Reciprocity of higher conserved charges in the $\mathfrak{sl}(2)$ sector of $\mathcal{N} = 4$ SYM

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Abstract
We extend the analysis of the generalized Gribov-Lipatov reciprocity to the higher conserved charges of type IIB superstring on $AdS_5 \times S^5$. The property is shown to hold for twist $L = 2$, and 3 operators in the $\mathfrak{sl}(2)$ subsector.

1 Introduction and Discussion

In the last years, integrability emerged as a powerful tool in the investigation of the AdS/CFT correspondence. The integrable spin chain description of the dilatation operator led to the all-loop conjectured Bethe Ansatz equations for the $\mathfrak{psu}(2,2|4)$ algebra [1,2,3], completely describing (once supplemented with the dressing phase [4,5]) the anomalous dimensions (and the full tower of conserved charges) of the model, up to wrapping effects. The presence of an infinite set of conserved charges $q_k$, forcing the factorizability of the scattering matrix for elementary excitations, is indeed a key manifestation of the integrability of a quantum model.

On the string side of the duality, the corresponding (classical) $\sigma$-model living on the string worldsheet in $AdS_5 \times S^5$ is also integrable, and the tower of non-local conserved charges was derived in [6,7,8].

Despite the physical relevance of $q_2$, identified with the anomalous scaling dimension - string energy, the first charge does not play a special role from the point of view of the integrability, and all the charges are on equal footing. In [9] the weak-strong coupling correspondence of the full tower of charges in the $\mathfrak{su}(2)$ sector has been studied, but the physical meaning and properties of the higher conserved charges remains less understood.

In this work we investigate the reciprocity properties (see [10] for a review) of the first higher conserved charges in the $\mathfrak{sl}(2)$ sector. Reciprocity has its far origin in QCD in a symmetric treatment of the Deep Inelastic Scattering (DIS) and electron-positron into hadrons. The modified symmetric DGLAP kernel $P(N)$ in the evolution equation obeys the relation: $\gamma(N) = P(N + \frac{1}{2}\gamma(N))$, where $\gamma(N)$ is the lowest anomalous dimension, and the reciprocity can be recast in the form of an asymptotic, large spin condition $P(N) = \sum_{\ell \geq 0} \frac{a_{\ell} (\log J^2)}{J^{2\ell}}$, where
$J^2 = N(N + 1)$, $a_\ell$ are coupling-dependent polynomials and $J^2$ is the Casimir of the collinear subgroup $SL(2,\mathbb{R}) \subset SO(2,4)$. This condition can also be interpreted as a parity invariance $J \rightarrow -J$ in the large spin regime.

The $\mathfrak{sl}(2)$ sector, spanned by single trace operators $O \sim \text{Tr}(D^{n_1}Z \ldots D^{n_k}Z)$, is a closed subsector of the theory under perturbative renormalization; $N = \sum n_i$ is the total spin and $L$ is the classical dimension minus the spin (twist) of the operator. The relevant dual string state is the classical folded ($S,J$) string solution, describing a string extended in the radial direction of $AdS_5$ and rotating in $AdS_5$, with center of mass moving on a circle of $S^5$ \cite{11,12}. This solution is linked via analytic continuation $E \rightarrow -J_1, S \rightarrow J_2, J \rightarrow -E$ to the string configuration $(J_1, J_2)$ with two angular momenta on $S^5$, for which the higher charges at strong coupling have been constructed in \cite{9} by using the Bäcklund transformations of the integrable classical string $\sigma$-model.

Analysing the first two charges at weak coupling $q_{4,6}$ (respectively at 3-loop plus 4-loop dressing part and 2-loop level) and the first charges at strong coupling at classical level, we find that the reciprocity condition can be consistently generalized for all the tested higher charges.

2 Higher Charges and Reciprocity at Weak Coupling

In the weak coupling regime the closed formulae for multi-loops higher charges can be efficiently computed following the Baxter approach \cite{13,14,15} together with the maximum transcendentality Ansatz (and then completed by the dressing factor starting from the four-loop order), resulting in a combination of harmonic sums of definite transcendentality \cite{16}. The reciprocity condition for the full tower of conserved charges can be generalized from the condition for $q_2$ defining the kernel $P_r(N)$ as

$$q_r(N) = P_r \left( N + \frac{1}{2} q_2(N) \right).$$

(1)

This equation emphasizes the role of the renormalized conformal spin, as also suggested by light cone quantization. Reciprocity implies a constraint on the form of the expansion of $P_r$ at large $N$, which should involve only integer inverse powers of $N(N+1)$. The check of this property is easier after a rewriting of the charges in terms of the $\Omega$ basis \cite{17}, where the reciprocity simply means that the $\Omega$ must have odd positive or even negative indices. We report here only the first, parity respecting results for the higher charge $q_4$:

$L = 2$, three-loops reciprocity of $q_4$

$$P_{4}^{(1)} = 16 (\Omega_3 + 6\Omega_{-2,1}),$$

$$P_{4}^{(2)} = \frac{-16}{5} (\pi^4 \Omega_1 + 120\Omega_{-4,1} + 20\pi^2 \Omega_{-2,1} + 60\Omega_{-2,3} + 60\Omega_{1,-4} + 20\pi^2 \Omega_{1,-2} + 120\Omega_{-2,1,-2} + 120\Omega_{1,-2,-2} - 480\Omega_{1,-2,1,1}),$$

$$P_{4}^{(3)} = \frac{32}{15} (180\zeta(3)\Omega_{-4} + 2\pi^6 \Omega_1 + 3\pi^4 \Omega_3 - 30\pi^2 \Omega_5 - 720\Omega_7 + 900\Omega_{-6,1} + 240\pi^2 \Omega_{-4,1} + 540\Omega_{-4,3} + 30\pi^4 \Omega_{-2,1} + 60\pi^2 \Omega_{-2,3} + 720\Omega_{1,-6} + 240\pi^2 \Omega_{1,-4} + 36\pi^4 \Omega_{1,-2} + 180\Omega_{3,-4} + 60\pi^2 \Omega_{3,-2} - 180\Omega_{5,-2} + 2520\Omega_{-4,-2,1} + 2160\Omega_{-4,1,-2})$$

(3)
Introducing the function $f$ptoniously, reciprocity is translated in the absence of inverse odd powers of $\eta$.

For the comparison with the gauge theory results we are interested in the slow string limit; as:

$$\sigma = 2L - 2 + 1440\Omega_{1,1,5} + 2160\Omega_{3,-2,-2} + 720\Omega_{5,1,1}$$

$$-1440\Omega_{2,1,1,1} + 2160\Omega_{2,-2,-2,1}$$

$$+1440\Omega_{2,-2,1,-2} + 720\Omega_{2,1,-2,-2} - 2880\Omega_{1,-4,1}$$

$$+1440\Omega_{1,-2,-2} - 960\pi^2\Omega_{1,-2,1,1} - 1440\Omega_{1,-2,1,3} - 1440\Omega_{1,-2,3,1} - 1440\Omega_{1,1,-4,1}$$

$$-960\pi^2\Omega_{1,1,-2,1} - 1440\Omega_{1,1,-2,3} - 1440\Omega_{3,-2,1,1} - 2880\Omega_{2,-2,1,1}$$

$$-2880\Omega_{2,-1,1,-2,1} - 5760\Omega_{1,1,-2,1,1} - 2880\Omega_{2,-1,1,1,1} - 11520\Omega_{1,1,-2,2,1}$$

$$-5760\Omega_{1,1,-2,1,2} + 11520\Omega_{1,1,-2,1,1} + 360\Omega_{1,1}\zeta(5) - 240\pi^2\Omega_{1,1}\zeta(3)$$

$$-720\Omega_{2,-1,1,\zeta(3)} - 720\Omega_{1,1,\zeta(3)} - 720\Omega_{1,1,-2,3}$$

$$= 3072\Omega_{6} + 3072\Omega_{2,-4} + 3072\Omega_{5,1} - 18432\Omega_{4,1,1}$$

$$-12288\Omega_{2,-1,3} - 12288\Omega_{2,-3,1} - 6144\Omega_{1,-4,1} - 6144\Omega_{1,-2,3}$$

$$-24576\Omega_{2,-2,1,1} - 12288\Omega_{2,-1,1,1} - 12288\Omega_{1,-2,2,1} + 98304\Omega_{1,1,1,1} + 24576\Omega_{1,-2,1,1}.$$ 

### 3 Higher Charges and Reciprocity at Strong Coupling

The string state dual of the gauge operators is the semiclassical sl(2) folded string. As anticipated, it is related to the $(J_1, J_2)$ string by an analytic continuation, mapping one into another the $\sigma$-models describing the strings on $AdS_3 \times S^3$ and $R \times S^3$, as well as the relative equations of motion, their solutions and the conserved charges. Energy $E = E/\sqrt{\lambda}$, spin $S = S/\sqrt{\lambda}$, and angular momentum $J = J/\sqrt{\lambda}$ for the folded string are related by

$$\sqrt{\kappa^2 - J^2} = \frac{1}{\sqrt{\eta}} F_1 \left( \frac{1}{2}, \frac{1}{2}, 1, -\frac{1}{\eta} \right), \quad \omega^2 - J^2 = (1 + \eta)(\kappa^2 - J^2),$$

$$S = \frac{\omega}{\sqrt{\kappa^2 - J^2}} \frac{1}{2\eta \sqrt{\eta}} F_1 \left( \frac{1}{2}, \frac{3}{2}, 2, -\frac{1}{\eta} \right), \quad \mathcal{E} = \frac{\kappa}{\sqrt{\kappa^2 - J^2}} \frac{1}{\sqrt{\eta}} F_1 \left( -\frac{1}{2}, -\frac{1}{2}, 1, -\frac{1}{\eta} \right)$$

and for the comparison with the gauge theory results we are interested in the slow string limit; using $J$ as an expansion parameter, the quantum contribution to the energy can be computed as:

$$\eta(S, J) = \eta^{(0)}(S) + \eta^{(2)}(S) J^2 + \eta^{(4)}(S) J^4 + \cdots,$$

$$\Delta = \mathcal{E} - S = \Delta^{(0)}(S) + \Delta^{(2)}(S) J^2 + \Delta^{(4)}(S) J^4 + \cdots.$$ 

Introducing the function $f$ defined as $\Delta(S) = \mathcal{E}(S) - S = f \left( S + \frac{1}{2} \mathcal{E}(S) \right)$ and expanding perturbatively, reciprocity is translated in the absence of inverse odd powers of $S$ in the expansions.
Higher charges $\mathcal{E}_{4,6,...}$ can be constructed in the $\mathfrak{su}(2)$ sector by using the Bäcklund transformation method [9], and then analytically continued ($t \to -1/\eta$, $\mathcal{E}_2 \to J$). As an example, for the first non-vanishing charge $\mathcal{E}_4$ we get:

$$
\mathcal{E}_4 = -\frac{16}{\pi^2 \eta^2} Z_1(t) + \frac{32}{\pi^2 \eta^2} Z_2(t),
$$

$$
Z_1(t) = \mathbb{K}(t)[\mathbb{E}(t) + (t - 1)\mathbb{K}(t)], \quad Z_2(t) = t(t - 1)\mathbb{K}(t)^4 \quad (10)
$$

where $t$ is a modular parameter. In analogy with the case of the energy, we propose to test reciprocity on the functions $f_k$ defined by

$$
Z_k(S) = f_k \left( S + \frac{1}{2} \mathcal{E}(S) \right), \quad Z_k(S) \equiv Z_k(-1/\eta(S)). \quad (11)
$$

Using the Lagrange-Bürmann formula [18]

$$
f(S) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{d}{dS} \right)^{k-1} \left( -\frac{\Delta(S)}{2} \right)^k Z'(S) = Z(S) - \frac{1}{2} \Delta(S) Z'(S) + \cdots \quad (12)
$$

from $\eta = \eta(S,J)$ we obtain an expansion for $f_k(S) = f_k^{(0)}(S) + f_k^{(2)}(S) J^2 + f_k^{(4)}(S) J^4 + \cdots$ and computing 0-th order correction for $Z_1$ and $Z_2$ we find the result

$$
f_1^{(0)} = -\frac{1}{4} \left( \log \bar{S} - 2 \right) \log \bar{S} + \left[ \log \bar{S} \right] + \left( 2 - 3 \log \bar{S} \right) \log \bar{S} - \frac{1}{\bar{S}^2} + \left[ \log \bar{S} \right] \left[ \frac{1}{\bar{S}^3} \right] + \cdots \quad (13)
$$

$$
f_2^{(0)} = \frac{1}{16} \left( \log \bar{S} \right)^4 + \left[ \log \bar{S} \right] + \left[ \log \bar{S} \right] \left[ \frac{1}{\bar{S}^2} \right] + \left[ \log \bar{S} \right] \left[ \frac{1}{\bar{S}^3} \right] \frac{1}{\bar{S}^4} + \left[ \log \bar{S} \right] \left[ \frac{1}{\bar{S}^5} \right], \quad (14)
$$

where the absence of inverse odd powers of $S$, highlighted by the boxes, clearly supports parity invariance. The procedure can be straightforwardly extended to the next conserved charges, showing parity invariance in all the tested cases.

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