ON BOUNDED APPROXIMATE IDENTITIES AND EXISTENCE OF DENSE IDEALS IN REAL LOCALLY C*- AND LOCALLY JB-ALGEBRAS

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Abstract. It has been established by Inoue that a complex locally C*-algebra with a dense ideal possesses a bounded approximate identity which belongs to that ideal. It has been shown by Fritzsche that if a unital complex locally C*-algebra has an unbounded element then it also has a dense one-sided ideal. In the present paper we obtain analogues of the aforementioned results of Inoue and Fritzsche for real locally C*-algebras (projective limits of projective families of real C*-algebras), and for locally JB-algebras (projective limits of projective families of JB-algebras).

1. Introduction

Banach associative regular *-algebras over \( \mathbb{C} \), so called \( C^* \)-algebras, were first introduced in 1940’s by Gelfand and Naimark in the paper [7]. Since then these algebras were studied extensively by various authors, and now, the theory of C*-algebras is a big part of Functional Analysis with applications in almost all branches of Modern Mathematics and Theoretical Physics. For the basics of the theory of C*-algebras, see for example Pedersen’s monograph [14].

The real analogues of complex C*-algebras, so called real C*-algebras, which are real Banach *-algebras with regular norms such that their complexifications are complex C*-algebras, were studied in parallel by many authors. For the current state of the basic theory of real C*-algebras, see Li’s monograph [11].

The real Jordan analogues of complex C*-algebras, so called JB-algebras, were first defined by Alfsen, Schultz and Størmer in [1] as the real Banach–Jordan algebras satisfying for all pairs of elements \( x \) and \( y \) the inequality of fineness

\[
\|x^2 + y^2\| \geq \|x\|^2,
\]

and regularity identity

\[
\|x^2\| = \|x\|^2.
\]

The basic theory of JB-algebras is fully treated in monograph of Hanche-Olsen and Størmer [8]. If \( A \) is a C*-algebra, or a real C*-algebra, then the self-adjoint part
A \text{sa} of \( A \) is a JB-algebra under the Jordan product
\[ x \circ y = \frac{(xy + yx)}{2}. \]

Closed subalgebras of \( A_{\text{sa}} \), for some C*-algebra or real C*-algebra \( A \), become relevant examples of JB-algebras, and are called JC-algebras.

Complete locally multiplicatively-convex algebras or equivalently, due to Arens-Michael Theorem, projective limits of projective families of Banach algebras, were first studied by Arens in [3] and Michael in [13]. They were since studied by many authors under different names. In particular, projective limits of projective families of C*-algebras were studied by Inoue in [9], Apostol in [2], Schm"udgen in [16], Phillips in [15], Bhatt and Karia in [4], Fritzche in [6], etc. We will follow Inoue [9] in the usage of the name locally C*-algebras for these topological algebras. The current state of the basic theory of locally C*-algebras is treated in the monograph of Fragoulopoulou [5].

In particular, Inoue in [9] proved that a complex locally C*-algebra with a dense ideal possesses a bounded approximate identity which belongs to the aforementioned ideal. On the other hand, Fritzche showed in [6] that if a unital complex locally C*-algebra has an unbounded element then it also has a dense one-sided ideal.

Katz and Friedman in [10] introduced topological algebras which are projective limits of projective families of real C*-algebras under the name of real locally C*-algebras, and projective limits of projective families of JB-algebras under the name of locally JB-algebras.

The present paper is aimed to the presentation of analogues of the cited above results of Inoue and Fritzche for real locally C*-algebras and locally JB-algebras.

2. Preliminaries

Let us first recall basic facts on JB algebras and fix the notation. (for details see the monograph [8]). Let \( A \) be JB-algebra endowed with a product " \( \circ \) " . We write
\[ A_1 = \{ a \in A; \| a \| \leq 1 \}, \]
\[ A^+ = \{ a^2; a \in A \}, \]
\[ A_1^+ = A_1 \cap A^+. \]

For \( a \in A \), the mapping \( T_a : A \to A \),
is defined by putting \( T_a b = a \circ b \), and \( U_a : A \to A \),
is defined by putting \( U_a b = 2a \circ (a \circ b) - a^2 \circ b \).

It is well known that \( U_a(A^+) \subseteq A^+ \).

A closed subspace \( I \) of \( A \) is called a Jordan ideal if \( T_a(A) \subseteq I \),
for all $a \in I$. Similarly, a closed subspace $Q$ of $A$ is said to be a \textit{quadratic ideal} if

$$U_a(A) \subseteq Q,$$

for all $a \in Q$. Both the Jordan ideal and the quadratic ideal of a Jordan algebra are subalgebras. We will use the symbol $B[a_1,\ldots,a_n]$ to denote the JB-subalgebra $B$ of $A$ generated by elements $a_1,\ldots,a_n$. The elements $a,b \in A$ are said to be \textit{operator commuting} if

$$T_aT_b = T_bT_a.$$

The JB-algebra $A$ is called \textit{associative or abelian} if it consists of operator commuting elements. The associative subalgebra $B[a]$ is said to be \textit{singly generated}.

The following two identities are corollaries from Shirshov-MacDonald theorem (see [8] for details):

$$(U_xy)^2 = U_xU_yx^2;$$

$$U_Uy^z = U_xU_yU_z.$$

The second identities is known by the name of \textit{MacDonald Identity}.

Now, let us briefly recall some more basic material from the aforementioned sources one needs to comprehend what follows.

A Hausdorff topological vector space over the field of $\mathbb{R}$ or $\mathbb{C}$, in which any neighborhood of the zero element contains a convex neighborhood of the zero element; in other words, a topological vector space is a \textit{locally convex space} if and only if the topology of is a Hausdorff locally convex topology.

A number of general properties of locally convex spaces follows immediately from the corresponding properties of locally convex topologies; in particular, subspaces and Hausdorff quotient spaces of a locally convex space, and also products of families of locally convex spaces, are themselves locally convex spaces. Let $\Lambda$ be an upward directed set of indices and a family

$$\{E_\alpha, \alpha \in \Lambda\},$$

of locally convex spaces (over the same field) with topologies

$$\{\tau_\alpha, \alpha \in \Lambda\}.$$

Suppose that for any pair $(\alpha,\beta)$,

$$\alpha \leq \beta,$$

$\alpha,\beta \in \Lambda$, there is defined a continuous linear mapping

$$g_\alpha^\beta : E_\beta \to E_\alpha.$$

A family

$$\{E_\alpha, \alpha \in \Lambda\}$$

is called \textit{projective}, if for each triplet $(\alpha,\beta,\gamma)$,

$$\alpha \leq \beta \leq \gamma,$$

$\alpha,\beta,\gamma \in \Lambda$,

$$g_\gamma^\gamma = g_\gamma^\beta \circ g_\beta^\alpha,$$

and for each $\alpha \in \Lambda$,

$$g_\alpha^\alpha = Id.$$
Let $E$ be the subspace of the product
\[ \prod_{\alpha \in \Lambda} E_{\alpha}, \]
whose elements
\[ x = (x_{\alpha}), \]
satisfy the relations
\[ x_{\alpha} = g_{\alpha}^{\beta}(x_{\beta}), \]
for all $\alpha \leq \beta$. The space $E$ is called the projective limit of the projective family $E_{\alpha}$, $\alpha \in \Lambda$, with respect to the family $(g_{\alpha}^{\beta})$, $\alpha, \beta \in \Lambda$ and is denoted by
\[ \lim \rightarrow E_{\beta}, \]
or
\[ \lim \leftarrow E_{\alpha}. \]
The topology of $E$ is the projective topology with respect to the family
\[ (E_{\alpha}, g_{\alpha}^{\beta}, \pi_{\alpha}), \]
$\alpha \in \Lambda$, where $\pi_{\alpha}$, $\alpha \in \Lambda$, is the restriction to the subspace $E$ of the projection
\[ \hat{\pi}_{\alpha} : \prod_{\beta \in \Lambda} E_{\beta} \to E_{\alpha}, \]
and
\[ \pi_{\beta} = g_{\alpha}^{\beta} \circ \pi_{\alpha}, \]
$\forall \alpha, \beta \in \Lambda$.

When you take instead of $E_{\alpha}$, $\alpha \in \Lambda$, a projective family of algebras, *-algebras, Jordan algebras, etc., you naturally get a correpsondent algebra, *-algebra or Jordan algebra structure in the projective limit algebra
\[ E = \lim \rightarrow E_{\alpha}. \]

Let $E$ be a vector space. A real function $p : E \to \mathbb{R}$ on $E$ is called a seminorm, if:
1. $p(x) \geq 0$, $\forall x \in E$;
2. $p(\lambda x) = |\lambda| p(x)$, $\forall \lambda \in \mathbb{R}$ or $\mathbb{C}$, and $x \in E$;
3. $p(x + y) \leq p(x) + p(y)$, $\forall x, y \in E$.
One can see that
\[ p(0) = 0. \]
If
\[ p(x) = 0, \]
implies
\[ x = 0, \]
seminorm is called a norm and is usually denoted by $\| \|$. If a space with a norm is complete, it is called a Banach space.

Let $(E, p)$ be a seminormed space, and
\[ N_{p} = \ker(p) = p^{-1}\{0\}. \]
The quotient space $E/N_{p}$ is a linear space and the function
\[ \|\|_{p} : E/N_{p} \to \mathbb{R}_{+} : \]
\[ x_{p} = x + N_{p} \to \|x_{p}\|_{p} = p(x), \]
is a well defined norm on $E/N_p$ induced by the seminorm $p$. The corresponding quotient normed space will be denoted by $E/N_p$, and the Banach space completion of $E/N_p$ by $E_p$. One can easily see that $E_p$ is the Hausdorff completion of the seminormed space $(E, p)$.

The algebras considered below will be without the loss of generality unital. If the algebra does not have an identity, it can be adjoint by the usual unitialization procedure.

A Jordan algebra is an algebra $E$ in which the identities

$$x \circ y = y \circ x,$$
$$x^2 \circ (y \circ x) = (x^2 \circ y) \circ x,$$

hold.

If $E$ is an algebra, the seminorm $p$ on $E$ compatible with the multiplication of $E$, in the sense that

$$p(xy) \leq p(x)p(y),$$

$\forall x, y \in E$, is called submultiplicative or m-seminorm.

For submultiplicative seminorm on a Jordan algebra $E$, the following inequality holds:

$$p(x \circ y) \leq p(x)p(y),$$

$\forall x, y \in E$. A seminorm on a Jordan algebra $E$ is called fine, if the following inequality holds:

$$p(x^2 + y^2) \geq p(x^2),$$

$\forall x, y \in E$.

A Banach-Jordan algebra is Jordan algebra which is as well a Banach algebra.

Let $E$ be an algebra. A subset $U$ of $E$ is called multiplicative or idempotent, if

$$UU \subseteq U,$$

in the sense that $\forall x, y \in U$, the product

$$xy \in U.$$

If $p$ is an m-seminorm on $E$ the unit semiball $U_p(1)$ corresponding to $p$, that is

$$U_p(1) = \{x \in E : p(x) \leq 1\},$$

and one can see that this set is multiplicative. Moreover, $U_p(1)$ is an absolutely-convex (balanced and convex),absorbing subset of $E$. It is known that given an absorbing absolutely-convex subset

$$U \subset E,$$

the function

$$p_U : E \rightarrow \mathbb{R}_+ :$$
$$x \rightarrow p_U(x) = \inf\{\lambda > 0 : x \in \lambda U\},$$

called gauge or Minkowski functional of $U$, is a seminorm. One can see that a real-valued function $p$ on the algebra $E$ is an m-seminorm iff

$$p = p_U,$$

for some absorbing, absolutely-convex and multiplicative subset

$$U \subset E.$$
In fact, one can take

\[ U = U_p(1). \]

By topological algebra we mean a topological vector space which is also an algebra, such that the ring multiplication is separately continuous. A topological algebra \( E \) is often denoted by \((E, \tau)\), where \( \tau \) is the topology of the underlying topological vector space of \( E \). The topology \( \tau \) is determined by a fundamental 0-neighborhood system, say \( \mathcal{B} \), consisting of absorbing, balanced sets with the property

\[ \forall V \in \mathcal{B} \exists U \in \mathcal{B}, \]

satisfying the condition \( U + U \subseteq V \). Since translations by \( y \) in \((E, \tau)\), i.e. the maps

\[ x \mapsto x + y : (E, \tau) \to (E, \tau), \]

\( y \in E \), are homomorphisms, an \( x \)-neighborhood in \((E, \tau)\) is of the form

\[ x + V, \]

with \( V \in \mathcal{B} \). A closed, absorbing and absolutely-convex subset of a topological algebra \((E, \tau)\) is called barrel. An \( m \)-barrel is a multiplicative barrel of \((E, \tau)\).

A locally convex algebra is a topological algebra in which the underlying topological vector space is a locally convex space. The topology \( \tau \) of a locally convex algebra \((E, \tau)\) is defined by a fundamental 0-neighborhood system consisting of closed absolutely-convex sets. Equivalently, the same topology \( \tau \) is determined by a family of nonzero seminorms. Such a family, say

\[ \Gamma = \{p_{\alpha}\}, \]

\( \alpha \in \Lambda \), or, for distinction purposes

\[ \Gamma_E = \{p_{\alpha}\}, \]

\( \alpha \in \Lambda \), is always assumed without a loss of generality saturated. That is, for any finite subset

\[ F \subseteq \Gamma, \]

the seminorm

\[ p_F(x) = \max_{p \in F} p(x), \]

\( x \in E \), again belongs to \( \Gamma \). Saying that

\[ \Gamma = \{p_{\alpha}\}, \]

\( \alpha \in \Lambda \), is a defining family of seminorms for a locally convex algebra \((E, \tau)\), we mean that \( \Gamma \) is a saturated family of seminorms defining the topology \( \tau \) on \( E \). That is

\[ \tau = \tau_\Gamma, \]

with \( \tau_\Gamma \) completely determined by a fundamental 0-neighborhood system given by the \( \varepsilon \)-semiballs

\[ U_p(\varepsilon) = \varepsilon U_p(\varepsilon) = \{x \in E : p(x) \leq \varepsilon\}, \]

\( \varepsilon > 0, p \in \Gamma \). More precisely, for each 0-neighborhood

\[ V \subseteq (E, \tau), \]

there is an \( \varepsilon \)-semiball \( U_p(\varepsilon) \), \( \varepsilon > 0, p \in \Gamma \), such that

\[ U_p(\varepsilon) \subseteq V. \]
The neighborhoods $U_p(\varepsilon)$, $\varepsilon > 0$, $p \in \Gamma$, are called **basic 0-neighborhoods**.

A locally C*-algebra (real locally C*-algebra, resp. locally JB-algebra) is a projective limit of projective family of C*-algebras (real C*-algebras, resp. JB-algebras). This is equivalent for locally C*- and real locally C*-algebras to the requirement that the family of defining continuous seminorms be regular:

$$p(x^*x) = p(x)^2,$$

as well as for the real locally C*-algebra $R$:

$$R \cap iR = \{0\}.$$

In the case of locally JB-algebras this is equivalent to the requirement that the family of defining continuous seminorms be fine and regular:

$$p(x^2 + y^2) \geq p(x^2),$$

$$p(x^2) = p(x)^2,$$

$\forall p \in \Gamma, x, y \in E$.

For a locally C*-algebra (real locally C*-algebra, resp. locally JB-algebra) $E$, by the bounded part we mean the subalgebra

$$E_b = \{x \in E : \|x\|_\infty = \sup_{p \in \Gamma(E)} p(x) < \infty\}.$$

### 3. BAI in real locally C*-algebras and locally JB-algebras with dense ideals

Let $(A, \tau)$ be a real or complex associative topological algebra and $(a_\lambda), \lambda \in \Lambda$, be a net in $(A, \tau)$ such that

$$\lim_\lambda x a_\lambda = x = \lim_\lambda a_\lambda x,$$

for any $x \in A$.

Such a net is called **approximate identity** (abbreviated to ai) of $(A, \tau)$. If only the left or the right equality above is valid, then we speak of a **left** (resp. **right**) ai. In the case an ai, when $(a_\lambda), \lambda \in \Lambda$, of $(A, \tau)$ is a bounded subset of $(A, \tau)$, we speak about a **bounded approximate identity** (abbreviated to bai) of $(A, \tau)$. In the case an left ai (resp. right ai), when $(a_\lambda), \lambda \in \Lambda$, of $(A, \tau)$ is a bounded subset of $(A, \tau)$, we speak about a **bounded left approximate identity** (bounded right approximate identity), abbreviated to **blai** (resp. **brai**) of $(A, \tau)$.

Let now $(J, \tau)$ be a real Jordan topological algebra and $(a_\lambda), \lambda \in \Lambda$, be a net in $(J, \tau)$ such that

$$\lim_\lambda a_\lambda \circ x = x,$$

for any $x \in J$.

Such a net is called **approximate identity** (abbreviated to ai) of $(J, \tau)$. In the case an ai, when $(a_\lambda), \lambda \in \Lambda$, of $(J, \tau)$ is a bounded subset of $(J, \tau)$, we speak about a **bounded approximate identity** (abbreviated to bai) of $(J, \tau)$.

Let now $(a_\lambda), \lambda \in \Lambda$, be a net in $(J, \tau)$ such that

$$\lim_\lambda U_{a_\lambda} x = x,$$
for any $x \in J$, then we speak of a \textit{quadratic ai} (abbreviated to \textit{qai}). In the case a \textit{qai}, $(a_\lambda), \lambda \in \Lambda$, of $(J, \tau)$ is a bounded subset of $(J, \tau)$, we speak about a \textit{bounded quadratic approximate identity} (abbreviated to \textit{bqai}) of $(J, \tau)$.

Taking a completion of an associative real or complex topological algebra $(A, \tau)$, or a completion of a real Jordan topological algebra $(J, \tau)$ (that is taking the completion of the underlying topological vector space of $(A, \tau)$ (resp. $(J, \tau)$)), one may fail to get a topological algebra, unless the multiplication in $(A, \tau)$ (resp. $(J, \tau)$) is jointly continuous (see, for example for \cite{12} details). If

$$\tau = \tau_\Gamma,$$

the respective completion of $(A, \tau)$ (resp. $(J, \tau)$), when it exists, will be denoted by $(\tilde{A}, \tilde{\tau}_\Gamma)$ (resp. $(\tilde{J}, \tilde{\tau}_\Gamma)$), where $\Gamma$ consists of the (unique) extensions of the elements of $\tau$ to the corresponding completion of $(A, \tau)$ (resp. $(J, \tau)$).

The following three lemmas present some properties of approximate identities. The first one talks about \textit{bai}'s for completion algebras and reveals the fact that the squares of the elements of \textit{bai} make up a \textit{bai} as well.

\textbf{Lemma 1.} Let $(a_\lambda), \lambda \in \Lambda$, be a \textit{bai} of a real or complex associative topological algebra $(A, \tau)$, or Jordan topological algebra $(J, \tau)$, with continuous multiplication. Then:

\begin{itemize}
  \item[i).] $(a_\lambda), \lambda \in \Lambda$, is also \textit{bai} for the completion $(\tilde{A}, \tilde{\tau})$ of $(A, \tau)$ (resp. $(\tilde{J}, \tilde{\tau})$ of $(J, \tau)$);
  \item[ii).] $(a_\lambda^2), \lambda \in \Lambda$, is a \textit{bai} of $(A, \tau)$ (resp. $(J, \tau)$).
\end{itemize}

\textit{Proof.} Immediately follows from considerations in \cite{12}. \hfill $\square$

The second one talks about \textit{bai} made up out of adjoint elements of another \textit{bai} in a topological *-algebra.

\textbf{Lemma 2.} Let $(A, \tau)$ be a topological *-algebra with an \textit{ai} $(a_\lambda), \lambda \in \Lambda$. Then $(a_\lambda^*), \lambda \in \Lambda$, is an \textit{ai} of $(A, \tau)$. Moreover, $(a_\lambda^*), \lambda \in \Lambda$, is a \textit{bai} whenever $(a_\lambda), \lambda \in \Lambda$, is a \textit{bai}.

\textit{Proof.} Immediately follows from considerations in \cite{12}. \hfill $\square$

The third one talks about the properties of \textit{bai} in certain factor-algebras of some topological *-algebras and topological Jordan algebras. Let $(A, \tau)$ be a real or complex topological *-algebra (or $(J, \tau)$ be a real Jordan topological algebra) with a saturated separating family of seminorms. It is called an \textit{m-convex algebra}, if each seminorm satisfies the \textit{submultiplicativity} inequality

$$p(xy) \leq p(x)p(y),$$

for every $x, y \in A$ (resp.

$$p(x \circ y) \leq p(x)p(y),$$

for every $x, y \in J$). It is called an \textit{m*-convex algebra}, if it is m-convex and each seminorm satisfies the identity

$$p(x^*) = p(x),$$

for every $x \in A$. 

Lemma 3. Let \((A, \tau)\) be a \(m^*-\)convex algebra (or a Jordan topological \(m\)-convex algebra \((J, \tau)\)). Then, if \((a_\lambda), \lambda \in \Lambda, \) is an ai, the net \((a_{\lambda,p}), \lambda \in \Lambda, \) with
\[ a_{\lambda,p} \equiv a_\lambda + N_p, \]
\((a_{\lambda,p}) \in A \) (resp. \((a_{\lambda,p}) \in J)\), \(p \in \Gamma, \lambda \in \Lambda, \) is an ai for both \((A,p)/N_p \) and \(A_p \) (resp. for both \((J,p)/N_p \) and \(J_p \)), for every \(p \in \Gamma. \) Moreover, \((a_{\lambda,p}), \lambda \in \Lambda, \) is bounded, whenever \((a_\lambda), \lambda \in \Lambda, \) is bounded.

Proof. Immediately follows from considerations in \[12]. \)

The following result is a real version of the theorem of Inoue (see \[9\] for details).

Theorem 1. Let \((A, \tau)\) be a real locally \(C^*\)-algebra, and \(I\) be a dense ideal in \((A, \tau)\). Then, \((A, \tau)\) has an ai \((a_\lambda), \lambda \in \Lambda, \) consisting of elements of \(I, \) such that:
1). The net \((a_\lambda), \lambda \in \Lambda, \) is increasing, in the sense that
\[ a_\lambda \geq 0, \]
for every \(\lambda \in \Lambda, \) and
\[ a_\lambda \leq a_\nu, \]
for any \(\lambda \leq \nu \) in \(\Lambda. \)

2). \[ p(a_\lambda) \leq 1, \]
for all \(p \in \Gamma, \lambda \in \Lambda. \)

Proof. Let \((A, \tau)\) be a real locally \(C^*\)-algebra. Then, from \[10\] it follows, that \((A_C, \widehat{\tau})\) is a complex locally \(C^*\)-algebra, where
\[ A_C = A + iA, \]
is the complexification of \(A. \) One can easily see that if \(I\) be a dense ideal in \((A, \tau), \) then \(I_C\) is a dense ideal of \((A_C, \widehat{\tau}),\) where
\[ I_C = I + iI, \]
the complexification of \(I\) in \((A_C, \widehat{\tau}). \) According to \[9\], there exists an ai \((u_\lambda), \lambda \in \Lambda, \) in \((A_C, \widehat{\tau})\) consisting of elements of \(I_C, \) such that:
1). The net \((u_\lambda), \lambda \in \Lambda, \) is increasing, in the sense that
\[ u_\lambda \geq 0, \]
for every \(\lambda \in \Lambda, \) and
\[ u_\lambda \leq u_\nu, \]
for any \(\lambda \leq \nu \) in \(\Lambda; \)
2). \[ \widehat{p}(u_\lambda) \leq 1, \]
for all \(\widehat{p} \in \widehat{\Gamma}, \lambda \in \Lambda, \) where for each \(p \in \Gamma, \) defined on \(A, \) its extention \(\widehat{p} \in \widehat{\Gamma} \) on all \(A_C, \) is defined as:
\[ \widehat{p}(x + iy) = \sqrt{p(x)^2 + p(y)^2}, \]
for every \(x, y \in A. \)

Let now
\[ u_\lambda = a_\lambda + ib_\lambda, \]
\(\lambda \in \Lambda, \) where \(a_\lambda, b_\lambda \in A. \) One can see that the net \((a_\lambda), \lambda \in \Lambda, \) satisfies the required conditions. \(\square\)
The following result is a real version of Inoue’s theorem from [9] on existence of left (resp. right) ai in real locally C*-algebras with dense left (resp. right) ideals.

**Theorem 2.** Let \((A, \tau)\) be a real locally C*-algebra, and \(I\) be a dense left (resp. right) ideal in \((A, \tau)\). Then, \((A, \tau)\) has a left (resp. right) ai \((a_\lambda), \lambda \in \Lambda\), consisting of elements of \(I\), such that:

1. The net \((a_\lambda), \lambda \in \Lambda\), is increasing, in the sense that
   \[ a_\lambda \geq 0, \]
   for every \(\lambda \in \Lambda\), and
   \[ a_\lambda \leq a_\nu, \]
   for any \(\lambda \leq \nu\) in \(\Lambda\).

2. \[ p(a_\lambda) \leq 1, \]
   for all \(p \in \Gamma, \lambda \in \Lambda\).

**Proof.** One can note that a complexification of a dense left (resp. right) ideal in in \((A, \tau)\) is a dense left (resp. right) ideal in \((A_\mathbb{C}, \hat{\tau})\). With that in mind, the rest of the proof repeats the proof of the preceding theorem. □

Now we turn our attention to the case of Jordan algebras. The next result is a version of the theorem of Inoue from [9] on existence of \(b\)ai for locally JB-algebras with dense Jordan ideals.

**Theorem 3.** Let \((J, \tau)\) be a locally JB-algebra, and \(I\) be a dense ideal in \((J, \tau)\). Then, \((J, \tau)\) has an ai \((a_\lambda), \lambda \in \Lambda\), consisting of elements of \(I\), such that:

1. The net \((a_\lambda), \lambda \in \Lambda\), is increasing, in the sense that
   \[ a_\lambda \geq 0, \]
   for every \(\lambda \in \Lambda\), and
   \[ a_\lambda \leq a_\nu, \]
   for any \(\lambda \leq \nu\) in \(\Lambda\).

2. \[ p(a_\lambda) \leq 1, \]
   for all \(p \in \Gamma, \lambda \in \Lambda\).

**Proof.** Let us first consider the set
\[ \Lambda = \{F \subseteq I : F - finite\}, \]
ordered by inclusion. For each
\[ \lambda = \{x_1, x_2, ..., x_n\}, \]
we put
\[ b_\lambda = \sum_{i=1}^{n} x_i^2. \]
For what follows we need a definition and a few lemmas about positive elements and spectrum in \(J\).

If \(J\) is a unital locally JB-algebra, and \(x \in J\), we denote by \(C(x)\) the smallest locally JB-subalgebra containing \(x\) and \(1\). According to Shirshov-Cohn theorem ([8]), \(C(x)\) is associative. We define the spectrum of \(x\) in \(J\), denoted by \(sp_J(x)\),
\[ sp_J(x) = \{\alpha \in \mathbb{R} : (x - \alpha 1) doesn't have an inverse in C(x)\}. \]
When the algebra is not unital, we first adjoin a unit ([8]), and then compute the spectrum in the unitization.

An element \( x \in J \) is called positive, and we write \( x \geq 0 \), if
\[
sp_J(x) \subseteq [0, \infty).
\]
We denote by \( J_+ \) the set of all positive elements in \( J \).

**Lemma 4.** Let \((J, \tau)\) be a locally JB-algebra, and
\[
J = \lim_{\leftarrow} J_p,
p \in \Gamma,
\]
be the Arens-Michael decomposition of \( J \) as a projective limit of a projective family of JB-algebras, and
\[
\pi_p : J \rightarrow J_p,
\]
be the continuous projection of \( J \) onto \( J_p \), for each \( p \in \Gamma \). The following conditions are equivalent for \( x \in J \):
1). \( x \geq 0 \);
2). \( x = y^2 \),
for some \( y \in J \);
3). \( x_p \geq 0_p \),
for each \( x_p = \pi_p(x) \in J_p \), \( p \in \Gamma \), and \( 0_p \) is the zero-element of \( J_p \).

**Proof.** Easily follows from Arens-Michael decomposition and correspondent properties of JB-algebras (see [8], [10]). \( \square \)

**Corollary 1.** Let \((J, \tau)\) be a locally JB-algebra. Then
\[
J_+ = \{ x^2 : x \in J \}.
\]

**Proof.** Evident. \( \square \)

**Corollary 2.** Let \((J, \tau)\) be a locally JB-algebra. Then \( J \) is formally real.

**Proof.** Easily follows from Arens-Michael decomposition and the correspondent property of JB-algebras. \( \square \)

**Lemma 5.** Let \((J, \tau)\) be a locally JB-algebra. Then \( J_+ \) is a closed convex cone, such that
\[
J_+ \cap (\neg J_+) = \{ 0 \}.
\]

**Proof.** Clearly follows from Arens-Michael decomposition and correspondent fact for JB-algebras ([8], [10]). \( \square \)
Corollary 3. Let \((J, \tau)\) be a locally JB-algebra, and \(x, y \in J\). The following statements hold:

1). \(x \leq y \Rightarrow U_z x \leq U_z y\), for all \(z \in J\);

2). \(0 \leq x \leq y \Rightarrow p(x) \leq p(y)\), for all \(p \in \Gamma\);

3). \(0 \leq x \leq y \Rightarrow 0 \leq x^{1/2} \leq y^{1/2}\).

In the case when \((J, \tau)\) is unital, one has:

4). \(x > 0 \Rightarrow x \in G_J\), where \(G_J\) is the set of invertible elements in \(J\);

5). \(x \geq 1 \Rightarrow x^{-1} \leq 1\);

6). \(0 < x \leq y \Rightarrow y^{-1} \leq x^{-1}\).

Proof. All properties follow from Arens-Michael decomposition of the algebra \((J, \tau)\) and corresponding properties of JB-algebras ([8], [10]).

Now, based on the presiding lemmas and corollaries, one can easily see that

\[ b_\lambda \in I \cap J_+ \]

for every \(\lambda \in \Lambda\).

Now, we need the following lemma.

Lemma 6. Let \((J, \tau)\) be a non-unital locally JB-algebra, and

\[ J = \lim_{\leftarrow} J_p, \]

\(p \in \Gamma\), be the Arens-Michael decomposition of \(J\) as a projective limit of a projective family of JB-algebras, and

\[ \pi_p : J \longrightarrow J_p, \]

be the continuous projection of \(J\) onto \(J_p\), for each \(p \in \Gamma\). There exits a unique unital locally JB-algebra \((J_1, \tau_1')\), such that \((J, \tau)\) is a locally JB-subalgebra, and

\[ J_1 = \lim_{\leftarrow} J_{1,p'}, \]

\(p' \in \Gamma'\), is the Arens-Michael decomposition of \(J_1\) as a projective limit of a projective family of unital JB-algebras \(J_{1,p}\), and each \(J_{1,p}\) is the unitization of a correspondent \(J_p\), for each \(p' \in \Gamma'\) is the extension of a correspondent \(p \in \Gamma\).

Proof. Easily obtained using a combination of arguments in [10] and [8].

Let now \(M\) be the locally JB-subalgebra of \((J_1, \tau_1')\), generated by two elements- \(b_\lambda\) and \(1\). Accoring to Shirshov-Cohn theorem (see [8]), this subalgebra is associative. If

\[ S = sp_J(b_\lambda) = sp_{J_1}(b_\lambda) \subseteq [0, \infty), \]

and

\[ f(t) = t(t + \frac{1}{n})^{-1}, \]
for every $t \in \mathbb{R}, n \in \mathbb{N}$, we obtain that
\[ f|_S \in C(S). \]

According to Spectral theorem for locally JB-algebras (see [10]), $C(S)$ is embedded in $M$ by means of a unique topological injective morphism $\Phi$, such that
\[ \Phi(1_{C(S)}) = 1, \]
and
\[ \Phi(id_S) = x, \]
where $1_{C(S)}$ is the constant function 1 on $S$, and $id_S$ is the identity map of $S$. Therefore, we can define
\[ a_\lambda = \Phi(f|_S) = b_\lambda \circ (b_\lambda + \frac{1}{n})^{-1} \in M, \]
\[ \lambda \in \Lambda. \]

Now we need the following lemma.

**Lemma 7.** Let $(P, \tau_P)$ and $(Q, \tau_Q)$ be two locally JB-algebras, and
\[ \varphi : (P, \tau_P) \longrightarrow (Q, \tau_Q), \]
be a Jordan morphism. Then,
\[ \varphi(P) = Q_+ \cap \varphi(P). \]

**Proof.** Obvious. \hfill \Box

**Corollary 4.** Let $(P, \tau_P)$ be a locally JB-algebra, and $Q$ be a closed Jordan subalgebra of $(P, \tau_P)$, with
\[ \tau_Q = \tau_P|_Q. \]
Then
\[ Q_+ = P_+ \cap Q. \]

**Proof.** Obvious. \hfill \Box

The presiding corollary implies that
\[ b_\lambda + \frac{1}{n} > 0, \]
in $M$. From Corollary 3.4 it follows that
\[ (b_\lambda + \frac{1}{n}) \in M, \]
is invertible in $M$. Since
\[ 0 \leq f|_S \leq 1_{C(S)}, \]
the presiding lemma implies that
\[ 0 \leq a_\lambda \leq 1, \]
\[ \lambda \in \Lambda. \]
One now can see that
\[ a_\lambda \in I \cap J_+, \]
for all $\lambda \in \Lambda$, and
\[ p'(a_\lambda) = p(a_\lambda) \leq 1, \]
for all $p \in \Gamma, p' \in \Gamma'$, and $\lambda \in \Lambda$. 
A computation shows that
\[ \sum_{i=1}^{n} ((a_\lambda - 1) \circ x_i)^2 = U_{(a_\lambda - 1)} b_\lambda = (a_\lambda - 1) \circ b_\lambda \circ (a_\lambda - 1) = n^{-2} b_\lambda \circ (b_\lambda + \frac{1}{n})^{-2}. \]

Now, taking a function
\[ g(t) = t(t + \frac{1}{n})^{-2}, \]
for every \( t \in \mathbb{R}, \ n \in \mathbb{N}, \) one can see that it has a maximum value at
\[ t = \frac{1}{n}, \]
so that
\[ 0 \leq g|S \leq \frac{n}{4} 1_{C(S)}. \]

Therefore, we get
\[ 0 \leq \Phi(g|S) = b_\lambda \circ (b_\lambda + \frac{1}{n})^{-2} \leq \frac{n}{4} 1, \]
using presiding calculations, we obtain
\[ ((a_\lambda - 1) \circ x_i)^2 \leq \frac{1}{4n}, \]
for every \( i = 1, ..., n. \) Applying Corollary 3.2 we get
\[ p'((a_\lambda - 1) \circ x_i)^2 = p(a_\lambda \circ x_i - x_i)^2 \leq \frac{1}{4n}, \]
for all \( p \in \Gamma, \ p' \in \Gamma', \ i = 1, ..., n. \)

Let now \( \varepsilon \) be an arbitrary small positive real number, and \( x \in I. \) Let \( \lambda(\varepsilon) \) be a finite subset of \( I \) with \( n \) elements, such that \( x \in \lambda(\varepsilon), \) and
\[ n > \frac{1}{\varepsilon^2}. \]
Then, based on presiding inequalities, we get that
\[ p(a_\lambda \circ x - x) < \varepsilon, \]
for every
\[ \lambda \geq \lambda(\varepsilon), \]
and \( p \in \Gamma. \) Thus we obtain that
\[ \lim_\lambda a_\lambda \circ x = x, \]
for every \( x \in I, \) and, because \( I \) is dense in \( (J, \tau), \) and
\[ p(a_\lambda) \leq 1, \]
for any \( \lambda \in \Lambda, \) and \( p \in \Gamma; \) we get that
\[ \lim_\lambda a_\lambda \circ x = x, \]
for every \( x \in J. \)
It remains to show that
\[ a_\lambda \leq a_\nu, \]
for any
\[ \lambda \leq \nu, \]
\( \lambda, \nu \in \Lambda. \) Let
\[ \lambda = \{x_1, ..., x_n\}, \]
and

\[ \nu = \{x_1, \ldots, x_m\}, \]

be in \( \Lambda \) with \( n \leq m \). Then

\[ b_\nu - b_\lambda = \sum_{i=n+1}^{m} x_i^2 \in J_+. \]

Moreover,

\[ 0 < b_\lambda + \frac{1}{n} \leq b_\nu + \frac{1}{n}, \]

therefore, based on presiding considerations we obtain that

\[ (b_\nu + \frac{1}{n})^{-1} \leq (b_\lambda + \frac{1}{n})^{-1}. \]

One can notice now that for real non-negative \( t \), since \( n \leq m \),

\[ \frac{1}{n}(t + \frac{1}{n})^{-1} \geq \frac{1}{m}(t + \frac{1}{m})^{-1}. \]

Therefore, from the Spectral theorem it follows that

\[ \frac{1}{n}(b_\nu + \frac{1}{n})^{-1} \geq \frac{1}{m}(b_\nu + \frac{1}{m})^{-1}, \]

and finally

\[ a_\lambda = 1 - \frac{1}{n}(b_\lambda + \frac{1}{n})^{-1} \leq 1 - \frac{1}{m}(b_\nu + \frac{1}{m})^{-1} \leq 1 - \frac{1}{m}(b_\nu + \frac{1}{m})^{-1} = a_\nu. \]

\[ \square \]

The following result is a version of Inoue’s theorem for existence of \( bqai \) in locally JB-algebras with dense quadratic ideals.

**Theorem 4.** Let \((J, \tau)\) be a locally JB-algebra, and \( I \) be a dense quadratic ideal in \((J, \tau)\). Then, \((J, \tau)\) has an \( qai \) \((a_\lambda), \lambda \in \Lambda \), consisting of elements of \( I \), such that:

1. The net \((a_\lambda), \lambda \in \Lambda \), is increasing, in the sense that

\[ a_\lambda \geq 0, \]

for every \( \lambda \in \Lambda \), and

\[ a_\lambda \leq a_\nu, \]

for any \( \lambda \leq \nu \) in \( \Lambda \).

2. \( p(a_\lambda) \leq 1, \)

for all \( p \in \Gamma, \lambda \in \Lambda \).

**Proof.** Follows step-by-step the proof of the presiding Theorem 3. \[ \square \]
4. **Existence of dense ideals in real locally C*-algebras and locally JB-algebras with unbounded elements**

In [6] Fritzsche established that if a complex unital locally C*-algebra has an unbounded element then it also has a proper dense left (resp. right) ideal.

The next result is a real analogue of the theorem of Fritzsche from [6] for real locally C*-algebras.

**Theorem 5.** Let \((A, \tau)\) be a real unital locally C*-algebra, and \(x \in A\), be such that \(x \notin A_b\), where
\[
A_b = \{x \in A : \|x\|_\infty = \sup_{p \in \Gamma(E)} p(x) < \infty\}.
\]
Then \((A, \tau)\) has a proper dense left (resp. right) ideal \(I\).

**Proof.** Let \((A, \tau)\) be a real locally C*-algebra. Then, from [10] it follows, that \((A_C, \hat{\tau})\) is a complex locally C*-algebra, where
\[
A_C = A + iA,
\]
is the complexification of \(A\). If \(x \in A\), be such that \(x \notin A_b\), then there exists at least one unbounded element in \((A_C, \hat{\tau})\), for example, one can take
\[
(1 + i)x = x + ix.
\]
Therefore, from Fritzsche theorem for complex locally C*-algebras (see [6]) it follows that there exists a dense left (resp. right) ideal \(I_C\) in \((A_C, \hat{\tau})\). Then, from the theorem of Inoue ([9]) it follows that there exists a left (resp. right) approximative identity \((u_\lambda)\), \(\lambda \in \Lambda\), consisting of elements of \(I_C\), such that:

1). The net \((u_\lambda)\), \(\lambda \in \Lambda\), is increasing, in the sense that
\[
u_\lambda \geq 0,
\]
for every \(\lambda \in \Lambda\), and
\[
u_\lambda \leq \nu_\nu,
\]
for any \(\lambda \leq \nu\) in \(\Lambda\).

2). \[
\hat{p}(u_\lambda) \leq 1,
\]
for all \(\hat{p} \in \hat{\Gamma}, \lambda \in \Lambda\) where for each \(p \in \Gamma\), defined on \(A\), its extention \(\hat{p} \in \hat{\Gamma}\) on all \(A_C\), is defined as:
\[
\hat{p}(x + iy) = \sqrt{p(x)^2 + p(y)^2},
\]
for every \(x, y \in A\).

Let now
\[
u_\lambda = a_\lambda + ib_\lambda,
\]
\(\lambda \in \Lambda\), where \(a_\lambda, b_\lambda \in A\). Let us now show that \((a_\lambda), \lambda \in \Lambda\), is a left (resp. right) approximate identity in \(A\). one can easily see that \((a_\lambda), \lambda \in \Lambda\), is an increasing net in \(A_+\), and, in fact, on one hand, we have
\[
p(a_\lambda) \leq \sqrt{p(a_\lambda)^2 + p(b_\lambda)^2} = \hat{p}(a_\lambda + ib_\lambda) = \hat{p}(u_\lambda) \leq 1,
\]
On the other hand, we have
\[
p(xa_\lambda - x) \leq \hat{p}(xa_\lambda - x) \longrightarrow 0,
\]
(resp.
\[
p(a_\lambda x - x) \leq \hat{p}(u_\lambda x - x) \longrightarrow 0),
\]
which proves that \((a_\lambda), \lambda \in \Lambda\), is a left (resp. right) approximate identity in \(A\).

Let now

\[
I = \{ x : x \in A, \ x = yz, \text{ where } y \in \{(a_\lambda) \setminus 1\}, \lambda \in \Lambda, \text{ and } z \in A \}
\]

(resp.

\[
I = \{ x : x \in A, \ x = yz, \text{ where } y \in \{(a_\lambda) \setminus 1\}, \lambda \in \Lambda, \text{ and } z \in A \}.
\]

Due to associativity of multiplication in \(A\), \(I\) is obviously a left (resp. right) ideal. On the other hand, one can see that \(I\) is dense in \(A\) due to the fact that \((a_\lambda), \lambda \in \Lambda\), is a left (resp. right) approximate identity in \(A\). In addition, one can see that \(I\) is proper, because for the unbounded element \(x \in A\),

\[
p(yx - x) \longrightarrow 0,
\]

when

\[
y \in \{(a_\lambda) \setminus 1\}, \lambda \in \Lambda,
\]

(resp.

\[
p(xy - x) \longrightarrow 0,
\]

when

\[
y \in \{(a_\lambda) \setminus 1\}, \lambda \in \Lambda,
\]

but

\[
p(yx - x) \neq 0,
\]

(resp.

\[
p(xy - x) \neq 0,
\]

due to the fact that \(1 \notin \{(a_\lambda) \setminus 1\}\).


Let us now turn again our attention to Jordan algebras. The next result is a Jordan-algebraic analogue of Fritzseche’s theorem from [6] for locally JB-algebras.

**Theorem 6.** Let \((J, \tau)\) be a real unital locally JB-algebra, and \(x \in J\), be such that \(x \notin J_b\), where

\[
J_b = \{ x \in J : \|x\|_\infty = \sup_{p \in \Gamma(E)} p(x) < \infty \}.
\]

Then \((A, \tau)\) has a proper dense quadratic ideal \(I\).

**Proof.** Let \((J, \tau)\) be a locally JB-algebra. Let us consider \(J\) as a dense quadratic ideal of itself. From Theorem 4 above it follows that \((J, \tau)\) has an qai \((a_\lambda), \lambda \in \Lambda\), consisting of elements of \(J\), such that:

1). The net \((a_\lambda), \lambda \in \Lambda\), is increasing, in the sense that

\[
a_\lambda \geq 0,
\]

for every \(\lambda \in \Lambda\), and

\[
a_\lambda \leq a_\nu,
\]

for any \(\lambda \leq \nu\) in \(\Lambda\).

2).

\[
p(a_\lambda) \leq 1,
\]

for all \(p \in \Gamma, \lambda \in \Lambda\).

Let now

\[
I = \{ x : x \in J, \ x = U_yz, \text{ where } y \in \{(a_\lambda) \setminus 1\}, \lambda \in \Lambda, \text{ and } z \in J \}.
\]
Due to MacDonald Identity one can see that $I$ is a quadratic ideal of $J$. One can easily see that $I$ is dense in $(J, \tau)$ because $(a_\lambda), \lambda \in \Lambda,$ is an qai of $(J, \tau)$. On the other hand, one can see that $I$ is a proper quadratic ideal of $(J, \tau)$, because for the unbounded element $x \in A$,

$$p(U_yx - x) \longrightarrow 0,$$

when

$$y \in \{(a_\lambda) \setminus 1\}, \lambda \in \Lambda,$$

but

$$p(U_yx - x) \neq 0,$$

due to the fact that

$$1 \notin \{(a_\lambda) \setminus 1\}.$$ 

□

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ON BAI AND EXISTENCE OF DENSE IDEALS IN REAL AND JORDAN ALGEBRAS

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