We revisit the time-adiabatic theorem of quantum mechanics and show that it can be extended to weakly nonlinear situations, that is to nonlinear Schrödinger equations in which either the nonlinear coupling constant or, equivalently, the solution is asymptotically small. To this end, a notion of criticality is introduced at which the linear bound states stay adiabatically stable, but nonlinear effects start to show up at leading order in the form of a slowly varying nonlinear phase modulation. In addition, we prove that in the same regime a class of nonlinear bound states also stays adiabatically stable, at least in terms of spectral projections.

1. Introduction

The time-adiabatic theorem of quantum mechanics is concerned with systems governed by a slowly varying time-dependent (self-adjoint) Hamiltonian operator $H = H(\varepsilon \tau)$, where $0 < \varepsilon \ll 1$ is a small adiabatic parameter, controlling the timescales on which $H$ varies. The associated Cauchy problem, governing the time evolution of the quantum mechanical wave function $\Psi = \Psi(\tau, x)$, with $x \in \mathbb{R}^d$, reads

$$i\partial_\tau \Psi = H(\varepsilon \tau)\Psi, \quad \Psi|_{\tau=\tau_0} = \Psi_{\text{in}}(x). \quad (1.1)$$

In the following, it will be more convenient to rewrite the system using the (slow) macroscopic time variable $t = \varepsilon \tau$. In this case, (1.1) becomes a singularly perturbed problem of the form

$$i\varepsilon \partial_t \Psi^\varepsilon = H(t)\Psi^\varepsilon, \quad \Psi^\varepsilon|_{t=t_0} = \Psi_{\text{in}}^\varepsilon(x), \quad (1.2)$$

where $\Psi^\varepsilon(t,x) \equiv \Psi(t/\varepsilon, x)$. A typical example for the time-dependent Hamiltonian $H(t)$, and the one we will be concerned with, is given by

$$H(t) := -\frac{1}{2} \Delta + V(t, x), \quad (1.3)$$

where $V(t, x)$ describes some time-dependent (real valued) potential.
It is well known that in the case where \( V = V(x) \) is time independent, the spectral theorem of self-adjoint operators allows for a precise description of the time evolution associated to (1.2). In particular, it implies that if the initial data \( \Psi_{\text{in}}^\varepsilon \) is concentrated in a given spectral subspace of \( H \), then it will remain so for all times. However, as soon as \( H = H(t) \), the spectral subspaces (in general) start to mix during the time evolution, and thus we do not have any precise information on the solution \( \Psi(t, \cdot) \).

However, one might hope that for small \( 0 < \varepsilon \ll 1 \) there is a remedy to the situation. To this end, let us assume that the spectral subspaces of \( H(t) \) vary smoothly in time for \( t \in [0, T] \), and that the initial wave function \( \Psi_{\text{in}}^\varepsilon \) is concentrated in one of these subspaces. Then, the classical time-adiabatic theorem of quantum mechanics states that, for sufficiently small \( \varepsilon \ll 1 \), the solution \( \Psi^\varepsilon(t, \cdot) \) approximately (i.e., up to a certain error which vanishes as \( \varepsilon \to 0 \)) remains within the same subspace, provided the latter stays isolated from the rest of the spectrum of \( H(t) \) for all \( t \in [0, T] \), see below. In this situation, the spectral subspace is said to be adiabatically stable under the time evolution. Note that in the unscaled variable \( \tau \) this result corresponds to an approximation on timescales of order \( \tau \sim O(1/\varepsilon) \). The first adiabatic result for quantum systems appeared as early as 1928, cf. [5]. Since then, many mathematical extensions and developments have taken place, see, e.g., [1,2,14–17,25], and the references therein. For a general introduction to this subject, we refer to [32].

A possible way of introducing the slow parameter \( \varepsilon \) is to think about a quantum mechanical experiment in which the experimentalist is allowed to slowly tune the external potential \( V = V(\varepsilon \tau, x) \). With this in mind, it is worth noting that modern quantum mechanical experiments are often performed on ultra-cold quantum gases in the state of their Bose–Einstein condensation [21]. Indeed, ultra-cold quantum gases offer a superb level of control, unprecedented in several respects, which has triggered a vast amount of scientific activity, both theoretical and experimental. It is well known that within a mean-field approximation the (macroscopic) wave function of the condensate is accurately described by a nonlinear Schrödinger (or, Gross–Pitaevskii) equation, cf. [21] for a general discussion [9,19], and the references therein for a rigorous mathematical justification. It therefore seems a natural question to ask, whether one can extend the results of time-adiabatic perturbation theory to the case of nonlinear Schrödinger equations (NLS). This work is a first, modest attempt in this direction, although one should mention that there exist some non-rigorous works in the physics literature, cf. [36]. Moreover, one should distinguish our time-adiabatic setting from the one in [24], which studies solitary wave solutions to nonlinear Schrödinger equations in a space-adiabatic situation, i.e., with a potential of the form \( V = V(t, \varepsilon x) \). In addition, we mention [7,8] both of which include rigorous results for related nonlinear adiabatic situations.

To be more concrete, we shall study the Cauchy problem corresponding to the following class of NLS:

\[
i \varepsilon \partial_t \Psi^\varepsilon = -\frac{1}{2} \Delta \Psi^\varepsilon + V(t, x) \Psi^\varepsilon + \lambda |\Psi^\varepsilon|^{2\sigma} \Psi^\varepsilon, \quad \Psi^\varepsilon|_{t=t_0} = \Psi_{\text{in}}^\varepsilon(x),
\] (1.4)