ON EDGE-PRIMITIVE 3-ARC-TRANSITIVE GRAPHS

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Abstract. This paper begins the classification of all edge-primitive 3-arc-transitive graphs by classifying all such graphs where the automorphism group is an almost simple group with socle an alternating or sporadic group, and all such graphs where the automorphism group is an almost simple classical group with a vertex-stabiliser acting faithfully on the set of neighbours.

Edge-primitive graphs, that is, graphs whose automorphism group acts primitively on the set of edges, were first studied by Weiss in 1973 [33] who classified all the edge-primitive graphs of valency three. Many famous graphs are edge-primitive such as the Heawood graph, Tutte-Coxeter graph, Hoffman-Singleton graph and Higman-Sims graph. Another motivation for the study of edge-primitive graphs is the study of graph decompositions [8]: Given a graph Γ and a group $G$ of automorphisms, a partition $\mathcal{P}$ of the set of edges of Γ is called a $G$-transitive decomposition of Γ if $\mathcal{P}$ is $G$-invariant and $G$ acts transitively on $\mathcal{P}$. If $G$ acts transitively on the edge set of Γ then Γ has no $G$-transitive decompositions if and only if $G$ acts primitively on the set of edges.

The study of edge-primitive graphs was reinvigorated in 2010 by the first author and Li [7] by providing a general structure theorem of such graphs and classifying all edge-primitive graphs whose automorphism group contains $\text{PSL}_2(q)$ as a normal subgroup. This has led to all edge-primitive graphs of valencies 4 [10] and 5 [11] being classified, and all those of prime valency and having a soluble edge-stabiliser [24]. Moreover, all edge-primitive graphs of prime power order [22] or which are Cayley graphs on abelian and dihedral groups [23] have been classified.

For $s \geq 1$, an $s$-arc in a graph Γ is an $(s + 1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices such that $v_i \sim v_{i+1}$ but $v_i \neq v_{i+2}$. We say that Γ is $s$-arc-transitive if the automorphism group of Γ acts transitively on the set of $s$-arcs. If a graph is 1-arc-transitive then we simply refer to it as being arc-transitive. If all vertices of Γ have valency at least two then an $s$-arc-transitive graph is also $(s - 1)$-arc-transitive. The study of $s$-arc-transitive graphs originated in the seminal work of Tutte [31, 32], who showed that a graph of valency three is at most 5-arc-transitive. This was later extended by Weiss [34] who showed that a graph of valency at least three is at most 7-arc-transitive. The vertex-primitive 4-arc-transitive graphs were classified by Li [16] and all edge-primitive 4-arc-transitive graphs were classified by Li and Zhang [17]. These classifications were enabled by the classification of all vertex-stabiliser, edge-stabiliser pairs for 4-arc-transitive graphs by Weiss [34]. Moreover, the examples arising in the edge-primitive case are the point-line incidence graphs of the Desarguesian projective planes, the generalised quadrangles associated with the symplectic groups $\text{PSp}_4(q)$, the generalised hexagons associated with the groups $G_2(q)$ of Lie type,
and five other sporadic examples. A key part of the classification in the edge-primitive case is that the edge-stabiliser is always soluble. Han, Liao and Lu \cite{12} have subsequently classified all edge-primitive graphs for almost simple groups with soluble edge-stabilisers.

We say that an edge-primitive graph is \textit{nontrivial} if it is connected, arc-transitive and has valency at least 3. Let $\Gamma$ be a nontrivial edge-primitive graph and let $G = \text{Aut}(\Gamma)$. For a vertex $v$ of $\Gamma$, denote the stabiliser of $v$ in $G$ by $G_v$. Suppose that $\{u, v\}$ is an edge. Then the edge-stabiliser $G_{\{u,v\}}$ is maximal in $G$. Moreover, the arc-stabiliser $G_{uv}$ is an index two subgroup of $G_{\{u,v\}}$ and also contained in two other subgroups, namely the vertex-stabilisers $G_v$ and $G_w$. Suppose further that $\Gamma$ is 2-arc-transitive. Then Lu \cite{20} has shown that if $\Gamma$ is not complete bipartite then $G$ is almost simple, that is, $G$ has a unique minimal normal subgroup $T$ and $T$ is a nonabelian simple group. We refer to $T$ as the socle of $G$ and denote it by $\text{soc}(G)$. He further showed that if $\Gamma$ is 3-arc-transitive then either the graph has valency 7 and $G_v = A_7$ or $S_7$, or $G_v$ acts unfaithfully on the set $\Gamma(v)$ of neighbours of $v$. These observations make a classification of edge-primitive 3-arc-transitive graphs feasible, whereas it would appear that we are far from a classification of all vertex-primitive 3-arc-transitive graphs. This paper is the first in a series aiming to classify all edge-primitive 3-arc-transitive graphs.

Let $G$ be a group with core-free subgroup $H$, that is $\bigcap_{x \in G} H^x = 1$, and let $g \in G$ be such that $g^2 \in H$ and $g$ does not normalise $H$. Then we can construct the coset graph $\Gamma = \text{Cos}(G, H, HgH)$ whose vertices are the right cosets of $H$ in $G$ and $Hx \sim Hy$ if and only if $xy^{-1} \in HgH$. Then $G$ acts faithfully as an arc-transitive group of automorphisms of $\Gamma$. Indeed all arc-transitive graphs arise in this manner by taking $G$ to be a group of automorphisms that acts transitively on the set of arcs, $H$ to be the stabiliser in $G$ of a vertex $v$ and $g \in G$ to be an element interchanging the two vertices of an edge $\{v, u\}$. See for example \cite{28}. We can use the coset graph construction to determine precisely when for a group $G$ with maximal subgroup $E$, there is an edge-primitive graph with group of automorphisms $G$ and edge-stabiliser $E$, see Lemma \ref{lem:1.1}.

Our first result deals with almost simple groups whose socle is either an alternating or a sporadic simple group. Our group theory notation in Table \ref{table:1} follows Atlas \cite{5} notation, which will be defined in Section \ref{sec:1}.

**Theorem 1.** Let $\Gamma$ be a nontrivial edge-primitive 3-arc-transitive graph with $G = \text{Aut}(\Gamma)$ such that $G$ is an almost simple group whose socle is either an alternating or a sporadic simple group. Then there is a quadruple $(G, E, A, H)$ as listed in Table \ref{table:1} such that $E$ is a maximal subgroup of $G$, $A = H \cap E$ and $\Gamma = \text{Cos}(G, H, HgH)$ for some $g \in H \setminus A$.

Our second result looks at classical groups where the stabiliser of a vertex $v$ acts faithfully on the set of neighbours of $v$. Classical groups with an unfaithful vertex-stabiliser will be dealt with in a subsequent paper.

**Theorem 2.** Let $\Gamma$ be a nontrivial edge-primitive 3-arc-transitive graph with $G = \text{Aut}(\Gamma)$ such that $G$ is an almost simple classical group and for a vertex $v$, the vertex-stabiliser $G_v$ acts faithfully on the set of neighbours of $v$. Then there is a quadruple $(G, E, A, H)$ as listed
Table 1. Edge-primitive 3-arc-transitive graphs with Γ and G as in Theorem 1

| G       | E          | A          | H                   | Notes                                |
|---------|------------|------------|---------------------|-------------------------------------|
| Aut(A_6) | [2^4]     | [2^4]      | S_4 \times S_2     | Tutte’s 8-cage [31], bipartite     |
| M_{12.2} | 3^{1+2} : D_8 | 3^{1+2} : 2^2 | 3^2 : 2S_4         | Weiss [35], bipartite              |
| J_3,2   | [2^6] : (S_3)^2 | [2^6] : ((S_3)^2 \cap A_6) | [2^4] : (3 \times A_5).2 | Weiss [36]                         |
| Ru      | 5^{1+2} : [2^5] | 5^{1+2} : [2^4] | 5^2 : GL_2(5)      | Ru graph [30]                      |
| O’N.2   | PGL_2(9)   | A_6        | A_7                 | Lu [20], bipartite                 |

in Table 2 such that E is a maximal subgroup of G, A = H \cap E and \Gamma = \text{Cos}(G, H, HgH) for some g \in H \setminus A.

Remark 1. The notation for the classical groups is defined in Section 4 and for the outer automorphisms \delta, \delta', \phi and \gamma we follow the notation of [2]. Moreover, (2^2)_{122} is the outer automorphism group of PSU_4(3) isomorphic to C_2 given in [5]. The graph obtained from the first row of Table 2 is the Hoffman-Singleton graph [13] and is the only vertex-primitive graph obtained. We believe that all of the remaining examples are new.

Remark 2. Table 2 also lists a subgroup S of G for each graph. If G acts primitively on vertices then S = G while if \Gamma is bipartite and the stabiliser G^+ in G of each bipartite half acts primitively on each bipartite half then S = G^+. In all other cases, S is a maximal subgroup of G or G^+ that contains \Lambda.

Table 2. Edge-primitive 3-arc-transitive graphs with \Gamma and G as in Theorem 2 and T = \text{soc}(G), where \delta', \delta, \phi and \gamma follow the notation of [2]

| G                  | E          | A          | H                   | Conditions                                                   |
|--------------------|------------|------------|---------------------|--------------------------------------------------------------|
| PSU_3(5).\langle \gamma \rangle | Aut(A_6)  | S_6        | S_7                 | G                                                           |
| PSU_6(q).\langle \gamma, \delta^3 \rangle  | Aut(A_6)  | S_6        | S_7                 | PSU_4(3).(2^2)_{122} : q = p \equiv 7, 13 (mod 24)           |
| PSU_6(q).\langle \gamma \rangle | PGL_2(9)  | A_6        | A_7                 | T : q = p \equiv 1, 31 (mod 48)                              |
| PSU_6(q).\langle \delta \rangle | PGL_2(9)  | A_6        | A_7                 | T : q = p \equiv 7, 25 (mod 48)                              |
| PSU_3(5).\langle \phi, \gamma, \delta^3 \rangle  | Aut(A_6)  | S_6        | S_7                 | T.\langle \phi \rangle : q = p^2, p \equiv 5, 11 (mod 24)    |
| PSU_6(q).\langle \phi, \gamma, \delta^3 \rangle  | Aut(A_6)  | S_6        | S_7                 | T.\langle \phi \gamma \rangle : q = p^2, p \equiv 13, 19 (mod 24) |
| PSU_6(q).\langle \gamma \rangle | PGL_2(9)  | A_6        | A_7                 | T : q = p \equiv 11, 17 (mod 24)                             |
| PSU_6(q).\langle \gamma \rangle | PGL_2(9)  | A_6        | A_7                 | T : q = p \equiv 17, 47 (mod 48)                             |
| PSU_6(q).\langle \gamma \rangle | PGL_2(9)  | A_6        | A_7                 | T : q = p \equiv 23, 41 (mod 48)                             |
| PSU_6(q).\langle \gamma \rangle | PGL_2(9)  | A_6        | A_7                 | M_{22}.2, PSp_4(7).2                                          |

We briefly outline the structure of the paper. First we list some preliminary results on edge-primitive graphs and outline our method for proving Theorems 1 and 2. The method involves considering each family of finite almost simple group G of alternating, sporadic or classical type and each type of maximal subgroup E \leq G. In Sections 2 and 3 we consider the case where G is alternating and sporadic respectively. We consider each type
of maximal subgroup of \( G \) using the O’Nan-Scott theorem if \( G \) is alternating and the Atlas \([5]\) if \( G \) is sporadic. We conclude Section 3 with the proof of Theorem \([1]\). In Section 4 we consider the case where \( G \) is a finite almost simple classical group: we make the additional assumption that the local action is faithful, and end with the proof of Theorem \([2]\).

We note that our analysis unveils some new edge-primitive 2-arc-transitive graphs: one with \( G = S_8 \) (Construction \([2.6]\)) and one with \( G = J_1 \) (see case (8) in the proof of Proposition \([3.1]\)).

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# 1. Preliminaries

For a group \( G \) with subgroup \( M \) we write \( \overline{M} \) max \( G \) if \( M \) is a maximal subgroup. Denote by \( \frac{1}{2}G \) any subgroup of \( G \) of index 2 if such a subgroup exists, and let \( \frac{1}{2}G = G \) otherwise. For the subgroup structures in Table 1 we follow the notation of \([3]\); in particular, for an integer \( n \) and prime \( r \) denote by \([n]\) an arbitrary group and \( n \) a cyclic group of order \( n \), and let \( r^n \) denote an extraspecial \( r \)-group of order \( r^{1+2n} \) of type \( \epsilon = \pm \). Furthermore, given groups \( A \) and \( B \) we use \( A : B \) to denote a semidirect product of \( A \) and \( B \), where \( A \) is normal. Finally, if a group \( G \) acts on a set \( \Omega \), then the permutation group induced by \( G \) on \( \Omega \) is denoted by \( G_{\Omega} \).

Let \( \Gamma = (V(\Gamma), E(\Gamma)) \) be a graph with \( e = \{u, v\} \in E(\Gamma) \) and let \( G \leqslant Aut(\Gamma) \). If \( G \) acts primitively on the set of vertices or edges of \( \Gamma \), we say \( \Gamma \) is \( G \)-vertex-primitive or \( G \)-edge-primitive respectively. We call \( \Gamma \) vertex-primitive if \( \Gamma \) is \( Aut(\Gamma) \)-vertex-primitive, and similarly if \( \Gamma \) is \( Aut(\Gamma) \)-edge-primitive. If \( \Gamma \) is bipartite and \( G \) acts transitively on the set of vertices of \( \Gamma \) then \( G \) has a normal subgroup \( G^+ \) of index 2 fixing each bipartite half setwise. We call a transitive group biprimitive if it is imprimitive and if all nontrivial systems of imprimitivity have precisely two parts. We say that \( \Gamma \) is \( G \)-vertex-biprimitive if \( G \) acts biprimively on the set of vertices of \( \Gamma \); furthermore, we say \( \Gamma \) is vertex-biprimitive if \( \Gamma \) is \( Aut(\Gamma) \)-vertex-biprimitive. For an integer \( s \), we say \( \Gamma \) is \((G, s)\)-arc-transitive if \( G \) acts transitively on the set of \( s \)-arcs of \( \Gamma \), and say \( \Gamma \) is \( s \)-arc-transitive if \( \Gamma \) is \((Aut(\Gamma), s)\)-arc-transitive.

Let \( \Gamma \) be a graph with \( G \leqslant Aut(\Gamma) \) acting transitively on the set of arcs of \( \Gamma \). Then \( G \) acts transitively on the set of 2-arcs of \( \Gamma \) if and only if \( G_{\Gamma(v)} \), the permutation group induced by \( G_v \) on \( \Gamma(v) \), is 2-transitive (see \([27, \text{Lemma 9.4}]\)). This gives an easy test to see if an edge-primitive graph is 2-arc-transitive.

By \([7, \text{Lemma 4.1}]\), if \( \Gamma \) is a disconnected edge-primitive graph then either \( \Gamma \) is the union of isolated vertices and single edges or \( \Gamma \) is the union of isolated vertices and a connected edge-primitive graph, and so to study edge-primitive graphs it suffices to consider connected edge-primitive graphs. By \([7, \text{Lemma 3.4}]\), a connected edge-primitive graph \( \Gamma \) is either a star, a cycle, or \( \Gamma \) is arc-transitive.

We say that a graph \( \Gamma \) is a spread of a graph \( \Gamma_0 \) if there is a partition \( P \) of \( V(\Gamma) \) such that
(a) $\Gamma_0$ is isomorphic to the quotient graph $\Gamma_\mathcal{P}$ with vertex set $\mathcal{P}$ and two parts $P_1, P_2 \in \mathcal{P}$ are adjacent if there exist $v \in P_1$ and $w \in P_2$ such that $v$ and $w$ are adjacent in $\Gamma$; and
(b) if $P_1$ and $P_2$ are adjacent in $\Gamma_\mathcal{P}$ then there is a unique $v \in P_1$ and $w \in P_2$ such that $v$ and $w$ are adjacent in $\Gamma$.

Note that there may be more than one $\Gamma_0$. By [7, Theorem 1], a nontrivial edge-primitive graph $\Gamma$ is either vertex-primitive, vertex-biprimitive or a spread of an edge-primitive graph $\Gamma_0$ that is vertex-primitive or vertex-biprimitive. If $\Gamma$ is a spread of $\Gamma_0$ and $\text{Aut}(\Gamma)$ acts faithfully on the partition $\mathcal{P}$, then $\text{Aut}(\Gamma) \leq \text{Aut}(\Gamma_0)$ and the stabiliser in $\text{Aut}(\Gamma)$ of a vertex of $\Gamma_0$ is the stabiliser of a part of the partition $\mathcal{P}$. For those graphs in Table 2 that are neither vertex-primitive nor vertex-biprimitive, the subgroup $S$ listed is such a stabiliser.

We now state the following characterisation of edge-primitive graphs as coset graphs.

**Lemma 1.1.** [7, Proposition 2.5] Let $G$ be a group with a maximal subgroup $E$. Then there exists a $G$-edge-primitive, $G$-arc-transitive graph $\Gamma$ with edge-stabiliser $E$ if and only if $E$ has a subgroup $A$ of index two, and $G$ has a core-free subgroup $H$ such that $A < H \neq E$; in this case $\Gamma \cong \text{Cos}(G, H, HgH)$ for some $g \in E\backslash A$.

The following result reduces the study of edge-primitive graphs with $G, E$ and $H$ as in Lemma 1.1 to $\text{Aut}(G)$-conjugacy classes of $E$ and $H$.

**Lemma 1.2.** Let $G, E, A$ and $H$ be as in Lemma 1.1 and let $g, g_1 \in E\backslash A$ and $\phi \in \text{Aut}(G)$. If $\Gamma = \text{Cos}(G, H, HgH)$ and $\Gamma_1 = \text{Cos}(G, H, Hg_1H)$ then $\Gamma = \Gamma_1$. Moreover, if $E_1 = E^\phi, A_1 = A^\phi, g_2 \in E_1\backslash A_1, H_1 = H^\phi$ and $\Gamma_2 = \text{Cos}(G, H_1, H_1g_2H_1)$ then $\Gamma \cong \Gamma_2$.

**Proof.** Since $|E : A| = 2$ we have that $g_1 = ag$ for some $a \in A \leq H$. Hence $Hg_1H = HagH = HgH$ and so $\Gamma = \Gamma_1$. Moreover, the automorphism $\phi$ of $G$ gives a bijection from the set of right cosets of $H$ in $G$ to the set of right cosets of $H_1$ in $G$ and maps $HgH$ to $H_1g_2H_1$. Thus $\phi$ provides an isomorphism from $\Gamma$ to $\Gamma_2$. \hfill $\Box$

For an almost simple group $G$ acting edge-primitively on a graph $\Gamma$, the next lemma allows us to consider the action of the subgroups $\text{soc}(G) \leq G \leq G$ on edges.

**Lemma 1.3.** Let $G$ be an almost simple group and let $\Gamma$ be a $G$-edge primitive, $G$-arc-transitive graph with edge-stabiliser $G_e$. If $\text{soc}(G) \leq G_1 \leq G$ such that $(G_e \cap G_1) \text{max} G_1$, then $G_1$ is edge-primitive and arc-transitive.

**Proof.** The special case $G_1 = \text{soc}(G) = \text{PSL}_2(q)$ is proved in [7, Lemma 8.3], and the proof in the general case is nearly identical. \hfill $\Box$

We now begin to consider the types of maximal subgroups $E$ of $G$ that yield edge-primitive graphs $\Gamma$ as in Lemma 1.1. The following results allow us to eliminate certain families of maximal subgroups. The first is essentially proved in [9, Lemma 2.14], and we omit the proof.

**Lemma 1.4.** Let $\Gamma$ be a connected $G$-arc-transitive graph, and let $\{u, v\} \in E(\Gamma)$ with $g \in G$ such that $v^g = u$. Then any nontrivial normal subgroup of $G_v$ is not normalised by $g$. 

5
Lemma 1.4. Let $\Gamma$ be a nontrivial $G$-edge-primitive graph and $e = \{u, v\} \in E(\Gamma)$. Assume $G_e$ acts unfaithfully on $\Gamma(v)$, and let $N \trianglelefteq G_e$ be a nonabelian normal subgroup such that $G_e/N$ is soluble and $N$ is a minimal normal subgroup of $G_{uv}$. Then $C_{G_e}(N)$ is noncyclic.

Proof. Suppose for a contradiction that $C_{G_e}(N)$ is cyclic. Let $K \trianglelefteq G_e$ be the kernel of the action of $G_e$ on $\Gamma(v)$, so that $1 \neq K \trianglelefteq G_{uv} \trianglelefteq G_e$. As $N$ is a direct product of nonabelian simple groups, $G_e/N$ is soluble and $|G_e : G_{uv}| = 2$, we have $N = G_{e}^{(\infty)} = G_{uv}^{(\infty)}$, where for a group $X$ we denote the last term in the derived series by $X^{(\infty)}$. As $K \cap N \leq N$ and $K \cap N \leq G_{uv}$, either $N \leq K$ or $K \cap N = 1$. In the first case, this implies $G_{e}^{(\infty)} \leq K \leq G_e$, and so $K^{(\infty)} = G_{e}^{(\infty)} \leq G_e$ and $K^{(\infty)} \text{char} K \leq G_v$. Thus $K^{(\infty)} = N$ is a nontrivial normal subgroup of $G_v$ that is also normal in $G_e$, contradicting Lemma 1.4. In the second case this implies $K \leq C_{G_e}(N)$, and since $C_{G_e}(N)$ is cyclic we have $K \text{char} C_{G_e}(N) \leq G_e$. Thus $K$ is a nontrivial normal subgroup of $G_v$ that is also normal in $G_e$, again contradicting Lemma 1.4.

Using Lemma 1.5, we obtain the following useful result.

Lemma 1.6. Let $\Gamma$ be a nontrivial connected $G$-edge-primitive graph and $e = \{u, v\} \in E(\Gamma)$. If $G_e$ acts unfaithfully on $\Gamma(v)$, then neither $G_v$ nor $G_e$ is almost simple.

Proof. Let $K \trianglelefteq G_v$ be the kernel of the action of $G_v$ on $\Gamma(v)$ so that $K \trianglelefteq G_{uv}$, and suppose for a contradiction that $G_v$ is an almost simple group. Since $K \trianglelefteq G_v$ we have $K \geq \text{soc}(G_v)$. As $K \trianglelefteq G_{uv} < G_v$, it follows that $G_{uv}$ is also an almost simple group with $\text{soc}(G_{uv}) = \text{soc}(G_v)$. Now $\text{soc}(G_{uv})$ is a characteristic subgroup of $G_{uv}$, which in turn is normal in $G_v$. This $\text{soc}(G_v) = \text{soc}(G_{uv})$ is normalised by both $G_v$ and $G_e$, contradicting Lemma 1.4.

Next suppose that $G_e$ is an almost simple group. Then $G_{uv}$ is also an almost simple group with $\text{soc}(G_{uv}) = \text{soc}(G_e)$, and by the Schreier Conjecture, $N = \text{soc}(G_{uv})$ satisfies the hypotheses of Lemma 1.5. However, $C_{G_v}(N) = 1$ contradicting the fact that $C_{G_v}(N)$ must be noncyclic.

The next result is [20, Lemma 4.1].

Lemma 1.7. [20] Let $\Gamma$ be a connected $d$-regular graph for $d \geq 3$, $e = \{u, v\} \in E(\Gamma)$ and $G \leq \text{Aut}(\Gamma)$. If $\Gamma$ is $(G, 2)$-arc-transitive and $G_v$ is faithful on $\Gamma(v)$, then $\Gamma$ is $(G, 3)$-arc-transitive if and only if $d = 7$, $\text{soc}(G_v) = A_7$ and $G_e \neq S_6$, i.e. $G_e = \text{PGL}_2(9), \text{M}_{10}$ or $\text{Aut}(A_6)$.

Combining Lemmas 1.6 and 1.7 provides a useful corollary:

Corollary 1.8. Let $\Gamma$ be a nontrivial $G$-edge-primitive graph with $e = \{u, v\} \in E(\Gamma)$, and suppose that $\Gamma$ is $(G, 3)$-arc-transitive. If $G_e$ or $G_v$ is an almost simple group, then $\text{soc}(G_v) = A_6, G_e \neq A_6$ or $S_6, \text{soc}(G_v) = A_7$ and $|G_v : G_{uv}| = 7$.

Proof. If the local action is unfaithful, then neither $G_e$ nor $G_v$ is almost simple by Lemma 1.6. If the local action is faithful, then the result follows from Lemma 1.7. □
We present one final lemma which will prove useful in ruling out certain possibilities for $G_v$.

**Lemma 1.9.** Let $\Gamma$ be a nontrivial $G$-edge-primitive graph and $e = \{u, v\} \in E(\Gamma)$. Then $|G_v| > |G_e|$.

**Proof.** This is immediate from the fact that $\Gamma$ is nontrivial, as this implies the valency of $\Gamma$ is $|G_v : G_{uv}| \geq 3$. □

We now outline our method of proving Theorems 1 and 2. Let $\Gamma$ and $G$ be as in either theorem. By Lemmas 1.1 and 1.2, to classify such graphs it suffices to classify subgroup lattices $L = (G, E, A, H)$ up to $\text{Aut}(G)$-conjugation, where $E = \text{max} G$, $A < E$ with $|E : A| = 2$ and $A < H < G$ with $T = \text{soc}(G) \not\leq H$. Moreover, by Corollary 1.8 if either $E$ or $H$ is almost simple then both are with $\text{soc}(E) = A_6, \text{soc}(H) = A_7, |H : A| = 7$ and $E \not= A_6$ or $S_6$. For each such lattice $L$, we consider the coset graph $\Gamma$ obtained from $L$. Clearly $G \leq \text{Aut}(\Gamma)$, and so to determine if $G$ is the full automorphism group it suffices to check [18] (which lists containments of primitive groups). If indeed $G = \text{Aut}(\Gamma)$, we then check if $\Gamma$ is 3-arc-transitive: if either $E$ or $H$ is almost simple then both must be as in Lemma 1.7, and if this is the case then $\Gamma$ is indeed 3-arc-transitive; if neither $E$ nor $H$ is almost simple, then we investigate the graph further. In most cases we can rule out the lattice $L$ by proving that the local action of $\Gamma$ cannot be 2-transitive (this is equivalent to showing that the action of $H$ on the set of right cosets of $A$ is not 2-transitive), and so $\Gamma$ is not 2-arc-transitive.

2. **Alternating groups**

In this section, we consider the case where $G$ is almost simple with $\text{soc}(G) = A_n$ for $n \geq 5$. The main result of this section is the following:

**Proposition 2.1.** Let $\Gamma$ be a nontrivial connected edge-primitive 3-arc-transitive graph with $G = \text{Aut}(\Gamma)$ and $\text{soc}(G) = A_n$ for some $n \geq 5$. Then $\Gamma$ is isomorphic to Tutte’s 8-cage (described in Construction 2.2 below) and $G = \text{Aut}(A_6)$.

**Construction 2.2** (Tutte’s 8-cage). Let $G = \text{Aut}(A_6)$. We define a coset graph $\Gamma_0 = \text{Cos}(G, H, HgH)$ as in Lemma 1.7 where $G, E, A$ and $H$ are as in Line 10 of Table 3 and $g \in E \setminus A$. We note that $\Gamma_0$ is defined in [17] Line 1, Table 2], and from here we see that $\Gamma_0$ is 5-arc-transitive.

To prove Proposition 2.1 we follow the method outlined at the end of Section 1. Many of our preliminary lemmas are only for edge-primitive graphs instead of edge-primitive 3-arc-transitive graphs. We will use the following general set up.

**Hypothesis 2.3.** Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a nontrivial connected edge-primitive graph such that $G = \text{Aut}(\Gamma)$ is an almost simple group with $\text{soc}(G) = A_n$. Let $e = \{u, v\}$ be an edge in $\Gamma$ and define the subgroups $E = G_e$, $H = G_v$ and $A = G_{uv} = E \cap H$. Note that $E \leq G, |E : A| = 2$ and $\text{soc}(G) \not\leq H$. 7
We proceed by considering each type of maximal subgroup $E$ of $G$. First assume $n \neq 6$. By the O'Nan Scott theorem, $E$ is either intransitive, imprimitive or primitive in its natural action, and if $E$ is primitive then it is either of affine, diagonal, product or almost simple type (see for example [18]). The case where $E$ is intransitive or imprimitive is dealt with in Section 2.2 and the remaining cases are considered in Section 2.3. The case where $n = 6$ is different due to an exceptional outer automorphism, and we consider this case separately in Section 2.1.

2.1. The case $\text{soc}(G) = A_6$. Using Magma [4], we list in Table 3 all quadruples $(G, E, A, H)$ (up to $\text{Aut}(G)$-conjugacy) such that $G$ is almost simple with $\text{soc}(G) = A_6$, $A < E \text{ max } G$, $|E : A| = 2$ and $A \leq H$ where $\text{soc}(G) \nsubseteq H$. In all cases we find that either $H \text{ max } G$ or $H \text{ max } \frac{1}{2}G$; we list the groups $S$ such that $H \text{ max } S$.

| $G$      | $E$       | $A$       | $H$       | $S$ | Notes                          |
|----------|-----------|-----------|-----------|-----|--------------------------------|
| $A_6$    | $S_4$     | $A_4$     | $A_5$     | $G$ |                                |
| $S_6$    | $S_4 \times S_2$ | $S_4$ | $S_5$ | $G$ |                                |
| $M_{10}$ | $5 : 4$  | $5 : 2$  | $A_5$     | $A_6$ |                                |
|          | $8 : 2$  | $D_8$ | $S_4$ | $A_6$ |                                |
|          | $8 : 2$  | $Q_8$ | $3^2 : Q_8$ | $G$ |                                |
| $\text{PGL}_2(9)$ | $D_{20}$ | $D_{10}$ | $A_5$ | $A_6$ |                                |
|          | $D_{16}$ | $D_8$ | $S_4$ | $A_6$ |                                |
|          | $8$      | $3^2 : 8$ | $G$ | |                                |
| $\text{Aut}(A_6)$ | $10 : 4$ | $\text{AGL}_1(5)$ | $S_5$ | $S_6$ | Tutte’s 8-cage [31] |
|          | $[2^5]$ | $[2^4]$ | $S_4 \times S_2$ | $S_6$ |                                |
|          | $[2^5]$ | $[2^4]$ | $3^2 : [2^4]$ | $G$ |                                |

**Lemma 2.4.** Suppose that $\Gamma$ is as in Hypothesis 2.3 with $n = 6$. Then $\Gamma$ is isomorphic to either $K_6$ or Tutte’s 8-cage. In particular, if $\Gamma$ is also 3-arc-transitive, then $\Gamma$ is isomorphic to Tutte’s 8-cage.

**Proof.** We have that $(G, E, A, H)$ is isomorphic to a quadruple from Table 3 and from such a quadruple we can recover $\Gamma$ by Lemma 1.1. If $L$ is as in Line 1 or 2 then $G$ is 2-transitive on the set of right cosets of $H$, and so $\Gamma \cong K_6$. Similarly, if $L$ is as in Line 5, 8 or 11 then $\Gamma \cong K_{10}$. However, in this case $\text{Aut}(\Gamma) = S_{10}$. In lines 3, 6, and 9, $\Gamma$ is bipartite with bipartition stabiliser $\frac{1}{2}G$ isomorphic to $A_6$ or $S_6$. Moreover, if $\{\Delta_1, \Delta_2\}$ is the bipartition and $H$ is the stabiliser of a vertex in $\Delta_1$ then $H$ is transitive on $\Delta_2$. Thus $\Gamma \cong K_{6,6}$. However, in this case $\text{Aut}(\Gamma) = S_6 \wr S_2$.

For the remaining lines 4, 7 and 10 of Table 3 we use Magma [4] to construct these graphs (using the CosetGeometry and Graph commands) and show that these graphs are isomorphic to Tutte’s 8-cage as in Construction 2.2 above. This completes the proof. □
2.2. Intransitive and imprimitive subgroups. By Section 2.1 we assume \( n \neq 6 \) for the remainder of this section. We begin by considering the case where \( E \) is an intransitive subgroup.

**Lemma 2.5.** Let \( \Gamma \) be as in Hypothesis 2.3 with \( n \neq 6 \) and suppose that \( E \) is intransitive, so that \( E = (S_k \times S_{n-k}) \cap G \). Then \( k = 2 \), \( H \cong S_{k-1} \cap G \) and \( \Gamma \cong K_n \).

**Proof.** If \( n = 5 \), then the result follows using MAGMA [4]. We may therefore assume \( n \geq 7 \).

We have \( E = (S_k \times S_{n-k}) \cap G \) for some \( 1 \leq k < \frac{n}{2} \). Without loss, we may assume \( E \) preserves the partition \( \{1, \ldots, k\} \cup \{k+1, \ldots, n\} \). Therefore \( A \cong A_k \times A_{n-k} \) (where we define \( A_1 = A_2 = 1 \)). Hence \( A \) contains a 3-cycle fixing at least 4 points, and so by a theorem of Jordan (see [6, Theorem 3.3E]) \( H \) is not primitive. Therefore \( H \) is either intransitive or imprimitive.

First suppose that \( H \) is imprimitive, so that \( H \leq S_a \wr S_b \) for some \( 1 < a, b \leq \frac{n}{2} \) such that \( n = ab \). Observe that \( A \) is \((n-k-2)\)-transitive on \( \{k+1, \ldots, n\} \). In particular, \( A \) is primitive on \( \{k+1, \ldots, n\} \), and so either \( a > n-k \) or \( b > n-k \), a contradiction.

We therefore have \( H \) intransitive. If \( k \neq 2 \) then \( A \) is transitive on \( \{1, \ldots, k\} \) and \( \{k+1, \ldots, n\} \), and so \( H \leq E \), contradicting Lemma 1.9. Therefore \( k = 2 \) and \( A \) is intransitive on \( \{1, 2\} \). As \( A_{n-2} \leq A^{(3\ldots n)} \) and \( |H| > |E| \) we must have \( H = (S_1 \times S_{n-1}) \cap G \). Since \( G \) is 2-transitive on the set of right cosets of \( H \) this implies \( \Gamma \cong K_n \), completing the proof. \( \square \)

We now consider the case where \( E \) is a transitive and imprimitive subgroup. We first give an example of such a graph.

**Construction 2.6.** Let \( G = S_8 \) and \( E = S_2 \wr S_4 \). Let \( A = E \cap A_8 \cong C_2^3 : S_4 \). Then \( |E : A| = 2 \) and there is a subgroup \( H \cong AGL_3(2) \) such that \( A < H < A_8 \). Thus \( (G, E, A, H) \) is a quadruple as in Lemma 1.7. We therefore have a nontrivial connected \( G \)-edge-primitive arc-transitive graph \( \Gamma_1 = \text{Cos}(G, H, HgH) \) for some \( g \in E \setminus A \). We see that \( \Gamma_1 \) has order \( |G : H| = 30 \) and valency \( |H : A| = 7 \). Moreover, as \( H < A_8 \) and \( |G : A_8| = 2 \) we have that \( \Gamma_1 \) is bipartite. Furthermore, the action of \( H \) on the set of right cosets of \( A \) is permutationally isomorphic to the action of \( GL(3, 2) \) on 7 points, which is 2-transitive. Hence \( \Gamma_1 \) is 2-arc-transitive. However, using MAGMA [4] we find that \( \Gamma_1 \) has girth equal to 4 and so there is a 3-arc \((v_0, v_1, v_2, v_0)\). Since \( \Gamma_1 \) is bipartite and vertex-transitive, each bipartite half has size 15. As \( \Gamma_1 \) has valency 7 it follows that \( \Gamma_1 \) has diameter at least 3 and so there is a 3-arc \((v_0, v_1, v_2, v_3)\) with \( v_3 \neq v_0 \). Thus \( \Gamma_1 \) is not 3-arc-transitive.

**Lemma 2.7.** Let \( \Gamma \) be as in Hypothesis 2.3 with \( n \neq 6 \) and suppose that \( E \) is imprimitive, so that \( E = (S_m \wr S_k) \cap G \). Then \( n = 8 \), \( m = 2 \) and \( \Gamma \cong \Gamma_1 \) as in Construction 2.6.

**Proof.** Let \( E = (S_m \wr S_k) \cap G \), where \( 1 < k, m \leq \frac{n}{2} \) and \( n = mk \). Without loss of generality, we may assume that \( E \) preserves blocks of imprimitivity of the form \( B_i = \{(i-1)m+1, \ldots, im\} \) for \( 1 \leq i \leq k \). Let \( B = (S^k_m) \cap G < E \) be the base of the wreath product. As neither \( A^k_m \) nor \( A_k \) contains a subgroup of index 2, \( A \geq A_m \wr A_k \).
First suppose \( m > 2 \). As \( A \) is transitive on \( \{1, \ldots, n\} \), so is \( H \). Suppose \( H \) is imprimitive, preserving a system of imprimitivity with blocks \( C_j \) for \( 1 \leq j \leq \ell \). Then \( B \cap A \) preserves the partition \( B_i = \bigcup_{1 \leq j \leq \ell} \langle B_i \cap C_j \rangle \) for all \( 1 \leq i \leq k \). But as \( B \cap A \geq A_m^k \) acts primitively on \( B_i \) we must have \( B_i \cap C_j = B_i \) or \( \emptyset \) for all \( 1 \leq i \leq k \), or \( |B_i \cap C_j| = 1 \) or \( 0 \) for all \( 1 \leq i \leq k \). In the first case, since \( A_k \) acts primitively on the set of blocks \( \{ B_i \} \) we have \( H = E \), contradicting Lemma 1.9. In the second case, if \( B_{i_1} \cap C_j \neq \emptyset \) and \( B_{i_2} \cap C_j \neq \emptyset \) for some \( i_1, i_2 \) and \( j \), then \( B \) contains elements acting transitively on \( B_{i_1} \) whilst fixing \( B_{i_2} \cap C_j \), implying \( B_i \subseteq C_j \), a contradiction. Therefore for each \( 1 \leq j \leq \ell \), \( B_i \cap C_j \neq \emptyset \) for exactly one value of \( i \). But this implies that \( |C_j| = 1 \), a contradiction. Therefore \( H \) is primitive. But \( A \) contains a 3-cycle fixing at least 3 points, contradicting [6, Theorem 3.3E].

We must therefore have \( m = 2 \), and so \( k \geq 4 \). Observe that \( A \geq A_k \), and so \( A \) is transitive on the set of blocks \( B_i \) for \( 1 \leq i \leq k \); in particular, \( (B \cap A)^{B_i} \cong (B \cap A)^{B_j} \) for all \( i \) and \( j \), and so we must have \( (B \cap A)^{B_i} \cong S_2 \) (otherwise \( |E : A| \geq 2^{k-1} \)). Hence \( A \), and therefore \( H \), is transitive on \( \{1, \ldots, n\} \). As before, \( H \) is not imprimitive, and so \( H \) is primitive. Observe that \( E \) contains the elements \( (1 \ 2)(3 \ 4), (1 \ 2)(5 \ 6) \) and \( (3 \ 4)(5 \ 6) \), and so \( A \) contains at least one of these elements since \( |E : A| = 2 \). By [25, Lemma 1.2], we must therefore have \( n = 8 \) and \( H = AGL_3(2) \). As \( E \) max \( G \), we must have \( G = S_8 \) by [18]. As \( A = E \cap H \), we now have \( (G, E, A, H) \) as in \( \Gamma_1 \) above, and so \( \Gamma \cong \Gamma_1 \). This completes the proof.  

\[ \square \]

2.3. Primitive subgroups. We now consider the case where \( E \) is primitive. By the O’Nan-Scott theorem, \( E \) is of affine, diagonal, product or almost simple type. We consider each type of primitive group, starting with the affine case.

**Lemma 2.8.** Let \( \Gamma \) be as in Hypothesis 2.3 with \( n \neq 6 \) and suppose further that \( \Gamma \) is a 3-arc-transitive graph. Then \( E \) is not of affine type.

**Proof.** Suppose otherwise for a contradiction, and let \( E = AGL_d(p) \cap G \) where \( n = p^d \). Let \( N = \text{soc}(E) \cong p^d \). As \( N \) is the unique minimal normal subgroup of \( E \), we have \( N \leq A \). Therefore \( A \) contains \( N \rtimes S_L(p) \); in particular, \( A \) is 2-transitive on \( \{1, \ldots, n\} \) if \( d > 1 \) and \( A \) is primitive in all cases. By [26], one of the following holds:

1. \( H \leq AGL_d(p) \) (a natural inclusion);
2. \( H \leq S_{n_0} \rtimes S_m \) for some \( n_0 = p^e \) such that \( n = n_0^m \) in product action (a blow-up of an exceptional inclusion);
3. \( (A, H) \) is listed in [26, Table 2] (an exceptional inclusion).

In the first case we have \( |H| \leq |E| \), contradicting Lemma 1.9, and the second case is not possible as either \( A \) is 2-transitive or \( n \) is a prime. Therefore \( H \) lies in [26, Table 2]. However, this implies that \( H \) is an almost simple group not isomorphic to \( A_7 \) or \( S_7 \), contradicting Corollary 1.8.  

\[ \square \]

Next we consider the case where \( E \) is of product type. We first require a preliminary result.

**Lemma 2.9.** Let \( G \) be an almost simple group with \( \text{soc}(G) = A_n \) and \( n = m^2 \) for some \( m \geq 5 \), and let \( M \) max \( G \) be of product type with \( M = (S_m \wr S_2) \cap G \). Suppose that \( Y = \frac{1}{2}M \)
is imprimitive and \( Y \leq X \leq G \) such that \( X \neq M \). Then either \( X \) is primitive of almost simple type, or \( X \) belongs to one of two chains

\[
Y \max N \max L \max G \text{ if } m \text{ is odd},
\]

\[
Y \max N \max S \max L \max G \text{ if } m \text{ is even},
\]

where \( L = (S_m \wr S_m) \cap G \) is an imprimitive subgroup,

\[
S = \left\{ (g_1, \ldots, g_m)h \in L \mid (g_1, \ldots, g_m) \in S_m, h \in S_m, \sum_{1 \leq i \leq m} \text{sgn}(g_i) \equiv 0 \pmod{2} \right\}
\]

and

\[
N = \{(g_1, \ldots, g_m)h \in L \mid (g_1, \ldots, g_m) \in S_m, h \in S_m, \text{sgn}(g_i) = \text{sgn}(g_j) \text{ for all } 1 \leq i, j \leq m\}.
\]

Proof. Suppose \( Y \leq X \leq G \). Since \(|M : Y| = 2\), the group \( Y \), and therefore \( X \), acts transitively on \( \{1, \ldots, n\} \). As \( Y \) is imprimitive, the projection of \( Y \) onto \( S_2 \) is trivial, and so \( Y = (S_m \times S_m) \cap G \). Therefore \( Y \) preserves two systems of imprimitivity \( \{\mathcal{B}_i\} \) and \( \{\mathcal{B}_i^c\} \) interchanged by \( M \wr Y \), each with \( m \) blocks \( \mathcal{B}_i^c \) of size \( m \) for \( 1 \leq i \leq m \) and \( 1 \leq s \leq 2 \), and so \( Y \) is contained in two imprimitive maximal subgroups of the form \( L = (S_m \wr S_m) \cap G \). Note that one factor of \( S_m \times S_m \) occurs as a diagonal subgroup of the base group \( S_m \), and the other permutes the factors. Observe that the suborbits of \( Y \) have lengths \( 1, m - 1 \) (occurs twice) and \( (m - 1)^2 \), and that the two suborbits of length \( m - 1 \) correspond to the two systems of imprimitivity.

As \( Y \cap S_m^m \) is a diagonal subgroup, \( Y \) is contained in \( N \), and since \( m \geq 5 \) and \( Y \) acts primitively on the \( m \) direct factors of \( S_m^m \) it is elementary to show that \( Y \) is in fact maximal in \( N \). Also, \( T = A_m \wr S_m \leq N \leq L \), and subgroups of \( L \) containing \( T \) correspond to subgroups of \( S_m^m/A_m^m \cong C_2^m \) normalised by \( S_m \), and these correspond to submodules of the permutation module \( F_m^m \) of \( S_m \). The only nonzero proper submodules of \( F_m^m \), are the constant module (corresponding to \( N \)) and the deleted permutation module (corresponding to \( S \)), and \( N \leq S \) if and only if \( m \) is even. Therefore \( N \max L \) if \( m \) is odd and \( N \max S \max L \) if \( m \) is even. Moreover, \( Y < S \) if and only if \( m \) is even. Finally \( L \max G \) by \([18]\), and so \( Y \) is indeed contained in the two maximal chains stated. It remains to prove that, other than \( M \) and almost simple primitive groups, there are no other overgroups of \( Y \).

First consider the case where \( X \) acts imprimitively on \( \{1, \ldots, n\} \), and assume that \( X \) preserves a system of imprimitivity with blocks \( \mathcal{C}_j \) for \( 1 \leq j \leq \ell \). Fix one of the two systems of imprimitivity \( \{\mathcal{B}_i\} = \{\mathcal{B}_i^c\} \) described above. Arguing in an identical manner to the proof of Lemma 2.7 we find that one of the following holds:

1. \(|\mathcal{B}_i \cap \mathcal{C}_j| = 0 \) or \( m \) for all \( 1 \leq i \leq m \) and all \( 1 \leq j \leq \ell \);
2. \(|\mathcal{B}_i \cap \mathcal{C}_j| = 0 \) or \( 1 \) for all \( 1 \leq i \leq m \) and all \( 1 \leq j \leq \ell \).

In case (1), since \( Y \) acts primitively on the set of blocks \( \{\mathcal{B}_i\} \), for each \( 1 \leq i \leq m \) we have \( \mathcal{B}_i = \mathcal{C}_j \) for some \( j \), and so \( X \leq L \). Now suppose (2) holds. Note that this implies \( m \leq \ell \), since \( \mathcal{B}_i = \bigcup_{1 \leq j \leq \ell} (\mathcal{B}_i \cap \mathcal{C}_j) \). Let \( \alpha_1 \in \mathcal{B}_i \cap \mathcal{C}_j \) and \( \alpha_2 \in \mathcal{B}_i \cap \mathcal{C}_j \). Recall that the suborbits of \( Y \) have length \( 1, m - 1 \) (occurs twice) and \( (m - 1)^2 \). If \( \alpha_2 \) belongs to the suborbit of length \( (m - 1)^2 \) then this implies \( |\mathcal{C}_j| \geq (m - 1)^2 \), a contradiction since
\[(m - 1)^2 > \frac{1}{2}m^2 \text{ for } m \geq 5. \] Therefore \(\alpha_2\) belongs to one of the two suborbits of length \(m - 1,\) and so \(|E_j| \geq m.\) Since \(m < \ell\) we have \(m = \ell = |E_j|,\) and this implies that the system of imprimitivity \(\{E_j\}\) is either equal to \(\{\mathcal{R}_i\}\) or the image of \(\{\mathcal{R}_i\}\) by an element of \(M'Y.\) As we are assuming (2) the second case holds, and this completes the proof that the only systems of imprimitivity preserved by \(Y\) are \(\{\mathcal{R}_1\}\) and \(\{\mathcal{R}_2\}\).

We now consider the case where \(X\) acts primitively on \(\{1, \ldots, n\}.\) Assume for the moment that \(X\ max G,\) so that \(X\) is of affine, diagonal, product or almost simple type. First suppose that \(X\) is affine, so that \(n = p^d\) for some prime \(p\) and \(X = \text{AGL}_d(p)\cap G.\) Let \(\alpha \in \{1, \ldots, n\}\) so that \(S_{m-1} \cap Y \cong Y_\alpha \leq \text{GL}_d(p).\) Let \(\beta \in \{1, \ldots, n\}\) be contained in an orbit of \(Y_\alpha\) of size \(m - 1.\) Then \(Y_{\alpha\beta}\) has orbits of size 1 (occurs twice), \(m - 2, m - 1\) (occurs twice) and \((m - 1)(m - 2).\) However, if \(\beta\) corresponds to a non-zero vector \(v_\beta \in V_d(p),\) the natural module of \(\text{GL}_d(p),\) then \(\text{GL}_d(p)\beta\) has \(p\) orbits of size 1, namely \(\{\lambda v_\beta\}\) for \(\lambda \in \mathbb{F}_p.\) This implies \(p = 2.\) Now choose \(\gamma\) from the same suborbit as \(\beta.\) Then \(Y_{\alpha\beta\gamma}\) has orbits of length 1 (occurs thrice), \(m - 3, m - 1\) (occurs thrice) and \((m - 1)(m - 3).\) However, if \(\gamma\) corresponds to \(v_\gamma \in V_d(p),\) \(\text{GL}_d(p)\beta\gamma\) fixes the 4 points \(0 = v_\alpha, v_\beta, v_\gamma\) and \(v_\beta + v_\gamma,\) and this yields a contradiction. Hence \(X\) is not of affine type.

Suppose now that \(X\) is of diagonal type, so that \(X = (T^k. (\text{Out}(T) \times S_k)) \cap G\) for some nonabelian simple group \(T\) and \(k \geq 2\) with \(n = |T|^{k-1}.\) Note that in this case we have \(m > 9,\) as otherwise this forces \(n = |A_5| = 60\) which is not square. Observe that \(Y \geq A_5^m,\) and so \(Y\) contains a 3-cycle in one of the factors which permutes exactly \(3m\) points. By \([19,\) Theorem 2], the minimal degree of \(X\) (that is, the smallest number of points moved by any nontrivial element of \(X\)) is \(\mu(X) \geq \frac{3}{5} n.\) However, this is a contradiction since \(3m < \frac{1}{3} n.\) Hence \(X\) is not of diagonal type.

We now suppose that \(X\) is of product type, with \(X = (S_\ell \ast S_k) \cap G\) for \(\ell \geq 5, k \geq 2\) and \(n = \ell^k.\) Now \(X\) has suborbits of lengths \(1\) and \((\ell - 1)^k,\) and so the remaining suborbits have lengths at most \((\ell^k - (\ell - 1)^k - 1.\) Recall that \(Y\) has suborbits of lengths \(1, m - 1\) (occurs twice) and \((m - 1)^2.\) This forces \(k = 2,\) as otherwise \((m - 1)^2 > 1, (\ell - 1)^k\) and \((\ell^k - (\ell - 1)^k - 1.\) Hence \(X\) is of the same type as \(M,\) and we have \(X = N_G(Y) = M.\)

We have therefore shown that if \(Y\) is contained in any primitive group \(X,\) then either \(X\) is of almost simple type or \(X = M.\) This completes the proof. \(\square\)

**Lemma 2.10.** Let \(\Gamma\) be as in Hypothesis 2.3 and suppose further that \(\Gamma\) is a \(3\)-arc-transitive graph. Then \(E\) is not of product type.

**Proof.** Let \(E\) be of product type, so that \(E = (S_m \ast S_k) \cap G\) where \(n = m^k\) for \(m \geq 5\) and \(k \geq 2.\) Then \(A \geq A_m \ast A_k,\) and so either \(A\) is primitive of product type or \(k = 2.\) First suppose that \(A\) is primitive. With the terminology of \([26,\) one of the following holds:

1. \(H \leq S_m \ast S_k\) (a natural inclusion);
2. \(m = 8\) and \(A \leq \text{PSL}_2(7) \ast S_k\) (an exceptional inclusion as listed in \([26, Table 1]);
3. \(m = 8^\ell\) for some \(\ell > 1\) and \(A \leq (\text{PSL}_2(7) \ast S_\ell) \ast S_k\) (a blow-up of an exceptional inclusion).

As \(H\) is not contained in \(E\) by Lemma 1.9, the first case does not hold. However, it is clear that the second and third cases also cannot hold as \(A_m \ast A_k \leq A,\) a contradiction.
Therefore $A$ is not primitive, and so we must have that $k = 2$ and $A$ projects trivially onto $S_2$. Thus $A$ is as in Lemma 2.9. Since $H \nleq E$ and $H$ is not almost simple by Lemma 1.6 (as $|A| > |S_6|$), $H$ is imprimitive and belongs to one of the two chains described in the lemma. However, the action of $H$ on the set of cosets of $A$ is not 2-transitive, a contradiction. Hence $E$ is not of product type.

**Lemma 2.11.** Let $\Gamma$ be as in Hypothesis 2.3. Then $E$ is not of diagonal type.

**Proof.** Let $E$ be of diagonal type with $E = (T^k.(\text{Out}(T) \times S_k)) \cap G$ for a nonabelian simple group $T$ such that $k \geq 2$ and $n = |T|^{k-1}$. Then $A \geq T^k$, and the image of $A$ under the projection $E \rightarrow S_k$ contains $A_k$. Hence $A$ is also a primitive group of diagonal type. By [26], $H$ is contained in $N_G(T^k) = E$, contradicting Lemma 1.9. Hence $E$ is not of diagonal type. □

We now consider the case where $E$ is almost simple. We require the following lemma:

**Lemma 2.12.** Let $G$ be an almost simple group with $\text{soc}(G) = A_n$ for $n \geq 5$, and suppose $G$ has subgroups $M$ and $N$ satisfying the following:

1. $M$ and $N$ are both almost simple, with $\text{soc}(M) = A_7$ and $\text{soc}(N) = A_6$;
2. $\text{soc}(G) \nleq M$;
3. $N \text{ max } G$;
4. $\text{soc}(N) \leq M$.

Then $G = A_8$, $M \cong A_7$ and $N \cong S_6$.

**Proof.** First suppose $n \geq 10$. We consider $G$ as a permutation group on $n$ letters in the natural way. If $N$ is an intransitive subgroup, then for some $m$ and $k$ such that $n = m + k$ we have $(S_m \times S_k) \cap A_n \leq N$, a contradiction since $\frac{1}{2}|S_m \times S_k| > |\text{Aut}(A_6)|$ for $n \geq 10$. If $N$ acts imprimitively, then for some $m, k$ such that $n = mk$ we have $(S_m^n) \cap A_n \leq N$, a contradiction. Therefore $N$ acts primitively, and so $\text{soc}(N)$ is transitive. Suppose for the moment that $M$, and therefore $\text{soc}(N)$, acts imprimitively. Using MAGMA [4], we find that this is only possible if $N \neq S_6$ and $n = 36$ or $45$; however, $M$ has no transitive action on $n$ points, a contradiction. Therefore $M$ also acts primitively. The only possible degree of a primitive action for both $M$ and $N$ is 15 (this can easily be seen using MAGMA). However, neither $S_{15}$ nor $A_{15}$ contains $S_7$ or $A_7$ as a maximal subgroup.

Therefore $n < 10$. Clearly $n \geq 8$, and the result follows using MAGMA. □

**Corollary 2.13.** Let $\Gamma$ be as in Hypothesis 2.3 and suppose further that $\Gamma$ is a 3-arc-transitive graph. Then $E$ is not an almost simple primitive group.

**Proof.** Suppose $E$ is an almost simple group. Then by Corollary 1.8, $\text{soc}(E) = A_6$ and $H$ is also an almost simple group with $\text{soc}(H) = A_7$. Therefore $G, E$ and $H$ satisfy the hypothesis of Lemma 2.12. However, this implies $E = S_6$, contradicting Corollary 1.8. □

2.4. **Proof of Proposition 2.1**

**Proof of Proposition 2.1** If $n = 6$ then the result follows from Lemma 2.4. We may therefore assume $n \neq 6$. We follow the method outlined at the end of Section 1 and classify
lattices \( L = (G, E, A, H) \). By the O’Nan Scott theorem, \( E \max G \) is either intransitive, imprimitive or primitive, and if \( E \) is primitive then \( E \) is of affine, diagonal, product or almost simple type. Each of these types is considered in Lemmas 2.3 to 2.11 and Corollary 2.13 and the result follows as \( K_n \) is not 3-arc-transitive.

3. Sporadic groups

In this section, we consider the case where \( G \) is an almost simple sporadic group and complete the proof of Theorem 1. As usual we let \( \Gamma = (V(\Gamma), E(\Gamma)) \) be a nontrivial connected edge-primitive graph with \( G = \Aut(\Gamma), e = \{u, v\} \in E(\Gamma) \), vertex-stabiliser \( G_v \) and edge-stabiliser \( G_e \) so that \( G_e \max G \) and \( G_{uv} = G_v \cap G_e \) is a subgroup of \( G_e \) of index 2.

The main result of this section is the following:

**Proposition 3.1.** Let \( \Gamma \) be a nontrivial connected edge-primitive 3-arc-transitive graph with \( G = \Aut(\Gamma) \) an almost simple sporadic group. Then \( \Gamma = \Cos(G, H, HgH) \) where \( g \in E \setminus A \) and \( (G, E, A, H) \) are listed in Table 4 below.

**Remark 3.** In Table 4 we record the group \( S \leq G \) where \( H \max S \) and either \( S = G \) or \( |G : S| = 2 \). We note that the first three graphs in Table 4 are in fact 4-arc-transitive and listed in [17, Table 2]. The fourth graph is [20, Example 4.2].

| \( G \)       | \( E \)            | \( A \)  | \( H \)         | \( S \)        | Notes  |
|--------------|-------------------|--------|-----------------|---------------|--------|
| \( M_{12} \) | \( 3^{1+2} : D_8 \) | \( 3^{1+2} : 2^2 \) | \( 3^2 : 2S_4 \) | \( M_{12} \)  | Weiss [35] |
| \( J_3 \)    | \( [2^6] : (S_3)^2 \) | \( [2^6] : ((S_3)^2 \cap A_6) \) | \( [2^4] : (3 \times A_5).2 \) | \( G \)       | Weiss [36] |
| Ru           | \( 5^{1+2} : [2^5] \) | \( 5^{1+2} : [2^7] \) | \( 5^2 : \GL_2(5) \) | \( G \)       | Ru graph [30] |
| O’N.2        | \( \PGL_2(9) \)    | \( A_6 \) | \( A_7 \)       | \( O’N \)     | Lu [20] |

**Proof of Proposition 3.1.** We follow the method outlined at the end of Section 1 and classify lattices \( L = (G, E, A, H) \) up to \( \Aut(G) \)-conjugation. Let \( G \) be an almost simple sporadic group with socle \( T \), and suppose \( G \) contains subgroups \( E, A \) and \( H \) as in Lemma 1.1. Information on the maximal subgroups of \( G \) can almost always be found in the Atlas [5], and a more complete list can be found in the survey article [37]. All maximal subgroups of \( G \) are known except if \( G = M \) is the Monster group; however, by [37] the remaining possibilities for maximal subgroups \( E \) of \( G \) are all almost simple with \( \soc(E) = \PSL_2(13) \) or \( \PSL_2(16) \), and so by Corollary 1.8 these cannot be edge-stabilisers of an edge-primitive 3-arc-transitive graph. Moreover, the order of a subgroup of index 2 of a maximal subgroup \( E \max M \) does not divide the order of an almost simple group with one of these socles, and so the vertex-stabiliser \( H \) of an edge-primitive graph with automorphism group \( M \) and edge-stabiliser \( E \) cannot be contained in any of these unknown maximal subgroups.

We now outline our method for finding lattices \( L \).
(1) Using [37], for each Aut(T)-class representative E of maximal subgroups of G we list the pairs of subgroups (E, M) of G such that M max G or M max 1/2 G and |M| is divisible by |E|/2. (It is perhaps interesting to note that all almost simple sporadic groups contain at least one such pair). We suppose that H ≤ M.

(2) If E does not contain a subgroup of index 2 then E does not occur in a lattice as above, and we remove the pairs containing E.

(3) If E is almost simple, then we must have E and H as in Corollary 1.8 with H ≤ M for some M. If this is the case, and E contains a subgroup of index 2 contained in H, then Γ is indeed 3-arc-transitive by Lemma 1.7 and we list L = (G, E, A, H) in Table 4 (this yields the fourth row of Table 4). Otherwise this does not yield a lattice, and we remove (E, M).

(4) For each (E, M) we consider the simple sections of M. Suppose Q ⊴ R ≤ M with X = R/Q a simple group. For each subgroup A of E of index 2, if A ≤ M then we have (A ∩ R) / (A ∩ Q) ≅ (A ∩ R) Q/Q ≤ X. Using MAGMA, [5] and [37] (and possibly [21] and [38]), in most cases we find that there exists a simple section X of M such that X contains no subgroup of the form (A ∩ R) / (A ∩ Q) for all subgroups A of E of index 2. If this is the case, we remove the pair (E, M) from our list.

(5) For each remaining pair (E, M) we list the pairs of subgroups (E, H) where H ≤ M is a subgroup that possibly contains A. We repeat steps (3) and (4) for H: if H is almost simple then we must have E and H as in Corollary 1.8 and we also consider the simple sections of H. If (E, H) fails either of these steps then we remove (E, H).

(6) We now suppose that each remaining pair (E, H) gives rise to a lattice L = (G, E, A, H) as above. We consider the action of H on the set of right cosets of A. If Γ is 2-arc-transitive then this action is 2-transitive. Therefore, using [3, Tables 7.3, 7.4], if it is not possible for H to have a 2-transitive action on the set of right cosets of A then we remove the pair (E, H). It is useful to note that if d = |H : A| and H acts 2-transitively on the set of right cosets of A, then d(d − 1) divides |H|. This fact can often be used to rule out certain pairs (E, H).

(7) In a number of remaining cases (E, H) the valency equals 4, and we check the classification of tetravalent edge-primitive 2-arc-transitive graphs given in [10] to see if such a lattice exists.

(8) This leaves the lattices listed in [17, Table 2] (which we add to our table) and one other possible lattice L = (J1, 7 : 6, 7 : 3, 23 : 7 : 3). Using MAGMA, we find that L does indeed exist and gives rise to an edge-primitive 2-arc-transitive graph Γ. However, in this case Γ is not 3-arc-transitive, as the valency is |H : A| = 8 and the stabiliser of a 2-arc has order 3. This completes Table 4.

We note that if G is sufficiently small then all lattices L of the above form can be completely determined using MAGMA, and in general this is much faster than using the method outlined above.
As an example, we show that $G = \text{Co}_1$ is not the automorphism group of an edge-
primitive 3-arc-transitive graph using the method above (note that generators of all max-
imal subgroups of $G$ are not known, and so in this case we cannot simply use MAGMA to
determine all lattices). By [37, p. 68], there are 22 classes of maximal subgroups of $G$. For
ease of notation, we refer to a class of maximal subgroups, or a representative of a class,
by its number using the ordering given in [37, p. 68].

(1) The orders of the maximal subgroups are listed in [5] (note the corrections given in
[37]). We find that there are 33 pairs $(E, M)$ where $E \max G, M \max G$ and $|E|/2$
divides $|M|$. 

(2) Without knowing further information on the structure of the maximal subgroups,
we cannot rule out any of our pairs by (2).

(3) We cannot rule out any pairs by (3), as $E$ is not equal to group number 1, 4 or 6.

(4) For each $(E, M)$ we consider the simple sections of $M$. For example, consider the
pair $(7,2)$. Here $E = (A_4 \times G_2(4)) : 2$ and $M = 3.Suz : 2$, and so $A = A_4 \times G_2(4)$.
Let $Q = 3 \triangleleft R = 3.Suz \leq M$ and $X = R/Q \cong Suz$. Then $A \cap R = A$, and
since $A$ has no normal subgroup of order 3 we have $A \cap Q = 1$ and so $(A \cap R) :
(A \cap Q) = 2^{14}.3^4.5^2.7.13$. However, this yields a contradiction using [3], as no
maximal subgroup of $X$ has order divisible by $2^{14}.3^4.5^2.7.13$. We therefore remove
the pair $(7,2)$. Using similar reasoning, we remove all but one pair.

(5) We are left to consider the pair $(18,12)$, where $E \cong (D_{10} \times (A_5 \times A_5)) : 2$ and
$M \cong (A_5 \times J_2) : 2$. Suppose $H \leq M$. Let $Q = A_5$ and $R = A_5 \times J_2 \triangleleft M$
so that $Q \triangleleft R$ and $X = R/Q = J_2$. Then $Y = (A \cap R)/(A \cap Q)$ satisfies $2^3.3.5^2 \leq$
$|Y| \leq |A|$. From [5] we must have $|Y| = 2^3.3.5^2$ and $Y \max X$, and so either
$(H \cap R)/(H \cap Q) = Y$ or $(H \cap R)/(H \cap Q) = X$. The first case implies $H = A$,
a contradiction. The second case implies that $H = M$, and so this is the only
possibility for the case $(18,12)$.

(6) We now suppose $(E, H)$ gives rise to a lattice $(G, E, A, H)$ for some $E \cong
(D_{10} \times (A_5 \times A_5)) : 2$ and $H \cong (A_5 \times J_2) : 2$ and some subgroup $A < E$ of
index 2. Then $d = |H : A| = 2^4.3^2.7$. Observe that $|H|$ is not divisible by $d(d - 1)$,
and so the action of $H$ on the set of right cosets of $A$ is not 2-transitive. Therefore
this lattice (if it exists) does not yield a 2-arc-transitive graph. This removes the
final pair, and completes the proof for $G = \text{Co}_1$.

The remaining cases are proved in a similar manner. 

Proof of Theorem [7]. This follows immediately from Propositions 2.1 and 3.1 

4. Classical groups: faithful local action

In this section we prove Theorem [2]. Throughout we assume that $G = \text{Aut}(\Gamma)$ is a finite
almost simple classical group such that $G_v$ acts faithfully on $\Gamma(v)$. We begin by fixing
some notation. Let $T = \text{soc}(G)$ be a finite simple classical group with natural module $V$
of dimension $n$ over $F_{q^\delta}$ of characteristic $p$, where $\delta = 2$ if $T$ is unitary and $\delta = 1$
otherwise. For the notation of $T$ and $G$ we follow [17, § 2]: in particular, we let $\Gamma L_n(q^\delta)$ be
the semilinear group and $\text{PGL}_n(q^d)$ its projective version. To refer to outer automorphisms of $T$ we use the conventions of [2 § 1.7.1]: we denote by $\delta$ a diagonal automorphism, $\phi$ a field automorphism and $\gamma$ a graph automorphism (in the sense of the Dynkin diagram).

Aschbacher’s theorem [1] states that if $X \leq G$ and $G$ does not contain an exceptional outer automorphism (in the case where $T = \text{PSp}_d(q)$ and $q$ is even or $T = \text{PO}_8^-$), then either $X$ is contained in a member of an Aschbacher class $\mathcal{C}_i$ for some $1 \leq i \leq 8$ or $X \in \mathcal{C}_9$. Details of the structure of the Aschbacher classes can be found in [15, § 4]. In particular, we note that if $X \in \mathcal{C}_9$ then $X$ is an almost simple group such that $\text{soc}(X)$ acts absolutely irreducibly on the natural module $V$ of $G$, and $\text{soc}(X)$ is not contained in a member of $\mathcal{C}_i$ for $i = 1, 3, 5$ or 8. This will be useful in the following lemma.

**Lemma 4.1.** Let $G$ be a finite almost simple classical group with subgroups $M$ and $N$ satisfying the following:

1. $M$ and $N$ are almost simple, with $\text{soc}(M) = A_7$ and $\text{soc}(N) = A_6$;
2. $N \triangleright G$;
3. $\text{soc}(G) \not\leq M$;
4. $\text{soc}(N) \leq M$.

Then $\text{soc}(G)$ is isomorphic to $\text{PSU}_3(5)$, $\text{PSL}_4(2)$, $\text{PSL}_6^+(q)$ or $\text{PO}_{10}^-(7)$.

**Proof.** Let $T = \text{soc}(G)$ be as above. Also let $X = \text{soc}(M) = A_7$ and $Y = \text{soc}(N) = A_6$. For a subgroup $S \leq G \leq \text{PGL}_n(q^d)$, denote by $\hat{S} \leq \text{GL}_n(q^d)$ the full preimage of $S$. Conversely, for a subgroup $S \leq \text{GL}_n(q^d)$ let $\overline{S} = S/(S \cap Z(\text{GL}_n(q^d))) \leq \text{PGL}_n(q^d)$. Observe that $N/(N \cap T) \leq G/T$ which is soluble by the Schreier conjecture, and so $T \triangleright Y$. Similarly $T \triangleright X$.

First suppose $n \leq 9$. The maximal subgroups of $G$ are listed in [2, Tables 8.1–8.59]. In the case where Aschbacher’s theorem applies, we first prove that

1. if $N \in \mathcal{C}_9$ then $M \in \mathcal{C}_9$.

Suppose for a contradiction that $N \in \mathcal{C}_9$ and $M \not\in \mathcal{C}_9$. From the definition of $\mathcal{C}_9$ and the fact $\text{soc}(N) \leq M$, we see that $M$ is not contained in a member of $\mathcal{C}_i$ for $i = 1, 3, 5$ or 8. If $M$ is contained in a member of $\mathcal{C}_2$ then $M$ preserves a subspace decomposition $\mathcal{D}$ of the form $V = U_1 \oplus \cdots \oplus U_t$ where $\text{dim}(U_i) = m$ and $n = mt$. As $Y$ acts absolutely irreducibly on $V$, $Y$ is not contained in the subgroup $M(\emptyset) \leq M$ fixing each component $U_i$ for $1 \leq i \leq t$, and so there exists a homomorphism from $Y$ into $M/M(\emptyset) \leq S_t$ such that the image of $Y$ acts transitively on the $t$ components. As $Y$ has no transitive action on 7, 8 or 9 points we must have $n = t = 6$. But this yields a contradiction, as the only insoluble composition factor of a group in $\mathcal{C}_2$ in this case is $A_6$.

Next suppose $M$ is contained in a member of $\mathcal{C}_4$ or $\mathcal{C}_7$, so that $M$ preserves a decomposition $\mathcal{D}$ of the form $V = U_1 \otimes \cdots \otimes U_t$ where $\text{dim}(U_i) = n_i$ for $1 \leq i \leq t$ and $n = n_1n_2\ldots n_t$. The insoluble composition factors of groups in these classes are classical groups with natural modules of dimension at most 3. It follows from [15, Proposition 5.3.7] that $t = 2$, and from the structure of $M$ we have $X$ and $Y$ contained in the direct product of two classical groups with natural modules of dimensions $(n_1, n_2) = (2, 4)$ or $(3, 3)$. In the first
instance the projection of $X$ onto the first factor must be trivial, and so $X$ is contained in the second factor. But this implies $Y \leq X$ is reducible, a contradiction. In the second case we have $n = 9$, and there exists no maximal subgroup $N \in \mathcal{C}_9$.

Finally consider the case where $M$ is contained in a member of $\mathcal{C}_6$, the normaliser of an extraspecial $r$-group for a prime $r$. Inspection of the tables in [2] shows that the only insoluble composition factors of such groups containing $Y$ are $\text{Sp}_6(2)$ and $\text{A}_8$ in the cases $T = \text{PSL}_8(q)$ and $\text{PΩ}_8^+(q)$ respectively. However, neither of these cases has a maximal subgroup $N \in \mathcal{C}_9$. This competes the proof of (1).

Using (1), we use the tables in [2] and find that $T$ is listed above, taking care to inspect the tables for exceptional isomorphisms involving almost simple groups with socle $A_6$, such as

$$A_6 \cong \text{PSL}_2(9) \cong \text{PSp}_2(9) \cong \text{PSU}_2(9) \cong \Omega_3(9) \cong \text{Sp}_4(2)' \cong \Omega_4^-(3).$$

If there exists a maximal subgroup $N$ with $N \not\in \mathcal{C}_9$ (such as in the case where $T = \text{PSU}_4(3)$ and $N \in \mathcal{C}_5$) then MAGMA is often useful for showing that $Y$ is not contained in an almost simple group with socle $A_7$.

Now suppose $n \geq 10$. Here Aschbacher’s theorem applies and so $N$ lies in one of $\mathcal{C}_1, \ldots, \mathcal{C}_9$. Either $T = \text{PSL}_n(q)$ and $G$ contains a graph automorphism or $G \leq \text{PGL}_n(q)$, and so $G$ acts on the set of subspaces of $V$.

We first prove that

(2) \hspace{1cm} N \in \mathcal{C}_9.

We begin by noting that if $S$ is a finite simple classical group with natural module of dimension at least 5, then

(3) \hspace{1cm} |N| < |S|.

First suppose $N \in \mathcal{C}_1$. If $N$ stabilises a decomposition $V = U \oplus W$ with $\dim(U) = m$, $\dim(W) = n - m$ and $m > \frac{n}{2} \geq 5$ then by [15, Lemma 4.1.1] we have $(\Omega(U) \times \Omega(W)) \leq N \cap T$, contradicting (3). Therefore either $N$ stabilises a totally singular $m$-space, or $T = \text{PSL}_n(q)$ and $N$ stabilises a pair of subspaces $\{U, W\}$, or $T = \text{PΩ}_n^+(q), n$ is even and $N$ stabilises a non-singular 1-space. In the first two cases, [15, Propositions 4.1.17-4.1.22] imply that $N \cap T$ is $p$-local, a contradiction. In the remaining case we have $\text{soc}(N \cap T) \cong \text{Sp}_{n-2}(q)$ by [15, Proposition 4.1.7], again a contradiction. Hence $N \not\in \mathcal{C}_1$.

Next suppose that $N \in \mathcal{C}_2$, so that $N$ stabilises a decomposition $V = U_1 \oplus \cdots \oplus U_t$ where $t \geq 2, \dim(U_i) = m$ for $1 \leq i \leq t$ and $n = mt$. By [15, Propositions 4.2.4-4.2.7, 4.2.9-4.2.11, 4.2.14-4.2.16], either $N \cap T$ has a normal subgroup of index $t!$ or $N$ is of type $O_2^+(q)^2$, and so we must have $t = 2$. The same results also imply that $\Omega(U_1) \leq N \cap T$, contradicting (3) since $m = \frac{n}{2} \geq 5$. Hence $N \not\in \mathcal{C}_2$.

We now consider $N \in \mathcal{C}_3$, so that $N$ is of the form $\text{Cl}_m(q^t)$ for some classical group and prime $t$ such that $n = mt$. By [15, Propositions 4.3.1, i=6,7,10,14,16,17], $N \cap T$ contains a normal subgroup of index $t$, and so $t = 2$. However, the same results imply that $N \cap T$ contains a classical group $\text{Cl}_2^+(q^2)$, contradicting (3).
Next suppose $N \in \mathcal{C}_4$, stabilizing a tensor product decomposition $V = U \otimes W$ with $\dim(U) = n_1, \dim(W) = n_2$ and $n = n_1 n_2$. However, since $N \cap T$ is non-local we have $\text{soc}(N \cap T) \cong \text{P}\Omega(U_1') \times \text{P}\Omega(W_1')$ by [15] Lemma 4.4.9, a contradiction. Therefore $N \notin \mathcal{C}_4$.

Now suppose $N \in \mathcal{C}_5$, so that $N$ is of the form $\text{Cl}_n(q^{1/2})$ for some classical group and some prime $t$. Since $N \cap T$ is non-local, by [15] Proposition 4.5.2 we have $\text{soc}(N \cap T) \cong S$ for some finite simple classical group $S$ with natural module of dimension $n$, contradicting (3). Hence $N \notin \mathcal{C}_5$.

We next suppose $N \in \mathcal{C}_6$. However, [15] Propositions 4.6.5–4.6.9] imply that $N \cap T$ is $r$-local for some prime $r$, a contradiction.

If $N \in \mathcal{C}_7$ then $N$ stabilises a tensor product decomposition $V = U_1 \otimes \cdots \otimes U_t$ where $\dim(U_i) = m$ for $1 \leq i \leq t$ and $n = m^t$. However, [15] Lemma 4.7.1] implies that $\text{soc}(N \cap T) \cong \text{P}\Omega(U_1') \times \cdots \times \text{P}\Omega(U_t')$, a contradiction. Hence $N \notin \mathcal{C}_7$.

Finally consider the case $N \in \mathcal{C}_8$. Since $N$ is non-local, [15] Lemma 4.8.1] implies that $\text{soc}(N \cap T) \cong S$ for some finite simple classical group $S$ with natural module of dimension $n$, a contradiction. This proves $N \notin \mathcal{C}_8$.

We therefore have $N \in \mathcal{C}_9$, and so we have proved (2).

From the description of $\mathcal{C}_9$ given in [15] § 1.2), $Y$ (and therefore $X$) acts irreducibly on $V$, and so $\hat{X}$ and $\hat{Y}$ are irreducible. By [11] 33.3] we have $\hat{X} = \hat{X}' \circ Z(\hat{X})$ and $\hat{Y} = \hat{Y}' \circ Z(\hat{Y})$, and so $\hat{X}'$ and $\hat{Y}'$ act irreducibly on $V$. As $\hat{X}'$ and $\hat{Y}'$ are perfect central extensions of $A_7$ and $A_6$ respectively, $\hat{X}' \cong A_7, 2.A_7, 3.A_7$ or $6.A_7$, and similarly $\hat{Y}' \cong A_6, 2.A_6, 3.A_6$ or $6.A_6$. Using [5] and [14], we find that the degrees of the irreducible representations of $\hat{X}'$ and $\hat{Y}'$ with $n \geq 10$ only coincide for $n = 10$ or 15.

Using [2] Tables 8.60–8.69] for $n = 10$ and [29] Tables 11.0.17–11.0.22] for $n = 15$, a similar proof to (1) shows that $M \in \mathcal{C}_9$. We find that the only possibility is $n = 10$ and $T = \text{P}\Omega_{10}(q)$ (noting the exceptional isomorphisms listed above) with $M \leq R \max G, R \in \mathcal{C}_9$ and $\text{soc}(R) = \text{P}\text{Sp}_4(q), \text{M}_{22}, A_{11}$ or $A_{12}$. By [2] Theorem 4.3.3], $\hat{Y}'$ only has an absolutely irreducible representation of degree 10 preserving a quadratic form in characteristic $p = 7$ (and in this case there is a unique representation), and so from [2] Tables 8.67, 8.69] we have $\epsilon = -$ and $q = 7$. This rules out the case $\text{soc}(R) = A_{12}$. We can also discard the case $\text{soc}(R) = A_{11}$, as the 10-dimensional module for $A_{11}$ is the deleted permutation module, and the restriction to $A_7$ is reducible. Therefore $\text{soc}(R) = \text{M}_{22}$ or $\text{P}\text{Sp}_4(7)$. In fact, in both of these cases we have $Y$ contained in a subgroup of $\text{soc}(R)$ isomorphic to $A_7$ by [2] Propositions 4.9.60, 4.9.63, 6.2.13], and so we list $T$ above.

Proof of Theorem\textsuperscript{2} As usual we follow the method outlined in Section 1 and classify subgroup lattices $L = (G, E, A, H)$ up to $\text{Aut}(G)$-conjugation, where $G$ is an almost simple classical group, $E \max G$, $A < E$ with $|E : A| = 2$ and $A < H < G$ with $T = \text{soc}(G) \notin H$. Moreover, by Lemma 1.4] $E$ and $H$ are almost simple with $\text{soc}(E) = A_6, \text{soc}(H) = A_7$, $|H : A| = 7$ and $E \neq A_6$ or $S_6$. Note that in order for $A$ to be a proper subgroup of $H$ we must have $A \cong A_6$ or $S_6$. We then consider the coset graph $\Gamma$ obtained from each such lattice $L$. Clearly $G \leq \text{Aut}(\Gamma)$, and so to determine if $G$ is the full automorphism group
of $\Gamma$ it suffices to check \[18\]. By construction and Lemma 4.4, $\Gamma$ is edge-primitive and 3-arc-transitive with automorphism group $G$, and we list the lattice $L$ in Table 2.

Let $G$ be an almost simple classical group with socle $T$, and suppose $G$ contains subgroups $E, A$ and $H$ as above. Observe that $E$ and $H$ satisfy the hypotheses of Lemma 4.1 with $E = N$ and $H = M$, and therefore $T = \text{PSU}_3(5), \text{PSL}_4(2), \text{PSL}_6(q)$ or $\text{PO}_{10}^- (7)$. Since $\text{PSL}_4(2) \cong A_8$ and Theorem 1 shows that there are no edge-primitive 3-arc-transitive graphs whose automorphism group has socle $A_8$, we have already eliminated the case where $T = \text{PSL}_4(2)$.

First suppose $T = \text{PSU}_3(5)$. By [2] Tables 8.5–8.6 we have $T \preceq G \preceq T.2$, where $T.2$ is an extension by a graph automorphism, and there are unique $\text{Aut}(T)$-classes of subgroups $E$ and $H$ of $G$ such that $\text{soc}(E) = A_6, \text{soc}(H) = A_7$ and $E \max G$ (here $E$ and $H \in \mathcal{C}_9$ and $H \max G$). Using MAGMA we find that, for both $G = T$ and $G = T.2$, there exists a unique subgroup $A < E$ of index 2 such that $A < H_0 < G$ where $H_0$ is Aut($T$)-conjugate to $H$. Therefore we have a single lattice, and this yields an edge-primitive 3-arc-transitive graph $\Gamma$ with automorphism group containing $G = T.2$. By [18], $G$ is indeed the full automorphism group, and so we have a lattice as listed in Table 2. The associated graph is the Hoffman-Singleton graph.

Next suppose $T = \text{PO}_{10}^- (7)$. By [2] Tables 8.68, 8.69] we have $T.2 \preceq G \preceq T.2^2$ where $T.2^2 = T.\langle \gamma, \delta' \rangle = \text{PO}_{10}^- (q)$ and $T.2 \neq T.\langle \delta' \rangle$. There are unique $\text{Aut}(T)$-class of subgroups $E$ and $H$ of $G$ such that $\text{soc}(E) = A_6, \text{soc}(H) = A_7$, $E \max G$ and $\text{soc}(E) < H$ (here $E \in \mathcal{C}_9$, and the result for $H$ follows from [2, Theorem 4.3.3]). Since $(E \cap T).2 \max T.2$ the graph is $T.2$-edge-primitive. We have $(E \cap T).2 \cong \text{PGL}_2 (9) \text{ or } M_{10}$, and from the final paragraph of the proof of Lemma 4.1 we have $A \cap T = \text{soc}(E)$ contained in a subgroup $H_0 \cong A_7$ that is $\text{Aut}(T)$-conjugate to $H \cap T$. From [2, Propositions 4.9.60, 6.2.13], $H_0 < M_{22} \cap \text{PSp}_4(7)$, and we therefore have a single lattice, and this yields an edge-primitive 3-arc-transitive graph $\Gamma$ with $T.2 \leq \text{Aut}(\Gamma)$. The lattice extends to $G = T.2^2$ with $E = \text{Aut}(A_6), A = S_6 = (E \cap T).\langle \delta' \rangle$ and $H = S_7 \leq \text{PSp}_4(7).2 \max T.\langle \delta' \rangle$, and by [18] this is the full automorphism group of $\Gamma$.

Finally we consider the case where $T = \text{PSL}_6(q)$. This case is more involved, as by [2] Tables 8.24–8.27] we see that there are infinitely many values of $q$ such that $G$ contains almost simple subgroups $E$ and $H$ with $\text{soc}(E) = A_6, \text{soc}(H) = A_7$ and $E \max G$. First suppose $\epsilon = +$. We shall work in the quasisimple group with $T = \text{SL}_6(q)$. There are at most two $\text{Aut}(G)$-classes of subgroups $E \max G$ with $\text{soc}(E) = A_6$, depending on if $q = p$ or $q = p^2$ and the value of $q \pmod{48}$. For each class, we consider if $E$ has a subgroup of index 2 contained in some almost simple group $H$ as above.

We first note that, from [2, Theorem 4.3.3], absolutely irreducible representations of quasisimple groups of type $A_7$ in dimension 3 only occur in characteristic 5. Therefore, if we have $E, A$ and $H$ as above (in the quasisimple group) and $H \leq (q - 1, 6) \circ \text{SL}_3(q) \in \mathcal{C}_9$, then by the conditions on the maximality of $E$ we have $q = 25$ and $E \cap T = 6 \cdot A_6$. However, a MAGMA calculation reveals that in $\text{SL}_6(25)$ the subgroup $6 \circ \text{SL}_3(25)$ does not have a subgroup isomorphic to $6 \cdot A_6$, and so this yields a contradiction. Therefore $H \nleq (q - 1, 6) \circ \text{SL}_3(q)$.  

20
First suppose \( E \cap T = 2 \times 3 A_6.2_3 \) is of novelty type N1 as in [2] Table 8.25] so that \( q = p \equiv 1 \pmod{24} \). Then \( G = T.2 \), an extension of \( G \) by a graph automorphism, \( E = 2 \times 3 A_6.2^2 \) and \( A = 2 \times 3 A_6.2_1 = 2 \times 3 S_6 \). Therefore \( A \not\leq T \) and so \( H \not\leq T \). Therefore the only possible maximal overgroups of \( H \) are 6, \( PSU_4(3).2_1 \). However, from [5] we have \( S_7 \not\leq PSU_4(3).2_1 \), a contradiction. Therefore this case does not give rise to a lattice.

The proof where \( E \cap T = 2 \times 3 A_6.2_3 \) is a novelty of type N2 is very similar, and so we now consider the case where \( E \cap T = 2 \times 3 A_6 \) is of type N3. In this case \( q = p \equiv 7 \pmod{24} \), and either \( G = T.\langle \delta^3, \gamma \rangle \) or \( p \equiv 2 \pmod{5} \) and \( G = T.\langle \gamma \rangle \). First suppose \( G = T.\langle \delta^3, \gamma \rangle \), so that \( E = 2 \times 3 \text{Aut}(A_6) \) and \( A = 2 \times 3 S_6 \). However, from [2] Propositions 4.8.9, 4.8.12], there is a unique \( \text{Aut}(T) \)-class of absolutely irreducible representations of \( 3^2 A_7 \) by [2] Theorem 4.3.3], and so this yields a single lattice which we add to our table. This gives rise to an edge-primitive 3-arc-transitive graph \( \Gamma \) with automorphism group \( G \), and \( \Gamma \) is the spread of an edge-primitive vertex-primitive graph with vertex-stabiliser \( PSU_4(3).2^2 \) and the same automorphism group and edge-stabiliser. The case with \( p \equiv \pm 2 \pmod{5} \) gives the same result.

The case where \( E \) is of novelty of type N4 is dealt with in a similar manner, and so we now consider the case where \( E \cap T = 6 \cdot A_6 \) is of novelty type N1. Here \( q = p \equiv 1, 31 \pmod{48} \) and \( G = T.2 \) is an extension of \( T \) by a graph automorphism. We have \( E = 6 \cdot A_6.2_2 \) and \( A = 6 \cdot A_6 = E \cap T \). From [2] Proposition 4.8.8], \( A \not< 6 \cdot A_7 \cdot \text{max} T \), and this is the only possibility for \( H \). We therefore have a lattice as desired, yielding a coset graph \( \Gamma \). In this case \( G \) is the full automorphism group of \( \Gamma \) by [18], and so this yields the lattice in Table [2].

The proofs for the remaining \( \text{Aut}(T) \)-classes of subgroups \( E \) of \( G \) are very similar (using [2] Propositions 4.8.8, 4.8.12]), and the results are listed in Table [2].

The case \( T = PSU_6(q) \) is almost identical to the proof in the linear case.

\[ \square \]

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