SEQUENTIAL DERIVATIVES

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ABSTRACT. Consider a real valued function defined, but not differentiable at some point. We use sequences approaching the point of interest to define and study sequential concepts of secant and cord derivatives of the function at the point of interest. If the function is the celebrated Weierstrass function, it follows from some of our results that the set cord derivatives at any point coincides with the extended real line.

1. Introduction

Very basic finite difference formulas in numerical analysis approximates the derivative $f'(x)$ using a sequence $h_n > 0$ such that $h_n \to 0$. The two basic formulas are

$$
\frac{f(x + h_n) - f(x)}{h_n} \to f'(x) \quad \text{and} \quad \frac{f(x + h_n) - f(x - h_n)}{2h_n} \to f'(x).
$$

The first formula is Newton’s difference quotient and determines the slope of a secant line of the graph of $f$. Roughly the Newton difference quotient approximates the slope of the tangent with an error proportional to $h_n$. In Newton’s difference quotient we could also use $h_n < 0$. The second formula is the symmetric difference quotient and determines the slope of a cord of the graph of $f$. Roughly the symmetric difference quotient approximates the slope of the tangent with an error proportional to $h_n^2$.

In this note we study the limits of the Newton difference quotients and of the symmetric difference quotients, when the function $f$ is continuous at $x$, but fails to have a derivative at $x$. Let $N_{f,x}$ be the set of limits of the Newton difference quotient for all $h_n$ such that the limit exists in the extended real numbers. And let $S_{f,x}$ be the set of limits of the difference quotient

$$
\frac{f(x + h_n) - f(x - k_n)}{h_n + k_n}
$$

for all $h_n, k_n > 0$ such that the limit exists in the extended real numbers. When $k_n = h_n$ this is the symmetric difference quotient. Among our results are (a) $N_{f,x}$ and $S_{f,x}$ are closed subsets of the extended real numbers, (b) any closed subset of the extended real numbers equals $N_{f,0}$ for some $f$, (c) $N_{f,x}$ is a subset of $S_{f,x}$, and (d) if $f$ is continuous on an interval then $N_{f,x}$ and $S_{f,x}$ are intervals. In part of Section 4 we assume $f$ is defined on a set of the form $\{0, h_1, -k_1, h_2, -k_2, \ldots\}$, $f(0) = 0$, and we assume the Newton difference quotients

$$
\frac{f(h_n)}{h_n} \to R \quad \text{and} \quad \frac{f(-k_n)}{-k_n} \to L
$$

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converge to real numbers. We show that the set of limits $S$ of the sequences
\[ \frac{f(x + h_n) - f(x - k_n)}{h_n + k_n} \]
onobtained by considering subsequences $h_{i_n}, k_{j_n}$ of the sequences $h_n, k_n$, depends on properties of the sequences $h_n, k_n$. For example, we show, $(e)$ if $h_n, k_n$ both decay to zero at the same polynomial rate, then $S$ is the interval with endpoints $R$ and $L$, $(f)$ if $h_n, k_n$ decay at the same exponential rate, then $S$ is a discrete set whose only accumulation points are $R$ and $L$, and $(g)$ if $h_n, k_n$ decay at different exponential rates, then whether $S$ is a discrete set or an interval depend on the rates of decay.

2. Sequential Secant Derivatives

We consider derivatives of real valued functions, our derivatives are defined in terms of sequences and we allow them to be infinite. Denote the real line by $\mathbb{R}$ and the extended real line $\mathbb{R} \cup \{\pm \infty\}$ by $\mathbb{R}$.

**Definition 2.1.** Let $f$ be a real valued function defined on the subset $D$ of $\mathbb{R}$ and let $x \in D$. We say $L$ in $\mathbb{R}$ is a sequential secant derivative of $f$ at $x$, if there is a sequence $h_n \neq 0$, such that $h_n \to 0$, $x + h_n \in D$, and
\[ Df(x, h_n) := \frac{f(x + h_n) - f(x)}{h_n} \to L. \] (2.1)

We say $L$ is a right hand sequential secant derivative of $f$ at $x$, if $h_n > 0$, and a left hand sequential secant derivative of $f$ at $x$, if $h_n < 0$. We will abbreviate $h_n > 0$ and $h_n \to 0$ as $n \to \infty$ by writing $h_n \to_0 0$.

Clearly, $f$ is differentiable at $x$ with derivative $L$ if and only if (2.1) holds for every $h_n \to 0$ with $x + h_n \in D$. The details can be found in any beginning analysis book, e.g., [Str00] or [Ped15].

**Remark 2.2.** The definition of sequential secant derivative is motivated by Weierstrass’ proof, see [Wei86] or [Ped15], that the Weierstrass functions
\[ W(x) := \sum_{k=0}^{\infty} a^n \cos(b^n \pi x), \]
where $0 < a < 1$ is a real number, $b$ is an odd integer and $ab > 1 + \frac{3\pi}{2}$, have $\pm \infty$ (in our terminology) as sequential secant derivatives at any point $x$. More precisely, Weierstrass showed there are sequences $h_n^L, h_n^R \to_0 0$, such that $|DW(x, -h_n^L)| \to \infty$, $|DW(x, h_n^L)| \to \infty$, and $DW(x, -h_n^R)$ and $DW(x, h_n^R)$ have different signs for all sufficiently large $n$.

To state and prove our results we need some terminology about subsets of the extended real numbers, this terminology is introduced in the following definition.

**Definition 2.3.** Let $S$ be a subset of the extended real line $\mathbb{R}$.

(a) We say $S$ is closed, if any $L \in \mathbb{R}$ for which there is a sequence of real numbers $L_n \in S \cap \mathbb{R}$ such that $L_n \to L$ must be in $S$.

(b) We say a set of real numbers $A$ is dense in $S$, if for any $L \in S$, there is a sequence $a_k$ in $A$, such that $a_k \to L$.

(c) A point $L$ in $S$ is isolated in $S$, if no sequence $a_k$ of points in $S$ with $a_k \neq L$ satisfies $a_k \to L$. 

(d) A closed interval in $\mathbb{R}$ is a set of the form $[a, b] := \{ t \in \mathbb{R} : a \leq t \leq b \}$, where $a < b$ are in $\mathbb{R}$.

For a bounded set $S$ this notion of “closed” agrees with the usual notion of a closed subset of the real line and for unbounded sets $S$ it agrees with the notion of a closed subset of the two-point-compactification of the real line. Similar remarks apply to the other terms in Definition 2.3.

We begin by showing that the set of all secant derivatives at a point is a closed subset of the extended real numbers. Conversely, we show that any non-empty closed subset of the extended real numbers is the set of secant derivatives at 0 of some function defined on the closed interval $[0, 1]$.

**Theorem 2.4.** Let $f$ be a real valued function defined a the subset $D$ of $\mathbb{R}$ and let $x \in D$. The set of sequential secant derivatives of $f$ at $x$ is a closed subset of $\mathbb{R}$.

**Proof.** Suppose the real numbers $L_n$ are sequential secant derivatives of $f$ at $x$, $L \in \mathbb{R}$, and $L_n \to L$. We must show $L$ is a secant derivative of $f$ at $x$. For each $n$, let $h_{n,m} \neq 0$ be such that $h_{n,m} \to 0$ as $m \to \infty$ and

$$\lim_{m \to \infty} \frac{f(x + h_{n,m}) - f(x)}{h_{n,m}} = L_n.$$  

Pick $N_n$ such that

$$\left| \frac{f(x + h_{n,N_n}) - f(x)}{h_{n,N_n}} - L \right| < \frac{1}{n}.$$  

It follows that

$$\lim_{n \to \infty} \frac{f(x + h_{n,N_n}) - f(x)}{h_{n,N_n}} = L.$$  

Hence $L$ is sequential secant derivative of $f$ at $x$. \hfill \blacksquare

Similarly, the set of right hand (and the set of left hand) sequential secant derivatives of $f$ at a point $x$ are closed subsets of $\mathbb{R}$.

In the following we explore the structure of the sets of sequential secant derivatives of functions defined on intervals. For simplicity we state the results for right hand sequential derivatives at 0 for functions defined on the closed interval $[0, 1]$.

**Theorem 2.5.** Given any non-empty closed subset $S$ of the extended real numbers, there is a real valued function $f$ defined on the closed interval $[0, 1]$, such that the set of right hand sequential secant derivatives of $f$ at 0 equals the set $A$.

**Proof.** Let $S$ be a closed subset of the extended real numbers. There are several cases depending on whether or not $\pm \infty$ are in $S$ or are isolated points of $S$. We will give a proof in case for the situation where $\infty$ is an isolated point of $S$ and either $-\infty$ is not in $S$ or $-\infty$ is not an isolated point of $S$. The modifications needed for the other cases are left for the reader.

Let $a_1, a_2, \ldots$ be a countable dense subset of $S \setminus \{ \infty \}$ consisting of real numbers. Let $f(0) = 0$. Let $\xi_n > 0$ be a strictly decreasing sequence such that $\xi_1 = 1$ and $\xi_n \to 0$. For each $n$, partition the interval $(\xi_{n+1}, \xi_n]$ into $n + 1$ subintervals $(\xi_{n,k-1}, \xi_n, \xi_{n,k}]$, $k = 1, 2, \ldots, n + 1$. For $x$ in an interval of the form $(\xi_{n,k-1}, \xi_n, \xi_{n,k}]$ with $1 \leq k \leq n$ let $f(x) = a_kx$, for $x$ in an interval of the form $[\xi_{n,n}, \xi_{n,n+1}]$ let $f(x) = \sqrt{x}$. Then the graph of $f$ contains segments of the graph of the equation $y = a_kx$ arbitrarily close to the origin, hence all the numbers $a_1, a_2, \ldots$ are right hand sequential secant derivatives of $f$ at 0. Similarly, the graph of $f$ contains
segments of the graph of \( y = \sqrt{x} \) arbitrarily close to the origin, hence \(+\infty\) is a right hand sequential derivative at \( 0 \).

Suppose \( L \) is a right hand sequential derivative of \( f \) at \( 0 \). Then there is a sequence \( h_m \searrow 0 \) such that \( \frac{f(h_m)}{h_m} \to L \). Now each \( h_m \) is in one of the intervals in \( (\xi_{n,k-1}, \xi_{n,k}) \). If \( k \leq n \), then \( \frac{f(h_m)}{h_m} = a_k \), where \( k = k(m) \) depends on \( m \). If \( k = n + 1 \), then \( \frac{f(h_m)}{h_m} = \frac{1}{\sqrt{h_m}} \). Since \( \frac{1}{\sqrt{h_m}} \to \infty \) and \( \infty \) is isolated in \( S \), it follows that, either \( \frac{f(h_m)}{h_m} = \frac{1}{\sqrt{h_m}} \) for all but a finite number of \( m \) or \( \frac{f(h_m)}{h_m} = a_k(m) \) for all but a finite number of \( m \). In the first case \( L = \infty \in S \), in the second case \( L \in S \), since \( a_k(m) \to L \) as \( m \to \infty \) and \( S \setminus \{\infty\} \) is closed.

By Theorem 2.5 any closed set is the set of sequential derivatives at a point of a real valued function defined on an interval. The following theorem shows that if \( f \) is continuous on the interval the conclusion is completely different, in fact, then the set of sequential secant derivatives must be a single point or a closed interval.

**Theorem 2.6.** If \( f : [0, 1] \to \mathbb{R} \) is continuous, then the set of right hand sequential derivatives of \( f \) at \( 0 \) is either a single point or a closed interval in \( \mathbb{R} \).

**Proof.** Replacing \( f \) by \( f(x) - f(0) \), if necessary, we may assume \( f(0) = 0 \). Suppose \( L < M \) are right hand derivatives of \( f \) at \( 0 \). Let \( L < K < M \). We must show \( K \) is a right hand sequential derivative of \( f \) at \( 0 \). Suppose \( h_n \searrow 0 \), \( k_n \searrow 0 \), \( \frac{f(h_n)}{h_n} \to L \) and \( \frac{f(k_n)}{k_n} \to M \). By passing to subsequences, if necessary, we may assume

\[
h_1 > k_1 > h_2 > k_2 > h_3 > k_3 > \cdots
\]

and

\[
\frac{f(h_n)}{h_n} < K < \frac{f(k_n)}{k_n}
\]

for all \( n \). Since \( \frac{f(x)}{x} \) is continuous on the interval \([k_n, h_n]\), it follows from the Intermediate Value Theorem, that there are \( \ell_n \), such that \( k_n = \ell_n < h_n \) and \( \frac{f(\ell_n)}{\ell_n} = K \). This completes the proof. 

The functions constructed in the proof of Theorem 2.5 are not continuous. However, we have the following analog of Theorem 2.5 for continuous functions.

**Theorem 2.7.** Any closed subinterval of the extended real line is the set of right hand sequential derivatives at \( 0 \) of some continuous function defined on the closed interval \([0, 1] \).

**Proof.** If \( f(x) = \sqrt{x} \sin \left( \frac{1}{x} \right) \) when \( x > 0 \) and \( f(0) = 0 \), then \( f \) is continuous and the collection of all right hand secant derivatives equals the extended real line.

Suppose \( a < b \) are real numbers. If \( |a| \leq |b| \), let \( f(0) = 0 \) and for \( x > 0 \) let

\[
f_{a,b}(x) := \begin{cases} bx \sin \left( \frac{1}{x} \right) & \text{when } b \sin \left( \frac{1}{x} \right) \geq a, \\ ax & \text{when } b \sin \left( \frac{1}{x} \right) < a. \end{cases}
\]

If \( |b| < |a| \), let \( f(0) = 0 \) and for \( x > 0 \) let

\[
f_{a,b}(x) := \begin{cases} ax \sin \left( \frac{1}{x} \right) & \text{when } b \sin \left( \frac{1}{x} \right) \geq a, \\ bx & \text{when } b \sin \left( \frac{1}{x} \right) < a. \end{cases}
\]

In either case, \( f \) is continuous on the closed interval \([0, 1] \) and the set of right hand sequential derivatives of \( f \) at the origin equals the closed interval \([a, b] \).
The cases where one endpoint is infinite and the other endpoint is finite is left for the reader.

3. Sequential Cord Derivatives

In Section 2 we considered the slopes of secant lines with endpoints \((x, f(x))\) and \((x + h_n, f(x + h_n))\) for sequences \(h_n \neq 0\). In this section we consider the slopes of cords with endpoints \((x - k_n, f(x - k_n))\) and \((x + h_n, f(x + h_n))\) for sequences \(h_n, k_n > 0\).

Definition 3.1. We say \(L \in \mathbb{R}\) is a sequential cord derivative of \(f\) at \(x\), if there are sequences \(h_n \searrow 0\), and \(k_n \searrow 0\), such that \(x + h_n \in D, x - k_n \in D\), and

\[
\frac{f(x + h_n) - f(x - k_n)}{h_n + k_n} \to L.
\]

For the sake of brevity, we will often say cord derivative in place of sequential cord derivative.

We begin by showing that, if \(f\) is differentiable at \(x\), then all the sequential cord derivatives of \(f\) at \(x\) exist and are equal to the derivative \(f'(x)\) of \(f\) at the point \(x\).

**Proposition 3.2.** If \(f'(x)\) exists, \(h_n \searrow 0\), and \(k_n \searrow 0\), then

\[
\frac{f(x + h_n) - f(x - k_n)}{h_n + k_n} \to f'(x).
\]

**Proof.** Suppose \(f'(x)\) is a real number. Let \(\varepsilon > 0\) be fixed. Pick \(n\) such that \(f(x + h_n) - f(x)\) and \(f(x - k_n) - f(x)\) both are within \(\varepsilon\) of \(f'(x)\). Since

\[
\frac{f(x + h_n) - f(x - k_n)}{h_n + k_n} = h_n \frac{f(x + h_n) - f(x)}{h_n} + k_n \frac{f(x - k_n) - f(x)}{k_n}.
\]

Combining this with \(\frac{h_n}{h_n + k_n} \geq 0\), \(\frac{k_n}{h_n + k_n} \geq 0\), and \(\frac{h_n}{h_n + k_n} + \frac{k_n}{h_n + k_n} = 1\) we see that

\[
\left| \frac{f(x + h_n) - f(x - k_n)}{h_n + k_n} - f'(x) \right| \leq \frac{h_n}{h_n + k_n} \left| \frac{f(x + h_n) - f(x)}{h_n} - f'(x) \right| + \frac{k_n}{h_n + k_n} \left| \frac{f(x - k_n) - f(x)}{k_n} - f'(x) \right| < \frac{h_n}{h_n + k_n} \varepsilon + \frac{k_n}{h_n + k_n} \varepsilon = \varepsilon.
\]

The cases where \(f'(x) = \pm\infty\) are left for the reader. This completes the proof.

The following provides a converse to the previous result, when \(f\) is assumed to be continuous at the point of interest.

**Theorem 3.3.** If \(f\) is continuous at \(x\) and all the cord derivatives of \(f\) at \(x\) exists and equals \(L \in \mathbb{R}\), then \(f'(x)\) exists and equals \(L\).

**Proof.** By assumption

\[
\frac{f(x + h_n) - f(x - k_n)}{h_n + k_n} \to L
\]
for all \( h_n \searrow 0 \), and \( k_n \searrow 0 \). Suppose \( h_n \searrow 0 \). Since \( f \) is continuous at \( x \) we can pick \( k_n \searrow 0 \) such that \( k_n \leq h_n^2 \) and \( |f(x - k_n) - f(x)| \leq h_n^2 \). Using
\[
\frac{f(x + h_n) - f(x)}{h_n} = \frac{h_n + k_n}{h_n} \left( \frac{f(x + h_n) - f(x - k_n)}{h_n + k_n} + \frac{f(x - k_n) - f(x)}{h_n} \right)
\]
we conclude, \( \frac{f(x + h_n) - f(x)}{h_n} \to L \). Hence, the right hand derivative of \( f \) at \( x \) exists and equals \( L \). Similarly, the left hand derivative of \( f \) at \( x \) exists and equals \( L \). □

Our next result shows that the set of sequential cord derivatives at a point is a closed interval. It is the analog of Theorem 2.4 for cord derivatives.

**Theorem 3.4.** The set of sequential cord derivatives at \( x \) is a closed subset of \( \mathbb{R} \).

**Proof.** Suppose \( L_n \in \mathbb{R} \) is a sequence of cord derivatives at \( x \) and \( L_n \to L \) as \( n \to \infty \). We must show \( L \) is a cord derivative. For each \( n \), let \( h_{n,m} \searrow 0 \) and \( k_{n,m} \searrow 0 \) be such that
\[
\frac{f(x + h_{n,m}) - f(x - k_{n,m})}{h_{n,m} + k_{n,m}} \to L_n
\]
Pick \( N_n \) such that
\[
\left| \frac{f(x + h_{n,N_n}) - f(x - k_{n,N_n})}{h_{n,N_n} + k_{n,N_n}} - L \right| < \frac{1}{n}
\]
then
\[
\frac{f(x + h_{n,N_n}) - f(x - k_{n,N_n})}{h_{n,N_n} + k_{n,N_n}} \to L
\]
This completes the proof. □

In Theorem 2.6 we showed that the set of one-sided secant derivatives of a continuous function is a closed interval. Our next result establishes an appropriate version of this for cord derivatives.

**Theorem 3.5.** If \( f : [-1, 1] \to \mathbb{R} \) is continuous, then the set of sequential cord derivatives of \( f \) at 0 is either the empty set, a single point, or a closed interval in \( \mathbb{R} \).

**Proof.** Replacing \( f \) by \( f(x) - f(0) \), if necessary, we may assume \( f(0) = 0 \). Suppose \( L^- < L^+ \) are cord derivatives of \( f \) at 0. Let \( L^- < K < L^+ \). We must show \( K \) is a sequential cord derivative of \( f \) at 0. Suppose \( h_n^+ \searrow 0 \), \( k_n^- \searrow 0 \), \( \frac{f(h_n^+) - f(-k_n^-)}{h_n^+ + k_n^-} \to L_n^+ \), for \( \alpha = \pm \). Hence for sufficiently large \( n \)
\[
\frac{f(h_n^-) - f(-k_n^-)}{h_n^- + k_n^-} < K < \frac{f(h_n^+) - f(-k_n^+)}{h_n^+ + k_n^+}
\]
Let
\[
\phi_n(t) = \frac{f(th_n^- + (1-t)h_n^+) - f(-tk_n^- - (1-t)k_n^+)}{th_n^- + (1-t)h_n^+ + tk_n^- + (1-t)k_n^+}
\]
The \( \phi_n \) is continuous on \([0, 1] \), \( \phi_n(0) = \frac{f(h_n^+) - f(-k_n^+)}{h_n^+ + k_n^+} \) and \( \phi_n(1) = \frac{f(h_n^-) - f(-k_n^-)}{h_n^- + k_n^-} \).
It follows from the Intermediate Value Theorem, that \( \phi_n(t_n) = K \) for some \( t_n \) between 0 and 1. Setting \( h_n' = t_nh_n^- + (1-t_n)h_n^+ \) and \( k_n' = tk_n^- + (1-t_n)k_n^+ \) it follows that \( h_n' \searrow 0 \), \( k_n' \searrow 0 \) and \( \frac{f(x + h_n') - f(x - k_n')}{h_n' + k_n'} = K \) for all sufficiently large \( n \). This completes the proof. □
4. Interactions Between Cord and Secant Derivatives

In this section we establish some relationships between the cord and secant derivatives. Our first result gives a condition under which the set of secant derivatives at a point \( x \) is a subset of the set of cord derivatives at \( x \). We show that it may be a proper subset and apply the inclusion to the Weierstrass function.

**Theorem 4.1.** If \( f \) is continuous at \( x \), then the set of secant derivatives of \( f \) at \( x \) is a subset of the set of cord derivatives of \( f \) at \( x \).

**Proof.** Let \( L \) be a right hand secant derivative of \( f \). By assumption

\[
\frac{f(x + h_n) - f(x)}{h_n} \to L \quad \text{as} \quad n \to \infty
\]

for some \( h_n \searrow 0 \). Since \( f \) is continuous at \( x \) we can pick \( k_n \searrow 0 \) such that \( k_n \leq h_n^2 \) and \( |f(x - k_n) - f(x)| \leq h_n^2 \). By the choice of \( k_n \) we have \( \frac{h_n + k_n}{h_n} \to 1 \) and

\[
\frac{f(x - k_n) - f(x)}{h_n} \to 0. \quad \text{Hence, using}
\]

\[
\frac{f(x + h_n) - f(x)}{h_n} = \frac{h_n + k_n}{h_n} \cdot \frac{f(x + h_n) - f(x)}{h_n} + \frac{f(x + h_n) - f(x)}{h_n} + \frac{f(x - k_n) - f(x)}{h_n}
\]

we conclude, \( \frac{f(x + h_n) - f(x) - f(x - k_n) - f(x)}{h_n + k_n} \to L \). Hence, a cord derivative of \( f \) at \( x \) exists and equals \( L \).

The case where \( L \) is a left hand secant derivative of \( f \) is similar. \( \blacksquare \)

**Example 4.2.** The Weierstrass function revisited. Weierstrass showed that \( \pm \infty \) are secant derivatives of \( W \) at any point \( x \). By Theorem 4.1 \( \pm \infty \) are also cord derivatives of \( W \) at any point \( x \). Since \( W \) is continuous on the real line, it follows from Theorem 3.5 that at any point \( x \), the set of cord derivatives of \( W \) equals the extended real line \( \mathbb{R} \).

In light of Theorem 4.1 a natural question is: Can the set of secant derivatives be a proper subset of the set of cord derivatives? By considering simple examples it is easy to see that the answer is yes. A simple example is provided by considering \( f(x) = |x| \) another example is provided in Example 4.6.

**Example 4.3.** If \( f(x) := |x| \) and \(-1 \leq L \leq 1 \), then there exists \( h_n \searrow 0 \), and \( k_n \searrow 0 \), such that

\[
\frac{f(h_n) - f(-k_n)}{h_n + k_n} \to L.
\]

**Proof.** This follows from Theorem 4.5. We provide a simple direct proof, the proof introduces some ideas used below.

(a) If \( L = 0 \), let \( h_n = k_n = \frac{1}{n} \).

(b) Suppose \( 0 < L < 1 \). Consider \( h_n = \frac{1}{n} \) and \( k_n = \frac{b}{n} \), then

\[
\frac{1}{n} - \frac{b}{n} = 1 - \frac{1}{1 + b} = L.
\]

Solving for \( b \) we see \( b = (1 - b)/(1 + L) \) does the job.

(c) If \( L = 1 \) setting \( h_n = \frac{1}{n} \) and \( k_n = \frac{1}{n^2} \) does the job, since

\[
\frac{1}{n} - \frac{1}{n^2} = 1 - \frac{1}{1 + \frac{1}{n}} \to 1.
\]
This also follows from Proposition 4.1.

(d) We leave the cases $-1 \leq L < 0$ for the reader.

Below we calculate the set of cord derivatives assuming the right hand and left hand secant derivatives exists. To simplify the notation we assume the point of interest is $x = 0$ and $f(0) = 0$. We can always arrange this by considering $g(t) = f(t + x) - f(x)$ in place of $f$. Suppose $h_n \downarrow 0$ and $k_n \downarrow 0$. The basis for our calculations is the formula

$$
\frac{f(h_n) - f(-k_n)}{h_n + k_n} = \frac{h_n}{h_n + k_n} \cdot \frac{f(h_n)}{h_n} + \frac{k_n}{h_n + k_n} \cdot \frac{f(-k_n)}{-k_n}.
$$

(4.1)

Suppose

$$
\frac{h_n}{h_n + k_n} \to r, \quad \frac{f(h_n)}{h_n} \to R \quad \text{and} \quad \frac{f(-k_n)}{-k_n} \to L \quad \text{as} \quad n \to \infty
$$

(4.2)

where $R$ and $L$ are real numbers and $h_n$ is a subsequence of $h_n$ and $k_n$ is a subsequence of $k_n$, then $0 \leq r \leq 1$ and

$$
\frac{f(h_n) - f(-k_j)}{h_n + k_j} \to rR + (1 - r)L \quad \text{as} \quad n \to \infty.
$$

(4.3)

Where Equation (4.3) follows from (4.2) by replacing the sequences $h_n$ and $k_n$ by the appropriate subsequences in (4.1).

In particular,

**Proposition 4.4.** If $h_n, k_n \downarrow 0$, $f$ is defined on the set $\{0, h_1, -k_1, h_2, -k_2, \ldots\}$, $L, R$ are real numbers, and

$$
\frac{f(h_n)}{h_n} \to R \quad \text{and} \quad \frac{f(-k_n)}{-k_n} \to L,
$$

then any cord derivative of $f$ at 0 is a real number between $R$ and $L$.

Below we explore the converse of this statement. When $h_n$ and $k_n$ decay at the same polynomial rate, then any real number between $R$ and $L$ is a cord derivative. When $h_n$ and $k_n$ decay at the same exponential rate, then the only accumulation points of the set of cord derivatives are $R$ and $L$, in particular, the set of cord derivatives is not an interval.

**Theorem 4.5.** Let $a, b, m > 0$ be real numbers. Suppose $p(n), q(n)$ are increasing functions and

$$
p(n) \to a \quad \text{and} \quad q(n) \to b.
$$

Let $h_n := \frac{1}{p(n)}$ and $k_n := \frac{1}{q(n)}$. Let $f$ be a function defined on $\{0, h_1, -k_1, h_2, -k_2, \ldots\}$ and let $R, L$ be real numbers. If $f(0) = 0$ and

$$
\frac{f(h_n)}{h_n} \to R \quad \text{and} \quad \frac{f(-k_n)}{-k_n} \to L
$$

then every real number between $L$ and $R$ is a cord derivative of $f$ at 0.

**Proof.** By Equation (4.3) and Theorem 3.4 it is sufficient to show that given any $0 < r < 1$ we can find subsequences $h_{n_i}$ and $k_{j_n}$ such that

$$
\frac{h_{n_i}}{h_{n_i} + k_{j_n}} \to r.
$$
For integers $i, j$ we have
\[
\frac{p(in)}{n^m} = i^m \frac{p(in)}{(in)^m} \to i^m a \quad \text{and} \quad \frac{q(jn)}{n^m} \to j^m b.
\]
Hence
\[
\frac{h_{in}}{h_{in} + k_{jn}} = \frac{q(jn)}{q(jn) + p(in)} \to j^m b + i^m a.
\]
It remains to show we can pick $i, j$ arbitrarily large such that $\frac{j^m b}{j^m b + i^m a}$ is within $\varepsilon > 0$ of $r$. To this end, let $j'$ be so large that $\frac{1}{j'} < \varepsilon$ and $r < \frac{j^m b}{j^m b + i^m a}$. Let $i$ be such that $\frac{j^m b}{j^m b + i^m a} < r$ and let $j > j'$ be such that
\[
\frac{j^m b}{j^m b + i^m a} < \leq \frac{(j + 1)^m b}{(j + 1)^m b + i^m a}.
\]
We complete the proof by showing
\[
\frac{(j + 1)^m b}{(j + 1)^m b + i^m a} - \frac{j^m b}{j^m b + i^m a} < \varepsilon.
\]
Let
\[
f(t) = \frac{bt}{bt + i^m a},
\]
then we must show $f(j + 1) - f(j) < \varepsilon$. Now
\[
0 < f'(t) = \frac{b a i^m}{b t + i^m a} < \frac{b a i^m}{2 b t i^m a} = \frac{1}{2 t}
\]
uniformly in $i$. By the Mean Value Theorem
\[
f(j + 1) - f(j) = f'(c) < \frac{1}{2 c} < \frac{1}{2 j} < \frac{1}{2 j'} < \varepsilon.
\]
This completes the proof.

\[\textbf{Example 4.6.}\] Let $a < b$ and $c < d$ be real numbers. Let $f_{a,b}$ be as in the proof of Theorem 2.7. The the right hand secant derivatives of $f_{a,b}$ at 0 equals the interval $[a,b]$ and the set of left hand secant derivatives of $x \to f_{-d,-c}(-x) = f_{c,d}(x)$ at 0 equals the interval $[c,d]$. Let
\[
g(x) := \begin{cases} f_{a,b}(x) & \text{when } x \geq 0 \\ f_{-d,-c}(-x) & \text{when } x < 0 \end{cases}.
\]
We claim that the set of cord derivatives of $g$ is the the convex hull $[\min \{a,c\}, \max \{b,d\}]$ of the intervals $[a,b]$ and $[c,d]$.

\[\textbf{Proof.}\] If one of $[a,b]$ and $[c,d]$ is a subinterval, the claim follows from Theorem 4.1 and Proposition 4.4.

Since $\sin \left(\frac{x}{k}\right) = y$ has solutions $x = \arcsin(y) + 2\pi k$ where $k$ is an integer, there are harmonic progressions $\alpha_n = \frac{1}{\pi + \pi n}$ and $\beta_n = \frac{1}{\pi + \pi n}$ such that $f_{a,b}(\alpha_n) = a \alpha_n$ and $f_{a,b}(\beta_n) = b \beta_n$ and similarly for $f_{-d,-c}$. Consequently, the claim follows from Theorem 4.5.

If follows from Theorem 4.5 that if $h_n$ and $k_n$ decay at the same polynomial rate, then the set of cord derivatives is an interval. It follows from our next result that, if the sequences $h_n$ and $k_n$ decay exponentially, then the set of cord derivatives need not be an interval.
Theorem 4.7. Suppose $a, b > 1$ are real numbers. Let $h_n := \frac{1}{a^n}$ and $k_n := \frac{1}{b^n}$. Let $f$ be a function defined on $\{0, h_1, -k_1, h_2, -k_2, \ldots\}$ and let $R, L$ be real numbers. Assume $f(0) = 0$ and
\[
\frac{f(h_n)}{h_n} \rightarrow R \text{ and } \frac{f(-k_n)}{-k_n} \rightarrow L.
\]
- If $\frac{\log(a)}{\log(b)}$ is a rational number, then $R$ and $L$ are the only accumulation points of the set of cord derivatives of $f$ at 0.
- If $\frac{\log(a)}{\log(b)}$ is an irrational number, then every real number between $L$ and $R$ is a cord derivative of $f$ at 0.

Proof. Note
\[
\frac{h_n}{h_n + k_n} = \frac{b^j}{b^j + a^i} = \frac{1}{1 + \frac{a^i}{b^j}}.
\]
Let $s := \frac{1-r}{r}$. Then
\[
\frac{h_n}{h_n + k_n} \rightarrow r \iff \frac{a^i}{b^j} \rightarrow s \iff i_n \log(a) - j_n \log(b) \rightarrow \log(s).
\]
If $\frac{\log(a)}{\log(b)}$ is an irrational number, then the set
\[
\left\{ \frac{\log(a)}{\log(b)} - j : i, j \in \mathbb{N} \right\}
\]
is dense in the real line.
On the other hand, if $\frac{\log(a)}{\log(b)} = \frac{p}{q}$ is a rational number, then the set in Equation (4.5) is a subset of the fractions with denominator $q$. Using (4.4) and that the set of cord derivatives is a closed set (by Theorem 3.4) the result follows.

The density of the set in Equation (4.5) was first proved by Nicole Oresme around 1360 in his paper *De commensurabilitate vel incommensurabilitate motuum celii*. For an English translation of Oresme’s proof see [Gra61]. A detailed analysis of Oresme’s proof is in [vP93]. More contemporary proofs and additional historical discussion can be found in [Pet83].

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