On a problem of A. Weil

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Abstract

A topological invariant of the geodesic laminations on a modular surface is constructed. The invariant has a continuous part (the tail of a continued fraction) and a combinatorial part (the singularity data). It is shown, that the invariant is complete, i.e. the geodesic lamination can be recovered from the invariant. The continuous part of the invariant has geometric meaning of a slope of lamination on the surface.

Key words and phrases: modular surface, geodesic lamination

MSC: 57M50 (geometric structures on low-dimensional manifolds)

1 Introduction

Let $S$ be a connected complete hyperbolic surface. By a geodesic $g$ on $S$ we understand the maximal arc consisting of the locally shortest sub-arcs; the geodesic $g$ is called simple if it has no self-crossing points. It is well known, that simple geodesic is either (i) a closed geodesic, (ii) a non-closed spiral geodesic tending to a closed geodesic or (iii) a non-closed geodesic, whose limit set is a perfect (Cantor) subset of $S$ [Casson & Bleiler 1988] [6]. A geodesic lamination is a closed subset of the surface $S$, which is the union of disjoint simple geodesics $g$.

The Chabauty topology (also known as the Gromov-Hausdorff topology) turns the set of minimal laminations into an important topological space $\Lambda = \Lambda(S)$. For example, certain compactifications of the Teichmüller space of $S$ are homeomorphic to $\Lambda$, modulo the laminations with more than one independent ergodic measure [Thurston 1997] [13]. The space $\Lambda$ is a compact
Hausdorff topological space; it has a metric measuring the angles between the asymptotic direction (a slope) of the laminations on the surface $S$. In the simplest case $S = T^2$ (a flat torus), all possible slopes are exhausted by the irrationals $\theta \in [0, 2\pi[$; the latter are known as the Poincaré rotation numbers.

In 1936 A. Weil asked about a generalization of the rotation numbers to the case of the higher genus surfaces [Weil 1936] [14]. Namely, the Weil problem consists in an explicit construction of the rotation numbers for laminations on surfaces of genus $g \geq 1$; the numbers must satisfy all formal properties of the Poincaré rotation numbers. It was conjectured, that the hyperbolic plane might be critical to a solution of the problem, *ibid*. An excellent survey of [Anosov 1995] [1] gives an account of the Weil’s problem after 1936. Many important contributions to a solution of the problem are due to [Anosov & Zhuzhoma 2005] [2], [Moeckel 1982] [9], [Schwartzman 1957] [11], [Series 1985] [12] and others, see [Anosov 1995] [1]. Let us mention an influential and spiritually close work of [Artin 1924] [3]. Needless to say, a solution to the Weil problem is significant for hyperbolic geometry and low-dimensional topology.

In this note we define a slope $\theta$ of lamination $\lambda$ on a surface $S$. The slope measures an asymptotic direction of $\lambda$ on $S$. To formalize our result, we assume that $S \cong \mathbb{H}^*/G$, where $G$ is a finite index subgroup of the modular group $SL(2, \mathbb{Z})$ and $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ is the extended Lobachevsky plane [Gunning 1962] [7]. According to [Moeckel 1982] [9] et al. such a choice of $S$ is the most natural for applications in number theory, e.g. the continued fractions. For the sake of clarity, we let $G \cong \Gamma_0(N)$, where $N \geq 1$ is an integer and $\Gamma_0(N) = \{(a, b, c, d) \in SL(2, \mathbb{Z}) \mid c \equiv 0 \text{ mod } N\}$ is the Hecke subgroup of $SL(2, \mathbb{Z})$. The modular surface $X_0(N) = \mathbb{H}^*/\Gamma_0(N)$ is a complete hyperbolic surface.

Recall, that an *axis* is a closed geodesic $g \in S$, which is covered by a half-circle $\tilde{g} \in \mathbb{H}^*$ fixed by a hyperbolic transformation, see [Casson & Bleiler 1988] [6] and [Gunning 1962] [7]. The lamination is called a singleton, if it consists of a unique simple closed geodesic $g$; the singletons are dense in the space $\Lambda(S)$ [Canary, Epstein & Green 1987] [5], Lemma 4.2.15. The type (iii) lamination $\lambda \in \Lambda(X_0(N))$ will be called a Legendre lamination, if there exists a regular continued fraction $[p_0, p_1, \ldots]$, such that for $k \geq 1$ the axis of the hyperbolic transformation

$$
\begin{pmatrix}
1 & p_0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & p_1
\end{pmatrix}
\cdots
\begin{pmatrix}
0 & 1 \\
1 & p_{2k}
\end{pmatrix}
$$

(1)
is a singleton \( g_k \) and \( \lambda = \lim_{k \to \infty} g_k \). We show in Lemma 2 that the Legendre laminations exist and form an uncountable subset of \( \Lambda(X_0(N)) \). The slope of the Legendre lamination is defined as

\[
\theta = p_0 + \frac{1}{p_1 + \frac{1}{p_2 + \ldots}} := [p_0, p_1, p_2, \ldots]
\]

To recover a lamination from \( \theta \), one needs to specify the number and type of the boundary components of the lamination [Casson & Bleiler 1988] [6]. Let \( g = g(N) \) be the genus of the modular surface and \( \Delta = (k_1, \ldots, k_m) \) a finite set of the positive integers and half-integers, such that \( \sum k_i = 2g - 2 \). The \( \Delta \) is called a singularity data of the lamination; see Section 2.3 for the details. Our main result is the following existence and rigidity theorem.

**Theorem 1** For each \( N \geq 1 \) there exists a continuum of the Legendre laminations \( \lambda \in \Lambda(X_0(N)) \); their topological types are bijective with the pairs \( (\Theta, \Delta) \), where \( \Theta \equiv \theta \mod GL(2, \mathbb{Z}) \) is the equivalence class of irrationals modulo the action of the matrix group \( GL(2, \mathbb{Z}) \) and \( \Delta \) the singularity data of \( \lambda \).

The article is organized as follows. In Section 2 the geodesic laminations are reviewed. Theorem 1 is proved in Section 3.

## 2 Geodesic laminations

This section is a brief review of the geodesic laminations following [Casson & Bleiler 1988] [6] and contains no original results. We refer the reader to the above cited monograph for a detailed account.

### 2.1 The Chabauty topology

Let \( S \) be a connected complete hyperbolic surface. By a geodesic \( g \in S \) we understand the maximal arc consisting of the locally shortest sub-arcs. The geodesic \( g \) is called simple if it has no self-crossing points. (We leave aside the non-trivial question of existence of simple geodesics; such geodesics make an uncountable set on any hyperbolic surface, albeit of measure zero [Artin 1924] [3].) A geodesic lamination on \( S \) is a closed subset \( \lambda \) of \( S \), which is a disjoint union of the simple geodesics. The geodesics are called the leaves of
The lamination \( \lambda \) is called \textit{minimal} if no proper subset of \( \lambda \) is a geodesic lamination. The following lemma gives an idea of the minimal laminations (our main object of study).

**Proposition 1** The minimal lamination \( \lambda \) in a complete hyperbolic surface \( S \) is either a singleton (simple closed geodesic) or an uncountable nowhere dense (Cantor) subset of \( S \).

**Proof.** See Lemma 4.2.2 of [Canary, Epstein & Green 1987] [5] and Lemma 3.3 of [Casson & Bleiler 1988] [6]. \( \square \)

The set of all minimal laminations on the surface \( S \) will be denoted by \( \Lambda(S) \). The laminations \( \lambda, \lambda' \) on \( S \) are said to be \textit{topologically conjugate} if there exists a homeomorphism \( \varphi : S \to S \), such that each leaf of \( \lambda \) through the point \( x \in S \) goes to the leaf of \( \lambda' \) through the point \( \varphi(x) \). Clearly, the set \( \Lambda(S) \) splits into the equivalence classes of topological conjugacy (or, topological \textit{types}).

Since \( S \) is a complete hyperbolic surface, its universal cover is the unit disk \( D \). Any \( \lambda \in \Lambda(S) \) lifts to a lamination \( \tilde{\lambda} \) on \( D \), which is invariant under the action of the covering transformations. Every leaf of \( \tilde{\lambda} \) is given by an unordered pair of points at the boundary of \( D \), and therefore the space of geodesics is homeomorphic to the Möbius band, \( M \).

Let \( C(M) \) be the set of all closed subsets of \( M \). The \textit{Chabauty topology} \( \square \) on \( C(M) \) is given by the Hausdorff distance \( d(X, Y) \leq \varepsilon \) iff \( X \subseteq N_\varepsilon(Y) \) and \( Y \subseteq N_\varepsilon(X) \), where \( N_\varepsilon(X) \) (\( N_\varepsilon(Y) \)) is a \( \varepsilon \)-neighbourhood of the closed set \( X \in C(M) \) (\( Y \in C(M) \)). The function \( d \) turns the set \( \tilde{\Lambda} \) of the laminations in \( D \) into a compact metrizable Hausdorff space. The Chabauty topology on \( \Lambda(S) \) can be defined as a factor-topology of the topology on \( \tilde{\Lambda} \) under the covering map. Everywhere in below, the standard topology on the set \( \Lambda(S) \) will be the Chabauty topology. The following statement, mentioned in the introduction, will be critical.

**Proposition 2** The subspace \( \Lambda^*(S) \) made of the singletons (i.e. the simple closed geodesics in \( S \)) is dense in the space \( \Lambda(S) \).

**Proof.** See [Canary, Epstein & Green 1987] [5], Lemma 4.2.15. \( \square \)

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1It is a tradition to reserve the term Chabauty topology for a topology on the set of all closed subgroups of a locally compact group. However, Canary, Epstein and Green [Canary, Epstein & Green 1987] [5] differ from the convention but point out that on metrizable space this is nothing but the Hausdorff topology.
2.2 The principal regions and boundary leaves

If \( \lambda \in \Lambda(S) \), then a component of \( S - \lambda \) is called a principal (complementary) region for \( \lambda \). (Note that \( S - \lambda \) may have several connected components.) The leaves of \( \lambda \), which form the boundary of a principal region, are called the boundary leaves. If \( \lambda \) is minimal (which we always assume to be the case), then each boundary leaf is a dense leaf of \( \lambda \), isolated from one side. Note that by the proposition \([1]\) the area of a minimal lamination is zero, hence \( \text{Area}(S - \lambda) = \text{Area} S \) and the principal region is a complete hyperbolic surface of the area \(-2\pi \chi(S)\), where \( \chi(S) = 2 - 2g \) is the Euler characteristic of the surface \( S \). (More precisely, if one takes the closure of the lift and quotient by the isometry group, then one gets a hyperbolic surface with geodesic boundary.) If \( U \) is a component of the preimage of the principal region in \( D \), then \( U \) is a union of the ideal polygons \( U_i \) in \( D \) (see Fig.1).

The hyperbolic area of an ideal \( n_i \)-gon \( U_i \) is equal to \((n_i - 2)\pi\) [Casson & Bleiler 1988] [6]. Since \( \sum \text{Area} U_i = (4g - 4)\pi \), the number of the ideal polygons in \( U \) is finite.

2.3 The singularity data

To capture combinatorial structure of the lamination \( \lambda \in \Lambda(S) \), we shall need the following collection of data. Denote by \( U_i^{(n_i)} \) an \( i \)-th ideal \( n_i \)-gon in the principal region of the lamination \( \lambda \); here \( n_i \geq 3 \) is an integer. It is known, that \( \text{Area} U_i^{(n_i)} = (n_i - 2)\pi \). On the other hand, the total area of all ideal polygons must be equal to the hyperbolic area of the surface \( S \), i.e.
\[
\sum_{i=1}^{m} \text{Area } U_i^{(n_i)} = (4g - 4)\pi. \text{ Thus, one arrives at the equation}
\]
\[
\sum_{i=1}^{m} \frac{n_i - 2}{2} = 2g - 2. \tag{3}
\]
For simplicity, we let \(k_i = \frac{1}{2}(n_i - 2)\); since \(n_i \geq 3\), the numbers \(k_i\) take integer and half-integer positive values. In the above notation, our formula becomes \(\sum_{i=1}^{m} k_i = 2g - 2\).

Let \(\{U_1^{(n_1)}, \ldots, U_m^{(n_m)}\}\) be a collection of the ideal polygons in the principal region of the lamination \(\lambda \in \Lambda(S)\); let \(\{k_1, \ldots, k_m\}\) be the corresponding collection of \(k_i\). The unordered tuple \(\Delta = (k_1, \ldots, k_m)\) will be called a singularity data of the lamination \(\lambda\). For example, the singularity data of laminations with the principal regions shown in Fig.1 (i) and (ii) are \(\Delta_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\) and \(\Delta_2 = (2)\), respectively.

Conversely, given a set of positive integers and half-integers, such that \(\sum_{i=1}^{m} k_i = 2g - 2\), there exists a lamination \(\lambda\) on surface \(S\) of genus \(g\), which realizes the singularity data \((k_1, \ldots, k_m)\); this fact follows from [Hubbard & Masur 1979] [8].

Finally, the term “singularity” is justified by the fact, that minimal geodesic laminations are bijective with the measured foliations on the same surface [Thurston 1997] [13]: under the bijection each ideal polygon \(U_i^{(n_i)}\) corresponds to a singular point of index \(-k_i\) of the foliation.

### 3 Proof of theorem 1

The proof is arranged into a series of lemmas, starting with the following elementary

**Lemma 1** The continued fraction of the Legendre lamination is unique.

**Proof.** Recall a classical bijection between the rationals \(r_k\) and finite continued fractions \([p_0, \ldots, p_{2k}]\), given by the formula:

\[
r_k = p_0 + \cfrac{1}{\left(p_1 + \cfrac{1}{\left(p_2 + \cfrac{1}{p_3 + \cdots}ight)}\right)}, \tag{4}
\]
see [Perron 1954] [10]; the bijection extends to the singletons, since each $[p_0, \ldots, p_{2k}]$ defines a singleton $g_k$. Moreover, such a bijection extends to the half-circles $\tilde{g}_k \in \mathbb{H}^*$ which lie in the same orbit of the group $\Gamma_0(N)$. We refer the reader to [Artin 1924] [3].

To the contrary, let $\tilde{\lambda}$ be the preimage of a Legendre lamination on $\mathbb{H}^*$, such that $\tilde{\lambda} = \lim_{k \to \infty} \tilde{g}_k = \lim_{k \to \infty} \tilde{g}_k'$, where $\tilde{g}_k \neq \tilde{g}_k'$. By the Artin bijection, one gets a pair of the regular continued fractions convergent to the same limit. It is known to be false, see e.g. [Perron 1954] [10], Satz 2.6, p. 33.

**Lemma 2** The Legendre laminations form an uncountable subset of $\Lambda(X_0(N))$.

**Proof.** (i) The set of the Legendre laminations is non-empty. Indeed, let $tr : \Gamma_0(N) \to \mathbb{Z}$ be the trace function. We denote by $Z_0$ the subset of $tr(\Gamma_0(N))$ consisting of traces of the hyperbolic transformations, whose axes are singletons. It is known, that $Z_0$ is an infinite set; moreover, each arithmetic progression contains an infinite number of elements of $Z_0$, e.g. [Birman & Series 1984] [4].

Denote by $\gamma_k \in \Gamma_0(N)$ the product (1). The matrix multiplication (in the first few terms) gives us:

\[
\begin{align*}
tr(\gamma_0) &= 2 \\
tr(\gamma_1) &= p_0 + p_1 \\
tr(\gamma_2) &= 2 + p_0p_1 + p_1p_2 \\
tr(\gamma_3) &= p_0 + p_1 + p_2 + p_3 + p_0p_1p_2 + p_1p_2p_3,
\end{align*}
\]

(5)

where $p_0 \in \mathbb{N} \cup \{0\}$ and $p_i \in \mathbb{N}$. It is clear, that one can find $p_0, p_1, p_2$, so that $tr(\gamma_1)$ and $tr(\gamma_2)$ belong to $Z_0$; indeed, it suffices to take $p_0 = z_1 - 1, p_1 = 1$ and $p_2 = z_2 - z_1 - 1$ for any two points $z_1, z_2 \in Z_0$, such that $2 \leq z_1 \leq z_2 - 2$.

In general, assume (by induction) that $p_0, p_1, \ldots, p_n$ satisfy the condition $tr(\gamma_i) \in Z_0$ for all $0 \leq i \leq n$. We want to choose $p_{n+1}$, such that the matrix $\gamma_{n+1}$ defined by (1) is hyperbolic. Consider an arithmetic progression $ap_{n+1} + b = tr(\gamma_{n+1})$, where $a$ and $b$ are integers depending only on $p_0, \ldots, p_n$. The arithmetic progression $\{a + b, 2a + b, \ldots\}$ contains infinitely many elements of $Z_0$; let $z_{n+1}$ be one of them with the index $p_{n+1}$. Then, $tr(\gamma_{n+1}) \in Z_0$ and the induction is completed.

In this way, one obtains an infinite sequence $[p_0, p_1, \ldots]$, such that the axes of hyperbolic transformations $\gamma_i$ are singletons; their hyperbolic length.
grows and, by compactness of $\Lambda(X_0(N))$, converges to a type (iii) lamination $\lambda$. By construction, $\lambda$ is the Legendre lamination.

(ii) The Legendre laminations are uncountable; indeed, by a modification of the argument of (i), one can choose, at each step of the induction, infinitely many $p_{n+1}$, which satisfy the equation $ap_{n+1} + b = tr(\gamma_{n+1})$. Thus, one gets an infinite sequence $[p_0, p_1, \ldots]$, where each $p_i$ runs a countable infinite set; the set of all such sequences is also infinite, but uncountable.

In view of lemma 1, the corresponding Legendre laminations are also uncountable; lemma 2 follows. □

Lemma 3 Two Legendre laminations are topologically conjugate, if and only if their singularity data coincide and their continued fractions coincide, except a finite number of terms.

Proof. Let $\varphi : X_0(N) \to X_0(N)$ be an automorphism, which conjugates laminations $\lambda$ and $\lambda' = \varphi(\lambda)$. Notice that the automorphism $\varphi$ preserves the singularity data of $\lambda$ and $\lambda'$, see Section 2.3. The action of $\varphi$ extends to the singletons; by the Artin bijection, it extends to the positive rationals $r_k$, cf. proof of lemma 1. But each automorphism of the rational numbers is given by the formula:

$$\varphi(r_k) = \frac{ar_k + b}{cr_k + d}, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in GL(2, \mathbb{Z}). \quad (6)$$

In the last formula, we can pass to the limit $\theta = \lim_{k \to \infty} r_k$; thus, whenever the $\theta$ is given by the continued fraction $[p_0, p_1, \ldots]$, then $\theta' = \varphi(\theta)$ is given by continued fraction $[q_1, \ldots, q_k; p_0, p_1, \ldots]$, where

$$\left( \begin{array}{cc} 0 & 1 \\ 1 & q_1 \end{array} \right) \cdots \left( \begin{array}{cc} 0 & 1 \\ 1 & q_k \end{array} \right) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right). \quad (7)$$

Notice, that the Legendre laminations $\lambda$ and $\lambda'$ are given by the fractions $\theta$ and $\theta'$, respectively. The converse statement can be proved similarly. Lemma 3 follows. □

Lemma 4 For each $\theta \in \mathbb{R} - \mathbb{Q}$ and an abstract data $\Delta$ there exists a Legendre lamination of slope $\theta$, which realizes the singularity data $\Delta$. 

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Proof. Let $D$ be the unit disk and $\Delta = (k_1, \ldots, k_m)$ a singularity data compatible with the modular surface $X_0(N)$; denote by $U_1, \ldots, U_m$ the ideal polygons corresponding to $\Delta$. (Notice that the polygons $U_i$ are not uniquely defined; yet their isometry class in the group $\Gamma_0(N)$ depends solely on $\Delta$.) If $\theta = [p_0, p_1, \ldots]$, we shall write $C_i$ to denote an axis of transformation $\gamma_i$ computed by formula (1); we denote the corresponding singleton by $\lambda_i^*$. For each $\gamma_i$ consider a region $O_{\gamma_i}$ of $D$ defined by the formula:

$$O_{\gamma_i} = \bigcup_{g\in\Gamma_0(N)} \bigcup_{|tr(h)| \leq |tr(\gamma_i)|} ghg^{-1} \left( \sum_{i=1}^{m} U_i \right); \quad (8)$$

notice that region $O_{\gamma_i}$ is invariant of the conjugacy class of transformation $\gamma_i$. We denote the maximal subset of $D - O_{\gamma_i}$ containing $C_i$ (and all isometric images of $C_i$) by $\Omega_i$.

Let us show that $D \supset \Omega_0 \supset \Omega_1 \supset \ldots$ are strict inclusions. Indeed, in view of formulas (5), we have $tr(\gamma_i+1) > tr(\gamma_i)$ for all $i \geq 0$; thus for the cosets $H_i := \{ghg^{-1} | g \in \Gamma_0(N), |tr(h)| \leq |tr(\gamma_i)|\}$, one gets an inclusion $H_i \subset H_{i+1}$. In view of formula (8), one obtains an inclusion $O_{\gamma_i} \subset O_{\gamma_{i+1}}$; thus, for the complement sets $\Omega_i$, we have a strict inclusion $\Omega_i \supset \Omega_{i+1}$, which holds for all $i \geq 0$.

Denote by $\Omega = \cap_{i=0}^{\infty} \Omega_i$. It is easy to see, that $\Omega$ is a non empty closed set as limit of a decreasing sequence of non empty closed sets.

By our construction, each $\Omega_i$ contains the arc $C_i$; therefore, the set $C = \lim_{i \to \infty} C_i$ is contained in the set $\Omega$. But $\Omega$ is a closed set (of measure zero) of the unit disk $D$; thus $C$ covers a type (iii) lamination $\lambda = \lim_{i \to \infty} g_i$ on the surface $X_0(N)$. The lamination $\lambda$ is a Legendre lamination of slope $\theta$, which realizes the singularity data $\Delta$. Lemma 4 is proved. □

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Now we use the above four lemmas to prove theorem 1. Namely, let $\Lambda_0 \subset \Lambda(X_0(N))$ be the space of all Legendre lamination on $X_0(N)$; in view of lemmas 1 and 3 each conjugacy class of $\lambda \in \Lambda_0$ defines an invariant $(\Theta, \Delta)$, where $\Theta = \{\theta' : \theta' = \frac{ad+bc}{cd+d} ; \ a, b, c, d \in \mathbb{Z}, \ ad - bc = \pm 1\}$. Conversely, lemma 4 says that any abstractly given invariant $(\Theta, \Delta)$ admits a realization by a lamination $\lambda \in \Lambda_0$. This argument finishes the proof of theorem 1. □

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