Landau-Ginzburg Topological Theories
in the Framework of GKM and Equivalent Hierarchies

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Abstract

We consider the deformations of “monomial solutions” to Generalized Kontsevich Model and establish the relation between the flows generated by these deformations with those of $N=2$ Landau-Ginzburg topological theories. We prove that the partition function of a generic Generalized Kontsevich Model can be presented as a product of some “quasiclassical” factor and non-deformed partition function which depends only on the sum of Miwa transformed and flat times. This result is important for the restoration of explicit $p-q$ symmetry in the interpolation pattern between all the $(p,q)$-minimal string models with $c<1$ and for revealing its integrable structure in $p$-direction, determined by deformations of the potential. It also implies the way in which supersymmetric Landau-Ginzburg models are embedded into the general context of GKM. From the point of view of integrable theory these deformations present a particular case of what is called equivalent hierarchies.

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1. In [1, 2] a new model was introduced, which naturally incorporates all the non-perturbative partition functions of $c < 1$ minimal string models and parameterizes them by unique potential which allows one to interpolate smoothly between the models.

On one hand, this potential if polynomial describes rolling among different reductions of KP hierarchy so that the whole space of these reductions falls out into different non-connected “orbits” (or “domains” or “universality” classes) marked by the higher degree of the polynomial. Thus, one problem is of explicit description of this rolling.

On the other hand, it seems that GKM has a universal nature, containing information about all subjects related to 2-dimensional gravity. One example of this phenomena has been already discussed in [3], where it was shown that discrete matrix models have a nice description in terms of GKM. There is another interesting problem – the relation between GKM and topological Landau-Ginzburg Models (LGM) [4], which were the subject of a very interesting recent development [4, 5, 6, 7, 8].

It turns out that the resolution to both above mentioned problems can be found along the same line of investigation. Indeed, the connection to LGM is naturally related to the question of interpolation between various $(p, q)$-solutions to $c < 1$ 2d gravity [1, 2, 9], i.e. different reductions of KP system, and quasiclassical limit of GKM. In this letter we concentrate on the most important point in this connection – the appearance and relation between corresponding integrable hierarchies which in the language of integrable theories is nothing but what is called “equivalent hierarchies” [10].

In order to do this, we first compute the derivatives with respect to first several “Miwa times” and show that they are naturally expressed through corresponding derivatives with respect to the coefficients of the potential. Moreover, it turns out that there are special combinations of these coefficients — so called flat or $p$-times [6, 11, 12] — which naturally occur in the framework of GKM [3] and are just those variables in what two above mentioned problems are simultaneously resolved.

In the most elegant way all the formulas may be rewritten using reparameterizations of the spectral parameter, which naturally lead to the description in terms of equivalent hierarchies. Different equivalent hierarchies in the GKM context are parameterized by polynomials of the same degree, and these polynomials are just superpotentials of LGM. Thus, we derive our main result — that “the topologically-deformed” GKM is expressed through the non-deformed one, being equivalent solution to the same integrable hierarchy (or, put differently, corresponding to different reductions of KP hierarchy which lie at the same “orbit”).

To be more precise, first we remind that the partition function of GKM is defined as a matrix integral

$$Z_{GKM}^{(N)}[V|M] \equiv C_{GKM}^{(N)}[V|M] e^{Tr V(M) - Tr MV'(M)} \int DX \ e^{-Tr V(X) + Tr V'(M) X}$$

1 In this letter by topological LGM we mean the LGM interacting with topological gravity.

2 We call the particular linear combinations of the coefficients of the potential $p$-times because they describe the deformation in one of possible directions — “$p$-direction” in deforming $(p, q)$-models, while deformation in the other “$q$-direction” is described by Miwa times. To avoid misunderstanding, in [3] the opposite notations were used.
over $N \times N$ “Hermitean” matrices, with normalization factor given by the Gaussian integral
\[
C_{GKM}^{(N)}[V|M]^{-1} \equiv \int DY \ e^{-TrV_2[M,Y]},
\]
\[
V_2 \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} Tr[V(M + \epsilon Y) - V(M) - \epsilon YV'(M)]. \tag{2}
\]

The partition function $Z_{GKM}$ actually depends on $M$ through the invariant variables
\[
T_k = \frac{1}{k} Tr \ M^{-k}, \quad k \geq 1,
\]
moreover, if rewritten in terms of $T_k$, $Z_{GKM}[V|T] = Z_{GKM}^{(N)}[V|M]$ is essentially independent of the size $N$ of the matrices: the only origin of $N$-dependence is that of $T_k$ themselves (for finite $N$ not all the $T_k$ are algebraically independent).

As a function of $T_k$ $Z_{GKM}[V|T_k]$ is a $\tau$-function of KP-hierarchy, $Z_{GKM}[V|T_k] = \tau_V[T_k]$, while “potential” $V$ specifies the relevant point of the infinite-dimensional Grassmannian.

For various choices of the “potential” $V(X)$ $GKM$ reproduces continuum limits of all multimatrix models: $V(X) = \frac{X^{p+1}}{p+1}$ is associated with the $(p-1)$-matrix model and thus with the entire set of $(p,q)$-minimal string models with all possible $q$’s. In order to specify $q$ one needs to make a special choice of $T$-variables: all $T_k = 0$, except for $T_1$ and $T_{p+q}$. The symmetry between $p$ and $q$ is implicit in this kind of formulation and is to be revealed somehow in the future studies. However, it immediately implies that since integrability occurs in $q$-direction (i.e. $Z$ is a $\tau$-function as a function of $T$’s), it should also appear in $p$-direction (i.e. in some sense $Z$ should also be a $\tau$-function as a function of the coefficients of the potential $V$).

In this letter we demonstrate that this is indeed the case, moreover, that the $\tau$-function is essentially the same in both directions. Indeed, from the considerations of the spherical limit of matrix models \[1\, 2\] as well as of LGM it is known that the dynamics over several first Miwa times and $p$-times coincide. As this dynamics, besides integrable equations, is consistent with the string equation which is a first-derivative equation, one can expect that the derivatives of the (proper re-defined) $\tau$-function with respect to both Miwa and $p$-times are equal. This is really the case in all genera, and the main statement we are going to prove reads
\[
Z_{GKM}[V|T_k] = \tau_V[T_k] = \exp \left( -\frac{1}{2} \sum_{i,j} A_{ij}(t)(\tilde{T}_i + t_i)(\tilde{T}_j + t_j) \right) \tau_p[\tilde{T}_k + t_k], \tag{4}
\]
where
\[
V(x) = \sum_{k=1}^{p+1} \frac{v_k}{k} x^k, \tag{5}
\]
\[
\tilde{T}_k = \frac{1}{k} Tr \ M^{-k}, \tag{6}
\]
\[
\tilde{M}^p = V'(M) \equiv W(M), \tag{7}
\]
\[
A_{ij} = Res W^{1/p} dW^{1/p}_+ = \frac{\partial^2 \log \tau_0}{\partial t_i \partial t_j}, \tag{8}
\]

\[3\] Proper changing the variable $X$ in the matrix integral, one can always choose $v_{p+1} = 1$, $v_p = 0$, what is convenient normalization and usually implies throughout the paper.
\( \tau_0 \) is the quasiclassical \( \tau \)-function and parameters \( \{ t_k \} \) are certain linear combinations of the coefficients \( \{ v_k \} \) of the potential

\[
t_k = \frac{p}{k(p-k)} \text{Res} W^{1-k/p}(\mu) d\mu
\]  

i.e. that a generic GKM \( \tau \)-function is expressed through \( \tau \)-function of \( p \)-reduction, depending only on the sum of time-variables \( \tilde{T}_k \) and \( t_k \), associated with deformations in “\( q \)” and “\( p \)-directions” respectively.

The change of the spectral parameter in (5) \( M \rightarrow \tilde{M} \) (and corresponding times \( T_k \rightarrow \tilde{T}_k \)) is a natural step from the point of view of equivalent hierarchies. As to original GKM, its \( p \)-dependence is not seen in eq.(1), though it can be also introduced implicitly through the choice of integration contour.

We omitted from this letter some proofs which will be published in the separate publication [13], together with more detailed discussion of some points discussed in the last section of this letter.

**2. Time derivatives.** The first step of our calculations concerns the derivatives of \( Z_{GKM} \) with respect to the time-variables \( T_k \). Such derivatives define nonperturbative correlators in string models and are of their own interest for the theory of GKM. The derivatives with respect to \( T_k \) with \( k \geq p+1 \) (responsible for the correlators of irrelevant operators) are not very easy to evaluate, things are simpler for \( T_k \) with \( 1 \leq k \leq p \). This is due to the fact mentioned in the previous section that these derivatives should be simple expressed through derivatives with respect to \( p \)-times. Using the obvious notation of average so that \( Z_{GKM} = \langle 1 \rangle \), we have

\[
\frac{\partial Z_{GKM}}{\partial T_k} \bigg|_V = \langle Tr M^k - Tr X^k \rangle, \quad 1 \leq k \leq p
\]  

It is implied that the derivative in the l.h.s. is taken under constant values of \( v_m \), i.e. preserving the form of the potential \( V \).

The r.h.s. of (10) can be also represented as

\[
- \frac{\partial Z_{GKM}}{\partial T_k} \bigg|_V = \left\langle Tr \frac{\partial V(X)}{\partial v_k} - Tr \frac{\partial V(M)}{\partial v_k} \right\rangle, \quad 1 \leq k \leq p
\]  

which looks similar but actually is different from \( -\frac{\partial}{\partial v_k} Z_{GKM} \), as it would be if (1) does not contain \( t \)-dependence of the coefficients \( A_{ij} \) of the quadratic form. The problem is that \( \frac{\partial}{\partial v_k} Z_{GKM} \) gets contributions not only from differentiating \( V(X) - V(M) \) in exponentials in (1) but also from the term \( V'(M)(X-M) \) as well as from the pre-exponential \( C[V|M] \). Let us keep this in mind for a while and now we turn to a slightly different question.

**Change of \( M \)-variables.** Another question which can be asked about \( Z_{GKM}[V|T] \) is whether it corresponds to anything reasonable if the KP-times are introduced in a way different from (3). For example, let us define, instead of (3), new variables

\[
T_k^{(f)} = \frac{1}{k} Tr[f(M)]^{-k}
\]  

If \( f(m) = m(1+o(1/m)) \) this can be considered as an allowed change of spectral parameter and it is not difficult to preserve the statement that \( Z_{GKM} \) is a KP \( \tau \)-function of \( T_k^{(f)} \) – variables: in fact, it is enough
to modify slightly the definition of $C[V|M]$. Instead of (3) it should be now

$$C^{(f)}[V|M] = C[V|f(M)].$$

(13)

Then

$$Z^{(f)}_{GKM} = \frac{C^{(f)}}{C} Z_{GKM} = \tau(T^{(f)}_k).$$

(14)

Of course, this procedure changes also the point of the Grassmannian.

The next step will be to find out an analog of the results (10) and (11) with $T_k$’s replaced by $T^{(f)}_k$’s. It is easy to do using eq. (10)

$$\frac{\partial}{\partial T^{(f)}_k} Z_{GKM}\bigg|_V = \sum_i \frac{\partial T_i}{\partial T^{(f)}_k} \frac{\partial Z_{GKM}}{\partial T_i} = \left( \sum_i \frac{\partial T_i}{\partial T^{(f)}_k} \left( T_f X^i - Tr M^i \right) \right) = \left( Tr[f^+_k(X)] - Tr[f^+_k(M)] \right), \ 1 \leq k \leq p,$n

(15)

where we used the following transformation of times:

$$T^{(f)}_k = \frac{1}{\lambda} \sum_j j T_j Res \lambda^{-1} f^{-k}(\lambda) d\lambda,$n

(16)

$$T_k = \sum_j T^{(f)}_j Res \lambda^{-k} f^j(\lambda) d\lambda.$n

(17)

Note that in the l.h.s. of (13) we differentiate $Z_{GKM}$ and not $Z^{(f)}_{GKM}$, in the latter case we get an additional correction of $\frac{\partial}{\partial T}[\log C^{(f)}/C]$ which is a sort of “quantum” correction, since $C[V|M]$ can be considered as a quantum correction to the classical one given by terms in the exponentials [7].

Now, in order to find the analog of (11) let us introduce new parameters $v^{(f)}_k$ in such a way that

$$\frac{\partial V(X)}{\partial v^{(f)}_k} = f^+_k(X).$$

(18)

With such definitions (13) takes the form

$$-\frac{\partial}{\partial T^{(f)}_k} Z_{GKM}\bigg|_V = \left( Tr \frac{\partial V(X)}{\partial v^{(f)}_k} - Tr \frac{\partial V(M)}{\partial v^{(f)}_k} \right),$$

(19)

Again, in general the r.h.s. of (13) is not the same as

$$-\frac{\partial}{\partial v^{(f)}_k} Z^{(f)}_{GKM} = \left( Tr \frac{\partial V(X)}{\partial v^{(f)}_k} - Tr \frac{\partial V(M)}{\partial v^{(f)}_k} \right) + Tr \left[ \frac{\partial V'(M)}{\partial v^{(f)}_k} (X - M)_f \right] - \frac{\partial \log C}{\partial v^{(f)}_k} (1)_f$$

(20)

Now, let us allow the matrix $M$ itself to change when $v^{(f)}_k$ are varied, this gives an additional contribution to (20) of the form

$$Tr V''(M) \frac{\partial M}{\partial v^{(f)}_k} (X - M)_f - Tr \frac{\partial M}{\partial v^{(f)}_k} \frac{\partial \log C}{\partial M} (1)_f$$

(21)

The obvious notation is $(\ldots)_f \equiv \frac{d^{(f)}}{C} (\ldots)$, e.g. $Z^{(f)}_{GKM} = (1)_f$. 
The second term in the r.h.s. of (21) and the first term in (20) can be now combined into

\[
Tr \left[ \frac{\partial V'(M)}{\partial v_k^{(f)}} + V''(M) \frac{\partial M}{\partial v_k^{(f)}} \right] (X - M)_f = Tr \frac{dV'(M)}{dv_k^{(f)}} (X - M)_f
\]

(22)

and we conclude that

\[
\left. \left( \frac{\partial}{\partial T_k^{(f)}} - \frac{\partial}{\partial v_k^{(f)}} \right) Z_{GKM}^{(f)} \right|_{\hat{T}_k^{(f)} = 0} = 0,
\]

(23)

only if the expression (22) is equal to zero and all Miwa times \( \hat{T}_k^{(f)} = 0 \). The latter requirement is the direct consequence of the fact that all normalization contributions are bilinear or linear forms of Miwa times (this point is discussed in details in the section 4).

3. \( p \)-times. Now let us consider what means that there are no corrections to (23) or the contribution of (22) is equal to zero. It implies first, that \( V'(M) \equiv W(M) \) is a fixed function of the new variable \( \tilde{M} = f(M) \), and second, the leading degree of this function is \( p \) (to dive asymptotic expansion of \( f(M) \)). Thus, it allows one to choose \( f(M) \) in the monomial form of degree \( p \):

\[
W(M) = f(M)^p \equiv \tilde{M}^p.
\]

(24)

Now, provided (22) is equal to zero, one obtains:

\[
\frac{\partial}{\partial T_k^{(f)}} \log Z_{GKM}^{(f)} = \frac{\partial}{\partial v_k^{(f)}} \log Z_{GKM}^{(f)}.
\]

(25)

This is almost the relation we need. Below, we will explain how the partition function can be redefined in order to have the equality of derivatives like (25) having the same objects in both sides of the equation.

Thus, we are led to special time variables induced by a special transformation of the spectral parameter \( \mu \rightarrow f(\mu) = W(\mu)^{1/p} \). These \( p \)-times are just those appeared in the paper [6] in the context of Landau-Ginzburg topological theories, and in the papers [11, 12] in the framework of quasiclassical (or dispersionless) hierarchies. We are going to demonstrate that these times are also natural in our approach and acquire a nice interpretation. Indeed, the explicit expression,

\[
t_k = \frac{p}{k(p - k)} \text{Res} W^{1-k/p} d\mu
\]

(26)

can be easily continued to negative values of the index \( k \) (the negative times will be discussed in detailed publication [13], see also the sect.6). Then we get two following formulas:

\[
\mu = -\frac{1}{p} \sum_{-\infty}^{p+1} k t_k \mu^{k-p},
\]

(27)

\[
V(\mu) - \mu V'(\mu) = \sum_{-\infty}^{p+1} t_k \mu^k.
\]

(28)

The first of these formulas can be easily modified for any variables \( v_k^{(f)} \), but the second one is specific for \( p \)-times and implies the natural interpretation of the exponential pre-factor in eq.(1) as the standard essential singularity factor in the Baker-Akhiezer function of \( p \)-time variables.
This can be done more transparently by the following procedure. Let us consider the equation for the Baker-Akhiezer function with all times \( T_k = 0 \) except for \( T_1 = x \):\[ [W(\partial) + x]\Psi(\mu, x) = W(\mu)\Psi(\mu, x). \tag{29} \]

One can look at this equation from two different points of view. On one hand, it can be considered as an initial boundary condition for the standard KP (\( V' \)-reduced) dynamics over Miwa times. Then this dynamics is described by the isospectral deformations and can be transformed to the standard KP dynamics by the re-expressing the spectral parameter \( \mu \) through a pseudo-differential operator using (29):\[ \mu \Psi(\mu, x) = [\partial + \sum_{i=1}^{\infty}u_{i+1}\partial^{-i}]\Psi(\mu, x) \equiv L\Psi(\mu, x). \tag{30} \]

Then (29) is a statement about reduction – i.e., pseudo-differential operator (namely \( W(L) \)) is the differential one.

On the other hand, the eq.(29) can be considered as describing the (consistent) dynamics over \( p \)-times, with zero Miwa times and \( p \)-th KdV initial boundary condition:\[ [\partial^p + x]\Psi(\mu, x) = \mu^p\Psi(\mu, x). \tag{31} \]

This dynamics corresponds to a special non-isospectral deformations in \( p \)-times and describes the flows between different reductions (or equivalent hierarchies, see the sect.4) and \textit{a priori} has nothing to do with the KP hierarchy-structure of GKM.

Thus, it turns out that these two dynamics are essentially the same in a special (matrix model) point of the Grassmannian determined by string equation. This amusing fact was previously proved in spherical limit in the papers \([11, 12]\), and will be explained in details in the sect.4–5. In other words the exact (non-perturbative) solution in this special point of the Grassmannian equals to a quasi-classical one, i.e. the quasi-classical approximation is in a sense exact. Leaving the correct formulation of this statement to sect.4–5, now we are going to discussion of the quasi-classical hierarchies and to demonstrate, what is remarkable in the chosen point of the Grassmannian.

\textit{Quasiclassical hierarchies.} Let us see what is specific in eq.(29) from general point of view of integrable theories. The point of the Grassmannian is determined by the set of coefficient functions of Lax operator of \( p \)-reduced KP hierarchy. In our case, this is given by the l.h.s. of the eq.(29) and its defining property is that it does not depend on \( x \) except for the constant term (i.e. \( \text{Res}_\partial \tilde{W}\partial^{-1} \)) depends linearly on \( x \). Now let us look at the KP hierarchy with Lax operator obeying this property.

The Lax representation of \( p \)-reduced KP hierarchy is\[ \frac{\partial \tilde{W}(\partial)}{\partial k} = [\tilde{W}^{k/p}(\partial), \tilde{W}(\partial)], \tag{32} \]
where $\tilde{W}(\partial)$ is a differential operator $\partial \equiv \partial/\partial x$, and the coefficients of the operator are allowed to depend on the first time $x$ as well as on other times $t_k$. Particular solution (or specific point of the Grassmannian) defining matrix models is distinguished by the requirement that all except for the constant term in $\tilde{W}$ are, in fact, $x$-independent. This is a consistent requirement provided only by the linear $x$-dependence of this constant term, i.e.

$$\tilde{W}(\partial) = W(\partial) + x,$$  \hspace{1cm} (33) 

where $W(\partial)$ at the r.h.s. is supposed to have no $x$-dependence at all. Then $(W(\partial) + x)^{k/p}$ does not depend on $x$ for $1 \leq k \leq p - 1$, and just equals to $W^{k/p}(\partial)$, so that the commutator at the r.h.s. of (33) acquires the only contribution from $x$-term in $\tilde{W}$, and (32) turns exactly into the equation

$$\frac{\partial W(\lambda)}{\partial t_k} = \frac{\partial W^{k/p}(\lambda)}{\partial \lambda}, \quad 1 \leq k \leq p - 1,$$  \hspace{1cm} (34) 

where $\lambda$ is a formal parameter.

This hierarchy (34), indeed, was obtained as a quasiclassical (or dispersionless) hierarchy satisfying the string equation [11, 12, 14]. However, the same equations (34) are exactly equivalent to the KP hierarchy on particular class of solutions (of the type of (33)). Thus, we state that the specific matrix model point of the Grassmannian gives rise to the dynamics with respect to the first $p$-times (if all Miwa times are equal to zero) which is simultaneously KP and dispersionless dynamics, i.e. the quasiclassical approximation is exact. Moreover, this dynamics is the same as the dynamics with respect to the first Miwa times. Therefore, we reproduce the result (23) in other terms.

### 4. Equivalent hierarchies

In this section we would like to discuss in details the general framework of integrability for the problems discussed above, in particular the notion of equivalent hierarchies.

The notion of equivalent solution of the KP hierarchy was introduced in [10] and was based on the particular transformations of the time variables. This concept was further developed in [15] for the general Zakharov-Shabat equations and the Toda lattice hierarchy (the KP case, as cited in [15], was considered in the unpublished paper by M. Noumi). Let us consider the general Zakharov-Shabat system

$$\partial B_i/\partial T_j - \partial B_j/\partial T_i + [B_i, B_j] = 0,$$  \hspace{1cm} (35) 

where Hamiltonians $B_i$ are the differential polynomials of $i$-th degree and are not restricted generally to be $(L^i)_+$, where $L$ is a pseudo-differential operator (30) giving a solution to the KP hierarchy. Then the system (35) contains the equations of the KP hierarchy as the subset since in general the first two polynomials have the form $B_2 = \partial^2 + 2u_2$, $B_3 = \partial^3 + 3(u_2 + a) + 3b$ with three independent functions while in the KP case $B_2 = \partial^2 + 2u_2$, $B_3 = \partial^3 + 3u_2 + 3(u_3 + u_{2,x})$ have only two independent functions. At the same time the system (33) contains more equations which restrict the functional dependence of the additional functions on the time variables. For example, $a(T)$ is $x$-independent and, therefore, $u_2$ is independently dependent on $x$.

\[\text{Of course, it is also possible to obtain this equation immediately by taking a } \mu \text{-derivative of the eq. (18).}\]
satisfy the usual Kadomtsev-Petviashvili equation. This freedom is the reflection of the fact that zero
curvature form is covariant under the arbitrary upper-triangle (gauge) transformations of the times
\[ T_i \to \tilde{T}_i = T_i + \xi_i(T_{i+1}, T_{i+2}, \ldots) \]  
and therefore the new differential operators defined by
\[ B_i(T) = \sum_j \frac{\partial \tilde{T}_j}{\partial T_i} \tilde{B}_j(\tilde{T}) \]  
also satisfy (35). In this sense the given time transformation (37) defines the equivalent hierarchy. The
covariance of (35) under the transformation (36) gives a possibility to eliminate the functional freedom
in the definition of the polynomials \( B_i \). Indeed, in [15] it has been proven that for an arbitrary \( B_i \) there
exists the unique transformation (36) such that \( \tilde{B}_i \) determine the KP hierarchy, i.e. \( \tilde{B}_i \) can be represented
in the form of \( (L^i)_+ \) for some operator \( L \) and satisfy the Lax equations:
\[ \frac{\partial L}{\partial T_i} = [B_i, L]. \]  
Only in this case the solution of these equations can be described by a single \( \tau \)-function.

Here we consider only the very restricted class of above transformations which are induced by the
variation of the spectral parameter \( \mu \) and, thus, still preserve the notion of the \( \tau \)-function. Let us introduce
an arbitrary function \( f(\lambda) \) which is expandable in the formal Laurent series \( f(\lambda) = \sum f_i \lambda^i \) \( (f_1 \equiv 1) \) and perform the transformation of the spectral parameter
\[ \tilde{\mu} = f(\mu) \]  
or, equivalently, define a new Lax operator
\[ \tilde{L} = f(L) \equiv L + \sum_{i=-\infty}^0 f_i L^i \]  
We should note that the transformation (39) respects the KP structure, i.e. maps the given KP hierarchy
onto equivalent one:
\[ \frac{\partial \tilde{L}}{\partial \tilde{T}_i} = [\tilde{B}_i, \tilde{L}] \]  
where new times \( \tilde{T} \) are introduce by the eq.(12) and
\[ \tilde{B}_i(\tilde{T}) = \sum_j \frac{\partial \tilde{T}_j}{\partial T_i} B_j(T). \]  
Now let us consider the \( \tau \)-function given in Miwa variables. It can be represented in the determinant
form (2)
\[ \tau(T) = \frac{\det \phi_i(\mu_j)}{\Delta(\mu)}, \]  
where times \( \{T\} \) are parameterized in the Miwa form (3) and \( \{\phi_i(\mu)\} \) are the basic vectors determining
the point of Grassmannian [16].
The relation between \( \tau \)-functions of the equivalent hierarchies can be easily derived from the eq.\((43)\) by an identical transformation:

\[
\tau (T) = \frac{\Delta(\tilde{\mu})}{\Delta(\mu)} \prod_i [f'(\mu_i)]^{1/2} \tilde{\tau}(\tilde{T})
\]

(44)

where \( \tilde{\tau}(\tilde{T}) \) as function of times \( \tilde{T} \) has the determinant form \((43)\) with the basic vectors

\[
\tilde{\phi}(\tilde{\mu}) = \left[ f'(\mu(\tilde{\mu})) \right]^{1/2} \phi_i(\mu(\tilde{\mu}))
\]

(45)

By a direct calculation one can show that pre-factor in eq.\((44)\) may be represented in the form

\[
\frac{\Delta(\tilde{\mu})}{\Delta(\mu)} \prod_i [f'(\mu_i)]^{1/2} = \exp \left( -\frac{1}{2} \sum_{i,j} A_{ij} \tilde{T}_i \tilde{T}_j \right)
\]

(46)

where

\[
A_{ij} = \text{Res} f^i(\lambda) d\lambda f^j_+ (\lambda).
\]

(47)

The notion of equivalent hierarchies is very useful in the context of GKM. In \([2]\) we proved that the \( \tau \)-function of GKM corresponds to \( V' \)-reduced KP hierarchy (in the case of polynomial \( V(\lambda) \)), i.e. \( V'(L) \) is a differential operator. Therefore, it is reasonable to consider the transformation \((39)\) with

\[
f(\mu) = [V'(\mu)]^{1/p} \equiv [W(\mu)]^{1/p}.
\]

(48)

In this case the equivalent hierarchy determined by the operator \( \tilde{L} = V'(L) \) is \( p \)-reduced one and we can treat the partition function of GKM in the terms of \( p \)-reduced \( \tau \)-function \( \tilde{\tau}(\tilde{T}) \) with the suitable deformations. These deformations are of two kinds. The first deformation corresponds to the transformation of times \( T_i \to \tilde{T}_i \) (see \((16),(17)\)); the second one corresponds to the multiplication of \( p \)-reduced \( \tau \)-function by the exponential pre-factor which is quadratic in times \( \{\tilde{T}\} \) and depends non-linearly on the coefficients of the potential \( V(\lambda) \). In the series of papers \([11, 14]\) it was shown that the matrix \( A_{ij} \) determined by the eqs.\((17), (18)\) can be represented in the form

\[
A_{ij} = \left( \frac{\partial^2}{\partial t_i \partial t_j} \log \tau_0(t) \right)_{t_{p+1} = \ldots = 0} ;
\]

(49)

where \( \tau_0(t) \) is the \( \tau \)-function of the quasiclassical \( p \)-reduced KP hierarchy restricted on “small phase space” \([17]\). Here \( \{t\} = \{t_1, \ldots, t_{p-1}, t_{p+1} = -\frac{p}{p+1}\} \) are \( p \)-times determining by eq.\((26)\) and corresponding quasiclassical “Lax operator” has the form

\[
\mathcal{L}(\lambda) = [W(\lambda)]^{1/p} ;
\]

(50)

this is essentially corresponds to the transformation of the spectral parameter \((48)\). Thus we can see that \( \tau \)-functions of the equivalent hierarchies (which are induced by variation of the spectral parameter) can be transformed to each other along quasiclassical flows.

We should remark also that the quasiclassical \( \tau \)-function satisfies the homogeneity condition \([11, 14]\)

\[
\sum_i t_i \frac{\partial}{\partial t_i} \log \tau_0(t) = 2 \log \tau_0(t).
\]

(51)
5. **Main results.** From the eqs. (44)-(48) we see that

$$\tau(T(\tilde{T})) = \tilde{\tau}(\tilde{T}) \exp \left( - \frac{1}{2} \sum_{i,j} A_{ij} \tilde{T}_i \tilde{T}_j \right)$$

(52)

where $A_{ij}$ is determined through the quasiclassical $\tau$-function $\tau_0(t)$ (eq. (49)). Let us introduce by definition the new $\tau$-function $\hat{\tau}(\tilde{T})$ of the $p$-reduced KP hierarchy:

$$\hat{\tau}(\tilde{T}) \equiv \frac{\tilde{\tau}(\tilde{T})}{\tau_0(t)} \exp \left( \sum_j j t_j \tilde{T}_j \right),$$

(53)

where

$$\frac{\tilde{\tau}(\tilde{T})}{\tau_0(t)} = \frac{\det \hat{\phi}_i(\tilde{\mu}_j)}{\Delta(\tilde{\mu})}$$

(54)

and the point of the Grassmannian is determined now by the basic vectors

$$\hat{\phi}_i(\tilde{\mu}) = \left[ p \tilde{\mu}_i - \frac{1}{2} \right] \int x^{i-1} e^{V(x) - x\tilde{\mu}_i} dx.$$  

(55)

Then it is easy to show that $\hat{\tau}(\tilde{T})$ satisfies the $L_{-1}$-constraint with shifted KP-times:

$$\sum_{k=1}^{p-1} k(p - k)(\tilde{T}_k + t_k)(\tilde{T}_{p-k} + t_{p-k}) + \sum_{k=1}^{\infty} (p + k)(\tilde{T}_{p+k} + t_{p+k}) \frac{\partial}{\partial \tilde{T}_k} \log \hat{\tau}(\tilde{T}) = 0,$$

(56)

where $t_i$ are equal to zero for $i \geq p + 2$ (see eq. (28)) so $\hat{\tau}(\tilde{T})$ actually depends only on the sum of (“quantum”) Miwa times and (quasiclassical) $p$-times. Moreover, from eqs. (25) and (52), (53) it directly follows that

$$\left( \frac{\partial}{\partial \tilde{T}_i} - \frac{\partial}{\partial t_i} \right) \hat{\tau}(\tilde{T}) = 0 \quad i = 1, 2, ..., p - 1.$$  

(57)

Therefore, our final answer is that in the polynomial case ($V(X) = \sum_{k=1}^{p+1} \frac{v_k}{k} X^k$) one can represent the GKM partition function in the following form:

$$\tau(T) = \hat{\tau}(\tilde{T} + t) \exp \left( - \frac{1}{2} \sum_{i,j} \left( \frac{\partial^2}{\partial \tilde{T}_i \partial \tilde{T}_j} \log \tau_0(t) \right)(\tilde{T}_i + t_i)(\tilde{T}_j + t_j) \right)$$

(58)

(in order to obtain the last formula we have used the homogeneity condition (41)), where $\hat{\tau}(\tilde{T} + t)$ is the standard $\tau$-function corresponding to monomial $V(x) = \frac{X^{p+1}}{p+1}$ and

$$\hat{\tau}(t) = \tau_0(t)$$

(59)

is the $\tau$-function of the quasiclassical $p$-reduced KP hierarchy with the “Lax operator” (40) (in fact, in the second part of the section 3 we have already met this property that the quasiclassical function is exact when Miwa times are equal to zero).

As $\hat{\tau}$ satisfies the standard string equation for multi-matrix model, one can also write it as GKM integral with the monomial potential [1, 2], but with $\tilde{T} + t$ playing the role of Miwa times. Therefore, $\hat{\tau}$ describes simultaneously $(p - 1)$-matrix model, and there is no smooth transition from this to another
Thus, $\hat{r}$ is singular under changing $p$, as well as exponential in (58). At the same time the complete answer corresponding to GKM integral is regular, and this is one of main advantages of GKM approach.

6. Discussion. In this letter we have proposed a way how two a priori different objects like $N = 2$ topological LG theories and conformal $c < 1$ minimal models coupled to Polyakov 2d gravity can be unified in the framework of GKM, exploiting integrable structure appeared in these objects. However, this connection deserves further understanding.

The main formula (4) describes the interpolating flow between two $(p, q)$ and $(p', q)$ models in terms related to LGM. Of course, this is only a local description around a given “critical point”. It shows that the flow has a complicated “phase”-structure with the LG theory being responsible for its first phase — on a particular $p$-orbit determined by the order of the potential $V'(X) \equiv W(X) = X^p + ...$

Formula (4) demonstrates that such flow can be more or less absorbed into the redefinition of times: $T \rightarrow \tilde{T}$ by

$$\tau_p(T) \rightarrow \frac{\tau_p(\tilde{T} + t)}{Z_{cl}(\tilde{T} + t|t)}.$$ (60)

Here $Z_{cl}$ denotes the classical contributions to the partition function, the $\tau$-function in the numerator corresponds to the same point of the Grassmannian (satisfies the same string equation) and the parameters of the flow just add to the corresponding KP times. It means that the LG flow is generated by first $p$ primaries of minimal conformal model plus 2d gravity theory being equivalent to the primary LG fields.

On the “boundary” of this phase $p$-times diverge thus being non-adequate parameters for the description of the “phase”-transition between different $p$-orbits, what is typical for the phase transitions.

The “classical” partition function in the denominator of (60) has to be better understood from the LG point of view. The derivative

$$A_{ij}(t) = \frac{\partial^2}{\partial t_i \partial t_j} \log \tau_0 = Res W^{i/p} dW^{j/p}$$ (61)

(see also [14]) is an object of similar nature to those appearing in LGM on the so called Grothendick residue formulas. Indeed, the scalar product defining the Grothendick residue [5, 6]:

$$\langle \phi_i \phi_j \rangle = Res \frac{\phi_i \phi_j}{W^{r}} d\mu$$ (62)

together with

$$\phi_i(\mu) = \frac{\partial}{\partial t_i} W(\mu) = \frac{\partial}{\partial \mu} W^{i/p}$$ (63)

gives

$$\langle \phi_i \phi_j \rangle \sim Res W^{(i-p)/p} dW^{j/p}$$ (64)

which is formally

$$\frac{\partial^2}{\partial t_i \partial t_j} \log \tau_0$$ (65)

if we introduce negative $p$-times. The negative times should correspond to a deformation under arbitrary (non-polynomial) change of spectral parameter $\tilde{\mu} = f(\mu)$ though the integrable structure (with respect to $p$-times $\{t\}$) is not yet completely clear.
The basic feature of topological theories is ring structure [4]. In our case a sort of ring appears immediately from the reduction condition

\[ W(\mu) \varphi_i(\mu) = \sum_j C_{ij} \varphi_j(\mu) \]  

(66)

with \( \mu \)-independent \( C_{ij} \) which is a direct consequence of generalized Airy equation [2]. In new basis the last relation turns to be

\[ \hat{\mu}^p \hat{\varphi}_i(\hat{\mu}) = \sum_j \hat{C}_{ij} \hat{\varphi}_j(\hat{\mu}) \]  

(67)

and the only difference with conventional LGM is that operations (66) and (67) are determined on non-polynomial functions. It also deserves noting that the Grothendick residue formula (62) acquires an especially simple form in the basis

\[ \Phi_i(\mu) = W'(\mu) \phi_i(W(\mu)). \]  

(68)

Indeed

\[ \langle \Phi_i \Phi_j \rangle = \text{Res} \frac{\Phi_i(\mu) \Phi_j(\mu)}{W'(\mu)} d\mu = \text{Res} W'(\mu) \phi_i(W(\mu)) \phi_j(W(\mu)) d\mu = \text{Res} \phi_i(W) \phi_j(W) dW \]  

(69)

which is also natural in the framework of topological theories [7].

Now let us stress that we can not immediately identify a concrete object from the GKM theory with the LGM partition function. This is due to the fact that there is still a little information about the latter one. Moreover, only its \( p \)-time dependence is known while the dependence on Miwa times is much more subtle point. It can be determined only having explicit expressions for all correlators, but there are no formulas for 5-point and higher correlators even of the primary fields (see [7]). Therefore, the LGM partition function at the moment can be identified with GKM objects only hypothetically. It is reasonable to conjecture that this partition function might coincide with \( \hat{Z} \). This is consistent with the fact that LGM can not smoothly interpolate between different \( p \) orbits.

Thus, we see that there exists a deep connection between the structure of LGM and matrix model formulation of 2\( d \) gravity which should be revealed better and we are going to return to this problem in the separate publication [13].

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