ASYMPTOTIC EXPANSION OF THE NONLOCAL HEAT CONTENT

TOMASZ GRZYWNY AND JULIA LENCZEWSKA

Abstract. Let \((p_t)_{t \geq 0}\) be a convolution semigroup of probability measures on \(\mathbb{R}^d\) defined by
\[
\int_{\mathbb{R}^d} e^{i\langle \xi, x \rangle} p_t(dx) = e^{-t \psi(\xi)}, \quad \xi \in \mathbb{R}^d,
\]
and let \(\Omega\) be an open subset of \(\mathbb{R}^d\) with finite Lebesgue measure. In this article we consider the quantity
\[
H_{\Omega}(t) = \int_{\Omega} \int_{\Omega-x} p_t(dy)dx,
\]
which is called the heat content. We study its asymptotic expansion under mild assumptions on \(\psi\), in particular in the case of the \(\alpha\)-stable semigroup.

1. Introduction

Let \(d \in \mathbb{N}\). We consider a semigroup of probability measures \((p_t)_{t \geq 0}\) given by
\[
\int_{\mathbb{R}^d} e^{i\langle \xi, x \rangle} p_t(dx) = e^{-t \psi(\xi)}, \quad \xi \in \mathbb{R}^d,
\]
where \(\psi\) is a symbol defined by
\[
\psi(\xi) = \int_{\mathbb{R}^d} \left(1 - e^{i\langle \xi, z \rangle}\right) \nu(dz), \quad \xi \in \mathbb{R}^d,
\]
and \(\nu(dz)\) is a Borel measure satisfying
\[
\nu(\{0\}) = 0, \quad \int_{\mathbb{R}^d} (1 \wedge |z|) \nu(dz) < \infty.
\]
Let \(\{P_t\}_{t \geq 0}\) be the convolution semigroup of operators on \(C_0(\mathbb{R}^d)\) defined by \((p_t)_{t \geq 0}\) and let \(\mathcal{L}\) denote its infinitesimal generator, which for \(f \in C^2_c(\mathbb{R}^d)\) is given by the formula
\[
\mathcal{L} f(x) = \int_{\mathbb{R}^d} (f(x + z) - f(x)) \nu(dz).
\]

Let \(\Omega\) be a non-empty, open subset of \(\mathbb{R}^d\) such that its Lebesgue measure \(|\Omega|\) is finite. We consider the following quantity associated with the semigroup \((p_t)_{t \geq 0}\),
\[
H_{\Omega}(t) = \int_{\Omega} \int_{\Omega-x} p_t(dy)dx,
\]
which we will call heat content.

We note that the function \(u(t, x) = \int_{\Omega-x} p_t(dy)\) is the weak solution of the initial value problem
\[
\frac{\partial}{\partial t} u(t, x) = \mathcal{L} u(t, x), \quad t > 0, \quad x \in \mathbb{R}^d,
\]
\[
u(0, x) = 1_{\Omega}(x).
\]
Therefore, the quantity \(H_{\Omega}(t)\) can be interpreted as the amount of heat in \(\Omega\) if its initial temperature is one whereas the initial temperature of \(\Omega^c\) is zero.

2020 Mathematics Subject Classification. 60G51, 60G52, 60J76, 35K05.

Key words and phrases. asymptotic expansion, characteristic exponent, convolution semigroup, fractional Laplacian, heat content, Hölder space, Lévy measure, nonlocal operator, perimeter, regular variation.

This research was partially supported by National Science Centre (Poland) grant 2019/33/B/ST1/02494.
Our main goal is to study the asymptotic expansion of $H_\Omega(t)$ for small $t$. We observe that

$$H_\Omega(t) = |\Omega| - H(t),$$

where

$$H(t) = \int_\Omega \int_{\Omega^c} p_t(dy)dx,$$

and hence it suffices to work with the function $H(t)$. One of the main results of [7] states that, for small $t$,

$$H_\Omega(t) = |\Omega| - t \text{Per}_\nu(\Omega) + o(t),$$

where $\text{Per}_\nu(\Omega)$ is the nonlocal perimeter related to the measure $\nu$, defined as

$$\text{Per}_\nu(\Omega) = \int_\Omega \int_{\Omega^c} \nu(dy)dx.$$

For instance, if $\nu$ is the $\alpha$-stable Lévy measure with $\alpha \in (0,1)$, denoted by $\nu^{(\alpha)}(dz) = \mathcal{A}_{d,-\alpha} |z|^{-d-\alpha} dz$, where

$$\mathcal{A}_{d,-\alpha} = \frac{2\alpha \Gamma \left( \frac{d+\alpha}{2} \right)}{\pi^{d/2} \left| \Gamma \left( -\frac{\alpha}{2} \right) \right|},$$

then $\text{Per}_{\nu^{(\alpha)}}(\Omega) = \mathcal{A}_{d,-\alpha} \text{Per}_{(\alpha)}(\Omega)$, with $\text{Per}_{(\alpha)}(\Omega)$ being the well-known $\alpha$-perimeter [6], given for $0 < \alpha < 1$ by

$$\text{Per}_{(\alpha)}(\Omega) = \int_\Omega \int_{\Omega^c} \frac{dy}{|x-y|^{d+\alpha}}.$$

In the present paper, we shall establish the next terms of the asymptotic expansion of the heat content related to convolution semigroups. Such result is new even for the fractional Laplacian $(-\Delta)^{\alpha/2}$ (in our setting we consider $\alpha \in (0,1)$). For instance, if $1/\alpha$ is a natural number, we prove the following expansion of the heat content for the fractional Laplacian:

$$H_\Omega(t) = |\Omega| + \sum_{n=1}^{1/\alpha-1} \frac{(-1)^n}{n!} t^{n\alpha} \text{Per}_{\nu^{(\alpha)}}(\Omega) + \frac{(-1)^{1/\alpha}}{(1/\alpha - 1)!\pi} t^{1/\alpha} \log(1/t) \text{Per}(\Omega) + o(t^{1/\alpha} \log(1/t)),$$

where $\text{Per}$ is the classical perimeter of the set, see (8). A natural question arises: will the next term be the mean curvature or its non-local counterpart?

The key observation to obtain the asymptotic expansion is that the heat content can be expressed as the action of the semigroup on the covariance function of a set. We give more general results, concerning the asymptotic expansion of $P_t f$ for functions $f$ belonging to Hölder space. Our standing assumption is the weak upper scaling of the symbol $\psi^{\ast}$, or equivalently, certain scaling properties of the concentration function of a Lévy measure, see Theorem 1. For a class of convolution semigroups, for instance for the semigroup associated to $\log(1 + \Delta)$, we get the full expansion, see Theorem 2. We apply these results to obtain the expansion of heat content, which are stated in Corollaries 1 and 2. Using asymptotic expansion of the heat kernel of the fractional Laplacian, we give more explicit asymptotic expansion in the case of $\alpha$-stable semigroups, see Theorems 3 and 4.

Heat content related to the Gaussian semigroup ($\mathcal{L} = \frac{1}{2} \Delta$) of a set at time $t$ was defined by van den Berg [20] by means of the heat semigroup. Van den Berg and Gilkey [21] proved that the heat content, regarded as a function of variable $t$, has an asymptotic expansion as $t$ tends to 0. The first three terms in the expansion include the volume of the set, its perimeter and its mean curvature. The short time behavior of heat semigroup in connection with the geometry of sets with finite perimeter was also studied by Angiuli, Massari and Miranda [5]. The concept of the heat content was extended to the nonlocal setting of $\alpha$-stable semigroups in 2016 by Acuña Valverde [1], who described the small-time asymptotic behavior of the nonlocal heat content in this case. In the one-dimensional case, the number of terms of the expansion depends on the
parameter $\alpha$, and in the multidimensional case, there are the first two terms of the expansion. The same author found first three terms of the asymptotic expansion for the Poisson heat content over the unit ball [2] and over convex bodies [3]. In 2017, Cygan and Grzywny [7] introduced the notion of a nonlocal heat content related to general probabilistic convolution semigroups and generalized the mentioned results of Acuña Valverde. Later, they proved similar results for the generalized heat content related to convolution semigroups [8]. Mazón, Rossi and Toledo [18] found the full asymptotic expansion of the heat content for nonlocal diffusion with nonsingular kernels. Recently, in a more general setting, the heat content related to fractional Laplacian in Carnot groups was studied by Ferrari, Miranda, Pallara, Pinamonti, and Sire [9].

2. Preliminaries

2.1. Convolution semigroups. For $f : \mathbb{R}^d \to \mathbb{R}$, let

$$ P_t f(x) = \int_{\mathbb{R}^d} f(x + y) p_t(dy), \quad t \geq 0, \ x \in \mathbb{R}^d. $$

The generator $L$ of the semigroup $\{P_t\}_{t \geq 0}$ is defined as

$$ Lf(x) = \lim_{t \to 0^+} \frac{P_t f(x) - f(x)}{t}, $$

for functions for which the above limit exists.

We denote by $C_0(\mathbb{R}^d)$ the space of continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ vanishing at infinity. For $\beta \in (0, 1]$ we define

$$ |||f|||_{\beta} := \sup_{|x-y| \leq 1} \frac{|f(x) - f(y)|}{|x-y|^{\beta}}. $$

We will consider the Hölder space

$$ C_0^\beta = \left\{ f \in C_0(\mathbb{R}^d) : \|f\|_{\beta} := |||f|||_{\beta} + \|f\|_{\infty} < \infty \right\}. $$

[15, Theorem 3.2] implies that, for a fixed $\beta \in (0, 1]$, if $\int_{|y|<1} |y|^\beta \nu(dy) < \infty$, then for $f \in C_0^\beta$,

$$ Lf(x) = \int_{\mathbb{R}^d} (f(x + y) - f(x)) \nu(dy). $$

The real part of the symbol $\psi$ equals $\text{Re}[\psi(\xi)] = \int_{\mathbb{R}^d} (1 - \cos(\xi, z)) \nu(dz)$. We will consider its radial, continuous and non-decreasing majorant defined by

$$ \psi^*(r) = \sup_{|\xi| \leq r} \text{Re}[\psi(\xi)], \quad r > 0. $$

For $r > 0$ we define the concentration function

$$ h(r) = \int_{\mathbb{R}^d} \left( 1 \wedge \frac{|x|^2}{r^2} \right) \nu(dx). $$

By [11, Lemma 4], for all $r > 0$,

$$ \frac{1}{8(1 + 2d)} h(1/r) \leq \psi^*(r) \leq 2h(1/r). $$

Hence $h$ is a more tractable version of $\psi^*$. By [12, Lemma 2.1],

$$ \int_{|z| \geq r} \nu(dz) \leq h(r) \quad \text{for all} \ r > 0. $$
By [12, Lemma 2.7], if \( f : [0, \infty) \to [0, \infty) \) is differentiable, \( f(0) = 0, f' \geq 0 \) and \( f' \in L_{\text{loc}}^1([0, \infty)) \), then for all \( r > 0 \),

\[
\int_{|z|<r} f(|z|) \nu(dz) = \int_0^r f'(s) \nu(|x| \geq s) \, ds - f(r)\nu(|x| \geq r).
\]

Let \( \theta_0 \geq 0 \) and \( \phi : (\theta_0, \infty) \to [0, \infty) \). We say that \( \Omega \) is of finite perimeter if \( \text{Per}(\Omega) < \infty \).

Then, \( \phi(\lambda \theta) \leq \overline{C}\lambda^\alpha \phi(\theta) \) for \( \lambda \geq 1, \theta > \theta_0 \).

In short, \( \phi \in \text{WUSC} (\alpha, \theta_0, \overline{C}) \). This condition will be our standing assumption on the symbol \( \psi^* \) throughout the paper.

The following auxiliary result is a consequence of (4), (5) and (6).

**Lemma 1.** Let \( \alpha \in (0, 2) \), \( \overline{C} \in [1, \infty) \) and \( \theta_0 \in [0, \infty) \). Consider

(A1) \( \psi^* \in \text{WUSC} (\alpha, \theta_0, \overline{C}) \)

(A2) There is \( C > 0 \) such that for all \( \lambda \leq 1 \) and \( r < 1/\theta_0 \),

\[
h(\lambda r) \leq C\lambda^{-\alpha} h(r).
\]

Then, (A1) implies (A2) with \( C = c_d\overline{C} \), where \( c_d = 16(1 + 2d) \), while (A2) gives (A1) with \( \overline{C} = c_d C \). If additionally \( \theta_0 \in [0, 1) \), then (A1) and (A2) each imply

(A3) For all \( \varepsilon > 0 \),

\[
\int_{|y|<1} |y|^{\alpha+\varepsilon} \nu(dy) < \infty.
\]

**Proof.** We will show that (A1) implies (A2). Using (4), (7) and again (4), we obtain

\[
h(\lambda r) \leq 8(1 + 2d)\psi^*_*(\lambda r)^{-1}) \leq 8(1 + 2d)\overline{C}\lambda^{-\alpha} \psi^*(r^{-1})
\]

\[
\leq 16(1 + 2d)\overline{C}\lambda^{-\alpha} h(r).
\]

The converse implication can be proved analogously. It remains to show that (A1) implies (A3). By (6) with \( f(s) = s^{\alpha+\varepsilon} \) and \( r = 1 \),

\[
\int_{|y|<1} |y|^{\alpha+\varepsilon} \nu(dy) = (\alpha + \varepsilon) \int_0^1 s^{\alpha+\varepsilon-1} \nu(|x| \geq s) \, ds - \nu(|x| \geq 1)
\]

\[
\leq (\alpha + \varepsilon) \int_0^1 s^{\alpha+\varepsilon-1} h(s) \, ds
\]

\[
\leq (\alpha + \varepsilon) Ch(1) \int_0^1 s^{\varepsilon-1} \, ds = \frac{C(\alpha + \varepsilon)}{\varepsilon} h(1) < \infty.
\]

\[ \square \]

### 2.2. Heat content.

Following [4, Section 3.3], for any measurable set \( \Omega \subset \mathbb{R}^d \) we define its perimeter \( \text{Per}(\Omega) \) as

\[
\text{Per}(\Omega) = \sup \left\{ \int_{\mathbb{R}^d} \mathbb{1}_\Omega(x) \text{div} \phi(x) \, dx : \phi \in C_c^1(\mathbb{R}^d, \mathbb{R}^d), \|\phi\|_{\infty} \leq 1 \right\}.
\]

We say that \( \Omega \) is of finite perimeter if \( \text{Per}(\Omega) < \infty \). We mention that, by [4, Proposition 3.62], for any open \( \Omega \) with Lipschitz boundary \( \partial \Omega \) and finite Hausdorff measure \( \sigma(\partial \Omega) \) we have

\[
\text{Per}(\Omega) = \sigma(\partial \Omega).
\]
For any $\Omega \subset \mathbb{R}^d$ with finite Lebesgue measure $|\Omega|$, we define the covariance function $g_\Omega$ of $\Omega$ as follows

\[ g_\Omega(y) = |\Omega \cap (\Omega + y)| = \int_{\mathbb{R}^d} 1_\Omega(x) 1_\Omega(x - y) dx, \quad y \in \mathbb{R}^d. \]

We recall some important properties of the function $g_\Omega$. Let $\Omega \subset \mathbb{R}^d$ have finite measure. Then, by [10, Proposition 2, Theorem 13 and Theorem 14], $g_\Omega$ is symmetric, nonnegative, bounded from above by $|\Omega|$, and $g_\Omega \in C_0(\mathbb{R}^d)$. Moreover, if $\mathrm{Per}(\Omega) < \infty$, then $g_\Omega$ is Lipschitz with

\[ 2\|g_\Omega\|_{\mathrm{Lip}} \leq \mathrm{Per}(\Omega). \]

Moreover, for all $r > 0$ the limit $\lim_{r \to 0^+} \frac{g_\Omega(0) - g_\Omega(ru)}{r}$ exists, is finite and

\[ \mathrm{Per}(\Omega) = \frac{\Gamma((d + 1)/2)}{\pi^{(d-1)/2}} \int_{S^{d-1}} \lim_{r \to 0^+} \frac{g_\Omega(0) - g_\Omega(ru)}{r} \sigma(du). \]

In particular, there is a constant $C = C(\Omega) > 0$ such that

\[ 0 \leq g_\Omega(0) - g_\Omega(y) \leq C(1 \wedge |y|). \]

By Cygan, Grzywny [7, Lemma 3], the related function $H(t)$ has the following form

\[ H(t) = \int_{\mathbb{R}^d} (g_\Omega(0) - g_\Omega(y)) \rho_t(dy), \]

and by [7, Proof of Lemma 1],

\[ \mathrm{Per}_\nu(\Omega) = \int_{\mathbb{R}^d} (g_\Omega(0) - g_\Omega(y)) \nu(dy). \]

By [7, Theorem 3], if $\Omega \subset \mathbb{R}^d$ is an open set such that $|\Omega| < \infty$ and $\mathrm{Per}(\Omega) < \infty$ (i.e. $1_{\Omega} \in \mathrm{BV}(\mathbb{R}^d)$), then

\[ t^{-1}H(t) = t^{-1}(g_\Omega(0) - P_t g_\Omega(0)), \]

which converges to $-\mathcal{L}g_\Omega(0) = \mathrm{Per}_\nu(\Omega)$ as $t$ tends to 0.

3. Main results and proofs

3.1. Convolution semigroups for nonlocal operators on $\mathbb{R}^d$.

**Lemma 2.** Assume that $\psi^* \in \mathrm{WUSC}(\alpha, 1, \overline{C})$ for some $\alpha \in (0, 1)$. If $f \in C^\beta_0$ for some $\beta \in (\alpha, 1]$, then $\mathcal{L}f \in C^{\beta - \alpha}_0$ and $\|\mathcal{L}f\|_{\beta^{\alpha}} \leq C_1(1 - \alpha/\beta)^{-1}h(1)\|f\|_{\beta}$, where $C_1 = C_1(c_d, \overline{C})$. In particular, if $\beta \in (2\alpha, 1]$, then $\mathcal{L}f \in \mathcal{D}(\mathcal{L})$.

**Proof.** By Lemma 1, for all $\lambda \leq 1$ and $r \leq 1$,

\[ h(\lambda r) \leq c\lambda^{-\alpha}h(r), \]

where $c = c_d\overline{C}$, and for all $\varepsilon > 0$,

\[ \int_{|y| < 1} |y|^{\alpha + \varepsilon} \nu(dy) < \infty. \]

First, we will deal with $\|\mathcal{L}f\|_{\beta^{\alpha}}$. Consider $|x - y| \leq 1$. By (3),

\[ |\mathcal{L}f(x) - \mathcal{L}f(y)| = \left| \int_{\mathbb{R}^d} (f(x + z) - f(x) - (f(y + z) - f(y))) \nu(dz) \right| \]

\[ \leq \int_{\mathbb{R}^d} |f(x + z) - f(x) - f(y + z) + f(y)| \nu(dz). \]
We split the integral above as follows
\[ \int_{\mathbb{R}^d} = \int_{|z| \leq |x-y|} + \int_{|z| > |x-y|} =: I_1 + I_2. \]

We will first deal with $I_1$. Denote $L = \|f\|_\beta$. We have
\[ |f(x) - f(y)| \leq L|x - y|^{\beta}. \]

By (19), Fubini theorem, (5) and (16) we have
\[ I_1 \leq \int_{|z| \leq |x-y|} (|f(x+z) - f(x)| + |f(y+z) - f(y)|) \nu(dz) \leq 2L \int_{|z| \leq |x-y|} |z|^{\beta} \nu(dz) \]
\[ = 2L \int_{|z| \leq |x-y|} s^{\beta} \nu(dz) = 2L \int_{0}^{s^{1/\beta}} \int_{|z| \leq |x-y|} \nu(dz) ds \leq 2L \int_{0}^{s^{1/\beta}} \nu(dz) ds \]
\[ \leq 2L \int_{0}^{s^{1/\beta}} h(s^{1/\beta}) ds \leq 2Lch(1) \int_{0}^{s^{1/\beta}} s^{-\alpha/\beta} ds = 2Lch(1)(1 - \alpha/\beta)^{-1}|x - y|^{\beta - \alpha}. \]

Now we will estimate $I_2$. Using again (19), (5) and (16) we get
\[ I_2 \leq \int_{|z| > |x-y|} (|f(x+z) - f(x)| + |f(x+z) - f(y)|) \nu(dz) \]
\[ \leq 2L|x - y|^{\beta} \int_{|z| > |x-y|} \nu(dz) \leq 2L|x - y|^{\beta} h(|x - y|) \leq 2Lch(1)|x - y|^{\beta - \alpha}. \]

Hence
\[ \|\mathcal{L}f\|_{\beta - \alpha} \leq (1 + (1 - \alpha/\beta)^{-1}) 2ch(1)\|f\|_\beta. \]

Furthermore,
\[ |\mathcal{L}f(x)| \leq \int_{\mathbb{R}^d} |f(x+y) - f(x)| \nu(dy). \]

We split the integral above as follows
\[ \int_{\mathbb{R}^d} = \int_{|y| < 1} + \int_{|y| \geq 1} =: I_3 + I_4. \]

Proceeding as in the case of $I_1$, we obtain
\[ I_3 \leq \|f\|_\beta \int_{|y| < 1} |y|^\beta \nu(dy) \leq \|f\|_\beta \frac{c}{1 - \alpha/\beta} h(1). \]

Next,
\[ I_4 \leq 2\|f\|_\infty \int_{|y| \geq 1} \nu(dy) \leq 2\|f\|_\infty h(1). \]

Therefore
\[ \|\mathcal{L}f\|_\infty \leq \frac{c}{1 - \alpha/\beta} h(1)\|f\|_\beta + 2h(1)\|f\|_\infty \leq \left( \frac{c}{1 - \alpha/\beta} + 2 \right) h(1)\|f\|_\beta \]
\[ \leq \frac{3c}{1 - \alpha/\beta} h(1)\|f\|_\beta. \]

(21)
We have contained in the domain of $L \in \mathcal{D}(L)$ and $(\leq)$, and $(\geq)$, and $(\leq)$. By Taylor's theorem applied to $L \in \mathcal{D}(L)$ and $(\leq)$, and $(\geq)$, and $(\leq)$. The proof is complete. □

**Corollary 1.** Assume that $\psi^* \in WUSC(\alpha, 1, C)$ for some $\alpha \in (0, 1)$. If $f \in C_0^\beta$ for some $\beta \in (\alpha, 1)$, then $\mathcal{L}^k f \in C_0^{\beta-\alpha}$ for $k \in \{1, \ldots, n\}$ and $\mathcal{L}^k f \in \mathcal{D}(\mathcal{L})$ for $k \in \{1, \ldots, n-1\}$.

It is well-known that for $f \in \mathcal{D}(\mathcal{L})$ and $t \geq 0$, $P_t$ is differentiable and $\frac{d}{dt}P_t f = L P_t f = P_t L f$, see e.g. Pazy [19, Theorem 1.2.4 c)]. Therefore, if $\mathcal{L}^k f \in \mathcal{D}(\mathcal{L})$ for $k \in \{1, \ldots, n-1\}$, then $\frac{d^n}{dt^n} P_t f = \mathcal{L}^n P_t f = P_t \mathcal{L}^n f$. To apply this result, we will use the fact that for $t_0 > 0$, $P_{t_0}(C_0^\beta) \subset C_0^\beta$. Indeed,

$$|P_{t_0} f(x) - P_{t_0} f(y)| \leq \int_{\mathbb{R}^d} |f(x+z) - f(x) - f(y+z) + f(y)| p_{t_0}(dz) \leq \int_{\mathbb{R}^d} (|f(x+z) - f(y+z)| + |f(y) - f(x)|) p_{t_0}(dz) \leq 2L|x-y|^\beta \int_{\mathbb{R}^d} p_{t_0}(dz) = 2L|x-y|^\beta.$$

**Theorem 1.** Assume that $\psi^* \in WUSC(\alpha, 1, C)$ for some $\alpha \in (0, 1)$. If $f \in C_0^\beta$ for some $\beta \in (\alpha, 1)$, then

$$\lim_{t \to 0^+} t^{-n} \left( P_t f - \sum_{k=0}^{n-1} \frac{t^k}{k!} \mathcal{L}^k f \right) = \frac{1}{n!} \mathcal{L}^n f.$$

**Proof.** By Corollary 1, $\mathcal{L}^k f \in \mathcal{D}(\mathcal{L})$ for $k \in \{1, \ldots, n-1\}$, hence $P_t f$ is $n$ times differentiable. By Taylor’s theorem applied to $t \mapsto P_t f$,

$$P_t f = \sum_{k=0}^{n-1} \frac{t^k}{k!} \mathcal{L}^k f + \frac{t^n}{n!} P_{\theta_0} \mathcal{L}^n f$$

for some $\theta_0 \in (0, t)$. The thesis follows from the right continuity of $P_t$ at $t = 0$. □

Theorem 1, Lemma 1 and (10) give the following result.

**Corollary 2.** Assume that there exists $C > 0$ such that for all $\lambda \leq 1$ and $r < 1$, $h(\lambda r) \leq C\lambda^{-\alpha} h(r)$ for some $\alpha \in (0, 1)$. Let $n \geq 2$. If $\alpha \leq 1$, then

$$\lim_{t \to 0^+} t^{-n} \left( H(t) - t \operatorname{Per}_\nu(\Omega) + \sum_{k=2}^{n-1} \frac{t^k}{k!} \mathcal{L}^k g_\Omega(0) \right) = -\frac{1}{n!} \mathcal{L}^n g_\Omega(0).$$

**Example 1.** If $\nu(\alpha)$ is an $\alpha$-stable Lévy measure, $\alpha \in (0, 1)$, then the Hölder space $C_0^\beta$ is contained in the domain of $\mathcal{L} = -(-\Delta)^{\beta/2}$ for any $\beta \in (\alpha, 1]$, and we have

$$\mathcal{L} f(x) = \int_{\mathbb{R}^d \setminus \{0\}} (f(x+y) - f(x)) \nu(\alpha)(dy), \quad f \in C_0^\beta, \quad x \in \mathbb{R}^d.$$

The associated semigroup $(p_t)_{t \geq 0}$ is the $\alpha$-stable semigroup in $\mathbb{R}^d$, determined by $\psi(\xi) = |\xi|^\alpha$. We have $\psi(r\xi) = r^\alpha \psi(\xi)$, so in particular $\psi^* \in WUSC(\alpha, 0, 1)$. If $f \in C_0^\beta$ for some $\beta \in (\alpha, 1]$,
then by Corollary 1, \( \mathcal{L}^k f \in \mathcal{D}(\mathcal{L}) \) for \( k \in \{1, \ldots, n-1\} \) and \( \mathcal{L}^k f \in \mathcal{C}_{\alpha}^{\beta-k\alpha} \) for \( k \in \{1, \ldots, n\} \), and by Theorem 1,

\[
(23) \quad \lim_{t \to 0^+} t^{-n} \left( -P_t f + \sum_{k=0}^{n-1} \frac{(-1)^{k-1} t^k}{k!} (-(-\Delta)^{k\alpha/2}) f \right) = \frac{(-1)^n}{n!} (-(-\Delta)^{n\alpha/2}) f,
\]

since \( ((-\Delta)^{\alpha/2})^n = (-\Delta)^{n\alpha/2} \) for \( n\alpha < 2 \), see [16; (1.1.12)]. By Corollary 2,

\[
\lim_{t \to 0^+} t^{-n} \left( H(t) - \sum_{k=1}^{n-1} \frac{(-1)^{k-1} t^k}{k!} \text{Per}_{\nu(\beta)}(\Omega) \right) = \frac{(-1)^{n-1}}{n!} \text{Per}_{\nu(\beta)}(\Omega).
\]

**Theorem 2.** Assume that for all \( \alpha \in (0, 1) \), \( \psi^* \in WUSC(\alpha, 1, C\alpha^{-1}) \) for some \( C > 0 \). If \( f \in \mathcal{C}^\beta_0 \) for some \( \beta \in (0, 1] \), then there exists \( t_0 > 0 \) such that for all \( t \in (0, t_0) \),

\[
P_tf = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathcal{L}^k f
\]

in \( \mathcal{C}_0(\mathbb{R}^d) \).

**Proof.** Without loss of generality, we can assume \( h(1) = 1 \). For any \( N \in \mathbb{N} \), let \( \alpha = \alpha(N) = \beta/(2N) \). For \( n \in \{1, 2, \ldots, N-1\} \), let \( \beta_n = \beta - n\alpha \). Since \( N\alpha = \beta/2 < \beta \), by the Proof of Theorem 1,

\[
P_tf(x) = \sum_{n=0}^{N-1} \frac{t^n \mathcal{L}^n f(x)}{n!} + \tilde{R}_N,
\]

where

\[
\tilde{R}_N = \frac{t^N P_{t_0} \mathcal{L}^N f(x)}{N!}.
\]

By (21), for any \( N \in \mathbb{N} \),

\[
\|\mathcal{L}^N f\|_\infty \leq \frac{7C/\alpha}{1 - \alpha/\beta N^{-1}} \|\mathcal{L}^{N-1} f\|_{\beta N^{-1}}.
\]

By (22), for \( k \in \{1, 2, \ldots, N-1\} \),

\[
\|\mathcal{L}^k f\|_{\beta_k} \leq \frac{7C/\alpha}{1 - \alpha/\beta k^{-1}} \|f\|_{\beta_k^{-1}}.
\]

Using (24) and applying (25) \( N \) times we get

\[
\|\mathcal{L}^N f\|_\infty \leq \frac{7C/\alpha}{1 - \alpha/\beta N^{-1}} \|\mathcal{L}^{N-1} f\|_{\beta N^{-1}} \leq \left( \prod_{k=0}^{N-1} \frac{7C/\alpha}{1 - \alpha/\beta k} \right) \|f\|_{\beta} \leq (7C)^N \left( \prod_{k=0}^{N-1} \frac{1}{1 - 2\alpha/\beta} \right) \|f\|_{\beta} = (7C)^N \left( \prod_{k=0}^{N-1} \frac{2N/\beta}{1 - 1/N} \right) \|f\|_{\beta} = (14C/\beta)^N e \|f\|_{\beta}.
\]

(26)
By the contractivity of $P_{\theta_0}$, (26) and Stirling’s formula,
\[
|\hat{R}_N| \leq \frac{t^N P_{\theta_0}^{N} L^N f}{N!} \leq \frac{t^N \| L^N f \|_\infty}{N!} \leq \frac{eC' N^N t^N}{\sqrt{2\pi N(N/e)^N}} \| f \|_\beta \leq \frac{eC' N^N t^N}{\sqrt{N}} \| f \|_\beta
\]
which tends to 0 as $N \to \infty$. The proof is complete. \qed

**Example 2.** Let $\psi(\xi) = \log(1 + |\xi|^2)$, i.e. $\mathcal{L} = -\log(1 - \Delta)$. Let $\alpha \in (0, 1]$. Since, for $\lambda \geq 1$ and $x \geq 1$,
\[
\log(1 + \lambda x) \leq \log(\lambda(1 + x)) = \frac{1}{\alpha} \log(\lambda^\alpha(1 + x)) \leq \frac{1}{\alpha} \log(\lambda^\alpha(1 + x)),
\]
and
\[
\frac{\log(\lambda^\alpha(1 + x))}{\log(1 + x)} \leq \frac{\lambda^\alpha}{\log 2},
\]
we have $\log(1 + \cdot) \in WUSC(\alpha, 1, 2/\alpha)$. Hence $\psi \in WUSC(\alpha, 1, 4/\alpha)$, that is $\psi$ satisfies the assumptions of Theorem 2.

**Corollary 3.** Assume that for all $\alpha \in (0, 1)$ and $\lambda \leq 1$, $h(\lambda r) \leq C\alpha^{-1}\lambda^{-1} h(r)$ for some $C > 0$. Then, there exists $t_0 > 0$ such that for all $t \in (0, t_0)$,
\[
H(t) = t\text{Per}_\nu(\Omega) - \sum_{k=2}^{\infty} \frac{t^k}{k!} \mathcal{L}^k g_\Omega(0).
\]

**Example 3.** Let $\nu$ be a finite measure on $\mathbb{R}^d$ and let $(p_t)_{t \geq 0}$ be determined by
\[
\psi(\xi) = \int_{\mathbb{R}^d} (1 - e^{i\langle \xi, z \rangle}) \nu(dz).
\]
In this case
\[
\mathcal{L} f(x) = \int_{\mathbb{R}^d} (f(x + y) - f(x)) \nu(dy).
\]
Generator $\mathcal{L}$ can be expressed as a convolution operator
\[
\mathcal{L} f = (\nu - \nu(\mathbb{R}^d)\delta_0) * f,
\]
therefore
\[
\mathcal{L}^n f = (\nu - \nu(\mathbb{R}^d)\delta_0)^n * f = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \nu(\mathbb{R}^d)^{n-i} \nu^{*i} * f,
\]
where for $k \in \mathbb{N}$, $\mu^{*k}$ denotes the $k$-fold iteration of the convolution of measure $\mu$ with itself, i.e. $\mu^{*0} = \delta_0$ and $\mu^{*k} = \mu^{*(k-1)} * \mu$ for $k \geq 1$. It is well-known that, since $\nu$ is finite and $\mathcal{L}$ is bounded, we have
\[
P_t = e^{t\mathcal{L}}.
\]
Therefore
\[
P_t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{L}^n = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{t^n}{i!(n-i)!} \nu(\mathbb{R}^d)^{n-i} \nu^{*i}
\]
\[
= \sum_{i=0}^{\infty} \frac{t^i}{i!(n-i)!} \nu(\mathbb{R}^d)^{n-i} \nu^{*i}
\]
\[
= \sum_{j=0}^{\infty} \frac{(-t\nu(\mathbb{R}^d))^j}{j!} \sum_{i=0}^{\infty} \frac{t^i}{i!} \nu^{*i} = e^{-t\nu(\mathbb{R}^d)} \exp(t\nu),
\]
where $\exp(\nu) = \sum_{n=0}^{\infty} \frac{1}{n!} \nu^{*n}$. This expansion follows also from Theorem 2. Applying this result to $f = g_\Omega$, we extend \cite[Theorem 1.2]{J}, which holds for compactly supported probabilistic measures with radial density, to general finite measures.
3.2. Heat content for the fractional Laplacian on $\mathbb{R}^d$. Let $(p_t)_{t\geq 0}$ be the $\alpha$-stable semigroup in $\mathbb{R}^d$, $\alpha \in (0,2)$. We recall that in this case $\psi(\xi) = |\xi|^\alpha$ and the corresponding Lévy measure $\nu$ is the $\alpha$-stable Lévy measure $\nu^{(\alpha)}$. The related function $h$ turns into $h(r) = c/r^\alpha$, for some $c > 0$.

Let
\[
an := \frac{1}{\pi^{d/2}} \frac{(-1)^{n-1}}{n!} 2^{n\alpha} \Gamma\left(\frac{n\alpha}{2} + 1\right) \Gamma\left(\frac{n\alpha + d}{2}\right) \sin\left(\frac{\pi n\alpha}{2}\right).
\]

For $\frac{n\alpha}{2} \notin \mathbb{N}$,
\[
a_n = \frac{(-1)^{n-1}}{n!} \mathcal{A}_{d,n\alpha}.
\]

By Hiraba [14, Remark 2.b]), for $\alpha < 1$ and $x \in \mathbb{R}^d \setminus \{0\}$,
\[
p_1(x) = \sum_{n=1}^{\infty} a_n |x|^{-n\alpha-d}.
\]

Our following two results extend [1, Theorem 1.2]. Note that they provide more detailed expansion than the one resulting from Corollary 2, compare with Example 1.

**Theorem 3.** Let $\alpha \in (0,1)$ be such that $1/\alpha \notin \mathbb{N}$ and let $\Omega \subset \mathbb{R}^d$ be an open set of finite Lebesgue measure and perimeter. Then
\[
\lim_{t \to 0^+} t^{-1/\alpha} \left( H(t) - \sum_{n=1}^{\left\lceil \frac{d}{\alpha} \right\rceil - 1} \frac{(-1)^{n-1}}{n!} t^n \text{Per}_{\nu^{(\alpha)}}(\Omega) \right)
\]
\[
= \frac{\pi^{d-1}}{\Gamma\left(\frac{d+1}{2}\right)} \text{Per}(\Omega) \left( \int_0^1 r^d p_1(1) \, dr \right) - \sum_{n=1}^{\infty} \frac{a_n}{1-n\alpha}.
\]

**Proof.** Without loss of generality, we can assume $\text{diam}(\Omega) = 1$. For $1/\alpha \notin \mathbb{N}$, $[1/\alpha] = [1/\alpha] - 1$, and we will use this formula in order to avoid repeating similar calculations in the next proof. By (13), (14) and the scaling property of $p_t$,
\[
H(t) - \sum_{n=1}^{\left\lceil \frac{d}{\alpha} \right\rceil - 1} \frac{(-1)^{n-1}}{n!} t^n \text{Per}_{\nu^{(\alpha)}}(\Omega)
\]
\[
= \int_{\mathbb{R}^d} \left( g_\alpha(0) - g_\alpha(x) \right) \left( p_t(x) - \sum_{n=1}^{\left\lceil \frac{d}{\alpha} \right\rceil - 1} \frac{(-1)^{n-1}}{n!} t^n \mathcal{A}_{d,n\alpha} |x|^{-d-n\alpha} \right) \, dx
\]
\[
= \int_{\mathbb{R}^d} \left( g_\alpha(0) - g_\alpha(t^{1/\alpha} x) \right) \left( p_t(x) - \sum_{n=1}^{\left\lceil \frac{d}{\alpha} \right\rceil - 1} \frac{(-1)^{n-1}}{n!} \mathcal{A}_{d,n\alpha} |x|^{-d-n\alpha} \right) \, dx.
\]

We split the above integral into
\[
\int_{|x| \leq 1} + \int_{1 < |x| \leq t^{-1/\alpha}} + \int_{|x| > t^{-1/\alpha}} =: I_1 + I_2 + I_3.
\]

We have
\[
I_1 = \int_{|x| \leq 1} \left( g_\alpha(0) - g_\alpha(t^{1/\alpha} x) \right) \left( p_t(x) - \sum_{n=1}^{\left\lceil \frac{d}{\alpha} \right\rceil - 1} \frac{(-1)^{n-1}}{n!} \mathcal{A}_{d,n\alpha} |x|^{-d-n\alpha} \right) \, dx
\]
\[
= t^{1/\alpha} \int_0^1 \int_{|y| < 1} r^d g_\alpha(0) - g_\alpha(t^{1/\alpha} ry) \left( p_t(1) - \sum_{n=1}^{\left\lceil \frac{d}{\alpha} \right\rceil - 1} \frac{(-1)^{n-1}}{n!} \mathcal{A}_{d,n\alpha} r^{-d-n\alpha} \right) \sigma(du) \, dr.
\]
By (10), (11) and Dominated Convergence Theorem,

$$\lim_{t \to 0^+} t^{-1/\alpha} I_1 = \frac{n \frac{d-1}{\alpha}}{\Gamma \left( \frac{d-1}{\alpha} \right)} \text{Per}(\Omega) \left( \int_0^1 r^d p_1(r e_d) \, dr - \sum_{k=1}^{\frac{1}{\alpha}} \frac{(-1)^{n-1}}{n!} A_{d-n\alpha} \right).$$

Next,

$$\frac{I_3}{g_\Omega(0)} = \int_{|x| > t^{-1/\alpha}} \left( p_1(x) - \sum_{n=1}^{\frac{1}{\alpha}} \frac{(-1)^{n-1}}{n!} A_{d-n\alpha} x^{-d-n\alpha} \right) \, dx$$

$$= \int_{|x| > t^{-1/\alpha}} \sum_{n=\frac{1}{\alpha}}^{\infty} a_n |x|^{-n\alpha-d} \, dx$$

$$= \sum_{n=\frac{1}{\alpha}}^{\infty} a_n \int_{|x| > t^{-1/\alpha}} |x|^{-n\alpha-d} \, dx = \omega_{d-1} \sum_{n=\frac{1}{\alpha}}^{\infty} \frac{a_n}{n\alpha} t^n.$$

We have

$$|I_3| \leq g_\Omega(0) \omega_{d-1} \sum_{n=\frac{1}{\alpha}}^{\infty} \frac{|a_n|}{n\alpha} t^n = O(t^{\frac{1}{\alpha}})$$

for $t < 1$, thus

$$\lim_{t \to 0^+} t^{-1/\alpha} I_3 = 0.$$

We have

$$I_2 = \int_{1<|x|<t^{-1/\alpha}} \left( g_\Omega(0) - g_\Omega(t^{1/\alpha} x) \right) \left( p_1(x) - \sum_{n=1}^{\frac{1}{\alpha}} \frac{(-1)^{n-1}}{n!} A_{d-n\alpha} x^{-d-n\alpha} \right) \, dx$$

$$= t^{1/\alpha} \int_{1}^{t^{-1/\alpha}} \int_{S^{d-1}} r^d g_\Omega(0) - g_\Omega(t^{1/\alpha} r u) \left( p_1(r e_d) - \sum_{n=1}^{\frac{1}{\alpha}} \frac{(-1)^{n-1}}{n!} A_{d-n\alpha} r^{-d-n\alpha} \right) \sigma(du) \, dr$$

$$= t^{1/\alpha} \int_{1}^{t^{-1/\alpha}} \int_{S^{d-1}} g_\Omega(0) - g_\Omega(t^{1/\alpha} r u) \frac{t^{1/\alpha}}{t^{1/\alpha}} \sum_{n=\frac{1}{\alpha}}^{\infty} a_n \sigma(du) r^{-n\alpha} \, dr.$$

By (10),

$$|I_2| \leq \frac{\text{Per}(\Omega)}{2} t^{1/\alpha} \sum_{n=\frac{1}{\alpha}}^{\infty} |a_n| \int_{1<|x| t^{-1/\alpha}} |x|^{1-n\alpha-d} \, dx$$

$$\leq \frac{\text{Per}(\Omega)}{2} t^{1/\alpha} \sum_{n=\frac{1}{\alpha}}^{\infty} |a_n| \int_{|x| > 1} |x|^{1-n\alpha-d} \, dx$$

$$= \frac{\text{Per}(\Omega)}{2} \omega_{d-1} t^{1/\alpha} \sum_{n=\frac{1}{\alpha}}^{\infty} \frac{|a_n|}{n\alpha - 1},$$

hence

$$I_2 = \sum_{n=\frac{1}{\alpha}}^{\infty} a_n \int_{1<|x| \leq t^{-1/\alpha}} \left( g_\Omega(0) - g_\Omega(t^{1/\alpha} x) \right) |x|^{-n\alpha-d} \, dx.$$
and

\[ \lim_{t \to 0^+} t^{-1/\alpha} I_2 = \sum_{n=\lceil \frac{1}{\alpha} \rceil}^{\infty} a_n \lim_{t \to 0^+} t^{-1/\alpha} \int_{1 < |x| \leq t^{-1/\alpha}} (g_\Omega(0) - g_\Omega(t^{1/\alpha} x)) |x|^{-n\alpha - d} \, dx. \]

We get

\[ t^{-1/\alpha} I_2 = \sum_{n=\lceil \frac{1}{\alpha} \rceil}^{\infty} a_n \int_{1}^{t^{-1/\alpha}} M_\Omega(t, r) r^{-n\alpha} \, dr, \]

where

\[ M_\Omega(t, r) = \int_{\mathbb{S}^{d-1}} \frac{g_\Omega(0) - g_\Omega(\frac{rt^{1/\alpha}}{1})}{rt^{1/\alpha}} \sigma(du). \]

We claim that

\[ \lim_{t \to 0^+} \int_{1}^{t^{-1/\alpha}} M_\Omega(t, r) r^{-n\alpha} \, dr = \frac{\pi^{(d-1)/2}}{\Gamma((d+1)/2)} \frac{1}{n\alpha - 1}. \]

Indeed, by (10) and (11),

\[ 0 \leq M_\Omega(t, r) \leq \frac{1}{2} \text{Per}(\Omega) \sigma(\mathbb{S}^{d-1}) \]

and, for any \( r > 0 \),

\[ \lim_{t \to 0^+} M_\Omega(t, r) = \frac{\pi^{(d-1)/2}}{\Gamma((d+1)/2)} \text{Per}(\Omega). \]

Moreover,

\[ \int_{1}^{t^{-1/\alpha}} r^{-n\alpha} \, dr \leq \int_{1}^{\infty} r^{-n\alpha} \, dr = \frac{1}{n\alpha - 1} \]

and hence (30) follows by Dominated Convergence Theorem.

\[ \square \]

**Theorem 4.** Let \( \alpha \in (0, 1) \) be such that \( 1/\alpha \in \mathbb{N} \) and let \( \Omega \subset \mathbb{R}^d \) be an open set of finite Lebesgue measure and perimeter. Then

\[ \lim_{t \to 0^+} (t^{1/\alpha} \log(1/t))^{-1} \left( H(t) - \sum_{n=1}^{1/\alpha - 1} (-1)^{n-1} n! t^n \text{Per}_{\nu(n\alpha)}(\Omega) \right) = \frac{(-1)^{1/\alpha - 1}}{(1/\alpha - 1)!} \pi \text{Per}(\Omega). \]

**Proof.** By the Proof of Theorem 3,

\[ H(t) - \sum_{n=1}^{1/\alpha - 1} (-1)^{n-1} n! t^n \text{Per}_{\nu(n\alpha)}(\Omega) = I_1 + I_2 + I_3, \]

where

\[ I_1 = t^{1/\alpha} \int_{0}^{1} \int_{\mathbb{S}^{d-1}} |\cdot| \frac{g_\Omega(0) - g_\Omega(t^{1/\alpha} r u)}{t^{1/\alpha} r} \left( p_1(\cdot|u) - \sum_{n=1}^{1/\alpha - 1} (-1)^{n-1} n! \mathcal{A}_{d-n\alpha} r^{-d-n\alpha} \right) \sigma(du) \, dr, \]

\[ I_2 = \int_{1 < |x| \leq t^{-1/\alpha}} (g_\Omega(0) - g_\Omega(t^{1/\alpha} x)) \sum_{n=1/\alpha}^{\infty} a_n |x|^{-n\alpha - d} \, dx, \]

and

\[ I_3 = |g_\Omega(0)| \omega_{d-1} \sum_{n=1/\alpha}^{\infty} \frac{a_n}{n\alpha} t^n. \]
By (10), (11) and Dominated Convergence Theorem,
\[ \lim_{t \to 0^+} (t^{1/\alpha} \log(1/t))^{-1} I_1 = 0. \]
Next,
\[ \lim_{t \to 0^+} (t^{1/\alpha} \log(1/t))^{-1} I_3 = 0. \]
By (10),
\[ |I_2| \leq \frac{\text{Per}(\Omega)}{2} t^{1/\alpha} \sum_{n=1/\alpha}^{\infty} |a_n| \int_{1 < |x| < t^{-1/\alpha}} |x|^{-n\alpha - d} \, dx \]
\[ = \frac{\text{Per}(\Omega)}{2} \left( \sum_{n=1/\alpha+1}^{\infty} |a_n| t^{1/\alpha} \frac{n^{1/\alpha} - 1}{1 - n\alpha} + \frac{a_{1/\alpha}}{1/\alpha} t^{1/\alpha} \log(1/t) \right) \]
\[ \leq \frac{\text{Per}(\Omega)}{2} \left( \sum_{n=1/\alpha+1}^{\infty} |a_n| \frac{t^{1/\alpha}}{n\alpha - 1} + \frac{a_{1/\alpha}}{1/\alpha} t^{1/\alpha} \log(1/t) \right). \]
Therefore
\[ I_2 = \sum_{n=1/\alpha}^{\infty} a_n \int_{1 < |x| < t^{-1/\alpha}} \left( g_\Omega(0) - g_\Omega(t^{1/\alpha} x) \right) |x|^{-n\alpha - d} \, dx. \]
We have
\[ (t^{1/\alpha} \log(1/t))^{-1} \int_{1 < |x| < t^{-1/\alpha}} \left( g_\Omega(0) - g_\Omega(t^{1/\alpha} x) \right) |x|^{-n\alpha - d} \, dx \]
\[ = \log(1/t)^{-1} \int_{1}^{t^{-1/\alpha}} \int_{S^{d-1}} \frac{g_\Omega(0) - g_\Omega(t^{1/\alpha} u)}{t^{1/\alpha} r} \sigma(du) r^{-n\alpha} \, dr \]
\[ = \log(1/t)^{-1} \int_{1}^{t^{-1/\alpha}} \mathcal{M}_\Omega(t, r) r^{-n\alpha} \, dr. \]
We claim that
\[ (33) \quad \lim_{t \to 0^+} \log(1/t)^{-1} \int_{1}^{t^{-1/\alpha}} \mathcal{M}_\Omega(t, r) r^{-n\alpha} \, dr = \frac{\pi^{d-1}}{\alpha \Gamma \left( \frac{d+1}{2} \right)} \text{Per}(\Omega), \]
where
\[ \mathcal{M}_\Omega(t, r) = \int_{S^{d-1}} \frac{g_\Omega(0) - g_\Omega(rt^{1/\alpha} u)}{rt^{1/\alpha}} \sigma(du). \]
Indeed, by substitution,
\[ \log(1/t)^{-1} \int_{1}^{t^{-1/\alpha}} \mathcal{M}_\Omega(t, r) r^{-n\alpha} \, dr = \log(1/t)^{-1} \int_{0}^{\log(1/t)/\alpha} \mathcal{M}_\Omega(t, e^r) \, dr \]
\[ = \int_{0}^{1/\alpha} \mathcal{M}_\Omega(t, t^{-r}) \, dr, \]
and by (10), (11) and Dominated Convergence Theorem,
\[ \lim_{t \to 0^+} \int_{0}^{1/\alpha} \mathcal{M}_\Omega(t, t^{-r}) \, dr = \int_{0}^{1/\alpha} \int_{S^{d-1}} \lim_{t \to 0^+} \frac{g_\Omega(0) - g_\Omega(rt^{1/\alpha} u)}{t^{1/\alpha - r}} \sigma(du) \, dr \]
\[ = \frac{\pi^{(d-1)/2}}{\alpha \Gamma \left( \frac{(d+1)/2} \right)} \text{Per}(\Omega). \]
We claim that for \( n \geq 1/\alpha + 1 \) we have
\[
\lim_{t \to 0^+} \log(1/t)^{-1} \int_1^{t^{-1/\alpha}} \mathcal{M}_\Omega(t, r)r^{-\alpha} \, dr = 0.
\]
Indeed, by (31), (31), and since (10) and (11),
\[
0 \leq \mathcal{M}_\Omega(t, r) \leq \frac{1}{2} \text{Per}(\Omega) \sigma(S^{d-1})
\]
and, for any \( r > 0 \),
\[
\lim_{t \to 0^+} \mathcal{M}_\Omega(t, r) = \frac{\pi^{(d-1)/2}}{\Gamma((d+1)/2)} \text{Per}(\Omega).
\]
Moreover,
\[
\int_1^{t^{-1/\alpha}} \log(1/t)^{-1}r^{-\alpha} \, dr \leq \int_1^\infty r^{-\alpha} \, dr = \frac{1}{\alpha - 1}
\]
for \( t < 1/e \), and hence (34) follows by Dominated Convergence Theorem. (33) and (34) yield
the thesis of the theorem. \( \square \)

3.3. Heat content for general stable operators on \( \mathbb{R} \). Let \( \alpha \in (0, 1) \cup (1, 2) \), \( \beta \in [-1, 1] \)
and \( \gamma > 0 \). We consider convolution semigroup \( (p_t)_{t \geq 0} \) on \( \mathbb{R} \) such that
\[
\psi(\xi) = \gamma |\xi|^\alpha \left(1 - i\beta \tan \left(\frac{\pi \alpha}{2}\right) \text{sgn}(\xi)\right), \quad \xi \in \mathbb{R}.
\]
The corresponding Lévy measure on \( \mathbb{R} \) is given by
\[
\nu(dx) = \frac{c_+ 1_{x > 0} + c_- 1_{x < 0}}{|x|^{1+\alpha}} \, dx,
\]
where
\[
c_+ = -\frac{1 + \beta}{2\Gamma(-\alpha) \cos(\frac{\pi \alpha}{2})} \quad \text{and} \quad c_- = -\frac{1 - \beta}{2\Gamma(-\alpha) \cos(\frac{\pi \alpha}{2})}.
\]
Let \( \Omega = (a, b) \subset \mathbb{R} \). We have \( g_\Omega(x) = (b - a - |x|)1_{[0,b-a]}(|x|) \). For \( \alpha \in (0, 1) \), \( \text{Per}_\alpha(\Omega) = (c_+ + c_-)^{-1}(1 - \alpha)^{-1}(b - a)^{1-\alpha} \).

We can fix the parameter \( \gamma \) without loss of generality. Assume that \( \gamma = \cos \left(\frac{\pi \beta \alpha}{2}\right) \) if \( \alpha < 1 \)
and \( \gamma = \cos \left(\frac{\pi \beta^2 - \alpha}{2}\right) \) if \( \alpha > 1 \).

Let
\[
b_n := \frac{(-1)^{n-1} \Gamma(n\alpha + 1)}{\pi^{3/2} n!} \sin \left(\pi n \alpha \rho\right),
\]
where \( \rho = \frac{1 + \beta}{2} \) if \( \alpha < 1 \), and \( \rho = \frac{1 - \beta(2 - \alpha)\alpha}{2} \) if \( \alpha > 1 \). By [22, (2.5.1)], for \( \alpha < 1 \),
\[
p_1(x) = \sum_{n=1}^{\infty} b_n x^{-n\alpha - 1}
\]
as \( x \to \infty \). By [22, (2.5.4)], for \( \alpha > 1 \) and \( \beta \neq -1 \), and any \( N \in \mathbb{N} \),
\[
p_1(x) = \sum_{n=1}^{N} b_n x^{-n\alpha - 1} + O(x^{-(N+1)\alpha - 1})
\]
as \( x \to \infty \). Let
\[
d_n := \frac{(-1)^{n-1} 2\Gamma(n\alpha + 1)}{\pi^{3/2} n!} \sin \left(\frac{\pi n \alpha}{2}\right) \cos \left(\frac{\pi n \alpha \beta}{2}\right).
\]
This constant will appear in the following proposition, which complements [1, Theorem 1.1].
We generalize the results for \( \alpha < 1 \) to the non-symmetric case. The last result is new even for
the symmetric case, since previously only the first two terms of the expansion were known.
Proposition 1. Let $\Omega = (a,b), |\Omega| = b - a$.

(i) Let $0 < \alpha < 1$ and $0 < t < \min\{\left|\Omega\right|^{\alpha}, e^{-1}\}$.

(a) If $1/\alpha \notin \mathbb{N}$, then there is a constant $C_\alpha$ independent of $\Omega$ such that

$$H(t) = \frac{2}{\pi} \sum_{n=1}^{\left[\frac{1}{\alpha}\right]} (-1)^{n-1} \frac{\Gamma(n\alpha)}{(1-n\alpha)n!} \sin \left( \frac{\pi n\alpha}{2} \right) \cos \left( \frac{\pi n\alpha t}{2} \right) \left|\Omega\right|^{1-n\alpha} t^n + C_\alpha t^{1/\alpha} + R_\alpha(t),$$

where $C_\alpha = \int_0^1 \int_\omega (p_1(x) + p_1(-x)) \, dx \, dw - \sum_{n=1}^\infty \frac{d_n}{n(1-n\alpha)}$ and $|R_\alpha(t)| \leq ct^{1/\alpha + 1}$.

(b) If $\alpha = 1/N$ for some $N \in \mathbb{N}$, then there is a constant $C_N(\Omega)$ such that

$$H(t) = \frac{2}{\pi} \sum_{n=1}^{N-1} (-1)^{n-1} \frac{\Gamma(n/N)}{(1-n/N)n!} \sin \left( \frac{\pi n}{2N} \right) \cos \left( \frac{\pi n t}{2N} \right) \left|\Omega\right|^{1-n/N} t^n$$

$$+ \frac{2}{\pi(N-1)!} \cos \left( \frac{\pi t}{2} \right) t^N \ln \left( \frac{1}{t} \right) + C_N(\Omega) t^N + R_{1/N}(t),$$

where $C_N(\Omega) = \int_0^1 \int_\omega (p_1(x) + p_1(-x)) \, dx \, dw + d_N \ln(|\Omega|) - \sum_{n \neq N} \frac{d_n}{n(1-n/N)}$.

(ii) If $1 < \alpha < 2$, $|\beta| \neq 1$, then, for any $N \in \mathbb{N},$

$$H(t) = t^{1/\alpha} \int_\mathbb{R} |x| p_1(x) \, dx + \frac{2}{\pi} \sum_{n=1}^{N} (-1)^{n-1} \frac{\Gamma(n\alpha + 1)}{n!} \sin \left( \frac{\pi n\alpha}{2} \right) \cos \left( \frac{\pi n \left( \frac{1}{\alpha} - \frac{1}{\alpha} \right)}{2} \right) \frac{1}{n\alpha(1-n\alpha)} \left|\Omega\right|^{1-n\alpha} t^n$$

$$+ R_N(t)$$

as $t \to 0^+$, where $|R_N(t)| \leq ct^{N+1}$.

Note that by [13, Proposition 1.4]

$$\int_\mathbb{R} |x| p_1(x) \, dx = \frac{2}{\pi} \Gamma \left( 1 - \frac{1}{\alpha} \right) \text{Re}\left( 1 + i \beta \tan \left( \frac{\pi}{2} \right) \right)^{1/\alpha}. $$

Proof. (i) can be proved analogously to [1, Theorem 1.1], using (35). We will prove (ii). We have

$$H(t) = \int_\mathbb{R} (|\Omega| \wedge |x|) p_1(x) \, dx = \int_\mathbb{R} (|\Omega| \wedge t^{1/\alpha}|x|) p_1(x) \, dx$$

$$= t^{1/\alpha} \int_{|x|<|\Omega|^{-1/\alpha}} |x| p_1(x) \, dx + \int_{|x| \geq |\Omega|^{-1/\alpha}} (|\Omega| - t^{1/\alpha}|x|) p_1(x) \, dx$$

$$= t^{1/\alpha} \int_{\mathbb{R}} |x| p_1(x) \, dx + \int_{|x| \geq |\Omega|^{-1/\alpha}} (|\Omega| - t^{1/\alpha}|x|) p_1(x) \, dx.$$
as $x \to \infty$. Therefore

$$\int_{|\Omega|^{t^{-1/\alpha}}}^\infty \left(|\Omega| - t^{1/\alpha} x \right) p_1(x) \, dx$$

$$= \int_{|\Omega|^{t^{-1/\alpha}}}^\infty \left( \sum_{n=1}^N (-1)^{n-1} b_n x^{-n-1} + O(x^{-(N+1)\alpha-1}) \right) \left(|\Omega| - t^{1/\alpha} x \right) \, dx$$

$$= \sum_{n=1}^N \frac{b_n}{n\alpha(1-n\alpha)} |\Omega|^{1-n\alpha} t^n + \int_{|\Omega|^{t^{-1/\alpha}}}^\infty O(x^{-(N+1)\alpha-1}) \left(|\Omega| - t^{1/\alpha} x \right) \, dx.$$  

We also have

$$\left| \int_{|\Omega|^{t^{-1/\alpha}}}^\infty O(x^{-(N+1)\alpha-1}) \left(|\Omega| - t^{1/\alpha} x \right) \, dx \right| \leq C \int_{|\Omega|^{t^{-1/\alpha}}}^\infty \left( t^{1/\alpha} x - |\Omega| \right) x^{-(N+1)\alpha-1} \, dx$$

$$= C \frac{1}{(N+1)\alpha((N+1)\alpha-1)} |\Omega|^{1-(N+1)\alpha} t^{N+1}.$$  

The calculations for $\int_{|\Omega|^{t^{-1/\alpha}}}^\infty \left(|\Omega| + t^{1/\alpha} x \right) p_1(x) \, dx$ are analogous since $p_1(-x)$ corresponds to $p_1(x)$ with parameters $(\alpha, -\beta, \gamma)$. \hfill \square

References

[1] L. Acuña Valverde. Heat content for stable processes in domains of $\mathbb{R}^d$. J. Geom. Anal., 27(1):492–524, 2017.

[2] L. Acuña Valverde. On the heat content for the Poisson kernel over the unit ball in the euclidean space. Bull. Lond. Math. Soc., 52(6):1093–1104, 2020.

[3] L. Acuña Valverde. On the heat content for the Poisson heat kernel over convex bodies. J. Math. Anal. Appl., 494(2):Paper No. 124655, 15, 2021.

[4] L. Ambrosio, N. Fusco, and D. Pallara. Functions of bounded variation and free discontinuity problems. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[5] L. Angiuli, U. Massari, and M. Miranda, Jr. Geometric properties of the heat content. Manuscripta Math., 140(3-4):497–529, 2013.

[6] L. Caffarelli, J.-M. Roquejoffre, and O. Savin. Nonlocal minimal surfaces. Comm. Pure Appl. Math., 63(9):1111–1144, 2010.

[7] W. Cygan and T. Grzywny. Heat content for convolution semigroups. J. Math. Anal. Appl., 446(2):1393–1414, 2017.

[8] W. Cygan and T. Grzywny. A note on the generalized heat content for Lévy processes. Bull. Korean Math. Soc., 55(5):1463–1481, 2018.

[9] F. Ferrari, M. Miranda, Jr., D. Pallara, A. Pinamonti, and Y. Sire. Fractional Laplacians, perimeters and heat semigroups in Carnot groups. Discrete Contin. Dyn. Syst. Ser. S, 11(3):477–491, 2018.

[10] B. Galerne. Computation of the perimeter of measurable sets via their covariogram. Applications to random sets. Image Anal. Stereol., 30(1):39–51, 2011.

[11] T. Grzywny. On Harnack inequality and H"older regularity for isotropic unimodal Lévy processes. Potential Anal., 41(1):1–29, 2014.

[12] T. Grzywny and K. Szczypkowski. Lévy processes: concentration function and heat kernel bounds. Bernoulli, 26(4):3191–3223, 2020.

[13] C. D. Hardin. Skewed stable variables and processes. Technical reports of Center for Stochastic Processes UNC, Dept. of Statistics, 1984.

[14] S. Hiraba. Asymptotic behaviour of densities of multi-dimensional stable distributions. Tsukuba J. Math., 18(1):223–246, 1994.

[15] F. Kühn and R. L. Schilling. On the domain of fractional Laplacians and related generators of Feller processes. J. Funct. Anal., 276(8):2397–2439, 2019.

[16] N. S. Landkof. Foundations of modern potential theory. Springer-Verlag, New York-Heidelberg, 1972. Translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180.

[17] J. M. Mazón, J. D. Rossi, and J. Toledo. The heat content for nonlocal diffusion with non-singular kernels. Adv. Nonlinear Stud., 17(2):255–268, 2017.
[18] J. M. Mazón, J. D. Rossi, and J. J. Toledo. *Nonlocal perimeter, curvature and minimal surfaces for measurable sets*. Frontiers in Mathematics. Birkhäuser/Springer, Cham, 2019.

[19] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of Applied Mathematical Sciences. Springer-Verlag, New York, 1983.

[20] M. van den Berg. Heat flow and perimeter in $\mathbb{R}^m$. *Potential Anal.*, 39(4):369–387, 2013.

[21] M. van den Berg and P. B. Gilkey. Heat content asymptotics of a Riemannian manifold with boundary. *J. Funct. Anal.*, 120(1):48–71, 1994.

[22] V. M. Zolotarev. *One-dimensional stable distributions*, volume 65 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 1986. Translated from the Russian by H. H. McFaden, Translation edited by Ben Silver.

Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland.

Email address: tomasz.grzywny@pwr.edu.pl

Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland.

Email address: julia.lenczewska@pwr.edu.pl