VANISHING POLYHEDRON AND COLLAPSING MAP

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ABSTRACT. In this paper we give a detailed proof that the Milnor fiber \( X_t \) of an analytic complex isolated singularity function defined on a reduced \( n \)-equidimensional analytic complex space \( X \) is a regular neighborhood of a polyhedron \( P_t \subset X_t \) of real dimension \( n-1 \). Moreover, we describe the degeneration of \( X_t \) onto the special fiber \( X_0 \), by giving a continuous collapsing map \( \Psi_t : X_t \rightarrow X_0 \) which sends \( P_t \) to \( \{0\} \) and which restricts to a homeomorphism \( X_t \setminus P_t \rightarrow X_0 \setminus \{0\} \).

INTRODUCTION

Let \( f : (X, x) \rightarrow (\mathbb{C}, 0) \) be a germ of complex analytic function \( f \) at a point \( x \) of a reduced equidimensional complex analytic space \( X \subset \mathbb{C}^N \) (with arbitrary singularity). In \cite{9} the first author proved that there exist sufficiently small positive real numbers \( \epsilon \) and \( \eta \) with \( 0 < \eta \ll \epsilon \ll 1 \) such that the restriction:
\[
f|_{B_\epsilon(x) \cap X \cap f^{-1}(D^*_\eta)} : B_\epsilon(x) \cap X \cap f^{-1}(D^*_\eta) \rightarrow D^*_\eta
\]
is a locally trivial topological fibration, where \( B_\epsilon(x) \) is the ball around \( x \in \mathbb{C}^N \) with radius \( \epsilon \), \( D_\eta \) is the disk around \( 0 \in \mathbb{C} \) with radius \( \eta \) and \( D^*_\eta := D_\eta \setminus \{0\} \).

For any \( t \in D^*_\eta \) the topology of \( X_t := B_\epsilon(x) \cap X \cap f^{-1}(t) \) does not depend on \( \epsilon \) small enough (see Theorem 2.3.1 of \cite{12}). We call \( X_t \) the Milnor fiber of \( f \), with boundary \( \partial X_t := X_t \cap S_\epsilon(x) \). Also set \( X_0 := B_\epsilon(x) \cap X \cap f^{-1}(0) \).

Let \( S = (S_\alpha)_{\alpha \in A} \) be a Whitney stratification of \( X \) \cite{21, 13}. According to Definition 1.1 of \cite{11}, we say that \( f : (X, x) \rightarrow (\mathbb{C}, 0) \) has an isolated singularity at \( x \in X \) in the stratified sense, i.e., relatively to the Whitney stratification \( S \), if there is an open neighborhood \( U \) of \( x \) in \( X \) such that the restriction of \( f \) to each stratum \( S_\alpha \cap U \) that does not contain \( x \) is a submersion and if the restriction of \( f \) to the stratum \( S_{a(x)} \cap U \) that contains \( x \) has an isolated singularity at \( x \).

The first author sketched a proof of the following theorem in \cite{10}:

**Theorem 1.** Let \( X \subset \mathbb{C}^N \) be a reduced equidimensional complex analytic space and let \( S = (S_\alpha)_{\alpha \in A} \) be a Whitney stratification of \( X \). Let \( f : (X, x) \rightarrow (\mathbb{C}, 0) \) be a germ of complex analytic function \( f \) at a point \( x \in X \). If \( f \) has an isolated singularity at \( x \) relatively to \( S \) and if \( \epsilon \) and \( \eta \) are sufficiently small positive real numbers as above, then for each \( t \in D^*_\eta \) there exist:

(i) a polyhedron \( P_t \) of real dimension \( \dim_{\mathbb{C}} X_t \) in the Milnor fiber \( X_t \), compatible with the Whitney stratification \( S \), and a continuous simplicial map:
\[
\tilde{\xi}_t : \partial X_t \rightarrow P_t
\]
compatible with \( S \), such that \( X_t \) is homeomorphic to the mapping cylinder of \( \tilde{\xi}_t \).
a continuous map \( \Psi_t : X_t \to X_0 \) that sends \( P_t \) to \( \{0\} \) and that restricts to a homeomorphism \( X_t \setminus P_t \to X_0 \setminus \{0\} \).

The purpose of this paper is to give a complete and detailed proof of Theorem 1. This theorem was conjectured by R. Thom in the early 70’s when \( X \) is smooth.

In section 1 we construct the relative polar curve of \( f \), which is the main tool to prove Theorem 1. In section 2 we prove the Theorem when \( X \) is two-dimensional. Then in section 3 we prove the Theorem in the general case, using two propositions (Proposition 14 and Proposition 15). In section 4 we prove those Propositions by finite induction on the dimension of \( X \). Finally, in section 5 we make the detailed construction of a vector field (Lemma 18) that is used in section 4.

1. Polar curves

The notion of polar curve of a complex analytic function defined on an open neighborhood of \( \mathbb{C}^n \) relatively to a linear form \( \ell \) were introduced by B. Teissier and the first author in [7] and [17]. Later, in [8] M. Kato and the first author extended this concept to a germ of complex analytic function \( f : (X,x) \to (\mathbb{C},0) \) relatively to a Whitney stratification \( S = (S_\alpha)_{\alpha \in A} \) of a reduced equidimensional complex analytic space \( X \).

Notice that by now we are not supposing that \( f \) has an isolated singularity in the stratified sense. This hypothesis will be asked after the lemma below.

Let \( f : X \to \mathbb{C} \) be a representative of the germ of function at \( x \) such that \( X \) is closed in an open neighborhood \( U \) of \( x \) in \( \mathbb{C}^N \). For any linear form:

\[ \ell : \mathbb{C}^N \to \mathbb{C} \]

the function \( f \) and the restriction of \( \ell \) to \( X \) induce the analytic morphism:

\[ \phi_\ell : X \to \mathbb{C}^2 \]

defined by \( \phi_\ell(z) = (\ell(z), f(z)) \), for any \( z \in X \).

We have the following lemma:

Lemma 2. There is a representative \( X \) of \( (X,x) \) in an open neighborhood \( U \) of \( \mathbb{C}^n \) and a non-empty Zariski open set \( \Omega \) in the space of non-zero linear forms of \( \mathbb{C}^N \) to \( \mathbb{C} \) that take \( x \in \mathbb{C}^N \) to \( 0 \in \mathbb{C} \) such that, for any \( \ell \in \Omega \), the analytic morphism \( \phi_\ell : X \to \mathbb{C}^2 \) satisfies:

(i) The part of the critical locus of the restriction of \( \phi_\ell \) to a stratum \( S_\alpha \) that lies outside \( f^{-1}(0) \) is either empty or a smooth reduced complex curve, whose closure in \( X \) is denoted by \( \Gamma_\alpha \).

(ii) The image \( (\Delta_\alpha,0) \) of \( (\Gamma_\alpha, x) \) by \( \phi_\ell \) is the germ of a complex curve.

Proof. Let us choose an open neighborhood \( U \) in \( \mathbb{C}^n \) such that the intersection \( U \cap S_\alpha \) is not empty for finite many indices \( \alpha \). Furthermore, we may assume that the closure \( \overline{S}_\alpha \) in \( U \) is defined by an ideal \( I(\overline{S}_\alpha) \) generated by complex analytic functions \( g_1, \ldots, g_m \) defined on \( U \), that is, \( I(\overline{S}_\alpha) = (g_1, \ldots, g_m) \).

Now consider a linear form \( \ell = a_1 x_1 + \cdots + a_N x_N \) and a stratum \( S_\alpha \) such that \( 0 \in \overline{S}_\alpha \). Let \( C_{\ell,\alpha} \) be the critical set of the restriction of \( \phi_\ell \) to \( S_\alpha \setminus f^{-1}(0) \). We will show that \( C_{\ell,\alpha} \) is contained in a member of a linear system parametrized by \( \ell \). Consider the matrix:
\[ J_\alpha = \begin{pmatrix}
\frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_N} \\
\vdots & \ddots & \vdots \\
\frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_N}
\end{pmatrix}. \]

A point \( z \) of \( S_\alpha \) is a point where the rank of \( J_\alpha \) at \( z \) is \( \rho := \max_{z \in S_\alpha} \text{rank}(J_\alpha(z)) \), since it is a non-singular point of \( S_\alpha \). A point of \( C_{\ell,\alpha} \) is a point of \( S_\alpha \setminus f^{-1}(0) \) where the matrix:

\[ J_{\phi,\alpha} = \begin{pmatrix}
\frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_N} \\
\vdots & \ddots & \vdots \\
\frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_N} \\
\frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_N} \\
a_1 & \cdots & a_N
\end{pmatrix} \]

has rank at most \( \rho + 1 \). So the determinants of the \( (\rho + 2) \)-minors:

\[ \begin{pmatrix}
\frac{\partial g_i}{\partial x_j} & \cdots & \frac{\partial g_i}{\partial x_{j+\rho+2}} \\
\vdots & \ddots & \vdots \\
\frac{\partial g_i}{\partial x_j} & \cdots & \frac{\partial g_i}{\partial x_{j+\rho+2}} \\
\frac{\partial f}{\partial x_j} & \cdots & \frac{\partial f}{\partial x_{j+\rho+2}} \\
a_i & \cdots & a_i
\end{pmatrix} \]

are zero, that is:

\[ \sum_{k=1}^{\rho+2} (-1)^{k+1} a_{ik} \cdot \det \begin{pmatrix}
\frac{\partial g_1}{\partial x_j} & \cdots & \frac{\partial g_1}{\partial x_{j+\rho+2}} \\
\vdots & \ddots & \vdots \\
\frac{\partial g_1}{\partial x_j} & \cdots & \frac{\partial g_1}{\partial x_{j+\rho+2}} \\
\frac{\partial f}{\partial x_j} & \cdots & \frac{\partial f}{\partial x_{j+\rho+2}} \\
\frac{\partial g_i}{\partial x_{j+k}} & \cdots & \frac{\partial g_i}{\partial x_{j+k+\rho+2}} \\
\vdots & \ddots & \vdots \\
\frac{\partial g_i}{\partial x_{j+k}} & \cdots & \frac{\partial g_i}{\partial x_{j+k+\rho+2}} \\
\frac{\partial f}{\partial x_{j+k}} & \cdots & \frac{\partial f}{\partial x_{j+k+\rho+2}} \\
\frac{\partial g_{i'}}{\partial x_{j+k}} & \cdots & \frac{\partial g_{i'}}{\partial x_{j+k+\rho+2}} \\
\vdots & \ddots & \vdots \\
\frac{\partial g_{i'}}{\partial x_{j+k}} & \cdots & \frac{\partial g_{i'}}{\partial x_{j+k+\rho+2}} \\
\frac{\partial f}{\partial x_{j+k}} & \cdots & \frac{\partial f}{\partial x_{j+k+\rho+2}}
\end{pmatrix} = 0. \]

An analytic version of a classical theorem of Bertini (see [1] and [2]) states that if \( h_1, \ldots, h_r \) are holomorphic functions defined on a complex analytic space \( Y \) and if \( \lambda_i \) are sufficiently generic, for \( i = 1, \ldots, r \), then the singular locus of the subvariety \( \{ \sum_{i=1}^r \lambda_i h_i = 0 \} \) is contained in the union of the singular set of \( Y \) and the set:

\[ \{ h_1 = \cdots = h_r = 0 \}. \]

So it follows from the analytic theorem of Bertini that there exists a non-empty Zariski open set \( \Omega_\alpha \) in the space of non-zero linear forms of \( \mathbb{C}^N \) to \( \mathbb{C} \) that take \( x \in \mathbb{C}^N \) to \( 0 \in \mathbb{C} \) such that for any \( \ell \in \Omega_\alpha \) one has that the singular points \( \Sigma_{C_{\ell,\alpha}} \) of \( C_{\ell,\alpha} \) are contained in the union of the set of the points where the determinants above are zero and the singular locus of \( S_\alpha \). That is:

\[ \Sigma_{C_{\ell,\alpha}} \subset (\text{Crit } f|_{S_\alpha}) \cup \Sigma_{S_\alpha} \cap S_\alpha. \]

Since this intersection is contained in \( f^{-1}(0) \cap S_\alpha \) and \( C_{\ell,\alpha} \) is contained in \( S_\alpha \setminus f^{-1}(0) \), it follows that \( C_{\ell,\alpha} \) is smooth.

Now, since \( \phi_{\ell}^{-1}(0,0) \cap (\Gamma_\alpha, x) \subset \{ x \} \), it follows from the geometric version of the Weierstrass preparation theorem given in [3] that the restriction of \( \phi_\ell \) to the germ \( (\Gamma_\alpha, x) \) is finite. In particular, a theorem of Remmert (see [5], page 60, for instance) implies that the image \( \Delta_\alpha \) of the analytic set \( \Gamma_\alpha \) by the finite morphism \( \phi_\ell \) is an analytic set of the same dimension, and hence it is a complex curve.
Finally, notice that there is a finite number of indices \( \alpha \in A \) such that \( x \) is contained in \( S_\alpha \). Let \( A_x \) be the finite subset of \( A \) formed by such indices. So the set:

\[
\Omega := \bigcap_{\alpha \in A_x} \Omega_\alpha
\]

is the desired non-empty Zariski open set in the space of non-zero linear forms of \( \mathbb{C}^N \) to \( \mathbb{C} \) that take \( x \in \mathbb{C}^N \) to \( 0 \in \mathbb{C} \).

\[\square\]

For any \( \ell \in \Omega \) we say that the germ of curve at \( x \) given by:

\[
\Gamma_\ell := \bigcup_{\alpha \in A} \Gamma_\alpha
\]

is the polar curve of \( f \) relatively to \( \ell \) at \( x \) and that the germ of curve at \( 0 \) given by:

\[
\Delta_\ell := \bigcup_{\alpha \in A} \Delta_\alpha
\]

is the polar discriminant of \( f \) relatively to \( \ell \) at \( 0 \). In fact, \( \Delta_\ell \) behaves as a discriminant of the germ \( \phi_\ell \) as indicated in Proposition 6 below.

From now on, we fix a linear form \( \ell \in \Omega \) and we set \( \phi := \phi_\ell \), \( \Gamma := \Gamma_\ell \) and \( \Delta := \Delta_\ell \).

Also, from this point we will assume that \( f \) has an isolated singularity at \( x \) relatively to the stratification \( S \).

Now let us recall some definitions and a result that gives us an important tool to prove Theorem 1 (see [12] for instance).

**Definition 3.** Let \( X \) and \( Y \) be analytic spaces with Whitney stratifications \( (X_\alpha)_{\alpha \in A} \) and \( (Y_\beta)_{\beta \in B} \) respectively. A map \( h : X \to Y \) is a stratified map if:

(i) \( h \) is continuous;
(ii) \( h \) sends each stratum \( X_\alpha \) to a unique stratum \( Y_{\beta(\alpha)} \), for some \( \beta(\alpha) \in B \);
(iii) the restriction of \( h \) to each stratum \( X_\alpha \) induces a smooth map \( h_\alpha : X_\alpha \to Y_{\beta(\alpha)} \).

We say that a stratified map \( h \) as above is a stratified submersion if each \( h_\alpha \) is a submersion.

We say that a stratified map \( h \) as above is a stratified homeomorphism if \( h \) is a homeomorphism and each \( h_\alpha \) is a smooth diffeomorphism.

**Definition 4.** Let \( X \) be an analytic space with Whitney stratification \( (X_\alpha)_{\alpha \in A} \). We say that a continuous vector field \( v \) in \( X \) is a stratified vector field if for each \( \alpha \in A \), the restriction \( v_\alpha \) of \( v \) to \( X_\alpha \) is a smooth vector field.

We have:

**Lemma 5.** Let \( X \) be a complex analytic space with a Whitney stratification \( (X_\alpha)_{\alpha \in A} \), and let \( h : X \to U \) be a proper stratified submersion over an open subset \( U \) of \( \mathbb{C}^k \). If \( v \) is a smooth vector field in \( U \), then \( h \) lifts \( v \) to a stratified vector field in \( X \) that is integrable.

The proof comes from the fact that, under the hypothesis of the first isotopy theorem of Thom-Mather ([13]), a smooth vector field lifts to a rugose vector field, and hence integrable. For this, we notice that for a complex analytic space, Theorem 1.2 of chapter V of [18] gives that the Whitney conditions imply the strict Whitney conditions, and then our claim follows from Proposition 4.6 of [20].
As we said before, Lemma 5 will be used many times in the next sections of this paper.

Now let us go back to the map \( \phi = (\ell, f) : X \to \mathbb{C}^2 \) defined above. Notice that it is stratified and that it induces a stratified submersion \( \phi^{-1}(U \setminus \Delta) \to U \setminus \Delta \), where \( U \) is an open subset of \( \mathbb{C}^2 \).

By an analytic version of Corollary 2.8 of [14], there exist \( \epsilon \) and \( \eta_2 \) small enough positive real numbers with \( 0 < \eta_2 \ll \epsilon \ll 1 \) such that, for any \( t \in \mathbb{D}_{\eta_2} \), the sphere \( S_\epsilon(x) \) around \( x \) of radius \( \epsilon \) intersects \( f^{-1}(t) \cap S_\alpha \) transversally, for any \( \alpha \in A \).

We can also choose the linear form \( \ell \) in such a way that there exists \( \eta_1 \) sufficiently small, with \( 0 < \eta_2 \ll \eta_1 \ll \epsilon \ll 1 \), such that \( \phi^{-1}(s,t) \cap S_\alpha = \ell^{-1}(s) \cap f^{-1}(t) \cap S_\alpha \) intersect \( S_\epsilon(x) \) transversally, for any \( (s,t) \in \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \), where \( \mathbb{D}_{\eta_1} \) and \( \mathbb{D}_{\eta_2} \) are the closed disks in \( \mathbb{C} \) centered at 0 and with radii \( \eta_1 \) and \( \eta_2 \), respectively.

So we have:

**Proposition 6.** The map \( \phi = (\ell, f) \) induces a stratified submersion:
\[
\phi| : \mathbb{B}_\epsilon(x) \cap X \cap \phi^{-1}(\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \setminus \Delta) \to \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \setminus \Delta.
\]

In particular, the first isotopy theorem of Thom-Mather gives:

**Corollary 7.** The restriction \( \phi| \) above is a topological locally trivial fibration.

Therefore the curve \( \Delta \) plays the role of a local topological discriminant for the stratified map \( \phi \).

For any \( t \) in the disk \( \mathbb{D}_{\eta_2} \) set:
\[
D_t := \mathbb{D}_{\eta_1} \times \{ t \}.
\]

If \( t \neq 0 \), the Milnor fiber \( f^{-1}(t) \cap \mathbb{B}_\epsilon(x) \) of \( f \) is homeomorphic to \( \phi^{-1}(D_t) \cap \mathbb{B}_\epsilon(x) \) (see Theorem 2.3.1 of [12]). So, in order to simplify notation, we reset:
\[
X_t := \phi^{-1}(D_t) \cap \mathbb{B}_\epsilon(x).
\]

Notice that with this notation, the boundary \( \partial X_t \) of \( X_t \) is given by the union of \( \phi^{-1}(\partial D_t) \cap S_\epsilon(x) \) and \( \phi^{-1}(\partial D_t) \cap \mathbb{B}_\epsilon(x) \).

Next we give a lemma that will be implicitly used many times in the paper:

**Lemma 8.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two analytic spaces in \( \mathbb{C}^N \) such that \( \mathcal{X} \) has a Whitney stratification \( (X_\alpha)_{\alpha \in A} \) and such that \( \mathcal{Y} \) is non-singular. If \( \mathcal{Y} \) intersects each stratum \( X_\alpha \) transversally in \( \mathbb{C}^N \), then the Whitney stratification of \( \mathcal{X} \) induces a Whitney stratification \( (Y_\alpha)_{\alpha \in A} \) of \( \mathcal{X} \cap \mathcal{Y} \), where \( Y_\alpha := X_\alpha \cap \mathcal{Y} \), for each \( \alpha \in A \).

In particular, the Whitney stratification \( \mathcal{S} \) of \( X \) induces a Whitney stratification \( \mathcal{S}(t) \) of \( X_t \). Precisely, the strata of such Whitney stratification are the following intersections, for \( \alpha \in A \):

1. \( S_\alpha \cap (X_t \setminus \partial X_t) \);
2. \( S_\alpha \cap \phi^{-1}(\partial D_t) \cap S_\epsilon(x) \);
3. \( S_\alpha \cap \phi^{-1}(\partial D_t) \cap \mathbb{B}_\epsilon(x) \);
4. \( S_\alpha \cap \phi^{-1}(\partial D_t) \cap S_\epsilon(x) \).
Now, for any $t \in \mathbb{D}_{\eta_2}^*$ set $D_t := \mathbb{D}_{\eta_1} \times \{t\}$. Then $\phi$ induces a stratified map:
\[ \ell_t : X_t \rightarrow D_t. \]
By construction, the restriction of $\ell_t$ to each stratum of $X_t$ is a submersion onto the image of the stratum at any point out of $\Gamma$. Therefore it induces a locally trivial fibration over $D_t \setminus (\Delta \cap D_t)$. That is, if we set:
\[ \Delta \cap D_t = \{y_1(t), \ldots, y_k(t)\} \]
then the restriction of $\ell_t$ given by:
\[ \varphi_t : X_t \setminus \ell_t^{-1}\{y_1(t), \ldots, y_k(t)\} \rightarrow D_t \setminus \{y_1(t), \ldots, y_k(t)\} \]
is a stratified submersion (see Definition 3) and a locally trivial fibration, by Thom-Mather first isotopy theorem.

**Remark 9.** In the case that $\Gamma$ is empty, one has that:
\[ \phi_t : \mathbb{B}_\varepsilon(x) \cap X \cap \phi^{-1}(\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2}) \rightarrow \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \]
is a locally trivial topological fibration, which implies a locally trivial topological fibration $\ell_t : X_t \rightarrow D_t$. Hence in this case the Milnor fiber $X_t$ is homeomorphic to the product of $D_t$ and the general fiber of $\ell_t$.

So from now on we shall assume that the polar curve $\Gamma$ is not empty.

### 2. The two-dimensional case

We shall prove Theorem 1 by induction on the dimension $n$ of the analytic space $X$. We could start by proving the theorem for $n = 1$ and then proceed by induction for $n \geq 2$, but we choose to start with the 2-dimensional case, in order to provide the reader a better intuition of the constructions.

So in this section we prove our theorem when $\mathcal{X} = (X, 0)$ is a 2-dimensional reduced equidimensional germ of complex analytic space and $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ has an isolated singularity at 0 in the stratified sense.

One particularity of this 2-dimensional case is that the singular set $\Sigma$ of $X$ has dimension at most one. If $\Sigma$ has dimension one, we can put it inside the curve $\Gamma$. Precisely, (only) in this section we denote by $\Gamma$ the union of the polar curve of $f$ with $\Sigma$. We also denote by $\Delta$ the union of the polar discriminant of $f$ with $\phi(\Sigma)$. Notice that if $\Delta$ is not empty, it is a complex curve.

#### 2.1. First step: constructing the vanishing polyhedron.

For any $t \in \mathbb{D}_{\eta_2}^*$ fixed, we remind that:
\[ \Delta \cap D_t = \{y_1(t), \ldots, y_k(t)\}. \]
Let $\lambda_t$ be a point in $D_t \setminus \{y_1(t), \ldots, y_k(t)\}$ and for each $j = 1, \ldots, k$, let $\delta(y_j(t))$ be the line segment in $D_t$ starting at $\lambda_t$ and ending at $y_j(t)$. We can choose $\lambda_t$ in such a way that any two of these line segments intersect only at $\lambda_t$.

Set:
\[ Q_t := \bigcup_{j=1}^{k} \delta(y_j(t)) \]
and
\[ P_t := \ell_t^{-1}(Q_t). \]

Since \( \ell_t \) is finite, one can see that \( P_t \) is a one-dimensional polyhedron in \( X_t \). And since the map \( \varphi_t : X_t \setminus \ell_t^{-1}(\{y_1(t), \ldots, y_k(t)\}) \rightarrow D_t \setminus \{y_1(t), \ldots, y_k(t)\} \) is a stratified submersion, each 1-simplex of \( P_t \) is contained in some stratum \( X_t \cap S_\alpha \) of \( X_t \), so \( P_t \) is adapted to the stratification \( S \).

We say that \( P_t \) is a vanishing polyhedron for \( f \).

Notice that, in this 2-dimensional case, the vanishing polyhedron \( P_t \) contains the 0-dimensional strata of the Whitney stratification of \( X_t \) induced by \( S \), which are the points of \( \Sigma \cap X_t \). In particular, \( X_t \setminus P_t \) is a smooth manifold.

**Lemma 10.** There exists a continuous vector field \( v_t \) in \( D_t \) such that:

1. It is smooth on \( D_t \setminus Q_t \);
2. It is null on \( Q_t \);
3. It is transversal to \( \partial D_t \) and points inwards;
4. The associated flow \( q_t : [0, \infty) \times (D_t \setminus Q_t) \rightarrow D_t \setminus Q_t \) defines a map:

\[ \xi_t : \partial D_t \longrightarrow Q_t \]
\[ u \longmapsto \lim_{\tau \to \infty} q_t(\tau, u), \]

such that \( \xi_t \) is continuous, simplicial and surjective.

**Proof.** Let \( d_t : D_t \rightarrow \mathbb{R} \) be the function given by the distance to the set \( Q_t \), that is \( d_t(x) := d(x, Q_t) \). Consider the small closed neighborhood of \( Q_t \) in \( D_t \) given by:
\[ R_t := (d_t)^{-1}([0, r]) \]
for some small \( r > 0 \).

Notice that the boundary of \( R_t \) is a Jordan curve. In fact, let \( d_j^t \) be the function given by the distance to \( \delta(y_j(t)) \), which is smooth outside \( \delta(y_j(t)) \) since \( \delta(y_j(t)) \) is a smooth manifold (see [4] for instance). So \( (d_j^t)^{-1}((0, r]) \) is a smooth manifold \( R_t(j) \) with boundary diffeomorphic to the circle. Clearly, \( R_t = \bigcup_{j=1}^k R_t(j) \) and its boundary is a piecewise linear closed curve.

We endow \( R_t \) with the vector field \( v_1 \) given as follows:

Let us decompose \( R_t \) in the sets \( R^1_t, \ldots, R^k_t, M^1_t, \ldots, M^k_t \) as indicated in Figure 1 below.

![Figure 1](image_url)

We endow each “rectangular” region \( R^j_t \) with an integrable vector field \( \omega_j \) as follows:
Let $I$ denote the closed interval $[0, 1]$. Using the Riemann mapping and Carathéodory’s theorem (see sections 17.3 and 17.20 of [10], for instance, or see [3] for the original proof), we can consider a homeomorphism $h_j : R^j_t \rightarrow I \times I$ such that:

- $h_j$ takes the side $Q_t \cap R^j_1$ onto $I \times \{0\}$;
- $h_j$ takes the side $a_j$ onto $\{0\} \times I$;
- $h_j$ restricts to a diffeomorphism $h_j : \text{int}(R^j_t) \rightarrow (0, 1) \times (0, 1)$, where $\text{int}(R^j_t)$ denotes the interior of $R^j_t$.

Also let $\rho : I \times I \rightarrow \mathbb{R}$ be the function given by the distance to $I \times \{0\}$, that is, $\rho(u, v) := v$. Since $\rho$ is smooth, the vector field $\omega$ on $I \times I$ given by the opposite of the gradient vector field associated to $\rho$ is continuous, smooth, non-zero outside $I \times \{0\}$, zero on $I \times \{0\}$, and each orbit associated to it has a limit point on $I \times \{0\}$.

So the pull-back of $\omega$ by $h_j$ gives a vector field $\omega_j$ on $R^j_t$ that is continuous, smooth and non-zero outside $Q_t \cap R^j_t$, zero on $Q_t \cap R^j_t$, and each orbit associated to it has a limit point on $Q_t \cap R^j_t$.

Since $h_j^{-1}$ sends each integral curve associated to $\omega$ to an integral curve associated to $w_j$ and since $h_j^{-1}$ sends the side $\{0\} \times I$ to the side $a_j$, then the fact that $\{0\} \times I$ coincides with an integral curve implies that the line segment $a_j$ coincides with an integral curve associated to $w_j$.

Putting all the rectangles $R^j_t$ together, each of them endowed with the corresponding vector field $\omega_j$, we get a subset $R_t$ of $R_t$ endowed with a continuous vector field $w$ that is smooth and non-zero outside $Q_t$, zero on $Q_t$, and each orbit associated to it has a limit point on $Q_t$. Moreover, each line segment $\tilde{a}_j = a_j \cup a_{j+1}$ coincides with an integral curve associated to $w$ (for consistency of the notation, we set $a_{k+1} := a_1$).

Now we do a similar process to endow each component $M^j_t$ with a vector field $w_j$ that is continuous, smooth and non-zero outside $Q_t \cap M^j_t$, zero on $Q_t \cap M^j_t$, and each orbit associated to it has a limit point on $Q_t \cap M^j_t$. Even more, we will do it in such a way that $w_j$ “glues smoothly” with the vector field $w$ of $R_t$.

Consider the set $A \subset \mathbb{R}^2$ given by the union of the rectangle $I \times I$ and a semi-disk as in Figure 2. Also consider a diffeomorphism $g_j : M^j_t \rightarrow A$ that takes the side $\tilde{a}_j$ onto $\{0\} \times I$ and that takes $Q_t \cap M^j_t$ onto $I \times \{1/2\}$.

Now consider the subsets $A_1 := [0, 2/3] \times [0, 1]$ and $A_2 := A \setminus ([0, 1/3] \times [0, 1])$ of $A$.

Recall that the side $\tilde{a}_j$ of $M^j_t$ is endowed with a vector field given by the restriction of the vector field $\omega$ to $\tilde{a}_j \subset R_t$. This naturally gives a vector field $v_1$ in $A_1$ that is continuous, smooth and non-zero outside $[0, 2/3] \times \{1/2\}$, zero on $[0, 2/3] \times \{1/2\}$, and each integral curve associated to it is a line segment $\{u\} \times I$, for some $u \in [0, 2/3]$, which has a limit point $(u, 0)$.

On the other hand, we endow $A_2$ with the vector field $v_2$ given by the opposite of the gradient vector field of the function distance to the line segment $[1/3, 1] \times \{1/2\}$. It is continuous, smooth and non-zero outside $[1/3, 1] \times \{1/2\}$, zero on $[1/3, 1] \times \{1/2\}$, and each integral curve associated to it has a limit point in $[1/3, 1] \times \{1/2\}$.

Then we glue the vector fields $v_1$ and $v_2$, using a partition of the unity, to obtain a vector field $v$ on $A$ that is continuous, smooth and non-zero outside $I \times \{1/2\}$, zero on $I \times \{1/2\}$, and each orbit associated to it has a limit point on $I \times \{1/2\}$.
Then we define the vector field \( w_j \) on \( M^j_t \) as the pull-back of \( v \) by \( g_j \).

\[
\begin{array}{c}
\{0\} \times 1 \\
\phantom{1} \downarrow \\
1 \times \{1/2\} \\
\phantom{1} \downarrow \\
\phantom{1} \downarrow v
\end{array}
\]

**Figure 2.**

So gluing the vector field \( w \) and the vector fields \( w_j \) we obtain a vector field \( \hat{v}_1 \) on \( \mathcal{R}_t \) that is continuous, smooth and non-zero outside \( Q_t \), zero on \( Q_t \), and each orbit associated to it has a limit point on \( Q_t \).

On the other hand, let \( r' \) be a small real number with \( 0 < r' < r \) and with \( r-r' \ll 1 \), and set \( \mathcal{R}'_t := (d_r)^{-1}([0, r']) \), so \( \mathcal{R}'_t \subset \mathcal{R}_t \). Also denote by \( \text{int}(\mathcal{R}_t) \) the interior of \( \mathcal{R}_t \) and by \( \text{int}(\mathcal{R}'_t) \) the interior of \( \mathcal{R}'_t \). We endow \( D_t \setminus \text{int}(\mathcal{R}'_t) \) with the vector field \( v_2 \) given by the opposite of the gradient vector field of the function on \( D_t \setminus \text{int}(\mathcal{R}_t) \) given by the distance to the point \( \lambda_t \).

Finally, one can check that the vector fields \( v_1 \) and \( v_2 \) never have opposite directions. So the vector field \( v_t \) is obtained by gluing the vector fields \( v_1 \) and \( v_2 \), using a partition of unity. That is, we consider a pair \( (\rho_1, \rho_2) \) of continuous functions from the compact disk \( D_t \) to the closed unit interval \([0, 1]\) such that:

- for every point \( p \in D_t \) one has that \( \rho_1(p) + \rho_2(p) = 1 \)
- the support of \( \rho_1 \) is contained in \( D_t \setminus \mathcal{R}'_t \)
- the support of \( \rho_2 \) is contained in \( \text{int}(\mathcal{R}_t) \)

Hence for any \( p \in D_t \setminus \text{int}(\mathcal{R}_t) \) we have that \( (\rho_1(p), \rho_2(p)) = (1, 0) \) and for any \( p \in \mathcal{R}'_t \) we have that \( (\rho_1(p), \rho_2(p)) = (0, 1) \). So we set \( v_t := \rho_1 v_1 + \rho_2 v_2 \).

Clearly, \( v_t \) is a continuous vector field on \( D_t \) which is smooth and non-zero on \( D_t \setminus Q_t \), zero on \( Q_t \) and transversal to \( \partial D_t \), pointing inwards. Moreover, each orbit associated to \( v_t \) has a limit point in \( Q_t \).

So the flow \( q_t : [0, \infty) \times (D_t \setminus Q_t) \to D_t \setminus Q_t \) associated to \( v_t \) defines a continuous and surjective map:

\[
\xi_t : \partial D_t \longrightarrow Q_t, \quad u \longmapsto \lim_{\tau \to \infty} q_t(\tau, u).
\]

**Remark 11.** Lemma 10 is still true if the set \( Q_t \) is takes as the union of simple paths that intersect only at a point \( \lambda_t \), instead of considering line segments. It is enough to consider an homeomorphism of the disk \( D_t \) on itself which is a diffeomorphism outside the point \( \lambda_t \).

One has:

**Proposition 12.** We can choose a lifting of the vector field \( v_t \) to a continuous vector field \( E_t \) in \( X_t \) so that:

1. It is tangent to each stratum of the interior of \( X_t \) (see Lemma 8);
2. It is smooth on each stratum of the induced stratification of \( X_t \setminus P_t \);
3. It is integrable on \( X_t \setminus P_t \);
Recall from Proposition 6 that the restriction \( q \) of the linear form \( l \) to the Milnor fiber \( X_t \) induces a stratified submersion:

\[
\varphi_t : X_t \setminus \ell_t^{-1}(\{y_1(t), \ldots, y_k(t)\}) \to D_t \setminus \{y_1(t), \ldots, y_k(t)\}.
\]

So we can lift the vector field \( v_t \) to a continuous vector field \( E_t \) in \( X_t \) that satisfies properties (1) to (5). See Lemma 5.

Let us show that we can choose \( E_t \) satisfying also (6). Fix \( z \in \partial X_t \). We want to show that \( \lim_{\tau \to \infty} q_t(\tau, z) \) exists, that is, that there exists a point \( \tilde{p} \in P_t \) such that for any open neighborhood \( \tilde{U} \) of \( \tilde{p} \) in \( X_t \) there exists \( \theta > 0 \) such that \( \tau > \theta \) implies that \( \tilde{q}_t(\tau, z) \in \tilde{U} \).

From Lemma 10 we know that there exists \( p \in Q_t \) such that \( \lim_{\tau \to \infty} q_t(\tau, \ell_t(z)) = p \), where \( q_t : [0, \infty) \times (D_t \setminus Q_t) \to D_t \) is the flow associated to the vector field \( v_t \). So for any small open neighborhood \( U \) of \( p \) in \( D_t \) there exists \( \theta > 0 \) such that \( \tau > \theta \) implies that \( q_t(\tau, \ell_t(z)) \in U \). Setting \( \{\tilde{p}_1, \ldots, \tilde{p}_r\} := \ell_t^{-1}(p) \), we can consider \( U \) sufficiently small such that there are disjoint connected components \( \tilde{U}_1, \ldots, \tilde{U}_r \) of \( \ell_t^{-1}(U) \) such that each \( \tilde{U}_j \) contains \( \tilde{p}_j \).

Since \( E_t \) is a lifting of \( v_t \), we have that \( \tilde{q}_t(\tau, \ell_t(z)) = \ell_t(q_t(\tau, z)) \) for any \( \tau \geq 0 \). So \( \tau > \theta \) implies that \( \ell_t^{-1}(\ell_t(q_t(\tau, z))) \subset \ell_t^{-1}(U) \). Hence for some \( j \in \{1, \ldots, r\} \) we have that \( \tilde{q}_t(\tau, z) \in \tilde{U}_j \). Therefore \( \lim_{\tau \to \infty} \tilde{q}_t(\tau, z) = \tilde{p}_j \). This proves (6).

Now we show that \( X_t \) is homeomorphic to the mapping cylinder of \( \tilde{\xi}_t \). In fact, the integration of the vector field \( E_t \) on \( X_t \setminus P_t \) gives a surjective continuous map:

\[
\alpha : [0, \infty] \times \partial X_t \to X_t
\]

that restricts to a homeomorphism:

\[
\alpha| : [0, \infty) \times \partial X_t \to X_t \setminus P_t.
\]

Since the restriction \( \alpha_\infty : \{\infty\} \times \partial X_t \to P_t \) is equal to \( \tilde{\xi}_t \), which is surjective, it follows that the induced map:

\[
[\alpha_\infty] : (\{\infty\} \times \partial X_t)/\sim \to P_t
\]

is a homeomorphism, where \( \sim \) is the equivalent relation given by identifying \((\infty, z) \sim (\infty, z')\) if \( \alpha_\infty(z) = \alpha_\infty(z') \). Hence the map:

\[
[\alpha] : ([0, \infty] \times \partial X_t)/\sim \to X_t
\]

induced by \( \alpha \) defines a homeomorphism between \( X_t \) and the mapping cylinder of \( \tilde{\xi}_t \). This proves (7).
2.2. Second step: constructing the collapsing map.

First, let us recall that \( X_t := \mathbb{B}_\epsilon \cap \phi^{-1}(\mathbb{D}_{\eta_1} \times \{ t \}) \) and that the map:
\[
\phi = (\ell, f) : X \to \mathbb{C}^2
\]
induces a stratified submersion \( \phi_1 : \mathbb{B}_\epsilon \cap X \cap \phi^{-1}(\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \setminus \Delta) \to \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \setminus \Delta \) (see Proposition 6).

Let \( \gamma \) be a simple path in \( \mathbb{D}_{\eta_2} \) joining 0 and some \( t_0 \in \partial \mathbb{D}_{\eta_2} \), such that \( \gamma \) is transverse to \( \partial \mathbb{D}_{\eta_2} \). We want to describe the collapsing of \( f \) along \( \gamma \), that is, how \( X_t \) degenerates to \( X_0 \) as \( t \in \gamma \) goes to 0.

The first step is to construct the vanishing polyhedron \( P_\gamma \) simultaneously, for all \( t \) in \( \gamma \).

Recall from Lemma 2 that the polar curve \( \Gamma \) is not contained in \( f^{-1}(0) \). Hence \( \Delta \) is not contained in \( \mathbb{D}_{\eta_1} \times \{ 0 \} \), and so the natural projection \( \pi : \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \to \mathbb{D}_{\eta_2} \) restricted to \( \Delta \) induces a ramified covering:
\[
\pi| : \Delta \to \mathbb{D}_{\eta_2}
\]
of degree \( k \), whose ramification locus is \( \{ 0 \} \subset \Delta \). Hence the inverse image of \( \gamma \setminus \{ 0 \} \) by this covering defines \( k \) disjoint simple paths in \( \Delta \), and each one of them is diffeomorphic to \( \gamma \setminus \{ 0 \} \).

Let \( \Lambda \) be a simple path in \( \mathbb{D}_{\eta_1} \times \gamma \) such that \( \pi(\Lambda) = \gamma \) and such that \( \Lambda \cap \Delta = \{ 0 \} \). Recall that \( \Delta \cap (\mathbb{D}_{\eta_1} \times \{ t \}) = \{ y_1(t), \ldots, y_k(t) \} \). Hence for each \( t \in \gamma \) the point \( \lambda_t = \Lambda \cap (\mathbb{D}_{\eta_1} \times \{ t \}) \) is a point in \( (\mathbb{D}_{\eta_1} \times \{ t \}) \setminus \{ y_1(t), \ldots, y_k(t) \} \).

We can choose the simple paths \( \delta(y_j(t)) \) joining \( \lambda_t \) and \( y_j(t) \), for each \( j = 1, \ldots, k \), in such a way that the set:
\[
T_j := \bigcup_{t \in \gamma} \delta(y_j(t)),
\]
forms a triangle and \( T_j \setminus \{ 0 \} \) is differentially immersed in \( \mathbb{D}_{\eta_1} \times \gamma \), for each \( j \in \{ 1, \ldots, k \} \).

The intersection of any two triangles \( T_j \) and \( T_{j'} \) is the path \( \Lambda \), for \( j, j' \in \{ 1, \ldots, k \} \) with \( j \neq j' \).

Set \( Q := \bigcup_{j=1}^k T_j \) and \( P_\gamma := \phi^{-1}(Q) \), which we call a collapsing cone for \( f \) along \( \gamma \).

See Figure 3

So \( P_\gamma \) is a polyhedron adapted to the stratification induced by \( \mathcal{S} \) such that the intersection \( P_\gamma \cap X_t \) is a vanishing polyhedron \( P_t \), for any \( t \in \gamma \setminus \{ 0 \} \), and \( P_\gamma \cap X_0 = \{ 0 \} \).

The second step is to construct a system of closed neighborhoods \( V_A(Q) \) of \( P_\gamma \) in \( \mathbb{D}_{\eta_1} \times \gamma \), for any \( A \geq 0 \), such that:

(i) The boundary \( \partial V_A(Q) \) of \( V_A(Q) \) is smooth, for any \( A \geq 0 \);
(ii) \( V_0(Q) = \mathbb{D}_{\eta_1} \times \gamma \);
(iii) For any \( A_1 > A_2 \) one has \( V_A_1(Q) \subset V_A_2(Q) \);
(iv) For any open neighborhood \( U \) of \( P_\gamma \) in \( \mathbb{D}_{\eta_1} \times \gamma \), there exists \( A_U \geq 0 \) sufficiently big such that \( V_{A_U}(Q) \) is contained in \( U \).
This system of neighborhoods will be used in the third step.

To construct such a system of neighborhoods, we first consider a vector field \( V \) in \( D_{\eta_1} \times \gamma \) that deformation retracts \( D_{\eta_1} \times \gamma \) onto \( Q \), just like in Lemma 10. That is, we consider a continuous vector field \( V \) in \( D_{\eta_1} \times \gamma \) such that:

- It is smooth (and hence integrable) on \( (D_{\eta_1} \times \gamma) \setminus Q \);
- It is zero on \( Q \);
- It is transversal to \( \partial D_{\eta_1} \times \gamma \);
- The projection of \( V \) onto \( \gamma \) is zero.

Just as we did in Proposition 12, we can use Proposition 6 to obtain:

**Proposition 13.** The vector field \( V \) can be lifted to a continuous vector field \( E_{\gamma} \) in \( X_{\gamma} \) such that:

(i) For any \( t \in \gamma \setminus \{0\} \), the restriction of \( E_{\gamma} \) to \( X_t \) gives a vector field \( E_t \) as in the proposition above, relatively to the polyhedron \( P_t = P_\gamma \cap X_t \);

(ii) The vector field \( E_{\gamma} \) is integrable, stratified and non-zero on \( X_\gamma \setminus P_\gamma \), tangent to each stratum of the interior of \( X_\gamma \setminus P_\gamma \), zero on \( P_\gamma \) and transversal to \( \partial X_\gamma \), points inwards;

(iii) The flow \( w : [0, \infty) \times ((D_{\eta_1} \times \gamma) \setminus Q) \rightarrow D_{\eta_1} \times \gamma \) associated to \( V \) defines a map:

\[
\xi : \partial D_{\eta_1} \times \gamma \rightarrow Q \\
\quad z \quad \mapsto \lim_{\tau \to \infty} w(\tau, z)
\]

that is continuous, simplicial and surjective.

Now, for any real number \( A > 0 \) set:

\[
V_A(Q) := (D_{\eta_1} \times \gamma) \setminus w([0, A) \times \partial D_{\eta_1} \times \gamma),
\]

which is a closed neighborhood of \( Q \) in \( D_{\eta_1} \times \gamma \).

Notice that \( \partial V_A(Q) \) is a smooth manifold that fibers over \( \gamma \) with fiber a circle, by the restriction of the projection \( \pi : D_{\eta_1} \times D_{\eta_2} \rightarrow D_{\eta_2} \). Moreover, \( D_{\eta_1} \times \gamma \) is clearly the mapping cylinder of \( \xi \).
The third step is to construct a stratified and integrable vector field $\mathcal{E}$ in $X_\gamma \setminus P_\gamma$, whose flow gives the degeneration of $X_{t_0}$ to $X_0$.

Since we are assuming that $X$ has dimension two and since the restriction:

$$\phi_1 : \phi^{-1}((D_{\eta_1} \times D_{\eta_2}) \setminus Q) \to (D_{\eta_1} \times D_{\eta_2}) \setminus Q$$

is a smooth covering, it follows that $\phi^{-1}(\partial V_A(Q)) \cap B_\epsilon$ is a smooth submanifold of $\phi^{-1}(D_{\eta_1} \times D_{\eta_2}) \cap B_\epsilon$ that is a proper locally trivial fibration over $\gamma$.

Let $\theta$ be a vector field on $\gamma$ that goes from $t_0$ to 0 in time $a > 0$ and fix $A > 0$. We are going to construct a smooth and integrable vector field $\mathcal{E}$ in $X_\gamma \setminus P_\gamma$ that lifts $\theta$ and such that $\mathcal{E}$ is tangent to $\phi^{-1}(\partial V'_A(Q))$, for any $A' \geq A$. We will construct it locally, that is, for each point $p \in X_\gamma \setminus P_\gamma$ we will construct a vector field $\mathcal{E}_p$ in some neighborhood $U_p$ of $p$, and then we will glue all of them using a partition of unity associated to the covering given by the neighborhoods $U_p$ (for the proof of the existence of a partition of unity associated to an infinite covering, see Lemma 41.6 of [15], for instance).

Each $\mathcal{E}_p$ is constructed in the following way:

(a) If $p \notin \phi^{-1}(V_A(Q)) \cap B_\epsilon$, there is an open neighborhood $U_p$ of $p$ in $X_\gamma$ that does not intersect the closed set $\phi^{-1}(V_A(Q)) \cap B_\epsilon$. Then we define a smooth vector field $\mathcal{E}_p$ on $U_p$ that lifts $\theta$.

(b) If $p \in \phi^{-1}(V_A(Q)) \cap B_\epsilon \setminus P_\gamma$, there is an open neighborhood $U_p$ of $p$ in $X_\gamma$ that does not intersect $P_\gamma$. We define a smooth vector field $\mathcal{E}_p$ on $U_p$ that lifts $\theta$ and that is tangent to $\phi^{-1}(\partial V'_A(Q))$, for any $A' \geq A$.

Then, as we said before, the vector field $\mathcal{E}$ is obtained by gluing the vector fields $\mathcal{E}_p$ using a partition of unity. Notice that $\mathcal{E}$ lifts $\theta$.

Hence the flow $h : [0, a] \times X_\gamma \setminus P_\gamma \to X_\gamma \setminus P_\gamma$ associated to $\mathcal{E}$ defines a stratified homeomorphism $\Psi$ from $X_{t_0} \setminus P_{t_0}$ to $X_0 \setminus \{0\}$ that extends to a continuous map from $X_{t_0}$ to $X_0$ and that sends $P_{t_0}$ to $\{0\}$.

So we have proved that if $(X, 0)$ is a 2-dimensional reduced equidimensional germ of complex analytic space and if $f : (X, 0) \to (\mathbb{C}, 0)$ has an isolated singularity at 0, then there exist:

(i) a vanishing polyhedron $P_t$ of real dimension one in the Milnor fiber $X_t$ of $f$ such that $X_t$ is a regular neighborhood of $P_t$;

(ii) a continuous map $\Psi_t : X_t \to X_0$ that sends $P_t$ to $\{0\}$ and that restricts to a homeomorphism $X_t \setminus P_t \to X_0 \setminus \{0\}$.

3. Proof of the main theorem

Now we go back to the general case of a germ of complex analytic function

$$f : (X, x) \to (\mathbb{C}, 0)$$

at a point $x$ of a reduced equidimensional complex analytic space $X \subset \mathbb{C}^N$ of any dimension. Let $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ be a Whitney stratification of $X$ and suppose that $f$ has an isolated singularity at $x$ in the stratified sense.

In order to simplify the notations, suppose further that $x$ is the origin in $\mathbb{C}^N$. 

We observe that in the following sections Γ denotes the polar curve of f relatively to a generic linear form ℓ and that ∆ denotes the polar discriminant of f relatively to ℓ, that is, \( \Delta := \phi(\Gamma) \) where \( \phi \) is the stratified map:

\[
\phi := (\ell, f) : (X, 0) \to (\mathbb{C}^2, 0).
\]

As before, we assume that the polar curve Γ is non-empty.

In the next section, we will prove the following propositions:

**Proposition 14.** For any \( t \in \mathbb{D}_{n_2}^* \) there exist:

(i) A polyhedron \( P_t \) in the Milnor fiber \( X_t := \mathbb{B}_e(x) \cap f^{-1}(t) \), with dimension \( \dim_{\mathbb{C}} X - 1 \), adapted to the stratification of \( X_t \) induced by \( S \), i.e., the interior of each simplex of \( P_t \) is contained in a stratum of \( S \).

(ii) A continuous vector field \( E_t \) in \( X_t \), tangent to each stratum of the interior of \( X_t \), so that:

1. It is smooth on each stratum of the induced stratification of \( X_t \setminus P_t \);
2. It is integrable on \( X_t \setminus P_t \);
3. It is zero on \( P_t \);
4. It is transversal to \( \partial X_t \) (in the stratified sense) and points inwards;
5. The flow \( \tilde{q}_t : [0, \infty) \times (X_t \setminus P_t) \to X_t \) associated to \( E_t \) defines a map:

\[
\tilde{\xi}_t : \partial X_t \to P_t
\]

\[
z \mapsto \lim_{t \to \infty} \tilde{q}_t(t, z)
\]

such that \( \tilde{\xi}_t \) is continuous, stratified, simplicial and surjective;

6. The Milnor fiber \( X_t \) is homeomorphic to the mapping cylinder of \( \tilde{\xi}_t \).

We say that the polyhedron \( P_t \) above is a vanishing polyhedron for \( f \).

The idea of the construction of \( P_t \) is quite simple and we will briefly describe it here. First recall the stratified map \( \ell_t : X_t \to D_t \) given by the restriction of \( \phi \) to \( X_t \).

By induction hypothesis, we have a vanishing polyhedron \( P_t' \) for the restriction of \( f \) to the hyperplane section \( X \cap \{ \ell = 0 \} \).

For each point \( y_j(t) \) in the intersection of the polar discriminant \( \Delta \) with the disk \( D_t := \mathbb{D}_{n_2} \times \{ t \} \) as above, let \( x_j(t) \) be a point in the intersection of the polar curve \( \Gamma \) with \( \ell_t^{-1}(y_j(t)) \). To simplify, we can assume that \( x_j(t) \) is the only point in such intersection.

Also by the induction hypothesis, we have a collapsing cone \( P_j \) for the restriction of the map \( \ell_t \) to a small neighborhood of \( x_j(t) \). The “basis” of such cone is the polyhedron \( P_j(a_j) := P_j \cap \ell_t^{-1}(a_j) \), where \( a_j \) is a point in \( \delta(y_j(t)) \setminus y_j(t) \) close to \( y_j(t) \).

Since \( \ell_t \) is a locally trivial fiber bundle over \( \delta(y_j(t)) \setminus y_j(t) \), we can “extend” the cone \( P_j \) until it reaches the “central” polyhedron \( P_t' \). This gives a “wing” that we denote by \( C_j \). The union of all the wings \( C_j \) together with \( P_t' \) gives our vanishing polyhedron \( P_t \).

The detailed construction of \( P_t \) is given in the next section, where we also show that the construction of the polyhedron \( P_t \) and the vector field \( E_t \) can be done simultaneously, for any \( t \) along any simple path \( \gamma \) in \( \mathbb{D}_{n_2} \) joining 0 and some \( t_0 \in \mathbb{D}_{n_2} \).

Precisely, we will prove the following:
Proposition 15. Let $\gamma$ be a simple path in $\mathbb{D}_{\eta_2}$ joining 0 and some $t_0 \in \mathbb{D}_{\eta_2}$. There exist a polyhedron $P_\gamma$ in $X_\gamma := X \cap f^{-1}(\gamma) \cap \mathbb{B}_\epsilon$, adapted to the stratification induced by $S$, and a continuous vector field $E$ in $X_\gamma$, tangent to each stratum of the interior of $X_\gamma$, such that:

(i) the intersection $P_\gamma \cap X_t$ is a vanishing polyhedron $P_t$ as in the proposition above, for any $t \in \gamma \setminus \{0\}$, and $P_\gamma \cap X_0 = \{0\}$;

(ii) for any $t \in \gamma \setminus \{0\}$, the restriction of $E$ to $X_t$ gives a vector field $E_t$ as in the proposition above, relatively to the polyhedron $P_t = P_\gamma \cap X_t$;

(iii) the vector field $E$ is integrable on $X_\gamma \setminus P_\gamma$, smooth on each stratum of the induced stratification of $X_\gamma \setminus P_\gamma$, non-zero outside $P_\gamma$, zero on $P_\gamma$, transversal to $\partial X_\gamma := X_\gamma \cap S_\epsilon$ in the stratified sense, and points inwards.

We say that the polyhedron $P_\gamma$ above is a collapsing cone for $f$ along the path $\gamma$.

One can check that the flow $\tilde{q} : [0, \infty) \times (X_\gamma \setminus P_\gamma) \to X_\gamma$ given by the integration of the vector field $E$ on $X_\gamma \setminus P_\gamma$ defines a continuous, simplicial and surjective map:

$$\tilde{\xi} : \partial X_\gamma \longrightarrow P_\gamma \quad z \longmapsto \lim_{\tau \to \infty} \tilde{q}(\tau, z)$$

such that $X_\gamma$ is homeomorphic to the mapping cylinder of $\tilde{\xi}$ (see Proposition 12).

Assuming that Propositions 14 and 15 are true, we can easily prove Theorem 1 of the Introduction as follows:

Fix $t \in \mathbb{D}_{\eta_2}^*$ and let $\gamma$ be a simple path in $\mathbb{D}_{\eta_2}$ connecting $t$ and 0. Consider the polyhedron $P_\gamma$ and the vector field $E$ in $X_\gamma$ given by Propositions 14 and 15 as well as the flow $\tilde{q}$ given by the integration of $E$.

For any positive real $A > 0$ set:

$$\tilde{V}_A(P_\gamma) := X_\gamma \setminus \tilde{q}([0, A) \times \partial X_\gamma),$$

which is a closed neighborhood of $P_\gamma$ in $X_\gamma$. Notice that using the first isotopy theorem of Thom-Mather, the boundary $\partial \tilde{V}_A(P_\gamma)$ of $\tilde{V}_A(P_\gamma)$ is a locally trivial topological fibration over $\gamma$.

Following the steps (a) and (b) of the end of section 2.2 and using the first isotopy theorem of Thom-Mather, we can construct a similar continuous vector field $E$ on $X_\gamma \setminus P_\gamma$ such that:

1. it lifts a smooth vector field $\theta$ on $\gamma$ that goes from $t_0$ to 0 in a time $a > 0$;
○ it is smooth on each stratum of $X \gamma \setminus P_\gamma$;
○ it is rugose and hence integrable;
○ it is tangent to $\partial \tilde{V}_A(P_\gamma)$, for any $A > 0$.

In fact, the vector fields constructed here are rugose because under the hypothesis of the first isotopy theorem of Thom-Mather, a smooth vector field lifts to a rugose vector field. See Lemma 5.

So the flow $g : [0, a] \times X \gamma \setminus P_\gamma \to X \gamma \setminus P_\gamma$ associated to $E$ defines a homeomorphism $\Psi_t$ from $X_t \setminus P_t$ to $X_0 \setminus \{0\}$ that extends to a continuous map from $X_t$ to $X_0$ and that sends $P_t$ to $\{0\}$, for any $t \in \gamma \setminus \{0\}$. This proves Theorem 1.

4. Proof of Propositions 14 and 15

In this section, we shall prove Propositions 14 and 15.

Actually, the same arguments used in the proof of Proposition 15 give a proof for the following stronger proposition, where instead of constructing the polyhedron $P_t$ and the vector field $E_t$ simultaneously for $t$ in a path $\gamma$, we construct them simultaneously for $t$ in a closed semi-disk $D^+$ of $D_{\eta_2}$.

**Proposition 16.** Let $D^+$ be a closed semi-disk in $D_{\eta_2}$. There exist a polyhedron $P^+$, adapted to the stratification induced by $S$, and a continuous vector field $E$ in $X^+ := X \cap f^{-1}(D^+) \cap B_\epsilon$, tangent to each stratum of the interior of $X^+$, such that:

(i) the intersection $P^+ \cap X_t$ is a vanishing polyhedron $P_t$ as in Proposition 14, for any $t \in D^+ \setminus \{0\}$, and $P^+ \cap X_0 = \{0\}$;

(ii) for any $t \in D^+ \setminus \{0\}$, the restriction of $E$ to $X_t$ gives a vector field $E_t$ as in Proposition 14, relatively to the polyhedron $P_t = P^+ \cap X_t$;

(iii) the vector field $E$ is integrable on $X^+ \setminus P^+$, smooth on each stratum of the induced stratification of $X^+ \setminus P^+$, non-zero outside $P^+$, zero on $P^+$, transversal to $\partial X^+ := X^+ \cap S_\epsilon$ in the stratified sense, and points inwards.

We say that the polyhedron $P^+$ is a **collapsing cone** for $f$ along the semi-disk $D^+$.

We will prove Propositions 14 and 15 by finite induction on the dimension $n$ of $X$. Proposition 16 will be used (as induction hypothesis) in the construction of the vector field $E_t$ of Proposition 14.

We have already proved Propositions 14 and 15 when $n = 2$, and one can check that it is easy to extend it to a proof of Proposition 16 in the two-dimensional case. Now we will prove that if these results are true whenever $\dim_C X \leq n - 1$, then they are true when $\dim_C X = n$.

4.1. Proof of Proposition 14: constructing the vanishing polyhedron.

As we said above, the polyhedron $P_t$ will consist of a “central” polyhedron $P'_t$ on which we will attach “wings” $C_j$. The first step will be to construct the central polyhedron $P'_t$. Then we will construct the extremity of each wing $C_j$. Finally, we will extend the extremity of each wing until it hits $P'_t$.

Recall that we have fixed a linear form $\ell : \mathbb{C}^N \to \mathbb{C}$ that takes $0 \in \mathbb{C}^N$ to $0 \in \mathbb{C}$ and that satisfies the conditions of Lemma 2. Then $\Gamma$ is the polar curve of $f$ relatively to $\ell$ at $0$ and $\Delta$ is the polar discriminant of $f$ relatively to $\ell$ at $0$. 
Also recall from Proposition [0] that the map $\phi = (\ell, f)$ induces a stratified submersion:

$$\phi_t : B_t \cap X \cap \phi^{-1}(D_{\eta_1} \times D_{\eta_2} \setminus \Delta) \to D_{\eta_1} \times D_{\eta_2} \setminus \Delta$$

and that for each $t \in D_{\eta_2}^*$ fixed, the restriction $\ell_t$ of $\ell$ to the Milnor fiber $X_t$ induces a topological locally trivial fibration:

$$\varphi_t : X_t \setminus \ell_t^{-1}\{\{y_1(t), \ldots, y_k(t)\}\} \to D_t \setminus \{y_1(t), \ldots, y_k(t)\},$$

where $D_t = D_{\eta_1} \times \{t\}$ and $\{y_1(t), \ldots, y_k(t)\} = \Delta \cap D_t$.

For any $t \in D_{\eta_2}$ set $\lambda_t := (0, t)$. Since the complex line $\{0\} \times \mathbb{C}$ is not a component of $\Delta$, we can suppose that $\lambda_t \notin \{y_1(t), \ldots, y_k(t)\}$.

For each $j = 1, \ldots, k$, let $\delta(y_j(t))$ be a simple path in $D_t$ starting at $\lambda_t$ and ending at $y_j(t)$, such that two of them intersect only at $\lambda_t$.

**First step: constructing the central polyhedron $P'_t$:**

Consider $f'$ the restriction of $f$ to the intersection $X \cap \{\ell = 0\}$, which has complex dimension $n - 1$. Then we can apply the induction hypothesis to $f'$ to obtain a vanishing polyhedron $P'_t$ in the fiber $X_t \cap \{\ell = 0\}$ and an integrable vector field $E'_t$ in $X_t \cap \{\ell = 0\}$, stratified on $(X_t \cap \{\ell = 0\}) \setminus P'_t$, that deformation retracts $X_t \cap \{\ell = 0\}$ onto $P'_t$.

**Second step: constructing the extremity of the wings $C_j$:**

First of all, in order to make it easier for the reader to understand the constructions, we will suppose that $\Gamma$ intersects $\ell_t^{-1}\{y_j(t)\}$ in only one point, which we call $x_j(t)$. The proof of the general case follows the same steps. In fact, we do the following conjecture:

**Conjecture 17.** For $\ell$ general enough, the map-germ $\phi_t = (\ell, f) : (X, x) \to (\mathbb{C}^2, 0)$ induces a bijective morphism between $\Gamma$ and $\Delta$.

Now recall that $\ell_t$ induces a locally trivial fibration over $\delta(y_j(t))\{y_j(t)\}$. If we look at the local situation at $x_j(t)$, we can apply the induction hypothesis to the germ $\ell_{t|} : (X_t, x_j(t)) \to (D_t, y_j(t))$, which has an isolated singularity at $x_j(t)$ in the stratified sense, in lower dimension. That is, considering $B_j$ a small ball in $\mathbb{C}^N$ centered at $x_j(t)$; $D_s$ a small disk in $D_t$ centered at $y_j(t)$ and $D_s^+$ a semi-disk of $D_s$ which contains $\delta(y_j(t)) \cap D_s$ in its interior, we obtain:

- a collapsing cone $P^+_j$ for $\ell_t$ along the semi-disk $D^+_s$;
- a collapsing cone $P_j$ for $\ell_t$ along the path $D_s \cap \delta(y_j(t))$; and
- a vector field $E_j$ in $\ell_t^{-1}(D_s) \cap B_j$;

which give the collapsing of the map $\ell_{t|} : B_j \cap \ell_t^{-1}(D_s) \to D_s$. See Figure [5]

So the extremity of the wing $C_j$ is given by the collapsing cone $P_j$.

**Third step: extending the extremity of each wing until it hits $P'_t$:**

First we need to construct the following vector fields on $A_j := \ell_t^{-1}(\delta(y_j(t)) \setminus \{y_j(t)\})$ that will be used to extend the cone $P_j$ and glue it on the central polyhedron $P'_t$:

- **Vector Field $\Xi$:** Let $\xi$ be a smooth non-singular vector field on $\delta(y_j(t)) \setminus \{y_j(t)\}$ that goes from $y_j(t)$ to $\lambda_x = (0, t)$. Since the restriction of $\ell_t$ to each Whitney stratum $S_\alpha$ has maximum rank over $\delta(y_j(t)) \setminus \{y_j(t)\}$, we can lift $\xi$ to an integrable stratified vector field $\Xi$ on $A_j$ (see Lemma [5]). In particular, for any
\[ u \in \delta(y_j(t))\setminus\{y_j(t)\} \] we can use the vector field \( \Xi \) to obtain a stratified homeomorphism \( \alpha_u : \ell_t^{-1}(\lambda_t) \to \ell_t^{-1}(u) \), which takes \( P'_t \) to a polyhedron \( \alpha_u(P'_t) \) in \( \ell_t^{-1}(u) \).

- **Vector Field \( \mathcal{V} \):** We can transport the vector field \( E'_t \) of \( \ell_t^{-1}(\lambda_t) = X_t \cap \{ \ell = 0 \} \) given by the induction hypothesis to all the fibers \( \ell_t^{-1}(u) \), for any \( u \in \delta(y_j(t))\setminus\{y_j(t)\} \). The transportation of \( E'_t \) to \( \ell_t^{-1}(u) \) is the vector field on \( \ell_t^{-1}(u) \) given by the flow obtained as image by \( \alpha_u \) of the flow given by \( E'_t \). So we obtain a vector field \( \mathcal{V} \) on \( A_j \) whose restriction to \( \ell_t^{-1}(\lambda_t) \) is \( E'_t \). The flow associated to \( \mathcal{V} \) takes a point \( z \in \ell_t^{-1}(u) \) to the polyhedron \( \alpha_u(P'_t) \).

- **Vector Field \( \mathcal{V}_1 \):** Let \( \theta \) be a smooth function on \( \delta(y_j(t)) \) such that \( \theta(\lambda_t) = 0 \) and such that \( \theta \) is non-singular and positive on \( \delta(y_j(t))\setminus\{\lambda_t\} \). It induces a function \( \bar{\theta} := \theta \circ \ell_t \) defined on \( A_j \). Set:

\[ \mathcal{V}_1 := \mathcal{V} + \bar{\theta} \cdot \Xi, \]

which is an integrable vector field, tangent to the strata of the interior of \( A_j \) induced by \( S \). Furthermore, this vector field \( \mathcal{V}_1 \) is pointing inwards on the boundary \( \partial A_j \), i.e. transversal in \( A_j \) to the strata of \( \partial A_j \) induced by \( S \).

Since the vectors \( \tilde{\mathcal{V}}(z) \) and \( \Xi(z) \) are not parallel for any \( z \in A_j\setminus P'_t \), the vector field \( \mathcal{V}_1 \) is zero only on the vanishing polyhedron \( P'_t \) of \( \ell_t^{-1}(\lambda_t) \). Then if \( z \) is a point in \( A_j\setminus \ell_t^{-1}(\lambda_t) \), the orbit of \( \mathcal{V}_1 \) that passes through \( z \) has its limit point \( z'_1 \) in \( P'_t \).

Moreover, since the orbit of \( \mathcal{V} \) that passes through \( z \) has its limit point \( z' \) in the transportation \( \alpha_{\ell_t(z)}(P'_t) \) of \( P'_t \) to \( \ell_t^{-1}(\ell_t(z)) \) by \( \Xi \), it follows that \( z'_1 \) is the point corresponding to \( z' \) by \( \Xi \), that is, \( z'_1 = \alpha_{\ell_t(z)}(z') \). In fact, if \( u := \ell_t(z) \) and if \( w := (\alpha_u)^{-1}(z) \) is the corresponding point in \( \ell_t^{-1}(\lambda_t) \), then by construction the integral curve \( \mathcal{C}_\mathcal{V}(z) \) associated to \( \mathcal{V} \) that contains \( z \) is given by \( \alpha_u(\mathcal{C}(w)) \), where \( \mathcal{C}(w) \) is the integral curve associated to \( E'_t \) that contains \( w \).

Set \( a_j := \partial D_s \cap \delta(y_j(t)) \) and \( P_j(a_j) := P_j \cap \ell_t^{-1}(a_j) \), where \( P_j \) is the collapsing cone for \( \ell_t \) at \( x_j(t) \) along the path \( D_s \cap \delta(y_j(t)) \), as defined above. By the previous paragraph, \( \mathcal{V}_1 \) takes \( P_j(a_j) \) to \( P'_t \).

Since the action of the flow given by \( \mathcal{V} \) is simplicial, we can assume that the action of the flow given by \( \mathcal{V}_1 \) is simplicial. Then the image of \( P_j(a_j) \) by the flow of \( \mathcal{V}_1 \) is a sub-polyhedron \( P'_j \) of \( P'_t \). Moreover, the orbits of the points in \( P_j(a_j) \) give a polyhedron \( R_j \). See Figure 6.
Set:

\[ C_j := P_j \cup R_j \cup P_j'. \]

We call \( C_j \) a \textit{wing} of the polyhedron \( P_t \). In the case when \( X \) is smooth, it corresponds to a Lefschetz thimble.

Then the polyhedron we are going to consider is:

\[ P_t := P'_t \bigcup_{j=1}^{k} C_j. \]

It is adapted to the stratification \( S \), since \( P_j \) is adapted to \( S \) and the vector field \( V_1 \) is tangent to the strata of \( S \).

Now we have:

**Lemma 18.** There is a continuous vector field \( E_t \) on \( X_t \) such that:

1. It is tangent to each stratum of \( \hat{X}_t \setminus P_t \) induced by \( S \), where \( \hat{X}_t := X_t \setminus \partial X_t \);
2. It is transversal to the strata of \( \partial X_t \) and points inwards;
3. It is smooth on each stratum of \( X_t \setminus P_t \);
4. It is integrable on \( X_t \setminus P_t \);
5. It is non-zero on \( X_t \setminus P_t \) and it is zero on \( P_t \);
6. The orbits of \( E_t \) have a limit point at \( P_t \) when the parameter goes to the infinity.

The vector field \( E_t \) is obtained by gluing several vector fields on \( X_t \) given by the lifting by \( \varphi_t \) of suitable vector fields on \( D_t \). The detailed proof of Lemma 18 is quite annoying since it contains too many technical steps and constructions, so we present it separately in the next section.

The flow defined by the vector field \( E_t \) of Lemma 18 gives a continuous, surjective and simplicial map \( \tilde{\xi}_t \) from \( \partial X_t \) to \( P_t \) such that \( X_t \) is homeomorphic to the simplicial map cylinder of \( \tilde{\xi}_t \) (see the proof of Proposition 12). This proves Proposition 14.

4.2. **Proof of Proposition 15:** constructing the collapsing cone.

Given a simple path \( \gamma \) in \( \mathbb{D}_{\eta_2} \) joining \( 0 \in \mathbb{D}_{\eta_2} \) and some \( t_0 \in \mathbb{D}_{\eta_2} \), we have to construct a polyhedron \( P_\gamma \), adapted to the stratification induced by \( S \), and a continuous vector
field $E$ in $X_{\gamma} := X \cap f^{-1}(\gamma) \cap \mathbb{B}_\varepsilon$, tangent to each stratum of $X_{\gamma}$, satisfying the conditions (i), (ii) and (iii) of Proposition 15.

We are going to construct $P_{\gamma}$. The construction of the vector field $E$ follows the same steps of the construction of the vector field $E_t$ of the subsection above, which is described with details in the next section.

First, consider the polyhedron $P_{t_0}$, contained in the Milnor fiber:
$$X_{t_0} = X \cap f^{-1}(t_0) \cap \mathbb{B}_\varepsilon$$
and constructed as in Proposition 14. It is given by the union:
$$P_{t_0} = P_{t_0}' \bigcup_{j=1}^k C_j,$$
where each $C_j$ is a wing glued to $P_{t_0}'$ along a sub-polyhedron $(P_j)'_{t_0}$ of $P_{t_0}'$ and $P_{t_0}'$ is the vanishing polyhedron for the restriction $f'$ of $f$ to $X \cap \{\ell = 0\}$.

By the induction hypothesis, we also have a collapsing cone $P_{\gamma}$ in $X_{\gamma} \cap \{\ell = 0\}$ and a continuous vector field $G'$ in $X_{\gamma} \cap \{\ell = 0\}$ that gives the degeneration of $f'$. That is, the restriction of $G'$ to $(X_{\gamma} \cap \{\ell = 0\}) \backslash P_{\gamma}$ is an integrable vector field whose associated flow defines a homeomorphism from $(X_{t_0} \cap \{\ell = 0\}) \backslash P_{t_0}'$ to $(X_{t_0} \cap \{\ell = 0\}) \backslash \{0\}$ that extends to a continuous map from $X_{t_0} \cap \{\ell = 0\}$ to $X_0 \cap \{\ell = 0\}$ and that sends $P_{t_0}'$ to $\{0\}$.

Recall from Proposition 6 that the map $\phi = (\ell, f)$ induces a stratified submersion:
$$\phi : \mathbb{B}_\varepsilon \cap X \cap \phi^{-1}(\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \backslash \Delta) \to \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \backslash \Delta,$$
where $\Delta \subset \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2}$ is the polar discriminant of $f$ relatively to the linear form $\ell$.

Set $\Lambda := \{0\} \times \gamma$, which we suppose that intersects $\Delta$ only at $0 \in \mathbb{C}^2$. Then we define the 2-dimensional polyhedra $T_j$ in $\mathbb{D}_{\eta_1} \times \gamma$, for each $j = 1, \ldots, k$, as in subsection 2.2. That is:
$$T_j := \bigcup_{t \in \gamma} \delta(y_j(t)),$$
where each $\delta(y_j(t))$ is a simple path connecting $y_j(t)$ and $\lambda_t = (0, t)$, and $\delta(y_j(0))$ is the origin.

For each $x_j(t)$ over $y_j(t)$, with $t \in \gamma$, choose a small radius $r(t)$ such that the set:
$$B_j := \bigcup_{t \in \gamma} \mathbb{B}_{r(t)}(x_j(t))$$
is a neighborhood of $\cup_{t \in \gamma^*} \{x_j(t)\}$ conic from 0, where $r(t)$ is a real analytic function of $t$ with $r(0) = 0$ and $\gamma^* := \gamma \backslash \{0\}$.

To each $B_j$ one can associate a neighborhood:
$$A_j := \bigcup_{t \in \gamma} \mathbb{D}_{s(t)}(y_j(t))$$
in $\mathbb{D}_{\eta_1} \times \gamma$, where $s(t)$ is an analytic function of $t \in \gamma$ with $0 < s(t) \ll r(t)$, if $t \neq 0$, and $s(0) = 0$.

Also let $\mathcal{U}$ be a neighborhood of $\Lambda \backslash \{0\}$, conic from 0, that meets all the $A_j$’s, but not containing any $y_j(t)$. See Figure 7.
Notice that the stratified vector field $G'$ that gives the degeneration of the restriction of $f$ to $X \cap \{ \ell = 0 \}$ is defined on $\phi^{-1}(\Lambda)$. Set:

$$\tilde{U} := \phi^{-1}(U) \cap X_\gamma.$$  

Since $U$ is contractible, we can extend the vector field $G'$ to a integrable stratified vector field $G_U$, defined on $\tilde{U}$. Notice that the flow given by the vector field $G_U$ sends the intersection $P_{t_0} \cap \tilde{U}$ to $0$, where $P_{t_0}$ is the polyhedron in $X_{t_0}$ previously constructed.

One can also construct an integrable vector field $G_j$ on each $B_j \setminus \{0\}$ that trivializes it over $\gamma \setminus \{0\}$.

Then, using a partition of unity $(\rho_U, \rho_1, \ldots, \rho_k)$ adapted to $\tilde{U}, B_1, \ldots, B_k$, we glue all the vector fields $G_j$'s and $G_U$ together. We obtain a continuous trivializing vector field:

$$G := \rho_U G_U + \sum_{j=1}^{k} \rho_j G_j$$

in $\tilde{U} \cup_{j=1}^{k} B_j$ such that:

- it is smooth on each stratum of $X_\gamma \cap (\tilde{U} \cup_{j=1}^{k} B_j)$ ;
- it is rugose and hence integrable;
- it projects to a radial vector field in $\gamma$ that converges to $0$.

Then we construct the vanishing cone $P_\gamma$ from the vanishing polyhedron $P_{t_0}$ using the flow of $G$.

5. **Proof of Lemma 18: constructing the vector field $E_t$**

In this section we give the detailed construction of the vector field $E_t$ of Lemma 18 whose flow gives a continuous, surjective and simplicial map $\tilde{\xi}_t$ from the boundary of the Milnor fiber $\partial X_t$ to the polyhedron $P_t$ constructed in the previous section, such
that the Milnor fiber $X_t := \phi^{-1}(\mathbb{D}_{\eta_1} \times \{t\}) \cap \mathbb{B}_e(x)$ is homeomorphic to the simplicial map cylinder of $\xi_t$.

Recall that we have fixed a linear form $\ell : \mathbb{C}^N \to \mathbb{C}$ that takes $0 \in \mathbb{C}^N$ to $0 \in \mathbb{C}$ and that satisfies the conditions of Lemma 2. Then $\Gamma$ is the polar curve of $f$ relatively to $\ell$ at $0$ and $\Delta$ is the polar discriminant of $f$ relatively to $\ell$ at $0$.

Also recall from Proposition 3 that the map $\phi = (\ell, f)$ induces a stratified submersion:

$$\phi : \mathbb{B}_\ell \cap X \cap \phi^{-1}(\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \setminus \Delta) \to \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \setminus \Delta$$

and that for each $t \in \mathbb{D}_{\eta_2}$ fixed, the restriction $\ell_t$ of $\ell$ to the Milnor fiber $X_t$ induces a topological locally trivial fibration:

$$\varphi_t : X_t \setminus \ell_t^{-1}\{(y_1(t), \ldots, y_k(t))\} \to D_t \setminus \{y_1(t), \ldots, y_k(t)\},$$

where $D_t := \mathbb{D}_{\eta_1} \times \{t\}$ and $\{y_1(t), \ldots, y_k(t)\} = \Delta \cap D_t$.

As before, take a point $\lambda_t$ in $D_t$ such that $\lambda_t \notin \{y_1(t), \ldots, y_k(t)\}$. Also, for each $j = 1, \ldots, k$, let $\delta(y_j(t))$ be a simple path in $D_t$ starting at $\lambda_t$ and ending at $y_j(t)$, such that two of them intersect only at $\lambda_t$. We defined the set $Q_t := \cup_{j=1}^k \delta(y_j(t))$.

Also recall that we can apply the induction hypothesis to the restriction $f'$ of $f$ to the intersection $X \cap \{\ell = 0\}$, which has complex dimension $n - 1$. We obtain a vanishing polyhedron $P'_t$ in the intersection $X_t \cap \{\ell = 0\}$ and a vector field $E'_t$ that deformation retracts it onto $P'_t$.

The vector field $E_t$ is obtained by gluing several vector fields on $X_t$ given by the lift by $\varphi_t : X_t \setminus \ell_t^{-1}\{(y_1(t), \ldots, y_k(t))\} \to D_t \setminus \{y_1(t), \ldots, y_k(t)\}$ of suitable vector fields on the disk $D_t$. This is possible thanks to the first isotopy lemma of Thom-Mather, since the restriction of $\varphi_t$ to each stratum of the Whitney stratification of $X_t$ induced by the Whitney stratification of $X$ is of maximum rank. The resulting vector fields are rugose in the sense of [20], and hence they are integrable.

Recall that the polyhedron $P_t$ is the union of the wings $C_j$ and the polyhedron $P'_t$ given by the induction hypothesis (see subsection 4.1). Moreover, each wing $C_j$ consists of a collapsing cone $P_j$, a product $R_j$ and the gluing polyhedron $P'_j$ on $P'_t$, that is:

$$C_j = P_j \cup R_j \cup P'_j.$$ 

See Figure 6

Then it is natural that the construction of the vector field $E_t$ concerns at least three subsets of the Milnor fiber $X_t$: the points that are taken to $P'_t \setminus P'_j$ by the flow associated to $E_t$; the points that are taken to $P'_j$ and the points that are taken to $C_j \setminus P'_j$. This justifies the complexity of the construction given below.

5.1. First step: decomposing $D_t$.

Let $v_t$ be the continuous vector field on $D_t$, smooth and non-zero outside $Q_t$, transversal to $\partial D_t$ and zero on $Q_t$, defined in Lemma 10. If $q_t : [0, \infty[ \times \partial D_t \to D_t$ is the flow associated to $v_t$, then set:

$$V := D_t - q_t([0, A] \times \partial D_t),$$

for some $A \gg 0$, which is a closed neighborhood of $Q_t$ whose boundary:

$$\partial V = q_t([A] \times \partial D_t).$$
is smooth and transversal to each $\partial D_s(y_j(t))$ as in Figure 8.

**Figure 8.**

Then we construct the vector field $E_t$ on $X_t$ by gluing a vector field $\tilde{E}_t$ in $\ell_t^{-1}(D_t \setminus V')$, where $V' := D_t - q_t([0,A'] \times \partial D_t)$, with $A' > A$, $A' - A \ll 1$; and a vector field $\tilde{E}_t$ in $\ell_t^{-1}(V)$, using a partition of unity.

The vector field $\tilde{E}_t$ in $\ell_t^{-1}(D_t \setminus V')$ is the lifting of the vector field $v_t$. It is integrable (see Lemma 5), and it is tangent to the strata of $\phi^{-1}(D_t \setminus V') \cap \mathbb{B}_c$ and transversal (and pointing inwards) to the strata of $\partial X_t$.

The construction of the vector field $\tilde{E}_t$ in $\ell_t^{-1}(V)$ is much more complicated. We are going to do it in the rest of this subsection.

### 5.2. Second step: decomposing $V$.

We first decompose $V$ into “branches” $V_j$ as follows: each “branch” $V_j$ is a closed neighborhood of $\delta(y_j(t)) \setminus \{0\}$ whose boundary is composed by $\partial V \cap V_j$ and two simple paths that one can suppose to be orbits of the vector field $v_t$ constructed above. See Figure 9.

**Figure 9.**

We will construct the vector field $\tilde{E}_t$ by gluing the vector fields $\tilde{E}_{t,j}$ that we are going to construct on each $\ell_t^{-1}(V_j)$. In other words, we will construct a vector field $\tilde{E}_{t,j}$ on $\ell_t^{-1}(V_j)$, for each $j$ fixed, which is continuous, integrable, tangent to the strata of $S$, non-zero and smooth on $\ell_t^{-1}(V_j) \setminus C_j$, and zero on $C_j$, where $C_j$ is the polyhedron defined in subsection 4.1.
5.3. Third step: covering $\ell_t^{-1}(V_j)$ by open sets $W_{j,i}$.

Fix $j \in \{1, \ldots, k\}$. The approach of the construction of each vector field $\tilde{E}_{t,j}$ will be the following: we will cover $\ell_t^{-1}(V_j)$ by open sets $W_{j,1}$, $W_{j,2}$, $W_{j,3}$ and $W_{j,4}$. Then we will construct the vector fields $\tilde{E}_{t,j,i}$ on $W_{j,i}$, for $i = 1, \ldots, 4$, in such a way that each orbit of the vector field $\tilde{E}_{t,j}$ obtained by gluing them with a partition of unity have a limit point at $P_t$.

As before, given positive real numbers $r$ and $s$, let $B_r$ denote the ball around $x_j(t)$ in $\mathbb{C}^N$ of radius $r$ and let $D_s$ denote the disk around $y_j(t)$ in $D_t$ of radius $s$.

Let $r$ and $r'$ be small enough positive real numbers such that $r' < r$ and $r - r' \ll 1$. Let us cover $\ell_t^{-1}(V_j)$ by the open sets $W_{j,1}$, $W_{j,2}$, $W_{j,3}$ and $W_{j,4}$ defined as follows:

- $W_{j,1} := \ell_t^{-1}((\hat{D}_s \cap \hat{V}_j) \cap \hat{B}_r)$

and

- $W_{j,2} := \ell_t^{-1}((\hat{D}_s \cap \hat{V}_j) \setminus B_{r'})$.

To define $W_{j,3}$ and $W_{j,4}$ we have to do a construction first. Set:

$$W_{j,3}' := \ell_t^{-1}(V_j \setminus D_{s'})$$

where $s' < s$ and $s - s' \ll 1$.

We can construct a vector field $\omega_j$ in $V_j \setminus D_{s'}$ which is smooth, non zero outside $\{0\}$, zero on $\{0\}$, with trajectories transversal to $\partial V \cap (V_j \setminus D_{s'})$ and to $\partial D_{s'} \cap V_j$, as in Figure 10.

![Figure 10](image)

Since the Whitney stratification $(S_\alpha)_{\alpha \in A}$ of $X$ induces a Whitney stratification on $\ell_t^{-1}(V_j \setminus D_{s'})$ and since the restriction of $\ell_t$ to $\ell_t^{-1}(V_j \setminus D_{s'}) \cap S_\alpha$ is a submersion for each $\alpha \in A$, we can lift $\omega_j$ to a continuous vector field $H_j$ in $\ell_t^{-1}(V_j \setminus D_{s'})$ that is rugose, smooth and tangent to each stratum, and that trivializes $\ell_t^{-1}(V_j \setminus D_{s'})$ over $V_j \setminus D_{s'}$.

Recall that the induction hypothesis applied to the restriction of $f$ to $X \cap \{\ell = 0\}$ gives a vanishing polyhedron $P_t'$ in $X_t \cap \{\ell = 0\}$ and a vector field $E_t'$ that deformation retracts $X_t \cap \{\ell = 0\}$ onto $P_t'$.

Then one can transport the vector field $E_t'$ on $\ell_t^{-1}(0)$ to all the fibers $\ell_t^{-1}(u)$ for $u \in V_j \setminus D_{s'}$. This way we obtain a vector field $\mathcal{V}$ in $\ell_t^{-1}(V_j \setminus D_{s'})$ which is integrable, tangent to each stratum of $\ell_t^{-1}(u) \cap \mathbb{B}_\epsilon$ and transversal to each stratum of $\ell_t^{-1}(u) \cap S_\epsilon$, for any $u \in V_j \setminus D_{s'}$. 


Now consider the vector field $\vartheta_1$ in $\ell^{-1}_t(V_j \setminus \hat{D}_s)$ given by:

$$\vartheta_1 := \mathcal{V} + H_j,$$

which is integrable, tangent to the strata of the stratification induced by $S$, transversal to the strata of $S_e \cap \ell^{-1}_t(V_j \setminus \hat{D}_s)$, non-zero outside $P'_t$ and zero on $P'_t$.

One can see (as in the case of the vector field $\vartheta_1$ of Subsection 4.1) that the orbits of $\vartheta_1$ have a limit point in $P'_t$. The orbits $A(V_j, r)$ that intersect the set $\ell^{-1}_t(V_j \cap \partial D_s) \cap B_r$ by the action of $\vartheta_1$ is a set that we call $W'_{j,4}$. Then we define the set:

- $W_{j,4} := W'_{j,4} \cup W_{j,1}$.

Finally, the set $W_{j,3}$ is given by:

- $W_{j,3} := W'_{j,3} \setminus A(V'_j, r')$,

where $r' < r$, with $r - r' \ll 1$, and $V'_j := D_t \setminus q_t([0, A'\times \partial D_t])$, with $A' < A$ and $A - A' \ll 1$. One can check that both $W_{j,3}$ and $W'_{j,4}$ are open sets.

5.4. Fourth step: constructing the vector fields $\tilde{E}_{t,j,i}$.

(1) Construction of $\tilde{E}_{t,j,1}$:

We can consider a smooth vector field $\tilde{\vartheta}$ on $V_j$ which is zero on $\delta(y_j(t))$, transversal to $\partial V_j$ and tangent to $\partial D_s \cap V_j$, like in Figure 11.

Let $D^+_s$ be a semi-disk of $D_s$ which contains $\delta(y_j(t)) \cap \hat{D}_s$ in its interior.

We will lift $\tilde{\vartheta}$ to a rugose vector field $\tilde{H}_j$ in $\ell^{-1}_t(\hat{D}_s \cap \hat{V}_j) \cap B_r$, which is zero on $\ell^{-1}_t(\hat{D}_s \cap \delta(y_j(t)))$, tangent to the strata of $S$ and of $\ell^{-1}_t(\hat{D}_s \cap \hat{V}_j) \cap S_r$, where $S_r := \partial B_r$, in the following way:

Recall that we can apply the induction hypothesis to the restriction:

$$(\ell_t)_! : \ell^{-1}_t(D^+_s \cap \hat{D}_s \cap \hat{V}_j) \cap \hat{B}_r \to D^+_s \cap \hat{D}_s \cap \hat{V}_j,$$

which has an isolated singularity at $x_j(t)$ in the stratified sense, since $\ell^{-1}_t(D^+_s \cap \hat{D}_s \cap \hat{V}_j)$ has complex dimension $n - 1$, where $n$ is the dimension of $X$.

Then we obtain a vector field $E_j$ and a collapsing cone $P^+_j$. Let:

$$\psi : [0, +\infty[ \times (\ell^{-1}_t(D^+_s \cap \hat{V}_j) \cap S_r) \to \ell^{-1}_t(D^+_s \cap \hat{V}_j) \cap B_r$$
be the flow associated to $E_j$ and set:

$$P_j(u) := \psi(\{u\} \times \ell_t^{-1}(\hat{D}_s^+ \cap \hat{V}_j) \cap S_r),$$

where $u \geq 0$.

The Whitney stratification $S$ induces a Whitney stratification of $P_j(u)$ (see Lemma 8) and notice that $P_j(0) = \ell_t^{-1}(\hat{D}_s^+ \cap \hat{V}_j) \cap S_r$ is the intersection of $\ell_t^{-1}(\hat{D}_s^+ \cap \hat{V}_j)$ with the stratified space $X_t \cap S_r$. Moreover, the restriction of $\ell_t$ to each stratum has maximum rank. So by Lemma 5 we can lift the vector field $\tilde{\omega}$ over $\hat{D}_s^+ \cap \hat{V}_j$ to a vector field that is tangent to the strata of $P_j(u)$.

On the other hand, for any point in $\ell_t^{-1}((\hat{D}_s \setminus \hat{D}_s^+) \cap \hat{V}_j)$ we just ask the vector field $\tilde{H}_j$ to be tangent to the strata of $S$ and to lift $\tilde{\omega}$. This can be done locally and then $\tilde{H}_j$ is obtained by a partition of unity.

Notice that at any point of $\hat{B}_r \cap \ell_t^{-1}(\hat{D}_s^+ \cap \hat{V}_j) - \{x_j(t)\}$ and at any point of $(\hat{B}_r \setminus B_{r'}) \cap \ell_t^{-1}((\hat{D}_s \setminus \hat{D}_s^+) \cap \hat{V}_j)$, for $r' < r$ with $r - r' \ll 1$, one can extend the vector field $E_j$ on a small open neighborhood.

Now we construct $\tilde{E}_{t,j,1}$ as follows:

- Over a small open neighborhood $U_{x_j(t)}$ of $x_j(t)$, consider the zero vector field.
- For any $z \in \hat{B}_r \cap \ell_t^{-1}(\hat{D}_s^+ \cap \hat{V}_j) \setminus \{x_j(t)\}$, take an open neighborhood $U_z$ of $z$ small enough such that it does not contain $x_j(t)$, it is contained in $\hat{B}_r \cap \ell_t^{-1}(\hat{D}_s \cap \hat{V}_j)$ and $E_j$ is well defined on it. Then in $U_z$ we define the vector field:

$$E_{1,z} := E_j|_{U_z} + \tilde{H}_{j|_{U_z}}.$$

This vector field is rugose, tangent to the strata of $S$, non-zero outside the intersection of $U_z$ and $P_j$ and zero on $P_j \cap U_z$, where:

$$P_j := P_j^+ \cap \ell_t^{-1}(\delta(y_j(t))).$$

- For any $z \in \hat{B}_r \cap \ell_t^{-1}((\hat{D}_s \setminus \hat{D}_s^+) \cap \hat{V}_j)$, take a small open neighborhood $U_z$ of $z$ and set

$$E_{1,z} := \tilde{H}_{j|_{U_z}}.$$

- For any $z \in (\hat{B}_r \setminus B_{r'}) \cap \ell_t^{-1}((\hat{D}_s \setminus \hat{D}_s^+) \cap \hat{V}_j)$, take a small open neighborhood $U_z$ of $z$ contained in $(\hat{B}_r \setminus B_{r'}) \cap \ell_t^{-1}((\hat{D}_s \setminus \hat{D}_s^+) \cap \hat{V}_j)$ and set:

$$E_{1,z} := E_j|_{U_z} + \tilde{H}_{j|_{U_z}}.$$

- Then considering a partition of unity ($\theta_z$) associated to the covering $(U_z)$, we set the vector field:

$$\tilde{E}_{t,j,1} := \sum \theta_z E_{1,z}$$

in $\ell_t^{-1}(\hat{D}_s \cap \hat{V}_j) \cap \hat{B}_r$, which is continuous, rugose outside the point $x_j(t)$ (and therefore in $W_{j,1}|P_j$), tangent to the strata of $S$, non-zero outside $P_j$ and zero on $P_j \cap (\ell_t^{-1}(\hat{D}_s \cap \hat{V}_j)) \cap \hat{B}_r$.

Notice that if $z \in \ell_t^{-1}(\hat{D}_s \setminus \hat{D}_s^+) \cap \hat{B}_r$, its orbit by $\tilde{E}_{t,j,1}$ has $\{x_j(t)\}$ as limit point, and the orbit by $\tilde{E}_{t,j,1}$ of a point $z \in \ell_t^{-1}(\hat{D}_s^+) \cap \hat{B}_r$ has its limit point in $P_j$. 
(2) Construction of $\tilde{E}_{t,j,2}$:

Consider a smooth non-zero vector field $\tilde{\omega}_j$ in $\tilde{V}_j \cap \tilde{D}$ as Figure 12 and such that, for any $u \in \tilde{V}_j \cap \tilde{D}$ one has the following implication:

$$\lambda \tilde{\omega}(u) + \mu \tilde{\omega}_j(u) = 0, \quad \lambda \geq 0, \quad \mu \geq 0 \implies \lambda = \mu = 0,$$

where $\tilde{\omega}$ is the vector field defined above.

![Figure 12](image)

Then $\tilde{E}_{t,j,2}$ is the lift of $\tilde{\omega}_j$ in $W_{j,2}$, which is rugose and tangent to the strata of $S$ and of $\ell_t^{-1}(\tilde{D}_s) \cap S_r$.

(3) Construction of $\tilde{E}_{t,j,3}$:

We set $\tilde{E}_{t,j,3}$ to be the restriction of the vector field $\vartheta_1$ constructed above to $W_{j,3}$.

(4) Construction of $\tilde{E}_{t,j,4}$:

Over $W_{j,4}'$, consider the continuous vector field $E_4'$ obtained by the restriction of $E$ to $\ell_t^{-1}(D_s^+) \cap \tilde{B}$, transported by the action of $\vartheta_1$. It is rugose and tangent to the strata of $S$.

Over $W_{j,4} = W_{j,4}' \cup W_{j,1}$, the vector field $E_4'$ glues with $\tilde{I}_{j,1}$, resulting a vector field $\tilde{E}_{t,j,4}$, which is continuous, rugose and non-zero on $W_{j,4} \setminus P_t$. The orbits of the points of $W_{j,4}$ by $\tilde{E}_{t,j,4}$ has limit points in $P_t$.

5.5. Fifth step: gluing all the vector fields to obtain $E_t$.

Now, considering $\psi_2$, $\psi_3$ and $\psi_4$ a partition of unity associated to $W_{j,2}$, $W_{j,3}$ and $W_{j,4}$, we obtain the vector field:

$$\tilde{E}_{t,j} := \psi_2 \tilde{E}_{t,j,2} + \psi_3 \tilde{E}_{t,j,3} + \psi_4 \tilde{E}_{t,j,4}$$

in $\ell_t^{-1}(\tilde{V}_j)$, which is continuous, rugose, non-zero on $\ell_t^{-1}(V_j) \setminus P_t$ and zero on $P_t$.

Gluing these vector fields $\tilde{E}_{t,j}$, for $j = 1, \ldots, k$, we get the vector field $\tilde{E}_t$.

Finally, gluing the vector field $\tilde{E}_t$ in $\ell_t^{-1}(V)$ and the vector field $\tilde{E}_t$ in $\ell_t^{-1}(D_t \setminus V')$ constructed in Subsection 5.1, we obtain a continuous vector field $\tilde{E}_t$ in $X_t$ with the properties (i) to (v) of Proposition 18. We just have to check that the orbits of this vector field have a limit point when the parameter goes to the infinity:
(a) If \( z \in \ell_t^{-1}(D_t \setminus V') \), the orbit of \( z \) arrives to \( W_{j,2} \cup W_{j,3} \cup W_{j,4} \) after a finite time.
(b) If \( z \in W_{j,2} \), the orbit of \( z \) arrives to \( W_{j,3} \cup W_{j,4} \) after a finite time.
(c) If \( z \in W_{j,3} \setminus W_{j,4} \), it has a limit point on \( \bigcup_{j=1}^{k} C_j \).
(d) If \( z \in W_{j,4} \setminus W_{j,3} \), it has a limit point on \( P'_t \).
(e) If \( z \in W_{j,3} \cap W_{j,4} \), we have that the orbit passing through \( z \) has a limit point that is the limit point by \( \vartheta_1 \) of the limit point of the orbit of \( z \) by \( E'_4 \). Hence this limit point is on \( P'_j = P'_t \cap \overline{C_j} \).

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