Hurwitz numbers and BKP hierarchy

S.M. Natanzon∗ A. Yu. Orlov†

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Abstract

We consider $d$-fold branched coverings of $\mathbb{RP}^2$ with arbitrary ramification type over $\infty \in \mathbb{RP}^2$, an arbitrary number of simple ramifications, plus a fixed number of ramifications with given profile lengths and a colored group with arbitrary ramification type provided the condition that the sum of the profile lengths inside of each group is given. We prove that the generating function for the signed sums of such coverings is a tau function of the BKP hierarchy of Kac and van de Leur. We compare TL and BKP tau functions generating Hurwitz numbers.

Key words: Hurwitz numbers, Schur functions, tau functions, hypergeometric functions, BKP, random partitions,

1 Introduction

In the beautiful paper of A. Okounkov [1] ramified coverings of the Riemann sphere with arbitrary ramification type over 0 and $\infty$ and simple ramifications elsewhere was considered and it was proven that the generating function for the related Hurwitz numbers (numbers of nonequivalent coverings with given ramification type) is a tau-function for the Toda lattice hierarchy. In further works [15–19] other examples of tau-functions for 2D Toda and KP hierarchy generating Hurwitz numbers of the sphere were constructed. In recent work [41] there was considered rather general TL tau function which gives some new examples of what was called the composite signed Hurwitz numbers and includes previous examples.

In the present paper we consider Hurwitz numbers of the projective plane $\mathbb{RP}^2$. Technically our work is partially related to [41] but now the base surface is not the Riemann sphere but the projective plane. For our purpose we use the BKP hierarchy of integrable equations introduced by V.Kac and J. van de Leur in [21]. (A brief explanation what is going on in case we change the hierarchy is given in the next paragraph.) Our result is that the BKP tau function (43) is the generating function for the certain linear combinations (23) of numbers of (possibly disjoint) coverings of the projective plane (22).

The paper is organized as follows. First we recall some facts about BKP and TL hierarchies. We need a special class of tau functions we call hypergeometric type. In the Section 3 the precise problem we consider is posed. In Section 4 we review TL tau functions generating composite signed Hurwitz numbers according to [41]. However we need a modification caused by semiinfinity of TL which we need to compare results with the BKP case later in Section 6. The result of the paper is written down in Section 5.

In Section C of the Appendix we show that one can consider the further case where many ramification profiles are fixed. The generating functions for these Hurwitz numbers are not tau functions at all but certain integrals of tau functions of the hypergeometric type. In the Appendix D we mention the case of Euler characteristics exceeding 2 on the background of BKP tau functions.

∗National Research University Higher School of Economics, Moscow, Russia; Institute for Theoretical and Experimental Physics, Moscow, Russia; Laboratory of Quantum Topology, Chelyabinsk State University, Chelyabinsk, Russia; email: natanzons@mail.ru

†Institute of Oceanology, Nahimovskii Prospekt 36, Moscow 117997, Russia, and National Research University Higher School of Economics, International Laboratory of Representation Theory and Mathematical Physics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russia, email: orlovs@ocean.ru
An outlook. For a reader who is familiar with the topic let us briefly explain the difference between TL and BKP tau functions in the context of generating of Hurwitz numbers and the Euler characteristic of the base - just looking at formalae a priori going to details and definitions.

The Frobenius formula for the Hurwitz numbers enumerating $d$-fold branched coverings of the Riemann surfaces contains the sum over irreducible representations $\lambda$ of the symmetric group

$$H_{\Omega}(\Delta^{(1)}, \ldots, \Delta^{(k)}) = C \sum_{|\lambda| = d} \left( \prod_{i=1}^{k} \frac{\chi_{\lambda}(\Delta^{(i)})}{\chi_{\lambda}(1)} \right) \chi_{\lambda}(1)^{e}$$  \hspace{1cm} (1)

see for instance \[20\], where (an even) $e$ is the Euler characteristic of a base Riemann surface $\Omega$, $d$ is the weight of profiles $\Delta^{(i)}$ over ramification points on $\Omega$, and $\chi_{\lambda}(\Delta)$ is the character of the symmetric group $S_{d}$ evaluated at a cycle type $\Delta$. A profile $\Delta^{(i)}$ is a partition of $d$ - the set of nonegative nonincreasing numbers $(d^{(1)}, d^{(2)}, \ldots)$ which describes the ramification over the point number $i$ on the base. In the context of string theory formula (1) is used by physicists, see for instance \[2\].

Here and below we write $\chi_{\lambda}(1)$ having in mind the evaluation of the irreducible character of the symmetric group $\chi_{\lambda}$ at the unity element in the symmetric group, which is given by the partition $(1^{d})$, otherwise $\chi_{\lambda}(1) = \dim \lambda$.

It is well-known that Schur functions $s_{\lambda}$ and characters of the symmetric group $\chi_{\lambda}$ are linearly dependent \[30\]. Soliton theory provides various series of products of the Schur functions over partitions for tau functions of various hierarchies of integrable equations. Okounkov found in \[1\] that the following sum

$$\sum_{\lambda} e^{\beta n|\lambda|} \exp \left( \beta |C_{\Gamma}| \frac{\chi_{\lambda}(\Gamma)}{\chi_{\lambda}(1)} \right) s_{\lambda}(p)s_{\lambda}(\bar{p})$$  \hspace{1cm} (2)

may be considered as a tau function of the Toda lattice (where power sum variables $p = (p_{1}, p_{2}, \ldots)$ and $\bar{p} = (\bar{p}_{1}, \bar{p}_{2}, \ldots)$ the integer $n$ play the role of higher times), and generates a certain class of Hurwitz numbers \(1\). This class describes coverings of the Riemann sphere ($e = 2$) with arbitrary given profiles over two given points (say, $0$ and $\infty$) and any number of simple ramifications described by the Young diagram $\Gamma = \Gamma_{\lambda} := (1^{d-2})$. One can recognize it thanks to the the relation between the Schur functions and the characters of the symmetric group \[30\]:

$$s_{\lambda}(p) = \chi_{\lambda}(1) \left( p_{\lambda} + \sum_{\Delta \neq \lambda |d} \frac{p_{\Delta}}{\Delta} \frac{\chi_{\lambda}(\Delta)}{\chi_{\lambda}(1)} \right)$$  \hspace{1cm} (3)

where the summation ranges over all partitions $\Delta = (d_{1}, d_{2}, \ldots)$ of the number $d = |\lambda|$, and $p_{\lambda}$ is the product $p_{d_{i}}p_{d_{j}} \cdots$, while for the brief outlook the numbers $\Delta$ and $|C_{\Gamma}|$ in (2) are unimportant, we only need they depend only on $d = |\lambda|$, not on $\lambda$, which enters the defying relation (1).

Now, it is clear from (1) that formula (2) is a generating function for Hurwitz numbers \[1\] where $\beta$, $p$ and $\bar{p}$ play the role of formal parameters. Basically, the Taylor coefficients in the terms $p_{\Delta} p_{\Delta}^{\beta \lambda}$ up to a factor coincide with the number of covers with the ramification type $\Delta, \Delta'$ over two points and further of type $1^{(d-2)}$ over $b$ points.

One may present (see \[1\]) a larger class of tau functions which generates combinations of Hurwitz numbers \[1\], namely

$$r^{2K_{p}}(n, p, \bar{p}) = \sum_{\lambda} r_{\lambda}(n) s_{\lambda}(p)s_{\lambda}(\bar{p})$$  \hspace{1cm} (4)

where now

$$r_{\lambda}(n) = e^{\beta n|\lambda|} \exp \left( \beta \frac{\chi_{\lambda}(1^{d-2})}{\chi_{\lambda}(1)} \right) \prod_{i=1}^{k} \frac{s_{\lambda}(p(a_{i} + n, q_{i}))}{s_{\lambda}(p(\infty, q_{i}))} \prod_{i=k+1}^{k+s} \frac{s_{\lambda}(p(n, q_{i}))}{s_{\lambda}(p(b_{i} + n, q_{i}))}$$  \hspace{1cm} (5)

and $p(a_{i}, q_{i}) = (p_{1}(a_{i}, q_{i}), p_{2}(a_{i}, q_{i}), \ldots)$ with

$$p_{m}(a_{i}, q_{i}) := \frac{1 - q_{i}^{m a_{i}}}{1 - q_{i}^{m}}, \quad p_{m}(\infty, q_{i}) := \frac{1}{1 - q_{i}^{m}}$$  \hspace{1cm} (6)
where \(a_i, q_i\) are arbitrary chosen. On a relation of these \(p(a_i, q_i)\) to Macdonald polynomials see the Appendix A.

Such TL tau functions were found in \([10]\). Certain specifications of series \((4)-(6)\) are known as hyper-geometric functions of matrix arguments \([27]\) (where all \(q_i \rightarrow 1\)) and further as Milne’s hypergeometric functions \([28]\) (where all \(q_i = q\)).

Now, let us replace the Schur functions in the expression for \(r_\lambda\) by their expansions \([3]\). We see that the common prefactor \(\chi_\lambda(1)\) is canceled in \(r_\lambda\), and that each character evaluated on a cycle \(\Delta\) enters the final expression for the tau function only as the ratio \(\frac{\chi_\lambda(\Delta)}{\chi_\lambda(1)}\), as in \([1]\). Then, the product of two Schur functions in the right hand side of formula \([1]\) produces the common factor \(\chi_\lambda(1)^2\) in each term of the sum labeled by \(\lambda\). As a result one may conclude that, similar to the Okounkov tau function, the TL tau function \([1]\) generates linear combinations of number of covering of the sphere. These specific combinations of Hurwitz numbers (“composite signed Hurwitz numbers”) in case \(q_i \rightarrow 1\) are written down in \([11]\) and for our convenience are also reproduced in the present text.

The following series

\[
\tau^{BKP}(n, p) = \sum_\lambda r_\lambda(n) s_\lambda(p) \tag{7}
\]

with the same \(r_\lambda(n)\) is also a tau function \([31]\) where the set \(p\) plays the role of higher times, but now it is a different hierarchy, namely, the BKP hierarchy introduced by Kac and van de Leur \([21]\). This case will be considered in the present paper. It is easy to see that now \(e = 1\). As it was shown in the papers \([1, 12, 13, 14]\) (see also the Remark to the Appendix A in \([20]\)) the Frobenius formula is still true for odd values of \(e\) which means that the base is non-orientable surface, then the case \(e = 1\) describes the coverings of the projective plane \(\mathbb{RP}^2\).

If we consider the further case \(e = 0\) (the coverings of the elliptic curve) with the help of the series

\[
\tau = \sum_{n, \lambda} r_\lambda(n) \tag{8}
\]

then we see it is not a tau function as there are no time variables here. This expression may be related to the trace of the certain diagonal \(GL_\infty\) element in the fermionic Fock space.

Thus, on the formal level we can explain the appearance of different hierarchies of integrable equations in the description of Hurwitz counting problem.

Series \((4), (7)\) and \((8)\) are related as follows

\[
\tau = \sum_n \left[ e^{L_\infty(\tilde{\partial})} \cdot \tau^{BKP}(n, p) \right]_{p=0}, \quad \tau^{BKP}( p) = \left[ e^{L_\infty(\tilde{\partial})} , \tau^{TL}(p, p) \right]_{p=0} \tag{9}
\]

where \(L_\infty(\tilde{\partial})\) is the Laplace operator \(\sum m \geq 0 \left( \frac{m}{2} \frac{\partial^2}{\partial p_m^2} + \frac{\partial}{\partial p_{m-1}} \right)\), see Section 6.

This paper is the detalization of the consideration above. We shall study only the case all \(q_i \rightarrow 1\).

## 2 BKP tau functions

There are two different BKP hierarchies of integrable equations, one was introduced by Kyoto group in \([23]\), the other was introduced by V. Kac and J. van de Leur in \([21]\). We need the last one. This hierarchy includes the celebrated KP one as a particular reduction. In a certain way (see \([29]\)) the BKP hierarchy may be related to the three-component KP hierarchy introduced in \([23]\) (earlier described in \([24]\) with the help of L-A pairs with matrix valued coefficients). For the detailed description of the BKP we send readers to the original work \([21]\), and here we write down first non-trivial equations for the BKP tau function (Hirota equations). These are

\[
\frac{1}{2} \frac{\partial \tau(N, n, p)}{\partial p_2} \tau(N + 1, n, p) - \frac{1}{2} \tau(N, n, p) \frac{\partial \tau(N + 1, n, p)}{\partial p_2} + \frac{1}{2} \frac{\partial^2 \tau(N, n, p)}{\partial^2 p_1} \tau(N + 1, n, p)
\]

\[
+ \frac{1}{2} \tau(N, n, p) \frac{\partial^2 \tau(N + 1, n, p)}{\partial^2 p_1} - \tau(N, n, p) \frac{\partial \tau(N + 1, n, p)}{\partial p_1} \frac{\partial \tau(N + 1, n, p)}{\partial p_1} = \tau(N + 2, n, p) \tau(N - 1, n, p) \tag{10}
\]
The BKP tau functions depend on the set of higher times $t_m = \frac{1}{m}p_m$, $m \geq 1$ and the discrete parameter $N$. In [31] the second discrete parameter $n$ was added and equation (11) relates BKP tau functions with neighboring $n$. The complete set of the Hirota equations with two discrete parameters is written down in the Appendix.

The general solution to Hirota equations may be written as

$$\tau_{\text{BKP}}(N, n, p) = \sum_{\lambda \in P} A_{\lambda}(N, n) s_{\lambda}(p)$$

(12)

where $P$ is the set of all partitions and where $A_{\lambda}$ solves Plucker relations for isotropic Grassmannian and may be written in a pfaffian form. The Schur function is defined as follows

$$s_{\lambda}(p) = \det (s_{\lambda-i+j}(p))_{i,j}, \quad e_{m > 0} \frac{1}{m} z^m =: \sum_{m \geq 0} z^m s_m(p)$$

(13)

**TL tau function.** Here we recall few facts about the different hierarchy, Toda lattice one, and for details we refer to the paper [33]. The simplest Hirota equation for the Toda lattice is

$$\frac{\partial^2 \tau_{\text{TL}}(n, p, \bar{p})}{\partial p_1 \partial \bar{p}_1} \tau_{\text{TL}}(n, p, \bar{p}) - \frac{\partial \tau_{\text{TL}}(n, p, \bar{p})}{\partial p_1} \frac{\partial \tau_{\text{TL}}(n, p, \bar{p})}{\partial \bar{p}_1} = - \tau_{\text{TL}}(n + 1, p, \bar{p}) \tau_{\text{TL}}(n - 1, p, \bar{p})$$

(14)

TL tau function may be written in form

$$\tau_{\text{TL}}(n, p, \bar{p}) = \sum_{\lambda \in P} s_{\lambda}(p) g_{\lambda, \mu}(n) s_{\mu}(p)$$

(15)

see [34].

We are interested in a certain subclass of the BKP tau functions [12] written down in [31] and called BKP hypergeometric tau functions, and also in the similar class of TL tau functions [15] found in [25, 26].

**BKP tau function of the hypergeometric type** For a given partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ and a function on the one-dimensional lattice $r(x)$, $x \in \mathbb{Z}$, we introduce the generalized Pochhammer symbol $r_n(x)$ as

$$r_n(x) = r(x)r(x+1) \cdots r(x+n-1)$$

(16)

and the generalized Pochhammer symbol, $r_{\lambda}$, related to a partition $\lambda$ as

$$r_{\lambda}(x) = r_{\lambda_1}(x)r_{\lambda_2}(x-1) \cdots r_{\lambda_l}(x-l+1)$$

(17)

It may be written also as a content product as follows

$$r_{\lambda}(x) = \prod_{i,j \in \lambda} r(x + j - i)$$

where $j - i$ is a content of the node in $i$-th row and $j$-th column of the Young diagram of a partition $\lambda$, see [29] for more details.

For $r(x) = x$ the generalized Pochhammer symbol coincides with the familiar one:

$$r_{\lambda}(x) = (x)_\lambda, \quad (x)_\lambda := (x)_{\lambda_1}(x-1)_{\lambda_2} \cdots (x-l+1)_{\lambda_l}, \quad (x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$$

Remark 1. If $r = fg$, then $r_{\lambda}(x) = f_{\lambda}(x)g_{\lambda}(x)$. 

4
We shall denote the length of a partition \( \lambda \) by \( \ell(\lambda) \) and the weight of \( \lambda \) by \( |\lambda| \), see [30].

We consider sums over partitions of form

\[
g(n) \sum_{\lambda \in \Sigma} r_\lambda(n) s_\lambda(p) =: \tau_r^{BKP}(N, n, p) \tag{18}
\]

where \( \Sigma \) is the set of all partitions, \( s_\lambda \) are the Schur functions [30] and the semi-infinite set \( p = (p_1, p_2, \ldots) \) is related to the called higher times in the soliton theory \( t = (t_1, t_2, \ldots) \) via \( p_m = mt_m \). The Schur function is defined as follows

\[
s_\lambda(p) = \det (s_{\lambda_i - j}(p))_{i,j}, \quad e^{\sum_{m>0} \frac{1}{m} z^m p_m} =: \sum_{m \geq 0} z^m s_m(p) \tag{19}
\]

The constant \( g(n) \) is not important and may be found in Appendix D, see (73), (74).

**TL tau function of the hypergeometric type.** These are

\[
g(n) \sum_{\lambda \in \Sigma} r_\lambda(n) s_\lambda(p) s_\lambda(p) =: \tau_r^{TL}(n, p) \tag{20}
\]

In case \( r \) vanishes at certain site number \( M \) it is better to speak about

**Semi-infinite TL tau function** with the origin at the site number \( M \). These are

\[
g(n) \sum_{\lambda \in \Sigma \setminus P} r_\lambda(n) s_\lambda(p) s_\lambda(p) =: \tau_r^{TL}(M, n, p) \tag{21}
\]

This tau function is a particular case of the previous one if we choose \( r(M) = 0 \).

**Remarks.** According to [26], [31]

(1) \( \tau_r^{TL} \) solves certain linear equation generalizing Gauss equation for Gauss hypergeometric function which may also be referred as a "string equation"

(2) There are various determinantal formulæ to present \( \tau_r^{TL} \), and pfaffian formulæ to present \( \tau_r^{BKP} \)

(3) Both \( \tau_r^{TL} \) and \( \tau_r^{BKP} \) may be obtained by the action of vertex operators on certain simple functions

Also [31], [35], [36] :

(4) Sums (18) and (20) may be considered as partition functions for models of random partitions where a partition \( \lambda \) contributes the weights \( r_\lambda(n) s_\lambda(p) \), or \( r_\lambda(n) s_\lambda(p) s_\lambda(p) \) respectively

(5) For certain specifications of \( r \) and \( p \) sums (18) and (20) may be viewed as multisoliton tau functions. Similarly, for the same specifications, they may be viewed as discrete versions of matrix models

3 Special Hurwitz numbers

For a partition \( \Delta \) of a number \( d = |\Delta| \) denote by \( \ell(\Delta) \) the number of the non-vanishing parts. For the Young diagram, corresponding to \( \Delta \), the number \( |\Delta| \) is the weight of the diagram and \( \ell(\Delta) \) is the number of rows. Denote by \([d_1, \ldots, d_\ell]\) the Young diagram with rows of length \( d_1, \ldots, d_\ell \) and corresponding partition of \( \sum d_i \).

Hurwitz number \( H_\Omega(d, \Delta^{(1)}, \ldots, \Delta^{(k)}) \) is defined by a connected surface \( \Omega \) and partitions \( \Delta^{(1)}, \ldots, \Delta^{(k)} \) of the number \( d = |\Delta^{(i)}|, i = 1, \ldots, k \). The Hurwitz number \( H_\Omega(d, \Delta^{(1)}, \ldots, \Delta^{(k)}) \) is the weighted number of branched coverings of the surface \( \Omega \) by other surfaces (connected or non-connected) with fixed critical values \( z_1, \ldots, z_k \in \Omega \) of topological types \( \Delta^{(1)}, \ldots, \Delta^{(k)} \). More precisely, \( z \in \Omega \) is the critical value of the branched covering \( f : \Sigma \rightarrow \Omega \) if \( z = f(p) \), where \( p \in \Sigma \) is a critical point of \( f \). Consider
degrees \( d_1, \ldots, d_k \) of \( f \) in all preimages \( f^{-1}(z) \). The partition \( (d_1, \ldots, d_k) \) of \( d = \deg(f) \) is called the topological type of the critical value \( z \). We say that branched coverings \( f' : \Sigma' \to \Omega \) and \( f'' : \Sigma'' \to \Omega \) are the same, if there exists a homeomorphism \( g : f' \to f'' \) such that \( f' = f''g \). Then

\[
H_\Omega(d, \Delta^{(1)}, \ldots, \Delta^{(k)}) = \sum_{f} \frac{1}{|\text{Aut}(f)|},
\]

where the sum is taken over all branched covering of \( \Omega \), with the critical values \( z_1, \ldots, z_k \in \Omega \) of the topological types \( \Delta^{(1)}, \ldots, \Delta^{(k)} \) respectively. This number is independent of the positions of the branching points \( z_i \).

The Hurwitz numbers arise in different fields of mathematics: from algebraic geometry to integrable systems. They are well studied for orientable \( \Omega \). In this case the Hurwitz number coincides with Hurwitz numbers as a number of morphisms of complex algebraic curves.

In this work we consider the Hurwitz numbers for non-orientable \( \Omega \) without boundary. They have also two other interpretations: as the number of the branched coverings of a Klein surface without boundary and over all partitions \( \Delta = \sum_{i=1}^{s} l_i \Delta_i \) let us consider the sum \( \sum_{\Delta} \chi(\Sigma_1) \cdots \chi(\Sigma_k) \),

\[
H_\Omega(d, \Delta^{(1)}, \ldots, \Delta^{(k)}) = (dl)^{-1} \sum_{\chi} \chi(\Sigma_1) \cdots \chi(\Sigma_k),
\]

(22)

where \( e = \dim H_\Omega(\Omega, \mathbb{Z}/2\mathbb{Z}) \) and \( \chi \) ranges over the irreducible complex characters of \( S_d \), associated with Young diagrams \( \lambda \) of weight \( d \).

We consider the sums of Hurwitz numbers which correspond to the following options:

- \( d, k, s \in \mathbb{Z} \), where \( d > 0, b, k, s \geq 0 \);
- \( l_1, \ldots, l_k \in \mathbb{Z} \), where \( 0 \leq l_i \leq d \);
- \( l'_1, \ldots, l'_s \in \mathbb{Z} \), where \( 0 < l'_i \leq d \) (the first inequality is strict);
- partitions \( \Delta^{(1)}, \ldots, \Delta^{(s)} \), where \( |\Delta^{(s)}| = d \);
- partitions \( \Delta_1, \ldots, \Delta_k \), where \( |\Delta_i| = d \);
- \( s \) sets of partitions \( \Delta_1^{(1)}, \Delta_1^{(1)} \cdots, \Delta_1^{(s)} \), \( i = 1, \ldots, s \), where \( |\Delta_i^{(s)}| = d \) for each \( i, j \).

Denote by

\[
S_\Omega(d|b|l_1, \ldots, l_k|l'_1, \ldots, l'_s|\Delta^{(1)}, \ldots, \Delta^{(s)}) = \sum_{\Delta} (-1)^e H_\Omega(d, \Delta^{(1)}, \ldots, \Delta^{(s)}, \Gamma_1, \ldots, \Gamma_b, \Delta_1, \ldots, \Delta_k, \Delta_1^{(1)}, \ldots, \Delta_1^{(s)}, \Delta_2^{(1)}, \ldots, \Delta_2^{(s)})
\]

(23)

where the sum is taken over partitions \( \Delta \) of the weight \( d \) and over all partitions \( \Delta_i^{(s)} \) of the same weight that are not equal to \((1, \ldots, 1)\), and

- \( \Gamma_1 = \cdots = \Gamma_b = [2, 1, \ldots, 1] \);
- \( d - \ell(\Delta_i) = l_i \), \( i=1, \ldots, k \);
Hurwitz \( \tau \)

\[ \sum_{j=1}^{m_i} (d - \ell(\Delta_j^i)) = l_i^s, \quad i = 1, \ldots, s; \]
\[ \varepsilon = m_1 + \cdots + m_s. \]

In particular numbers \( S_{\Omega}(d|b|1, \ldots, 1|1, \ldots, 1|\Delta^{(1)}, \Delta^{(2)}) \) are the 2-Hurwitz numbers from [1] with profiles \( \Delta^{(1)} \) and \( \Delta^{(2)} \) in two given points with additional \( b + k + s \) simple ramification points.

It follows from [22] that

\[
S_{\Omega}(d|b|l_1, \ldots, l_k|l_1^s, \ldots, l_s^s|\Delta^{(1)}, \ldots, \Delta^{(t)})
\]

we can present as the sum, where summands correspond to Young diagram \( \lambda \) of weight \( d \). Denote by

\[
S^2_{\Omega}(d|b|l_1, \ldots, l_k|l_1^s, \ldots, l_s^s|\Delta^{(1)}, \ldots, \Delta^{(t)})
\]

the subsum of the last sum, that consists of summand, corresponding to \( \ell(\lambda) \leq N \leq \infty \). In particular

\[
S_{\Omega}(d|b|l_1, \ldots, l_k|l_1^s, \ldots, l_s^s|\Delta^{(1)}, \ldots, \Delta^{(t)}) = S^2_{\Omega}(d|b|l_1, \ldots, l_k|l_1^s, \ldots, l_s^s|\Delta^{(1)}, \ldots, \Delta^{(t)}).
\]

Precisely \( S^2_{\Omega}(d|b|l_1, \ldots, l_k|l_1^s, \ldots, l_s^s|\Delta^{(1)}, \ldots, \Delta^{(t)}) \) is given by [23] if Hurwitz numbers

\[
H_{\Omega}(d, \Delta^{(1)}, \ldots, \Delta^{(p)}) = \prod_{i=1}^{p} \frac{|C(\Delta^{(i)})|}{(d!)^c} \sum_{\lambda \in \mathcal{P}} (\chi(\lambda(1^d))^\epsilon \prod_{i=1}^{p} \chi(\lambda(\Delta^{(i)})) \chi(\lambda(1^d))
\]

by

\[
H_{\Omega}^N(d, \Delta^{(1)}, \ldots, \Delta^{(p)}) = \prod_{i=1}^{p} \frac{|C(\Delta^{(i)})(d!)^c} \sum_{\lambda \in \mathcal{P}} (\chi(\lambda(1^d))^\epsilon \prod_{i=1}^{p} \chi(\lambda(\Delta^{(i)})) \chi(\lambda(1^d))
\]

where \( C(\Delta) \) is the cardinality of the class \( \Delta \).

Let us note that for \( d \leq N \) numbers \( H_{\Omega}^N(d, \Delta^{(1)}, \ldots, \Delta^{(p)}) = H_{\Omega}(d, \Delta^{(1)}, \ldots, \Delta^{(p)}) \) while for \( d > N \) numbers \( H_{\Omega}^N(d, \Delta^{(1)}, \ldots, \Delta^{(p)}) \) have no geometrical interpretation.

## 4 Hurwitz \( \tau \)-functions for the semiinfinite Toda hierarchy

Here we basically review the result of [11] with certain modifications, namely, we need not infinite but semiinfinite Toda lattice to compare with our main result in the next section. We do not consider the combinatorial interpretation of hypergeometric tau functions in terms of counting paths problem in Cayley graph related to the symmetric group worked out in [22], [11].

Consider

\[
\tau^{TL}(M, n|q, \beta|a_1, \ldots, a_k|b_1, \ldots, b_s|p, \bar{p}) = q e^{\beta n} \prod_{i=1}^{k} a_i^{-d} (a_i + n)^{-i} \prod_{i=1}^{s} b_i^{-d} (b_i + n)^{-i} S^N_{CP}(B|\Delta^{(1)}, \Delta^{(2)}) p_{\Delta^{(1)}} \bar{p}_{\Delta^{(2)}}
\]

where \( n > M \) and the sum is taken over all \( B = (d|l_1, \ldots, l_k|l_1^s, \ldots, l_s^s|\Delta^{(1)}, \Delta^{(2)}) \). Here \( p_{\Delta^{(i)}} = p_{d_1^{(i)}, \ldots, d_s^{(i)}} \) with \( \Delta^{(i)} = (d_1^{(i)}, \ldots, d_s^{(i)}) \), \( d_1^{(i)} + d_2^{(i)} + \cdots = d, i = 1, 2 \).

**Theorem 1.** The function \( \tau^{TL}(M, n|a_1, \ldots, a_k|b_1, \ldots, b_s|p, \bar{p}) \) is a \( \tau \) functions for semiinfinite 2DToda hierarchy for any complex numbers \( (a_1, \ldots, a_k|b_1, \ldots, b_s) \).

**Remark 2.** The special case \( s = 0 \) was pointed out in [19].

**Proof.** Let us consider tau function [24] where we choose

\[
\tau(x) = q e^{\beta x} \prod_{i=1}^{k} (a_i + x) \prod_{i=1}^{s} (b_i + x), \quad x > M
\]
and \( r(x) = 0 \) for \( x \leq M \). We denote such hypergeometric tau function \( \tau^{TL}(N, n|q, \beta|a_1, \ldots, a_k|b_1, \ldots, b_s| \mathbf{p}, \bar{\mathbf{p}}) \).

This gives rise to the following value of the generalized Pochhammer symbol:

\[
    r_\lambda(x) = q^{|\lambda|} e^{\beta f_2(\lambda, x)} \prod_{i=1}^{k} \frac{(a_i + x)_{\lambda}}{(b_i + x)_{\lambda}}, \quad \ell(\lambda) \leq x - M
\]  

(29)

and \( r_\lambda(x) = 0 \) otherwise. If we take into account that

\[
    (a)_\lambda = \frac{s_\lambda(p(a))}{s_\lambda(p_{\infty})}
\]  

(30)

which was used in [26] and may be derived from Example 4 in Chapter I section 3 of [30], we may present \( r_\lambda \) as

\[
    r_\lambda(x) = q^{|\lambda|} e^{\beta f_2(\lambda, x)} s_\lambda(p_{\infty})^{k - \ell} \prod_{i=1}^{k} \frac{s_\lambda(p(a_i + x))}{s_\lambda(p(b_i + x))}, \quad \ell(\lambda) \leq x - M
\]  

(31)

This form is convenient for us.

In [30]

\[
    p(a) = (a, a, a, \ldots), \quad p_{\infty} = (1, 0, 0, \ldots)
\]  

(32)

and

\[
    f_2(\lambda, x) = \frac{1}{2} \sum_{i} \left[ (x + \lambda_i - i + \frac{1}{2})^2 - (x - i + \frac{1}{2})^2 \right]
\]  

(33)

\[
    = f_2(\lambda, 0) + x|\lambda|
\]  

(34)

Below \( f_2(\lambda, 0) = f_2(\lambda) \).

In this case

\[
    \tau^{TL}(N, n|q, \beta|a_1, \ldots, a_k|b_1, \ldots, b_s| \mathbf{p}, \bar{\mathbf{p}}) = \sum_{\ell(\lambda) \leq n - M} (q e^{\beta n})^{|\lambda|} e^{\beta f_2(\lambda)} s_\lambda(p_{\infty})^{k - \ell} \prod_{i=1}^{k} \frac{s_\lambda(p(a_i + n))}{s_\lambda(p(b_i + n))} s_\lambda(p)s_\lambda(\bar{p})
\]  

(35)

(36)

Following A. Okounkov we use the characteristic map to relate combinatorial problems to integrable systems, namely, we use

\[
    s_\lambda(p) = \sum_{\Delta \in \mathcal{P}} \frac{1}{z_\Delta} \chi_\Delta^\lambda p_\Delta
\]  

(37)

see, for instance, [30], where

\[
    z_\Delta = \prod_{i=1}^{\infty} i^{m_i}, \quad p_\Delta = \prod_{i=1}^{\infty} p_i^{m_i}
\]  

(38)

where \( m_i \) denotes the number of parts equal to \( i \) of the partition \( \Delta \) (then a partition \( \Delta \) is often denoted by \( 1^{m_1}, 2^{m_2}, \ldots \)). In [37] \( \chi_\Delta^\lambda \) coincides with the value of the irreducible character \( \chi^\lambda \) of the symmetric group \( S_n \), \( n = |\lambda| \), at elements of cycle-type \( \Delta \). In the relation [37] \( |\lambda| = |\Delta| \). From [37] we have

\[
    \dim \lambda = \chi_{(1^{|\lambda|})}^\lambda = |\lambda|! s_\lambda(p_{\infty})
\]  

(39)

From [32] one may write

\[
    \prod_{i=1}^{k} p_\Delta(a_i + n) = \prod_{i=1}^{k} (a_i + n)^{\ell(\Delta^{(i)})}
\]  

(40)

Let us denote the conjugation class of \( S_d \) related to a partition \( \Delta \) by \( C_\Delta \), \( |\Delta| = d \). The cardinality of \( C_\Delta \) is equal to

\[
    |C_\Delta| = \frac{|\Delta|!}{z_\Delta}
\]  

(41)
As it is known (for instance, see [1], also [5,9])

\[ f_2(\lambda) = \frac{|Cr_{\lambda}|^2}{\dim \lambda} \]

(42)

where \( \Gamma \) is the partition \( 1^{[\lambda]-1}2 \) (we choose this notation because the Young diagram of the partition \( 1^{[\lambda]-1}2 \) resembles the Greek letter \( \Gamma \)).

We use the Taylor series of

\[ \frac{1}{(b_i)_\lambda} = b_i^{-d} \left( 1 + \sum_{\Delta \neq (1^d)} |C_\Delta|/|\Delta|! \Delta b_i^{(\Delta)-d} \right)^{-1} \]

which follows from (30), (3), (40) and the definition of Hurwitz numbers for \( \mathbb{RP}^1 \).

Taking into account remarks (37), (39)-(42) we get the formula (26).

\[ \square \]

**Remark 3.** The corollary of the various properties of the generating functions like

\[ \tau^{S\mathbb{L}}(N, n|a_1, \ldots, a_k|b_1, \ldots, b_s|p, \bar{p})|_{a_1=\bar{a}_1} = \tau^{S\mathbb{L}}(N, n|a_2, \ldots, a_k|b_2, \ldots, b_s|p, \bar{p}) \]

or

\[ \tau^{S\mathbb{L}}(M; n, p[p,q]); \alpha_1, \ldots, a_k, b_1, \ldots, b_s) = \tau^{S\mathbb{L}}(M - n; 0, p[p,e^{\beta n}q, \beta]; a_1 + n, \ldots, a_k + n; b_1 + n, \ldots, b_s + n) \]

will be considered in the more detailed text.

## 5 Hurwitz τ-functions for BKP hierarchy

Construct now generating function for \( S_{\mathbb{RP}^2}^N(d|[l_1, \ldots, l_k]|_1, \ldots, l_s|_\Delta) \). Put

\[ \tau^{\mathbb{BKP}}(N, n|a_1, \ldots, a_k|b_1, \ldots, b_s|p, \bar{p}) = \sum_B (qe^{\beta n})^d J_i b_i^{-d} (b_1 + n)^{-t_1} \cdots \cdot (b_i + n)^{-t_i} \cdot \cdot \cdot (b_s + n)^{-t_s_\Delta} s_{\Delta}(\bar{p}) \]

(43)

where the sum is taken over all \( B = (d|[l_1, \ldots, l_k]|_1, \ldots, l_s|_\Delta) \). Here \( qe^{\beta n} = q_1 q_2 \cdots \) with \( \Delta = (d_1, d_2, \ldots) \) and \( d_1 + d_2 + \cdots = d \).

**Theorem 2.** The function \( \tau^{\mathbb{BKP}}(N, n|a_1, \ldots, a_k|b_1, \ldots, b_s|p) \) is a \( \tau \)-functions for BKP hierarchy for any complex numbers \( (a_1, \ldots, a_k|b_1, \ldots, b_s|p) \) and integer \( n > 0 \).

**Proof.** We chose the same \( r \) as in the case of the infinite Toda lattice \( (M \to -\infty) \) and insert it into [18]. We denote such tau function \( \tau^{\mathbb{BKP}}(N, n|a_1, \ldots, a_k|b_1, \ldots, b_s|p) \). We obtain

\[ \tau^{\mathbb{BKP}}(N, n|a_1, \ldots, a_k|b_1, \ldots, b_s|p) = \sum_{\lambda \in \Delta, \lambda \in N} (qe^{\beta n})^{|\lambda|} e^{\beta f_2(\lambda)} \prod_{i=1}^{k} (a_i + n) \lambda \prod_{i=1}^{l} (b_i + n) \lambda \]

(44)

\[ \cdot \sum_{s \in \mathbb{S}^N} s_\lambda(p) \]

or

\[ \tau^{\mathbb{BKP}}(N, n|a_1, \ldots, a_k|b_1, \ldots, b_s|p) = \sum_{\lambda \in \Delta, \lambda \in N} (qe^{\beta n})^{|\lambda|} e^{\beta f_2(\lambda)} s_\lambda(p) \prod_{i=1}^{k} \prod_{i=1}^{l} s_\lambda(p(a_i + n)) \]

(45)

Using (29), (33), (34) and the definition of Hurwitz numbers for \( \mathbb{RP}^2 \) we obtains the right hand side of (43) in a way similar to the TL case.

\[ \square \]

**Remark 4.** We have

\[ \tau^{S\mathbb{L}}(N, n|a_1, \ldots, a_k|b_1, \ldots, b_s|p)|_{a_1=\bar{a}_1} = \tau^{S\mathbb{L}}(N, n|a_2, \ldots, a_k|b_2, \ldots, b_s|p) \]

(46)

\[ \tau^{\mathbb{BKP}}(N, n|p|q, \beta, a_1, \ldots, a_k|b_1, \ldots, b_s) = \tau^{\mathbb{BKP}}(N, 0|p|e^{\beta n}q, \beta; a_1 + n, \ldots, a_k + n; b_1 + n, \ldots, b_s + n) \]

(47)
6 Transformation of Hurwitz \( \tau \)-functions for the semi-infinite 2DToda hierarchy to \( \tau \)-functions for BKP hierarchy

The sum (18) is an example of the BKP hypergeometric tau function [31] (Hirota equations for the BKP tau functions may be found in Appendix B). It may be obtained from the hypergeometric tau function of the semi-infinite Toda lattice \( n \geq M \) (here a given \( M \) is the origin of the lattice):

\[
\frac{\partial \phi_n}{\partial p_1 \partial \bar{p}_1} = r'(n)e^{\phi_{n-1}-\phi_n} - r'(n+1)e^{\phi_{n}-\phi_{n+1}}, \quad e^{-\phi_n} = \frac{\tau_{r'}(M; n+1, p, \bar{p})}{\tau_{r'}(M; n, p, \bar{p})} g(n) g(n+1)
\]

where \( r'(n) = r(n) \delta(n) \) (\( \delta(M) = 0, \delta(n) = 1 \) otherwise). The multiplication by \( \delta \) provides the restriction of the summation region by the condition \( r'_\lambda(n) \) for \( \ell(\lambda) \leq n - M \) for the tau function of the Toda lattice, this we see from the definition of \( r_\lambda(n) \). Fixing \( n \) and choosing \( N = n - M \) we obtain

\[
\tau_{BKP}(N,n,p) = \left[e^{L_\infty} \cdot \tau_{TL}(n-N; n, p, \bar{p})\right]_{p=0} \tag{48}
\]

where \( L_\infty \) is the following Laplacian operator

\[
L_\infty = \sum_{m \geq 1} m \frac{\partial^2}{\partial p_m^2} + 2 \sum_{m \geq 1, \text{odd}} m \frac{\partial}{\partial p_m}
\]

The operator \( e^{\frac{1}{2}L_\infty} \) and the evaluation at \( \bar{p} = 0 \) 'eliminate' one Schur function in each term of (20). The proof of (48) follows from

\[
\sum_{\lambda \in P} s_\lambda(p) = e^{\frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} p_m^2 + \sum_{m=1}^{\infty} p_{2m-1}}
\]

and

\[
\left[s_\mu(\tilde{\partial}) \cdot s_\lambda(p)\right]_{p=0} = \delta_{\mu,\lambda}
\]

which may be derived from Examples in Chapter I section 5 of [30]. Here \( s_\lambda(\tilde{\partial}) \) denotes the Schur function as the function defined by (19) where each \( p_m \) is replaced by \( m \frac{\partial}{\partial p_m} \).

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This Appendix is a result of a discussion with John Harnad.

One can write down the scalar product where Macdonald polynomials $P_\lambda(q^a, q; p)$ are orthonormal, by the integral over all power sums variables $p$ as follows

$$
\langle f, g \rangle = \int_{\mathbb{C}^\infty} f(p)g(p^*) \prod_{m=1}^{\infty} e^{-\frac{|p_m|^2}{4} + \frac{2p_m(a, q) \overline{p_m(a, q)}}{2\pi im}} dp_m \wedge dp_m^*
$$

where $p_m^*$ is the complex conjugate to $p_m^*$. In this basis

$$
\langle p_{\Delta} p_{\Delta'} \rangle = \frac{1}{w_{\Delta}(a, q)} \delta_{\Delta, \Delta'}, \quad w_{\Delta}(a, q) = \frac{p_{\Delta}(a, q)}{z_{\Delta}}
$$
Exactly this ratio appears in the character expansion formula

\[ s_\lambda(p(a, q)) = \sum_\Delta \chi_\lambda(\Delta) w_\Delta(a, q) \]  

(51)

Also one can write

\[ \sum_{\Delta \in \mathcal{P}} \frac{p_\Delta(a, q)}{z_\Delta} p_{\Delta}(p) p_{\Delta}(p) = e^{\sum_{m>0} \frac{1}{m!} P_m(a, q)} = \sum_{\Delta \in \mathcal{P}} \frac{p_\Delta(a, q)}{z_\Delta} \sum_{\lambda \in \mathcal{P}} s_\lambda(pp) \chi_\lambda(\Delta) \]  

(52)

where \( pp \) denotes the set \((p_1, p_1, p_2, \ldots)\).

Equation (52) corresponds to the case \( q \rightarrow 1 \) where Macdonald polynomials convert to Jack ones.

The relation of hypergeometric tau functions to the quantum integrable systems is unclear. The combinatorial and geometric interpretations of hypergeometric tau functions parametrized by pairs \( a_i, q_i \) in the TL case will be considered in [42].

B Hirota equations for the BKP tau function with two discrete time variables.

The BKP hierarchy we are interested in was introduced in [21]. It was used to construct various matrix models [37], [31], [38]. Hirota equations for the BKP hierarchy of Kac-van de Leur were presented in [21]. However in our case we need more general version which includes both discrete variables \( N \) and \( n \), see [31]. The BKP tau function we need has the following form

\[ \tau^{BKP}(N, n, p, q) = (N + n|e^{\sum_{m>0} \frac{1}{m!} P_m J_m} g|n) \]  

(53)

where Clifford algebra element \( g \) may be considered as an element of \( \mathcal{O}(2\infty + 1) \) group which specifies the choice of the BKP tau function,

\[ J_m = \sum_{i \in \mathbb{Z}} \psi_i \psi_i^\dagger \]  

are Fourier modes of current operators, see details in [31]. Hirota equations for tau function [53] may be obtained by a certain specification of the Hirota equations for the two-sided BKP tau function

\[ \tau^{BKP}(N, n, p, q) = (N + n|e^{\sum_{m>0} \frac{1}{m!} P_m J_m} g e^{\sum_{m>0} \frac{1}{m!} P_m J_m} |n) \]  

see [31], which in our notations are

\[ \int \frac{dz}{2\pi i} z^N e^{N-n-2} e^{V(p - \bar{z})} \tau(N - 1, n', p - [z^{-1}], \bar{p}) \tau(N + 1, n, p + [z^{-1}], \bar{p}) \]  

+ \[ \int \frac{dz}{2\pi i} z^N e^{N-n-2} e^{V(p - \bar{z})} \tau(N + 1, n', p' + [z^{-1}], \bar{p'}) \tau(N - 1, n, p - [z^{-1}], \bar{p}) \]  

= \[ \int \frac{dz}{2\pi i} z^n e^{V(p' - \bar{z})} \tau(N' - 1, n' + 1, p' - [z], \bar{p}) \tau(N + 1, n - 1, p, \bar{p} + [z]) \]  

+ \[ \int \frac{dz}{2\pi i} z^n e^{V(p' - \bar{z})} \tau(N' + 1, n' - 1, p' + [z], \bar{p}') \tau(N - 1, n + 1, p, \bar{p} - [z]) \]  

\[ \frac{(-1)^{n+n'} 2}{1 - (-1)^{N+N'}} \tau(N', n, p', \bar{p}') \tau(N, n, p, \bar{p}) \]  

(54)

see also [30]. Here \( p = (p_1, p_2, \ldots) \), \( p' = (p'_1, p'_2, \ldots) \), \( \bar{p} = (\bar{p}_1, \bar{p}_2, \ldots) \), \( \bar{p}' = (\bar{p}'_1, \bar{p}'_2, \ldots) \), and

\[ V(z, p) = \sum_{m>0} \frac{1}{m!} z^m p_m \]

The notation \( p + [z^{-1}] \) denotes the set \((p_1 + z^{-1}, p_2 + z^{-2}, p_3 + z^{-3}, \ldots)\).

Remark 5. Actually up to some simple factor the two-sided BKP tau function of [31] coincides with the two-component BKP tau function of [21] and Hirota equations [53] basically coincide with the Hirota equations for the two-component BKP, see Appendix in [30].
To obtain Hirota equations for (53) we chose \( p = p' = 0 \).

For \( n' = n + 1 \), we obtain (see (31))

\[
\int \frac{dz}{2\pi i} z^{N' - N - 1} e^{V(p' - p, z)} \tau(N' - 1, n + 1, p' - [z^{-1}]) \tau(N + 1, n, p + [z^{-1}])
\]

\[
+ \int \frac{dz}{2\pi i} z^{N' - N - 2} e^{V(p' - p, z)} \tau(N' + 1, n + 1, p' + [z^{-1}]) \tau(N - 1, n, p - [z^{-1}])
\]

\[
= \tau(N' + 1, n, p') \tau(N - 1, n, p) - \frac{1}{2} (1 - (-1)^{N' + N}) \tau(N', n + 1, p') \tau(N, n, p) \tag{55}
\]

For \( n' = n \), we obtain Hirota equations as in (21)

\[
\int \frac{dz}{2\pi i} z^{N' - N - 2} e^{V(t' - t, z)} \tau(N' - 1, n, t' - [z^{-1}]) \tau(N + 1, n, t + [z^{-1}])
\]

\[
+ \int \frac{dz}{2\pi i} z^{N' - N - 2} e^{V(t' - t, z)} \tau(N' + 1, n, t' + [z^{-1}]) \tau(N - 1, n, t' - [z^{-1}])
\]

\[
= \frac{1}{2} (1 - (-1)^{N' + N}) \tau(N', n, t') \tau(N, n, t) \tag{56}
\]

Let us write down some of them. Taking \( N' = N + 1 \) and all \( p_i = p'_i, i \neq 1 \) in (55) and picking up the terms linear in \( p'_1 - p_1 \) we obtain

\[
\frac{1}{2} \tau(N + 1, n, p) \frac{\partial \tau(N + 1, n, p)}{\partial p_1} - \frac{1}{2} \tau(N, n + 1, p) \frac{\partial \tau(N, n + 1, p)}{\partial p_1} = \frac{\partial \tau(N + 2, n, p)}{\partial p_1} \tau(N - 1, n + 1, p) - \frac{\partial \tau(N + 1, n + 1, p)}{\partial p_1} \tau(N, n, p) \tag{57}
\]

Taking \( N' = N + 1 \) and all \( p_i = p'_i, i \neq 2 \) in (56) and picking up the terms linear in \( p'_2 - p_2 \) we obtain

\[
\frac{1}{2} \frac{\partial \tau(N, n, p)}{\partial p_2} \tau(N + 1, n, p) - \frac{1}{2} \frac{\partial \tau(N, n, p)}{\partial p_2} \tau(N - 1, n + 1, p) + \frac{1}{2} \frac{\partial^2 \tau(N, n, p)}{\partial p_1^2} \tau(N + 1, n, p)
\]

\[
+ \frac{1}{2} \frac{\partial^2 \tau(N, n, p)}{\partial p_1 \partial p_2} \tau(N + 1, n, p) - \frac{\partial \tau(N + 1, n, p)}{\partial p_1} \frac{\partial \tau(N, n + 1, p)}{\partial p_1} = \tau(N + 2, n, p) \tau(N - 1, n, p) \tag{58}
\]

C Matrix integrals as generating functions for Hurwitz numbers

Let us consider the following integral

\[
\int e^{\sum_{i=1}^K \text{tr} V_i (A_i) + \sum_{m>0} m \cdot \text{tr} (A_1 \cdots A_K)} \prod_{i=1}^K dA_i \tag{59}
\]

where \( V_i \) are functions of eigenvalues of normal (diagonalizable) \( N \times N \) matrices \( A_i \).

Suppose we want to re-write the integral over the matrix entries as an integral over eigenvalues of matrices \( A_i \) and re-writing the integral as an integral over eigenvalues \( x_i^{(1)} \). In this case we need to consider

\[
e^{\text{tr} U_1 A_1 U_1^* \cdots U_K A_K U_K^*}
\]

and integrate over all \( U_i \) (in fact over \( V_1 := U_1^* U_2, \ldots, V_{K-1} := U_1^* U_2 \)).

\[
I := \int_{U(N)^{K}} e^{\text{tr} U_1 A_1 U_1^* U_2 A_2 U_2^* \cdots U_K A_K U_K^*} \prod_{i=1}^K dU_i
\]

where we get the \( K - 1 \)-fold integral over \( V_1 := U_1^* U_2 \) multiplied by the volume of unitary group.

Then we use the known formula (for instance see (30))

\[
\int_{U(N)} s_A(A U B U^*) dU = \frac{s_A(A) s_A(B)}{s_A(I_N)} \tag{60}
\]

\[
14
\]
where $I_N$ is the unit matrix (see for instance [30]) and Cauchy-Littlewood formula
\[ e^{\sum_{m>0} \frac{1}{m} p_m \tau_m} = \sum_{\lambda} s_\lambda(p)s_\lambda(p^*) \] (61)
where
\[ p_m = \text{tr}(AUBU^\dagger)^m \] (62)
like it was done in [10]. Writing the product of $K$ matrices as $A_1B_1$ where $B_1 = A_2 \cdots A_K$ diagonalizing, then repeating $K - 1$ times we obtain
\[ I = \text{Vol}U(N) \sum_{i(\lambda) \leq N} \frac{s_\lambda(p) \prod_{k=1}^K s_\lambda(A_i)}{(s_\lambda(I_N))^{K-1}} \] (63)
which may be related to more complicated Hurwitz numbers with $K + 1$ arbitrary profiles, namely to the sums $S_{\mathbb{C}P^1}(d|b|l_1, \ldots, l_K|l_1^*, \ldots, l_K^*|\Delta^{(1)}, \ldots, \Delta^{(K+1)})$.

Similarly, we can integrate hypergeometric $\tau_T^L(p, A_1 \cdots A_K)$ and $\tau_T^{BKP} A_1 \cdots A_K$ (instead of the simplest TL tau function given by Itsykson-Zuber $e^{\tau_T^{AUBU^\dagger}}$).

We obtain

**Proposition 1.** The generating function $S_{\mathbb{C}P^1}(d|b|l_1, \ldots, l_K|l_1^*, \ldots, l_K^*|\Delta^{(1)}, \ldots, \Delta^{(K+1)})$ is constructed as the following matrix integral
\[ \int_{U(N) \times \cdots \times U(N)} \tau_T^L(n, p, U_1A_1U^\dagger_1U_2A_2U^\dagger_2 \cdots U_KA_KU^\dagger_K) \prod_{i=1}^K dU_i = \sum_B (qe^{\beta n}) \frac{\prod_{i=1}^K (a_i + n)^i!}{\prod_{i=1}^K b_i! (b_i + n)!^i} S_{\mathbb{C}P^1}(B) \prod_{i=1}^{K+1} \mathcal{P}^{(i)} \] (64)
where the sum is taken by all $B = (d|b|l_1, \ldots, l_K|l_1^*, \ldots, l_K^*|\Delta^{(1)}, \ldots, \Delta^{(K+1)})$ and where $p^{(i)} = (p_1^{(i)}, p_2^{(i)}, \ldots)$ and
\[ p_m^{(i)} = \text{tr}A_i^m, \quad i = 1, \ldots, K, \quad p_m^{(K+1)} = p_m \]

**Proof.** From (60)-(62) we obtain
\[ \int_{U(N) \times \cdots \times U(N)} \tau_T^L(n, p, U_1A_1U_1^\dagger U_2A_2U_2^\dagger \cdots U_KA_KU^\dagger_K) \prod_{i=1}^K dU_i = \sum_{i(\lambda) \leq N} \tau_\lambda(n) \frac{s_\lambda(p) \prod_{k=1}^K s_\lambda(A_i)}{(s_\lambda(I_N))^{K-1}} \] (66)
Then we apply the same steps as the Theorem 2.

**Proposition 2.** Construct now generating function for $S_{\mathbb{R}P^2}(d|b|l_1, \ldots, l_K|l_1^*, \ldots, l_K^*|\Delta^{(1)}, \ldots, \Delta^{(K)})$. Put
\[ \int_{U(N) \times \cdots \times U(N)} \tau_T^{BKP}(n, U_1A_1U_1^\dagger U_2A_2U_2^\dagger \cdots U_KA_KU^\dagger_K) \prod_{i=1}^K dU_i = \sum_B (qe^{\beta n}) \frac{\prod_{i=1}^K b_i^{-d} (b_i + n)^i!}{\prod_{i=1}^K a_i^{-d} (a_i + n)^i!} S_{\mathbb{R}P^2}(B) \prod_{i=1}^{K+1} \mathcal{P}^{(i)} \] (67)
where the sum is taken by all $B = (d|b|l_1, \ldots, l_K|l_1^*, \ldots, l_K^*|\Delta^{(1)}, \ldots, \Delta^{(K)})$ and where $p_i = (p_1^{(i)}, p_2^{(i)}, \ldots)$ and
\[ p_m^{(i)} = \text{tr}A_i^m, \quad i = 1, \ldots, K \]

**Proof.** From (60)-(62) we obtain
\[ \int_{U(N) \times \cdots \times U(N)} \tau_T^{BKP}(U_1A_1U_1^\dagger U_2A_2U_2^\dagger \cdots U_KA_KU^\dagger_K) \prod_{i=1}^K dU_i = \sum_{i(\lambda) \leq N} \tau_\lambda(n) \frac{\prod_{k=1}^K s_\lambda(A_i)}{(s_\lambda(I_N))^{K-1}} \] (69)
Then we apply the same steps as the Theorem 2.
D Fermionic formulae

Details may be found in [20,31]. Let \( \{ \psi_i, \psi_i^\dagger, i \in \mathbb{Z} \} \) are Fermi creation and annihilation operators that satisfy the usual anticommutation relations and vacuum annihilation conditions

\[
[\psi_i^{(a)}, \psi_j^{(b)}]_+ = \delta_{ij} \delta_{a,b}, \quad \psi_i^{(1)}|n, *\rangle = \psi_i^{(1)}|n, *\rangle = 0, \quad \psi_i^{(2)}|*, n\rangle = \psi_i^{(2)}|*, n\rangle = 0 \quad \text{if} \quad i < n,
\]

(70)

Sometimes we will omit the superscript (1) in particular write \( \psi \) instead of \( \psi^{(1)} \).

The hypergeometric tau functions may be written as follows

\[
\tau_r^{\text{TL}}(n, p, \mathbf{p}) = g(n)|n\rangle \langle n| e^{\sum_{m>a} \frac{1}{m} J_m p_m} e^{-\sum_{m>a} \frac{1}{m} A_m} |n\rangle
\]

where \( J_m = \sum_{i \in Z} \psi_i^{\dagger} \psi_{i+m} \) and \( A_m = \sum_{i \in Z} r(i) \ldots r(i-m) \psi_i^{\dagger} \psi_{i-m} \). The semi-infinite TL may be described either putting by \( r(N) = 0 \), or, it is may be suitable to present it in form

\[
\tau_r^{\text{TL}}(M, n, p, \mathbf{p}) = (-1)^{M(M+1)} g(n) |M + n, -M - n| e^{\sum_{m>a} \frac{1}{m} J_m p_m} e^{-\sum_{m \in Z} U_i \psi_i^{\dagger} \psi_i} |a, n\rangle
\]

For BKP [21] one needs to introduce an additional Fermi mode \( \phi \) which anticommutes with each other Fermi operator except itself: \( \phi^2 = \frac{1}{2} \), and \( \phi|0\rangle = \frac{1}{\sqrt{2}}|0\rangle \). Then

\[
\tau_r^{\text{BKP}}(N, n, p, \mathbf{p}) = g(n) |N + n| e^{\sum_{m>a} \frac{1}{m} B_m p_m} e^{\omega^\dagger}|0\rangle = g(n) \sum_{\lambda \in \mathcal{P}} r_\lambda(0) s_\lambda(p)
\]

(71)

and

\[
\tau_r^{\text{BKP}}(N = \infty, 0, p) = (0) e^{\sum_{m>a} \frac{1}{m} B_m p_m} e^{\omega^\dagger}|0\rangle = g(n) \sum_{\lambda \in \mathcal{P}} r_\lambda(0) s_\lambda(p)
\]

(72)

where

\[
r(i) = e^{U_{i-1} - U_i}\]

(73)

and

\[
\omega = \sum_{i,j} \psi_i \psi_j - \sqrt{2} \phi \sum_{i \in \mathbb{Z}} \psi_i, \quad \omega_- = \sum_{i,j \geq 0} \psi_i \psi_j - \sqrt{2} \phi \sum_{i \geq 0} \psi_i, \quad \omega_+ = \sum_{i,j \geq 0} (-)^{i+j} \psi_j^{\dagger} \psi_{i-1}^{\dagger} + \sqrt{2} \phi \sum_{i \geq 0} \psi_i^{\dagger},
\]

\[
B_m = \sum_{i \in \mathbb{Z}} \frac{r(i)}{r(i + m)} \psi_{i+m}^{\dagger}
\]

and

\[
g(n) = |n\rangle \langle n| e^{\sum_{i \in \mathbb{Z}} U_i \psi_i^{\dagger} \psi_i} |n\rangle =
\]

\[
e^{-U_0 + \cdots - U_{n-1}} \quad \text{if} \quad n > 0
\]

(74)

\[
= 1 \quad \text{if} \quad n = 0
\]

(75)

\[
e^{U_{n-1} + \cdots + U_0} \quad \text{if} \quad n < 0
\]

(76)

Quasi tau functions. One can consider the following series in the Schur functions (compare to [16])

\[
\tau^{[\omega]}(n, \{ p^i \}) := g(n) \sum_{\lambda \in \mathcal{P}} r_\lambda(n) \prod_{i=1}^c s_\lambda(p^{(i)})
\]

(77)

It may be presented in forms

\[
\tau^{[\omega]}(n, \{ p^i \}) = g(n) |n + N| e^{\sum_{m>a} p^{(i)}_m B_m + \sum_{i \in \mathbb{Z}} r^{(i)}_m J_m} e^{\omega \omega^\dagger}|0\rangle = |n + N| e^{\sum_{i=1} c p^{(i)}_m J_m} e^{-\sum_{i \in \mathbb{Z}} U_i \psi_i^{\dagger} \psi_i} e^{-\omega \omega^\dagger}|0\rangle
\]

(78)

where

\[
\omega^{[\omega]} := \sum_{i,j \geq 1} \prod_{a=1}^c \left( \psi_i^{(a)} \psi_j^{(a)} \right) - \sqrt{2} \phi \sum_{i \in \mathbb{Z}} \prod_{a=1}^c \psi_i^{(a)}
\]

(79)
which may be viewed as a sum of commutative elements in the tensor product $\delta(2\infty+1) \otimes \cdots \otimes \delta(2\infty+1)$.

Let us note that
\[
e^{\omega(e)} |n\rangle = 1 + \sum_{N>0} c_e \left( \lambda, N + n \right) \otimes \cdots \otimes \left( \lambda, N + n \right)
\]  
where $c_e = 1$ for even $e$ and $c_e = (2e - 1)!!$ for $e$ odd.

By coupling (80) with the vector
\[
\langle N + n | e^{\sum_{a=1}^e \sum_{m>0} (-1)^e \frac{a}{m \lambda(m)} P_{\lambda m}^{(e)}} e^{-\sum_{i \in \mathbb{Z}} U_i \psi_i \psi_i^\dagger}:
\]
we obtain the right hand side of (77).

For $e = 1$ we obtain hypergeometric $\tau_r^{BKP}(N, n, p)$. For $e = 2$ we obtain $\tau_r^{TL}(N, n, p^{(1)}, p^{(2)})$.

Using the commutativity of all factors $\psi_i^{(a)} \psi_j^{(a)}$ in (79) one can verify that for each $a$
\[
\left[ e^{\omega(e)} \otimes e^{\omega(e)} , \phi \otimes \phi + \sum_{i \in \mathbb{Z}} \left( \psi_i^{(a)} \otimes \psi_i^{(a)} + \psi_i^{(a)} \otimes \psi_i^{(a)} \right) \right] = 0
\]
This is the fermionic form for the BKP Hirota equation [21] with respect to the component $a$. It means that (77) solves multicomponent BKP Hirota equations where $t_m(a) = \frac{1}{m} p_m^{(a)}$, $a = 1, \ldots, e$, play the role of the BKP higher times.

Let us note that for $e > 2$ formula (77) generates Hurwitz numbers for disjoint base surfaces $\Omega$. 