Hall-magneto hydrodynamic surface waves in solar wind flow-structures

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Abstract. This paper investigates the parallel propagation of magneto hydrodynamic (MHD) surface waves travelling along an ideal steady plasma slab surrounded by a steady plasma environment in the framework of Hall magnetohydrodynamics. The magnitudes of the ambient magnetic field, plasma density and flow velocity inside and outside the slab are different. Two possible directions of the relative flow velocity (in a frame of reference co-moving with the ambient flow) have been studied. In contrast to the conventional MHD surface waves which are usually assumed to be pure surface or pseudo-surface waves, the Hall-MHD approach makes it necessary to treat the normal MHD slab’s modes as \textit{generalized} surface waves. The latter have to be considered as a superposition of two partial waves, one of which is a pure/pseudo-surface-wave whereas the other constitutive wave is a leaky one. From the two kinds of surface-wave modes that can propagate, notably \textit{sausage} and \textit{kink} ones, the dispersion behaviour of the kink mode turns out to be more complicated than that of the sausage mode. In general, the flow increases the waves’ phase velocities comparing with their magnitudes in a static Hall-MHD plasma slab. The applicability of the results to real solar wind flow-structures is briefly discussed.
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### 1. Introduction

It is now well established that the solar atmosphere, from the photosphere to the corona and the solar wind, is a highly structured medium. Satellite observations have confirmed the existence of photospheric flux tubes, coronal holes, coronal loops and magnetic arcades. Another key observation of the solar atmosphere is the presence of steady flows. Bulk motions are registered along or nearly along the magnetic field lines which outline the magnetic structures [1]. Recent observations made by two _HELIOS_ spacecraft have revealed fine structures in high-speed solar wind flows. These structures are in the form of thin flow layers (or tubes) that are adjacent to each other with differences in their plasma parameters (density, magnetic field, steady flow-speed) [2]. The structures can be separated by tangential discontinuities in the magnetic field, across which the total (kinetic plus magnetic) pressure is continuous. Parker [3] was the first to predict the existence of such long and thin spaghetti-like stream structures as constituent units of the solar wind high-speed streams. These structures originate in the Sun, from where they spread out in the heliosphere, thus maintaining their identity even at a distance of 1AU from the Sun.

Satellite measurements of plasma characteristics, such as the magnetic field, the thermal and flow velocity and density of the plasma or plasma compositions, are important to understand the various plasma wave modes which may occur. However, wave analysis requires further information and special tools to identify which set of modes is contributing to observed wave features. In practice, one may use filters to perform the so-called _pattern recognition_ to detect the various kind of modes that may propagate in the plasma and to determine their contributions to the wave energy [4].

The magnetohydrodynamic formalism widely used in studying wave phenomena in the solar atmosphere is a plasma model which is basically classical one-fluid dynamics with the additional complication that the fluid is assumed to be electrically conducting and so capable of generating and interacting with magnetic fields. Magnetohydrodynamics gives a good description of large-scale slow phenomena in a magnetized plasma. Therefore, it is only applicable to frequencies well below the ion cyclotron frequency $\omega_{ci}$ [5]. Multi-fluid models are plasma theories whose validity does not exhibit any frequency limit; however, their applicability to the mode filter concept seems to be practically very tough [4]. An important step in expanding the frequency range is the change from conventional magnetohydrodynamics to the Hall magnetohydrodynamics.
The Hall-magnetohydrodynamic approach in studying wave phenomena in fusion and astrophysical plasmas is relatively new. It accounts for the ion-cyclotron/Hall term in the generalized Ohm’s law (reflecting some aspects of plasma kinetic theory) on the dispersion characteristics and the damping of hydromagnetic waves. The Hall magnetohydrodynamics (Hall MHD) is defined to be conventional magnetohydrodynamics plus the Hall term in Faraday’s law [i.e. $\nabla \times (j \times B)/n_e e$] [6]. In this way, it is possible to describe waves with frequencies up to $\omega \approx \omega_{ci}$. Since the model still neglects the electron mass, it is limited to frequencies well below the lower hybrid frequency: $\omega \ll \omega_{LH}$. Generally speaking, the theory of Hall MHD is relevant to plasma dynamics occurring on length scales shorter than an ion inertial length ($L < l_{Hall} = c/\omega_{pi}$) and time scales shorter than an ion cyclotron period ($t < \omega_{ci}^{-1}$) [6]. One expects that the Hall MHD is applicable in studying not only various space and laboratory plasma wave processes, but also the magnetic evolution of white dwarfs and neutron stars [7], magnetic screening in accreting neutron stars [8], interstellar-medium turbulence [9], Alfvénic collisionless magnetic reconnection [10,11], generation of magnetic fields in astrophysical environments through dynamo activity [12]–[14], and the evolution of the magnetorotational instability in protostellar disks [15], accretion disks [16]–[18], and dwarf nova disks [19].

Hydromagnetic surface waves in the solar atmosphere in the framework of conventional magnetohydrodynamics have been discussed over the last two decades by several authors (e.g. [20]–[26]). Surface waves differ distinctly from the bulk waves in amplitude of vibrations. At a plane interface, the amplitude of a surface wave decreases exponentially, in the direction normal to the interface, while the amplitude remains constant along the interface. Inside a slab, the amplitude of a surface wave behaves as $\cosh(x)$ or $\sinh(x)$ whilst decreases exponentially outside the slab—this is a pure surface wave. When inside the slab the wave field is oscillatory, the wave is called pseudo-surface (or body) wave, and while inside the slab the wave amplitude is evanescent and outside the layer the wave field turns out to be oscillatory, the perturbation is termed a leaky wave. The inclusion of the Hall term in the generalized Ohm’s law changes substantially the propagation characteristics of hydromagnetic surface waves as well as their spatial structure [27]–[30]. In particular, for a finite-$\beta$ Hall-MHD plasma slab the transverse wave structure is specified by two pairs of attenuation coefficients in contrast with the one pair for the traditional MHD surface modes. As has been shown recently by Shukla and Sharma [31], the compressible Hall-MHD model equations govern the coupling of the right and left circularly polarized Alfvén waves whose mutual nonlinear interaction controls, e.g. the filament formation in the solar wind, more specifically the critical size of the filaments and the threshold for parametric instabilities of Alfvén waves. Furthermore, the possibility of acceleration of solar wind plasma by nonlinear Hall-MHD waves (Korteweg–de Vries–Burgers’ solitons) is another application [32]. Thus, these are additional supporting arguments in favour of studying the propagation of hydromagnetic surface waves in flowing plasma slabs by means of the Hall MHD.

The first examination of hydromagnetic surface waves in an isolated incompressible plasma layer in steady state, surrounded by a stationary medium, was carried out by Geronicolas [33], who has shown that the MHD surface mode becomes unstable when the squared flow speed is roughly twice as large as the squared Alfvén speed in the slab. A similar conclusion for a cylindrical incompressible plasma flow surrounded by an incompressible inviscid, and infinitely conducting fluid has been drawn by Satya Narayanan and Somasundaram [34]. The dispersive and stability properties of Alfvén surface waves along the boundary of a moving cylindrical
plasma column, surrounded by a stationary medium, embedded in a parallel magnetic field for an incompressible plasma has been performed by Somasundaram and Satya Narayanan [35]. Later studies of surface waves in a steady cylindrical compressible plasma surrounded by a static [36,37] or moving [40] plasma medium were applied to isolated photospheric flux tubes, coronal streamers and solar wind flow tubes. The problem for the hydromagnetic stability of a plasma jet moving in an external magnetized plasma also has been investigated by Singh and Talwar [38], assuming the fluids to be infinitely conducting, compressible plasmas and allowing for different magnetic fields and their different relative orientation within and outside the jet. The authors obtain numerically the growth rates both for kink and sausage perturbations for an incompressible jet, as well as for a compressible flow in the limit of the wide and slender slab approximation. The dispersion characteristics of magnetosonic waves in structured atmospheres with steady flows have been studied by Nakariakov and Roberts [39] (magnetic slab) and Terra-Homem et al [40] (cylindrical geometry). They show that flows change the characteristics of magnetosonic modes both qualitatively and quantitatively. The flow may lead to the appearance of a new type of trapped modes, notably backward waves. These waves are the usual slab modes propagating in the direction opposite to the internal flow, but advected with the flow. The disappearance of some modes due to the flow is also demonstrated. The results are applied to coronal and photospheric magnetic structures. The MHD modes specifically of solar wind flowing slabs have also been investigated [41]—the authors show that for realistic conditions in a flow tube, corresponding to a ‘spaghetti structure’, there are two types of trapped waves propagating away from the Sun, fast and slow waves. A more general theoretical examination of magnetosonic waves’ propagation in high-speed solar wind streams, taking into account the non-parallel propagation of such waves, was recently completed by Joarder [42]. The author finds that the non-parallel propagation introduces new magnetoacoustic wave modes, such as the forward-propagating slow surface modes and the backward-propagating fast surface modes that propagate outward in a direction away from the Sun; and the oppositely propagating (slow and fast) pseudo-surface and surface modes that propagate towards the Sun in a frame co-moving with the ambient flow external to the the solar wind flow-sheet.

The first study on surface-wave parallel propagation in a flowing ideal MHD flux tube surrounded by a static plasma environment (both embedded in a constant magnetic field $B_0$) in the framework of the Hall MHD was recently performed [43]. It has been shown that while in a static plasma slab the hydromagnetic surface waves (sausage and kink modes) are Alfvénic waves (their phase velocities are close to the Alfvén speed in the layer), in slabs with steady flows they become super-Alfvénic waves. Moreover, as it is logical to expect, there exist two type of waves: forward and backward ones, bearing in mind that the flow velocity defines the positive (forward) direction. In this paper, we consider a more general (and more realistic) plasma configuration, namely a steady MHD plasma jet bounded by flowing (with a different speed) environment both embedded in different magnetic fields. This new geometry offers a larger variety of hydromagnetic surface waves that can be observed/detected in the solar wind.

From a solar wind point of view, sufficiently away from the Sun, a more realistic Hall-MHD plasma is that with a plasma $\beta = (\mu_0 n_e k_B T_e / B_0^2)$ of order unity. Another important consideration for choosing such a value for $\beta$ is the circumstance that the MHD surface waves whose speeds are closer to the sound speed $v_s$ are less attenuated than those which possess phase speeds equal to or higher than the Alfvén speed $v_A$ [2]. In the following sections, we investigate the influence of the flow velocity on the dispersion characteristics of hydromagnetic surface waves (sausage and kink modes) travelling along a compressible, infinitely conducting,
magnetized jet moving past an also steady (with a different speed) compressible, infinitely conducting, magnetized plasma. For simplicity, we consider only a planar jet of width $2x_0$, allowing for different plasma densities and magnetic fields within and outside the jet. The magnitudes of Alfvén and sound speeds inside the jet are taken to be equal, namely $v_{A0} = v_o = 65 \text{ km s}^{-1}$, while in the surrounding environment their values are correspondingly $v_{Ae} = 100 \text{ km s}^{-1}$ and $v_{se} = 70 \text{ km s}^{-1}$. With these values of Alfvén and sound speeds in both media, the ratio of plasma densities is $\rho_o/\rho_e = 1.708$, and that of the ambient magnetic fields being $B_{0o}/B_{0e} = \alpha = 1.177$. The plasma $\beta$ (calculated as the ratio $v_s^2/v_A^2$) inside the layer is equal to 1, while outside it is 0.49. If we take $n_e = 3 \times 10^6 \text{ m}^{-3}$ and $B_{0o} = 5 \times 10^{-9} \text{T}$ as typical values inside the slab at 1 AU [44,45], the ion cyclotron frequency $\omega_{ci}/2\pi = 76 \text{ mHz}$, and the Hall scale length ($=v_{A0}/\omega_{ci}$, which is equivalent to $c/\omega_{pi}$) is $l_{\text{Hall}} \approx 140 \text{ km}$. This scale length is small, but not negligible compared with layer’s width of a few hundred kilometres. The flow speeds of the jet and its environment (which are generally rather irregular) are chosen to be $U_o = 500 \text{ km s}^{-1}$ and $U_e = 480 \text{ km s}^{-1}$, respectively. This means that the corresponding Alfvénic Mach numbers ($=U/v_A$) are $M_o = 7.7$ and $M_e = 4.8$.

The present paper is organized in the following way. In section 2, by means of the basic equations of Hall magnetohydrodynamics, with appropriate boundary conditions, we derive the wave dispersion relations. The dispersion curves of sausage and kink hydromagnetic surface waves in the flowing planar plasma jet are presented and discussed in section 3. Section 4 contains a summary and possible applications of our results to solar wind physics.

2. Basic equations and dispersion relation

Consider a steady (with speed $U_o$) plasma slab with uniform density $\rho_o$ and thickness $2x_0$, bounded by plasmas of densities $\rho_e$ moving with speed $U_e$, the interfaces being the surfaces $x = \pm x_0$. The uniform magnetic fields $B_{0o,e}$ and the steady flow velocities $U_{o,e}$ point in the $z$ direction. The wave vector $k$ lies also along the $z$-axis and its direction is the same as that of $B_{0o,e}$ and $U_{o,e}$ for forward waves and opposite for backward waves, respectively. The basic equations for Hall-MHD waves are the linearized equations governing the evolution of the perturbed plasma density $\delta \rho$, pressure $\delta p$, fluid velocity $\delta \mathbf{v}$ and wave magnetic field $\delta \mathbf{B}$ [27]:

$$\frac{\partial}{\partial t} \delta \rho + (\mathbf{U} \cdot \nabla) \delta \rho + \rho_o \nabla \cdot \delta \mathbf{v} = 0,$$

(1)

$$\rho_o \frac{\partial}{\partial t} \delta \mathbf{v} + \rho_o (\mathbf{U} \cdot \nabla) \delta \mathbf{v} + \nabla \left( \delta p + \frac{1}{\mu_0} B_0 \cdot \delta \mathbf{B} \right) - \frac{1}{\mu_0} (B_0 \cdot \nabla) \delta \mathbf{B} = 0,$$

(2)

$$\nabla \cdot \delta \mathbf{B} = 0,$$

(3)

$$\frac{\partial}{\partial t} \delta \mathbf{B} - (B_0 \cdot \nabla) \delta \mathbf{v} + B_0 \nabla \cdot \delta \mathbf{v} + (\mathbf{U} \cdot \nabla) \delta \mathbf{B} + l_{\text{Hall}} \frac{v_A}{B_0} B_0 \cdot \nabla \nabla \times \delta \mathbf{B} = 0,$$

(4)

$$\frac{\partial}{\partial t} \delta p + (\mathbf{U} \cdot \nabla) \delta p + \gamma p_0 \nabla \cdot \delta \mathbf{v} = 0,$$

(5)

where $v_A = B_0/(\mu_0 \rho_0)^{1/2}$ and $\gamma = \frac{5}{3}$. The pressure perturbation $\delta p$ is related to the mass density perturbation $\delta \rho$ via $\delta p = v_s^2 \delta \rho$, where $v_s = (\gamma p_0/\rho_0)^{1/2}$ is the speed of the sound. Following the way of solving the above set of equations, developed in [27], after Fourier transforming all
perturbed quantities \( \propto g(x) \exp(-i\omega t + ikz) \), we derive two coupled second-order differential equations for \( \delta v_x \) and \( \delta v_y \), namely

\[
\frac{k^2 v_A^2}{\omega_c^2(\omega - k \cdot U)} v_A^2 \left( \frac{d^2}{dx^2} - k^2 \right) \delta v_x - i \left[ (\omega - k \cdot U)^2 - k^2 v_A^2 + \frac{k^2 v_A^2}{\omega_c^2} \left( \frac{d^2}{dx^2} - k^2 \right) \right] \delta v_y = 0
\]

and

\[
\left[ (\omega - k \cdot U)^2 + v_A^2 \left( \frac{d^2}{dx^2} - k^2 \right) \right] \left( \frac{d^2}{dx^2} - k^2 \right) \delta v_x - \frac{i}{\omega_c} \left( \omega - k \cdot U \right) v_A^2 \left( \frac{d^2}{dx^2} - k^2 \right) \delta v_y = 0.
\]

By introducing the notations

\[
\begin{align*}
\epsilon &= \frac{\omega - k \cdot U}{\omega_c}, \\
a &= \frac{(\omega - k \cdot U)^2}{k^2 v_A^2}, \\
A &= \frac{a}{1 + (a - 1)k^2/q^2}, \\
q^2 &= \frac{((\omega - k \cdot U)^2 - k^2 v_A^2)(\omega - k \cdot U)^2 - k^2 v_A^2}{k^2 v_A^2 v_s^2 - (\omega - k \cdot U)^2(v_s^2 + v_A^2)},
\end{align*}
\]

the above two governing equations become

\[
\left( \frac{d^2}{dx^2} - Ak^2 \right) \delta v_x - \frac{a - 1}{\epsilon} Ak^2 \delta v_y = 0,
\]

and

\[
\left( \frac{d^2}{dx^2} - q^2 \right) \delta v_x + \frac{\epsilon}{a - 1} \frac{q^2}{k^2} \left( \frac{d^2}{dx^2} - Ak^2 \right) \delta v_y = 0.
\]

We seek the solutions to these coupled equations in the form

\[
\begin{align*}
\delta v_x(x) &= f \left[ \exp(-\kappa x) \mp \exp(\kappa x) \right], \\
\delta v_y(x) &= i h \left[ \exp(-\kappa x) \mp \exp(\kappa x) \right],
\end{align*}
\]

and obtain the set of equations

\[
\begin{align*}
(k^2 - Ak^2) f + \frac{a - 1}{\epsilon} Ak^2 h &= 0, \\
(k^2 - q^2) f - \frac{\epsilon}{a - 1} \frac{q^2}{k^2} (k^2 - q^2) h &= 0,
\end{align*}
\]

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which yield the following expression for $\kappa$:

$$
\kappa_{1,2} = \frac{k}{\sqrt{2}} \left\{ 1 + A \left[ 1 - \left( \frac{k a - 1}{q \varepsilon} \right)^2 \right] \right\}^{1/2} 
\pm \sqrt{\left\{ 1 + A \left[ 1 - \left( \frac{k a - 1}{q \varepsilon} \right)^2 \right] \right\}^2 - 4A \left[ 1 - \left( \frac{k a - 1}{q \varepsilon} \right)^2 \right]} \right\}^{1/2}. \quad (10)
$$

This means that there are in fact two pairs of attenuation coefficients: $(\kappa_{o1}, \kappa_{o2})$ inside the slab and $(\kappa_{e1}, \kappa_{e2})$ outside the jet, respectively.

As is known, two types of waves may exist on a bounded MHD plasma waveguide (cylinder or layer). Recall that for a slab geometry the general solutions to the equations governing $\delta v_x$ and $\delta v_y$ are sought in the form of superpositions of $\cosh \kappa_o x$ and $\sinh \kappa_o x$. Those solutions contain waves whose shape is defined by the $\cosh$ function (they are called *kink* waves) and another type of waves associated with the $\sinh$ function (which are termed *sausage* waves). The transverse structure of both waves inside the slab is determined by the two attenuation coefficients $\kappa_{o1,2}$. Thus, the solutions for $\delta v_x$ and $\delta v_y$ inside the slab ($|x| < x_0$), anticipating a sausage wave form, accordingly, are

$$
\delta v_x(x) = f_1 \frac{\sinh \kappa_{o1} x}{\sinh \kappa_{o1} x_0} + f_2 \frac{\sinh \kappa_{o2} x}{\sinh \kappa_{o2} x_0}
$$

and

$$
\delta v_y(x) = i f_1 G_{o1} \frac{\sinh \kappa_{o1} x}{\sinh \kappa_{o1} x_0} + i f_2 G_{o2} \frac{\sinh \kappa_{o2} x}{\sinh \kappa_{o2} x_0},
$$

where

$$
G_{o1,2} = -\frac{\varepsilon}{A_0 (A_0 - 1)} \frac{\kappa_{o1,2}^2 - A_0 k^2}{k^2}.
$$

For a kink surface-wave form, the expressions for perturbed fluid velocity components have the same description—it is only necessary to replace $\sinh$ with $\cosh$. The solutions outside the layer (identical for both modes) are

$$
\delta v_x(x) = \begin{cases} 
\sum_{i=1,2} \alpha_i \exp[-\kappa_{ei}(x - x_0)] & \text{for } x > x_0, \\
\sum_{i=1,2} \beta_i \exp[\kappa_{ei}(x + x_0)] & \text{for } x < -x_0 
\end{cases}
$$

and

$$
\delta v_y(x) = \begin{cases} 
i \sum_{i=1,2} \alpha_i G_{ei} \exp[-\kappa_{ei}(x - x_0)] & \text{for } x > x_0, \\
i \sum_{i=1,2} \beta_i G_{ei} \exp[\kappa_{ei}(x + x_0)] & \text{for } x < -x_0. 
\end{cases}
$$

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Here, as above,
\[ G_{e1,2} = -\frac{\varepsilon}{A_e} \left( \frac{k_e^2 - A_e k^2}{k^2} \right). \]

Having derived the expressions for the perturbed fluid velocity components \( \delta v_x \) and \( \delta v_y \), one can calculate the perturbed total pressure (the sum of kinetic and magnetic pressures) and arrive at
\[
\delta p_{\text{total}}(x) = \delta p + \frac{1}{\mu_0} B_0 \cdot \delta B = i \frac{\rho_0}{\omega - k \cdot U} v_A^2 \times \left\{ \frac{k^2}{q^2} \left[ \frac{(\omega - k \cdot U)^2}{k^2 v_A^2} - 1 \right] \frac{d}{dx} \delta v_x(x) + i \frac{\omega - k \cdot U}{\omega_{ci}} \frac{d}{dx} \delta v_y(x) \right\}.
\]

The perturbed wave electric field \( \delta E \) can be obtained from the generalized Ohm’s law
\[
E = -v \times B + \frac{m_i}{e\rho} j \times B,
\]
which, by means of Ampère’s law (multiplied vectorially by \( B \)), yields
\[
\delta E = B_0 \times \left( \delta v - l_{\text{Hall}} \frac{v_A}{B_0} \nabla \times \delta B \right) - U \times \delta B.
\]

Its components are
\[
\delta E_x(x) = -\frac{\omega}{\omega - k \cdot U} \left( \frac{\omega - k \cdot U}{kv_A} \right)^2 B_0 \delta v_y(x)
\]
and
\[
\delta E_y(x) = \frac{\omega}{\omega - k \cdot U} B_0 \left[ \delta v_x(x) + i \frac{\omega - k \cdot U}{\omega_{ci}} \delta v_y(x) \right].
\]

The above expressions for perturbed quantities are used when implementing the boundary conditions.

It follows from the solutions to the basic equations (8) and (9) for the perturbed fluid velocity components \( \delta v_x \) and \( \delta v_y \) that the number of integration constants is 6. However, because of the symmetry (or antisymmetry), the two \( \beta_{1,2} \) amplitudes are in fact directly obtainable from the \( x > x_0 \) solutions—indexed as and \( \beta s \) are not independent. Thus, we can derive the dispersion relations by applying only four boundary conditions at one interface, for example, at \( x = x_0 \). These boundary conditions are widely discussed in [27]. We can borrow them except the first one, the continuity of \( \delta v_x \) across the interface, as in the present case of a flowing slab this condition must be replaced by the continuity of \( \delta v_x / (\omega - k \cdot U) \) [46]. The rest of the boundary conditions are the continuity of the full perturbed pressure \( \delta p_{\text{total}} \), the y-component of perturbed wave electric field \( \delta E_y \), and the x-component of perturbed electric displacement \( \delta D_x = \varepsilon_0 (K_{xx} \delta E_x + K_{xy} \delta E_y) \) at the interface. In the last boundary condition, \( K_{xx} \) and \( K_{xy} \) are the low-frequency components of the plasma dielectric tensor [47]
\[
K_{xx} \approx \frac{e^2}{v_A^2} \quad \text{and} \quad K_{xy} \approx i \frac{\omega - k \cdot U}{\omega_{ci}} \frac{e^2}{v_A^2}.
\]
By imposing the boundary conditions at the interface \( x = x_0 \), and after some straightforward algebra, we finally arrive at the dispersion relation for parallel propagation of sausage and kink waves in a planar jet, surrounded by steady plasma media, in the form of

\[
D(\omega, k, U_{o,e}, B_{0o,e}, l_{\text{Hall}}, x_0, \beta_{o,e}, \rho_{o,e}) = 0, \tag{11}
\]

where the input parameters are introduced via seven numbers, namely two plasma \( \beta \)s, the ratios \( \rho_{o,e}/\rho_{e} \), and \( B_{0e}/B_{0o} = \alpha \), the parameter \( l_{\text{Hall}}/x_0 \), and the Alfvénic Mach numbers \( M_{o,e} = U_{o,e}/v_{A0,e} \). The explicit form of the dispersion relation (11) for sausage and kink waves is

\[
\tilde{\omega} k^2 q_o^2 \left[ \left( \frac{\omega - k \cdot U_o}{kv_{A0}} \right)^2 - 1 \right] \left( -Y_0 \kappa_{o1} \left( \frac{\coth}{\tanh} \right) \kappa_{o1} x_0 + \kappa_{o2} \left( \frac{\coth}{\tanh} \right) \kappa_{o2} x_0 \right) \\
+ \alpha \kappa^2 q_c^2 \left[ \left( \frac{\omega - k \cdot U_e}{kv_{Ae}} \right)^2 - 1 \right] \left[ \kappa_{e1} (A_{12} - Y_0 A_{11}) - \kappa_{e2} (A_{22} - Y_0 A_{21}) \right] \\
- \alpha \varepsilon_e \left\{ -G_{o1} Y_0 \kappa_{o1} \left( \frac{\coth}{\tanh} \right) \kappa_{o1} x_0 + G_{o2} \kappa_{o2} \left( \frac{\coth}{\tanh} \right) \kappa_{o2} x_0 \right. \\
+ \left. \alpha \left[ G_{e1} \kappa_{e1} (A_{12} - Y_0 A_{11}) - G_{e2} \kappa_{e2} (A_{22} - Y_0 A_{21}) \right] \right\} = 0. \tag{12}
\]

Here

\[
\tilde{\omega} = \frac{\omega - k \cdot U_e}{\omega - k \cdot U_o}, \quad \varepsilon_{o,e} = \frac{\omega - k \cdot U_{o,e}}{\omega_{ci}^{o,e}}, \\
Y_0 = \frac{\alpha - C_1 G_{o2} + C_2 (G_{e1} A_{12} - G_{e2} A_{22})}{\alpha - C_1 G_{o1} + C_2 (G_{e1} A_{11} - G_{e2} A_{21})},
\]

where

\[
C_1 = \frac{1}{\varepsilon_o} \left( \frac{\omega - k \cdot U_o}{kv_{A0}} \right)^2 \left[ 1 - \tilde{\omega} \alpha^3 (\rho_e/\rho_o) \right]^{-1}, \\
C_2 = \frac{1}{\varepsilon_o} \left( \frac{\omega - k \cdot U_o}{kv_{A0}} \right)^2 \frac{\tilde{\omega} \alpha^3 (\rho_e/\rho_o)^2}{1 - \tilde{\omega} \alpha^3 (\rho_e/\rho_o)} + \varepsilon_o,
\]

and

\[
A_{11} = \frac{1}{G_{e1} - G_{e2}} \left( \frac{1 - \alpha}{\varepsilon_o} + G_{o1} - \tilde{\omega} G_{e2} \right), \\
A_{12} = \frac{1}{G_{e1} - G_{e2}} \left( \frac{1 - \alpha}{\varepsilon_o} + G_{o2} - \tilde{\omega} G_{e2} \right),
\]
As can be seen, the wave frequency $\omega$ is Doppler-shifted (to a different extent) inside and outside the layer. This dispersion relation generalizes corresponding equation (12) in [43], which holds for a steady jet surrounded by a static environment ($U_e = 0$ and $B_{0e} = B_{0o}$). We note also that for a static plasma slab ($U_o = 0$) both embedded in a magnetic field $B_0$, the above dispersion relation (12), coincides with equation (17) obtained in [27]. When the Hall term in the governing equations is also dropped ($l_{Hall} = 0$), we recover the ‘classical’ dispersion relations of Edwin and Roberts [25]. It is worth pointing out that due to the complexity of the wave attenuation coefficients $\kappa_1, \kappa_2$ in both media (see equation (10)), there is no way for obtaining a simpler explicit form of dispersion relation (12), say, in the long-wavelength limit $kx_0 \ll 1$, which is sometimes possible for such waves in the framework of the conventional magnetohydrodynamics.

3. Numerical results and discussion

The dispersion relations (12) are transcendental equations and they can be solved only numerically. We shall obtain the normalized (with respect to Alfvén speed $v_{A0}$ inside the slab) phase velocity $V_{ph} = v_{ph}/v_{A0}$ as a function of the dimensionless wave number $K = kx_0$. As we have already said, the sausage and kink surface waves depend upon seven input parameters: the two plasma $\beta$ ($\beta_o$ and $\beta_e$), the plasma density ratio $\rho_o/\rho_e$, the magnetic field ratio $\alpha = B_{0e}/B_{0o}$, the parameter $l_{Hall}/x_0$ (which comes from $\varepsilon_o \equiv (\omega - k \cdot U_o)/\omega_{ci}^2 = K(l_{Hall}/x_0)(V_{ph} - M_o)$ and $\varepsilon_e \equiv (\omega - k \cdot U_e)/\omega_{ci}^2 = K(l_{Hall}/x_0)(V_{ph} - M_e\alpha(\rho_o/\rho_e)^{1/2})/\alpha$), and the Alfvénic Mach numbers $M_{o,e} = U_{o,e}/v_{A0,e}$. The choice of the first four input parameters is constrained by our basic assumption that the plasma slab under consideration is structured owing to different plasma densities and magnetic fields inside and outside the layer. Recall that the equilibrium magnetic fields and plasma parameters are subject to the constraint [27]

$$\frac{\rho_e}{\rho_o} = \frac{v_{A0}^2 + \frac{1}{2}v_{Ae}^2}{v_{Ae}^2 + \frac{1}{2}v_{A0}^2},$$

which is satisfied by the magnitudes of $\alpha, \beta_{o,e}$ and $\rho_o/\rho_e$ already specified in section 1. The value of $l_{Hall}/x_0$ depends upon the slab’s width: for a steady flux tube of few hundred kilometre width it is a number smaller than unity (our choice is 0.4); however for a filament [31] $l_{Hall}/x_0$ can be of the order of 1. Concerning the flow velocities (and corresponding Alfvénic Mach numbers) we have two possibilities. In the first case as the flow inside the slab is faster than the one of the surrounding plasma, the direction of the corresponding relative velocity (with respect to a frame of reference associated with the flowing environment) is positive (i.e. the flow is forward, outwards the Sun). When the two flow velocities are interchanged in magnitude, the relative flow velocity becomes negative and the flow is backward (i.e. towards the Sun). For each of these

3 In the same reference, tanh and coth in equation (17) must be interchanged.
Figure 1. Dispersion curves of forward sausage waves in a forward flow for $\beta_o = 1, \beta_e = 0.49, \alpha = 1.177, \rho_o/\rho_e = 1.708, l_{Hall}/x_0 = 0.4, M_o = 0–8.5$ and $M_e = 0–5.3$. The curves corresponding to $M_e$ lying in the interval $0–7$ are plotted by a step $\Delta M_o = 0.5$, whereas in the range of $7–8.5$, $\Delta M_o = 0.1$.

two cases, one can observe both forward ($k = k^+\hat{z}$) and backward ($k = -k^+\hat{z}$) sausage and kink waves [39]. All the possible (eight) waves’ dispersive diagrams will be presented in this paper.

With these input parameters the wave attenuation coefficients $\kappa_{o1,2}$ and $\kappa_{e1,2}$ can be real or imaginary quantities. In the case when either of the two attenuation coefficients $\kappa_{o1,2}$ are imaginary the hyperbolic functions become periodic trigonometric functions (i.e. cot or tan) and besides the so-called principal/fundamental waves there also occur higher harmonic modes. Furthermore, we shall select (and plot) the dispersion curves of the fundamental waves only.

3.1. Sausage and kink waves in a forward flowing slab

We start with the case when the flow speed in the slab is $500\text{ km s}^{-1}$ and that outside the slab is $480\text{ km s}^{-1}$ (or in terms of internal and external Alfvénic Mach numbers $M_o = 7.7$ and $M_e = 4.8$). The relative flow velocity is $20\text{ km s}^{-1}$. With these specified values of the flow speeds it becomes clear that we are going to deal with relatively slowly moving flow-structures which are mostly present in the ecliptic plane. Although the flow is chosen to be forward one, as we have mentioned, there are still two possible directions of the wave to propagate: along the relative flow velocity (forward waves) and opposite to it (backward waves). Bearing in mind that sausage and kink waves can occur in spatially bounded plasmas, we have finally four different cases to study, namely forward and backward sausage waves and forward and backward kink modes, all of them propagating in a forward flowing layer.

In figure 1 we present the dispersion diagrams for the forward sausage mode, whereas in figure 2 we show the dispersion curves for the backward sausage waves. In all figures presented, the first (blue) curve at the bottom of the plot corresponds to a static Hall-MHD plasma slab ($M_o = M_e = 0$). From both figures it is seen that the flow increases the normalized waves’ phase velocities—both modes become super-Alfvénic waves. There are, however, two distinctive differences: (i) the phase velocity of the forward sausage mode increases with increasing the
dimensionless wave number, while for the backward sausage wave this behaviour is just the opposite; and (ii) looking at figure 2, one observes that the normalized phase velocities near the top region (in green colour) are diminished compared with the phase velocities of forward waves for the same Mach numbers. Moreover, these waves (in figure 2) are actually backward ones only for very small Mach numbers. With increasing the flow speed, the normalized phase velocity becomes positive, i.e. the wave is advected with the flow and continues to propagate in the positive direction.

The dispersion curves of forward and backward kink surface waves are shown in figures 3 and 4, respectively. We see that here one encounters similar phenomena, i.e. an increase in normalized phase velocities and advection of the backward kink wave. It is interesting to notice, however, that both kink modes are practically dispersionless.

3.2. Sausage and kink waves in a backward flowing slab

In the case of a backward flowing slab, we interchange the aforementioned magnitudes of the two flow velocities, namely setting $U_o = 480 \text{ km s}^{-1}$ and $U_e = 500 \text{ km s}^{-1}$. This leads to new Alfvénic Mach numbers; accordingly $M_o = 7.4$ inside the slab and $M_e = 5.0$ outside the layer. Now the relative flow velocity becomes $-20 \text{ km s}^{-1}$. The negative sign denotes an opposite direction of the relative flow velocity with respect to the direction of the magnetic fields. Thus, we have forward and backward sausage waves as well as forward and backward kink waves along a backward flowing flux tube (see figures 5 and 6).

Here, as in the case of a forward flowing slab, both types of sausage surface waves exhibit a similar evolution with increasing dimensionless wave number $kx_0$, notably associated with an increase/decrease in the normalized wave phase velocity $v_{ph}/v_{Ao}$ of forward/backward sausage waves with increasing dimensionless wave number.

The forward kink surface waves in a backward flowing flux tube are aligned smoothly, as shown in figure 7. In contrast, the backward kink waves (figure 8), starting from the Alfvénic
Mach number $M_o = 4.0$, exhibit some complexity: each dispersive curve consists of two curves merging at around $kx_0 \approx 0.9$ (for $M_o = 4.0$) with the merging point moving to the left to $kx_0 = 0.5$ for $M_o = 8.0$. This complex structure of dispersion curves is illustrated in figure 9. One can conclude from figure 9 that in a very narrow $kx_0$ interval, we might have three different normalized phase velocities for the same Alfvénic Mach number: two of them belonging to the left-hand-side curve, and one to the right-hand-side curve. Such kind of merging parts of a given ‘global’ dispersion curve is a typical phenomenon for surface waves in Hall-MHD-bounded plasma structures [29, 30].
Figure 5. Dispersion curves of forward sausage waves in a backward flow for $\beta_0 = 1$, $\beta_e = 0.49$, $\alpha = 1.177$, $\rho_0/\rho_e = 1.708$, $I_{\text{Hall}}/x_0 = 0.4$, $M_o = 0–8$, and $M_e = 0–5.4$. The curves corresponding to $M_o$ lying in the interval $0–7$ are plotted by a step $\Delta M_o = 0.5$, whereas in the range of $7–8$, $\Delta M_o = 0.1$.

Figure 6. Dispersion curves of backward sausage waves in a backward flow for the same parameter set as in figure 5.

4. Conclusion

The dispersion properties of the hydromagnetic surface waves travelling in a Hall-MHD plasma jet surrounded by steady (with a different speed) plasmas (or in short Hall-MHD surface waves) can be summarized as follows. Generally, the flow increases the phase velocity of the waves to an extent depending upon the Alfvénic Mach numbers $M_{oe}$ and the kind of the wave (forward
or backward one). As seen from dispersion diagrams shown in figures 1–8, for parallel wave propagation in a static waveguide ($M_{0,e} = 0$) the waves can be identified as Alfvénic ones—their phase velocities normalized with respect to the internal Alfvén speed $v_{Ao}$ are of order unity. The magnetoacoustic ‘cusp speed’, defined as [22]

$$v_{To} = \frac{v_{so} v_{Ao}}{(v_{so}^2 + v_{Ao}^2)^{1/2}},$$

Figure 7. Dispersion curves of forward kink waves in a backward flow for the same parameter set as in figure 5.

Figure 8. Dispersion curves of backward kink waves in a backward flow for the same parameter set as in figure 5.
is sub-sonic and sub-Alfvénic. Its normalized value inside the slab is 0.71. Here we recall that for parallel wave propagation one can distinguish two types of MHD waves: acoustic (the slow mode) and Alfvénic (the intermediate/fast mode) waves [44]. To the best of our knowledge, the slow waves in spatially bounded Hall-MHD plasmas have not yet been investigated.

In a planar jet, the forward Hall-MHD surface waves become super-Alfvénic ones. For the backward waves the situation is somewhat different. The wave modes are actually backward only for small Mach numbers. Moreover, with increasing the flow speeds (in both media), the magnitude of the wave phase velocity reduces, passes the zero and starts to increase. This corresponds to an overturning the direction of the wave’s propagation—the wave is advected with the flow and from a backward wave it becomes a forward one with a diminished phase velocity in comparison with that of a true forward wave. The shift of the phase velocity of a particular forward/backward Hall-MHD surface wave may be expressed by [46]

\[
v_{\text{ph}} \equiv \frac{\omega}{k} = \frac{\rho_0}{\rho_o + \rho_e} U_{\text{rel}} \pm v_{\text{Hall-MHD}}^{\text{ph}},
\]

where \(U_{\text{rel}}\) is the relative flow velocity, \(v_{\text{Hall-MHD}}^{\text{ph}}\) is the phase velocity of the wave in a static plasma configuration, and the \(\pm\) sign is associated with a forward/backward wave. The above equation is in concordance with equation (31) in [4], which is however relevant to unbounded Hall-MHD plasmas and as a result of this circumstance the coefficient in front of \(U_{\text{rel}}\) is simply 1.

As we have already discussed, the transverse structure of the sausage and kink surface-wave perturbations in a spatially bounded Hall-MHD plasma is determined by two pairs of attenuation coefficients, \(\kappa_{o1,2}\) and \(\kappa_{e1,2}\), respectively. In most cases, one of the \(\kappa_{o,e}\)'s is real and other is imaginary. Moreover, the particular values of the attenuation coefficients depend on the dimensionless wave number \(kx_0\)—for small wave numbers usually one of \(\kappa\)s is an imaginary number, later on (with increasing wave number) becoming a real one. Thus, in

![Figure 9. Zoomed-in area of the dispersion curves of backward kink waves in a backward flow in the range \(M_o = 7–8\).](http://www.njp.org/)
general, the Hall-MHD surface waves are a mixture/hybrid of/between the main type of waves: pure surface, pseudo-surface and leaky ones. Hence, these waves have to be treated as generalized surface waves. The exact determination of the transverse structure of all surface waves considered here is however a tough, but generally solvable, task that needs an individual treatment (see e.g. figures 8 and 9 in [30]).

Another important aspect of this study is related to the seven input parameters. During the numerical solving of the dispersion relations we kept five of them unaltered (plasmas’ $\beta_s$, the magnetic fields ratio $\alpha$, the plasma densities ratio $\rho_o/\rho_e$ and the parameter $l_{\text{Hall}}/x_0$), whereas the two Alfvénic Mach numbers were allowed to grow. A reasonable question that immediately arises is: what might occur if one of the constant input parameters varies? For small variations of $\beta_o$ (and respectively in $\beta_e$) near 1 the shapes of the dispersion curves remain similar to the ones obtained. The same is valid for any small deviation in the chosen value of $l_{\text{Hall}}/x_0$.

The magnitude of the plasma densities ratio, as well as that of the ambient magnetic fields, is restricted by equation (13) as we pointed out previously. For any drastic change in those parameters it is natural however to expect a distinctive modification of dispersion curves’ shape that is especially noticeable for significant variations of plasmas’ $\beta_s$. The choice of the flow speeds (or more importantly of the corresponding Alfvénic Mach numbers) turns out to be rather critical for the shape of the dispersion curves. For example, a considerable increase in the relative flow can lead to very complex dispersion curves for some Mach numbers as recently demonstrated in [43]—see figures 4, 6 and 8 therein, which are more complicated than our figure 9. The explanation to such strange dispersion curves is still an open question.

Another difficult aspect of our study is the question of the stability of the Hall-MHD modes travelling on flux tubes. Because of the extreme complexity of the wave attenuation coefficients and dispersion relations, the applicability of the existing criteria for stability (e.g. those used in [33,38]) is not possible—one cannot obtain an explicit expression for the wave frequency as a function of the wave number and plasmas’ and flows’ parameters. Obviously, this problem needs a separate treatment which is beyond the scope of this work. In any case, the Kelvin–Helmholtz instability [46] could play a dominant role in the amplification of the waves and in the excitation of MHD turbulence in the solar wind.

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