Orthospectra of Geodesic Laminations and Dilogarithm Identities on Moduli Space

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Abstract

Given a measured lamination $\lambda$ on a finite area hyperbolic surface we consider a natural measure $M_\lambda$ on the real line obtained by taking the push-forward of the volume measure of the unit tangent bundle of the surface under an intersection function associated with the lamination. We show that the measure $M_\lambda$ gives summation identities for the Rogers dilogarithm function on the moduli space of a surface.

1 Introduction

Let $S$ be a closed hyperbolic surface and $\lambda$ a geodesic lamination on $S$. We let $\Omega$ be the volume measure on the unit tangent bundle $T_1(S)$. We let $\alpha(v)$ be the longest geodesic arc containing $v$ as a tangent vector and which does not intersect $\lambda$ transversely in its interior. Generically $\alpha(v)$ will be a geodesic arc with endpoints on $\lambda$.

We define the function $L : T_1(S) \to \overline{\mathbb{R}}$ by letting $L(v) = \text{Length}(\alpha(v))$. We note that $L(v)$ is measurable but can be infinite. We define measure $M_\lambda$ on the real line by $M_\lambda = L_*\Omega$. Then $M_\lambda$ is a measure describing the distribution of the lengths of $\alpha(v)$.

We cut $S$ along $\lambda$ to obtain a surface with boundary denoted $S_\lambda$. A $\lambda-$cusp of $S$ is an ideal vertex of a component of $S_\lambda$. We let $N_\lambda$ be the number of $\lambda-$cusps of $S$. We denote by $\{\alpha_i\}$ the geodesic arcs in $S_\lambda$ which have endpoints perpendicular to $\partial S_\lambda \subseteq \lambda$ and denote the length of $\alpha_i$ by $l_i$. We note that if a component of $S_\lambda$ is an ideal $k-$gon then there are a finite number of geodesics $\alpha_i$ in this component. Otherwise there are an infinite number. We call the set $\{l_i\}$ (with multiplicities) the $\lambda-$orthospectrum. By doubling $S - \lambda$ we see that the $\lambda-$orthospectrum corresponds to a subset of the closed geodesics of a finite area surface and therefore is a countable set.

We prove the following length spectrum identity

$$\sum_i L\left(\frac{1}{\cosh^2 \frac{l_i}{2}}\right) = \frac{\pi^2}{12}(6|\chi(S)| - N_\lambda)$$

(1)

where $L$ is a Rogers dilogarithm function (described below).
2 Dilogarithms and Polylogarithms

The $k^{th}$ polylogarithm function $\text{Li}_k$ is defined by the Taylor series

$$\text{Li}_k(z) = \sum_{i=1}^{\infty} \frac{z^n}{n^k}$$

for $|z| < 1$ and by analytic continuation to $C$. In particular

$$\text{Li}_0(z) = \frac{1}{1-z}, \quad \text{Li}_1(z) = -\log(1-z).$$

Also

$$\text{Li}'_k(z) = \frac{\text{Li}_{k-1}(z)}{z}, \quad \text{giving} \quad \text{Li}_k(z) = \int_0^z \frac{\text{Li}_{k-1}(z)}{z} \, dz.$$

Also the functions $\text{Li}_k$ are related to the Riemann $\zeta$ function by $\text{Li}_k(1) = \zeta(k)$.

The dilogarithm function is the function $\text{Li}_2(z)$ and is given by

$$\text{Li}_2(z) = -\int_0^z \frac{\log(1-z)}{z} \, dz.$$

Below is a brief description of some properties of the dilogarithm function. They can all be found in 1991 survey "Structural Properties of Polylogarithms" by L. Lewin (see [3]). From the power series representation, it is easy to see that the dilogarithm function satisfies the functional equation

$$\text{Li}_2(z) + \text{Li}_2(-z) = \frac{1}{2} \text{Li}_2(z^2).$$

Other functional relations of the dilogarithm can be best described by normalizing the dilogarithm function. The (extended) Rogers $\mathcal{L}$–function (see [5]) is defined by

$$\mathcal{L}(x) = \text{Li}_2(x) + \frac{1}{2} \log |x| \log(1-x) \quad x \leq 1.$$

In terms of the Rogers $\mathcal{L}$–function, Euler’s reflection relations for the dilogarithm are

$$\mathcal{L}(x) + \mathcal{L}(1-x) = \mathcal{L}(1) = \frac{\pi^2}{6} \quad 0 \leq x \leq 1$$

$$\mathcal{L}(-x) + \mathcal{L}(-x^{-1}) = 2\mathcal{L}(-1) = -\frac{\pi^2}{6} \quad x > 0 \quad (2)$$

Also in terms of $\mathcal{L}$, Landen’s identity is

$$\mathcal{L} \left( \frac{-x}{1-x} \right) = -\mathcal{L}(x) \quad 0 < x < 1 \quad (3)$$

and Abel’s functional equation is

$$\mathcal{L}(x) + \mathcal{L}(y) = \mathcal{L}(xy) + \mathcal{L} \left( \frac{x(1-y)}{1-xy} \right) + \mathcal{L} \left( \frac{y(1-x)}{1-xy} \right) \quad (4)$$

Also a closed form for $\mathcal{L}(x)$ is known for certain values of $x$ including

$$\mathcal{L}(1) = \frac{\pi^2}{6} \quad \mathcal{L} \left( \frac{1}{2} \right) = \frac{\pi^2}{12} \quad \text{(Euler)} \quad \mathcal{L}(\phi^{-1}) = \frac{\pi^2}{10} \quad \mathcal{L}(1 - \phi^{-1}) = \frac{\pi^2}{15} \quad \text{(Landen)}$$

where $\phi$ is the golden ratio.
3 Statement of Results

The main result of the paper is the following;

Main Theorem There exists a function $\rho : \mathbb{R}^2 \to \mathbb{R}$ such that infinitesimally

$$dM_\lambda = \left(\frac{4N_\lambda x^2}{\sinh^2 x} + \sum_i \rho(l_i, x)\right)dx$$

where $N_\lambda$ is the number of $\lambda$-cusps of $S$. Furthermore the total mass of the measure $\rho(l, x)dx$ on the real line is given by

$$F(l) = \int_0^\infty \rho(l, x) \, dx = 8\mathcal{L}\left(\frac{1}{\cosh^2 \frac{l}{2}}\right)$$

In particular the measure $M_\lambda$ depends only on the $\lambda$-orthospectrum.

4 Length Spectrum Identity

As $M_\lambda = L_*\Omega$, $M_\lambda$ has total mass equal to the volume of $T_1(S)$. Therefore $M_\lambda(\mathbb{R}) = \Omega(T_1(S)) = 4\pi^2|\chi(S)|$. Summing up the masses of measures in the Main Theorem we immediately obtain the following.

Length Spectrum Identity Theorem Let $\lambda$ be a geodesic lamination on a finite area hyperbolic surface $S$. Then the $\lambda$-orthospectrum satisfies the following

$$\sum_i \mathcal{L}\left(\frac{1}{\cosh^2 \frac{l_i}{2}}\right) = \frac{\pi^2(6|\chi(S)| - N_\lambda)}{12}$$

or equivalently

$$\sum_i \mathcal{L}\left(-\frac{1}{\sinh^2 \frac{l_i}{2}}\right) = \frac{\pi^2(6\chi(S) + N_\lambda)}{12}$$

By Landen’s identity (see equation 3) we have

$$\mathcal{L}\left(\frac{1}{\cosh^2 \frac{l}{2}}\right) = -\mathcal{L}\left(-\frac{1}{\sinh^2 \frac{l}{2}}\right).$$

Thus we can see that the second form of the Length Spectrum Identity corresponds to the first via Landen’s identity.
5 Length Spectrum Identity on Moduli Space

We note that if $S$ is a connected hyperbolic surface of finite area with non-empty geodesic boundary, letting $\lambda = \partial S$ then the Length Spectrum Identity gives a summation identity on the Moduli space $\text{Mod}(S)$ of $S$. In this case the Euler characteristic $\chi(S)$ can be a fraction and is defined such that $2\pi\chi(S)$ is the negative of the area of $S$. This relation is an infinite relation except in the case when $S$ is an ideal polygon. In this case we will show that these finite identities include the classical dilogarithm identities described above.

5.1 Classical Identities and the Moduli space of ideal polygons

For $S$ an ideal n-gon, the Length Spectrum Identity is a finite summation relation. We will show that the associated relations give an infinite list of finite relations including the classical identities stated in the previous section.

If $\{l_i\}$ is a $\lambda$–orthospectrum, we will define two parameterizations by letting

$$a_i = -\frac{1}{\sinh^2 \frac{l_i}{2}}, \quad b_i = \frac{1}{\cosh^2 \frac{l_i}{2}}.$$ 

We now consider the Poincaré disk model and let $x_i, i = 1, \ldots, n$ be the vertices in anticlockwise cyclic ordering around the circle. Let $s_i$ be the side $x_i x_{i+1}$. Let $l_{ij}$ be the length of the diagonal between $s_i$ and $s_j$ for $|i - j| \geq 2$. We define the cross-ratio by

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_3)(z_4 - z_2)}.$$

As the cross ratio is invariant under Möbius transformations, we map the quadruple $(x_i, x_{i+1}, x_j, x_{j+1})$ to $(-1, 1, e^{l_{ij}}, -e^{l_{ij}})$. Then

$$[x_i, x_{i+1}, x_j, x_{j+1}] = [-1, 1, e^{l_{ij}}, -e^{l_{ij}}] = \frac{-1 - 1}{(1 - e^{l_{ij}})(1 - e^{l_{ij}})} = \frac{4e^{l_{ij}}}{(e^{l_{ij}} + 1)^2} = \frac{1}{\cosh^2 \frac{l_{ij}}{2}}.$$

As $S$ has area $(n-2)\pi$ and $n$ cusps, $\chi(S) = (n-2)/2$ and $N_\lambda = n$. Thus the Length Spectrum identity becomes

$$\sum_{i,j} L([x_i, x_{i+1}, x_j, x_{j+1}]) = \frac{(n-3)\pi^2}{6}$$

where the sum is over all ordered pairs $i, j$ such that the sides $s_i, s_j$ are disjoint (at infinity). In terms of dilogarithms we get

$$\sum_{i,j} \text{Li}_2([x_i, x_{i+1}, x_j, x_{j+1}]) = \frac{(n-3)\pi^2}{6} - \frac{1}{2} \sum_{i,j} \log (1 - [x_i, x_{i+1}, x_j, x_{j+1}]) \log ([x_i, x_{i+1}, x_j, x_{j+1}])$$

(6)
5.2 Some Cases

**Quadrilateral:** The ideal quadrilateral has 4 cusps and two ortholengths \( l_1, l_2 \). By elementary hyperbolic geometry we have \( \sinh(l_1/2) \sinh(l_2/2) = 1 \). Therefore \( a_1 a_2 = 1 \) and letting \( a = a_1 \) the Length Spectrum identity is equivalent to the classical reflection identity of Euler.

\[
\mathcal{L}(a) + \mathcal{L}(a^{-1}) = -\frac{\pi^2}{6}.
\]  

(7)

Also we have

\[
b_2 = \frac{1}{\cosh^2(l_2/2)} = \frac{1}{1 + \sinh^2(l_2/2)} = \frac{1}{1 + \frac{1}{\sinh^2(l_1/2)}} = \frac{\sinh^2(l_1/2)}{\cosh^2(l_1/2)} = 1 - \frac{1}{\cosh^2(l_1/2)} = 1 - b_1
\]

Thus letting \( b = b_1 \), the Length Spectrum identity is equivalent to the Euler reflection identity

\[
\mathcal{L}(b) + \mathcal{L}(1 - b) = \frac{\pi^2}{6}.
\]  

(8)

**Pentagon and Abel’s Identity:** If we choose a general ideal pentagon then there are 5 diagonals and therefore 5 parameters \( a_i \). We send three of the vertices to 0, 1, \( \infty \) and the other two to \( u, v \) with \( 0 < u < v < 1 \). Then the cross ratios in terms of \( u, v \) are

\[
u, 1 - v, \quad \frac{v - u}{v}, \quad \frac{v - u}{1 - u}, \quad \frac{u(1 - v)}{v(1 - u)}.
\]

Putting into the equation we obtain the following equation.

\[
\mathcal{L}(u) + \mathcal{L}(1 - v) + \mathcal{L}\left(\frac{v - u}{v}\right) + \mathcal{L}\left(\frac{v - u}{1 - u}\right) + \mathcal{L}\left(\frac{u(1 - v)}{v(1 - u)}\right) = \frac{\pi^2}{3}.
\]  

(9)

Letting \( x = u/v, y = v \), then we get

\[
\mathcal{L}(xy) + \mathcal{L}(1 - y) + \mathcal{L}(1 - x) + \mathcal{L}\left(\frac{y(1 - x)}{1 - xy}\right) + \mathcal{L}\left(\frac{x(1 - y)}{1 - xy}\right) = \frac{\pi^2}{3}.
\]  

(10)

Now by applying Euler’s reflection identities for \( x, y \), we obtain Abel’s identity for the Rogers \( \mathcal{L} \)–function.

\[
\mathcal{L}(x) + \mathcal{L}(y) = \mathcal{L}(xy) + \mathcal{L}\left(\frac{y(1 - x)}{1 - xy}\right) + \mathcal{L}\left(\frac{x(1 - y)}{1 - xy}\right).
\]  

(11)

**General equation:** We obtain similar finite identities in the general ideal \( n \)--gon case. In general we note that equation \( [5] \) will have \( (n - 3) \) independent variables and will be given by the summation of evaluating \( \mathcal{L} \) on \( \frac{n(n - 3)}{2} \) rational functions in the \( (n - 3) \) variables.
5.3 Regular Ideal n-gon relation

We now consider the dilogarithm equation for the specific case of a regular ideal n-gon. In this case the cross ratios can be calculated and the dilogarithm formulas for specific values of the dilogarithm function.

We consider a regular ideal \( n \)-gon in with center 0 in the Poincaré disk model and vertices at \( v_k = u_k, k = 0, \ldots, n - 1 \) for \( u = e^{2\pi i/n} \). Then equation 5 can be thought of as an equation on the roots of the polynomial \( z^n = 1 \). We have

\[
[v_0, v_1, v_r, v_{r+1}] = -\frac{(1 - u)(u^{r+1} - u^r)}{(1 - u^r)(u^{r+1} - u)} = \frac{u^r(u - 1)^2}{u.(u^r - 1)^2} = \frac{\sin^2\left(\frac{\pi}{n}\right)}{\sin^2\left(\frac{r\pi}{n}\right)}
\]

For \( r < n/2 \) there are exactly \( n \) distinct perpendiculars between sides separated by \( r \) sides and for \( r = n/2 \) there are \( n/2 \) such sides. To take care of the even and odd case simultaneously we let \( e_n \) be 1 if \( n \) is even and 0 if \( n \) is odd. Therefore we have

\[
\sum_{r=2}^{[n/2]-1} n.L\left(\frac{\sin^2\left(\frac{\pi}{n}\right)}{\sin^2\left(\frac{r\pi}{n}\right)}\right) + e_n n/2.L\left(\sin^2\left(\frac{\pi}{n}\right)\right) = \frac{(n - 3)\pi^2}{6n}
\]

(12)

Limiting case: We let \( n \) go to infinity and obtain the equation

\[
\lim_{n \to \infty} \sum_{r=2}^{[n/2]-1} L\left(\frac{\sin^2\left(\frac{\pi}{n}\right)}{\sin^2\left(\frac{r\pi}{n}\right)}\right) + e_n n/2.L\left(\sin^2\frac{\pi}{n}\right) = \lim_{n \to \infty} \frac{(n - 3)\pi^2}{6n} = \frac{\pi^2}{6}
\]

This gives a Rogers \( L \)-function series relation due to Lewin (see p. 298 of [3])

\[
\sum_{r=2}^{\infty} L\left(\frac{1}{r^2}\right) = \frac{\pi^2}{6}
\]

Regular ideal quadrilateral: This case is trivial \( a_1 = a_2 = -1, b_1 = b_2 = 1/2 \) and equations 7,8 give the classical evaluations

\[ L(-1) = -\frac{\pi^2}{12} \quad \text{and} \quad L\left(\frac{1}{2}\right) = \frac{\pi^2}{12}. \]

Regular ideal pentagon, Golden Mean: For the regular ideal pentagon, the orthospectrum consists of 5 geodesics each of the same length \( l \). Using the formula above for \( n = 5, r = 2 \) we obtain that \( l \) satisfies

\[
cosh^2\left(\frac{l}{2}\right) = \frac{2}{\sqrt{5} + 3} = \phi^2
\]
where $\phi$ is the golden mean. Therefore as $\phi^2 = \phi + 1$

$$\sinh^2\left(\frac{l}{2}\right) = \phi^2 - 1 = \phi$$

and we have $a = -\phi^{-1}$. Thus the Length Spectrum Identity gives the classical relations of Landen

$$\mathcal{L}(-\phi^{-1}) = -\frac{\pi^2}{15}, \quad \mathcal{L}(\phi^{-2}) = \frac{\pi^2}{15}.$$ 

Applying the quadrilateral relations \[7, 8\] we also get

$$\mathcal{L}(\phi) = -\frac{\pi^2}{6} - \mathcal{L}(-\phi^{-1}) = -\frac{\pi^2}{10}.$$ 

**Regular ideal Hexagon:** For a regular ideal hexagon, there are 9 elements of the orthospectrum, with the 6 being perpendicular to sides one apart and three being perpendicular to opposite sides. Putting $n = 6$ into equation \[12\] above then gives

$$6\mathcal{L}(\frac{1}{3}) + 3\mathcal{L}(\frac{1}{4}) = \frac{\pi^2}{2}.$$

Before we prove the main theorem, we first consider the geometry of ideal quadrilaterals in the hyperbolic plane.

### 6 Intersections with ideal quadrilaterals

Given two disjoint geodesics $g_1, g_2$ with perpendicular distance $l$ between them, let $Q$ be the ideal quadrilateral with opposite sides $g_1, g_2$. Then we can map $Q$ by a M"obius transformation to the ideal quadrilateral $Q_a$ in the upper half-plane with vertices $a, 0, 1, \infty \in \mathbb{R}$ where $a < 0$. Similarly we can map $Q$ to the ideal quadrilateral $Q_b$ in the upper half-plane with vertices $0, b, 1, \infty \in \mathbb{R}$ where $b > 0$. Using cross-ratios we have that

$$a = -\frac{1}{\sinh^2 \frac{l}{2}}, \quad b = \frac{1}{\cosh^2 \frac{l}{2}} \quad \text{(13)}$$

The choice of normalization $Q_a, Q_b$ leads to the equivalent forms of the Length Spectrum Identity. We choose normalization $Q_a$ for our calculations.

If $x, y \in \mathbb{R}, x \neq y$, we let $g(x, y)$ be the geodesic in the upper half plane with end points $x, y$. Then for $(x, y) \in (a, 0) \times (1, \infty)$, the geodesic $g(x, y)$ intersects $Q_a$ in a definite length denoted $L(x, y)$. 

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Lemma 1 The map \( L : (a,0) \times (1,\infty) \rightarrow \mathbb{R} \) is given by the formula

\[
L(x,y) = \frac{1}{2} \ln \left( \frac{y(y-a)(x-1)}{x(x-a)(y-1)} \right) = \frac{1}{2} \ln \left( \frac{f(y)}{f(x)} \right).
\]

where \( f(x) = \frac{x(x-a)}{x-1} \).

Proof: Let \( T \) be the ideal triangle with vertices 0,1,\( \infty \). Let \( l_1 : (-\infty,0) \times (0,1) \rightarrow \mathbb{R} \) and \( l_2 : (-\infty,0) \times (1,\infty) \rightarrow \mathbb{R} \) be given by letting \( l_1(x,y) \) be the length of the intersection of \( g(x,y) \) with \( T \) and \( l_2(x,y) \) be the length of the intersection of \( g(x,y) \) with \( T \). By a previous paper (see [1]) the functions \( l_i \) are given by

\[
l_1(x,y) = \frac{1}{2} \ln \left( \frac{1-x}{1-y} \right) \quad l_2(x,y) = \frac{1}{2} \ln \left( \frac{y(x-1)}{x(y-1)} \right).
\]

To calculate \( L \), we split the quadrilateral \( Q_a \) by the vertical line at \( x = 0 \) into two ideal triangles \( T_1, T_2 \) where \( T_1 \) has vertices 0,1,\( \infty \) and \( T_2 \) has vertices \( a,0,\infty \). Then \( T_1 = T \) and \( f_2(z) = z/a \) sends \( T_2 \) to \( T \). Therefore

\[
L(x,y) = l_2(x,y) + l_1(y/a,x/a)
\]

Therefore

\[
L(x,y) = \frac{1}{2} \ln \left( \frac{y(x-1)}{x(y-1)} \right) + \frac{1}{2} \ln \left( \frac{1-y/a}{1-x/a} \right) = \frac{1}{2} \ln \left( \frac{y(x-1)(a-y)}{x(y-1)(a-x)} \right).
\]
We consider the rational function \( f(x) \) defined above. Differentiating we have

\[
    f'(x) = \frac{(2x - a)(x - 1) - 1.(x^2 - ax)}{(x - 1)^2} = \frac{x^2 - 2x + a}{(x - 1)^2}
\]

Therefore \( f(x) \) has two critical points \( 1 \pm \sqrt{1 - a} \). We label the critical points \( x_0 = 1 - \sqrt{1 - a} \) and \( y_0 = 1 + \sqrt{1 - a} \) and note that \( x_0 \) is a maximum and \( y_0 \) a minimum.

7 Proof of Summation Identity

By definition

\[
    (L_\star \Omega)(\phi) = \int_{T_1(S)} \phi(L(v)) \, d\Omega.
\]

Let \( \alpha, \beta \) be two arcs in \( S_l \) with endpoints on \( \partial S_l \). Then we say \( \alpha \sim \beta \) if they are homotopic relative to the boundary \( \partial S_l \).

We define the sets \( A_i = \{ v \in T_1(S) | \alpha(v) \sim \alpha_i \} \). Also for each \( \lambda \)-cusp \( c \) we define \( A_c = \{ v \in T_1(S) | \alpha(v) \sim c \} \) where \( \alpha(v) \sim c \) if \( \alpha(v) \) can be homotoped (rel boundary) out the cusp \( c \). Note that for \( v \in A_i \) or \( v \in A_c \), \( L(v) \) is finite. Finally we define the set \( A_\infty \) to be all \( v \) not in any \( A_i \) or \( A_c \). By definition, the sets \( A_i, A_c, A_\infty \) form a partition of \( T_1(S) \). If we double \( S_l \) along its boundary, the geodesic arcs \( \alpha_i \) correspond to a subset of the geodesics of the doubled surface. Therefore as the length spectrum of the doubled surface is countable, so is the collection of arcs \( \alpha_i \) in \( S_l \). Also, by ergodicity of geodesic flow on \( S \) (see [2]), the set \( A_\infty \) is a measure zero.
Therefore

$$(L_\ast \Omega)(\phi) = \sum_i \int_{A_i} \phi(L(v)) \, d\Omega + \sum_c \int_{A_c} \phi(L(v)) \, d\Omega.$$  

We let

$$a_i = -\frac{1}{\sinh^2 \frac{l_i}{2}}.$$  

Then setting $Q_i = Q_{\alpha_i}$, we have that $Q_i$ is a quadrilateral with perpendicular of length $l_i$. We lift $\alpha_i$ to the upper half plane so that it is the perpendicular of length $l_i$ in $Q_i$. We lift each $\lambda-$cusp $c$ to the ideal vertex at infinity between the vertical geodesics $x = 0, x = 1$. Let $T$ be the ideal triangle with vertices $0, 1, \infty \in \mathbb{R}$.

If $v \in T_1(\mathbb{H}^2)$ in the upper half plane, we define $g(v)$ to be the geodesic with tangent vector $v$. We also denote the endpoints of $g(v)$ by $(x(v), y(v))$.

We lift the set $A_i$ to the set $A'_i \subseteq T_1(Q_i)$. Then for $v \in A'_i$ the geodesic arc $\alpha'(v) = Q_i \cap g(v)$ is a lift of $\alpha(v)$. Similarly we lift $A_c$ to the set $A'_c \subseteq T_1(T).$ Then for $v \in A'_c$ the geodesic arc $\alpha'(v) = T \cap g(v)$ is a lift of $\alpha(v)$. By abuse of notation we also let $\Omega$ be the volume measure on $T_1(\mathbb{H}^2)$. We parameterize $T_1(\mathbb{H}^2)$ by $(x, y, l) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ where $(x, y, l)$ corresponds to the vector $v$ such that $g(v)$ has ordered endpoints $(x, y)$ and $v$ has basepoint on $g(v)$ a distance $l$ from the highest point of $g(v)$ in the upper half-plane. Then the volume form $\Omega$ can be written as (see [4])

$$d\Omega = \frac{2dx dy dl}{(x - y)^2}.$$  

Therefore

$$\int_{A_c} \phi(L(v)) \, d\Omega = \int_{A'_c} \frac{2\phi(L(v)) \, dx dy dl}{(x - y)^2}.$$  

We note that $L(v)$ only depends on the endpoints and therefore we can write $L(v) = L(x, y)$. If $v \in A'_c$ then either $(x, y)$ or $(y, x) \in (-\infty, 0) \times (1, \infty)$. Integrating over $l$ we have

$$\int_{A'_c} \frac{2\phi(L(v)) \, dx dy dl}{(x - y)^2} = \int_{-\infty}^0 \int_1^\infty \frac{4\phi(L(x, y))L(x, y) \, dx dy}{(x - y)^2}.$$  

By our previous paper [1]

$$\int_{-\infty}^0 \int_1^\infty \frac{4\phi(L(x, y))L(x, y) \, dx dy}{(x - y)^2} = \int_0^\infty \frac{4\phi(L)L^2 dL}{\sinh^2 L}.$$  

Thus as there are $N_\lambda \lambda-$cusps we have

$$\sum_c \int_{A_c} \phi(L(v)) \, d\Omega = N_\lambda \int_0^\infty \frac{4\phi(L)L^2 dL}{\sinh^2 L} = M_\infty(\phi)$$

where $M_\infty$ is the measure with infinitesimal

$$dM_\infty = \frac{4N_\lambda x^2 dx}{\sinh^2 x}.$$  

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Similarly we have by lifting \( A_i \) to \( A'_i \) that
\[
\int_{A_i} \phi(L(v)) \, d\Omega = \int_{A'_i} \frac{2 \phi(L(v)) \, dxdydl}{(x-y)^2}.
\]
If \( v \in A'_i \) then either \((x, y)\) or \((y, x)\) \(\in (a_i, 0) \times (1, \infty)\). Integrating over \( l \) we have
\[
\int_{A'_i} \frac{2 \phi(L(v)) \, dxdydl}{(x-y)^2} = \int_{a_i}^{\infty} \int_1^\infty \frac{4 \phi(L(x, y)) L(x, y) \, dxdy}{(x-y)^2}.
\]
For \( a < 0 \) we define \( M_a(\phi) \) to be the righthandside of the above equation. Then
\[
M_a(\phi) = \int_{a}^{\infty} \int_1^{\infty} \frac{4 \phi(L(x, y)) L(x, y) \, dxdy}{(x-y)^2}.
\]
Then
\[
M_{\lambda} = M_{\infty} + \sum_i M_{a_i}
\]
As \( M_{\lambda} = L_{\lambda} \Omega \) it has total mass equal to the volume of \( T_1(S) \) which is \( 4 \pi^2 |\chi(S)| \). Therefore
\[
\Omega(T_1(S)) = 4 \pi^2 |\chi(S)| = M_{\lambda}(1) = M_{\infty}(1) + \sum_i M_{a_i}(1)
\]
(14)

By an elementary calculation (see [1])
\[
\int_{0}^{\infty} \frac{x^2 \, dx}{\sinh^2 x} = \frac{\pi^2}{6}.
\]
Therefore
\[
M_{\infty}(1) = \int_{0}^{\infty} \frac{4 N \lambda x^2 \, dx}{\sinh^2 x} = 4 N \lambda \int_{0}^{\infty} \frac{x^2 \, dx}{\sinh^2 x} = 4 N \lambda \cdot \frac{\pi^2}{6} = \frac{2 N \lambda \pi^2}{3}.
\]
Using lemma [4] we substitute the formula for \( L(x, y) \) to obtain
\[
M_{a}(1) = \int_{a}^{\infty} \int_1^\infty \frac{2 \log \frac{y(y-a)(x-1)}{x(x-a)(y-1)}}{(x-y)^2} \, dxdy.
\]
Then by equation [14] above we obtain
\[
4 \pi^2 |\chi(S)| = M_{\infty}(1) + \sum_i M_{a_i}(1) = \frac{2 N \lambda \pi^2}{3} + \sum_i F(l_i)
\]
giving the summation identity
\[
\sum_i F(l_i) = 4 \pi^2 |\chi(S)| - \frac{2 N \lambda \pi^2}{3} = \frac{2 \pi^2}{3} (6 |\chi(S)| - N \lambda)
\]
(15)
8 Integral Calculation

In this section we find a formula for $F(I)$ by calculating an integral. We note that by the previous section, we already know that the function $F$ satisfies the functional equation \[15\]. We will make use of this to reduce $F$ to the form we wish independent of using any classical dilogarithm relations.

Lemma 2 For $a < 0$

\[
\int_a^0 \int_1^\infty \frac{\log \left| \frac{y-a}{x-a} \right|}{x-y} \frac{dy}{x-y^2} = -4 \mathcal{L}(a)
\]

Proof: We let

\[
G(a) = \int_a^0 \int_1^\infty \frac{\log \left| \frac{y-a}{x-a} \right|}{x-y} \frac{dy}{x-y^2}
\]

Integrating by parts we get

\[
\int \frac{\log \left| \frac{y-a}{x-a} \right|}{x-y} dx = -\log \left| \frac{y-a}{x-a} \right| x-y + \int \frac{1}{x-y} \left( \frac{1}{x-1} - \frac{1}{x} - \frac{1}{x-a} \right) dx.
\]

Using

\[
\int \frac{1}{(x-a)(x-b)} dx = \frac{1}{a-b} \left( \log |x-a| - \log |x-b| \right)
\]

we get

\[
\int \frac{\log \left| \frac{y-a}{x-a} \right|}{x-y} dx = \log \left| \frac{y-a}{x-a} \right| \frac{x-y}{y-1} + \frac{1}{y-1} \left( \log |x-y| - \log |x-1| \right) + \frac{1}{y} \left( \log |x-y| - \log |x| \right) - \frac{1}{y-a} \left( \log |x-y| - \log |x-a| \right)
\]

\[
= \frac{\log \left| \frac{y-a}{x-a} \right|}{y-x} - \log |x-1| y-1 + \frac{\log |x|}{y-1} + \frac{\log |x-a|}{y-a} + \log |x-y| \left( \frac{1}{y-1} - \frac{1}{y} - \frac{1}{y-a} \right)
\]

We define

\[
I(y) = \int_a^0 \frac{\log \left| \frac{y-a}{x-a} \right|}{x-y^2} dx
\]

To evaluate the improper integral $I(y)$ we gather the divergent terms to find their limits. Therefore

\[
I(y) = \lim_{x \to 0^-} \log |x| \left( \frac{1}{y} - \frac{1}{y-x} \right) - \lim_{x \to a^+} \log |x-a| \left( \frac{1}{y} - \frac{1}{y-x} \right) + \\
\log \left| \frac{y-a}{y-1} \right| + \log |y| \left( \frac{1}{y-1} - \frac{1}{y} - \frac{1}{y-a} \right)
\]

\[
- \frac{\log \left| \frac{y-a}{y} \right|}{y-a} + \log |a-1| y-1 - \log |y| \left( \frac{1}{y-1} - \frac{1}{y} - \frac{1}{y-a} \right)
\]

\[
= \log \left| \frac{y-a}{y} \right| - \log |a-1| y-1 - \log |y| \left( \frac{1}{y-1} - \frac{1}{y} - \frac{1}{y-a} \right)
\]
By elementary calculus, both limits are zero. As \( y > 1 \) and \( a < 0 \), when we gather the remaining terms by common denominators and get

\[
I(y) = \frac{-2 \log(-a) + 2 \log(y-a) - \log(y-1)}{y} + \frac{\log(1-a) + \log(y) - \log(y-a)}{y-1} + \\
\frac{2 \log(-a) - \log(1-a) - 2 \log(y) + \log(y-1)}{y-a}
\]

We now rewrite in the following form

\[
I(y) = \left( \frac{\log(y)}{y-1} - \frac{\log(y-1)}{y} \right) + 2 \left( \frac{\log \left( \frac{y-a}{a} \right)}{y} - \frac{\log \left( \frac{y}{y-a} \right)}{y-a} \right) + \left( \frac{\log \left( \frac{y-1}{1-a} \right) - \log \left( \frac{y-a}{y-1} \right)}{y-a} \right).
\]  \hspace{1cm} (16)

Before we calculate the integral of \( I(y) \) we note some properties of the dilogarithm. As the dilogarithm function \( \text{Li}_2 \) satisfies

\[
\text{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt
\]

Then \( L \) has derivative

\[
L'(x) = \frac{d}{dx} \left( \text{Li}_2(x) + \frac{1}{2} \log |x| \log(1-x) \right) = -\frac{\log(1-x)}{x} + \frac{1}{2} \left( \frac{\log(1-x)}{x} - \frac{\log |x|}{1-x} \right) = -\frac{1}{2} \left( \frac{\log(1-x)}{x} + \frac{\log |x|}{1-x} \right)
\]

Now if \( a < b \), then on the interval \( x > b \), we have \( (b-x)/(b-a) < 0 \). We define

\[
J(x, a, b) = 2L \left( \frac{b-x}{b-a} \right).
\]

Then differentiating \( J \) we get

\[
J'(x, a, b) = 2L' \left( \frac{b-x}{b-a} \right) \cdot \frac{-1}{b-a} = \left( \frac{\log \left( \frac{x-a}{b-a} \right)}{b-x} + \frac{\log \left( \frac{x-b}{b-a} \right)}{x-a} \right) \left( \frac{\log \left( \frac{x-b}{b-a} \right) - \log \left( \frac{x-a}{b-a} \right)}{x-b} \right)
\]

We set

\[
J(y) = -J(y, 0, 1) - 2J(y, a, 0) + J(y, a, 1).
\]

Then from equation \( 16 \) we have that

\[
J'(y) = -J'(y, 0, 1) - 2J'(y, a, 0) + J'(y, a, 1) = I(y).
\]

Therefore we have an antiderivative for \( I \) and integrate to find \( G \) to get

\[
G(a) = \int_1^\infty I(y) dy = J(y)|_1^\infty = \lim_{y \to \infty} J(y) - \lim_{y \to 1^+} J(y).
\]

We let \( L_\infty \) be the limit if \( L(x) \) as \( x \) tends to \( -\infty \). Therefore

\[
\lim_{y \to 1^+} J(y) = -2L(0) - 4L(a^{-1}) + 2L(0) = -4L(a^{-1}) \quad \lim_{y \to \infty} J(y) = -4L_\infty.
\]

Thus

\[
G(a) = -4L_\infty + 4L(a^{-1}) = -4(L_\infty - L(a^{-1}))
\]

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It follows immediately from Euler’s reflection identity that \( G(a) = -4L(a) \) but for completeness we derive it independently. From the formula we have \( G(0) = -4(L_\infty - L_\infty) = 0 \). Also by equation \[ G(a) \] must satisfy a summation identity

\[
G(a) + G(a^{-1}) = G_\infty
\]

where \( G_\infty \) is the limit of \( G \) as \( a \) tends to \(-\infty\). Therefore

\[
G(a) = G_\infty - G(a^{-1}) = (G_\infty + 4L_\infty) - 4L(a)
\]

But as \( G(0) = 0 \) we have

\[
G(a) = -4L(a) \quad \text{and finally} \quad F(l) = -8L\left(-\frac{1}{\sinh^2(l/2)}\right).
\]

We note that by performing the integral over the quadrilateral \( Q_b \) where \( b = 1/\cosh^2(l/2) \), the above can be repeated to show

\[
F(l) = 8L\left(\frac{1}{\cosh^2(l/2)}\right).
\]

Equivalently we note it also follows from Landen’s identity.

9 Volume interpretation of \( L \)

Let \( g_1, g_2 \) be disjoint geodesics in \( \mathbb{H}^2 \) with perpendicular distance \( l \) and endpoints \( x_1, y_1 \) and \( x_2, y_2 \) respectively on \( S^1 \). Given \( v \in T_1(S) \) let \( g_v \) be the associated oriented geodesic with tangent \( v \). Then we define the set

\[
C(g_1, g_2) = \{ v \in T_1(S) \mid g_v \cap g_1 \neq \emptyset, \; g_v \cap g_2 \neq \emptyset \}
\]

Let \( t = [x_1, y_1, x_2, y_2] \), then depending on the ordering of the points on the circle we have

\[
t = [-1, 1, e^l, -e^l] = \frac{1}{\cosh^2(l/2)} \quad \text{or} \quad t = [-1, 1, -e^l, e^l] = -\frac{1}{\sinh^2(l/2)}.
\]

It follows from the invariance of volume on \( T_1(S) \), that the volume of \( C(g_1, g_2) \) in \( T_1(S) \) only depends on \( t \). We therefore define \( V(t) = \text{Volume}(S(g_1, g_2)) \).

Then it follows from the main theorem that

\[
L(t) = \pm \frac{1}{8} V(t)
\]

where the sign is given by the sign of \( t \). Therefore we can interpret the Rogers \( L \)–function as a signed volume function on \( T_1(S) \) for the sets \( G(g_1, g_2) \).
10 Integral Formula for $\rho$

We let

$$L(x, y) = \frac{1}{2} \log \left( \frac{y(y-a)(x-1)}{x(x-a)(y-1)} \right) = \frac{1}{2} \log \left( \frac{f(y)}{f(x)} \right) \quad \text{for} \quad f(x) = \frac{x(x-a)}{x-1}.$$  

Taking derivatives of the length function $L(x, y)$ we have

$$\frac{\partial L}{\partial x} = -\frac{f'(x)}{2f(x)} \quad \frac{\partial L}{\partial y} = \frac{f'(y)}{2f(y)}.$$  

By the previous section, the function $f$ has critical points $x_0, y_0$. Furthermore on $(a, 0)$ the function $f(x)$ has global maximum at $x_0$ and on $(1, \infty)$, $f$ has global minimum at $y_0$. Therefore fixing $x$, the function $u : (1, \infty) \to \mathbb{R}$ given by $u(y) = L(x, y)$ is decreasing on $(1, y_0)$ and increasing on $(y_0, \infty)$. Therefore we make the change of variable $t = L(x, y), x = x$. Finding inverses for $f$ we define the two function $g_+, g_-$ by

$$g_{\pm}(x) = \frac{(a + x) \pm \sqrt{(a + x)^2 - 4x}}{2}.$$  

Then solving $t = L(x, y)$ gives $f(y) = f(x)e^{2t}$. Therefore on $(1, y_0)$ we have $y = g_-(f(x)e^{2t})$ and on $(y_0, \infty)$ we have $y = g_+(f(x)e^{2t})$. Therefore

$$M_\alpha(\phi) = \int_a^0 \left( \int_1^{y_0} + \int_{y_0}^{\infty} \frac{4.\phi(L(x, y))L(x, y) \, dy}{(x-y)^2} \right) \, dx.$$  

and

$$\int_1^{y_0} \frac{4.\phi(L(x, y))L(x, y) \, dy}{(x-y)^2} = \int_{L(x,y_0)}^{L(x,y_0)} \frac{4.\phi(t)t.g_-'(f(x)e^{2t})2f(x)e^{2t}dt}{(x-g_-(f(x)e^{2t}))^2}$$  

$$\int_{y_0}^{\infty} \frac{4.\phi(L(x, y))L(x, y) \, dy}{(x-y)^2} = \int_{L(x,y_0)}^{L(x,y_0)} \frac{4.\phi(t)t.g_+'(f(x)e^{2t})2f(x)e^{2t}dt}{(x-g_+(f(x)e^{2t}))^2}$$  

Therefore combining we have

$$M_\alpha(\phi) = \int_a^0 \int_{L(x,y_0)}^{\infty} 8.\phi(t).t.e^{2t}.f(x) \left( \frac{g_-'(f(x)e^{2t})}{(x-g_-(f(x)e^{2t}))^2} - \frac{g_+'(f(x)e^{2t})}{(x-g_+(f(x)e^{2t}))^2} \right) \, dt \, dx.$$  

We switch the order of integration. The function $L(x, y_0)$ is minimum at $x_0$ with minimum value $l = L(x_0, y_0)$ being the length of the perpendicular (see figure 2). Thus we integrate $t$ from $l$ to infinity. The integral in the $x$ direction is between the two $x$ solutions of $t = L(x, y_0)$ which are solutions to $f(x) = f(y_0)e^{-2t}$. Thus we integrate $x$ from $g_-(f(y_0)e^{-2t})$ to $g_+(f(y_0)e^{-2t})$ giving

$$M_\alpha(\phi) = \int_{L(x,y_0)}^{\infty} 8.\phi(t).t.e^{2t}dt \left( \int_{g_-(f(y_0)e^{-2t})}^{g_+(f(y_0)e^{-2t})} \left( \frac{g_-'(f(x)e^{2t})}{(x-g_-(f(x)e^{2t}))^2} - \frac{g_+'(f(x)e^{2t})}{(x-g_+(f(x)e^{2t}))^2} \right) f(x) \, dx \right)$$  

Therefore

$$M_\alpha(\phi) = \int_0^{\infty} \phi(t).\rho(l, t) \, dt.$$  

where
\[ \rho(l,t) = 8t e^{2t} x(l, \infty) \left( \int_{g^-(f(y_0)e^{-2t})}^{g^+(f(y_0)e^{-2t})} \left( \frac{g'_+(f(x)e^{2t})}{(x - g_+(f(x)e^{2t}))^2} - \frac{g'_-(f(x)e^{2t})}{(x - g_-(f(x)e^{2t}))^2} \right) f(x) dx \right) \]
and \( f(x) = \frac{x(x-a)}{x-1} \) where \( a = -\frac{1}{\sinh^2 l/2} \)

Therefore
\[ (L_* \Omega)(\phi) = \int_0^\infty \phi(x) \rho(x) dx \]
where
\[ \rho(x) = \frac{4N \lambda x^2}{\sinh^2 x} + \sum_i \rho(l_i, x) \]

11 Asymptotic behavior

In this section we study the asymptotic behavior of the function \( \rho(l,t) \) for large \( t \).

For functions of a single variable, we write \( f(x) \simeq g(x) \) as \( x \) tends to \( x_0 \) if
\[ \lim_{x \to x_0} \frac{f(x)}{g(x)} = 1. \]
Furthermore for functions of more than one variable, we write \( f(x,y) \simeq_x g(x,y) \) as \( x \) tends to \( x_0 \) if
\[ \lim_{x \to x_0} \frac{f(x,y)}{g(x,y)} = 1. \]

**Theorem 3** The measure \( \rho(l,t) dx \) on the real line satisfies
\[ \lim_{t \to \infty} \frac{\rho(l,t)}{16t^2 e^{-2t}} = r(l) \]
uniformly on compact subsets of \((0, \infty)\) where
\[ r(l) = \frac{-2a^2 + 5a - 2}{a(1-a)} \quad \text{for} \quad a = -\frac{1}{\sinh^2 \left( \frac{l}{2} \right)} \]

**Proof:** We now show \( \lim_{t \to \infty} \rho(l,t) = r(l) \) converges uniformly on compact subsets of \((0, \infty)\). Let \( I \subseteq (0, \infty) \) be a compact interval. Now let \( t \in I \). As before we let \( a = -1/\sinh^2(l/2) \) and define \( f(x) = x(x-a)/(x-1) \) with inverses \( g_{\pm} \) and critical values \( x_0, y_0 \). Let
\[ G(t,x) = 8te^{2t} \left( \frac{g'_+(f(x)e^{2t})}{(x - g_+(f(x)e^{2t}))^2} - \frac{g'_-(f(x)e^{2t})}{(x - g_-(f(x)e^{2t}))^2} \right) f(x) \]
Then for \( t > l \) we have

\[
\rho(l, t) = \int_{g_-(f(y_0)e^{-2t})}^{g_+(f(y_0)e^{-2t})} G(t, x) \, dx
\]

For \( C > 0 \), we further define

\[
\rho(C, l, t) = \int_{g_-(f(y_0)Ce^{-2t})}^{g_+(f(y_0)Ce^{-2t})} G(t, x) \, dx
\] (17)

On the interval \([a, 0]\) \( f \) has maximum at \( x_0 \). Therefore \( \rho(C, l, t) \) is defined for all \( t \) such that \( f(y_0)Ce^{-2t} < f(x_0) \) or

\[
 t > K_0(C) = \frac{1}{2} \ln C + \frac{1}{2} \ln \left( \frac{f(y_0)}{f(x_0)} \right) = l + \frac{1}{2} \ln C
\]

Considering \( g_\pm(x) \) for large \( x \) we have

\[
g_\pm(x) = \frac{(a + x) \pm \sqrt{(a + x)^2 - 4x}}{2} \approx \frac{(a + x)}{2} \left( 1 \pm \left( 1 - \frac{2x}{(a + x)^2} \right) \right)
\]

Therefore

\[
g_-(x) \approx \frac{(a + x)}{2} \left( 1 - \frac{2x}{(a + x)^2} \right) = \frac{x}{a + x} \approx 1 - \frac{a}{x}
\]

and

\[
g_+(x) \approx \frac{(a + x)}{2} \left( 1 + \frac{2x}{(a + x)^2} \right) = (a + x) - \frac{x}{a + x} \approx (a - 1) + x + \frac{a}{x}
\]

Taking leading terms we have

\[
g_-(x) \approx 1 \quad g'_-(x) \approx \frac{a}{x^2} \quad g_+(x) \approx x \quad g'_+(x) \approx 1 \quad (18)
\]

We let \( I_C = [g_-(f(y_0)Ce^{-2t}), g_+(f(y_0)Ce^{-2t})] \). Then for \( x \in I_C \) we have \( f(x)e^{2t} \geq C \cdot f(y_0) \). Therefore for \( C \) sufficiently large we use the above approximations to approximate \( G(t, x) \) on \( I_C \). We substitute the approximations into the formula for \( G(t, x) \) to define

\[
G_1(t, x) = 8t e^{2t} \left( \frac{1}{x - f(x)e^{2t}} - \frac{a}{(f(x)e^{2t})^2} \right) f(x)
\]

Simplifying we have

\[
G_1(t, x) = 8t e^{-2t} \left( \frac{1}{(1 - \frac{x}{f(x)e^{2t}})^2} - \frac{a}{(x - 1)^2} \right) \frac{1}{f(x)}.
\]

Noting that \( f(x)e^{2t} > Cf(y_0) \) on \( I_C \), then for large \( C \) the quantity \( \frac{x}{f(x)e^{2t}} \) is small and we obtain the approximation

\[
G_2(t, x) = 8t e^{-2t} \left( 1 - \frac{a}{(x - 1)^2} \right) \frac{1}{f(x)}.
\]

Therefore given an \( \epsilon > 0 \) we can find a \( K_1(\epsilon) \) such that

\[
\frac{G(t, x)}{G_2(t, x)} \in [1 - \epsilon, 1 + \epsilon] \quad \text{for all } C > K_1(\epsilon), t > K_0(C), x \in I_C.
\]
Therefore integrating
\[
\frac{1}{\rho(C, l, t)} \left( 8te^{-2t} \cdot \int_{g_-(f(y_0)e^{-2t})}^{g_+(f(y_0)e^{-2t})} \left( 1 - \frac{a}{(x-1)^2} \right) \frac{1}{f(x)} dx \right) \in [1 - \epsilon, 1 + \epsilon]
\]
for \( C > K_1(\epsilon) \) and \( t > K_0(C) \). We fix a \( K > K_1(\epsilon) \) and define
\[
\rho_K(l, t) = 8te^{-2t} \left( \int_{g_-(f(y_0)Ke^{-2t})}^{g_+(f(y_0)Ke^{-2t})} \left( 1 - \frac{a}{(x-1)^2} \right) \frac{1}{f(x)} dx \right)
\]
\[
= 8te^{-2t} \left( \int_{g_-(f(y_0)e^{-2t})}^{g_+(f(y_0)e^{-2t})} \left( \frac{x-1}{x(x-a)} - \frac{a}{x(x-a)(x-1)} \right) dx \right)
\]
Integrating we have
\[
\int \left( \frac{x-1}{x(x-a)} - \frac{a}{x(x-a)(x-1)} \right) dx = \left( \frac{1-a}{a} \ln |x| - \frac{a}{1-a} \ln |x-1| - \frac{a^2-3a+1}{a(1-a)} \ln |x-a| \right)\bigg|_{g_+(f(y_0)e^{-2t})}^{g_+(f(y_0)Ke^{-2t})}
\]
Therefore
\[
\rho_K(l, t) = 8te^{-2t} \left( \frac{1-a}{a} \ln |x| - \frac{a}{1-a} \ln |x-1| - \frac{a^2-3a+1}{a(1-a)} \ln |x-a| \right)\bigg|_{g_+(f(y_0)e^{-2t})}^{g_+(f(y_0)Ke^{-2t})}
\]
For \( x \) small we have
\[
g_\pm(x) = (a+x) \pm \sqrt{(a+x)^2 - 4x} \approx \frac{(a+x)}{2} \left( 1 + \left( 1 - \frac{2x}{a+x} \right)^2 \right)
\]
Therefore
\[
g_-(x) \approx (a+x) - \frac{x}{a+x} \approx a - \frac{(1-a)x}{a} \quad \quad g_+(x) \approx \frac{x}{a+x} \approx \frac{x}{a}
\]
Therefore
\[
\rho_K(l, t) \approx t \cdot 8te^{-2t} \left( \frac{1-a}{a} \ln \frac{Kf(y_0)e^{-2t}}{a^2} - \frac{a}{1-a} \ln \frac{1}{a-1} - \frac{a^2-3a+1}{a(1-a)} \ln \frac{a^2}{(1-a)f(y_0)Ke^{-2t}} \right)
\]
Taking limits as we have
\[
\rho_K(l, t) \approx t \cdot (16t^2e^{-2t}) \left( -\frac{1-a}{a} - \frac{a^2-3a+1}{a(1-a)} \right) = (16t^2e^{-2t}) \cdot \frac{-2a^2+5a-2}{a(1-a)}
\]
Therefore given \( \epsilon > 0 \) there exists \( K_1(\epsilon) > 0 \) such that for any \( C > K_1(\epsilon) \) both
\[
\liminf_{t \to \infty} \frac{\rho(C, l, t)}{16t^2e^{-2t}r(a)} \quad \text{and} \quad \limsup_{t \to \infty} \frac{\rho(C, l, t)}{16t^2e^{-2t}r(a)} \quad \text{are in} \quad [1 - \epsilon, 1 + \epsilon].
\]
where
\[
r(a) = \frac{-2a^2+5a-2}{a(1-a)}
\]
We now define
\[
\rho_-(C, l, t) = \int_{g_-(f(y_0)e^{-2t})}^{g_-(f(y_0)Ce^{-2t})} G(t, x) dx \quad \text{and} \quad \rho_+(C, l, t) = \int_{g_+(f(y_0)e^{-2t})}^{g_+(f(y_0)Ce^{-2t})} G(t, x) dt.
\]
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Then by definition
\[ \rho(t, l) = \rho(C, l, t) + \rho_-(C, l, t) + \rho_+(C, l, t). \]
We now bound the functions \( \rho_\pm(C, l, t) \). Let \( I_-^C, I_+^C \) be the given intervals.

On the interval \( I, g_\pm(f(x)e^{2t}) > 1 \) and \( x < 0 \) so \( (x - g_\pm(f(x)e^{2t}))^2 > 1 \). Also as \( g'_-(f(x)e^{2t}) < 0 \) we have
\[
|G(t, x)| = (8t.e^{2t}). \left( \frac{g'_+(f(x)e^{2t})}{(x - g_+(f(x)e^{2t}))^2} - \frac{g'_-(f(x)e^{2t})}{(x - g_-(f(x)e^{2t}))^2} \right) f(x).
\leq 8t.e^{2t}. (g'_+(f(x)e^{2t}) - g'_-(f(x)e^{2t})) f(x).
\]
The derivative of \( g_\pm(x) \) is given by
\[ g'_\pm(x) = \frac{1}{2} \pm \frac{1}{2} \frac{x + a - 2}{(x + a)^2 - 4x}. \]
Therefore
\[ g'_+(x) - g'_-(x) = \frac{x + a - 2}{\sqrt{(x + a)^2 - 4x}}. \]
As \( f \) has critical values \( f(x_0) \) and \( f(y_0) \) we have that
\[ g'_+(x) - g'_-(x) = \frac{x + a - 2}{\sqrt{(x - f(x_0))(x - f(y_0))}}. \]
We note that on \( I_\pm^C \) we have \( f(y_0) < f(x)e^{2t} < Cf(y_0) \) then
\[
g'_+(f(x)e^{2t}) - g'_-(f(x)e^{2t}) \leq \frac{Cf(y_0) + a - 2}{\sqrt{(f(y_0) - f(x_0))(f(x)e^{2t} - f(y_0))}}.
\leq \left( \frac{Cf(y_0) + a - 2}{\sqrt{f(y_0) - f(x_0)}} \right) \frac{e^{-t}}{\sqrt{f(x) - f(y_0)e^{-2t}}}
\]
The function \( f(x) = x(x-a)/(x-1) \) has maximum at \( x_0 \) on \( (a, 0) \). Therefore for \( b < f(x_0) \)
\[ f(x) - b = \frac{(x - g_-(b))(x - g_+(b))}{(x - 1)} \]
As \( x \in (a, 0) \) we have
\[ f(x) - b \geq (x - g_-(b))(g_+(b) - x) \]
Therefore
\[
g'_+(f(x)e^{2t}) - g'_-(f(x)e^{2t}) \leq \left( \frac{Cf(y_0) + a - 2}{\sqrt{f(y_0) - f(x_0)}} \right) \frac{e^{-t}}{\sqrt{(x - g_-(f(y_0)e^{-2t}))(g_+(f(y_0)e^{-2t}) - x)}}
\]
Now restricting to \( I_+^C \) we have \( x > g_+(f(y_0)Ce^{-2t}) \). Therefore for \( x \in I_+^C \),
\[
g'_+(f(x)e^{2t}) - g'_-(f(x)e^{2t}) \leq \left( \frac{Cf(y_0) + a - 2}{\sqrt{(f(y_0) - f(x_0))(g_+(f(y_0)e^{-2t}) - g_-(f(y_0)e^{-2t}))}} \right) \frac{e^{-t}}{g_+(f(y_0)e^{-2t}) - x}
\]
Therefore we have
\[ \rho_+(C, l, t) \leq \int_{t_C}^b |G(t, x)| dx \leq D(t)8e^t \int_{t_C}^b \frac{f(x)}{\sqrt{g_+(f(y_0)e^{-2t}) - x}} dt \]
where \( D(t) \) is the constant
\[ D(t) = \left( \frac{Cf(y_0) + a - 2}{\sqrt{(f(y_0) - f(x_0))(g_+(f(y_0)Ce^{-2t}) - g_-(f(y_0)e^{-2t}))}} \right) \]
As \( f(x) = \frac{x(x - a)}{x - 1} \) then, \( 0 < f(x) \leq ax \) on \((a, 0)\) we have
\[ \rho_+(C, l, t) \leq \int_{t_C}^b |G(t, x)| dx \leq D(t)8e^t \int_{t_C}^b \frac{x}{\sqrt{g_+(f(y_0)e^{-2t}) - x}} dx \]
By integration we have
\[ \int_a^b \frac{x}{\sqrt{b - a}} dx = \frac{2}{3}(2b + a)\sqrt{b - a} \]
Therefore
\[ \rho_+(C, l, t) \leq 16D(t)8e^t \int_{t_C}^b \frac{x}{\sqrt{g_+(f(y_0)e^{-2t}) - x}} dx \]
Now for \( t \) large we have
\[ \lim_{t \to \infty} D(t) = \left( \frac{Cf(y_0) + a - 2}{\sqrt{(f(y_0) - f(x_0)).|a|}} \right) = D. \]
We note for \( x \) small \( g_+(x) \approx x/a \). Therefore
\[ \limsup_{t \to \infty} \left| \frac{\rho_+(C, l, t)}{t^2e^{-2t}} \right| \leq \limsup_{t \to \infty} \frac{16D.a.t.e^t \left( \frac{2(f(y_0)e^{-2t} + f(y_0)Ce^{-2t})}{a} \right) \sqrt{f(y_0)e^{-2t} - f(y_0)Ce^{-2t}}}{t^{2}e^{-2t}}. \]
\[ \limsup_{t \to \infty} \left| \frac{\rho_+(C, l, t)}{t^2e^{-2t}} \right| \leq \limsup_{t \to \infty} \frac{16D.f(y_0)^{3/2}(C + 2)\sqrt{C - 1}}{t.\sqrt{-a} = 0.} \]
Thus
\[ \lim_{t \to \infty} \frac{\rho_+(C, l, t)}{t^2e^{-2t}} = 0. \]
Similarly for \( \rho_-(C, l, t) \) we once again have that
\[ \lim_{t \to \infty} \frac{\rho_-(C, l, t)}{t^2e^{-2t}} = 0. \]
Therefore given \( \epsilon > 0 \) we can find \( K(\epsilon) \) such that for \( C > K(\epsilon) \) by equations \[19\]
\[ \limsup_{t \to \infty} \frac{\rho(l, t)}{16t^2e^{-2t}r(a)} = \limsup_{t \to \infty} \left( \frac{\rho_-(C, l, t)}{16t^2e^{-2t}r(a)} + \frac{\rho(C, l, t)}{16t^2e^{-2t}r(a)} \right) = \limsup_{t \to \infty} \frac{\rho(C, l, t)}{16t^2e^{-2t}r(a)} \in [1 - \epsilon, 1 + \epsilon] \]
As \( \epsilon \) is arbitrary we have
\[ \limsup_{t \to \infty} \frac{\rho(l, t)}{16t^2e^{-2t}r(a)} = 1 \]
Similarly
\[ \liminf_{t \to \infty} \frac{\rho(l, t)}{16t^2e^{-2t}r(a)} = 1 \]
References

[1] Martin Bridgeman, David Dumas, Distribution of intersection lengths of a random geodesic with a geodesic lamination. *Ergodic Theory and Dynamical Systems*, 27(4), 2007

[2] Eberhard Hopf. Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung. *Ber. Verh. Sächs. Akad. Wiss. Leipzig*, 91:261–304, 1939.

[3] L. Lewin, (Ed.). Structural Properties of Polylogarithms, *Mathematical Surveys and Monographs*, AMS, Providence, RI, 1991.

[4] Peter J. Nicholls. *The Ergodic Theory of Discrete Groups*, volume 143 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1989.

[5] L.J. Rogers. On Function Sum Theorems Connected with the Series $\sum_{1}^{\infty} \frac{x^n}{n^2}$ *Proc. London Math. Soc.* 4, 169-189, 1907