On the approximation of a polytope by its dual $L_p$-centroid bodies

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Abstract

We show that the rate of convergence on the approximation of volumes of a convex symmetric polytope $P \in \mathbb{R}^n$ by its dual $L_p$-centroid bodies is independent of the geometry of $P$. In particular we show that if $P$ has volume 1,

$$\lim_{p \to \infty} \frac{p}{\log p} \left( \frac{|Z_p^{\circ}(P)|}{|P^\circ|} - 1 \right) = n^2.$$  

We provide an application to the approximation of polytopes by uniformly convex sets.

1 Introduction

Let $K$ be a convex body in $\mathbb{R}^n$ of volume 1 and, for $\delta \in (0, 1)$, let $K_\delta$ be the convex floating body of $K$ [22]. It is the intersection of all halfspaces $H^+$ whose defining hyperplanes $H$ cut off a set of volume $\delta$ from $K$. Note that $K_\delta$ converges to $K$ in the Hausdorff metric as $\delta \to 0$. C. Schütt and the second name author showed an exact formula for the convergence of volumes [22],

$$\lim_{\delta \to 0} \frac{|K| - |K_\delta|}{\delta^{n+1}} = a_{1}(K),$$

which involves the affine surface area of $K$, $a_{1}(K)$. The same phenomenon (and similar formulas) has been observed for other types of approximation using instead of floating bodies, convolution bodies [21], illumination bodies [27] or Santaló bodies [18]. We refer to e.g. [2, 4-9, 12-17, 23-26, 28-30] for further details, extensions and applications. Another family of bodies that approximate a given

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convex body $K$ are the $L_p$-centroid bodies of $K$ introduced by Lutwak and Zhang [17]. For a symmetric convex body $K$ of volume 1 in $\mathbb{R}^n$ and $1 \leq p \leq n$, the $L_p$-centroid body $Z_p(K)$ is the convex body that has support function

$$h_{Z_p(K)}(\theta) = \left( \int_K |\langle x, \theta \rangle|^p \, dx \right)^{\frac{1}{p}}, \quad \theta \in S^{n-1}.$$ 

Note that $Z_p(K)$ converges to $K$ in the Hausdorff metric as $p \to \infty$. It has been shown in [19] that the family of $L_p$-centroid bodies is isomorphic to the family of the floating bodies: $K_\delta$ is isomorphic to $Z_{\log \frac{1}{\delta}}(K)$. However, it was proved in [19] that in the case of $C^2_+$ bodies, the convergence of volume of the $L_p$-centroid bodies is independent of the “geometry” of $K$: For any symmetric convex body in $\mathbb{R}^n$ of volume 1 that is $C^2_+$ (i.e. $K$ has $C^2$ boundary with everywhere strictly positive Gaussian curvature),

$$\lim_{p \to \infty} \frac{p}{\log p} \left( |Z_p^\circ(K)| - |K^\circ| \right) = \frac{n(n+1)}{2} |K^\circ|.$$ 

In this work we show that the same phenomenon occurs also in the case of polytopes. We show the following

**Theorem 1.1.** Let $K$ be a symmetric polytope of volume 1 in $\mathbb{R}^n$. Then

$$\lim_{p \to \infty} \frac{p}{\log p} \left( |Z_p^\circ(K)| - |K^\circ| \right) = \frac{n^2}{2} |K^\circ|.$$ 

As an application of this result we get bounds for the approximation of a polytope by a uniformly convex body with respect to the symmetric difference metric:

**Theorem 1.2.** Let $P$ be a symmetric polytope in $\mathbb{R}^n$. Then there exists $p_0 = p_0(P)$ such that for every $p \geq p_0$, there exists a $p$-uniformly convex body $K_p$ such that

$$d_s(P, K_p) \leq 2n^2 |P| \frac{\log p}{p},$$

where $d_s$ is the symmetric difference metric.

The statements and proofs are for symmetric convex bodies only. If $K$ is not symmetric, then $Z_p(K)$ does not converge to $K$ since the $Z_p(K)$ are centrally symmetric by definition. However, all results can be extended to the non-symmetric case with minor modifications of the proofs by using the non-symmetric version of the $L_p$-centroid bodies from [12] (see also [6]).

The paper is organized as follows. In section 2 we give some bounds for the approximation of volume in the case of a general convex body. In section 3 we consider the case of polytopes and we give the proof of Theorem 1.1. Finally, in section 4, we discuss approximation of a polytope by $p$-uniformly convex bodies (see [11]) and we give the proof of Theorem 1.2.
Notation.

We work in \( \mathbb{R}^n \), which is equipped with a Euclidean structure \( \langle \cdot, \cdot \rangle \). We denote by \( \| \cdot \|_2 \) the corresponding Euclidean norm, and write \( B^n_2 \) for the Euclidean unit ball and \( S^{n-1} \) for the unit sphere. Volume is denoted by \( | \cdot | \). We write \( \sigma \) for the rotationally invariant surface measure on \( S^{n-1} \).

A convex body is a compact convex subset \( C \) of \( \mathbb{R}^n \) with non-empty interior. We say that \( C \) is symmetric, if \( x \in C \) implies that \( -x \in C \). We say that \( C \) has center of mass at the origin if \( \int_C \langle x, \theta \rangle \, dx = 0 \) for every \( \theta \in S^{n-1} \). The support function \( h_C : \mathbb{R}^n \to \mathbb{R} \) of \( C \) is defined by \( h_C(x) = \max \{ \langle x, y \rangle : y \in C \} \).

The polar body of \( C \) is defined as \( C^\circ = \{ y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in C \} \).

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2 General Bounds

Let \( K \) be a symmetric convex body in \( \mathbb{R}^n \) of volume 1. Let \( \theta \in S^{n-1} \). We define the parallel section function \( f_{K,\theta} : [-h_K(\theta), h_K(\theta)] \to \mathbb{R}^+ \) by

\[
 f_{K,\theta}(t) := |K \cap (\theta^\perp + t\theta)|.
\]

By Brunn’s principle, \( f_{K,\theta}^{-1/\theta} \) is concave and attains its maximum at 0. So we have that

\[
 \left( 1 - \frac{t}{h_K(\theta)} \right)^{n-1} f_{K,\theta}(0) \leq f_{K,\theta}(t) \leq f_{K,\theta}(0).
\]

The right-hand side inequality is sharp if and only if \( K \) is a cylinder in the direction of \( \theta \) and the left-hand side inequality is sharp if and only if \( K \) is a double cone in the direction of \( \theta \).

The next proposition is well known. There, for \( x, y > 0 \), \( B(x, y) = \int_0^1 \lambda^{x-1}(1 - \lambda)^{y-1}d\lambda = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \) is the Beta function and \( \Gamma(x) = \int_0^\infty \lambda^{x-1}e^{-\lambda}d\lambda \) is the Gamma function.

Proposition 2.1. Let \( K \) be a symmetric convex body in \( \mathbb{R}^n \) of volume 1. Let \( 1 \leq p < \infty \) and \( \theta \in S^{n-1} \). Then

\[
 B(p+1,n)^\frac{1}{p} \leq \frac{h_{Z_p(K)}(\theta)}{h_K(\theta)} \leq \left( \frac{n}{p+1} \right)^\frac{1}{p}.
\]

Proof. As \( |K| = 1 \),

\[
 \frac{2}{n} h_K(\theta) f_{K,\theta}(0) \leq 1 \leq 2 h_K(\theta) f_{K,\theta}(0).
\]
Hence, on the one hand, with (1),

\[ h_{Z_p(K)}^p(\theta) = 2 \int_0^{h_K(\theta)} t^p f_{K,\theta}(t) dt \leq 2 f_{K,\theta}(0) \int_0^{h_K(\theta)} t^p dt = \frac{2}{p+1} f_{K,\theta}(0) h_{K}^{p+1}(\theta) \leq \frac{n}{p+1} h_{K}^p(\theta). \]

On the other hand, also with (1),

\[ h_{Z_p(K)}^p(\theta) = 2 \int_0^{h_K(\theta)} t^p f_{K,\theta}(t) dt \geq 2 f_{K,\theta}(0) \int_0^{h_K(\theta)} t^p \left(1 - \frac{t}{h_K(\theta)}\right)^{n-1} dt \geq 2 f_{K,\theta}(0) h_{K}^{p+1}(\theta) \int_0^1 s^p (1-s)^{n-1} ds \geq B(p + 1, n) h_{K}^p(\theta). \]

The proof is complete. \(\square\)

As it was mentioned in the introduction, it was proved in [19] that if \(K\) is a \(C^2\) symmetric convex body of volume 1, then

\[ \lim_{p \to \infty} \frac{p}{\log p} \left( |Z_p^o(K)| - |K^o| \right) = \frac{n(n+1)}{2} |K^o|. \]

Before we consider the case of polytopes, we show that for every convex body we have that

\[ n|K^o| \leq \lim_{p \to \infty} \frac{p}{\log p} \left( |Z_p^o(K)| - |K^o| \right) \leq n^2|K^o|. \]

**Proposition 2.2.** Let \(K\) be a symmetric convex body in \(\mathbb{R}^n\) of volume 1. Then

\[ n|K^o| \leq \lim_{p \to \infty} \frac{p}{\log p} \left( |Z_p^o(K)| - |K^o| \right) \leq n^2|K^o|. \]

**Proof.** We have that

\[ |Z_p^o(K)| - |K^o| = \frac{1}{n} \int_{S^{n-1}} \frac{1}{h_{Z_p(K)}^p(\theta)} - \frac{1}{h_{K}^p(\theta)} d\sigma(\theta) \]

\[ = \frac{1}{n} \int_{S^{n-1}} \frac{1}{h_{Z_p(K)}^p(\theta)} \left( \frac{h_{K}^p(\theta)}{h_{Z_p(K)}^p(\theta)} - 1 \right) d\sigma(\theta), \]

where \(\sigma\) is the usual surface area measure on \(S^{n-1}\). By Proposition 2.1,

\[ \frac{h_{K}^p(\theta)}{h_{Z_p(K)}^p(\theta)} \geq \left( \frac{n}{p+1} \right)^{-\frac{2}{p}} = 1 + \frac{n \log p}{p} \pm o\left( \frac{p}{\log p} \right) \]

and

\[ \frac{h_{K}^p(\theta)}{h_{Z_p(K)}^p(\theta)} \leq B(p + 1, n)^{-\frac{2}{p}} = 1 + \frac{n^2 \log p}{p} \pm o\left( \frac{p}{\log p} \right). \]

For the last equality see e.g. [19], Lemma 4.3 - which is also stated here as Lemma 3.3. Lebesgue’s convergence theorem completes the proof. \(\square\)
3 Polytopes

Let $K$ be a convex polytope in $\mathbb{R}^n$ with vertices $v_1, \ldots, v_M$. For $0 \leq k \leq n - 1$, let $A_k = \{ F_k : F_k$ is a $k$-dimensional face of $K \}$. For $\theta \in S^{n-1}$ and $0 \leq s \leq h_k(\theta)$ let

$$g(\theta, s) = \text{card} \{(v_i : v_i \in K \cap (\langle v_i, \theta \rangle \geq s))\}.$$

Let

$$B_K = \{ \theta \in S^{n-1} : \forall s \leq h_K(\theta) : g(\theta, s) > 1 \} \quad (2)$$

and

$$G_K = \{ \theta \in S^{n-1} : \exists s < h_K(\theta) : g(\theta, s) = 1 \} \quad (3)$$

Finally, for $\theta \in G_K$, let

$$s_\theta = \min\{s > 0 : g(\theta, s) = 1\} \quad (4)$$

Remarks. Let $\theta \in G_K$.

(i) Then there is a vertex $v_i$ such that for all $s_\theta \leq s \leq h_K(\theta)$

$$\{x \in K : \langle x, \theta \rangle \geq s\} = \text{co}[K \cap (\theta^+ + s\theta), v_i]$$

(ii) Recall that $f_{K,\theta}(s) = |K \cap (\theta^+ + s\theta)|$. We have for all $s_\theta \leq s \leq h_K(\theta)$

$$f_{K,\theta}(s) = f_{K,\theta}(s_\theta) \left(1 - \frac{s}{h_K(\theta)}\right)^{n-1} \left(1 - \frac{s_\theta}{h_K(\theta)}\right)^{n-1} \quad (5)$$

For a convex body $K$, let $H_K = \max_{\theta \in S^{n-1}} h_K(\theta)$.

For $1 \leq k \leq n$, let $K$ be a $k$-dimensional convex body in a $k$-dimensional affine space of $\mathbb{R}^n$. Let

$$r(K) = \sup\{r > 0 : \exists x \in K \text{ such that } x + rB_2^k \subseteq K\} \quad (6)$$

be the inradius of $K$. Let

$$r_0 = \min_{1 \leq k \leq n-1} \min_{F_k \in A_k} r(F_k)$$

Note that $r_0 > 0$. We also put $h_0 = \max_{u \in B_K} h_K(u)$. For $\delta > 0$, we define

$$A(\delta) = \{ \theta \in S^{n-1} : \exists u \in B_K : \|\theta - u\| < \delta \} \quad (7)$$

and

$$s(\delta) = \sup_{\theta \in S^{n-1} \setminus A(\delta)} \frac{s_\theta}{h_K(\theta)} \quad (8)$$

Remark. $s(\delta) < 1$ and if $\theta \to \phi$ where $\phi \in B_K$, then by continuity, $\frac{s_\theta}{h_K(\theta)} \to 1$. Hence we may assume that for $\delta > 0$ small enough, $s(\delta)$ is attained on the “boundary” of $S^{n-1} \setminus A(\delta)$. 

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Lemma 3.1. Let $K$ be a 0-symmetric polytope in $\mathbb{R}^n$ of volume 1. Then for $\delta$ small enough,

$$s(\delta) = \sup_{\theta \in S^{n-1} \setminus A(\delta)} \frac{s_\theta}{h_K(\theta)} \leq 1 - \frac{\delta r_0}{2h_0}$$

Proof. Let $\delta \leq \frac{h_0}{h_K}$. By the above Remark, for $\delta > 0$ small enough, there exists $\phi \in S^{n-1} \setminus A(\delta)$ such that $s(\delta) = \frac{s_\phi}{h_K(\phi)}$.

As $\phi \in S^{n-1} \setminus A(\delta)$, there exists $u \in B_K$, such that $\|u - \phi\| = \delta$. Let $v \in \partial K$ be that vertex of $K$ such that $\langle \phi, v \rangle = \max_{x \in K} \langle \phi, x \rangle$. Let

$$x_0 = \{\alpha \phi : \alpha \geq 0\} \cap \partial K, \quad z_0 = \{\alpha u : \alpha \geq 0\} \cap \partial K,$$

and

$$d_1 = \|x_0 - z_0\|, \quad d_2 = \|x_0 - v\|.$$

$x_0$, $v$ and $z_0$ lie in the $n-1$-dimensional face $F$ orthogonal to $u$. As $\phi \in G_K$, we may also assume that $\delta$ is small enough such that $s_\phi = \|x_0\|$, and hence $s(\delta) = \frac{\|x_0\|}{h_K(\phi)}$.

Let $\omega$ be the angle between $\phi$ and $u$. Then

$$\tan \omega = \frac{d_1}{h_K(u)} \quad \text{and} \quad \sin \omega = \frac{h_K(\phi) - s_\phi}{d_2}.$$

Hence

$$\frac{h_K(\phi) - s_\phi}{d_2} = \frac{d_1 \cos \omega}{h_K(u)}$$

and thus

$$\frac{s_\phi}{h_K(\phi)} = 1 - \frac{d_1 d_2 \cos \omega}{h_K(u) h_K(\phi)}.$$

As $d_2 \geq r_0$ and as $\delta \leq \frac{d_1 \cos \omega}{h_K(u)}$, we get that

$$\frac{s_\phi}{h_K(\phi)} \leq 1 - \frac{\delta r_0}{h_K(\phi)}.$$

Now observe that

$$h_k(\phi) = h_K(\phi - u) + h_K(u) \leq \delta H_K + h_K(u) \leq 2h_0.$$

Therefore,

$$\frac{s_\phi}{h_K(\phi)} \leq 1 - \frac{\delta r_0}{2h_0}. \quad \Box$$

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a $C^2$ log-concave function with $\int_{\mathbb{R}_+} f(t)dt < \infty$ and let $p \geq 1$. Let $g_p(t) = t^p f(t)$ and let $t_p = t_p(f)$ the unique point such that $g'(t_p) = 0$. We make use of the following Lemma due to B. Klartag [10] (Lemma 4.3 and Lemma 4.5).
Lemma 3.2. Let $f$ be as above. For every $\varepsilon \in (0,1)$,
\[
\int_0^\infty t^p f(t) dt \leq \left(1 + Ce^{-cp^2}\right) \int_{t_p(1-\varepsilon)}^{t_p(1+\varepsilon)} t^p f(t) dt
\]
where $C > 0$ and $c > 0$ are universal constants.

We will use Lemma 3.2 for the function $f_K,\theta(s) = |K \cap (\theta^\perp + s\theta)|$ in the proof of the next lemma. First we observe

Remark 1. Let $\theta \in G_K$. As above, let $g_p(t) = t^p f_{K,\theta}(t)$ and let $t_p$ be the unique point such that $g_p'(t_p) = 0$. Note that, since $t_p \to h_K(\theta)$, as $p \to \infty$ (see e.g. [19], Lemma 4.5), for $p$ large enough - namely $p$ so large that $t_p \geq s_{\theta}$ - we can use (5) and compute $t_p$.

\[
t_p = \frac{p}{p + n - 1} h_K(\theta)
\] (9)

We will also use (see e.g. [19], Lemma 4.3).

Lemma 3.3. Let $p > 0$. Then
\[
(B(p + 1, n))^\frac{p}{p} = 1 - \frac{n^2}{p} \log p + \frac{n}{p} \log (\Gamma(n)) + \frac{n^4}{2p^2}(\log p)^2 - \frac{n^3}{2p^2} \log (\Gamma(n)) \log p + o(p^2).
\]

Lemma 3.4. Let $K$ be a $0$-symmetric polytope in $\mathbb{R}^n$ of volume 1. For all sufficiently small $\delta$, for all $\theta \in S^{n-1} \setminus A(\delta)$ and for all $p \geq \alpha_n(K)$, we have
\[
\left(\frac{h_{Z_p(K)}(\theta)}{h_K(\theta)}\right)^n \leq 1 - n^2 \frac{\log p}{p} + (n-1)n \frac{\log \frac{1}{3}}{p} + c_{K,n}.
\]

$\alpha_n(K) = \frac{4(n-1)h_0}{h_K}$ and $c_{K,n}$ are constants that depend on $K$ and $n$ only.

Proof. Let $0 < \delta \leq \frac{h_0}{h_K}$ be as in Lemma 3.1. Let $\theta \in S^{n-1} \setminus A(\delta)$. Hence, in particular, $\theta \in G_K$. By Lemma 3.2 we have for all $\varepsilon \in (0,1)$
\[
\frac{h_{Z_p(K)}(\theta)}{h_K(\theta)} = 2 \int_0^{h_K(\theta)} t^p f_{K,\theta}(t) dt \leq 2 \left(1 + Ce^{-cp^2}\right) \int_{t_p(1-\varepsilon)}^{t_p(1+\varepsilon)} t^p f_{K,\theta}(t) dt
\]
Since $t_p \to h_K(\theta)$, as $p \to \infty$ (see e.g. [19], Lemma 4.5), there exists $p_\varepsilon > 0$ (which we will now determine), such that for all $p \geq p_\varepsilon$,
\[
(1 - \varepsilon)t_p \geq s_{\theta}.
\] (10)
By \[(9), (10)\] holds for all \(p \geq p_\varepsilon\) with
\[
p_\varepsilon \geq \frac{(n - 1) s_{\theta K}}{1 - \varepsilon - \frac{s_{\theta K}}{h_{K}(\theta)}}.
\]

By Lemma 3.1, \(\frac{s(\theta)}{h_{K}(\theta)} \leq 1 - \frac{\delta r_0}{2h_0}\) and thus \((10)\) holds for all \(p \geq p_\varepsilon\) with
\[
p_\varepsilon \geq \frac{n - 1}{\delta} \frac{2h_0 - \delta r_0}{r_0 - 2h_0\varepsilon/\delta}.
\]

We choose \(\varepsilon = \frac{r_0\delta}{4h_0}\). Then for
\[
p_\varepsilon \geq \frac{n - 1}{\delta} \frac{4h_0}{r_0}
\]
the estimate \((10)\) holds for all \(p \geq p_\varepsilon\) uniformly for all \(\theta \in S^{n-1} \setminus A(\delta)\). Thus, using also \((5)\),
\[
\begin{align*}
\frac{h^p_{Z_{p}(K)\theta}}{h_{K}(\theta)} & \geq 2 \left(1 + Ce^{-c_p\varepsilon^2}\right) \int_{(1-\varepsilon)t_p}^{h_{K}(\theta)} t^p f_{K,\theta}(t) dt \\
& \leq 2 \left(1 + Ce^{-c_p\varepsilon^2}\right) \int_{s_0}^{h_{K}(\theta)} t^p f_{K,\theta}(t) dt \\
& = 2 \left(1 + Ce^{-c_p\varepsilon^2}\right) \frac{h_{K}^{p+1}(\theta) f_{K,\theta}(s_\theta)}{1 - \frac{s_{\theta K}}{h_{K}(\theta)}} \int_{\frac{s_{\theta K}}{h_{K}(\theta)}}^{1} u^p (1 - u)^{n-1} du \\
& \leq 2 \left(1 + Ce^{-c_p\varepsilon^2}\right) \frac{h_{K}^{p+1}(\theta) f_{K,\theta}(0)}{1 - \frac{s_{\theta K}}{h_{K}(\theta)}} \int_{0}^{1} u^p (1 - u)^{n-1} du \\
& \leq n \left(1 + Ce^{-c_p\varepsilon^2}\right) B(p + 1, n) h_{K}^p(\theta) \left(\frac{2h_0}{\delta r_0}\right)^{n-1}.
\end{align*}
\]

In the last inequality we have used that \(1 - \frac{s_{\theta K}}{h_{K}(\theta)} \geq \frac{\delta r_0}{2h_0}\) and that \(\frac{2h_0}{\delta r_0} f_{K,\theta}(0) \leq |K| = 1\). Equivalently, \((11)\) becomes
\[
\left(\frac{h_{Z_{p}(K)\theta}}{h_{K}(\theta)}\right)^n \leq n^{\frac{p}{n}} \left(1 + Ce^{-c_p\varepsilon^2}\right)^{\frac{n}{p}} \frac{2h_0}{\delta r_0} B(p + 1, n)^{\frac{1}{n}}.
\]

With Lemma 3.3 we then get
\[
\left(\frac{h_{Z_{p}(K)\theta}}{h_{K}(\theta)}\right)^n \leq 1 - n^{\frac{p}{p}} \log p + (n - 1)n \log \frac{1}{p} + C_{K,n} p.
\]

\(\square\)
Let $\delta \in [0, 1)$ and $\theta \in S^{n-1}$. We define the cap $C(\theta, \delta)$ of the sphere $S^{n-1}$ around $\theta$ by
\[
C(\theta, \delta) := \{ \phi \in S^{n-1} : \| \phi - \theta \|_2 \leq \delta \}.
\]
We will estimate the surface area of a cap, and to do so we will make use of the following fact which follows immediately from e.g. Lemma 1.3 in [23].

**Lemma 3.5.** Let $\theta \in S^{n-1}$ and $\delta < 1$. Then
\[
\operatorname{vol}_{n-1}(B_2^{n-1}) (1 - \frac{\delta^2}{4})^{\frac{n-1}{2}} \delta^{n-1} \leq \sigma(C(\theta, \delta)) \leq \operatorname{vol}_{n-1}(B_2^{n-1}) (1 - \frac{\delta^2}{4})^{\frac{n-1}{2}} \frac{(1 + \frac{\delta^4}{4})}{(1 - \frac{\delta^2}{4})} \delta^{n-1}.
\]

**Proof of Theorem 1.1.**

For $p$ given, let $\delta = \frac{1}{\log p}$. Let $A(\delta)$ as defined in (2.10). Let $p_0$ be such that $p_0$ and $\delta = \frac{1}{\log p}$ satisfy the assumptions of Lemma 3.4, i.e. $\frac{p_0}{\log p_0} \geq \frac{4(n-1)h_n}{r_n}$. By Lemma 3.4 we have for all $p \geq p_0$,
\[
\frac{p}{\log p} (|Z_p^n(K)| - |K^n|) \geq \frac{1}{n} \int_{S^{n-1} \setminus A(\delta)} \frac{1}{h_{Z_p^n(K)}(\theta)} \left( \left( 1 - \frac{h_{Z_p^n(K)}(\theta)}{h_K^n(\theta)} \right) \right) d\sigma(\theta)
\]
\[
\geq \frac{1}{n} \int_{S^{n-1} \setminus A(\delta)} \frac{1}{h_{Z_p^n(K)}(\theta)} \left( \frac{n^2 \log p}{p} - (n-1)n \frac{\log \log p}{p} + c_{K,n} \right) d\sigma(\theta)
\]
\[
= \frac{1}{n} \int_{S^{n-1}} \frac{1}{h_{Z_p^n(K)}(\theta)} \left( \frac{n^2 \log p}{p} - (n-1)n \frac{\log \log p}{p} + c_{K,n} \right) d\sigma(\theta)
\]
\[
- \frac{1}{n} \int_{A(\delta)} \frac{1}{h_{Z_p^n(K)}(\theta)} \left( \frac{n^2 \log p}{p} - (n-1)n \frac{\log \log p}{p} + c_{K,n} \right) d\sigma(\theta).
\]

Hence,
\[
\frac{p}{\log p} (|Z_p^n(K)| - |K^n|) \geq \frac{1}{n} \int_{S^{n-1}} \frac{1}{h_{Z_p^n(K)}(\theta)} \left( n^2 - (n-1)n \frac{\log \log p}{\log p} + c_{K,n} \right) \frac{d\sigma(\theta)}{\log p}
\]
\[
- \frac{1}{n} \int_{A(\delta)} \frac{1}{h_{Z_p^n(K)}(\theta)} \left( n^2 - (n-1)n \frac{\log \log p}{\log p} + c_{K,n} \right) \frac{d\sigma(\theta)}{\log p}.
\]

Note that, since $K$ is centrally symmetric, $r(K) = \inf_{\theta \in S^{n-1}} h_K(\theta)$. Also, since $Z_p(K)$ converges to $K$, for $p$ sufficiently large, $h_{Z_p^n(K)}(\theta) \geq \left( \frac{r(K)}{2} \right)^n$ for every
\( \theta \in S^{n-1} \). Together with Lemma 3.5 we thus get
\[
\frac{1}{n} \int_{A(\delta)} \frac{1}{h_{Z_p(K)}(\theta)} d\sigma(\theta) \leq \frac{2^{n+1}}{n r(K)^n} \text{card}(B_K) \text{vol}_{n-1}(B_2^{n-1}) \delta^{n-1} \left(1 - \frac{\delta^2}{4}\right)^{\frac{n-1}{2}} \frac{(1 + \frac{\delta^2}{4})^{\frac{n}{2}}}{(1 - \frac{\delta^2}{4})^{\frac{n}{2}}}
\]
\[
\leq \frac{2^{n+1} \text{card}(B_K) \text{vol}_{n-1}(B_2^{n-1})}{n r(K)^n} \left(\frac{\log p}{\log p}\right)^{n-1}.\]

By Proposition 2.2 and Lebesgue’s convergence theorem we can interchange integration and limit and get
\[
\lim_{p \to \infty} \frac{p}{\log p} \left(\frac{|Z_p^n(K)| - |K^n|}{|K^n|}\right) \geq \frac{1}{n} \int_{S^{n-1}} \lim_{p \to \infty} \frac{1}{h_{Z_p(K)}(\theta)} \left(n^2 - \frac{(n-1)n \log \log p}{\log p} + \frac{c_{K,n}}{(\log p)^n}\right) d\sigma(\theta)
\]
\[
- \frac{2^{n+1} \text{card}(B_K) \text{vol}_{n-1}(B_2^{n-1})}{n r(K)^n} \lim_{p \to \infty} \left(\frac{n^2}{(\log p)^{n-1}} - \frac{(n-1)n \log \log p}{(\log p)^n} + \frac{c_{K,n}}{(\log p)^n}\right)
\]
\[
= n^2 |K^n|.
\]

Here, we have also used that \( \lim_{p \to \infty} h_{Z_p(K)}(\theta) = h_K(\theta) \).

The inequality from above follows by Proposition 2.2.

\(\square\)

### 4 Approximation with uniformly convex bodies

Let \( K \) be a symmetric convex body in \( \mathbb{R}^n \) and \( 2 \leq p < \infty \). We say that \( K \) is \( p \)-uniformly convex (with constant \( C_p \) (see e.g. \cite{3, 11}), if for every \( x, y \in \partial K \),
\[
\|x + y\|_K \leq 1 - C_p \|x - y\|_K^p.
\]

We will need the following Proposition. The proof is based on Clarkson inequalities and can be found in e.g. (\cite{3}, pp. 148).

**Proposition 4.1.** Let \( K \) be a compact set in \( \mathbb{R}^n \) of volume 1. Then for \( p \geq 2 \), \( Z_p^n(K) \) is \( p \)-uniformly convex with constant \( C_p = \frac{1}{p2^p} \).

The symmetric difference metric between two convex bodies \( K \) and \( C \) is
\[
d_s(C, K) = |(C \setminus K) \cup (K \setminus C)|.
\]

**Proof of Theorem 1.2**
Let $P_1 = \frac{P^o}{|P^o|}$. Then $P_1^o = |P^o|\frac{1}{P} P$ and $|P_1^o| = |P||P^o|$. Let $K_p = |P^o|^{-\frac{1}{p}} Z^o_p(P_1)$. Then by Proposition 4.1 we have that $K_p$ is uniformly convex. Note that $P \subseteq K_p$.

By Theorem 1.1 we have that
\[
\lim_{p \to \infty} \frac{p}{\log p} (|Z^o_p(P)| - |P^o|) = n^2 |P_1^o|.
\]

So, for every $\varepsilon > 0$, there exists $p_0(\varepsilon, P)$ such that
\[
d_s(P, K_p) = |K_p| - |P| = \frac{1}{|P^o|} (|Z^o_p(P_1)| - |P^o_1|) \leq (1 + \varepsilon)n^2 |P^o| \log p \frac{p}{p} = (1 + \varepsilon)n^2 |P^o| \log p \frac{p}{p}.
\]

We choose $\varepsilon = 1$ and the proof is complete. \hfill \Box

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