Inhomogeneous approximation with coprime integers and lattice orbits

Michel Laurent & Arnaldo Nogueira

ABSTRACT – Let \((\xi, y)\) be a point in \(\mathbb{R}^2\) and \(\psi : \mathbb{N} \to \mathbb{R}^+\) a function. We investigate the problem of the existence of infinitely many pairs \(p, q\) of coprime integers such that

\[|q\xi + p - y| \leq \psi(|q|).\]

We prove that the existence is ensured for every point \((\xi, y)\) whose coordinate \(\xi\) is irrational when \(\psi(|q|) = c|q|^{-1/2}\) for some suitable coefficient \(c = c(\xi, y)\). An optimal metrical statement is also established. We link the subject with density exponents of lattice orbits in \(\mathbb{R}^2\).

1 Introduction and results

Minkowski has proved that for every real irrational number \(\xi\) and every real number \(y\) not belonging to \(\mathbb{Z}\xi + \mathbb{Z}\), there exist infinitely many pairs of integers \(p, q\) such that

\[(1) \quad |q\xi + p - y| \leq \frac{1}{4|q|}.\]

See for instance Theorem II in Chapter 3 of Cassels’ monograph [4]. The statement is optimal in the sense that the approximating function \(\ell \mapsto (4\ell)^{-1}\) cannot be decreased. Note that the restriction \(y \not\in \mathbb{Z}\xi + \mathbb{Z}\) can be dropped at the cost of replacing the upper bound \((4|q|)^{-1}\) by \(c|q|^{-1}\) for any constant \(c\) greater than \(1/\sqrt{5}\). When \(y = 0\), the primitive point \((\frac{p}{\gcd(p, q)}, \frac{q}{\gcd(p, q)})\) remains a solution to (1), therefore we may moreover require that the pair of integers \(p, q\) satisfying (1) be coprime. However, for a non-zero real number \(y\), this extra requirement is far from being obvious to satisfy. We obtain such a result with a weaker accuracy of order \(1/\sqrt{|q|}\).

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**Theorem 1.** Let $\xi$ be an irrational real number and $y$ a real number. There exist infinitely many pairs of coprime integers $(p,q)$ such that

$$|q\xi + p - y| \leq \frac{c}{\sqrt{|q|}}$$

with $c = 2\sqrt{3} \max(1, |\xi|)^{1/2}|y|^{1/2}$.

Theorem 1 will be deduced in Section 2 from our results [9] of effective density for $SL(2,\mathbb{Z})$-orbits in $\mathbb{R}^2$. However, it will clearly appear that our method for proving Theorem 1 provides more than necessary. Theorem 1 may probably be improved. We address the following

**Problem.** Can we replace the approximating function $\psi(\ell) = c\ell^{-1/2}$ occurring in Theorem 1 by a smaller one, possibly $\psi(\ell) = c\ell^{-1}$?

We shall further discuss this problem in Section 4 for the function $\psi(\ell) = 2\ell^{-1}$, offering some hints and indicating the difficulties which then arise. It turns out that the approximating function $\psi(\ell) = \ell^{-1}$ is permitted for almost all pairs $(\xi,y)$ of real numbers relatively to Lebesgue measure and that $-1$ is the critical exponent. The last assertion follows from the following metrical statement:

**Theorem 2.** Let $\psi : \mathbb{N} \mapsto \mathbb{R}^+$ be a function. Assume that $\psi$ is non-increasing, tends to 0 at infinity and that for every positive integer $c$ there exist positive real numbers $c_1 < c_2$ satisfying

$$c_1\psi(\ell) \leq \psi(c\ell) \leq c_2\psi(\ell), \quad \forall \ell \geq 1.$$

Assume furthermore that

$$\sum_{\ell \geq 1} \psi(\ell) = +\infty.$$

Then, for almost all pairs $(\xi,y)$ of real numbers there exist infinitely many primitive points $(p,q)$ such that

$$q \geq 1 \quad \text{and} \quad |q\xi + p - y| \leq \psi(q).$$

If $\sum_{\ell \geq 1} \psi(\ell)$ converges, the pairs $(\xi,y)$ satisfying (4) for infinitely many primitive points $(p,q)$ form a set of null Lebesgue measure.

Note that (4) gives an additional information on the sign of $q$, and that we could have equivalently required that $q$ be negative. Such a refinement could as well be achieved in the frame of Theorem 1, with a weaker approximating function of the form $\psi(\ell) = \ell^{-\mu}$ for any given real number $\mu < 1/3$, by employing alternatively Theorem 5 in Section 9 of [9]. We leave the details of proof to the interested reader, arguing as in Section 2. For questions of density involving signs, see also [6].
The proof of Theorem 2 is given in Section 3. It combines standard tools from metrical number theory with the ergodic properties of the linear action of $SL(2, \mathbb{Z})$ on $\mathbb{R}^2$. We refer to Harman’s book [7] for closely related results. See also the recent overview [1] and the monographs [13], [14].

Theorem 2 is a metrical statement about pairs $(\xi, y)$ of real numbers. A natural question is to understand what happens on each fiber when we fix either $\xi$ or $y$. In this direction, here is a partial result which will be deduced from the explicit construction displayed in Section 4.

**Theorem 3.** Let $\xi$ be an irrational number and let $(p_k/q_k)_{k \geq 0}$ be the sequence of its convergents. Assume that the series

$$
\sum_{k \geq 0} \frac{1}{\max(1, \log q_k)}
$$

diverges. Then for almost every real number $y$ there exist infinitely many primitive points $(p, q)$ satisfying

$$
|q\xi + p - y| \leq \frac{2}{|q|}.
$$

Moreover the series (5) diverges for almost every real number $\xi$.

We now turn to the second part of the paper devoted to density exponents for lattice orbits in $\mathbb{R}^2$. As already mentioned, the approximating function $\psi(\ell) = c \ell^{-1/2}$ occurring in Theorem 1 is directly connected to the density exponent $1/2$ for $SL(2, \mathbb{Z})$-orbits. We intend to show that this exponent $1/2$ is best possible in general. Thus our method for proving Theorem 1 fails with an approximating function $\ell^{-\mu}$ for $\mu > 1/2$.

We work in the more general setting of lattices $\Gamma$ in $SL(2, \mathbb{R})$. Recall that a lattice $\Gamma$ in $SL(2, \mathbb{R})$ is a discrete subgroup for which the quotient $\Gamma \setminus SL(2, \mathbb{R})$ has finite Haar measure. We view $\mathbb{R}^2$ as a space of column vectors on which the group of matrices $\Gamma$ acts by left multiplication. We equip $\mathbb{R}^2$ with the supremum norm $| |$, and for any matrix $\gamma \in \Gamma$, we denote as well by $|\gamma|$ the maximum of the absolute values of the entries of $\gamma$. Let us first give a

**Definition.** Let $\mathbf{x}$ and $\mathbf{y}$ be two points in $\mathbb{R}^2$. We denote by $\mu_\Gamma(\mathbf{x}, \mathbf{y})$ the supremum, possibly infinite, of the exponents $\mu$ such that the inequality

$$
|\gamma \mathbf{x} - \mathbf{y}| \leq |\gamma|^{-\mu}
$$

has infinitely many solutions $\gamma \in \Gamma$.

Note that for a fixed $\mathbf{x} \in \mathbb{R}^2$, the function $\mathbf{y} \mapsto \mu_\Gamma(\mathbf{x}, \mathbf{y})$ is $\Gamma$-invariant. By the ergodicity of the action of $\Gamma$ on $\mathbb{R}^2$, see [12], this function is therefore constant almost everywhere on $\mathbb{R}^2$. We denote by $\mu_\Gamma(\mathbf{x})$ its generic value and we call $\mu_\Gamma(\mathbf{x})$ the generic density exponent of the orbit $\Gamma \mathbf{x}$. 


Theorem 4. The upper bound $\mu_\Gamma(x) \leq 1/2$ holds true for any point $x \in \mathbb{R}^2$ such that the orbit $\Gamma x$ is dense in $\mathbb{R}^2$.

In an equivalent way, Theorem 4 asserts that the upper bound $\mu(x, y) \leq 1/2$ holds for almost all points $y \in \mathbb{R}^2$. This bound was already known in the case of the unimodular group $\Gamma = SL(2, \mathbb{Z})$ as a consequence of Theorem 3 in [9].

One may optimistically conjecture that $\mu_\Gamma(x) = 1/2$ for every point $x$ such that $\Gamma x$ is dense in $\mathbb{R}^2$, or at least for almost every point $x \in \mathbb{R}^2$. In this direction, it follows from [9] that the lower bound

$$\mu_{SL(2, \mathbb{Z})}(x) \geq \frac{1}{3}$$

holds for all points $x$ in $\mathbb{R}^2 \setminus \{0\}$ with irrational slope. Weaker lower bounds can as well be deduced from [11] which are valid for any lattice $\Gamma \subset SL(2, \mathbb{R})$. Note that the function $x \mapsto \mu_\Gamma(x)$ is $\Gamma$-invariant since the quantity $\mu_\Gamma(x)$ obviously depends only on the orbit $\Gamma x$. Thus, the generic density exponent $\mu_\Gamma(x)$ takes the same value for almost all points $x \in \mathbb{R}^2$.

2 Proof of Theorem 1

We first state a result obtained in [9]. In this section, we denote by $\Gamma$ the lattice $SL(2, \mathbb{Z})$. For any point $x = \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right)$ in $\mathbb{R}^2$ with irrational slope $x_1/x_2$, the orbit $\Gamma x$ is dense in $\mathbb{R}^2$. We have obtained in [9] effective results concerning the density of such an orbit. In particular, our estimates are essentially optimal when the target point $y$ has rational slope.

Lemma 1. Let $x$ be a point in $\mathbb{R}^2$ with irrational slope and $y = \left( \begin{array}{c} y \\ y \end{array} \right)$ a point on the diagonal with $y \neq 0$. Then, there exist infinitely many matrices $\gamma \in \Gamma$ such that

$$|\gamma x - y| \leq \frac{c}{|\gamma|^{1/2}} \quad \text{with} \quad c = 2\sqrt{3}|x|^{1/2}|y|^{1/2}. \quad (7)$$

Proof. The point $y$ has rational slope 1. Apply Theorem 1 (ii) of [9] with $a = b = 1$.

Put $x = \left( \begin{array}{c} \xi \\ 1 \end{array} \right)$. The point $x$ has irrational slope $\xi$ so that Lemma 1 may be applied.
Write $\gamma = \begin{pmatrix} q_1 & p_1 \\ q_2 & p_2 \end{pmatrix}$ a matrix provided by Lemma 1. Then, the inequality (7) gives
\[
\max \left( \left| q_1 \xi + p_1 - y \right|, \left| q_2 \xi + p_2 - y \right| \right) \leq \frac{c}{\max(|p_1|, |p_2|, |q_1|, |q_2|)^{1/2}} \leq \frac{c}{\max(|q_1|, |q_2|)^{1/2}}.
\]
Therefore, both points $(p, q) = (p_1, q_1)$ and $(p, q) = (p_2, q_2)$ satisfy (2), and since the determinant $q_1 p_2 - q_2 p_1 = 1$, the two integer points $(p_1, q_1)$ and $(p_2, q_2)$ are primitive. As there exist infinitely many matrices $\gamma$ verifying (7), we thus find infinitely many coprime solutions to (2).

**Remark.** The choice of another fixed rational direction would give the same kind of estimate, with a different constant $c$. Notice however that our approach gives more than required, since we get two solutions of (2) forming a matrix $\gamma$ of determinant 1. A better understanding of the shrinking target problem for the dense orbit $\Gamma x$, not to a point $y$ as in [9] but to a line in $\mathbb{R}^2$, may possibly lead to the expected exponent $-1$.

### 3 Proof of Theorem 2

It is convenient to view the pairs $(\xi, y)$ occurring in Theorem 2 as column vectors $\begin{pmatrix} \xi \\ y \end{pmatrix}$ in $\mathbb{R}^2$. We are concerned with the set $\mathcal{E}(\psi)$ of vectors $\begin{pmatrix} \xi \\ y \end{pmatrix} \in \mathbb{R}^2$ for which there exist infinitely many primitive integer points $(p, q)$ such that
\[
q \geq 1 \quad \text{and} \quad |q\xi + p - y| \leq \psi(q).
\]
For fixed $p, q$, denote by $\mathcal{E}_{p,q}(\psi)$ the strip
\[
\mathcal{E}_{p,q}(\psi) := \left\{ \begin{pmatrix} \xi \\ y \end{pmatrix} \in \mathbb{R}^2; \quad |q\xi + p - y| \leq \psi(q) \right\},
\]
and for every positive integer $q$, let
\[
\mathcal{E}_q(\psi) := \bigcup_{\gcd(p, q) = 1} \mathcal{E}_{p,q}(\psi)
\]
be the union of all relevant strips involved in (8) for fixed $q$. Without loss of generality, we shall assume that $\psi(q) \leq 1/2$, so that the above union is disjoint. Then $\mathcal{E}(\psi)$ is equal to the lim sup set
\[
\mathcal{E}(\psi) = \bigcap_{Q \geq 1} \bigcup_{q \geq Q} \mathcal{E}_q(\psi).
\]
As usual when dealing with lim sup set in metrical theory, we first estimate Lebesgue measure of pairwise intersections of the subsets $E_q(\psi), q \geq 1$. We establish next a new kind of zero-one law.

### 3.1 Measuring intersections

In this section, we restrict our attention to points located in the unit square $[0, 1]^2$. We denote by $\varphi$ the Euler totient function and by $\lambda$ the Lebesgue measure on $\mathbb{R}^2$.

**Lemma 2.** Let $\psi : \mathbb{N} \rightarrow [0, 1/2]$ be a function.

(i) For every positive integer $q$, we have

$$\lambda(E_q(\psi) \cap [0, 1]^2) = \frac{2\varphi(q)\psi(q)}{q}.$$ 

(ii) Let $q$ and $s$ be distinct positive integers. Then, we have the upper bound

$$\lambda(E_q(\psi) \cap E_s(\psi) \cap [0, 1]^2) \leq 4\psi(q)\psi(s).$$

**Proof.** Denote by $\chi_q$ the characteristic function of the interval $[-\psi(q), \psi(q)]$. Then the characteristic function $\chi_{E_q(\psi)}$ of the subset $E_q(\psi) \subset \mathbb{R}^2$ is equal to

$$\chi_{E_q(\psi)}(\xi, y) = \sum_{p \in \mathbb{Z}, \gcd(p, q) = 1} \chi_q(q\xi + y).$$

Observe that if $\begin{pmatrix} \xi \\ y \end{pmatrix}$ belongs to $[0, 1]^2$, the indices $p$ of non-vanishing terms occurring in the last sum are located in the interval $-1 \leq p \leq q$. Integrating first with respect to $x$, we find

$$\lambda(E_q(\psi) \cap [0, 1]^2) = \int_0^1 \int_0^1 \chi_{E_q(\psi)}(x, y) dxdy$$

$$= \sum_{-1 \leq p \leq q, \gcd(p, q) = 1} \int_0^1 \int_0^1 \chi_q(q\xi - p - y) dxdy$$

$$= \int_{1-\psi(q)}^1 \frac{-1 + y + \psi(q)}{q} dy + \sum_{s \leq q-2, \gcd(s, q) = 1} \int_0^1 \frac{2\psi(q)}{q} dy$$

$$+ \int_0^{1-\psi(q)} \frac{2\psi(q)}{q} dy + \int_{1-\psi(q)}^1 \frac{1 - y + \psi(q)}{q} dy$$

$$= \frac{2\varphi(q)\psi(q)}{q}.$$
The first term appearing in the third equality of the above formula corresponds to the summation index \( p = -1 \) and the two last ones to \( p = q - 1 \). We have thus proved (i).

For the second assertion, we majorize

\[
\lambda(\mathcal{E}_q(\psi) \cap \mathcal{E}_s(\psi) \cap [0, 1]^2) = \int_0^1 \int_0^1 \chi_{\mathcal{E}_q(\psi)}(x, y) \chi_{\mathcal{E}_s(\psi)}(x, y) dxdy \\
\leq \int_0^1 \int_0^1 \left( \sum_{p \in \mathbb{Z}} \chi_q(qx + p - y) \right) \left( \sum_{r \in \mathbb{Z}} \chi_s(sx + r - y) \right) dxdy \\
= \int_0^1 \int_0^1 \chi_q(\|qx - y\|) \chi_s(\|sx - y\|) dxdy,
\]

where \( \|\cdot\| \) stands as usual for the distance to the nearest integer. Now, (ii) follows from the probabilistic independence formula

\[
\int_0^1 \int_0^1 \chi_q(\|qx - y\|) \chi_s(\|sx - y\|) dxdy = 4\psi(q)\psi(s),
\]

obtained by Cassels on page 124 of [4] (see Proof (ii)).

\[\square\]

3.2 A zero-one law

We say that a subset of \( \mathbb{R}^2 \) is a null set if it has Lebesgue measure 0. A set whose complementary is a null set is called a full set. The goal of this section is to prove the

**Proposition.** Let \( \psi \) be an approximating function as in Theorem 2. Then the subset \( \mathcal{E}(\psi) \) is either a null set or a full set.

For proving the proposition, it is convenient to introduce the larger subset

\[
\mathcal{E}'(\psi) = \bigcup_{k \geq 1} \mathcal{E}(k\psi).
\]

In other words, \( \mathcal{E}'(\psi) \) is the set of all points \( \begin{pmatrix} \xi \\ y \end{pmatrix} \) in \( \mathbb{R}^2 \) for which there exist a positive real number \( \kappa \), depending possibly on \( \begin{pmatrix} \xi \\ y \end{pmatrix} \), and infinitely many primitive points \((p, q)\) satisfying

\[
q \geq 1 \quad \text{and} \quad |q\xi + p - y| \leq \kappa \psi(q). \quad (9)
\]

Observe that \( \mathcal{E}(k\psi) \subseteq \mathcal{E}(k'\psi) \) if \( 1 \leq k \leq k' \). In particular, \( \mathcal{E}(\psi) \) is contained in \( \mathcal{E}'(\psi) \).
Lemma 3. Let \( \psi : \mathbb{N} \to \mathbb{R}^+ \) be a function tending to zero at infinity. Then the difference \( \mathcal{E}'(\psi) \setminus \mathcal{E}(\psi) \) is a set of null Lebesgue measure.

**Proof.** We show that all sets \( \mathcal{E}(k\psi), k \geq 1 \), have the same Lebesgue measure. For every real number \( y \), denote by \( \mathcal{E}(\psi, y) \subseteq \mathbb{R} \) the section of \( \mathcal{E}(\psi) \) on the horizontal line \( \mathbb{R} \times \{ y \} \), i.e.

\[
\mathcal{E}(\psi, y) = \left\{ \xi \in \mathbb{R} ; \left( \begin{array}{c} \xi \\ y \end{array} \right) \in \mathcal{E}(\psi) \right\}.
\]

Then, using (8), we can express

\[
\mathcal{E}(\psi, y) = \bigcap_{Q \geq 1} \bigcup_{q \geq Q} \bigcup_{p \in \mathbb{Z} \setminus \{p \in \mathbb{Z} : \gcd(p, q) = 1\}} \left[ \frac{-p + y - \psi(q)}{q}, \frac{-p + y + \psi(q)}{q} \right]
\]

as a limsup set of intervals. If we restrict to a bounded part of \( \mathcal{E}(\psi, y) \), the above union over \( p \) reduces to a finite one. Observe that the centers \( \frac{-p + y}{q} \) of these intervals do not depend on \( \psi \), and that their length is multiplied by the constant factor \( k \) when replacing \( \psi \) by \( k\psi \). Appealing now to a result due to Cassels [5], we infer that all lim sup sets \( \mathcal{E}(k\psi, y), k \geq 1 \), have the same Lebesgue measure. See also Corollary of Lemma 2.1 on page 30 of [7]. Notice that for fixed \( k \), the length \( \frac{2k\psi(q)}{q} \) of the intervals \( \left[ \frac{-p + y - k\psi(q)}{q}, \frac{-p + y + k\psi(q)}{q} \right] \) tend to 0 as \( q \) tends to infinity, as required by Lemma 2.1. By Fubini, the fibered sets

\[
\mathcal{E}(k\psi) = \prod_{y \in \mathbb{R}} \left( \mathcal{E}(k\psi, y) \times \{ y \} \right), \quad k \geq 1,
\]

have as well the same Lebesgue measure in \( \mathbb{R}^2 \).

Lemma 4. Let \( \psi : \mathbb{N} \to \mathbb{R}^+ \) be a function satisfying the conditions (3). Then \( \mathcal{E}'(\psi) \) is either a null or a full set.

**Proof.** It is based on the following observation. Let \( \left( \begin{array}{c} \xi \\ y \end{array} \right) \) belong to \( \mathcal{E}'(\psi) \) and let

\[
\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)
\]

be a matrix in \( SL(2, \mathbb{Z}) \) such that \( c\xi + d > 0 \). Then the point \( \left( \begin{array}{c} \xi' \\ y' \end{array} \right) \) with coordinates

\[
\xi' = \frac{a\xi + b}{c\xi + d} \quad \text{and} \quad y' = \frac{y}{c\xi + d}
\]

belongs to \( \mathcal{E}'(\psi) \). Indeed, substituting

\[
q = aq' + cp', \quad p = bq' + dp'
\]

(10)
in (9) and dividing by $c\xi + d$, we obtain the inequalities

\[(11) \quad q' \geq 1 \quad \text{and} \quad \left| q'\xi' + p' - y' \right| \leq \frac{\kappa}{c\xi + d} \psi(q) \leq \kappa' \psi(q),\]

for some $\kappa' > 0$ independent of $q'$. The positivity of $q'$ is proved as follows. Note that (9) implies the estimate

\[p = -q\xi + O_{\gamma,\xi}(1).\]

Then, inverting the linear substitution (10), we find

\[q' = dq - cp = q(c\xi + d) + O_{\gamma,\xi}(1).\]

Since we have assumed that $c\xi + d > 0$, the term $q(c\xi + d)$ is arbitrarily large when $q$ is large enough. The conditions (3) now show that $\psi(q) \propto \psi(q')$. Thus (11) is satisfied for infinitely many primitive points $(p', q')$, since the linear substitution (10) is unimodular. We have shown that $\left(\frac{\xi'}{y'}\right)$ belongs to $E'(\psi)$.

We now prove that the intersection $E'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)$ is either a full or a null subset of the half plane $\mathbb{R} \times \mathbb{R}^+$. To that purpose, we consider the map

\[\Phi : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R} \times \mathbb{R}^+, \quad \text{defined by} \quad \Phi \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = \left(\frac{x}{y} \frac{1}{y}\right).\]

Clearly $\Phi$ is a continuous involution of $\mathbb{R} \times \mathbb{R}^+$. The image

\[\Omega := \Phi \left( E'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+) \right) \]

is formed by all points of the type

\[\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \xi \\ y \end{pmatrix},\]

where $\begin{pmatrix} \xi \\ y \end{pmatrix}$ ranges over $E'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)$. Now, the above condition $c\xi + d > 0$ is obviously equivalent to $cu + dv > 0$ since $y$ is positive. Then, the point

\[\Phi \left(\begin{pmatrix} au + bv \\ cu + dv \end{pmatrix} \right) = \begin{pmatrix} \frac{au + bv}{cu + dv} \\ \frac{c\xi + d}{cu + dv} \end{pmatrix} = \begin{pmatrix} \frac{a\xi + b}{c\xi + d} \\ \frac{y}{c\xi + d} \end{pmatrix}\]

belongs to $E'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)$, by the preceding observation. Applying the involution $\Phi$, we find that

\[\Phi \left(\begin{pmatrix} \frac{a\xi + b}{c\xi + d} \\ \frac{y}{c\xi + d} \end{pmatrix} \right) = \begin{pmatrix} au + bv \\ cu + dv \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}\]
belongs to $\Omega$. In other words, setting $\Gamma = SL(2, \mathbb{Z})$, we have established the inclusion

$$(\Gamma \Omega) \cap (\mathbb{R} \times \mathbb{R}^+) \subseteq \Omega.$$ 

Since the reversed inclusion is obvious, the equality $\Omega = (\Gamma \Omega) \cap (\mathbb{R} \times \mathbb{R}^+)$ holds in fact. Assuming that $\Omega$ is not a null set, the ergodicity of the linear action of $\Gamma$ on $\mathbb{R}^2$ \cite{12} shows that $\Gamma \Omega$ is a full set in $\mathbb{R}^2$. Hence $\Omega$ is a full set in the half plane $\mathbb{R} \times \mathbb{R}^+$. Transforming now $\Omega$ by $\Phi$, we find that

$$\Phi(\Omega) = \mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+),$$

is as well a full set in $\mathbb{R} \times \mathbb{R}^+$, thus proving the claim.

We finally use another transformation to carry the zero-one law from the positive half plane $\mathbb{R} \times \mathbb{R}^+$ to the negative one $\mathbb{R} \times \mathbb{R}^-$. Writing (9) in the equivalent form

$$q \geq 1 \quad \text{and} \quad |q(-\xi) + (-p) - (-y)| \leq \kappa \psi(q),$$

shows that $\mathcal{E}'(\psi)$ is invariant under the symmetry $(\xi, y) \mapsto (-\xi, -y)$ which maps $\mathbb{R} \times \mathbb{R}^+$ onto $\mathbb{R} \times \mathbb{R}^-$. Therefore $\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^-)$ is a null or a full set in $\mathbb{R} \times \mathbb{R}^-$ when $\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)$ is accordingly a null or a full set in $\mathbb{R} \times \mathbb{R}^+$. \hfill \Box

Now, the combination of Lemma 3 and Lemma 4 obviously yields our proposition.

3.3 Concluding the proof of Theorem 2

Assume first that $\sum \psi(\ell)$ converges. We have to show that the set

$$\mathcal{E}(\psi) = \limsup_{q \to +\infty} \mathcal{E}_q(\psi)$$

has null Lebesgue measure. Lemma 2 shows that the partial sums

$$\sum_{q=1}^{Q} \lambda(\mathcal{E}_q(\psi) \cap [0, 1]^2) = 2 \sum_{q=1}^{Q} \frac{\varphi(q) \psi(q)}{q} \leq 2 \sum_{q=1}^{Q} \psi(q)$$

converge (*). Then, Borel-Cantelli Lemma ensures that the lim sup set $\mathcal{E}(\psi) \cap [0, 1]^2$ is a null set. Thus $\mathcal{E}(\psi)$ cannot be a full set. Now, the above proposition tells us that $\mathcal{E}(\psi)$ is a null set.

(*) Here again we assume without loss of generality that $\psi(q) \leq 1/2$ for every $q \geq 1$, so that Lemma 2 may be applied.
We now consider the case of a divergent series \( \sum \psi(\ell) \). Observe that the estimate
\[
\frac{1}{2} \sum_{q=1}^{Q} \psi(q) \leq \sum_{q=1}^{Q} \frac{\varphi(q)\psi(q)}{q} \leq \sum_{q=1}^{Q} \psi(q)
\]
holds true for any large integer \( Q \), since the sequence \( \psi(\ell)_{\ell \geq 1} \) is non-increasing. The right inequality is obvious, while the left one easily follows from Abel summation process. See for instance Chapter 2 of [7], where full details are provided. By Lemma 2 and (12), the sums
\[
\sum_{q=1}^{Q} \lambda(E_q(\psi) \cap [0,1]^2) = 2 \sum_{q=1}^{Q} \frac{\varphi(q)\psi(q)}{q} \geq \sum_{q=1}^{Q} \psi(q)
\]
are then unbounded. Then, using a classical converse to Borel-Cantelli Lemma, we have the lower bound
\[
\lambda\left(E(\psi) \cap [0,1]^2\right) = \lambda\left(\limsup_{q \to +\infty} (E_q(\psi) \cap [0,1]^2)\right)
\geq \limsup_{Q \to +\infty} \frac{\left(\sum_{q=1}^{Q} \lambda(E_q(\psi) \cap [0,1]^2)\right)^2}{\sum_{q=1}^{Q} \sum_{s=1}^{Q} \lambda(E_q(\psi) \cap E_s(\psi) \cap [0,1]^2)}.
\]
See for instance Lemma 2.3 in [7]. Lemma 2 and (12) now show that the numerator on the right hand side of (13) equals
\[
4 \left(\sum_{q=1}^{Q} \frac{\varphi(q)\psi(q)}{q}\right)^2 \geq \left(\sum_{q=1}^{Q} \psi(q)\right)^2,
\]
when \( Q \) is large, while the denominator is bounded from above by
\[
4 \sum_{q=1, s=1}^{Q} \psi(q)\psi(s) + 2 \sum_{q=1}^{Q} \psi(q) \leq 4 \left(\sum_{q=1}^{Q} \psi(q)\right)^2 + 2 \sum_{q=1}^{Q} \psi(q).
\]
Thus (13) yields the lower bound
\[
\lambda\left(E(\psi) \cap [0,1]^2\right) \geq \frac{1}{4}.
\]
Hence \( E(\psi) \) is not a null set; it is thus a full set according to our proposition.
4 An approach to our problem

In this section, we apply a transference principle between homogeneous and inhomogeneous approximation, as displayed in Chapter V of [4] and in [3], for constructing explicit integer solutions of the inequality

\[(14) \quad |q\xi + p - y| \leq \frac{2}{|q|}.\]

Let \((p_k/q_k)_{k\geq 0}\) be the sequence of convergents to the irrational number \(\xi\). The theory of continued fractions, see for instance the monograph [8], tells us that

\[(15) \quad |q_k\xi - p_k| \leq \frac{1}{q_{k+1}} \quad \text{and} \quad p_kq_{k+1} - p_{k+1}q_k = (-1)^{k+1},\]

for any \(k \geq 0\). Setting \(\nu_k = (-1)^{k+1}qKy\), we thus have the relations

\[(16) \quad \nu_kq_{k+1} + \nu_{k+1}q_k = 0 \quad \text{and} \quad \nu_k(q_{k+1}\xi - p_{k+1}) + \nu_{k+1}(q_k\xi - p_k) = y.\]

Now, let \(n_k\) be anyone of the two integers \([\nu_k]\) and \([\nu_k]\) (†). Then,

\[(17) \quad |\nu_k - n_k| < 1,\]

and \(n_k\) is either equal to \((-1)^{k+1}[yq_k]\) or to \((-1)^{k+1}[yq_k]\). Setting

\[(18) \quad p = -n_kp_{k+1} - n_{k+1}p_k \quad \text{and} \quad q = n_kq_{k+1} + n_{k+1}q_k,\]

we deduce from (16) the expressions

\[(19) \quad q\xi + p - y = n_k(q_{k+1}\xi - p_{k+1}) + n_{k+1}(q_k\xi - p_k) - y \]

\[= (n_k - \nu_k)(q_{k+1}\xi - p_{k+1}) + (n_{k+1} - \nu_{k+1})(q_k\xi - p_k)\]

and

\[(20) \quad q = (n_k - \nu_k)q_{k+1} + (n_{k+1} - \nu_{k+1})q_k.\]

Recall that \(q_k\xi - p_k\) and \(q_{k+1}\xi - p_{k+1}\) have opposite signs. Assuming that \(n_k - \nu_k\) and \(n_{k+1} - \nu_{k+1}\) have the same sign, we infer from the formulas (19), (20) and from (15), (17) that

\[(21) \quad |q\xi + p - y| < \frac{1}{q_{k+1}} \quad \text{and} \quad |q| < 2q_{k+1}.\]

(†) As usual \([x]\) and \([x]\) stand respectively for the integer part and the upper integer part of the real number \(x\). Then \([x] = [x] + 1\), unless \(x\) is an integer in which case \([x] = [x] = x\).
Otherwise, we have

\[ |q\xi + p - y| < \frac{2}{q_{k+1}} \quad \text{and} \quad |q| < q_{k+1}. \]

The inequalities (21) and (22) obviously imply (14).

Since the linear substitution (18) is unimodular, the integers \( p \) and \( q \) are coprime if and only if \( n_k \) and \( n_{k+1} \) are coprime. Recall that the two choices \( n_k = \lfloor \nu_k \rfloor \) and \( n_{k+1} = \lceil \nu_k \rceil \) are admissible, both for \( n_k \) and \( n_{k+1} \). It thus remains to find indices \( k \) for which at least one of the coprimality conditions

\[ \gcd(\lfloor y q_k \rfloor, \lfloor y q_{k+1} \rfloor) = 1 \quad \text{or} \quad \gcd(\lceil y q_k \rceil, \lceil y q_{k+1} \rceil) = 1, \]

is verified. Note that (23) obviously fails for all \( k \geq 0 \) when \( y \) is an integer not equal to 1 or to \(-1\). Otherwise, the contingent existence of infinitely many indices \( k \) satisfying (23) is a non-trivial problem that we leave hanging.

### 4.1 Proof of Theorem 3

We quote the following metrical result due to Harman (Theorem 8.3 in [7]). Assume that the series (5) diverges. Then for almost all positive real numbers \( y \), there exist infinitely many indices \( k \) such that the integer part \( \lfloor y q_k \rfloor \) is a prime number. These indices \( k \) fulfill (23) since, assuming for simplicity that \( y \) is irrational, either \( \lfloor y q_{k+1} \rfloor = \lfloor y q_k \rfloor + 1 \) is not divisible by \( \lfloor y q_k \rfloor \) and is thus relatively prime with \( \lfloor y q_k \rfloor \). Hence (14) has infinitely many coprime solutions \((p, q)\) for almost every positive real number \( y \). Writing now (14) in the equivalent form

\[ |(-q)\xi + (-p) - (-y)| \leq \frac{2}{|q|} \]

shows that, \( \xi \) being given, the set of all real numbers \( y \) for which (14) has infinitely many coprime solutions is invariant by the symmetry \( y \mapsto -y \). The first assertion is thus established. To complete the proof, note that

\[ \lim_{k \to +\infty} \frac{\log q_k}{k} = \frac{\pi^2}{12 \log 2} \]

for almost every \( \xi \) by Khintchine-Levy Theorem (see equation (4.18) in [2]). Thus the series (5) diverges for almost every \( \xi \).
5 Generic density exponents

We prove in this section Theorem 4, as a consequence of Borel-Cantelli Lemma combined with the following counting result.

**Lemma 5.** Let \( x \) be a point in \( \mathbb{R}^2 \) whose orbit \( \Gamma x \) is dense in \( \mathbb{R}^2 \). For every symmetric compact set \( \Omega \) in \( \mathbb{R}^2 \setminus \{0\} \) there exists \( c > 0 \) such that

\[
\text{Card}\{\gamma \in \Gamma; \gamma x \in \Omega, |\gamma| \leq T\} \leq cT
\]

for any real number \( T \geq 1 \).

**Proof.** Ledrappier [10] has shown that the limit formula

\[
\lim_{T \to +\infty} \frac{1}{T} \sum_{\gamma \in \Gamma, |\gamma| \leq T} f(\gamma x) = \frac{4}{|x| \text{vol}(\Gamma \setminus \text{SL}(2, \mathbb{R}))} \int_{\mathbb{R}^2} \frac{f(y)}{|y|} dy,
\]

holds for any even continuous function \( f : \mathbb{R}^2 \to \mathbb{R} \) having compact support on \( \mathbb{R}^2 \setminus \{0\} \), with a suitable normalisation of Haar measure on \( \text{SL}(2, \mathbb{R}) \). Approximating uniformly from above and from below the characteristic function of \( \Omega \) by even continuous functions, we deduce that

\[
\lim_{T \to +\infty} \frac{\text{Card}\{\gamma \in \Gamma; \gamma x \in \Omega, |\gamma| \leq T\}}{T} = \frac{4}{|x| \text{vol}(\Gamma \setminus \text{SL}(2, \mathbb{R}))} \int_{\Omega} \frac{dy}{|y|}.
\]

Lemma 5 immediately follows. \( \Box \)

For any point \( y \in \mathbb{R}^2 \) and any positive real number \( r \), we denote by

\[
B(y, r) = \{z \in \mathbb{R}^2; |z - y| \leq r\}
\]

the closed disc centered at \( y \) with radius \( r \).

**Lemma 6.** Let \( x \) be a point in \( \mathbb{R}^2 \) whose orbit \( \Gamma x \) is dense, \( \Omega \) a symmetric compact set in \( \mathbb{R}^2 \setminus \{0\} \) and \( \mu \) a real number \( > 1/2 \). For every integer \( n \geq 1 \), put

\[
\mathcal{B}_n = \bigcup_{\gamma \in \Gamma, |\gamma| = n, \gamma x \in \Omega} B(\gamma x, n^{-\mu}).
\]

Then the set

\[
\mathcal{B} := \limsup_{n \to +\infty} \mathcal{B}_n = \bigcap_{N \geq 1} \bigcup_{n \geq N} \mathcal{B}_n = \bigcap_{N \geq 1} \bigcup_{\gamma \in \Gamma, |\gamma| \geq N, \gamma x \in \Omega} B(\gamma x, |\gamma|^{-\mu})
\]

has null Lebesgue measure.
Proof. We apply Borel-Cantelli Lemma and we prove that the series \( \sum_{n \geq 1} \lambda(B_n) \) converges if \( \mu > 1/2 \).

For every positive integer \( n \), set
\[
M_n = \text{Card}\{ \gamma \in \Gamma; \gamma x \in \Omega, |\gamma| = n \}.
\]
Lemma 5 gives us the upper bound
\[
M_1 + \cdots + M_n = \text{Card}\{ \gamma \in \Gamma; \gamma x \in \Omega, |\gamma| \leq n \} \leq cn,
\]
for some \( c > 0 \) independent of \( n \geq 1 \). Since a ball of radius \( r \) has Lebesgue measure \( 4r^2 \), we trivially bound from above
\[
\lambda(B_n) \leq \sum_{\gamma \in \Gamma; |\gamma| = n, \gamma x \in \Omega} 4n^{-2\mu} = 4M_n n^{-2\mu}.
\]
Summing by parts, we deduce from (24) that
\[
\sum_{n=1}^{N} \frac{M_n}{n^{2\mu}} = \sum_{n=1}^{N-1} (M_1 + \cdots + M_n) \left( \frac{1}{n^{2\mu}} - \frac{1}{(n+1)^{2\mu}} \right) + \frac{M_1 + \cdots + M_N}{N^{2\mu}} 
\leq c \sum_{n=1}^{N-1} n \left( \frac{1}{n^{2\mu}} - \frac{1}{(n+1)^{2\mu}} \right) + \frac{cN}{N^{2\mu}} = c \sum_{n=1}^{N} \frac{1}{n^{2\mu}}.
\]
The partial sums
\[
\sum_{n=1}^{N} \lambda(B_n) \leq 4 \sum_{n=1}^{N} \frac{M_n}{n^{2\mu}} \leq 4c \sum_{n=1}^{N} \frac{1}{n^{2\mu}}
\]
thus converge if \( \mu > 1/2 \).

5.1 Proof of Theorem 4

We argue by contradiction and suppose on the contrary that \( \mu_{\Gamma}(x) > 1/2 \). Fix a real number \( \mu \) with \( 1/2 < \mu < \mu_{\Gamma}(x) \). Then for almost all points \( y \in \mathbb{R}^2 \), we have \( \mu(x, y) > \mu \). This means that there exist infinitely many \( \gamma \in \Gamma \) satisfying (6), or equivalently that \( y \) belongs to infinitely many balls of the form \( B(\gamma x, |\gamma|^{-\mu}) \). We now restrict our attention to points \( y \) with \( \mu(x, y) > \mu \) lying in an annulus
\[
\Omega' = \{ z \in \mathbb{R}^2; a' \leq |z| \leq b' \},
\]
where \( b' > a' > 0 \) are arbitrarily fixed. Since \( y \) belongs to the intersection \( \Omega' \cap B(\gamma x, |\gamma|^{-\mu}) \), we deduce from the triangle inequality the estimate
\[
a' - |\gamma|^{-\mu} \leq |\gamma x| \leq b' + |\gamma|^{-\mu}.
\]
Fixing $a < a'$ and $b > b'$, the center $\gamma x$ then lies in the larger annulus

$$\Omega = \{ z \in \mathbb{R}^2; a \leq |z| \leq b \},$$

provided that $|\gamma|$ is large enough. It follows that $y$ falls inside the union of balls

$$\bigcup_{|\gamma| \geq N, \gamma x \in \Omega} B(\gamma x, |\gamma|^{-\mu})$$

considered in Lemma 6 for every integer $N$ large enough, and thus $y$ belongs to $\mathcal{B}$. However, Lemma 6 asserts that $\mathcal{B}$ is a null set which is a contradiction.

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Institut de Mathématiques de Luminy, Case 907, 163 avenue de Luminy, 13288, Marseille Cédex 9.

michel-julien.laurent@univmed.fr,
araldo.nogueira@univmed.fr