A MARKOV CHAIN APPROACH TO RENORMALIZATION GROUP
TRANSFORMATIONS

MEI YIN

December 6, 2012

Abstract. We aim at an explicit characterization of the renormalized Hamiltonian after decimation transformation of a one-dimensional Ising-type Hamiltonian with a nearest-neighbor interaction and a magnetic field term. To facilitate a deeper understanding of the decimation effect, we translate the renormalization flow on the Ising Hamiltonian into a flow on the associated Markov chains through the Markov-Gibbs equivalence. Two different methods are used to verify the well-known conjecture that the eigenvalues of the linearization of this renormalization transformation about the fixed point bear important information about all six of the critical exponents. This illustrates the universality property of the renormalization group map in this case.

Keywords: Markov-Gibbs equivalence, renormalization, universality

1. Introduction

The discovery of the equivalence of Markov random fields and Gibbs random fields was a major breakthrough in the interchange of ideas between probability and physics. A Markov random field is a natural generalization of the familiar concept of a Markov chain, which is a collection of random variables with the property that, given the present, the future is (conditionally) independent of the past. If we look at the chain itself as a very simple graph and ignore the directionality implied by “time”, then a Markov chain may alternatively be viewed as a chain graph of stochastic variables, where each variable is independent of all other variables (both future and past) given its two neighbors. A Markov random field is the same thing, only that rather than a chain graph, we allow the relationship between the variables to be defined by any graph structure, and each variable is independent of all the others given its neighbors in the graph. A Gibbs random field, on the other hand, is formed by a set of random variables whose configurations obey a Gibbs distribution, which is a probability distribution that factorizes over all possible cliques, i.e. complete subgraphs in the graph, and the factors are conveniently referred to as “clique potentials”. These two ways of defining a random configuration are apparently quite different [1]: A Markov random field is characterized by its local property (the Markovianity) whereas a Gibbs random field is specified by its global property (the Gibbs distribution).

The rigorous study of the relationship between these two seemingly unrelated fields was initiated by Dobrushin [2] in the context of statistical physics, who considered the questions of existence and uniqueness of a random field subject to a Markovian conditional distribution. Further investigations quickly ensued. Averintsev [3] and Spitzer [4] independently proved that the class of two-state Markov chains is identical to the class of Gibbs ensembles on the simple cubic lattice. Hammersley and Clifford [5] showed that the same equivalence holds between a multi-state Markov field and a generalized Gibbs ensemble over an arbitrary finite graph.
The celebrated Hammersley-Clifford theorem states that each Markov field with a system of neighbors and the associated system of cliques is also a Gibbs field with the same system of cliques, and vice versa, each Gibbs field is also a Markov field with the corresponding system of neighbors. This implies that the joint probability and the conditional probability can specify each other, and serves as a theoretical basis for many modeling applications, where the global characteristic is captured and represented through a set of tractable local characteristics. The original method of proof, however, did not have great intuitive appeal, and many alternative proofs of this theorem were developed. Sherman [6] verified the equivalence of Markov fields and Gibbs ensembles under more relaxed conditions by the repeated use of the inclusion-exclusion principle. Preston [7] adopted a direct approach to the two-state problem and presented an explicit formula for the pair potential. Grimmett [8] showed that the equivalence of structure follows immediately from an application of the Möbius inversion theorem. A final improvement was done by Besag [9], who applied methods of statistical analysis and gave a much simpler, analytical proof of the general result.

The nearest-neighbour Ising model in one dimension is commonly used to demonstrate the powerful Markov-Gibbs equivalence. Though an ordered phase only emerges at zero temperature, this classic model is physically important in that it has a fixed point (the so-called “zero temperature phase transition”) where the critical exponents may be sensibly defined as in higher dimensions. There is the astonishing empirical fact that these critical exponents depend only on overall features of the system, and are related to eigenvalues of the linearized renormalization group map near the fixed point [10]. This universality conjecture has generated continued interest in the scientific community, and various approaches to the renormalization effect on the one-dimensional Ising model have been explored [11, 12].

Consider a one-dimensional Ising model with \( N \) spins \( \sigma_i = \pm 1 \), labelled successively \( i = 0, \ldots, N-1 \). We take the system size \( N \) to be very large (strictly speaking, infinite). The Gibbs field of this model is described by a Hamiltonian \( H \), consisting of a nearest-neighbor interaction \( J \) and a magnetic field term \( m \):\
\[
H = - \left( J \sum_{i=0}^{N-1} \sigma_i \sigma_{i+1} + m \sum_{i=0}^{N-1} \sigma_i \right),
\]
(1)

where periodic boundary condition is imposed so that \( \sigma_N = \sigma_0 \), a standard setup to ensure that \( H \) is translation-invariant. We focus on a specific renormalization group transformation, namely decimation transformation with blocking factor \( b \). To avoid unnecessary technicalities, we assume that \( b \) divides \( N \). The decimation procedure is straightforward: Fix the spins \( \sigma_{bi} \) for \( i = 0, \ldots, N/b - 1 \), and integrate out the remaining ones. This will generate a renormalized Gibbs field with a Hamiltonian \( H' \) having the same form as the original Hamiltonian \( H \), but containing a nearest-neighbor interaction \( J' \) and a magnetic field term \( m' \):

\[
H' = - \left( J' \sum_{i=0}^{N/b-1} \sigma_{bi} \sigma_{b(i+1)} + m' \sum_{i=0}^{N/b-1} \sigma_{bi} \right).
\]
(2)

The renormalized spin coefficients \((J', m')\) and the original spin coefficients \((J, m)\) are related by the decimation map:

\[
\exp \left( C + J' \sigma_0 \sigma_b + m' \frac{m}{2} (\sigma_0 + \sigma_b) \right)
\]
where $C$ is a normalization constant. Notice that to avoid double counting, we have assigned a “half” of the magnetic field $m$ ($m'$) to each spin.

We would like to obtain an explicit characterization of the renormalized model, but as the blocking factor $b$ gets large, solving for $(J', m')$ directly from (3) becomes very difficult. We thus take an alternative approach and investigate the decimation effect on the associated Markov chains. As there is no finite phase transition in one dimension, we follow the common practice and measure the nearest-neighbor interaction strength $J$ ($J'$) by the Boltzmann factor $k = e^{-2J}$ ($k' = e^{-2J'}$) instead. An explicit solution for $(k', m')$ then follows from the Markov-Gibbs equivalence (Hammersley-Clifford theorem). The diagram below illustrates these ideas:

![Diagram showing original and renormalized Hamiltonians and Markov chains](image)

where:
- (II) and (IV) indicate the Markov-Gibbs equivalence (Hammersley-Clifford theorem).
- (I) is the decimation map on the Ising Hamiltonian (cf. (3)).
- (III) is the decimation map on the associated Markov chains (to be examined).

A key tenet of the renormalization group is its explanation of universality [13]. Thus we would also like to verify the widely-believed universality conjecture in this special case, which states that the linearization of the decimation transformation with blocking factor $b$ about the two-dimensional fixed point $(k = m = 0)$ has two real eigenvalues $b^y_T$ and $b^y_H$, where $y_T = y_H = 1$. Suppose we start with a Hamiltonian that is close to critical. The decimation map will first drive it towards the fixed point for a large number of iterations, but eventually will drive it away. The singular behavior of the model arises from iterating the map infinitely many times, and the critical properties are determined by how much time the Hamiltonian spends near the fixed point, when its behavior is governed by the linearization. In fact, it is observed that there are exact non-trivial relations between the six critical exponents (specific heat $\alpha$, spontaneous magnetization $\beta$, magnetic susceptibility $\gamma$, response to magnetic field at zero temperature $\delta$, correlation length $\nu$, and correlation function at zero temperature $\eta$) and the two eigenvalues (more precisely $y_T$ and $y_H$) of the linearization:

$$\alpha = 2 - \frac{d}{y_T} = 1,$$
$$\beta = \frac{d - y_H}{y_T} = 0,$$
$$\gamma = \frac{2y_H - d}{y_T} = 1,$$
$$\delta = \frac{y_H}{d - y_H} = \infty,$$
$$\nu = \frac{1}{y_T} = 1,$$
$$\eta = d + 2 - 2y_H = 1.$$

(5)

(More discussions may be found in [11] and [14].)

**Theorem 1.1 (Universality Conjecture).** At the fixed point $(k = m = 0)$, the Jacobian matrix of the renormalized spin coefficients $(k', m')$ with respect to the original spin coefficients $(k, m)$ is given by

$$Jac = \left( \begin{array}{cc} \frac{\partial k'}{\partial k} & \frac{\partial k'}{\partial m} \\ \frac{\partial m'}{\partial k} & \frac{\partial m'}{\partial m} \end{array} \right) = \left( \begin{array}{cc} b & 0 \\ 0 & b \end{array} \right).$$

(7)
The rest of this paper is organized as follows. In Section 2 we verify the universality conjecture by analyzing the decimation map on the Ising Hamiltonian directly (First Proof of Theorem 1.1). In Section 3 the statistical physics model is transformed into a probability model through the Markov-Gibbs equivalence (Theorems 3.1 and 3.2). We investigate the decimation effect on the associated Markov chains and give an explicit characterization of the renormalized Hamiltonian (Theorem 3.3). An alternative proof of the universality conjecture from this point of view is also provided (Second Proof of Theorem 1.1). Finally, Section 4 is devoted to concluding remarks.

2. Renormalization group approach

In this section we will examine the renormalization group equation (3) directly. Although it is difficult to find an explicit solution to (3) for a large blocking factor \( b \), the Jacobian matrix of partial derivatives (7) may be computed via implicit differentiation.

First Proof of Theorem 1.1. The decimation map (3) consists of 4 equations.

1. Corresponding to \( \sigma_0 = \sigma_b = 1 \):

\[
\exp \left( C + J' + m' \right) = \sum_{\sigma_1, \ldots, \sigma_{b-1}} \exp \left( J \left( \sigma_1 + \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} + \sigma_{b-1} \right) + m \sum_{i=1}^{b-1} \sigma_i + m \right). \tag{8}
\]

2. Corresponding to \( \sigma_0 = \sigma_b = -1 \):

\[
\exp \left( C + J' - m' \right) = \sum_{\sigma_1, \ldots, \sigma_{b-1}} \exp \left( J \left( -\sigma_1 + \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} - \sigma_{b-1} \right) + m \sum_{i=1}^{b-1} \sigma_i - m \right). \tag{9}
\]

3. Corresponding to \( \sigma_0 = 1, \sigma_b = -1 \):

\[
\exp \left( C - J' \right) = \sum_{\sigma_1, \ldots, \sigma_{b-1}} \exp \left( J \left( \sigma_1 + \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} - \sigma_{b-1} \right) + m \sum_{i=1}^{b-1} \sigma_i \right). \tag{10}
\]

4. Corresponding to \( \sigma_0 = -1, \sigma_b = 1 \):

\[
\exp \left( C - J' \right) = \sum_{\sigma_1, \ldots, \sigma_{b-1}} \exp \left( J \left( -\sigma_1 + \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} + \sigma_{b-1} \right) + m \sum_{i=1}^{b-1} \sigma_i \right). \tag{11}
\]

Due to symmetry, (10) and (11) are equivalent. We may therefore assume that (3) breaks down into 3 equations: (8), (9), and (10). To compute the Jacobian matrix of the decimation transformation at the fixed point \((k = m = 0)\), we perform implicit differentiation on these equations at \((J = \infty, m = 0)\). As an example, we differentiate both sides of (8) with respect to \( m \), which gives

\[
\frac{\partial C}{\partial m} + \frac{\partial J'}{\partial m} + \frac{\partial m'}{\partial m} - 1
\]
we will abbreviate by $s$ where $n$ is the single “dominating terms”. We have

$$
\sum_{s_1,\ldots,s_{b-1}} \exp \left( J \left( \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} + \sigma_{b-1} \right) + m \sum_{i=1}^{b-1} \sigma_i \right) \left( \sum_{i=1}^{b-1} \sigma_i \right).
$$

Because lower order terms become insignificant at “$J = \infty$”, it suffices to keep track of the “dominating terms”. We have

$$
\frac{\partial C}{\partial m} + \frac{\partial J'}{\partial J} + \frac{\partial m'}{\partial m} = \frac{s(\sum_{i=1}^{b-1} \sigma_i)}{n(\sigma_1 + \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} + \sigma_{b-1})} - 1,
$$

where $n(f)$ counts the number of $\sigma$ configurations that maximize $f(\sigma)$, and $s(g_{\max}(f))$ (which we will abbreviate by $s(g)$) is the sum of $g(\sigma)$ over the maximizers of $f(\sigma)$. Repeating this “dominating” procedure provides us with 6 independent equations for the partial derivatives:

$$
\frac{\partial C}{\partial J} + \frac{\partial J'}{\partial J} + \frac{\partial m'}{\partial m} = \frac{s(\sigma_1 + \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} + \sigma_{b-1})}{n(\sigma_1 + \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} + \sigma_{b-1})},
$$

$$
\frac{\partial C}{\partial m} + \frac{\partial J'}{\partial m} + \frac{\partial m'}{\partial m} = \frac{s(\sum_{i=1}^{b-1} \sigma_i)}{n(\sigma_1 + \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} + \sigma_{b-1})} + 1,
$$

$$
\frac{\partial C}{\partial J} + \frac{\partial J'}{\partial J} - \frac{\partial m'}{\partial m} = \frac{s(\sigma_1 + \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} - \sigma_{b-1})}{n(\sigma_1 + \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} - \sigma_{b-1})},
$$

$$
\frac{\partial C}{\partial m} + \frac{\partial J'}{\partial m} - \frac{\partial m'}{\partial m} = \frac{s(\sum_{i=1}^{b-1} \sigma_i)}{n(\sigma_1 + \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} - \sigma_{b-1})} - 1,
$$

$$
\frac{\partial C}{\partial J} - \frac{\partial J'}{\partial J} = \frac{s(\sigma_1 + \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} - \sigma_{b-1})}{n(\sigma_1 + \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} - \sigma_{b-1})},
$$

$$
\frac{\partial C}{\partial m} - \frac{\partial J'}{\partial m} = \frac{s(\sum_{i=1}^{b-1} \sigma_i)}{n(\sigma_1 + \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} - \sigma_{b-1})}.
$$

It is quite clear that $\max(\sigma_1 + \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} + \sigma_{b-1}) = b$ is achieved only when $\sigma_1 = \cdots = \sigma_{b-1} = 1$, and that $\max(-\sigma_1 + \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} - \sigma_{b-1}) = -b$ is achieved only when $\sigma_1 = \cdots = \sigma_{b-1} = -1$. The harder task it to determine when $\max(\sigma_1 + \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} - \sigma_{b-1})$ is obtained. Because of the single “−” sign, it is not possible for all the $b$ terms in this sum $(\sigma_1, \sigma_1 \sigma_2, \sigma_2 \sigma_3, \cdots, \sigma_{b-2} \sigma_{b-1}, -\sigma_{b-1})$ to be 1 as in the previous two cases. An ideal maximizer should have $b - 1$ terms with value 1 and only one term with value −1. We claim that each one of the $b$ locations of −1 corresponds to exactly one $\sigma$ configuration: Suppose the $i$th term has value −1 ($\sigma_i = -1$ for $i = 1$, $\sigma_{i-1} \sigma_i = -1$ for $2 \leq i \leq b - 1$, or $\sigma_{b-1} = 1$ for $i = b$), then we must have $\sigma_1 = \cdots = \sigma_{i-1} = 1$ and $\sigma_i = \cdots = \sigma_{b-1} = -1$. (14)–(19) are thus simplified:

$$
\frac{\partial C}{\partial J} + \frac{\partial J'}{\partial J} + \frac{\partial m'}{\partial m} = b, \quad \frac{\partial C}{\partial m} + \frac{\partial J'}{\partial m} + \frac{\partial m'}{\partial m} = b,
$$

$$
\frac{\partial C}{\partial J} + \frac{\partial J'}{\partial J} - \frac{\partial m'}{\partial m} = b, \quad \frac{\partial C}{\partial m} + \frac{\partial J'}{\partial m} - \frac{\partial m'}{\partial m} = -b,
$$

$$
\frac{\partial C}{\partial J} - \frac{\partial J'}{\partial J} = b - 2, \quad \frac{\partial C}{\partial m} - \frac{\partial J'}{\partial m} = 0.
$$
Solving (20)—(22) yields
\[ \frac{\partial J'}{\partial J} = 1, \quad \frac{\partial J'}{\partial m} = \frac{\partial m'}{\partial J} = 0, \quad \frac{\partial m'}{\partial m} = b, \] (23)
which further implies that the Jacobian matrix \( \text{Jac} \) (7) is diagonal, i.e., \( \frac{\partial k'}{\partial m} = \frac{\partial m'}{\partial k} = 0 \). To complete the proof of the universality conjecture, it remains to verify that \( \frac{\partial k'}{\partial k} = b \). We perform the “dominating” procedure as before. For notational convenience, we temporarily denote \( \sigma_1 + \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} - \sigma_{b-1} \) by \( f(\sigma) \), and \( \sigma_1 + \sum_{i=1}^{b-2} \sigma_i \sigma_{i+1} + \sigma_{b-1} \) by \( g(\sigma) \). Dividing (10) by (8) at the fixed point \( (J = \infty, m = 0) \), we have
\[ k' = \exp(-2J') = \frac{n(f) \exp (J \cdot \max(f))}{n(g) \exp (J \cdot \max(g))} = \frac{b \exp((b-2)J)}{\exp(bJ)} = b \exp(-2J) = bk. \] (24)

3. Markov chain approach

In this section we will transform the statistical physics model into a probability model and investigate the decimation effect on the associated Markov chains. This is a special case of Hammersley-Clifford theorem where the exact correspondence between the Markov field and the Gibbs field may be worked out explicitly. The idea is to regard the Ising system as a two-state Markov chain with transition probability matrix
\[ P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}, \] (25)
where
\[ p = P(\sigma_1 = 1|\sigma_0 = -1), \] (26)
and
\[ q = P(\sigma_1 = -1|\sigma_0 = 1). \] (27)

**Theorem 3.1** (Hammersley-Clifford). *The Ising Hamiltonian\( H \) (1) is fully characterized by the transition probabilities\( p \) and\( q \).*

**Remark.** The transition probabilities\( p \) and\( q \) and the spin coefficients\( k \) and\( m \) are related by (36), (37), (41), and (42). The spin coefficients fixed point\( (k = m = 0) \) thus corresponds to the transition probabilities fixed point\( (p = q = 0) \).

**Proof.** Baxter [14] showed that the mean and covariance of the Ising spins in the infinite-volume limit are functions of the spin coefficients\( k \) and\( m \):
\[ E\sigma_0 = \frac{\sinh m}{\sqrt{\sinh^2 m + k^2}}, \] (28)
\[ \text{Cov}(\sigma_0, \sigma_1) = \frac{k^2 \cosh m - \sqrt{\sinh^2 m + k^2}}{\sinh^2 m + k^2 \cosh m + \sqrt{\sinh^2 m + k^2}}. \] (29)

Through the Markov-Gibbs equivalence, we show that (28) and (29) may alternatively be viewed as functions of the transition probabilities\( p \) and\( q \). Recall that the Markov chain has a stationary distribution:
\[ P(\sigma_0 = -1) = \frac{q}{p + q}, \quad P(\sigma_0 = 1) = \frac{p}{p + q}, \] (30)
which readily displays the dependence of the mean on the transition probabilities,

\[ E\sigma_0 = E\sigma_1 = \frac{p - q}{p + q}. \]  

To obtain an analogous expression for the covariance, we resort to the tower property of conditional expectation,

\[
E\sigma_0\sigma_1 = E(E_0E_1|\sigma_0) \\
= P(\sigma_1 = 1|\sigma_0 = 1)P(\sigma_0 = 1) - P(\sigma_1 = -1|\sigma_0 = 1)P(\sigma_0 = 1) \\
- P(\sigma_1 = 1|\sigma_0 = -1)P(\sigma_0 = -1) + P(\sigma_1 = -1|\sigma_0 = -1)P(\sigma_0 = -1) \\
= \frac{4pq}{(p + q)^2} + (1 - p - q)\frac{4pq}{(p + q)^2},
\]

which then gives

\[
\text{Cov}(\sigma_0, \sigma_1) = E\sigma_0\sigma_1 - E\sigma_0E\sigma_1 = (1 - p - q)\frac{4pq}{(p + q)^2},
\]

The two characterizations of the Ising Hamiltonian \( H \) are thus connected by:

\[
\frac{p - q}{p + q} = \frac{\sinh m}{\sqrt{\sinh^2 m + k^2}}.
\]

\[
(1 - p - q)\frac{4pq}{(p + q)^2} = \frac{k^2}{\sinh^2 m + k^2} \cosh m - \sqrt{\sinh^2 m + k^2}.
\]

It is not hard to derive an explicit expression of \( p \) and \( q \) in terms of \( k \) and \( m \) from (34) and (35):

\[
p = \frac{\sqrt{\sinh^2 m + k^2} + \sinh m}{\cosh m + \sqrt{\sinh^2 m + k^2}}.
\]

\[
q = \frac{\sqrt{\sinh^2 m + k^2} - \sinh m}{\cosh m + \sqrt{\sinh^2 m + k^2}}.
\]

The reverse direction, however, requires more work. For computational convenience, we make a change of variables, \( A = \sinh m, B = \sinh^2 m + k^2 \). Then (36) and (37) become

\[
\sqrt{B} + A = p\sqrt{A^2 + 1} + q\sqrt{B},
\]

\[
\sqrt{B} - A = q\sqrt{A^2 + 1} + q\sqrt{B}.
\]

Dividing (39) into (38), we have

\[
\frac{A - p\sqrt{A^2 + 1}}{A - q\sqrt{A^2 + 1}} = \frac{p - 1}{q - 1}.
\]

This is an equation for \( A \) only, and an explicit expression of \( k \) and \( m \) in terms of \( p \) and \( q \) follows easily:

\[
k = \sqrt{\frac{pq}{(1 - p)(1 - q)}},
\]

\[
m = \frac{1}{2} \log \left( \frac{1 - q}{1 - p} \right).
\]
Theorem 3.2. The renormalized Ising Hamiltonian $H'$ (2) is fully characterized by the renormalized transition probabilities $p'$ and $q'$, where

$$p' = P(\sigma_b = 1|\sigma_0 = -1), \quad (43)$$

and

$$q' = P(\sigma_b = -1|\sigma_0 = 1). \quad (44)$$

Remark. The renormalized transition probabilities $p'$ and $q'$ and the renormalized spin coefficients $k'$ and $m'$ are similarly related as in (36), (37), (41), and (42).

Proof. This follows from Theorem 3.1 once we realize that site 0 and site $b$ are nearest neighbors after decimation transformation with blocking factor $b$. The $b$-step transition probability matrix $P^b$ represents the decimation map on the associated Markov chains, and is given by

$$P^b = \left( \begin{array}{cc} \frac{p}{p+q} & \frac{q}{p+q} \\ \frac{1}{p+q} & \frac{1}{p+q} \end{array} \right)^b,$$

where the first equality is simply the spectral decomposition of the matrix $P$. This then implies that

$$p' = \frac{p}{p+q}(1 - (1 - p - q)^b), \quad (46)$$

and

$$q' = \frac{q}{p+q}(1 - (1 - p - q)^b). \quad (47)$$

Theorem 3.3. The decimation map (3) identifies the connection between the renormalized Hamiltonian $H'$ (2) and the original Hamiltonian $H$ (1).

Proof. We follow (II), (III), and (IV) as shown in (4). The original Ising model is described by a Hamiltonian $H$ with spin coefficients $k$ and $m$. (II) indicates the alternative view of this system as a two-state Markov chain with transition probabilities $p$ and $q$ (cf. (36) and (37)). (III) then transforms this Markov chain into a renormalized Markov chain with renormalized transition probabilities $p'$ and $q'$ (cf. (46) and (47)). Finally, (IV) recovers the renormalized spin coefficients $k'$ and $m'$ of the renormalized Hamiltonian $H'$ (cf. (41) and (42)).

$$ (k, m) \xrightarrow{(II)} (p, q) \xrightarrow{(III)} (p', q') \xrightarrow{(IV)} (k', m') \quad (48)$$

Second Proof of Theorem 1.1. Theorem 3.3 establishes an explicit expression of the renormalized spin coefficients $k'$ and $m'$ in terms of the original spin coefficients $k$ and $m$ (cf. (48)). To evaluate the Jacobian matrix $Jac (7)$ at the fixed point $(k = m = 0)$, we start by considering $\frac{\partial k'}{\partial k}$ and $\frac{\partial m'}{\partial k}$ with $m$ held fixed at zero. By (II), on the $m = 0$ curve,

$$ p = q = \frac{k}{1+k}. \quad (49)$$

(III) then gives

$$ p' = q' = \frac{1}{2} \left(1 - (1 - 2p)^b\right). \quad (50)$$
which further implies, by (IV), that
\[
k' = \frac{p'}{1 - p'},
\]
\[
m' = 0.
\]
We conclude that \( \frac{\partial m'}{\partial k} = 0 \) from (52), and by applying the chain rule to (49), (50), and (51), that \( \frac{\partial k'}{\partial m} = b \).

We proceed with the calculations for \( \frac{\partial m'}{\partial m} \) and \( \frac{\partial k'}{\partial m} \) with \( k \) held fixed at zero. By (II), on the \( k = 0 \) curve, either \( p \) or \( q \) is zero, depending on the sign of \( m \). Without loss of generality, assume \( m \geq 0 \). In this case,
\[
p = \frac{2 \sinh m}{\cosh m + \sinh m},
\]
\[
q = 0.
\]
(III) then gives
\[
p' = 1 - (1 - p)^b,
\]
\[
q' = 0,
\]
which further implies, by (IV), that
\[
k' = 0,
\]
\[
m' = -\frac{1}{2} \log(1 - p').
\]
We conclude that \( \frac{\partial k'}{\partial m} = 0 \) from (57), and by applying the chain rule to (53), (55), and (58), that \( \frac{\partial m'}{\partial m} = b \).

4. Concluding remarks

This paper aims at an explicit characterization of the renormalized Hamiltonian after decimation transformation of a one-dimensional Ising-type Hamiltonian with a nearest-neighbor interaction and a magnetic field term. We transform the statistical physics model into a probability model through the Markov-Gibbs equivalence and analyze the decimation effect on the associated Markov chains. As the Ising model is a prototype for a wide variety of spin models, it is expected that the exploitation of Markov-Gibbs equivalence in this special case will shed light on the application of renormalization group ideas in a more general setting. Two different proofs of the universality conjecture are presented, one based directly upon the renormalization group equation, and the other from the Markov chain point of view. Although the first proof does not employ advanced mathematical methods, it provides a new perspective on the renormalization flow. For example, it has been verified, following similar ideas, that one-dimensional \( q \)-state Potts model \( (q \geq 2) \) exhibits the same eigenvalue statistics \( y_T \) and \( y_H \), independent of the number of states \( q \) and the blocking factor \( b \). (The percolation limit \( q \to 1 \), however, remains open, and is believed to display different critical features.) The second proof uses ideas from Markov chains, and is expected to work with higher-dimensional \( q \)-state Potts models as well, where the covariant matrices may be expressed in terms of the random cluster representation of Fortuin and Kasteleyn \[15\]. As the number of dimensions \( d \) and the number of states \( q \) get large, it will be harder to write down exact formulas for the transition probabilities in the covariant matrices, but the Metropolis and Glauber algorithms should provide a reasonable approximation scheme. Since Markov chains may take both discrete and continuous values, an advantage of exploring this second perspective is that we can also consider decimation with spin scaling applied to continuous spin systems, not just discrete systems like Potts and Ising.
models, and hence avoid the lack of spin rescaling, a common problem encountered in a “pure” decimation. In summary, we hope this rigorous investigation will provide insight into the intrinsic structure of the renormalization group transformation and help us better understand the nature of universality.

ACKNOWLEDGEMENTS

The author owes deep gratitude to her PhD advisor Bill Faris for his continued help and support. She appreciated the opportunity to talk about an early version of this work in the 2010 Arizona School of Analysis with Applications, organized by Bob Sims and Daniel Ueltschi. This research was supported in part by the R. H. Bing Fellowship at University of Texas at Austin.

REFERENCES

[1] J. Honerkamp, Statistical Physics: An Advanced Approach with Applications, Springer, Berlin, 2002.
[2] P. Dobrushin, The description of a random field by means of conditional probabilities and conditions of its regularity, Theory Probab. Appl. 13 (1968) 197-224.
[3] M. Averintsev, On a method of describing discrete parameter random fields, Problemy Peredači Informacii 6 (1970) 100-109.
[4] F. Spitzer, Markov random fields and Gibbs ensembles, Amer. Math. Monthly 78 (1971) 142-154.
[5] J. Hammersley, P. Clifford, Markov fields on finite graphs and lattices, [http://www.statslab.cam.ac.uk/~grg/books/hammfest/hamm-cliff.pdf (1971)].
[6] S. Sherman, Markov random fields and Gibbs random fields, Israel J. Math. 14 (1973) 92-103.
[7] C. Preston, Generalized Gibbs states and Markov random fields, Adv. Appl. Probab. 5 (1973) 242-261.
[8] G. Grimmett, A theorem about random fields, Bull. Lond. Math. Soc. 5 (1973) 81-84.
[9] J. Besag, Spatial interaction and the statistical analysis of lattice systems, J. Roy. Statist. Soc. Ser. B 36 (1974) 192-236.
[10] K. Wilson, The renormalization group: Critical phenomena and the Kondo problem, Rev. Mod. Phys. 47 (1975) 773-840.
[11] D. Nelson, M. Fisher, Soluble renormalization groups and scaling fields for low-dimensional Ising systems, Ann. Phys. 91 (1975) 226-274.
[12] M. Nauenberg, Renormalization group solution of the one-dimensional Ising model, J. Math. Phys. 16 (1975) 703-705.
[13] A.C.D. van Enter, R. Fernández, A.D. Sokal, Regularity properties and pathologies of position-space renormalization-group transformations: Scope and limitations of Gibbsian theory, J. Stat. Phys. 72 (1993) 879-1167.
[14] R. Baxter, Exactly Solved Models in Statistical Mechanics, Academic Press, London, 1982.
[15] C.M. Fortuin, P.W. Kasteleyn, On the random cluster model: I. Introduction and relation to other models, Physica 57 (1972) 536-564.

Department of Mathematics, University of Texas, Austin, TX, 78712, USA
E-mail address: myin@math.utexas.edu