Whitney’s index formula in higher dimensions and Laplace integrals

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1 Introduction

Let \( f = (f_1, f_2) : \mathbb{R} \rightarrow \mathbb{R}^2 \) be a long curve, i.e. a smooth immersion such that \( f_1(x) = x \) and \( f_2(x) = 0 \) for \( x \leq -1 \) and \( x \geq 1 \). By the famous theorem of Whitney (see [1]) long curves are classified by a single integral invariant, an index (rotation number). Informally saying, index is a number of full rotations made by the tangent vector \( Df(x) = (f'_1(x), f'_2(x)) \neq 0 \) as \( x \) goes from \(-\infty\) to \(+\infty\) (or, equally, from \(-1\) to \(1\)). The explicit formula for the index is as follows:

\[
I(f) = \int_{-\infty}^{\infty} \frac{f''_1(x)f'_2(x) - f''_2(x)f'_1(x)}{(f'_1(x))^2 + (f'_2(x))^2} \, dx
\]

The integral in the right-hand side of (1) is \( \int_{\mathbb{R}} (Df)^* \omega \) where

\[
\omega(z_1, z_2) = \frac{z_2 dz_1 - z_1 dz_2}{z_1^2 + z_2^2}
\]

is the closed form in \( \mathbb{R}^2 \setminus \{0\} \) forming a basis in its de Rham cohomologies \( H^1(\mathbb{R}^2 \setminus \{0\}) = \mathbb{R} \).

Suppose now that \( f \) is generic. By Thom’s transversality theorem (see [2]) \( f \) has only a finite number of self-intersection points. All these points are

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simple (i.e. only two branches of the curve meet) and transversal. A point \( a \in \mathbb{R}^2 \) of simple transversal self-intersection can be equipped with a sign \( \sigma(a) = \pm 1 \) by the following rule. Let \( f(x^{(1)}) = f(x^{(2)}) = a \) and \( x^{(1)} < x^{(2)} \). By transversality, the tangent vectors \( D_f(x^{(1)}) \) and \( D_f(x^{(2)}) \) are not parallel, and therefore form a basis in \( \mathbb{R}^2 \). Choose an orientation of \( \mathbb{R}^2 \) and take

\[
\sigma(a) = \begin{cases} 
+1, & \text{if the basis } (D_f(x^{(1)}), D_f(x^{(2)})) \text{ gives a chosen orientation of } \mathbb{R}^2, \\
-1, & \text{if the basis } (D_f(x^{(1)}), D_f(x^{(2)})) \text{ gives an opposite orientation.} 
\end{cases}
\]

Then there holds Whitney’s index formula (see [I]):

\[
I(f) = \sum_a \sigma(a) \tag{4}
\]

where the sum is taken over all the self-intersection points of the curve \( f \).

The paper is devoted to the generalization of (1)–(4) to the case of smooth immersions \( f: \mathbb{R}^n \to \mathbb{R}^{2n} \) such that \( f(x) = (x, 0) \) for \( |x| \) large. In Section 2, we define an index and prove a “sum of signs” formula for generic immersions. Section 3 contains a proof of an explicit integral formula for the index (for \( n \) even); the proof makes use of asymptotic expansions of certain Laplace integrals. The formula obtained looks like \( I(f) = \int_{\mathbb{R}^n} (D_f)^* \omega \) where \( \omega \) is an \( n \)-form on the Stiefel variety \( V(n, 2n) \) and \( D_f: \mathbb{R}^n \to V(n, 2n) \) is a differential. It is proved in Section 4 that \( \omega \) is a generator of the de Rham cohomology group \( H^n(V(n, 2n)) \).

2 Index and self-intersection points

Recall that the map \( f = (f_1, \ldots, f_{2n}): \mathbb{R}^n \to \mathbb{R}^{2n} \) is called an immersion if for any \( x \in \mathbb{R}^n \) the \((n \times 2n)\)-matrix

\[
D_f(x) = \begin{pmatrix} \frac{\partial f_i}{\partial x_j} \end{pmatrix} \tag{5}
\]

has the rank \( n \) (i.e., belongs to the Stiefel variety \( V(n, 2n) \)). We will always suppose that the immersion \( f \) is “fixed at infinity”, i.e. \( f(x) = (x, 0) \) as soon as \( x \) lies outside the unit cube of \( \mathbb{R}^n \). Generic immersions \( \mathbb{R}^n \to \mathbb{R}^{2n} \) fixed
at infinity have only a finite number of self-intersection points, all of which are simple and transversal.

The homotopy classes of immersions \( \mathbb{R}^n \to \mathbb{R}^{2n} \) form an abelian group \( \text{Imm}_n \). Definition of the multiplication resembles that of a homotopy group: take
\[
(f_1 \ast f_2)(x) = \begin{cases} 
  f_1(2x_1 + 1, x_2, \ldots, x_n), & \text{if } x_1 \leq 0, \\
  f_2(2x_1 - 1, x_2, \ldots, x_n), & \text{if } x_1 \geq 0.
\end{cases}
\]

One of the formulations of the famous Smale classification theorem (see [3]) is

**Theorem 1** The mapping \( D_f : \text{Imm}_n \to \pi_n(V(n,2n)) \) is a group isomorphism.

So to classify immersions we need to know the group \( \pi_n(V(n,2n)) \). The answer is the following well-known lemma (see, for example, [4], page 173):

**Lemma 1**
\[
\begin{align*}
\pi_i(V(n,2n)) &= 0 \text{ for } 1 \leq i \leq n - 1, \\
\pi_n(V(n,2n)) &= \begin{cases} 
\mathbb{Z} & \text{for } n = 1 \text{ and } n \text{ even}, \\
\mathbb{Z}_2 & \text{for odd } n \geq 3.
\end{cases}
\end{align*}
\]

Thus, smooth immersions \( f : \mathbb{R}^n \to \mathbb{R}^{2n} \) are classified by a single invariant, an index \( I(f) \) which is either an integer (for \( n = 1 \) and \( n \) even) or an element of \( \mathbb{Z}_2 \) (for odd \( n \geq 3 \)). The case \( n = 1 \) is covered by the results of Whitney, and in what follows we will always assume that \( n > 1 \).

For \( n \) even, a point \( a \in \mathbb{R}^{2n} \) of simple transversal self-intersection of immersion \( f \) can be equipped with a sign \( \sigma(a) = \pm 1 \) by the following rule: if \( f(x^{(1)}) = f(x^{(2)}) = a \) (and \( x^{(1)} \neq x^{(2)} \)), then take
\[
\sigma(a) = \text{sgn det} \left( \frac{D_f(x^{(1)})}{D_f(x^{(2)})} \right)
\]
(\( \text{the right-hand side contains a determinant of the matrix } 2n \times 2n \)). Geometrically this means that we choose positively oriented bases in the tangent spaces to the image of \( f \) at points \( x^{(1)} \) and \( x^{(2)} \). Since \( f \) is immersion, these \( 2n \) vectors form a basis in \( \mathbb{R}^{2n} \), and \( \sigma(a) \) is taken \( \pm 1 \) depending on the orientation of this basis (orientation of \( \mathbb{R}^{2n} \) is fixed beforehand). Since \( n \) is supposed
to be even, the result will not depend on the order in which the preimages \( x^{(1)}, x^{(2)} \in f^{-1}(a) \) are taken (for \( n = 1 \) we supposed that \( x^{(1)} < x^{(2)} \), but there is no way to choose this order for \( n > 1 \)).

**Theorem 2**

\[
I(f) = \sum_a \sigma(a)
\]

where the sum is taken over all the self-intersection points of \( f \), and the summation is performed in \( \mathbb{Z} \) for \( n \) even and in \( \mathbb{Z}_2 \) for odd \( n \geq 3 \).

**Proof** Let \( f_t \) be a generic continuous family of smooth immersions \( \mathbb{R}^n \to \mathbb{R}^{2n} \) fixed at infinity. By Thom’s transversality theorem the set of self-intersection points of \( f_t \) is changes continuously with \( t \), except for a finite number of points \( t_1, \ldots, t_k \) where two “catastrophes” may occur: either two self-intersection points merge and disappear, or, conversely, emerge. It can be easily observed that for \( n \) even the pair of points to emerge or to perish always have opposite signs. It follows from this that (for any \( n \)) the right-hand side of (7) depends only on the homotopic class of the immersion \( f \). Thus, this right-hand side defines a mapping \( S \) from \( \text{Imm}_n \) to \( \mathbb{Z} \) or \( \mathbb{Z}_2 \), depending on parity of \( n \).

Definition of multiplication in \( \text{Imm}_n \) shows that \( S \) is a homomorphism. By Theorem 1, to prove equation (7) it is enough to show that \( S \) is an isomorphism, i.e. that its image is the whole \( \mathbb{Z} \) or \( \mathbb{Z}_2 \). In other words it should be proved that \( 1 \in \text{Im} \, S \), i.e. that there exists an immersion \( f^{(n)} : \mathbb{R}^n \to \mathbb{R}^{2n} \) with only one self-intersection point.

Such immersion is easily constructed by induction on \( n \). For \( n = 1 \) take a smooth plane curve \( f^{(1)} \) such that \( f^{(1)}(-1/2) = f^{(1)}(1/2) \), and there are no more self-intersection points. Suppose now that immersion \( f^{(n)} \) is already constructed, and the self-intersection is \( f^{(n)}(-1/2, 0, \ldots, 0) = f^{(n)}(1/2, 0, \ldots, 0) \).

Let \( \varphi(t) \) be a smooth function such that \( \varphi(-1/2) \neq \varphi(1/2) \) and \( \varphi(t) = 0 \) for \( |t| \geq 1 \). Then it can be easily seen that the formula

\[
f^{(n+1)}(x_1, \ldots, x_{n+1}) = (f^{(n)}_1(x_1, \ldots, x_n), \ldots, f^{(n)}_n(x_1, \ldots, x_n), x_{n+1},
\]

\[
f^{(n)}_{n+1}(x_1, \ldots, x_n), \ldots, f^{(n)}_{2n}(x_1, \ldots, x_n), x_{n+1} \varphi(x_1))
\]

defines an immersion \( \mathbb{R}^{n+1} \to \mathbb{R}^{2n+2} \) whose sole self-intersection is \( f^{(n+1)}(-1/2, 0, \ldots, 0) = f^{(n+1)}(1/2, 0, \ldots, 0) \), so that induction step is made. \( \blacksquare \)
3 An integral formula for the index

Let us derive an explicit integral formula for the index \( I(f) \). For the reasons explained in the previous Section we restrict ourselves to the case of \( n \) even (an element of \( \mathbb{Z}_2 \) hardly can appear as an integral).

Start with a technical result. Let \( f : \mathbb{R}^n \to \mathbb{R}^{2n} \) be an arbitrary smooth map (even not immersion) fixed at infinity. Consider a mapping \( A_f : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) given by the formula

\[
A_f(x, y) = f(x) - f(y) \quad (x, y \in \mathbb{R}^n)
\]

and recall that degree of a smooth mapping \( g : M \to N \) (\( M \) and \( N \) being oriented manifolds of the same dimension) is defined as

\[
\deg g = \sum_{g(x) = y} \text{sgn}(x).
\]

Here \( y \in N \) is a generic point, and \( \text{sgn}(x) \) is taken 1 or \( -1 \) depending on whether the mapping \( Df : T_x M \to T_y N \) preserves or inverts orientation.

The right-hand side of this equation can be called multiplicity of \( g \) at \( y \).

**Lemma 2** The degree of the mapping \( A_f \) is zero.

**Proof** Prove first that the degree of \( A_f \) is defined. To do this denote \( E \subset \mathbb{R}^{2n} \) the subspace of vectors \( \langle a_1, \ldots, a_n, 0, \ldots, 0 \rangle \). Since \( n > 1 \), any two points \( z_0, z_1 \in \mathbb{R}^{2n} \setminus E \) can be connected with the path \( z_t \) that does not intersect \( E \). Notice now that since \( f \) is fixed at infinity, for every compact set \( B \subset \mathbb{R}^{2n} \setminus E \) its full preimage \( A_f^{-1}(B) \) is compact, and therefore multiplicity of \( A_f \) at \( z_t \) does not depend on \( t \). So, multiplicity is the same for all generic points, and \( \deg A_f \) is defined. But then \( \deg A_f = 0 \) because the image of \( A_f \) is not the whole \( \mathbb{R}^{2n} \) (for example, if \( z \in \mathbb{R}^{2n} \) is large enough and normal to the subspace \( E \) then certainly \( A_f^{-1}(z) = \emptyset \)).

Consider not \( A_f \) as a change of variable in \( \mathbb{R}^{2n} \) and apply it to the 2n-form \( \nu = \exp(-\lambda |z|^2 / 2) dz_1 \wedge \ldots \wedge dz_{2n} \). Using Lemma 2, obtain the following result:

\[
J(f) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \exp\left(-\lambda |f(x) - f(y)|^2 / 2\right) \det\left( \begin{array}{c} D_f(x) \\ D_f(y) \end{array} \right) \, dx \, dy
\]

\[
= \int_{\mathbb{R}^{2n}} A_f^* \nu = \deg A_f \int_{\mathbb{R}^{2n}} \nu = 0.
\]

(8)
Our idea is now to find the asymptotics of the integral \( \int \) for \( \lambda \to +\infty \) using the general results about asymptotics of Laplace integrals. Then we consider the principal term of the asymptotics and equalize it to zero.

To formulate the result introduce first some notations. Let \( \varphi = (\varphi^i_j) \in V(n, 2n) \) (i.e. \( \text{Rk} \varphi = n, 1 \leq j \leq 2n, 1 \leq i \leq n \)). For an arbitrary set of integers \( J = \{j_1, \ldots, j_n\} \) such that \( 1 \leq j_1 < \ldots < j_n \leq 2n \) denote

\[
\mu(J) = n(n - 1)/2 + j_1 + \ldots + j_n,
\]

\[
\mathfrak{M}_J \overset{\text{def}}{=} \det(\varphi_\alpha^\sigma) \quad s = 1, \ldots, n \quad \alpha_s \notin J
\]

Denote also

\[
U \overset{\text{def}}{=} \left( \sum_{k=1}^{2n} \varphi^k_1 \varphi^k_j \right),
\]

\[
(u_{ij}(\varphi)) \overset{\text{def}}{=} U^{-1},
\]

\[
\det U = 1/u^2(\varphi).
\]

(by the general theorem about Gram determinant, \( \det U \) equals the sum of squares of the \( n \)-th order minors of the matrix \( \varphi \), and is therefore positive).

**Theorem 3** Let \( n \) be even, \( f : \mathbb{R}^n \to \mathbb{R}^{2n} \) be a smooth immersion fixed at infinity, and \( D_f : \mathbb{R}^n \to V(n, 2n) \) be its differential. Then the index of \( f \) is given by the formula

\[
I(f) = \int_{\mathbb{R}^n} (D_f)^* \omega
\]

where the \( 2n \)-form \( \omega \) at the point \( \varphi = (\varphi^j_i) \in V(n, 2n) \) is

\[
\omega(\varphi) = -\frac{1}{2n+1 \pi^{n/2} (n/2)!} u(\varphi) \sum_{i_1, \ldots, i_n=1}^n u_{i_1i_2}(\varphi) \ldots u_{i_n-1i_n}(\varphi)
\]

\[
\times \sum_{\sigma \in \Sigma_n} \sum_{J = \{j_1, \ldots, j_n\}} (-1)^{\mu(J)} \mathfrak{M}_J(\varphi) \, d\varphi^j_{i_1(1)} \wedge \ldots \wedge d\varphi^j_{i_n(2n)}. \quad (10)
\]

(\( \Sigma_n \) means the symmetric group).
To prove Theorem 3 we will need the following asymptotical expansion found in [5] (Statement 4.1 at page 125):

**Lemma 3** Let \( a, S \) be smooth real-valued functions at \( \mathbb{R}^k \), \( S \) achieving its minimum at the origin (and only at the origin). Let the Hessian \( G = \frac{\partial^2 S}{\partial u_i \partial u_j}(0) \) be nondegenerate. Denote \( L \) the 2nd order differential operator \( L = \sum_{i,j=1}^{k} (G^{-1})_{ij} \frac{\partial^2}{\partial u_i \partial u_j} \), and \( R(u) \stackrel{\text{def}}{=} S(u) - S(0) - \frac{1}{2}(Gu, u) \). Then for \( \lambda \to +\infty \) there is an asymptotic expansion up to the \( O(\lambda^{-\infty}) \):

\[
\int_{\mathbb{R}^k} a(u) \exp(-\lambda S(u)) \, du \sim \exp(-\lambda S(0)) \left( \frac{2\pi}{\lambda} \right)^{k/2} |\det G|^{-1/2} \\
\times \sum_{p=0}^{\infty} \frac{1}{p!(2\lambda)^p} L^p \left( a(u) \exp(-\lambda R(u)) \right) \bigg|_{u=0}.
\]

(11)

**Proof of Theorem 3** The left-hand side \( J(f) \) of (8) is exactly the Laplace integral mentioned in Lemma 3 with \( k = 2n, u = (x, y), S(x, y) = |f(x) - f(y)|^2 / 2 \), and \( a(x, y) = \det \left( \frac{\partial f}{\partial x} \right) \). Minimum point of \( S \) is not unique but the general localization principle for Laplace integrals (see [5]) tells that one should consider expansions (11) for all such points and sum them up (points other then local minima make \( O(\lambda^{-\infty}) \) contributions).

Local minima of \( S(x, y) \) are:

1. Self-intersection points of \( f \): \( f(x) = f(y) \) and \( x \neq y \).
2. Diagonal points: \( x = y \).
3. Other (“sporadic”) minima.

At sporadic minima \( s = S(x, y) > 0 \). So expansion (11) contains the term \( \exp(-\lambda s) \) and therefore is already \( O(\lambda^{-\infty}) \). Thus, sporadic minima will be neglected.

To self-intersection points \( (x, y) \) Lemma 3 can be applied directly. The principal term here is \( p = 0 \). Denote \( g_i \) \( (i = 1, \ldots, 2n) \) the vector \( \frac{\partial f_i}{\partial x_1}(x), \ldots, \frac{\partial f_i}{\partial x_n}(x) \),
\[-\frac{\partial f}{\partial y_1}(y), \ldots, -\frac{\partial f}{\partial y_n}(y)\) and observe that \(G_{ij} = (g_i, g_j)\). By the general theorem about Gram determinant we have \(\det G = \left|\det \begin{pmatrix} D_f(x) \\ D_f(y) \end{pmatrix}\right|^2\), and the principal term is

\[
\left(\frac{2\pi}{\lambda}\right)^n |\det G|^{-1/2} a(x, y) = \left(\frac{2\pi}{\lambda}\right)^n \text{sgn} \det \begin{pmatrix} D_f(x) \\ D_f(y) \end{pmatrix}.
\]

It follows now from Theorem 4 that total contribution to the principal term of asymptotics from all the self-intersection points is equal to

\[
C_{\text{self-intersections}} = 2 \left(\frac{2\pi}{\lambda}\right)^n I(f).
\]

The factor 2 is present because every self-intersection point appears twice as a minimum of \(S\): the first time as \((x, y)\) and the second time as \((y, x)\).

Consider now the contribution from the diagonal. The minimum of \(S\) here is not isolated (and therefore degenerate) but it is easy to see that we should consider asymptotic expansion of the integral over \(y\) (with \(x\) fixed) and then integrate it over \(x\) (necessary theorems about uniformity of asymptotic expansion (11) are contained in [5]). If \(x\) is fixed then \(y = x\) is a nondegenerate minimum of \(S\), and \(G_{ij} = (h_i, h_j)\) where \(h_i = \left(\frac{\partial f_1}{\partial x_i}(x), \ldots, \frac{\partial f_n}{\partial x_i}(x)\right)\). By the general theorem about Gram determinant

\[
g(x) \overset{\text{def}}{=} |\det G|^{-1/2} = \left|\det \begin{pmatrix} \sum_{k=1}^{2n} \frac{\partial f_k}{\partial x_i}(x) \frac{\partial f_k}{\partial x_j}(x) \end{pmatrix}\right|^{-1/2} = \left(\sum_{1 \leq i_1 < \ldots < i_n \leq 2n} \left|\det \begin{pmatrix} \frac{\partial f_{i_1}}{\partial x_{i_1}}(x) \end{pmatrix}\right|^2\right)^{-1/2}.
\]

The right-hand side contains the sum of squares of all the \(n\)-th order minors of the \((2n \times n)\)-matrix \(\left(\frac{\partial f_k}{\partial x_j}(x)\right)\).

First we are to find the leading term of expansion. Let \(v(t), w(t)\) be smooth functions of one variable. Then it is easy to see that

\[
\frac{d^m}{dt^m} \left(v(t) \exp(\lambda w(t))\right) = \exp(\lambda w(t)) \sum_{u} \lambda^u
\]
× \sum_{r, k_1, \ldots, k_u \geq 0, \atop r + k_1 + \ldots + k_u = m} b_{r,k_1,\ldots,k_u} v^{(r)}(t) w^{(k_1)}(t) \ldots w^{(k_u)}(t) \tag{14}

for some integers \( b_{r,k_1,\ldots,k_u} \geq 0 \) (which we need not specify). Apply this formula with \( v = a, w = R \) to expansion (11). Here \( a(x, y) = O(|x - y|^n) \) and \( R(x, y) = O(|x - y|^3) \), and the degree of the operator \( L \) is 2 (so, \( L^p \) contains differentiations of the order \( 2p \)). Thus, the \( p \)-th term of expansion will contain \( \lambda^{u-p-n/2} \) where

\[
2p = r + k_1 + \ldots + k_u, \quad r \geq n, \quad k_i \geq 3.
\]

Thus \( 2p \geq n + 3u \), and therefore \( u - p \leq -n/2 \). So, the leading term is

\[
\frac{\pi^{n/2}}{(n/2)!} \frac{1}{\lambda^n} \int_{\mathbb{R}^n} g(x) (L^{n/2} a(x, y)) \big|_{y=x} dx \tag{15}
\]

where \( g(x) \) is given by (13).

Denote \( G^{-1} \equiv (g_{ij}(x)) \). Then, by definition,

\[
L = \sum_{i,j=1}^{n} g_{ij}(y) \frac{\partial^2}{\partial y_i \partial y_j},
\]

and

\[
L^{n/2} = \sum_{i_1, \ldots, i_n=1}^{n} g_{i_1 i_2}(y) \ldots g_{i_{n-1} i_n}(y) \frac{\partial^n}{\partial y_{i_1} \ldots \partial y_{i_n}}.
\]

This yields:

\[
(L^{n/2} a(x, y)) \big|_{y=x} = \sum_{i_1, \ldots, i_n=1}^{n} g_{i_1 i_2}(x) \ldots g_{i_{n-1} i_n}(x) \frac{\partial^n}{\partial y_{i_1} \ldots \partial y_{i_n}} \det \left( \frac{\partial f_{i_k}(x)}{\partial x_j} \right) \big|_{y=x} = \sum_{i_1, \ldots, i_n=1}^{n} g_{i_1 i_2}(x) \ldots g_{i_{n-1} i_n}(x) \sum_{\sigma \in \Sigma_n} \det \left( \frac{\partial f_{i_k}(x)}{\partial x_j} \right) \frac{\partial f_{i_k}(x)}{\partial y_{\sigma(k)}}(x),
\]

Here \( \Sigma_n \) means the symmetric group. The subscript \( j \) in determinants runs from 1 to \( 2n \), the subscript \( k \), from 1 to \( n \). Thus, total contribution to the
principal term of the asymptotics of $J(f)$ from the diagonal $y = x$ equals:

$$C_{\text{diagonal}} = \frac{\pi^{n/2}}{(n/2)!} \lambda^n \int_{\mathbb{R}^n} g(x) \sum_{i_1, \ldots, i_n = 1}^n g_{i_1 i_2}(x) \cdots g_{i_{n-1} i_n}(x) \times$$

$$\times \sum_{\sigma \in \Sigma_n} \det \left( \frac{\partial f_j}{\partial x_k}(x) \frac{\partial^2 f_j}{\partial x_{\sigma(k)} \partial x_{\sigma(i)}}(x) \right).$$

(16)

But it follows from (8) that $C_{\text{diagonal}} + C_{\text{self-intersections}} = 0$. Thus, combining (12) and (16), one has

$$I(f) = -\frac{1}{2^{n+1} \pi^{n/2} (n/2)!} \int_{\mathbb{R}^n} g(x) \sum_{i_1, \ldots, i_n = 1}^n g_{i_1 i_2}(x) \cdots g_{i_{n-1} i_n}(x) \times$$

$$\times \sum_{\sigma \in \Sigma_n} \det \left( \frac{\partial f_j}{\partial x_k}(x) \frac{\partial^2 f_j}{\partial x_{\sigma(k)} \partial x_{\sigma(i)}}(x) \right).$$

(17)

Notice now that for any $1 \leq j_1, \ldots, j_n \leq 2n$, $1 \leq i_1, \ldots, i_n \leq n$

$$\det \left( \frac{\partial^2 f_{j_s}}{\partial x_{is} \partial x_{i_s}} \right) dx_1 \wedge \cdots \wedge dx_n = (Df)^*d\varphi^{j_1}_{i_1} \wedge \cdots \wedge d\varphi^{j_n}_{i_n}$$

(18)

Now to get (9) expand the determinant in (17) in the first $n$ rows. $lacksquare$

4 An explicit formula for the generator of $H^n(V(n, 2n))$

Take up again Lemma 1. By Gurevich theorem (see e.g. [4]) for $n$ even, $H^n_{\text{DR}}(V(n, 2n)) = \mathcal{R}$ is the first nontrivial cohomology group of the Stiefel variety $V(n, 2n)$. Let $\omega_0$ be a closed $n$-form at $V(n, 2n)$ whose integral over the fundamental cycle (generator of the $H_n(V(n, 2n), \mathbb{Z})$) is 1. Then the homotopic class $[\gamma] \in \pi_n(V(n, 2n)) = \mathbb{Z}$ of the spheroid $\gamma : S^n \to V(n, 2n)$ equals to $\int_{S^n} \gamma^*\omega_0$. Now Theorems 3 and 4 suggest that $\omega = \omega_0$, i.e.,

**Theorem 4** The cohomology class of the $n$-form $\omega$ given by (17) forms the basis in $H^n_{\text{DR}}(V(n, 2n))$. The value of this class on the generator of the $H_n(V(n, 2n), \mathbb{Z})$ is 1.

The main step in the proof of Theorem 4 is the following

**Lemma 4** The form $\omega$ is closed.
**Proof** By Theorem [3], the equality
\[ \int_{\mathbb{R}^n} \gamma^* \omega = \int_{\mathbb{R}^n} \gamma^* \omega_0 \] (19)
holds for any holonomic \( \gamma = (\gamma^i_j) : \mathbb{R}^n \to V(n, 2n) \). The holonomity means that \( \gamma = D_f \) for some immersion \( f : \mathbb{R}^n \to \mathbb{R}^{2n} \) fixed at infinity, or, equivalently, that the equations
\[ \frac{\partial \gamma^i_j(x)}{\partial x_k} = \frac{\partial \gamma^i_k(x)}{\partial x_j} \quad \text{for any } i, j, k \] (20)
and
\[ \gamma^i_j(x) = \begin{cases} 1, & i = j \leq n, \\ 0, & \text{otherwise}. \end{cases} \] (21)
(for \( x \) lying outside the unit cube of \( \mathbb{R}^n \)) hold.

Consider a one-parameter family of mappings \( \gamma_t = (\gamma^i_j + t \delta^i_j) \). The definition of Lie derivative \( \mathcal{L} \) and the Cartan formula \( \mathcal{L}_X = \iota_X d + d \iota_X \) imply that
\[
\frac{d}{dt} \int_{\mathbb{R}^n} \gamma^*_t \omega = \sum_{i=1}^{2n} \sum_{j=1}^{n} \int_{\mathbb{R}^n} \delta^i_j \gamma^*_t \mathcal{L}_{\partial/\partial \varphi^i_j} \omega \\
= \sum_{i=1}^{2n} \sum_{j=1}^{n} \int_{\mathbb{R}^n} \delta^i_j \gamma^*_t \iota_{\partial/\partial \varphi^i_j} d\omega. \] (22)
Taking in (22) \( t = 0 \) and \( \gamma = \gamma_0 = D_f \), and taking (19)–(21) into consideration, one obtains the equality
\[
\sum_{i=1}^{2n} \sum_{j=1}^{n} \int_{\mathbb{R}^n} \delta^i_j (D_f)^* \iota_{\partial/\partial \varphi^i_j} d\omega = 0 \] (23)
for every \( f \) and every \( \delta = (\delta^i_j(x)) \) such that
\[ \frac{\partial \delta^i_j(x)}{\partial x_k} = \frac{\partial \delta^i_k(x)}{\partial x_j} \quad \text{for any } i, j, k \] (24)
and
\[ \delta^i_j(x) = 0 \] (25)
for $x$ lying outside the unit cube of $\mathbb{R}^n$.

Take some immersion $f : \mathbb{R}^n \to \mathbb{R}^{2n}$ fixed at infinity, and denote $P^i_j(x)$ the function such that $P^i_j(x) \, dx_1 \wedge \ldots \wedge dx_n = (D_f)^* \iota_{\partial / \partial \varphi^j} d\omega$. Denote also $Q^i_{jk}$ a function such that $P^i_j = \frac{\partial Q^i_{jk}}{\partial x_k}$. $Q^i_{jk}$ is defined uniquely up to addition of a function $C^i_{jk}$ independent of $x_k$. If (24) and (25) are fulfilled then

$$\sum_{i=1}^{2n} \sum_{j=1}^{n} \int_{\mathbb{R}^n} \delta^i_j P^i_j \, dx = \frac{1}{n} \sum_{i=1}^{2n} \sum_{j,k=1}^{n} \int_{\mathbb{R}^n} \delta^i_j \frac{\partial Q^i_{jk}}{\partial x_k} \, dx$$

$$= -\frac{1}{n} \sum_{i=1}^{2n} \sum_{j,k=1}^{n} \int_{\mathbb{R}^n} Q^i_j \frac{\partial \delta^i_j}{\partial x_k} \, dx. \quad (26)$$

Since (26) is zero for $\delta$ satisfying (24) and (25), the standard Riesz lemma implies that

$$\frac{\partial P^i_j}{\partial x_k} + \frac{\partial P^i_k}{\partial x_j} = 0 \quad \text{for any } i, j, k. \quad (27)$$

In particular, it holds for $j = k$, i.e.

$$\frac{\partial P^i_j}{\partial x_j} = 0,$$

or

$$\mathcal{L}_{\partial / \partial x_j}(D_f)^* \iota_{\partial / \partial \varphi^j} d\omega = 0. \quad (28)$$

Equation (28) and boundary condition (21) imply that $(D_f)^* \iota_{\partial / \partial \varphi^j} d\omega = 0$. Since $f$ is arbitrary, it is possible only if $\iota_{\partial / \partial \varphi^j} d\omega = 0$ for any $i, j$, which means that $d\omega = 0$. Lemma is proved. ■

**Proof of Theorem 4** Since $\omega$ is closed and the cohomologies $H^n(V(n, 2n))$ are one-dimensional, there exist $\lambda \in \mathbb{R}$ and the $(n-1)$-form $\nu$ such that $\omega = \lambda \omega_0 + d\nu$. Then for any $\gamma : \mathbb{R}^n \to V(n, 2n)$ fixed at infinity one obtains

$$\int_{\mathbb{R}^n} \gamma^* \omega = \lambda \int_{\mathbb{R}^n} \gamma^* \omega_0 \quad (29)$$

Comparing (24) with (19) one obtains $\lambda = 1$. Theorem is proved. ■

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