Three-Nucleon Low-Energy Constants from the Consistency of Interactions and Currents in Chiral Effective Field Theory

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The chiral low-energy constants $c_D$ and $c_E$ are constrained by means of accurate ab initio calculations of the $A=3$ binding energies and, for the first time, of the triton $\beta$ decay. We demonstrate that these low-energy observables allow a robust determination of the two undetermined constants, a result of the surprising fact that the determination of $c_D$ depends weakly on the short range correlations in the wave functions. These two- plus three-nucleon interactions, originating in chiral effective field theory and constrained by properties of the $A=2$ system and the present determination of $c_D$ and $c_E$, are successful in predicting properties of the $A=3$, and 4 systems.

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The fundamental connection between nuclear forces and the underlying theory of quantum chromodynamics (QCD) remains one of the greatest contemporary theoretical challenges, due to the non-perturbative character of QCD in the low-energy regime relevant to nuclear phenomena. However, the last two decades of theoretical developments provide us with a bridge to overcome this obstacle, in the form of chiral perturbation theory (χPT) [1]. The χPT Lagrangian, constructed by integrating out degrees of freedom of the order of $\Lambda_\chi \sim 1$ GeV and higher (nucleons and pions are thus the only explicit degrees of freedom), is an effective Lagrangian of QCD at low energies. As such, it retains all conjectured symmetry principles, particularly the approximate chiral symmetry, of the underlying theory. Furthermore, it can be organized in terms of a perturbative expansion in positive powers of $Q/\Lambda_\chi$ where $Q$ is the generic momentum in the nuclear process or the pion mass [1]. Though the subject of an ongoing debate about its validity [2, 3], the naive extension of this expansion to non-perturbative phenomena provides a practical interface with existing many-body techniques, and clearly holds a significant value for the study of the properties of QCD at low energy and its chiral symmetry.

The chiral symmetry dictates the operator structure of each term of the effective Lagrangian, whereas the coupling constants (not fixed by the symmetry) carry all the information on the integrated-out degrees of freedom. A theoretical evaluation of these coefficients, or low-energy constants (LECs), is equivalent to solving QCD at low-energy. Recent lattice QCD calculations have allowed a theoretical estimate of LECs of single- and two-nucleon diagrams [4], while LECs of diagrams involving more than two nucleons are out of the reach of current computational resources. Alternatively, the undetermined constants can be constrained by low-energy experiments.

The strength of χPT is that the chiral expansion is used to derive both nuclear potentials and currents from the same Lagrangian. Therefore, the electroweak currents in nuclei (which determine reaction rates in processes involving external probes) and the strong interaction dynamics ($\pi N$ scattering, the $NN$ interaction, the $NNN$ interaction, etc.) are all based on the same theoretical grounds and rooted in the low-energy limits of QCD. In particular, χPT predicts, along with the $NN$ interaction at the leading order (LO), a three-nucleon ($NNN$) interaction at the next-to-next-to-leading order or $N^3$LO [5, 6], and even a four-nucleon force at the fourth order ($N^4$LO) [7]. At the same time, the LO nuclear current consists of (the standard) single-nucleon terms, while two-body currents, also known as meson-exchange currents (MEC), make their first appearance at $N^2$LO [8]. Up to $N^3$LO both the $NNN$ potential and the current are fully constrained by the parameters defining the $NN$ interaction, with the exception of two “new” LECs, $c_D$ and $c_E$. The latter, $c_E$, appears only in the potential as the strength of the $NNN$ contact term [see Fig. 1(a)]. On the other hand, $c_D$ manifests itself both in the contact term part of the $NN-\pi N$ three-nucleon interaction of Fig. 1(a) and in the two-nucleon contact vertex with an external probe of the exchange currents [see Fig. 1(b)].

First attempts to determine $c_D$ and $c_E$ have used...
the triton binding energy (b.e.) alongside an additional strong observable, either the nd doublet scattering length [6], or the $^4\text{He}$ b.e. [8]. However, this led to a substantial uncertainty in the values of the LECs, due to correlations between the $^3\text{H}$ b.e. and these observables, known respectively as the Phillips and Tjon lines. The fine-tuning of these observables is very sensitive to the structure of the adopted $N\bar{N}N$ force. Hence small variations of the cutoff, different regularization schemes, missing terms of the interaction, etc., tend to produce large swings in the extracted of $c_D$ and $c_E$. A different approach was adopted in Ref. [9]. There, a preferred choice for the two LEC’s was obtained by complementing the constraint on the $A=3$ b.e. with a sensitivity study on the radius of $^4\text{He}$ and on various properties of p-shell nuclei. The same interaction was then successfully used to predict the $^4\text{He}$ total photo-absorption cross section [10].

The need for a complemental determination of these LECs is two-fold. First, it would be desirable to perform such a determination within the $A\leq3$ systems, so to suppress any additional many body contribution. Second, despite recent progress [11], the $N\bar{N}N$ potential has been fully worked out only up to $N^3\text{LO}$, leaving inconsistency in the calculation of the wave functions when combined with the NN force at $N^3\text{LO}$. Clearly, it would be preferable to adopt an observable with minimal dependence on the short-range part of the wave function. In this respect, the relation (mandated by the chiral symmetry of QCD) between electroweak processes and $N\bar{N}N$-force effects offers venues to achieve these goals. This relation was established in the context of effective field theory [12–14], and manifests itself in $\chi$PT via the appearance of $c_D$ in both the $NN-N-N$ diagram of Fig. 1(a) and the one in Fig. 1(b). In particular, Gérardstig and Phillips [13], suggested the triton beta-decay as one of the electroweak processes that could be used as input to fix the strength of the $N\bar{N}N$ force. It is the purpose of this Letter to undertake this task and show that by using the triton half life, as well as the $A=3$ b.e., one can constrain the two undetermined LECs within the three-nucleon sector by means of fully converged $ab\ initial$ calculations. We demonstrate that this determination is robust. The resulting chiral Lagrangian predicts, without any free parameters, various $A=3$, and 4 properties.

The triton is an unstable nucleus, which undergoes $\beta$-decay with a “comparative” half-life of $(fT_{1/2})_t = (1129.6 \pm 3)$ s [15]. This quantity can be used to extract an empirical value for $\langle E^A_1 \rangle = \langle |^3\text{He}| |E^A_1|^2 |^4\text{He}\rangle$ [16,17], the reduced matrix element of the $J=1$ electric multiple of the axial vector current, through

$$fT_{1/2})_t = \frac{K/G^2_V}{1 - \delta_c} + 3\pi \frac{f_A}{f_V} \langle E^A_1 \rangle^2.$$  \hspace{1cm} (1)

Here, $K = 2\pi^3 \ln 2/m_e^5$ (with $m_e$ the electron mass), $G_V$ is the weak interaction vector coupling constant (such that $K/G^2_V = 6146.6 \pm 0.68$ [18]), $f_A/f_V = 1.00529 \pm$ 0.0011 accounts for the small difference in the statistical rate function between vector and axial-vector transitions, and $\delta_c = 0.13\%$ [17] is a small correction to the reduced matrix element of the Fermi operator, calculated between the $A=3$ wave functions (which is 1 for this specific case) due to isospin-breaking in the nuclear interaction. One can use these values to extract $\langle E^A_1 \rangle_{\text{emp}} = 0.6848 \pm 0.0011$. The weak axial current adopted in this work is the Nöther current built from the axial symmetry of the chiral Lagrangian up to order $N^3\text{LO}$ [19]. At LO this current consists of the standard single-nucleon part, which at low momentum transfer is proportional to the Gamow-Teller (GT) operator, $E^A_1|_{\text{LO}} = i g_A (6\pi)^{-1/2} \sum_i^A \sigma_i \bar{\tau}_i^+ \bar{\sigma}_i$, where $\sigma_i, \tau_i^+$ are spin and isospin-raising operators of the ith nucleons, and $g_A = 1.2695 \pm 0.0029$ is the axial constant [20]. For this reason, the quantity $\sqrt{3\pi} g^{-1}_A \langle E^A_1 \rangle_{\text{emp}}$ is often referred to as “experimental”, or “empirical”, GT.

Corrections to the single-nucleon current appear at $N^3\text{LO}$ in the form of MEC and relativistic terms. The MEC are formed by a one-(charged)-pion exchange, and a contact term. While the relativistic corrections are negligible for the triton half life, the MEC have a substantial influence on this $\beta$-decay rate. This is a reflection of the fact that $E^A_1$ is a chirally unprotected operator [21]. Moreover, the strength of the MEC contact term, usually denoted by $\delta_R$, is related to $c_D$ through:

$$\delta_R \equiv \frac{M_N}{A_X g_A} c_D + \frac{1}{3} M_N (c_3 + 2c_4) + \frac{1}{6}$$ \hspace{1cm} (2)

Here, $M_N$ is the nucleon mass, and $c_3$ and $c_4$ are LECs of the dimension-two $\pi N$ Lagrangian, already part of the chiral $N N$ potential at NLO. Therefore, one can use $\langle E^A_1 \rangle_{\text{emp}}$ as second constraint for the determination of $c_D$ and $c_E$.

Following the $c_D+c_E$ trajectory which reproduces, on average, the $A=3$ b.e., as discussed in Ref [9], here, we (i) calculate the $^3\text{H}$ and $^3\text{He}$ g.s. wave functions by solving the Schrödinger equation for three nucleons interact-
The ratio \( \langle E_1^{4} \rangle_{\text{th}} / \langle E_1^{4} \rangle_{\text{emp}} \) using the N\(^3\)LO NN potential \cite{22} and with and without the local N\(^2\)LO NNN interaction \cite{24}, and the axial current with and without MEC (\( c_D, c_E \)) are varied along the averaged trajectory of Fig. 2. The shaded area is twice the experimental uncertainty. Also shown: results (without NNN force) for the phenomenological AV18 potential (with \( \Lambda = 500 \) MeV imposed in the current), and for the N\(^3\)LO NN potential of Epelbaum et al. \cite{40} (with \( \Lambda = 450, 600 \) MeV, and a 700 MeV spectral-function cutoff in the two-pion exchange).

The present calculations are performed in the framework of the no-core shell model (NCSM) approach \cite{8, 9, 23, 24}. This method looks for the eigenvectors of the Hamiltonian in the form of expansions over a complete set of harmonic oscillator (HO) basis states up to a maximum excitation of \( N_{\text{max}} \hbar \Omega \) above the minimum energy configuration, where \( \Omega \) is the HO parameter. Thanks to the large model-space size adopted (\( N_{\text{max}} = 40 \)), \( A = 3 \) b.e. and reduced matrix element of \( E_1^{4} \) are converged to less than 0.05%. Note that the same regulator \( F_A(q^2) = \exp(-q^2/\Lambda^2) \) is used for both NNN terms of the interaction and MEC, a process resulting in a local chiral NNN force (for relevant parameters and definitions see Ref. \cite{23}). The \( A = 3, 4 \) calculations of Ref. \cite{24} were later confirmed by the results of Ref. \cite{26}, providing a benchmark for the local chiral NNN force. The MEC utilized in this work were validated against those of Park et al. \cite{19}. Finally, we tested the implementation of the MEC within the NCSM approach by reproducing (within 0.1%) the AV18 \cite{27} results for \( \langle E_1^{4} \rangle \) obtained using the effective-interaction hyper-spherical harmonics approach \cite{19}.

The theory to empirical value ratio for the \( E_1^{4} \) reduced matrix element along the averaged constraint of Fig. 2 (which reproduces the \( A = 3 \) b.e. to about 10 keV \cite{9}) is presented in Fig. 3. The 1.08% tolerance band highlighted by the shaded area (obtained by doubling the error bar) is mainly due to the uncertainties on \( \langle E_1^{4} \rangle_{\text{emp}} \) and \( q_A \). Besides the full calculation, which appears as a solid line, we report also the results of several tests, aimed to analyze the sensitivity of the triton half life to NNN force and/or MEC.

First we note the fundamental importance of the axial two-body currents in reaching agreement with experiment. By suppressing the MEC, in the whole investigated \( c_D-c_E \) range, the calculations under-predict \( \langle E_1^{4} \rangle_{\text{emp}} \) by about 2%. The same, almost constant, behavior is found when adding to the single-nucleon current only the long-range one-pion-exchange term of the MEC, which corresponds to setting \( d_R = 0 \). In this case, the theoretical results over-predict \( \langle E_1^{4} \rangle_{\text{emp}} \) by \( \sim 11\% \). Only when adding the contact part of the MEC, which is related to the short range weak correlations of axial character, can the half-life reach its experimental value. In particular, we find that agreement within \( \pm 0.54\% \) of the empirical value is obtained for \( -0.3 \leq c_D \leq -0.1 \). The corresponding \( c_E \) values lie in the range \([ -0.220, -0.189 ]\). These results are summarized by the dotted lines in Fig. 2.

In a similar spirit, we now study the effect of the suppression of the NNN force. If we try to calibrate \( c_D \) to reproduce the measured half-life, we obtain a curve in close agreement with the results of the full calculation \cite{19} (for completeness we show also the curve corresponding to the suppression of both MEC and NNN force). Moreover, a quantitatively similar \( c_D \) dependence of \( \langle E_1^{4} \rangle \) can be obtained using \( A = 3 \) wave functions produced by the phenomenological AV18 NNN potential (without NNN force). It is therefore clear that the half life of triton presents a very weak sensitivity to the NNN force, and hence to the strength of the spin-orbit interaction. Thanks to this feature, which might be unique to s-shell nuclei, we are confident that the determination of \( c_D \) and \( c_E \) obtained in this way is robust. Incidentally, the weak dependence of the half life of triton upon the NNN force can also explain the success of recent calculations done in a hybrid approach, coined EFT* \cite{19}.

As the values of the \( c_3 \) and \( c_4 \) LECs are somewhat uncertain (see, e.g., Ref. \cite{8}), it is important to assess to which extent they would influence the determination of \( c_D \) from the triton half life. While very sensitive to the smallest change in \( c_3 \), the N\(^3\)LO fit of the NN data of Ref. \cite{22} does not deteriorate dramatically for \( 3.4 \text{GeV}^{-1} \leq c_4 \leq 5.4 \text{GeV}^{-1} \). Fig. 3 shows calculations (without NNN force) carried out by setting \( c_4 \) to \( 3.4 \text{GeV}^{-1} \) (\( \pi N \) value \cite{29}) in the axial current, while the \( A = 3 \) wave functions are still obtained from the N\(^3\)LO \( NN \) potential of Ref. \cite{22} (where, in GeV\(^{-1} \), \( c_3 = -3.2 \) and \( c_4 = 5.4 \)). We find that the use of the lower \( c_4 \) value produces a shift \( \sim 0.3 \) towards more positive \( c_D \) values.

Finally, Fig. 3 shows two additional curves (without NNN force) obtained using the N\(^3\)LO \( NN \) potential of Ref. \cite{30} with \( \Lambda = 450 \) and 600 MeV (700 MeV two-pion exchange spectral-function cutoff) and the parameters defining it (particularly, in GeV\(^{-1} \), \( c_3 = -3.4 \) and
c_4 = 3.4). As one could expect, the extracted value of the NNN LEC c_D depends on the choice of the cutoff. However, we observe that the \Lambda = 450/700 MeV potential gives comparable results as the N^3LO NN potential of Ref. [22] and AV18, indicating that the determination of c_D depends mainly on the MEC cutoff and weakly on the cutoff imposed in the nuclear potential.

With this calibration of c_D and \epsilon_E, for this potential, in principle, any other calculation is a prediction of \chi PT. In Table I we present a collection of A = 3 and 4 data, obtained with and without inclusion of the NNN force for c_D = −0.2 (\epsilon_E = −0.205), a choice in the middle of the constrained interval. Besides triton and ^3He g.s. energies, which are by construction within few keV from experiment, the NN + NNN results for the ^4He are in good agreement with measurement. Note that \alpha particle g.s. energy and point-proton radii change minimally with respect to variations of c_D in the interval [−0.3, −0.1], and the results at the extremes are both within the numerical uncertainties quoted in Table I. This result is not inconsistent with the study of Ref. [11], which showed preference for c_D ~ −1, since for p-shell nuclei one expects the (neglected) higher-order NNN force terms to affect, probably through a shift, the value of c_D [11].

Summarizing, we have used the A = 3 b.e. and the half-life of triton to constrain the undetermined N^3LO \chi PT parameters of the NNN force. We have demonstrated the robustness of the constraint on c_D by showing the weak sensitivity of the \langle E_1^A \rangle matrix element with respect to the NNN force. In particular, we find −0.3 ≤ c_D ≤ −0.1, and, correspondingly, −0.220 ≤ \epsilon_E ≤ −0.189. The latter is expected to change due to N^3LO terms of the NNN interaction, which were not included thus far. In conclusion, we have identified a clear path towards determining the NNN force that, once the NN interaction will be pinned down, will pave the way to parameter-free predictions of QCD in the consistent approach provided by \chi PT.

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|        | ^3H | ^3He | ^4He |
|--------|-----|------|------|
| E_{\text{g.s.}} | \langle r_p^{2\,1/2} \rangle | E_{\text{g.s.}} | \langle r_p^{2\,1/2} \rangle | E_{\text{g.s.}} | \langle r_p^{2\,1/2} \rangle |
| NN | −7.852(4) | 1.651(5) | −7.124(4) | 1.847(5) | −25.39(1) | 1.515(2) |
| NN + NNN | −8.473(4) | 1.605(5) | −7.727(4) | 1.786(5) | −28.50(2) | 1.461(2) |
| Expt. | −8.482 | 1.60 | 7.178 | 1.77 | −28.296 | 1.467(13) |
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