well-defined free variable with values in \( \text{Rel} \). (Like \( f \), the domain of \( \preceq_f \) is unspecified.) As such, \( \preceq_f \) seems to be an accurate representation of a forecaster that either has no experience or that chooses to ignore all her experience. Our purpose is to describe a framework that describes how a forecaster might exploit her experience and impose constraints on the values that the variable \( \preceq_C \) can take given that she subscribes to the basic axioms of [GS].

Conceivable cases  Like [GS], our framework is general enough to accommodate a forecaster that goes beyond her current memory and includes hypothetical cases that she has not experienced, but which, through reasoning, interpolation or resampling, she can clearly describe. These hypothetical cases are formally constant, like members of \( D^* \). Whilst resampled copies of \( f \) lie beyond the experience of the forecaster and cannot be fully described (c.f. Karni and Viero [8] and Halpern and Rêgo [6, 7]), this does not mean they are inconceivable. The sense in which copies of \( f \) are conceivable can be understood by drawing an analogy with physical sectors (the minimal storage unit of a hard drive) and their content. Indeed, a more elaborate model might model a case as a pair. The first dimension being the label or address of some potentially “empty” physical sector and the second being the content that the predictor assigns to that physical sector. Cases to which no content is assigned then offer the predictor the opportunity to explore extensions of her present forecast.

Let \( A \) denote the resulting set of all cases that are relevant to
the current prediction problem. Let \([f]\) denote the set of copies of \(f\) in \(A\). Finally, the set \(D \overset{\text{def}}{=} A \setminus [f]\) consists of the set of constant or deterministic cases.

**Databases**  
Like [GS], the present model allows for every finite sized database. With the canonical example of case resampling and subsampling from the literature on bootstrapping in mind, we let

\[ D \overset{\text{def}}{=} \{ D \subseteq \mathbb{D} : \#D < \infty \} \]

denote the set of determinate or constant databases. (These are referred to as memories in [GS].)

Like \(D^*\), each \(D \in D\) contains no copies of \(f\). Perhaps through experience, in-sample reasoning, or some algorithm the forecaster possesses a well-defined plausibility ranking \(\preceq_D \in \text{Rel}\) for each \(D \in D\).

The primitive data we shall use to derive a similarity representation is the set of rankings \(\{ \preceq_D : D \in D \}\). The notation

\[ \preceq_D \]

is the point \(\langle \preceq_D : D \in D \rangle\) in \(\text{Rel}^D\),

and \(<_D\) and \(\approx_D\) denote the asymmetric and symmetric parts of \(\preceq_D\).

Let \(\mathcal{A}\) denote the corresponding set of all finite subsets of \(A\). For each \(A \in \mathcal{A} \setminus D\), the fact that for some \(a \in [f]\), \(a \in A\) means that \(\preceq_A\) is variable, indeed a well-defined free variable in \(\text{Rel}\). Although, in isolation each such \(\preceq_A\) is free, when the axioms we introduce hold, the potential values of the vector \(\preceq_A \overset{\text{def}}{=} \langle \preceq_A : A \in \mathcal{A} \rangle\) may well be constrained by the actual values of \(\preceq_D\).

**Case types**  
As in [GS], two past cases \(c, d \in D\) are of the same case type if, and only if, the marginal information of \(c\) is everywhere equal to the marginal information of \(d\). Formally, \(c \sim^* d\) if, and only if, for
every $D \in \mathcal{D}$ such that $c, d \notin D$, $\preceq_{D \cup \{c\}} = \preceq_{D \cup \{d\}}$. [GS] confirm that $\sim^*$ is an equivalence relation on $\mathcal{D}$. The set $\mathcal{D}/\sim^*$ of equivalence classes generated by $\sim^*$ is therefore a partition of $\mathcal{D}$.

We extend $\sim^*$ to $\mathcal{A}$ by taking $[f]$ to be an equivalence class of its own, so that, for every $c \in \mathcal{D}$, $f \not\sim^* c$.

**Richness Assumption.** For every case type $t \in \mathcal{A}/\sim^*$, $\# t = \infty$.

For each $A \in \mathcal{A}$, let $t \mapsto I_A(t) = \#(A \cap t)$ denote the vector that counts the number of cases each case type in $A$. As in [GS], take $A, B \in \mathcal{A}$ to be equivalent, written $A \sim^* B$, if, and only if, $I_A = I_B$.

**Potential extensions** We model the potential impact of novel cases using potential extensions of $\preceq_{\mathcal{D}}$.

**Definition 1.** $\mathcal{R} = \langle \mathcal{R}_A : A \in \mathcal{A} \rangle$ is a (potential) extension of $\preceq_{\mathcal{D}}$ if, and only if, for some nonempty $Y \subseteq X$ all of the following hold.

1. For every $A \in \mathcal{A}$, $\mathcal{R}_A$ is a binary relation on $Y$ with symmetric part $\mathcal{G}_A$ and asymmetric part $\mathcal{P}_A$.
2. For every $D \in \mathcal{D}$, $\mathcal{R}_D$ is the restriction of $\preceq_{\mathcal{D}}$ to $Y$.\(^4\)
3. For every $a, a' \in \mathcal{A}$ such that $a \sim^* a'$ and every $A \in \mathcal{A}$ such that $a, a' \notin A$, $\mathcal{R}_{A \cup a} = \mathcal{R}_{A \cup a'}$.

Let $\mathcal{R}$ be an extension. Two cases $a, a' \in \mathcal{D}$ are equivalent with respect to $\mathcal{R}$, written $a \sim^{\mathcal{R}} a'$ if, for every $D \in \mathcal{D}$ such that $a, a' \notin D$, $\mathcal{R}_{D \cup a} = \mathcal{R}_{D \cup a'}$. When explicit reference to $Y$ is necessary, we refer to $\mathcal{R}$ as a $Y$-extension and to $\text{Ext}(Y, \preceq_{\mathcal{D}})$ as the set of such extensions.

We partition the set of extensions as follows.

\(^4\)That is, $\mathcal{R}_D = \preceq_{\mathcal{D}} \cap Y^2$, or, equivalently, for every $D \in \mathcal{D}$ and every $x, y \in Y$, $x \mathcal{R}_D y$ if, and only if, $x \preceq_{\mathcal{D}} y$. 

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Definition. An extension $R$ is either regular or novel. It is novel whenever it holds that, for every $d \in \mathcal{D}$, $\notin^{R} d$.

As a consequence of definition 1, an extension $R$ is novel if, and only if, $[\notin]_R$ is a distinct equivalence class of $\sim^{R}$. For novel extensions, $\notin$ mimics the role of new information, or, in the terminology of Halpern and Rêgo [5], $\notin$ is a placeholder for information that the predictor cannot currently describe.

For regular extensions, $\notin$ is equivalent, in terms of the information it provides, to some case in $\mathcal{D}$. It is essentially a copy of some past case. Whilst there are many regular extensions, the following observation establishes that every regular $Y$-extension $R$ such that $Y = X$ is equivalent to $\leq_{D}$, written $R \equiv \leq_{D}$. By this we mean that, for every $A \in \mathcal{A}$, there exists $D \in \mathcal{D}$ such that $D \sim^{R} A$ and $\leq_{D} = R_A$. (The converse follows from item 2 of definition 1.) We extend this notion of equivalence to pairs of extensions.

Observation 1 (Proof on page 6). Every regular $Y$-extension $R$ is equivalent to the restriction of $\leq_{D}$ to $Y$.

Proof of observation 1. W.l.o.g., take $A \in \mathcal{A} \setminus \mathcal{D}$, so that $A$ contains at least one copy of $\notin$. For any $a \in A \cap [\notin]$, the fact that $R \in \text{reg}(X, \leq_{D})$ implies that $a \sim^{R} a_1$ for some $a_1 \in \mathcal{D}$. The richness assumption ensures that we may choose $a_1$ from the complement of $A$. Then $A_1 \overset{\text{def}}{=} A \setminus a$ contains neither $a$ nor $a_1$, we conclude that $R_A = R_{A_1 \cup a_1}$. If $a$ is the unique member of $A \cap [\notin]$, then the proof is complete. Otherwise, using the fact that $A$ is finite, we may proceed by induction until we obtain a set $A_n$ such that $A_n \cap [\notin]_R$ is empty and $A' \overset{\text{def}}{=} A_n \cup \{a_1, \ldots, a_n\}$ satisfies $A' \in \mathcal{D}$, $R_{A'} = R_A$ and $A' \sim^{R} A$. Item 2 completes the argument because $R_{A'} \equiv \leq_{A'}$ whenever $A' \in \mathcal{D}$ and $R \in \text{ext}(X, \leq_{D})$. 
For each $Y \subseteq X$, the corresponding statement holds by the same argument, with $\preceq_D \cap Y^2$ replacing $\preceq_D$. \hfill \square

2 Axioms and main theorem

We first restate the axioms of [GS] in terms of extensions.

2.1 The basic axioms of [GS]

A1 (Transitivity). For every $A \in \mathcal{A}$, $\mathcal{R}_A$ is transitive on $Y$.

A2 (Completeness). For every $A \in \mathcal{A}$, $\mathcal{R}_A$ is complete on $Y$.

A3 (Combination). For every disjoint $A, B \in \mathcal{A}$ and every $x, y \in Y$, $[x \mathcal{R}_A y$ and $x \mathcal{R}_B y]$ and $[x \mathcal{P}_A y$ and $x \mathcal{R}_B y]$ respectively imply

$$x \mathcal{R}_{A \cup B} y \quad \text{and} \quad x \mathcal{P}_{A \cup B} y.$$

A4 (Archimedeanity). For every disjoint $A, B \in \mathcal{A}$ and every $x, y \in Y$, if $x \mathcal{P}_A y$, then there exists $k \in \mathbb{Z}_+$ such that, for every pairwise disjoint $\{A_j \in \mathcal{A} : A_j \sim^\mathcal{R} A$ and $A_j \cap B = \emptyset\}_1^k$, $x \mathcal{P}_{A_1 \cup \ldots \cup A_k \cup B} y$.

Observation 1 ensures that it is meaningful to say that $\preceq_D$ satisfies a basic axiom if, and only if, either some (and therefore every) regular $Y$-extension with $Y = X$ satisfies that axiom.\footnote{Note that, in the case that $D$ is empty, the set $\text{reg}(X, \preceq_D)$ is also empty.} We adopt this generalised form so as to accommodate our main axiom, prudence.

2.2 Diversity and Prudence

For $k = 4$, the following axiom is a restatement of the diversity axiom of [GS]. (Our main theorem holds with $k = 2$.) First some notation.
For any extension \( \mathcal{R} \), let total(\( \mathcal{R} \)) denote the total orders that arise in \( \mathcal{R} \) (i.e. those that are antisymmetric, complete and transitive).\(^6\)

**Diversity (k-diversity).** *For every* \( Y \subseteq X \) *of cardinality* \( n = 2, \ldots, k \), *and every regular* \( Y \)-extension \( \mathcal{R} \) *of* \( \preceq_D \), *\#total(\( \mathcal{R} \)) = n!.*

Before introducing our main axiom, we introduce a minimal subclass of novel extensions that are suitable for checking or testing.

**Definition 2.** A novel extension \( \mathcal{R} \) is testworthy if it satisfies A2–A4 and if \( \mathcal{R}_i = \mathcal{R}_D^{-1} \) for some \( D \in \mathcal{D} \) such that \( \mathcal{R}_D \) is total.\(^7\)

When \( C^* \) is sufficiently rich, that, in addition to the basic axioms, \( \preceq_D \) satisfies 4-diversity, the predictor need not engage in higher-order sampling or testing of novel extensions. Testing, that is, to see if the arrival of novel cases will force her to either be dogmatic (and exclude accurate plausibility rankings/predictions) or violate transitivity. We will assume the following holds for \( k = 4 \).

**Prudence (k-prudence).** *For every* \( Y \subseteq X \) *with* \( 3 \leq \#Y \leq k \) *and every testworthy* \( Y \)-extension \( \mathcal{R} \) *of* \( \preceq_D \), *there exists an extension* \( \mathcal{R} \) *of* \( \preceq_D \) *that satisfies A1–A4, \( \mathcal{R}_i = \mathcal{R}_i \) *and \#total(\( \mathcal{R} \)) \leq \#total(\( \mathcal{R} \)).*

Given our definition of extension, it is natural to ask whether A1–A4 are superfluous in the presence of 4-prudence. As we will see in the proof of the main theorem, one issue is that, when \( T \) is infinite, there may exist \( Y \subseteq X \) such that the set of testworthy \( Y \)-extensions is empty. For every such \( Y \), 4-prudence holds vacuously. As we show,

\[^6\]Thus, if \( \mathcal{R} \) is an extension that satisfies A1 and A2, then

\[
\text{total}(\mathcal{R}) \overset{\text{def}}{=} \{ R : \text{for some } A \in \mathcal{A}, R = \mathcal{R}_A \text{ is total} \}.
\]

\[^7\]Recall that the inverse \( \mathcal{R}_D^{-1} \) of \( \mathcal{R}_D \) satisfies \( x \mathcal{R}_D^{-1} y \) if, and only if, \( y \mathcal{R}_D x \).
for such $Y$ the regular $Y$-extensions of $\preceq_D$ are $k$-diverse. One may also ask whether $4$-prudence is simply requiring that $4$-diversity holds for novel $Y$-extensions such that $\preceq_D$ fails to satisfy $4$-diversity on $Y$. In one of the steps in our proof, we show that $4$-prudence guarantees a representation even when, for some $Y$, there is no $Y$-extension (novel or regular) that satisfies $k$-diversity for $k = 3, 4$.

2.3 $D$-distinctness relative to a reference eventuality

Finally, we present an axiom where a possibly unknown reference eventuality plays a subtle role in simultaneously expanding the domain of the model and simplifying the statement of the main theorem. In combination with the other axioms, $A_0$ is the weakest condition that yields uniqueness of the representation and allows us to avoid restrictions on the cardinality of $X$. We demonstrate this claim through our proof of the main theorem and ??.

A0 ($D$-distinctness relative to a reference eventuality). There exists $x_0 \in X$ such that, for every distinct $y, z \in X$, it is not the case that

$$
\text{for every } D \in D, x_0 \preceq_D y \text{ if, and only if, } x_0 \preceq_D z.
$$

We note that $A_0$ is implied by $4$-diversity. Indeed, $4$-diversity implies that the same condition holds for every $x_0 \in X$. In ??, we summarise the implications of a reference-point-free version of $A_0$ via ??, which is a special case of the main theorem that now follows. In settings where no eventuality forms an obvious point of reference, it may well be more natural to adopt the alternative, reference-point-free model. On the other hand, in such situations, we may identify the set
of potential reference eventualities by eliciting $\not\leq_D$ to be those $x_0 \in X$ that satisfy $A_0$.

2.4 Existence

Any function $v : X \times D \to \mathbb{R}$ is a real-valued matrix on $X \times D$ and $v(x, \cdot)$ denotes one of its rows. Recall that $v(x, \cdot)$ is (directly) proportional to $v(y, \cdot)$ if there exists a nonzero constant $\lambda$ such that $v(x, \cdot) = \lambda v(y, \cdot)$. Also that $v(x, \cdot)$ is weakly dominated by $v(y, \cdot)$ whenever $v(x, \cdot) \leq v(y, \cdot)$. (This inequality holds pointwise on $D$.)

**Theorem 1** (Part I, Existence). *Let there be given $X$, $A$ and $\not\leq_D$, as above, such that the richness condition holds. Then (1.a) and (1.b) are equivalent.*

- (1.a) $T = D/_{\sim}$ and $\not\leq_D$ satisfies $A_0$–$A_4$, 2-diversity and 4-prudence.
- (1.b) $T$ is the coarsest partition of $D$ such that, for some matrix $v : X \times T \to \mathbb{R}$ and $x_0 \in X$ with $v(x_0, \cdot) = 0$, both the following hold: no row is weakly dominated by, or proportional to, any other row; and

\[
(*) \left\{ \begin{array}{l}
\text{for every } x, y \in X \text{ and every } D \in D, \\
\quad x \not\leq_D y \iff \sum_{t \in T} v(x, t)I_D(t) \leq \sum_{t \in T} v(y, t)I_D(t).
\end{array} \right.
\]

To allow for a straightforward comparison, we now present a restatement of the corresponding theorem of [GS].

**Theorem** ([GS], Existence). *Let there be given $X$, $D$ and $\not\leq_D$, as above, such that the richness condition holds. Then (1.a) and (1.b) are equivalent.*

- (1.a) $T = D/_{\sim}$ and $\not\leq_D$ satisfies $A_1$–$A_4$ and 4-diversity.
- (1.b) $T$ is the coarsest partition of $D$ such that, for some matrix $v$ on $X \times T$, both of the following hold: no row is weakly dominated by any the affine combination of any three other rows; and $(*)$. 

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2.5 Uniqueness

A key consequence of assuming that $\preceq_D$ satisfies A0 and 2-diversity is the following uniqueness property. Although these conditions are not necessary for uniqueness, they are the minimal conditions that are not context-specific.

**Theorem 1** (Part II, Uniqueness). *If there exists a matrix $v$ satisfying (1.b), then it is unique in the following sense: for every other matrix $v$ that satisfies (1.b), there is a scalar $\lambda > 0$ such that $v = \lambda v$.*

We discuss the counterpart to of theorem 1 part II in [GS]

**Discussion of diversity and prudence (to go somewhere else)** Like [GS], we appreciate the technical nature role of the diversity axiom, yet we also view it as another form of richness condition on the set $C^*$ of past cases. Our main contention is that $C^*$ may not be so rich as to satisfy 4-diversity. That is to say, there may exist $Y \subseteq X$ such that $\#Y = 4$, and such that the data does not support all $4! = 24$ strict rankings.

**Remark 1.** *(To go somewhere else)*

*The main contribution of this paper is to show that, in the case where the predictor engages in higher-order sampling and explores novel extensions, this absence of rich data does not preclude a similarity representation of the consistent form that [GS] derive. This is feasible provided the predictor is prudent enough to check that the arrival of novel cases will not force her into the dilemma of choosing between being dogmatic and violating transitivity.*
3 Proof of theorem 1

We begin with a proof of the fact that $T = \mathbb{D}_{\sim}$ is the coarsest partition that is rich and satisfies (1.b).

**Lemma 1** (Proof on page 12). In theorem 1, (1.b) holds only if $T$ is at least as fine as $\mathbb{D}_{\sim}$.

*Proof of lemma 1.* By way of contradiction, suppose that there exists $T'$ satisfying (1.b) and such that $T'$ is either coarser than $\mathbb{D}_{\sim}$, or it is incomparable with $\mathbb{D}_{\sim}$. In either scenario, for some $c, d \in t' \in T'$, $c \in s$ and $d \in t$, where $s$ and $t$ are distinct members of $\mathbb{D}_{\sim}$. Then $c \not\sim^* d$, so that there exists $D$ satisfying $c, d \notin D$ and $\preceq_{D \cup c} \neq \preceq_{D \cup d}$. Fix some such $D$ and take $v' : X \times T' \to \mathbb{R}$ to be a 2-diversified matrix satisfying (1.b). Then, since $c, d \in t'$, $I_c(t') = I_d(t') = 1$, and

$$I_{D \cup c}(t') = I_D(t') + I_c(t') = I_D(t') + I_d(t') = I_{D \cup d}(t').$$

Then, for every $x, y \in X$, $x \preceq_D y$ if, and only if, $\sum_{t'' \in T'} (-v(x, t'') + v(y, t'')) I_D(t'') \geq 0$. In turn, for every $x, y \in X$,

$$\sum_{t'' \in T'} (-v(x, t'') + v(y, t'')) I_{D \cup c}(t'') \leq 0 \text{ iff } \sum_{t'' \in T'} (-v(x, t'') + v(y, t'')) I_{D \cup d}(t'') \leq 0.$$

But this contradicts the fact that, $\preceq_{D \cup c} \neq \preceq_{D \cup d}$. \hfill \Box

We now translate the model into one where databases are represented by counting vectors, the dimensions of which are case types. Via this translation, we arrive at theorem 2 which holds if, and only if, theorem 1 does. The proof of theorem 2 can be found in section 4.

3.1 Translation to counter vectors

Take $\mathbb{Z}_{+}$ to denote the nonnegative integers and $\mathbb{Z}_{+}$ to denote those that are (strictly) positive. Let $2^X$ denote the family of nonempty
subsets $Y \subseteq X$ and let $\mathcal{J}$ denote the family of partitions $T$ that satisfy the richness condition and are at least as fine as $\mathbb{D}/\prec\prec$.

For now, let $T$ denote an arbitrary partition of $\mathbb{D}$ and let $T^f \overset{\text{def}}{=} T \cup [f]$. For any such $T$, let $\mathcal{T}$ denote a free variable in $\{T, T^f\}$. When no possible confusion should arise, we work on the understanding that $f$ is shorthand for $[f]$. Let $t^T \subseteq \mathbb{Z}_+^T$ denote the set of counting vectors:

\[
\{ I : T \to \mathbb{Z}_+ \text{ such that } \{ s : I(s) \neq 0 \} \text{ is finite} \}.
\]

**Construction of rankings indexed by counting vectors**

For each $D \in \mathbb{D}$, there exists a unique $I \in t^T$, such that,

for each $t \in T$, $I(t) = \#(D \cap t)$. 

(1)

Provided $T$ is rich, a partial converse of this latter statement also holds. That is, for each $I \in t^T$, there exists $D \in \mathbb{D}$ such that $I(t) = \#(D \cap t)$. However, since $D$ is nonunique in this respect, so we need to ensure that, for every $I \in t^T$, $\preceq_I$ is well-defined according to

**Definition 3.** For each $I \in t^T$, let $\preceq_I = \preceq_D$, if, and only if, $D \in \mathbb{D}$ satisfies eq. (1).

**Lemma 2** (Proof on page 13). $\preceq_{t^T} \overset{\text{def}}{=} \langle \preceq_I : I \in t^T \rangle$ is well-defined if, and only if, $T \in \mathcal{J}$.

**Proof of lemma 2: necessity of $T \in \mathcal{J}$.** First recall that a necessary condition for an expression to be well-defined is that it is defined. If $\#t < \infty$ for some $t \in T$, then there exists $I \in t^T$, such that $I(t) > \#t$, so that (1) fails to hold and $\preceq_I$ is not defined. Thus a necessary condition for $\preceq_{t^T}$ to be defined is that, for every $t \in T$, $\#t = \infty$.

If $T$ is coarser than $\mathbb{D}/\prec\prec$, then for some $s, t \in \mathbb{D}/\prec\prec$ and $t' \in T$, $s, t \subseteq t'$. Then, by the definition of $\prec\prec$, there exists $c \in s$ and $d \in t$
such that, for some \( A \in \mathcal{A} \) such that \( c, d \notin A \), \( \preceq_{A \cup c} \neq \preceq_{A \cup d} \). Since \( s, t \subset t' \), for \( C = A \cup c \) and \( D = A \cup d \), we have \( \#(C \cap t') = \#(D \cap t') \).

Thus, by definition 3, for \( I \in \mathcal{I} \) such that (1) holds for both \( C \) and \( D \), \( \preceq_I \) fails to be well-defined. A similar argument shows that \( \preceq_{\mathcal{T} \cap I} \) fails to be well-defined for every \( T \) that is incomparable with \( \mathcal{D}/\prec \). Thus a necessary condition for \( \preceq_{\mathcal{T} \cap I} \) to be defined is that \( T \) is at least as fine as \( \mathcal{D}/\prec \).

**Proof of lemma 2:** sufficiency of \( T \in \mathcal{I} \). Consider \( T \) such that, for every \( t \in T \), \( \#t = \infty \). In this case, for each \( I \in \mathcal{I} \), the existence of \( D \in \mathcal{D} \) such that (1) holds is ensured. It remains to be shown that the nonuniqueness of \( D \) is not an issue when \( T \) is at least as fine as \( \mathcal{D}/\prec \).

Consider

**Proposition 1.** For every \( C, D \in \mathcal{D} \), \( C \sim^* D \) implies \( \preceq_C = \preceq_D \).

**Proof of Proposition 1.** For the case where \( C = \{c\} \) and \( D = \{d\} \), \( \preceq_C = \preceq_D \) follows directly from the definition of \( \sim^* \). For the case where \( \#D > 1 \), the proof proceeds by induction. Suppose that the lemma holds for pairs of databases of cardinality \( k \). Take \( C \) and \( D \) to be of cardinality \( k + 1 \) and such that \( C \sim^* D \). Let \( f : C \to D \) be the bijection that satisfies \( c \sim^* f(c) \) for each \( c \in C \). By the induction hypothesis, \( C' \sim^* f(C') \) for some \( C' \subset C \) such that \( \#C' = k \). Then \( C \setminus C' = \{c'\} \) for some \( c' \) such that \( c' \sim^* f(c') \). Since \( C \) is the disjoint union of \( C' \) and \( c' \) and \( D \) is the disjoint union of \( f(C') \) and \( f(c') \), it follows that \( \preceq_C = \preceq_D \), as required.

Let \( C \) and \( D \) be such that \( \#(t \cap C) = \#(t \cap D) \) for every \( t \in T \). Then, since \( T \) is at least as fine as \( \mathcal{D}/\prec \), each \( s \in \mathcal{D}/\prec \) is the union of members of \( T \). This implies \( \#(s \cap C) = \#(s \cap D) \) for every \( s \in \mathcal{D}/\prec \).

This allows us to apply proposition 1 and obtain \( \preceq_C = \preceq_D \).
As a consequence of lemma 1 and lemma 2, we henceforth take $T = \mathbb{D}_{/\sim}$, but note that the following construction and proof would work for any member of $\mathcal{T}$.

**Construction of extensions indexed by counting vectors**

For every nonempty $Y \subseteq X$, let $\text{reg}(Y, \preceq_\bullet)$, $\text{nov}(Y, \preceq_\bullet)$ and $\text{test}(Y, \preceq_\bullet)$ respectively denote the set of regular, novel and testworthy $Y$-extensions of $\preceq_\bullet$.

For each $Y \in 2^X$, let the *regular $Y$-extension of $\preceq_{tT}$* simply be the restriction of the latter to $Y$. Then, $\text{reg}(Y, \preceq_{tT})$ is a singleton and equal to $\{\preceq_{tT} \cap Y^2\}$. By observation 1, for every such $Y$, the unique regular extension $R_{tT}$ is equivalent to every $R_A$ in $\text{reg}(Y, \preceq_{\mathcal{A}})$, written $R_{tT} \equiv R_A$.

Next, take $\text{ext}(Y, \preceq_{tT})$ to be the union of $\text{reg}(Y, \preceq_{tT})$ and the set $\text{nov}(Y, \preceq_{tT})$ of proper $Y$-extensions of $\preceq_{tT}$, where the latter are the subject of the definition that now follows. First, for each $t \in T$, let $\delta_t$ be the canonical basis vector for dimension $t$ in $\mathbb{Z}^T$, so that $\delta_t(s) = 1$ if $s = t$ and $\delta_t(s) = 0$ otherwise. For each $t \in T^f$, we define $\delta^f_t : T^f \to \mathbb{Z}$ analogously.

**Definition 4.** For every $Y \in 2^X$, $\mathcal{R} = \langle R_J : J \in t^T \rangle$ belongs to $\text{nov}(Y, \preceq_{tT})$ if, and only if, it satisfies all of the following.

1. Item 1 of definition 1: $\mathcal{I}_J \equiv \mathcal{R}_J \cap \mathcal{R}_J^{-1}$ and $\mathcal{P}_J \equiv \mathcal{R}_J \setminus \mathcal{R}_J^{-1}$.
2. For every $J = I \times 0$ in $t^T$ such that $I \in t^T$, $\mathcal{R}_J = \preceq_I \cap Y^2$.
3. For every $s \in T$, there exists $I \in t^T$ such that, for $J = I \times 0 \in t^T$,

$$
\mathcal{R}_{J+\delta_s^f} \neq \mathcal{R}_{J+\delta_I^f}.
$$

Recall (from observation 1) that $\mathcal{R}_{tT} \equiv \mathcal{R}_{\mathcal{A}}$ whenever it holds that for every $I \in t^T$, there exists $A \in \mathcal{A}$ such that $R_I = R_A$ and vice versa.
Take \( \eta : \mathcal{A} \to \mathcal{T}^I \) to be the map \( A \mapsto \eta(A) = I \) where \( I \) is such that, for each \( t \in \mathcal{T}^I \), \( I(t) = \#(A \cap t) \). Then, for each \( Y \), the sets \( \text{nov}(Y, \leq_{1^T}) \) and \( \text{nov}(Y, \leq_\mathcal{D}) \) are isomorphic (written \( \text{nov}(Y, \leq_{1^T}) \cong \text{nov}(Y, \leq_\mathcal{D}) \)) in the following sense. For each member \( R \) of the former, there exists a member \( R' \) of the latter such that \( R' \equiv R \), and vice versa.

**Claim 1** (Proof on page 16). For every \( Y \in 2^X \), \( \text{nov}(Y, \leq_{1^T}) \cong \text{nov}(Y, \leq_\mathcal{D}) \).

**Proof of Claim 1.** We show that there exists a canonical embedding (a structure preserving injection) of \( \text{nov}(Y, \leq_{1^T}) \) into \( \text{nov}(Y, \leq_\mathcal{D}) \). The fact that this map is also surjective follows from the fact that \( \text{nov}(Y, \leq_\mathcal{D}) \) can be embedded in \( \text{nov}(Y, \leq_{1^T}) \) in the same way.

Take \( R \in \text{nov}(Y, \leq_{1^T}) \) and define \( R = \langle R_A : A \in \mathcal{A} \rangle \) in the following way. For each \( A \in \mathcal{A} \), let \( R_A \overset{\text{def}}{=} R_J \) if, and only if, \( A \in \eta^{-1}(J) \).

Clearly, if we take some \( R' \neq R \in \text{nov}(Y, \leq_{1^T}) \) and define a corresponding \( R' \in \text{nov}(Y, \leq_\mathcal{D}) \) in the same way, then \( R' \neq R \), so that our mapping is injective. Thus, if we can show that \( R \) belongs to \( \text{nov}(Y, \leq_\mathcal{D}) \), then we have indeed constructed the required embedding.

The fact that \( R \) satisfies items 1 and 2 of definition 1 follows immediately from definition 4. The proof that item 3 of definition 1 holds is as follows. Take any \( a, a' \in \mathcal{A} \) and \( A \in \mathcal{A} \) such that \( a \sim^* a' \) and \( a, a' \notin A \). First, observe that \( A \cup a \sim^* A \cup a' \), and moreover, for some \( t \in \mathcal{T}^I \) we have \( a, a' \in t \). Then, by the definition of \( \eta' \), both \( A \cup a \) and \( A \cup a' \) belong to \( \eta^{-1}(J) \) for some \( J \in \mathcal{T}^I \). Thus \( R_{A \cup a} = R_{A \cup a'} \), as required for \( R \) to be an extension of \( \leq_\mathcal{D} \). Finally, item 3 of definition 4 ensures that \( a \not\sim_R a' \) for every \( a \in \mathcal{D} \). Since \( \sim_R \) inherits this property, \( R \) is indeed novel.
Axioms and theorem in terms of counting vectors

We now restate our axioms and main theorem in terms of members of \(\text{ext}(Y, \preceq_{t^T})\). In these axioms, \(R\) and \(R^f\) are extensions of \(\preceq_{t^T}\). In the axioms and results that follow, we proceed on the understanding that

\[
T = \begin{cases} 
T & \text{when the statement refers to regular extensions, and} \\
T^f & \text{otherwise.}
\end{cases}
\]

A1* For every \(I \in t^T\), \(R_I\) is transitive on \(Y\),

A2* For every \(I \in t^T\), \(R_I\) complete on \(Y\),

A3* For every \(I, J \in t^T\) and every \(x, y \in Y\), if \(x R_I y\), then \(x R_J y\) implies \(x R_{I+J} y\) and \(x P_J y\) implies \(x P_{I+J} y\).

A4* For every \(I, J \in t^T\) and every \(x, y \in Y\), if \(x P_I y\), then there exists \(j \in Z_+\) such that \(x P_{jI+J} y\).

Similar to the main section, \(\preceq_{t^T}\) satisfies one of the basic axioms A1*–A4* if, and only if, the (unique) regular extension that satisfies \(R = \preceq_{t^T}\) satisfies that axiom.

Just as in definition 2, a novel extension \(R\) of \(\preceq_{t^T}\) is testworthy if it satisfies A2*–A4* and \(R_I\) is both strict (i.e. antisymmetric) and, for some \(I \in t^T\) and \(I \times 0 \in t^{2T}\), \(R_I = R_{I \times 0}^{-1}\). Let \(\text{test}(Y, \cdot)\) denote the set of testworthy \(Y\)-extensions of \(\preceq_{t^T}\), \(\preceq_{t^T}\) or \(\preceq_{t^T}\) (the latter is introduced below). As a consequence of claim 1 and the construction of \(\preceq_{t^T}\), for each \(Y \in 2^X\), \(\text{test}(Y, \preceq_{t^T}) \simeq \text{test}(Y, \preceq_{t^T})\).

4-Pru* For every \(Y \subseteq X\), \(#Y = 3, 4\), and every testworthy \(Y\)-extension \(R\) of \(\preceq_{t^T}\), there exists an extension \(R\) of \(\preceq_{t^T}\) that satisfies A1*–A4*, \(R_I = R_I\) and \(\#\text{total}(R) \leq \#\text{total}(R)\).

2-Div* For every \(Y \subseteq X\) of cardinality 2, and every regular \(Y\)-extension \(R\) of \(\preceq_{t^T}\), \(\#\text{total}(R) = 2\).
For $k \geq 3$, the $k$-diversity axiom is identical except that it now applies to extensions of $\preceq_{i^T}$. The above translation and related results ensure that theorem 1 is a consequence of the following theorem. By lemmas 1 and 2, we are able to modify the statement so that, as in the corresponding result of [GS], the fact that $T$ coincides with $D_{/\sim}$, is given.

**Theorem 2.** Let $T = D_{/\sim}$ and let there be given $X$ and $\preceq_{i^T}$ as above. Then (2.a) and (2.b) are equivalent.

(2.a) $\preceq_{i^T}$ satisfies A1*–A4*, 4-Pru* and 2-Div*.

(2.b) There exists a matrix $v : X \times T \rightarrow \mathbb{R}$ such that no row dominates any other, and,

$$(**) \text{ for every } x, y \in X \text{ and every } I \in i^T,$$

$$x \preceq_I y \text{ iff } \sum_{t \in T} v(x, t)I(t) \leq \sum_{t \in T} v(y, t)I(t).$$

**4 Proof of theorem 2**

In the proof of the corresponding theorem (see theorem 2) of [GS], the authors first translate from $i^T$ to a suitable subset $i^T$ of rational vectors $J : T \rightarrow \mathbb{Q}_+$. We begin the proof with a similar translation to show that $\preceq_{i^T}$ is equivalent to $\preceq_{i^T}$. We then derive axioms A1*–A4*, which apply to extensions of the $\preceq_{i^T}$ and establish that 4-prudence holds for $\preceq_{i^T}$ if, and only if, it holds for $\preceq_{i^T}$. In section 4.2, via lemma 4 obtain a characterisation of A2*–A4* and the set of novel extensions. Although this result is of interest in its own right, it is also essential because it will allow us to work with vector representations $v^{Y^2} \triangleq \{v^\zeta \in \mathbb{R}^T : \zeta \in Y^2\}$ instead of extensions. That is, for any $Y$-extension $\mathcal{R}$, $v^\zeta$ such that $v^\zeta \cdot I \geq 0$ if, and only if, $\zeta \in \mathcal{R}_I$. We
refer to vector representations that are inseparable across pairs in $X$ as \textit{pairwise representations}. In this respect, section 4.2 extends lemma of 1 of [GS].

The main part of the proof then follows. We first establish the conditions under which our proof by induction on $X$ will succeed. This step is closely related to lemma 3 and claim 9 of [GS]. There the authors establish that a necessary and sufficient condition for the existence of a matrix satisfying item (**) is that the Jacobi identity holds on $X$. Before continuing our outline of the proof, we provide the definition and relevant terminology for this key concept.

**Definition 5.** For $Y \in 2^X$, $u^{Y^2} \overset{\text{def}}{=} \{ u^{xy} \in \mathbb{R}^n : x, y \in Y \}$ satisfies the Jacobi identity \textit{whenever},

$$
\text{for every } x, y, z \in Y, \, u^{xz} = u^{xy} + u^{yz}.
$$

(2)

If $u^{Y^2}$ is a pairwise representation of $R$ that satisfies the Jacobi identity, we say that $u^{Y^2}$ is a \textit{Jacobi representation} of $R$. Furthermore, if $R$ is a $Y$-extension, then the \textit{Jacobi identity holds for} $R$ whenever there exists Jacobi representation of $R$; when $R$ is regular, we say that the \textit{Jacobi identity holds on} $Y$.

The following condition is closely related to $k$-prudence.

\textbf{$k$-Jac.} For $\leq_{1^X}$ and every $Y \subseteq X$ such that $3 \leq \#Y \leq k$, the Jacobi identity holds on $Y$.

We show that, when $2$-diversity holds (on $X$), a sufficient (and necessary) condition for the Jacobi identity to hold (on $X$) is that $4$-Jac holds. We note that this is a novel result. Example 1 shows that, in the absence of $2$-diversity, the proof by induction breaks down because even though $4$-Jac holds, the Jacobi identity may still fail to hold on sets of cardinality 5.
In ??, we turn to 4-prudence with the goal of showing that it is equivalent to the requirement that 4-Jac holds. The first step in this argument is to show that 4-diversity is complementary to 4-prudence in the following sense.

If, for some \( Y \in 2^X \) such that \( \#Y = k \), there are no test-worthy \( Y \)-extensions of \( \preceq_{T^Y} \), then \( k \)-diversity holds on \( Y \).

On the one hand, this ensures that, when 4-prudence holds vacuously, we are in the setting of [GS], where item (2.b) holds. On the other hand, we are able to exploit the contrapositive of this statement in our proof that (2.a) implies (2.b). Suppose that the Jacobi identity fails to hold for some regular \( Y \)-extension, then, by [GS], \( k \)-diversity fails to hold on \( Y \). In turn, if \( k \)-diversity fails to hold on \( Y \), then the set of testworthy \( Y \)-extensions is nonempty. From there, we are able to show that 4-prudence fails on \( Y \).

Finally, we show that (2.b) implies (2.a) by showing that, if the Jacobi identity holds on \( Y \), for every \( Y \) such that \( \#Y = 3,4 \), then the 4-prudence holds.

### 4.1 Translation to rationals

In the present translation, we maintain the assumption that \( T \) is an arbitrary member of \( \mathcal{T} \). Essential to the present translation is the following result that, along rays of counting vectors, plausibility rankings are homogeneous.

**Claim 2.** For every extension \( R_{1^T} \), if \( R_{1^T} \) satisfies A3*, then for every \( I \in 1^T \) and every \( k \in \mathbb{Z}_+ \), \( R_{kI} = R_I \).

**Proof of claim 2.** Since \( T \in \mathcal{T} \), lemma 2 and claim 1 ensure that \( R_{1^T} \) is well-defined for \( T = T, T^\dagger \). Fix \( I \in 1^T \) and proceed by induction.
For the initial step, A3* implies $\preceq_{2I} = \preceq_I$. For the inductive step, suppose that $\preceq_{(k-1)I} = \preceq_I$ and apply A3* once more. 

Let $I^T \subseteq \overline{Q}_+^T$ denote the set of nonnegative rational-valued vectors $I$ such that $\{t \in T : I(t) \neq 0\}$ is finite. In turn, let $I^T_I$ denote the corresponding subset of $\overline{Q}_+^T$.

For each $J \in I^T$, there exists a minimal $k \in \mathbb{Z}_+$ such that $kJ$ belongs to $I^T$. In the presence of claim 2, this implies that, for each $J \in I^T$, $\preceq_J = \preceq_{kJ}$ is well-defined, and so is $\preceq_{I^T} = \langle \preceq_J : J \in I^T \rangle$. Once again, in the sense of observation 1, $\preceq_{I^T}$ is equivalent to $\preceq_{I^T}$ and, for any nonempty $Y \subseteq X$ the set $\text{reg}(Y, \preceq_{I^T})$ of regular extensions that are indexed by rational vectors is simply the restriction of $\preceq_{I^T}$ to $Y$. Similar to before, let $\zeta : I^T \to I^T_I$ be the map $J \mapsto \kappa(J) \overset{\text{def}}{=} k_J$ such that, for each $J$, $k_J \in \mathbb{Z}_+$ is minimal. Then, for each $J \in I^T_I$, let $R_J \overset{\text{def}}{=} R_{\kappa(J)}$, so that $R_{I^T_I} = \langle R_J : J \in I^T_I \rangle$ is well-defined.

**Claim 3.** If $R_{I^T_I}$ and $R_{I^T}$ are equivalent and the latter satisfies A3*, then for every $J \in I^T_I$ and every positive rational $q$

$$R_{qJ} = R_J.$$ 

**Proof of claim 3.** The fact that $R_{I^T}$ satisfies A3* ensures that we can appeal to claim 2. Fix $R_{I^T}, J$ and $q$ as in claim 3 and let $L \overset{\text{def}}{=} qJ$. Take $\kappa$ to be the minimal member of $\mathbb{Z}_+$ such that $\kappa J = I \in I^T$. Then, by the construction of $R$, $R_J = R_I$. Similarly, let $k$ be the minimal member of $\mathbb{Z}_+$ such that $kL \in I^T$, so that $R_L = R_{kL}$ by construction. Finally, since $kL$ is proportional to $I$, claim 2 yields $R_{kL} = R_I$. 

**Axioms in terms of rational vectors** We now restate the axioms for extensions $R$ of $\preceq_{I^T}$. 

**A1** For every $J \in I^T$, $R_J$ is transitive on $X$. 

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A2* For every $J \in I^T$, $R_J$ complete on $X$.

A3* For every $I, J \in I^T$, every $x, y \in X$ and every $\lambda, \mu \in \mathbb{Q}_+$, together $x R_I y$ and $x R_J y$ imply $x R_{\lambda I + \mu J} y$, and, if either premise holds strictly then $x R_{\lambda I + \mu J} y$.

A4* For every $I, J \in I^T$ and every $x, y \in X$ if $x R_I y$, then there exists $0 < \lambda < 1$ such that, for every $\mu \in \mathbb{Q} \cap [0, \lambda)$, $x R_{(1-\mu)I + \mu J} y$.

For the prudence axiom, recall that a testworthy extension of $\preceq_I^T$ is defined just in the same way as a testworthy extension of $\preceq_I^T$.

$\mathcal{A}_4$-Pru* For every $Y \subseteq X$, $\#Y = 3, 4$, and every testworthy $Y$-extension $\mathcal{R}$ of $\preceq_I^T$, there exists an extension $\mathcal{R}$ of $\preceq_I^T$ that satisfies $A1^*$--$A4^*$, $R_I = R_I$ and $\#total(\mathcal{R}) \leq \#total(\mathcal{R})$.

The following two results imply that $\preceq_I^T$ satisfies $\mathcal{A}_4$-Pru* if, and only if, $\preceq_I^T$ satisfies $\mathcal{A}_4$-Pru*.

**Claim 4** (Proof on page 22). For every nonempty $Y \subseteq X$,

$$\text{nov}(Y, \preceq_I^T) \simeq \text{nov}(Y, \preceq_I^T).$$

*Proof of claim 4.* This follows directly from the construction of $\preceq_I^T$ and its rational extensions, and the same argument as claim 1. \(\square\)

**Claim 5** (Proof on page 22). If $R_I^T$ is equivalent to $R_I^T$, then each of $A1^*$, $A2^*$, and $A3^*$ holds for $R_I^T$ if, and only if, the corresponding axiom holds for $R_I^T$. Furthermore, $R_I^T$ satisfies $A3^*$--$A4^*$ if, and only if, $R_I^T$ satisfies $A3^*$--$A4^*$.

*Proof of claim 5.* Fix $R_I^T$ and $R_I^T$ as in claim 5. If $R_I^T$ satisfies $A1^*$, then, by the construction of rational extensions, so does $R_I^T$. Moreover, the converse also holds, by construction. The same is true of $A2^*$ and $A2^*$. 22
In the present paragraph, we assume the $Y$-extension $R_ι$ satisfies $A3^*$ and show that $R_ι$ satisfies $A3^*$. Fix $x, y ∈ Y$ and $J ∈ T^ι$ such that $x R_ι y$ and $x R_ι y$. Fix $λ, μ ∈ Q_+$ and let $κ$ be the smallest positive integer such that both $I := κλJ$ and $I' := κμJ'$ belong to $T^ι$. Then, by claim 3, $x R_I y$ and $x R_{I'} y$. Moreover, by the construction of rational extensions, $x R_{I'} y$ and, by $A3^*$, $x R_{I + J'} y$. Finally, since $I + I' = (κλJ + μJ')$, by the construction of rational extensions, we have $x R_{κλJ + μJ'} y$, as required for $A3^*$.

Once again, the converse of the statement proved in the preceding paragraph holds by the construction of rational extensions.

In the present paragraph, we assume $R_ι$ satisfies $A3^*$ and $A4^*$ and prove that $R_ι$ satisfies $A4^*$ (the proof that $R_ι$ also satisfies $A3^*$ is above). Fix $x, y ∈ X$ such that $x R_J y$ for some $J ∈ T^ι$ and take any $J' ∈ T^ι$. Then, by the construction of $R_{J'}$, there exists $I, I' ∈ T$ such that $jJ = I$ and $j'J' = I'$ for some $j, j' ∈ Z_+$. By claim 3, $R_I = R_J$ and $R_{I'} = R_{J'}$. Moreover, by the construction of $R_{J^ι}$, $R_J = R_{J'}$ and $R_{I'} = R_{J'}$. Since $x R_{I'} y$, $A4^*$ implies the existence of $κ ∈ Z_+$ such that $x R_{κI + J'} y$. Then, by the construction of $R_{J'}$, $x R_{κI + J'} y$. Let $ν := \frac{1}{κj'}$ and take $λ = νj'$, so that $0 < λ < 0$ and $1 − λ = νκj$. In fact, since $λ ∈ Q$, we have

$$K := (1 − λ)J + λJ' ∈ T^ι.$$ Simplifying, we obtain $K = ν(κI + I')$. Since $ν ∈ Q_+$ and $κI + I' ∈ T^ι$, claim 3 implies $R_K = R_{κI + I'}$. This allows us to conclude that $x R_K y$. Finally, take any $μ ∈ Q ∩ (0, λ)$. From basic properties of the real numbers, there exists $ξ < 1$ such that $μ = ξλ$ and, moreover, $ξ$ is rational. Next, note that the definition of $K$ implies $ξ(K − J) = ξλ(J' − J)$. Adding $J$ to each side of the latter and applying the
definition of $\mu$ yields

$$(1 - \xi)J + \xi K = (1 - \mu)J + \mu J' .$$

Then, since $x \mathcal{P}_{J} y$ and $x \mathcal{P}_{K} y$, $A3^*$ implies $x \mathcal{P}_{(1-\mu)J + \mu J'} y$, as required for $A4^*$.

In this paragraph, we assume that $\mathcal{R}_{J}$ satisfies $A3^*$ and $A4^*$ and prove that $A4^#$ holds. Take $I, I' \in i^T$ such that $x \mathcal{P}_{I} y$ and any other $I' \in i^T$. Then, by construction, $x \mathcal{P}_{I} y$ and, by $A4^*$, there exists $\lambda \in \mathbb{Q} \cap (0,1)$ such that $x \mathcal{P}_{(1-\lambda)J + \lambda I'} y$. Then, since $\mu$ is rational, $
\mu = j/k$ for some $j, k \in \mathbb{Z}_+$. Let $q := (1 - \mu)/\mu = (k - j)/j$ and let $\kappa = jq$, so that $\kappa = k - j$. The fact that $0 < \mu < 1$ ensures that $\kappa \in \mathbb{Z}_+$. To complete the proof, we show that $x \mathcal{P}_{\kappa I + I'} y$, for then $x \mathcal{P}_{\kappa I + I'}$ follows. Together $x \mathcal{P}_{(1-\kappa)I + \kappa I'} y$ and claim 3 imply $x \mathcal{P}_{qI + I'} y$. Similarly, together $x \mathcal{P}_{I} y$ and claim 3 imply $x \mathcal{P}_{(j-1)qI} y$. Then, since $(j - 1)qI + (qI + I') = jqI + I'$ and $\kappa = jq$, an application of $A3^*$ yields the desired result. 

4.2 A characterisation of $A2^* - A4^*$

We begin by extending lemma 1 of [GS] to accommodate $Y$-extensions. For the case where $T$ is infinite, the inner product $J \cdot u^x y$ is shorthand for the sum $\sum \{J(t)u^x y(t) : J(t) > 0\}$ which is well-defined by virtue of the fact that every $J \in j^T$ has finite support.

Lemma 3 (Proof on page 24). For every $\mathcal{R} \in \text{ext}(Y, \preceq_{j^T})$, if $\mathcal{R}$ satisfies $A2^* - A4^*$, then, for every $x, y \in Y$, there exists $v^{x y}$ and $v^{y x}$ in $\mathbb{R}^T$ such that

1. $F^{xy}_{\mathcal{R}} \overset{\text{def}}{=} \{J \in j^T : x \mathcal{R}_{J} y \} = \{J : 0 \preceq J \cdot v^{xy}\}$
2. $G^{xy}_{\mathcal{R}} \overset{\text{def}}{=} \{J \in j^T : x \mathcal{P}_{J} y \} = \{J : 0 < J \cdot v^{xy}\}$

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3. \( v^{yx} = -v^{xy} \).

**Proof of lemma 3.** Fix \( R \in \text{ext}(Y, X) \) and \( x \neq y \) in \( X \). Suppressing reference to \( R \), if \( G^{xy} \) and \( G^{yx} \) are both nonempty, then 2-diversity holds for \( x, y \) and our proof follows from that of lemma 1 of [GS]. Note that, neither lemma 1 of [GS], nor the following argument rely on A1*: together with A2*-A4*, it suffices that, for each \( J \in \mathcal{T} \), \( \mathcal{G}_J \) and \( \mathcal{P}_J \) are defined to be the symmetric and asymmetric parts of \( R_J \). W.l.o.g., we may therefore assume that \( G^{yx} = \emptyset \). If \( x \mathcal{I}_J y \) for every \( J \in \mathcal{T} \), then it is easy to verify that \( v^{xy} = 0 \in \mathbb{R}^\mathcal{T} \) satisfies the lemma. Henceforth, we assume that \( G^{xy} \) is nonempty.

Recall the definition of the basis vectors \( \{\delta_s : s \in \mathcal{T}\} \) of definition 4. By A2* and the assumption that \( G^{yx} = \emptyset \), \( \mathcal{T} \) is the disjoint union of \( \mathcal{T}_\mathcal{P} \overset{\text{def}}{=} \{ s : x \mathcal{P}_\delta y \} \) and \( \mathcal{T}_\mathcal{G} \overset{\text{def}}{=} \{ s : x \mathcal{G}_\delta y \} \). Take any \( v^{xy} \) such that \( v^{xy}(s) > 0 \) if \( s \in \mathcal{T}_\mathcal{P} \) and \( v^{xy}(s) = 0 \) otherwise. Since \( J : v^{xy} \geq 0 \) for every \( J \in \mathcal{T} \), item 1 of the lemma holds.

For item 2 of the lemma we will show that \( J : v^{xy} = 0 \), if, and only if, \( x \mathcal{I}_J y \). Fix an arbitrary \( J \in \mathcal{T} \) and let \( \mathcal{T}_J \overset{\text{def}}{=} \{ s : J(s) > 0 \} \) and note that \( \mathcal{T}_J \) is nonempty because \( J \in \mathcal{T} \). Note that \( J : v^{xy} = 0 \) if, and only if, the set \( \mathcal{T}_J \cap \mathcal{T}_\mathcal{P} \) is empty. Thus, it suffices to show that \( \mathcal{T}_J \subseteq \mathcal{T}_\mathcal{G} \) if, and only if, \( x \mathcal{I}_J y \). Let \( 1, \ldots, k \) be an enumeration of \( \mathcal{T}_J \), and let \( \delta_1, \ldots, \delta_k \) be the corresponding basis vectors. Then \( J = q_1 \delta_1 + \cdots + q_k \delta_k \) for some \( q_1, \ldots, q_k \). If \( \mathcal{T}_J \subseteq \mathcal{T}_\mathcal{G} \), then \( x \mathcal{I}_{\delta_j} y \) for every \( j = 1, \ldots, k \) and \( k-1 \) applications of A3* yield \( x \mathcal{I}_J y \). Now suppose that \( J(s) > 0 \) for some \( s \in \mathcal{T}_\mathcal{P} \), so that \( \mathcal{T}_J \nsubseteq \mathcal{T}_\mathcal{G} \). Then, for some \( j, x \mathcal{P}_{\delta_j} y \). Since \( G^{yx} = \emptyset \), we have \( x \mathcal{R}_{\delta_i} y \) for every \( i \neq j \) and, via \( k-1 \) applications of A3*, we obtain \( x \mathcal{P}_J y \), so that by item 1 definition 4, \( -\mathcal{I}_J y \). We conclude that \( -\mathcal{I}_J y \).

For item 3 of the lemma, let \( v^{yx} \overset{\text{def}}{=} -v^{xy} \). Then \( v^{yx} \) satisfies
Lemma 4 (Proof on page 26). For every $Y \in 2^X$, a $Y$-extension $R_{Y^x}$ satisfies $A2^*-A4^*$ if, and only if, there exists $\{v^{xy} : x, y \in Y\} \subset \mathbb{R}^{Y}$ such that the following condition, henceforth (3), holds.

For every $x, y \in Y$, $v^{yz} = -v^{xy}$ and, for every $J \in I^T$, $x \not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\not{\note
is positive. Moreover, for every \( q \in [0, r] \), \((1 - q)i + qj\) is positive. Then, by linearity of the inner product, it follows that
\[
((1 - q)I + qJ) \cdot v^x y > 0.
\]

Finally, to obtain A4*, apply 3 once more.

The following argument accounts for the additional content of the theorem that arises when \( R_1 \) is novel. Let \( R_1 \) satisfy A2*–A4* and be novel. Fix arbitrary \( t \neq f \). Then definition 4 implies that there exists \( J \in j^T \) and \( L = J \times 0 \in J^T \) such that \( R_{L+\delta_t^I} \neq R_{L+\delta_f^I} \). Consider the case where, for some \( x, y \in Y \), it holds that both \( y R_{L+\delta_t^I} x \) and \( x R_{L+\delta_f^I} y \). By 3, this is the case if, and only if, \((L + \delta_t^I) \cdot v^x y\) is nonpositive and \((L + \delta_f^I) \cdot v^x y\) is positive. Next note that, for every \( s \in T^I \), \((L + \delta_t^I) \cdot v^x y = L \cdot v^x y + v^x y(s) \). Thus,
\[
L \cdot v^x y + v^x y(t) \leq 0 < L \cdot v^x y + v^x y(f),
\]
which is equivalent to \( v^x y(t) \leq -L \cdot v^x y < v^x y(f) \).

The preceding paragraph ensures that \( v^x y(t) < v^x y(f) \). Let \( \mu = v^x y(t) \) and \( \nu = v^x y(f) \). If \( \nu = 0 \), then it immediately follows that \( \mu < 0 \), so that \( \nu \mu \leq 0 < \mu^2 \). Then the required \( \rho \) exists since we may take \( \rho(\cdot) \overset{\text{def}}{=} v^x y(\cdot) \mu \). The same is true whenever \( \mu \leq 0 \leq \nu \), so let us consider the case where both \( \mu \) and \( \nu \) are positive. Then since \(-L \cdot v^x y\) is also positive and \( L \) has nonnegative entries, there exists \( s \) in the support of \( L \) such that \( v^x y(s) \) is negative. Note that the support of \( L \) is a subset of \( T \). Then the proof for this case is complete since \( \rho(\cdot) \overset{\text{def}}{=} v^x y(\cdot)v^x y(s) \) is neither positive, nor constant. Finally, in the case where both \( \mu \) and \( \nu \) are negative, \(-L \cdot v^x y\) is negative, and there exists \( s \) in the support of \( L \) such that \( v^x y(s) \) is positive and, once again, the required \( \rho \) exists.
The preceding argument accounts for all the possible cases where 
\( y^{R_{I+\delta}} x \) and \( x^{P_{I+\delta}} y \). If instead \( y^{R_{L+\delta}} x \) and \( x^{P_{L+\delta}} y \), then using

the fact that \( v^{yx} = -v^{xy} \), we arrive at the inequalities

\[
L \cdot v^{yx} + v^{yx}(t) < 0 \leq L \cdot v^{yx} + v^{yx}(f),
\]

which is equivalent to \( v^{yx}(t) < -L \cdot v^{yx} \leq v^{yx}(f) \). Via a relabeling of \( x \) and \( y \), we conclude that, for this \( t \), there exists \( x, y \in Y \) such that 
\( v^{xy}(t) < v^{xy}(f) \). The arguments of the preceding paragraph then yield

the desired inequalities for some \( s \in T \).

It remains to be shown that we have a sufficient condition for \( R \) to be novel. Suppose that \( \{v^{xy} : x, y \in Y\} \) satisfies (3) for some extension \( R \). Moreover suppose that for every \( t \neq \bar{t} \), there exists \( x, y \in Y \) and

\( s \in T \) such that \( \rho \), defined as in the theorem, is neither positive nor constant. Choose arbitrary \( t \in T \). Then the fact that \( \rho \) is nonconstant implies that \( v^{xy}(t) \neq v^{xy}(\bar{f}) \). Take \( \mu = v^{xy}(t) \) and \( \nu = v^{xy}(\bar{f}) \), and consider the case where \( \mu < \nu < 0 \). Then, since \( \rho \) is nonconstant, \( v^{xy}(s) \neq 0 \), and, since \( \rho \) is nonpositive, it follows that \( v^{xy}(s) \) is positive.

Then, for some \( \lambda \in \mathbb{Q}_+ \), \( -\lambda v^{xy}(s) \in (\mu, \nu) \). Let \( L \overset{\text{def}}{=} \lambda \delta_{\bar{f}} \) and observe that \( \mu < -L \cdot v^{xy} \leq \nu \), where at least one inequality holds strictly.

By retracing the steps of the second paragraph of the present proof (in reverse order), we arrive at the conclusion that \( R_{L+\delta} \neq R_{L+\delta} \), as required.

The case where both \( \mu \) and \( \nu \) are positive is similar to the above and therefore the argument is omitted. If \( \mu \leq 0 \leq \nu \), then let \( L \) be the zero vector in \( T^T \). Then (3) implies that \( R_{L+\delta} \neq R_{L+\delta} \), as required.

Since the preceding argument holds for every \( t \neq \bar{t} \), \( R \) is indeed novel. \( \square \)
4.3 When 4-prudence holds vacuously

Clearly, if there is some $Y \in 2^X$, of cardinality $3$ or $4$, such that $\text{test}(Y, \preceq_{JT})$ is empty, then 4-Pru* holds vacuously on $Y$. When $T$ is infinite, it is possible that some such $Y$ exists. It therefore important to establish that 4-diversity holds whenever $\text{test}(Y, \preceq_{JT})$ is empty. For then theorem 2 of [GS] guarantees that $(\ast\ast)$ holds.

To facilitate our discussion of the case where $\preceq_{JT}$ fails to satisfy 2-diversity, in the present section we adopt the following, considerably weaker, assumption. $Y \in 2^X$ is pairwise $D$-distinct whenever it holds that, for every distinct $x, y \in Y$, there exists $D \in D$ such that $-(x \approx_D y)$.

**Lemma 5.** Let $\preceq_{JT}$ satisfy $A1^*–A4^*$ and let $Y \in 2^X$ be pairwise $D$-distinct and such that $\#Y = 4$. If $\text{test}(Y, \preceq_{JT})$ is empty, then 4-diversity holds on $Y$. (A similar statement holds if $\#Y = 3$.)

**Proof of lemma 5.** Let $Y = \{x, y, z, w\}$. (The proof for case where $\#Y = 3$ is similar and omitted.) Let $R$ be a regular $Y$-extension. The assumptions lemma 5 imposes are sufficient for lemma 6, below. As a consequence, $\text{total}(R)$ is nonempty. Choose $I \in i^T$ such that $R_I$ is total and let $Z = \mathcal{P}_I$. Then, for every distinct $x, y \in Y$, either $\zeta = x \times y \in Z$, or $\zeta^{-1} = y \times x \in Z$.

Since $\preceq_{JT}$ satisfies $A2^*–A4^*$ and $R$ is a regular extension, lemma 4 ensures the existence of a family of vectors $\{u^\zeta : \zeta \in Y^2\} \subset \mathbb{R}^T$ satisfying (3). Then $\zeta \in Z$ if, and only if, $u^\zeta \cdot I > 0$. By the properties of the inner product (or by $A3^*$), for every $\zeta \in Z$, there exists $s^\zeta \in T$ such that $u^\zeta(s^\zeta) > 0$. Let $S^Z$ denote the set of such $s^\zeta$. In the proof of the claim that now follows, we will use this notation and exploit the fact that $\text{test}(Y, \preceq_{JT}) = \emptyset$. 

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Claim 6 (Proof on page 29). For every vector $M \overset{\text{def}}{=} \langle \mu^\zeta : \zeta \in Z \rangle$ in $\mathbb{R}^Z$ that is either negative or positive, there exists $t \in T$ such that $\langle u^\zeta(t) : \zeta \in Z \rangle = M$.

Proof of claim 6. By way of contradiction, suppose that some such $M$ contradicts the claim. That is, there exists $M$, as in claim 6 and negative such that, for every $t \in T$, there exists $\zeta \in Z$ such that $\mu^\zeta \neq u^\zeta(t)$. We seek a contradiction of the assumption that $\text{test}(Y, \preceq_{1^T})$ is empty.

We first define a novel extension as follows. For each $\zeta \in \mathcal{P}_I$, let $v^\zeta \overset{\text{def}}{=} u^\zeta \times \mu^\zeta \in \mathbb{R}^{\mathcal{P}_I}$. For $\xi = \zeta^{-1}$ such that $\zeta \in Z$, let $v^\xi = -v^\zeta$. Finally, for every remaining $\zeta \in Y^2$, let $v^\zeta = 0$. To see that it does indeed generate a novel $Y$-extension, we appeal to lemma 4. In particular, note that the product of $v^\zeta(f)$ and $v^\xi(s^\zeta)$ is negative for every $s^\zeta \in S^Z$. This ensures that, for each $t \in T$, we have a function

$$\rho(\cdot) = v^\zeta(\cdot)v^\xi(s^\zeta)$$

on $\{t, f\}$ that is neither positive nor constant.

Finally, in contradiction of the fact that $\text{test}(Y, \preceq_{1^T}) = \emptyset$, let $\mathcal{R}$ denote the $Y$-extension generated by $\{v^\zeta : \zeta \in Y^2\}$. For every $\zeta \in \mathcal{P}_I$, we have

$$v^\zeta(f) = \mu^\zeta < 0 < v^\zeta \cdot I.$$ 

Thus, $\mathcal{R}_f = \mathcal{R}_f^{-1}$, as required.

This contradiction allows us to conclude that there exists $J \in 1^T$ such that $\mathcal{R}_J = \mathcal{R}_I^{-1}$. Then noting that $\mathcal{P}_J = Z^{-1}$, we may also apply the preceding arguments, with $J$ replacing $I$, to conclude that, for every positive vector $M \overset{\text{def}}{=} \langle \mu^\zeta : \zeta \in Z^{-1} \rangle$ in $\mathbb{R}^Z$, there exists $t \in T$ such that $M = \langle u^\zeta(t) : \zeta \in Z^{-1} \rangle$. Since $\zeta \in Z^{-1}$ if, and only if, $\zeta^{-1} \in Z$, a relabelling allows to rewrite the latter statement as follows.
For every positive $M \stackrel{def}{=} \langle \mu^\zeta : \zeta \in Z \rangle$, there exists $t \in T$ such that $M = \langle u^\zeta(t) : \zeta \in Z \rangle$. \hfill \Box

To complete the proof of lemma 5, let $R$ denote an arbitrary total ordering of $Y$. We prove that there exists $J \in J^T$ such that $u^\zeta \cdot J \geq 0$ if, and only if, $\zeta \in R$. Claim 6 ensures that we can choose $s \in T$ such that
\[
 u^\zeta(s) \in \begin{cases} 
 (0,1) & \text{if } \zeta \in R^{-1} \cap Z, \\
 (1,\infty) & \text{if } \zeta \in R \cap Z.
\end{cases}
\]
Note that, since $R$ is a total order and $Z$ is the asymmetric part of a total order, for every $\zeta$ that does not belong to $(R \cup R^{-1}) \cap Z$, either $\zeta \in Z^{-1}$ or $\zeta = x \times x$. By 3, for every $\zeta \in Z^{-1}$, $\zeta^{-1} \in Z$ satisfies $u^\zeta(s) = -u^{\zeta^{-1}}(s)$ (and, for every $\zeta$ such that $\zeta = x \times x$, $u^\zeta(s) = 0$). Claim 6 also ensures that we can choose $t$ such that, for every $\zeta \in Z^{-1}$, $u^\zeta(t) = 1$, so that, by 3, for every $\zeta \in Z$, $u^\zeta = -1$.

Finally, we show that for $J := \delta_s + \delta_t \in J^T$ satisfies the desired property: for every $\zeta$ such that $\zeta = x \times y$ for distinct $x,y \in Y$,
\[
 \zeta \in R \text{ if, and only if, } u^\zeta \cdot J > 0.
\]
If $\zeta \in R \cap Z$, then $u^\zeta(s) > 1$ and, since $\zeta \in Z$, $u^\zeta(t) = -1$. On the other hand, if $\zeta \in R \cap Z^{-1}$, then $u^\zeta(t) = 1$. Moreover, since $\zeta \in R \cap Z^{-1}$ if, and only if $\zeta^{-1} \in R^{-1} \cap Z$, we observe that $u^{\zeta^{-1}} \in (0,1)$, so that $u^\zeta \in (-1,0)$. Thus, for every such $\zeta$, $u^\zeta \cdot J = u^\zeta(s) + u^\zeta(t) > 0$. \hfill \Box

In our proof of lemma 5 we appeal to the following result, which is also useful when $\text{test}(Y, \preceq_{j^T})$ is nonempty. Indeed, a necessary condition for $\text{test}(Y, \preceq_{j^T})$ nonempty is that, for the unique regular $Y$-extension $\mathcal{R}$, $\text{total}(\mathcal{R})$ is nonempty.

**Lemma 6** (Proof on page 31). Let $\preceq_{j^T}$ satisfy $A1^* \cdot A4^*$. If $Y \in 2^X$ is pairwise $D$-distinct and of finite cardinality, and $\mathcal{R}$ is a regular $Y$-extension, then $\text{total}(\mathcal{R})$ is nonempty.

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Proof of lemma 6. Fix $Y \subset X$ such that $1 < \#Y < \infty$ and let $\mathcal{R}$ be a regular $Y$-extension of $\leq_{1T}$. We proceed by induction on the cardinality of $Y$. For the initial step suppose $\#Y = 2$ and fix $x \neq y$ in $Y$. Since $Y$ is pairwise $\mathcal{D}$-distinct, there exists $J \in i^T$ such that $\neg(x \mathcal{R}_J y)$, and since $\#Y = 2$, A2* ensures that $\mathcal{R}_J$ is total.

For the inductive step, take $\#Y = n > 2$. For $z \in Y$, let $Z := Y \setminus \{w\}$ and let $\mathcal{R}$ be a regular $Z$-extension. Then, since $\mathcal{R}$ and $\mathcal{R}$ are both regular and $Z \subseteq Y$, $\mathcal{R}$ and $\mathcal{R}$ agree on $Z$. By the induction hypothesis, there exists $J \in i^T$ such that $\mathcal{R}_J$ is total, so that for every $x, y \in Z$, $\neg(x \mathcal{R}_J y)$. It suffices to consider the case where, for some $x \in Z$, $x \mathcal{R}_J w$. Then A1* and the fact that $\mathcal{R}_J = \mathcal{R}_J \cap Z^2$, we obtain $\neg(y \mathcal{R}_J w)$ for every $y \in Z \setminus \{x\}$.

Since $Y$ is pairwise $\mathcal{D}$-distinct, there exists $L \in i^T$ such that $\neg(x \mathcal{R}_L w)$. By A3*, for every $0 < q < 1$ in $\mathbb{Q}$, $\neg(x \mathcal{R}_{(1-q)J+qL} w)$. Since $Y$ is finite, repeated application of A4* yields the following conclusion: for every $y \in Z \setminus \{x\}$, there exists $0 < r^y < 1$ such that, for every $0 < q < r^y$ in $\mathbb{Q}$ and every $z \in Y \setminus \{y\}$, $\neg(y \mathcal{R}_{(1-q)J+qL} z)$. Since $Z$ is finite, let $r = \min \{r^y : y \in Z \setminus \{x\}\}$. Then, for $0 < q < r$ in $\mathbb{Q}$ and $M := (1-q)J + qL$, $\mathcal{R}_M$ is total, so that total($\mathcal{R}$) is nonempty.

4.4 Induction on $X$

The following lemma proves that, when 2-Div* holds, 4-Jac is sufficient (and obviously necessary) for the the Jacobi identity to hold (on $X$). This result is closely related to lemma 3 of [GS]. It is distinguished by the fact that our axioms do not yield the following condition.

4-Independence. There exists a pairwise representation $\{u^\zeta : \zeta \in X^2\}$ of $\leq_{1T}$ such that, for every $x, y, z, w \in X$, the triple $\{u^{xy}, u^{yz}, u^{zw}\}$ is linearly independent.
Remark 2. The logic of the proof of lemma 3 of [GS] is the following. 4-Div* is sufficient for 4-independence. In turn, [GS] show that, when 4-independence holds, 3-Jac is sufficient for the Jacobi identity to hold. (In essence, [GS] contains a proof of the contrapositive of each of these steps.) Clearly, 4-Jac is necessary for Jacobi identity to hold. Thus, together 4-independence and 3-Jac are sufficient for 4-Jac. Via ?? we show that the converse is not true.

Lemma 7. If $\zeta_{3T}$ satisfies $A_0$, 2-Div* and 4-Jac, then there exists a Jacobi representation $v^X$ of $\zeta_{3T}$. Moreover, $v^X$ is unique, for every other Jacobi representation $v^X$ of $\zeta_{3T}$, there exists $\lambda > 0$ such that $v^{xy} = \lambda v^{xy}$ for every $x, y \in X$.

Proof of lemma 7. In the case that $\# X \leq 4$, we only need to show that $v^X$ is unique. (This will also account for the initial step in the proof by induction then follows.) Let $v^X$ denote another representation. By lemma 4, for every distinct $x, y \in X^2$, there exists $\lambda^{xy} > 0$ such that $v^{xy} = \lambda^{xy} v^{xy}$. We need to show that $\lambda^{xy} = \lambda$ for every distinct $x, y \in X$. Let $X = \{x_0, y, z, w\}$. By $A_0$, no vector in the set $\{v^{x_0y}, v^{x_0z}, v^{x_0w}\}$ is directly proportional to any other. By 2-Div*, no vector in this set is equal to zero. This implies that they form a linearly independent set. Then, since 4-Jac holds for both $v^X$ and $v^X$, we derive the equation

$$(1 - \lambda^{x_0y})v^{x_0y} + (1 - \lambda^{yz})v^{yz} = (1 - \lambda^{x_0z})v^{x_0z} \quad (4)$$

Suppose that $1 - \lambda^{yz} = 0$. Then, either the other coefficients in eq. (4) are both equal to zero (and our proof is complete), or we obtain a contradiction of $A_0$. Thus, $1 - \lambda^{yz}$ is nonzero and we may divide through by this term and solve for $v^{yz}$. First note that, since $v^X$ is a Jacobi representation, $v^{y_0x} + v^{x_0y} = v^{y_0y} = 0$. Then, since $v^{y_0x} = -v^{x_0y}$,

$$v^{yz} = \frac{1 - \lambda^{x_0y}}{1 - \lambda^{yz}} v^{y_0x} + \frac{1 - \lambda^{x_0y}}{1 - \lambda^{yz}} v^{x_0z}. $$
Then the fact that both of the latter coefficients are equal to one follows from linear independence of $v^{y_{x_{0}}} \text{ and } v^{x_{0}z}$ together with the Jacobi identity $v^{yz} = v^{y_{x_{0}}} + v^{x_{0}z}$. Thus, $\lambda ^{x_{0}y} = \lambda ^{yz} = \lambda ^{x_{0}z}$, as required. Repeated application of the same argument and the fact that the same coefficients appear in multiple Jacobi identities, (e.g. $\lambda ^{y_{x_{0}}} = \lambda ^{yw} = \lambda ^{x_{0}w}$), we conclude that $v^{x}$ is suitably unique.

For future reference, we note that, by 2-diversity, $v^{yz} \neq 0$, and the above argument implies that $v^{yz}$ is not directly proportional to either $v^{x_{0}y}$ or $v^{x_{0}z}$.

Take any $Y \subseteq W$ such that $\#Y \geq 4$ and $x_{0} \in Y$ and let

$$u^{Y} \overset{\text{def}}{=} \left\{ u^{\zeta} \in u^{W} : \zeta \in Y^{2} \right\}.$$ 

By the induction hypothesis, there exists a pairwise representation $v^{Y} \overset{\text{def}}{=} \left\{ v^{\zeta} : \zeta \in Y^{2} \right\}$ that satisfies the Jacobi identity and is unique (upto a positive scalar multiple that is uniform in $\zeta \in Y^{2}$). In other words, there is only one degree of freedom associated with $v^{Y}$. We prove that, for every $w \in W \setminus Y$ and $Y \cup \{w\}$, the Jacobi identity holds on $Y \cup \{w\}$. Since this holds regardless of whether $w$ is a successor ordinal, the proof also accounts for the possibility that $w$ is a limit ordinal, as required for the case where $W$ is infinite.

For the remainder of the proof of ?? take $x \overset{\text{def}}{=} x_{0}$. For arbitrary distinct $x', y, z \in Y$ and $w \in W \setminus Y$, let $Z \overset{\text{def}}{=} \{x, y, z, w\}$ and let $Z' \overset{\text{def}}{=} \{x', y, z, w\}$. By 4-Jac, the Jacobi identity holds on each of these sets, and our goal is to show that it also holds on their union, the five-element set we denote by $Z''$. The key to this proof is to show that, for the Jacobi representation on $Z''$, $v^{yw}$ and $v^{zw}$ can be chosen independently of $Z$ and $Z'$. The following observation then completes the proof.

For every distinct $y, z \in Y$ and $w \in W \setminus Y$, there exist
vectors $v^{yw}$ and $v^{zw}$ that are independent of every $x \in Y \setminus \{y, z\}$ and that extend $v^Y$ to $v^{Y \cup \{w\}}$, so that, for every $v^{yz} \in v^Y$, $v^{yz} = v^{yw} + v^{wz}$.

Since each vector in the pairwise representation $u^Z$ is associated with one degree of freedom, $4$-Jac implies the existence of positive scalars $\alpha, \beta, \gamma, \sigma$ and $\tau$ such that

$$\alpha u^{xw} + \beta u^{wy} = \sigma u^{xy},$$  \hspace{1cm} (5)

and

$$\beta u^{yw} + \gamma u^{wz} = \tau u^{yz},$$  \hspace{1cm} (6)

and

$$\sigma u^{xy} + \tau u^{yz} = u^{xz}. \hspace{1cm} (7)$$

By the induction hypothesis, $v^Y$ satisfies the Jacobi identity, and since $x, y, z \in Y$, there is a unique, known positive scalar $\phi$ such that $\phi u^{xz} = v^{xz}$. Moreover, if $\text{span}\{u^{xy}, u^{yz}\}$ is two-dimensional, then, the linear system eq. (7) in the two unknowns, $\sigma$ and $\tau$ has a unique solution. This, together with the induction hypothesis (which yields $v^{xy} + v^{yz} = v^{xz}$), implies that $(\phi \sigma)u^{xy} = v^{xy}$ and $(\phi \tau)u^{yz} = v^{yz}$. If $\text{span}\{u^{xy}, u^{yz}\}$ is one-dimensional, we have one degree of freedom and may choose $\sigma$ such that $(\phi \sigma)u^{xy} = v^{xy}$. Then, $(\phi \tau)u^{yz} = v^{yz}$ follows from eq. (7) and the induction hypothesis. The remainder of the proof takes these values of $\sigma$ and $\tau$ as given.

Similarly, for $Z' = \{x', y, z, w\}$, there exist $\alpha', \beta', \gamma', \sigma', \tau' > 0$ such that

$$\alpha' u^{x'w} + \beta' u^{wy} = \sigma' u^{x'y},$$  \hspace{1cm} (8)

and

$$\beta' u^{yw} + \gamma' u^{wz} = \tau' u^{yz},$$  \hspace{1cm} (9)
and

\[ \sigma' u^{x'y} + \tau' u^{yz} = u^{x'z}. \] (10)

By the induction hypothesis, since \( v^Y \) is a Jacobi representation and \( x', y, z \in Y \), there is a known positive scalar \( \phi' \) such that \( \phi' u^{x'z} = v^{x'z} \).

Moreover, by symmetry with the arguments involving \( \sigma \) and \( \tau \), we may take \( \sigma' \) and \( \tau' \) to be known positive scalars such that \( \phi' \sigma' u^{x'y} = v^{x'y} \) and \( \phi' \tau' u^{yz} = v^{yz} \). An immediate consequence of this fact is the equality \( \phi' \tau' = \phi \tau \).

To obtain a Jacobi representation on \( Z'' \) that extends \( v^Y \), we need to verify that \( \beta, \gamma, \beta' \) and \( \gamma' \) can be chosen so that \( \phi \beta u^{x'y} = \phi' \beta' u^{x'y} \) and \( \phi \gamma u^{yz} = \phi' \gamma' u^{yz} \). To this end, we use the fact that \( \phi' \tau' = \phi \tau \) to divide the terms in eq. (6) and eq. (9) by \( \phi \tau \) and \( \phi' \tau' \) respectively and obtain

\[ \frac{\beta}{\tau} u^{y} + \frac{\gamma}{\tau} u^{w} = u^{yz} = \frac{\beta'}{\tau'} u^{y} + \frac{\gamma'}{\tau'} u^{w}. \] (11)

Now, if \( \text{span}\{u^{yw}, u^{zw}\} \) is two-dimensional, linear independence of these two vectors implies that \( \beta/\tau = \beta'/\tau' \) and \( \gamma/\tau = \gamma'/\tau' \), as required.

Now consider the case where \( \text{span}\{u^{yw}, u^{zw}\} \) is one-dimensional.

Recall that A0 implies \( \text{span}\{u^{yw}, u^{zw}\} \) is two-dimensional, and consider the case where \( \text{span}\{u^{yz}, u^{x'w}\} \) is two-dimensional. Solving eq. (5) and eq. (8) for \( u^{yw} \) we obtain

\[ \frac{\sigma}{\beta} u^{x'} + \frac{\alpha}{\beta} u^{x} = u^{yw} = \frac{\sigma'}{\beta'} u^{x'} + \frac{\alpha'}{\beta'} u^{x'.} \]

By the induction hypothesis (and in particular the fact that \( v^{x'y}, v^{yz} \) and \( v^{x'z} \) are given) \( \sigma, \tau \) and \( \phi \) are known. Then, since \( \text{span}\{u^{y'}, u^{zw}\} \) is two dimensional, \( \alpha \) and \( \beta \) are uniquely determined. Via eq. (6), this in turn implies that \( \gamma \) is also unique. By the same token, \( \beta' \) and \( \gamma' \) are also unique and the desired equality holds.
The remaining possibility is \( \text{span}\{u^{yw}, u^{wz}\} \) and \( \text{span}\{u^{yx'}, u^{x'w}\} \), both one-dimensional. By 4-Jac, \( u^{yw} \) belongs to \( \text{span}\{u^{yx'}, u^{x'w}\} \) and all three vectors, \( u^{yx'}, u^{x'w} \) and \( u^{wz} \), belong to \( \text{span}\{u^{yw}\} \). Repeated application of 4-Jac allows us to conclude that in fact all the vectors in \( u^Z \) belong to the same one-dimensional subspace. In this case, there is one degree of freedom in the solution to eq. (8) and eq. (9). (Once we choose \( \beta' \), eq. (8) and eq. (9) determine \( \alpha' \) and \( \gamma \) respectively.) In contrast, by A0 (which implies uniqueness of the coefficients of eqs. (5) to (7)), \( \beta \) and \( \gamma \) are uniquely determined. Moreover, given our choice of \( v^Y \), the induction hypothesis implies that \( \tau \) and \( \tau' \) are also uniquely determined. Then eq. (11) uniquely determines \( \beta' \) and \( \gamma' \).

We now show that, when one or more vectors \( u^\zeta \) in a pairwise representation of \( \preceq_j \) are nonunique (up to multiplication by a positive scalar), the following example demonstrates that the requirement that a Jacobi representation exists on every subset of cardinality four is insufficient. We first introduce some notation.

For every \( \zeta \in W \), let \( u^\zeta \) denote the cone of vectors \( u^\zeta \) such that \( u^\zeta \cdot j \geq 0 \) if, and only if, \( \zeta \preceq_j \). Next, let \( v^\zeta_{\emptyset} \overset{\text{def}}{=} u^\zeta \) and consider the collection

\[
\{ v^y_B \subseteq u^y \colon B \subset W \text{ and } y \neq z \text{ in } W \setminus B \}
\]

where each \( v^y_B \) denotes the set of vectors \( v \) such that, for every \( x \in B \), there exist \( v^{yx} \in v^y_B \setminus x \) and \( v^{xz} \in v^z_B \setminus x \) such that \( v = v^{yx} + v^{xz} \in u^y \).

Note that, each \( v^\zeta_B \) inherits the property of being a cone from the cones \( u^\zeta \). Clearly, the dimension of \( v^\zeta_B \) is weakly decreasing in the cardinality of \( B \).\(^9\) Finally, \( v^\zeta_B = \emptyset \) for some \( \zeta \) and \( B \), if, and only if, for every \( y \in 2^X \) such that \( B \cup \zeta \subseteq Y \), there is no Jacobi representation on \( Y \).

\(^9\) by dimension, we mean the span of the vectors in \( v^\zeta_B \).
Example 1. Fix distinct $y, z \in W$ and suppose that $v_{x}^{yz}, v_{w}^{yz}$ and $v_{w}^{yz}$ are all two dimensional. (This is the canonical set up in the proof of ?? and the setting of [GS].) Now note that, if the Jacobi identity holds on every set of cardinality 4, then $v_{xw}^{yz} = v_{x}^{yz} \cap v_{w}^{yz}$ is nonempty, and the same can be said of $v_{wx'}^{yz}$ and $v_{xx'}^{yz}$. Suppose moreover that the latter three sets are one-dimensional. (Once again, this is the canonical set up and agrees with the case where the $u^\zeta$ are one dimensional.) In contrast with the canonical set up, now suppose that $u^{yz}$ is at least three-dimensional. That is to say, there exists $T^\prime \overset{\text{def}}{=} \{t_{xw}, t_{wx'}, t_{xx'}\} \subseteq T$ such that, w.l.o.g., every $u \in u^{yz}$ is positive on $t^\prime$ and nonnegative otherwise. Choose an arbitrary $u \in u^{yz}$. For each $t_\zeta \in T^\prime$, let $u_\zeta = u + \delta_\zeta$, so that $u_\zeta$ also belongs to $u^{yz}$. Finally, suppose that, $u_\zeta \in v_\zeta^{yz}$ for $\zeta = xw, wx', xx'$. Then, as figure ?? shows, $v_{xwx'}^{yz}$ is empty and the Jacobi identity fails to hold on $\{x, y, z, w, x^\prime\}$.

4.5 Equivalence of $4$-prudence and $4$-Jac

In the present section, we show that, when $\preceq_{1T}$ satisfies $A1^*–A4^*$, $4$-prudence is equivalent to $4$-Jac.

**Theorem 3** (Proof on page 38). Let $\preceq_{1T}$ satisfy $A1^*–A4^*$. Then $4$-prudence holds if, and only if, $4$-Jac holds.

**Proof of theorem 3.** We begin by dispensing with the cases where $4$-prudence holds vacuously. Suppose that $X = \{x, y\}$, so that there is no $Y \subset X$ such that $\#Y \geq 3$, and $4$-prudence holds vacuously. Then $4$-Jac holds follows directly from lemma 4. In particular, since $\preceq_{1T}$ satisfies $A2^*–A4^*$, there exists $u^{xy} \in \mathbb{R}^{T}$ satisfying 3, so that for every $x', x'', x''' \in X$, at most two of these elements are distinct. W.l.o.g., suppose that $x''' = x''$, so that $u^{23} = u^{22} = 0$. Then $u^{12} + u^{23} =
\( u_{12} + u_{23} = u_{13} \), as required. In section 4.3, we established that if, for some \( Y \subseteq X \) such that \( 3 \leq \#Y \leq 4 \), \( \text{test}(Y, \preceq_{Y}) = \emptyset \), then 4-diversity holds on \( Y \). Thus, for every such \( Y \), although 4-prudence holds vacuously on \( Y \), the results of [GS] ensure that 4-Jac holds on \( Y \).

Henceforth, we consider \( Y \subseteq X \) such that \( 3 \leq \#Y \leq 4 \) and \( \text{test}(Y, \preceq_{Y}) \) is nonempty. Throughout, \( \mathcal{R} \) will denote a testworthy \( Y \)-extension and \( \mathcal{R} \) will denote one that is regular.

**Step 1 ( \( Y = \{x, x', x''\} \): 4-Jac implies 4-prudence ).** We begin by assuming that 4-Jac holds for some pairwise representation \( u^{Y} \) of \( \mathcal{R} \). Since \( \text{test}(Y, \preceq_{Y}) \) is nonempty, there exists a testworthy extension \( \mathcal{R} \) and \( J^{*} \in J^{T} \) such that \( \mathcal{R}_{J^{*} \times 0} \) is total. Moreover, \( \mathcal{R}_{J^{*} \times 0} = \mathcal{R}_{J^{*}} \). Thus, for every distinct \( x, y \in Y \), \( u^{xy} \cdot J^{*} \neq 0 \). W.l.o.g., we suppose that \( \mathcal{R} \) fails to satisfy A1*. (For otherwise \( \mathcal{R} \) itself satisfies each of the conditions of extension we now seek.)

The next result ensures that we can always choose a representation

\[ v^{Y}_{\mu} \overset{\text{def}}{=} \{ v^{xy} : T^{f} \rightarrow \mathbb{R}, x, y \in Y \} \]

of \( \mathcal{R} \) so that its restriction to \( T \) coincides with \( u^{Y} \). That is, for every \( x, y \in Y \), \( v^{xy}_{\mu} = u^{xy} \times \mu^{xy} \) for some \( \mu^{xy} \in \mathbb{R} \).

**Proposition 2.** Let \( u^{Y} \) be a pairwise representation of the regular \( Y \)-extension \( \mathcal{R} \) such that each member of \( u^{Y} \) is unique up to multiplication by a scalar. For every testworthy \( Y \)-extension \( \mathcal{R} \), there exists \( \mu^{Y} = \{ \mu^{xy} : x, y \in Y \} \subset \mathbb{R} \) such that \( v^{Y}_{\mu} \overset{\text{def}}{=} (u^{x} \times \mu^{y})^{Y} \) is a pairwise representation of \( \mathcal{R} \).

**Proof.** Since \( \mathcal{R} \) is testworthy, it satisfies A2*-A4*, and lemma 4 ensures it has a pairwise representation \( u^{Y} = (u^{x} \times \eta^{y})^{Y} \). Moreover, there exists \( J \in J^{T} \) such that \( \mathcal{R}_{J} = \mathcal{R}^{-1}_{J^{*} \times 0} \) is total. Then, for every...
distinct \( x, y \in Y \), \( \eta^{xy} < 0 \) if, and only if, \( 0 < J \cdot u^{xy} \). Fix \( x \neq y \) and, w.l.o.g., suppose that \( x \not\in J \times 0 \), so that \( 0 < J \cdot u^{xy} \). Then, for some \( 0 < \lambda < 1 \),
\[
(1 - \lambda)J \cdot u^{xy} + \lambda\eta^{xy} = 0.
\]
Equivalently, \( \eta^{xy} = -\frac{1 - \lambda}{\lambda}J \cdot u^{xy} \). Indeed, since \( v^{xy} \) defines a hyperplane in \( \mathbb{R}^T \), for every \( \kappa \in (0, 1) \cap \mathbb{Q} \),
\[
x \not\in J(1 - \kappa) \times \kappa \, y \, \text{if, and only if,} \, \kappa < \lambda.
\]
Since \( R \) is regular, it agrees with \( R \) on \( J^T \). Since \( u^Y \) is a pairwise representation of \( R \), \( J \cdot u^{xy} \) is also positive. Thus, for some \( \alpha > 0 \),
\[
u^{xy} = \alpha u^{xy}.
\]
Let \( \mu^{xy} \overset{\text{def}}{=} -\frac{1 - \lambda}{\lambda}J \cdot u^{xy} \). Then, for every \( \kappa \in (0, 1) \cap \mathbb{Q} \),
\[
u^{xy} \overset{\text{def}}{=} u^{xy} \times \mu^{xy} \text{ satisfies}
\[
0 < ((1 - \kappa)J \times \kappa) \cdot v^{xy} \, \text{if, and only if,} \, \kappa < \lambda.
\]
Since the preceding argument is independent of \( J \), it holds for every \( J' \in J^T \) such that \( x \not\in J' \times 0 \). Since the preceding argument holds for every distinct \( x, y \in Y \), we observe that \( v^Y \) is a pairwise representation of \( R \). \( \square \)

Since \( R \) does not satisfy \( A1^* \), there exists \( L' \in T^f \) such that, for some \( x, y, z \in Y \), \( x \not\in R_{L' \times Y} z \), and \( z \not\in R_{L' \times x} \). Equivalently, \( v^{xy} \cdot L' \) and \( v^{yz} \cdot L' \) are both nonnegative whereas \( v^{xz} \cdot L' \) is negative.

4-Jac implies 4-prudence when \( u \) is unique Let \( Y \equiv \{1, 2, 3, 4\} \) and suppose that \( J^* \) is such that \( i \not\in J^* \times 0 \) if, and only if \( i < j \). Since \( R_{J^* \times 0} = \leq_{J^* \times 0} \), it follows that \( 0 \leq u^{ij} \cdot J^* \) if, and only if \( i < j \). Moreover, since \( u^Y \) satisfies 4-Jac, we seek \( v^Y \overset{\text{def}}{=} \{u^{xy} \times \mu^{xy} : x, y \in Y\} \) such that all of the following hold:

1. for every \( i, j \in Y \), \( \mu^{ij} \leq 0 \) if, and only if, \( i < j \);
2. \( \mu^Y \) satisfies the Jacobi identity;

3. if there exists \( L \in T^I \) such that \( R_L \) is transitive and \( x \mathcal{P}_L y \mathcal{P}_L z \mathcal{P}_L w \), then, for every for \( \alpha, \beta, \gamma \geq 0 \) with \( \alpha + \beta + \gamma = 1 \), there exists \( t \in T^I \) such that

\[
\alpha \nu^{xy}(t) + \beta \nu^{yz}(t) + \gamma \nu^{zw}(t) > 0.
\]

For item 1, it suffices to take any \( 0 < \lambda < 1 \) and define \( \mu^Y \) such that, for every \( i, j \in Y \), \( \mu^{ij} \) solves \( (1 - \lambda)u^{ij} \cdot J^* + \lambda \mu^{ij} = 0 \). For then \( \mu^{ij} = -\frac{1 - \lambda}{\lambda} u^{ij} \cdot J^* \) is nonpositive if, and only if \( i \leq j \).

Since \( u^Y \) satisfies 4-Jac, \( \mu^Y \) necessarily fails to satisfy the Jacobi identity. For, if \( \mu^Y \) satisfies the Jacobi identity, then so does \( v^Y \) and \( R \) transitive at \( L \). (This follows directly from the equality \( v^{xy} + v^{yz} = v^{xz} \).)

We seek \( \eta^Y = \{ \eta^{xy} \in \mathbb{R} : x, y \in Y \} \) satisfying the Jacobi identity and such that, for each \( x, y \in Y \), the sign of \( \eta^{xy} \) is the same as \( \mu^{xy} \). Consider the case where the linear hull \( \text{Lin}N^Y \) of \( N^Y \) \( \{ J \in T^I : x' \approx_J x'' \approx_J x''' \} \) is a hyperplane in \( \mathbb{R}^T \). In this case, for every distinct \( x, y \in Y \), \( N^{xy} \) \( \{ J \in T^I : x \approx_J y \} \) is equal to \( N^{xy} \) and the vectors \( u^{xy} \in u^Y \) such that \( x \neq y \) are collinear. ***Proof of this*** Then, assuming \( x' <_J x'' <_J x''' \),

First consider the case where, for every \( i, j = 1, 2, 3, 4 \) such that \( i < j \), there exists \( \alpha^{ij} > 0 \) such that \( u^{ij} = \alpha^{ij} u^{12} \), and, in particular, \( \alpha^{12} = 1 \). Since \( \alpha^{12} u^{12} + \alpha^{23} u^{12} = \alpha^{13} u^{12} \), we observe that \( 1 + \alpha^{23} = \alpha^{13} \). Moreover, since, for every \( i < j \), \( \alpha^{ij} u^{12} \cdot J^* \) is positive, we observe that \( \alpha^{13} > \max\{1, \alpha^{23}\} \).

For every \( i < j \), we seek \( \eta^{ij} \) such that, for some \( 0 < \lambda < 1 \),

\[
\eta^{ij} \overset{\text{def}}{=} -\frac{1 - \lambda}{\lambda} u^{ij} \cdot J^*.
\]
Note that, for every $0 < \lambda < 1$, $\eta^{ij}_{\lambda}$ has the same sign as $\mu^{ij}$, and, moreover for such $\eta^{ij}_{\lambda}$, the inner product of $L(\lambda) \overset{\text{def}}{=} (1 - \lambda)J^* \times \lambda$ and $\nu^{ij} = u^{ij} \times \eta^{ij}_{\lambda}$ has the property $\nu^{ij} \cdot L(\lambda) = 0$.

For every $i < j$, let $\eta^{ij}_{\lambda} \overset{\text{def}}{=} \frac{1}{\lambda^2} - \frac{1}{\lambda^4} \alpha^{ij} u^{12} \cdot J^*$, so that $\eta^{ij}_{\lambda} < 0$ and let $\theta^{ij}_{\lambda} = (1 - \lambda^{ij})/\lambda^{ij}$. Then $\eta^{12}_{\lambda} + \eta^{23}_{\lambda} = \eta^{13}_{\lambda}$ if, and only if,

$$\theta^{12}_{\lambda} + \theta^{23}_{\lambda} \alpha^{23} = \theta^{13}_{\lambda} \alpha^{13}.$$  

By choosing $\kappa_Y$ to also be pairwise distinct, we obtain $\nu^{ij}_{\lambda} \overset{\text{def}}{=} u^{ij} \times \eta^{ij}_{\lambda}$ that, not only satisfies the Jacobi identity, but also #total$(\nu^Y)$ = 4.

To see this, let $R$ denote the extension that $\nu^Y$ generates (so that $R$ and $\nu^Y$ satisfy 3). Next, note that, in addition to the total rankings $R_f$ and $R_{J*}$, we can use the fact that $\theta^Y$ is pairwise distinct. In particular, suppose that $\lambda^{23} < \lambda^{13} < \lambda^{12}$. Then, for every $\xi$ such that $\lambda^{23} < \xi < \lambda^{13}$, we have the nonextremal ranking $x' P_{L(\xi)} x'' P_{L(\xi)} x''$. For every $\xi$ such that $\lambda^{13} < \xi < \lambda^{12}$, we have the nonextremal ranking $x'' P_{L(\xi)} x' P_{L(\xi)} x''$.

It remains to be shown that #total$(R) \leq$ #total$(R)$. Suppose that, for some $L, M \in \mathcal{J}^f$, $R_L = \mathcal{R}^{-1}_M$, so that $R_L$ and $R_M$ are pairwise extremal. Then, for every $i \neq j$, $(u^{ij} \times \gamma^{ij}) \cdot L$ is positive if, and only if, $(u^{ij} \times \gamma^{ij}) \cdot M$ is negative. Moreover, suppose that $R_L \neq \mathcal{R}_i \neq \mathcal{R}_M$, so that $R_L, R_M$ is distinct from $(\mathcal{R}_{J*}, \mathcal{R}_f)$. Then, w.l.o.g., suppose that $x' P_L x'' P_L x''$ and $x'' P_M x'' P_L x'$. We now appeal to lemma 4 of [GS]. There it is shown that these conditions imply that there is no of $0 \leq \beta \leq 1$ such that either of the convex combinations $(1 - \beta)v^{13}_\eta + \beta v^{23}_\eta$ and $(1 - \beta)v^{23}_\eta + \beta v^{31}_\eta$ belong to $\overline{\mathbb{R}}^T_\bot$. Equivalently, for every $0 \leq \beta \leq 1$, there exists $t, t' \in \mathbb{T}^f$ such that $(1 - \beta)v^{13}_\eta(t) + \beta v^{23}_\eta(t)$ and $(1 - \beta)v^{23}_\eta(t') + \beta v^{31}_\eta(t')$ are both positive. But note that neither of the rankings $R_L$ and $R_M$ feature in the regular $Y$-extension $R'$. This is because $N^Y$ is a hyperplane that separates total rankings that coincide

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with $\mathcal{R}_{j*}'$ from total rankings (if any) that coincide with $\mathcal{R}_{j*}'^{-1}$. Then, lemma 4 of [GS] implies that, for every $s \in T$, there exists a convex combination of the form $(1 - \beta)u^{13}(s) + \beta u^{32}(s)$ that is nonpositive. This implies

First note that, if there exists $L \in i^{Tf} \setminus i^{T}$ such that $L \in N_{R}^{Y}$ (so that $x' J L x'' J L x'''$), then $\text{Lin } N_{R}^{Y}$ is a hyperplane in $\mathbb{R}^{Tf}$. In this case, there are only two rankings in $\mathcal{R}$, $\mathcal{R}_{j*}$ and $\mathcal{R}_{j}$. The remaining possibility is that $N_{R}^{Y} = N^{Y}$. In this case, $N_{R}^{Y}$ belongs to the boundary of $i^{Tf}$. Indeed $N_{R}^{Y}$ is a subset of the hyperplane $\mathbb{R}^{T} \times \{0\}$ in $\mathbb{R}^{Tf}$. Note that, if, for some $L \in i^{Tf}$, $\mathcal{R}_{L}$ is total and distinct from both $\mathcal{R}_{j*}$ and $\mathcal{R}_{j}$, then $\nu$

This follows directly from the fact that $\text{Lin } N^{Y}$ is a hyperplane in $\mathbb{R}^{T}$. For then

Let $\mathcal{R}'$ be a regular $Y$-extension and let

$$u^{Y} = \{ u^{yz} \in i^{T} : y, z \in Y \}$$

be a pairwise representation of $\mathcal{R}'$. First suppose that $N^{Y} \overset{\text{def}}{=} \{ J : x \approx_{J} x' \approx_{J} x'' \}$ is equal to $N^{yz}$ for every distinct $y, z \in Y$. Moreover, suppose that the linear hull of $N^{yz}$ is a hyperplane in $\mathbb{R}^{T}$. In this case, for any testworthy $Y$-extension $\mathcal{R}$, if $N_{R}^{Y} \overset{\text{def}}{=} \{ L \in i^{Tf} : L J L y \}$ satisfies $N^{Y} \subseteq N_{R}^{Y}$, then the linear hull of $N_{R}^{Y}$ in $\mathbb{R}^{Tf}$ is a hyperplane and $N_{R}^{yz} = N_{R}^{Y}$ for every distinct $x$ and $y$. For any such $\mathcal{R}$, the fact that $N_{R}^{Y}$ separates $\mathbb{R}^{Tf}$ into two halfspaces implies that at most two antisymmetric rankings of $Y$ feature in $\mathcal{R}$.

**Step 2 ( $\# Y = 4$).**
4.6 Necessity of 4-Pru* when X = 3

Lemma 8. For every pairwise \(D\)-distinct \(Y \subseteq X\) of cardinality 3 and every \(\preceq_L\) in Prop\((Y, \preceq_3)\), if \(\mathcal{R}\) satisfies A2*-A4* but not A1*, then

- \(2 \leq \#\text{ran}(\mathcal{R}) \leq 8\)
- \(0 \leq \#\text{intran}(\mathcal{R}) \leq 2 \leq \#\text{total}(\mathcal{R}) \leq 6\)

Moreover, all the bounds in the present lemma are tight in the sense that there exists \(\mathcal{R} \in \text{Prop}(Y, \preceq_3)\) that attains each one.

Proof. By Orlik and Terao [9, p.1], the maximum number of chambers in an arrangement of three hyperplanes is \(1 + 3 + \binom{3}{2} + \binom{3}{3} + \binom{3}{4} + \cdots = 8\). This ensures that \(\text{ran}(\preceq_{\Delta_T}) \leq 8\). ?? then ensures that \(1 \leq \#\text{total}(\preceq_{\Delta_T})\). Next recall that, since \(\preceq_{\Delta_T} \in \text{Prop}(Y, \preceq_3)\), we have \(\preceq_f \in \text{total}(\preceq_{\Delta_T})\) and \(\preceq_f \neq \preceq_J \times 0\) for every \(J \in \Delta_T\), so that \(2 \leq \#\text{total}(\preceq_{\Delta_T}) \leq \#\text{ran}(\preceq_{\Delta_T})\).

Next, the fact that \(\#\text{total}(\preceq_{\Delta_T}) \leq 6\) follows from the fact that there are at most two distinct intransitive members of \(\text{ran}(\preceq_{\Delta_T})\). To see this, note that one possible strict cycle on \(Y\) is \([x \prec y, y \prec z \prec x]\). We claim that the only other strict cycle on \(Y\) is the inverse cycle \([x \prec P^{-1} z, z \prec y \prec P^{-1} x]\). This follows from the fact that if \(T^{xy}(P)\) is the transposition operator that reverses the strict preference of \(x \prec y\) to \(y \prec x\) and leaves \(P\) otherwise unchanged, then \(T^{xy}(P)\) is a transitive binary relation. The same is true of \(T^{yz}(P)\) and \(T^{xz}(P)\). Next consider the composition operator \(T^{xy} \circ T^{yz}\). Let \(P' := (T^{xy} \circ T^{yz})(P)\). Then \(P'\) is transitive because \(z \prec P' y, y \prec P' z\) and \(z \prec P' x\). Similarly \(P' := (T^{xy} \circ T^{xz})(P)\) is transitive because \(y \prec P' x, x \prec P' z\) and \(y \prec P' z\). The same is true for the only remaining composition of two transposition operators. Moreover, since \((T^{xy} \circ T^{yz} \circ T^{xz})(P)\) is intransitive and equal to \(P^{-1}\), proof of the claim is complete. \(\square\)
Proposition 3. Let $Y \subseteq X$ be pairwise $D$-distinct and of cardinality 3 and let $\mathcal{R} \in \text{Prop}(Y, \lesssim_3)$ satisfy $A2^* - A4^*$ but not $A1^*$. If Hyp($\mathcal{R}$) is not pairwise distinct, then $2 \leq \#\text{ran}(\mathcal{R}) \leq 4$. If Hyp($\mathcal{R}$) is pairwise distinct, then

- $3 \leq \#\text{ran}(\mathcal{R}) \leq 8$.
- $5 \leq \#\text{ran}(\mathcal{R})$ implies $1 \leq \#\text{intran}(\mathcal{R})$ and $5 \leq \#\text{total}(\mathcal{R})$.
- $\#\text{intran}(\mathcal{R}) = 2$ implies $4 \leq \#\text{total}(\mathcal{R})$.

Proof. We begin by assuming that Hyp($\lesssim_{\Delta_T}$) is not pairwise distinct. The case where $H = H^{xy} = H^{yz} = H^{xz}$ does not arise because then $\mathcal{R}_{L}$ is transitive for every $L \in H$ and $2 \leq \#\text{ran}(\lesssim_{\Delta_T}) \leq 2$, so that $\text{ran}(\lesssim_{\Delta_T}) = \text{total}(\lesssim_{\Delta_T})$. That is to say, we obtain a contradiction of the assumption that $\mathcal{R}$ does not satisfy $A1^*$. W.l.o.g. therefore, suppose $H = H^{xy} = H^{yz} \neq H^{xz}$. The case where $\#\text{intran}(\mathcal{R}) = 0$ arises, for example, when total($\mathcal{R}$) consists of two rankings $(x, z, y)$ and $(y, x, z)$. For $(T^{xy} \circ T^{yz}) (x, z, y) = (y, x, z)$ and for any $L \in \Delta_{T_1} \cap H$, we have $x \not\in L y \not\in L z$ and $x \not\in L z$.

At most two hyperplanes cut $\Delta_{T_1}$, so that Orlik and Terao [9, p. 1] implies $\#\text{ran}(\lesssim_{\Delta_T}) \leq 4$.

Let $P$ and $P^{-1}$ (defined in the proof of lemma 8) be the two intransitive orderings of $Y$ and let $P, P^{-1} \in \text{ran}(\mathcal{R})$.

then $4 \leq \#\text{total}(\lesssim_{\Delta_T})$ if $\{H^{-}\}$ is pairwise distinct and $\#\text{total}(\lesssim_{\Delta_T}) = 2$ otherwise. If Hyp($\lesssim_{\Delta_T}$) is not pairwise distinct, the case where $H^{xy} = H^{yz} = H^{xz}$ does not arise because then $\#\text{ran}(\lesssim_{\Delta_T}) = 2$, so that $\text{ran}(\lesssim_{\Delta_T}) = \text{total}(\lesssim_{\Delta_T})$. W.l.o.g. suppose $H^{xy} = H^{yz} \neq H^{xz}$.

Since $4 \leq \#\text{ran}(\lesssim_{\Delta_T})$, and at most two hyperplanes cut $\Delta_{T_1}$, Orlik and Terao [9, p. 1] implies $\#\text{ran}(\lesssim_{\Delta_T}) = 4$. Henceforth, take Hyp($\lesssim_{\Delta_T}$) to be pairwise distinct. Let $L, L' \in \Delta_{T_1}$ satisfy $\lesssim_L = P$
and \( \preceq_{L'} = P^{-1} \). Since \( P \) and \( P^{-1} \) are strict, we may w.l.o.g. assume that \( L \) and \( L' \) belong to the interior of \( \Delta_{T^f} \). Then for some \( 0 < \lambda, \lambda' < 1 \), \( L = (1 - \lambda)J \times \lambda \) and \( L' = (1 - \lambda')J' \times \lambda' \). If \( \text{conv}(L, L') \) contains a point of \( H^{xy} \cap H^{yz} \cap H^{xz} \), then that point is interior to \( \Delta_{T^f} \) and hence total \( (\preceq_{\Delta_{T^f}}) \) is maximal, so that \#total \( (\preceq_{\Delta_{T^f}}) = 6 \). So suppose that, for every such \( L, L' \), \( \text{conv}(L, L') \cap H^{xy} \cap H^{yz} \cap H^{xz} \) is empty. Recall that \( \preceq_{f} \in \text{total}(\preceq_{\Delta_{T^f}}) \). If \( \preceq_{J \times 0} \notin \text{total}(\preceq_{\Delta_{T^f}}) \), then \( x' \sim_J x'' \) for some \( x', x'' \in Y \). Then, by the arguments provided for proof of ??, there exists \( \epsilon > 0 \), sufficiently close to zero, such that \( \preceq_{J \times \epsilon} \in \text{total}(\preceq_{\Delta_{T^f}}) \) and such that \( \preceq_{J \times \epsilon} \neq \preceq_{f} \). For sufficiently small \( \epsilon \), the same is true of \( J' \times \epsilon \). If \( \dim \Delta_{T^f} = 1 \), then \( J = J' \) as \#T = 1. In turn, \( \preceq_{J \times \epsilon} = \preceq_{J' \times \epsilon} \). Since \#ran \( (\preceq_{\Delta_{T^f}}) = 4 \) and the rankings \( \preceq_{f} \) and \( \preceq_{J \times \epsilon} \) are extremal on the interval \( \text{conv}(J \times \epsilon, \delta_{f}) \), \( P \) and \( P^{-1} \) are adjacent. But this contradicts the fact that \( \text{Hyp}(\preceq_{\Delta_{T^f}}) \) is pairwise distinct and \( P^{-1} = (T^{xy} \circ T^{yz} \circ T^{xz}) (P) \). Thus, \( \dim \Delta_{T^f} \geq 2 \).

W.l.o.g., suppose \( x \preceq_{f} y \preceq_{f} z \), so that \( T^{xz}(P) = \preceq_{f} = (T^{xy} \circ T^{yz}) (P^{-1}) \). First suppose that \( \preceq_{J \times \epsilon} = \preceq_{J' \times \epsilon} \), so that \( 2 \leq \#\text{total}(\preceq_{\Delta_{T^f}}) \). Then since \( (T^{xy} \circ T^{yz}) (P^{-1}) = \preceq_{f} \), there exists \( 0 < \mu, \nu < 1 \) such that \( x \sim_{M_{\mu}} y \) and \( y \sim_{M_{\nu}} z \). If \( \mu < \nu \), then there exists \( \mu < \xi < \nu \) such that \( \preceq_{M_{\xi}} = T^{xy}(P^{-1}) \in \text{total}(\Delta_{T^f}) \). Since \( \preceq_{M_{\xi}} \) is distinct from \( \preceq_{f} \) and \( \preceq_{J} \), \( 3 \leq \#\text{total}(\preceq_{\Delta_{T^f}}) \). We reach a similar conclusion if \( \nu < \mu \), the only difference being that, in this case, \( \preceq_{M_{\xi}} = T^{yz}(P^{-1}) \). If \( \mu = \nu \), then \( M_{\mu} \in H^{xy} \cap H^{yz} \). Since \( H^{xy} \) and \( H^{yz} \) are distinct, we may perturb \( J \times \epsilon \) to find a new \( J'' \) (and if necessary choosing a smaller \( \epsilon \) ) such that \( M''_{\epsilon} \overset{\text{def}}{=} (1 - \epsilon)J'' \times \epsilon \) satisfies \( \preceq_{M''_{\epsilon}} = \preceq_{M_{\epsilon}} \). Then for \( M''_{\epsilon} \) there exists \( \mu \neq \nu \) such that \( x \sim_{M''_{\mu}} y \) and \( y \sim_{M''_{\nu}} z \). This in turn yields the existence of \( \xi \) such that \( \preceq_{M_{\xi}} \in \text{total}(\preceq_{\Delta_{T^f}}) \), so that \( 3 \leq \#\text{total}(\preceq_{\Delta_{T^f}}) \).

Next, note that, since \( \preceq_{f} = (T^{xy} \circ T^{xz}) (P^{-1}) \) and \( \preceq_{M_{\epsilon}} = T^{xz}(P^{-1}) \),
we have

\[ \preceq_{M_\epsilon} = (T^{xz} \circ T^{xy} \circ T^{yz}) (\preceq_f). \]

Then, since transposition operators commute and \(T^{xz}(\preceq_f) = P\), we observe that \(\preceq_{M_\epsilon} = (T^{xy} \circ T^{yz}) (P)\). Then, recalling that we have assumed \(\preceq_{M_\epsilon} = \preceq_{M_\lambda}\), we conclude that \(\preceq_{M_\epsilon} = (T^{xy} \circ T^{yz}) (P)\). Then, applying to \(M'\) the argument that we have just applied to \(M\), we have

\[ \preceq_{M_\psi} \neq \preceq_{M_\psi}, \text{ then } 4 \leq \#\text{total}(\preceq_{\Delta_f}). \]

Let

\[ d : \text{ran}(\preceq_{\Delta_f}) \times \text{ran}(\preceq_{\Delta_f}) \to \mathbb{Z}_+ \]  

(12)

be the metric that counts the number of transpositions needed to obtain one ranking from another.\(^\text{10}\)

Then, whereas \(d(P^{-1}, P) = 3\), we have \(d(\preceq_{M_\epsilon}, P^{-1}) = 1 = d(\preceq_{M_\psi}, P).\) \(\Box\)

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\(^{10}\)We can confirm that \(d\) is a metric by verifying that the triangle inequality holds. Let \(P', P''\) and \(P'''\) be arbitrary members of \(\text{ran}(\preceq_{\Delta_f})\). If \(d(P', P'') \leq 1\), the fact that \(1 \leq d(P', P''') + d(P'', P''')\) is clear. If \(d(P', P'') = 2\), then, for every \(P''' \neq P', P''\), \(d(P'', P')\) and \(d(P''', P''')\) are both greater than one. If \(d(P', P'') = 3\), then \(P'\) and \(P''\) are pairwise extremal (that is \(P''\) is the inverse ranking of \(P'\)). Thus, for every \(P''' \neq P', P''\), \(d(P', P') = 1\) implies \(d(P', P'') = 2\) and \(d(P'', P') = 2\) implies \(d(P'', P'') = 1\).
A Proofs

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