ON A 2D STOCHASTIC EULER EQUATION OF TRANSPORT TYPE: 
EXISTENCE AND ENERGY TRANSFER

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Abstract. We prove weak existence of Euler equation (or Navier-Stokes equation) perturbed by a multiplicative noise on bounded domains of \( \mathbb{R}^2 \) with Dirichlet boundary conditions and with periodic boundary conditions. Solutions are \( H^1 \) regular. The equations are described geometrically as perturbations of geodesics in infinite dimensions and the transfer of energy between Fourier modes in the periodic case.

1. Introduction

We consider a stochastic partial differential equation which can be considered as a random perturbation of the Euler as well as the Navier-Stokes equation on a two-dimensional bounded domain where we consider Dirichlet boundary conditions (or periodic boundary conditions).

In the second section we formulate the problem and state the weak existence of the stochastic p.d.e. in the space \( H^1 \). We define in section 3 the finite-dimensional approximations of the solution and complete the proof in section 4.

Our next result (section 5) is the definition of a Girsanov transformation: under the new measure solutions of the non-linear s.p.d.e. become solutions of a linear stochastic transport equation.

This linear equation is actually a stochastic parallel transport over Brownian paths and can be characterized in terms of the geometry defined by the \( L^2 \) metric in the space of measure-preserving diffeomorphisms of the underlying two-dimensional domain.

Section 6 is devoted to the periodic boundary conditions case. We explain the geometric formulation of our equations in Section 7. Finally, in the last section, we describe the energy transfer between Fourier modes for solutions of the stochastic p.d.e’s. considered before in the periodic case.

2. Euler equation perturbed by a multiplicative noise

We consider the following stochastic Euler equation in dimension 2:

\[
\begin{aligned}
\left\{
\begin{array}{l}
du(t,\theta) = - (u(t,\theta) \cdot \nabla)u(t,\theta) \, dt + \sum_{i=1}^{2} \partial_i u(t,\theta) \circ dB^i(t) & \quad \text{in } ]0,T[ \times \Theta, \\
\text{div } u(t,\theta) = 0 & \quad \text{in } ]0,T[ \times \Theta, \\
u(t,\theta) = 0 & \quad \text{on } ]0,T[ \times \Gamma, \\
u(0,\theta) = u_0(\theta) & \quad \text{in } \Theta,
\end{array}
\right.
\end{aligned}
\]

(1)

where \( \nabla \) denotes the gradient, \( \text{div } u = \sum_{i=1}^{2} \partial_i u \). We suppose that \( \Theta \) is a bounded simply connected domain in \( \mathbb{R}^2 \), \( \Gamma = \partial \Theta \) is sufficiently regular. The term \( \sum_{i=1}^{2} \partial_i u(t,\theta) \circ dB^i(t) \) is
considered as a stochastic perturbation of the deterministic equation where $B = (B^1, B^2)$ is a 2-dimensional Brownian motion in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the differential is taken in the Stratonovich sense.

Notice that using the relation between Stratonovich and Itô differentials, we obtain

\[
\begin{cases}
  du(t, \theta) = \left\{ \frac{1}{2} \Delta u(t, \theta) - (u(t, \theta) \cdot \nabla) u(t, \theta) \right\} dt \\
  + \sum_{i=1}^{2} \partial_t u(t, \theta) dB^i(t) & \text{in }]0, T[ \times \Theta, \\
  \text{div } u(t, \theta) = 0 & \text{in }]0, T[ \times \Theta, \\
  u(t, \theta) = 0 & \text{on }]0, T[ \times \Gamma, \\
  u(0, \theta) = u_0(\theta) & \text{in } \Theta,
\end{cases}
\]

where $\Delta$ denotes the Laplacian. Observe that Equation (2) is a stochastic Navier-Stokes equation in dimension 2. In addition, we can view $B$ as a cylindrical Wiener process in $\mathbb{R}^2$. In fact, we could think Equation (2) as a particular case of Equation (1.1) in [13] with

\[
\{a^{ij}\}_{1 \leq i, j \leq 2} = \begin{pmatrix}
  1/2 & 0 \\
  0 & 1/2
\end{pmatrix}, \\
\sigma^1 = (1 \ 0), \quad \sigma^2 = (0 \ 1),
\]

\[
p = \tilde{p} = f^j = g^j = 0, \quad j = 1, 2.
\]

However, in our case, $a^{ij} - 1/2 \sigma^i \cdot \sigma^j = 0$, for all $1 \leq i, j \leq 2$, that is, this matrix is not uniformly nondegenerated. Thus, we cannot apply directly Mikulevicius’ results [13] (see also [15]) on the existence and uniqueness of solution to Equation (2).

We introduce the basic spaces in this note:

- $V = \{ v \in C_0^\infty(\Theta)^2, \text{div } v = 0 \}$
- $H = \text{the closure of } V \text{ in } [L^2(\Theta)]^2$,
- $V = \text{the closure of } V \text{ in } [H^1_0(\Theta)]^2$.

The space $H$ is equipped with the scalar product $\langle \cdot, \cdot \rangle_0$ and associated norm $\| \cdot \|_0$ induced by $[L^2(\Theta)]^2$; the space $V$ is a Hilbert space with the scalar product

\[
\langle u, v \rangle_1 = \sum_{i=1}^{2} \langle \partial_i u, \partial_i v \rangle_0,
\]

and associated norm $\| \cdot \|_1$. Note that this norm is equivalent to the $[H^1(\Theta)]^2$-norm by Poincaré’s inequality. The space $V$ is contained in $H$, is dense in $H$, and the injection is continuous. Let $H'$ and $V'$ denote the dual space of $H$ and $V$, respectively. We have the dense, continuous embedding

\[ V \hookrightarrow H = H' \hookrightarrow V'. \]

The main result is the following existence result for the solution to Equation (2):

**Theorem 2.1.** Let $u_0 \in V$. Then there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a right-continuous filtration $\mathbb{F} = \{ \mathcal{F}_t \}$ of $\sigma$-algebras, a real 2-dimensional Brownian motion $B_t$,
and an \([L^2(\Theta)]^2\)-valued weakly continuous \(\mathbb{F}\)-adapted process \(u(t)\) such that
\[
\sup_{0 \leq t \leq T} \|u(t)\|_2^2 =: K < +\infty
\]
with probability 1,
\[
\sup_{0 \leq t \leq T} \mathbb{E}\|u(t)\|_2^2 =: K' < \infty
\]
and (2) holds. In addition, \(u(t)\) is strongly continuous in \(t\).

To prove Theorem 2.1 we shall follow the methods in [15] and [13].

3. FAEDO-GALERKIN APPROXIMATIONS

Following the arguments of Chapter III in [19], we consider the weak formulation of Equation (2), namely:
\[
\begin{align*}
&d\langle u(t), v \rangle = - \left\{ \frac{1}{2} \langle u(t), v \rangle_1 + \langle (u(t) \cdot \nabla) u(t), v \rangle_0 \right\} dt \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + \sum_{l=1}^2 \langle \partial_t u(t), v \rangle_0 dB^l(t) \\
&u(0) = u_0
\end{align*}
\]
for all \(v \in V\). Notice that Equation (3) is equivalent to the following stochastic evolution equation in \(V'\)
\[
\begin{align*}
&du(t) = - \left\{ \frac{1}{2} \mathcal{A} u(t) + \mathcal{B} u(t) \right\} dt + \sum_{l=1}^2 \partial_t u(t) dB^l(t) \\
&u(0) = u_0
\end{align*}
\]
where \(\mathcal{A}\) and \(\mathcal{B}\) are defined as
\[
\langle \mathcal{A} u, v \rangle = \langle u, v \rangle_1, \quad \langle \mathcal{B} u, v \rangle = \langle (u \cdot \nabla) u, v \rangle_0,
\]
for all \(u, v \in V\).

It is well known that there exists an orthonormal basis \(\{e_j\}\) for \(H\), that is also orthogonal for \(V\). In addition, this basis verifies
\[
\langle e_j, e_k \rangle_1 = \lambda_j \langle e_j, e_k \rangle_0,
\]
\(\text{div } e_j = 0\) in \(\Theta\) and \(e_j = 0\) on \(\Gamma\), for all \(j\), where \(\lambda_j > 0\) and \(\lambda_j \to +\infty\) when \(j \to +\infty\). For each \(n\) we define an approximate solution \(u_n\) of (3) as follows:
\[
u_n(t) = \sum_{i=1}^n g_i^n(t)e_i
\]
and
\[
\begin{align*}
&d\langle u_n(t), e_j \rangle = - \left\{ \frac{1}{2} \langle u_n(t), e_j \rangle_1 + \langle (u_n(t) \cdot \nabla) u_n(t), e_j \rangle_0 \right\} dt \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + \sum_{l=1}^2 \langle \partial_t u_n(t), e_j \rangle_0 dB^l(t) \\
u_n(0) = u_{0n},
\end{align*}
\]
for \( t \in [0, T] \), \( j = 1, \ldots, n \), where \( u_{0n} \) is the orthogonal projection in \( H \) of \( u_0 \) onto the space \( H_n := \text{span} \{e_1, \ldots, e_n\} \), that is, \( u_{0n} = \sum_{i=1}^{n} (u_0, e_j)_{0} e_j \). The equations (5) form a stochastic differential equation system for the functions \( g^1_n, \ldots, g^n_n \):

\[
\begin{align*}
 dg^l_n(t) &= -\left\{ \frac{1}{2} g^l_n(t) + \sum_{i=1}^{n} \sum_{k=1, k \neq j}^{n} (e_i \cdot \nabla) e_k, e_j \right\} g^l_n(t) \, dt \\
 & \quad + \sum_{l=1}^{2} \sum_{i=1}^{n} \langle \partial_i e_i, e_j \rangle_{0} g^l_n(t) \, dB^l(t) \tag{6}
\end{align*}
\]

Notice that the system (6) has a unique strong solution in \( C([0, T]; \mathbb{R}) \) because its coefficients are defined by Lipschitz functions. Thus, \( u_n \in C([0, T]; H_n) \).

Let us obtain a priori estimates for \( u_n \) which are independent on \( n \).

**Lemma 3.1.** Let \( u_0 \in V \). Then for each \( T > 0 \)

\[
\sup_n \sup_0 \leq t \leq T \| u_n(t) \|_0^2 \leq \| u_0 \|_0^2, \tag{7}
\]

with probability 1, and

\[
\sup_n \mathbb{E} \| u_n(t) \|_1^2 \leq \| u_0 \|_1^2 \tag{8}
\]

for any \( t \in [0, T] \).

**Proof.** Applying Itô’s formula (see for instance Theorem 4.3. in [3]) to \( u_n \) in the evolution formulation (4) and to \( \nabla u_n \), respectively, and taking in account the identities

\[
\langle (u \cdot \nabla) v, v \rangle_0 = 0, \quad \langle \nabla[(u \cdot \nabla) u], \nabla u \rangle_0 = 0,
\]

that hold for all \( u, v \in V \), we obtain

\[
\begin{align*}
\| u_n(t) \|_0^2 &= \| u_{0n} \|_0^2 - 2 \int_0^t \left\langle \frac{1}{2} A u_n(s), u_n(s) \right\rangle \, ds - 2 \int_0^t \langle B u_n(s), u_n(s) \rangle \, ds \\
& \quad + 2 \sum_{l=1}^{2} \int_0^t \langle \partial_l u_n(s), u_n(s) \rangle_0 \, dB^l(s) + \int_0^t \| u_n(s) \|_1^2 \, ds \\
& = \| u_{0n} \|_0^2 - \int_0^t \langle u_n(s), u_n(s) \rangle_1 \, ds - 2 \int_0^t \langle (u_n(s) \cdot \nabla) u_n(s), u_n(s) \rangle_0 \, ds \\
& \quad + \sum_{l=1}^{2} \int_0^t \left( \int \partial_l \left( \sum_{i=1}^{2} \langle u_n(s, \theta) \rangle_0^2 \right) \, d\theta \right) \, dB^l(s) + \int_0^t \| u_n(s) \|_1^2 \, ds \\
& = \| u_{0n} \|_0^2 + \sum_{l=1}^{2} \int_0^t \left( \int \eta_l \left( \sum_{i=1}^{2} \langle u_n(s, \theta) \rangle_0^2 \right) \, dS \right) \, dB^l(s) = \| u_{0n} \|_0^2,
\end{align*}
\]
where $\eta = (\eta^1, \eta^2)$ denotes the unit exterior normal vector, and

\[
\| \nabla u_n(t) \|^2_0 = \| \nabla u_{0n} \|^2_0 - 2 \int_0^t \left( \frac{1}{2} \nabla [A u_n(s)], \nabla u_n(s) \right) \, ds
\]

\[
- 2 \int_0^t \langle \nabla [B u_n(s)], \nabla u_n(s) \rangle \, ds + 2 \sum_{l=1}^2 \int_0^t \langle \partial_t u_n(s), \nabla u_n(s) \rangle_0 \, dB^l(s)
\]

\[
+ \int_0^t \sum_{j,l=1}^2 \| \partial_j \partial_l u_n(s) \|^2_0 \, ds
\]

\[
= \| \nabla u_{0n} \|^2_0 - \int_0^t \sum_{j,l=1}^2 \| \partial_j \partial_l u_n(s) \|^2_0 \, ds
\]

\[
- 2 \int_0^t \langle \nabla [(u_n(s) \cdot \nabla) u_n(s)], \nabla u_n(s) \rangle_0 \, ds
\]

\[
- 2 \sum_{l=1}^2 \int_0^t \langle \partial_t u_n(s), \text{div}[\nabla u_n(s)] \rangle_0 \, dB^l(s)
\]

\[
+ \int_0^t \sum_{j,l=1}^2 \| \partial_j \partial_l u_n(s) \|^2_0 \, ds
\]

\[
= \| \nabla u_{0n} \|^2_0 - 2 \sum_{l=1}^2 \int_0^t \langle \partial_t u_n(s), \Delta u_n(s) \rangle_0 \, dB^l(s).
\]

To sum up,

\[
\| u_n(t) \|^2_0 = \| u_{0n} \|^2_0,
\]

\[
\| u_n(t) \|^2_1 = \| u_{0n} \|^2_1 - 2 \int_0^t \langle \partial_t u_n(s), \Delta u_n(s) \rangle_0 \, dB^l(s).
\]

Thus,

\[
\sup_{0 \leq t \leq T} \| u_n(t) \|^2_0 \leq \| u_0 \|^2_0,
\]

\[
\sup_{0 \leq t \leq T} E \| u_n(t) \|^2_1 = \| u_{0n} \|^2_1 \leq \| u_0 \|^2_1.
\]

This finishes the proof of this lemma. \qed

4. Existence of weak solutions

For each $n$, the solution $u_n$ of (15) induces a measure $\mathbb{P}^n$ on a trajectory space determined by the estimates of Lemma 3.1.

For $\kappa \in [-\infty, \infty]$, write $\Lambda^\kappa = \Lambda_{\eta}^\kappa = (1 - \Delta)^{\kappa/2}$ and define the space $[H^\kappa(\mathbb{R}^2)]^2$ as the space of generalized functions $v$ with the finite norm $\| v \|_{\kappa} = \| \Lambda^\kappa v \|_0$. The spaces $H^\kappa(\mathbb{R}^2)$ are a particular case of Bessel potential spaces $W^{\kappa,p}(\mathbb{R}^2)$ with $p = 2$. Notice also that if $\kappa > 0$ these spaces are known as fractional order Sobolev spaces, and if $\kappa$ is a non-negative integer, they are Sobolev spaces. For a smooth bounded domain $G \subset \mathbb{R}^2$, we denote $[H^\kappa(G)]^2$ the space of all generalized functions $v$ on $G$ that can be extended to a generalized functions in $[H^\kappa(\mathbb{R}^2)]^2$ with the norm

\[
\| u \|_\kappa = \inf \left\{ \| \tilde{v} \|_\kappa : \tilde{v} \in [H^\kappa(\mathbb{R}^2)]^2, \tilde{v} = v \text{ a.e. in } G \right\}.
\]
The duality between $[H^\kappa(G)]^2$ and $[H^{-\kappa}(G)]^2$, $k \in ]-\infty, \infty[$, is defined by

$$\langle \phi, \psi \rangle = \langle \Lambda^\kappa \phi, \Lambda^{-\kappa} \psi \rangle_0,$$

for any $\phi \in [H^\kappa(G)]^2$, $\psi \in [H^{-\kappa}(G)]^2$.

Fix $\mathcal{U} = [H^\kappa(\Theta)]^2$, $\kappa > 2$. Denote by $\mathcal{U}'$ its dual space with a topology defined by the seminorm

$$\|\varphi\|_{\mathcal{U}'} = \sup \{ |\varphi(v)| : v \in C^\infty_0(\Theta), \|v\|_\kappa \leq 1 \}.$$

**Lemma 4.1.** The embedding $L^2(\Theta) \to \mathcal{U}'$ is compact.

**Proof.** See the proof of Lemma 2.6 in [15].

Let $\mathcal{X}_1 = C([0, T], \mathcal{U}')$ be the set of $\mathcal{U}'$-valued trajectories with the topology $\mathcal{T}_1$ of the uniform convergence on $[0, T]$. Let $\mathcal{X}_2 = C([0, T], [L^2(\Theta)]^2)$ be the set of $[L^2(\Theta)]^2$-valued weakly continuous functions with the topology $\mathcal{T}_2$ of the uniform weak convergence on $[0, T]$. Let $\mathcal{X}_3 = L^2_w(0, T; [H^1(\Theta)]^2)$ be the set of $[H^1(\Theta)]^2$-valued square integrable functions on $[0, T]$ with a topology $\mathcal{T}_3$ of weak convergence on finite intervals. Finally consider $\mathcal{X}_4 = L^2(0, T; [L^2(\Theta)]^2)$ with the topology $\mathcal{T}_4$ associated with the norm of this space.

**Lemma 4.2.** Let $\mathcal{X} = \cap_{i=1}^4 \mathcal{X}_i$ and $\mathcal{T}$ be the supremum of the corresponding topologies. Then $K \subset \mathcal{X}$ is relatively compact with respect to $\mathcal{T}$ if the following conditions hold:

(i) $\sup_{\varphi \in K} \sup_{0 \leq t \leq T} \|\varphi(t)\|_0 < \infty$,

(ii) $\sup_{\varphi \in K} \int_0^T \|\varphi(s)\|_1^2 \, ds < \infty$,

(iii) $\lim_{\delta \to 0} \sup_{\varphi} \sup_{|t-s| \leq \delta, 0 \leq s, t \leq T} \|\varphi(t) - \varphi(s)\|_{\mathcal{U}'} = 0$.

**Proof.** See the proof of Lemma 2.7 in [15].

Denote $u$ the canonical process in $\mathcal{X}$: $u(t) = u(t, w) = w(t, \theta), w \in \mathcal{X}$. Let $\mathcal{D}_t = \sigma \{ u(s), s \leq t \}, \mathcal{D} = \{ \mathcal{D}_t \}_{0 \leq t \leq T}, \mathcal{D}_T = \mathcal{D}_T$. For each $n$, the approximation $u_n$ (satisfying (7), (8)) defines a measure $\mathbb{P}^n$ on $(\mathcal{X}, \mathcal{D})$.

**Corollary 4.3.** The set $\{ \mathbb{P}^n, n \geq 1 \}$ is relatively weakly compact on $(\mathcal{X}, \mathcal{T})$.

**Proof.** The proof is obtained by following the arguments used in the proof of Corollary 2.8 in [15] (see also Corollary 3.4 in [13]) and our a priori estimates (7) and (8), together with Lemmas 4.4, 4.1 and 4.2.

**Lemma 4.4.** There exists a constant $C$ independent of $n$ such that for all $u \in V$,

$$\|Au\|_{-1} \leq C\|u\|_1, \quad \sum_{l=1}^2 \|\partial_l u\|_{-1} \leq C\|u\|_0, \quad \|Bu\|_{-\kappa} \leq C\|u\|_0^2.$$

**Proof.** $\|Au\|_{-1} \leq C\|u\|_1$ holds by definition of the operator $A$.

For any $v \in V$ we obtain that

$$\left| \sum_{l=1}^2 \langle \partial_l u, v \rangle \right| = \left| - \sum_{l=1}^2 \langle u, \partial_l v \rangle \right| = \left| \sum_{l=1}^2 \langle u, \partial_l v \rangle_0 \right| \leq \|u\|_0\|v\|_1.$$

For any $v \in [C^\infty_0(\Theta)]^2$, applying integration by parts and Sobolev’s embedding theorem,

$$|\langle Bu, v \rangle| = |\langle (u \cdot \nabla) u, v \rangle| = | - \langle (u \cdot \nabla)v, u \rangle | \leq \sup_{\theta \in \Theta} \{ |\nabla v(\theta)| \} \|u\|_0^2 \leq C\|v\|_{\kappa}\|u\|_0^2.$$

□
Now we shall identify $\mathbb{P}^n$ to a solution of a martingale problem. Let us give the definition of our martingale problem.

For any $v \in C_0^\infty(\Theta)$, we denote

$$\varphi^v(u(s)) = \frac{1}{2} \left\langle -\frac{1}{2} A u(s) - B u(s), v \right\rangle - \frac{1}{2} \sum_{i=1}^2 \left\langle \partial_i u(s), v \right\rangle^2,$$  \hspace{1cm} (9)

where $i^2 = -1$. We recall that $\langle \cdot, \cdot \rangle$ denotes the duality between $[H^s(\Theta)]^2$ and $[H^{-s}(\Theta)]^2$.

However, observe that

$$\langle \frac{1}{2} A u(s), v \rangle = \frac{1}{2} \langle \nabla u(s), \nabla v \rangle_0$$

$$\langle B u(s), v \rangle = \langle (u(s) \cdot \nabla) u(s), v \rangle_0 = - \langle (u(s) \cdot \nabla)v, u(s) \rangle_0$$

$$\langle \partial_i u(s), v \rangle = \langle \partial_i u(s), v \rangle_0.$$

**Definition 4.1.** We say a probability measure $\mathbb{P}$ on $\mathcal{X}$ is a solution of the martingale problem $(u_0, -\frac{1}{2} A - B, \mathcal{E})$, where $\mathcal{E} = \{ \partial_1 u \cdot \partial_2 u \}$, if for each $v \in C_0^\infty(\Theta)$,

$$L_t^v \doteq \exp \{ i \langle u(t), v \rangle \} - \int_0^t \exp \{ i \langle u(s), v \rangle \} \varphi^v(u(s)) \, ds \in \mathcal{M}^c_{\text{loc}}(\mathbb{D}, \mathbb{P}^n),$$

and $u(0) = u_0$, $\mathbb{P}$-a.s.

For any $v \in C_0^\infty(\Theta)$, applying Itô’s formula to the scalar semimartingale

$$\langle u_n(t), v \rangle = \langle u_n(t), v \rangle_0 = \langle u_0 n, v \rangle + \int_0^t \left\langle \frac{1}{2} A u_n(s) - B u_n(s), v \right\rangle \, ds$$

$$+ \sum_{i=1}^2 \int_0^t \langle \partial_i u_n(s), v \rangle \, dB^i(s),$$

and the function $f(x) = \exp \{ ix \}$, we obtain

$$\exp \{ i \langle u_n(t), v \rangle \} = \exp \{ i \langle u_0 n, v \rangle \}$$

$$+ \int_0^t i \exp \{ i \langle u_n(s), v \rangle \} \left\langle \frac{1}{2} A u_n(s) - B u_n(s), v \right\rangle \, ds$$

$$+ \sum_{i=1}^2 \int_0^t i \exp \{ i \langle u_n(s), v \rangle \} \langle \partial_i u_n(s), v \rangle \, dB^i(s)$$

$$+ \frac{1}{2} \int_0^t i^2 \exp \{ i \langle u_n(s), v \rangle \} \sum_{i=1}^2 \langle \partial_i u_n(s), v \rangle^2 \, ds$$

$$= \int_0^t \exp \{ i \langle u_n(s), v \rangle \} \varphi^v(u_n(s)) \, ds$$

$$+ \exp \{ i \langle u_0 n, v \rangle \} + \sum_{i=1}^2 \int_0^t i \exp \{ i \langle u_n(s), v \rangle \} \langle \partial_i u_n(s), v \rangle \, dB^i(s).$$

Thus, we have shown the following result
Lemma 4.5. For each $n$, $\mathbb{P}^n$ is a measure on $\mathcal{X}$ such that for each test function $v$ belonging to $C_0^\infty(\Theta)$,

$$L_t^{n,v} = \exp\{i\langle u_n(t), v \rangle\} - \int_0^t \exp\{i\langle u_n(s), v \rangle\} \varphi^v(u_n(s)) \, ds$$

$$= \exp\{i\langle u_0, v \rangle\} + \sum_{l=1}^2 \int_0^t i \exp\{i\langle u_n(s), v \rangle\} \langle \partial_t u_n(s), v \rangle \, dB^l(s),$$

that is, $L_t^{n,v} \in \mathcal{M}^c_{\text{loc}}(\mathbb{D}, \mathbb{P}^n)$. Therefore, $\mathbb{P}^n$ is a solution of the martingale problem $(u_0, -\frac{1}{2}A - B, \mathcal{E})$.

To prove Theorem 2.1 we need the following result

Theorem 4.6. For each $u_0 \in V$ there exists a measure $\mathbb{P}$ on $\mathcal{X}$ solving the martingale problem $(u_0, -\frac{1}{2}A - B, \mathcal{E})$ such that

$$\sup_{0 \leq t \leq T} \|u(t)\|^2_0 < \infty \text{ with probability } 1 \text{ and } \sup_{0 \leq r \leq T} \mathbb{P}\{\|u(r)\|^2_1\} < \infty. \tag{10}$$

In addition, $\mathbb{P}$ - a.s.

$$\int_0^T \left\| \frac{1}{2}Au(s) + Bu(s) \right\|_1^2 \, ds < \infty. \tag{11}$$

Notice that in (10) we make a slight abuse of notation as in [15], that is, we write $\mathbb{P}\{F\}$ for an integral of a measurable function $F$ with respect to the measure $\mathbb{P}$.

Proof. The proof follows from the arguments used in the proof of Theorem 2.10 in [15]. For sake of completeness we shall sketch some of them. Owing to Corollary 4.3 we can suppose that a sequence of measures $\{\mathbb{P}^n\}$ converges weakly to some measure $\mathbb{P}$. Let $\omega_n \to \omega$ in $\mathcal{X}$. By Lemma 4.2 the sequence $\{\omega_n(t)\}$ is weakly relatively compact in $L^2\left([0,T]; [H^1(\Theta)]^2\right)$. Thus, using the weakly relatively compactness of $\{\omega_n(t)\}$ and Lemma 4.4 it is possible to prove that the sequence $\{L_t^{n,v}(\omega_n)\}$ is equicontinuous in $t$ with respect to $n$. Next, one shows that

$$\sup_{s \leq T} |L_t^{n,v}(\omega_n) - L_t^v(\omega)| \xrightarrow{n \to \infty} 0.$$

Thus for each compact set $K \subseteq \mathcal{X}$,

$$\sup_{s \leq T} \sup_{\omega \in K} |L_t^{n,v}(\omega_n) - L_t^v(\omega)| \xrightarrow{n \to \infty} 0.$$

Then the probability $\mathbb{P}$, as the limit of the sequence $\{\mathbb{P}^n\}$, is shown to be a solution of the martingale problem $(u_0, -\frac{1}{2}A - B, \mathcal{E})$ and the estimates (10) and (11) can be checked. \qed

Finally, we shall give the proof of our main result:

Proof of Theorem 2.1 Using Theorem 4.6 the proof can be completed by borrowing the arguments of that of Theorem 2.1 in [15]. Again, for sake of completeness we shall give the details. Owing to Theorem 4.6 there exists a measure $\mathbb{P}$ on $\mathcal{X}$ such that (10) holds and $\mathbb{P}$-a.s. for each $v \in C_0^\infty$,

$$\langle u(t), v \rangle_0 = \langle u_0, v \rangle_0 + \int_0^t \left\langle \frac{1}{2}Au(s) + Bu(s), v \right\rangle \, ds + M^v(t),$$
where $M^v_t \in \mathcal{M}_{loc}(\mathbb{D}, \mathbb{P})$ and
\[
\|M^v(t)\|_0^2 - \sum_{l=1}^{2} \int_0^t \langle \partial_t u(s), v \rangle^2 ds \in \mathcal{M}_{loc}(\mathbb{D}, \mathbb{P}).
\]
Since
\[
P\left\{ \sum_{l=1}^{2} \int_0^t \langle \partial_t u(s), v \rangle^2_0 ds \right\} < \infty,
\]
there is an $[L^2(\Theta)]^2$-valued continuous martingale $M(t)$ such that $\mathbb{P}$-a.s. $\langle M(t), v \rangle = M^v(t)$ for all $t$. In fact, taking an $[L^2(\Theta)]^2$-basis $\{v_k\}$, we define $M(t) := \sum_{k=1}^{\infty} M^v(t) v_k$.

Therefore, $\mathbb{P}$-a.s.,
\[
\begin{cases}
du(t) = \left\{ \frac{1}{2} \Delta u(t) + \mathcal{B}u(t) \right\} dt + dM(t), \\
u(0) = u_0.
\end{cases}
\]
According to Lemma 3.2 in [14], there exists a cylindrical Wiener process $\tilde{B}$ in $\mathbb{R}^2$ (possibly in some extension of the probability space $(\Omega, \mathcal{D}_T, \mathbb{P})$) such that
\[
M(t) = \sum_{l=1}^{2} \int_0^t \partial_t u(s) dB^l(s).
\]
This completes the proof of our main result.

\section{Girsanov Transformations}

We can write Equation (2) in the following way:
\[
du(t, \theta) = \frac{1}{2} \Delta u(t, \theta) dt + \sum_{l=1}^{2} \partial_t u(t, \theta) dB^l(t, \theta),
\]
where
\[
\tilde{B}(t, \theta) := B(t) - \int_0^t u(r, \theta) dr.
\]
We would like to prove that $\tilde{B}$ is a Wiener process with respect to a new probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$. The following proposition holds:

\begin{proposition}
Assume hypotheses of Theorem 2.1.
For every $\delta > 0$ there exists a subset $\Theta(\delta) \subset \Theta$ such that $\mu(\Theta(\delta)^c) \leq \delta$ and such that the stochastic process
\[
\tilde{B}(t, \cdot) := B(t) - \int_0^t u(r, \cdot) dr
\]
is $[L^\infty(\Theta(\delta))]^2$-valued Wiener process with respect to a new probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{P}}_\delta)$, where
\[
\frac{d\tilde{\mathbb{P}}_\delta}{d\mathbb{P}} = \exp \left( \sum_{l=1}^{2} \int_0^T u^l(r, \theta, \omega) dB^l(r) - \frac{1}{2} \int_0^T \left[ \sum_{l=1}^{2} (u^l(r, \theta, \omega))^2 \right] dr \right).
\]
\end{proposition}
Proof. By Theorem 2.1 we know that the solution $u$ of Equation (2) satisfies that

$$\sup_{0 \leq r \leq T} \|u(r, \omega)\|_0^2 =: \mathcal{K} < \infty,$$

a.e. $\omega \in \Omega$.

Fix $\delta > 0$. Define the set

$$\Theta(\delta) := \{ \theta \in \Theta : \int_0^T \left[ (u^1(r, \theta, \omega))^2 + (u^2(r, \theta, \omega))^2 \right] dr \leq \frac{TK}{\delta} \}. $$

We apply Chebyshev’s inequality for the set $\Theta(\delta)^c$ and the Lebesgue measure $\mu$. We get

$$\mu(\Theta(\delta)^c) \leq \frac{\delta}{TK} \int_{\Theta(\delta)^c} \left( \int_0^T \left[ (u^1(r, \theta, \omega))^2 + (u^2(r, \theta, \omega))^2 \right] dr \right) d\theta$$

$$= \frac{\delta}{TK} \int_0^T \left( \int_{\Theta} \left[ (u^1(r, \theta, \omega))^2 + (u^2(r, \theta, \omega))^2 \right] d\theta \right) dr$$

$$\leq \delta,$$

a.e. $\omega \in \Omega$.

Now, notice that on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we have that

$$\sup_{\theta \in \Theta(\delta)} \left\{ \int_0^T \left[ (u^1(r, \theta, \omega))^2 + (u^2(r, \theta, \omega))^2 \right] dr \right\} \leq \frac{TK}{\delta},$$

a.e. $\omega \in \Omega$.

Then, Novikov’s condition

$$\mathbb{E} \left( \exp \left\{ \frac{1}{2} \int_0^T \left[ (u^1(t, \theta))^2 + (u^2(t, \theta))^2 \right] dt \right\} \right) \leq \exp \left\{ \frac{TK}{2\delta} \right\} < +\infty,$$

holds, for any $\theta \in \Theta(\delta)$. Therefore, we can assume that

$$\tilde{B}(t, \cdot) := B(t) - \int_0^t u(r, \cdot) dr$$

is a $[L^\infty(\Theta(\delta))]^2$-valued Wiener process with respect to a new probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{P}}_\delta)$.

The measure $\tilde{\mathbb{P}}_\delta$ is absolutely continuous with respect to $\mathbb{P}$ with density given by

$$\exp \left( \sum_{l=1}^2 \int_0^T u^l(r, \theta, \omega) dB^l(r) - \frac{1}{2} \int_0^T \left[ \sum_{l=1}^2 (u^l(r, \theta, \omega))^2 \right] dr \right).$$

\[ \square \]

We remark that similar (linear) stochastic heat equations with multiplicative noise as (13) have been studied by Pardoux [16, 17, 18], Krylov and Rozovskii [10, 11, 12], and Funaki [9] among others. We refer also to the more recent work [8].
6. 2D stochastic Euler equations on the torus

In Equation (11) we replace the 2-dimensional Brownian motion by the following Wiener process on $\Theta = [0, 2\pi]^2$ with free divergence and periodic boundary conditions:

$$W(t, \theta) = \frac{1}{\sqrt{c_W}} \sum_{k \in \mathbb{Z}^2} q_k^{1/2} [c_k(\theta) B_k^1(t) + s_k(\theta) B_k^2(t)],$$

(14)

where

$$q_{(0,0)} = 1, \quad q_k = \frac{1}{|k|^{2(2\beta - 1)}} \text{ for } k \in \mathbb{Z}^2 \setminus \{(0,0)\},$$

$$c_W = 1 + \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{(k^1)^2}{|k|^{2\beta}} = 1 + \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{(k^2)^2}{|k|^{2\beta}}, \quad |k|^2 = (k^1)^2 + (k^2)^2, \quad \beta > 3$$

$$c_{(0,0)} = (1, 0), \quad s_{(0,0)} = (0, 1),$$

$$c_k(\theta) = \frac{1}{|k|} (k^2, -k^1) \cos(k \cdot \theta), \quad s_k(\theta) = \frac{1}{|k|} (k^2, -k^1) \sin(k \cdot \theta) \text{ for } k \in \mathbb{Z}^2 \setminus \{(0,0)\},$$

$$k \cdot \theta = k^1 \theta_1 + k^2 \theta_2$$

$B_k = (B_k^1, B_k^2)$ is a sequence of independent 2-dimensional standard Brownian motions.

(See [5] for details.) Notice that $\{c_k, s_k : k \in \mathbb{Z}^2\}$ is a complete system of vectors for $[L^2(\Theta)]^2$ of divergence free and with periodic boundary conditions, which are the eigenvectors of the operator $-\Delta$ with eigenvalues $\{|k|^2 : k \in \mathbb{Z}^2\}$, respectively.

This stochastic process is a centered Gaussian process on the Sobolev space $[H^\alpha(\Theta)]^2$, $0 < \alpha < \beta - 2$, with covariance function

$$\mathbb{E} \left( W(t) \otimes W(t) \right) = t Q,$$

where $Q$, defined by the eigenvalues $\{q_k\}_{k \in \mathbb{Z}^2}$ in $\{c_k, s_k : k \in \mathbb{Z}^2\}$, is of trace class. In addition, we can define

$$\partial_i W(t, \theta) = \frac{1}{\sqrt{c_W}} \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} q_k^{1/2} k^i \left[ -s_k(\theta) B_k^1(t) + c_k(\theta) B_k^2(t) \right],$$

for $1 \leq i \leq 2$.

To be more precise, we want to study the following stochastic Euler equation in dimension 2:

$$
\begin{align*}
\frac{du(t, \theta)}{dt} &= -(u(t, \theta) \cdot \nabla)u(t, \theta) \, dt + \sum_{i=1}^2 \partial_i u(t, \theta) \circ dW^i(t, \theta) \quad \text{ in } [0, T] \times \Theta, \\
\text{div} \, u(t, \theta) &= 0 \quad \text{ in } [0, T] \times \Theta, \\
u(t, (\theta^1, 0)) &= u(t, (\theta^1, 2\pi)), \\
u(t, (0, \theta^2)) &= u(t, (2\pi, \theta^2)), \text{ for } \theta^1, \theta^2 \in [0, 2\pi], \, t \in [0, T], \\
u(\theta, 0) &= u_0(\theta) \quad \text{ in } \Theta,
\end{align*}
$$

(15)
in the Stratonovich formulation, or

\[
\begin{align*}
  du(t, \theta) &= \left\{ \frac{1}{2} \Delta u(t, \theta) - (u(t, \theta) \cdot \nabla) u(t, \theta) \right\} \, dt \\
  \text{div } u(t, \theta) &= 0 \quad \text{in } ]0, T[ \times \Theta, \\
  \partial_t u(t, \theta) &= 0 \quad \text{in } ]0, T[ \times \Theta, \\
  u(t, (\theta^1, 0)) &= u(t, (\theta^1, 2\pi)), \\
  u(t, (0, \theta^2)) &= u(t, (2\pi, \theta^2)), \quad \text{for } \theta^1, \theta^2 \in [0, 2\pi], \; t \in ]0, T[, \\
  u(\theta, 0) &= u_0(\theta) \quad \text{in } \Theta,
\end{align*}
\]

in the Itô formulation, respectively, where we also assume that the initial condition \(u_0(\theta)\) is also a periodic function. Indeed, using the expressions of \(\partial_t W^j(t, \theta)\)'s, \(1 \leq i, j \leq 2\), we obtain that (cf. [5])

\[
\sum_{l=1}^{2} \partial_l u(t, \theta) \circ dW^l(t, \theta) = \sum_{l=1}^{2} \partial_l u(t, \theta) dW^l(t, \theta) + \frac{1}{2} \Delta u(t, \theta).
\]

The stochastic term of \(\partial_t u(t, \theta)\) is

\[
\sum_{j=1}^{2} \partial_{jl}^2 u(t, \theta) dW^j(t, \theta) + \sum_{j=1}^{2} \partial_j u(t, \theta) d\partial_t W^j(t, \theta),
\]

and the respective joint quadratic variations give

\[
\langle W^j(t, \theta), W^l(t, \theta) \rangle = \delta_{jl}, \quad \langle \partial_t W^j(t, \theta), W^l(t, \theta) \rangle = 0.
\]

The basic spaces \(H\) and \(V\) are defined as

\[
H = \left\{ v \in [L^2(\Theta)]^2 : \text{div } v = 0; \; v(\theta^1, 0) = v(\theta^1, 2\pi), \right\}
\]

\[
V = \left\{ v \in [H^1(\Theta)]^2 : \text{div } v = 0; \; v(\theta^1, 0) = v(\theta^1, 2\pi), \right\}
\]

respectively.

In the space \(H\) consider the Stokes operator \(\mathcal{A} : D(\mathcal{A}) \subset H \to H\), defined as \(\mathcal{A} v = -P_H \Delta v\), for all \(v \in D(\mathcal{A})\), where \(P_H\) is the Leray projector. Denoting by \(\langle \cdot, \cdot \rangle\) the inner product in \(H\), the operator \(\mathcal{A}\) is defined as

\[
\langle \mathcal{A} u, v \rangle = \int_{\Theta} \nabla u \cdot \nabla v,
\]

for all \(u, v \in V\).

We also define \(\mathcal{B} : V \to V'\) as \(\mathcal{B} u = (u \cdot \nabla) u\), that is,

\[
\langle \mathcal{B} u, v \rangle = \int_{\Theta} (u \cdot \nabla) u \cdot v,
\]

for all \(u, v \in V\). Here \(V'\) denotes the topological dual of \(V\).
In terms of $\mathcal{A}$, $\mathcal{B}$ we can write Equation (19) as the following stochastic evolution equation in $V'$:
\[
\begin{aligned}
    du(t) &= - \left\{ \frac{1}{2} \mathcal{A}u(t) + \mathcal{B}u(t) \right\} dt + \sum_{l=1}^{2} \partial_t u(t) dW^l(t), \\
    u(0) &= u_0.
\end{aligned}
\]  
(19)

The formulation of Equation (19) is equivalent to the following weak or variational form:
\[
\begin{aligned}
    d\langle u(t), v \rangle &= - \left\{ \frac{1}{2} \mathcal{A}u(t) + \mathcal{B}u(t), v \right\} dt + \sum_{l=1}^{2} \langle \partial_t u(t), dW^l(t), v \rangle \quad \text{in } [0, T], \\
    \langle u^v(0), v \rangle &= \langle u_0, v \rangle,
\end{aligned}
\]  
(20)

for all $v \in V$.

Following analogous arguments as those in Section 3 we can define the corresponding Faedo-Galerkin approximations of Equation (20). Let $H_n := \text{span} \left\{ c_k, s_k : k \in I_n^2 \right\}$, where $I_n^2 = \{-n, \ldots, -1, 0, 1, \ldots, n\}^2$, and define
\[
u_n(t) = \sum_{k \in I_n^2} \left[ u_n^{1k}(t) c_k + u_n^{2k}(t) s_k \right],
\]

where $u_n^{1k} = \langle u_n^v(t), c_k \rangle$, $u_n^{2k} = \langle u_n^v(t), s_k \rangle$, as the solution of the following stochastic differential equation: For each $v \in H_n$
\[
    d\langle u_n(t), v \rangle = - \left\{ \frac{1}{2} \mathcal{A}u_n(t) + \mathcal{B}u_n(t), v \right\} dt + \sum_{l=1}^{2} \langle \partial_t u_n(t), dW^l_n(t), v \rangle,
\]  
(21)

where
\[
    W_n(t, \theta) = \frac{1}{\sqrt{C_W}} \sum_{k \in I_n^2} q_k^{1/2} \left[ c_k B_1^k(t) + s_k B_2^k(t) \right],
\]

with $u_{0n} = \sum_{k \in I_n^2} \langle u_0, c_k \rangle c_k + \langle u_0, s_k \rangle s_k$ as initial condition.

Let us consider a priori estimates for $u_n$ independent on $n$.

**Lemma 6.1.** Let $u_0 \in V$. Then for each $T > 0$
\[
    \sup_n \sup_{0 \leq t \leq T} \| u_n(t) \|^2 \leq \| u_0 \|^2_0
\]  
(22)

with probability 1, and
\[
    \sup_n \mathbb{E} \| u_n(t) \|^2 \leq \| u_0 \|^2_1 e^{Ct},
\]  
(23)

for any $t \in [0, T]$, where $C > 0$ is a constant not depending on $n$.

**Proof.** Firstly, notice that we can write the process $u_n$ in the following way:
\[
    u_n(t) = u_{0n} - \int_0^t \left\{ \frac{1}{2} \mathcal{A}u_n(s) + \mathcal{B}u_n(s) \right\} ds \\
    + \frac{1}{\sqrt{C_W}} \sum_{k \in I_n^2} q_k^{1/2} \int_0^t \left[ (c_k \cdot \nabla) u_n(s) dB_1^k(s) + (s_k \cdot \nabla) u_n(s) dB_2^k(s) \right],
\]
where

\((\mathbf{c}_{(0,0)} \cdot \nabla)u_n(s) = \partial_1 u_n(s), \quad (\mathbf{g}_{(0,0)} \cdot \nabla)u_n(s) = \partial_2 u_n(s)\),

and

\((\mathbf{c}_k \cdot \nabla)u_n(s) = \frac{k^2}{|k|} \cos(k \cdot \ast) \partial_1 u_n(s) - \frac{k^1}{|k|} \cos(k \cdot \ast) \partial_2 u_n(s), \quad (\mathbf{g}_k \cdot \nabla)u_n(s) = \frac{k^2}{|k|} \sin(k \cdot \ast) \partial_1 u_n(s) - \frac{k^1}{|k|} \sin(k \cdot \ast) \partial_2 u_n(s),\)

for \(k \neq (0,0)\).

On the other hand,

\[
\begin{align*}
\nabla u_n(t) &= \nabla u_0 - \int_0^t \left\{ \frac{1}{2} \nabla [A u_n(s)] + \nabla [B u_n(s)] \right\} \, ds \\
&
+ \frac{1}{\sqrt{c_W}} \sum_{k \in \mathcal{I}_2} q_k^{1/2} \int_0^t \left[ \nabla [(\mathbf{c}_k \cdot \nabla)u_n(s)] dB_k^1(s) + \nabla [(\mathbf{g}_k \cdot \nabla)u_n(s)] dB_k^2(s) \right],
\end{align*}
\]

where

\[
\begin{align*}
\nabla [(\mathbf{c}_k \cdot \nabla)u_n(s)] &= (\partial_1 \partial_1 u_n(s), \partial_2 \partial_1 u_n(s)), \\
\nabla [(\mathbf{g}_k \cdot \nabla)u_n(s)] &= (\partial_1 \partial_2 u_n(s), \partial_2 \partial_2 u_n(s)),
\end{align*}
\]

and

\[
\begin{align*}
\nabla [(\mathbf{c}_k \cdot \nabla)u_n(s)] &= -\frac{(k^2 k^1, (k^2)^2)}{|k|} \sin(k \cdot \ast) \partial_1 u_n(s) + \frac{k^2}{|k|} \cos(k \cdot \ast) (\partial_1 \partial_1 u_n(s), \partial_2 \partial_1 u_n(s)) \\
&+ \frac{(k^1 k^2)}{|k|} \sin(k \cdot \ast) \partial_2 u_n(s) - \frac{k^1}{|k|} \cos(k \cdot \ast) (\partial_1 \partial_2 u_n(s), \partial_2 \partial_2 u_n(s)),
\end{align*}
\]

\[
\begin{align*}
\nabla [(\mathbf{g}_k \cdot \nabla)u(s)] &= \frac{(k^2 k^1, (k^2)^2)}{|k|} \cos(k \cdot \ast) \partial_1 u_n(s) + \frac{k^2}{|k|} \sin(k \cdot \ast) (\partial_1 \partial_1 u_n(s), \partial_2 \partial_1 u_n(s)) \\
&- \frac{(k^1 k^2)}{|k|} \cos(k \cdot \ast) \partial_2 u_n(s) - \frac{k^1}{|k|} \sin(k \cdot \ast) (\partial_1 \partial_2 u_n(s), \partial_2 \partial_2 u_n(s)),
\end{align*}
\]

for \(k \neq (0,0)\).

Then, using similar arguments as those in the proof of Lemma 3.1, in particular using Itô’s formula, we obtain

\[
\begin{align*}
\|u_n(t)\|^2_2 &= \|u_0\|^2_2 - \int_0^t \langle u_n(s), u_n(s) \rangle_1 \, ds - 2 \int_0^t \langle (u_n(s) \cdot \nabla)u_n(s), u_n(s) \rangle_0 \, ds \\
&+ \frac{2}{\sqrt{c_W}} \sum_{k \in \mathcal{I}_2} q_k^{1/2} \int_0^t \left[ \langle (\mathbf{c}_k \cdot \nabla)u_n(s), u_n(s) \rangle_0 dB_k^1(s) \\
&+ \langle (\mathbf{g}_k \cdot \nabla)u_n(s), u_n(s) \rangle_0 dB_k^2(s) \right] \\
&+ \frac{1}{c_W} \left( 1 + \sum_{k \in \mathcal{I}_2 \setminus \{(0,0)\}} (k^1 k^2) \right) \int_0^t \|u_n(s)\|^2_1 \, ds \\
&\leq\|u_0\|^2_2 - \int_0^t \|u_n(s)\|^2_2 \, ds + \int_0^t \|u_n(s)\|^2_1 \, ds = \|u_0\|^2_1,
\end{align*}
\]
for a.e. $\omega \in \Omega$, and
\[
\|\nabla u_n(t)\|^2_0 = \|\nabla u_0\|^2_0 - \int_0^t \sum_{j,l=1}^2 \|\partial_j \partial_l u_n(s)\|^2_0 \, ds
- 2 \int_0^t \langle \nabla[(u_n(s) \cdot \nabla) u_n(s)], \nabla u_n(s) \rangle_0 \, ds
- \frac{2}{\sqrt{cW}} \sum_{k \in I^2_n \setminus \{(0,0)\}} \frac{1}{|k|^{2\beta-2}} \int_0^t \left[ \langle (c_k \cdot \nabla) u_n(s), \Delta u_n(s) \rangle_0 \, dB_k^1(s) \right.
+ \langle (\mathfrak{s}_k \cdot \nabla) u_n(s), \Delta u_n(s) \rangle_0 \, dB_k^2(s) \bigg]
+ \frac{1}{cW} \left( 1 + \sum_{k \in I^2_n \setminus \{(0,0)\}} \frac{(k^1)^2}{|k|^{2\beta}} \right) \int_0^t \sum_{j,l=1}^2 \|\partial_j \partial_l u_n(s)\|^2_0 \, ds
+ \frac{1}{cW} \left( \sum_{k \in I^2_n \setminus \{(0,0)\}} \frac{(k^1)^2}{|k|^{2\beta-2}} \right) \int_0^t \|\nabla u_n(s)\|^2_0 \, ds
\leq \|\nabla u_0\|^2_0 + \frac{\epsilon'_W}{cW} \int_0^t \|\nabla u_n(s)\|^2_0 \, ds
- \frac{2}{\sqrt{cW}} \sum_{k \in I^2_n \setminus \{(0,0)\}} \frac{1}{|k|^{2\beta-1}} \int_0^t \left[ \langle (c_k \cdot \nabla) u_n(s), \Delta u_n(s) \rangle_0 \, dB_k^1(s) \right.
+ \langle (\mathfrak{s}_k \cdot \nabla) u_n(s), \Delta u_n(s) \rangle_0 \, dB_k^2(s) \bigg],
\]
where
\[
\epsilon'_W = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{(k^1)^2}{|k|^{2\beta-2}}.
\]
Thus, applying expectations to the expression of $\|\nabla u_n(t)\|^2_0$ and next Gronwall inequality, we obtain
\[
\mathbb{E}\|\nabla u_n(t)\|^2_0 \leq \|\nabla u_0\|^2_0 e^{\frac{\epsilon'_W}{cW} t}.
\]
It ends the proof of this Lemma. $\square$

Observe that using Lemma 6.1 we can adapt the arguments of Section 3 in order to give an existence result for the solution of Equation (16) as a solution of a martingale problem. The statement of the result that can be proven is the following:

**Theorem 6.2.** Let $u_0 \in V$. Then there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a right-continuous filtration $\mathbb{F} = \{\mathcal{F}_t\}$ of $\sigma$-algebras, a Wiener process $W$ on $[0, 2\pi]^2$ defined by (14), and an $L^2([0, 2\pi]^2)^2$-valued weakly continuous $\mathbb{F}$-adapted process $u(t)$ such that
\[
\sup_{0 \leq t \leq T} \|u(t)\|^2_0 \leq \|u_0\|^2_0 =: K < +\infty
\]
with probability 1,
\[
\sup_{0 \leq t \leq T} \mathbb{E}\|u(t)\|^2_1 \leq \|u_0\|^2_1 e^{CT} =: K' < \infty,
\]
where $C$ is a positive constant, and (16) holds. In addition, $u(t)$ is strongly continuous in $t$. 
Observe that if in the definition of the Brownian motion $W(t)$ we only sum on a finite number of indices this Theorem still holds.

Concerning Girsanov transformation a result analogous to Proposition 5.1. holds, by the same methods, in the simplest case where the Brownian motion is space-independent, namely when

$$W(t) = c_{(0,0)} B_{(0,0)}^1(t) + s_{(0,0)} B_{(0,0)}^2(t) = (B^1(t), B^2(t))$$

The stochastic transport equation for this case has been studied recently in [20].

7. Geometric formulation of the equation

V. I. Arnold (cf. [2]) showed that the deterministic non-viscous incompressible Euler equation, namely

$$\frac{d}{dt} u(t, \theta) = -(u(t, \theta) \cdot \nabla) u(t, \theta) - \nabla p(t, \theta)$$
$$\text{div } u(t, \theta) = 0$$

corresponds to the equation of the geodesic flow defined on the group $G(M)$ of measure-preserving diffeomorphisms over a compact oriented Riemannian manifold $M$ with respect to the $L^2$ metric for the volume measure. Such a group is infinite-dimensional but, nevertheless, can be endowed with some Riemannian structure, as it was proved and developed by Ebin and Marsden ([11]) following Arnold’s work. The metric is right-invariant and we have practically a Lie algebra (except for some regularity conditions). In particular Euler equation can be written as a geodesic equation,

$$\frac{d}{dt} u(t) = -\sum_{i,j} \Gamma_{i,j} u^i(t) u^j(t)$$

where $\Gamma$ denote the Christoffel symbols of the corresponding connection, $u^*$ the components of the vector $u$ in the corresponding tangent space to the identity (or Lie algebra, which is identified with the space of vector fields with vanishing divergence). The equation here should be interpreted in the sense of distributions for $L^2$, which explains that there is no pressure term.

Actually the formulation of Euler equation as a geodesic flow is a particular case of the so-called Euler-Poincaré reduction in Geometrical Mechanics, that has been generalized to the stochastic framework in [1].

When $M = T^2$ is the two-dimensional torus we refer to [3] for an explicit computation of the Lie brackets of the vector fields $c_k, s_k$. The corresponding Levi-Civita-Christoffel symbols, defined by

$$\Gamma_{k,l}^m = \frac{1}{2}(c_{k,l}^m - c_{l,m}^k + c_{m,k}^l)$$

where

$$[e_k, e_l] = \sum_m c_{k,l}^m e_m$$

and $e_k$ is a generic notation for either $c_k$ or $s_k$, have also been computed in detail in [3]. In this context the stochastic Euler equation ([11]) reads,
\[ du(t) = - \sum_{l,j} \Gamma_{e_l,e_j} u^l(t) u^j(t) \circ dB^l(t) \]

and the linear equation (13), obtained after a Girsanov transformation,

\[ du(t) = - \Gamma_{e_l,e_j} u^j(t) \circ d\tilde{B}^l(t), \]

where we have also used the notation \( \Gamma_{e_k,e_l} \) for \( \Gamma_{k,l} \).

Using a space-independent Brownian motion corresponds to let the component \( l \) take only the value \((0,0)\).

8. Energy transfer

Let us consider the case of the two-dimensional torus and suppose we perturb Euler equation by a space-independent Brownian motion only. We want to describe how the energy is transferred between Fourier modes.

Define the energy modes of \( u(t) \) by

\[ \xi(t,k) = \mathbb{E}(\langle u(t), c_k \rangle^2 + \langle u(t), s_k \rangle^2) \]

For the space-invariant Brownian motion we considered, we have, writing \( e_k \) as a generic notation for \( c_k \) or \( s_k \),

\[ d\langle u(t), e_k \rangle = \sum_{i=1}^2 \langle \partial_i u(t) dB^i(t), e_k \rangle + \frac{1}{2} \langle \Delta u(t), e_k \rangle dt. \]

By Itô calculus,

\[ d\langle u(t), e_k \rangle^2 = 2\langle u(t), e_k \rangle \sum_{i=1}^2 \langle \partial_i u(t) dB^i(t), e_k \rangle + \langle u(t), e_k \rangle \langle \Delta u(t), e_k \rangle dt \]

\[ + \sum_{i=1}^2 \langle \partial_i u(t), e_k \rangle^2 dt \]

Therefore,

\[ \frac{d}{dt} \mathbb{E}(\langle u(t), c_k \rangle^2 + \langle u(t), s_k \rangle^2) = -|k|^2 \mathbb{E}(\langle u(t), c_k \rangle^2 + \langle u(t), s_k \rangle^2) \]

\[ + \mathbb{E}((k^1)^2 \{ \langle u(t), c_k \rangle^2 + \langle u(t), s_k \rangle^2 \}) \]

\[ + \mathbb{E}((k^2)^2 \{ \langle u(t), c_k \rangle^2 + \langle u(t), s_k \rangle^2 \}), \]

where the expectation being taken with respect to the measure \( \tilde{P} \).

Therefore we have, for every \( k \),

\[ \frac{d}{dt} \xi(t,k) = 0 \]

and the energy transfer is therefore trivial.

For a space-invariant Brownian motion on the two-dimensional torus the energy transfer was computed for the linear equation in [13] (Theorem 4.4): the process is still conserved but the transfer is much more elaborated. The question remains whether we can obtain in this case the linear equation from the non-linear one by a Girsanov transformation: this
seems to be a hard problem, since the regularity we obtain for the solutions that we have obtained is not sufficient.

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