Strong polygamy of quantum correlations in multi-party quantum systems

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We propose a new type of polygamy inequality for multi-party quantum entanglement. We first consider the possible amount of bipartite entanglement distributed between a fixed party and any subset of the rest parties in a multi-party quantum system. By using the summation of these distributed entanglements, we provide an upper bound of the distributed entanglement between a party and the rest in multi-party quantum systems. We then show that this upper bound also plays as a lower bound of the usual polygamy inequality, therefore the strong polygamy of multi-party quantum entanglement. For the case of multi-party pure states, we further show that the strong polygamy of entanglement implies the strong polygamy of quantum discord.

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I. INTRODUCTION

Entanglement is one of the most remarkable features in the field of quantum information and computation theory with many useful applications such as quantum teleportation and quantum key distribution [13]. One of the essential differences of quantum entanglement from other classical correlations is in its restricted shareability; if a pair of parties in a multi-party quantum system share maximal entanglement, they cannot have any entanglement nor classical correlations with the rest. This restricted shareability of entanglement is known as the monogamy of entanglement (MoE) [4, 5], which does not have any classical counterpart, and this makes quantum physics fundamentally different from classical physics.

In the seminal paper by Coffman, Kundu and Wooters [3], MoE was mathematically characterized in forms of a trade-off inequality; for a three-qubit state \( \rho_{ABC} \) with two-qubit reduced density matrices \( \rho_{AB} = \text{tr}_C \rho_{ABC} \) and \( \rho_{AC} = \text{tr}_B \rho_{ABC} \),

\[
\tau(\rho_{A(BC)}) \geq \tau(\rho_{AB}) + \tau(\rho_{AC}),
\]

where \( \tau(\rho_{ABC}) \) is the entanglement of \( \rho_{ABC} \) with respect to the bipartition between A and BC measured by tangle [4], and \( \tau(\rho_{AB}) \) and \( \tau(\rho_{AC}) \) are the tangles of \( \rho_{AB} \) and \( \rho_{AC} \) respectively. Later, Inequality (1) was generalized for multi-qubit systems [7] and some classes of multi-qubit systems in terms of various entanglement measures [8]. It was recently shown that squashed entanglement is a faithful entanglement measure, which also shows the monogamy inequality of entanglement in arbitrary dimensional quantum systems [9].

Whereas MoE shows the restricted shareability of bipartite entanglement in multi-party quantum systems, the possible amount of bipartite entanglement distribution assisted by the third party is known to have a dually monogamous (thus polygamous) property in multi-party quantum systems; for three-qubit systems, polygamy of entanglement (PoE) was first characterized as a polygamy inequality

\[
\tau(\rho_{A(BC)}) \leq \tau_a(\rho_{AB}) + \tau_a(\rho_{AC}),
\]

where \( \tau_a(\rho_{AB}) \) and \( \tau_a(\rho_{AC}) \) are the tangle of assistance of \( \rho_{AB} \) and \( \rho_{AC} \) respectively [10, 11]. Inequality (2) was generalized for various classes of multi-party, higher dimensional quantum systems [12], and a general polygamy inequality of entanglement was recently shown in terms of entanglement of assistance in arbitrary dimensional multi-party quantum systems [13].

The study of shareability and distribution of quantum correlations, especially quantum entanglement, in multi-party quantum systems is the key ingredient of many quantum information and communication protocols. For example, due to the mutually-exclusive relation of entanglement sharing characterized by monogamy inequality, one can possibly quantify how much information an eavesdropper could potentially obtain about the secret key to be extracted in quantum cryptography [14]. In other words, the security of quantum key distribution protocols that prohibits an eavesdropper from obtaining any information without disturbance is guaranteed by MoE, the law of quantum physics, rather than assumptions on the difficulty of computation.

Here, we propose a new type of polygamy inequality for quantum entanglement; in multi-party quantum systems, we first consider the possible amount of bipartite entanglement distributed between a fixed party and any subset of the rest parties. By using the summation of these distributed entanglements, we provide an upper bound of the distributed entanglement between a party and the rest. We then show that this upper bound also plays as a lower bound of the general polygamy inequality of multi-party quantum entanglement; therefore the strong polygamy of multi-party quantum entanglement. For the case of multi-party pure states, we further show that the strong polygamy of entanglement implies that of quantum discord.

This paper is organized as follows. In Section III, we briefly recall the definitions and some properties of bi-
partite quantum correlations such as entanglement of assistance, quantum discord, one-way unlocalizable entanglement and one-way unlocalizable quantum discord. In Section II we establish the strong polygamy of distributed entanglement in terms of EoA, and we also show a close relation between the strong polygamy of entanglement and quantum discord for multi-party pure states in Section III B. In Section IV we summarize our results.

II. BIPARTITE QUANTUM CORRELATIONS

For a bipartite quantum state $\rho_{AB}$, its one-way classical correlation $J^+ (\rho_{AB})$ is

$$J^+ (\rho_{AB}) = \max_{\{M_x\}} \left[ S(\rho_A) - \sum_x p_x S(\rho_A^x) \right], \quad (3)$$

where $p_x = \text{tr}[(I_A \otimes M_x)\rho_{AB}]$ is the probability of the outcome $x$, $\rho_A^x = \text{tr}_B[I_A \otimes M_x]\rho_{AB}$ is the state of system $A$ when the outcome was $x$, and the maximum is taken over all the measurements $\{M_x\}$ applied on system $B$.

For a tripartite pure state $|\psi\rangle_{ABC}$ with reduced density matrices $\rho_A = \text{tr}_{BC}|\psi\rangle_{ABC}\langle\psi|$, $\rho_{AB} = \text{tr}_{C}|\psi\rangle_{ABC}\langle\psi|$, and $\rho_{AC} = \text{tr}_{B}|\psi\rangle_{ABC}\langle\psi|$, a trade-off relation between quantum entanglement and classical correlation was shown:

$$S(\rho_A) = J^+ (\rho_{AB}) + E_f(\rho_{AC}), \quad (4)$$

where

$$E_f(\rho_{AC}) = \min \sum_i p_i S(\rho_A^i) \quad (5)$$

is the entanglement of formation (EoF) of $\rho_{AC}$, whose minimization is taken over all pure state decompositions of $\rho_{AC}$,

$$\rho_{AC} = \sum_i p_i |\phi^i\rangle_{AC}\langle\phi^i|, \quad (6)$$

with $\text{tr}_C|\phi^i\rangle_{AC}\langle\phi^i| = \rho_A^i$.

From the definition, $E_f(\rho_{AC})$ is considered as the minimum averaged entanglement needed to prepare $\rho_{AC}$, and the term formation naturally arises. Furthermore, Eq. (4) can be interpreted as follows: for any tripartite pure state $|\psi\rangle_{ABC}$ (a three-party closed quantum system), the total correlation between subsystems $A$ and $BC$ quantified by the entropy $S(\rho_A)$ consists of the classical correlation $J^+ (\rho_{AB})$ between subsystems $A$ and $B$, and the formation of entanglement $E_f(\rho_{AC})$ between $A$ and $C$.

As a dual quantity to EoF, the entanglement of assistance (EoA) is defined as the maximum average entanglement

$$E_a(\rho_{AC}) = \max \sum_i p_i S(\rho_A^i), \quad (7)$$

over all possible pure state decompositions of $\rho_{AC}$.

EoA is clearly a mathematical dual to EoF because one takes the maximum average entanglement whereas the other takes the minimum.

We also note that for a pure state $|\psi\rangle_{ABC}$, all possible pure state decompositions of $\rho_{AC}$ can be realized by rank-1 measurements of subsystem $B$, and conversely, any rank-1 measurement can be induced from a pure state decomposition of $\rho_{AC}$. Thus $E_a(\rho_{AC})$ can be considered as the possible maximum average entanglement that can be distributed between $A$ and $C$ with the assistance of the environment $B$. This makes the duality between EoF and EoA clearer because one is the formation of entanglement whereas the other is the possible entanglement distribution.

Similarly to the duality between EoF and EoA, we have a dual quantity to $J^+ (\rho_{AB})$: for a bipartite state $\rho_{AB}$, the one-way unlocalizable entanglement (UE) is defined as

$$E_u^+ (\rho_{AB}) := \min_{\{M_i\}} \left[ S(\rho_A) - \sum_x p_x S(\rho_A^x) \right], \quad (8)$$

where the minimum is taken over all possible rank-1 measurements $\{M_i\}$ applied on system $B$.

Moreover, the trade-off relation in Eq. (4) was also shown to have a dual relation in terms of EoA and UE in three-party quantum systems. For a three-party pure state $|\psi\rangle_{ABC}$,

$$S(\rho_A) = E_u^+(\rho_{AB}) + E_a(\rho_{AC}). \quad (9)$$

For a bipartite state $\rho_{AB}$, quantum discord (QD) is defined as the difference between the mutual information and one-way classical correlation,

$$\delta^+ (\rho_{AB}) = I(\rho_{AB}) - J^+ (\rho_{AB}), \quad (10)$$

where

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \quad (11)$$

is the mutual information of $\rho_{AB}$ with reduced density matrices $\rho_A$ and $\rho_B$ onto its subsystems $A$ and $B$ respectively.

Based on the duality between one-way classical correlation and UE, a dual quantity to QD was introduced; for a bipartite quantum state $\rho_{AB}$ its one-way unlocalizable quantum discord (UD) is defined as

$$\delta_u^+ (\rho_{AB}) = I(\rho_{AB}) - E_u^+(\rho_{AB}), \quad (12)$$

where $E_u^+(\rho_{AB})$ is the UE of $\rho_{AB}$ in Eq. (8).

III. STRONG POLYGAMY OF QUANTUM CORRELATIONS

A. Strong Polygamy of Quantum Entanglement

In multi-party quantum systems, the distribution of bipartite entanglement quantified by EoA has a polygamous relation as follows; for an $n + 1$-party quantum
state $\rho_{AB_1\cdots B_n}$ with reduced density matrices $\rho_{BA_i}$ on bipartite subsystems $AB_i$ for $i = 1, \cdots, n$,

\[
E_a (\rho_{A(B_1\cdots B_n)}) \leq \sum_{i=1}^{n} E_a (\rho_{AB_i}) \leq \sum_{i=1}^{n} E_a (\rho_{AB_i}) ,
\]

where $E_a (\rho_{A(B_1\cdots B_n)})$ is EoA of $\rho_{AB_1\cdots B_n}$ with respect to the bipartition between $A$ and the rest, and $E_a (\rho_{BA_i})$ is EoA of $\rho_{BA_i}$ for $i = 1, \cdots, n$.

Let us denote $\mathcal{B} = \{B_1, \cdots, B_n\}$, that is, the set of subsystems $B_i$’s, and consider a nonempty proper subset $\mathcal{X} = \{B_i, \cdots, B_k\}$ of $\mathcal{B}$ for $1 \leq k \leq n - 1$. Together with the complement $\mathcal{X}' = \mathcal{B} - \mathcal{X}$ of $\mathcal{X}$ in $\mathcal{B}$, $\rho_{AB_1\cdots B_n}$ can also be considered as a three-party quantum state $\rho_{A\mathcal{X}\mathcal{X}'}$. Furthermore, the polygamy inequality in (13) implies

\[
E_a (\rho_{A\mathcal{B}}) = E_a (\rho_{A\mathcal{X}\mathcal{X}'}) \leq E_a (\rho_{A\mathcal{X}}) + E_a (\rho_{A\mathcal{X}'}) ,
\]

where $E_a (\rho_{A\mathcal{X}})$ and $E_a (\rho_{A\mathcal{X}'})$ are EoA of reduced density matrices $\rho_{A\mathcal{X}}$ and $\rho_{A\mathcal{X}'}$, respectively. Because Inequality (14) holds for any proper subset $\mathcal{X}$ of $\mathcal{B}$, we consider all possible nonempty proper subsets $\mathcal{X}$ of $\mathcal{B}$, which lead us to the following inequality,

\[
E_a (\rho_{A\mathcal{B}}) \leq \frac{1}{2^n - 2} \sum_{\mathcal{X}} (E_a (\rho_{A\mathcal{X}}) + E_a (\rho_{A\mathcal{X}'})) ,
\]

where the summation is over all possible nonempty proper subsets $\mathcal{X}$’s.

Here we note that the set of all nonempty proper subsets of $\mathcal{B}$ is the same with the set of their complements;

\[
\{\mathcal{X}|\mathcal{X} \subset \mathcal{B}\} = \{\mathcal{X}'|\mathcal{X}' \subset \mathcal{B}\} ,
\]

thus we have

\[
\sum_{\mathcal{X}} E_a (\rho_{A\mathcal{X}'}) = \sum_{\mathcal{X}} E_a (\rho_{A\mathcal{X}}) ,
\]

and Eq. (15) becomes

\[
E_a (\rho_{A\mathcal{B}}) \leq \frac{1}{2^n - 1} \sum_{\mathcal{X}} E_a (\rho_{A\mathcal{X}}) .
\]

By considering all possible nonempty proper subsets $\mathcal{X}$ of $\mathcal{B}$ and using Eqs. (15) and (17), we have

\[
\frac{1}{2^n - 1} \sum_{\mathcal{X}} E_a (\rho_{A\mathcal{X}}) \leq \sum_{i=1}^{n} E_a (\rho_{AB_i}) .
\]

From inequalities (18) and (20), we have the following strong polygamy inequalities of distributed entanglement in multi-party quantum systems; for any multi-party state $\rho_{AB_1\cdots B_n}$, (pure or mixed)

\[
E_a (\rho_{A\mathcal{B}}) \leq \frac{1}{2^n - 1} \sum_{\mathcal{X}} E_a (\rho_{A\mathcal{X}}) \leq \sum_{i=1}^{n} E_a (\rho_{AB_i}) ,
\]

where the first summation is over all nonempty proper subsets $\mathcal{X}$ of $\mathcal{B} = \{B_1, \cdots, B_n\}$.

Here, the term strong is twofold. First, Inequality (21) is in fact tighter than the usual polygamy inequality in (13). Moreover, we have considered the entanglement distribution (EoA) between the single party $A$ and all possible subsets $\mathcal{X}$’s of $\mathcal{B}$ to obtain a tighter polygamy inequality whereas the usual polygamy inequality only considers EoA between $A$ and each single party ($B_i$’s) in $\mathcal{B}$.

### B. Strong Polygamy of Quantum Discord

Let us now consider strong polygamy inequality of quantum discord in multi-party quantum systems in terms of UD. We first note that the definition of UD in Eq. (12) and the relation between EU and EoA in Eq. (9) lead us to the following relation between ED and EoA; for a three-party pure state $|\psi\rangle_{ABC}$ with its reduced density matrices $\rho_{AB}$ and $\rho_{AC}$,

\[
E_a (\rho_{AB}) = \delta_u^{++} (\rho_{AC}) + S (\rho_{A|C}) ,
\]

where $S (\rho_{A|C}) = S (\rho_{AC}) - S (\rho_C)$ is the conditional entropy of $\rho_{AC}$. For a multi-party pure state $|\psi\rangle_{AB,\cdots, A_n}$ and a nonempty proper subset $\mathcal{X}$ of $\mathcal{B}$, Eq. (22) implies

\[
E_a (\rho_{A\mathcal{X}}) = \delta_u^{++} (\rho_{A\mathcal{X}'}) + S (\rho_{A|\mathcal{X}'}) ,
\]

where $\rho_{A\mathcal{X}}$ and $\rho_{A\mathcal{X}'}$ are the reduced density matrices of $|\psi\rangle_{\mathcal{A}\mathcal{B},\cdots, A_n}$ on to subsystems $\mathcal{A}\mathcal{X}$ and $\mathcal{A}\mathcal{X}'$, respectively.

Now we consider above equality for all possible nonempty proper subsets $\mathcal{X}$ of $\mathcal{B} = \{B_1, \cdots, B_n\}$ to obtain

\[
\sum_{\mathcal{X}} E_a (\rho_{A\mathcal{X}}) = \sum_{\mathcal{X}} (\delta_u^{++} (\rho_{A\mathcal{X}'}) + S (\rho_{A|\mathcal{X}'})) = \sum_{\mathcal{X}} \delta_u^{++} (\rho_{A\mathcal{X}'}) + \sum_{\mathcal{X}} S (\rho_{A|\mathcal{X}'}) = \sum_{\mathcal{X}} \delta_u^{++} (\rho_{A\mathcal{X}}) + \sum_{\mathcal{X}} S (\rho_{A|\mathcal{X}}) ,
\]

For a nonempty proper subset $\mathcal{X} = \{B_1, \cdots, B_n\}$ of $\mathcal{B}$ and its complement $\mathcal{X}' = \{B_{n+1}, \cdots, B_n\}$, Inequality (13) also implies

\[
E_a (\rho_{A\mathcal{X}}) + E_a (\rho_{A\mathcal{X}'}) \leq \sum_{j=1}^{k} E_a (\rho_{AB_j}) + \sum_{j=k+1}^{n} E_a (\rho_{AB_j}) = \sum_{i=1}^{n} E_a (\rho_{AB_i}) .
\]
where the last equality is due to Eq. (10). Furthermore, due to the complementary property of conditional entropy, we have

$$S\left(\rho_{A|X}\right) + S\left(\rho_{A|X'}\right) = 0$$

for any three-party pure state $|\psi\rangle_{AXX'}$, and this implies

$$\sum_{X} S\left(\rho_{A|X}\right) = 0,$$

where the summation is over all nonempty proper subsets of $B$. From Eqs. (24) and (26), we have

$$\sum_{X} E_a\left(\rho_{A|X}\right) = \sum_{X} \delta_u^{-}\left(\rho_{A|X}\right),$$

for any multi-party pure state $|\psi\rangle_{AB}$ and its reduced density matrix $\rho_{A|X}$.

Let us now consider UD of a bipartite pure state $|\psi\rangle_{AB}$; the definition of UD in Eq. (12) leads us to

$$\delta_u^{-}\left(|\psi\rangle_{AB}\right) = I\left(|\psi\rangle_{AB}\right) - E_u^{-}\left(|\psi\rangle_{AB}\right).$$

For a bipartite pure state $|\psi\rangle_{AB}$, we have

$$I\left(|\psi\rangle_{AB}\right) = S(\rho_A) + S(\rho_B) - S\left(|\psi\rangle_{AB}\right)$$

$$= 2S(\rho_A),$$

thus Eq. (28) becomes

$$\delta_u^{-}\left(|\psi\rangle_{AB}\right) = 2S(\rho_A) - E_u^{-}\left(|\psi\rangle_{AB}\right).$$

We note that any purification of $|\psi\rangle_{AB}$ in three-party quantum systems $ABC$ is trivially a product state $|\psi\rangle_{AB} \otimes |\phi\rangle_C$ for some pure state $|\phi\rangle_C$. From the definition of EU in Eq. (5), we have

$$E_u^{-}\left(|\psi\rangle_{AB}\right) = S(\rho_A) - E_a\left(\rho_{AC}\right),$$

where $\rho_{AC}$ is the reduced density matrix of $|\psi\rangle_{AB} \otimes |\phi\rangle_C$ on subsystems $AC$, which is

$$\rho_A \otimes |\phi\rangle_C\langle\phi|.$$

Because $E_a(\rho_{AC}) = 0$ for the product state $\rho_{AC}$, we have

$$E_u^{-}\left(|\psi\rangle_{AB}\right) = S(\rho_A),$$

for the bipartite pure state $|\psi\rangle_{AB}$, therefore Eqs. (30) and Eq. (33) lead us to

$$\delta_u^{-}\left(|\psi\rangle_{AB}\right) = S(\rho_A).$$

We also note that EoA of $|\psi\rangle_{AB}$ is just the entropy of subsystems, thus

$$\delta_u^{-}\left(|\psi\rangle_{AB}\right) = E_a\left(\rho_{AB}\right).$$

Now, from Eqs. (27) and (35) together with Inequality (18), we have

$$\delta_u^{-}\left(|\psi\rangle_{AB}\right) \leq \frac{1}{2n-1-1} \sum_{X} \delta_u^{-}\left(\rho_{A|X}\right),$$

where the summation is over all non-empty proper subsets of $B$. In other words, strong polygamy inequality of entanglement in (18) also implies the strong polygamy of quantum discord for the case of multi-party pure states $|\psi\rangle_{AB}$, that is, closed quantum systems.

IV. SUMMARY

We have proposed a strong polygamy inequality for multi-party quantum entanglement; by considering the possible amount of entanglement distribution in terms of EoA between a fixed party and any subset of the rest parties in a multi-party quantum system, we have provided an upper bound of the distributed entanglement between a party and the rest. We have also shown that this upper bound plays as a lower bound of the usual polygamy inequality. We have further shown that the strong polygamy of entanglement implies that of quantum discord for the case of multi-party pure states.

Our results strengthen the characterization of the polygamous nature of entanglement in multi-party quantum systems. Moreover, our results shows a closed relation between PoE and quantum discord, which provides a strong clue for possible relations between PoE and other quantum correlation measures. Noting the importance of the study on multipartite quantum correlations, our results can provide a rich reference for future work on the study of quantum correlations in multi-party quantum systems.

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