FANO MANIFOLDS WITH LONG EXTREMAL RAYS

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Abstract. Let $X$ be a Fano manifold of pseudoindex $i_X$ whose Picard number is at least two and let $R$ be an extremal ray of $X$ with exceptional locus $\text{Exc}(R)$. We prove an inequality which bounds the length of $R$ in terms of $i_X$ and of the dimension of $\text{Exc}(R)$ and we investigate the border cases. In particular we classify Fano manifolds $X$ of pseudoindex $i_X$ obtained blowing up a smooth variety $Y$ along a smooth subvariety $T$ such that $\dim T < i_X$.

1. Introduction

A smooth complex projective variety of dimension $n$ is called Fano if its anticanonical bundle $-K_X = \wedge^n TX$ is ample. The index of $X$, $r_X$, is the largest natural number $m$ such that $-K_X = mH$ for some (ample) divisor $H$ on $X$, while the pseudoindex $i_X$ is defined as the minimum anticanonical degree of rational curves on $X$ and it is an integral multiple of $r_X$.

The pseudoindex is related to the Picard number $\rho_X$ of $X$ by a conjecture which claims that $\rho_X(i_X - 1) \leq n$, with equality if and only if $X \cong (\mathbb{P}^{i_X-1})^{\rho_X}$; this conjecture appeared in [8] as a generalization of a similar one (with the index in place of the pseudoindex) proposed by Mukai in 1988.

A first step towards the proof of this conjecture was made by Wiśniewski in [24], where he proved that if $i_X > \frac{n+2}{2}$ then $\rho_X = 1$. More recently several authors ([8], [21], [1], [10]) dealt with this problem but the general case is still open.

In this paper we investigate a related problem.

Let $X$ be a Fano manifold with $\rho_X > 1$ and let $R$ be an extremal ray of $X$. Let $l(R) := \min\{-K_X \cdot C | C \text{ a rational curve in } R\}$ be the length of $R$ and $\text{Exc}(R) = \{x \in X : x \in C \text{ a rational curve in } R\}$ be its exceptional locus. We first prove the following bound:

$$i_X + l(R) \leq \dim \text{Exc}(R) + 2,$$

which is an improved statement of the conjecture in the case $\rho_X = 2$.

Then we investigate the cases in which equality holds. Equivalently we ask if on a Fano variety of pseudoindex $i_X$ an extremal ray $R$ of maximal length does determine the structure of the variety. We prove the following

Theorem 1.1. Let $X$ be a Fano manifold of dimension $n$, pseudoindex $i_X$ and Picard number $\rho_X \geq 2$, and let $R$ be a fiber type or divisorial extremal ray such that

$$i_X + l(R) = \dim \text{Exc}(R) + 2.$$

Then $X \cong \mathbb{P}^k \times \mathbb{P}^{n-k}$ or $X \cong Bl_{\mathbb{P}^t}(\mathbb{P}^n)$ with $0 \leq t \leq \frac{n-3}{2}$.

We do not know how to prove a similar theorem if $R$ is an extremal ray whose associated contraction is small (i.e. $\dim \text{Exc}(R) \leq n - 2$). However if we replace in the assumptions the pseudoindex $i_X$ with the index $r_X$ then we have the following
Theorem 1.2. Let $X$ be a Fano manifold of dimension $n$, index $r_X$, and Picard number $\rho_X \geq 2$, and let $R$ be an extremal ray such that
\[ r_X + l(R) = \dim \text{Exc}(R) + 2. \]
Then, denoted by $e$ the dimension of $\text{Exc}(R)$, we have
\[ X = \mathbb{P}^{\rho_k}(\mathcal{O}^{\oplus e-k+1} \oplus \mathcal{O}(1)^{\oplus n-e}), \]
with $k = n - r + 1$.

Finally we consider the next step, namely the case
\[ i_X + l(R) = \dim \text{Exc}(R) + 1. \]
For a fiber type or divisorial extremal ray $R$ we prove that $\rho_X \leq 3$, describing the Kleiman-Mori cone of $X$ and classifying the varieties with $\rho_X = 3$, (Theorem 5.1).

If we assume moreover that $R$ is the ray associated to a smooth blow-up, we have a complete classification:

Theorem 1.3. Let $X$ be a Fano manifold and let $R$ an extremal ray whose associate contraction $\varphi_R : X \to Y$ is the blow up of a smooth subvariety $T \subset Y$, such that
\[ i_X + l(R) \geq n \quad \text{or equivalently} \quad i_X \geq \dim T + 1. \]
Then $X$ is one of the following
\begin{itemize}
  \item[a)] $\text{Bl}_{p^t}(\mathbb{P}^n)$, with $\mathbb{P}^t$ a linear subspace of dimension $\leq \frac{n}{2} - 1$.
  \item[b)] $\text{Bl}_{p^t}(\mathbb{Q}^n)$, with $\mathbb{P}^t$ a linear subspace of dimension $\leq \frac{n}{2} - 1$.
  \item[c)] $\text{Bl}_{Q^t}(\mathbb{Q}^n)$, with $\mathbb{Q}^t$ a smooth quadric of dimension $\leq \frac{n}{2} - 1$ not contained in a linear subspace of $\mathbb{Q}^n$.
  \item[d)] $\text{Bl}_p(V_d)$ where $V_d$ is $\text{Bl}_Y(\mathbb{P}^n)$ and $Y$ is a submanifold of dimension $n - 2$ and degree $\leq n$ contained in an hyperplane $H$ such that $p \notin H$.
  \item[e)] $\text{Bl}_{P^1 \times \{p\}}(P^1 \times \mathbb{P}^{n-1})$.
\end{itemize}
Note that if $T$ is a point the condition $i_X \geq \dim T + 1 = 1$ is empty. In this case the theorem is actually the main theorem of [7], where Fano varieties which are the blow-up at a point of a smooth variety are classified (those varieties correspond to cases a) and b) with $t = 0$ and d) of the above theorem). That paper has been for us a very important source of inspiration.

In the appendix we propose a slight variation of a result of [6] relating the pseudoindexes of two Fano manifolds one of which is the image of the other through a birational contraction.

2. Background material

In (2.1) and (2.2) we recall basic definitions and facts concerning Fano-Mori contractions and families of rational curves; our notation is consistent with the one in [17] to which we refer the reader.

Afterwards, in (2.3), for the reader’s convenience we recall some results of [11] and [14] which are frequently used in the rest of the paper.

2.1. Fano-Mori contractions. Let $X$ be a smooth complex Fano variety of dimension $n$ and let $K_X$ be its canonical divisor. By the Cone Theorem the cone of effective 1-cycles which is contained in the $\mathbb{R}$-vector space of 1-cycles modulo numerical equivalence, $\text{NE}(X) \subset N_1(X)$, is polyhedral; a face of $\text{NE}(X)$ is called an
extremal face and an extremal face of dimension one is called an extremal ray.

From the structure of the cone follows that

**Lemma 2.1.** Let $X$ be a Fano variety and $D$ an effective divisor on $X$. Then there exists an extremal ray $R \subset \text{NE}(X)$ such that $D \cdot R > 0$.

To an extremal face $\sigma$ is associated a morphism with connected fibers $\varphi_\sigma : X \to W$ onto a normal variety, which contracts the curves whose numerical class is in $\sigma$; $\varphi_\sigma$ is called an extremal contraction or a Fano-Mori contraction.

A Cartier divisor $H$ such that $H = \varphi_\sigma^* A$ for an ample divisor $A$ on $W$ is called a good supporting divisor of the map $\varphi_\sigma$ (or of the face $\sigma$).

An extremal ray $R$ (and the associated extremal contraction $\varphi_R$) is called numerically effective (nef for short) or of fiber type if $\dim W < \dim X$, otherwise the ray (and the contraction) is non nef or birational. This terminology is due to the fact that there exists an effective divisor $E$ such that $E \cdot R < 0$ if and only if the ray is not nef. If the codimension of the exceptional locus of a birational ray $R$ is equal to one the ray and the associated contraction are called divisorial, otherwise they are called small.

### 2.2. Families of rational curves.

Let $X$ be a normal projective variety and let $\text{Hom}(P^1, X)$ be the scheme parametrizing morphisms $f : P^1 \to X$. We consider the open subscheme $\text{Hom}_{\text{bir}}(P^1, X) \subset \text{Hom}(P^1, X)$, corresponding to those morphisms which are birational onto their image, and its normalization $\text{Hom}_{\text{bir}}^n(P^1, X)$. The group $\text{Aut}(P^1)$ acts on $\text{Hom}_{\text{bir}}^n(P^1, X)$ and the quotient exists.

**Definition 2.2.** The space $\text{Ratcurves}^n(X)$ is the quotient of $\text{Hom}_{\text{bir}}^n(P^1, X)$ by $\text{Aut}(P^1)$, and the space $\text{Univ}(X)$ is the quotient of the product action of $\text{Aut}(P^1)$ on $\text{Hom}_{\text{bir}}^n(P^1, X) \times P^1$.

**Definition 2.3.** We define a family of rational curves to be an irreducible component $V \subset \text{Ratcurves}^n(X)$. Given a rational curve $f : P^1 \to X$ we will call a family of deformations of $f$ any irreducible component $V \subset \text{Ratcurves}^n(X)$ containing the equivalence class of $f$.

Given a family $V$ of rational curves, we have the following basic diagram:

$$p^{-1}(V) =: U \xrightarrow{i} X$$

where $i$ is the map induced by the evaluation $ev : \text{Hom}_{\text{bir}}^n(P^1, X) \times P^1 \to X$ and $p$ is a $P^1$-bundle. We define $\text{Locus}(V)$ to be the image of $U$ in $X$; we say that $V$ is a covering family if $\text{Locus}(V) = X$.

If we fix a point $x \in X$ everything can be repeated starting from the scheme $\text{Hom}(P^1, X; 0 \mapsto x)$ parametrizing morphisms $f : P^1 \to X$ which send $0 \in P^1$ to $x$. Given a family $V \subseteq \text{Ratcurves}^n(X)$, we can consider the subscheme $V \cap \text{Ratcurves}^n(X, x)$ parametrizing curves in $V$ passing through $x$; we usually denote by $V_x$ a component of this subscheme.

**Definition 2.4.** Let $V$ be a family of rational curves on $X$. Then

(a) $V$ is unsplint if it is proper;
(b) $V$ is locally unsplit if for the general $x \in \text{Locus}(V)$ every component $V_x$ of $V \cap \text{Ratcurves}^n(X, x)$ is proper;
(c) $V$ is generically unsplit if there is at most a finite number of curves of $V$ passing through two general points of $\text{Locus}(V)$.

**Proposition 2.5.** \cite{IV.2.6} Let $X$ be a smooth projective variety and $V$ a family of rational curves. Assume either that $V$ is generically unsplit and $x$ is a general point in $\text{Locus}(V)$ or that $V$ is unsplit and $x$ is any point in $\text{Locus}(V)$ or that $x$ is a point such that $V_x$ is unsplit. Then
\begin{itemize}
  \item[(a)] $\dim X + \deg V \leq \dim \text{Locus}(V) + \dim \text{Locus}(V_x) + 1$;
  \item[(b)] $\deg V \leq \dim \text{Locus}(V_x) + 1$.
\end{itemize}

This last proposition, in case $V$ is the unsplit family of deformations of a minimal extremal rational curve, i.e. a curve of minimal degree in an extremal face of $X$, gives the fiber locus inequality:

**Proposition 2.6.** \cite{15}, \cite{25} Let $\varphi$ be a Fano-Mori contraction of $X$ and let $E = E(\varphi)$ be its exceptional locus; let $S$ be an irreducible component of a (non trivial) fiber of $\varphi$. Then
$$
\dim E + \dim S \geq \dim X + l - 1
$$
where
$$
l = \min\{-K_X \cdot C \mid C \text{ is a rational curve in } S\}.
$$
If $\varphi$ is the contraction of a ray $R$, then $l$ is called the length of the ray.

**Definition 2.7.** We define a Chow family of rational curves to be an irreducible component $V \subset \text{Chow}(X)$ parametrizing rational and connected 1-cycles. If $V$ is a family of rational curves, the closure of the image of $V$ in $\text{Chow}(X)$ is called the Chow family associated to $V$.

We say that $V$ is quasi-unsplit if every component of any reducible cycle in $V$ is numerically proportional to $V$.

Let $X$ be a smooth variety, $V^1, \ldots, V^k$ Chow families of rational curves on $X$ and $Y$ a subset of $X$.

**Definition 2.8.** We denote by $\text{Locus}(V^1, \ldots, V^k)_Y$ the set of points that can be joined to $Y$ by a connected chain of $k$ cycles belonging respectively to the families $V^1, \ldots, V^k$.

We denote by $\text{ChLocus}_m(V^1, \ldots, V^k)_Y$ the set of points that can be joined to $Y$ by a connected chain of at most $m$ cycles belonging to the families $V^1, \ldots, V^k$.

**Definition 2.9.** We define a relation of rational connectedness with respect to $V^1, \ldots, V^k$ on $X$ in the following way: $x$ and $y$ are in $\text{rc}(V^1, \ldots, V^k)$ relation if there exists a chain of rational curves in $V^1, \ldots, V^k$ which joins $x$ and $y$, i.e. if $y \in \text{ChLocus}_m(V^1, \ldots, V^k)_x$ for some $m$.

To the $\text{rc}(V^1, \ldots, V^k)$ relation we can associate a fibration, at least on an open subset:

**Theorem 2.10.** \cite{9}, \cite{IV.4.16} There exist an open subvariety $X^0 \subset X$ and a proper morphism with connected fibers $\pi : X^0 \to Z^0$ such that
\begin{itemize}
  \item[(a)] the $\text{rc}(V^1, \ldots, V^k)$ relation restricts to an equivalence relation on $X^0$;
  \item[(b)] the fibers of $\pi$ are equivalence classes for the $\text{rc}(V^1, \ldots, V^k)$ relation;
\end{itemize}
Theorem 2.11. \cite{20} Through every point of a Fano variety $X$ there exists a rational curve of anticanonical degree $\leq \dim X + 1$.

Remark 2.12. The families $\{V^i \subset \text{Ratcurves}^n(X)\}$ containing rational curves with degree $\leq \dim X + 1$ are only a finite number, so, for at least one index $i$, we have that $\text{Locus}(V^i) = X$. Among these families we choose one with minimal anticanonical degree, and we call it a minimal covering family.

Let $X$ be a Fano variety and $\pi : X^0 \to Z^0$ a proper surjective morphism on a smooth quasiprojective variety $Z^0$ of positive dimension.

A relative version of Mori’s theorem, \cite[Theorem 2.1]{13}, states that, for a general point $z \in Z^0$, there exists a rational curve $C$ on $X$ of anticanonical degree $\leq \dim X + 1$ which meets $\pi^{-1}(z)$ without being contained in it (an horizontal curve, for short).

As in remark 2.12 we can find a family $V$ of horizontal curves such that $\text{Locus}(V)$ dominates $Z^0$ and $\text{deg} V$ is minimal among the families with this property. Such a family is called a minimal horizontal dominating family for $\pi$.

Lemma 2.13. \cite[Lemma 6.5]{11} Let $X$ be a Fano variety, let $\pi : X \dashrightarrow Z$ be the fibration associated to a $\text{rc}(V^1, \ldots, V^k)$ relation and let $V$ be a minimal horizontal dominating family for $\pi$. Then

\begin{itemize}
  \item[(a)] curves parametrized by $V$ are numerically independent from curves contracted by $\pi$;
  \item[(b)] $V$ is locally unsplit;
  \item[(c)] if $x$ is a general point in $\text{Locus}(V)$ and $F$ is the fiber containing $x$, then $\dim(F \cap \text{Locus}(V_x)) = 0$.
\end{itemize}

2.3. Chains of rational curves, numerical equivalence and cones. In this subsection we present some results concerning the dimension, the maximum number of numerically independent curves and the cone of curves of subsets of the form $\text{Locus}(V^1, \ldots, V^k)_Y$ or $\text{ChLocus}(V^1, \ldots, V^k)_Y$ when $V^1, \ldots, V^k$ are unsplit families and $Y$ is chosen in a suitable way.

Definition 2.14. Let $V^1, \ldots, V^k$ be unsplit families on $X$. We will say that $V^1, \ldots, V^k$ are numerically independent if the numerical classes $[V^1], \ldots, [V^k]$ are linearly independent in the vector space $N^1(X)$. If moreover $C \subset X$ is a curve we will say that $V^1, \ldots, V^k$ are numerically independent from $C$ if in $N^1(X)$ the class of $C$ is not contained in the vector subspace generated by $[V^1], \ldots, [V^k]$.

The following lemma is a generalization of proposition 2.5 and of \cite[Theoreme 5.2]{8}.

Lemma 2.15. \cite[Lemma 5.4]{11} Let $Y \subset X$ be a closed subset and $V$ an unsplit family. Assume that curves contained in $Y$ are numerically independent from curves in $V$, and that $Y \cap \text{Locus}(V) \neq \emptyset$. Then for a general $y \in Y \cap \text{Locus}(V)$

\begin{itemize}
  \item[(a)] $\dim \text{Locus}(V)_Y \geq \dim(Y \cap \text{Locus}(V)) + \dim \text{Locus}(V_y)$;
  \item[(b)] $\dim \text{Locus}(V)_Y \geq \dim Y + \text{deg} V - 1$.
\end{itemize}
Moreover, if $V^1, \ldots, V^k$ are numerically independent unsplit families such that curves contained in $Y$ are numerically independent from curves in $V^1, \ldots, V^k$ then either
\[ \text{Locus}(V^1, \ldots, V^k)_Y = \emptyset \]
or
\[ \text{dim} \text{Locus}(V^1, \ldots, V^k)_Y \geq \text{dim} Y + \sum \text{deg} V^i - k. \]

**Notation:** Let $S$ be a subset of $X$. We write $N_1(S) = \langle [V^1], \ldots, [V^k] \rangle$ if the numerical class in $X$ of every curve $C \subset S$ can be written as $[C] = \sum a_i [C_i]$, with $a_i \in \mathbb{Q}$ and $C_i \subset V^i$. We write $\text{NE}(S) = \langle [V^1], \ldots, [V^k] \rangle$ (or $\text{NE}(S) = \langle [R^1], \ldots, [R^k] \rangle$) if the numerical class in $X$ of every curve $C \subset S$ can be written as $[C] = \sum a_i [C_i]$, with $a_i \in \mathbb{Q}_{\geq 0}$ and $C_i \subset V^i$ (or $[C_i] \in R_i$).

**Lemma 2.16.** [21] Lemma 1] Let $Y \subset X$ be a closed subset and $V$ an unsplit family of rational curves. Then every curve contained in $\text{Locus}(V)_Y$ is numerically equivalent to a linear combination with rational coefficients
\[ \lambda C_Y + \mu C_V, \]
where $C_Y$ is a curve in $Y$, $C_V$ belongs to the family $V$ and $\lambda \geq 0$.

**Corollary 2.17.** [11] Corollary 4.4] If $X$ is rationally connected with respect to some (quasi) unsplit families $V^1, \ldots, V^k$ then $N_1(X) = \langle [V^1], \ldots, [V^k] \rangle$.

**Proposition 2.18.** [11] Corollary 4.2, [11] Corollary 2.23]
(a) Let $V$ be a quasi-unsplit family of rational curves and $x$ a point in $\text{Locus}(V)$. Then $\text{NE}(\text{ChLocus}_m(V)_x) = \langle [V] \rangle$.
(b) Let $V$ be a family of rational curves and $x$ a point in $X$ such that $V_x$ is unsplit. Then $\text{NE}(\text{Locus}(V_x)) = \langle [V] \rangle$.
(c) Let $\sigma$ be an extremal face of $\text{NE}(X)$, $F$ a fiber of the associated contraction and $V$ an unsplit family independent from $\sigma$. Then $\text{NE}(\text{ChLocus}_m(V)_F) = \langle [\sigma], [V] \rangle$.

**Corollary 2.19.** Let $D \subset X$ be an effective divisor and $V$ an unsplit family numerically independent from curves in $D$ such that $D \cdot V > 0$; then, for every $x \in \text{Locus}(V)$ we have $\dim \text{Locus}(V_x) = 1$; in particular, if $V$ is the family of deformations of a minimal extremal rational curve in a ray $R$ then every non trivial fiber of $\varphi_R$ is one dimensional.

**Proof.** Since $D \cdot V > 0$, for every $x \in \text{Locus}(V)$ we have $D \cap \text{Locus}(V_x) \neq \emptyset$, and so $\dim (D \cap \text{Locus}(V_x)) \geq \dim \text{Locus}(V_x) - 1$. It follows that $\dim \text{Locus}(V_x) = 1$, since a curve in the intersection would be a curve in $D$ not independent from $V$. □

## 3. Some technical results

In order to make the exposition clearer, we collect in this section some technical lemmata we will use in the proofs of the main theorems.

**Lemma 3.1.** Let $X$ be a smooth projective variety, $A \subset X$ a subvariety of dimension $m$ and $V$ a covering family of rational curves for $X$.
Suppose that for a point $x \in X \setminus A$ the family $V_x$ is unsplit and that $\dim \text{Locus}(V_x) \geq m + 2$. Then the general curve in $V_x$ does not meet $A$.

**Proof.** We can assume that $V_x$ is irreducible; consider the diagram
and the inverse image $i^{-1}(A)$; since $V_x$ is unsplit the map $i$ is finite to one away from $i^{-1}(x)$, so it is finite to one when restricted to $i^{-1}(A)$ and so $\dim i^{-1}(A) = \dim A = m$. It follows that $i^{-1}(A)$ cannot dominate $V_x$ which has dimension $= \dim \text{Locus}(V_x) - 1 \geq m + 1$. □

**Corollary 3.2.** Suppose that a Fano variety $X$ is the blow up of a Fano variety $Y$ along a smooth subvariety $T$ such that $\dim T \leq i_X + l(R) - 3$ and denote by $E$ the exceptional divisor. Then, if $V_Y$ is a minimal dominating family of rational curves for $Y$ and $V^*$ is a family of deformations of the strict transform of a general curve in $V_Y$ we have $E \cdot V^* = 0$.

**Proof.** Suppose by contradiction that $E \cdot V^* > 0$. By the canonical bundle formula we have

$$-K_Y \cdot V_Y = -K_X \cdot V^* + l(R)E \cdot V^*,$$

hence

$$-K_Y \cdot V_Y \geq i_X + l(R) \geq \dim T + 3.$$ 

We can therefore apply lemma 3.1 and obtain that the general curve of $V_Y$ does not meet $T$, a contradiction with $E \cdot V^* > 0$. □

**Lemma 3.3.** Let $X$ be a Fano variety whose cone of curves is generated by a divisorial extremal ray $R_1$ with exceptional locus $E$ and a fiber type extremal ray $R_2$, and let $V$ be a quasi unsplit covering family of rational curves. Then $[V] \in R_2$; in particular $E \cdot V^* > 0$.

**Proof.** Consider the rcV fibration $X \dashrightarrow Z$. By proposition 2.18 we have $\dim Z > 0$ since $V$ is quasi unsplit and $\rho_X = 2$.

Then $V$ is extremal by [11] Lemma 2.28 since $X$ has not small contractions.

The last assertion follows from lemma 2.7 since $E \cdot R_1 < 0$. □

**Lemma 3.4.** Let $X$ be a Fano variety of dimension $n$ and pseudoindex $i_X \geq 2$ whose cone of curves is generated by a divisorial extremal ray $R_1$ with exceptional locus $E$ and by a fiber type extremal ray $R_2$. Suppose that $l(R_1) + i_X \geq n$ and that there exists a covering family $V$ of rational curves of degree $\leq n + 1$ such that $E \cdot V = 0$.

Then $V$ is not quasi unsplit and all the reducible cycles in the associated Chow family $V$ have two irreducible components, $C_1$ and $C_2$, where $C_1$ and $C_2$ are curves in the rays $R_1$ and $R_2$ respectively.

**Proof.** First of all we note that, since $E \cdot V = 0$, by lemma 3.3 $V$ is not quasi unsplit.

Let $C = \sum C_i$ be a reducible cycle in $V$. At least one of the components of $C$, let it be $C_1$, has negative intersection with $E$; in fact, if $E \cdot C_i = 0$ for every $i$ the effective divisor $E$ would be numerically trivial on the whole NE($X$) since $\rho_X = 2$.

Denote by $V^1$ a family of deformations of $C_1$; if $V^1$ is not unsplit then there exists
a reducible cycle $\sum C_i$ in $\mathcal{V}$, and for at least one of the components, call it $C_{11}$, we have $E \cdot C_{11} < 0$.

Denote by $V^{11}$ a family of deformations of $C_{11}$. If $V^{11}$ is not unsplit, we repeat the argument, and the procedure terminates because $-K_X \cdot V > -K_X \cdot V^1 > -K_X \cdot V^{11} > \cdots > 0$.

Therefore every reducible cycle $\sum C_i$ in $\mathcal{V}$ has an irreducible component on which $E$ is negative and such that its family of deformations is unsplit.

Let $\Gamma$ be one of these components and $W$ a family of deformations of $\Gamma$; since $E \cdot \Gamma < 0$ we have $\text{Locus}(W) \subset E$. We claim that $[W] \in R_1$. Assume by contradiction that $W$ is independent from $R_1$.

Denoted by $F$ a fiber of $\varphi_{R_1}$ meeting $\text{Locus}(W)$, by lemma 2.15 we have

$$n - 1 \geq \dim \text{Locus}(W)_F \geq i_X - 1 + \dim F \geq n - 1.$$ 

This forces $\text{Locus}(W) = E$, so $F \subset \text{Locus}(W)$ and we can apply part a) of lemma 2.15 and get

$$n - 1 = i_X - 1 + \dim F = \dim \text{Locus}(W)_F \geq \dim F + \dim \text{Locus}(W_y)$$

which implies that dim $\text{Locus}(W_y) = i_X - 1$ so $W$ is covering, a contradiction.

Therefore $[W] \in R_1$ and for every reducible cycle $\sum C_k$ in $\mathcal{V}$ we have

$$1 + \dim \text{Locus}(W)_F \geq \dim F + \dim \text{Locus}(W_y)$$

(1)

$$n + 1 \geq -K_X \cdot V = -K_X \cdot \sum C_k \geq \rho(R_1) + (k - 1)i_X \geq n.$$ 

Hence $k = 2$ and every reducible cycle has two components, $C_1$, which belongs to $R_1$ and $C_2$.

From (1) it also follows that $-K_X \cdot V \geq n$, and this implies that $V$ is not locally unsplit. To prove this fact we assume by contradiction that $V$ is locally unsplit.

If $-K_X \cdot V = n + 1$, then $\rho_X = 1$ by proposition 2.18 b), while if $-K_X \cdot V = n$ for a general $x \in X$ $D_x = \text{Locus}(V)_x$ is a divisor; this divisor is zero on $R_1$ by corollary 2.19 since $R_1$ has fibers of dimension $\geq 2$ and, by proposition 2.18 b), $\text{NE}(D_x) = ([V])$.

On the other hand $\varphi(D_x)$ is an effective, hence ample divisor on $Y$, so it meets the center of the blow up which has positive dimension. It follows that $D \cap E \neq \emptyset$; this, together with $D \cdot R_1 = 0$ implies that $D$ contains fibers of $\varphi_{R_1}$, a contradiction with $\text{NE}(D_x) = ([V])$.

Since $V$ is covering, not locally unsplit and $\text{Locus}(V_1) = E$, the family $V_2$ of deformations of $C_2$ is a covering family; we have $-K_X \cdot V_2 \leq i_X + 1 < 2i_X$, so $V_2$ is an unsplit family and therefore, by lemma 2.3 its numerical class belongs to the ray $R_2$.

4. A BOUND ON THE LENGTH

Lemma 4.1. Let $X$ be a Fano manifold with $\rho_X \geq 2$, let $R$ be an extremal ray of $X$ and denote by $\text{Exc}(R)$ its exceptional locus. Then there exists a family of rational curves $V$ independent from $R$ such that, for some $x \in \text{Exc}(R)$, $V_x$ is unsplit.

Moreover, if $R$ is not nef and $W$ is a minimal covering family, then, among the families of deformations of irreducible components of cycles in $W$, there is a family $V$ as above and one of the following happens

a) $\text{Exc}(R) \subset \text{Locus}(V)$.

b) There exists a reducible cycle $C_R + \sum_{i=1}^k C_i$ in $W$ with $[C_R] \in R$.
Proof. If $R$ is a nef ray it’s enough to choose $V$ as the family of deformation of a minimal extremal rational curve in any ray $R_1 \neq R$, so we can assume that $R$ is not nef.

Let $W$ be a minimal covering family for $X$. Note that, since $W$ is covering, it is certainly independent from $R$: in fact, since $R$ is not nef there exists an effective divisor $H$ such that $H \cdot R < 0$ so curves whose numerical class is in $R$ are contained in $H$.

If there exists $x \in \text{Exc}(R)$ such that $W_x$ is unsplit then we are done, otherwise for every $x \in \text{Exc}(R)$ there exists in $W$ a reducible cycle $\sum_i C_i$, with rational components, passing through $x$.

Denote by $T'$ the families of deformations of the curves $C_i$; since the number of such families is finite, for at least one index $j$ we have $\text{Exc}(R) \subseteq \text{Locus}(T')$. If $T'$ is independent from $R$ then let $W' = T'$, otherwise let $C_j + \sum_{i \neq j} C_i$ be a reducible cycle in $W$ passing through a point $x \in \text{Exc}(R)$. Since $[W] = [C_j + \sum_{i \neq j} C_i]$ is independent from $R$ and every component which is proportional to $R$ is contained in $\text{Exc}(R)$ there exists an irreducible component $C_k$ independent from $R$ which meets $\text{Exc}(R)$. In this case denote by $W^1$ the family of deformations of $C_k$.

We have thus found a family $W^1$ which is independent from $R$ such that $\text{Locus}(W^1) \cap \text{Exc}(R) \neq \emptyset$. Moreover either we can choose $W^1$ such that $\text{Exc}(R) \subseteq \text{Locus}(W^1)$ or there exists a reducible cycle in $W$ with one component belonging to $R$. Let $x_1 \in \text{Locus}(W^1) \cap E$. If $W_{x_1}$ is unsplit we are done, otherwise we repeat the argument.

Since $n + 1 > \deg W > \deg W^1 > \cdots > 0$ the procedure terminates. \hfill \Box

Proof of inequality (*). Let $x \in \text{Exc}(R)$ and $V$ be as in lemma 4.1. Let $\varphi_R : X \to Y$ be the extremal contraction associated to $R$ and let $F_x$ be the fiber of $\varphi_R$ which contains $x$. The numerical class of every curve in $F_x$ is in $R$ and, by proposition 2.18, b) the numerical class of every curve in $\text{Locus}(V_x)$ is proportional to $[V]$ so, since $V$ is independent from $R$, we have $\dim \text{Locus}(V_x) \cap F_x = 0$.

Moreover, by inequalities 2.5 and 2.6, we have $\dim \text{Locus}(V_x) \geq i_X - 1$ and $\dim F_x \geq \dim X - \dim \text{Exc}(R) + l(R) - 1$. Combining these inequalities we get

$$
(2) \quad \dim X \geq \dim \text{Locus}(V_x) + \dim F_x \\
\geq i_X + \dim X - \dim \text{Exc}(R) + l(R) - 2
$$

which gives

$$
i_X + l(R) \leq \dim \text{Exc}(R) + 2,$$

and the proposition is proved. \hfill \Box

5. The border cases

Proof of 4.1. First of all note that, since the length of a fiber type extremal ray is $\leq n + 1$, equality holding if and only if $X \simeq \mathbb{P}^n$, and the length of a birational extremal ray is $\leq n - 1$, the assumptions of the theorem imply $i_X \geq 2$.

Let $V$ the family given by lemma 4.1 let $x \in \text{Exc}(R)$ be a point such that $V_x$ is unsplit and let $F_x$ be the fiber of $\varphi_R$ containing $x$. If equality holds in (*), then
equality holds everywhere in (2); in particular we have

\begin{align*}
(3) & \quad \dim F_x = l(R) + \dim X - \dim \text{Exc}(R) - 1 \\
(4) & \quad \dim \text{Locus}(V_x) = i_X - 1.
\end{align*}

The last equality, together with inequality (2.5) yields that \( \dim \text{Locus}(V) = n \), so \( V \) is a covering family, and that \( \deg V = i_X \), so \( V \) is unsplit.

If \( \varphi_R : X \to Y \) is of fiber type then we can apply [21 Theorem 1] to get that \( X \cong \mathbb{P}^{1x-1} \times \mathbb{P}^{l(R)-1} \).

Suppose now that \( \varphi_R : X \to Y \) is divisorial and call \( E \) the divisor \( \text{Exc}(R) \).

By (3) we have \( \dim F_x = l(R) \); note that, since \( V \) is covering and unsplit, this equality holds for every \( F \), hence we can apply [2 Theorem 5.1] and we obtain that \( \varphi_R \) is the blow up of \( Y \) along a smooth subvariety \( T \).

Let \( F \) be any fiber of \( \varphi_R \); by lemma [21, b) we have

\[ \dim \text{Locus}(V)_F \geq \dim F + i_X - 1 \geq n, \]

so, by proposition [21, c) we have \( NE(X) = \langle [R], [R_V] \rangle \), where \( R_V \) is the ray spanned by the numerical class of \( V \).

The target \( Y \) of \( \varphi_R \) is a smooth variety with \( \rho_Y = 1 \) covered by rational curves, hence a Fano variety; let \( V_Y \) be a minimal dominating family of rational curves for \( Y \) and let \( V^* \) be the family of deformations of the strict transform of a general curve in \( V_Y \). By corollary [16, we have \( E \cdot V^* = 0 \), hence, by lemma [16], the family \( V^* \) is not quasi unsplit and all the reducible cycles in the associated Chow family \( V^* \) have two irreducible components, \( C_R \) and \( C_V \), where \( C_R \) and \( C_V \) are curves in the rays \( R \) and \( R_V \), respectively. In particular

\begin{align*}
(5) & \quad n + 1 \geq -K_Y \cdot V_Y = -K_X \cdot V^* = -K_X \cdot (C_R + C_V) \geq l(R) + i_X = n + 1, \quad \text{and} \quad Y \cong \mathbb{P}^n \text{ by the proof of [16 Theorem 1.1].} \tag{Note that the assumptions of the quoted result are different, but the proof actually works in our case, since for a very general } y \text{ the pointed family } V_{Y_y} \text{ has the properties 1-3 in [16 Theorem 2.1]).}
\end{align*}

By equation (5) we also have \( -K_X \cdot C_R = l(R) \) and \( -K_X \cdot C_V = i_X \), so \( C_R \) and \( C_V \) are minimal extremal rational curves; in particular \( E \cdot C_R = -1 \) and therefore, since \( E \cdot V^* = 0 \) we have \( E \cdot C_V = 1 \).

Let \( \psi : X \to Z \) be the contraction of the ray \( R_V \); we know that \( E \cdot C_V > 0 \), so every fiber of \( \psi \) meets a fiber \( F \) of \( \varphi_R \) and therefore its dimension is \( n - \dim F = i_X - 1 \), since fibers of different extremal ray contractions can meet only in points.

Let now \( G \) be a general fiber of \( \psi \); \( G \) is smooth, and, by adjunction

\[ K_G + (\dim G + 1)E_G = \mathcal{O}_G, \]

so \( G \) is a projective space and \( E \cap G \) is an hyperplane which dominates \( T \). Therefore \( T \) is a projective space by [19 Theorem 4.1].

The bound on the dimension of \( T \) follows from the fact that \( \dim T = n - l(R) - 1 = i_X - 2 \) and \( 2i_X \leq l(R) + i_X = n + 1. \quad \square \)

**Theorem 5.1.** Let \( X \) be a Fano manifold of Picard number \( \rho_X \geq 2 \), and let \( R \) a fiber type or divisorial extremal ray such that

\[ i_X + l(R) = \dim \text{Exc}(R) + 1. \]

Then \( \rho_X \leq 3 \) and \( \rho_X = 3 \) if and only if \( X \) is...
Since projective bundle globally by \([13, \text{Lemma 2.12}]\).

Let \(V\) be the fiber of \(X\). A straightforward computation shows that \(\mathcal{R}\) is a projective bundle. By \([12, \text{Corollary 0.4}]\) or \([16, \text{Theorem 1.1}]\),

\(\mathcal{R}\) is a projective bundle. By \([12, \text{Theorem 5.1}]\),

we have \(\dim \mathcal{R} \geq 1\).

Proof. If \(i_X = 1\) and \(R\) is divisorial we have \(\dim \mathcal{R} \geq n - 1\) so, by \([2, \text{Theorem 1.1}]\), \(X\) is the blow up at a point of a variety \(X'\); by \([11, \text{Theorem 1.1}]\) we are in case c) or in case d).

If \(i_X = 1\) and \(R\) is of fiber type then \(\dim \mathcal{R} = n\); in particular \(\varphi_R : X \to B\) is equidimensional with \(n - 1\)-dimensional fibers over a smooth curve \(B\). The general fiber of \(\varphi_R\) is a projective space by \([12, \text{Corollary 0.4}]\) or \([10, \text{Theorem 1.1}]\).

Over an open Zariski subset \(U\) of \(B\) the morphism \(p\) is a projective bundle. By taking the closure in \(X\) of a hyperplane section of \(p\) defined over the open set \(U\) we get a global relative hyperplane section divisor (we use \(\rho(X/B) = 1\) hence \(p\) is a projective bundle globally by \([13, \text{Lemma 2.12}]\).

Since \(X\) is a Fano manifold \(B \simeq \mathbb{P}^1\). Write \(X = \mathbb{P}^1(\oplus \mathcal{O}(a_i))\) with \(0 \leq a_0 \leq a_1 \leq a_{n-1}\). A straightforward computation shows that \(X\) is Fano if and only if either all the \(a_i\) are zero or all the \(a_i\) but the last are zero and \(a_{n-1} = 1\). In the first case \(X = \mathbb{P}^1 \times \mathbb{P}^{n-1}\) and \(i_X = 1\), in the second case \(X = Bl_{\mathbb{P}^{n-1}}(\mathbb{P}^n)\).

From now on we can assume \(i_X \geq 2\).

Let \(V\) the family given by \([11, \text{Lemma 2.11}]\) let \(x \in \text{Exc}(\mathcal{R})\) be a point such that \(V_x\) is unsplit and let \(F_x\) be the fiber of \(\varphi_R\) containing \(x\). First of all we prove that \(V\) is an unsplit family. In fact, if \(V\) were not unsplit then \(-K_X \cdot V \geq 2i_X - 1\) and \(\dim \text{Locus}(V_x) \geq 2i_X - 1\).

In this case we would have

\[
\dim \text{Locus}(V_x) + \dim F_x \geq 2i_X - 1 + \dim \mathcal{R} - 1 \geq n + i_X - 1 > n
\]

and so \(\dim \text{Locus}(V_x) \cap F_x \geq 1\), a contradiction, since \(V\) is independent from \(R\).

Now we divide the proof in two cases, according to the type of \(R\).

Case 1: \(R\) is nef.

Recall that, according to the proof of lemma \([11, \text{Lemma 2.11}]\) in this case \(V\) is the family of deformations of a minimal extremal rational curve in a ray \(R_1\) different from \(R\).

Suppose that \(R_1\) is not nef; by inequality \([2, \text{Inequality 2.6}]\) if \(F\) is a fiber of the associated contraction we have \(\dim F \geq i_X\) and, by lemma \([2, \text{Lemma 2.16}]\)

\[
\dim \text{Locus}(R)_F \geq \dim F + \dim \mathcal{R} - 1 \geq i_X + \dim \mathcal{R} - 1 = n.
\]

It follows that \(\dim F = i_X\) and \(X = \text{Locus}(R)_F\), so \(\text{NE}(X) = \langle [R], [R_1]\rangle\) by proposition \([2, \text{Proposition 2.15}]\c)

Since \(\dim F = i_X = l(R_1)\) for every fiber of the contraction associated to \(R_1\), this contraction is a smooth blow up by \([2, \text{Theorem 5.1}]\).
We can repeat the second part of the proof of theorem 1.1 exchanging \( R_1 \) and \( R \) and obtain that \( X = Bl_{[p(R) - 2]}(\mathbb{P}^n) \), so we are in case e).

Suppose now that \( R_1 \) is nef and consider the rc\((R, R_1)\) fibration \( \pi_{R,R_1} : X \rightarrow Z \). Let \( F \) be a general fiber of \( \pi_{R,R_1} \) and \( x \in F \) a point; \( F \) contains \( \text{Locus}(R, R_1)_x \) which has dimension \( \geq i_X + l(R) - 2 = n - 1 \) by lemma 2.10, so \( \text{dim} Z \leq 1 \).

Suppose that \( \text{dim} Z = 1 \) and let \( V' \) be a minimal horizontal dominating family for \( \pi_{R,R_1} \); by lemma 2.13 c) \( \text{dim} \text{Locus}(V')_x = 1 \) and so \( -K_X \cdot V' = i_X \).

In particular \( V' \) is unsplit and, by 2.5 covering. We can apply [21, Theorem 1] to conclude that \( X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^{n-2} \) and we are in case a).

If \( \text{dim} Z = 0 \) then \( X \) is rc\((R, R_1)\)-connected and \( \rho_X = 2 \) by corollary 2.17 in this case we clearly have \( \text{NE}(X) = \langle [R], [R_1] \rangle \).

Case 2: \( R \) is not nef.

Let \( W \) be a minimal covering family for \( X \) and let \( V \) be a family as in lemma 1.1 chosen among the families of deformations of irreducible components of cycles in \( \mathcal{W} \).

Step 1 \( V \) is an unsplit covering family.

Let \( x \in \text{Exc}(R) \) be a point such that \( V_x \) is unsplit and let \( F_x \) be the fiber of \( \phi_x \) containing \( x \). Since \( V \) is independent from \( R \), we have \( \text{dim} \text{Locus}(V_x) \cap F_x = 0 \), hence \( \text{dim} \text{Locus}(V_x) \leq n - \text{dim} F_x \), giving rise to the following chain of inequalities:

\[-K_X \cdot V - 1 \leq \text{dim} \text{Locus}(V_x) \leq n - \text{dim} F_x \leq n - l(R) \leq i_X.\]

This implies that \( -K_X \cdot V \leq i_X + 1 \) and therefore that \( V \) is unsplit, since we are assuming that \( i_X \geq 2 \).

Suppose that \( V \) is not a covering family. Then, by inequality 2.4 \( \text{dim} \text{Locus}(V_x) \geq 2 \) and therefore \( E := \text{Exc}(R) \) is not contained in \( \text{Locus}(V) \). In fact, in this case, by lemma 2.15 a) we would have \( \text{dim} \text{Locus}(V)_F \geq \text{dim} F + \text{dim} \text{Locus}(V_x) = n \), a contradiction.

So we are in case b) of lemma 1.1 and there exists a reducible cycle \( C_R = \sum C_i \) in \( \mathcal{W} \) with \([C_R] \in R \). Hence we have

\[n \geq K_X \cdot W \geq -K_X \cdot (C_R + \sum_{i=1}^k C_i) \geq l(R) + k i_X \geq n + (k - 1) i_X\]

forcing \( -K_X \cdot W = n \) and \( k = 1 \).

We have thus proved that in \( \mathcal{W} \) there exists a reducible cycle \( C_R + C_V \), with \( C_R \) in \( R \) and \( C_V \) in \( V \).

Let \( D = \text{Locus}(W_x) \) for a general \( x \in X \); by proposition 2.18 \( \text{NE}(D) = \langle [W] \rangle \).

By corollary 2.10 since the fibers of \( \phi_x \) are at least two dimensional we have \( D \cdot R = 0 \); by the same corollary, since \( \text{dim} \text{Locus}(V_x) \geq 2 \) we have \( D \cdot V = 0 \). This implies also that \( D \cdot W = D \cdot (C_R + C_V) = 0 \).

By lemma 2.4 there exists an extremal ray \( R_1 \) such that \( D \cdot R_1 > 0 \); let \( V^1 \) be a family of deformations of a minimal curve in \( R_1 \). By lemma 2.19 b) we have \( \text{dim} \text{Locus}(V^1)_D \geq \text{dim} D + i_X - 1 \geq n \), hence \( X = \text{Locus}(V^1)_D \) and \( \rho_X = 2 \).

This is a contradiction, since \( D \) is zero on \( R \) and \( V \) and so, if \( \rho_X = 2 \) it would be zero on the entire cone. Therefore \( V \) is a covering family as claimed. \( \square \)

Step 2 \( \rho_X \leq 3 \).
Let $F$ be a fiber of $\varphi_R$; by lemma 2.15 b) we have
\[
\dim \text{Locus}(V)_F \geq \dim F + i_X - 1 \geq n - 1
\]
If $X = \text{Locus}(V)_F$ then, by proposition 2.18 c) $\text{NE}(X) = \langle [R], [V] \rangle$ and we are done. Note that this is always the case if $\dim F > l(R)$, so we assume from now on that $\varphi_R$ is equidimensional with fibers of dimension $l(R)$, hence it is a smooth blow up by [2] Theorem 5.1.

An irreducible component of $\text{Locus}(V)_F$ is thus a divisor $D \subset X$ such that $\text{NE}(D) = \langle [R], [V] \rangle$. If $D \cdot V > 0$ then $X = \text{ChLocus}_2(V)_F$ and $\text{NE}(X) = \langle [R], [V] \rangle$ again by proposition 2.18 c), so we can assume $D \cdot V = 0$.

By lemma 2.1 there exists an extremal ray $R_1$ such that $D \cdot R_1 > 0$.

If $R_1 \not\subset \text{NE}(D)$ then, by lemma 2.16 b), denoted by $V^1$ a family of deformations of a minimal extremal rational curve in $R_1$, we have $\dim \text{Locus}(V^1)_D = n$. By lemma 2.16 $N_1(X) = \langle [R], [V], [V^1] \rangle$, so $\rho_X \leq 3$, equality holding if and only if $R_1$ is not contained in the vector subspace of $N_1(X)$ spanned by $R$ and $[V]$.

If $R_1 \subset \text{NE}(D)$ then $R_1 = R$ because $D \cdot V = 0$. It follows that $\text{Locus}(R)_D = E$, so $N_1(E) = \langle [R], [V] \rangle$.

If $E \cdot V > 0$ then $\text{Locus}(V)_E = X$ and $N_1(X) = \langle [R], [V] \rangle$ by lemma 2.16, so $\rho_X = 2$. We claim that we cannot have $E \cdot V = 0$; in fact, in this case every curve of $V$ which meets $E$ is entirely contained in $E$, so $E = \text{Locus}(V)_F = D$ and we have $D \cdot R < 0$. Recalling that $D \cdot V = 0$ we have that $D$ is not positive on $\text{NE}(D)$, a contradiction, since we are assuming $R_1 \subset \text{NE}(D)$ and $D \cdot R_1 > 0$.

**Step 3** $\rho_X = 2$, description of the cone.

We have to prove that $\text{NE}(X) = \langle [R], [R_1] \rangle$ where $R_1$ is a fiber type extremal ray. By step two this is the case if for a fiber $F$ of $\varphi_R$ either we have $X = \text{Locus}(V)_F$ or an irreducible component of $\text{Locus}(V)_F$ is a divisor $D$ such that $D \cdot V > 0$. We can therefore assume that an irreducible component of $\text{Locus}(V)_F$ is a divisor $D$ such that $D \cdot V = 0$; moreover we know that there exists an extremal ray $R_1$ of $X$ on which $D$ is positive.

If $R_1 \not\subset \text{NE}(D)$ then $\text{NE}(X) = \langle [R], [R_1] \rangle$ and moreover, by corollary 2.19 the contraction associated to $R_1$ has one dimensional fibers, and so it is of fiber type, since $i_X \geq 2$.

If $R_1 \subset \text{NE}(D)$ then $R_1 = R$ thus, if $V$ is not extremal, $D$ is negative on an extremal ray $R_2$, and so $\text{Exc}(R_2) \subset D$, against $\text{NE}(D) = \langle [R], [V] \rangle$. Therefore $V$ is extremal and $\text{NE}(X) = \langle [R], [V] \rangle$.

**Step 4** $\rho_X = 3$, description of the cone.

By step two, if $\rho_X = 3$, then $\text{Locus}(V)_F$ has dimension $n - 1$; moreover, denoted by $D$ one irreducible component of $\text{Locus}(V)_F$ we have $D \cdot V = 0$ and $D \cdot R_1 > 0$ for a ray $R_1$ not contained in the vector subspace of $N_1(X)$ spanned by $R$ and $[V]$. Since $\text{NE}(D) = \langle [R], [V] \rangle$, by corollary 2.19 every fiber of the contraction associated to $R_1$ is one dimensional. Combining this with $i_X \geq 2$, by inequality 2.6 we have that $V^1$ is a covering unsplit family.

By lemma 2.16 denoting again by $F$ a fiber of $\varphi_R$ we have $\dim \text{Locus}(V^1)_F = \dim \text{Locus}(V^1, V)_F = n$, so $X = \text{Locus}(V^1)_F = \text{Locus}(V^1, V)_F$.

We can write $X = \text{Locus}(V^1)_F = \text{Locus}(V)_{\text{Locus}(V^1)_F}$ and therefore, by lemma 2.16 and proposition 2.18 the numerical class of every curve in $X$ can be written as
a linear combination $a[V] + b[V^1] + c[R]$ with $b,c \geq 0$.
On the other hand $X = \text{Locus}(V^1, V)_F = \text{Locus}(V^1)\text{Locus}(V)_F$, so the numerical class of every curve in $X$ can be written as a linear combination $a[V] + b[V^1] + c[R]$ with $a,c \geq 0$. By the uniqueness of the decomposition it follows that $\text{NE}(X) = \langle [V], [V^1], [R] \rangle$.

**Step 5** If $\rho_X = 3, i_X \geq 2$ and $R$ is not nef then $X \simeq Bl_{\mathbb{P}^1 \times \{p\}}(\mathbb{P}^1 \times \mathbb{P}^{n-1})$.

We have thus proved that the cone of curves of $X$ is generated by $R$, which is the ray associated to a smooth blow up $\varphi_R : X \to Y$, and by other two fiber type extremal rays, call them $R_1$ and $R_2$, which both have length two. In particular we have $i_X = 2$, so $l(R) = n - 2$ and $\dim F_R = n - 2$ for every fiber of $\varphi_R$.
Moreover, since $E = \text{Exc}(R)$ is non negative on $R_1$ and $R_2$, by [25 Proposition 3.4] $Y$ is a Fano variety.
The effective divisor $E$ is positive on at least one of the rays $R_i$ by lemma [24] let us assume that $E \cdot R_1 > 0$. Let $\sigma$ be the extremal face spanned by $R$ and $R_1$ and consider the associated contraction $\varphi_\sigma$.
Let $x \in X$ be a point, let $\Gamma_1$ be a curve in $R_1$ through $x$ and let $F$ be a fiber of $\varphi_R$ meeting $\Gamma_1$. The fiber of $\varphi_\sigma$ through $x$ contains $\text{Locus}(R_1)_F$, which has dimension $n - 1$ by lemma [2.15], so the target of $\varphi_\sigma$ is a smooth curve, which has to be rational since $X$ is Fano. We have a commutative diagram

![Diagram](image)

The general fiber $F_\sigma$ of $\varphi_\sigma$ is, by adjunction, a Fano variety of index $\geq 2$ which has a divisorial extremal ray of dimension $F_\sigma - 1$, so, by theorem [14] $F_\sigma \simeq Bl_Y \mathbb{P}^{n-1}$.
It follows that the general fiber of $\varphi_\sigma$ is $\mathbb{P}^{n-1}$. The Fano variety $Y$ has a fiber type extremal ray $\psi_\sigma$ of length $\dim Y$ while the other ray is of fiber type, since the associated contraction contracts the images of curves in $R_2$. Therefore $i_Y \geq 2$.
We can thus apply theorem [14] to conclude that $Y \simeq \mathbb{P}^1 \times \mathbb{P}^{n-1}$. Let $T \simeq \mathbb{P}^1$ be the center of the blow up; we claim that $T$ is a fiber of the projection $Y \to \mathbb{P}^{n-1}$.
By contradiction, assume that this is not the case. Let $C \simeq \mathbb{P}^1$ be a fiber of the projection $Y \to \mathbb{P}^{n-1}$ meeting $T$ and let $C$ be the strict transform of $C$.
By the canonical bundle formula we have

$$-K_X \cdot \tilde{C} = -K_Y \cdot C - l(R)E \cdot \tilde{C} \leq 2 - l(R) \leq 0,$$
and so $X$ is not a Fano variety, a contradiction. \qed

6. **Blow ups**

**Proof of [13]** If $i_X + l(R) = n + 1$, by theorem [11] we have that $X = Bl_p(\mathbb{P}^n)$, with $t \leq \frac{n-3}{2}$.

We can thus assume that $i_X + l(R) = n$. By theorem [24] if $\rho_X \geq 3$, then $X$ is either $Bl_{\mathbb{P}^1 \times \{p\}}(\mathbb{P}^1 \times \mathbb{P}^{n-1})$ or $Bl_p(V_d)$ where $V_d$ is $Bl_Y(\mathbb{P}^n)$ and $Y$ is a submanifold of dimension $n - 2$ and degree $\leq n$ contained in an hyperplane which does not contain $p$. Note that case a) of theorem [13] has been excluded since it is not a blow up.
We can thus assume, from now on, that \( \rho_X = 2 \); again by theorem 5.41, either \( X \cong B_1(P^n) \) or \( \rho_X = 2 \), \( i_X \geq 2 \) and the cone of curves of \( X \) is generated by \( R \) and by a fiber type extremal ray \( R_Y \). (Case e) of theorem 5.41 has been excluded since in that case \( R \) is a fiber type ray).

The target \( Y \) of \( \varphi_R \) is a smooth variety with \( \rho_Y = 1 \) covered by rational curves, hence a Fano variety; let \( V_Y \) be a minimal dominating family of rational curves for \( Y \) and let \( V^* \) be the family of deformations of the strict transform of a general curve in \( V_Y \). The center of the blow up \( T \), has dimension \( \leq \dim Y - 3 \) since

\[
\text{codim } T - 1 = l(R) \geq i_X \geq 2,
\]

therefore we can apply corollary 5.32 and obtain that \( E \cdot V^* = 0 \).

Since \( E \cdot V^* = 0 \), by lemma 5.1, the family \( V^* \) is not quasi unsplit and all the reducible cycles in the associated Chow family \( V^* \) have two irreducible components, \( C_1 \) and \( C_2 \), where \( C_1 \) and \( C_2 \) are curves in the rays \( R \) and \( R_Y \) respectively.

Let \( \Gamma_R \) and \( \Gamma_Y \) be curves in \( R \) and \( R_Y \) respectively with minimal anticanonical degree. Since \( \varphi_R \) is a smooth blow up \( E \cdot \Gamma_R = -1 \), hence the numerical class of every curve in \( R \) is an integral multiple of \( [\Gamma_R] \); in particular we can write \( [C_1] = m_1[\Gamma_R] \) with \( m_1 \) a positive integer. By the canonical bundle formula

\[
(6) \quad n + 1 \geq -K_Y \cdot V_Y = -K_X \cdot V^* = -K_X \cdot (C_1 + C_2) \geq m_1 l(R) + i_X \geq (m_1 - 1) l(R) + n.
\]

Recalling that \( l(R) \geq i_X \geq 2 \) we have \( m_1 = 1 \), i.e. \( [C_1] = [\Gamma_R] \). It follows that \( E \cdot C_2 = 1 \), so \( [C_2] = [\Gamma_Y] \) and \( [V^*] = [\Gamma_R + \Gamma_Y] \).

Consider now the contraction of \( R_Y \), \( \psi : X \to Z \) and let \( F \) be any fiber of \( \psi \). Since \( E \cdot \Gamma_Y > 0 \) the fiber \( F \) meets a fiber \( F_R \) of \( \varphi_R \) and therefore \( \dim F \leq n - \dim F_R = i_X \).

On the other hand, by inequality 5.5 \( \dim F \geq l(R_Y) - 1 \geq i_X - 1 \), so the length of \( R_Y \) is either \( i_X \) or \( i_X + 1 \). In the first case, by equation (6) we have \(-K_Y \cdot V_Y = n \), while in the second we have \(-K_Y \cdot V_Y = n + 1 \).

The contraction \( \psi \) is supported by \( K_X + i_X E \) in the first case and by \( K_X + (i_X + 1) E \) in the second; in both cases, since for every fiber of \( \psi \) we have \( i_X - 1 \leq \dim F \leq i_X \), the target variety \( Z \) is smooth by [3, Theorem 4.1].

The general fiber of \( \psi \) has dimension either \( i_X - 1 \) or \( i_X \), so the dimension of \( Z \) is either \( l(R) + 1 \) or \( l(R) \). We divide the proof in two cases, accordingly.

**Case 1** \( \dim Z = l(R) + 1 \).

In this case \( \psi \) is supported by \( K_X + i_X E \), its general fiber has dimension \( i_X - 1 \) and it is a projective space \( P^{i_X - 1} \) by [3, Theorem 4.1], while jumping fibers, if they exist, have dimension \( i_X \) and are projective spaces \( P^{i_X} \), again by [3, Theorem 4.1].

We claim that, for at least one fiber \( F \) of \( \psi \), we have \( E \cap F = P^{i_X - 1} \). The claim is clearly true if either \( E \) contains a fiber of dimension \( i_X - 1 \) or, being \( E \cdot \Gamma_Y = 1 \), if \( \psi \) has a jumping fiber (\( E \) cannot contain a jumping fiber \( F \), otherwise, by lemma 2.15 a) we will have \( \dim E \geq \dim \text{Locus}(R)_F \geq i_X + l(R) \geq n \)).

Suppose by contradiction that neither of these two possibilities happens. The restriction of \( \psi \) to \( E \) is thus an equidimensional morphism with general fiber a projective space, such that \( E \) restricted to the general fiber is \( O_{P}(1) \), so \( \psi \) makes \( E \) a projective bundle over \( Z \).
Therefore $E$, which is also a projective bundle over $T$, has two projective bundle structures and $\rho_E = 2$, so, by [22, Theorem 2], $E$ is the projectivization of the tangent bundle of a projective space, but this is impossible since the two fibrations of $E$ have fibers of dimension $i_X - 2$ and $l(R)$ and these two dimensions are different, being $l(R) \geq i_X$, so the claim is proved.

It follows that either $\psi$ has a jumping fiber or $E$ contains a fiber of $\psi$; in both cases $T$, the center of the blow up, is dominated by the intersection of $E$ with this fiber, and so it is a projective space of dimension $i_X - 1$ by [19, Theorem 4.1].

To finish the proof, we have to show that $Y \simeq \mathbb{Q}^n$, and we will do this proving the existence of a line bundle $L_Y \in \text{Pic}(Y)$ such that $-K_Y = nL_Y$ and applying the Kobayashi-Ochiai theorem.

Take a line $l$ in $T$ and denote by $Y_l$ the inverse image $\varphi_{-1}^{-1}(l)$; $Y_l$ is a projective bundle over a smooth rational curve, so a toric variety. The restriction $\psi|_{Y_l} : Y_l \to Z$ is thus a surjective morphism from a toric variety to a smooth variety with Picard number one, so $Z$ is a projective space by [22, Theorem 1].

Let $L$ be the line bundle $\psi^*O_Y(1) + E$; we have $L \cdot R = 0$ and therefore there exists $L_Y \in \text{Pic}(Y)$ such that $\varphi_{R}^*L_Y = L$.

Moreover, since $L \cdot V^* = 1$ we have $L_Y \cdot V_Y = 1$, so, recalling that $-K_Y \cdot V_Y = \dim Y$ we get $-K_Y = nL_Y$ with and we conclude that $Y \simeq \mathbb{Q}^n$ by the Kobayashi-Ochiai theorem.

**Case 2** \quad $\dim Z = l(R)$.

In this case, as noted above, every fiber of $\psi$ has dimension $i_X + 1$. The contraction $\psi$ is supported either by $K_X + i_XE$ and it is a projective bundle or by $K_X + (i_X + 1)E$ and it is a quadric bundle, by [8, Theorem 4.1].

Every fiber of $\varphi_R$ dominates $Z$ so, by [19, Theorem 4.1] $Z$ is a projective space.

Let $L$ be the line bundle $\psi^*O_Y(1) + E$; we have $L \cdot R = 0$ so there exists $L_Y \in \text{Pic}(Y)$ such that $\varphi_{R}^*L_Y = L$.

Moreover, since $L \cdot V^* = 1$ we have $L_Y \cdot V_Y = 1$.

**Case 2a** \quad $\psi : X \to Z$ is a projective bundle.

In this case $-K_Y \cdot V_Y = n + 1$, so $-K_Y = (n + 1)L_Y$ and $Y$ is a projective space. The intersection of $E$ with the general fiber of $\psi$ is thus a projective space and therefore the center $T$ of the blow up is a linear space by [19, Theorem 4.1].

**Case 2b** \quad $\psi : X \to Z$ is a quadric bundle.

In this case $-K_Y \cdot V_Y = n$, so $-K_Y = nL_Y$ and $Y$ is a smooth quadric by the Kobayashi-Ochiai theorem.

The intersection of $E$ with the general fiber of $\psi$ is thus a smooth quadric, so the center $T$ of the blow up is either a linear space or a smooth quadric by [23].

Actually the first case can be excluded by direct computation, since the blow up of a quadric along a linear subspace is not a quadric bundle over $\mathbb{P}^r$.

In the second case let $\Pi \simeq \mathbb{P}^1$ be the linear subspace of dimension $i_X$ of $\mathbb{P}^{n+1}$ which contains $T \simeq \mathbb{Q}^{i_X-1}$.

Two cases are possible: either $Y \supseteq \Pi$ or $Y \cap \Pi = T$. The first case has to be excluded because, if $Y \supseteq \Pi$ the blow up of $\mathbb{Q}^n$ along $T$ does not give rise to a Fano variety.
To see this, take a line \( l \subset \Pi \) not contained in \( T \); by the canonical bundle formula, if \( X = Bl_T \mathbb{P}^n \) we have
\[
-K_X \cdot \tilde{l} = -K_Y \cdot l - l(R)E \cdot \tilde{l} \leq n - 2l(R) \leq 0.
\]

Finally note that in both cases the bound on the dimension of the center follows from the fact that \( i_X \leq l(R) \) and so \( 2i_X \leq l(R) + i_X \leq n \). \( \square \)

7. Varieties with a polarization

**Proof of 1.2** Let \( V \) the family given by lemma [2.11] let \( x \in \text{Exc}(R) \) be a point such that \( V_x \) is unsplit and let \( F_x \) be the fiber of \( \varphi_R \) containing \( x \).

First of all we prove that \( \rho_X = 2 \) and that the cone of curves of \( X \) is generated by \( R \) and by the ray spanned by \([V]\).

We are assuming that equality holds in (*), so equality holds everywhere in (2); in particular we have
\[
\text{(7)} \quad \dim F_x = l(R) + \dim X - \dim \text{Exc}(R) - 1 = \dim X - r_X + 1
\]
\[
\text{(8)} \quad \dim \text{Locus}(V_x) = r_X - 1.
\]

This forces \( \deg V = r_X \), so the family \( V \) is unsplit. Moreover, by inequality (2.4) \( V \) is a covering family.

Therefore, by lemma 2.15 we have \( \dim \text{Locus}(V)_{F_x} \geq \dim F_x + r_X - 1 = \dim X \), so, by proposition 2.18 c), we have \( NE(X) = [\{\text{int}(X), |R|\}] \).

Let \( \psi : X \to Z \) be the contraction of the ray \( R_V \) spanned by \([V]\), which is of fiber type since \( V \) is a covering family; curves parametrized by \( V \) have anticanonical degree \( r_X \), so they are minimal extremal curves in \( R_V \) which has length \( r_X \).

By inequality (2.4) every fiber of \( \psi \) has dimension \( \geq l(R_V) - 1 = r_X - 1 \), so \( Z \leq n - r_X + 1 \). Again by inequality (2.4) the fibers of \( \varphi_R \) have dimension \( \geq n - e + 1 - l - 1 = n - r_X + 1 \), so they dominate \( Z \). In particular every fiber of \( \psi \) meets a fiber \( F_R \) of \( \varphi_R \) and so its dimension is \( \leq \dim X - \dim F_R = r_X - 1 \); therefore the contraction \( \psi : X \to Z \) is equidimensional.

Moreover we also have that the dimension of every fiber of \( \varphi_R \) is \( \leq \dim Z \leq n - r_X + 1 \), so \( \varphi_R \) is equidimensional with fibers of dimension \( n - r_X + 1 \) and \( \dim Z = n - r_X + 1 \).

Denote by \( H \) the divisor such that \( -K_X = r_X H \). The general fiber \( G \) of \( \psi \) is, by generic smoothness and adjunction, a projective space \( \mathbb{P}^{r_X - 1} \) and \( H_G \simeq \mathcal{O}(1) \), so, by [2] Lemma 2.12, \( \psi \) is a projective bundle over \( Z, X = \mathbb{P}_Z(\mathcal{E}) \), with \( \mathcal{E} = \varphi_R H \).

In particular \( Z \) is a smooth Fano variety of Picard number one.

The canonical bundle formula yields
\[
\psi^*(K_Z + \det \mathcal{E}) = K_X + r_X H = \mathcal{O}_X,
\]
and so \( -K_Z = \det \mathcal{E} \). Note also that, if \( C_R \) is a curve in \( R \) then
\[
\text{(9)} \quad H \cdot C_R = \frac{-K_X \cdot C_R}{r_X} \geq \frac{l(R)}{r_X}.
\]

Let \( V_Z \) be a minimal covering family for \( Z \) and \( C \) a curve in \( V_Z \); Let \( \nu : \mathbb{P}^1 \to C \subset Z \) be the normalization of \( C \) and let \( Z_C \) be the fiber product \( Z_C = \mathbb{P}^1 \times_C X \).
The cone of curves \( \text{NE}(Z) \) following way

Denote by \( m \) the maximum index \( \nu \) such that \( a_i \leq a_{i+1} \) \( \forall i \).

The variety \( Z \) is a projective bundle over \( \mathbb{P}^1 \), \( Z = \mathbb{P}^1(\nu\mathcal{E}) \); the vector bundle \( \nu\mathcal{E} \) is ample, so we can write \( \nu\mathcal{E} \cong \bigoplus_{i=0}^{r_X-1} \mathcal{O}(a_i) \) with \( a_i > 0 \) and \( a_i \leq a_{i+1} \) \( \forall i \).

The cone of curves \( \text{NE}(Z) \) is generated by the class of a line in a fiber of \( p \) and by the class of a section \( \nu \mathcal{E} \).

Therefore \( \nu \mathcal{E} \) is ample, so we can write \( \nu\mathcal{E} \cong \bigoplus_{i=0}^{r_X-1} \mathcal{O}(a_i) \).

It follows that the dimension of \( \text{NE}(Z) \) is in \( \text{NE}(X) \).

The morphism \( \bar{\nu} \) induces a map of spaces of cycles \( N_1(Z_C) \to N_1(X) \) which allows us to identify \( \text{NE}(Z) \) with a subcone of \( \text{NE}(X) \).

Since \( \bar{\nu}(Z_C) \) contains lines in the fibers of \( \psi \) and contains curves in the fibers of \( \varphi_R \) (since for dimensional reasons \( \dim(\bar{\nu}(Z) \cap F_R) \geq 1 \)), we have an identification \( \text{NE}(Z) \cong \text{NE}(X) \).

In particular \( F_R \cap \bar{\nu}(Z_C) \), which is a curve whose numerical class in \( X \) is a multiple of \( [\Gamma_R] \), is the image of a curve \( \Gamma \) whose numerical class in \( Z_C \) is a multiple of \( [C_0] \).

By [21] Lemma 3 | the curve \( \Gamma \) is the union of disjoint minimal sections, so \( \bar{\nu}(Z_C) \cap F_R \) consists of the images via \( \bar{\nu} \) of disjoint minimal sections.

On the other hand, if \( C_0 \) is a minimal section, then \( \bar{\nu}(C_0) \) is a curve whose numerical class is in \( R \), so it is contained in a fiber of \( \varphi_R \).

It follows that the dimension of \( \varphi_R(\text{Exc}(R)) \) is the dimension of the space parametrizing minimal sections, which is \( m \).

Therefore

\[
m = \dim \text{Exc}(R) - \dim F_R = l(R) + r_X - 2 - \dim F_R.
\]

Moreover, since \( [C_0] \in R \) we have, by equation 9

\[
a_0 = H \cdot C_0 \geq \frac{l(R)}{r_X},
\]

hence \( a_i \geq a_0 + 1 \geq \frac{l(R)}{r_X} + 1 \) for \( i = m + 1, \ldots, r_X - 1 \).

It follows that

\[
\dim Z + 1 \geq -K_Z \cdot C = \det \mathcal{E} \cdot C =
\]

\[
= (m + 1)a_0 + \sum_{m+1}^{r_X-1} a_i \geq (m + 1) \frac{l(R)}{r_X} + (r_X - m - 1) \left( \frac{l(R)}{r_X} + 1 \right) =
\]

\[
= l(R) + r_X - m - 1 = \dim F_R + 1 = \dim Z + 1.
\]

Therefore \( Z \) admits a minimal dominating family of degree \( \dim Z + 1 \), hence \( Z \) is a projective space of dimension \( n - r_X + 1 \) by the proof of [10] Theorem 1.1].

Since equality holds everywhere we also have \( a_0 = 1, a_i = 2 \) \( i = m + 1, \ldots, r_X - 1 \), so the splitting type of \( \mathcal{E} \) on lines of \( Z \) is uniform.

If \( \dim \text{Exc}(R) \leq \dim X - 2 \) then \( \text{rk} \mathcal{E} = r_X \leq l(R) < n - r + 1 = \dim Z \), therefore \( \mathcal{E} \) is decomposable by [14] and \( \mathcal{E} \cong \bigoplus_{i=0}^{r_X-1} \mathcal{O}(1) \bigoplus \mathcal{O}(2) \).
If \( \dim \text{Exc}(R) = \dim X - 1 \) then \( \text{rk} \mathcal{E} = \dim Z \) and the splitting type of \( \mathcal{E} \) is \( (1, \ldots, 1, 2) \), so, by [14], either \( \mathcal{E} \) is decomposable or \( \mathcal{E} \) is the tangent bundle of \( Z = \mathbb{P}^{\dim Z} \), but the second case has to be excluded since \( X \) has a divisorial contraction. Finally, if \( \text{Exc}(R) = X \) then the splitting type of \( \mathcal{E} \) is \( (1, \ldots, 1) \), so \( \mathcal{E} \) is decomposable by [4] Proposition 1.2 and \( X \) is a product of projective spaces. \( \square \)

**Proposition 7.1.** Let \( X \) be a Fano variety of Picard number \( \rho_X = 2 \), index \( r_X \geq 2 \), and let \( R \) a fiber type or divisorial extremal ray such that \( r_X + l(R) = \dim \text{Exc}(R) + 1 \). Then, if \( R \) is divisorial either \( X \) is as in theorem 1.3 or \( X \) has the structure of a projective bundle over a smooth variety.

If \( R \) is of fiber type then \( X \) is a projective bundle or a quadric bundle or the projectivization of a \( \tilde{\text{B}}\text{änică} \) sheaf over a smooth variety \( Y \).

**Proof.** By theorem 5.1 either \( \text{Bl}_{\mathbb{P}(l(R) - 2)(\mathbb{P}^n)} \) or the cone of curves \( \text{NE}(X) \) is generated by \( R \) and by a fiber type extremal ray; let \( \psi : X \to Z \) be the contraction of this ray. Let \( H \) be the line bundle such that \( -K_X = r_X H \), let \( A \in \text{Pic}(Z) \) be an ample divisor and let \( H' = H + \psi^* A \). The contraction \( \psi \) is supported by \( K_X + r_X H' \).

If \( R \) is divisorial then every fiber of \( \varphi_R \) has dimension \( \geq l(R) \). If equality holds for every fiber, \( \varphi_R \) is a smooth blow up by [2] Theorem 5.1, so \( X \) is as in theorem 1.3. We can therefore assume that there exists a fiber \( F \) of \( \varphi_R \) of dimension \( \geq l(R) + 1 \). The contraction \( \psi : X \to Z \) has fibers of dimension \( \geq r_X - 1 \geq n - l(R) - 1 \), so \( \dim Z \leq l(R) + 1 \). It follows that \( F \) dominates \( Z \) and meets every fiber of \( \psi \), forcing the equidimensionality of \( \psi \).

We can now conclude that \( X \) is a projective bundle over \( Z \) by [13] Lemma 2.12] since \( H \cdot V = 1 \).

If \( R \) is of fiber type then every fiber of \( \varphi_R \) has dimension \( \geq l(R) - 1 \) and so the contraction \( \psi : X \to Z \) has fibers of dimension \( \leq n - l(R) + 1 \leq r_X \), so we can conclude by [3] Theorem 4.1] and [5] Proposition 2.5]. \( \square \)

8. **Appendix**

The results in theorem 1.3 show that if a Fano variety \( X \) is the blow-up of a smooth variety \( Y \) along a smooth subvariety \( T \) and \( i_X \geq \dim T + 1 \) then also \( Y \) is a Fano variety and \( i_Y \geq i_X \).

In general these two facts are not true; in [25] Section 3] the question whether \( Y \) has to be a Fano variety was posed and some answers were given in [25] Propositions 3.4 and 3.6].

In particular the examples [25] 3.7, 3.8] show that \( i_T \geq \dim T + 1 \) is the best possible bound which guarantees that \( Y \) is a Fano manifold.

The second problem, i.e. - assuming that \( Y \) is Fano can the pseudoindex of \( Y \) be less than the pseudoindex of \( X \)? - has been studied in [4]. The following example of that paper shows that the answer can be positive:

**Example 8.1.** Let \( Y_n = \mathbb{P}_{\mathbb{P}^m}(\mathcal{O}^{\otimes m} \oplus \mathcal{O}(1)) \) and let \( T_n \subset Y_n \) be the submanifold defined by the subbundle \( \mathcal{O}^{\otimes m} \). Note that \( \dim Y_n = n = 2m \) and \( \dim T_n = m \). Let \( \varphi_n : X_n = \text{Bl}_{T_n}(Y_n) \to Y_n \) be the blow-up of \( Y_n \) along \( T_n \).

One can easily prove that \( X_n \) and \( Y_n \) are Fano manifolds, if \( n \geq 4 \), and moreover \( i_{Y_n} = 1 \) and, if \( n \geq 6 \), \( i_{X_n} = 2 \) (while \( i_{X_4} = 1 \)).
The following are the main results of [6]:

**Theorem 8.2.** Let $X$ be a Fano manifold of dimension $n$ which is the blow up $\varphi_R : X \to Y$ of a smooth Fano manifold $Y$ along a smooth subvariety $T$.

- If $2 \dim T \leq n + i_Y - 1$ then $i_X \leq i_Y$ unless $n \geq 6$ is even and $X = X_n$, $Y = Y_n$, $T = T_n$ are as in the above example.
- If $i_Y \geq \frac{n}{3} - 1$ then $i_X \leq i_Y$ unless $n = 6$ and $X = X_6$, $Y = Y_6$, $T = T_6$ are as in the above example.

We propose here a slight variation of the results of [6], considering birational contractions between smooth Fano manifolds:

**Proposition 8.3.** Let $X$ be a Fano manifold, let $\varphi_R : X \to Y$ be the contraction of a birational extremal ray $R$ such that $Y$ is a smooth Fano manifold and let $T = \varphi_R(\text{Exc}(R))$. If $i_Y > 2 \dim T + 1 - n$ or if $i_Y > \frac{n}{3} - 1$ then $i_X \leq i_Y$.

**Proof.** Since $\varphi$ is a birational map between smooth varieties the exceptional locus $\text{Exc}(R)$ is a divisor and we have the formula:

$$K_X = K_Y + G,$$

where $G$ is a divisor supported on $\text{Exc}(R)$.

Let $C \subset Y$ be a rational curve such that $i_Y = -K_Y \cdot C$ and let $V$ be a family of rational curves on $Y$ containing $C$.

By inequality $2.5$ we have $2 \dim \text{Locus}(V) \geq n + i_Y - 1$, therefore if $i_Y > 2 \dim T + 1 - n$ we have that $\dim \text{Locus}(V) > \dim T$ and this implies that there exists a curve $C$ in $V$ not contained in $T$.

The strict transform of it, call it $\tilde{C}$, is a rational curve on $X$ satisfying $G \cdot \tilde{C} \geq 0$, therefore, by the canonical bundle formula, $i_X \leq -K_X \cdot \tilde{C} \leq i_Y$.

Assume now that $i_Y > \frac{n}{3} - 1$ and, by contradiction, that $i_X > i_Y$; by the first part we can assume that $i_Y \geq 2 \dim T + 1 - n$.

Denote by $F$ a general fiber of the map $\varphi$; from $2.5$ we have $\dim F \geq i_X$ and therefore

$$i_Y \leq 2 \dim T + 1 - n = n - 1 - 2 \dim F \leq n - 1 - 2i_X \leq n - 3 - 2i_Y,$$

that is $i_Y \leq \frac{n}{3} - 1$, which is a contradiction. \hfill $\Box$

**Corollary 8.4.** Let $X$ be a Fano manifold, let $\varphi_R : X \to Y$ be the contraction of a birational extremal ray $R$ such that $Y$ is a smooth Fano manifold and let $T = \varphi_R(\text{Exc}(R))$. If $i_X \geq \dim T$ then $i_X \leq i_Y$.

**Proof.** Let $F$ be a general fiber of $\varphi_R$; we have

$$\dim T \leq \dim \text{Exc}(R) - \dim F \leq n - 1 - i_X \leq n - \dim T + 1$$

so that

$$2 \dim T - n + 1 \leq 0 < i_Y.$$

We can thus apply proposition $2.5$ to conclude. \hfill $\Box$
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