BOGDANOV-TAKENS BIFURCATION IN A SIRS EPIDEMIC
MODEL WITH A GENERALIZED NONMONOTONE
INCIDENCE RATE

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Abstract. In this paper, we study a SIRS epidemic model with a generalized
nonmonotone incidence rate. It is shown that the model undergoes two different
topological types of Bogdanov-Takens bifurcations, i.e., repelling and attract-
ing Bogdanov-Takens bifurcations, for general parameter conditions. The ap-
proximate expressions for saddle-node, Homoclinic and Hopf bifurcation curves
are calculated up to second order. Furthermore, some numerical simulations,
including bifurcations diagrams and corresponding phase portraits, are given
to illustrate the theoretical results.

1. Introduction. In 1927, Kermack and McKendrick [11] proposed a classic in-
fected disease compartmental model, where the population is divided into three
classes labeled $S(t)$, $I(t)$ and $R(t)$, which denote the number of susceptible indi-
viduals, the number of infected individuals, the number of individuals who have
been infected and then recovered or removed at time $t$, respectively. Assuming the
recovered individuals have temporary immunity, the classical Kermack-McKendrick
model has the following form:

$$
\frac{dS}{dt} = b - dS - g(I)S + \delta R,
$$

$$
\frac{dI}{dt} = g(I)S - (d + \mu)I,
$$

$$
\frac{dR}{dt} = \mu I - (d + \delta)R,
$$

where $b$ is the recruitment rate of the population, $d$ is the natural death rate of the
population, $\mu$ is the natural recovery rate of the infective individuals, $\delta$ is the rate
at which recovered individuals lose immunity and return to the susceptible class,
g(I)S is called the incidence rate, and $g(I)$ is a function to measure the infection
force of a disease.

In [11], Kermack and McKendrick assumed that infection force $g(I)$ is a linear
function of $I$ (see Figure 1(a)), i.e., the incidence rate $g(I)S$ is bilinear, which may be inconsistent with the reality when $I(t)$ is getting larger.

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11871235).
In order to study the cholera epidemic spread in Bari, Italy, in 1973, Capasso et al. [6] and Capasso and Serio [5] proposed a saturated incidence rate

\[ g(I)S = \frac{kIS}{1 + \alpha I}, \quad (2) \]

where the infection force \( g(I) = \frac{kI}{1 + \alpha I} \) eventually tends to a saturation level \( \frac{k}{\alpha} \) when \( I \) is getting larger (see Figure 1(b)).

Liu et al. [16] introduced the following general nonlinear incidence rate

\[ g(I)S = \frac{k^p I^p S}{1 + \alpha^q I^q}, \quad (3) \]

which was used by a number of authors, see, for example, [1, 2, 7, 8, 19, 14, 15], etc.

Ruan and Wang [19] studied system (1) with a specific nonlinear saturated incidence rate

\[ g(I)S = \frac{kI^2 S}{1 + \alpha I^2}, \quad (4) \]

where \( g(I) = \frac{kI^2}{1 + \alpha I^2} \) is a monotone function when \( I > 0 \) (see Figure 1(b)). They showed the existence of Hopf bifurcation and Bogdanov-Takens bifurcation in system (1) with incidence rate (4). Tang et al. [21] also studied system (1) with incidence rate (4) and showed the existence of degenerate Hopf bifurcation.

To model the effects of psychological factor when the infection number is getting larger, Xiao and Ruan [22] proposed a specific incidence rate

\[ g(I)S = \frac{kIS}{1 + \alpha I^2}, \quad (5) \]

where \( g(I) \) is nonmonotone and eventually tends to zero (see Figure 1(c)). They showed that the disease-free equilibrium of system (1) with incidence rate (5) is globally asymptotically stable if the basic reproduction number \( R_0 \leq 1 \), and the unique positive equilibrium is globally asymptotically stable if \( R_0 > 1 \). Then system (1) with nonmonotone incidence rate (5) has no complex dynamics and bifurcation phenomena.

Xiao and Zhou [23] considered a generalized form of the nonmonotone incidence rate (5) as follows:

\[ g(I)S = \frac{kIS}{1 + \beta I + \alpha I^2}, \quad (6) \]

where \( \beta \) is a parameter satisfied \( \beta > -2\sqrt{\alpha} \) such that \( 1 + \beta I + \alpha I^2 > 0 \) for all \( I \geq 0 \). They presented qualitative analysis for model (1) with generalized nonmonotone incidence rate (6) and showed the existence of a cusp of codimension 2, bistable phenomenon and periodic oscillation. Later, Zhou et al. [24] further studied the existence of different kinds of bifurcations, such as Bogdanov-Takens bifurcation and Hopf bifurcation, by choosing several set of specific parameter values. Their results showed that model (1) with generalized nonmonotone incidence rate (6) can exhibit complex dynamics and bifurcation phenomena. However, the set of parameter values they chosen to unfold the Bogdanov-Takens bifurcation is biologically meaningless.
In this paper, we continue to consider model (1) with generalized nonmonotone incidence rate (6) as follows:

\[
\begin{align*}
\frac{dS}{dt} &= b - dS - \frac{kSI}{1 + \beta I + \alpha I^2} + \delta R, \\
\frac{dI}{dt} &= \frac{kSI}{1 + \beta I + \alpha I^2} - (d + \mu)I, \\
\frac{dR}{dt} &= \mu I - (d + \delta)R,
\end{align*}
\]

where \(b, d, \delta, \mu, k, \alpha\) are all positive parameters and \(\beta > -2\sqrt{\alpha}\). We can see that \(g(I) = \frac{kI}{1 + \beta I + \alpha I^2}\) increases when \(I\) is small and decreases when \(I\) is large, then tends to zero as \(I\) tends to infinite (see Figure 2). When \(\alpha = \frac{9}{4}\) and \(k = 1\), the difference of the curves of \(g(I)\) when \(\beta \geq 0\) and \(-2\sqrt{\alpha} < \beta < 0\) can be seen from Figure 2(a) and (b), where \(g(I)\) has sigmoid shape when \(-2\sqrt{\alpha} < \beta < 0\) and \(I\) is small. We will not choose specific parameter values, and show that model (7) undergoes two different topological types of Bogdanov-Takens bifurcations, i.e., repelling and attracting Bogdanov-Takens bifurcations, for general parameter conditions. The approximate expressions for saddle-node, Homoclinic and Hopf bifurcation curves are calculated up to second order. Furthermore, some numerical simulations, including repelling
and attracting Bogdanov-Takens bifurcations diagrams and corresponding phase portraits, are given to illustrate the theoretical results.

\[ g(I) = \frac{kI}{1 + \beta I + \alpha I^2} \]

Figure 2. The curves of infection force \( g(I) = \frac{kI}{1 + \beta I + \alpha I^2} \) with \( \alpha = \frac{9}{2} \) and \( k = 1 \). (a) \( \beta \geq 0 \); (b) \( -2\sqrt{\alpha} < \beta < 0 \).

The organization of this paper is as follows. In section 2, we show that model (7) undergoes repelling and attracting Bogdanov-Takens bifurcations of codimension 2 for general parameter conditions. In section 3, some numerical simulations are given to illustrate the theoretical results. A brief discussion is given in the last section.

2. Bogdanov-Takens bifurcation. As is shown in Xiao and Zhou [23] and Zhou et al. [24], the limit set of system (7) is on the plane \( S + I + R = \frac{b}{d} \). Thus, we focus on the reduced system

\[ \frac{dI}{dt} = \frac{kI}{1 + \beta I + \alpha I^2} (\frac{b}{d} - I - R) - (d + \mu)I, \]
\[ \frac{dR}{dt} = \mu I - (d + \delta)R. \]

(8)

For simplicity, we rescale system (8) by

\[ I = \frac{d + \delta}{k} x, \quad R = \frac{d + \delta}{k} y, \quad t = \frac{1}{d + \delta} \tau, \]

then system (8) becomes (we still denote \( \tau \) by \( t \))

\[ \frac{dx}{dt} = \frac{x}{1 + mx + nx^2} (A - x - y) - px, \]
\[ \frac{dy}{dt} = qx - y, \]

(9)

where

\[ m = \frac{\beta(d + \delta)}{k}, \quad n = \frac{\alpha(d + \delta)^2}{k^2}, \quad A = \frac{bk}{d(d + \delta)}, \quad p = \frac{d + \mu}{d + \delta}, \quad q = \frac{\mu}{d + \delta}, \]

and \( q < p \leq q + 1, m > -2\sqrt{n}, A, p, q, n > 0 \).
We first define
\[
\begin{align*}
n_* &= -\frac{(1 + q + mp)(1 + q + 2pq - mp)}{4p^2}, \\
A_* &= \frac{2p(1 + q + pq)}{1 - mp + q + 2pq}, \\
m_* &= -\frac{1 + p + q + 2pq}{p^2},
\end{align*}
\]
\[
\Gamma = \{(A, m, n, p, q) : A, n, p, q > 0, q < p \leq q + 1, -2\sqrt{n} < m < -\frac{1 + q}{p}\}.
\] (10)

The following lemma is from Lemma 2.7 in Xiao and Zhou [23].

Lemma 2.1. When \((A, m, n, p, q) \in \Gamma, n = n_*, A = A_*\) and \(m \neq m_*\), system (9) has a unique positive equilibrium \(E_*(x_*, y_*)\), which is a cusp of codimension 2, where \(x_* = -\frac{1+q+mp}{2pn}, y_* = qx_*\). The phase portraits are shown in Figure 3.

Next, we study the existence of Bogdanov-Takens bifurcation of codimension 2 around \(E_*(x_*, y_*)\) in system (9).

Theorem 2.2. When \((A, m, n, p, q) \in \Gamma, n = n_*, A = A_*\) and \(m \neq m_*\) hold, system (9) has a cusp \(E_*(x_*, y_*)\) of codimension 2 (i.e., Bogdanov-Takens singularity). If we choose \(A\) and \(q\) as bifurcation parameters, then system (9) undergoes Bogdanov-Takens bifurcation of codimension 2 in a small neighborhood of the unique positive equilibrium \(E_*(x_*, y_*)\). Moreover,

(i): If \(m < m_*\), then there exist a repelling Bogdanov-Takens bifurcation of codimension 2. Hence system (9) exhibits an unstable limit cycle, or an unstable homoclinic loop for various parameter values;

(ii): If \(m > m_*\), then there exist an attracting Bogdanov-Takens bifurcation of codimension 2. Hence system (9) exhibits a stable limit cycle, or a stable homoclinic loop for various parameter values.
Proof. Consider the following unfolding system
\[
\begin{align*}
\frac{dx}{dt} &= \frac{x}{1 + mx + nx^2}(A + \lambda_1 - x - y) - px, \\
\frac{dy}{dt} &= (q + \lambda_2)x - y, \\
\end{align*}
\] (11)
where \((\lambda_1, \lambda_2)\) is a parameter vector in a small neighborhood of \((0, 0)\), \((A, m, n, p, q) \in \Gamma, n = n_*, A = A_*\) and \(m \neq m_*\). We are interested only in the dynamics and bifurcations of system (11) when \((x, y)\) varies in a small neighborhood of the interior equilibrium \(E_*(x_*, y_*)\).

Firstly, we translate \(E_*(x_*, y_*)\) into the origin by \(X = x - x_*, Y = y - y_*\) and use Taylor expansion, system (11) can be rewritten as (for simplicity, we still denote \(X, Y\) by \(x, y\), respectively)
\[
\begin{align*}
\frac{dx}{dt} &= a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 x y + P_1(x, y, \lambda_1, \lambda_2), \\
\frac{dy}{dt} &= a_6 + a_7 x - y, \\
\end{align*}
\] (12)
where \(P_1(x, y, \lambda_1, \lambda_2)\) is a \(C^\infty\) function at least of the third order with respect to \((x, y)\), whose coefficients depend smoothly on \(\lambda_1\) and \(\lambda_2\), and
\[
\begin{align*}
a_1 &= \frac{\lambda_1}{q}, \quad a_2 = 1 + \frac{\lambda_1(1 - mp + q + 2pq)(1 + q + pq)}{2p^2q^2}, \quad a_3 = -\frac{1}{q}, \\
a_4 &= \frac{1 - mp + q + 2pq}{8p^2q^2}(2p^2q^2(2 + p + mp^2 + 2q + 3pq) + \lambda_1(1 - mp + q + 2pq)(2 + 4q + 3pq + mp^2q + 2q^2 + 3pq^2)), \\
a_5 &= -\frac{(1 - mp + q + 2pq)(1 + q + pq)}{2p^2q^2}, \quad a_6 = \frac{2p\lambda_2}{1 - mp + q + 2pq}, \quad a_7 = q + \lambda_2,
\end{align*}
\]
where we have eliminated \(n\) and \(A\) by \(n = n_*\) and \(A = A_*\), respectively.

Secondly, we let
\[
\begin{align*}
X &= x, \\
Y &= a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 x y + P_1(x, y, \lambda_1, \lambda_2),
\end{align*}
\]
system (12) becomes (we still denote \(X, Y\) by \(x, y\), respectively)
\[
\begin{align*}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= b_1 + b_2 x + b_3 y + b_4 x^2 + b_5 x y + b_6 y^2 + Q_1(x, y, \lambda_1, \lambda_2),
\end{align*}
\] (13)
where \(Q_1(x, y, \lambda_1, \lambda_2)\) has the same property as \(P_1(x, y, \lambda_1, \lambda_2)\), and
\[
\begin{align*}
b_1 &= a_1 + a_3 a_6, \quad b_2 = \frac{a_2 a_5^2 + a_2^5 a_6 + a_3^2 a_7}{a_3^2}, \quad b_3 = 0, \\
b_4 &= \frac{a_2^3 a_4 - a_1 a_2 a_4 - a_1^2 a_2^2 + a_3^2 a_6 a_5 + a_3^2 a_7}{a_3^2}, \quad b_5 = \frac{a_2^3 a_4 - a_2 a_5 a_4 - a_1^2 a_2^2}{a_3^2}, \quad b_6 = \frac{a_5}{a_3}.
\end{align*}
\]
Thirdly, we let \(dt = (1 - b_6 x)\) \(d\tau\), system (13) becomes (we still denote \(\tau\) by \(t\))
\[
\begin{align*}
\frac{dx}{dt} &= y(1 - b_6 x), \\
\frac{dy}{dt} &= (1 - b_6 x)(b_1 + b_2 x + b_3 y + b_4 x^2 + b_5 x y + b_6 y^2 + Q_1(x, y, \lambda_1, \lambda_2)).
\end{align*}
\] (14)
Next we let $X = x$, $Y = y(1 - b_6 x)$, then system (14) becomes (we still denote $X$, $Y$ by $x$, $y$, respectively)

\[
\begin{align*}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= c_1 + c_2 x + c_3 x^2 + c_4 xy + Q_2(x, y, \lambda_1, \lambda_2),
\end{align*}
\]

(15)

where $Q_2(x, y, \lambda_1, \lambda_2)$ has the same property as $P_1(x, y, \lambda_1, \lambda_2)$, and

\[c_1 = b_1, \quad c_2 = b_2 - b_1 b_6, \quad c_3 = b_4 - 2 b_2 b_6 + b_1 b_6^2, \quad c_4 = b_5.\]

Notice that when $\lambda_1 = \lambda_2 = 0$, we have

\[c_1 = 0, \quad c_2 = 0, \quad c_3 = \frac{(1 - mp + q + 2pq)(1 + q + mp)}{4pq} < 0, \quad c_4 = \frac{(1 - mp + q + 2pq)(1 + p + mp^2 + q + 2pq)}{2p^2q} \neq 0,
\]

since $m < -\frac{1+q}{p} < 0$ and $m \neq m_\ast$.

Fourthly, we let $X = x + \frac{c_2}{2c_3}$, $Y = y$, system (15) becomes (we still denote $X$, $Y$ by $x$, $y$, respectively)

\[
\begin{align*}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= d_1 + d_2 y + d_3 x^2 + d_4 xy + Q_3(x, y, \lambda_1, \lambda_2),
\end{align*}
\]

(16)

where $Q_3(x, y, \lambda_1, \lambda_2)$ has the same property as $P_1(x, y, \lambda_1, \lambda_2)$, and

\[d_1 = c_1 - \frac{c_2^2}{4c_3}, \quad d_2 = \frac{c_2 c_4}{2c_3}, \quad d_3 = c_3, \quad d_4 = c_4.
\]

Making the final change of variables by

\[X = \frac{d_2^2}{d_3} x, \quad Y = \frac{d_4^3}{d_3} y, \quad t = \frac{d_4}{d_3} \tau,
\]

we obtain (we still denote $X$, $Y$, $\tau$ by $x$, $y$, $t$, respectively)

\[
\begin{align*}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= \mu_1 + \mu_2 y + x^2 + xy + Q_4(x, y, \lambda_1, \lambda_2),
\end{align*}
\]

(17)

where $Q_4(x, y, \lambda_1, \lambda_2)$ has the same property as $P_1(x, y, \lambda_1, \lambda_2)$, and

\[\mu_1 = \frac{d_1 d_4^2}{d_3^4}, \quad \mu_2 = \frac{d_2 d_4}{d_3}.
\]

We next express $\mu_1$ and $\mu_2$ in terms of $\lambda_1$ and $\lambda_2$ as follows

\[
\begin{align*}
\mu_1 &= s_1 \lambda_1 + s_2 \lambda_2 + s_3 \lambda_1^2 + s_4 \lambda_1 \lambda_2 + s_5 \lambda_2^2 + o(\lambda_1, \lambda_2^2), \\
\mu_2 &= t_1 \lambda_1 + t_2 \lambda_2 + t_3 \lambda_1^2 + t_4 \lambda_1 \lambda_2 + t_5 \lambda_2^2 + o(\lambda_1, \lambda_2^2),
\end{align*}
\]

(18)
where
\[ s_1 = \frac{4(1 - mp + q + 2pq)(1 + p + mp^2 + q + 2pq)^4}{p^3q^3(1 + q + mp)^3}, \]
\[ s_2 = \frac{-8(1 + p + mp^2 + q + 2pq)^4}{p^3q^3(1 + q + mp)^3}, \]
\[ s_3 = \frac{2(1 - mp + q + 2pq)^2(1 + p + mp^2 + q + 2pq)^3}{p^3q^3(1 + q + mp)^4} \]
\[ \times ((-1 + q)^3 + p(1 + q)^2(5 + 8q) \]
\[ + p^2(1 + q)(6 + 5m - mq + 20q + 29q^2) + p^3(2m(6 + 17q + 8q^2) + q(6 + 13q) \]
\[ + 18q^2)) + mp^4(q(12 + 13q) + m(6 + 8q)) + 6m^2qp^5), \]
\[ s_4 = \frac{-4(1 + p + mp^2 + q + 2pq)^3}{p^3q^3(1 + q + mp)^4} \]
\[ \times (5(1 + q)^4 + p(1 + q)^3(8 - 5m + 43q) + p^2(1 + q)^2(3 \]
\[ + 44q - 28mq + 129q^2) + p^3(1 + q)(q(9 + 75q + 160q^2) + m(3 + 23q - 27q^2) \]
\[ - m^2(8 + 5q)) + p^4(2q^2(3 + 19q + 34q^2) + mq(15 + 76q + 37q^2) - m^2(3 + 28q \]
\[ + 22q^2)) + mp^5(2q^2(6 + 19q) + m(3 - 8q)q - m^2(3 + 4q)) - 3m^2p^6(m - 2q)), \]
\[ s_5 = \frac{8(1 + p + mp^2 + q + 2pq)^4}{p^3q^3(1 + q + mp)^4} \]
\[ \times (11 + 22 + 36p)q + (11 + 36p + 24p^2)q^2), \]
\[ t_1 = \frac{(1 + q + pq)(1 - mp + q + 2pq)(1 + p + mp^2 + q + 2pq)^2}{p^3q^3(1 + q + mp)^2}, \]
\[ t_2 = \frac{-2(1 + q)(1 + p + mp^2 + q + 2pq)^2}{p^3q^3(1 + q + mp)^2}, \]
\[ t_3 = \frac{(1 + q + pq)^2(1 - mp + q + 2pq)^2(1 + p + mp^2 + q + 2pq)(1 + 4q + 3q^2)}{p^3q^3(1 + q + mp)^3} \]
\[ \times + p(1 + m + 3q - mq + 6q^2) + mp^2(2 + 3q) + m^2p^3), \]
\[ t_4 = \frac{-2(1 + q + pq)(1 + p + mp^2 + q + 2pq)}{p^3q^3(1 + q + mp)^3} \]
\[ \times (2(1 + q)^4 - p(-3 + 2m - 16q) \]
\[ + p^2(1 + q)^2(1 - 5(-3 + 2m)q + 44q^2)) + p^3(1 + q)(-m^2(3 + 2q) + m(1 + 8q - 7q^2) \]
\[ + 2q(1 + 11q + 24q^2) + p^4(8q^2(1 + 2q) + 2mq(2 + 11q + 7q^2) - m^2(1 + 9q + 8q^2)) \]
\[ - mp^5(2m(q - 1)q - 8q^3 + m^2(1 + q))), \]
\[ t_5 = \frac{16(1 + q)(1 + p + mp^2 + q + 2pq)^2(1 + (2 + 3p)q + (1 + 3p + 2p^2)q^2)}{p^3q^3(1 + q + mp)^3}. \]

Note that
\[ \left| \frac{\partial(\mu_1, \mu_2)}{\partial(\lambda_1, \lambda_2)} \right|_{\lambda = 0} = \frac{8(1 - mp + q + 2pq)(1 + p + mp^2 + q + 2pq)^5}{p^3q^3(1 + q + mp)^5} \neq 0 \]
since \( p, q > 0, m < \frac{-1 + p}{p} < 0 \) and \( m \neq m_\ast \). Thus the parameter transformation (18) is a homomorphism in a small neighborhood of the origin, and \( \mu_1 \) and \( \mu_2 \) are independent parameters.

The results in Bogdanov [3, 4] and Takens [20] now imply that system (17) undergoes Bogdanov-Takens bifurcation when \((\lambda_1, \lambda_2)\) vary in a small neighborhood of \((0, 0)\), i.e., system (17) is the versal unfolding of Bogdanov-Takens singularity of codimension 2, thus system (11) (i.e., system (9)) can undergo Bogdanov-Takens bifurcation. The representations of the bifurcation curves are as follows:

(i) The saddle-node bifurcation curve \( SN = \{ (\mu_1, \mu_2) | \mu_1 = 0, \mu_2 \neq 0 \} \); 
(ii) The Hopf bifurcation curve \( H = \{ (\mu_1, \mu_2) | \mu_2 = -\mu_1, \mu_1 < 0 \} \); 
(iii) The homoclinic bifurcation curve \( HL = \{ (\mu_1, \mu_2) | \mu_2 = \frac{5}{7}\sqrt{-\mu_1}, \mu_1 < 0 \} \).

From the expression of \( c_3, c_4 \) and \( t = \frac{d\lambda}{dt} = \frac{c_3}{c_3} \tau \), which was used to get system (17), we also have the following results (see [12, 9, 10]):
(i.1) If \( m < m_* \), then \( \frac{c_3}{c_1} > 0 \). Therefore, there exists a repelling Bogdanov-Takens bifurcation of codimension 2 at \( E_*(x_*, y_*) \) for system (11) (i.e., system (9));

(i.2) If \( m > m_* \), then \( \frac{c_3}{c_1} < 0 \). Therefore, there exists an attracting Bogdanov-Takens bifurcation of codimension 2 at \( E_*(x_*, y_*) \) for system (11) (i.e., system (9)).

Remark 1. Zhou et al. [24] showed that system (9) exhibits an attracting Bogdanov-Takens bifurcation of codimension 2 by choosing a set of specific parameter values, which is biologically meaningless because it does not satisfy \( p > q \). We have shown that system (9) undergoes two different topological types of Bogdanov-Takens bifurcations, i.e., repelling and attracting Bogdanov-Takens bifurcations, for general parameter conditions.

3. Numerical simulations. Next, we illustrate our theoretical conclusions by numerical simulations. Firstly, we fixed \( p = 3, q = 2, m = -3 < m_* = -2 \), then get \( A = \frac{9}{4} \) and \( n = 4 \) from \( A = A_* \) and \( n = n_* \), respectively. This set of specific parameter values satisfy the conditions in the case (i) of Theorem 2.2. Then system (17) undergoes repelling Bogdanov-Takens bifurcation when \( (\lambda_1, \lambda_2) \) vary in a small neighborhood of \( (0, 0) \) with this set of specific parameter values. From (18), we can get

\[
\begin{align*}
\mu_1 &= -3\lambda_1 + \frac{3}{4}\lambda_2 - \frac{277}{2}\lambda_1^2 + \frac{47}{8}\lambda_1\lambda_2 + \frac{179}{32}\lambda_2^2 + o(|\lambda_1, \lambda_2|^2), \\
\mu_2 &= \frac{3}{2}\lambda_1 - \frac{1}{8}\lambda_2 + 25\lambda_1^2 + \frac{7}{3}\lambda_1\lambda_2 - \frac{5}{8}\lambda_2^2 + o(|\lambda_1, \lambda_2|^2),
\end{align*}
\]

and the approximate expressions for saddle-node, Homoclinic and Hopf bifurcation curves up to second order are given as follows:

(i) The saddle-node bifurcation curve \( SN = \)
\[
\{(\lambda_1, \lambda_2) : -3\lambda_1 + \frac{3}{4}\lambda_2 - \frac{277}{2}\lambda_1^2 + \frac{47}{8}\lambda_1\lambda_2 + \frac{179}{32}\lambda_2^2 = 0, \quad \mu_2 \neq 0\};
\]

(ii) The Hopf bifurcation curve \( H = \)
\[
\{(\lambda_1, \lambda_2) : -3\lambda_1 + \frac{3}{4}\lambda_2 - \frac{545}{4}\lambda_1^2 + \frac{11}{2}\lambda_1\lambda_2 + \frac{359}{64}\lambda_2^2 = 0, \quad \mu_1 < 0\};
\]

(iii) The homoclinic bifurcation curve \( HL = \)
\[
\{(\lambda_1, \lambda_2) : \frac{75}{49}\lambda_1 + \frac{75}{196}\lambda_2 - \frac{13409}{196}\lambda_1^2 + \frac{257}{98}\lambda_1\lambda_2 + \frac{8999}{3136}\lambda_2^2 = 0, \quad \mu_1 < 0\}.
\]

The repelling Bogdanov-Takens bifurcation diagram of codimension 2 and corresponding phase portraits of system (11) with \( p = 3, q = 2, m = -3, A = \frac{9}{4}, n = 4 \) are given in Figure 4. These bifurcation curves \( H, HL \) and \( SN \) divide the small neighborhood of the origin in the parameter \((\lambda_1, \lambda_2)\)-plane into four regions (see Figure 4(a)).

(a) When \( (\lambda_1, \lambda_2) = (0, 0) \), the unique positive equilibrium is a cusp of codimension 2 (see Figure 3(a)).
(b) There are no positive equilibria when the parameters lie in region I (see Figure 4(b)), we can see that the disease will die out for all positive initial populations.
(c) When the parameters lie on the curve \( SN \), there is a unique positive equilibrium, which is a saddle-node.
Figure 4. The repelling Bogdanov-Takens bifurcation diagram and phase portraits of system (11) with \( p = 3, \quad q = 2, \quad m = -3, \quad A = \frac{9}{4}, \quad n = 4 \). (a) Bifurcation diagram; (b) No equilibria when \((\lambda_1, \lambda_2) = (0.03, 0.25)\) lies in the region I; (c) An unstable focus when \((\lambda_1, \lambda_2) = (0.03, 0.12097)\) lies in the region II; (d) An unstable limit cycle when \((\lambda_1, \lambda_2) = (0.03, 0.1197)\) lies in the region III; (e) An unstable homoclinic loop when \((\lambda_1, \lambda_2) = (0.03, 0.119045)\) lies on the curve HL; (f) A stable focus when \((\lambda_1, \lambda_2) = (0.03, 0.115)\) lies in the region IV.
(d) Two positive equilibria, one is an unstable focus and the other is a saddle, will occur through the saddle node bifurcation when the parameters cross $SN$ into region II (see Figure 4(c)).

(e) An unstable limit cycle will appear through the subcritical Hopf bifurcation when the parameters cross $H$ into region III (see Figure 4(d)), where the focus is stable, whereas the focus is an unstable one with multiplicity one when the parameters lie on the curve $H$.

(f) An unstable homoclinic cycle will occur through the homoclinic bifurcation when the parameters pass region III and lie on the curve $HL$ (see Figure 4(e)).

(g) The relative location of one stable and one unstable manifold of the saddle will be reverse when the parameters cross III into region IV (compare Figure 4(c) and Figure 4(f)).

Secondly, we fixed $p = 3$, $q = 2$, and $m = \frac{-3}{2} > m_\ast = -2$, then get $A = \frac{36}{13}$ and $n = \frac{13}{16}$ from $A = A_\ast$ and $n = n_\ast$, respectively. This set of specific parameter values satisfy the conditions in the case (ii) of Theorem 2.2. Then system (17) undergoes attracting Bogdanov-Takens bifurcation when $(\lambda_1, \lambda_2)$ vary in a small neighborhood of $(0, 0)$ with this set of specific parameter values. From (18), we can get

$$\mu_1 = -\frac{39}{4} \lambda_1 + 3 \lambda_2 + \frac{12337}{16} \lambda_1^2 - \frac{4277}{16} \lambda_1 \lambda_2 + \frac{179}{2} \lambda_2^2 + o(|\lambda_1, \lambda_2|^2),$$

$$\mu_2 = \frac{39}{8} \lambda_1 - \frac{1}{2} \lambda_2 - \frac{16055}{64} \lambda_1^2 + \frac{2899}{48} \lambda_1 \lambda_2 - 10 \lambda_2^2 + o(|\lambda_1, \lambda_2|^2),$$

and the approximate expressions for saddle-node, Homoclinic and Hopf bifurcation curves up to second order are given as follows:

(i) The saddle-node bifurcation curve $SN = \{(\lambda_1, \lambda_2) : \frac{39}{4} \lambda_1 + 3 \lambda_2 + \frac{12337}{16} \lambda_1^2 - \frac{4277}{16} \lambda_1 \lambda_2 + \frac{179}{2} \lambda_2^2 = 0, \mu_2 \neq 0\}$;

(ii) The Hopf bifurcation curve $H = \{(\lambda_1, \lambda_2) : \frac{39}{4} \lambda_1 + 3 \lambda_2 + \frac{50869}{64} \lambda_1^2 - \frac{4355}{16} \lambda_1 \lambda_2 + \frac{359}{4} \lambda_2^2 = 0, \mu_1 < 0\}$;

(iii) The homoclinic bifurcation curve $HL = \{(\lambda_1, \lambda_2) : \frac{975}{196} \lambda_1 + \frac{75}{49} \lambda_2 + \frac{130829}{3136} \lambda_1^2 - \frac{15821}{112} \lambda_1 \lambda_2 + \frac{8999}{196} \lambda_2^2 = 0, \mu_1 < 0\}$.

The attracting Bogdanov-Takens bifurcation diagram of codimension 2 and corresponding phase portraits of system (11) with $p = 3$, $q = 2$, $m = \frac{-3}{2}$, $A = \frac{36}{13}$, $n = \frac{13}{16}$ are given in Figure 5. These bifurcation curves $H, HL$ and $SN$ divide the small neighborhood of the origin in the parameter $(\lambda_1, \lambda_2)$-plane into four regions (see Figure 5(a)).

(i) When $(\lambda_1, \lambda_2) = (0, 0)$, the unique positive equilibrium is a cusp of codimension 2 (see Figure 3(b)).

(ii) There are no positive equilibria when the parameters lie in region I (see Figure 5(b)), we can see that the disease will die out for all positive initial populations.

(iii) When the parameters lie on the curve $SN$, there is a unique positive equilibrium, which is a saddle-node.

(iv) Two positive equilibria, one is a stable focus and the other is a saddle, will occur through the saddle node bifurcation when the parameters cross $SN$ into region II (see Figure 5(c)).
Figure 5. The attracting Bogdanov-Takens bifurcation diagram and phase portraits of system (9) with $p = 3$, $q = 2$, $m = -\frac{3}{2}$, $A = \frac{36}{13}$, $n = \frac{13}{16}$. (a) Bifurcation diagram; (b) No equilibria when $(\lambda_1, \lambda_2) = (0.03, 0.11)$ lies in the region I; (c) A stable focus when $(\lambda_1, \lambda_2) = (0.03, 0.098)$ lies in the region II; (d) A stable limit cycle when $(\lambda_1, \lambda_2) = (0.03, 0.096)$ lies in the region III; (e) A stable homoclinic loop when $(\lambda_1, \lambda_2) = (0.03, 0.09342)$ lies on the curve HL; (f) An unstable focus when $(\lambda_1, \lambda_2) = (0.03, 0.09)$ lies in the region IV.
(v) A stable limit cycle will appear through the subcritical Hopf bifurcation when the parameters cross \( H \) into region III (see Figure 5(d)), where the focus is unstable, whereas the focus is a stable one with multiplicity one when the parameters lie on the curve \( H \).

(vi) A stable homoclinic cycle will occur through the homoclinic bifurcation when the parameters pass region III and lie on the curve \( HL \) (see Figure 5(e)).

(vii) The relative location of one stable and one unstable manifold of the saddle will be reverse when the parameters cross III into region IV (compare Figure 5(c) and Figure 5(f)).

4. Discussion. In some epidemic diseases, when the infectious number is getting larger, the psychological factor can play an important influence on the disease, since people and government can take a series of protection measures and intervention policies to control the disease ([22]). The psychological effect can be modelled by a nonmonotone incidence rate, i.e., the infection force \( g(I) \) increases firstly when the infection number is small, and then decreases when the infection number is getting larger. In [23], Xiao and Zhou proposed a generalized nonmonotone incidence rate (6) to model the psychological effect. For model (1) with generalized nonmonotone incidence rate (6) (i.e., model (9)), they showed the existence of a cusp of codimension 2, bistable phenomenon and periodic oscillation. Later, Zhou et al. [24] further studied the existence of Bogdanov-Takens bifurcation by choosing a set of specific parameter values. However, the set of parameter values they chosen to unfold the Bogdanov-Takens bifurcation is biologically meaningless because it does not satisfy \( p > q \). In this paper, we continue to consider model (1) with generalized nonmonotone incidence rate (6), for general parameter conditions, we showed that model (9) undergoes two different topological types of Bogdanov-Takens bifurcations, i.e., repelling and attracting Bogdanov-Takens bifurcations. We calculated the approximate expressions for saddle-node, Homoclinic and Hopf bifurcation curves up to second order. Furthermore, some numerical simulations, including bifurcations diagrams and corresponding phase portraits, are also given to illustrate the theoretical results. Our results can be seen as a complement to the work in [23] and [24].

The stable limit cycle arising from the supercritical Hopf bifurcation implies the existence of sustained periodic oscillation for the disease, and the disease tends to periodic outbreak when the initial population lies in the attracting region of the stable limit cycle. It is very important to understand the underlying mechanics of periodic oscillations, which have been observed in the real world [1, 7, 8, 16, 13, 17, 18].

From Lemma 2.1, if \((A,m,n,p,q) \in \Gamma, n = n_*, \ A = A_* \) and \( m \neq m_* \), then the unique positive equilibrium \( E_*(x_*,y_*) \) of system (9) is a cusp of codimension 2. Moreover, we find that \( E_*(x_*,y_*) \) is a cusp of codimension at least 3 when \((A,m,n,p,q) \in \Gamma, n = n_*, \ A = A_* \) and \( m = m_* \), then system (9) may undergo Bogdanov-Takens bifurcation of codimension more than 2. We will consider these problems in the future.

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