Walks and the spectral radius of graphs

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Abstract

Given a graph $G$, write $\mu (G)$ for the largest eigenvalue of its adjacency matrix, $\omega (G)$ for its clique number, and $w_k (G)$ for the number of its $k$-walks. We prove that the inequalities

$$w_{q+r} (G) \leq \mu^r (G) \leq \frac{\omega (G) - 1}{\omega (G)} w_r (G)$$

hold for all $r > 0$ and odd $q > 0$. We also generalize a number of other bounds on $\mu (G)$ and characterize pseudo-regular and pseudo-semiregular graphs in spectral terms.

Keywords: number of walks, spectral radius, pseudo-regular graph, pseudo-semiregular graph, clique number

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1 Introduction

Our graph-theoretic notation is standard (e.g., see [2]); in particular, we assume that graphs are defined on the vertex set $\{1, 2, ..., n\} = [n]$. Given a graph $G$, a $k$-walk is a sequence of vertices $v_1, ..., v_k$ of $G$ such that $v_i$ is adjacent to $v_{i+1}$ for all $i = 1, ..., k - 1$; we write $w_k (G)$ for the number of $k$-walks in $G$. The eigenvalues of the adjacency matrix $A (G)$ of $G$ are ordered as $\mu (G) = \mu_1 \geq \ldots \geq \mu_n$.

Various bounds of $\mu (G)$ in terms of $w_k (G)$ are known; the earliest one, due to Collatz and Sinogowitz [4], reads as

$$\mu (G) \geq \frac{2e (G)}{v (G)} = \frac{w_2 (G)}{w_1 (G)}.$$  \hspace{1cm} (1)

This inequality was strengthened by Hofmeister ([9], [10]) to

$$\mu^2 (G) \geq \frac{1}{v (G)} \sum_{u \in V (G)} d^2 (u) = \frac{w_3 (G)}{w_1 (G)},$$  \hspace{1cm} (2)
in turn, improved by Yu, Lu, and Tian [18] to
\[ \mu^2(G) \geq \frac{w_5(G)}{w_3(G)}, \]
and by Hong and Zhang [13] to
\[ \mu^2(G) \geq \frac{w_7(G)}{w_5(G)}. \]

In this note we prove that, in fact, the inequality
\[ \mu^r(G) \geq \frac{w_{q+r}(G)}{w_q(G)} \]
holds for all \( r > 0 \) and odd \( q > 0 \).

Let \( \omega(G) \) be the clique number of \( G \). Wilf [17] gave the bound
\[ \mu(G) \leq \frac{\omega(G) - 1}{\omega(G)} \nu(G) = \frac{\omega(G) - 1}{\omega(G)} w_1(G), \]
and Nikiforov [15] showed that
\[ \mu^2(G) \leq 2\frac{\omega(G) - 1}{\omega(G)} e(G) = \frac{\omega(G) - 1}{\omega(G)} w_2(G), \]
generalizing earlier results in [5], [7], [12], [16], and [17].

In this note we prove that, in fact, the inequality
\[ \mu^r(G) \leq \frac{\omega(G) - 1}{\omega(G)} w_r(G) \]
holds for every \( r \geq 1 \).

We generalize also a number of other upper and lower bounds on \( \mu(G) \) in terms of walks and characterize pseudo-regular and pseudo-semiregular graphs in terms of their eigenvectors.

The rest of the paper is organized as follows. In Section 2 we recall some basic notions used further, in Section 3 we investigate lower bounds on \( \mu(G) \), and in Section 4 we investigate upper bounds on \( \mu(G) \).

2 Some preliminary results

Given a graph \( G \) and a vertex \( u \in V(G) \), write \( \Gamma(u) \) for the set of neighbors of \( u \) and \( w_k(u) \) for the number of the \( k \)-walks starting with \( u \); for every two vertices \( u, v \in V(G) \), write \( w_k(u, v) \) for the number of the \( k \)-walks starting with \( u \) and ending with \( v \).

We state below some basic results related to walks in graphs.
2.1 The number of \(k\)-walks in a graph

Let \(G\) be a graph of order \(n\) with eigenvalues \(\mu_1 \geq \ldots \geq \mu_n\) and \(u_1, \ldots, u_n\) be corresponding orthogonal unit eigenvectors. For every \(i \in [n]\), let \(u_i = (u_{i1}, \ldots, u_{in})\) and set \(c_i = \left(\sum_{j=1}^n u_{ij}\right)^2\).

The number of \(k\)-walks in \(G\) (see, e.g., \[3\], p. 44, Theorem 1.10) is given as follows.

**Theorem 1** For every \(k \geq 1\), \(w_k(G) = c_1 \mu_1^{k-1} + \ldots + c_n \mu_n^{k-1}\). \(\square\)

In particular, for \(k = 1\),

\[
\sum_{i=1}^n c_i = n. \tag{3}
\]

We also list several equalities that we will use later without reference.

\[
\begin{align*}
\sum_{u \in V(G)} d^2(u) &= w_3(G); \\
\sum_{uv \in E(G)} d(u) d(v) &= w_4(G); \\
\sum_{u \in V(G)} w_p^2(u) &= w_{2p-1}(G); \\
\sum_{u \in V(G)} w_p(u) w_q(u) &= w_{p+q-1}(G); \\
\sum_{u,v \in V(G)} w_r(u,v) w_p(u) &= w_{p+r}(u); \\
\sum_{u,v \in V(G)} w_r(u,v) w_p(u) w_q(v) &= w_{p+q+r-2}(G).
\end{align*}
\]

2.2 The inequality of Motzkin and Straus

The following result of Motzkin and Straus \[14\] will be used in Section 4.

**Theorem 2** For any graph \(G\) of order \(n\) and real numbers \(x_1, \ldots, x_n\) with \(x_i \geq 0\), \((1 \leq i \leq n)\), and \(x_1 + \ldots + x_n = 1\),

\[
\sum_{ij \in E(G)} x_i x_j \leq \frac{\omega(G) - 1}{\omega(G)}. \tag{4}
\]

Equality holds iff the subgraph induced by the vertices corresponding to nonzero entries of \(x\) is a complete \(\omega(G)\)-partite graph such that the sum of the \(x_i\)'s in each part is the same. \(\square\)

Wilf \[17\] was the first to apply inequality \[3\] to graph spectra, obtaining, in particular, the following result.

**Theorem 3** Let \(x = (x_1, \ldots, x_n)\) be an eigenvector to \(\mu(G)\) with \(\|x\| = 1\). Then

\[
\mu(G) = \sum_{ij \in E(G)} x_i x_j \leq \frac{\omega(G) - 1}{\omega(G)} \left(\sum_{i=1}^n x_i\right)^2. \tag{5}
\]

\(\square\)
It is rather entertaining to find the connected graphs for which equality holds in (5).

We note without a proof that for $G = K_{4n,4n,n}$ equality holds in (4) - it is enough to consider the vector $x = (x_1, ..., x_{9n})$ defined as

$$x_i = \begin{cases} (12n)^{-1/2}, & 1 \leq i \leq 8n \\ (3n)^{-1/2}, & 8n < i \leq 9n \end{cases}.$$

Here we state only a partial result.

**Theorem 4** Let $G$ be a connected graph and $x = (x_1, ..., x_n)$ be a unit eigenvector to $\mu(G)$ such that

$$\mu(G) = \frac{\omega(G) - 1}{\omega(G)} \left( \sum_{i=1}^{n} x_i \right)^2.$$

Then $G$ is a complete $\omega(G)$-partite graph.

**Proof** Since $G$ is connected, $x_i > 0$ for every $i \in [n]$. The assertion follows from the case of equality in (4). $\square$

## 3 Lower bounds on $\mu(G)$

Given a graph with no isolated vertices and a vertex $v$, call the value $\sum_{u \in \Gamma(v)} d(v)/d(u)$ the **average degree** of $u$. A graph $G$ with no isolated vertices is called:

- **pseudo-regular** if its vertices have the same average degree;
- **semiregular** if it is bipartite and vertices belonging to the same part have the same degree;
- **pseudo-semiregular** if it is bipartite and vertices belonging to the same part have the same average degree.

In this section we first prove Theorem 5 and then show that its hypothesis cannot be relaxed. Next we describe pseudo-regular and pseudo-semiregular graphs in terms of their eigenvectors, and finally we extend two other lower bounds on $\mu(G)$.

The following theorem generalizes results stated in [18] and [13].

**Theorem 5** For every graph $G$,

$$\mu^r(G) \geq \frac{w_{q+r}(G)}{w_q(G)}$$

for all $r > 0$ and odd $q > 0$.

If $q > 1$, equality holds in (6) if and only if each component of $G$ has spectral radius $\mu(G)$ and is pseudo-regular or, if $r$ is even, pseudo-semiregular.

If $q = 1$, equality holds in (6) if and only if each component of $G$ has spectral radius $\mu(G)$ and is regular or, if $r$ is even, semiregular.
Proof Let \( v(G) = n \). Theorem 1 implies (4) by

\[
\frac{w_{q+r}(G)}{w_q(G)} = \frac{\sum_{i=1}^{n} c_i \mu_i^{q+r-1}}{\sum_{i=1}^{n} c_i \mu_i^{q-1}} = \mu^r(G) \frac{\sum_{i=1}^{n} c_i \left( \frac{\mu_i}{\mu_1} \right)^{q+r-1}}{\sum_{i=1}^{n} c_i \left( \frac{\mu_i}{\mu_1} \right)^{q-1}} \leq \mu^r(G). \tag{7}
\]

Suppose now that

\[
\mu^r(G) = \frac{w_{q+r}(G)}{w_q(G)}. \tag{8}
\]

Assume first that \( G \) is connected and let \( M \) be the set of all \( i \in [2,n] \) such that \( c_i \neq 0 \) and \( \mu_i \neq 0 \). We shall show that if \( G \) is nonbipartite, then \( M = \emptyset \). From (7) we find that

\[
\sum_{i=2}^{n} c_i \left( \frac{\mu_i}{\mu_1} \right)^{q+r-1} = \sum_{i=2}^{n} c_i \left( \frac{\mu_i}{\mu_1} \right)^{q-1}, \tag{9}
\]

and so, \( |\mu_i| = \mu_1 \) for every \( i \in M \), contradicting that \( G \) connected and nonbipartite.

Hence, \( w_k(G) = c_1 \mu_1^{k-1} \) for every \( k > 0 \). In particular, \( w_4(G) = \sqrt{w_3(G) w_5(G)} \), and so

\[
\sum_{u \in V(G)} d(u) w_3(u) = w_4(G) = \sqrt{w_3(G) w_5(G)} = \sqrt{\sum_{u \in V(G)} d^2(u) \sum_{u \in V(G)} w_2^2(u)}.
\]

The condition for equality in Cauchy-Schwarz inequality implies that \( w_3(u)/d(u) \) is constant for all vertices \( u \), i.e., that \( G \) is pseudo-regular.

If \( q = 1 \), then (3) implies that \( c_i = 0 \) for all \( 1 < i \leq n \); hence \( c_1 = n, \mu_1 = w_2(G)/n \), and so \( G \) is regular.

Let now \( G \) be bipartite. Since the spectrum of \( G \) is symmetric with respect to 0, from (3) it follows that either \( M = \emptyset \) or \( M = \{n\} \). If \( M = \emptyset \) (i.e., \( c_n = 0 \)), the case reduces to the previous one. If \( c_n > 0 \), equality (3) may hold only if \( r \) is even. Also, we have

\[
\sum_{u \in V(G)} d^2(u) = w_3(G) = c_1 \mu_1^2 + c_n \mu_1^2 = (c_1 + c_n) \mu_1^2,
\]

\[
\sum_{u \in V(G)} d(u) w_4(u) = w_5(G) = c_1 \mu_1^4 + c_n \mu_1^4 = (c_1 + c_n) \mu_1^4,
\]

\[
\sum_{u \in V(G)} w_4^2(u) = w_7(G) = c_1 \mu_1^6 + c_n \mu_1^6 = (c_1 + c_n) \mu_1^6.
\]

Therefore,

\[
\sum_{u \in V(G)} d^2(u) \sum_{u \in V(G)} w_4^2(u) = \sum_{u \in V(G)} d(u) w_4(u);
\]

the condition of equality in Cauchy-Schwarz’s inequality implies that \( w_4(u)/d(u) = w_4(v)/d(v) \) for every \( u, v \in V(G) \). We borrow the following argument from [9]. Letting \( u \) to be a vertex of minimum average degree \( w_3(u)/d(u) = \delta \) and \( v \) be a vertex of
maximum average degree \( \frac{w_3(v)}{d(v)} = \Delta \), we see that
\[
\frac{w_4(u)}{d(u)} = \frac{\sum_{t \in \Gamma(u)} w_3(t)}{d(u)} \leq \frac{\Delta \sum_{t \in \Gamma(u)} d(t)}{d(u)} = \Delta \delta
\]
\[
= \delta \frac{\sum_{t \in \Gamma(v)} d(t)}{d(v)} \leq \sum_{t \in \Gamma(v)} w_3(t) \leq \frac{w_4(v)}{d(v)} = \frac{w_4(u)}{d(u)};
\]
thus, every vertex of average degree \( \delta \) is adjacent only to vertices of average degree \( \Delta \) and vice versa. Since \( G \) is connected, it follows that it is pseudo-semiregular.

If \( q = 1 \), then
\[
\sum_{i=2}^{n-1} c_i \left( \frac{\mu_i}{\mu_1} \right)^r = \sum_{i=2}^{n-1} c_i,
\]
and so \( c_i = 0 \) for all \( 1 < i < n \); hence, from (\[\text{[raw]}\]), \( c_1 + (-1)^r c_n = n \). If \( c_n = 0 \), the case reduces to the previous one. Otherwise, \( r \) is even and so \( \mu_1^2 n = w_3(G) \). Since
\[
\mu^2(G) > \frac{1}{n} \|-A^2\|_1 = \frac{1}{n} w_3(G),
\]
unless all row sums of \( A^2(G) \) are equal, we deduce that \( w_3(u) \) is constant for every \( u \), and so \( G \) is semiregular.

If the graph is not connected, say let \( G_1, ..., G_k \) be its components, we have
\[
\mu^r(G) = \frac{\sum_{i=1}^k w_{q+r}(G_i)}{\sum_{i=1}^k w_q(G_i)} \leq \frac{\sum_{i=1}^k \mu^r(G_i) w_q(G_i)}{\sum_{i=1}^k w_q(G_i)} \leq \frac{\mu^r(G) \sum_{i=1}^k w_q(G_i)}{\sum_{i=1}^k w_q(G_i)} \leq \mu^r(G).
\]
Thus, (\[\text{[raw]}\]) implies that \( \mu^r(G) = \mu^r(G_i) = \frac{w_{q+r}(G_i)}{w_q(G_i)} \) for each component of \( G_i \).

We omit the straightforward proof of the converse of the case of equality. \( \square \)

### 3.1 The case of even \( q \)

Observe that if \( G \) is connected and nonbipartite, then the ratio \( \frac{w_{q+r}(G)}{w_q(G)} \) tends to \( \mu^r(G) \) as \( q \) tends to infinity. Indeed, from (\[\text{[raw]}\]) and \( |\mu_i|/\mu_1 < 1 \) holding for every \( i = 2, ..., n \), we obtain the following theorem.

**Theorem 6** For every connected nonbipartite graph \( G \) and every \( \varepsilon > 0 \), there exists \( q_0(\varepsilon) \) such that if \( q > q_0(\varepsilon) \) then
\[
(1 - \varepsilon) \frac{w_{q+r}(G)}{w_q(G)} \leq \mu^r(G) \leq (1 + \varepsilon) \frac{w_{q+r}(G)}{w_q(G)}
\]
for every \( r > 0 \).
Inequality (6) may fail for \( q \) even as shown by the following example for \( q = 2k \) and odd \( r \). Let \( 0 < a < b \) be integers and \( G = K_{a,b} \) be the complete bipartite graph with parts of size \( a \) and \( b \). We see that

\[
\begin{align*}
    w_{2k} (G) &= 2a^k b^k, \\
    w_{2k+r} (G) &= a (ba)^{k+(r-1)/2} + b (ba)^{k+(r-1)/2}, \\
    \frac{w_{2k+r} (G)}{w_{2k} (G)} &= \frac{a + b}{2} (ba)^{(r-1)/2} > (ab)^{r/2} = \mu^r (G).
\end{align*}
\]

Therefore, for bipartite \( G \), \( q \) even and \( r \) odd, \( \mu^r (G) \) may differ considerably from \( w_{q+r} (G) / w_q (G) \), no matter how large \( q \) is. We are not able to answer the following natural question.

**Problem 7** Let \( G \) be a connected bipartite graph. Is it true that

\[
\mu^r (G) \geq \frac{w_{q+r} (G)}{w_q (G)}
\]

for every even \( q \geq 2 \) and \( r \geq 2 \)?

We also note without a proof that the graph \( G = K_{2t,2t} \) satisfies \( \mu^2 (G) < w_4 (G) / w_2 (G) \).

### 3.2 Characterization of pseudo-regular and pseudo-semiregular graphs

Write \( \mathbf{i} \) for the vector \((1, \ldots, 1) \in \mathbb{R}^n\). As a by-product of the proof of Theorem 5 we obtain characterizations of pseudo-regular and pseudo-semiregular graphs.

**Theorem 8** If \( G \) is a pseudo-regular graph and \( \mu_s \) is an eigenvalue of \( G \) such that \( 0 < |\mu_s| < \mu (G) \), then every eigenvector to \( \mu_s \) is orthogonal to \( \mathbf{i} \). If \( G \) has no bipartite component, then the converse is also true.

**Proof** Let \( v (G) = n, \ u_1, \ldots, u_n \) be orthogonal unit eigenvectors of \( G \) to \( \mu_1, \ldots, \mu_n \) and \( c_1, \ldots, c_n \) be as defined in Section 2.1. Suppose \( 0 < |\mu_s| < \mu_1 \). If \( G \) is pseudo-regular, then \( w_4 (G) = \sqrt{w_3 (G) w_5 (G)} \) and so

\[
\sum_{i=1}^{n} c_i \left( \frac{\mu_i}{\mu_1} \right)^3 = \sqrt{\sum_{i=1}^{n} c_i \left( \frac{\mu_i}{\mu_1} \right)^2 \sum_{i=1}^{n} c_i \left( \frac{\mu_i}{\mu_1} \right)^4}.
\]

The condition for equality in Cauchy-Schwarz’s inequality implies that \( |\mu_i / \mu_1| = \mu_1 / \mu_1 = 1 \) whenever \( |\mu_i| > 0 \) and \( c_i > 0 \). Hence, \( c_s = (\sum_{i=1}^{n} u_{is})^2 = 0 \), i.e., \( u_s \) is orthogonal to \( \mathbf{i} \).

If \( G \) has no bipartite component, we see that

\[
w_k (G) = \mu^{k-1} (G) \sum_{\mu_i = \mu (G)} c_i
\]
for every \( k \geq 1 \). In particular, \( w_3(G) \mu^2(G) = w_5(G) \); from the case of equality of Theorem 3 we see that \( G \) is pseudo-regular, completing the proof.

**Theorem 9** Let \( G = G(n) \) be a bipartite graph with eigenvalues \( \mu_1 \geq \ldots \geq \mu_n \). If \( G \) is pseudo-semi-regular, then for all \( s \in [n] \) such that \( 0 < |\mu_s| < \mu(G) \) every eigenvector to \( \mu_s \) is orthogonal to \( i \). If \( G \) is connected, the converse is also true.

**Proof** Let \( u_1, \ldots, u_n \) be orthogonal unit eigenvectors to \( \mu_1, \ldots, \mu_n \), and \( c_1, \ldots, c_n \) be as defined in Section 2.1. If \( G \) is pseudo-semi-regular, \( w_4(u)/d(u) = w_4(v)/d(v) \) for every \( u, v \in V(G) \). Letting \( t = w_4(u)/d(u) \), we see that \( w_{2k+3}(u) = tw_{2k+1}(u) \) for every integer \( k > 0 \). Hence, \( t = \mu^2(G) \) and so \( w_5(G) = \mu^2(G) w_3(G) \), implying in turn

\[
\sum_{i=2}^{n-1} c_i \left( \frac{\mu_i}{\mu_1} \right)^4 = \sum_{i=2}^{n-1} c_i \left( \frac{\mu_i}{\mu_1} \right)^2.
\]

We see that \( c_s = 0 \) for every \( s \) such that \( 0 < |\mu_s| < \mu(G) \), and so \( u_s \) is orthogonal to \( i \).

If \( G \) is connected and for every \( s \) such that \( 0 < |\mu_s| < \mu(G) \) every eigenvector to \( \mu_s \) is orthogonal to \( i \), then

\[
w_k(G) = \left( c_1 + (-1)^{k-1} c_n \right) \mu^{k-1}(G),
\]

for every integer \( k > 0 \). In particular, \( w_5(G) = \mu^2(G) w_3(G) \); from the case of equality in Theorem 3 it follows that \( G \) is pseudo-semi-regular.

### 3.3 More lower bounds

A common device for finding lower bounds on \( \mu(G) \) is the Rayleigh principle applied with carefully chosen vectors.

Let \( p \geq 0 \), \( r \geq 1 \) be integers and \( G \) be a graph of order \( n \) with no isolated vertices. Setting \( x_i = w_p(i)/\sqrt{w_{2p-1}(G)} \) for all \( i \in [n] \) and letting \( x = (x_1, \ldots, x_n) \), the Rayleigh principle gives another proof of inequality (6) by

\[
\mu^r(G) \geq \langle A^r(G) x, x \rangle = \frac{1}{w_{2p-1}(G)} \sum_{u,v \in V(G)} w_{r+1}(u,v) w_p(u) w_p(v) = \frac{w_{2p+r-1}(G)}{w_{2p-1}(G)}.
\]

Set now \( y_i = \sqrt{w_p(i)/w_p(G)} \) for all \( i \in [n] \) and let \( y = (y_1, \ldots, y_n) \). By the Rayleigh principle we obtain the following general bound

\[
\mu^r(G) \geq \langle A^r(G) y, y \rangle = \frac{1}{w_p(G)} \sum_{u,v \in V(G)} w_{r+1}(u,v) \sqrt{w_p(u) w_p(v)}.
\] (10)
Since by Cauchy-Schwarz’s inequality we have

\[
\sum_{u,v \in V(G)} \frac{w_{r+1}(u,v) \sqrt{w_p(u) w_p(v)}}{\sqrt{w_p(u) w_p(v)}} \geq \left( \sum_{u,v \in V(G)} w_{r+1}(u,v) \right)^2 = w_{r+1}^2(G),
\]

inequality (10) implies also that

\[
\mu^r(G) \sum_{u,v \in V(G)} \frac{w_{r+1}(u,v)}{\sqrt{w_p(u) w_p(v)}} \geq \frac{w_{r+1}^2(G)}{w_p(G)}. \tag{11}
\]

Setting \( p = 2, r = 1 \), we obtain the following inequalities proved by Favaron, Mahéo, and Saclé [8], and in a wider context also by Hoffman, Wolfe, and Hofmeister [11],

\[
\mu(G) \geq \frac{1}{2e(G)} \sum_{u \in V(G)} \sqrt[d(u)]{d(v)}, \tag{12}
\]

\[
\mu(G) \geq \frac{2e(G)}{\sum_{u \in V(G)} \sqrt[d(u)]{d(v)}}. \tag{13}
\]

As shown in [8] and [11] equality holds in (12) and (13) iff \( G \) is regular or semiregular. The case of equality in (10) and (11) is an open question.

### 4 Upper bounds on \( \mu(G) \)

In this section we present two general upper bounds on \( \mu(G) \). Theorem 12 below gives the first bound in terms of the clique number and the number of walks. The bound of the second type is given in Section 4.1. The proof of Theorem 12 relies on two simple preliminary results.

**Lemma 10** For every \( r > 0 \) and every graph \( G \),

\[
w_{2r}(G) \leq \frac{\omega(G) - 1}{\omega(G)} w_r^2(G).
\]

**Proof** Indeed, we have

\[
w_{2r}(G) = \sum_{uv \in E(G)} w_r(u) w_r(v) \leq \frac{\omega(G) - 1}{\omega(G)} \left( \sum_{u \in V(G)} w_r(u) \right)^2 = \frac{\omega(G) - 1}{\omega(G)} w_r^2(G).
\]

Applying Lemma 10 several times, we generalize it as follows.
Corollary 11 For every graph $G$ and $k, r > 0$,
\[
\frac{\omega(G) - 1}{\omega(G)} w_{2k \cdot r} (G) \leq \left( \frac{\omega(G) - 1}{\omega(G)} w_r (G) \right)^{2k}.
\]

We are ready now to prove the main result of this section.

Theorem 12 For every graph $G$ and $r \geq 1$,
\[
\mu^r (G) \leq \frac{\omega(G) - 1}{\omega(G)} w_r (G) \quad (14)
\]

Proof Clearly it suffices to prove inequality (14) for connected graphs. We shall assume first that $G$ is nonbipartite. Assume that (14) fails, i.e.,
\[
\mu^r (G) > (1 + c) \frac{\omega(G) - 1}{\omega(G)} w_r (G)
\]

for some $G, r > 0, c > 0$. Then, by Corollary 11 for every $k > 0$,
\[
\mu^{2k \cdot r} (G) > \left( (1 + c) \frac{\omega(G) - 1}{\omega(G)} w_r (G) \right)^{2k} \geq (1 + c)^{2k} \frac{\omega(G) - 1}{\omega(G)} w_{2k \cdot r} (G). \quad (15)
\]

Note that Theorem 1 implies that for every $\varepsilon$,
\[
c_1 \mu^{q - 1} (G) < (1 + \varepsilon) w_q (G) \quad (16)
\]

for all sufficiently large $q$. Hence, for $q = 2k \cdot r$ and $k$ sufficiently large, Theorem 8 and inequality (16) imply that
\[
\mu^{2k \cdot r} (G) \leq \frac{\omega(G) - 1}{\omega(G)} c_1 \mu^{2k \cdot r - 1} (G) < (1 + \varepsilon) \frac{\omega(G) - 1}{\omega(G)} w_{2k \cdot r} (G),
\]
contradicting (15).

Finally we have to prove (14) for bipartite $G$. Then $\omega(G) = 2$, so we have to prove that $\mu^r (G) \leq w_r (G) / 2$ for every $r \geq 2$. If $r$ is odd, Theorem 8 and Theorem 11 imply
\[
\mu^r (G) \leq \frac{1}{2} c_1 \mu^{r - 1} \leq \frac{1}{2} w_r (G).
\]

Let now $r$ be even. Write $cw_k (G)$ for the number of closed walks on $k$ vertices in $G$ (i.e., $k$-walks with the same start and end vertex.) It is known that
\[
cw_{k+1} (G) = tr (A^k (G)) = \mu_1^k + \ldots + \mu_n^k. \quad (17)
\]

The spectrum of bipartite graphs is symmetric with respect to 0, thus $2\mu^r (G) \leq cw_{r+1} (G) \leq w_r (G)$, completing the proof. \hfill \Box
Theorem 13 Suppose that $G$ is a graph such that equality holds in (14) for some $r \geq 1$. If $r = 1$, then $G$ is a regular complete $\omega(G)$-partite graph. If $r > 1$, then $G$ has a single nontrivial component $G_1$. If $\omega(G) > 2$, then $G_1$ is a regular complete $\omega(G)$-partite graph. If $\omega(G) = 2$, then $G_1$ is a complete bipartite graph, and if $r$ is odd, then $G_1$ is regular.

Proof Assume

$$
\mu^r(G) = \frac{\omega(G) - 1}{\omega(G)} w_r(G)
$$

and let $c_i$ be defined as in Section 2.1.

If $r = 1$ then

$$
\frac{\omega(G) - 1}{\omega(G)} v(G) = \mu(G) \leq \sqrt{2 \frac{\omega(G) - 1}{\omega(G)} e(G)};
$$

from the case of equality in Turán’s theorem (see, e.g., [2]) it follows that $G$ is regular complete $\omega(G)$-partite graph.

Assume now $r \geq 2$; let $G_1$ be a component of $G$ with $\mu(G) = \mu(G_1)$. If $G_2$ is another nontrivial component of $G$, then

$$
\mu^r(G_1) = \mu^r(G) = \frac{\omega(G) - 1}{\omega(G)} \left( w_r(G_2) + w_r(G_1) \right) > \frac{\omega(G) - 1}{\omega(G)} w_r(G_1),
$$

a contradiction; thus $G_1$ is the only nontrivial component of $G$. We also see that the equality (18) holds for $G_1$, so for simplicity we shall assume that $G$ is connected. From Corollary 11 and (18) we deduce that

$$
\mu^{2kr}(G) = \frac{\omega(G) - 1}{\omega(G)} w_{2kr}(G) = \frac{\omega(G) - 1}{\omega(G)} \mu^{2kr-1}(G) \sum_{i=1}^{n} c_i \left( \frac{\mu_i}{\mu_1} \right)^{2k-1} \quad (19)
$$

for every integer $k > 0$. Assume $G$ is nonbipartite; therefore, $|\mu_n(G)| < \mu(G)$ and, letting $k$ tend to infinity, we find that

$$
\mu(G) = \frac{\omega(G) - 1}{\omega(G)} c_1.
$$

From Theorem 4 it follows that $G$ is a complete $\omega(G)$-partite graph, and thus $G$ has no positive eigenvalues other than $\mu(G)$. Hence, from (19), any $c_i$ corresponding to a negative eigenvalue must be 0. Therefore,

$$
n = w_1(G) = c_1 \mu(G) = \frac{\omega(G)}{\omega(G) - 1} \mu(G),
$$

a case that is settled above.

Let now $G$ be bipartite. If $r$ is odd, we have

$$
\mu^r(G) = \frac{1}{2} w_r(G) = \frac{1}{2} \sum_{i=1}^{n} c_i (\mu_i)^{r-1};
$$
so, by Theorem 3, \( c_1 = 1/2 \). Moreover, either \( c_i = 0 \) or \( \mu_i = 0 \) for \( i = 2, \ldots, n \). We have again

\[
n = w_1 (G) = c_1 \mu (G) = 2 \mu (G),
\]

implying that \( G \) is a regular complete bipartite graph.

For even \( r \) we have

\[
2 \mu^r (G) \leq cw_{r+1} (G) \leq w_r (G) = 2 \mu^r (G),
\]

and, in view of (17), we conclude that \( G \) has only two nonzero eigenvalues - \( \mu_1 \) and \( \mu_n \). Hence, in our case, Smith’s theorem implies that \( G \) is a complete bipartite graph. \( \square \)

### 4.1 More upper bounds

It is known that the Perron root of a nonnegative matrix does not exceed its maximal row sum. This idea has been exploited to obtain the following bounds

\[
\mu (G) \leq \max_{u \in V (G)} \sqrt{w_3 (u)}, \quad (20)
\]

\[
\mu (G) \leq \max_{u \in V (G)} \frac{w_3 (u)}{d (u)}, \quad (21)
\]

\[
\mu (G) \leq \max_{uv \in E (G)} \sqrt{\frac{d (u) d (v)}{w_3 (u) w_3 (v)}}, \quad (22)
\]

\[
\mu (G) \leq \max_{uv \in E (G)} \sqrt{\frac{w_3 (u) w_3 (v)}{d (u) d (v)}}, \quad (23)
\]

Inequalities (20) and (21) are proved in [8], inequality (22) is proved in [1], and inequality (23) in [6]. As an attempt to interrupt this monotonic sequence we propose the following general result.

**Theorem 14** For every integers \( p \geq 1, r \geq 1 \) and any graph \( G \),

\[
\mu^r (G) \leq \max_{u \in V (G)} \frac{w_{r+p} (u)}{w_p (u)}.
\]

**Proof** Set \( b_n = w_p (i) \) for each \( i \in [n] \) and let \( B \) be the diagonal matrix with main diagonal \((b_1, \ldots, b_n)\). Since \( B^{-1}A^r (G)B \) has the same spectrum as \( A^r (G) \), \( \mu^r (G) \) is bounded from above by the maximum row sum of \( B^{-1}A^r (G)B \) - say the sum of the \( k \)th row - and so,

\[
\mu^r (G) \leq \sum_{v \in V (G)} w_r (k, v) \frac{w_p (v)}{w_p (k)} = \frac{w_{r+p} (k)}{w_p (k)} \leq \max_{u \in V (G)} \frac{w_{r+p} (u)}{w_p (u)}.
\]

\( \square \)
Setting \( p = 1, r = 2 \), we obtain (20); the case \( p = 2, r = 1 \) implies (21). Furthermore, (22) follows from (20) by

\[
\mu^2(G) \leq \max_{u \in V(G)} w_3(u) = \max_{u \in V(G)} d(u) \left( \frac{1}{d(u)} \sum_{v \in \Gamma(u)} d(v) \right) \leq \max_{uv \in E(G)} d(u) d(v),
\]

and (23) follows by

\[
\mu^2(G) \leq \max_{u \in V(G)} \frac{w_4(u)}{d(u)} = \max_{u \in V(G)} \frac{w_3(u) w_4(u)}{d(u) w_3(u)} \leq \max_{u \in V(G)} \frac{w_3(u)}{d(u)} \left( \frac{\sum_{v \in \Gamma(u)} w_3(v)}{\sum_{v \in \Gamma(u)} d(v)} \right) \leq \max_{uv \in E(G)} \frac{w_3(u)}{d(u)} \frac{w_3(v)}{d(v)}
\]

with plenty of room.

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