The Continuum Pólya-Like Random Walk

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Abstract

The Pólya urn scheme is a discrete-time process concerning the addition and removal of colored balls. There is a known embedding of it in continuous-time, called the Pólya process. We introduce a generalization of this stochastic model, where the initial values and the entries of the transition matrix (corresponding to additions or removals) are not necessarily fixed integer values as in the standard Pólya process. In one of our scenarios, we even allow the entries of the matrix to be random variables. As a result, we no longer have a combinatorial model of “balls in an urn”, but a broader interpretation as a random walk, in a possibly high number of dimensions. In this paper, we study several parametric classes of these generalized continuum Pólya-like random walks.

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1 Introduction

We introduce a generalization of the Pólya process on c colors by embedding each element of the replacement matrix into \( \mathbb{R} \), or by even treating the elements of the replacement matrix as random variables themselves. We are no longer restricted to thinking about integer-valued quantities (such as counts of balls in an urn), but rather we can now handle real-valued quantities, such as a random walk in c dimensions, and we can view the replacement matrix itself as random.

1.1 Background

We first review the standard Pólya process. The Pólya urn scheme is a process underlying an urn that evolves in discrete time. The urn contains balls of up to c colors. The colors are numbered, say the set of colors is \( C = \{1, \ldots, c\} \). At each discrete epoch in time, a ball is sampled from the urn. It is then put back in the urn, together with a number of other balls in various colors. If the sampled ball has color \( i \in C \), then we add to the urn \( A_{i,j} \) balls of color \( j \), for \( j \in C \). If \( A_{i,j} \) is negative, we remove \( A_{i,j} \) balls of color \( j \). It is customary to represent these dynamics by a replacement matrix

\[
A = \begin{pmatrix}
A_{1,1} & A_{1,2} & \cdots & A_{1,c} \\
A_{2,1} & A_{2,2} & \cdots & A_{2,c} \\
\vdots & \vdots & \ddots & \vdots \\
A_{c,1} & A_{c,2} & \cdots & A_{c,c}
\end{pmatrix}
\]

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In the standard Pólya process, each element of $A$ is an integer. (We will extend this view to $A$ having real valued entries in Section 1.2.) It is usually assumed that the urn is “tenable,” in the sense that the selection of balls can be continued ad infinitum, no matter which stochastic path is followed, i.e., the process will never get stuck.

The Pólya process is an embedding of the Pólya urn scheme in real time. Embedding in real time (poissonization) was suggested by Kac [7] as a general methodology for understanding discrete probability problems. In the context of urns, poissonization was introduced by Athreya and Karlin [1] to understand discrete-time Pólya urn schemes. Poissonization was thus meant as a transform. The inverse transform—to translate results in the continuous domain back to results in the discrete domain—is fraught with difficulty.

Some authors developed interest in the continuous-time Pólya process for its own sake (see [2, 3, 4, 10]). In the Pólya process, each ball carries an internal clock that rings in Exp(1) time (a random amount of time, according to an exponential random variable with mean 1), independently of the behavior of all other clocks. Whenever a clock rings, it is instantaneously reset to ring again in Exp(1) time (independently of the clocks on all the other balls). In other words, each ball has the ability to generate a new Poisson process with intensity 1. When the clock associated with a ball of color $i$ rings, the addition and removal of balls corresponds to picking a colored ball from a Pólya urn and using the $i$th row of the replacement matrix $A$ to determine which balls to add or remove. All replacements are assumed to occur instantaneously, and each new ball is given an independent clock that rings in Exp(1) time.

1.2 Generalized Pólya process

In this paper, we introduce a generalized Pólya process that can be viewed as a second layer of embedding of the Pólya urn scheme. The starting numbers and the (possibly random) numbers added are in $\mathbb{R}$ (no longer restricted to $\mathbb{Z}$), so the numbers are no longer counts of balls. A natural (more general) interpretation is a random walk in $c$ dimensions. At time $t$, the current position of the walk in $\mathbb{R}^c$ is a $c$-dimensional column vector $X(t) := (X_1(t), \ldots, X_c(t))^T$.

At any point $t$ in time, the next renewal occurs after a random amount of time, according to a master clock, the waiting time of which follows an exponential random variable with mean $\sum_{i \in C} X_i(t)$. Given that a renewal occurs, the probability that the renewal corresponds to color $i$ is $X_i(t)/\sum_{j \in C} X_j(t)$. In other words, the probability of a renewal of type $i$ is proportional to the amount of quantity $i$ present when the transition occurs. In such a case, the $i$th row of the matrix $A$ dictates the $c$-dimensional direction in which to move, i.e., we add $A_{i,j}$ units to $X_j(t)$, for each $j \in C$. Thus, it is appropriate then to call $A$ the navigation matrix instead of the replacement matrix.

To avoid trivialities, we only consider starting values $X_j(0)$ and matrices $A_{i,j}$ in which the walk is tenable, i.e., the walk always avoids the origin, and each of the coordinates is always nonnegative. We call the row vector $X(t) = (X_1(t), \ldots, X_c(t))$ a continuum Pólya-like process or random walk.

This stochastic process is not a Poisson process, because the rate of the process itself is random, i.e., the rate of replacement is not simply a function of time. The current rate of replacement depends on the number of replacements that have taken place beforehand.

Such a process could model any general physical process (e.g., interaction of gasses), in which the time until the next change in the process is exponential, and the parameter of which remains always in proportion to the current quantities involved in the process.
1.3 Organization of the paper

The paper is organized as follows. Section 2 specifies the probability model for the stochastic process. In Section 3, we derive a fundamental partial differential equation that governs the behavior of the generalized Pólya process. In Section 4, we derive a functional equation for the moment generating function of the position of the random walk at time $t$. In Section 5, we show how several classical probability models can be generalized with this approach; in all of these classical cases, the partial differential equations from Section 4 can be solved. Section 6 is perhaps the most novel part of the paper, because we solve the partial differential equations for a balanced upper-triangular case, a case that proved difficult in the usual urn setting.

2 The probabilistic model

In order to be able to establish the partial differential equations of Section 3, we need to precisely describe the number of renewals, say $N(t, \Delta t)$, that occur in the processes during the interval $(t, t + \Delta t]$. We use the notation $C_i$ to indicate that exactly one renewal occurs in the interval $(t, t + \Delta t]$ and that renewal is induced by a renewal in the $i$th coordinate. To study the behavior of $X(t + s)$, for $s > 0$, we condition on the vector $X(t)$.

Lemma 2.1. We have

$$\mathbb{P}(N(t, \Delta t) = 0 \mid X(t)) = \exp\left(-\Delta t \sum_{j \in C} X_j(t)\right),$$

$$\mathbb{P}(\{N(t, \Delta t) = 1\} \cap C_i \mid X(t)) = \Delta t X_i(t) \exp\left(-\Delta t \sum_{j \in C} X_j(t)\right) + O\left((\Delta t)^2\right),$$

$$\mathbb{P}(N(t, \Delta t) \geq 2 \mid X(t)) = O\left((\Delta t)^2\right),$$

as $\Delta t \to 0$.

Proof. Given the values $X(t)$, the master clock does not ring during the interval $(t, t + \Delta t]$ with probability

$$\mathbb{P}(N(t, \Delta t) = 0 \mid X(t)) = \prod_{j \in C} \left(\frac{(\Delta t)^0 e^{\Delta t}}{0!}\right)^{X_j(t)} = \exp\left(-\Delta t \sum_{j \in C} X_j(t)\right).$$

For fixed $t$, once we are given the values $X(t)$, the next ring of the master clock, after time $t$, occurs at a random time $x$ (with $t < x$) with probability density function

$$\left(\sum_{j \in C} X_j(t)\right) \exp\left(-(x - t) \sum_{j \in C} X_j(t)\right).$$

When such a clock ring occurs at time $x$, it is a ring of type $i$ with probability $X_i(t)/\sum_{j \in C} X_j(t)$. Then, for $t < x \leq t + \Delta t$, there are no additional subsequent rings before time $t + \Delta t$ with probability

$$\prod_{j \in C} \left(\frac{(t + \Delta t - x)^0 e^{t + \Delta t - x}}{0!}\right)^{X_j(t) + A_{i,j}} = \exp\left(-(t + \Delta t - x) \sum_{j \in C} (X_j(t) + A_{i,j})\right).$$
Putting all of this together, we see that $\mathbb{P}((N(t, \Delta t) = 1) \cap C_i \mid X(t))$ is equal to

$$
\int_t^{t+\Delta t} \left( \sum_{j \in C} X_j(t) \right) \exp\left(-x - t \sum_{j \in C} X_j(t) \left( \frac{X_i(t)}{\sum_{j \in C} X_j(t)} \right) \right) \times \exp\left(-(t + \Delta t - x) \sum_{j \in C} (X_j(t) + A_{i,j}) \right) dx.
$$

This simplifies to

$$
\int_t^{t+\Delta t} X_i(t) \exp\left(-\Delta t \sum_{j \in C} X_j(t) - (t + \Delta t - x) \sum_{j \in C} A_{i,j} \right) dx = \frac{X_i(t) \exp\left(-\Delta t \sum_{j \in C} X_j(t) \right) \left( \Delta t \sum_{j \in C} A_{i,j} + O((\Delta t)^2) \right)}{\sum_{j \in C} A_{i,j}}
$$

$$
= \Delta t X_i(t) \exp\left(-\Delta t \sum_{j \in C} X_j(t) \right) + O((\Delta t)^2).
$$

Finally, we get

$$
\mathbb{P}(N(t, \Delta t) \geq 2 \mid X(t)) = 1 - \mathbb{P}(N(t, \Delta t) = 0 \mid X(t)) - \sum_{i \in C} \mathbb{P}((N(t, \Delta t) = 1) \cap C_i \mid X(t))
$$

$$
= 1 - \exp\left(-\Delta t \sum_{j \in C} X_j(t) \right) - \sum_{i \in C} \Delta t X_i(t) \exp\left(-\Delta t \sum_{j \in C} X_j(t) \right) + O((\Delta t)^2)
$$

$$
= O((\Delta t)^2),
$$

and this completes the proof of the lemma.

3 The fundamental partial differential equation

We formulate here a partial differential equation for the continuum Pólya-like random walk. We set $u = (u_j)_{j \in C}$ to mark our colors $C = \{1, \ldots, c\}$. Let

$$
\phi(t, u) = \mathbb{E}\left[ \exp\left( \sum_{j \in C} u_j X_j(t) \right) \right]
$$

be the joint moment generating function of the coordinates of the random walk $X(t)$. For $i \in C$, let

$$
\psi_i(u) = \mathbb{E}\left[ \exp\left( \sum_{j \in C} u_j A_{i,j} \right) \right]
$$

be the joint moment generating function of the random variables on row $i$ of the navigation matrix $A$.

**Theorem 3.1.** The joint moment generating function $\phi(t, u)$ satisfies

$$
\frac{\partial \phi}{\partial t} + \sum_{i \in C} (1 - \psi_i) \frac{\partial \phi}{\partial u_i} = 0.
$$
Proof. We use conditional expectation to calculate

$$E := \mathbb{E}\left[\exp\left(\sum_{j \in C} u_j X_j(t + \Delta t)\right) \mid X(t)\right],$$

i.e., the expectation conditioned on the status $X(t)$ at time $t$. We do this by first conditioning on whether there are 0, 1, or at least 2 renewals in the interval from $t$ to $\Delta t$ and, if there is exactly 1 renewal, on the color of the chosen ball. The probabilities of these events are calculated in Lemma 2.1. As in Section 2, let

$$\phi(t) = \mathbb{E}\left[\exp\left(\sum_{j \in C} u_j X_j(t + \Delta t)\right) \mid X(t)\right].$$

We obtain

$$E = \mathbb{E}\left[\exp\left(\sum_{j \in C} u_j X_j(t + \Delta t)\right) \mid X(t)\right] \times \mathbb{P}(N(t, \Delta t) = 0)$$

$$+ \sum_{i \in C} \mathbb{E}\left[\exp\left(\sum_{j \in C} u_j X_j(t + \Delta t)\right) \mid X(t), (N(t, \Delta t) = 1) \cap C_i\right] \times \mathbb{P}(N(t, \Delta t) = 1, C_i)$$

$$+ \mathbb{E}\left[\exp\left(\sum_{j \in C} u_j X_j(t + \Delta t)\right) \mid X(t)\right] \times \mathbb{P}(N(t, \Delta t) \geq 2).$$

In the last equality, we have allowed for the possibility that the $A_{i,j}$ themselves (i.e., the entries of the navigation matrix) may be random variables, since we need this generality in Section 5.1. We only assume that these matrix entries are independent of the current state of the process $X(t)$.

Collecting all these facts and utilizing Lemma 2.1 we obtain

$$E = \exp\left(\sum_{j \in C} u_j X_j(t)\right) \exp\left(-\Delta t \sum_{i \in C} X_i(t)\right)$$

$$+ \sum_{i \in C} \mathbb{E}\left[\exp\left(\sum_{j \in C} u_j A_{i,j}\right)\right] \exp\left(\sum_{j \in C} u_j X_j(t)\right) \Delta t X_i(t) \exp\left(-\Delta t \sum_{j \in C} X_j(t)\right) + O((\Delta t)^2).$$

A local expansion of the exponentials gives

$$E = \exp\left(\sum_{j \in C} u_j X_j(t)\right) \left(1 - \Delta t \sum_{i \in C} X_i(t) + O\left((\Delta t)^2\right)\right)$$

$$+ \Delta t (1 + O(\Delta t)) \exp\left(\sum_{j \in C} u_j X_j(t)\right) \sum_{i \in C} X_i(t) \mathbb{E}\left[\exp\left(\sum_{j \in C} u_j A_{i,j}\right)\right] + O((\Delta t)^2).$$

Taking expectations over $X(t)$ yields

$$\phi(t + \Delta t, \mathbf{u}) = \phi(t, \mathbf{u}) - \Delta t \sum_{i \in C} \mathbb{E}\left[X_i(t) \exp\left(\sum_{j \in C} u_j X_j(t)\right)\right]$$

$$+ \Delta t (1 + O(\Delta t)) \mathbb{E}\left[\exp\left(\sum_{j \in C} u_j X_j(t)\right) \sum_{i \in C} X_i(t)\right] \psi_i(\mathbf{u}) + O((\Delta t)^2).$$

So, we can now write the limiting form

$$\frac{\partial \phi(t, \mathbf{u})}{\partial t} = \lim_{\Delta t \to 0} \frac{\phi(t + \Delta t, \mathbf{u}) - \phi(t, \mathbf{u})}{\Delta t}$$

$$= - \sum_{i \in C} \mathbb{E}\left[X_i(t) \exp\left(\sum_{j \in C} u_j X_j(t)\right)\right] (1 - \psi_i(\mathbf{u}))$$

$$= - \sum_{i \in C} \frac{\partial \phi(t, \mathbf{u})}{\partial u_i} (1 - \psi_i(\mathbf{u})).$$

\[\square\]
Remark 3.2. The proof of Theorem 3.1 is a generalization of the proof in Balaji and Mahmoud [2]. However, it needed some new techniques. In [2], there is a conditional argument that uses a sum on the number of balls of a color, given that number. Of course, the number of balls is an integer and such a sum can be carried out. Here, the counterpart of a number of balls of a certain color is a coordinate, which is continuous, and such a conditional sum cannot be written.

4 Moments

We derive functional equations for the moments. For example, we derive a functional equation for the mean position (see Theorem 4.1 at the end of this section) by differentiating (with respect to $u_j$) on both sides of the partial differential equation in Theorem 3.1. This yields

$$\frac{\partial}{\partial u_j} \left( \frac{\partial \phi}{\partial t} \right) + \sum_{i \in C} (1 - \psi_i) \frac{\partial^2 \phi}{\partial u_j \partial u_i} - \sum_{i \in C} \frac{\partial \psi_i}{\partial u_j} \times \frac{\partial \phi}{\partial u_i} = 0.$$  

Evaluation of the summands at $u_i = 0$, for all $i \in C$, yields

$$\frac{\partial}{\partial u_j} \left( \frac{\partial \phi}{\partial t} \right) = \frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial u_j} \right) \bigg|_{u=0} = \frac{\partial}{\partial t} \mathbb{E}[X_j(t) \exp \left( \sum_{i \in C} u_i X_i(t) \right)] \bigg|_{u=0} = \frac{d}{dt} \mathbb{E}[X_j(t)].$$

We also have

$$\sum_{i \in C} (1 - \psi_i) \frac{\partial^2 \phi}{\partial u_j \partial u_i} \bigg|_{u=0} = 0,$$

as $(1 - \psi_i)\big|_{u=0} = 0$, and

$$\sum_{i \in C} \frac{\partial \psi_i}{\partial u_j} \times \frac{\partial \phi}{\partial u_i} \bigg|_{u=0} = \sum_{i \in C} \mathbb{E}[A_{i,j}] \mathbb{E}[X_i(t)].$$

Hence, for the $j$th coordinate, we get the ordinary differential equation

$$\frac{d}{dt} \mathbb{E}[X_j(t)] = \sum_{i \in C} \mathbb{E}[A_{i,j}] \mathbb{E}[X_i(t)].$$

Putting the differential equations, for $j \in C$, together in matrix form we get the functional equation

$$\frac{d}{dt} \mathbb{E}[X(t)] = \mathbb{E}[A^T]X(t),$$

where $A^T$ is the transpose of the navigation matrix $A$. This first-order functional equation has a standard solution; this yields the following theorem.

**Theorem 4.1.** Let $A$ be the navigation matrix of a continuum Pólya-like random walk. At time $t$, the expected value of the coordinates of the walk are

$$\mathbb{E}[X(t)] = e^{\mathbb{E}[A^T]t} X(0).$$

Note that, for a matrix $M$, we have used the notation $e^M = \sum_{n=0}^{\infty} M^n/n!$, and we can compute this by using the Jordan form of $M$. 
5 Illustrative examples

We give here some examples. Some of them resemble and extend standard Pólya processes to the random walk counterpart. Some have no solved equivalent in the Pólya world (neither the discrete- or continuous-time versions).

5.1 Pólya–Eggenberger-like random walk

Suppose the navigation matrix $A$ is in the diagonal matrix form

$$
\begin{pmatrix}
A_1 & 0 & \cdots & 0 \\
0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_c
\end{pmatrix},
$$

where $A_i$ is a nonnegative random variable. Let us focus on the $i$th component; we set $u = u_i$ and $u_j = 0$ for $j \neq i$. Let $\psi_i(u) = \mathbb{E}[e^{uA_i}]$ be the moment generating function of $A_i$, and let $\phi_i(t, u) = \mathbb{E}[e^{uX_i(t)}]$ be the moment generating function of $X_i(t)$. The partial differential equation simplifies to

$$
\frac{\partial \phi_i(t, u)}{\partial t} + (1 - \psi_i(u)) \frac{\partial \phi_i(t, u)}{\partial u} = 0.
$$

This equation can be solved for several standard distributions of the $A_i$. Two examples are discussed below.

**Component-wise almost surely constant**

Suppose $A_i = \alpha_i \in \mathbb{R}^+$ almost surely. This gives rise to

$$
\phi_i(t, u) = \left(1 - e^{\alpha_i t}(1 - e^{-\alpha_i u})\right)^{-X_i(0)/\alpha_i}.
$$

We set $u = se^{-\alpha_i t}$ and take the limit, to obtain

$$
\lim_{t \to \infty} \mathbb{E}\left[\exp\left(\frac{sX_i(t)}{e^{\alpha_i t}}\right)\right] = \lim_{t \to \infty} \phi_i\left(t, \frac{s}{e^{\alpha_i t}}\right) = (1 - \alpha_i s)^{-X_i(0)/\alpha_i}.
$$

The latter moment generating function is that of a Gamma($X_i(0)/\alpha_i$, $\alpha_i$) random variable. That is, we have

$$
\frac{X_i(t)}{e^{\alpha_i t}} \overset{D}{\to} \text{Gamma}\left(\frac{X_i(0)}{\alpha_i}, \alpha_i\right).
$$

Note that the displacements along the $i$th coordinate affect only changes in that direction. In other words, the limit multivariate distribution has independent marginals, with the $i$th marginal having the latter gamma limit distribution. Also observe that this random walk has a very long memory. It never forgets where it starts. Even the limit is influenced by the initial position vector, which comes in as a parameter in the joint limit distribution.

**Component-wise exponential distribution**

We illustrate with another instance, in which the navigation matrix itself has random elements. Suppose the Pólya–Eggenberger-like random walk operates under exponentially distributed displacements. That is, the $A_i$ (for $i \in C$) are independent Exp(1) random variables. Thus, for $u < 1$,
we have \( \psi_i(u) = 1/(1 - u) \) and we can solve \([5.1]\). In this case, we have

\[
\phi_i(t, u) = e^{-W(-ue^{t-u})},
\]

where \( W(\cdot) \) is Lambert’s \( W \) function (defined implicitly as any complex solution of \( z = W(z) e^{W(z)} \)). Note that \( T(z) := -W(-z) \) is called the tree function and appears in the enumeration of trees.

Set \( u = se^{-t} \), and take the limit

\[
\lim_{t \to \infty} E\left[ \exp\left( sX_i(t) \frac{e^{t-u}}{e^t} \right) \right] = \lim_{t \to \infty} \phi_i\left( t, \frac{s}{e^t} \right) = \frac{T(s)}{s}.
\]

The right-hand side in the latter equation is the moment generating function of a Lambert random variable \( W^* \). Thus, in the limit we have

\[
e^{-t} \begin{pmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_c(t) \end{pmatrix} \xrightarrow{D} \begin{pmatrix} W_1^* \\ W_2^* \\ \vdots \\ W_c^* \end{pmatrix},
\]

and the components \( W_i^* \) of the limiting vector are independent Lambert random variables, each of which is distributed like \( W^* \).

### 5.2 Ehrenfest-like random walk

In this example we take the Ehrenfest navigation matrix

\[
A = \begin{pmatrix} -\gamma & \gamma \\ \gamma & -\gamma \end{pmatrix}
\]

for some positive \( \gamma \). This Ehrenfest-like random walk is in two dimensions. Let us call the two coordinates of the walk \( X(t) \) and \( Y(t) \), i.e., in the previously used notation, \( X(t) = (X(t), Y(t))^T \).

Note that the two vectors for the choice of movement are in opposite directions and aligned along the 45-degree line

\[
X(t) + Y(t) = \lambda. \tag{5.2}
\]

for some intercept \( \lambda \). Thus, the movement is constrained to a linear subspace. For this walk to be tenable, both \( X(0)/\gamma \) and \( Y(0)/\gamma \) have to be nonnegative integers (or alternatively, \( \lambda/\gamma \) and \( X(0)/\lambda \) must be a positive integer). With \( u \) and \( v \) being variables of the moment generating function, the equation to solve is

\[
\frac{\partial \phi(t; u, v)}{\partial t} + (1 - e^{-\gamma u + \gamma v}) \frac{\partial \phi(t; u, v)}{\partial u} + (1 - e^{\gamma u - \gamma v}) \frac{\partial \phi(t; u, v)}{\partial v} = 0.
\]

Set \( v = 0 \) and \( \eta(t, u) := \phi(t; u, 0) \), and note that \( \eta(t, u) = E[e^{uX(t)}] \) is the moment generating function of \( X(t) \). We can rewrite the partial differential equation as

\[
\frac{\partial \eta(t, u)}{\partial t} + (1 - e^{-\gamma u}) \frac{\partial \eta(t, u)}{\partial u} + (1 - e^{\gamma u}) E\left[ Y(t) e^{uX(t)} \right] = 0.
\]

Using the invariant in equation \([5.2]\), we write the latter equation as

\[
\frac{\partial \eta(t, u)}{\partial t} + (1 - e^{-\gamma u}) \frac{\partial \eta(t, u)}{\partial u} + (1 - e^{\gamma u}) E\left[ (\lambda - X(t)) e^{uX(t)} \right] = 0.
\]
We thus have the simplified equation
\[
\frac{\partial \eta(t,u)}{\partial t} + (e^{\gamma u} - e^{-\gamma u}) \frac{\partial \eta(t,u)}{\partial u} + \lambda(1 - e^{\gamma u}) \eta(t,u) = 0.
\]
This equation has the solution
\[
\eta(t,u) = \frac{1}{(1 - e^{-\gamma t - \gamma u} + e^{\gamma u} + e^{-2\gamma t})^2} \left(1 - e^{-2\gamma t + \gamma u} + e^{\gamma u} + e^{-2\gamma t}\right)^{\lambda/\gamma} X(0).
\]
As \( t \to \infty \) we have the limit
\[
\lim_{t \to \infty} \eta(t,u) = \left(\frac{1 + e^{\gamma u}}{2}\right)^{\lambda/\gamma}.
\]
Recall that the tenability requires that \( \lambda/\gamma \) is a positive integer. Therefore, the limit of the moment generating function is that of Bin(\( \lambda/\gamma \), \( 1/2 \)), a binomial random variable that counts the number of successes in \( \lambda/\gamma \) independent, identically distributed trials, with rate of success \( 1/2 \) per trial.

Unlike the Pólya–Eggenberger-like random walk, the Ehrenfest-like random walk is not much affected by where it starts in the first quadrant of the \( XY \)-plane. At any \( t \), the exact distribution does have \( X(0) \) in it; however, its influence is attenuated exponentially fast in time, and in the limit it is completely obliterated.

### 5.3 Walking along a 45-degree hill

For \( \gamma \in \mathbb{R}^+ \), the navigation matrix
\[
A = \begin{pmatrix} -\gamma & -\gamma \\ \gamma & \gamma \end{pmatrix}
\]
takes a walk along an oblique line, like climbing a 45-degree hill. That walk will remain tenable so long as \( Y(0) > X(0) \). Even if the navigator walks all the way down to the bottom of the hill (hitting the \( Y \)-axis at a positive point), the probability is 1 that he/she comes back up along the 45-degree line, staying in the first quadrant. In this walk, we always add or subtract increments in the two dimensions that are in the same amount; the difference \( Y(t) - X(t) = Y(0) - X(0) = \lambda > 0 \) remains the same at all times.

As with the Ehrenfest-like random walk of the previous section, specializing the partial differential equation of Theorem 3.1 to this walk along a 45-degree hill, we again use \( v = 0 \) and \( \eta(t,u) := \phi(t,u,0) \). We note that
\[
\frac{\partial \eta(t,u)}{\partial t} + (1 - e^{-\gamma u}) \frac{\partial \eta(t,u)}{\partial u} + (1 - e^{\gamma u}) \left( \frac{\partial \eta(t,u)}{\partial u} + \lambda \eta(t,u) \right) = 0.
\]
Rearranging, we get
\[
\frac{\partial \eta(t,u)}{\partial t} + (2 - e^{\gamma u} - e^{-\gamma u}) \frac{\partial \eta(t,u)}{\partial u} + \lambda(1 - e^{\gamma u}) \eta(t,u) = 0.
\]
This differential equation has the solution
\[
\eta(t,u) = \frac{1}{(1 - e^{\gamma t(e^{\gamma u} - 1)})^\lambda/\gamma} \left(1 - e^{\gamma t(e^{\gamma u} - 1)}\right)^{X(0)/\gamma} \\
\times \left(1 - \gamma t(e^{\gamma u} - 1)\right)^{Y(0)/\gamma}.
\]

\[\tag{5.3}\]

From the exact moment generating function we get the exact mean and variance for the coordinates by taking derivatives at \( u = 0 \); these quantities are

\[
\begin{align*}
E[X(t)] &= \lambda \gamma t + X(0), \\
E[Y(t)] &= \lambda \gamma t + Y(0), \\
\text{Var}(X(t)) &= \text{Var}(Y(t)) = \lambda \gamma t^2 + (2X(0) + \lambda)\gamma^2 t.
\end{align*}
\]

Next, we put \( u = s/t \) in (5.3) and use the local expansion

\[
e^\gamma u = e^{\gamma s/t} = 1 + \frac{\gamma s}{t} + O\left(\frac{1}{t^2}\right).
\]

We obtain

\[
E[e^{sX(t)/t}] = \frac{1}{(1 - \gamma^2 s + O(1/t))^{\lambda/\gamma}} \left(\frac{1 - \gamma^2 s + O(1/t)}{1 - \gamma^2 s + O(1/t)}\right)^{X(0)/\gamma}
\]

which implies

\[
\lim_{t \to \infty} E[e^{sX(t)/t}] = \frac{1}{(1 - \gamma^2 s)^{\lambda/\gamma}}.
\]

The limiting moment generating function is that of a Gamma\((\lambda/\gamma, \gamma^2)\) random variable, i.e., \(X(t)/t\) converges in distribution to a Gamma\((\lambda/\gamma, \gamma^2)\) random variable. We note that \(Y(t)\) has a similar behavior.

### 6 Walking according to a balanced triangular scheme

Next, we consider a walk \( \mathbf{X}(t) = (X(t), Y(t))^T \) following a balanced triangular scheme with the transition matrix

\[
\mathbf{A} = \begin{pmatrix} \alpha & \delta - \alpha \\ 0 & \delta \end{pmatrix},
\]

where \( \alpha < \delta \) are numbers in \( \mathbb{R}^+ \). We are excluding the case \( \alpha = \delta \), as it has already been handled as a Pólya–Eggenberger like random walk (Section 5.1).

The case has some historical significance. Pólya urn models came about in the first decades of the 20th Century. Soon thereafter, a theory was developed for many types of urns. However, the triangular flavor remained defiant until very recently. The triangular case has been handled in [6] and limit distributions have been characterized. Alternative characterizations are given in [5, 8, 11].

**Theorem 6.1.** Suppose we have a balanced triangular scheme as described above. Then the moment generating function is

\[
\phi(t; u, v) = E[e^{uX(t) + vY(t)}] = e^{-X(0)/t} \left( e^{-\alpha u} - e^{-\alpha v} + (e^{-\delta v} - 1 + e^{-\delta t})^{\alpha/\delta} \right)^{-X(0)/\alpha} \\
\times e^{-Y(0)/t} \left( e^{-\delta v} - 1 + e^{-\delta t} \right)^{-Y(0)/\delta}.
\]

From this result we can compute the moments in a straightforward way. See Section 6.6 for details.

To prove Theorem 6.1 we solve the corresponding partial differential equation of Theorem 3.1 by using the method of characteristics. To improve readability, the proof is split up into several sections.
6.1 Characteristic curves

We derive the characteristic curves belonging to the partial differential equation by establishing the following lemma.

**Lemma 6.2.** The functions

\[ x_c = e^{\alpha t}(e^{-\alpha u} - e^{-\alpha v}) \quad \text{and} \quad y_c = e^{\delta t}(e^{-\delta v} - 1) \]

are characteristic curves for the partial differential equation that corresponds to the balanced triangular scheme specified above.

To prove this lemma, consider the more general matrix

\[ A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \]

We want to find a solution \( \phi(t; u, v) \) of the partial differential equation

\[ \frac{\partial \phi}{\partial t} + \left(1 - e^{\alpha u}e^{\beta v}\right) \frac{\partial \phi}{\partial u} + \left(1 - e^{\gamma u}e^{\delta v}\right) \frac{\partial \phi}{\partial v} = 0. \]

**Parameterizing**

As a first step in the method of characteristics, we introduce a new parameter \( s \). We set

\[ \frac{dt}{ds} = 1, \]

\[ \frac{du}{ds} = 1 - e^{\alpha u}e^{\beta v}, \]

and

\[ \frac{dv}{ds} = 1 - e^{\gamma u}e^{\delta v}. \]

By using the chain rule and inserting (6.1), we obtain

\[ \frac{d}{ds} \phi(t(s); u(s), v(s)) = 0, \]

so our function is constant along the characteristics.

From (6.1t) it follows that \( s = t + t_0 \). We choose \( t_0 = 0 \), thus \( s = t \), which we use from now on.

**Specialization to the upper triangular case**

Since we are interested in the upper triangular case, we now specialize to \( \beta = \delta - \alpha \) and \( \gamma = 0 \). Thus, (6.1v) becomes

\[ \frac{dv}{dt} = 1 - e^{\delta v}, \]

and we can easily solve it by the separation of variables. This gives

\[ v - \frac{1}{\delta} \log \left(1 - e^{\delta v}\right) = t - y_0, \]
for some (constant) initial condition $y_0$. This is equivalent to
\[
e^{\delta(t-y_0)} = \frac{e^{\delta v}}{1-e^{\delta v}} = \frac{1}{e^{-\delta v} - 1},
\]
\[
e^{\delta y_0} = e^{\delta(t-v)} \left(1 - e^{\delta v}\right) = e^{\delta t} \left(e^{-\delta v} - 1\right),
\]
and to
\[
e^{\delta v} = \frac{1}{e^{-\delta(t-y_0)} + 1}
\]
as well. In particular, $y_c = e^{\delta y_0}$ (together with Equation (6.2b)) is the second of our two characteristic curves of Lemma 6.2.

It remains to derive the first characteristic curve. Inserting (6.2c) into (6.1u) yields the differential equation
\[
\frac{du}{dt} = 1 - \frac{e^{\alpha u}}{(e^{-\delta(t-y_0)} + 1)^{\beta/\delta}}.
\]
With the help of a symbolic algebra system, we can solve this; we obtain
\[
\alpha x_0 = e^{\alpha(t-u)} - \frac{1}{\alpha + \beta} e^{\alpha t} \left(e^{\delta(t-y_0)}\right)^{\beta/\delta} H,
\]
for some (constant) initial condition $x_0$.

Solving for this $x_0$ gives
\[
x_0 = \frac{1}{\alpha + \beta} e^{\alpha(t-u)} - \frac{1}{\alpha + \frac{\alpha + \beta}{\delta}} e^{\alpha t} \left(e^{\delta(t-y_0)}\right)^{\beta/\delta} H.
\]

Balancing our triangular scheme

At this point we consider the balanced triangular case and specialize to $\beta = \delta - \alpha$. The characteristic curve (6.3) is now
\[
x_0 = \frac{1}{\alpha} e^{\alpha(t-u)} - \frac{1}{\delta} e^{\alpha t} \left(-Z\right)^{1-\alpha/\delta} 2F_1 \left(1 - \frac{\alpha}{\delta}; 1; 2; Z\right)
\]
with $Z = -e^{\delta(t-y_0)} = 1/ \left(1 - e^{-\delta v}\right)$, cf. (6.2a).

We can simplify the hypergeometric function, namely,
\[
2F_1 \left(\mu; 1; 2; Z\right) = \frac{1 - (1 - Z)^{1-\mu}}{(1-\mu)Z}.
\]

By using $1 - \mu = \alpha/\delta$, since $(Z - 1)/Z = e^{-\delta v}$, we obtain
\[
x_0 = \frac{1}{\alpha} e^{\alpha(t-u)} + \frac{1}{\alpha} e^{\alpha t} \frac{1 - (1 - Z)^{\alpha/\delta}}{(-Z)^{\alpha/\delta}} = \frac{e^{\alpha t}}{\alpha} \left(e^{-\alpha u} - e^{-\alpha v} + (e^{-\delta v} - 1)^{\alpha/\delta}\right).
\]

Setting
\[
x_c = \alpha x_0 - e^{\alpha y_0} = e^{\alpha t} \left(e^{-\alpha u} - e^{-\alpha v}\right)
\]
(we also used (6.2b)) completes the proof of Lemma 6.2.
6.2 The General Solution

As we have both characteristic curves \( x_c \) and \( y_c \) now (cf. Lemma 6.2), we can write down the general solution to our partial differential equation. For some function \( \tilde{\phi} \), we have

\[
\phi(t; u, v) = \tilde{\phi} \left( e^{\alpha t} \left( e^{-\alpha u} - e^{-\alpha v} \right), e^{\delta t} \left( e^{-\delta v} - 1 \right) \right). \tag{6.4}
\]

To determine \( \tilde{\phi} \) we need to take care of the initial conditions.

6.3 Initial Conditions

We have two possible cases to consider for the initial conditions of the solution (6.4). The case covered in this section is \( u = v \). Then, regardless of the type of transition we have, we are adding \( \delta \) balls to the urn, each time a transition takes place. In the other case (see Section 6.4), we set \( u = 0 \), so that we are only considering transitions with the second direction of navigation.

Suppose we have \( i := X(0) + Y(0) \) balls at time zero. The time between the \( \ell \)th and \((\ell + 1)\)st transition is exponential, with parameter \( i + \ell \delta \), i.e., with expected time \( 1/(i + \ell \delta) \). When the \((\ell + 1)\)st transition occurs (for any \( i \geq 0 \)), we add \( \delta \) more balls to the urn at that time. So there are always \( i + \ell \delta \) balls in the urn, after \( \ell \) transitions have taken place.

**Lemma 6.3.** If \( u = v \), then the probability-generating function of \( Y(t) \) is

\[
\mathbb{E} \left[ e^{sY(t)} \right] = \left( 1 - e^{\delta t} + e^{\delta(t-v)} \right)^{(X(0)+Y(0))/\delta}.
\]

Note that

\[
\mathbb{E} \left[ e^{sY(t)} \right] = (1 + y_c)^{-(X(0)+Y(0))/\delta},
\]

where \( y_c \) is the characteristic curve of Lemma 6.2.

To prove this, we will setup and solve Kolmogorov’s Forward Equations.

**Kolmogorov’s Forward Equations**

Let \( P_{i,j}(t) \) denote the probability that, starting with \( i \) balls in the urn at a certain time, then \( t \) time units later, we have \( j \) balls in the urn. In particular, \( P_{i,i+\ell \delta}(t) \) is the probability that, starting with \( i \) balls, we have exactly \( \ell \) transitions during the next \( t \) time units. We follow some of the notation of Ross [9]. We let \( v_i \) denote the rate for the exponential distribution of time until the next transition occurs, when there are currently \( i \) balls in the urn. In our case, since the balls act independently, and each ball has exponential rate 1 of being chosen, we have \( v_i = i \). We define \( q_{i,j} = v_i P_{i,j}, \) so \( q_{i,i+\delta} = v_i = i \), and \( q_{i,j} = 0 \) otherwise.

We have, as in Ross’s Lemma 5.4.1,

\[
\lim_{t \to 0} \frac{1 - P_{i,i}(t)}{t} = v_i \quad \text{and} \quad \lim_{t \to 0} \frac{P_{i,j}(t)}{t} = q_{i,j}, \quad \text{for} \ i \neq j.
\]

So, now we set up Kolmogorov’s Forward Equations, following Theorem 5.4.4 of Ross. In our case, these equations are

\[
P'_{i,i}(t) = -v_i P_{i,i}(t) = -i P_{i,i}(t), \tag{6.5a}
\]

and for \( \ell \geq 1, \)

\[
P'_{i,i+i+\ell \delta}(t) = (i + (\ell - 1) \delta) P_{i,i+(\ell-1)\delta}(t) - (i + \ell \delta) P_{i,i+i+\ell \delta}(t). \tag{6.5b}
\]

We use the rising factorial notation \( (i/\delta)^{\ell} := \prod_{k=0}^{\ell-1} (i/\delta + k) \) in the statement of the lemma. The initial conditions on the \( P_{i,i+\ell \delta}(t) \) are \( P_{i,i}(0) = 1 \) and \( P_{i,i+\ell \delta}(0) = 0 \) for \( \ell \geq 1. \)
Lemma 6.4. The functions

\[ P_{i,i+\ell\delta}(t) = \left( \frac{i}{\ell!} \right) \ell e^{-it(1-e^{-\delta t})^\ell} \]

are the solutions to the Kolmogorov system of differential equations \((6.5)\).

Before proving Lemma 6.4, we first note this indeed is a probability distribution, since \(\sum_{\ell \geq 0} P_{i,i+\ell\delta}(t) = 1\) (follows from Lemma 6.3) and since all these summands are nonnegative.

In particular, we have the solutions

\[ P_{i,i}(t) = e^{-it} \quad \text{and} \quad P_{i,i+\delta}(t) = \frac{i}{\delta} e^{-it(1-e^{-\delta t})}. \]

Proof of Lemma 6.4. Inserting \(t = 0\) shows that the initial conditions are satisfied. We used the conventions that \(0^0 = 1\), and that the empty product equals 1. The case \(\ell = 0\) follows by a direct calculation.

When \(\ell \geq 1\) we have

\[
P'_{i,i+\ell\delta}(t) = \prod_{k=0}^{\ell-1} \frac{(i + k\delta)}{\ell! \delta^t} \left( -ie^{-it(1-e^{-\delta t})^\ell} + \ell\delta e^{-it(1-e^{-\delta t})^{\ell-1}} \right)
\]

\[
= -i \prod_{k=0}^{\ell-1} \frac{(i + k\delta)}{\ell! \delta^t} e^{-it(1-e^{-\delta t})^\ell} 
- \ell\delta \prod_{k=0}^{\ell-1} \frac{(i + k\delta)}{\ell! \delta^t} e^{-it(1-e^{-\delta t})(1-e^{-\delta t})^{\ell-1}} 
+ (i + (\ell - 1)\delta) \prod_{k=0}^{\ell-1} \frac{(i + k\delta)}{(\ell - 1)! \delta^t (1-e^{-\delta t})^{\ell-1}} e^{-it(1-e^{-\delta t})^{\ell-1}} 
= -(i + \ell\delta) P_{i,i+\ell\delta}(t) + (i + (\ell - 1)\delta) P_{i,i+(\ell-1)\delta}(t),
\]

which proves the lemma.

Probability Generating Function

By using the solutions to Kolmogorov’s Forward Equations (Lemma 6.4), it is not hard anymore to derive the moment generating function \(\mathbb{E}[e^{\nu Y(t)}]\).

Proof of Lemma 6.3. To calculate \(\mathbb{E}[e^{\nu Y(t)}]\), we insert \(i = X(0) + Y(0)\) at the end of this proof; but for the moment we still write the \(i\). Taking Lemma 6.4 and summing yields

\[
\mathbb{E}[e^{\nu Y(t)}] = \sum_{\ell \geq 0} P_{i,i+\ell\delta}(t)e^{u(i+\ell\delta)} = e^{i(u-v)} \sum_{\ell \geq 0} \frac{(i/\delta)\ell}{\ell!} \left( e^{\delta v(1-e^{-\delta t})} \right)^\ell.
\]

Since \((1-Z)^{-\mu} = \sum_{\ell \geq 0} \mu^{\ell} Z^{\ell}/\ell!\) (again using the rising factorial notation \(\mu^{\ell} := \mu(\mu+1)\ldots(\mu+\ell-1)\)), we obtain

\[
\mathbb{E}[e^{\nu Y(t)}] = e^{-i(t-v)} \left( 1 - e^{\delta v(1-e^{-\delta t})} \right)^{-i/\delta},
\]

and the result follows by rearranging the terms.
6.4 Another Initial Condition

For this initial condition, we consider the situation at time $t = 0$. Again, we have $X(0)$ units of the first type and $Y(0)$ units of the second type. This translates to the probability generating function

$$
E \left[ e^{uX(0)+vY(0)} \right] = e^{uX(0)e^{vY(0)}}
$$

$$
= \left( x_c|_{t=0} + (y_c|_{t=0} + 1)^{\alpha/\delta} \right)^{-X(0)/\alpha} (y_c|_{t=0} + 1)^{-Y(0)/\delta},
$$

which we have by rewriting in terms of the characteristic curves $x_c$ and $y_c$ of Lemma 6.2.

6.5 Solution to the Partial Differential Equation

We are now ready to determine the function $\tilde{\phi}$ of the general solution (6.4) and thus proving Theorem 6.1.

Proof of Theorem 6.1. We use the initial conditions provided by Lemma 6.3 and Section 6.4. We compute

$$
\phi(t; u, v) = E \left[ e^{uX(t)+vY(t)} \right]
$$

$$
= (x_c + (y_c + 1)^{\alpha/\delta})^{-X(0)/\alpha} (y_c + 1)^{-Y(0)/\delta}
$$

$$
= \left( e^{\alpha t} (e^{-\alpha u} - e^{-\alpha v}) + (e^{\delta t} (e^{-\delta v} - 1) + 1)^{\alpha/\delta} \right)^{-X(0)/\alpha} \left( e^{\delta t} (e^{-\delta v} - 1) + 1 \right)^{-Y(0)/\delta}
$$

$$
= e^{-X(0)t} \left( e^{-\alpha u} - e^{-\alpha v} + (e^{-\delta v} - 1 + e^{-\delta t})^{\alpha/\delta} \right)^{-X(0)/\alpha} e^{-Y(0)t} \left( e^{-\delta v} - 1 + e^{-\delta t} \right)^{-Y(0)/\delta}.
$$

This solution fulfills our initial conditions, and our proof is complete.

6.6 Moments

Theorem 6.1 provides us with the moment generating function. The moments now follow by differentiation and by setting $u = 0$ and $v = 0$. The first moments are

$$
E[X(t)] = X(0)e^{\alpha t},
$$

$$
E[Y(t)] = (X(0) + Y(0))e^{\delta t} - X(0)e^{\alpha t}.
$$

Note that the two coordinates of the walk grow at different rates, and the drift is much stronger in the vertical direction.

We can also compute the second moments,

$$
E[X^2(t)] = X(0)(\alpha + X(0))e^{2\alpha t} - \alpha X(0)e^{\alpha t},
$$

$$
E[X(t)Y(t)] = X(0)(\alpha + X(0) + Y(0))e^{(\alpha + \delta) t} - X(0)(\alpha + X(0))e^{2\alpha t},
$$

$$
E[Y^2(t)] = (X(0) + Y(0))(\delta + X(0) + Y(0))e^{2\delta t} - 2X(0)(\alpha + X(0) + Y(0))e^{(\alpha + \delta) t}
$$

$$
+ \delta(X(0) + Y(0))e^{\delta t} + X(0)(\alpha + X(0))e^{\alpha t} + \alpha X(0)e^{\alpha t},
$$
and consequently
\[ \text{Var}(X(t)) = \alpha X(0)e^{2\alpha t} - \alpha X(0)e^{\alpha t}, \]
\[ \text{Cov}(X(t), Y(t)) = \alpha X(0)e^{(\alpha+\delta)t} - \alpha X(0)e^{2\alpha t}, \]
\[ \text{Var}(Y(t)) = \delta(X(0) + Y(0))e^{2\beta t} - 2\alpha X(0)e^{(\alpha+\delta)t} \]
\[ - \delta(X(0) + Y(0))e^{\beta t} + \alpha X(0)e^{2\alpha t} + \alpha X(0)e^{\alpha t}. \]

**Remark 6.5.** Of course, the first moments follow by Theorem 4.1 as well: We have
\[ E[X(t)] = e^{\left( \begin{array}{cc} \alpha & 0 \\ \delta & \alpha \end{array} \right) t} X(0). \]

The matrix in the exponent is diagonal, which makes it an easy computation and we obtain
\[ \begin{pmatrix} E[X(t)] \\ E[Y(t)] \end{pmatrix} = \begin{pmatrix} e^{\alpha t} & 0 \\ e^{\delta t} & e^{\alpha t} \end{pmatrix} \begin{pmatrix} X(0) \\ Y(0) \end{pmatrix}. \]

Pursuing a similar approach to the one we used to derive the mean in Theorem 4.1, we can try to go forward with the second moment. We only highlight the salient steps. Take the partial derivatives \( \partial^2/\partial u^2, \partial^2/\partial u \partial v, \) and \( \partial^2/\partial v^2 \) of the moment generating function of Theorem 6.1 and evaluate each equation at \( u = 0 \) and \( v = 0. \) We obtain the system of ordinary differential equations
\[
\frac{d}{dt} E[X^2(t)] = 2\alpha E[X^2(t)] + \alpha^2 E[X(t)], \\
\frac{d}{dt} E[X(t)Y(t)] = (\alpha + \delta) E[X(t)Y(t)] + \beta E[X^2(t)] + \alpha \beta E[X(t)], \\
\frac{d}{dt} E[Y^2(t)] = 2\delta E[Y^2(t)] + 2\beta E[X(t)Y(t)] + \beta^2 E[X(t)] + \delta^2 E[Y(t)],
\]
with \( \beta = \delta - \alpha. \) This system is to be solved under the initial conditions \( E[X^2(0)] = X(0)^2, \)
\( E[X(0)Y(0)] = X(0)Y(0), \) and \( E[Y^2(0)] = Y(0)^2. \)

We can solve it sequentially, starting with \( E[X^2(t)] \), as its differential equation is self contained. We can then plug in the solution of \( E[X^2(t)] \) into the differential equation for \( E[X(t)Y(t)] \), which at this point would have only known components on the right-hand side. Finally, we plug in all the known functions of averages and mixed moments in the differential equation for \( E[Y^2(t)]. \)

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