Perihelion advance for orbits with large eccentricities in the Schwarzschild black hole

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Abstract

We deduce a new formula for the perihelion advance $\Theta$ of a test particle in the Schwarzschild black hole by applying a newly developed non-linear transformation within the Schwarzschild space-time. By this transformation we are able to apply the well-known formula valid in the weak-field approximation near infinity also to trajectories in the strong-field regime near the horizon of the black hole. The resulting formula has the structure $\Theta = c_1 - c_2 \ln(c_3^2 - e^2)$ with positive constants $c_{1,2,3}$ depending on the angular momentum of the test particle. It is especially useful for orbits with large eccentricities $e < c_3 < 1$ showing that $\Theta \to \infty$ as $e \to c_3$.

Keyword(s): Perihelion precession, perihelion advance, Schwarzschild black hole

1 Introduction

Surprisingly little is known about the value of the perihelion advance in the strong-field region of the Schwarzschild space-time, especially at large eccentricities of the closed orbit. Recent interest in calculations and measurements
of orbital characteristics like the relativistic pericenter precession, also called relativistic perihelion advance, see e.g. [1], [2], [3] and [4], show the necessity to have good closed-form expressions for it in the most prominent example of a space-time, the Schwarzschild black hole. Also the calculation of perihelion precession in classical mechanics is subject to recent interest, see [5] and section 5 of [6]. For latest work on analogous questions in axially symmetric space-times see e.g. [7]. In section VII of [8], the nearly circular motion of a particle within \( f(R) \)-gravity has been discussed, whereas in [9], the orbital motion in \( f(R) \)-gravity with quadrupole radiation was calculated.

It is the aim of the present paper to apply the ideas and formulas deduced in section 10 of [6] to the calculation of the perihelion advance \( \Theta \) of periodic orbits in the Schwarzschild black hole.\(^1\) Especially, we want to find out, how \( \Theta \) depends on the eccentricity \( e \) of the orbits. To have a stricter posed question, we ask: what is the change of \( \Theta \) if we change the eccentricity \( e \) of the orbit but keep the angular momentum \( h = r^2 \dot{\varphi} \) of the particle\(^2\) constant?

The core of the deduction is a newly developed non-linear transformation within the Schwarzschild space-time. Due to its importance we have chosen to present this transformation in two independent versions, one by the set of eqs. (2.16) to (2.36), which directly applies to the geodesics in the Schwarzschild space-time, the other one is given in the appendix, which is a self-contained and more abstract deduction of the primarily unexpected symmetry of eq. (2.16).

The notation is as follows: Let the orbit be the periodic but non-constant function \( r(\varphi) \) with invariantly defined Schwarzschild radius \( r \), then \( \Theta \) is defined by the period \( 2\pi + \Theta \) of the function \( r \), i.e. \( r(\varphi) = r(\varphi + 2\pi + \Theta) \). The eccentricity \( e \) is defined by

\[
e = \frac{r_2 - r_1}{r_2 + r_1}
\]

where \( r_2 = \max r(\varphi) \) and \( r_1 = \min r(\varphi) \). Thus, \( e \) and \( \Theta \) are invariantly

\(^1\)Of course, the exact formula is well-known: it contains elliptic integrals, but in practice this formula is of minor use only. And, probably more importantly: these elliptic integrals which can be evaluated numerically to every degree of accuracy, do not easily lead to the identification of the physically interesting quantities we are trying to find out in the present paper.

\(^2\)The dot denotes the derivative with respect to the eigentime of the particle.
defined quantities, even for orbits which may be very far from being elliptic ones. Moreover, $e$ does not change if $r$ is replaced by $c \cdot r$ with a constant $c$. If we identify $e$ and $-e$, then also the replacement of $r$ by its inverse $1/r$ in eq. (1.1) leads to the same eccentricity.

By continuously changing $r_1$ and $r_2$ we get circular orbits at $r_1 = r_2$, and interchanging $r_1$ and $r_2$ leads to the same geometry of the orbit, hence the same value of $\Theta$. This means, we expect $\Theta$ to be an even function of $e$ at constant angular momentum. From now on we restrict $e$ to the interval $0 < e < 1$, nevertheless, this consideration is useful, as the Taylor development of $\Theta$ around $e = 0$ should only contain even powers of $e$.

The reason why we parametrize the periodic non-circular orbits by angular momentum is the following one: At fixed value $h$, these orbits can be uniquely be parametrized by $e$, but can also be uniquely be parametrized by the perihelion $r_1$ of the orbits, and can also be uniquely parametrized by the total energy $E$ of the test particle. This is not a trivial statement, as for the general case, given $h$ and $E$, more than one orbit exists: e.g. one hyperbolic orbit and another one leading towards the horizon.

2 Geodesics in the Schwarzschild black hole

We take the Schwarzschild solution in Schwarzschild coordinates\textsuperscript{3} with mass parameter $m > 0$ as usual:

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{dr^2}{1 - 2m/r} - r^2 d\Omega^2$$  \hspace{1cm} (2.1)

where $d\Omega^2$ is the metric of the standard 2-sphere. We apply units such that light velocity $c = 1$ and Newton’s gravitational constant $G = 1$.\textsuperscript{4} To calculate time-like geodesics we may assume without loss of generality that they are situated in the equatorial plane. The angular coordinate is denoted by $\varphi$ and the proper time along the time-like geodesic $(t(\tau), r(\tau), \varphi(\tau))$ is denoted by

\textsuperscript{3}But see [10] for clarifying historical notes to this notion.

\textsuperscript{4}In principle, we could also apply units such that $G = 1/m$, and under these circumstances, we have one less parameter in all the calculations, but then the departure from the usual well-known formulas is becoming even larger.
τ, a dot denotes $d/dr$. We assume $\dot{t} > 0$. Then we get from eq. (2.1)

$$1 = \left(1 - \frac{2m}{r}\right) \dot{t}^2 - \frac{\dot{r}^2}{1 - 2m/r} - r^2 \dot{\varphi}^2.$$  

(2.2)

We restrict to the region outside the horizon, i.e. to $r > 2m$.

$$h = r^2 \dot{\varphi}$$  

(2.3)

is the conserved angular momentum of the test particle. We exclude purely radial motion which is characterized by $h = 0$ and choose the orientation of space such that $h > 0$. Inserting eq. (2.3) into eq. (2.2) we get

$$1 = \left(1 - \frac{2m}{r}\right) \dot{t}^2 - \frac{\dot{r}^2}{1 - 2m/r} - \frac{h^2}{r^2}.$$  

(2.4)

A further conserved quantity is the energy $E$ defined by

$$E = \left(1 - \frac{2m}{r}\right) \dot{t} > 0.$$  

(2.5)

Inserting eq. (2.5) into eq. (2.4) we get

$$1 = \frac{E^2 - \dot{r}^2}{1 - 2m/r} - \frac{h^2}{r^2}.$$  

(2.6)

To remove $\dot{r}$, the third of the three terms containing a $\tau$-derivative from eq. (2.6), we describe the path of the particle as $r(\varphi)$ and get via $\frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}}$ by the help of eq. (2.3)

$$\dot{r} = \frac{h}{r^2} \cdot \frac{dr}{d\varphi}.$$  

(2.7)

Inserting eq. (2.7) into eq. (2.6) we get after multiplication with $1 - 2m/r$

$$\left(1 + \frac{h^2}{r^2}\right) \cdot \left(1 - \frac{2m}{r}\right) = E^2 - \frac{h^2}{r^4} \cdot \left(\frac{dr}{d\varphi}\right)^2.$$  

(2.8)

To get complete information from the geodesic equation, we still need the radial part of it, we take it from eq. (9.6) of [6]:

$$0 = \frac{\ddot{r}}{1 - 2m/r} - \frac{h^2}{r^3} + \frac{m}{r^2} \cdot \frac{E^2 - \dot{r}^2}{(1 - 2m/r)^2}.$$  

(2.9)
Following [11], we introduce the nondimensionalized inverted Schwarzschild radius $u$ via
\[ u = \frac{h^2}{m \cdot r} \]  
and the dimensionless parameter $\varepsilon$ via
\[ \varepsilon = \frac{3m^2}{h^2} > 0. \]  
This leads to $r = h^2/(m \cdot u)$ and
\[ \frac{dr}{d\varphi} = -\frac{h^2}{m \cdot u^2} \cdot \frac{du}{d\varphi}. \]  
Inserting eq. (2.12) into eq. (2.8) we get
\[ \left(1 + \frac{m^2u^2}{h^2}\right) \cdot \left(1 - \frac{2m^2u}{h^2}\right) = E^2 - \frac{m^2}{h^2} \cdot \left(\frac{du}{d\varphi}\right)^2. \]  
With eq. (2.11) we finally get
\[ \left(1 + \frac{\varepsilon u^2}{3}\right) \cdot \left(1 - \frac{2\varepsilon u}{3}\right) = E^2 - \frac{\varepsilon}{3} \left(\frac{du}{d\varphi}\right)^2. \]  
We multiply by $3/(2\varepsilon)$ and get
\[ \frac{1}{2} \left(\frac{du}{d\varphi}\right)^2 - u + \frac{u^2}{2} - \frac{\varepsilon u^3}{3} = \mu = \frac{3}{2\varepsilon} \left(E^2 - 1\right). \]  
Derivating this equation we get
\[ \frac{d^2u}{d\varphi^2} + u = 1 + \varepsilon u^2. \]  
Eqs. (2.15)/(2.16) represent the motion of a particle $u$ in the potential
\[ V(u) = -u + \frac{u^2}{2} - \frac{\varepsilon u^3}{3} \]  
with $\mu$ interpreted as energy. A dash denoting $\frac{d}{du}$ we get
\[ V'(u) = -1 + u - \varepsilon u^2 \]  
and
\[ V''(u) = 1 - 2\varepsilon u. \]
To shift the turning point of this potential to the origin, we define

\[ v = u - \frac{1}{2\varepsilon} \]  

(2.20)
as new variable instead of \( u \). Eqs. (2.17)/(2.20) lead to

\[ V(u) = -\frac{1}{2\varepsilon} + \frac{1}{12\varepsilon^2} - v + \frac{v}{4\varepsilon} - \varepsilon v^3/3. \]  

(2.21)

Defining further

\[ \mu_2 = \mu + \frac{1}{2\varepsilon} - \frac{1}{12\varepsilon^2} \]  

(2.22)
then eq. (2.15) now reads

\[ \frac{1}{2} \left( \frac{dv}{d\varphi} \right)^2 + v \left( \frac{1}{4\varepsilon} - 1 \right) - \frac{\varepsilon v^3}{3} = \mu_2. \]  

(2.23)

Let us note the special solution \( v \equiv 0 \) for \( \varepsilon = 1/4 \) and \( \mu_2 = 0 \). It corresponds to a special semistable circular orbit. Apart from this special solution, eq. (2.23) possesses periodic solutions \( v(\varphi) \) only for the parameter range \( 0 < \varepsilon < 1/4 \). This we will always assume in the following.

With the notation

\[ V_2(v) = v \left( \frac{1}{4\varepsilon} - 1 \right) - \frac{\varepsilon v^3}{3} \]  

(2.24)
we get

\[ V'_2(v) = \frac{1}{4\varepsilon} - 1 - \varepsilon v^2 \]  

(2.25)
and

\[ V''_2(v) = -2\varepsilon v. \]  

(2.26)
The equation \( V'_2(v) = 0 \) possesses the solutions

\[ \pm \frac{\sqrt{1 - 4\varepsilon}}{2\varepsilon} \]  

(2.27)
representing one maximum and one minimum. This forces us to introduce the new variable

\[ w = \frac{2v\varepsilon}{\sqrt{1 - 4\varepsilon}} \]  

(2.28)
instead of \( v \). With

\[ \mu_3 = \frac{4\varepsilon^2\mu_2}{1 - 4\varepsilon} \]  

(2.29)
we get now from eq. (2.23)
\[ \frac{1}{2} \left( \frac{dw}{d\varphi} \right)^2 + \frac{w}{2} \cdot \sqrt{1 - 4\varepsilon} - \frac{w^3}{6} \cdot \sqrt{1 - 4\varepsilon} = \mu_3. \] (2.30)

In the next step we replace \( \varphi \) by a new angular coordinate
\[ \psi = \varphi \cdot \sqrt{1 - 4\varepsilon}. \] (2.31)

It is to be observed, that \( \varphi = 0 \) is identified with \( \varphi = 2\pi \), so \( \psi = 0 \) is identified with \( \psi = 2\pi \cdot \sqrt{1 - 4\varepsilon} \). Eqs. (2.30)/(2.31) together with
\[ \mu_4 = \frac{\mu_3}{1 - 4\varepsilon} \] (2.32)
lead to
\[ \frac{1}{2} \left( \frac{dw}{d\psi} \right)^2 + \frac{w}{2} - \frac{w^3}{6} = \mu_4. \] (2.33)

This is the equation we are going to solve now. Derivating eq. (2.33) we arrive at the simple equation
\[ \frac{d^2w}{d\psi^2} = \frac{w^2 - 1}{2}. \]

Remarkably enough, it contains no \( \varepsilon \). The function
\[ f(w) = \frac{w}{2} - \frac{w^3}{6} \] (2.34)
has zeroes at \( w = 0 \) and \( w = \pm \sqrt{3} \). For the derivative we get
\[ f'(w) = \frac{1}{2} - \frac{w^2}{2} \] (2.35)
possessing zeroes at \( w = \pm 1 \). \( f(-1) = -1/3 \) is the local minimum and \( f(1) = 1/3 \) is the local maximum of \( f \). We note that also \( f(-2) = 1/3 \). The constant solutions of eq. (2.33) are \( w(\psi) \equiv -1 \) with \( \mu_4 = -1/3 \) representing the stable circular orbits and \( w(\psi) \equiv 1 \) with \( \mu_4 = 1/3 \) representing the unstable circular orbits. Besides these exceptions it holds: Every periodic solution \( w(\psi) \) of eq. (2.33) is a non-constant one and is completely confined in the interval \(-2 < w(\psi) < 1\). This is related to the energy parameter \( \mu_4 \) being confined to \(-1/3 < \mu_4 < 1/3\). Let \( w_1 = \max w(\psi) \) and \( w_2 = \min w(\psi) \),
then $\mu_4 = f(w_1) = f(w_2)$. We have $-1 < w_1 < 1$ and $-2 < w_2 < -1$. Let us parametrize these periodic orbits by the parameter $e_4$ defined by

$$w_1 = 2e_4 - 1 \quad \text{with} \quad 0 < e_4 < 1. \quad (2.36)$$

In the limit $e_4 \to 0$ we get the stable circular orbits.

In the other limit $e_4 \to 1$ we come arbitrarily close to the unstable circular orbits, that means, that the perihelion advance tends to infinity in this limit.

Using eqs. (2.34)/(2.36) we get

$$\mu_4(e_4) = f(w_1) = -\frac{4}{3}e_4^3 + 2e_4^2 - \frac{1}{3}. \quad (2.37)$$

Due to $d\mu_4/de_4 = 4e_4(1-e_4) > 0$ this represents a one-to-one correspondence between $e_4$ and $\mu_4$. Solving now the equation $\mu_4 = f(w_2)$ for $w_2$ we get with eq. (2.37)

$$w_2 = \frac{1}{2} - e_4 - \sqrt{3} \cdot \sqrt{1 - \left(\frac{1}{2} - e_4\right)^2}. \quad (2.38)$$

Let $\psi_0 = k(e_4)$ be the complete period of the function $w(\psi)$. We calculate it by solving eq. (2.33) with $w(0) = w_2$ and $w(\psi_0/2) = w_1$ and get

$$\psi_0 = k(e_4) = 2 \int_{w_2}^{w_1} \frac{dw}{\sqrt{2\mu_4 + w^3/3 - w}}. \quad (2.39)$$

Though the exact solutions for such integrals can be found in the literature, see e.g. [12], page 355, for the general theory and [13], [14] for its concrete application, they are of little use as the elliptic integrals can be evaluated by the Weierstrass function only, and not in the usual closed-form presentation, which would allow for a physical interpretation. The three cases $e_4 \in \{1/2, 0, 1\}$ will now be considered in detail.

Let $e_4 = 1/2$, then $\mu_4 = w_1 = 0$ and $w_2 = -\sqrt{3}$. Eq. (2.39) simplifies to

$$\psi_0 = k(1/2) = 2 \int_{-\sqrt{3}}^{0} \frac{dw}{\sqrt{w^3/3 - w}}. \quad (2.40)$$

The substitution $x = \sqrt{-w/\sqrt{3}}$ leads to

$$k(1/2) = 4 \cdot \sqrt{3} \cdot I \quad (2.41)$$
where
\[ I = \int_0^1 \frac{dx}{\sqrt{1 - x^4}} = \frac{1}{4\sqrt{2\pi}} \cdot (\Gamma(1/4))^2 \] (2.42)

according to standard tables, the Gamma-function has \( \Gamma(1/4) = 3.62561 \), and we get \( I = 1.31103 \) and

\[ k(1/2) = \psi_0 = 6.90164 = 2\pi \cdot 1.09843. \] (2.43)

To get higher accuracy, the 1.09843 in eq. (2.43) has to be replaced by \( \alpha \) with

\[ \alpha = \frac{\sqrt{3}}{\sqrt{8\pi^3}} \cdot (\Gamma(1/4))^2. \]

The limit \( e_4 \to 0 \) corresponds to \( w_1 \to -1 \), and \( f''(-1) = 1 \), so the system represents a harmonic oscillator with unit frequency:

\[ w(\psi) = -1 + e_4 \cdot \cos \psi, \quad k(0) = \psi_0 = 2\pi. \] (2.44)

The limit \( e_4 \to 1 \) can similarly be solved: it corresponds to \( w_1 \to 1 \), and \( f''(1) = -1 \), so we have to replace \( \cos \) by \( \cosh \) in eq. (2.44) and \( e_4 \) by \( 1 - e_4 \):

\[ w(\psi) = 1 - (1 - e_4) \cdot \cosh \psi, \quad \psi_0 \to \infty \quad \text{as} \quad e_4 \to 1. \] (2.45)

To quantify the diverging part we replace \( \cosh \psi \) by \( \exp(\psi)/2 \) and \( 1 - e_4 \) by \( (1 - e_4^2)/2 \), then eq. (2.45) reads

\[ w(\psi) = 1 - \exp(\psi)(1 - e_4^2)/4. \]

\( w(\psi) = 0 \) will be reached at \( \psi = \ln(4/(1 - e_4^2)) \), so the general structure must be approximately

\[ k(e_4) = \psi_0 = c_1 - c_2 \cdot \ln \left(1 - e_4^2\right) \] (2.46)

with certain positive constants \( c_1 \) and \( c_2 \) of order 1. We will fix them by the condition that eq. (2.46) holds exactly true for \( e_4 = 1/2 \) and \( e_4 = 0 \) according to eqs. (2.43) and (2.44) resp. This leads to \( c_1 = 2\pi \) and \( c_2 = 2\pi \cdot 0.34215 \).

So eq. (2.46) reads

\[ k(e_4) = 2\pi \cdot \left(1 - 0.34215 \cdot \ln \left(1 - e_4^2\right)\right). \] (2.47)
For $e_4 \ll 1$ we get approximately

$$k(e_4) = 2\pi \cdot \left(1 + 0.34215 \cdot e_4^2\right).$$

(2.48)

To get higher accuracy, the $0.34215$ in eqs. (2.47), (2.48), (2.51) and (2.52) has to be replaced by $\beta$ with $\alpha$ from above and

$$\beta = \frac{\alpha - 1}{\ln 4 - \ln 3}.$$

Let us now return from the $\psi$-picture to the $\varphi$-picture. Let $\varphi_0$ be the complete period in the $\varphi$-picture, then we get by eq. (2.31)

$$\varphi_0 = \frac{\psi_0}{\sqrt{1 - 4\varepsilon}} = \frac{k(e_4)}{\sqrt{1 - 4\varepsilon}}.$$  

(2.49)

The perihelion advance is $\Theta = \varphi_0 - 2\pi$, i.e.

$$\Theta = \frac{k(e_4)}{\sqrt{1 - 4\varepsilon}} - 2\pi.$$  

(2.50)

Inserting eq. (2.47) into eq. (2.50) we get

$$\Theta = 2\pi \cdot \left(1 - 0.34215 \cdot \ln \left(1 - e_4^2\right) - 1\right).$$  

(2.51)

This eq. (2.51) is useful for all $e_4$ with $0 < e_4 < 1$ and a strict result for $e_4 = 1/2$; also both the limiting behaviours $e_4 \to 0$ and $e_4 \to 1$ represent strict ones.

Inserting the approximation eq. (2.48) into eq. (2.50) we get

$$\Theta = 2\pi \cdot \left(1 + 0.34215 \cdot \frac{e_4^2}{\sqrt{1 - 4\varepsilon}} - 1\right).$$  

(2.52)

This eq. (2.52) is useful for all $e_4$ with $0 < e_4 \ll 1$. Inserting the expression (2.11) for $\varepsilon$, we get the final formula, expressing the perihelion advance in dependence on angular momentum $h$ and the parameter $e_4$, which is correlated to the eccentricity of the orbit:

$$\Theta = 2\pi \cdot \left(1 - 0.34215 \cdot \ln \left(1 - e_4^2\right) - \frac{12m^2/h^2}{\sqrt{1 - 12m^2/h^2}} - 1\right).$$  

(2.53)
3 Properties of the orbits

To be able to interpret eq. (2.53) we need a better knowledge of the orbits. The formula (1.1) for the eccentricity is invariant with respect to multiplication of \( r \) by a constant factor, and also (after identification of \( e \) with \(-e\)) invariant with respect to a replacement of \( r \) by \( 1/r \), but not invariant with respect to adding a constant to \( r \). But just this addition of the constant has been done in eq. (2.20), so it must be expected, that the parameter \( e_4 \) will depend both in \( e \) and on \( h \), and not only on \( e \).

Due to eq. (2.11), the condition \( 2m < r < \infty \) reads \( 0 < u < 3/(2\varepsilon) \), and we restrict to the interval \( 0 < \varepsilon < 1/4 \) because only for these \( \varepsilon \)-values non-circular periodic orbits exist. The limit \( \varepsilon \to 0 \) is the Newtonian limit; for this case we know the non-circular periodic orbits to be exact elliptic ones, i.e. \( \Theta = 0 \) for all values of \( h \) and \( e \). For \( \varepsilon = 1/4 \), only one periodic orbit exists, it is a circular one. For \( \varepsilon > 1/4 \), the angular momentum is too small to allow for periodic orbits, the particle goes to \( r \to \infty \) or to \( r \leq 2m \) after sufficient long time.

To find the real properties of the perihelion advance, we have therefore to restart at the point just before the translation (2.20) has been applied. The function \( V(u) \), eq. (2.17), has the following zeroes: \( V(0) = 0 \), for \( \varepsilon > 3/16 \) this is the only one, for \( \varepsilon \leq 3/16 \) the other zeroes of \( V \) are calculated via

\[
u_{5,6} = \frac{3}{4\varepsilon} \left( 1 \pm \sqrt{1 - \frac{16\varepsilon}{3}} \right)
\]

where \( 0 < u_6 \leq 3/(4\varepsilon) \leq u_5 < 3/(2\varepsilon) \), and in the special case \( \varepsilon = 3/16 \) we have the double root \( u_5 = u_6 = 3/(4\varepsilon) \).

Let \( u_1 \) be the maximal value of \( u(\varphi) \) and \( u_2 \) be the minimal one. Then we get

\[
u_{1,2} = \frac{h^2}{mr_{1,2}} = \frac{m(1 \pm e)}{p - m(3 + e^2)}
\]

leading to

\[
u_1 + u_2 = \frac{m}{p - m(3 + e^2)}
\]

The parameter \( p \) has its usual meaning, its relation to \( r_{1,2} \) and \( e \) can be seen e.g. from eq. (3.9) below.
and
\[ \frac{u_1 - u_2}{2} = me \left( \frac{p}{m(3 + e^2)} \right). \] (3.4)

Solutions with constant value \( u \) fulfil \( u = 1 + \varepsilon u^2 \), see eq. (2.18), i.e. the function \( dV(u)/du \) has the two zeroes:
\[ u_{3,4} = \frac{1}{2\varepsilon} \cdot \left( 1 \pm \sqrt{1 - 4\varepsilon} \right) \] (3.5)

where \( 0 < u_4 < 1/(2\varepsilon) < u_3 < 1/\varepsilon \). The second derivative of \( V(u) \), see eq. (2.19), vanishes for \( u = 1/(2\varepsilon) \) only. Therefore, at \( u = u_4 \) we have a minimum of \( V(u) \), and at \( u = u_3 \) a maximum. For the special case \( \varepsilon = 3/16 \) we have \( u_3 = u_5 = u_6 = 4 \) and \( V(4) = 0 \).

We get
\[ V(u_4) = \frac{1}{12\varepsilon^2} \cdot \left( 1 - 6\varepsilon - (1 - 4\varepsilon)^{3/2} \right) \] (3.6)
and have obviously always \( V(u_4) < 0 \). To find the sign of \( V(u_3) \) more calculations are necessary. We get
\[ V(u_3) = \frac{1}{12\varepsilon^2} \cdot \left( 1 - 6\varepsilon + (1 - 4\varepsilon)^{3/2} \right) \] (3.7)
which is positive for \( 0 < \varepsilon < 3/16 \) and negative for \( 3/16 < \varepsilon < 1/4 \). Thus, to get the set of non-constant periodic bounded orbits, we have to distinguish three cases:\(^6\) First case: For \( 0 < \varepsilon < 3/16 \) such orbits exist for \( 0 < -\mu < -V(u_4) \) with \( \mu \) taken from eq. (2.15). Second case: For \( \varepsilon = 3/16 \) such orbits exist for \( 0 < -\mu < 16/27 \). Third case: For \( 3/16 < \varepsilon < 1/4 \) such orbits exist for \( -V(u_3) < -\mu < -V(u_4) \).

We parametrize the bounded solutions by \( r_1 \) and \( r_2 \), where \( r_1 \) is the perihelion and \( r_2 \) the aphelion. In the present calculations we restrict to the parameter values \( 2m < r_1 < r_2 < \infty \), i.e. to motion completely outside the horizon \( r = 2m \).

We pose as additional restriction to the possible values of \( r_1 \) and \( r_2 \) the property, that a bounded orbit with these values really exists.

Second case: \( \varepsilon = 3/16 \) implies \( h = 4m \). All values \( 0 < -\mu < 16/27 \) are possible, leading to \( 25/27 < E^2 < 1 \), with \( E \) from eqs. (2.5)/(2.14) and all

\(^6\)The second case can be subsumed to the first one by allowing \( \varepsilon \leq 3/16 \) as well as under the third one by allowing \( \varepsilon \geq 3/16 \). But as the second case possesses other special properties it is simpler to deal with it in an extra case.
values of the eccentricity $0 < e < 1$ are possible. In the $u$-picture we get: All values $0 < u_2 < 4/3$ an all values $4/3 < u_1 < 4$ are possible; and this also exhausts the set of all possible values. In the $r$-picture we get: perihelion $r_1$ has $4m < r_1 < 12m$, and aphelion $r_2$ has $r_2 > 12m$. In the limit $e \to 0$ we get the perihelion advance $\Theta \to 2\pi(\sqrt{2} - 1)$ and in the limit $e \to 1$ we get the perihelion advance $\Theta \to \infty$.

First case: $0 < \varepsilon < 3/16$, i.e. $h > 4m$, and all values of the eccentricity $0 < e < 1$ are possible. Physically, this first case can be interpreted as follows: Further increasing the energy of the particle at constant angular momentum would have the consequence that its path goes to $r \to \infty$, so no periodic orbit appears. This behaviour we know already from Euclidean geometry: if the eccentricities of a set of ellipses tend to 1, then the result is a parabola.

Third case: $3/16 < \varepsilon < 1/4$, i.e. $2\sqrt{3}m < h < 4m$, and all values of the eccentricity $0 < e < e(\varepsilon)$ are possible, where $e(\varepsilon) < 1$ is the following expression:

$$e(\varepsilon) = -\frac{3}{1 - 2/\sqrt{1 - 4\varepsilon}}. \quad (3.8)$$

In the limit $\varepsilon \to 3/16$ we get, as expected, $e(\varepsilon) \to 1$. It is, however, quite surprising, that in the other limit $\varepsilon \to 1/4$ we get $e(\varepsilon) \to 0$. The perihelion advance tends to infinity if the eccentricity tends to $e(\varepsilon)$. For $\varepsilon$-values being only slightly below $1/4$, the perihelion advance becomes quite large even for extremely small eccentricities. Physically, this third case can be interpreted as follows: Further increasing the energy of the particle at constant angular momentum would have the consequence that its path goes to $r < 2m$ inside the horizon, so no periodic orbit appears.

To get an easy comparison with other deductions we also define the arithmetic mean $a = (r_1 + r_2)/2$ of $r_1$ and $r_2$ and call it semimajor axis. The geometric mean of them is denoted by $b = \sqrt{r_1 \cdot r_2}$ and we call it semiminor axis. Even if the orbit is not an exact ellipse, we use the usual formula (1.1) for defining the eccentricity $e = (r_2 - r_1)/(r_2 + r_1)$ having the range $0 < e < 1$. Further, we use the parameter $p = b^2/a$ which is also sometimes used in dealing with ellipses, it holds $p = a(1 - e^2) = 2r_1r_2/(r_1 + r_2)$, so $p$ is just the harmonic mean of $r_1$ and $r_2$. 

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For the exact ellipse in the Euclidean plane we have the equation

\[ r = \frac{p}{1 + e \cos \varphi} \]

thus

\[ r_1 = \frac{p}{1 + e} \quad \text{and} \quad r_2 = \frac{p}{1 - e}. \quad (3.9) \]

Therefore, the parametrization of the set of orbits discussed here with \((r_1, r_2)\) can also be done with \((p, e)\), and the latter one eases the comparison with the literature.

In the next step we prescribe the values of \(r_1\) and \(r_2\) and calculate the energy \(E\) and the angular momentum \(h > 0\) by use of eq. (2.8). At the extrema, \(dr/d\varphi = 0\), so we get as one of the conditions

\[ \left(1 + \frac{h^2}{r_1^2}\right) \cdot \left(1 - \frac{2m}{r_1}\right) = \left(1 + \frac{h^2}{r_2^2}\right) \cdot \left(1 - \frac{2m}{r_2}\right). \quad (3.10) \]

Inserting eqs. (3.9) into eq. (3.10) we arrive at

\[ mp^2 = h^2 \left(p - m(3 + e^2)\right) \quad (3.11) \]

which makes sense for \(p > m(3 + e^2)\) only. This inequality will always be assumed to hold in the following, it is equivalent to

\[ r_1 > m \cdot \frac{3 + e^2}{1 + e}, \quad (3.12) \]

meaning that the minimally allowed value for \(r_1\) depends on the eccentricity, it holds

\[ 2m < m \cdot \frac{3 + e^2}{1 + e} < 3m. \quad (3.13) \]

Analogously we get

\[ r_2 > m \cdot \frac{3 + e^2}{1 - e} > 3m. \quad (3.14) \]

Inequality (3.12) can also be expressed as \(ab^2 > m(4a^2 - b^2)\). From eq. (3.11) we get

\[ h = \frac{p\sqrt{m}}{\sqrt{p - m(3 + e^2)}} \quad (3.15) \]

and then

\[ E = \frac{\sqrt{(p - 2m)^2 - 4m^2e^2}}{\sqrt{p(p - m(3 + e^2))}}. \quad (3.16) \]
Further, the parameter $\epsilon$ calculates to

$$\epsilon = \frac{3m^2}{h^2} = \frac{3m}{p} - \frac{3m^2(3 + e^2)}{p^2}. \quad (3.17)$$

One easily calculates

$$0 < \epsilon \leq \frac{3}{4(3 + e^2)} < \frac{1}{4} \quad (3.18)$$

and for $p = 2m(3 + e^2)$ one has equality in the middle of this chain of inequalities.

4 Discussion

Let us interpret eq. (2.53), i.e.

$$\Theta = 2\pi \cdot \left( \frac{1 - 0.34215 \cdot \ln (1 - \frac{e^2}{4})}{\sqrt[4]{1 - 12m^2/h^2}} - 1 \right). \quad (4.1)$$

This fourth root in the denominator seems at least a little bit dubious, so we compare with the literature. First we concentrate on the orbits with negligible eccentricity but allow strong fields, i.e., the orbit may be close to the horizon. Then eq. (4.1) reduces to

$$\Theta = 2\pi \cdot \left( \frac{1}{\sqrt[4]{1 - 12m^2/h^2}} - 1 \right) = 2\pi \cdot \left( \frac{1}{\sqrt[4]{1 - 4\epsilon}} - 1 \right). \quad (4.2)$$

Second, we apply $\frac{1}{\sqrt[4]{1 - \delta}} = 1 + \frac{\delta}{4} + \frac{5\delta^2}{32} + \ldots$ and get the weak-field limit by inclusion of the first two terms to

$$\Theta = 2\pi \cdot \left( \frac{3m^2}{h^2} + \frac{45m^4}{2h^4} \right). \quad (4.3)$$

Eq. (4.40) from [14] reads in our notation

$$\Theta = 2\pi \cdot \left( \frac{3m^2}{h^2} + \frac{15m^4(6 + e^2)}{4h^4} \right). \quad (4.4)$$

As one can see, eqs. (4.3) and (4.4) become identical in the limit $e \to 0$, thus confirming the correctness of our calculation at least in this order of approximation.
To have a better comparison with other known results, we continue the discussion with the exact circular orbits. In our notation they can be calculated by inserting \( \dot{r} = 0 \) into eqs. (2.6) and (2.9), i.e.,

\[
\left(1 + \frac{h^2}{r^2}\right) \cdot \left(1 - \frac{2m}{r}\right) = E^2
\]

and

\[
\frac{h^2}{r^3} = \frac{mE^2}{r^2(1 - 2m/r)^2}
\]

together with

\[
\frac{\sqrt{m/r}}{\sqrt{1 - 2m/r}}
\]

resp. These equations can be solved for \( h \) and \( E \) by

\[
h = \frac{\sqrt{m/r}}{\sqrt{1 - 3m/r}} \quad \text{and} \quad E = \frac{1 - 2m/r}{\sqrt{1 - 3m/r}}.
\]

Therefore, only for \( r > 3m \) such orbits are possible, and in the limit \( r \to 3m \), a light-like circular orbit appears: The velocity of the particle is

\[
\frac{\sqrt{m/r}}{\sqrt{1 - 2m/r}}
\]

which tends to 1 as \( r \to 3m \). For every \( r > 3m \), a circular orbit exists, and with \( h \) from eq. (4.7) we calculate the parameter \( \varepsilon = 3m^2/h^2 \) to

\[
\varepsilon = \frac{3m}{r} \cdot \left(1 - \frac{3m}{r}\right).
\]

If we insert the expression \( \varepsilon \) eq. (4.9) from the circular orbits into eq. (4.2) we get

\[
\Theta = 2\pi \cdot \left(\frac{1}{\sqrt[4]{1 - 12m(1 - 3m/r)/(r - 1)}} - 1\right).
\]

This is exactly the same as

\[
\Theta = 2\pi \cdot \left(\frac{1}{\sqrt{1 - 6m/r}} - 1\right)
\]

which represents the expression for orbits close to circular ones already deduced in [6], eq. (11.2). For \( m \ll r \) we develop eq. (4.11) to

\[
\Theta = \frac{6\pi m}{r} \approx \frac{6\pi m}{a(1 - e^2)}
\]
a form which can be found in the majority of texts, e.g. as eq. (1) of [13].

In the weak-field approximation applied to eq. (4.8) we get velocity $\sqrt{m/r}$, and for the unit mass test particle $E_{\text{kin}} = m/(2r)$, $E_{\text{pot}} = -m/r$, hence total energy $E = 1 - m/(2r)$.

Let us finally discuss the case $e_4 = 1/2$ in more details. Then the perihelion advance reads exactly also for strong fields, see eqs. (2.43)/(2.53):

$$\Theta = 2\pi \cdot \left( \frac{1.09843}{\sqrt{1 - 12m^2/h^2}} - 1 \right).$$

(4.13)

We want to calculate the properties of the orbits to which this case belongs, the eccentricity should be something like $e = 1/2$, but, as already said, the relation between $e$ and $e_4$ may slightly depend on $h$. Looking back to eq. (2.40) we see that $\mu_4 = 0$ and $w_1 = \max w = 0$ and $w_2 = \min w = -\sqrt{3}$. With $\varepsilon = 3m^2/h^2$ and eqs. (2.29)/(2.32) we have $\mu_2 = \mu_3 = 0$. With eq. (2.28) we then get:

$$v_1 = \max v = 0 \quad \text{and} \quad v_2 = \min v = -\sqrt{3 - 12\varepsilon}/2\varepsilon.$$

With eq. (2.20) we then get:

$$u_1 = \max u = \frac{1}{2\varepsilon} \quad \text{and} \quad u_2 = \min u = \frac{1}{2\varepsilon} \cdot \left( 1 - \sqrt{3 - 12\varepsilon} \right).$$

To represent a closed orbit, $u_2 > 0$ is necessary, i.e. $1/6 < \varepsilon < 1/4$. Applying $r = h^2/(mu)$ leads to the perihelion

$$r_1 = 6m$$

(4.14)

and aphelion

$$r_2 = 6m/ \left( 1 - \sqrt{3 - 12\varepsilon} \right).$$

(4.15)

This gives rise to the eccentricity

$$e = \frac{1}{2/\sqrt{3 - 12\varepsilon} - 1}$$

(4.16)

thus having $e \to 0$ and $\Theta \to \infty$ as $\varepsilon \to 1/4$; and $e \to 1$ and $\Theta \to 2\pi \cdot 0.4456$ as $\varepsilon \to 1/6$. This means: For all periodic orbits with perihelion $r = 6m$, eq. (4.13) represents an exact formula for the perihelion advance.
To find out, how the perihelion advance changes with eccentricity, we solve eq. (4.16) for $\varepsilon$:

$$\varepsilon = \frac{1}{4} - \frac{1}{3} \cdot \frac{e^2}{(e + 1)^2}.$$ 

Then eq. (4.13) reads

$$\Theta = 2\pi \cdot \left(\sqrt[4]{3} \cdot \frac{1.09843 \cdot \sqrt{1 + 1/e - 1}}{\sqrt{2}}\right),$$

(4.17)

where 1.09843 is an abbreviation for

$$\frac{\sqrt[4]{3}}{\sqrt{8\pi^3}} \cdot (\Gamma(1/4))^2.$$ 

This shows that at least in this range, a Taylor development around $e = 0$ is not possible.

Next, we want to find out, how the perihelion advance changes with aphelion

$$r_2 = 6m \cdot \frac{1 + e}{1 - e}$$

(4.18)

leading to

$$\Theta = 2\pi \cdot \left(\sqrt[4]{3} \cdot 1.09843/\sqrt{1 - 6m/r_2 - 1}\right).$$

(4.19)

The formulas deduced here allow to evaluate orbital properties, especially the perihelion advance for all periodic orbits in the Schwarzschild field, and in regions near the horizon they are much better and easier to handle than other methods used in the literature. The method used has the potential also to be applied to alternative theories of gravitation, see e.g. [15] and [16] and the references cited there, to get the theoretical background for possible experimental tests of the theories.

5 Appendix

The main idea of this paper is outlined by the set of transformations between eqs. (2.16) and (2.36). To find out the internal structure of this idea, we present in this appendix another and independent deduction of the unexpected non-linear transformation of eq. (2.16), i.e.

$$\frac{d^2u}{d\varphi^2} + u = 1 + \varepsilon u^2,$$

(5.1)
which was the core of the calculations in the main part of this paper. It is
a self-contained deduction, so it can be read independently from the main
text, and shows the internal symmetry of eq. (2.16)/(5.1).

Here we concentrate solely on this eq. (5.1) and how solutions at different
values $\varepsilon$ are related to each other. Here, $\varepsilon < 1/4$ is a positive real parameter
and $u = u(\varphi)$ is a non-constant periodic function of the independent variable
$\varphi$. Let the smallest period of $u$ be $\vartheta > 0$. Due to the simple structure of
the equation one can easily show that $\vartheta$ can also be defined as follows: If
$u(\varphi_1)$ is a local minimum and $u(\varphi_2)$ is the next one, then $u(\varphi_1) = u(\varphi_2)$ and
$\vartheta = \varphi_2 - \varphi_1$. Thus, the perihelion shift for this orbit is

$$\Theta = \vartheta - 2\pi.$$ (5.2)

Now we fix a further real parameter $\lambda > 1$ and define the function $v = v(\varphi)$
as follows:

$$v(\varphi) = B \cdot u(\sqrt{\lambda} \varphi) - A$$ (5.3)

where $A$ and $B$ depend on $\varepsilon$ and $\lambda$ only, and $B > 0$. Clearly, $v(\varphi)$ is
also a non-constant periodic function. The function $v$ has smallest period
$\tilde{\vartheta} = \vartheta/\sqrt{\lambda}$ and perihelion shift $\tilde{\Theta} = \tilde{\vartheta} - 2\pi$. With eq. (5.2) we get

$$\tilde{\Theta} = \Theta/\sqrt{\lambda} - 2\pi(1 - 1/\sqrt{\lambda}).$$ (5.4)

Thus, the perihelion shift of the function $v$ is smaller than simply the perihe-
lion shift of the function $u$ divided by $\sqrt{\lambda}$ as one would have expected from
a first glance. In this sense the transformation discussed here is a non-linear
one. Now we want to fix $A$ and $B$ such that

$$\frac{d^2 v}{d \varphi^2} + v = 1 + \tilde{\varepsilon} v^2$$ (5.5)

with a parameter $\tilde{\varepsilon} = \tilde{\varepsilon}(\varepsilon, \lambda)$. This means, that $v$ shall solve essentially the
same equation as $u$ does, only the value of the parameter $\varepsilon$ may differ. To
find the form of $A$, $B$ and $\tilde{\varepsilon}$ we insert (5.1) and (5.3) into (5.5). This leads via

$$\frac{d^2 v}{d \varphi^2} = b \cdot \frac{d^2 u(\sqrt{\lambda} \varphi)}{d \varphi^2}$$

to

$$B\lambda[1 + \varepsilon u^2 - u] + Bu(\sqrt{\lambda} \varphi) - A = 1 + \tilde{\varepsilon}[B^2 u^2 + A^2 - 2ABu].$$ (5.6)

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Here, \( u \) means \( u(\sqrt{\lambda} \varphi) \). This equation (5.6) must be identically fulfilled. The vanishing of the terms proportional to \( u^2 \) leads to \( B\lambda \varepsilon = B^2 \bar{\varepsilon} \), i.e. to

\[
\bar{\varepsilon} = \frac{\lambda \varepsilon}{B} .
\]  
(5.7)

Inserting (5.7) into (5.6) we get

\[
B\lambda - B\lambda u(\sqrt{\lambda} \varphi) + Bu(\sqrt{\lambda} \varphi) - A = 1 + A^2 \lambda \varepsilon / B - 2A\lambda \varepsilon u(\sqrt{\lambda} \varphi) .
\]  
(5.8)

The vanishing of the terms proportional to \( u \) leads to

\[-B\lambda + B = -2A\lambda \varepsilon ,
\]
i.e. to

\[
A = \frac{B(\lambda - 1)}{2\lambda \varepsilon} .
\]  
(5.9)

One gets \( A > 0 \). Finally, inserting (5.9) into (5.8) we get

\[
B\lambda - \frac{B(\lambda - 1)}{2\lambda \varepsilon} = 1 + \frac{\lambda \varepsilon}{B} \cdot \left( \frac{B(\lambda - 1)}{2\lambda \varepsilon} \right)^2 ,
\]
i.e.

\[
B = \frac{4\lambda \varepsilon}{1 - \lambda^2(1 - 4\varepsilon)} .
\]  
(5.10)

To ensure \( B > 0 \) we restrict to the region where \( \lambda^2(1 - 4\varepsilon) < 1 \), i.e. where

\[
1 < \lambda < \frac{1}{\sqrt{1 - 4\varepsilon}} .
\]  
(5.11)

Inserting (5.10) into (5.7) and (5.9) we get

\[
\bar{\varepsilon} = \frac{1}{4} \cdot [1 - \lambda^2(1 - 4\varepsilon)] > 0
\]  
(5.12)

and

\[
A = \frac{2(\lambda - 1)}{1 - \lambda^2(1 - 4\varepsilon)}
\]  
(5.13)

resp. To find the upper limit for \( \bar{\varepsilon} \) we calculate

\[
\varepsilon - \bar{\varepsilon} = \left( \frac{1}{4} - \varepsilon \right) \cdot (\lambda^2 - 1)
\]
(5.14)

which leads to \( 0 < \bar{\varepsilon} < \varepsilon \).
The limit $\tilde{\epsilon} \to 0$ is a singular one, as in this limit, the non-linear equation (5.5) becomes a linear one. Nevertheless, it leads to the following result: All non-constant solutions of eq. (5.5) with $\tilde{\epsilon} = 0$ have the period $\tilde{\vartheta} = 2\pi$, i.e. perihelion shift $\tilde{\Theta} = 0$. With eq. (5.4) we get for this case

$$\Theta = 2\pi \left( \sqrt{\lambda} - 1 \right). \quad (5.15)$$

With eq. (5.12) we get for $\tilde{\epsilon} = 0$

$$\lambda = (1 - 4\tilde{\epsilon})^{-1/2}. \quad (5.16)$$

Combining eqs. (5.15) and (5.16) we finally get for the perihelion shift

$$\Theta = 2\pi \left( \frac{1}{\sqrt{1 - 4\tilde{\epsilon}}} - 1 \right). \quad (5.17)$$

This is exactly the same as eq. (4.2).

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