CARTAN CALCULUS FOR QUANTUM DIFFERENTIALS ON BICROSSPRODUCTS

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Abstract. We provide the Cartan calculus for bicovariant differential forms on bicrossproduct quantum groups $k(M)\triangleright\triangleleft kG$ associated to finite group factorizations $X = GM$ and a field $k$. The irreducible calculi are associated to certain conjugacy classes in $X$ and representations of isotropy groups. We find the full exterior algebras and show that they are inner by a bi-invariant 1-form $\theta$ which is a generator in the noncommutative de Rham cohomology $H^1$. The special cases where one subgroup is normal are analysed. As an application, we study the noncommutative cohomology on the quantum codouble $D^*(S_3) \cong k(S_3)\triangleleft\triangleright k\mathbb{Z}_6$ and the quantum double $D(S_3) = k(S_3)\triangleright\triangleleft kS_3$, finding respectively a natural calculus and a unique calculus with $H^0 = k$.

1. Introduction

There has been a lot of interest in recent years in finite groups $M$, say, as noncommutative differential geometries (even though the algebra of functions $k(M)$, $k$ a field, is commutative), see [1, 2, 3, 4, 5, 6]. The bicovariant differential calculi on $k(M)$ are defined by conjugacy classes $C \subset M$ not containing the group identity and defined in practice by the Cartan calculus consisting of a basis $\{e_a : a \in C\}$ of left-invariant differential 1-forms and the bimodule and exterior derivative relations

\[
df = \sum_{a \in C}(R_a(f) - f)e_a, \quad e_afa = R_a(f)e_a, \quad \forall f \in k(M)
\]

where $R_a$ denotes right multiplication on the group. It turns out in this way that there is an entire geometry and Lie theory of finite groups. Another feature is that the calculus is inner in the sense that there exists an element $\theta = \sum e_a$ such that $df = [\theta, f]$. Graded commutator with $\theta$ similarly defines the differential in higher degree, while a certain braiding $\Psi$ describes the skew-symmetrization of basic 1-forms.

Since the suitable dual of a Hopf algebra is also a Hopf algebra, one has another class of models where the ‘coordinate’ algebra is the group algebra $kG$, say, for a finite group $G$. If $G$ is nonAbelian this is now genuinely noncommutative. Such objects provide the first examples of noncommutative geometry which is strictly noncommutative in both the quantum groups approach and the Connes and operator theory approach (as for example in the Baum-Connes theory for the K-theory of $CG$ in terms of $EG$, [3]). Differential calculi in this case were classified in [7] and are given by irreducible right-representations $V$ and vectors $\theta \in V/k$ (only the class
of $\theta \in V$ controls the calculus). Here the invariant 1-forms are labelled by a basis $e_i \in V$ and the calculus has the form

\[
du = u\theta \ast (u - 1), \quad e_i u = u(e_i \ast u), \quad \forall u \in G
\]

where $\ast$ denotes the right action. The calculus is inner via the chosen $\theta$. Such models in the Lie setting would be the Hopf algebra $U(g)$ where $g$ is a Lie algebra, for example $U(su_2)$ leads to the ‘fuzzy sphere’. The Abelian discrete group case is also useful e.g. after twisting to describe Clifford algebras as noncommutative spaces and to describe noncommutative tori at the algebraic level.

In the present paper we extend the above formulae to the next more complicated finite noncommutative geometry in this family, namely to bicrossproduct quantum groups $k(M)\triangleright\triangleleft kG$ where the above two models are ‘smashed together’. These are now genuine noncommutative and noncocommutative quantum groups. They also have the self-dual-type feature namely the dual is $kM\triangleleft\triangleright k(G)$ of the same bicrossproduct type. For Lie groups they were proposed as nontrivial noncommutative geometries (in connection with quantum gravity) in [12] and as quantum Poincaré groups of noncommutative spacetimes in [13]. More recently they have played a role in computing cyclic cohomology [14] as well as in the renormalisation of quantum field theories [15]. Here they play a role linked to diffeomorphism invariance. The finite group case is intimately linked to set-theoretic solutions of the Yang-Baxter equations and over $\mathbb{C}$ was characterised by Lu as Hopf algebras with positive basis [16]. For all these reasons it is clear that such bicrossproduct quantum groups should be an important next most complicated and truly ‘quantum’ source of examples after the finite group cases. Their noncommutative differential geometry, however, is very little explored and explicit formulae for their differential structure, a prerequisite for any actual computations and applications of the geometry, have been totally lacking. We provide these now, in Section 3. Sections 4, 5 cover special semidirect cases where either $M$ or $G$ are normal. The semidirect case in Section 4 also includes the important case of the quantum codouble $D^*(G) = k(G)\triangleright\triangleleft kG$ of a finite group, where we find a natural calculus induced from one on $k(G)$ defined by a conjugacy class in $G$. Section 6 applies our Cartan calculus to explicit computations of noncommutative de Rham cohomology, which turns out to be nontrivial. The noncommutative differential geometry of the quantum double $D(S_3)$ in Section 6.3, particularly, should be physically interesting in connection with finite conformal field theory and finite versions of fuzzy spheres. We find a unique calculus with the connectedness property $H^0 = k$.

Our starting point, in the preliminary Section 2, is the known but nonconstructive classification theorem [17] for bicovariant differentials on bicrossproducts due to E. Beggs and one of the present authors. From the Woronowicz theorem [18] one knows that bicovariant calculi are classified by Ad-stable right ideals in the augmentation ideal of the Hopf algebra. It was shown in [17] that these are in 1-1 correspondence with certain equivalence classes in the group $X = GM$ which determines the bicrossproduct. We recall that if $X$ is a group factorization (in the sense of two subgroups $G, M$ such that the product $G \times M \to X$ is bijective) then each group acts on the other by actions $\triangleright, \triangleleft$ defined by $su = (s\triangleright u)(s\triangleleft u)$ for $u \in G$
and $s \in M$. They obey
\begin{equation}
\begin{aligned}
s \epsilon e &= s, \quad \epsilon \triangleright u = u, \quad s \triangleright e = e, \quad e \triangleright u = e \\
(su \triangleright )v &= su(uv), \quad s \triangleright (tvu) = (st) \triangleright u \\
\triangleright (uv) &= (\triangleright (u)((s \triangleright u) \triangleright v), \quad (st) \triangleright u = (s \triangleright (tvu))(t \triangleright u)
\end{aligned}
\end{equation}
and conversely such a matched pair of actions allows to reconstruct $X = G \rtimes M$ by a double cross product construction\cite{19}. Moreover, at least in the finite case it means that the group algebra $kG$ acts on $k(M) \triangleright \triangleleft kG$ is by definition the cross product algebra $\triangleright \triangleleft$ by the action and cross coproduct coalgebra $\triangleright \triangleleft$ by the coaction. Section 2 recalls the Beggs-Majid result with a slightly more explicit description as the decomposition into conjugacy classes of a certain $Z \subset X$. We also make a shift of conventions from left modules to right modules which is not straightforward. Our goal from this starting point is then to find a suitable basis for the invariant differential forms and the Cartan calculus for the differential structure. We find (Theorem 3.2) that there is indeed a natural choice of such a basis $\{e_a\}$ dual to a basis $\{f_a\}$ of the quantum tangent space $L$, identified with a subrepresentation under an action of $D(X)$ on $kX$. Hence there is the induced $X$-graduation $\| \cdot \|$ on $L$ which factorizes as
\[ \|f_a\| = \langle f_a \rangle^{-1}|f_a|, \]
say, in $MG$, and an induced right action $\ast$ of $X$ on $\{e_a\}$. Then (Theorem 3.2)
\[ e_a f = R(f_a)(f)e_a, \quad e_a u = \langle f_a \rangle \triangleright u e_a \ast u \]
\[ df = \sum_a e_a(R(f_a)(f) - f)e_a, \quad du = \sum_a e_a((f_a) \triangleright u)e_a \ast u - ue_a, \]
where $e_a = \delta(f_a)\langle f_a \rangle$ is defined by the pairing between $kX$ and $k(X)$. We also find that the calculus is again inner. These structures, and $\theta = \sum_a e_a e_a$ ‘unify’ the two extreme cases above when either $G$ or $M$ is trivial. Note that there is no ‘algorithm’ from\cite{17} leading from the classification to a suitable basis and resulting Cartan calculus needed for practical applications, so that the work in the present sequel is required. Further new results are the inner property and that $\theta$ is a generator of the noncommutative de Rham cohomology.

**Preliminaries.** Here we collect all the basic definitions needed in the paper. We work over a field $k$ of characteristic zero. Let $X = GM$ be a finite group factorization. The bicrossproduct Hopf algebra $A = k(M) \triangleright kG$ has basis $\delta_s \otimes u$ where $s \in M$, $u \in G$ and $\delta_s$ is the Kronecker delta-function in $k(M)$. The product, coproduct $\Delta : A \rightarrow A \otimes A$, counit $\epsilon : A \rightarrow k$ and ‘coinverse’ or antipode $S : A \rightarrow A$ for a Hopf algebra are
\begin{equation}
(\delta_s \otimes u)(\delta_t \otimes v) = \delta_{stu,t}(\delta_s \otimes uv), \quad \Delta(\delta_s \otimes u) = \sum_{ab=s} \delta_a \otimes b \triangleright u \otimes \delta_b \otimes u
\end{equation}
\begin{equation}
1 = \sum_s \delta_s \otimes e, \quad \epsilon(\delta_s \otimes u) = \delta_{s,e}, \quad S(\delta_s \otimes u) = \delta_{(stu)^{-1}} \otimes (s \triangleright u)^{-1}. \quad (5)
\end{equation}
We use here the conventions and notations for Hopf algebras in\cite{19}. Thus, $\Delta, \epsilon$ are algebra maps and coassociative (they define an algebra on the dual) and $S$ obeys
\[ \sum (Sa_{(1)})a_{(2)} = \epsilon(a)1 = \sum a_{(1)}(Sa_{(2)}) \text{ for all } a \text{ if we use the ‘Sweedler notation’ } \Delta a = \sum a_{(1)} \otimes a_{(2)}. \] The point of view in the paper is that $A$ is like functions on a
group and $\Delta, \varepsilon, S$ encode the ‘group’ structure. Similarly, an action of this ‘group’ is expressed as a coaction of $A$, which is like an action but with arrows reversed. Meanwhile, the dual $H = A^* = kM\triangleright k(G)$ is also a bicrossproduct, with

$$\Delta(s \otimes \delta_u)(t \otimes \delta_v) = \delta_{u,t \triangleright v}(s \otimes \delta_v), \quad \Delta(s \otimes \delta_u) = \sum_{xy = u} s \otimes \delta_x \otimes s \triangleleft x \otimes \delta_y$$

$$1 = \sum_u \varepsilon \otimes \delta_u, \quad \varepsilon(s \otimes \delta_u) = \delta_{u,e}, \quad S(s \otimes \delta_u) = (s \triangleleft u)^{-1} \otimes \delta_{(\delta \triangleright u)^{-1}}$$

We use the Drinfeld quantum double $D(H) = H^{*_{op}} \bowtie H$ built on $H^* \otimes H$ in the double cross product form [11], see [10]. In the present case of $H = kM\triangleright k(G)$, the double was computed in [20] and the cross relations between $H$ and $H^{*_{op}}$ are

$$1 \otimes t \otimes \delta_v)(\delta_s \otimes u \otimes 1) = \delta_{s'} \otimes u' \otimes t' \otimes \delta_{v'}$$

where

$$s' = (t \triangleleft (s \triangleright u)^{-1})s(t \triangleleft vu^{-1})^{-1}, \quad u' = (t \triangleleft vu^{-1}) \triangleright u$$

$$t' = t \triangleleft (s \triangleright u)^{-1}, \quad v' = (s \triangleright u)vu^{-1}$$

obeying

$$t' \triangleleft u' = t \triangleleft vu^{-1}, \quad s' \triangleright u' = (t \triangleright (s \triangleright u)^{-1})^{-1}$$

$$t' \triangleright v' = (s' \triangleright u')(t \triangleright vu^{-1}), \quad s' \triangleright u' = t(s \triangleright u)(t \triangleright vu)^{-1}$$

We use, and will freely use basic identities such as:

$$t^{-1} \triangleright (t \triangleright u)^{-1} = (t \triangleleft u)^{-1}, \quad (t \triangleleft u) \triangleright u^{-1} = (t \triangleright u)^{-1}$$

$$t^{-1} \triangleleft (t \triangleright u)^{-1} = u^{-1}, \quad (t \triangleright u)^{-1} \triangleright (t \triangleright u)^{-1} = t^{-1}$$

It was shown in [20] that $D(H)$ is a cocycle twist of the double $D(X) = k(X) \bowtie k(X)$, meaning in particular that its category of modules is equivalent to that of $X$-crossed modules in the sense of Whitehead.

Next, we need the notion of a bicovariant differential calculus over any Hopf algebra $A$. A differential calculus over any algebra $A$ is an $A - A$-bimodule $\Omega^1$ and a linear map $d : A \rightarrow \Omega^1$ such that $d(ab) = adb + (da)b$ for all $a, b \in A$ and such that the map $A \otimes A \rightarrow \Omega^1$ defined by $adb$ is surjective. In the Hopf algebra case we require bicovariance in the sense that $\Omega^1$ is also an $A - A$-bicomodule via bimodule maps and $d$ is a bicomodule map [13], in which case one may identify $\Omega^1 = A \otimes \Lambda^1$ where $\Lambda^1$ is the space of invariant 1-forms. It forms a right $A$-crossed module (i.e. a compatible right $A$-module and $A$-comodule or right module of the Drinfeld double $D(A)$ in the finite dimensional case). The (co)action on $\Omega^1$ from the left are via the (co)product of $A$, while from the right it is the tensor product of that on $A$ and on $\Lambda^1$. Then the classification amounts to that of $\Lambda^1$ as quotient crossed modules of $A^+ = \ker \varepsilon \subset A$. Also, a calculus is irreducible (more precisely one should say ‘coirreducible’) if it has no proper quotients. Then as in [7] we actually classify the duals $L = \Lambda^{1*}$, which we call ‘quantum tangent spaces’, as irreducible crossed submodules of $H^+ = \ker \varepsilon \subset H$ under $D(H)$, where $H$ is a Hopf algebra dual to $A$. Finally, we note that the category of $A$-crossed modules is a braided one (since the Drinfeld double is quasitriangular) and hence there is an induced braiding $\Psi : \Lambda^1 \otimes \Lambda^1 \rightarrow \Lambda^1 \otimes \Lambda^1$ which can be used to define an entire ‘exterior algebra’ $\Omega(A) = A \otimes \Lambda$. The invariant forms $\Lambda$ are generated by $\Lambda^1$ with
‘antisymmetrization’ relations defined by \( \Psi \). We will use these notations and concepts throughout the paper.

2. Classification of differentials by conjugacy classes in \( X \)

In this section we provide a concise but self-contained account of the classification theory in [17]. We unfortunately need to recall it in detail before we can derive the Cartan calculus associated to each classification datum in Section 3. We will, however, take the opportunity to reformulate the theory of [17] more directly in terms of conjugacy classes and to change to what are now more standard left-invariant forms. This is not a routine left-right reversal of all formulae as the bicrossproduct is not itself being reversed, and in fact leads to cleaner results.

2.1. Modules of the quantum double of a bicrossproduct. According to the Woronowicz theory [18] in the form recalled above, the first step to the classification is to understand the \( D(H) \)-modules where \( H = kM \rtimes k(G) \), and in particular the canonical one on \( H^+ \). We begin by recalling what is known about these, from [20] [17] but with a necessary switch from left to right modules. This is again not routine, but we omit the proofs. Note that a \( D(H) \) right module means a compatible right module of \( H \) and left module of \( H^* \) (or right module of \( H^{\text{op}} \)).

**Proposition 2.1.** [20] Prop. 4.1] The right modules of \( D(kM \rtimes k(G)) \) are in one-one correspondence with vector spaces \( W \) which are:

\( (i) \) \( G \)-graded right \( M \)-module such that \( |w| = t^{-1} |w| \), for all \( t \in M \), where \( \langle , \rangle \) denotes the \( G \)-degree of a homogeneous element \( w \in W \).

\( (ii) \) \( M \)-graded left \( G \)-module such that \( \langle w, u \rangle = \langle w \rangle u^{-1} \), for all \( u \in G \), where \( \langle , \rangle \) denotes the \( M \)-degree of a homogeneous element \( w \in W \).

\( (iii) \) Bigraded by \( G, M \) together and mutually “cross modules” according to

\[
\langle w, u \rangle = t^{-1} \langle w \rangle (t^{-1} |w|)^{-1}, \quad |w, u| = (\langle w \rangle u^{-1} - 1 |w|)^{-1}
\]

\( (iv) \) \( G - M \) “bimodules” according to

\[
((t^{-1} |w| u) |u|) = (u^{-1} |w|) a (t |t^{-1} |w| u) \]

The corresponding action of the quantum double is given by

\[
w \delta (t \otimes \delta_v) = \delta t^{-1} |w| \otimes t \delta w, \quad (\delta_s \otimes u) \otimes u = \delta (s, (w u^{-1} \otimes u) w) \]

and the induced braiding is

\[
\Psi_{L, W} (l \otimes u) = w \delta (l^{-1} |w|^{-1})^{-1} \otimes (l^{-1} |w|^{-1})^{-1} \otimes l
\]

In particular, \( D(H) \) acts on \( H \) by the standard right quantum adjoint action of \( H^* \):

\[
g \delta h = Sh_{(1)} g h_{(2)}, \quad a \delta h = \sum h_{(1)} \otimes h_{(2)}, a >
\]

where \( g, h \in H, a \in H^* \), and \( \Delta h = h_{(1)} \otimes h_{(2)} \) is the Sweedler notation. A routine computation from the Hopf algebra structure of \( kM \rtimes k(G) \) yields these as

\[
(s \otimes \delta_u) \delta (t \otimes \delta_v) = \delta (s |w| u (uw)) (t^{(1)} \otimes tl^{(1)}) \otimes \delta (s |w| u (uw)), \quad t'' = (t |w| u)^{-1} - 1 \delta u
\]

\[
(\delta_l \otimes v) \delta (s \otimes \delta_u) = \delta (s u^{-1} \otimes s \otimes \delta u)
\]
Comparing these with the form of the actions in Proposition 2.1, we find easily that the gradings, the $M - G$ actions for the right canonical representation of $D(kM \bowtie k(G))$ on $W = kM \bowtie k(G)$ and the induced braiding are

$$|s \otimes \delta_u| = (s \triangleright u)^{-1}u, \quad <s \otimes \delta_u> = s \triangleleft u$$

$$(s \otimes \delta_u)\triangleleft t = \bar{t} s^{-1} \otimes \delta_{\triangleright tu}, \quad tv(s \otimes \delta_u) = s \otimes \delta_{uv^{-1}}$$

where $\bar{t} = t^{-1}(s \triangleright u)^{-1}$ and $s'' = (s \triangleleft u)^{-1}q^{-1}$.

Following the spirit of [20] we can also give right $D(H)$-modules in terms of the right modules of the quantum double $D(X) = k(X) \bowtie kX$ of the group $X$, where the action is by Ad. Explicitly, its Hopf algebra structure is

$$(\delta_x \otimes y)(\delta_u \otimes b) = \delta^{-1}_{y^{-1}xy, a}(\delta_x \otimes yb), \quad \Delta(\delta_x \otimes y) = \sum_{a \in x} \delta_a \otimes y \otimes \delta_b \otimes y$$

and suitable formulae for the counit and antipode. It was shown in [20] that there is an algebra isomorphism $\Theta : D(H) \rightarrow D(X)$ defined by

$$\Theta(\delta_x \otimes u \otimes t \otimes \delta_v) = \delta_{u^{-1}s^{-1}(tv)u} \otimes u^{-1}(tvu)$$

A straightforward computation shows that its inverse is

$$\Theta^{-1}(\delta_{su} \otimes tv) = \delta_{s^{-1}(tv)u} \otimes (tv)^{-1} \otimes (t \alpha) \otimes \delta_{u^{-1}v}$$

where $\alpha = t^{-1}v^{-1}(s^{-1}tv)$. Hence $D(H)$ and $D(X)$ modules correspond under these isomorphisms.

On the other hand, it is known that $D(X)$-modules $W$ are nothing other than crossed modules in the sense of Whitehead, see [19], i.e. given by $X$-graded $X$-modules with grading $|| | |$ and (right) action $\triangleright$, say, compatible in the sense

$$||w \triangleright x|| = x^{-1}||w||x$$

for all $x \in X$ acting on homogeneous $w \in W$. The corresponding action is of course

$$w \triangleright (\delta_x \otimes y) = \delta_x, ||w\triangleright y||, \quad \forall x, y \in X.$$

It is easy to see that the correspondence with the gradings and actions in Proposition 2.1 is

$$||w|| = (w)^{-1}||w||, \quad w \triangleright us = (u^{-1} \triangleright w) \triangleright (s^{-1} \triangleright |u^{-1} \triangleright w|^{-1})^{-1}$$

$\forall w \in W, us \in X$.

Therefore the canonical representation of $D(H)$ that we are interested in can be identified with such an $X$-crossed module. Before giving it, following [17], we identify the vector space $kX$ spanned by $X$ with the vector space $W = kM \bowtie k(G)$ via $vt \equiv t \otimes \delta_v$. Then

**Proposition 2.2.** [17] The right canonical representation of $D(H)$ can be identified with $kX$ as an $X$-crossed module

$$||vt|| = ||t \otimes \delta_v|| = v^{-1}t^{-1}v, \quad vt \triangleright us = (\tilde{s} \triangleright vu)(\tilde{t} \triangleright s^{-1}), \quad \tilde{s} = s^{-1} \triangleright (vu)^{-1}.$$
Finally, we are actually interested in the canonical action not on $H$ but on $H^+$. This is the right quantum adjoint action as before and $h\langle a \rangle = \sum h_{(1)} < h_{(2)}, a > - < a, h > 1$ for all $h \in H^+$. It is arranged so that the counit projection to $H^+$ is an intertwiner. Therefore in our case

$$\Pi : kX \to H^+, \quad vt \mapsto t \otimes \delta_v - \varepsilon(t \otimes \delta_v)1 = t \otimes \delta_v - \delta_{v,e}$$

is an intertwiner between this action $\langle$ (viewed as an action of $D(X))$ and the action $\delta$ defined by the crossed module structure.

2.2. Quantum tangent spaces in $kM\blacktriangleright k(G)$. We are now ready briefly to re-formulate the classification [17] for the quantum tangent spaces $L \subset H^+$ of bicrossproduct quantum groups $H = kM\blacktriangleright k(G)$. The minor technical innovation is to rework the theory in terms of a subset $Z \subset X$ stable under conjugation in $X$. Here

$$Z = \text{image}(\mathcal{N}), \quad \mathcal{N} : X \to X, \quad \mathcal{N}(vt) = ||vt||$$

is manifestly stable since $(us)^{-1}||vt||us = ||vt\tilde{u}s||$, for all $vt, us \in X$ as an expression of the $X$-crossed module structure of $kX$ in Proposition 2.2. Working with $Z$ is obviously equivalent to working as in [17] with the quotient $X/\sim$, where $x \sim y$ if $\mathcal{N}(x) = \mathcal{N}(y)$. Moreover, orbits under $\tilde{a}$ as in [17] now correspond to conjugacy classes in $Z$. We denote respectively by $X_z$ and $C_z$ the centralizer and the conjugacy class in $X$ of an element $z \in Z$. Clearly, $Z$ is the partition into conjugacy classes of its elements. All results in this section are along the lines of [17] with such differences.

**Proposition 2.3.** For each $z \in Z$ we set $J_z = k\mathcal{N}^{-1}(z)$.

(i) The space $J_z$ is a right $X_z$ representation

(ii) $M_{C_z} = \bigoplus_{z \in C_z} J_z \subset kX$ is a subrepresentation under the right action of $k(X)\blacktriangleright kX$ from Proposition 2.2. Moreover, $kX = \bigoplus_{C_z} M_{C_z}$ is the decomposition of $kX$ into such subrepresentations.

**Proof.** Statement (i) is immediate. We now prove (ii). The action of $\delta_z \in D(X)$ denoted by $\tilde{\delta}_z$ is a projection operator that projects $kX$ onto $J_z$. Then we have $kX = \bigoplus_{z \in Z} J_z$. Since $Z$ is a partition by the conjugacy classes $C_z$, we have

$$kX = \bigoplus_{C_z} \bigoplus_{z \in C_z} J_z = \bigoplus_{C_z} M_{C_z}.$$ 

For a chosen conjugacy class $C$, let us set

$$\pi_C = \sum_{z \in C} (\tilde{\delta}_z).$$

The operator $\pi_C$ is a projection of $kX$ onto $M_C$. To show that $M_C$ is a right $D(X)$ representation, it is enough to show that the action $\tilde{\delta}(\delta_z \otimes y)$ of any $\delta_z \otimes y \in D(X)$ commutes with $\pi_C$, i.e.

$$\pi_C \circ (\tilde{\delta}(\delta_z \otimes y)) = (\tilde{\delta}(\delta_z \otimes y)) \circ \pi_C.$$ 

This is an easy computation using the crossed relation $y\delta_z = \delta_{yz^{-1}}y$ in $D(X)$.

From now we fix a conjugacy class $C_0$ of an element $z_0 \in Z$, denote by $X_0$ the centralizer of $z_0$ in $X$ and set $J_0 = k\mathcal{N}^{-1}(z_0)$. 

Proposition 2.4. Let $J_0 = J_1 \oplus J_2 \ldots \oplus J_n$ be the decomposition into irreducibles under the action of $X_0$. For each $z = \bar{z}^{-1}z_0\bar{z} \in C_0$, we set $J_{iz} = J_i\bar{z}\bar{e}$ (this does not depend on the choice of $\bar{e}$), then

$$M_i = \oplus_{z \in C_0} J_{iz} \subset M_{C_0}, \quad 1 \leq i \leq n$$

are irreducible subrepresentations under the right action of $k(X) \triangleright \delta_kX$. Moreover, $M_{C_0} = \oplus_{i=1}^n M_i$ is a decomposition of $M_{C_0}$ into irreducibles.

Proof. First of all we prove that $J_{iz}$ does not depend on the choice of $\bar{e}$. Indeed suppose that $\bar{z}^{-1}z_0\bar{z} = y^{-1}z_0y = z'$. Then we have $y\bar{z}^{-1} \in X_0$ which implies that $J_i\bar{z}y\bar{z}^{-1} = J_i$ hence

$$J_{iy} = J_i\bar{z}y = (J_i\bar{z}y\bar{z}^{-1})\bar{z}\bar{e} = J_i\bar{z}\bar{e} = J_{iz}$$

Next, by equivariance of $N$ one shows easily that $J_{iz} \cap J_{iy} = \{0\}$ if $\bar{z}^{-1}z_0\bar{z} \neq y^{-1}z_0y$. So $M_i$ as shown is a direct sum. Reasoning as in [17] with suitable care, one shows that $M_i$ is a right $k(X) \triangleright \delta_kX$-module: the essential steps are the following: Let $P_i : J_0 \to J_0$ be a right $X_0$-map which projects to $J_i \subset J_0$ with all other $J_j$ contained in its kernel. Let us define the map $Q_i : M_{C_0} \to M_{C_0}$ by

$$(18) \quad Q_i = \sum_{z \in C_0} (\bar{z}\bar{e}) \circ P_i \circ (\bar{z}\bar{e}^{-1}) \circ (\bar{z}\bar{e})$$

It is clear that $Q_i$ is a projection onto $M_i$. The similar computations as in [17] yield

$$Q_i \circ (\bar{z}(\delta_a \otimes b)) = (\bar{z}(\delta_a \otimes b)) \circ Q_i,$$

proving that $M_i$ is a $D(X)$-module. Moreover it is clear that $\sum_{z \in C_0} (J_0\bar{z}\bar{e}) = \sum_{i=1}^n M_i$ and since $Q_i Q_j = 0$ for $i \neq j$ we have $\sum_{z \in C_0} (J_0\bar{z}\bar{e}) = \bigoplus_{i=1}^n M_i$. Finally one may verify that $M_i$ is irreducible as $D(X)$-right module. \(\diamondsuit\)

We therefore have a decomposition of $kX$ into irreducibles, for every choice of conjugacy class $C$ of an element $z_0 \in Z$ and every irreducible subrepresentation of the centralizer of $z_0$ in $X$. The converse also holds:

Proposition 2.5. Let $\mathcal{M} \subset kX$ be an irreducible right $k(X) \triangleright \delta_kX$ representation under the action from Proposition 2.2. Then as vector space, $\mathcal{M}$ is of the form

$$\mathcal{M} = \bigoplus_{z \in C} (M_{C_0}\bar{z}\bar{e})$$

For some conjugacy class $C$ in $X$ of $z_0 \in Z$ and some irreducible subrepresentation $M_{C_0} \subset J_{z_0}$ of the centralizer $X_0$ of $z_0$ in the group $X$.

Proof. We choose $z_0 \in Z$ such that $M_{C_0} := M\bar{z}\delta_{z_0}$ is nonzero. Hence $M_{C_0} \subset J_{z_0}$. Moreover $M_{C_0}$ is a $X_0$ subrepresentation of $J_{z_0}$. Indeed let $m\bar{z}\delta_{z_0}, m \in M$ be an element of $M_{C_0}$ and $g \in X_0$. We note that $m\bar{z}g \in M$ since $M$ is a $D(X)$-module. We note also that $g\delta_{z_0} = \delta_{(g^{-1}z_0)g} = \delta_{z_0} g$. Hence

$$(m\bar{z}\delta_{z_0})\bar{z}g = m\bar{z}\delta_{z_0} g = m\bar{z}g\delta_{z_0} \in M_{C_0}$$

which shows that $M_{C_0}$ is a $X_0$-subrepresentation of $J_{z_0}$. Next if $J_1$ is an irreducible subrepresentation of $M_{C_0}$ under the action of $X_0$, then by the preceding proposition $\bigoplus_{z \in C} (J_1\bar{z}\bar{e}) \subset \mathcal{M}$ is an irreducible right representation of $D(X)$. And since $\mathcal{M}$ is irreducible we have $\mathcal{M} = \bigoplus_{z \in C} (J_1\bar{z}\bar{e})$. Finally note that $J_1$ is in fact $M_{C_0}$, so
that \(M_0\) is irreducible as \(X_0\)-module, indeed by Proposition \(\mathbb{24}\) two distinct sub-representations \(J_1\) and \(J_2\) of \(X_0\) should give distinct irreducible subrepresentations \(\sum_{z \in C} (J_1 z) \subset M\) and \(\sum_{z \in C} (J_2 z) \subset M\). This is not possible as \(M\) is irreducible. \(\diamondsuit\)

Application of \(\Pi : kX \to H^+\) from Section 2.1 then tells us that we obtain subrepresentations of \(H^+\) under the action of \(D(X)\) by projecting via \(\Pi\) the subrepresentations of \(kX\). We can now give the total description of the irreducible quantum tangent spaces of \(H\). Then cf. \([17]\).

**Theorem 2.6.** The irreducible quantum tangent spaces \(L \subset H^+\) are all given by the following 2 cases:

(a) For a conjugacy class \(C \neq \{e\}\) of an element \(z_0 \in Z\), for each irreducible right subrepresentation \(M_0 \subset J_{z_0}\) of the centralizer of \(z_0\), we have an irreducible right \(D(H)\)-module \(M = \bigoplus_{z \in C} (M_0 z)\) and an isomorphic irreducible right subrepresentation \(L = \Pi(M) \subset H^+\).

(b) For \(C = \{e\}, J_e = kG, X_e = X\) and for any nontrivial nonzero irreducible right subrepresentation \(M_0 \subset kG\) we obtain an irreducible right \(D(H)\)-module \(M = \bigoplus_{z \in C} (M_0 z)\) = \(M_0\) and the isomorphic \(D(H)\)-subrepresentation \(L = \Pi(M_0) \subset H^+\).

**Proof.** These steps are the same as in \([17]\). Briefly, if \(M = \bigoplus_{z \in C} (M_0 z)\) is an irreducible representation of the unprojected action then by equivariance, the map \(\Pi : M \to L\) is a map of representations. It is surjective. If it is 1-1 then the two representations are isomorphic. The unique case where \(\Pi\) is not 1-1 is where \(\bigoplus_{z \in C} (M_0 z)\) is an irreducible right subrepresentation of \(H^+\) under \(k(X) \triangleright kX\) then the inverse image \(\Pi^{-1}(L) \subset kX\) is also a representation of \(k(X) \triangleright kX\) and it contains \(k\bar{1}\). If \(L \neq 0\) then \(\Pi^{-1}(L)\) contains at least one other irreducible representation \(M\) such that \(k\bar{1} \oplus M \subset \Pi^{-1}(L)\) then \(M\) must be of the form described above and by irreducibility of \(L, \Pi(M) = L\). \(\diamondsuit\)

We note that the element \(z_0\) is not strictly part of the classification of the differential calculi. In fact an irreducible bicovariant differential calculus is defined by a conjugacy class \(C\) and a irreducible \(D(X)\)-subrepresentation \(M \subset kX\) such that \(|M| = C\), where \(|M|\) denotes the set of images by \(|.|\) of homogeneous elements of \(M\). It does not depends on the chosen element in \(C\). In the other words if

\[
M = \bigoplus_{z \in C} (M_0 z)
\]

with \(M_0\) an irreducible subrepresentation of \(J_{z_0}\) under the action of the centralizer of \(z_0\) then for any \(z_1 \in C\) we can write also \(M\) as

\[
M = \bigoplus_{z' \in C} (M_1 z')
\]

where \(M_1\) an irreducible subrepresentation of \(J_{z_1}\) under the action of the centralizer of \(z_1\). This follows from Proposition \(\mathbb{24}\). Indeed giving \(M = \bigoplus_{z \in C} (M_0 z)\), and \(z_1 \in C\), we set \(M_1 = M z_1\). This is nonzero since \(M_0 z_1 \subset M z_1\), \(M_1 = M z_1 \neq 0\) implies by the proof of Proposition \(\mathbb{24}\) that \(M_1\) is an irreducible
subrepresentation of $kN^{-1}(z_1)$ under the action of the centralizer $G_{z_1}$ of $z_1$ and moreover $\mathcal{M} = \bigoplus_{z \in C}(\mathcal{M}_1 \bar{\otimes} z)$.  

With this characterization of the quantum tangent spaces in terms of conjugacy classes and centralizers, we recover the well known cases where $H = kM$ or $H = k(G)$:

**Proposition 2.7.** (i) Set $G = \{e\}$ then $X = M$ and Theorem 2.6 recovers the usual classification of the irreducible bicovariant calculus on $A = k(M)$ by nontrivial conjugacy classes in $M$.

(ii) Set $M = \{e\}$ then $X = G$ and we recover the classification for calculi on $A = kG$ by nontrivial irreducible subrepresentations $V \subset kG$ under the regular right action of $G$ on itself as in [7].

(iii) Set $X = G \times M$ with trivial actions. Then $A = k(M) \otimes kG$. An irreducible bicovariant calculus in case (i) the action of $X$ on $kX$ is $v \bar{\otimes} s = s^{-1} vs$, $Z = M$. For any conjugacy class $C_0$ of an element $t \in M$ we denote by $C$ the conjugacy class of $t^{-1}$ and we have $J_t = k\{t^{-1}\}$, since $\|b\| = b^{-1}$, $\forall b \in M$. Hence the corresponding irreducible subrepresentation $\mathcal{M} \subset kM$ under the action of $D(H)$ is $\mathcal{M} = \sum_{z \in C_t}(\mathcal{M}_1 \bar{\otimes} z) = \sum_{z \in C_t} k(z^{-1}t^{-1}z) = kC$ and $L = k\{a - e, a \in C\}$, i.e the basis of $L^*$ is labelled by a conjugacy class as usual.

For case (ii) the action of $X$ on $kX$ is $v \bar{\otimes} u = vu$, $\|v\| = e$, $\forall v \in G$. Hence $Z = \{e\}$ so that we are in case (b) of the theorem. Therefore the quantum tangent spaces $L \subset H^+$ are isomorphic to the irreducible subrepresentations $V \subset kG$ as stated.

For case (iii) we have $Z = M$. The action of $X$ on itself is $v \bar{\otimes} s = vu.s^{-1}ts$. Let us consider a conjugacy class $C^{-1}_{t_0}$ of $t_0$ in Z. The centralizer of $t_0$ in $X = G.cent_M(t_0)$, where $cent_M(t_0)$ is the centralizer of $t_0$ in $M$, $J_0 = kN^{-1}(t_0^{-1}) = kG.t_0$. The action of $X_0$ on $J_0$ is $v.s = vu.t_0$, $\forall v \in G, us \in X_0$ which leads to $\mathcal{M}_0$ of the form $\mathcal{M}_0 = V.t_0$, where $V$ is as mentioned, hence $\mathcal{M} = \bigoplus_{t \in C^{-1}_{t_0}} (V.t_0 \cdot \bar{\otimes} t) = V.C_{t_0}$ where $C_{t_0}$ is the conjugacy class of $t_0$ in $M$. ∗

The calculus in case (iii) is a product of calculi on $G, M$ for the cases (i) and (ii) and has the product of their dimensions.

3. **Cartan calculus on $k(M)\bar{\otimes} kG$**

We are now ready to proceed to our main results. Let $A = k(M)\bar{\otimes} kG$ be the dual of $H = kM\bar{\otimes} kG$. Our goal is to find an explicit description for the calculus corresponding to each choice of classification datum. This amounts to a description of the differential forms and the commutation relations with functions and $d$, i.e. a 'Cartan calculus' for the associated noncommutative differential geometry.
We fix a conjugacy class $C$ of an element $z_0 \in Z$, an irreducible right subrepresentation $M_0 \subset J_{z_0}$ of the centralizer of $z_0$, and the corresponding nontrivial irreducible right $D(H)$-module $M = \bigoplus_{z \in C} (M_0 \hat{\circ} z)$ as in Theorem 2.6 above. For each $z \in C$ we fix one element $\bar{z}$ so that $z = \bar{z}^{-1} z_0 \bar{z}$ and we set $\bar{C} = \{ \bar{z} \mid z \in C \}$. As we saw above, $M = \bigoplus_{\bar{z} \in \bar{C}} (M_0 \hat{\circ} \bar{z})$. We now choose a basis $(f_i)_{i \in I}$ of $M_0$ ($I$ is finite) and set

$$f_{iz} := f_i \hat{\circ} \bar{z}$$

We recall that here $\hat{\circ}$ is the action of $X$ on itself defined in Proposition 2.2.

**Lemma 3.1.** The vectors $(f_{iz}), i \in I$ form a basis of $M$ with homogeneous $X$-degree $z$.

**Proof.** : By definition it is clear that $(f_{iz})$ generate $M$ since $(f_i)$ generate $M_0$. Using the direct sum in the decomposition of $M$ and the fact that $(f_i)$ are linearly independent, one checks easily that $(f_{iz})$ are linearly independent too. By definition, $(f_i)$ are homogeneous of degree $z_0$. This implies that each $f_{iz}$ is homogeneous of degree $z$ since for homogeneous $w$, $||w \hat{\circ} x|| = x^{-1} ||w|| x$, $\forall x \in X$. \hfill ⋄

In what follows, we identify $M$ with the quantum tangent space $L$ as isomorphic vector spaces via $\Pi$. The dual $\Lambda^1$ of $L$ is equipped with the dual basis $(e_{iz})$ of the basis $(f_{iz})$.

To simplify we relabel these basis by $(e_a)_{a \in I}$ and $(f_a)_{a \in I}$ respectively for the space of invariant 1-forms and the quantum tangent space. We recall the factorization (14) of an $X$-grading into an $G$-grading $|$ and an $M$ grading $\langle \rangle$.

We are now ready to follow the Woronowicz construction explained in the preliminaries to build $(\Omega^1, d)$ as a differential bimodule, namely we set $\Omega^1(A) = A \otimes \Lambda^1$,

$$da = \sum (\id \otimes \Pi_{\Lambda^1})(a_{(1)} \otimes (a_{(2)} - \varepsilon(a_{(2)}))),$$

$$a.x = a \otimes x, \quad x.a = \sum a_{(1)} \otimes x \triangleleft a_{(2)}$$

for all $a \in A$ and $x \in \Lambda^1$, where $\Pi_{\Lambda^1}$ denotes the projection of $A^+$ on $\Lambda^1$ adjoint to the injection $L \subset H^+$. We have the following :

**Theorem 3.2.** With the chosen basis of $L$ as above, the differential calculus in Theorem 2.6 is explicitly defined by:

(i) The left $A$-module of 1-forms $\Omega^1(A) = A \otimes \Lambda^1$.

(ii) The right module structure according to commutation relations between “functions” and 1-forms:

$$e_a \delta_s = \delta_s (f_a) - 1 e_a, \quad e_a u = (f_a) \circ u) e_a * u$$

where

$$e_a * x = \sum_{b \in I} < e_a, f_b \hat{\circ} x^{-1} > e_b, \quad \forall x \in X.$$

is the right action of $X$ on $\Lambda^1$ adjoint to the left action $x * f_a := f_a \hat{\circ} x^{-1}$ on $L$. 

The exterior differential:

\[ d\delta_s = \sum_a <\delta(f_a), f_a (\delta_s) > - \delta_s e_a \]

\[ du = \sum_a <\delta(f_a), f_a > \delta(f_a) e_a - \sum_a <\delta(f_a), f_a > u e_a \]

where \(<\delta_v t, f_a >\) for all \(v t \in X\), is the pairing between \(k(X)\) and its dual \(k^X\).

**Proof.** We first of all note the following facts easily obtained from (12) and the factorization of \(X\) grading in (14) and that we freely use in the proof:

\[ <f_a \ VARU, f_a > = <f_a \ua, f_a \ua >, \forall u \in G, \ |f_a \ua| = (s^{-1}|f_a|^{-1})^{-1}, \forall s \in M.\]

We note also that the right action of \(A\) in (20) is the restriction of the action of \(D(A^*)\) on \(\Lambda^1\), we view it via the isomorphism \(\Theta\) as action of \(D(X)\) on \(\Lambda^1\), adjoint to a left action of \(\Theta(A)\) on \(L \cong M \subset k^X\). Clearly equation (15) expresses both right action of \(\Theta(A^*)\) and left action of \(\Theta(A)\) on \(M\), thus:

\[ e_a \delta_b \ VARU = e_a \Theta(\delta_b \ VARU) = \sum_{e A} <e_a, f_c \ua \ua(\delta_b \ VARU) > e_c \]

for all \(\delta_b \ VARU \in A\). On the other hand

\[ \Theta(\delta_b \ VARU) = \sum_{e U} \delta_{u^{-1}svu} \ua^{-1} \]

then by (11) and (20) we have

\[ e_a \delta_s = \sum_{e U, b \in M} \delta_{u^{-1}svu} \ua^{-1} \]

We compute

\[ e_a \delta_{b^{-1}v} = \sum_{e A} <e_a, f_c \ua \ua\delta_{b^{-1}v} > e_c = \sum_{e A} \delta_{|f_a|, b^{-1}v} \delta_{|f_a|, b^{-1}v} e_a = \delta_{|f_a|, b^{-1}v} e_a \]

so that

\[ e_a \delta_s = \sum_{u, b} \delta_{u^{-1}svu} \ua^{-1} \]

Next, let \(u \in G \subset A\). From (11) and (20) again, we have

\[ e_a u = \sum_{b \in M} (ku) \ua \ua e_a \Theta(\delta_b \ VARU) \]

To compute \(e_a \Theta(\delta_b \ VARU)\), we first note that if we change the basis \((f_a)\) to \(f'_a =: f_a \ua\) then its dual \((e_a)\) transforms as

\[ e'_a =: e_a \ua = \sum_{e A} <e_a, f_c \ua^{-1} > e_c.\]
Then
\[ e_a \Theta(\delta_b \otimes u) = \sum_{c \in I, v \in G} \langle e_a, f'_c \delta_{u^{-1}v} \otimes u^{-1} \rangle > e'_c \]
\[ = \sum_{c \in I, s \in G} \delta([f'_c], u^{-1}v) < e_a, f'_c \delta_{u^{-1}v} > e'_c \]
\[ = \sum_{c \in I, s \in G} \delta([f'_c], b \delta_u \delta([f'_c], v^{-1}b \delta_u) < e_a, f'_c e - u \]
\[ = \delta([f_a], b e_a * u) \]
from which we deduce \( e_a u = ((f_a) \circ u) e_a * u \) as required.

We now prove the formulae for differentials. Writing \( \bar{a} = a - e(a) \in A^+ \), we write the projection as
\[ \Pi_A(\bar{a}) = \sum_{c \in I} < \Pi_A(\bar{a}), f_c > e_c = \sum_{c \in I} < \bar{a}, i(f_c) > e_c = \sum_{c \in I} < \bar{a}, \Pi(f_c) > e_c \]
where \( i \) is the injection \( L \subset H^+ \) which in our case, viewing \( L \) as \( M \subset kX \), is just the restriction on \( M \) of the map \( \Pi : kX \rightarrow H^+ \). Denoting its adjoint map \( \Pi^* : H^* \rightarrow k(X) \) we have therefore \( \Pi_A(\bar{a}) = \sum_{c \in I} < \Pi^*(\bar{a}), f_c > e_c \). In our case,
\[ \Pi^*(\delta_s \otimes u - \delta_{s,e} 1 \otimes e) = \delta_{us} - \delta_{s,e} \sum_{t \in M} \delta_{e,t} \]

Let \( s \in M \). From (14) we have
\[ d\delta_s = \sum_{b \in M} \delta_{b^{-1}} \otimes \Pi_A(\delta_b - \delta_{b,e} 1_A) = \sum_{b \in M} \delta_{b^{-1}} \otimes \delta_{e,b} - \delta_{b,e} \sum_{t \in M} \delta_{e,t} \]
\[ = \sum_{b \in M, c \in I} \delta_{b^{-1}} \otimes < \delta_{e,b} - \delta_{b,e} \sum_{t \in M} \delta_{e,t}, f_c > e_c \]
\[ = \sum_{a \in I} < \delta_{(f_a)}, f_a > (\delta_{s(f_a)} - \delta_s e_a), \]
where we used
\[ < \delta_{e,b}, f_a > = < \delta_{(f_a)}, f_a > \delta_{b,(f_a)}, \forall b \in M, \]
as one may see by expanding \( f_a = \sum a_i v^i t^i \), say. This pairing also equals \(< \delta_{ub}, f_a \delta_u > \) for all \( u \in G \) since \( < vt \delta u, \delta_{ub} > = < v t, \delta_{e,b} > = \delta_{e, \delta} \delta_{t,b} \). Hence, using (14) and (15), we similarly have
\[ du = \sum_{b \in M} (b \delta u) \otimes \Pi_A(\delta_b \otimes u - \delta_{b,e} 1_A) \]
\[ = \sum_{b \in M, a \in I} (b \delta u) \otimes < \Pi^*(\delta_b \otimes u - \delta_{b,e} 1_A), f'_a > e'_a \]
\[ = \sum_{a \in I, b \in M} (b \delta u) < \delta_{ub}, f'_a > e'_a - \sum_{a \in I, t \in M} u < \delta_{e,t}, f_a > e_u \]
\[ = \sum_{a \in I} < \delta_{(f_a)}, f_a > (\langle f_a \rangle \circ u) e_a * u - \sum_{a \in I} < \delta_{(f_a)}, f_a > u e_a. \]
This ends the proof of Theorem \ref{thm:braiding}

\begin{corollary}
All irreducible bicovariant differential calculi on a bicrossproduct \(A = k(M)\bowtie kG\) are inner in the sense
\[
da = [\theta, a], \quad \forall a \in A, \quad \text{where} \quad \theta = \sum_{a \in I} e_a c_a, \quad e_a = \langle f_a, \delta_a \rangle.
\]
\end{corollary}

\begin{proof}
The relations \(\theta \delta_s - \delta \theta = d \delta_s\) and \(\theta u - u \theta = du\) are obtained from the definitions in Theorem \ref{thm:braiding}.
\end{proof}

Once the first order differential calculus is defined explicitly, we need also the braiding \(\Psi\) induced on \(\Lambda^1 \otimes \Lambda^1\) (then on \(\Omega^1(A) \otimes_A \Omega^1(A)\)) to determine \(\Omega^n(A), n \geq 2\). Thus \(\Omega^2(A) = A \otimes \Lambda^2\) where \(\Lambda^2\) is the space of invariant 2-forms defined as the quotient of \(\Lambda^1 \otimes \Lambda^1\) by ker(id - \(\Psi\)).

\begin{proposition}
The braiding \(\Psi\) induced on \(\Omega^1(A)\) by the action of the quantum double \(D(A^*)\) is given by
\[
\Psi(e_a \otimes e_b) = e_b \ast (\langle f_a | \langle f_b \rangle \rangle^{-1} \otimes e_a \ast | f_b \rangle).
\]
\end{proposition}

\begin{proof}
The formula of the braiding on a basis \((e_a \otimes e_b)\) of \(\Lambda^1 \otimes \Lambda^1\) is
\[
(25) \quad \Psi(e_a \otimes e_b) = \sum_i \beta^i \ast e_b \otimes e_a \ast \alpha_i
\]
where \((\alpha_i)\) is a basis of \(A\) with dual basis \((\beta^i)\) of \(A^*\). The right action of \(A\) on \(\Lambda^1\) is given by \ref{eq:right_action}
\[
(26) \quad e_a \ast \delta_v = \delta_{t, (f_a)} e_a \ast v
\]
for all \(t \in M\) and \(u \in G\). We now compute \((t \otimes \delta_v) \circ e_a\), the adjoint of the action \(f_a \circ (t \otimes \delta_v)\). We have
\[
\Theta(t \otimes \delta_v) = \sum_{s \in M} \delta_{s^{-1}(tbv)} \otimes t \circ v
\]
so that
\[
f_b \circ (t \otimes \delta_v) = \sum_{s \in M} f_b \circ (s^{-1}(tbv) \otimes (t \circ v)) = \sum_{s \in M} \delta_{\langle f_a | | s^{-1}(tbv) \rangle} f_b \circ (t \circ v).
\]
Then considering the basis \(f_b'' = f_b \circ (t \circ v)^{-1}\) whose dual basis is \(e_b'' = e_b \ast (t \circ v)^{-1}\), we compute
\[
(t \otimes \delta_v) \circ e_a = \sum_{b \in I} < e_a, f_b'' \circ (t \otimes \delta_v) > e_b'' = \sum_{b \in I, s \in M} \delta_{\|f_a''\|, s^{-1}(tbv)} < e_a, f_b'' \circ (t \circ v) > e_b'' = \sum_{s \in M} \delta_{\|f_a''\|, s^{-1}(tbv)} e_a \ast (t \circ v)^{-1}.
\]
It is easy to check that
\[
\|f_a''\| = (\langle f_a | \circ (t \circ v)^{-1} \rangle^{-1}(t \circ v) \ast | f_a \rangle^{-1})^{-1}
\]
so that
\[
(27) \quad (t \otimes \delta_v) \circ e_a = \delta_{(tbv)^{-1}, (t \circ v)} \ast e_a \ast (t \circ v)^{-1} = \delta_{(tbv)^{-1}} e_a \ast (t \circ v)^{-1}.
\]
Finally, combining equations (20) and (21) gives the formula for the braiding as stated.  

\[ \text{Corollary 3.5.} \] The left-invariant 1-form \( \theta \) obeys \( \theta \wedge \theta = 0 \) and is closed and nontrivial in the first noncommutative de Rham cohomology \( H^1 \). The basic 1-forms obey the Maurer-Cartan relations \( de_a = \{ \theta, e_a \} \).

\[ \text{Proof.} \] We need only to prove that \( \theta \) is right-invariant (the rest then follows by general arguments). This is equivalent to invariance under the left action (27) of \( H \), which is a modest computation. Alternatively, the relevant coaction on \( A^1 \) is the projection of the adjoint one on \( A^+ \). At least for \( C^X \neq \{ e \} \), we have \( \theta = -\Pi A^1(\delta_e \otimes e - \sum_{t \in M} \delta_t \otimes e) \) by (24) and similar computations as there. This representative element of \( A^+ \) is then more obviously Ad-invariant. Since this coaction also enters into \( \Psi \), invariance then implies that \( \Psi(\theta \otimes \theta) = \theta \otimes \theta \). This is in any case true when \( C^X = \{ e \} \) since \( \Psi \) is then the usual flip. Hence \( \theta \otimes \theta = 0 \) in the exterior algebra. On the other hand, for the Woronowicz construction for any Hopf algebra one may show that if the first order calculus is inner by a left-invariant 1-form \( \theta \) obeying \( \theta \otimes \theta = 0 \) then the entire exterior calculus is inner, i.e. \( dp = [\theta, \omega] \) for any form \( \omega \in \Omega \). The graded commutator here denotes commutator in degree 0 and anticommutator in degree 1. Hence the last part of the Corollary is automatic. It implies then that \( d\theta = 0 \).

It remains only to show that \( \theta \) is not exact. This is actually true for any left-invariant 1-form on a left-covariant calculus when the Hopf algebra is semisimple. Precisely such Hopf algebras have a (say) right-invariant integral \( \int : A \to k \) such that \( \int 1 = 1 \) (for our bicrossproduct \( A \) it is \( \int (\delta_u \otimes u) = |M|^{-1}\delta_{u,x} \) as in [19]). In this case suppose \( da \in A^1 \) for some \( a \in A \), so that \( \Delta_L(da) = a(1) \otimes da(2) = 1 \otimes da \), then \( (\int a(1)) da(2) = \int (1) da = \int (a)d(1) = 0 \) by right-invariance of the integral and \( d(1) = 0 \). Hence \( \theta \) is necessarily nontrivial in the noncommutative de Rham cohomology.  

\[ \text{Proposition 3.6.} \] We recover the results known in the cases of the group algebra and functions algebra of a finite group.

\[ \text{Proof.} \] (i) For \( G = \{ e \} \), \( A = k(M) \) and \( L = \Pi(kC) \) for a conjugacy class \( C \). Here a basis of \( kC \) is \( \{ f_a = a \}_{a \in C} \) since the action is \( ts = s^{-1}ts \). Moreover \( \langle f_a \rangle = a, \ |f_a| = e \) and \( e_b \ast a^{-1} = e_{aba^{-1}} \), \( \forall a, b \in C \). Then Theorem 3.2, Corollary 3.3 and Proposition 3.4 read

\[ \Omega^1(A) = A \otimes (kC)^*; \quad e_a \delta_s = \delta_{sa^{-1}} e_a \]

\[ d(\delta_s) = \sum_{a \in C} \langle \delta_a, f_a \rangle (\delta_{sa^{-1}} - \delta_s)e_a = \sum_{a \in C} (\delta_{sa^{-1}} - \delta_s)e_a \]

which is exactly 11 on a general function \( f \in k(M) \). Moreover,

\[ \theta = \sum_{a \in C} \langle \delta_a, f_a \rangle e_a = \sum_{a \in C} e_a, \quad \Psi(e_a \otimes e_b) = e_{aba^{-1}} \otimes e_a \]

for \( a, b \in C, s \in M \) and \( R_s(f)(x) = f(xa), \forall x \in M \). (ii) For \( M = \{ e \}, A = kG, L \) is an irreducible subrepresentation of \( kG \) under the right multiplication
in \( G \). Let \( (f_j) \) be a basis of \( L \) with the dual basis \( (e_j) \). Here we have \( \|f_j\| = e, \forall j \).

Then Theorem 3.2, Corollary 3.3 and Proposition 3.4 read

\[
\Omega^1(A) = A \otimes L^*; \quad e_i u = ue_i * u
\]

\[
du = \sum_i <\delta_e, f_i> (ue_i * u - ue_i) = u\theta * (u - 1)
\]

\[
\theta = \sum_i <\delta_e, f_i> e_i, \quad \Psi(e_i \otimes e_j) = e_j \otimes e_i
\]
as in [2].

Hence the Cartan calculus in theorem 5.2 indeed generalizes the ones on the group algebra and on the algebra of functions of a finite group.

4. Differential calculi on cross coproducts \( k(M)\rhd<kG \).

Now that we have the Cartan calculus for general bicrossproduct Hopf algebras, we specialize to the semidirect case where \( X = G \ltimes M \) or \( A = k(M)\rhd<kG \), a cross coproduct. These are the ‘coordinate’ algebras of semidirect product quantum groups \( H \). In this case some further simplifications are possible.

We start with a general observation about the structure of \( Z \) for general \( X \) = \( GM \). As usual, \( u,v,g... \) are elements of \( G \) and \( s,t,\tilde{s}... \) are those of \( M \).

Proposition 4.1. (i) For general \( X = G.M \), the set \( Z \) is given in terms of conjugacy classes \( C^M \) of \( M \) by

\[
Z = \bigcup_{C^M \subset M} \bigcup_{u \in G} (u^{-1}C^M u)
\]

and for any fixed conjugacy class \( C^M \subset M \), the set

\[
C^X = \bigcup_{u \in G} (u^{-1}C^M u)
\]
is a conjugacy class in \( X \).

(ii) In the semidirect case \( X = G\ltimes M \), the map

\[
C^Z : C^M \to \bigcup_{u \in G} (u^{-1}C^M u)
\]

from the set of conjugacy classes of \( M \) to that of conjugacy classes of \( X \) contained in \( Z \) is one to one.

Proof. We first note that the map \( C^Z \) is not one to one in general (e.g. for the \( \mathbb{Z}_6,\mathbb{Z}_6 \) example, \( C^M_i \) and \( C^M_{i+1} \) are different and have the same image through \( C^Z \)).

In the semidirect case \( X = G\ltimes M \) this map is one to one since

\[
(u^{-1}t_1u) = (v^{-1}t_2v) \iff (u^{-1}(t_1\triangleright u)).t_1 = (v^{-1}(t_2\triangleright v)).t_2 \iff t_1 = t_2.
\]

The other assertions are easily obtained too.  

\[ \diamond \]
4.1. Canonical calculus for the case $X = G > \triangleleft M$. Now we specialize to the semidirect case $X = G > \triangleleft M$. As we saw above, an irreducible differential calculus on $A$ is defined by a conjugacy class (of $t_0^{-1} \in M$ say) $C_0^X \subset Z$ and a choice of an irreducible subrepresentation $M_0$ of $J_0 = k \mathcal{N}^{-1}(t_0^{-1})$ under the action of the centralizer of $t_0$ in $X$. In the semidirect case, we have

**Proposition 4.2.**

(i) $C_0^X = \{ u^{-1}(u^\epsilon),.t \in X, \ t \in C_0^M, u \in G \}$

(ii) $\mathcal{N}^{-1}(t_0^{-1}) = N_0.t_0$

where $N_0 = \{ u \in G, t_0^\triangleright u = u \}$ is a subgroup of $G$.

(iii) The centralizer $X_0$ of $t_0$ in $X$ is $X_0 = N_0.\text{cent}(t_0)$.

(iv) The action of $X_0$ on $J_0 = k N_0.t_0$ is given by

$$v.t_0 \triangleright us = (s^{-1} \triangleright vu).t_0, \ \forall v, u \in N_0, s \in \text{cent}(t_0).$$

For $(v)$, the element

$$m_0 = \sum_{v \in N_0} v.t_0$$

generates a one-dimensional trivial $N_0.\text{cent}(t_0)$-irreducible subrepresentation of $J_0 = k N_0.t_0$, since for all $u.s \in N_0.\text{cent}(t_0)$

$$m_0 \triangleright us = \sum_{v \in N_0} v.t_0 \triangleright us = \sum_{v \in N_0} (s^{-1} \triangleright vu).t_0 = \sum_{v \in N_0} v.t_0 = 0$$

where the penultimate equality is by freeness of $\triangleright \circ R_u$, with $R_u = \text{right multiplication}$. Hence if $t_0 \neq e$ then

$$M = \bigoplus_{z \in C_0^X} k.m_0 \triangleright z$$

is the corresponding quantum tangent space with dimension $|C_0^X|$. Hence, to any conjugacy class of $M$ (or any irreducible differential calculus on $k(M)$) corresponds a canonical irreducible differential calculus on $A$. Here, we made a convention that the null calculus corresponds to $t_0 = e$. ⊤

As an important subcase, we consider now $X = G > \triangleleft G$ where the action is by conjugation. In this case $A = k(G)\triangleright kG = D^*(G)$ is the dual of the quantum double of the group algebra $kG$. Then Proposition 4.2 reads

**Corollary 4.3.** When $M = G$ and $X = G > \triangleleft G$ by conjugation, we have

(i) $C_0^X = \bigcup_{s \in C_0^M} (C_0^M \triangleright s^{-1}).s$

(ii) $\mathcal{N}^{-1}(t_0^{-1}) = \text{cent}(t_0).t_0$

(iii) $X_0 = \text{cent}(t_0).\text{cent}(t_0)$

(iv) The action of $u.s \in \text{cent}(t_0).\text{cent}(t_0)$ on $v.t_0 \in \mathcal{N}^{-1}(t_0^{-1})$ is

$$v.t_0 \triangleright us = s^{-1} vu.s.t_0 = Ad_{s^{-1}} \circ R_u(v).t_0$$
Proof: We check easily that \( N_0 \) becomes \( \text{cent}(t_0) \) and the results stated follow immediately from Proposition 4.2.

Moreover, part (v) of Proposition 4.2 says that any irreducible differential calculus on \( k(G) \) extends to a canonical irreducible differential calculus on \( A = D^*(G) \). We describe it explicitly. Let

\[
C_0^M = \{ s_0 = t_0^{-1}, s_1, ..., s_N \}
\]

be a conjugacy class (of \( t_0^{-1} \)) in \( M = G \) and \( C_0^X \) be the corresponding conjugacy class of \( t_0^{-1} \) in \( X \) as above. For each \( 0 \leq i \leq N \), we fix \( s_i \) in \( M \) such that

\[
s_i = \tilde{s}_i t_0^{-1} \tilde{s}_i.
\]

To avoid confusion we use here the following notation: \( s_i \) is always in \( M \) and we let \( \tilde{s}_i \) denote the identity map from \( M \) to \( G \), so \( s_i \) denotes the same element in \( G \). As usual, in any expression \( g.t \in X \), we have \( \tilde{g} \in G \) and \( t \in M \). Then by (i) of Corollary 4.3, each element \( z_{ij} \) of \( C_0^X \) is of the form

\[
z_{ij} = s_i \tilde{s}_j^{-1} s_j, \quad s_i, s_j \in C_0^M
\]

The elements \( \tilde{s}_i \) define \( \tilde{z}_{ij} \in X \) such that \( z_{ij} = \tilde{z}_{ij} t_0^{-1} \tilde{z}_{ij} \) and we have

\[
\tilde{z}_{ij} = \tilde{s}_i \tilde{s}_j^{-1} \tilde{s}_j.
\]

Indeed if we set \( g_{ij} = \tilde{s}_i \tilde{s}_j^{-1} e = \tilde{s}_i \tilde{s}_j \) then we have

\[
\tilde{z}_{ij} t_0^{-1} \tilde{z}_{ij} = \tilde{s}_j^{-1} g_{ij} t_0^{-1}(g_{ij} \tilde{s}_j) = (\tilde{s}_j^{-1} v g_{ij}^{-1}) (e \tilde{s}_j^{-1} t_0^{-1})(e \tilde{s}_j) = (\tilde{s}_j^{-1} v g_{ij}^{-1}) (\tilde{s}_j^{-1} t_0^{-1} \tilde{s}_j) = (\tilde{s}_j^{-1} v g_{ij}^{-1}) (t_0^{-1} \tilde{s}_j) = s_i \tilde{s}_j^{-1} \tilde{s}_j = z_{ij}
\]

We are now in position to compute the Cartan relations for the calculus defined by \( \mathcal{M}_0 = k m_0 = \sum_{v \in \text{cent}(t_0)} k(v,t_0) \) and \( C_0^X \). We label the basis of \( \mathcal{M} \) using elements of \( C_0^X \) as

\[
f_{z_{ij}} : = m_0 \tilde{z}_{ij} = \sum_{v \in \text{cent}(t_0)} \tilde{s}_j^{-1} v \tilde{s}_i \tilde{s}_j^{-1}
\]

and then denote by \( (e_{z_{ij}}) \) the dual basis of \( (f_{z_{ij}}) \).

Lemma 4.4. The action \( * \) on the basis \( (e_{z_{ij}}) \) is

\[
e_{z_{im}} * (e_{s_j}) = e_{z_{ij} z_{im} s_j}, \quad e_{z_{im}} * (u.e) = e_{u^{-1} z_{im} u}, \quad u, s_j \in G
\]

for all \( 0 \leq j, m, l \leq N \), i.e. \( X \) acts by the right adjoint action on the indexes.

Proof. From the definition of \( \tilde{z} \) we have for all \( 0 \leq p, q, j \leq N \)

\[
f_{z_{pq}} \tilde{s}_j^{-1} = f_{z_{pq}} \tilde{s}_j^{-1} \tilde{s}_j = \sum_{v \in \text{cent}(t_0)} (\tilde{s}_j^{-1} v \tilde{s}_i \tilde{s}_j^{-1}) \tilde{s}_j^{-1} = \sum_{v \in \text{cent}(t_0)} s_j \tilde{s}_q^{-1} v \tilde{s}_p \tilde{s}_j^{-1} \tilde{s}_j s_j^{-1} = \sum_{v \in \text{cent}(t_0)} s_j \tilde{s}_q^{-1} v \tilde{s}_p \tilde{s}_j^{-1} \tilde{s}_j s_j^{-1} \tilde{s}_j^{-1}
\]

\[
(28)
\]
On the other hand, \( f_{z_{pq}} \tilde{s}_j^{-1} \) is homogeneous and should be linear combination of \( f_{z_{ij}}, 0 \leq i, j \leq N \). But the latter have different degrees then we deduce that \( f_{z_{pq}} \tilde{s}_j^{-1} \) is linear combination of only one of them, the one whose degree is \( ||f_{z_{pq}} \tilde{s}_j^{-1}|| = s_j z_{pq} s_j^{-1} \), explicitly, \( f_{z_{pq}} \tilde{s}_j^{-1} = c f_{s_j z_{pq} s_j^{-1}} \), where \( c \) is a constant. In fact this constant is 1 since in the expansion of \( f_{z_{pq}} \tilde{s}_j^{-1} \) in equation (28) the nonzero coefficients of \( u \in X \) equal 1. Therefore \( f_{z_{pq}} \tilde{s}_j^{-1} = f_{s_j z_{pq} s_j^{-1}} \) and

\[
e_{z_{im}} * (e, s_j) = \sum_{p,q} < e_{z_{im}}, f_{z_{pq}} \tilde{s}_j^{-1} > e_{z_{pq}} = \sum_{p,q} < e_{z_{im}}, f_{s_j z_{pq} s_j^{-1}} > e_{z_{pq}} = e_{s_j^{-1} z_{im} s_j}
\]
as stated. One follows the same reasoning to prove the second assertion of the lemma. ⊤

We can now explicitly give the differential calculus of dimension \( |C^G|^2 \) defined by \( (m_0, C^G) \) as above for each conjugacy class \( C^G \) of \( G \). We use Theorem 3.2 and Proposition 3.4.

**Proposition 4.5.** The Cartan calculus and braiding for the canonical differential calculus on \( D^*(G) \) defined by \( (m_0, C^G) \) are given by:

(i) Commutations relations

\[
e_{z_{ij}} f = R_i(f) e_{z_{ij}}, \quad e_{z_{ij}} u = (s_j^{-1} u s_j) e_{u^{-1} z_{ij}} u
\]

(ii) Differentials

\[
df = \sum_i \partial_i(f) e_{z_{ii}}, \quad du = \sum_i (s_i^{-1} u s_i) e_{u^{-1} z_{ii}} u - \sum_i u e_{z_{ii}}
\]

(iii) The element

\[
\theta = \sum_i e_{z_{ii}}
\]

(iv) The braiding

\[
\Psi(e_{z_{ij}} \otimes e_{z_{im}}) = e_{s_j^{-1} z_{im} s_j} \otimes e_{s_i^{-1} u z_{ij} s_i^{-1} u}
\]

for \( u \in G, f \in k(M) = k(G) \), where \( R_i(f)(g) = f(g s_i^{-1}) \) for all \( g \in G \), and \( \partial_i = R_i - \text{id} \).

**Proof.** Since

\[
||f_{z_{ij}}|| = z_{ij} = s_j s_j^{-1} \cdot s_j,
\]

we have

\[
\langle f_{z_{ij}} \rangle = s_j^{-1} \delta(s_i s_j^{-1})^{-1} = e. s_j^{-1}, \quad |f_{z_{ij}}|^{-1} = s_j^{-1} \delta(s_j^{-1})^{-1} = s_j^{-1} s_j e
\]

then

\[
\langle \delta(f_{z_{ij}}), f_{z_{ij}} \rangle = \sum_{v \in \text{cent}(t_{ij})} \langle \delta_{s_j^{-1} s_j^{-1} v s_j s_j^{-1} \delta_{s_j^{-1} s_j^{-1}}} \rangle = \sum_{v \in \text{cent}(t_{ij})} \delta_{s_j s_j^{-1} s_j^{-1} v s_j s_j^{-1}} = \delta_{s_i, s_j} = \delta_{i, j}
\]
where we use the fact that for \( v \in \text{cent}(t_0) \), \( s_j = v s_i \implies s_j^{-1} t_0^{-1} s_j = s_i^{-1} t_0^{-1} s_i = s_i \) and by definition of \( s_i \) we deduce \( s_j = s_i \). We then rewrite the results in Theorem 3.2, Corollary 3.3 and Proposition 3.4 using the previous Lemma 4.4 the \( G-M \) bigrading and pairing above to obtain the results as stated.

One may verify that the restriction to \( k(G) \) of the differential calculus \( (m_0, C_{t_0}) \) on \( D^*(G) \) is exactly the differential calculus defined on \( k(G) \) by \( C_{t_0} \) in Proposition 3.6(i) after suitable matching of the conventions. These results from our theory for bicrossproducts are in agreement with calculi on \( D^*(G) \) that can be constructed by entirely different methods \([7]\) via its coquasitriangular structure.

### 4.2. The case \( X = G \triangleleft M \) with \( G \) Abelian.

It is known \([19]\) that if \( G \) is Abelian then \( kG \cong k(\hat{G}) \) and equivalently \( k(G) \cong k(G) \), where \( \hat{G} \) is the group of characters of \( G \). Then

\[
A = k(M) \triangleright \triangleright kG = k(M) \triangleright \triangleright k(G) \cong k(M \triangleright \triangleright \hat{G}).
\]

The product in \( M \triangleright \triangleright \hat{G} \) is

\[
(t, \psi)(s, \phi) = (ts, (\psi \triangleright s)\phi)
\]

where we denote the element \((t, \psi)\) by \( t, \psi \), using factorization notation. The action of \( M \) on \( \hat{G} \) is

\[
(\psi \triangleright s)(u) = \psi(s^{-1}u), \quad \forall s \in M, \forall u \in G.
\]

Explicitly an element \( f \in k(G) \) is viewed as

\[
\hat{f} = \sum_{\phi \in \hat{G}, u \in G} \frac{1}{|G|} \phi(u^{-1}) f(u) \phi \in k\hat{G}
\]

while \( \phi \in \hat{G} \) is viewed as

\[
\hat{\phi} = \sum_{u \in G} \phi(u) \delta_u \in k(G)
\]

This induces Hopf algebras (Fourier) isomorphisms

\[ F : k(M) \triangleright \triangleright kG \rightarrow k(M \triangleright \triangleright \hat{G}), \quad F^* : k(M \triangleright \triangleright \hat{G}) \rightarrow kM \triangleright \triangleright k(G) \]

defined by

\[
F(\delta_t \otimes v) = \sum_{\chi \in \hat{G}} \chi(v) \delta_{t, \chi}, \quad F^{-1}(\delta_{t, \chi}) = \sum_{u \in G} \frac{1}{|G|} \chi(u^{-1}) \delta_t \otimes u
\]

and

\[
F^*(s, \chi) = \sum_{u \in G} \chi(u) s \otimes \delta_u, \quad F^{*-1}(s \otimes \delta_u) = \sum_{\chi \in \hat{G}} \frac{1}{|G|} \chi(u^{-1}) s \chi
\]

\( \forall t, s \in M, \forall u \in G, \forall \chi \in \hat{G} \).

Since \( A \) is isomorphic to the algebra of functions on a group, it follows that the irreducible bicovariant differential calculi on \( A \) from the general theory above must correspond to nontrivial conjugacy classes of \( M \triangleright \triangleright \hat{G} \). We now exhibit this correspondence as follows:

For the first direction, let \( \hat{C}_0 \) be a nontrivial conjugacy class of \( t_0, \psi_0 \) in \( M \triangleright \triangleright \hat{G} \). This class defines an irreducible bicovariant differential calculus on \( A = k(M \triangleright \triangleright \hat{G}) \) whose quantum tangent space is

\[
L = k.\{ a - e, a \in \hat{C}_0 \} \subset \ker \varepsilon \subset A^* = kM \triangleright \triangleright k(G).
\]
From this we determine $\mathcal{M} \subset X$ as
\[ k\hat{1} \oplus \mathcal{M} = \Pi^{-1}(L) \quad \text{(by Theorem 2.6)} \]
Then we take as conjugacy class $C_{\mathcal{M}}X$ in $Z$ that determined by the conjugacy class $C_{0}^{\mathcal{M}}$ of $t_{0}^{-1}$ in $M$, namely $C_{0}^{\mathcal{M}}$, and we define $\mathcal{M}_{0}$ using Proposition 2.5 as
\[ \mathcal{M}_{0} = \mathcal{M}_{0} \triangleright \delta t_{0}^{-1}. \]
One then verifies easily that $\mathcal{M}_{0} \triangleright \delta t_{0}^{-1}$ is nonzero as required in Lemma 2.5 and $C_{\mathcal{M}}X$ defined above does not depend on the chosen element in $\hat{C}_{0}$.

For the second direction, we suppose that we are given a nonzero irreducible bicovariant differential calculus on $A$ defined (say) by an irreducible subrepresentation $\mathcal{M} \subset kX$ under the action of $D(X)$. We need to construct a conjugacy class $\hat{C} \subset M \triangleleft <\hat{G}$ such that the differential calculus defined on $A$ by $\hat{C}$ coincides with that defined by $\mathcal{M}$, i.e.,

\[ k.(\hat{C} - e) := k\{a - e, a \in \hat{C}\} = \Pi(\mathcal{M}) \]
as quantum tangent spaces in $H = A^\star$. First of all, we note that $H$ is the group algebra $k.M \triangleleft <\hat{G}$ so that $\ker \varepsilon_{H}$ is generated as vector space by the set
\[ B_{\varepsilon_{H}} = \{t.\psi - e, \ t \in M, \psi \in \hat{G}\}. \]
Since $\Pi(\mathcal{M}) \subset \ker \varepsilon_{H}$, for all $m \in M$, $\Pi(m)$ is linear combination of elements of $B_{\varepsilon_{H}}$. In general, not all of such elements are necessary to span $\Pi(\mathcal{M})$, so let us denote by
\[ B_{\mathcal{M}} = \{t_{i}.\psi_{j} - e, \ t_{i} \in M, \psi_{j} \in \hat{G}, \ (i, j) \in I \times J\} \]
a minimal set of elements of $B_{\varepsilon_{H}}$ such that
\[ \Pi(\mathcal{M}) \subset k.B_{\mathcal{M}}. \]
A long but not difficult computation using Fourier isomorphisms above shows that $k.B_{\mathcal{M}} = \Pi(\mathcal{M})$ and any conjugacy class $C$ in $M \triangleleft <\hat{G}$, of an element $t_{1}.\psi_{1}$ such that $t_{1}.\psi_{1} - e \in \Pi(\mathcal{M})$ obeys
\[ k.(\hat{C} - e) = \Pi(\mathcal{M}) \]
as expected.

5. Canonical calculi on crossproducts $k(M) \triangleleft <kG$

We now consider the complementary special case where $X = G \triangleleft <M$ and $A = k(M) \triangleleft <kG$, a cross product Hopf algebra. We show that conjugacy classes in $M$ which are invariant under the right action of $G$ define canonical bicovariant differential calculi on both $k(M)$ and $A$ such that the calculus on $A$ is an extension of the one on $k(M)$. This gives a natural way to define bicovariant differential calculi on the double $D(G)$ of any finite group $G$.

**Proposition 5.1.** Let $X = G \triangleleft <M$ be a semi-direct factorization. For any $G$-invariant conjugacy class $C$ of $M$, when it exists, we set
\[ \mathcal{M} = \bigoplus_{a \in C} k\{\sum_{v \in G} v.a^{-1}.\psi^{-1}\} \subset kX \]
Then
\[ (i) \quad \text{The vector space } \mathcal{M} \text{ is isomorphic to an irreducible quantum tangent space in } kM \triangleleft <k(G), \text{ precisely it is of the form } \mathcal{M} = \bigoplus_{z \in C} \mathcal{M}_{0} \triangleright z \text{ as above.} \]
Let \( \mathcal{C} \) be a nontrivial \( G \)-invariant conjugacy class in \( M \) and let \( t_0 \in C \) and \( X_{t_0} \) the centralizer of \( t_0 \). Then
\[
\eta^{-1}(t_0) = \{ v \in X, t^{-1}v = t_0 \} = \{ v(t_0^{-1}uv^{-1}), \ v \in G \}
\]
Let us \( \in X_{t_0} \) we have
\[
ust_0 = t_0us \implies t_0^{-1}uv^{-1} = ut_0^{-1}u^{-1} = (s^{-1}uv^{-1})t_0^{-1}(svu^{-1})
\]
so that the action of \( us \in X_{t_0} \) on \( v(t_0^{-1}uv^{-1}) \in \eta^{-1}(t_0) \) is
\[
v(t_0^{-1}uv^{-1})us = vu((s^{-1}uv^{-1})v^{-1})(t_0^{-1}uv^{-1})(svu^{-1}uv^{-1}) = vu(v(t_0^{-1}uv^{-1})v^{-1}) \in \eta^{-1}(t_0)
\]
This proves that the one-dimensional vector space
\[
\mathcal{M}_{t_0} = k \sum_{v \in G} v(t_0^{-1}uv^{-1})
\]
is an irreducible \( X_{t_0} \)-module.
For any \( a \in C \subset M \), we fix \( \bar{a} \in M \) such that \( a = \bar{a}^{-1}t_0\bar{a} \) and set
\[
\mathcal{M} = \bigoplus_{a \in C} \mathcal{M}_{t_0}\bar{a}.
\]
It is clear that the \( G \)-invariance of \( C \) implies that \( C \) is also a conjugacy class in \( X \) hence by Proposition 2.4 and Theorem 2.6 \( \mathcal{M} \) is isomorphic to a quantum tangent space in \( kM\triangleright \langle k \rangle \).
Furthermore we easily obtain
\[
\mathcal{M}_{t_0}\bar{a} = k \sum_{v \in G} v(a^{-1}t_0^{-1}a^{-1}uv^{-1}) = k \sum_{v \in G} v(a^{-1}uv^{-1})
\]
then
\[
\mathcal{M} = \bigoplus_{a \in C} k \sum_{v \in G} v(a^{-1}uv^{-1}).
\]
For the Cartan calculus of the differential calculus associated to \( \mathcal{M} \), we choose the canonical basis \( (f_a)_{a \in C} \) defined by
\[
f_a =: \sum_{v \in G} v(a^{-1}uv^{-1})
\]
then it is clear that
\[
(f_a) = a^{-1}, \quad |f_a| = e, \quad 0 < \delta(f_a) \! = \! f_a \! = \! 1, \quad \forall a \in C.
\]
On the other hand, for \( u \in G \) and \( c \in C \),
\[
f_c\bar{a}^{-1} = \sum_{v \in G} vu^{-1}(c^{-1}uv^{-1}) = \sum_{w \in G} w((c^{-1}uv^{-1})vw^{-1}) = f_{c^{-1}uv^{-1}} = f_{ucu^{-1}}
\]
so that 
\[ e_a * u = \sum_{c \in C} e_c \cdot u_{c^{-1}} - 1 e = e_{u^{-1} a u}. \]

Then the Cartan calculus from Theorem 3.2 reads 
\[ e_a \delta s = \delta s e_a, \quad e_a u = u e_{u^{-1} a u} \]
\[ d \delta s = \sum_{a \in C} (\delta s a - \delta s) e_a, \quad du = \sum_{a \in C} u(e_{u^{-1} a u} - e_{a}) \]

It is now clear that this canonical differential calculus on \( k(M) > kG \) is an extension of the differential calculus defined on \( k(M) \) as in Proposition 3.6 by the opposite conjugacy class \( C^{-1} \) of \( C \).

6. Exterior algebra and cohomology computations

We have already seen that \( \theta \) in Corollary 3.5 is a nontrivial element of the noncommutative de Rham cohomology for any bicrossproduct. In this section we will glean more insight into the cohomology through a close look at particular bicrossproducts. From a physical point of view this is the beginning of 'electromagnetism’ on such spaces. From a noncommutative geometers point of view it is the 'differential topology' of the algebra equipped with the differential structure. Note that very little is known in general about the full noncommutative de Rham cohomology even for finite groups, but insight has been gained through examples such as in [14]. We are extending this process here.

In particular, just as for a Lie algebra there is a unique differential structure giving a connected and simply connected Lie group, so we might hope for a ‘natural’ if not unique choice of calculus such that at least \( H^0 = k \cdot 1 \), which is a connectedness condition (so that a constant function is a multiple of the identity) and with small \( H^1 \). By looking at several examples and using our explicit Cartan relations for bicrossproducts, we find that a phenomenon of this type does appear to hold. In particular, as a main result of the paper from a practical point of view, we find a unique such calculus on the quantum double \( D(S_3) \) viewed as a bicrossproduct, i.e. a natural choice for its differential geometry. We also cover the codouble \( D^*(S_3) \) as another bicrossproduct.

In each case studied here, we describe the factorizing groups, the set \( Z \) and hence the classification of calculi. We then compute the first order calculi in each case using the theory above, and the braiding on basic forms \( \{ e_a \} \) dual to the basis \( \{ f_a \} \) stated in each case of the quantum tangent space yielded by the classification. In each case,
\[ \Omega^1(A) = A \otimes \Lambda^1, \quad \Lambda^1 := \langle e_a \rangle_k \]
where \( A \) is the bicrossproduct Hopf algebra and \( < >_k \) denotes the \( k \)-span. In describing the exterior derivative we use the translation and ‘finite difference’ operators
\[ R_s(f)(r) = f(rs); \quad (\partial_s f)(r) = f(rs) - f(r), \quad \forall r \in M, \quad f \in k(M) \]
for the relevant group \( M \) and relevant \( s \in M \), as already used elsewhere.

From the braiding we then compute the higher order differential calculus using the braided factorial matrices \( A_n \) given by
\[ A_n = (\text{id} \otimes A_{n-1})(n, -\Psi), \quad [n, -\Psi] = \text{id} - \Psi_{12} + \Psi_{12}\Psi_{23} + \ldots + (-1)^{(n-1)}\Psi_{12}...\Psi_{n-1,n} \]
where $\Psi_{i,i+1}$ denotes $\Psi$ acting in the $i, i + 1$ positions in $\Lambda^1 \otimes \Lambda^n$. The space $\Lambda^n$ of invariant $n$-forms is then the quotient of $(\Lambda^1)^{\otimes n}$ by $\ker A_n$. This is the computationally efficient braided groups approach used in \[\text{[4, 5, 6]}\] and equivalent to the original Woronowicz description of the antisymmetrizers in \[\text{[18]}\]. These braided integer matrices have also been adopted by other authors, such as \[\text{[21]}\].

6.1. **Calculi and cohomology on** $k(\mathbb{Z}_2)\quad <\quad k\mathbb{Z}_3$. This baby example $k(\mathbb{Z}_2)\quad <\quad k\mathbb{Z}_3$ is actually a semidirect coproduct isomorphic to $k(S_3)$ and among other things demonstrates the Fourier theory in Section 4.2. From the theory of calculi on finite groups, we know that there are two irreducible calculi of dimensions 2, 3 respectively, according to the nontrivial conjugacy classes of $S_3$. We illustrate how this known result comes about in our bicrossproduct theory.

Here, $X = S_3$ factorizes into $M = \mathbb{Z}_2 = \{e, s\}$ and $G = \mathbb{Z}_3 = \{e, u, u^2\}$, where $s = (12), u = (123)$. The right action of $G$ on $M$ is trivial and the left action of $M$ on $G$ is defined by $\sigma = (u, u^2)$ (the permutation). The set $Z$ of elements $|x|$ is $Z = \{e, s, us, u^2s\}$ which splits into two conjugacy classes $C^X = \{e\}$ and $C^X = \{s, us, u^2s\}$. This leads to the following irreducible bicovariant calculi.

\[(i) \quad C^X = \{e\}, M = \langle f_1, f_2 \rangle_k, \text{ where } q = e^{2\pi i / 3} \text{ and} \]
\[f_1 = e + q^2 u + qu^2, \quad f_2 = e + qu + q^2 u^2 \]
\[e_a f = f e_a, \quad \forall f \in k(\mathbb{Z}_2); \quad a = 1, 2 \]
\[e_1 u = q^2 u e_1, \quad e_1 u^2 = qu e_1, \quad e_2 u = que_2, \quad e_2 u^2 = q^2 u^2 e_2 \]
\[\theta = e_1 + e_2 \]
\[df = 0, \forall f \in k(\mathbb{Z}_2), \quad du = u(q^2 - 1)e_1 + u(q - 1)e_2, \quad du^2 = u^2(q - 1)e_1 + u^2(q^2 - 1)e_2 \]
\[\Psi(e_a \otimes e_b) = e_b \otimes e_a, \quad a, b = 1, 2. \]

The exterior algebra has the usual relations and dimensions
\[e_a^2 = 0, \quad e_1 \wedge e_2 = - e_2 \wedge e_1, \quad \dim(\Omega) = 1 : 2 : 1. \]

The cohomology can be identified with
\[H^0 = k(\mathbb{Z}_2), \quad H^1 = k(\mathbb{Z}_2)e_1 \oplus k(\mathbb{Z}_2)e_2, \quad H^2 = k(\mathbb{Z}_2)e_1 \wedge e_2 \]
with dimensions 2:4:2.

\[(ii) \quad C^X = \{s, us, u^2s\}, M = \langle f_1, f_2, f_3 \rangle_k, \text{ where} \]
\[f_1 = s, \quad f_2 = us, \quad f_3 = u^2s \]
\[e_a f = R_s(f)e_a, \quad \forall f \in k(\mathbb{Z}_2), \quad a = 1, 2, 3 \]
\[e_1 u = u^2 e_2, \quad e_1 u^2 = u e_3, \quad e_2 u = u^2 e_3, \quad e_2 u^2 = u e_1, \quad e_3 u = u^2 e_1, \quad e_3 u^2 = u e_2 \]
\[\theta = e_1, \quad df = \partial_s(f)e_1, \quad du = u^2 e_2 - u e_1, \quad du^2 = u e_3 - u^2 e_1 \]
\[\Psi(e_1 \otimes e_1) = e_1 \otimes e_1, \quad \Psi(e_2 \otimes e_1) = e_1 \otimes e_2, \quad \Psi(e_3 \otimes e_1) = e_1 \otimes e_3 \]
\[\Psi(e_1 \otimes e_2) = e_2 \otimes e_3, \quad \Psi(e_2 \otimes e_2) = e_3 \otimes e_1, \quad \Psi(e_3 \otimes e_2) = e_3 \otimes e_2 \]
\[\Psi(e_1 \otimes e_3) = e_2 \otimes e_2, \quad \Psi(e_2 \otimes e_3) = e_2 \otimes e_3, \quad \Psi(e_3 \otimes e_3) = e_2 \otimes e_1. \]
The exterior algebra is quadratic with relations
\[ e_1 \land e_1 = 0, \quad e_2 \land e_3 = 0, \quad e_3 \land e_2 = 0, \quad e_1 \land e_2 + e_2 \land e_1 + e_1^2 = 0, \quad e_1 \land e_3 + e_3 \land e_1 + e_1^2 = 0 \]
and has dimensions and cohomology:
\[ \dim(\Omega) = 1 : 3 : 4 : 3 : 1 \]
\[ H^0 = k.1, \quad H^1 = k.e, \quad H^2 = 0, \quad H^3 = k.e^3, \quad H^4 = k.e_1 \land e_2 \land e_2 \land e_2. \]

This is isomorphic to the cohomology and calculus on \( k(S_3) \) studied in [4]. We see that this is the unique choice \((ii)\) with \( H^0 = k.1 \). We also see that both calculi exhibit Poincaré duality.

6.2. Calculi and cohomology on \( k(S_3) \rhd kZ_6 \). The second example \( k(S_3) \rhd kZ_6 \) is a nontrivial bicrossproduct [20] but is (nontrivially) isomorphic to a version of the dual of a quantum double \( D^*(S_3) = k(S_3) \rhd kS_3 \). Among other things, it demonstrates our results for the codouble in Section 4.1.

Here \( X = S^3 \times S^3 \) factorizes differently into groups
\[ G = Z_6 = \{ u^0, u, u^2, u^3, u^4, u^5 \}, \quad M = S_3 = \{ e, s, t, t^2, st, st^2 \} \]
where \( s = (12), t = (132) \) and \( u \) is the generator of \( Z_6 \). The right action of \( u \) on \( M \) is the permutation
\[ u = (st, st^2)(t, t^2) \]
while the left action of \( M \) on \( G \) is given completely in terms of permutations by
\[
\begin{align*}
\varepsilon & = \text{id}, \quad s \triangleright = (u^0, u^5)(u^2, u^4), \quad t \triangleright = (u^5, u^3, u), \quad t^2 \triangleright = (u, u^3, u^5) \\
st \triangleright & = (u^2, u^4)(u^0, u^3), \quad st \triangleright = (u^2, u^4)(u, u^3)
\end{align*}
\]
For \( X = Z_6, S_3 \) the set of the values of \( ||.|| \) is
\[ Z = \{ e, s, t, t^2, st, st^2, u^2 s, u^4 s, u^2 t, u^4 t, u^2 s t, u^4 s t, u^2 s t^2, u^4 s t^2 \} \]
which splits into three conjugacy classes
\[ C^X = \{ 0 \}, \quad C^X = \{ t^2, u^2 t, u^4 t \}, \quad C^X = \{ s, st, st^2, u^2 s, u^4 s, u^2 s t, u^4 s t, u^2 s t^2, u^4 s t^2 \}. \]
If we choose the respective basis points to be \( e = e, t, s \) then we have
\[ N^{-1}(e) = G, \quad N^{-1}(t) = \{ t^2, u^2 t, u^4 t \}, \quad N^{-1}(s) = \{ s, u^3 s \} \]
The centralizers of \( e, t, s \) in \( X \) are respectively
\[ X_e = X, \quad G_t = \{ e, t, t^2, u^2 s, u^4 s, u^2 t, u^4 t \}, \quad G_s = \{ e, s, u^3 s, u^3 s \} \]
Applying the general theory of Sections 4 and 5 to these data leads to the irreducible bicovariant differential calculi on \( A = k(S_3) \rhd kZ_6 \) as follows:

\[ (i) \quad C^X = \{ e \}, \quad M = \langle f_1 \rangle, \quad \text{where } f_1 = u^0 - u + u^2 - u^3 + u^4 - u^5.
\]
\[ e_1 f = f e_1, \quad \forall f \in k(S_3), \quad e_1 u^i = (-1)^i u^i e_1, \quad \forall u^i \in G
\]
\[ \theta = e_1, \quad df = 0, \quad \forall f \in k(S_3), \quad du^i = (-1 + (-1)^i) u^i e_1 \]
\[ \Psi(e_1 \otimes e_1) = e_1 \otimes e_1 \]

\[ (ii) \quad C^X = \{ e \}, \quad M = \langle f_1, f_2, f_3, f_4 \rangle, \quad \text{where setting } q = e^{-2u^2},
\]
\[ f_1 = u^0 + qu + q^2 u^2 - u^3 - qu^4 - q^2 u^5, \quad f_2 = u^0 + q^2 u - qu^2 + u^3 + q^2 u^4 - qu^5
\]
\[ f_3 = u^0 - qu + q^2 u^2 + u^3 - qu^4 + q^2 u^5, \quad f_4 = u^0 - q^2 u - qu^2 - u^3 + q^2 u^4 + qu^5
\]
\[ e_a f = f e_a, \quad \forall f \in k(S_3), \quad a = 1, 2, 3, 4, \quad e_1 u^j = q^j u^j e_1, \quad e_2 u^j = q^{2j} u^j e_2,
\]
\[e_3 u^j = q^{4j} u^j e_3, \quad e_4 u^j = q^{5j} u^j e_4, \quad \theta = e_1 + e_2 + e_3 + e_4\]

\[df = 0, \forall f \in k(S_3), \quad du^j = u^j ((q^j - 1)e_1 + (q^{2j} - 1)e_2 + (q^{4j} - 1)e_3 + (q^{5j} - 1)e_4)\]

\[\Psi(e_a \otimes e_b) = e_b \otimes e_a, \quad a, b = 1, 2, 3, 4\]

\[(iii)-(v) \quad C^X = \{t, t^2, u^2 t, u^4 t^2\}, M = \langle f_1, f_2, f_3, f_4 \rangle_k, \text{ where } q = 1, e^{2\pi i}, e^{-2\pi i}\]

for the three cases and

\[f_1 = t^2 + qu^2 t^2 + q^2 u^4 t^2, \quad f_2 = t + q^2 u^2 t + qu^4 t,\]

\[f_3 = u^2 t + q^3 u^2 t^2 + q^2 u^5 t^2, \quad f_4 = u^5 t + q^2 u t + qu^3 t\]

\[e_1 f = R_t(f)e_1, \quad e_2 f = R_{t^2}(f)e_2, \quad e_3 f = R_{t^2}(f)e_3, \quad e_4 f = R_t(f)e_4, \quad \forall f \in k(S_3)\]

\[e_1 u^{2j} = q^{-2j} u^{2j} e_1, \quad e_1 u^{2j+1} = q^{-2j} u^{2j+3} e_3,\]

\[e_2 u^{2j} = q^{-j} u^{2j} e_2, \quad e_2 u^{2j+1} = q^{-j} u^{2j+5} e_4,\]

\[e_3 u^{2j} = q^{-2j} u^{2j} e_3, \quad e_3 u^{2j+1} = q^{1-2j} u^{2j+5} e_1,\]

\[e_4 u^{2j} = q^{-j} u^{2j} e_4, \quad e_4 u^{2j+1} = q^{-j} u^{2j+3} e_2\]

\[\theta = e_1 + e_2, \quad df = \partial_t(f)e_1 + \partial_{t^2}(f)e_2\]

\[du^{2j} = (q^{-2j} - 1)u^{2j} e_1 + (q^{-j} - 1)u^{2j} e_2,\]

\[du^{2j+1} = q^{-2j} u^{2j+3} e_3 + q^{-j} u^{2j+5} e_4 - u^{2j+1}(e_1 + e_2)\]

The resulting exterior algebra and cohomology depend on the braiding. In case (iii) we have:

\[\dim(\Omega) = 1 : 4 : 6 : 4 : 1, \quad H^0 = \langle \delta_i u^{2j} \rangle_i = 0, 1; \quad j = 0, 1, 2 \rangle_k\]

\[H^1 = \langle \delta_i u^{2j} e_a \rangle_i = 0, 1; \quad j = 0, 1, 2; \quad a = 1, 2, 3, 4 \rangle_k\]

where the cohomology is 6-dimensional in degree 0 and 24 dimensional in degree 1.

The relations in the exterior algebra are that the forms \(\{e_a\}\) anticommute as usual.

In case (iv) we have the 6 relations

\[e_2^2 = 0, \quad e_1 \wedge e_2 + e_2 \wedge e_1 = 0, \quad e_3 \wedge e_4 + e_4 \wedge e_3 = 0\]

and

\[\dim(\Omega) = 1 : 4 : 10 : 53 : \cdots, \quad H^0 = \langle \delta_i \phi_j u^{2j} \rangle_i = 0, 1; \quad j = 0, 1, 2 \rangle_k\]

\[H^1 = \langle \delta_i \phi_j u^{2j} e_a \rangle_i = 0, 1; \quad j = 0, 1, 2; \quad a = 1, 2 \rangle_k\]

where

\[\phi_i = \sum_{j=0}^{j=2} \delta_i q^{ij}.\]

Here the dimensions of the cohomology are 6 in degree 0 and 12 in degree 1. The case (v) is identical with \(q\) replaced by \(q^{-1}\).

\[(vi) - (vii) \quad C^X = \{s, st, st^2, u^2 s, u^4 s, u^2 st, u^4 st, u^2 st^2, u^4 st^2\},\]

\[M = \langle f_{ai} \rangle a = 1, 2, 3; \quad i \in \mathbb{Z}_3 \rangle_k\]
where \( q = \pm 1 \) for the two cases and:

\[
\begin{align*}
   f_{10} &= s + q u^3 s, & f_{11} &= u^2 s + q u^5 s, & f_{12} &= u s + q u^4 s, & f_{20} &= s t + q u^5 s t^2, & f_{21} &= u^2 s t + q u s t^2 \\
   f_{22} &= u^4 s t + q u^3 s t^2, & f_{30} &= s t^2 + q u s t, & f_{31} &= u^3 s t + q u^2 s t^2, & f_{32} &= u^5 s t + q u^4 s t^2.
\end{align*}
\]

For brevity, we give the details only for the case (the other is similar). Then

\[
\begin{align*}
   e_{1i} f &= R_s(f) e_{1i}, & e_{2i} f &= R_s(f) e_{2i}, & e_{3i} f &= R_{st^2}(f) e_{3i}, & \forall f \in k(S_3) \\
   e_{1i} u^j &= u^{-j} e_{1i-j}, & e_{2i} u^{2k} &= u^{4k} e_{2i+k}, & e_{2i} u^{2k+1} &= u^{4k+1} e_{3i+k} \\
   e_{3i} u^{2k} &= u^{4k} e_{3i+k}, & e_{3i} u^{2k+1} &= u^{4k+3} e_{2i+k+1} \\
   \theta &= e_{10} + e_{20} + e_{30}, & df &= \partial_s(f) e_{10} + \partial_{st}(f) e_{20} + \partial_{st^2}(f) e_{30} \\
   du^{2k} &= u^{-2k} e_{1,k} + u^{4k} e_{2,k} + u^{4k} e_{3,k} - u^{2k} \theta \\
   du^{2k+1} &= u^{-(2k+1)} e_{1,-(2k+1)} + u^{4k+3} e_{2,k+1} + u^{4k+1} e_{3,k} - u^{2k+1} \theta \\
   \Psi(e_{1i} \otimes e_{aj}) &= e_{(23)a-j} \otimes e_{1i-j} \\
   \Psi(e_{2i} \otimes e_{aj}) &= e_{(12)a-j} \otimes e_{2i-j} \\
   \Psi(e_{3i} \otimes e_{aj}) &= e_{(12)a-j} \otimes e_{3i-j}.
\end{align*}
\]

The resulting exterior algebra has relations

\[
\begin{align*}
   e_{aj} \land e_{a,-j} &= 0, & e_{ai}^2 + \{e_{a,i-1}, e_{a,i+1}\} &= 0 \\
   e_{1i} \land e_{2j} + e_{2j-i} \land e_{3,-i} + e_{3,-j} \land e_{1,i-j} &= 0 \\
   e_{2j} \land e_{1i} + e_{3,-i} \land e_{2j-i} + e_{1,i-j} \land e_{3,-j} &= 0
\end{align*}
\]

for \( a = 1, 2, 3 \) and \( i, j \in \mathbb{Z}_3 \). The dimensions of the exterior algebra and cohomology in low degree are

\[
\dim(\Omega) = 1:9:48:198: \cdots, \quad H^0 = k,1, \quad H^1 = k,\theta.
\]

From these explicit computations we conclude in particular:

**Proposition 6.1.** Only the 9-dimensional calculi (vi)–(vii) have \( H^0 = k,1 \)

The natural one here is (vi) where \( q = 1 \) with the other as a signed variant. We also have Poincaré duality at least for all cases where the exterior algebra was small enough to be fully computed. According to [17] this bicrossproduct is a coquasi triangular Hopf algebra, isomorphic to the quantum double \( D^*(kZ_2) \) of our first example in Section 6.1, hence also to the quantum double \( D^*(S_3) = k(S_3) \) of the type covered in Section 4.1. The canonical calculi given in Theorem 4.5 correspond to (iii) and (vi), with (vii) indeed the canonical extension of the natural (3-dimensional) calculus on \( S_3 \) (as in Section 6.1). The other cases fit in their number and dimensions with a completely different classification theorem for factorizable coquasi triangular Hopf algebras [17], which implies that calculi can be classified by representations of the quantum double \( D(S_3) \), with dimension the square of that of the representation. These are labelled by conjugacy classes in \( S_3 \) and representations of the centralizer, giving calculi of dimensions 1, 4, 4, 4 and 9, 9 for the three classes. We see that we obtain isomorphic results from our bicrossproduct classification (the isomorphism is nontrivial, however).
6.3. Calculus and cohomology on $D(S_3)$. Finally we consider the dual example to the preceding one, with “coordinate ring” the cross product $k(S_3)\times kS_3 = D(S_3)$ in the same conventions as for $D(X)$ in Section 2. This corresponds to the semidirect factorization in Section 5, namely $X=S_3\times S_3$ with action by conjugation.

Here $G = M = S_3$, thus we use the same notation as in Section 6.1 namely elements of $S_3.e$ are underlined, those of $e.S_3$ are not, so that a general element of $X = S_3.S_3$ is of the form $v,t$. We set again $S_3 = \{e, u, u^2, s, us, u^2s\}$. The right adjoint action of $G = S_3.e$ on $M = e.S_3$ is given by

\[
\varrho_u = (s, us, u^2s), \quad \varrho_s = (u, u^2)(us, u^2s)
\]

while the left action of $e.S_3$ on $S_3.e$ is trivial. The set $Z$ is $e.S_3$. It splits into three conjugacy classes $C^X$, namely $\{e\}$ and $\{u, u^2\}$ and $\{s, us, u^2s\}$.

Following the general theory in Sections 2, we obtain in fact eight non-isomorphic irreducible bicovariant differential calculi on $k(S_3)\times kS_3$ as follows: (i)–(ii) for $C^X = \{e\}$ we have one calculus of dimension 1 and one of dimension 2. (iii)–(v) for $C^X = \{u, u^2\}$ we have two calculi of dimension 2 and one of dimension 4. (iv)–(viii) for $C^X = \{s, us, u^2s\}$ we have two calculi of dimension 3 and one of dimension 6. We omit details for most of these calculi since they are similar in complexity and flavour to Section 6.3, limiting ourselves to the most interesting one (viii) only:

\[
\begin{align*}
\{(viii)\} & \quad C^X = \{s, us, u^2s\}, \quad M = <f_0, f_1, f_2, f_3, f_4, f_5>_k, \quad \text{where } q = e^{2\pi i}\ \\
& \quad f_0 = e.s + q^2 u.s + u^2.sus, \quad f_1 = u.s + q^2 u^2.s + qe.us \\
& \quad f_2 = u^2.s + q^2 e.s + u^2.sus, \quad f_3 = s.s + q^2 s.us + q^2 us.s^2 \\
& \quad f_4 = u^2.s.s + qus.us + q^2 s.s^2 + u^2.s^2us \\
\end{align*}
\]

We have commutation relations

\[
e_i f = R_{u^is}(f)e_i, \quad i \in \mathbb{Z}_6, \quad f \in k(S_3)
\]

\[
e_0 s = se_3, \quad e_3 s = se_0, \quad e_i s = se_{i-1}, \text{ for } i \neq 0, 3
\]

\[
e_2 u = ue_0, \quad e_3 u = ue_3, \quad e_i u = ue_{i+1}, \text{ for } i \neq 2, 5
\]

and exterior differentials

\[
d(f) = \partial_s(f)e_0 + q\partial_{u^s}(f)e_1 + q^2\partial_{us}(f)e_2 \\
\theta = e_0 + qe_1 + q^2 e_2, \quad d(u^s) = (q^{2i} - 1)u^s\theta + q^{2i}s(e_3 - e_0) + q^{2i+1}s(e_5 - e_1) + q^{2i+2}s(e_4 - e_2), \quad i = 0, 1, 2.
\]

The braiding is

\[
\begin{align*}
\Psi(e_i \otimes e_0) = qe_0 \otimes e_i, & \quad \Psi(e_i \otimes e_1) = q^2 e_0 \otimes e_i, & \quad \Psi(e_i \otimes e_2) = e_2 \otimes e_i, \\
\Psi(e_i \otimes e_3) = q^2 e_4 \otimes e_i, & \quad \Psi(e_i \otimes e_4) = qe_3 \otimes e_i, & \quad \Psi(e_i \otimes e_5) = e_5 \otimes e_i, \quad i = 2, 5 \\
\Psi(e_i \otimes e_0) = q^2 e_2 \otimes e_i, & \quad \Psi(e_i \otimes e_1) = e_1 \otimes e_i, & \quad \Psi(e_i \otimes e_2) = qe_0 \otimes e_i, \\
\Psi(e_i \otimes e_3) = qe_5 \otimes e_i, & \quad \Psi(e_i \otimes e_4) = e_4 \otimes e_i, & \quad \Psi(e_i \otimes e_5) = qe_3 \otimes e_i, \quad i = 1, 4 \\
\Psi(e_i \otimes e_0) = e_0 \otimes e_i, & \quad \Psi(e_i \otimes e_1) = qe_2 \otimes e_i, & \quad \Psi(e_i \otimes e_2) = q^2 e_1 \otimes e_i, \\
\Psi(e_i \otimes e_3) = e_3 \otimes e_i, & \quad \Psi(e_i \otimes e_4) = qe_5 \otimes e_i, & \quad \Psi(e_i \otimes e_5) = qe_4 \otimes e_i, \quad i = 0, 3
\end{align*}
\]

and yields the degree 2 exterior algebra $\Omega^2(D(S_3))$ as 21-dimensional with relations

\[
e_i \wedge e_i = 0, \quad i \in \mathbb{Z}_6, \quad \{e_i, e_{i+3}\} = 0, \quad i = 0, 1, 2 \quad e_0 \wedge e_2 + qe_2 \wedge e_1 + q^2 e_1 \wedge e_0 = 0, \quad e_1 \wedge e_2 + qe_0 \wedge e_1 + q^2 e_2 \wedge e_0 = 0, \\
e_5 \wedge e_3 + qe_3 \wedge e_4 + q^2 e_4 \wedge e_5 = 0, \quad e_5 \wedge e_4 + qe_3 \wedge e_5 + q^2 e_4 \wedge e_3 = 0
\]
\[ e_5 \wedge e_0 + e_3 \wedge e_4 + e_1 \wedge e_2 + q(e_0 \wedge e_4 + e_1 \wedge e_5 + e_2 \wedge e_3) = 0 \]
\[ e_0 \wedge e_5 + e_1 \wedge e_3 + e_2 \wedge e_4 + q(e_4 \wedge e_0 + e_5 \wedge e_1 + e_3 \wedge e_2) = 0. \]

There are further relations in degree 3, i.e. the entire Woronowicz exterior algebra in this example is not quadratic. Its dimensions and cohomology in low degree are

\[ \dim(\Omega) = 1 : 6 : 21 : 60 : 152 : \cdots, \quad H^0 = k.1 \quad H^1 = k.\theta \oplus k.\bar{\theta} \]

where \( \bar{\theta} = e_3 + q^{-1}e_4 + q^{-2}e_5. \)

From these and similar computations for all the other calculi (along the lines in Section 6.2) we find:

**Proposition 6.2.** The 6-dimensional calculus (viii) is the unique irreducible calculus with \( H^0 = k.1. \)

We also have Poincaré duality at least where the exterior algebra was small enough to be fully computed. For example, for (vi)-(vii) the dimensions of \( \Omega \) are 1:3:4:3:1 and the dimensions of the cohomology are 6 : 6 : 0 : 6 : 6.

The quantum double \( D(S_3) \) is interesting for many reasons. Let us note that being quasitriangular, it has a universal \( R \)-matrix or quasitriangular structure \( R \) which controls the noncocommutativity. This in turn is the nonAbelianness of the underlying noncommutative group if one views \( D(S_3) \) as a function algebra, so should correspond to Riemannian curvature in the setting of [3]. Our result is that there is a unique irreducible calculus to take for this geometry. The ensuing noncommutative Riemannian geometry will be developed elsewhere.

Also, from a mathematical point of view, \( D(S_3) \) is a cotwist by a multiplication-altering cocycle of the tensor product \( k(S_3) \otimes kS_3 \), its differential calculi can also be obtained from those of the tensor product \( k(S_3) \otimes kS_3 \) by cotwisting the exterior algebra according to the cotwisting theorem in [22]. This means that the classification of differential calculi and their cohomology for \( D(S_3) \) is exactly the same as for the tensor product covered in Proposition 2.7. Also note that until now the main example of a nontrivial bicrossproduct in [17] was \( k(Z_6) \triangleright kZ_6 \). We find, however, that this is actually isomorphic as a Hopf algebra to \( (k(Z_2) \triangleright kZ_3) \otimes (k(Z_3) \triangleright kZ_2) \)

i.e. to the tensor product \( k(S_3) \otimes kS_3 \) again, hence has the same features via twisting as \( D(S_3) \). Similarly, replacing \( Z_6.S_3 \) in Section 6.2 by the opposite factorization \( S_3.Z_6 \) leads to the dual bicrossproduct Hopf algebra \( k(Z_6) \triangleright kS_3 \) which by Proposition 2.1 in [17] is isomorphic to the quantum double \( D(S_3) \) again. Therefore several other known bicrossproducts reduce to or have the same features as \( D(S_3) \) above.

**Acknowledgements.** Main results and part of the writing up was done when F.N was visiting Queen Mary, University of London in the summer of 2002; he thanks the department there, D. Lambert from FUNDP Belgium and J-P. Antoine from UCL, Belgium for organizing the visit for collaboration.

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