Thermodynamic Behavior of Field Equations for $f(R)$ Gravity

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Abstract

Recently it has shown that Einstein’s field equations can be rewritten into a form of the first law of thermodynamics both at event horizon of static spherically symmetric black holes and apparent horizon of Friedmann-Robertson-Walker (FRW) universe, which indicates intrinsic thermodynamic properties of spacetime horizon. In the present paper we deal with the so-called $f(R)$ gravity, whose action is a function of the curvature scalar $R$. In the setup of static spherically symmetric black hole spacetime, we find that at the event horizon, the field equations of $f(R)$ gravity can be written into a form

$$dE = TdS - PdV + Td\bar{S},$$

where $T$ is the Hawking temperature and $S = Af'(R)/4G$ is the horizon entropy of the black hole, $E$ is the horizon energy of the black hole, $P$ is the radial pressure of matter, $V$ is the volume of black hole horizon, and $d\bar{S}$ can be interpreted as the entropy production term due to nonequilibrium thermodynamics of spacetime. In the setup of FRW universe, the field equations can also be cast to a similar form

$$dE = TdS + WdV + Td\bar{S},$$

at the apparent horizon, where $W = (\rho - P)/2$, $E$ is the energy of perfect fluid with energy density $\rho$ and pressure $P$ inside the apparent horizon. Compared to the case of Einstein’s general relativity, an additional term $d\bar{S}$ also appears here. The appearance of the additional term is consistent with the argument recently given by Eling et al. (gr-qc/0602001) that the horizon thermodynamics is non-equilibrium one for the $f(R)$ gravity.

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1 Introduction

The four laws of black hole mechanics within Einstein’s theory of general relativity are very closely analogous to the four laws of the usual thermodynamics [1]. The quantities that provide a basis for the black hole thermodynamics are the Hawking temperature \( T = \kappa/2\pi \), entropy \( S = A/4G \) and energy \( E \), where \( \kappa \) is the surface gravity, \( A \) is the area of the event horizon and \( E = M \) is the mass of the black hole. The Hawking temperature together with black hole entropy is connected through the identity \( TdS = dE \) and is usually called the first law of black hole thermodynamics [1, 2, 3]. In general, the first law of black hole thermodynamics is related with the energy change, \( dE \) when a black hole varies from one stationary state to another by

\[
dE = \frac{\kappa}{8\pi} dA + \text{work terms}, \quad \text{or} \quad dE = TdS + \text{work terms.} \tag{1}
\]

The “work terms” are given differently depending upon the type of the black holes. For Kerr-Newman black hole family, the first law of black hole thermodynamics states that the differences in mass \( M \), the horizon area \( A \), the angular momentum \( J \), and the electric charge \( Q \) of two nearby black holes are connected by

\[
dM = TdS + \Omega dJ + \Phi dQ,
\]

where \( \Omega = \frac{\partial M}{\partial J} \) is the angular velocity and \( \Phi = \frac{\partial M}{\partial Q} \) is the electric potential. The first law of black hole thermodynamics seemingly indicates that there should be some relationship between thermodynamics and Einstein field equations because black hole solutions are derived from the Einstein equations and the geometric quantities (horizon area, surface gravity) of the spacetime metric are related with thermodynamical quantities (entropy, temperature) of the thermal system. In this regard, Jacobson [4] is the first one to explore such a relation. He found that it is indeed possible to derive the Einstein’s equations from the proportionality of entropy to the horizon area together with the fundamental relation \( \delta Q = TdS \), assuming the relation holds for all local Rindler causal horizons through each spacetime point. Here \( \delta Q \) and \( T \) are the energy flux and Unruh temperature seen by an accelerated observer just inside the horizon. For the so-called \( f(R) \) gravity, however, Eling, Guedens, and Jacobson [5] recently have shown that in order to derive the corresponding field equations by using the procedure in [1], a treatment with nonequilibrium thermodynamics of spacetime is needed, in which an additional entropy production term \( d\tilde{S} \) has to be added to the Clausius relation, \( \delta Q/T = dS + d\tilde{S} \). More recently, we have shown that a similar entropy production term is also required for the scalar-tensor theory, while it is not for the Lovelock gravity [6].

If one identifies the black hole mass \( M \) as an energy \( E \), compared to the standard first law of thermodynamics \( (dE = TdS - PdV) \), obviously a work term related to the volume change is absent in the first law of black hole thermodynamics \( dE = TdS \). More recently, by generalizing earlier works [7], Paranjape, Sarkar and Padmanabhan [8] have considered a special kind of
spherically symmetric black hole spacetimes and found that it is indeed possible to interpret Einstein’s equations as the thermodynamic identity $TdS = dE + PdV$ by considering black hole horizon as a boundary of the system, here $P$ is the radial pressure of matter at the horizon. They have also shown that the field equations for Gauss-Bonnet gravity and in more general Lanczos-Lovelock action in spherically symmetric space-time can also be expressed as $TdS = dE + PdV$.

These results provide a deeper relation between the thermodynamics of horizon and Einstein’s field equations of static, spherical symmetric black hole spacetimes.

The thermodynamical properties of the black holes horizon can be extended to the space-time horizons other than black hole horizon. For example, the de Sitter space time with radius $\ell$, there exists a cosmological event horizon. This horizon, like a black hole horizon, can be regarded as a thermodynamical system \[9\] associated with the Hawking temperature $T = 1/2\pi \ell$ and entropy $S = A/4G$, where $A = 4\pi \ell^2$ is the cosmological horizon area. For an asymptotic de Sitter space, like Schwarzschild-de Sitter space-time, there still exists the cosmological horizon which behaves like a black hole horizon with entropy proportional to the area of the cosmological horizon and whose Hawking temperature is given by $T = \kappa/2\pi$, where $\kappa$ is the surface gravity of the cosmological horizon. It is easy to verify that the cosmological horizons of these space-times satisfy the first law of black hole thermodynamics of the form $TdS = -dM$ \[10\], where the minus appears due to the fact that when the black hole mass $M$ increases, the cosmological horizon radius decreases. In this framework, the first law $TdS = -dE$ of cosmological horizon thermodynamics does not involve the work term as well. By applying the first law of thermodynamics ($TdS = -dE$) to the apparent horizon of the an $(n+1)$-dimensional Friedman-Robertson-Walker (FRW) universe and by working out the heat flow through the apparent horizon, one of the present authors and Kim \[11\] are able to derive the Friedmann equations of the FRW universe with any spatial curvature, during which one assumes that the apparent horizon has temperature and entropy expressed by

$$T = \frac{1}{2\pi \tilde{r}_A}, \quad S = \frac{A}{4G},$$

respectively, where $A$ is the area of the apparent horizon with radius $\tilde{r}_A$. Also by using the entropy expression of a static spherically symmetric black hole in the Gauss-Bonnet gravity and in more general Lovelock gravity, they reproduce the corresponding Friedmann equations in each gravity. The possible extensions to the scalar-tensor gravity and $f(R)$ gravity theory have been studied in reference \[12\]. In the cosmological setting, related discussions see also, \[13\] \[14\] \[15\] \[16\].

On the other hand, we know that the Friedmann equations of FRW universe are the field equations with a source of perfect fluid. In this cosmological setup, different from the case of black hole spacetimes discussed in \[8\], there is a well-defined concept of pressure $P$ and energy density $\rho$. Therefore, it is required to establish a new relationship between the first law
of thermodynamics with work term and the Friedmann equations of FRW universe. In this respect, the authors of the present work found [17] that by employing Misner-Sharp energy relation inside a sphere of radius \( \tilde{r}_A \) of apparent horizon, it is possible to rewrite the differential form of the Friedman equations as a universal form \( TdS = dE + WdV \), where \( W = (\rho - P)/2 \). This is nothing but the unified first law of trapping horizon firstly suggested by Hayward [18]. We extended this procedure to the Gauss-Bonnet gravity and in more general Lovelock gravity, and verified that in both cases, the Friedmann equations near apparent horizon can also be interpreted as \( TdS = dE + WdV \), where \( E \) is the total matter energy \( (\rho V) \) inside the apparent horizon. The thermodynamics of apparent horizon in the brane world scenario also obeys such a formula [19]. However, it is interesting to investigate to what extent the horizon thermodynamics can be developed in a spacetime against quantum gravitational effects. A feasible way to proceed in this direction is to weigh up the effects of higher curvature corrections to Einstein gravity, since such corrections naturally occur due to quantum effects [20]. As a special case of considering the effect of higher curvature corrections, we will deal with in this paper the so-called \( f(R) \) gravity, whose action is an arbitrary function of curvature scalar \( R \). When \( f(R) = R \), the Einstein’s general relativity is recovered.

This paper is organized as follows. In Sec. 2, we will discuss the field equations of a general class of spherical symmetric black hole spacetimes at the horizon and rewrite them into a form of the first law of thermodynamics by using Hawking temperature and entropy associated with the black hole horizon. In Sec. 3, we will discuss, in the cosmological setup, the corresponding field equations for the \( f(R) \) gravity. We will show that, at the apparent horizon of FRW universe, Friedmann equations can also be written into a form of first law of thermodynamics. But in both cases, an additional entropy production term will appear. Finally in Sec. 4 we will discuss our results.

2 Thermodynamic behavior of field equations in the black hole spacetime

We consider a class of static, spherically symmetric spacetimes expressed by the line element

\[
 ds^2 = -U(r)dt^2 + \frac{1}{U(r)}dr^2 + L^2(r)(d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \( U(r) \) and \( L(r) \) are two functions of \( r \). We assume that the function \( U(r) \) has a simple zero at \( r = a \) and \( U'(a) \neq 0 \) but has a finite value at \( r = a \). This defines a space-time horizon at \( r = a \). Note that \( L(r) \) is a finite and continuous function of \( r \), therefore the horizon has a finite area \( A = 4\pi L^2(a) \) and hence the associated entropy of the horizon is determined by the
location of the zero at \( r = a \) of the function \( U(r) \). The associated Hawking temperature with the horizon can be obtained from the periodicity consideration of the Euclidean time. One can obtain a Euclidean metric by Wick rotating the time coordinate \( t \to -i\tau \). This metric will be regular at the horizon, \( r = a \), if \( \tau \) is taken to be an angular variable with a period \( \beta = 4\pi/U'(a) \). This period is just the inverse Hawking temperature \( T = 1/\beta \) of the black hole. So the non-zero surface gravity \( \kappa = U'(a)/2 \) is related to the Hawking temperature via \( T = \kappa/2\pi \).

The components of the Einstein tensor for the metric (3) are given by

\[
G^0_0 = \frac{1}{L^2}(-1 + UL'^2 + L(U'L' + 2UL'')),
\]

\[
G^1_1 = \frac{1}{L^2}(-1 + UL'L' + UL'^2),
\]

and

\[
G^2_2 = G^3_3 = \frac{1}{2L}(2U'L' + LU'' + 2UL''),
\]

where prime stands for the derivative with respect to \( r \). It can be seen readily that the components \( G^0_0 \) and \( G^1_1 \) of Einstein tensor are not equal but at the spacetime horizon where \( U(a) = 0 \), they become equal and are given by

\[
G^0_0|_{r=a} = G^1_1|_{r=a} = \frac{1}{L^2}(-1 + LU'L'),
\]

where \( U' \) and \( L' \) are evaluated at \( r = a \). The explicit evaluation of curvature scalar reads

\[
R = -\frac{1}{L^2}(-2 + 2UL'^2 + L^2U'' + 4L(U'L' + UL'')).
\]

At the horizon, the curvature scalar reduces to

\[
R|_{r=a} = -\frac{1}{L^2}(-2 + L^2U'' + 4LU'L').
\]

Now let us consider the Lagrangian of the \( f(R) \) gravity

\[
L = \frac{1}{16\pi G} f(R) + L_m,
\]

where \( f(R) \) is a continuous function of curvature scalar \( R \) and \( L_m \) is the Lagrangian density of matter fields. Varying the action yields corresponding equations of motion

\[
f_{,R}(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} f(R) - \nabla_\mu \nabla_\nu f_{,R}(R) + g_{\mu\nu} \nabla^2 f_{,R}(R) = 8\pi G T^m_{\mu\nu},
\]

where \( T^m_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta I_m}{\delta g^{\mu\nu}} \) is the energy-momentum tensor for matter fields, \( \nabla \) represents the covariant derivative defined with the Livi-Civita connection of the metric, \( R_{\mu\nu} \) is the Ricci tensor and
the subscript \(^\prime\), \(R\) denotes the derivative with respect to the curvature scalar \(R\). The above equations can be cast to a form

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G \left( \frac{1}{f_{,R}(R)} T_{\mu\nu}^{m} + \frac{1}{8\pi G} T_{\mu\nu}^{\text{cur}} \right),
\]

where \(T_{\mu\nu}^{\text{cur}}\) is a stress-energy tensor for the effective curvature fluid and is given by

\[
T_{\mu\nu}^{\text{cur}} = \frac{1}{f_{,R}(R)} \left( \frac{1}{2} g_{\mu\nu} (f(R) - R f_{,R}(R)) + \nabla_{\mu} \nabla_{\nu} f_{,R}(R) - g_{\mu\nu} \nabla^{2} f_{,R}(R) \right).
\]

Setting \(f_{,R}(R) = F(R)\) and using the relations \(\nabla^{2} F = \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu\nu} \partial_{\nu} F)\) and \(\nabla_{\mu} \nabla_{\nu} F = \partial_{\mu} \partial_{\nu} F - \Gamma_{\mu\nu}^{\lambda} \partial_{\lambda} F\) along with metric (3), one can find the components of \(T_{\mu\nu}^{\text{cur}}\) given by

\[
T_{0}^{0(\text{cur})} = \frac{1}{F(R)} \left( \frac{1}{2} (f(R) - RF(R)) + \frac{U'F'}{2} - \frac{1}{L^2} (2LL'UF' + L^2 U'F' + L^2 UF'' \right),
\]

and

\[
T_{1}^{1(\text{cur})} = \frac{1}{F(R)} \left( \frac{1}{2} (f(R) - RF(R)) + (UF'' + \frac{U'F'}{2}) - \frac{1}{L^2} (2LL'UF' + L^2 U'F' + L^2 UF'') \right).
\]

Note that here the prime stands for the derivative with respect to \(r\). From equations (14) and (15), it is easy to see that the components \(T_{0}^{0(\text{cur})}\) and \(T_{1}^{1(\text{cur})}\) are not equal in general, but at the spacetime horizon, where \(U(a)\) vanishes, we have

\[
T_{0}^{0(\text{cur})} = T_{1}^{1(\text{cur})} = \frac{1}{F(R)} \left( \frac{1}{2} (f(R) - RF(R)) - \frac{U'F'}{2} \right).
\]

It is evident from equations (7) and (16) that the field equations are consistent with \(f(R)\) gravity at the horizon of the spherically symmetric metric (3) provided that the stress-energy tensor of matter fields has the form

\[
T_{0}^{0(m)} = T_{1}^{1(m)},
\]

at the horizon.

Note that \(\nabla^{2} F(R) = \frac{1}{L^2} (2LL'UF' + L^2 U'F' + L^2 UF'')\). At the horizon, namely, \(U(a) = 0\), one then has \(\nabla^{2} F(R)|_{r=a} = U'(a)F'\) and the 0–0 component of equations of motion takes the form

\[
-\frac{1 + LL'U'}{L^2} = 8\pi G \left( \frac{1}{F(R)} T_{0}^{0(m)} + \frac{1}{8\pi G F(R)} \left( \frac{1}{2} (f(R) - RF(R)) - \frac{1}{2} U'F' \right) \right),
\]

which can be rewritten as

\[
-\frac{F(R)}{2G} + \frac{U'LL'F}{2G} = 4\pi \left( \frac{1}{T_{0}^{0(m)}} + \frac{1}{8\pi G} \left( \frac{1}{2} (f(R) - RF(R)) - \frac{1}{2} U'F' \right) \right) L^2.
\]

Note that here the curvature scalar \(R\) is taken its value at the horizon \(R = R(r)|_{r=a}\). If one has a close look at the above equation, it throws light on the notion of horizon entropy, temperature,
and energy. So a thermodynamic interpretation of this equation is possible. Multiplying by an infinitesimal displacement of horizon \( da \) on both sides of this equation, it is trivial to rewrite this equation in the form

\[
-\frac{1}{2G}F(R)da + \frac{U'}{4\pi}d(\frac{4\pi L^2 F(R)}{4G}) = T_0^{0(m)}(4\pi L^2 da) + \frac{1}{4G}(f(R) - RF(R))L^2 da.
\] (20)

Note that \( F(R) = \frac{1}{2}d(F(R)L^2) - \frac{1}{2}L^2F' da \). Substituting it into the above equation, we finally get

\[
-\frac{1}{2G}F(R)da + \frac{U'}{4\pi}d(\frac{4\pi L^2 F(R)}{4G}) = T_0^{0(m)}(4\pi L^2 da) + \frac{1}{4G}(f(R) - RF(R))L^2 da.
\] (21)

Now we first concentrate on the left hand side of the above equation. We consider the first term \( \frac{1}{2G}F(R)da = \frac{1}{2G}f(R)(R)da \) of this equation and convert it to the limit of general relativity by setting \( f(R) = R \), we get \( \frac{1}{2G}da \) as the energy change \( dE = \frac{1}{2G}da \) during the infinitesimal displacement \( da \) within the Einstein gravity \( E = a/2G \) is just the Misner-Sharp energy inside the horizon \( [17] \); For a Schwarzschild black hole solution, the such defined energy \( E = a/2G \) gives the Schwarzschild mass). Naturally, we may regard the term \( \frac{1}{2G}F(R)da \equiv dE \) as the energy change within the \( f(R) \) gravity during a infinitesimal horizon displacement \( da \). The second term on the left hand side can be written as \( \frac{U'}{4\pi}d(\frac{4\pi L^2 F(R)}{4G}) \). Here one can identify \( \frac{U'}{4\pi} = T \) as the temperature of the black hole and the term \( d(\frac{4\pi L^2 F(R)}{4G}) \) is nothing but the entropy change \( dS \) of the black hole in the \( f(R) \) gravity \( [21, 22] \). So the above equation can be rewritten as

\[
T dS - dE = T_0^{0(m)}4\pi L^2 da + \frac{1}{4G}(f(R) - RF) L^2 da
\] (22)

The first term of the right hand side of above equation is \( T_0^{0(m)}4\pi L^2 da \), where one can recognize \( 4\pi L^2 \) as a surface area of horizon with a multiple displacement \( da \) of the horizon radius \( a \), which corresponds to a change of volume. We write it as \( 4\pi L^2 da = dV \). Since at the horizon one has \( T_0^{0(m)} = T_1^{1(m)} \equiv P \), which represents the radial pressure of matter fields at the horizon, therefore this term turns out to be the work term \( PdV \) against the pressure. Hence the above equation can be written as

\[
T dS - dE = PdV + T\frac{A}{4G}(\frac{f(R) - RF}{U'})da,
\] (23)

where \( A = 4\pi L^2(a) \) is the area of the black hole horizon. It has been shown in references \([7, 8]\) that the field equations in Einstein, Gauss-Bonnet and Lovelock gravities can be rewritten as the thermodynamical identity \( T dS = dE + PdV \) at the black hole horizon of a special class of spherical symmetric black hole spacetimes, but here it is evident from the above equation that the field equation of \( f(R) \) gravity can no longer be cast to the thermodynamical identity \( T dS =
\[ dE + PdV \] at the black hole horizon if one defines the horizon energy as \( E = \frac{1}{2G} \int_0^a F(R)da \).

It is reminiscent of the argument recently given by Eling et al. [5]. In that reference, they argued that in order to derive the field equations for the \( f(R) \) gravity by using relation \( \delta Q = TdS \) and assuming the relation holds for all local Rindler causal horizons through each spacetime point, here \( \delta Q \) and \( T \) are the energy flux and Unruh temperature seen by an accelerated observer just inside the horizon, and \( S = f_RA/4G \) is the horizon entropy, one has to add an additional entropy term to the Clausius relation so that \( \delta Q/T = dS + d\bar{S} \), here the expression of \( d\bar{S} \) is given in [5]. The appearance of the additional entropy term is interpreted as entropy production term in the nonequilibrium thermodynamics. That is, in their setting, the horizon thermodynamics for the \( f(R) \) gravity is a nonequilibrium one. In the spirit of [5], it is then natural to regard the last term in (23) as the entropy production term in our setup. In other words, if one includes the effects of higher curvature correction to the Einstein gravity and the curvature scalar \( R \) in general relativity is replaced by a generic function \( f(R) \) then the usual thermal behavior of the field equations at black hole horizon [7, 8] will not be sustained by the system but will shift from an equilibrium setup to a non-equilibrium one [5]. Instead of satisfying the thermodynamical identity \( TdS = dE + PdV \) at the black hole horizon, we have to put in the effects of higher curvature correction to the horizon entropy grown up internally because of the system being out of equilibrium [5]. So one may have a balance thermal identity for higher curvature gravity theory

\[ \begin{align*}
    dE &= TdS - PdV + Td\bar{S} \\
    \text{(24)}
\end{align*} \]

where \( Td\bar{S} \) is just the last term of (23), which is the entropy production term added to balance the inequality \( TdS > dE + PdV \). We note that the entropy production term for the \( f(R) \) gravity has the form

\[ d\bar{S} = \frac{A}{4G} \left( \frac{R - f(R)}{\dot{U}} \right) da, \]

in our setup of spherically symmetric black hole horizon. It is evident from this equation that the entropy production term vanishes at the limit \( f(R) = R \) of the Einstein’s gravity, and the usual thermal behavior \( TdS = dE + PdV \) of the field equation at the horizon is recovered.

Clearly the above interpretation of nonequilibrium thermodynamics of black hole horizon is not unique. One may combine the last term in (21) to the first term so that one can define a new energy associated with the black hole horizon as

\[ \begin{align*}
    d\bar{E} &= \frac{1}{2G} Fda + \frac{1}{4G} (f(R) - RF)L^2 da. \\
    \text{(26)}
\end{align*} \]

In this way, we can rewrite (21) to the standard first law of thermodynamics

\[ d\bar{E} = TdS - PdV. \]

\[ \text{(27)} \]
Take an example, let us consider a constant curvature black hole solution of the $f(R)$ gravity without matter fields. In that case, we have from (11) that $f_{,R}R_0 = 2f(R_0)$, where the $R_0$ is the curvature scalar of the black hole solution. The field equations (11) then can be cast to the standard Einstein’s equations with an effective cosmological constant $\Lambda \equiv 3/\ell^2 = R_0/4$. Thus we have a simple Schwarzschild-de Sitter (anti-de Sitter) black hole solution depending on the sign of the effective cosmological constant, which has the metric functions, $L(r) = r$ and

$$U(r) = 1 - \frac{2GM}{r} - \frac{r^2}{\ell^2},$$

(28)

where $M$ is the mass of the black hole in the Einstein gravity. In terms of the black hole horizon radius $U(r)|_{r=a} = 0$, the mass can be expressed by

$$M = \frac{a}{2G} \left( 1 - \frac{a^2}{\ell^2} \right).$$

(29)

On the other hand, it is interesting to note that integrating (26) yields

$$\bar{E} = F(R_0)M.$$  

(30)

Clearly, the black hole mass $M$ does not satisfy the equation (27) with $P = 0$, but $\bar{E}$ does. Therefore defining $\bar{E}$ as the horizon energy of the black hole is reasonable.

### 3 Thermodynamics of apparent horizon of FRW universe

In the previous section we have discussed the behavior of the field equations for a static, spherically symmetric black hole spacetimes at the black hole horizon. In this section we generalize the thermodynamics of black hole horizon to a dynamical apparent horizon in a FRW universe within the $f(R)$ gravity. We consider an $(n+1)$-dimensional, spatially homogenous and isotropic FRW universe described by the metric

$$ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega_{n-1}^2 \right),$$

(31)

where $a(t)$ is the scale factor of the universe with $t$ being cosmic time and $d\Omega_{n-1}^2$ is the metric of $(n-1)$-dimensional sphere with unit radius and the spatial curvature constant $k = 1$, 0 and $-1$ correspond to a closed, flat and open universe, respectively. Using the spherical symmetry, the metric (31) can be rewritten as

$$ds^2 = h_{ab}dx^a dx^b + \tilde{r}^2 d\Omega_{n-1}^2,$$

(32)

where $\tilde{r} = a(t)r$ and $x^0 = t$, $x^1 = r$ and the two dimensional metric $h_{ab} = \text{diag}(-1, a^2/1 - kr^2)$. The dynamical apparent horizon is determined by the relation $h_{ab} \partial_a \tilde{r} \partial_b \tilde{r} = 0$, which implies that
the vector $\nabla \tilde{r}$ is null on the apparent horizon surface. The explicit evaluation of the apparent horizon gives the apparent horizon radius

$$\frac{1}{\tilde{r}_A^2} = H^2 + \frac{k}{a^2},$$

(33)

where $H$ denotes for the Hubble parameter. The associated temperature $T = \kappa/2\pi$ with the apparent horizon is defined through the surface gravity

$$\kappa = \frac{1}{2\sqrt{-h}} \partial_a (\sqrt{-h} h^{ab} \partial_b \tilde{r}),$$

(34)

which gives

$$\kappa = -\frac{1}{\tilde{r}_A} (1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}).$$

(35)

From this equation one can see that when $\dot{\tilde{r}}_A < 2H\tilde{r}_A$, the surface gravity $\kappa$ is negative. If one still uses $T = \kappa/2\pi$ to define a temperature of the apparent horizon, we obtain a negative temperature! This case is quite similar to the case of cosmological horizon for the Schwarzschild-de Sitter spacetime. In this case, one should use $T = |\kappa|/2\pi$ to define the temperature of the apparent horizon.

The total matter energy inside a sphere of radius $\tilde{r}_A$ of the apparent horizon is given by

$$E = \Omega_n \tilde{r}_A^n \rho,$$

(36)

where $V = \Omega_n \tilde{r}_A^n$ is the volume of the sphere within the apparent radius, and $\rho$ is the energy density of the perfect fluid in the universe. We consider FRW universe as a thermodynamical system with apparent horizon surface as a boundary of the system. In general the radius of the apparent horizon $\tilde{r}_A$ is not constant but changes with time. Let $d\tilde{r}_A$ be an infinitesimal change in radius of the apparent horizon during a time of interval $dt$. This small displacement $d\tilde{r}_A$ in the radius of apparent horizon will cause a small change $dV$ in the volume $V$ of the apparent horizon. Since the energy $E$ inside the apparent horizon of FRW universe is directly related with the apparent radius $\tilde{r}_A$ therefore an infinitesimal change $d\tilde{r}_A$ will result in a change $dE$ in energy $E$ and is given by

$$dE = n\Omega_n \tilde{r}_A^n \rho d\tilde{r}_A - n\Omega_n \tilde{r}_A^n (\rho + P)H dt.$$

(37)

Here, we have utilized the continuity equation (39).

Now we consider the field equations (12), the stress-energy tensor $T_{\mu\nu}^m$ is taken to be the one of the perfect fluid with time dependent energy density $\rho(t)$ and pressure $P(t)$

$$T_{\mu\nu}^m = (\rho + P) U_\mu U_\nu + P g_{\mu\nu},$$

(38)
where $U_\mu$ is the four velocity of the fluid. The conservation of the stress-energy tensor, $T^{\mu\nu(m)} = 0$, leads to the continuity equation

$$\dot{\rho} + nH(\rho + P) = 0,$$

where the dot represents derivative with respect to cosmic time $t$.

Using the relations $\nabla^2 F = \frac{1}{\sqrt{-g}} \partial_\mu(\sqrt{-g} g^{\mu\nu} \partial_\nu F)$ and $\nabla_\mu F = \partial_\mu F - \Gamma^\lambda_{\mu\nu} \partial_\nu F$ along with the $(n+1)$-dimensional metric (31) of the FRW universe, one can find the components of $T^{\text{cur}}_{\mu\nu}$

\[ T^{\text{cur}}_{00} = -\frac{1}{F(R)} \left( \frac{1}{2} (f(R) - RF(R)) + nHF,R(R) \dot{R} \right), \]

\[ T^{\text{cur}}_{ii} = \frac{1}{F(R)} \left( \frac{1}{2} (f(R) - RF(R)) + F,R(R) \dot{R} + F,R,R(R) \dot{R}^2 + (n-1)H F,R(R) \dot{R} \right) g_{ii}, \]

where $F(R) = df/dR$, $F,R = dF/dR$ and $F,R,R = d^2F/dR^2$. The field equations become

\[ H^2 + \frac{k}{a^2} = \frac{16\pi G}{n(n-1)} \left( \frac{1}{F(R)} - \frac{1}{8\pi G} \left( \frac{1}{2} (f(R) - RF(R)) + nHF,R(R) \dot{R} \right) \right), \]

\[ \dot{H} - \frac{k}{a^2} = -\frac{8\pi G}{(n-1)F(R)} \left( (\rho + P) + \frac{1}{8\pi G} \left( d(F,R(R) \dot{R}) - HF,R(R) \dot{R} \right) \right), \]

where $d(F,R(R)) \equiv d(F,R(R))/dt$. These are two Friedmann equations for the $f(R)$ gravity \cite{23}. Note that, as mentioned in the previous section, in this gravity theory the black hole entropy has a relation to its horizon area $S = \frac{A}{4G} F(R)|_{\text{Horizon}}$. We assume it also holds for the apparent horizon. Using relation $H^2 + \frac{k}{a^2} = \frac{1}{\mathcal{V}_A}$ and taking differential of it, one gets

\[ -\frac{1}{\mathcal{V}_A} d\mathcal{V}_A = H(\dot{H} - \frac{k}{a^2}) dt. \]

Substituting it into (43) and then multiplying in both sides with a factor $n\Omega_n \mathcal{V}_A^{n-3}$, we reach

\[ \frac{1}{2\pi \mathcal{V}_A} (n(n-1)\Omega_n \mathcal{V}_A^{n-2} F(R) d\mathcal{V}_A)/4G = n\Omega_n \mathcal{V}_A^n H \left( (\rho + P) + \frac{1}{8\pi G} (d(F,R(R) \dot{R}) - HF,R(R) \dot{R}) \right) dt. \]

Now consider the differential

\[ d(n\Omega_n \mathcal{V}_A^{n-1} F(R)/4G) = n(n-1)\Omega_n \mathcal{V}_A^{n-2} F(R) d\mathcal{V}_A/4G + n\Omega_n \mathcal{V}_A^{n-1} F,R(R) \dot{R} dt/4G, \]

and substitute it into (45), one gets

\[ \frac{1}{2\pi \mathcal{V}_A} d(n\Omega_n \mathcal{V}_A^{n-1} F(R)/4G) = n\Omega_n \mathcal{V}_A^n H \left( (\rho + P) + \frac{1}{8\pi G} (d(F,R(R) \dot{R}) - HF,R(R) \dot{R}) \right) dt \]

\[ + \frac{n\Omega_n \mathcal{V}_A^{n-2} F,R(R) \dot{R} dt}{8\pi G}. \]
Multiplying both sides of this equation by a factor \(-1 - \dot{r}_A/2H\dot{r}_A\) and identifying that the surface gravity \(\kappa = \left(-1/\dot{r}_A\right)(1 - \dot{r}_A/2H\dot{r}_A)\) at the apparent horizon, one can obtain

\[
\frac{\kappa}{2\pi}d\left(\frac{AF(R)}{4G}\right) = \left(1 - \frac{\dot{r}_A}{2H\dot{r}_A}\right)n\Omega_n\tilde{r}_A^nH \left((\rho + P) + \frac{1}{8\pi G}\left(d(F,R)\dot{R} - HF,R\dot{R}\right)\right) dt \\
- \left(1 - \frac{\dot{r}_A}{2H\dot{r}_A}\right)n\Omega_n\tilde{r}_A^{n-2}F,R\dot{R} dt,
\]

where \(A = n\Omega_n\tilde{r}_A^{n-1}\) is the area of the apparent horizon and \(\Omega_n = \pi^{n/2}/\Gamma(n/2 + 1)\) being the volume of an \(n\)-dimensional unit ball. In addition, let us note that all quantities in the above equation are evaluated at the apparent horizon. Recognizing \(\kappa/2\pi\) as the temperature \(T\) of the apparent horizon and the quantity inside the parentheses on the left hand side of this equation, is just the entropy \(S = \frac{A}{4G}F(R)\dot{r}_A\) of the apparent horizon in the \(f(R)\) gravity. Hence, having identified temperature \(T\) and entropy \(S\), the above equation can be written as

\[
TdS = -n\Omega_n\tilde{r}_A^n(\rho + P)H dt + \frac{n}{2}\Omega_n\tilde{r}_A^{n-1}(\rho + P)d\dot{r}_A \\
+ T\frac{A}{4G} \left(H\tilde{r}_A^2(d(F,R)\dot{R} - HF,R(R)\dot{R}) + F,R(R)\dot{R}\right) dt.
\]

Now we turn to the total matter energy \(E = \Omega_n\tilde{r}_A^2\rho\) inside the apparent horizon. Using (37), the above equation becomes

\[
TdS - dE = -\frac{1}{2}(\rho - P)dt(\Omega_n\tilde{r}_A^n) + T\frac{A}{4G} \left(H\tilde{r}_A^2(d(F,R)\dot{R} - HF,R(R)\dot{R}) + F,R(R)\dot{R}\right) dt.
\]

Note that \((\rho - P)/2 = W\) is the work density [18]. Hence the equation (50) can be further rewritten as

\[
dE = TdS + WdV - T\frac{A}{4G} \left(H\tilde{r}_A^2(d(F,R)\dot{R} - HF,R(R)\dot{R}) + F,R(R)\dot{R}\right) dt.
\]

In the cosmological setting, we have already shown [17] that the field equations obey the universal form \(dE = TdS + WdV\) at the apparent horizon of FRW universe in Einstein, Gauss-Bonnet and Lovelock gravities. But it no longer holds for the scalar-tensor theory [6]. Here we see from the equation (51) that the universal form \(dE = TdS + WdV\) also does not hold for the \(f(R)\) gravity at the apparent horizon of FRW universe because of the appearance of the additional term \(T\frac{A}{4G} \left(H\tilde{r}_A^2(d(F,R)\dot{R} - HF,R(R)\dot{R}) + F,R(R)\dot{R}\right) dt\). Of course when \(f(R) = R\), the additional term is absent, the usual result for the Einstein gravity is recovered [17]. In the spirit of the argument of [6], once again, the additional term can be interpreted as follows. This additional term is developed internally in the system as a result of being out of equilibrium. In order to get a balance relation for a universal form, one may take

\[
dE = TdS + WdV + Td\tilde{S}
\]
The term $d\bar{S}$ is the entropy production term grown up internally due to the non-equilibrium setup within the $f(R)$ gravity at the apparent horizon of FRW universe. Comparing equation (51) and (52), we get

$$d\bar{S} = -\frac{A}{4G} \left( H\tilde{r}_A^2 (d(F, R)\dot{R}) - HF, R(R)\dot{R} + F, R(R)\dot{R} \right) dt. \quad (53)$$

If one further defines $d\tilde{S} = d(S + \bar{S})$ as an effective entropy change during the infinitesimal displacement $d\tilde{r}_A$ of the apparent horizon of FRW universe in interval $dt$, the above equation (52) can be rewritten as

$$dE = Td\tilde{S} + WdV, \quad (54)$$

where $\tilde{S} = S + \bar{S}$ is the effective entropy associated with the apparent horizon of FRW universe.

## 4 Conclusion and Discussion

Black hole thermodynamics implies that there must exist some relation between gravity theory and the laws of thermodynamics. Indeed, it has been shown that at the horizon of spherical symmetric black hole spacetime, field equations can be written into the form of the first law of thermodynamics, $dE = TdS - PdV$, not only for the Einstein gravity, but also Gauss-Bonnet gravity and more general Lovelock gravity [7, 8]. In the cosmological setting, applying the Clausius relation $\delta Q = TdS$ to the apparent horizon of a FRW universe, one can derive corresponding Friedmann equations of the universe with any spatial curvature in the Einstein, Gauss-Bonnet and Lovelock gravity theories [11]. Further we have shown that at the apparent horizon, the field equations of gravity can also be cast to a similar form of the first law, $dE = TdS + WdV$, in the Einstein, Gauss-Bonnet, and Lovelock gravity theories [17]. To what extent does such a formulism hold? In the present paper we have discussed the so-called $f(R)$ gravity.

In the setup of static, spherically symmetric black hole spacetime, we have shown that the field equations of the $f(R)$ gravity can be rewritten as $dE = TdS - PdV + Td\bar{S}$, at the black hole horizon. Here $E = 1/2G \int_0^a F(R)da$ is regarded as the horizon energy; when $f(R) = R$, the horizon energy reduces to the one given in [7] for Einstein gravity. $T$ is the Hawking temperature of the black hole and $S = F(R)A/4G$ is the horizon entropy of the black hole, and $P$ is the radial pressure of matter fields at the horizon. The additional term $d\bar{S}$ given in (25) can be regarded as an entropy production term due to a nonequilibrium thermodynamics setup [5], where it is argued that in the setup of a class of Rindler causal horizons, in order to derive the field equations for the $f(R)$ gravity, an entropy production term has to be added to the Clausius relation $\delta Q = TdS + Td_\gamma S$. However, in our case, we have argued that the additional term can be combined to the horizon energy term by defining a new horizon energy (26). In
this way, the field equations of $f(R)$ gravity can be cast into the standard form of the first law of thermodynamics (27): $d\bar{E} = TdS - PdV$, which reduces to the first law of black hole thermodynamics when $P = 0$.

In the setup of cosmology of FRW universe, compared to the cases of Einstein gravity, Gauss-Bonnet gravity and Lovelock gravity, we have found that an additional term appears when the Friedmann equations of the $f(R)$ gravity are rewritten into a form of the first law of thermodynamics at the apparent horizon of the universe, $d\bar{E} = TdS + WdV + Td\bar{S}$. As usual, here $E$ is the total energy of matter inside the apparent horizon, $T$ and $S$ are the associated temperature and entropy with the horizon, and $W = (\rho - P)/2$ is the work density. The additional term in this case is given by (53). Once again, the additional term can be interpreted as the entropy production term associated with the apparent horizon in the nonequilibrium thermodynamics within the $f(R)$ gravity in the spirit of [5].

The authors of [5] used the modified Clausius relation

$$\delta Q = TdS + Td_\bar{S},$$

(55)

to a Rindler horizon of spacetime with the assumption that the horizon has a temperature $T = 1/2\pi$ and entropy $S = AF(R)/4G$, where $A$ is the horizon area. They defined the heat as the mean flux of the boost energy current of matter across the horizon

$$\delta Q = \int T_{ab}\chi^a d\Sigma^b.$$

(56)

In this framework, in order to get the correct field equations of the $f(R)$ gravity, they worked out that the entropy production term $d_\bar{S}$ is

$$d_\bar{S} = -\frac{3}{8G} \int F(R)\theta^2 \lambda d\lambda d^2A$$

(57)

where $\theta = d(ln d^2A)/d\lambda$ is the expansion of the congruence of null geodesics generating the horizon and $\lambda$ is an affine parameter.

We note from (25), (53) and (57) that in the different settings, the additional entropy term has a different from for the $f(R)$ gravity. The same happens in the scalar-tensor gravity [6]. It would be of great interest to further understand this issue. In addition, why does such an additional term appears for the $f(R)$ gravity and scalar-tensor gravity, while it does not in the Einstein’s gravity, Gauss-Bonnet gravity and more general Lovelock gravity. This is a quite interesting question, which deserves further investigation. This might be related to the observation that the field equations for the Einstein’s gravity, Gauss-Bonnet gravity and Lovelock gravity can also be derived from a holographic surface term [24].
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