STRUCTURE THEOREMS FOR THE
SYMMETRIC GROUPS ACTING ON ITS
NATURAL MODULE

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December 11, 2013

Abstract

This paper gives an explicit structure theorem for the symmetric group acting on the symmetric algebra of its natural module. Let $G$ be the symmetric group on $x_1, \ldots, x_n$ and let $d_i$ be the $i$th elementary symmetric polynomial in the $x_i$’s. We show that if we take monomial representations discussed in [7, Section 3] to be the modules $V_I$, then we have an isomorphism of $kG$-modules 

$$k[x_1, \ldots, x_n] \cong \bigoplus_{\{n\} \subseteq I \subseteq \{1, \ldots, n\}} k[d_I] \otimes_k V_I.$$ 

This paper gives a structure theorem for the symmetric group, $G$, acting on its natural module, which gives us a $kG$-decomposition of the graded components of $S = k[x_1, \ldots, x_n]$, where $k$ is a unital ring such that $ab = 0$ implies $a = 0$ or $b = 0$ for $a, b \in k$. Which is to say, for $d_1, \ldots, d_n$ the elementary symmetric polynomials in $x_1, \ldots, x_n$, we give $kG$-submodules of $S$, $V_I$ for $I \subseteq \{1, \ldots, n\}$ $n \in I$, such that the multiplication map 

$$\bigoplus_{\{n\} \subseteq I \subseteq \{1, \ldots, n\}} k[d_I] \otimes_k V_I \rightarrow S$$

is a $kG$-isomorphism.

In fact the monomial representations discussed in [7, Section 3] may be taken as the modules, $V_I$, occurring in a structure theorem. Many of the intermediate steps will be similar to those from [7], but the fact that we get a structure theorem is new as is the observation that we may use $e_I$, rather than the $e'_I$ used by Kemper. Note that although the ring $k$ need not be commutative, we require that $ax_i = x_ia$ for $i = 1, \ldots, n$ and for all $a \in k$.

It will turn out that in this example of a structure theorem all $V_I$ with $n \notin I$ are zero, this was also true for the upper triangular structure theorem.

For more information on structure theorems see [4], [5], [6], [10] and [9]. A more verbose exposition of this material and additional examples of structure theorems can be found in [8].

1 Definition and Results in the Literature

Let $k$ be a unital ring such that $ab = 0$ implies $a = 0$ or $b = 0$, which need not be commutative. Let $R = k[d_1, \ldots, d_n]$ be the $\mathbb{N}$ graded polynomial $k$-algebra in the indeterminants $d_1, \ldots, d_n$, with $\deg(d_i) > 0$ but not necessarily
with \( \deg(d_1) = 1 \). Let \( G \) be any finite group and let \( S \) be a finitely generated \( \mathbb{Z} \)-graded \( RG \)-module.

**Definition 1.1.** With notation as above, a Structure Theorem for \( S \) over \( RG \) is a set of finitely generated \( kG \)-submodules, \( X_I \subseteq S \), one for each \( I \subseteq \{1, \ldots, n\} \), such that the map:

\[
\phi : \bigoplus_{I \subseteq \{1, \ldots, n\}} k[d_i | i \in I] \otimes_k X_I \to S
\]

\[
\phi : d \otimes_k x \mapsto dx
\]

is an isomorphism of \( kG \)-modules.

Note that the map \( \phi \) is split over \( kG \), as it is a \( kG \)-isomorphism. As the module being mapped from is not an \( R \)-module, it cannot hope to be an \( R \)-map, however the following lemma is straightforward.

**Lemma 1.2.** For each component of the sum, the map:

\[
\phi_I : k[d_i | i \in I] \otimes_k X_I \to S
\]

\[
\phi_I : d \otimes_k x \mapsto dx
\]

is a \( k[d_i | i \in I]G \)-homomorphism.

If we insist that \( k \) is a field, then we know that a structure theorem exists for the symmetric group acting on its natural module by the following arguments.

**Theorem 1.3** (Symonds 2006). \([9]\) Let \( k \) be a field and let \( R = k[d_1, \ldots, d_n] \) be the graded polynomial ring with \( \deg(d_i) > 0 \) for all \( i \), let \( G \) be a finite group graded in degree 0 and let \( S \) be a finitely generated \( \mathbb{Z} \)-graded \( RG \)-module. A structure theorem for \( S \) exists exactly when only finitely many isomorphism classes of indecomposable \( kG \)-modules occur as summands of \( S \).

Note that since we insisted that the \( X_I \) are finitely generated it is not the case that every \( S \) trivially has a structure theorem given by \( X_\emptyset = S |_{kG} \).

Let \( S = k[x_1, \ldots, x_n] \) be a polynomial ring in \( n \) variables graded in degree 1. With respect to the basis \( x_1, \ldots, x_n \) of the degree 1 component of \( S \), let \( P \) denote a finite subgroup of the upper triangular of matrices with 1’s on the diagonal.

**Theorem 1.4** (Karagueuzian and Symonds 2007). \([5, \text{Theorem 1.1}]\) For \( k \) a finite field, \( S \) and \( P \) as immediately above and \( R \subseteq S^P \) a particular Noether normalization of \( S^P \), \( S \) has a structure theorem over \( RP \).

Any group acting on \( S \) with grading preserving algebra automorphisms is defined by its action on the degree 1 component of \( S \). Let \( P \) be any Sylow-\( p \)-subgroup of \( G \). It can be shown that we may chose a basis of the degree 1 component of \( S \) such that the elements of \( P \) are represented by upper triangular matrices with 1’s on the diagonal. A similar argument to Theorem 1.4 (found
in the proof of [13 Corollary 4.2]) tells us that $S$ has a structure theorem over $R_p$. Since $P$ is a Sylow-$p$-subgroup of $G$, this tells us that $S$ has finitely many isomorphism classes of indecomposable $kG$-summands. Hence $S$ has a structure theorem over $RG$. All together this shows:

**Corollary 1.5.** Let $k$ be a field of characteristic $p$. For $S = k[x_1, \ldots, x_n]$, with $\deg(x_i) = 1$ for $i = 1, \ldots, n$, $G$ a finite group of grading preserving algebra automorphisms and $R \subseteq S^G$ a polynomial ring such that $S$ is a finite $R$-module, $S$ has a structure theorem over $RG$. cf. [12 Corollary 1.2].

## 2 Notation

We now fix notation which we will use for the rest of the paper.

Take $k$ to be any unital ring such that $ab = 0$ implies $a = 0$ or $b = 0$. Fix an $n \in \mathbb{N}_{>0}$, let $S = k[x_1, \ldots, x_n]$ and let $G = \text{Sym}(x_1, \ldots, x_n)$ be the symmetric group on the variables $x_1, \ldots, x_n$. Let $d_i$ be the $i$th elementary symmetric polynomial in $x_1, \ldots, x_n$ e.g. $d_1 = x_1 + \cdots + x_n$, $d_n = x_1x_2 \cdots x_n$ and for $i \in \{1, \ldots, n\}$:

$$d_i = \sum_{g \in G/\text{stab}_G(x_1 \cdots x_i)} g(x_1 \cdots x_i).$$

Let $R = k[d_1, \ldots, d_n]$. It is well known that the $d_i$ are algebraically independent (a result sometimes called the fundamental theorem of symmetric polynomials), so $R$ is a polynomial $k$-algebra.

Note that $\text{stab}_G(x_1 \cdots x_i)$ is the stabilizer of the monomial $x_1 \cdots x_i$, which is the same as the stabilizer of the set $\{x_1, \ldots, x_i\}$. Elements of this group are made up of a permutation of $x_1, \ldots, x_i$ and a permutation of $x_{i+1}, \ldots, x_n$.

For any $m \in \mathbb{N}$ let $[m] = \{1, 2, \ldots, m\}$. For $I = \{i_1, \ldots, i_m\} \subseteq [n]$, let $d_I = d_{i_1} \cdots d_{i_m} = \prod_{i \in I} d_i$, we let $d_\emptyset = 1$. Let $d_I$ denote the set $\{d_i|i \in I\}$ and $k[d_I]$ denote the polynomial ring $k[d_i|i \in I]$.

Let $\text{lm}^{\text{lex}}$ denote the leading monomial in the usual lexicographical ordering on monomials in $x_1, \ldots, x_n$. For $I = \{i_1, \ldots, i_m\} \subseteq [n]$ with $n \in I$, set $e'_I = \text{lm}^{\text{lex}}(d_I)/d_i$ and let $V'_I$ be the $kG$-module generated by $e'_I$. An element of $R$ of the form $d_{i_1}^{e'_i} \cdots t_{i_m}^{e'_m}$ we will call a $d_l$-monomial, and if $I = [n]$ we may shorten this to a $d$-monomial. Likewise $x_I$ and $x$-monomials, are elements of $S$ of the form $x_{i_1}^{e'_i} \cdots x_{i_m}^{e'_m}$, with $i_j \in I$ and $[n]$ respectively.

Note that if we defined $e''_I = \text{lm}^{\text{lex}}(d_I)$ and $V''_I = \langle e''_I \rangle$ for any $I \subseteq [n]$, then for any $I$ with $n \notin I$ we would have $V''_I \cong V''_{I \cup \{n\}}$, and for any $I$ with $n \in I$ we would have $V''_I \cong V''_{I \cup\{n\}}$. So no new isomorphism classes of module occur for $V''_I$ with $n \notin I$. In both cases the isomorphism is given by multiplication by $d_n$.

This notation is summarized in the top part of table below, for now ignore the bottom two rows as $G$-$\text{lm}$ has not yet been defined.
- **$k$**: unital ring such that $ab = 0 \implies a = 0$ or $b = 0$ for all $a, b \in k$
- **$S$**: $k[x_1, \ldots, x_n]$
- **$G$**: $\text{Sym}(x_1, \ldots, x_n)$
- **$d_i$**: $i^{th}$ elementary symmetric polynomial in $x_1, \ldots, x_n$
- **$R$**: $k[d_1, \ldots, d_n]$
- **$[m]$**: $\{1, 2, \ldots, m\}$
- **$I \subseteq [n]$**: $I = \{i_1, \ldots, i_{|I|}\}$
- **$\hat{d}_I$**: $d_{i_1} \cdots d_{i_{|I|}} = \prod_{i \in I} d_i$ for $\{n\} \subseteq I \subseteq [n]$
- **$d_I$**: $\{d_i | i \in I\}$
- **$e'_I$**: $\text{lm}_{\text{lex}}(\hat{d}_I)/d_n$ for $\{n\} \subseteq I \subseteq [n]$
- **$V'_I$**: the $kG$-module generated by $e'_I$ for $\{n\} \subseteq I \subseteq [n]$
- **$e_I$**: an element of $S$ such that $G\text{-im}(e_I) = \{\text{lm}_{\text{lex}}(d_I)/d_n\}$, $\text{stab}_G(e_I) = \text{stab}_G(\text{lm}_{\text{lex}}(e_I))$ for $\{n\} \subseteq I \subseteq [n]$
- **$V_I$**: the $kG$-module generated by $e_I$ for $\{n\} \subseteq I \subseteq [n]$

The result we are aiming for is:

**Theorem 2.1.** With notation as above, we have a structure theorem:

$$S \cong \bigoplus_{\{n\} \subseteq I \subseteq [n]} k[d_I] \otimes_k V'_I$$

Where the map from right to left is the $kG$-homomorphism $d \otimes_k v \mapsto dv$.

This is an immediate corollary of Theorem 5.2, where $e'_I$ and $V'_I$ are replaced by $e_I$ and $V_I$.

Using $e_I$, rather than $e'_I$, does make the notation a little more messy but being able to use $e_I$ allows more flexibility. It may also be useful for considering localizations of $S$. For example, assume the $e_I$ version of the theorem holds and fix $r \in [n]$, then the following choices for $e_I$ are allowed:

$$e_I = \begin{cases} 
  e'_I & \text{if } r \not\in I \\
  d_r e'_{I \setminus \{r\}} & \text{if } r \in I 
\end{cases}$$

If $r \not\in I$ then $d_r V_I \subseteq \text{Im}(1_R \otimes_k V_{I \cup \{r\}})$. On the other hand if $r \in I$, since the theorem holds, we have $d_r V_I = \text{Im}(d_r \otimes_k V_I)$. So for all $I \subseteq [n]$ with $n \in I$, we have $d_r V_I \subseteq \text{Im}(k[d_{I \cup \{r\}}] \otimes_k V_{I \cup \{r\}})$. This tells you that for $S_{d_r}$, the localization of $S$ by $d_r$, we have a split isomorphism of $kG$-modules:

$$S_{d_r} \cong \bigoplus_{\{I \subseteq [n] | r \in I\}} k[d_I][d_r^{-1}] \otimes_k V_I$$

where the isomorphism from right to left is given by multiplication.

The two main tools we use are the $>$-leading monomials and the reduced form.
3 Leading Monomials

The following definitions are similar to [7, Section 3 Definition 13].

**Definition 3.1** ($\succ, \succeq, \approx, M(-), G$-lm). For two $x$-monomials, $y, z \in S$, pick $g, h \in G$ such that $gy \succeq_{\text{lex}} g'y$ for all $g' \in G$ and $hz \succeq_{\text{lex}} h'z$ for all $h' \in G$, we say that $y \succ z$ if $gy \succeq_{\text{lex}} hz$, otherwise $y \prec z$.

We say that $x \approx y$ if $x \prec y$ and $y \prec x$, i.e. if there exists $g, h \in G$ such that $gx = hy$.

For $u \in S$, define $M(u)$ to be the set of $x$-monomials occurring in $u$ (with non-zero coefficient) and define

$$G$-lm$(u) := \{x \in M(u)|x \succ y \text{ for all } y \in M(u)\}$

For a set $X$ such that $x \approx y$ for all $x, y \in X$, write $X \approx m$ if $\forall x \in X, x \approx m$. Note that $\forall x, y \in G$-lm$(u), x \approx y$, so $G$-lm$(u) \approx m$ makes sense.

Note the distinction between $G$-lm$(u) \approx m$ and $G$-lm$(u) = \{m\}$ for $u, m \in S$, $m$ an $x$-monomial. The former says that the leading monomials of $u$ in the $\succ$ ordering are all equal to $gm$ for some $g \in G$. The latter says that there is exactly one $n \in M(u)$ which is maximal in the $\succ$ ordering and this $n$ is equal to $m$.

Let $e_I$ and $V_I$ be as in the box from Section 2, i.e. for $I \subseteq [n]$ with $n \in I$:

- $e_I$ is an element of $S$ such that $G$-lm$(e_I) = \{\text{lm}_{\text{lex}}(d_I)/d_n\} = \{e'_I\}$, stab$_G(e_I) = \text{stab}_G(e'_I)$ and the coefficient of the $\succ$-leading monomial is a unit; $V_I$ is the module $kGe_I$.

The condition that $G$-lm$(e_I) = \{e'_I\}$ could be relaxed to $G$-lm$(e_I) = \{g \cdot e'_I\}$, or we could say $G$-lm$(e_I) \approx \{e'_I\}$ and $|G$-lm$(e_I)| = 1$. We gain no benefit from this as the next lemma tells us that for such an $e_I$ we would have $G$-lm$(g^{-1}e_I) = \{e'_I\}$, so the $V_I$ obtained in this way are the same. So we insist that $G$-lm$(e_I) = \{e'_I\}$.

**Lemma 3.2.** Let $d$ be a $d$-monomial considered as an element of $S$ and $u, v, w$ be any elements of $S$ then:

1. $\text{lm}_{\text{lex}}(uv) = \text{lm}_{\text{lex}}(u)\text{lm}_{\text{lex}}(v)$
2. $G$-lm$(d) \approx \text{lm}_{\text{lex}}(d)$
3. $G$-lm$(d) = \{g \cdot \text{lm}_{\text{lex}}(d) | g \in G\}$
4. For any $g \in G$ we have $G$-lm$(u) \approx G$-lm$(gu)$.
5. Let $m$ be an $x$-monomial with $m \in G$-lm$(u)$, $G$-lm$(u) = Gm \cap M(u)$
6. For $G$-lm$(u)$ and $G$-lm$(v)$ disjoint, $G$-lm$(u + v) \subseteq G$-lm$(u) \cup G$-lm$(v)$, in particular $G$-lm$(u + v) \approx G$-lm$(u)$ or $G$-lm$(u + v) \approx G$-lm$(v)$.
7. $G$-lm$(gu) = gG$-lm$(u)$ for all $g \in G$. 
Proof. (1) follows from $a \preceq_{\text{lex}} b \implies ac \preceq_{\text{lex}} bc$ for $x$-monomials $a, b, c$.

(2) and (3) are because $d$ is a $d$-monomial.

(4) is because $m \approx gm$ and $m \succ n \iff gm \succ gn$, for $x$-monomial $m, n$.

(5) if $n \in G$-lm$(u)$, then $n \nsucceq m'$ for all $m' \in M(u)$, so in particular $n \nsucceq m$. We already know $m \nsucceq n$, so $m \succeq n$, i.e. $n \in Gm$. Clearly $G$-lm$(u) \subseteq M(u)$, so $G$-lm$(x) \subseteq Gm \cap M(x)$.

Conversely, if $n \in Gm \cap M(u)$, then $n \succeq m$ and $m \succeq m'$ for all $m' \in M(u)$. So $n \succeq m'$, for all $m' \in M(u)$, and $n \in M(u)$, so $n \in G$-lm$(u)$.

(6) for $m \in G$-lm$(u)$, $n \in G$-lm$(v)$, without loss of generality let $m \succeq n$. Then $m \in M(u + v)$, as $m \notin M(v)$, and $m \succeq m'$ for all $m' \in M(u + v)$.

(7) suppose $m \in G$-lm$(u)$, then $gm \in M \cap M(gu)$ so $gm \in G$-lm$(gu)$ by part (5). This shows that $G$-lm$(u) \subseteq G$-lm$(gu)$ and $g^{-1}$G-elm$(gu) \subseteq G$-lm$(g^{-1}gu)$. \hfill \square

Lemma 3.3. For $e_I$ and $V_I$ as in the box and $u \in V_I$ with $u \neq 0$, we have $G$-lm$(u) \approx e_I$

Proof. As $V_I = kGe_I$, for $T$ a transversal of stab$_G(e_I)$ in $G$, any non-zero element of $V_I$ may be expressed uniquely as a sum:

$$\sum_{g \in T} \lambda_ge_{gI},$$

for $\lambda_g \in k$, with at least one $\lambda_g$ non-zero.

Since stab$_G(e_I) = \text{stab}_G(e'_I)$, the $T$ we chose above is a transversal of stab$_G(e'_I)$ in $G$. By definition $G$-lm$(e_I) = \{e'_I\}$, hence by Lemma 3.2(7), $G$-lm$(ge_I) = gG$-lm$(e_I) = \{g \cdot e'_I\}$. So for $g, h \in T$ we have that $G$-lm$(ge_I)$ and $G$-lm$(he_I)$ are disjoint when $g \neq h$.

By repeated application of Lemma 3.2(6), for $\lambda_g \in k$ with at least one of the $\lambda_g \neq 0$ we have:

$$G \text{-lm} \left( \sum_{g \in T} \lambda_g ge_I \right) \approx G \text{-lm}(g'e_I),$$

for some $g' \in T$ and by Lemma 3.2(4) $G$-lm$(g'e_I) \approx e_I'$ for all $g' \in T$. \hfill \square

Lemma 3.4. For $e_I$ and $V_I$ as in the box, $d$ a $d_I$-monomial and $u \in S$, if $G$-lm$(u) \approx e'_I$, then $G$-lm$(du) \approx \text{lm}_{\text{lex}}(d)e'_I$.

In particular, for $u \in V_I - \{0\}$ we have: $G$-lm$(du) \approx \text{lm}_{\text{lex}}(d)e'_I$.

Proof. Take $m \in G$-lm$(u)$, there exists a $g \in G$ such that $gm = e'_I$. By Lemma 3.2(7) $gG$-lm$(u) = G$-lm$(gu)$, and by Lemma 3.2(4), $G$-lm$(gu) \approx G$-lm$(u)$. So we may assume $e'_I \in G$-lm$(u)$ and $\text{lm}_{\text{lex}}(u) = e'_I$. Hence $\text{lm}_{\text{lex}}(d)e'_I = \text{lm}_{\text{lex}}(du)$ by Lemma 3.2(1), in particular $\text{lm}_{\text{lex}}(d)e'_I \in M(du)$. So it is sufficient to show that $\text{lm}_{\text{lex}}(d)e'_I \geq n$ for all $n \in M(du)$. 

\hfill 6
For $d = d_1 \ldots d_n$:

$$M(du) = \left\{ \left( \prod_{i=1}^{n} \prod_{j=1}^{t_i} g_{i,j} \text{lm}_{\text{lex}}(d_i) \right) a \mid a \in M(u), g_{i,j} \in G \right\}$$

That $G\text{-lm}(u) \approx e'_I$ implies that for all $h \in G$ and all $a \in M(u)$, we have $e'_I \geq_{\text{lex}} ha$. Clearly $\text{lm}_{\text{lex}}(d_i) \geq_{\text{lex}} g_{i,j} \text{lm}_{\text{lex}}(d_i)$, so $\text{lm}_{\text{lex}}(d) e'_I \geq_{\text{lex}} h n$ for all $n \in M(du)$.

The “in particular” statement follows from Lemma 3.3.

\[\square\]

### 4 Reduced Form

The following definition is equivalent to [17 Section 3 Definition 10], where it is described as a generalization of Göbel’s concept of “special” terms.

**Definition 4.1.** For an $x$-monomial $m \in S$, $m = x_1^{r_1} \ldots x_n^{r_n}$, the reduced form of $m$, $\text{Red}(m)$, is the $x$-monomial $x_1^{r'_1} \ldots x_n^{r'_n}$ where:

- $|\{r'_i| i = 1, \ldots, n\}| = |\{r_i| i = 1, \ldots, n\}| = a \leq n$
- $\{r'_i| i = 1, \ldots, n\} = \{0, 1, \ldots, a - 1\}$
- $r'_i < r'_j \iff r_i < r_j$ for all $i, j$.

We say that an $x$-monomial, $m$, is in reduced form if $m = \text{Red}(m)$.

Note that for every $x$-monomial in $S$, $m$, there exists a $g \in G$ such that $qm$ is the leading $x$-monomial of some $d$-monomial. This is simply the observation that every $x$-monomial $m = x_1^{m_1} \ldots x_n^{m_n}$ with $m_1 \geq m_2 \geq \cdots \geq m_n$ can be written as $x_1^{a_1}(x_1 x_2)^{a_2}(x_1 x_2 x_3)^{a_3} \ldots (x_1 \ldots x_n)^{a_n}$. The idea of this definition is that the reduced form of $m$ tells us which $d_i$ occur at least once in this $d$-monomial by looking at when the powers change. For example: the reduced form of $x_1^2 x_2^3 x_3^4$ is $x_1^2 x_2^3 x_3$, and this is the leading monomial of $d_2 d_3$. Another example is $\text{Red}(x_2^3 x_3^4) = x_2 x_3$, which the group element $(x_1, x_3)$ applied to the leading monomial of $d_1 d_2$.

We show, in Corollary 4.8 that one way to think of $\text{Red}(d)$, for $d$ a $d$-monomial, is to write out the product of the leading monomials of the $d_i$’s vertically, then get rid of the repetitions and the $d_n$’s. For example, let $I =$
\{i_1, \ldots, i_a\}, i_1 < i_2 < \cdots < i_a = n \text{ and } d = d_{i_1} \ldots d_{i_a} \text{ we may write } \text{lm}_{\text{lex}}(d) \text{ as:}

\[
\begin{align*}
\text{lm}_{\text{lex}}(d_{i_1})^{t_1} & \{x_1 \ldots x_{i_1} \\
\text{lm}_{\text{lex}}(d_{i_2})^{t_2} & \{x_1 \ldots x_{i_1} \ldots x_{i_2} \\
\vdots & \ddots \\
\text{lm}_{\text{lex}}(d_{i_{a-1}})^{t_{a-1}} & \{x_1 \ldots x_{i_1} \ldots x_{i_2} \ldots x_{i_{a-1}} \\
\text{lm}_{\text{lex}}(d_n)^{t_n} & \{x_1 \ldots x_{i_1} \ldots x_{i_2} \ldots x_{i_{a-1}} \ldots x_n
\end{align*}
\]

So the reduced form is just:

\[
\begin{align*}
&x_1 \ldots x_{i_1} \\
&x_1 \ldots x_{i_1} \ldots x_{i_2} \\
&x_1 \ldots x_{i_1} \ldots x_{i_2} \ldots x_{i_k} \\
&x_1 \ldots x_{i_1} \ldots x_{i_2} \ldots x_{i_{a-1}}
\end{align*}
\]

which is clearly \(\text{lm}_{\text{lex}}(d_{i_1} \ldots d_{i_{a-1}})\).

\textbf{Lemma 4.2.} For \(x\)-monomials \(m = x_1^{m_1} \ldots x_n^{m_n}\) and \(r = x_1^{r_1} \ldots x_n^{r_n}\), \(\text{Red}(m) = \text{Red}(r)\) if and only if we have \((m_i > m_j) \iff (r_i > r_j)\).

\textit{Proof.} Let \(\text{Red}(m) = x_1^{m_1} \ldots x_n^{m_n}\) and \(\text{Red}(r) = x_1^{r_1} \ldots x_n^{r_n}\). If \(\text{Red}(m) = \text{Red}(r)\) then \(m_i = r_i\) and \((m_i > m_j) \iff (m_i' > m_j') \iff (r_i' > r_j') \iff (r_i > r_j)\). For the converse: if \((m_i > m_j) \iff (r_i > r_j)\), then \((m_i' > m_j') \iff (r_i' > r_j')\), and the longest increasing chain of \(m_i\) is the same length as the longest increasing chain of \(r_i'\). Hence \(\{r_i'| i = 1, \ldots, n\} = \{m_i' | i = 1, \ldots, n\}\). \(\square\)

\textbf{Lemma 4.3.} For an \(x\)-monomial, \(m\), \(\text{Red}(m) \approx e_i'\) for some \(I \subseteq [n], n \in I\).

\textit{Proof.} This follows directly from the definition. Let \(m'\) be a monomial in reduced form, \(m' = x_1^{m_1'} \ldots x_n^{m_n'}\) and \(\{m_1', \ldots, m_n'\} = \{0, \ldots, a\}\). Then there exists a \(g \in G\) such that \(gm' = m'' = x_1^{m_1''} \ldots x_n^{m_n''}\) with \(m_1'' \geq m_2'' \geq \cdots \geq m_n'' = 0\). We may write:

\[
m'' = (x_1 \ldots x_{i_1})^{a}(x_{1+i_1} \ldots x_{i_2})^{a-1} \ldots (x_{1+i_{a-1}} \ldots x_{i_a})^{1}(x_{1+i_a} \ldots x_n)^0.
\]

But this is equal to: \(\text{lm}_{\text{lex}}(d_{i_1})^{l_1}\text{lm}_{\text{lex}}(d_{i_2})^{l_2} \ldots \text{lm}_{\text{lex}}(d_{i_a}),\) and so: \(m' \approx m'' = e_{\{i_1, \ldots, i_a, n\}}\). \(\square\)

\textbf{Lemma 4.4.} For \(x\)-monomials \(x, y \in S\): \(\text{Red}(gx) = g\text{Red}(x)\) and \(x \approx y\) implies \(\text{Red}(x) \approx \text{Red}(y)\). cf. \cite{[7] Section 3 Lemma 12}.  

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Proof. We first show that $\text{Red}(g^{-1}x) = g^{-1}\text{Red}(x)$ for any $g \in G$, this of course shows that $\text{Red}(gx) = g\text{Red}(x)$. Let $x = x_1^{r_1} \cdots x_n^{r_n}$ and $\text{Red}(x) = x_1^{t_1} \cdots x_n^{t_n}$. For $\text{Red}(g^{-1}x) = x_1^{s_1} \cdots x_n^{s_n}$ and $g^{-1}\text{Red}(x) = x_1^{t_1} \cdots x_n^{t_n}$, we must have that \( \{r_i \mid i = 1, \ldots, n\} = \{s_i \mid i = 1, \ldots, n\} = \{t_i \mid i = 1, \ldots, n\} = \{0, 1, \ldots, a - 1\} \), so by Lemma 4.2 it remains to show that $s_i > s_j$ if and only if $t_i > t_j$.

We defined $G$ as acting on $\{x_1, \ldots, x_n\}$, this gives us an action on $\{1, \ldots, n\}$ via $gx_i = x_{g(i)}$. In this notation $g^{-1}(x_1^{r_1} \cdots x_n^{r_n}) = x_1^{s_1(\cdots)} \cdots x_n^{s_n(\cdots)}$. Hence $t_i = r_i^{g(i)}$ so $t_i > t_j$ if and only if $r_i^{g(i)} > r_j^{g(i)}$ and by the definition of $\text{Red}(x)$ this is if and only if $r_g(i) > r_g(j)$. Likewise the definition of $\text{Red}(g^{-1}x)$ states that $s_i > s_j$ if and only if $r_g(i) > r_g(j)$. Hence $\text{Red}(gx) = g\text{Red}(x)$ by Lemma 4.2.

To show that $x \approx y$ implies $\text{Red}(x) \approx \text{Red}(y)$, note that if $gx = hy$ then $\text{Red}(gx) = \text{Red}(hy)$. So by the above, $g\text{Red}(x) = h\text{Red}(y)$, which is the same as saying $\text{Red}(x) \approx \text{Red}(y)$. $\blacksquare$

Definition 4.5. If $X$ is a set of $x$-monomial such that $x \approx y$ for all $x, y \in X$ (e.g. $X = G\text{-Im}(u)$), then by $\text{Red}(X)$ we mean $\{\text{Red}(x) \mid x \in X\}$.

Note that by Lemma 4.4 if $\forall x, y \in X, x \approx y$ then $\forall x', y' \in \text{Red}(X), x' \approx y'$, so it makes sense to talk about $\text{Red}(X) \approx m$ when $x \approx y$ for all $x, y \in X$.

Lemma 4.6. Let $e'_t$ be as in the box and let $u \in S$ be such that $\text{Red}(\text{Im}_{\text{lex}}(u)) = e'_t$. Then for all $t \in I$ we have $\text{Red}(\text{Im}_{\text{lex}}(d_t u)) = e'_t$.

Proof. Let $m = \text{Im}_{\text{lex}}(u)$ with $m = x_1^{m_1} \cdots x_n^{m_n}$ and $e'_t = x_1^{t_1} \cdots x_n^{t_n}$. $\text{Red}(m) = e'_t$ implies $m_i > m_j \iff r_i > r_j$ by Lemma 4.2.

For $I = \{i_1, \ldots, i_a, n\}$ with $1 \leq i_1 < i_2 < \cdots < i_a < n$, by definition we have $e'_t = \text{Im}_{\text{lex}}(\prod_{i=1}^{a} d_{i_j})$. By Lemma 3.2(1) this is equal to $(x_1 \cdots x_{i_1})(x_1 \cdots x_{i_2}) \cdots (x_1 \cdots x_{i_{a-1}})$.

Collecting all the powers of $x_i$ together we get

$$e'_t = (x_1 \cdots x_{i_1})^{a}(x_{i_1+1} \cdots x_{i_2})^{a-2} \cdots (x_{i_{a-2}+1} \cdots x_{i_{a}})^{1}$$

for $i, j \in [n]$ with $i > j$, we have $r_i > r_j$ if and only if $\forall l \in I$ such that $i \geq l > j$. Hence $m_i > m_j$ if and only if $\forall l \in I$ such that $i \geq l > j$.

By Lemma 3.3(1) $\text{Im}_{\text{lex}}(d_t u) = \text{Im}_{\text{lex}}(d_t)\text{Im}_{\text{lex}}(u) = \text{Im}_{\text{lex}}(d_t)m$. Let $\text{Im}_{\text{lex}}(d_t u) = x_1^{m_1} \cdots x_n^{m_n}$, then:

$$x_1^{m_1} \cdots x_n^{m_n} = \text{Im}_{\text{lex}}(d_t u) = (x_1 \cdots x_t)m = x_1^{1+m_1} \cdots x_t^{1+m_t}x_{t+1}^{m_{t+1}} \cdots x_n^{m_n}.$$  

We now compare $(m_i, m_j)$ and $(m'_i, m'_j)$ for any pair of $i, j \in [n]$.

For $i, j \leq t$: we have $m'_i = m_i + 1$ and $m'_j = m_j + 1$, so we have $(m'_i > m'_j) \iff (m_i > m_j)$.

For $t < i, j$: likewise we have $m'_i = m_i$ and $m'_j = m_j$, so we have $(m'_i > m'_j) \iff (m_i > m_j)$.

For $i \leq t < j$: we have $m'_i = m_i + 1$ and $m'_j = m_j$, so $m'_i > m'_j$. But, by the observation following Equation 4.7 and the fact that $t \in I$, we also have $m_i > m_j$.

Hence by Lemma 4.2 $\text{Red}(\text{Im}_{\text{lex}}(d_t u)) = \text{Red}(\text{Im}_{\text{lex}}(u)) = e'_t$. $\blacksquare$
Proof. In this proof we show that the result holds for any $r$, $r'$ and $I$. Let $m = 0$ for any $d_I$-monic $d$, $\text{Red}(\text{lm}_{\text{lex}}(de_I)) = e'_I$. By Lemma 3.2 \text{(1)} $\text{lm}_{\text{lex}}(de_I) = \text{lm}_{\text{lex}}(d)e'_I$. \hfill $\square$

5 Main Theorem

We now draw together the results of the previous sections to prove that we have a structure theorem.

Lemma 5.1. For $e_I,e'_I$ and $V_I$ as in the box, given distinct $d_I$-monic $r, r_1, \ldots, r_m$, we have $rV_I \cap (\sum_{i=1}^m r_iV_I) = \{0\}$. In particular we have $rV_I \cap r'V_I = 0$ for any $d_I$-monic $r \neq r'$.

For $u \in V_I - \{0\}$ and $d \in k[d_I]$ we have $\text{Red}(G\text{-lm}(du)) \approx e'_I$.

Conversely, if $m$ is an $x$-monic then there exists an $I \subseteq [n]$ with $n \in I$, a $d_I$-monic, $r$, and a $g \in G$ such that $G\text{-lm}(rge_I) = \{m\}$.

Proof. In this proof we show that the result holds for any $d_I$-monic, then use Lemma 3.2 \text{(6)} to get the result about an arbitrary element of $k[d_I]$.

First we make a general observation. By Lemma 3.4 for any $d_I$-monic and any $u, v \in V_I - \{0\}$, we have $G\text{-lm}(ru) \approx G\text{-lm}(rv) \approx \text{lm}_{\text{lex}}(r)e'_I$. For $r'$ a $d_I$-monic $r \neq r'$, we have $\text{lm}_{\text{lex}}(r)e'_I \neq \text{lm}_{\text{lex}}(r')e'_I$. Hence $G\text{-lm}(ru) \neq G\text{-lm}(rv)$.

To prove that $rV_I \cap (\sum_{i=1}^m r_iV_I) = \{0\}$, let $r, r_1, \ldots, r_m \in R$ be distinct $d_I$-monic faithfuls $u, u_1, \ldots, u_m$ be distinct elements of $V_I$. Then by the above observation, $G\text{-lm}(ru_i) \neq G\text{-lm}(ru_j)$ for any $i, j \in [m]$ with $i \neq j$. So by repeated application of Lemma 3.2 \text{(6)}, $G\text{-lm}(\sum_{i=1}^m r_iu_i) \approx G\text{-lm}(ru_j)$ for some $j \in [m]$. Hence, by the above observation, $G\text{-lm}(ru) \neq G\text{-lm}(ru_j)$, so $G\text{-lm}(ru) \neq G\text{-lm}(\sum_{i=1}^m r_iu_j)$. So $rV_I \cap (\sum_{i=1}^m r_iV_I) = \{0\}$. This proves the first statement, the “in particular” statement follows as a special case or from the observation at the start of the proof.

Now we prove that for $d \in k[d_I]$ we have $\text{Red}(G\text{-lm}(du)) \approx e'_I$. For $r$ a $d_I$-monic, by Lemma 3.4 $G\text{-lm}(ru) \approx \text{lm}_{\text{lex}}(r)e'_I$. By Corollary 4.8 $\text{Red}(\text{lm}_{\text{lex}}(r)e'_I) = e'_I$. So by Lemma 4.4 $\text{Red}(G\text{-lm}(ru)) \approx e'_I$. This deals with the case when $d = r$ is a $d_I$-monic.

For $d = \sum_{i=1}^m \lambda_ir_i$, with $\lambda_i \in k - \{0\}$ and $r_i$ $d_I$-monics, the $G\text{-lm}(\lambda_ir_iu)$ are pairwise disjoint. Hence by Lemma 3.2 \text{(6)}, there exists an $j \in \{1, \ldots, m\}$ such that $G\text{-lm}(du) \approx G\text{-lm}(rju)$. So may prove that $\text{Red}(G\text{-lm}(du)) \approx e'_I$ using the “$d$ is a $d_I$-monic case” proved above.

For the converse: By Lemma 3.2 \text{(7)}, it is sufficient to find $I, r$ and $g_m$ for $m$ with the property that $m \geq \text{lex} g_m$ for all $g \in G$. So for the rest of the proof we assume that $m$ has this property.

Now $m = \prod_{i=1}^m \text{lm}_{\text{lex}}(d_i)^{t_i}$, for some $t_i \in \mathbb{N}$. Let $I = \{i | t_i \neq 0\} \cup \{n\}$, so that $m = d_n^t \prod_{i \in I} \text{lm}_{\text{lex}}(d_i)^{t_i}$. Then $e_I$ divides $m$ and $m = \text{lm}_{\text{lex}}(e_I d_n^t \prod_{i \in I} \text{lm}_{\text{lex}}(d_i)^{t_i-1})$. Let $r = d_n^t \prod_{i \in I} d_i^{t_i-1}$ then by Lemma 3.2 $G\text{-lm}(e_I r) = \{m\}$, \hfill $\square$
Theorem 5.2. Let $e_I$ be elements of $S$ such $G$-$\text{lm}(e_I) = \{\text{lm}_{\text{lex}}(d_I)/d_n\}$ and $\text{stab}_G(e_I) = \text{stab}_G(\text{lm}_{\text{lex}}(e_I))$. Let $V_I$ be the $kG$-module generated by $e_I$. Then as $kG$-modules we have:

$$S \cong \bigoplus_{\{n\} \subseteq I \subseteq \{n\}} k[d_I] \otimes_k V_I$$

is a structure theorem for $S$, i.e. the map from right to left is the $kG$-homomorphism $d \otimes_k v \mapsto dv$.

Proof. It is clear that the map is a $kG$-map as inclusion and multiplication by $d_i$ are $kG$-maps. It remains to show that the map is a bijection.

**Injection:** By induction on subsets of $[n]$ containing $n$. The base case is just the observation that for every $I \subseteq [n]$ with $n \in I$, the map $k[d_I] \otimes_k V_I \to S$ is injective. To see this suppose that $u = \sum_{i=1}^m r_i \otimes u_i \to 0$, where $r_i \neq r_j$ for $i \neq j$ and $u_i \in V_I - \{0\}$. Then $\sum_{i=1}^m r_i u_i = 0$. So by the first statement of Lemma 5.1, $r_1 u_1 = \sum_{i=2}^m r_i u_i = 0$ and thus $u = 0$. So $k[d_I] \otimes_k V_I \to S$ is an injective map.

For the inductive hypothesis, suppose that given, $A$, a set of subsets of $[n]$ all of which contain $n$ (i.e. $A \subset P([n])$ and $\forall I \in A, n \in I$), the map $\bigoplus_{I \in A} k[d_I] \otimes_k V_I \to S$ is injective. We want to show that $(\bigoplus_{I \in A} k[d_I] \otimes_k V_I) \oplus (k[d_J] \otimes_k V_J) \to S$ is injective for $J \subseteq [n], n \in J$ and $J \notin A$.

By Lemma 5.1 for all $v \in \phi(k[d_J] \otimes_k V_J)$, $\text{Red}(G$-$\text{lm}(\phi(u))) \approx e'_J$. So it is sufficient to show that for $u = \sum_{I \in A} u_I$ with $u_I \in k[d_I] \otimes_k V_I$, $\text{Red}(G$-$\text{lm}(\phi(u))) \neq e'_J$.

By Lemma 5.1 $\text{Red}(G$-$\text{lm}(\phi(u_I))) \approx e'_I$, it is clear that $e'_J \neq e'_I$ for $I \neq J$. Hence by Lemma 3.2(6), $G$-$\text{lm}(\{\phi(u_I)\})$ are disjoint. By Lemma 3.2(6), $G$-$\text{lm}(\sum_{I \in A} \phi(u_I)) \approx G$-$\text{lm}(\phi(u_{I_0}))$ for some $I_0 \in A$. By Lemma 5.1 again, $\text{Red}(G$-$\text{lm}(\phi(u))) \approx \text{Red}(G$-$\text{lm}(\phi(u_{I_0}))) \approx e'_{I_0}$, and $e'_I \neq e'_J$ as $I_0 \neq J$. This shows that the map is in injection.

**Surjection:** To show that the map is surjective we argue by induction on $G$-$\text{lm}$, where $G$-$\text{lm}(u) > G$-$\text{lm}(v)$ if $G$-$\text{lm}(u) > G$-$\text{lm}(v)$ or if $G$-$\text{lm}(u) \approx G$-$\text{lm}(v)$ and $G$-$\text{lm}(u) \geq G$-$\text{lm}(v)$.

The least $G$-$\text{lm}(u)$ is $\{0\}$, which is clearly mapped onto. For $u \in S$ assume every $v \in S$ with $G$-$\text{lm}(v) < G$-$\text{lm}(u)$ is mapped onto. Pick $m \in G$-$\text{lm}(u)$, $\text{Red}(m) = g \cdot e'_I$ by Lemma 4.3. Then by Lemma 5.1 $\exists r \in k[d_I]$ s.t. $G$-$\text{lm}(rge_I) = \{m\}$, hence $G$-$\text{lm}(u) > G$-$\text{lm}(u - rge_I)$.

$$rge_I = \phi(r \otimes_k ge_I),$$
and by the inductive hypothesis $u - rge_I = \phi(\tilde{u})$ for some $\tilde{u}$, so $\phi(\tilde{u} + r \otimes_k ge_I) = u$.

Note that the modules $V_I$ are not indecomposable. In fact it may be interesting to calculate the vertices of their indecomposable summands as in the example of the upper triangular group, [5] above Corollary 9.5, the modules which occur in the structure theorem, $X_I$ (written as $X_I(j)$ for $j \in I \subseteq \{1, \ldots, n\}$ in the notation of that paper), are induced from a subgroup, $U_J$, which depends on the set of invariants $\{d_I | I \in J\}$. Be warned that we have adopted different conventions to [5], in particular, for us structure theorems are a sum of $k[d_I]_i \in I \otimes_k X_I$, but in [5] they are a sum of $k[d_I]_i \notin I \otimes_k X_I$. 

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It is worth noting that if $k$ were a field, in principal, we could have shown the map in Theorem 5.2 was either injective or surjective and then compared the Hilbert series of the two modules. However this proved somewhat complicated as for $I = \{i_1, \ldots, i_m\}$ with $i_1 < \cdots < i_m = n$, the dimension over $k$ of $V_I$ is

$$\frac{|G|}{|\text{stab}_G(e_I)|} = \frac{n!}{i_1!(i_2-i_1)!(\cdots)(n-i_m-1)!}.$$ 

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