Abstract

We solve multidimensional SDEs with distributional drift driven by symmetric, $\alpha$-stable Lévy processes for $\alpha \in (1, 2]$ by studying the associated (singular) martingale problem and by solving the Kolmogorov backward equation. We allow for drifts of regularity $(2 - 2\alpha)/3$, and in particular we go beyond the by now well understood “Young regime”, where the drift must have better regularity than $(1 - \alpha)/2$. The analysis of the Kolmogorov backward equation in the low regularity regime is based on paracontrolled distributions. As an application of our results we construct a Brox diffusion with Lévy noise.

Keywords: Singular diffusions, stable Lévy noise, distributional drift, paracontrolled distributions, Brox diffusion

1. Introduction

We study the weak well-posedness of Lévy-driven stochastic differential equations with distributional drift,

$$dX_t = V(t, X_t)dt + dL_t, \quad X_0 = x \in \mathbb{R}^d,$$

where $L$ is a non-degenerate, symmetric, $\alpha$-stable Lévy process for $\alpha \in (1, 2]$, and $V(t, \cdot)$ is a distribution in the space variable for each $t \geq 0$.

The special case where $L$ is a Brownian motion has received lots of attention in recent years, since such singular diffusions arise as models for stochastic processes in random media. For example, as random directed polymers [AKQ14, DD16, CSZ17], self-attracting Brownian motion in a random medium [CC18], or as continuum analogue of Sinai’s random walk in random environment (Brox diffusion, [Bro86]). Singular diffusions also arise as “stochastic characteristics” of singular SPDEs, for example the KPZ equation [GP17] or the parabolic Anderson model [CC18].

Of course, for distributional $V$ the point evaluation $V(t, X_t)$ is not meaningful, so a priori it is not clear how to even make sense of (1.1). The right perspective is not to consider $V(t, X_t)$ at a fixed time, but rather to work with the integral $\int_0^t V(s, X_s)ds$. Because of small scale oscillations of $X$, which are induced by the oscillations of $L$, we only “see an averaged version” of $V$ and this gives rise to some regularization. At least for a Brownian motion or for a sufficiently “wild” Lévy jump process, on the other hand we would not expect any regularization from a Poisson process. In the Brownian case, this intuition can be made rigorous in different ways, for example via a Zvonkin transform which removes the drift [Zvo74, Ver81, BC01, KR05, FGP10, FIR17], by considering the associated martingale problem and by constructing a domain for the singular generator [FRW03, DD16, CC18], or by Dirichlet forms [Mat94]. In the one-dimensional case it is also possible to apply an Itô-McKean construction based on space and time transformations [Bro86].
Here we follow the martingale problem approach, in the spirit of [DD16, CC18] who considered the Brownian case. Formally, \( X \) solves \([1,1]\) if and only if it solves the martingale problem for the generator \( \mathcal{G}^V = \partial_t - (-\Delta)^{\alpha/2} + V \cdot \nabla \), where the fractional Laplacian is the generator of \( L \) (and later we will consider slightly more general \( L \)). That is, for all functions \( u \) in the domain of \( \mathcal{G}^V \), the process \( u(t, X_t) - u(0, x) - \int_0^t \mathcal{G}^V u(s, X_s) \, ds, \ t \geq 0 \), is a martingale. The difficulty is that the domain of \( \mathcal{G}^V \) necessarily has trivial intersection with the smooth functions: If \( u \) is smooth, then \( (\partial_t - (-\Delta)^{\alpha/2}) u \) is smooth as well, while for non-constant \( u \) the product \( V \cdot \nabla u \) is only a distribution and not a continuous function. If we want \( \mathcal{G}^V u \) to be a continuous function, then \( u \) has to be non-smooth so that \( (\partial_t - (-\Delta)^{\alpha/2}) u \) is also a distribution which has appropriate cancellations with \( V \cdot \nabla u \) and the sum of these terms is a continuous functions.

We can find such \( u \) by solving the Kolmogorov backward equation
\[
\partial_t u = (-\Delta)^{\alpha/2} u - V \cdot \nabla u + f, \quad u(T, \cdot) = u_T, \tag{1.2}
\]
for suitable continuous functions \( f \) and \( u_T \), so that \( \mathcal{G}^V u = f \) by construction. Given \( V \in C([0, T], \mathcal{C}_\beta(\mathbb{R}^d, \mathbb{R}^d)) \), where \( \mathcal{C}_\beta = B^\beta_{\infty, \infty} \), the regularization obtained from \( (-\Delta)^{\alpha/2} \) suggests that \( u(t, \cdot) \in \mathcal{C}^{\alpha+\beta} \). Therefore, \( \nabla u(t, \cdot) \in \mathcal{C}^{\alpha+\beta-1} \), and since the product \( V(t, \cdot) \cdot \nabla u(t, \cdot) \) is well posed if and only if the sum of the regularities of the factors is strictly positive, we need \( \alpha + 2\beta - 1 > 0 \), or \( \beta > (1 - \alpha)/2 \). We call this the Young regime, in analogy with the regularity requirements that are needed for the construction of the Young integral.

There have been several results on singular Lévy SDEs in the Young regime in recent years. Athreya, Butkovsky and Mytnik [ABM20] consider the time-homogeneous one-dimensional case and construct weak solutions via a Zvonkin transform, before establishing strong uniqueness and existence by a Yamada-Watanabe type argument (which in particular is restricted to \( d = 1 \)). Two nearly simultaneous works Ling and Zhao [LZ19] respectively de Raynal and Menozzi [dRM19] consider the multi-dimensional (time-homogeneous resp. time-inhomogeneous) case and they prove existence and uniqueness for the martingale problem. They even allow nearly simultaneous works Ling and Zhao [LZ19] respectively de Raynal and Menozzi [dRM19]

To go beyond the Young regime we use techniques from singular SPDEs. More precisely, following the ideas of [DD16, CC18] in the Brownian case, we use paracontrolled distributions [GIP15] to solve \([1,2]\) for \( \beta > (2 - 2\alpha)/3 \) and \( \alpha \in (1, 2) \). The idea is to treat \( u \) as a perturbation of the linearized equation with additive noise, \( \partial_t w = (-\Delta)^{\alpha/2} w - V \), and to leverage this to gain some regularity. This works as long as the nonlinearity \( V \cdot \nabla u \) is of lower order than the linear operator \( (-\Delta)^{\alpha/2} \), i.e. if \( \alpha > 1 \). And indeed in that case we have \( (2 - 2\alpha)/3 < (1 - \alpha)/2 \), and we can go beyond the Young regime.

Being able to go beyond the Young regime is important for our main application, the construction of a “Brox jump diffusion” with \( \alpha \)-stable Lévy noise. Here \( d = 1 \) and \( V \) is a (periodic) space white noise, so in particular we can only take \( \beta = -1/2 - \varepsilon \) for \( \varepsilon > 0 \), which is never in the Young regime, not even in the Brownian case \( \alpha = 2 \). We also indicate how to adapt our constructions in order to treat a non-periodic white noise. On the other hand, we do not study the qualitative behavior of the solution and we leave this for future research.

**Structure of the paper** In Section 2 we collect some background material on Besov spaces and \( \alpha \)-stable Lévy processes, and we discuss the Schauder estimates for the fractional Laplacian. In Section 3 we then solve the Kolmogorov backward equation. Our main theorem concerning the
existence and uniqueness of a solution to the martingale problem is proven in Section 4 while in Section 3 we construct the Brox diffusion with Lévy noise.

2. Preliminaries

In this section, we introduce some technical ingredients that we will need in the sequel.

Let \((p_j)_{j \geq -1}\) be a smooth dyadic partition of unity, i.e. a family of functions \(p_j \in C_c^\infty(\mathbb{R}^d)\) for \(j \geq -1\), such that

1.) \(p_{-1}\) and \(p_0\) are non-negative radial functions (they just depend on the absolute value of \(x \in \mathbb{R}^d\)), such that the support of \(p_{-1}\) is contained in a ball and the support of \(p_0\) is contained in an annulus;
2.) \(p_j(x) := p_0(2^{-j} x), x \in \mathbb{R}^d, j \geq 0;\)
3.) \(\sum_{j=-1}^{\infty} p_j(x) = 1\) for every \(x \in \mathbb{R}^d;\) and
4.) \(\text{supp}(p_i) \cap \text{supp}(p_j) = \emptyset\) for all \(|i - j| > 1\).

We then define the Besov spaces

\[ B^\theta_{p,q,j} := \{ u \in \mathcal{S}' : \| u \|_{B^\theta_{p,q,j}} = \|(2^{\theta j} \| \Delta_j u \|_{L^p})_{j \geq -1} \|_{\ell_q} < \infty \}, \]  

where \(\Delta_j u = \mathcal{F}^{-1}(p_j \mathcal{F} u)\) are the Littlewood-Paley blocks, and the Fourier transform is defined with the normalization \(\hat{\varphi}(y) := \mathcal{F} \varphi(y) := \int_{\mathbb{R}^d} \varphi(x) e^{-2\pi i (x,y)} dx\) (and \(\mathcal{F}^{-1} \varphi(x) = \hat{\varphi}(-x)\)); moreover, \(\mathcal{S}\) are the Schwartz functions and \(\mathcal{S}'\) are the Schwartz distributions. For \(p = q = \infty\), the space \(B^\theta_{\infty,\infty}\) has the unpleasant property that \(C_b^\infty \subset B^\theta_{\infty,\infty}\) is not dense. Therefore, we rather work with the following space:

\[ \mathcal{C}^\theta := \{ u \in \mathcal{S}' : \lim_{j \to \infty} 2^{\theta j} \| \Delta_j u \|_{\ell_q} = 0 \} \]

equipped the norm \(\| \cdot \|_{\theta} := \| \cdot \|_{B^\theta_{\infty,\infty}}\), for which \(C_b^\infty\) is a dense subset. We also write \(\mathcal{C}^\theta_{\mathbb{R}^d} = (\mathcal{C}^\theta)^d\) and \(\mathcal{C}^\theta := \cap_{\gamma \in \mathbb{R}} \mathcal{C}^{\theta + \gamma}\).

We recall from Bony’s paraproduct theory (cf. [BCD11, Section 2]) that in general the product \(uv := u \circ v + u \circ v + u \circ v\) of \(u \in \mathcal{C}^\theta\) and \(v \in \mathcal{C}^\beta\) for \(\theta, \beta \in \mathbb{R}\), is well defined if and only if \(\theta + \beta > 0\). Here, we use the notation of [MP19, MW17] for the para- and resonant products \(\circ, \circ, \circ\), which satisfy the following estimates:

\[\begin{align*}
\| u \circ v \|_{\theta + \beta} & \lesssim \| u \|_{\theta} \| v \|_{\beta}, \quad \text{if } \theta + \beta > 0, \\
\| u \circ v \|_{\beta} & \lesssim \| u \|_{L^\infty} \| v \|_{\beta}, \quad \text{if } \theta > 0, \\
\| u \circ v \|_{\beta + \theta} & \lesssim \| u \|_{\theta} \| v \|_{\beta}, \quad \text{if } \theta < 0.
\end{align*}\]  

So if \(\theta + \beta > 0\) we have \(\| uv \|_{\gamma} \lesssim \| u \|_{\theta} \| v \|_{\beta}\) for \(\gamma := \min(\theta, \beta + \beta)\).

For \(T > 0, \rho \in (0, 1)\) and for a Banach space \(X\) we write \(C^{\rho}_{\mathcal{T}} X := C^\rho([0,T], X)\), with

\[ \| u \|_{C^{\rho}_{\mathcal{T}} X} := \sup_{0 \leq s < t \leq T} \left( \frac{\| u(t) - u(s) \|_{X}}{(t-s)^\rho} + \sup_{t \in [0,T]} \| u(t) \|_{X} \right) \]

and \(C_{\mathcal{T}} X := C([0,T], X)\) with norm \(\| u \|_{C_{\mathcal{T}} X} := \sup_{t \in [0,T]} \| u(t) \|_{X}\). Analogously, we define for \(\overline{T} \in (0,T]\) the space \(C^{\rho}_{\mathcal{T},\overline{T}} X := C([T-\overline{T}, T], X)\).
Next, we collect some facts about $\alpha$-stable Lévy processes and their generators and semigroups. A symmetric $\alpha$-stable Lévy process $L$ is a Lévy process, satisfying the scaling property $(L_{tk})_{t \geq 0} \equiv k^{1/\alpha}(L_t)_{t \geq 0}$ for any $k > 0$ and $L \equiv -L$, where $\equiv$ denotes equality in law. These properties determine the jump measure $\mu$ of $L$, see [Sat99, Theorem 14.3]. That is, the Lévy jump measure $\mu$ of $L$ is given by

$$\mu(A) := E \left[ \sum_{0 \leq t \leq 1} 1_A(\Delta L_t) \right] = \int_S \int_{\mathbb{R}^+} 1_A(k\xi) \frac{1}{k^{1+\alpha}} dk \tilde{\nu}(d\xi), \quad A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),$$

(2.3)

where $\tilde{\nu}$ is a finite, symmetric, non-zero measure on the unit sphere $S \subset \mathbb{R}^d$. We also define for $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ and $t \geq 0$

$$\pi(A \times [0,t]) = \sum_{0 \leq s \leq t} 1_A(\Delta L_s),$$

which is a Poisson random measure with intensity measure $dt \mu(dy)$. Let $\tilde{\pi}(dr,dy) = \pi(dr,dy) - dr \mu(dy)$ be the compensated Poisson random measure of $L$. We refer to the book by Peszat and Zabczyk for the integration theory against Poisson random measures and for the Burkholder-Davis-Gundy inequality [PZ07, Lemma 8.21 and 8.22], which we will both use in the sequel. The generator $A$ of $L$ satisfies $C_b^\infty(\mathbb{R}^d) \subset \text{dom}(A)$ and it is given by

$$A \varphi(x) = \int_{\mathbb{R}^d} (\varphi(x+y) - \varphi(x) - 1_{\{|y| \leq 1\}}(y) \nabla \varphi(x) \cdot y) \mu(dy)$$

(2.4)

for $\varphi \in C_b^\infty(\mathbb{R}^d)$. If $(P_t)_{t \geq 0}$ denotes the semigroup of $L$, the convergence $t^{-1}(P_t f(x) - f(x)) \to A f(x)$ is uniform in $x \in \mathbb{R}^d$ (see [PZ07, Theorem 5.4]).

To derive Schauder estimates for $(P_t)$ it will be easier to work with another representation of the generator $A$. For that purpose we first introduce an operator $\mathcal{L}_\nu^\alpha$ via Fourier analysis, and then we show that it agrees with $A$.

**Definition 2.1.** Let $\alpha \in (0,2]$ and let $\nu$ be a symmetric (i.e. $\nu(A) = \nu(-A)$), finite and non-zero measure on the unit sphere $S \subset \mathbb{R}^d$. We define the operator $\mathcal{L}_\nu^\alpha$ as

$$\mathcal{L}_\nu^\alpha \varphi = \mathcal{F}^{-1}(\psi_\nu^\alpha \varphi) \quad \text{for} \quad \varphi \in C_b^\infty,$$

(2.5)

where $\psi_\nu^\alpha(z) := \int_S |(z,\xi)|^\alpha \nu(d\xi)$.

**Remark 2.2.** If we take $\nu$ as a suitable multiple of the Lebesgue measure on the sphere, then $\psi_\nu^\alpha(z) = |2\pi z|^\alpha$ and thus $\mathcal{L}_\nu^\alpha$ is the fractional Laplace operator $-(-\Delta)^{\alpha/2}$. And if moreover $\alpha = 2$, then the fractional Laplacian of course agrees with the usual Laplacian.

**Lemma 2.3.** For $\varphi \in C_b^\infty$ we have $-\mathcal{L}_\nu^\alpha \varphi = A \varphi$, where $A$ is the generator of the symmetric, $\alpha$-stable Lévy process $L$ with characteristic exponent $E[\exp(2\pi i(z, L_t))] = \exp(-t \psi_\nu^\alpha(z))$. The process $L$ has the jump measure $\mu$ as defined in Equation (2.3), with $\tilde{\nu} = C \nu$ for some $C > 0$.

**Proof.** By Fourier inversion, $L_t$ has the density $\rho_t(x) = \mathcal{F}^{-1}(\exp(-t \psi_\nu^\alpha))$ w.r.t. the Lebesgue measure (note that $\psi_\nu^\alpha(z) = \psi_\nu^\alpha(-z)$). So for the semigroup $(P_t)$ of $L$ we have $P_t \varphi(x) = \int \rho_t(y) \varphi(x+y) dy$ with $\partial_t P_t \varphi|_{t=0} = \mathcal{F}^{-1}(-\psi_\nu^\alpha \hat{\varphi}) = -\mathcal{L}_\nu^\alpha \varphi$ for any $\varphi \in C_b^\infty$. The identity $\tilde{\nu} = C \nu$ is shown in the proof of [Sat99, Theorem 14.10].

**Assumption 2.4.** Throughout the paper we assume that the measure $\nu$ from Definition 2.1 has $d$-dimensional support, in the sense that the linear span of its support is $\mathbb{R}^d$. This means that the process $L$ can reach every open set in $\mathbb{R}^d$ with positive probability.
So far we defined $L^\alpha_\nu$ on $C^\infty_b$, so in particular on Schwartz functions. But the definition of $L^\alpha_\nu$ on Schwartz distributions by duality is problematic, because for $\alpha \in (0, 2)$ the function $\psi^\alpha_\nu$ has a singularity in 0. This motivates the next proposition.

**Proposition 2.5. (Continuity of the operator $L^\alpha_\nu$)**

Let $\alpha \in (0, 2]$. Then for $\beta \in \mathbb{R}$ and $u \in C^\infty_b$ we have

$$\|L^\alpha_\nu u\|_{\beta-\alpha} \lesssim \|u\|_{\beta}.$$  

In particular, $L^\alpha_\nu$ can be uniquely extended to a continuous operator from $\mathcal{C}^\beta$ to $\mathcal{C}^{\beta-\alpha}$.

**Proof.** For $j \geq 0$ it follows from [BCD11, Lemma 2.2], that $\|L^\alpha_\nu \Delta_j u\|_{L^\infty} \lesssim 2^{-j(\beta-\alpha)} \|u\|_{\beta}$, as $\psi^\alpha_\nu$ is infinitely often continuously differentiable in $\mathbb{R}^d \setminus \{0\}$ with $|\partial^\mu \psi^\alpha_\nu(z)| \lesssim |z|^{-|\mu|}$ for a multiindex $\mu \in \mathbb{N}_0^n$ with $|\mu| \leq \alpha$ and $\Delta_j u$ has a Fourier transform, which is supported in $2^j \mathcal{A}$, where $\mathcal{A}$ is the annulus, where $\rho_0$ is supported. For $j = -1$ we use that $-L^\alpha_\nu = A$ for $A$ as in Equation (2.4), and therefore

$$-L^\alpha_\nu \Delta_{-1} u(x) = \int_{\mathbb{R}^d} (\Delta_{-1} u(x + y) - \Delta_{-1} u(x) - \nabla \Delta_{-1} u(x) \cdot y \mathbb{1}_{\{|y| \leq 1\}}) \mu(dy)$$

$$\lesssim \int_{B(0, 1)} \|D^2 \Delta_{-1} u\|_{L^\infty} |y|^2 \mu(dy) + \|\Delta_{-1} u\|_{L^\infty} \mu(B(0, 1)^c) \lesssim \|u\|_{\alpha},$$

where $B(0, 1) = \{|y| \leq 1\}$ and the last step follows from the Bernstein inequality in [BCD11, Lemma 2.1].

**Remark 2.6.** One can show that the operators $A$ and $-L^\alpha_\nu$ even agree on $\bigcup_{\epsilon > 0} \mathcal{C}^{2+\epsilon}$. Indeed, for $\varphi \in \bigcup_{\epsilon > 0} \mathcal{C}^{2+\epsilon}$ we have that $\varphi$ and its partial derivatives up to order 2 are uniformly continuous, and thus it follows from [PZ07, Theorem 5.4] that $A \varphi$ has the same expression as in (2.4). Then we can use that $C^\infty_b$ is dense in $\mathcal{C}^{2+\epsilon}$ for all $\epsilon > 0$ and apply a continuity argument to deduce that $A \varphi = -L^\alpha_\nu \varphi$ for $\varphi \in \bigcup_{\epsilon > 0} \mathcal{C}^{2+\epsilon}$.

For $z \in \mathbb{R}^d \setminus \{0\}$ we also have

$$\psi^\alpha_\nu(z) = |z|^\alpha \int_{S} |\frac{z}{|z|} \cdot \xi|^\alpha \nu(d\xi) \geq |z|^\alpha \min_{|y| = 1} \int_{S} |\langle y, \xi \rangle|^\alpha \nu(d\xi),$$

and by Assumption (2.4) the minimum on the right hand side is strictly positive. Otherwise, there would be some $y_0 \neq 0$ with $\int_{S} |\langle y_0, \xi \rangle|^\alpha \nu(d\xi) = 0$ and this would mean that the support of $\nu$ (and thus also its span) is contained in the orthogonal complement of span($y_0$). Therefore, $e^{-\psi^\alpha_\nu}$ decays faster than any polynomial at infinity and outside of 0 it even behaves like a Schwartz function.

**Lemma 2.7.** Let $\nu$ be a finite, symmetric measure on the sphere $S \subset \mathbb{R}^d$ satisfying Assumption (2.4). Let $P_t \varphi := \mathcal{F}^{-1}(e^{-t\nu^\alpha} \hat{\varphi}) = \rho_t * \varphi$, where $t > 0$, $\rho_t = \mathcal{F}^{-1} e^{-t\nu^\alpha} \in L^1$, and $\varphi \in C^\infty_b$. Then we have for $\vartheta \geq 0$, $\beta \in \mathbb{R}$

$$\|P_t \varphi\|_{\beta + \theta} \lesssim (t^{-\vartheta/\alpha} \vee 1) \|\varphi\|_{\beta},$$

and for $\vartheta \in [0, \alpha)$

$$\|(P_t - 1d) \varphi\|_{\beta - \theta} \lesssim t^{\vartheta/\alpha} \|\varphi\|_{\beta}.$$  

Therefore, if $\vartheta \geq 0$, then $P_t$ has a unique extension to a bounded linear operator in $L(\mathcal{C}^\beta, \mathcal{C}^{\beta+\theta})$ and this extension satisfies the same bounds.
Proof. This follows from [GIP15] Lemma A.5, see also [GIP15] Lemma A.7, Lemma A.8. □

Corollary 2.8. (Schauder Estimates)

Let \((P_t)\) and \(\nu\) be as in Lemma 2.7. Let \(T > 0, T \in (0, T]\), and \(\beta \in \mathbb{R}\). For \(v \in C_T, T \in \mathbb{R}^\beta\) and \(t \in [T - T, T]\) we define \(J^T v(t) := \int_T^T P_{t-r}v(r)dr\). Then we have for \(\theta \in [0, \alpha]\)

\[
\|J^T v\|_{C_T, T, T^{\beta+\theta}} \lesssim T^{1-\alpha/\theta} \|v\|_{C_T, T, T^{\beta}}. \tag{2.8}
\]

If moreover \(\beta < 0\) and \(\theta \in (-\beta, \alpha)\), then

\[
\|J^T v\|_{C_T, T, T^{\beta+\theta}} \lesssim T^{1-\alpha/\theta} \|v\|_{C_T, T, T^{\beta}}. \tag{2.9}
\]

Proof. This follows from the same arguments as [GIP15] Lemma A.9. In that lemma only the most difficult case \(\theta = \alpha\) is treated, but the case \(\theta < \alpha\) follows directly from Lemma 2.7 since then \(\int_T^T (r - t)^{-\theta/\alpha}dr \lesssim (T - t)^{1-\theta/\alpha} \leq T^{-\theta/\alpha}\). □

3. The Kolmogorov backward equation

Our goal is to define and construct weak solutions (or better: martingale solutions) to the Lévy SDE

\[
dX_t = V(t, X_t)dt + dL_t, \quad X_0 = x \in \mathbb{R}^d, \tag{3.1}
\]

where \(L\) is a \(d\)-dimensional, symmetric, \(\alpha\)-stable Lévy process for \(\alpha \in (1, 2]\), and where \(V \in C_T, T \in \mathbb{R}^{\beta}\) for \(\beta < 0\). For that purpose we follow [DD16] [CC18] in formulating the martingale problem for \(X\), which is based on the operator

\[
\mathcal{L}_\nu := \partial_t + \mathcal{L}_\nu^\alpha + V \cdot \nabla, \tag{3.2}
\]

where the \(\mathcal{L}_\nu^\alpha\) is the generator of \(L\) (see Definition 2.1 and Lemma 2.3). To make sense of the martingale problem, we have to solve the Kolmogorov backward equation

\[
\mathcal{L}_\nu^\alpha u = f, \quad u(T, \cdot) = u^T \iff \partial_t u = -\mathcal{L}_\nu^\alpha u - V \cdot \nabla u + f, \quad u(T, \cdot) = u^T, \tag{3.3}
\]

for \(f \in C_T, T \in \mathbb{R}^{\beta}\) for \(\epsilon > 0\), and \(u^T \in \mathcal{C}^{2+\beta}\). Here we need sufficient regularity of \(V\): Since at best \(V \cdot \nabla u \in \mathcal{C}^\beta\) and since inverting \(\partial_t + \mathcal{L}_\nu^\alpha\) gains \(\alpha\) degrees of regularity, we expect that \(u \in C_T, T \in \mathbb{R}^{\beta+\alpha}\). Thus, we need \(\beta + (\alpha + \beta - 1) > 0\) in order for \(V \cdot \nabla u\) to be well defined, i.e. \(\beta > \frac{1-\alpha}{2}\) (we call this the Young case, by analogy to the Young integral). To allow for more irregular \(V\) we follow [CC18] in using paracontrolled distributions [GIP15]. Then we need to postulate the existence of certain resonant products of \(V\), and under that assumption we obtain the existence and uniqueness of a paracontrolled solution \(u\) for \(\beta > \frac{2-2\alpha}{3}\).

The solution theory for Equation (3.3) is similar to the Brownian case, where \(-\mathcal{L}_\nu^\alpha\) is replaced by the Laplacian, and which is treated in [CC18]. For completeness we include the proofs, but readers familiar with [CC18] could skip most of this section and only have a look at Theorem 3.1, Definition 3.3 Theorem 3.8 and Theorem 3.10 where Theorem 3.10 carries out the arguments for proving the continuity of the solution map in the rough case.

Let us start with the Young case. We call \(u\) a mild solution to Equation (3.3) if

\[
u_t = P_{T-t}u^T + \int_T^T P_{s-t}(V_s \cdot \nabla u_s - f_s)ds =: P_{T-t}u^T + J^T(V \cdot \nabla u - f)(t),
\]

for \(t \in [0, T]\), where \((P_t)\) is the semigroup generated by \(-\mathcal{L}_\nu^\alpha\), as defined in Lemma 2.7.
Theorem 3.1. Let $\alpha \in (1,2]$, $\beta \in (\frac{1-\alpha}{2},0)$ and $\theta \in (1-\beta,\beta+\alpha)$. Let $V \in C_{T}^{\beta}C_{\mathbb{R}^{d}}^{\beta}$ and $u^{T} \in C_{T}^{\theta}$. Then the PDE
\begin{equation}
\partial_{t} u = \mathcal{L}^{\alpha} u - V \cdot \nabla u + f, \quad u(T,\cdot) = u^{T}, \tag{3.4}
\end{equation}
admits a unique mild solution $u \in C_{T}^{\theta} \cap C_{T}^{\beta/\alpha}L^{\infty}$. Moreover, the solution map
\begin{equation}
\mathcal{C}^{\theta} \times C_{T}^{\beta} \times C_{T}^{\beta} \ni (u^{T}, f, V) \mapsto u \in C_{T}^{\theta} \cap C_{T}^{\beta/\alpha}L^{\infty}.
\end{equation}
is continuous.

Proof. The proof follows from the Banach fixed point theorem applied to the map
\[ \Phi^{T};u(t) = P_{T-t}u^{T} + J^{T}(\nabla u \cdot V - f)(t), \]
where $J^{T}(t) = \int_{t}^{T} P_{t-r}v(r)dr$. Replacing first the interval $[0,T]$ by $[T-\mathcal{T},T]$ for $\mathcal{T} \in (0,T]$ sufficiently small, the estimates for $P$ and $J$ from Lemma 2.7 and Corollary 2.8, together with the estimates for the product in (2.2) show that $\mathcal{T}$ is sufficiently small, then $\Phi^{T};u$ is a contraction on $C_{T}^{\theta} \cap C_{T}^{\beta/\alpha}L^{\infty}$. Moreover, $\mathcal{T}$ does not depend on the terminal condition $u^{T}$ and therefore we can iterate this construction and patch the solutions together to obtain a solution on $[0,T]$.

The continuity of the solution map follows from the linearity of the equation and from Gronwall’s inequality for locally finite measures, cf. [EK86, Appendix, Theorem 5.1].

Our next aim is to go beyond the Young case. If $\beta \leq \frac{1-\alpha}{2}$, then the sum of the regularities of $\nabla u$ and $V$ is negative ($\theta - 1 + \beta < \beta + \alpha - 1 + \beta \leq 0$), and therefore the resonant product $\nabla u \odot V$ is ill defined. To overcome this problem, we use the paracontrolled ansatz
\begin{equation}
u = \nabla u \odot J^{T}(V) + u^{\sharp}, \tag{3.5}
\end{equation}
where the paraproduct is defined as $\nabla u \odot J^{T}(V) = \sum_{j=1}^{d} \partial_{j} u \odot J^{T}V_{j}$, and where $u^{\sharp}$ will be more regular than $u$.

Remark 3.2. The intuition behind the paracontrolled ansatz is as follows. Assume that we found a solution $u \in C_{T}^{\beta/\alpha}L^{\infty} \cap C_{T}^{\theta}$ for $\theta = \beta + \alpha - \varepsilon$ for some (small) $\varepsilon > 0$, and that we can make sense of the resonant product $\nabla u \odot V$ in such a way that it has its natural regularity $C_{T}^{\beta+\theta-1}$, despite the fact that $\beta + \theta - 1 \leq 0$. Then we would get that
\[ u^{\sharp} = u - \nabla u \odot J^{T}(V) = P_{T-\cdot}u^{T} - J^{T}(\nabla u \odot V) + J^{T}(\nabla u \odot V) + (J^{T}(\nabla u \odot V) - \nabla u \odot J^{T}(V)) \]
is more regular than $u$ (in fact $2\theta - 1$ regular in space, if $u^{T} \in C_{T}^{\beta+\theta-1}$ and $f \in C_{T}^{\beta\varepsilon}$ for $\varepsilon > 0$) by Schauder estimates for the first four terms and by the commutator estimate from Lemma 3.7 below for last term on the right hand side. This explains why the paracontrolled ansatz might be justified. The reason why the ansatz is useful is that it isolates the singular part of $u$ in a paraproduct, and then we can use commutator estimates to handle the paraproduct.

Therefore, we have to show that assuming the paracontrolled ansatz we can make sense of the product $\nabla u \odot V$ (by moreover postulating the existence of certain extrinsically given resonant products of $V$) and that the paracontrolled ansatz is stable under the Banach fixed point map. To make this precise, we need to define the Banach space of paracorrented distributions. From now on we fix $\alpha \in (1,2]$ and $\beta \in (\frac{2-2\alpha}{3},0)$ and we define paracorrented distributions as follows:
Definition 3.3. Let $T > 0$ let $V \in C_T^\beta \mathbb{R}^d$ be fixed. For $\theta \in ((2 - \beta)/2, \beta + \alpha)$ and $T \in (0, T]$, we define the space of paracontrolled distributions $\mathcal{D}_T^\theta = \mathcal{D}_T^{\theta}(V)$ as the set of tuples $(u, u') \in (C_{T,T}^{\beta} \cap C_{T,T}^{\beta/\alpha} L^\infty) \times C_{T,T}^{\beta-1}$ such that $u^T := u - u' \otimes J^T(V) \in C_{T,T}^{\theta} \mathbb{R}^{2d-1}$. We define a norm on $\mathcal{D}_T^\theta$ by setting

$$
\|(u, u') - (v, v')\|_{\mathcal{D}_T^\theta} := \|u - v\|_{C_{T,T}^{\theta} L^\infty} + \|u - v\|_{C_{T,T}^{\beta/\alpha} L^\infty} + \|u' - v'\|_{C_{T,T}^{\beta-1} \mathbb{R}^{2d-1}}.
$$

Then, $(\mathcal{D}_T^\theta, \|\cdot\|_{\mathcal{D}_T^\theta})$ is a Banach space. If moreover $W \in C_T^\beta \mathbb{R}^d$ and $(v, v') \in \mathcal{D}_T^\theta(W)$, then we use the same notation $\|(u, u') - (v, v')\|_{\mathcal{D}_T^\theta}$ or $\|u - v\|_{\mathcal{D}_T^\theta}$ with the same definition, despite the fact that $(u, u')$ and $(v, v')$ do not live in the same space.

Remark 3.4. In contrast to the definition of Cannizzaro and Chouk, we included the norm $\|u\|_{C_T^{\beta/\alpha} L^\infty}$ instead of $\|\nabla u\|_{C_T^{(\beta-1)/\alpha} L^\infty}$, as it will be easier to show continuity of the solution map w.r.t. the $C_T^{\beta/\alpha} L^\infty$-norm, which will be needed below. Moreover, our space of paracontrolled distributions does not depend on the right hand side $f$.

If we assume that $u$ is paracontrolled, then we can make sense of the problematic term $\nabla u \otimes V$, despite the fact that $u$ has insufficient regularity: We have

$$
\partial_i u \otimes V^j = \sum_{i=1}^d (u'^i \otimes J^T(\partial_j V^i)) \otimes V^j + U^{i,j} \otimes V^j
$$

where we define

$$
\mathcal{R}(f, g, h) := (f \otimes g) \circ h - f(g \circ h)
$$

and $U^{i,j} = \partial_i u' + \sum_{i=1}^d \partial_j u'^i \otimes J^T(V^i) \in C_T^\beta \mathbb{R}^{2d-2}$. By the commutator lemma [GIP14] Lemma 2.4 the term $\mathcal{R}(u'^i, J^T(\partial_j V^i), V^j)$ is well defined and in $C_T^\beta \mathbb{R}^{2d-2+\beta}$. The term $J^T(\partial_j V^i) \otimes V^j$ is still ill defined, but it only depends on $V$. So let us assume that we are extrinsically given for all $i$ and $j$ the resonant products $J^T(\partial_j V^i) \otimes V^j \in C_T^\beta \mathbb{R}^{2+\alpha-1}$. Then the product $u'^i (J^T(\partial_j V^i) \otimes V^j)$ is well defined since $\theta - 1 + 2\beta + \alpha - 1 > 0$ and thus the product $\nabla u \otimes V = \sum_{j=1}^d \partial_j u \otimes V^j$ is well defined in $C_T^\beta \mathbb{R}^{2d-2+\beta}$.

This discussion motivates the following definition:

Definition 3.5. (Enhanced drift)

Let $\beta \in ((2 - 2\alpha)/3, 0)$ and $T > 0$. For $\beta \in ((2 - 2\alpha)/3, 0)$ we define the space $\mathcal{X}^\beta := C_T^\beta \mathbb{R}^d$. For $\beta \in ((2 - 2\alpha)/3, 1/2]$ we define $\mathcal{X}^\beta$ as the closure of

$$
\{K(\eta) := (\eta, (J^T(\partial_j \eta^i) \otimes \eta^j)_{i,j=1,\ldots,d}) \in C_T C_0^\infty(\mathbb{R}^d, \mathbb{R}^d) : \eta \in C_T C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)\}
$$

in $C_T^\beta \mathbb{R}^d \times C_T^{\beta+\alpha-1} \mathbb{R}^d$. In that case we will also denote the elements of $\mathcal{X}^\beta$ by $\mathcal{Y} = (\mathcal{Y}_1, \mathcal{Y}_2)$, and we say that $\mathcal{Y}$ is a lift or an enhancement of $V$ if $\mathcal{Y}_1 = V$. 
Proposition 3.6. Let $T > 0$, $\mathbf{T} \in (0, T)$ and $(2 - \beta)/2 < \theta < \beta + \alpha$ and $\beta \in (\frac{2 - 2\alpha}{3}, \frac{1 - \alpha}{2}]$. For $(u, u') \in \mathcal{D}_T^\theta$ and $\mathcal{Y} = (\mathcal{Y}_1, \mathcal{Y}_2) \in \mathcal{X}^\beta$, we define

$$\nabla u \circ \mathcal{Y} := \sum_{i,j=1}^{d} u^{i,j} \mathcal{Y}_2^{i,j} + \sum_{i,j=1}^{d} \mathcal{R}(u^{i,j}, J^T(\partial_j \mathcal{Y}_1^i), \mathcal{Y}_1^j) + \sum_{j=1}^{d} U^{2,j} \circ \mathcal{Y}_1^j.$$

Here, the commutator $\mathcal{R}$ is as in (3.7) and $U^{1,j} = \partial_i u^i + \sum_{i=1}^{d} \partial_j u^{i,i} \otimes J^T(\mathcal{Y}_1^i) \in C_{\mathbf{T}, T}^{2\beta - 2}$.

Then, the map $\mathcal{D}_T^\theta \ni (u, u') \mapsto \nabla u \circ \mathcal{Y} \in C_{\mathbf{T}, T}^{2\beta + \alpha - 1}$ is Lipschitz continuous, more precisely

$$\|\nabla u \circ \mathcal{Y} - \nabla u \circ \mathcal{Y}'\|_{C_{\mathbf{T}, T}^{2\beta + \alpha - 1}} \lesssim \|\mathcal{Y}' - \mathcal{Y}\|_{\mathcal{X}^\beta}(1 + \|\mathcal{Y}\|_{\mathcal{X}^\beta})\|(u, u') - (v, v')\|_{\mathcal{D}_T^\theta}. \quad (3.8)$$

Moreover, the product $\nabla u \cdot \mathcal{Y} := \nabla u \circ \mathcal{Y}_1 + \nabla u \circ \mathcal{Y}_1 + \nabla u \circ \mathcal{Y}$, where $\nabla u \circ \mathcal{Y}$ is defined as above, is well defined in $C_{\mathbf{T}}^{2\beta}$.

Proof. The products $u^{i,j} \mathcal{Y}_2^{i,j}$ are well defined because the sum of the regularities is $\theta - 1 + 2\beta + \alpha - 1 > -1 + \frac{2}{3} \beta + \alpha > 0$. The commutators $\mathcal{R}(u^{i,j}, J^T(\partial_j \mathcal{Y}_1^i), \mathcal{Y}_1^j)$ are well defined because the sum of the regularities is $\theta - 1 + \beta + \alpha - 1 + \beta > -1 + \frac{2}{3} \beta + \alpha > 0$. The resonant products $U^{2,j} \circ \mathcal{Y}_1^j$ are well defined since the sum of the regularities is $2\theta - 2 + \beta > 2 - \beta - 2 + \beta = 0$. \hfill $\square$

We already motivated in Remark 3.2 that we will need the following commutator lemma concerning the action of the $J^T$ operator on the $\otimes$-paraproduct. Its proof can be found in Appendix A.1.

Lemma 3.7. Let $T > 0$, $0 < \sigma < 1$, $\varepsilon \in \mathbb{R}$ with $-1 \leqslant \sigma - \varepsilon + 1 < \alpha$ and $h \in C_{\mathbf{T}}^{2\sigma}$. For $\mathbf{T} \in (0, T]$, let $g \in C_{\mathbf{T}, T}^{2\sigma} \cap C_{\mathbf{T}, T}^{2\sigma}$. Then the following inequality holds

$$\|J^T(g \otimes h) - g \otimes J^T(h)\|_{C_{\mathbf{T}, T}^{2\sigma + 1}} \lesssim T^\kappa\|g\|_{C_{\mathbf{T}, T}^{2\sigma}} + \|h\|_{C_{\mathbf{T}, T}^{2\sigma}}\|h\|_{C_{\mathbf{T}, T}^{2\sigma}}$$

where $\kappa = 1 - \frac{\sigma + 1 - \varepsilon}{\alpha} > 0$.

For fixed $\mathcal{Y} \in \mathcal{X}^\beta$ and $f \in C_{\mathbf{T}}^{2\sigma}$, $\varepsilon > 0$, or $f = \mathcal{Y}^j_1$ for some $j$, the contraction mapping will now be defined as

$$\Phi_{\mathbf{T}, T} : \mathcal{D}_T^\theta \rightarrow \mathcal{D}_T^\theta; \quad (u, u') \mapsto (v, v'),$$

where

$$v := -J^T(f) + J^T(\nabla u \cdot \mathcal{Y}) + \psi^T$$

for $\psi^T = P_{T-1}u^T$ and

$$v' := \begin{cases} \nabla u & \text{if } f \in C_{\mathbf{T}}^{2\sigma} \\ \nabla u - e_j & \text{if } f = \mathcal{Y}^j_1 \end{cases},$$

where $(e_j)$ is the canonical basis of $\mathbb{R}^d$.

Theorem 3.8. Let $T > 0$, $\beta \in (\frac{2 - 2\alpha}{3}, \frac{1 - \alpha}{2}]$ and $\theta \in ((2 - \beta)/2, \beta + \alpha)$. Let $u^T \in \mathcal{E}^{2\beta - 1}$, $f \in C_{\mathbf{T}}^{2\sigma}$ or $f = \mathcal{Y}^j_1$ for some $j$ and let $\mathcal{Y} \in \mathcal{X}^\beta$. Then there exists a unique fixed point of the map $\Phi_{\mathbf{T}, T}$ in $\mathcal{D}_T^\theta$, that is, a unique (mild) solution of the Kolmogorov backward equation

$$\mathcal{G}(\mathcal{Y}_1, \mathcal{Y}_2) = f, \quad u(T, \cdot) = u^T, \quad (3.10)$$

where $\mathcal{G}(\mathcal{Y}_1, \mathcal{Y}_2) := \partial_t + J^T + \mathcal{Y} \cdot \nabla$. Moreover, for $V \in C_{\mathbf{T}}(C_b^{\infty})$ and $f \in C_{\mathbf{T}}C_b$ the solution $u$ of $\mathcal{G}(V, \mathcal{K}(V)) = f$, $u(T, \cdot) = u^T$, agrees with the classical solution of the PDE.
Proof. We first consider \( f \in C_T \mathcal{E}^\varepsilon \) for \( \varepsilon > 0 \), and we show that for \( T \in (0, T] \) and for \((u, u') \in \mathcal{D}_{T,T}^\theta \), we have \( \Phi^T,T(u, u') \in \mathcal{D}_{T,T}^\theta \), and that there exists \( \kappa > 0 \), depending only on \( \theta \) and \( \beta \), such that

\[
\| \Phi^T,T(u, u') - \Phi^T,T(\tilde{u}, \tilde{u}') \|_{\mathcal{D}_{T,T}^\theta} \lesssim (1 + ||Y||_{X^{\beta}} ||Y||_{X^{\beta}} T^\kappa \| (u, u') - (\tilde{u}, \tilde{u}') \|_{\mathcal{D}_{T,T}^\theta}),
\]

so in particular that \( \Phi^T,T \) is a strict contraction for sufficiently small \( T \). By linearity of \( \Phi^T,T \) it suffices to estimate \( \| \Phi^T,T(u, u') \|_{\mathcal{D}_{T,T}^\theta} \).

So let \( \Phi^T,T(u, u') = (v, v') \). We need to bound the norms

\[
\|v\|_{C_{T,T}^{\beta,\varepsilon}}, \quad \|v\|_{C_{T,T}^{\beta,\varepsilon}L^\infty}, \quad \|u'\|_{C_{T,T}^{\beta,\varepsilon}L^\infty}, \quad \|\Phi^T,T(u, u')\|_{C_{T,T}^{\beta,\varepsilon}L^{2^\beta - 1}},
\]

where \( \Phi^T,T(u, u')^2 = v - v' \circ J^T(\gamma_1) = v - \nabla u \circ J^T(\gamma_1) \). We only show the estimate for \( \|v\|_{C_{T,T}^{\beta,\varepsilon}L^\infty} \). The other terms can be estimated using the same arguments as in [CC18, Proposition 3.9], the only difference is that we use the Schauder estimates for \(-\mathcal{L}^\alpha_D\) instead of those for the Laplacian. For \( T - T \leq r < t \leq T \) we have by (2.7) and (2.9)

\[
\|v(t) - v(r)\|_{L^\infty} \lesssim \|v(t) - v_T\|_{L^\infty} + \|J^T(\nabla u \cdot \gamma)(t) - J^T(\nabla u \cdot \gamma)(r)\|_{L^\infty} + \|\gamma(t) - \gamma(r)\|_{L^\infty} + \|f(t) - f(r)\|_{L^\infty} \lesssim T^{\kappa_1} |t - r|^{\beta + \alpha} \|u\|_{\mathcal{D}_{T,T}^{\beta,\varepsilon}} + \|f\|_{\mathcal{D}_{T,T}^{\beta,\varepsilon}},
\]

where \( \kappa_1 = \frac{\beta + \alpha}{\alpha} \) and where we used the estimate for the product \( \nabla u \cdot \gamma = \nabla u \circ \gamma_1 + \nabla u \circ \gamma_1 \). And on \( [0, T] \) we have by (3.11) \( \|v\|_{C_{T,T}^{\beta,\varepsilon}L^\infty} \leq \|v(t)\|_{C_{T,T}^{\beta,\varepsilon}L^\infty} + \|v(0)\|_{C_{T,T}^{\beta,\varepsilon}L^\infty} + \|\gamma(0)\|_{L^\infty} + \|f(0)\|_{L^\infty} + \|f(0)\|_{L^\infty} \lesssim T^{\kappa_2} \|v(0)\|_{C_{T,T}^{\beta,\varepsilon}L^\infty} + \|f(0)\|_{C_{T,T}^{\beta,\varepsilon}L^\infty} \|u\|_{\mathcal{D}_{T,T}^{\beta,\varepsilon}} \).}

To obtain a mild solution \( u \) on \([0, T]\), we solve the equation iteratively first on \( [T - T, T] \) with terminal condition \( u^T \in \mathcal{E}^{2^\beta - 1} \), then on \( [T - 2T, T - T] \), and so on. There is a small subtlety because also on \( [T - 2T, T - T] \) we will consider solutions that are paracontrolled by \( J^T(V) \) and not by \( J^T(V) \). Moreover, the terminal condition \( u(T - T, \cdot) \) is only in \( \mathcal{E}^\theta \) and not in \( \mathcal{E}^{2^\beta - 1} \). But we only needed \( u^T \in \mathcal{E}^{2^\beta - 1} \) in order to obtain a regular terminal condition \( u^T(T) \) in \( \mathcal{E}^{2^\beta - 1} \). And on the interval \( [T - 2T, T - T] \) we have the terminal condition \( u(T - T) - \nabla u(T - T) \circ \mathcal{J}^T V(T - T) \), which is in \( \mathcal{E}^{2^\beta - 1} \) since \( u \) is paracontrolled on \([T - T, T]\). By iterating this, we obtain a unique fixed point of the map \( \Phi^T,T \) and thus the unique paracontrolled solution of the equation on \([0, T]\).

The case \( f = \gamma_2^j \) for some \( j \in \{1, \ldots, d\} \) is similar and we omit the argument.

In the case of \( V \in C_T(C^{\infty}_b)^d \) and \( f \in C_T C_b \) the solution \( u \) of \( \mathcal{G}^{(V, X(V))} u = f, u(T, \cdot) = u^T \) agrees with the classical solution of the PDE, as the product \( X(V) = V \circ J^T(\nabla V) \) is well-defined (in any Besov space with positive regularity) and the product \( \gamma \cdot \nabla u \) agrees with the usual product \( V \cdot \nabla u \) by the derivation in (3.9). \( \Box \)

**Remark 3.9.** If \( u \in C_T^{\theta / \alpha} L^\infty \cap C_T \mathcal{E}^\theta \), then \( \nabla u \in C_T^{(\theta - 1) / \alpha} L^\infty \). Indeed, we estimate the Littlewood-Paley blocks in two different ways, once using the time regularity of \( u \) and then the space regularity to interpolate between the two bounds. That is, we have

\[
\|\Delta_j(u_t - u_s)\|_{L^\infty} \lesssim |t - s|^{\beta / \alpha} \|u\|_{C_T^{\theta / \alpha} L^\infty} \wedge 2^{-j\theta} \|u\|_{C_T \mathcal{E}^\theta}
\]
and thus for $|t-s| \leq 1$

$$\| \nabla u_t - \nabla u_s \|_{L^\infty} \lesssim \sum_j \| \Delta_j (\nabla u_t - \nabla u_s) \|_{L^\infty} \lesssim \sum_j 2^j \| \Delta_j (u_t - u_s) \|_{L^\infty} \lesssim \sum_{j: 2^{-j} \geq |t-s|^{1/\alpha}} |t-s|^{-j/\alpha} 2^j + \sum_{j: 2^{-j} < |t-s|^{1/\alpha}} 2^{-j(\theta-1)} \lesssim |t-s|^{-\theta/\alpha - 1/\alpha} + |t-s|^{(\theta-1)/\alpha} = 2|t-s|^{(\theta-1)/\alpha},$$

using that $\theta > 1$ for the convergence of the geometric series in the estimate of the second summand.

**Theorem 3.10.** In the setting of Theorem 3.8, the solution map

$$(u^T, \gamma, f) \in \mathcal{E}^{2\theta-1} \times \mathcal{X}^\beta \times (C_T^\theta \mathcal{E}^\epsilon \cup \{ \mathcal{Y}_1^1, \ldots, \mathcal{Y}_1^n \}) \mapsto u \in C_T^\theta \mathcal{E}^\epsilon \cap C_T^{\theta/\alpha} L^\infty,$$

where $u$ is the solution of (3.10) and $C_T^\theta \mathcal{E}^\epsilon \cap C_T^{\theta/\alpha} L^\infty$ is equipped with the sum of the respective norms, is locally Lipschitz continuous.

**Proof.** We only show the continuity for $f \in C_T^\epsilon \mathcal{E}^\epsilon$, the case $f = \gamma_1^i$ is handled analogously. The continuity of the solution map is a bit subtle, because the space $\mathcal{D}_T^\theta \mathcal{E}^\epsilon(V)$ depends on $V$.

Let $u$ be the solution of the PDE for $\gamma \in \mathcal{X}^\beta$, $f \in C_T^\epsilon \mathcal{E}^\epsilon$ and $u^T \in \mathcal{E}^{2\theta-1}$ and $V$ the solution corresponding to the data $\gamma \in \mathcal{X}^\beta$, $g \in C_T^\theta \mathcal{E}^\epsilon$ and $vT \in \mathcal{E}^{2\theta-1}$. By the fixed point property we have $\Phi^T_T(u, u') = (u, u')$ and $\Phi^T_T(v, v') = (v, v')$. We want to estimate $\|u - v\|_{\mathcal{D}_T^\theta \mathcal{E}^\epsilon}$. For that purpose we estimate using the definition of the product from Proposition 3.6 and rebracketing like $ab - cd = a(b - d) + (a - c)d$.

$$\| \nabla u \cdot \gamma - \nabla v \cdot \gamma \|_{\mathcal{X}^\beta} \lesssim (1 + \| \gamma \|_{\mathcal{X}^\beta} + \| u \|_{\mathcal{D}_T^\theta \mathcal{E}^\epsilon}) (1 + \| u \|_{\mathcal{D}_T^\theta \mathcal{E}^\epsilon}) (\| u - v \|_{\mathcal{D}_T^\theta \mathcal{E}^\epsilon} + \| \gamma - \gamma \|_{\mathcal{X}^\beta}).$$

Since the solution $u$ can be bounded in terms of $u^T, f, \gamma$ by Gronwall’s inequality for locally finite measures (cf. [EKS86, Appendix, Theorem 5.1]), and similarly for $v$, we conclude that

$$\| \nabla u \cdot \gamma - \nabla v \cdot \gamma \|_{\mathcal{X}^\beta} \lesssim C(\| u \|_{\mathcal{D}_T^\theta \mathcal{E}^\epsilon}, \| u^T \|_{\mathcal{D}_T^\theta \mathcal{E}^\epsilon}, \| f \|_{\mathcal{D}_T^\theta \mathcal{E}^\epsilon}, \| g \|_{\mathcal{D}_T^\theta \mathcal{E}^\epsilon}) (\| u - v \|_{\mathcal{D}_T^\theta \mathcal{E}^\epsilon} + \| \gamma - \gamma \|_{\mathcal{X}^\beta}),$$

where $C = C(\| u \|_{\mathcal{D}_T^\theta \mathcal{E}^\epsilon}, \| u^T \|_{\mathcal{D}_T^\theta \mathcal{E}^\epsilon}, \| f \|_{\mathcal{D}_T^\theta \mathcal{E}^\epsilon}, \| g \|_{\mathcal{D}_T^\theta \mathcal{E}^\epsilon})$ is a constant, that depends on the norms of the input data on $[0, T]$. Therefore, we obtain from Lemma 2.7 together with the Schauder estimates Corollary 2.8 and Remark 3.9 (recall that $u' = \nabla u$ and $v' = \nabla v$):

$$\| u - v \|_{C_T^\theta \mathcal{E}^\epsilon} + \| u - v \|_{C_T^{\theta/\alpha} L^\infty} + \| u' - v' \|_{C_T^{\theta/\alpha - 1} \mathcal{E}^\epsilon} \lesssim \| u^T - v^T \|_{\mathcal{D}_T^\theta \mathcal{E}^\epsilon} + \| f - g \|_{C_T^\theta \mathcal{E}^\epsilon} + \| J^T (\nabla u \cdot \gamma - \nabla v \cdot \gamma) \|_{C_T^\theta \mathcal{E}^\epsilon \cap C_T^{\theta/\alpha} L^\infty} \lesssim \| u^T - v^T \|_{\mathcal{D}_T^\theta \mathcal{E}^\epsilon} + \| f - g \|_{C_T^\theta \mathcal{E}^\epsilon} + T^\kappa_1 C(\| u - v \|_{\mathcal{D}_T^\theta \mathcal{E}^\epsilon} + \| \gamma - \gamma \|_{\mathcal{X}^\beta}),$$

where $\kappa_1$ is a constant depending on the norms of the input data on $[0, T]$. Therefore, we conclude that

$$\| u - v \|_{C_T^\theta \mathcal{E}^\epsilon} + \| u - v \|_{C_T^{\theta/\alpha} L^\infty} + \| u' - v' \|_{C_T^{\theta/\alpha - 1} \mathcal{E}^\epsilon} \lesssim \| u^T - v^T \|_{\mathcal{D}_T^\theta \mathcal{E}^\epsilon} + \| f - g \|_{C_T^\theta \mathcal{E}^\epsilon} + \| J^T (\nabla u \cdot \gamma - \nabla v \cdot \gamma) \|_{C_T^\theta \mathcal{E}^\epsilon \cap C_T^{\theta/\alpha} L^\infty} \lesssim \| u^T - v^T \|_{\mathcal{D}_T^\theta \mathcal{E}^\epsilon} + \| u - v \|_{C_T^\theta \mathcal{E}^\epsilon} + \| \gamma - \gamma \|_{\mathcal{X}^\beta} + \| \gamma - \gamma \|_{\mathcal{X}^\beta},$$

which is the desired estimate.
where \( \kappa_1 = \frac{\beta + \alpha - \theta}{\alpha} > 0 \) and where the \( C_T^{\beta/\alpha} L^\infty \)-norm of \( u - v \) is estimated using the fixed point and an estimate as in (3.12). Moreover, using the fixed point property and analogue estimates for the term \( \| \Phi^{T,T}(u, u') \|_{C_T^{\beta/\alpha}} \) as in the proof of [CC18, Proposition 3.9] using Lemma 3.7 we obtain

\[
\| u^T - v^T \|_{C_T^{\beta/\alpha}} \lesssim \| u^T - v^T \|_{\mathcal{D}^{\beta/\alpha}} + \| f - g \|_{C_T^{\beta/\alpha}} + (1 + C) \| \nu - \nu' \|_{\mathcal{D}^{\beta/\alpha}} + (T^{2\kappa_2} \wedge T^{\kappa_1}) C \| u - v \|_{\mathcal{D}^{\beta/\alpha}}.
\]

where \( \kappa_2 = \frac{2(\alpha + \beta - \theta)}{\alpha} > 0 \) and \( C > 0 \) is again a (possibly different) constant depending on the norms of the input data. So overall

\[
\| u - v \|_{\mathcal{D}^{\beta/\alpha}} \lesssim \| u^T - v^T \|_{\mathcal{D}^{\beta/\alpha}} + \| f - g \|_{C_T^{\beta/\alpha}} + (1 + C) \| \nu - \nu' \|_{\mathcal{D}^{\beta/\alpha}} + (T^{\kappa_1} \wedge T^{\kappa_2}) C \| u - v \|_{\mathcal{D}^{\beta/\alpha}}.
\]

Assume for the moment that \( T \) is small enough so that \( T^{\kappa_1} \wedge T^{\kappa_2} \) times the implicit constant on the right hand side is \( < 1 \). Then we can take the last term to the other side and divide by a positive factor, obtaining

\[
\| u - v \|_{\mathcal{D}^{\beta/\alpha}} \lesssim \tilde{C} (\| u^T - v^T \|_{\mathcal{D}^{\beta/\alpha}} + \| f - g \|_{C_T^{\beta/\alpha}} + \| \nu - \nu' \|_{\mathcal{D}^{\beta/\alpha}}),
\]

where \( \tilde{C} > 0 \) is a constant that depends on the input data. Thus, the map \( (u^T, f, \nu') \mapsto (u, u^T) \subset C_T^{\beta/\alpha} \cap C_T^{\beta/\alpha} \) is locally Lipschitz continuous, which implies that the solution map is continuous with values in \( C_T^{\beta/\alpha} \).

If \( T \) is such that \( T^{\kappa_1} \wedge T^{\kappa_2} \) times the implicit constant is \( \geq 1 \), then we apply the same estimates for \( \mathcal{D}^{\beta/\alpha}_{T,T} \), where \( T \) is small enough, and then bound \( \| u - v \|_{C_T^{\beta/\alpha}} \leq \sum_{i=1}^n \| u - v \|_{C_T^{\beta/\alpha}} \), where \( n \) is the smallest integer such that \( T - nT \leq 0 \). The same argument also works for the \( C_T^{\beta/\alpha} L^\infty \)-norm with \( \| u - v \|_{C_T^{\beta/\alpha} L^\infty} \lesssim \sum_{i=1}^n \| u - v \|_{C_T^{\beta/\alpha}} \wedge T^{\beta/\alpha} \) for the chosen \( n \) and we obtain also local Lipschitz continuity of the solution map w.r.t. this norm. \( \square \)

4. Existence and uniqueness for the martingale problem

Recall the definition of \( \mathcal{X}^{\beta} \) from Definition 3.5. For \( \beta \in (\frac{1}{2\alpha}, 0) \) we have \( \mathcal{X}^{\beta} = C_T^{\beta} \mathcal{E}^{\beta}_{\mathbb{R}^d} \), while for \( \beta \in (\frac{2-2\alpha}{\alpha}, \frac{1}{2}) \) the space \( \mathcal{X}^{\beta} \) is the closure of

\[
\{ K(\eta) := (\eta, (J^T (\partial_j \eta^i) \circ \eta^i)_{i,j=1,\ldots,d}) : \eta \in C_T C^\infty_\beta (\mathbb{R}^d, \mathbb{R}^d) \}.
\]

in \( C_T^{\beta} \mathcal{E}^{\beta}_{\mathbb{R}^d} \times C_T^{\beta/\alpha} \mathcal{E}^{\beta/\alpha}_{\mathbb{R}^d} \). For \( V \in \mathcal{X}^{\beta} \) we define solutions to the SDE

\[
dx_t = V(t, x_t) dt + dL_t, \quad x_0 = x \in \mathbb{R}^d,
\]
as solutions to the corresponding martingale problem.

We consider the Skorokhod space \( (\Omega, \mathcal{F}) := (D([0, T], \mathbb{R}^d), \mathcal{B}(D([0, T], \mathbb{R}^d))) \) with canonical filtration \((\mathcal{F}_t)_{t \geq 0}\), i.e. \( \mathcal{F}_t = \sigma(X_s : s \leq t) \) where \((X_t)_{t \geq 0} \) is the canonical process with \( X_t = \omega(t) \) for \( \omega \in \Omega \).

Definition 4.1. (Martingale Problem)

Let \( \alpha \in (1, 2) \) and \( \beta \in (\frac{2-2\alpha}{\alpha}, 0) \), and let \( T > 0 \) and \( V \in \mathcal{X}^{\beta} \). Then, we call a probability measure \( P \) on the Skorokhod space \((\Omega, \mathcal{F})\) a solution of the martingale problem for \((\mathcal{F}^V, \delta_x)\), if

1.) \( P(X_0 = x) = 1 \) (i.e. \( P^{X_0} = \delta_x \), and
2.) for all \( f \in C_T^\infty \) with \( \varepsilon > 0 \) and for all \( u^T \in \mathcal{C}^3 \), the process \( M = (M_t)_{t \in [0,T]} \) is a martingale under \( P \) with respect to \( (\mathcal{F}_t) \), where
\[
M_t = u(t, X_t) - u(0, x) - \int_0^t f(s, X_s)ds
\]  
(4.1)
and where \( u \) solves the Kolmogorov backward equation \( \mathcal{G}^V u = f \) with terminal condition \( u(T, \cdot) = u^T \).

This is a generalization of the classical notion of a weak solution, in the sense that if \( V^n \) is a bounded and measurable function, then \((X^n_t)_{t \in [0,T]} \) is a weak solution to
\[
dX^n_t = V^n(t, X^n_t)dt + dL_t, \quad X^n_0 = x,
\]  
(4.2)
if and only if it solves the martingale problem of Definition 1.1.

Our main result is:

**Theorem 4.2.** Let \( \alpha \in (1,2] \) and \( L \) be a symmetric, \( \alpha \)-stable Lévy process, such that the measure \( \nu \) satisfies Assumption 2.4. Let \( T > 0 \) and \( \beta \in ((2 - 2\alpha)/3, 0) \) and let \( V \in \mathcal{C}^\beta \) be as in Definition 3.5. Then for all \( x \) problem for \( \hat{C} > 0 \) we have uniformly in \( 0 \)
\[
\text{E} \left[ \left( \int_0^t \int_{|y| \leq C} |y|^2 \pi(ds, dy) \right)^n \right] \lesssim \sum_{\omega \in \mathbb{N}^n : |\omega| = n} \prod_{i=1}^n \left( t - r \right) \int_{|y| \leq C} |y|^{2\alpha \mu(dy)} \omega_i
\]
We give the proof of the theorem, we first establish some auxiliary results.

**Lemma 4.3.** Let \( \alpha \in (1,2) \) and let \( \pi \) be the Poisson random measure of the \( \alpha \)-stable Lévy process \( L \). We define for a multi-index \( \omega \in \mathbb{N}_0^n \) with \( n \in \mathbb{N} \):
\[
|\omega| := \omega_1 + 2\omega_2 + \cdots + n\omega_n.
\]
Then we have for all \( C > 0 \) and \( t > r \):
\[
\text{E} \left[ \left( \int_r^t \int_{|y| \leq C} |y|^2 \pi(ds, dy) \right)^n \right] \lesssim \sum_{\omega \in \mathbb{N}^n : |\omega| = n} \prod_{i=1}^n \left( t - r \right) \int_{|y| \leq C} |y|^{2\alpha \mu(dy)} \omega_i
\]
We give the proof of the lemma, we first establish some auxiliary results.

**Lemma 4.4.** Let \( \alpha \in (1,2) \), \( \theta \in (1, \alpha) \) and \( u \in C_T^\infty \cap C_T^{\beta/\alpha} \), \( L^\infty \), and let \( \rho \in 2\mathbb{N} \). Let moreover \( \hat{\pi} \) be the compensated Poisson random measure of the \( \alpha \)-stable Lévy process \( L \). Then we have uniformly in \( 0 \leq r \leq t \leq T \):
\[
\text{E} \left[ \left( \int_r^t \int_{\mathbb{R}^d} ((u(t, X_s + y) - u(t, X_s - y)) - u(s, X_{s+} + y) - u(s, X_{s-})) \hat{\pi}(ds, dy) \right)^\rho \right] \lesssim |t - r|^\rho/\rho.
\]

**Proof.** To abbreviate the notation we write \( \Delta_y u(s, x) := u(s, x + y) - u(s, x) \). By the Burkholder-Davis-Gundy inequality together with [PZ07] Lemma 8.21 we get for any \( \rho \geq 1 \) and for \( C > 0 \) to be chosen later
\[
\text{E} \left[ \left( \int_r^t \int_{\mathbb{R}^d} (\Delta_y u(t, X_{s-}) - \Delta_y u(s, X_{s-}) \hat{\pi}(ds, dy) \right)^\rho \right] \lesssim \text{E} \left[ \left( \int_r^t \int_{\mathbb{R}^d} (\Delta_y u(t, X_{s-}) - \Delta_y u(s, X_{s-}))^2 \pi(ds, dy) \right)^{\rho/2} \right]
\]
(4.3)
where we used that 

\[ \|\nabla u\|_{L^\infty} \lesssim |t-r|^{\rho/\alpha} \] 

The integral inside the expectation is a Poisson distributed random variable with parameter \((t-r)\mu\{y : |y| > C\} \simeq (t-r)^{\alpha-\alpha}.\) This motivates the choice \(C = (t-r)^{1/\alpha},\) for which this term is of the claimed order. For the first term on the right hand side of (4.3), we estimate by the mean value theorem and using the time regularity of \(\nabla u\) (cf. also Remark 3.9): 

\[ \mathbb{E}\left[ \left| \int_r^t \int_{|y| > C} \left( \partial_y u(t, X_s) - \partial_y u(s, X_s) \right)^2 \pi(ds, dy) \right|^{\rho/2} \right] \leq |t-r|^{\rho/\alpha} \|\nabla u\|_{L^\infty}^{\rho/\alpha} \mathbb{E}\left[ \left| \int_r^t \int_{|y| < C} \pi(ds, dy) \right|^{\rho/2} \right]. \]

Now by Lemma 4.3 and by the choice \(C = (t-r)^{1/\alpha},\) we obtain 

\[ \mathbb{E}\left[ \left( \int_r^t \int_{|y| < C} |y|^2 \pi(ds, dy) \right)^{\rho/2} \right] \leq \sum_{\omega \in \mathbb{N}_0^d : |\omega| = \rho/2} \prod_{i=1}^{\rho/2} \left( t-r \int_{|y| < C} |y|^{2i} \mu(dy) \right)^{\omega_i} \lesssim |t-r|^{\rho/\alpha}, \]

where we used that \(\int_{|y| < C} |y|^k \mu(dy) \simeq C^{k-\alpha}\) for \(k \geq 2.\) Together this yields for any \(\rho \in 2\mathbb{N}\)

\[ \mathbb{E}\left[ \left| \int_r^t \int_{\mathbb{R}^d} \left( \partial_y u(t, X_s) - \partial_y u(s, X_s) \right) \pi(ds, dy) \right|^{\rho} \right] \lesssim |t-r|^{\rho/\alpha} + |t-r|^{\rho(\theta-1)/\alpha} |t-r|^{\rho/\alpha} \lesssim |t-r|^{\rho/\alpha}. \]

\[ \square \]

**Corollary 4.5.** In the setting of Theorem 4.2, let \((V^n)_{n \in \mathbb{N}} \subset C_T C_b^\infty(\mathbb{R}^d, \mathbb{R}^d)\) be a smooth approximation with \((V^n, K(V^n)) \to \mathcal{Y}^\beta)\). Let \((X^n_t)_{t \in [0,T]}\) be the strong solution of the SDE 

\[ dX^n_t = V^n(t, X^n_t) dt + dL_t, \quad X^n_0 = x \in \mathbb{R}^d. \]

Let \(\theta \in (2-\beta)/2, \alpha + \beta)\) and \(\rho \in 2\mathbb{N}.\) Then, we have uniformly in \(n \in \mathbb{N},\) and \(0 \leq r \leq t \leq T:\)

\[ \sup_n \mathbb{E}\left[ \left| \int_r^t V^n(s, X^n_s) ds \right|^{\rho} \right] \lesssim |t-r|^{\theta \rho/\alpha}. \quad (4.4) \]

**Proof.** Let \(t \in (0,T)\) and consider the solution \(u^{n,t} \in C_T C_b^\infty(\mathbb{R}^d, \mathbb{R}^d)\) of the system of equations

\[ aV^n u^{n,t,i} = V^{n,i}, \quad u^{n,t,i}(t,\cdot) = 0, \quad \text{for } i = 1, \ldots, d. \]

For \(\beta \in (2-2\alpha, 1/\alpha)\) this equation is not exactly of the same type as the equation in Theorem 3.10 because we prescribe the terminal condition at time \(t \leq T\) and not in \(T.\) We still use the paracontrolled ansatz \(u^{n,t} = (u^{n,t})^T \in J_T(V^n) + (u^{n,t})^T,\) i.e. we do not replace \(J_T V^n\) by \(J_T^T V^n,\)
because as $n \to \infty$ we only control $\nabla J^TV^n \odot V^n$ but not $\nabla J^TV^n \odot V^n$. This means there is a blowup of $\| (u^{n,t})^2(s) \|_{2\theta-1}$ as $s \to t$. We discuss below how to deal with this singularity, and we will see that

$$\sup_{n \in \mathbb{N}, t \in [0,T]} \| u^{n,t} \|_{C_t^\theta L^\infty_{x,d}} < \infty. \quad (4.5)$$

Let first $\alpha \in (1,2)$. Then we apply Itô’s formula to $u^{n,t}(t, X^n_r) - u^{n,t}(r, X^n_r)$ and we use that $X^n$ solves the SDE with drift $V^n$ and that $\mathcal{G}^V u^n = V^n$ to obtain

$$\int_r^t V^n(s, X^n_s)ds = u^{n,t}(t, X^n_t) - u^{n,t}(r, X^n_r) + \int_r^t \int_{\mathbb{R}^d} \left( u^{n,t}(s, X^n_s + y) - u^{n,t}(s, X^n_s) \right) \hat{\pi}(ds, dy).$$

As $u^{n,t}(t) = 0$ and by (4.5) we obtain

$$|u^{n,t}(t, X^n_t) - u^{n,t}(r, X^n_r)| = |u^{n,t}(t, X^n_t) - u^{n,t}(r, X^n_r)| \leq |t - r|^\theta/\alpha \| u \|_{C_t^\theta L^\infty_{x,d}} \lesssim |t - r|^\theta/\alpha.$$

Using once more that $u^{n,t}(t) = 0$, we obtain from Lemma 4.4

$$\mathbb{E} \left[ \left| \int_r^t \int_{\mathbb{R}^d \setminus \{0\}} \left( u^{n,t}(s, X^n_{s-} + y) - u^{n,t}(s, X^n_{s-}) \right) \hat{\pi}(ds, dy) \right|^\rho \right]$$

$$= \mathbb{E} \left[ \left| \int_r^t \int_{\mathbb{R}^d \setminus \{0\}} \left( (u^{n,t}(s, X^n_{s-} + y) - u^{n,t}(s, X^n_{s-}) - (u^{n,t}(s, X^n_{s-} + y) - u^{n,t}(s, X^n_{s-})) \right) \hat{\pi}(ds, dy) \right|^\rho \right]$$

$$\lesssim |t - r|^\theta/\alpha,$$

so (4.4) holds for $\alpha \in (1,2)$. For $\alpha = 2$ the argument is essentially the same, except much easier: Then we only have to replace the jump martingale $\int_r^t \int_{\mathbb{R}^d \setminus \{0\}} \left( u^{n,t}(s, X^n_{s-} + y) - u^{n,t}(s, X^n_{s-}) \right) \hat{\pi}(ds, dy)$ by $\int_r^t \nabla u^{n,t}(s, X^n_s) dB_s$ and apply the Burkholder-Davis-Gundy inequality.

Therefore, the proof is complete once we show (4.5). For that purpose we introduce the singular spaces

$$\mathcal{M}_t^{\theta,\mathcal{E}^\gamma} := \{ f \in C([0,t], \mathcal{P}^\rho) \mid s \mapsto (t-s)\gamma f(s) \in C_t \mathcal{E}^\gamma \},$$

and we adapt the definition of paraprocontrolled distributions by requiring $u^{n,t} = (u^{n,t})' \odot J^TV^n + (u^{n,t})^2$, $u^{n,t} \in C_t \mathcal{E}^\gamma \cap C_t^{\theta/\alpha} L^\infty$, $(u^{n,t})' \in C_t \mathcal{E}^{\theta}$, $(u^{n,t})^2 \in C_t \mathcal{E}^\gamma \cap \mathcal{M}_t^{\theta-1/\alpha,2\theta-1}$. Since the blow-up $(\theta-1)/\alpha$ is less than 1, we can then use techniques for paraprocontrolled distributions with such singularities (see e.g. [GP17, Section 6]) to see that the paraprocontrolled norm of $u^{n,t}$ is bounded in $n$ and $t$, so in particular (4.5) holds.

Proof of Theorem 4.2. Let $(V^n)_{n \in \mathbb{N}} \subset C_T C^\infty_{\theta,\mathcal{E}^\gamma}(\mathbb{R}^d, \mathbb{R}^d)$ be such that $(V^n, \mathcal{K}(V^n)) \rightarrow \mathcal{V}$ in $\mathcal{Z}^\beta$ and let $X^n$ be the unique strong solution of the SDE

$$dX^n_t = V^n(t, X^n_t) dt + dL_t, \quad X^n_0 = x. \quad (4.6)$$

To prove the existence of a solution to the martingale problem for $(\mathcal{G}V, \delta_x)$ we follow the usual strategy: We show tightness of $(X^n)_{n \in \mathbb{N}}$, and then we show that every limit point solves the martingale problem for $(\mathcal{G}V, \delta_x)$. Then we show that the solution to that martingale problem is unique in law, and therefore $(X^n)$ converges weakly.

Step 1: Tightness of $(\mathbb{P}^X^n)$ on $D([0,T], \mathbb{R}^d)$. We apply (4.4) from Corollary 4.3 for $\rho \in 2\mathbb{N}$ large enough so that $\theta \rho/\alpha > 1$, which shows that the drift term $A^n := \int_0^t V^n(s, X^n_s) ds$ satisfies Kolmogorov’s tightness criterion. Therefore, $(A^n)$ is tight in $C([0,T], \mathbb{R}^d)$ and thus in particular $C$-tight in $D([0,T], \mathbb{R}^d)$ (meaning that every limit
We consider a weakly convergent subsequence, also denoted by \((P^{X_n})\), and we write \(Q\) for its limit. Let \(X\) be the canonical solution of \(D([0, T], \mathbb{R}^d)\) and let \(E_n[\cdot]\) (resp. \(E_Q[\cdot]\)) denote integration w.r.t. \(P^{X_n}\) (resp. \(Q\)). Let \(f \in C_T \mathcal{C}^2\) and \(u^f \in \mathcal{C}^3\), and let \((f^n)_{n \in \mathbb{N}} \subset C_T C_b^\infty\) be such that \(f^n\) converges to \(f\) in \(C_T \mathcal{C}^2\). Let \(u^n\) be the solution of \(\mathcal{G}^{V_n} u^n = f^n\) with terminal condition \(u^n(T, \cdot) = u^T\). Since \(f^n\) and \(V^n\) are smooth we have \(u^n \in C^{1,2}([0, T] \times \mathbb{R}^d)\) and \(u^n\) is a strong solution of the Kolmogorov backward equation. We can thus apply Ito’s formula for càdlàg processes to \(u^n(t, X_t)\) under the measure \(P^{X_n}\) and obtain as the operators \(-L^\alpha\) and \(A\) from [2,4] agree on \(C_b^\infty\) (and in fact \(u^n \in C_b^\infty\)), that in the jump case \(\alpha \in (1, 2)\)

\[
M^n_t := u^n(t, X_t) - u^n(0, x) - \int_0^t f^n(s, X_s) ds \\
= u^n(t, X_t) - u^n(0, x) - \int_0^t \mathcal{G}^{V^n} u^n(s, X_s) ds \\
= \int_0^t \int_{\mathbb{R}^d} \left( u^n(r, X_{r-} + y) - u^n(r, X_{r-}) \right) \hat{\pi}(dr, dy)
\]

is a martingale in the canonical filtration. Indeed, \(M^n\) is a local martingale because it is a stochastic integral against a compensated Poisson random measure, and it is a true martingale because \(u^n(s, X_{r-} + y) - u^n(s, X_{r-})\) is square-integrable w.r.t. \(\mathbb{P} \otimes dr \otimes \mu\), where we use the boundedness of \(u^n\) for the big jump part and the boundedness of \(\nabla u^n\) for the small jump part. In the Brownian case (\(\alpha = 2\)) we have \(M^n = \int_0^t \nabla u^n(s, X^n_s) dB_s\), which is a martingale because \(\nabla u^n\) is bounded.

Let now \(u\) be the solution to \(\mathcal{G}^V u = f\) with terminal condition \(u(T) = u^T\). By the continuity of the solution map, \((u^n)\) converges to \(u\) in the spaces \(C_T \mathcal{C}^\theta\) and \(C_T^{\beta/\alpha} L^\infty\), for \(\theta \in ((2 - \beta)/\beta, \alpha + \beta)\). We show that \((M_t)_{t \in [0, T]}\) is a martingale under \(Q\), where

\[
M_t = u(t, X_t) - u(0, x) - \int_0^t f(s, X_s) ds.
\]

For that purpose let \(0 \leq r \leq t \leq T\) and let \(F : D([0, r], \mathbb{R}^d) \to \mathbb{R}\) be continuous and bounded. Since \(M^n\) is a martingale under \(P^{X_n}\), we have

\[
E_n[(M^n_t - M^n_r) F((X_u)_{u \leq r})] = 0.
\]

We define for \(x \in D := D([0, T], \mathbb{R}^d)\)

\[
M^n_{r,t}(x) := \left( u^n(t, x(t)) - u^n(r, x(r)) - \int_r^t f^n(u, x(u)) du \right),
\]

and \(M_{r,t}(x)\) analogously with \(u^n, f^n\) replaced by \(u, f\). We further define \(M^n_{0,t}(x) := M^n_t(x)\) and \(M_{0,t}(x) := M_t(x)\). We want to let \(n \to \infty\) in (1.8). Therefore, we first note that \(\sup_{x \in D}|M^n_{r,t}(x) - M_{r,t}(x)| \to 0\) for \(n \to \infty\), by the convergence of \((u^n, f^n)\) to \((u, f)\) in \(C_T C_b \times C_T C_b \subset C_T \mathcal{C}^\theta \times C_T \mathcal{C}^\varepsilon\). Thus, we obtain, by boundedness of \(F\), that

\[
\lim_{n \to \infty} E_n[M_{r,t} F((X_u)_{u \leq r})] = 0.
\]

Now, by [JS03 Proposition VI.2.1], we know that the map \(D \ni x \mapsto \int_0^t f(s, x(s)) ds\) is continuous w.r.t. the \(J_1\)-topology and it is bounded by boundedness of \(f\). Moreover, if we know that
\( \mathbb{Q}(\Delta X_t = \Delta X_r = 0) = 1 \), then by [JS03, Proposition VI.3.14] and since \( X^n \to X \) in distribution in \( D \), we have that \( X^n_t \to X_t \) and \( X^n_r \to X_r \) in distribution. Together this gives (as \( \mathbb{R} \ni y \mapsto u(t, y) - u(r, y) \) is continuous and bounded)

\[
0 = \lim_{n \to \infty} \mathbb{E}_n[M_{t,r}F((X_u)_{u \in [0,T]})] = \mathbb{E}_Q[M_{t,r}F((X_u)_{u \in [0,T]})],
\]
and since \( 0 \leq r \leq t \leq T \) and \( F \) were arbitrary, we obtain that \( \mathbb{Q} \) solves the martingale problem for \( (\mathcal{G}^V, \delta_\varepsilon) \). So it remains to show that indeed \( \mathbb{Q}(\Delta X_t = \Delta X_r = 0) = \mathbb{P}(\Delta L_t = \Delta L_r = 0) = 1 \), and this shows that \( \mathbb{Q} \) indeed solves the martingale problem for \( (\mathcal{G}^V, \delta_\varepsilon) \).

**Step 3: Uniqueness for the martingale problem and strong Markov property.**

Let \( \mathbb{Q}_1 \) and \( \mathbb{Q}_2 \) be two solutions of the martingale problem for \( \mathcal{G}^V \) with the same initial distribution \( \mu = \mathbb{Q}_1^0 = \mathbb{Q}_2^0 \). Let \( f \in C_T \mathcal{E}^\varepsilon \) and let \( u \) be the solution of \( \mathcal{G}^V u = f \), \( u(T) = 0 \). Then we obtain for \( i = 1, 2 \),

\[
\int_{\mathbb{R}^d} u(0, x)\mu(dx) = \mathbb{E}_{\mathbb{Q}_i} \left[u(T, X_T) - \int_0^T f(s, X_s)ds\right] = -\mathbb{E}_{\mathbb{Q}_i} \left[\int_0^T f(s, X_s)ds\right].
\]

Thus, we have for all \( f \in C_T \mathcal{E}^\varepsilon \)

\[
\mathbb{E}_{\mathbb{Q}_1} \left[\int_0^T f(s, X_s)ds\right] = \mathbb{E}_{\mathbb{Q}_2} \left[\int_0^T f(s, X_s)ds\right].
\]

Therefore, \( \mathbb{Q}_1^{X_t} = \mathbb{Q}_2^{X_t} \) for all \( t \in [0, T] \), that is, the one dimensional marginal distributions of \( \mathbb{Q}_1 \) and \( \mathbb{Q}_2 \) agree. Indeed, this follows by taking \( f_\delta(s, x) = \delta^{-1} h_\delta(s) g(x) \) for \( h_\delta \simeq 1_{[t, t+\delta]} \) and \( g \in \mathcal{E}^\varepsilon \) and letting \( \delta \to 0 \). Now [EK86, Theorem 4.4.3] shows that \( \mathbb{Q}_1 = \mathbb{Q}_2 \) and that under the solution \( \mathbb{Q} \) to the martingale problem for \( (\mathcal{G}^V, \delta_\varepsilon) \) the canonical process is a strong Markov process. \( \square \)

**5. Brox diffusion with Lévy noise**

The Brox diffusion is the solution \( X \) of the SDE

\[
dX_t = W(X_t)dt + dB_t, \quad X_0 = x \in \mathbb{R}, \quad (5.1)
\]

where \( B \) is a standard Brownian motion and \( (W(x))_{x \in \mathbb{R}} \) is a two-sided standard Brownian motion that is independent of \( B \). This model was introduced by Brox [Bro86] as a continuous analogue of Sinai’s random walk, with the motivation that when studying \( X \) we can exploit the scaling properties of \( W \) and \( B \). Brox’s construction is based on time and space transformations as in the Itô-McKean construction of diffusions. It is natural to replace \( W \) or \( B \) by \( \alpha \)-stable Lévy processes, which also have nice scaling properties. The construction of the process with \( W \) replaced by a Lévy process is not much of a problem, as the Itô-McKean approach still works [Tan87, Car97, KTT17]. On the other hand, replacing \( B \) by an \( \alpha \)-stable Lévy process is
more delicate and it is not obvious if the Ito-McKean construction could work. But using our approach we can hope to solve the martingale problem for the SDE
\[ dX_t = W(X_t)dt + dL_t, \quad X_0 = x \in \mathbb{R}. \quad (5.2) \]
To be precise, the white noise \( W \) is not actually an element of any Besov space, but only of weighted Besov spaces: With \( \langle x \rangle = (1 + |x|^2)^{1/2} \) we have \( \langle \cdot \rangle^{-\alpha} W \in \mathcal{C}^{\alpha-1/2} \) for all \( \alpha > 0 \). It is possible to extend our analysis of the martingale problem to allow for a drift term in a suitable weighted Besov space, and at the end of this section we discuss how this could be done. But to simplify the presentation we consider a periodic white noise \( \dot{X} \).

The Fourier basis of \( J \) remains to construct \( (J^T(\nabla \xi) \circ \xi)(\omega) \in C_T \mathcal{C}^{\alpha-1/2} \) for almost all \( \omega \). It is possible to extend our analysis of the martingale problem to allow for a drift term in a suitable weighted Besov space, and at the end of this section we discuss how this could be done. But to simplify the presentation we consider a periodic white noise \( \dot{X} \).

We choose \( \xi \) independently of the Lévy process \( L \), and we consider a fixed “typical” realization \( \xi(\omega) \). To apply the theory that we developed in this paper, we need to construct a canonical enhancement of \( \xi(\omega)^R \) in such a way that we obtain an enhanced drift in the sense of Definition 3.3.

We first note that almost surely \( \xi \in C_T \mathcal{C}^{\alpha-1/2}(\mathbb{T}) \) (so we let \( \beta = -1/2 - \varepsilon \) for some very small \( \varepsilon > 0 \), see e.g. [GP15, Exercise 11]). Therefore, \( \xi(\omega)^R \in C_T \mathcal{C}^{\alpha-1/2}(\mathbb{T}) \) for almost all \( \omega \). It remains to construct \( (J^T(\nabla \xi) \circ \xi)(\omega) \in C_T \mathcal{C}^{\alpha-1/2}(\mathbb{T}) \) for almost all \( \omega \), which we will do in the next lemma.

**Lemma 5.1.** Let \( \alpha \in (3/2, 2] \), \( \vartheta < \alpha - 2 \) and \( J^T(u)(t) = \int_t^T P_{\tau-t} u(r) dr \) for the semigroup \( (P_t) \) generated by \( \mathcal{L}_\vartheta \), \( P_t \varphi = \mathcal{F}^{-1}(e^{-t\vartheta \xi^n} \mathcal{F} \varphi) \). Let \( \xi^n = \sum_{|k|\leq n} \xi(k) e_k \) where \( (e_k)_{k \in \mathbb{Z}} = (e^{-2\pi ik})_{k \in \mathbb{Z}} \) is the Fourier basis of \( L^2(\mathbb{T}) \). Then \( (J^T(\nabla \xi^n) \circ \xi^n) \) converges in probability in \( C_T \mathcal{C}^{\vartheta}(\mathbb{T}) \) to a limit denoted by \( J^T(\nabla \xi) \circ \xi \in C_T \mathcal{C}^{\vartheta}(\mathbb{T}) \).

**Proof.** We carry out the computations for \( n = \infty \) and show that \( J^T(\nabla \xi) \circ \xi \in C_T \mathcal{C}^{\vartheta}(\mathbb{T}) \) can be constructed as a random variable in the second Wiener-Itô chaos generated by \( \xi \). Since the kernel appearing in the definition of \( J^T(\nabla \xi) \circ \xi \) provides a uniform bound for the kernels that appear in the chaos representation of \( (J^T(\nabla \xi^n) \circ \xi^n) \), the claimed convergence then follows from the dominated convergence theorem.

To bound \( J^T(\nabla \xi) \circ \xi \), note that \( J^T(\nabla \xi)(t) = \varrho_t * \xi \), where \( \varrho_t = \nabla \mathcal{F}^{-1}(\int_t^T e^{-t} \xi^n e \mathcal{F} \varphi) \). We first derive a bound on the expectation of the \( B_{p,p}^\xi \)-norm (for \( \xi \) to be chosen afterwards) of the increment \( (\varrho_t * \xi) \circ \xi - (\varrho_s * \xi) \circ \xi \) for almost all \( \omega \). Using this bound, our claim will follow from the Besov embedding theorem together with Kolmogorov’s continuity criterion. We have

\[
\mathbb{E}[\|((\varrho_t - \varrho_s) * \xi \circ \xi)^p \|_{B_{p,p}^\xi}] = \mathbb{E}\left[ \sum_j 2^{kj} \int_T \mathbb{E}[\|\Delta_j((\varrho_t - \varrho_s) * \xi \circ \xi)(x\|^p) dx \right] \\
= \sum_j 2^{kj} \int_T \mathbb{E}[\|\Delta_j((\varrho_t - \varrho_s) * \xi \circ \xi(x\|^p) dx \\
\lesssim \sum_j 2^{kj} \int_T \mathbb{E}[\|\Delta_j((\varrho_t - \varrho_s) * \xi \circ \xi(x\|^2)^{p/2} dx,
\]

18
where in the last step we used that the random variable $\Delta_j((\xi_t - \xi_s) \ast \xi \otimes \xi)(x)$ is in the second (inhomogeneous) Wiener-Itô chaos and therefore all its moments are comparable by Gaussian hypercontractivity $^{[Jan97]}$ Theorem 5.10. It remains to estimate
\[
E[|\Delta_j((\xi_t - \xi_s) \ast \xi \otimes \xi)(x)|^2] = E[|((\xi_t - \xi_s) \ast \xi \otimes \xi)(\kappa_j(x - \cdot))|^2],
\]  
where $\kappa_j = \mathcal{F}_Z^{-1}p_j = \sum_{k \in \mathbb{Z}} e^{2\pi i k} p_j(k)$. Let now $\psi_{\otimes}(x, y) = \sum_{|l_1 - l_2| \leq 1} \kappa_1(x) \kappa_2(y)$. Then with formal notation:
\[
(\xi_t - \xi_s) \ast \xi \otimes \xi(x) = \int \int \psi_{\otimes}(x - y_1, x - y_2)((\xi_t - \xi_s) \ast \xi)(y_1)(\xi)(y_2)dy_1dy_2,
\]
and thus
\[
(\xi_t - \xi_s) \ast \xi \otimes \xi(\kappa_j(x - \cdot)) = p_1 \int \int \kappa_j(x - z) \psi_{\otimes}(z - y_1, z - y_2)((\xi_t - \xi_s)(y_1 - 1))\xi(\delta(y_2 - \cdot))dy_1dy_2dz.
\]
To derive the chaos decomposition of the right hand side, we introduce the kernel
\[
A_j^{\psi}(x, r_1, r_2) = \int \int \int \kappa_j(x - z)\psi_{\otimes}(z - y_1, z - y_2)(\xi_t - \xi_s)(y_1 - r_1)\delta(y_2 - r_2)dy_1dy_2dz,
\]
with which
\[
(\xi_t - \xi_s) \ast \xi \otimes \xi(\kappa_j(x - \cdot)) = W_2(A_j^{\psi}(x, \cdot, \cdot)) + E[(\xi_t - \xi_s) \ast \xi \otimes \xi(\kappa_j(x - \cdot))],
\]  
where $W_2$ denotes a second order Wiener-Itô integral. We start by estimating the first term on the right hand side: Using the symmetrization $A_j^{\psi}(x, r_1, r_2) = E(A_j^{\psi}(x, r_1, r_2) + A_j^{\psi}(x, r_2, r_1))$, we have
\[
E[|W_2(A_j^{\psi}(x, \cdot, \cdot))|^2] \leq 2\|A_j^{\psi}(x, \cdot, \cdot)\|_{L^2(\mathbb{R}^2)}^2 \leq 2\|A_j^{\psi}(x, \cdot, \cdot)\|_{L^2(\mathbb{R}^2)}^2
\]  
where the last equality is Parseval’s identity. Now, we obtain by computing each integral iteratively
\[
\int A_j(x, r_1, r_2)e^{-2\pi i (k_1 r_1 + k_2 r_2)} dr_1 dr_2 = \kappa_j(-k_1 + k_2)e^{-2\pi i (k_1 + k_2)} \hat{\psi}_{\otimes}(-k_1, -k_2)(\hat{\xi_t} - \hat{\xi_s})(-k_1),
\]
where $\hat{f}(k) = \int_T f(x)e^{-2\pi i kx} dx$ is the Fourier transform on the torus and
\[
\hat{\psi}_{\otimes}(k_1, k_2) := \int \int \psi_{\otimes}(y_1, y_2)e^{-2\pi i (k_1 y_1 + k_2 y_2)} dy_1 dy_2 = \sum_{|l_1 - l_2| \leq 1} p_{l_1}(k_1)p_{l_2}(k_2).
\]
As $|\hat{\psi}_{\otimes}(k)| \geq |k|^\alpha$ and $1 - e^{-x} \leq e^x$ for $x \geq 0$, we have for $s < t$ and $\varepsilon \in [0, 1]$
\[
|\hat{\xi_t} - \hat{\xi_s}| \leq |k| \int_s^t e^{-(r-s)\hat{\psi}_{\otimes}(k)} dr + \int_t^T e^{-(r-t)\hat{\psi}_{\otimes}(k)} (1 - e^{-(t-s)\hat{\psi}_{\otimes}(k)}) dr \lesssim T |t - s|^\varepsilon |k|^{1 - \alpha + \varepsilon}.
\]
This leads to
\[
\left| \iint A_j^{l,s}(x, r_1, r_2) e^{-2\pi i (k_1 r_1 + k_2 r_2)} dr_1 dr_2 \right|^2 \lesssim |t-s|^{2\varepsilon} \left| \sum_{|l_1-l_2| \leq 1} p_{l_1}(k_1) p_{l_2}(k_2) \right|^2 |k_1|^{2-2\alpha(1-\varepsilon)}.
\]

Let now \( \tilde{p}_{l_1} := \sum_{|l_1-l_2| \leq 1} p_{l_2} \). Since for fixed \( k_1 \) there are at most three \( l_1 \) with \( p_{l_1}(k_1) \neq 0 \), we can bound \( \sum_{|l_1-l_2| \leq 1} p_{l_1}(k_1)^2 \tilde{p}_{l_1}(k_2)^2 \) and thus we obtain in (5.5)
\[
E[|W_2(A_j^{l,s}(x, \cdot, \cdot))|^2] \lesssim |t-s|^{2\varepsilon} \sum_{k_1, k_2 \leq 2^l} \sum_{l_1} 2^l p_{l_1}(k_1) \tilde{p}_{l_1}(k_2)^2 |k_1|^{2-2\alpha(1-\varepsilon)}
\]
\[
= |t-s|^{2\varepsilon} \sum_{l_1:2 \leq 2^l} \sum_{k_1} 2^l p_{l_1}(k_1)^2 |k_1|^{2-2\alpha(1-\varepsilon)}
\]
\[
\lesssim |t-s|^{2\varepsilon} \sum_{l_1:2 \leq 2^l} 2^{2l} 2^{l_1(2-2\alpha(1-\varepsilon))} \lesssim |t-s|^{2\varepsilon} 2^{-l(4-2\alpha(1-\varepsilon))},
\]
where we used that \( p_{l}(k) \neq 0 \) for \( O(2^l) \) values of \( k \), with \( i = j \) respectively \( i = l_1 \), and we choose \( \varepsilon \in (0, 1) \) so that \( 3 - 2\alpha(1 - \varepsilon) < 0 \) to obtain the convergence of the series in the last estimate (recall that we assume \( \alpha > 3/2 \)).

Let \( e_k(x) = e^{2\pi i k x} \) so that \( \int e_k(x) e_l(x) dx = \delta_{k=-l} \). Then the second term on the right hand side of (5.4) is
\[
E[(\varrho_t - \varrho_s) * \xi \otimes \xi(k_j(x - \cdot))]^2
\]
\[
= \left( \iint \kappa_j(x-z) \psi_\varrho(z-y_1, z-y_2) (\varrho_t - \varrho_s)(y_1-y_2) dy_1 dy_2 dz \right)^2
\]
\[
= \left( \sum_{k, l, k', l'} \kappa_j(k) \psi_\varrho(k', l') (\varrho_t - \varrho_s)(l) \iint e_k(x-z) e_{k'}(z-y_1) e_{l'}(z-y_2) dy_1 dy_2 dz \right)^2
\]
\[
= \left( \sum_{k'} \kappa_j(0) \psi_\varrho(k', -k') (\varrho_t - \varrho_s)(k') \right)^2
\]
\[
\lesssim \delta_{j=-1} |t-s|^{2\varepsilon} \left( \sum_{k'} \psi_\varrho(k', -k') |k'|^{1-\alpha(1-\varepsilon)} \right)^2
\]
\[
\lesssim \delta_{j=-1} |t-s|^{2\varepsilon} \lesssim \delta_{j=-1} |t-s|^{2\varepsilon},
\]
by orthogonality of the Fourier basis \( (e_k) \) and where again we assume that \( \varepsilon \in (0, 1] \) is small enough so that \( 3 - 2\alpha(1 - \varepsilon) < 0 \) to guarantee that the series in \( l \) converges.

Combining this estimate with (5.4) and (5.6), we get via the Besov embedding theorem that for all \( \vartheta' < \alpha - 2 \) there exists \( \varepsilon > 0 \) such that for all \( p > 1 \) (by taking \( \varrho = \vartheta' \)),
\[
E[\| (\varrho_t - \varrho_s) * \xi \otimes \xi \|^p_{L^p_{\varrho', 1/p}}] \lesssim E[\| (\varrho_t - \varrho_s) * \xi \otimes \xi \|^p_{B_{\varrho', p}}] \lesssim |t-s|^p.
\]
Aftering choosing \( p \) large enough so that \( \varphi p > 0 \) we obtain from Kolmogorov’s continuity criterion that \( J^T(\nabla \varrho) \otimes \xi \in CT^\varphi C^{\vartheta'-1/p} \). Given \( \vartheta < \alpha - 2 \) as in the statement of the theorem, it now suffices to take \( \vartheta' \in (\vartheta, \alpha - 2) \) and then \( p \) large enough so that \( \vartheta' - 1/p > \vartheta \). 

By freezing a “typical” realization of \( \xi(\omega) \), we obtain the following corollary of Lemma 5.1 and Theorem 4.2.
Theorem 5.2. Let $\alpha \in (7/4, 2]$ and let $\xi$ be a periodic white noise on a probability space $(\Omega, \mathcal{F}, p)$. Then for almost all $\omega$ there exists a unique solution to the “quenched martingale problem” associated to the Brox diffusion with symmetric, $\alpha$-stable Lévy process $L$,

$$dX_t = \xi(\omega)(X_t)dt + dL_t, \quad X_0 = x \in \mathbb{R}.$$ 

If we denote the distribution of $X$ by $P_\omega$, then the “annealed measure” $\int P_\omega(\cdot)\mathbb{P}(d\omega)$ is the distribution of a Brox diffusion in a white noise potential, driven by an independent symmetric $\alpha$-stable Lévy process $L$.

Remark 5.3. By analogy with rough path regularities, the constraint $\alpha > 7/4$ corresponds to an “$\alpha > 1/3$ condition” in rough paths, and we expect that it is possible to treat $\alpha \in (3/2, 7/4]$ by considering higher order expansions of the Kolmogorov backward equation. To carry out this analysis we would need to use regularity structures [Hair14] or the higher order paracontrolled calculus of [BB19]. The constraint $\alpha > 3/2$ appears in the construction of the resonant product $J^T(\nabla \xi) \odot \xi$, so it seems to be of a similar nature as the constraint $H > 1/4$ for the Hurst index of a fractional Brownian motion that is required to construct its iterated integrals [CQ02]. But in fact not only the probabilistic construction fails at $\alpha = 3/2$: At that value the equation is critical in the sense of Hairer [Hair14] and we cannot solve it with perturbative techniques such as paracontrolled distributions or regularity structures.

Remark 5.4. To avoid dealing with weighted function spaces, we restricted our attention to periodic $\xi$. But we expect that it is also possible to treat the white noise $\xi$ on $\mathbb{R}$ with our approach, at the price of a slightly more involved analysis. In that case we have $\langle \cdot \rangle^{1/2-} \in \mathcal{C}^{1/2-}$ and $\langle \cdot \rangle^{1/2} J^T(\nabla \xi) \odot \xi \in \mathcal{C}^{1/2-}$ for all $\alpha > 0$, where $\langle x \rangle = (1 + |x|^2)^{1/2}$. With the techniques of [Beb16, HL13, MP19] it is still possible to solve the Kolmogorov backward equation for such $\xi$, by working in weighted function spaces with a time-dependent weight. Roughly speaking, if the terminal condition $u_T$ grows like $e^{\delta |x|^4}$ as $x \to \infty$, where $\delta \in (0, 1)$ and $l \in \mathbb{R}$, then $u(T - t)$ grows like $e^{(l+t)|x|^4}$. This might look dangerous because for $\alpha < 2$ our Lévy noise does not even have finite second moments, let alone finite (sub-)exponential moments. But we can take $l \in \mathbb{R}$ arbitrary, and in particular $l \leq -T$ is allowed and then $u(t)$ is bounded for all $t$. In that way it should be possible to extend our results to construct a Brox diffusion with Lévy noise in a non-periodic white noise potential.

A. Appendix

A.1. Commutator estimates

The following commutator estimate between the semigroup generated by $-\mathcal{L}_\nu^\alpha$ and the para-product will be used in the proof of Lemma 5.7 below.

Lemma A.1. Let $(P_t)$ be as in Lemma 2.7. Then, for $\gamma < 1$, $\beta \in \mathbb{R}$ and $\vartheta \geq -1$ the following commutator estimate holds:

$$\|P_t(u \otimes v) - u \otimes P_t v\|_{\gamma, \beta + \vartheta} \lesssim t^{-\vartheta/\alpha} \|u\|_{\gamma} \|v\|_{\beta}. \quad (A.1)$$

Proof. This is [Per14] Lemma 5.3.20 and Lemma 5.5.7, applied to $\varphi(z) = \exp(-\psi_\nu^\alpha(z))$.

Proof of Lemma 2.7. We write $J^T(g \otimes h)(t) - g(t) \otimes J^T(h)(t) = I_1(t) + I_2(t)$, where

$$I_1(t) = \int_t^T (P_{t-r}(g(r) \otimes h(r)) - g(r) \otimes P_{t-r}h(r))dr,$$

$$I_2(t) = \int_t^T (g(r) - g(t)) \otimes P_{t-r}h(r)dr.$$
For $I_1$ we apply (A.1) and obtain for $t \in [T - T, T]$ as $\sigma < 1$ and $-1 \leq \sigma - \gamma + 1 < \alpha$,

$$
\|I_1(t)\|_{2\sigma+1} \lesssim \int_t^T \|g(r)\|_{C^{\sigma}_{T,T} L^1_{R^d}} \|h(r)\|_{C^{\gamma}_\tau} dr \lesssim T^{\sigma} \|g\|_{C^{\sigma}_{T,T} L^1_{R^d}} \|h\|_{C^{\gamma}_\tau},
$$

where $\kappa := 1 - \frac{\sigma - \gamma + 1}{\alpha} > 0$. Now it follows from the estimates for the paraproduct (2.2), and from the estimate (2.4) for the regularizing effect of $P_t$ as $\sigma > 0$ that

$$
\|I_2(t)\|_{2\sigma+1} \lesssim \int_t^T \|g(r) - g(t)\|_{L^\infty_{R^d}} \|P_{r-t}h(r)\|_{C^{2\sigma+1}_{T,T}} dr \\
\lesssim \|g\|_{C^{\sigma}_{T,T} L^1_{R^d}} \|h\|_{C^{\gamma}_\tau} \int_t^T (r-t)^{2\sigma+1-\gamma} dr \\
\lesssim T^{\sigma} \|g\|_{C^{\sigma}_{T,T} L^1_{R^d}} \|h\|_{C^{\gamma}_\tau},
$$

where $\kappa = 1 - \frac{\sigma - \gamma + 1}{\alpha} > 0$. This is the claimed bound.

A.2. An application of Campbell’s formula

Here we are in the setting of Lemma 4.3, i.e. $\pi$ is the Poisson random measure of the $\alpha$-stable Lévy process $L$, $|\omega| := \omega_1 + 2\omega_2 + \cdots + n\omega_n$, and $C > 0$ and $0 \leq r < t$. Lemma 4.3 follows by plugging $\lambda = 0$ into Equation (A.2) below.

Lemma A.2. For $\lambda \in \mathbb{R}$ we define the following moment generating function:

$$
\Phi(\lambda) := \mathbb{E} \left[ \exp \left( \int_r^t \int_{|y| \leq C} \lambda |y|^2 \pi(ds, dy) \right) \right].
$$

Then the derivatives of $\Phi$ satisfy

$$
\Phi^{(n)}(\lambda) = \Phi(\lambda) \sum_{\omega \in \mathbb{N}_0^n : |\omega|=n} c(\lambda, \omega) \prod_{i=1}^n \left( (t-r) \int_{|y| \leq C} |y|^2 e^{\lambda |y|^2} \mu(dy) \right)^{\omega_i}
$$

for suitable integers $c(\lambda, \omega)$.

Proof. We prove this by induction. For $n = 0$ the claim is obviously true, so we assume that it holds for $n$ and establish it also for $n + 1$. We get with Campbell’s formula (see [Kin93 Section 3.2]):

$$
\Phi(\lambda) = \exp \left( \int_r^t \int_{|y| \leq C} (e^{\lambda |y|^2} - 1) \mu(dy) ds \right) = \exp \left( (t-r) \int_{|y| \leq C} (e^{\lambda |y|^2} - 1) \mu(dy) \right),
$$

and therefore

$$
\Phi^{(n+1)}(\lambda) = \partial_\lambda \Phi^{(n)}(\lambda)
$$

$$
= \partial_\lambda \left( \Phi(\lambda) \sum_{\omega \in \mathbb{N}_0^n : |\omega|=n} c(\lambda, \omega) \prod_{i=1}^n \left( (t-r) \int_{|y| \leq C} |y|^2 e^{\lambda |y|^2} \mu(dy) \right)^{\omega_i} \right)
$$

$$
= \Phi(\lambda) \left( (t-r) \int_{|y| \leq C} |y|^2 e^{\lambda |y|^2} \mu(dy) \right) \sum_{\omega \in \mathbb{N}_0^n : |\omega|=n} c(\lambda, \omega) \prod_{i=1}^n \left( (t-r) \int_{|y| \leq C} |y|^2 e^{\lambda |y|^2} \mu(dy) \right)^{\omega_i}
$$

$$
+ \Phi(\lambda) \sum_{\omega \in \mathbb{N}_0^n : |\omega|=n} c(\lambda, \omega) \partial_\lambda \left( \prod_{i=1}^n \left( (t-r) \int_{|y| \leq C} |y|^2 e^{\lambda |y|^2} \mu(dy) \right)^{\omega_i} \right).
$$
The first term on the right hand side is of the claimed form with \( \tilde{\omega} = (\omega_1 + 1, \omega_2, \ldots, \omega_n, 0) \in \mathbb{N}_{0}^{n+1} \) such that \( |\tilde{\omega}| = n + 1 \). For the second term on the right hand side we get by Leibniz’s rule

\[
\partial_\lambda \left( \prod_{i=1}^n \left( (t-r) \int_{|y| \leq C} |y|^{2i} e^{\lambda|y|^2} \mu(dy) \right)^{\omega_i} \right)
\]

\[
= \sum_{j=1}^n \prod_{i \neq j} \left( (t-r) \int_{|y| \leq C} |y|^{2i} e^{\lambda|y|^2} \mu(dy) \right)^{\omega_i} \times \omega_j \left( (t-r) \int_{|y| \leq C} |y|^{2j} e^{\lambda|y|^2} \mu(dy) \right)^{\omega_j-1}
\]

\[
= \sum_{j=1}^{n+1} \omega_j \prod_{i=1}^{n+1} \left( (t-r) \int_{|y| \leq C} |y|^{2i} e^{\lambda|y|^2} \mu(dy) \right)^{\tilde{\omega}_j^i},
\]

with \( \tilde{\omega}_j^i \in \mathbb{N}_{0}^{n+1} \) defined by

\[
\tilde{\omega}_j^i = \begin{cases} 
\omega_i, & i \neq j, j + 1, \\
\omega_j - 1, & i = j, \\
\omega_j + 1, & i = j + 1.
\end{cases}
\]

As required we have \( |\tilde{\omega}^j| = |\omega| - j + (j + 1) = |\omega| + 1 = n + 1 \), and thus the proof is complete. \( \square \)

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