Lie group classification of second-order delay ordinary differential equations

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Abstract

This article is part of a research program the aim of which is to turn Lie group theory into an efficient tool for classifying, transforming and solving a further class of equations, namely Delay Ordinary Differential Systems (DODSs). Such a system consists of two equations involving one independent variable \(x\) and one dependent variable \(y\). As opposed to ODEs the variable \(x\) figures in more than one point (we consider the case of two points, \(x\) and \(x\)\textsuperscript{−}). The dependent variable \(y\) and its derivatives figure in both \(x\) and \(x\)\textsuperscript{−}. This is not a discretization since both \(x\) and \(y\) remain continuous. Two previous articles were devoted to first-order DODSs, here we concentrate on a large class of second-order ones. We show that within this class the symmetry algebra can be of dimension \(n\) with \(0 \leq n \leq 6\) for genuinely nonlinear DODSs and \(n = \infty\) for linear or linearizable ones. The symmetry algebras can be used to perform symmetry reduction and obtain exact particular (invariant) solutions.

1 Introduction

Two previous articles were devoted to the adaptation of Lie group and Lie algebra theory to the study of delay ordinary differential equations \cite{3, 4}. In these articles we restricted ourselves to the case of first-order DODEs, supplemented by a general delay equation. Thus we considered first-order delay ordinary differential systems (DODSs) of the form

\[
\ddot{y} = f(x, y, y\downarrow), \quad \frac{\partial f}{\partial y\downarrow} \neq 0, \quad x \in I,
\]

\[
x\downarrow = g(x, y, y\downarrow), \quad x\downarrow < x, \quad g(x, y, y\downarrow) \neq \text{const}. \quad (1.1)
\]

In (1.1) \(I\) is a finite or semifinite interval and \(f\) and \(g\) are arbitrary smooth functions. For details and motivation we refer to our articles \cite{3, 4}.

Our main results were the following:
1. We classified DODSs of the form (1.1) into conjugacy classes under arbitrary Lie point transformations and found that their Lie point symmetry groups can have dimension \( n = 0, 1, 2, 3 \) or \( n = \infty \). If \( n = \infty \) the DODE is linear and we have \( g = g(x) \) or can be transformed into this linear form by a point transformation. In general the Lie algebra of the infinite-dimensional symmetry group is a solvable Lie algebra with an infinite-dimensional nilradical and is realized by vector fields of the form 
\[
X = \eta(x, y) \frac{\partial}{\partial y}.
\]

2. If the symmetry algebra of a DODS contains a 2-dimensional subalgebra realized by linearly connected vector fields, then this DODS is linearizable (or already linear) with \( g = g(x) \).

3. The symmetry algebra for genuinely nonlinear DODEs has dimension \( n \leq 3 \). For algebras with \( n = 2 \) or 3 we presented a method for obtaining exact particular solutions.

The purpose of this article is to perform a similar analysis for second-order DODSs and to show how the previously obtained results can be generalized to more general DODSs.

The paper is organized as follows. In the next section we formulate the problem. We define a DODS which consists of a DODE and a delay equation that specifies the position of the delay point \( x_- \). In section 3 we provide the general theory and outline the method for constructing invariant DODSs. Section 4 is devoted to linear DODEs with solution-independent delay relations. They all admit infinite-dimensional symmetry groups. The Lie group classification of nonlinear second-order DODSs is obtained in section 5. It is given in Table 1 in the Appendix. Finally, the concluding remarks are presented in section 6.

2 Formulation of the problem

The purpose of this article is to perform a symmetry classification of second-order delay ordinary differential equations (DODEs)

\[
\ddot{y} = f(x, y, y_-, \dot{y}, \dot{y}_-), \quad \left( \frac{\partial f}{\partial y_-} \right)^2 + \left( \frac{\partial f}{\partial \dot{y}_-} \right)^2 \neq 0, \quad x \in I, \tag{2.1}
\]

where \( I \subset \mathbb{R} \) is some finite or semifinite interval. We will be interested in symmetry properties of this DODE, which is considered locally, independently of initial conditions. For equation (2.1) we have to specify the delayed point \( x_- \) where the delayed function value \( y_- = y(x_-) \) and the derivative value \( \dot{y}_- = \dot{y}(x_-) \) are taken, otherwise the problem is not fully determined. Therefore, we supplement the DODE with a delay relation

\[
x_- = g(x, y, y_-, \dot{y}, \dot{y}_-), \quad x_- < x, \quad g(x, y, y_-, \dot{y}, \dot{y}_-) \neq \text{const}. \tag{2.2}
\]

The two equations (2.1) and (2.2) together will be called a delay ordinary differential system (DODS). Here \( f \) and \( g \) are arbitrary smooth functions. Sometimes for convenience we shall write (2.2) in the equivalent form

\[
\Delta x = x - x_- = \tilde{g}(x, y, y_-, \dot{y}, \dot{y}_-), \quad \tilde{g}(x, y, y_-, \dot{y}, \dot{y}_-) = x - g(x, y, y_-, \dot{y}, \dot{y}_-).
\]

In most of the existing literature the delay parameter \( \Delta x \) is considered to be constant

\[
\Delta x = \tau > 0, \quad \tau = \text{const}. \tag{2.3}
\]
An alternative is to impose a specific form of the function (2.2) to include some physical features of the delay $\Delta x$.

In the classification that we are performing we will find all special cases of functions $f$ and $g$, when the DODS under consideration will possess a nontrivial symmetry group. For all such cases of $f$ and $g$ the corresponding group will be presented.

We will be interested in group transformations, leaving the Eqs. (2.1), (2.2) invariant. That means that the transformation will transform solutions of the DODS into solutions. They leave the set of all solutions invariant. Let us stress that we need to consider these two equations together. This makes our approach similar to one of the approaches for considering symmetries of discrete equations (see, for example, [1, 2]), where the invariance is required for both the discrete equation and the equation for the lattice on which the discrete equation is considered. Here, we have the DODE (2.1) instead of a discrete equation and the delay relation (2.2) instead of a lattice equation.

In order to solve a DODE on some interval $I$ one must add initial conditions to the DODS (2.1), (2.2). Contrary to the case of ordinary differential equations, the initial condition must be given by a function $\varphi(x)$ on an initial interval $I_0 \subset \mathbb{R}$, e.g.

$$y(x) = \varphi(x), \quad x \in [x_{-1}, x_0].$$

(2.4)

We assume that $\varphi(x)$ is continuously differentiable so that

$$\dot{y}(x) = \dot{\varphi}(x), \quad x \in [x_{-1}, x_0].$$

For $x - x_- = \tau = x_0 - x_{-1}$ constant this leads to the method of steps [12] for solving the DODE either analytically or numerically. Thus for $x_0 \leq x \leq x_1 = x_0 + \tau$ we replace

$$y_-(x) = y(x_-) = \varphi(x-\tau), \quad \dot{y}_-(x) = \dot{\varphi}(x-\tau),$$

and this reduces the DODE (2.1) to an ODE

$$\dot{y}(x) = f(x, y(x), \varphi(x-\tau), \dot{y}(x), \dot{\varphi}(x-\tau)), \quad x_0 \leq x \leq x_1 = x_0 + \tau$$

(2.5)

which is solved with initial conditions $y(x_0) = \varphi(x_0)$, $\dot{y}(x_0) = \dot{\varphi}(x_0)$.

On the second step we consider the same procedure: we solve the ODE

$$\dot{y}(x) = f(x, y(x), y(x-\tau), \dot{y}(x), \dot{y}(x-\tau)), \quad x_1 \leq x \leq x_2 = x_1 + \tau,$$

(2.6)

where $y(x - \tau)$, $\dot{y}(x - \tau)$ and the initial conditions are known from the first step. Thus we continue until we cover the entire interval $I$, at each step solving the ODE with input from the previous step. This procedure provides a solution that is in general continuous in the points $x_n = x_0 + n\tau$, $n = 0, 1, 2, \ldots$ but not smooth.

The condition (2.4) is very restrictive and rules out most of the symmetries that could be present for ODEs. Hence we consider the more general case of the DODS (2.1), (2.2) and we must adapt the method of steps to this case. We again use the initial condition (2.4), which is required to satisfy the condition

$$x_{-1} = g(x_0, \varphi(x_0), \varphi(x_{-1}), \dot{\varphi}(x_0), \dot{\varphi}(x_{-1})).$$

(2.7)

We will also assume that

$$x_{-1} \leq x_- < x.$$

(2.7)

On the first step we solve the system

$$\dot{y}(x) = f(x, y(x), \varphi(x_-), \dot{y}(x), \dot{\varphi}(x_-)), \quad x_- = g(x, y(x), \varphi(x_-), \dot{y}(x), \dot{\varphi}(x_-)),$$

(2.8a)

(2.8b)
for \( y(x) \) and \( x_\pm(x) \) from the point \( x_0 \) with initial conditions \( y(x_0) = \varphi(x_0), \dot{y}(x_0) = \dot{\varphi}(x_0) \). The system is solved forward till a point \( x_1 \) such that

\[
x_0 = g(x_1, y(x_1), y(x_0), \dot{y}(x_1), \dot{y}(x_0)).
\]

Thus, we obtain the solution \( y(x) \) on the interval \([x_0, x_1]\). In the general case of delay relation (2.2), point \( x_1 \) can depend on the particular solution \( y(x) \) generated by the initial values (2.4).

Once we know the solution on the interval \([x_0, x_1]\), we can proceed to the next interval \([x_1, x_2]\) such that

\[
x_1 = g(x_2, y(x_2), y(x_1), \dot{y}(x_2), \dot{y}(x_1))
\]

and so on. Thus the method of steps for the solution of the initial value problem introduces a natural sequence of intervals

\[
[x_n, x_{n+1}], \quad n = -1, 0, 1, 2, \ldots \tag{2.9}
\]

We emphasize that the introduction of the intervals (2.9) is not a discretization. The variable \( x \) varies continuously over the entire region where Eqs. (2.1), (2.2) are satisfied.

### 3 Invariant second-order DODSs

In this section we describe the method to be used to classify all second-order delay ordinary differential systems (2.1), (2.2) that are invariant under some nontrivial Lie group of point transformations into conjugacy classes under local diffeomorphisms. For each class we propose a representative DODS. This is an extension of the method used in [3, 4] for first-order DODEs.

We realize the Lie algebra \( L \) of the point symmetry group \( G \) of the system (2.1), (2.2) by vector fields of the same form as in the case of ordinary differential equations, namely

\[
X_\alpha = \xi_\alpha(x, y) \frac{\partial}{\partial x} + \eta_\alpha(x, y) \frac{\partial}{\partial y}, \quad \alpha = 1, \ldots, n. \tag{3.1}
\]

The prolongation of these vector fields acting on the system (2.1), (2.2) will have the form

\[
prX_\alpha = \xi_\alpha \frac{\partial}{\partial x} + \eta_\alpha \frac{\partial}{\partial y} + \xi^-_\alpha \frac{\partial}{\partial x} - \eta^-_\alpha \frac{\partial}{\partial y} + \zeta^1_\alpha \frac{\partial}{\partial \dot{x}} + \zeta^-_\alpha \frac{\partial}{\partial \dot{y}} + \zeta^2_\alpha \frac{\partial}{\partial \ddot{y}} \tag{3.2}
\]

with

\[
\xi_\alpha = \xi_\alpha(x, y), \quad \eta_\alpha = \eta_\alpha(x, y), \quad \xi^-_\alpha = \xi_\alpha(x_-, y_-), \quad \eta^-_\alpha = \eta_\alpha(x_-, y_-),
\]

\[
\zeta^1_\alpha(x, y, \dot{y}) = D(\eta_\alpha) - \dot{y}D(\xi_\alpha), \quad \zeta^-_\alpha(x_-, y_-, \dot{\dot{y}}) = D(\eta^-_\alpha) - \dot{\dot{y}}D(\xi^-_\alpha),
\]

\[
\zeta^2_\alpha(x, y, \dot{y}, \ddot{y}) = D(\zeta_\alpha) - \ddot{y}D(\zeta_\alpha).
\]

where \( D \) is the total derivative operator. Let us note that Eq. (3.2) combines prolongation for shifted discrete variables \{\( x_-, y_- \)\} [11, 2] with standard prolongation for the continuous derivatives \( \dot{y} \) and \( \ddot{y} \) [15, 14].

All finite-dimensional complex Lie algebras of vector fields of the form (3.1) were classified by S. Lie [6, 8]. The real ones were classified more recently in [4] (see also [14, 13]). The classification is performed under the local group of diffeomorphisms

\[
\bar{x} = \bar{x}(x, y), \quad \bar{y} = \bar{y}(x, y).
\]
In section 5 we construct the invariant delay ordinary differential equations supplemented by invariant delay relations, proceeding by dimension of the Lie algebra.

In the paper we will use following notations

\[ \Delta x = x - x_-, \quad \Delta y = y - y_-, \quad y_x = \frac{\Delta y}{\Delta x}, \]

In the “no delay” limit we have \( \Delta x \to 0, \Delta y \to 0 \) and \( y_x \to \dot{y} \). We shall run through the list of representative algebras given in [5] and for each of them construct all group invariants in the space with local coordinates \((x, y, x-, y-, \dot{y}, \dot{y}-)\). We shall construct both strong invariants (invariant in the entire space) and weak invariants (invariant on some manifolds). The invariants will be used to write invariant differential delay systems (2.1), (2.2). The usual restriction \( x- = x- \tau, \) \( \tau = \text{const} \) excludes most symmetries. We however use the delay relation (2.2) instead. The situation is similar to that of first-order ordinary delay differential equations considered in the earlier articles [3, 4].

Let us assume that a Lie group \( G \) is given and that its Lie algebra \( L \) is realized by the vector fields of the form (3.1). If we wish to construct a second-order DODE with a delay relation that are invariant under this group, we proceed as follows. We choose a basis of the Lie algebra, namely \( \{ X_\alpha, \alpha = 1, \ldots, n \} \), and impose the equations

\[ \text{pr}_X \Phi(x, y, x-, y-, \dot{y}, \dot{y}-) = 0, \quad \alpha = 1, \ldots, n, \quad (3.3) \]

with \( \text{pr}_X \) given by Eq. (3.2). Using the method of characteristics, we obtain a set of elementary invariants \( I_1, \ldots, I_k \). Their number is

\[ k = \dim M - (\dim G - \dim G_0), \quad (3.4) \]

where \( M \) is the manifold that \( G \) acts on and \( G_0 \) is the stabilizer of a generic point on \( M \). In our case we have \( M \sim (x, y, x-, y-, \dot{y}, \dot{y}-) \) and hence \( \dim M = 7 \).

Practically, it is convenient to express the number of invariants as

\[ k = \dim M - \text{rank } Z, \quad k \geq 0, \quad (3.5) \]

where \( Z \) is the matrix

\[ Z = \left( \begin{array}{cccccc} \xi_1 & \eta_1 & \xi_1^- & \eta_1^- & \zeta_1^1 & \zeta_1^2 \\ \vdots & & \vdots & & \vdots & \\ \xi_n & \eta_n & \xi_n^- & \eta_n^- & \zeta_n^1 & \zeta_n^2 \end{array} \right). \quad (3.6) \]

The rank of \( Z \) is calculated at a generic point of \( M \).

The invariant DODE and delay relation are written as

\[ F(I_1, \ldots, I_k) = 0, \quad (3.7a) \]

\[ G(I_1, \ldots, I_k) = 0, \quad (3.7b) \]

where \( F \) and \( G \) satisfy

\[ \det \left( \frac{\partial(F, G)}{\partial(\dot{y}, x-)} \right) \neq 0. \]

Note that Eqs. (3.7) can be rewritten in the form (2.1), (2.2). Equation (3.7a) with delay relation (3.7b) obtained in this manner is ”strongly invariant”, i.e. \( \text{pr}_X F = 0 \) and \( \text{pr}_X G = 0 \) are satisfied identically.

Further invariant equations are obtained if the rank of \( Z \) is less than maximal on some manifold described by the equations

\[ F(x, y, x-, y-, \dot{y}, \dot{y}-) = 0, \quad (3.8a) \]

\[ G(x, y, x-, y-, \dot{y}, \dot{y}-) = 0, \quad (3.8b) \]
\[
\det \left( \frac{\partial (F, G)}{\partial (y, x-)} \right) \neq 0,
\]

which satisfy the conditions

\[
\text{pr} X_\alpha F |_{F=0, G=0} = 0, \quad \alpha = 1, \ldots, n, \tag{3.9a}
\]
\[
\text{pr} X_\alpha G |_{F=0, G=0} = 0, \quad \alpha = 1, \ldots, n. \tag{3.9b}
\]

Thus we obtain ”weakly invariant” DODS (3.8), i.e. equations (3.9) are satisfied on the solutions of the system \( F = 0, G = 0 \).

Note that we must discard trivial cases when the obtained equations (2.1), (2.2) do not satisfy the conditions

\[
\left( \frac{\partial f}{\partial y} - \frac{\partial f}{\partial \dot{y}} \right)^2 + \left( \frac{\partial f}{\partial y} - \frac{\partial f}{\partial \dot{y}} \right)^2 \neq 0, \quad x_- < x \quad \text{or} \quad g(x, y, y_-) \neq \text{const} \quad (3.10)
\]
as expected for a DODS.

We shall make use of two different existing classifications of real finite dimensional Lie algebras. One is the complete classification of the finite-dimensional Lie subalgebras of \( \text{diff}(2, \mathbb{R}) \), realized by vector fields of the form (3.1). This classification into conjugacy classes under the group of inner automorphisms, i.e. under arbitrary local transformations of variables was presented in [5]. An earlier classification of subalgebras of \( \text{diff}(2, \mathbb{C}) \) is due to Sophus Lie [6, 8]. The other is the classification of low-dimensional Lie algebras into isomorphism classes. This classification is complete for \( \dim L \leq 6 \). The work is due to many authors [7, 9, 10, 11, 16, 19].

The classification together with many results on the classification of certain series of solvable Lie algebras are summed up in the book [18].

These two classifications are combined together for algebras satisfying \( 1 \leq \dim L \leq 4 \) in our previous article [3]. Here we use these results and supplement them by algebras of dimensions \( \dim L = 5, 6 \) for which there exist DODSs of the second order.

Let us make a few comments on the notations of Lie algebras, which we use in the paper. The detailed results are given in the Table 1 in the Appendix. Indecomposable Lie algebras precede the decomposable ones (like \( 2\mathfrak{n}_{1,1} \) or \( \mathfrak{n}_{1,1} \oplus \mathfrak{s}_{2,1} \)) in the list. Isomorphic Lie algebras can be realized in more than one manner by vector fields. Already at dimension \( \dim L = 2 \) we see that \( \mathfrak{s}_{2,1} \) and \( 2\mathfrak{n}_{1,1} \) have two inequivalent realizations each. In dimension 3 for instance \( \mathfrak{s}l(2, \mathbb{R}) \) is represented in 4 different inequivalent manners. For nilpotent Lie algebras elements of the derived algebra precede a semicolon, e.g. \( X_1, X_2 \) in \( \mathfrak{n}_{1,1} \). For solvable Lie algebras the nilradical precedes a semicolon, e.g. \( X_1, X_2, X_3 \) in \( \mathfrak{s}_{4,6} \).

Two symmetries of the form (6.1) are called linearly connected if they (more precisely, their coefficients \( \xi(x, y) \) and \( \eta(x, y) \)) are proportional with a nonconstant proportionality coefficient. The vector fields in \( \mathbf{A}_{2,1} \) and \( \mathbf{A}_{2,3} \) (see Table 1) are “linearly connected”: \( X_1 \) and \( X_2 \) are both proportional to \( \partial/\partial y \). In \( \mathbf{A}_{2,2} \) and \( \mathbf{A}_{2,4} \) they span the entire tangent space \( \{ \partial/\partial x, \partial/\partial y \} \). Such operators \( X_1 \) and \( X_2 \) are called ”linearly nonconnected”. In the next section linearly connected symmetries will be used to characterize linear DODEs.

**4 Linear DODSs**

We recall that we are considering DODSs for functions \( f(x, y) \) where \( (x, y) \) are the coordinates of points on a two-dimensional manifold locally isomorphic to \( \mathbb{R}^2 \). The vector fields that form the considered local Lie algebras are realized by first-order linear differential operators of the form (3.1). A basis of an \( n \)-dimensional Lie
algebra may consist of "linearly connected" vector fields $X_\alpha$. By definition they satisfy
\begin{equation}
\sum_{\alpha=1}^{n} A_\alpha(x, y)X_\alpha = 0,
\end{equation}
where the coefficients $A_\alpha(x, y)$ are not all constants (since the basis elements $X_\alpha$, $\alpha = 1, ..., n$, must be linearly independent). In the present two-dimensional setting this means that in a generic point local coordinates can be so chosen that all the linearly connected vector fields simultaneously have the form
\begin{equation}
X_\alpha = \eta_\alpha(x, y) \frac{\partial}{\partial y},
\end{equation}
where $\eta_1, ..., \eta_n$ are linearly independent functions.

In this section we consider DODSs which admit four-dimensional Lie algebras of linearly connected vector fields. In this case the DODEs are linear and the delay relations are solution-independent (or can be brought into such form by an invertible transformation). Such DODSs actually admit infinite-dimensional symmetry algebras due to the linear superposition principle. It is convenient to consider the most general linear DODS in the considered class, namely
\begin{equation}
\ddot{y} = a_1(x)\dot{y} + a_2(x)\dot{y}_- + a_3(x)y + a_4(x)y_- + b(x), \quad x_- = g(x),
\end{equation}
where $a_i(x)$, $b(x)$ and $g(x)$ are arbitrary real functions, smooth in some interval $x \in I$, satisfying
\begin{equation}
a_2^2(x) + a_3^2(x) \neq 0, \quad g(x) < x, \quad g(x) \neq \text{const}. \tag{4.4}
\end{equation}
We will also need its homogeneous counterpart
\begin{equation}
\ddot{y} = a_1(x)\dot{y} + a_2(x)\dot{y}_- + a_3(x)y + a_4(x)y_- \quad x_- = g(x). \tag{4.5}
\end{equation}

We will refer to (4.3) and (4.5) as linear DODSs. They can be related to invariance properties according to the following theorem.

**Theorem 4.1** Let the DODE (2.1) with the delay relation (2.2) admit four linearly connected symmetries. Then it can be transformed into the linear equation
\begin{equation}
\ddot{y} = a_1(x)\dot{y} + a_2(x)\dot{y}_- + a_3(x)y + a_4(x)y_- + b(x), \quad a_2^2(x) + a_3^2(x) \neq 0. \tag{4.6}
\end{equation}
This DODE is supplemented by the delay relation
\begin{equation}
x_- = g(x), \quad g(x) < x, \quad g(x) \neq \text{const}, \tag{4.7}
\end{equation}
which does not depend on the solutions of the considered DODE.

**Proof.** For two variables there are two possibilities for four-dimensional algebras with four linearly connected operators [5][3].

1. Solvable algebra $\mathfrak{g}_{4,3}$ with operators $A_{4,5}$
\begin{equation}
X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = \chi(x) \frac{\partial}{\partial y}, \quad X_4 = y \frac{\partial}{\partial y}, \quad \chi(x) \neq 0. \tag{4.8}
\end{equation}
In this case there are three invariants in the space $(x, y, x_-, y_-, y, y_-)$:
\begin{equation}
\begin{bmatrix}
\dot{y} & \dot{y}_- & y & y_- \\
0 & 0 & 1 & 1 \\
0 & 1 & x & x_- \\
\dot{\chi}(x) & \dot{\chi}(x_-) & \chi(x) & \chi(x_-) \\
0 & 0 & 1 & 1 \\
1 & 1 & x & x_- \\
\dot{\chi}(x) & \dot{\chi}(x_-) & \chi(x) & \chi(x_-)
\end{bmatrix}
\end{equation}

\begin{align*}
I_1 &= x, \quad I_2 = x_-, \quad I_3 = \begin{bmatrix}
\dot{y} & \dot{y}_- & y & y_- \\
0 & 0 & 1 & 1 \\
0 & 1 & x & x_- \\
\dot{\chi}(x) & \dot{\chi}(x_-) & \chi(x) & \chi(x_-) \\
0 & 0 & 1 & 1 \\
1 & 1 & x & x_- \\
\dot{\chi}(x) & \dot{\chi}(x_-) & \chi(x) & \chi(x_-)
\end{bmatrix}
\end{align*}
We obtain the invariant linear homogeneous DODE

\[
\begin{pmatrix}
\dot{y} & \dot{y} & y & y_- \\
0 & 0 & 1 & 1 \\
0 & 1 & x & x_- \\
\dot{\chi}(x) & \dot{\chi}(x) & \chi(x) & \chi(x_-)
\end{pmatrix}
= f(x)
\begin{pmatrix}
\dot{y} & \dot{y}_- & y & y_- \\
0 & 0 & 1 & 1 \\
1 & 1 & x & x_- \\
\dot{\chi}(x) & \dot{\chi}(x_-) & \chi(x) & \chi(x_-)
\end{pmatrix}
\]

with the invariant delay relation

\[x_- = g(x).\] (4.9)

Note that the DODE has second order if

\[
\begin{pmatrix}
0 & 1 & 1 \\
1 & x & x_- \\
\dot{\chi}(x) & \chi(x) & \chi(x_-)
\end{pmatrix}
= \chi(x) - \chi(g(x)) - \dot{\chi}(x)(x - g(x)) \neq 0.
\]

2. Abelian algebra 4m1,1 with operators \( A_{4,22} \), namely

\[
x_1 = \frac{\partial}{\partial y}, \quad x_2 = x \frac{\partial}{\partial y}, \quad x_3 = \chi_1(x) \frac{\partial}{\partial y}, \quad x_4 = \chi_2(x) \frac{\partial}{\partial y}.
\] (4.11)

where \( \{1, x, \chi_1(x), \chi_2(x)\} \) are linearly independent functions.

In the space \((x, x_-)\), there are three invariants

\[
I_1 = x, \quad I_2 = x_- \quad I_3 = \begin{pmatrix}
\dot{y} & \dot{y} & \dot{y}_- & y & y_- \\
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & x & x_- \\
\dot{\chi}_1(x) & \dot{\chi}_1(x) & \dot{\chi}_1(x_-) & \chi_1(x) & \chi_1(x_-) \\
\dot{\chi}_2(x) & \dot{\chi}_2(x) & \dot{\chi}_2(x_-) & \chi_2(x) & \chi_2(x_-)
\end{pmatrix}.
\]

We obtain the invariant linear inhomogeneous DODE

\[
\begin{pmatrix}
\dot{y} & \dot{y} & \dot{y}_- & y & y_- \\
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & x & x_- \\
\dot{\chi}_1(x) & \dot{\chi}_1(x) & \dot{\chi}_1(x_-) & \chi_1(x) & \chi_1(x_-) \\
\dot{\chi}_2(x) & \dot{\chi}_2(x) & \dot{\chi}_2(x_-) & \chi_2(x) & \chi_2(x_-)
\end{pmatrix}
= f(x)
\]

with the same invariant delay relation

\[x_- = g(x).\] (4.13)

The obtained DODE has second-order if

\[
\begin{pmatrix}
0 & 0 & 1 & 1 \\
1 & 1 & x & x_- \\
\dot{\chi}_1(x) & \dot{\chi}_1(x_-) & \chi_1(x) & \chi_1(x_-) \\
\dot{\chi}_2(x) & \dot{\chi}_2(x_-) & \chi_2(x) & \chi_2(x_-)
\end{pmatrix}
\neq 0.
\]

In both cases we obtain the statement of the theorem provided that we get a second-order DODE.

Thus, DODSs which admit four linearly connected symmetries are linearizable.

\[\square\]

**Corollary 4.2** Any Lie algebra containing a four-dimensional subalgebra of linearly connected vector fields will provide an invariant DODS that can be transformed into the linear DODE with a solution-independent delay relation. The larger Lie algebra will at most put constraints on the functions involved in the DODS.
In the rest of the section we consider symmetry properties of the linear DODS in greater detail.

**Proposition 4.3** The change of variables

\[
\bar{x} = x, \quad \bar{y} = y - \sigma(x),
\]

where \(\sigma(x)\) is an arbitrary solution of the inhomogeneous DODS \((4.3)\), transforms the inhomogeneous DODS \((4.3)\) into its homogeneous counterpart \((4.5)\).

**Theorem 4.4** Consider the linear DODS \((4.3)\). For all functions \(a_i(x), b(x)\) and \(g(x)\) the DODS admits an infinite-dimensional symmetry algebra represented by the vector fields

\[
X(\rho) = \rho(x) \frac{\partial}{\partial y}, \quad Y(\sigma) = (y - \sigma(x)) \frac{\partial}{\partial y},
\]

where \(\rho(x)\) is the general solution of the homogeneous DODS \((4.3)\) and \(\sigma(x)\) is any one particular solution of the DODS \((4.3)\).

**Proof.** Application of the symmetry operator

\[
X = \xi(x,y) \frac{\partial}{\partial x} + \eta(x,y) \frac{\partial}{\partial y}
\]

to the delay equation in \((4.3)\) gives

\[
\xi(x-,y-) = \dot{g}(x)\xi(x,y).
\]

Since \(x, y, y-, \dot{y}\) and \(\dot{y}-\) can be considered as independent while \(x-\) and \(\dot{y}\) are related to them via equations \((4.3)\) we get

\[
\xi = \xi(x), \quad \xi(g(x)) = \dot{g}(x)\xi(x).
\]

Now we consider the application of the symmetry to the DODE in \((4.3)\) on the solutions of this equation:

\[
\eta_{xx}(x,y) + \left(2\eta_{xy}(x,y) - \dot{\xi}(x)\right) \dot{y} + \eta_{yy}(x,y) \dot{y}^2
\]

\[
+ \left(\eta_y(x,y) - 2\dot{\xi}(x)\right) \left(a_1(x)\dot{y} + a_2(x)\dot{y}- + a_3(x)y + a_4(x)y- + b(x)\right)
\]

\[
= \xi(x) \left(\dot{a}_1(x)y + \dot{a}_2(x)y^- + \dot{a}_3(x)y + \dot{a}_4(x)y- + \dot{b}(x)\right)
\]

\[
+ a_1(x) \left[\eta_{xx}(x,y) + \left(\eta_y(x,y) - \dot{\xi}(x)\right) \dot{y}\right]
\]

\[
+ a_2(x) \left[\eta_{x-}(x-,y-) + \left(\eta_{y-}(x-,y-) - \dot{\xi}(x-)\right) \dot{y}-\right]
\]

\[
+ a_3(x)\eta(x,y) + a_4(x)\eta(x-,y-).
\]

Splitting equation \((4.18)\) for terms with \(\dot{y}^2, \dot{y}, \dot{y}-\) and terms without derivatives, we obtain the equations

\[
\eta_{yy}(x,y) = 0,
\]

\[
2\eta_{xy}(x,y) - \dot{\xi}(x) + a_1(x) \left(\eta_y(x,y) - 2\dot{\xi}(x)\right)
\]

\[
= \dot{a}_1(x)\xi(x) + a_1(x) \left(\eta_y(x,y) - \dot{\xi}(x)\right),
\]

\[
a_2(x) \left(\eta_y(x,y) - 2\dot{\xi}(x)\right) = \dot{a}_2(x)\xi(x) + a_2(x) \left(\eta_{y-}(x-,y-) - \dot{\xi}(x-)\right),
\]
\[ \eta_{xx}(x, y) + \left( \eta_y(x, y) - 2\dot{\xi}(x) \right) (a_3(x)y + a_4(x)y_- + b(x)) = \xi(x) \left( \dot{a}_3(x)y + \dot{a}_4(x)y_- + \dot{b}(x) \right) + a_1(x)\eta_x(x, y) + a_2(x)\eta_{-x}(x, y_-) + a_3(x)\eta(x, y) + a_4(x)\eta(x, y_-). \]  

(4.22)

From equation (4.19) we obtain
\[ \eta(x, y) = A(x)y + B(x). \]  

(4.23)

Substituting this result into equations (4.20)-(4.22), we get the system
\[ 2\ddot{A}(x) - \ddot{\xi}(x) + a_1(x) \left( A(x) - 2\dot{\xi}(x) \right) = \dot{a}_1(x)\xi(x) + a_1(x) \left( A(x) - \dot{\xi}(x) \right), \]  

(4.24)
\[ a_2(x) \left( A(x) - 2\dot{\xi}(x) \right) = \dot{a}_2(x)\xi(x) + a_2(x) \left( A(x) - \dot{\xi}(x) \right), \]  

(4.25)
\[ \ddot{A}(x)y + \ddot{B}(x) + \left( A(x) - 2\dot{\xi}(x) \right) (a_3(x)y + a_4(x)y_- + b(x)) = \xi(x) \left( \dot{a}_3(x)y + \dot{a}_4(x)y_- + \dot{b}(x) \right) + a_1(x) \left( \dot{A}(x)y + \dot{B}(x) \right) + a_1(x) \left( \dot{A}(x) - \dot{B}(x) \right) + a_3(x) \left( A(x)y + B(x) \right) + a_4(x) \left( A(x)y_- + B(x)_- \right). \]  

(4.26)

The last equation splits for terms with \( y, y_- \) and the remaining terms as
\[ \ddot{A}(x) + a_3(x) \left( A(x) - 2\dot{\xi}(x) \right) = \ddot{a}_3(x)\xi(x) + a_1(x)\dot{A}(x) + a_3(x)A(x), \]  

(4.27)
\[ a_4(x) \left( A(x) - 2\dot{\xi}(x) \right) = \ddot{a}_4(x)\xi(x) + a_2(x)\dot{A}(x)_- + a_4(x)A(x)_-, \]  

(4.28)
\[ \ddot{B}(x) + b(x) \left( A(x) - 2\dot{\xi}(x) \right) = \ddot{b}(x)\xi(x) + a_1(x)\dot{B}(x) + a_2(x)\dot{B}(x)_- + a_3(x)B(x) + a_4(x)B(x)_-. \]  

(4.29)

The equation (4.24) can be integrated as
\[ A(x) = \frac{\dot{\xi}(x)}{2} + \frac{a_1(x)\xi(x)}{2} + A_0, \quad A_0 = \text{const.} \]  

(4.30)

Now we have to consider two cases

1. \( a_2(x) \neq 0 \)

   In this case we substitute (4.30) into (4.25) and obtain
\[ \dot{\xi}(x) = \left[ \frac{a_1(x)}{2} - \frac{a_1(x)g(x)}{2} - \frac{\ddot{a}_2(x)}{a_2(x)} + \frac{\ddot{g}(x)}{2\dot{g}(x)} \right] \xi(x). \]  

(4.31)

Here we used
\[ \dot{\xi}(g(x)) = \frac{\ddot{g}(x)}{\dot{g}(x)} \xi(x) + \dot{\xi}(x), \]

obtained by differentiation of (4.17). For general \( a_1(x), a_2(x) \neq 0 \) and \( g(x) \) equations (4.17) and (4.31) have only one solution, namely \( \xi(x) \equiv 0. \)
2. \(a_2(x) \equiv 0, a_4(x) \neq 0\)

In this case we substitute (4.30) into (4.28) and obtain

\[
\dot{\xi}(x) = \left[ \frac{a_1(x)}{4} - \frac{a_1(g(x))\dot{g}(x)}{4} - \frac{\dot{a}_4(x)}{2a_4(x)} - \frac{\ddot{g}(x)}{4g(x)} \right] \xi(x). 
\]

(4.32)

For general \(a_1(x), a_4(x) \neq 0\) and \(g(x)\) equations (4.17) and (4.32) have only one solution, namely \(\xi(x) \equiv 0\).

Thus for both considered cases we obtain symmetries with coefficients

\[
\xi(x, y) \equiv 0, \quad \eta(x, y) = A_0 y + B(x), \quad A_0 = \text{const},
\]

(4.33)

where \(B(x)\) solves

\[
\ddot{B}(x) = a_1(x)\dot{B}(x) + a_2(x)\dot{B}(x) + a_3(x)B(x) + a_4(x)B(x) - A_0 b(x), \\
x_\text{g} = g(x).
\]

(4.34)

Finally, we can rewrite the admitted symmetries as given in (4.15).

□

Remark 4.5 For the homogeneous linear DODS (4.3) the theorem gets simplified since we can use \(\sigma(x) = 0\). We get an infinite-dimensional symmetry algebra represented by the vector fields

\[
X(\rho) = \rho(x) \frac{\partial}{\partial y}, \quad Y = y \frac{\partial}{\partial y},
\]

(4.35)

where \(\rho(x)\) is the general solution of the homogeneous DODS (4.3).

Let us continue to investigate the determining equations (4.17), (4.24), (4.25), (4.27), (4.28), (4.29). We are interested only in symmetries with \(\xi(x) \neq 0\). The following result will be needed.

Lemma 4.6 The overdetermined differential-discrete system of equations (4.17) and

\[
\dot{\xi}(x) = K(x)\xi(x) 
\]

(4.36)

has nontrivial solutions if and only if the compatibility condition

\[
K(g(x))(\dot{g}(x))^2 = \ddot{g}(x) + K(x)\dot{g}(x)
\]

(4.37)

is satisfied.

Proof. Differentiating (4.17) with respect to \(x\), we get

\[
\dot{\xi}(g(x))\dot{g}(x) = \ddot{g}(x)\xi(x) + \dot{g}(x)\dot{\xi}(x).
\]

Using (4.17) and (4.36), we obtain

\[
K(g(x))(\dot{g}(x))^2\xi(x) = \ddot{g}(x)\xi(x) + \dot{g}(x)K(x)\xi(x).
\]

From this equation it follows that either the compatibility condition (4.37) is satisfied or \(\xi(x) \equiv 0\). The compatibility condition is required for existence of nontrivial solution \(\xi(x) \neq 0\) which solve the overdetermined system (4.17) and (4.36).

□
Remark 4.7 We can present the solution of the system (4.17), (4.36) as
\[ \xi(x) = e^{\int K(x)dx}. \] (4.38)

Though \( \xi(x) \) is given as a solution of the ODE (4.36) it must also satisfy the equation (4.17) provided that the compatibility condition (4.37) holds.

We consider two cases.

1. \( a_2(x) \neq 0 \)

   In this case we take (4.31) as an equation of the form (4.36). If functions \( a_1(x), a_2(x) \) and \( g(x) \) satisfy the corresponding compatibility condition, there is a nontrivial solution \( \xi(x) \).

   Thus we obtain an additional symmetry of the form
   \[ Z = \xi(x) \frac{\partial}{\partial x} + (A(x)y + B(x)) \frac{\partial}{\partial y}, \quad \xi(x) \neq 0, \] (4.39)

   where
   \[ \xi(x) = e^{\int K(x)dx}, \quad A(x) = \frac{\dot{\xi}(x)}{2} + \frac{a_1(x)\xi(x)}{2}, \] (4.40)

   provided that functions \( a_3(x) \) and \( a_4(x) \) satisfy the equations (4.27), (4.28).

   Here the function \( B(x) \) is a particular solution of the DODS
   \[ \ddot{B}(x) = a_1(x)\dot{B}(x) + a_2(x)B(x) - a_3(x)B(x_-) + a_4(x)(B(x_-) + b(x)\frac{\dot{\xi}(x)}{2} + \left(\dot{b}(x) - \frac{a_1(x)b(x)}{2}\right)\xi(x), \quad x_- = g(x). \] (4.41)

2. \( a_2(x) \equiv 0, \, a_4(x) \neq 0 \)

   In this case we get (4.32) as an equation of the form (4.36). If functions \( a_1(x), a_4(x) \) and \( g(x) \) satisfy the corresponding compatibility condition, there is a nontrivial solution \( \xi(x) \).

   We obtain a symmetry of the form (4.39), (4.40) provided that the function \( a_2(x) \) satisfies the equations (4.27). The function \( B(x) \) is a particular solution of the DODS (4.41).

Below we will refer to the conditions which provide existence of nontrivial solutions \( \xi(x) \) for the overdetermined system of equations (4.17), (4.24), (4.25), (4.27), (4.28) as compatibility conditions for \( \xi(x) \neq 0 \). We just described how they are obtained for the two subcases, but we will not write them down because of the size of the expressions.

Let us formulate the results obtained as the following theorem.

Theorem 4.8 Consider the linear DODS (4.3). For specific choices of the arbitrary functions \( a_i(x) \) and \( g(x) \), namely for functions satisfying the compatibility conditions for \( \xi(x) \neq 0 \), the symmetry algebra is larger. It contains one additional basis element of the form
\[ Z = \xi(x) \frac{\partial}{\partial x} + (A(x)y + B(x)) \frac{\partial}{\partial y}, \quad \xi(x) \neq 0, \] (4.42)

where
\[ A(x) = \frac{\dot{\xi}(x)}{2} + \frac{a_1(x)\xi(x)}{2} \] (4.43)

and the function \( B(x) \) is a particular solution of the DODS (4.41).
Remark 4.9: For the homogeneous linear DODS (4.5), Theorem 4.8 gets simplified and we can present the admitted symmetries as

\[ X(\rho) = \rho(x) \frac{\partial}{\partial y}, \quad Y = y \frac{\partial}{\partial y}, \quad Z = \xi(x) \frac{\partial}{\partial x} + A(x)y \frac{\partial}{\partial y}, \quad (4.44) \]

where \( \xi(x) \neq 0 \) is a solution of the overdetermined system (4.17), (4.24), (4.25), (4.27), (4.28), \( A(x) \) is given by (4.43), and \( \rho(x) \) is the general solution of the homogeneous DODS (4.5).

We can use the results established above to provide simplifications of the linear DODSs.

Theorem 4.10: If the linear DODS (4.3) has functions \( a_i(x) \) and \( g(x) \) satisfying the compatibility conditions for \( \xi(x) \neq 0 \), the DODS can be transformed into the representative form

\[ \ddot{y} = \alpha \dot{y} - \beta y + \gamma y - h(x), \quad x_\cdot = x - C, \quad (4.45) \]

where \( \alpha, \beta, \gamma \) and \( C \) are constants such that \( \alpha^2 + \gamma^2 \neq 0, \quad C > 0 \).

This DODS admits symmetries

\[ X(\rho) = \rho(x) \frac{\partial}{\partial y}, \quad Y(\sigma) = (y - \sigma(x)) \frac{\partial}{\partial y}, \quad Z = \frac{\partial}{\partial x} + B(x) \frac{\partial}{\partial y}, \quad (4.46) \]

where \( \sigma(x) \) is any one particular solution of the inhomogeneous DODS (4.45), \( \rho(x) \) is the general solution of the corresponding homogeneous DODS and \( B(x) \) is a particular solution of the DODS

\[ \ddot{B}(x) = \alpha \dot{B}(x) + \beta B(x) + \gamma B(x_\cdot) + \hat{h}(x), \quad x_\cdot = x - C. \]

Proof. Let us consider a linear DODS (4.3) which admits a symmetry of the form (4.42) with \( \xi(x) \neq 0 \).

Variable change

\[ \bar{x} = \int \frac{1}{\xi(x)} dx \quad (4.47) \]

brings this DODS into the form

\[ \ddot{y} = \alpha_1(x) \dot{y} + \alpha_2(x) \dot{y}_- + \alpha_3(x) y + \alpha_4(x) y_- + \bar{b}(x), \quad x_- = x - C. \quad (4.48) \]

After that the variable change

\[ \bar{y} = e^{-\frac{1}{2} \int \alpha_1(x) dx} y \quad (4.49) \]

brings the DODS (4.48) into the form (4.45). \[ \square \]

Further simplification is possible if we know one particular solution of the DODS and can bring the DODE into the homogeneous form.

Corollary 4.11: If the linear homogeneous DODS (4.5) has functions \( a_i(x) \) and \( g(x) \) satisfying the compatibility conditions for \( \xi(x) \neq 0 \), the DODS can be transformed into the representative form

\[ \ddot{y} = \alpha \dot{y} - \beta y + \gamma y_-, \quad x_- = x - C \quad (4.50) \]
with \( \alpha^2 + \gamma^2 \neq 0, \quad C > 0 \).

It admits symmetries

\[
X(\rho) = \rho(x) \frac{\partial}{\partial y}, \quad Y = y \frac{\partial}{\partial y}, \quad Z = \frac{\partial}{\partial x},
\]

where \( \rho(x) \) is the general solution of the DODS (4.3).

**Theorem 4.12** If the linear DODS (4.3) has functions \( a_i(x) \) and \( g(x) \) which do not satisfy the compatibility conditions for \( \xi(x) \neq 0 \), the DODS can be transformed into the representative form

\[
\ddot{y} = \alpha(x) \dot{y}_- + \beta(x) y + \gamma(x) y_+ + \delta(x), \quad x_- = x - C, \quad C > 0,
\]

where \( \alpha(x), \beta(x) \) and \( \gamma(x) \) are functions such that the compatibility conditions for \( \xi(x) \neq 0 \) are not satisfied.

**Proof.** Let us consider a linear DODS (4.3) which does not possess an additional symmetry of the form (4.42).

We can straighten the delay relation and transform the DODS into the form

\[
\ddot{y} = \tilde{a}_1(x) \dot{y}_- + \tilde{a}_2(x) \dot{y}_- + \tilde{a}_3(x)y + \tilde{a}_4(x)y_+ + \tilde{b}(x), \quad x_- = x - C, \quad C > 0.
\]

Then we can employ the variable change

\[
\bar{x} = x, \quad \bar{y} = e^{-\frac{1}{2} \int \tilde{a}_1(x) dx} y
\]

to bring the DODS (4.53) into the form (4.52).

Since there can be no additional symmetry of the form (4.42) the compatibility conditions for \( \xi(x) \neq 0 \) must not hold. \square

**Corollary 4.13** If the linear homogeneous DODS (4.5) has functions \( a_i(x) \) and \( g(x) \) which do not satisfy the compatibility conditions for \( \xi(x) \neq 0 \), the DODS can be transformed into the representative form

\[
\ddot{y} = \alpha(x) \dot{y}_- + \beta(x) y + \gamma(x) y_-, \quad x_- = x - C, \quad C > 0,
\]

where \( \alpha(x), \beta(x) \) and \( \gamma(x) \) are functions such that the compatibility conditions for \( \xi(x) \neq 0 \) are not satisfied.

## 5 Classification of nonlinear invariant DODSs

In this section we perform a Lie group classification of nonlinear (and nonlinearizable) DODSs. Nonlinear DODSs should have nonlinear DODEs or solution-dependent delay relations (or both). To find invariant DODEs with invariant delay relations we need to compute invariants in the space \( (x, y, x-, y-, \dot{y}, \dot{y}-, \ddot{y}) \) as explained in section 3. We present only one case for each dimension of Lie algebra. The other cases are given in Table 1 in the Appendix.
5.1 Dimension one

We start with the simplest case of a symmetry group, namely a one-dimensional one. Its Lie algebra is generated by one vector field of the form \( (5.1) \). By an appropriate change of variables we take this vector field into its rectified form (locally in a nonsingular point \((x, y)\)). Thus we have \( A_{1,1} \):

\[
X_1 = \frac{\partial}{\partial y} \tag{5.1}
\]

In order to write a second-order DODE and a delay relation invariant under this group we need the invariants annihilated by the prolongation \( (3.2) \) of \( X_1 \) to the prolonged space \((x, y, x_-, y_-, \dot{y}, \dot{y}_-, \ddot{y})\). A basis for the invariants is

\[
I_1 = x, \quad I_2 = x_-, \quad I_3 = \Delta y, \quad I_4 = \dot{y}, \quad I_5 = \dot{y}_-, \quad I_6 = \ddot{y}.
\]

The most general second-order DODE with a delay relation which are invariant under the corresponding group can be written as

\[
\ddot{y} = f(x, \Delta y, \dot{y}, \dot{y}_-), \quad (5.2a)
\]
\[
x_- = g(x, \Delta y, \dot{y}, \dot{y}_-), \quad (5.2b)
\]

where \( f \) and \( g \) are arbitrary functions.

5.2 Dimension two

\( A_{2,1} \): The non-Abelian Lie algebra \( s_{2,1} \) with linearly connected basis elements

\[
X_1 = \frac{\partial}{\partial y}, \quad X_2 = y \frac{\partial}{\partial y} \tag{5.3}
\]

provides us with the invariants

\[
I_1 = x, \quad I_2 = x_-, \quad I_3 = \frac{\dot{y}}{\Delta y}, \quad I_4 = \frac{\dot{y}_-}{\Delta y}, \quad I_5 = \frac{\ddot{y}}{\dot{y}}.
\]

The general invariant DODS can be written as

\[
\ddot{y} = \dot{y}f \left( x, \frac{\dot{y}}{\Delta y}, \frac{\dot{y}_-}{\Delta y} \right), \quad (5.4a)
\]
\[
x_- = g \left( x, \frac{\dot{y}}{\Delta y}, \frac{\dot{y}_-}{\Delta y} \right). \quad (5.4b)
\]

There are three more realizations of two-dimensional Lie algebras by vector fields of the form \( (5.1) \): one more realization of algebra \( s_{2,1} \) which has nonconnected basis elements and two realization of Abelian Lie algebra \( 2n_{1,1} \): one with linearly connected basis elements, the other with linearly nonconnected. These three cases are treated similarly to \( A_{2,1} \). We present the results in Table 1.

5.3 Dimension three

\( A_{3,1} \): The nilpotent Lie algebra \( n_{3,1} \) can, up to equivalence, be realized in one way only:

\[
X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial x} \tag{5.5}
\]

The invariants are

\[
I_1 = \Delta x, \quad I_2 = \dot{y} - \frac{\Delta y}{\Delta x}, \quad I_3 = \dot{y} - \dot{y}_-, \quad I_4 = \ddot{y}.
\]
It provides us with an invariant DODS

\[
\ddot{y} = f (\dot{y} - y_x, \dot{y} - y_\gamma), \quad (5.6a)
\]
\[
\Delta x = g (\dot{y} - y_x, \dot{y} - y_\gamma). \quad (5.6b)
\]

The other realizations of three-dimensional Lie algebras by vector fields of the form \[\text{(5.1)}\] are treated similarly. The results are in Table 1.

**Remark 5.1 (Alternative version)** Lie algebra \(A_{3,8}\) is often (and equivalently) realized by the vector fields

\[
X_1 = \frac{\partial}{\partial x}, \quad X_2 = 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}. \quad (5.7)
\]

There are four invariants

\[
I_1 = \frac{\Delta x}{yy_\gamma}, \quad I_2 = y (\dot{y} - \frac{\Delta y}{\Delta x}), \quad I_3 = y_\gamma (\dot{y} - \frac{\Delta y}{\Delta x}), \quad I_4 = y^3 \ddot{y},
\]

which provide us with the invariant DODS

\[
\ddot{y} = \frac{1}{y} f (y (\dot{y} - y_x), y_\gamma (\dot{y} - y_x)), \quad (5.8a)
\]
\[
\Delta x = yy_\gamma g (y (\dot{y} - y_x), y_\gamma (\dot{y} - y_x)). \quad (5.8b)
\]

**Remark 5.2 (Alternative version)** For Lie algebra \(A_{3,10}\) we can use another realization

\[
X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3 = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}. \quad (5.9)
\]

We obtain invariants

\[
I_1 = \left(\frac{y - y_\gamma}{y - x}\right) (x - x_\gamma), \quad I_2 = \left(\frac{x - x_\gamma}{y - x_\gamma}\right)^2 \ddot{y}, \quad I_3 = \left(\frac{x - x_\gamma}{y - x_\gamma}\right)^2 \dot{y}_\gamma,
\]
\[
I_4 = \frac{(x - y) \ddot{y}}{2 |y|^{3/2}} + \frac{1}{\sqrt{|y|}} + \sqrt{|y|}
\]

and the invariant DODS

\[
\ddot{y} = \frac{2(\dot{y} + \dot{y}_\gamma^2)}{y - x} + \frac{2|y|^{3/2}}{y - x} f \left(\frac{x - x_\gamma}{y - x_\gamma}\right)^2 \dot{y}_\gamma, \quad (5.10a)
\]
\[
\frac{(y - y_\gamma)(x - x_\gamma)}{(y - x)(y_\gamma - x_\gamma)} = g \left(\frac{x - x_\gamma}{y - x_\gamma}\right)^2 \ddot{y}_\gamma, \quad (5.10b)
\]

### 5.4 Dimension four

Two realizations of four-dimensional Lie algebras, namely realizations of algebras \(A_{4,5}\) and \(A_{4,22}\) which have four linearly connected vector fields and provide us with linear DODEs, were considered in section \[\text{5.3}\]. Here we investigate the other cases.

**\(A_{4,1}\)**: The nilpotent Lie algebra \(n_{4,1}\) can be realized as

\[
X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = x^2 \frac{\partial}{\partial y}, \quad X_4 = \frac{\partial}{\partial x}. \quad (5.11)
\]
A set of invariants

\[ I_1 = \Delta x, \quad I_2 = \dot{y} + \dot{y}_- - 2\frac{\Delta y}{\Delta x}, \quad I_3 = \ddot{y} - \frac{\dot{y} - \dot{y}_-}{\Delta x} \]

provides us with invariant DODE and the invariant delay relation

\[ \begin{align*}
\dot{y} &= \frac{\dot{y} - \dot{y}_-}{\Delta x} + f (\dot{y} + \dot{y}_- - 2y_x), \\
\Delta x &= g (\dot{y} + \dot{y}_- - 2y_x).
\end{align*} \]

(5.12a) (5.12b)

The other realizations of four-dimensional Lie algebras by vector fields of the form (3.1) are similarly treated to the case \( A_{4,1} \). They are given in Table 1.

**Remark 5.3 (Alternative version)** Lie algebra \( A_{4,19} \) can also be realized by the vector fields

\[ X_1 = \frac{\partial}{\partial x}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = y \frac{\partial}{\partial y}, \quad X_4 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}. \]

(5.13)

There are three invariants

\[ I_1 = \Delta x \left( \dot{y} - \frac{\Delta y}{\Delta x} \right), \quad I_2 = \Delta x \left( \dot{y}_- - \frac{\Delta y}{\Delta x} \right), \quad I_3 = (\Delta x)^2 \frac{y\dot{y}_-}{y_-}, \]

which provide us with the invariant DODS

\[ \begin{align*}
\dot{y} &= \frac{y^2}{(\Delta x)^2 y} f \left( \frac{\Delta x}{y} (\dot{y}_- - y_x) \right), \\
\Delta x &= \frac{\Delta x}{y} (\dot{y}_- - y_x) = g \left( \frac{\Delta x}{y} (\dot{y}_- - y_x) \right).
\end{align*} \]

(5.14a) (5.14b)

**5.5 Dimension five**

For five and higher dimensional Lie algebras we consider only realizations which do not contain four-dimensional subalgebras of linearly connected vector fields. Only such cases can lead to nonlinear invariant DODSs.

**\( A_{5,1}(24) \)**: Lie algebra \( s_{5,33} \) has a realization

\[ X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = x^2 \frac{\partial}{\partial y}, \quad X_4 = \frac{\partial}{\partial x}, \quad X_5 = x \frac{\partial}{\partial x}. \]

(5.15)

A complete set of invariants is

\[ I_1 = \Delta x \left( \dot{y} + \dot{y}_- - 2\frac{\Delta y}{\Delta x} \right), \quad I_2 = (\Delta x)^2 \left( \dot{y} - \frac{2}{\Delta x} \left( \dot{y} - \frac{\Delta y}{\Delta x} \right) \right). \]

The invariant DODE with the delay relation can be written as

\[ \begin{align*}
\dot{y} &= 2\frac{\dot{y} - y_x}{\Delta x} + \frac{C_1}{(\Delta x)^2}, \\
\Delta x &= \frac{C_2}{\dot{y} + \dot{y}_- - 2y_x}.
\end{align*} \]

(5.16a) (5.16b)

The other realizations of five-dimensional Lie algebras by vector fields of the form (5.1) are examined similarly to \( A_{5,1} \). We present the results in Table 1.
5.6 Dimension six

\( A_{6,1}(7) \): There exist a realizations of the algebra \( \mathfrak{so}(3, 1) \) which can be taken as

\[
X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_4 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y},
\]

\[
X_5 = (x^2 - y^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y}, \quad X_6 = 2xy \frac{\partial}{\partial x} + (y^2 - x^2) \frac{\partial}{\partial y}.
\] (5.17)

We find one invariant

\[
I = \arctan(\dot{y}) + \arctan(\dot{y} - y_x) - 2 \arctan(y_x)
\]

and the invariant manifold (we are interested only in invariant manifolds which involve \( \ddot{y} \))

\[
\ddot{y} - 2 \frac{y_x^2 + 1}{y_x^2 + 1} \frac{\ddot{y} - y_x}{\Delta x} = 0.
\]

Thus, we obtain the most general invariant DODS

\[
\ddot{y} - 2 \frac{y_x^2 + 1}{y_x^2 + 1} \frac{\ddot{y} - y_x}{\Delta x} = 0, \quad (5.18a)
\]

\[
\arctan(\dot{y}) + \arctan(\dot{y} - y_x) - 2 \arctan(y_x) = C. \quad (5.18b)
\]

Two more realizations of six-dimensional Lie algebras by vector fields of the form (3.1) and the corresponding DODSs are given in Table 1.

In addition to the six-dimensional algebras given in Table 1 there are two more realizations of six-dimensional Lie algebras by vector fields of the form (3.1). They are realizations of algebra \( \mathfrak{gl}(2, \mathbb{R}) \ltimes 2n_{1,1} \):

\[
A_{6,4}(6) : \quad X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x \frac{\partial}{\partial x}, \quad X_4 = y \frac{\partial}{\partial x},
\]

\[
X_5 = x \frac{\partial}{\partial y}, \quad X_6 = y \frac{\partial}{\partial y}, \quad X_5 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \quad X_6 = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}.
\] (5.19)

and

\[
A_{6,5}(28) : \quad X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x \frac{\partial}{\partial x}, \quad X_4 = x \frac{\partial}{\partial y},
\]

\[
X_5 = y \frac{\partial}{\partial y}, \quad X_6 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}. \quad (5.20)
\]

Direct computation shows that algebras \( A_{6,4} \) and \( A_{6,5} \) do not provide invariant second-order DODSs.

5.7 Dimensions \( n \geq 7 \)

For higher dimensional Lie algebras there is only one realization without four linearly connected vector fields. It is a realization of Lie algebra \( \mathfrak{sl}(3, \mathbb{R}) \)

\[
A_{8,1}(8) : \quad X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x \frac{\partial}{\partial x}, \quad X_4 = y \frac{\partial}{\partial x},
\]

\[
X_5 = x \frac{\partial}{\partial y}, \quad X_6 = y \frac{\partial}{\partial y}, \quad X_5 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \quad X_6 = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} \quad (5.21)
\]

This vector fields contains the realization \( A_{6,4} \) as a subalgebra. Consequently, there is no DODS for the case \( A_{8,1} \).

We arrive at the following Theorem.
Theorem 5.4 The only DODSs of the form (2.1), (2.2) that have symmetry algebras of dimension $n \geq 7$ are equivalent to linear DODSs with solution-independent delay equations.

6 Conclusion

In the present paper we gave a Lie group classification of delay ordinary differential systems, which consist of second-order DODEs and delay relations. The classification was obtained similarly to our previous paper [3] devoted to first-order delay ordinary differential systems. We remark that the approach has some common features with the Lie group classification of ordinary difference equations supplemented by lattice equations (ordinary difference systems). However, the solutions of delay differential systems are continuous while solutions of ordinary difference systems exist only in lattice points.

The dimension of a symmetry algebra admitted by a DODS can be $0 \leq n \leq 6$ or $n = \infty$. In the case $n = \infty$, the DODS consists of a linear DODE and a solution-independent delay relation, or it can be transformed into such a form by an invertible transformation. A genuinely nonlinear DODS can admit symmetry algebras of dimension $0 \leq n \leq 6$. The Lie group classification of nonlinear DODSs is given in the Table 1.

Analyzing invariant DODSs, we obtained several theoretical results. It was proved that the symmetry algebra with four linearly connected symmetries provides a delay differential system which can be transformed into a linear DODE supplemented by a solution-independent delay relation (see Theorem 4.1). Such linear delay differential systems admit infinite-dimensional symmetry groups, since they allow the linear superposition of solutions (Theorem 4.4).

Linear DODSs (4.3) can be split into two classes. The first class consists of DODSs which in addition to the infinite-dimensional symmetry algebra corresponding to the superposition principle admit one further element (4.42). This class of linear DODSs is characterized by the fact that the coefficients $a_i(x)$ and $g(x)$ satisfy the compatibility conditions for $\xi(x) \neq 0$, which were described in section 4. The superposition group acts only on the dependent variable $y$, whereas the additional symmetry also acts on the independent variable $x$. Such DODSs can be brought into the form (4.45). If we know one particular solution, we can transform the DODE into the homogeneous form and bring the DODS to the form (4.50).

The other class contains DODSs which do not have any symmetry not related to the linear superposition principle. They can be transformed into the form (4.52). If we know one particular solution, we can transform the DODE into the homogeneous form and bring the DODS to the form (4.55).

Symmetries of DODSs can be used to find particular solutions, namely invariant solutions. A procedure for calculating particular solutions, which are invariant solutions with respect to one-dimensional subgroups of the symmetry group, is the same as in the case of first-order DODSs [3] [4].

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Appendix

Table 1. Classification of nonlinear invariant DODSs.

First three columns give Lie algebras and their realizations. In Column 1 we give the isomorphism class using the notations of [18]. Thus \( n_{i,k} \) denotes the \( k \)-th nilpotent Lie algebra of dimension \( i \) in the list. The only nilpotent algebras in the table are \( n_{1,1}, n_{3,1}, n_{4,1} \). Similarly, \( s_{i,k} \) is the \( k \)-th solvable Lie algebra of dimension \( i \) in the list. Simple Lie algebras are identified by their usual names \( (\mathfrak{sl}(2,\mathbb{R}), \mathfrak{o}(3)) \). In Column 2 \( A_{i,k} \) runs through algebras in the list of subalgebras of \( \text{diff}(2,\mathbb{R}) \) and \( i \) is again the dimension of the algebra. The numbers in brackets correspond to notations used in the list of Ref. [5]. In Column 3 we give vector fields spanning each representative algebra.

Two last columns give invariant DODEs and invariant delay relations. For dimensions 4, 5 and 6 only algebras which provide with invariant DODSs are given.
| Lie algebra | Case | Operators | DODE | Delay relation |
|-------------|------|-----------|------|----------------|
| $n_{1,1}$   | $A_{1,1}(9)$ | $X_1 = \partial_y$ | $\dddot{y} = f(x, \Delta y, \dot{y}, \ddot{y}-)$ | $x_- = g(x, \Delta y, \dot{y}, \ddot{y})$ |
| $s_{2,1}$   | $A_{2,1}(10)$ | $X_1 = \partial_y; X_2 = y \partial_y$ | $\dddot{y} = \dot{y} f\left(x, \frac{\dot{y}}{\Delta y}, \frac{\dddot{y}}{\Delta y}\right)$ | $x_- = g\left(x, \frac{\dot{y}}{\Delta y}, \frac{\dddot{y}}{\Delta y}\right)$ |
|             | $A_{2,2}(22)$ | $X_1 = \partial_y; X_2 = x \partial_x + y \partial_y$ | $\dddot{y} = \frac{1}{x} f\left(y_x, \dot{y}, \ddot{y}-\right)$ | $x_- = x g\left(y_x, \dot{y}, \ddot{y}^-\right)$ |
| $2n_{1,1}$  | $A_{2,3}(20)$ | $\{X_1 = \partial_y\}, \{X_2 = x \partial_y\}$ | $\dddot{y} = f\left(x, \dot{y} - y_x, \dddot{y} - \dot{y}\right)$ | $x_- = g\left(x, \dot{y} - y_x, \dddot{y} - \dot{y}\right)$ |
|             | $A_{2,4}(22)$ | $\{X_1 = \partial_x\}, \{X_2 = \partial_y\}$ | $\dddot{y} = f(\Delta y, \dot{y}, \ddot{y})$ | $\Delta x = g(\Delta y, \dot{y}, \ddot{y})$ |
| Lie algebra | Case | Operators | DODE | Delay relation |
|-------------|------|-----------|------|---------------|
| \( \mathfrak{n}_{3,1} \) | \( \mathbf{A}_{3,1}^{(22)} \) | \( X_1 = \partial_y; \ X_2 = x\partial_y, \ X_3 = \partial_x \) | \( \dot{y} = f(\dot{y} - y_x, \dot{y} - \dot{y}_-) \) | \( \Delta x = g(\dot{y} - y_x, \dot{y} - \dot{y}_-) \) |
| \( \mathfrak{g}_{3,1} \) | \( \mathbf{A}_{3,2}^{(12)} \) | \( X_1 = \partial_x, \ X_2 = \partial_y; \ X_3 = x\partial_x + ay\partial_y, \ 0 < |a| \leq 1 \) | \( \dot{y} = |\Delta x|^{a-2}f(\frac{\dot{y}}{|\Delta x|^{a-1}}, \frac{\dot{y}}{|\Delta x|^{a-1}}) \) | \( \Delta x = |\Delta y|^\frac{a}{2}g(\frac{\dot{y}}{|\Delta x|^{a-1}}, \frac{\dot{y}}{|\Delta x|^{a-1}}) \) |
| \( \mathfrak{g}_{3,2} \) | \( \mathbf{A}_{3,3}^{(21, 22)} \) | \( X_1 = \partial_y, \ X_2 = x\partial_y; \ X_3 = (1 - a)x\partial_x + y\partial_y, \ i) \ a \neq 1 \) | \( \dot{y} = |x|^\frac{2a-1}{a-1}f(\frac{\dot{y} - y_x}{|x|^{a-1}}, \frac{\dot{y} - \dot{y}_-}{|x|^{a-1}}) \) | \( x_- = xg(\frac{\dot{y} - y_x}{|x|^{a-1}}, \frac{\dot{y} - \dot{y}_-}{|x|^{a-1}}) \) |
| \( \mathfrak{g}_{3,2} \) | \( \mathbf{A}_{3,4}^{(25)} \) | \( X_1 = \partial_x, \ X_2 = \partial_y; \ X_3 = x\partial_x + (x + y)\partial_y \) | \( \dot{y} = \frac{1}{\Delta x}f(\dot{y} - y_x, \dot{y} - \dot{y}_-) \) | \( \Delta x = e^{y_x}g(\dot{y} - y_x, \dot{y} - \dot{y}_-) \) |
| \( \mathfrak{g}_{3,2} \) | \( \mathbf{A}_{3,5}^{(22)} \) | \( X_1 = \partial_y, \ X_2 = x\partial_y; \ X_3 = \partial_x + y\partial_y \) | \( \dot{y} = e^{\xi}f(e^{-\xi}(\dot{y} - y_x), e^{-\xi}(\dot{y} - \dot{y}_-)) \) | \( \Delta x = g(e^{\xi}(\dot{y} - y_x), e^{-\xi}(\dot{y} - \dot{y}_-)) \) |
| Lie algebra | Case | Operators | DODE | Delay relation |
|-------------|------|-----------|------|----------------|
| $\mathfrak{g}_{3,3}$ | $A_{3,6}^b(1)$ | $X_1 = \partial_x$, $X_2 = \partial_y$; $X_3 = (bx + y)\partial_x + (by - x)\partial_y$, $b \geq 0$ | $\dot{y} = (1 + y^2)^{3/2}e^b\arctan y$ | $\Delta x e^b\arctan y \sqrt{1 + y^2}$ |
| $\mathfrak{g}_{3,3}$ | $A_{3,7}^b(22)$ | $X_1 = \partial_y$, $X_2 = x\partial_y$; $X_3 = (1 + x^2)\partial_x + (x + b)y\partial_y$, $b \geq 0$ | $\dot{y} = e^b\arctan x \left(1 + x^2\right)^{3/2}f(u, v)$ | $x = x - g(u, v)$ |
| $\mathfrak{sl}(2, \mathbb{R})$ | $A_{3,8}(18)$ | $X_1 = \partial_y$, $X_2 = x\partial_x + y\partial_y$, $X_3 = 2xy\partial_x + y^2\partial_y$ | $\dot{y} = -\frac{\dot{y}}{2x} + \frac{\dot{y}^3}{x} \left(1 - \frac{2x}{\Delta y}, \frac{1}{y}, + \frac{2x}{\Delta y}\right)$ | $x = \frac{(\Delta y)^2}{x}g \left(1 - \frac{2x}{\Delta y}, \frac{1}{y}, + \frac{2x}{\Delta y}\right)$ |
| $\mathfrak{g}_{3,9}(2)$ | $X_1 = \partial_y$, $X_2 = x\partial_x + y\partial_y$, $X_3 = 2xy\partial_x + (y^2 - x^2)\partial_y$ | $\dot{y} = \left(1 + y^2\right)\dot{y} + \left(1 + y^2\right)^{3/2}f(u, v)$ | $(x - x_-)^2 + (\Delta y)^2 = xx_-g(u, v)$ |
| $\mathfrak{g}_{3,10}(17)$ | $X_1 = \partial_y$, $X_2 = x\partial_x + y\partial_y$, $X_3 = 2xy\partial_x + (y^2 + x^2)\partial_y$ | $\dot{y} = \left|1 - y^2\right|\dot{y} + \left|1 - y^2\right|^{3/2}f(u, v)$ | $(x - x_-)^2 - (\Delta y)^2 = xx_-g(u, v)$ |
| $\mathfrak{g}_{3,11}(11)$ | $X_1 = \partial_y$, $X_2 = y\partial_y$, $X_3 = y^2\partial_y$ | $\dot{y} = 2\frac{\dot{y}^2}{\Delta y} + \dot{y}f \left(x, \frac{(\Delta y)^2}{yy_-}\right)$ | $x = g \left(x, \frac{(\Delta y)^2}{yy_-}\right)$ |
| Lie algebra | Case  | Operators | DODE | Delay relation |
|-------------|-------|-----------|------|----------------|
| $\mathfrak{so}(3, \mathbb{R})$ | $\mathbf{A}_{3,12}(3)$ | \begin{align*} X_1 &= (1 + x^2)\partial_x + xy\partial_y, \\ X_2 &= xy\partial_x + (1 + y^2)\partial_y, \\ X_3 &= y\partial_x - x\partial_y \end{align*} | $\ddot{y} = \left(1 + y^2 + (y - xy)^2\right)^{3/2} \frac{1}{1 + x^2 + y^2} f(u_{12}, v_{12})$ | $(x - x_-)^2 \left(1 + y_x^2 + (y - xy_x)^2\right) \left(1 + x^2 + y^2\right) = g(u_{12}, v_{12})$ |
| $\mathfrak{n}_{1,1} \oplus \mathfrak{s}_{2,1}$ | $\mathbf{A}_{3,13}(23)$ | \begin{align*} \{X_1 = \partial_x\}, \\ \{X_2 = \partial_y; X_3 = y\partial_y\} \end{align*} | $\ddot{y} = \dot{y} f \left(\frac{\dot{y}}{\Delta y}, \frac{\dot{y}_-}{\Delta y}\right)$ | $\Delta x = g \left(\frac{\ddot{y}}{\Delta y}, \frac{\ddot{y}_-}{\Delta y}\right)$ |
| | $\mathbf{A}_{3,14}(22)$ | \begin{align*} \{X_1 = x\partial_y\}, \\ \{X_2 = \partial_y; X_3 = x\partial_x + y\partial_y\} \end{align*} | $\ddot{y} = \frac{1}{x} f (\dot{y} - y_x, \dot{y} - \dot{y}_-)$ | $x_- = x g (\dot{y} - y_x, \dot{y} - \dot{y}_-)$ |
| | $\mathbf{A}_{3,15}(20)$ | \begin{align*} \{X_1 = \partial_y\}, \{X_2 = x\partial_y\}, \{X_3 = \chi(x)\partial_y\} \end{align*} | \begin{align*} \ddot{y} &= \frac{\ddot{\chi}(x)}{\chi(x) - x\chi}(\dot{y} - y_x) \\ + \chi(x) f \left(x, \frac{\dot{y} - y_x}{\chi(x) - x\chi} - \frac{\dot{y}_- - \dot{y}_-}{\chi(x) - \chi(x_-)}\right) \end{align*} | $x_- = g \left(x, \frac{\ddot{y} - y_x}{\chi(x) - x\chi} - \frac{\ddot{y}_- - \dot{y}_-}{\chi(x) - \chi(x_-)}\right)$ |
There were used the following notations

\[ u_7 = \sqrt{1 + x^2} e^{-b \arctan x} (\dot{y} - y_x), \quad v_7 = \sqrt{1 + x^2} e^{-b \arctan x} (\dot{y}_- - y_x); \]

\[ u_9 = \frac{2 \Delta y x \dot{y} - (\Delta y)^2 - x_+^2 + x_+^2}{2 \Delta y x + ((\Delta y)^2 + x_+^2 - x_+^2) \dot{y}}, \quad v_9 = \frac{2 \Delta y x_+ \dot{y} - (\Delta y)^2 + x_+^2 - x_-^2}{-2 \Delta y x_+ + ((\Delta y)^2 + x_+^2 - x_-^2) \dot{y}_-}; \]

\[ u_{10} = \frac{2 \Delta y x \dot{y} - (\Delta y)^2 - x_-^2 + x_-^2}{2 \Delta y x - ((\Delta y)^2 + x_-^2 - x_-^2) \dot{y}}, \quad v_{10} = \frac{2 \Delta y x_- \dot{y} - (\Delta y)^2 + x_-^2 - x_+^2}{2 \Delta y x_- + ((\Delta y)^2 + x_-^2 - x_+^2) \dot{y}_-}; \]

\[ u_{12} = \frac{(x - x_-)(\dot{y} - y_x)}{\sqrt{1 + y_-^2 + (y - y_+)^2} \sqrt{1 + x_-^2 + y_-^2}}, \quad v_{12} = \frac{(x - x_-)(\dot{y}_- - y_x)}{\sqrt{1 + y_-^2 + (y_- - y_-)^2} \sqrt{1 + x_-^2 + y_-^2}}. \]
| Lie algebra | Case | Operators | DODE | Delay relation |
|-------------|------|-----------|------|----------------|
| $\mathfrak{g}_{4,1}$ | $A_{4,1}(22)$ | $X_1 = \partial_y$, $X_2 = x\partial_y$; $X_3 = x^2\partial_y$, $X_4 = \partial_x$; | $\ddot{y} = \frac{\dot{y} - \dot{y}_-}{\Delta x} + f(\dot{y} + \dot{y}_- - 2y_x)$ | $\Delta x = g(\dot{y} + \dot{y}_- - 2y_x)$ |
| $\mathfrak{g}_{4,1}$ | $A_{4,2}(22)$ | $X_1 = \partial_y$, $X_2 = x\partial_y$, $X_3 = e^x\partial_y$; $X_4 = \partial_x$ | $\ddot{y} = \frac{e^{\Delta x}}{\Delta x - 1}(\dot{y} - \dot{y}_- + f(z_2))$ | $\Delta x = g(z_2)$ |
| $\mathfrak{g}_{4,2}$ | $A_{4,3}(22)$ | $X_1 = \partial_y$, $X_2 = x\partial_y$, $X_3 = x^2\partial_y$; $X_4 = \partial_x + y\partial_y$ | $\ddot{y} = \frac{\dot{y} - \dot{y}_-}{\Delta x} + e^x f(e^{-x}(\dot{y} + \dot{y}_- - 2y_x))$ | $\Delta x = g(e^{-x}(\dot{y} + \dot{y}_- - 2y_x))$ |
| $\mathfrak{g}_{4,3}$ | $A_{4,4}^\alpha(22)$ | $X_1 = \partial_y$, $X_2 = x\partial_y$, $X_3 = |x|^\alpha\partial_y$; $X_4 = (1 - a)x\partial_x + y\partial_y$, $a \in [-1, 0) \cup (0, 1)$, $\alpha \neq \{0, \frac{1}{1 - a}, 1\}$ (see [18] for additional restriction on $a$ and $\alpha$) | $\ddot{y} = \frac{\alpha}{\sin(\Delta x)^{\alpha - 1}}(\dot{y} - \dot{y}_-) + e^x f(z_4)$ | $x_- = xg(z_4)$ |
| $\mathfrak{g}_{4,4}$ | $A_{4,5}^\alpha(22)$ | $X_1 = \partial_y$, $X_2 = x\partial_y$, $X_3 = e^{ax}\partial_y$; $X_4 = \partial_x + y\partial_y$, $a \neq 0, 1$ | $\ddot{y} = \frac{\alpha e^{\Delta x}}{\sin(\Delta x) - 1}(\dot{y} - \dot{y}_-) + e^x f(z_6)$ | $\Delta x = g(z_6)$ |
| $\mathfrak{g}_{4,5}$ | $A_{4,7}^{\alpha,\beta}(22)$ | $X_1 = \partial_y$, $X_2 = e^{ax}\cos(\beta x)\partial_y$, $X_3 = e^{ax}\sin(\beta x)\partial_y$; $X_4 = \partial_x + y\partial_y$, $\beta \neq 0$ | $\ddot{y} = \left(\frac{\beta\cos(\beta \Delta x)}{\sin(\beta \Delta x)} + \alpha\right)\dot{y} - \frac{\beta e^{\alpha \Delta x}}{\sin(\beta \Delta x)}\dot{y}_- + e^x f(z_7)$ | $\Delta x = g(z_7)$ |
| Lie algebra | Case | Operators | DODE | Delay relation |
|-------------|------|-----------|------|----------------|
| $g_{4,6}$   | $A_{4,8}(24)$ | $X_1 = \partial_y, X_2 = \partial_x, X_3 = x\partial_y; X_4 = x\partial_x$ | $\ddot{y} = \frac{1}{(\Delta x)^2} f (\Delta x (\dot{y} - y_x))$ | $\Delta x = g (\Delta x (\dot{y} - y_x)) / \dot{y} - y_x$ |
| $g_{4,8}$   | $A_{4,9}(24)$ | $X_1 = \partial_y, X_2 = \partial_x, X_3 = x\partial_y; X_4 = x\partial_x + a y \partial_y, a \neq 0, 1$ | $\ddot{y} = |\Delta x|^{-2} f (|\Delta x|^{-\alpha} (\dot{y} - y_x))$ | $|\Delta x|^{1-\alpha} = g (|\Delta x|^{1-\alpha} (\dot{y} - y_x)) / \dot{y} - y_x$ |
| $g_{4,10}$  | $A_{4,10}(25)$ | $X_1 = \partial_y, X_2 = \partial_x, X_3 = x\partial_y; X_4 = x\partial_x + (2y + x^2)\partial_y$ | $\ddot{y} = \frac{\dot{y} - \dot{y}_x}{\Delta x} + f (\Delta x \exp \left[ \frac{\dot{y} - y_x}{\Delta x} \right])$ | $\Delta x \exp \left[ \frac{\dot{y} - y_x}{\Delta x} \right] = g (\Delta x \exp \left[ \frac{\dot{y} - y_x}{\Delta x} \right])$ |
| $g_{4,11}$  | $A_{4,11}(24)$ | $X_1 = \partial_y, X_2 = \partial_x, X_3 = x\partial_y; X_4 = x\partial_x + y \partial_y$ | $\ddot{y} = \frac{1}{\Delta x} f (\dot{y} - y_x)$ | $\dot{y} - y_x = g (\dot{y} - y_x)$ |
| $A_{4,12}(23)$ | $X_1 = \partial_y, X_2 = x\partial_y, X_3 = \partial_x; X_4 = y \partial_y$ | $\ddot{y} = (\dot{y} - y_x) f \left( \frac{\dot{y} - y_x}{\dot{y} - y_x} \right)$ | $\Delta x = g (\dot{y} - y_x)$ |
| $g_{4,12}$  | $A_{4,13}(4)$ | $X_1 = \partial_x, X_2 = \partial_y; X_3 = x\partial_x + y \partial_y, X_4 = y \partial_x - x \partial_y$ | $\ddot{y} = \frac{(1 + \dot{y}^2)^2}{\Delta x (\dot{y} - y_x)} f \left( \frac{\dot{y} - y_x}{1 + y \dot{y}_x} \right)$ | $\dot{y} - y_x = g \left( \frac{\dot{y} - y_x}{1 + y \dot{y}_x} \right)$ |
| $A_{4,14}(23)$ | $X_1 = \partial_y, X_2 = x \partial_y; X_3 = y \partial_y, X_4 = (1 + x^2) \partial_x + x y \partial_y$ | $\ddot{y} = \frac{\dot{y} - y_x}{(x^2 + 1)} f(z), \ z = \sqrt{\frac{x^2 + 1}{x^2 + 1}}$ | $x_0 = \frac{x - g(z)}{1 + x g(z)}$ |
| Lie algebra  | Case   | Operators                                                                 | DODE                                                                                                                                                                                                 | Delay relation   |
|-------------|--------|---------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------|
| \(n_{1,1} \oplus \mathfrak{s}_{3,1}\) | \(A_{4,15}^{(22)}\) | \(\begin{cases} X_1 = |x|^{\frac{1}{\alpha_1}} \partial_y \\ X_2 = \partial_y, \, X_3 = x \partial_y, \, X_4 = (1 - a)x \partial_x + y \partial_y \end{cases}\), \(a \in [-1, 0) \cup (0, 1)\) | \(\ddot{y} = \frac{a}{1-a} |x|^{\frac{2a-1}{\alpha_1}} (\dot{y} - \dot{y}_-)}{\frac{|x|^{\frac{2a-1}{\alpha_1}} \text{sgn}(x) - |x_-|^{\frac{2a-1}{\alpha_1}} \text{sgn}(x_-)}{}} + x \frac{2a-1}{\alpha_1} f(z_{15})\) | \(x_- = xg(z_{15})\) |
| \(n_{1,1} \oplus \mathfrak{s}_{3,2}\) | \(A_{4,16}^{(22)}\) | \(\begin{cases} X_1 = e^{\xi} \partial_y \end{cases}\), \(X_2 = \partial_y, \, X_3 = x \partial_y, \, X_4 = \partial_x + y \partial_y \) | \(\ddot{y} = \frac{e^{\Delta x}}{e^{\Delta x} - 1} (\dot{y} - \dot{y}_-) + e^{\xi} f(z_{16})\) | \(\Delta x = g(z_{16})\) |
| \(n_{1,1} \oplus \mathfrak{s}_{3,3}\) | \(A_{4,17}^{(22)}\) | \(\begin{cases} X_1 = \partial_y \end{cases}\), \(X_2 = \partial_x, \, X_3 = e^{\alpha \xi} \cos x \partial_y, \, X_4 = e^{\alpha \xi} \sin x \partial_y \), \(\alpha \geq 0\) | \(\ddot{y} = \left(\frac{\cos(\Delta x)}{\sin(\Delta x)} + \alpha\right) \dot{y} - \frac{e^{\alpha \xi}}{\sin(\Delta x)} \dot{y}_- + f(z_{17})\) | \(\Delta x = g(z_{17})\) |
| Lie algebra | Case | Operators | DODE | Delay relation |
|-------------|------|-----------|------|----------------|
| $n_{1,1} \oplus \mathfrak{sl}(2, \mathbb{R})$ | $A_{4,18}(14)$ | $\{X_1 = \partial_x\}$, $\{X_2 = \partial_y, X_3 = y\partial_y, X_4 = y^2\partial_y\}$ | $\ddot{y} = 2 \frac{\dot{y}^2}{\Delta y} + \dot{y} f \left( \frac{(\Delta y)^2}{\dot{y}^2} \right)$ | $\Delta x = g \left( \frac{(\Delta y)^2}{\dot{y}^2} \right)$ |
| $A_{4,19}(19)$ | $\{X_1 = x\partial_x\}$, $\{X_2 = \partial_y, X_3 = x\partial_x + y\partial_y, X_4 = 2xy\partial_x + y^2\partial_y\}$ | $\ddot{y} = -\frac{\dot{y}}{2x} + \frac{x\dot{y}^3}{(\Delta y)^2}$ \times $f \left( \frac{(\Delta y)^2}{xx} \left( \frac{1}{\dot{y}} + 2x - x \frac{\Delta y}{\dot{y}} \right) \right)$ | $\frac{(\Delta y)^2}{xx} \left( \frac{1}{\dot{y}} + 2x - x \frac{\Delta y}{\dot{y}} \right)^2$ | $\frac{(\Delta y)^2}{xx} \left( \frac{1}{\dot{y}} + 2x - x \frac{\Delta y}{\dot{y}} \right)^2$ |
| $2s_{2,1}$ | $A_{4,20}(13)$ | $\{X_1 = \partial_x\}$, $\{X_2 = x\partial_x\}$, $\{X_3 = \partial_y, X_4 = y\partial_y\}$ | $\ddot{y} = \frac{\dot{y}}{\Delta x} f \left( \frac{\dot{y}}{\bar{y}} \right)$ | $\Delta x = \frac{\Delta y}{\dot{y}} g \left( \frac{\dot{y}}{\bar{y}} \right)$ |
| $A_{4,21}(23)$ | $\{X_1 = \partial_y\}$, $\{X_2 = x\partial_x + y\partial_y\}$, $\{X_3 = x\partial_y, X_4 = x\partial_x\}$ | $\ddot{y} = \frac{\dot{y} - y_x}{x} f \left( \frac{\dot{y} - y_x}{\bar{y} - y_x} \right)$ | $x_- = x g \left( \frac{\dot{y} - y_x}{\bar{y} - y_x} \right)$ |
There were used the notations

\[
z_2 = \left( \frac{e^{\Delta x} - 1}{\Delta x} - 1 \right) \dot{y} + \left( e^{\Delta x} - \frac{e^{\Delta x} - 1}{\Delta x} \right) \dot{y}_- - (e^{\Delta x} - 1)y_x,
\]

\[
z_4 = x^{\frac{2a-1}{a-1}} \alpha |x_-|^{\alpha - 1} \text{sgn}(x_-)(\dot{y} - y_x) - \alpha |x_-|^{\alpha - 1} \text{sgn}(x_-)(\dot{y}_- - y_x),
\]

\[
z_6 = e^{-x} \left[ \left( \frac{e^{\alpha \Delta x} - 1}{\Delta x} - a \right) \dot{y} + \left( a e^{\alpha \Delta x} - \frac{e^{\alpha \Delta x} - 1}{\Delta x} \right) \dot{y}_- - a(e^{\alpha \Delta x} - 1)y_x \right],
\]

\[
z_7 = e^{-x} \left[ \left( \frac{\beta \cos(\beta \Delta x) - \beta e^{-\alpha \Delta x}}{\sin(\beta \Delta x)} - \alpha \right) \dot{y} + \left( \frac{\beta \cos(\beta \Delta x) - \beta e^{\alpha \Delta x}}{\sin(\beta \Delta x)} + \alpha \right) \dot{y}_- + (\alpha^2 + \beta^2)(y - y_-) \right],
\]

\[
z_{15} = x^{\frac{2a}{a-1}} \left[ |x_-|^{\frac{a}{a-1}} \text{sgn}(x_-)(\dot{y} - y_x) - |x_-|^{\frac{a}{a-1}} \frac{|x|}{x-x_-} (\dot{y} - y_-) \right],
\]

\[
z_{16} = e^{-x} \left[ \left( \frac{e^{\Delta x} - 1}{\Delta x} - 1 \right) \dot{y} + \left( e^{\Delta x} - \frac{e^{\Delta x} - 1}{\Delta x} \right) \dot{y}_- - (e^{\Delta x} - 1)y_x \right],
\]

\[
z_{17} = \left( \frac{\cos(\Delta x) - e^{-\alpha \Delta x}}{\sin(\Delta x)} - \alpha \right) \dot{y} + \left( \frac{\cos(\Delta x) - e^{\alpha \Delta x}}{\sin(\Delta x)} + \alpha \right) \dot{y}_- + (\alpha^2 + 1)(y - y_-).
\]
| Lie algebra | Case | Operators | DODE | Delay relation |
|-------------|------|-----------|------|----------------|
| $5_{5,33}$ | A$_{5,1}(24)$ | $X_1 = \partial_y$, $X_2 = x\partial_y$, $X_3 = x^2\partial_y$, $X_4 = \partial_x$; $X_5 = x\partial_x$ | $\ddot{y} = 2\frac{\dot{y} - y_x}{\Delta x} + \frac{C_1}{(\Delta x)^2}$ | $\Delta x = \frac{C_2}{\dot{y} + \dot{y}_- - 2y_x}$ |
| $5_{5,34}$ | A$_{5,2}(25)$ | $X_1 = \partial_y$, $X_2 = x\partial_y$, $X_3 = x^2\partial_y$, $X_4 = \partial_x$; $X_5 = x\partial_x + (3y + x^3)\partial_y$ | $\ddot{y} = 4\frac{\dot{y} + 2\dot{y}_- - 6y_x}{\Delta x} + C_1\Delta x$ | $\frac{1}{(\Delta x)^2}(\dot{y} + \dot{y}_- - 2y_x) - \ln|\Delta x| = C_2$ |
| $5_{5,35}$ | A$_{5,3}(24)$ | $X_1 = \partial_y$, $X_2 = x\partial_y$, $X_3 = x^2\partial_y$, $X_4 = \partial_x$; $X_5 = x\partial_x + \alpha y\partial_y$, $\alpha \neq \{0, 2\}$ | $\ddot{y} = 2\frac{\dot{y} - y_x}{\Delta x} + C_1|\Delta x|^{\alpha - 2}$ | $|\Delta x|^{\alpha - 1} = C_2 (\dot{y} + \dot{y}_- - 2y_x)$ |
| $5_{5,36}$ | A$_{5,4}(24)$ | $X_1 = \partial_y$, $X_2 = x\partial_y$, $X_3 = x^2\partial_y$, $X_4 = \partial_x$; $X_5 = x\partial_x + 2y\partial_y$ | $\ddot{y} = 2\frac{\dot{y} - y_x}{\Delta x} + C_1$ | $\Delta x = C_2 (\dot{y} + \dot{y}_- - 2y_x)$ |
| $5_{5,44}$ | A$_{5,6}(26)$ | $X_1 = \partial_y$, $X_2 = x\partial_y$, $X_3 = \partial_x$; $X_4 = x\partial_x$, $X_5 = y\partial_y$ | $\ddot{y} = C_1\frac{\dot{y} - y_x}{\Delta x}$ | $\Delta x = \frac{(1 - C_2)\Delta y}{\dot{y} - C_2\dot{y}_-}$ |
| Lie algebra | Case | Operators | DODE | Delay relation |
|------------|------|-----------|------|----------------|
| **sl(2, R) ⊕ 2n_{1,1}** | A_{5,7}(27) | \{X_1 = \partial_x, X_2 = 2x\partial_x + y\partial_y, X_3 = x^2\partial_x + xy\partial_y\} \& \{X_4 = \partial_y\}, \{X_5 = x\partial_y\} | \ddot{y} = C_1 (\dot{y} - y_x)^3 | \Delta x = \frac{C_2}{(\dot{y} - y_x)(\dot{y} - y_x)} |
| | A_{5,8}(5) | \{X_1 = x\partial_x - y\partial_y, X_2 = y\partial_x, X_3 = x\partial_y\} \& \{X_4 = \partial_x\}, \{X_5 = \partial_y\} | \ddot{y} = C_1(\Delta x)^3(\dot{y} - y_x)^3 | (\Delta x)^2 = \frac{C_2(\dot{y} - \dot{y} - y_x)}{(\dot{y} - y_x)(\dot{y} - y_x)} |
| **g_{2,1} \oplus sl(2, R)** | A_{5,9}(15) | \{X_1 = \partial_x, X_2 = x\partial_x\}, \{X_3 = \partial_y, X_4 = y\partial_y, X_5 = y^2\partial_y\} | \ddot{y} = 2 \frac{\dot{y}^2}{\Delta y} + C_1 \frac{\dot{y}}{\Delta x} | (\Delta x)^2 = C_2 \frac{(\Delta y)^2}{\dot{y} \dot{y} -} |
| **so(3, 1)** | A_{6,1}(7) | \begin{align*} X_1 &= \partial_x, X_2 = \partial_y, X_3 = x\partial_x + y\partial_y, X_4 = y\partial_x - x\partial_y, \\
X_5 &= (x^2 - y^2)\partial_x + 2xy\partial_y, X_6 = 2xy\partial_x + (y^2 - x^2)\partial_y \end{align*} | \ddot{y} - 2 {\dot{y}^2 + 1 \frac{\dot{y} - y_x}{\Delta x}} = 0 | arctan(\dot{y}) + arctan(\dot{y} -) - 2 arctan(y_x) = C |
| **sl(2, R) \times 3n_{1,1}** | A_{6,2}(27) | \{X_1 = \partial_x, X_2 = x\partial_x + y\partial_y, X_3 = x^2\partial_x + 2xy\partial_y\} \& \{X_4 = \partial_y\}, \{X_5 = x\partial_y\}, \{X_6 = x^2\partial_y\} | \ddot{y} - 2 \frac{\dot{y} - y_x}{\Delta x} = 0 | \dot{y} + \dot{y} - 2y_x = C |
| **sl(2, R) \oplus sl(2, R)** | A_{6,3}(16) | \begin{align*} X_1 &= \partial_x, X_2 = x\partial_x, X_3 = x^2\partial_x\}, \\
X_4 &= \partial_y, X_5 = y\partial_y, X_6 = y^2\partial_y\} | \ddot{y} - 2 \frac{\dot{y}^2}{\Delta y} + \frac{2\dot{y}}{\Delta x} = 0 | (\Delta x)^2 = C \frac{(\Delta y)^2}{\dot{y} \dot{y} -} |