RATIONAL POINTS ON THE NOETHER-LEFSCHETZ LOCUS OF K3 MODULI SPACES

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ABSTRACT. Let $L$ be an even hyperbolic lattice, and let $\mathcal{F}_L/\mathbb{Q}$ be the moduli space of $L$-quasipolarized K3 surfaces. Let $K$ be a number field and let $x \in \mathcal{F}_L(K)$ be a $K$-rational point. We show how one can use maps between Shimura varieties to understand whether $x$ belongs to the Noether-Lefschetz locus of $\mathcal{F}_L$ or not, that is, whether the K3 surface represented by $x$ has geometric Néron-Severi lattice isometric to $L$. Our arguments generalize to all orthogonal modular varieties associated to lattices of signature $(2, n-2)$.

1. INTRODUCTION

The moduli spaces of lattice polarized K3 surfaces were introduced by Nikulin [Nik79a] and Dolgachev [Dol96] in order to generalize the classical moduli spaces of polarized K3 surfaces. Given a hyperbolic lattice $L$ which embeds primitively into the K3 lattice $\Lambda \cong U^3 \oplus E_8(-1)^2$, the moduli space $\mathcal{F}_L$ parametrizes pairs $(X, \iota)$ where $X$ is a complex K3 surface and $\iota: L \hookrightarrow \text{NS}(X)$ is a primitive embedding of lattices, satisfying a certain ampleness condition. It is a quasiprojective variety of dimension $\dim(\mathcal{F}_L) = 20 - \text{rank}(L)$ which is the quotient of a hermitian symmetric domain by an arithmetic subgroup. The Noether-Lefschetz locus of $\mathcal{F}_L$ is denoted by $\mathcal{F}_{NL}$ and consists of all the pairs $(X, \iota)$ for which the map $\iota: L \hookrightarrow \text{NS}(X)$ is a proper inclusion. It is a countable union of proper subvarieties of $\mathcal{F}_L$.

Let now $K$ be a number field. In this paper, we characterize the $K$-rational points of $\mathcal{F}_L$ which lie in the Noether-Lefschetz locus, by studying their behaviour under some natural branched coverings of $\mathcal{F}_L$ belonging to the same Shimura tower. We determine these coverings using elementary lattice theory.

More precisely, under some mild condition on $L$, the moduli space $\mathcal{F}_L$ is a geometrically irreducible variety defined over $\mathbb{Q}$. For any chosen $N > 0$, there are geometrically irreducible Shimura varieties $\mathcal{F}_i/\mathbb{Q}$ for $i = 1, \cdots, n$ (where $n$ depends on $N$ and on the choice of a prime number $p$) and finite quotient maps $\pi_i: \mathcal{F}_i \to \mathcal{F}_L$ which satisfy the following:

1. For every $i = 1, \cdots, n$ we have that $\deg(\pi_i) > N$;
2. For any finite field extension $K'/\mathbb{Q}$ and any rational point $x \in \mathcal{F}^{\text{red}}_L(K)$, there is a $1 \leq i \leq n$ such that the fibre $\pi_i^{-1}\{x\}$ contains a $K'$-rational point with $[K': K] | 32$.

Thus, if $N > 32$ and if $\pi_i^{-1}\{x\}$ is $K$-irreducible for $1 \leq i \leq n$, the K3 surface associated to $x$ must have geometric Néron-Severi lattice isomorphic to $L$. We refer the reader to Theorem 4.6 and the discussion afterwards for the general statement. In many cases, the constant $32$ can be reduced up to $8$. For instance, the theorem above with constant
8 works for any hyperbolic lattice satisfying \( \text{rank}(L) \leq 8 \). The stated version works more generally for very stable lattices, see Definition 3.9.

We hope that our result can be applied in practical computations, as in finding Picard generic specializations in families of K3 surfaces. Determining the Néron-Severi lattice of a K3 surface over a number field is a notoriously difficult problem, see for instance [Kuw00], [EL07], [Klo07], [HKT13], [ES13], [EJ11], [Cha14] among many others. In some cases, one works with a (modular) family of lattice polarized K3 surfaces and one is interested in finding Picard generic specializations. This could then be accomplished by using the characterization above, up to writing the maps \( \pi_i \) algebraically.

Finally, the following uniform bound conjectured by Várilly-Alvarado makes the characterization of \( \mathcal{F}_L^{nl} \) effective:

**Conjecture 1.1** ([VA17]). Let \( L \) be an even hyperbolic lattice and let \( X/K \) be a K3 surface over a number field such that \( \text{NS}(X) \cong L \). Then, \( |\text{Br}(X)^G| \) can be bounded only in terms of \( [K: \mathbb{Q}] \).

A large body of literature can be found around this subject, consider for instance [SZ12], [HKT13], [New16], [VAV17], [CFTTV18], [CC20], [Val21] as well as the references in there.

**Theorem 1.1.** Assume that Conjecture 1.1 is true for a very stable lattice \( L \). Let \( N > 0 \) be an integer. Then, there is a finite set of maps \( \pi_i: \mathcal{F}_i \to \mathcal{F}_L \) as before such that \( x \in \mathcal{F}_L^{nl}(K) \) if and only if there is an \( i \) such that \( \pi_i^{-1}\{x\} \) contains a \( K'/K \) rational point with \( [K': K] = 32 \). This works over any number field \( K \) such that \( [K: \mathbb{Q}] \leq N \).

### 1.1. Structure of the paper.

In Section 2 we recall the construction of the moduli spaces of lattice polarized K3 surfaces, following the original paper of Dolgachev and the considerations carried out in the more recent one of Alexeev and Engel [AE21]. We prove in Theorem 2.1 that all these spaces have natural models over \( \mathbb{Q} \). This is probably well known, but we could not find a reference. In Section 3 we compare the spaces with Shimura varieties of orthogonal type. In Section 4 we study maps between these moduli spaces induced by maps of lattices, and we prove Proposition 4.2 which is the key tool for our main result. Finally, in 4.1 we prove our main result.

### 1.2. Idea of the proof.

Let \( L \) be an even hyperbolic lattice and let \( f: L \to \Lambda \) be a primitive embedding into the K3 lattice. Let \( T \) denote its orthogonal complement, which is an even lattice of signature type \((2,+,n_-)\). Let \( D_T^+ \) denote the Grassmanian of positive oriented planes in \( T_\mathbb{R} \). This is a hermitian symmetric domain with two connected components exchanged by complex conjugation. Let \( O(T)^* \) denote the kernel of the natural map \( O(T) \to O(T^\vee/T) \) where \( T^\vee \) is the dual of \( T \). In many cases, the moduli space \( \mathcal{F}_L \) is isomorphic to \( \mathbb{D}_T^+/O(T)^* \). We note that the association \( T \to M_T := \mathbb{D}_T^+/O(T)^* \) is functorial in the sense that every map of lattices \( T \to S \) of the same signature type (not necessarily primitive) induces a map \( M_T \to M_S \) due to Proposition 4.1.

Let now \( M \) be a connected component of \( M_T \). In Proposition 4.2 we show that for any primitive inclusion of lattices \( T \hookrightarrow S \), the degree of the induced map \( M \to M_S \) onto its image can be bounded only in terms of \( \text{rank}(S) - \text{rank}(T) \). If, moreover, \( T \) has corank one in \( S \), then this degree is always a divisor of 4.

In order to use this fact, we fix a prime number \( p \) and let \( S' \subset S \) be any sublattice such that \( S/S' \cong \mathbb{Z}/p\mathbb{Z} \). The functoriality mentioned before yields a branched covering \( M_{S'} \to M_S \) whose degree (on every connected component) can be made arbitrarily large only depending on \( p \). Note that there are only finitely many such sublattices for a given \( p \).
Choose \( x \in \mathcal{F}_L^0(K) \). Then, there is a primitive sublattice \( T \hookrightarrow S \) of corank one such that \( x \) belongs to the image of \( M_T \to M_S \). Choose any sublattice \( S' \subset S \) of index \( p \) such that \( S' \cap T = T \). We then obtain a commutative triangle of maps

\[
\begin{array}{ccc}
M_T & \xrightarrow{f} & M_{S'} \\
\downarrow f & & \downarrow \pi' \\
M_S & \xrightarrow{\pi} & M_S \\
\end{array}
\]

The discussion above says that both diagonal maps have degree onto their image which is a divisor of 4, whereas the vertical map has arbitrarily large degree. Note that in order to make all these maps and varieties defined over \( \mathbb{Q} \) one may need to add more components. After we do this we arrive at our main results simply by picking a preimage of \( x \) under \( f \) and then considering its image under \( f' \). The collection of maps \( \{\pi_i\}_{i=1}^n \) is then given by a slight modification of the coverings \( M_{S'} \to M_S \) associated to the finitely many sublattices \( S' \subset S \) of index \( p \), for \( p \) a given big enough prime.

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2. Moduli spaces of lattice polarized K3 surfaces

In this section we recall the construction of moduli spaces of lattice-polarized K3 surfaces. The standard reference is Dolgachev [Dol96], we also follow the more recent version of Alexeev and Engel [AE21]. We refer the reader to these papers concerning most of the proofs in this chapter.

2.1. Stable orthogonal groups. A lattice \( L \) is an abelian group \( L \cong \mathbb{Z}^n \) together with a non-degenerate symmetric pairing \( L \times L \to \mathbb{Z} \). All the lattices in this paper will be even, meaning that \((x, x) \in 2\mathbb{Z}\) for any \( x \in L \). The signature of \( L \) is the signature of \( L_\mathbb{R} \) and the discriminant form of an even lattice \( L \) is the group \( A_L = L^\vee / L \) endowed with its natural quadratic form \( q: A_L \to \mathbb{Q}/2\mathbb{Z} \) (see [Nik79]). One denotes by \( O(A_L) \) the finite group of isometries of \( A_L \). The orthogonal group of \( L \) is denoted as \( O(L) \) (later, \( O(L) \) will mean \( O_L(\mathbb{Z}) \), the \( \mathbb{Z} \)-points of the orthogonal group seen as a \( \mathbb{Z} \)-group scheme). We denote by \( \ell(A_L) \) the length of the abelian group \( A_L \) (minimal number of generators) and finally we say that a lattice \( L \) is unimodular if \( \text{discr}(L) = \pm 1 \), that is, if the natural inclusion \( L \hookrightarrow L^\vee \) is an isomorphism.

Definition 2.1. The stable isometry group of \( L \) is \( O(L)^* := \ker(O(L) \to O(A_L)) \).

The group \( O(L)^* \) enjoys many useful properties.

Lemma 2.2. Let \( L \to U \) be a primitive embedding of even lattices with \( U \) unimodular. Let \( \Gamma(L, U) := \{g \in O(U): g|_{L^\vee} = \text{Id}\} \). Then, the natural map \( \Gamma(L, U) \to O(L) \) induces an isomorphism \( \Gamma(L, U) \cong O(L)^* \).

Proof. Let \( K \) be the orthogonal complement of \( L \) in \( U \). By [Nik79][Proposition 1.6.1], there is a natural isomorphism \( A_L \cong -A_K \) where \( -A_K \) denotes the group \( A_K \) with opposite quadratic form. It follows that if \( g \in \Gamma(L, U) \) then the induced isometry \( g|_L \in \)
Definition 2.4. A small cone of \(O(L)\) must act trivially on \(A_L\). On the other hand, let \(\tilde{g} \in O(L)^*\) be a stable isometry. Due to [Nik79b][Corollary 1.5.2] one sees that \(\tilde{g} \oplus \text{Id}_K : L \oplus K \rightarrow L \oplus K\) can be extended to a global isometry \(g \in \Gamma(L, U)\). \(\square\)

Said differently, an isometry of \(L\) is stable if and only if it can be extended to an isometry of \(U\) by requiring it to be the identity on \(L^\perp\). Note that this works for every unimodular lattice \(U\) which contains \(L\) primitively.

Proposition 2.3. Let \(\tilde{L} \hookrightarrow N\) be any inclusion of lattices that respect the quadratic form. Then, there is a natural map \(O^*(L) \rightarrow O^*(N)\) induced by extension by \(1\).

Proof. Let \(\tilde{L}\) be the saturation of \(L\) in \(N\). The map \(L \rightarrow \tilde{L} \rightarrow N\) where the first arrow exhibits \(\tilde{L}\) as an overlattice of \(L\) (that is, the quotient \(\tilde{L}/L\) is finite) and the second arrow is a primitive embedding of lattices. It is sufficient to prove the proposition for these two kinds of map. So assume that \(N\) is an overlattice of \(L\). As explained in [Nik79b][Proposition 1.4.1], \(N\) is determined by an isotropic subgroup \(H \subset A_L\), that is, under the natural quotient map \(\pi : L^\perp \rightarrow A_L\), one has that \(\pi^{-1}(H) = N\). Clearly, if an isometry \(g \in O(L)\) acts trivially on \(A_L\), then \(g\) extends uniquely to an isometry of \(N\). Moreover, there is a natural isomorphism \(A_N \cong H^\perp/H\), and it follows that such extension must belong to \(O(N)^*\). Consider now a primitive embedding \(\tilde{N} \rightarrow N\), so that \(N/L\) is torsion free. Let \(U\) be a unimodular lattice with a primitive embedding \(N \hookrightarrow U\). We obtain a chain \(L \subset N \subset U\). Let \(g \in O(L)^*\) be a stable isometry. By the lemma above, we can extend \(g \oplus \text{Id}_{L_L}\) to an isometry \(\tilde{g} \in O(U)\). This isometry clearly belongs to \(\Gamma(N, U) \cong O(N)^*\), and the proposition is proved. \(\square\)

Remark 1. All these properties holds in general for lattices over PID, e.g., for \(p\)-adic lattices.

2.2. Moduli spaces. Let \(\Lambda\) denote the K3 lattice, the unique even, unimodular lattice of rank 22 and of signature \((3+, \ldots, 19-\)\). An even lattice \(L\) is called hyperbolic if its signature pair is of the form \((1+, \ldots, 19-\)\). Two primitive embedding \(f_1, f_2 : L \rightarrow \Lambda\) are equivalent if there exists \(g \in O(\Lambda)\) such that \(f_1 = g \circ f_2\). We denote by \(\mathcal{E}(L)\) the set of equivalence classes of primitive embedding of \(L\) into \(\Lambda\).

Remark 2. It is known that \(\mathcal{E}(L)\) is always finite, e.g., it follows from Nikulin’s description of this set [Nik79b][Proposition 1.6.1] and the finiteness of the genus of a lattice. Moreover, for any even hyperbolic lattice \(L\) of rank \(\text{rank}(L) \leq 20\) we have that if \(\ell(A_L) \leq 20 - \text{rank}(L)\) then \(\mathcal{E}(L) = \{*\}\) due to [Nik79b][Corollary 1.13.3], so that every lattice of rank smaller than 10 embeds uniquely into the K3 lattice.

For an embedding \(f\) we denote by \([f]\) its class in \(\mathcal{E}(L)\). For an even hyperbolic lattice \(L\), write \(V = \{x \in L_\mathbb{R} : (x, x) > 0\}\) and fix a primitive embedding \(f : L \rightarrow \Lambda\). Consider the set \(\Delta(f) := \{\delta \in \Lambda \setminus L^\perp : (\delta, \delta) = -2\text{ and } (\delta, L) \text{ is hyperbolic}\}\). For each \(\delta \in \Delta(f)\) we obtain a hyperplane \(V_\delta \subset L_\mathbb{R}\) given by the condition \((x, \delta) = 0\). A small cone \(C\) of \(L\) relative to \(f\) is a connected component of the complement \(V \setminus \bigcup_{\delta \in \Delta(f)} V_\delta\) (see [AE21][Definition 2.21]). Note that the set \(\bigcup_{\delta \in \Delta(f)} V_\delta\) depends only on the class \([f] \in \mathcal{E}(L)\).

Definition 2.4. A small cone \(C\) of \(L\) is a connected component of

\[
V \setminus \bigcup_{\delta \in \Delta(e) \atop e \in \mathcal{E}(L)} V_\delta.
\]
Since the set $E(L)$ is finite, it follows from [AE21][Proposition 2.23] that the small cones are locally rational polyhedral, that is, they are given locally by the intersection of finitely many rational half spaces.

Fix an even hyperbolic lattice $L$ that embeds primitively into the K3 lattice and a small cone $C$ of $L$. The following is [AE21][Definition 2.16] and [AE21][Proposition 2.24].

**Definition 2.5.** For $e \in E(L)$, an $(L, e)$-quasipolarized K3 surface is a pair $(X, \iota)$ where $X$ is a complex K3 surface and $\iota$ is a primitive embedding $\iota: L \hookrightarrow \text{NS}(X)$ such that $\iota(C)$ contains a big and nef class, and such that the composition $L \to \text{NS}(X) \to H^2(X, \mathbb{Z}(1)) \cong \Lambda$ represents $e$ (note that this is well defined). Two $(L, e)$-quasipolarized K3 surfaces $(X, \iota), (X', \iota')$ are isomorphic if there exists an isomorphism $h: X \overset{\sim}{\to} X'$ such that $\iota = h^* \circ \iota'$ where $h^*: \text{NS}(X') \to \text{NS}(X)$ is the natural pullback map on Néron-Severi lattices (in particular, the isomorphism is not required to preserve the small cone.)

Note that $\iota(C)$ contains automatically an ample class whenever $\iota(L)^{-}$ does not contain $(-2)$-classes. This is due to the definition of small cones: if $\ell \in \iota(C)$ is a big and nef, then it is ample if and only if $(\ell, \delta) > 0$ for any effective $(-2)$-class $\delta \in \text{NS}(X)$. If $\delta \notin L^+$, then $(\ell, \delta) \neq 0$ because $\ell \in \check{C}$, so it follows that $(\delta, \ell) > 0$.

For a hyperbolic lattice $L$, its roots are given by $\Delta(L) = \{ \delta \in L: (\delta, \delta) = -2 \}$. The Weyl group $W(L) \subset O(L)$ is the group generated by the reflections in elements of $\Delta(L)$. Note that $W(L) \subset O(L)^*$ always, because any such reflection can be extended (as a reflection) on any even unimodular lattice $U$ such that $L \subset U$ primitively. A Weyl chamber is a connected component of $V \setminus \bigcup_{\delta \in \Delta(L)} V_\delta$. Clearly, small cones are contained in Weyl chambers. It is a classical fact that $\{ \pm 1 \} \times W(L)$ acts freely on $V$ with a Weyl chamber $\mathcal{K}$ as a fundamental domain, so that the set of Weyl chambers is a torsor under $\{ \pm 1 \} \times W(L)$.

The construction of the moduli space of $(L, e)$-quasipolarized K3 surfaces then follows Dolgachev’s paper. Let $\mathcal{M}$ be the fine moduli space of marked K3 surfaces. This is a 20-dimensional complex manifold which is not Hausdorff, and which parametrizes K3 surfaces $X$ together with a marking $\phi: H^2(X, \mathbb{Z}(1)) \overset{\sim}{\to} \Lambda$, up to the natural notion of isomorphism. Over $\mathcal{M}$ one has a universal family $(X, \phi)$, and the period map

$$p: \mathcal{M} \to \mathcal{D} := \{ \sigma \in \mathbb{P}(\Lambda_C): (\sigma, \sigma) = 0, (\sigma, \sigma') > 0 \}$$

takes a point $m \in \mathcal{M}$ to the line generated by $\phi_m(H^{2,0}(X_m))$. The open subset $\mathcal{D}$ is the period domain of $\Lambda$: it is not yet a hermitian symmetric domain (because $\Lambda$ has the wrong signature) and it consists of two connected components exchanged by the complex conjugation $\sigma \mapsto \bar{\sigma}$. For any $x \in \mathcal{D}$ seen as a line in $\Lambda_C$, we denote by $W_x \subset O(\Lambda)$ its Weyl group, the group generated by reflection in elements $\lambda \in \Lambda \cap x^\perp$ such that $(\lambda, \lambda) = -2$. The period map $p$ is surjective [Fod80] and its fibres $p^{-1}(x)$ are naturally torsors under $\{ \pm \text{Id} \} \times W_x$, where $g \in \{ \pm \text{Id} \} \times W_x$ acts by $(X, \phi) \mapsto (X, g \circ \phi)$. It follows that the fibres are in one-to-one correspondence to Weyl chambers in $\text{NS}(X)$ whenever $X$ is projective.

In order to define marked $(L, e)$-quasipolarized K3 surfaces, we fix an embedding $L \hookrightarrow \Lambda$ which represents $e$ and a small cone $C$ on $L$.

**Definition 2.6.** A marked $(L, e)$-quasipolarized K3 surface is a K3 surface $X$ together with a marking $\phi: H^2(X, \mathbb{Z}(1)) \overset{\sim}{\to} \Lambda$ such that $L \subset \phi(\text{NS}(X))$ and such that the induced map $L \to \text{NS}(X)$ is quasipolarized (the image of $C$ contains a big and nef divisor). Two marked $(L, e)$-quasipolarized K3 surfaces $(X, \phi), (X', \phi')$ are isomorphic if there exists an
isomorphism $h: X \cong X'$ such that $\iota = h^* \circ \iota'$ where $h^*: H^2(X', \mathbb{Z}(1)) \to H^2(X, \mathbb{Z}(1))$ is the natural pullback map in cohomology.

To any marked $(L, e)$-quasipolarized K3 surface $(X, \phi)$ one can associate a $(L, e)$-quasipolarized K3 surface by putting $\iota_\phi = \phi_{1}\!\!: L \hookrightarrow \text{NS}(X)$. Consider now the period domain

$$\mathcal{D}_{L^+} := \{ \sigma \in \mathbb{P}(L_+^2) : (\sigma, \sigma) = 0 \text{ and } (\sigma, \sigma) > 0 \}.$$ 

There is a natural embedding $\mathcal{D}_{L^+} \subset \mathcal{D}$, and we put $\mathcal{M}_L = \mathcal{D}_{L^+}^{-1}\{ \mathcal{D}_{L^+} \}$; this set consists of all the marked K3 surfaces $(X, \phi)$ such that $L \subset \phi(\text{NS}(X))$. Restricting $\mathcal{M}_L$ further

$$\mathcal{M}_{L^p}^q := \{(X, \phi) \in \mathcal{M}_L : (X, \phi_{L^1}) \text{ is quasipolarized}\}$$

yields the fine moduli space of marked $(L, e)$-quasipolarized K3 surfaces. For a point $x \in \mathcal{D}_{L^+}$ denote by $W_x(L^+) \subset O(\Lambda)$ the group generated by reflections in vectors $\delta \in x^+ \cap L^+$ such that $(\delta, \delta) = -2$. Note that $x^+ \cap L^+$ is a definite lattice (it corresponds to the orthogonal of $L$ inside $\text{NS}(X)$) so that $W_x(L^+)$ is finite.

**Theorem 2.7.** The restriction of the period map to $\mathcal{M}_{L^p}^q$ is surjective, and the fibres $p^{-1}\{x\}$ are naturally torsors under the finite group $\{ \pm 1 \} \times W_x(L^+)$. 

**Proof.** In fact, let $x \in \mathcal{D}_{L^+}$ be any period and pick a preimage $(X, \phi)$. Since we have that $W(\text{NS}(X)) \subset O(\text{NS}(X))^*$ we get a natural embedding $W(\text{NS}(X)) \subset O(H^2(X, \mathbb{Z}))$ so that we let $\{ \pm 1 \} \times W(\text{NS}(X))$ act on the fibres of $\mathcal{M}_L \to \mathcal{D}_{L^+}$. Then we simply choose $w \in \{ \pm 1 \} \times W(L)$ so that $(X, \phi \circ w)$ contains the fixed small cone $\mathcal{C}$ so that $(X, \phi \circ w) \in \mathcal{M}_{L^p}^q$. This proves surjectivity.

For the other statement, let $(X, \phi)$ and $(X, \psi)$ be two different markings that yield the same $(L, e)$-quasipolarized K3 surface. Consider the Hodge isometry $\phi^{-1} \circ \psi$ of $H^2(X, \mathbb{Z}(1))$, which is the identity on $L$. If this maps is not induced by an isomorphism, then there must exist an effective $(-2)$-class $\delta \in \text{NS}(X)$ such that $\phi^{-1} \circ \psi(\delta)$ is not effective. But then $\delta \in \text{NS}(X)$ must belong to $L^+$ by the definition of small cones, i.e., that there is $w \in W(L^+)$ such that $(X, \psi \circ w)$ and $(X, \phi)$ are isomorphic. It is not difficult to show that if $w \neq w'$ then $(X, \psi \circ w) \neq (X, \phi \circ w')$. This concludes the proof. \qed

**Remark 3.** If in Definition 2.5 we replace small cones by Weyl chamber as in [Dol96] the same conclusion would not hold, see [AE21][Remark 3.3].

In order to construct the space $\mathcal{M}_{L,e} \subset \mathcal{M}_L$ we had to make the choice of a small cone and of an embedding $f: L \hookrightarrow \Lambda$ representing $e$. As explained in [AE21], for any other choice one obtains a (non-canonically) isomorphic fine moduli space of marked quasipolarized K3 surfaces. As we shall see, this ambiguity disappears when we get rid of the marking. Consider the stable group $\Gamma = \{ g \in O(\Lambda) : g|_L = \text{Id}_L \}$. Note that $\Gamma \cong O(L^+)^*$ acts both on $\mathcal{M}_{L,e}^q$ and on $\mathcal{D}_{L^+}$. For the following, see [AE21][Corollary 2.27] and the discussion after [Dol96][Proposition 3.3].

**Proposition 2.8.** The period map induces an isomorphism

$$\mathcal{F}_{L,e} := \mathcal{M}_{L,e}^q / \Gamma \cong \mathcal{D}_{L^+} / \Gamma.$$ 

The points of $\mathcal{F}_{L,e}$ are in one-to-one correspondence to isomorphism classes of $(L, e)$-quasipolarized K3 surfaces.

It follows that for any other choice of the embedding $L \hookrightarrow \Lambda$ representing $e$ and for any other choice of a small cone, the resulting moduli spaces are naturally isomorphic. In fact, if instead of $j: L \subset \Lambda$ we choose $g \circ j$ for some $g \in O(\Lambda)$, then the moduli spaces $\mathcal{M}_{L,e}^q$
and $\mathcal{M}_{qg(L)}^{gp}$ are clearly isomorphic via the map on period domains induced by $g$. If $C'$ is another small cone on $L$, then the fine moduli spaces $\mathcal{M}_{qg,L}^{gp}$ and $\mathcal{M}_{qg,C}^{gp}$, are not canonically isomorphic, but they are identified after the quotient by $\Gamma$.

The manifold $\mathcal{D}_{L,e}$ consists of at most two connected components, and it is a hermitian domain of type IV. The group $O(L^+) \cong \Gamma$ is an arithmetic subgroup, and from Baily-Borel it follows that $\mathcal{F}_{L,e}$ is a quasi-projective normal variety with at most two connected components (note that $O(L^+)^*$ is not neat in general, as it may contain torsion, e.g., reflections).

We observe that the definition of $L$-quasipolarized K3 surfaces can be made over families. Let $S/C$ be a noetherian scheme, and let $\pi: X \to S$ be a smooth projective family of K3 surfaces. Let $\text{Pic}(X/S)$ be the relative Picard sheaf of the family (see [Riz06] Chapter 3) and let $L_S$ denote the locally constant sheaf on $S$ induced by the lattice $L$.

**Definition 2.9.** We say that $X/S$ is $(L,e)$-quasipolarized if there an embedding of sheaves $\iota: L_S \hookrightarrow \text{Pic}(X/S)$ which is primitive and $(L,e)$-quasipolarized for every geometric fibre of $S$.

To show that this is well defined, denote by $j_s: L \hookrightarrow \text{Pic}(X_s)$ the induced primitive embedding for any geometric point $s \in S(\mathbb{C})$. Since $S$ is connected the class $[L \hookrightarrow \text{Pic}(X_s) \hookrightarrow H^1(X_\mathbb{C}, \mathbb{Z}(1))] \in \mathcal{E}(L)$ does not depend on the chosen $s$. Note also that a class $\ell \in \mathcal{E}$ that is big and nef on one fibre, it is big and nef on every fibre. Using the argument of [Do96] Remark 3.4, one construct a unique map $S \to \mathcal{F}_{L,e}$ such that $s \mapsto [(X_s, \ell_s)]$, showing that $\mathcal{F}_{L,e}$ also coarsely represents the associated moduli functor. Finally, we get rid of the embedding $e$:

**Definition 2.10.** A $L$-quasipolarized K3 surface is a pair $(X, \iota)$ where $X$ is a complex K3 surface and $\iota$ is a quasipolarized embedding $L \hookrightarrow \text{NS}(X)$.

The space $\mathcal{F}_L := \bigcup_{\ell \in \mathcal{E}(L)} \mathcal{F}_{L,e}$ is then the coarse moduli space of $L$-quasipolarized K3 surfaces.

**Theorem 2.11.** The space $\mathcal{F}_L$ has a natural model over $\mathbb{Q}$.

**Proof.** The best way to prove this would be to follow Rizov’s steps [Riz05], which would also yield a model of the moduli space over some open subset of $\text{Spec}(\mathbb{Z})$. Due to the length of the argument, we proceed using different but standard techniques.

For a prime number $p > 2$ coprime to $\text{discr}(\text{NS}(X))$ consider the group

$$\Gamma_p := \ker(\Gamma \to \text{O}(T/pT))$$

where $\Gamma$ is as in Proposition 2.8 and $T = L^+$. Note that $\Gamma_p$ is neat, so that $\mathcal{F}_p = \mathcal{D}_T/\Gamma_p$ is smooth and quasi projective.

It follows from standard arguments that one has a family of $(L,e)$-quasipolarized K3 surfaces over $\mathcal{F}_p$, endowed with an isometry $\ell: \iota(L)^+ \cong T \otimes \mathbb{Z}/p\mathbb{Z}$. Rigidity arguments due to Faltings [Fal84], see also [Pet17], assure that the variety $\mathcal{F}_p$ can be defined over $\mathbb{Q}$. Then [Mil05] Theorem 3.21 also says that $\text{Aut}(\mathcal{F}_p)$ is finite, so that every model over $\mathbb{Q}$ is $\mathbb{Q}$-isomorphic. This also means that there is a canonically defined subset in $\mathcal{F}_p(\mathbb{C})$ of algebraic points. We denote such $\mathbb{Q}$-model by $\mathcal{F}_p$, so that the original one over $\mathbb{C}$ becomes $\mathcal{F}_{p,c}$. Let $(X, \iota)$ be a complex $(L,e)$-quasipolarized K3 surface and let $x \in \mathcal{F}_p(\mathbb{C})$ be any point that maps to the period point of $(X,\iota)$. We want to show that $X$ is defined over $\mathbb{Q}$ if and only if $x$ maps to $\mathcal{F}_p(\mathbb{Q})$. One direction is obvious due to the existence of the family over $\mathcal{F}_{p,c}$, for the other direction we use [Bal19] Lemma 4.3 which says that a K3
surface is defined over \( \overline{\mathbb{Q}} \) if and only if the set \( \{ X^\sigma : \sigma \in \text{Aut}(\mathbb{C}/\overline{\mathbb{Q}}) \} = \{ X \} \) (otherwise, it is uncountably infinite). Since the same characterization holds for the algebraic points in \( \mathcal{T}_p(\mathbb{C}) \), we prove our claim.

The group \( \Gamma_p \) is normal in \( \Gamma \), so that \( \Gamma/\Gamma_p \) acts on \( \mathcal{T}_p \). Taking the quotient yields a normal quasiprojective variety which is a \( \mathbb{Q} \)-form of \( \mathcal{T}_{L,e} \). We denote this variety by \( \mathcal{T}_e \).

Then, by our discussion, \( \mathcal{T}_e(\mathbb{Q}) \) parametrizes isomorphism classes of \( (L,e) \)-quasipolarized K3 surfaces that can be defined over \( \mathbb{Q} \subset \mathbb{C} \) (note that we always work with a fixed \( \mathbb{C} \)).

In order to show that this is also a coarse moduli space, one notes that for any family of \( (L,e) \)-quasipolarized K3 surfaces \( X/S \) where \( S/\mathbb{Q} \) is a noetherian base, one has an induced map \( S_C \to \mathcal{T}_{e,C} \) and this is defined over \( \overline{\mathbb{Q}} \) because \( S(\overline{\mathbb{Q}}) \subset S(\mathbb{C}) \) must be mapped to \( \mathcal{T}_e(\overline{\mathbb{Q}}) \).

Define \( \mathcal{T}_L = \bigsqcup_e \mathcal{T}_{L,e} \). In order to conclude the proof we need to show that \( \mathcal{T}_L \) descends over \( \mathbb{Q} \). There is a natural \( G_\mathbb{Q} \)-action on \( \mathcal{T}_L(\overline{\mathbb{Q}}) \) defined as follows: for any \( \sigma \in G_\mathbb{Q} \) and any \( (X,i) \in \mathcal{T}_L(\overline{\mathbb{Q}}) \) let \( (X,i)^\sigma = (X^\sigma, \sigma \circ i) \) where \( \sigma \circ : \text{NS}(X) \to \text{NS}(X^\sigma) \) is the natural induced map. Note that \( \sigma \circ \) behaves as if it was induced by an isomorphism on Néron-Severi groups because it sends ample classes to ample class, which implies that also \( (X,i)^\sigma \) is quasipolarized. At the level of points of \( \mathcal{T}_L(\overline{\mathbb{Q}}) \), we denote this by \( x \mapsto \sigma(x) \).

Let \( \sigma \in G_\mathbb{Q} \) be any element, and consider the space \( \mathcal{T}_L^\sigma \) together with the map \( \sigma_\mathcal{L} : \mathcal{T}_L \to \mathcal{T}_L^\sigma \). For any family \( X/S \) let \( p_S : S \to \mathcal{T}_e \) be the period map and consider the composition

\[
S \xrightarrow{\sigma_\mathcal{L}} S^{\sigma_\mathcal{T}} \xrightarrow{p_{S(\sigma_\mathcal{T})}} \mathcal{T}_L^\sigma \xrightarrow{\sigma_\mathcal{L}} \mathcal{T}_L^\sigma.
\]

This induced map is \( \overline{\mathbb{Q}} \)-linear and exhibits \( \mathcal{T}_L^\sigma \) as a coarse moduli space of \( L \)-quasipolarized K3 surfaces. Since coarse moduli spaces are unique up to unique isomorphism, we obtain a unique isomorphism \( \pi_\mathcal{T} : \mathcal{T}_L^\sigma \cong \mathcal{T}_L^\sigma \). We leave to the reader to check that this defines a Galois descent data in the sense of \[ \text{Proposition 4.4.4.} \] By \[ \text{Corollary 4.4.6.} \] and \[ \text{Remark 4.4.8.} \] the space \( \mathcal{T}_L \) is defined over \( \mathbb{Q} \). We call this model again by \( \mathcal{T}_L \).

Let \( (X,i) \) be a \( L \)-quasipolarized K3 surface over \( \overline{\mathbb{Q}} \) and let \( x \) be its period point, \( x = p(X,i) \). Then, its period point for \( \mathcal{T}_L \) is given by applying \( \sigma_\mathcal{L} \) to \( \sigma_\mathcal{T}^{-1}(x) \), where this latter action is the one described in the paragraph above. The composition

\[
\mathcal{T}_L(\overline{\mathbb{Q}}) \xrightarrow{\mathcal{L}_\mathcal{T}} \mathcal{T}_L^\sigma(\overline{\mathbb{Q}}) \xrightarrow{\mathcal{L}_\mathcal{T}} \mathcal{T}_L(\overline{\mathbb{Q}})
\]

describes by construction the Galois action on \( \mathcal{T}_L(\overline{\mathbb{Q}}) \), and it gives back the natural \( G_\mathbb{Q} \)-action on \( \mathcal{T}_L(\overline{\mathbb{Q}}) \).

The theorem also shows that for any number field \( K \) we have

\[
\mathcal{T}_L(K) = \{ (X,i) \in \mathcal{T}_L(\overline{\mathbb{Q}}) : (X^\sigma, i^\sigma) \cong (X,i) \text{ for every } \sigma \in G_K \}.
\]

Note that \( \mathcal{T}_L \) is usually not geometrically irreducible, and we know that we can describe its components using the set \( \mathcal{E}(L) \). It is interesting to determine how \( G_\mathbb{Q} \) permutes these components. For a concrete example, take any K3 surface \( X/\overline{\mathbb{Q}} \) and \( \sigma \in G_\mathbb{Q} \) such that \( T(X) \) and \( T(X^\sigma) \) are not isometric (once we base change both surfaces to \( \mathbb{C} \)). In this case, \( \text{NS}(X) \cong \text{NS}(X^\sigma) \) but the two embeddings \( \text{NS}(X) \hookrightarrow \Lambda \) and \( \text{NS}(X^\sigma) \hookrightarrow \Lambda \) cannot be isomorphic, since their orthogonal complements are not isometric. We can use the result of Schütt \[ \text{Sch10} \] to provide many such examples of Picard rank 20. On the other hand, this is not completely satisfactory because the surfaces involved have zero-dimensional moduli and because their transcendental lattices are positive definite. One may still wonder whether \( G_\mathbb{Q} \) acts always transitively on \( \mathcal{E}(L) \). This could be in principle understood by
studying the Galois action on the connected components of a Shimura variety, as we shall see later.

2.3. Discussion. In order to construct the moduli spaces of lattice polarized K3 surfaces, one makes the choice of a small cone at the beginning, and then one sees how this choice does not influence the construction. In this section we quickly explain the reason behind this. Consider the set of pairs \((\Lambda, j)\) where \(\Lambda\) denotes a Hodge structure of K3 type on the K3 lattice and \(j: L \to \Lambda^{0,0}\) is a primitive embedding. One says that another such pair \((\Lambda', j')\) is isomorphic to the previous one if there is a Hodge isometry \(f: \Lambda \xrightarrow{\sim} \Lambda'\) such that \(f \circ j = j'\). Note that to any such pair \((\Lambda, j)\) there is associated an element \(e \in \mathcal{E}(L)\). One shows that the set of isomorphism classes of these Hodge theoretical pairs can be naturally identified with \(\mathcal{F}_L\). To any \((L, e)\)-quasipolarized K3 surface one can associate the pair \((H^2(X, \mathbb{Z})(1), i)\). Small cones are needed in order to go the other way around:

**Lemma 2.12.** Choose a small cone \(\mathcal{C}\) on \(L\). To any pair \((\Lambda, j)\) there is associated a well defined isomorphism class of \(L\)-quasipolarized K3 surfaces.

**Proof.** By the surjectivity of the period map and the global Torelli theorem, to any pair \((\Lambda, j)\) one can associate a well defined K3 surface \(X\) with a (class of) primitive embedding \(j: L \to \text{NS}(X)\). Pick any \(w \in W(L)\) such that \(\iota := j \circ w\) is quasipolarized with respect to \(\mathcal{C}\) and consider the corresponding \(L\)-quasipolarized K3 surface \((X, \iota)\). Note that since \(w \in W(L)\) extends to the whole \(H^2(X, \mathbb{Z})(1)\) we have that \((H^2(X, \mathbb{Z}), \iota \circ w) \cong (H^2(X, \mathbb{Z}), \iota)\). Thus, we only need to show that this construction is well defined. Let \((X', \iota')\) be another \(L\)-quasipolarized K3 surface obtained via the same procedure. Choose any Hodge isometry \(f: H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})\) such that \(f \circ \iota = \iota'\). Assume that \(L^\perp\) does not contain \((-2)\)-classes, and let \(\delta \in \text{NS}(X)\) be any effective \((-2)\)-class and \(c \in \mathcal{C}\) be any element. Then \(\langle \iota(c), \delta \rangle > 0\) and therefore \(\langle \iota'(c), f(\delta) \rangle > 0\). This shows that \(f\) is induced by a unique isomorphism of \(L\)-quasipolarized K3 surfaces. If \(L^\perp\) contains \((-2)\)-classes, then \(f\) is not necessarily induced by an isomorphism. On the other hand, we can pick any element \(w\) in the finite group \(W(L^\perp) \subset \text{O}(H^2(X, \mathbb{Z}))\) such that \(f \circ w\) is induced by a unique isomorphism. Also in this case, the two \(L\)-quasipolarized K3 surfaces are isomorphic. 

Thus, the choice of a small cone is needed in order to pass from Hodge theoretical data to actual K3 surfaces and line bundles on them.

2.4. Fields of moduli and fields of definition. Let \((X, \iota)\) be a \(L\)-quasipolarized K3 surface over \(\overline{\mathbb{Q}}\). The field generated by the point in \(\mathcal{F}_L(\overline{\mathbb{Q}})\) representing \((X, \iota)\) is called the field of moduli of \((X, \iota)\). This can also be defined as the fixed field of

\[\{\sigma \in G_{\mathbb{Q}}: (X, \iota) \cong (X^{\sigma}, \iota^{\sigma})\}\subset G_{\mathbb{Q}}.\]

In general, the fields of moduli is smaller than the fields of definition. For a lattice polarized K3 surfaces, most of the times the two fields coincide. Denote by \(\mu(X)\) be the set of Hodge isometries of \(T(X)\). Note that this corresponds to the roots of unity in the field \(E_X = \text{End}_{\text{Hdg}}(T(X))_{\mathbb{Q}}\) and, in particular, \(\mu(X) = \{\pm 1\}\) whenever \(E_X\) is a totally real field.

**Proposition 2.13.** Let \((X, \iota)\) be a \(L\)-quasipolarized K3 surface, and assume that \(X\) can be defined over \(\overline{\mathbb{Q}}\). Assume that \(\iota\) is an isomorphism between \(L\) and \(\text{NS}(X)\). If the natural map \(\mu(X) \to \text{O}(A_X)\) is injective, then \((X, \iota)\) can be defined over its field of moduli.
Proof. Let \((X, \iota)\) and \((Y, \iota')\) be \((L, c)\)-quasipolarized K3 surfaces, and assume that both \(\iota\) and \(\iota'\) are isomorphisms and that \((X, \iota) \cong (X', \iota')\). We begin by showing that there is a unique isomorphism between \((X, \iota)\) and \((X', \iota')\). Pick two isomorphisms \(f, f'\) and consider \(g = f' \circ \pi^{-1} \in \text{Aut}(X)\). Then, \(g\) is an automorphism of \(X\) which acts trivially on \(\text{NS}(X)\) and therefore on \(A_X\). Its action on \(T(X)\) is given by an integral Hodge isometry, that is, by an element of \(\mu \in \mu(X)\). Since the action of \(\mu\) on \(A_X\) must be the identity due to the fact that \(T(X) / T(X)\) and \(\text{NS}(X) / \text{NS}(X)\) are naturally isomorphic, we conclude thanks to the assumption that \(\mu(X) \to O(A_X)\) is injective.

Thus, for any \(\sigma \in G_q\) such that \((X, \iota) \cong (X', \iota')\) there is a unique isomorphism \(f_\sigma\) of polarized K3 surfaces \(X \to X'\). One checks that this gives a Galois descent data for \(X\) over its field of moduli (see [Val22a] and [Val22b] where similar techniques are used). □

3. Shimura variety interpretation

In this section we study the spaces \(\mathcal{F}_L\) and their Shimura varieties analogue. In particular, we generalize the construction of \(\mathcal{F}_L\) outside the context of moduli spaces of K3 surfaces, and we compare such spaces with Shimura varieties of orthogonal type.

Let \(T\) be any even lattice of signature \((2_+, n_-)\), and let \(D^\pm_T \subset \mathbb{H}(T_\mathbb{C})\) be the associated period domain. Then, \((D^\pm_T, SO(T_\mathbb{Q}))\) is a Shimura datum ([Mil05]). For a compact-open subgroup \(K \subset SO(T_\mathbb{Q})(\mathbb{A}_f)\) one can associate the corresponding Shimura variety

\[
\text{Sh}_K(D_T, SO(T_\mathbb{Q})) := SO(\mathbb{Q}) \backslash D_T / SO_T(\mathbb{A}_f) / K,
\]

where the action of \(q \in SO(\mathbb{Q})\) is given by \(q(\sigma, g) = (q\sigma, qg)\) and the action of \(K\) is the right multiplication on the right coordinate. The product can be taken both with \(SO(\mathbb{A}_f)\) endowed with the adelic topology, or with the discrete topology: this is does not make a difference when one goes to the quotient. The space \(\text{Sh}_K(D_T, SO(T_\mathbb{Q}))\) consists of finitely many connected components, and each one is the quotient of a connected component of \(D^\pm_T\) by an arithmetic subgroup of \(SO_T(\mathbb{Q})\). Denote the kernel of \(SO_T(\mathbb{Z}) \to O(A_L)\) by \(\text{SO}_T(\mathbb{Z})^*\). We are then interested in the variety \(\text{Sh}_K(D_T, SO(T_\mathbb{Q}))\) when \(K = \text{SO}_T(\mathbb{Z})^*\) and its relation to the moduli spaces constructed before. We introduce the following notation:

\[
(3.0.1) \quad M_T := O(\mathbb{Q}) \backslash D^\pm_T / O(\mathbb{A}_f) / O(\mathbb{Z})^*,
\]

and

\[
(3.0.2) \quad S_T := SO(\mathbb{Q}) \backslash D^\pm_T / SO_T(\mathbb{A}_f) / SO(\mathbb{Z})^*.
\]

Note that we have a natural map \(S_T \to M_T\). Note also that in the definition of Shimura varieties it is required that the relevant reductive group is connected. Thus, the space \(M_T\) is not technically a Shimura variety and we cannot use many results that are known to hold for Shimura varieties (canonical models etc). On the other hand, we will prove that \(M_T\) is isomorphic to the spaces \(\mathcal{F}_L\) for an appropriate \(L\), whenever \(T\) embeds primitively into the K3 lattice.

3.1. Equivalence of \(M_T\) and \(\mathcal{F}_L\). We begin with the following definition. Let \(T\) be an even lattice of signature \((2_+, n_-)\).

**Definition 3.1.** A \(T\)-polarized K3 structure is a tuple \((T', \gamma')\) where \(T'\) is an even lattice in the same genus of \(T\) endowed with a Hodge structure of weight zero of K3 type, for which the quadratic form of \(T'_\mathbb{Q}\) induces a polarization, and \(\gamma' : A_T \xrightarrow{\sim} A_{T'}\) is an isomorphism of finite quadratic forms. Another tuple \((T'', \gamma'')\) is isomorphic to the first, if there is a
integral Hodge isometry $f : T' \to T''$ such that, if $\bar{f} : A_{T'} \to A_{T''}$ denotes the natural induced isometry on the discriminant forms, then $\gamma' = \gamma'' \circ \bar{f}$.

Let $\mathcal{H}_T$ be the set of all isomorphism classes of $T$-polarized K3 structures. We define $\mathcal{H}_0(T)$ as the isomorphism classes of pairs $(T', \gamma')$ where $T'$ is now only an even lattice in the same genus of $T$ and $\gamma' : A_T \xrightarrow{\sim} A_{T'}$, and two such pairs are equivalent if there is an isometry $f : T' \to T''$ such that $\bar{f} \circ \gamma' = \gamma''$.

**Lemma 3.2.** Let $L$ be an even hyperbolic lattice and let $f : L \hookrightarrow \Lambda$ be a primitive embedding. Let $T$ be its orthogonal complement. There is a natural isomorphism $\mathcal{H}_0(T) \cong \mathcal{H}(L)$.

**Proof.** Due to [Nik79b][Proposition 1.6.1] we can identify $\mathcal{H}(L)$ with isomorphism classes of pairs $(T', \rho')$ where $\rho : A_{T'} \xrightarrow{\sim} -A_L$ is an isometry and $T'$ is a lattice in the same genus as $T$. Two such pairs give isomorphic primitive embedding if and only if there is an isometry $f : T' \to T''$ such that $\rho' = \rho'' \circ \bar{f}$. We recall briefly how this correspondence is constructed: the graph of the isometry $\Gamma_\rho \subset A_{T'} \oplus A_L$ is an isotropic subgroup, and $\Lambda$ is the overlattice of $T' \oplus L$ associated to $\Gamma_\rho$. This is easily seen to be the K3 lattice due to its uniqueness in its genus.

Using the natural isomorphism $g : A_T \xrightarrow{\sim} -A_L$ induced by the identification $T = L^\perp$, we can associate to any $(T', \gamma') \in \mathcal{H}_0(T)$ the element $(T', g \circ \gamma^{-1}) \in \mathcal{H}(L)$. This is a bijection.

One has a natural surjective map $\mathcal{H}_T \to \mathcal{H}_0(T)$. The next proposition basically reinterprets the Discussion 2.3 into saying that $\mathcal{H}_L(\mathbb{C})$ parametrizes $T$-polarized Hodge structures of K3 type:

**Proposition 3.3.** Let $L$ be an even hyperbolic lattice with a primitive embedding $L \hookrightarrow \Lambda$ such that $L^\perp = T$. There is a natural identification $\mathcal{H}_T \cong \mathcal{H}_L(\mathbb{C})$ such that the following square commutes:

\[
\begin{array}{ccc}
\mathcal{H}_T & \xrightarrow{\sim} & \mathcal{H}_L(\mathbb{C}) \\
\downarrow & & \downarrow \\
\mathcal{H}_0(T) & \xrightarrow{\sim} & \mathcal{H}(L)
\end{array}
\]

**Proof.** To any $L$-quasipolarized K3 surface $(X, \iota)$ one let $(\iota(L)^\perp, \gamma_X)$ be the pair where $\gamma_X$ is given by the composition $A_{\iota(L)^\perp} \cong -A_L \cong A_T$. It is easy to see that this defines an injection $\mathcal{H}_L(\mathbb{C}) \hookrightarrow \mathcal{H}_T$. To go the other way around, we showed before that to any element $(T', \gamma') \in \mathcal{H}_T$ one can associate a primitive embedding of $L$ into $\Lambda$ with orthogonal complement equals to $T''$. Note that $\Lambda$ is constructed as an overlattice of $L \oplus T''$. We endow $\Lambda$ with the Hodge structure coming from $T'$ and we then use the discussion in Section 2.3.

We now show now that $M_T$ is isomorphic to $\mathcal{H}_L$ when $L$ embeds primitively into the K3 lattice with orthogonal complement isomorphic to $T$, basically by showing that also $M_T$ parametrizes $T$-polarized Hodge structures of K3-type. We begin with a lemma:

**Lemma 3.4.** There is a natural isomorphism $\mathcal{H}_0(T) \cong O_T(\mathbb{Q}) \setminus O_T(\mathbb{A}_F)/O_T(\mathbb{Z})^*$.  

**Proof.** Consider $g \in O_T(\mathbb{A}_F)$. Then, $T^g := g(T \otimes \mathbb{Z}) \cap T_\mathbb{Q}$ is the unique lattice that satisfies $T^g \otimes \mathbb{Z} = g(T \otimes \mathbb{Z})$ where the equality is taken in $T \otimes \mathbb{A}_F$. The map $g$ also
induces a natural isomorphism \( \gamma_g : A_T \to A_{T^g} \), so we obtain a map \( O(A_f) \to \mathcal{X}_0^*(T) \) sending \( g \) to \( (T^g, \gamma_g) \). The surjectivity of this map is due to the fact that \( O(\hat{\mathbb{Z}}) \to O(A_T) \) is surjective [Nik79b][Corollary 1.9.6] and it is straightforward to check that it induces an isomorphism between the double coset and \( \mathcal{X}_0^*(T) \).

**Proposition 3.5.** For any even lattice \( T \) of signature type \( (2_+, n_-) \) there is a natural identification
\[
M_T(\mathbb{C}) \cong \mathcal{X}^*(T)
\]
and a natural isomorphism
\[
M_T(\mathbb{C}) \cong \bigsqcup_{(T', \gamma') \in \mathcal{X}^*_0(T)} D_T^\frac{1}{2} / O(T')^*.
\]

**Proof.** We put \( V = T_0 \). Let \( (\sigma, g) \in \Omega^\pm \times O(A_f) \) and denote by \( [\sigma, g] \in M_T(\mathbb{C}) \) its equivalence class. Denote by \( V_\sigma \) the polarized Hodge structure induced by \( \sigma \in \mathbb{D}_T \subset \mathbb{P}(T_\mathbb{C}) \), that is, \( V_{1,1} = \sigma, V_{-1,1} = \bar{\sigma} \) and \( V^{0,0} \cap V_\mathbb{R} = ((\sigma \otimes \bar{\sigma}) \cap T_\mathbb{R})^\perp \). If there is a Hodge isometry \( f : V_\sigma \to V_\tau \), then \( f \) is an element of \( O(\mathbb{Q}) \) such that \( f(\sigma) = \tau \). Consider now the unique lattice \( T^g \subset V \) such that \( T^g \otimes \hat{\mathbb{Z}} = g(T \otimes \hat{\mathbb{Z}}) \) with equality occurring in \( T \otimes A_f \). Denote by \( T^g \) the integral Hodge structure induced by the inclusion \( T^g \subset V_\sigma \). Note that \( g \) induces an isomorphism \( \gamma_g : A_T \to A_{T^g} \), so we obtain an element \( (T^g, \gamma_g) \in \mathcal{X}^*_0(T) \). Clearly \( T^g k = T^k \) for any \( k \in O_T(\hat{\mathbb{Z}}) \) and \( \gamma_g k = \gamma_g \) for any \( k \in O_T(\hat{\mathbb{Z}})^* \).

This shows that the map is well defined. To construct the inverse, one picks an element \((T', \gamma')\) and one chooses any isometry \( T'_0 \cong V \). The image of \( T' \subset V \) is a lattice in the same genus of \( T \), because \( T' \) has the same signature and the same discriminant form of \( T \). We can find both a Hodge structure \( \sigma \) and an element \( g \in O(A_f) \) such that \( T' \cong T^g \) as integral polarized Hodge structures. We only need to show that we can choose \( g \) such that \((T^g, \gamma_g) \cong (T', \gamma')\). This follows from the fact that the map \( O(\hat{\mathbb{Z}}) \to O(A_L) \) is surjective [Nik79b][Corollary 1.9.6] and from the fact that for any \( h \in O(\hat{\mathbb{Z}}) \) one has \( T^g h = T^g \). These two constructions are one the inverse of the other.

We now prove the last part of the theorem. The discussion above shows how to any \([\sigma, g] \in M_T(\mathbb{C})\) we can associate an element \( (T^g, \gamma_g) \in \mathcal{X}^*_0(T) \). Assume that for another \([\sigma', g']\) we have that \((T^g, \gamma_g) \cong (T^g, \gamma_g)\). Note that now we are considering only lattices, without Hodge structures. This means that we can find \( q \in O(\mathbb{Q}) \) and \( k \in O(\hat{\mathbb{Z}})^* \) such that \( qg'k = g \); hence, the fibre of the map \( M_T(\mathbb{C}) \to \mathcal{X}^*_0(T) \) are represented by \((\sigma, g)\) for a fixed \( g \in O(A_f) \) and \( \sigma \) varying in \( \mathbb{D}_T^\frac{1}{2} \). Now, two elements \((\sigma, g)\) and \((\sigma', g)\) map to the same one in \( M_T \) if and only if there is \( q \in O(\mathbb{Q}) \) and \( k \in O(\hat{\mathbb{Z}})^* \) such that \( q' = q \sigma \) and \( qgg'k = g \), i.e., if and only there is \( q \in O_T(\mathbb{Q}) \) such that \( g^{-1} q g \in O_T(\mathbb{Z})^* \) and \( \sigma' = q \sigma \).

We can see \( g \) as an isometry between \( T \otimes \hat{\mathbb{Z}} \) and \( T^g \otimes \hat{\mathbb{Z}} \), so the equation above says that \( q \) must act integrally on \( T^g \), i.e., \( q \in O_{T^g}(\mathbb{Z}) \). Moreover, since \( k \) acts trivially on \( A_T \) one deduces from the same equation that \( q \in O_{T^g}(\mathbb{Z})^* \). Now, note that for any \( q \in O_{T^g}(\mathbb{Z})^* \) there is \( k \in O_T(\mathbb{Z})^* \) such that \( qgk = g \), simply by putting \( k = ggg^{-1} \). So the fibre over \((T^g, \gamma_g) \in \mathcal{X}^*_0(T) \) is naturally isomorphic to \( D_T^\frac{1}{2} / O(T^g)^* \).

Putting together the previous two propositions, we have arrived at the following. Let \( L \) be an even hyperbolic lattice, and fix a primitive embedding \( L \subset \Lambda \) with orthogonal complement \( T \). Then, the choice of such embedding induces a natural isomorphisms \( \mathcal{E}(L) \cong \mathcal{X}^*_0(L) \) and \( \mathcal{F}_L \cong M_T \) such that
involution on \( \tilde{\mathcal{H}} \). We now introduce the set \( \tilde{\mathcal{H}} \) analogous situation happens). That is, the set \( \tilde{\mathcal{H}} \) which also keeps track of the global orientation (see also [OS18] and [MP15] where the analogous situation happens). That is, the set \( \tilde{\mathcal{H}} \) parametrizes isomorphism classes of triples \((T', \gamma', d_2)\) where \((T', \gamma')\) is \(T\)-polarized K3 structure and \(d_2\): \(\text{det}(T' \otimes \mathbb{Z}_2) \sim\text{det}(T \otimes \mathbb{Z}_2)\) is an isometry, up to the natural notion of isomorphism. There is an evident involution on \(\tilde{\mathcal{H}}\) which we denote by \(\tau\), and which sends \((T', \gamma', d_2)\) to \((T', \gamma', -d_2)\). The ‘quotient’ by this involution is evidently \(\mathcal{H}\).

**Lemma 3.6.** Let \(T\) be any even lattice such that \(\ell(A_T) < \text{rank}(L)\). Then, the natural map \(\text{SO}_T(\hat{\mathbb{Z}}) \to \text{O}(A_T)\) is surjective.

**Proof.** It is enough to prove this for \(p\)-adic lattices (even if \(p = 2\)). In fact, \(\ell(A_T) = \max\{\ell(A_{T_p})\} \) where \(T_p = T \otimes \mathbb{Z}_p\). Let \(T_p\) be a \(p\)-adic lattice which is even if \(p = 2\), then the map \(\text{O}(T_p)(\mathbb{Z}_p) \to \text{O}(A_T)\) is surjective by [Nik79b] Corollary 1.9.6]. In order to prove the statement it is thus enough to find \(f \in \text{O}(T_p)(\mathbb{Z}_p)\) such that \(f\) acts trivially on \(A_T\) and \(\text{det}(f) = -1\). Under the condition that \(\ell(T_p) < \text{rank}(T_p)\) we can find \(x \in T_p\) such that \((x, x)\mathbb{Z}_p = \mathbb{Z}_p\) if \(p \neq 2\) and \((x, x)\mathbb{Z}_2 = 2\mathbb{Z}_2\). This follows from the fact that any \(p\)-adic lattice (even, if \(p = 2\)) can be decomposed as an orthogonal sum \(T_p = U_0 \perp U_1(p) \perp \cdots \perp U_n(p^n)\) where each \(U_i\) is unimodular and \(U(p)\) denotes the lattice \(U\) with form scaled by \(p^k\). Since \(\ell(A_{T_p}) < \text{rank}(T_p)\) we see that \(\text{rank}(U_0) > 0\).

As showed in [Shi10] Lemma 29.6, the reflection \(r_x \in \text{O}(L)(\mathbb{Q}_p)\) induced by \(x\) is actually integral \(r_x \in \text{O}(L)(\mathbb{Z}_p)\). Let \(U_p\) be any unimodular lattice, even if \(p = 2\), such that \(L_p\) embeds in \(U_p\) primitively. By considering \(x\) as an element of \(U_p\) we can construct analogously the involution \(r'_p \in \text{O}(U_p)(\mathbb{Z}_p)\). This proves that \(r_x \oplus \text{id}: L_p \oplus L_p^+ \to L_p \oplus L_p^+\) extends to an isometry of \(U_p\). It follows from Lemma 2.2 that \(r_x \in \text{O}(L)(\hat{\mathbb{Z}})^*\).

**Corollary 3.7.** Let \(T\) be any even lattice of signature type \((2, n)\). Assume that \(\ell(A_T) < \text{rank}(T)\). Then, the natural map \(\text{ST}_T(\mathbb{C}) \to M_T(\mathbb{C})\) is surjective.

**Proof.** Let \([\sigma, g] \in M_T(\mathbb{C})\), and for any prime \(p\) denote by \(g_p \in \text{O}(T_p)\) the \(p\)-component of \(g\), so that \(g_p \in \text{O}(T_p)(\mathbb{Z}_p)\) for almost every \(p\). For any \(p\) such that \(\text{det}(g_p) = -1\) choose a reflection \(r_p\) as in Lemma 3.6 and consider the element \(k \in \text{O}(L)(\hat{\mathbb{Z}})^*\) given by \(k_p = 1\) if \(\text{det}(g_p) = 1\) and \(k_p = r_p\) if \(\text{det}(g_p) = -1\). It follows that \([\sigma, g] = [\sigma, gk]\) and that \(gk \in \text{SO}(A_T)\).

**Lemma 3.8.** Let \(T\) be any even lattice such that \(\ell(A_T) < \text{rank}(L)\). One has a natural injection \(S_T \hookrightarrow \mathcal{H}(T)\) such that \(\text{im}(S_T(\mathbb{C})) \cup \tau(\text{im}(S_T(\mathbb{C}))) = \mathcal{H}(T)\). This union may not be disjoint.

**Proof.** To any \((\sigma, g) \in \mathcal{D}_I^+ \times \text{SO}(A_T)\) one associates the element \((T^g, \gamma_g, d_2) \in \tilde{\mathcal{H}}(T)\) as in the proof on Proposition 3.5 where the map \(d_2\) is given by the natural isomorphism \(\text{det}(T \otimes \mathbb{Z}_2) \cong \text{det}(g(T) \otimes \mathbb{Z}_2)\) induced by \(g_2\). One readily checks that that this defines an injection \(S_T \hookrightarrow \mathcal{H}(T)\). The last statement follows from Corollary 3.7.
In particular, the natural map $S_T \to M_T$ is surjective and has degree (over each connected component) at most 2.

Finally, the connected components of $S_T$ can be analyzed using the same techniques as in Proposition 3.5 see also Mil05. One introduces the double coset

$$c = \SO(\mathbb{Q}) \backslash \SO(\mathbb{A}_f) / \SO(\mathbb{Z})^*$$

and one shows that

$$S_T = \bigcup_{g \in c} D_T^\pm / \SO_T^*,$$

where $T^g$ denotes the lattice $g(T) \cap T_0$ for $g \in \SO(\mathbb{A}_f)$. The inclusion $\SO \subset O$ induces a natural map $c \to \mathcal{K}_0^*(T)$ and the map $S_T \to M_T$ makes the following square commute:

$$\begin{array}{ccc}
\bigcup_{g \in c} D_T^\pm / \SO_T^* & \longrightarrow & \bigcup_{T' \in \mathcal{K}_0^*(T)} D_{T'}^\pm / O(T')^* \\
\downarrow & & \downarrow \\
c & \longrightarrow & \mathcal{K}_0^*(T).
\end{array}$$

3.3. Connected components. We study the properties of the spaces $M_T$ and $S_T$ more deeply. In particular, we are interested in their connected components, and to find conditions on $T$ which ensure that those spaces are indeed geometrically irreducible.

Definition 3.9. An even lattice $S$ of signature $(2_+, n_-)$ is called stable if $n_- \geq 1$ and $\ell(A_S) \leq \rank(S) - 2$ and very-stable if $n_- \geq 2$ and $\ell(A_S) \leq \rank(S) - 3$. An even hyperbolic lattice $L$ that embeds primitively into the K3 lattice is called (very) stable if its orthogonal complement in $\Lambda$ is (very) stable.

Note that if $L$ is an even hyperbolic lattice and $\rank(L) \leq 9$ then $L$ is automatically very stable.

Lemma 3.10. Let $S$ be a stable lattice, then $O(S) \to O(A_S)$ is surjective and $\mathcal{K}_0^*(S) = \{\ast\}$. If $S$ is also very stable, then $O(S) \to O(A_S)$ is surjective and, if $T \subset S$ is primitive of corank one, then $T$ is stable.

Proof. The first two claims follow from Nik79b[Theorem 1.14.2]. If $S$ is very stable, it follows from Nikulin Nik79b[Corollary 1.13.5] that $S \cong S' \perp U$ where $U$ is the hyperbolic plane. In particular, $S$ contains $(2)$-classes and therefore there are elements $r \in O(S)^*$ such that $r^2 = 1$ and $\det(r) = -1$. Since the map $O(T) \to O(A_T)$ is surjective, we conclude the proof of the third claim. For the last one, note that $\ell(A_T) \leq \ell(A_S) + 1$ whenever $\rank(T) = \rank(S) - 1$. \qed

In the following discussion, and until the end of the chapter, we write $O$ for $O_T$, and similar for the other groups. As explained in Mil05[Lemma 5.13], the connected components of $S_T(\mathbb{C})$ are in one to one correspondence to the double coset $\SO(\mathbb{Q})_+ \backslash \SO(\mathbb{A}_f) / \SO(\mathbb{Z})^*$, where $\SO(\mathbb{Q})_+$ is defined as follows. Recall that $O(\mathbb{R})$ has four connected components: choose a decomposition $T_\mathbb{R} = T^+ \perp T^-$ where $T^+$ is a positive plane. For any isometry $f \in O(\mathbb{R})$ one can consider the induced maps $f^+: T^+ \to T \xrightarrow{f} T \to T^+$ where the last arrow is the orthogonal projection, and similarly for the minus sign. The connected components of $O(\mathbb{R})$ are then determined by the signs of the determinants $(\det(f^+), \det(f^-))$, so that $O(\mathbb{R}) = O(\mathbb{R})^{+,+} \cup O(\mathbb{R})^{-,-}$. The center of $SO_T$ is $Z = \mu_2$ if $T$ has even rank, and $Z = \{1\}$ otherwise, and $\SO(\mathbb{Q})_+ \subset \SO(\mathbb{Q})$ is defined as the elements that are mapped to the identity component of $(\SO_T / \mathbb{Z})(\mathbb{R})$. Since $\SO(\mathbb{Q})$ is dense in $\SO(\mathbb{R})$ because $\SO$ is connected, we see that $\SO(\mathbb{Q})_+ = O(\mathbb{R})^{+,+} \cap \SO(\mathbb{Q})$ in both cases.
Corollary 3.11. If $T$ is stable then $S_T$ has at most four connected components.

Proof. If $T$ is stable then $\mathcal{X}_0(T) = \{\ast\}$ so that $M_T$ has at most two components, and we conclude via the proposition above. \qed

Proposition 3.12. Assume that $T$ is very stable, then $|SO(\mathbb{Q})_+ \setminus SO(A_f)/SO(\hat{\mathbb{Z}})^*| = 1$ and therefore $S_T$ is geometrically irreducible. Moreover, the natural map $S_T \rightarrow M_T$ has degree precisely 2.

Proof. Assume that $T$ is very stable. We begin by studying $SO(\mathbb{Q}) \setminus SO(A_f)/SO(\hat{\mathbb{Z}})^*$. We know that $O(\mathbb{Q}) \setminus O(A_f)/O(\hat{\mathbb{Z}}) = \{\ast\}$ because this consists of the genus of $T$, and $T$ is unique in its genus due to [Nik79b, Theorem 1.14.2] whenever $T$ is (very) stable.

Thus, for any $g \in SO(A_f)$ we can find $q \in O(\mathbb{Q})$ and $k \in O(\hat{\mathbb{Z}})$ such that $qk = g$. Taking determinants, this show that $\det(q) \cdot \det(k) = 1$. Note that $\det(q) = \pm 1$ so that also $\det(k) = \pm 1$. In case both of them are $-1$, we can write $qr^{-1}rk = g$ for any reflexion $r$ as in Lemma 3.10. This shows that $SO(\mathbb{Q}) \setminus SO(A_f)/SO(\hat{\mathbb{Z}}) = \{\ast\}$.

Let $g \in SO(A_f)$ be any element, and consider any $q \in SO(\mathbb{Q})$ such that $qg \in SO(\hat{\mathbb{Z}})$. Pick any element $f \in SO(\mathbb{Z})$ such that $f$ induces the same map on $A_T$ as $qg$, which exists due to Lemma 3.10. It follows that $\pi^{-1}qg \in SO(\hat{\mathbb{Z}})^*$, and this shows that $SO(\mathbb{Q}) \setminus SO(A_f)/SO(\hat{\mathbb{Z}})^* = \{\ast\}$.

The fact that also $SO(\mathbb{Q})_+ \setminus SO(A_f)/SO(\hat{\mathbb{Z}})^* = \{\ast\}$ follows then from the existence of the decomposition $T = U \oplus T'$. Let $i : T \rightarrow T'$ be the map induced by $-\text{Id}$ on $U$ and $\text{Id}$ on $T'$. Then $i \in SO(T)^*$ and $i \in O(\mathbb{R})^{-,-}$. We know that for any $g \in SO(A_f)$ we can find $q \in SO(\mathbb{Q})$ and $k \in SO(\hat{\mathbb{Z}})^*$ such that $qk = g$. If $q \in SO(\mathbb{Q})_+$ then there is nothing to prove. If not, $(qk)(i^{-1}k) = qg$ so that we conclude.

We now show that the map $S_T \rightarrow M_T$ has degree two. Note that both the varieties are irreducible (over $\mathbb{C}$). Put $SO_+(\mathbb{Z})^* = SO(\mathbb{Q})_+ \cap SO(\hat{\mathbb{Z}})^*$, so that we have an isomorphism $S_T \cong \Omega^+/SO_+(\mathbb{Z})^*$. On the other hand, consider the subgroup $O(\mathbb{Q}) \subset O(\mathbb{Q})$ as the subgroup that preserves the orientation on the positive two planes. This corresponds to $O(\mathbb{Q}) = (O(\mathbb{R})^{+,+} \cup O(\mathbb{R})^{-,-}) \cap O(\mathbb{Q})$. Put $O(\mathbb{Z})^* = O(\mathbb{Q}) \cap O(\hat{\mathbb{Z}})^*$. Then, the space $M_T$ is isomorphic to $D_T^+/\hat{0}(\mathbb{Z})^*$ because $i \in O(T)^*$ exchanges $D_T^+$ with $D_T$. Finally, $SO_+(\mathbb{Z})^* \subset O(\mathbb{Z})^*$ has index 2 because the reflections $r$ in $(-2)$-classes of $T$ have signature $(+, -, -)$.

Our last proposition follows directly by the work of Rizov [Riz05].

Proposition 3.13. For any even hyperbolic lattice $L$ and any primitive embedding $L \hookrightarrow \Lambda$ with $L^\perp = T$, the map $S_T \rightarrow \mathcal{F}_L$ is defined over $\mathbb{Q}$.

Proof. We give a sketch of the argument. In order to prove that this map is defined over $\mathbb{Q}$ it is enough to show that it commutes with the Galois action on CM points. So let $x \in S_T$ be a CM point with CM field $E$ and let $X/\mathbb{Q}$ be the corresponding K3 surface. We know that the reflex field of $X$ is $E$ itself (see for instance the computation appearing in [Val21]). For any $\sigma \in G_E = \text{Gal}(\overline{E}/E)$ let $r_x(\sigma) \in A_{E, \mathbb{F}}$ be as in [Mil05, Chapter 12]. By the main theorem of complex multiplication proved by Rizov, there is a unique Hodge isometry $f : T(X)_E \rightarrow T(X^\sigma)_E$ such that the natural map $\sigma^* : T(X)_\mathbb{Z} \rightarrow T(X^\sigma)_\mathbb{Z}$ is given by the composition of $T(X)_{A_f} \xrightarrow{r_x(\sigma)} T(X)_{A_T} \xrightarrow{f} T(X^\sigma)_{A_T}$. Unraveling the Galois action on the set $\mathcal{F}_L(\overline{\mathbb{Q}})$ one shows that if $[x, g]$ is the moduli point of $(X, \iota)$ then $[x, r_x(\sigma)g]$ is the moduli point of $(X^\sigma, \iota^\sigma)$. Since the canonical models of Shimura varieties are defined
by prescribing the same action on special (= CM) points ([Mil05][Definition 12.8]) one concludes.

4. Maps between moduli spaces

The aim of this section is to study some basic maps between the spaces constructed in the previous sections. The maps we are interested in are essentially maps of Shimura datum (i.e., induced by maps of the relevant reductive groups) but we find it more convenient to think of them in terms of lattices. Attached to any lattice $T$ (i.e., induced by maps of the relevant reductive groups) we find that we are interested in are essentially maps of Shimura datum $\text{Sh}_T$. Now, to any (non necessarily primitive) inclusion of lattices $T \hookrightarrow S$ of the same signature type, we show how there are associated map $M_T \to M_S$ and $\text{Sh}_T \to \text{Sh}_S$ in a functorial way. A straightforward example of such maps is constructed as follows. Assume that we are given a primitive embedding $T \subset S$ and assume also that $S$ embeds primitively into the $T$ lattice. Let $\Lambda$ be any map of lattices (not necessarily primitive). Then, there are natural induced maps $\tilde{\Lambda}(\text{Id})S \to \tilde{\Lambda}(\text{Id})S$. We begin by constructing the natural map $\Lambda T \to \tilde{\Lambda}S$. Proof. We begin by constructing the natural map $\tilde{\Lambda}T \to \tilde{\Lambda}S$. Let $T$ be the orthogonal complement of $T$ in $S$. Then, $S$ is an overlattice of $T \oplus K$, meaning $T \oplus K \subset S$ is of finite index. We recall from Nikulin’s theory that an overlattice is determined by an isotropic subgroup $H \subset A_T \oplus A_K = A_{T \oplus K}$. We also have a natural isomorphism $A_S \xrightarrow{\sim} H^+ / H$. Given an element $(T', \gamma) \in \mathcal{X}^*(T)$, we can in this way obtain a subgroup $H' \subset A_{T'} \oplus A_K$ simply by putting $H' = (\gamma' \oplus \text{Id}_{A_{T'}})(H)$. This determines an overlattice $S'$ of $T' \oplus K$ which belongs all to the same genus of $S$, since they have the same signature and discriminant form. Moreover, we have isomorphisms $A_S \xrightarrow{\sim} H^+ / H$ and $\tilde{\Lambda}(\text{Id})S \oplus \tilde{\Lambda}(\text{Id})K \cong \tilde{\Lambda}(\text{Id})S$, thus obtaining an element of $\mathcal{X}^*(S)$. We finally endow $S'$ with the Hodge structure induced by the one of $T$ and the trivial Hodge structure on $K$.

In order to construct a similar map for $\tilde{\mathcal{X}}^*$ we only need to show how to deal with $d_2$ part of the data. This is easily done using that $\det(K \oplus T) = \det(K) \otimes \det(T)$ naturally.

The next proposition is our key observation:

Proposition 4.2. For any primitive embedding $T \hookrightarrow S$ and any connected component $M_T$ of $M_T$, the degree of the induced map $M_T^0 \to M_S$ onto its image can be bounded only in terms of $\text{rank}(S) - \text{rank}(T)$. If $T$ has corank one, then the degree is a divisor of $A$.

Proof. Let $M_S^0$ be the connected component of $M_S$ to which $M_T^0$ is mapped. As explained in the proof of Proposition 4.1, this is determined via lattice theory: if $p_0(M_T)$ is determined (up to an indeterminacy of at most two) by an isomorphism class of a pair $(T', \gamma') \in \mathcal{X}_0(T)$, then the construction carried out in the Proposition above determines another tuple $(S', \gamma'_S)$. Note that $S'$ is constructed as an overlattice of $T' \oplus K$. This means
that there is a primitive embedding \( T' \hookrightarrow S' \) such that the map \( M^0_T \to M^0_S \) is induced by \( \mathcal{D}^+_T / O^*(T) \to \mathcal{D}^+_S / O^*(S') \).

Let \( O(T)^*_+ \) denote the stabilizer in \( O(T)^* \) of \( \mathcal{D}^+_T \) (this corresponds to what we introduced before Corollary 3.11). We have to study the map \( \mathcal{D}^+_T / O(T)^*_+ \to \mathcal{D}^+_S / O(S)^*_+ \). Let \( \text{Stab}(T, S)^* := \{ g \in O(S)^*: g(T) = T \} \) and write \( \Gamma \subseteq O(T) \) for the image of \( \text{Stab}(T, S)^* \) in \( O(T) \) under the natural restriction map. We have \( O(T)^* \subset \Gamma \) due to Lemma 2.2.

We put \( \text{Stab}(T, S)^*_+ := \text{Stab}(T, S)^* \cap O(S)^*_+ \) and similarly \( \Gamma_+ \subset O(T)^* \) for the image of \( \text{Stab}(T, S)^*_+ \) in \( O(T) \). The natural injection \( O(T)^* \hookrightarrow O(S)^* \) given by extension by one clearly respects the induced decomposition \( O(T)^* \cap O_T(\mathbb{R})^{\pm, \pm} \). This implies that \( O(T)^*_+ \subset \text{Stab}(T, S)^*_+ \) and hence that \( O(T)^*_+ \subset \Gamma_+ \) as normal subgroups.

The degree of \( \mathcal{D}^+_T / O(T)^*_+ \to \mathcal{D}^+_S / O(S)^*_+ \) is then given by \( 2^{-1}|\Gamma_+: O(T)^*_+| \) if \(-\text{Id} \in \Gamma \setminus O(T)^* \) or \( |\Gamma_+: O(T)^*_+| \) otherwise.

Let now \( K = T^\perp \) be the orthogonal complement of \( T \) in \( S \). It follows that \( K \) is a definite lattice and therefore that \( O(K) \) is finite. Moreover, the cardinality of \( |O(K)| \) can be bounded only in terms of \( \text{rank}(K) \), due to the fact that every finite group acting on some \( \mathbb{Z}^n \) embeds in \( \text{Aut}((\mathbb{Z}/2\mathbb{Z})^n) \) under the natural quotient map. One has an injection

\[
\text{Stab}(T, S)^* \hookrightarrow O(T) \times O(K),
\]

and we consider the induced map

\[
\text{Stab}(T, S)^* \to O(K).
\]

An element \( g \in \text{Stab}(T, S)^* \) that maps to the identity of \( O(K) \) is an element of \( O(S)^* \) that stabilizes \( T \) and that acts trivially on \( K \). By embedding \( S \) into an even unimodular lattice \( U \), we know that \( g \) extends to an isometry of \( U \) by requiring it to be the identity on the orthogonal complement of \( S \). It follows that this isometry preserves \( T \) and acts as the identity also on the orthogonal complement of \( T \), and using Lemma 2.2 this shows that \( \ker(\text{Stab}(T, S)^* \to O(K)) = O(T)^* \) and therefore that \( |\Gamma: O(T)^*| \leq |O(K)| \). This is enough for the first part of the theorem.

If \( T \) has corank one, then \( K \cong \mathbb{Z} \) and therefore \( O(K) = \{ \pm 1 \} \), and hence the degree is at most 4 in this case. \( \square \)

Similarly, we have the analogous statement for Shimura varieties. We omit the proof as it is the same.

**Proposition 4.3.** For any primitive embedding \( T \hookrightarrow S \) and any connected component \( S^0_T \) of \( S_T \), the degree of the induced map \( S^0_T \to S^0_S \) onto its image can be bounded only in terms of \( \text{rank}(S) - \text{rank}(T) \). If \( T \) has corank one, then the degree is a divisor of 4.

Assume now that \( T \subset S \) has finite cokernel, so that \( \mathcal{D}_T = \mathcal{D}_S \). The induced map \( M_T \to M_S \) is finite, again of degree \( |O(T)^*: O(S)^*| \) or \( 2^{-1}|O(T)^*: O(S)^*| \) (recall that \( O(T)^* \subset O(S)^* \) naturally). For our proof we need the following numerical lemma:

**Lemma 4.4.** For any integer \( N > 0 \) and any prime \( p \) big enough and any \( S' \subset S \) such that \( S/S' \cong \mathbb{Z}/p\mathbb{Z} \) we have that \( |O(S)^*: O(S')^*| > N \).

**Proof.** Pick any prime that is coprime to \( \text{discr}(S) \). Then, finite subgroups of \( S \) of index \( p \) correspond to linear subspaces of \( S \otimes \mathbb{F}_p \) of codimension one. Since \( p \) is coprime to \( \text{discr}(S) \) the pairing induces a duality between codimension one spaces and lines in \( S \otimes \mathbb{F}_p \).

Let \( \text{Spin}_p \) be the spin group associated to \( T_0 \), which is simply connected and semisimple, and it is the universal cover of \( \text{SO}_T \). Strong approximation [PR94][Theorem 7.12] asserts that \( \text{Spin}_T(\mathbb{Q}) \) is dense in \( \text{Spin}_T(\mathbb{A}_f) \), which implies that \( K \cap \text{Spin}_T(\mathbb{Q}) \) is dense in
for any compact-open subgroup $K \subset \text{Spin}_T(A_f)$. By choosing $K = \text{Spin}_T(\hat{\mathbb{Z}})$ we see that $\text{Spin}_T(\mathbb{Z})$ is dense in $\text{Spin}_T(\hat{\mathbb{Z}})$. Since for $p$ odd and coprime to $\text{disc}(T)$ both the groups $\text{SO}_T$ and $\text{Spin}_T$ have good reduction at $p$, we obtain natural isomorphisms $\text{SO}_T \otimes \mathbb{F}_p \cong \text{SO}(T \otimes \mathbb{F}_p)$ and similarly for $\text{Spin}_T$. But by the strong approximation property we know that $\text{Spin}_T(\mathbb{Z}) \to \text{Spin}_T(\mathbb{F}_p)$ is surjective. This implies that the image of $O_T(\mathbb{Z}) \to O_T(\mathbb{F}_p)$ has index at most four. Let $\mathcal{L}_{p,0}$ be the set of isotropic lines in $T \otimes \mathbb{F}_p$, $\mathcal{L}_{p,+}$ be the set of lines generated by elements $x$ such that $(x, x) \in \mathbb{F}_p^{\times 2}$ and $\mathcal{L}_{p,-}$ the analogous set for non-squares. We know that $O_T(\mathbb{F}_p)$ acts transitively on each of these sets due to Witt’s theorem. This implies that if a subgroup stabilizes an element of some set above, say $\mathcal{L}_{p,-}$, then it is a subgroup of $O_T(\mathbb{F}_p)$ of index at least $4^{-1}|\mathcal{L}_{p,-}|$. Since $O(S')^\circ$ is always contained in such stabilizer, and the cardinalities of the sets $\mathcal{L}_{p,\pm}$ are unbounded, we conclude.

Finally, we note that for any map of lattices $T \to S$ of rank greater than two, the induced map $S_T \to S_S$ are defined over $\mathbb{Q}$, because both Shimura datum have reflex field $\mathbb{Q}$. In some cases this can also be seen directly at the level of moduli spaces. We showed before how primitive embeddings of hyperbolic lattices $f : L \hookrightarrow N$ induce maps $f_* : \mathcal{F}_L \to \mathcal{F}_N$, which can be described by associating to a $L$-quasi-polarized K3 surface $(X, \iota)$ the $N$-quasi-polarized K3 surface $f_*(X, \iota) = (X, \iota \circ f)$ (after the choice of some coherent small cones). For any $\sigma \in G_\mathbb{Q}$ we see that

$$(f_*(X, \iota))^{\sigma} = (X, \iota \circ f)^{\sigma} = (X^{\sigma}, (\iota \circ f)^{\sigma}) = (X^{\sigma}, \sigma \circ \iota \circ f) = f_*((X, \iota)^{\sigma}).$$

**Definition 4.5.** Let $S$ be any even lattice of signature $(2_+, n_-)$. The Noether-Lefschetz loci of $M_S$ and $S_S$ are defined as follows:

$$M^\text{nl}_S = \bigcup_{T \to S} \text{im}(M_T \to M_S),$$

$$S^\text{nl}_S = \bigcup_{T \to S} \text{im}(S_T \to S_S),$$

where both unions are taken over all the even lattices $T$ of same signature type and such that rank$(T) = \text{rank}(S) - 1$ and over all the primitive embeddings.

It is clearly enough to consider lattices of corank one, since any other primitive sublattice of $S$ embeds into such one. It is straightforward to show that there are only finitely many components of the Noether-Lefschetz locus associated to a given sublattice.

### 4.1. Proofs

**Theorem 4.6.** Let $T$ be any even lattice of signature type $(2_+, n_-)$ and assume that $T$ is very stable and that rank$(T) \geq 5$. Consider the Shimura variety $S_T/\mathbb{Q}$ introduced in \ref{2.0.2} which is always geometrically irreducible under our conditions. For any $N > 0$, there are geometrically irreducible Shimura varieties $S_i/\mathbb{Q}$ for $i = 1, \cdots, n$ and branched covering maps $\pi_i : S_i \to S$ with the following property:

1. $\deg(\pi_i) \geq N$;
2. For any finite field extension $K/\mathbb{Q}$ and any rational point $x \in S_T(K)$ that belongs to the Noether-Lefschetz locus, there is an $i$ such that the fibre $\pi_i^{-1}(x)$ contains a $K'$-rational point for $K'/K$ an extension with $[K': K] \leq 16$. If $\ell(A_S) \leq \text{rank}(S) - 3$ as well, then the constant 16 can be reduced to 4.
Proof. Under our assumption we have that \( S' \) is geometrically irreducible. Let \( p \) be a prime big enough and coprime to \( \text{discr}(S) \), and consider the finite set \( \mathcal{L}_p \) of sublattices of \( S \) of index \( p \). Note that the lattices \( S' \) are very stable if \( \text{rank}(S') > 4 \), and therefore \( S' \) is geometrically irreducible as well. In fact, since \( p \) is coprime to \( \text{discr}(S) \) the group \( A_{S'} \) splits canonically as \( A_S \oplus (S' \otimes \mathbb{Z}_p)^\vee / (S' \otimes \mathbb{Z}_p) \). Now, \( (S' \otimes \mathbb{Z}_p)^\vee / (S' \otimes \mathbb{Z}_p) \) is either isomorphic to \( (\mathbb{Z}/p\mathbb{Z})^2 \) or \( \mathbb{Z}/p^2\mathbb{Z} \). This means that the length of \( A_{S'} \) is given by \( \max\{2, \ell(A_S)\} \) and since \( \ell(A_S) \leq \text{rank}(S) - 3 \), this proves our point.

Pick any primitive \( T \subset S \) with \( \text{rank}(T) = \text{rank}(S) - 1 \). Pick any \( S'' \in \mathcal{L}_p \) such that \( S'' \cap T = T \), and consider the triangle of morphisms of \( \mathbb{Q} \)-varieties

\[
\begin{array}{ccc}
S_T & \longrightarrow & S_{S''} \\
\uparrow f & & \downarrow \pi \\
S_S & \longrightarrow & S_S
\end{array}
\]

induced by the inclusions \( T \hookrightarrow S \) and \( T \hookrightarrow S'' \). Note that \( \ell(A_T) \leq \text{rank}(T) - 2 \), and in particular \( S_T \) consists of at most four connected components due to Corollary \([3.11]\). Let \( x \in \text{Im}(f)(K) \). Then there exists a point \( y \in S_T(K') \) where \( K'/K \) is an extension of degree at most \( 16 = 4 \cdot 4 \) that is mapped to \( x \). One factor of 4 is needed in order to split the eventual components of \( S_T \), whereas the other comes from Proposition \([4.3]\). Then, \( z = f(y) \in S(S')(K') \) is a point such that \( \pi(z) = x \). We can then simply take the maps \( \pi \) to be the ones induced by all the sublattices of \( S \) of index \( p \) big enough.

Finally, if \( \ell(A_S) \leq \text{rank}(S) - 3 \) then it follows that also \( T \) is automatically very stable, hence we can drop a factor of 4 in the computation above. \( \square \)

The analogous statement for moduli spaces of K3 surfaces follows from Proposition \([3.12]\) and Corollary \([3.11]\). The only caveat is that we cannot use directly the maps \( M_{S''} \to \hat{M}_S \) because we do not know that the varieties \( M_{S''} \) have natural models over \( \mathbb{Q} \), although most probably they do. Instead of proving this, we rather use the coverings given by the composition \( S_{S''} \to M_{S''} \to M_S \) to prove the analogous statement, so to obtain a diagram

\[
\begin{array}{ccc}
S_T & \longrightarrow & S_{S''} \\
\uparrow f & & \downarrow \pi \\
M_S & \longrightarrow & M_S
\end{array}
\]

at the cost of multiplying the constant 16 (or 4, if \( \ell(A_S) \leq \text{rank}(S) - 3 \)) by another factor of 2.

Proof of Theorem \([17]\). Pick a cover \( \hat{\mathcal{S}}_L \to \mathcal{S}_L \) of degree \( d \) which supports a family of \( L \)-quasipolarized K3 surfaces. Let \( S \) be the generic transcendental lattice of the family, so that \( S \) is a very stable lattice by assumption, and let \( S_{S''} \to S_S \) be the map of Shimura varieties associated to a sublattice \( S'' \subset S \) such that \( S/S'' \cong \mathbb{Z}/p\mathbb{Z} \). Let \( x \in S_{S''}(K) \) be a \( K \)-rational point, and let \( y \) be its image in \( \mathcal{S}_L(K) \) under the natural map \( S_{S''} \to \mathcal{S}_L \).

Then, there is an extension \( K'/K \) of degree \( [K'/K] \leq d \) and a \( L \)-quasipolarized K3 surface \( X/K' \) representing \( y \). Assume now that \( x \) is Picard generic. Due to the description of these moduli spaces as Shimura varieties, we have a map \( S'' \subset T(X_C) \) of cokernel isomorphic to \( \mathbb{Z}/p\mathbb{Z} \). The fact that \( (X, \iota) \) is the image of \( x \) implies that \( G_{K'} \) preserves the lattice \( S'' \otimes \hat{\mathbb{Z}} \subset T(X_C) \otimes \hat{\mathbb{Z}} \), and that it acts trivially on \( A_{S''} \). But there exists a unique Brauer class (up to multiples) \( \alpha \in \text{Br}(X)[p] \cong \text{Hom}(T(X_C), \mathbb{Z}/p\mathbb{Z}) \) such that

\[
\text{Picard generic} = \{ \alpha \}.
\]
$S' = \ker(\alpha)$. Since $\langle \alpha \rangle \cong \mathbb{Z}/p\mathbb{Z}$ is a quotient of $A_{S'}$, we see that $Br(X)^{G_{K'}}$ contains a class of order $p$.

Now we take $p$ big enough such that there are no K3 surfaces $X/K$ with $\text{NS}(X) \cong L$, $p$ divides $|Br(X)^{G_{K}}|$, and $[K : \mathbb{Q}] \leq 33Nd$. Such $p$ exists because we are assuming Várilly-Alvarado conjecture. Let $K'/\mathbb{Q}$ be any extension of degree $[K : \mathbb{Q}] \leq N$, and let $y \in \mathcal{F}_L(K)$ be a $K$-rational point. Assume that there is $\pi_i$ such that $\pi_i^{-1}\{y\}$ contains a $K'$-rational point for $[K' : K] \leq 32$, and assume that $y$ has generic Picard rank.

As explained before, this means that there is an extension $K''/K$ of degree smaller than $32d$ and a K3 surface $X/K''$ with $\text{NS}(X) = L$ and $Br(X)^{G_{K''}}$ contains a class of order $p$. Since $[K : \mathbb{Q}] \leq N$, this contradicts our assumption on $p$, and it follows that $y$ must belong to the Noether-Lefschetz locus. 

\[\square\]

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