ON THE DIMENSION OF GLOBAL ATTRACTOR FOR THE CAHN-HILLIARD-BRINKMAN SYSTEM WITH DYNAMIC BOUNDARY CONDITIONS

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ABSTRACT. The objective of this paper is to study the fractal dimension of the global attractor for the Cahn-Hilliard-Brinkman system with dynamic boundary conditions. Inspired by the idea of the ℓ-trajectory method, we prove the existence of a finite dimensional global attractor in an auxiliary normed space for the Cahn-Hilliard-Brinkman system with dynamic boundary conditions and estimate the fractal dimension of the global attractor in the original phase space for this system by defining a Lipschitz mapping from the auxiliary normed space into the original phase space.

1. Introduction. In this paper, we consider the fractal dimension of the global attractor for the following Cahn-Hilliard-Brinkman system:

\[
\begin{align*}
\frac{\partial \phi}{\partial t} + \nabla \cdot (u \phi) &= \nabla \cdot (M \nabla \mu), \quad (x,t) \in \Omega \times \mathbb{R}^+, \\
\mu &= -\epsilon \Delta \phi + \frac{1}{2} f(\phi), \quad (x,t) \in \Omega \times \mathbb{R}^+, \\
-\nu \Delta u + \eta u &= -\nabla p - \gamma \phi \nabla \mu, \quad (x,t) \in \Omega \times \mathbb{R}^+, \\
\nabla \cdot u &= 0, \quad (x,t) \in \Omega \times \mathbb{R}^+.
\end{align*}
\]

Equations (1.1) is subject to the following dynamic boundary conditions

\[
\begin{align*}
u(x,t) = 0, \quad (x,t) \in \Gamma \times \mathbb{R}^+, \\
\frac{\partial \mu}{\partial n} &= 0, \quad (x,t) \in \Gamma \times \mathbb{R}^+, \\
\frac{1}{\kappa} \frac{\partial \phi}{\partial t} &= \alpha \Delta \Gamma \phi - \beta \phi + \beta \phi, \quad (x,t) \in \Gamma \times \mathbb{R}^+
\end{align*}
\]

and initial conditions

\[
\begin{align*}
\phi(x,0) &= \phi_0(x), \quad x \in \Omega, \\
\phi(x,0) &= \theta_0(x), \quad x \in \Gamma,
\end{align*}
\]

where \(\Omega \subset \mathbb{R}^3\) is a bounded domain with smooth boundary \(\Gamma\) and \(\mathbb{R}^+ = [0, +\infty)\), \(\nu > 0\) is the viscosity, \(\eta > 0\) is the fluid permeability, \(M > 0\) stands for the...
mobility, \( \epsilon > 0 \) is related to the diffuse interface thickness, \( \gamma > 0 \) is a surface tension parameter, \( d > 0 \) \( \alpha > 0 \) \( \beta > 0 \) are constants, \( p \) is the fluid pressure, \( \vec{n} \) is the outward normal vector on \( \Gamma \), \( \Delta \Gamma \) is the Laplace-Beltrami operator on the surface \( \Gamma \) of \( \Omega \) and \( f \) is the derivative of a double well potential \( F(s) = \frac{1}{4} (s^2 - 1)^2 \) describing phase separation.

Dynamic boundary conditions were recently proposed by physicists to describe spinodal decomposition of binary mixtures where the effective interaction between the wall (i.e., the boundary) and two mixture components is short-ranged. This type of boundary conditions is very natural in many mathematical models such as heat transfer in a solid in contact with a moving fluid, thermoelasticity, diffusion phenomena, heat transfer in two medium, problems in fluid dynamics. Therefore, from the viewpoints of mathematics, the long-time behavior of solutions for many equations with dynamical boundary conditions have been studied extensively, see [14, 37] for the reaction-diffusion equation, [8, 15, 27, 31] for the Cahn-Hilliard equation and [36] for the Cahn-Hilliard-Brinkman system.

Cahn-Hilliard-Brinkman (1.1) system consists of a convective Cahn-Hilliard equation for the phase field \( \phi \), i.e., the difference of the relative concentrations of the two phases, and a Brinkman equation, a modified Darcy equation proposed by H. C. Brinkman [5] in 1947 for the fluid velocity \( u \), which has been proposed to model phase separation of incompressible binary fluids in a porous medium (see [28]). The Brinkman equation incorporates a diffuse interface surface force proportional to \( \phi \nabla \mu \), where \( \mu \) is the so-called chemical potential which is the variational derivative of the free energy functional

\[
E(\phi) = \int_{\Omega} \frac{\epsilon}{2} |\nabla \phi|^2 + \frac{1}{\epsilon} F(\phi) \, dx.
\]

For this reason, such a system belongs to a class of diffuse interface models which are used to describe the behavior of multi-phase fluids. Cahn-Hilliard-Brinkman system (1.1) with \( M, \nu, \) and \( \eta \) possibly depending on \( \phi \) has been analyzed from the numerical point of view in [9, 10]. From the analytical point of view, the authors in [4] have proved the well-posedness of solutions and the existence of a global attractor in \( H^1(\Omega) \) for the system (1.1) with the Neumann boundary conditions and more general \( f(u) \), and also estimated the convergence rate of a given weak solution to a single equilibrium via Lojasiewicz-Simon inequality. Furthermore, the authors considered the behavior of the solutions as the viscosity goes to zero, i.e., the existence of a weak solution for the Cahn-Hilliard-Hele-Shaw system was proved as the limit of solutions for the Cahn-Hilliard-Brinkman system when the viscosity goes to zero. Recently, the authors in [22] further proved the regularity and the finite fractal dimensionality of the global attractor for the Cahn-Hilliard-Brinkman system with the Neumann boundary conditions and double well potential functional. To the best of our knowledge, in contrast to the Cahn-Hilliard-Brinkman system with Neumann boundary conditions, there is no results on the fractal dimension of the global attractor for the Cahn-Hilliard-Brinkman system with dynamical boundary conditions.

It is well known that the dimension of attractor is one of most important topics in studying the asymptotic behavior of infinite-dimensional dynamical systems. In the past several decades, many mathematicians have devoted work to understanding the dimension of global attractors for some autonomous partial differential equations (see [1, 2, 3, 7, 11, 24, 32, 33, 35]). While, unlike the Hausdorff dimension, the finite fractal dimension of attractor plays a very important role in the finite dimensional
reduction theory of infinite dimensional dynamical systems basing on the fact that a compact set \( A \) in a metric space with fractal dimension \( d_F(A) \) less than \( \frac{m}{2} \) for some \( m \in \mathbb{N} \) can be placed in the graph of Hölder continuous mapping which maps a compact subset of \( \mathbb{R}^m \) onto \( A \) (see [21, 23]). There are mainly three kinds of valid methods to estimate the finite fractal dimension of attractor: volume contraction method by calculating the Lyapunov exponents in the case that the semigroup is differentiable (see [32, 35]); the method basing on the squeezing property or the smoothing property of the difference of two solutions in the non-differentiable case (see [3, 12, 13, 20]); the \( \ell \)-trajectory method in the minimal regularity on the solutions case (see [18, 24, 25, 29, 30]).

In this paper, we will consider the existence of finite-dimensional global attractor for the Cahn-Hilliard-Brinkman system with dynamical boundary conditions by the \( \ell \)-trajectory method. Comparing to the Cahn-Hilliard-Brinkman system with the usual Neumann boundary conditions, to obtain the finite fractal dimension of the global attractor for the system (1.1)-(1.3), we will encounter the following difficulty: because of

\[
\int_{\Omega} \phi_t \Delta^2 \phi \, dx = \int_{\Gamma} \phi_t \frac{\partial \Delta \phi}{\partial n} - \int_{\Gamma} \Delta \phi \frac{\partial \phi_t}{\partial n} + \cdots,
\]

it is very tricky to deal with these two terms on the right hand side. That’s why we cannot choose \( \Delta^2 \phi \) as a test function to prove the smooth property of the difference of two solutions and the differentiability of the corresponding semigroup on the global attractor for problem (1.1)-(1.3). To overcome this difficulty, motivated by the idea of the method of \( \ell \)-trajectories for any small \( \ell > 0 \) proposed in [24], in this paper, we first define a semigroup \( \{L_t\}_{t \geq 0} \) on some subset \( X_\ell \) of \( L^2(0, \ell; V_I) \) induced by the semigroup \( \{S_I(t)\}_{t \geq 0} \) generated by problem (1.1)-(1.3), and then, we prove the existence of a finite dimensional global attractor \( \mathcal{A}_\ell \) in \( X_\ell \) for the semigroup \( \{L_t\}_{t \geq 0} \) by the method of \( \ell \)-trajectories. Finally, by defining a Lipschitz continuous operator and using the uniqueness of global attractor, we obtain the finite fractal dimension of the global attractor \( \mathcal{A} \) in the original phase space \( V_I \) for problem (1.1)-(1.3).

Throughout this paper, for the sake of simplicity, we assume \( M = \epsilon = \gamma = \nu = \eta = d = 1 \). Let \( C \) be a generic constant that is independent of the initial datum for \( \phi \). Define the average of function \( \phi(x) \) over \( \Omega \) as

\[
m_\phi = \frac{1}{|\Omega|} \int_{\Omega} \phi(x) \, dx.
\]

2. Preliminaries. In order to study problem (1.1)-(1.3), we introduce the space of divergence-free functions defined by

\[
\mathcal{V} = \{ u \in (C^\infty_c(\Omega))^3 : \nabla \cdot u = 0 \}.
\]

Denote by \( H \) and \( V \) the closure of \( \mathcal{V} \) with respect to the norms in \( (L^2(\Omega))^3 \) and \( (H^1(\Omega))^3 \), respectively.

We define the Lebesgue spaces on \( \Gamma \) as follows

\[
L^p(\Gamma) = \left\{ v : \|v\|_{L^p(\Gamma)} < \infty \right\},
\]

where

\[
\|v\|_{L^p(\Gamma)} = \left( \int_\Gamma |v|^p \, dS \right)^{\frac{1}{p}}.
\]
for \( p \in [1, \infty) \). Moreover, we define
\[
L^p(\Omega) \oplus L^p(\Gamma) = L^p(\bar{\Omega}, d\sigma), \quad p \in [1, \infty)
\]
and
\[
\|U\|_{L^p(\bar{\Omega}, d\sigma)} = \left( \int_{\Omega} |u|^p \, dx + \int_{\Gamma} |v|^p \, d\sigma \right)^{\frac{1}{p}}
\]
for any \( U = (u, v) \in L^p(\bar{\Omega}, d\sigma) \), where the measure \( d\sigma = dx|_{\Omega} \oplus d\sigma|_{\Gamma} \) on \( \bar{\Omega} \) is defined by \( \sigma(A) = |A \cap \Omega| + S(A \cap \Gamma) \) for any measurable set \( A \subset \bar{\Omega} \).

We also introduce the functional spaces
\[
H^1(\Omega, d\sigma) = \left\{ (\phi, \psi) \in H^1(\Omega) \times H^1(\Gamma) : \psi = T\phi \in H^1(\Gamma) \right\}
\]
equipped with norms \( \| (\cdot, \cdot) \|_{H^1(\Omega, d\sigma)} \) defined as follows
\[
\| (\phi, \psi) \|_{H^1(\Omega, d\sigma)} = \left( \int_{\Omega} |\nabla \phi|^2 \, dx + \int_{\Gamma} \alpha|\nabla \Gamma \psi|^2 + \beta|\psi|^2 \, d\sigma \right)^{\frac{1}{2}}.
\]
Denote by \( X' \) the dual space of \( X \) and let \( H^s(\Omega), H^s(\Gamma) \) \( (s \in \mathbb{R}) \) be the usual Sobolev spaces. In general, any vector \( \theta \in L^p(\bar{\Omega}, d\sigma) \) will be of the form \( (\theta_1, \theta_2) \) with \( \theta_1 \in L^p(\Omega, dx) \) and \( \theta_2 \in L^p(\Gamma, dS) \), and there need not be any connection between \( \theta_1 \) and \( \theta_2 \).

**Remark 2.1.** ([17]) \( C(\Omega) \) is a dense subspace of \( L^2(\bar{\Omega}, d\sigma) \) and a closed subspace of \( L^\infty(\bar{\Omega}, d\sigma) \).

**Lemma 2.2.** ([16, 27]) Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary \( \Gamma \). Consider the following linear problem
\[
\begin{align*}
-\Delta \phi &= j_1, \quad x \in \Omega, \\
-\alpha \Delta \Gamma \psi + \frac{\partial \phi}{\partial n} + \beta \psi &= j_2, \quad x \in \Gamma.
\end{align*}
\]
Assume that \( (j_1, j_2) \in H^s(\Omega, d\sigma) \), \( s \geq 0, s + \frac{1}{2} \notin \mathbb{N} \). Then the following estimate holds
\[
\| (\phi, \psi) \|_{H^{s+2}(\Omega, d\sigma)} \leq C(\| j_1 \|_{H^s(\Omega)} + \| j_2 \|_{H^s(\Gamma)})
\]
for some constant \( C > 0 \).

**Lemma 2.3.** ([6, 19, 24, 26, 34]) Assume that \( p_1 \in (1, \infty), p_2 \in [1, \infty) \). Let \( X \) be a Banach space and \( X_0, X_1 \) be separable and reflexive Banach spaces such that \( X_0 \subset \subset X \subset X_1 \). Then
\[
Y = \{ u \in L^{p_1}(0, \ell; X_0) : u' \in L^{p_2}(0, \ell; X_1) \} \subset \subset L^{p_1}(0, \ell; X),
\]
where \( \ell \) is a fixed positive constant.

**Definition 2.4.** ([32, 35]) Let \( \{ S(t) \}_{t \geq 0} \) be a semigroup on a Banach space \( X \). A set \( A \subset X \) is said to be a global attractor if the following conditions hold:

(i) \( A \) is compact in \( X \).

(ii) \( A \) is strictly invariant, i.e., \( S(t)A = A \) for any \( t \geq 0 \).

(iii) For any bounded subset \( B \subset X \) and for any neighborhood \( \mathcal{O} = \mathcal{O}(A) \) of \( A \) in \( X \), there exists a time \( \tau_0 = \tau_0(B) \) such that \( S(t)B \subset \mathcal{O}(A) \) for any \( t \geq \tau_0 \).

**Lemma 2.5.** ([26]) Let \( X \) be a (subset of) Banach space and \( (S(t), X) \) be a dynamical system. Assume that there exists a compact set \( K \subset X \) which is uniformly absorbing and positively invariant with respect to \( S(t) \). Let moreover \( S(t) \) be continuous on \( K \). Then \( (S(t), X) \) has a global attractor.
Definition 2.6. ([32, 35]) Let $H$ be a separable real Hilbert space. For any non-empty compact subset $K \subset H$, the fractal dimension of $K$ is the number
\[
d_F(K) = \limsup_{\epsilon \to 0^+} \frac{\log(N_\epsilon(K))}{\log(\frac{1}{\epsilon})},
\]
where $N_\epsilon(K)$ denotes the minimum number of open balls in $H$ with radii $\epsilon > 0$ that are necessary to cover $K$.

Lemma 2.7. ([26]) Let $X, Y$ be norm spaces such that $X \subset Y$ and $A \subset Y$ be bounded. Assume that there exists a mapping $L$ such that $L:A = A$ and $L: Y \to X$ is Lipschitz continuous on $A$. Then $d_F(A)$ is finite.

Lemma 2.8. ([26]) Let $X$ and $Y$ be two metric spaces and $f : X \to Y$ be $\alpha$-Hölder continuous on the subset $A \subset X$. Then
\[
d_F(f(A), Y) \leq \frac{1}{\alpha}d_F(A, X).
\]
In particular, the fractal dimension does not increase under a Lipschitz continuous mapping.

3. The existence of global attractors.

3.1. The well-posedness of weak solutions. In [36], the authors have proved the well-posedness of weak solutions for problem (1.1)-(1.3) by the Faedo-Galerkin method. Now, we just state it as follows.

Theorem 3.1. Assume that $(\phi_0, \theta_0) \in H^1(\bar{\Omega}, d\sigma)$. Then there exists a unique weak solution $(u(t), \phi(t))$ for the Cahn-Hilliard-Brinkman system with dynamic boundary conditions (1.1)-(1.3) such that $m\phi(t) = m\phi_0$, which depends continuously on the initial data $(\phi_0, \theta_0)$ with respect to the norm in $H^1(\bar{\Omega}, d\sigma)$.

For every fixed $I \in \mathbb{R}$, let $V_I = \{(\phi, \psi) \in H^1(\bar{\Omega}, d\sigma) : m\phi = I\}$, by Theorem 3.1, we can define the operator semigroup $\{S_I(t)\}_{t \geq 0}$ in $V_I$ by
\[
S_I(t)(\phi_0, \theta_0) = \phi(t) = \phi(t; \phi_0, \theta_0)
\]
for all $t \geq 0$, which is $(V_I, V_I)$-continuous, where $(u(t), \phi(t))$ is the unique weak solution of problem (1.1)-(1.3) with $(\phi(x, 0), \psi(x, 0)) = (\phi_0, \theta_0) \in H^1(\bar{\Omega}, d\sigma)$.

3.2. The existence of a global attractor in $X_\ell$. In this subsection, we will be concerned with the existence of global attractors for problem (1.1)-(1.3) by using the $\ell$-trajectory method. From Theorem 3.1, we know that there exists a unique weak solution $(u(t), \phi(t))$ of problem (1.1)-(1.3) for any fixed initial data $(\phi_0, \theta_0)$ in $V_I$. Therefore, for any $\ell > 0$ and any $(\phi_0, \theta_0) \in V_I$, there is only one solution defined on the time interval $[0, \ell]$ with the initial data $(\phi_0, \theta_0) \in V_I$, denoted by $\chi(\tau, \phi_0, \theta_0))$ for the sake of simplicity. Let $X_\ell$ be the set of all the solution trajectories defined on the time interval $[0, \ell]$ equipped with the topology of $L^2(0, \ell; V_I)$. Since $X_\ell \subset C_w([0, \ell]; V_I)$, it makes sense to talk about the point values of trajectories. On the other hand, it is not clear whether $X_\ell$ is closed in $L^2(0, \ell; V_I)$. Hence $X_\ell$ in general is not a complete metric space. In what follows, we first introduce some operators.

For any $t \in [0, 1]$, we define the mapping $e_t : X_\ell \to V_I$ by
\[
e_t(\chi) = \chi(t\ell)
\]
for any $\chi \in X_\ell$. 

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The mapping $b : V_I \rightarrow X_\ell$ is given by
$$b((\phi_0, \theta_0)) = (\phi(\tau; (\phi_0, \theta_0)), \psi(\tau; (\phi_0, \theta_0))) = S_t(\tau)(\phi_0, \theta_0), \ \tau \in [0, \ell]$$
for any $(\phi_0, \theta_0) \in V_I$ and the operators $L_t : X_\ell \rightarrow X_\ell$ is defined by the relation
$$L_t b((\phi_0, \theta_0)) = (\phi(t + \tau; (\phi_0, \theta_0)), \psi(t + \tau; (\phi_0, \theta_0))) = S_t(t + \tau)(\phi_0, \theta_0), \ \tau \in [0, \ell]$$
for any $(\phi_0, \theta_0) \in V_I$, where $(u, \phi)$ is the unique weak solution of problem (1.1)-(1.3) with initial data $(\phi_0, \theta_0)$, we can easily prove the operators $\{L_t\}_{t \geq 0}$ is a semigroup on $X_\ell$.

Next, we will carry out some a priori estimates to obtain the existence of absorbing sets for problem (1.1)-(1.3).

**Theorem 3.2.** There exists a positive constant $\rho_1$, satisfying that for any bounded subset $B \subset V_I$, there exists a time $\tau_1 = \tau_1(B) > 0$ such that for any weak solutions of problem (1.1)-(1.3) with initial data $(\phi_0, \theta_0) \in B$, we have
$$\| (\phi(t), \psi(t)) \|^2_{H^1(\overline{\Omega}, d\sigma)} \leq \rho_1$$
and
$$\int_0^\ell \| (\phi(t+s), \psi(t+s)) \|^2_{H^1(\overline{\Omega}, d\sigma)} \, ds \leq \rho_1$$
for any $t \geq \tau_1$.

**Proof.** Taking the inner product of the first equation, the second equation and the fourth equation of (1.1) with $\mu, \delta \phi$ and $u$, respectively, where $\delta$ is a small positive constant will be determined later, we find
$$\frac{d}{dt} \left( \frac{1}{2} \| (\phi, \psi) \|^2_{H^1(\overline{\Omega}, d\sigma)} + \int_\Omega F(\phi) \, dx \right) + \| \psi_t \|^2_{L^2(\Gamma)} + \| \nabla \mu \|^2_{L^2(\Omega)}$$
$$+ \| u \|^2_{H^1(\Omega)} + \delta \| (\phi, \psi) \|^2_{H^1(\overline{\Omega}, d\sigma)} + \delta \int_\Omega f(\phi) \phi \, dx$$
$$= - \int_\Omega \nabla \cdot (u \phi) \mu - \int_\Omega (u \phi) \cdot \nabla \mu + \delta \int_\Omega \mu \phi \, dx - \delta \int_\Gamma \psi_t \, dS$$
$$= \delta \int_\Omega \mu \phi \, dx - \delta \int_\Gamma \psi_t \, dS,$$
which implies that
$$\frac{d}{dt} \left( \frac{1}{2} \| (\phi, \psi) \|^2_{H^1(\overline{\Omega}, d\sigma)} + \int_\Omega F(\phi) \, dx \right) + \| \psi_t \|^2_{L^2(\Gamma)} + \| \nabla \mu \|^2_{L^2(\Omega)}$$
$$+ \| u \|^2_{H^1(\Omega)} + \delta \| (\phi, \psi) \|^2_{H^1(\overline{\Omega}, d\sigma)} + \delta \int_\Omega f(\phi) \phi \, dx$$
$$\leq \delta \| \mu - m \mu \|_{L^2(\Omega)} \| \phi \|_{L^2(\Omega)} + \delta |\Omega| |m \mu| |m \phi| + \delta \| \psi \|_{L^2(\Gamma)} \| \psi \|_{L^2(\Gamma)}$$
$$\leq C \delta \| \nabla \mu \|_{L^2(\Omega)} \| \phi \|_{L^2(\Omega)} + \delta |\Omega| |m \mu| |I| + \delta \| \psi \|_{L^2(\Gamma)} \| \psi \|_{L^2(\Gamma)}.$$

(3.1)

Thanks to
$$\left| \int_\Omega \mu \, dx \right| \leq \| \psi_t \|_{L^1(\Gamma)} + \beta \| \psi \|_{L^1(\Gamma)} + \| \phi \|^3_{L^3(\Omega)} + |\Omega| |I|,$$
we get
$$|\Omega| |m \mu| |I| \leq C \| \psi_t \|_{L^2(\Gamma)} + C \| \psi \|_{L^2(\Gamma)} + C \| \phi \|^3_{L^3(\Omega)} + |\Omega| |I|^2.$$
(3.2)
Note that
\[ f(s)s \geq 2F(s) - \frac{1}{2}, \quad \forall s \in \mathbb{R}, \quad (3.3) \]
\[ s^4 \leq 8F(s) + 2, \quad \forall s \in \mathbb{R}. \quad (3.4) \]

Let \( \delta = \frac{\beta}{2} \), combining (3.1)-(3.4) with Young inequality, we yield that
\[
\frac{d}{dt} \left( \| (\phi, \psi) \|^2_{H^1(\Omega, dx)} + \int_\Omega 2F(\phi) \, dx \right) + \delta \left( \| (\phi, \psi) \|^2_{H^1(\Omega, dx)} + \int_\Omega 2F(\phi) \, dx \right)
+ \| \psi_t \|^2_{L^2(\Gamma)} + \| \nabla \mu \|^2_{L^2(\Omega)} + 2\| u \|^2_{H^1(\Omega)}
\leq C_\beta. \quad (3.5)
\]

We infer from the classical Gronwall inequality that
\[
\| (\phi(t), \psi(t)) \|^2_{H^1(\Omega, dx)} + \int_\Omega 2F(\phi(t)) \, dx
\leq e^{-\delta t} \left( \| (\phi_0, \theta_0) \|^2_{H^1(\Omega, dx)} + \int_\Omega 2F(\phi_0) \, dx \right) + \frac{1}{\delta} C_\beta. \quad (3.6)
\]

From Young inequality and Hölder inequality, it is concluded that
\[
\| (\phi, \psi) \|^2_{H^1(\Omega, dx)} \leq \| (\phi, \psi) \|^2_{H^1(\Omega, dx)} + \int_\Omega 2F(\phi) \, dx \leq \left( \| (\phi, \psi) \|^2_{H^1(\Omega, dx)} + 1 \right)^2. \quad (3.7)
\]

By virtue of (3.6)-(3.7), we obtain
\[
\| (\phi(t), \psi(t)) \|^2_{H^1(\Omega, dx)} \leq e^{-\delta t} \left( \| (\phi_0, \theta_0) \|^2_{H^1(\Omega, dx)} + 1 \right)^2 + \frac{1}{\delta} C_\beta \quad (3.8)
\]
for any \( (u_0, \theta_0) \in V_I \), which implies that there exists a positive constant \( \mathcal{R}_1 = \frac{1}{\delta} C_\beta + 1 \) satisfying that for any bounded subset \( B \subset V_I \), there exists a positive time \( \tau_0 = \tau_0(B) \) depending on the \( V_I \)-norm of \( B \) such that
\[
\| (\phi(t), \psi(t)) \|^2_{H^1(\Omega, dx)} \leq \mathcal{R}_1^2
\]
for any \( t \geq \tau_0 \).

Integrating (3.5) from 0 to \( \ell \) and combining (3.6), we get
\[
\int_0^\ell \| \psi_t(r) \|^2_{L^2(\Gamma)} + \| \nabla \mu(r) \|^2_{L^2(\Omega)} + 2\| u(r) \|^2_{H^1(\Omega)} \, dr
+ \delta \int_0^\ell \left( \| (\phi(r), \psi(r)) \|^2_{H^1(\Omega, dx)} + \int_\Omega 2F(\phi(r)) \, dx \right) \, dr
\leq C_\beta \ell + \| (\phi_0, \theta_0) \|^2_{H^1(\Omega, dx)} + \int_\Omega 2F(\phi_0) \, dx. \quad (3.9)
\]

Integrating (3.5) from \( r \) to \( t+r \) and integrating the resulting inequality with respect to \( r \) over \( (0, \ell) \), we obtain
\[
\int_0^\ell \left( \| (\phi(t+r), \psi(t+r)) \|^2_{H^1(\Omega, dx)} + \int_\Omega 2F(\phi(t+r)) \, dx \right) \, dr
\leq e^{-\delta t} \int_0^\ell \left( \| (\phi(r), \psi(r)) \|^2_{H^1(\Omega, dx)} + \int_\Omega 2F(\phi(r)) \, dx \right) \, dr + \ell \frac{C_\beta}{\delta} (1 - e^{-\delta t})
\leq e^{-\delta t} \frac{1}{\delta} \left( C_\beta \ell + \| (\phi_0, \theta_0) \|^2_{H^1(\Omega, dx)} + \int_\Omega 2F(\phi_0) \, dx \right) + \ell \frac{C_\beta}{\delta},
\]
and
\[
\int_0^t \| (\phi(t + r), \psi(t + r)) \|_{H^1(\Omega, d\sigma)}^2 \, dr \\
\leq e^{-\delta t} \frac{1}{\delta} \left( C_\beta \ell + \| (\phi_0, \theta_0) \|_{H^1(\Omega, d\sigma)}^2 + \int_{\Omega} 2F(\phi_0) \, dx \right) + \ell \frac{C_\beta}{\delta}.
\]

Therefore, for any bounded subset \( B \subset V_I \), there exists a time \( \tau_1 = \tau_1(B) > \tau_0 \) such that
\[
\int_0^\ell \| (\phi(t + r), \psi(t + r)) \|_{H^1(\Omega, d\sigma)}^2 \, dr \leq 1 + \ell \frac{C_\beta}{\delta}
\]
for any \( t \geq \tau_1 \).

Let
\[
B_0 = \left\{ (\phi, \psi) \in V_I : \| (\phi, \psi) \|_{H^1(\Omega, d\sigma)}^2 \leq \rho_1 \right\},
\]
we infer from Theorem 3.2 that there exists a time \( t_0 = t_0(B_0) \geq 0 \) such that for any \( t \geq t_0 \), we have
\[
S_I(t)B_0 \subset B_0.
\]

Define
\[
B_1 = \bigcup_{\ell \geq t_0} S_I(t)B_0 \subset B_0
\]
and
\[
B^\ell_0 = \{ \chi \in X_\ell : \epsilon_0(\chi) \in B_1 \},
\]
from the continuity of \( S_I(t) \) and Theorem 3.2, we deduce that \( B_1 \) is a bounded subset of \( V_I \),
\[
S_I(t)B_1 \subset B_1
\]
and
\[
L_\ell B^\ell_0 \subset B^\ell_0
\]
for any \( t \geq 0 \).

From Theorem 3.2, we immediately obtain the following result.

**Corollary 3.3.** For any bounded subset \( B^\ell \subset X_\ell \), there exists a time \( t_1 = t_1(B^\ell) > 0 \) such that for any weak solutions of problem (1.1)-(1.3) with short trajectory \( \chi \in B^\ell \), we have
\[
\| (\phi(t), \psi(t)) \|_{H^1(\Omega, d\sigma)}^2 \leq \rho_1
\]
and
\[
\int_0^\ell \| (\phi(t + s), \psi(t + s)) \|_{H^1(\Omega, d\sigma)}^2 \, ds \leq \rho_1
\]
for any \( t \geq t_1 \).

In what follows, we prove the existence of a compact absorbing set in \( X_\ell \) of the semigroup \( \{L_\ell\}_{\ell \geq 0} \).
Theorem 3.4. There exists a positive constant $\rho_2$ satisfying for the subset $B_0^\ell$, there exists a time $\tau_2 = \tau_2(B_0^\ell) > 0$ such that for any weak solutions of problem (1.1)-(1.3) with short trajectory $\chi \in B_0^\ell$, we have

$$\int_0^\ell \|\phi(t + r), \psi(t + r)\|^2_{H^2(\Omega, d\sigma)} + \|\phi_t(t + r)\|^2_{H^1(\Omega)} + \|\psi_t(t + r)\|^2_{L^2(\Gamma)} \, dr \leq \rho_2$$

for any $t \geq \tau_2$.

Proof. From the proof of Theorem 3.2, we know that there exists a $t_0 = t_0(B_0^\ell)$ such that

$$\|\phi(t), \psi(t)\|^2_{H^1(\Omega, d\sigma)} + \int_0^\ell \left(\|\phi(t + r), \psi(t + r)\|^2_{H^2(\Omega, d\sigma)} + \int_\Omega 2F(\phi(t + r)) \, dx\right) \, dr \leq 2 + (1 + \ell)\frac{C_\beta}{2}$$

(3.11)

for any $t \geq t_0$.

Integrating (3.5) between $t - s$ and $t + \ell$ with $t \geq t_0 + \frac{\ell}{2}$, $s \in (0, \frac{\ell}{2})$, we obtain

$$\int_0^\ell \|\phi_t(t + r)\|^2_{L^2(\Gamma)} + \|\nabla \mu(t + r)\|^2_{L^2(\Omega)} + 2\|u(t + r)\|^2_{H^1(\Omega)} \, dr \leq C_\beta(\ell + s) + \left(\|\phi(t - s), \psi(t - s)\|^2_{H^1(\Omega, d\sigma)} + \int_\Omega 2F(\phi(t - s)) \, dx\right).$$

(3.12)

After integrating (3.12) with respect to $s$ over $(0, \frac{\ell}{2})$ and combining (3.11), we have

$$\int_0^\ell \|\psi_t(t + r)\|^2_{L^2(\Gamma)} + \|\nabla \mu(t + r)\|^2_{L^2(\Omega)} + 2\|u(t + r)\|^2_{H^1(\Omega)} \, dr \leq g_1$$

(3.13)

for any $t \geq t_0 + \frac{\ell}{2}$.

It follows from Lemma 2.2 and (3.11), (3.13) that

$$\int_0^\ell \|\phi(t + r), \psi(t + r)\|^2_{H^2(\Gamma, d\sigma)} \, dr \leq g_2$$

(3.14)

holds for any $t \geq t_0 + \frac{\ell}{2}$.

Thanks to

$$\|\phi_t\|_{H^1(\Omega)}' \leq \|\phi\|_{L^4(\Omega)}\|\psi\|_{L^4(\Omega)} + \|\nabla \mu\|_{L^2(\Omega)},$$

(3.15)

combining (3.11) with (3.13), we yield that

$$\int_0^\ell \|\phi_t(t + r)\|^2_{H^1(\Omega)}' \, dr \leq g_3$$

(3.16)

for any $t \geq t_0 + \frac{\ell}{2}$.

Let

$$Y = \left\{\chi \in X_\ell : \chi \in L^2(0, \ell; H^2(\Omega, d\sigma)), \chi_t \in L^1(0, \ell; (H^1(\Omega))' \times L^2(\Gamma))\right\}$$

equipped with the following norm

$$\|\chi\|_Y = \left\{\int_0^\ell \|\chi(r)\|^2_{H^2(\Gamma, d\sigma)} \, dr + \left(\int_0^\ell \|\chi_t(r)\|^2_{(H^1(\Omega))' \times L^2(\Gamma)} \, dr\right)^2\right\}^\frac{1}{2}.$$
From Theorem 3.2 and Theorem 3.4, we know that $L_tB^t_0 \subset B^t_0$ for any $t \geq 0$ as well as $L_tB^t_0 \subset B^t_1$ for any $t \geq t_2$.

**Lemma 3.5.** We have the following conclusion: $\overline{L_tB^t_0} \subset B^t_0$ for any $t \geq 0$.

**Proof.** Thanks to $L_tB^t_0 \subset B^t_0$ for any $t \geq 0$, it is enough to prove that

$$\overline{B^t_0} \subset B^t_0.$$

For any $\chi \in \overline{B^t_0}$, there exists a sequence of trajectories $\chi_n \in B^t_0$ such that $\chi_n \to \chi$ in $L^2(0, t; V_I)$, which implies that $e_t(\chi_n) \to e_t(\chi_0)$ in $V_I$ for almost all $t \in [0, 1]$. Since $e_0(\chi_n) \in B_1$ for any $n \in \mathbb{N}$, there exists a subsequence of $\{e_0(\chi_n)\}_{n=1}^{\infty}$, $\{e_0(\chi_n)\}_{n=1}^{\infty}$ of $\{e_0(\chi_n)\}_{n=1}^{\infty}$ and $(\phi_0, \theta_0) \in V_I$ such that $e_0(\chi_n) \to (\phi_0, \theta_0)$ in $V_I$. From the proof of the existence of weak solutions for the problem (1.1)-(1.3), we deduce that for any $T > 0$, there exists a subsequence converging (**) weakly in spaces $\{(\phi, \psi) \in L^\infty(0, T; H^1(\bar{\Omega}, d\sigma)) \cap L^2(0, T; H^2(\bar{\Omega}, d\sigma)) \)$, $(\phi_1, \psi_1) \in L^2(0, T; (H^1(\bar{\Omega}))', \times L^2(\Gamma))$ to a certain function $(\phi(0), \psi(0)) = (\phi_0, \theta_0)$. Therefore, $\chi_0 \in X_r$. It remains to show that $e_0(\chi) \in B_1$. Since $B_1$ is closed, $e_t(\chi_0) \in B_1$ for almost all $t \in [0, 1]$. In particular, $e_{t_n}(\chi_0) \in B_1$ for any sequence $t_n$ with $t_n \to 0$. From the continuity of $\chi_0 : [0, t] \to H \times V_I$ and the closedness of $B_1$, we deduce that $e_0(\chi_0) \in B_1$ which means $\chi_0 \in B^t_0$.

**Lemma 3.6.** The mapping $L_t : X_t \to X_t$ is locally Lipschitz continuous on $B^t_0$ for all $t \geq 0$.

**Proof.** For any fixed $t > 0$ and any $\chi_1, \chi_2 \in B^t_0$, let $(u_1(t + \tau), \phi_1(t + \tau)) = L_t\chi_1$, $(u_2(t + \tau), \phi_2(t + \tau)) = L_t\chi_2$ and let $u = u_1 - u_2$, $\phi = \phi_1 - \phi_2$, $p = p_1 - p_2$. Then $(u, \phi, p)$ satisfies the following equations

$$\begin{cases}
\frac{\partial u}{\partial t} + u \cdot \nabla \phi + u_2 \cdot \nabla \phi - \Delta \mu = 0, & (x, t) \in \Omega \times \mathbb{R}^+,
\mu = \mu_1 - \mu_2 = -\Delta \phi + f(\phi_1) - f(\phi_2), & (x, t) \in \Omega \times \mathbb{R}^+,
-\Delta u + u \cdot \nabla p = -\phi_1 \nabla \mu - \phi \nabla \mu_2, & (x, t) \in \Omega \times \mathbb{R}^+,
\nabla \cdot u = 0, & (x, t) \in \Omega \times \mathbb{R}^+.
\end{cases} \quad (3.17)$$

Equations (3.17) is subject to the following boundary conditions

$$\begin{cases}
u(x, t) = 0, & (x, t) \in \Gamma \times \mathbb{R}^+,
\frac{\partial u}{\partial n} = 0, & (x, t) \in \Gamma \times \mathbb{R}^+,
\frac{\partial \phi}{\partial t} - \alpha \Delta r \psi + \frac{\partial \phi}{\partial n} + \beta \psi = 0, & (x, t) \in \Gamma \times \mathbb{R}^+.
\end{cases} \quad (3.18)$$

Multiplying the first equation and the third equation of (3.17) by $-\Delta \phi$ and $u$, respectively, and integrating by parts, we find

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\phi(t, \psi(t))\|_{L^2(\Omega)}^2 &+ \|\psi(t)\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 + \|\nabla \Delta \phi\|_{L^2(\Omega)}^2 \\
&= \int_{\Omega} \nabla (f(\phi_1) - f(\phi_2)) \cdot \nabla \Delta \phi \, dx + \int_{\Omega} (f(\phi_1) - f(\phi_2)) (u \cdot \nabla \phi_1) \, dx + \int_{\Omega} (u_2 \cdot \nabla \phi) \Delta \phi \, dx \\
&\quad - \int_{\Omega} (u_2) \cdot \nabla \mu_2 \, dx \\
&\leq \|\nabla (f(\phi_1) - f(\phi_2))\|_{L^p(\Omega)} \|\Delta \phi\|_{L^q(\Omega)} + \|\nabla (f(\phi_1) - f(\phi_2))\|_{L^p(\Omega)} \|u\|_{L^q(\Omega)} \|\phi_1\|_{L^q(\Omega)} \\
&\quad + \|u_2\|_{L^q(\Omega)} \|\phi\|_{L^q(\Omega)} \|\Delta \phi\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)} \|\phi\|_{L^q(\Omega)} \|\nabla \mu_2\|_{L^q(\Omega)}. \quad (3.19)
\end{align*}$$
Due to
\[
\|\nabla (f(\phi_1) - f(\phi_2))\|_{L^2(\Omega)} \\
\leq \|f'(\phi_1)\|_{L^\infty(\Omega)} \|\nabla \phi\|_{L^2(\Omega)} + 3\|\phi_1 + \phi_2\|_{L^\infty(\Omega)} \|\phi\|_{L^6(\Omega)} \|\nabla \phi_2\|_{L^6(\Omega)},
\]
we infer from (3.19)-(3.20) and Young inequality that
\[
\frac{d}{dt} \| (\phi(t), \psi(t)) \|_{H^1(\Omega)}^2 + \|\psi(t)\|^2_{L^2(\Gamma)} + \|u\|^2_{H^1(\Omega)} + \|\nabla \Delta \phi\|_{L^2(\Omega)} \\
\leq L(t) \| (\phi(t), \psi(t)) \|_{H^1(\Omega)}^2,
\]
where
\[
L(t) = C(1 + \|\phi_1\|^2_{L^2(\Omega)})(\|f'(\phi_1)\|^2_{L^\infty(\Omega)} + \|\phi_1 + \phi_2\|^2_{L^6(\Omega)} \|\nabla \phi_2\|_{L^6(\Omega)}) \\
+ C(\|u_2\|^2_{H^1(\Omega)} + \|\nabla u_2\|^2_{L^2(\Omega)}).
\]
Let \( s \in (0, \ell) \). Integrating (3.21) from \( s \) to \( t + s \), we obtain
\[
\| (\phi(t + s), \psi(t + s)) \|_{H^1(\Omega, \partial \Omega)}^2 \\
\leq \int_s^{t+s} L(r) \| (\phi(r), \psi(r)) \|_{H^1(\Omega, \partial \Omega)}^2 \, dr + \| (\phi(s), \psi(s)) \|_{H^1(\Omega, \partial \Omega)}^2.
\]
From the classical Gronwall inequality, we deduce
\[
\| (\phi(t + s), \psi(t + s)) \|_{H^1(\Omega, \partial \Omega)}^2 \leq \| (\phi(s), \psi(s)) \|_{H^1(\Omega, \partial \Omega)}^2 \exp(\int_s^{t+s} L(r) \, dr) \\
\leq M_\ell(t) \| (\phi(s), \psi(s)) \|_{H^1(\Omega, \partial \Omega)}^2,
\]
where
\[
M_\ell(t) = \exp(\int_0^{t+\ell} L(r) \, dr).
\]
Since \( e_0(\chi^1) \) and \( e_0(\chi^2) \) is uniformly bounded in \( V_\ell \) for any \( \chi^1, \chi^2 \in B_0^\ell \), from the proof of Theorem 3.2, we conclude \( M_\ell(t) \) is a finite number depending on \( e_0(\chi_1) \) and \( e_0(\chi_1) \).

Integrating (3.23) with respect to \( s \) for \( 0 \) to \( \ell \), we get
\[
\int_0^\ell \| (\phi(t + s), \psi(t + s)) \|_{H^1(\Omega, \partial \Omega)}^2 \, ds \leq M_\ell(t) \int_0^\ell \| (\phi(s), \psi(s)) \|_{H^1(\Omega, \partial \Omega)}^2 \, ds,
\]
which implies the mapping \( L_\ell : X_\ell \to X_\ell \) is locally Lipschitz continuous on \( B_0^\ell \) for all \( t \geq 0 \).

Thanks to the positively semi-invariance of \( B_1 \), Lemma 2.3, Theorem 3.2 and Theorem 3.4, we easily deduce that \( K = L_\ell B_0^{L^2(0,\ell;V_\ell)} \) is a positive semi-invariant, uniformly absorbing compact subset of \( X_\ell \). Therefore, we can immediately obtain the existence of a global attractor in \( X_\ell \) from Lemma 2.5 stated as follows.

**Theorem 3.7.** The semigroup \( \{L_\ell\}_{\ell \geq 0} \) generated by problem (1.1)-(1.3) possesses a global attractor \( A_\ell \) in \( X_\ell \) and \( e_\ell(A_\ell) \) is uniformly bounded in \( V_\ell \) with respect to \( t \in [0, 1] \), where
\[
e_\ell(A_\ell) = \{e_\ell(\chi) : \chi \in A_\ell\}
\]
for any \( t \in [0, 1] \).

In what follows, we prove the smooth property of the semigroup \( \{L_\ell\}_{\ell \geq 0} \) to estimate the fractal dimension of the global attractor \( A_\ell \).
Theorem 3.8. Let \( \chi^1 \) and \( \chi^2 \) be two short trajectories belonging to \( A_\ell \). Then there exists a positive constant \( \kappa \) independent of \( t \) such that for any \( t \geq \ell \), we have
\[
\|L_\ell \chi^1 - L_\ell \chi^2\|_V^2 \leq \kappa M_\ell(t) \int_0^\ell \|\chi^1(r) - \chi^2(r)\|_{V_5}^2 \, dr,
\]
where \( M_\ell(t) \) is given in (3.24).

Proof. For any \( \chi^1, \chi^2 \in A_\ell \), let \( u_1(t + \tau), \phi_1(t + \tau) = L_\ell \chi^1 \), \( u_2(t + \tau), \phi_2(t + \tau) = L_\ell \chi^2 \) and \( u = u_1 - u_2, \phi = \phi_1 - \phi_2 \). Since \( \epsilon_\ell(\chi^1) \) and \( \epsilon_\ell(\chi^2) \) is uniformly bounded in \( V_\ell \) with respect to \( t \in [0,1] \) for any \( \chi^1, \chi^2 \in A_\ell \), we conclude from the proof of Lemma 3.6 that
\[
\frac{d}{dt} \|(\phi(t), \psi(t))\|_{H^1(\bar{\Omega},d\sigma)}^2 + \|\psi(t)\|_{L^2(\Gamma)}^2 + \|u(t)\|_{H^1(\Omega)}^2 + \|\nabla \Delta \phi(t)\|_{L^2(\Omega)}^2
\leq L(t) \|(\phi(t), \psi(t))\|_{H^1(\bar{\Omega},d\sigma)}^2.
\] (3.26)

For any \( t \geq \ell \), integrating (3.7) from \( t - s \) to \( t + \ell \) with \( s \in [0, \frac{\ell}{2}] \), we yield
\[
\|(\phi(t + \ell), \psi(t + \ell))\|_{H^1(\bar{\Omega},d\sigma)}^2 + \int_{t-s}^{t+\ell} \|\psi(r)\|_{L^2(\Gamma)}^2 + \|u(r)\|_{H^1(\Omega)}^2 + \|\nabla \Delta \phi(r)\|_{L^2(\Omega)}^2 \, dr
\leq \int_{t-s}^{t+\ell} L(r) \|(\phi(r), \psi(r))\|_{H^1(\bar{\Omega},d\sigma)}^2 \, dr + \|(\phi(t-s), \psi(t-s))\|_{H^1(\bar{\Omega},d\sigma)}^2.
\] (3.27)

It follows from the classical Gronwall inequality that
\[
\|(\phi(t + \ell), \psi(t + \ell))\|_{H^1(\bar{\Omega},d\sigma)}^2 + \int_{t-s}^{t+\ell} \|\psi(r)\|_{L^2(\Gamma)}^2 + \|u(r)\|_{H^1(\Omega)}^2 + \|\nabla \Delta \phi(r)\|_{L^2(\Omega)}^2 \, dr
\leq \exp(\int_{t-s}^{t+\ell} L(\zeta) \, d\zeta) \|(\phi(t-s), \psi(t-s))\|_{H^1(\bar{\Omega},d\sigma)}^2.
\] (3.28)

For any \( t \geq \ell \) and any \( s \in [0, \frac{\ell}{2}] \), integrating (3.26) from \( s \) to \( t - s \), we obtain
\[
\|(\phi(t-s), \psi(t-s))\|_{H^1(\bar{\Omega},d\sigma)}^2
\leq \int_s^{t-s} L(r) \|(\phi(r), \psi(r))\|_{H^1(\bar{\Omega},d\sigma)}^2 \, dr + \|(\phi(s), \psi(s))\|_{H^1(\bar{\Omega},d\sigma)}^2.
\]

We deduce from the classical Gronwall inequality that
\[
\|(\phi(t-s), \psi(t-s))\|_{H^1(\bar{\Omega},d\sigma)}^2
\leq \|(\phi(s), \psi(s))\|_{H^1(\bar{\Omega},d\sigma)}^2 \exp(\int_s^{t-s} L(r) \, dr)
\leq \|(\phi(s), \psi(s))\|_{H^1(\bar{\Omega},d\sigma)}^2 \exp(\int_0^{t-s} L(r) \, dr).
\] (3.28)

Combining (3.27) with (3.28), we get
\[
\int_0^\ell \|\psi_{t}(t + \ell)\|_{L^2(\Gamma)}^2 + \|u(t + \ell)\|_{H^1(\Omega)}^2 + \|\nabla \Delta \phi(t + \ell)\|_{L^2(\Omega)}^2 \, dr
\leq \exp(\int_0^{t+\ell} L(\zeta) \, d\zeta) \|(\phi(s), \psi(s))\|_{H^1(\bar{\Omega},d\sigma)}^2
= M_\ell(t) \|(\phi(s), \psi(s))\|_{H^1(\bar{\Omega},d\sigma)}^2.
\]
Integrating the above inequality over \((0, \xi)\) with respect to \(s\), we find
\[
\int_0^\xi \left\| \psi_t(t + r) \right\|^2_{L^2(\Gamma)} + \left\| u(t + r) \right\|^2_{H^1(\Omega)} + \left\| \nabla \Delta \phi(t + r) \right\|^2_{L^2(\Omega)} \, dr
\leq \frac{2M(t)}{\ell} \int_0^\xi \left\| (\phi(s), \psi(s)) \right\|^2_{H^1(\Omega, \partial \Omega)} \, ds.
\]
Thanks to the boundedness of \(M(t)\) for any fixed \(t \in [\ell, s]\), we obtain
\[
\int_0^\ell \left\| \psi_t(t + r) \right\|^2_{L^2(\Gamma)} + \left\| u(t + r) \right\|^2_{H^1(\Omega)} + \left\| \nabla \Delta \phi(t + r) \right\|^2_{L^2(\Omega)} \, dr
\leq \frac{2M(t)}{\ell} \int_0^\ell \left\| (\phi(s), \psi(s)) \right\|^2_{H^1(\Omega, \partial \Omega)} \, ds. \tag{3.29}
\]
Since
\[
\int_{\Omega} \Delta \phi \, dx = \int_{\Gamma} \psi_t - \alpha \Delta \Gamma \psi + \beta \psi \, dS 
\leq C \left( \left\| \psi_t \right\|_{L^2(\Gamma)} + \left\| \psi \right\|_{L^2(\Gamma)} \right)
\]
we deduce from Lemma 2.2 that
\[
\left\| (\phi, \psi) \right\|_{H^1(\Omega, \partial \Omega)} \leq C \left( \left\| \Delta \phi \right\|_{L^2(\Omega)} + \left\| \psi_t \right\|_{L^2(\Gamma)} \right) 
\leq C \left( \left\| \nabla \Delta \phi \right\|_{L^2(\Omega)} + \left\| (\phi, \psi) \right\|_{H^1(\Omega, \partial \Omega)} + \left\| \psi_t \right\|_{L^2(\Gamma)} \right). \tag{3.30}
\]
Combining (3.25) with (3.29)-(3.30), we get
\[
\int_0^\ell \left\| \psi_t(t + r) \right\|^2_{L^2(\Gamma)} + \left\| u(t + r) \right\|^2_{H^1(\Omega)} + \left\| (\phi(t + r), \psi(t + r)) \right\|^2_{H^1(\Omega, \partial \Omega)} \, dr
\leq \kappa_1 M(t) \int_0^\ell \left\| (\phi(s), \psi(s)) \right\|^2_{H^1(\Omega, \partial \Omega)} \, ds \tag{3.31}
\]
for some positive constant \(\kappa_1\).

Thanks to
\[
\left\| \phi_t \right\|_{H^1(\Omega, \partial \Omega)} \leq \left\| u \right\|_{L^2(\Omega)} \left\| \phi_1 \right\|_{L^2(\Omega)} + \left\| u_2 \right\|_{L^2(\Omega)} \left\| \phi_2 \right\|_{L^2(\Omega)} + \left\| \nabla \mu \right\|_{L^2(\Omega)} 
\leq \left\| u \right\|_{H^1(\Omega)} \left\| \phi_1 \right\|_{L^2(\Omega)} + \left\| u_2 \right\|_{H^1(\Omega)} \left\| \phi_2 \right\|_{L^2(\Omega)} + \left\| \nabla \mu \right\|_{L^2(\Omega)}, \tag{3.32}
\]
we infer from (3.13)-(3.14), (3.20), (3.25),(3.29) that
\[
\left( \int_0^\ell \left\| \psi_t(t + r) \right\|_{H^1(\Omega, \partial \Omega)} \, dr \right)^2 \leq \kappa_2 M(t) \int_0^\ell \left\| (\phi(s), \psi(s)) \right\|^2_{H^1(\Omega, \partial \Omega)} \, ds \tag{3.33}
\]
for some positive constant \(\kappa_2\). The proof of Theorem 3.8 is completed.

From Lemma 2.3 and Lemma 2.7, Theorem 3.7 and Theorem 3.8, we immediately obtain the following result.

**Theorem 3.9.** The fractal dimension of the global attractor \(A_\ell\) in \(X_\ell\) of the semigroup \(\{L_\ell\}_{\ell \geq 0}\) generated by problem (1.1)-(1.3) established in Theorem 3.7 is finite.

3.3. **The existence of a global attractor in** \(V_\ell\). In this subsection, we prove the existence of a finite dimensional global attractor in \(V_\ell\) of the semigroup generated by problem (1.1)-(1.3).
Theorem 3.10. The mapping $e_1: \mathcal{A}_\ell \to \mathcal{A} = e_1(\mathcal{A}_\ell)$ is Lipschitz continuous. That is, for any two short trajectories $\chi^1, \chi^2 \in \mathcal{A}_\ell$, there exists a positive constant $\theta$ dependent on $\ell$ such that
\[
\|e_1(\chi^1) - e_1(\chi^2)\|_{H^1(V_t)} \leq \theta \int_0^\ell \|\chi^1(r) - \chi^2(r)\|_{H^1(V_t)} dr.
\]

Proof. For any $\chi^1, \chi^2 \in \mathcal{A}_\ell$, let $(u_1(t + \tau), \phi_1(t + \tau)) = L_\ell \chi^1$, $(u_2(t + \tau), \phi_2(t + \tau)) = L_\ell \chi^2$ and let $u = u_1 - u_2$, $\phi = \phi_1 - \phi_2$. Thanks to the uniformly boundedness of $e_0(\chi^1)$ and $e_0(\chi^2)$ in $V_t$ for any $\chi^1, \chi^2 \in \mathcal{A}_\ell$, from the proof of Lemma 3.6, we obtain
\[
\frac{d}{dt} \|(\phi(t), \psi(t))\|^2_{H^1(\Omega, \mu, \sigma)} + \|\psi(t)\|^2_{H^2(\Gamma)} + \|u(t)\|^2_{H^1(\Omega)} + \|
abla \Delta \phi(t)\|^2_{L^2(\Omega)} \\
\leq L(t) \|(\phi(t), \psi(t))\|^2_{H^1(\Omega, \mu, \sigma)}.
\]
(3.34)
For $s \in (0, \ell)$, we infer from the classical Gronwall inequality and (3.34) that
\[
\|(\phi(s), \psi(s))\|^2_{H^1(\Omega, \mu, \sigma)} \leq \|\|(\phi(s), \psi(s))\|^2_{H^1(\Omega, \mu, \sigma)} \exp(\int_s^\ell L(r) dr) \\
\leq \|\|(\phi(s), \psi(s))\|^2_{H^1(\Omega, \mu, \sigma)} \exp(\int_0^\ell L(r) dr).
\]
(3.35)
Integrating (3.35) over $(0, \ell)$, we yield
\[
\|(\phi(s), \psi(s))\|^2_{H^1(\Omega, \mu, \sigma)} \leq \frac{1}{\ell} \exp(\int_0^\ell L(r) dr) \int_0^\ell \|(\phi(s), \psi(s))\|^2_{H^1(\Omega, \mu, \sigma)} ds.
\]
From Lemma 2.2 and (3.9), we know that
\[
\mathcal{M}_\ell(0) = \exp(\int_0^\ell L(r) dr) < +\infty,
\]
which implies that the mapping $e_1: \mathcal{A}_\ell \to \mathcal{A}$ is Lipschitz continuous.

Theorem 3.11. The semigroup $\{S_t(t)\}_{t \geq 0}$ generated by problem (1.1)-(1.3) possesses a global attractor $\mathcal{A} = e_1(\mathcal{A}_\ell)$ in $V_t$. Furthermore, the fractal dimension of the global attractor $\mathcal{A}$ is finite.

Proof. From Lemma 2.8, Theorem 3.7, Theorem 3.9 and Theorem 3.10, we know that $\mathcal{A}$ is compact and the fractal dimension of $\mathcal{A}$ is finite. As a result of $L_\ell \mathcal{A}_\ell = \mathcal{A}_\ell$, we have
\[
S_t(t) \mathcal{A} = S_t(t) e_1(\mathcal{A}_\ell) = e_1(L_\ell \mathcal{A}_\ell) = e_1(\mathcal{A}_\ell) = \mathcal{A}
\]
for any $t \geq 0$. From the definition of $B_1$, we deduce that for any bounded subset of $V_t$, there exists some time $t = t(B)$ such that for any $t \geq t$, we have
\[
S_t(t) B \subset B_1.
\]
Therefore, we only need to prove that
\[
\lim_{t \to +\infty} \text{dist}_{V_t}(S_t(t) B_1, \mathcal{A}) = 0.
\]
Otherwise, there exist some positive constant $\epsilon_0$, some sequence $\{(\phi_n, \psi_n)\}_{n=1}^\infty \subset B_1$ and some $\{t_n\}_{n=1}^\infty$ with $t_n \to +\infty$ as $n \to +\infty$ such that
\[
\text{dist}_{H^1(V_t)}(S(t_n)(\phi_n, \psi_n), \mathcal{A}) \geq \epsilon_0.
\]
(3.36)
From the definition of $B_1$, we deduce that there exists $\chi_n \in B_0$ such that 
$$(\phi_n, \psi_n) = e_0(\chi_n).$$

Since $\{\chi_n\}_{n=1}^\infty$ is bounded in $X_\ell$ and $A_\ell$ is a global attractor in $X_\ell$ of the semigroup $\{L_t\}_{t\geq 0}$ generated by problem (1.1)-(1.3), there exist a subsequence $\{\chi_{n_j}\}_{j=1}^\infty$ of $\{\chi_n\}_{n=1}^\infty$ and a subsequence $\{t_{n_j}\}_{j=1}^\infty$ of $\{t_n\}_{n=1}^\infty$ such that 
$L_{t_{n_j}} \to \chi \in A_\ell$ in $X_\ell$ for $j \to +\infty$.

Thanks to the continuity of $e_1$, we have 
$S_I(t_{n_j}) (\phi_{n_j}, \psi_{n_j}) = e_1(L_{t_{n_j}} - \ell \chi_{n_j}) \to e_1(\chi) \in A$ in $H \times V_I$ as $j \to +\infty$, which contradicts with (3.36).

\begin{remark}
For any $I \in \mathbb{R}$, from the uniqueness of the global attractor for the semigroup $\{S_I(t)\}_{t\geq 0}$, we infer that the global attractor for problem (1.1)-(1.3) obtained in Theorem 3.11 coincide with the global attractor for problem (1.1)-(1.3) established by using the Aubin-Lions compactness lemma in [36].
\end{remark}

Combining Theorem 3.11 with Remark 3.12, we immediately conclude the following result.

\begin{corollary}
For any $I \in \mathbb{R}$, the fractal dimension of the global attractor for problem (1.1)-(1.3) established in [36] is finite.
\end{corollary}

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