Coupling Constant Dependence in the Thermodynamics of $N = 4$ Supersymmetric Yang-Mills Theory

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Abstract

The free energy of the maximally supersymmetric $SU(N)$ gauge theory at temperature $T$ is expected to scale, in the large $N$ limit, as $N^2 T^4$ times a function of the ’t Hooft coupling, $f(g_{YM}^2 N)$. In the strong coupling limit the free energy has been deduced from the near-extremal 3-brane geometry, and its normalization has turned out to be $3/4$ times that found in the weak coupling limit. In this paper we calculate the leading correction to this result in inverse powers of the coupling, which originates from the $R^4$ terms in the tree level effective action of type IIB string theory. The correction to $3/4$ is positive and of order $(g_{YM}^2 N)^{-3/2}$. Thus, $f(g_{YM}^2 N)$ increases as the ’t Hooft coupling is decreased, in accordance with the expectation that it should be approaching 1 in the weak coupling limit. We also discuss similar corrections for other conformal theories describing coincident branes. In particular, we suggest that the coupling-independence of the near extremal entropy for D1-branes bound to D5-branes is related to the vanishing of the Weyl tensor of $AdS_3 \times S^3$.

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1 Introduction

A quarter of a century ago ’t Hooft showed that non-Abelian gauge theories simplify in
the limit where the number of colors, \( N \), is taken to infinity \([1]\). If \( g_{YM}^2 N \) is kept fixed,
then the perturbative expansion reduces to planar diagrams only. ’t Hooft speculated
that these planar diagrams are to be thought of as the world sheets of a string. This
idea has been among the dominant themes in theoretical physics ever since.

Recently, a great deal of progress in this direction has been taking place. In \([2]\) it was
suggested that the “confining string” may be defined by a two-dimensional conformal
sigma model with a 5-dimensional target space. The 5-th dimension was argued to
arise from the conformal factor of the world sheet geometry.\(^1\)

Another, seemingly unrelated, development is connected with the Dirichlet brane \([3]\) description of black 3-branes in \([4, 5]\). The essential observation is that, on the one
hand, the black branes are solitons which curve space \([4, 5]\) and, on the other hand,
the world volume of \( N \) parallel D-branes is described by supersymmetric \( U(N) \) gauge
theory with 16 supercharges \([12]\). A particularly interesting system is provided by the
limit of a large number \( N \) of coincident D3-branes \([4, 5]\), whose world volume is
described by \( \mathcal{N} = 4 \) supersymmetric \( U(N) \) gauge theory in 3 + 1 dimensions. In \([7]\)
is was shown that their description by the 3-brane solution of type IIB supergravity
becomes reliable in the “double-scaling limit,”

\[
\frac{L^4}{\alpha'^2} = 2g_{YM}^2 N \to \infty, \quad \omega^2 \alpha' \to 0, \tag{1}
\]

where \( L \) is the radius of the 3-brane throat, \( g_{YM}^2 = 2\pi g_s \), and \( \omega \) is a characteristic
excitation energy. An equivalent statement is that \( \alpha' \) is being taken to 0.

In an important development Maldacena, with some motivation from the results
in \([4, 5, 6, 7]\) from work on other black holes in \([13]\) and from studies of the throat
geometry in \([14, 15]\), made a fairly concrete “confining string” proposal \([16]\). Taking
the limit \( \alpha' \to 0 \) directly in the 3-brane metric, he found that the “universal region”
describing the gauge theory is the throat, whose geometry is the space \( AdS_5 \times S^5 \)
with equal radii \( L \) of the factors. The proposal \([16]\) was sharpened in \([17, 18]\) where
it was shown how to calculate the correlation functions of certain gauge theory vertex
operators from the response of the type IIB theory on \( AdS_5 \times S^5 \) to boundary conditions.
Many other interesting results were obtained recently, but they are outside the scope
of this paper.

An interesting ingredient the proposal \([16]\) adds to the sigma model formulation in
\([2]\) is the presence of a Ramond-Ramond self-dual 5-form background field. Though the

\(^1\)A similar phenomenon is known to occur for random surfaces embedded in 1 dimension and turns
them into a string theory with a 2-dimensional target space \([3, 4]\).
corresponding action for the type IIB superstring was recently constructed in [19], little is known to date about properties of sigma models with Ramond-Ramond backgrounds. In the absence of their exact solution, one can still extract a host of useful information about strongly coupled gauge theories by using the $\alpha'$ expansion of the type IIB string effective action. The study of leading $\alpha'$ corrections was recently initiated in [20].

The specific aspect of the gauge theory – string theory connection that we will pursue here is the thermodynamics of the maximally supersymmetric $SU(N)$ gauge theory in the large $N$ limit. In [6] it was pointed out that the near extremal black 3-brane of Hawking temperature $T$ may be used to study the large $N$ SYM theory heated up to the same temperature. The metric of the black 3-brane is

$$ds^2 = H^{-1/2}(r) \left[ -f(r)dt^2 + d\mathbf{x}^2 \right] + H^{1/2}(r) \left[ f^{-1}(r)dr^2 + r^2d\Omega_5^2 \right], \quad (2)$$

where

$$H(r) = 1 + \frac{L^4}{r^4}, \quad f(r) = 1 - \frac{r_0^4}{r^4}.$$  

The horizon is located at $r = r_0$ and the extremality is achieved in the limit $r_0 \to 0$ where the Hawking temperature vanishes. In order to study the conformal limit of the world volume theory of coincident 3-branes we need to keep $T L \ll 1$. This leads to the requirement that the metric is near-extremal, i.e. $r_0 \ll L$. Here the Hawking temperature

$$T = \frac{r_0}{\pi L^2} \ll \frac{1}{L}. \quad (3)$$

In [6] the entropy of the $SU(N)$ SYM theory was identified with the Bekenstein-Hawking entropy of the geometry (2). The result turned out to have a remarkable form,

$$S_{BH} = \frac{\pi^2}{2} N^2 V_3 T^3. \quad (4)$$

The $T^3$ scaling is in agreement with the conformal invariance of the theory. Indeed, the extensivity of the entropy, and the absence of another scale in the theory, requires the $V_3 T^3$ dependence. Furthermore, the factor $N^2$ indicates that one is dealing with a theory of $N^2$ unconfined massless degrees of freedom. As in [6], it is instructive to compare (4) with the entropy of the weak coupling limit of the SYM theory, where it reduces to that of $8N^2$ free massless bosons and fermions. Now the answer turns out to be

$$S_0 = \frac{2\pi^2}{3} N^2 V_3 T^3. \quad (5)$$

Thus, $S_{BH} = \frac{3}{4} S_0$.

Is there a definite contradiction here? The answer is no, because $S_{BH}$ is relevant to the SYM theory in the limit of $N \to \infty$ and large 't Hooft coupling $g_{\text{YM}}^2 N$. The
relations $L^4/\ell^4_{\text{Planck}} \sim N$ and $L^4/\ell^4_{\text{string}} \sim g^2_{\text{YM}}N$ reveal that only in this limit are both the $\alpha'$ and the loop corrections to the solution (4) negligible. In this paper our focus will be on the $\alpha'$ corrections. We suppress the loop corrections, which amounts in the gauge theory to taking the $N \to \infty$ limit first with fixed ’t Hooft coupling.

On general grounds, one expects the following expression for the entropy of the large $N$ gauge theory,

$$S(g^2_{\text{YM}}N) = f(g^2_{\text{YM}}N) \frac{2\pi^2}{3} N^2 V_3 T^3,$$

where $f(g^2_{\text{YM}}N)$ is (hopefully) a smooth function. Its weak coupling limit is $f(0) = 1$. The supergravity approach [6] predicts that, in the limit $g^2_{\text{YM}}N \to \infty$, $f$ should approach $\frac{3}{4}$.

In order to study precisely how this limiting value is approached, it is necessary to take into account the leading higher derivative correction in the type IIB tree level effective action. As discussed in [17, 18] the corrections to the effective action in the $AdS_5 \times S^5$ background are measured by the dimensionless parameter

$$\frac{\alpha'}{L^2} = (2g^2_{\text{YM}}N)^{-1/2}. \quad (7)$$

Thus, the $\alpha'$ expansion in the type IIB theory translates into the strong ’t Hooft coupling expansion in the SYM theory. This will help us identify the leading correction. Our main result is that the strong coupling expansion for $f$ is

$$f(g^2_{\text{YM}}N) = \frac{3}{4} + \frac{45}{32} \zeta(3)(2g^2_{\text{YM}}N)^{-3/2} + \ldots. \quad (8)$$

In section 2 we derive this result by first order perturbation theory on the supergravity action. A priori one might expect this direct treatment to be inadequate because it ignores perturbations in the geometry which change, for instance, the relation between the temperature and the non-extremality parameter $r_0$. We show in section 3 that taking the perturbations to the geometry into account does not, however, change the final answer for the free energy to first order.

Section 4 summarizes our results for other CFT’s describing coincident branes. In [23], the entropy of the non-dilatonic near-extremal p-branes (i.e. D3, M2, M5 branes) was computed from the leading-order supergravity solution. The free energy has the form expected of a CFT in $p$ dimensions:

$$F = -k_p N^{\frac{p+1}{2}} V_p T^{p+1}, \quad (9)$$

where $N$ is the quantized p-brane charge or the number of coincident branes, and $k_p$ is a constant of order unity. The conformal invariance of the corresponding world-volume theories fixes the temperature dependence, so that the higher-order corrections

$^2$The string loop effects give rise to corrections of order $g^2_s \alpha'^4/L^8 \sim 1/N^2$. Work on such corrections to the entropy was initiated in [21, 22].
to the supergravity action are expected to change only the $N$-dependent coefficient by subleading terms. In particular, we derive the first $1/N$ corrections to the free energies in the case of M5-branes and M2-branes, generalizing the leading-order results of [23].

For the near-horizon geometry which describes D1-branes bound to D5-branes at strong coupling we find that the leading $\alpha'$ correction to the free energy vanishes. This is due to the fact that the leading $\alpha'$ correction may be expressed in terms of the Weyl tensor [20], while the geometry is locally $AdS_3 \times S^3$ which is conformally flat so that the Weyl tensor vanishes. Based on this, we further argue that there are no corrections to all orders in $\alpha'$. This seems to explain why the near-extremal entropy is independent of the coupling in this case [24, 25].

2 The free energy from the gravitational action

Since the horizon of a near-extremal 3-brane is located far down its throat, the same answer for the Bekenstein-Hawking entropy is obtained if we replace the 3-brane metric by the throat approximation, $r \ll L$. The resulting metric [16, 26]

$$ds^2 = \frac{r^2}{L^2} \left[ -(1 - \frac{r_0^4}{r^4}) dt^2 + dx^2 \right] + \frac{L^2}{r^2} (1 - \frac{r_0^4}{r^4})^{-1} dr^2 + L^2 d\Omega_5^2 ,$$

(10)

is a product of $S^5$ with a certain limit of the Schwarzschild black hole in $AdS_5$ [27]. The Euclidean Schwarzschild black hole is asymptotic to $S^1 \times S^3$, and the required limit is achieved as the volume of $S^3$ is taken to infinity. Thus, the Euclidean continuation of the metric (10) is asymptotic to $S^1 \times R^3$, and the circumference of $S^1$ is $\beta = 1/T$. Indeed, if we set $r = r_0 (1 + L^{-2} \rho^2)$ near $r_0$, then the relevant 2d part of the Euclidean metric becomes:

$$ds^2 = d\rho^2 + \frac{4r_0^2}{L^4} \rho^2 d\tau^2 , \quad \tau = it ,$$

(11)

so that the period of the Euclidean time required by the regularity of the metric is $\beta = \pi L^2 / r_0$.

Following [28], one can identify the free energy $F$ of the theory with the Euclidean gravitational action times the temperature, i.e.

$$I = \beta F .$$

(12)

The 5-dimensional gravitational action obtained from the $D = 10$ type IIB supergravity action by compactifying on $S^5$ is

$$I_5 = - \frac{1}{16 \pi G_5} \int d^5 x \sqrt{g_5} \left( R_5 + \frac{12}{L^2} \right) .$$

(13)
Calculation of the contributions from the 2-derivative terms is divergent at large distances and requires a subtraction. The end result is that, as expected, the entropy is the horizon area divided by $4G_5$ [27]. The leading term in the free energy is

$$ F_0 = -\frac{\pi^2}{8} N^2 V T^4 . $$

(14)

We will focus on the contribution to $F$ coming from the $\alpha'^3 R^4$ string correction [29, 30] to the supergravity action. In the Einstein frame, and using the convention of including $(F_5)^2$ in the action and imposing self-duality after the equations of motion are derived, the tree level type IIB string effective action has the following structure:

$$ I = -\frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{g} \left[ R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{4 \cdot 5!} (F_5)^2 + \cdots + \gamma e^{-\frac{3}{2} \phi} W + \cdots \right] , $$

(15)

$$ \gamma = \frac{1}{8} \zeta(3)(\alpha')^3 , $$

$$ W = R^{hmnk} R_{pmnq} R^{rs} R^{q}_{rsk} + \frac{1}{2} R^{hkmm} R_{pqmm} R^{rs} R^{q}_{rsk} + \text{terms depending on the Ricci tensor} . $$

(16)

Dots stand for other terms depending on antisymmetric tensor field strengths and derivatives of dilaton [3]. The field redefinition ambiguity [30, 31] allows one to change the coefficients of terms involving the Ricci tensor (in essence, ignoring other fields, one may use $R_{mn} = 0$ to simplify the structure of $W$ as the graviton legs in the 4-point string amplitude are on-shell). Thus there exists a scheme where $W$ depends only on the Weyl tensor

$$ W = C^{hmnk} C_{pmnq} C^{rs} C^{q}_{rsk} + \frac{1}{2} C^{hkmm} C_{pqmm} C^{rs} C^{q}_{rsk} . $$

(17)

In general, there are other terms of the same order which accompany $C^4$ by supersymmetry. The Riemann tensor dependent part of $W$ (the first line in (16)) may be written as

$$ W = \frac{1}{3 \cdot 2^8} J_0 , \quad J_0 = t_8 \cdot t_8 RRRR - \frac{1}{4} \epsilon_8 \cdot \epsilon_8 RRRR , $$

(18)

which is the bosonic part of a superinvariant [32, 33, 34]. Here the $t_8 \cdot t_8 R^4$ term has the structure $24 \text{STr}[R^4 - \frac{1}{4}(R^2)^2]$ while the $\epsilon_8 \cdot \epsilon_8 RRRR$ term is defined in $D$-dimensional

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3As follows from [29], there exists a scheme in which the string-frame metric and dilaton dependent terms in the $O(\alpha'^3)$ action are given by $\int d^{10}x \sqrt{g} e^{-2\phi} \left[ R + 4(\partial \phi)^2 + \gamma W(R) \right]$, where $W(R)$ is the Riemann tensor part of $W$.  

4Note that while the on-shell N=1, D=10 superinvariant [32, 20] $\int d^{10}x d^{16}\theta \Phi^4 \rightarrow \int d^{10}x d^{16}\theta \left( \theta \gamma^{mnk} \theta \gamma_{pq} R_{mnpq} \right)^4$ depends only on the Weyl tensor (because of the identity $\gamma^{mnk} \theta \gamma_{mn} \theta = 0$) and is proportional to $W$ in [17], its off-shell extension (13) contains the Riemann tensor [33], i.e. it should involve also the Ricci tensor or derivatives of the dilaton and other fields.
Euclidean space as
\[ \frac{1}{(D - 8)!} \epsilon_D \cdot \epsilon_D R R R R = 8! \delta^{m_1}_{[m_1} \cdots \delta^{m_8}_{m_8]} R^{m_1 m_2} \cdots R^{m_7 m_8} . \]

The choice (17) of \( W \) is special in that the leading-order \( AdS_5 \times S^5 + F_5 \)-strength solution [33], which has a conformally flat metric, is then not modified by the \( R^4 \) correction [20]. Strictly speaking, the above discussion does not rule out that the additional \( F_5 \)-dependent terms (ignored in [20]) may lead to a modification of the solution, but more general arguments based on maximal supersymmetry of this background [19, 36] suggest that this does not happen.\(^5\)

The non-extremal background with the metric (10) (and constant dilaton \( \phi \) and self-dual \( F_5 \) field being the same as in the extremal case \( r_0 = 0 \)) will, however, suffer a modification, just like the Schwarzschild solution is modified by \( R^2 \) corrections present in the bosonic or heterotic string theory [37]. We will discuss the detailed form of this modification in the next section.

In this section we will use a simpler approach, which is essentially the traditional first order perturbation theory for the free energy. To evaluate the leading correction to the free energy, we will substitute the unperturbed metric (10) into the action term \( W \). A more detailed calculation in the next section provides a check on this simple procedure. Possible extra \( F_5 \)-dependent terms should not affect our computation since the 5-form field for the non-extremal background (10) is the same as in the case of \( r_0 = 0 \). The shift of the dilaton from its constant value (which we choose to be \( \phi_0 = 0 \)) changes the value of the action only at the next order of perturbation theory in \( \gamma \).

The ‘\( AdS_5 \)’ part of the (Euclidean) metric (14) has the following Ricci and Weyl tensors (\( m, n = \tau, r, 1, 2, 3; \ a, b = \tau, r; \ i, j, k = 1, 2, 3 \))
\[ R_{mn} = -\frac{4}{L^2} g_{mn}, \quad R = -\frac{20}{L^2}, \quad (19) \]
\[ C^a_{\ cd} = 3X \epsilon^{ab}_{\ cd}, \quad C^{ai}_{\ bj} = -X \delta^a_{\ b} \delta^i_{\ j}, \quad C^{ij}_{\ kl} = X (\delta^i_k \delta^j_l - \delta^i_l \delta^j_k) , \quad X \equiv \frac{1}{L^2} \frac{r_0^4}{r^4}, \]
so that \( W \) in (17) is given by\(^6\)
\[ W = \frac{180}{L^8} \frac{r_0^{16}}{r^{16}}. \quad (20) \]

\(^5\)As in the case of the group space compactification (the WZW model), the special scheme choice is crucial also for having the parameters of \( AdS_n \times S^m \) backgrounds being unchanged by higher-order corrections (for example, the field redefinitions like \( g_{mn} \rightarrow g_{mn} + a R_{mn} + \ldots \) rescale the factors in the metric).

\(^6\)The same result is found for the full 10-dimensional metric (10). This is related to the fact that the the value of the Ricci scalar of this metric is zero in both the extremal and the non-extremal cases.
The simplicity of this result is obviously a consequence of the conformal flatness of the \( AdS_5 \) metric which is the \( r_0 \to 0 \) limit of (10). As a result, the correction to the action

\[
\delta I = T^{-1} \delta F = -\frac{1}{16\pi G_5} \int d^5x \sqrt{g_5} \gamma W
\]

is a perfectly convergent integral at large \( r \). Since \( \text{Vol}(S^5) = \pi^3 L^5 \), we have

\[
\frac{1}{16\pi G_5} = \frac{\pi^3 L^5}{16\pi G_{10}} = \frac{\pi^3 L^5}{2\kappa^2}.
\]

(22)

Thus, using \( \sqrt{g_5} = \left(\frac{r}{L}\right)^3 \), we arrive at the following form of the correction to the free energy,

\[
\delta F = -\frac{\pi^3 L^5}{2\kappa^2} V_5 \frac{\alpha^3}{L^{11}} \frac{45}{2} \zeta(3) \int_{r_0}^\infty dr \ r^3 \frac{r^{16}}{r^{16}}.
\]

(23)

For the unperturbed solution, the temperature is \( T = \frac{r_0}{\pi L} \). Combining this with the relation based on the charge quantization rule for 3-branes [6, 17],

\[
L^4 = \frac{N\kappa}{2\pi^{5/2}} = 2g_{\text{YM}}^2 N\alpha'^2,
\]

(24)

we find

\[
\delta F = -\frac{\pi^2}{8} N^2 V_3 T^4 \left[ 1 + \frac{15}{8} \zeta(3)(2g_{\text{YM}}^2 N)^{-3/2} \right].
\]

(25)

Thus,

\[
F = F_0 + \delta F = -\frac{\pi^2}{8} N^2 V_3 T^4 \left[ 1 + \frac{15}{8} \zeta(3)(2g_{\text{YM}}^2 N)^{-3/2} \right],
\]

(26)

so that the leading correction is \textit{positive}.

Thus, if we write

\[
F = -f(g_{\text{YM}}^2 N) \frac{\pi^2}{6} N^2 V_3 T^4,
\]

(27)

so that \( f \) approaches 1 for small \( g_{\text{YM}}^2 N \), then for large \( g_{\text{YM}}^2 N \) we have

\[
f(g_{\text{YM}}^2 N) = \frac{3}{4} + \frac{45}{32} \zeta(3) (2g_{\text{YM}}^2 N)^{-3/2} + \ldots,
\]

(28)

\section{The perturbed solution}

In order to find the perturbed solution, we need to generalize the calculation of the action to arbitrary static metrics with same symmetry properties. This turns out to be surprisingly simple. If we consider the \((p + 2)\)-dimensional metric

\[
ds^2 = H^2 \left( K^2 dr^2 + P^2 d\tau^2 + \sum_{i=1}^{p} dx_i^2 \right)
\]

(29)
where \( H, K, \) and \( P \) are functions of \( r \) only, then the invariant in (17) is given by

\[
W = d_{p+2} \frac{1}{K^4H^8P^4} \left[ \left( \frac{K'}{P} \right)^4 \right],
\]

where for the most interesting cases of \( p + 2 = 4, 5, 7 \)

\[
d_4 = \frac{1}{18}, \quad d_5 = \frac{5}{36}, \quad d_7 = \frac{292}{1125}.
\]

Primes will always denote derivatives with respect to \( r \).

Because of the symmetry of the metric, we may reduce the action (15) to one dimension:

\[
I = -\frac{N^2}{8\pi^2} V_3 \beta \int_{r_0}^{\infty} dr \sqrt{g_5} \left( R_5 + 12 - \frac{1}{2} (\partial_r \phi)^2 + \gamma e^{-\frac{3}{2} \phi} W \right),
\]

where \( \gamma = \frac{1}{8} \zeta(3) \alpha^3 \). In this section we shall set \( L = 1 \) (the dependence on \( L \) of \( \gamma \)-dependent corrections can be restored by \( \gamma \to \gamma L^{-6} \)).

There is a slight subtlety in the derivation of (31): the 5-form field strength is required to be self-dual in the full, \( \alpha' \)-corrected, ten-dimensional metric; and as in the unperturbed solution, there are \( N \) units of 5-form flux through the \( S^5 \). The condition of self-duality is easily made explicit when the ten-dimensional metric has no components mixing the \( S^5 \) and the \( AdS_5 \) directions, which is the case even when the \( \alpha' \) corrections are taken into account. Then for each of the 5-dimensional factors we have \( F_{abcde} = \sqrt{h_5} \epsilon_{abcde} \), where \( h_{5ab} \) is the 5-dimensional part of the metric, and there are no ‘mixed’ components of the 5-form field strength. This choice solves the 5-form equations of motion. Moreover, the corrections to these equations coming from possible derivative \( DF_5 \) terms which accompany the \( R^4 \) terms also vanish. Using the ten-dimensional Einstein equation, one can verify that, with this choice of the 5-form background, the cosmological constant in (31) receives no explicit \( \alpha' \) corrections.\(^8\)

To leading order in \( \gamma \) the dilaton perturbation \( \phi_1 = \phi - \phi_0 \) enters the action as

\[
I(\phi_1) = -\frac{N^2}{8\pi^2} V_3 \beta \int_{r_0}^{\infty} dr \sqrt{g_5} \left( -\frac{1}{2} \phi_1'^2 - \frac{3}{2} \gamma \phi_1 W + ... \right).
\]

To this order the dilaton perturbation does not mix with the metric perturbation, so the dilaton can be ignored altogether in the computation of the correction to the action.

\(^7\)Let us note that for this class of metrics \( W \sim (C_{mnkl}C^{mnkl})^2 \), and

\[
C_{mnkl}C^{mnkl} = q_{p+2} \frac{1}{K^2H^4P^2} \left[ \left( \frac{K'}{P} \right)^4 \right]^2, \quad q_4 = 2, \quad q_5 = \frac{4}{3}, \quad q_7 = \frac{8}{3}.
\]

\(^8\) We are grateful to E. Kiritsis for raising the question about our solution that prompted us to add the above explanation.
To express the action (31) most simply, it helps to set
\[ H = r, \quad K = e^{a+4b}, \quad P = e^b. \] (33)

Then we can write
\[
\ell = \sqrt{g_5} (R_5 + 12) = -2re^{a+3b}(2 + ra') + 12r^5e^{a+5b} - \frac{d}{dr} \left[ (a' + 4b')r^3e^{a+3b} \right],
\]
\[ w = \sqrt{g_5} W = \frac{5}{36} \frac{e^{a-3b}}{r^3} \left( a'^2 + 7a'b' + 12b'^2 + a'' + 4b'' \right)^4, \] (34)
\[ I = -\frac{N^2}{8\pi^2} V_3 \beta \int_r^\infty dr \left[ \ell(a, a', b) + \gamma w(a, a', a'', b, b') \right]. \]

The Euler-Lagrange equations which follow from this action,
\[
\frac{\partial \ell}{\partial a} - \frac{d}{dr} \frac{\partial \ell}{\partial a'} = -\gamma \left( \frac{\partial w}{\partial a} - \frac{d}{dr} \frac{\partial w}{\partial a'} + \frac{d^2}{dr^2} \frac{\partial w}{\partial a''} \right),
\]
\[
\frac{\partial \ell}{\partial b} = -\gamma \left( \frac{\partial w}{\partial b} - \frac{d}{dr} \frac{\partial w}{\partial b'} + \frac{d^2}{dr^2} \frac{\partial w}{\partial b''} \right),
\] (35)

can be manipulated into quite tractable equations, namely
\[
b' + (2r^3 + \gamma w_a)e^{2b} = 0, \]
\[
a' + \frac{2}{r} - (10r^3 + \gamma w_b)e^{2b} = 0, \] (36)

where
\[
w_a = \frac{1}{6r^2e^{a+5b}} \left( \frac{\partial w}{\partial a} - \frac{d}{dr} \frac{\partial w}{\partial a'} + \frac{d^2}{dr^2} \frac{\partial w}{\partial a''} \right)
\]
\[ = 90r_0^{12}r_1^{13}(-16r^4 + 19r_0^4) + O(\gamma), \]
\[
w_b = \frac{1}{6r^2e^{a+5b}} \left( \frac{\partial w}{\partial b} - \frac{d}{dr} \frac{\partial w}{\partial b'} + \frac{d^2}{dr^2} \frac{\partial w}{\partial b''} \right)
\]
\[ = 90r_0^{12}r_1^{13}(-64r^4 + 79r_0^4) + O(\gamma). \] (37)

The first equation in (36) is separable when \( w_a \) is regarded as a function of \( r \) only. We may take \( e^{-2b} \to 0 \) at the horizon as a boundary condition on \( b \) (then the position of the horizon \( r = r_0 \) is not shifted). Thus
\[
e^{-2b} = \int_{r_0}^r ds \left[ 4s^3 + 2\gamma w_a(s) \right], \] (38)

and the second equation in (36) can be integrated directly. The result is
\[
b = -\frac{1}{2} \log(r^4 - r_0^4) + \frac{15}{2} \gamma \left( 5 \frac{r_0^4}{r^4} + 5 \frac{r_0^8}{r^8} - 19 \frac{r_0^{12}}{r^{12}} \right) + O(\gamma^2), \]
\[
a = -2 \log r + \frac{5}{2} \log(r^4 - r_0^4) - \frac{15}{2} \gamma \left( 25 \frac{r_0^4}{r^4} + 25 \frac{r_0^8}{r^8} - 79 \frac{r_0^{12}}{r^{12}} \right) + O(\gamma^2). \] (39)
There is an arbitrary choice of additive constant in the \( O(\gamma) \) term in \( a \), corresponding to rescalings of the time variable. The choice made in (39) leads to the same normalization of the time variable as in the extremal \( \text{AdS}_5 \) metric. Different normalizations of the time variable, \( t \to \lambda t \), change the final relation (45) by sending \( F \to \lambda F \), \( T \to \lambda T \).

Clearly, our choice of normalization is preferred once one chooses the particular timelike Killing vector field \( \partial/\partial t \) in the extremal solution.

For completeness let us note that the equation for the dilaton perturbation,

\[
\frac{1}{r^3} \left[ r(r^4 - r_0^4) \phi_1' \right]' = \frac{3}{2} \gamma W, \tag{40}
\]

has the following solution regular at the horizon,

\[
\phi_1 = -\frac{45}{8} \frac{\gamma}{r} \left( \frac{r_0^4}{r^4} + \frac{r_0^8}{2r^8} + \frac{r_0^{12}}{3r^{12}} \right) . \tag{41}
\]

Thus, the leading \( \alpha' \) correction makes the dilaton and the effective string coupling \( e^\phi \) decrease towards the horizon.

The temperature is defined through the absence of a conical singularity in the periodic Euclidean metric: the surface gravity at the horizon is

\[
\hat{\kappa} = 2\pi T = \sqrt{g^{rr}} \frac{d}{dr} \sqrt{g_{00}} \bigg|_h = e^{a + 3b} \left( a' + 4b' + \frac{1}{r} \right) \bigg|_{r=r_0} = 2r_0(1 + 15\gamma) . \tag{42}
\]

We may now compute the \( O(\gamma) \) correction to the free energy as a function of the gravitational action of the perturbed geometry and then using (41). The integral defining the action, \( I \), must be regulated by subtracting off its zero temperature limit, \( I_0 \). Following \cite{27} we accomplish this by first cutting off the integrals at a large radius \( r_{\text{max}} \). Thus, to leading nontrivial order in \( \gamma \),

\[
I = -\frac{N^2}{8\pi^2} \beta V_3 \int_{r_0}^{r_{\text{max}}} dr \sqrt{g_5} \left( R_5 + 12 + \gamma W \right) = \frac{N^2}{4\pi^2} \beta V_3 (r_{\text{max}}^4 - r_0^4) \left( 1 - 75\gamma \left[ \frac{r_0^4}{r_{\text{max}}^4} + O \left( \frac{r_0^8}{r_{\text{max}}^8} \right) \right] \right) , \tag{43}
\]

\[
I_0 = -\frac{N^2}{8\pi^2} \beta' V_3 \int_0^{r_{\text{max}}} dr \sqrt{g_5} \left( R_5 + 12 \right) = -\frac{N^2}{4\pi^2} \beta' V_3 r_{\text{max}}^4 .
\]

In the expression for \( I_0 \) we have noted that the Weyl tensor term vanishes, and we have defined the periodicity

\[
\beta' = e^{a + 4b} \bigg|_{r_{\text{max}}} = \beta \left[ 1 - \frac{1}{2} (1 + 75\gamma) \frac{r_0^4}{r_{\text{max}}^4} + O \left( \frac{r_0^8}{r_{\text{max}}^8} \right) \right] . \tag{44}
\]

10
for the Euclidean time so that the proper length of the circle which it parametrizes is the same at the radius $r_{\text{max}}$ as for the near-extremal metric. After taking the difference of $I$ and $I_0$ we may send $r_{\text{max}} \to \infty$ thus finding the following expression for the free energy,

$$F = \frac{I - I_0}{\beta} = -\frac{N^2}{8\pi^2} V_3 r_0^4 (1 + 75\gamma) = -\frac{\pi^2}{8} N^2 V_3 T^4 (1 + 15\gamma) \ .$$  

(45)

Surprisingly, this is equal (after $\gamma \to \gamma L^{-6}$) to the result (26) obtained in the previous section by ignoring all perturbations to the metric and computing only the change in the action from the Weyl tensor term. That means that the features of the geometry relevant to its thermodynamics are miraculously unchanged at order $\alpha'^3$. Notice that the ratios $T/r_0$, $\beta'/\beta$, and the action $I$ have all changed from the values used in section 2. But the basic thermodynamic relation (26) remained the same.

Using standard thermodynamics, it is easy to derive the corresponding expression for the entropy. It has the same $\gamma$ dependent factor as the free energy,

$$S = \frac{\pi^2}{2} N^2 V_3 T^3 (1 + 15\gamma) \ .$$  

(46)

Let us observe that the $O(\gamma)$ correction to the entropy cannot be obtained by evaluating the horizon area of the perturbed solution divided by $4G_5$. The latter gives instead

$$\frac{A_h}{4G_5} = \frac{\pi^2}{2} N^2 V_3 T^3 (1 - 45\gamma) \ .$$  

(47)

The fact that the entropy is not directly related to the horizon area in higher-derivative gravity was already noted in [37, 38, 39, 40].

So far we have discussed the corrections to the entropy in inverse powers of the 't Hooft coupling, which originate from the $\alpha'^m$ corrections in the tree level string effective action. Another interesting direction is to consider the $1/N^2$ effects which originate from the string loop corrections. In fact, the leading term induced at one loop is of the same $R^4$ form (14) that determines the leading planar correction. Following [20] we can actually replace the tree-level $\zeta(3)$-coefficient of the $R^4$ term by the function [41] containing tree-level, one-loop and non-perturbative D-instanton corrections

$$2\zeta(3)(2g_{\text{YM}}^2 N)^{-3/2} \quad \to \quad 2\zeta(3)(2g_{\text{YM}}^2 N)^{-3/2} + \frac{1}{24N^2} (2g_{\text{YM}}^2 N)^{1/2}$$

$$+ \ 1 \ 2 \ (e^{-4\pi^2/g_{\text{YM}}^2} (1 + o(g_{\text{YM}}^2)) \ ,$$  

(48)

where $h$ represents infinite series of instanton corrections (our definition of $g_{\text{YM}}^2 = 2\pi g_s$ is different from the one used in [20] by factor of 1/2). In particular, we find the following string one-loop contribution to the entropy,

$$\delta S = \frac{5\pi^2}{256} (2g_{\text{YM}}^2 N)^{1/2} V_3 T^3 \ .$$  

(49)
Since it comes from the one-loop correction with the smallest number of derivatives, this should be the dominant term at strong coupling. One-loop contributions to the entropy that scale as $T^3$ were recently discussed in [22].

Another interesting question is whether the $SL(2, Z)$ symmetry imposes simple constraints on the function $f(g_{YM}^2 N)$. In [20] it was pointed out that, although the underlying theory is certainly S-dual, the term-by-term expansion of the function of the 't Hooft coupling does not satisfy any simple constraints. Hence, there is no paradox associated with the appearance of the fractional power of $g_{YM}^2 N$ at strong coupling. Indeed, in choosing a fixed 't Hooft coupling we automatically choose a small $g_{YM}$ which masks the constraints imposed by the $SL(2, Z)$ in the 't Hooft limit (cf. [12]).

4 Free energy of other CFT’s

The low-energy limit of the world volume theory of coincident D3 branes is in many ways of special importance because it is related to four-dimensional Yang-Mills theory. However, there are other interesting brane configurations related to conformal field theories in other dimensions, and in this section we discuss their thermodynamic properties.

4.1 D5+D1 and black hole entropy

One such interesting configuration involves $N_1$ D1-branes bound to $N_5$ D5-branes [13, 24]. This defines a conformal field theory in $1 + 1$ dimensions whose coupling constant is measured by $g_s$. When $g_s N_1$ and $g_s N_5$ are much smaller than one, then this field theory can be studied using standard D-brane methods and its entropy may be counted. In the limit $g_s N_1, g_s N_5 \to \infty$ the conformal field theory describes a black string in six dimensions, whose near-horizon geometry is (as in the S-dual case [43, 45]) the $AdS_3 \times S^3 \times T^4$ background of type IIB supergravity. In [13] the entropy of BPS states at weak coupling was compared with the Bekenstein-Hawking entropy, which is applicable at strong coupling, and perfect agreement was found. An obvious reason for this agreement is supersymmetry. In [24, 25] this agreement was extended to near-extremal entropy. Here the supersymmetry argument cannot be used, and the exact agreement of the coefficients is puzzling.

Motivated by our analysis of the $\alpha'$ corrections, we propose an explanation. From the point of view of string theory on $AdS_3 \times S^3 \times T^4$, the leading corrections in inverse powers of $g_s N_1$ and $g_s N_5$ should come from the $C^4$ terms, in complete analogy with our $AdS_5 \times S^5$ calculation. The answer is particularly simple here because the Weyl tensor vanishes even for the near-extremal solution! The space-time contains a product of two three-dimensional spaces, which separately cannot have any Weyl curvature. It turns
out that the Weyl tensor for the whole space factorizes (and therefore vanishes) only if the radii of the $AdS_3$ and $S^3$ factors are equal. This requirement is fulfilled here, just as in the $AdS_5 \times S^5$ case.

Furthermore, if we assume that all the $\alpha'$ corrections can be written in an appropriate scheme in terms of the Weyl tensor and superpartners, then it would seem that the action and the geometry are not corrected, even away from extremality. Indeed, the argument is much strengthened by observing that in the near-extremal limit the $r_0$-deformed $AdS_3$ factor is actually the BTZ black hole which is locally still $AdS_3$. Thus even for $r_0 \neq 0$ we have locally $AdS_3 \times S^3$, i.e. a space conformal to a flat space because of equal radii of the factors.

This implies that all $\alpha'$ corrections vanish, and thus the free energy in the CFT of the D1-branes bound to the D5-branes is completely independent of the string coupling! Thus, we have found a plausible explanation for the agreement of the near-extremal entropy of the 6d black string with the weakly coupled D-brane calculation. Let us also note that a completely analogous argument applies to near-extremal black holes in $D = 4$ whose near-horizon geometry is $AdS_2 \times S^2$, and facilitates string theoretic studies of their entropy. Black holes of this type include the classic Reissner-Nordstrom solution.

Our line of reasoning relies completely on having the non-extremality parameter $r_0$ much smaller than the scale of the $AdS_3$ geometry, because only the near-horizon limit of the geometry decomposes into a product of three-dimensional or flat factors. One would therefore expect any corrections to the entropy of the full D1+D5 solution (which no longer enjoys this product structure) to contain inverse factors of the extremal mass.

### 4.2 Free energy of M-branes

We now discuss the conformal theories on multiple coincident M5 and M2 branes. Because of the conformal invariance, the higher-order corrections to the $D = 11$ supergravity action are expected to change the leading-order expressions only by replacing the coefficient $k_p N^{p+1}$ by a function $f_p(N)$ which approaches $k_p N^{p+1}$ for large $N$. In the case $D = 10$, $p = 3$, discussed in the previous two sections, there were two independent dimensionful quantities ($\alpha'$ and the Planck length) which could be combined in appropriate power with the ra-

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9For related discussion of higher-order terms in the string effective action and possible subtleties see 10.

10This is not so surprising when the 5 + 1 bound state is realized in terms of solitonic five-branes and fundamental strings: the (orbifold of) $SL(2,R) \times SU(2)$ WZW model is an exact conformal theory. See also 11 for a related discussion.

11The above argument applies also to the rotating versions of the $D = 5, 4$ black holes (for a discussion of their near-horizon geometry and entropy see 12).
radius of curvature of the geometry to obtain dimensionless parameters \( g^2_{YM} N \) and \( N \) which parametrize these subleading corrections. Now the situation is somewhat different because there is no \( \alpha' \) parameter in M-theory, only the Planck length \( l_{11} \sim \kappa_{11}^{2/9} \). Correspondingly, we shall demonstrate explicitly that there are only \( 1/N \) corrections to the free energy. This will be done by performing the direct analog of the computation of the leading correction \((26)\) to the D3-brane free energy in the case of M5 and M2 branes. We will not work out the perturbed geometry in these cases, though this is straightforward using the same methods as in section \(3\).

The \( D = 11 \) supergravity has a cubic \( R^4 \) 1-loop UV divergence \((\ref{eq:1loop})\) which, when computed with a specific cutoff \( \Lambda_{11} = \pi^{2/3} l_{11}^{-1} \) motivated by string theory \((\ref{eq:cutoff})\), leads to the following correction to the leading Einstein action

\[
I = - \int d^{11} x \sqrt{g} \left( \frac{1}{2\kappa_{11}^2} R + \frac{1}{\kappa_{11}^{2/3}} \xi W + \ldots \right),
\]

where \( W \sim RRRR \) has the same structure as \((\ref{eq:W})\). The coefficient of this \( R^4 \) term is expected to be universal (as it may be related to the ‘anomaly’ \( \int C_3 R^4 \) term \((\ref{eq:anomaly})\) by supersymmetry \((\ref{eq:susy})\)).

The near-horizon limits of the extremal M2 and M5 solutions, i.e. the backgrounds \((\text{AdS}_7)_2 L \times (S^4)_L \) and \((\text{AdS}_4)_{1/2} L \times (S^7)_L \) (where we have indicated the values of the radii of the factors) have maximal supersymmetry and should not be modified by higher-order corrections to the effective action of M-theory \cite{36}. Since these \( D = 11 \) spaces are not conformally flat (because of the different radii of the two factors), their exactness should be manifest in a ‘\( D = 11 \) supersymmetric’ scheme which may be different from the one in which \( W \) is expressed in terms of the Weyl tensor only \((\ref{eq:W})\). We shall assume that such scheme in which extremal solutions are exact indeed exists and concentrate on corrections to the non-extremal \( (r_0 \text{-dependent}) \) solutions.

The important observation that allows us to use the \((p + 2)\)-dimensional analogue of the expression \((\ref{eq:W})\) to compute corrections to the free energy of Mp-branes is that the sphere \((S^{9-p})\) part of the metric and the 4-form field strength of the near-horizon backgrounds are not modified by the non-extremality parameter \( r_0 \). As a result, the \( r_0 \) dependent correction to the action is effectively determined by the \( p + 2 \) dimensional \((S^{9-p} \text{ compactified})\) theory. Since \( \text{AdS}_{p+2} \) is conformally flat (and, by assumption, must not be modified by the \( R^4 \) term), the correction is again described by the Weyl-tensor dependent term \((\ref{eq:W})\) only.\footnote{As in the D3-brane case, we are ignoring possible additional terms depending on the \( F_4 \) field strength. We expect that such terms which have the same dimension as \( R^4 \) and have ‘reducible’ contractions of indices only are scheme-dependent and thus are adjusted to have \( \text{AdS}_{p+2} \times S^{9-p} \)}
Let us start with the M5-brane case. The throat limit of black M5-brane metric (see, e.g., [58]) is

$$ds^2 = \frac{r}{L} (-f dt^2 + \sum_{i=1}^{5} dx_i^2) + \frac{L^2}{r^2} f^{-1} dr^2 + L^2 d\Omega_4^2,$$

(51)

where \( f = 1 - \frac{r_0^3}{r^3} \). Introducing the variable \( y = \sqrt{Lr} \), we bring the metric into the form used in [27] to describe a black hole in \( AdS_{p+2} \) (cf. (10))

$$ds^2 = \frac{y^2}{L^2} \left[ - (1 - \frac{y_0^6}{y^6}) dt^2 + \sum_{i=1}^{5} dx_i^2 \right] + 4 \frac{L^2}{y^2} (1 - \frac{y_0^6}{y^6})^{-1} dy^2 + L^2 d\Omega_4^2, \quad y_0 \equiv (Lr_0)^{1/2}.$$

(52)

Computing the Weyl tensor for the 7-dimensional part of this metric one finds that the invariant in (17) has the following value

$$W = \frac{3285}{64L^8} \frac{y_0^{24}}{y^{24}}.$$

(53)

The corresponding correction to the Euclidean action then takes the form

$$\delta I = - \frac{3285}{64L^8} \xi \kappa_{11}^{-2/3} \text{Vol}(S^4) \beta V_5 \int_{y_0}^{\infty} dy \sqrt{g_7} \frac{y_0^{24}}{y^{24}},$$

(54)

where \( \text{Vol}(S^4) = \frac{8\pi^2}{3} L^4 \) and \( \sqrt{g_7} = 2 \frac{y^2}{y_7} \). As follows from the condition of regularity of the Euclidean metric, the temperature is

$$T = \beta^{-1} = \frac{3y_0}{4\pi L^2}.$$

(55)

As a result, the leading correction to the free energy is given by

$$\delta F = - k \kappa_{11}^{-2/3} L^3 V_5 T^6, \quad k = \frac{8\pi^2}{27} \cdot \xi \cdot (\frac{4\pi}{3})^6 \cdot \frac{3285}{64}.$$

(56)

Using the charge quantization for M5 branes we find the relation [7]

$$L^9 = N^3 \frac{\kappa_{11}^2}{27\pi^5}.$$

(57)

Substituting it into \( \delta F \), we finally obtain

$$\delta F = -a_1 NV_5 T^6,$$

(58)

as an exact solution, while ‘irreducible’ terms vanish in the case of our backgrounds which have \( r_0 \)-independent \( S^{9-p} \) and \( F_4 \) parts.

13This computation can be done using either (29), (30) or, directly, the curvature of (52). The Euclidean 7-metric has the Ricci tensor \( R_{mn} = -\frac{\kappa_{11}}{27} g_{mn} \), \( R = -\frac{21}{27} \), and the Weyl tensor (\( i = 1, ..., 5 \))

$$C^\tau_{\tau j} = \frac{2}{9} X, \quad C^\tau_{\tau j} = -\frac{1}{9} X \delta_i^j, \quad C_{01}^{ij} = -\frac{1}{9} X, \quad C^{ij}_{kl} = \frac{1}{9} X (\delta^i_k \delta^j_l - \delta^i_l \delta^j_k), \quad X \equiv \frac{y_0^6}{y^6}. $$
To this order the free energy of the $d = 6$ world-volume $(0, 2)$ theory is thus predicted to be given by

$$F = -V_5 T^6 \left(a_0 N^3 + a_1 N\right), \quad (59)$$

where $a_0 = 2^6 3^{-7} \pi^3$ and $a_1$ are positive numerical coefficients.

In the M2 brane case the throat metric is

$$ds^2 = \frac{r^4}{L^4}(-f dt^2 + \sum_{i=1}^2 dx_i^2) + \frac{L^2}{y^2} f^{-1} dr^2 + L^2 d\Omega_7^2, \quad (60)$$

where $f = 1 - \frac{r_0^2}{r^6}$. The variable $y = \frac{r^2}{L}$ brings (60) into the standard form for a black hole in $AdS_4$:

$$ds^2 = \frac{y^2}{L^2} \left[-(1 - \frac{y_0^3}{y^3}) dt^2 + \sum_{i=1}^2 dx_i^2 \right] + \frac{L^2}{4y^2} (1 - \frac{y_0^3}{y^3})^{-1} dy^2 + L^2 d\Omega_7^2, \quad y_0 \equiv \frac{r_0^2}{L}. \quad (61)$$

The product (17) of the Weyl tensors of the 4-dimensional part of this metric is found to be equal to (cf.(30))

$$14 W^4 = 1152 \frac{y_0^{12}}{L^8} y_0^{12}. \quad (62)$$

The correction to the Euclidean action is then

$$\delta I = -1152 \frac{\kappa^{2/3}}{L^8} \xi \, \text{Vol}(S^7) \beta \, V_2 \int_{y_0}^{\infty} dy \sqrt{g_4} \, \frac{y_0^{12}}{L^8 y^{12}}, \quad (63)$$

where $\text{Vol}(S^7) = \frac{2}{3} L^7$, $\sqrt{g_4} = \frac{1}{2} \frac{y^2}{L^2}$, and

$$T = \beta^{-1} = \frac{3y_0}{2\pi L^2}. \quad (64)$$

The leading correction to the free energy is thus

$$\delta F = -m \kappa^{2/3} L^3 V_2 T^3, \quad m = 1152 \cdot \frac{\pi^4}{18} \cdot \frac{(2\pi/3)^5}{L^8}. \quad (65)$$

Using the relation that follows from the M2 brane charge quantization [4],

$$L^9 = N^{3/2} \frac{\kappa^{11} \sqrt{2}}{\pi^5}, \quad (66)$$

we finally have

$$\delta F = -b_1 N^{1/2} V_2 T^3, \quad (67)$$

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14 This Euclidean 4-metric has the following Ricci tensor $R_{mn} = -\frac{12}{\pi^2} g_{mn}$, $R = -\frac{48}{\pi^2}$, and Weyl tensor $(i = 1, 2) C^{\tau y}_{\tau y} = 4X$, $C^{\tau}_{\tau j} = -2X \delta_j$, $C^{y j}_{y j} = -2X$, $C^{i j}_{k l} = 4X (\delta_i^k \delta_j^l - \delta_i^l \delta_j^k)$, $X \equiv \frac{\kappa^{11}}{L^9 y^9}$. 

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\[ b_1 = 64 \left( \frac{2\pi}{3} \right)^5 2^{1/6} \pi^{7/3}. \]

To this order the free energy of the conformal theory on \( N \) coincident M2-branes is thus given by

\[ F = -V_2 T^3 \left( b_0 N^{3/2} + b_1 N^{1/2} \right), \tag{68} \]

where \( b_0 = 2^{7/2} 3^{-4} \pi^2 \) and \( b_1 \) are positive.

We end this section with several comments. It was suggested in [56] that not only \( R^4 \) but all the terms \( R^{3n+1} \) \((n = 1, 2, \ldots)\) may play a similar special role in M-theory. In particular, on dimensional grounds, they appear in the effective action multiplied by integer powers of the inverse M2-brane tension, \( T_2^{-1} = 2\pi l_1^3 \). Including such terms and repeating the above analysis we find the following generalizations of the expressions for the free energy in the M5-brane case

\[ F = -V_5 T^6 N^3 \left( a_0 + \sum_{n=1}^{\infty} a_n \frac{N^{2n}}{N^{n}} \right), \tag{69} \]

and the M2-brane case

\[ F = -V_2 T^3 N^{3/2} \left( b_0 + \sum_{n=1}^{\infty} b_n \frac{N^n}{N^{n}} \right), \tag{70} \]

where \( a_n \) and \( b_n \) are numerical coefficients. Although we expect \( a_1 \) and \( b_1 \) to be given reliably by the first-order perturbation to the action, the perturbed geometry will almost certainly affect the \( a_n \) and \( b_n \) with \( n > 1 \).

In the case of \( SU(N) \) SYM theory, comparison with standard field theoretic methods may become possible if one manages to extrapolate the series in inverse powers of the ‘t Hooft coupling, calculated from type IIB string theory, to weak coupling. Is there an analogue of this statement for the CFT’s on M-branes? Here, the only parameter is \( N \) and the coupling cannot be dialed separately. The details of the field theory are rather poorly known for \( N > 1 \). The M2-brane CFT is expected to be effectively described by an IR fixed point of \( D = 3, U(N) \) SYM, while the M5 brane theory – by an UV fixed point of \( D = 5, U(N) \) SYM. The world volume theory becomes free in the conformal limit for \( N = 1 \) and the coefficient in the free energy can be fixed at this point. However, even if we knew the full asymptotic \( 1/N \) expansion of the free energy in supergravity, it is not clear that a reliable comparison could be made at \( N = 1 \).

Note Added

After this paper was completed a question was raised [56] concerning the precise meaning of the metric derived in section 3.\(^{15}\) Here we clarify this issue. We consider the

\(^{15}\)We are grateful to S. Theisen for sending us the draft of their paper prior to publication.
following ansatz for the 10-dimensional metric in the Einstein frame,

\[ ds_{10}^2 = e^{-\frac{10\nu(x)}{3}}g_{5mn}(x)dx^m dx^n + e^{2\nu(x)}d\Omega_5^2, \]  

(71)

where we set \( L = 1 \). Taking the standard ansatz for the 5-form field and compactifying on \( S^5 \), we find the following 5-dimensional effective action,

\[
I_5 = -\frac{1}{16\pi G_5} \int d^5x \sqrt{g_5} \left[ R_5 - \frac{1}{2}(\partial\phi)^2 - \frac{40}{3}(\partial\nu)^2 - V(\nu) + \gamma e^{10\nu - \frac{2}{3}\phi} \left( W + O((\partial\nu)^2) \right) \right],
\]

(72)

where

\[
V(\nu) = 8e^{-\frac{40}{3}\nu} - 20e^{-\frac{16}{3}\nu}.
\]

This shows that the field \( \nu \), which is the logarithm of the 5-sphere radius, acquires a mass; hence, it is a “fixed scalar.” If \( \gamma = 0 \), then the classical value of \( \nu \) can be chosen to be zero. This field receives a source of order \( \gamma \) and the classical equation becomes

\[
\nabla^2 \nu - 32\nu + \frac{3}{8}\gamma W + O(\nu^2) + O(\gamma^2) = 0.
\]

(73)

Its solution is [10]

\[
\nu = \frac{15\gamma r_0^8}{32} \left( 1 + \frac{r^4}{r_0^4} \right) + O(\gamma^2).
\]

(74)

The 5-dimensional metric \( g_{5mn} \) is precisely what we determined in section 3 (both the metric and the dilaton corrections do not mix with \( \nu \) to leading order in \( \gamma \)). This is the perturbed metric of a black hole in \( AdS_5 \) which governs the dynamics of fields which do not depend on the 5-sphere coordinates. However, the variation of the 5-sphere radius in the 10-dimensional metric, given by (74), does affect the modes that come from higher Kaluza-Klein harmonics.

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