Improved LP-based Approximation Algorithms for Facility Location with Hard Capacities

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Abstract

We present LP-based approximation algorithms for the capacitated facility location problem (CFL), a long-standing problem with intriguing unsettled approximability in the literature dated back to the 90s. We present an elegant iterative rounding scheme for the MFN relaxation that yields an approximation guarantee of \((10 + \sqrt{67})/2 \approx 9.0927\), a significant improvement upon the previous LP-based ratio of 288 due to An et al. in 2014.

For CFL with cardinality facility cost (CFL-CFC), we present an LP-based 4-approximation algorithm, which not only surpasses the long-standing ratio of 5 due to Levi et al. that ages up for decades since 2004 but also unties the long-time match to the best approximation for CFL that is obtained via local search in 2012. Our result considerably deepens the current understanding for the CFL problem and indicates that an LP-based ratio strictly better than 5 in polynomial time for the general problem may still be possible to pursue.

1 Introduction

We consider the facility location problem with hard capacities (CFL), a long-standing problem with intriguing unsettled approximability and literature dated back to the 90s. In this problem, we are given a set \(F\) of facilities, a set \(D\) of clients, and a distance metric \(c\) defined over \(F \cup D\). Each \(i \in F\) is associated with an open cost \(o_i\) and a capacity \(u_i\), which is the number of clients it can serve when opened up. The cost of serving a client \(j\) using a facility \(i\) equals the distance between them. The goal is to determine a set of facilities to open up and an assignment of the clients to the opened facilities which respects their capacity limits so that the total cost is minimized.

The CFL problem was first considered by Shmoys, Tardos, and Aardal [13]. For facilities with uniform capacities, Koropolu et al. [8] showed that the local search heuristic proposed by Kuehn and Hamburger [9] yields a constant factor approximation. Chudak and Williamson [6] improved the analysis of Korupolu et al. [8] and obtained a \((6, 5)\)-approximation, i.e., a solution whose cost is bounded by 6 times the facility cost plus 5 times the service cost of an optimal solution. Aggarwal et al. [2] introduced the idea of taking suitable linear combinations of inequalities which captures the local optimality and obtained a 3-approximation.

For facilities with non-uniform capacities, i.e., the general CFL problem, Pal et al. [12] presented a \((9, 5)\)-approximation based on local search algorithm. The ratio was improved by Mahdian and Pal [11] to an \((8, 7)\)-approximation. Zhang et al. [15] introduced the idea of multi-exchange local operations and further improved the ratio to \((6, 5)\). The algorithm was later modified by Bansal et al. [4] to achieve a 5-approximation, which is the best ratio known for the CFL problem.
In contrast to the rich LP-based toolboxes developed for the uncapacitated facility location problem (UFL), the fact that no LP-based algorithms with constant approximation guarantee were known for CFL was intriguing and surprising. In fact, devising an LP-based approximation with constant guarantee for CFL was listed as one of ten open problems in the textbook on approximation algorithm due to Williamson and Shmoys [14]. This problem was resolved by the notable work of An, Singh, and Svensson [3], in which a strong multi-commodity flow network (MFN) relaxation with constant integrality gap is presented. The approximation guarantee obtained in this work, however, is in the order of couple hundreds, and it remains an open problem to devise a better LP-based guarantee for CFL or a better integrality gap for the MFN relaxation.

In the pursuit of settling down the approximability of CFL, an important variation between the general problem and the case with uniform capacities is when we have cardinality-type facility costs, i.e., uniform facility cost, This was studied by Levi, Shmoys, and Swamy [10], in which an LP-based 5-approximation was presented. Interestingly, the approximation ratio of 5 not only remains the best known ratio for a long time since 2004, but also coincide with the best known approximation result obtained later for the general CFL problem [4] using local search heuristic.

1.1 Our Contributions

We present improved LP-based approximation algorithms for the general CFL problem and the case with cardinality facility costs. Our first result is the following theorem.

**Theorem 1.** There is an LP-based algorithm for CFL that produces a \((10 + \sqrt{67}) / 2 \approx 9.0927\)-approximation in polynomial-time.

This yields a significant improvement upon the previous LP-based ratio of 288 due to An et al. [3] in 2014. Our approximation algorithm for CFL is built on an elegant iterative rounding scheme for the MFN relaxation that combines the ideas from several previous works [1,3,10]. We believe that our result considerably deepens the current understanding for the CFL problem.

The second result we present is the following theorem for capacitated facility location problem with cardinality facility cost (CFL-CFC).

**Theorem 2.** There is an LP-based algorithm for CFL-CFC that produces a 4-approximation in polynomial-time.

This surpasses the decades-old ratio of 5 due to Levi et al. [4] which ages up since 2004 and unitiess the long-time match to the best approximation for CFL that is obtained via local search in 2012. This sends a message that, an LP-based ratio strictly better than 5 for the general CFL problem may still be possible to pursue. Our algorithm for CFL-CFC is built on a two-staged rounding scheme that combines a novel set of ideas with the results developed in the past for both facility location and capacitated covering problems [1,5,7,10].

**Overview of the Algorithms**

The core part of our results can be seen as rounding procedures for small instances incurred in the LP relaxations, i.e., the rounding decisions for the small facilities along with the assignments made to them. Our proposed procedures aim at fractionally serving the clients while making sure that the rounded facilities of interests (if not all) are reasonably sparsely-loaded by the assignments.

Our algorithm for CFL inherits the “round-or-separate” framework from An et al. [3] with a distinguishable set of ideas and philosophies behind. Inside each iteration incurred in this framework, we are given as input a candidate solution and the task is either to round this candidate
solution integrally, or to produce a separating hyperplane so that Ellipsoid method can proceed. Our result for this part is the following theorem.

**Theorem (Sketch)**. Given a candidate solution \((x, y)\) for the MFN relaxation and a target parameter \(\alpha\) with \(0 < \alpha \leq 1/3\), we can compute in polynomial-time either (i) a hyperplane separating \((x, y)\) and the feasible region, or (ii) an integral solution \((x^*, y^*)\) rounded from \((x, y)\) such that

\[
\text{cost}(x^*, y^*) \leq \max \left\{ \frac{3}{2\alpha}, \frac{7 - 4\alpha}{(1 - \alpha)^2} \right\} \times \text{cost}(x, y).
\]

In our algorithm, instead of guaranteeing a firm fraction of demand be sent to the small instances, as was did in the previous work by An et al. [3], we emphasize that, with proper construction, the large facilities are sparsely-loaded by the flow sent to them and therefore are ready for a reasonable rounding blow-up of \(1/(1 - \alpha)\) when necessary.

Our rounding procedure for the small facilities is built on the idea due to Abraham et al. [1] and a couple of prior works developed for uncapacitated facility location. In each iteration, the facility with the least “per-unit-flow-rerouting-cost”, i.e., the cost incurred if we relocate flow *simultaneously* and *proportionally* from *all facilities in the vicinity* to the facility, is selected to be rounded. After the rounding process is done, we guarantees that at least \((1 - \alpha)\)-fraction of each client is assigned.

To bound the extra rerouting cost incurred during the process, in the previous works [1, 3] a filtering technique is applied to ensure a low assignment radius for each client throughout the process. However, this technique inevitably causes a tremendous blowup in the resulting guarantee. In our result, we apply an implicit primal-dual schema for an exact pricing on the cost incurred. This bounds the assignment radius tightly for each client that gets reassigned. To ensure that correct pricing is obtained for the need of our rounding process in each iteration, we apply this technique in an iterative manner. Together this yields our approximation for CFL.

Our approximation algorithm for CFL-CFC is built on a two-staged rounding scheme that combines a novel set of ideas with the results developed in the past for both facility location and covering problems [1, 5, 7, 10]. As is done in our first result, we apply primal-dual schema to ensure a tightly-bounded assignment radius for each client throughout the rounding process. However, the large facilities are no longer sparsely-loaded by the assignment given by natural LPs.

To overcome this issue, when the residue demand of a client drops below the target threshold, we discard the residue client and redistribute part of it to the large facilities in the vicinity to form the so-called “outlier clients.” The outlier clients participate in the rounding process and act as normal clients except for that, there is no threshold for the outlier clients to be discarded, and we guarantee that the outlier clients will be fully-assigned for the final feasibility.

To achieve this guarantee, in the second stage of the rounding process, we formulate the assignment problem of the remaining outlier clients as a carefully designed assignment LP. We apply a technique, which is originally developed for the capacitated covering problems [5, 7], to show that, basic feasible solutions of this simple LP corresponds naturally to an implicit matching between the large facilities at which the outlier clients reside and the non-integral facilities given by this LP. Together this yields a bound for our final rounding.

**Organization of this paper.** The rest of this paper is organized as follows. In the rest of this section we describe the preliminaries necessary to present our approximation algorithms. We present our approximation algorithm for CFL in Section 2, page 5, and our approximation algorithm for CFL-CFC in Section 3, page 8.

The additional content is organized as follows. We establish our approximation guarantee for CFL in Section 4, page 11, and our guarantee for CFL-CFC in Section 5, page 26.
1.2 Preliminaries

In the CFL problem, we are given a set $F$ of facilities, a set $D$ of clients, and a distance metric $c$ defined over $F \cup D$. Each $i \in F$ is associated with an open cost $o_i$ and a capacity $u_i$, which is the number of clients it can serve when opened up. A facility multiplicity function $y: F \rightarrow [0, 1]$ denotes the decision whether each facility is selected to be opened up. A client assignment $x: F \times D \rightarrow [0, 1]$ specifies the decision whether client $j$ is assigned to facility $i$ (served by facility $i$). A solution for CFL consists of a multiplicity function $y$ and an assignment function $x$ such that the following conditions are met: (a) $\sum_{i \in F} x_{i,j} \geq 1$, for each $j \in D$. (b) $\sum_{j \in D} x_{i,j} \leq u_i \cdot y_i$, for each $i \in F$. (c) $x_{i,j} \leq y_i$, for each $i \in F, j \in D$. The cost of a solution $(x, y)$ is defined to be

$$\psi(x, y) := \sum_{i \in F} o_i \cdot y_i + \sum_{i \in F, j \in D} c_{i,j} \cdot x_{i,j}.$$ 

Given an instance $\Psi = (F, D, c, o, u)$ of CFL, the goal of this problem is to compute an integral solution $(x, y)$ such that $\psi(x, y)$ is minimized.

The MFN Relaxation. As the natural LP formulation for CFL is known to have an unbounded integrality gap even for simple settings, An, Singh, and Svensson [3] designed a strong LP relaxation based on multicommodity flow networks (MFN). The idea is to impose Knapsack-cover type constraints, formulated as reassignable partial assignments given as free in each qualifying test on both the facility values and the assignments of the clients. In the following we introduce the framework and the construction of this relaxation.

Definition 1 (Partial Assignments). A partial assignment $g$ is a function $g: F \times D \rightarrow [0, 1]$. The partial assignment $g$ is said to be valid if (i) $\sum_{i \in F} g_{i,j} \leq 1$, for each $j \in D$, and (ii) $\sum_{j \in D} g_{i,j} \leq u_i$, for each $i \in F$.

Given a candidate fractional solution $(x, y)$ and a valid partial assignment $g$, An, Singh, and Svensson [3] designed the following multi-commodity flow network, denoted $\text{MFN}_\Psi(x, y, g)$, as a qualifying test for the candidate solution $(x, y)$.

Definition 2 (Multi-commodity Flow Network). For a valid partial assignment $g$ and a candidate solution $(x, y)$, $\text{MFN}_\Psi(x, y, g)$ is a multi-commodity flow network defined as follows.

- Each client $j \in D$ corresponds to two nodes $j^s$ and $j^t$ in the network and is associated with a commodity $j$ with source-sink pair $(j^s, j^t)$ and demand $r_j^{(g)} := 1 - \sum_{i \in F} g_{i,j}$.
- Each facility $i \in F$ corresponds to two nodes $i^s$ and $i^t$ that are connected by an arc $(i, i')$ of capacity $u_i^{(g)} := y_i \cdot \left(u_i - \sum_{j \in D} g_{i,j}\right)$.
- For each $j \in D$ and each $i \in F$, there is an arc $(j^s, i)$ of capacity $x_{i,j}$, an arc $(i, j^s)$ of capacity $g_{i,j}$, and an arc $(i', j^t)$ of capacity $r_{j}^{(g)} \cdot y_i$.

For any $i \in F, j \in D$, let $P_{(x,y)}^{(g)}(i, j)$ to denote the set of paths in $\text{MFN}_\Psi(x, y, g)$ for commodity $j$ to sink via $i^t$. The superscript $(g)$ and the subscript $(x, y)$ is omitted when there is no confusion in the context. Let $P := \cup_{i \in F, j \in D} P_{(x,y)}^{(g)}(i, j)$ to denote the set of possible paths. See also Figure 1 for an illustration on the construction of $\text{MFN}_\Psi(x, y, g)$ and the corresponding constraints.

Lemma 3 (An, Singh, Svensson [3]). Given an instance $\Psi = (F, D, c, o, u)$ of CFL, the constraints defined by $\text{MFN}_\Psi(x, y) := \{ \text{MFN}_\Psi(x, y, g) \text{ feasible : } \forall \text{ valid } g \}$ is a valid relaxation for integral solutions on $\Psi$. Furthermore, the separation problem for the feasibility of $\text{MFN}_\Psi(x, y, g)$ for any valid $g$ can be answered in weakly polynomial-time, and a basic feasible flow can be obtained.
2 LP-based Approximation for CFL

In this section we describe our approximation algorithm for CFL and prove Theorem 1. Our algorithm inherits the round-or-separate framework used in [3] for producing candidate solutions that can either be rounded or separated. For completeness we first describe the details of the framework in the following. Then we present our modified construction for obtaining sparsely-loaded flow and our iterative rounding scheme.

The Round-or-Separate Framework. Let Ψ = (F, D, o, c, u) be an instance of CFL. The algorithm starts by guessing the cost of the optimal solution using a standard binary search. For each guess, say, γ, the algorithm applies the Ellipsoid algorithm on LP-(MFN) in Figure 2 with cost γ. For each separation problem incurred in the Ellipsoid algorithm, say, for (x′, y′), we apply Theorem 4, which is stated below, for either a separating hyperplane or an integral solution (x(γ), y(γ)) with the claimed approximation guarantee.

When an integral solution is successfully rounded or when the Ellipsoid algorithm concludes the infeasibility of (x′, y′), the algorithm proceeds to the next iteration of the binary search process until the desired precision is attained. When the binary search process terminates, the algorithm outputs (x(γ0), y(γ0)) for the smallest γ0 for which an integral solution is successfully rounded when γ0 is considered.

Our result for CFL is the following theorem.

Theorem 4. Given a candidate solution (x′, y′) for LP-(MFN) and a target parameter α with 0 < α ≤ 1/3, we can compute in polynomial-time either (i) a separating hyperplane for (x′, y′) and LP-(MFN), or (ii) an integral solution (x*, y*) rounded from (x′, y′) for Ψ with

\[ \psi(x^*, y^*) \leq \max \left\{ \frac{3}{2\alpha}, \frac{7 - 4\alpha}{(1 - \alpha)^2} \right\} \times \psi(x', y'). \]
To prove Theorem 1, it suffices to prove the statement of Theorem 4 and observe that the specific choice of \( \alpha := \frac{\sqrt{67} - 10}{11} \) gives the claimed result. We also note that, it suffices to consider the MFN-type constraints as the remaining constraints in LP-(MFN) can be verified directly.

In the rest of this section, we describe our iterative rounding process that establishes Theorem 4. We complete the proof in Section 4, page 11.

**Obtaining a Sparsely-Loaded Flow or a Separating Hyperplane.** Let \((x', y')\) be a candidate solution and \(0 < \alpha \leq 1/3\) be the target parameter. We classify the facilities as follows. Let \( I := \{ i \in F : 0 < y'_i < \alpha \} \) and \( U := \{ i \in F : y'_i \geq \alpha \} \). The facilities in \( U \) are further classified into two categories. Let

\[
U^{(>)} := \left\{ i \in U : \sum_{j \in D} x'_{i,j} > (1 - \alpha) \cdot u_i \right\}
\]

and \( U^{(\leq)} := U \setminus U^{(>)} \). Provided that the facilities in \( U \) are to be rounded up in the approximate solution, we know that the assignments to \( U^{(\leq)} \) are ready for rounded up as well.

Consider the maximum b-matching problem in the bipartite graph \( G = (D, U^{(>)} , E) \), where for each \( j \in D \) and \( i \in U^{(>)} \), there exists an edge \((j, i)\) in \( E \) with capacity \( x'_{i,j}/(1 - \alpha) \). Solve the maximum b-matching problem on \( G \) for an optimal assignment \( h \).

We say that a client \( j \in D \) is partially-assigned in \( h \) if \( \sum_{i \in U^{(>)} } h_{i,j} < 1 \) and fully-assigned otherwise. We say that an alternating path \( P \) between \( D \) and \( U^{(>)} \) that starts at a partially-assigned client \( j \in D \) is an augmenting path if the following holds.

- For each \((j', i') \in P\), where \( j' \in D, i' \in U^{(>)} \), we have \( h_{i',j'} < x'_{i,j'}/(1 - \alpha) \).
- For each \((i', j') \in P\), where \( i' \in U^{(>)} , j' \in D \), we have \( h_{i',j'} > 0 \).

Intuitively, an augmenting path that starts from a partially-assigned \( j \) is a way of increasing its assignment without altering the optimality of maximum b-matching between \( D \) and \( U^{(>)} \).

We say that a facility \( i \in U^{(>)} \) is tightly-occupied if there exists an augmenting path between a partially-assigned client and \( i \). For each \( i \in F, j \in D \), define the partial assignment \( g_{i,j} \) as

\[
g_{i,j} := \begin{cases} 
 h_{i,j}, & \text{if } i \in U^{(>)} \text{ is tightly-occupied}, \\
 0, & \text{otherwise}.
\end{cases}
\]

Apply Lemma 3 for either a basic feasible flow \( f \) or a separating hyperplane for \( \text{MFN}_\Psi(x', y'', g) \), where \( y'' := 1 \) for all \( i \in U \) and \( y'' := y'_i \) otherwise. If a violated constraint is found, report it to the Ellipsoid algorithm. Otherwise, we proceed to round the flow \( f \).

Since feasible solutions of \( \text{MFN}_\Psi(x', y', g) \) are also feasible for \( \text{MFN}_\Psi(x', y'', g) \), hyperplanes separating \( \text{MFN}_\Psi(x', y'', g) \) also separate \( \text{MFN}_\Psi(x', y', g) \). Furthermore, since \( f \) is a basic feasible solution, it follows that the number of paths on which nonzero flow are sent in \( f \) is at most the number of constraints in \( \text{MFN}_\Psi(x', y'', g) \), which is polynomial in \( F \) and \( D \).

**Rounding the Small Instance via Sparsely-Loaded Flow.** In the following we describe our iterative rounding process for facilities in \( I \) using the flow \( f \) obtained in the previous step.

During the rounding process, the algorithm maintains a parameter tuple \( \Psi' = (F', D', r') \) which denotes the remaining instance to be processed, where \( r' \) is the residue demand of each \( j \in D' \). The algorithm updates the parameter tuple gradually during the process until \( D' \) becomes empty.
Initially, for each $j \in D$, the residue demand of $j$ is defined to be $r_j' := \sum_{i \in I, p \in P'(i,j)} f_{ip}$, where $P'(i,j)$ is the set of paths for which nonzero flow for commodity $j$ is sent to sink via $i$ in $f$. The algorithm sets $F' := I$ and $D' := \{ j \in D : r_j' > \alpha \cdot r_j \}$, where $r_j := 1 - \sum_{i \in U(>)} g_{i,j}$ is the demand to be sent in $\text{MFN}_\Psi(x', y'', g)$ for commodity $j \in D$.

For any $i \in F'$ with $\sum_{j \in D'} x_{i,j} > 0$ and any $(x, y)$ of interests, define the per-unit-assignment rerouting cost to $i$ as

$$\theta(i, D', x, y) := \frac{1}{\sum_{j \in D'} x_{i,j}} \cdot \left( 3 \cdot o_i \cdot y_i + 2 \cdot \sum_{j \in D'} c_{i,j} \cdot x_{i,j} \right).$$

Provided that $(x, y)$ is optimal for LP-(M) given in Figure 3, standard complementary slackness guarantees that $c_{i,j} \leq \lambda_j$ for all $x_{i,j} > 0$, where $\lambda$ comes from an optimal solution for LP-(DM). Furthermore, it follows that $3 \cdot o_i \cdot y_i + 2 \cdot \sum_{j \in D'} c_{i,j} \cdot x_{i,j} = o_i \cdot y_i + 2 \cdot \sum_{j \in D'} \lambda_j \cdot x_{i,j}$, and $\theta(i, D', x, y)$ corresponds implicitly and naturally to the cost incurred if we reroute the assignments which are originally assigned to facilities in the vicinity of $i$ simultaneously and proportionally to $i$.

![Table: Natural LP Formulations for Iterative Rounding Process](image)

| Minimize $\psi(x, y)$ | LP-(M) | Maximize $\sum_{j \in D'} r_j' \cdot \lambda_j - \sum_{i \in F'} \frac{1}{2} \cdot (1 - \alpha) \cdot \eta_i$ | LP-(DM) |
|-----------------------|--------|-------------------------------------------------|--------|
| $\sum_{i \in F'} x_{i,j} = r_j', \quad \forall j \in D'$ | | $\sum_{j \in D'} r_j' \cdot \lambda_j - \sum_{i \in F'} \frac{1}{2} \cdot (1 - \alpha) \cdot \eta_i$ | |
| $\sum_{j \in D'} x_{i,j} \leq u_i \cdot y_i, \quad \forall i \in F'$ | | $\lambda_j \leq \beta_i + \Gamma_{i,j} + c_{i,j}, \quad \forall i \in F', j \in D'$ | |
| $x_{i,j} \leq \frac{2\alpha}{1-\alpha} \cdot r_j \cdot y_i, \quad \forall i \in F', j \in D'$ | | $u_i \cdot \beta_i + \frac{2\alpha}{1-\alpha} \cdot \sum_{j \in D'} r_j \cdot \Gamma_{i,j} \leq o_i + \eta_i, \quad \forall i \in F'$ | |
| $y_i \leq \frac{1-\alpha}{2}, \quad \forall i \in F'$ | | $\lambda_j \in \mathbb{R}, \beta_i, \Gamma_{i,j}, \eta_i \geq 0, \quad \forall i \in F', j \in D'$ | |
| $x_{i,j} \geq 0, y_i \geq 0, \quad \forall i \in F', j \in D'$ | |

Figure 3: The natural LP formulations for our iterative rounding process.

In the following, we describe the rounding process in details. In each iteration, the algorithm solves LP-(M) on the current parameter tuple $\Psi'$ for optimal $(x^\dagger, y^\dagger)$ and selects a facility $i \in F'$ to form a cluster. Depending on the status of the facilities in $F'$, we have two cases.

- If $y_i^\dagger = (1 - \alpha)/2$ for some $i \in F'$, then the algorithm picks an arbitrary $i$ with $y_i^\dagger = (1 - \alpha)/2$ and sets $o_k^\dagger = 0$ for all $k \in F' \setminus \{i\}$.
- If $y_i^\dagger < (1 - \alpha)/2$ for all $i \in F'$, then the algorithm selects among the facilities $i \in F'$ with $\sum_{j \in D'} x_{i,j}^\dagger > 0$, the facility $i$ with the minimum $\theta(i, D', x^\dagger, y^\dagger)$. Intuitively, the facility $i$ has the lowest per-unit-assignment rerouting cost, and any other facility in $F'$ can afford the rerouting cost within the cluster centered at $i$.

For each $j \in D'$, let $o_j^{(i)} := \left( (1 - \alpha)/(2y_i^\dagger) - 1 \right) \cdot x_{i,j}^\dagger$ denote the amount of assignment to be gathered from the vicinity of facility $i$ to $i$ via client $j$. 
For each $k \in F' \setminus \{i\}$, define the contribution (fraction) of $k$ towards $i$ to be

$$\sigma_k^{(i)} := \frac{1}{\sum_{\ell \in D'} x_{k,\ell}^{\top}} \cdot \sum_{j \in D'} \sigma_{k,j}^{(i)};$$

where $\sigma_{k,j}^{(i)} := \frac{x_{k,j}^{\top}}{\sum_{\ell \in F^{\prime \setminus \{i\}}} x_{\ell,j}^{\top}} \cdot \delta_j^{(i)}$ is the contribution of $k$ to be sent via client $j$. Note that, from the definition it follows that $\sum_{k \in F^{\prime \setminus \{i\}}} \sigma_{k,j}^{(i)} = \delta_j^{(i)}$, and the amount of flow to be gathered via $j$ can be fulfilled.

The algorithm updates the parameter tuple $\Psi'$ as follows. The algorithm removes $i$ from $F'$ and sets $r_j' := \sum_{k \in F'} (1 - \sigma_k^{(i)}) x_{k,j}^T$ for all $j \in D'$. For each $j \in D'$ with $r_j' \leq \alpha \cdot r_j$, the algorithm removes $j$ from $D'$. Then the algorithm proceeds to the next iteration until $D'$ becomes empty.

**Final Output.** When $D'$ becomes empty, the algorithm forms the integral multiplicity function $y^*$ by setting $y_i^* := 1$ for all $i \in U \cup (I \setminus F')$ and $y_i^* := 0$ otherwise. The algorithm solves the min-cost assignment problem on $D$ and $F^* := \{ i \in F : y_i^* = 1 \}$ for an optimal integral assignment $x^*$ and reports $(x^*, y^*)$ as the claimed integral solution.

### 3 4-Approximation for CFL-CFC

In the following, we describe our approximation algorithm $A$ for CFL-CFC and prove Theorem 2. Let $\Psi = (F, D, c, u)$ be an instance of CFL-CFC.

For the ease of presentation, for any $A \subseteq F$, any assignment $x$, and any $j \in D$, we use $N_{(A,x)}(j) := \{ i \in A : x_{i,j} > 0 \}$ to denote the set of facilities in $A$ to which $j$ is assigned in $x$. Similarly, for any $B \subseteq D$ and any $i \in F$, we use $N_{(B,x)}(i) := \{ j \in B : x_{i,j} > 0 \}$ to denote the set of clients in $B$ that is assigned to $i$ in $x$.

Solve $LP-(N)$ and its dual LP, given below in Figure 4, on $\Psi$ for optimal primal and dual solutions $(x^*, y^*)$ and $(\alpha, \beta, \Gamma, \eta)$. It follows that, $\alpha_j \geq \alpha_{i,j}$ for each $i \in F, j \in D$ with $x_{i,j}^T > 0$. We will use the fact that $\alpha_j$ is a valid estimation on the assignment radius for each $j \in D$ in $x^*$.

\[
\begin{align*}
\text{min} & \sum_{i \in F} y_i + \sum_{i \in F, j \in D} c_{i,j} \cdot x_{i,j} & \text{LP-(N)} \\
\sum_{i \in F} x_{i,j} & \geq 1, \quad \forall j \in D, \\
\sum_{j \in D} x_{i,j} & \leq u_i \cdot y_i, \quad \forall i \in F, \\
0 & \leq x_{i,j} \leq y_i, \quad \forall i \in F, j \in D, \\
0 & \leq y_i \leq 1, \quad \forall i \in F,
\end{align*}
\]

\[
\begin{align*}
\text{max} & \sum_{j \in D} \alpha_j - \sum_{i \in F} \eta_i & \text{LP-(DN)} \\
\alpha_j & \leq \beta_i + \Gamma_{i,j} + c_{i,j}, \quad \forall i \in F, j \in D, \\
u_i \cdot \beta_i + \sum_{j \in D} \Gamma_{i,j} & \leq 1 + \eta_i, \quad \forall i \in F, \\
\alpha_j, \beta_i, \Gamma_{i,j}, \eta_i & \geq 0, \quad \forall i \in F, j \in D.
\end{align*}
\]

Figure 4: The natural LP formulations for CFL-CFC.

**Initial Classification.** Let $I := \{ i \in F : 0 < y_i^* < \frac{1}{2} \}$ and $U := \{ i \in F : y_i^* \geq \frac{1}{2} \}$. The clients in $D$ are divided into three categories, namely, those that are served merely by $I$, those
that are served jointly by $I$ and $U$, and those that are served merely by $U$. Let

$$J^{(I)} := \left\{ j \in D : x'_{i,j} = 0 \text{ for all } i \in U \right\}, \quad J^{(I')} := \left\{ j \in D : \min_{i \in I} \left( \max_{i \in I} x'_{i,j}, \max_{i \in U} x'_{i,j} \right) > 0 \right\},$$

and $J^{(U)} := D \setminus \left( J^{(I)} \cup J^{(I')} \right)$ denote the clients in the three categories, respectively. Note that, in addition to the rounding problem for facilities in $I$, we need to resolve the rounding problem for assignments of the clients in the first two categories as well.

Our Rounding Process. Let $F'$ and $D'$ be the set of facilities and the set of clients yet to be processed, and $x^*$ be an intermediate assignment function our algorithm will maintain during the process. Initially, $F' := I$, $D' := J^{(I)} \cup J^{(I')}$, and $x^* := 0$.

Our rounding algorithm consists of two phases. In the first phase, it proceeds in iterations to form clusters. In each of such iterations, the algorithm checks if $\sum_{i \in F'} x'_{i,j} \geq 1/2$ holds for all $j \in D'$. If not, the algorithm makes it so by repeatedly removing small clients from $D' \cap J^{(I')}$ and redistributing their demand to facilities in $U$ to form a set of outlier clients. We use $H$ to denote the set of outlier clients created in this step and $H' \subseteq H$ to denote those that are created but not yet processed by the rounding algorithm. Initially $H := \emptyset$ and $H' := \emptyset$.

When $\sum_{i \in F'} x'_{i,j} \geq 1/2$ holds for all $j \in D'$, a cluster centered at a client is formed and possibly rounded, depending on whether or not the client forming the cluster is outlier, and the corresponding parts of the cluster are removed from $F'$, $D'$, and $H'$, respectively. The cluster forming process repeats until $D' \cup H'$ becomes empty.

In the second phase the algorithm rounds the remaining clusters centered at the outlier clients using an assignment LP to form an integral multiplicity function. In the following we describe the three components of our rounding algorithm in details.

Creating the Outlier Clients. When $\sum_{i \in F'} x'_{i,j} < 1/2$ for some $j \in J^{(I')} \cap D'$, the algorithm discards $j$ and relocates some of the remaining demand to facilities in $N_{(U,x')}(j)$ to form outlier clients in a way as if the demand were originated from these facilities.

Let $r'_{j} := \min \{ \sum_{i \in F'} x'_{i,j}, \sum_{i \in U} x'_{i,j} \}$ be the amount of residue demand of $j$ to be redistributed. For each $w \in N_{(U,x')}(j)$, we create a client $j_w$ at the facility $w$ with demand $d_{j_w} := r'_{j} \cdot x'_{w,j} / \sum_{i \in U} x'_{i,j}$ and add $j_w$ to both $H$ and $H'$. Set $\alpha_{j_w} := \alpha_{j} + c_{w,j}$. For each $i \in N_{(F',x')}(j)$, we further set $x'_{i,j_w} := d_{j_w} \cdot x'_{i,j} / \sum_{i \in F'} x'_{i,j}$.

See also Figure 5 for an illustration on the construction of $j_w$. It follows by definition that $d_{j_w} \leq x'_{w,j}$, and the residue demand of $j$ is fully redistributed since $\sum_{w \in N_{(U,x')}(j)} d_{j_w} = r'_{j}$. We note that $\sum_{i \in N_{(F',x')}(j)} x'_{i,j_w} = d_{j_w}$, and the demand of $j_w$ is fully-assigned.

After the outlier client $j_w$ is created for each $w \in N_{(U,x')}(j)$, the algorithm removes $j$ from $D'$ and set $x'_{i,j}$ to be zero for all $i \in F'$.

Forming Clusters and Rounding. When $\sum_{i \in F'} x'_{i,j} \geq 1/2$ holds for all $j \in D'$, the algorithm selects a client $j \in D' \cup H'$ that minimizes $\alpha_j$ to form a cluster. Depending on the set to which $j$ belongs, the algorithm proceeds differently.
• If \( j \in H' \), then a cluster centered at \( j \) with satellite facilities \( N_{(F',x')}(j) \) is formed. We use \( B(j) := N_{(F',x')}(j) \) to denote the set of satellite facilities at this moment. The algorithm then removes \( j \) from \( H' \) and \( B(j) \) from \( F' \). The rounding problem for this cluster is handled later in the second phase of the algorithm.

• If \( j \in D' \), the algorithm further selects a facility \( i \in N_{(F',x')}(j) \) with the maximum \( u_i \). The algorithm reroutes the assignment and facility values from the facilities in \( N_{(F',x')}(j) \) to \( i \) as follows. Let \( \delta_i := (\frac{2}{x} - 2) / \sum_{k \in N_{(F',x')}(j) \setminus \{i\}} y_k' \) be the factor to reroute from \( N_{(F',x')}(j) \setminus \{i\} \).

For each \( \ell \in N_{(F',x')}(j) \setminus \{i\} \), the algorithm scales \( y_{\ell}' \) down by \( (1 - \delta_i) \). For each \( k \in N_{(D' \cup H',x')}(\ell) \), the algorithm further scales \( x_{\ell,k}' \) down by \( (1 - \delta_i) \) and increases \( x_{i,k}' \) by the same amount \( x_{i,k}' \) has decreased in this step.

The algorithm increases \( x_{i,k}' \) by \( x_{i,k}' \) for each \( k \in D' \) and then removes \( i \) from \( F' \).

When the above updates are done, for each \( k \in J^{(l)} \cap D' \) with \( \sum_{i \in F'} x_{i,k}' < 1/2 \), the algorithm removes \( k \) from \( D' \) and sets \( x_{i,k}' \) to be zero for all \( i \in F' \). Then the algorithm proceeds to the next iteration until \( D' \cup H' \) becomes empty.

**Rounding the Outlier Clusters.** When \( D' \cup H' \) becomes empty, the algorithm proceeds to handle the rounding problem for clusters centered at clients in \( H \), which is formulated as an instance of CFL-CFC with facility set \( G := \bigcup_{j \in H} B(j) \) and client set \( U \) as follows. Each \( w \in U \) is associated with a demand \( d_w \) defined as

\[
d_w := \sum_{k \in H, \ell \in B(k), \ell \in D \cup H} t_{\ell}' \cdot x_{\ell,k}',
\]

where the scaling factor \( t_{\ell}' \) is defined as

\[
t_{\ell}' := \frac{1}{\sum_{k \in I} x_{k,\ell}' + \sum_{k \in G} x_{k,\ell}'} \left( 1 - \sum_{i \in U} x_{i,\ell}' - r_{\ell}' \right)
\]

if \( \ell \in D \) and \( \sum_{k \in I} x_{k,\ell}' + \sum_{k \in G} x_{k,\ell} > 0 \), and \( t_{\ell}' := 1 \) otherwise. Intuitively, \( t_{\ell}' \) is the factor for which the assignments made for \( \ell \) should be scaled up, and the demand \( d_w \) of each \( w \in U \) is the total demand of the clusters centered at the outlier clients located at \( w \).

The algorithm solves the assignment LP for the formulated instance, denoted LP-(O) and given in Figure 6, for a basic optimal solution \((x'', y'')\).

**Final Output.** Define the multiplicity function \( y^* \) as follows. Let \( y_i^* := 1 \) for \( i \in I \setminus F' \), \( y_i^* := \lceil y_i'' \rceil \) for \( i \in G \), and \( y_i^* := 0 \) otherwise. The algorithm solves the min-cost assignment problem on \( D \) and \( F^* := \{ i \in F : y_i^* = 1 \} \) for an optimal integral assignment \( x^\dagger \), and reports \((x^\dagger, y^*)\) as the approximation solution.

The following theorem, which is proved in Section 5, page 26, bounds the cost incurred by \((x^\dagger, y^*)\) and completes the proof for Theorem 2.

**Theorem 5.** Let \( \Psi \) be the input instance of CFL-CFC and \((x', y')\) be an optimal solution for LP-(N) on \( \Psi \). Algorithm \( A \) computes in polynomial time a feasible integral solution \((x^\dagger, y^*)\) with

\[
\psi(x^\dagger, y^*) \leq 4 \cdot \psi(x', y').
\]
4 Proof of Theorem 4.

It is clear that the first phase of the algorithm runs in polynomial time. Since \( y' \leq y'' \), the feasible region of \( \text{MFN}_\Psi(x', y', g) \) is contained in that of \( \text{MFN}_\Psi(x', y'', g) \). Hence, any hyperplane separating \((x', y')\) from \( \text{MFN}_\Psi(x', y'', g) \) also separates \((x', y')\) from \( \text{MFN}_\Psi(x', y', g) \). To complete the proof of Theorem 4, it suffices to prove the following lemma.

**Lemma 6.** Provided that \( \text{MFN}_\Psi(x', y', g) \) is feasible, our iterative rounding process in the second phase is well-defined and terminates in polynomial time. Furthermore, the feasible region of the min-cost assignment problem on \( D \) and \( F^* \) is non-empty, and

\[
\psi(x^*, y^*) \leq \max \left\{ \frac{3}{2\alpha}, \frac{7 - 4\alpha}{(1 - \alpha)^2} \right\} \times \psi(x', y').
\]

The proof for Lemma 6 is outlined as follows. First, we define in Section 4.1 the intermediate function \( f' \) which corresponds to the assignments made to facilities in \( U \) and in Section 4.2 the intermediate assignment \( x'' \) which corresponds to the assignments made by our iterative rounding process to facilities in \( I \setminus F' \), respectively. We prove in the same section that the iterative rounding process is well-defined and runs in polynomial time.

In Section 4.3, page 18, we show that, the assignment \( x''' \) defined jointly from \( f' \) and \( x'' \), together with \( y^* \) forms a feasible solution for \( \Psi \) after properly rounded up. This shows that the feasible region of the min-cost assignment problem on \( D \) and \( F^* \) is nonempty.

For the approximation guarantee, in Section 4.4, page 20, we analyze the cost incurred by \( x''' \) and \( y^* \), and show that it satisfies the claimed approximation guarantee. Note that, this completes the proof of this lemma since \( x^* \) is the optimal min-cost assignment between \( D \) and \( F^* \).

![Figure 7: (Restate for further reference) The construction of \( \text{MFN}_\Psi(x, y, g) \) and the corresponding LP constraints.](image)
4.1 The Pseudo-Flow Function $f'$ for Assignments to $U$

In this section we define the intermediate function $f'$ which corresponds to the assignments made to facilities in $U$\footnote{In the rest of the proof, we will refer $f''$ to as a flow function for convenience despite the fact that it uses edges that do not exist in $\text{MFN}_q(\mathbf{x}', \mathbf{y}'', \mathbf{g})$.}. Consider the first phase of the algorithm for obtaining the partial assignment $g$ and the corresponding feasible flow $f$ for $\text{MFN}_q(\mathbf{x}', \mathbf{y}'', \mathbf{g})$.

Let $D^{(0)}$ denote the set of clients that are fully-assigned in $h$ and unreachable from any partially-assigned clients using augmenting paths. Let $D^{(p)} := D \setminus D^{(0)}$ denote the set of clients that are either partially-assigned or reachable by partially-assigned clients via augmenting paths.

The following proposition shows that, in the flow $f$, none of $j^s \in D^{(p)}$ has sent nonzero flow that passes through $j^s$ for any $j \in D^{(0)}$, which implies that for any $i \in F$ and $j \in D^{(0)}$, the arc $(j^s, i)$ carries only flow for commodity $j$.

**Proposition 7.** For any $j \in D^{(0)}$, $i \in F$, and any $p \in P$ with $(j^s, i) \in p$, we have $f_p > 0$ implies that $p \in \bigcup_{i' \in F} P(i', j)$, that is, $f_p > 0$ only when $p$ is a path for commodity $j$.

**Proof.** Since $j \in D^{(0)}$, it follows that $g_{k,j} = 0$ for all $k \in F$ by the definition of $g$. Hence, $p$ must start from $j^s$ and is thereby a path for commodity $j$. \hfill $\square$

Define the intermediate flow function $f'$ as follows.

- For any $i \in U^{(>)}$ that is tightly-occupied and any $j \in D$, we define
  
  $$f'_p := (1 - \alpha) \cdot g_{i,j}, \text{ where } p = j^s \rightarrow i \rightarrow i' \rightarrow j^t.$$

- For any $i \in U^{(>)}$ that is not tightly-occupied, we further consider two cases.
  
  - For any $j \in D^{(0)}$, we define $f'_p := (1 - \alpha) \cdot h_{i,j}$, where $p = j^s \rightarrow i \rightarrow i' \rightarrow j^t$.
  
  - For any $j \in D^{(p)}$ and any $p \in P(i, j)$, we define $f'_p := f_p$.

- Finally, for any $i \in U^{(\leq)}$, any $j \in D$, and any $p \in P(i, j)$, we define $f'_p := f_p$.

For any $i \in U$, $j \in D$, let $\Phi'(i, j)$ denote the set of paths with nonzero flow for commodity $j$ to sink via $i'$ in the flow functions $f'$. The following lemma, which is valid from the construction of $f'$, shows that the facilities in $U$ are sparsely-loaded.

**Lemma 8.** For any $i \in U$, we have

$$\sum_{j \in D, p \in \Phi'(i,j)} f'_p \leq (1 - \alpha) \cdot u_i.$$

**Proof of Lemma 8.** Depending on the category to which each $i$ belongs, we consider the following three cases separately.

- $i \in U^{(>)}$ is tightly-occupied. By the construction rules of $f'$ and the definition of $g$, we have
  
  $$\sum_{j \in D, p \in \Phi'(i,j)} f'_p = \sum_{j \in D^{(0)}} f'_p = (1 - \alpha) \cdot \sum_{j \in D^{(0)}} g_{i,j} = (1 - \alpha) \cdot \sum_{j \in D} h_{i,j} = (1 - \alpha) \cdot u_i,$$

  where the last equality follows from the fact that there exists an augmenting path connecting a partially-assigned client to $i$, which implies that $\sum_{j \in D} h_{i,j} = u_i$ by the optimality of $h$.\hfill $\square$
• $i \in U^{(>)}$ is not tightly-occupied. Since $i$ is not tightly-occupied, by the optimality of $h$ it follows that, $h_{i,j} = x_{i,j}'/(1 - \alpha)$ holds for any $j \in D'(p)$. By the construction rules of $f'$ and Proposition 7, we have

$$
\sum_{j \in D'(p), p \in \Phi'(i,j)} f'_p = \sum_{j \in D'(p), p \in P(i,j)} f_p \leq \sum_{j \in D'(p), p \in P(i,j) \text{ with } (j^*,i) \in p} f_p \leq \sum_{j \in D'(p)} x_{i,j}' = (1 - \alpha) \cdot \sum_{j \in D'(p)} h_{i,j}, \quad (3)
$$

where in the second last inequality we apply Constraint (2.b) due to the fact that $f$ is a feasible flow for $\text{MFN}_p(x', y'', g)$. From the second construction rule of $f'$, we have

$$
\sum_{j \in D'(i)} f'_{j^*i \rightarrow i^*j} = (1 - \alpha) \cdot \sum_{j \in D'(i)} h_{i,j}. \quad (4)
$$

Combining (3) and (4) with the definition of $\Phi'(i,j)$, we have

$$
\sum_{j \in D, p \in \Phi'(i,j)} f'_p = \sum_{j \in D'(p), p \in \Phi'(i,j)} f'_p + \sum_{j \in D'(i), p \in \Phi'(i,j)} f'_p \leq (1 - \alpha) \cdot u_i,
$$

where the last inequality follows from the feasibility of $h$, i.e., $\sum_{j \in D} h_{i,j} \leq u_i$.

• $i \in U^{(\leq)}$. By the third construction rule of $f'$ and the feasibility of $f$, we have

$$
\sum_{j \in D, p \in \Phi'(i,j)} f'_p = \sum_{j \in D, p \in P(i,j)} f_p \leq \sum_{j \in D, p \in P(i,j)} x_{i,j}' \leq (1 - \alpha) \cdot u_i,
$$

where in the last inequality we apply the definition of $U^{(\leq)}$.

This completes the proof of this lemma.

4.2 The Assignment $x''$ for the Rounded Facilities in $I \setminus F''$

Consider the rounding process in the second phase of the algorithm. With a slight abuse of notations, let $F'$ denote the particular set $F'$ at the moment when the rounding process ends, i.e., $F'$ denotes the set of facilities in $I$ that are not selected to be rounded up by the algorithm. For each $i \in I \setminus F'$, let $\Psi^{(i)} = (F^{(i)}, D^{(i)}, r^{(i)})$ denote the parameter tuple the algorithm maintains in the beginning of the iteration when facility $i$ is selected.

Let $\Psi^{(0)} = (F^{(0)}, D^{(0)}, r^{(0)})$ denote the initial parameter tuple the algorithm constructs in the beginning of the second phase. The following lemma, which shows that the feasible region of LP-(M) on $\Psi^{(0)}$ is nonempty, is proved by verifying the corresponding constraints of LP-(M) combined with the fact that $0 < \alpha \leq 1/3$.

**Lemma 9.** The solution $(x^{(i)}, y^{(i)})$ defined by

$$x_{i,j}^{(i)} := \sum_{p \in P(i,j)} f_p \quad \text{for all } i, j \in D \quad \text{and} \quad y_i^{(i)} := \frac{1 - \alpha}{2\alpha} \cdot y_i' \quad \text{for all } i \in I$$

is feasible for LP-(M) on the initial parameter tuple $\Psi^{(0)}$. 

13
\[
\begin{align*}
\min & \quad \sum_{i \in F'} o_i \cdot y_i + \sum_{i \in F', j \in D'} c_{i,j} \cdot x_{i,j} \\
\text{s.t.} & \quad \sum_{i \in F'} x_{i,j} = r_j', \quad \forall j \in D' \quad (M-1) \\
& \quad \sum_{j \in D'} x_{i,j} \leq u_i \cdot y_i, \quad \forall i \in F' \quad (M-2) \\
& \quad 0 \leq x_{i,j} \leq \frac{2\alpha}{1-\alpha} \cdot r_j \cdot y_i, \quad \forall i \in F', j \in D' \\
& \quad 0 \leq y_i \leq \frac{1-\alpha}{2}, \quad \forall i \in F' \quad (M-4)
\end{align*}
\]

\[
\begin{align*}
\max & \quad \sum_{j \in D'} r_j' \cdot \lambda_j - \sum_{i \in F'} \frac{1}{2} \cdot (1-\alpha) \cdot \eta_i \\
\text{s.t.} & \quad \lambda_j \leq \beta_i + \Gamma_{i,j} + c_{i,j}, \quad \forall i \in F', j \in D' \quad (DM-1) \\
& \quad u_i \cdot \beta_i + \frac{2\alpha}{1-\alpha} \cdot \sum_{j \in D'} r_j \cdot \Gamma_{i,j} \leq o_i + \eta_i, \quad \forall i \in F' \quad (DM-2) \\
& \quad \lambda_j \in \mathbb{R}, \beta_i, \Gamma_{i,j}, \eta_i \geq 0, \quad \forall i \in F', j \in D' \quad (DM-3)
\end{align*}
\]

Figure 8: (Restate for further reference) The natural LP formulation and its dual LP for our iterative rounding process.

**Proof of Lemma 9.** We prove this lemma by verifying the constraints of LP-(M) separately.

- For Constraint (M-1), consider any \( j \in D'(0) \). Since \( F'(0) = I \) by definition, further applying the definition of \( r_j'(0) \), we have

\[
\sum_{i \in F'(0)} x_{i,j}^{(0)} = \sum_{i \in I, p \in P(i,j)} f_p = r_j^{(0)}.
\]

- For Constraint (M-2), consider any \( i \in F'(0) \). By the definition of \( D'(0) \) and Constraint (2.d) from the fact that \( f \) is a feasible flow for \( MFN_{\psi}(x', y'', g) \), it follows that

\[
\sum_{j \in D'(0)} x_{i,j}^{(0)} = \sum_{j \in D'(0), p \in P(i,j)} f_p \leq \sum_{j \in D, p \in P(i,j)} f_p \leq u_i^{(g)} \cdot y_{i'}.
\]

Since \( F'(0) := I \) by definition and \( i \in F'(0) \), it follows that \( \sum_{j \in D} g_{i,j} = 0 \) by the way \( g \) is defined and therefore \( u_i^{(g)} = u_i \). Moreover, observe that \( (1-\alpha)/(2\alpha) \) is strictly decreasing for \( \alpha > 0 \). Since \( 0 < \alpha \leq 1/3 \), it follows that

\[
y_i^{(0)} := \frac{1-\alpha}{2\alpha} \cdot y_{i'} \geq \frac{1-1/3}{2/3} \cdot y_{i'} = y_{i'}.
\]
Combining the above, we obtain
\[
\sum_{j \in D(0)} x_{i,j}^{(0)} \leq u_i \cdot y_i^{(0)}.
\]

- For Constraint (M-3), consider any \( i \in F'(0) \) and any \( j \in D'(0) \). Apply Constraint (2.e) by the fact that \( f \) is a feasible flow for \( \text{MFN}_\Psi(x', y'', g) \) and the definition of \( y_i^{(0)} \) and \( r_j \), we have
\[
x_{i,j}^{(0)} = \sum_{p \in F(i,j)} f_p \leq r_j^{(g)} \cdot y_i' = \frac{2\alpha}{1 - \alpha} \cdot r_j \cdot y_i^{(0)}.
\]

- For Constraint (M-4), consider any \( i \in F'(0) \). By the definition of \( y_i^{(0)} \) and the fact that \( F'(0) = I \), which implies that \( y_i' < \alpha \), it follows that
\[
y_i^{(0)} = \frac{1 - \alpha}{2\alpha} \cdot y_i' \leq \frac{1 - \alpha}{2}.
\]

This proves the lemma.

Consider any \( i \in I \setminus F' \) and the iteration when \( i \) is processed. Suppose for the moment that the feasible region of LP-(M) on \( \Psi^{(i)} \) is nonempty. Let \( (x^{(i)}, y^{(i)}) \) denote the optimal solution the algorithm computes for LP-(M) on \( \Psi^{(i)} \). The following lemma shows that the scaled-down operation in each iteration is well-defined.

**Lemma 10.** For any \( i \in I \setminus F' \) with \( y_i^{(i)} < (1 - \alpha)/2 \), the following holds.

- For any \( j \in D^{(i)} \), we have
  \[
  0 \leq \delta_j^{(i)} \leq \sum_{\ell \in F^{(i)} \setminus \{i\}} x_{\ell,j}^{(i)}.
  \]

- For any \( k \in F^{(i)} \setminus \{i\} \), we have \( \sigma_k^{(i)} \leq 1 \).

**Proof of Lemma 10.** Consider any \( j \in D^{(i)} \). Since \( (x^{(i)}, y^{(i)}) \) is feasible for LP-(M) on \( \Psi^{(i)} \), we have \( y_i^{(i)} \leq (1 - \alpha)/2 \), which implies that \( (1 - \alpha)/(2 \cdot y_i^{(i)}) \geq 1 \). Hence
\[
\delta_j^{(i)} := \left( \frac{1 - \alpha}{2 \cdot y_i^{(i)}} - 1 \right) \cdot x_{i,j}^{(i)} \geq 0.
\]

On the other hand, applying Constraint (M-3) of LP-(M), it follows that
\[
\delta_j^{(i)} := \frac{1 - \alpha}{2 \cdot y_i^{(i)}} \cdot x_{i,j}^{(i)} - x_{i,j}^{(i)} \leq \alpha \cdot r_j - x_{i,j}^{(i)} < r_j^{(i)} - x_{i,j}^{(i)} = \sum_{\ell \in F^{(i)} \setminus \{i\}} x_{\ell,j}^{(i)},
\]
where in the second last inequality we apply the fact that \( r_j^{(i)} > \alpha \cdot r_j \) holds for all \( j \in D^{(i)} \) by the design of the algorithm, and in the last equality we apply Constraint (M-1) of LP-(M). This proves the first part of this lemma.
Consider any \( k \in F^{(i)} \setminus \{i\} \). By the conclusion of the first part, for any \( j \in D^{(i)} \), we have

\[
\sigma_{k,j}^{(i)} := \frac{\delta_j^{(i)}}{\sum_{\ell \in F^{(i)} \setminus \{i\}} x_{\ell,j}^{\dagger(i)}} \cdot x_{k,j}^{\dagger(i)} \leq x_{k,j}^{(i)},
\]

which implies that

\[
\sigma_k^{(i)} := \frac{1}{\sum_{\ell \in D^{(i)} \setminus \{i\}} x_{\ell,k}^{\dagger(i)}} \cdot \sum_{j \in D^{(i)} \setminus \{i\}} \sigma_{k,j}^{(i)} \leq 1.
\]

Let \( \Psi^{(i)} = (F^{(i)}, D^{(i)}, r^{(i)}) \) denote the updated parameter tuple the algorithm maintains at the end of the iteration \( i \). Note that, by the algorithm design, \( F^{n(i)} := F^{(i)} \setminus \{i\} \). The following lemma shows that the feasible region of LP-(M) on the updated tuple \( \Psi^{(i)} \) is nonempty and establishes the feasibility of our iterative rounding process.

**Lemma 11.** For any \( i \in I \setminus F' \), the solution \( (x^{(i)}, y^{(i)}) \) defined by

\[
x_{k,j}^{(i)} := \left( 1 - \sigma_k^{(i)} \right) \cdot x_{k,j}^{\dagger(i)} \quad \text{for any } k \in F^{(i)}, j \in D^{(i)},
\]

\[
y_k^{(i)} := \left( 1 - \sigma_k^{(i)} \right) \cdot y_k^{\dagger(i)} \quad \text{for any } k \in F^{(i)},
\]

is feasible for LP-(M) on the updated parameter tuple \( \Psi^{(i)} \).

**Proof of Lemma 11.** We prove this lemma by verifying the corresponding constraints in LP-(M).

First, by Lemma 10, \( 1 - \sigma_k^{(i)} \geq 0 \) holds for all \( k \in F^{(i)} \), and \( (x^{\dagger(i)}, y^{\dagger(i)}) \) is thereby nonnegative.

- For Constraint (M-1), consider any \( j \in D^{(i)} \). By the definition of \( x_{k,j}^{\dagger(i)} \) for any \( k \in F^{(i)} \) and the definition of \( r_j^{(i)} \), we have

\[
\sum_{k \in F^{(i)}} x_{k,j}^{\dagger(i)} = \sum_{k \in F^{(i)}} \left( 1 - \sigma_k^{(i)} \right) \cdot x_{k,j}^{\dagger(i)} = r_j^{(i)}.
\]

- For Constraint (M-2) and (M-3), consider any \( k \in F^{(i)} \) and any \( j \in D^{(i)} \). Since \( (x^{\dagger(i)}, y^{\dagger(i)}) \) is feasible for LP-(M) on \( \Psi^{(i)} \) and since \( F^{(i)} \supseteq F^{n(i)}, D^{(i)} \supseteq D^{n(i)} \), we have

\[
\sum_{j \in D^{(i)}} x_{k,j}^{\dagger(i)} \leq \sum_{j \in D^{(i)}} x_{k,j}^{\dagger(i)} \leq u_k \cdot y_k^{\dagger(i)} \quad \text{and} \quad x_{k,j}^{(i)} \leq \frac{2\alpha}{1 - \alpha} \cdot r_j \cdot y_k^{\dagger(i)}.
\]

Multiplying both inequalities by \( \left( 1 - \sigma_k^{(i)} \right) \) and applying the definition of \( (x^{\dagger(i)}, y^{\dagger(i)}) \), we obtain

\[
\sum_{j \in D^{(i)}} x_{k,j}^{\dagger(i)} \leq u_k \cdot y_k^{\dagger(i)} \quad \text{and} \quad x_{k,j}^{\dagger(i)} \leq \frac{2\alpha}{1 - \alpha} \cdot r_j \cdot y_k^{\dagger(i)}.
\]

- For Constraint (M-4), consider any \( k \in F^{(i)} \). Applying the definition of \( y_k^{\dagger(i)} \) and the fact that \( (x^{\dagger(i)}, y^{\dagger(i)}) \) is feasible for LP-(M), we have

\[
y_k^{(i)} = \left( 1 - \alpha_k^{(i)} \right) \cdot y_k^{\dagger(i)} \leq y_k^{\dagger(i)} \leq \frac{1 - \alpha}{2}.
\]
Lemma 9, Lemma 10, and Lemma 11 jointly establish that our iterative rounding process is well-defined. Since $F'(0) := I$ and since exactly one facility is selected to be removed from $F'$ in each iteration, it follows that the process repeats for at most $|I|$ iterations before the set $D'$ becomes empty. Therefore, our iterative rounding process terminates in polynomial time.

For any $i \in I \setminus F'$ and $j \in D'$, define the assignment $x''_{i,j}$ as

$$x''_{i,j} := \sum_{k \in F'(i)} \sigma^{(i)}_k \cdot x^\dagger_{k,j},$$

where in the above we extend the original definition of $\sigma^{(i)}$ and define $\sigma^{(i)}_j := 1$ for notational brevity. Recall that, for any $j \in D$, the initial residue demand of $j$ is defined as

$$r^0_j = \sum_{i \in I, p \in P'(i,j)} f_p = \sum_{i \in I, p \in P(i,j)} f_p.$$

The following lemma, which is valid from the algorithm design, shows that the residue demand of each $j$ is mostly assigned by our rounding process to facilities in $I \setminus F'$.

**Lemma 12.** For any $j \in D'$, we have

$$\sum_{i \in I \setminus F'} x''_{i,j} \geq r^0_j - \alpha \cdot r_j.$$

**Proof of Lemma 12.** By Lemma 9, Lemma 11, the definition of $r^0_j$ and the definition of $x''_{i,j}$ for each $i \in I \setminus F'$, it follows that, each unit of assignment of any $j \in D'$ that is scaled down during the iteration for $i \in I \setminus F'$ is included in $\sum_{i \in I \setminus F'} x''_{i,j}$. Since the algorithm removes a client $j$ from $D'$ only when its residue demand $r_j^0$ is no greater than $\alpha \cdot r_j$ and since the initial residue demand of $j$ is $r^0_j$, it follows that $\sum_{i \in I \setminus F'} x''_{i,j} \geq r^0_j - \alpha \cdot r_j$. \hfill \qed

The following lemma shows that the facilities in $I \setminus F'$ is also sparsely-loaded under the assignments given in $x''$.

**Lemma 13.** For any $i \in I \setminus F'$, we have

$$\sum_{j \in D} x''_{i,j} = \frac{1 - \alpha}{2} \cdot \frac{1}{y^{(i)}_i} \cdot \sum_{\ell \in D'} x^\dagger_{i,\ell} \leq \frac{1 - \alpha}{2} \cdot u_i.$$

**Proof of Lemma 13.** By the definition of $x''$ and the definition of $\sigma^{(i)}_k$ for all $k \in F'(i)$, we have

$$\sum_{j \in D} x''_{i,j} = \sum_{j \in D', k \in F'(i)} \sigma^{(i)}_k \cdot x^\dagger_{k,j} = \sum_{j \in D'} x^\dagger_{i,j} + \sum_{k \in F'(i) \setminus \{i\}} \frac{1}{\ell \in D'} x^\dagger_{k,\ell} \sum_{\ell \in D'} \sigma^{(i)}_k \cdot \sum_{j \in D'} x_{k,j} = \sum_{j \in D'} x^\dagger_{i,j} + \sum_{k \in F'(i) \setminus \{i\}, \ell \in D'} \sigma^{(i)}_{k,\ell}. \hfill (5)$$
Consider the second item in (5). Further applying the definition of \(\sigma_{k,\ell}^{(i)}\) for all \(k \in F'(i) \setminus \{i\}\), \(\ell \in D'(i)\) and the definition of \(\delta_{\ell}^{(i)}\), we obtain

\[
\sum_{k \in F'(i) \setminus \{i\}} \sigma_{k,\ell}^{(i)} = \sum_{k \in F'(i) \setminus \{i\}, \ell \in D'(i)} \sum_{p \in F'(i) \setminus \{i\}} x_{p,\ell}^{(i)} \cdot \delta_{\ell}^{(i)}
= \sum_{\ell \in D'(i)} \delta_{\ell}^{(i)} = \sum_{\ell \in D'(i)} \left( \frac{1 - \alpha}{2} \cdot \frac{1}{y_{i}^{(i)}} - 1 \right) \cdot x_{i,\ell}^{(i)}.
\]

(6)

Combining (5) and (6), and applying applying (M-2) in LP-(M) by the fact that \((x_{i}^{(i)}, y_{i}^{(i)})\) is a feasible solution for LP-(M) for \(\Psi(i)\), we have

\[
\sum_{j \in D} x_{i,j}''' = 1 - \frac{\alpha}{2} \cdot \frac{1}{y_{i}^{(i)}} \cdot \sum_{\ell \in D'(i)} x_{i,\ell}^{(i)} \leq 1 - \frac{\alpha}{2} \cdot u_{i}.
\]

This completes the proof of this lemma.

\[\square\]

4.3 The Overall Assignment \(x'''\) and the Feasibility

Define the assignment \(x'''\) as follows. For any \(i \in F^*\) and \(j \in D\), let

\[
x_{i,j}''' := \begin{cases} \sum_{p \in \Phi'(i,j)} f_{p}', & \text{if } i \in U, \\ x_{i,j}'', & \text{if } i \in I \setminus F'. \end{cases}
\]

The following lemma establishes the feasibility of our rounding algorithm.

Lemma 14. For any \(j \in D\), we have

\[
\sum_{i \in F^*} x_{i,j}''' \geq 1 - \alpha.
\]

Proof of Lemma 14. By the definition of \(x'''\), to prove this lemma, it suffices to prove that

\[
\sum_{i \in F^*} x_{i,j}''' = \sum_{i \in U, p \in \Phi'(i,j)} f_{p}' + \sum_{i \in I \setminus F'} x_{i,j}'' \geq 1 - \alpha.
\]

Depending on the category to which \(j\) belongs in the first phase of the algorithm, we consider the following two cases.

- \(j \in D'(\ell)\), i.e., \(j\) is fully-assigned by \(h\) to facilities in \(U^{(\ell)}\) and unreachable from any partially-assigned client via augmenting paths.

Since \(j\) is fully-assigned by \(h\) to \(U^{(\ell)}\), we have

\[
\sum_{i \in U^{(\ell)}} h_{i,j} = 1.
\]

Furthermore, since \(j\) is unreachable from any partially-assigned client via augmenting paths, it follows that for any \(i \in U^{(\ell)}\), \(h_{i,j} > 0\) implies that \(i\) is not tightly-occupied.

Hence, by the construction rules of \(f'\), it follows that

\[
\sum_{i \in U, p \in \Phi'(i,j)} f_{p}' + \sum_{i \in I \setminus F'} x_{i,j}'' \geq \sum_{i \in U^{(\ell)}} f_{j\leftarrow i \rightarrow i'\rightarrow j'} = (1 - \alpha) \sum_{i \in U^{(\ell)}} h_{i,j} = 1 - \alpha.
\]
• For the other case, \( j \in D^{(i)} \), i.e., \( j \) is either partially-assigned or reachable from partially-assigned clients via augmenting paths, since \( f \) is feasible for \( \text{MFN}_\Psi(x', y'', g) \), it follows from constraint (2.a) and the definition of \( \text{MFN}_\Psi(x', y'', g) \) that
\[
\sum_{i \in U, p \in P(i, j)} f_p + \sum_{i \in I, p \in P(i, j)} f_p = r_j := 1 - \sum_{i \in U} g_{i, j}. \tag{7}
\]

Consider the two items in the LHS of (7) separately. By the construction of \( \text{MFN}_\Psi(x', y'', g) \), no flow is sent in \( f \) to sink via tightly-occupied facilities in \( U^{(\triangleright)} \) since \((i, i')\) has zero capacity for each tightly-occupied \( i \in U^{(\triangleright)} \).

Hence, by the second and third construction rules of \( f' \), we know that the flow sent in the first item, \( \sum_{i \in U, p \in P(i, j)} f_p, \) is included entirely in \( f' \), i.e.,
\[
\sum_{i \in U^{(\triangleright)}, \quad \text{i not tightly-occupied}} \sum_{p \in \Phi'(i, j)} f_p' + \sum_{i \in U^{(\triangleleft)}, \quad p \in \Phi'(i, j)} f_p' = \sum_{i \in U, \quad p \in P(i, j)} f_p. \tag{8}
\]

For the second item in the LHS of (7), \( \sum_{i \in I, \quad p \in P(i, j)} f_p \), consider the rounding process for the clusters in \( I \setminus F' \). Applying Lemma 12 and the definition of \( r_j^{(0)} \), we have
\[
\sum_{i \in I \setminus F'} x''_{i, j} \geq r_j^{(0)} - \alpha \cdot r_j = \sum_{i \in I, \quad p \in P(i, j)} f_p - \alpha \cdot r_j. \tag{9}
\]

By the first construction rule of \( f' \), we know that exactly \((1 - \alpha) \cdot \sum_{i \in U} g_{i, j}\) units of flow is included in \( f' \). Therefore,
\[
\sum_{i \in U^{(\triangleright)}, \quad \text{i tightly-occupied}} \sum_{p \in \Phi'(i, j)} f_p' = (1 - \alpha) \cdot \sum_{i \in U^{(\triangleright)}, \quad \text{i tightly-occupied}} g_{i, j} = (1 - \alpha) \cdot \sum_{i \in U} g_{i, j}. \tag{10}
\]

Combining the definition of \( U \) with (8), we obtain
\[
\sum_{i \in U, \quad p \in \Phi'(i, j)} f_p' + \sum_{i \in I \setminus F'} x''_{i, j} = \sum_{i \in U^{(\triangleright)}, \quad p \in \Phi'(i, j)} f_p' + \sum_{i \in U, \quad p \in P(i, j)} f_p + \sum_{i \in I \setminus F'} x''_{i, j}
\geq (1 - \alpha) \cdot \sum_{i \in U} g_{i, j} + (1 - \alpha) \cdot r_j = 1 - \alpha,
\]

where in the second inequality we apply (10), (9), and then (7), respectively, and in the last equality we apply the definition of \( r_j \).

This completes the proof of this lemma.

It follows directly from Lemma 14, Lemma 8, and Lemma 13 that \((x''/(1 - \alpha), y^*)\) is a feasible solution for \( \Psi \), and the feasible region of the min-cost assignment problem on \( D \) and \( F^* \) is thereby nonempty. We have the following corollary.

**Corollary 15.** \( \left( \frac{1}{1 - \alpha} \cdot x'', y^* \right) \) is a feasible solution for \( \Psi \).
4.4 Approximation Guarantee

In the following we bound the cost incurred by facilities in $F^*$ and the assignment cost incurred by $x^{m′′}/(1−α)$. Note that this establishes an upper-bound on the cost incurred by $(x^*, y^*)$ since $x^*$ is the optimal solution of the min-cost problem between $F^*$ and $D$.

For any $i ∈ U, j ∈ D$ and any $p ∈ Ψ(i, j) ∪ P(i, j)$, define the absolute length of path $p$ as

$$|p| := \sum_{i′∈F, j′∈D, (j′, i′)∈p} c_{i′, j′} + \sum_{i′∈F, j′∈D, (i′, j′)∈p} c_{i′, j′}.$$  

By the definition of $x^{m′′}$ and the triangle inequality $c_{i, j} \leq |p|$ which holds for any $i ∈ U, j ∈ D$, and any $p ∈ Ψ(i, j)$, we have

$$\sum_{i∈F^*, j∈D} c_{i, j} \cdot x^{m′′}_{i, j} = \sum_{i∈U, j∈D} \sum_{p∈Ψ(i, j)} c_{i, j} \cdot f_p + \sum_{i∈F^*, j∈D} c_{i, j} \cdot x^{m′′}_{i, j} \leq \sum_{i∈U, j∈D} \sum_{p∈Ψ(i, j)} |p| \cdot f_p + \sum_{i∈F^*, j∈D} c_{i, j} \cdot x^{m′′}_{i, j}.  \tag{11}$$

In (11), we decompose the assignment cost of $x^{m′′}$ into two parts, namely, (i) the assignment cost made by $f$, $g$, and $h$ to facilities in $U$, and (ii) the assignment cost incurred by our iterative rounding process to facilities in $I \setminus F'$. In the following we consider the two parts separately.

We begin with the cost incurred by assignments to facilities in $I \setminus F'$. For any $i ∈ I \setminus F'$, let $(\lambda_{(i)}, \beta_{(i)}, \Gamma_{(i)}, \eta_{(i)})$ be an optimal dual solution for LP-(DM) on $Ψ(i)$. The following lemma, which is proven by standard complementary slackness conditions, establishes the equivalence between the cost incurred around a non-extremal facility and the dual values it receives.

**Lemma 16.** For any $i ∈ I \setminus F'$ and any $k ∈ F^{(i)}$ with $0 < y_{k}^{(i)} < (1−α)/2$, we have

$$\sum_{j∈D^{(i)}} \lambda_{j}^{(i)} \cdot x_{k, j}^{(i)} = o_k \cdot y_{k}^{(i)} + \sum_{j∈D^{(i)}} c_{k, j} \cdot x_{k, j}^{(i)}.$$  

**Proof of Lemma 16.** Consider the complementary slackness conditions which follow from the fact that $(x^{(i)}, y^{(i)})$ and $(\lambda^{(i)}, \beta^{(i)}, \Gamma^{(i)}, \eta^{(i)})$ are optimal primal and dual solutions for LP-(M) and LP-(DM), respectively. Since $x_{k, j}^{(i)} > 0$ implies that corresponding dual inequality (DM-1), $\lambda_j ≤ \beta_k + \Gamma_k + c_{k, j}$, must be tight, we have

$$\sum_{j∈D^{(i)}} \lambda_{j}^{(i)} \cdot x_{k, j}^{(i)} = \beta_k^{(i)} \cdot \sum_{j∈D^{(i)}} x_{k, j}^{(i)} + \sum_{j∈D^{(i)}} \Gamma_{k, j}^{(i)} \cdot x_{k, j}^{(i)} + \sum_{j∈D^{(i)}} c_{k, j} \cdot x_{k, j}^{(i)}.$$  

$\beta_k^{(i)} > 0$ and $\Gamma_{k, j}^{(i)} > 0$ imply that the corresponding inequalities (M-2) and (M-3) hold with equality. Hence the above equality becomes

$$\beta_k^{(i)} \cdot u_k \cdot y_{k}^{(i)} + \frac{2α}{1−α} \sum_{j∈D^{(i)}} r_j \cdot y_{k}^{(i)} \cdot \Gamma_{k, j}^{(i)} + \sum_{j∈D^{(i)}} c_{k, j} \cdot x_{k, j}^{(i)}.$$  

The assumption that $y_{k}^{(i)} > 0$ and $y_{k}^{(i)} < (1−α)/2$ imply that the corresponding inequality (DM-2) holds with equality and the dual variable $\eta_k^{(i)}$ must be zero. The above equality becomes

$$y_{k}^{(i)} \cdot \left( u_k \cdot \beta_k^{(i)} + \frac{2α}{1−α} \sum_{j∈D^{(i)}} r_j \cdot \Gamma_{k, j}^{(i)} \right) + \sum_{j∈D^{(i)}} c_{k, j} \cdot x_{k, j}^{(i)} = o_k \cdot y_{k}^{(i)} + \sum_{j∈D^{(i)}} c_{k, j} \cdot x_{k, j}^{(i)}.$$
and this lemma is proved.

The following lemma bounds the cost incurred by each individual facility \( i \in I \setminus F' \) selected in our iterative rounding process.

**Lemma 17.** For any \( i \in I \setminus F' \), we have

\[
(1 - \alpha) \cdot o_i + \sum_{j \in D(i)} c_{i,j} \cdot x_{i,j}'' \leq 3 \cdot \sum_{k \in F''} \sigma_k^{(i)} \cdot \left( o_k \cdot y_k^{(i)} + \sum_{j \in D(i)} c_{k,j} \cdot x_{k,j}^{(i)} \right),
\]

where we extend the definition of \( \sigma_k^{(i)} \) and define \( \sigma_i^{(i)} := 1 \) for notational brevity.

**Proof of Lemma 17.** If \( y_i^{(i)} = (1 - \alpha)/2 \) holds in the beginning of the iteration when \( i \) is selected for rounding, then \( \sigma_k^{(i)} = 0 \) for all \( k \in F'' \setminus \{i\} \) and this lemma holds trivially.

In the following we assume that \( y_i^{(i)} < (1 - \alpha)/2 \). Note that this implies that \( y_k^{(i)} < (1 - \alpha)/2 \) for all \( k \in F'' \) by the design of the algorithm. By the algorithm design, we have \( \sum_{j \in D(i)} x_{i,j}^{(i)} > 0 \), which implies that \( y_i^{(i)} > 0 \). Apply the definition of \( x'' \) and the triangle inequality, we have

\[
\sum_{j \in D(i)} c_{i,j} \cdot x_{i,j}'' = \sum_{j \in D(i)} c_{i,j} \cdot \sum_{k \in F''} \sigma_k^{(i)} \cdot x_{k,j}^{(i)} \leq \sum_{j \in D(i), k \in F''} \sigma_k^{(i)} \cdot (c_{i,k} + c_{k,j}) \cdot x_{k,j}^{(i)}
\]

\[
= \sum_{j \in D(i), k \in F''} \sigma_k^{(i)} \cdot c_{i,k} \cdot x_{k,j}^{(i)} + \sum_{k \in F'', j \in D(i)} \sigma_k^{(i)} \cdot c_{k,j} \cdot x_{k,j}^{(i)}. \tag{12}
\]

By (12), to prove this lemma, it suffices to show that

\[
(1 - \alpha) \cdot o_i + \sum_{j \in D(i), k \in F''} \sigma_k^{(i)} \cdot c_{i,k} \cdot x_{k,j}^{(i)} \leq \sum_{k \in F''} \sigma_k^{(i)} \cdot \left( 3 \cdot o_k \cdot y_k^{(i)} + 2 \sum_{j \in D(i)} c_{k,j} \cdot x_{k,j}^{(i)} \right). \tag{13}
\]

In the following we prove Inequality (13). First, since \( (x^{(i)}, y^{(i)}) \) and \( (\lambda^{(i)}, \beta^{(i)}, \Gamma^{(i)}, \eta^{(i)}) \) are optimal primal and dual solutions for LP-(M) and LP-(DM), it follows that,

\[
\lambda_j^{(i)} = \beta_k^{(i)} + \Gamma_{k,j}^{(i)} + c_{k,j} \geq c_{k,j} \quad \text{for all} \quad k \in F'', \quad j \in D(i) \quad \text{with} \quad x_{k,j}^{(i)} > 0. \tag{14}
\]

Therefore, applying the definition of \( \sigma_k^{(i)} \), the fact that \( c_{i,j} = 0 \), and the triangle inequality, we have

\[
\sum_{j \in D(i), k \in F''} \sigma_k^{(i)} \cdot c_{i,k} \cdot x_{k,j}^{(i)} = \sum_{j \in D(i), k \in F'' \setminus \{i\}} \frac{1}{\sum_{\ell \in D(i)} x_{k,\ell}^{(i)}} \cdot \sum_{\ell \in D(i)} \sigma_k^{(i)} \cdot c_{i,k} \cdot x_{k,j}^{(i)}
\]

\[
= \sum_{k \in F'' \setminus \{i\}, \ell \in D(i)} c_{i,k} \cdot \sigma_k^{(i)} \leq \sum_{k \in F'' \setminus \{i\}, \ell \in D(i)} (c_{i,k} + c_{\ell,k}) \cdot \sigma_k^{(i)}
\]

\[
\leq \sum_{k \in F'' \setminus \{i\}, \ell \in D(i)} \left( c_{i,\ell} + \lambda_{\ell}^{(i)} \right) \cdot \sigma_k^{(i)}. \tag{15}
\]

where in the last inequality we apply the bound obtained in (14).
Further applying the definition of $\sigma_{k,\ell}^{(i)}$ for all $k \in F^{(i)} \setminus \{i\}$ and $\ell \in D^{(i)}$, we obtain

$$
\sum_{k \in F^{(i)} \setminus \{i\}, \ell \in D^{(i)}} \left( c_{i,\ell} + \lambda_{\ell}^{(i)} \right) \cdot \sigma_{k,\ell}^{(i)} = \sum_{k \in F^{(i)} \setminus \{i\}, \ell \in D^{(i)}} \left( c_{i,\ell} + \lambda_{\ell}^{(i)} \right) \cdot \frac{x_{k,\ell}^{(i)}}{\sum_{p \in F^{(i)} \setminus \{i\}} x_{p,\ell}^{(i)}} \cdot \delta_{\ell}^{(i)}
$$

$$
= \sum_{\ell \in D^{(i)}} \left( c_{i,\ell} + \lambda_{\ell}^{(i)} \right) \cdot \delta_{\ell}^{(i)} \leq \frac{1 - \alpha}{2} \cdot \frac{1}{y_{i}^{(i)}} \cdot \sum_{j \in D^{(i)}} \left( c_{i,j}^{(i)} + \lambda_{j}^{(i)} \right) \cdot x_{i,j}^{(i)},
$$

(16)

where in the last inequality we apply the definition of $\delta_{\ell}^{(i)}$ for all $\ell \in D^{(i)}$. Combining (15) and (16), we obtain

$$
(1 - \alpha) \cdot o_{i} + \sum_{j \in D^{(i)}, k \in F^{(i)}} \sigma_{k}^{(i)} \cdot c_{i,k} \cdot x_{k,j}^{(i)} \leq \frac{1 - \alpha}{2} \cdot \frac{1}{y_{i}^{(i)}} \cdot \left( 3 \cdot o_{i} \cdot y_{i}^{(i)} + 2 \cdot \sum_{j \in D^{(i)}} c_{i,j} \cdot x_{i,j}^{(i)} \right)
$$

$$
= \frac{1 - \alpha}{2} \cdot \frac{1}{y_{i}^{(i)}} \cdot \theta \left( i, D^{(i)}, x^{(i)}, y^{(i)} \right) \cdot \sum_{j \in D^{(i)}} x_{i,j}^{(i)},
$$

(17)

where in the second last equality we apply Lemma 16 with the fact that $0 < y_{i}^{(i)} < (1 - \alpha)/2$, and in the last equality we apply the definition of $\theta \left( i, D^{(i)}, x^{(i)}, y^{(i)} \right)$.

By Lemma 13 and the definition of $x''$, we have

$$
\frac{1 - \alpha}{2} \cdot \frac{1}{y_{i}^{(i)}} \cdot \sum_{j \in D^{(i)}} x_{i,j}^{(i)} = \sum_{j \in D^{(i)}} x_{i,j}'' = \sum_{j \in D^{(i)}, k \in F^{(i)}} \sigma_{k}^{(i)} \cdot x_{k,j}^{(i)}.
$$

(18)

Moreover, by the design of the rounding algorithm, for any $k \in F^{(i)}$, we have

$$
\theta \left( i, D^{(i)}, x^{(i)}, y^{(i)} \right) \leq \theta \left( k, D^{(i)}, x_{k}^{(i)}, y_{k}^{(i)} \right).
$$

Combining this property with (18), we have

$$
\frac{1 - \alpha}{2} \cdot \frac{1}{y_{i}^{(i)}} \cdot \theta \left( i, D^{(i)}, x^{(i)}, y^{(i)} \right) \cdot \sum_{j \in D^{(i)}} x_{i,j}^{(i)}
$$

$$
\leq \sum_{k \in F^{(i)}} \sigma_{k}^{(i)} \cdot \theta \left( k, D^{(i)}, x_{k}^{(i)}, y_{k}^{(i)} \right) \cdot \sum_{j \in D^{(i)}} x_{k,j}^{(i)}
$$

$$
= \sum_{k \in F^{(i)}} \sigma_{k}^{(i)} \cdot \left( 3 \cdot o_{k} \cdot y_{k}^{(i)} + 2 \cdot \sum_{j \in D^{(i)}} c_{k,j} \cdot x_{k,j}^{(i)} \right),
$$

(19)

where in the last inequality we apply the definition of $\theta \left( k, D^{(i)}, x_{k}^{(i)}, y_{k}^{(i)} \right)$ for all $k \in F^{(i)}$ with $\sum_{j \in D^{(i)}} x_{k,j}^{(i)} > 0$. Combining (17) and (19) proves Inequality (13) and proves this lemma.

Applying Lemma 9, Lemma 11, and Lemma 17, we have the following lemma which establishes the guarantee for our iterative rounding process.
Lemma 18.

\[
\sum_{i \in I \setminus F'} \left( o_i \cdot y_i^* + \frac{1}{1 - \alpha} \cdot \sum_{j \in D(i)} c_{i,j} \cdot x_{i,j}'' \right) 
\leq \frac{3}{2} \cdot \frac{1}{\alpha} \cdot \sum_{i \in I} o_i \cdot y_i' + \frac{3}{1 - \alpha} \cdot \sum_{i \in I, j \in D, p \in P(i,j)} |p| \cdot f_p.
\] (20)

Proof of Lemma 18. Let \((x^{\tau(i)}, y^{\tau(i)})\) be an optimal solution for LP LP-(M) on the initial parameter tuple \(\Psi^{(0)}\). By Lemma 9 and triangle inequality, it follows that

\[
\sum_{i \in I} o_i \cdot y_{i}^* + \sum_{i \in I, j \in D} c_{i,j} \cdot x_{i,j}^* \leq \sum_{i \in I} o_i \cdot y_{i}^* + \sum_{i \in I, j \in D} c_{i,j} \cdot x_{i,j}^* 
\leq \frac{1 - \alpha}{2 \cdot \alpha} \cdot \sum_{i \in I} o_i \cdot y_i' + \sum_{i \in I, j \in D} \sum_{p \in P(i,j)} |p| \cdot f_p.
\] (21)

For any \(i \in I \setminus F'\), let \(\Psi^{n(i)} = (F^{n(i)}, D^{n(i)}, r^{n(i)})\) be the updated parameter tuple the algorithm maintains at the end of the iteration for rounding \(i\), and \((x^{\tau(i)}, y^{\tau(i)})\) be an optimal solution for LP-(M) on \(\Psi^{n(i)}\). By Lemma 11, it follows that

\[
\sum_{k \in F'^{(i)}} o_k \cdot y_{k}^{\tau(i)} + \sum_{k \in F'^{(i)}, j \in D'^{(i)}} c_{k,j} \cdot x_{k,j}^{\tau(i)} \leq \sum_{k \in F'^{(i)}} \left( 1 - \sigma_k^{(i)} \right) \cdot \left( o_k \cdot y_k^{\tau(i)} + \sum_{j \in D'^{(i)}} c_{k,j} \cdot x_{k,j}^{\tau(i)} \right).
\]

Combining the above with Lemma 17 and apply the fact that \(F'^{(i)} = F^{(i)} \setminus \{i\}\), we have

\[
\left( (1 - \alpha) \cdot o_i + \sum_{j \in D^{(i)}} c_{i,j} \cdot x_{i,j}' \right) + 3 \cdot \left( \sum_{i \in F'^{(i)}} o_i \cdot y_{i}^{\tau(i)} + \sum_{i \in F'^{(i)}, j \in D'^{(i)}} c_{i,j} \cdot x_{i,j}^{\tau(i)} \right) 
\leq 3 \cdot \sum_{k \in F'^{(i)}} \left( o_k \cdot y_{k}^{\tau(i)} + \sum_{j \in D'^{(i)}} c_{k,j} \cdot x_{k,j}^{\tau(i)} \right).\]

(22)

Inequality (22) shows that, the total cost incurred by \(i\) can be bounded within three times the difference between the optimal values of the successive iterations. Taking the summation over \(i \in I \setminus F'\) and applying Inequality (21), we obtain

\[
\sum_{i \in I \setminus F'} \left( (1 - \alpha) \cdot o_i + \sum_{j \in D^{(i)}} c_{i,j} \cdot x_{i,j}' \right) \leq 3 \cdot \left( \sum_{i \in I} o_i \cdot y_{i}^{\tau(0)} + \sum_{i \in I, j \in D} c_{i,j} \cdot x_{i,j}^{\tau(0)} \right) 
\leq 3 \cdot \frac{1 - \alpha}{2 \cdot \alpha} \cdot \sum_{i \in I} o_i \cdot y_i' + 3 \cdot \sum_{i \in I, j \in D, p \in P(i,j)} |p| \cdot f_p.
\]

Multiplying the above by \(1/(1 - \alpha)\) completes the proof of this lemma. 

\[ \square \]

23
Combining Lemma 18 and Inequality (11), it follows that

$$
\Psi\left(\frac{1}{1-\alpha} \cdot x'''' , \ y^*\right) = \sum_{i \in U \cup (I \setminus F')} o_i \cdot y^*_i + \frac{1}{1-\alpha} \cdot \sum_{i \in F^*, j \in D} c_{i,j} \cdot x''''_{i,j}
$$

$$
\leq \frac{1}{\alpha} \cdot \sum_{i \in U} o_i \cdot y'_i + \frac{1}{1-\alpha} \cdot \sum_{i \in U, j \in D} \sum_{p \in \Phi'(i,j)} |p| \cdot f'_p
$$

$$
+ \frac{3}{2\alpha} \cdot \sum_{i \in I} o_i \cdot y'_i + \frac{3}{1-\alpha} \cdot \sum_{i \in I, j \in D} \sum_{p \in \Phi(i,j)} |p| \cdot f_p.
$$

(23)

Recall that, in the definition of absolute length $|p|$ of any path $p \in \Phi'(i,j) \cup P(i,j)$, we take into account the length of the edges between $j^*$ and $i'$ for any $j' \in D$ and any $i' \in F$. Therefore, to bound the assignment cost incurred in (23) and (24), it suffices to consider the contribution of each possible pair of $j^*$ and $i$ for any $j \in D$ and $i \in F$.

We have the following lemma.

Lemma 19. We have

$$
\frac{1}{1-\alpha} \cdot \sum_{i \in U, j \in D, p \in \Phi'(i,j)} |p| \cdot f'_p + \frac{3}{1-\alpha} \cdot \sum_{i \in I, j \in D, p \in P(i,j)} |p| \cdot f_p \leq \frac{7 - 4\alpha}{(1-\alpha)^2} \cdot \sum_{i \in F, j \in D} c_{i,j} \cdot x'_{i,j}.
$$

(25)

Proof of Lemma 19. Let $i \in F$ and $j \in D$ be an arbitrary pair to be analyzed. For notational brevity, define $P' := \bigcup_{i^* \in U, j^* \in D} \Phi'(i^*, j^*)$ and $P(I) := \bigcup_{i^* \in I, j^* \in D} P(i^*, j^*)$ to denote the set of paths considered in the statement of this lemma. Furthermore, let

$$
a_{i,j}^{(\Rightarrow)} := \frac{1}{1-\alpha} \cdot \sum_{p \in P', (j^*,i) \in p} f'_p + \frac{3}{1-\alpha} \cdot \sum_{p \in P(I), (i^*,j) \in p} f_p
$$

and

$$
b_{i,j}^{(\Rightarrow)} := \frac{1}{1-\alpha} \cdot \sum_{p \in P', (i,i^*) \in p} f'_p + \frac{3}{1-\alpha} \cdot \sum_{p \in P(I), (i,j^*) \in p} f_p
$$

denote the total flow that is sent on the edge $(j^*,i)$ and $(i,j^*)$ by $f'$ and $f$, respectively.

In the following we will argue that

$$
a_{i,j}^{(\Rightarrow)} + b_{i,j}^{(\Rightarrow)} \leq \frac{7 - 4\alpha}{(1-\alpha)^2}.
$$

(25)

Note that (25) implies the statement of this lemma directly.

Depending on the category to which $i$ belongs, consider the following cases. We also refer the reader to Section 4.1, page 12, for the definition of the function $f'$.

• $i \in U^{(>)}$ is tightly-occupied.
In this case, by the definition of $f'$, we have

$$\sum_{p \in P', (j^*, i) \in p} f'_p = (1 - \alpha) \cdot g_{i,j} + \sum_{i^* \in U^{(>)}, j^* \in D_{|p|}, p \in P(i^*, j^*), (j^*, i) \in p} f_p + \sum_{i^* \in U^{(<)}, j^* \in D, p \in P(i^*, j^*), (j^*, i) \in p} f_p,$$

and

$$\sum_{p \in P', (i,j') \in p} f'_p = \sum_{i^* \in U^{(>)}, j^* \in D_{|p|}, p \in P(i^*, j^*), (i,j') \in p} f_p + \sum_{i^* \in U^{(<)}, j^* \in D, p \in P(i^*, j^*), (i,j') \in p} f_p.$$

It follows that

$$a_{i,j}^{(\Rightarrow)} + b_{i,j}^{(\Leftarrow)} \leq g_{i,j} + \frac{3}{1 - \alpha} \left( \sum_{p \in P, (j^*, i) \in p} f_p + \sum_{p \in P', (i,j') \in p} f_p \right)$$

$$\leq g_{i,j} + \frac{3}{1 - \alpha} \left( x'_{i,j} + g_{i,j} \right) \leq \frac{7 - 4\alpha}{(1 - \alpha)^2} \cdot x'_{i,j},$$

where the second last inequality follows from the feasibility of $f$ for $\text{MFN}_\Psi(x', y'', g)$ and the last inequality follows from the fact that $g_{i,j} = h_{i,j} \leq x'_{i,j}/(1 - \alpha)$.

- $i \in U^{(>)}$ is not tightly-occupied, $i \in U^{(<)}$, or $i \in I$.

In these cases, we have $g_{i,j} = 0$, which implies that $b_{i,j}^{(\Leftarrow)} = 0$. Furthermore,

$$\sum_{p \in P', (j^*, i) \in p} f'_p \leq (1 - \alpha) \cdot h_{i,j} + \sum_{j^* \in D, p \in P(i,j^*), (j^*, i) \in p} f_p \leq 2 \cdot x'_{i,j}$$

by the feasibility of $f$ for $\text{MFN}_\Psi(x', y'', g)$ and the fact that $h_{i,j} \leq x'_{i,j}/(1 - \alpha)$.

Hence, we have

$$a_{i,j}^{(\Rightarrow)} + b_{i,j}^{(\Leftarrow)} \leq \frac{5}{1 - \alpha} \cdot x'_{i,j} \leq \frac{7 - 4\alpha}{(1 - \alpha)^2} \cdot x'_{i,j}.$$

In either case, (25) holds. This proves the lemma.

Combining (23) and (24) with Lemma 19, we obtain

$$\Psi \left( \frac{1}{1 - \alpha} \cdot x'', y' \right) \leq \frac{3}{2\alpha} \cdot \sum_{i \in F} a_i \cdot y'_i + \frac{7 - 4\alpha}{(1 - \alpha)^2} \cdot \sum_{i \in F, j \in D} c_{i,j} \cdot x'_{i,j}$$

$$\leq \max \left\{ \frac{3}{2\alpha}, \frac{7 - 4\alpha}{(1 - \alpha)^2} \right\} \cdot \Psi(x', y').$$

This completes the proof of Lemma 6 and proves Theorem 4.
\[
\begin{align*}
\text{min} & \quad \sum_{i \in F} y_i + \sum_{i \in F, j \in D} c_{i,j} \cdot x_{i,j} \\
\text{s.t.} & \quad \sum_{i \in F} x_{i,j} \geq 1, \quad \forall j \in D \quad (N-1) \\
& \quad \sum_{j \in D} x_{i,j} \leq u_i \cdot y_i, \quad \forall i \in F \quad (N-2) \\
& \quad 0 \leq x_{i,j} \leq y_i, \quad \forall i \in F, j \in D \quad (N-3) \\
& \quad 0 \leq y_i \leq 1, \quad \forall i \in F. \quad (N-4)
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad \sum_{j \in D} \alpha_j - \sum_{i \in F} \eta_i \\
\text{s.t.} & \quad \alpha_j \leq \beta_i + \Gamma_{i,j} + c_{i,j}, \quad \forall i \in F, j \in D \quad (D-1) \\
& \quad u_i \cdot \beta_i + \sum_{j \in D} \Gamma_{i,j} \leq 1 + \eta_i, \quad \forall i \in F \quad (D-2) \\
& \quad \alpha_j, \beta_i, \Gamma_{i,j}, \eta_i \geq 0, \quad \forall i \in F, j \in D. \quad (D-3)
\end{align*}
\]

Figure 9: (Restate for further reference) The natural LP formulations for CFL-CFC.

5 Proof of Theorem 5.

The proof of this theorem is outlined as follows. First we define the notations and notions that are used throughout this section. In Section 5.1, page 27, we show that our rounding algorithm is well-defined and terminates in polynomial time. We define in the same section an intermediate assignment \(x^\circ\) and establish the feasibility of \((x^\circ, y^\ast)\) for \(\Psi\). This shows that the feasible region of the min-cost assignment problem between \(D\) and \(F^\ast\) is nonempty. We establish the approximation guarantee for \((x^\circ, y^\ast)\) in Section 5.2, page 33. This completes the proof since \(x^\dagger\) is the optimal solution for the min-cost assignment problem between \(D\) and \(F^\ast\).

Notations. Consider the cluster-forming process. Let \(C_{D'}\) and \(C_{H'}\) denote the sets of clusters centered at the non-outlier clients and outlier clients, respectively. For each \(q \in C_{D'}\), we use \(j(q)\) to denote the center client of \(q\) and \(i(q)\) denote the facility that is selected to be rounded up in the iteration when \(q\) is formed. Let \(F_{D'}^\ast := \{ i(q) : q \in C_{D'} \}\) denote the set of facilities rounded up for the clusters in \(C_{D'}\). Note that, by the setting of the algorithm, \(F_{D'}^\ast\) and \(G\) are mutually exclusive, and the set of satellite facilities \(B(j)\) for each \(j \in H\) jointly forms a partition of \(G\).

To prevent notational ambiguity, we will use \((x', y')\) to denote the initial solution the algorithm has for \(\Psi\). For each \(q \in C_{D'}\), we use \(D(q), F(q), H(q), H'(q), x'(q), \) and \(y'(q)\) to denote the set \(D'\), the set \(F'\), the sets \(H\), the set \(H'\), the assignment \(x'\), and the multiplicity \(y'\) the algorithm maintains at the moment when the cluster \(q\) is formed. We use \(B(q)\) to denote the set \(N_{(F',x')}((j(q))\) of satellite
facilities at that moment. Similarly, we use \( x^{(\text{II})} \) and \( y^{(\text{II})} \) to denote the assignment \( x' \) and the multiplicity \( y' \) the algorithm maintains when it enters the second phase.

For each outlier client \( j \in H \), we use \( w(j) \) to denote the facility in \( U \) at which \( j \) is located. We use \( p(j) \) to denote the specific parent client in \( J^{(++)} \) from which \( j \) is created. On the contrary, for any \( w \in U \), we use \( H(w) \) to denote the set of outlier clients located at \( w \). For any \( j \in J^{(++)} \), we use \( H(j) \) to denote the set of outlier clients that are created from \( j \).

For notational brevity, for any assignment function \( x \) of interest, we will use \( x|_{A,B} \) to denote the assignments made in \( x \) between \( A \subseteq F \) and \( B \subseteq D \). Similarly, for any multiplicity function \( y \) of interest, we will use \( y|_{A} \) to denote the multiplicity of facilities in \( A \subseteq F \) in \( y \). The value of zero is assumed for pairs and facilities not in the defining domain of \( x \) and \( y \).

5.1 The Feasibility

We first consider our rounding process for clusters in \( C_{D'} \) in the first phase and show that the rounding algorithm is well-defined and runs in polynomial time. We consider in page 29 the rounding process for clusters in \( C_{H'} \) in the second phase. In page 31 we define the assignment \( x^o \) and show that \((x^o, y^o)\) forms a feasible solution for \( \Psi \).

The rounding process for \( C_{D'} \). In the following we consider the first phase of our rounding process. To make the presentation precise, we will further use \((\tilde{x}', \tilde{y}')\), \( \tilde{F}', \tilde{D}', \) and \( \tilde{H} \) to denote the solution \((x', y')\), the sets \( F', D', \) and \( H \) the algorithm maintains at any particular moment, referred to by the context, during the rounding process in the first phase.

In order to characterize the behavior of the rounding process, we consider two types of constraints for \((\tilde{x}', \tilde{y}')\) throughout the process, namely, the capacity constraints for facilities in \( \tilde{F}' \) and constraint (N-3) from LP-(N) for facilities in \( F' \) and clusters in \( D' \), listed as follows.

\[
\sum_{j \in \tilde{D}' \cup \tilde{H}} \tilde{x}'_{i,j} \leq u_i \cdot \tilde{y}'_i, \quad \forall i \in \tilde{F}' \quad \text{(MN-2)}
\]
\[
\tilde{x}'_{i,j} \leq \tilde{y}'_i, \quad \forall i \in \tilde{F}', j \in \tilde{D}' \quad \text{(N-3)}
\]

It is clear that (MN-2) and (N-3) hold in the beginning of the rounding process since initially \( F' := I, D' := J^{(1)} \cup J^{(++)}, H := \emptyset \), and \((\tilde{x}', \tilde{y}')\) is a feasible solution for LP-(N).

The following two lemmas establish that, throughout the first phase of the rounding process, the constraints (MN-2) and (N-3) remain valid with respect to \((x', y')\), \( \tilde{F}', \tilde{D}', \) and \( \tilde{H} \).

Lemma 20. The process of creating outlier clients does not render (MN-2) invalid.

Proof of Lemma 20. Consider the process of creating outlier clients from a client, say, \( j \in J^{(++)} \cap D' \). By the algorithm design, this happens when \( \sum_{i \in F'} \tilde{x}'_{i,j} < 1/2 \).

For each \( i \in N_{(F', \tilde{F}')} (j) \), consider the assignments the algorithm has made from \( H(j) \) to \( i \) after \( H(j) \) is created. By the algorithm setting and the definition of \( d_\ell \) for any \( \ell \in H(j) \), we have

\[
\sum_{\ell \in H(j)} x_{i,\ell} := \sum_{\ell \in H(j)} d_\ell \cdot \frac{\tilde{x}'_{i,j}}{\sum_{k \in F'} \tilde{x}'_{k,j}} = \sum_{k \in F'} \frac{\tilde{x}'_{i,j}}{\tilde{x}'_{k,j}} \cdot \sum_{\ell \in H(j)} r'_i \cdot \frac{x_{w(\ell),j}}{\sum_{w \in U} x_{w,j}}.
\]
By the definition of \( r'_j \), we have \( r'_j \leq \sum_{k \in F'} x'_{k,j} \). Hence, the above becomes

\[
\sum_{\ell \in H(j)} x'_{i,\ell} \leq \tilde{x}'_{i,j} \cdot \sum_{\ell \in H(j)} \frac{\tilde{x}'_{w(\ell),j}}{\tilde{x}'_{w,\ell}} = \tilde{x}'_{i,j},
\]

where in the last equality we apply the fact that \( \sum_{\ell \in H(j)} \tilde{x}'_{w(\ell),j} = \sum_{w \in U} \tilde{x}'_{w,j} \). Since the algorithm resets \( \tilde{x}_{i,j} \) to be zero after \( H(j) \) is created, it follows from (26) that the LHS of (MN-2) for \( i \) does not increase after \( H(j) \) is created and the new assignments to \( i \) are made.

**Lemma 21.** We have \( 0 < \delta_{i(q)} \leq 1 \) for any \( q \in C_{D'} \). Furthermore, the scaled-down operation the algorithm performs when rounding \( i(q) \) does not render (MN-2) and (N-3) invalid.

**Proof of Lemma 21.** Consider any \( q \in C_{D'} \). We will show that, provided that constraints (MN-2) and (N-3) are valid in the beginning of the iteration for which \( q \) is formed, we have \( 0 < \delta_{i(q)} \leq 1 \) and the rounding process for \( i(q) \) does not render (MN-2) and (N-3) invalid. Note that this proves the lemma together with Lemma 20 since (MN-2) and (N-3) are valid in the beginning of the process.

Since \( i(q) \in F'^{(q)} \), we know that \( y_{i(q)}^{(q)} < 1/2 \). Furthermore, since \( j(q) \) is selected as the center client and since \( j(q) \in D'^{(q)} \) by assumption, it follows that \( \sum_{i \in F'^{(q)}} x_{i,j(q)}^{(q)} \geq 1/2 \). This implies that

\[
\sum_{i \in B(q) \setminus \{i(q)\}} y_{i(q)}^{(q)} \geq \sum_{i \in B(q) \setminus \{i(q)\}} x_{i,j(q)}^{(q)} \geq \frac{1}{2} - x_{i,j(q)}^{(q)} \geq \frac{1}{2} - y_{i(q)}^{(q)} > 0,
\]

where in the above inequalities we apply constraint (N-3) for \( i \in B(q) \) and \( j(q) \). This shows that

\[
\delta_{i(q)} := \left( \frac{1}{2} - y_{i(q)}^{(q)} \right) \cdot \frac{1}{\sum_{i \in B(q) \setminus \{i(q)\}} y_{i(q)}^{(q)}} > 0.
\]

On the contrary, by (N-3) we have \( \sum_{i \in B(q)} y_{i(q)}^{(q)} \geq \sum_{i \in B(q)} x_{i,j(q)}^{(q)} \geq 1/2 \). This implies that \( 1/2 - y_{i(q)}^{(q)} \leq \sum_{i \in B(q) \setminus \{i(q)\}} y_{i(q)}^{(q)} \) and \( \delta_{i(q)} \leq 1 \).

To see that constraints (MN-2) and (N-3) remain valid at the end of this iteration, observe that for each \( i \in B(q) \setminus \{i(q)\} \) and any \( j \in D'^{(q)} \), both \( x_{i,j}^{(q)} \) and \( y_{i(q)}^{(q)} \) are scaled down simultaneously by the constant \( (1 - \delta_{i(q)}) \). This completes the proof of this lemma.

By Lemma 20 and Lemma 21, we have the following corollary.

**Corollary 22.** Throughout the first phase of the rounding process, the constraints (MN-2) and (N-3) remain valid with respect to \( (\tilde{x}', \tilde{y}', \bar{F}', \bar{D}', \bar{H}) \).

Lemma 21 shows that the scaled-down operation in our rounding process is well-defined. Since constraint (N-3) holds throughout the process, it follows that, for any particular moment,

\[
\sum_{i \in F'} \tilde{x}'_{i,j} \leq \sum_{i \in F'} \tilde{y}'_j \quad \text{holds for any } j \in \bar{D}'.
\]

By the algorithm design, we know that at least one facility is removed from \( F' \) after each iteration in the first phase. Therefore, the rounding process repeats for at most \( \left| F' \right| := |I| \) iterations before \( F' \) becomes empty, which in turn implies that \( D' \cup H' \) is empty by (27). This shows that the rounding algorithm terminates in polynomial time.

The following lemma, which shows that the rounded facility is sparsely loaded by the rerouted assignments, is straightforward to verify.
Lemma 23. We have $\sum_{j \in D \cup H} x_{i,j}^*(q) \leq u_i(q)/2$ for any $q \in C_{D'}$.

Proof of Lemma 23. Consider any $q \in C_{D'}$. By the design of the algorithm and the fact that constraint (N-3) holds throughout the process, we have

$$\sum_{j \in D \cup H} x_{i,j}^*(q) = \sum_{j \in D \cup H} x_{i,j}^{n(q)} + \sum_{i \in B(q) \setminus \{i(q)\}} \sum_{j \in D \cup H} \delta_{i(q)} \cdot x_{i,j}^{n(q)}$$

$$\leq u_i(q) \cdot y_{i(q)}^{n(q)} + \sum_{i \in B(q) \setminus \{i(q)\}} \delta_{i(q)} \cdot u_i \cdot y_i^{n(q)}$$

$$\leq u_i(q) \cdot y_{i(q)}^{n(q)} + u_i(q) \cdot \left(\frac{1}{2} - y_{i(q)}^{n(q)}\right) = \frac{1}{2} \cdot u_i(q),$$

where in the second last inequality we use apply fact that $u_i(q) \geq u_i$ for all $i \in B(q)$ by the way $i(q)$ is selected and the definition of $\delta_{i(q)}$. \qed

The rounding process for $C_{H'}$. In the following we consider the rounding process for clusters in $C_{H'}$ in the second phase. The following lemma summarizes the status of the facilities and clients when the algorithm enters the this phase.

Lemma 24. When the algorithm enters the second phase, the following holds.

- For any $i \in G$, $\sum_{j \in D \cup H} x_{i,j}^{n(II)} \leq u_i \cdot y_i^{n(II)}$.
- For any $j \in J(I)$, $\sum_{i \in I} x_{i,j}^* + \sum_{i \in G} x_{i,j}^{n(II)} > 1/2$.
- For any $j \in J^{(++)}$, $\sum_{i \in I} x_{i,j}^* + \sum_{i \in G} x_{i,j}^{n(II)} + \sum_{i \in U} x_{i,j}' > 1/2$.
- For any $j \in H$, $\sum_{i \in F_{p,j}} x_{i,j}^* + \sum_{i \in G} x_{i,j}^{n(II)} = \sum_{i \in I} x_{i,j}'$.

Proof of Lemma 24. The first statement of this lemma follows directly from Corollary 22 and the definition of $(x^{n(II)}, y^{n(II)})$. The remaining of this lemma follows from the way how the algorithm handles the residue demand of each client. Consider the moment for which each $j \in J(I) \cup J^{(++)} \cup H$ is removed from consideration in the first phase, and the fact that $G := \bigcup_{h \in H} B(j)$.

For $j \in J(I) \cup J^{(++)}$, it is removed when $\sum_{i \in F_{p,j}} x_{i,j}' < 1/2$. It follows that the assignments rerouted to clusters in $C_{D'}$, the assignments taken into clusters in $C_{H'}$, and possibly the original assignments to facilities in $U$, account for at least $1/2$.

For $j \in H$, it is removed when selected as the center of a cluster that is possibly empty. When this happens, all of the remaining assignments for $j$ are taken into this cluster. \qed

Lemma 24 leads to the following corollary.

Corollary 25. $0 \leq t_{j}' \leq 2$ for all $j \in D$.

Proof of Corollary 25. It suffices to prove the statement for $j \in D$ with $\sum_{i \in I} x_{i,j}^* + \sum_{i \in G} x_{i,j}^{n(II)} > 0$. Since $\sum_{i \in I} x_{i,j}^* + \sum_{i \in G} x_{i,j}^{n(II)} > 0$ implies that $j \in J(I) \cup J^{(++)}$, by the definition of $r_{j}'$, we have

$$1 - \sum_{i \in U} x_{i,j}' - r_{j}' \geq 1 - \sum_{i \in I} x_{i,j}' - \sum_{i \in I} x_{i,j}' \geq 0,$$

which implies that $t_{j}' > 0$.

In the following we show that $t_{j}' \leq 2$. Since $j \in J(I) \cup J^{(++)}$, it suffices to prove the statement for the following cases.
• If \( j \in J^{(I)} \), then \( t'_j < 2 \) directly from the conclusion of Lemma 24 since \( 1 \sum_{i \in I} x^*_{i,j} + \sum_{i \in G} x_{i,j}^{(II)} > 1/2 \) by the algorithm design and \( (1 - \sum_{i \in U} x'_{i,j} + r'_j) \leq 1 \).

• If \( j \in J^{(++)} \) and \( r'_j \neq \sum_{i \in U} x'_{i,j} \), then all of the residue demand of \( j \) has been redistributed as outlier clients when \( j \) is to be removed from \( D' \). It follows that

\[
1 - \sum_{i \in U} x'_{i,j} - r'_j = \sum_{i \in I} x^*_{i,j} + \sum_{i \in G} x_{i,j}^{(II)} \quad \text{and} \quad t'_j = 1.
\]

• If \( j \in J^{(++)} \) and \( r'_j := \sum_{i \in U} x'_{i,j} \), then by the conclusion of Lemma 24 we have

\[
1 - \sum_{i \in U} x'_{i,j} - r'_j = 1 - 2 \cdot \sum_{i \in U} x'_{i,j} < 2 \cdot \left( \sum_{i \in I} x^*_{i,j} + \sum_{i \in G} x_{i,j}^{(II)} \right)
\]

and \( t'_j < 2 \).

In all cases we have \( t'_j \leq 2 \). \( \square \)

The bundled assignment \( g \) for LP-(O). In the following we consider the assignment LP for rounding the clusters in \( C_H' \). For any \( w \in U \) and \( i \in G \) such that \( i \in B(k) \) for some \( k \in H(w) \), i.e., \( i \) belongs to the clusters centered at some \( k \in H(w) \), define the bundled assignment \( g_{i,w} \) as

\[
g_{i,w} := \sum_{\ell \in D \cup H} t'_\ell \cdot x_{i,\ell}^{(II)}.
\]

The following lemma shows that the feasible region of LP-(O) is nonempty, and the basic optimal solution \((x'', y'')\) exists.

**Lemma 26.** \(( g|_{G,U} , 2 y^{(II)}|_{G} \) is a feasible solution for LP-(O).

**Proof of Lemma 26.** We prove by verifying the constraints of LP-(O).
• Consider (O-1) in LP-(O).
  For any \( w \in U \), apply the definition of \( g \) and the definition of \( d_w \), we have
  \[
  \sum_{i \in G} g_{i,w} = \sum_{k \in H(w), i' \in B(k)} \sum_{\ell \in D \cup H} t'_{\ell} \cdot x^\text{(II)}_{i',\ell} = d_w \geq d_w.
  \]

• Consider (O-3) in LP-(O).
  For any \( i \in G \), since \( G \subseteq I \), it follows that \( y^\text{(II)}_i \leq y'_i \leq 1/2 \), and we have \( 2 \cdot y^\text{(II)}_i \leq 1 \).

• Consider (O-2) in LP-(O).
  For any \( i \in G \), let \( k \in H \) be the outlier client such that \( i \in B(k) \). Applying the definition of \( g \) and Corollary 25, we have
  \[
  \sum_{w \in U} g_{i,w} = g_{i,w(k)} = \sum_{\ell \in D \cup H} t'_{\ell} \cdot x^\text{(II)}_{i,\ell} \leq \sum_{\ell \in D \cup H} 2 \cdot x^\text{(II)}_{i,\ell} \leq 2 \cdot u_i \cdot y^\text{(II)}_i,
  \]
  where in the last inequality we apply the conclusion of Corollary 22 which states that constraint (MN-2) holds for \( i \) when \( i \) is removed from \( F' \) in the first phase.

This proves the lemma. \( \square \)

**The unbundled assignment \( h \) from \( x'' \).** For each \( i \in G \) and \( j \in D \cup H \), define the unbundled assignment \( h_{i,j} \) from \( x'' \) as
\[
  h_{i,j} := \sum_{w \in U} x''_{i,w} \cdot \frac{1}{d_w} \cdot \sum_{k \in H(w), i' \in B(k)} t'_{i,j} \cdot x^\text{(II)}_{i',j}.
\]

Intuitively, in \( h \) we redistribute the bundled assignment \( x'' \) back for the original clients, and it follows that for any \( j \in D \cup H \),
\[
  \sum_{i \in G} h_{i,j} = \sum_{i \in G, w \in U} x''_{i,w} \cdot \frac{1}{d_w} \cdot \sum_{k \in H(w), i' \in B(k)} t'_{i,j} \cdot x^\text{(II)}_{i',j}
  = \sum_{w \in U} \sum_{k \in H(w), i' \in B(k)} t'_{i,j} \cdot x^\text{(II)}_{i',j} = \sum_{i \in G} t'_{i,j} \cdot x^\text{(II)}_{i,j}, \tag{28}
\]
where in the second equality we apply (O-1) from LP-(O) for the fact that \( \sum_{i \in G} x''_{i,w} = d_w \) for all \( w \in U \) since \((x'', y'')\) is a feasible solution for LP-(O) and in the last equality we use the fact that the set of satellite facilities for each \( j \in H \) forms a partition of \( G \).

**The Assignment \( x^o \).** Provided above, the assignment \( x^o \) made for each \( j \in D \) is defined as
\[
x^o \big|_j := x' \big|_{U,\{j\}} + t'_{j} \cdot x^\star \big|_{F^\star_{D' \setminus \{j\}}} + x^\star \big|_{F^\star_{D' \cup H \setminus \{j\}}} + h \big|_{G,\{j\} \cup H \setminus \{j\}}.
\]

More precisely, for any \( i \in F \), \( j \in D \),
\[
x^o_{i,j} := \begin{cases} 
  x'_{i,j}, & \text{if } i \in U, \\
  t'_{j} \cdot x^\star_{i,j} + \sum_{k \in H(j)} x^\star_{i,k}, & \text{if } i \in F^\star_{D'}, \\
  h_{i,j} + \sum_{k \in H(j)} h_{i,k}, & \text{if } i \in G, \\
  0, & \text{otherwise}.
\end{cases}
\]

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Intuitively, the assignment of each $j \in D$ in $x^\circ$ consists of its original assignments to $U$ and the rounded assignments for clients in $\{j\} \cup H(j)$ to facilities in $F^\ast_{D'} \cup G$.

The following lemma, which asserts the feasibility of $x^\circ$, is straightforward to verify.

Lemma 27. $(x^\circ, y^\ast)$ is feasible for LP-(N) on the input instance $\Psi$.

Proof of Lemma 27. In the following we show that $(x^\circ, y^\ast)$ is feasible for LP-(N) on $\Psi$. Since $y^\ast$ is already integral and takes values only from $\{0, 1\}$, it suffices to argue that $x^\circ$ fully-assigns each $j \in D$ and respects the capacity constraints given by $y^\ast$.

For the latter part, since $x^\circ$ keeps the assignments of $D$ to $U$ unchanged, it suffices to argue for the assignments to $F^\ast_{D'} \cup G$. By the definition of $x^\circ$, Corollary 25, and Lemma 23, for any $i \in F^\ast_{D'}$, we have

$$
\sum_{j \in D} x^\circ_{i,j} = \sum_{j \in D} t'_j \cdot x^\ast_{i,j} + \sum_{j \in H} x^\ast_{i,j} \leq \sum_{j \in D \cup H} 2 \cdot x^\ast_{i,j} \leq u_i = u_i \cdot y^\ast_i.
$$

Similarly, for any $i \in G$, applying the definition of $x^\circ$ and $h$, and the fact that $(x''', y'')$ is feasible for LP-(O), we have

$$
\sum_{j \in D} x^\circ_{i,j} = \sum_{j \in D \cup H} h_{i,j} = \sum_{j \in D \cup H} \sum_{w \in U} x''_{i,w} \cdot \frac{1}{d_w} \cdot \sum_{k \in H(w), v \in B(k)} t'_j \cdot x''_{v,j} = \sum_{w \in U} x''_{i,w} \leq u_i \cdot y''_i,
$$

where in the second last equality we apply the definition of $d_w$ for each $w \in U$.

In the following, we show that $x^\circ$ fully-assigns each $j \in D$. It suffices to argue for the clients in $J^{(I)} \cup J^{(I\ast)}$. For the former case, for any $j \in J^{(I)}$, we have $\sum_{i \in U} x'_{i,j} = 0$ and $H(j) = \emptyset$. Hence,

$$
\sum_{i \in F} x^\circ_{i,j} = \sum_{i \in F_{D'}} t'_j \cdot x^\ast_{i,j} + \sum_{i \in G} h_{i,j}
= \sum_{i \in F_{D'}} t'_j \cdot x^\ast_{i,j} + \sum_{i \in G} t'_j \cdot x''_{i,j} = t'_j \cdot \left( \sum_{i \in F_{D'}} x^\ast_{i,j} + \sum_{i \in G} x''_{i,j} \right) = 1,
$$

where in the second equality we apply Equality (28), and in the last equality we apply the definition of $t'_j$ with the fact that $\sum_{i \in U} x'_{i,j} = r'_j = 0$.

For $j \in J^{(I\ast)}$, we have

$$
\sum_{i \in F} x^\circ_{i,j} = \sum_{i \in U} x'_{i,j} + \sum_{i \in F_{D'}} \left( t'_j \cdot x^\ast_{i,j} + \sum_{k \in H(j)} x^\ast_{i,k} \right) + \sum_{i \in G} \left( h_{i,j} + \sum_{k \in H(j)} h_{i,k} \right) \tag{29}
$$

Applying Equality (28) and the definition of $t'_k$ for $k \in H(j)$, we have

$$
\sum_{i \in G} \left( h_{i,j} + \sum_{k \in H(j)} h_{i,k} \right) = \sum_{i \in G} t'_j \cdot x''_{i,j} + \sum_{k \in H(j), i \in G} x''_{i,k} \tag{30}
$$

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Combining (29) and (30), we have

\[
\sum_{i \in F} x_{i,j}^0 = \sum_{i \in U} x_{i,j} + \sum_{i \in F_D^p} t_j' \cdot x_{i,j}^* + \sum_{i \in G} t_j' \cdot x_{i,j}^{(\Pi)} + \sum_{k \in H(j)} \left( \sum_{i \in F_D^p} x_{i,k}^* + \sum_{i \in G} x_{i,k}^{(\Pi)} \right)
\]

\[= \sum_{i \in U} x_{i,j}' + \sum_{i \in F_D^p} t_j' \cdot x_{i,j}^* + \sum_{i \in G} t_j' \cdot x_{i,j}^{(\Pi)} + \sum_{k \in H(j)} \sum_{i \in I} x_{i,k}', \tag{31}\]

where in the last equality we apply the conclusion of Lemma 24. By the construction of outlier clients, each \(k \in H(j)\) is fully-assigned to facilities in \(I\) by \(x'\). Hence, we have \(\sum_{i \in I} x_{i,k}' = d_k\) for each \(k \in H(j)\). Further applying the definition of \(d_k\), we have

\[
\sum_{k \in H(j)} \sum_{i \in I} x_{i,k}' = \sum_{k \in H(j)} d_k = \sum_{k \in H(j)} r_j' \cdot \frac{x_{u(k),j}'}{\sum_{i \in U} x_{i,j}'} = r_j', \tag{32}\]

where the last equality follows from the fact that exactly one outlier client is created for each \(i \in U\) with \(x_{i,j}' > 0\). Combining (31) and (32) and applying the definition of \(t_j'\), we have

\[
\sum_{i \in F} x_{i,j}^0 = \sum_{i \in U} x_{i,j}' + \sum_{i \in F_D^p} t_j' \cdot x_{i,j}^* + \sum_{i \in G} t_j' \cdot x_{i,j}^{(\Pi)} + r_j' = 1.
\]

This completes the proof of this lemma.

\[\square\]

5.2 Approximation Guarantee

In the following we establish our approximation guarantee for CFL-CFC. We will bound the cost incurred by clusters in \(\mathcal{C}_{H'}\) and \(\mathcal{C}_{D'}\) separately. First we bound below the cost incurred by \(\mathcal{C}_{H'}\). We bound the cost incurred by \(\mathcal{C}_{D'}\) in page 37 and establish the overall guarantee in page 40.

Recall that we use \(p(j)\) for each \(j \in H\) to denote the client in \(\mathcal{D}\) from which \(j\) is created. In the following, we extend this definition and define \(p(k) := k\) for each \(k \in \mathcal{D}\) for notational brevity.

The Clusters in \(\mathcal{C}_{H'}\). Consider the cost incurred by clusters in \(\mathcal{C}_{H'}\). We begin with the following lemma regarding the assignment radius of the outlier clients in \(H\).

Lemma 28. For any \(j \in H\) and \(i \in G\) such that \(x_{i,j}^{(\Pi)} > 0\), we have \(c_{i,j} \leq \alpha_j\).

Proof of Lemma 28. By the algorithm design, the outlier client \(j\) is created and assigned to \(i\) in \(x'\) only when \(x_{i,p(j)}' > 0\) holds in the beginning of the rounding process, which implies that \(c_{i,p(j)} \leq \alpha_{p(j)}\) by complementary slackness condition. By triangle inequality and the definition of \(\alpha_j\), it follows that \(c_{i,j} = c_{i,w(j)} \leq c_{w(j),p(j)} + c_{i,p(j)} \leq c_{w(j),p(j)} + \alpha_{p(j)} = \alpha_j\).

The following lemma, which bounds the assignment cost of \(x^0|_{G,D}\) in terms of that in \(x''\) and \(x^{(\Pi)}|_{G,D\cup H}\), follows from the way the clusters in \(\mathcal{C}_{H'}\) are formed and the assignments are bundled.

Lemma 29.

\[
\sum_{i \in G, j \in D} c_{i,j} \cdot x_{i,j}^0 \leq \sum_{i \in G, j \in U} c_{i,j} \cdot x_{i,j}'' + \sum_{i \in G} \sum_{j \in D \cup H} t_j' \cdot (c_{i,p(j)} + \alpha_j) \cdot x_{i,j}^{(\Pi)}.
\]
Proof of Lemma 29. By the definition of $x^\circ$ and $h$, we have

$$\sum_{i \in G, j \in D} c_{i,j} \cdot x^\circ_{i,j} = \sum_{i \in G, j \in D} \sum_{k \in (j) \cup H(j)} c_{i,j} \cdot h_{i,k} = \sum_{i \in G} \sum_{j \in D \cup H} c_{i,p(j)} \cdot h_{i,j}$$

$$= \sum_{i \in G} \sum_{j \in D \cup H} c_{i,p(j)} \sum_{w \in U} x^\prime_{i,w} \cdot \frac{1}{d_w} \cdot \sum_{k \in H(w), \ j' \in B(k)} t_j' \cdot x^\prime_{i',j}.$$  \hspace{1cm} (33)

By triangle inequality, for any $i \in G, j \in D \cup H, w \in U, k \in H(w)$, and $i' \in B(k)$ such that $x^\prime_{i',j} > 0$, we have

$$c_{i,p(j)} \leq c_{i,w} + c_{i',w} + c_{i',p(j)} \leq c_{i,w} + \alpha_k + c_{i',p(j)},$$

where the last inequality follows from Lemma 28 and the fact that $i' \in B(k)$ implies that $x^\prime_{i',k} > 0$. See also Figure 11 for an illustration.

Applying the above inequality on (33) with proper rearrangement, we obtain

$$\sum_{i \in G, w \in U} c_{i,w} \cdot x^\prime_{i,w} \cdot \frac{1}{d_w} \cdot \sum_{k \in H(w), \ j' \in B(k)} \sum_{j \in D \cup H} t_j' \cdot x^\prime_{i',j'}$$

$$+ \sum_{i \in G, w \in U} x^\prime_{i,w} \cdot \frac{1}{d_w} \cdot \sum_{k \in H(w), \ j' \in B(k)} \sum_{j \in D \cup H} \left( c_{i',p(j)} + \alpha_k \right) \cdot t_j' \cdot x^\prime_{i',j}.$$  \hspace{1cm} (33)

Applying the definition of $d_w$ on the former term and the fact that $\sum_{i \in G} x^\prime_{i,w} = d_w$ by the fact that $(x'', y'')$ is a feasible solution for LP-(O) on the latter term, the above becomes

$$\sum_{i \in G, w \in U} c_{i,w} \cdot x^\prime_{i,w} + \sum_{w \in U} \sum_{k \in H(w), \ j' \in B(k)} \sum_{j \in D \cup H} \left( c_{i',p(j)} + \alpha_k \right) \cdot t_j' \cdot x^\prime_{i',j}.$$  \hspace{1cm} (33)

By the design of the algorithm, for any $w \in U, k \in H(w), i' \in B(k)$, and for any $j \in D \cup H$ with $x^\prime_{i',j} > 0$, we have $\alpha_k \leq \alpha_j$, since $k$ is selected as cluster center because of having the lowest $\alpha$ value. Therefore, the above is further bounded by

$$\sum_{i \in G, w \in U} c_{i,w} \cdot x^\prime_{i,w} + \sum_{w \in U} \sum_{k \in H(w), \ j' \in B(k)} \sum_{j \in D \cup H} \left( c_{i',p(j)} + \alpha_j \right) \cdot t_j' \cdot x^\prime_{i',j}.$$  \hspace{1cm} (33)

Figure 11: An illustration on the bundled assignment from $w \in U$ to $i \in G$ and unbundled assignments for $k \in H(w), i' \in B(k)$ such that $x^\prime_{i',j} > 0$. 


Applying the fact that the satellite facilities of clusters in $C_H$ forms a partition of $G$, the above is exactly

$$
\sum_{i \in G, j \in U} c_{i,j} \cdot x''_{i,j} + \sum_{i \in G} \sum_{j \in D \cup H} t'_j \cdot \left( c_{i,p(j)} + \alpha_j \right) \cdot x''_{i,j}.
$$

The following lemma bounds the cost incurred by $(x'', y'')$.

**Lemma 30.** We have

$$
\sum_{i \in G} \left[ y'_i \right] + \sum_{i \in G, j \in U} c_{i,j} \cdot x''_{i,j} \leq 2 \cdot \sum_{i \in G} y''_{i} + \sum_{i \in G, j \in U} c_{i,j} \cdot g_{i,j},
$$

where $L := \{ i \in G : 0 < y''_i < 1 \}$.

**Proof of Lemma 30.** Since $(x'', y'')$ is optimal for LP-(O), by Lemma 26, the cost of $(x'', y'')$ is no more than that of $\left( g\vert_{G,U} \right) \cdot 2 \left( y'\vert_G \right)$. Hence,

$$
\sum_{i \in G} y''_i + \sum_{i \in G, j \in U} c_{i,j} \cdot x''_{i,j} \leq 2 \cdot \sum_{i \in G} y''_{i} + \sum_{i \in G, j \in U} c_{i,j} \cdot g_{i,j} = 2 \cdot \sum_{i \in G} y''_{i} + \sum_{i \in G, w \in U, \ell \in D \cup H \text{ for some } k \in H(w)} t'_\ell \cdot x''_{i,\ell}.
$$

By the algorithm setting and Lemma 28, for any $i \in G$ and $w \in U$ such that $i \in B(k)$ for some $k \in H(w)$, i.e., facility $i$ belongs to some cluster centered at some $k \in H(w)$, we have $c_{i,w} \leq \alpha_k \leq \alpha_\ell$, for any $\ell \in D \cup H$ with $x''_{i,\ell} > 0$. Therefore,

$$
\sum_{i \in G} y''_i + \sum_{i \in G, j \in U} c_{i,j} \cdot x''_{i,j} \leq 2 \cdot \sum_{i \in G} y''_{i} + \sum_{i \in G, w \in U, \ell \in D \cup H \text{ for some } k \in H(w)} t'_\ell \cdot x''_{i,\ell} = 2 \cdot \sum_{i \in G} y''_{i} + \sum_{i \in G, j \in D \cup H} t'_j \cdot \alpha_j \cdot x''_{i,j},
$$

where the last equality follows from the fact that the set of satellite facilities of clusters in $C_H$ forms a partition of $G$. Applying the definition of $L$ competes the proof of this lemma. $
$

The following lemma follows from the fact that $(x'', y'')$ is a basic solution for LP-(O).

**Lemma 31.**

$$
|L| \leq |U|, \quad \text{where } L := \{ i \in G : 0 < y''_i < 1 \}.
$$

**Proof of Lemma 31.** Consider the set of constraints in LP-(O) that hold with equality at $(x'', y'')$, for which we denote by $E^{(=)}$ in the following. Let $M_3$ and $M_4$ denote the set of constraints in $E^{(=)}$ of the types (O-3) and (O-4), respectively. Formally,

$$
M_3 := \{ i : y_i \leq 1 \in E^{(=)} \}.
$$
and
\[ M_4 := \left\{ (i, j) : \begin{cases} x_{i,j} \geq 0 & \in \mathcal{E}(=) \\ y_i \geq 0 & \in \mathcal{E}(=) \end{cases} \right\}. \]

Let \( X \) be the number of variables in LP-(O). Since \((x'', y'')\) is a basic solution for LP-(O), it follows that, the coefficient matrix of \( \mathcal{E}(=) \) is of full-rank, i.e., has rank \( X \). Since \( M_3 \) and \( M_4 \) are linearly independent, there exists a subset \( \mathcal{E}' \subseteq \mathcal{E}(=) \) of linearly independent constraints such that \( M_3 \cup M_4 \subseteq \mathcal{E}' \) and \( |\mathcal{E}'| = X \).

Let \( \mathcal{E}'' := \mathcal{E}' \setminus (M_3 \cup M_4) \) and modify the constraints in \( \mathcal{E}'' \) by setting the variable \( y_i \) to be 1 for all \( i \in M_3 \). Similarly, modify \( \mathcal{E}'' \) by setting \( x_{i,j} \) to be zero for all \((i, j) \in M_4 \) and \( y_i \) to be zero for all \( i \in M_4 \). By doing so, we removed equally many constraints and variables from \( \mathcal{E}' \). Since \( M_3 \) and \( M_4 \) are linearly independent, it follows that
\[ \text{rank}(\mathcal{E}'') = \text{rank}(\mathcal{E}') - |M_3 \cup M_4|, \]
and the coefficient matrix of \( \mathcal{E}'' \) is still of full-rank.

Let \( M_1 := \left\{ j : \sum_{i \in G} x_{i,j} \geq d_j \in \mathcal{E}'' \right\} \) and \( M_2 := \left\{ i : \sum_{j \in U} x_{i,j} \leq u_i \cdot y_i \in \mathcal{E}'' \right\} \).

Also let \( H := \left\{ (i, j) : x_{i,j}'' \neq 0 \right\} \). It follows by the above setting that, \( L \cup H \) corresponds exactly to the set of variables in \( \mathcal{E}'' \). Since the coefficient matrix of \( \mathcal{E}'' \) has full rank, the pivot in each row of the matrix defines a one-to-one mapping \( \phi : L \cup H \to M_1 \cup M_2 \) between the variables and the constraints.

Consider each \( i \in L \). Since the variable \( y_i \) appears exactly in one constraint in \( M_2 \), the mapping must map \( y_i \) to the constraint it corresponds to, i.e., \( \phi(i) = i \). Since the constraint \( i \) corresponds to in \( M_2 \) contributes one rank, it is non-degenerated and contains at least one variable in \( H \). Let \( x_{i,j} \) be one such variable. Since \( x_{i,j} \) appears in exactly two constraints, i.e., in the one \( i \) corresponds to in \( M_2 \) and the one \( j \) corresponds to in \( M_1 \), and since \( \phi(i) = i \), it follows that \( \phi((i, j)) = j \). Since the mapping \( \phi \) is one-to-one, \( j \) cannot be mapped to by other pairs.

Applying the above argument for each \( i \in L \) results in a set consisting of distinct clients \( j \) from \( U \) with the same cardinality. This shows that \(|L| \leq |U|\). \( \square \)

Applying Lemma 29, Lemma 30, Lemma 31, and the fact that \( y_i' \geq 1/2 \) for all \( i \in U \), we obtain the following bound for the cost incurred by \((x^0_{|G,D}, y^0_{|G})\).
\[
\sum_{i \in G} y_i^* \quad + \sum_{i \in G, j \in D} c_{i,j} \cdot x_{i,j}^0 \quad \leq \quad 2 \cdot \sum_{i \in G} y_i^{(II)} \quad + \quad 2 \cdot \sum_{i \in G} y_i' \quad + \quad \sum_{i \in G, j \in H} t_j' \cdot (c_{i,p(j)} + 2 \cdot \alpha_j) \cdot x_{i,j}^{(II)} \\
\quad + \quad \sum_{i \in G, j \in D} t_j' \cdot (c_{i,j} + 2 \cdot \alpha_j) \cdot x_{i,j}^{(II)} \\
\quad \leq \quad 2 \cdot \sum_{i \in G} y_i^{(II)} \quad + \quad 2 \cdot \sum_{i \in U} y_i' \quad + \quad \sum_{i \in G, j \in H} (c_{i,p(j)} + 2 \cdot \alpha_j) \cdot x_{i,j}^{(II)} \\
\quad + \quad \sum_{i \in G, j \in D} (2 \cdot c_{i,j} + 2 \cdot t_j' \cdot \alpha_j) \cdot x_{i,j}^{(II)},
\]

where in the last inequality we apply Corollary 25 the fact that \( t_j' \leq 2 \) for all \( j \in D \) and the definition that \( t_j' = 1 \) for all \( j \in H \).
The Clusters in $C_{D'}$. In the following we consider the the clusters in $C_{D'}$. The following lemma bounds the cost incurred by each individual cluster $q$ in $C_{D'}$.

**Lemma 32.** For any $q \in C_{D'}$, we have

(i) \[ y_{i(q)}^* \leq 2y_{i(q)}^{l(q)} + 2\delta_{i(q)} \cdot \sum_{k \in B(q) \setminus \{i(q)\}} y_k^{l(q)}, \] and

(ii) \[ \sum_{j \in D} c_{i(q),j} \cdot x_{i(q),j}^o \leq \sum_{j \in D} 2 \cdot c_{i(q),j} \cdot x_{i(q),j}^{l(q)} + \sum_{j \in H} c_{i(q),p(j)} \cdot x_{i(q),j}^{l(q)} \]

\[ + \delta_{i(q)} \cdot \sum_{k \in B(q) \setminus \{i(q)\}} \left( \sum_{j \in D} 2 \cdot c_{k,j} \cdot x_{k,j}^{r(q)} + \sum_{j \in H} c_{k,p(j)} \cdot x_{k,j}^{r(q)} \right) \]

\[ + \sum_{j \in D} 2 \cdot t_{j}^* \cdot \alpha_j \cdot x_{i(q),j}^* + \sum_{j \in H} 2 \cdot \alpha_j \cdot x_{i(q),j}^*. \]

**Proof of Lemma 32.** The lemma follows directly from the rounding process for clusters in $C_{D'}$. Consider the iteration for which $q$ is formed and ready to be rounded. By the algorithm design, the total facility value that has been removed from $F'$ due to the rounding process for $q$ is

\[ y_{i(q)}^{l(q)} + \delta_{i(q)} \cdot \sum_{k \in B(q) \setminus \{i(q)\}} y_k^{l(q)} = \frac{1}{2} = \frac{1}{2} \cdot y_{i(q)}^*, \]

where in the first equality we apply the definition of $\delta_{i(q)}$. Attributing the cost of $y_{i(q)}^*$ to the facility value that is removed from $F'$ due to cluster $q$ proves the first part of this lemma.

For the second part, by the definition of $x^o$ and the way how the algorithm reroutes the assignments from facilities in $B(q) \setminus \{i(q)\}$ to $i(q)$, we have

\[ \sum_{j \in D} c_{i(q),j} \cdot x_{i(q),j}^o = \sum_{j \in D} c_{i(q),j} \cdot \left( t_{j} \cdot x_{i(q),j}^* + \sum_{\ell \in H(j)} x_{i(q),\ell}^* \right) \]

\[ = \sum_{j \in D} c_{i(q),j} \cdot t_{j} \cdot x_{i(q),j}^{l(q)} + \sum_{\ell \in H} c_{i(q),p(\ell)} \cdot x_{i(q),\ell}^{l(q)} \]

\[ + \delta_{i(q)} \cdot \sum_{k \in B(q) \setminus \{i(q)\}} \left( \sum_{j \in D} c_{i(q),j} \cdot t_{j} \cdot x_{k,j}^{l(q)} + \sum_{\ell \in H} c_{i(q),p(\ell)} \cdot x_{k,\ell}^{l(q)} \right). \quad \text{(35)} \]

By the algorithm setting, for any $k \in B(q) \setminus \{i(q)\}$ and any $\ell \in H$ with $x_{k,\ell}^{l(q)} > 0$, we have

\[ c_{i(q),p(\ell)} \leq c_{k,p(\ell)} + c_{k,j(\ell)} + c_{i(q),j(\ell)} \leq c_{k,p(\ell)} + 2\alpha_{j(\ell)} \leq c_{k,p(\ell)} + 2\alpha_{\ell}, \]

where in the second inequality we apply the fact that $i(q)$ and $k$ are in $B(q)$, which implies that $x_{i(q),j(\ell)}^{l(q)} > 0$, $x_{k,j(\ell)}^{l(q)} > 0$, and $\max (c_{i(q),j(\ell)}, c_{k,j(\ell)}) \leq \alpha_{j(\ell)}$ by complementary slackness, and in the last inequality we apply the assumption that $x_{k,\ell}^{l(q)} > 0$, which implies that $\ell \in H^{l(q)}$ and $\alpha_{j(\ell)} \leq \alpha_{\ell}$ by the way $j(\ell)$ is selected. By a similar argument, we have $c_{i(q),j} \leq c_{k,j} + 2\alpha_j$ for any $k \in B(q) \setminus \{i(q)\}$ and any $j \in D$ with $x_{k,j}^{l(q)} > 0$. 

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By the above conclusion, Corollary 25, and the way the assignment $x^*$ is formed during the rounding process, we have

$$
\delta_i(q) \cdot \sum_{k \in B(q) \setminus \{i(q)\}, j \in D} c_{i(q),j} \cdot t'_j \cdot x_{k,j}^{l(q)} \leq \delta_i(q) \cdot \sum_{k \in B(q) \setminus \{i(q)\}, j \in D} (c_{k,j} + 2 \cdot \alpha_j) \cdot t'_j \cdot x_{k,j}^{l(q)}
$$

$$
\leq \delta_i(q) \cdot \sum_{k \in B(q) \setminus \{i(q)\}, j \in D} 2 \cdot c_{k,j} \cdot x_{k,j}^{l(q)} + \sum_{j \in D} 2 \cdot t'_j \cdot \alpha_j \cdot x_{i(q),j}^*.
$$

Similarly, we have

$$
\delta_i(q) \cdot \sum_{k \in B(q) \setminus \{i(q)\}, \ell \in H} c_{i(q),p(\ell)} \cdot x_{k,\ell}^{l(q)} \leq \delta_i(q) \cdot \sum_{k \in B(q) \setminus \{i(q)\}, \ell \in H} c_{k,p(\ell)} \cdot x_{k,\ell}^{l(q)} + \sum_{\ell \in H} 2 \cdot \alpha_{\ell} \cdot x_{i(q),\ell}^*.
$$

Combining the above two inequalities with (35) and further applying Corollary 25, we have

$$
\sum_{j \in D} c_{i(q),j} \cdot x_{i,j}^0 \leq \sum_{j \in D} 2 \cdot c_{i(q),j} \cdot x_{i,j}^{l(q)} + \sum_{\ell \in H} c_{i(q),p(\ell)} \cdot x_{i,\ell}^{l(q)} + \sum_{j \in D} 2 \cdot t'_j \cdot \alpha_j \cdot x_{i(q),j}^* + \sum_{\ell \in H} 2 \cdot \alpha_{\ell} \cdot x_{i(q),\ell}^*.
$$

Take summation on the cost given in Lemma 32 over all clusters in $C_{D'}$ (in the reverse order they are formed) and consider the contribution of each facility in $F_{D'}^* \cup G$ into the RHS of the summation. We have the following lemma which bounds the cost incurred by $x^*|_{F_{D'}^*} \cup D$ and $y^*|_{F_{D'}^*}$.

**Lemma 33.**

(i) \[ \sum_{i \in F_{D'}^*} y_i^* \leq \sum_{i \in \Gamma \setminus G} y_i' + 2 \cdot \sum_{i \in \Gamma \setminus G} \left( y_i' - y_i'^{(II)} \right). \] (36)

(ii) \[ \sum_{i \in F_{D'}^*, j \in D} c_{i,j} \cdot x_{i,j}^0 \leq \sum_{i \in F_{D'}^*, j \in D} 2 \cdot t'_j \cdot \alpha_j \cdot x_{i,j}^* + \sum_{i \in F_{D'}^*, j \in H} 2 \cdot \alpha_j \cdot x_{i,j}^* + \sum_{i \in \Gamma \setminus G, j \in D} 2 \cdot c_{i,j} \cdot \left( x_{i,j}^0 - \sum_{\ell \in H(j)} x_{i,j}^* - x_{i,j}^{l(q)} \right) + \sum_{i \in \Gamma \setminus G, j \in H} c_{i,p(j)} \cdot \left( x_{i,j}^0 - x_{i,j}^{l(q)} \right)
\]

$$
+ \sum_{i \in \Gamma \setminus G, j \in D} 2 \cdot c_{i,j} \cdot \left( x_{i,j}^0 - \sum_{\ell \in H(j)} x_{i,j}^* \right) + \sum_{i \in \Gamma \setminus G, j \in H} c_{i,p(j)} \cdot x_{i,j}^*.
$$ (37)

**Proof of Lemma 33.** Consider the first part of Lemma 32. We have

$$
\sum_{i \in F_{D'}^*} y_i^* = \sum_{q \in C_{D'}} y_{i(q)}^* \leq \sum_{q \in C_{D'}} \left( 2y_{i(q)}^{l(q)} + 2\delta_i(q) \cdot \sum_{k \in B(q) \setminus \{i(q)\}} y_{k}^{l(q)} \right). \quad (38)
$$
We charge the cost of each item in the RHS to the facilities in \( I \). By the design of the scaled-down operation for rounding each \( q \in C_{D'} \), we know that, for each \( k \in (I \setminus G) \cap D'(q) \), the facility value \( y_k^{(q)} \) decreases exactly by \( \delta_{i(q)} \cdot y_k^{(q)} \) if \( k \neq i(q) \) and decreases by \( y_k^{(q)} \) followed by taken from consideration otherwise. For each \( k \in G \cap D'(q) \), the facility value \( y_k^{(q)} \) decreases exactly by \( \delta_{i(q)} \cdot y_k^{(q)} \). Hence, by charging the cost of each item in the RHS of (38) to the value decreased for each facility, we obtain

\[
\sum_{i \in F^*_{D'}} y_i' \leq 2 \cdot \sum_{i \in I \setminus G} y_i' + 2 \cdot \sum_{i \in G} \left( y_i' - y_i^{(II)} \right).
\]

Note that, for each \( i \in G \), its facility value \( y_i^{(II)} \) is preserved for the second phase when it is included into a cluster in \( C_{H'} \), and thereby receives no charge from the RHS of (38) by the charging scheme.

The second part of this lemma follows from an analogous charging argument. Consider the second part of Lemma 32. We have

\[
\sum_{i \in F^*_{D'}} \sum_{j \in D} c_{i,j} \cdot x_{i,j}^0 = \sum_{q \in C_{D'}} \sum_{j \in D} \sum_{j \in D} c_{i(q),j} \cdot x_{i(q),j}^0
\]

\[
\leq \sum_{q \in C_{D'}} \left( \sum_{j \in D} 2 \cdot c_{i(q),j} \cdot x_{i(q),j}^0 + \sum_{j \in H} c_{i(q),p(j)} \cdot x_{i(q),j}^0 \right)
\]

\[
+ \sum_{q \in C_{D'}} \delta_{i(q)} \cdot \sum_{k \in B(q) \setminus \{i(q)\}} \left( \sum_{j \in D} 2 \cdot c_{k,j} \cdot x_{k,j}^0 + \sum_{j \in H} c_{k,p(j)} \cdot x_{k,j}^0 \right)
\]

\[
+ \sum_{q \in C_{D'}} \left( \sum_{j \in D} 2 \cdot t_j' \cdot \alpha_j \cdot x_{i(q),j}^* + \sum_{j \in H} 2 \cdot \alpha_j \cdot x_{i(q),j}^* \right). 
\]

(39)

For the last item in (39), we have

\[
\sum_{q \in C_{D'}} \left( \sum_{j \in D} 2 \cdot t_j' \cdot \alpha_j \cdot x_{i(q),j}^* + \sum_{j \in H} 2 \cdot \alpha_j \cdot x_{i(q),j}^* \right)
\]

\[
= \sum_{i \in F^*_{D'}, j \in D} 2 \cdot t_j' \cdot \alpha_j \cdot x_{i,j}^* + \sum_{i \in F^*_{D'}, j \in H} 2 \cdot \alpha_j \cdot x_{i,j}^*
\]

by definition. For the remaining items in the RHS of (39), we charge the cost to the assignment values decreased due to the scaled-down operation when rounding each \( q \in C_{D'} \). For each \( i \in I, j \in D \), the assignment value lost due to the creation of outlier clients, if present, can be partly charged except for those that become the demand of outlier clients, which is exactly \( \sum_{i \in H(j)} x_{i,j}' \). For \( i \in G \), \( j \in C \cup H \), the assignment value that is included into a cluster in \( C_{H'} \) and preserved for the second phase, does not receive charges from the RHS of (39). Therefore, the second part of this lemma follows. \( \square \)
The Overall Guarantee. In the following we establish the overall guarantee. Combining Inequality (34), Inequality (36), and Inequality (37), we obtain

\[
\psi(x^*, y) = \psi \left( x^*_U |_{D}, y^* |_{U} \right) + \psi \left( x^*_F |_{F_D}, y^* |_{F_D} \right) + \psi \left( x^*_G |_{G}, y^* |_{G} \right)
\]

\[
\leq 4 \cdot \sum_{i \in U} y'_i + 2 \cdot \sum_{i \in I} y'_i + \sum_{i \in U, j \in \mathcal{D}} c_{i,j} \cdot x'_{i,j}
\]

\[
+ \sum_{i \in I, j \in \mathcal{D}} 2 \cdot c_{i,j} \cdot \left( x'_{i,j} - \sum_{\ell \in H(j)} x'_{i,\ell} \right) + \sum_{i \in I, j \in H} c_{i,p(j)} \cdot x'_{i,j}
\]

\[
+ \left( \sum_{j \in \mathcal{D}} 2 \cdot t'_j \cdot \alpha_j + \sum_{j \in H} 2 \cdot \alpha_j \right) \cdot \left( \sum_{i \in F'_{D'}} x^*_i + \sum_{i \in G} x^*_{i,j} \right). \tag{40}
\]

Consider the item \(\sum_{i \in I, j \in H} c_{i,p(j)} \cdot x'_{i,j}\) in (40). By the definition of \(H(j)\) for each \(j \in \mathcal{D}\), we have

\[
\sum_{i \in I, j \in H} c_{i,p(j)} \cdot x'_{i,j} = \sum_{i \in I, j \in \mathcal{D}, \ell \in H(j)} c_{i,j} \cdot x'_{i,\ell}. \tag{41}
\]

Consider the last item in (40). Applying the definition of \(t'_j\) for each \(j \in \mathcal{D}\) with \(\sum_{i \in F'_{D'}} x^*_i + \sum_{i \in G} x^*_{i,j} > 0\), we have

\[
\sum_{j \in \mathcal{D}} 2 \cdot t'_j \cdot \alpha_j \cdot \left( \sum_{i \in F'_{D'}} x^*_i + \sum_{i \in G} x^*_{i,j} \right) = \sum_{j \in \mathcal{D}} 2 \cdot \alpha_j \cdot \left( 1 - \sum_{i \in U} x'_i - r'_j \right). \tag{42}
\]

By the algorithm design, we have \(\alpha_j = \alpha_{p(j)} + c_{w(j),p(j)}\) for any \(j \in H\). Further applying the fact that the demand \(d_j\) of any outlier client \(j \in H\) is fully-assigned when created and remains fully-assigned during the rounding process, it follows that

\[
\sum_{j \in H} 2 \cdot \alpha_j \cdot \left( \sum_{i \in F'_{D'}} x^*_i + \sum_{i \in G} x^*_{i,j} \right) = \sum_{j \in H} 2 \cdot \left( \alpha_{p(j)} + c_{w(j),p(j)} \right) \cdot d_j
\]

\[
\leq \sum_{j \in \mathcal{D}} 2 \cdot \alpha_j \cdot r'_j + \sum_{j \in H} 2 \cdot c_{w(j),p(j)} \cdot x'_{w(j),p(j)}
\]

\[
\leq \sum_{j \in \mathcal{D}} 2 \cdot \alpha_j \cdot r'_j + 2 \cdot \sum_{i \in U, j \in \mathcal{D}} c_{i,j} \cdot x'_{i,j}, \tag{43}
\]

where in the second last inequality follows from the fact that

\[
\sum_{k \in H(j)} \alpha_{p(j)} \cdot d_k = \alpha_j \cdot \sum_{k \in H(j)} d_k = \alpha_j \cdot \sum_{k \in H(j)} r'_j \cdot \frac{x'_{w(j),p(j)}}{\sum_{i \in U} x'_{i,p(j)}} = \alpha_j \cdot r'_j
\]

for all \(j \in \mathcal{D}\) and the fact that \(d_j \leq x'_{w(j),p(j)}\) for any \(j \in H\) by the definition of \(d_j\).
Combining (42) and (43), we have
\[
\left( \sum_{j \in D} 2 \cdot t_j' \cdot \alpha_j + \sum_{j \in H} 2 \cdot \alpha_j \right) \cdot \left( \sum_{i \in F_{j'}} x_{i,j}' + \sum_{i \in G} x_{i,j}^{(II)} \right) \\
\leq \sum_{j \in D} 2 \cdot \left( 1 - \sum_{i \in U} x_{i,j}' \right) \cdot \alpha_j + 2 \cdot \sum_{i \in U, j \in D} c_{i,j} \cdot x_{i,j}'.
\]
(44)
Further combining (41) and (44) with (40), we obtain
\[
\psi(x^o, y^o) \leq 4 \cdot \sum_{i \in U} y_i' + 3 \cdot \sum_{i \in I, j \in D} c_{i,j} \cdot x_{i,j}' \\
+ 2 \cdot \sum_{i \in I} y_i' + 2 \cdot \sum_{i \in I, j \in D} c_{i,j} \cdot x_{i,j}' + \sum_{j \in D} 2 \cdot \left( 1 - \sum_{i \in U} x_{i,j}' \right) \cdot \alpha_j.
\]
(45)
The following lemma follows from complementary slackness between \((x', y')\) and \((\alpha, \beta, \Gamma, \eta)\), and the fact that \(0 < y_i' < 1\) for all \(i \in I\).

**Lemma 34.**
\[
\sum_{j \in D} \left( 1 - \sum_{i \in U} x_{i,j}' \right) \cdot \alpha_j \leq \sum_{i \in I} y_i' + \sum_{i \in I, j \in D} c_{i,j} \cdot x_{i,j}'.
\]

**Proof of Lemma 34.** Consider any \(i \in I\) and the cost incurred. From the fact that \((x', y')\) and \((\alpha, \beta, \Gamma, \eta)\) are optimal primal and dual solutions for LP-(N) and LP-(DN), by complementary slackness conditions, we have
\[
y_i' + \sum_{j \in D} c_{i,j} \cdot x_{i,j}' = y_i' \cdot \left( u_i \cdot \beta_i + \sum_{j \in D} \Gamma_{i,j} \right) + \sum_{j \in D} c_{i,j} \cdot x_{i,j}'.
\]

since \(y_i' > 0\) implies that constraint (D-2) is tight, and \(y_i' < 1\) implies that the dual variable \(\eta_i\) must be zero. By the fact that \(\beta_i > 0\) implies that constraint (N-2) is tight and \(\Gamma_{i,j} > 0\) implies that constraint (N-3) is tight, the above becomes
\[
\beta_i \cdot \sum_{j \in D} x_{i,j}' + \sum_{j \in D} \Gamma_{i,j} \cdot x_{i,j}' + \sum_{j \in D} c_{i,j} \cdot x_{i,j}'.
\]
Finally, applying the fact that \(x_{i,j}' > 0\) implies that constraint (D-1) is tight, we obtain
\[
y_i' + \sum_{j \in D} c_{i,j} \cdot x_{i,j}' = \sum_{j \in D} \alpha_j \cdot x_{i,j}'.
\]
Taking summation over all facilities in \(I\), we obtain
\[
\sum_{i \in I} y_i' + \sum_{i \in I, j \in D} c_{i,j} \cdot x_{i,j}' = \sum_{j \in D, i \in I} \alpha_j \cdot x_{i,j}' \geq \sum_{j \in D} \left( 1 - \sum_{i \in U} x_{i,j}' \right) \cdot \alpha_j.
\]
where in the last equality we apply constraint (N-1) for each \(j \in D\). 

Applying Lemma 34 on Inequality (45), we have
\[
\psi(x^o, y^o) \leq 4 \cdot \sum_{i \in F} y_i' + 4 \cdot \sum_{i \in F, j \in D} c_{i,j} \cdot x_{i,j}',
\]
and Theorem 5 is proved.
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