Optimal Communication Complexity of Byzantine Consensus under Honest Majority

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Abstract

Communication complexity is one of the most important efficiency metrics for distributed algorithms, but considerable gaps in the communication complexity still exist for Byzantine consensus, one of the most fundamental problems in distributed computing. This paper provides three results that help close some of these gaps. (1) We present a Byzantine consistent broadcast (BCB) protocol with linear communication complexity when \( f \leq (1/2 - \varepsilon)n \) where \( \varepsilon \) is any positive constant. The new protocol relies on an expander graph and a threshold signature scheme. (2) The linear BCB protocol is used to obtain quadratic Byzantine broadcast (BB) and Byzantine agreement (BA) under the same \( f \leq (1/2 - \varepsilon)n \) resilience (which is close to optimal for BA). (3) We also show a quadratic communication lower bound of BCB with \( f \geq (1/2 + \varepsilon)n \).

1 Introduction

Byzantine consensus\(^1\) is one of the most fundamental problems in distributed algorithms [21]. It also serves as an important building block in cryptography and distributed systems. At a high level, Byzantine consensus is the problem for \( n \) parties to agree on a value, despite that up to \( f \) of them may behave arbitrarily (called Byzantine faults). Arguably the most important efficiency metric of Byzantine consensus is the communication complexity, since communication will be the bottleneck in applications like multi-party computation and cryptocurrency. Despite decades of research, there are still significant gaps in the lower and upper bounds of communication complexity of Byzantine consensus. This paper aims at filling some of these gaps.

Before describing our results, we first note that we will focus on the synchronous, authenticated (i.e., with digital signatures and public-key infrastructure), and deterministic setting in this paper. Focusing on deterministic protocols allows us to focus on the simpler and arguably more fundamental worst-case (as opposed to expected) communication complexity. It is then natural to assume synchrony: under asynchrony, many consensus problems have no deterministic solutions [14]. Lastly, the reason to focus on authenticated protocols is two-fold: (i) they are more widely used in practice [3] [27], and (ii) the communication complexity for unauthenticated Byzantine broadcast and agreement has already reached optimal [6] (though gaps still exist for weaker primitives).

The most widely studied Byzantine consensus problems are Byzantine Agreement (BA) and Byzantine Broadcast (BB). Their optimal communication complexity has been an open question

\(^1\) We use the term “consensus” to refer to all variants of agreement and broadcast problems in this paper.
for a very long time. Dolev and Reischuk proved that a quadratic number of messages are necessary for deterministic BB/BA protocols [10]. But for decades, the best known protocol in the synchronous, authenticated and deterministic setting is the classic Dolev-Strong protocol [12], which uses quadratic messages but cubic communication. The reason is that in Dolev-Strong, the messages can contain up to \( f + 1 \) signatures. While randomized BA/BB protocols can achieve expected quadratic [13, 17, 23, 4, 2] or even sub-quadratic [18, 9, 4] communication, they do not close the gap on the worst-case communication complexity. On a side note, most of the sub-quadratic randomized protocol reduce communication by sampling a smaller committee to run consensus; while effective, such techniques shed less insights on the “inherent” communication complexity of the consensus problems.

Only recently, Yin et al. [27] provided the necessary tools to achieve quadratic communication for deterministic BA/BB with \( f < n/3 \). This still leaves open whether BA/BB can be solved with quadratic worst-case communication if \( f \geq n/3 \). The main result of this paper is to answer this question in the affirmative if \( f \leq (\frac{1}{2} - \varepsilon)n \) for any positive constant \( \varepsilon \).

A key stepping stone for our new upper bounds is a linear communication protocol for Byzantine Consistent Broadcast (BCB) with \( f \leq (\frac{1}{2} - \varepsilon)n \). BCB is perhaps one of the easiest problems in the Byzantine consensus family. It only requires honest parties not to decide on conflicting values; it is OK that some honest parties decide but others do not. Our key new technique is to utilize an expander graph to check for inconsistent values. An expander has a constant number of edges per vertex—ensuring constant communication per party and linear communication in total—and good connectivity to detect inconsistent values effectively. Another important building block is the well-established use of threshold signature [7, 4, 27].

Back to BA/BB, we can now repeat the linear BCB \( f + 1 = O(n) \) times, each time with a different party serving as the BCB sender. This gives the desired \( O(n^2) \) communication. Some additional techniques including the linear view-change method of Yin et al. [27] are used to transition safely between BCB instances.

On a side note, we feel the importance of BCB may have been somewhat overlooked. The most important application for Byzantine consensus today is Byzantine fault tolerant (BFT) replication systems [26, 25, 8] (more recently called blockchains). Most BFT replication protocols (possibly without realizing it) run a sequence of BCB instances with a few extra steps [8, 19, 27]. In this sense, BB and BA turn out to be overkill for practical protocols. Therefore, we would like to characterize the communication complexity of BCB. Towards this end, besides the above linear protocol for \( f \leq (\frac{1}{2} - \varepsilon)n \), we also establish a lower bound showing that BCB cannot be solved with sub-quadratic communication if \( f \geq (\frac{1}{2} + \varepsilon)n \).

Lastly, toward potentially practical applications, we present a BFT replication protocol with \( f \leq (\frac{1}{2} - \varepsilon)n \) and amortized linear communication. The protocol builds on the new linear BCB protocol and adds some modifications to work without lock-step rounds.

**Summary of results.** Table 1 summarizes the current landscape of Byzantine consensus communication complexity and highlights our new results. We also list the new results below. Let \( \varepsilon \) be any positive constant.

1. A Byzantine consistent broadcast protocol with linear communication with \( f \leq (\frac{1}{2} - \varepsilon)n \);
2. A quadratic communication lower bound for Byzantine consistent broadcast if \( f \geq (\frac{1}{2} + \varepsilon)n \);
3. A Byzantine broadcast protocol and a Byzantine agreement protocol with quadratic communication with \( f \leq (\frac{1}{2} - \varepsilon)n \);
4. A practical BFT replication protocol with amortized linear communication with \( f \leq (\frac{1}{2} - \varepsilon)n \).
Table 1: Upper and lower bounds for worst-case communication complexity of Byzantine consensus. \( \varepsilon \) is any positive constant. We assume \( f = \Theta(n) \). New results of this paper are marked bold.

| Problem                              | Resilience                          | Upper Bound       | Lower bound       |
|--------------------------------------|-------------------------------------|-------------------|-------------------|
| Byzantine Consistent Broadcast (BCB) | \( f < n/3 \) \(< n/2 \) \( (\frac{1}{2} + \varepsilon)n \leq f < n - 1 \) | \( O(n) \) [25]  | \( \Omega(n) \)  |
|                                      |                                     | \( O(n^3) \)      | \( \Omega(n) \)  |
| Byzantine Reliable Broadcast (BRB)   | \( f < n/2 \) \( f < n - 1 \)      | \( O(n^2) \) [16] | \( \Omega(n^2) \) |
|                                      |                                     | \( O(n^3) \) [12] |                   |
| Byzantine Broadcast (BB)             | \( f \leq (\frac{1}{2} - \varepsilon)n \) \( f < n - 1 \) | \( O(n^2) \)      | \( \Omega(n^2) \) |
|                                      |                                     | \( O(n^3) \) [12] |                   |
| Byzantine Agreement (BA)             | \( f < n/3 \) \( f \leq (\frac{1}{2} - \varepsilon)n \) \( f < n/2 \) \( c \) | \( O(n^2) \) [6]  | \( \Omega(n^2) \) |
|                                      |                                     | \( O(n^3) \) [12] |                   |

\( a \) The lower bound for BB in [10] can be easily extended to BRB. We give a proof in Appendix A for completeness.

\( b \) The number of messages in [12] is \( O(n^2) \), but a message includes up to \( f + 1 \) signatures, making the communications cubic. We do not know a way for [12] to utilize threshold signatures because every round in the protocol requires a different threshold.

\( c \) BA is impossible if \( f \geq n/2 \) [15].

\( d \) Although [12] presents a BB protocol, it can be extended to BA with an initial round to multicast inputs. See more details in Section 5.

Organization. The rest of the paper is organized as follows. Section 2 introduces definitions, models and notations. Section 3 presents a BCB protocol with linear communications under honest majority. Section 4 provides a quadratic communication lower bound for BCB under honest minority. Section 5 presents BB and BA protocols with quadratic communication under honest majority. Section 6 provides a practical BFT replication protocol. Finally, we discuss future directions and conclude the paper in Section 7.

2 Preliminaries

Execution model. We define a protocol as an algorithm for a set of parties. There are a set of \( n \) parties, of which at most \( f < n \) are Byzantine faulty and behave arbitrarily. We assume \( f = \Theta(n) \). All presented protocols are secure against \( f \) adaptive corruption that can happen anytime during the protocol execution. A party that is not faulty throughout the execution is said to be honest and faithfully execute the protocol. We use the term quorum to mean the minimum number of all honest parties, i.e., \( n - f \). A protocol proceeds in synchronous rounds. If an honest party sends a message at the beginning of some round, an honest recipient receives the message at the end of that round. We assume digital signatures and public-key infrastructure (PKI), and use \( \langle x \rangle_r \) to denote a message \( x \) signed by party \( r \). As commonly done in Byzantine consensus, we abstract away the details of cryptography, namely, we assume the signature schemes enjoy ideal unforgeability. We further assume a threshold signature scheme [7, 22], in which a set of signatures \( \langle x \rangle_r \) for a message \( x \) from \( t \) (the threshold) distinct parties can be combined into a threshold signature for \( x \) with the same length as an individual signature.

Complexity metrics. The communication complexity of a protocol is the maximum number of bits sent by all honest parties combined across all executions. Since all messages in our protocols
are signed, we use the signature size \( \kappa \) as the unit of measure for communication. We assume the size of any input value is on the order of \( \kappa \). Thus, when we report \( O(n^2) \) communication complexity for a protocol, its bit complexity is \( O(n^2\kappa) \). The lower bounds (in this work and prior work \cite{10}), however, are lower bounds on the number of messages. With no assumptions on the message size, this leaves a gap of \( \kappa \) in a pair of matching upper and lower bounds. If we further assume that every message in authenticated protocols is signed, then matching bounds are tight.

**Byzantine consensus variants.** There are two main variants of Byzantine consensus: broadcast and agreement. In agreement, each party has an input value, and all parties try to decide on the same value. In broadcast, a designated sender denoted by \( r_s \) has an input \( v_{in} \) to broadcast to all parties. The broadcast problem further has several variants. The consistency and validity requirements of the three variants are the same. The difference lies in their termination requirement. BRB relaxes it over BB and allows that either all parties decide or no party decides. BCB further relaxes it and allows some parties to decide while others do not.

**Definition 1** (Byzantine Consistent Broadcast (BCB)). A Byzantine consistent broadcast protocol must satisfy (i) consistency: if two honest parties \( r \) and \( r' \) decide values \( v \) and \( v' \), then \( v = v' \), and (ii) validity: if the sender \( r_s \) is honest, then all honest parties decide the input value \( v_{in} \) and terminate.

**Definition 2** (Byzantine Reliable Broadcast (BRB)). A Byzantine reliable broadcast protocol must satisfy (i) consistency: same as above, (ii) validity: same as above, and (iii) totality: if an honest party decides a value, then every honest party decides a value.

**Definition 3** (Byzantine Broadcast (BB)). A Byzantine broadcast protocol must satisfy (i) consistency: same as above, (ii) validity: same as above, and (iii) termination: every honest party decides a value and terminates.

**Definition 4** (Byzantine Agreement (BA)). A Byzantine agreement protocol must satisfy (i) consistency: same as above, (ii) validity: if all honest parties have the same input value, then all honest parties decide that value, and (iii) termination: same as above.

### 3 Byzantine Consistent Broadcast with Linear Communication

In this section, we show that BCB can be solved with linear communications under honest majority, as formally stated in Theorem \[1\]. To complete the proof of Theorem \[1\], we present a BCB protocol \( O(n) \)-BCB in Figure \[1\]. \( O(n) \) communication is clearly tight because every party at least needs to receive the input value from the sender.

**Theorem 1.** For all constant \( \varepsilon > 0 \), there exists a Byzantine consistent broadcast protocol with communication complexity \( O(n) \) tolerating \( f \leq (\frac{1}{2} - \varepsilon)n \) faults for all \( n \).

**Intuitive overview.** As one can expect, achieving consistency is the hardest part here. Let us first comment on why linear BCB is easy if \( f < n/3 \) \[25 \cite{8} \cite{7} \cite{27]\. With \( n > 3f \), it is well known we can use quorums of \( n - f > 2f \), and then two quorums will intersect at \( \geq f + 1 \) parties. In other words, if we call a quorum of votes a certificate, there cannot exist conflicting certificates, because that would require at least one honest party to vote for both conflicting values. This gives rise to the following protocol where everyone interacts with the designated sender only. The sender proposes a value to everyone, collects their votes into a certificate, and then sends the certificate to
everyone. A party, upon seeing the certificate, can decide on that value safely. Using a threshold signature scheme for the certificate, the communication is linear \([7]\).

However, this method breaks down for \(f \geq n/3\) due to the well-known “split-brain” attack. Suppose \(n = 3f\) and parties are divided into three groups \(P, Q\) and \(R\) each of size \(f\). Suppose the sender is in \(R\) and the \(f\) parties (including the sender) in \(R\) are Byzantine. If \(P\) and \(Q\) never talk to each other, \(R\) can easily simulate one execution with \(P\) and another execution with \(Q\), leading them to decide conflicting values. Now it seems natural that parties need to echo the sender’s proposed value so that they can detect if the sender “equivocates”, i.e., proposing different values to different parties. This “everyone echo” step will trivially incur quadratic communication.

The central technique in the \(O(n)\)-BCB protocol is to let each party echo the sender’s proposed value to a constant set of neighbors in a predefined communication graph. We will show that if the graph is a good expander, then the equivocation detection power is sufficient to achieve consistency.

**Definition 5 (Expander).** Let \(\alpha\) and \(\beta\) be constants satisfying \(0 < \alpha < \beta < 1\). An \((n, \alpha, \beta)\)-expander is a graph of \(n\) vertices such that, for any set \(S\) of \(\alpha n\) vertices, the number of neighbors of \(S\) is more than \(\beta n\).

It is well-known that for any \(n\) and \(0 < \alpha < \beta < 1\), \((n, \alpha, \beta)\)-expanders with constant degrees exist. For our purpose, we need an \((n, 2\varepsilon, 1 - 2\varepsilon)\)-expander; in other words, we set \(\alpha = 2\varepsilon\) and \(\beta = 1 - 2\varepsilon\). Henceforth, we write an \((n, 2\varepsilon, 1 - 2\varepsilon)\)-expander as \(G_{n,\varepsilon}\). We give a simple randomized construction in Appendix [3]. It is important to note that even if we use a randomized expander construction, the BCB protocol is still deterministic, because all the randomization happens in the offline “BCB protocol design” phase. Once a required expander \(G_{n,\varepsilon}\) is found, it can be hardened into the BCB protocol. Nonetheless, deterministic expander constructions also exist, e.g., Cayley graphs and Ramanujan graphs [5].

With a required expander \(G_{n,\varepsilon}\), the protocol works as follows. Upon receiving the sender’s proposal, every party echos it to its neighbors in \(G_{n,\varepsilon}\) before sending a vote for the proposal (to the sender only). This way, with constant communication per party, we have the following guarantee. If a quorum of \(n - f\) parties vote for the proposal, at least \(n - 2f = 2\varepsilon n\) parties are honest. They will echo the proposal to their neighbors in \(G_{n,\varepsilon}\). Due to the expansion property of \(G_{n,\varepsilon}\), they have > \((1 - 2\varepsilon)n \geq 2f\) neighbors. Out of these, > \(f\) are honest and will not vote for a conflicting value. This prevents the creation of a conflicting certificate.

**Correctness of the protocol.** We prove the correctness of the protocol \(O(n)\)-BCB, which proves Theorem [1].

**Theorem 2 (Consistency).** If two honest parties \(r\) and \(r'\) decide values \(v\) and \(v'\), then \(v = v'\).

**Proof.** Suppose for the sake of contradiction two distinct values \(v\) and \(v'\) are both decided. Then two conflicting certificates \(C(v)\) and \(C(v')\) are created. If \(C(v)\) is created, then at least \(2\varepsilon n\) honest parties must have sent votes for \(v\) in round 3. Then, they must have propagated the proposal of \(v\) (signed by the sender) in round 2. Due to the expansion property of \(G_{n,\varepsilon}\), more than \((1 - 2\varepsilon)n \geq 2f\) parties must have received it at the end of round 3, out of which more than \(f\) are honest parties. Thus, less than \(n - f\) parties will vote for \(v' \neq v\), so \(C(v')\) cannot be created. We have obtained a contradiction.\[\square\]

**Theorem 3 (Validity).** If the sender \(r_s\) is honest, then all honest parties decide the input value \(v_{in}\) and terminate.
Let $r_s$ be the designated sender and $v_m$ be its input value. Let $r$ be a party. “Propagate” means sending to all neighbors in $G_{n,\varepsilon}$ and “multicast” means sending to all $n$ parties.

- **Round 1 (Propose):** $r_s$ multicasts the input value $v_m$ in a message $\langle \text{propose}, v_m \rangle_{r_s}$.
- **Round 2 (Echo):** If a party $r$ receives $\langle \text{propose}, v \rangle_{r_s}$, propagates the propose message.
- **Round 3 (Vote):** If $r$ propagates $\langle \text{propose}, v \rangle_{r_s}$ and does not receive $\langle \text{propose}, v' \rangle_{r_s}$ for a different value $v' \neq v$ in Round 2, then $r$ sends $r_s$ a vote for $v$ in the form of $\langle \text{vote}, v \rangle_r$.
- **Round 4 (Forward):** If $r_s$ receives a quorum $(n - f)$ of $\langle \text{vote}, v \rangle_{r'}$, denoted a certificate $C(v)$, then $r_s$ multicasts $C(v)$.

If $r$ receives $C(v)$, it decides $v$ and terminates.

$C(v)$ is batched into one message using a threshold signature scheme with a threshold of $n - f$ out of $n$.

Figure 1: Byzantine Consistent Broadcast with $O(n)$ Communications and $f \leq (\frac{1}{2} - \varepsilon)n$.

**Proof.** An honest sender multicasts a proposal for the input value $v_m$ (and no other value), collects enough votes, and multicasts $C(v_m)$. All honest parties decide $v_m$ and terminate.

**Theorem 4** (Communication Complexity). Communication complexity is $O(n)$.

**Proof.** The sender sends at most $2n$ messages and each party additionally sends at most $d + 1$ messages, where $d$ is the degree of the expander $G_{n,\varepsilon}$ and is a constant (Theorem 13). Therefore, the communication complexity is $2n + (d + 1)n = O(n)$.

4 A Quadratic Lower Bound for Byzantine Consistent Broadcast

In this section, we show in Theorem 5 that under Byzantine majority, BCB requires quadratic communication. This shows that if linear communication is desired, the $O(n)$-BCB protocol in the previous section achieves close to optimal resilience (modulo $\varepsilon$ terms).

**Theorem 5.** For all constant $\varepsilon > 0$, if a protocol solves BCB with $f \geq (\frac{1}{2} + \varepsilon)n$ for all $n$, its communication complexity is $\Omega(n^2)$.

**Proof.** Suppose for the sake of contradiction, for some positive constant $\varepsilon$, there exists a BCB protocol with a communication complexity of $\gamma(n) = o(n^2)$ and resilience $f \geq (\frac{1}{2} + \varepsilon)n$ for all $n$. Let $r_s$ be the sender. Let $S$ and $S'$ be two disjoint sets of non-sender parties of size $\lceil \frac{n-1}{2} \rceil$ and $\lfloor \frac{n-1}{2} \rfloor$, respectively. We consider three executions.

In the first execution (W1), all parties in $S'$ crash and all other parties are honest. This is clearly within the fault tolerance threshold as $|S'| = \lceil \frac{n-1}{2} \rceil < f$. Let the input value $v_m = 0$. By validity, all parties in $S$ decide 0. The second execution (W2) is symmetric. All parties in $S$ crash and all other parties are honest. The input value $v_m = 1$. By validity, all parties in $S'$ decide 1.

Let $d(n) = 8\gamma(n)/\varepsilon n$. Since $\gamma(n) = o(n^2)$, when $n$ is sufficiently large, we have $\gamma(n)/n^2 < \varepsilon^2/32$, which implies that $d(n) = 8\gamma(n)/\varepsilon n < \varepsilon n/4$. 


Now, we partition the parties as follows. We say two parties communicated in an execution if at least one of them sends a message to the other.

1. $r_s$: the designated sender.
2. $R$: the set of parties who communicate with more than $d(n)$ parties in either W1 or W2. Note that to keep the communication complexity at $\gamma(n)$, at most $2\gamma(n)/d(n)$ parties can communicate with more than $d(n)$ parties in one execution. Thus, $|R| \leq 4\gamma(n)/d(n) = \varepsilon n/2$.
3. $r' \in (S' - R)$: a party in $S'$ that is not in $R$.
4. $V' = S' - R - \{r'\}$: the set of remaining parties in $S'$.
5. $V_f \subset (S - R)$: the set of parties in $S - R$ who communicate with $r'$ in either W1 or W2. Note that $|V_f| \leq 2d(n)$ since $r' \notin R$.
6. $V_h = S - R - V_f$: the set of remaining parties in $S$.

Now we define a third execution W3: $r'$ and $V_h$ are honest, and the other parties (including the sender $r_s$) are all Byzantine. This is within the fault threshold, because the number of faults is,

$$n - |V_h| - 1 \leq n - (|S| - (|R| + |V_f|)) - 1 \leq n - (\lceil n - 1/2 \rceil) + |R| + 2d(n) - 1 \leq n/2 + |R| + 2d(n) - 1/2 < n/2 + \varepsilon n/2 + \varepsilon n/2 = n(1/2 + \varepsilon).$$

The last step is due to two facts we established earlier: $|R| < \varepsilon n/2$ and $d(n) < \varepsilon n/4$ when $n$ is sufficiently large.

In W3, faulty parties behave in the following manner. Parties in $S$ behave in the same way as in W1, except that (i) they do not send any messages to parties in $S'$, and (ii) they ignore all messages from $S'$. Parties in $S'$ behave in the same way as in W2, except that (i) they do not send any messages to parties in $S$, and (ii) they ignore all messages from $S$. The faulty sender $r_s$ behaves like in W1 towards $S$ and behaves like in W2 towards $S'$.

We will show that $V_h$ cannot distinguish W1 and W3 and that $r'$ cannot distinguish W2 and W3. Note that all parties other than $V_h$ and $r'$ are Byzantine and will behave towards them accordingly in W3 the way they did in W1 and W2, respectively. So we just need to further show that $V_h$ and $r'$ do not communicate with each other in W3. Suppose towards a contradiction that they do. Then there is a first message sent between them in W3. Before that message, $V_h$ cannot distinguish W1 and W3 because all parties other than $V_h$ communicate with them in the same manner in both executions. Thus, the source of the message cannot be $V_h$, because they do not send messages to $r'$ in W1. On the other hand, $r'$ cannot distinguish W2 and W3 before that message either, because all parties other than $V_h$ communicate with them in the same manner in both executions. Thus, the source of the message cannot be $r'$ for the same reason. Thus, $r'$ and $V_h$ do not communicate with each other.

Therefore, $V_h$ cannot distinguish W1 and W3, and decide 0 in W3. At the same time, $r'$ cannot distinguish W2 and W3, and decides 1 in W3. Thus, consistency is violated in W3, which is a contradiction to the existence of such a protocol. \qed
5 Byzantine Broadcast and Agreement

In Section 3, we introduced a BCB protocol with $O(n)$ communication complexity. In this section, we extend it into BB/BA protocols with quadratic communication complexity. We name the protocols $O(n^2)$-BB and $O(n^2)$-BA.

**Theorem 6.** For all constant $\varepsilon > 0$, there exists Byzantine broadcast and agreement protocols with $O(n^2)$ communication complexity tolerating $f \leq (\frac{1}{2} - \varepsilon)n$ faults for all $n$.

5.1 Byzantine Broadcast

The $O(n^2)$-BB protocol is described in Figure 2.

**Definitions and notations.** The protocol proceeds in iterations with a constant number of rounds per iteration. Each iteration has a leader. We use $L_i$ to denote the leader of iteration $i$. The leader of the very first iteration is the designated sender, i.e., $L_1 = r_s$. The leader schedule can be follow a simple round-robin order (or any non-repeating order) after the first iteration.

Similar as before, a quorum of votes for a value $v$ in iteration $i$ is called an iteration-$i$ certificate for $v$, and is denoted $C_i(v)$. We say $v$ is certified in iteration $i$ if $C_i(v)$ exists. As before, $C_i(v)$ are batched into one message using a threshold signature scheme with a threshold of $n - f$ out of $n$. The certificates are ranked by the iteration in which they are formed, i.e., the rank of $C_i(v)$ is $i$.

For convenience, we sometimes compare the rank of a certificate with $\bot$, which we define to have a rank of $-1$, i.e., lower than all certificates.

**Intuitive overview.** Intuitively, each iteration runs an instance of $O(n)$-BCB (round 1–4), where the leader serves as the BCB sender. If the leader is honest, all honest parties can decide the value proposed by the leader, by the validity of BCB. Furthermore, conflicting values cannot be decided in the same iteration, by the consistency of BCB. The additional challenge here is to ensure decisions are consistent across iterations. Towards this end, we add another propagation step and a locking mechanism (which is pioneered by Paxos [20] and common in BFT [8, 27]).

We define a new message type echoed. A quorum of echoed messages for an object $b$ in iteration $i$ is denoted by $E_i(b)$. A party sends an echoed message for an object $b$ when it propagates $b$ in the expander graph. Thus, $E_i(b)$ serves as a proof that enough honest parties have propagated $b$. We remark that the object $b$ will be a certificate itself in our protocol. A party decides a value $v$ if it receives $E_i(C_i(v))$. This means more than $n - 2f = 2\varepsilon n$ parties propagated $C_i(v)$. Again due to the expander graph, more than $f$ honest parties will receive $C_i(v)$. These parties will “lock” on $C_i(v)$ by setting a local variable $\text{lock}_{i,r} = C_i(v)$ in round 7. When the next leader $L_{i+1}$ proposes a value $v'$ along with a corresponding certificate $C_i'(v')$, a party $r$ does not vote for $v'$ if the certificate $C_i'(v')$ in the proposal is lower than its lock $\text{lock}_{i,r}$. This ensures that whenever an honest party decides, certificates for conflicting values cannot be created in subsequent iterations, guaranteeing consistency.

**Correctness of the protocol.** We prove the correctness of the protocol $O(n^2)$-BB, which proves Theorem 6.

**Lemma 1.** If an honest party decides a value $v$ in iteration $i$, then for all iteration $j \geq i$, if $C_j(v')$ exists, then $v' = v$. 

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Let \( r_s \) be the designated sender and \( v_{in} \) be its input value. Let \( r \) be a party. “Propagate” means sending to all neighbors in \( G_{n,e} \) and “multicast” means sending to all \( n \) parties. The protocol proceeds in iterations \( 1 \leq i \leq f + 1 \). The leader of iteration \( i \) is denoted \( L_i \). \( \text{lock}_{0,r} \) is initialized to \( \perp \). The protocol for iteration \( i \) is as follows.

- **Round 1 (Propose):** \( L_i \) multicasts a proposal \( \langle \text{propose}, v, P, i \rangle \) to \( L_i \), where \( v \) and \( P \) are selected as follows.
  (a) If \( i = 1 \), then \( v = v_{in} \) and \( P = \perp \);
  (b) Else if \( \text{lock}_{i-1,r} \neq \perp \), then \( v \) is the value certified in \( \text{lock}_{i-1,r} \) and \( P = \text{lock}_{i-1,r} \);
  (c) Else, then \( v \) is picked freely and \( P = \perp \).

- **Round 2 (Echo):** If a party \( r \) receives \( \langle \text{propose}, v, P, i \rangle \) from \( L_i \), \( r \) propagates the proposal.

- **Round 3 (Vote):** If a party \( r \) propagates \( \langle \text{propose}, v, P, i \rangle \) to \( L_i \) and does not receive \( \langle \text{propose}, v', P', i \rangle \) for a different value \( v' \neq v \) in Round 2, and if the rank of \( P \) is not lower than \( \text{lock}_{i-1,r} \), then \( r \) sends \( r_s \) a value for \( v \) in the form of \( \langle \text{vote}, v, i \rangle \).

- **Round 4 (Forward):** If \( L_i \) receives \( n - f \langle \text{vote}, v, i \rangle \), denoted \( C_i(v) \), then multicast \( C_i(v) \).

- **Round 5 (Echo):** If \( r \) receives \( C_i(v) \), \( r \) propagates \( C_i(v) \) and sends \( \langle \text{echoed}, C_i(v), i \rangle \) to \( L_i \).

- **Round 6 (Forward):** If \( L_i \) receives \( n - f \langle \text{echoed}, C_i(v), i \rangle \), denoted \( E_i(C_i(v)) \), then multicast \( E_i(C_i(v)) \).

  At the end of the round, if \( r \) receives \( E_i(C_i(v)) \), \( r \) decides \( v \).

  If \( r \) has received \( C_i(v) \) in any of Round 3, 4, 5, or 6 in this iteration, \( r \) sets \( \text{lock}_{i,r} = C_i(v) \); else \( r \) sets \( \text{lock}_{i,r} = \text{lock}_{i-1,r} \).

- **Round 7 (Status):** \( r \) sends \( \text{lock}_{i,r} \) to \( L_{i+1} \).

  At the end of the round, \( L_{i+1} \) picks a highest certificate it receives, and sets \( \text{lock}_{i,r} \) to it.

Figure 2: Byzantine Broadcast with \( O(n^2) \) communication and \( f \leq (\frac{1}{2} - \varepsilon)n \).

**Proof.** We prove it by induction on the iteration number. The base case \( (j = i) \) follows from the consistency of BCB (Theorem 2).

Before moving on to the inductive case, we show that there is a set \( R \) of honest parties with \( |R| > f \) that all lock on \( C_i(v) \) by the end of iteration \( i \). As some honest party decides \( v \) in iteration \( i \), that honest party must have observed \( E_i(C_i(v)) \) at the end of round 6. Then, in round 5, at least \( 2\varepsilon n \) honest parties must have propagated \( C_i(v) \), and more than \( 2f \) parties must have received \( C_i(v) \) by round 6. Thus, more than \( f \) honest parties have received \( C_i(v) \). This will be the desired set \( R \).

Since there is no certificate for any different value in iteration \( i \), in round 7, every party \( r \in R \) sets \( \text{lock}_{i,r} = C_i(v) \).

Next, suppose the inductive hypothesis holds up to \( j \), we will prove it for \( j + 1 \). Note that at the beginning of iteration \( j + 1 \), parties in \( R \) are still locked on certificates for \( v \) (possibly from an iteration higher than \( i \)), because due to the inductive hypothesis, no certificate for a different value can be higher than \( C_i(v) \) up till that point. Furthermore, each \( r \in R \) will not vote for a value \( v' \neq v \) because doing so requires observing at the beginning of Round 3, a certificate \( C_{v'}(v') \) that is
higher than $\text{lock}_{i,r}$ (which is at least as high as $\mathcal{C}_i(v)$). Such a $\mathcal{C}_i'(v')$, again, does not exist up till that point due to the inductive hypothesis. Therefore, a value $v' \neq v$ will not get enough $(n - f)$ votes to have a certificate in iteration $j + 1$.

**Theorem 7** (Consistency). *If two honest parties $r$ and $r'$ decide values $v$ and $v'$, respectively, then $v = v'$.*

**Proof.** Suppose two values $v$ and $v'$ are both decided. Let $i$ and $i'$ the iterations in which $v$ and $v'$ are decided, respectively. Without loss of generality, we assume $i \leq i'$. Since all decided values are certified, by Lemma $\square$ $v = v'$.

**Theorem 8** (Validity). *If the sender $r_s$ is honest, all honest parties decide the input value $v_{in}$ and terminate.*

**Proof.** Termination is clear as all parties terminate after the last round (Round 7) of the last iteration (iteration $f + 1$). We next prove all parties decide $v_{in}$. Note that $L_1 = r_s$. If $r_s$ is honest, it proposes $v_{in}$, collects and multicasts $\mathcal{C}_1(v_{in})$, collects and multicasts $\mathcal{E}_1(\mathcal{C}_1(v_{in}))$, and makes all honest parties decide $v_{in}$ in iteration 1.

**Theorem 9** (Termination). *Every honest party decides a value and terminates.*

**Proof.** Termination is clear as all parties terminate after the last round (Round 7) of the last iteration (iteration $f + 1$). We next prove all parties decide.

If the sender is honest, all honest parties decide $v_{in}$ by validity. If a leader $L_i$ of an iteration $i > 1$ is honest, then $L_i$ proposes the value of $\text{lock}_{i-1,L}$ if it is not $\bot$, or a freely selected value otherwise. Since all honest parties $r$ send their $\text{lock}_{i-1,r}$ to $L_i$ in Round 7 of the previous iteration $i - 1$ and $L_i$ picks a highest one to be its own $\text{lock}_{i-1,L_i}$, we have that $\text{lock}_{i-1,L_i}$ must be at least as high as any $\text{lock}_{i-1,r}$. Therefore, all honest parties will vote for the value $L_i$ proposes and decide the value. Since at most $f$ parties can be faulty, there exists an iteration with an honest leader. Thus, all honest parties decide a value by iteration $f + 1$.

**Theorem 10** (Communication Complexity). *Communication complexity is $O(n^2)$.*

**Proof.** In each iteration, the leader sends at most $3n$ messages, and each party additionally sends at most $2d + 3$ messages, where $d$ is the degree of the expander $G_n,\varepsilon$ and is a constant (Theorem $\square$). Therefore, the communication complexity is $(3n + (2d + 3)n)(f + 1) = O(n^2)$.

**Round complexity.** Though round complexity is not the focus of this paper, we briefly discuss it. The round complexity of the above protocol is $O(f)$. We could easily improve the round complexity to be $O(t)$ where $t$ is the number of actual faults, i.e., achieving an “early stopping” property $\square$. When a party $r$ decides a value $v$, multicasts $\langle \text{decide}, v \rangle_r$. Upon receiving more than $f$ decide messages for $v$, a party batches them into one threshold signature and multicasts it, and then terminates. By iteration $t + 1$, an honest leader emerges and thus all honest parties decide some value $v$, send decide messages for $v$, receive enough of them, send out a threshold signature of it, and terminate.
Let \( r \) be a party and let \( v_{in} \) be its input value. “Propagate” means sending to all neighbors in \( G_{n,ε} \) and “multicast” means sending to all \( n \) parties.

The protocol proceeds in iterations \( 1 \leq i \leq f + 1 \). The leader of iteration \( i \) is denoted \( L_i \). But the protocol has one extra round before the first iteration.

- **Round of preprocessing:** Multicast the input \( v_{in} \) in the form of \( \langle \text{vote}, v_{in}, 0 \rangle_r \). At the end of this round, if \( r \) receives \( C_0(v) \), set \( \text{lock}_{0,r} \) to \( C_0(v) \); else, set \( \text{lock}_{0,r} \) to \( \perp \).

The protocol for iteration \( i \) is as follows.

- **Round 1 (Propose):** \( L_i \) multicasts a proposal \( \langle \text{propose}, v, P, i \rangle_{L_i} \), where \( v \) and \( P \) are selected as follows.
  
  (a) If \( \text{lock}_{i-1,r} \neq \perp \), then \( v \) is the value certified in \( \text{lock}_{i-1,r} \) and \( P = \text{lock}_{i-1,r} \);
  
  (b) Else, then \( v \) is picked freely and \( P = \perp \).

- **Round 2–7:** identical to \( O(n^2) \)-BB.

Figure 3: Byzantine Agreement with \( O(n^2) \) communication and \( f \leq (\frac{1}{2} - ε)n \).

### 5.2 Byzantine Agreement

Next, we change \( O(n^2) \)-BB into a Byzantine agreement protocol \( O(n^2) \)-BA with two simple modifications. The full description of \( O(n^2) \)-BA is in Figure 3.

1. Before starting the iterations, we add a preprocessing round in which each party \( r \) votes for its input \( v \) in the form of \( \langle \text{vote}, v_{in}, 0 \rangle_r \). At the end of this round, if a party receives \( C_0(v) \), it sets \( \text{lock}_{0,r} \) to \( C_0(v) \). Note that \( \perp \) (with rank \( −1 \)) is still lower than \( C_0(v) \).

2. The first rule in proposal value selection (related to the designated sender) is removed naturally. The first leader \( L_1 \) (which is now an ordinary party) proposes \( v \) if it has set \( \text{lock}_{0,L_1} = C_0(v) \).

**Correctness of the Protocol.** Consistency, termination, and asymptotic complexity of \( O(n^2) \)-BA are identical to those of \( O(n^2) \)-BB. Only the validity proof changes slightly: If all honest parties have the same input value \( v \), then all honest parties lock on \( C_0(v) \) before the iterations start; then, no other value can become certified, guaranteeing validity.

**Theorem 11 (Validity).** If all honest parties has a same input value \( v \), then all honest parties decide \( v \) and terminate.

**Proof.** If all honest parties has a same input value \( v \), then at the end of the preprocessing round, all honest parties receive \( C_0(v) \) and set \( \text{lock}_{0,r} = C_0(v) \). Therefore, in iteration 1 and inductively all iterations \( i \), no honest party votes for any values \( v' \neq v \). So \( C_i(v') \) will not be created and \( v' \) cannot be decided. Similar to the proof of Theorem 9, one of the leaders will be honest. In an honest leader’s iteration, all honest parties decide and they decide \( v \). \( \square \)
6 BFT Replication with Linear Communications

This section presents a BFT replication protocol $O(n)$-BFT with linear communication, based on $O(n)$-BCB. Before showing the protocol, we briefly review BFT replication. BFT replication (BFT for short) achieves a replicated state machine, where requests from external clients lead to state transitions. Overall, it provides clients with a fault-tolerant distributed service that has the same abstraction as a single non-faulty server. To provide a consistent and available service, parties continuously decide and agree on a totally-ordered sequence of requests, called a log. These requirements are formalized as safety and liveness in Appendix C.

Our protocol $O(n)$-BFT relies on the stable-leader approach. That is, an honest leader continuously proposes values to append to the log. When the leader becomes faulty, it will be replaced. A reign of a leader is called a view, which is analogous to an iteration in the BB/BA protocol in Section 5. Consistency within the same view is provided by BCB. Consistency across views is ensured by the locking mechanism similar to the BB/BA protocol.

We will also extend the $O(n)$-BCB protocol to work without lock-step rounds to make the protocol more practical. This creates a new challenge as parties now are no longer perfectly synchronized. The concrete challenge for our protocol is to ensure that when the first honest party votes for a value, propagation has already completed, i.e., more than $f$ honest parties have already observed the value. Without lock-step rounds, parties may vote at slightly different times and the first honest party may vote before propagation completes. Thus, a naive extension of $O(n)$-BCB fails to achieve consistency. To make sure parties vote after propagation completes, we make two modifications: (i) in the echo phase, a party sends the leader an echoed message for the value $v$, and (ii) a party votes $\Delta$ time (a known upper bound on the communication delay) after receiving $E(v)$. Since $E(v)$ serves as a proof of propagation as in $O(n^2)$-BB, waiting for this $\Delta$ time ensures that the propagation is completed. The detailed protocol description and correctness proof are given in Appendix C.

7 Conclusion

In this paper, we provided three new results: (1) a BCB protocol with linear communications under honest majority, (2) a quadratic communication lower bound to solve BCB under fault majority, (3) BB and BA protocols with quadratic communication. Towards practical applications, we give a BFT replication protocol with linear communication.

Since our protocols require $f < (1/2 - \varepsilon)n$, some gaps in communication complexity remain, e.g., BA with $(1/2 - \varepsilon)n < f < n/2$, and BCB/BB with $f > (1/2 - \varepsilon)n$. These are intriguing open questions for future work.

References

[1] Ittai Abraham, TH Hubert Chan, Danny Dolev, Kartik Nayak, Rafael Pass, Ling Ren, and Elaine Shi. Communication complexity of byzantine agreement, revisited. In ACM Symposium on Principles of Distributed Computing (PODC), pages 317–326, 2019.

[2] Ittai Abraham, Srinivas Devadas, Danny Dolev, Kartik Nayak, and Ling Ren. Synchronous byzantine agreement with expected $o(1)$ rounds, expected $o(n^2)$ communication, and optimal resilience. In Financial Cryptography and Data Security (FC), pages 320–334. Springer, 2019.
[3] Ittai Abraham, Dahlia Malkhi, Kartik Nayak, Ling Ren, and Maofan Yin. Sync hotstuff: Simple and practical synchronous state machine replication. *IACR Cryptology ePrint Archive, Report 2019/270*, 2019. [https://eprint.iacr.org/2019/270](https://eprint.iacr.org/2019/270).

[4] Ittai Abraham, Dahlia Malkhi, and Alexander Spiegelman. Validated asynchronous byzantine agreement with optimal resilience and asymptotically optimal time and word communication. *arXiv preprint arXiv:1811.01332*, 2018.

[5] Noga Alon. *Tools from higher algebra*.

[6] Piotr Berman, Juan A Garay, and Kenneth J Perry. Bit optimal distributed consensus. In *Computer science*, pages 313–321. Springer, 1992.

[7] Christian Cachin, Klaus Kursawe, Frank Petzold, and Victor Shoup. Secure and efficient asynchronous broadcast protocols. In *Annual International Cryptology Conference (CRYPTO)*, pages 524–541. Springer, 2001.

[8] Miguel Castro, Barbara Liskov, et al. Practical byzantine fault tolerance. In *3rd Symposium on Operating Systems Design and Implementation (OSDI)*, pages 173–186. USENIX, 1999.

[9] Jing Chen and Silvio Micali. Algorand. *arXiv preprint arXiv:1607.01341*, 2016.

[10] Danny Dolev and Rüdiger Reischuk. Bounds on information exchange for byzantine agreement. *Journal of the ACM (JACM)*, 32(1):191–204, 1985.

[11] Danny Dolev, Ruediger Reischuk, and H Raymond Strong. Early stopping in byzantine agreement. *Journal of the ACM (JACM)*, 37(4):720–741, 1990.

[12] Danny Dolev and H. Raymond Strong. Authenticated algorithms for byzantine agreement. *SIAM Journal on Computing*, 12(4):656–666, 1983.

[13] Paul Feldman and Silvio Micali. Optimal algorithms for byzantine agreement. In *Proceedings of the twentieth annual ACM symposium on Theory of computing*, pages 148–161, 1988.

[14] Michael J Fischer, Nancy A Lynch, and Michael S Paterson. Impossibility of distributed consensus with one faulty process. *Journal of the ACM (JACM)*, 32(2):374–382, 1985.

[15] Matthias Fitzi. *Generalized communication and security models in Byzantine agreement*. PhD thesis, ETH Zurich, 2002.

[16] Yue Guo, Rafael Pass, and Elaine Shi. Synchronous, with a chance of partition tolerance. In *Annual International Cryptology Conference (CRYPTO)*, pages 499–529. Springer, 2019.

[17] Jonathan Katz and Chiu-Yuen Koo. On expected constant-round protocols for byzantine agreement. *Journal of Computer and System Sciences*, 75(2):91–112, 2009.

[18] Valerie King and Jared Saia. Breaking the $o(n^2)$ bit barrier: scalable byzantine agreement with an adaptive adversary. *Journal of the ACM (JACM)*, 58(4):1–24, 2011.

[19] Ramakrishna Kotla, Lorenzo Alvisi, Mike Dahlin, Allen Clement, and Edmund Wong. Zyzzyva: speculative byzantine fault tolerance. In *Proceedings of twenty-first ACM SIGOPS Symposium on Operating Systems Principles (SOSP)*, pages 45–58, 2007.
[20] Leslie Lamport. The part-time parliament. In Concurrency: the Works of Leslie Lamport, pages 277–317. 2019.

[21] Leslie Lamport, Robert Shostak, and Marshall Pease. The byzantine generals problem. ACM Transactions on Programming Languages and Systems, 4(3):382–401, 1982.

[22] Benoît Libert, Marc Joye, and Moti Yung. Born and raised distributively: Fully distributed non-interactive adaptively-secure threshold signatures with short shares. Theoretical Computer Science, 645:1–24, 2016.

[23] Silvio Micali. Byzantine agreement, made trivial, 2016.

[24] Atsuki Momose, Jason Paul Cruz, and Yuichi Kaji. Hybrid-bft: Optimistically responsive synchronous consensus with optimal latency or resilience. IACR Cryptology ePrint Archive, Report 2020/406, 2020. https://eprint.iacr.org/2020/406.

[25] Michael K Reiter. Secure agreement protocols: Reliable and atomic group multicast in ram-part. In ACM Conference on Computer and Communications Security (CCS), pages 68–80, 1994.

[26] Fred B Schneider. Implementing fault-tolerant services using the state machine approach: A tutorial. ACM Computing Surveys (CSUR), 22(4):299–319, 1990.

[27] Maofan Yin, Dahlia Malkhi, Michael K Reiter, Guy Golan Gueta, and Ittai Abraham. Hot-stuff: Bft consensus with linearity and responsiveness. In ACM Symposium on Principles of Distributed Computing (PODC), pages 347–356. ACM, 2019.

A A Quadratic Lower Bound on Communication Complexity for Byzantine Reliable Broadcast

It is well known that quadratic communications are necessary to achieve BB [10]. In this section, we will show in Theorem 12 that BRB, which is much easier than BB, has the same lower bound on communication complexity. The proof is a straightforward extension from [10]. The only difference is that our proof relies on totality, while the proof in [10] relies on termination.

**Theorem 12.** There does not exist a BRB protocol with communication complexity of at most $f^2/4$.

**Proof.** Suppose for the sake of contradiction, there exists a BRB protocol with communication complexity of at most $f^2/4$. Without loss of generality, we assume that there exists a set of $\lceil (n - 1)/2 \rceil$ parties different from a sender $r_s$ (say $R$) that do not decide a certain value $v$ if they receive no messages. There must exist such a value since otherwise, there exist two different values that at least one party decides both if it receives no messages. We consider a network that has the following two partitions.

1. $A$: a set of $\lfloor f/2 \rfloor$ parties, which is a subset of $R$.
2. $B$: all remaining parties including a sender $r_s$. 

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We consider two executions that violate either consistency or totality. In the first execution (W1), all parties in \( A \) are Byzantine, and the sender’s input value is \( v \). Parties in \( A \) do not communicate with each other. Toward \( B \), parties in \( A \) execute honestly except that they ignore the first \( \lceil f/2 \rceil \) messages from \( B \). By validity, all parties in \( B \) decide \( v \). Here, since the communication complexity of the protocol is at most \( f^2/4 \), there exists a party (say \( p \)) in \( A \) that receives at most \( \lceil f/2 \rceil \) messages from \( B \). Let \( B_p \) be a set of all parties that send messages to \( p \).

In the second execution (W2), all parties in \( A \) except \( p \) are Byzantine and all parties in \( B_p \) are Byzantine. This gives a total of \( (\lceil f/2 \rceil - 1) + \lceil f/2 \rceil \leq f \) faults as allowed. The sender’s input value is also \( v \). All parties in \( B \) \( B_p \) execute in the same way as in W1 except that they do not send any messages to \( p \). \( A_p \) execute in the same way as in W1. Here, since \( p \) cannot receive any messages, \( p \) executes in the same way as in W1. Therefore, parties in \( B \setminus B_p \) cannot distinguish W1 and W2, thus decide \( v \). On the other hand, \( p \) does not receive any messages, and since \( p \in R \), it does not decide \( v \). If \( p \) decides a certain value \( v' \neq v \), then consistency is violated. If \( p \) does not decide, then totality is violated because honest parties in \( B \setminus B_p \) have decided. Thus, it is a contradiction and such a BRB protocol does not exist.

\[ \square \]

B Expander

We show that an expander \( G_\varepsilon \) in the Definition 5 exists for all positive constant \( \varepsilon \). We use \( \Gamma(V,G) \) to denote a set of all neighbors of \( V \) in a graph \( G \).

**Theorem 13** (Existence of Expander). For all positive integer \( n \) and positive constant \( \varepsilon \), there exists an expander \( G_{n,\varepsilon} \).

**Proof.** Let \( c = 2\varepsilon \). Consider a random \( d \) degree graph \( G \) taking the union of random \( d \) perfect matchings (if \( n \) is odd, the first party has two links). In each perfect matching \( P \), for any set of \( cn \) parties (say \( S \)), and any set of \((1 - c)n \) parties (say \( T \)), the probability that \( \Gamma(S,P) \subseteq T \) is bounded above by,
\[
\Pr[\Gamma(S,P) \subseteq T] \leq (\frac{1 - c}{n})^{cn} = (1 - c)^{\frac{cn}{n}}.
\]

Thus, the probability that any set of \( cn \) parties does not expand in the graph, i.e., \( |\Gamma(S,G)| \leq (1 - c)n \) for any \( S \), is bounded above by,
\[
\left( \frac{n}{cn} \right) \left( \frac{n}{(1 - c)n} \right) (1 - c)^{\frac{cdn}{n}}
\leq \left( \frac{e}{c^n} \right)^{cn} (1 - c)(1 - c)^{\frac{cdn}{n}}
\leq (e(\frac{1}{c^n}))(1 + (\frac{d}{n^2} - 1))n
\]

For a sufficiently large constant \( d \), the above probability is exponentially small, which means there is non-zero (in fact, overwhelmingly large) probability that a randomly chosen graph is an expander. Thus, \( G_{n,\varepsilon} \) exists.

\[ \square \]
C BFT Replication with Linear Communication

C.1 Model
In this section, we do not assume lock-step execution of rounds following recent practical synchronous BFT protocols [3]. Instead, parties proceed in real time. In this model, the synchrony assumption states that a message sent by an honest party at time $t$ will be received by an honest recipient by time $t + \Delta$, where $\Delta$ is the known upper bound on message delay. The replication protocol additionally uses a cryptographic hash function $H$.

**BFT replication.** BFT replication (BFT for short) achieves a replicated state machine. External clients send requests to all parties, and all honest parties continuously decide a sequence of requests in a linearizable log, with the two requirements below.

**Definition 6 (Byzantine Fault Tolerant Replication).** A Byzantine fault tolerant replication protocol must satisfy (i) safety: if two honest parties decide requests $c$ and $c'$ at the same log position, then $c = c'$, and (ii) liveness: all client requests are eventually decided by all honest parties.

**Communication complexity.** Since a BFT replication protocol runs forever, its communication complexity is typically (and naturally) defined in an amortized sense: the amount of communication needed per consensus decision (i.e., per log record).

C.2 Protocol Description

The description of the protocol $O(n)$-BFT in detail is in Figure 4.

**Definitions and notations.** The protocol proceeds in views identified by a monotonically increasing integers starting from 1. In each view, client requests are decided in a process called steady-state, directed by a party selected as leader. When a leader is faulty and honest parties fails to process client requests, a fallback process called view-change is invoked to change the view and the leader.

In steady-state, client requests are batched into a block and proposed by a leader. Each block $B_k$ includes (i) $b_k$: a batch of client requests with a total-order, and (ii) $h_{k-1}$: a hash of a block which $B_k$ refers to. The hash references, starting from the first block called genesis block hard-coded in the protocol denoted by $B_0 = (\perp, \perp)$, form a chain of blocks so-called blockchain. The height $k$ of a block $B_k$ is naturally defined as its position in the blockchain. We say a block $B_k = (b_k, h_{k-1})$ is valid if (i) there is a valid block $B_{k-1}$ and $h_{k-1} = H(B_{k-1})$, or (ii) $B_k$ is the genesis block. We say a block $B$ extends $B'$ if $B = B'$ or $B$ is a descendant of $B'$. If two blocks $B$ and $B'$ do not extend one another, we say $B$ and $B'$ conflict with each other.

Similar to the Byzantine consensus protocols presented in the previous sections, each non-leader sends vote/echoed message for a block proposed by a leader. A quorum of votes for a value $v$ in view $i$ is called a certificate for $v$ in view $i$, and denoted by $C_i(v)$. We say a value $v$ is certified in view $i$ if $C_i(v)$ exists. The genesis block is considered certified automatically. All certificates are ranked first by view number, and then by height. For example, a certificate $C_i(B_k)$ is higher in rank than $C_{i-1}(B_{k+1})$ but lower in rank than $C_i(B_{k+1})$. A quorum of echoed messages for an object $b$ in view $i$ is denoted as $E_i(b)$; note that $b$ may be a certificate.

Each party multicasts a blame message if it fails to decide blocks in a predefined time or detects any faulty behavior by the leader. A set of $f + 1$ blame messages in view $i$ is denoted by $B(i)$, which triggers a view-change.
Let $L$ be a leader of view $i$, $L'$ be a leader of view $i + 1$, and $r$ be a replica. “Propagate” means sending to all neighbors in $G_{n,e}$ and “multicast” means sending to all $n$ parties. Party $r$ executes the following steady-state process. If $r$ receives an invalid block, ignores it.

1. **Propose:** $L$ multicasts $\langle \text{propose}, B_k, P, i \rangle_L$, (i) at the beginning of the iteration, or (ii) upon receiving a new highest certificate of the current iteration $C_i(B_{k-1})$. $P$ is a highest certificate $C_j(B_{k-1})$ in case (i), or $C_i(B_{k-1})$ in case (ii). $B_k = (b_k, h_{k-1})$ and $b_k$ is a batch of client requests and $h_{k-1}$ is a hash of $B_{k-1}$.

2. **Echo:** If $r$ receives $\langle \text{propose}, B_k, C_j(B_{k-1}), i \rangle_L$, propagates it, and sends $\langle \text{echoed}, B_k, i \rangle_r$ to $L$.

3. **Vote:** If $L$ receives $n - f$ of $\langle \text{echoed}, B_k, i \rangle_{r'}$, denoted by $E_i(B_k)$, multicasts $E_i(B_k)$. If $r$ receives $E_i(B_k)$ and corresponding proposal, wait for $\Delta$, and sends $\langle \text{vote}, B_k, i \rangle_r$ to $L$, if $B_k$ is a descendant of a certified block as high in rank as $\text{lock}_{i-1,r}$.

4. **Echo:** If receives $n - f$ of $\langle \text{vote}, B_k, i \rangle_{r'}$, denoted by $C_i(B_k)$, propagates it, and sends $\langle \text{echoed}, C_i(B_k), i \rangle_r$ to $L$. $L$ multicasts $C_i(B_k)$.

5. **Decide:** If $r$ receives $n - f$ of $\langle \text{echoed}, C_i(B_k), i \rangle_{r'}$, denoted by $E_i(C_i(B_k))$, decide $B_k$ and all its ancestors. $L$ multicasts $E_i(C_i(B_k))$

Party $r$ simultaneously executes the following view-change process.

1. **Blame:** If (i) less than $p$ blocks are decided within $(11 + 2(p - 1))\Delta$ time in view $i$, or (ii) two conflicting blocks are proposed in view $i$, multicasts $\langle \text{blame}, i \rangle_r$. In the case of (ii), multicasts the two blocks and stop all processes in the steady-state.

2. **Quit & Lock:** If $r$ receives $f + 1$ of $\langle \text{blame}, i \rangle_{r'}$, denoted by $B(i)$, multicasts it, stop all processes in the steady-state, and wait for $2\Delta$ ($4\Delta$ if $L'$). Set $\text{lock}_{i,r}$ a highest certified block, send it to $L'$ and start the next view $i + 1$.

Figure 4: BFT Replication with $O(n)$ Communications and $f \leq (\frac{1}{2} - \varepsilon)n$

### C.3 Correctness of the Protocol

We prove the correctness of the protocol $O(n)$-BFT with $f \leq (\frac{1}{2} - \varepsilon)n$ for all $n$ and for any positive constant $\varepsilon$.

**Lemma 2** (Certified without Equivocation). *If two certificates $C_i(B)$ and $C_i(B')$ are both created in the same view $i$, then $B$ and $B'$ do not conflict with each other.*

**Proof.** Suppose for the sake of contradiction, two conflicting certificates $C_i(B)$ and $C_i(B')$ are both created in the same view $i$. Let $t_c$ be the time when the first honest party votes for $B$, and $t_q$ be the time when $E_i(B)$ is created. For $B'$, define $t'_c$ and $t'_q$ in the same way as above. We have $t_c \geq t_q + \Delta$, because an honest party votes for a block after waiting for $\Delta$. Then, $E_i(B')$ could not have been created by $t_q$, i.e., $t_q < t'_q$. Otherwise, there is at least $2\varepsilon n$ honest parties who must have
propagated \( B' \) by \( t_q \). Due to the expansion property of \( G_{n,x} \), more than \( f \) honest parties must have received \( B' \) by time \( t_c \). It would prevent \( C_i(B) \) from being created. Similarly, \( E_i(B) \) could not have been created by \( t'_q \), i.e., \( t'_q < t_q \). These propositions contradict each other.

We say a party \( r \) directly decides a block \( B \) in view \( i \), if \( r \) has not decided its descendants at the time \( r \) decides \( B \).

**Lemma 3.** If an honest party directly decides a block \( B_k \) in view \( i \), then for all view \( j \geq i \), if a block \( B \) as high in rank as \( B_k \) is certified in view \( j \), then \( B \) extends \( B_k \).

**Proof.** We prove it by induction on the view number. The base case \( (j = i) \) is clear by Lemma 2.

Before moving on to the inductive step, we prove that there is a set \( R \) of more than \( f \) honest parties, such that for each party \( r \in R \), \( \text{lock}_{i,r} \) extends \( B_k \). Suppose an honest party directly decides a block \( B_k \) in view \( i \), then this party must have received \( E_i(C_i(B_k)) \), which implies at least a quorum \( (n - f) \) of party parties (hence at least \( 2\varepsilon n \) honest parties) must have propagated \( C_i(B_k) \). Let \( t \) be the time when the \( 2\varepsilon n \)-th honest party (call it \( p \)) propagates \( C_i(B_k) \). Then, more than \( 2f \) parties must have received \( C_i(B_k) \) by \( t + \Delta \). Out of these, more than \( f \) are honest, and this is the desired set \( R \). Next, we show that no party \( r \in R \) sets \( \text{lock}_{i,r} \) before \( t + \Delta \). This is because, if any \( r \in R \) does, then it must have observed \( B(v) \) before \( t - 2\Delta \), and multicast it. Then, all honest parties must have received \( B(v) \) and stopped all processes in the steady-state by time \( t \). This would prevent \( p \) from sending an echoed message for \( C_i(B_k) \) at time \( t \), which is a contradiction. Since all certified blocks in view \( i \) do not conflict with each other, every parties \( r \in R \) sets \( \text{lock}_{i,r} \) to a certified block that extends \( B_k \).

Next, we prove the inductive step (view \( j + 1 \)). Here, by inductive hypothesis, all certificates as high in rank as \( C_i(B_k) \) extend \( B_k \). Thus, all parties in \( R \) still lock on a descendant of \( B_k \) at the beginning of view \( j + 1 \). Suppose for the sake of contradiction, there exists a certificate \( C_{j+1}(B) \) as high in rank as \( C_i(B_k) \) and \( B \) does not extends \( B_k \). Let \( C_{j+1}(B_{min}) \) be the lowest ranked certificate in view \( j + 1 \) that \( B \) extends. There exists such a \( B_{min} \) since \( B \) itself satisfies these conditions. Then, there exist a set of \( n - f \) parties (say \( R' \)) that have voted for \( B_{min} \). Then, there exists a certificate \( C_j'(B_{min}^{-1}) \) where \( j' \leq j \) and \( B_{min} \) extends \( B_{min}^{-1} \), such that for every party \( r' \in R' \), \( C_j'(B_{min}^{-1}) \) is as high in rank as \( \text{lock}_{j,r'} \). Then, \( C_j'(B_{min}^{-1}) \) is as high in rank as \( C_i(B_k) \). Otherwise, there exists an honest party \( h \in R' \) such that \( \text{lock}_{j,h} \) is higher in rank than \( C_j'(B_{min}^{-1}) \) because \( R \) and \( R' \) intersect. Then, by inductive hypothesis, \( B_{min}^{-1} \) extends \( B_k \), which contradicts the hypothesis that \( B \) does not extend \( B_k \).

**Theorem 14 (Safety).** If two honest parties decide requests \( c \) and \( c' \), respectively, at the same log position, then \( c = c' \).

**Proof.** We first prove that honest parties will not decide different blocks at the same height. Suppose two distinct blocks \( B_k \) and \( B'_k \) are both decided by honest parties. Suppose each decision is a result of direct decision of \( B_l \) and \( B'_l \). Here, all directly decided block are certified. Therefore, by Lemma 2 and Lemma 3, \( B_l \) and \( B'_l \) do not conflict with each other, thus \( B_k = B'_k \).

Since all client requests in a block are totally-ordered, and two different blocks are not decided at the same height, two different requests are not decided at the same log position.

In order to achieve liveness strictly, we also need to deal with censorship resistance i.e., no client’s requests are being blocked by the leader. It is not hard to prevent censorship using additional mechanisms such as changing leaders every certain number of blocks or having time-outs for each request. However, for simplicity, we omit these details and assume that faulty leaders that censor client requests will be replaced.
Theorem 15 (Liveness). All clients requests are eventually decided by all honest parties.

Proof. We first prove that all honest parties continuously decide blocks. Suppose all honest parties permanently stay in view $i$. Then, at least an honest party decides blocks continually in view $i$, otherwise all honest parties multicast blame and invoke view-change. If an honest party decides a block $B$ in view $i$ and all honest parties do not invoke view-change, all honest parties receive $E_i(C_i(B))$ and decide $B$. And thus, all honest parties can continuously decide blocks.

Suppose the first honest party receives $B(v)$ at time $t$, then all honest parties receive it by time $t + \Delta$, and start the next view $i + 1$ after $2\Delta$ if it is not the leader $L_{i+1}$ of view $i + 1$. Therefore, all honest parties other than $L_{i+1}$ send their lock$_v$, and start view $i + 1$ within $[t + 2\Delta, t + 3\Delta]$. Since $L_{i+1}$ wait for $4\Delta$ after receiving $B(v)$, $L_{i+1}$ start view $i + 1$ at least after $t + 4\Delta$. Thus, $L_{i+1}$ can receive lock$_v$ of all honest parties before starting the next view. If $L_{i+1}$ is honest, it proposes blocks extending a block as high in rank as lock$_v$ of all honest parties, every $\alpha$ time at least after $t + 5\Delta$. And then, all honest parties can decide at least $p$ blocks by $t + (13 + 2(p - 1))\Delta$, and do not multicast blame, and thus $B(v + 1)$ is not created. Even if some leaders are faulty, after at most $f$ view-change, all honest parties can continuously decide blocks under an honest leader. Since faulty leaders that censor requests will be replaced, all requests are eventually decided by all honest parties. \hfill $\square$

Theorem 16 (Communication Complexity). Communication complexity is $O(n)$.

Proof. Suppose a leader of view $i$ is honest. Then, to decide each block, the leader sends at most $4n$ messages, and each party additionally sends at most $2d + 3$ messages, where $d$ is the degree of the expander $G_{n,\varepsilon}$ and is a constant. Thus, the communication required per decision is $O(n)$. A total of $O(n^3)$ communication may be spent on replacing up to $f$ faulty leaders. However, as parties can decide a growing number of decisions, the communication required per decision (or the amortized communication complexity) converges to $O(n)$. \hfill $\square$