Classical and Quantum Chaotic Behaviors of Two Colliding Harmonic Oscillators

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Abstract

We have systematically studied both classical and quantum chaotic behaviors of two colliding harmonic oscillators. The classical case falls in Kolmogorov-Arnold-Moser class. It is shown that there exists an energy threshold, above which the system becomes nonintegrable. For some values of the initial energy near the threshold, we have found that the ratio of frequencies of the two oscillators affects the Poincaré sections significantly. The largest Lyapunov character exponent depends linearly on the ratio of frequencies of the two oscillators away from the energy threshold in some chaotic regions, which shows that the chaotic behaviors of the system are mainly determined by the ratio.

In the quantum case, for certain parameters, the distribution of the energy level spacings also varies with the ratio of frequencies of the two oscillators. The relation between the energy spectra and the ratio of frequencies of the two oscillators, the interaction constant, and the semi-classical quantization constant, is also investigated respectively.
I. INTRODUCTION

Many simple systems show rather complicated classical and quantum chaotic behaviors due to collision. A well-known example is the billiard system which describes the collision of a free mass point with various shapes of boundaries[1]. Another example is the occurrence of the chaotic scattering due to collisions[2]. We know that there are many processes related to the collision phenomena in the real world, such as chemical reactions[3], celestial mechanics[4], charged particles in an electromagnetic field[5], processes in hydrodynamics[6], scattering in atomic physics[7], and transportation problems in mesoscopic physics[8], etc. A lot of papers are devoted to investigate collision problems in literature. The study of colliding phenomena, however, is far from complete, even for the elastic collision.

The elastic collision can be assumed to be described by a so-called point contact interaction or δ-interaction. In the regime of quantum mechanics, several one-dimensional systems with δ -interaction potential are exactly solved[9, 10, 11]. Especially, the solutions of many body systems with δ -interaction have deep influences on physics even on mathematics[10, 11]. A lot of people have studied the following Hamiltonian

\[
H = \sum_i \frac{1}{2m} P_i^2 + \sum_{ij} \lambda_{ij} \delta(x_i - x_j), \tag{1}
\]

with \( P_i \) the momentum of the ith particle. A simple generalization of eq.(1) is that each particle has its own potential \( V(x) \) such that

\[
H = \sum_i \left( \frac{1}{2m} P_i^2 + V_i(x_i) \right) + \sum_{ij} \lambda_{ij} \delta(x_i - x_j). \tag{2}
\]

This Hamiltonian system, in general, can be exactly solved neither in classical mechanics nor in quantum mechanics. The system of two colliding harmonic oscillators is just the nontrivial reduction of the Hamiltonian eq.(2).

In this paper, we will study systematically, both in the sense of classical mechanics and quantum mechanics, the simplest nontrivial reduction of the extended Hamiltonian eq.(2): two colliding harmonic oscillators, i.e., two harmonic oscillators with δ-interaction. This system is conserved, and the classical case falls in Kolmogorov-Arnold-Moser(KAM) class[12].
Besides, there exists an energy threshold, above which the system becomes nonintegrable. For some values of the initial energy, we have found that the ratio of frequencies of the two oscillators affects the Poincaré sections significantly. The largest Lyapunov character exponent (LCE) depends linearly on the ratio of frequencies of the two oscillators away from the energy threshold in some chaotic regions. In the quantum case, we have studied the distribution of the energy level spacings and the density of probability determined by the wave function, and the relation between the energy spectra and the ratio of frequencies of the two oscillators, the interaction constant and the semi-classical quantization constant, respectively. The paper is organized as follows: The classical case of the two colliding oscillators will be discussed in Sec. II, and the quantum case will be studied in detail in Sec. III. The discussions and concluding remarks will be given finally.

II. THE CLASSICAL CASE

As stated above, we will study the two colliding oscillators, the simplest nontrivial reduction of the Hamiltonian eq.(2). The Hamiltonian of the system can be written as

\[ H = H_1 + H_2 + H', \] (3)

where

\[ H_1 = \frac{p_1^2}{2m_1} + \frac{1}{2}k_1(x_1 + b)^2, \] (4)

\[ H_2 = \frac{p_2^2}{2m_2} + \frac{1}{2}k_2(x_2 - b)^2, \] (5)

and \( H' \) is the point contact interaction

\[ H' = V \delta(x_1 - x_2), \] (6)

where \( k_1, k_2 > 0 \) are stiffness constants, \( 2b \) is the distance between two oscillators, and \( V \) is the interacting constant.

The classical dynamic equations of the system can be in principle written down, but the resulting equations contain the first order derivative of \( \delta \)-function explicitly. Since numerical
methods cannot deal with such an indefinite quantity directly, an approximate expression of $\delta$-function must be used in the situation. Whereas in the quantum case it is not so, i.e., the $\delta$-function can be treated exactly. Based on this consideration and for the consistency of the two cases, we will use the collision condition to replace the $\delta$-interaction in classical case. Such a replacement is equivalent to the case that the $\delta$-function is directly used. Therefore, the equations of motion of our system are written down as follows

$$\begin{align*}
\dot{x}_1 &= \frac{P_1}{m_1} \\
\dot{P}_1 &= -k_1(x_1 + b) \\
\dot{x}_2 &= \frac{P_2}{m_2} \\
\dot{P}_2 &= -k_2(x_2 - b),
\end{align*}$$

(7)

for $x_1 \neq x_2$, and for $x_1 = x_2$ with an additional condition: the exchange of $P_1$ and $P_2$ at $x_1 = x_2$. In this way, it is easy to understand that the behaviors of the system are independent of the nonzero parameter $V$. For simplicity, we will take the values of parameters as

$$b = 1, \quad m_1 = m_2 = 1, \quad \text{and} \quad k_2 = 1$$

(8)

in subsequent discussions.

If the two oscillators, with fixed total energy, are separated larger enough, apparently, there might be no collision at all. Hence, there should exist an energy threshold $E_c$. For a given energy $E$, the system is integrable as $E < E_c$, and the system is in general nonintegrable as $E > E_c$. The energy threshold corresponds to the case of $x_1 = x_2 = x$ and $P_1 = P_2 = 0$, at which the collision just takes place, without exchanging the momentum. We know that the energy in this case can be written as a function of colliding position $x$ and the separation parameter $b = 1$:

$$E(x, b = 1) = \frac{1}{2}k_1(x + 1)^2 + \frac{1}{2}k_2(x - 1)^2.$$  

(9)

For the parameter $b = 1$ fixed, $E(x, 1)$ has a minimum value with respect to $x = x_c$. We thus have

$$k_1(x_c + 1) + k_2(x_c - 1) = 0.$$  

(10)
$x_c$ can be obtained

$$x_c = \frac{(k_2 - k_1)}{k_1 + k_2}.$$  \hspace{1cm} (11)

Therefore, substituting $x_c$ into eq.(9), we have

$$E_c \equiv E(x_c, 1) = \frac{2k_1}{k_1 + 1} < 2,$$  \hspace{1cm} (12)

where the conditions in eq.(8) are used. The system will be nonintegrable if the initial energy $E \geq 2$ for any $k_1$. This result is in agreement with the conclusion given in Ref. [13]. We will only consider the case for $E \geq 2$.

To investigate the classical behaviors of the system, it is needed to obtain the corresponding Poincaré sections. To realize this, we have to integrate out eqs. (7) numerically. Here it should be noted that the ratio of frequencies of the two oscillators is related to the ratio $k_1/k_2$ by

$$\frac{\omega_1}{\omega_2} = \sqrt{\frac{k_1}{k_2}}.$$  \hspace{1cm} (13)

One may infer that the ratio $\omega_1/\omega_2$ is rational or irrational, will give rise to different dynamic behaviors, or equivalently, different shapes of the Poincaré sections. For this purpose, let us now present our numerical results. By integrating out eqs.(7) numerically, we have obtained the Poincaré sections for $k_1/k_2 = 3, 4, 8, 9$ with $E = 2$ and 5, and $k_1/k_2 = 3, 4$ with $E = 10$ and 30, respectively. In the calculation, we have kept the energy of the system to at least six effective digits. The Poincaré sections for $k_1/k_2 = 3, 4$ and $E = 2$ are shown in Fig. 1(a) and (b). Notice that in the calculation we have taken 60 different initial values, in the same region, for both $k_1/k_2 = 3$ and 4. It is clear that the former shows some chaotic behaviors, while the latter shows the quasiperiodic behaviors, at least for the present initial values. This seems to exhibit that the ratio of $\omega_1/\omega_2$ is integer or irrational, indeed results in different dynamic behaviors. However, further investigation shows that when the energy becomes larger, the differences between the integer ratios and the irrational ratios are vague. To show this, the Poincaré sections for $k_1/k_2 = 3, 4$ and $E = 5$ are given in Fig.1(c) and (d) respectively. Unlike the cases of $E = 2$, the differences between the Poincaré sections
are ambiguous for $E = 5$, i.e., the ratio of $\omega_1/\omega_2$ is integer or irrational, may not affect significantly the dynamic behaviors of the two colliding oscillators as the energy becomes larger. To verify above statement, we have still calculated the Poincaré sections for the other values of $k_1/k_2$ and $E$. For instance, the Poincaré sections for $k_1/k_2 = 8, 9$ and $E = 2, 5$ are obtained, as shown in Fig.2. One may observe that it shows the similar behaviors as $k_1/k_2 = 3, 4$. The Poincaré sections for $k_1/k_2 = 3, 4$ and $E = 10, 30$ are also calculated, but they show nothing new except similar behaviors as $E = 5$. All above analyses mean that the ratio of frequencies of the two oscillators is integer or irrational, will influence the dynamic behaviors of the system drastically only when the energy is near the threshold $E_c$ (notice that $E_c = 1.6$ for $k_1 = 4$ and 1.8 for $k_1 = 9$), i.e., the integer ratio exhibits some regular orbits, and the irrational ratio presents some chaotic orbits, as the energy is near $E_c$. Whereas the integer or irrational ratio $\omega_1/\omega_2$ does not play essential role in the classical dynamic behaviors of the system, as the energy is far away from the threshold $E_c$, i.e., the system can exhibit some chaotic behaviors either for the irrational ratio or for the integer ratio in the case, as far as we have studied. It should be pointed out that in above discussion we usually start from the same initial conditions for given energy $E$.

In addition to, to study the chaotic behaviors of the system in detail, we have also calculated the largest Lyapunov character exponent (LCE) [14] for different $k_1/k_2$ and different energies away from the threshold $E_c$ ($E > 2$) in some chaotic regions. The result, as shown in Fig.3, shows that the largest LCE depends on $k_1/k_2$ linearly

$$\lambda_{\text{max}} = \beta \frac{k_1}{k_2},$$

(14)

with $\beta$ a constant, and is independent of $E$. This suggests that the classical chaotic behaviors of the system are mainly controlled by the ratio of frequencies of the two oscillators, as the initial energy is away from the threshold.
III. THE QUANTUM CASE

Now let us study the quantum mechanical behaviors of the two colliding harmonic oscillators. The Hamiltonian of the system is

\[ H = H_0 + H', \tag{15} \]

where \( H_0 = H_1 + H_2, \ x_i, \ P_i \ (i = 1, 2) \) satisfy the canonical commuting relations, and eq.(7) should be replaced by its corresponding quantum partner. Evidently, \( H_0 \) can be exactly solved:

\[ H_0 |nm> = E_{nm} |nm>, \tag{16} \]

with the eigenfunction

\[ |nm> \equiv \varphi_n(x_1 + 1)\varphi_m(x_2 - 1), \tag{17} \]

where

\[ \varphi_l(x_j) = N_l \exp(-\frac{\alpha_j}{2} x_j^2) H_l(\alpha_j x_j), \]

\[ N_l = \sqrt{\frac{\alpha_j}{\sqrt{\pi2}l!}}, \quad \alpha_j = \sqrt{\frac{k_j}{\hbar}}, \quad (j = 1, 2), \]

and \( H_l(x) \) is the Hermite polynomials. \{ |nm> \} thus constitutes a complete orthonormal set. The eigenvalue \( E_{nm} \) is

\[ E_{nm} = \hbar \omega_1 (n + \frac{1}{2}) + \hbar \omega_2 (m + \frac{1}{2}), \quad n, m = \text{integer} \tag{18} \]

where \( \omega_1 = \sqrt{k_1} \) and \( \omega_2 = \sqrt{k_2} \). However, we must solve, at present, the following Schrödinger’s equation

\[ H\psi(x_1, x_2) = E\psi(x_1, x_2), \tag{19} \]

with \( E \) the eigen energy of the system. To achieve this, we can write \( \psi(x_1, x_2) \) as the linear combination of \( |nm> \):

\[ \psi(x_1, x_2) = \sum_{n,m} a_{nm} |nm>. \tag{20} \]
Because of the orthogonal property of $|nm>$, we have
\[
\sum_{n,m}^N (H_{nm,lk} - E\delta_{nl}\delta_{mk})a_{nm} = 0, \tag{21}
\]
where
\[
H_{nm,lk} = <nm|H'|lk> + E_{nm}\delta_{nl}\delta_{mk}, \tag{22}
\]
and
\[
\delta_{nm} = \begin{cases} 
0, & \text{if } n \neq m \\
1, & \text{if } n = m 
\end{cases}
\]

Since it is difficult to obtain the analytic solution of eq.(21), we solve it numerically. The integration in eq.(22) can be carried out by means of the properties of $\delta$-function and the Gauss-Hermite integration methods. Note that the matrix which consists of $H_{nm,lk}$ is real symmetric. To find the energy spectra, the eigen equation must be solved. The maximum number of energy levels we have solved is 961. We find that the statistical properties of 961 levels are qualitatively same as those of 200 levels. In the following, we will give the numerical results in detail, in order to show our consequences clearly.

A. $P(S)$ versus $S$

The distribution $P(S)$ of the energy level spacings is very interesting. We take $\hbar^2 = 0.2$. For $k_1/k_2 = 3$ (The ratio of frequencies is thus an irrational number), the distribution of the energy level spacings obeys the Wigner distribution for different interaction parameter $V$, and exhibits the energy repulsion[15]. However, for $k_1/k_2 = 4$ (The ratio of frequencies is an integer), the distribution of the energy level spacings shows the transition from Poisson distribution to Wigner distribution if $V$ varies from 0.0001 to 0.1. Fig.4 gives a comparison for the above two cases. Notice that all figures in Fig.4 are given for 200 levels. For $k_1/k_2 = 8,9$, the similar phenomena occur. These results can be compared with the case of two dimensional harmonic oscillator where the $P(S)$ does not exist if the frequencies are commensurable, and is peaked about a nonzero value of $S$ if the frequencies are incommensurable[16].
B. $|\psi(x_1, x_2)|^2$

The density of probability of the system at $(x_1, x_2)$ plane is given by $|\psi(x_1, x_2)|^2$. Here $\hbar^2$ is fixed to be 0.2. The surface plots and the related contour plots are given in Fig.5. The system shows some localization. It seems that the density of probability is more local for the case of $k_1/k_2 = 4$ than for the case of $k_1/k_2 = 3$. The scars in Fig. 5 almost lie in the region which is very near the accessible region for the classical orbits, but there is no clear relation between the density of the probability and the classical orbits, like the case of billiard systems[17]. For different $V$ and $k_1/k_2$, both the eigenvalues and the eigenvectors are so different that it is difficulty to compare the density of the probabilities for different parameters. The density of the probabilities has no explicit transition when the energy (i.e., the eigenvalues) crosses the classical chaotic energy threshold.

C. $E(V)$ versus $V$

The relation between the eigenvalues and the interaction constant $V$ is also studied. $\hbar^2$ is still fixed to be 0.2. We find that the eigenvalues are only sensitive to very small values of $V$. When $V$ is larger than 1, the eigenvalues do not change with respect to the interaction parameter $V$. When $V$ approaches zero, there is no degenerate energy level for $k_1/k_2 = 3$, while there exist clearly the degenerate energy levels for $k_1/k_2 = 4$, as shown in Fig.6. In Fig. 6, except $V = 0$, there is no energy level crossing, even though several lines are very close. For $k_1/k_2 = 8, 9$, the similar phenomena occur. We have also calculated the corresponding energy curvature [16, 17] $\left\langle -\frac{d^2 E}{dV^2} \right\rangle$ numerically where the average is for 50 different levels, as shown in Fig.6. The curvature seems to be independent of $k_1/k_2$. This result can be fitted by $\frac{a_1}{(1+v^2)^2}$ for small $V$, as given in [18].
D. $E(k_1/k_2)$ versus $k_1/k_2$

Now let us investigate the relation between the eigenvalues and the ratio $k_1/k_2$. $\hbar^2$ is yet fixed to be 0.2. $V$ is fixed to be 1. The energy repulsion is very clear, as shown in Fig.7, where $k_1$ takes the values from 1 to 25. In Fig.7, there is no energy crossing in the given region as well. We have calculated the curvature $\langle -\frac{d^2E}{dk_1^2} \rangle (k_2 = 1)$ numerically where the average is taken for 50 different levels. The result can also be fitted by $\frac{a_2}{(1+k_1)^2}$.

E. $E(\hbar)$ versus $\hbar$

The relation between the eigenvalues and the quantization constant $\hbar$ is studied also. In this case, the parameter $V$ takes two values: 0.001 and 1.0. The parameter $k_1/k_2$ also takes two values: 3 and 4. For $k_1/k_2 = 3$, the energy as the function of $\hbar$ is almost independent of the values of the parameter $V$. However, for $k_1/k_2 = 4$, the energy as the function of $\hbar$ is quite different for different values of the parameter $V$, as shown in Fig.8. For $k_1/k_2 = 4$, when $V$ is small, the degenerate energy level occurs as $\hbar$ is large, and when $V$ is large, the degenerate energy level occurs as $\hbar$ is small. We still can not fully understand this surprising phenomena for the moment. As expected, the semiclassical quantization constant $\hbar$ plays a crucial role in discussions of the transition from quantum systems to classical systems.

IV. DISCUSSION AND CONCLUSION

We have systematically studied both the classical and quantum chaotic behaviors of two colliding harmonic oscillators. In the classical case, there exists an energy threshold above which the system can be chaotic. The system is conserved, and the classical behaviors of the system belongs to the KAM class. Moreover, the ratio of the frequencies of two harmonic oscillators affects the Poincaré sections significantly, as the initial energy is near the threshold. The largest LCE depends linearly on the ratio $k_1/k_2$ of the two oscillators, and is independent of the initial energy in some chaotic regions, as the energy is away from the
threshold. This implies that the chaotic behaviors of the system are mainly controlled by the ratio $k_1/k_2$, not by $E$ in the situation. In the quantum case, the ratio of frequencies of the two harmonic oscillators is also important. Like in the two dimensional harmonic oscillators, the incommensurable frequencies play a special role. When we deal with the distribution of the energy level spacings, the integer ratio of frequencies shows the transition from the Poisson distribution to the Wigner distribution with increasing $V$. If the interaction parameter $V$ is not very small, the properties of the system are independent of $V$. The relation between the energy levels and the ratio of frequencies of the two harmonic oscillators, and the semi-classical quantization constant $\hbar$ is investigated, respectively. The curvature of the energy levels with respect to $V$ or $k_1/k_2$ shows universal properties. As one notes that, the parameter $V$ plays no role in the classical case, because the potential barrier at $x_1 = x_2$ is infinite, in which makes $V$ playing no role. While in quantum case, there are possibilities for a particle transmission over an $\delta$-function-type potential barrier, and $V$ should survive. Our present results just confirm this fact. Because of the numerical difficulty, the total behaviors of the system in the processes of $\hbar$ approaching zero have not been obtained. Nevertheless, the partial results have already shown some very interesting phenomena. One may observe that such an apparently simple system indeed exhibits rather complicated dynamical behaviors, and a few phenomena can not be sensibly explained for the moment. This system may yet contain rich exotic behaviors, and deserves to further study. How to understand the transition from a microscopic system, through a mesoscopic system, to a macroscopic system, is still a fascinating topic.

In summary, for some values of the initial energy near $E_c$, we have found that the ratio of frequencies of the two oscillators affects the Poincaré sections significantly. The largest Lyapunov character exponent depends linearly on the ratio of frequencies of the two oscillators away from the energy threshold in some chaotic regions, which shows that the chaotic behaviors of the system are mainly determined by the ratio. In the quantum case, for certain parameters, the distribution of the energy level spacings also varies with the ratio of frequencies of the two oscillators. The relation between the energy spectra and the
ratio of frequencies of the two oscillators, the interaction constant, and the semi-classical quantization constant, is also investigated respectively, and some interesting phenomena are presented.

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**Figure Captions**

Fig.1. The Poincaré sections for $k_1 = 3, 4$  
Fig.2. The Poincaré sections for $k_1 = 8, 9$  
Fig.3. The largest Lyapunov character exponent versus $k_1/k_2 (k_2 = 1)$ for different energies $E > 2$  
Fig.4. The distribution of level spacings for different $V$ and $k_1$.  
Fig.5. $|\psi(x_1, x_2)|^2$ versus $x_1, x_2$ (The wave functions are not normalized.)  
Fig.6. The Energy $E$ and the curvature $\langle -\frac{d^2E}{dV^2} \rangle$ versus the interaction constant $V$  
Fig.7. The energy $E$ and the curvature $\langle -\frac{d^2E}{dk_1^2} \rangle$ versus $k_1$  
Fig.8. The energy $E(h)$ versus $\hbar$
$k_1 = 3$, $V = 0.005$
$k = 3, V = 0.01$
$V = 1$, $E = 2.98$, $K_1 = 3$
$V = 0.0001$, $E = 3.05$, $K_1 = 3$
$V = 0.0001$, $E = 2.91$, $K_1 = 4$
$E=2, K_1=8$
$k_1 = 3$

Diagram showing the relationship between $E(V)$ and $V$. The graph has multiple curves indicating different values of $E(V)$ for various values of $V$. The axes are labeled $E(V)$ and $V$. The horizontal axis extends from 0 to 0.5, and the vertical axis from 0.5 to 6.
