Hawking temperature and the emergent cosmic space

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Abstract

As proposed by T. Padmanabhan [10] cosmic space emerges from the difference between the degrees of freedom and of the bulk and the surface. He managed to derive Friedmann and Raychaudhuri equations for flat FLRW. We generalize Padmanabhan’s idea to non-flat cases. Our approach, unlike other works on the subject, completely respects Padmanabhan’s idea about horizon’s temperature. The generalization covers Einstein gravity’s extension to higher dimensions, Gauss-Bonnet and general Lovelock gravity.

Keywords: Holographic principle, Emergent cosmic space, Lovelock gravity.

1 Introduction

The famed Nobel laureate, A. Sakharov in 1967 [1] proposed that gravity appears as an emergent phenomenon (the so called induced gravity). Space-time background emerges as a mean field approximation of some underlying microscopic degrees of freedom, similar to hydrodynamics or the theory of continuum elasticity from molecular physics. The laws of black hole mechanics were discovered in the 1970 [2, 3, 4, 5]. Black hole thermodynamics indicate that a black hole’s entropy is proportional to its horizon’s area and its temperature is proportional to its surface gravity at the horizon. A worth mentioning study on the connection between thermodynamics and gravity was done by T. Jacobson [6] who was able to derive Einstein’s field equations from Clausius’ relation $\delta Q = TdS$ along with the equivalence principle. He considered $T$ as the Unruh temperature [7] seen by an accelerated observer just inside the horizon and $\delta Q$ as the energy flux across the horizon. The considerable consequence of this
approach is that the Einstein’s field equations are nothing but the equations of state of the space-time. In 2010 Verlinde [8] remarked that gravity is not a fundamental interaction but an entropic force caused by changes in the entropy with the information on the holographic screen. This way, he was able to derive Newton law of gravity using the entropic force, holographic principle and the equipartition law for energy. He also derived Newton’s second law of mechanics using the entropic force and Unruh temperature. Padmanabhan [13] simultaneously derived Newton’s law of gravity by assuming the equipartition law of energy for the horizon degrees of freedom and $S = E/2T$, where $E$ is an active gravitational mass. Besides laws of classical mechanics, Cai [18] observed that the equipartition law of energy for the horizon’s degrees of freedom, the holographic principle with the Unruh temperature lead to Friedmann’s equations for flat space-time. Very recently Friedmann’s equations in Lovelock gravity with arbitrary spatial curvature obtained in Ref. [9]. Padmanabhan showed in all the Lanczos-Lovelock models, the bulk and the surface terms in the lagrangian are related through

$$\sqrt{-g} L_{\text{surf}} = -\partial_a \left( g_{ij} \frac{\delta (\sqrt{-g} L_{\text{bulk}})}{\delta (\partial_a g_{ij})} \right), \quad (1)$$

the boundary term of the action gives the entropy while the bulk term gives the energy. In the emergent perspective, it is the entropy density, rather than the energy density of matter, which plays the crucial role. He [19] proposed that the gravitational action acts as the free energy of space-time for static metric which posses a horizon ($L = TS - E$). Naturally the standard action principle in theories of gravity does nothing but extremising the free energy of space-time. In this approach, it seems necessary to treat time differently from space, which contradicts the spirit of general covariance. Moreover, it seems incorrect to argue that space is emergent around finite gravitating systems (such as the Sun-Earth system). These dilemmas at least in the context of cosmology with special choice of time (cosmic time) disappear. Padmanabhan has pointed out that “Cosmic space is emergent as cosmic time progresses”. Following Padmanabhan’s idea, the expansion of the Universe continues till the holographic equipartition takes place. He has proposed that in an infinitesimal interval $dt$ of cosmic time, the increase $dV$ of the cosmic volume is given by

$$\frac{dV}{dt} = L_p^2 (N_{\text{surf}} - N_{\text{bulk}}). \quad (2)$$

2 Thermodynamics of the Apparent Horizon

The standard spatial homogenous and isotropic Universe line element can be written as

$$ds^2 = h_{\mu\nu} dx^\mu dx^\nu + \tilde{r}^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (3)$$

where $\tilde{r} = a(t) r$, $x^0 = t$, $x^1 = r$ and $h_{\mu\nu} = \text{diag}(-1, a^2/(1 - kr^2))$ is the two dimensional metric. The dynamical apparent horizon [14,15], a marginally
trapped surface with vanishing expansion \((\theta_l\theta_n = 0\) where \(l, n\) are the outgoing and ingoing radial null geodesics\([15]\)), is determined by the relation \(h^{\mu\nu}\partial_\mu\tilde{r}\partial_\nu\tilde{r} = 0\). Straightforward calculations yield the radius of the apparent horizon as

\[
\tilde{r}_A = \left( H^2 + \frac{k}{a^2} \right)^{-\frac{1}{2}}, \tag{4}
\]

where \(H = \frac{\dot{a}}{a}\) is the Hubble parameter and \(k\) is +1, −1 or 0, corresponding to the closed, open, or flat Universes. The Hubble horizon \((\tilde{r}_H = \frac{1}{H})\) only provides the order of magnitude of the radius of curvature of FLRW space and is used as an estimate of the the event horizon radius during inflation, when the Universe is close to a de Sitter space. This concept seems unnecessary, in spite of existing the apparent horizon. The apparent horizon is defined locally using null geodesic congruences and their expansions, and there is no reference to the global causal structure.

The associated temperature \(T\) with the apparent horizon is calculated in the literature with different methods\([15]\) and references therein). Similar to the thermodynamics of black hole horizon, we assume the Hawking temperature is also valid in the cosmological context

\[
\kappa = \frac{1}{2\sqrt{-h}}\partial_\nu \left( \sqrt{-h} h^{\mu\nu}\partial_\nu \tilde{r} \right) = -\frac{1}{\tilde{r}_A} \left( 1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A} \right). \tag{5}
\]

The zeroth law of the black hole thermodynamics tells us for a static or stationary black hole, the surface gravity is a constant. Evidently this law with the first law of the black hole thermodynamics have motivated authors to deal with the apparent horizon radius as a fixed value. Hence it seem it is convincing to use \(\kappa \approx \frac{1}{\tilde{r}_A}\)\([14]\). However, it is easily seen that the above result is contradicting with the emerging cosmic space (like \([10, 11, 9, 12]\)). We know \(N_{\text{sur}} = N_{\text{bulk}}\) for the pure de Sitter Universe, but our Universe is asymptotically de Sitter due to late time acceleration. The difference between degrees of freedom of the bulk and the surface will change the apparent volume sphere, and the mechanism which proposed in equation (2) enforces the Universe to decrease the difference between the degrees of freedom of the bulk and the surface. At the final stage, our Universe will reach the holographic equipartition. The key idea is that the difference between degrees of freedom on the surface and inside the bulk changes the apparent volume of the Universe. While focusing on the non-zero difference between degrees of freedom, the apparent horizon radius definitely can not be a fixed value. Because, we do not have any volume change (whether increase or decrease) for a fixed value of the apparent horizon, it is not allowed to ignore the second term in the equation (5). The important conclusion is that the equations (2) and (10) offered in \([9, 12, 10, 11]\) seem not consistent with Padmanabhan’s proposal.

According to the Hawking temperature formula \(k_B T = \frac{\hbar}{2\pi}\kappa\), the apparent temperature would be negative which is kind of odd, therefore we need to employ
the absolute value of the Hawking temperature in the context of cosmology

\[ T = \frac{|\kappa|}{2\pi} = \frac{\dot{r}_A}{4\pi} \left( \dot{H} + 2H^2 + \frac{k}{a^2} \right) = \frac{1}{2\pi \dot{r}_A} \left( 1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A} \right). \]  \( \text{(6)} \)

3 Friedmann and Raychaudhuri equations in Einstein gravity

The original idea of Padmanabhan [10, 11] “Cosmic space is emergent as cosmic time progresses” is valid at least in the context of cosmology. Hence, we just need to know appropriate form of \( N_{\text{sur}} \) and the relevant temperature of horizon. By assuming the number of surface degrees of freedom is given by [10]

\[ N_{\text{sur}} = \frac{A}{L_p^2} = \frac{4\pi \tilde{r}_A^2}{L_p^2}, \]  \( \text{(7)} \)

where \( L_p \) is the Planck length, \( A = 4\pi \tilde{r}_A^2 \) denotes the cosmic sphere surface with the apparent radius \( \tilde{r}_A \), and the Hawking temperature associated with the apparent horizon is given by equation (6). Also the Komar energy contained inside the bulk is [20]

\[ E_{\text{Komar}} = |\rho + 3p|V, \]  \( \text{(8)} \)

where \( V = \frac{4}{3} \pi \tilde{r}_A^3 \) and the bulk degrees of freedom obey the equipartition law of energy

\[ N_{\text{bulk}} = -\frac{2}{T}E_{\text{Komar}}. \]  \( \text{(9)} \)

As suggested in [10, 11] one needs to add a minus sign in front of \( E_{\text{Komar}} \), in order to have \( N_{\text{bulk}} > 0 \), which makes sense only in the accelerating phase \( \rho + 3p < 0 \). For normal matters the minus sign is not needed.

Padmanabhan’s idea is that the difference between the degrees of freedom of the surface and the bulk has to tend zero. This simple law indicates that the rate of change of cosmic volume over cosmic time, is a function of the difference between the degrees of freedom of the surface and bulk. Up to the first order we have two suggestions until now. The first one is mentioned in equation (2) and another one as in [9]

\[ \frac{dV}{dt} = L_p^2 H\tilde{r}_A (N_{\text{sur}} - N_{\text{bulk}}). \]  \( \text{(10)} \)

We propose the following formula for the Universe with an arbitrary spatial curvature, with correct form of the Hawking temperature

\[ \frac{dV}{dt} = L_p^2 4\pi \tilde{r}_A^3 TH (N_{\text{sur}} - N_{\text{bulk}}) = L_p^2 4 \tilde{r}_A^3 TH (N_{\text{sur}} - N_{\text{bulk}}). \]  \( \text{(11)} \)

\footnote{For simplicity, the natural units \( k_B = c = \hbar = 1 \) are used throughout this paper.}
Replacing (6), (7), (8) and (9) into (11), we obtain

\[
\dot{\tilde{r}}_A = \frac{L_p^2 H}{2 \pi \tilde{r}_A} \left( 1 - \frac{\dot{\tilde{r}}_A}{2 H \tilde{r}_A} \right) \left[ \frac{4 \pi \tilde{r}_A^2}{L_p^2} + \frac{32 \pi^2 \tilde{r}_A^5 H (\rho + 3 \rho)}{3 (2 H \tilde{r}_A - \dot{\tilde{r}}_A)} \right].
\]  

(12)

Reorganizing the terms, gives Raychaudhuri equation

\[
\dot{H} + H^2 = -\frac{4 \pi}{3} L_p^2 (\rho + 3 \rho).
\]  

(13)

Now, using the continuity equation \( \dot{\rho} + 3H(\rho + p) = 0 \) and the equation (12) we obtain

\[
\ddot{r}_A^2 - \frac{\tilde{r}_A^3 \ddot{r}_A}{H} = \frac{4 \pi}{3} L_p^2 (\dot{\rho} + 2 \rho),
\]  

(14)

which can be simplified as

\[
\frac{d}{dt} \left( a^2 \tilde{r}_A^{-2} \right) = \frac{d}{dt} \left( a^2 (H^2 + \frac{k}{a^2}) \right) = \frac{8 \pi}{3} L_p^2 \frac{d}{dt} (a^2 \rho).
\]  

(15)

Integrating with integration constant equals zero, we get

\[
H^2 + \frac{k}{a^2} = \frac{8 \pi}{3} L_p^2 \rho.
\]  

(16)

which happens to be the well-known Friedmann equation of FLRW Universe. So we succeeded to derive Friedmann equation with arbitrary curvature by Shekhi’s method. Our suggestion does not suffer from ill-defined temperature at all. We are going to extend these calculations to the \((n + 1)\)-dimensional Einstein gravity, Gauss-Bonnet and Lovelock gravity.

In order to generalize, equation (11) is modified into \((n + 1)\)-dimensional Universe as following (Ref. [8])

\[
N_{\text{sur}} = \frac{\alpha A}{L_p^{n-1}},
\]  

(17)

where \(A = n \Omega_n \tilde{r}_A^{n-1}, \alpha = \frac{n-1}{2(n-2)} \) and \(\Omega_n\) is the volume of the unit \(n\)-dimensional sphere. In this case, equation (11) will be modified as

\[
\frac{dV}{dt} = \alpha^{-1} L_p^{n-1} 4 \pi \tilde{r}_A^2 TH \left( N_{\text{sur}} - N_{\text{bulk}} \right),
\]  

(18)

where the volume \(V = \Omega_n \tilde{r}_A^n\). Therefor the \((n + 1)\)-dimensional form of bulk Komar energy is

\[
E_{\text{Komar}} = \frac{(n - 2) \rho + np}{n - 2} V,
\]  

(19)

and the bulk degrees of freedom is given by (9), similar to the 4D case. By substituting Komar energy (19) into the equation (18) and the continuity equation in \((n + 1)\)-dimensions, \(\dot{\rho} + nH(\rho + p) = 0\), one gets

\[
\frac{d}{dt} \left( a^2 \tilde{r}_A^{-2} \right) = \frac{d}{dt} \left( a^2 (H^2 + \frac{k}{a^2}) \right) = \frac{16 \pi L_p^{n-1} d}{n(n-1)} \frac{d}{dt} (a^2 \rho).
\]  

(20)
Integrating with zero integration constant yields the Friedmann equation in the $(n + 1)-$dimensions

$$H^2 + \frac{k}{a^2} = \frac{16\pi L_p^{n-1}}{n(n-1)} \rho. \quad (21)$$

One can yield Raychaudhuri equation directly from equation (18)

$$\dot{H} + H^2 = \frac{-8\pi L_p^{n-1}}{n(n-1)} [(n-2)\rho + np]]. \quad (22)$$

Note that, equations (11) and (18) could be simplified as

$$L_p^{n-1} TH (N_{sur} - N_{bulk}) = \frac{\alpha}{4\pi \bar{r}_A} \frac{dV_n}{dt} = \frac{\alpha^2}{n-1} \frac{dV_{n-2}}{dt}. \quad (23)$$

It seems that projection of the cosmic space on the $(n-2)$-dimensional hypersurface plays a crucial rule towards holographic equipartition. This point is worth further studying.

## 4 Friedmann equation in Gauss-Bonnet and Lovelock gravity

The entropy of the Gauss-Bonnet gravity does not obey the Bekenstein-Hawking area law. An additional correction term in the Gauss-Bonnet gravity is given by [12]

$$S = \frac{A}{4L_p^{n-1}} \left[ 1 + \frac{n-1}{n-3} \frac{2\tilde{\alpha}}{\bar{r}_A} \right], \quad (24)$$

where $\tilde{\alpha} = (n-2)(n-3)\alpha$, and other quantities are the same as in the previous section. The area law formula can be rescued by redefinition of area of holographic surface. The effective area is

$$\tilde{A} = A \left[ 1 + \frac{n-1}{n-3} \frac{2\tilde{\alpha}}{\bar{r}_A} \right], \quad (25)$$

and the effective volume rate is

$$\frac{d\tilde{V}}{dt} = \frac{\tilde{r}_A}{n-1} \frac{d\tilde{A}}{dt} = n\Omega_n \tilde{r}_A^{n-1} \tilde{r}_A (1 + 2\tilde{\alpha} \frac{\bar{r}_A}{n-3}) \tilde{r}_A. \quad (26)$$

The number of degrees of freedom on the holographic surface in the Gauss-Bonnet gravity is proposed as

$$N_{sur} = \frac{\alpha n \Omega_n \tilde{r}_A^{n+1}}{L_p^{n-1}} \left( \tilde{r}_A^{-2} + 2\tilde{\alpha} \tilde{r}_A^{-4} \left( \frac{H\tilde{r}_A - \dot{\tilde{r}}_A}{2H\tilde{r}_A - \dot{\tilde{r}}_A} \right) \right), \quad (27)$$
so that, the origin of the last term in parentheses is the extra term in the temperature definition (6), and other terms are defined similar to [9, 12]. Now using equation (18), we obtain

\[
\frac{16\pi L_p^{n-1}}{n(n-1)} \frac{d}{dt}(a^2 \rho) = \frac{d}{dt} \left[ a^2 \left( \hat{\beta}_A^{-2} + \hat{\alpha} \hat{\gamma}_A^{-4} \right) \right] = \frac{d}{dt} \left[ a^2 \left( H^2 + \frac{k}{a^2} + \hat{\alpha} (H^2 + \frac{k}{a^2})^2 \right) \right].
\] (28)

Integrating yields

\[
H^2 + \frac{k}{a^2} + \hat{\alpha} \left( H^2 + \frac{k}{a^2} \right)^2 = \frac{16\pi L_p^{n-1}}{n(n-1)} \rho.
\] (29)

Once again we have correct form of Friedmann equation in the \((n+1)\)-dimensional Gauss-Bonnet gravity.

In the more general Lovelock gravity [16], the spherically symmetric black hole has entropy in the following form [17]

\[
S = \frac{A_+}{4L_p^{n-1}} \sum_{i=1}^{m} \frac{i(n-1)}{n-2i+1} \hat{c}_i \hat{r}_A^{2-2i},
\] (30)

where \(m = \left\lceil \frac{n}{2} \right\rceil\), and \(\hat{c}_i\) are dimensional dependent coefficients. The effective area of the holographic surface is

\[
\hat{A} = n\Omega_n \hat{r}_A^{-n+1} \sum_{i=1}^{m} \frac{i(n-1)}{n-2i+1} \hat{c}_i \hat{r}_A^{2-2i},
\] (31)

and the effective volume rate is given by

\[
\frac{d\hat{V}}{dt} = \frac{\hat{r}_A}{n-1} \frac{d\hat{A}}{dt} = n\Omega_n \hat{r}_A^{n+1} \sum_{i=1}^{m} \hat{c}_i \hat{r}_A^{2i}.
\] (32)

We also propose the number of degrees of freedom on the holographic surface in the Lovelock gravity as

\[
N_{\text{sur}} = \frac{\hat{A}}{L_p^{n-1}} \sum_{i=1}^{m} \frac{1}{2H \hat{r}_A - \hat{r}_A} \hat{c}_i \hat{r}_A^{2i} \left( 2H \hat{r}_A - \hat{r}_A \right).
\] (33)

So that, the origin of the last term in parentheses is the extra term in the temperature definition (6), and other terms are defined similar to [9, 12]. Now using equation (18), we obtain

\[
\frac{16\pi L_p^{n-1}}{n(n-1)} \frac{d}{dt}(a^2 \rho) = \frac{d}{dt} \left[ a^2 \sum_{i=1}^{m} \hat{c}_i \hat{r}_A^{2i} \right] = \frac{d}{dt} \left[ a^2 \sum_{i=1}^{m} \hat{c}_i (H^2 + \frac{k}{a^2})^i \right].
\] (34)

Integrating yields

\[
\sum_{i=1}^{m} \hat{c}_i (H^2 + \frac{k}{a^2})^i = \frac{16\pi L_p^{n-1}}{n(n-1)} \rho.
\] (35)

As one can see, we arrive at the same result as Cai did using his thermodynamical approach [13, 14].
5 Concluding Remarks

Padmanabhan has recently proposed that the emergence of space and expansion of the Universe are related to the difference between the number of degrees of freedom on the surface of horizon and the one in the emerged bulk. Padmanabhan used the Hubble horizon to derive Friedmann equation for flat FLRW. Cai approach applies to all dimensions and also the general Lovelock gravity. Very recently Sheykhi used the apparent horizon to rederive Friedman equation in any dimension with an arbitrary spatial curvature. We realized that the omitted term of the horizon temperature plays a crucial rule. We generalize Padmanabhan’s formula and derive Friedmann equation not only for \((n+1)\)-dimensional Einstein gravity but also for the more general Lovelock gravity for any spatial curvature.

References

[1] A. D. Sakharov, Sov. Phys. Dokl. 12, 1040 (1968) [Gen. Rel. Grav. 32, 365 (2000)].
[2] J. M. Bardeen, B. Carter and S.W. Hawking, Commun. Math. Phys. 31, 161 (1973).
[3] J. D. Bekenstein, Phys. Rev. D 7, 2333 (1973).
[4] J. D. Bekenstein, Phys. Rev. D 9, 3292 (1974).
[5] S. W. Hawking, Nature 248, 30 (1974).
[6] T. Jacobson, Phys. Rev. Lett. 75, 1260 (1995), arXiv:gr-qc/9504004.
[7] W. G. Unruh, Phys. Rev. D 14, 870 (1976).
[8] E. Verlinde, J. High Energy Phys. 4, 029 (2011), arXiv:1001.0785.
[9] A. Sheykhi, Phys. Rev. D 87, 061501 (2013), arXiv:1304.3054.
[10] T. Padmanabhan, arXiv: 1206.4916.
[11] T. Padmanabhan, arXiv: 1207.0505.
[12] R. G. Cai, J. High Energy Phys. 11, 016 (2012), arXiv:1207.0622.
[13] T. Padmanabhan, Mod Phys. Lett. 25, 1129 (2010), arXiv:0912.3165.
[14] R. G. Cai and S. P. Kim, J. High Energy Phys. 02, 050 (2005), arXiv:hep-th/0501055.
[15] V. Faraoni, Phys. Rev. D 84, 024003 (2011), arXiv:1106.4427.
[16] D. Lovelock, J. Math. Phys. (N.Y.) 12, 498 (1971).
[17] R. G. Cai, Phys. Lett. B 582, 237 (2004), arXiv:hep-th/0311240

[18] R. G. Cai et al., Phys. Rev. D 81, 061501 (2010), arXiv:1001.3470

[19] S. Kolekar et al., Phys. Rev. D 85, 064031 (2012), arXiv:1111.0973

[20] T. Padmanabhan, Class. Quant. Grav. 21, 4485 (2004), arXiv:gr-qc/0310027