Nonlinear turbulent magnetic diffusion and mean-field dynamo

Igor Rogachevskii and Nathan Kleeorin
Department of Mechanical Engineering, The Ben-Gurion University of the Negev, POB 653, Beer-Sheva 84105, Israel
(Dated: February 12, 2002)

The nonlinear coefficients defining the mean electromotive force (i.e., the nonlinear turbulent magnetic diffusion, the nonlinear effective velocity, the nonlinear $\kappa$-tensor, etc.) are calculated for an anisotropic turbulence. A particular case of an anisotropic background turbulence (i.e., the turbulence with zero mean magnetic field) with one preferential direction is considered. It is shown that the toroidal and poloidal magnetic fields have different nonlinear turbulent magnetic diffusion coefficients. It is demonstrated that even for a homogeneous turbulence there is a nonlinear effective velocity which exhibits diamagnetic or paramagnetic properties depending on anisotropy of turbulence and level of magnetic fluctuations in the background turbulence. The diamagnetic velocity results in the field is pushed out from the regions with stronger mean magnetic field, while the paramagnetic velocity causes the magnetic field tends to be concentrated in the regions with stronger field. Analysis shows that an anisotropy of turbulence strongly affects the nonlinear turbulent magnetic diffusion, the nonlinear effective velocity and the nonlinear $\alpha$-effect. Two types of nonlinearities (algebraic and dynamic) are also discussed. The algebraic nonlinearity implies a nonlinear dependence of the mean electromotive force on the mean magnetic field. The dynamic nonlinearity is determined by a differential equation for the magnetic part of the $\alpha$-effect. It is shown that for the $\alpha\Omega$ axisymmetric dynamo the algebraic nonlinearity alone (which includes the nonlinear $\alpha$-effect, the nonlinear turbulent magnetic diffusion, the nonlinear effective velocity, etc.) cannot saturate the dynamo generated mean magnetic field while the combined effect of the algebraic and dynamic nonlinearities limits the mean magnetic field growth.

PACS numbers: 47.65.+a; 47.27.-i

I. INTRODUCTION

Generation of magnetic fields by turbulent flow of conducting fluid is a fundamental problem which has a large number of applications in solar physics and astrophysics, geophysics and planetary physics, etc. In recent time the problem of nonlinear mean-field magnetic dynamo is a subject of active discussions (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]). It was suggested in [11] that the quenching of the nonlinear $\alpha$-effect is very strong and causes a very weak saturated mean magnetic field. However, the later suggestion is in disagreement with observations of galactic and solar magnetic fields (see, e.g., [12, 13, 14, 15, 16, 17]) and with numerical simulations (see, e.g., [18, 19, 20]).

Saturation of the dynamo generated mean magnetic field is caused by the nonlinear effects, i.e., by the back reaction of the mean magnetic field on the $\alpha$-effect, turbulent magnetic diffusion, differential rotation, etc. The evolution of the mean magnetic field $B$ is determined by the following equation

$$\partial B/\partial t = \nabla \times (V \times B + \mathcal{E} - \eta \nabla \times B),$$

where $V$ is a mean velocity (e.g., the differential rotation), $\eta$ is the magnetic diffusion due to the electrical conductivity of fluid. The mean electromotive force $\mathcal{E} = (u \times b)$ in an anisotropic turbulence is given by

$$\mathcal{E}_i = \alpha_{ij} B_j + (\mathbf{V}^{\text{eff}} \times \mathbf{B})_i - \eta_{ij} (\nabla \times \mathbf{B})_j - \kappa_{ijk} (\partial B)_j - [\delta \times (\nabla \times \mathbf{B})]_i,$$

where $(\partial B)_{ij} = (1/2)(\nabla_i B_j + \nabla_j B_i)$, $u$ and $b$ are fluctuations of the velocity and magnetic field, respectively, angular brackets denote averaging over an ensemble of turbulent fluctuations, the tensors $\alpha_{ij}$ and $\eta_{ij}$ describe the $\alpha$-effect and turbulent magnetic diffusion, respectively, $\mathbf{V}^{\text{eff}}$ is the effective diamagnetic (or paramagnetic) velocity, $\kappa_{ijk}$ and $\delta$ describe a nontrivial behavior of the mean magnetic field in an anisotropic turbulence. Nonlinearities in the mean-field dynamo imply dependencies of the coefficients ($\alpha_{ij}, \eta_{ij}, \mathbf{V}^{\text{eff}}$, etc.) defining the mean electromotive force on the mean magnetic field. The $\alpha$-effect and the differential rotation are the sources of the generation of the mean magnetic field, while the turbulent magnetic diffusion and the $\kappa$-effect (which is determined by the tensor $\kappa_{ijk}$) contribute to the dissipation of the mean magnetic field.
In the present paper we derived equations for the nonlinear turbulent magnetic diffusion, the nonlinear effective velocity, the nonlinear \( \kappa \)-effect, etc. for an anisotropic turbulence. The obtained results for the nonlinear mean electromotive force are specified for an anisotropic background turbulence with one preferential direction. The background turbulence is the turbulence with zero mean magnetic field. We demonstrated that toroidal and poloidal magnetic fields have different nonlinear magnetic diffusion coefficients. It is shown that even for a homogeneous turbulence there is a nonlinear effective velocity which can be a diamagnetic or paramagnetic velocity depending on anisotropy of turbulence and level of magnetic fluctuations in the background turbulence.

II. THE GOVERNING EQUATIONS

In order to derive equations for the nonlinear turbulent magnetic diffusion and other nonlinear coefficients defining the mean electromotive force we will use a mean field approach in which the magnetic, \( \mathbf{H} \), and velocity, \( \mathbf{v} \), fields are divided into the mean and fluctuating parts: \( \mathbf{H} = \mathbf{B} + \mathbf{b} \), \( \mathbf{v} = \mathbf{V} + \mathbf{u} \), where the fluctuating parts have zero mean values, \( \mathbf{V} = \langle \mathbf{v} \rangle = \text{const} \), and \( \mathbf{B} = \langle \mathbf{H} \rangle \). The momentum equation and the induction equation for the turbulent fields \( \mathbf{u} \) and \( \mathbf{b} \) in a frame moving with a local velocity of the large-scale flows \( \mathbf{V} \) are given by

\[
\frac{\partial \mathbf{u}}{\partial t} = -\frac{\nabla \mathbf{b}'}{\rho} - \frac{1}{\mu_0 \rho} \left[ \mathbf{b} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{b}) \right] + \mathbf{T} + \nu \Delta \mathbf{u} + \frac{\mathbf{F}}{\rho},
\]

\[
\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \eta \nabla \times \nabla \times \mathbf{b} + \mathbf{G},
\]

and \( \nabla \cdot \mathbf{u} = 0 \), where \( \mathbf{b}' \) are the fluctuations of the hydrodynamic pressure, \( \mathbf{F} \) is a random external stirring force, \( \nu \) is the kinematic viscosity, \( \eta \) is the magnetic diffusion due to the electrical conductivity of fluid, \( \rho \) is the density of fluid, \( \mu_0 \) is the magnetic permeability of the fluid, the nonlinear terms \( \mathbf{T} \) and \( \mathbf{G} \) are given by \( \mathbf{T} = (\mathbf{u} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{u} + [\mathbf{b} \times (\nabla \times \mathbf{b}) - \mathbf{b} \times (\nabla \times \mathbf{b})] / (\mu_0 \rho) \), and \( \mathbf{G} = \nabla \times (\mathbf{u} \times \mathbf{b}) - (\mathbf{u} \times \mathbf{b}) \). We consider the case of large hydrodynamic (\( \text{Re} = l_0 u_0 / \nu \gg 1 \)) and magnetic (\( \text{Rm} = l_0 u_0 / \eta \gg 1 \)) Reynolds numbers, where \( u_0 \) is the characteristic velocity in the maximum scale \( l_0 \) of turbulent motions.

A. The procedure of the derivation of equation for the nonlinear mean electromotive force

The procedure of the derivation of equation for the nonlinear mean electromotive force is as follows (for details, see Appendix A).

(a) By means of Eqs. (3) and (4) we derive equations for the second moments:

\[
f_{ij}(\mathbf{k}, \mathbf{R}) = \int \langle u_i(k + \mathbf{K}/2)u_j(-k + \mathbf{K}/2) \rangle \times \exp(i \mathbf{K} \cdot \mathbf{r}) \frac{d \mathbf{K}}{4 \pi^2} = f_{ij}(-k, \mathbf{R}),
\]

\[
h_{ij}(\mathbf{k}, \mathbf{R}) = \int \langle b_i(k + \mathbf{K}/2)b_j(-k + \mathbf{K}/2) \rangle \times \exp(i \mathbf{K} \cdot \mathbf{r}) \frac{d \mathbf{K}}{4 \pi^2} = h_{ij}(-k, \mathbf{R}),
\]

\[
g_{ij}(\mathbf{k}, \mathbf{R}) = \int \langle b_i(k + \mathbf{K}/2)u_j(-k + \mathbf{K}/2) \rangle \times \exp(i \mathbf{K} \cdot \mathbf{r}) \frac{d \mathbf{K}}{4 \pi^2}.
\]

where \( \mathbf{R} \) and \( \mathbf{K} \) correspond to the large scales, and \( \mathbf{r} \) and \( k \) to the small ones, i.e., \( \mathbf{R} = (x + y)/2 \), \( \mathbf{r} = x - y \), \( \mathbf{K} = k_1 + k_2 \), \( k = (k_1 - k_2)/2 \).

(b) We split all correlation functions (i.e., \( f_{ij}, h_{ij}, g_{ij} \)) into two parts, e.g., \( h_{ij} = h_{ij}^{(N)} + h_{ij}^{(S)} \), where the tensor \( h_{ij}^{(N)} = [h_{ij}(\mathbf{k}, \mathbf{R}) + h_{ij}(-\mathbf{k}, \mathbf{R})]/2 \) describes the nonhelical part of the tensor and \( h_{ij}^{(S)} = [h_{ij}(\mathbf{k}, \mathbf{R}) - h_{ij}(-\mathbf{k}, \mathbf{R})]/2 \) determines the helical part of the tensor. Such splitting is caused, e.g., by different times of evolution of the helical and nonhelical parts of the magnetic tensor. In particular, the characteristic time of evolution of the tensor \( h_{ij}^{(N)} \) is of the order \( \tau_0 = l_0 / u_0 \), while the relaxation time of the tensor \( h_{ij}^{(S)} \) is of the order of \( \tau_0 \text{Rm} \).

(c) Equations for the second moments contain higher moments and a problem of closing the equations for the higher moments arises. Various approximate methods have been proposed for the solution of problems of this type (see, e.g., [29-31]). The simplest procedure is the \( \tau \)-approximation, which is widely used in the theory of kinetic equations. For magnetohydrodynamic turbulence this approximation was used in [25] (see also [11, 23]). In the simplest variant, it allows us to express the third moments in terms of the second moments:

\[
M_{ij} - M_{ij}^{(0)} = -(f_{ij} - f_{ij}^{(0)}) / \tau(k),
\]

\[
R_{ij}^{(N)} = -(h_{ij}^{(N)} - h_{ij}^{(0N)}) / \tau(k),
\]

\[
C_{ij} = -g_{ij} / \tau(k),
\]

where \( M_{ij}, R_{ij}, C_{ij} \) are the third moments in equations for \( f_{ij}, h_{ij} \) and \( g_{ij} \), respectively (see Eqs. (A3)-(A5) in Appendix A). The superscript \( (0) \) corresponds to the background magnetohydrodynamic turbulence (it is a turbulence with zero mean magnetic field, \( \mathbf{B} = 0 \)), \( h_{ij}^{(0N)} \) is the nonhelical part of the tensor of magnetic fluctuations of the background turbulence, and \( \tau(k) \) is the characteristic relaxation time of the statistical moments. We applied the \( \tau \)-approximation only for the nonhelical part \( h_{ij}^{(N)} \) of the tensor of magnetic fluctuations because the corresponding helical part \( h_{ij}^{(S)} \) is determined by an evolutionary equation (see, e.g., [32]).
and Section III-C). We took into account here magnetic fluctuations which can be generated by a stretch-twist-fold mechanism when a mean magnetic field is zero (see, e.g., [54–56]). This implies that $h_{ij}^{(0)} \neq 0$. In the range of background turbulence $R_{ij}(B = 0) = 0$ and $C_{ij}(B = 0) = 0$. We also took into account that the cross-helicity tensor $g_{ij}$ for $B = 0$ is zero, i.e., $g_{ij}(B = 0) = 0$.

The $\tau$-approximation is in general similar to Eddy Damped Quasi Normal Markovian (EDQNM) approximation. However some principle difference exists between these two approaches (see [24, 25]). The EDQNM closures do not relax to equilibrium, and this procedure does not describe properly the motions in the equilibrium state in contrast to the $\tau$-approximation. Within the EDQNM theory, there is no dynamically determined relaxation time, and no slightly perturbed steady state can be approached [24]. In the $\tau$-approximation, the relaxation time for small departures from equilibrium is determined by the random motions in the equilibrium state, but not by the departure from equilibrium [26]. We use the $\tau$-approximation, but not the EDQNM approximation because we consider a case with $\tau_b > \tau_h$ for all turbulence scales.

(c). Following to [21] we use an identity $B_{j,k} = (\partial \bar{B})_{jk} - \varepsilon_{ijk}(\nabla \times B)_j/2$ which allows us to rewrite Eq. (11) for the electromagnetic force in the form

$$E_i = \alpha_{ij}B_j + (U \times B)_i - \eta_{ij}(\nabla \times B)_j - \kappa_{ijk}(\partial \bar{B})_{jk}$$

where

$$\alpha_{ij}(B) = (a_{ij} + a_{ji})/2, \quad U_k(B) = \varepsilon_{kji}a_{ij}/2,$$

$$\eta_{ij} = (\varepsilon_{ikp}b_{kjp} + \varepsilon_{jkp}b_{ikp})/4,$$

$$\kappa_{ijk}(B) = -(b_{ijk} + b_{ikj})/2.$$ (14)

B. The model for the background turbulence

For the integration in $k$-space in Eqs. (12) and (13) we have to specify a model for the background turbulence (i.e., turbulence with zero mean magnetic field). We assume that the background turbulence is anisotropic and incompressible. The second moments for turbulent velocity and magnetic fields of the background turbulence are given by

$$\tau c_{ij}(k) = (5/4)\{P_{ij}(k)(2/5)\eta_T^{(a)}(k) - \mu_{mn}^{(a)}(k)k_{mn} + 2[\varepsilon_{ijk}^{(a)}(k)k_{mn} + \mu_{ij}^{(a)}(k) - \mu_{im}^{(a)}(k)k_{mj}]\} \kappa_{im}^{(a)}(m^3k^3)}.$$ (17)

(see [27]), where $c_{ij} = f_{ij}^{(0N)}$ when $a = v$, and $c_{ij} = h_{ij}^{(0N)}$ when $a = \eta$, and $\eta_{ij}^{(a)}(k) = \tau f_{pp}^{(0N)}(k), \eta_{ij}^{(h)}(k) = \tau h_{pp}^{(0N)}(k), P_{ij}(k) = \delta_{ij} - k_{ij}, \delta_{mn}$ is the Kronecker tensor. The anisotropic part of this tensor $\mu_{mn}^{(a)}(k)$ has the properties: $\mu_{mn}^{(a)}(k) = \mu_{mn}^{(a)}(k)$ and $\mu_{pp}^{(a)}(k) = 0$. Inhomogeneity of the background turbulence is assumed to be weak, i.e., in Eq. (17) we dropped terms $\sim O[\nabla \varphi^{(pp)} : \mu_{ij}^{(a)}]$, where $\eta_T^{(v)} = \tau_0\nu_0^2/3, \eta_T^{(h)} = \tau_0\nu_0^2/3\mu_0\rho$ and $\nu_0$ is the characteristic value of the magnetic fluctuations in the background turbulence. To integrate over $k$ in Eqs. (12) and (13) we use the Kolmogorov spectrum of the background turbulence, i.e., $\tau f_{pp}^{(0N)}(k) = \theta_{pp}^{(h)}(k)\varphi(k)$, $\tau h_{pp}^{(0N)}(k) = \theta_{pp}^{(h)}(k)\varphi(k)$ and $\mu_{mn}^{(a)}(k) = \mu_{mn}^{(a)}(R)\varphi(k)/3$, where $\varphi(k) = (\pi k^2k_0^2)^{-1}(k/k_0)^{-3/4}, \tau_0 = 2\tau_0(k_0/k_0)^{-3/4}$. We take into account that the inertial range of the turbulence exists in the scales: $l_d < l < l_0$. Here the maximum scale of the turbulence $l_0 \ll L_B$, and $l_0 = l_0/Re^{1/4}$ is the viscous scale of turbulence, and $L_B$ is the characteristic scale of variations of the nonuniform mean magnetic field.

In the next section we present results for the nonlinear coefficients defining the mean electromotive force.

III. NONLINEAR COEFFICIENTS DEFINING THE MEAN ELECTROMOTIVE FORCE

The procedure described in Section II (see also, for details Appendix A) allows us to calculate the nonlin-
ear turbulent magnetic diffusion tensor, the nonlinear $\kappa$-tensor, the nonlinear $\alpha$-tensor and the nonlinear effective drift velocity.

### A. Nonlinear turbulent magnetic diffusion tensor and nonlinear $\kappa$-tensor

The general form of the turbulent magnetic diffusion tensor $\eta_{ij}(B)$ contains all possible tensors: $\delta_{ij}$, $\mu^{(a)}_{ij}$, $\beta_{ij}$ and their symmetric combination $\tilde{\mu}^{(a)}_{ij} = \mu^{(a)}_{ij} + \beta_{nm}^{(a)} n_{nm}$ [see Eq. (A50) in Appendix A], where $\beta_{ij} = \beta_i \beta_j / \beta^2$, $\beta_i = 4B_i / (\omega_0 \sqrt{2 \rho_0})$. For an isotropic background turbulence (when $\mu^{(a)}_{ij} = 0$) the turbulent magnetic diffusion tensor $\eta_{ij}(B)$ is given by

$$\eta_{ij}(B) = \delta_{ij} \{ A_1(\sqrt{2} \beta) \eta^{(v)}_{ij} + [A_1(\beta) - A_1(\sqrt{2} \beta)] \eta^{(h)}_{ij} \} + \frac{1}{2} \beta_{ij} A_2(\beta) (\eta^{(v)}_{ij} + \eta^{(h)}_{ij}) ,$$

(18)

where the functions $A_k(\beta)$ are defined in Appendix B. For $\beta \ll 1$ Eq. (18) reads

$$\eta_{ij}(B) = \delta_{ij} \{ \eta^{(v)}_{ij} - (2\beta^2/5)(2\eta^{(v)}_{ij} - \eta^{(h)}_{ij}) \} - (2/5) \beta_{ij} \eta^{(v)}_{ij} + \eta^{(h)}_{ij} ,$$

(19)

and for $\beta \gg 1$ it is given by

$$\eta_{ij}(B) = (3\pi/10) \eta^{(v)}_{ij} + \eta^{(h)}_{ij} (\sqrt{2} - 1) - \beta_{ij} (\eta^{(v)}_{ij} + \eta^{(h)}_{ij}) .$$

(20)

The mean magnetic field causes an anisotropy of the turbulent magnetic diffusion tensor which is determined by the tensor $\beta_{ij}$. Magnetic fluctuations of the background turbulence contribute to the turbulent magnetic diffusion tensor $\eta_{ij}(B)$ in the nonlinear case. It follows from Eq. (24) that for $\beta \gg 1$ the tensor $\eta_{ij} \propto 1/\beta$.

The $\kappa$-tensor describes a nontrivial behavior of the mean magnetic field in an anisotropic turbulence. For an isotropic background turbulence the $\kappa$-tensor vanishes in spite of an anisotropy caused by the mean magnetic field. For an anisotropic background turbulence a general form of the $\kappa$-tensor is given by Eq. (A51) in Appendix A. For $\beta \ll 1$ this tensor is given by

$$\kappa_{ijk} = -(1/6) (3\tilde{L}^{(v)}_{ijk} + \tilde{L}^{(h)}_{ijk}) + (1/7) \beta^2 (5\tilde{L}^{(v)}_{ijk} + \tilde{L}^{(h)}_{ijk}) - 4\tilde{N}^{(v)}_{ijk} + 2\tilde{N}^{(h)}_{ijk} ,$$

(21)

and for $\beta \gg 1$ it reads

$$\kappa_{ijk} = -(\pi/16\beta)(\sqrt{2} - 1) [\tilde{L}^{(v)}_{ijk} + \tilde{L}^{(h)}_{ijk} + 3(\tilde{N}^{(v)}_{ijk} + \tilde{N}^{(h)}_{ijk})] ,$$

(22)

where $\tilde{L}^{(a)}_{ijk} = \varepsilon_{ijn} \mu^{(a)}_{nk} + \varepsilon_{ikn} \mu^{(a)}_{jn}$ and $\tilde{N}^{(a)}_{ijk} = \mu^{(a)}_{ij} (\varepsilon_{ijn} \beta_{nk} + \varepsilon_{ikn} \beta_{jm})$. Note that for $\beta \gg 1$ the tensor $\kappa_{ijk} \propto 1/\beta$. The $\kappa$-tensor contributes to the turbulent magnetic diffusion of the toroidal and poloidal mean magnetic fields (see Section V).

### B. The hydrodynamic part of the nonlinear $\alpha$-tensor

Using Eqs. (13) and (17) we get

$$\alpha_{ij}^{(v)}(B, R) = \int \alpha_{ij}^{(v)}(0, k, R) \frac{dk}{1 + \psi(B, k)} ,$$

(23)

where hereafter $\alpha_{ij}^{(v)}(0, k, R) \equiv \alpha_{ij}^{(v)}(B = 0, k, R)$. Analysis in [22] shows that a form of the tensor $\alpha_{ij}^{(v)}(0, k, R)$ in an anisotropic turbulence can be constructed using the tensors $k_{ij}$, $k_{ijmn}$ and $\nu_{ij}$, where $k_{ijmn} = k_{ij} k_{mn}$ and $\nu_{ij}$ is the anisotropic part of the hydrodynamic contribution of the $\alpha$-tensor. Thus we use the following model for the tensor $\alpha_{ij}^{(v)}(0, k, R)$

$$\alpha_{ij}^{(v)}(0, k, R) = \{ 2\alpha_0^{(v)}(R)k_{ij} + 5\epsilon k_{ijmn} \nu_{mn}(R) \} + (1 - \epsilon) [\nu_{ip}(R)k_{pj} + \nu_{jp}(R)k_{pi}] \varphi(k)/2 ,$$

(24)

where the parameter $\epsilon$ describes an anisotropy of the helical component of turbulence and it changes in the interval: $0 \leq \epsilon \leq 1$. Here $\alpha_{ij}^{(v)}(0, R) = \int \alpha_{ij}^{(v)}(0, k, R) \frac{dk}{1 + \psi(B, k)}$. For $\epsilon = 0$ Eq. (24) coincides with that derived in [22]. The asymptotic formulas for $\alpha_{ij}^{(v)}$ for $\beta \ll 1$ and $\beta \gg 1$ are given by Eqs. (A52) and (A56) in Appendix A.

### C. The mean electromotive force and the nonlinear magnetic $\alpha$-tensor

Using Eqs. (A40), (A44), (A46) we calculate the electromotive force $E$

$$E_i = \alpha_{ij} B_j + (V^{eff} \times B)_i - \eta_{ij} (\nabla \times B)_j - \kappa_{ijk} (\partial B)_jk ,$$

(26)

where the nonlinear effective drift velocity $V^{eff} = U + V^{(N)}$, and the velocity $U_i(B) = -(1/2) \varepsilon_{imn} q_{mn} = -(1/2) \nabla_p A^{(M)}_p (\sqrt{2} \beta)$ (see [7]), the velocity $V^{(N)}$ is given by Eq. (A44), the tensor of turbulent magnetic diffusion
\( \eta_{ij} \) is given by Eq. (A50), the tensor \( \kappa_{ijk} \) is determined by Eq. (A51), the tensor \( \Lambda^{(M)}_{ij} \) is defined in Eqs. (A22) and (A33). In the kinematic dynamo the effective drift velocity (turbulent diamagnetic velocity) is caused by an inhomogeneity of turbulence. The effective drift velocity \( \mathbf{U}(\mathbf{B}) \) is determined by the tensor \( a_{ij} \) and is due to an induced inhomogeneity of turbulence caused by the nonuniform mean magnetic field. This implies that the nonuniform mean magnetic field modifies turbulent velocity field and creates the inhomogeneity of turbulence. The effective velocity \( \mathbf{V}^{(N)}(\mathbf{B}) \) is determined by tensor \( \delta_{ijk} \) and is caused by the nonuniform mean magnetic field.

The \( \alpha \)-tensor is determined by the hydrodynamic and magnetic contributions, i.e., \( \alpha_{ij}(\mathbf{B}) = \alpha^{(v)}_{ij}(\mathbf{B}) + \alpha^{(h)}_{ij}(\mathbf{B}) \) with

\[
\alpha^{(h)}_{ij}(\mathbf{B}) = \alpha^{(h)}_0(\mathbf{B}) \Phi(\beta) \delta_{ij} \tag{27}
\]

(see [8]), where the tensor \( \alpha^{(v)}_{ij}(\mathbf{B}) \) is determined by Eq. [23], the function \( \Phi(\beta) = (3/\beta^3)[1 - \arctan(\beta)/\beta] \), and the magnetic part \( \alpha^{(h)}_0(\mathbf{B}) \) of the \( \alpha \)-effect is determined by the dynamic equation

\[
\frac{\partial \alpha^{(h)}_0(\mathbf{B})}{\partial t} + \frac{\alpha^{(h)}_0(\mathbf{B})}{T} + \nabla \cdot (\mathbf{W} \alpha^{(h)}_0(\mathbf{B}) + \mathbf{F}_{\text{flux}}) = -\frac{4}{9 \eta_T \mu_0 \rho} \mathbf{E}(\mathbf{B}) \cdot \mathbf{B} \tag{28}
\]

(see [8], [23], [24], [32], [33]), where \( W_i = c_{ij} V_j \) is the velocity which depends on the mean fluid velocity \( \mathbf{V} \) for an isotropic turbulence the tensor \( c_{ij} = \delta_{ij} \) and for an anisotropic turbulence with one preferential direction, say in the direction \( \mathbf{e} \), the tensor \( c_{ij} = (23/30) \delta_{ij} + (7/10) e_i e_j \), see [23]); the flux

\[
\mathbf{F}_{\text{flux}} \propto \tau \alpha^{(v)}(\mathbf{B}) \frac{\nabla \rho}{\rho} \left( \frac{\eta_T^{(v)}(\mathbf{B}) B^2}{\eta_T^{(v)}(\mathbf{B} = 0) \mu_0 \rho} \right) \tag{29}
\]

is related with the flux of the magnetic helicity and is independent of the mean fluid velocity \( \mathbf{V} \) (see also [8]), and \( T \sim \tau_0 \mathrm{Rm} \) is the characteristic time of relaxation of magnetic helicity. The asymptotic formulas for \( \alpha^{(h)}(\mathbf{B}) \) for \( \beta \ll 1 \) and \( \beta \gg 1 \) are given by Eqs. (A54) and (A57) in Appendix A.

\section*{IV. ANISOTROPIC BACKGROUND TURBULENCE WITH ONE PREFERENTIAL DIRECTION}

Now we consider an anisotropic background turbulence with one preferential direction, say along an unit vector \( \mathbf{e} \). Thus the tensor \( \eta^{(v)}_{ij}(\mathbf{B} = 0) = \langle \tau_{ij} \rangle v \) is given by \( \eta^{(v)}_{ij}(\mathbf{B} = 0) = \eta_T^{(v)} \delta_{ij} + \mu_{ij}^{(v)} = \eta_0^{(v)} \delta_{ij} + \varepsilon^{(v)}_{ij} e_i e_j \), where the trace \( \eta_T^{(v)}(\mathbf{B} = 0) \) in this equation yields \( \eta_0^{(v)} = \eta_T^{(v)} - (1/3) \varepsilon^{(v)}_{ij} \) and \( e_i e_j = e_i e_j \). Therefore, the anisotropic part \( \mu_{ij}^{(v)}(\mathbf{B} = 0) \) is given by \( \mu_{ij}^{(v)} = \varepsilon^{(v)}_{ij} (e_j - (1/3) \delta_{ij}) \) and \( \mu_{ij}^{(v)} = \mu_{ij}^{(v)} + \mu_{ij}^{(v)} = \mu_0^{(v)} \delta_{ij} + \varepsilon^{(v)}_{ij} e_i e_j \), where the trace \( \eta_T^{(v)}(\mathbf{B} = 0) \) in this equation yields \( \eta_0^{(v)} = \eta_T^{(v)} - (1/3) \varepsilon^{(v)}_{ij} \) and \( e_i e_j = e_i e_j \). Therefore, the

\begin{align*}
\eta^{(v)}_{ij}(\mathbf{B}) = M_\alpha \delta_{ij} + M_\beta e_i e_j + M_\gamma \beta_{ij}, \tag{30} \\
F^{\text{eff}}(\mathbf{B}) = B^{-2}[M_1^{(1)} \nabla B^2 + M_2^{(2)} e(\mathbf{e} \cdot \nabla) B^2], \tag{31} \\
\kappa_{ijk}(\mathbf{B})(\partial \hat{B})_{jk} = M_\alpha e(\mathbf{e} \cdot \nabla) B_i, \tag{32} \\
\end{align*}

(see Appendix C), where we assumed that \( \mathbf{e} \cdot \hat{\mathbf{B}} = 0, \), the functions \( M_\alpha, M_\beta, M_\gamma, M_1^{(1)} \), and \( M_2^{(2)} \) are given by Eqs. (24)–(25) in Appendix C. The tensor \( \eta^{(v)}_{ij}(\mathbf{B}) \) contains three tensors \( \delta_{ij}, e_i e_j \) and \( \beta_{ij} \) since here there two preferred directions, along the vectors \( \mathbf{e} \) and \( \mathbf{B} \).

Now we consider the hydrodynamic part of the \( \alpha \)-effect for an anisotropic background turbulence with one preferential direction. The tensor \( \alpha^{(v)}_{ij}(\mathbf{B} = 0) \) in this case can be rewritten in the form \( \alpha^{(v)}_{ij}(\mathbf{B} = 0) = \alpha_0^{(v)} \delta_{ij} + \nu_{ij} = [\alpha_0^{(v)}(1 - (1/3) \varepsilon^{(v)}_{ij} e_i e_j, \) where \( \varepsilon^{(v)}_{ij} \) is a degree of an anisotropy of the \( \alpha \)-tensor. Thus, the anisotropic part \( \nu_{ij} \) is given by \( \nu_{ij} = \varepsilon^{(v)}_{ij} - (1/3) \delta_{ij} \).

The electromagnetic force contains the tensor \( \alpha_{ij} \) in the form \( \alpha_{ij} B_j = \alpha_0^{(v)} \delta_{ij} + \nu_{ij} = \alpha_0^{(v)}[e_i e_j - (1/3) \delta_{ij}] \).

\begin{align*}
\alpha^{(v)}_{ij}(\mathbf{B}) = \delta_{ij} \left[ \sum_{i=1}^{3} \alpha_i \beta_i \right] + \alpha^{(v)}_{ij} - (1/3) \varepsilon^{(v)}_{ij} \right]\right] e_i e_j, \tag{33} \\
\end{align*}

where we assumed that \( \mathbf{e} \cdot \hat{\mathbf{B}} = 0, \) for \( \varepsilon \neq 0 \) the tensor \( \alpha^{(v)}_{ij}(\mathbf{B}) \) can change its sign at some value \( B_* \) of the mean magnetic field [see Eqs. (C17) and (C23) in Appendix C]. Thus the point \( B = B_* \) can determine a steady state configuration of the mean magnetic field for \( \varepsilon \neq 0 \).
V. APPLICATIONS: MEAN-FIELD EQUATIONS FOR THE THIN-DISK AXISYMMETRIC \( \alpha \Omega \)-DYNAMO

Here we apply the obtained results for the nonlinear mean electromotive force to the analysis of the thin-disk axisymmetric \( \alpha \Omega \)-dynamo. Using Eqs. (30)–(32) we derive the mean-field equations for the thin-disk axisymmetric \( \alpha \Omega \)-dynamo:

\[
\begin{align*}
\frac{\partial B}{\partial t} &= \frac{\partial}{\partial z} \left( \eta_B \frac{\partial B}{\partial z} \right) + \dot{G} \frac{\partial A}{\partial z}, \\
\frac{\partial A}{\partial t} &= \eta_A \frac{\partial^2 A}{\partial z^2} - V_A \frac{\partial A}{\partial z} + \alpha B,
\end{align*}
\]

where \( r, \varphi \) and \( z \) are cylindrical coordinates, \( B = Be_\varphi + \nabla \times (Ae_\varphi) \), \( \dot{\Omega} = -r(\partial \Omega/\partial r) \), and

\[
\begin{align*}
\eta_A(B) &= M_\eta + M_\kappa + M_\delta, \\
\eta_B(B) &= M_\eta + M_\kappa - 2M_V, \\
V_A &= (\eta_A - \eta_B)(\ln |B|)' , \\
\alpha(B) &= \Phi_\alpha(B)\alpha_1^{(v)} + \Phi(B)\alpha_0^{(h)}(B),
\end{align*}
\]

and \( F' = \partial F/\partial z, M_V = M_V^{(1)} + M_V^{(2)} \). In the axisymmetric problem \( \partial B/\partial \varphi = 0 \). The thin-disk approximation implies that the spatial derivatives of the mean magnetic field with respect to \( z \) are much larger than the derivatives with respect to \( r \). It is seen from Eqs. (30)–(32) and (36)–(37) that the contributions to the turbulent diffusion coefficients \( \eta_A(B) \) and \( \eta_B(B) \) are from the tensor of turbulent diffusion \( \eta_{ij}(B) \), the tensor \( \kappa_{ij}(B) \) and the nonlinear velocity \( \mathbf{U}(B) + \mathbf{V}^{(N)}(B) \). On the other hand, contributions to the effective velocity \( V_A(B) \) are from the tensor of turbulent diffusion \( \eta_{ij}(B) \) and the nonlinear velocity \( \mathbf{U}(B) + \mathbf{V}^{(N)}(B) \). The functions \( \eta_A(B), \eta_B(B) \) and \( V_A(B) \) are given by Eqs. (31)–(33) in Appendix C.

The nonlinear dependencies: (A) of the turbulent magnetic diffusion coefficients \( \eta_A(B)/\eta_T^{(v)} \) and \( \eta_B(B)/\eta_T^{(v)} \); (B) of the effective velocity \( V_A(B)/\langle |B| \rangle' \); and (C) of the nonlinear dynamo number \( D(B)/D_* \) are presented in Figures 1–3. Here \( D_* = \alpha_s G h^3/\eta_V^2, D(B) = \alpha^{(v)}(B)G h^3/[\eta_A(B)\eta_B(B)] \), \( \eta_s = \eta_T^{(v)} + (2/3)\epsilon^{(v)}_\mu \), \( \alpha_s \) is the maximum value of the hydrodynamic part of the \( \alpha \) effect, \( h \) is the disc thickness and \( \alpha^{(v)}(B) = \alpha_0^{(v)}\Phi_\alpha(B) \). For simplicity we consider the case \( \epsilon = 0 \).

In order to separate the study of the algebraic and dynamic nonlinearities we defined the nonlinear dynamo number \( D(B) \) using only the hydrodynamic part of the \( \alpha \) effect. We considered three cases: two types of an anisotropic background turbulence \( \epsilon^{(v)}_\mu = \pm 1.35\eta_T^{(v)} \); \( \epsilon^{(h)}_\mu = 0 \) without magnetic fluctuations (Fig. 1 and Fig. 3) and an isotropic \( \epsilon^{(v)}_\mu = \epsilon^{(h)}_\mu = 0 \) background turbulence with equipartition of hydrodynamic and magnetic fluctuations (Fig. 2). The negative degree of anisotropy \( \epsilon^{(v)}_\mu \) implies that the vertical (along axis \( z \)) size of turbulent elements is less than the horizontal size and positive \( \epsilon^{(v)}_\mu \) means that the horizontal size is less than the vertical size.

\[
\epsilon^{(v)}_\mu \text{ means that the horizontal size is less than the vertical size.}
\]

Figures 1-3 and the equations for \( \eta_A(B) \) and \( \eta_B(B) \) show that the toroidal and poloidal magnetic fields have different nonlinear turbulent magnetic diffusion coefficients. In isotropic background turbulence (Fig. 2) the nonlinear effective velocity \( V_A(B) \) is negative. The latter implies that it is diamagnetic velocity, which results in the field is pushed out from the regions with stronger mean magnetic field. In the anisotropic background turbulence (Fig. 1 and Fig. 3) the nonlinear effective velocity is positive, \( i.e., \) paramagnetic velocity which causes the magnetic field tends to be concentrated in the regions with stronger field). The sign of \( \epsilon^{(v)}_\mu \) affects the value of \( \eta_A(B), \eta_B(B) \) and \( V_A(B) \), \( e.g., \) for positive parameter of anisotropy the functions \( \eta_A(B), \eta_B(B) \) and \( V_A(B) \) are
larger at least in one order of magnitude than those for negative \( \varepsilon^{(v)}_\mu \).

The dependencies of the nonlinear dynamo number \( D(B)/D_* \) on the mean magnetic field \( B/B_{eq} \) demonstrate that the algebraic nonlinearity alone (i.e., quenching of both, the nonlinear \( \alpha \) effect and the nonlinear turbulent diffusion coefficients) cannot saturate the growth of the mean magnetic field (where \( B_{eq} = \sqrt{\mu_0 \rho_0 u_0} \)). Indeed, for anisotropic background turbulence without magnetic fluctuations (Fig. 1 and Fig. 3) the nonlinear dynamo number \( D(B)/D_* \) is a nonzero constant for \( \beta \gg 1 \), i.e., it is independent on \( \beta \). This is because for \( \beta \gg 1 \) the functions \( \eta_A \propto 1/\beta \), \( \eta_B \propto 1/\beta \) and \( \alpha \propto 1/\beta^2 \) [see Eqs. (18)-(21) in Appendix C]. In the case of isotropic background turbulence with equipartition of hydrodynamic and magnetic fluctuations (Fig. 2) the nonlinear dynamo number \( D(B)/D_* \propto \beta \) for \( \beta \gg 1 \) because in this case the functions \( \eta_A \propto 1/\beta^2 \), \( \eta_B \propto 1/\beta \) and \( \alpha \propto 1/\beta^2 \) [see Eqs. (18)-(21) in Appendix C]. Note that the saturation of the growth of the mean magnetic field can be achieved when the derivative of the nonlinear dynamo number \( dD(B)/dB < 0 \). Thus, the algebraic nonlinearity alone cannot saturate the growth of the mean magnetic field. We will show below that the combined effect of the algebraic and dynamic nonlinearities can limit the growth of the mean magnetic field.

Equation (28) in nondimensional form is given by

\[
\frac{\partial \alpha^{(h)}_0}{\partial t} + \frac{\alpha^{(h)}_0}{T} = 4 \left( \frac{h}{l_0} \right)^2 [\eta_B B' A' - (\eta_A A'' - V_A A' + \alpha B)] + [C |\alpha^{(v)}_0(z)| f_\eta(z) \Phi_\alpha(B) \eta_A(B) B^2]' , \tag{38}
\]

where \( C \) is a coefficient, \( f_\eta(z) \) describes the inhomogeneity of the turbulent magnetic diffusion, and we define

![FIG. 2: (A). The nonlinear turbulent magnetic diffusion coefficients; (B). The nonlinear effective velocity; (C). The nonlinear dynamo number for \( \eta^{(h)}_A = \eta^{(h)}_B ; \varepsilon^{(v)}_\mu = \varepsilon^{(h)}_\mu = 0 \).](image1)

![FIG. 3: (A). The nonlinear turbulent magnetic diffusion coefficients; (B). The nonlinear effective velocity; (C). The nonlinear dynamo number for \( \eta^{(h)}_A = \eta^{(h)}_B ; \varepsilon^{(v)}_\mu = 1.35 \eta^{(v)}_\mu ; \varepsilon^{(h)}_\mu = 0 \).](image2)
f(z) = \alpha_1^{(v)}(z)f_1(z). We use here the standard dimensionless form of the galactic dynamo equation (see, e.g., [10]), in particular, the length is measured in units of the disc thickness h, the time is measured in units of h²/ηF, and B is measured in units of the equipartition energy \( B_{eq} = \sqrt{\rho\mu u_0} \). Here \( u_0 \) is the characteristic turbulent velocity in the maximum scale \( l_0 \) of turbulent motions, \( \eta_F = l_0 u_0/3 \) and \( \alpha_0^{(v)} \) and \( \alpha_0 \) are measured in units of \( \alpha \) (the maximum value of the hydrodynamic part of the \( \alpha \) effect). For galaxies \( h/l_0 \approx 5 \) and \( C \approx 0.05 - 0.1 \).

Nondimensional equations for \( A \) and \( B \) are given by

\[
\frac{\partial B}{\partial t} = (\eta_B B')' - D_0 A', \tag{39}
\]

\[
\frac{\partial A}{\partial t} = \eta_A A'' - V_A A' + \alpha B, \tag{40}
\]

where \( D_0 = \alpha_ch^3/\eta_f^2 \) and \( B_\sigma = -A'(z) \). In a steady state Eqs. (38)-(41) yield

\[
[\eta_B(B')^2 + 2CD_0\Phi_\alpha(B)\eta_A(B)B^2]f(z) = 0, \tag{41}
\]

where we used the following boundary conditions \( B(z = \pm 1) = 0, B'(z = 0) = 0 \) and \( f(z = 0) = 0 \). The solution of Eq. (41) for negative \( D_0 \) is given by

\[
B(z) = \sqrt{2C|D_0|}\left(\int |z| \sqrt{f(\tilde{z})}d\tilde{z}\right)^2, \tag{42}
\]

where we used that for \( \beta \gg 1 \) the functions \( \eta_A(B) \sim 3/5\beta, \eta_B(B) \sim 2/\beta^2, \Phi_\alpha(B) \sim 2/\beta^2, \) and \( \chi(B) \sim 2/\sqrt{3} \). Here for simplicity we considered the case \( \varepsilon^{(v)} = 0 \). In a steady state \( A(z) = -\eta_B(B')/|D_0| \). Now we specify the profile of the function \( f(z) \). For example, \( f(z) = f_*[\sin(\pi z/2)]^{2k+1}[\cos(\pi z/2)]^2, \) where \( k = 1; 2; 3; \ldots \) and

\[
f_* = \left(\frac{2k + 3}{2}\right)^{2k+1/2} \left(\frac{2k + 3}{2k + 1}\right)^{(2k+1)/2}.
\]

The function \( f(z) \) changes in the interval \( 0 \leq f(z) \leq 1 \) and it has a maximum \( f(z = z_m) = \frac{2}{\pi} \arctan\sqrt{(2k + 1)/2} \). Equation (43) for this profile \( f(z) \) with \( k = 2 \) yields

\[
B(z) \approx 0.4C|D_0|\left\{1 - [\sin(\pi z/2)]^7/2\right\}^2. \tag{44}
\]

Equation (44) describes the equilibrium configuration of the mean toroidal magnetic field. Thus, the saturation of the growth of the mean magnetic field is caused by both, the algebraic and dynamic nonlinearities. The dynamic nonlinearity is determined by the dynamic equation (28) whereas the algebraic nonlinearity implies the nonlinear dependences of the turbulent magnetic diffusion coefficients \( \eta_A(B) \) and \( \eta_B(B) \) and of the effective velocity \( V_A(B) \) on the mean magnetic field [see Eqs. (31)-(33)].

\[\text{VI. DISCUSSION}\]

In this study we calculated the nonlinear tensor of turbulent magnetic diffusion, the nonlinear \( \kappa \)-tensor, the nonlinear effective velocity, and other coefficients defining the mean electromotive force for an anisotropic turbulence. The obtained results were specified for an anisotropic background turbulence with one preferential direction. We found that the turbulent magnetic diffusion coefficients for the toroidal and poloidal magnetic fields are different. We demonstrated that even for a homogeneous turbulence there is the nonlinear effective velocity which can be a diamagnetic or paramagnetic velocity depending on anisotropy of turbulence and level of magnetic fluctuations in the background turbulence. The diamagnetic velocity implies that the field is pushed out from the regions with stronger mean magnetic field, while the paramagnetic velocity causes the magnetic field tends to be concentrated in the regions with stronger field.

Note that dependencies of the \( \alpha \)-effect, the turbulent magnetic diffusion coefficient and the effective drift velocity on the mean magnetic field for an isotropic turbulence have been found in [36, 37, 38] using a modified second order correlation approximation. Our results are different from that obtained in [36, 37, 38]. The reason is that in [36, 37, 38] a phenomenological procedure was used. In particular, in the first step of the calculations the nonlinear terms in the magnetohydrodynamic equations were dropped (which is valid for small hydrodynamic and magnetic Reynolds numbers or in a highly conductivity limit and small Strouhal numbers). In the next step of the calculations in [36, 37, 38] it was assumed that \( \rho = \eta = l_0^2/\tau_c \), where \( l_c \) and \( \tau_c \) are the correlation length and time of turbulent velocity field. The latter is valid when the hydrodynamic and magnetic Reynolds numbers are of the order of unit. In the present paper we use a different procedure (the \( \tau \)-approximation) for large hydrodynamic and magnetic Reynolds numbers.

In this study we also demonstrated an important role of two types of nonlinearities (algebraic and dynamic) in the mean-field dynamo. The algebraic nonlinearity is determined by a nonlinear dependence of the mean electromotive force on the mean magnetic field. The dynamic nonlinearity is determined by a differential equation for the magnetic part of the \( \alpha \)-effect. This equation is a consequence of the conservation of the total magnetic helicity (which includes both, the magnetic helicity of the mean magnetic field and the magnetic helicity of small-scale magnetic fluctuations). We found that at least for the \( \alpha \Omega \) axisymmetric dynamo the algebraic nonlinearity alone [i.e., the nonlinear functions \( \alpha(B), \eta_A(B), \eta_B(B) \)] and \( V_A(B) \) cannot saturate the dynamo generated mean magnetic field. The important parameter which characterizes the algebraic nonlinearity is the nonlinear dynamo number \( D(B) \). The saturation of the growth of the dynamo generated mean magnetic field by the algebraic nonlinearity alone is possible when the derivative \( dD(B)/dB < 0 \). We found that for the \( \alpha \Omega \) axisymmetric
dynamo the nonlinear dynamo number $D(B)$ is either a constant or $D(B) \propto B$ for $B > B_{eq}/3$ depending on the model of the background turbulence. Therefore, in this case the algebraic nonlinearity alone cannot saturate the dynamo generated mean magnetic field.

The situation is changed when the dynamic nonlinearity is taken into account. The crucial point is that the dynamic equation for the magnetic part of the $\alpha$-effect (i.e., the dynamic nonlinearity) includes the flux of the magnetic helicity. Without the flux, the total magnetic helicity is conserved locally and the level of the saturated mean magnetic field is very low [5]. The flux of the magnetic helicity results in that the total magnetic helicity is not conserved locally because the magnetic helicity of small-scale magnetic fluctuations is redistributed by a helicity flux. In this case an integral of the total magnetic helicity over the disc is conserved. The equilibrium state is given by a balance between magnetic helicity production and magnetic helicity transport [3]. These two types of the nonlinearities (algebraic and dynamic) results in an agreement with observations of the galactic magnetic order that of the equipartition field

$$\langle \frac{\delta B}{B} \rangle = \frac{1}{2 \mu_0 \nu} \frac{\partial \langle B \rangle}{\partial t}$$

of the nonlinearities (algebraic and dynamic) results in that the total magnetic helicity results in that the total magnetic helicity.

**Acknowledgments**

We have benefited from stimulating discussions on nonlinear dynamo with A. Brandenburg, D. Moss, K.-H. Rädler, P. H. Roberts, A. Ruzmaikin, D. Sokoloff and E. T. Vishniac. We are also grateful to the anonymous referee for very useful and important comments which strongly improved our paper. This work was partially supported by INTAS Program Foundation (Grant No. 99-348).

**APPENDIX A: CALCULATION OF THE MEAN ELECTROMOTIVE FORCE**

Let us derive equations for the second moments. For this purpose we rewrite Eqs. (3) and (4) in a Fourier space and repeat twice the vector multiplication of Eq. (3) by the wave vector $\mathbf{k}$. The result is given by

$$\frac{d u_i(k,t)}{d t} = \frac{1}{\mu_0 \rho} \left[ (2P_{ij}(k) - \delta_{ij}) \tilde{S}_i^{(c)}(u;B) + \tilde{S}_i^{(b)}(u;B) \right] - \tilde{T}_i - \nu k^2 u_i - \tilde{F}_i,$$

$$\frac{d b_i(k,t)}{d t} = \tilde{S}_i^{(b)}(u;B) - \tilde{S}_i^{(c)}(u;B) + G_i - \eta k^2 b_i,$$

where

$$\tilde{S}_i^{(c)}(a;A) = i \int a_p(k - Q)Q_pA_i(Q)dQ,$$

$$\tilde{S}_i^{(b)}(a;A) = ik_p \int a_i(k - Q)A_p(Q)dQ,$$

$$\tilde{T}_i = \mathbf{k} \times (\mathbf{k} \times \mathbf{T})/k^2,$$

$$\tilde{F}(k,R,t) = \mathbf{k} \times (\mathbf{k} \times \mathbf{F}(k,R))/k^2,$$

$$P_{ij}(k) = \delta_{ij} - \delta_{ij}, \delta_{ij}$$ is the Kronecker tensor and $k_{ij} = k_i k_j/k^2$. We use the two-scale approach, i.e., a correlation function

$$\langle u_i(x) u_j(y) \rangle = \int \langle u_i(k_1) u_j(k_2) \rangle \exp\{i(k_1 \cdot x + k_2 \cdot y)\} dk_1 dk_2$$

$$= \int f_{ij}(k,R) \exp\{i(k \cdot r + iK \cdot R)\} dk dR,$$

$$f_{ij}(k,R) = \int \langle u_i(k + K/2) u_j(-k + K/2) \rangle \exp\{i(k \cdot R)\} dK,$$

where $R = (x + y)/2, \mathbf{r} = x - y, \mathbf{K} = k_1 + k_2, k = (k_1 - k_2)/2$, $\mathbf{R}$ and $\mathbf{r}$ correspond to the large scales, and $\mathbf{r}$ and $\mathbf{k}$ to the small ones (see, e.g., [39, 40]). The others second moments have the same form, e.g.,

$$h_{ij}(k,R) = \int \langle b_i(k + K/2) b_j(-k + K/2) \rangle \exp\{i(k \cdot R)\} dK / \mu_0 \rho,$$

$$g_{ij}(k,R) = \int \langle b_i(k + K/2) u_j(-k + K/2) \rangle \exp\{i(k \cdot R)\} dK.$$

The two-scale approach is valid when $(1/B)(dB/dR) \ll l_0^{-1}$, where $B = |\mathbf{B}|$. Now we derive the equations for the correlation functions $f_{ij}(k,R)$, and $h_{ij}(k,R)$, and $g_{ij}(k,R)$

$$\partial f_{ij}/\partial t = i\langle \mathbf{k} \cdot \mathbf{B} \rangle \Phi_{ij} + M_{ij} + F_{ij} - 2\nu k^2 f_{ij},$$

$$\partial h_{ij}/\partial t = -i\langle \mathbf{k} \cdot \mathbf{B} \rangle \Phi_{ij} + R_{ij} - 2\eta k^2 h_{ij},$$

$$\partial g_{ij}/\partial t = I_{ij} + C_{ij} - (\nu + \eta) k^2 g_{ij},$$

$$I_{ij} = i\langle \mathbf{k} \cdot \mathbf{B} \rangle (f_{ij} - h_{ij}) + (1/2)\langle \mathbf{B} \cdot \nabla \rangle (f_{ij} + h_{ij})$$

$$- f_{pi} B_{pi} + h_{pi} (2P_{ij}(k) - \delta_{ij}) B_{i,p} - B_{p,q} k_p f_{ij} + h_{ij},$$

where $\nabla \cdot \partial / \partial R, f_{ij} = (1/2) \partial f_{ij} / \partial k_{q}, h_{ij} = (1/2) \partial h_{ij} / \partial k_{q}$, and $F_{ij}(k,R) = \langle \tilde{F}(k,R) u_j(-k,R) \rangle + \langle u_i(k,R) \tilde{F}(k,-R) \rangle, B_{ij} = B_{i,j}/B_{R,j}$, and

$$\Phi_{ij}(k,R) = [g_{ij}(k,R) - g_{ij}(-k,R)]/\mu_0 \rho.$$

The third moments are given by $M_{ij}(k,R) = -\langle \tilde{F}(k) u_j(-k) \rangle - \langle u_i(k) \tilde{T}(k) \rangle, R_{ij}(k,R) = \langle \tilde{G}(k) b_j(-k) \rangle + \langle b_i(k) \tilde{G}(k) \rangle$ and $C_{ij}(k,R) = \langle \tilde{G}(k) u_j(-k) \rangle - \langle b_i(k) \tilde{T}(k) \rangle$.

For the derivation of Eqs. (A3)-(A6) we performed several calculations that are similar to the following, which arose in computing $\partial g_{ij}/\partial t$. The other calculations follow similar lines and are not given here. Let us define $Y_{ij}(k,R)$ by

$$Y_{ij}(k,R) = \int \langle \tilde{S}_i^{(b)}(u;B; k + K/2) u_j(-k + K/2) \rangle \exp\{iK \cdot R\}$$

$$\times \exp\{iK \cdot R\} dK = i \int (k_p + K_p/2) B_p(Q) \exp\{iK \cdot R\}$$

$$\times \langle u_i(k + K/2 - Q) u_j(-k + K/2) \rangle dK dQ.$$
Next, we introduce new variables: \( \tilde{k}_1 = k + K/2 - Q \), \( \tilde{k}_2 = -k + K/2 \), and \( \tilde{k} = (k_1 - k_2)/2 = -k/2 \). Therefore,

\[
Y_{ij}(k, R) = \left( \int f_{ij}(k - Q/2, K - Q)(k_p + K_p/2)B_p(Q) \times \exp(ik \cdot R)dQ \right)
\]

Since \(|Q| \ll |k|\) we use the Taylor expansion

\[
f_{ij}(k - Q/2, K - Q) \approx f_{ij}(k, K - Q) - \frac{1}{2} \frac{\partial f_{ij}(k, K - Q)}{\partial k_q}Q_q + O(Q^2)
\]

and the following identities:

\[
[f_{ij}(k, R)B_p(R)]_K = \int f_{ij}(k, K - Q)B_p(Q)dQ,
\]

\[
\nabla_q[f_{ij}(k, R)B_p(R)] = \int iK_p[f_{ij}(k, R)B_p(R)]_K \times \exp(ik \cdot R)dQ,
\]

Therefore, Eqs. (A8)-(A10) yield

\[
Y_{ij}(k, R) \approx \left[ [i(k \cdot B) + (1/2)(B \cdot \nabla)]f_{ij}(k, R) - \frac{1}{2} \frac{\partial f_{ij}(k, K - Q)}{\partial k_q}Q_q + O(Q^2) \right] \cdot \Phi_{ij}.
\]

In Eqs. (A3) and (A4) we neglected the terms \( \propto (B \cdot \nabla)g_{ij} \) and \( \propto B_{ip}g_{ij} \) because they contribute to the modification of the mean Lorentz force by the turbulence effect (see, e.g., [24, 25]). In Eq. (A5) we neglected the second and higher derivatives over \( R \). We also neglected in Eq. (A6) the terms which are of the order of \( R \nu^{-1} \nabla(B_i, f_{ij}, h_{ij}) \) and \( R \nu^{-1} \nabla(B_i, f_{ij}, h_{ij}) \). When the mean magnetic field is zero Eq. (A6) reads

\[
\frac{\partial f_{ij}^{(0)}}{\partial t} = M_{ij}^{(0)} + F_{ij}^{(0)} - 2\nu k^2 f_{ij}^{(0)},
\]

We assume that \( F_{ij} \) is not changed during the generation of the mean magnetic field, i.e., \( F_{ij} = F_{ij}^{(0)} \). This implies an assumption of a constant power of the source of turbulence.

Now we split all correlation functions (i.e., \( f_{ij}, h_{ij}, g_{ij}, \Phi_{ij} \)) into two parts, e.g., \( f_{ij} = f_{ij}^{(N)} + f_{ij}^{(S)} \), where

\[
f_{ij}^{(N)} = \frac{[f_{ij}(k, R) + f_{ij}(-k, R)]}{2}
\]

and \( f_{ij}^{(S)} = [f_{ij}(k, R) - f_{ij}(-k, R)]/2 \). Next, we use \( \tau \) approximation which is determined by Eqs. (B1)-(B10). We assume that \( \eta k^2 \ll \tau^{-1} \) and \( \nu k^2 \ll \tau^{-1} \) for the inertial range of turbulent fluid flow. We also assume that the characteristic time of variation of the mean magnetic field \( B \) is substantially longer than the correlation time \( \tau(k) \) for all turbulence scales. Thus, Eqs. (A3)-(A6) yield

\[
f_{ij}^{(N)} \approx f_{ij}^{(0N)} + i\tau(k \cdot B)\Phi_{ij}^{(S)}, \quad (A13)
\]

\[
h_{ij}^{(N)} \approx h_{ij}^{(0N)} - i\tau(k \cdot B)\Phi_{ij}^{(S)}, \quad (A14)
\]

\[
f_{ij}^{(S)} \approx f_{ij}^{(0S)} + i\tau(k \cdot B)\Phi_{ij}^{(S)}, \quad (A15)
\]

\[
g_{ij} \approx \tau I_{ij}, \quad (A16)
\]

where \( \psi = 2(k \cdot B \tau)^2/\mu_0 \rho, k_{ij} = k_i k_j/k^2, f_{ij}^{(0N)} \) and \( f_{ij}^{(0S)} \) describe the nonhelical and helical tensors of the background turbulence. The tensor \( h_{ij}^{(S)} \) is determined by an evolutionary equation (see, e.g., [24, 25] and Section III-C). Now we calculate \( \Phi_{ij}^{(N)} \) and \( \Phi_{ij}^{(S)} \). The definition of \( \Phi_{ij}^{(N)} \), given by Eq. (A7), and Eq. (A16) yield

\[
\Phi_{ij}(k, R) \approx \tau(\mu_0 \rho)^{-1} f_{ij}(k, R) - f_{ij}(-k, R).
\]

Substituting Eq. (A6) into Eq. (A17) and using Eqs. (3) and (4) we obtain

\[
\Phi_{ij} \approx \frac{\tau}{(1 + \psi)\mu_0 \rho} [2i(k \cdot B)(f_{ij}^{(0S)} - h_{ij}^{(0S)}) + B_{ip} f_{ip}(f_{ij}^{(0N)} + h_{ij}^{(0N)}) + 2B_{ip}(h_{ij}^{(0N)} - h_{ij}^{(0S)} - h_{ij}^{(0S)}) + O(B_{ij})].
\]
Hereafter we use the following definitions:

\[ X_{ijk...}^{(C)}(\beta) = X_{ijk...}^{(v)}(\beta) - X_{ijk...}^{(h)}(\sqrt{2}\beta) + X_{ijk...}^{(b)}(\sqrt{2}\beta), \]  
\[ X_{ijk...}^{(M)}(\beta) = X_{ijk...}^{(v)}(\beta) - X_{ijk...}^{(b)}(\sqrt{2}\beta), \]  
\[ X_{ijk...}^{(P)}(\beta) = X_{ijk...}^{(v)}(\beta) + X_{ijk...}^{(h)}(\beta), \]  

and

\[ \chi_{ij}^{(a)}(\beta) = \frac{\int c_{ij}(k)\tau(k)\,dk}{1 + \psi(\beta, k)}, \]  
\[ \zeta_{ijmn}^{(a)}(\beta) = \frac{\int c_{ij}(k)\tau(k)\,dk}{1 + \psi(\beta, k)}, \]

and \( \beta_i = 4B_i/\sqrt{2\mu_0 u_0}\), \( \psi(\beta, k) = [(\beta-k)u_0\tau/2]^2 \), \( c_{ij} = f^{(0N)}_{ij} \) for \( \lambda_{ij}^{(v)} \) and \( c_{ij} = f^{(0N)}_{ij} \) for \( \lambda_{ij}^{(b)} \).

For the calculation of the tensor \( b_{ijk} \) we specified a model of the background turbulence (i.e., turbulence with zero mean magnetic field). The turbulent velocity and magnetic fields of the background turbulence are determined by Eq. (47). To integrate over the angles in \( k \)-space in Eqs. (A27) and (A28) we use the following identities:

\[ \int \frac{k_{ij}\sin\theta}{1 + a\cos^2\theta} \,dk = \hat{A}_1 \delta_{ij} + \hat{A}_2 \beta_{ij}, \]  
\[ \int \frac{k_{ijmn}\sin\theta}{1 + a\cos^2\theta} \,dk = \hat{C}_1 (\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + \hat{C}_2 \beta_{ijmn} + \hat{C}_3 (\delta_{ij}\beta_{mn} + \delta_{im}\beta_{jn} + \delta_{in}\beta_{jm} + \delta_{jm}\beta_{in} + \delta_{jn}\beta_{im} + \delta_{mn}\beta_{ij}), \]

where \( a = [\beta u_0 k\tau(k)/2]^2 \), \( \beta_i = \beta_i/\beta \), \( \beta_{ij} = \beta_i \beta_j \beta_{ij} = \beta_i \beta_j \beta_{ij} \ldots \), and

\[ \hat{A}_1 = \frac{2\pi}{a} \left[ (a + 1) \frac{\arctan(\sqrt{a})}{\sqrt{a}} - 1 \right], \]  
\[ \hat{A}_2 = -\frac{2\pi}{a} \left[ (a + 3) \frac{\arctan(\sqrt{a})}{\sqrt{a}} - 3 \right], \]  
\[ \hat{C}_1 = \frac{\pi}{2a^2} \left[ (a + 1)^2 \frac{\arctan(\sqrt{a})}{\sqrt{a}} - 5a/3 - 1 \right], \]  
\[ \hat{C}_2 = \frac{\pi}{2a^2} \left[ (3a^2 + 30a + 35) \frac{\arctan(\sqrt{a})}{\sqrt{a}} - 55a/3 - 35 \right], \]  
\[ \hat{C}_3 = \frac{\pi}{2a^2} \left[ (a^2 + 6a + 5) \frac{\arctan(\sqrt{a})}{\sqrt{a}} - 13a/3 - 5 \right]. \]

To integrate over \( k \) in Eqs. (A27) and (A28) we use the Kolmogorov spectrum of the background turbulence, i.e., \( \tau_{ij}^{(0N)}(k) = \eta_{ij}^{(v)} \varphi(k), \eta_{ij}^{(0N)}(k) = \eta_{ij}^{(b)} \varphi(k) \), and \( \eta_{ij}^{(b)}(k) = \mu_{ij}^{(b)}(R) \varphi(k)/3 \), where \( \varphi(k) = (\pi k^2 k_0)^{-1}(k/k_0)^{-7/3}, \tau(k) = 2\tau_0(k/k_0)^{-2/3} \), where \( k_0 \leq k \leq k_d \), \( k_0 = l_0^{-1} \), \( l_0 \) is the maximum scale of turbulent motions and \( k_d = k_0 \text{Re}^{3/4} \) is determined by the Kolmogorov’s viscous scale of turbulence. The integration in \( k \)-space in Eqs. (A27) and (A28) yields

\[ \lambda_{ij}^{(a)}(\beta) = \Lambda_{ij}^{(a)}(\beta) + \beta_i \beta_j \dot{\beta}_{ij} + \Psi_2(\beta) \mu_{ij}^{(a)} \hat{\beta}, \]  
\[ \gamma_{ij}^{(a)}(\beta) = \frac{5}{12} C_2(\beta) \mu_{ij}^{(a)} \hat{\beta}, \]  
\[ \Gamma_{ij}^{(a)}(\beta) = \frac{5}{12} C_2(\beta) \mu_{ij}^{(a)} + \frac{1}{2} A_2(\beta) \eta_{ij}^{(a)}, \]  
\[ U_{ijm}^{(a)}(\beta) = \frac{5}{6} [\{ A_2(\beta) - C_3(\beta) \} \mu_{ij}^{(a)} \beta_m - C_3(\beta) \mu_{im}^{(a)} \beta_j - \mu_{ipm}^{(a)} \beta_{pj} - C_2(\beta) \mu_{ip}^{(a)} \beta_{jm} - C_3(\beta) \mu_{im}^{(a)} \beta_{jn} - C_3(\beta) \mu_{ip}^{(a)} \beta_{jm} \}, \]  
\[ \hat{\zeta}_{ijmn}^{(a)}(\beta) = \frac{1}{2} \delta_{mn} [\Lambda_{ij}^{(a)}(\beta) + \frac{5}{6} C_3(\beta) \mu_{ij}^{(a)} \beta_j], \]

and \( \mu_{ij}^{(a)} = \mu_{ij}^{(b)} \psi_{ij}(\beta), \psi_{ij}(\beta) = (5/6)[A_1(\beta) + A_2(\beta) + C_1(\beta)], \psi_{ij}(\beta) = (5/6)[C_1(\beta) - A_2(\beta), \psi_{ij}(\beta) = (5/3)[A_2(\beta) + C_3(\beta)]. \) The functions \( \Lambda_{ij}(\beta) = \int_{k_0}^{\infty} \Lambda_{ij}(a) \varphi(k) k^2 \,dk = (3a^4/\pi) \int_{\infty}^{\infty} (\hat{A}_n(\beta)/X^5) \) \( a_k \). Similarly for \( C_n(\beta) \), where \( a = [\beta u_0 k\tau(k)/2]^2 = X^2 = \beta^3(k/k_0)^{2/3} \), and we took into account that the inertial range of the turbulence exists in the scales: \( l_d \leq \beta \leq l_0 \). Here the maximum scale of the turbulence \( l_0 \ll L_B \), \( l_d = l_0^{1/3} \) is the viscous scale of turbulence, and \( L_B \) is the characteristic scale of variations of the nonuniform mean magnetic field. For very large Reynolds numbers \( k_d = l_0^{1/3} \) is very large and the turbulent hydrodynamic and magnetic energies are very small in the viscous dissipative range of the turbulence \( 0 \leq \beta \leq l_0 \). Thus we integrated in \( A_n \) over \( k_0 \) from \( k_0 = l_0^{1/3} \) to \( \infty \). The functions \( A_n(\beta) \) and \( C_n(\beta) \) are given in Appendix B. In Eqs. (A27)–(A37) we omitted terms which are symmetric in indexes \( i \) and \( n \) because after multiplication \( \zeta_{ijmn}^{(a)}(\beta) \) by \( \epsilon_{ijn} \), these symmetric terms vanish [see Eq. (A23)].

In order to extract terms \( \epsilon_{ijn} \hat{\beta}_m \) which contribute to the nonlinear diamagnetic and paramagnetic velocities, we split \( b_{ijk} \) into two parts, i.e., \( b_{ijk} = b_{ijk}^{(1)} + b_{ijk}^{(2)} \), where

\[ b_{ijk}^{(1)} = \epsilon_{ijn} \hat{\beta}_m \left[ \delta_{ij} \gamma_{i}^{(P)} \hat{\beta}_j + \Psi_2(\beta) \mu_{kp}^{(P)} \hat{\beta}_p \right] + 2[\Gamma^{(C)}(\beta)\delta_{nkj} + U_{nkj}^{(C)}(\beta)], \]
\[
b^{(2)}_{ijk} = \varepsilon_{ijn} A^{(P)}_{nk}(\beta) + 2\varepsilon_{ijnm} \epsilon^{(C)}_{nmkj}(\beta) \tag{A39}
\]

[see the definitions given by Eqs. (A24)-(A26)]. Next, we calculate \( b_{ijk} B_{j,k} \). Using Eqs. (A11), (A38) and (A39) we also split the electromotive force into two parts
\[
\mathcal{E} = \mathcal{E}^{(1)} + \mathcal{E}^{(2)},
\]
\[
\mathcal{E}^{(1)} = \mathcal{E}^{(1)} \mathcal{B}_{j,k},
\]
\[
\mathcal{E}^{(2)} = a_{ij} B_j + b^{(2)}_{ijk} B_{j,k}. \tag{A42}
\]

Using Eqs. (A38) and (A41) we obtain
\[
\mathcal{E}^{(1)} = (\mathbf{V}^{(N)} \times \mathbf{B})_i - \eta^{(1)}_{ij} (\nabla \times \mathbf{B})_j, \tag{A43}
\]
where
\[
\mathbf{V}^{(N)}(\mathbf{B}) = \frac{1}{2B^2} [\gamma^{(P)}(\beta) + 2\Gamma^{(C)}(\beta)] \nabla_i B^2
\]
\[
+ \frac{1}{B} [2\Gamma^{(C)}(\beta) + \Psi_2(\beta) \mu_{kp} \delta_{p} \delta_{ij}] \nabla_k B_j, \tag{A44}
\]
\[
\eta^{(1)}_{ij} = \gamma^{(P)}(\beta) P_{ij}(\beta), \tag{A45}
\]
and \( P_{ij}(\beta) = \delta_{ij} - \beta_{ij} \). For the calculation of the terms \( \alpha^{(P)}(\beta) \) in these equations we used an identity \( \varepsilon_{ijnm} \beta_{mp} B_{m,p} \equiv -[\mathbf{B} \times (\mathbf{B} \cdot \nabla) \mathbf{B}]/B^2 = -[\mathbf{B} \times \nabla (B^2/2)]/B^2 - P_{ij}(\beta) (\nabla \times \mathbf{B})_j \), which follows from the formula \( \mathbf{B} \cdot \nabla \mathbf{B} = (1/2) \nabla B^2 - \mathbf{B} \times (\nabla \times \mathbf{B}) \).

Following to (2) we use an identity \( B_{i,k} = (\partial B)_{j,k} - \varepsilon_{jkl} (\nabla \times \mathbf{B})_{l} /2 \) in order to rewrite Eq. (A42) in the form
\[
\mathcal{E}^{(2)} = a_{ij} B_j + (\mathbf{U} \times \mathbf{B})_i - \eta^{(2)}_{ij} (\nabla \times \mathbf{B})_j
\]
\[
- \kappa_{ij} (\partial B)_{j,k}, \tag{A46}
\]
where
\[
\eta^{(2)}_{ij} = (\varepsilon_{ijkp} b^{(2)}_{kp} + \varepsilon_{jikp} b^{(2)}_{kp}) /4, \tag{A47}
\]
\[
\kappa_{ij} (\mathbf{B}) = - (b^{(2)}_{ij} + b^{(2)}_{ki}) /2. \tag{A48}
\]

Using Eqs. (A43) and (A46) we obtain the equation for the electromotive force \( \mathcal{E} = \mathcal{E}^{(1)} + \mathcal{E}^{(2)} \) which is given by Eq. (A49). The tensor of turbulent magnetic diffusion, \( \eta_{ij} (\mathbf{B}) = \eta^{(1)}_{ij} + \eta^{(2)}_{ij}, \tag{A49} \)
is given by
\[
\eta_{ij} (\mathbf{B}) = \delta_{ij} [A_1(\beta) \eta^{(P)}_T + \frac{5}{12} C_2(\beta) + 2C_3(\beta)] \mu_{\beta}^{(P)}
\]
\[
- [A_1 \eta_T]^{(C)} (\beta) - \frac{5}{12} C_3 \beta [\mu_{\beta}^{(P)}]^{(C)} - \frac{5}{6} [2\Psi(\beta) \mu_{ij}^{(P)}]
\]
\[
+ \Psi_2(\beta) \mu_{ij}^{(P)} + \frac{5}{6} [(A_1 + C_1) \mu_{ij}]^{(C)}
\]
\[
+ \frac{5}{12} C_3 \mu_{ij}^{(C)} + \frac{1}{2} \beta_{ij} [A_2(\beta) \eta^{(P)}_T + \frac{5}{6} C_2(\beta) \mu_{\beta}^{(P)}], \tag{A50}
\]
where \( \mu_{ij}^{(a)} = \mu_{in}^{(a)} \beta_{nj} + \beta_{in} \mu_{nj}^{(a)} \), and we used Eqs. (A43), (A44) and the definitions (A24)-(A26). In particular, \( [X]^{(C)}(\beta) = X^{(v)}(\beta) - X^{(v)}(\sqrt{2} \beta) + X^{(h)}(\sqrt{2} \beta) \) which implies, e.g., \( [A_1 \eta_T]^{(C)} = A_1(\beta) \eta_T^{(v)} - A_1(\sqrt{2} \beta) \eta_T^{(v)} + A_1(\sqrt{2} \beta) \eta_T^{(h)} \).

Using Eqs. (A43) and (A44) we calculate \( \kappa_{ij} (\mathbf{B}) \):
\[
\kappa_{ij} (\mathbf{B}) = - \frac{1}{2} [\psi(\beta) \hat{L}_{ij}^{(P)} + \psi_2(\beta) \hat{N}_{ij}^{(P)}]
\]
\[
+ \frac{5}{6} [(A_1 - 3C_1) \hat{L}_{ij}^{(C)}] - \frac{5}{2} [C_3 \hat{N}_{ij}^{(C)}], \tag{A51}
\]
where \( \hat{L}_{ij}(\beta) = \varepsilon_{ijn} \mu_{nk}^{(a)} + \varepsilon_{ihn} \mu_{nj}^{(a)} \), \( \hat{N}_{ij}(\beta) = \mu_{np}^{(a)} \varepsilon_{ijn} \beta_{pk} + \varepsilon_{ikh \beta_{pj}} \) and \( [C_3 \hat{N}_{ij}^{(C)}] = C_3(\beta) \hat{N}_{ij}^{(v)} - C_3(\sqrt{2} \beta) \hat{N}_{ij}^{(v)} - \hat{N}_{ij}^{(h)} \) and similarly for \( [(A_1 - 3C_1) \hat{L}_{ij}^{(C)}] \) see Eq. (A24).

The asymptotic formulas for the nonlinear coefficients defining the mean electromotive force for \( \beta \ll 1 \) are given by
\[
\eta_{ij} (\mathbf{B}) = \frac{3\pi}{5\beta} \left[ \delta_{ij} \left( \eta_T^{(M)} + \eta_T^{(h)} + \frac{5}{48} \mu_{\beta}^{(P)} (3 - \sqrt{2}) \right) + \mu_{\beta}^{(h)} (\sqrt{2} + 1) \right] + \frac{5}{48} \mu_{ij}^{(v)} (9 - 5 \sqrt{2})
\]
\[
+ \mu_{ij}^{(h)} (5 \sqrt{2} - 1) - \frac{3}{10} \mu_{ij}^{(v)} (5 - \sqrt{2})
\]
\[
+ \frac{5}{8} \mu_{ij}^{(P)} \right], \tag{A54}
\]
and for \( \beta \gg 1 \) they are given by:
\[
\eta_{ij} (\mathbf{B}) = \frac{3\pi}{10 \beta} \left[ \delta_{ij} (1 - \epsilon) \mu_{\beta}^{(1)} + \mu_{\beta}^{(1)} (1 + 9 \beta) \right]
\]
\[
+ \frac{2 \alpha_{ij}^{(v)}}{\beta} \delta_{ij}, \tag{A55}
\]
\[
\alpha_{ij}^{(v)} (\mathbf{B}) = - \frac{3\pi}{10 \beta} \left[ \delta_{ij} (1 - \epsilon) \mu_{\beta}^{(1)} + \mu_{\beta}^{(1)} (1 + 9 \beta) \right]
\]
\[
+ \frac{2 \alpha_{ij}^{(v)}}{\beta} \delta_{ij}, \tag{A56}
\]
\[
\alpha_{ij}^{(h)} (\mathbf{B}) = \frac{3\pi}{2 \beta} \alpha_{ij}^{(h)} (\mathbf{B}) \delta_{ij}. \tag{A57}
\]

The asymptotic formulas for the tensor \( \kappa_{ij} \) for \( \beta \ll 1 \) and \( \beta \gg 1 \) are given by Eqs. (21) and (22).
APPENDIX B: THE FUNCTIONS $A_\alpha(\beta)$ AND $C_\alpha(\beta)$

The functions $A_\alpha(\beta)$ and $C_\alpha(\beta)$ are given by

\begin{align*}
A_1(\beta) &= \frac{6}{5} \left[ \arctan \frac{\beta}{\beta} \left(1 + \frac{5}{7\beta^2} \right) + \frac{1}{14} L(\beta) - \frac{5}{7\beta^2} \right], \\
A_2(\beta) &= -\frac{6}{5} \left[ \arctan \frac{\beta}{\beta} \left(1 + \frac{15}{7\beta^2} \right) - \frac{2}{7} L(\beta) - \frac{15}{7\beta^2} \right], \\
C_1(\beta) &= \frac{3}{10} \left[ \arctan \frac{\beta}{\beta} \left(1 + \frac{10}{7\beta^2} + \frac{5}{9\beta^2} \right) + \frac{2}{63} L(\beta) \\
&\quad - \frac{235}{189\beta^2} - \frac{5}{9\beta^4} \right], \\
C_2(\beta) &= \frac{3}{2} \left[ \arctan \frac{\beta}{\beta} \left(\frac{3}{5} + \frac{30}{7\beta^2} + \frac{35}{9\beta^4} \right) + \frac{16}{315} L(\beta) \\
&\quad - \frac{565}{189\beta^2} - \frac{35}{9\beta^4} \right], \\
C_3(\beta) &= -\frac{3}{2} \left[ \arctan \frac{\beta}{\beta} \left(\frac{1}{5} + \frac{6}{7\beta^2} + \frac{5}{9\beta^4} \right) - \frac{8}{315} L(\beta) \\
&\quad - \frac{127}{189\beta^2} - \frac{5}{9\beta^4} \right],
\end{align*}

where $L(\beta) = 1 - 2\beta^2 + 2\beta^4 \ln(1 + \beta^{-2})$. For $\beta \ll 1$ these functions are given by

\begin{align*}
A_1(\beta) &\sim 1 - \frac{2}{5} \beta^2, & A_2(\beta) &\sim -\frac{4}{5} \beta^2, \\
C_1(\beta) &\sim \frac{1}{5} \left[ 1 - \frac{2}{7} \beta^2 \right], & C_2(\beta) &\sim -\frac{32}{105} \beta^4 \ln \beta, \\
C_3(\beta) &\sim -\frac{4}{35} \beta^2,
\end{align*}

and for $\beta \gg 1$ they are given by

\begin{align*}
A_1(\beta) &\sim \frac{3\pi}{5} - \frac{2}{\beta^2}, & A_2(\beta) &\sim -\frac{3\pi}{5} + \frac{4}{\beta^2}, \\
C_1(\beta) &\sim \frac{3\pi}{20\beta}, & C_2(\beta) &\sim \frac{9\pi}{20\beta}, \\
C_3(\beta) &\sim -\frac{3\pi}{20\beta}.
\end{align*}

Since the function $A_1(\beta) + A_2(\beta) \sim O(\beta^{-2})$ for $\beta \gg 1$ (it describes an isotropic part of the $\alpha$-effect) we took into account in the functions $A_1(\beta)$ and $A_2(\beta)$ the terms which are of the order of $\sim O(\beta^{-2})$. Here we also used that for $\beta \ll 1$ the function $L(\beta) \sim 1 - 2\beta^2 + 4\beta^4 \ln \beta$, and for $\beta \gg 1$ the function $L(\beta) \sim 2/3\beta^2$.

APPENDIX C: DERIVATION OF THE NONLINEAR DEPENDENCIES $\eta_\alpha(B)$, $\eta_\mu(B)$ AND $V_\alpha(B)$

Now we consider an anisotropic background turbulence with one preferential direction, say along unit vector $e$, where $e \cdot \hat{\beta} = 0$. In this case

\begin{align*}
V^{(N)} &= B^{-2}[V^{(1)} \nabla B^2 + V^{(2)} (e \cdot \nabla) B^2 + V^{(3)} (B \cdot \nabla) B] , \\
\kappa_{ijk} (\partial \hat{B})_{jk} &= -B^{-2}[\hat{W} \nabla B^2 + 2(B \cdot \nabla) B \times B] - M_\kappa (e \times (e \cdot \nabla) B),
\end{align*}

where

\begin{align*}
V^{(1)} &= -\frac{1}{4} \left[ A_2(\beta) \eta_\mu^{(P)} + \frac{5}{18} C_2(\beta) \xi_\mu^{(P)} \right. \\
&\quad \left. + \frac{5}{9} (C_2 + 2A_2) \xi_\mu^{(C)} - \frac{1}{2} [A_2 \eta_\mu^{(C)}] \right] , \\
V^{(2)} &= \frac{5}{6} \left[ (A_2 - C_3) \xi_\mu^{(C)} \right] , \\
V^{(3)} &= \frac{5}{9} (C_3 \xi_\mu^{(C)} - \frac{1}{3} \Psi_2 (\beta) \xi_\mu^{(P)} , \\
U^{(1)} &= -(\sqrt{2} / 48) \Psi (\sqrt{2} \beta) , \\
U^{(2)} &= -(\sqrt{2} / 48) \Psi_1 (\sqrt{2} \beta) \xi_\mu^{(M)} , \\
U^{(3)} &= \frac{1}{6} \Psi_2 (\sqrt{2} \beta) \xi_\mu^{(M)} ,
\end{align*}

\[\hat{W} = -\frac{1}{12} \left[ \Psi_2 (\beta) \xi_\mu^{(P)} + 5(C_3 \xi_\mu^{(C)} \right].\]

In order to derive Eq. (C3) we used the following identities:

\[L_{ijk}^{\alpha}(\partial \hat{B})_{jk} = -e_\mu^{(P)} (e \times (e \cdot \nabla) B),\]

\[N_{ijk}^{\alpha}(\partial \hat{B})_{jk} = -\frac{1}{6} B^2 \xi_\mu^{(P)} ((e \cdot \nabla) B^2 + 2(B \cdot \nabla) B),\]

Using Eqs. (C4)-(C3) and (A50) we calculate the functions $M_\eta, M_\mu, M_\beta, M_\kappa, M_\mu^{(1)}$ and $M_\mu^{(2)}$ in Eqs. (B1)-(B2):

\begin{align*}
M_\eta &= A_1(\beta) \eta_\mu^{(P)} + \frac{5}{36} [A_1(\beta) + 4A_2(\beta) + C_1(\beta) \\
&\quad - C_2(\beta) - 5C_5(\beta)] \xi_\mu^{(P)} - \frac{5}{18} (C_1 + A_1) \xi_\mu^{(C)} \\
&\quad - [A_1 \eta_\mu^{(C)}] + \frac{1}{6} \Psi_2 (\sqrt{2} \beta) \xi_\mu^{(M)} , \\
M_\mu &= \frac{1}{2} \Psi_1 (\beta) \xi_\mu^{(P)} + \frac{5}{6} (3C_1 - A_1) \xi_\mu^{(C)} , \\
M_\beta &= \frac{1}{6} \left[ 3A_2(\beta) \eta_\mu^{(P)} + \frac{5}{6} C_2(\beta) + 4 \Psi_2 (\beta) \xi_\mu^{(P)} \right. \\
&\quad - \Psi_2 (\sqrt{2} \beta) \xi_\mu^{(M)} \right] , \\
M_\kappa &= \frac{1}{2} \Psi_1 (\beta) \xi_\mu^{(P)} + \frac{5}{6} (3C_1 - A_1) \xi_\mu^{(C)} , \\
M_\mu^{(1)} &= V^{(1)} + U^{(1)} + 2W + \frac{1}{2} (V^{(3)} + U^{(3)}) \\
&= -\frac{1}{4} A_2(\beta) \eta_\mu^{(P)} - \frac{5}{72} C_2(\beta) + 4C_5(\beta) \\
&\quad - 4A_2(\beta) \xi_\mu^{(P)} + \frac{1}{12} \Psi_2 (\sqrt{2} \beta) \xi_\mu^{(M)}
\end{align*}
Now we take into account that \( \mathbf{V}^{(N)} \), \( \mathbf{U} \) and \( \kappa \) contribute into the tensor \( \eta_{ij} \). This implies that in order to calculate \( M_\eta, M_\varepsilon, \) and \( M_\beta \) we perform the change

\[
\eta_{ij} \to \eta_{ij} + P_{ij}(\beta)[V^{(3)} + U^{(3)} + 2W], \tag{C10}
\]

where the second term in \( (C10) \) [which is proportional to \( P_{ij}(\beta) \)] describes a contribution \( \mathbf{V}^{(N)} \), \( \mathbf{U} \) and \( \kappa \) into the tensor \( \eta_{ij} \). Using Eqs. \( (B1), (B2) \) and \( (C4)-(C9) \) we calculate the functions \( \eta_A(B), \eta_B(B) \) and \( V_A(B) \):

\[
\eta_A(B) = \tilde{\eta}(B) + (10/9)[(2C_1 - A_1)\varepsilon_\mu]^{(C)} + [-A_1\eta_T]^{(C)}, \tag{C11}
\]

\[
\eta_B(B) = \tilde{\eta}(B) + \frac{5}{18}[(8C_1 + C_2 + 10C_3 - 4A_1 - 4A_2)\varepsilon_\mu]^{(C)} - [(A_1 + A_2)\eta_T]^{(C)} + \frac{\sqrt{2\beta}}{24}\Psi(\sqrt{2\beta}), \tag{C12}
\]

\[
V_A(B) = \left( \frac{5}{18}[(4A_2 - C_2 - 10C_3)\varepsilon_\mu]^{(C)} + [A_2\eta_T]^{(C)} - \frac{\sqrt{2\beta}}{24}\Psi(\sqrt{2\beta})(\ln |B|)' \right)^{\prime}, \tag{C13}
\]

where \( \Psi(x) = 12[A_1(x) + (1/2)A_2(x)]\eta_T^{(M)} + \Psi_0(x)\eta_T^{(M)} \), \( \eta_{T'}(B) = [A_1(\beta) + (1/2)A_2(\beta)]\eta_T^{(P)} + (\varepsilon_\mu^{(P')}/12)\Psi_0(\beta), \)

\( \eta_\alpha(\beta) = (5/3)[4A_1(\beta) + 3A_2(\beta) + 4C_1(\beta) - C_3(\beta)] \) and \( [X]^{(C)} \) is defined by Eq. \( (A2) \). The asymptotic formulas for the functions \( \eta_A, \eta_B, V_A \) and \( \alpha_{ij}^{(v)} \) for \( \beta \ll 1 \) are given by

\[
\eta_A(B) = -\frac{2}{5}\beta^2[3\eta_T^{(v)} - 8\eta_T^{(h)} + \frac{10}{63}(14\varepsilon_\mu^{(v)} - \varepsilon_\mu^{(h)})], \tag{C14}
\]

\[
\eta_B(B) = -\frac{2}{5}\beta^2[8\eta_T^{(v)} - 4\eta_T^{(h)} + \frac{5}{7}(3\varepsilon_\mu^{(v)} - 4\varepsilon_\mu^{(h)})], \tag{C15}
\]

\[
V_A(B) = \frac{2}{5}\beta^2[3\eta_T^{(v)} - 4\eta_T^{(h)} + \frac{5}{7}(3\varepsilon_\mu^{(v)} - 4\varepsilon_\mu^{(h)})](\ln |B|)', \tag{C16}
\]

\[
\alpha_{ij}^{(v)}(B) = \delta_{ij}[(\alpha_0^{(v)} - (1/3)\varepsilon_\alpha)(1 - \frac{2}{5}\beta^2\varepsilon_\alpha)] - \frac{2}{105}\beta^2\varepsilon_\alpha \epsilon, \tag{C17}
\]

and for \( \beta \gg 1 \) they are given by

\[
\eta_A(B) = \frac{\pi}{6\beta^3}[\frac{9}{5}(\sqrt{2} - 1)\eta_T^{(M)} + \frac{\sqrt{2} - \frac{7}{8}}{\frac{9}{8}}\varepsilon_\mu^{(v)} - (\sqrt{2} - \frac{7}{8})\varepsilon_\mu^{(h)}], \tag{C18}
\]

\[
\eta_B(B) = \frac{\pi}{4\sqrt{2}\beta}\left\{ \frac{3}{5}(2\sqrt{2} - 1)\eta_T^{(h)} + (2\sqrt{2} + 1)\eta_T^{(l)} \right\} + \frac{1}{24}[(22\sqrt{2} - 13)\varepsilon_\mu^{(v)} + (18\sqrt{2} + 13)\varepsilon_\mu^{(h)}], \tag{C19}
\]

\[
V_A(B) = -\frac{3\pi}{4\sqrt{2}\beta}\left\{ \left( \frac{4\sqrt{2}}{5} - 1 \right)\eta_T^{(v)} + \frac{5}{8}\varepsilon_\mu^{(v)} + \eta_T^{(h)} \right\} + \frac{5}{8}\varepsilon_\mu^{(h)}(\ln |B|)', \tag{C20}
\]

\[
\alpha_{ij}^{(v)}(B) = -\delta_{ij}\left( \frac{\pi}{\beta^2}\varepsilon_\alpha \epsilon - \frac{2}{5}\beta^2[\alpha_0^{(v)} - \frac{1}{3}\varepsilon_\alpha(1 - \varepsilon_\alpha)] \right). \tag{C21}
\]
[18] G. Glatzmaier and P. Roberts, Nature 377, 203 (1995).
[19] A. Brandenburg, A. Nordlund, R. Stein and U. Torkelson, Astroph. J. 446, 741 (1995); 458, L45 (1996).
[20] A. Brandenburg, R. L. Jennings, A. Nordlund, M. Rieutord, R. F. Stein and I. Tuominen, J. Fluid Mech. 306, 325 (1996).
[21] K.-H. Rädler, Astron. Nachr. 301, 101 (1980); Geophys. Astrophys. Fluid Dynamics 20, 191 (1982).
[22] K.-H. Rädler, N. Kleeorin and I. Rogachevskii, in preparation.
[23] N. Kleeorin, and A. Ruzmaikin, Magnetohydrodynamics No. 2, 17 (1982).
[24] N. Kleeorin, I. Rogachevskii, and A. Ruzmaikin, Solar Phys. 155, 223 (1994); Astron. Astrophys. 297, 159 (1995).
[25] N. Kleeorin and I. Rogachevskii, Phys. Rev. E 59, 6724 (1999).
[26] S. A. Orszag, J. Fluid Mech. 41, 363 (1970).
[27] A. S. Monin and A. M. Yaglom, Statistical Fluid Mechanics (MIT Press, Cambridge, Massachusetts, 1975), Vol. 2.
[28] W. D. McComb, The Physics of Fluid Turbulence (Clarendon, Oxford, 1990).
[29] A. Pouquet, U. Frisch, and J. Leorat, J. Fluid Mech. 77, 321 (1976).
[30] N. Kleeorin, I. Rogachevskii, and A. Ruzmaikin, Sov. Phys. JETP 70, 878 (1990).
[31] N. Kleeorin, M. Mond, and I. Rogachevskii, Astron. Astrophys. 307, 293 (1996).
[32] S. I. Vainshtein and L. L. Kichatinov, Geophys. Astrophys. Fluid Dynamics 24, 273 (1983).
[33] Ya. B. Zeldovich, A. A. Ruzmaikin, and D. D. Sokoloff, The Almighty Chance (Word Scientific Publ., London, 1990), and references therein.
[34] I. Rogachevskii and N. Kleeorin, Phys. Rev. E 56, 417 (1997); N. Kleeorin and I. Rogachevskii, Phys. Rev. E 50, 493 (1994).
[35] I. Rogachevskii and N. Kleeorin, Phys. Rev. E 59, 3006 (1999).
[36] L. L. Kichatinov, Astron. Astroph. 243, 483 (1991).
[37] G. Rüdiger and L. L. Kichatinov, Astron. Astroph. 269, 581 (1993).
[38] L. L. Kichatinov, V. V. Pipin and G. Rüdiger, Astron. Nachr. 315, 157 (1994).
[39] P. H. Roberts and A. M. Soward, Astron. Nachr. 296, 49 (1975).
[40] N. Kleeorin and I. Rogachevskii, Phys. Rev. E 50, 2716 (1994).