On axisymmetric MHD equilibria with incompressible flows under side conditions

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Abstract

Axisymmetric equilibria with incompressible flows of arbitrary direction are studied in the framework of magnetohydrodynamics under a variety of physically relevant side conditions. To this end a set of pertinent non-linear ODEs are transformed to quasilinear ones and the respective initial value problem is solved numerically with appropriately determined initial values near the magnetic axis. Several equilibria are then constructed surface by surface. The non field aligned flow results in novel configurations with a single magnetic axis, toroidal shell configurations in which the plasma is confined within a couple of magnetic surfaces and double shell-like configurations. In addition, the flow affects the elongation and triangularity of the magnetic surfaces.
1. Introduction

In a previous paper \cite{1} the first two authors derived a generalized Grad-Shafranov equation governing the magnetohydrodynamic (MHD) equilibrium states of an axisymmetric plasma with incompressible flows [Eq. (1) in section 2]. By assignment of the free functions contained in (1), known solutions of the Grad-Shafranov equation, i.e. the Solovév solution \cite{2,3} and the Hernegger-Maschke solution \cite{4,5}, were extended in Refs. \cite{6} and \cite{7}, respectively. For both extended solutions the flow can change the magnetic field topology, thus resulting in a variety of new configurations of astrophysical and laboratory concern.

Instead of assigning the free surface functions of (1), it may be of physical or mathematical importance to introduce side conditions, e.g. isodynamicity: $B^2 = B^2(\psi)$, where $\psi$ and $B$ are the poloidal magnetic flux function and the magnetic field modulus, respectively. In the quasi-static case, viz. when the flow is neglected in the momentum equation but it is kept in Ohm's law, it was proved in Ref. \cite{8} that there is a unique configuration of this kind with circular magnetic surface cross-sections near the magnetic axis. This equilibrium was fully constructed in Ref. \cite{9}. The same configuration persists in the case of flows aligned to the magnetic field \cite{1}. In the case of non field aligned flows satisfying the side condition $P + B^2/2 = f(\psi)$, where $P$ is the thermal pressure and $f$ is an arbitrary smooth function of $\psi$, the magnetic surfaces near axis become elliptical with elongation perpendicular to the axis of symmetry \cite{10}.

The aim of the present study is to construct axisymmetric steady states with incompressible flows of generic direction under a variety of side conditions and to examine the impact of the flow on the equilibrium characteristics and particularly in connection with the magnetic topology. A preliminary investigation was conducted in Ref. \cite{11}. Here, the construction is carried out numerically on the basis of a procedure suggested in Refs. \cite{8,9} and \cite{1}. The main conclusion is that the flow results in a variety of novel equilibria and opens up the possibility of changing the magnetic field topology.

The side conditioned equilibrium equations are briefly reviewed in section 2 along with the solving procedure of Refs. \cite{8} and \cite{1}. In addition, the original ODEs of concern are mapped to quasi-linear ones by a transformation which for the quasi-static case was employed in Ref. \cite{9}. After establishing initial values of the unknown functions near the magnetic axis the problem becomes well posed and is solved numerically. The various kinds of config-
urations associated with numerical solutions are presented in section 3 and are compared with the existing ones in the literature. Section 4 summarizes the study and the conclusions.

2. Side conditioned equilibria

The MHD equilibrium states of an axisymmetric magnetized plasma with incompressible flows are determined by the generalized Grad-Shafranov equation [1],

\[ (1 - M^2) \Delta^* \psi - \frac{1}{2}(M^2)'|\nabla \psi|^2 + \frac{1}{2} \left( \frac{X^2}{1 - M^2} \right)' + R^2 P_s' + \frac{R^4}{2} \left( \frac{\rho(\Phi)^2}{1 - M^2} \right)' = 0, \]

(1)
along with the Bernoulli relation for the pressure

\[ P = P_s(\psi) - \rho \left[ \frac{v^2}{2} - \frac{R^2(\Phi')^2}{1 - M^2} \right]. \]

(2)

Here, \((z, R, \phi)\) are cylindrical coordinates with \(z\) corresponding to the axis of symmetry; the function \(\psi(R, z)\) labels the magnetic surfaces; \(M(\psi)\) is the Mach function of the poloidal velocity with respect to the poloidal-magnetic-field Alfvén velocity; \(\rho(\psi)\) and \(\Phi(\psi)\) are the density and the electrostatic potential; \(X(\psi)\) relates to the toroidal magnetic field; for vanishing flow the surface function \(P_s(\psi)\) coincides with the pressure; \(v\) is the velocity modulus which can be expressed in terms of surface functions and \(R\); \(\Delta^* = R^2 \nabla \cdot (\nabla / R^2)\); and the prime denotes a derivative with respect to \(\psi\). Derivation of (1) and (2) is provided in Ref. [1]. The surface quantities \(M(\psi), \Phi(\psi), X(\psi), \rho(\psi)\) and \(P_s(\psi)\) are free functions for each choice of which (1) is fully determined and can be solved whence the boundary condition for \(\psi\) is given.

By using the transformation

\[ u(\psi) = \int_0^\psi [1 - M^2(g)]^{1/2} dg, \]

(3)

(1) and (2), respectively, reduce to

\[ \Delta^* u + \frac{1}{2} \frac{d}{du} \left( \frac{X^2}{1 - M^2} \right) + R^2 \frac{dP_s}{du} + \frac{R^4}{2} \left( \rho \frac{d\Phi}{du} \right)^2 = 0, \]

(4)

\[ P = P_s(\psi) - \rho \left[ \frac{v^2}{2} - R^2 \left( \frac{d\Phi}{du} \right)^2 \right]. \]

(5)
Note that no quadratic term as $|\nabla u|^2$ appears anymore in (4). The forms of (4) and (5) indicate one to introduce the new surface quantities

$$N(u) = \frac{X}{\sqrt{1 - M^2}}, \quad L(u) = \sqrt{\rho} \frac{d\Phi}{du},$$

which are helpful in reducing the number of the explicit free functions by one.

Instead of specifying the free functions to determine (1), one can introduce side conditions as those quoted in table 1. They can be expressed in terms of the thermal pressure $P$, the magnetic pressure $B^2/2$, the flow energy density $\rho v^2/2$ or combinations of them and consist in that these quantities remain uniform on magnetic surfaces. Such conditions lead, in general, to an additional relation between $(\nabla u)^2$, $u$ and $R$ as already accomplished in Ref. [1]. Specifically, (4) and (5) can be put in the respective forms

$$(\nabla u)^2 = 2[i(u) + R^2 j(u) + R^4 k(u)]$$

$$\Delta^* u = -f(u) - R^2 g(u) - R^4 h(u)$$

where

$$i(u) = -\frac{N(u)^2}{2},$$

$$f(u) = \frac{1}{2} \frac{dN(u)^2}{du} = -\frac{di(u)}{du},$$

$$g(u) = \frac{dP_s(u)}{du},$$

$$h(u) = \frac{1}{2} \frac{dL(u)^2}{du}.$$
however, stemming from the non-linearity of the ODEs. Here, to overcome this difficulty, the function $u$ is mapped to a new function $w$ through the transformation:

$$
\frac{du}{dw} = -\left( g + \frac{dj}{du} \right) \equiv \mathcal{F}(w),
$$

(13)

where $\mathcal{F}(w)$ is an arbitrary smooth function.

Under this transformation the compatibility condition leads to the following set of ODEs

$$
4Q + 2\Theta Y' - Y(Y + \Theta') = 0,
$$

(14)

$$
8\Xi + Y(4 - Q') + 2QY' + \Theta' = 0,
$$

(15)

$$
-3 - 2\Theta H' + Q' + 2\Xi Y' + H(6Y + \Theta' - Y\Xi') = 0,
$$

(16)

$$
-2QH' + H(Q' - 8) + \Xi' = 0,
$$

(17)

$$
-5H^2 - 2\Xi H' + H\Xi' = 0,
$$

(18)

where the functions $Q(w), \Theta(w), Y(w), \Xi(w)$ and $H(w)$ can be expressed in terms of the initial physical surface quantities. Note that (14)-(18) are quasi-linear, viz. the derivatives appear linearly, and therefore Picard’s theorem guarantees existence and uniqueness of the respective initial value problem. Because of indefiniteness of some of (14)-(18) on the magnetic axis when they are put in solved forms, initial values of the unknown functions near the magnetic axis can be obtained on the basis of Mercier expansions around axis along with a l’Hospital-like procedure. Details will be given elsewhere. Also, the integral relation for $z(w, x)$ is given by

$$
\frac{\partial z}{\partial x} \bigg|_w = -\frac{p}{q} = \pm 14 \left[ Hx^2 + x - Y \right] \left[ \Theta + Qx + x^2\Xi - \frac{1}{4}x(Hx^2 + x - Y)^2 \right]^{1/2},
$$

(19)

where $x = R^2$, $p \equiv \partial w/\partial x$, $q \equiv \partial w/\partial z$, $r \equiv \partial^2 w/\partial x^2$ and $q \equiv \partial^2 w/\partial z^2$. Once initial values are established the problem is well posed and can be solved numerically. Accordingly, we have developed a programme in Mathematica 5.1. For up-down symmetric configurations to be considered here there are three free parameters ($R_0$, $\Xi_0$ and $H_0$) associated with the radial distance of the magnetic axis and the functions $\Xi$ and $H$ thereon. It is also noted that the problem can be solved by the transformation, alternative to (13)

$$
\frac{du}{dw} = -\frac{1}{2} \left( h + \frac{dk}{du} \right) \equiv \mathcal{F}(w).
$$

(20)
3. Magnetic configurations and impact of flow

For the solutions to be presented here we have chosen, without loss of generality, \( R_0 = 1 \) and \( w_0 = 0 \), where \( w_0 \) is the value of \( w \) on axis. Depending on the sign of \( \Xi_0 \), there are two kinds of configurations:

1. Toroidal configurations for \( \Xi_0 < 0 \) having a single magnetic axis [located on \((z = 0, R = R_0)\) where \( w = w_0 = 0 \)]. These configurations have magnetic field topology similar to those of Refs. [9] and [10]. \( w \)-contours of such a solution are shown in Fig. 1. In general for \( \Xi_0 < 0 \) the magnetic surfaces are more elongated horizontally (parallel to the mid-plane \( z = 0 \)) up to the magnetic axis as compared with the quasi-static ones corresponding to \( \Xi_0 = H_0 = 0 \).

2. Toroidal shells for \( \Xi_0 > 0 \) in which the plasma is contained within two toroidal surfaces. \( w \)-contours for a solution of this kind are shown in Fig. 2. Although mathematically the solution has an extremum on \( w = 0 \), the physically acceptable part is restricted to non positive values of \( \Theta(w) \), thus resulting in a toroidal vertically elongated shell.

The various kinds of configurations can also be classified in terms of the parameters \( \Xi_0 \) and \( H_0 \) as follows.

1. \( \Xi_0 = 0 \) and \( H_0 \neq 0 \):
   Let us first note that \( \Xi(w) = 0 \) implies parallel flows (or the quasi-static equilibrium) because then it follows from (18) that \( H(w) = 0 \). This should not be confused with the case of \( \Xi_0 = 0 \) (on axis) in this paragraph which involves non parallel flows when \( H_0 \neq 0 \). As in the case of parallel flows, however, the magnetic surfaces near axis have circular cross sections. For \( H_0 > 0 \) the surfaces far from axis are less parallel elongated than those of the quasi-static equilibrium. For \( H < 0 \) the triangularity can change drastically. As an example, a configuration with inverse triangularity is shown in Fig. 3.

2. \( \Xi_0 \neq 0 \) and \( H_0 = 0 \):
   Equilibria for \( \Xi_0 < 0 \) and \( H_0 = 0 \) have been constructed in Ref. [10] by a different method. Unlike in Ref. [10], however, no restriction on the elongation of the magnetic surfaces (parallel to the mid-plane \( z = 0 \)
was found here. As a matter of fact such a configuration with very elongated surfaces is presented in Fig. 4 (in purple). Thus, the limitation on the elongation reported in [10] may be due to the particular method of solution in that paper. For $\Xi_0 > 0$ the equilibrium becomes a toroidal shell with magnetic surfaces elongated perpendicular to the mid-plane $z = 0$.

3. $\Xi_0 \neq 0$ and $H_0 \neq 0$:
In this generic case it is particularly interesting to examine whether there are configurations with two magnetic axes. This requires two roots in the numerator of (19) and four roots in the denominator appropriately located with respect to the roots of the numerator. (Note that for a quasi-static equilibrium ($H = \Xi = 0$) only configurations with a single magnetic axis are possible.) For this reason we first examined this requirement by applying the Sturm theorem and Descartes rule [12]. The former determines the exact number of real roots of a polynomial with real coefficients; the latter determines the maximum number of positive roots of such a polynomial. It turns out that the requirement is compatible with the Sturm theorem and Descartes rule. Then, by inspection we found that the requirement can be fulfilled for $\Xi_0 > 0$ and $H_0 < 0$. An equilibrium of this kind shown in Fig. 5 consists of a toroidal shell reaching the axis of symmetry, similar to those reported in paragraph 3.2, and a second thin shell-like configuration located farther from the axis of symmetry. The distance between the two configurations decreases as $|\Xi_0/H_0|$ takes larger values. As can be seen in Fig. 5, however, the magnetic surfaces of the shell-like configuration do not close. Closeness does not improve either by varying $\Xi_0$ and $H_0$ or by using a different numerical method in FORTRAN. Therefore, the existence of double toroidal shell configurations with closed magnetic surfaces remains an open question.

For the other three combinations of signs of $\Xi_0$ and $H_0$ one can obtain configurations similar to those presented in paragraph 3.2. As in the case of $H_0 = 0$, for $\Xi_0 < 0$ there is no limitation on the elongation of the magnetic surfaces parallel to the mid-plane $z = 0$ as $|\Xi_0|$ increases. Also, it is worth to mention the strong change in the triangularity of the configuration for $H_0 < 0$ as it is illustrated in Fig. 6.

4. Summary and Conclusions
We have studied axisymmetric equilibria with incompressible flows under side conditions of physical relevance by a procedure introduced in Refs. [8], [9] and [1]. This procedure reduces the problem to a set of ODEs for certain surface functions and an integral relation determining the points of the cross section of a magnetic surface with the poloidal plane. Because of the nonlinearity of the original ODEs, we have employed transformation (13) mapping the original ODEs to quasilinear ones [Eqs. (14)-(18)] and containing equal number of unknown surface functions; thus, existence and uniqueness of the respective initial value problem is guaranteed. After determining appropriately initial values near axis, because of indefiniteness of the ODEs thereon, the problem has been solved numerically surface by surface with two free parameters ($\Xi_0$ and $H_0$) associated with the non-field aligned flow.

The flow results in the following novel kinds of up-down symmetric equilibria:

1. Configurations with a single magnetic axis for $\Xi_0 \leq 0$ with magnetic surfaces near axis elongated parallel to the mid-plane $z = 0$. For $\Xi_0 = 0$ the magnetic surfaces near axis become circular as the quasi-static ones corresponding to $\Xi_0 = H_0 = 0$. The special case of equilibria constructed by a different method in Ref. [10] are recovered for $H_0 = 0$. Unlike in Ref. [10], however, no restriction on the elongation has been found in the present study.

2. Toroidal shells for $\Xi_0 > 0$ and $H_0 \geq 0$ in which the plasma is confined in the interior of two nested magnetic surfaces. The magnetic surfaces are elongated perpendicular to the mid-plane $z = 0$ compared with the quasi-static ones.

3. Equilibria consisting of a toroidal shell reaching the axis of symmetry and a second shell-like configuration for $\Xi_0 > 0$ and $H_0 < 0$. Thus, the flow opens up the possibility of changing the magnetic field topology.

Also, the shape of the magnetic surfaces far from axis is affected by the value and sign of $H_0$; specifically: a) they become less elongated parallel to the mid-plane $z = 0$ as $H_0$ takes larger positive values and b) the triangularity of those surfaces is affected drastically for negative values of $H_0$.

It is emphasized that the above reported conclusions hold irrespective of the particular condition of table [1] except for $P + B^2/2$ being uniform on surfaces which corresponds to $H_0 = 0$. The properties of particular equilibria
in connection with profiles of the (original) physical quantities, i.e. pressure, magnetic field, velocity etc, deserves further investigation. Also, in view of the tough and in general unsolved stability problem of steady states with flow, the stability of the equilibria constructed here remains an open question.

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