Stein-Weiss inequality on product spaces

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Abstract

We give the classification between weighted norm inequalities of strong fractional integral operators and their associated multi parameter Muckenhoupt characteristics, by considering the weights to be power functions. As a result, we extend the classical Stein-Weiss theorem to product spaces.

1 Introduction

Let $0 < \alpha < N$. A fractional integral operator $I_\alpha$ is defined by

$$
(I_\alpha f)(x) = \int_{\mathbb{R}^N} f(y) \left(\frac{1}{|x-y|}\right)^{N-\alpha} dy.
$$

(1.1)

In 1928, Hardy and Littlewood [1] first established a weighted norm inequality for $I_\alpha$ in one dimensional space, by considering the weights to be suitable power functions. This result has been extended to higher dimensions by Stein and Weiss [3] and now bears the name of Stein-Weiss inequality.

**Theorem A:** Stein and Weiss (1958) Let $\omega(x) = |x|^{-\gamma}, \sigma(x) = |x|^\delta, \gamma, \delta \in \mathbb{R}$. We have

$$
\|\omega I_\alpha f\|_{L^q(\mathbb{R}^N)} \leq C_{p, q, \gamma, \delta, N} \|f\sigma\|_{L^p(\mathbb{R}^N)}
$$

(1.2)

for $1 < p \leq q < \infty$, if

$$
\gamma < \frac{N}{q}, \quad \delta < N \left(\frac{p-1}{p}\right), \quad \gamma + \delta \geq 0
$$

(1.3)

and

$$
\frac{\alpha}{N} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma + \delta}{N}.
$$

(1.4)

Throughout, we regard $C$ as a generic constant depending on its subindices.

In the case of $\gamma = \delta = 0$, Theorem A was proved in $\mathbb{R}^N$ by Sobolev [2]. This is known today as Hardy-Littlewood-Sobolev inequality.

The weighted norm inequalities of fractional integrals have been extensively studied, i.e: by Muckenhoupt and Wheeden [5], Coifman and Fefferman [9], Fefferman and Muckenhoupt [8], Pérez [10] and Sawyer and Wheeden [6].
Let \( Q \) denote a cube in \( \mathbb{R}^N \). It is well known that the norm inequality (1.2) implies

\[
\sup_{Q \subset \mathbb{R}^N} |Q|^{\frac{1}{p} - \frac{1}{q}} \left\{ \frac{1}{|Q|} \int_Q \omega^\theta(x) dx \right\} \left\{ \frac{1}{|Q|} \int_Q \left( \frac{1}{|\sigma|} \right)^{\frac{p}{p-1}}(x) dx \right\}^{\frac{p}{p-1}} < \infty. \tag{1.5}
\]

The supremum (1.5) is called the Muckenhoupt characteristic, as was first introduced by Muckenhoupt for which \( \omega^\theta \) and \( \sigma^{-\frac{p}{p-1}} \) are nonnegative and locally integrable functions.

By taking into account \( \omega(x) = |x|^{-\gamma}, \sigma(x) = |x|^{\delta}, \gamma, \delta \in \mathbb{R} \), we find that (1.5) implies the constraints in (1.3)-(1.4). Hence, (1.2), (1.3)-(1.4) and (1.5) are equivalent conditions.

Consider \( \mathbb{R}^N \) as a product space, by writing \( \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \cdots \times \mathbb{R}^{N_n} \), \( n \geq 2 \). Let

\[
0 < \alpha_i < N_i, \quad i = 1, 2, \ldots, n \quad \text{and} \quad \alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n. \tag{1.6}
\]

In this paper, we give an extension of Theorem A on product spaces by studying so-called the strong fractional integral operator \( I_\alpha \) defined by

\[
(I_\alpha f)(x) = \int_{\mathbb{R}^n} f(y) \prod_{i=1}^n \left( \frac{1}{|x_i - y_i|} \right)^{N_i - \alpha_i} dy, \tag{1.7}
\]

whose kernel has singularity appeared on every coordinate subspace.

Study of certain operators that commute with a multi-parameter family of dilations, dates back to the time of Jessen, Marcinkiewicz and Zygmund. During the past several decades, a number of pioneering results have been accomplished, for example, by Robert Fefferman [12]-[13], Chang and Fefferman [16], Cordoba and Fefferman [11], Fefferman and Stein [14], Müller, Ricci and Stein [15], Journé [17] and Pipher [18]. The area remains largely open for fractional integration.

\section{Statement of main result}

\textbf{Theorem A*:} Let \( \omega(x) = |x|^{-\gamma}, \sigma(x) = |x|^{\delta}, \gamma, \delta \in \mathbb{R} \). For \( 1 < p \leq q < \infty \), the following conditions are equivalent:

1. Let \( Q = Q_1 \times Q_2 \times \cdots \times Q_n \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \cdots \times \mathbb{R}^{N_n} = \mathbb{R}^N \) where \( Q_i \) denotes a cube in \( \mathbb{R}^{N_i} \), for every \( i = 1, 2, \ldots, n \).

\[
\sup_{Q \subset \mathbb{R}^N} \prod_{i=1}^n |Q_i|^{\frac{1}{p} - \frac{1}{q}} \left\{ \frac{1}{|Q|} \int_Q \omega^\theta(x) dx \right\} \left\{ \frac{1}{|Q|} \int_Q \left( \frac{1}{|\sigma|} \right)^{\frac{p}{p-1}}(x) dx \right\}^{\frac{p}{p-1}} < \infty. \tag{2.1}
\]

2. \( \gamma < \frac{N}{q}, \quad \delta < \frac{p-1}{p}, \quad \gamma + \delta \geq 0 \) \quad (2.2)

and

\[
\frac{\alpha}{N} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma + \delta}{N}. \tag{2.3}
\]
For $\gamma \geq 0, \delta \leq 0$,
\[
\alpha_i - \frac{N_i}{p} < \delta, \quad i = 1, 2, \ldots, n. \tag{2.4}
\]

For $\gamma \leq 0, \delta \geq 0$,
\[
\alpha_i - N_i \left(\frac{q-1}{q}\right) < \gamma, \quad i = 1, 2, \ldots, n. \tag{2.5}
\]

For $\gamma > 0, \delta > 0$,
\[
\sum_{i \in U} \alpha_i - \frac{N_i}{p} < \delta, \quad U = \left\{ i \in \{1, 2, \ldots, n\} : \alpha_i - \frac{N_i}{p} \geq 0 \right\},
\]
\[
\sum_{i \in V} \alpha_i - \left(\frac{q-1}{q}\right)N_i < \gamma, \quad V = \left\{ i \in \{1, 2, \ldots, n\} : \alpha_i - N_i \left(\frac{q-1}{q}\right) \geq 0 \right\}. \tag{2.6}
\]

3. Let $I_{\alpha}$ to be defined in (1.6)-(1.7). We have
\[
\|\alpha I_{\alpha}f\|_{L^q(\mathbb{R}^n)} \leq C p q \alpha \gamma \delta n N \|f\sigma\|_{L^p(\mathbb{R}^n)}. \tag{2.7}
\]

**Remark 2.1** In the 2-parameter setting ($n = 2$), **Theorem A** is first proved in the joint work by Sawyer and Wang [7]. For $\gamma \geq 0, \delta \leq 0$ or $\gamma \leq 0, \delta \geq 0$, the “sandwiching” idea introduced in [7] applies to the general multi-parameter situation. However, the difficult case occurs when $\gamma > 0, \delta > 0$, whereas the method used in [7] relies on solving a system of algebraic equations, which is no longer solvable for $n > 2$.

**Sketch of Proof:** In section 3, we introduce a new framework, where the product space is decomposed into an infinitely many of dyadic cones. Every partial sum operator defined on a dyadic cone is essentially an one-parameter fractional integral operator, satisfying the desired regularity.

In section 4, by taking into account $\omega(x) = |x|^{-\gamma}, \sigma(x) = |x|^\delta$, $\gamma, \delta \in \mathbb{R}$, we prove that the Muckenhoupt characteristic (2.1) implies the constraints in (2.2)-(2.6).

In section 5, by using (2.2)-(2.6), we show that
\[
\prod_{i=1}^n |Q|^{-\left(\frac{1}{r}-\frac{1}{p}\right)} \left(\frac{1}{|Q|} \int_Q \omega^{pr}(x) \, dx\right)^{\frac{1}{p}} \left(\frac{1}{|Q|} \int_Q \left(\frac{1}{\sigma}\right)^{\frac{p}{r}}(x) \, dx\right)^{\frac{r-1}{p}}, \quad r > 1 \tag{2.8}
\]
decays exponentially, as the eccentricity of $Q$ getting large, for $\alpha_i > N_i \left(\frac{1}{p} - \frac{1}{q}\right), i = 1, 2, \ldots, n$.

On the other hand, we handle the case $\alpha_i = N_i \left(\frac{1}{p} - \frac{1}{q}\right), i = 1, 2, \ldots, n$ in section 6.

We prove **Theorem A** in the last section, by decomposing $I_{\alpha}$ so that the resulting estimates can be reduced to either of the above two cases.

For dealing with such convolution operators with positive kernels, it is suffice to assume $f \geq 0$ in the rest of the paper.
3 Cone decomposition on product spaces

Let $t$ denote an $n$-tuple $(2^{-t_1}, 2^{-t_2}, \ldots, 2^{-t_n})$ where $t_i, \ i = 1, 2, \ldots, n$ are nonnegative integers. We require $t_\nu = \min\{t_i: i = 1, 2, \ldots, n\} = 0$.

Define

$$\left(\Delta_I t f\right)(x) \doteq \int_{\Gamma_t(x)} f(y) \prod_{i=1}^n \left(\frac{1}{|x_i - y_i|}\right)^{N_i - \alpha_i} dy$$

(3. 1)

where

$$\Gamma_t(x) \doteq \bigotimes_{i=1}^n \left\{ y_i \in \mathbb{R}^{N_i}: 2^{-t_i} \leq \frac{|x_i - y_i|}{|x_\nu - y_\nu|} < 2^{-t_i+1} \right\}.$$  

(3. 2)

Observe that $\Gamma_t(x)$ in (3. 2) is a dyadic cone with vertex on $x$ whose eccentricity depends on $t$. In particular, we write

$$\Gamma_0(x) \doteq \Gamma_t(x), \quad t_1 = t_2 = \cdots = t_n = 0.$$  

(3. 3)

![Figure 1: dyadic cones in a 2-parameter setting.](image)

Denote an $n$-parameter dilation

$$t x = \left(2^{-t_1}x_1, 2^{-t_2}x_2, \ldots, 2^{-t_n}x_n\right).$$

(3. 4)
Let $Q^i$ be a dilated of $Q$ such that $|Q^i|^{\frac{1}{n}} = 2^{-t_i} |Q|^{\frac{1}{n}}, i = 1, 2, \ldots, n$. We have

\[
\prod_{i=1}^{n} |Q|^{\frac{\alpha}{n}} \left( \frac{1}{t} \right)^{\frac{1}{p} \left( 1 - \frac{\alpha}{n} \right)} \left\{ \frac{1}{|Q|} \int_{Q} \omega^{\alpha} (tx) \, dx \right\}\{ \frac{1}{|Q|} \int_{Q} \left( \frac{1}{t} \right)^{\frac{\alpha q}{n}} (tx) \, dx \} \leq \prod_{i=1}^{n} |Q^i|^{\frac{\alpha}{n}} \left( \frac{1}{t} \right)^{\frac{1}{p} \left( 1 - \frac{\alpha}{n} \right)} \left\{ \frac{1}{|Q^i|} \int_{Q^i} \omega^{\alpha} (x) \, dx \right\}\{ \frac{1}{|Q^i|} \int_{Q^i} \left( \frac{1}{t} \right)^{\frac{\alpha q}{n}} (x) \, dx \} \]

for every $Q \subset \mathbb{R}^N$.

Given $t$, consider

\[
t^Q \subset \mathbb{R}^N : |Q|^{\frac{1}{n}} / |Q^i|^{\frac{1}{n}} = 2^{-t_i}, \; i = 1, 2, \ldots, n.
\]

For $r \geq 1$, we define

\[
A_{pqr}^\alpha (t : \omega, \sigma) = \sup_{t^Q} \prod_{i=1}^{n} |Q^i|^{\frac{\alpha}{n}} \left( \frac{1}{t} \right)^{\frac{1}{p} \left( 1 - \frac{\alpha}{n} \right)} \left\{ \frac{1}{|Q^i|} \int_{Q^i} \omega^{\alpha} (x) \, dx \right\}\{ \frac{1}{|Q^i|} \int_{Q^i} \left( \frac{1}{t} \right)^{\frac{\alpha q}{n}} (x) \, dx \} \leq \prod_{i=1}^{n} 2^{t_i (\alpha - \frac{N}{p} + \frac{N}{q})} A_{pqr}^\alpha (t : \omega, \sigma) \]  

by (3.5).

Suppose that $Q$ satisfies $|Q_1|^{\frac{1}{n}} = |Q_2|^{\frac{1}{n}} = \cdots = |Q_n|^{\frac{1}{n}}$. We have $Q^i = t^Q$ and

\[
|Q|^{\frac{N}{n} - \left( \frac{1}{p} - \frac{1}{q} \right)} \left\{ \frac{1}{|Q|} \int_{Q} \omega^{\alpha} (tx) \, dx \right\}\{ \frac{1}{|Q|} \int_{Q} \left( \frac{1}{t} \right)^{\frac{\alpha q}{n}} (tx) \, dx \} \leq \prod_{i=1}^{n} 2^{t_i (\alpha - \frac{N}{p} + \frac{N}{q})} A_{pqr}^\alpha (t : \omega, \sigma) \]  

by (3.7).

Now, recall Sawyer-Wheeden theorem for one-parameter fractional integral operators in weighted norms, stated as Theorem 1 in [6]:

\[
\int_{\mathbb{R}^N} \left\{ \int_{\mathbb{R}^N} f(y) \left( \frac{1}{|x - y|} \right)^{N - \alpha} \, dy \right\}^{q} \omega^{\alpha}(x) \, dx \leq C_{p, q, r, \alpha, N} A_{pqr}^\alpha (\omega, \sigma) \left\{ \int_{\mathbb{R}^N} (\omega) \, dx \right\}^{\frac{1}{p}}
\]

for $1 < p \leq q < \infty$, if

\[
A_{pqr}^\alpha (\omega, \sigma) = \sup_{Q : |Q|^{\frac{1}{n}} = \cdots = |Q^i|^{\frac{1}{n}}} \prod_{i=1}^{n} |Q^i|^{\frac{\alpha}{n}} \left( \frac{1}{t} \right)^{\frac{1}{p} \left( 1 - \frac{\alpha}{n} \right)} \left\{ \frac{1}{|Q^i|} \int_{Q^i} \omega^{\alpha} (x) \, dx \right\}\{ \frac{1}{|Q^i|} \int_{Q^i} \left( \frac{1}{t} \right)^{\frac{\alpha q}{n}} (x) \, dx \} \leq \infty \quad r > 1.
\]
Remark 3.1 The constant \( C_{p,q,r,n} A_{pqr}^\alpha(\omega, \sigma) \) in (3.9) is not written explicitly in the statement of Theorem 1 by Sawyer and Wheeden [6]. But it can be computed directly by carrying out the proof given in section 2 of [6].

By applying (3.9)-(3.10) and using the estimate in (3.8), we have

\[
\left\{ \int_{\mathbb{R}^n} \left( \frac{1}{|x - y|} \right)^{N-\alpha} \right\}^q \omega^q(t x) dx \leq C_{p,q,r,n} \prod_{i=1}^n 2^{t_i \left( \alpha_i + \frac{N_i}{r} + \frac{N_i}{q} \right)} A_{pqr}^\alpha(t : \omega, \sigma) \left\{ \int_{\mathbb{R}^n} (f \sigma)^p(t x) dx \right\}^{\frac{1}{p}}
\]

for \( 1 < p \leq q < \infty \) and every \( t \).

Recall from (3.1)-(3.2). By changing dilations \( x \rightarrow tx, y \rightarrow ty \), we have

\[
\left\{ \int_{\mathbb{R}^n} (\Delta t f(x))^q(x) \omega^q(x) dx \right\}^{\frac{1}{q}} \leq C_{p,q,r,n} \prod_{i=1}^n 2^{-t_i \left( \alpha_i + \frac{N_i}{r} + \frac{N_i}{q} \right)} \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} f(t x) \left( \frac{1}{|x - y|} \right)^{N-\alpha} dy \right\}^q \omega^q(t x) dx \prod_{i=1}^n 2^{-t_i N_i} dx \]

\[
\leq C_{p,q,r,n} A_{pqr}^\alpha(t : \omega, \sigma) \left\{ \int_{\mathbb{R}^n} (f \sigma)^p(t x) dx \right\}^{\frac{1}{p}} \leq C_{p,q,r,n} A_{pqr}^\alpha(t : \omega, \sigma) \left\{ \int_{\mathbb{R}^n} (f \sigma)^p(x) dx \right\}^{\frac{1}{p}} \]

by (3.11)

\[
= C_{p,q,r,n} A_{pqr}^\alpha(t : \omega, \sigma) \left\{ \int_{\mathbb{R}^n} (f \sigma)^p(x) dx \right\}^{\frac{1}{p}}
\]

Observe that \( \Delta t f \) is essentially an one-parameter fractional integral operator, satisfying

\[
\left\| (\Delta t f) x \right\|_{L^q(\mathbb{R}^n)} \leq C_{p,q,r,n} A_{pqr}^\alpha(t : \omega, \sigma) \left\| f \sigma \right\|_{L^p(\mathbb{R}^n)}
\]

for \( 1 < p \leq q < \infty \).
By applying Minkowski inequality, provided that
\[ \sum_{t} A_{pq}^{\alpha}(t : \omega, \sigma) < \infty, \] (3. 14)
the norm inequality holds in (2. 7).

4 Necessary constraints

First, it is well known that the norm inequality (2. 7) implies
\[ A_{pq}^{\alpha}(\omega, \sigma) \leq \sup_{Q \in \mathbb{R}^N} \prod_{i=1}^{n} |Q_i|^{\alpha} \left( \int_{Q_i} \left( \frac{1}{|Q_i|} \int_{Q_i} \omega^q(x) dx \right)^{\frac{1}{q}} \left( \int_{Q_i} \left( \frac{1}{|Q_i|} \int_{Q_i} \frac{1}{\sigma} dx \right)^{\frac{p}{p}} dx \right)^{\frac{p-1}{p}} \right) \] (4. 1)

Let \( \omega(x) = |x|^{-\gamma}, \sigma(x) = |x|^\delta, \gamma, \delta \in \mathbb{R} \). We aim to show the Muckenhoupt characteristic (4. 1) implying the constraints in (2. 2)-(2. 6).

Let \( Q^\lambda \) denote a dilated variant of \( Q \) for \( \lambda > 0 \), such that \( Q^\lambda = Q_1^\lambda \times Q_2 \times \cdots \times Q_n^\lambda \) and \( |Q_i^\lambda|^{\frac{1}{n}} = \lambda |Q_i|^{\frac{1}{n}}, i = 1, 2, \ldots, n \). Suppose \( \omega(x) = |x|^{-\gamma}, \sigma(x) = |x|^\delta, \gamma, \delta \in \mathbb{R} \). From (4. 1)-(??), we have
\[ \prod_{i=1}^{n} |Q_i|^{\alpha} \left( \int_{Q_i} \left( \frac{1}{|Q_i|} \int_{Q_i} \omega^q(x) dx \right)^{\frac{1}{q}} \left( \int_{Q_i} \left( \frac{1}{|Q_i|} \int_{Q_i} \frac{1}{\sigma} dx \right)^{\frac{p}{p}} dx \right)^{\frac{p-1}{p}} \right) \]
\[ = \lambda^{\gamma + \delta - \alpha + N(\frac{1}{p} - \frac{1}{q})} \prod_{i=1}^{n} |Q_i^\lambda|^{\alpha} \left( \int_{Q_i^\lambda} \left( \frac{1}{|Q_i^\lambda|} \int_{Q_i^\lambda} \omega^q(x) dx \right)^{\frac{1}{q}} \left( \int_{Q_i^\lambda} \left( \frac{1}{|Q_i^\lambda|} \int_{Q_i^\lambda} \frac{1}{\sigma} dx \right)^{\frac{p}{p}} dx \right)^{\frac{p-1}{p}} \right) \]
\[ \leq \lambda^{\gamma + \delta - \alpha + N(\frac{1}{p} - \frac{1}{q})} A_{pq}^{\alpha} \left( |x|^{-\gamma}, |x|^\delta \right) < \infty. \]

Consider \( |Q_1|^{\frac{1}{n}} = |Q_2|^{\frac{1}{n}} = \cdots = |Q_n|^{\frac{1}{n}} = 1 \). The first line of (4. 2) is bounded from below.
Suppose \( \gamma + \delta - \alpha + N \left( \frac{1}{p} - \frac{1}{q} \right) \neq 0 \). By either taking \( \lambda \to 0 \) or \( \lambda \to \infty \), the last line of (4. 2) is vanished. Hence that we must have \( \gamma + \delta - \alpha + N \left( \frac{1}{p} - \frac{1}{q} \right) = 0 \) which is (2. 3).

We write \( x = (x_i, x_i^+) \in \mathbb{R}^N \times \mathbb{R}^{N-n}, i = 1, 2, \ldots, n \) and \( Q_i^+ = \bigotimes_{j \neq i} Q_j \). Let \( Q_i \) shrink to some \( x_i \in Q_i \) and \( |Q_i|^{\frac{1}{n}} = 1, j \neq i \) in (4. 1). Suppose \( x_i \neq 0 \) in \( \mathbb{R}^{N-i} \). By applying Lebesgue Differentiation Theorem, we have
\[ \left\{ \lim_{|Q_i| \to 0} \int_{Q_i} \left( \frac{1}{|Q_i|} \int_{Q_i} \omega^q(x) dx \right)^{\frac{1}{q}} \left( \int_{Q_i} \left( \frac{1}{|Q_i|} \int_{Q_i} \frac{1}{\sigma} dx \right)^{\frac{p}{p}} dx \right)^{\frac{p-1}{p}} dx \right\} \leq A_{pq}^{\alpha} \left( |x|^{-\gamma}, |x|^\delta \right), \quad i = 1, 2, \ldots, n. \] (4. 3)
Note that $|Q_i^\gamma| = 1$ in (4.3). The boundedness of $A_{pq}^\alpha(|x|^{-\gamma}, |x|^0)$ requires
\[
\frac{\alpha_i}{N_i} \geq \frac{1}{p} - \frac{1}{q} \quad i = 1, 2, \ldots, n.
\] (4.4)
By putting together (4.4) and (2.3), we find $\gamma + \delta \geq 0$. On the other and, it is essential to require $\gamma q < N$ and $\delta \left( \frac{p}{p - 1} \right) < N$ for the local integrability of $|x|^{-\gamma q}$ and $|x|^{-\delta \left( \frac{p}{p - 1} \right)}$ respectively. These are the constraints in (2.2).

In the remaining section, we assume $Q$ centered on the origin of $\mathbb{R}^N$.

Let $S$ be a proper subset of $\{1, 2, \ldots, n\}$. We define the truncated cube $Q_i^\epsilon = Q_i \cap \{|x| \geq \epsilon\}$ for $\epsilon > 0$ and every $i \in S$. Denote $Q^\epsilon = \bigotimes_{i \in S} Q_i^\epsilon \times \bigotimes_{i \in S^c} Q_i$ and $Q_S = \bigotimes_{i \in S} Q_i$, $Q_{S^c} = \bigotimes_{i \in S^c} Q_i$. Moreover, we write $x = (x_S, x_{S^c}) \in \mathbb{R}^{NS} \times \mathbb{R}^{N-NS}$ for which $N_S = \sum_{i \in S} N_i$.

Suppose that there exists at least one $i \in S^c$ such that $\alpha_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) > 0$. Let $0 < \lambda < 1$.

Consider $|Q_i|^{\lambda N_i} = 1$ for $i \in S$ and $|Q_i|^{\lambda N_i} = \lambda$ for $i \in S^c$. We have
\[
\prod_{i=1}^n |Q_i|^{\alpha_i \left( \frac{1}{N_i} - \frac{1}{n} \right)} \left\{ \frac{1}{|Q|} \int_Q \left( \frac{1}{|x|} \right)^{\gamma q} \, dx \right\} \frac{1}{\lambda} \left\{ \frac{1}{|Q|} \int_Q \left( \frac{1}{|x|} \right)^{\delta p} \, dx \right\}^{\frac{p-1}{p}}
\]
\[
= \lim_{\epsilon \to 0} \prod_{i=1}^n |Q_i|^{\alpha_i \left( \frac{1}{N_i} - \frac{1}{n} \right)} \left\{ \frac{1}{|Q|} \int_Q \left( \frac{1}{|x|} \right)^{\gamma q} \, dx \right\} \frac{1}{\lambda} \left\{ \frac{1}{|Q|} \int_Q \left( \frac{1}{|x|} \right)^{\delta p} \, dx \right\}^{\frac{p-1}{p}}
\]
\[
= \lim_{\epsilon \to 0} \lambda^{\sum_{i \in S^c} \alpha_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right)} \left\{ \frac{1}{|Q|} \int_Q \left( \frac{1}{|x|} \right)^{\gamma q} \, dx \right\} \frac{1}{\lambda} \left\{ \frac{1}{|Q|} \int_Q \left( \frac{1}{|x|} \right)^{\delta p} \, dx \right\}^{\frac{p-1}{p}}
\]
\[
= \lim_{\epsilon \to 0} 0 \times \left\{ \int \cdots \int_{\bigotimes_{i \in S} Q_i^\epsilon} \left( \frac{1}{\sum_{i \in S} |x_i|^2} \right)^{\frac{1}{2}} \prod_{i \in S} dx_i \right\} \frac{1}{\lambda} \left\{ \int \cdots \int_{\bigotimes_{i \in S} Q_i^\epsilon} \left( \frac{1}{\sum_{i \in S} |x_i|^2} \right)^{\frac{1}{2}} \prod_{i \in S} dx_i \right\}^{\frac{p-1}{p}}
\]
\[
= \lim_{\epsilon \to 0} 0 = 0.
\] (4.5)

Suppose $\alpha_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) = 0$ for every $i \in S^c$. Let $Q_i$ shrink to the origin of $\mathbb{R}^{N_i}$ for every $i \in S^c$ in (4.1). By applying Lebesgue differentiation theorem, we have
\[
A_{pq}^\alpha(|x|^{-\gamma}, |x|^0) \Rightarrow \prod_{i=1}^n |Q_i|^{\alpha_i \left( \frac{1}{N_i} - \frac{1}{n} \right)} \left\{ \frac{1}{|Q|} \int_Q \left( \frac{1}{|x|} \right)^{\gamma q} \, dx \right\} \frac{1}{\lambda} \left\{ \frac{1}{|Q|} \int_Q \left( \frac{1}{|x|} \right)^{\delta p} \, dx \right\}^{\frac{p-1}{p}}
\]
\[
= \prod_{i \in S} |Q_i|^{\alpha_i \left( \frac{1}{N_i} - \frac{1}{n} \right)} \left\{ \frac{1}{|Q_S|} \int_{Q_S} \left( \frac{1}{|x_S|} \right)^{\gamma q} \, dx_S \right\} \frac{1}{\lambda} \left\{ \frac{1}{|Q_S|} \int_{Q_S} \left( \frac{1}{|x_S|} \right)^{\delta p} \, dx_S \right\}^{\frac{p-1}{p}}
\] (4.6)
where $\gamma q < N_S$ and $\delta \left(\frac{p}{p+1}\right) < N_S$ become necessities.

**Case One:** Consider $\gamma \geq 0, \delta \leq 0$. Let $|Q|^{\gamma q} = 1$ for $i \in \{1, 2, \ldots, n\}$ and $|Q|^{\gamma q} = \lambda$ for all $j \neq i$. Suppose $a_j - N_i \left(\frac{1}{p} - \frac{1}{q}\right) = 0$ for every $j \neq i$. We have

\[
\prod_{i=1}^{n} |Q_i|^{\frac{\alpha_j - \frac{1}{p}}{\gamma q}} \left\{ \frac{1}{|Q|} \int_Q \left(\frac{1}{|x|}\right)^{\gamma q} dx \right\} \left\{ \frac{1}{|Q|} \int_Q \left(\frac{1}{|x|}\right)^{\frac{\beta q}{p+1}} dx \right\}^{\frac{p-1}{p}} \]

\[
\leq C_{q \gamma n} \left\{ \int_Q \left(\frac{1}{\lambda + |x|}\right)^{\gamma q} dx \right\} \left\{ \int_Q \left(\frac{1}{|x|}\right)^{\frac{\beta q}{p+1}} dx \right\} \] \quad (\delta \leq 0) \tag{4.7}

\[
\leq C_{p q \gamma \delta n} \left\{ \int_{\lambda < |x| \leq 1} \left(\frac{1}{\lambda + |x|}\right)^{\gamma q} dx \right\} \] \quad (\delta \leq 0) \tag{4.8}

where

\[
\int_{\lambda < |x| \leq 1} \left(\frac{1}{\lambda + |x|}\right)^{\gamma q} dx \leq C_N \ln \left(\frac{\lambda + 1}{1}\right) \quad \text{if} \quad \gamma = \frac{N_i}{q},
\]

\[
\int_{\lambda < |x| \leq 1} \left(\frac{1}{\lambda + |x|}\right)^{\gamma q} dx \leq C_N \left\{ \left(\frac{1}{\lambda + 1}\right)^{\gamma q - N_i} - \left(\frac{1}{\lambda + 1}\right)^{\gamma q - N_i} \right\} \quad \text{if} \quad \gamma > \frac{N_i}{q}.
\]

From (4.7)-(4.8), as $\lambda \to 0$, we need

\[
\gamma < \frac{N_i}{q}, \quad i = 1, 2, \ldots, n \tag{4.9}
\]

in order to satisfy the inequality in (4.2).

Suppose that there exists $j \neq i$ such that $a_j - N_i \left(\frac{1}{p} - \frac{1}{q}\right) > 0$. We have

\[
\prod_{i=1}^{n} |Q_i|^{\frac{\alpha_j - \frac{1}{p}}{\gamma q}} \left\{ \frac{1}{|Q|} \int_Q \left(\frac{1}{|x|}\right)^{\gamma q} dx \right\} \left\{ \frac{1}{|Q|} \int_Q \left(\frac{1}{|x|}\right)^{\frac{\beta q}{p+1}} dx \right\}^{\frac{p-1}{p}} \]

\[
\geq C_{q \gamma n} \prod_{j \neq i} \lambda^{a_j - N_i \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \int_Q \left(\frac{1}{\lambda + |x|}\right)^{\gamma q} dx \right\} \left\{ \int_Q \left(\frac{1}{|x|}\right)^{\frac{\beta q}{p+1}} dx \right\} \] \quad (\delta \leq 0) \tag{4.10}

\[
\geq C_{p q \gamma \delta n} \prod_{j \neq i} \lambda^{a_j - N_i \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \int_{0 < |x| \leq \lambda} \left(\frac{1}{\lambda}\right)^{\gamma q} dx \right\} \] \quad (\delta \leq 0) \tag{4.11}

Recall the estimate in (4.5) and take $\mathcal{S} = \{i\}$. We have (4.10) equal to zero at $\lambda = 0$. Together with (4.9), we find

\[
\gamma < \frac{N_i}{q} + \sum_{j \neq i} a_j - N_i \left(\frac{1}{p} - \frac{1}{q}\right), \quad i = 1, 2, \ldots, n. \tag{4.11}
\]
Case Two: Consider $\gamma \leq 0, \delta \geq 0$. Let $|Q_i|^{\frac{1}{N_i}} = 1$ for $i \in \{1, 2, \ldots, n\}$ and $|Q_i|^{\frac{1}{N_j}} = \lambda$ for all $j \neq i$. Suppose $\alpha_j - N_j\left(\frac{p}{p} - \frac{1}{q}\right) = 0$ for every $j \neq i$. We have

$$
\prod_{i=1}^{n} |Q_i|^{\frac{\alpha_j - (\frac{p}{p} - \frac{1}{q})}{N_j}} \left\{ \frac{1}{|Q|} \int_{Q} \left( \frac{1}{|x|} \right)^{\gamma q} dx \right\}^\frac{1}{\delta} \left\{ \frac{1}{|Q|} \int_{Q} \left( \frac{1}{\lambda + |x|} \right)^{\frac{\delta p}{p-1}} dx \right\}^\frac{p-1}{p} 
$$

$$
\geq C_{p, \delta, n} \left\{ \int_{Q} \left( \frac{1}{|x|} \right)^{\gamma q} dx \right\}^\frac{1}{\delta} \left\{ \int_{Q} \left( \frac{1}{\lambda + |x|} \right)^{\frac{\delta p}{p-1}} dx \right\}^\frac{p-1}{p} \quad (\gamma \leq 0) \quad (4.12)
$$

$$
\geq C_{p, q, \gamma, \delta, n} \left\{ \int_{\lambda < |x| \leq 1} \left( \frac{1}{\lambda + |x|} \right)^{\frac{\delta p}{p-1}} dx \right\}^\frac{p-1}{p}
$$

where

$$
\int_{\lambda < |x| \leq 1} \left( \frac{1}{\lambda + |x|} \right)^{\frac{\delta p}{p-1}} dx \leq C_N \ln \left( \frac{1 + \lambda}{2\lambda} \right) \quad \text{if} \quad \delta = N_i \left( \frac{p-1}{p} \right),
$$

$$
\int_{\lambda < |x| \leq 1} \left( \frac{1}{\lambda + |x|} \right)^{\frac{\delta p}{p-1}} dx \leq C_N \frac{1}{\delta \left( \frac{p}{p-1} \right) - N_i} \left[ \left( \frac{p}{p-1} \right) - N_i \right]^{-N_i} \quad (4.13)
$$

$$
\text{if} \quad \delta > N_i \left( \frac{p-1}{p} \right).
$$

From (4.12)-(4.13), as $\lambda \to 0$, we need

$$
\delta < N_i \left( \frac{p-1}{p} \right), \quad i = 1, 2, \ldots, n \quad (4.14)
$$

in order to satisfy the inequality in (4.2).

Suppose that there exists $j \neq i$ such that $\alpha_j - N_i\left(\frac{p}{p} - \frac{1}{q}\right) > 0$. We have

$$
\prod_{i=1}^{n} |Q_i|^{\frac{\alpha_j - (\frac{p}{p} - \frac{1}{q})}{N_i}} \left\{ \int_{Q_i} \left( \frac{1}{|x|} \right)^{\gamma q} dx \right\}^\frac{1}{\delta} \left\{ \int_{Q_i} \left( \frac{1}{\lambda + |x|} \right)^{\frac{\delta p}{p-1}} dx \right\}^\frac{p-1}{p} 
$$

$$
\geq C_{p, \delta, n} \prod_{j \neq i} \lambda^{\alpha_j - N_i\left(\frac{p}{p} - \frac{1}{q}\right)} \left\{ \int_{Q_i} \left( \frac{1}{|x|} \right)^{\gamma q} dx \right\}^\frac{1}{\delta} \left\{ \int_{Q_i} \left( \frac{1}{\lambda + |x|} \right)^{\frac{\delta p}{p-1}} dx \right\}^\frac{p-1}{p} \quad (\gamma \leq 0) \quad (4.15)
$$

$$
\geq C_{p, q, \gamma, \delta, n} \prod_{j \neq i} \lambda^{\alpha_j - N_i\left(\frac{p}{p} - \frac{1}{q}\right)} \left\{ \int_{0 < |x| \leq \lambda} \left( \frac{1}{\lambda} \right)^{\frac{\delta p}{p-1}} dx \right\}^\frac{p-1}{p}
$$

$$
= C_{p, q, \gamma, \delta, n} \lambda^{\left(\frac{p-1}{p}\right)N_i - \delta + \sum_{j \neq i} \alpha_j - N_i\left(\frac{p}{p} - \frac{1}{q}\right)}.
$$
Recall the estimate in (4.5) and take $S = \{i\}$. We have (4.15) equal to zero at $\lambda = 0$. Together with (4.14), we find

$$\delta < N_i \left( \frac{p - 1}{p} \right) + \sum_{j \neq i} \alpha_j - N_j \left( \frac{1}{p} - \frac{1}{q} \right),$$

(4.16)

$i = 1, 2, \ldots, n$.

**Case Three:** Consider $\gamma > 0, \delta > 0$. Note that (4.1) is invariant by changing dilations in one-parameter as shown in (4.2), because of (2.3).

Recall the definition of $U$ and $V$ from (2.6). We write $x_U \in \mathbb{R}^{N_U}$ and $x_V \in \mathbb{R}^{N_V}$ where $\mathbb{R}^{N_U} = \bigotimes_{i \in U} \mathbb{R}^{N_i}$ and $\mathbb{R}^{N_V} = \bigotimes_{i \in V} \mathbb{R}^{N_i}$.

Let $|Q_i|^{\frac{1}{N_i}} = \lambda^{-1}$ for every $i \in U$ and $|Q_i|^{\frac{1}{N_i}} = 1$ for all other $i \notin U$. We have

$$\prod_{i=1}^n |Q_i|^\frac{\alpha_i - N_i}{p} \left\{ \int_{Q_i} \left( \frac{1}{|x|} \right)^{\gamma q} dx \right\} \left\{ \int_{Q_i} \left( \frac{1}{|x|} \right)^{\delta q} dx \right\}$$

$$\geq C_{p,q,\gamma,\delta} n \prod_{i \in U} \left( \frac{1}{\lambda} \right)^{\frac{\alpha_i - N_i}{q}} \left\{ \int_{Q_i} \left( \frac{1}{1 + \sum_{i \in U} |x_i|} \right)^{\gamma q} dx \right\} \prod_{i \in U} \left( \frac{\lambda^{\delta p}}{p} \right)^{\frac{1}{q}}$$

$$\geq C_{p,q,\gamma,\delta} n \prod_{i \in U} \left( \frac{1}{\lambda} \right)^{\frac{\alpha_i - N_i}{p}} \left\{ \int_{Q_i} \left( \frac{1}{1 + \sum_{i \in U} |x_i|} \right)^{\gamma q} dx \right\} \prod_{i \in U} \left( \frac{\lambda^{\delta p}}{p} \right)^{\frac{1}{q}}$$

$$\geq C_{p,q,\gamma,\delta} n \left( \frac{1}{\lambda} \right)^{\frac{1}{\lambda}} \sum_{i \in U} \alpha_i - \frac{N_i}{p} - \delta.$$

(4.17)

In the case of $U = \{1, 2, \ldots, n\}$, since $\gamma$ satisfies the first strict inequality in (2.2), we find

$$\delta = \frac{N}{q} - \gamma + \sum_{i=1}^n \alpha_i - \frac{N_i}{p} \quad \text{by (2.3)}$$

(4.18)

$$\geq \sum_{i=1}^n \alpha_i - \frac{N_i}{p} = \sum_{i \in U} \alpha_i - \frac{N_i}{p}.$$
Suppose that \( U \) is a proper subset of \([1, 2, \ldots, n]\) and there exists at least one \( i \in U \) such that \( \alpha_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) > 0 \). By applying the estimate in (4.5) with \( S = U \), we have (4.17) equal to zero at \( \lambda = 0 \). The last line of (4.17) implies

\[
\sum_{i \in U} \alpha_i - \frac{N_i}{p} < \delta. \tag{4.19}
\]

Suppose that \( U \) is a proper subset of \([1, 2, \ldots, n]\) where \( \alpha_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) = 0 \) for every \( i \in U \). Let \( S = U \). We have \( |x|^{-\gamma} \) and \(|x|^0\) satisfying the Muckenhoupt characteristic (4.6) on \( \mathbb{R}^N \setminus \bigotimes_{i \in U} \mathbb{R}^{N_i} \). Denote \( \alpha_U = \sum_{i \in U} \alpha_i \). By carrying out the same estimate in (4.2), we find

\[
\frac{\alpha_U}{N_U} = \frac{1}{p} - \frac{1}{q} + \gamma + \delta, \tag{4.20}
\]

This further implies

\[
\delta = \frac{N_U}{q} \gamma - \sum_{i \in U} \alpha_i - \frac{N_i}{p} > \sum_{i \in U} \alpha_i - \frac{N_i}{p}. \tag{4.21}
\]

Let \( |Q_i|^{-\frac{1}{N_i}} = \lambda^{-1} \) for every \( i \in \mathcal{V} \) and \( |Q_i|^{-\frac{1}{N_i}} = 1 \) for all other \( i \notin \mathcal{V} \). We have

\[
\prod_{i=1}^n \left| Q_i \right|^{-\frac{1}{N_i}} \left\{ \left( \frac{1}{|Q_i|} \int_Q \left( \frac{1}{|x|} \right)^{q_q} \, dx \right)^{\frac{1}{q_q}} \right\}^{p-1} \left\{ \left( \frac{1}{|Q_i|} \int_Q \left( \frac{1}{|x|} \right)^{q_p} \, dx \right)^{\frac{1}{q_p}} \right\}^{p-1}
\]

\[
\geq C_{p, q, \gamma} \delta n \left( \frac{1}{\lambda} \right)^{\sum_{i \in \mathcal{V}} \alpha_i \left( \frac{1}{q} - \frac{1}{q_q} \right) N_i \left\{ \prod_{i \in \mathcal{V}} \lambda^{N_i} \int_{Q_i} \cdots \int_{Q_i} \lambda^{q_q} \prod_{i \in \mathcal{V}} dx_i \right\}^{\frac{1}{q_q}}^{p-1}
\]

\[
\geq C_{p, q, \gamma} \delta n \left( \frac{1}{\lambda} \right)^{\sum_{i \in \mathcal{V}} \alpha_i \left( \frac{1}{q} - \frac{1}{q_q} \right) N_i \left\{ \prod_{i \in \mathcal{V}} \lambda^{N_i} \int_{Q_i} \cdots \int_{Q_i} \lambda^{q_q} \prod_{i \in \mathcal{V}} dx_i \right\}^{\frac{1}{q_q}}^{p-1}
\]

\[
\geq C_{p, q, \gamma} \delta n \left( \frac{1}{\lambda} \right)^{\sum_{i \in \mathcal{V}} \alpha_i \left( \frac{1}{q} - \frac{1}{q_q} \right) N_i \left\{ \prod_{i \in \mathcal{V}} \lambda^{N_i} \int_{Q_i} \cdots \int_{Q_i} \lambda^{q_q} \prod_{i \in \mathcal{V}} dx_i \right\}^{\frac{1}{q_q}}^{p-1}
\]

\[
\geq C_{p, q, \gamma} \delta n \left( \frac{1}{\lambda} \right)^{\sum_{i \in \mathcal{V}} \alpha_i \left( \frac{1}{q} - \frac{1}{q_q} \right) N_i \left\{ \prod_{i \in \mathcal{V}} \lambda^{N_i} \int_{Q_i} \cdots \int_{Q_i} \lambda^{q_q} \prod_{i \in \mathcal{V}} dx_i \right\}^{\frac{1}{q_q}}^{p-1}
\]

\[
\geq C_{p, q, \gamma} \delta n \left( \frac{1}{\lambda} \right)^{\sum_{i \in \mathcal{V}} \alpha_i \left( \frac{1}{q} - \frac{1}{q_q} \right) N_i \left\{ \prod_{i \in \mathcal{V}} \lambda^{N_i} \int_{Q_i} \cdots \int_{Q_i} \lambda^{q_q} \prod_{i \in \mathcal{V}} dx_i \right\}^{\frac{1}{q_q}}^{p-1}
\]
In the case of $V = \{1, 2, \ldots, n\}$, since $\delta$ satisfies the second strict inequality in (2.2), we find

$$
\gamma = \left(\frac{p-1}{p}\right)N - \delta + \sum_{i=1}^{n} \alpha_i - N_i \left(\frac{q-1}{q}\right)
$$

by (2.3) \hfill (4.23)

$$
> \sum_{i=1}^{n} \alpha_i - N_i \left(\frac{q-1}{q}\right) = \sum_{i \in V} \alpha_i - N_i \left(\frac{q-1}{q}\right).
$$

Suppose that $V$ is a proper subset of $\{1, 2, \ldots, n\}$ and there exists at least one $i \in V^c$ such that $\alpha_i - N_i \left(\frac{1}{p} - \frac{1}{q}\right) > 0$. By applying the estimate in (4.5) with $S = V$, we have (4.22) equal to zero at $\lambda = 0$. The last line of (4.22) implies

$$
\sum_{i \in V} \alpha_i - N_i \left(\frac{q-1}{q}\right) < \gamma.
$$

Suppose that $V$ is a proper subset of $\{1, 2, \ldots, n\}$ where $\alpha_i - N_i \left(\frac{1}{p} - \frac{1}{q}\right) = 0$ for every $i \in V^c$. Let $S = V$. We have $|x_V|^{-\gamma}$ and $|x_V|^0$ satisfying the Muckenhoupt characteristic (4.6) on $\mathbb{R}^{N_V} = \bigotimes_{i \in V} \mathbb{R}^{N_i}$. Denote $\alpha_V = \sum_{i \in V} \alpha_i$. By carrying out the same estimate in (4.2), we find

$$
\gamma < \frac{N_V}{q}, \quad \delta < N_V \left(\frac{p-1}{p}\right), \quad \frac{\alpha_V}{N_V} = \frac{1}{p} - \frac{1}{q} + \gamma + \delta.
$$

This further implies

$$
\gamma = \left(\frac{p-1}{p}\right)N_V - \delta + \sum_{i \in V} \alpha_i - N_i \left(\frac{q-1}{q}\right)
$$

$$
> \sum_{i \in V} \alpha_i - N_i \left(\frac{q-1}{q}\right).
$$

**Remark 4.1** By using the formula in (2.3), we can verify that the constraints in (4.11) and (4.16) are equivalent to (2.4) and (2.5) respectively. Namely,

for $\gamma \geq 0, \delta \leq 0$,

$$
\gamma < \frac{N_i}{q} + \sum_{j \neq i} \alpha_j - N_j \left(\frac{1}{p} - \frac{1}{q}\right) \quad \iff \quad \alpha_i - \frac{N_i}{p} < \delta, \quad i = 1, 2, \ldots, n,
$$

(4.27)

for $\gamma \leq 0, \delta \geq 0$,

$$
\delta < N_i \left(\frac{p-1}{p}\right) + \sum_{j \neq i} \alpha_j - N_j \left(\frac{1}{p} - \frac{1}{q}\right) \quad \iff \quad \alpha_i - N_i \left(\frac{q-1}{q}\right) < \gamma, \quad i = 1, 2, \ldots, n.
$$

(4.28)
5 Decay estimate on varying eccentricities

Principal Lemma: Let \( \gamma, \delta \) satisfying (2.2)-(2.6). Suppose

\[
\alpha_i / N_i > \frac{1}{p} - \frac{1}{q}, \quad i = 1, 2, \ldots, n. \tag{5.1}
\]

For \( 0 < \lambda_i \leq 1, i = 1, 2, \ldots, n \), define

\[
\lambda Q \subset \mathbb{R}^N : |Q|^{\frac{1}{N}} / |Q_\nu|^{\frac{1}{N}} = \lambda_\nu. \tag{5.2}
\]

There exists an \( \varepsilon > 0 \) such that

\[
\sup_{\lambda Q} \prod_{i=1}^n |Q_i|^{\frac{1}{N_i} - \frac{1}{\alpha_i}} \left\{ \frac{1}{|Q|} \int_Q \left( \frac{1}{|x|} \right)^{qr} dx \right\}^{\frac{1}{2}} \left\{ \frac{1}{|Q|} \int_Q \left( \frac{1}{|x|} \right)^{pr} dx \right\}^{\frac{p}{2}} \leq C_{p, q, r, \gamma, \delta, n} \prod_{i=1}^n (\lambda_i)^\varepsilon \tag{5.3}
\]

for some \( r > 1 \). The values of \( \varepsilon \) and \( r \) depend only on \( p, q, \gamma, \delta, \alpha, n, N \).

Remark 5.1 Without the condition (5.1), we can only show that the Muckenhoupt characteristic in (5.3) is bounded.

Proof: By carrying out the same estimate in (4.2) and using the formula (2.3), we find that the \( r\)-bump characteristic (5.3) is invariant by changing dilations in one-parameter. Therefore, it is sufficient to consider \( |Q_i|^{\frac{1}{N_i}} = 1 \).

Let \( Q_i^0 \) and \( Q_i^* \subset \mathbb{R}^N \) to be centered on the origin of \( \mathbb{R}^N \) and

\[
|Q_i^0|^{\frac{1}{N_i}} = |Q_i|^{\frac{1}{N_i}}, \quad |Q_i^*|^{\frac{1}{N_i}} = 3|Q_i|^{\frac{1}{N_i}} = 3\lambda_i, \quad i = 1, 2, \ldots, n. \tag{5.4}
\]

Remark 5.2 Suppose \( Q_i \cap Q_i^0 = \emptyset \). We must have \( |x_i| \geq |x_i^0| / \sqrt{n} \) for every \( x_i \in Q_i \) and every \( x_i^0 \in Q_i^0 \). Otherwise, if \( Q_i \) intersects \( Q_i^0 \), then \( Q_i \subset Q_i^* \).

After a permutation on indices \( i = 1, 2, \ldots, n \), we can assume \( \nu = 1 \) and

\[
1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n. \tag{5.5}
\]

Case One: Let \( \gamma \geq 0, \delta \leq 0 \) satisfy (2.2)-(2.4). By adjusting the value of \( r \), we assume

\[
\sum_{i=1}^{m-1} N_i < \gamma qr < \sum_{i=1}^m N_i, \quad \delta \leq 0, \quad 1 \leq m \leq n. \tag{5.6}
\]
Suppose that $Q$ is centered on $z \in \mathbb{R}^N$ for some $|z| \leq 3$. We have

$$
\prod_{i=1}^{n} \left|Q_{i}\right|^{\frac{a_i}{p} - \left(\frac{1}{p} - \frac{1}{q} \right)} \left\{ \frac{1}{|Q|} \int_{Q} \left( \frac{1}{|x|} \right)^{\frac{np}{q}} \, dx \right\}^{\frac{1}{p'}} \left\{ \frac{1}{|Q|} \int_{Q} \left( \frac{1}{|x|} \right)^{\frac{np}{q}} \, dx \right\}^{\frac{p-1}{mp}}
\leq C_{p q r \gamma \delta} \prod_{i=1}^{n} \left( \lambda_i \right)^{a_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right)} \prod_{i=1}^{n} \left( \frac{1}{\lambda_i} \right)^{N_i} \left\{ \int \cdots \int_{\bigotimes_{j=1}^{n} Q_j} \left( \frac{1}{|x|} \right)^{\frac{np}{q} \cdot m \cdot \frac{1}{|x_m|}} \, dx_1 \cdots dx_n \right\}^{\frac{1}{p'}}
\leq C_{p q r \gamma \delta} \prod_{i=1}^{n} \left( \lambda_i \right)^{a_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right)} \prod_{i=1}^{n} \left( \frac{1}{\lambda_i} \right)^{N_i} \left\{ \int \cdots \int_{\bigotimes_{j=1}^{n} Q_j} \left( \frac{1}{|x|} \right)^{\frac{np}{q} \cdot m \cdot \frac{1}{|x_m|}} \, dx_1 \cdots dx_n \right\}^{\frac{1}{p'}}
\leq C_{p q r \gamma \delta} \prod_{i=1}^{n} \left( \lambda_i \right)^{a_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right)} \prod_{i=1}^{n} \left( \frac{1}{\lambda_i} \right)^{N_i} \left\{ \int_{Q_m} \left( \frac{1}{|x_m|} \right)^{\frac{np}{q} \cdot m \cdot \frac{1}{|x_m|}} \, dx_m \right\}^{\frac{1}{p'}}
\leq C_{p q r \gamma \delta} \prod_{i=1}^{n} \left( \lambda_i \right)^{ \frac{1}{p} \sum_{j=1}^{m} N_j - \gamma} \prod_{i=1}^{n} \left( \frac{1}{\lambda_i} \right)^{\frac{N_i}{p'}} \prod_{i=1}^{n} \left( \lambda_i \right)^{a_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right)}.
$$

From direct computation, the formula in the last line of (5.7) can be rewritten as

$$
\left( \lambda_m \right)^{ \frac{1}{p} \sum_{j=1}^{m} N_j - \gamma} \prod_{i=2}^{m} \left( \lambda_i \right)^{a_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right)} \prod_{i=m+1}^{n} \left( \lambda_i \right)^{a_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right)} \quad \lambda_1 = 1
$$

(5.8)

$$
= \left( \lambda_m \right)^{ \frac{N_1}{p} + \sum_{i=2}^{m} \left( a_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) \right)} \prod_{i=2}^{m} \left( \frac{\lambda_i}{\lambda_m} \right)^{a_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right)} \prod_{i=m+1}^{n} \left( \frac{\lambda_i}{\lambda_m} \right)^{a_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right)}.
$$

Recall Remark 4.1. $\gamma \geq 0$, $\delta \leq 0$ satisfy the two equivalent strict inequalities in (4.27).

Define $0 \leq \delta \leq 1$ implicitly by letting $\lambda_m = (\lambda_n)^{\delta}$. For $r$ sufficiently close to 1, we have

$$
\delta \left[ \frac{N_1}{p r} + \sum_{i=2}^{m} \left( a_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) \right) - \gamma \right] + (1 - \delta) \left[ \alpha_n - N_n \left( \frac{1}{p} - \frac{1}{q} \right) \right] > 0
$$

(5.9)
and

\[ \alpha_i - \frac{N_i}{p} + \left(1 - \frac{1}{r}\right) \frac{N_i}{q} < 0, \quad i = 1, 2, \ldots, n. \quad (5.10) \]

Note that \( \alpha_n - N_n \left(\frac{1}{p} - \frac{1}{q}\right) > 0 \). By using (5.9)-(5.10), we find that (5.8), is bounded by \( C_{pqr} \delta \odot n \lambda \) for some \( \varepsilon = \varepsilon(p,q,r,\alpha,\gamma,\delta,n,N) > 0 \).

**Case Two:** Let \( \gamma \leq 0, \delta \geq 0 \) satisfy (2.2)-(2.3) and (2.5). By adjusting the value of \( r \), assume

\[ \gamma \leq 0, \quad \sum_{i=1}^{m-1} N_i < \delta \left(\frac{pr}{p-1}\right) < \sum_{i=1}^{m} N_i, \quad 1 \leq m \leq n. \quad (5.11) \]

Suppose that \( Q \) is centered on \( z \in \mathbb{R}^N \) for some \( |z| \leq 3 \). We have

\[
\prod_{i=1}^{n} \left| \mathcal{Q}_{ij} \right|^{-\frac{1}{N_i} \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \frac{1}{|Q|} \int_{Q} \left(\frac{1}{|x|}\right)^{\gamma pr} \, dx \right\} \left\{ \frac{1}{|Q|} \int_{Q} \left(\frac{1}{|x|}\right)^{\delta \left(\frac{pr}{p-1}\right)} \, dx \right\}^{\frac{p-1}{pr}}
\]

\[ \leq C_{pqr} \delta \odot n \prod_{i=1}^{n} \left(\lambda_i\right)^{\alpha_i - N_i \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \prod_{i=1}^{n} \left(\frac{1}{\lambda_i}\right)^{N_i} \int \cdots \int_{\mathbb{R}^N} \left(\frac{1}{|x_1| + \cdots + |x_n|}\right)^{\delta \left(\frac{pr}{p-1}\right)} \, dx_1 \cdots dx_n \right\}^{\frac{p-1}{pr}}
\]

\[ \leq C_{pqr} \delta \odot n \prod_{i=1}^{n} \left(\lambda_i\right)^{\alpha_i - N_i \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \prod_{i=1}^{n} \left(\frac{1}{\lambda_i}\right)^{N_i} \int \cdots \int_{\mathbb{R}^N} \left(\frac{1}{|x_1| + \cdots + |x_n|}\right)^{\delta \left(\frac{pr}{p-1}\right) - \sum_{i=1}^{m-1} N_i} \, dx_1 \cdots dx_n \right\}^{\frac{p-1}{pr}}
\]

\[ \leq C_{pqr} \delta \odot n \prod_{i=1}^{n} \left(\lambda_i\right)^{\alpha_i - N_i \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \prod_{i=1}^{n} \left(\frac{1}{\lambda_i}\right)^{N_i} \int \cdots \int_{\mathbb{R}^N} \left(\frac{1}{|x_m| + \cdots + |x_n|}\right)^{\delta \left(\frac{pr}{p-1}\right) - \sum_{i=1}^{m-1} N_i} \, dx_m \cdots dx_n \right\}^{\frac{p-1}{pr}}
\]

\[ \leq C_{pqr} \delta \odot n \prod_{i=1}^{n} \left(\lambda_i\right)^{\alpha_i - N_i \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \prod_{i=1}^{n} \left(\frac{1}{\lambda_i}\right)^{N_i} \int \cdots \int_{\mathbb{R}^N} \left(\frac{1}{|x_m|}\right)^{\delta \left(\frac{pr}{p-1}\right) - \sum_{i=1}^{m} N_i} \, dx_m \right\}^{\frac{p-1}{pr}} \text{ by Remark 5.2}
\]

\[ \leq C_{pqr} \delta \odot n \left(\lambda_m\right)^{\frac{p-1}{pr}} \sum_{i=1}^{m} \left(\frac{1}{\lambda_i}\right)^{\delta \odot N_i} \prod_{i=1}^{m-1} \left(\lambda_i\right)^{\alpha_i - N_i \left(\frac{1}{p} - \frac{1}{q}\right)} \prod_{i=1}^{n} \left(\lambda_i\right)^{\alpha_i - N_i \left(\frac{1}{p} - \frac{1}{q}\right)}.
\]

(5.12)
From direct computation, the formula in the last line of (5.12) can be rewritten as

\[
\lambda_m\left(\frac{p-1}{pr}\right)\sum_{i=1}^{m} N_i - \delta \prod_{i=2}^{m} \lambda_i \prod_{i=m+1}^{n} \lambda_i N_i \left(\frac{1}{p-1}\right) \frac{1}{\lambda_m} \prod_{i=m+1}^{n} N_i \left(\frac{1}{p-1}\right) = \lambda_m\left(\frac{p-1}{pr}\right)\sum_{i=1}^{m} N_i - \delta \prod_{i=2}^{m} \lambda_i \prod_{i=m+1}^{n} \lambda_i N_i \left(\frac{1}{p-1}\right) \frac{1}{\lambda_m} \prod_{i=m+1}^{n} N_i \left(\frac{1}{p-1}\right) (5.13)
\]

Recall Remark 4.1. \(\gamma \leq 0, \delta \geq 0\) satisfy the two equivalent strict inequalities in (4.28). Define \(0 \leq \delta' \leq 1\) implicitly by letting \(\lambda_m = (\lambda_n)^\delta\). For \(r\) sufficiently close to 1, we have

\[
\delta \left[ \frac{1}{p} \left( \frac{p-1}{pr} \right) + \sum_{i=2}^{n} \alpha_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) - \delta \right] + (1 - \delta) \left[ \alpha_n - N_n \left( \frac{1}{p} - \frac{1}{q} \right) \right] > 0 \tag{5.14}
\]

and

\[
\alpha_i - N_i \left( \frac{q-1}{q} \right) + \left( 1 - \frac{1}{r} \right) \left( \frac{p-1}{p} \right) N_i < 0, \quad i = 1, 2, \ldots, n. \tag{5.15}
\]

Note that \(\alpha_n - N_n \left( \frac{1}{p} - \frac{1}{q} \right) > 0\). By using (5.14)-(5.15), we find that (5.13) is bounded by \(c_{\gamma \delta n N} (\lambda_n)^\epsilon\) for some \(\epsilon = \epsilon(p, q, r, \alpha, \gamma, \delta, n, N) > 0\).

Suppose that \(Q\) is centered on \(x \in \mathbb{R}^N\) for which \(|x| > 3\). Since \(Q\) has a diameter 1, we have

\[
\frac{1}{2} |x| \leq |x| \leq 2|x| \tag{5.16}
\]

whenever \(x \in Q\). From (5.16), we have

\[
\prod_{i=1}^{n} \left( \frac{1}{|x_i|} \int_{Q} \left( \frac{1}{|x_i|} \right)^{\gamma qr} dx \right) \left[ \frac{1}{|x|} \int_{Q} \left( \frac{1}{|x|} \right)^{\gamma (p-1)} dx \right] \leq c_{\gamma \delta} \prod_{i=1}^{n} (\lambda_i)^{\gamma qr} N_i \left( \frac{1}{p-1} \right) \leq c_{\gamma \delta} \prod_{i=1}^{n} (\lambda_i)^{\gamma qr} N_i \left( \frac{1}{p-1} \right), \quad (\gamma + \delta \geq 0) \tag{5.17}
\]

Case Three: Let \(\gamma > 0, \delta > 0\) satisfy (2.2)-(2.3) and (2.6). By adjusting the value of \(r\), assume

\[
\sum_{i=1}^{m-1} N_i < \gamma qr < \sum_{i=1}^{m} N_i, \quad 1 \leq m \leq n, \tag{5.18}
\]

\[
\sum_{i=1}^{l-1} N_i < \delta \left( \frac{pr}{p-1} \right) < \sum_{i=1}^{l} N_i, \quad 1 \leq l \leq n. \tag{5.19}
\]
We have
\[
\prod_{i=1}^{n} |Q_i|^{\frac{n}{N} - \left( \frac{1}{p} - \frac{1}{q} \right)} \left\{ \frac{1}{|Q|} \int_{Q} \left( \frac{1}{|x|} \right)^{\alpha qr} dx \right\}^{\frac{1}{\theta'}} \left\{ \frac{1}{|Q|} \int_{Q} \left( \frac{1}{|x|} \right)^{\alpha'(\frac{p}{\theta'})} dx \right\}^{\frac{p-1}{\theta'}}
\]
\[
\leq C_{p,q} \gamma \delta n \prod_{i=1}^{n} (\lambda_i)^{\gamma qr - N_i \left( \frac{1}{p} - \frac{1}{q} \right)} \prod_{i=1}^{n} \left( \frac{1}{\lambda_i} \right)^{\frac{N_i}{p'}} \prod_{i=1}^{n} \left( \frac{1}{\lambda_i} \right)^{N_i \left( \frac{p-1}{\theta'} \right)}
\]
\[
\left\{ \int \cdots \int_{Q} \left( \frac{1}{|x_1| + \cdots + |x_n|} \right)^{\alpha qr - \sum_{i=1}^{m-1} N_i} dx_1 \cdots dx_n \right\}^{\frac{1}{\theta'}}
\]
\[
\left\{ \int \cdots \int_{Q} \left( \frac{1}{|x_1| + \cdots + |x_n|} \right)^{\alpha'(\frac{p}{\theta'}) - \sum_{i=1}^{m-1} N_i} dx_1 \cdots dx_n \right\}^{\frac{p-1}{\theta'}}
\]
\[
\leq C_{p,q} \gamma \delta n \prod_{i=1}^{n} (\lambda_i)^{\gamma qr - N_i \left( \frac{1}{p} - \frac{1}{q} \right)} \prod_{i=1}^{n} \left( \frac{1}{\lambda_i} \right)^{\frac{N_i}{p'}} \prod_{i=1}^{n} \left( \frac{1}{\lambda_i} \right)^{N_i \left( \frac{p-1}{\theta'} \right)}
\]
\[
\left\{ \int \cdots \int_{Q} \left( \frac{1}{|x_1| + \cdots + |x_n|} \right)^{\alpha qr - \sum_{i=1}^{m-1} N_i} dx_1 \cdots dx_n \right\}^{\frac{1}{\theta'}} \left\{ \int \cdots \int_{Q} \left( \frac{1}{|x|} \right)^{\alpha'(\frac{p}{\theta'}) - \sum_{i=1}^{m-1} N_i} dx_1 \cdots dx_n \right\}^{\frac{p-1}{\theta'}}
\]
\[
\leq C_{p,q} \gamma \delta n \prod_{i=1}^{n} (\lambda_i)^{\gamma qr - N_i \left( \frac{1}{p} - \frac{1}{q} \right)} \prod_{i=1}^{n} \left( \frac{1}{\lambda_i} \right)^{\frac{N_i}{p'}} \prod_{i=1}^{n} \left( \frac{1}{\lambda_i} \right)^{N_i \left( \frac{p-1}{\theta'} \right)}
\]
\[
\left\{ \int_{Q_m} \left( \frac{1}{|x_m|} \right)^{\gamma qr - \sum_{i=m}^{m-1} N_i} dx_m \right\}^{\frac{1}{\theta'}} \left\{ \int_{Q_l} \left( \frac{1}{|x|} \right)^{\alpha'(\frac{p}{\theta'}) - \sum_{i=1}^{m-1} N_i} dx_l \right\}^{\frac{p-1}{\theta'}}
\]
\[
\leq C_{p,q} \gamma \delta n \prod_{i=1}^{n} (\lambda_m)^{\gamma qr - N_m \left( \frac{1}{p} - \frac{1}{q} \right)} \prod_{i=1}^{m} \left( \frac{1}{\lambda_i} \right)^{\frac{N_i}{p'}} \prod_{i=1}^{m} \left( \frac{1}{\lambda_i} \right)^{N_i \left( \frac{p-1}{\theta'} \right)}
\]
\[
\leq C_{p,q} \gamma \delta n \prod_{i=1}^{n} (\lambda_m)^{\gamma qr - N_m \left( \frac{1}{p} - \frac{1}{q} \right)} \prod_{i=1}^{m} \left( \frac{1}{\lambda_i} \right)^{\frac{N_i}{p'}} \prod_{i=1}^{m} \left( \frac{1}{\lambda_i} \right)^{N_i \left( \frac{p-1}{\theta'} \right)}
\]
by Remark 5.2

(5.20)
Let $0 \leq k \leq n - 1$. From direct computation, we have

$$
\frac{1}{r} \left( \frac{1}{q} + \frac{p-1}{p} \right) \sum_{i=1}^{k} N_i - (\gamma + \delta) + \sum_{i=k+1}^{n} \alpha_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right)
$$

$$
= \frac{N}{r} - \frac{1}{r} \left( \frac{1}{p} - \frac{1}{q} \right) N - (\gamma + \delta) + \sum_{i=k+1}^{n} \alpha_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) - N_i \left( 1 - \frac{1}{r} \right) \left( \frac{1}{p} - \frac{1}{q} \right)
$$

by (2.3)

$$
= \frac{N}{r} - \alpha + N \left( 1 - \frac{1}{r} \right) \left( \frac{1}{p} - \frac{1}{q} \right) + \sum_{i=k+1}^{n} \alpha_i - N_i \left( 1 - \frac{1}{r} \right) \left( \frac{1}{p} - \frac{1}{q} \right)
$$

$$
= \sum_{i=1}^{k} \frac{N_i}{r} - \alpha_i + N_i \left( 1 - \frac{1}{r} \right) \left( \frac{1}{p} - \frac{1}{q} \right). \tag{5.21}
$$

Suppose $l \leq m$. The formula in the last line of (5.20) can be rewritten as

$$
(\lambda_m)^{\frac{1}{q} \left( \frac{p-1}{p} \right) \sum_{i=1}^{m} N_i - (\gamma + \delta)} \left( \frac{\lambda_i}{\lambda_m} \right)^{\frac{p-1}{p} \sum_{i=1}^{n} N_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) \prod_{i=1}^{m} \left( \lambda_i \right)^{n_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) \prod_{i=1}^{m} \left( \frac{\lambda_i}{\lambda_m} \right)^{N_i - 1} \prod_{i=1}^{n} \left( \frac{1}{\lambda_i} \right)^{\frac{1}{q} \left( \frac{p-1}{p} \right) N_i}}
$$

$$
= (\lambda_m)^{\frac{1}{q} \left( \frac{p-1}{p} \right) \sum_{i=1}^{m} N_i - (\gamma + \delta)} \left( \frac{\lambda_i}{\lambda_m} \right)^{\frac{p-1}{p} \sum_{i=1}^{n} N_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) \prod_{i=1}^{m} \left( \lambda_i \right)^{n_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) \prod_{i=1}^{m} \left( \frac{\lambda_i}{\lambda_m} \right)^{N_i - 1} \prod_{i=1}^{n} \left( \frac{1}{\lambda_i} \right)^{\frac{1}{q} \left( \frac{p-1}{p} \right) N_i}}
$$

by (5.21)

$$
= (\lambda_m)^{\frac{1}{q} \left( \frac{p-1}{p} \right) \sum_{i=1}^{m} N_i - \sum_{i=m+1}^{n} \alpha_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) \prod_{i=1}^{m} \left( \lambda_i \right)^{n_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) \prod_{i=1}^{m} \left( \frac{\lambda_i}{\lambda_m} \right)^{N_i - 1} \prod_{i=1}^{n} \left( \frac{1}{\lambda_i} \right)^{\frac{1}{q} \left( \frac{p-1}{p} \right) N_i}}
$$

by (5.21)

$$
= \prod_{i=m+1}^{n} \left( \frac{\lambda_i}{\lambda_m} \right)^{n_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) \prod_{i=1}^{m} \left( \lambda_i \right)^{n_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) \prod_{i=1}^{m} \left( \frac{\lambda_i}{\lambda_m} \right)^{N_i - 1} \prod_{i=1}^{n} \left( \frac{1}{\lambda_i} \right)^{\frac{1}{q} \left( \frac{p-1}{p} \right) N_i}}
$$

by (5.21)

$$
(\lambda_m)^{\frac{1}{q} \left( \frac{p-1}{p} \right) \sum_{i=1}^{m} N_i - \sum_{i=m+1}^{n} \alpha_i - (\gamma + \delta) \prod_{i=1}^{m} \left( \lambda_i \right)^{n_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) \prod_{i=1}^{m} \left( \frac{\lambda_i}{\lambda_m} \right)^{N_i - 1} \prod_{i=1}^{n} \left( \frac{1}{\lambda_i} \right)^{\frac{1}{q} \left( \frac{p-1}{p} \right) N_i}}
$$

Recall the subset $\mathcal{U}$ defined in (2.6) where $\alpha_i - N_i/p < 0$ for every $i \notin \mathcal{U}$. 

(5.22)
Notice that $\lambda_m \leq \lambda_l$ when $l \leq m$. For $r$ sufficiently close to 1, we have

$$
\left( \frac{\lambda_m}{\lambda_l} \right)^{\sum_{i=1}^{m-1} N_i \frac{r}{\alpha_i} - \left( \frac{p}{\alpha_i} - 1 \right) N_i} \prod_{i=l}^{m} \left( \frac{\lambda_m}{\lambda_l} \right)^{N_i \frac{r}{\alpha_i} - \left( \frac{p}{\alpha_i} - 1 \right) N_i} \leq \left( \frac{\lambda_m}{\lambda_l} \right)^{\sum_{i=1}^{m} N_i \frac{r}{\alpha_i} - \left( \frac{p}{\alpha_i} - 1 \right) N_i} \prod_{i=l}^{m} \left( \frac{\lambda_m}{\lambda_l} \right)^{N_i \frac{r}{\alpha_i} - \left( \frac{p}{\alpha_i} - 1 \right) N_i} + \delta
$$

(5.23)

By bringing the estimates in (5.22)-(5.23) back to (5.20), we find

$$
\left( \frac{\lambda_m}{\lambda_l} \right)^{\frac{1}{r} \sum_{i=1}^{m} N_i \frac{r}{\alpha_i} - \left( \frac{p}{\alpha_i} - 1 \right) N_i} \leq \left( \frac{\lambda_m}{\lambda_l} \right)^{\frac{1}{r} \sum_{i=1}^{m} N_i \frac{r}{\alpha_i} - \left( \frac{p}{\alpha_i} - 1 \right) N_i} \prod_{i=l}^{m} \left( \frac{\lambda_m}{\lambda_l} \right)^{N_i \frac{r}{\alpha_i} - \left( \frac{p}{\alpha_i} - 1 \right) N_i} + \delta
$$

(5.24)

Recall that $\delta > 0$ satisfies the first strict inequality in (2.6). From (5.1) and (1.6), we also have $(\frac{1}{p} - \frac{1}{q}) N_i < \alpha_i < N_i$ for every $i = 1, 2, \ldots, n$. Define implicitly $0 \leq \delta_1 \leq \delta_2 \leq 1$ by letting $\lambda_l = (\lambda_m)^{\delta_1}$ and $\lambda_m = (\lambda_m)^{\delta_2}$.

For $r$ sufficiently close to 1, we have

$$
\delta_1 \left[ \frac{N_1}{r} - \alpha_1 + \left( \frac{1}{1 - \frac{1}{p}} \right) \left( \frac{1}{1 - \frac{1}{q}} \right) N_1 \right] + (1 - \delta_2) \left( \alpha_n - N_n \left( \frac{1}{p} - \frac{1}{q} \right) \right)
$$

(5.25)

The estimate in (5.25) implies that (5.24) is bounded by a constant multiple of $(\lambda_m)^{\varepsilon}$ for some $\varepsilon = \varepsilon(p, q, r, \alpha, \gamma, \delta, n N) > 0$. 

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On the other hand, suppose \( m \leq l \). The last line of (5.20) can be rewritten as

\[
(\lambda_m)^{\frac{1}{\theta}} \sum_{i=1}^{\frac{m-1}{\theta}} N_i \gamma (\lambda_i) \left( \frac{(\frac{m-1}{\theta})}{\gamma} \right) \sum_{i=1}^{n} (\lambda_i)^{\alpha_i} - N_i (\frac{1}{\frac{m}{\theta}}) \prod_{i=1}^{m-1} \left( \frac{1}{\lambda_i} \right) \prod_{i=m}^{l} \left( \frac{1}{\lambda_i} \right)^{\frac{m-1}{\theta} N_i} 
\]

\[
= (\lambda_l)^{\frac{1}{\theta}} \left( \frac{m-1}{\theta} \right) \sum_{i=1}^{n} (\lambda_i)^{\alpha_i} - N_i (\frac{1}{\frac{m}{\theta}}) \prod_{i=1}^{m} \left( \frac{1}{\lambda_i} \right) \prod_{i=m}^{l} \left( \frac{1}{\lambda_i} \right)^{\frac{m-1}{\theta} N_i} 
\]

Recall the subset \( V \) defined in (2.6) where \( \alpha_i - N_i (\frac{q-1}{q}) < 0 \) for every \( i \notin V \).

Notice that \( \lambda_l \leq \lambda_m \) when \( m \leq l \). For \( r \) sufficiently close to 1, we have

\[
\left( \frac{\lambda_l}{\lambda_m} \right)^{\sum_{i=1}^{\frac{m-1}{\theta}} N_i - \alpha_i \left( 1 - \frac{1}{1} \right) \left( \frac{q-1}{q} \right) N_i} \prod_{i=m}^{l} \left( \frac{1}{\lambda_i} \right)^{\frac{m-1}{\theta} N_i} 
\]

\[
\leq \left( \frac{\lambda_l}{\lambda_m} \right)^{\sum_{i \in V \cap \{1,...,m-1\}} \left( \frac{q-1}{q} \right) N_i - \alpha_i - \alpha_j \left( 1 - \frac{1}{1} \right) \left( \frac{q-1}{q} \right) N_i} \prod_{i=m}^{l} \left( \frac{1}{\lambda_i} \right)^{\frac{m-1}{\theta} N_i} 
\]

(5.27)

\[
\leq \left( \frac{\lambda_l}{\lambda_m} \right)^{\sum_{i \in V \cap \{1,...,m-1\}} \left( \frac{q-1}{q} \right) N_i - \alpha_i - \alpha_j \left( 1 - \frac{1}{1} \right) \left( \frac{q-1}{q} \right) N_i} \prod_{i=m}^{l} \left( \frac{1}{\lambda_i} \right)^{\frac{m-1}{\theta} N_i} 
\]
By bringing the estimates in (5.26)-(5.27) back to (5.20), we find
\[
\lambda_m^{\frac{1}{p} \sum_{i=1}^m N_i - \gamma} \left( \lambda_i \right)^{\left( \frac{1}{p'} - \frac{1}{q'} \right) \sum_{i=1}^n N_i - \delta} \prod_{i=1}^n \left( \lambda_i \right)^{\left( \delta - 1 \right) \left( \frac{1}{p} - \frac{1}{q} \right) N_i} = \prod_{i=1}^n \left( \lambda_i \right)^{\left( \delta - 1 \right) \left( \frac{1}{p} - \frac{1}{q} \right) N_i}
\]
\[
\left( \lambda_i \right)^{\left( \delta - 1 \right) \left( \frac{1}{p} - \frac{1}{q} \right) N_i}
\]
\[
\left( \lambda_i \right)^{\left( \delta - 1 \right) \left( \frac{1}{p} - \frac{1}{q} \right) N_i - \left( \frac{1}{p} - \frac{1}{q} \right) \sum_{i=1}^n N_i - \gamma} \prod_{i=1}^n \left( \lambda_i \right)^{\left( \delta - 1 \right) \left( \frac{1}{p} - \frac{1}{q} \right) N_i} \prod_{i=1}^n \left( \lambda_i \right)^{\left( \delta - 1 \right) \left( \frac{1}{p} - \frac{1}{q} \right) N_i}
\]

Recall that \( \gamma > 0 \) satisfies the second strict inequality in (2.6). From (5.1) and (1.6), we also have \( \left( \frac{1}{p} - \frac{1}{q} \right) N_i < \alpha_i < N_i \) for every \( i = 1, 2, \ldots, n \). Define implicitly \( 0 \leq \delta_1 \leq \delta_2 \leq 1 \) by letting \( \lambda_m = \left( \lambda_n \right)^{\delta_1} \) and \( \lambda_i = \left( \lambda_n \right)^{\delta_2} \).

For \( r \) sufficiently close to 1, we have
\[
\delta_1 \left[ \frac{N_1}{r} - \alpha_1 + \left( 1 - \frac{1}{r} \right) \left( \frac{1}{p} - \frac{1}{q} \right) N_1 \right] + \left( 1 - \delta_2 \right) \left( \alpha_n - N_n \left( 1 - \frac{1}{r} \right) \left( \frac{1}{r} - \frac{1}{q} \right) \right)
\]
\[
+ \left( \delta_2 - \delta_1 \right) \left[ \sum_{i \in \psi \cup \{1, \ldots, n\}} \left( \frac{q-1}{q} \right) N_i - \alpha_i - \left( 1 - \frac{1}{r} \right) \left( \frac{p-1}{p} \right) N_i + \gamma \right] > 0.
\]

The estimate in (5.29) implies that (5.28) is bounded by a constant multiple of \( \lambda_n^\varepsilon \) for some \( \varepsilon = \varepsilon(p, q, r, \alpha, \gamma, \delta, n, \mathbb{N}) > 0 \).

\[ \square \]

6 One-weight inequality on product spaces

Let \( \omega(x) = |x|^{-\gamma}, \sigma(x) = |x|^{\delta}, \gamma, \delta \in \mathbb{R} \). Consider \( \gamma + \delta = 0 \) so that \( \omega = \sigma \).

From (2.3) and (4.4), we must have
\[
\frac{\alpha_i}{N_i} = \frac{1}{p} - \frac{1}{q}, \quad i = 1, 2, \ldots, n.
\]

Write \( x = (x_i, x_i^+) \in \mathbb{R}^{N_i} \times \mathbb{R}^{N_i} \) and \( Q_i = \bigotimes_{j \neq i} Q_i \) for every \( i = 1, 2, \ldots, n \).

Let \( Q_i \) shrink to \( x_i^+ \) in (2.1). By applying the Lebesgue Differentiation Theorem, we have
\[
\left\{ \frac{1}{|Q_i|} \int_{Q_i} \omega^\vartheta (x_i, x_i^+) dx_i \right\}^{\frac{1}{\vartheta}} \left\{ \frac{1}{|Q_i|} \int_{Q_i} \left( \frac{1}{\omega} \right)^{q-1} (x_i, x_i^+) dx_i \right\}^{\frac{p-1}{q}} < \infty.
\]
for every \( Q_i \subset \mathbb{R}^N \) and \( a \cdot e x_i^+ \in \mathbb{R}^{N-N_i}, i = 1, 2, \ldots, n \).

Observe that (6.1)-(6.2) are sufficient conditions of the Muckenhoupt-Wheeden Theorem [5] which implies

\[
\left\{ \int_{\mathbb{R}^N} \left( \frac{1}{|x_i - y_i|} \right)^{N_i - \alpha_i} dy_i \right\}^{\frac{q}{q}} \leq \mathbb{C}_{p, q, N_i, \omega} \left\{ \int_{\mathbb{R}^{N-N_i}} \left( f(y_i) \right)^p (x_i, x_i^+) dx_i \right\}^{\frac{q}{p}}
\]

(6.3)

for \( 1 < p < q < \infty \) and \( a.e x_i^+ \in \mathbb{R}^{N-N_i}, i = 1, 2, \ldots, n \).

By using (6.3), we have

\[
\left\{ \int_{\mathbb{R}^N} \left( \omega \right)^p (x) dx \right\}^{\frac{1}{q}} = \left\{ \int_{\mathbb{R}^N} f(y) \prod_{i=1}^n \left( \frac{1}{|x_i - y_i|} \right)^{N_i - \alpha_i} dy \right\}^{\frac{q}{q}} \omega(x) dx \right\}^{\frac{1}{q}} \leq \mathbb{C}_{p, q, N_i, \omega} \left\{ \int_{\mathbb{R}^{N-N_i}} \left( f(y_i) \right)^p (x_i, x_i^+) dx_i \right\}^{\frac{q}{p}}
\]

\[
\leq \mathbb{C}_{p, q, N_i, \omega} \left\{ \int_{\mathbb{R}^{N-N_i}} \left( f(y_i) \right)^p (x_i, x_i^+) dx_i \right\}^{\frac{q}{p}} \leq \mathbb{C}_{p, q, N} \omega \left\{ \int_{\mathbb{R}^N} \left( f(x) \right)^p dx \right\}^{\frac{1}{p}}, \quad 1 < p < q < \infty.
\]

(6.4)

7 Proof of Theorem A*

Let \( \{1, 2, \ldots, n\} = I \cup J \) where

\[
I = \left\{ i \in \{1, 2, \ldots, n\} : \frac{\alpha_i}{N_i} = \frac{1}{p} - \frac{1}{q} \right\}, \quad J = \left\{ i \in \{1, 2, \ldots, n\} : \frac{\alpha_i}{N_i} > \frac{1}{p} - \frac{1}{q} \right\}. \quad (7.1)
\]

Define

\[
\alpha_I = \sum_{i \in I} \alpha_i, \quad Q_I = \bigotimes_{i \in I} Q_i, \quad \mathbb{R}^N_I = \bigotimes_{i \in I} \mathbb{R}^{N_i},
\]

\[
\alpha_J = \sum_{i \in J} \alpha_i, \quad Q_J = \bigotimes_{i \in J} Q_i, \quad \mathbb{R}^N_J = \bigotimes_{i \in J} \mathbb{R}^{N_i}, \quad (7.2)
\]

We write \( x = (x_I, x_J) \in \mathbb{R}^N_I \times \mathbb{R}^N_J \) and denote the cardinality of \( I \) and \( J \) by \(|I|\) and \(|J|\).
Suppose \( \omega(x) = |x|^{-\gamma}, \sigma(x) = |x|^\delta, \gamma, \delta \in \mathbb{R} \) satisfy the Muckenhoupt characteristic (2. 1). Consider \( Q_i \) centered on the origin of \( \mathbb{R}^N \) for every \( i \in I \). Let \( Q_i, i \in I \) shrink to the origin. By applying Lebesgue Differentiation Theorem, we have

\[
\sup_{Q \subset \mathbb{R}^N} \prod_{i \in J} |Q|^{\frac{\omega_i}{N}} \left\{ \frac{1}{|Q_i|} \int_{Q_i} \left( \frac{1}{|x_i|} \right)^{\gamma} \, dx \right\} \geq \left\{ \frac{1}{|Q|} \int_{Q} \left( \frac{1}{|x_i|} \right)^{\gamma} \, dx \right\} \frac{p-1}{p} < \infty.
\]

(7. 3)

The boundedness of \( A_{pq}^\alpha (|x|^{-\gamma}, |x|^\delta) \) requires \( \gamma q < N_f \) and \( \delta \left( \frac{p}{p-1} \right) < N_f \).

**Proposition 7.1** Let \( \omega(x_f) = |x_f|^{-\gamma}, \sigma(x_f) = |x_f|^\delta \) for \( \gamma, \delta \in \mathbb{R} \) satisfying (7. 3). For a.e. \( x_f \in \mathbb{R}^N_f \), we have

\[
\left\{ \int_{\mathbb{R}^N_f} \left( \int_{\mathbb{R}^N_f} f(x_f, y_f) \prod_{i \in J} \left( \frac{1}{|x_i - y_i|} \right)^{N_i - \alpha_i} \, dy \right)^{\frac{1}{q}} \omega^\alpha(x_f) \, dx_f \right\} \leq C_{p, q, \alpha, \gamma, \delta, |J|} N_f \left( \int_{\mathbb{R}^N_f} \left( f(x_f, x_f) \right)^{\mu} \sigma^\mu(x_f) \, dx_f \right)^{\frac{1}{\mu}}, \quad 1 < p \leq q < \infty.
\]

(7. 4)

**Proof:** From section 4, we have the Muckenhoupt characteristic (7. 3) implying \( \gamma, \delta \) to satisfy (2. 2)-(2. 6) with \( \alpha, \beta, N \) replaced by \( \alpha_f, |J|, N_f \) respectively. Suppose \( |J| = 1 \). **Theorem A** by Stein and Weiss [3] shows that these constraints are sufficient conditions to imply (7. 4).

Consider \( |J| \geq 2 \). By applying **Principal Lemma** in the beginning of section 5, we have \( \omega(x_f) = |x_f|^{-\gamma}, \sigma(x_f) = |x_f|^\delta \) satisfying the decay estimate (5. 2)-(5. 3) for every \( Q_f \subset \mathbb{R}^N_f \) where \( \alpha_i > N_i \left( \frac{1}{p} - \frac{1}{q} \right), i \in J \).

Let \( t \) denote the \( |J| \)-tuple \( \left( 2^{-t_1}, 2^{-t_2}, \ldots, 2^{-t_{|J|}} \right) \). We have

\[
\sum_t A_{pq}^{\alpha_f, \beta_f} (t : |x_f|^{-\gamma}, |x_f|^\delta) < \infty
\]

(7. 5)

as required in (3. 14) for every \( 0 < s < 1 \). \( \Box \)

Let \( \gamma > 0, \delta > 0 \) satisfy (2. 2)-(2. 3) and (2. 6). In particular, we have

\[
\omega(x) = |x|^{-\gamma} \leq |x_f|^{-\gamma} = \omega(x_f), \quad \sigma(x_f) = |x_f|^\delta \leq |x|^{\delta} = \sigma(x).
\]

(7. 6)
From (7.6), we have

$$\left\{ \int_{\mathbb{R}^n} (\omega I_{\alpha f})^q(x)dx \right\}^{\frac{1}{q}} \leq \left\{ \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} f(y) \prod_{i=1}^n \left( \frac{1}{|x_i - y_i|} \right)^{N_{1-\alpha}} dy \right)^q \omega^q(x_J)dx \right\}^{\frac{1}{q}}$$

$$\leq C_{p, q, \gamma, \delta, J, N} \left\{ \int_{\mathbb{R}^n_J} \left( \int_{\mathbb{R}^n_J} f(y_I, x_J) \prod_{i \in J} \left( \frac{1}{|x_i - y_i|} \right)^{N_{1-\alpha}} dy_I \right)^p \omega^p(x_J)dx_J \right\}^{\frac{1}{p}}$$

by Proposition 7.1

$$\leq C_{p, q, \gamma, \delta, J, N} \left\{ \int_{\mathbb{R}^n_J} \left( \int_{\mathbb{R}^n_J} f(y_I, x_J) \prod_{i \in J} \left( \frac{1}{|x_i - y_i|} \right)^{N_{1-\alpha}} dy_I \right) dx_I \right\}^{\frac{p}{q}}$$

by Minkowski integral inequality

$$\leq C_{p, q, \gamma, \delta, J, N} \left\{ \int_{\mathbb{R}^n_J} (f \sigma)^p(x)dx \right\}^{\frac{1}{p}}, \quad 1 < p < q < \infty.$$ \hfill (7.7)

Consider \(\gamma \geq 0, \delta \leq 0\) satisfying (2.2)-(2.4) or \(\gamma \leq 0, \delta \geq 0\) satisfying (2.2)-(2.3) and (2.5). Note that it is suffice to study one of these two cases because \(I_{a}\) is self-adjoint and

$$\|\omega I_{\alpha f}\|_{L^q(\mathbb{R}^n)} \leq \|f\sigma\|_{L^q(\mathbb{R}^n)} \quad \text{if and only if} \quad \|\sigma I_{\alpha f}\|_{L^{q'}}(\mathbb{R}^n) \leq \|\omega^{-1}\|_{L^{q'}(\mathbb{R}^n)}.$$ \hfill (7.8)

Let \(\gamma \geq 0, \delta \leq 0\). Suppose that \(f\) is supported in the region where \(|x_I| \leq |x_J|\). By using (7.6) and carrying out the same estimate (7.7), we have

$$\left\{ \int_{\mathbb{R}^n} \left( \int_{|x_I| \leq |x_J|} \frac{1}{|x_i - y_i|} dy \right)^{q} d\omega(x) \right\}^{\frac{1}{q}} \leq \left\{ \int_{\mathbb{R}^n_J} \left( \int_{\mathbb{R}^n_J} f(y_I, x_J) \prod_{i \in J} \left( \frac{1}{|x_i - y_i|} \right)^{N_{1-\alpha}} dy_I dx_I dx_J \right)^q \omega^q(x_J)dx \right\}^{\frac{1}{q}}$$

$$\leq C_{p, q, \gamma, \delta, n} \left\{ \int_{|x_I| \leq |x_J|} (f(x_I, x_J))^p \sigma^p(x_J)dx_I dx_J \right\}^{\frac{1}{p}}$$

$$\leq C_{p, q, \gamma, \delta, n} \left\{ \int_{\mathbb{R}^n} (f \sigma)^p(x)dx \right\}^{\frac{1}{p}}.$$ \hfill (7.9)

The last inequality holds in (7.9) because \(\sigma(x_J) = |x_J|^\delta \approx |x|^\delta = \sigma(x)\) for \(|x_I| \leq |x_J|\).

On the other hand, suppose \(f\) supported in the region \(|x_I| > |x_J|\). Recall that \(\gamma \geq 0, \delta \leq 0\)
By putting together (7.10) and (7.11), we find

$$\gamma + \delta = \sum_{i=1}^{n} \alpha_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) = \sum_{i \in J} \alpha_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) \quad \text{by (7.1)}$$

(7.10)

and

$$\alpha_i - \frac{N_i}{p} < \delta \leq 0 \quad \text{for every} \quad i \in [1, 2, \ldots, n] = I \cup J.$$ 

(7.11)

By putting together (7.10) and (7.11), we find

$$0 \leq \gamma + \delta = \sum_{i \in J} \alpha_i - N_i \left( \frac{1}{p} - \frac{1}{q} \right) < \frac{N_J}{q}, \quad 0 < \frac{N_J}{q} \left( \frac{p}{p-1} \right).$$

(7.12)

**Proposition 7.2** Let \( \rho \left( x_J \right) = |x_J|^{-(\gamma + \delta)}, \eta \left( x_J \right) \equiv 1. \) For a.e \( x_J \in \mathbb{R}^N, \) we have

$$\left\{ \int_{\mathbb{R}^N_J} \left\{ \int_{\mathbb{R}^N_J} f(x_J, y_J) \prod_{i \in J} \left( \frac{1}{|x_i - y_i|} \right)^{N_i - \alpha_i} dy_J \right\}^q \rho^q(x_J)dx_J \right\}^{\frac{1}{q}} \leq C_p q \ a_J \ y \ |J| \ N_J \ \left( \int_{\mathbb{R}^N_J} \left( f(x_J, x_J) \right)^p \eta^p(x_J)dx_J \right)^{\frac{1}{p}}, \quad 1 < p \leq q < \infty.$$ 

(7.13)

**Proof:** Observe that (7.10)-(7.12) imply the constraints in (2.2)-(2.4) with \( \gamma, \delta, \alpha, n, N \) replaced by \( \gamma + \delta, 0, \alpha_J, |J|, N_J \) respectively.

Suppose \( |J| = 1, \) From Theorem A, it follows that (7.10)-(7.12) are sufficient conditions to imply (7.13).

Consider \( |J| \geq 2. \) By applying Principal Lemma, \( \rho(x_J) = |x_J|^{-(\gamma + \delta)}, \eta(x_J) \equiv 1 \) satisfy the decay estimate in (5.2)-(5.3) for every \( Q_J \subset \mathbb{R}^N \) where \( \alpha_i > N_i \left( \frac{1}{p} - \frac{1}{q} \right), i \in J. \)

Let \( t \) denote the \( |J| \)-tuple \( \left( 2^{-t_1}, 2^{-t_2}, \ldots, 2^{-t_{|J|}} \right). \) We have

$$\sum_{t} A_{pqs}^{\alpha_J} \left( t : |x_J|^{-(\gamma + \delta)}, 1 \right) < \infty$$ 

(7.14)

as required in (3.14) for every \( 0 < s < 1. \)

**Proposition 7.3** Let \( \omega(x_J) = \sigma(x_J) = |x_J|^\delta. \) For a.e \( x_J \in \mathbb{R}^N \), we have

$$\left\{ \int_{\mathbb{R}^N_J} \left\{ \int_{\mathbb{R}^N_J} f(y_J, x_J) \prod_{i \in J} \left( \frac{1}{|x_i - y_i|} \right)^{N_i - \alpha_i} dy_J \right\}^q \omega^q(x_J)dx_J \right\}^{\frac{1}{q}} \leq C_p q \ a_J \ y \ |J| \ N_J \ \left( \int_{\mathbb{R}^N_J} \left( f(x_J, x_J) \right)^p \omega^p(x_J)dx_J \right)^{\frac{1}{p}}, \quad 1 < p \leq q < \infty.$$ 

(7.15)
Proof: Recall (7. 1) and (7. 11). We have

\[-\delta + \delta = 0 = \sum_{i \in I} \alpha_i - N_i \left(\frac{1}{p} - \frac{1}{q}\right), \quad -\delta < \frac{N_i}{p} - \alpha = \frac{N_i}{q} \text{ for } i \in I. \tag{7. 16}\]

Note that the constraints in (7. 16) are sufficient conditions for Theorem A on every subspace \(\mathbb{R}^N, i \in I\). The norm inequality (7. 15) can be obtained by following the iteration argument given in section 6.

\[\Box\]

Let \(\rho(x_F) = |x_F|^{\gamma + \delta}\) and \(\sigma(x_F) = |x_F|^\delta\) where \(\gamma + \delta \geq 0\) and \(\delta \leq 0\). It is clear that

\[\omega(x) = |x|^{-\gamma} \leq \rho(x_F) \sigma(x_F). \tag{7. 17}\]

We have

\[\left\{ \int_{\mathbb{R}^N} \left\{ \int_{|y_i| > |y_j|} f(y) \prod_{i=1}^N \left(\frac{1}{|x_i - y_i|}\right)^{N_i - \alpha_j} dy \right\}^{\frac{q}{p}} \alpha^q(x) dx \right\}^{\frac{1}{q}} \leq \left\{ \int_{\mathbb{R}^N} \left\{ \int_{|y_i| > |y_j|} f(y) \prod_{i=1}^N \left(\frac{1}{|x_i - y_i|}\right)^{N_i - \alpha_j} dy \right\}^{\frac{q}{p}} \rho^q(x_F) \sigma^q(x_F) dx \right\}^{\frac{1}{q}} \text{ by (7. 17)}\]

\[= \left\{ \int_{\mathbb{R}^N} \left\{ \int_{|y_i| > |y_j|} f(y_I, y_J) \prod_{i \in I \cup J} \left(\frac{1}{|x_i - y_i|}\right)^{N_i - \alpha_j} dy_I dy_J \right\}^{\frac{q}{p}} \rho^q(x_F) \sigma^q(x_F) dx_I dx_J \right\}^{\frac{1}{q}} \]

\[\leq C_{p, q, \gamma, \delta, \epsilon} \left\{ \int_{\mathbb{R}^N} \left\{ \int_{|y_i| > |y_j|} f(y_I, x_J) \prod_{i \in I} \left(\frac{1}{|x_i - y_i|}\right)^{N_i - \alpha_j} dy_I \right\}^{\frac{q}{p}} \rho^q(x_F) \sigma^q(x_F) dx_I dx_J \right\}^{\frac{1}{q}} \text{ by Proposition 7.2} \]

\[\leq C_{p, q, \gamma, \delta, \epsilon} \left\{ \int_{\mathbb{R}^N} \left\{ \int_{|y_i| > |x_j|} f(y_I, x_J) \prod_{i \in I} \left(\frac{1}{|x_i - y_i|}\right)^{N_i - \alpha_j} dy_I \right\}^{\frac{q}{p}} \rho^q(x_F) \sigma^q(x_F) dx_I dx_J \right\}^{\frac{1}{q}} \text{ by Minkowski integral inequality} \]

\[\leq C_{p, q, \gamma, \delta, \epsilon} \left\{ \int_{|x_i| > |x_j|} \left( f(x_I, x_J) \right)^p \sigma^p(x_F) dx_I dx_J \right\}^{\frac{1}{p}} \text{ by Proposition 7.3} \]

\[\leq C_{p, q, \gamma, \delta, \epsilon} \left\{ \int_{\mathbb{R}^N} \left( f(x) \right)^p dx \right\}^{\frac{1}{p}}, \quad 1 < p \leq q < \infty. \tag{7. 18}\]

The last inequality holds because \(\sigma(x_F) \approx \sigma(x)\) for \(|x_F| > |x|\).

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