New solutions of initial conditions in general relativity

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Abstract
We find new classes of exact solutions of the initial momentum constraint for vacuum Einstein’s equations. Considered data are either invariant under a continuous symmetry or they are assumed to have the exterior curvature tensor of a simple form. In general the mean curvature $H$ is non-constant and $g$ is not conformally flat. In the generic case with the symmetry we obtain general solution in an explicit form. In other cases solutions are given up to quadrature. We also find a class of explicit solutions without symmetries which generalizes data induced by the Kerr metric or other metrics related to the Ernst equation. The conformal method of Lichnerowicz, Choquet-Bruhat and York is used to prove solvability of the Hamiltonian constraint if $H$ vanishes. Existence of marginally outer trapped surfaces in initial manifold is discussed.

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1. Introduction

Initial data for the vacuum Einstein equations consists of a three-dimensional manifold $S$ together with a Riemannian metric $g_{ij}$ and a symmetric tensor $K_{ij}$. They have to satisfy the following constraint equations

\begin{align}
\nabla_i (K^{ij} - g^{ij} H) &= 0 \quad (1) \\
R + H^2 - K_{ij} K^{ij} &= 0 \quad (2)
\end{align}

where $\nabla_i$ are covariant derivatives, $R$ is the Ricci scalar of $g_{ij}$ and $H = K^i_i$. Tensors $g_{ij}$ and $K_{ij}$ are interpreted, respectively, as the induced metric and the external curvature of $S$ embedded in four-dimensional spacetime $M$ with metric $g^{(4)}$

\begin{align}
g_{ij} \, dx^i \, dx^j = e^\theta g^{(4)}, \quad 2K_{ij} \, dx^i \, dx^j = e^\vartheta L_4 g^{(4)} \quad (3)
\end{align}
(\mathcal{L}_A is the Lie derivative along the future unit normal vector of \( S \) and \( e^* \) denotes the pullback under an embedding \( e : S \rightarrow M \)).

In order to analyze constraints (1) and (2) one can employ the method of transverse-traceless decomposition of \( K_{ij} \) into its trace, a longitudinal part and a divergence free traceless tensor [1]. Combining this decomposition with a conformal transformation of the initial data yields a system of differential equations equivalent to (1) and (2). The main advantage of this approach (known as the conformal method) is that in the new system one can distinguish dynamical variables from free functions (note that system (1)–(2) is under-determined). In this approach initial data sets with the vanishing mean curvature \( H \) are constructed by means of the conformal transformation

\[
g'_{ij} = \psi^4 g_{ij}, \quad K'_{ij} = \psi^{-2} K_{ij}, \quad \psi > 0.
\]

(4)

If \( H = 0 \) and tensor \( K_{ij} \) satisfies the momentum constraint (1) then this transformation yields another traceless (\( H' = 0 \)) solution of this constraint. The new fields satisfy the Hamiltonian constraint (2) if

\[
\Delta \psi = \frac{1}{8} R \psi - \frac{1}{8} K_{ij} K^{ij} \psi^{-7},
\]

(5)

where \( \nabla_i \) denotes the covariant derivative and \( \Delta \) is the covariant Laplace operator for metric \( g_{ij} \). The pair \( (g'_{ij}, K'_{ij}) \) constitutes unconstrained initial data for the vacuum Einstein equations.

All known exact solutions of the constraint equations are based on the above scheme. Among them there are solutions of Brill and Lindquist [2], Bowen and York [3] and Brandt and Brugmann [4]. In all of them metric \( g_{ij} \) is flat, hence the momentum constraint can be easily solved. In general, equation (5) is not solvable analytically. The existence of its solutions was determined in cases where \( (S, g_{ij}) \) is closed [5, 6], asymptotically flat [7–9] or asymptotically hyperbolic [10, 11]. Also particular results are known for \( (S, g_{ij}) \) asymptotically flat with an interior boundary [9, 12]. A detailed discussion of the problem can be found in [13].

In this paper we construct new solutions of the momentum constraint with constant or non-constant mean curvature without assuming that the initial metric is conformally flat. For brevity we write (1) in the form

\[
\nabla_i T^{ij} = 0
\]

(6)

where \( T^{ij} = K^{ij} - H g^{ij} \). In section 2 we assume that data are preserved by a continuous symmetry (which is the Killing symmetry of the corresponding four-dimensional metric). In generic case all the solutions of the momentum constraint are found explicitly. They depend on six free functions (including 3 degrees of freedom of the metric tensor), which cannot be gauged away. They generalize solutions found in [14, 15] and contain data induced by the Kerr metric (and other stationary axially symmetric metrics) what is not possible in the case of conformally flat initial metrics [16]. In section 3 we consider non-symmetric data but we make an algebraic assumption about \( T_{ij} \). We obtain a particular class of solutions which depend on three free functions of three coordinates (the same number as in the case of the flat metric). It also contains data for axially symmetric vacuum metrics related to a solution of the Ernst equation.

In principle, free functions in the data can be used to fulfill the Hamiltonian constraint. We shortly discuss this possibility in the beginning of section 4, however we don’t have rigorous results in this direction. If \( T_{ij} = 0 \) then \( H = 0 \) and one can use the conformal method to construct implicitly data which satisfy all initial constraints (in practical applications the Hamiltonian constraint has to be solved numerically). In section 4 we prove the existence of solutions of the Lichnerowicz equation (5) following results of Maxwell [9] for asymptotically flat data. Maxwell’s approach for manifolds with a boundary is used in section 5 in order to construct initial data with marginally outer trapped surfaces.

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2. Symmetric solutions of the momentum constraint

In this section we assume that initial data is preserved by a vector field \( v \). This field extends to the Killing vector of the corresponding four-dimensional metric \( g^{(4)} \). A class of data of this kind was constructed in [14, 15] with the exterior curvature equivalent to (8) and (35). Its generalization was recently considered in [17].

The following propositions describe general solution of the momentum constraint with the symmetry \( v \). We use nonholonomic basis written in coordinates \( x^a = x^a, \varphi \), with \( a = 1, 2 \), such that \( v = \partial \varphi \).

**Proposition 2.1.** Let
\[
g = g_{ab} \, dx^a \, dx^b + \alpha^2 (d\varphi + \beta_a \, dx^a)^2
\]
and components of metric and the exterior curvature be independent of \( \varphi \). Then

- In the basis \( \theta^a = dx^a, \theta^3 = d\varphi + \beta_a \, dx^a \) the momentum constraint is equivalent to an explicit formula for \( T_a^3 \)
\[
T_a^3 = \alpha^{-1} \eta^{ab} \omega_b
\]
and equation
\[
(\alpha T^b_a)_b = T_a^3 \lambda + \lambda \omega_a, \quad \lambda = \eta^{ab} \beta_{a,b},
\]
where the covariant derivative \( \partial_b \) and the Levi-Civita tensor \( \eta^{ab} \) correspond to \( g_{ab} \) and \( \omega \) is a function of \( x^a \).

- In complex coordinates \( x^a = \zeta, \bar{\zeta} \) such that \( g_{ab} = \gamma^2 \delta_{ab} \)
\[
2(\alpha T_\zeta \bar{\zeta})_\zeta = \gamma^2 T_\zeta \bar{\zeta} \bar{\zeta} - \gamma^2 (\alpha T^\zeta_\zeta) + \gamma^2 \lambda \omega \bar{\zeta}.
\]

Given functions \( \alpha, \lambda, \gamma, \omega, T_\zeta \bar{\zeta} \) and \( T_\zeta \bar{\zeta} \) equation (10) defines \( \alpha T_\zeta \bar{\zeta} \) as an integral.

**Proof.** Metric (7) is the most general metric with a single continuous symmetry \( \partial \varphi \). If data are independent of \( \varphi \) then in the basis \( \theta^a = dx^a, \theta^3 = d\varphi + \beta_a \, dx^a \) the momentum constraint \( \nabla_j T_j^a = 0 \) takes very simple form
\[
(\alpha \sqrt{|\tilde{g}|} T_3^a)_{\partial \varphi} = 0,
\]
where \( |\tilde{g}| = \det g_{ab} \). General solution of (11) is given by (8). Substituting it into \( \nabla_j T_j^a = 0 \) yields equation (9).

In dimension 2 one can always find coordinates \( x^a = x, y \) such that
\[
g_{ab} = \gamma^2 \delta_{ab}.
\]
In this case a complex combination of equations (9) written in complex coordinates \( \zeta = x + iy, \bar{\zeta} = x - iy \) yields equation (10). The assumption of analyticity allows to integrate the rhs of (10) over \( \zeta \). Note that \( T_\zeta \bar{\zeta} \) is a complex function and \( T_\zeta \bar{\zeta} \bar{\zeta} \) is real. For non-analytic fields one can use the formula
\[
2\alpha T_\zeta \bar{\zeta} = -\frac{1}{4\pi} \int d^2\zeta' \frac{F(\xi')}{(\xi' - \xi)} + h(\xi)
\]
where \( F \) denotes the rhs of (10).

An advantage of formula (10) is that it treats all cases on equal footing. However, in order to obtain a more explicit description of solutions it is useful to consider separately the cases \( \alpha = \text{const} \) and \( \alpha \neq \text{const} \).
Proposition 2.2. If $\alpha \neq \text{const}$ then

- Tensor $T^{ab}$ uniquely corresponds to functions $T^a$, $T^0$ such that
  \[
  T^{ab} = \eta^a T^b + \xi^a \eta^b T^\xi + \xi^b \xi^a T^0
  \]
  (13)
  \[
  T^a = T^{ab} \eta_b, \quad T^0 = \xi_a T^{ab} \xi_b,
  \]
  (14)
  where
  \[
  \xi_b = \frac{\alpha_b}{\nabla \alpha}, \quad \eta^a = \eta^{ab} \xi_b, \quad \nabla \alpha = (\alpha^{bc} \alpha_{bc})^{1/2}.
  \]

- If functions $\alpha$ and $\nabla \alpha$ are independent then equation (9) determines $T^0$ and $T_3^3$ in terms of other data
  \[
  \alpha \eta^0 = (\alpha T^a)_{|a} + \alpha \xi_b \eta^a \xi_{[b} - \lambda \eta^a \omega_{a,}\]
  (16)
  \[
  (\nabla \alpha)T_3^3 = (\alpha T^0 \xi^b + \alpha \xi_a T^a \eta^b)_{|b} - \alpha \eta^b \xi_{[b} \eta^a T^3 - \lambda \xi^a \omega_{a,},
  \]
  (17)
  where
  \[
  \kappa = \eta^{bc} \alpha_{cb} (\nabla \alpha)^{-2}.
  \]

- If functions $\alpha$ and $\nabla \alpha$ are independent then equation (9) is equivalent to (17) and the equation
  \[
  (\alpha \eta_b T^b \eta^a)_{|a} = -(\alpha \xi_b T^b \xi^a)_{|a} - \alpha \xi_b T^b \xi_{[a} + \lambda \eta^a \omega_{a,}
  \]
  (19)
  which yields an integral expression for one of the functions $\eta_b T^b$, $\xi_b T^b$ or $\omega$.\hfill \Box

Proof. Vectors $\xi^a$ and $\eta^a$ form an orthonormal basis on surfaces $\varphi = \text{const}$. If $T^a$ and $T^0$ are defined by (14) then (13) can be easily proved by taking contractions of $T^{ab}$ with $\eta_a$ and $\xi_a$. Similar contractions of equation (9) lead, respectively, to

\[
\eta^a (\alpha T^a)_{|b} = \lambda \eta^a \omega_{a,}
\]
(20)
and

\[
\xi^a (\alpha T^a)_{|b} = (\nabla \alpha)T_3^3 + \lambda \xi^a \omega_{a,}.
\]
(21)
Substituting (13) into (20) and (21) yields (16) and (17). Since $\alpha$, $\nabla, \neq 0$ equation (17) can be treated as a definition of $T_3^3$. If functions $\alpha$ and $\nabla \alpha$ are independent then $\kappa \neq 0$ and (16) defines $T^0$. If they are dependent then $\kappa = 0$ and equation (16), with $T^0$ decomposed in the basis $(\xi^a, \eta^a)$, reduces to equation (19). In order to show that the latter equation can be easily integrated let us choose coordinates $x^a = x$, $y$ such that $\alpha = y$ and

\[
g = y^2 dx^2 + \sigma^2 dy^2 + y^2 (dp + \beta^i dx^i)^2.
\]
(22)
Then $\xi_a T^a = \sigma \gamma T^{\gamma}$, $\eta_a T^a = \rho^2 T^{x^3}$ and equation (19) reads

\[
y (\sigma T_x)_{|x} = -\gamma^{-1} (\sigma \gamma T_x)_{|x} + \sigma \lambda \omega_x.
\]
(23)
Equation (23) is equivalent to an integral expression for $T^{x^3}$, $T^{\gamma}$ or $\omega$. Alternatively one can derive from it an explicit expression for $\sigma$ or $\gamma$ in terms of other variables. \hfill \Box

The most general symmetric data satisfying the momentum constraint are given by (7), (8), (16) and (17). They depend on nine functions of two coordinates. Three of them can be fixed by means of a transformation of coordinates $x^a$ and a shift of $\varphi$. Thus, there are 6 functions which cannot be gauged away. Their number decreases to 5 if the Hamiltonian constraint is imposed.

Now, let us assume that $\alpha = \text{const}$. Without a loss of generality one can set $\alpha = 1$. Function $T_3^3$ is arbitrary since now it is not present in (9). Since $\sigma_{x,a} = 0$ there is no hint how to choose coordinates $x$, $y$ in order to represent solutions of equation (9) in a simple way. The following proposition shows how it can be done. We omit the proof which is straightforward.
Proposition 2.3. If $\alpha = 1$ then in coordinates such that
\[ g = dx^2 + \sigma^2 dy^2 + (d\varphi + \beta_a dx^a)^2 \] (24)
equation (9) is equivalent to
\[ (\sigma T^y_x)_{;y} = -(\sigma^3 T^{xy})_{;x} + \lambda \sigma \omega_y \] (25)
and
\[ (\sigma T^y_x)_{;x} = \sigma_y T^y_x - (\sigma T^{xy})_{;y} + \lambda \sigma \omega_x. \] (26)
Equation (25) defines $T_y^x$ and, consecutively, equation (26) defines $T_x^y$.

Formulas (8), (16) and (17) allow to obtain generic solutions of the momentum constraint in an explicit way. Irrespective of that one can find particularly simple solutions of (9) making an ansatz. For instance, let $\alpha \neq \text{const}$ then
\[ g = \sigma^2(dx^2 + dy^2) + \alpha^2 d\varphi^2. \] (27)
Then equation (9) reads
\[ (\alpha \tilde{T}_{ab})_{;b} = 0, \] (28)
where $\tilde{T}_{a}^{\ b}$ is the traceless part of $T_{a}^{\ b}$. It follows from (28) that there is a function $f$ such that
\[ \alpha T_{ab} = f_{ab} + c g_{ab}, \quad f_{xx} + f_{yy} = 0, \quad c = \text{const}. \] (29)
One can slightly generalize these solutions assuming
\[ \alpha T_{ab} = f_{ab} + h(\sigma)\delta_{ab}, \] (30)
\[ f_{xx} + f_{yy} = 2\sigma^3 h(\sigma), \] (31)
where $h$ is a function of $\sigma$. These data can be also obtained from (10) under the assumption that $\alpha T_{a}^{\ i}$ is a function of $\sigma$. Note that for a nontrivial dependence of the rhs of (31) on $\sigma$ this equation can be considered as a definition of $\sigma$ in terms of the function $f$.

Other simple solutions of (9) with $\alpha \neq \text{const}$ can be obtained if $\alpha T_{a}^{\ i}$ is a function of $\sigma$. Note that for a nontrivial dependence of the rhs of (31) on $\sigma$ this equation can be considered as a definition of $\sigma$ in terms of the function $f$.

Other simple solutions of (9) with $\alpha \neq \text{const}$ can be obtained if $T^{33} = 0$ and $\pm \alpha T_{ab}$ has the form of the energy-momentum tensor of a scalar field $f$ with a potential $V(f)$
\[ \pm \alpha T_{ab} = f_{ab} f_{,b} - \left(\frac{1}{2} f_{,c} f_{,c} + V\right) g_{ab}. \] (32)
In this case equation (9) is satisfied provided that the scalar field equation $f_{,a}^{\ i} = V_{,i}$ is fulfilled. For metric $g_{ab}$ of the form (27) this equation reads
\[ f_{,aa} = \sigma^2 V_{,f}. \] (33)
If $V_{,f} \neq 0$ equation (33) defines $\sigma$ in terms of $f$. If $V_{,f} = 0$ then $f = \Re \, h(\zeta)$, where $h$ is a holomorphic function of $\zeta = x + iy$. In both cases the data can be also obtained from (10) under the assumption that $\alpha T_{a}^{\ i}$ is a function of $f$ and $\alpha T_{\zeta}^{\ \zeta} = \pm (f_{,\zeta})^2$.

Formulas (30)–(31) or (32)–(33) define also particular solutions of the momentum constraint if $\alpha = \text{const}$. In this case the function $T^{33}$ can be arbitrary.

Corollary 1. Simple solutions of equation (9) are given by (27) and (30)–(31) or (32)–(33). If $\alpha \neq \text{const}$ then $T^{33} = 0$.

In order to solve the Hamiltonian constraint by means of the conformal method it is important to have asymptotically flat solutions of the momentum constraint with $H = 0$. Then, by means of a conformal transformation one can obtain either $\alpha = 1$ or $\alpha = \text{const}$, $\kappa \neq 0$ or $\alpha \neq \text{const}, \kappa = 0$. In all these cases condition $H = 0$ imposes an extra constraint on the free functions. The simplest situation is for $\alpha = 1$ since then one can adjust $T^{33}$ to get
\( T^i_0 = 0 \). In order to obtain an asymptotically flat metric another conformal transformation is necessary. For instance, one can multiply metric (24) by \( e^{2\xi} \) and interpret \( e^\xi \) as a distance from the symmetry axis. The new metric is asymptotically flat if \( \sigma e^\xi \to 1 \) for \( e^\xi \to \infty \) or \( y \to \infty \).

Vanishing of the new tensor \( T^{ij}_0 \) at infinity can be assured by an appropriate condition on \( T^{ij} \) and \( \omega \).

If \( \alpha \neq \text{const} \) then condition \( H = 0 \) is equivalent to

\[
T^3_3 + T^0_0 + \eta_0 T^0_i = 0. \tag{34}
\]

If \( \alpha \) and \( \nabla \alpha \) are independent then this equation becomes a second order linear equation for one of the functions \( T^a \). This is more complicated situation than in the case \( \alpha = 1 \), however, now one can expect that the data are asymptotically flat without a necessity of a conformal transformation. If \( \alpha \neq \text{const} \) and \( \kappa = 0 \) then condition (34) can be treated as an equation for \( T^0_0 \). If metric is in the form (22) then this equation leads to an expression for \( (\nabla^2 \sigma^{-1} \eta T^0_0) \), hence \( T^0_0 \) can be determined. Metric (22) may be asymptotically flat without performing a conformal transformation. For instance, it is sufficient that \( \sigma \to 1 \) and \( y \to 1 \) if \( x^2 + y^2 \to \infty \).

Equation (9) is trivially satisfied and \( H = 0 \) if

\[
T^{ab} = T^{33} = 0, \quad \lambda = 0. \tag{35}
\]

In this case the symmetry vector \( \partial_\phi \) is nontwisting and we can change the coordinate \( \varphi \) to obtain \( \beta_\phi = 0 \) in metric (7). Condition (35) is realized on constant time surfaces in the Kerr metric and in other stationary axially symmetric metrics defined by solutions of the Ernst equation [18]. Indeed, these metrics read

\[
g^{(4)} = g_{AB} \, dx^A \, dx^B + g_{ab} \, dx^a \, dx^b, \tag{36}
\]

where \( x^A = t, \varphi \) and the metric components depend only on \( x^B = r, \theta \). On the surface \( t = \text{const} \) metric (36) takes the form (7). Since \( k = k^A \partial_A \) formula (3) yields the exterior curvature tensor of the form \( u_A \, dx^a \, d\varphi \), where \( u_A \) are functions of \( x^B \). Thus, \( K^{ab} = K^{33} = 0 \), hence also \( H = 0 \) and condition (35) follows. If the vacuum Einstein equations are satisfied components \( K^{ab} \) must have form (8). For instance, in the case of the Kerr metric one obtains

\[
g = \rho^2 \Delta^{-1} \, dr^2 + \rho^2 \, d\theta^2 + \rho^2 \Sigma^2 \sin^2 \theta \, d\varphi^2 \tag{37}
\]

and (8) with

\[
\omega = 4aM \rho^{-2} \bigl(2(r^2 + a^2) + (r^2 - a^2) \sin^2 \theta \bigr) \cos \theta, \tag{38}
\]

where

\[
\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2, \quad \Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta. \tag{39}
\]

Metric (37) tends to the flat metric if \( r \to \infty \) and function \( \omega \) satisfies

\[
\omega \to 4aM(2 + \sin^2 \theta) \cos \theta \quad \text{if} \quad r \to \infty. \tag{40}
\]

Condition (40) assures fast vanishing of tensor \( T^{ij}_0 \) when \( r \) increases. It can be used to define the Kerr-like asymptotical behavior of the initial data.

If \( \partial_\theta \) is interpreted as the axial symmetry then initial data should be regular on the symmetry axis. Hence, in spherical like coordinates \( r, \theta, \varphi \) metric should behave like

\[
g \simeq g_{rr} \, dr^2 + g_{\theta\theta}(d\theta^2 + \sin^2 \varphi \, d\varphi^2) \quad \text{if} \quad \sin \theta \to 0, \tag{41}
\]

where \( g_{rr} \) and \( g_{\theta\theta} \) are positive functions. A more general expression is allowed in the case of the exterior curvature. In particular, \( \omega \) should satisfy

\[
\omega_r \sim \sin^4 \theta, \quad \omega_\varphi \sim \sin^3 \theta \quad \text{if} \quad \sin \theta \to 0. \tag{42}
\]

Obviously, conditions (41) and (42) are satisfied by the Kerr data.
In general, symmetric data undergoing condition (35) are not related to stationary axially symmetric metrics of the form (36). Note that these metrics are fully determined by the complex Ernst potential which has to satisfy the Ernst equation. Hence, the corresponding initial data on \( t = \text{const} \) is also determined by this potential. Generic axially symmetric data satisfying (35) do not share this property.

3. Nonsymmetric solutions of the momentum constraint

Tensor \( T^{ij} \) can be written in the form
\[
T^{ij} = \rho u^i v^j + p g^{ij},
\]
where vectors \( u = u^i \partial_i \) and \( v = v^i \partial_i \) are real or complex conjugated, \( u = \bar{v} \). To this end we define \( p \) as a real solution of the equation
\[
|\tilde{\gamma}| = \left( \det \gamma_{AB} \right)^{\frac{1}{2}},
\]
and
\[
\gamma_{AC} \beta^C = \beta, C \sigma^C_A,
\]
where \( x^A = x, z \) and \( \sigma^A_B = \text{diag}(1, -1) \).

This notation helps to describe the following class of solutions of the momentum constraint.

Definition 3.1. Given functions \( \beta \) and \( \gamma \) such that matrix \( \gamma_{AB} \) is nondegenerate we define \( \gamma_{AB}, |\tilde{\gamma}| \) and \( \beta^A \) by
\[
\gamma_{AB} = \gamma_{CD} \sigma^C_A \sigma^D_B, \quad |\tilde{\gamma}| = (\det \gamma_{AB})^{\frac{1}{2}}, \quad \gamma_{AC} \beta^C = \beta, C \sigma^C_A,
\]
where \( x^A = x, z \) and \( \sigma^A_B = \text{diag}(1, -1) \).

Under assumption (45) or (46) solutions of the momentum constraint can be expressed in terms of free functions and integrals over them. In this section we will present only a family of solutions which can be written in a relatively simple way. In general they have no symmetry, however, they contain a subclass of data obtained in section 2.

In order to describe solutions satisfying (45) let us first introduce some notation.

Proposition 3.2. The momentum constraint is satisfied by
\[
g = (\alpha \rho |\tilde{\gamma}|)^{-\frac{1}{2}} [\alpha^2 (x^2 + \gamma_{AB} (dx^A + \beta^A dy) (dx^B + \beta^B dy))] \]
\[
T^{ij} = \rho \delta^{i}_{i} \delta^{j}_{j} + p g^{ij}
\]
provided that $\gamma_{AB}$ has the Euclidean signature and functions $\gamma$, $\beta$, $p$, $\alpha > 0$, $\rho > 0$ satisfy either

$$p = p(g_{\alpha\beta}), \quad \rho = 2 \frac{dp}{dg_{\alpha\beta}}$$

or

$$p = \text{const,} \quad \gamma_{\alpha\beta} = c(\alpha \rho |\tilde{\gamma}|)^2, \quad c = \text{const.}$$

All coordinates and functions are either real or imaginary. Three of the functions are arbitrary up to positivity conditions.

**Proof.** In the case (45) the momentum constraint reads

$$(\rho |g|g_{\alpha\beta})_{,\alpha} + (\rho |g|g_{\alpha\beta})_{,\beta} = |g|(\rho g_{\alpha\beta,\alpha} - 2p_{,\alpha}).$$

In order to simplify (52) we assume that the rhs of (52) vanishes

$$2p_{,\alpha} = \rho g_{\alpha\beta,\alpha}.$$  

In this case it follows from (52) that there exist functions $\gamma_i$ such that

$$\rho |g|g = \gamma_{\alpha\beta}, \quad \rho |g|g_{\alpha\beta} = -\gamma_{\alpha\beta}.$$  

These equations are compatible iff $\gamma_{\alpha\beta} = -\gamma_{\alpha\beta}$, hence there exists a function $\gamma$ such that $\gamma_\alpha = \gamma_\beta$, and $\gamma_\gamma = -\gamma_{\alpha\beta}$. Substituting these expressions into (54) and denoting $\gamma$ by $\beta$, yields

$$\rho |g|g = \alpha^2 dy^2 + \gamma_{AB}(dx^A + \beta^A dy)(dx^B + \beta^B dy),$$

where $\gamma_{AB}$ and $\beta^A$ are given by definition 2.2. Function $\alpha$ can be arbitrary positive since $g_{\gamma\gamma}$ is not present in (52). From (55) one can calculate $g_{ij}$ and obtain metric in the form (48). Still equation (53) has to be satisfied. If $g_{\alpha\beta} \neq \text{const}$ it yields (50). If $g_{\alpha\beta} = \text{const}$ one obtains (51).

In the case (50) functions $\alpha$, $\beta$ and $\gamma$ are arbitrary up to $\alpha > 0$ and positivity of $\gamma_{AB}$. It is also true in the case (51) with $c \neq 0$ since then $\rho$ can be expressed in terms of other functions. In the case (51) with $c = 0$ function $\gamma$ splits into the sum of functions of 2 variables but then $\alpha$, $\beta$ and $\rho$ are arbitrary. Thus, in each case there are 3 functions of 3 coordinates which are arbitrary up to conditions of positivity.

In order to prepare ground for studies of the Hamiltonian constraint let us identify solutions from proposition 3.2 with $H = 0$. If $H = 0$ and $\gamma_{\alpha\beta} \neq 0$ we can apply a conformal transformation to obtain

$$|\tilde{\gamma}|^2 \alpha^2 = |\gamma_{\alpha\beta}|^2, \quad \rho = 1, \quad p = \frac{1}{2} \text{sgn}(|\gamma_{\alpha\beta}|).$$

Metric and $T^{ij}$ now read

$$g = \frac{\gamma_{\alpha\beta}}{|\tilde{\gamma}|^2} dy^2 + |\gamma_{\alpha\beta}|^{-1} \gamma_{AB}(dx^A + \beta^A dy)(dx^B + \beta^B dy),$$

$$T^{ij} = \delta^{ij}_x \delta^y_z + \frac{1}{2} \text{sgn}(|\gamma_{\alpha\beta}|) g^{ij}.$$  

If $\gamma_{\alpha\beta} = p = 0$ then by means of a conformal transformation one obtains

$$T^{ij} = (\alpha f h)^{-1} \delta^{ij}_x \delta^y_z$$

and

$$g = \alpha^2 dy^2 + f^2(x, y)(dx + \beta_x dy)^2 + h^2(y, z)(dz - \beta_z dy)^2.$$  

Here $f$ and $h$ are functions of two coordinates as indicated. A change of coordinates $x$ and $z$ allows to transform $f$ and $h$ to any pair of nonzero functions of this type, e.g. to constant functions. However, it is convenient to keep them arbitrary until conditions of asymptotical flatness are imposed.
Corollary 2. The momentum constraint with $H = 0$ is satisfied by data given by (58)–(59) or (60)–(61), where $\gamma_{AB}$ and $\beta^A$ are related to functions $\beta$ and $\gamma$ according to definition 2.2. Functions $f$ and $h$ can be transformed to $f = h = 1$.

Data (60)–(61) include (up to conformal transformations) a subclass of invariant data satisfying condition (35). For them $z = \phi$ and $T^{ij} \sim \delta^{(i}(\delta^{j)}_z$ in coordinates $x'$ such that $\omega = F(y)$. Metric (7) is conformally equivalent to metric of the form (61) with $\alpha_z = \beta_z = h_z = 0$.

Thanks to this property one can deduce conditions which assure asymptotical flatness of data (60)–(61)

$$f^2 = \frac{\sin y}{x^2}, \quad h^2 = \sin^3 y$$

$$\alpha^2 \to \sin y, \quad \beta_x \to 0, \quad \beta_z \to 0 \quad \text{if} \quad x \to \infty.$$  

Under condition (62) the conformal transformation (4) with $\psi^2 = x\alpha^{-1}$ yields

$$T^{ij} = \frac{\tilde{\alpha}^2}{x^2} \delta^{(i}(\delta^{j)}_z$$

and

$$g = x^2 dy^2 + \tilde{\alpha}(dx + \beta_x dy)^2 + x^2 \tilde{\alpha} \sin^2 y(dz - \beta_z dy)^2,$$

where $\tilde{\alpha} = \alpha^2 / \sin y$. Conditions (63) can be replaced by

$$\tilde{\alpha} \to 1, \quad \beta \to 0 \quad \text{if} \quad x \to \infty.$$  

It is clear from the above assumptions that, asymptotically, $x, y, z$ become spherical coordinates $r, \theta, \phi$ of the flat metric. Data given by (64) and (65) generalize the Kerr data. They contain two free functions ($\tilde{\alpha}$ and $\beta$) of three coordinates. In general, these data have no symmetries. They are asymptotically flat under conditions (66) provided that derivatives of $\tilde{\alpha}$ and $\beta$ vanish sufficiently fast.

Now, we turn to initial data obeying condition (46).

Proposition 3.3. The momentum constraint is satisfied by the data given by

$$\pm T^{ij} = \delta^{(i}(\delta^{j)} + \left( c + \frac{1}{2} \beta |\tilde{g}|^{-\frac{1}{2}} \right) \tilde{g}^{ij}$$

$$g = \beta |\tilde{g}|^{-\frac{1}{2}} (dz + \beta_x dx^2) + g_{ab} dx^a dx^b$$

or

$$\pm T^{ij} = \beta |\tilde{g}|^{-\frac{1}{2}} \delta^{(i}(\delta^{j)} + cg^{ij}$$

$$g = (dz + \beta_x dx^2) + g_{ab} dx^a dx^b,$$

where $\beta_0, z = \beta, z = 0, \beta = 0, c = \text{const}$, metric $g_{ab}$ has the Euclidean signature and $|\tilde{g}|$ denotes $\det g_{ab}$. Function $\beta$ can be transformed to 1.

Proof. In case (46) the momentum constraint reads

$$(p|g|g_{zz},) = \frac{1}{2} \rho |g|g_{zz,l} - |g|p_{,l}$$

where $|g| = (\det g_{ij})^{1/2}$. It can be solved in full generality in terms of integrals. In order to obtain simpler solutions we first assume

$$p = \text{const} + \frac{1}{2} \rho g_{zz}, \quad \rho = \pm 1.$$  


In this case it follows from (71) that
\[ g = |g|^{-1} \beta^3 (dz + \beta_a dx^a)^2 + g_{ab} dx^a dx^b, \quad \beta_{a,z} = \beta_{z} = 0. \] (73)

Taking determinant of this metric yields
\[ |g| = \beta^2 |\tilde{g}|^{-\frac{3}{2}}. \] (74)

From (73) and (74) one obtains metric in the form (68). Equation (67) follows from (72) and (68).

If we assume \( p = \text{const} \) it is convenient to work with the unit vector \( v^i = \partial_z \). Then \( g_{zz} = 1 \) and from (71) one obtains \( g_{a,z} = 0 \) and \( \rho \sqrt{|g|} = \beta(x^a) \). Hence, expressions (69) and (70) follow.

In solutions described by this proposition components of the metric \( g_{ab} \) cannot be in general gauged away since coordinate transformations are strongly restricted by the form of \( T^{ij} \). One can accommodate \( \beta \) in \( \det g_{ab} \) by a change of coordinates \( x^a \). We keep arbitrary \( \beta \) since then the asymptotical flatness conditions are simpler. □

Data with \( H = 0 \) are present in both classes of solutions presented in proposition 3.3. They are jointly given by

**Corollary 3.** The data
\[ T^{ij} = c \left( g^{ij}_{1} \tilde{g}^{ij}_{1} \right), \] (75)

\[ g = (dz + \beta_a dx^a)^2 + g_{ab} dx^a dx^b, \] (76)

where \( c = \text{const} \) and \( \beta_{a,z} = (\det g_{ab})_z = 0 \), satisfy the momentum constraint and \( H = 0 \).

Above solutions can be asymptotically flat. For instance, let \( z = \ln r \), \( \beta_a = 0 \) and \( x^a = \theta, \phi \) be spherical angles. If
\[ \det g_{ab} = \sin^2 \theta, \] (77)

and
\[ g_{ab} dx^a dx^b \to d\theta^2 + \sin^2 \theta d\phi^2 \] if \( r \to \infty \) (78)

then conformal transformation (4) with \( \psi^2 = r \) yields asymptotically flat metric.

**4. Hamiltonian constraint**

In this section and in the next one we will apply the Lichnerowicz–Choquet-Bruhat–York conformal method to solutions of the momentum constraint from sections 2 and 3. Before we do it let us shortly discuss how to exploit free functions in these solutions in order to solve the Hamiltonian constraint if \( H \neq 0 \). For instance, this can be easily done for solutions with \( g_{\phi\phi} = 1 \) described by proposition 2.3. Then the function \( T_{\phi}^{\phi} \) is not involved in the momentum constraint and it can be defined by the following equation equivalent to (2)

\[ (T_{\psi}^{\psi} - T_{\phi}^{\phi})^2 = 2R + 3(T_{\psi}^{\psi})^2 - 2T_{ab}T^{ab} - 4T^{op}T_{op}, \] (79)

provided that the rhs of (79) is nonnegative. Unfortunately, since \( H \neq 0 \) and conformal transformations are not allowed, initial metric cannot be asymptotically flat in this case. Another simple example concerns solutions described in the last point of proposition 2.2. Now, the Hamiltonian constraint can be considered as an ordinary differential equation for the function \( T^0 \). In coordinates (22) it reads
\[ \gamma^{-1}(\gamma T^0)_{\gamma} = F \pm 2\gamma^{-1}(G + T^0 \eta_{ab} T^{ab})^\frac{1}{2}, \] (80)
where $F$ and $G$ are expressions independent of $T^0$

$$F = (y^{-1}y_y + y^{-1})n_aT^a - (\beta y)^{-1}(\beta^2l_aT^a)_x$$

$$G = \frac{1}{2}R - (l_aT^a)^2 - T_{\alpha\beta}T^{\alpha\beta}.$$  (82)

Unfortunately, we are not able to prove existence of global solutions of equation (80). If they exist one can obtain asymptotically flat data in this case ($\beta$ and $\gamma$ should tend to 1 at infinity).

From now on we will assume that $H = 0$ and we focus on asymptotically flat data. In this case the conformal method is effective if one can show that the Lichnerowicz equation (5) has a solution $\psi$ which is positive everywhere and tends to 1 at infinity. Our considerations are mainly based on the existence theorems of Maxwell [9] (note that the exterior curvature in [9] has the opposite sign with respect to that defined by (3)). They are applicable if components of seed data and the exterior curvature belong to appropriate weighted Sobolev spaces $W^{k,p}_g$. Metric doesn’t have to be conformally flat. Existence and uniqueness of $\psi$ depends crucially on the positivity of the Yamabe type invariant

$$\lambda_\delta := \inf_{f \in C_0^\infty}\frac{\int_S (4|\nabla f|^2 + Rf^2)dv_g}{\|f\|^2_{L^2}} > 0,$$  (83)

where $dv_g$ (also present in the norm of $f$) is the volume element corresponding to seed metric $g$.

In order to formulate the results of Maxwell we define asymptotically flat initial data in the following way.

**Definition 4.1.** Let an initial surface $S$ be a union of a compact set and so-called asymptotically flat ends, which are diffeomorphic to a completion $E$ of a ball in $R^3$. We say that data $(g, K)$ are asymptotically flat of class $W^{k,p}_g$ if the following conditions are satisfied

$$g \in W^{k,p}_{loc}(S), \quad (g_{ij} - \delta_{ij}) \in W^{k,p}_g(E), \quad K_{ij} \in W^{k-1,p}_{g-1}(S).$$  (84)

where $k \geq 2$, $kp > 3$, $\delta < 0$ and indices $i, j$ correspond to the Cartesian coordinates of $E$.

For instance, conditions (84) are satisfied, with $\delta > -\epsilon$ and arbitrary $p$, if $g$ and $K$ are fields of class $C^\omega$ and $C^{\omega-1}$, respectively, and they satisfy the following conditions which are often used in relativity

$$g_{ij} = \delta_{ij} + 0_k(r^{-\epsilon}), \quad K_{ij} = 0_{k-1}(r^{-1-\epsilon}).$$  (85)

Here $\epsilon$ is a positive constant, $r$ is the radial distance in $R^3$ and we write $f = o_k(r^{-\epsilon})$ if derivatives of $f$ of order $n \leq k$ fall off as $r^{-n-\epsilon}$ when $r \to \infty$. Note that conditions (84) with $\delta < -1/2$ or (85) with $\epsilon > 1/2$ are sufficient to define the ADM energy–momentum [19].

The existence theorem of Maxwell can be formulated in the following form

**Theorem 4.2 (Maxwell).** Let $(S, g)$ be a complete Riemannian manifold without boundary and let $(g, K)$ be a traceless $(H = 0)$ solution of the momentum constraint which is asymptotically flat of class $W^{k,p}_g$. Then there exists a solution of the Lichnerowicz equation (5) such that $\psi > 0$ and $(\psi - 1) \in W^{k,p}_g$ if and only if $\lambda_\delta > 0$. If it exists it is unique and the conformally transformed data (4) satisfy all constraint equations and are asymptotically flat of class $W^{k,p}_g$.

A direct verification of condition (83) is rather difficult. We will show that it can be replaced by a simpler condition if the Euclidean type Sobolev inequality is satisfied on $S$. Let us decompose the Ricci scalar of $g$ into a positive and negative part

$$R = R_+ + R_-,$$  (86)

where $R_- = 0$ at points where $R \geq 0$ and $R_+ = 0$ if $R \leq 0$. 

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Proposition 4.3. If the Sobolev inequality
\[ \|f\|_{L^6} \leq A \|
abla f\|_{L^2}, \quad A = \text{const} > 0 \] (87)
is satisfied and
\[ \|R - \|_{L^{3/2}} < \frac{8}{A^2} \] (88)
then \( \lambda_g > 0 \).

Proof. From the Hölder inequality one obtains
\[ \int_S |R - |^2 dv_g \leq \|R - \|_{L^{3/2}} \|f\|_{L^6}^2 . \] (89)
Since \( R \geq R - \) inequality (89) leads to
\[ \int_S R f^2 dv_g \geq -\|R - \|_{L^{3/2}} \|f\|_{L^6}^2 . \] (90)
From (90) and (87) it follows that
\[ \int_S (8|\nabla f|^2 + R f^2) dv_g \geq \left( \frac{8}{A^2} - \|R - \|_{L^{3/2}} \right) \|f\|_{L^6}^2 . \] (91)
Hence,
\[ \lambda_g \geq \left( \frac{8}{A^2} - \|R - \|_{L^{3/2}} \right) \] (92)
and \( \lambda_g \) is positive if (88) is satisfied. \( \square \)

In the three-dimensional flat Euclidean space the Sobolev inequality (87) is satisfied and \( R = 0 \), hence condition (88) is void. Similar situation occurs if \((S, g)\) is complete and asymptotically flat and \( R \geq 0 \) [20]. Condition \( R \geq 0 \) is necessarily satisfied if \( g \) is induced by a solution of the Einstein equations and \( H = 0 \) on the initial surface. This property follows from the Hamiltonian constraint which is one of the Einstein equations. For instance, let \( g \) be the metric induced by the Kerr solution on the surface \( t = \text{const} \). If \( g \) together with a traceless tensor \( K \), different from that for the Kerr data, satisfy the momentum constraint one can apply theorem 4.2 without bothering about condition (83).

In what follows we show how to generate axially symmetric initial data given a seed metric \( g \) on \( S \) such that the Sobolev inequality (87) is satisfied.

Theorem 4.4. Consider axially symmetric metric
\[ g^{(u)} = \alpha^2 d\phi^2 + e^{2u} g_{ab} dx^a dx^b \] (93)
related to a complete and asymptotically flat metric \( g = g^{(0)} \) of class \( W^{k,p}_k \). Assume that \((S, g)\) admits the Sobolev inequality (87) and that \( u \) satisfies
\[ \|(R - 2\tilde{\Delta} u) - \|_{L^{3/2}} < \frac{8}{A} . \] (94)
where the Ricci scalar \( R \) and the norm refer to \( g \) and \( \tilde{\Delta} \) is the covariant Laplacian of metric \( g_{ab} dx^a dx^b \).

If axially symmetric data \((g^{(u)}, K)\) with \( H = 0 \) satisfy the momentum constraint and are asymptotically flat of class \( W^{k,p}_k \) then there exist conformal data of the same class which satisfy all the constraint equations.
Proof. Equation (5) corresponding to (93) takes the form
\[ \Delta^{(a)} \psi = \frac{1}{8} e^{-2u} (R - 2\Delta u) \psi - \frac{1}{8} K^{ij} K_{ij} \psi^{-7}. \] (95)
with the covariant Laplacian \( \Delta^{(a)} \) defined by \( g^{(a)} \). If \( \psi \) doesn’t depend on \( \varphi \) equation (95) is equivalent to
\[ \Delta \psi = \frac{1}{8} (R - 2\Delta u) \psi - \frac{1}{8} e^{2u} K^{ij} K_{ij} \psi^{-7}. \] (96)
where \( \Delta \) is the Laplacian corresponding to the seed metric \( g \). Let us consider equation (96) without assuming the axial symmetry of \( \psi \). Inspection of the proof of theorem 1 in [9] shows that this theorem is still valid if \( R \) is replaced by another function from the same Sobolev space. Equation (96) has a unique solution \( \psi \) since (94) implies
\[ \lambda^{(a)} := \inf_{f \in C^\infty(S)} \frac{\int_S (8|\nabla f|^2 + (R - 2\Delta u) f^2) \, dv_g}{\|f\|^2_{L^6}} > 0, \] (97)
where the measure in the integral and the norm correspond to the metric \( g \). The function \( \psi \) cannot depend \( \varphi \) since otherwise \( \psi \) with shifted coordinate \( \varphi \) would be another solution of equation (96). It follows that \( \psi \) satisfies also equation (95). In order to prove that there is no other solutions of this equation let us make the conformal transformation (4). Then the Hamiltonian constraint (2) is satisfied by new (primed) fields, hence \( R' \geq 0 \). If \( \psi \) was not the unique solution of (95) there must be a positive function \( \xi \) (a ratio of two solutions of (95)) which is not identically 1 but tends to 1 at infinity and satisfies
\[ \Delta' \xi = \frac{1}{8} R' \left( \xi - \frac{1}{\xi^7} \right). \] (98)
If we multiply (98) by \( (\xi - \xi^{-7}) \) and integrate it over \( S \) we obtain
\[ \int_S \left( 1 + \frac{7}{\xi^8} \right) (\nabla \xi)^2 \, dv_g + \frac{1}{8} \int_S R' \left( \xi - \frac{1}{\xi^7} \right)^2 \, dv_g = 0. \] (99)
Since both integrated expressions are nonnegative they have to vanish. Taking into account the asymptotic behavior of \( \xi \) we obtain \( \xi = 1 \) everywhere on \( S \). Thus, \( \psi \) defined originally as a solution of (96) is also a unique solution of equation (95). □

Note that condition (94) is satisfied if
\[ \| (\Delta u)_+ \|_{L^{2/3}} < \frac{4}{3A}. \] (100)
If the lhs of (100) is finite one can achieve (100) via transformation \( u \to Cu \) with a suitably small value of constant \( C \). If \( u = 0 \) condition (100) is trivially satisfied and the only problem is to solve the momentum constraint with respect to \( K \) (see proposition 2.1). For instance, one can take the Kerr initial metric (37) and \( K \) given by (8) and (35) with any \( \omega \) satisfying asymptotic condition (40).

5. Hamiltonian constraint and horizons

From the point of view of the black hole theory it is important that initial data admit a surface \( S_0 \) which can be considered as a black hole horizon. Conditions on \( S_0 \) are usually formulated in terms of functions \( \theta_{\pm} \) defined on \( S_0 \) by
\[ \theta_{\pm} = H - K(n, n) \pm 2h, \quad 2h = n'_{ij}, \] (101)
where \( n \) is the unit normal of \( S_0 \) oriented outside \( S_0 \). From the four-dimensional point of view, \( \theta_{\pm} \) are expansions of null geodesics emerging from \( S_0 \) in the direction \( k \pm n \), respectively. In
order to interpret $S_0$ as a black hole horizon condition $\theta_+ = 0$ is commonly assumed. Then $S_0$ is called marginally outer trapped surface (MOTS). Unfortunately, in general, theorem 4.2 does not allow to control existence of MOTS or other trapped surfaces for initial data, even if data $(g, K)$ admit such a surface.

There is an exception to this rule if $g$ has a $Z_2$ symmetry preserving $S_0$ and if equation

$$K_{ij} n^i n^j = 0 \quad (102)$$

is satisfied on $S_0$. Then, from the uniqueness assured by theorem 4.2 solution $\psi$ is also $Z_2$ symmetric, hence its normal derivative vanishes on $S_0$. All components of the exterior curvature of $S_0$ embedded in $S$ vanish for both metrics $g$ and $g'$. Hence, $\theta_+ = \theta_- = 0$ for the ultimate initial data.

An example of this type is again provided by the Kerr metric. In this case the surface $t = \text{const}$ crosses the bifurcation surface (the Einstein–Rosen bridge) and on its other side the Boyer–Lindquist radial coordinate $r$ grows again up to infinity. Thus, $r$ is not a global coordinate on $S$. A better coordinate $\tilde{r}$ is related to $r$ by

$$r = M + \sqrt{M^2 - a^2 \cosh \tilde{r}}, \quad \tilde{r} \in [-\infty, \infty]. \quad (103)$$

Metric (37) is symmetric with respect to $\tilde{r} \to -\tilde{r}$ and the exterior curvature of the surface $\tilde{r} = 0$ vanishes. One can modify this metric according to theorem 4.4 with $u$ and $\omega$ being even functions of $\tilde{r}$. The corresponding conformal factor $\psi$ will be also $Z_2$ symmetric and the surface $\tilde{r} = 0$ will have vanishing null expansions $\theta_\pm$ with respect to the conformally transformed data.

A construction of initial data with MOTS is given in [9]. In this approach $S$ has an inner boundary $S_0$ and equation (5) is supplemented by the boundary condition

$$n^i \partial_i \psi + \frac{1}{2} h - \frac{1}{4} K(n, n) \psi^{-3} = 0 \quad (104)$$

(we recall that we use fields $K_{ij}$ and $h$ which differ by sign from fields $K_{ij}$ and $h$ in [9]). Condition (104) guarantees that $\theta_+ = 0$ upon the conformal transformation (4). Unfortunately, the key existence theorem in [9] (theorem 1) refers to properties of conformally equivalent data satisfying $R = 0$. Since these data are not known explicitly, it is highly nontrivial to satisfy assumptions of this theorem unless condition (102) is satisfied. In the latter case applying theorem 1 from [9] to the data obtained by means of corollary 1 from [9] yields

**Theorem 5.1** (Maxwell). Let $(S, g)$ be a Riemannian manifold with an inner boundary $S_0$ and $(g, K)$ be a traceless solution of the momentum constraint which is asymptotically flat of class $W^{k,p}$. If $K(n, n) = 0$ and

$$\lambda_g := \inf_{f \in C^\infty} \frac{\int_S (8|\nabla f|^2 + Rf^2) du - \int_{S_0} hf^2 d\sigma}{\|f\|_{L^2}^2} > 0 \quad (105)$$

then equation (5) with the boundary condition

$$n^i \partial_i \psi + \frac{1}{2} h \psi = 0 \quad (106)$$

possesses a solution $\psi > 0$. The conformally transformed data (4) satisfy all the constraint equations and are asymptotically flat of class $W^{k,p}$. The boundary $S_0$ is a marginally outer trapped surface with $\theta_+ = \theta_- = 0$.

As in preceding section we are going to replace condition (105) by a simpler one under the assumption that the Sobolev inequality (87) is satisfied. For instance, it follows from proposition 4.3 that $\lambda_g > 0$ if $h \leq 0$ and (88) is satisfied. In order to find less restrictive
conditions let us introduce a compact Riemannian submanifold \( S' \subset S \) with a boundary containing \( S_0 \). The following identities are satisfied on \( S' \) \cite{21, 22}

\[
||f||_{L^2(S')} \leq B||f||_{L^2(S)} \quad (107)
\]

\[
||f||_{L^2(S')} \leq C||\nabla f||_{L^2(S')} + D||f||_{L^2(S)},
\]

where \( B, C, D \) are positive constants which depend on a choice of \( S' \).

**Proposition 5.2.** Let the flat Sobolev inequality (87) be satisfied on \( S \). Then there is a constant \( E \) such that inequality

\[
A^2||R_-||_{L^2(S)} + E^2||h_+||_{L^2(S_0)} \leq 8
\]

implies (105).

**Proof.** Since \( ||f||_{L^2(S_0)} \leq ||f||_{L^2(S)} \) and \( ||\cdot||_{L^2(S')} \leq ||\cdot||_{L^2(S)} \) it follows from (107) and (108) that

\[
||f||_{L^2(S_0)} \leq C||\nabla f||_{L^2(S)} + BD||f||_{L^2(S)}
\]

and consequently

\[
||f||_{L^2(S_0)}^2 \leq 2C^2||\nabla f||_{L^2(S)}^2 + 2B^2D^2||f||_{L^2(S)}^2.
\]

Let us decompose \( R \) and \( h \) into positive and negative parts following (86). From \( h \leq h_+ \) and the Hölder inequality on \( S_0 \) one obtains

\[
- \int_{S_0} h^2 d\nu \geq -||h_+||_{L^2(S_0)} ||f||^2_{L^2(S)}.
\]

It follows from (111), (112) and (90) that

\[
\int_S (8||f||^2 + Rf^2) d\nu \geq \int_{S_0} h^2 d\nu - (8 - 2C^2||h_+||_{L^2(S_0)}) ||\nabla f||^2_{L^2(S)}
\]

\[
- (||R_-||_{L^2(S)} + 2B^2D^2||h_+||_{L^2(S_0)}) ||f||^2_{L^2(S)}.
\]

Condition (105) is satisfied if the rhs of (113) is greater or equal to \( \lambda'||f||^2_{L^2(S)} \), where \( \lambda' \) is a positive constant. The latter condition takes the form

\[
(8 - 2C^2||h_+||_{L^2(S_0)}) ||\nabla f||^2_{L^2(S)} \geq (\lambda' + ||R_-||_{L^2(S)} + 2B^2D^2||h_+||_{L^2(S_0)}) ||f||^2_{L^2(S)}.
\]

The Sobolev inequality (87) implies (114) with some \( \lambda' > 0 \) provided that the norms of \( R_- \) and \( h_+ \) satisfy (109) with

\[
E^2 = 2C^2 + 2A^2B^2D^2.
\]

□

Now, we will apply theorem 5.1 and proposition 5.2 to axially symmetric data from section 2. If condition (35) is satisfied and \( S_0 \) is axially symmetric then the normal vector \( n \) has no axial component and \( K(n, n) = 0 \). Moreover one can choose a conformal representative of metric such that equation (5) takes the form characteristic for the flat metric. This property facilitates a possible way to prove the Sobolev inequality (87) and simplifies condition (109).

**Theorem 5.3.** Let \( S \) be a connected unbounded subset of the Euclidean space \( \mathbb{R}^3 \) with an axially symmetric compact boundary \( S_0 \) such that the Sobolev inequality (87) is satisfied. Let \( u \) and \( \omega \) be functions of \( r \) and \( \theta \) and

\[
g^{(u)} = r^2 \sin^2 \theta \, d\psi^2 + e^{2u}(d r^2 + r^2 \, d\theta^2)
\]

(116)
be asymptotically flat data of class $W^{k,p}_g$. Let
\[4A^2\|\langle \hat{\Delta} u \rangle_+\|_{L^2(S)} + E^2\|h^{(0)}\|_{L^2(S_0)} \leq 8\]
where $\hat{\Delta}$ is the Laplacian of the metric $dr^2 + r^2 d\theta^2$, the norms correspond to flat metric, $h^{(0)}$ is the mean curvature of $S_0$ with respect to flat metric and $E$ is given by (115).

Then there exist conformally equivalent data which satisfy all the constraint equations and are asymptotically flat of class $W^{k,p}_g$. The boundary $S_0$ is MOTS with $\theta_+ = \theta_- = 0$.

**Proof.** It is easy to show that equation (5) for data (116)–(117) is equivalent to
\[\Delta \psi = -\frac{1}{2} (\hat{\Delta} u) \psi - \frac{1}{8} r^{-4} \sin^{-4} \theta (\omega^a \omega_b) \psi^{-7}\]
with the flat Laplacian $\Delta$. Let $\tilde{n}'$ be the normal vector of $S_0$ with unit length with respect to flat metric $g^{(0)}$. Boundary condition (106) takes the form
\[\tilde{n}' \partial_\nu \psi + \frac{1}{2} h^{(0)} \psi = 0, \quad 2h^{(0)} = \tilde{n}'_{ii}\]
where $\tilde{n}'_{ii}$ is defined by means of $g^{(0)}$. From theorem 5.1 equation (119) and condition (120) have a unique solution $\psi$ if
\[\lambda^g := \inf_{f \in C^\infty(S)} \left( \int_S (8|\nabla f|^2 - 2 f^2 \hat{\Delta} u) dv - \int_{S_0} h^{(0)} f^2 ds \right) \|f\|^2_{L^2_g} > 0\]
where the integrals and the norm are defined by means of $g^{(0)}$. According to proposition 5.2, assumption (118) implies (121), so $\psi$ exists. Following the proof of theorem 4.4 one can show that $\psi$ doesn’t depend on $\varphi$ and it is also a unique solution of equation (5) with condition (106), both corresponding to metric (116) (note that equation (99) is still true since the normal derivative of $\psi'$ on $S_0$ has to vanish).

Locally every metric (7) can be conformally transformed to the form (116). In the case of the initial Kerr metric one can simply substitute (103) and $\tilde{r} = \ln r$ into (37). Then a conformal transformation leads to (116) with $r'$ instead of $r$. The Kerr horizon corresponds to $r' = 1$. In this case the initial surface $S$ is given by $R^3$ with a removed ball. We prove in appendix that for such $S$ the Sobolev inequality (87) is satisfied. Thus, theorem 5.3 gives tools to generalize the Kerr initial data.

A drawback of the approach with an internal boundary is that, in general, we cannot control prolongation of initial data throughout the boundary. Even if the metric $g$ before the conformal transformation can be continued to another asymptotically flat region it is not known whether the conformal factor $\psi$ can be. This is because theorem 5.1 can be applied to the exterior and interior regions independently but it says nothing about values of $\psi$ on the boundary surface.

If $S_0$ is the two-dimensional sphere a particular continuation is provided by the Bowen–York puncture method [3]. Let us consider metric (116) with boundary at $r = 1$. If
\[u_r = 0, \quad \psi_r + \frac{1}{2} \psi = 0\]
at $r = 1$, then this boundary has the vanishing exterior curvature tensor corresponding to the final metric
\[g' = \psi^4 g\]
(123)
One can continue $g'$ through the surface $r = 1$ putting $g'(1/r) = g'(r)$. An equivalent method is first to make the coordinate transformation $r = \exp \tilde{r}$ and then to assume that $g'(\tilde{r}) = g'(r)$. We can complete so defined metric by the exterior curvature (117) with $\omega$ being an even function of $\tilde{r}$. In this way one obtains initial data with two asymptotically flat ends. These data are also available by use of theorem 4.4 with $g$ given by e.g. the initial Schwarzschild metric, but then the Sobolev inequality (87) is more difficult to prove.
6. Summary

We have been studying solutions of the vacuum constraints in general relativity such that, in general, the initial metric $g$ is not conformally flat and the mean exterior curvature is not constant. Section 2 concerns with the momentum constraint for data with a continuous symmetry. If the length $\alpha$ of the symmetry vector and function $\alpha, \beta, \alpha, \beta$ are independent then all solutions are given explicitly (proposition 2.2). In other cases solutions are given partly explicitly and partly in terms of integrals (propositions 2.1, 2.2 and 2.3). Several simple families of solutions are presented. For instance, condition (35) defines a class of solutions which contains data for stationary axially symmetric metrics and also for nonstationary solutions. Among them there are solutions which are asymptotically flat.

Data without symmetries are investigated in section 3 under assumption (43) about algebraic structure of the exterior curvature tensor. Special solutions of the momentum constraint are described by propositions 3.2 and 3.3. Among them there are asymptotically flat data with $H = 0$ (see corollaries 2 and 3 and hereafter). These solutions are nonsymmetric generalization of the class of axially symmetric data which contains the Kerr initial data.

In order to prove solvability of the Hamiltonian constraint for asymptotically flat data (section 4) we assume $H = 0$ and use the results of Maxwell [9] on the conformal method of Lichnerowicz, Choquet-Bruhat and York. We show that the crucial inequality (83) follows from a simpler one if the flat Sobolev inequality is satisfied (proposition 4.3). More definite results are obtained if data are axially symmetric (theorem 4.4).

In order to encode marginally trapped surfaces into initial data (section 5) we follow again the approach of Maxwell. Now the initial surface has an inner boundary which is supposed to become a marginally outer trapped surface after the conformal transformation solving the Hamiltonian constraint. We show again that the most important condition (105) follows from a simpler one (proposition 5.2) and we present a version of the existence theorem for a particular class of axially symmetric data (theorem 5.3).

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Appendix

It is known [22] that

$$|u|_{L^\infty(\mathbb{R}^3)} \leq A|\nabla u|_{L^2(\mathbb{R}^3)}$$

(A.1)

for every $u \in C_1^2(\mathbb{R}^3)$. Let $M = \mathbb{R}^3 \setminus B$, where $B = B(0, b)$ is an open ball of a radius $b$ with a center at 0. Given $u \in C_1^2(M)$ and a parameter $\alpha \geq 1$ we define the following function $u_\alpha$ in the ball

$$u_\alpha(r, \theta, \phi) = u(r^{-\alpha}b^{\alpha+1}, \theta, \phi),$$

where $r, \theta$ and $\phi$ are the spherical coordinates of $\mathbb{R}^3$. In order to obtain inequality of type (A.1) in $M$ we first prove the following estimation.

**Lemma A.1.**

$$\|\nabla u_\alpha\|_{L^2(B)}^2 \leq \alpha \|\nabla u\|_{L^2(M)}^2.$$  

(A.2)
Proof. The reasoning is purely computational. Denote by $d^2\Omega$ the standard volume form on the unit sphere. Now
\[
\|\nabla u\|_{L^2(B)}^2 = \int_0^b dr \int_{S^2} r^2 d^2\Omega \left( \frac{1}{r^2} \left( \left| \frac{\partial u}{\partial r} \right|^2 + \frac{1}{r^2} \frac{\partial u}{\partial \theta} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left| \frac{\partial u}{\partial \phi} \right|^2 \right) + \frac{1}{r^2 \sin^2 \theta} \left( \frac{\partial u}{\partial \phi} \right)^2,
\]
where $s = b(\frac{r}{b})^\alpha$.

Changing the variables $r \mapsto s$ we get
\[
r = b \left( \frac{s}{b} \right)^{\frac{1}{\alpha}}, \quad \frac{\partial s}{\partial r} = \frac{1}{\alpha} \left( \frac{s}{b} \right)^{\frac{1}{\alpha} - 1}, \quad dr = -\frac{1}{\alpha} \left( \frac{s}{b} \right)^{\frac{1}{\alpha} - 1} ds.
\]
Consequently
\[
\|\nabla u\|_{L^2(B)}^2 = \int_b^\infty ds \frac{1}{\alpha} \left( \frac{s}{b} \right)^{\frac{1}{\alpha} + 1} \int_{S^2} s^2 d^2\Omega \left( \frac{1}{s^2} \frac{\partial u}{\partial s} \left| \frac{\partial u}{\partial s} \right|^2 + \frac{1}{s^2} \frac{\partial u}{\partial \theta} \left| \frac{\partial u}{\partial \theta} \right|^2 + \frac{1}{s^2 \sin^2 \theta} \left| \frac{\partial u}{\partial \phi} \right|^2 \right) + \frac{1}{s^2 \sin^2 \theta} \left( \frac{\partial u}{\partial \phi} \right)^2.
\]
Since $s \geq b$ and $\alpha \geq 1$, the latter expression is not greater than
\[
\alpha \int_b^\infty ds \int_{S^2} s^2 d^2\Omega \left( \frac{1}{s^2} \frac{\partial u}{\partial s} \left| \frac{\partial u}{\partial s} \right|^2 + \frac{1}{s^2} \frac{\partial u}{\partial \theta} \left| \frac{\partial u}{\partial \theta} \right|^2 + \frac{1}{s^2 \sin^2 \theta} \left| \frac{\partial u}{\partial \phi} \right|^2 \right) = \alpha \|\nabla u\|_{L^2(M)}^2.
\]

Theorem A.2. For $u \in C^1_c(M)$, where $M = R^3 \setminus B(0, 1)$, the following inequality holds
\[
\|u\|_{L^p(M)} \leq (2 + \sqrt{2})A \|\nabla u\|_{L^2(M)},
\]
where $A$ is the constant in the Sobolev inequality (A.1) in $R^3$.

Proof. Let us consider the following prolongation $\tilde{u}$ of $u$:
\[
\tilde{u}(r, \theta, \phi) = \begin{cases} u(r, \theta, \phi) & \text{for } r \geq b \\ 3u_1(r, \theta, \phi) - 2u_2(r, \theta, \phi) & \text{for } r < b. \end{cases}
\]
It is easy to check that $\tilde{u}$ is a well defined function of class $C^1_c(R^3)$ such that $\tilde{u}|_M = u$. Moreover,
\[
\|\nabla \tilde{u}\|_{L^2(R^3)} = \|\nabla u\|_{L^2(M)} + \|\nabla u\|_{L^2(B)} = \|\nabla u\|_{L^2(M)} + 3\|u_1\|_{L^2(B)} + 2\|u_2\|_{L^2(B)} \leq 2(2 + \sqrt{2})\|\nabla u\|_{L^2(M)},
\]
where lemma 5.1 was used in the last estimation. From this and (A.1) one obtains
\[
\|u\|_{L^p(M)} \leq \|\tilde{u}\|_{L^p(R^3)} \leq A \|\nabla \tilde{u}\|_{L^2(R^3)} \leq 2(2 + \sqrt{2})A \|\nabla u\|_{L^2(M)}.
\]
References

[1] York J W Jr 1973 Conformally invariant orthogonal decomposition of symmetric tensor on Riemannian manifolds and the initial-value problem of general relativity J. Math. Phys. 14 456–64
[2] Brill D R and Lindquist R W 1963 Interaction energy in geometromechanics Phys. Rev. 131 471–6
[3] Bowen J M and York J W Jr 1980 Time-asymmetric initial data for black holes and black hole collisions Phys. Rev. D 21 2047–55
[4] Brandt S and Brugmann B 1997 A simple construction of initial data for multiple black holes Phys. Rev. Lett. 78 3606–9
[5] Choquet-Bruhat Y and York J W Jr 1980 The Cauchy problem General Relativity and Gravitation ed A Held (New York: Plenum) pp 99–172
[6] Isenberg J 1995 Constant mean curvature solutions of the Einstein constraint equations on closed manifolds Class. Quantum Grav. 12 2249–74
[7] Cantor M and Brill D R 1981 The Laplacian on asymptotically flat manifolds and the specification of scalar curvature Compos. Math. 43 317–25
[8] Cantor M 1977 The existence of non-trivial asymptotically flat initial data for vacuum spacetimes Commun. Math. Phys. 57 83–96
[9] Maxwell D 2005 Solutions of the Einstein constraint equations with apparent horizon boundaries Commun. Math. Phys. 253 561–83
[10] Anderson A and Chruściel P T 1996 On asymptotic behavior of solutions of the constraints equations in general relativity with ‘hyperboloidal boundary conditions’ Dissert. Math. 355 1–100
[11] Anderson A, Chruściel P T and Friedrich H 1992 On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einstein’s field equations Commun. Math. Phys. 149 587–612
[12] Dain S 2004 Trapped surfaces as boundaries for the constraint equations Class. Quantum Grav. 21 555–71
[13] Bartnik R and Isenberg J 2004 The constraint equations The Einstein Equations and the Large Scale Behavior of Gravitational Fields: 50 years of the Cauchy Problem in General Relativity ed P T Chruściel and H Friedrich (Berlin: Birkhäuser)
[14] Baker J and Puzio R 1999 New method for solving the initial value problem with application to multiple black holes Phys. Rev. D 59 044030
[15] Dain S 2001 Initial data for a Head-On collision of two Kerr-like black holes with close limit Phys. Rev. D 64 124002
[16] Garat A and Price R H 2000 Nonexistence of conformally flat slices of the Kerr spacetime Phys. Rev. D 61 124011
[17] Conboye R and Murchadha N O 2013 Potentials for transverse trace-free tensors arXiv:1306.1363
[18] Stephani H, Kramer D, MacCallum M, Hoensalers C and Herlt E 2003 Exact Solutions of Einstein’s Field Equations 2nd edn (Cambridge: Cambridge University Press)
[19] Bartnik R 1986 The mass of an asymptotically flat manifold Commun. Pure Appl. Math. 39 661–93
[20] Salo–Coste L 2009 Sobolev inequalities in familiar and unfamiliar settings Sobolev Spaces in Mathematics I (Int. Math. Ser. vol 8) (New York: Springer) pp 299–343
[21] Adams R A and Fournier J F 2003 Sobolev Spaces 2nd ed (New York: Academic)
[22] Hebey E 2000 Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities (Courant Lecture Notes in Mathematics vol 5) (New York: Courant/AMS)