On the conductivity imaging by MREIT: available resolution and noisy effect

J J Liu¹, J K Seo², M Sini²,³ and E J Woo⁴

¹Department of Mathematics, Southeast University, Nanjing, 210096, P.R.China
²Department of Mathematics, Yonsei University, Seoul, 120-749, Korea
⁴College of Electronics and Information, Kyung Hee University, Kyungki, 449-701,Korea

E-mail: jjliu@seu.edu.cn, seoj@yonsei.ac.kr, sini@yonsei.ac.kr, ejwoo@khu.ac.kr

Abstract. Magnetic resonance electrical impedance tomography (MREIT) is a new technique in medical imaging, which aims to provide electrical conductivity images of biological tissue. Compared with the traditional electrical impedance tomography (EIT), MREIT reconstructs the interior conductivity from the deduced magnetic field information inside the tissue. Since the late 1990s, MREIT imaging techniques have made significant progress experimentally and numerically. However, the theoretical analysis on the MREIT algorithms is still at the initial stage. This paper aims to give a state of the art of the MREIT technique and to concern the convergence property as well as the numerical implementation of harmonic $B_z$ algorithm, one of the well-implementation MREIT algorithms. We present some late advances in the convergence issues of MREIT algorithm. Some open problems related to the noisy effects and the numerical implementations are also given.

1. Introduction

Magnetic resonance electrical impedance tomography (MREIT) is a new medical imaging technique that aims to provide electrical conductivity image for biological tissues such as human body with sufficiently high spatial resolution and accuracy. In MREIT, we inject current $I$ into a conductive object $\Omega \subset \mathbb{R}^3$ through a pair of surface electrodes to produce current density $J = (J_x, J_y, J_z)$ inside the object. The internal current generates the distribution of magnetic flux density $B = (B_x, B_y, B_z)$. MREIT takes advantage of an MRI scanner as a tool to capture one component of $B$, say, $z$-component $B_z$, where $z$ is the direction of the main magnetic field of the MRI scanner. Conductivity imaging in MREIT is based on the nonlinear relationship between medium conductivity and magnetic field information due to Maxwell’s equations.

Compared with the traditional EIT technique ([2, 3, 4, 16, 25, 28]), which uses the boundary measurement of current and voltage to detect the interior conductivity of a conductive domain, MREIT uses the internal current distribution information corresponding to boundary input current to reconstruct the tissue conductivity ([5, 22, 30]). The physical possibility of this new technique is based on the indirect measurement of internal magnetic field. By measuring the internal magnetic field along one direction using MR scanner, the internal current distribution can be obtained ([6, 8]). MREIT technique aims to use the internal current (magnetic field

³ Present address: Johann Radon Institute for Computational and Applied Mathematics, Altenberger Strasse 69, A-4040, Linz, Austria.
flux) information to detect the tissue conductivity. From the mathematical point of view, the application of interior information of biological tissue weakens the ill-posedness of the inverse problem, and therefore a high resolution of conductivity image is expected.

Since the late 1990’s, imaging techniques in MREIT have made significant progress ([13, 21, 23]). Recent published numerical simulations and phantom experiments show that high spatial resolution of conductivity images are available as long as we inject current $I$ from the boundary with enough strength to induce large SNR (signal to noise ratio) of $B_z$ data ([10, 12, 14, 15, 20, 29]). In fact, the decrease of the strength of current $I$ results in a decrease in SNR of $B_z$ data due to the non-ideal data acquisition system of an MR scanner. Hence, if we limit the amount of injection current according to the safety regulations, the measured $B_z$ data tend to have lower SNR and get usually degraded in their accuracy, and below a certain level of SNR of $B_z$, MREIT algorithm may not provide clinically useful conductivity image. Since the MREIT algorithm solves this nonlinear problem in an iterative style, it is also expected that satisfactory results are obtained in a finite iteration steps with finite computing time. Mathematically, this issue is essentially the convergence property of MREIT algorithms related to the signal noise level and the computing efficiency ([17]). That is, to what extent of input noise level, the iteration algorithm can generate convergent numerical results.

The aim of this paper is to introduce some results on the rigorous mathematical analysis on the MREIT inversion algorithms. To this end, we firstly describe an mathematical model of MREIT that agrees with a practical medical imaging system. Then we pay our attention to the latest advance in the convergence property of harmonic $B_z$ algorithm for MREIT problem. This algorithm is proposed in [19, 24] which is essentially an iteration procedure to approximate conductivity $\sigma(r)$. The uniqueness for this inverse problem in 2-dimensional case has been given in [11]. The convergence difference between 2-dimensional model problem and 3-dimensional practical problem is stated in this paper. Finally, we give some numerical results to show the convergence property. Some open problems with respect to the noisy input data are also stated.

For the practical purpose, the other important issue is the inversion performance for noisy data $B_z$. The practical $B_z$ from measurement contains too much noise, which is indeed amplified moreover due to the differentiation operation used in the harmonic $B_z$ algorithm. So the final results of harmonic-$B_z$ scheme depend essentially on the convergence property and the efficient computation of 2-order derivatives for noisy $B_z$ applied in this iteration procedure. Some regularizing techniques are applied in this paper to deal with the noisy input data so that a stable solution can be established.

This paper is organized as follows. In section 2, we describe the basic principle of EIT and MREIT technique. Then in section 3 we survey one of the main inversion algorithm for MREIT, namely, the harmonic-$B_z$ algorithm. The algorithm itself as well as some recently-developed convergence properties of the iteration procedure are given, which partially explain the applicable scope of this algorithm. Finally, we give some numerical examples for this inversion algorithm in section 4. Especially, we test the harmonic-$B_z$ algorithm for the noisy input data, which give a quantitative description of the available resolution of this algorithm.

2. EIT and MREIT techniques
Both EIT and MREIT technique depend on Maxwell’s equations describing the relation between electrical field and magnetic field. Let the object to be imaged occupy a three-dimensional bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial \Omega$. We assume that the conductivity distribution of $\Omega$, denoted by a scalar function $\sigma$, is isotropic. For anisotropic medium $\Omega \subset \mathbb{R}^3$, $\sigma$ should be a
3 × 3 matrix function. The time harmonic Maxwell’s equations are given by

\[
\begin{align*}
\nabla \times \mathbf{E}(\mathbf{r}) &= -i\omega \mu \mathbf{H}(\mathbf{r}), \\
\nabla \times \mathbf{H}(\mathbf{r}) &= i\omega \epsilon \mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r}),
\end{align*}
\]

where \( \mathbf{r} = (x, y, z) \in \Omega \), \( \omega \) is the frequency of time harmonic wave, \( \mathbf{E}(\mathbf{r}), \mathbf{H}(\mathbf{r}) \) are electrical field and magnetic field, respectively, while the total internal current

\[
\mathbf{J} = \mathbf{J}_c + \mathbf{J}_s
\]

with source current \( \mathbf{J}_s \) and the conduction current

\[
\mathbf{J}_c = \sigma \mathbf{E}.
\]

In EIT, we usually assume that the source term \( \mathbf{J}_s \approx 0 \) and \( \omega \mu \mathbf{H} \approx 0 \) at low frequency \( \omega \). It is assumed that \( i\omega \mu \mathbf{H}(\mathbf{r}) \approx 0 \) due to the smallness of \( \omega \), and then we can express the electrical field \( \mathbf{E}(\mathbf{r}) = -\nabla u(\mathbf{r}) \) for curl free field \( \mathbf{E}(\mathbf{r}) \). Therefore, it follows from (1) that

\[
\nabla \cdot ((\sigma + i\omega \epsilon) \nabla u(\mathbf{r})) = -\nabla \cdot (\nabla \times \mathbf{H}(\mathbf{r})) \equiv 0.
\]

In EIT problems, we inject current \( g \) from the boundary of conductor domain \( \Omega \). Then it follows from (4) that, for small \( \omega \), the potential \( u(\mathbf{r}) \) in \( \Omega \) meets

\[
\begin{align*}
\nabla \cdot (\sigma \nabla u) &= 0, & \mathbf{r} \in \Omega \\
-\sigma \frac{\partial u}{\partial \nu} &= g, & \mathbf{r} \in \partial \Omega.
\end{align*}
\]

In EIT, we measure the Dirichlet data on \( \partial \Omega \) which can be used to reconstruct \( \sigma \). Indeed, the inverse problem of EIT is to reconstruct \( \sigma \) from the Neumann to Dirichlet map

\[
\Lambda_\sigma : g \mapsto u|_{\partial \Omega}.
\]

Electrical Impedance Tomography (EIT) has not been very successful in producing high-resolution images of cross-sectional tissue conductivity distributions mainly due to the ill-posed characteristics of the corresponding inverse problem. Many studies have been performed to appropriately handle or overcome this inherent technical difficulty in EIT. These include difference imaging techniques, regularization, utilization of a priori information and so on. However, even with all of these techniques, the image reconstruction problem in EIT still remains ill-posed. Also, from the physical point of view, it is impossible to give this map as inversion input, since only the finite number of input current forms as well as finite number of measuring points can be specified. For given \( g \) satisfying \( \int_{\partial \Omega} g ds = 0 \) due to conservation of current, (5) determines \( u \) up to a constant.

The uniqueness of reconstruction of \( \sigma \) from \( \Lambda_\sigma \) has been studied thoroughly, for example, see [1, 9]. Also it is well-known that this kind of inverse problem is severe ill-posed. That is, the boundary data are not sensitive to the local change of \( \sigma \) in \( \Omega \). The disadvantage of EIT technique is its severe ill-posedness, since only the boundary data are used to reconstruct \( \sigma \).

We notice that the magnetic field information \( \mathbf{H} \) in \( \Omega \) is ignored in deriving (5) and (6). MREIT technique uses the interior magnetic flux information \( \mathbf{B} = \mu \mathbf{H} \) to weaken the ill-posedness in determining \( \sigma \). Now we describe an mathematical model of MREIT that agrees
Figure 1. MREIT system at Impedance Imaging Research Center in Korea and image reconstruction software.

with a practical medical imaging system, so that we can understand how the given data, the injection current and the measured $B_z$ data, are generated. The configuration of the MREIT system is shown in Figure 1.

In MREIT, we inject a current $I$ through the pair of copper electrodes $\mathcal{E}_1$ and $\mathcal{E}_2$ attached on $\partial \Omega$. The injection current $I$ produces an internal current density $\mathbf{J}(\mathbf{r}) = (J_x, J_y, J_z)$ inside $\Omega$, which can be expressed as $\mathbf{J} = -\sigma \nabla u$ where $u$ is the corresponding voltage. That is, $u(\mathbf{r})$ meets the following mixed boundary value problem

$$
\begin{array}{l}
\nabla \cdot (\sigma \nabla u) = 0, \quad \mathbf{r} \in \Omega \\
\int_{\mathcal{E}_1} \sigma \frac{\partial u}{\partial \nu} \, ds = -\int_{\mathcal{E}_2} \sigma \frac{\partial u}{\partial \nu} \, ds = I, \\
-\sigma \frac{\partial u}{\partial \nu} = 0, \quad \mathbf{r} \in \partial \Omega \setminus (\mathcal{E}_1 \cup \mathcal{E}_2).
\end{array}
$$

(7)

Since the electrodes are perfect conductor, the potential $u$ should be constant in $\mathcal{E}_1, \mathcal{E}_2$, that is,

$$
u(\mathbf{r}) = C_i, \quad \mathbf{r} \in \mathcal{E}_i, \ i = 1, 2.
$$

(8)

Compared the boundary conditions in (5) and (7), it can be seen that only the total amount of current in the electrodes can be given. Generally, it is very difficult to specify the input current density $g$ in the electrodes in a point-wise sense in practice.

This potential $u$ under condition (8), up to a constant, is a scalar multiple of the solution $\tilde{u} \in C(\Omega)$ of the following mixed boundary value problem

$$
\begin{array}{l}
\nabla \cdot (\sigma \nabla \tilde{u}) = 0 \quad \text{in} \quad \Omega \\
\tilde{u}|_{\mathcal{E}_1} = 1, \quad \tilde{u}|_{\mathcal{E}_2} = 0 \\
-\sigma \nabla \tilde{u} \cdot \nu = 0 \quad \text{on} \quad \partial \Omega \setminus \mathcal{E}_1 \cup \mathcal{E}_2,
\end{array}
$$

(9)

where $\nu$ is the unit outward normal vector to the boundary $\partial \Omega$. To be precise,

$$
u(\mathbf{r}) = \frac{I}{\int_{\mathcal{E}_1} \sigma \nabla \tilde{u} \cdot \nu \, ds} \tilde{u}(\mathbf{r}), \quad \mathbf{r} \in \Omega,
$$

(10)
see [18, 27]. Here the notation $I$ is also used for the amount of injection current. The representation formula (10) gives a way to the solution of direct problem (7) and (8) at each iteration steps in MREIT inversion algorithms.

Now, let us consider the inverse problem in MREIT. The presence of the internal current density $J$ generates a magnetic flux density $B = (B_x, B_y, B_z)$ and Ampere’s law $J = \nabla \times B/\mu_0$ holds inside $\Omega$. Using the MRI scanner, we can measure $B_z$, the $z$-component of $B$. According to the Biot-Savart law, $B_z$ can be expressed as $B_z = \Lambda_I[\sigma, u](r)$ where

$$
\Lambda_I[\sigma, u](r) := \frac{\mu_0}{4\pi} \int_{\Omega} \sigma(r) \left[ (x - x') \frac{\partial u}{\partial y}(r') - (y - y') \frac{\partial u}{\partial x}(r') \right] \frac{dy'}{|r - r'|^3} \, dr', \quad r \in \Omega. \tag{11}
$$

In order to reconstruct an isotropic conductivity $\sigma$, at least two different injection currents are needed. For the uniqueness, we also need to fix the scaling uncertainty of $\sigma$ by measuring voltage difference $|u_1(\mathbf{r}_1) - u_1(\mathbf{r}_2)| = a \neq 0$ at any two fixed boundary points $\mathbf{r}_1, \mathbf{r}_2 \in \partial \Omega$. This is due to the observation that, without knowledge of this voltage difference, we may have the same $B_z$ from many different scaled conductivities:

$$
\Lambda_I[\sigma, u](\mathbf{r}) = \Lambda_I[c\sigma, \frac{1}{c}u](\mathbf{r}), \quad \mathbf{r} \in \Omega \quad \text{for all } c > 0.
$$

For simplicity only, we assume $\mathbf{r}_1 \in \mathcal{E}_1^1$ and $\mathbf{r}_2 \in \mathcal{E}_2^1$ so that

$$
|u_1|_{\mathcal{E}_1^1} - |u_2|_{\mathcal{E}_2^1} = a.
$$

Indeed, it is technically difficult to measure the voltage difference along the injection electrodes, and we use the other electrodes to measure the voltage difference in practice.

Now, we are ready to explain the mathematical model for MREIT. Let $(I_j, B^j_z)$ for $j = 1, \cdots, N$ and $a$ be given data. The goal of MREIT is to reconstruct the conductivity distribution $\sigma$ satisfying the following three conditions:

(i) Each $u_j$ satisfies

$$
\begin{align*}
\nabla \cdot (\sigma \nabla u_j) &= 0 \quad \text{in } \Omega, \\
u \mathbf{u}_j \Big|_{\mathcal{E}_1^j} &= \text{a constant}, \quad u_j \Big|_{\mathcal{E}_2^j} = 0, \\
-\sigma \nabla u_j \cdot \nu &= 0 \quad \text{on } \partial \Omega \setminus \mathcal{E}_1^j \cup \mathcal{E}_2^j, \\
\int_{\mathcal{E}_1^j} \sigma \nabla u_j \cdot \nu \, ds &= I_j, \quad \int_{\mathcal{E}_2^j} \sigma \nabla u_j \cdot \nu \, ds = -I_j,
\end{align*}
\tag{12}
$$

where $\mathcal{E}_1^j$ and $\mathcal{E}_2^j$ are the pair of surface electrodes through which the current $I_j$ is injected.

(ii) $|u_1|_{\mathcal{E}_1^1} = a$.

(iii) $\Lambda_I[\sigma](\mathbf{r}) = B^j_z(\mathbf{r})$ for $\mathbf{r} \in \Omega$.

A more thorough explanation on this algorithm may be found in [27].

The MREIT technique applies the interior information about magnetic flux to reconstruct the conductivity. In this way, the ill-posedness of inverse problem is weakened. The difference between these two inversion models can be roughly described in Figure 2.
3. Harmonic $B_z$ algorithm and convergence

Recently, there have been great advances in developing MREIT techniques including theory, algorithms and experimental methods. The first realistic reconstruction algorithm in MREIT is the harmonic $B_z$-algorithm that successfully reconstructs the conductivity image with a high resolution and accuracy in numerical simulations and experimental studies.

In this section, we firstly state the harmonic $B_z$ algorithm, and then give the convergence results obtained recently. We only deal with the harmonic $B_z$ algorithm with two injection currents $I_1$ and $I_2$. By using Ampère’s law, we know

$$\mu_0 \nabla \times J = \nabla \times (\nabla \times B) = -\Delta B + \nabla(\nabla \cdot B) = -\Delta B,$$

noticing $\nabla \cdot B \equiv 0$. Taking $z$ component and noticing $J = -\sigma \nabla u$ lead to

$$\Delta B_z = \mu_0 (\sigma_x u_y - \sigma_y u_x),$$

which yields for two input currents $I_1, I_2$ that

$$\begin{bmatrix}
\frac{\partial \sigma}{\partial x} \\
\frac{\partial \sigma}{\partial y}
\end{bmatrix} = \frac{1}{\mu_0} \begin{bmatrix}
\frac{\partial u_1[\sigma]}{\partial y}, & -\frac{\partial u_1[\sigma]}{\partial x} \\
-\frac{\partial u_2[\sigma]}{\partial y}, & -\frac{\partial u_2[\sigma]}{\partial x}
\end{bmatrix}^{-1} \begin{bmatrix}
\Delta B_1^z \\
\Delta B_2^z
\end{bmatrix},$$

(13)

where $u_i[\sigma]$ is the solution of (12). Notice, two $B_z$’s data generate the derivative of conductivity along $x$ and $y$ direction, but no information about $\sigma_z$ is given. Due to this reason, the harmonic $B_z$ algorithm reconstructs $\sigma$ at each slice $z = z_0$ of $\Omega$ respectively. More precisely, using the identity (13) and

$$\Delta x', y' \left( \frac{1}{2\pi} \ln \sqrt{(x' - x)^2 + (y' - y)^2} \right) = \delta(x' - x, y' - y),$$

Figure 2. Comparison of physical configuration between EIT and MREIT, the arrow in black represents the internal current used in MREIT.
we have the following representation formula in each slice $\Omega \cap \{z = z_0\}$:

$$
\sigma(x, y, z_0) = H_{\sigma}(x, y, z_0) - \frac{1}{2\pi} \int_{\Omega \cap \{z = z_0\}} \frac{(x-x', y-y')}{\sqrt{|x-x'|^2 + |y-y'|^2}} \cdot A[\sigma] \left[ \frac{\Delta B^1_z}{\Delta B^2_z} \right] dx' dy', \tag{14}
$$

where

$$
A[\sigma] := \frac{1}{\mu_0} \begin{bmatrix} \frac{\partial u_1[\sigma]}{\partial y} & -\frac{\partial u_1[\sigma]}{\partial x} \\ \frac{\partial u_2[\sigma]}{\partial y} & -\frac{\partial u_2[\sigma]}{\partial x} \end{bmatrix}^{-1}, \tag{15}
$$

and

$$
H_{\sigma}(x, y, z_0) := \int_{\partial(\Omega \cap \{z = z_0\})} \frac{(x-x', y-y') \cdot \nu}{2\pi \sqrt{|x-x'|^2 + |y-y'|^2}} \sigma(x', y', z_0) \, dl,
$$

where $\nu$ is the two dimensional unit outward normal vector to the boundary $\partial\Omega_{z_0}$ and $dl$ is the line element. Note that $H_{\sigma}$ is harmonic in two dimensional slice $\Omega \cap \{z = z_0\}$.

The harmonic $B_z$ algorithm for reconstructing $\sigma$ is a natural iterative procedure of the key identity (14) that can be described as follows.

1. Let $n = 0$ and assume an initial conductivity distribution $\sigma_0 = 1$.
2. Compute $u^n_j, j = 1, 2$ by solving (12). Scale $\sigma_n(r) := \sigma_n(r) \frac{|u^n_1(r) - u^n_2(r)|}{\sigma}$ and $u^n_j(r) := u^n_j(r) \frac{\sigma}{|u^n_1(r) - u^n_2(r)|}$ for $r \in \Omega$.
3. Compute $\sigma_{n+1}$ using (14) with $u_j$ replaced by $u^n_j$.
4. If $\|\sigma_{n+1} - \sigma_n\|_2 < \epsilon$, go to Step 5. Here, $\epsilon$ is a given tolerance. Otherwise, set $n \leftarrow (n + 1)$ and go to Step 2.
5. If it is needed, compute current density images as $J_j \leftarrow -\sigma_{n+1} \nabla u^n_j$, where $u^n_j$ is a solution of the boundary value problem in (12) with $\sigma_{n+1}$ replacing $\sigma$.

We consider the special case where $\Omega$ is cylindrical in its shape and the conductivity distribution $\sigma$ does not change in $z$-direction. An appropriate choice of injection current and electrodes using long longitudinal electrodes which is independent $z$-direction, we could produce a transversal internal current density $J$, that is, $J = (J_x, J_y, 0)$, see Figure 3. In this special case, the reconstruction problem can be reduced to find the two dimensional conductivity $\sigma(x, y, z_0)$.

Although the above algorithm has been tested with satisfactory numerical performance, the theoretical analysis on the convergence property of this algorithm is still needed. Recently, we obtain the following results.

**Result 1:** Gradient convergence property in special cases of the 2-dimensional model.

We use the following iteration scheme

$$
\sigma^0 = 1, \quad \left[ \frac{\partial \sigma^{n+1}}{\partial x}, \frac{\partial \sigma^{n+1}}{\partial y} \right] = A[\sigma^n] \left[ \frac{\Delta B^1_z}{\Delta B^2_z} \right], \tag{16}
$$

to generate $\{\nabla \sigma^n\}$ due to (13). After obtaining $\nabla \sigma^{n+1}$, (14) is used to obtain $\sigma^{n+1}$ provided that the boundary value be specified.
Theorem 3.1. Assume that the 2-dimensional target domain is $\Omega = \{(x, y) : -1 < x, y < 1\}$ with conductivity $\sigma^*$ satisfying $0 < \sigma^* < \infty$. Two pairs of electrodes

$$E_1^\pm = \{(\pm 1, y) : |y| < 1\}, \quad E_2^\pm = \{(x, \pm 1) : |x| < 1\}$$

are applied to input current $I$ respectively from the boundary.

1. If $\sigma^*$ is depending only on $x$–variable (or $y$–variable), then $\nabla \sigma_1 = \nabla \sigma^*$. This means that the contrast of $\sigma^*$ is recovered exactly by only one iteration.

2. If $\sigma^*$ is a small perturbation of a constant background such that $\|\sigma^* - 1\|_{C^1(\Omega)} \leq \epsilon$, $\sigma^* = 1$ near $\partial \Omega$ with $\epsilon$ small enough. Then the sequence $\{\sigma^n\}$ constructed by (3.4) with $\sigma^n|_{\partial \Omega} = 1$ converges to $\sigma^*$. Moreover, it follows that

$$\|\nabla(\sigma^n - \sigma^*)\|_{L^\infty(\Omega)} = O(\epsilon^{n+1}), \quad n = 1, 2, \ldots$$

This result can be considered as some direct observations on the convergence property of harmonic $B_z$ algorithm in very special case. For the proof, see [17].

**Result 2:** Convergence property of 2-dimensional model.

Theorem 3.2. Assume that the target conductivity $\sigma^*(x, y) \in C^1(\Omega)$ meets

$H1.$ $0 < \sigma^* \leq \sigma^*(x) \leq \sigma^*_+$ for known constants $\sigma^*_\pm$;

$H2.$ there exists $\tilde{\Omega} \subset \subset \Omega$ such that $\sigma^*$ is a known constant in $\Omega \setminus \tilde{\Omega}$;

$H3.$ $|\det A[\sigma^*]^{-1}(x, y)| \geq d^*_\pm > 0$ in $\tilde{\Omega}$ where $d^*_\pm$ is a known constant.

Under these hypotheses, there exist constants $\epsilon = \epsilon(\sigma^*_\pm, d^*_\pm) > 0$ small enough and $\theta = \theta(\epsilon, \sigma^*_\pm, d^*_\pm) \in (0, 1)$ such that if we take the initial guess $\sigma^0$ as the constant $\sigma^*|_{\Omega \setminus \tilde{\Omega}}$, then the sequence $\{\sigma^n\}$ given by the harmonic $B_z$ iteration scheme holds that for $\|\nabla \sigma^*\|_{C^1(\Omega)} \leq \epsilon$,

$$\sigma^n \equiv \sigma^* \text{ in } \Omega \setminus \tilde{\Omega}, \quad \|\sigma^n - \sigma^*\|_{C^1(\tilde{\Omega})} \leq K \theta^n \epsilon, \quad n = 1, 2, \ldots$$

where $K := \text{diam} (\Omega) + 1$. 

Figure 3. Axially symmetric cylindrical configuration.
For the proof, see [18].

Since the harmonic $B_z$ algorithm reconstructs the conductivity $\sigma^*(x,y,z)$ in general 3-dimensional medium $\Omega \subset \mathbb{R}^3$ at each slice $\Omega \cap \{z = z_0\}$, the above results give the main convergence property of this algorithm. However, for general 3-dimensional medium, when we generate $\sigma^{n+1}(x,y,z_0)$ at each slice, we need $\sigma^n(x,y,z)$ in the whole domain $\Omega$, not only in the same slice $\Omega \cap \{z = z_0\}$. Moreover, as explained previously, the harmonic $B_z$ algorithm does not give any gradient information of $\sigma^{n+1}$ along $z$ direction. This property leads to some essential difference on the convergence property between 2-dimensional and 3-dimensional cases.

**Result 3:** Convergence property of the 3-dimensional model.

**Theorem 3.3.** Assume the target conductivity $\sigma^* \in C^1(\overline{\Omega})$ with $\Omega \subset \mathbb{R}^3$ which satisfies the following conditions:

A1. $0 < \sigma^* \leq \sigma^*_\epsilon$ with known constants $\sigma^*_\epsilon$;

A2. There exists $\Omega \subset \subset \Omega$ such that $\sigma^*$ is a known constant in $\Omega \setminus \overline{\Omega}$;

A3. $|\det A(\sigma^*)^{-1}(x,y,z)| \geq d^*_\epsilon > 0$ at all slices $z = z_0$ in $\Omega$, where $d^*_\epsilon$ is a known constant. Under these hypotheses, there exist constants $\epsilon = \epsilon(\sigma^*_\epsilon, d^*_\epsilon) > 0$ small enough and $\theta = \theta(\epsilon, \sigma^*_\epsilon, d^*_\epsilon) \in (0,1)$ such that if we take the initial guess $\sigma^0$ as the constant $\sigma^1(\Omega, \overline{\Omega})$, then it holds that the sequence $\{\sigma^n := \sigma^n(x,y,z_0)\}$ for all $z_0 \in \Omega$, where $\sigma^n(x,y,z_0)$ is constructed by the harmonic $B_z$ iteration for every $z_0$, converges to the true conductivity $\sigma^*$ in $\Omega$ for $\sigma^*$ satisfying

\[ \|\nabla \sigma^*\|_{C(\overline{\Omega})} \leq \epsilon. \]  

More precisely, it holds that

\[ \|\sigma^n - \sigma^*\|_{C(\overline{\Omega})} \leq K\theta^n \epsilon, \quad \|\nabla_{x,y}(\sigma^n - \sigma^*)\|_{C(\overline{\Omega})} \leq K\theta^n \epsilon, \quad n = 1,2,\ldots \]  

where $K := diam(\Omega) + 1$.

For the proof, see [18].

In this three-dimensional setting, the estimate is given by the $C$–norm while the one in the two-dimensional case is given by the $C^1$–norm. We can not improve the derivative estimate $\nabla_{x,y}(\sigma^n - \sigma^*)$ in three-dimensional case by $\nabla(\sigma^n - \sigma^*)$ since we do not know $\partial_z(\sigma^n - \sigma^*)$, although we have the full three gradient estimates for $\sigma^*$. The main difficulty in this case is due to the fact that in the iteration process, we get $\nabla_{x,y}\sigma^{n+1}$ at each slice with no information about $\partial_z\sigma^{n+1}$. That is, the harmonic $B_z$ method approximates the three-dimensional conductivity function $\sigma^*(x,y,z)$ in $\Omega \subset \mathbb{R}^3$ at each two-dimensional slice $\Omega_{z_0} := \Omega \cap \{x,y,z : z = z_0\} \subset \mathbb{R}^2$. Then $\sigma^n$ in $\Omega \subset \mathbb{R}^3$, at each iteration, is constructed as $\sigma^n(x,y,z) := \bigcup_{z_0} \sigma^n(x,y,z_0)$.

The key to the harmonic $B_z$ algorithm is that, we firstly recover $\nabla_{x,y}\sigma$ at each slice $z = z_0$ by iteration and then construct $\sigma^{n+1}$ in this slice using $\nabla_{x,y}\sigma^{n+1}(x,y,z_0)$ and $\sigma^{n+1}$ on the boundary of the slice. When we compute $\nabla_{x,y}\sigma^{n+1}(x,y,z_0)$ at each iteration, we need to compute $\Delta_{x,y}B_z$. In the case of noisy input data for $B_z$, it is well-known that $\Delta_{x,y}B_z$ will amplify the noise and therefore contaminates the iteration results. Therefore some efficient denoising technique for measurement of $B_z$ is needed for the efficiency of harmonic $B_z$ algorithm.

The other possible way to the efficient application of noisy input data $B_z$ is some inversion algorithm without the Laplacian operation on the input $B_z$ data. This can be done by the
integral version of Biot-Savart’s law, that is,
\[
B_i^z(r) := \frac{\mu_0}{4\pi} \int_{\Omega} \frac{\sigma(r) [(x - x') \frac{\partial u_i}{\partial y}(r') - (y - y') \frac{\partial u_i}{\partial x}(r')]}{|r - r'|^3} \, dr', \quad r \in \Omega
\]
for two input current \( I_i \) with \( i = 1, 2 \) in suitable form. \( (21) \) constitutes a nonlinear coupled integral equations of the first kind. The nonlinearity of this equation can be treated by iteration
\[
B_i^{z+1}(r) := \frac{\mu_0}{4\pi} \int_{\Omega} \frac{\sigma^{n+1}(r) [(x - x') \frac{\partial u_i^{n+1}}{\partial y}(r') - (y - y') \frac{\partial u_i^{n+1}}{\partial x}(r')]}{|r - r'|^3} \, dr', \quad r \in \Omega
\]
for given \( \sigma^n \). At each iteration, this equation is a linear integral equation of the first kind with respect to \( \sigma^{n+1} \). To deal with the ill-posedness of this equation, some regularization such as the Tikhonov regularization can be used.

The advantage of this algorithm is that the differentiation operation on \( B_z \) is not needed directly. So it is expected that this algorithm can deal with the noisy input data in an efficient way. However, the amount of computation is relatively large. On the other hand, this algorithm recovers the conductivity directly on the whole domain \( \Omega \), rather than slice by slice. The numerical tests as well as the theoretical analysis of this inversion scheme are still open.

4. Numerical implementations

There have been much works on the numerical tests of inversion algorithm for MREIT technique. Here we only give some numerics for the harmonic \( B_z \) algorithm for two models. From these numerics, we can see that the harmonic \( B_z \) algorithm is efficient for exact input data. But for the noisy input data, the denoising technique is required for the satisfactory iteration results.

**Example 1.** Consider rectangle domain \( \Omega := [-1, 1] \times [-2, 2] \) with target conductivity
\[
\sigma^*(x, y) = \begin{cases} 
3, & 0 \leq r \leq 0.4 \\
-10r^2 + 4.6, & 0.4 \leq r \leq 0.6 \\
1, & \text{otherwise},
\end{cases}
\]
where \( r = \sqrt{x^2 + y^2} \). We apply two pairs of electrodes \( E_1^\pm, E_2^\pm \) in \( \partial \Omega \) with width 0.2 to inject currents with density \( g^j \):
\[
g^j|_{E_j^\pm} = \pm 1; \ g^j|_{\partial \Omega \setminus E_j^\pm} = 0, \quad j = 1, 2.
\]

For this configuration (see Figure 4), we firstly simulate the magnetic field \( B_z \) as inversion input data by solving direct problem. Then we test the harmonic \( B_z \) algorithm.

**Case 1:** Using exact input data.
To check the performance of the algorithm, we generate \( \Delta B_z \) directly by
\[
\Delta B_z = \mu_0 (\sigma_x u_y - \sigma_y u_x),
\]
rather than \( B_z \) itself. In this way, we avoid the numerical error due to the Laplacian operation on \( B_z \). The iteration performance for different iterations with initial guess \( \sigma^0 = 1 \) are given in Figure 5.
When we do the iterations using an initial guess with relatively large error for exact input data, the convergence of the algorithm is still satisfactory. The numerical results for a initial guess
\[
\sigma^0(x, y) = \frac{4}{5} - \frac{1}{5} \cos \left( \frac{x^2 + y^2}{2} \right),
\] (25)
which has completely different shape from the exact conductivity data, can be found in [18].

In [18], the numerical tests are given only for the exact input data $B_z$. In the next example, we report our recent numerics for noisy $B_z$. Since the Laplacian operation is needed in the harmonic-$B_z$ algorithm based on the identity (14), some regularization techniques are required to prevent the amplification of the noisy. In our numerical test for the noisy input data, we apply the spline function together with the Tikhonov regularization to compute the Laplacian of noisy $B_z$ with regularizing parameter $\alpha$. Such a technique can be found in [26] for computing the derivative of a function with single variable.

**Case 2:** Numerical performance for noisy input data.

In this case, we simulate the magnetic field by (21) for given $\sigma^*$. Then we generate the random noisy data of $B_z$ at the discrete points in the following way:
\[
B_z^\delta(i, j) := B_z(i, j) + \max_{l,m \in S} |B_z(l, m)| \times \delta \times \text{rand}(i, j), \quad (i, j) \in S,
\] (26)
where $\mathbb{S}$ is the discrete point set of $\Omega$, $\text{rand}$ is the random number generator in $(-1,1)$. In computing $\Delta B_z^\delta$, we firstly use the spline function to construct the approximation of $B_z$ from $B_z^\delta$, which is a minimizer of some regularization functional. Then we apply the Laplacian to this approximate function. For details of constructing this approximate function in one-dimensional case, the readers are refereed to [7, 26]. By using this technique, we try to weaken the ill-posedness of the differentiation computations for noisy data. Using different Tikhonov regularization parameters, the reconstruction with 1% perturbation (that is, $\delta = 0.01$) by 6 iterations are given in Figure 6.

**Figure 6.** Result for noisy input data with $\alpha = 1E-04$ and $\alpha = 1E-05$, where $\alpha$ is the regularizing parameter for computing $\Delta B_z$.

**Example 2.** We keep the configuration in Example 1 unchanged and consider the conductivity distribution

$$
\sigma^*(x, y) = \begin{cases} 
3.1, & f_1(x, y) \leq 0.2 \\
-10f_1(x, y)^2 + 3.5, & 0.2 \leq f_1(x, y) \leq 0.5, \\
1.3, & f_2(x, y) \leq 0.1 \\
-10f_2(x, y)^2 + 1.4, & 0.1 \leq f_2(x, y) \leq 0.2, \\
1, & \text{otherwise}
\end{cases}
$$

with

$$
f_1(x, y) = \sqrt{(x - 0.2)^2 + (y - 0.8)^2}, \quad f_2(x, y) = \sqrt{(x + 0.4)^2 + (y + 1)^2}.
$$

This conductivity distribution contains two inhomogeneous parts under the constant background, which is more practical. Using the exact $\Delta B_z$ data, the reconstruction by 2 and 6 iterations are shown in Figure 7.

Now let us consider the perturbation case. In this model, we add 1%, 3% error respectively in the same way as that in Example 1. The results by 6 iterations are shown in Figure 8.

From these figures, we see that under the noisy case, the small inhomogeneous part is hard to be identified, even if some regularizing procedure has been taken to smooth the error data. This phenomenon is reasonable, since the small conductivity implies that the magnetic filed in that part is also weak for fixed boundary input current, and therefore the SNR is lower. These results reveal the sensitivity of the MREIT inversion algorithms from the numerical point
of view. Therefore, in order to apply MREIT techniques in the practical models with error measurements, some efficient \textit{a-priori} denoising techniques as well as the regularizing method should be taken.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7}
\caption{Results for different iterations with exact data}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8}
\caption{Results for 1\% and 3\% perturbation by 6 iterations}
\end{figure}

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**References**

[1] Alessandrini G, Isakov V and Powell J 1995 Local uniqueness of the inverse conductivity problem with one measurement \textit{Trans. Amer. Math. Soc.} \textbf{347} 3031-3041

[2] Bertero M and Boccaci P 1998 \textit{Introduction to Inverse Problems in Imaging} (London: IOP Publishing)

[3] Blue R S, Isaacson D and Newell J C 2000 Real-time three dimensional electrical impedance imaging \textit{Physiol. Meas.} \textbf{21} 15-26
[4] Cheny M, Isaacson D and Newell J C 1999 Electrical impedance tomography SIAM Review 41 85-101
[5] Eyuboglu M, Reddy R and Leigh J S 1998 Imaging electrical current density using nuclear magnetic resonance Elektrik 6 201-214
[6] Gambda H R, Bayford D and Holder D 1999 Measurement of electrical current density distribution in a simple head phantom with magnetic resonance imaging Phys. Med. Biol. 44 281-291
[7] Hank M and Scherzer O 2001 Inverse problems light: numerical differentiation Amer. Math. Monthly 108 512-521
[8] Ider Y Z and Birgul O 1998 Use of the magnetic field generated by the internal distribution of inject currents for electrical impedance tomography (MR-EIT) Elektrik 6 215-225
[9] Isokov V 1997 Inverse Problems for Partial Differential Equations (Berlin: Springer-Verlag)
[10] Khang H S, Lee B I, Oh S H, Woo E J, Lee S Y, Cho M H, Kwon O I, Yoon J R and Seo J K 2002 J-substitution algorithm in magnetic resonance electrical impedance tomography (MREIT): phantom experiments for static resistivity images IEEE Trans. Med. Imaging 21 No.6 695-702
[11] Kim Y J, Kwon O, Seo J K and Woo E J 2003 Uniqueness and convergence of conductivity image reconstruction in magnetic resonance electrical impedance tomography Inverse Problems 19 No.5 1213-1225
[12] Kwon O, Woo E J, Yoon J R and Seo J K 2002 Magnetic resonance electrical impedance tomography (MREIT): simulation study of J-substitution algorithm IEEE Trans. Biomed. Eng. 49 No.3 160-167
[13] Lee B I, Oh S H, Woo E J, Lee S Y, Cho M H, Kwon O, Seo J K, Lee J Y and Baek W S 2003 Three-dimensional forward solver and its performance analysis in magnetic resonance electrical impedance tomography (MREIT) using recessed electrodes Phys. Med. Biol. 48 1971-1986
[14] Lee B I 2003 Three dimensional forward solver and its applications in magnetic electric impedance tomography Thesis of Ph.D in Department of Electronic Engineering, Kyung Hee University
[15] Lee B I, Oh S H, Woo E J, Lee S Y, Cho M H, Kwon O, Seo J K and Baek W S 2003 Static resistivity image of a cubic saline phantom in magnetic resonance electrical impedance tomography (MREIT) Physiol. Meas. 24 579-89
[16] Lionheart W 2005 The reconstruction problem, D.S.Holder ed, Electrical Impedance Tomography, Methods, History and Applications (London: IOP Publishing)
[17] Liu J J, Pyo H C, Seo J K and Woo E J 2006 Convergence properties and stability issues in MREIT algorithm Contemporary Mathematics 408 201-218
[18] Liu J J, Seo J K, Sini M and Woo E J On the convergence of the harmonic Bz algorithm in magnetic resonance electrical impedance tomography SIAM J. Appl. Math. (to appear)
[19] Oh S H, Lee B I and Woo E J et al 2003 Reconstruction of conductivity and current density images using only one component of magnetic field measurements IEEE Trans. Biomed. Eng. 50 No.9 1121-1124
[20] Oh S H, Lee B I, Woo E J, Lee S Y, Cho M H, Kwon O and Seo J K 2003 Conductivity and current density image reconstruction using harmonic Bz algorithm in magnetic resonance electrical impedance tomography Phys. Med. Biol. 48 3101-3106
[21] Park C, Kwon O, Woo E J and Seo J K 2004 Electrical conductivity imaging using gradient Bz decomposition algorithm in magnetic resonance electrical impedance tomography (MREIT) IEEE Trans. Med. Imag. 23 388-394
[22] Scott G C, Joy M L G, Armstrong R L and Henkelman R M 1991 Measurement of nonuniform current density by magnetic resonance IEEE Trans. Med. Imag. 10 362-374
[23] Scott G C, Joy M L G, Armstrong R L and Henkelman R M 1992 Sensitivity of magnetic current density imaging J. of Mag. Reson. 10 235-254
[24] Seo J K, Yoo J R, Woo E J and Kwon O 2003 Reconstruction of conductivity and current density images using only one component of magnetic field measurements IEEE Trans. Biomed. Eng. 50 No.9 1121-1124
[25] Vauhkonen M, Vadasz D, Karjalainen P A, Somersalo E and Kaipio J P 1998 Tikhonov regularization and prior information in electrical impedance tomography IEEE Trans Med. Imaging 19 285-293
[26] Wang Y B, Jia X Z and Cheng J 2002 A numerical differentiation method and its application to reconstruction of discontinuity Inverse Problems 18 1461-1476
[27] Woo E J, Seo J K and Lee S Y 2005 Magnetic resonance electrical impedance tomography, D.S.Holder ed, Electrical Impedance Tomography, Methods, History and Applications (London: IOP Publishing)
[28] Woo E J 1990 Finite element method and reconstruction algorithms in electrical impedance tomography Ph.D Thesis in University of Wisconsin-Madison
[29] Woo E J, Lee S Y, Seo J K, Kwon O, Oh S H and Lee B I 2004 Conductivity images of biological tissue phantoms using a 3.0T Tesla MREIT system *Proc. Int. Conf. IEEE Engi. Med. Biol. Soc.*

[30] Zhang N 1992 Electrical impedance tomography based on current density imaging *MS Thesis in Department of Electrical Engineering, University of Toronto, Canada*