Can Our Spacetime Emerge from Anti–de Sitter Space?

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Abstract—We present a model showing that our four-dimensional spacetime with the signature (\(+, -, -, -\)) and almost vanishing positive curvature may have originated from a \(b\)-ary tree-like branching of a single discrete entity, and the \(AdS_5\) space related to this branching process.

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The standard cosmology, the \(\Lambda\)CDM model, is based on the Big Bang theory. The latter explains the existence of our observable Universe by a sequential expansion of a hot and high-density matter from quantum gravity scales to modern size of the Universe [1]. At quantum gravity scales the spacetime is believed to be discrete, possibly in a form of spin network [2, 3]. For this reason the importance of the question of how our continuous differentiable spacetime has emerged from a discrete one cannot be overestimated [2].

The “reverse engineering” of the expanding Universe driven by dark energy, cold dark matter and the ordinary matter, according to the Einstein equations

\[
R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = -\kappa T_{\alpha\beta},
\]

leads to the Big Bang singularity.

It is possible, however, to put the problem up side down. Suppose we have a single entity, which breaks up into fragments according to a tree-like graph (see Fig. 1). Can we construct a four-dimensional de Sitter spacetime with the Minkowski signature by doing so for some given value of branching factor \(b\)?

The \(b\)-ary trees, i.e., the trees with branching factor \(b\), are intimately related with hyperbolic spaces. Indeed, if we define the “circumference” of a tree as the number of nodes at the distance of exactly \(r\) hops from the root, this circumference is proportional to \(b^r\) [4]. At the sufficiently large \(r\), this dependence coincides with the circle length in the hyperbolic space \(l(r) = 2\pi \sinh(\zeta r)\), if we admit \(\zeta = \ln b\). Strictly speaking, the regularity of the tree, i.e., the exact self-similarity of the branching process, is not required: as soon as some classification of nodes, elements of a set, is at least approximately a tree the corresponding space is negatively curved [5]. This fact is widely used in studies of different networks in computer science, social networks, etc., see, e.g. [4] and references therein.

The Friedmann–Lemaître–Robertson–Walker (FLRW) solution of the Einstein equations (1) provides a homogeneous and isotropic metrics

\[
dl^2 = c^2 dt^2 - S^2(t),
\]

which determines the expanding \((k > 0, \Lambda = 0)\) Universe in terms of an ODE system

\[
\begin{align*}
2\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2} &= \frac{8\pi G}{c^2} T_i^i, \quad i = 1, 3, \\
\frac{\dot{S}^2 + kc^2}{S^2} &= \frac{8\pi G \rho_0}{3c^2},
\end{align*}
\]

Fig. 1. Self-similar fragmentation of a single entity (\(\ast\)) into \(b\) parts according to a \(b\)-ary tree. The case of \(b = 2\) is shown.
where $T^\beta_\alpha$ is the energy-momentum tensor of the matter fields, and the upper dots mean the derivatives with respect to the time argument $t$, as usual.

The $S(t)$ function is understood as an expansion rate of the Universe, experienced by any local observer at time $t$ anywhere in space. However, beyond Eq. (2)-model it is not self-evident that the expansion rate should be a unique function of local time. For instance, the Universe may be connected, but not simply-connected [6], and the evaluation of $S$ along different paths may give different results (see Fig. 2). For this reason, it just seems more general to consider the radius (or the curvature) of the Universe as an independent coordinate, which traces the global history of the Universe.

In this sense, we may live on a four-dimensional boundary of a five-dimensional gravity, which arises, for instance, in the low-dimensional limit of heteroic string theory [7, 8]. If the Universe expansion is ceased (gedanken experiment!), our local spacetime should be Minkowskian for we observe the speed of light to be the same in any inertial frame of reference.

Our goal is to explain the positively curved local spacetime with the signature $+−−−$ in terms of some global branching process. The situation resembles the inflation of a balloon, the surface of which is our spacetime. The changes in the matter fields, living on the balloon’s surface, take place due to: (i) local time evolution, taking place in the local frame of reference; (ii) branching processes, which result in the balloon expansion. Thus the number of “hops” from the beginning of inflation plays the role similar to the local time, but not identical to it. The measurements of local time intervals are accessible to the local observers on the balloon’s surface, but the number of hops, i.e., the bulk geometry, is not accessible, unless somebody stops the inflation by ceasing the branching process at a given number of hops from the root. In such a case our toy Universe will be frozen at a surface of constant curvature.

Since the above described dynamical structure is a tree-like, it should be described by a negatively curved five-dimensional space—we need an extra “time” coordinate $\rho$ to describe the branching. Such five-dimensional space can be constructed by the embedding of the two-time analogue of the Minkowski sphere

$$x_0^2 + x_0^2 - \sum_{i=1}^{4} x_i^2 = L^2$$

into 6D Minkowski space with the metrics

$$dl^2 = dx_0^2 + dx_0^2 - \sum_{i=1}^{4} dx_i^2.$$

The resulting space is known as $AdS_5$ and is widely known for $AdS/CFT$ correspondence in quantum field theory [9, 10]. The $AdS_5$ space is invariant under $SO(2, 4)$ symmetry group, which provides a natural choice of coordinates on it:

$$x_0 = L \cosh \rho \cos \tau,$$
$$x_0 = L \cosh \rho \sin \tau,$$
$$x_i = L \sinh \rho \Omega_i, \quad i = 1, 4,$$

where $\Omega_i$ are the coordinates on the unit sphere in four dimensions. The hyperbolic coordinate $\rho$ is the branching coordinate we are looking for. So, we expect the slice $\rho = \text{const}$ to be our four-dimensional spacetime frozen at a given branching step.

Casting the metrics on $AdS_5$ in the coordinates (5) and assuming $L = 1$ hereafter, we get:

$$dl^2 = \cosh^2 \rho d\tau^2 - \rho^2 - \sinh^2 \rho d\Omega_3^2,$$
$$d\Omega_3^2 = d\chi^2 + \sin^2 \chi d\theta^2 + \sin^2 \chi \sin^2 \theta d\phi^2,$$

where $d\Omega_3^2$ is the squared interval on the the unit sphere $S^3$:

$$x_4 = \cos \chi, \quad x_1 = \sin \chi \cos \theta,$$
$$x_2 = \sin \chi \sin \theta \cos \phi, \quad x_3 = \sin \chi \sin \theta \sin \phi.$$

Using the coordinates $[\tau, \chi, \theta, \phi]$ with the metrics (6) on the ($\rho = \text{const}$)-hypersurface we derive the local metrics of the four-dimensional universe:
The Christoffel symbols for the metrics (7) are:

\[
\Gamma^\chi_{00} = -\cos \chi \sin \chi, \quad \Gamma^\chi_{0\phi} = -\cos \chi \sin \chi \sin^2 \theta,
\]

\[
\Gamma^0_{\chi\phi} = \cot \chi, \quad \Gamma^\phi_{0\phi} = -\cos \theta \sin \theta, \quad \Gamma^\phi_{00} = \cot \theta.
\]

Metrics (7) defines a positively-curved spacetime with the non-zero components of the Riemann tensor being

\[
R^{\theta\chi}_{\chi\phi} = R^{\theta\phi}_{\chi\phi} = 1,
\]

\[
R^{\phi\chi}_{00} = -R^{\phi\theta}_{00} = -\sin^2 \chi,
\]

\[
R^{\phi\theta}_{00} = -R^{\phi\phi}_{00} = -\sin^2 \chi \sin^2 \theta,
\]

where

\[
R^j_{\ell jk} = \Gamma^j_{\ell k,j} - \Gamma^j_{\ell j,k} + \Gamma^m_{\ell j} \Gamma^j_{mj} - \Gamma^m_{\ell k} \Gamma^j_{mk}.
\]

The scalar curvature \( R = g^{ik} R_{ik} \) is positive for the metrics (7):

\[
R = \frac{6}{\sinh^2 \rho}, \quad \text{being}
\]

\[
R^{\chi\chi} = -2, \quad R^{\phi0} = -2 \sin^2 \chi, \quad R^{\phi\phi} = -2 \sin^2 \chi \sin^2 \theta.
\]

The non-zero components of the Einstein’s tensor

\[
G^{\chi\chi} = g^{\chi\chi} R - \frac{1}{2} g^{\chi\chi} R
\]

have the form

\[
G^{\chi\chi}_\chi = -\frac{3}{\sinh^2 \rho}, \quad G^{\phi\phi} = G^{\phi0} = G^{\phi\chi} = -\frac{1}{\sinh^2 \rho}.\quad (9)
\]

In the limit of \( \rho \to \infty \) the curvature (8) vanishes and we yield a flat Minkowski space, free of matter fields. For any finite value of \( \rho \) the Einstein tensor (9) implies the expansion of the Universe due to the initial branching process. From the dimensional consideration we infer that for an \( AdS_n \) space the branching factor is \( b = n \), and hence \( b = 5 \) in our case.

This is not a strict proof, but if we make Euclidean change \( \tau = it \) in (6), we can map the (\( \rho = \text{const} \))-hypersurface onto \( S^4 \) sphere. The \( S^4 \) sphere admits a tessellation with four-dimensional simplices, each of those has five vertices and five hyperfaces, and can be subdivided into five parts according to the branches of a \((b = 5)\)-tree. This corresponds to the \((b = 5)\)-tree fragmentation.

The charm of the \( AdS/CFT \)-correspondence is due to the fact that quantum theory gravity can be described using conformal field theory on the boundary, which is a flat space. Classically, conformal invariance can be understood as a scaling invariance: the fields behave identically at different scales, as it happens for fractals, generated by a tree-like fragmentation process [11]. In quantum field theory the conformal invariance is possible for massless fields. Thus the gravity is a kind of memory of such self-similar fragmentation, a kind of hologram of our world on the boundary [12, 13].

Our approach extends this point of view. Instead of considering a flat boundary only, we consider a collection of different slices of the discrete fragmentation process, labelled by the number of hops \( r \) from the root of the branching tree, or the hyperbolic coordinate \( \rho \) in the continuous approximation. Each slice represents our four-dimensional universe at a frozen instant of inflation. To understand the dynamics of the whole Universe we need not only the dynamics on current slice with respect to the local time \( \tau \), but also the history of the fragmentation process, i.e. which regions of the previous slices were the “parents” of the current slice regions. This is deeply related to the entanglement entropy [14], and can be considered only within the framework of quantum field theory. In our approach the “static” solution \( \rho = \text{const} \) does make sense, since we can fill our four-dimensional space-time with quantum fields \( \phi(t, x, y, z) \) and consider their dynamics with respect to the local time \( t \).

We do not neglect the standard FLRW solution here, instead, using the Weyl postulate on the trajectories from the Big Bang, we can study the case of

\[
dS = \frac{dS}{d\tau} d\tau, \quad (10)
\]

where

\[
S = \sinh \rho, \quad (11)
\]

according to the metrics (6). Equality (10) in our context is rather strong physical assertion. It says that any change in the local time \( \tau \), and hence any evolution of local physical fields, happens synchronously and due to the changes in \( \rho \) —the Universe expansion in our case. We are not going to dig into philosophical problems related to the assumption (10), instead we have to think of it as a strict fact. 

Therefore, the ricci tensor is

\[
R^{\chi\chi} = R^{\phi\phi} = R^{\phi0} = R^{\phi\chi} = -\frac{1}{\sinh^2 \rho}.
\]

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will study some formal consequences of it in a five-dimensional world with the metrics (6). Since the restriction (10) makes the metrics (6) into four-dimensional metrics

\[ dl^2 = \left[ 1 + S^2 - \frac{S^2}{1 + S^2} \right] d\tau^2, \]

\[ -S^2 \left[ \frac{dr^2}{1 - r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \tag{12} \]

where \( r = \sin \chi \), and \( dp = \frac{dS}{\sqrt{1 + S^2}} \), up to conformal transformation, the metrics (12) is equivalent to the standard FLRW metrics (2). Its classical dynamics in the original time \( \tau \) can be studied using the Einstein tensor

\[
G_i^i = \frac{(S^2 + 1)^3 S^2 + 3S^2 S^4 - 2S(S^2 + 1)^3 S - (S^2 + 1)^4}{S^2 \left[ S^2 - (S^2 + 1)^2 \right]^2}, \tag{13}
\]

\[
G_0^0 = \frac{3 \left( S^2 \right)^2 + (S^2 + 1)^2}{S^2 - (S^2 + 1)^2}, \quad i = 1, 3,
\]

which follows from the metrics (12). The analysis of the classical dynamics may be the subject of a separate study, but is beyond the scope of this paper. Planning the quantum field theory studies for future research, we can still say within presented classical framework that the Friedmann model description in terms of the four-dimensional spacetime coordinated is too small, and we may need a new independent inflation coordinate \( \rho \) to describe our expanding Universe.

Having this note almost prepared the authors became aware of a new paper [15], proposing an exponential tree-like growth of the dark matter (\( \chi \)) number density by using the energy of the heat bath particles (\( \psi \)): \( \chi \psi \rightarrow \chi \chi \). In some sense our paper presents an alternative to this approach to the dark matter abundance: the exponential growth may be attributed to the evolution of spacetime from a discrete to a continuous one. The matter fields then should be added to this background.

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**CONFLICT OF INTEREST**

The authors declare that they have no conflicts of interest.

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