Applying the Kövári-Sós-Turán Theorem to a Question in Group Theory

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Abstract. Let \( m \leq n \) be positive integers and \( \mathcal{X} \) a class of groups which is closed for subgroups, quotient groups and extensions. Suppose that a finite group \( G \) satisfies the condition that for every two subsets \( M \) and \( N \) of cardinalities \( m \) and \( n \), respectively, there exist \( x \in M \) and \( y \in N \) such that \( \langle x, y \rangle \in \mathcal{X} \). Then either \( G \in \mathcal{X} \) or \( |G| \leq \left( \frac{180}{53} \right)^m (n - 1) \).

Let \( m, n \) be positive integers and \( \mathcal{X} \) be a class of groups. We say that a group \( G \) satisfies the condition \( \mathcal{X}(m, n) \) if for every two subsets \( M \) and \( N \) of cardinalities \( m \) and \( n \), respectively, there exist \( x \in M \) and \( y \in N \) such that \( \langle x, y \rangle \in \mathcal{X} \). If \( G \) satisfies the condition \( \mathcal{X}(m, n) \), then we write \( G \in \mathcal{X}(m, n) \).

In [5] M. Zarrin proposed the following question.

Question 1. Let \( G \) be a finite group and \( G \notin \mathcal{X} \). Does there exist a bound (depending only on \( m \) and \( n \)) for the size of \( G \) if \( G \) satisfies the condition \( \mathcal{X}(m, n) \)?

An affirmative answer is given in [5] for the class of nilpotent groups. In an earlier paper R. Bryce gave a positive solution for the class of supersoluble groups, under the additional condition \( n = m \). In this short note we prove that an affirmative question can be given whenever \( \mathcal{X} \) is a class of finite groups which is closed for subgroups, quotient groups and extensions. Our argument relies on the Kövári-Sós-Turán theorem [3], stating that, if \( m \leq n \) are two positive integers, then a graph with \( t \) vertices and at least \( \left( \frac{(n - 1)^{1/m}2^{1/m} + (m - 1)2}{2} \right) \) edges, contains a copy of the complete bipartite graph \( K_{m,n} \). The crucial observation is the following:

Theorem 1. Let \( \mathcal{X} \) be a class of groups and suppose that there exists a real positive number \( \gamma \) with the following property: if \( \mathcal{X} \) is a finite group and the probability that two randomly chosen elements of \( \mathcal{X} \) generate a group in \( \mathcal{X} \) is greater than \( \gamma \), then \( \mathcal{X} \) is in \( \mathcal{X} \). If \( m \leq n \), then

\[
|G| \leq \left( \frac{2}{1 - \gamma} \right)^m (n - 1)
\]

for any \( G \in \mathcal{X}(m, n) \setminus \mathcal{X} \).

Proof. Let \( G \in \mathcal{X}(m, n) \setminus \mathcal{X} \). Consider the graph \( \Gamma_\mathcal{X}(G) \) whose vertices are the elements of \( G \) and in which two vertices \( x_1 \) and \( x_2 \) are joined by an edge if and only if \( \langle x_1, x_2 \rangle \notin \mathcal{X} \) and let \( \eta \) the number of edges of \( \Gamma_\mathcal{X}(G) \). Since \( G \notin \mathcal{X} \), the probability that two vertices of \( \Gamma_\mathcal{X}(G) \) are joined by an edge is at least \( 1 - \gamma \), so we must have

\[
\eta \geq \frac{(1 - \gamma)|G|^2}{2}.
\]
On the other hand, since \( G \in \mathcal{X}(m, n) \), \( \Gamma_X(G) \) cannot contain the complete bipartite graph \( K_{m,n} \) as a subgraph. By the Kövári-Sós-Turán theorem,

\[
\eta \leq \frac{(n-1)^{1/m}|G|^{2-1/m} + (m-1)|G|}{2}
\]

Combining (0.1) and (0.2) we deduce

\[
\left(\frac{n-1}{|G|}\right)^{1/m} + \frac{n-1}{|G|} \geq \left(\frac{n-1}{|G|}\right)^{1/m} + \frac{m-1}{|G|} \geq 1 - \gamma.
\]

We may assume \( |G| \geq n-1 \). This implies \( \left(\frac{n-1}{|G|}\right)^{1/m} \geq \frac{n-1}{|G|} \) and therefore it follows from (0.3) that

\[
\left(\frac{n-1}{|G|}\right)^{1/m} \geq \frac{1 - \gamma}{2}.
\]

This implies

\[
|G| \leq \left(\frac{2}{1 - \gamma}\right)^m (n-1). \quad \Box
\]

Guralnick and Wilson [2], using the classification of the finite simple groups, proved the following result. There exists a real number \( \kappa \), strictly between 0 and 1, with the following property: let \( \mathcal{X} \) be any class of finite groups which is closed for subgroups, quotient groups and extensions, and let \( G \) be a finite group; if the probability that two randomly chosen elements of \( G \) generate a group in \( \mathcal{X} \) is greater than \( \kappa \), then \( G \) is in \( \mathcal{X} \). Combining [2, Proposition 5] with [4, Theorem 1.1], one may deduce that \( \kappa \) can be taken to be \( \frac{47}{90} = \max\left(1 - \frac{53}{90}, \frac{5}{18}\right) \). This allows us to deduce our main result.

**Corollary 2.** Let \( \mathcal{X} \) be any class of finite groups which is closed for subgroups, quotient groups and extensions, and let \( G \) be a finite group. If \( m \leq n \) are positive integers and \( G \in \mathcal{X}(m, n) \setminus \mathcal{X} \), then \( |G| \leq \left(\frac{180}{35}\right)^m (n-1) \).

With the same argument, combining Theorem 1 with [2, Theorem A] (see also the remark in [2] following the statement of Theorem A), we deduce the following results, the first of which is an improvement of [5, Theorem 3.6].

**Corollary 3.** Let \( m \leq n \) be positive integers and \( G \) a finite group.

1. If \( \mathcal{X} \) is the class of nilpotent groups and \( G \in \mathcal{X}(m, n) \setminus \mathcal{X} \), then
   \[
   |G| \leq 4^m (n-1).
   \]

2. If \( \mathcal{X} \) is the class of soluble groups and \( G \in \mathcal{X}(m, n) \setminus \mathcal{X} \), then
   \[
   |G| \leq \left(\frac{60}{19}\right)^m (n-1).
   \]

3. If \( \mathcal{X} \) is the class of finite groups of odd order and \( G \in \mathcal{X}(m, n) \setminus \mathcal{X} \), then
   \[
   |G| \leq \left(\frac{8}{3}\right)^m (n-1).
   \]
References

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