On the maximal dimension of a completely entangled subspace for finite level quantum systems

by

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Summary: Let $\mathcal{H}_i$ be a finite dimensional complex Hilbert space of dimension $d_i$ associated with a finite level quantum system $A_i$ for $i = 1, 2, \ldots, k$. A subspace $S \subset \mathcal{H} = \mathcal{H}_{A_1A_2\ldots A_k} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_k$ is said to be completely entangled if it has no nonzero product vector of the form $u_1 \otimes u_2 \otimes \ldots \otimes u_k$ with $u_i$ in $\mathcal{H}_i$ for each $i$. Using the methods of elementary linear algebra and the intersection theorem for projective varieties in basic algebraic geometry we prove that

$$\max_{S \in \mathcal{E}} \dim S = d_1d_2\ldots d_k - (d_1 + \cdots + d_k) + k - 1$$

where $\mathcal{E}$ is the collection of all completely entangled subspaces.

When $\mathcal{H}_1 = \mathcal{H}_2$ and $k = 2$ an explicit orthonormal basis of a maximal completely entangled subspace of $\mathcal{H}_1 \otimes \mathcal{H}_2$ is given.

We also introduce a more delicate notion of a perfectly entangled subspace for a multipartite quantum system, construct an example using the theory of stabilizer quantum codes and pose a problem.

Key Words: finite level quantum systems, separable states, entangled states, completely entangled subspaces, perfectly entangled subspace, stabilizer quan-
tum code.

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# 1 Completely Entangled Subspaces

Let $H_i$ be a complex finite dimensional Hilbert space of dimension $d_i$ associated with a finite level quantum system $A_i$ for each $i = 1, 2, \ldots, k$. A state $\rho$ of the combined system $A_1A_2\ldots A_k$ in the Hilbert space

$$\mathcal{H} = \mathcal{H} \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_k$$

(1.1)

is said to be *separable* if it can be expressed as

$$\rho = \sum_{i=1}^{m} p_i \rho_{i1} \otimes \rho_{i2} \otimes \ldots \otimes \rho_{ik}$$

(1.2)

where $\rho_{ij}$ is a state of $A_j$ for each $j$, $p_i > 0$ for each $i$ and $\sum_{i=1}^{m} p_i = 1$ for some finite $m$. A state which is not separable is said to be *entangled*. Entangled states play an important role in quantum teleportation and communication [3]. The following theorem due to Horodecki et al [2] suggests a method of constructing entangled states.

**Theorem 1.1** (Horodecki et al) Let $\rho$ be a separable state in $\mathcal{H}$. Then the range of $\rho$ is spanned by a set of product vectors.

For the sake of readers’ convenience and completeness we furnish a quick proof.

**Proof**: Let $\rho$ be of the form (1.2). By spectrally resolving each $\rho_{ij}$ into one
dimensional projections we can rewrite (1.2) as

$$\rho = \sum_{i=1}^{n} q_i |u_{i1} \otimes u_{i2} \otimes \ldots \otimes u_{ik}\rangle \langle u_{i1} \otimes u_{i2} \otimes \ldots \otimes u_{ik}|$$  \hspace{1cm} (1.3)

where $u_{ij}$ is a unit vector in $H_j$ for each $i, j$ and $q_i > 0$ for each $i$ with $\sum_{i=1}^{n} q_i = 1$. We shall prove the theorem by showing that each of the product vectors $u_{i1} \otimes u_{i2} \otimes \ldots \otimes u_{ik}$ is, indeed, in the range of $\rho$. Without loss of generality, consider the case $i = 1$. Write (1.3) as

$$\rho = q_1 |u_{11} \otimes u_{12} \otimes \ldots \otimes u_{1k}\rangle \langle u_{11} \otimes u_{12} \otimes \ldots \otimes u_{1k}| + T$$  \hspace{1cm} (1.4)

where $q_1 > 0$ and $T$ is a nonnegative operator. Suppose $\psi \neq 0$ is a vector in $H$ such that $T|\psi\rangle = 0$ and $\langle u_{11} \otimes u_{12} \otimes \ldots \otimes u_{1k}|\psi\rangle \neq 0$. Then $\rho|\psi\rangle$ is a nonzero multiple of the product vector $u_{11} \otimes u_{12} \otimes \ldots \otimes u_{1k}$ and $u_{11} \otimes u_{12} \otimes \ldots \otimes u_{1k} \in R(\rho)$, the range of $\rho$. Now suppose that the null space $N(T)$ of $T$ is contained in $\{u_{11} \otimes u_{12} \otimes \ldots \otimes u_{1k}\}^\perp$. Then $R(T) \supset \{u_{11} \otimes u_{12} \otimes \ldots \otimes u_{1k}\}$ and therefore there exists a vector $\psi \neq 0$ such that

$$T|\psi\rangle = |u_{11} \otimes u_{12} \otimes \ldots \otimes u_{1k}\rangle.$$

Note that $\rho|\psi\rangle \neq 0$, for otherwise, the positivity of $\rho, T$ and $q_1$ in (1.4) would imply $T|\psi\rangle = 0$. Thus (1.4) implies

$$\rho|\psi\rangle = (q_1 \langle u_{11} \otimes u_{12} \otimes \ldots \otimes u_{1k}|\psi\rangle + 1) |u_{11} \otimes u_{12} \otimes \ldots \otimes u_{1k}\rangle.$$

\textbf{Corollary} If a subspace $S \subset \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_k$ does not contain any nonzero product vector of the form $u_1 \otimes u_2 \otimes \ldots \otimes u_k$ where $u_i \in \mathcal{H}_i$ for each $i$, then any state with support in $S$ is entangled.

\textbf{Proof :} Immediate.
Definition 1.2 A nonzero subspace $S \subset \mathcal{H}$ is said to be completely entangled if $S$ contains no nonzero product vector of the form $u_1 \otimes u_2 \otimes \ldots \otimes u_k$ with $u_i \in \mathcal{H}_i$ for each $i$.

Denote by $\mathcal{E}$ the collection of all completely entangled subspaces of $\mathcal{H}$. Our goal is to determine $\max_{S \in \mathcal{E}} \dim S$.

Proposition 1.3 There exists $S \in \mathcal{E}$ satisfying

$$\dim S = d_1 d_2 \ldots d_k - (d_1 + d_2 + \ldots + d_k) + k - 1.$$ (1.6)

Proof: Let $N = d_1 + d_2 + \cdots + d_k - k + 1$. Without loss of generality, assume that $\mathcal{H}_i = \mathbb{C}^{d_i}$ for each $i$, with the standard scalar product. Choose and fix a set $\{\lambda_1, \lambda_2, \ldots, \lambda_N\} \subset \mathbb{C}$ of cardinality $N$. Define the column vectors

$$u_{ij} = \begin{bmatrix} 1 \\ \lambda_i \\ \lambda_i^2 \\ \vdots \\ \lambda_i^{d_j-1} \end{bmatrix}, \ 1 \leq i \leq N, \ 1 \leq j \leq k \quad (1.5)$$

and consider the subspace

$$S = \{u_{i1} \otimes u_{i2} \otimes \ldots \otimes u_{ik}, \ 1 \leq i \leq N\}^\perp \subset \mathcal{H}. \quad (1.6)$$

We claim that $S$ has no nonzero product vector. Indeed, let

$$0 \neq v_1 \otimes v_2 \otimes \ldots \otimes v_k \in S, \ v_i \in \mathcal{H}_i.$$ (1.7)
If
\[ E_j = \{ i \mid \langle v_j | u_{ij} \rangle = 0 \} \subset \{1, 2, \ldots, N\} \] (1.8)
then (1.7) implies that
\[ \{1, 2, \ldots, N\} = \bigcup_{j=1}^{k} E_j \]
and therefore
\[ N \leq \sum_{j=1}^{k} \#E_j. \]

By the definition of \( N \) it follows that for some \( j \), \( \#E_j \geq d_j \). Suppose \( \#E_{j_0} \geq d_{j_0} \). From (1.8) we have
\[ \langle v_{j_0} | u_{i_{j_0}} \rangle = 0 \text{ for } i = i_1, i_2, \ldots, i_{d_{j_0}} \]
where \( i_1 < i_2 < \cdots < i_{d_{j_0}} \). From (1.5) and the property of van der Monde determinants it follows that \( v_{j_0} = 0 \), a contradiction. Clearly, \( \dim S \geq d_1d_2\cdots d_k - (d_1 + \cdots + d_k) + k - 1. \) \( \blacksquare \)

**Proposition 1.4**  Let \( S \subset \mathcal{H} \) be a subspace of dimension \( d_1d_2\cdots d_k - (d_1 + \cdots + d_k) + k \). Then \( S \) contains a nonzero product vector.

**Proof:**  Identify \( \mathcal{H}_j \) with \( \mathbb{C}^{d_j} \) for each \( j = 1, 2, \ldots, k \). For any nonzero element \( v \) in a complex vector space \( \mathcal{V} \) denote by \([v]\) the equivalence class of \( v \) in the projective space \( \mathbb{P}(\mathcal{V}) \). Consider the map
\[ T : \mathbb{P}(\mathbb{C}^{d_1}) \times \mathbb{P}(\mathbb{C}^{d_2}) \times \cdots \times \mathbb{P}(\mathbb{C}^{d_k}) \to \mathbb{P}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_k}) \]
given by
\[ T([u_1], [u_2], \ldots, [u_k]) = [u_1 \otimes \cdots \otimes u_k]. \]
The map \( T \) is algebraic and hence its range \( \mathbb{P}(T) \) is a complex projective variety of dimension \( \sum_{i=1}^{k} (d_i - 1) \). By hypothesis \( \mathbb{P}(S) \) is a projective variety of
dimension $d_1d_2 \ldots d_k - (d_1 + \ldots + d_k) + k - 1$. Thus

$$\dim IP(S) + \dim R(T) = d_1d_2 \ldots d_k - 1$$

$$= \dim IP(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \ldots \otimes \mathbb{C}^{d_k}).$$

Hence by Theorem 6, page 76 in [4] we have

$$IP(S) \cap R(T) \neq \emptyset.$$

In other words $S$ contains a product vector.

**Theorem 1.5** Let $\mathcal{E}$ be the collection of all completely entangled subspaces of $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_k$. Then

$$\max_{S \in \mathcal{E}} \dim S = d_1d_2 \ldots d_k - (d_1 + d_2 + \cdots + d_k) + k - 1.$$

**Proof** : Immediate from Proposition 1.3 and Proposition 1.4.

2 An Explicit Orthonormal Basis for a Completely Entangled Subspace of Maximal Dimension in $\mathbb{C}^n \otimes \mathbb{C}^n$

Let $\{|x\}, x = 0, 1, 2, \ldots, n - 1\}$ be a labelled orthonormal basis in the Hilbert space $\mathbb{C}^n$. Choose and fix a set

$$E = \{\lambda_1, \lambda_2, \ldots, \lambda_{2n-1}\} \subset \mathbb{C}$$

of cardinality $2n - 1$ and consider the subspace

$$S = \{u_{\lambda_i} \otimes u_{\lambda_i}, 1 \leq i \leq 2n - 1\}^\perp$$
where

\[ u_{\lambda} = \sum_{x=0}^{n-1} \lambda^x |x\rangle, \lambda \in \mathbb{C}. \]

By the proof of Proposition 1.3 and Theorem 1.5 it follows that \( S \) is a maximal completely entangled subspace of dimension \( n^2 - 2n + 1 \). We shall now present an explicit orthonormal basis for \( S \).

First, observe that \( S \) is orthogonal to a set of symmetric vectors and therefore \( S \) contains the antisymmetric tensor product space \( \mathbb{C}^n \wedge \mathbb{C}^n \) which has the orthonormal basis

\[ B_0 = \left\{ \frac{|xy\rangle - |yx\rangle}{\sqrt{2}}, 0 \leq x < y \leq n - 1 \right\}. \quad (2.1) \]

Thus, in order to construct an orthonormal basis of \( S \), it is sufficient to search for symmetric tensors lying in \( S \) and constituting an orthonormal set. Any symmetric tensor in \( S \) can be expressed as

\[ \sum_{0 \leq x \leq n-1} \sum_{0 \leq y \leq n-1} f(x,y) |xy\rangle \quad (2.2) \]

where \( f(x,y) = f(y,x) \) and

\[ \sum_{0 \leq x \leq n-1} \sum_{0 \leq y \leq n-1} f(x,y) \lambda_x^{x+y} = 0, \quad 1 \leq i \leq 2n - 1, \]

which reduces to

\[ \sum_{0 \leq x \leq n-1} \sum_{0 \leq j-x \leq n-1} f(x,j-x) = 0 \forall 0 \leq j \leq 2n - 2. \quad (2.3) \]

Define \( \mathcal{K}_j \) to be the subspace of all symmetric tensors of the form (2.2) where the coefficient function \( f \) is symmetric, has its support in the set \( \{(x, j-x), 0 \leq x \leq n - 1, 0 \leq j - x \leq n - 1\} \) and satisfies (2.3). Simple algebra shows that \( \mathcal{K}_0 = \mathcal{K}_1 = \mathcal{K}_{2n-3} = \mathcal{K}_{2n-2} = 0 \) and

\[ S = \mathcal{H} \wedge \mathcal{H} \oplus \bigoplus_{j=2}^{2n-4} \mathcal{K}_j. \]
We shall now present an orthonormal basis $B_j$ for $\mathcal{K}_j$, $2 \leq j \leq 2n - 4$. This falls into four cases.

**Case 1 :** $2 \leq j \leq n - 1$, $j$ even

$$B_j = \left\{ \frac{1}{\sqrt{j(j+1)}} \left[ \sum_{m=0}^{\frac{j}{2}-1} (|m, j - m\rangle + |j - m, m\rangle) - j \frac{j}{2} \frac{j}{2} \right] \right\} \cup \left\{ \frac{1}{\sqrt{j}} \sum_{m=0}^{\frac{j}{2}-1} e^{\frac{4\pi i m p}{j}} (|m, j - m\rangle + |j - m, m\rangle), \quad 1 \leq p \leq \frac{j}{2} - 1 \right\}.$$

**Case 2 :** $2 \leq j \leq n - 1$, $j$ odd

$$B_j = \left\{ \frac{1}{\sqrt{j+1}} \sum_{m=0}^{\frac{j}{2}-1} e^{\frac{4\pi i m p}{j+1}} (|m, j - m\rangle + |j - m, m\rangle), \quad 1 \leq p \leq \frac{j-1}{2} \right\}.$$

**Case 3 :** $n \leq j \leq 2n - 4$, $j$ even

$$B_j = \left\{ \frac{1}{\sqrt{(2n-2-j)(2n-1-j)}} \left[ \sum_{m=0}^{\frac{2n-2-j}{2}-1} (|j - n + m + 1, n - m - 1\rangle + |n - m - 1, j - n + m + 1\rangle) - (2n - 2 - j) \frac{j}{2} \frac{j}{2} \right] \right\} \cup \left\{ \frac{1}{\sqrt{2n-2-j}} \sum_{m=0}^{\frac{2n-2-j}{2}-1} e^{\frac{4\pi i m p}{2n-2-j}} (|j - n + m + 1, n - m - 1\rangle + |n - m - 1, j - n + m + 1\rangle), \quad 1 \leq p \leq \frac{2n-2-j}{2} - 1 \right\}.$$

**Case 4 :** $n \leq j \leq 2n - 4$, $j$ odd

$$B_j = \left\{ \frac{1}{\sqrt{2n-1-j}} \sum_{m=0}^{\frac{2n-1-j}{2}-1} e^{\frac{4\pi i m p}{2n-1-j}} (|j - n + m + 1, n - m - 1\rangle \right\}.$$
\[ + \left| n - m - 1 \right. \left. j - n + m + 1 \right) \), 1 \leq p \leq \frac{2n - 1 - j}{2} - 1 \right\}

The set \( B_0 \cup \bigcup_{j=2}^{2n-4} B_j \), where \( B_0 \) is given by (2.1) and the remaining \( B_j \)'s are given by the four cases above constitute an orthonormal basis for the maximal completely entangled subspace \( S \).

3 Perfectly Entangled Subspaces

As in Section 1, let \( \mathcal{H}_i \) be a complex Hilbert space of dimension \( d_i \) associated with a finite level quantum system \( A_i \) for each \( i = 1, 2, \ldots, k \). For any subset \( E \subset \{1, 2, \ldots, k\} \) let

\[ \mathcal{H}(E) = \otimes_{i \in E} \mathcal{H}_i \]

\[ d(E) = \prod_{i \in E} d_i \]

so that the Hilbert space \( \mathcal{H} = \mathcal{H}(\{1, 2, \ldots, k\}) \) of the joint system \( A_1 A_2 \ldots A_k \) can be viewed as \( \mathcal{H}(E) \otimes \mathcal{H}(E'), E' \) being the complement of \( E \). For any operator \( X \) on \( \mathcal{H} \) we write

\[ X(E) = Tr_{\mathcal{H}(E')} X \]

where the right hand side denotes the relative trace of \( X \) taken over \( \mathcal{H}(E') \). Then \( X(E) \) is an operator in \( \mathcal{H}(E) \). If \( \rho \) is a state of the system \( A_1 A_2 \ldots A_k \) then \( \rho(E) \) describes the marginal state of the subsystem \( A_{i_1} A_{i_2} \ldots A_{i_r} \) where \( E = \{i_1, i_2, \ldots, i_r\} \).

**Definition 3.1** A nonzero subspace \( S \subset \mathcal{H} \) is said to be perfectly entangled if for any \( E \subset \{1, 2, \ldots, k\} \) such that \( d(E) \leq d(E') \) and any unit vector \( \psi \in S \)
one has
\[
(\langle \psi | \psi \rangle) (E) = \frac{I_E}{d(E)}
\]
where \( I_E \) denotes the identity operator in \( \mathcal{H}(E) \).

For any state \( \rho \), denote by \( S(\rho) \) the von Neumann entropy of \( \rho \). If \( \psi \) is a pure state in \( \mathcal{H} \) then \( S((|\psi \rangle \langle \psi|) (E')) = S((|\psi \rangle \langle \psi|) (E')) \). Thus perfect entanglement of a subspace \( S \) is equivalent to the property that for every unit vector \( \psi \) in \( S \), the pure state \( |\psi \rangle \langle \psi| \) is maximally entangled in every decomposition \( \mathcal{H}(E) \otimes \mathcal{H}(E') \), i.e.,
\[
S((|\psi \rangle \langle \psi|)(E)) = S((|\psi \rangle \langle \psi|)(E')) = \log_2 d(E)
\]
whenever \( d(E) \leq d(E') \). In other words, the marginal states of \( |\psi \rangle \langle \psi| \) in \( \mathcal{H}(E) \) and \( \mathcal{H}(E') \) have the maximum possible von Neumann entropy.

Denote by \( \mathcal{P} \) the class of all perfectly entangled subspaces of \( \mathcal{H} \). It is an interesting problem to construct examples of perfectly entangled subspaces and also compute \( \max_{S \in \mathcal{P}} \dim S \).

Note that a perfectly entangled subspace \( S \) is also completely entangled. Indeed, if \( S \) has a unit product vector \( \psi = u_1 \otimes u_2 \otimes \cdots \otimes u_k \) where each \( u_i \) is a unit vector in \( \mathcal{H}_i \) then \( (|\psi \rangle \langle \psi|)(E) \) is also a pure product state with von Neumann entropy zero. Perfect entanglement of \( S \) implies the stronger property that every unit vector \( \psi \) in \( S \) is indecomposable, i.e., \( \psi \) cannot be factorized as \( \psi_1 \otimes \psi_2 \) where \( \psi_1 \in \mathcal{H}(E), \psi_2 \in \mathcal{H}(E') \) for any proper subset \( E \subset \{1,2,\ldots,k\} \).

**Proposition 3.2** Let \( S \subset \mathcal{H} \) be a subspace and let \( P \) denote the orthogonal projection on \( S \). Then \( S \) is perfectly entangled if and only if, for any proper
subset $E \subset \{1,2,\ldots,k\}$ with $d(E) \leq d(E')$

$$(PXP)(E) = \frac{Tr \ P X}{d(E)} I_E$$

for all operators $X$ on $\mathcal{H}$.

**Proof:** Sufficiency is immediate. To prove necessity, assume that $S$ is perfectly entangled. Let $X$ be any hermitian operator on $\mathcal{H}$. Then by spectral theorem and Definition 3.2 it follows that $(PXP)(E) = \alpha(X)I_E$ where $\alpha(X)$ is a scalar. Equating the traces of both sides we see that $\alpha(X) = d(E)^{-1}TrPX$. If $X$ is arbitrary, then $X$ can be expressed as $X_1 + iX_2$ where $X_1$ and $X_2$ are hermitian and the required result is immediate.

Using the method of constructing single error correcting 5 qudit stabilizer quantum codes in the sense of Gottesman [1], [3] we shall now describe an example of a perfectly entangled $d$-dimensional subspace in $h^\otimes 5$ where $h$ is a $d$-dimensional Hilbert space. To this end we identify $h$ with $L^2(A)$ where $A$ is an abelian group of cardinality $d$ with group operation $+$ and null element 0. Then $h^\otimes 5$ is identified with $L^2(A^5)$. For any $x = (x_0, x_1, x_2, x_3, x_4)$ in $A^5$ denote by $|x\rangle$ the indicator function of the singleton subset $\{x\}$ in $A^5$. Then $\{|x\rangle, x \in A^5\}$ is an orthonormal basis for $h^\otimes 5$. Choose and fix a nondegenerate symmetric bicharacter $\langle , , \rangle$ for the group $A$ satisfying the following:

$$|\langle a, b \rangle| = 1, \langle a, b \rangle = \langle b, a \rangle, \langle a, b + c \rangle = \langle a, b \rangle \langle a, c \rangle \forall a, b, c \in A$$

and $a = 0$ if and only if $\langle a, x \rangle = 1$ for all $x \in A$. Define

$$\langle x, y \rangle = \prod_{i=0}^{4} \langle x_i, y_i \rangle, \ x, y \in A^5.$$

(Note that $\langle x, y \rangle$ denotes the bicharacter evaluated at $x$, $y$ whereas $\langle x|y \rangle$ denotes the scalar product in $\mathcal{H} = L^2(A^5)$.) With these notations we introduce
the unitary Weyl operators $U_a, V_b$ in $\mathcal{H}$ satisfying

$$U_a|x\rangle = |a + x\rangle, \ V_b|x\rangle = \langle b, x | x\rangle, \ x \in A^5.$$  

Then we have the Weyl commutation relations:

$$U_a U_b = U_{a+b}, \ V_a V_b = V_{a+b}, V_b U_a = \langle a, b \rangle U_a V_b$$

for all $a, b \in A^5$. The family $\{d^{-\frac{1}{2}}U_a V_b, a, b \in A^5\}$ is an orthonormal basis for the Hilbert space of all operators $X, Y$ with the scalar product $\langle X|Y \rangle = Tr X^\dagger Y$

Introduce the cyclic permutation $\sigma$ in $A^5$ defined by

$$\sigma((x_0, x_1, x_2, x_3, x_4)) = (x_4, x_0, x_1, x_2, x_3). \hspace{1cm} (3.1)$$

Then $\sigma$ is an automorphism of the product group $A^5$ and

$$\sigma^{-1}((x_0, x_1, x_2, x_3, x_4)) = (x_1, x_2, x_3, x_4, x_0).$$

Define

$$\tau(x) = \sigma^2(x) + \sigma^{-2}(x). \hspace{1cm} (3.2)$$

Let $C \subset A^5$ be the subgroup defined by

$$C = \{x|x_0 + x_1 + x_2 + x_3 + x_4 = 0\}.$$  

Define

$$W_x = \langle x, \sigma^2(x) \rangle U_x V_{\tau(x)}, \ x \in A^5. \hspace{1cm} (3.3)$$

Then the correspondence $x \rightarrow W_x$ is a unitary representation of the subgroup $C$ in $\mathcal{H}$. Define the operator $P_C$ by

$$P_C = d^{-4} \sum_{x \in C} W_x. \hspace{1cm} (3.4)$$
Then $P_C$ is a projection satisfying $Tr P_C = d$. The range of $P_C$ is an example of a stabilizer quantum code in the sense of Gottesman. From the methods of [1] it is also known that $P_C$ is a single error correcting quantum code. The range $R(P_C)$ of $C$ is given by

$$R(P_C) = \{ |\psi\rangle W_\mathbf{x}|\psi\rangle = |\psi\rangle \text{ for all } \mathbf{x} \in C\}.$$ 

Our goal is to establish that $R(P_C)$ is perfectly entangled in $L^2(A)^{\otimes 5}$. To this end we prove a couple of lemmas.

**Lemma 3.3** For any $a, b \in A^5$ the following holds:

$$\langle a | P_C | b \rangle = \begin{cases} 0 & \text{if } \sum_{i=0}^4 (a_i - b_i) \neq 0, \\ d^{-4} \langle a, \sigma^2(a) \rangle \overline{\langle b, \sigma^2(b) \rangle} & \text{otherwise.} \end{cases}$$

**Proof**: We have from (3.1) - (3.4)

$$\langle a | P_C | b \rangle = d^{-4} \sum_{x_0 + x_1 + x_2 + x_3 + x_4 = 0} \langle x, \sigma^2(x) \rangle \langle \tau(x), b \rangle \langle a | x + b \rangle$$

which vanishes if $\sum_{i=0}^4 (a_i - b_i) \neq 0$. Now assume that $\sum_{i=0}^4 (a_i - b_i) = 0$. Then

$$\langle a | P_C | b \rangle = d^{-4} \langle a - b, \sigma^2(a - b) \rangle \overline{\langle b, \sigma^2(b) \rangle}$$

$$= d^{-4} \langle a, \sigma^2(a) \rangle \overline{\langle b, \sigma^2(b) \rangle}.$$ 

**Lemma 3.4** Consider the tensor product Hilbert space

$$L^2(A)^{\otimes 5} = \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4$$

where $\mathcal{H}_i$ is the $i$-th copy of $L^2(A)$. Then for any $\{i, j\} \subset \{0, 1, 2, 3, 4\}$ and $a, b \in A^5$ the operator $(P_C|a\rangle\langle b|P_C)$ ($\{i, j\}$) is a scalar multiple of the identity in $\mathcal{H}_i \otimes \mathcal{H}_j$. 

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Proof : By Lemma 3.2 and the definition of relative trace we have, for any \(x_0, x_1, y_0, y_1 \in A\),

\[
\langle x_0, x_1 | (P_C | a \rangle \langle b | P_C) (\{0, 1\}) | y_0, y_1 \rangle \\
= \sum_{x_2, x_3, x_4 \in A} \langle x_0, x_1, x_2, x_3, x_4 | P_C | a \rangle \langle b | P_C | y_0, y_1, x_2, x_3, x_4 \rangle \\
= d^{-8} \sum_{x_2 + x_3 + x_4 = \sum a_i - x_0 - x_1} \sum_{x_2 + x_3 + x_4 = \sum b_i - y_0 - y_1} \langle x, \sigma^2(x) \rangle \langle a, \sigma^2(a) \rangle \langle b, \sigma^2(b) \rangle \\
\times \langle y_0, y_1, x_2, x_3, x_4, \sigma^2(y_0, y_1, x_2, x_3, x_4) \rangle
\]

The right hand side vanishes if \(\sum (a_i - b_i) \neq x_0 + x_1 - y_0 - y_1\). Now suppose that \(\sum (a_i - b_i) = x_0 + x_1 - y_0 - y_1\). Then the right hand side is equal to

\[
d^{-8} \langle a, \sigma^2(a) \rangle \langle b, \sigma^2(b) \rangle \langle \sum a_i - x_0 - x_1, x_0 + x_1 - y_0 - y_1 \rangle \times \sum_{x_2, x_1 \in A} \langle x_2, y_1 - x_1 \rangle \langle x_4, y_0 - x_0 \rangle \\
= \begin{cases} 
0 & \text{if } x_0 \neq y_0 \text{ or } x_1 \neq y_1, \\
d^{-6} \langle a, \sigma^2(a) \rangle \langle b, \sigma^2(b) \rangle & \text{otherwise.}
\end{cases}
\]

This proves the lemma when \(i = 0, j = 1\). A similar (but tedious) algebra shows that the lemma holds when \(i = 0, j = 2\).

The cyclic permutation \(\sigma\) of the basis \(\{ |x\rangle, x \in A^5 \}\) induces a unitary operator \(U_\sigma\) in \(A^5\). Since \(\sigma\) leaves \(C\) invariant it follows that \(U_\sigma P_C = P_C U_\sigma\) and therefore

\[
U_\sigma P_C | a \rangle \langle b | P_C U_\sigma^{-1} = P_C | \sigma(a) \rangle \langle \sigma(b) | P_C,
\]

which, in turn, imples that

\[
\langle x_1, x_2 | (P_C | a \rangle \langle b | P_C) (\{1, 2\}) | y_1, y_2 \rangle \\
= \langle x_1, x_2 | P_C | \sigma^{-1}(a) \rangle \langle \sigma^{-1}(b) | P_C \rangle (\{0, 1\}) | y_1, y_2 \rangle.
\]

By what has been already proved the lemma follows for \(i = 1, j = 2\). A similar covariance argument proves the lemma for all pairs \(\{i, j\}\).
Theorem 3.4 The range of $P_C$ is a perfectly entangled subspace of $L^2(A^{\otimes 5})$ and $\dim P_C = \#A$.

Proof : Immediate from Lemma 3.3 and the fact that every operator in $L^2(A^{\otimes 5})$ is a linear combination of operators of the form $|a\rangle\langle b|$ as $a, b$ vary in $A^5$.

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