THE UNIVERSAL THETA DIVISOR OVER THE MODULI SPACE OF CURVES

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ABSTRACT. By computing the class of the universal antiramification locus of the Gauss map, we obtain a complete birational classification of the universal theta divisor \( \Theta_g \) over the moduli space of curves.

The universal theta divisor over the moduli space \( A_g \) of principally polarized abelian varieties of dimension \( g \) is the divisor \( \Theta_g \) inside the universal abelian variety \( X_g \to A_g \), characterized by two properties: (i) \( \Theta_g \mid [A, \Theta] = \Theta \), for every ppav \( [A, \Theta] \in A_g \), and (ii) the restriction \( s^*(\Theta_g) \) along the zero section \( s : A_g \to X_g \) is trivial on \( A_g \). The study of the geometry of \( \Theta_g \) mirrors that of \( A_g \) itself. Thus it is known that \( \Theta_g \) is unirational for \( g \leq 4 \); the case \( g \leq 3 \) is classical, for \( g = 4 \), we refer to [Ve1]. Whenever \( A_g \) is of general type (that is, in the range \( g \geq 7 \), cf. [Fr], [Mum], [T]), one can use Viehweg's additivity theorem [Vi] for the fibre space \( \Theta_g \to A_g \), to conclude that \( \Theta_g \) is of general type as well. Our first result concerns the birational type of \( \Theta_5 \):

**Theorem 0.1.** The universal theta divisor \( \Theta_5 \) is uniruled.

By making use of the generically finite Prym map \( P : R_6 \to A_5 \), we show that the pull-back of the universal theta divisor over the Prym moduli space \( R_6 \), that is,

\[
\Theta^P_5 := \Theta_5 \times_{A_5} R_6
\]

is uniruled. We sketch the idea of the proof and refer to Section 4 for details.

We fix a general element \([C, \eta] \in R_6\) inducing an étale double cover \( f : \tilde{C} \to C \). The canonical curve \( C \subset \mathbb{P}^5 \) can be viewed as a quadratic section of a smooth quintic del Pezzo surface \( S \subset \mathbb{P}^5 \). A general element \( L \) on the theta divisor \( \Xi(C, \eta) \) of the Prym variety \( P(C, \eta) \) is a line bundle \( L \) on \( \tilde{C} \) such that \( \text{Nm}_f(L) = K_C \) and \( h^0(\tilde{C}, L) = 2 \). We associate to this data a rank 3 quadric \( Q_L \in \text{PSym}^2 H^0(K_C) \) everywhere tangent to the canonical curve \( C \subset \mathbb{P}^5 \) along a Prym canonical divisor, that is,

\[
C \cdot Q_L = 2d_L,
\]

where \( d_L \in |K_C \otimes \eta| \). The pencil in \( |-2K_S| \) generated by the curves \( C \) and \( S \cdot Q_L \) induces a rational curve in the universal Prym theta divisor \( \Theta^P_5 \).

Next we obtain a complete birational classification of the universal theta divisor

\[
\mathcal{Th}_g := M_g \times_{A_g} \Theta_g
\]

over the moduli space of curves. If \([C] \in M_g\) is a smooth curve, the Abel-Jacobi map \( C_{g-1} \to \text{Pic}^{g-1}(C) \) provides a resolution of singularities of the theta divisor \( \Theta_C \) of the Jacobian of \( C \). Thus one may regard the degree \( g - 1 \) universal symmetric product

\[
\mathcal{M}_{g,g-1} := M_{g,g-1}/\mathcal{E}_{g-1}
\]

as a birational model of \( \mathcal{Th}_g \) (having only finite quotient singularities), and ask for the place of \( \mathcal{Th}_g \) in the classification of varieties. We provide a complete answer to this question. For small genus, \( \mathcal{Th}_g \) enjoys rationality properties:
Theorem 0.2. $\mathcal{H}_g$ is unirational for $g \leq 9$ and uniruled for $g \leq 11$.

The first part of the theorem is a consequence of Mukai’s work [M1, M2] on representing canonical curves with general moduli as linear sections of certain homogeneous varieties. When $g \leq 9$, there exists a Fano variety $V_g \subset P^{N_g}$ of dimension $n_g := N_g - g + 2$ and index $n_g - 2$, such that general 1-dimensional complete intersections of $V_g$ are canonical curves $[C] \in \mathcal{M}_g$ having general moduli. The correspondence

$$\Sigma := \{((x_1, \ldots, x_{g-1}), \Lambda) \in V_g^{g-1} \times G(g, N_g + 1) : x_i \in \Lambda, \text{ for } i = 1, \ldots, g - 1\}$$

maps dominantly onto $\mathcal{H}_g$ via the map $((x_1, \ldots, x_{g-1}), \Lambda) \mapsto [V_g \cap \Lambda, x_1 + \cdots + x_{g-1}]$. Since $\Sigma$ is a Grassmann bundle over the rational variety $V_g^{g-1}$, it follows that $\mathcal{H}_g$ is unirational in the range $g \leq 9$. The cases $g = 10, 11$ are settled by the observation that in this range the space $\overline{\mathcal{M}}_{g, g-1}$ is uniruled, see [FP], [FV2].

For the remaining genera, we achieve a complete classification. This is the main result of the paper:

Theorem 0.3. The universal theta divisor $\mathcal{H}_g$ is a variety of general type for $g \geq 12$.

We also have a birational classification theorem for the universal degree $n$ symmetric product $\overline{\mathcal{C}}_{g,n} := \overline{\mathcal{M}}_{g,n}/S_n$ for all $1 \leq n \leq g - 2$, and refer to Section 3 for details. Our results are complete in degree $g - 2$ and less precise as $n$ decreases. Similarly to Theorem 0.3, the nature of $\overline{\mathcal{C}}_{g,g-2}$ changes when $g = 12$:

Theorem 0.4. The universal degree $g - 2$ symmetric product $\overline{\mathcal{C}}_{g,g-2}$ is uniruled for $g < 12$ and a variety of general type for $g \geq 12$.

The proof of Theorem 0.3 relies on the calculation of the universal antiramification divisor class of the Gauss map. For a curve $C$ of genus $g$, let $\gamma : C_{g-1} \dasharrow (P^{g-1})^\vee$ be the Gauss given by $\gamma(D) := \langle D \rangle$ for $D \in C_{g-1} - C_{g-1}^1$. The branch divisor $\text{Br}(\gamma) \subset (P^{g-1})^\vee$ is the dual of the canonical curve $C \subset P^{g-1}$. The closure in $C_{g-1}$ of the ramification divisor $\text{Ram}(\gamma)$ is the locus of divisors $D \in C_{g-1}$ such that $\text{supp}(D) \cap \text{supp}(K_C(-D)) \neq \emptyset$. The antiramification divisor $\text{Antram}(\gamma)$, defined by the following equality of divisors

$$\gamma^*(\text{Br}(\gamma)) = \text{Antram}(\gamma) + 2 \cdot \text{Ram}(\gamma),$$

splits into the locus of non-reduced divisors $\Delta_C := \{2p + D : p \in C, D \in C_{g-3}\}$ and the locus of divisors $D \in C_{g-1}$ such that $K_C(-D)$ has non-reduced support. Globalizing this construction over $\mathcal{M}_g$, we are lead to consider the universal antiramification divisor $\overline{\text{Antram}}_g := \{[C, x_1, \ldots, x_{g-1}] \in \mathcal{M}_{g,g-1} : \exists p \in C \text{ with } H^0(K_C(-x_1 - \cdots - x_{g-1} - 2p)) \neq 0\}$.

We have the following formula for the class of $\overline{\text{Antram}}_g$:

Theorem 0.5. The closure in $\overline{\mathcal{M}}_{g,g-1}$ of the antiramification locus is linearly equivalent to,

$$[\overline{\text{Antram}}_g] = -4(g - 7)\lambda + 4(g - 2) \sum_{i=1}^{g-1} \psi_i - 2\delta_{\text{irr}} - (12g - 22)\delta_{0:2} - \sum_{i=0}^{g-1} \sum_{s=0}^{i-1} \left(2i^3 - 5i^2 - 3i + 4g - 4i^2 s + 14si - 6gs - s + 2s^2 g - 3s^2 + 2\right)\delta_{i:s} \in \text{Pic}(\overline{\mathcal{M}}_{g,g-1}).$$

\[\text{The description of the ramification divisor of the Gauss map given in [ACGH] p. 247 is erroneous.}\]
The universal theta divisor over the moduli space of curves

By construction, \( \text{Attrib} \) is \( \mathcal{S}_{g-1} \)-invariant, thus it descends to an effective divisor \( \mathcal{E}_g \) on \( \overline{\mathcal{C}}_{g,g-1} \) which, as it turns out, spans an extremal ray of the cone \( \text{Eff}(\overline{\mathcal{C}}_{g,g-1}) \). The universal theta divisor is equipped with the involution \( \tau : \overline{\mathcal{C}}_{g,g-1} \rightarrow \overline{\mathcal{C}}_{g,g-1} \),

\[
\tau([C, x_1 + \cdots + x_{g-1}]) := [C, y_1 + \cdots + y_{g-1}],
\]

where \( \mathcal{O}_C(y_1 + \cdots + y_{g-1} + x_1 + \cdots + x_{g-1}) = K_C \). Then \( \mathcal{E}_g \) is the pull-back of the boundary divisor \( \Delta_{0:2} \subset \overline{\mathcal{C}}_{g,g-1} \) under this map. Since the extremality of \( \Delta_{0:2} \) is easy to establish, the following result comes naturally:

**Theorem 0.6.** The effective divisor \( \mathcal{E}_g \) is covered by irreducible curves \( \Gamma_g \subset \overline{\mathcal{C}}_{g,g-1} \) such that \( \Gamma_g : \mathcal{E}_g < \mathcal{C}_g \) in particular \( \mathcal{E}_g \in \text{Eff}(\overline{\mathcal{C}}_{g,g-1}) \) is a non-movable extremal effective divisor.

The curves \( \Gamma_g \) have a simple modular construction. One fixes a general linear series \( A \in W^2_{g+1}(C) \), in particular \( A \) is complete and has only ordinary ramification points. The general point of \( \Gamma_g \) corresponds to an element \([C, D] \in \overline{\mathcal{C}}_{g,g-1} \), where \( D \in \mathcal{C}_{g-1} \) is an effective divisor such that \( H^0(C, A \otimes \mathcal{O}_C(-2p - D)) \neq 0 \), for some point \( p \in C \), that is, \( D \) is the residual divisor cut out by a tangent line to the degree \( g+1 \) plane model of \( C \) given by \( A \). We refer to Section 2 for details.

The proofs of Theorems 0.3 and 0.4 rely on two ingredients. First, we use our result \([FV2]\), stating that for \( g \geq 4 \), the singularities of \( \mathcal{C}_{g,n} \) impose no adjoint conditions, that is, plurcanonical forms defined on the smooth locus of \( \overline{\mathcal{C}}_{g,n} \) extend to a smooth model of the symmetric product. Precisely, if \( \epsilon : \overline{\mathcal{C}}_{g,n} \rightarrow \mathcal{C}_{g,n} \) denotes any resolution of singularities, then for any \( \ell \geq 0 \), there is a group isomorphism

\[
\epsilon^* : H^0((\mathcal{C}_{g,n})_{\text{reg}}, K^{\otimes \ell}_{\mathcal{C}_{g,n}}) \xrightarrow{\sim} H^0(\overline{\mathcal{C}}_{g,n}, K^{\otimes \ell}_{\overline{\mathcal{C}}_{g,n}}).
\]

In particular, \( \mathcal{H}_g \) is of general type when the canonical class \( K_{\overline{\mathcal{C}}_{g,g-1}} \in \text{Pic}(\overline{\mathcal{C}}_{g,g-1}) \) is big. This makes the problem of understanding the effective cone of \( \overline{\mathcal{C}}_{g,g-1} \) of some importance. If \( \pi : \overline{\mathcal{M}}_{g,g-1} \rightarrow \overline{\mathcal{C}}_{g,g-1} \) is the quotient map, the Hurwitz formula implies

\[
\pi^*(K_{\overline{\mathcal{C}}_{g,g-1}}) \equiv K_{\overline{\mathcal{M}}_{g,g-1}} - \delta_{0:2} \in \text{Pic}(\overline{\mathcal{M}}_{g,g-1}).
\]

The sum \( \sum_{i=1}^{g-1} \psi_i \in \text{Pic}(\overline{\mathcal{M}}_{g,g-1})^{\text{reg}} \) of cotangent tautological classes descends to a big and nef class on \( \overline{\mathcal{C}}_{g,g-1} \) (cf. Proposition 1.2), thus in order to conclude that \( \mathcal{H}_g \) is of general type, it suffices to exhibit an effective divisor \( \mathcal{D} \in \text{Eff}(\overline{\mathcal{C}}_{g,g-1}) \), such that

\[
\pi^*(K_{\overline{\mathcal{C}}_{g,g-1}}) \in \mathbb{Q}_{>0}\left( \sum_{i=1}^{g-1} \psi_i \right) + \phi^*\text{Eff}(\overline{\mathcal{M}}_g) + \mathbb{Q}_{\geq 0}\left( \lambda, \pi^*([\mathcal{D}]), \delta_{i,c} : i \geq 0, c \geq 2 \right).
\]

In this formula, \( \phi : \overline{\mathcal{M}}_{g,g-1} \rightarrow \overline{\mathcal{M}}_g \) denotes the morphism forgetting the marked points, and refer to Section 1 for the standard notation for boundary divisor classes on \( \overline{\mathcal{M}}_{g,n} \). Comparing condition (2) against the formula for \( K_{\overline{\mathcal{C}}_{g,g-1}} \) given by (1), if one writes

\[
\pi^*(\mathcal{D}) \equiv a\lambda - b_{\text{irr}}\delta_{\text{irr}} + c \sum_{i=1}^{g-1} \psi_i - \sum_i b_{i,c} \delta_{i,c} \in \text{Pic}(\overline{\mathcal{M}}_{g,g-1}),
\]

the following inequality

\[
3c < b_{0:2}
\]

is a necessary condition for the existence of a divisor \( \mathcal{D} \) satisfying (2). It is straightforward to unravel the geometric significance of the condition (3). If \( [C] \in \mathcal{M}_g \) is a general curve, there is a rational map \( u : C_{g-1} \rightarrow \overline{\mathcal{C}}_{g,g-1} \) given by restriction. Denoting by
Antram, M. 

that all moduli spaces \( \mathcal{D} \) can be chosen to be a Brill-Noether divisor.

Corollary 0.7. The closure inside \( \mathcal{M}_{g,g-1} \) playing this role is precisely \( \mathfrak{Intram}_g \).

We explain briefly how Theorem 0.5 implies the statement about the Kodaira dimension of \( \mathcal{M}_{g,g-1} \). We choose an effective divisor \( D \equiv a\lambda - \sum i b_i \delta_i \in \text{Eff}(\mathcal{M}_g) \) on the moduli space of curves, with \( a, b_i > 0 \), having slope \( s = s(D) := \frac{a}{\min b_i} \) as small as possible. Then note that the following linear combination

\[
\pi^*(K_{\mathcal{M}_{g,g-1}}) - \frac{1}{6g-11} (\frac{3}{2} [\mathfrak{Intram}_g] - (12g-25)\phi^*(D) - \sum_{i=1}^{g-1} \psi_i - ((84g-185) - (12g-25)s) \lambda)
\]

is expressible as a positive combination of boundary divisors on \( \mathcal{M}_{g,g-1} \). Since, as already pointed out, the class \( \sum_{i=1}^{g-1} \psi_i \in \text{Pic}(\mathcal{M}_{g,g-1}) \) descends to a big class on \( \overline{\mathcal{M}}_{g,g-1} \), one obtains the following:

**Corollary 0.7.** For all \( g \) such that the slope of the moduli space of curves satisfies the inequality

\[
s(\mathcal{M}_g) := \inf_{D \in \text{Eff}(\mathcal{M}_g)} s(D) < \frac{84g-185}{12g-25},
\]

the universal theta divisor \( \Theta_g \) is of general type.

The bound appearing in Corollary 0.7 holds precisely when \( g \geq 12 \); for \( g \) such that \( g + 1 \) is composite, the inequality \( s(\mathcal{M}_g) \leq 6 + 12/(g + 1) \) is well-known, and \( D \) can be chosen to be a Brill-Noether divisor \( \mathcal{M}_{g,d} \) corresponding to curves with a \( g_d \), when the Brill-Noether number \( \rho(g, r, d) = -1 \), cf. [FH1]. When \( g + 1 \) is prime and \( g \neq 12 \), then in practice \( g = 2k - 2 \geq 16 \), and \( D \) can be chosen to be the Gieseker-Petri \( \mathcal{G}^1 \) consisting of curves \( C \) possessing a pencil \( A \in W^1_k(C) \) such that the Petri map \( \mu_0(C, A) : H^0(C, A) \otimes H^0(C, K_C \otimes A) \to H^0(C, K_C) \) is not an isomorphism. When \( g = 12 \), one has to use the divisor constructed on \( \mathcal{M}_{12} \) in [FV1]. Finally, when \( g \leq 11 \) it is known that \( s(\mathcal{M}_g) \geq 6 + 12/(g + 1) \) and inequality (1.7) is not satisfied. In fact, as already pointed out \( \kappa(\Theta_g) = -\infty \) in this range.

The proof of Theorem 0.4 proceeds along similar lines, and relies on finding an explicit \( \mathfrak{G}_{g-2} \)-invariant extremal ray of the cone of effective divisors on \( \mathcal{M}_{g,g-2} \). A representative of this ray is characterized by the geometric condition that the marked points appear in the same fibre of a pencil of degree \( g - 1 \). One can construct such divisors on all moduli spaces \( \mathcal{M}_{g,n} \) with \( 1 \leq n \leq g - 2 \), cf. Section 3.

**Theorem 0.8.** The closure inside \( \mathcal{M}_{g,g-2} \) of the locus

\[
\mathcal{F}_{g,1} := \{ [C, x_1, \ldots, x_{g-2}] \in \mathcal{M}_{g,g-2} : \exists A \in W^1_g(C) \text{ with } H^0(C, A(-\sum_{i=1}^{g-2} x_i)) \neq 0 \}
\]
is a non-movable, extremal effective divisor on \( \mathcal{M}_{g,g-2} \). Its class is given by the formula:

\[
\mathcal{F}_{g,1} \equiv -(g-12)\lambda + (g-3) \sum_{i=1}^{g-2} \psi_i - \delta_{\text{int}} - \frac{1}{2} \sum_{s=2}^{g-2} s(g-4+sg-2s) \delta_{0,s} - \cdots \in \text{Pic}(\mathcal{M}_{g,g-2}).
\]
Note that again, inequality (3) is satisfied, hence $\mathcal{F}_{g,1}$ can be used to prove that $K_{\mathcal{M}_{g,g-2}}$ is big. Moreover, $\mathcal{F}_{g,1}$ descends to an extremal divisor $\mathcal{F}_{g,1} \in \operatorname{Eff}(\mathcal{M}_{g,g-2})$. In fact, we shall show that $\mathcal{F}_{g,1}$ is swept by curves intersecting its class negatively.

1. Cones of divisors on universal symmetric products

The aim of this section is to establish certain facts about boundary divisors on $\mathcal{M}_{g,n}$ and $\mathcal{C}_{g,n}$, see [AC] for a standard reference. We follow the convention set in [FV2], that is, if $\mathcal{M}$ is a Deligne-Mumford stack, we denote by $\mathcal{M}$ its coarse moduli space.

For an integer $0 \leq i \leq [g/2]$ and a subset $T \subset \{1, \ldots, n\}$, we denote by $\Delta_{i:T}$ the closure in $\mathcal{M}_{g,n}$ of the locus of $n$-pointed curves $[C_1 \cup C_2, x_1, \ldots, x_n]$, where $C_1$ and $C_2$ are smooth curves of genera $i$ and $g-i$ respectively meeting transversally in one point, and the marked points lying on $C_1$ are precisely those indexed by $T$. We define $\delta_{i:T} := [\Delta_{i:T}]_\mathbb{Q} \in \operatorname{Pic}(\mathcal{M}_{g,n})$. For $0 \leq i \leq [g/2]$ and $0 \leq s \leq g$, we set $\Delta_{i:s} := \sum_{#(T) = s}\delta_{i:T}$, $\delta_{i:s} := [\Delta_{i:s}]_\mathbb{Q} \in \operatorname{Pic}(\mathcal{M}_{g,n})$.

By convention, $\delta_{0:s} := \emptyset$, for $s < 2$, and $\delta_{i:s} := \delta_{g-i:n-s}$. If $\phi: \mathcal{M}_{g,n} \to \mathcal{M}_g$ is the morphism forgetting the marked points, we set $\lambda := \phi^*(\lambda)$ and $\delta_{\text{irr}} := \phi^*(\delta_{\text{irr}})$, where $\delta_{\text{irr}} := [\Delta_{\text{irr}}] \in \operatorname{Pic}(\mathcal{M}_g)$ denotes the class of the locus of irreducible nodal curves. Furthermore, $\psi_1, \ldots, \psi_n \in \operatorname{Pic}(\mathcal{M}_{g,n})$ are the cotangent classes corresponding to the marked points. The canonical class of $\mathcal{M}_{g,n}$ is computed via Kodaira-Spencer theory:

\begin{equation}
K_{\mathcal{M}_{g,n}} = 13\lambda - 2\delta_{\text{irr}} + \sum_{i=1}^n \psi_i - 2 \sum_{T \subset \{1, \ldots, n\}} \delta_{i:T} - [\delta_{1:0}] \in \operatorname{Pic}(\mathcal{M}_{g,n}).
\end{equation}

Let $\mathcal{C}_{g,n} := \mathcal{M}_{g,n}/\mathcal{S}_n$ be the universal symmetric product and $\pi: \mathcal{M}_{g,n} \to \mathcal{C}_{g,n}$ (respectively $\varphi: \mathcal{C}_{g,n} \to \mathcal{M}_g$) the projection (respectively the forgetful map), so that $\phi = \varphi \circ \pi$. We denote by $\bar{\lambda}, \bar{\delta}_{\text{irr}}, \bar{\delta}_{i:c} := [\bar{\Delta}_{i:c}] \in \operatorname{Pic}(\mathcal{C}_{g,n})$ the divisor classes on the symmetric product pulling-back to the same symbols on $\mathcal{M}_{g,n}$. Clearly, $\pi^*(\bar{\lambda}) = \lambda$, $\pi^*(\bar{\delta}_{\text{irr}}) = \bar{\delta}_{\text{irr}}$, $\pi^*(\bar{\delta}_{i:c}) = \bar{\delta}_{i:c}$; in the case $i = 0, c = 2$, this reflects the branching of the map $\pi$ along the divisor $\bar{\Delta}_{0:2} \subset \mathcal{C}_{g,n}$. Following [FV2], let $L$ denote the line bundle on $\mathcal{C}_{g,n}$, having fibre $L[C, x_1 + \cdots + x_n] := T^{\psi}_{x_1}(C) \otimes \cdots \otimes T^{\psi}_{x_n}(C)$, over a point $[C, x_1 + \cdots + x_n] := \pi([C, x_1, \ldots, x_n]) \in \mathcal{C}_{g,n}$. We set $\bar{\psi} := c_1(L)$, and note:

\begin{equation}
\pi^*(\bar{\psi}) = \sum_{i=1}^n (\psi_i - \sum_{i \in T \subset \{1, \ldots, n\}} \delta_{0:T}) = \sum_{i=1}^n \psi_i - \sum_{s=2}^n s \delta_{0:s} \in \operatorname{Pic}(\mathcal{M}_{g,n}).
\end{equation}

Proposition 1.1. For $g \geq 3$ and $n \geq 0$, the morphism $\pi^*: \operatorname{Pic}(\mathcal{C}_{g,n})_\mathbb{Q} \to \operatorname{Pic}(\mathcal{M}_{g,n})_\mathbb{Q}$ is injective. Furthermore, there is an isomorphism of groups $\operatorname{Pic}(\mathcal{C}_{g,n})_\mathbb{Q} \xrightarrow{\cong} N^1(\mathcal{C}_{g,n})_\mathbb{Q}$, coupled with the commutativity of the obvious diagrams relating the Picard and Néron-Severi groups of $\mathcal{M}_{g,n}$ and $\mathcal{C}_{g,n}$ respectively.

\begin{proof}
The first assertion is an immediate consequence of the existence of the norm morphism $\operatorname{Nm}_\pi: \operatorname{Pic}(\mathcal{M}_{g,n}) \to \operatorname{Pic}(\mathcal{C}_{g,n})$, such that $\operatorname{Nm}_\pi(\pi^*(L)) = L^{\otimes \deg(\pi)}$, for every $L \in \operatorname{Pic}(\mathcal{C}_{g,n})$. The second part comes from the isomorphism $\operatorname{Pic}(\mathcal{M}_{g,n})_\mathbb{Q} \xrightarrow{\cong} N^1(\mathcal{M}_{g,n})_\mathbb{Q}$, coupled with the commutativity of the obvious diagrams relating the Picard and Néron-Severi groups of $\mathcal{M}_{g,n}$ and $\mathcal{C}_{g,n}$ respectively.
\end{proof}
One may thus identify \( \text{Pic}(\tilde{\mathcal{C}}_{g,n})_\mathbb{Q} \cong \text{Pic}(\overline{\mathcal{M}}_{g,n})_\mathbb{Q} \). The Riemann-Hurwitz formula applied to the branched covering \( \pi : \overline{\mathcal{M}}_{g,n} \to \tilde{\mathcal{C}}_{g,n} \) yields,

\[
\pi^*(K_{\tilde{\mathcal{C}}_{g,n}}) = K_{\overline{\mathcal{M}}_{g,n}} - \delta_{0:2} \equiv 13\lambda + \sum_{i=1}^{n} \psi_i - 2\delta_{\text{irr}} - 3\delta_{0:2} - 2 \sum_{s=3}^{n} \delta_{0:s} - \cdots .
\]

As expected, the sum of cotangent classes descends to a big line bundle on \( \tilde{\mathcal{C}}_{g,n} \).

**Proposition 1.2.** The divisor class \( N_{g,n} := \tilde{\psi} + \sum_{s=2}^{n} s\delta_{0:s} \in \text{Eff}(\tilde{\mathcal{C}}_{g,n}) \) is big and nef.

**Proof.** The class \( N_{g,n} \) is characterized by the property that \( \pi^*(N_{g,n}) = \sum_{i=1}^{n} \psi_i \). This is a nef class on \( \overline{\mathcal{M}}_{g,n} \), in particular, \( N_{g,n} \) is nef on \( \tilde{\mathcal{C}}_{g,n} \). To establish that \( N_{g,n} \) is big, we express it as a combination of effective classes and the class \( \kappa_1 \in \text{Pic}(\tilde{\mathcal{C}}_{g,n}) \), where

\[
\pi^*(\kappa_1) = \kappa_1 = 12\lambda + \sum_{i=1}^{n} \psi_i - \delta_{\text{irr}} - \sum_{i=0}^{[g/2]} \delta_{i:1} \in \text{Pic}(\overline{\mathcal{M}}_{g,n}).
\]

Since \( \pi^*(\kappa_1) \) is ample on \( \overline{\mathcal{M}}_{g,n} \), it follows that \( \kappa_1 \) is ample as well. To finish the proof, we exhibit a suitable effective class on \( \overline{\mathcal{M}}_{g,n} \) having negative \( \lambda \)-coefficient. For that purpose, we choose \( \mathcal{W}_{g,n} \subset \tilde{\mathcal{C}}_{g,n} \) to be the locus of effective divisors having a Weierstrass point in their support. For \( i = 1, \ldots, n \), we denote by \( \sigma_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,1} \) the morphism forgetting all but the \( i \)-th point, and let

\[
\mathcal{W} \equiv -\lambda + \left( \frac{g+1}{2} \right) \psi - \sum_{i=1}^{g-1} \left( \frac{g+i+1}{2} \right) \delta_{i:1} \in \text{Eff}(\overline{\mathcal{M}}_{g,1}),
\]

be the class of the divisor of Weierstrass points on the universal curve. Then one finds

\[
\pi^*(\mathcal{W}) = \sum_{i=1}^{n} \sigma_i^*(\tilde{\mathcal{W}}) = -n\lambda + \left( \frac{g+1}{2} \right) \sum_{i=1}^{n} \psi_i - \left( \frac{g+1}{2} \right) \sum_{s=2}^{n} s\delta_{0:s} - \cdots \in \text{Pic}(\overline{\mathcal{M}}_{g,n}),
\]

and \( \mathcal{W}_{g,n} \equiv -g\lambda + \left( \frac{g+1}{2} \right) \tilde{\psi} - \sum_{i=1}^{[g/2]} \sum_{s=2}^{\lfloor g/s \rfloor} b_{i:s} \delta_{i:s} \), where \( b_{i:s} > 0 \). One checks that \( N_{g,n} \) can be written as a \( \mathbb{Q} \)-combination with positive coefficients of the ample class \( \kappa_1 \), the effective class \( [\mathcal{W}_{g,n}] \) and other boundary divisor classes. In particular, \( N_{g,n} \) is big. \( \square \)

2. THE UNIVERSAL ANTIRAMIFICATION LOCUS OF THE GAUSSEN MAP

We begin the calculation of the divisor \( \overline{\text{Antram}}_{g} \), and for a start we consider its restriction \( \overline{\text{Antram}}_{g} \) to \( \mathcal{M}_{g,g-1} \). Recall that \( \overline{\text{Antram}}_{g} \) is defined as the closure of the locus of pointed curves \( [C, x_1, \ldots, x_{g-1}] \in \mathcal{M}_{g,g-1} \), such that there exists a holomorphic form on \( C \) vanishing at \( x_1, \ldots, x_{g-1} \) and having an unspecified double zero.

Let \( u : \mathcal{M}_{g,g-1}^{(1)} \to \mathcal{M}_{g,g-1} \) be the universal curve over the stack of \( (g-1) \)-pointed smooth curves and we denote by \( ([C, x_1, \ldots, x_{g-1}], p]) \in \mathcal{M}_{g,g-1}^{(1)} \) a general point, where \( [C, x_1, \ldots, x_{g-1}] \in \mathcal{M}_{g,g-1} \) and \( p \in C \) is an arbitrary point. For \( i = 1, \ldots, g-1 \), let \( \Delta_{ip} \subset \mathcal{M}_{g,g-1}^{(1)} \) be the diagonal divisor given by the equation \( p = x_i \). Furthermore, for \( i = 1, \ldots, g-1 \) we consider as before the projections \( \sigma_i : \mathcal{M}_{g,g-1}^{(1)} \to \mathcal{M}_{g,1} \) (respectively \( \sigma_p : \mathcal{M}_{g,g-1}^{(1)} \to \mathcal{M}_{g,1} \), obtained by forgetting all marked points except \( x_i \) (respectively
and have the following exact sequence:

\[
\begin{array}{ccccc}
X & \xrightarrow{q} & M_{g,1}^{(1)} & \xrightarrow{f} & M_{g,1} \\
\downarrow & & \downarrow & & \\
M_{g,1} & \xrightarrow{\phi} & M_g
\end{array}
\]

in which all the morphisms are smooth and \( \phi \) (hence also \( q \)) is proper. For \( 1 \leq i \leq g-1 \) there are tautological sections \( r_i : M_{g,1}^{(1)} \to X \) as well as \( r_p : M_{g,1}^{(1)} \to X \), and set \( E_i := \text{Im}(r_i), E_p := \text{Im}(r_p). \) Thus \( \{E_i\}_{i=1}^{g-1} \) and \( E_p \) are relative divisors over \( q \).

For a point \( [(C, x_1, \ldots, x_{g-1}), p] \in M_{g,1}^{(1)} \), we denote \( D := \sum_{i=1}^{g-1} x_i + 2p \in C_{g+1} \), and have the following exact sequence:

\[
0 \to \frac{H^0(O_C(D))}{H^0(O_C)} \to H^0(O_D(D)) \xrightarrow{\alpha} H^1(O_C) \to H^1(O_C(D)) \to 0.
\]

In particular, the morphisms \( \alpha_D \) globalize to a morphism of vector bundles over \( M_{g,1}^{(1)} \)

\[
\alpha : A := q_* \left( O_X \left( \sum_{i=1}^{g-1} E_i + 2E_p \right) / O_X \right) \to R^1 q_* O_X.
\]

The subvariety \( Z := \{(C, x_1, \ldots, x_{g-1}), p) \in M_{g,1}^{(1)} : H^0(K_C(-2p - \sum_{i=1}^{g-1} x_i)) \neq 0 \} \) is the non-surjectivity locus of \( \alpha \) and \( \text{Anstram}_g := u_*(Z) \subset M_{g,1}^{(1)} \). The class of \( Z \) is equal to

\[
[Z] = c_2 \left( A^\vee - (R^1 q_* O_X)^\vee \right) = c_2 \left( -q_* O_X \left( \sum_{i=1}^{g-1} E_i + 2E_p \right) \right) \in A^2(M_{g,1}^{(1)}),
\]

where the last term can be computed by Grothendieck-Riemann-Roch:

\[
\chi_q \left( q_* O_X \left( \sum_{i=1}^{g-1} E_i + 2E_p \right) \right) = q_* \left[ \left( \sum_{k=0}^{g-1} \frac{(\sum_{i=1}^{g-1} E_i + 2E_p)^k}{k!} \right) \cdot \left( 1 - \frac{c_1(\omega_q)}{2} + \frac{c_2(\omega_q)}{12} + \cdots \right) \right],
\]

and we are interested in evaluating the terms of degree 1 and 2 in this expression. The result of applying GRR to the morphism \( q \), can be summarized as follows:

**Lemma 2.1.** One has the following relations in \( A^*(M_{g,1}^{(1)}) \):

(i)

\[
\chi_1 \left( q_* (O_X \left( \sum_{i=1}^{g-1} E_i + 2E_p \right) ) \right) = \lambda - \sum_{i=1}^{g-1} K_i - 3K_p + 2 \sum_{i=1}^{g-1} \Delta_{ip}.
\]

(ii)

\[
\chi_2 \left( q_* (O_X \left( \sum_{i=1}^{g-1} E_i + 2E_p \right) ) \right) = \frac{5}{2} K_p^2 + \frac{1}{2} \sum_{i=1}^{g-1} K_i^2 - 2 \sum_{i=1}^{g-1} (K_i + K_p) \cdot \Delta_{ip}.
\]

**Proof.** We apply systematically the push-pull formula and the following identities:

\[
E_i^2 = -E_i \cdot q^*(K_i), \quad E_p^2 = -E_p \cdot q^*(K_p), \quad E_i \cdot c_1(\omega_q) = E_i \cdot q^*(K_i), \quad E_p \cdot c_1(\omega_q) = E_p \cdot q^*(K_p),
\]

\[
E_i \cdot E_j = 0 \text{ for } i \neq j, \quad E_i \cdot E_p = E_i \cdot q^*(\Delta_{ip}), \quad \text{and } q_* (c_1^2(\omega_q)) = 12 \lambda.
\]
Proposition 2.2. One has $[\text{Antram}_g] = -4(g - 7)\lambda + (4g - 8)\sum_{i=1}^{g-1} \psi_i \in \text{Pic}(M_{g,g-1})$.

Proof. We apply the results of Lemma 2.1 as well as the formulas from [HM] p. 55, in order to estimate the push-forward under $u$ of the degree 2 monomials in tautological classes. Setting $F := q_* (O_C(\sum_{i=1}^{g-1} E_i + 2E_p))$, we obtain that 

$$u_* (\text{ch}_1^2(F)) = -(8g - 116)\lambda + (8g - 24)\sum_{i=1}^{g-1} \psi_i,$$

and $u_* (\text{ch}_2(F)) = 30\lambda - 4\sum_{i=1}^{g-1} \psi_i$, hence $[\text{Antram}_g] = u_* (\text{ch}_1^2(F) - 2\text{ch}_2(F))/2$, and the claimed formula follows at once.

We proceed now towards proving Theorem 0.5 and expand the class $[\text{Antram}_g]$ in the standard basis of the Picard group, that is, 

$$[\text{Antram}_g] = a\lambda + c\sum_{i=1}^{g-1} \psi_i - b_{irr}\delta_{irr} - \sum_{i=0}^{g-1} b_{i:s}\delta_{i:s}. $$

We have just computed $a = -4(g - 7)$ and $c = 4(g - 2)$. The remaining coefficients are determined by intersecting $[\text{Antram}_g]$ with curves lying in the boundary of $\overline{M}_{g,g-1}$ and understanding how $[\text{Antram}_g]$ degenerates. We begin with the coefficient $b_{0:2}$:

Proposition 2.3. One has the relation $(4g - 6)c - (g - 2)b_{0:2} = (4g - 2)(g - 2)$. It follows that $b_{0:2} = 12g - 22$.

Proof. We fix a general pointed curve $[C, x_1, \ldots, x_{g-2}] \in \mathcal{M}_{g,g-2}$ and consider the family 

$$C_{x_{g-1}} := \{ [C, x_1, \ldots, x_{g-2}, x_{g-1}] : x_{g-1} \in C \} \subset \overline{M}_{g,g-1}.$$ 

The curve $C_{x_{g-1}}$ is the fibre over $[C, x_1, \ldots, x_{g-2}]$ of the morphism $\overline{M}_{g,g-1} \to \overline{M}_{g,g-2}$ forgetting the point labeled by $x_{g-1}$. Note that $C_{x_{g-1}} \cdot \psi_i = 1$ for $i = 1, \ldots, g - 2$ and $C_{x_{g-1}} \cdot \psi_{g-1} = 3g - 4 = 2g - 2 + (g - 2)$. Obviously $C_{x_i} \cdot \delta_{0:2} = g - 2$ and the points in the intersection correspond to the case when $x_{g-1}$ collides with one of the fixed points $x_1, \ldots, x_{g-2}$. The intersection of $C_{x_i}$ with the remaining generators of $\text{Pic}(\overline{M}_{g,g-1})$ is equal to zero. We set $A := K_C \otimes O_C(-x_1 - \cdots - x_{g-2}) \in W^1_2(C)$. By the generality assumption, $h^0(C, A) = 2$, and all ramification points of $A$ are simple. Pointed curves in the intersection $C_{x_{g-1}} \cdot [\text{Antram}_g]$ correspond to points $x_{g-1} \in C$, such that there exists a (ramification) point $p \in C$ with $H^0(C, A \otimes O_C(-2p - x_{g-1})) \neq 0$. The pencil $A$ carries $4g - 2$ ramification points. For each of them there are $g - 2$ possibilities of choosing $x_{g-1} \in C$ in the same fibre as the ramification point, hence the conclusion follows.

Next we determine the coefficient $b_{irr}$. First we note that the relation

$$a - 12b_{irr} + b_{1:0} = 0$$

holds. Indeed, the divisor $[\text{Antram}_g]$ is disjoint from the curve in $\Delta_{1:0} \subset \overline{M}_{g,g-1}$, obtained from a fixed pointed curve $[C, x_1, \ldots, x_{g-1}, q] \in \overline{M}_{g-1,g}$, by attaching at the point $q$ a pencil of plane cubics along a section of the pencil induced by one of the 9 base points.

Proposition 2.4. One has the relation $b_{irr} = 2$. 

Proof. We fix a general curve \([C, q, x_1, \ldots, x_{g-1}] \in \mathcal{M}_{g-1,g}\) and we define the family

\[ C_{\text{irr}} := \{ [C/t \sim q, x_1, \ldots, x_{g-1}] : t \in C \} \subset \mathcal{M}_{g-1,g-1}. \]

Then \(C_{\text{irr}} \cdot \psi_i = 1\) for \(i = 1, \ldots, g - 1\), \(C_{\text{irr}} \cdot \delta_{i1} = -(\deg(K_C) + 2) = -2g + 2\), and finally \(C_{\text{irr}} \cdot \delta_{10} = 1\). All other intersection numbers with generators of \(\text{Pic}(\mathcal{M}_{g-1,g})\) equal zero.

We fix an effective divisor \(D \in C_e\) of degree \(e \geq g\) (for instance \(D = q + \sum_{i=1}^{g-1} x_i\)). For each pair of points \((t, p) \in C \times C\), there is an exact sequence on \(C\)

\[ 0 \rightarrow H^0(C, K_C(q + t - 2p - \sum_{i=1}^{g-1} x_i)) \rightarrow H^0(C, K_C(D + q + t - 2p - \sum_{i=1}^{g-1} x_i)) \rightarrow \]

\[ H^0(D, K_C(D + q + t - 2p - \sum_{i=1}^{g-1} x_i)) \rightarrow H^1(C, K_C(q + t - 2p - \sum_{i=1}^{g-1} x_i)) \rightarrow 0. \]

The intersection \(C_{\text{irr}} \cdot \mathfrak{Antram}_g\) corresponds to the locus of pairs \((t, p) \in C \times C\) such that the map \(\beta_{t,p}\) is not injective. On the triple product of \(C\), we consider two of the projections \(f : C \times C \times C \rightarrow C \times C\) and \(p_1 : C \times C \times C \rightarrow C\) given by \(f(x, t, p) = (t, p)\) and \(p_1(x, t, p) = x\), then set \(A := K_C(q - \sum_{i=1}^{g-1} x_i) \in \text{Pic}^{g-2}(C)\). We denote by \(\Delta_{12}, \Delta_{13} \subset C \times C \times C\) the corresponding diagonals, and finally, introduce the line bundle on \(C \times C \times C\)

\[ \mathcal{F} := p_1^*(A) \otimes O_{C \times C \times C}(\Delta_{12} - 2\Delta_{13}). \]

Applying the Porteous formula, one can write

\[ C_{\text{irr}} \cdot \mathfrak{Antram}_g = c_2(R^1 f_* \mathcal{F} - R^0 f_* \mathcal{F}) = \frac{\text{ch}_1^2(f_* \mathcal{F}) + 2\text{ch}_2(f_* \mathcal{F})}{2} \in A^2(C \times C). \]

We evaluate \(\text{ch}_i(f_* \mathcal{F})\) using GRR applied to the morphism \(f\), that is,

\[ \text{ch}(f_* \mathcal{F}) = f_* \left[ \left( \sum_{a \geq 0} \left( \frac{p_1^*(A) + \Delta_{12} - 2\Delta_{13}}{a!} \right) \cdot \left(1 - \frac{1}{a} p_1^*(K_C) \right) \right) \right]. \]

Denoting by \(F_1, F_2 \in H^2(C \times C)\) the class of the fibres, after calculations one finds that

\[ \text{ch}_1(f_* \mathcal{F}) = -(g - 2) F_1 - 4(g - 2) F_2 - 2 \Delta_C \in H^2(C \times C, \mathbb{Q}), \]

\[ \text{ch}_2(f_* \mathcal{F}) = -(2g - 2) \in H^4(C \times C, \mathbb{Q}), \]

that is, \(c_2(R^1 f_* \mathcal{F} - R^0 f_* \mathcal{F}) = 4(g - 2)(g - 1)\). Coupled with (6), this yields \(b_{\text{irr}} = 2\).

We are left with the task of determining the coefficient of \(\delta_{i,s}\). This requires solving a number of enumerative geometry problems in the spirit of de Jonquières’ formula. We fix integers \(0 \leq i \leq g\) and \(s \leq i - 1\) as well as general pointed curves \([C, x_1, \ldots, x_s] \in \mathcal{M}_{i,s}\) and \([D, q, x_{s+1}, \ldots, x_{g-1}] \in \mathcal{M}_{g-i,g-s}\), then construct a pencil of stable curves of genus \(g\), by identifying the fixed point \(q \in D\) with a variable point, also denoted by \(q\), on the component \(C\):

\[ C_{i,s} := \{ [C \cup_q D, x_1, \ldots, x_s, x_{s+1}, \ldots, x_{g-1}] : q \in C \} \subset \Delta_{i,s} \subset \mathcal{M}_{g,g-1}. \]

We summarize the non-zero intersection numbers of \(C_{i,s}\) with generators of \(\text{Pic}(\mathcal{M}_{g,g-1})\):

\[ C_{i,s} \cdot \psi_1 = \cdots = C_{i,s} \cdot \psi_s = 1, \ C_{i,s} \cdot \delta_{i,s-1} = i, \ C_{i,s} \cdot \delta_{i,s} = 2i - 2 + s. \]
Theorem 2.5. We fix integers $0 \leq i \leq g$ and $0 \leq s \leq i - 1$. Then, the following formula holds:

$$b_{i,s} = 2i^3 - 5i^2 - 3i + 4g - 4i^2 s + 14si - 6gs - s + 2s^2 g - 3s^2 + 2.$$ 

In the proof an essential role is played by the following calculation:

Proposition 2.6. Let $i, s$ be integers such that $0 \leq s \leq i - 1$, and $[C, x_1, \ldots, x_s] \in \mathcal{M}_{i,s}$ a general pointed curve. The number of pairs $(q, p) \in C \times C$ such that

$$H^0(C, K_C \otimes \mathcal{O}_C(-x_1 - \cdots - x_s - (i - s - 1)q - 2p)) \neq 0,$$

is equal to $a(i, s) := 2(i - s - 1)(2i^3 - 5i^2 - 2i^2 s + 3i s)$.

Remark 2.7. By specializing, one recovers well-known formulas in enumerative geometry. For instance, $a(3, 0) = 56$ is twice the number of bitangents of a smooth plane quartic, whereas $a(4, 0) = 324$ equals the number of canonical divisors of type $3g + 2p + x \in |K_C|$, where $|C| \in \mathcal{M}_4$. This matches de Jonquières’ formula, cf. [ACGH] p.359.

Proof of Theorem 2.5 We fix a general point $[C \cup_q D, x_1, \ldots, x_{g-1}] \in C_{i,s} \cdot \mathcal{Antram}_{q}$ corresponding to a point $q \in C$. We shall show that $q$ is not one of the marked points $x_1, \ldots, x_s$ on $C$, then give a geometric characterization of such points and count their number. Let

$$\omega_D \in H^0(D, K_D \otimes \mathcal{O}_D(2iq)) \quad \text{and} \quad \omega_C \in H^0(C, K_C \otimes \mathcal{O}_C(2g - 2i)q)$$

be the aspects of the section of the limit canonical series on $C \cup_q D$, which vanishes doubly at an unspecified point $p \in C \cup D$ as well as along the divisor $x_1 + \cdots + x_{g-1}$. The condition $\text{ord}_q(\omega_C) + \text{ord}_q(\omega_D) \geq 2g - 2$, comes from the definition of a limit linear series. We distinguish two cases depending on the position of the point $p$. If $p \in D$ then,

$$\text{div}(\omega_C) \geq x_1 + \cdots + x_s, \quad \text{div}(\omega_D) \geq x_{s+1} + \cdots + x_{g-1} + 2p.$$ 

Since the points $q, x_{s+1}, \ldots, x_{g-1} \in D$ are general, we find that $\text{ord}_q(\omega_D) \leq i + s - 2$. Moreover, $K_D \otimes \mathcal{O}_D(i - s + 2q - x_{s+1} - \cdots - x_{g-1}) \in W^1_{q-i+1}(D)$ is a pencil, and $p \in D$ is one of its (simple) ramification points. The Hurwitz formula gives $4(g - i)$ choices for such $p \in D$.

By compatibility, $\text{ord}_q(\omega_C) \geq 2g - i - s$. A parameter count implies that equality must hold. The condition $H^0(C, K_C \otimes \mathcal{O}_C(-x_1 - \cdots - x_s - (i - s)q)) \neq 0$, is equivalent to asking that $q \in C$ be a ramification point of $K_C \otimes \mathcal{O}_C(- \sum_{j=1}^s x_j) \in W^1_{2i - 2s - 1}(C)$. Since the points $x_1, \ldots, x_s \in C$ are chosen to be general, all ramification points of this linear series are simple and occur away from the marked points. From Plücker’s formula, the number of ramification points equals $(i - s)(i^2 - 1 - is)$. Multiplying this with the number of choices for $p \in D$, we obtain a total contribution of $4(g - i)(i - s)(i^2 - is - 1)$ to the intersection $C_{i,s} \cdot \mathcal{Antram}_{q}$, stemming from the case when $p \in D$. The proof that each of these points of intersection is to be counted with multiplicity 1 is standard and proceeds along the lines of [EH2] Lemma 3.4.

We assume now that $p \in C$. Keeping the notation from above, it follows that $\text{ord}_q(\omega_D) = i + s - 1$ and $\text{ord}_q(\omega_C) = 2g - i - s - 1$, therefore

$$0 \neq \sigma_C \in H^0(C, K_C \otimes \mathcal{O}_C(- \sum_{j=1}^s x_j - (i - s - 1)q - 2p)).$$
The section $ω_D$ is uniquely determined up to multiplication by scalars, whereas there are $a(i, s)$ choices on the side of $C$, each counted with multiplicity 1.

In principle, the double zero of the limit holomorphic form could specialize to the point of attachment $q ∈ C ∩ D$, and we prove that this would contradict our generality hypothesis. One considers the semistable curve $X := C ∪ q_1 E ∪ q_2 D$, obtained from $C ∪ D$ by inserting a smooth rational component $E$ at $q$, where $\{q_1\} := C ∩ E$ and $\{q_2\} := D ∩ E$. There also exist non-zero sections

$$\omega_D ∈ H^0(D, K_D(2i_q2)), \quad ω_E ∈ H^0(E, O_E(2g_2 – 2)), \quad ω_C ∈ H^0(C, K_C((2g_2 – 2)i_1)), $$

satisfying $\text{ord}_{q_1}(ω_C) + \text{ord}_{q_1}(ω_E) ≥ 2g_2 – 2$ and $\text{ord}_{q_2}(ω_E) + \text{ord}_{q_2}(ω_D) ≥ 2g_2 – 2$. Furthermore, $ω_E$ vanishes doubly at a point $p ∈ \{q_1, q_2\}^c$. Since $ω_C$ (respectively $ω_D$) also vanishes along the divisor $x_1 + \cdots + x_s$ (respectively $x_{s+1} + \cdots + x_{g_2 – 1}$), it follows that $\text{ord}_{q_1}(ω_C) ≤ 2g_2 – i – s$ and $\text{ord}_{q_2}(ω_D) ≤ i + s – 1$, hence by compatibility, $\text{ord}_{q_1}(ω_C) + \text{ord}_{q_2}(ω_D) ≥ 2g_2 – 3$. This rules out the possibility of a further double zero and shows that this case does not occur.

To summarize, keeping in mind that the $ψ$-coefficient of $[\text{Anstram}_{g}]$ is equal to $4g_2 – 8$, we find the relation

$$(7) \quad (2i_2 – 2s)b_{i_2} – sb_{i_2} + s(4g_2 – 8) = 4(g_i – i)(s_i – 2s) + a(i, s).$$

For $s = 0$, we have by convention $b_{i_2} = 0$, which gives $b_{i_2} = 2i^2 – 5i^2 – 3i + 4 + 1$. By induction, we find using recursion (7) the claimed formula for $b_{i, s}$. □

As already explained, having calculated the class $[\text{Anstram}_{g}] ∈ \text{Pic}(\overline{M}_{g, g – 1})$ and using known bound on the slope $s(M_g)$, one derives that $\mathcal{T}_g$ is of general type when $g ≥ 12$. We discuss the last cases in Theorem 0.2 and thus complete the birational classification of $\mathcal{T}_g$:

\textit{End of proof of Theorem 0.2} We noted in the Introduction that for $g ≤ 9$ the space $\mathcal{T}_g$ is unirational, being the image of a variety which is birational to a Grassmann bundle over the rational Mukai variety $V^{g_2 – 1}_g$. When $g ∈ \{10, 11\}$, the space $\overline{M}_{g, g – 1}$ is uniruled [FP]. This implies the uniruledness of $\mathcal{T}_g$ as well. □

3. The Kodaira Dimension of $\overline{C}_{g,n}$

In this section we provide results concerning the Kodaira dimension of the symmetric product $\overline{C}_{g,n}$, where $n ≤ g – 2$. There are two cases depending on the parity of the difference $g – n$. When $g – n$ is even, we introduce a subvariety inside $\overline{C}_{g,n}$ consisting of divisors $D ∈ C_n$ which appear in a fibre of a pencil of degree $(g + n)/2$ on a curve $[C] ∈ M_g$. We set integers $g ≥ 1$ and $1 ≤ m ≤ g/2$, then consider the locus

$$\mathcal{F}_{g,m} := \{[C, x_1, \ldots, x_{g – 2m}] ∈ \overline{M}_{g,g – 2m} : ∃ A ∈ W^1_{g – m}(C) \text{ with } H^0(C, A(– \sum_{j=1}^{g–2m} x_j)) ≠ 0\}.$$ 

A parameter count shows that $\mathcal{F}_{g,m}$ is expected to be an effective divisor on $\overline{M}_{g,g – 2m}$. We shall prove this, then compute the class of its closure in $\overline{M}_{g,g – 2m}$. 

□
Theorem 3.1. Fix integers $g \geq 1$ and $1 \leq m \leq g/2$, then set $n := g - 2m$ and $d := g - m$. The class of the compactification inside $\overline{M}_{g,g-2m}$ of the divisor $\mathcal{F}_{g,m}$ is given by the formula:

$$
\mathcal{F}_{g,m} = \left( \frac{10n}{g-2} \left( \frac{g-2}{d-1} \right) - \frac{n}{g} \left( \frac{g}{d} \right) \right) \lambda + \frac{n-1}{g-1} \left( \frac{g-1}{d-1} \right) \sum_{j=1}^{n} \psi_j - \frac{n}{g-2} \left( \frac{g-2}{d-1} \right) \delta_{irr} - \sum_{s=2}^{n} \left( \frac{s(n^2 - g + sgn - sn)}{2(g-1)(d-g)} \right) \left( \frac{g-1}{d} \right) \delta_{0.s} - \cdots \in \text{Pic}(\overline{M}_{g,n}).
$$

Proof. We fix a general curve $[C] \in M_g$ and consider the incidence correspondence

$$
\Sigma := \{ (D, A) \in C_{g-2m} \times W_{g-m}(C) : H^0(C, A \otimes \mathcal{O}_C(-D)) \neq 0 \},
$$

together with the projection $\pi_1 : \Sigma \to C_{g-2m}$. It follows from [F1] Theorem 0.5, that $\Sigma$ is pure of dimension $g - 2m - 1 = \rho(g,1,g-m) + 1$. To conclude that $\mathcal{F}_{g,m}$ is a divisor inside $\overline{M}_{g,g-2m}$, it suffices to show that the general fibre of the map $\pi_1$ is finite, which implies that $\phi^{-1}([C]) \cap \mathcal{F}_{g,m}$ is a divisor in $\phi^{-1}([C])$; we also note that the fibre $\phi^{-1}([C])$ is isomorphic to the $n$-th Fulton-Macpherson configuration space of $C$. We specialize to the case $D = (g - 2m) \cdot p$, where $p \in C$. One needs to show that for a general curve $[C] \in M_g$, there exist finitely many pencils $A \in W_{g-m}(C)$ with $h^0(C, A \otimes \mathcal{O}_C(-(g-m)p)) \geq 1$, for some point $p \in C$. This follows from [HM] Theorem B, or alternatively, by letting $C$ specialize to a flag curve consisting of a rational spine and $g$ elliptic tails, in which case the point $p$ specializes to a $(g-2m)$-torsion points on one of the elliptic tails (in particular it can not specialize to a point on the spine). For each of these points, the pencils in question are in bijective correspondence to points in a transverse intersection of Schubert cycles in $G(2, g-m+1)$. In particular their number is finite.

In order to compute the class $[\mathcal{F}_{g,m}]$, we expand it in the usual basis of $\text{Pic}(\overline{M}_{g,n})$

$$
\mathcal{F}_{g,m} \equiv a \lambda + c \sum_{i=1}^{g-2m} \psi_i - b_{irr} \delta_{irr} - \sum_{i,s \geq 0} b_{i,s} \delta_{i,s},
$$

then note that the coefficients $a, c$ and $b_{irr}$ respectively, have been computed in [F2] Theorem 4.9. The coefficient $b_{0,2}$ is determined by intersecting $\mathcal{F}_{g,m}$ with a fibral curve

$$
C_{x_n} := \{ [C, x_1, \ldots, x_{n-1}, x_n] : x_n \in C \} \subset \overline{M}_{g,n},
$$

corresponding to a general $(n-1)$-pointed curve $[C, x_1, \ldots, x_{n-1}] \in \overline{M}_{g,n-1}$. By letting the points $x_1, \ldots, x_{n-1} \in C$ coalesce to a point $q \in C$, points in the intersection $C_{x_n} \cap \mathcal{F}_{g,m}$ are in $1:1$ correspondence with points $x_n \in C$, such that $h^0(C, A(-(n-1)q - x_n)) \geq 1$. This number equals $(g - 2m - 1) \binom{g}{m}$, see [HM] Theorem A, that is,

$$
(2g + 2n - 4)c - (n - 1)b_{0,2} = C_{x_n} \cdot \mathcal{F}_{g,m} = (m + 1) \# \left\{ A \in W_{g-m}(C) : h^0(C, A \otimes \mathcal{O}_C(-(g-2m-1)q)) \geq 1 \right\} = (g - 2m - 1) \binom{g}{m},
$$

which determines $b_{0,2}$. The coefficients $b_{0,s}$ are computed recursively, by exhibiting an explicit test curve $\Gamma_{0,s} \subset \Delta_{0,s}$ which is disjoint from $\mathcal{F}_{g,m}$. We fix a general element
\[ [C, q, x_{s+1}, \ldots, x_n] \in \overline{M}_{g,n+1-s} \] and a general \( s \)-pointed rational curve \([\mathbf{P}^1, x_1, \ldots, x_s] \in \overline{M}_{0,s}\). We glue these curves along a moving point \( q \) lying on the rational component:
\[
\Gamma_{0,s} := \{ [\mathbf{P}^1 \cup q, C, x_1, \ldots, x_s, x_{s+1}, \ldots, x_n] : q \in \mathbf{P}^1 \} \subset \Delta_{0,s} \subset \overline{M}_{g,n}.
\]
Clearly, \( \Gamma_{0,s} \cdot \mathcal{F}_{g,m} = sc + (s - 2) b_{0,s} - s b_{0,s-1} \). We claim \( \Gamma_{0,s} \cap \mathcal{F}_{g,m} = \emptyset \). Assume that on the contrary, one can find a point \( q \in \mathbf{P}^1 \) and a limit linear series \( g_{ds} \) on \( \mathbf{P}^1 \cup q, C \),
\[
l = ((A, V_C), (O_{\mathbf{P}^1}(d), V_{\mathbf{P}^1})) \in G^1_d(C) \times G^1_d(\mathbf{P}^1),
\]
together with sections \( \sigma_C \in V_C \) and \( \sigma_{\mathbf{P}^1} \in V_{\mathbf{P}^1} \), satisfying \( \text{ord}_q(\sigma_C) + \text{ord}_q(\sigma_{\mathbf{P}^1}) \geq d \) and
\[
\text{div}(\sigma_C) \geq x_{s+1} + \cdots + x_n, \quad \text{div}(\sigma_{\mathbf{P}^1}) \geq x_1 + \cdots + x_s.
\]
Since \( \sigma_{\mathbf{P}^1} \neq 0 \), one finds that \( \text{ord}_q(\sigma_{\mathbf{P}^1}) \leq g - m - s \), hence by compatibility, \( \text{ord}_q(\sigma_C) \geq s \). We claim that this is impossible, that is, \( H^0(\mathcal{C}, A \otimes \mathcal{O}_C(-sq - x_1 - \cdots - x_n)) \neq 0 \), for every \( \mathcal{C} \in W^1_{g,m}(C) \). Indeed, by letting all points \( x_{s+1}, \ldots, x_n, q \in C \) coalesce, the statement \( H^0(\mathcal{C}, A \otimes \mathcal{O}_C(-(g - 2m) \cdot q)) = 0 \), for a general \([C, q] \in \overline{M}_{g,1}\) is a consequence of the “pointed” Brill-Noether theorem as proved in [EH1] Theorem 1.1. This shows that \( 0 = \Gamma_{0,s} \cdot \mathcal{F}_{g,m} = sc + (s - 2)b_{0,s} - sb_{0,s-1} \),
for \( 3 \leq s \leq n \), which determines recursively all coefficients \( b_{0,s} \). The remaining coefficients \( b_{0,i} \) with \( 1 \leq i \leq \lfloor g/2 \rfloor \) can be determined via similar test curve calculations, but we skip these details. \( \square \)

Keeping the notation from the proof of Theorem 3.1, a direct consequence is the calculation of the class of the divisor \( \mathcal{F}_{g,m}[C] := \pi_1(\Sigma) \) inside \( C_{g-2m} \). This offers an alternative proof of [Mus] Proposition III; furthermore the proof of Theorem 3.1 answers in the affirmative the question raised in loc.cit., concerning whether the cycle \( \mathcal{F}_{g,m}[C] \) has expected dimension, and thus, it is a divisor on \( C_{g-2m} \).

We denote by \( \theta \in H^2(C_{g-2m}, \mathbb{Q}) \) the class of the pull-back of the theta divisor, and by \( x \in H^2(C_{g-2m}, \mathbb{Q}) \) the class of the locus \( \{ p_0 + D : D \in C_{g-2m-1} \} \) of effective divisors containing a fixed point \( p_0 \in C \). For a very general curve \( [C] \in \mathcal{M}_g \), the group \( N^1(C_{g-2m}, \mathbb{Q}) \) is generated by \( x \) and \( \theta \), see [ACGH].

Let \( \overline{\mathcal{F}}_{g,m} \) be the effective divisor on \( \mathcal{C}_{g,g-2m} \) to which \( \mathcal{F}_{g,m} \) descends, that is, \( \pi^*(\overline{\mathcal{F}}_{g,m}) = \mathcal{F}_{g,m} \). The class of \( \overline{\mathcal{F}}_{g,m} \) is completely determined by Theorem 3.1.

**Corollary 3.2.** Let \( [C] \in \mathcal{M}_g \) be a general curve. The cohomology class of the divisor
\[
\mathcal{F}_{g,m}[C] := \{ D \in C_{g-2m} : \exists A \in W^1_{g,m}(C) \text{ such that } H^0(C, A \otimes \mathcal{O}_C(-D)) \neq 0 \}
\]
is equal to \((1 - \frac{2m}{g})(g \theta - \frac{a}{g-2m} x) \). In particular, the class \( \theta - \frac{a}{g-2m} x \in N^1(C_{g-2m}, \mathbb{Q}) \) is effective.

**Proof.** Let \( u : C_{g-2m} \dashrightarrow \mathcal{C}_{g,g-2m} \) be the rational map given by
\[
u(x_1 + \cdots + x_{g-2m}) = [C, x_1 + \cdots + x_{g-2m}].
\]
Note that \( u \) is well-defined outside the codimension 2 locus of effective divisors with support of length at most \( g - 2m - 2 \). We have that \( u^*(\overline{\delta}_{0,2}) = \delta_C \), where \( \delta_C := [\Delta_C]/2 \) is the reduced diagonal. Its class is given by the MacDonald formula, cf. [K1] Lemma 7:
\[
\delta_C \equiv -\theta + (2g - 2m - 1)x.
\]
Furthermore, \( u^*(\tilde{\psi}) \equiv \theta + \delta_C + (2m - 1)x \), see [K2] Proposition 2.7. Thus \( \mathcal{F}_{g,m}[C] \equiv u^*(\mathcal{F}_{g,m}) \), and the conclusion follows after some calculations. \( \square \)

The divisor \( \tilde{\mathcal{F}}_{g,m} \) is defined in terms of a correspondence between pencils and effective divisors on curves, and it is fibred in curves as follows: We fix a complete pencil \( A \in W_{g-m}^1(C) \) with only simple ramification points. The variety of secant divisors

\[
V_{g-2m}^1(A) := \{ D \in C_{g-2m} : H^0(C, A \otimes \mathcal{O}_C(-D)) \neq 0 \}
\]
is a curve (see [F1]), disjoint from the indeterminacy locus of the rational map \( u : C_{g-2m} \rightarrow \mathcal{T}_{g,g-2m}^n \). We set \( \Gamma_{g-2m}(A) := u(V_{g-2m}^1(A)) \cap \mathcal{T}_{g,g-2m}^n \). By varying \( [C] \in \mathcal{M}_g \) and \( A \in W_{g-m}^1(C) \), the curves \( \Gamma_{g-2m}(A) \) fill-up the divisor \( \tilde{\mathcal{F}}_{g,m} \). It is natural to test the extremality of \( \tilde{\mathcal{F}}_{g,m} \) by computing the intersection number \( \Gamma_{g-2m}(A) \cdot \tilde{\mathcal{F}}_{g,m} \). To state the next result in a unified form, we adopt the convention \( \binom{a}{b} := 0 \), whenever \( b < 0 \).

**Proposition 3.3.** For all integers \( 1 \leq m < g/2 \), we have the formula:

\[
\Gamma_{g-2m}(A) \cdot \tilde{\mathcal{F}}_{g,m} = (m-1) \binom{g-m-2}{m} \binom{g}{m}.
\]

In particular, \( \Gamma_{g-2}(A) \cdot \tilde{\mathcal{F}}_{g,1} = 0 \), and the divisor \( \tilde{\mathcal{F}}_{g,1} \in \text{Eff}(\mathcal{T}_{g,g-2}) \) is extremal.

**Proof.** This is an immediate application of Corollary 3.2. The class \([V_{g-2m}^1(A)]\) can be computed using Porteous’ formula, see [ACGH] p.342:

\[
[V_{g-2m}^1(A)] \equiv \sum_{j=0}^{g-2m-1} \binom{-m-1}{j} \frac{x^j \cdot g^{g-2m-j-1}}{(g-2m-1-j)!} \in H^{2(g-2m-1)}(C_{g-2m}, \mathbb{Q}).
\]

Using the push-pull formula, we write \( \Gamma_{g-2m}(A) \cdot \tilde{\mathcal{F}}_{g,m} = \mathcal{F}_{g,m}[C] \cdot [V_{g-2m}^1(A)] \), then estimate the product using the identity \( x^k g^{g-2m-k} = g!/((2m+k)! \in H^{2(g-2m)}(C_{g-2m}, \mathbb{Q}) \) for \( 0 \leq k \leq g-2m \). For \( m = 1 \), observe that \( \Gamma_{g-2}(A) \cdot \tilde{\mathcal{F}}_{g,1} = 0 \). Since the curves of type \( \Gamma_{g-2}(A) \) cover \( \tilde{\mathcal{F}}_{g,1} \), this implies that \( \tilde{\mathcal{F}}_{g,1} \) is extremal. \( \square \)

We can use Theorem 3.1 to describe the birational type of \( \overline{\mathcal{C}}_{g,n} \) when \( 12 \leq g \leq 21 \) and \( 1 \leq n \leq g-2 \). We recall that when \( g \leq 9 \), the space \( \overline{\mathcal{C}}_{g,n} \) is uniruled for all values of \( n \). The transition cases \( g = 10, 11 \), as well as the case of the universal Jacobian \( \overline{\mathcal{C}}_{g,g} \), are discussed in detail in [EV2]. Furthermore \( \overline{\mathcal{C}}_{g,n} \) is uniruled when \( n \geq g+1 \); in this case the symmetric product \( C_n \) of any curve \([C] \in \mathcal{M}_g \) is birational to a \( \mathbb{P}^{n-1} \)-bundle over the Jacobian \( \text{Pic}^n(C) \). Our main result is that, in the range described above, \( \overline{\mathcal{C}}_{g,n} \) is of general type in all the cases when \( \mathcal{M}_{g,n} \) is known to be of general type, see [Log], [F2]. We note however that the divisors \( \mathcal{F}_{g,m} \) only carry one a certain distance towards a full solution. The classification of \( \overline{\mathcal{C}}_{g,n} \) is complete only when \( n \in \{ g-1, g-2, g \} \).

**Theorem 3.4.** For integers \( g = 12, \ldots, 21 \), the universal symmetric product \( \overline{\mathcal{C}}_{g,n} \) is of general type for all \( f(g) \leq n \leq g-1 \), where \( f(g) \) is described in the following table.

| \( g \) | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|---|---|---|---|---|---|---|---|---|---|---|
| \( f(g) \) | 10 | 11 | 10 | 10 | 9 | 9 | 7 | 6 | 4 |
Proof. The strategy described in the Introduction to prove that $K_{\Theta_{g, g-1}}$ is big, applies to the other spaces $\bar{C}_{g,n}$ with $1 \leq n \leq g-2$ as well. To show that $\bar{C}_{g,n}$ is of general type, it suffices to produce an effective class on $\bar{C}_{g,n}$ which pulls back via $\pi$ to $a \lambda + c \sum_{i=1}^{n} \psi_{i} - b_{ir} \delta_{ir} - \sum_{i,s} b_{i,s} \delta_{i,s} \in \text{Eff}(\bar{M}_{g,n})^{3},$ such that the following conditions are fulfilled:

$$\frac{a + s(\bar{M}_{g})}{13c}(2c - b_{ir}) < 1 \quad \text{and} \quad \frac{b_{0,2}}{3c} > 1.$$

When $g - n$ is even, we write $g - n = 2m$, and for all entries in the table above one can express $K_{\bar{C}_{g,n}}$ as a positive combination of $\sum_{i=1}^{n} \psi_{i} \cdot [\bar{C}_{g,m}], \varphi^{*}(D)$, where $D \in \text{Eff}(\bar{M}_{g})$, and other boundary classes.

If $g - n = 2m + 1$ with $m \in \mathbb{Z}_{\geq 0}$, for each integer $1 \leq j \leq n + 1$, we denote by $\phi_{j} : \bar{M}_{g,n+1} \to \bar{M}_{g,n}$ the projection forgetting the $j$-th marked point and consider the effective $\mathbb{G}_{n}$-invariant effective $\mathbb{Q}$-divisor on $\bar{M}_{g,n}$

$$E := \frac{1}{n + 1} \sum_{j=1}^{n+1} \psi_{j}(\bar{C}_{g,m} : \delta_{0 : \{j,n+1\}}) \in \text{Eff}(\bar{M}_{g,n}).$$

Using Theorem 3.1 as well as elementary properties of push-forwards of tautological classes, $K_{\bar{C}_{g,n}}$ is expressible as a positive $\mathbb{Q}$-combination of boundaries, $[E]$, a pull-back of an effective divisor on $\bar{M}_{g}$, and the big and nef class $\sum_{i=1}^{n} \psi_{i}$ precisely in the cases appearing in the table. \hfill \Box

Remark 3.5. When $g \notin \{12, 16, 18\}$, the bound $s(\bar{M}_{g}) \leq 6 + 12/(g + 1)$, emerging from the slope of the Brill-Noether divisors, has been used to verify $(8)$. In the remaining cases, we employ the better bounds $s(\bar{M}_{12}) = 4415/642 < 6 + 12/13$ (see [FV1]), and $s(\bar{M}_{16}) = 407/61 < 6 + 12/17$ see [F2], coming from Koszul divisors on $\bar{M}_{12}$ and $\bar{M}_{16}$ respectively. On $\bar{M}_{18}$, we use the estimate $s(\bar{M}_{18}) \leq 302/45$ given by the class of the Petri divisor $P_{18,10}^{P}$, see [EH1]. Improvements on the estimate on $s(\bar{M}_{g})$ in the other cases, will naturally translate in improvements in the statement of Theorem 3.4.

4. THE UNIVERSAL Prym Theta DIVISOR IN GENUS 6

The aim of this last section is to establish the uniruledness of the universal theta divisor $\sigma : \Theta_{6} \to \mathcal{R}_{6}$, over the moduli space $\mathcal{R}_{6}$ classifying pairs $[C, \eta]$, where $C$ is a smooth curve of genus 6 and $\eta \in \text{Pic}^{0}(C)$ is a non-trivial 2-torsion point. It is proved in [DS] that the Prym map $P : \mathcal{R}_{6} \to A_{5}$ is generically finite of degree 27, thus in order to conclude that $\Theta_{5}$ is uniruled it suffices to establish the same conclusion for $\Theta_{6}$.

For a point $[C, \eta] \in \mathcal{R}_{g}$, we denote by $f : \tilde{C} \to C$ the unramified double cover induced by $\eta$ and by $i : \tilde{C} \to \bar{C}$ the involution interchanging the sheets of $f$. Setting $P := PH^{0}(K_{\tilde{C}})^{\vee}$, we view $P^{+} := PH^{0}(K_{C})^{\vee}$ as a subset of $P$. If

$$\text{Nm}_{f} : \text{Pic}^{2g-2}(\tilde{C}) \to \text{Pic}^{2g-2}(C)$$

is the norm map, then $P(C, \eta) := \text{Nm}_{f}^{-1}(K_{C})^{+}$ and one has the following realization for the Prym theta divisor:

$$\Xi(C, \eta) := \{ L \in \text{Pic}^{2g-2}(\tilde{C}) : \text{Nm}_{f}(L) = K_{C}, h^{0}(\tilde{C}, L) \geq 2, h^{0}(\bar{C}, L) \text{ is even} \}.$$

The universal Prym theta divisor $\Theta_{6}^{P}$ is the parameter space of triples $[C, \eta, L]$, where $[C, \eta] \in \mathcal{R}_{g}$ and $L \in \Xi(C, \eta)$ and $\sigma([C, \eta, L]) = [C, \eta]$. 
We fix a general element \([C, \eta, L] \in \Theta_p^6\), where \(h^0(\tilde{C}, L) = 2\). The set of divisors
\[
\{ f_*(D) : D \in |L| \} \subset |K_C|
\]
can be viewed as a conic in a \((g - 1)\)-dimensional projective space. Following \[Ve2\], to this conic one can associate its dual hypersurface \(Q_L \in \text{PSym}^2 H^0(K_C)\) which is a rank 3 quadric. Alternatively, viewing \(L \in W_{2g-2}(\tilde{C})\) as a singular point of the Riemann theta divisor \(\Theta_{\tilde{C}}\), we consider the projectivized tangent cone \(\hat{Q}_L \supset \tilde{C}\) of \(\Theta_{\tilde{C}}\) at the point \(\tilde{L}\), and then \(Q_L := \hat{Q}_L \cap P^6\). In coordinates, if \((s_1, s_2)\) is a basis of \(H^0(\tilde{C}, L)\), we have the following concrete description of the two quadrics:
\[
\hat{Q}_L : (s_1 \cdot \iota^*(s_1)) \cdot (s_2 \cdot \iota^*(s_2)) - (s_1 \cdot \iota^*(s_2))^2 = 0
\]
and
\[
Q_L : 4(s_1 \cdot \iota^*(s_1)) \cdot (s_2 \cdot \iota^*(s_2)) - (s_1 \cdot \iota^*(s_2) + s_2 \cdot \iota^*(s_1))^2 = 0.
\]
We summarize the properties of \(Q_L\), and refer to \[Ve2\] for details:

**Proposition 4.1.** For a general point \([C, \eta, L] \in \Theta_p^6\), there exists a rank 3 quadric \(Q_L \subset \P H^0(K_C)^2\) satisfying the following properties:

(i) The rulings of \(Q_L\) cut out the pencils \(L\) and \(\iota^*(L)\) on the curve \(\tilde{C}\).

(ii) \(Q_L : C = 2d_L\), for some Prym canonical divisor \(d_L \in |K_C \otimes \eta|\).

(iii) The divisor \(\iota^*(d_L)\) consists of points \(\tilde{x} \in \tilde{C}\) such that \(h^0(\tilde{C}, L(-\tilde{x} - \iota(\tilde{x})) = 1\).

(iv) For a general divisor \(d \in |K_C \otimes \eta|\), there exists \(L \in \mathcal{E}(C, \eta)\) such that \(d = \iota^*(d_L)\).

Specializing to the case \(g = 6\), we are in a position to prove that \(\Theta_p^6\) is uniruled.

**Proof of Theorem 0.1.** We fix a general point \([C, \eta, L] \in \Theta_p^6\). There exists a smooth del Pezzo surface \(S \subset P^5\) containing the canonical image of \(C\) and such that \(C \subset |-2K_S|\). Using Proposition 4.1 to \(L \in \mathcal{E}(C, \eta)\), one associates a Prym canonical divisor \(d_L \in |K_C \otimes \eta|\) and a quadric \(Q_L \subset P^5\). The curve \(C_L := S \cap Q_L\) is a smooth curve of genus 6; since it lies on a rank 3 quadric it is endowed with a vanishing theta-null induced from the ruling of \(Q_L\). Let \(P_L \subset |O_S(2)|\) be the pencil spanned by \(C\) and \(C_L\), or equivalently, the pencil on \(S\) that on the curve \(C\) cuts out the divisor \(2d_L\).

For each smooth curve \(C' \in P_L\), the fixed divisor \(d_L \in \text{Div}(C')\) can be viewed as belonging to the linear system \(|K_{C'} \otimes \eta'|\), for a certain point of order two \(\eta' \in \text{Pic}^0(C')[2]\). This induces a point \([C', \eta'] \in R_6\). Furthermore, if \(f' : \tilde{C}' \rightarrow C'\) is the induced covering, since \(\tilde{C}' \subset \hat{Q}_L\), the rulings of \(\hat{Q}_L\) cut out a line bundle \(L' \in \text{Pic}(\tilde{C}')\) such that \(\text{Nm}_{f'}(L') = K_{C'}\) and \(h^0(\tilde{C}', L') \geq 2\). Furthermore, \(h^0(\tilde{C}', L')\) is even, because the parity of line bundles with canonical norm do not mix in deformations, hence \(L' \in \mathcal{E}(C', \eta')\). It follows that the family of elements \([C', \eta', L']\) defines a rational curve in \(\Theta_p^6\) passing through a general point of the moduli space. 

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