CLASSIFYING REAL POLYNOMIAL PENCILS

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Abstract. Let \( \mathbb{R}P^n \) be the space of all homogeneous polynomials of degree \( n \) in two variables with real coefficients. The standard discriminant \( D_{n+1} \subset \mathbb{R}P^n \) is Whitney stratified according to the number and the multiplicities of multiple real zeros. A real polynomial pencil, that is, a line \( L \subset \mathbb{R}P^n \) is called generic if it intersects \( D_{n+1} \) transversally. Nongeneric pencils form the Grassmann discriminant \( D_{2,n+1} \subset G_{2,n+1} \), where \( G_{2,n+1} \) is the Grassmannian of lines in \( \mathbb{R}P^n \). We enumerate the connected components of the set \( \tilde{G}_{2,n+1} = G_{2,n+1} \setminus D_{2,n+1} \) of all generic lines in \( \mathbb{R}P^n \) and relate this topic to the Hawaii conjecture and the classical theorems of Obreschkoff and Hermite-Biehler.

1. Introduction and main results

In what follows by a pencil \( L = \{ \alpha P + \beta Q \} \) we will always mean a real polynomial pencil of degree \( n \) homogeneous polynomials in two real variables, i.e., a real line in \( \mathbb{R}P^n \) identified with the space of all homogeneous degree \( n \) real polynomials considered up to a constant factor. Here \( (\alpha : \beta) \in \mathbb{R}P^1 \) is a projective parameter. In order to use derivatives it will often be convenient to view homogeneous degree \( n \) polynomials in two variables as inhomogeneous polynomials of degree at most \( n \) in one variable. Any choice of a basis \( (P, Q) \) in \( L \) allows us to consider the real rational function \( P/Q \); a different choice of basis leads to a rational function of the form \( (AP + BQ)/(CP + DQ) \), which can be viewed as the postcomposition of the rational function \( P/Q \) with the real linear fractional transformation \( (Az + B)/(Cz + D) \) in the target \( \mathbb{C}P^1 \). Thus all properties of real rational functions which are invariant under real linear fractional transformations in the target space are naturally inherited by real polynomial pencils. For instance, the graph of a real rational function \( P/Q \) restricted to \( \mathbb{R}P^1 \) defines a finite branched covering \( \mathbb{R}P^1 \to \mathbb{R}P^1 \). We call two rational functions \( P_1/Q_1 \) and \( P_2/Q_2 \) graph-equivalent if there exist diffeomorphisms of the source \( \mathbb{R}P^1 \) and the target \( \mathbb{R}P^1 \) sending the graph of \( P_1/Q_1 \) to that of \( P_2/Q_2 \). As a property which is invariant under the postcomposition with a linear fractional transformation of the target space, the above graph-equivalence can be defined for the pencils \( \alpha P_1 + \beta Q_1 \) and \( \alpha P_2 + \beta Q_2 \).

\[ \text{Figure 1. Graph-equivalent and nonequivalent pencils} \]

1991 Mathematics Subject Classification. Primary 58K05; Secondary 12D10, 14P05, 26C10, 30C15.

Key words and phrases. Real polynomial pencils, Grassmann discriminant, boundary-weighted gardens.
The most classical notion of genericity for meromorphic functions/pencils requires that the function/pencil under consideration should have the maximal possible number of (simple) critical points with all distinct critical values. We will refer to this notion as Hurwitz-genericity, see §3. The classification of Hurwitz-generic real rational functions was carried out in details in [NSV]. The violation of Hurwitz-genericity essentially occurs for two basic reasons. Either several critical points collapse and form a degenerate critical point or some critical values collide but their corresponding critical points are still distinct. In the present paper we study a weaker notion of genericity than Hurwitz-genericity requiring only that all real critical points of the considered real rational functions/pencils stay simple, see Definition 1 below. This notion is the natural counterpart of the absence of the collapse of critical points in the realm of real algebraic geometry. It still keeps some important information about the behavior of real rational functions/pencils and is closely related to the natural analog of the classical discriminant for the Grassmannian of two dimensional subspaces. One more important observation is that the violation of such genericity is detected in the source space instead of the target which is always more difficult. In short, we forbid singularities and do not care about multisingularities.

Our notion of genericity allows us in particular to give a complete solution to the following problem.

Problem 1. For which pencils $L = \{\alpha P + \beta Q\}$ is the number of real zeros in this pencil constant, i.e., when is the number of real solutions (counted with multiplicities) of the equation $\alpha P + \beta Q = 0$ independent of $\alpha/\beta$?

An example of such a situation is provided by a well-known result of Obreschkoff, see [Ob], saying that a pencil $L = \{\alpha P + \beta Q\}$ consists of polynomials with only real (distinct) zeros if and only if both $P$ and $Q$ have real (distinct) and interlacing zeros. However, there exist pencils with a constant number of real zeros which are not covered by Obreschkoff’s result. For instance, one may consider the pencil $L = \{\alpha P + \beta P'\}$, where $P(x) = x^4 + x^2 - 5x - 4$.

An easy observation is that a pencil $\{\alpha P + \beta Q\}$ has a constant number of real zeros if and only if the Wronskian $W(P, Q) = PQ' - QP'$ has no real zeros, or in other words, $P$ and $Q$ form a fundamental system for some second order linear ordinary differential equation. Indeed, note that for any fixed $(\alpha : \beta)$ the number of real zeros of the equation $\alpha P + \beta Q = 0$ equals the number of the real intersection points of the rational curve $\gamma = (P, Q)$ with the line through the origin with slope $\alpha/\beta$. If the number of real intersection points is constant then there should be no real tangent lines to $\gamma$ passing through the origin. But the points on $\gamma$ where the tangent line to $\gamma$ passes through the origin correspond exactly to the real zeros of the Wronskian $W(P, Q)$, see Figure 2.

![Figure 2](image_url)

Figure 2. When is the number of real zeros constant?

Let $G_{2,n+1}$ denote as usual the Grassmannian of lines in $\mathbb{RP}^n$. The fact that the behavior of the number of real zeros in a pencil $L = \{\alpha P + \beta Q\}$ is closely related
to the properties of real zeros of the Wronskian \( W(P, Q) = PQ' - QP' \) justifies the following definition:

**Definition 1.** A real polynomial pencil \( L = \{\alpha P + \beta Q\} \) is called *generic* if the Wronskian \( W(P, Q) = PQ' - QP' \) has no multiple real zeros and it is called *non-generic* otherwise. The set \( D_{2, n+1} \) of all nongeneric real pencils in \( G_{2, n+1} \) is called the *Grassmann discriminant*.

Clearly, the degree of the Wronskian of almost any pencil in \( \mathbb{R}P^n \) equals \( 2n - 2 \). If the Wronskian of a pencil in \( \mathbb{R}P^n \) is of the degree \( 2n - 4 \) or less then we consider this pencil as degenerate (since its Wronskian has a double zero at \( \infty \)).

**Definition 2.** Two generic pencils are called *equivalent* if they can be connected by a path through generic pencils, i.e., if they belong to the same connected component of the set \( \tilde{G}_{2, n+1} = G_{2, n+1} \setminus D_{2, n+1} \) of all generic pencils in \( \mathbb{R}P^n \).

Note that as defined above, the equivalence of two generic real pencils does not necessarily imply their graph-equivalence since real critical values can collide. The main question that we address below is the following.

**Problem 2.** Enumerate the equivalence classes of all generic pencils in \( \mathbb{R}P^n \).

The study of this topic originated from our attempt to solve the following intriguing conjecture of Craven, Csordas and Smith (cf. [CCS]; see also [ShS]).

**Conjecture 1** (Hawaiian Conjecture). *If a real polynomial \( P \) has \( 2s \) nonreal zeros then the Wronskian \( W(P, P') = PP'' - (P')^2 \) has at most \( 2s \) real zeros.*

The main result of this paper – Theorem 1 below – completely solves Problem 2. The answer is given in terms of boundary-weighted gardens of total weight \( n \), a notion that we define and study in detail in §2 and §3. The notion of garden of a real rational function provides also a natural topological context for studying Conjecture 1 and related questions, see Conjectures 2 and 3 in §5.

**Theorem 1.** The connected components in the space \( \tilde{G}_{2, n+1} = G_{2, n+1} \setminus D_{2, n+1} \) of all generic pencils in \( \mathbb{R}P^n \) are in \( 1 \)–\( 1 \) correspondence with the set of equivalence classes of all boundary-weighted gardens of total weight \( n \).

From Theorem 1 and the arguments involving the Wronskian that we mentioned earlier we immediately deduce the following answer to Problem 1.

**Corollary 1.** There exist \( \left[ \frac{n+1}{2} \right] \) different components in \( \tilde{G}_{2, n+1} = G_{2, n+1} \setminus D_{2, n+1} \) where the Wronskian \( W(P, Q) \) has no real zeros at all.

The values of the number of connected components for small values of \( n \) are 1, 2, 4, 8, 14, 28 for \( n \) equal to 1, 2, 3, 4, 5, 6, respectively (see Figure 3). This sequence of integers was not recognized by the online encyclopedia of integer sequences.

The structure of the paper is as follows. In §2 we define the notions of garden, boundary-weighted garden and Morse perestroika and list some of their properties. We further study these notions in §3, where we prove the main results of the paper. In §4 we build on some of the aforementioned ideas and obtain a simple new proof of a generalization of the famous Hermite-Biehler theorem. Finally, §5 contains a number of conjectures and open problems.

**Acknowledgements.** The authors are grateful to I. Krasikov for numerous enlightening discussions. The second author is obliged to A. Eremenko, A. Gabrielov, S. Natanzon and A. Vainshtein for shaping his understanding of the topology of the space of real rational functions and the Wronski map. The financial support and the stimulating atmosphere of the program “Topological aspects of real algebraic geometry” held in Spring 2004 at MSRI Berkeley are also highly appreciated.
2. Preliminaries on Gardens and Gardening

Let us first recall the notion of garden of a real polynomial pencil as defined in [EGH1] and [NSV].

**Definition 3.** The garden $G(L)$ of a real polynomial pencil $L = \{\alpha P(z) + \beta Q(z)\}$ is the set of all $z \in \mathbb{C}P^1$ for which the rational function $f_L = \frac{P(z)}{Q(z)}$ attains real values.

Note that the defining property of $G(L)$ is actually independent of the choice of real basis $(P, Q)$ of the real polynomial pencil $L$.

Let $z$ be an affine coordinate on $\mathbb{C}P^1$. Observe that $G(L) \subset \mathbb{C}P^1$ is an algebraic curve in the coordinates $(\Re z, \Im z)$ which necessarily contains $\mathbb{R}P^1 \subset \mathbb{C}P^1$ and is invariant under the complex conjugation map $\tau : \mathbb{C}P^1 \to \mathbb{C}P^1$. Note that the singularities of any garden occur exactly at the critical points of $f_L = \frac{P(z)}{Q(z)}$ where
$f_L$ attains a real value. If such a critical point has multiplicity $m \geq 2$ then at that point the garden has a transversal intersection of $m$ nonsingular branches with angle $\frac{\pi}{m}$ between any two neighboring branches. A critical point with real critical value is called simple if its multiplicity equals $2$. A pencil $L = \{\alpha P(z) + \beta Q(z)\}$ is called nonsingular if the only critical points of $f_L$ with real critical values are real and simple. This implies that the only singularities of its garden $G(L)$ are transversal intersections of $\mathbb{R}P^1$ with other branches of $G(L)$. The garden $G(L)$ of a nonsingular pencil $L$ will be called nonsingular as well. The aforementioned transversal intersections are called the vertices of the garden. Note that the vertices of $G(L)$ correspond exactly to the real zeros of $W(P, Q)$.

We need to describe nonsingular gardens in more details. A nonsingular garden $G(L)$ is the disjoint union of two basic parts $G(L) = C(L) \cup O(L)$, the chord part $C(L)$ and the (possibly empty) oval part $O(L)$. The chord part $C(L)$ is the connected component of $G(L)$ containing $\mathbb{R}P^1$ while the oval part $O(L)$ is the complement $G(L) \backslash C(L)$. We call the edges connecting the vertices of $C(L)$ and not belonging to $\mathbb{R}P^1$ the chords. The oval part $O(L)$ consists of a number of $\tau$-invariant smooth closed curves called ovals. The connected components of $\mathbb{C}P^1 \setminus G(L)$ are called the faces of the garden $G(L)$. Let us fix the standard metric on the image $\mathbb{C}P^1$ such that the length of $\mathbb{R}P^1$ equals $1$. If we choose some basis $(P, Q)$ of the nonsingular pencil $L$ under consideration then by using the rational function $f_L = \frac{P(z)}{Q(z)} : \mathbb{C}P^1 \to \mathbb{C}P^1$ we can assign an extra piece of information to all elements of the garden $G(L)$.

**Definition 4.** The edge-weighted garden $EG(P/Q)$ of the rational function $P/Q$ is the garden $G(L)$ of the pencil $L = \{\alpha P(z) + \beta Q(z)\}$ together with all edges, chords and ovals, each of these objects being endowed with the weight given by the length of its respective image in the target $\mathbb{R}P^1$ under the rational function $P/Q$. By the total weight of an edge-weighted garden we mean the sum of the weights of all its edges, chords and ovals.

**Remark 1.** Note that the image of an edge, chord or oval can cover some interval of $\mathbb{R}P^1$ several times. The lengths/weights considered in Definition 4 are total lengths obtained by counting multiplicities. In particular, this implies that the total weight of the edge-weighted garden $EG(P/Q)$ equals the degree of $P/Q$ as a map from $\mathbb{C}P^1$ to $\mathbb{C}P^1$.

**Definition 5.** A boundary-weighted garden is a nonsingular garden with positive integer weights assigned to each boundary component of each face and satisfying the additional requirement that $\tau$-symmetric faces are assigned equal weights. The total weight of a boundary-weighted garden is the sum of the weights of all boundary components contained in the closed upper hemisphere other than ovals plus twice the weight of all ovals in the closed upper hemisphere.

There is an obvious map $\lambda$ from edge-weighted gardens to boundary-weighted gardens obtained by assigning to each boundary component the sum of the weights of the elements contained in this boundary component. Note that the latter sum is either the sum of the weights of all edges and chords if the boundary component contains them, or just the weight of an oval if the boundary component is an oval, see Figure 5. One can easily see that the image under $\lambda$ of an edge-weighted garden $EG(P/Q)$ is invariant under postcompositions of the rational function $P/Q$ with real linear fractional transformations. We may therefore associate to each nonsingular pencil a canonical boundary-weighted garden in the following way.

**Definition 6.** The boundary-weighted garden $FG(L)$ of a given nonsingular pencil $L$ is the image under the map $\lambda$ of the edge-weighted garden $EG(P/Q)$, where $(P, Q)$ is some basis of $L$. 
Note that the integer placed in each face on Figure 5 is the weight of the outer boundary component of the face (if the face is multiconnected). In order to describe connected components in the space of generic real pencils we need to introduce the following equivalence relation on the set of all boundary-weighted gardens of given total weight. In what follows we will work with the half of a garden contained in the upper hemisphere and assume that all operations are performed symmetrically.

**Definition 7.** By a *Morse perestroika* of a boundary-weighted garden we mean the following operation. Choose any face whose boundary contains either two chords, two ovals or a chord and an oval. Drag them together and cut and paste them. Under this operation two disjoint boundary components will be glued together into one whose weight is the sum of the weights of the former components. If the original face was simply connected then it will be cut into two new faces. Its boundary will be cut into two new boundary components whose weights are two arbitrary positive integers which add up to the weight of the former boundary. Two boundary-weighted gardens that can be obtained from each other by a sequence of Morse perestroikas are called *equivalent*.

It is not difficult to see that the equivalence relation introduced in Definition 7 preserves the total weight and the number of chords of a garden.

We have defined all the notions mentioned in Theorem 1 and are now ready to prove this theorem.

### 3. Proofs

We start with some generalities about $D_{2,n+1}$ which can be easily extended to linear polynomial families of higher dimensions. The following important mapping is called the *Wronski map*, see e.g. [EG2]. Let $\mathbb{K}$ denote $\mathbb{R}$ or $\mathbb{C}$. Introducing an affine coordinate $z$ on $\mathbb{K}P^1$ we can identify $\mathbb{K}P^n$ with the space of inhomogeneous polynomials of degree at most $n$ in the variable $z$. Consider now the map

$$W : G_{2,n+1} \to \mathbb{K}P^{2n-2}$$
that sends a 2-dimensional linear polynomial subspace of $\mathbb{P}^n$ to the linear span of its Wronskian, i.e., the determinant of the $2 \times 2$-matrix \[
abla_{\mathcal{W}} \left( \begin{array}{cc} P(z) & Q(z) \\ P'(z) & Q'(z) \end{array} \right), \]
where $(P(z), Q(z))$ is some basis of the chosen subspace. Note that a change of basis in the given subspace amounts to multiplying the Wronskian by a nonzero constant and that all such Wronskians are polynomials in $z$ of degree at most $2n - 2$.

Several important facts are known about the map $\mathcal{W}$. Over $\mathbb{C}$ the map $\mathcal{W}$ is finite and its degree equals the degree of $G_{2,n+1}$ under its Plücker embedding. The latter number equals the $n$-th Catalan number $C_n = \frac{1}{n} \binom{2n-2}{n}$, see [Go]. Moreover, the Wronski map is perfectly adjusted to the Schubert cell decomposition of $G_{2,n+1}$ constructed by using the natural complete flag in $\mathbb{P}^n$ whose $i$-dimensional subspaces consist of all polynomials of degree at most $i$, where $i = 0, 1, \ldots, n$. It turns out that over $\mathbb{C}$ the degree of the restriction of $\mathcal{W}$ to any of the above Schubert cells equals the degree of this cell under the Plücker embedding of $G_{2,n+1}$, see [EG2].

Denote by $\mathcal{D}_{2n-2} \subset \mathbb{P}^{2n-2}$ the standard discriminant in $\mathbb{P}^{2n-2}$, that is, the set of all polynomials having a multiple zero over $\mathbb{K}$. The Grassmann discriminant $\mathcal{D}_{2,n+1}$ introduced in Definition 4 may alternatively be characterized as follows.

**Definition 8.** The Grassmann discriminant $\mathcal{D}_{2,n+1} \subset G_{2,n+1}$ is the inverse image $W^{-1}(\mathcal{D}_{2n-2})$ of $\mathcal{D}_{2n-2}$ under the Wronski map $W$.

**Lemma 1.** The discriminant $\mathcal{D}_{2,n+1}$ consists of two irreducible components $U$ and $V$. The first component $U$ is the closure of the set of all lines in $\mathbb{P}^n$ tangent to $D_n \subset \mathbb{P}^n$ at its smooth points. The second component $V$ is the set of all lines passing through the stratum $\Sigma_3 \subset \mathcal{D}_n$, where $\Sigma_3$ consists of all polynomials having a root over $\mathbb{K}$ of multiplicity exceeding 2 (compare with [GS] and see Figure 4).

**Proof.** Take a pencil $L = \{\alpha P + \beta Q\}$ and consider the matrix

$$M_L = \begin{pmatrix} P(z) & P'(z) & P''(z) \\ Q(z) & Q'(z) & Q''(z) \end{pmatrix}. $$

If the Wronskian $W(P, Q) = \begin{vmatrix} P(z) & P'(z) \\ Q(z) & Q'(z) \end{vmatrix}$ has a multiple zero at some $z_0$ then

$$\begin{vmatrix} P(z_0) & P'(z_0) \\ Q(z_0) & Q'(z_0) \end{vmatrix} = \begin{vmatrix} P(z_0) & P''(z_0) \\ Q(z_0) & Q''(z_0) \end{vmatrix} = 0.$$ 

The latter conditions can be satisfied in two different ways. Either there exists $z_0$ such that $P(z_0) = Q(z_0) = 0$, i.e., the first column in $M_L$ vanishes at $z_0$, or the first column never vanishes but there exists $z_0$ such that the first and the second rows are linearly dependent. The first situation corresponds to the case when the rational curve $(P(z), Q(z))$ passes through the origin and the corresponding pencil in $\mathbb{P}^n$ is tangent to $D_n$. The second situation means that there exists a linear combination of $P$ and $Q$ which vanishes up to a cubic term, i.e., the pencil intersects $\Sigma_3$, which means geometrically that the curve $(P(z), Q(z))$ has a tangent line at some inflection point passing through the origin. \hfill \Box

For the sake of completeness let us present without proof yet another characterization of $D_{2,n+1}$. The **standard rational normal curve** $\rho : \mathbb{P}^1 \to \mathbb{P}^n$ is the curve consisting of all degree $n$ polynomials with an $n$-tuple root. Given a complete projective flag $f$ in $\mathbb{P}^n$ we associate to $f$ the standard Schubert cell decomposition $\mathcal{S}_f$ of $G_{2,n+1}$ whose cells consist of all 2-dimensional projective subspaces with a given set of dimensions of intersections with the subspaces of $f$. The cells are labeled by Young diagrams with at most two rows of length not exceeding $n - 1$. Given a rational curve $\gamma : \mathbb{P}^1 \to \mathbb{P}^n$ one defines its **flag lift** $\gamma_f : \mathbb{P}^1 \to F_{n+1}$
to be the curve consisting of all osculating flags to $\gamma$. As is well known, the same definition applies in fact to any projective algebraic curve.

**Proposition 1.** The component $U$ (respectively, $V$) of $D_{2,n+1}$ is the union of the Schubert cells $\bigcup_{f \in \rho^F} C_{1,1}(f)$ (respectively, $\bigcup_{f \in \rho^F} C_{2,0}(f)$), where $f$ runs over the flag lift $\rho^F$ of the standard rational curve $\rho$. Here $C_{1,1}(f)$ is the cell of codimension two in $G_{2,n+1}$ whose Young diagram with respect to $f$ is $(1, 1)$ while $C_{2,0}$ is the cell whose Young diagram with respect to $f$ is $(2, 0)$.

![Figure 7. Section of the $D_{2,4}$-discriminant transversal to the lift of $\rho$ to $G_{2,4}$ using tangent lines](image)

In order to complete the proof of Theorem 1 we need several additional definitions and constructions. Let us first recall the following classical definition.

**Definition 9.** A pencil $L = \{\alpha P(z) + \beta Q(z)\}$ of degree $n$ polynomials is called *Hurwitz-generic* if the rational function $f_L = \frac{P(z)}{Q(z)}$ has $2n - 2$ distinct critical points with distinct critical values and it is called *Hurwitz-nongeneric* otherwise.

**Remark 2.** As we already noted in the introduction, a real rational function of the form $(AP + BQ)/(CP + DQ)$ may be viewed as the postcomposition of the rational function $P/Q$ with the linear fractional transformation $(Az + B)/(Cz + D)$ in the target $\mathbb{C}P^1$. This shows that the property introduced in Definition 9 is independent of the choice of basis of the pencil $L$.

**Definition 10.** The *Hurwitz discriminant* is the subset $H_{2,n+1} \subset G_{2,n+1}$ consisting of all Hurwitz-nongeneric pencils in the Grassmannian of lines in $\mathbb{R}P^n$.

Clearly, any real Hurwitz-generic pencil $L$ is generic in the sense of Definition 9. Moreover, such a pencil is also nonsingular, i.e., it has a nonsingular garden $\mathcal{G}(L)$. Indeed, any complex critical point together with its complex conjugate form a pair that cannot have a real critical value. This proves the following lemma.

**Lemma 2.** The Grassmann discriminant $D_{2,n+1}$ is always contained in the Hurwitz discriminant $H_{2,n+1}$.

We say that a nonsingular garden is *directed* if its edges, chords and ovals are directed in such a way that the boundary of each face becomes a directed cycle. This means that any given face will lie either to the right of any of its boundary components or to the left of any such component whenever we follow the direction that has been assigned to a boundary component. The faces that lie to the left of all of their boundary components are called *positive* while faces lying to the right of their boundary components are called *negative*. All neighbors of positive faces are negative and vice versa. Obviously, in order to direct a garden it suffices to direct any one of its edges. Therefore, there exist exactly two possible ways of directing
Theorem 2. Let $\mathcal{H}_{2,n+1}$ denote the divisor of all Hurwitz-nongeneric pencils. The connected components in the space $\bar{\mathcal{H}}_{2,n+1} = G_{2,n+1} \setminus \mathcal{H}_{2,n+1}$ of all Hurwitz-generic pencils are in 1–1 correspondence with the set of all properly directed and cyclicly labeled gardens of weight $n$ modulo the action of the involution.

Recall from Lemma 2 that the Grassmann discriminant $\mathcal{D}_{2,n+1}$ and the Hurwitz discriminant $\mathcal{H}_{2,n+1}$ satisfy $\mathcal{H}_{2,n+1} \supset \mathcal{D}_{2,n+1}$. For our further purposes we need the following description of $\mathcal{H}_{2,n+1}$.

Theorem 3. The Hurwitz discriminant $\mathcal{H}_{2,n+1}$ is the union of four real discriminants $U$, $V$, $W$ and $Z$, where $U$ and $V$ are defined in Lemma 2 and $W$ and $Z$ are two real algebraic hypersurfaces with the same complexification, namely the hypersurface of all coinciding critical values. More precisely, $W$ is the set of all real pencils $L = \{\alpha P + \beta Q\}$ for which the rational function $f_L = \frac{P(z)}{Q(z)}$ has two real critical points with coinciding real critical value, while $Z$ is the set of all real pencils $L = \{\alpha P + \beta Q\}$ for which the rational function $f_L = \frac{P(z)}{Q(z)}$ has two complex conjugate critical points with coinciding (and therefore real) critical value.

Our plan is as follows. We will show that by crossing $W$ one can realize any admissible relabeling of a given cyclicly labeled boundary-weighted garden and that by crossing $Z$ we can realize any of its admissible Morse perestroikas. These two facts will be easy corollaries of the following statements.

Theorem 4. Any edge-weighted garden $\mathcal{G}$ of total weight $n$ is realized by a real rational function of degree $n$. Moreover, the set of all real rational functions with a given edge-weighted oriented garden is path-connected.

Here by an edge-weighted garden of total weight $n$ we understand an abstract embedded $\tau$-invariant “graph” containing $\mathbb{R}P^1$ with vertices only of even multiplicity and possibly containing a number of $\tau$-invariant ovals considered up to a diffeomorphism of the plane. All edges, chords and ovals of this “graph” are equipped with positive weights. Moreover, ovals have integer weights. Finally, for any boundary component the sum of all weights in this component is a positive integer and the sum of the weights of all elements in this “graph” equals $n$. It is important to note that in Theorem 4 we do not assume that $\mathcal{G}$ is a nonsingular garden and that we actually allow arbitrary complex critical points with real critical values.
Proof of Theorem 4. The proof is based on ideas similar to those used in the proof of Theorem 1 in [NSV] and so we will only sketch it here. (The only major difference compared to [NSV] is that we allow singular gardens.) We want to construct a topological branched covering \( \mathbb{C}P^1 \to \mathbb{C}P^1 \) which is invariant under complex conjugation and whose garden is isomorphic to \( \mathcal{G} \). This will prove the realization theorem, since by Riemann’s uniqueness theorem there exists a unique complex structure on \( \mathbb{C}P^1 \) for which this topological covering is holomorphic. The orientation of the garden uniquely specifies which of its faces should be mapped to the upper hemisphere and also which faces should be mapped to the lower hemisphere. (Neighboring faces are always mapped to opposite hemispheres.) Each open face of the garden is a topological surface of genus 0. The normalization of its closure is a closed topological surface with boundary. Now for any face of \( \mathcal{G} \) consider the total weight of its boundary components, that is, the number of times each boundary component should traverse \( \mathbb{R}P^1 \). The Riemann-Hurwitz formula determines the number of simple complex critical points the face under consideration should contain. We also know in which hemisphere the corresponding critical values should lie. Let us now recall some definitions from [NSV] §3.1. Denote by \( \Lambda^± \) the upper hemisphere \( \{ z \in \mathbb{C}P^1 \mid \text{Im}\, z \geq 0 \} \) and by \( P \) a genus \( g \) topological surface with a boundary consisting of \( k \) connected components. Consider the set \( \mathcal{N}_{g,m} \) of all generic degree \( m \) branched coverings of the form \( \phi : P \to \Lambda^+ \) and let \( a_1, \ldots, a_k \) be all the distinct connected components of \( \partial P \). Given a partition \((m_1, \ldots, m_k) \vdash m \) denote by \( \mathcal{N}_{g,m}^k(m_1, \ldots, m_k) \subset \mathcal{N}_{g,m}^k \) the subset of maps \( \phi : P \to \Lambda^+ \) such that \( \deg \phi|_{a_i} = m_i \) for \( i = 1, \ldots, k \). Obviously,

\[
\mathcal{N}_{g,m}^k \cap (m_1, \ldots, m_k) = \bigcup_{(m_1, \ldots, m_k) \vdash m} \mathcal{N}_{g,m}^k(m_1, \ldots, m_k).
\]

Let \( f \) be a face in the upper hemisphere of \( \mathbb{C}P^1 \setminus \mathcal{G} \) and consider the space \( \mathcal{N}_f \) of all branched coverings from the normalization of the closure of \( f \) to \( \Lambda^± \), where \( \Lambda^± \) is the upper or lower hemisphere depending on where \( f \) should be mapped according to the chosen orientation. Lemma 2 in [NSV] shows that for any partition \((m_1, \ldots, m_k) \vdash m \) the space \( \mathcal{N}_{g,m}^k(m_1, \ldots, m_k) \) is path-connected. In particular, this implies that each space \( \mathcal{N}_f \) is path-connected. We need the following result.

**Lemma 3.** Let \( \mathcal{G} \) be an edge-weighted oriented garden and fix an arbitrary set of (critical) values for the vertices belonging to its chords. Denote by \( \text{Rat}_G \) the set of all real rational functions with edge-weighted oriented garden \( \mathcal{G} \) and having these prescribed critical values. Then \( \text{Rat}_G \) is homeomorphic to \( \Pi_{f \in \text{Ind}_G} \mathcal{N}_f \times (\mathbb{R}P^1)^q \), where \( q \) is the number of different connected components of \( \mathcal{G} \) containing vertices – i.e., critical points with real critical values – and \( \text{Ind}_G \) is the index set of all faces \( f \) in the upper hemisphere of \( \mathbb{C}P^1 \setminus \mathcal{G} \).

**Remark 3.** Note that for a nonsingular garden with a positive number of vertices one has \( q = 1 \) since all its vertices belong to \( \mathbb{R}P^1 \). However, the singular gardens considered in Theorem 4 might contain singular ovals with vertices which are not connected to \( \mathbb{R}P^1 \).

**Proof of Lemma 3.** Let us show first that by assigning all real critical values and picking an arbitrary map \( \phi_f \) from each space \( \mathcal{N}_f \) for \( f \in \text{Ind}_G \) we can glue together all the \( \phi_f \)'s into precisely one half of a unique real rational function from \( \text{Rat}_G \). This follows simply from the fact that the real critical values determine exactly which parts of the boundary components of \( \phi_f \) and \( \phi_{f'} \) for any two neighboring faces \( f_1 \) and \( f_2 \) should be identified (glued together). Indeed, by gluing together all the \( \phi_f \)'s for all \( f \in \text{Ind}_G \) according to this recipe we get a unique map from \( \Lambda^+ \) to \( \mathbb{C}P^1 \). We may then take the conjugate copy of the latter map and glue the two
halves together along $\mathbb{RP}^1$ into a sphere $\mathbb{CP}^1$, thus obtaining a unique final map $\mathbb{CP}^1 \to \mathbb{CP}^1$. One can easily see that the final map is the topological branched covering that satisfies all the properties required. It just remains to notice that in order to assign all real critical values for an edge-weighted garden it is necessary and sufficient to assign arbitrarily just one real critical value for each connected component of $G$ containing vertices. The critical values of the remaining vertices in each such component will then be automatically restored from the set of weights of the chords and ovals in the respective component.

To finish the proof of Theorem 4 just notice that the Cartesian product of path-connected topological spaces is path-connected.

**Corollary 2.** Any admissible Morse perestroika of a given nonsingular boundary-weighted garden is realizable.

**Proof.** Any singular garden that occurs while performing an arbitrary generic perestroika contains just two simple complex conjugate critical points with a common real critical value. It follows from Theorem 4 that such a garden can be realized by a rational function. Any small generic 1-parameter deformation of this rational function will necessarily produce the required perestroika. Indeed, in any such deformation the imaginary part of the interesting critical value will necessarily change signs while the rest of the garden will topologically stay the same.

**Theorem 5.** The set of all real rational functions with a given boundary-weighted oriented garden is path-connected.

**Proof.** We use an argument similar to that of [EG1]. Let $G$ be an oriented boundary-weighted garden and denote by $\mathcal{ELG}$ the set of all possible edge-weighted gardens whose boundary-weighted gardens coincide with $G$, see Figure 5. Enumerating arbitrarily all chords and edges in $G$ and denoting the weight of the $i$-th chord by $w_{c,i}$, the weight of the $j$-th edge by $w_{e,j}$ and the weight of the $m$-th boundary component by $w(B_m)$ we get the following system of linear inequalities (one for each edge and chord) and linear equations (one for each boundary component other than an oval) satisfied by edge weights for all gardens in $\mathcal{ELG}$

\[
\begin{align*}
    w_{c,i} &> 0, \\
    w_{e,j} &> 0, \\
    \sum_{i \in B_m} w_{c,i} + \sum_{j \in B_m} w_{e,j} &= w(B_m).
\end{align*}
\]

Let $\text{Sol}_G$ denote the set of all solutions to system $\text{I}$. Obviously, $\text{Sol}_G$ is a nonempty convex polytope. For any solution of $\text{I}$ we get an edge-weighted oriented garden. By Theorem 3 the set of all real rational functions realizing such a garden is path-connected. Therefore, the set of real rational functions with a given oriented boundary-weighted garden is actually fibered over a contractible base with isomorphic path-connected fibers. (Note that by Lemma 3 the topology of the fiber does not depend on the particular weights of the chords.) Thus the total space of fibration is path-connected.

**Corollary 3.** Any admissible relabeling of a given boundary-weighted and cyclicly labeled garden is realizable.

**Proof.** Take any admissible labeling of a given boundary-weighted garden. Place its labels arbitrarily on $\mathbb{RP}^1$ in an order-preserving way, i.e., assign real critical values to all real critical points. Then one can restore the weights of all the chords and edges of the garden. These weights will necessarily satisfy system $\text{I}$ given above. Having done so for two different labelings and using the fact that the set of rational functions in Theorem 5 is path-connected we conclude that we can find a path from
Proposition 2. The classical Hermite-Biehler theorem asserts that given two polynomials $P$ and $Q$ with real coefficients and of degrees $n$ and $n-1$, respectively, the zeros of the complex polynomial $S = P + iQ$ have (nonzero) imaginary parts of the same sign if and only if $P$ and $Q$ have real distinct and interlacing zeros. In fact, if $\mu$ is an arbitrary complex number and $\sharp_+$ denotes the number of zeros of the polynomial $S_\mu := P + \mu Q$ lying in the upper half-plane then the following more general result is known to be true, see [Ga].

**Proposition 2.** In the above notation consider the plane real rational curve $\gamma_\mu$ given by $(P + R\mu \cdot Q, \Im \mu \cdot Q)$. Then $\sharp_+$ equals the winding number of $\gamma_\mu$ around the origin.

Below we give a new proof of the generalized Hermite-Biehler theorem for all pairs $(P, Q)$ of real polynomials. In particular, our method yields a simple proof of the main result in [HDB].

**Proposition 3.** For given polynomials $P$ and $Q$ with real coefficients the complex polynomial $S_\mu = P + \mu Q$ with $\mu \notin \mathbb{R}$ has a real zero if and only if $P$ and $Q$ have a common real zero.

**Proof.** Indeed, if $P(x)$ and $Q(x)$ have a common real zero $x_0$ then $S_\mu(x_0) = 0$. On the other hand, if for some $x_0 \in \mathbb{R}$ one has $S_\mu(x_0) = 0$ then $P(x_0) + R\mu \cdot Q(x_0) = 0$ and $\Im \mu \cdot Q(x_0) = 0$, which immediately imply $P(x_0) = Q(x_0) = 0$ since $\Im \mu \neq 0$. □

A convenient geometric reformulation of this statement is as follows. Denote by $\text{Pol}_n$ the space of all monic degree $n$ polynomials with complex coefficients of the form $S(z) = z^n + a_1z^{n-1} + a_2z^{n-2} + \ldots + a_n$ and let $\mathcal{RD} \subset \text{Pol}_n$ be the hypersurface of all polynomials $S$ that have at least one real zero. Finally, let $\text{Res} \subseteq \text{Pol}_n$ be the hypersurface of all $S = P + iQ$ such that $P$ and $Q$ have a real common zero. In the literature on singularities $\text{Res}$ is often called the (generalized) Whitney umbrella.

**Corollary 4.** The discriminant $\mathcal{RD}$ coincides with the resultant hypersurface $\text{Res}$.

**Remark 4.** In the definition of $\text{Res}$ we disregard the subvariety of real codimension two where $P$ and $Q$ have common complex zeros.

Given an arrangement of black and white distinct points on $\mathbb{R}$ we define its canonical reduction to be the interlacing (possibly empty) arrangement obtained in the following way: if our arrangement contains a pair of neighboring points of
the same color then we remove these points and we continue this procedure until no such removals can be performed. Note that the resulting canonical reduction depends only on the initial (relative) order of the points in the given arrangement and not on their exact locations on $\mathbb{R}$.

**Corollary 5** (cf. [KS]). The number of connected components in $Pol_n \setminus RD$ equals $n + 1$ and these components can be labeled by the canonical reductions as follows. Let $P$ and $Q$ be polynomials with real coefficients of degree $n$ and $n - 1$, respectively. Assume that they have no common real zeros and that the leading coefficient of $Q$ is positive. Then for the polynomial $S_\mu = P + \mu Q$ with $\Im \mu \neq 0$ one has $\sharp_+ - \sharp_- = \kappa T$, where $\sharp_+$ (respectively, $\sharp_-$) is the number of zeros of $S_\mu$ in the upper (respectively, lower) half-plane, $\kappa$ is the sign of $\Im \mu$ and $T$ is the number of zeros of $P$ appearing in the canonical reduction of the real zeros of $P$ and $Q$.

5. Final remarks

As we already mentioned in the introduction, the notion of garden of a real rational function provides a natural topological framework for investigating the Hawaii conjecture (Conjecture 1). Indeed, given a polynomial $P$ of degree $n$ with real coefficients let us consider the garden $G_P$ of the rational function $P'/P$ (cf. Definitions 3–4). Obviously, all zeros of $P$ lie on $G_P$. We make the following conjecture.

**Conjecture 2.** Each chord of $G_P$ contains at least one nonreal zero of $P$.

Note that Conjecture 2 would immediately imply the Hawaii conjecture since the real critical points of $P'/P$ are the same as the real zeros of the Wronskian $W(P, P')$ and the latter are precisely the endpoints of the chords in $G_P$ (cf. §2).

It is natural to ask whether the Hawaii conjecture extends to classes of rational functions other than logarithmic derivatives. Let $n$ be a positive integer and denote by $QP_n$ the set of all nonidentically vanishing rational functions of the form

$$f(x) = \sum_{i=1}^{n} c_i P_i^\alpha(x),$$

where $c_i \in \mathbb{R}$ for $1 \leq i \leq n$, $\alpha$ is a real number satisfying $\alpha \leq -1$ and $P_1, \ldots, P_n$ are second degree monic polynomials with real coefficients without real zeros. Based on extensive numerical experiments, we propose the following analog of Conjecture 1 for the class $QP_n$.

**Conjecture 3.** If $f \in QP_n$ then $f$ has at most $2n - 1$ real critical points. Moreover, if $\alpha$ is a negative integer then the following analog of Conjecture 2 holds: each chord of the garden $G_f$ of the real rational function $f$ contains at least one nonreal zero of the polynomial $\prod_{i=1}^{n} P_i(x)$.

A possible way to attack Conjecture 3 might be as follows. Let us first recall the definition of a Chebycheff system as given in e.g. [K].

**Definition 11.** A linear $n$-dimensional space $V$ of smooth real-valued functions defined on some interval $(a, b)$ ($a$ might be equal to $-\infty$ and $b$ to $+\infty$) is called a Chebycheff system if any nonidentically vanishing function $f \in V$ has at most $n - 1$ real zeros on $(a, b)$ counted with multiplicities.

**Problem 3.** Let $P_1, \ldots, P_n$ be as in (2) and $\alpha \in \mathbb{R}$ with $\alpha \leq -1$. Is it possible to extend the $n$-tuple of functions $(P_1^\alpha(x))', \ldots, (P_n^\alpha(x))'$ to a Chebycheff system of dimension $2n$ on $(-\infty, +\infty)$?

Note that an affirmative answer to Problem 3 would automatically confirm the validity of Conjecture 3.
The main question about the classification of generic pencils (Problem 2) extends straightforwardly to polynomial families with more than one parameter. However, a solution to the problem of enumerating connected components in other Grassmannians similar to Theorem 1 would first require an appropriate definition of the notion of garden in these cases.

To conclude, let us formulate some related questions.

**Problem 4.** What can one say about the topology of the space of generic pencils $\tilde{G}_{2,n+1} = G_{2,n+1} \backslash D_{2,n+1}$? For instance, are connected components in $\tilde{G}_{2,n+1}$ contractible? Note that this is true for polynomials without multiple real roots.

A real rational function of degree $n$ is called an $M$-function if all its $2n-2$ critical points and critical values are real and distinct. Any $M$-function of degree $n$ induces a degree $n$ map $\mathbb{RP}^1 \to \mathbb{RP}^1$ with exactly $2n - 2$ branching points.

**Problem 5.** What type of maps $\mathbb{RP}^1 \to \mathbb{RP}^1$ of degree $n$ with $2n - 2$ branching points can occur from $M$-functions?

More precisely, given a map $\mathbb{RP}^1 \to \mathbb{RP}^1$ of degree $n$ with $2n - 2$ simple branching points let us label its $n$ real critical points and the corresponding $n$ critical values cyclicly. Then we can associate to this map the unique cyclic permutation of length $2n - 2$ sending each critical point to its critical value. Problem 5 may therefore be reformulated as follows.

**Problem 6.** What cyclic permutation can an $M$-function have?

Note that Problem 5 is actually asking for a description of all possible shapes of the graphs of real rational $M$-functions – a topic which is standardly considered in elementary calculus courses if one omits “$M$-” in the above formulation. However, in the general case the answer to Problem 5 seems to be unknown and quite nontrivial.

**Problem 7.** Enumerate the connected components in the space of real rational functions having only simple real critical points with distinct critical values.

The arguments given in the introduction show that all real pencils are necessarily graph-equivalent in each such component. This project is the intermediate situation between the one covered in [NSV] and the one described in the present article.

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