ON HERMITE HADAMARD INEQUALITIES FOR PRODUCT OF TWO log-ϕ-CONVEX FUNCTIONS

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Abstract. In this paper, we introduce the notion of log-ϕ-convex functions and present some properties and representation of such functions. We obtain some results of the Hermite Hadamard inequalities for product log-ϕ-convex functions.

1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g., [4], [8, p.137]). These inequalities state that if \( f : I \to \mathbb{R} \) is a convex function on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \), then

\[
(1.1) \quad f \left( \frac{a + b}{2} \right) \leq \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

The inequality (1.1) has evoked the interest of many mathematicians. Especially in the last three decades numerous generalizations, variants and extensions of this inequality have been obtained, to mention a few, see ([8]-[10]) and the references cited therein.

The function \( f : [a, b] \subset \mathbb{R} \to \mathbb{R} \), is said to be convex if the following inequality holds

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)
\]

for all \( x, y \in [a, b] \) and \( \lambda \in [0, 1] \). We say that \( f \) is concave if \((-f)\) is convex.

A function \( f : I \to [0, \infty) \) is said to be log-convex or multiplicatively convex if \( \log f \) is convex, or, equivalently, if for all \( x, y \in I \) and \( t \in [0, 1] \) one has the inequality:

\[
(1.2) \quad f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}.
\]

We note that if \( f \) and \( g \) are convex and \( g \) is increasing, then \( g \circ f \) is convex; moreover, since \( f = \exp(\log f) \), it follows that a log-convex function is convex, but the converse may not necessarily be true [7]. This follows directly from (1.2) because, by the arithmetic-geometric mean inequality, we have

\[
[f(x)]^t [f(y)]^{1-t} \leq tf(x) + (1-t)f(y)
\]

for all \( x, y \in I \) and \( t \in [0, 1] \).

For some results related to this classical results, (see [4], [5], [9], [10]) and the references therein. Dragomir and Mond [6] proved the following Hermite-Hadamard type inequalities for the log-convex functions:

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\begin{align}
(1.3) \quad f \left( \frac{a + b}{2} \right) & \leq \exp \left[ \frac{1}{b - a} \int_a^b \ln [f(x)] \, dx \right] \\
& \leq \frac{1}{b - a} \int_a^b G(f(x), f(a + b - x)) \, dx \\
& \leq \frac{1}{b - a} \int_a^b f(x) \, dx \\
& \leq L(f(a), f(b)) \\
& \leq \frac{f(a) + f(b)}{2},
\end{align}

where $G(p, q) = \sqrt{pq}$ is the geometric mean and $L(p, q) = \frac{p - q}{\ln p - \ln q}$ ($p \neq q$) is the logarithmic mean of the positive real numbers $p, q$ (for $p = q$, we put $L(p, q) = p$).

Let us consider a function $\varphi : [a, b] \to [a, b]$ where $[a, b] \subset \mathbb{R}$. Youness have defined the $\varphi$-convex functions in [11]:

**Definition 1.** A function $f : [a, b] \to \mathbb{R}$ is said to be $\varphi$-convex on $[a, b]$ if for every two points $x, y \in [a, b]$ and $t \in [0, 1]$ the following inequality holds:

$$f(t\varphi(x) + (1 - t)\varphi(y)) \leq tf(\varphi(x)) + (1 - t)f(\varphi(y)).$$

In [2], Cristescu proved the following results for the $\varphi$-convex functions

**Lemma 1.** For $f : [a, b] \to \mathbb{R}$, the following statements are equivalent:

(i) $f$ is $\varphi$-convex functions on $[a, b]$,

(ii) for every $x, y \in [a, b]$, the mapping $g : [0, 1] \to \mathbb{R}$, $g(t) = f(t\varphi(x) + (1 - t)\varphi(y))$ is classically convex on $[0, 1]$.

Obviously, if function $\varphi$ is the identity, then the classical convexity is obtained from the previous definition. Many properties of the $\varphi$-convex functions can be found, for instance, in [11], [2], [11].

In this paper, we introduce the notion of log-$\varphi$-convex functions and we obtain a representation of log-$\varphi$-convex. Finally, a version of Hermite–Hadamard-type inequalities for log-$\varphi$-convex functions is presented.

2. Main Results

Let us consider a $\varphi : [a, b] \to [a, b]$ where $[a, b] \subset \mathbb{R}$ and $I$ stands for a convex subset of $\mathbb{R}$. We say that a function $f : I \to \mathbb{R}^+$ is a log-$\varphi$-convex if

\begin{align}
(2.1) \quad f(t\varphi(x) + (1 - t)\varphi(y)) & \leq [f(\varphi(x))]^t [f(\varphi(y))]^{1 - t} 
\end{align}

for all $x, y \in I$ and $t \in [0, 1]$. We say that $f$ is a log-$\varphi$-midconvex if $2.1$ is assumed only for $t = \frac{1}{2}$, that is

$$f \left( \frac{\varphi(x) + \varphi(y)}{2} \right) \leq \sqrt{f(\varphi(x))f(\varphi(y))}, \text{ for } x, y \in I$$

Obviously, if function $\varphi$ is the identity, then the classical logarithmic convexity is obtained from $2.1$. 
From the above definitions, we have

\[ f(t \varphi(x) + (1 - t) \varphi(y)) \leq \left[ f(\varphi(x)) \right]^t \left[ f(\varphi(y)) \right]^{1-t} \leq tf(\varphi(x)) + (1 - t)f(\varphi(y)) \leq \max \{ f(\varphi(x)), f(\varphi(y)) \}. \]

**Lemma 2.** For \( f : [a, b] \to \mathbb{R}^+ \), the following statements are equivalent:

(i) \( f \) is log-\( \varphi \)-convex functions on \([a, b]\),

(ii) for every \( x, y \in [a, b] \), the mapping

\[ g : [0, 1] \to \mathbb{R}^+, \quad g(t) = f(t \varphi(x) + (1 - t) \varphi(y)) \]

is classically log-convex on \([0, 1]\).

**Proof.** Let us consider two points \( x, y \in [a, b] \), \( \lambda \in [0, 1] \) and \( t_1, t_2 \in [0, 1] \). Then, we obtain

\[
\begin{align*}
g(\lambda t_1 + (1 - \lambda)t_2) &= f([\lambda t_1 + (1 - \lambda)t_2] \varphi(x) + [1 - \lambda t_1 - (1 - \lambda)t_2] \varphi(y)) \\
&= f(\lambda [t_1 \varphi(x) + (1 - t_1) \varphi(y)] + (1 - \lambda) [t_2 \varphi(x) + (1 - t_2) \varphi(y)]) \\
&\leq \left[ f(t_1 \varphi(x) + (1 - t_1) \varphi(y)) \right]^{\lambda} \left[ f(t_2 \varphi(x) + (1 - t_2) \varphi(y)) \right]^{1-\lambda} \\
&= \left[ g(t_1) \right]^{\lambda} \left[ g(t_2) \right]^{1-\lambda}
\end{align*}
\]

which gives that \( g \) is log-convex function.

Conversely, if \( g \) is log-convex function for \( x, y \in [a, b] \), \( \lambda \in [0, 1] \) and \( t_1 = 1, t_2 = 0 \), then we get

\[
\begin{align*}
f(\lambda \varphi(x) + (1 - \lambda) \varphi(y)) &= g(\lambda 1 + (1 - \lambda) 0)) \\
&\leq \left[ g(1) \right]^{\lambda} \left[ g(0) \right]^{1-\lambda} \\
&= \left[ f(\varphi(x)) \right]^{\lambda} \left[ f(\varphi(y)) \right]^{1-\lambda}
\end{align*}
\]

which shows that \( f \) is log-\( \varphi \)-convex. This completes to proof. \( \square \)

We give now a new Hermite–Hadamard-type inequalities for log-\( \varphi \)-convex functions:
Theorem 1. If \( f : [a, b] \to \mathbb{R}^+ \) is log-\( \varphi \)-convex for the continuous function \( \varphi : [a, b] \to [a, b] \), then

\[
(2.2) \quad f \left( \frac{\varphi(a) + \varphi(b)}{2} \right) \leq \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} G \left( f(x), f(\varphi(a) + \varphi(b) - x) \right) dx
\]

\[
\leq \frac{1}{\varphi(b) - \varphi(a)} \int f(x)dx
\]

\[
\leq \frac{f(\varphi(b)) - f(\varphi(a))}{\log f(\varphi(b)) - \log f(\varphi(a))} = L \left( f(\varphi(b)), f(\varphi(a)) \right)
\]

\[
\leq \frac{f(\varphi(a)) + f(\varphi(b))}{2}.
\]

Proof. Since \( f \) be log-\( \varphi \)-convex functions, we have that for all \( t \in [0, 1] \)

\[
f \left( \frac{\varphi(a) + \varphi(b)}{2} \right) = f \left( \frac{t\varphi(a) + (1-t)\varphi(b)}{2} \right) \leq \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} \sqrt{f(t\varphi(a) + (1-t)\varphi(b))} \sqrt{f((1-t)\varphi(a) + t\varphi(b))} dt
\]

Integrating the above inequality with respect to \( t \) over \([0, 1]\) and we also use the substitution \( x = (1-t)\varphi(a) + t\varphi(b) \), we obtain

\[
f \left( \frac{\varphi(a) + \varphi(b)}{2} \right)
\]

\[
\leq \int_{0}^{1} \sqrt{f(t\varphi(a) + (1-t)\varphi(b))} \sqrt{f((1-t)\varphi(a) + t\varphi(b))} dt
\]

\[
= \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} \sqrt{f(x)f(\varphi(a) + \varphi(b) - x)} dx
\]

\[
\leq \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} A \left( f(x), f(\varphi(a) + \varphi(b) - x) \right) dx
\]

and so for

\[
\int_{\varphi(a)}^{\varphi(b)} f(x)dx = \int_{\varphi(a)}^{\varphi(b)} f(\varphi(a) + \varphi(b) - x)dx
\]
From the log-$\varphi$-convexity of $f$, we have
\begin{equation}
\frac{1}{\varphi(b) - \varphi(a)} \int \varphi(b) f(x) dx \leq \frac{1}{\varphi(b) - \varphi(a)} \int \varphi(a) f(x) dx.
\end{equation}

Thus, from (2.3) and (2.4) we obtain required result (2.2). This completes to proof. \hfill \Box

\textbf{Theorem 2.} If $f, g : [a, b] \to \mathbb{R}^+$ is log-$\varphi$-convex for the continuous function $\varphi : [a, b] \to [a, b]$, then
\begin{equation}
\frac{1}{\varphi(b) - \varphi(a)} \int \varphi(b) f(x) g(x) dx \leq L (f(\varphi(b))g(\varphi(b)), f(\varphi(a))g(\varphi(a)))
\end{equation}

\begin{equation}
\leq \frac{1}{4} \left( (|f(\varphi(b))| + |f(\varphi(a))|) L([f(\varphi(b))], |f(\varphi(a))|) \\
+ \frac{1}{4} \left( (|g(\varphi(b))| + |g(\varphi(a))|) L([g(\varphi(b))], |g(\varphi(a))|) \right) \right).
\end{equation}
\textit{Proof.} Since \( f \) and \( g \) be log-\( \varphi \)-convex functions, we have that for all \( t \in [0, 1] \)
\[
f(t\varphi(a) + (1-t)\varphi(b)) \leq [f(\varphi(a))]^t [f(\varphi(b))]^{1-t}
\]
and
\[
g(t\varphi(a) + (1-t)\varphi(b)) \leq [g(\varphi(a))]^t [g(\varphi(b))]^{1-t}.
\]
Thus, it follows that
\[
\frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)g(x) \, dx
\]
\[
\leq \int_0^1 \left[ f(\varphi(a)) \right]^t \left[ f(\varphi(b)) \right]^{1-t} \left[ g(\varphi(a)) \right]^t \left[ g(\varphi(b)) \right]^{1-t} \, dt
\]
\[
= f(\varphi(b))g(\varphi(b)) \int_0^1 \left[ \frac{f(\varphi(a))g(\varphi(a))}{f(\varphi(b))g(\varphi(b))} \right]^t \, dt
\]
\[
= \frac{f(\varphi(b))g(\varphi(b))}{\log f(\varphi(b))g(\varphi(b)) - \log f(\varphi(a))g(\varphi(a))} \left[ \frac{f(\varphi(a))g(\varphi(a))}{f(\varphi(b))g(\varphi(b))} - 1 \right]
\]
\[
= \frac{f(\varphi(b))g(\varphi(b)) - f(\varphi(a))g(\varphi(a))}{\log f(\varphi(b))g(\varphi(b)) - \log f(\varphi(a))g(\varphi(a))}
\]
\[
= L( f(\varphi(b))g(\varphi(b)), f(\varphi(a))g(\varphi(a)) )
\]
\[
\leq \frac{1}{2} \int_0^1 \left( \left[ f(t\varphi(a) + (1-t)\varphi(b)) \right]^2 + \left[ g(t\varphi(a) + (1-t)\varphi(b)) \right]^2 \right) \, dt
\]
\[
\leq \frac{1}{2} \int_0^1 \left( \left[ f(\varphi(a)) \right]^{2t} \left[ f(\varphi(b)) \right]^{2-2t} + \left[ g(\varphi(a)) \right]^{2t} \left[ g(\varphi(b)) \right]^{2-2t} \right) \, dt
\]
\[
= \frac{1}{4} \left\{ \left[ f(\varphi(b)) \right]^2 \int_0^1 \left[ f(\varphi(a)) \right]^u \, du + \left[ g(\varphi(b)) \right]^2 \int_0^1 \left[ g(\varphi(a)) \right]^u \, du \right\}
\]
\[
= \frac{1}{4} \left\{ \left[ f(\varphi(b)) \right]^2 - \left[ f(\varphi(a)) \right]^2 \log f(\varphi(b)) - \log f(\varphi(a)) \right\} + \left[ g(\varphi(b)) \right]^2 - \left[ g(\varphi(a)) \right]^2 \log g(\varphi(b)) - \log g(\varphi(a)) \right\}
\]
\[
= \frac{1}{4} \left\{ \left( [f(\varphi(b))] + [f(\varphi(a))] \right) L( [f(\varphi(b))] , [f(\varphi(a))] ) \right\}
\]
\[
+ \frac{1}{4} \left\{ \left( [g(\varphi(b))] + [g(\varphi(a))] \right) L( [g(\varphi(b))] , [g(\varphi(a))] ) \right\}
\]
which is the required (2.5). This proves the theorem. \( \square \)
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