INTERIOR $C^{1,1}$ REGULARITY OF SOLUTIONS TO DEGENERATE MONGE-AMPÈRE TYPE EQUATIONS

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Abstract. In this paper, we study the interior $C^{1,1}$ regularity of viscosity solutions for a degenerate Monge-Ampère type equation $\det[D^2 u - A(x, u, Du)] = B(x, u, Du)$ when $B \geq 0$ and $B^{-1} \in C^{1,1}([\Omega \times \mathbb{R} \times \mathbb{R}^n])$. We prove that $u \in C^{1,1}(\Omega)$ under the $A3$ condition and $A3w^+$ condition respectively. In the former case, we construct a suitable auxiliary function to obtain uniform a priori estimates directly. In the latter case, the main argument is to establish the Pogorelov type estimates, which are interesting independently.

1. Introduction

In this paper, we shall study the following degenerate Monge-Ampère type equation (DMATE)

\[
det[D^2 u - A(\cdot, u, Du)] = B(\cdot, u, Du), \quad \text{in } \Omega,
\]

where $\Omega$ is a bounded domain, $Du$ and $D^2 u$ denote the gradient and Hessian matrix of second order derivatives of the unknown function $u : \Omega \to \mathbb{R}$ respectively, $A : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is a symmetric $n \times n$ matrix valued function and $A \in C^{2,1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $B : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+ \cup \{0\}$ is a nonnegative scalar function and $B^{-1} \in C^{1,1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$. We shall use $x, z$ and $p$ to denote the points in $\Omega, \mathbb{R}$ and $\mathbb{R}^n$, respectively.

We say that $A$ is strictly regular in $\Omega$, if

\[
\sum_{i,j,k,l=1}^n D^2_{p_kp_l}A_{ij}(x, z, p)\xi_i\xi_j\eta_k\eta_l \geq c_0|\xi|^2|\eta|^2,
\]

holds for all $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $\xi, \eta \in \mathbb{R}^n$ with $\xi \cdot \eta = 0$, and some positive constant $c_0$. If $c_0$ on the right hand side in (1.2) is replaced by 0, we say that $A$ is regular in $\Omega$. As usual, the strictly regular condition and regular condition are also said to be the $A3$ condition and the $A3w$ condition, respectively, see [16, 17]. If (1.2) holds for $c_0 = 0$ without the restriction $\xi \cdot \eta = 0$, we call (1.2) the regular condition without orthogonality or the $A3w$ condition without orthogonality. We introduce a particular form of $A3w$ condition, namely

\[
\sum_{i,j,k,l=1}^n D^2_{p_kp_l}A_{ij}(x, z, p)\xi_i\xi_j\eta_k\eta_l \geq \mu_0(\xi \cdot \eta)^2,
\]

holds for all $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $\xi, \eta \in \mathbb{R}^n$, and some constant $\mu_0$. We call (1.3) the $A3w^+$ condition. It is obvious that the $A3w^+$ condition implies the $A3w$ condition. The $A3w$ condition without orthogonality implies the $A3w^+$ condition when $\mu_0 \leq 0$.

The aim of this paper is to investigate interior regularity of solutions to the degenerate equation (1.1). It is well known that the Pogorelov estimate plays an important role in establishing interior regularity.
of solutions to Monge-Ampère equations. When \( A \equiv 0 \), the equation (1.1) reduces to the classical Monge-Ampère equation. For the case \( B \geq B_0 > 0 \) with a constant \( B_0 \), the Pogorelov estimate for the equation (1.1) together with the homogeneous Dirichlet boundary condition \( u = 0 \) on \( \partial \Omega \) was first proved by Pogorelov [18]. Various versions of Pogorelov estimates for nondegenerate Monge-Ampère equations can be found in [4, 5, 8, 20]. For the case \( B > 0 \), Blocki [1] proved

\[
(w - u)^\alpha |D^2u| \leq C, \quad \text{in } \Omega,
\]

where \( \alpha = n - 1 \) if \( n \geq 3 \) and \( \alpha > 1 \) if \( n = 2 \), \( w \in C^2(\Omega) \) is convex satisfying \( u \leq w \) in \( \Omega \) and \( \lim_{x \to \partial \Omega} (w(x) - u(x)) = 0 \), and the constant \( C \) is independent of the lower bound of \( B \). When \( A \neq 0 \), the Monge-Ampère type equations (1.1) arise in various aspects such as optimal mass transportation problems, geometric optics and conformal geometry etc (see, for instance [9, 11, 17, 19]). The Pogorelov type estimates of non-degenerate Monge-Ampère type equations were established under the assumptions of \( A3w \) and \( A \)-boundedness conditions in [13][15]. Without the \( A \)-boundedness condition, the interior second order derivative estimates of Pogorelov type were also shown to be valid in [9] by constructing a different barrier function with the help of an admissible function. In the optimal mass transportation setting, interior \( C^2 \) regularity for non-degenerate Monge-Ampère type equations was obtained under the A3 condition in [17].

In this paper, we investigate the interior regularity of a viscosity solution \( u \) to the degenerate Monge-Ampère type equation (1.1). By constructing a suitable auxiliary function to directly obtain uniform \textit{a priori} estimates of second order derivatives, we first prove that \( u \in C^{1,1}(\Omega) \) under the A3 condition. Then we relax the A3 condition to the A3w condition, by assuming some suitable additional conditions, we establish the Pogorelov type estimates, which are independently interesting, and further show that the solution \( u \) has interior \( C^{1,1} \) regularity.

More precisely, we have the following main results.

**Theorem 1.1.** Let \( u \in C^4(\Omega) \cap C^{1,1}(\Omega) \) be a solution of the equation (1.1) in a bounded domain \( \Omega \subset \mathbb{R}^n \), where \( B \) is a positive function and \( B^{1+\frac{1}{n}} \in C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n) \). Assume that

\[
D_{pp}\tilde{B} \geq -CBI,
\]

for some nonnegative constant \( C_B \), where \( I \) is the \( n \times n \) identity matrix and \( \tilde{B} = \log B \). Assume that \( A \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^{n \times n}) \) is strictly regular. Then, we have

\[
|D^2u(x)| \leq C,
\]

where \( C \) depends on \( n \), \( \text{dist}(x, \partial \Omega) \), \( \sup_{\Omega} |Du| \), \( ||B^{1+\frac{1}{n}}||_{C^{1,1}} \), \( ||A||_{C^2} \) and \( c_0 \).

Before stating the next theorem, we first define the viscosity solution of the equation (1.1). A function \( u \) is called a viscosity subsolution (supersolution) of the equation (1.1), if for any function \( \phi \in C^2(\Omega) \) such that \( u - \phi \) has a local maximum (minimum) at some point \( x_0 \in \Omega \), there holds

\[
\det[D^2\phi(x_0) - A(x_0, \phi(x_0), D\phi(x_0))] \geq (\leq)B(x_0, \phi(x_0), D\phi(x_0)).
\]

A function \( u \) is a viscosity solution of the equation (1.1) if it is both a viscosity subsolution and a viscosity supersolution of the equation (1.1).

**Theorem 1.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), and \( u \) be a viscosity solution of the equation (1.1). Assume that \( A \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^{n \times n}) \) is strictly regular, \( B \) is a nonnegative function, \( B^{1+\frac{1}{n}} \in C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n) \) and \( B \) satisfies the condition (1.5). Then, we have \( u \in C^{1,1}(\Omega) \).

Note that the constant \( c_0 \) in Theorem 1.1 is from the strictly regular condition (1.2) of the matrix \( A \). The second order derivative estimate (1.6) depends on \( c_0 \), which will blow up when \( c_0 \) tends to 0. In this sense, Theorems 1.1 and 1.2 are not valid for the interior second order derivative estimate under the A3w condition.
However, we can still obtain the interior $C^{1,1}$ regularity for the degenerate Monge-Ampère type equation (1.1) under the $A3w^+$ condition with the help of suitable barrier functions. In order to construct the barrier functions, we can assume either the $A$-boundedness condition or the existence of a strict subsolution.

First, we introduce the $A$-boundedness condition as in [14, 19]. We say that the $A$-boundedness condition holds, if there exists a function $\varphi \in C^2(\bar{\Omega})$ satisfying
\begin{equation}
[D_{ij}\varphi - D_{pq}A_{ij}(x,z,p)D_{k\ell}\varphi(x)]\xi_i\xi_j \geq |\xi|^2,
\end{equation}
for all $\xi \in \mathbb{R}^n, (x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$.

Next, we introduce the definition of a strict subsolution of the equation (1.1). A function $u \in C^2(\Omega)$ is called an elliptic (a degenerate elliptic) function when its augmented Hessian matrix $M[u] := D^2u - A(x,u,Du) > 0(\geq 0)$. If $u$ is also a solution of the equation (1.1), we call it an elliptic (a degenerate elliptic) solution. A function $\bar{u} \in C^2(\Omega)$ is said to be elliptic (degenerate elliptic) with respect to $u$ in $\Omega$, if $M_u[\bar{u}] := D^2\bar{u} - A(\cdot, u, Du) > 0(\geq 0)$ in $\Omega$. If such a function $\bar{u}$ also satisfies
\begin{equation}
\det(M_u[\bar{u}]) > B(\cdot, u, Du),
\end{equation}
at points in $\Omega$, we call $\bar{u}$ a strict subsolution of the equation (1.1).

We now formulate the Pogorelov type estimate under $A3w^+$ in the following theorem.

**Theorem 1.3.** Let $u \in C^4(\Omega) \cap C^{1,1}(\bar{\Omega})$ be a solution of the equation (1.1) in a bounded domain $\Omega \subset \mathbb{R}^n$, where $B$ is a positive function, $B^{\frac{1}{n+1}} \in C^1(\bar{\Omega} \times \mathbb{R}^n)$ and $B$ satisfies the condition (1.5). Assume that $A \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ satisfies the $A3w^+$ condition, and there exists a $C^{1,1}$ function $w$ satisfying $w \geq u$ in $\Omega$, $w = u$ on $\partial\Omega$, which is degenerate elliptic with respect to $u$ in $\Omega$. Assume also one of the following conditions:

(i) $A$-boundedness condition (1.8) holds;

(ii) there exists a strict subsolution $\bar{u} \in C^2(\Omega)$ of the equation (1.1) satisfying (1.9).

Then we have the estimate
\begin{equation}
(w-u)^\tau|D^2u| \leq C, \quad \text{in } \Omega,
\end{equation}
where $\tau = 2$ if $B_p \not\equiv 0$ and $\tau = 1$ if $B_p \equiv 0$, the constant $C$ depends on $n$, $\Omega$, $\|B^{\frac{1}{n+1}}\|_{C^{1,1}}$, $\|A\|_{C^2}$, $\sup_{\Omega}|Dw|$, $\sup_{\Omega}|Du|$. In case (ii), the constant $C$ depends in addition on $\bar{u}$.

There is a technical reason why we restrict our attention under the $A3w^+$ condition, see Remark 1.1 after the proof of Theorem 1.3.

**Remark 1.1.** We remark that, in Theorem 1.3 if $B$ satisfies a further condition $\frac{|B_p|}{B} \leq C$ for some nonnegative constant $C$, then the estimate (1.10) can be improved to $(w-u)|D^2u| \leq C$, which corresponds to the estimate (1.10) for the $B_p \equiv 0$ case as well.

From Theorem 1.3 we can have the following interior regularity result.

**Theorem 1.4.** Under the assumptions of Theorem 1.3, assume instead that $u$ is a viscosity solution of the equation (1.1) and $B$ is a nonnegative function, and assume further that $A$ and $B$ are nondecreasing in $z$. Then we have $u \in C^{1,1}(\Omega)$.

In order to guarantee the comparison principle, the monotonicity conditions for both $A$ and $B$ with respect to $z$ are assumed in Theorem 1.4.

**Remark 1.2.** We emphasize that the constants $C$ in both the estimates (1.6) in Theorem 1.1 and (1.10) in Theorem 1.3 are independent of the positive lower bound of $B$, so that they can be applied to obtain the interior $C^{1,1}$ regularity for the degenerate equation (1.1). The assumption $B^{\frac{1}{n+1}} \in C^{1,1}$ can be found in [6, 7], which is proved to be optimal in [21] when $A \equiv 0$ and $B$ is independent of $z$ and $p$. 

When $\mu_0 \leq 0$, the matrix $A \equiv 0$ satisfies the $A3w^+$ condition \text{(1.3)} automatically, so that Theorem \text{(1.3)} and \text{(1.4)} can apply to the standard Monge-Ampère equation $\det D^2u = B(\cdot, u, Du)$.

The organization of this paper is as follows. In Section 2 we introduce some properties of $B$ when $B^{\frac{1}{n-1}} \in C^{1,1}$, in Lemma 2.1 and Corollary 2.1 which are useful in deriving estimates independent of the lower bound of $B$. A fundamental barrier construction under the $A3w$ condition is also introduced in Section 2.2, which will be used in Section 4 when we only assume the $A3w^+$ condition. In Section 3 we obtain interior second order derivative estimates for the Monge-Ampère type equation (1.1) under $A3$ condition, and then show the interior $C^{1,1}$ regularity for viscosity solutions of the DMATE (1.1). In Section 4, under the $A3w^+$ condition, we establish the Pogorelov type estimates for the Monge-Ampère type equation (1.1) by using suitable barrier functions, and apply these estimates to obtain interior $C^{1,1}$ regularity for viscosity solutions of the DMATE (1.1).

2. Preliminaries

In this section, we introduce some properties of $B$ when $B^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, and a fundamental lemma of barrier construction, which will be used in later sections.

In the equation (1.1), we suppose $B > 0$ in $\Omega$, $\bar{u}_{ij} := u_{ij} - A_{ij}$ and $\{\bar{u}^{ij}\} := \{\bar{u}_{ij}\}^{-1}$. Then both matrices $\{\bar{u}_{ij}\}$ and $\{\bar{u}^{ij}\}$ are positive definite. We can rewrite the equation (1.1) in the form

\[
\log \det \{\bar{u}_{ij}\} = \bar{B}, \quad \text{in } \Omega,
\]

where $\bar{B} := \log B$. By differentiating the equation (2.1) in the direction $\xi \in \mathbb{R}^n$ once and twice respectively, we have

\[
\bar{u}^{ij}[D_\xi u_{ij} - D_\xi A_{ij} - (D_z A_{ij})D_\xi u - (D_p A_{ij}) D_\xi u_k] = D_\xi \bar{B},
\]

and

\[
\bar{u}^{ij}[D_{\xi \xi} u_{ij} - D_{\xi \xi} A_{ij} - (D_{p \xi} A_{ij}) D_\xi u_k D_\xi u_l - (D_{p \xi} A_{ij}) D_{\xi \xi} u_k - (D_z A_{ij}) D_{\xi \xi} u]
- (D_{zz} A_{ij})(D_\xi u)^2 - 2(D_{n \xi} A_{ij}) D_\xi u - 2(D_{p \xi} A_{ij}) D_\xi u D_\xi u_k - 2(D_{p p \xi} A_{ij}) D_\xi u_k
= \bar{u}^{il} \bar{u}^{jl} D_\xi \bar{u}_{ij} D_\xi \bar{u}_{kl} + D_{\xi \xi} \bar{B},
\]

where

\[
D_\xi \bar{B} = \frac{B_\xi + B_z D_\xi u + B_p D_\xi u_k}{B},
\]

and

\[
D_{\xi \xi} \bar{B} = \frac{B_{\xi \xi} + B_{zz}(D_\xi u)^2 + B_z D_{\xi \xi} u + B_p D_{\xi \xi} u_k + B_{p p \xi} (D_\xi u)(D_\xi u_k)}{B}
+ \frac{2B_{\xi \xi} D_\xi u + 2B_{p \xi} D_\xi u_k + 2B_{p p \xi} (D_\xi u)(D_\xi u_k)}{B}
- \frac{B_\xi^2 + B_z^2 (D_\xi u)^2 + B_p B_{p \xi} (D_\xi u_k)(D_\xi u_l)}{B^2}
- \frac{2B_{\xi \xi} B_z u_k + 2B_{\xi \xi} B_p u_k + 2B_{\xi \xi} B_{p \xi} (D_\xi u)(D_\xi u_k)}{B^2}.
\]

Note that we use the standard summation convention in the context that repeated indices indicate summation from 1 to $n$ unless otherwise specified.

We introduce the following lemma and its corollary, in order to deal with the right-hand side term of the equation (1.1).
Lemma 2.1. Assume \( B^{\frac{1}{n-t}}(x, u, Du) \in C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n) \) and \( B > 0 \), then we have

\[
(2.6) \quad \left| \frac{B_i}{B} \right|, \left| \frac{B_z}{B} \right|, \left| \frac{B_{pi}}{B} \right| \leq (n-1)\sqrt{2\|B^{\frac{1}{n-t}}\|_{C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)}B^{-\frac{1}{2(n-1)}}},
\]
in \( \tilde{\Omega} \times \mathbb{R} \times \mathbb{R}^n \), for \( i = 1, \cdots, n \), and

\[
(2.7) \quad \left| \frac{B_{ij}}{B} \right|, \left| \frac{B_{i2}}{B} \right|, \left| \frac{B_{ip}}{B} \right|, \left| \frac{B_{zi}}{B} \right|, \left| \frac{B_{z2}}{B} \right|, \left| \frac{B_{zpi}}{B} \right| \leq (n-1)(2n-3)\|B^{\frac{1}{n-t}}\|_{C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)}B^{-\frac{1}{4(n-1)}},
\]
in \( \tilde{\Omega} \times \mathbb{R} \times \mathbb{R}^n \), for \( i, j = 1, \cdots, n \).

Proof. By Taylor’s formula, for any given \( (x_0, z_0, p_0) \in \tilde{\Omega} \times \mathbb{R} \times \mathbb{R}^n \),

\[
0 \leq B^{\frac{1}{n-t}}(x, z, p)
\]

(2.8) \begin{align*}
&\leq B^{\frac{1}{n-t}}(x_0, z_0, p_0) + \nabla \left( B^{\frac{1}{n-t}} \right) (x_0, z_0, p_0) \cdot (x - x_0, z - z_0, p - p_0) \\
&\quad + \frac{1}{2}\|B^{\frac{1}{n-t}}\|_{C^{1,1}(\tilde{\Omega} \times \mathbb{R} \times \mathbb{R}^n)} \left[ |x - x_0|^2 + |z - z_0|^2 + |p - p_0|^2 \right],
\end{align*}
holds for any \((x, z, p) \in \tilde{\Omega} \times \mathbb{R} \times \mathbb{R}^n\), where \( \nabla := (D_x, D_z, D_p) \). Kirszbraun’s Theorem (in Section 12.10.43 in [3]) asserts that there exists an extension from \( \Omega \times \mathbb{R} \times \mathbb{R}^n \) to \( \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \) such that \( B^{\frac{1}{n-t}} \in C^{1,1}(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n) \) and \( \|B^{\frac{1}{n-t}}\|_{C^{1,1}(\tilde{\Omega} \times \mathbb{R} \times \mathbb{R}^n)} = \|B^{\frac{1}{n-t}}\|_{C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)} \), then (2.8) holds for all \((x, z, p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n\). Consequently, we have

\[
(2.9) \quad \left( B^{\frac{1}{n-t}} \right)_i(x_0, z_0, p_0) \leq 2\|B^{\frac{1}{n-t}}\|_{C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)}B^{\frac{1}{n-t}}(x_0, z_0, p_0) \leq 0, \quad \text{for } i = 1, \cdots, n,
\]

(2.10) \[
\left( B^{\frac{1}{n-t}}_z \right)_i(x_0, z_0, p_0) \leq 2\|B^{\frac{1}{n-t}}\|_{C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)}B^{\frac{1}{n-t}}(x_0, z_0, p_0) \leq 0,
\]
and

(2.11) \[
\left( B^{\frac{1}{n-t}} \right)_{pi}(x_0, z_0, p_0) \leq 2\|B^{\frac{1}{n-t}}\|_{C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)}B^{\frac{1}{n-t}}(x_0, z_0, p_0) \leq 0, \quad \text{for } i = 1, \cdots, n,
\]
namely,

(2.12) \[
\left| B^{\frac{1}{n-t}}_i(x_0, z_0, p_0) \right| \leq \sqrt{2\|B^{\frac{1}{n-t}}\|_{C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)}B^{\frac{1}{n-t}}(x_0, z_0, p_0)}, \quad \text{for } i = 1, \cdots, n,
\]

(2.13) \[
\left| B^{\frac{1}{n-t}}_z(x_0, z_0, p_0) \right| \leq \sqrt{2\|B^{\frac{1}{n-t}}\|_{C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)}B^{\frac{1}{n-t}}(x_0, z_0, p_0)},
\]
and

(2.14) \[
\left| B^{\frac{1}{n-t}}_{pi}(x_0, z_0, p_0) \right| \leq \sqrt{2\|B^{\frac{1}{n-t}}\|_{C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)}B^{\frac{1}{n-t}}(x_0, z_0, p_0)}, \quad \text{for } i = 1, \cdots, n.
\]

By (2.12), (2.13) and (2.14), we have

(2.15) \[
\left| B_i(x_0, z_0, p_0) \right| \leq (n-1)\sqrt{2\|B^{\frac{1}{n-t}}\|_{C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)}B^{\frac{1}{n-t}}(x_0, z_0, p_0)}, \quad \text{for } i = 1, \cdots, n,
\]

(2.16) \[
\left| B_z(x_0, z_0, p_0) \right| \leq (n-1)\sqrt{2\|B^{\frac{1}{n-t}}\|_{C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)}B^{\frac{1}{n-t}}(x_0, z_0, p_0)},
\]
and

(2.17) \[
\left| B_{pi}(x_0, z_0, p_0) \right| \leq (n-1)\sqrt{2\|B^{\frac{1}{n-t}}\|_{C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)}B^{\frac{1}{n-t}}(x_0, z_0, p_0)}, \quad \text{for } i = 1, \cdots, n.
\]

Since \((x_0, z_0, p_0)\) can be an arbitrary point in \( \tilde{\Omega} \times \mathbb{R} \times \mathbb{R}^n \), from (2.13), (2.16) and (2.17), conclusion (2.6) is proved.
Next, by a direct computation, we obtain

\[(2.18)\]

\[
D_{ij} \left( B^\frac{1}{n-1} \right) = \frac{1}{n-1} B^{-\frac{1}{n-1}} \left( \frac{B_{ij}}{B} - \frac{n-2 B_i B_j}{n-1 B^2} \right),
\]

in \( \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \), for \( i, j = 1, \cdots, n \). Therefore, we have from (2.18) that

\[(2.19)\]

\[
\left| \frac{B_{ij}}{B} \right| \leq (n-1) \left| \left( B^\frac{1}{n-1} \right)_{ij} \right| B^{-\frac{1}{n-1}} + \frac{n-2}{n-1} \left| \frac{B_i}{B} \right| \left| \frac{B_j}{B} \right| \leq (n-1)(2n-3)\| B^\frac{1}{n-1} \|_{C^{0,1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)} B^{-\frac{1}{n-1}},
\]

in \( \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \), where (2.6) is used in the last inequality. Then (2.19) completes the proof of the first inequality in (2.7). The other inequalities in (2.7) can be derived similarly to (2.19). We omit the remaining proof, in order to avoid too many repetitions. \( \square \)

**Remark 2.1.** In fact, we can have a relaxed version of the estimate (2.20),

\[(2.20)\]

\[
\left| \frac{B_i}{B} \right|, \left| \frac{B_z}{B} \right|, \left| \frac{B_{p_k}}{B} \right| \leq (n-1)\| B^\frac{1}{n-1} \|_{C^{0,1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)} B^{-\frac{1}{n-1}},
\]

in \( \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \), for \( i = 1, \cdots, n \), which can be readily verified by a direct calculation. Namely, we have

\[(2.21)\]

\[
\left| \frac{B_i}{B} \right| = (n-1) \left| D_i (B^\frac{1}{n-1}) \right| B^{-\frac{1}{n-1}} \leq (n-1)\| B^\frac{1}{n-1} \|_{C^{0,1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)} B^{-\frac{1}{n-1}},
\]

for \( i = 1, \cdots, n \). The estimates for \( \left| \frac{B_z}{B} \right| \) and \( \left| \frac{B_{p_k}}{B} \right| \) can be obtained exactly in the same way.

We have the following consequence of Lemma 2.1 and Remark 2.1.

**Corollary 2.1.** Assume \( B^\frac{1}{n-1} (x, u, Du) \in C^{1,1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n) \), \( B > 0 \) and \( \tilde{B} = \log B \). Then we have the following properties:

(i) \[(2.22)\]

\[
|D_i \tilde{B}| \leq C \left[ 1 + \max_j (|\tilde{u}_{ij}|) \right] B^{-\frac{1}{n-1}}
\]

holds for \( i = 1, \cdots, n \), where the constant \( C \) depends on \( n, \| B^\frac{1}{n-1} \|_{C^{1,1}}, A \) and \( \sup_{\bar{\Omega}} |Du| \).

(ii) If the condition (1.5) holds, then

\[(2.23)\]

\[
D_{ii} \tilde{B} \geq -C \left[ 1 + \max_j (|\tilde{u}_{ij}|) \right] B^{-\frac{1}{n-1}} - C' \left[ 1 + \max_j (|\tilde{u}_{ij}|) \right] + \sum_{k=1}^n \tilde{B}_{p_k} D_{ii} u_k
\]

holds for \( i = 1, \cdots, n \), where the constant \( C \) depends on \( n, \| B^\frac{1}{n-1} \|_{C^{1,1}}, A \) and \( \sup_{\bar{\Omega}} |Du| \), and the constant \( C' \) depends on \( C_B \) and \( A \).

**Proof.** Choosing \( \xi = e_i \) in (2.21), we have, for \( i = 1, \cdots, n \),

\[(2.24)\]

\[
D_i \tilde{B} = \frac{B_i + B_z D_i u + B_{p_k} D_{ii} u_k}{B}.
\]

It follows from (2.20) that

\[(2.25)\]

\[
\left| \frac{B_i + B_z D_i u}{B} \right| \leq C \left( \left| \frac{B_i}{B} \right| + \left| \frac{B_z}{B} \right| \right) \leq C B^{-\frac{1}{n-1}},
\]

where the constant \( C \) depends on \( n, \| B^\frac{1}{n-1} \|_{C^{1,1}} \) and \( \sup_{\bar{\Omega}} |Du| \). Since \( \tilde{u}_{ij} = u_{ij} - A_{ij} \), we obtain

\[(2.26)\]

\[
\left| \frac{B_{p_k} D_{ii} u_k}{B} \right| \leq C \left[ 1 + \max_j (|\tilde{u}_{ij}|) \right] B^{-\frac{1}{n-1}},
\]
where the constant $C$ depends on $n, \|B^{\frac{-1}{2}}\|_{C^{0,1}}$ and $A$. Combining (2.21), (2.25) and (2.26), we get (2.22) and finish the proof of conclusion (i).

Next, we turn to prove (ii). It follows from (2.6) and (2.7) that, for $i = 1, \cdots, n$,

$$\left| B_{ii} + B_{iz}(D_i u)^2 + B_z D_i u + 2B_{iz} D_j u + 2B_{ip} D_i u_l + 2B_{zp} D_i u D_j u_{ik} \right| B^{-\frac{i}{2}} \leq C \left[ 1 + \max_j (|u_{ij}|) \right] B^{-\frac{i}{2}},$$

and

$$\left| B_i^2 + B_z^2 u_i^2 + 2B_i B_z u_i + 2B_i B_p D_i u_l + 2B_z B_p u_i D_i u_{il} \right| B^{-\frac{i}{2}} \leq C \left[ 1 + \max_j (|u_{ij}|) \right] B^{-\frac{i}{2}},$$

where the constants $C$ depend on $n, \|B^{\frac{-1}{2}}\|_{C^{1,1}}$, $A$ and sup $\Omega [D u]$. By the condition (1.3), we have

$$\frac{B_{p_k} B - B_{p_k} B_p}{B^2} u_i u_{ik} \geq -C_B \delta_{ik}(\tilde{u}_{il} + A_{il})(\tilde{u}_{ik} + A_{ik}) \geq -C' \left[ 1 + \max_j (|u_{ij}|) \right]^2,$$

where $\delta_{ik}$ denotes the usual Kronecker delta, the constant $C'$ depends on $C_B$ and $A$. Taking $\xi = \epsilon_1$ in (2.14), and using (2.27), (2.28) and (2.29), we get (2.23) and finish the proof of conclusion (ii).

**Remark 2.2.** We remark that $\tilde{B} = \log B$ satisfies the condition (1.5), if it is semi-convex in $p$. The term $\sum_{k=1}^n B_{p_k} D_i u_{ik}$ on the right hand side of (2.23) can also be dealt with in the later discussion.

By the equation (1.1), we can build the relationship between $B^{-\frac{i}{2}}$ and $\sum_{i=1}^n \tilde{u}^{-1}, (\tilde{u}^{-1}) = (\tilde{u}^{-1})^{-1}$, if $\tilde{u}_{11} > 1$. Therefore, a suitable barrier function is necessary to control the term $C \sum_{i=1}^n \tilde{u}^{-1}$. We introduce the following barrier construction lemma under the A3w condition, which is a variant of Lemma 2.1(ii) in [10] when the operator $F$ is given by “log det”. Similar versions of such a lemma can also be found in [9,12].

**Lemma 2.2.** Let $u \in C^2(\bar{\Omega})$ be an elliptic solution of the equation (1.1) and $\underline{u} \in C^2(\bar{\Omega})$ be a strict subsolution of the equation (1.1) satisfying (1.9). Assume that $A \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ satisfies the A3w condition, $B \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ is a positive function satisfying (1.5). Then the inequality

$$L \left[ e^{\kappa(\underline{u} - u)} \right] \geq \epsilon_1 \sum_{i=1}^n \tilde{u}^{ii} - C,$$

holds in $\Omega$ for sufficiently large positive constant $\kappa$ and uniform positive constants $\epsilon_1$ and $C$, where

$$L = \sum_{i,j=1}^n \tilde{u}^{ij} \left( D_{ij} - \sum_{k=1}^n D_{p_k} A_{ij}(x, u, D u) D_k \right) - \sum_{k=1}^n \tilde{B}_{p_k} D_k.$$

**Proof.** Since $\underline{u}$ is a strict subsolution satisfying (1.9), by taking $F = \log \det$ in Lemma 2.1(ii) in [10], following (2.17) in [10] we have

$$L \left[ e^{\kappa(\underline{u} - u)} \right] \geq \epsilon_1 \left( \sum_{i=1}^n \tilde{u}^{ii} + 1 \right) + C \left[ \tilde{B}(\cdot, u, D u) - \tilde{B}(\cdot, u, D u) - \sum_{k=1}^n \tilde{B}_{p_k}(\cdot, u, D u) D_k (\underline{u} - u) \right].$$
for large positive constant $\kappa$ and uniform positive constant $\varepsilon_1$. By Taylor’s formula and the condition (1.3), we have

$$
\tilde{B}(\cdot, u, Du) - \tilde{B}(\cdot, u, D\bar{u}) - \sum_{k=1}^{n} \tilde{B}_{pk}(\cdot, u, D\bar{u}) D_k(\bar{u} - u)
$$

(2.33)

\[
= \frac{1}{2} \sum_{k,l=1}^{n} \tilde{B}_{pkpl}(\cdot, u, \bar{p}) D_k(\bar{u} - u) D_l(\bar{u} - u) \\
\geq -\frac{1}{2} C_B|D(\bar{u} - u)|^2,
\]

where $\bar{p} = \theta Du + (1 - \theta) D\bar{u}$ with $\theta \in (0, 1)$. Then the estimate (2.30) can be obtained by combining (2.32) and (2.33).

In Lemma 2.2, if the A3w condition holds without orthogonality, the inequality barrier inequality still holds by replacing the barrier function $e^{\kappa(u - u)}$ with $\kappa(u - u)$. Note also that if $C_B = 0$ in condition (1.3), namely $\tilde{B}$ is convex in $p$, then the barrier inequality (2.30) can be replaced by

$$
L \left[ e^{\kappa(u - u)} \right] \geq \varepsilon_1 \left( \sum_{i=1}^{n} \tilde{u}^{ii} + 1 \right),
$$

(2.34)

since the second term on the right hand side of (2.32) is nonnegative in this case.

3. Interior regularity for the DMATE (1.1) under the A3 condition

In this section, by constructing an auxiliary function, we obtain interior second order derivative estimates for the Monge-Ampère type equation (1.1) under the A3 condition and $B > 0$. We then use the estimates to obtain the interior regularity for the solution of the DMATE (1.1).

**Proof of Theorem 1.1.** We employ the auxiliary function

$$
G(x, \xi) = \eta^2(x) \tilde{u}_{11},
$$

(3.1)

where $\eta$ is a cut-off function in $\Omega$, $0 \leq \eta \leq 1$, $\tilde{u}_{11} = \tilde{u}_{ij} \xi_i \xi_j$, $\tilde{u}_{ij} = u_{ij} - A_{ij}(x, u, Du)$ and $\xi \in \mathbb{R}^n$ is a unit vector. We may assume that $G$ attains its maximum at $x_0 \in \Omega$ and $\xi = \xi_0$. Without loss of generality, we may assume $\{\tilde{u}_{ij}\}$ is diagonal at $x_0$ and $\xi = e_1$. Then the function

$$
G(x, \xi_0) = \eta^2(x) \tilde{u}_{11}
$$

attains its maximum at $x_0$. Denoting

$$
\tilde{G}(x) := \log G(x, \xi_0) = 2 \log \eta + \log \tilde{u}_{11},
$$

(3.2)

then $\tilde{G}(x)$ also attains its maximum at $x_0$. At $x_0$, we have

$$
\tilde{G}_i = 2 \frac{\eta_i}{\eta} + D_i \tilde{u}_{11} = 0,
$$

(3.3)

\[
\tilde{G}_{ij} = 2 \frac{\eta_{ij}}{\eta} - 2 \frac{\eta_i \eta_j}{\eta^2} + D_{ij} \tilde{u}_{11} - D_{ij} \tilde{u}_{11} D_{ij} \tilde{u}_{11}
\]

\[
= 2 \frac{\eta_{ij}}{\eta} - 2 \frac{\eta_i \eta_j}{\eta^2} + \frac{D_{ij} \tilde{u}_{11}}{\tilde{u}_{11}},
\]

(3.4)
for \( i, j = 1, \cdots, n \), and the matrix \( \{ \tilde{G}_{ij} \} \leq 0 \). From now on, we assume all the calculations are taken at \( x_0 \). Then it follows from \( \{ \tilde{u}^{ij} \} \geq 0 \), \( \tilde{u}_{11} \geq 0 \) and the first equality of (3.4) that

\[
0 \geq \tilde{u}_{11} L \tilde{G} = \tilde{u}_{11} \sum_{i,j=1}^{n} \tilde{u}^{ij} D_{ij} \tilde{G}
\]

(3.5)

\[
= \tilde{u}_{11} \sum_{i,j=1}^{n} \tilde{u}^{ij} \left[ 2 \frac{\eta_{ij}}{\eta} - 6 \frac{\eta_{ij}}{\eta^2} + \frac{D_{ij} \tilde{u}_{11}}{\tilde{u}_{11}} \right]
\]

\[
\geq -C \frac{\tilde{u}_{11}}{\eta^2} \sum_{i=1}^{n} \tilde{u}^{ii} + \sum_{i,j=1}^{n} \tilde{u}^{ij} D_{ij} \tilde{u}_{11},
\]

where \( L \) is the linearized operator defined in (2.31). Recalling that \( \tilde{u}_{11} = u_{11} - A_{11} \), we obtain

\[
\sum_{i,j=1}^{n} \tilde{u}^{ij} D_{ij} \tilde{u}_{11} = \sum_{i,j=1}^{n} \tilde{u}^{ij} D_{ij} (u_{11} - A_{11})
\]

(3.6)

\[
\geq \sum_{i,j,k,l=1}^{n} \tilde{u}^{ij} [u_{11ij} - (D_{pk} A_{11}) u_{kij} - (D_{pq} A_{11}) u_{qij}] - C \left( 1 + \sum_{i=1}^{n} \tilde{u}^{ii} \right),
\]

where \( C \) is a constant depending on \( A \) and \( \sup_{\Omega} |D\tilde{u}| \). By a direct computation, we have

\[
\sum_{i,j=1}^{n} \tilde{u}^{ij} u_{11ij} = \sum_{i,j=1}^{n} \tilde{u}^{ij} D_{11} u_{ij} = \sum_{i,j=1}^{n} \tilde{u}^{ij} D_{11} (\tilde{u}_{ij} + A_{ij})
\]

(3.7)

\[
\geq \sum_{i,j,k,l=1}^{n} \tilde{u}^{ij} [D_{11} \tilde{u}_{ij} + (D_{pk} A_{ij}) u_{k11} + (D_{pq} A_{ij}) u_{q11}] - C \left( 1 + \sum_{i=1}^{n} \tilde{u}^{ii} \right),
\]

where \( C \) is a constant depending on \( A \) and \( \sup_{\Omega} |D\tilde{u}| \). By differentiating equation (2.1) in the direction \( \xi \in \mathbb{R}^n \) once and twice, we get

\[
\tilde{u}^{ij} D_{\xi} \tilde{u}_{ij} = D_{\xi} B,
\]

(3.8)

and

\[
\tilde{u}^{ij} D_{\xi \xi} \tilde{u}_{ij} \geq D_{\xi \xi} B.
\]

(3.9)

Here the inequality (3.9) is obtained by using the concavity of “log det”. Inserting (3.6) and (3.7) into (3.5), we have

\[
0 \geq -C \frac{\tilde{u}_{11}}{\eta^2} \sum_{i=1}^{n} \tilde{u}^{ii} + D_{11} B - CD_{1} B + \sum_{i,j,k,l=1}^{n} \tilde{u}^{ij} [(D_{pk} A_{ij}) u_{k11} - (D_{pq} A_{ij}) u_{q11}]
\]

(3.10)

where (3.8), (3.9) and the first equality in (3.4) are used to deal with the terms \( \sum_{i,j,k=1}^{n} \tilde{u}^{ij} (D_{pq} A_{ij}) u_{q11} \), \( \sum_{i,j=1}^{n} \tilde{u}^{ij} D_{11} \tilde{u}_{ij} \) and \( \sum_{i,j,k=1}^{n} \tilde{u}^{ij} (D_{pq} A_{ij}) u_{q11} \) respectively, the terms \( -C (1 + \sum_{i=1}^{n} \tilde{u}^{ii}) \) in (3.6) and (3.7) are absorbed in the first term on the right hand side of (3.10) since we can always assume \( \tilde{u}_{11} \) and \( \sum_{i=1}^{n} \tilde{u}^{ii} \) as large as we want. Next, we estimate the last term in (3.10). Since both \( \{ \tilde{u}^{ij} \} \) and
\{\tilde{u}_{ij}\} are diagonal at \(x_0\), we get

\[
\sum_{i,j,k,l=1}^{n} \tilde{u}_{ij}[(D_{pkpr} A_{ij}) u_{ki}u_{lj} - (D_{pkpr} A_{11}) u_{ki}u_{lj}]
\]

\[
= \sum_{i\neq 1}^{n} \sum_{k,l=1}^{n} \tilde{u}_{ii}[(D_{pkpr} A_{ii}) (\tilde{u}_{k1} + A_{k1})(\tilde{u}_{l1} + A_{l1}) - D_{pkpr} A_{11} (\tilde{u}_{ki} + A_{ki})(\tilde{u}_{li} + A_{li})]
\]

\[
\geq \sum_{i\neq 1}^{n} \tilde{u}_{ii} (D_{p1p1} A_{ii}) \tilde{u}_{11}^2 - C \sum_{i=1}^{n} \tilde{u}_{ii} \tilde{u}_{11}.
\]

(3.11)

Using (2.22) and (2.23) in Corollary 2.1, then (3.10) becomes

\[
0 \geq \sum_{i\neq 1}^{n} \tilde{u}_{ii} (D_{p1p1} A_{ii}) \tilde{u}_{11}^2 - C \sum_{i=1}^{n} \tilde{u}_{ii} \tilde{u}_{11} - D_{11} \tilde{B} - CD_{11} \tilde{B}
\]

\[
\geq \sum_{i\neq 1}^{n} \tilde{u}_{ii} (D_{p1p1} A_{ii}) \tilde{u}_{11}^2 - C \tilde{u}_{11} \sum_{i=1}^{n} \tilde{u}_{ii} - C \tilde{u}_{11} \tilde{B}^{\frac{1}{n-1}} - C \tilde{u}_{11}^2 + \sum_{k=1}^{n} \tilde{B}_{pk} D_{11} u_k
\]

\[
\geq \sum_{i\neq 1}^{n} \tilde{u}_{ii} (D_{p1p1} A_{ii}) \tilde{u}_{11}^2 - C \tilde{u}_{11} \sum_{i=1}^{n} \tilde{u}_{ii} - C \tilde{u}_{11} \tilde{B}^{\frac{1}{n-1}} - C \tilde{u}_{11}^2,
\]

(3.12)

where the third order derivative term \(\sum_{k=1}^{n} \tilde{B}_{pk} D_{11} u_k\) is treated by using (2.20) and the first equality in (3.4). Note that the constant \(C\) changes from line to line in the context. Since \(\tilde{u}_{11} > 1\), we can get

\[
\sum_{i=1}^{n} \tilde{u}_{ii} \geq (n-1) \left( \prod_{i=2}^{n} \tilde{u}_{ii} \right)^{\frac{1}{n-1}}
\]

\[
= (n-1) \left( \prod_{i=1}^{n} \tilde{u}_{ii} \right)^{\frac{1}{n-1}} (\tilde{u}_{11})^{\frac{1}{n-1}}
\]

\[
\geq (n-1) B^{-\frac{1}{n-1}}.
\]

(3.13)

Plugging (3.13) into (3.12), we obtain

\[
0 \geq \sum_{i\neq 1}^{n} \tilde{u}_{ii} (D_{p1p1} A_{ii}) \tilde{u}_{11}^2 - C \tilde{u}_{11} \sum_{i=1}^{n} \tilde{u}_{ii} - C (\tilde{u}_{11})^2.
\]

(3.14)
By the A3 condition, choosing \( \tilde{\xi} = \tilde{u}_{11} e_1 \) and \( \tilde{\eta} = \frac{1}{n} \sum_{i=2}^{n} \sqrt{\tilde{u}^{ii}} e_i \), we have

\[
\sum_{i \neq 1} \tilde{u}^{ii} (D_{p_i p_i} A_{ii}) \tilde{u}^{2}_{11} = \sum_{i,j,k,l=1}^{n} D_{p_k p_l} A_{ij} \tilde{\xi} \tilde{\xi} \tilde{\eta} \tilde{\eta} \\
\geq c_0 \tilde{u}^{2}_{11} \sum_{i=2}^{n} \tilde{u}^{ii} \\
\geq \frac{1}{n} c_0 \tilde{u}^{2}_{11} \sum_{i=2}^{n} \tilde{u}^{ii} + \frac{n-1}{n} c_0 \tilde{u}^{2}_{11} \tilde{u}^{22} \\
\geq \frac{1}{n} c_0 \tilde{u}^{2}_{11} \sum_{i=2}^{n} \tilde{u}^{ii} + \frac{n-1}{n} c_0 \tilde{u}^{2}_{11} \tilde{u}^{11} \\
\geq \frac{1}{n} c_0 \tilde{u}^{2}_{11} \sum_{i=1}^{n} \tilde{u}^{ii}.
\]

(3.15)

Without loss of generality, we assume

\[
\frac{1}{2n} c_0 \sum_{i=1}^{n} \tilde{u}^{ii} \geq C.
\]

(3.16)

Otherwise we are done. Combining (3.14), (3.15) and (3.16), we have

\[
0 \geq \frac{1}{2n} c_0 \tilde{u}^{2}_{11} \sum_{i=1}^{n} \tilde{u}^{ii} - C \tilde{u}^{11} \sum_{i=1}^{n} \tilde{u}^{ii},
\]

which leads to

\[
\eta^2 \tilde{u}^{11} \leq C.
\]

(3.18)

We now complete the proof of Theorem 1.1.

Note that the constant \( C \) in (1.6) in Theorem 1.1 is independent of the positive lower bound of \( B \). Then the \( C^{1,1} \) regularity result under the A3 condition, Theorem 1.2, follows directly from the interior estimates in Theorem 1.1. Here we omit the proof of Theorem 1.2 since it is standard.

4. Interior regularity for the DMATE (1.1) under the A3w+ condition

In this section, we prove the Pogorelov type estimate in Theorem 1.3 under the A3w+ condition and suitable barrier conditions, which can be applied to the interior \( C^{1,1} \) regularity for solutions of the DMATE (1.1) in Theorem 1.4. Here we omit the proof of Theorem 1.2 since it is standard.

Proof of Theorem 1.3. First we note that under either (i) or (ii), we have

\[
L \varphi \geq \varepsilon_1 \sum_{i=1}^{n} \tilde{u}^{ii} - C,
\]

for some positive constants \( \varepsilon_1 \) and \( C \). In case (i), \( \varphi \) is the function in the \( A \)-boundedness condition (1.8), and (4.1) with \( \varepsilon_1 = 1 \) can be calculated directly from (1.8). While in case (ii), the inequality (4.1) with \( \varphi = e^{\kappa (\tilde{u} - u)} \) is proved in (2.30) in Lemma 2.2.

We construct the auxiliary function

\[
h(x, \xi) = \eta^2 \tilde{u}^{11} e_1^2 |Du|^2 + \gamma \varphi,
\]

(4.2)
where $\varphi$ is the barrier function in (4.1), $\tilde{u}_{ij} = \tilde{u}_{ij} \xi_i \xi_j$, $\xi = (\xi_1, \cdots, \xi_n)$ and $|\xi| = 1$, $\tilde{u}_{ij} = u_{ij} - A_{ij}$, $\eta = w - u$ and $\alpha, \beta, \gamma$ are positive constants to be determined.

Since $h \geq 0$ in $\Omega$ and $h = 0$ on $\partial \Omega$, we may assume that $h$ attains its maximum at the point $\bar{x} \in \Omega$ and some unit vector $\xi$. We may assume $u(\bar{x}) < w(\bar{x})$, namely $\eta(\bar{x}) > 0$. By taking the logarithm of $h$, we obtain

$$\bar{h}(x, \xi) := \log h(x, \xi) = \alpha \log \eta + \log(\tilde{u}_{11}) + \frac{1}{2} \beta |Du|^2 + \gamma \varphi.$$  

Thus, $\bar{h}$ also attains its maximum at the point $\bar{x} \in \Omega$ and the vector $\bar{\xi}$. We may assume that $\bar{\xi} = (1, 0, \cdots, 0)$ and $\{\tilde{u}_{ij}\}$ is diagonal at $\bar{x}$. We define

$$v(x) := \bar{h}(x, \xi)|_{\xi = \bar{\xi}} = \alpha \log \eta + \log(\tilde{u}_{11}) + \frac{1}{2} \beta |Du|^2 + \gamma \varphi.$$  

Since $\bar{x}$ is also the maximum point of $v$, we have

$$Dv(\bar{x}) = 0,$$

and

$$D^2v(\bar{x}) \leq 0.$$  

It follows from (4.5), (4.6) and $\{\tilde{u}_{ij}\} \geq 0$ that

$$Lv(\bar{x}) \leq 0,$$

where $L$ is the linearized operator defined in (2.31). By a direct computation, we have, at $\bar{x}$,

$$D_iv = \frac{\alpha D_i \eta}{\eta} + \frac{D_i \tilde{u}_{11}}{\tilde{u}_{11}} + \beta D_k u D_{ki} u + \gamma D_i \varphi,$$

and

$$D_{ii}v = \frac{\alpha D_{ii} \eta}{\eta} - \frac{\alpha (D_i \eta)^2}{\eta^2} + \frac{D_{ii} \tilde{u}_{11}}{\tilde{u}_{11}} - \frac{(D_i \tilde{u}_{11})^2}{\tilde{u}_{11}^2}$$

$$+ \beta \sum_{i,k=1}^{n} (D_{ik} u)^2 + (D_k u) D_{ik} u + \gamma D_{ii} \varphi,$$

for $i = 1, \cdots, n$. Inserting (4.8) and (4.9) into (4.10), we get

$$0 \geq Lv(\bar{x})$$

$$= \frac{\alpha}{\eta} L \eta - \frac{\alpha}{\eta^2} \sum_{i=1}^{n} \tilde{u}_{ii}(D_i \eta)^2 + \frac{1}{\tilde{u}_{11}} L \tilde{u}_{11} - \frac{1}{\tilde{u}_{11}^2} \sum_{i=1}^{n} \tilde{u}_{ii}(D_i \tilde{u}_{11})^2$$

$$+ \beta \sum_{k=1}^{n} D_k u L u_k + \beta \sum_{i,k=1}^{n} \tilde{u}_{ii} (D_{ik} u)^2 + \gamma L \varphi.$$  

Next, we estimate each term of (4.10). From now on, all calculations are made at the maximum point $\bar{x}$. We first consider the general case that $B$ depends on $p$, namely $B_p \neq 0$. By calculations, we
have

\[ L\eta = \sum_{i=1}^{n} \tilde{u}^{ii} \left[ D_{ii}w - \tilde{u}_{ii} - A_{ii}(x, u, Du) - \sum_{k=1}^{n} (D_{pk}A_{ii}(x, u, Du))D_k\eta \right] - \sum_{k=1}^{n} \tilde{B}_{pk}D_k\eta \]

\[
\geq - n + \sum_{i=1}^{n} \tilde{u}^{ii} \left[ A_{ii}(x, u, Dw) - \tilde{u}_{ii} - \sum_{k=1}^{n} (D_{pk}A_{ii}(x, u, Du))D_k\eta \right] - \sum_{k=1}^{n} \tilde{B}_{pk}D_k\eta
\]

(4.11)

\[
\geq - n - CB^{-\frac{2}{2(n-1)}} + \frac{1}{2} \sum_{i,k,l=1}^{n} \tilde{u}^{ii} A_{ii,kl}(x, u, \tilde{p}) D_k\eta D_l\eta
\]

\[
\geq - n - CB^{-\frac{2}{2(n-1)}} - \frac{1}{2} \mu_0 \sum_{i=1}^{n} \tilde{u}^{ii}(D_i\eta)^2,
\]

for \( \tilde{p} = (1 - \theta)Du + \theta Dw \) and \( \theta \in (0, 1) \), where \( D_{ii}w - A_{ii}(x, u, Dw) \geq 0 \) is used to obtain the first inequality, Taylor’s formula and (2.6) are used to obtain the second inequality, the A3w+ condition is used to obtain the third inequality, \( \mu_0 = -\min\{\mu_0, 0\} \) and \( \mu_0 \) is the constant in (1.3). Using the Cauchy’s inequality, it follows from (4.11) that

\[
\frac{\alpha}{\eta} L\eta \geq - \alpha \left[ \frac{n}{\eta} + \frac{C}{\eta} B^{-\frac{2}{n-1}} + \frac{\mu_0 n}{2\eta} \sum_{i=1}^{n} \tilde{u}^{ii}(D_i\eta)^2 \right]
\]

(4.12)

\[
\geq - \frac{an}{\eta} - \frac{\alpha^2 C}{\eta^2} - CB^{-\frac{2}{n-1}} \frac{\mu_0 n}{2} \sum_{i=1}^{n} \tilde{u}^{ii}(D_i\eta)^2,
\]

where we have assumed \( \eta(\bar{x}) \in (0, 1) \). We will show the trivial case when \( \eta(\bar{x}) > 1 \) at the end of the proof.

In order to estimate \( \frac{1}{u_{11}} L\tilde{u}_{11} \), we first calculate \( Lu_{11} \). We can assume \( \tilde{u}_{11} \geq 1 \), otherwise we are done. By a direct computation and using (2.3) with \( \xi = e_1 \), we have

\[
Lu_{11} \geq \sum_{i=1}^{n} \tilde{u}^{ii} \tilde{u}^{jj} (D_1 \tilde{u}_{ij})^2 + \sum_{i,k,l=1}^{n} \tilde{u}^{ii} A_{ii,kl} u_{k1} u_{l1} + D_{11} \tilde{B} - \tilde{B}_{pk} D_k u_{11} - C \sum_{i,j=1}^{n} [(1 + \tilde{u}_{jj}) \tilde{u}^{ii}]
\]

(4.13)

\[
\geq \sum_{i=1}^{n} \tilde{u}^{ii} \tilde{u}^{jj} (D_1 \tilde{u}_{ij})^2 - C \sum_{i=1}^{n} \tilde{u}^{ii} + D_{11} \tilde{B} - \tilde{B}_{pk} D_k u_{11} - C \sum_{i,j=1}^{n} [(1 + \tilde{u}_{jj}) \tilde{u}^{ii}],
\]

where the A3w+ condition is used to obtain the second inequality. With the help of (2.23) in Corollary 2.1 we can further get

\[
Lu_{11} \geq \sum_{i,j=1}^{n} \tilde{u}^{ii} \tilde{u}^{jj} (D_1 \tilde{u}_{ij})^2 - C \sum_{i=1}^{n} \tilde{u}^{ii} - C \sum_{i,j=1}^{n} [(1 + \tilde{u}_{jj}) \tilde{u}^{ii}]
\]

(4.14)

\[
- C(1 + \tilde{u}_{11}) B^{-\frac{1}{1-\alpha}} - C(1 + \tilde{u}_{11})^2
\]

\[
\geq \sum_{i,j=1}^{n} \tilde{u}^{ii} \tilde{u}^{jj} (D_1 \tilde{u}_{ij})^2 - C \sum_{i,j=1}^{n} \tilde{u}^{jj} \tilde{u}^{ii} - C \tilde{u}_{11} B^{-\frac{1}{1-\alpha}} - C \tilde{u}_{11}^2,
\]

where we assume \( \tilde{u}_{11} \geq 1 \) and \( \sum_{i=1}^{n} \tilde{u}^{ii} \geq 1 \) to obtain the second inequality. Note that the third order term \( -\tilde{B}_{pk} D_k u_{11} \) in (4.11) is eliminated by the last term of (2.23). Next, we calculate \( LA_{11} \). Using
the definition of \( L, \tilde{u}_{ij} = u_{ij} - A_{ij} \) and the \( C^2 \) smoothness of \( A \), we obtain

\[
LA_{11} \leq C + C \sum_{i,j=1}^{n} [(1 + \tilde{u}_{jj})\tilde{u}^{ii}] + \sum_{i,j,k,l=1}^{n} \tilde{u}^{ij} D_{p_k p_l} A_{11} \tilde{u}_{ki} \tilde{u}_{kj} + D_k \tilde{B}
\]

\[
\leq C \sum_{i,j=1}^{n} [(1 + \tilde{u}_{jj})\tilde{u}^{ii}] + \tilde{A}_{11} \tilde{B}
\]

(4.15)

\[
\leq C \sum_{i,j=1}^{n} \tilde{u}_{jj} \tilde{u}^{ii} + C \tilde{u}_{11} B^{-\frac{1}{n-1}},
\]

where we again assume \( \tilde{u}_{11} \geq 1 \) and \( \sum_{i=1}^{n} \tilde{u}^{ii} \geq 1 \). Recalling \( \tilde{u}_{11} = u_{11} - A_{11} \), we get from (4.14) and (4.15) that

(4.16)

\[
L \tilde{u}_{11} \geq \sum_{i,j=1}^{n} \tilde{u}^{ii} \tilde{u}^{jj} (D_1 \tilde{u}_{ij})^2 - C \sum_{i,j=1}^{n} \tilde{u}_{jj} \tilde{u}^{ii} - C \tilde{u}_{11} B^{-\frac{1}{n-1}} - C \tilde{u}_{11}^2.
\]

Therefore, we have

(4.17)

\[
\frac{1}{\tilde{u}_{11}} L \tilde{u}_{11} \geq \frac{1}{\tilde{u}_{11}} \sum_{i,j=1}^{n} \tilde{u}^{ii} \tilde{u}^{jj} (D_1 \tilde{u}_{ij})^2 - C \sum_{i=1}^{n} (\tilde{u}^{ii} + \tilde{u}_{ii}) - CB^{-\frac{1}{n-1}}.
\]

Choosing \( \xi = e_k \) in (2.2), we have

(4.18)

\[
Lu_k = \sum_{i=1}^{n} \tilde{u}^{ii} \left[ D_{ii} u_k - \sum_{l=1}^{n} (D_{p_l A_{ii}}) D_l u_k \right] - \sum_{l=1}^{n} \frac{B_{p_l}}{B} u_k
\]

\[
= \sum_{i=1}^{n} \tilde{u}^{ii} D_k A_{ii} + \sum_{i=1}^{n} \tilde{u}^{ii} (D_A u_{ij}) u_k + \sum_{k \neq i} \frac{B_k}{B} u_k,
\]

for \( k = 1, \cdots, n \). Hence, we have

(4.19)

\[
\beta \sum_{k=1}^{n} D_k u L u_k \geq -\beta C \sum_{i=1}^{n} \tilde{u}^{ii} - \beta C B^{-\frac{1}{n-1}}.
\]

By a direct calculation, we have

(4.20)

\[
\beta \sum_{i,k=1}^{n} \tilde{u}^{ii} (D_{ik} u)^2 = \beta \sum_{i=1}^{n} \tilde{u}^{ii} (\tilde{u}_{ii} + A_{ii})^2 + \beta \sum_{k \neq i} \tilde{u}^{ii} A_{ik}^2
\]

\[
\geq \beta \sum_{i=1}^{n} \tilde{u}_{ii} - \beta C \sum_{i=1}^{n} \tilde{u}^{ii}.
\]

From the barrier inequality (4.1) in both cases (i) and (ii), we can also have

(4.21)

\[
\gamma L \varphi \geq \frac{1}{2} \sum_{i=1}^{n} \tilde{u}^{ii},
\]
by assuming $\sum_{i=1}^n \tilde{u}_{ii} \geq \frac{2C}{\varepsilon_1}$. Now choosing $\alpha \geq 1$ and $\beta \geq 1$ and inserting (4.12), (4.17), (4.19), (4.20) and (4.21) into (4.10), we obtain
\begin{equation}
0 \geq -\frac{\alpha^2 C}{\eta^2} - \beta CB - \frac{1}{\varepsilon_1} + \left(\frac{1}{2} \gamma \varepsilon_1 - \beta C\right) \sum_{i=1}^n \tilde{u}_{ii} + (\beta - C) \sum_{i=1}^n \tilde{u}_{ii} = \sum_{i=1}^n \tilde{u}_{ii}
\end{equation}
(4.22)

\begin{equation}
-\alpha C \sum_{i=1}^n \tilde{u}_{ii} (D_i \eta)^2 \frac{\eta^2}{\tilde{u}_{i1}} + \frac{1}{\tilde{u}_{i1}} \sum_{i,j=1}^n \tilde{u}_{ii} \tilde{u}_{ij} (D_i \tilde{u}_{ij})^2 - \frac{1}{\tilde{u}_{i1}} \sum_{i=1}^n \tilde{u}_{ii} (D_i \tilde{u}_{i1})^2.
\end{equation}

Splitting $\sum_{i=1}^n \tilde{u}_{ii} (D_i \eta)^2 \frac{\eta^2}{\tilde{u}_{i1}}$ into two parts, we have
\begin{equation}
\sum_{i=1}^n \tilde{u}_{ii} (D_i \eta)^2 \frac{\eta^2}{\tilde{u}_{i1}} = (D_i \eta)^2 \frac{\eta^2}{\tilde{u}_{i1}} + \sum_{i=1}^n \tilde{u}_{ii} (D_i \eta)^2 \frac{\eta^2}{\tilde{u}_{i1}}.
\end{equation}
(4.23)

Observing that the first term on the right hand side of (4.23) can be absorbed by the first term on the right hand side of (4.22), we only need to estimate the last term in (4.23). From (1.30) and (4.8), we have
\begin{equation}
\alpha C \sum_{i=1}^n \tilde{u}_{ii} (D_i \eta)^2 \frac{\eta^2}{\tilde{u}_{i1}} = \alpha C \sum_{i=1}^n \tilde{u}_{ii} \left\{ \frac{1}{\alpha^2} \left[ \frac{D_i (\tilde{u}_{i1})}{\tilde{u}_{i1}} + \beta D_k u (\tilde{u}_{ki} - A_{ki}) + \gamma D_i \varphi \right] \right\}^2
\end{equation}
(4.24)

\begin{equation}
\leq \frac{C}{\alpha} \sum_{i=1}^n \tilde{u}_{ii} \left\{ \left( \frac{D_i (\tilde{u}_{i1})}{\tilde{u}_{i1}} \right)^2 + \beta^2 (\tilde{u}_{ii}^2 + 1) + \gamma^2 (D_i \varphi)^2 \right\}
\end{equation}
\begin{equation}
\leq \frac{1}{2 \tilde{u}_{i1}} \sum_{i=1}^n \tilde{u}_{ii} (D_i \tilde{u}_{i1})^2 + \frac{n}{\tilde{u}_{i1}} \sum_{i=1}^n \tilde{u}_{ii} + \tilde{u}_{ii},
\end{equation}

where we choose $\alpha = (\beta^2 + \gamma^2 + 2)C$. Thus, from (4.22), (4.23) and (4.24), we have
\begin{equation}
0 \geq -\frac{\alpha^2 C}{\eta^2} - \beta CB - \frac{1}{\varepsilon_1} + \left(\frac{1}{2} \gamma \varepsilon_1 - \beta C\right) \sum_{i=1}^n \tilde{u}_{ii} + (\beta - C) \sum_{i=1}^n \tilde{u}_{ii}
\end{equation}
(4.25)

\begin{equation}
-\frac{1}{2 \tilde{u}_{i1}} \sum_{i=1}^n \tilde{u}_{ii} (D_i \tilde{u}_{i1})^2 + \frac{n}{\tilde{u}_{i1}} \sum_{i,j=1}^n \tilde{u}_{ii} \tilde{u}_{ij} (D_i \tilde{u}_{ij})^2 - \frac{1}{\tilde{u}_{i1}} \sum_{i=1}^n \tilde{u}_{ii} (D_i \tilde{u}_{i1})^2.
\end{equation}

Using the Pogorelov term $\sum_{i,j=1}^n \tilde{u}_{ii} \tilde{u}_{ij} (D_i \tilde{u}_{ij})^2$, we have
\begin{equation}
-\frac{1}{2 \tilde{u}_{i1}} \sum_{i=1}^n \tilde{u}_{ii} (D_i \tilde{u}_{i1})^2 + \frac{n}{\tilde{u}_{i1}} \sum_{i,j=1}^n \tilde{u}_{ii} \tilde{u}_{ij} (D_i \tilde{u}_{ij})^2 - \frac{1}{\tilde{u}_{i1}} \sum_{i=1}^n \tilde{u}_{ii} (D_i \tilde{u}_{i1})^2
\end{equation}
(4.26)

\begin{equation}
\geq \frac{1}{2 \tilde{u}_{i1}} \sum_{i=1}^n \tilde{u}_{ii} (D_i \tilde{u}_{i1})^2 + \frac{2}{\tilde{u}_{i1}} \sum_{i=1}^n \tilde{u}_{ii} [(D_i \tilde{u}_{i1})^2 - (D_i \tilde{u}_{i1})^2]
\end{equation}
\begin{equation}
\geq \frac{1}{2 \tilde{u}_{i1}} \sum_{i=1}^n \tilde{u}_{ii} (D_i \tilde{u}_{i1})^2 + \frac{2}{\tilde{u}_{i1}} \sum_{i=1}^n \tilde{u}_{ii} (D_i A_{1i} - D_i A_{1i}) (2D_i \tilde{u}_{i1} + D_i A_{1i} - D_i A_{1i})
\end{equation}
\begin{equation}
\geq -\frac{C}{\tilde{u}_{i1}^2} \sum_{i=1}^n \tilde{u}_{ii} \geq -C \sum_{i=1}^n \tilde{u}_{ii},
\end{equation}
where Cauchy’s inequality is used in the second last inequality. Therefore, from (4.25) and (4.26), we have

\[
0 \geq -\frac{\alpha^2 C}{\eta^2} - \beta CB^{-\frac{1}{n-1}} + \left(\frac{1}{2} \gamma \varepsilon_1 - \beta C\right) \sum_{i=1}^{n} \tilde{u}_{ii} + (\beta - C) \sum_{i=1}^{n} \tilde{u}_{ii}.
\]

By using the key relationship (3.13) between \(B^{-\frac{1}{n-1}}\) and \(\sum_{i=1}^{n} \tilde{u}_{ii}\), we have from (4.27) that

\[
0 \geq -\frac{\alpha^2 C}{\eta^2} + \left(\frac{1}{2} \gamma \varepsilon_1 - \beta C\right) \sum_{i=1}^{n} \tilde{u}_{ii} + (\beta - C) \sum_{i=1}^{n} \tilde{u}_{ii}.
\]

By choosing \(\beta = C + 1\) and \(\gamma = \frac{2\beta C}{\varepsilon_1}\), (4.28) becomes

\[
0 \geq -\frac{\alpha^2 C}{\eta^2} + n \sum_{i=1}^{n} \tilde{u}_{ii} \geq -\frac{\alpha^2 C}{\eta^2} + \tilde{u}_{11},
\]

which leads to

\[
\eta^2 \tilde{u}_{11}(\bar{x}) \leq \alpha^2 C.
\]

We then immediately get the conclusion (1.10) in the \(B_p \not\equiv 0\) case.

For the case \(B_p \equiv 0\) or more general case \(|B_p| \leq C\), (4.12) can be replaced by

\[
\frac{\alpha}{\eta} L\eta \geq -\frac{\alpha C}{\eta} - C'B^{-\frac{1}{n-1}} - \frac{\alpha \mu_0}{2} \sum_{i=1}^{n} \tilde{u}_{ii} \frac{(D_i \eta)^2}{\eta^2},
\]

(the constant \(C' = 0\) when \(B_p \equiv 0\)), and correspondingly, (4.29) can be replaced by

\[
0 \geq -\frac{\alpha C}{\eta} + \tilde{u}_{11},
\]

which leads to

\[
\eta \tilde{u}_{11}(\bar{x}) \leq \alpha C.
\]

We then immediately get the conclusion (1.10) in the \(B_p \equiv 0\) case or more general \(|B_p| \leq C\) case. Then Theorem 1.3 is proved provided \(\eta(\bar{x}) \in (0, 1]\).

While if \(\eta(\bar{x}) > 1\), (4.12) still holds. Furthermore, \(\eta\) in the denominators on the right hand side of (4.12) can be replaced by 1. Following the above proof, we can have

\[
\tilde{u}_{11}(\bar{x}) \leq C,
\]

which also leads to the conclusion (1.10).

We now complete the proof of Theorem 1.3.

Remark 4.1. In the above proof, the A3w+ condition is crucial in the critical inequality (4.11), which is the reason why we restrict our study in the class of A satisfying A3w+. Alternative conditions to get through the inequality (4.11) can be found in (2.4), Remark 2.1 and Remark 2.2 in [15]. Note that the inequality (4.13), which is deduced from the A3w+ condition, can also be derived by just using the A3w condition and some other conditions, see [14, 15].

We are now ready to prove Theorem 1.4.

The proof of Theorem 1.4 Let \(\Omega_j\) be a sequence of \(C^\infty\) bounded domains such that \(\Omega_j \to \Omega\) as \(j \to \infty\). Note that if in case (i), these domains also need to satisfy the A-boundedness condition. We can find \(B_j \in C^\infty\) such that \(B_j > 0, B_j\) tends uniformly to \(B\) in \(\Omega\) and \(\|B_j\|_{C^{1,1}(\Omega_j \times \mathbb{R} \times \mathbb{R}^n)} \leq C\) for some uniform constant \(C\), (independent of \(j\)). From the existence result in [12], the Dirichlet problem \(\det(M[u_j]) = B_j\) in \(\Omega_j, u_j = w\) on \(\partial \Omega_j\), has a unique classical solution \(u_j \in C^3(\bar{\Omega}_j)\).
Since $A$ and $B$ are nondecreasing in $z$, from the strong maximum principle, either $u \equiv w$ in $\Omega$ or $u < w$ in $\Omega$. In the former case, since $w \in C^{1,1}(\Omega)$, we immediately have $u \in C^{1,1}(\Omega)$. Next, we only consider the latter case when $u < w$ in $\Omega$. Since $u_j$ is a degenerate elliptic solution, we can have the uniform gradient estimate from [13]. By applying the Pogorelov type estimate (1.10) in the domain $\{u_j < w - \varepsilon\}$ for any fixed small constant $\varepsilon > 0$, we have

$$\tag{4.35} (w - u_j - \varepsilon)^\top |D^2 u_j| \leq C, \quad \text{in } \{u_j < w - \varepsilon\},$$

where the constant $C$ is independent of $j$. Thus, we have

$$\tag{4.36} |D^2 u_j| \leq C, \quad \text{in } \{u_j < w - 2\varepsilon\},$$

where the constant $C$ is independent of $j$. From the stability property of viscosity solutions [2], we have $u_j \to u$ as $j \to \infty$, and

$$\tag{4.37} u \in C^{1,1}(\{u < w - 2\varepsilon\}),$$

for any fixed small constant $\varepsilon > 0$. Since the domain $\{u < w - 2\varepsilon\}$ tends to $\Omega = \{u < w\}$ as $\varepsilon$ to 0, from (137), we finally get $u \in C^{1,1}(\Omega)$. □

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