Sensitivity and stability of long periodic orbits of chaotic systems

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The relationship between the stability of unstable periodic orbits of dissipative chaotic systems and the sensitivity of period averages to parameter perturbations is clarified. Using Floquet theory, we show that period averages vary most rapidly when structural perturbations of the equations of motion are well expressed by adjoint Floquet eigenfunctions associated to Floquet exponents of small magnitude, along invariant subspaces closest to marginality. Building on this analysis, we then construct a large inventory of periodic orbits of the Lorenz equations and of the Kuramoto-Sivashinsky system in a minimal-domain configuration, focusing on long periodic orbits spanning a large fraction of the chaotic attractor. We examine how the statistical distributions of Floquet exponents, period averages and their sensitivities to parameter perturbations vary as the period \( T \) increases. We show that, at least for these two systems, the distribution of these quantities converges to a Dirac delta function as \( T \to \infty \). We observe that the limiting value of period averages and Floquet exponents tend to the same asymptotic value obtained on long chaotic trajectories. Conversely, the limiting value of the sensitivity is not consistent with the response of long-time averages to finite-amplitude parameter perturbations, due to the lack of a linear response for non-hyperbolic dynamics.

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Invariant solutions of the Navier-Stokes equations have proven particularly useful in unraveling the nature of turbulent shear flows in canonical geometries. Motivated by the prospect of using such solutions for optimization and design, we have previously specialized variational methods to unstable periodic orbits, with the aim of employing the sensitivity of period averaged quantities to parameter perturbations as a well-behaved proxy for the sensitivity of ergodic averages. With a infinite hierarchy of periodic solutions available, however, the recurrent question is: which one should be picked? Here, using numerical experiments on low-dimensional dynamical systems, we show that the probability distribution of the sensitivity of period averages converges to a Dirac delta function as the period increases. In other words, long periodic orbits, spanning large fraction of the attractor, provide converging estimates of the sensitivity, although care should be taken when using such estimates for non-hyperbolic dynamics.

\[ \text{I. INTRODUCTION} \]

Robust evidence has been recently offered\(^2\) to corroborate the intuition that temporally-recurrent invariant solutions of the Navier-Stokes equations – unstable periodic orbits – may provide a constitutive skeleton organizing spatiotemporal dynamics of turbulent shear flows in canonical geometries. Motivated by these advances, we have previously suggested\(^9\) how unstable periodic orbits might be used for control and optimization of shear flows, rather than serving as an analysis tool. In particular, we have specialized adjoint methods for time-periodic systems\(^5\) to unstable periodic orbits. We have shown that enforcing periodicity conditions on the adjoint problem, justified by the peculiar topology of these trajectories, prevents the growth of exponential instabilities that would otherwise feature prominently in the solution of the adjoint equations\(^1\). Operationally, the approach provides the sensitivity of period averaged quantities with respect to small perturbations of control/design variables parameterizing the equations of motion. Possessing such knowledge might facilitate, among others, control or optimization, two notably challenging tasks for nonlinear chaotic dynamics. Geometrically, small parameter perturbations can be pictured as producing a smooth, global state-space deformation of the unstable periodic orbits supporting and shaping the attractor, as opposed to causing exponential divergence. The conjecture is that the sensitivity of period averages to parameter perturbations may be used as a proxy for the sensitivity of ergodic averages. The approach is a special case of shadowing theory ideas\(^1\) recently introduced in the context of sensitivity analysis of chaotic systems\(^1\).

While complete knowledge of period averages of the short, fundamental cycles may be sufficient to compute ergodic averages using cycle averaging formulas\(^1\), a formalism that relies on the sensitivity of such cycles to compute the sensitivity of ergodic averages is, to the best of the author’s knowledge, not available in the literature. In fact, the existence of a perturbation theory for deterministic non-hyperbolic chaos is a subtle subject. For hyperbolic dynamics, Ruelle’s linear response theory\(^2\) provides the technical apparatus to analyze the response of statistical quantities to small, periodic perturbations.
of the parameters. However, systems of practical interest rarely satisfy hyperbolicity and the literature is rich of examples where infinite-time averaged quantities exhibit complex, non smooth behavior when parameters are varied. Such systems are infinitesimally close to internal structural bifurcations, and parameter perturbations might have a catastrophic impact on the invariant measure that supports ergodic averages, unless stochastic components are introduced or in the “thermodynamic” limit of high-dimensional systems.

In this paper, we do not focus on this issue, but rather aim to provide a better understanding of more practical aspects. The first original contribution of this paper is a thorough empirical study of the statistical distribution of period averages and their sensitivity to parameter perturbations as the period $T$ increase. While in previous work we focused on short cycles, here we focus on long orbits, spanning increasingly larger fraction of the attractor. We resort, by necessity, to low-dimensional systems, the Lorenz equations at standard parameters and a small-domain Kuramoto-Sivashinsky system in the antisymmetric subspace, where obtaining a sizable inventory of long periodic orbits is a feasible task. We will show that, similar to period averages, longer orbits provide converging estimates of the sensitivity of period averages to parameter perturbations as the period $T$ increases. In fact, the statistical distribution of the sensitivity tends asymptotically to a Dirac delta function, with standard deviation decreasing as $T^{-1/2}$. In addition, using classical results from bifurcation theory, we will derive analytically the functional form of the tails of the probability distribution of the sensitivity of period averages. The second contribution is that we clarify the relation between the stability of periodic orbits, analyzed using Floquet theory and the sensitivity of period averages. The goal is to better illustrate the inevitable effects of bifurcations that periodic orbits undergo in parameter space and elucidate the role of marginality. As opposed to classical variational methods, where the most unstable direct/adjoint covariant Lyapunov vector dominates asymptotically the solution of the linearized tangent/adjoint equations, we show that for a periodic orbit the direct/adjoint Floquet eigenfunctions associated to Floquet exponents of small magnitude (the eigenspace closest to marginality) play the largest role in determining the response of trajectories in state-space and of period averages.

The rest of this paper is organized as follows. In section, we lay down the general notation. This is followed by section, where we recall fundamental elements of Floquet theory for the linear stability of periodic orbits. We leave to appendix the description of the numerical technique used to obtain (part of) the spectrum of Floquet exponents for very long orbits. In section, we recall the variational technique for unstable periodic orbits originally introduced in Ref. With this introductory material, we clarify in section the connection between the stability and sensitivity. This insight provides more robust grounds to understand the practical role of bifurcations on sensitivity analysis as well the convergence of sensitivities as the period $T$ increases, for which numerical results are reported in section.

II. PRELIMINARIES

We consider dissipative dynamical systems of the form

$$\frac{d\mathbf{u}(t)}{dt} = \mathbf{f}(\mathbf{u}(t), \alpha),$$

(1)

governing the evolution of the state vector $\mathbf{u}(t) \in \mathbb{R}^n$, with $t$ being time. Without loss of generality, we restrict the attention to problems where the nonlinear vector field $\mathbf{f}$ depends only on one parameter $\alpha \in \mathbb{R}$, in regimes for which chaotic solutions are observed for typical initial conditions. We denote with $\mathbf{f}_\alpha(\mathbf{t}) \in \mathbb{R}^{n \times n}$ the stability matrix

$$\mathbf{f}_\alpha(\mathbf{t}) = \mathbf{f}_\alpha(\mathbf{t}, \alpha) = \frac{\partial \mathbf{f}(\mathbf{u}(t), \alpha)}{\partial \mathbf{u}},$$

(2)

and define the Jacobian matrices $\mathbf{J}(\mathbf{t}, \tau) \in \mathbb{R}^{n \times n}$, $t \geq \tau$, satisfying the initial value problem

$$\frac{d\mathbf{J}(\mathbf{t}, \tau)}{dt} = \mathbf{f}_\alpha(\mathbf{t})\mathbf{J}(\mathbf{t}, \tau), \quad \mathbf{J}(\tau, \tau) = \mathbf{I},$$

(3)

with $\mathbf{I}$ the identity matrix. The dot product of two vectors is denoted with $\mathbf{a}^\top \cdot \mathbf{b}$, with the script $^\top$ indicating transposition of vectors and matrices.

We focus on periodic solutions of $\mathbf{f}$, satisfying $\mathbf{u}(t + T) = \mathbf{u}(t)$ for an unknown period $T$ that is not set a priori, but depends implicitly on the parameter $\alpha$. We shall thus consider the space of smooth periodic functions

$$\mathcal{P}_T = \{ f(t) : \mathbb{R} \rightarrow \mathbb{R}, f(t) = f(t + T) \},$$

(4)

parametrized by $T$, and extend this space to vector-valued functions, denoted with $\mathcal{P}_T^n$. We will make use of the norm $\| \cdot \|_{\mathcal{P}_T^n}$ induced by the inner product

$$\| \mathbf{v}(t), \mathbf{w}(t) \| = \frac{1}{T} \int_0^T \mathbf{v}(t)^\top \cdot \mathbf{w}(t) dt,$$

(5)

for any two vector-valued functions $\mathbf{v}(t)$ and $\mathbf{w}(t)$ in $\mathcal{P}_T^n$.

III. ELEMENTS OF FLOQUET THEORY

In this section, we briefly recall some elements of Floquet theory, defining the direct and adjoint Floquet eigenfunctions. These provide invariant subspaces in which the solution of the sensitivity problem can be conveniently expanded, shedding light on the relation between stability and sensitivity of period averages.

For an arbitrary point on a periodic orbit, the eigen-decomposition of the monodromy matrix

$$\mathbf{J}(T, 0)\mathbf{e}_k(0) = \mu_k \mathbf{e}_k(0), \quad k = 1, \ldots, n,$$

(6)

produces the Floquet multipliers $\mu_k$ and the associated right eigenvectors $\mathbf{e}_k(0)$. To simplify the notation and the analysis of section, we assume here that multipliers are real, positive and distinct, so that the eigenvectors form collectively a basis of $\mathbb{R}^n$. More useful than the multipliers are the Floquet exponents $\lambda_k = \log(\mu_k)/T$, defining
the period averaged growth/decay of tangent perturbations initially aligned to the invariant subspaces $e_k(0)$.

We introduce the Floquet eigenfunctions $w_k(t)$, generated by the eigenvectors $e_k(0)$ as

$$w_k(t) = \exp(-\lambda_k t) J(t,0)e_k(0),$$

and satisfying the differential eigenproblem

$$\mathcal{L}w_k(t) \equiv \frac{dw_k(t)}{dt} - f_u(t)w_k(t) = -\lambda_k w_k(t),$$

with unit $\| \cdot \|_{P^2}$ norm. Because multipliers are assumed to be real and positive, the eigenfunctions $w_k(t)$ are elements of $P^2$. Note that the Floquet eigenfunctions of a periodic orbit have the same function that covariant Lyapunov vectors play for chaotic trajectories.

Sorting the Floquet exponents in descending order, we denote by $\chi - 1$ the number of positive exponents. It is well known that $w_\chi(t) = f(t)/\|f(t)\|_{P^2}$ is a marginal direction. In other words, the linear differential operator $\mathcal{L}$ is singular, with nullspace

$$\text{Null}\{\mathcal{L}\} = \text{span}\{w_\chi(t)\}. \quad (9)$$

The other useful element of Floquet theory, perhaps considered less extensively in the literature, are the adjoint Floquet eigenfunctions $w_k^+(t)$. These are elements of $P^2$ defined as

$$w_k^+(t) = \exp(-\lambda_k(T-t)) J(t,T)^\top e_k^+(T), \quad (10)$$

where the vectors $e_k^+(T)$ are the left eigenvectors of the monodromy matrix, satisfying

$$e_k^+(T)^\top J(0,T) = \mu_k e_k^+(T)^\top \quad (11)$$

for the same multipliers and exponents of the direct problem. The adjoint eigenfunctions satisfy the adjoint differential eigenproblem

$$\mathcal{L}^+ w_k^+(t) \equiv -\frac{dw_k^+(t)}{dt} - f_u^+(t)w_k^+(t) = -\lambda_k w_k^+(t),$$

where the operator $\mathcal{L}^+$ is the adjoint of $\mathcal{L}$ according to the inner product $\langle \cdot, \cdot \rangle$. These two operators share the same spectrum, and the adjoint operator $\mathcal{L}^+$ is thus singular, with nullspace

$$\text{Null}\{\mathcal{L}^+\} = \text{span}\{w_\chi^+(t)\}. \quad (13)$$

A final remark is that, at any time $t$, the direct or adjoint eigenfunctions do not form individually an orthogonal set of vectors but instead satisfy the bi-orthogonality relation

$$w_k^+(t)^\top \cdot w_j(t) = \delta_{kj} C_k \quad \forall t, \quad (14)$$

with $\delta_{kj}$ the Kronecker symbol and with $C_k \neq 0 \in \mathbb{R}$.

**IV. TANGENT AND ADJOINT SENSITIVITY**

We now briefly recall tangent and adjoint methods to compute the sensitivity of period averaged quantities. Consider an observable of interest, denoted by a function

$$J(t) = J(u(t)) : \mathbb{R}^n \mapsto \mathbb{R},$$

that, for the sake of simplicity, is assumed here not to depend explicitly on the parameter $\alpha$. Let also denote the partial derivative of the observable with respect to its argument as $J_u(t) = J_u(u(t))$, mapping $\mathbb{R}^n$ to $\mathbb{R}^n$.

Consider one periodic orbit $u(t) \in P^2_n$ and define the objective function $\mathcal{J}(\alpha) : \mathbb{R} \mapsto \mathbb{R}$

$$\mathcal{J}(\alpha) = \frac{1}{T} \int_0^T J(u(t)) dt$$

as the period average of the observable over the periodic orbit. This is a function of the parameter $\alpha$ since both $u(t)$ and $T$ depend, implicitly, on it. The goal is to compute the sensitivity of the objective function with respect to $\alpha$, the gradient $d\mathcal{J}/d\alpha$.

**A. Tangent approach**

For the tangent approach, linearization of (16) as reported in Ref. 12 shows that the gradient of the period average is given by the inner product

$$d\mathcal{J}/d\alpha = \langle J_u(t), v(t) \rangle,$$

where the perturbation $v(t) \in P^2$ satisfies the tangent equation

$$\mathcal{L}v(t) = f_u(t) - \tau f(t). \quad (18)$$

The perturbation $v(t)$ is the first order state-space deformation of the periodic orbit $u(t)$ when $\alpha$ is varied. The forcing term $f_u(t) \in P^2$ is the derivative of the right hand side of (1) with respect to the parameter

$$f_u(t) = f_u(u(t)) = \frac{\partial f(u(t), \alpha)}{\partial \alpha}, \quad (19)$$

while the scalar $\tau = (dT/d\alpha)/T$ is the (unknown) relative period gradient, producing an algebraically growing mode along $f(t)$, allowing $v(t)$ to be time periodic. The introduction of this term is akin to classical approaches in perturbation/continuation analysis of periodic problems, where time is rescaled by the period.

For an hyperbolic orbit, the differential operator $\mathcal{L}$ is singular with nullspace given by (13) and equation (18) has a one parameter family of solutions. Physically speaking, this is a reflection of the translational invariance along a periodic orbit: if $v(t)$ is a solution, then $v(t) + \sigma f(t)$ is a solution too, for any $\sigma \in \mathbb{R}$, with same gradient $d\mathcal{J}/d\alpha$, since

$$\langle J_u(t), f(t) \rangle = 0$$

for all possible cost functions when $u(t) \in P^2$. Hence, for (18) to have a solution, the scalar $\tau$ must have the unique value that shifts the right hand side $f_u(t) - \tau f(t)$ in the range of the operator $\mathcal{L}$, or, by Fredholm’s Alternative (cf. [14], Lemma 1.1, pg. 146) makes it orthogonal to the nullspace of its adjoint $\mathcal{L}^+$, i.e. by satisfying

$$\langle w_k^+(t), f_u(t) - \tau f(t) \rangle = 0.$$ (21)
Hence, the scalar \( \tau \) could be in principle determined as
\[
\tau = \frac{\left[ w^+_\chi(t), f_u(t) \right]}{C_\chi \| f(t) \|_{\mathcal{P}_T}}
\]
(22)
if \( w^+_\chi(t) \) was known. Numerically, it is more convenient to drop the singularity by adding the constraint
\[
\left[ v(t), f(t) \right] = 0,
\]
(23)
which fixes the component of \( v(t) \) along the nullspace and leads to the solution of (18) with minimum norm. In matrix form, the tangent problem reads
\[
\begin{bmatrix}
L & f(t) \\
\| f(t) \| & 0
\end{bmatrix}
\begin{bmatrix}
v(t) \\
\tau
\end{bmatrix}
\begin{bmatrix}
f_u(t) \\
0
\end{bmatrix} = 0
\]
(24)
and its solution provides the perturbation \( v(t) \) and the period gradient \( \tau \). Note that the left hand side of (24) has the same structure of the Newton-Raphson linear system, as we shall see in section V. With a bounded and similar discretization techniques can be employed.

Constraining \( v(t) \) to remain in \( \mathcal{P}_T^n \) by using an appropriate numerical method is the key to avoid exponential instabilities intrinsic to the tangent dynamics around an unstable periodic orbit. The solution \( v(t) \) will thus not grow exponentially along the most unstable subspace \( w_1(t) \), but will remain bounded, with magnitude and structure that depend on the complete stability spectrum, as we shall see in section V. With a bounded \( v(t) \), the gradient (17) is effectively the slope of the function \( J(\alpha) \) obtained from continuation.

**B. Adjoint approach**

For problems with several parameters an adjoint approach is computationally more economic. Expression (17) is first combined with the constraint (23) and the linearized equation (18), using a scalar \( \beta \) and an adjoint function \( v^+(t) \) restricted to \( \mathcal{P}_T^n \), leading to
\[
dJ/d\alpha = \left[ J_u(t), v(t) \right] + \left[ v(t), f(t) \right] \beta + 
\left[ L v(t) + \tau f(t) - f_u(t), v^+(t) \right],
\]
(25)
where the additional terms are all identically zero for all possible choices of \( v^+(t) \) and \( \beta \), because of equations (18) and (23). Using integration by parts on the third term, and noting that the resulting boundary term is zero since both \( v^+(t) \) and \( v(t) \) have been explicitly chosen to be periodic functions, leads, after rearranging terms, to
\[
dJ/d\alpha = \left[ \mathcal{L}^+ v^+(t) + J_u(t) + \beta f(t), v(t) \right] + 
\left[ v^+(t), f(t) \right] \tau - \left[ v^+(t), f_u(t) \right],
\]
(26)
Selecting now the adjoint function \( v^+(t) \) and \( \beta \) such as to cancel the first and second terms of this expression by solving the adjoint problem
\[
\mathcal{L}^+ v^+(t) = -J_u(t) - \beta f(t)
\]
(27)
with the constraint \( \left[ v^+(t), f(t) \right] = 0 \), or, using matrix notation,
\[
\begin{bmatrix}
L^+ & f(t) \\
\| f(t) \| & 0
\end{bmatrix}
\begin{bmatrix}
v^+(t) \\
\beta
\end{bmatrix}
\begin{bmatrix}
-J_u(t) \\
0
\end{bmatrix} = 0,
\]
(28)
we obtain an expression for the gradient of the period average that is independent of \( v(t) \) and \( \tau(t) \)
\[
dJ/d\alpha = - \left[ v^+(t), f_u(t) \right].
\]
(29)
Taking the inner product of the adjoint solution with each forcing term is less expensive than repeatedly solving (24) for many forcing terms when the vector field in (1) depends on many parameters. Moreover, constraining the adjoint solution to be periodic prevents the exponential growth along the leading unstable adjoint eigenfunction \( w^+_1(t) \) that would be otherwise observed by marching backwards in time.

A final technical remark is that, because the adjoint operator \( \mathcal{L}^+ \) is singular, Fredholm’s Alternative requires the right hand side of (27) to be orthogonal to \( f(t) \), the nullspace of the adjoint of \( \mathcal{L}^+ \). This condition is always met for all possible observables if \( \beta = 0 \), because of (20). Hence, including the constraint in (25) is not technically required, but it is added for symmetry with (24) so that the discretized form of (25) is a square, non-singular matrix that is precisely the transpose of the discretisation of the tangent problem in (24).

**V. STABILITY AND SENSITIVITY**

The relation between Floquet stability and the sensitivity of period averages is now clarified. The approach consists of projecting the tangent problem (18) onto the invariant subspaces formed by the Floquet eigenfunctions. The result is that a scalar ordinary differential equation (ODE) is obtained for each subspace, facilitating the interpretation.

First, the tangent solution \( v(t) \) is expanded in the Floquet eigenfunctions as
\[
v(t) = \sum_{k=1}^{n} w_k(t) a_k(t),
\]
(30)
with unknown expansion coefficients \( a_k(t) \in \mathcal{P}_T \). The forcing term in (18) is also similarly expanded
\[
f_u(t) = \sum_{k=1}^{n} w_k(t) b_k(t),
\]
(31)
where the functions \( b_k(t) \in \mathcal{P}_T \) can be determined by dotting (31) with the \( k \)-th adjoint eigenfunction
\[
b_k(t) = \frac{w^+_k(t) \cdot f_u(t)}{C_k}, \quad k = 1, \ldots, n,
\]
(32)
where the bi-orthogonality relation (14) is used.

Substituting the expansion (30) into the tangent equation (18) and using (8), produces
\[
\sum_{k=1}^{n} w_k(t) \left[ \frac{da_k(t)}{dt} - \lambda_k a_k(t) - b_k(t) + \tau \| f(t) \|_{\mathcal{P}_T} \delta_{k,1} \right] = 0.
\]
(33)
Since the Floquet eigenfunctions form a basis of \( \mathbb{R}^n \) for all \( t \) by assumption, the term in the square brackets in
must be zero. We thus obtain a set of decoupled linear ODEs with constant coefficients
\[
\frac{da_k(t)}{dt} = \lambda_k a_k(t) + b_k(t) - \tau \|f(t)\|_{\mathcal{P}} \delta_k, \tag{34}
\]
k = 1, \ldots, n, the tangent sensitivity problem expressed in the basis of the Floquet eigenfunctions. Along the neutral subspace \( f(t) \) the equation reads
\[
\frac{da_k(t)}{dt} = \frac{\mathbf{w}_k(t)^\top \cdot \mathbf{f}_a(t)}{C_\chi} - \tau \|f(t)\|_{\mathcal{P}}, \tag{35}
\]
In order for \( a_k(t) \) to remain in \( \mathcal{P}_T \), the right hand side must have zero integral over the period by Fredholm’s Alternative, since the equation is self adjoint and \( a_k(t) = 1 \) is a nontrivial solution of the homogeneous adjoint equation. This constraints fixes the relative period gradient \( \tau \) to a value that is the same as equation (22).

A particularly insightful expression can be derived for the expanding or contracting directions. The solution of the scalar ODE \( \frac{da_k(t)}{dt} = \lambda_k a_k(t) + b_k(t) \) can be expressed by its Green’s function (cf. [14], pg. 148) as
\[
a_k(t) = \int_0^T \frac{\mu_k}{1 - \mu_k} e^{-\lambda_k s} b_k(t + s) \, ds, \tag{36}
\]
and the upper bound
\[
\sup_\tau |a_k(t)| \leq \sup_\tau |b_k(t)|/|\lambda_k| = B_k/|\lambda_k| \tag{37}
\]
can be derived. This expression suggest several remarks.

First, it shows that the amplitude of the tangent solution along a particular Floquet eigenfunction \( \mathbf{w}_k(t) \) depends directly on the strength of the projection of the forcing \( \mathbf{f}_a(t) \) on the associated adjoint Floquet eigenfunctions, the coefficients \( b_k \). Hence, for spatially extended systems, detailed knowledge of the spatiotemporal dynamics of the adjoint Floquet eigenfunctions and not just the direct ones might provide an understanding of how physically relevant features of the solutions are influenced by problem parameters. One can then interpret the adjoint Floquet eigenfunctions as special directions where the forcing \( \mathbf{f}_a(t) \) can be particularly effective in modifying dynamical behavior, which is key, for instance, for control design.

Second, without further details on the coefficients \( b_k(t) \), generic parameter perturbations induce relatively small space-state changes along the highly contracting directions, while most of the “yield” occurs along the Floquet invariant subspace that is closest to marginality, regardless of the sign of \( \lambda_k \). When varying the system parameter \( \alpha \) towards a bifurcation, one (or a pair of) Floquet exponent approaches zero, the tangent solution displays a large amplitude along the corresponding eigendirection, resulting in large gradients of time averaged quantities. This is in stark contrast with classical variational methods for sensitivity analysis, where the largest response is, asymptotically, along the leading covariant Lyapunov vector.

Third, the boundedness of the forcing term \( \mathbf{f}_a(t) \) and of the Floquet eigenfunctions implies that the coefficients \( b_k(t) \), and thus the expansion coefficients \( a_k(t) \) and the tangent solution \( \mathbf{v}(t) \), have, on average, similar magnitude for long periodic orbits if the Floquet exponents converge as \( T \to \infty \). At this stage it is convenient to note that the Floquet exponents are the period averages of the “local exponents”
\[
\lambda_k(t) = \frac{\mathbf{w}_k(t)^\top [\mathbf{f}_a(t) + \mathbf{f}(t)] \mathbf{w}_k(t)}{||\mathbf{w}_k||^2}, \tag{38}
\]
uniquely defined functions of state space expressing the local growth rate of tangent perturbations along the invariant subspaces (here \( || \cdot || \) indicates the euclidean norm). By the Central Limit Theorem, the distribution of the \( k \)-th Floquet exponent across distinct orbits of similar period \( T \), must converge in law to a Dirac delta function with standard deviation decaying as \( T^{-1/2} \), assuming that the auto-correlation of time histories of \( \lambda_k \) decays sufficiently quickly. Hence, expression (37) indicates that the distribution of the sensitivity of period averages \( dJ/d\alpha \) will also converge to a delta function as \( T \) increases. In other words, while some scatter might be observed for short cycles, long periodic orbits will asymptotically provide the same sensitivity to parameter perturbations.

Fourth, recent investigations on the dynamics of covariant Lyapunov vectors in spatially extended systems have shown that the tangent space is hyperbolically split into a finite-dimensional subspace given the “physical modes”, spanned by leading covariant Lyapunov vectors and the remaining infinite-dimensional space spanned by the fast, dissipative scales associated to very negative exponents. The perspective does not change when Floquet eigenfunctions are utilized for periodic orbits. In this sense, the analysis of this section and more generally this work on sensitivity analysis of periodic orbits would suggests that an adequate description of the effect of parameter perturbations on long-time averages in dissipative systems might be obtained by a variational analysis restricted to finite-dimensional subspace spanned by the leading “physical” modes.

VI. NUMERICAL RESULTS

To address the questions raised in the introduction, and provide support for some of the claims made in section V, we now turn to numerical experiments and consider periodic orbits of two well known chaotic systems. The first is given by the Lorenz equations, where
\[
\begin{align*}
\frac{du_1}{dt} &= \sigma (u_2 - u_1), \\
\frac{du_2}{dt} &= \rho u_1 - u_2 - u_1 u_3, \\
\frac{du_3}{dt} &= u_1 u_2 - \beta u_3,
\end{align*} \tag{39}
\]
where standard parameters \( \sigma = 10, \beta = 8/3 \) and \( \rho = 28 \) are used throughout. As in other sensitivity studies on the Lorenz equations, we consider the sensitivity of the period average of the observable \( J(t) = u_3(t) \) with respect to perturbations of \( \rho \). Numerical integration of chaotic trajectories is performed using a classical fourth-order Runge-Kutta method with \( \Delta t = 0.005 \).
The second system is a finite-dimensional truncation of a spatially extended system, the Kuramoto-Sivashinsky equation
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^4 u}{\partial x^4} = 0, \tag{40}
\]

defining the evolution of zero-mean, spatially-periodic fields \( u \equiv u(x, t) \) over the domain \( x \in [0, 2\pi] \) with dynamics restricted to the invariant subspace of odd solutions. Here, we consider a relatively high diffusivity constant \( \nu = (2\pi/L)^2 \), with \( L = 39 \), the same value we considered in previous work. The spectral expansion
\[
u = \sum_{k=-n}^{n} \gamma_k(t) \exp(ikx), \tag{41}
\]

with \( u_k = -u_{-k}, u_0 = 0 \), is truncated at \( n = 28 \), leading to a system of ODEs approximating solutions of the original partial differential equation. Some of the numerical results reported in the next sections have been checked in a statistical sense for finer resolutions, with negligible quantitative changes. We take the energy density
\[
J[u(x, t)] = \frac{1}{4\pi} \int_{0}^{2\pi} u^2(x, t) \, dx \tag{42}
\]
as the functional of interest and then examine the sensitivity of its period average with respect to the diffusivity \( \nu \). Numerical integration of chaotic trajectories is performed using the fourth-order accurate implicit-explicit method IMEXRKC4 with time step \( \Delta t = 0.125\nu \).

### A. Search of periodic orbits

We use a global Newton-Raphson search algorithm developed in previous work based on the original method of Ref. 11 and classical techniques for nonlinear boundary value problems. Briefly, at iteration \( k \), this method solves a Newton-Raphson update equation to adjust a trial solution composed of a state-space loop \( u_k \in \mathcal{P}_{T_k}^n \) and a period \( T_k \), where the loop is not, at least initially, a solution of the equations, i.e. the residual
\[
\mathbf{r}_k(t) = \mathbf{d}u_k(t)/dt - \mathbf{f}(u_k(t), \alpha) \in \mathcal{P}_{T_k}^n \tag{43}
\]
is generally different from zero along the loop. The only significant modification that we have implemented is that the loop derivative operator \( d/dt \) is approximated using an eight-order accurate finite-difference stencil (instead of fourth order), enhancing the overall accuracy/cost ratio and allowing longer orbits to be found. The same high-order discretisation is used for the solution of the tangent and adjoint problems.

In previous work we attempted to locate exhaustively all short periodic orbits up to a fixed topological length and examined their sensitivity. The exponential growth of the number of periodic orbits with the period, however, quickly renders this approach infeasible when the analysis turns to long orbits. In this work, we adopt a different strategy where, for long periods, we located up to five thousand periodic orbits with period falling within

![FIG. 1. Number of unique periodic orbits found as a function of the reference period \( T \) for the Lorenz equations, panel (a), and KS system, panel (b). The red line is at 5000. For the KS system, the reference period is scaled by the viscosity.](image)

TABLE I. Reference temporal grid spacing for the finite-difference approximation of the derivative, period of the shortest admissible orbit and ratio between the largest reference period and \( T_{\text{min}} \).

|                | reference \( \Delta t \) | \( T_{\text{min}} \) | \( T_{\text{max}}/T_{\text{min}} \) |
|----------------|--------------------------|----------------------|-------------------------------|
| Lorenz        | 0.01                     | 1.5586               | \( \sim 645 \)                |
| KS 0.125\( \nu \)| 24.9080\( \nu \)        | 29.0800\( \nu \)       | \( \sim 201 \)               |

5% of a number of increasingly large and logarithmically spaced reference periods \( T \), as a balance between computing resources and statistical representativeness. For short periods, we search as many orbits as it is feasible, although we cannot guarantee that all periodic orbits are found. Initial guesses are obtained from near recurrences. Search results are reported in figure. A note of caution is that, given the impracticality of obtaining the complete set of periodic orbits for large periods, the statistical analysis and the discussion that is presented in the next sections will inevitably suffer of some form survivor bias, as they will be based on the subset of periodic orbits that “survived” the search process. In fact, Newton-Raphson techniques are known to converge at a slower rate or fail to converge at all when the orbit possesses multipliers close to the unit circle. Here, we take the perspective that this analysis might provide some useful guidance in a practical setting with turbulent shear flows, where one is bound to use/analyze only the few orbits that can be practically found given computational constraint.

We report in figure the shortest and longest periodic orbits found for the Lorenz equations, panel (a), and KS system, panel (b). While the short cycles are topologically simple, longer orbits wind around the attractor in a complicated fashion and are thus indistinguishable to the eye from long chaotic trajectories. The maximum reference period \( T_{\text{max}} \) is about two orders of magnitude larger than the period of the shortest admissible periodic orbit, as indicated in table, and is sufficient to reveal the asymptotic convergence properties of period averaged quantities and their sensitivities as \( T \) increases.

We build an inventory of unique solutions by ensuring that all periodic orbits found have period differing to at
least eight decimal places, which is few orders of magnitude larger than the accuracy at which this quantity is determined in the search, with the temporal discretisation settings reported in table I. For both systems, we have observed that relatively few Newton-Raphson iterations are required and that the convergence history is independent of the period, although the success rate appears to decline slightly for longer periods, likely as a result of the increasing condition number of the Newton-Raphson update problem. This is illustrated in figure [3] where we show the history of the norm of the residual \([13]\) for about a hundred test long period searches for the Lorenz and KS equations, in panels (a) and (b), respectively. Panels (c) and (d) shows how the success rate, estimated from these test searches, varies with the period. We have observed these trends to be independent of the temporal discretisation and, for the KS equations, of the spatial resolution. This gives us enough confidence that the long periodic orbits used in the rest of the paper are numerically reliable approximations of exact solutions of the equations, and not an artifact of the numerical method used for the search.

B. Statistics of time averages

We first examine how statistics of time averages over periodic and chaotic trajectories vary as a function of the reference period \(\hat{T}\). Results are reported in figure [4]. In this and subsequent figures, we denote by mean\([-\cdot]\] and std\([-\cdot]\) the mean and standard deviation of a quantity obtained over an ensemble of periodic or chaotic trajectories, respectively. We use the notation \(\mathcal{J}^{\hat{T}}\) to empha-
data, to the long-time average of chaotic trajectories.

The standard deviation of the time average over periodic and chaotic trajectories, bottom panels, decays asymptotically as $T^{-1/2}$, for both periodic and chaotic trajectories. This is the trend predicted by the Central Limit Theorem\cite{65} if correlations decay sufficiently fast. In fact, both systems considered here display ‘typical’ chaos (in the terminology of Ref.\cite{66}), with correlations dying out exponentially. On the other hand, for low periods the standard deviation of chaotic trajectories decays at a faster rate for both systems considered. We argue that this is an effect of correlations affecting the asymptotic behavior. For short periodic orbits, increasing $T$ does not necessarily result in a lower variance, in agreement with previous observations.\cite{53,55}

The probability distribution of time averages over periodic and chaotic trajectories for two different reference time spans $T$. Left, Lorenz equations, right, KS system. The gray parabola is the normal distribution.

C. Statistics of Floquet exponents

Because of the importance of Floquet analysis on the sensitivity of periodic averages, we first examine statistics of Floquet exponents across periodic orbits of increasing reference period, comparing these with the statistics of the finite-time Lyapunov exponents (FTLE) of same reference period. The FTLEs are calculated using classical numerical methods,\cite{68,69} involving propagating a set of vectors in tangent space and occasionally performing a re-orthogonalization using the Gram-Schmid procedure to counter the inevitable alignment to the most unstable subspace.

![Diagram](image-url)  
**FIG. 6.** The real part of the fifteen leading Floquet exponents of a long and a short orbit of the KS system. The inset shows the eight leading exponents. The ninth exponent has a much lower value, at around -82.

Computing the spectrum of Floquet exponents is, however, a notoriously challenging problem\cite{70,71}, even for orbits of moderate period. The exponential growth of the entries of the monodromy matrix\cite{69} renders its eigenvalue decomposition inaccurate in finite-precision arithmetic. In this work, where the focus is on long periodic solutions, we have devised an algorithm that is based on the recent work of Ref.\cite{71} and is inspired by standard methods to compute Lyapunov exponents.\cite{68,69} With this algorithm we have been able to compute accurately Floquet exponents for arbitrarily long orbits, corresponding to multipliers spanning thousands of orders of magnitude. We defer to appendix A the description of the algorithm and its validation.

For the KS system, all periodic orbits that we have found have only one unstable Floquet eigendirection, and with the present spatial resolution 26 contracting directions. For illustrative purposes, the leading part of the Floquet spectrum of one long and one short periodic orbit is shown in figure\cite{72}. The first eight exponents fall in the range ($-30, 5$), while the ninth exponent is sharply more negative and is followed by a long tail of negative exponents, corresponding to contracting “spurious” modes,\cite{55,56} with a value that is closely determined by the linear term of the governing equations. All orbits in our database have a spectrum with similar characteristics.

We report in figure\cite{72} the evolution of the mean (top panels) and standard deviation (bottom panels) of selected Floquet exponents (open circles) and FTLEs (filled
as a function of the reference period $\hat{T}$. We report data for the two non-trivial exponents of the Lorenz equations (panels a-d) and the first three non-trivial exponents for the KS system (e-l). Similar to figure 4, error bars define plus/minus three times the standard error and are shown only for Floquet exponents, since statistics of the FTLEs are computed over a sufficiently large collection of independent orbits to make the bars smaller than the symbols in the figure.

As the reference period increases, the average Floquet exponents converge, within the statistical relevance of our calculations (and perhaps less clearly for the first stable exponent of the KS system), to the corresponding infinite-time Lyapunov exponents, i.e. long periodic orbits have the same stability properties of long ergodic trajectories. Similar to period averages, we observe a distortion over short periodic trajectories. The standard deviation across samples in the collection of periodic and chaotic trajectories decays as $\hat{T}^{-1/2}$. For short periods, the standard deviation of FTLEs of the Lorenz equations decreases more rapidly, as $T^{-1}$, eventually transitioning to the Gaussian law. This is induced by exponential tails characterizing the distribution of short-time FTLEs, often observed for intermittent systems.

Probability distributions of the exponents are reported in figure 8, where we present data for the longest observed for intermittent systems. Continuation in $\rho$ of the extremal orbit found for $T = 10$ (with symbol sequence A14B in the notation of Ref. [72]), reported in panel (b), shows that points on trajectories near this region of state space move upwards by a lower amount than points at larger values of $x_3$, e.g. near the two unstable equilibria at $x_3 = \rho - 1$, thus causing a reduction of $\mathcal{J}_\rho$. For longer orbits, the fraction of the period spent in the neighbourhood of the origin diminishes in relative terms and approaches that of long chaotic trajectories. As a result, the left tail of the distributions in figure 7(a) shows a progressively faster decay and ultimately converges in law to a delta function, a Gaussian distribution (denoted with a dashed line). For large reference periods, the mean of the distributions tends to a value around $\mathcal{J}_\rho \hat{T} \simeq 1.017$.

Probability distributions of the gradient $\mathcal{J}_\nu^{\hat{T}}$ for periodic orbits of the KS system are reported in panel (a) of figure 10. Near the peak, the distributions can be approximated reasonably well by a Gaussian law. However, much higher/lower sensitivities are observed for a few orbits, resulting in a significant departure from normality and heavy tails. To characterize these tails more precisely, we find fifteen thousand more periodic orbits for $T = 500$ and show in panel (b) the probability distribution of the deviation from the mean of the distribu-
directly associated to structural bifurcations. This is illustrated in figure 11 where we report the results of continuation analysis on an orbit close to bifurcation at \( \hat{T} = 500 \) with large gradient \( J_\nu \simeq 1572.27 \). The average energy density \( J(\nu) \) and its gradient \( J_\nu(\nu) \) are reported as a function of the bifurcation parameter \( \nu \) in panels (a) and (b), respectively. Near the bifurcation point, denoted by \( \nu_b \), the period averaged energy density is well described by the functional form

\[
J(\nu) \simeq C_0 + C_1 \sqrt{\nu - \nu_b} + C_2 (\nu - \nu_b), \quad \nu > \nu_b, \tag{44}
\]

where the square root term is typical in normal forms of bifurcations for periodic orbits. \( \nu_b \) is a random variable that can take arbitrarily large values as \( \nu \) is increased, as expressed by (44). Assume also that the number of orbits with period near some large reference \( \hat{T} \) is large, so that \( J_\nu \) is measured sufficiently far away from the bifurcation point, is about \(-246.65\), in line with the high probability region of the distributions in figure 10(a). Panel (c) shows the evolution of the leading Floquet multipliers and is discussed later on.

This functional form is sufficient to explain the structure of the tails in figure 10. To achieve this, assume that periodic orbits appear in bifurcations at critical values \( \nu_b \) as \( \nu \) is increased, as expressed by (44). Assume also that the number of orbits with period near some large reference \( \hat{T} \) is large, so that \( J(\nu) \) can be thought of as a random variable, with the coefficients \( C_0, C_1, C_2 \) and the bifurcation point \( \nu_b \) being random variables with values differing from orbit to orbit. The gradient \( J_\nu(\nu) \) is then also a random variable that can take arbitrarily large values if \( \nu - \nu_b \) is small in magnitude. Now, the probability that the gradient \( J_\nu(\nu) \) is less than some large positive constant \( x \) can be expressed by introducing the cumulative distribution function \( P_{J_\nu}(x) \), defined as

\[
P_{J_\nu}(x) = \text{prob} \{ J_\nu(\nu) < x \} = 1 - \text{prob} \{ J_\nu(\nu) > x \} = 1 - \text{prob} \{ \nu_b - \nu_m < (C_1/2x)^2 \}, \tag{46}
\]

where we have used the definition (45) and neglected \( C_2 \), since \( x \gg 1 \), to develop the algebra in the last
step. The probability in the third line can be equivalently interpreted as the probability that bifurcation points are closer to the reference value than a distance \((C_1/2x)^2\). Assuming the points \(\nu_b\) not to be preferentially distributed on the real line near \(\nu\), this probability is then \(cx^{-2}\) for some constant \(c\). In other words, the larger \(x\), the less likely is that a periodic orbit bifurcates near \(\nu\). Hence, the cumulative distribution of \(P_J(x)\) must, asymptotically for large \(x\), obey

\[
P_J(x) = 1 - cx^{-2}.
\]

The probability distribution of the gradient \(J_{\nu}\) can then be obtained by differentiating the cumulative distribution with respect to its argument, leading to the power-law

\[
p(x) = cx^{-3}, \quad x \gg 1,
\]

the behavior observed in figure 9. More generally, sampling functions that have poles of the form \((\nu - \nu_b)^\gamma\), produces probability distributions with power-law tails of the form \(p(x) \sim x^{-n}, \quad x \gg 1\), with exponent \(n = (\gamma - 1)/\gamma\), leading to \(n = 3\) for the present case with \(\gamma = -1/2\) of equation 45.

We now discuss how statistics of sensitivities of periodic orbits vary with the reference period \(T\). For distributions with power-law tails of the form \(p(x) \sim x^{-n}\), central moments of order \(m\) are undefined for \(m \geq n - 1\) and do not converge upon increasing the number of samples. In the present case, with \(n = 3\), while the mean sensitivity across periodic orbits is defined (although convergence is weak), the standard deviation is not. Hence, we use the median and interquartile range for the KS system, as more robust quantities. Results are shown in figure 12 for the Lorenz equations, left panels, and the KS system, right panels. We observe that, as \(T \to \infty\) the mean/median converges to a value that is approximately \(J_\nu = 1.017\) and \(J_\nu = -155\), for the Lorenz and KS systems, respectively. However, for short cycles, the sensitivity of periodic orbits can be, in average terms over the inventory of available orbits, remarkably different to that of long cycles. This behavior is quite significant for the KS system, and is remarkably more pronounced than

the behavior of period averages in figure 4(b), although more similar to that of the Floquet exponents in figure 7(e-g). If we think of periodic orbits of spatially extended systems as global space-time solutions, the present observations would suggest that a “minimal domain” exists in the temporal direction for dynamics to be completely developed and statistics and sensitivities to be sufficiently well described, akin to the spatial “minimal flow unit” in numerical simulation of turbulent shear flow. The standard deviation and interquartile range of the sensitivity (denoted by iqr[\(\cdot\)] in panel (d)) follow the same asymptotic behavior of the period averages and decay.
asymptotically as \( \hat{T}^{-1/2} \). In summary, these results show that the sensitivity of period averaged quantities of periodic orbits converges as the period increases. In other words, longer orbits, spanning an increasingly larger fraction of the attractor provide converging estimates of the sensitivity of time averaged quantities.

We now compare the asymptotic value of the sensitivity of period averages with the response of long-time averages to finite-amplitude parameter perturbations using long chaotic simulations. Carefully conducted numerical approximations of the gradient using a finite-difference formula (see Ref. 24 for details) show that the response of the average of \( x_3 \) to perturbation of \( \rho \) is approximately \( \mathcal{J}_\nu \approx 1.002 \), well below the asymptotic value from long periodic orbits. Numerical evidence has been provided32 suggesting that the Lorenz equations have a linear response to perturbations of the parameter \( \rho \), despite not being hyperbolic (it is a singularly hyperbolic system in the terminology of Ref. 75) and despite orbits appearing and disappearing in homoclinic bifurcations. It has been speculated76 that some observables might vary continuously with parameters or that the bifurcating orbits have very long period and their effect of the invariant measure is negligible. How to reconcile the supposed existence of a linear response with the difference we observe between the prediction of the variational method and the actual response of the system is a question that deserves further analysis.

For the KS system, we show in figure 13 how the long-time averaged energy density varies with the viscosity. The data points are obtained by first computing averages over tens of thousands of independent segments of length \( T = 5000\nu \), and then reporting the median value, which is more robust to outliers arising from initial conditions leading to a non-chaotic state. Using a bootstrapping technique we have also computed the standard error on the median, which is typically smaller than the symbol size in the figure, and it is thus not shown. Panel (b) and (c) focus near the reference viscosity \( \nu = (2\pi/39)^2 \) in the area spanned by the two red lines in panels (a) and (b), respectively. The dashed line represents the asymptotic gradient \( d\mathcal{J}/d\nu \) from periodic orbits. The system clearly lacks a linear response, in the sense that the limit

\[
\lim_{\delta \nu \to \infty} \frac{\mathcal{J}^\infty(\nu + \delta \nu) - \mathcal{J}^\infty(\nu)}{\delta \nu}, \tag{49}
\]

is not defined, as the response of the system is not proportional to the perturbation in the parameter \( \nu \) at any scale. As the distributions of gradients in figure 14 show, the system is always infinitesimally close to bifurcation, with small parameter perturbations inducing some of the periodic orbits supporting the attractor to bifurcate. In such conditions, the meaning of gradients obtained from a linearized method, even for very long orbits, is unclear. It has been conjectured22 that for high-dimensional systems, which might exhibit a linear response even if dynamics are not hyperbolic, sensitivities from shadowing methods15 might reflect more closely the response of statistical properties to parameter perturbations. However, evidence to support this hypothesis is currently lacking and perhaps difficult to obtain with periodic orbits due to inherent challenges in the search of periodic solutions in high-dimensional systems.

E. Sensitivity of Floquet exponents

The last question that we focus on concerns the possibility that long periodic orbits could be somehow more prone to bifurcate than short cycles when parameters are varied. This would then suggest that tracking long orbits when parameters are optimized over might become a challenging task. The correct approach to quantify “robustness to parameter perturbations” would be to calculate for each orbit the least parameter perturbation required to send a second Floquet exponent on the imaginary axis of the complex plane, for instance, using a continuation technique. An answer to the question above would then be obtained by examining how the probability distribution of the least perturbation varies with the reference period.

Computing this minimal perturbation for all orbits in our database would be a computationally intensive task. Hence, we use a proxy quantity, the sensitivity of the leading Floquet exponent with respect to the bifurcation parameter, the gradient \( \partial \text{Re}(\lambda^T) / \partial \alpha \). We interpret this quantity as a measure of robustness to parameter perturbations based on the ‘speed’ at which exponents move, at least locally and on average, along the real axis. An increase of the average speed with the period would signal a higher proneness to bifurcation.

While variational techniques to calculate the sensitivity of the Floquet exponent exists8 here we calculate this quantity by using a forward finite-difference approxima-

![FIG. 13. The time averaged energy density as a function of \( \nu \). Data points denote the median time average across thousand of simulations from different initial conditions, with averaging time \( T = 5000\nu \). The dashed line represents the slope predicted by periodic orbits with reference period \( \hat{T} = 5000\nu \). Panels (b) and (c) focus on the area between the two red vertical lines in panels (a) and (b), respectively.](image-url)
Expanding the logarithm in a Taylor series at exponent $\lambda$ generally increases with the period. This is the behavior for the KS system, in panels (b) and (c). Panel (c), is a clearer view of the the tail of the distribution for $\hat{T}=500$, using the shift $\theta=1000$ to highlight the asymptotic behavior.

For an infinitesimal perturbation of the parameters around the bifurcation point, the denominator is dominated by the square-root term implying that, from the analysis of section VII, the distribution of $\partial \lambda/\partial \nu$, should have a power-law tail with exponent $n=3$. However, this trend only holds asymptotically. For orbits with large period, the linear term at the denominator has a larger weight, leading to power-laws with exponent $n=2$. This transition is observed in figure 14-(c). The qualitative structure of this argument does not change when an explicit dependence of the period $T$ on the viscosity, the sensitivity of the (real) Floquet exponent is obtained by differentiating (51), leading to

$$
\frac{\partial \lambda}{\partial \nu} \simeq \frac{c_\lambda}{2T e_\lambda (\nu - \nu_b) + 2\sqrt{\nu - \nu_b}}.
$$

For the KS equations, panel (b), the distributions do not collapse to a delta function. Instead, the mode increases with the reference period and the distributions eventually converge to a distribution characterized by a power-law tail. For all orbits, the gradient $\partial \text{Re}\{\lambda^T\}/\partial \nu$ is positive. In panel (c), we focus on the tail of the distribution at $\hat{T}=500$ at which more periodic orbits have been found. The tail is initially characterized by an exponent $n=2$ and transitions asymptotically to $n=3$.

Insight into the asymptotic form of the tails can be obtained from an analysis of normal forms of bifurcations of periodic orbit. Considering saddle-node or pitchfork bifurcations, the theory suggests that the positive (for instance) unstable Floquet multiplier changing its stability near the bifurcation point should have the functional form

$$
\mu(\nu) = 1 + c_\mu \sqrt{\nu - \nu_b}, \quad \nu > \nu_b,
$$

for some coefficient $c_\mu$ that varies from orbit to orbit and generally increases with the period. This is the behavior observed in figure 14-(c). The associated real Floquet exponent $\lambda(\nu)$ then reads as

$$
\lambda(\nu) = \log(1 + c_\mu \sqrt{\nu - \nu_b})/T.
$$

Expanding the logarithm in a Taylor series at $\nu_b$ shows that, for infinitesimal $\nu - \nu_b$, the exponent should satisfy

$$
\lambda(\nu) \simeq c_\lambda \sqrt{\nu - \nu_b},
$$

where we have introduced the coefficient $c_\lambda = c_\mu /T$, which should asymptotically not depend on average on the period. Neglecting the functional dependence of $T$ on the viscosity, the sensitivity of the (real) Floquet exponent is obtained by differentiating (51), leading to

VII. CONCLUSIONS

In this paper, we have considered the sensitivity of period averaged quantities of unstable periodic orbits of two dissipative chaotic systems, the Lorenz equations at standard parameters and a minimal-domain Kuramoto-Sivashinky system with dynamics restricted to the antisymmetric subspace. As opposed to our previous work where we have focused on short cycles, the question that we set out to address in this paper is whether periodic
orbits of sufficiently long period – long cycles spanning a large fraction of the attractor – may be used as statistically accurate proxies for both time averaged quantities and their sensitivity to parameter perturbations.

To answer this question, we have proceeded numerically and have used a global, damped Newton-Raphson search method\(\text{[4,42,77]}\) to build a large, but not exhaustive, inventory of thousands of periodic orbits of increasing period, with the longer orbits being at least two orders of magnitude longer than the shorter admissible cycle. Our motivation to consider such a large database was not to develop an exhaustive hierarchy of cycles, but rather to examine the statistical distribution of averages and sensitivities and develop an empirical understanding of how these quantities vary with the period.

To guide the analysis of the computational results, we have first elucidated the relation between the stability characteristics and the sensitivity of period averages. By solving the sensitivity problem in the space spanned by the Floquet eigenfunctions, we have shown that the response of a periodic orbit in state-space is largest along Floquet eigenfunctions corresponding to Floquet exponents of small magnitude, and relatively less pronounced along strongly unstable/stable directions. This also clarifies the role of structural bifurcations: at such points, the response is infinite and the gradient of the period average is undefined. In addition, as advocated in Ref. \(\text{[41]}\), the analysis of adjoint Floquet eigenfunctions of unstable orbits might reveal particularly effective means to modify dynamical behavior via control.

Second, we have examined the statistical distribution of period averaged quantities over collections of periodic orbits with increasing reference period \(T\). The empirical observation is that these distributions tend asymptotically to delta functions as \(T\) is increased, centered at the same limiting value obtained from long chaotic trajectories. For the two systems considered, displaying “typical chaos”, the limiting distribution appears to be normal. Floquet exponents, being the period averages of the local rate of growth of infinitesimal perturbations in tangent space, also exhibit the same behavior, so that the Floquet exponents of long orbits converge to the Lyapunov exponents calculated using standard method\(\text{[68]}\). The key to these results is a numerical technique that we have developed to find exponents of arbitrarily long orbits, inspired by classical methods to compute the spectrum of Lyapunov exponents\(\text{[74]}\).

Third, we have examined the sensitivity of period averaged quantities to parameter perturbations. One new result is that, for a given \(T\), the distribution of the sensitivity can display power-law tails of the form \(p(x) \approx x^{-3}\), where the exponent can be predicted using a statistical argument and classical normal forms of bifurcations of periodic orbits. We have also shown that the sensitivity of period averages converges as \(T\) is increased. However, as also observed in previous work using periodic orbits\(\text{[41]}\), as well as with other shadowing algorithms\(\text{[42,77]}\), the asymptotic value is not necessarily consistent with the response of the system to finite-amplitude parameter perturbations (if such a thing can be defined at all).

We conclude this paper by discussing open challenges ahead. First, ergodic averages are differentiable quantities only if dynamics are hyperbolic\(\text{[29]}\) and a linear response exists\(\text{[21]}\). In absence of these conditions, the meaning and value of sensitivities from periodic orbits (or other shadowing algorithms) is unclear and deserves further analysis. The conjecture is that for high-dimensional systems, where statistics behave as if dynamics where hyperbolic\(\text{[12,29]}\), the “thermodynamic limit” of Ruelle\(\text{[12]}\) may be invoked and a better consistency between sensitivities from shadowing ideas and the response of the system might be observed\(\text{[41,77]}\). Evidence in support of this conjecture is currently lacking. Second, the applicability of the ideas discussed in this paper to systems of high-dimensions, e.g. spatially extended systems characterized by small correlation length/domain size, remains unclear (see Ref. \(\text{[41]}\) for an case with the Navier-Stokes equations). One major challenge is that the increase of system dimension inevitably implies a decrease of near-recurrence events, which are key to generate good initial guesses. As advocated in Ref. \(\text{[4]}\) more robust search methods are required. Third, the size of the linear systems arising in the Newton-Raphson search iteration grows linearly with the period \(T\); regardless of the numerical method utilized, i.e. either for global search methods\(\text{[41,114]}\) or with multiple-shooting techniques\(\text{[62,80]}\). The condition number of these problems grows with \(T\), introducing errors in the Newton-Raphson corrections that might eventually prevent convergence. Hence, finding long periodic orbits might, eventually, prove too challenging. While we have made no attempt in this work to examine these trends, we feel this will be an important aspect to address in future work.

Appendix A: Floquet exponents of long periodic orbits

In this appendix we discuss the algorithm that we have used to compute the spectrum of Floquet exponents and which was the key of some of the numerical results presented in this paper. The approach is based on ideas introduced in Ref. \(\text{[71]}\), and is a specialization to periodic orbits of classical methods to compute Lyapunov exponents\(\text{[83]}\).

The algorithm exploits two fundamental facts. The first is that the Jacobian matrices \(\text{[3]}\) obey the multiplicative property

\[
J(t_2,t_0) = J(t_2,t_1)J(t_1,t_0), \tag{A1}
\]

for any times \(t_2 \geq t_1 \geq t_0\). Hence, the monodromy matrix \(J(T,0)\) can be equivalently expressed as the product of \(M\) short-time Jacobian matrices \(J_i = J(t_{i-1},t_i), i = 1, \ldots, M\), as

\[
J(T,0) = J_MJ_{M-1}\ldots J_1, \tag{A2}
\]

for a partitioning of the interval \([0,T]\) into \(M\) sub-intervals specified by times \(0 \equiv t_0 > t_1 > \ldots > t_{M-1} > t_M \equiv T\). Note that the jacobian matrices \(J_i\) obey the cyclic property \(J_{M+1} = J_1\).

The second fact is that a well-conditioned eigenvalue revealing decomposition exists for products of matrices such as \(\text{[A2]}\). This is the periodic real Schur decomposition\(\text{[31,52]}\) initially introduced in the context of Floquet
analysis in Ref. [70] for the computation of the multipliers and more recently extended [83, 84] to compute the eigenfunctions. This decomposition consists in factorizing the short-time Jacobian matrices using a set of orthogonal matrices $Q_{i}$, $i = 1, \ldots, M$, satisfying the cyclic property $Q_{0} = Q_{M}$, as

$$J_{i} = Q_{i} R_{i} Q_{i-1}^{T} \quad (A3)$$

where the factors $R_{i}, i = 1, \ldots, M - 1$ are upper triangular matrices and $R_{M}$ is in real Schur form, a block upper-triangular matrix with either $1 \times 1$ and $2 \times 2$ blocks on the diagonal, in case the monodromy matrix possesses pairs of complex conjugate multipliers.

Using these two facts, the monodromy matrix can be expressed in real Schur form as

$$J(T, 0) = Q_{0} R_{M} \cdots R_{2} R_{1} Q_{0}^{T}. \quad (A4)$$

The product $R_{M} \cdots R_{2} R_{1}$ and the monodromy matrix are unitarily similar and thus share the same spectrum of eigenvalues. However, because of the structure of the factors $R_{i}$, obtaining the spectrum is a straightforward computation, since the spectrum of a block triangular matrix is the union of the spectra of the blocks. The structure of the block upper triangular factor $R_{M}$ determines whether exponents are real or form complex conjugate pairs (see Ref. [83], Th. 7.4.1). For a $1 \times 1$ block at location $(i, i)$, a real Floquet exponent can be obtained as

$$\lambda_{i} = \log(\mu_{i})/T = \frac{1}{T} \log \prod_{j=1}^{M} |R_{j}|_{ii} = \frac{1}{T} \sum_{j=1}^{M} \log |R_{j}|_{ii} \quad (A5)$$

Computing the sum of the logarithms is recommended, as multiplication can quickly over/underflow before the logarithm is taken. For a $2 \times 2$ block, a pair of complex conjugate exponents can be obtained with a bit more work by recursively multiplying all $2 \times 2$ blocks of the factors $R_{i}$ at location $(i, i + 1)$, and accumulating the sum of the logarithms of scaling factors required to set the largest element in the partial products to have unitary magnitude. Overall, this algorithm only operates on well-conditioned short-time Jacobian matrices, instead of forming the monodromy matrix, and it is numerically robust.

The numerical algorithm required to obtain the factors $R_{i}$ and $Q_{i}$ in equation (A3) from the short-time Jacobian matrices is based on classical QR-based eigenvalue algorithms [83, 84]. Developing a robust implementation is a lengthy and delicate task. In this paper, we have adopted a different approach, introduced in Ref. [71], and is a specialization to periodic orbits of classical methods to compute Lyapunov exponents [83]. The approach works in a matrix-free fashion by only requiring the action of these matrices on a set of tangent vectors. This is computationally more efficient when only a handful of the leading Floquet exponents is required, and is necessary for high-dimensional systems described by partial differential equations. The algorithm is also simpler to implement and only requires minor modifications to an existing time-stepper code for the linearized equations.

The algorithm is iterative and we denote quantities at iteration $k$, with a superscript $(k)$. First, a set of $m$ linearly independent tangent vectors is defined, where $m$ is the number of desired exponents. For notational convenience, we arrange them as the columns of matrix $Q_{0}^{k} \in \mathbb{R}^{n \times m}$, where we use a hat to denote a matrix with reduced dimensions. The period is divided into $M$ subintervals, which need not have the same size. In each subinterval, the columns of $Q_{0}^{k}$ are: 1) propagated forward in time using a linearized time-stepping solver and 2) reorthogonalized in place using a Gram-Schmidt procedure. These two steps are formally equivalent to computing

$$J_{i} Q_{i-1}^{k} = \hat{Q}_{i}^{k} \hat{R}_{i}^{k} \quad i = 1, \ldots, M, \quad (A6)$$

which is akin to (A3). Note that forming $J_{i}$ is not necessary, only its action on the columns of $Q_{i-1}^{k}$ is required, making the approach suitable for PDE problems. The triangular factors $\hat{R}_{i}^{k} \in \mathbb{R}^{m \times m}$ from the orthogonalization are stored for later processing. The time between subsequent re-orthogonalizations depends on the expanding/contracting characteristics of the tangent space and should be chosen such that the columns of $\hat{Q}_{i}^{k}$ remain numerically linearly independent.

After the last sub-interval, the iterations are restarted by setting $Q_{0}^{k+1} = \hat{Q}_{M}^{k}$, and after $k$ iterations the monodromy matrix is formally equivalent to

$$J(T, 0) = Q_{0}^{k+1} \hat{R}_{M}^{k} \cdots \hat{R}_{2}^{k} \hat{R}_{1}^{k} Q_{0}^{k+1T}. \quad (A7)$$

If the first $m$ Floquet multipliers are all real and distinct, the columns of $Q_{0}^{k+1}$ converge geometrically to a basis for the subspace spanned by the leading $m$ Floquet eigenvectors. Hence, the difference $\|Q_{0}^{k+1} - Q_{0}^{k}\|$ converges to zero in any norm and (A7) is the real Schur form of the monodromy matrix, as in equation (A4). In fact, this iteration procedure is a form of subspace iteration (also known as orthogonal iteration [83]) and referred to as simultaneous iteration in Ref. [71]). The leading $m$ Floquet exponents can then be obtained from the diagonals of the factors $\hat{R}_{i}$, as discussed.

A simple adjustment of this approach can be introduced when some of the multipliers form complex conjugate pairs. The iterations still converge, in the sense that the subspace spanned by $\hat{Q}_{i}^{k}$ converges, but only the columns associated to real exponents converge individually [71]. The subspace spanned by a pair of columns of $Q_{0}^{k+1}$ corresponding by the space spanned by the Floquet eigenvector associated to complex conjugate multipliers also converges, but at every iteration the two columns are rotated by an angle in the subspace they span. Hence, we introduce a rotation matrix $D^{k}$ such that

$$Q_{0}^{k+1} = \hat{Q}_{0}^{k} D^{k}. \quad (A8)$$

For large $k$, this rotation converges to a matrix that has the structure of the product of Givens rotation matrices, each rotating one pair of columns of $Q_{0}^{k}$ to the corresponding pair of $Q_{0}^{k+1}$. With this modification, the product $D^{k} \hat{R}_{M}^{k} \cdots \hat{R}_{2}^{k} \hat{R}_{1}^{k}$ converges to a block upper triangular matrix and (A7) with (A8) is formally equivalent to (A4).
To the best of the author’s understanding, a procedure to compute this rotation matrix was not proposed Ref. [71] and the contribution of this appending is a simple strategy to obtain it. The rotation $\tilde{D}^k$ can be found as the solution of the orthogonal Procrustes problem\cite{83}
\begin{equation}
\arg\min_{\tilde{D}^k} \| \hat{Q}_0^{k+1} - \hat{Q}_0^k \tilde{D}^k \|_F = U^k V^{k\top},
\end{equation}
where $\| \cdot \|_F$ is the Frobenius norm and where the two matrices at the right hand side are obtained from the Singular Value Decomposition
\begin{equation}
\hat{Q}_0^{k\top} \hat{Q}_0^{k+1} = U^k \Sigma^k V^{k\top}.
\end{equation}

In our implementation, we compute the rotation $\tilde{D}^k$ along the iterations and use a simple heuristic to detect pairs of complex conjugate eigenvectors, or inverse hyperbolic directions, when a diagonal entry is close to $-1$. We then monitor the maximum absolute difference between estimates of the Floquet exponents and stop the iterations when such difference is lower than a user-defined tolerance. Panels (a-d) of figure 15 shows the progressive convergence of the rotation matrix $\tilde{D}_k$ for a calculation on the shortest periodic orbit of the KS system reported in figure 2. Iteration $k = 1, 3, 10$ and $33$ are shown. Except for the fifth and sixth column, the columns of $\hat{Q}_0^{k+1}$ converge to the columns of $\hat{Q}_0^k$ and all Floquet multipliers are real and positive. Panel (e) shows the convergence of the error on the estimate of a few selected Floquet exponents. In practice, we have found this method to be quite robust for the systems used in this paper, where Floquet exponents are typically well separated. We have observed that the number of iterations required for convergence decreases with the period, with exponents of the longest periodic orbits of the KS system requiring only three/four iterations to converge to machine accuracy. This can be attributed to the faster convergence of the columns of $\hat{Q}_0^k$ to the subspace spanned by the leading Floquet eigenmodes, as the integration time is proportionally longer, following the same pattern of convergence of algorithms to compute Lyapunov exponents from chaotic trajectories\cite{93}. We have therefore made no attempt at improving the convergence rate and computational cost by using shift and deflation techniques that are customarily used in state-of-the-art implementations of eigenvalue algorithms\cite{83,84}.
Sensitivity and stability of long periodic orbits

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