Ising parameterization of QCD Landau free energy and its dynamics

Lijia Jiang,1,2 Jun-Hui Zheng,2 and Horst Stöcker1,2,3

1 Frankfurt Institute for Advanced Studies, 60438 Frankfurt am Main, Germany
2 Institut für Theoretische Physik, Goethe-Universität, 60438 Frankfurt am Main, Germany
3 GSI Helmholtzzentrum für Schwerionenforschung GmbH, 64291 Darmstadt, Germany

We present a general linear parameterization scheme for the QCD Landau free energy in the vicinity of the critical point of chiral phase transition in the $\mu$-$T$ plane. Based on the parametric free energy, we show that, due to the finite size effects, the regions of fluctuations of the order parameter (i.e. the $\sigma$ field) are broadened, and the discontinuities of the first order phase transition are smoothed. Meanwhile, the kurtosis of the $\sigma$ field is universally negative around the critical point. Using the Fokker-Plank equation, we derive the dynamical corrections to the free energy. The dynamical cumulants of the $\sigma$ field on the freeze-out line, record earlier information in the first order phase transition region as compared to the crossover region. The typical behavior of dynamical cumulants can be understood from the equilibrium cumulants by considering the memory effects.

Keywords:

I. INTRODUCTION

The QCD phase structure has been of intensive theoretical and experimental interest for decades [1–4]. In various extreme conditions, the QCD matter could exist as quark-gluon plasma (QGP), hadronic resonance gas (HRG), or color superconductor, etc. A conjecture for the phase diagram in the $\mu$-$T$ plane is summarized in [5]. Of great interest is the chiral phase transition ($\chi$PT) between the hot QGP phase and the HRG phase [6–8]. For small baryon chemical potential, $\mu_B$, lattice simulations show that the transition is a broad crossover [9–12]. For large $\mu_B$, the sign problem of lattice QCD prevents full ab initio simulations. Hence, the information about the order of the phase transition can only be obtained through effective models, like the Nambu-Jona-Lasinio model [13, 14] and quark-meson model [15], or non-perturbative methods like Functional Renormalization Group [16] and Dyson-Schwinger equation [17]. These models show that the transition is of first order for large $\mu_B$. These phenomenological analysis predict the existence of a chiral critical point.

In the vicinity of the critical point, the correlation length of the $\sigma$ field is largely increased, leading to intense fluctuations of the net proton production according to perturbation theory [18]. The related observables are presently measured by the STAR collaboration at the Relativistic Heavy Ion Collider (RHIC) [19, 20]. However, these current experimental data can not even qualitatively be explained by theoretical calculations [21–23], due to the complexities of a realistic dynamical modeling of the experimental situation [24–29]. A main component of the dynamical modeling is the parameterization of the QCD free energy or the equation of state near the QCD critical point.

As the critical behaviors of the $\sigma$ field fall into the same universality class as the 3D Ising model [30], one can obtain the parametric equation of state by mapping the QCD parameters ($\mu, T$) onto the Ising variables ($r, h$) [31–34]. Combined with the dynamical evolution function, there are various interesting dynamical effects presented, like memory effects [22] and universal off-equilibrium scaling behaviors [32, 35]. However, the exact mapping from ($\mu, T$) to ($r, h$) is still unknown, which leads to large uncertainties in the static and dynamical study. Also, the parametric equation of state is obtained in the thermodynamic limit [36–38], which may not be applicable to the RHIC physics with a finite size [39, 40].

In the present paper, we develop a general linear parametric scheme for the QCD Landau free energy in the vicinity of the critical point, by comparing with the free energy of the Ising model. The method can include finite size effects naturally. Using the parametric free energy, we study the finite size effects and the equilibrium cumulants of the $\sigma$ field. The dynamical corrections to the free energy and the cumulants are evaluated by employing the Fokker-Plank equation.

Let us point out that for a finite size system, the discontinuities of the first order phase transition (FOPT) are smoothed, and the distribution of the $\sigma$ field is broadened. Consequently, the kurtosis of the $\sigma$ field is universally negative, both on the crossover side and on the FOPT side. For the dynamical evolution, the cumulants record earlier information in the FOPT region, as compared to the crossover region.

The article is structured as follows. In section II, we develop the Ising mapping of the QCD Landau free energy. The linear parameterization of the free energy is realized. In section III, the dynamical free energy is obtained through the Fokker-Plank equation. The static finite size effects are discussed in section IV and the dynamical cumulants are evaluated in section V. Finally, we give more discussions.
FIG. 1: A sketch of the QCD phase diagram. The colored 4 parts (I-IV) are sorted according to the sign of $\eta_1$ and $\eta_2$. The dash-dotted lines determined by $\Delta = 0$, are the boundary of the coexistence of two minima.

II. ISING PARAMETERIZATION

According to the Landau theory of phase transitions, the free energy in the critical region is supposed to be analytic and obey the symmetry of the Hamiltonian. The Landau free energy density of the $\chi$PT\cite{41, 42} can be written in terms of the $\sigma$ field as

$$\Omega[\sigma] = \alpha_1(\mu, T)\sigma + \frac{\alpha_2(\mu, T)}{2}\sigma^2 + \frac{\alpha_3(\mu, T)}{3}\sigma^3 + \frac{\alpha_4(\mu, T)}{4}\sigma^4,$$  

with the Taylor expansion of the $\sigma$ done up to the fourth order. The constant term is omitted here because it is irrelevant to the structure of the QCD phase diagram. A zero-momentum mode approximation of the $\sigma$ field is assumed. The distribution of the critical fluctuations is described by the probability function $P[\sigma] \propto \exp\{-\Omega[\sigma]/T\}$\cite{18}, where $V$ is the volume of the system. In the chiral limit, $\alpha_1 = \alpha_3 = 0$\cite{43}, while for the physical world, a finite term, $\alpha_1\sigma$, is introduced to handle the explicit chiral symmetry breaking of the quark masses. The cubic term, $\alpha_3\sigma^3$, emerges only after the renormalization contributed from the high-momentum modes of the $\sigma$ field. The coefficient of the quartic term, $\alpha_4$, is supposed to be positive, in order to sustain the stability of the system.

In the thermodynamic limit ($V \to \infty$), the phase structure is fully determined by the global minimum of the free energy (1). The information on the $\chi$PT is presented by performing a translation transformation for the sigma field, $\sigma = \tilde{\sigma} + \sigma_c$, with

$$\sigma_c(\mu, T) = -\frac{\alpha_3(\mu, T)}{3\alpha_4(\mu, T)}.$$ 

Consequently, the cubic term in eq. (1) can be eliminated, and we obtain the translated free energy density

$$\Omega[\tilde{\sigma}] = \eta_1 (\mu, T) \tilde{\sigma} + \frac{\eta_2 (\mu, T)}{2} \tilde{\sigma}^2 + \frac{\eta_4 (\mu, T)}{4} \tilde{\sigma}^4,$$ 

which has exactly the same form as the free energy density of the Ising model. The redefined coefficients, $\eta_i$, can be expressed as functions of the original coefficients $\alpha_i$ and the shift $\sigma_c$ by

$$\eta_1 = \alpha_1 + \alpha_2\sigma_c - 2\alpha_4\sigma_c^2,$$

$$\eta_2 = \alpha_2 - 3\alpha_4\sigma_c^2,$$

$$\eta_4 = \alpha_4.$$ 

Now the $\tilde{\sigma}$ field has become an effective order parameter, which behaves just like the spin density in the Ising model. With the translated free energy, we can divide the phase spaces in the $\mu$-$T$ plane into four parts, according to the sign of $\eta_1$ and $\eta_2$. As shown in Fig. 1, on the phase transition line, $\eta_1 = 0$, the sign of $\eta_2$ determines the order of the phase transition. In regions II and IV, $\eta_2 < 0$, two minima coexist in the coexistence region, which is determined by $\Delta \equiv (\frac{\eta_1}{\eta_2})^2 + (\frac{\eta_4}{\eta_2})^3 < 0$. The physical vacuum is the lower one, which is the typical character of the FOPT. On the other part of the $\mu$-$T$ plane, $\Delta > 0$, only one minimum exists,
and the system undergoes a continuous phase transition (crossover) at $\eta_1$ crosses its zero point. The sign of $\eta_1$ determines the location of the global minimum. In general, in the upper regions I and II, with $\eta_1 > 0$, the global minimum is at $\bar{\sigma} < 0$ (i.e. $\sigma < \sigma_c$); while in the lower regions III and IV, with $\eta_1 < 0$, the global minimum is at $\bar{\sigma} > 0$ (i.e. $\sigma > \sigma_c$). At the critical point $(\mu_c, T_c)$, both $\eta_1$ and $\eta_2$ vanish.

To parameterize the coefficients in $\Omega(\bar{\sigma})$ linearly in the $\mu$-$T$ plane, it is convenient to define two unit vectors $\mathbf{b} = (b_\mu, b_T)$ and $\mathbf{b}' = (b'_\mu, b'_T)$. The vector $\mathbf{b}$ is parallel to the tangent of the phase transition curve at the critical point. Note that $\mathbf{b}$ and $\mathbf{b}'$ are not necessarily orthogonal. The angle between them should be determined by experiment. Around the critical point, by projecting the vector $(\mu - \mu_c, T - T_c)$ onto the perpendicular vector of $\mathbf{b}$ and that of $\mathbf{b}'$, respectively, the coefficients are linearly parameterized as:

$$
\eta_1(\mu, T) = d_1 [b_T (\mu - \mu_c) - b_\mu (T - T_c)],
$$

$$
\eta_2(\mu, T) = d_2 [-b'_T (\mu - \mu_c) + b'_\mu (T - T_c)],
$$

$$
\eta_4(\mu, T) = d_4.
$$

With these projections, the sign of $\eta_1$ and $\eta_2$ in the different regions (I-IV) can be correctly expressed by constraining all the constants to $d_i > 0$ $(i = 1, 2, 4)$. Along the phase transition curve, $\eta_1(\mu, T) = 0$, the correlation length is $\xi = \eta_2^{-1/2}$, which diverges at the critical point, and is finite on both the FOPT side and the crossover side [3]. In the critical region, the variable $\sigma_c(\mu, T)$ is parameterized as a constant, $\sigma_c(\mu, T) = \sigma_c(\mu_c, T_c)$, with the zeroth order approximation. Then, the original free energy $\Omega[\sigma]$ (eq.(1)) can also be linearly parameterized, through the relations (2) and (4)-(6).

In the current Ising parameterization of the QCD free energy, the coefficients $\eta_1$ and $\eta_2$ in the translated free energy (3) correspond exactly to the Ising variables $h$ and $r$, respectively. The parameterization used here is universal. Hence, it can also be applied to the (hadronic) gas-liquid phase transition [44, 45] or to other similar scenarios, by replacing the $\sigma$ field with the corresponding order parameter. Moreover, it can be generalized to finite size systems (see the section 'finite size effects').

Through the transformation of eq. (1) to eq. (3), we present the connection between the QCD free energy and the Ising free energy. Different from the former approaches on the QCD equation of state [31–34], we provide a new parametric approach on the QCD free energy. Within linear approximation, it makes the direct mapping from the curved QCD phase transition lines in $\mu$-$T$ plane to the Ising phase transition lines in the $r$-$h$ plane.

Through the transformation of eq. (1) to eq. (3), we present the similarity between the QCD free energy and the Ising free energy. It provides a new mapping method different from the former ones based on the QCD equation of state [31–34]. The new mapping is rather direct. It makes the mapping from the curved QCD phase transition lines in $\mu$-$T$ plane to the Ising phase transition in the $r$-$h$ plane with linear approximation.

## III. DYNAMICAL FREE ENERGY

With the parameterized equilibrium free energy, we now develop the dynamical equations. The Fokker-Plank equation for the dynamical probability distribution (denoted by $P(\sigma; t)$) of the $\sigma$ field is [22]

$$
\partial_t P(\sigma; t) = -\frac{1}{m_{\sigma}^2 \tau_{\text{eff}}} \partial_\sigma \left( \partial_\sigma (\Omega(\sigma; t) - \Omega_0[\sigma]) P(\sigma; t) \right).
$$

Here, $\Omega[\sigma; t] \equiv -\theta^T \ln P[\Omega; t]$ is the dynamical free energy density. $\Omega_0[\sigma]$ is the equilibrium free energy density at $(\mu(t), T(t))$ (see eq. (1)). The parameter $\tau_{\text{eff}}$ is the effective relaxation rate. $m_{\sigma}$ is the equilibrium mass of the $\sigma$ field, defined as $m_{\sigma}^2 = \frac{d^4 \Omega_0[\sigma]}{d \sigma^4}|_{\sigma = \sigma_0}$, where $\sigma_0$ is the global minimum of $\Omega_0[\sigma]$. The following calculation assumes that, the dependence of the relaxation rate $\tau_{\text{eff}}$ on the equilibrium correlation length $\xi_{\text{eq}}(\mu, T) = m_{\sigma}^{-1}$ satisfies $\tau_{\text{eff}} = \tau_{\text{rel}}(\frac{\xi_{\text{eq}}}{\xi_{\text{ini}}})^3$ [22]. Here $\tau_{\text{rel}}$ and $\xi_{\text{ini}}$ are the initial relaxation rate and the initial equilibrium correlation length, respectively. The value $z = 3$ is given by the dynamical critical exponent of Model H [46, 47].

The evolution equation of $\Omega[\sigma; t]$ is obtained from eq. (10),

$$
\partial_t \Omega(\sigma; t) = \Omega(\sigma; t) \partial_t \left( \ln \frac{V}{T} \right) + \frac{T}{V} \partial^2_\sigma (\Omega(\sigma; t) - \Omega_0[\sigma]) \frac{1}{m_{\sigma}^2 \tau_{\text{eff}}} - \frac{\partial_\sigma (\Omega(\sigma; t) - \Omega_0[\sigma]) \partial_\sigma \Omega(\sigma; t)}{m_{\sigma}^2 \tau_{\text{eff}}}.
$$

(11)
Denoting $\Omega[\sigma; t] = \sum_i \alpha_i(t)\sigma^i/i$, the evolution of each coefficient $\alpha_i$ becomes,

$$\frac{d\alpha_i(t)}{dt} = \alpha_i(t)\partial_i\left(\ln \frac{T}{V}\right) + \frac{i(i + 1)T}{V} \frac{\alpha_{i+2}(t) - \alpha_{i+2}^0}{m_{\sigma_i}^2 r_{\text{eff}}} - \sum_{j=0}^{i-1} \frac{i\alpha_{i-j+1}(t)[\alpha_{j+1}(t) - \alpha_{j+1}^0]}{m_{\sigma_j}^2 r_{\text{eff}}}.$$ (12)

Here $\alpha_i^0$ is the coefficient of $\Omega_i[\sigma]$. The terms, $\alpha^i$, with power $i > 4$ emerge automatically during the time evolution from the above coupled equations. As before, the free energy is cut at the order $i = 4$, assuming that the contributions from higher power terms are negligible.

IV. FINITE SIZE EFFECTS

The discussion of the free energy density (1) is based on the assumption of the thermodynamic limit. In reality, the size of the heavy ion system is finite. This has a direct influence on the renormalization process. Consequently, the coefficients $\alpha_i$ in eq. (1) depend on the volume [48]. Hereafter, we will not focus in detail on these finite size corrections to the free energy density. Rather, we point out that the parameters $\mu$, $T$, $d_1$, $d_2$, $b_\mu(b_T)$, and $\sigma$ are volume dependent. The numerical calculations, are done for the following parameters: $(\mu, T) = (240, 170)$ MeV, $d_1 = 3 \times 10^4$ MeV$^2$, $d_2 = 400$ MeV, $d_4 = 15$, $\sigma = 50$ MeV, $b_T = -b_\mu = 0.99$, and $b_\mu \approx b_T = (1 - b_\mu^2)^{1/2}$. Here, $b \perp b'$ is set for simplicity. The values of these parameters are estimated so that the cumulants are $\kappa_n \sim 10^n$ MeV$^n$.

Even without considering its effects on the free energy density, the finite volume can still influence the fluctuations of the $\sigma$ field through the probability function $P(\sigma) \propto \exp[-\Omega(\sigma)V/T]$. Fig. 2 shows the cumulants of the $\sigma$ field with respect to the chemical potential is fixed at $\mu = 250$ MeV.

![The cumulants of the $\sigma$ field are shown with respect to the temperature, for different volumes. The chemical potential is fixed at $\mu = 250$ MeV.](image)

FIG. 2: The cumulants of the $\sigma$ field are shown with respect to the temperature, for different volumes. The chemical potential is fixed at $\mu = 250$ MeV.

The cumulants of the $\sigma$ field are defined as $\kappa_1 = \mu_1$, $\kappa_2 = \mu_2 - \mu_1^2$, $\kappa_3 = \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3$, and $\kappa_4 = \mu_4 - 4\mu_3\mu_1 - 3\mu_2^2 + 12\mu_2\mu_1^2 - 6\mu_1^4$, with the moments $\mu_n = \langle \sigma^n \rangle = \int d\sigma^n P(\sigma)/\int d\sigma P(\sigma)$. 

---

1 The cumulants of the $\sigma$ field are defined as $\kappa_1 = \mu_1$, $\kappa_2 = \mu_2 - \mu_1^2$, $\kappa_3 = \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3$, and $\kappa_4 = \mu_4 - 4\mu_3\mu_1 - 3\mu_2^2 + 12\mu_2\mu_1^2 - 6\mu_1^4$, with the moments $\mu_n = \langle \sigma^n \rangle = \int d\sigma^n P(\sigma)/\int d\sigma P(\sigma)$. 

---
V. DYNAMICAL CUMULANTS

The evolution equation (12) is employed to study the dynamical cumulants. Along the trajectory, both the chemical potential and the volume are supposed to be fixed [53]. The evolution of the temperature satisfies $T(t) = T_{\text{ini}} \left( \frac{t_{\text{ref}}}{t_{\text{ref}}} \right)^{-1}$, where $T_{\text{ini}}$ is the initial temperature, $t_{\text{ref}} = 10$ fm is the reference time and $\lambda = 0.45$ [22]. In Fig. 4, we plot the evolution of the cumulants for different relaxation rates, starting from the initial temperature $T_{\text{ini}} = 185$ MeV. The memory effect shown in Fig. 4 is similar for all chemical potentials around $\mu_c$. The peaks and dips of the cumulants are suppressed during the evolution with increasing relaxation rate. Fig. 5 shows the dynamical cumulants with respect to the chemical potential. The initial and the freeze-out temperature are supposed to be about 5.05 MeV above and 3.37 MeV below the phase transition temperature, respectively (i.e. $\eta_1 = 1.5 \times 10^3$ MeV$^3$ for the initial state and $\eta_1 = -1.0 \times 10^3$ MeV$^3$ for the final state, see the lower r.h.s. subfigure in Fig. 3). For different relaxation rates, the dynamical cumulants on the FOPT side significantly deviate from the equilibrium cumulants, and present rich structures. Especially, for $\tau_{\text{rel}} = 0.2$ fm, the change of sign for $\kappa_3$ and the presence of non-monotonic behavior for $\kappa_4$ compared to equilibrium ones are suggestive for the understanding of the STAR data [20].

The dynamical behavior of the cumulants can be understood from the equilibrium cumulants by considering the memory effects. Let us define an effective temperature where the equilibrium expectation of the sigma field, $\kappa_1$, equals to the dynamical $\kappa_1$ on the freeze-out line. The effective temperature lines for different relaxation rates are shown in the lower r.h.s. subfigure of Fig. 3. The equilibrium cumulants $\kappa_2$-$\kappa_4$ on the effective temperature lines, approximately reflect the sign of the dynamical cumulants in Fig. 5. Specifically, for $\kappa_3$, when the relaxation rate is large, say $\tau_{\text{rel}} = 0.2$ fm, the effective temperature in the crossover region is basically below the phase transition line ($\eta_1 < 0$), while in the FOPT region, it is still above the phase transition line ($\eta_1 > 0$), leading to a positive dynamical $\kappa_3$. For $\kappa_4$, when $\tau_{\text{rel}} = 0.1$ fm, the effective temperature falls into the negative kurtosis region; when $\tau_{\text{rel}} = 0.2$ fm ($\tau_{\text{rel}} = 0.05$ fm), the effective temperature line crosses the right (left) sign-change line of the kurtosis. In general, the memory effect in the FOPT region is more significant than that in the crossover region (i.e.
FIG. 4: The dynamical cumulants with respect to the temperature, for different relaxation rates. Here $\mu = 250$ MeV and $V = 10^3$ fm$^3$.

FIG. 5: The dynamical cumulants with respect to the chemical potential on the freeze-out line, for different relaxation rates.

the cumulants record earlier information). This is due to the barrier in the free energy in the FOPT region.

VI. DISCUSSION

Now we compare our method with the former studies on the dynamical cumulants in [22, 51, 52]. In [22], the dynamical evolution of cumulants are derived from the Fokker-Plank equation. The parametric QCD equation of state near the QCD critical point in the thermodynamic limit is taken as input. However, the mapping from Ising variables $(r, h)$ to the QCD variables $(T, \mu)$ is not clear, as pointed out in their paper. The advantage of this method is that it contains the quantum fluctuation correction to the critical indexes. However, the method can not be extended directly to the FOPT region when the finite size effect becomes significant. In [51, 52], the dynamical evolution of high order cumulants are studied based on event-by-event simulations of Langevin equation. The effective potential of $\sigma$ is obtained by evaluating the linear sigma model. It takes account of the
fluctuation effect (long wavelength modes) in the real space. However, the critical point in the model depends on the number of flavors of the quarks and the coupling strength of the interaction, and locates in the hadron resonance gas region, below the statistical freeze-out line.

In our calculations, the Ising mapping is built between the QCD free energy and Ising free energy by using their similarities. The dynamical free energy is derived from the Fokker-Plank equation. The corresponding dynamical cumulants are calculated based on the dynamical free energy. In this paper, we use the zero mode approximation, which is intrinsic a mean field approximation. The nonzero modes can be further taken into account if we employ the Ginzburg-Landau free energy. Thus the mapping is compatible with Langevin equation. This method can be easily extended to finite size system, and is also applicable in the FOPT region. Furthermore, the mapping is model independent and the parameterization of free energy is universal within the 3D Ising model universality class.

In summary, we have provided a new parametric approach to the QCD free energy near the critical point by comparing it to the Ising model. The parameterization can be applied in all phase transitions within the same universality class. The volume of the system significantly influences the fluctuations of the $\sigma$ field. A small volume smoothes the discontinuities in the FOPT region. In addition, compared to the crossover region, the cumulants in the first order phase transition region are enhanced, due to the broadening of $\sigma$'s distribution. The kurtosis is generally negative around the critical point. We also study the dynamical evolution of the free energy and the related cumulants. We find that earlier information about the cumulants are recorded in the FOPT region, rather than in the crossover region. The sign of the dynamical cumulants are reflected in general by the equilibrium cumulants, after considering the memory effects.

Note that the current parameterization of the QCD free energy is linear, it works in the region close to the critical point. In a wider region, higher order expansions of ($\mu$, $T$) should be taken into account. The parametric free energy and its dynamical evolution can be easily employed in a realistic dynamical model (like chiral hydrodynamics[25–28] or Hydro+[29]) in RHIC, with tunable location of the critical point and controllable range of the critical region as input. The realistic dynamical modeling with this parametric QCD free energy will be done in future studies.

VII. ACKNOWLEDGMENTS

L. Jiang thanks Volodymyr Vovchenko, Shanjin Wu and Huichao Song for reading the manuscript and comments. H. Stöcker acknowledges the support through the Judah M. Eisenberg Laureatus Chair at Goethe University. This work is supported by the GSI Helmholtzzentrum f¨ur Schwerionenforschung and by the Helmholtz International Center for the Facility for Antiproton and Ion Research (HIC for FAIR) within the framework of the Landes-O

...cumulants, after considering the memory effects.

Note that the current parameterization of the QCD free energy is linear, it works in the region close to the critical point. In a wider region, higher order expansions of ($\mu$, $T$) should be taken into account. The parametric free energy and its dynamical evolution can be easily employed in a realistic dynamical model (like chiral hydrodynamics[25–28] or Hydro+[29]) in RHIC, with tunable location of the critical point and controllable range of the critical region as input. The realistic dynamical modeling with this parametric QCD free energy will be done in future studies.

VII. ACKNOWLEDGMENTS

L. Jiang thanks Volodymyr Vovchenko, Shanjin Wu and Huichao Song for reading the manuscript and comments. H. Stöcker acknowledges the support through the Judah M. Eisenberg Laureatus Chair at Goethe University. This work is supported by the GSI Helmholtzzentrum für Schwerionenforschung and by the Helmholtz International Center for the Facility for Antiproton and Ion Research (HIC for FAIR) within the framework of the Landes-O

...cumulants, after considering the memory effects.

Note that the current parameterization of the QCD free energy is linear, it works in the region close to the critical point. In a wider region, higher order expansions of ($\mu$, $T$) should be taken into account. The parametric free energy and its dynamical evolution can be easily employed in a realistic dynamical model (like chiral hydrodynamics[25–28] or Hydro+[29]) in RHIC, with tunable location of the critical point and controllable range of the critical region as input. The realistic dynamical modeling with this parametric QCD free energy will be done in future studies.

VII. ACKNOWLEDGMENTS

L. Jiang thanks Volodymyr Vovchenko, Shanjin Wu and Huichao Song for reading the manuscript and comments. H. Stöcker acknowledges the support through the Judah M. Eisenberg Laureatus Chair at Goethe University. This work is supported by the GSI Helmholtzzentrum für Schwerionenforschung and by the Helmholtz International Center for the Facility for Antiproton and Ion Research (HIC for FAIR) within the framework of the Landes-O

...cumulants, after considering the memory effects.

Note that the current parameterization of the QCD free energy is linear, it works in the region close to the critical point. In a wider region, higher order expansions of ($\mu$, $T$) should be taken into account. The parametric free energy and its dynamical evolution can be easily employed in a realistic dynamical model (like chiral hydrodynamics[25–28] or Hydro+[29]) in RHIC, with tunable location of the critical point and controllable range of the critical region as input. The realistic dynamical modeling with this parametric QCD free energy will be done in future studies.

VII. ACKNOWLEDGMENTS

L. Jiang thanks Volodymyr Vovchenko, Shanjin Wu and Huichao Song for reading the manuscript and comments. H. Stöcker acknowledges the support through the Judah M. Eisenberg Laureatus Chair at Goethe University. This work is supported by the GSI Helmholtzzentrum für Schwerionenforschung and by the Helmholtz International Center for the Facility for Antiproton and Ion Research (HIC for FAIR) within the framework of the Landes-O

...cumulants, after considering the memory effects.

Note that the current parameterization of the QCD free energy is linear, it works in the region close to the critical point. In a wider region, higher order expansions of ($\mu$, $T$) should be taken into account. The parametric free energy and its dynamical evolution can be easily employed in a realistic dynamical model (like chiral hydrodynamics[25–28] or Hydro+[29]) in RHIC, with tunable location of the critical point and controllable range of the critical region as input. The realistic dynamical modeling with this parametric QCD free energy will be done in future studies.

VII. ACKNOWLEDGMENTS

L. Jiang thanks Volodymyr Vovchenko, Shanjin Wu and Huichao Song for reading the manuscript and comments. H. Stöcker acknowledges the support through the Judah M. Eisenberg Laureatus Chair at Goethe University. This work is supported by the GSI Helmholtzzentrum für Schwerionenforschung and by the Helmholtz International Center for the Facility for Antiproton and Ion Research (HIC for FAIR) within the framework of the Landes-O

...cumulants, after considering the memory effects.

Note that the current parameterization of the QCD free energy is linear, it works in the region close to the critical point. In a wider region, higher order expansions of ($\mu$, $T$) should be taken into account. The parametric free energy and its dynamical evolution can be easily employed in a realistic dynamical model (like chiral hydrodynamics[25–28] or Hydro+[29]) in RHIC, with tunable location of the critical point and controllable range of the critical region as input. The realistic dynamical modeling with this parametric QCD free energy will be done in future studies.

VII. ACKNOWLEDGMENTS

L. Jiang thanks Volodymyr Vovchenko, Shanjin Wu and Huichao Song for reading the manuscript and comments. H. Stöcker acknowledges the support through the Judah M. Eisenberg Laureatus Chair at Goethe University. This work is supported by the GSI Helmholtzzentrum für Schwerionenforschung and by the Helmholtz International Center for the Facility for Antiproton and Ion Research (HIC for FAIR) within the framework of the Landes-O

...cumulants, after considering the memory effects.

Note that the current parameterization of the QCD free energy is linear, it works in the region close to the critical point. In a wider region, higher order expansions of ($\mu$, $T$) should be taken into account. The parametric free energy and its dynamical evolution can be easily employed in a realistic dynamical model (like chiral hydrodynamics[25–28] or Hydro+[29]) in RHIC, with tunable location of the critical point and controllable range of the critical region as input. The realistic dynamical modeling with this parametric QCD free energy will be done in future studies.
[21] L. Jiang, P. Li and H. Song, Phys. Rev. C 94, 024918 (2016).
[22] S. Mukherjee, R. Venugopalan and Y. Yin, Phys. Rev. C 92, 034912 (2015).
[23] M. Bluhm, M. Nahrgang, S. A. Bass and T. Schaefer, Eur. Phys. J. C 77, 210 (2017).
[24] K. Paech, H. Stöcker and A. Dumitru, Phys. Rev. C 68, 044907 (2003).
[25] M. Nahrgang, S. Leupold, C. Herold and M. Bleicher, Phys. Rev. C 84, 024912 (2011).
[26] M. Nahrgang, C. Herold, S. Leupold, I. Mishustin and M. Bleicher, J. Phys. G 40, 055108 (2013).
[27] C. Herold, M. Nahrgang, I. Mishustin and M. Bleicher, Phys. Rev. C 87, 014907 (2013).
[28] C. Herold, M. Nahrgang, I. Mishustin and M. Bleicher, Nucl. Phys. A 925, 14 (2014).
[29] M. Stephanov and Y. Yin, arXiv:1712.10305 [nucl-th].
[30] S. Gavin, A. Gocksch and R. D. Pisarski, Phys. Rev. D 49, 3079 (1994).
[31] R. Guida and J. Zinn-Justin, Nucl. Phys. B 489, 626 (1997).
[32] B. Berdnikov and K. Rajagopal, Phys. Rev. D 61, 105017 (2000).
[33] C. Nonaka and M. Asakawa, Phys. Rev. C 71, 044904 (2005).
[34] M. A. Stephanov, Phys. Rev. Lett. 107, 052301 (2011).
[35] S. Mukherjee, R. Venugopalan and Y. Yin, Phys. Rev. Lett. 117, 222301 (2016).
[36] J. Cardy, Scaling and Renormalization in Statistical Physics, Cambridge University Press, ISBN 978-0-521-49959-0 (1996).
[37] F. Kos, D. Poland, D. Simmons-Duffin and A. Vichi, JHEP 2016: 36 (2016).
[38] Z. Komargodski, and D. Simmons-Duffin, J. Phys. A: Mathematical and Theoretical, 50 154001 (2017).
[39] C. N. Yang and T. D. Lee, Phys. Rev. 87, 404 (1952); T. D. Lee and C. N. Yang, Phys. Rev. 87, 410 (1952).
[40] R. D. Pisarski and F. Wilczek, Phys. Rev. D 29, 338 (1984).
[41] M. Gell-Mann and M. Levy, Nuovo Cimento 16, 705 (1960).
[42] N. Petropoulos, J. Phys. G: Nuclear and Particle Physics 25, 2225 (1999).
[43] V. Vovchenko, D. V. Anchishkin, M. I. Gorenstein and R. V. Poberezhnyuk, Phys. Rev. C 92, 054901 (2015).
[44] V. Vovchenko, M. I. Gorenstein and H. Stoecker, Phys. Rev. Lett. 118, 182301 (2017).
[45] P. C. Hohenberg, and B. I. Halperin, Rev. Mod. Phys. 49, 435 (1977).
[46] D. T. Son and M. A. Stephanov, Phys. Rev. D 70, 056001 (2004).
[47] E. Brezin and J. Zinn-Justin, Nucl. Phys. B 257, 867 (1985).
[48] Y. Imry, Phys. Rev. B 21, 2042 (1980).
[49] E. E. Zabrodin, L. V. Bravina, H. Stöcker and W. Greiner, Phys. Rev. C 59, 894 (1999).
[50] L. Jiang, S. Wu and H. Song, Nucl. Phys. A 967, 441 (2017).
[51] L. Jiang, S. Wu and H. Song, EPJ Web Conf. 17, 11600 (2018).
[52] The time of evolution around the phase transition curve turns out to be rather brief in this numerical calculation. Hence, the change of the volume during the expansion is omitted here.