Moduli spaces of topological calibrations, 
Calabi-Yau, HyperKähler, 
$G_2$ and Spin(7) structures

RYUSHI GOTO
Department of Mathematics, 
Graduate School of Science, 
Osaka University,

Abstract. We shall obtain unobstructed deformations of four geometric structures: Calabi-Yau, HyperKähler, $G_2$ and Spin(7) structures in terms of closed differential forms (calibrations). We develop a direct and unified construction of smooth moduli spaces of these four geometric structures and show that the local Torelli type theorem holds in a systematic way.

§0. Introduction

There has been considerable interest recently in Riemannian manifolds with vanishing Ricci tensor. The list of holonomy group of Ricci-flat manifolds includes four interesting classes of the holonomy groups: SU($n$), Sp($m$), $G_2$ and Spin(7) [1]. The Lie group SU($n$) arises as the holonomy group of Calabi-Yau manifolds and Sp($m$) is the holonomy group of HyperKähler manifolds. The exceptional Lie group $G_2$ and the Lie group Spin(7) respectively occur as the holonomy groups of 7 and 8 dimensional manifolds. There are many interesting common properties
between these four geometries. One of the most remarkable property is that the deformation spaces of these geometric structures are smooth, i.e., unobstructed. It is also intriguing that there are smooth moduli spaces of these geometric structures, in which the local Torelli type theorem hold, so that is, these moduli spaces are locally described in terms of cohomology groups. Bogomorov, Tian and Todorov show that the deformation space (Kuranishi space) of Calabi-Yau structures is smooth by using Kodaira-Spencer theory [2],[25],[27]. The moduli space of polarized Calabi-Yau manifolds is constructed by Fujiki-Schmacher [8] from complex geometric point of view. Joyce obtains smooth moduli spaces of $G_2$ and Spin(7) structures respectively [13],[14],[15]. The construction of moduli spaces of $G_2$ and Spin(7) structures are different from the one of Calabi-Yau structures since $G_2$ and Spin(7) manifolds are real manifolds, in which we can not apply the deformation theory of complex manifolds. Hitchin shows a significant and suggestive construction of deformation spaces of Calabi-Yau structures on real 6 manifolds and $G_2$ structures on 7 manifolds [12]. It must be noted that these four geometries are defined by certain closed differential forms on real manifolds. From this point of view we shall obtain a direct and unified construction of smooth moduli spaces of these geometric structures. In the case of Calabi-Yau manifolds, we consider a real compact $2n$ manifold with a pair consisting of a closed complex $n$ form $\Omega$ and a symplectic form $\omega$. We show that a certain pair $(\Omega, \omega)$ defines a Calabi-Yau metric (Ricci-flat Kähler metric) on $X$. Hence the deformation space of Calabi-Yau metrics on $X$ arises as the deformation space of such pairs of closed forms $(\Omega, \omega)$ (see section 4-2 for precise definition of Calabi-Yau structures). In section 1, we discuss a general deformation theory of geometric structures defined by closed differential forms. Let $V$ be a real $n$ dimensional vector space. Then we consider the linear action $\rho$ of $G=\text{GL}(V)$ on the direct sum of skew-symmetric tensors,

\[ \rho: \text{GL}(V) \to \bigoplus_{i=1}^{l} \text{End}(\wedge^{p_i} V^*). \]

Let $\Phi^0 = (\phi^0_1, \phi^0_2, \cdots, \phi^0_l)$ be an element of $\bigoplus_{i=1}^{l} \wedge^{p_i} V^*$. Then we have
the $G$-orbit $\mathcal{O} = \mathcal{O}_{\Phi_0^V}(V)$:

$$\mathcal{O}_{\Phi_0^V}(V) = \{ \rho_g \Phi_0^V = (\rho_g \phi_1^0, \cdots, \rho_g \phi_l^0) \in \bigoplus_{i=1}^l \wedge^p V^* \mid g \in G \}.$$  

Then the orbit $\mathcal{O} = \mathcal{O}_{\Phi_0^V}$ is regarded as a homogeneous space $G/H$, where $H$ is the isotropy group. If the isotropy group $H$ is a subgroup of the orthogonal group $O(V)$ for a metric $g_V$ on $V$, we call $\mathcal{O}$ a metrical orbit. Let $X$ be a real $n$ dimensional compact manifold. Then we define a homogeneous space bundle $A_\mathcal{O}(X) \to X$ by

$$A_\mathcal{O}(X) = \bigcup_{x \in X} \mathcal{O}_{\Phi_0^V}(T_x X).$$

Then we define $\mathcal{E}_\mathcal{O}(X)$ to be the set of global sections $\Gamma(X, A_\mathcal{O}(X))$. The moduli space $\mathcal{M}_\mathcal{O}(X)$ is defined as the quotient

$$\mathcal{M}_\mathcal{O}(X) = \widetilde{\mathcal{M}}_\mathcal{O}(X)/\text{Diff}_0(X),$$

where

$$\widetilde{\mathcal{M}}_\mathcal{O}(X) = \{ \Phi \in \mathcal{E}(X) \mid d\Phi = 0 \}$$

and $\text{Diff}_0(X)$ denotes the identity component of diffeomorphisms of $X$. Let $\Phi_0$ be an element of $\widetilde{\mathcal{M}}_\mathcal{O}(X)$. Then we shall obtain a deformation complex $\#_{\Phi_0}$ (see section 1):

$$\Gamma(E^0) \xrightarrow{d_0} \Gamma(E^1) \xrightarrow{d_1} \Gamma(E^2) \xrightarrow{d_2} \cdots$$

If $\#_{\Phi_0}$ is an elliptic complex, the orbit $\mathcal{O}$ is called an elliptic orbit (see definition 1-1). As we shall show that this complex $\#_{\Phi_0}$ is a subcomplex of the direct sum of de Rham complex (for simplicity we call this the de Rham complex):

$$\Gamma(E^0) \xrightarrow{d_0} \Gamma(E^1) \xrightarrow{d_1} \Gamma(E^2) \xrightarrow{d_2} \cdots$$

$$\cdots \xrightarrow{d} \Gamma(\bigoplus_i \wedge^{p_i-1}) \xrightarrow{d} \Gamma(\bigoplus_i \wedge^{p_i}) \xrightarrow{d} \Gamma(\bigoplus_i \wedge^{p_i+1}) \xrightarrow{d} \cdots$$
Hence we have the map $p^k$ from the cohomology group of the complex $\#\Phi^0$ to the cohomology group of the de Rham complex:

$$p^k: H^k(\#\Phi^0) \longrightarrow \bigoplus_i H^{p_i-1-k}_{dR}(X).$$

If the maps $p^1$ and $p^2$ are respectively injective for any $\Phi^0 \in \widetilde{\mathcal{M}}_\mathcal{O}(X)$ on every compact $n$ dimensional manifold $X$, we call $\mathcal{O}$ a topological orbit.

Then we have the following theorems:

**Theorem 1-8.** If an orbit $\mathcal{O}$ is metrical, elliptic and topological, then the corresponding moduli space $\mathcal{M}_\mathcal{O}(X)$ is a smooth manifold. (In particular $\mathcal{M}_\mathcal{O}(X)$ is Hausdorff.) Further $\mathcal{M}_\mathcal{O}(X)$ has canonical coordinates given by an open ball of the cohomology group $H^1(\#\Phi)$.

Since de Rham cohomology group is invariant under the action of $\text{Diff}_0(X)$, we have the map

$$P: \mathcal{M}_\mathcal{O}(X) \longrightarrow \bigoplus_i H^i_{dR}(X).$$

Then we have

**Theorem 1-9.** If an orbit $\mathcal{O}$ is metrical, elliptic and topological, then the map $P$ is locally injective.

Further under the assumption that $\mathcal{O}$ is metrical, elliptic and topological we have

**Theorem 1-11.** Let $\widetilde{\mathcal{M}}_\mathcal{O}(X)$ be the set of closed elements of $\mathcal{E}$. We denote by $\text{Diff}(X)$ the group of diffeomorphisms of $X$. There is the action of $\text{Diff}(X)$ on $\widetilde{\mathcal{M}}_\mathcal{O}(X)$. Then the quotient $\mathcal{M}_\mathcal{O}(X)/\text{Diff}(X)$ is an orbifold.

In order to obtain these theorems, we study the problem of obstruction to deformations in our situation. $\mathcal{E}_\mathcal{O}(X)$ is regarded as a infinite dimensional homogenous space (a Hilbert manifold). Hence we have the tangent space $T_{\Phi^0}\mathcal{E}_\mathcal{O}(X)$ of $\mathcal{E}_\mathcal{O}(X)$. We denote by $\mathcal{H}$ the Hilbert space consisting of closed forms in $\bigoplus_i \wedge^{p_i}$ Then the space $\widetilde{\mathcal{M}}_\mathcal{O}(X)$ is the intersection between the Hilbert space $\mathcal{H}$ and the Hilbert manifold $\mathcal{E}_\mathcal{O}(X)$. We define an infinitesimal tangent space of $\widetilde{\mathcal{M}}_\mathcal{O}$ by the intersection $\mathcal{H} \cap T_{\Phi^0}\mathcal{E}_\mathcal{O}(X)$. Then we shall discuss if the infinitesimal tangent space is regarded as the tangent space of actual deformations.
Definition 1-6. A closed element $\Phi^0 \in \mathcal{E}(X)$ is unobstructed if there exists an integral curve $\Phi_t(\alpha)$ in $\widetilde{\mathcal{M}}_\mathcal{O}(X)$ for each infinitesimal tangent vector $\alpha \in \mathcal{H} \cap T_{\Phi^0} \mathcal{E}_\mathcal{O}(X)$ such that

$$\frac{d}{dt} \Phi_t(\alpha)|_{t=0} = \alpha$$

An orbit $\mathcal{O}$ is unobstructed if every $\Phi^0 \in \widetilde{\mathcal{M}}_\mathcal{O}(X)$ is unobstructed for any compact $n$ dimensional manifold $X$

We shall prove the following criterion in section 2.

Theorem 1-7 (Criterion of unobstructedness). We assume that an orbit $\mathcal{O}$ is elliptic (see definition 1-1 in section one). If the map $p^2 : H^2(\#_{\Phi^0}) \to \bigoplus_i H^{p_i+1}_{DR}(X)$ is injective, then $\Phi^0$ is unobstructed (see section 1 for $p^2$).

At first we try to construct a deformation of calibrations as a formal power series in $t$. Then we encounter obstructions to deformation of calibrations. A primary obstruction is discussed in subsection 2-1, which is given by a generalization of the Nijenhuis tensor (see subsection 2-0). If the primary obstruction vanishes, then we have the second obstruction. Successively we have higher obstructions to deformations. Explicit description of higher obstructions are given in subsection 2-2. In subsection 2-3, we prove our criterion of unobstructedness (Theorem 1-5). If the criterion holds, then all obstruction vanish simultaneously. Hence we have a deformation of calibrations as a formal power series in $t$. Further we prove the power series uniformly converges. Section 3 is devoted to prove main theorems. Our discussion is based on [7] and [22]. We have a smooth family of closed forms $\mathcal{S}_{\Phi^0}$, which is given by the deformation space in section 2. The injectivity of the map $p^1$ is essential to show that $\mathcal{S}_{\Phi^0}$ gives coordinates of the moduli space $\mathcal{M}_\mathcal{O}(X)$ in subsection 3-2. We also obtain the Hausdorff property of the moduli space in subsection 3-3. We will give the proof of theorems in subsection 3-4. In section 4, 5, 6 and 7 we shall show that Calabi-Yau, HyperKähler, $G_2$ and Spin(7) structures are metrical, elliptic and topological respectively.
In section 4-1 we define an $SL_n(\mathbb{C})$ structure as a certain complex form $\Omega$, which defines the almost complex structure $I_\Omega$ with trivial canonical line bundle. Then the integrability of the almost complex structure $I_\Omega$ is given by a closeness of the complex differential form $\Omega$. We show that the orbit of $SL_n(\mathbb{C})$ structures is elliptic and satisfies the criterion. In section 4-2, we define a Calabi-Yau structure as a certain pair consisting $SL_n(\mathbb{C})$ structure $\Omega$ and a real symplectic form $\omega$. Then we prove that the orbit corresponding to a Calabi-Yau structure is elliptic and topological. Hence we obtain the smooth moduli space of Calabi-Yau structures. A reference of Calabi-Yau manifolds is [1]. Our primary obstruction of $SL_n(\mathbb{C})$ structures corresponds to the one of Kodaira-Spencer theory. Then our result is regarded as another proof of unobstructedness by using calibrations. Our direct proof reveals a geometric meaning of unobstructed deformations. (we do not use Calabi-Yau’s theorem to obtain a smooth deformation space of Calabi-Yau structures). It must be noted that Kawamata and Ran give algebraic proof of unobstructed deformations.[16],[23]. In section 5, we show the orbit corresponding to a HyperKähler structure is also elliptic and topological. In section 6 and 7 we discuss $G_2$ and Spin (7) structures respectively.

§1. Moduli spaces of calibrations

Let $V$ be a real vector space of dimension $n$. We denote by $\wedge^p V^*$ the vector space of $p$ forms on $V$. Let $\rho_p$ be the linear action of $G = GL(V)$ on $\wedge^p V^*$. Then we have the action $\rho$ of $G$ on the direct sum $\bigoplus_i \wedge^p_i V^*$ by

$$\rho: GL(V) \longrightarrow \bigoplus_{i=1}^l \text{End}(\wedge^p_i V^*),$$

$$\rho = (\rho_{p_1}, \cdots, \rho_{p_l}).$$

We fix an element $\Phi^0_V = (\phi^0_1, \phi^0_2, \cdots, \phi^0_l) \in \bigoplus_i \wedge^p_i V^*$ and consider the $G$-orbit $\mathcal{O} = \mathcal{O}_{\Phi^0_V}$ through $\Phi^0_V$:

$$\mathcal{O}_{\Phi^0_V} = \{ \Phi_V = \rho g \Phi^0_V \in \bigoplus_i \wedge^p_i V^* | g \in G \}$$
The orbit $O_{\Phi^0_V}$ can be regarded as a homogeneous space,

$$O_{\Phi^0_V} = G/H,$$

where $H$ is the isotropy group

$$H = \{ g \in G \mid \rho_g \Phi^0_V = \Phi^0_V \}.$$  

We denote by $A_O(V) = A(V)$ the orbit $O_{\Phi^0_V} = G/H$. The tangent space $E^1(V) = T_{\Phi^0}A(V)$ is given by

$$E^1(V) = T_{\Phi^0}A(V) = \{ \rho_\xi \Phi^0 \in \bigoplus_i \wedge^{p_i} V^* \mid \xi \in \mathfrak{g} \},$$

where $\rho$ denotes the differential representation of $\mathfrak{g}$. The vector space $E^1(V)$ is the quotient space $\mathfrak{g}/\mathfrak{h}$. We also define a vector space $E^0(V)$ by the interior product,

$$E^0(V) = \{ i_v \Phi^0 = (i_v \phi^0_1, \ldots, i_v \phi^0_l) \in \bigoplus_i \wedge^{p_i-1} V^* \mid v \in V \}.$$

$E^2(V)$ is define as a vector space spanned by the following set,

$$E^2(V) = \text{Span} \{ \alpha \wedge i_v \Phi^0 \in \bigoplus_i \wedge^{p_i+1} V^* \mid \alpha \in \wedge^2 V^*, i_v \Phi^0 \in E^0(V) \}.$$

We also define $E^k(V)$ for $k \geq 0$ by

$$E^k(V) = \text{Span} \{ \beta \wedge i_v \Phi^0 \in \bigoplus_i \wedge^{p_i+k-1} V^* \mid \beta \in \wedge^k V^*, i_v \Phi^0 \in E^0(V) \}.$$  

Let $\{e_1, \ldots, e_n\}$ be a basis of $V$ and $\{\theta^1, \ldots, \theta^n\}$ the dual basis of $V^\ast$. Then we see that $\rho_\xi \Phi^0_V$ is written as

$$\rho_\xi \Phi^0_V = \sum_{ij} \xi^j_i \theta^j \wedge i_{e_i} \Phi^0_V,$$

where $\xi = \sum_{ij} \xi^j_i \theta^j \otimes e_i$ and $i_{e_i}$ denotes the interior product. Hence we have the graded vector space $E(V) = \bigoplus_k E^k(V)$ generated by $E^0(V)$ over $\wedge^* V^*$. Then we have the complex by the exterior product of a nonzero $u \in V^*$,

$$E^0(V) \xrightarrow{\wedge u} E^1(V) \xrightarrow{\wedge u} E^2(V) \xrightarrow{\wedge u} \cdots.$$
Definition 1-1 (elliptic orbits). An orbit $O_{\Phi^0_V}$ is an elliptic orbit if the complex

$$E^0(V) \overset{\wedge u}{\longrightarrow} E^1(V) \overset{\wedge u}{\longrightarrow} E^2(V) \overset{\wedge u}{\longrightarrow} \cdots$$

is exact for any nonzero $u \in V^*$. In other words, if $\alpha \wedge u = 0$ for $\alpha \in E^k(V)$, then there exists $\beta \in E^{k-1}(V)$ such that $\alpha = \beta \wedge u$ for $k = 1, 2$.

Remark. If $\alpha \wedge u = 0$, then we have $\alpha = \beta \wedge u$ for some $\beta \in \bigoplus_i \wedge^{p_i - 1}$ since the de Rham complex is elliptic. However $\beta$ is not an element of $E^0(V)$ in general, (Note that $E^0(V)$ is a subspace of $\bigoplus_i \wedge^{p_i - 1}$). For instance, we take $\Phi^0_V$ as a real symplectic form $\omega$ on a real $2n$ dimensional vector space $V$. Then $E^0 = \wedge^1$ and $E^1 = \wedge^2$. Hence $O_{\Phi^0_V}$ is elliptic. However if $\Phi^0_V$ is a degenerate 2 form on $V$, i.e., $\omega^n = 0$, then $O_{\Phi^0_V}$ is not elliptic.

Definition 1-2 (metrical orbits). Let $O_{\Phi^0_V}$ be an orbit as before. An orbit $O_{\Phi^0_V}$ is metrical if the isotropy group $H$ is a subgroup of $O(V)$ with respect to a metric $g_V$ on $V$.

Let $X$ be a compact real manifold of dimension $n$. We define $A_{\circ}(T_xX)$ by using an identification $h: T_xX \cong V$. The subspace $A_{\circ}(T_xX) \subset \bigoplus_i \wedge^{p_i} T^*_x X$ is independent of a choice of an identification $h$. Hence we define the $\mathbb{G}/H$-bundle $A(X)(= A_{\circ}(X))$ by

$$A_{\circ}(X) = \bigcup_{x \in X} A(T_xX) \longrightarrow X.$$ 

We denote by $\mathcal{E}(= \mathcal{E}(X))$ the set of $C^\infty$ global sections of $A(X)$,

$$\mathcal{E}(X) = \Gamma(X, A(X)).$$

Let $\Phi^0$ be a closed element of $\mathcal{E}$. Then we have the vector spaces $E^k(T_xX)$ for each $x \in X$ and $k \geq 0$. We define the vector bundle $E^k_X(= E^k)$ over $X$ as

$$E^k_X := \bigcup_{x \in X} E^k(T_xX) \longrightarrow X.$$
for each $k \geq 0$. (Note that the fibre of $E^1$ is $g/h$.) Then we define
the graded module $\Gamma(E)$ over $\Gamma(\wedge^\ast)$ as $\oplus_k \Gamma(E^k)$, where $\Gamma$ denotes the set of
global $C^\infty$ sections and $\wedge^p$ is the sheaf of germs of smooth $p$ forms on
$X$.

**Theorem 1-3.** $\Gamma(E)$ is the differential graded module in $\oplus_k \Gamma(\oplus_i \wedge^{p_i+k-1})$
with respect to the exterior derivative $d$.

**Proof.** Since $\Gamma(E)$ is the graded module generated by $\Gamma(E^0)$, it is suffices
to prove that $di_v\Phi^0$ is an element of $\Gamma(E^1)$ for $v \in \Gamma(TX)$. We denote
by Diff($X$) the group of diffeomorphisms of $X$. Then there is the action
of Diff($X$) on differential forms on $X$ and we see that $\mathcal{E}(X)$ is invariant
under the action of Diff($X$). An element of $\Gamma(E^0)$ is given as $i_v\Phi^0 = (i_v\phi_1, \cdots, i_v\phi_l)$, where $v \in \Gamma(TX)$. Since $\Phi^0$ is closed, we have

$$di_v\Phi^0 = L_v\Phi^0.$$ 

The vector field $v$ generates the one parameter group of transformation
$f_t$. Then $L_v\Phi^0 = \frac{d}{dt} f_t^*\Phi^0|_{t=0}$. Since $\mathcal{E}(X)$ is invariant under the action
of Diff($X$), $f_t^*(\Phi^0) \in \mathcal{E}(X)$. Since the tangent space of $\mathcal{E}$ at $\Phi^0$ is $\Gamma(E^1)$,
$L_v\Phi^0 \in \Gamma(E^1)$. Hence $di_v\Phi^0 \in \Gamma(E^1)$. From definition of $E^k(V)$, we see
that $da \in \Gamma(E^k)$ for all $a \in \Gamma(E^{k-1})$ for all $k$. □

Then from theorem 1-3, we have a complex $\#_{\Phi^0}$

$$(\#_{\Phi^0}) \quad \Gamma(E^0) \xrightarrow{d_0} \Gamma(E^1) \xrightarrow{d_1} \Gamma(E^2) \xrightarrow{d_2} \cdots,$$

where $\Gamma(E^i)$ is the set of $C^\infty$ global sections for each vector bundle and
$d_i = d|_{E^i}$ for each $i = 0, 1, 2$. The complex $\#_{\Phi^0}$ is a subcomplex of the
direct sum of the de Rham complex (For simplicity, we call this complex
the de Rham complex):

$$\Gamma(E^0) \xrightarrow{d_0} \Gamma(E^1) \xrightarrow{d_1} \Gamma(E^2) \xrightarrow{d_2} \cdots$$

$$\cdots \xrightarrow{d} \Gamma(\oplus_i \wedge^{p_i-1}) \xrightarrow{d} \Gamma(\oplus_i \wedge^{p_i}) \xrightarrow{d} \Gamma(\oplus_i \wedge^{p_i+1}) \xrightarrow{d} \cdots.$$
If $\mathcal{O}$ is an elliptic orbit, the complex $\#_{\Phi^0}$ is an elliptic complex for all closed $\Phi^0 \in \mathcal{E}^1$ on any $n$ dimensional compact manifold $X$ (Note that the complex in definition 1-1 is the symbol complex of $\#_{\Phi^0}$). Then we have a finite dimensional cohomology group $H^k(\#_{\Phi^0})$ of the elliptic complex $\#_{\Phi^0}$. Since $\#_{\Phi^0}$ is a subcomplex of deRham complex, there is the map $p^k$ from the cohomology group of the complex $\#_{\Phi^0}$ to de Rham cohomology group:

$$p^k : H^k(\#_{\Phi^0}) \longrightarrow \bigoplus_i H^{p_i-k+1}(X, \mathbb{R}).$$

where

$$H^k(\#_{\Phi^0}) = \{ \alpha \in \Gamma(E^k) \mid d_k \alpha = 0 \}/\{ d\beta \mid \beta \in \Gamma(E^{k-1}) \}.$$

**Definition 1-4 (Topological calibrations and topological orbits).** A closed element $\Phi^0 \in \mathcal{E}(X)$ is a topological calibration if the map

$$p^k : H^k(\#_{\Phi^0}) \longrightarrow \bigoplus_i H^{p_i+k-1}(X, \mathbb{R})$$

is injective for $k = 1, 2$. An orbit $\mathcal{O}$ in $\bigoplus_i \wedge^i V^*$ is topological over a manifold $X$ if any closed element of $\mathcal{E}(X)$ is a topological calibration. An orbit $\mathcal{O}$ is topological if $\mathcal{O}$ is topological over any compact $n$ dimensional manifold $X$.

**Lemma 1-5.** Let $\mathcal{O}$ be a metrical orbit and $\Phi^0$ an element of $\mathcal{E} = \Gamma(X, \mathcal{A}_\mathcal{O}(X))$. Then there is a canonical metric $g_{\Phi^0}$ on $X$ corresponding to each $\Phi^0$.

**Proof.** The orbit $\mathcal{O}$ is defined in terms of $\Phi^0 \in \bigoplus_i \wedge^i V^*$ on $V$. We also have $\Phi^0(x) \in \mathcal{A}(T_xX)$ on each tangent space $T_xX$. Let $\text{Isom}(V, T_xX)$ be the set of isomorphisms between $V$ and $T_xX$. Then define $H_x$ by

$$H_x = \{ h \in \text{Isom}(V, T_xX) \mid \Phi^0 = h^*\Phi^0(x) \}.$$

Then we see that $H_x$ is isomorphic to the isotropy group $H$. $h^*g_V$ defines the metric on the tangent space $T_xX$ for $h \in H_x$. Since $H$ is a subgroup of $\text{O}(V)$, the metric $h^*g_V$ does not depend on a choice of $h \in H_x$. $\square$
Let $O$ be an orbit in $\bigoplus_i \wedge^{p_i} V^*$. Then we define the moduli space $\mathcal{M}_O(X)$ by

$$\mathcal{M}_O(X) = \{ \Phi \in \mathcal{E} \mid d\Phi = 0 \} / \text{Diff}_0(X),$$

where $\text{Diff}_0(X)$ is the identity component of the group of diffeomorphisms for $X$. We denote by $\tilde{\mathcal{M}}_O(X)$ the set of closed elements in $\mathcal{E}$:

$$\tilde{\mathcal{M}}_O(X) = \{ \Phi \in \mathcal{E}(X) \mid d\Phi = 0 \}.$$

We have the natural projection $\pi: \tilde{\mathcal{M}}_O(X) \to \mathcal{M}_O(X)$. Let $\Phi^0$ be an element of $\tilde{\mathcal{M}}_O(X)$. As we shall show in section 2, $\mathcal{E}(X)$ is regarded as an infinite dimensional homogenous space (a Hilbert manifold). Hence we have the tangent space $T_{\Phi^0}\mathcal{E}(X)$. We denote by $\mathcal{H}$ the Hilbert space consisting of closed forms. Then the space $\tilde{\mathcal{M}}_O(X)$ is the intersection between the Hilbert space $\mathcal{H}$ and the Hilbert manifold $\mathcal{E}(X)$. We define an infinitesimal tangent space of $\tilde{\mathcal{M}}_O$ by the intersection $\mathcal{H} \cap T_{\Phi^0}\mathcal{E}$. Since $T_{\Phi^0}\mathcal{E}(X) = E^1$, the infinitesimal tangent space is written as

$$\mathcal{H} \cap T_{\Phi^0}\mathcal{E}(X) = \mathcal{H} \cap E^1.$$

Then we shall discuss if the infinitesimal tangent space is regarded as the tangent space of actual deformations.

**Definition 1-6.** A closed element $\Phi^0 \in \mathcal{E}(X)$ is unobstructed if there exists an integral curve $\Phi_t(a)$ in $\tilde{\mathcal{M}}(X)$ for each $a \in \mathcal{H} \cap E^1$ such that

$$\frac{d}{dt} \Phi_t(a)|_{t=0} = a.$$

An orbit $O$ is unobstructed if any $\Phi^0 \in \tilde{\mathcal{M}}_O(X)$ is unobstructed for any compact $n$ dimensional manifold $X$. (see section 2 for the precise statement with respect to Sobolev norms.)

The following figures (i),(ii) and (iii) explain our situation well. If the Hilbert space $\mathcal{H}$ is in a generic position, the intersection $\tilde{\mathcal{M}}(X) = \mathcal{E} \cap \mathcal{H}$ is
smooth and every element $\Phi^0$ is unobstructed (see figure (i)). However if $\mathcal{H}$ is in a special position, $\widetilde{\mathcal{M}}(X)$ may be singular. (see figure (ii)). Then the infinitesimal tangent space may not gives actual deformations. In figure (ii), the point $p$ is singular and the infinitesimal tangent space at $p$ coincides with $\mathcal{H}$. Figure (iii) shows this problem is subtle. The intersection $\mathcal{E}$ is a line and it seems to be non-singular. However the infinitesimal tangent space is $\mathcal{H}$ at each point, which is obstructed.

We shall prove the following theorems in section 2.

**Theorem 1-7 (Criterion of unobstructedness).** We assume that an orbit $\mathcal{O}$ is elliptic. If the map $p^2 : H^2(\#_{x^0}) \to \oplus_i H^{p_i+1}_{DR}(X)$ is injective, then $\Phi^0$ is unobstructed.

We shall prove the following theorems in section 3.
**Theorem 1-8.** If an orbit \( \mathcal{O} \) is metrical, elliptic and topological, then the corresponding moduli space \( \mathcal{M}_\mathcal{O}(X) \) is a smooth manifold. (In particular \( \mathcal{M}_\mathcal{O}(X) \) is Hausdorff.) Further \( \mathcal{M}_\mathcal{O}(X) \) has canonical coordinates given by an open ball of the cohomology group \( H^1(\#_{\Phi}) \).

Since de Rham cohomology group is invariant under the action of \( \text{Diff}_0(X) \), we have the map

\[
P: \mathcal{M}_\mathcal{O}(X) \to \bigoplus_i H^i_{dR}(X).
\]

Then we have

**Theorem 1-9.** If an orbit \( \mathcal{O} \) is metrical, elliptic and topological, then the map \( P \) is locally injective.

Further under the assumption that \( \mathcal{O} \) is metrical, elliptic and topological we have

**Theorem 1-10.** Let \( I(\Phi) \) be the isotropy group,

\[
I(\Phi) = \{ f \in \text{Diff}_0(X) \mid f^*\Phi = \Phi \}.
\]

Then there is a sufficiently small slice \( S_{\Phi_0} \) at \( \Phi^0 \) such that the isotropy group \( I(\Phi^0) \) is a subgroup of \( I(\Phi) \) for each \( \Phi \in S_{\Phi_0} \), i.e.,

\[
I(\Phi^0) \subset I(\Phi).
\]

(Our definition of the slice will be given in section 2 and 3.)

**Theorem 1-11.** Let \( \tilde{\mathcal{M}}_\mathcal{O}(X) \) be the set of closed elements of \( \mathcal{E}^1 \). We denote by \( \text{Diff}_0(X) \) the group of diffeomorphism of \( X \). There is the action of \( \text{Diff}_0(X) \) on \( \tilde{\mathcal{M}}_\mathcal{O}(X) \). Then the quotient \( \tilde{\mathcal{M}}_\mathcal{O}(X)/\text{Diff}_0(X) \) is an orbifold.
§2. Deformations of calibrations

§2-0 Preliminary results. Let $X$ be a manifold and we denote by $\wedge^*$ the differential forms on $X$. Let $P$ be a linear operator acting on $\wedge^*$. Then the operator $P: \wedge^* \to \wedge^*$ is a derivative if $P$ satisfies the followings:

$$P(s + t) = P(s) + P(t),$$
$$P(s \wedge t) = P(s) \wedge t + s \wedge P(t),$$

where $s, t \in \wedge^*$. An anti-derivative $Q$ is also a linear operator defined by the following:

$$Q(s + t) = Q(s) + Q(t),$$
$$Q(s \wedge t) = Q(s) \wedge t + (-1)^{|s|} s \wedge Q(t),$$

where $|s|$ denotes the degree of a differential form $s$. Then the exterior derivative $d$ is the anti-derivative and the differential representation $\hat{\rho}_a$ is a derivative for each $a \in \text{End}(TX)$.

Lemma 2-0-1. The commutator $[\hat{\rho}_a, d] = \hat{\rho}_a \circ d - d \circ \hat{\rho}_a$ is the anti-derivative. We denote by $L_a$ the commutator $[\hat{\rho}_a, d]$.

Proof. In general the commutator of a derivative $P$ and an anti-derivative $Q$ is an anti-derivative if $Q$ preserves degrees of differential forms.

The operator $L_a$ is regarded as a generalizations of the Lie derivative. Indeed we have

Lemma 2-0-2. The commutator $L_a$ is expressed as

$$L_a: \wedge^n \longrightarrow \wedge^{n+1},$$

$$L_a \eta(u_0, u_1, \ldots, u_n) = \sum_{i=0}^{n} (-1)^i L_a u_i \eta(u_0, \tilde{i}, \ldots, u_n),$$

$$- \sum_{i<j} (-1)^{i+j} \eta(a[u_i, u_j], u_0, \tilde{i}, \tilde{j}, \ldots, u_n)$$
where $\eta$ is an $n$ form and $a \in \text{End}(TX)$ maps a vector $u_i$ to $au_i \in TX$ and we denote by $L_{au_i}$ the ordinary Lie derivative.

Proof. It is sufficient to show the lemma with respect to vectors $\{u_i\}$ satisfying $[u_i, u_j] = 0$. Then we have

\[
(\hat{\rho}_a d\eta)(u_0, \cdots u_n) = \sum_i (-1)^i (i_{au_i} d\eta)(u_0, \cdots, u_n)
\]

\[
(d\hat{\rho}_a \eta)(u_0, \cdots, u_n) = -\sum_i (-1)^i (di_{au_i} \eta)(u_0, \cdots, u_n).
\]

Hence from $L_{au_i} = di_{au_i} + i_{au_i} d$, we have the result.

we also have a description of the commutator between $L_a$ and $\hat{\rho}_a$.

Lemma 2-0-3.

\[
[L_a, \hat{\rho}_b] = i_{N(a,b)} - L_{ab},
\]

where $a, b \in \text{End}(TX) \cong \wedge^1 \otimes T$ and a tensor $N(a, b) \in \wedge^2 \otimes T$ is given by the following

\[
N(a, b)(u, v) = ab[u, v] + ba[u, v] + [au, bv] - [av, bu]
\]

\[
- a[bv, u] + a[bv, u] - b[au, v] + b[av, u],
\]

for $u, v \in TX$, and $i_{N(a,a)}$ is the composition of the interior product and the wedge product of the tensor $N(a, a) \in \wedge^2 \otimes TX$.

Remark. The tensor $N(a, b)$ is a generalization of the Nijenhuis tensor.

Proof of lemma 2-0-3. For $a, b \in \text{End}(TX)$, we have the tensor $N(a, b) \in \wedge^2 \otimes TX$. Then $i_{N(a,b)}$ is the linear operator from $\wedge^* \to \wedge^{*+1}$. We see that $i_{N(a,b)}$ is an anti-derivative. By lemma 2-0-1, $L_{ab}$ is an anti-derivative, where $ab$ denotes the composition of endmorphisms. As in proof of lemma 2-0-1, the commutator $[L_a, \hat{\rho}_a]$ is also an anti-derivative. Hence it sufficient to show that the identity in lemma 2-0-3 for functions and 1 forms. For a function $f$, we have $[L_a, \hat{\rho}_a]f = -\hat{\rho}_a L_a f = -L_{a^2} f$. 
Since $i_{N(a,b)}f = 0$, we have the identity. For a one form $\theta$ by lemma 2-0-2, we have

\[
L_a \hat{\rho}_b \theta(u,v) = (L_{au} \hat{\rho}_b \theta)(v) - (L_{av} \hat{\rho}_b \theta)(u) + \hat{\rho}_b \theta(\hat{\rho}_a[u,v]) \\
= au(\hat{\rho}_b \theta(v)) - av(\hat{\rho}_a \theta(u)) + \theta(ba[u,v]) \\
- \hat{\rho}_b \theta([au,v]) + \hat{\rho}_b \theta([av,u]).
\]

\[
\hat{\rho}_b L_a \theta(u,v) = (L_a \theta)(\hat{\rho}_b u,v) + (L_a \theta)(u, \hat{\rho}_b v) \\
= (L_{abu} \theta)(v) - (L_{av} \theta)(\hat{\rho}_b u) + \theta(a[bu,v]) \\
+ (L_{au} \theta)(\hat{\rho}_b v) - (L_{abv} \theta)(u) + \theta(a[u,bv]) \\
= (abu)(\theta v) - (av)(\theta(bu)) + \theta(a[bu,v]) \\
- \theta([abu,v]) + \theta([av,bu]) + \theta(a[u,bv]) \\
+ (au)\theta(bv) - (abv)\theta(u) \\
- \theta([au,bv]) + \theta([abv,u])
\]

Hence the commutator is given by

\[
[L_a, \hat{\rho}_b] \theta(u,v) = - (abu)(\theta v) + \theta([abu,v]) + (abv)\theta(u) - \theta([abv,u]) \\
+ \theta(ba[u,v]) + \theta([au,bv]) - \theta([av,bu]) \\
- \theta(a[bu,v]) + \theta(a[bv,u]) - \theta(b[au,v]) + \theta(b[av,u]) \\
= - L_{ab} \theta(u,v) + i_{N(a,b)}\theta
\]

\[\square\]

**Lemma 2-0-4.** We assume that $\Phi$ and $\hat{\rho}_a \Phi$ are closed forms respectively. Then $d\hat{\rho}_a \hat{\rho}_a \Phi$ is an element of $\Gamma(E^2)$.

**Proof.**

\[
d\hat{\rho}_a \hat{\rho}_a \Phi = \hat{\rho}_a d\hat{\rho}_a \Phi - L_a \hat{\rho}_a \Phi = - L_a \hat{\rho}_a \Phi \\
= - \hat{\rho}_a L_a \Phi - i_{N(a,a)}\Phi + L_a^2 \phi.
\]
Since $L_a \Phi = \hat{\rho}_a d\Phi - d\hat{\rho}_a \Phi = 0$, we have
$$d\hat{\rho}_a \hat{\rho}_a \Phi = -i_{N(a,a)} \Phi + L_a^2 \Phi.$$ Since $N(a, a) \in \wedge^2 \otimes T \cong \wedge^1 \otimes \text{End}(TX)$, then it follows from our definition of $E^2$ that
$$i_{N(a,a)} \Phi \in \Gamma(E^2).$$ Since $L_a^2 \Phi = -d\hat{\rho}_a^2 \Phi \in d\Gamma(E^1) \subset \Gamma(E^2)$. Hence we have the result. $\square$

We denote by $G = G(a,a)$ the operator $i_{N(a,a)} - L_a^2$. Then we consider the commutator $[\hat{\rho}_a, G(a,a)]$. For simplicity we write this by $Ad_{\hat{\rho}_a} G(a,a)(= Ad_{\hat{\rho}_a} G)$,
$$Ad_{\hat{\rho}_a} G(a,a) = [\hat{\rho}_a, G(a,a)].$$

The $k$th composition of commutator is denoted by
$$Ad^k_{\hat{\rho}_a} G = [\hat{\rho}_a, [\hat{\rho}_a, \cdots [\hat{\rho}_a, G(a,a)], \cdots]],$$ where $Ad_{\hat{\rho}_a} G(a,a)$ acts on differential forms.

**Lemma 2-0-5.** $Ad^k_{\hat{\rho}_a} G(a,a)\Phi^0$ is an element of $\Gamma(E^2)$.

**Proof.** At first we consider $Ad_{\hat{\rho}_a} G(a,a)\Phi^0$. By lemma 2-0-3, we have
$$Ad_{\hat{\rho}_a} G(a,a)\Phi^0 = [\hat{\rho}_a, G(a,a)]\Phi^0$$
$$= [\hat{\rho}_a, i_{N(a,a)}] \Phi^0 - [\hat{\rho}_a, L_a^2] \Phi^0$$
$$= [\hat{\rho}_a, i_{N(a,a)}] \Phi^0 + G(a^2, a)\Phi^0.$$ Since $N(a^2, a) \in \wedge^1 \otimes \text{End}(TX)$, as in lemma 2-0-4, $G(a^2, a)\Phi^0$ is an element of $\Gamma(E^2)$. We see that $[\hat{\rho}_a, i_{N(a,a)}]$ is given by the interior product of the tensor $\hat{\rho}_a(N(a,a)) \in \wedge^1 \otimes \text{End}(TX)$, where $\hat{\rho}_a$ acts on the tensor $N(a,a)$. Hence $[\hat{\rho}_a, i_{N(a,a)}] \Phi^0$ is an element of $\Gamma(E^2)$. Therefore $Ad_{\hat{\rho}_a} G(a,a)\Phi^0 \in \Gamma(E^2)$. By induction, we see that $Ad^k_{\hat{\rho}_a} G(a,a)\Phi^0$ is an element of $\Gamma(E^2)$. $\square$
§2-1 Primary obstruction. In this section we use the same notation as in section 1 and subsection 2-0. The references for analytic tools are found in [7],[9],[18], and [19]. Let $X$ be a real $n$ dimensional compact manifold. We fix a Riemannian metric $g$ on $X$. (Note that this metric does not depend on calibration $\Phi$.) We denote by $C^\infty(X, \wedge^p)$ the set of smooth $p$ forms on $X$. Let $C^\infty_\mathcal{O}(X, \wedge^p)$ be the set of smooth $p$ forms on $X$. Let $L^2_s(X, \wedge^p)$ be the Sobolev space and suppose that $s > k + \frac{n}{2}$, i.e., the completion of $C^\infty(X, \wedge^p)$ with respect to the Sobolev norm $\| \cdot \|_s$, where $k$ is sufficiently large (see [9] for instance). Then we have the inclusion $L^2_s(X, \wedge^p) \hookrightarrow C^k(X, \wedge^n)$. We define $\mathcal{E}_s$ by

\begin{equation}
\mathcal{E}_s = C^k(X, \mathcal{A}_\mathcal{O}(X)) \cap L^2_s(X, \oplus_{i=1}^l \wedge^{p_i}).
\end{equation}

Then we have

**Lemma 2-1-1.** $\mathcal{E}_s$ is a Hilbert manifold (see [19] for Hilbert manifolds). The tangent space $T_{\Phi^0} \mathcal{E}_s$ at $\Phi^0$ is given by

$$T_{\Phi^0} \mathcal{E}_s = L^2_s(X, E^1).$$

**Proof.** We denote by $\exp$ the exponential map of Lie group $G = \text{GL}(n, \mathbb{R})$. Then we have the map $k_x$

\begin{equation}
k_x : E^1(T_x X) \rightarrow \mathcal{A}(T_x X),
\end{equation}

by

\begin{equation}
k_x(\hat{\rho}_\xi \Phi^0(x)) = \rho_{\exp \xi} \Phi^0(x).
\end{equation}

for each tangent space $T_x X$. From 2-1-2, we have the map $k$

\begin{equation}
k : L^2_s(E^1) \rightarrow \mathcal{E}_s,
\end{equation}

by

$$k|_{E^1(T_x X)} = k_x.$$
The map $k$ defines local coordinates of $\mathcal{E}_s$. □

Let $\text{GL}(TX)$ be the group of gauge transformations, i.e., for $g \in \text{GL}(TX)$ we have the diagram:

$$
\begin{array}{ccc}
TX & \xrightarrow{g} & TX \\
\downarrow & & \downarrow \\
X & \xrightarrow{id} & X
\end{array}
$$

An element $g \in \text{GL}(TX)$ acts on $\mathcal{E}_\mathcal{O}(X)$ by

$$
\Phi \mapsto \rho_g(\Phi)
$$

The tangent space $T_{\Phi^0}\mathcal{E}(X)$ is given by $E^1(X)$,

$$
E^1(X) = \{ \hat{\rho}_a\Phi^0 \mid a \in \text{End}(TX) \}
$$

where $\hat{\rho}$ is the differential representation of $\rho$. We denote by $H(TX)$ be the gauge transformations with structure group $H$, i.e., the isotropy group. Then by lemma 2-1-1, $\mathcal{E}$ is regarded as the infinite dimensional homogenous space $\text{GL}(TX)/H(TX)$. Let $\mathcal{H}$ be the closed subspace of $L^2_s(X, \oplus_{i=1}^l \wedge^{p_i})$ consisting of closed forms. Then $\widetilde{\mathcal{M}}_s(X)$ is the intersection between $\mathcal{E}$ and $\mathcal{H}$. The image $dE^0(X)$ is given by

$$
dE^0(X) = \{ dv\Phi^0 = L_v\Phi^0 \mid v \in TX \},
$$

where $L_v$ is the Lie derivative with respect to $v \in TX$. Hence the cohomology $H^1(\#)$ of the complex $\#_{\Phi^0}$ is considered as the infinitesimal tangent space of the moduli space $\mathcal{M}(X) = \widetilde{\mathcal{M}}(X)/\text{Diff}_0(X)$. However, the moduli space may not be a manifold in general. This is because the infinitesimal tangent space may not be exponentiate the actual deformations. Then there exists an obstruction. This is a general problem of deformation. In our situation, we must study the intersection $\mathcal{E} \cap \mathcal{H}$. In order to obtain a deformation space, we shall construct a deformation of...
Φ₀ in terms of a power series in \( t \). We consider a formal power series in \( t \):

\[
(2-1-5) \quad a(t) = a₁t + \frac{1}{2!}a₂t^2 + \frac{1}{3!}a₃t^3 + \cdots \in \text{End}(TX)[[t]],
\]

where \( a_k \in \text{End}(TX) \). We define a formal power series \( g(t) \) by,

\[
g(t) = \exp a(t) \in GL(TX)[[t]]
\]

For simplicity, we put \( a = a(t) \). The gauge group \( GL(TX) \) acts on differential forms by \( ρ \). This action \( ρ \) is written in terms of the differential representation \( \hat{ρ} \),

\[
ρ_g(t)Φ₀ = Φ₀ + \hat{ρ}_aΦ₀ + \frac{1}{2!}\hat{ρ}_a\hat{ρ}_aΦ₀ + \frac{1}{3!}\hat{ρ}_a\hat{ρ}_a\hat{ρ}_aΦ₀ + \cdots
\]

(2-1-6)

\[
= Φ₀ + \hat{ρ}_{a₁}Φ₀t + \frac{1}{2}(\hat{ρ}_{a₂}Φ₀ + \hat{ρ}_{a₁}\hat{ρ}_{a₁}Φ₀)t² + \cdots,
\]

where \( \hat{ρ} \) is just written as

\[
\hat{ρ}_{a(t)}Φ₀ = \sum_{k=1}^{∞} \frac{1}{k!}\hat{ρ}_{a_k}Φ₀ t^k.
\]

The equation what we want to solve is ,

(eq*)

\[
dρ_{g(t)}Φ₀ = 0.
\]

We must find a power series \( a = a(t) \) satisfying the condition (eq*). At first we take \( a₁ \) such that \( d\hat{ρ}_{a₁}Φ₀ = 0 \). Then it remains to determine \( a₂, a₃, \cdots \) satisfying (eq*). \( dρ_{g(t)}Φ₀ \) is written as a power series,

(2-1-7)

\[
dρ_{g(t)}Φ₀ = \sum_{k=1} {\frac{1}{k!}dR_k t^k},
\]
where $R_k$ denotes the homogenous part of degree $k$. Hence the equality $d\rho_g(t)\Phi^0 = 0$ is reduced to the system of infinitely many equations

\[(eq_k) \quad dR_k = 0, \quad k = 1, 2, \ldots\]

By our assumption $d\hat{\rho}_{a_1} \Phi^0 = 0$, we already have $dR_1 = 0$ (see (2-1-6)). Thus in order to obtain $a(t)$, it suffices to determine $a_k$ satisfying (eq_k) by induction on $k$. By (2-1-6), the term of the second order $dR_2$ is given as

\[(2-1-8) \quad dR_2 = \frac{1}{2!} (d\hat{\rho}_{a_2} \Phi^0 + d\hat{\rho}_{a_1} \hat{\rho}_{a_1} \Phi^0)\]

We denote by $Ob_2(a_1)$ the quadratic term,

\[(2-1-9) \quad Ob_2(a_1) = \frac{1}{2!} (d\hat{\rho}_{a_1} \hat{\rho}_{a_1} \Phi^0)\]

Then by lemma 2-0-4 in section 2-0, $Ob_2$ is an element of $\Gamma(E^2)$, which is explicitly written as

\[(2-1-10) \quad Ob_2(a_1) = -\frac{1}{2!} (-i^*_N(a_1,a_1) + L_{a_1^2})\Phi^0,\]

Since $Ob_2(a_1)$ is a $d$-closed form, this defines a representative of the cohomology group $H^2(\#)$. In order to determine $a_2$ satisfying $dR_2 = 0$, we must solve the equation,

\[(eq_2) \quad \frac{1}{2!} d\hat{\rho}_{a_2} \Phi^0 = -Ob_2(a_1).\]

The L.H.S of (eq_2) cohomologically vanishes in $H^2(\#)$. Hence if the class $[Ob_2(a_1)] \in H^2(\#)$ does not vanishes, there exists no solution $a_2$ of eq_2 and no deformation with $a_1$. In this sense we call the class $[Ob_2(a_1)]$ the obstruction to deformation of $\Phi^0$ (the primary obstruction). If $[Ob_2(a_1)]$ vanishes, then we have a solution $a_2$ by

\[(2-1-11) \quad \frac{1}{2!} \hat{\rho}_{a_2} \Phi^0 = -d_1^* G_\#(Ob_2(a_1)),\]

where $G_\#$ denotes the Green operator of the complex $\#$. It is quite remarkable that the representative $Ob_2(a_1)$ is $d$-exact form. Hence $Ob_2(a)$ is in kernel of the map $p^2: H^2(\#) \to \bigoplus_i H^{p_i+1}(X)$. Hence we obtain a nice criterion of unobstructedness.
Theorem 2-1-2. If the map $p^2 : H^2(\#) \to \oplus_i H^{p_i+1}(X)$ is injective, the obstruction class $[Ob_2(a_1)]$ vanishes.

§2-2 Higher obstructions

Similarly we obtain infinitely many obstructions to deformation of $\Phi^0$. We define an operator $G(a, a)$ on $\wedge^*$ by

$$G(a, a) = i_{N(a, a)} - L_a^2,$$

where $a = a(t) \in \text{End}(TX)[t]$. We denote its $k$th homogenous part by $G(a, a)_k$. Then by lemma 2-0-4, we have

$$Ob_2(a_1) = -\frac{1}{2!} G(a, a)_2.$$  

We assume that $a_1, a_2, \cdots a_{k-1}$ are determined satisfying $dR_1 = 0, dR_2 = 0, \cdots, dR_{k-1} = 0$. Then $dR_k$ is written as a $d$-exact form:

$$dR_k = d\hat{\rho}_{a_k} \Phi^0 + \sum_{l=2}^{k} \frac{1}{l!} (d\hat{\rho}^l_{a})_k \Phi^0,$$

where $(d\hat{\rho}^l_{a})_k$ denotes the $k$th homogeneous part of $d\hat{\rho}^l_{a}$. We define $Ob_k(a_{<k})$ as $\sum_{l=2}^{k} \frac{1}{l!} (d\hat{\rho}^l_{a})_k \Phi^0$, where $a_{<k} = a_1 t + \frac{1}{2!} a_2 t^2 + \cdots + \frac{1}{(k-1)!} a_{k-1} t^{k-1}$. Then we have

Proposition 2-2-1.

$$dR_k = \frac{1}{k!} d\hat{\rho}_{a_k} \Phi^0 + Ob_k(a_{<k}),$$

where $Ob_k$ is written as

$$Ob_k(a) = \left( -\frac{1}{k!} G(a, a) \Phi^0 + \frac{1}{k!} [\hat{\rho}_{a, G(a, a)}] \Phi^0 - \cdots + (-1)^{k-1} \frac{1}{k!} [\hat{\rho}_{a, [\cdots [\hat{\rho}_{a, G(a, a)] \cdots]} \right)_k \Phi^0$$

$$= (f(Ad_{\hat{\rho}_a}) G(a, a))_k \Phi^0,$$
where \( f(x) \) is a convergent sequence,

\[
(2-2-3) \quad f(x) = -\frac{1}{2!} + \frac{1}{3!}x - \frac{1}{4!}x^2 - \cdots = -\frac{e^{-x} - 1 + x}{x^2}
\]

and \( \text{Ad}_{\hat{\rho}_a} \) is the adjoint operator \([\hat{\rho}_a, \cdot]\). Substituting \( \text{Ad}_{\hat{\rho}_a} \) into \( f(x) \), we have an operator \( f(\text{Ad}_{\hat{\rho}_a}) \). The higher obstruction \( \text{Ob}_k \) consists of commutators. Hence

\[
\text{Ob}_k = (f(\text{Ad}_{\hat{\rho}_a})G(a,a))_k \Phi^0
\]

is essentially the interior product of \( \Phi^0 \) in terms of the tensors of type \( \wedge^2 \otimes T \). Hence we see that \( \text{Ob}_k(a_{< k}) \in E^2 \).

**Proof.** In the case \( k = 1 \) we have the proposition. We shall prove the proposition by induction on \( k \). We assume that proposition holds for all \( l < k \). Then we have

\[
(2-2-4) \quad dR_l = -(L_a)_l \Phi^0 + (f(\text{Ad}_{\hat{\rho}_a})G(a,a))_l \Phi^0.
\]

We put \((L_a)_{< k}\) as

\[
(L_a)_{< k} = \sum_{l=2}^{k-1} (L_a)_l.
\]

If \( dR_l = 0 \) \((l < k)\), from our assumption we have

\[
(L_a)_{< k} \Phi^0 = - (f(\text{Ad}_{\hat{\rho}_a})G(a,a))_{< k} \Phi^0
\]

\[
= \left( -\frac{1}{2!}G(a,a) + \frac{1}{3!}[\hat{\rho}_a, G(a,a)] - \frac{1}{4!}[\hat{\rho}_a, [\hat{\rho}_a, G(a,a)]] + \cdots \right)_{< k} \Phi^0.
\]

\[
(2-2-5) \quad = \sum_{l=2}^{k} (-1)^{l-1} \frac{1}{l!} (\text{Ad}_{\hat{\rho}_a}^{l-2} G(a,a))_{< k} \Phi^0
\]
Then by using lemma 2-0-3, we have

\[
d(\rho e^a)_{k} \Phi^0 = \sum_{l=1}^{k} \frac{1}{l!} (d \hat{\rho}_a^l)_{k} \Phi^0
\]

\[
= -(L_a)_{k} \Phi^0 - \frac{1}{2!} (G(a, a) + 2\hat{\rho}_a L_a)_{k} \Phi^0
\]

\[
- \frac{1}{3!} (G(a, a)\hat{\rho}_a + 2\hat{\rho}_a G(a, a) + 3\hat{\rho}_a \hat{\rho}_a L_a)_{k} \Phi^0
\]

\[
- \frac{1}{4!} (G(a, a)\hat{\rho}_a \hat{\rho}_a + 2\hat{\rho}_a G(a, a)\hat{\rho}_a + 3\hat{\rho}_a \hat{\rho}_a G(a, a) + 4\hat{\rho}_a \hat{\rho}_a \hat{\rho}_a L_a)_{k} \Phi^0 - \cdots .
\]

(2-2-6)

Since the degree of \( a = a(t) \) is greater than or equal to one, we have

\[
(2-2-7) \quad (\hat{\rho}_a^m L_a)_{k} = (\hat{\rho}_a^m (L_a)_{<k})_{k},
\]

for a positive integer \( m \). Hence from (2-2-7), we substitute (2-2-5) into (2-2-6) and we have

\[
d(\rho e^a)_{k} \Phi^0 = -(L_a)_{k} \Phi^0 - \frac{1}{2!} G(a, a)_{k} \Phi^0
\]

\[
- \frac{1}{2!} 2(\hat{\rho}_a (-\frac{1}{2!} G(a, a) + \frac{1}{3!} \text{Ad} \hat{\rho}_a G(a, a) + \cdots ))_{k} \Phi^0
\]

\[
- \frac{1}{3!} (G(a, a)\hat{\rho}_a + 2\hat{\rho}_a G(a, a))_{k} \Phi^0 - \frac{1}{3!} 3(\hat{\rho}_a \hat{\rho}_a (-\frac{1}{2!} G(a, a) + \cdots ))_{k} \Phi^0 - \cdots .
\]
Then we calculate each homogeneous part with respect to \( a \) and we have

\[
d(\rho e^a)_k \Phi^0 = -(L_a)_k \Phi^0 - \frac{1}{2!} G(a, a)_k \Phi^0
\]

\[
+ \left( \frac{2}{2!} \hat{\rho}_a G(a, a) - \frac{2}{3!} \hat{\rho}_a G(a, a) - \frac{1}{3!} G(a, a) \hat{\rho}_a \right)_k \Phi^0
\]

\[
+ \left( -\frac{2}{2!3!} \hat{\rho}_a [\hat{\rho}_a, G(a, a)] + \frac{3}{3!2!} \hat{\rho}_a \hat{\rho}_a G(a, a) \right)_k \Phi^0
\]

\[
+ \frac{1}{4!} \left( -G(a, a) \hat{\rho}_a \hat{\rho}_a - 2 \hat{\rho}_a G(a, a) \hat{\rho}_a - 3 \hat{\rho}_a \hat{\rho}_a G(a, a) \right)_k \Phi^0 + \cdots
\]

\[
= -(L_a)_k \Phi^0 - \frac{1}{2!} G(a, a)_k \Phi^0 + \frac{1}{3!} [\hat{\rho}_a, G(a, a)]_k \Phi^0
\]

\[
+ \left( -\frac{1}{4!} G(a, a) \hat{\rho}_a \hat{\rho}_a + \left( \frac{2}{3!} - \frac{2}{3!2!} \right) \hat{\rho}_a G \hat{\rho}_a + \left( \frac{3}{3!2!} + \frac{2}{3!2!} - \frac{2}{3!2!} \right) \hat{\rho}_a \hat{\rho}_a G \right)_k \Phi^0 + \cdots
\]

\[
= -(L_a)_k \Phi^0 - \frac{1}{2!} G(a, a)_k \Phi^0 + \frac{1}{3!} [\hat{\rho}_a, G(a, a)]_k \Phi^0 - \frac{1}{4!} [\hat{\rho}_a, [\hat{\rho}_a, G(a, a)]]_k \Phi^0 + \cdots
\]

\[
= -(L_a)_k \Phi^0 + \sum_{l=2}^k (-1)^{l-1} \frac{1}{l!} \text{Ad}_{\hat{\rho}_a}^{l-2} G(a, a)_k \Phi^0
\]

\[
\square
\]

We determine \( a_k \) such that

\[
(eq_k) \quad \frac{1}{k!} d\hat{\rho}_{a_k} \phi^0 = -\text{Ob}_a(a_{<k})
\]

In order that there exists a solution of \( eq_k \), it is necessary that \([\text{Ob}_k] = 0 \in H^2(#)\). If \([\text{Ob}_k] = 0\), we define \( a_k \) by

\[
(2-2-8) \quad \frac{1}{k!} \hat{\rho}_{a_k} \phi^0 = -d^* G_\#(\text{Ob}_k(a_{<k})).
\]

Since \( \text{Ob}_k(a_{<k}) \) is d-exact, then we also have a criterion,

**Theorem 2-2-2.** If \( p^2 \) is injective, then \( \text{Ob}_k(a_{<k}) \) vanishes for all \( k \).

Thus we construct a power series \( a(t) \) satisfying \( d\rho_{g(t)} \Phi^0 = 0 \). Next we must prove that this power series \( a(t) \) converges for sufficiently small \( t \).
§2-3 THE CONVERGENCE

We rewrite definition 1-6 by using the Sobolev norm.

**Definition 1-6.** A closed element \( \Phi^0 \in \mathcal{E}_s(X) \) is unobstructed if there exists an integral curve \( \Phi_t(a) \) in \( \widetilde{\mathcal{M}}_s(X) \) for each \( a \in \mathcal{E}_s \cap \mathcal{H} \) such that

\[
\frac{d}{dt} \Phi_t(a) \big|_{t=0} = a
\]

An orbit \( \mathcal{O} \) is unobstructed if any \( \Phi^0 \in \widetilde{\mathcal{M}}_s(X) \) is unobstructed for every compact \( n \) dimensional manifold \( X \).

The rest of this subsection is devoted to the proof theorem 1-7 (criterion of unobstructedness). Our method is similar to the one of the Kodaira-Spencer theory. (See the extremely helpful book by Kodaira [18] for technical details.)

**Proof of theorem 1-7.** We already have a formal power series \( a(t) \) such that

\[
d\rho_{g(t)} \Phi^0 = 0.
\]

Hence it is sufficient to prove that \( a(t) \) uniformly converges with respect to the Sobolev norm \( \| \|_s \). Since \( (L_a)_k \Phi^0 = L_{a_k} \Phi^0 = -d\hat{\rho}_{a_k} \Phi^0 \) and \( dR_k = 0 \), \( a_k \) satisfies

\[
(2-3-1) \quad -\frac{1}{k!} d\hat{\rho}_{a_k} \Phi^0 = Ob_k.
\]

As in section 2-2, \( Ob_k \) is an element of \( \Gamma(E^2) \). \( Ob_k \) is also written as

\[
(2-3-2) \quad Ob_k = \frac{1}{2!} d\hat{\rho}_{a_{<k}}^2 \Phi^0 + \cdots + \frac{1}{(k-1)!} d\hat{\rho}_{a_{<k}}^{k-1} \Phi^0
\]

By (2-3-2), we see that \( Ob_k \) is an exact form. Hence if the map \( p^2 : H^2(\#) \to \bigoplus_i H^{p_i+1}(X) \) is injective, then the class \([Ob_k] \in H^2(\#)\) vanishes. Hence we obtain a solution of the equation (2-3-1) by

\[
(2-3-3) \quad \frac{1}{k!} \hat{\rho}_{a_k} \Phi^0 = -d^*_1 G_{\#}(Ob_k) \in E^1.
\]
We assume that $a_k$ belongs to the orthogonal complement of the Lie algebra of the isotropy group $H$. Hence $a_k$ is defined uniquely by $\hat{\rho}_{a_k} \Phi^0$ and we have the estimate

\[(2-3-4) \quad \|a_k\|_s = C_1 \|\hat{\rho}_{a_k} \Phi^0\|_s\]

Hence by (2-3-3), we have a formal power series,

\[a = \sum_{k=1}^{\infty} \frac{1}{k!} a_k t^k.\]

Given two power series $P(t) = \sum_k p_k t^k$ and $Q(t) = \sum_k q_k t^k$, if $p_k < q_k$ for all $k$, we denote it by

\[P(t) \ll Q(t).\]

We denote by $(P)_k$ the homogeneous part of degree $k$ of $P(t)$. Let $A(t)$ be a convergent series given by

\[A(t) = \frac{b}{16c} \sum_{k=1}^{\infty} \frac{c^k t^k}{k^2},\]

with $b > 0, c > 0$. $b$ and $c$ will be determined later. As regards $A(t)$ we have the following inequality (see section 5-3 in [18]),

\[A(t)^l \ll (\frac{c}{b})^{l-1} A(t).\]

Fix a natural number $s$. We shall show by induction on $k$ if we choose appropriate large $b$ and $c$,

\[(**_k) \quad \|a_{\leq k}\|_s \ll A(t),\]

where $\|a_{\leq k}\|_s = \sum_{l=1}^{k} \frac{1}{l!} \|a_l\|_s t^l$. We assume **$_{k-1}$ holds and make an estimate $\|a_k\|_s$. By (2-3-3) we have the inequalities for constants $C_2, C_3$,

\[
\frac{1}{k!}\|a_k\|_s = C_1 \frac{1}{k!} \|\hat{\rho}_{a_k} \Phi^0\|_s = C_1 \|d^1 G_\#(Ob_k)\|_s
\]<

\[
C_2 \|G_\#(Ob_k)\|_{s+1} < C_3 \|Ob_k\|_{s-1}
\]
By theorem 2-2-1, we have an estimate,

\[ \|O_{b,k}\|_{s-1} < \left( \frac{1}{2!} \|G(a,a)\Phi^0\|_{s-1} + \frac{1}{3!} \|A_{\hat{\rho}} G(a,a)\Phi^0\|_{s-1} + \cdots + \frac{1}{k!} \|A_{\hat{\rho}}^{k-2} G(a,a)\Phi^0\|_{s-1} \right)_k \]

\[ < C_4 \left( \frac{1}{2!} \|G(a,a)\|_{s-1} + \frac{2}{3!} \|a\|_{s-1} \|G(a,a)\|_{s-1} + \cdots + \frac{1}{k!} \|a\|_{s-1} \|G(a,a)\|_{s-1} \right)_k \]

\[ < C_4 (\tilde{f}(2\|a\|_{s-1})) \|G(a,a)\|_{s-1} \]

where \( \tilde{f}(x) = \frac{1}{x^2} (e^x - 1 - x) \). We have an estimate of \( G(a,a) \),

\[ \|G(a,a)\|_{s-1} < C_5 \|a\|_s \|a\|_s \]

Hence by (2-3-5),

\[ \|O_{b,k}\|_{s-1} < C_6 \left( \frac{1}{2!} + \frac{1}{3!} \|a_{<k}\|_s + \cdots + \frac{1}{k!} \|a_{<k}\|_{s-2} \|a_{<k}\|_s \right)_k \]

where \( C_6 \) is a constant. By the hypothesis of the induction,

\[ \|O_{b,k}\|_{s-1} < C_6 \left( \frac{1}{2!} + \frac{1}{3!} 2A(t) + \cdots + \frac{1}{k!} 2^{k-1} A(t) A(t) A(t) \right)_k \]

\[ < C_6 \left( \frac{1}{2!} \left( \frac{1}{2!} 2A(t) \right) + \frac{1}{3!} 2^2 \left( \frac{1}{2!} A(t) \right) + \cdots + \frac{1}{k!} 2^{k-1} \left( \frac{1}{2!} A(t) \right) \right)_k \]

\[ = C_6 \left( \frac{1}{2!} \left( \frac{1}{2!} A(t) \right) + \frac{1}{3!} 2^2 \left( \frac{1}{2!} A(t) \right) + \cdots + \frac{1}{k!} 2^{k-1} \left( \frac{1}{2!} A(t) \right) \right)_k \]

\[ < C_6 \frac{1}{2p} (e^{2p} - 1 - 2p) A_k(t), \]

where \( p = \frac{b}{c} \). We define \( p \) by \( C_6 \frac{1}{2p} (e^{2p} - 1 - 2p) = 1 \). Then we obtain

\[ \|O_{b,k}\|_{s-1} < A_k(t) \]

Therefore we have

\[ \frac{1}{k!} \|a_{<k}\|_s < C_3 A_k(t). \]

Since \( A(t) \) is a convergent series for sufficiently small \( t \), we see that \( a(t) \) uniformly convergents. \( \square \)
Further we assume that
\[ d\hat{\rho}_{a_1}\Phi^0 = 0, \]
\[ d^*_0\hat{\rho}_{a_1}\Phi^0 = 0, \]
where \( d^* \) is the adjoint operator and
\[ 0 \longrightarrow E^0 \xrightarrow{d_0} E^1 \xrightarrow{d_1} \cdots. \]
We also apply the elliptic regularity to \( \rho_{g(t)}\Phi^0 \). As in our construction, we have
\[ d\rho_{g(t)}\Phi^0 = \hat{\rho}_a\Phi^0 + \sum_{l=2}^{\infty} \frac{1}{l!}d\hat{\rho}_a^l\Phi^0 = 0 \]
\[ d^*_0\hat{\rho}_a\Phi^0 = 0 \]
Hence \( \hat{\rho}_a\Phi^0 \) is a weak solution of an elliptic differential equation,
\[ \triangle\#\hat{\rho}_a\Phi^0 + d^*_0\left(\sum_{l=2}^{\infty} \frac{1}{l!}d\hat{\rho}_a^l\Phi^0\right) = 0, \]
where \( \triangle\# \) is the Laplace operator of the complex \( \# \). Hence we obtain

**Theorem 2-3-1.** If \( p^2 \) is injective, then there exists a smooth solution of the equation (eq.*) for all tangent \( [\hat{\rho}_a\Phi^0] \in \mathbb{H}^1(\#_{\Phi^0}) \). i.e., There exists a smooth form \( \rho_{\exp a(t)}\Phi^0 \in \tilde{\mathcal{M}}(X) \) such that
\[ \left(\rho_{\exp a(t)}\Phi^0\right)'|_{t=0} = \hat{\rho}_a\Phi^0 \]

### §3 Proof of Theorems

In this section we assume that an orbit \( \mathcal{O} \in \oplus_i \wedge^p_i \) is metrical, elliptic and topological. We shall show that the moduli space \( \mathcal{M}_\mathcal{O}(X) \) is a manifold.
§3-1. In this subsection we explain preliminary results, which are related to functional analysis on manifolds. Our discussion will heavily depend on [7] and [22] and we use the same notation as in section one and two. Let \( X \) be a smooth \( n \)-dimensional manifold and \( F \) a smooth fibre bundle over \( X \). Then we have a Hilbert manifold \( L^2_s(F) \) consisting of those sections of \( F \) which in local coordinates are defined by functions square integrable up to order \( s \), i.e., Sobolev space \( L^2_s(X) \). We also define a Banach manifold \( C^k(F) \) by functions \( C^k(X) \) and for \( s > k + \frac{n}{2} \), we have a smooth inclusion \( L^2_s(F) \subset C^k(F) \). If we consider the case \( F = X \times X \), we find that the sections of \( F \) are exactly the maps of \( X \) to \( X \). Then \( C^1(F) \) is the set of \( C^1 \) maps from \( X \) to \( X \) with the topology of uniform convergence up to the first derivative. We define \( C^1 \operatorname{Diff}(X) \) by the \( C^1 \) diffeomorphisms:

\[
C^1 \operatorname{Diff}(X) = \{ f \in C^1(F) \mid f^{-1} \in C^1(F) \}.
\]

We denote by \( C^1 \operatorname{Diff}_0(X) \) the identity component of \( C^1 \operatorname{Diff}(X) \). Pick \( s + 1 > \frac{n}{2} + 1 \). We define \( \operatorname{Diff}^{s+1}_0(X) \) by

\[
\operatorname{Diff}^{s+1}_0(X) = C^1 \operatorname{Diff}_0(X) \cap L^2_{s+1}(F).
\]

Then \( \operatorname{Diff}^{s+1}_0(X) \) is the Hilbert manifold. In section 3 of [7] it is shown that \( \operatorname{Diff}^{s+1}_0(X) \) is a topological group under the operation of composition of mappings. Further we have the action \( A \) by using pull back:

\[
A : \mathcal{E}_s \times \operatorname{Diff}^{s+1}_0(X) \longrightarrow \mathcal{E}_s,
\]

where \( \mathcal{E}_s = C^1(\mathcal{A}(X)) \cap L^2_s(\oplus_i \Lambda^p i) \) (see 2-1-1 in section two). Then the action \( A \) is well defined and continuous. For an element \( \Phi \in \mathcal{E}_s \), we define \( A_\Phi : \operatorname{Diff}^{s+1}_0(X) \longrightarrow \mathcal{E}_s \) by \( A_\Phi(f) = f^* \Phi \). This map is continuous. Furthermore if \( \Phi \) is smooth, then \( A_\Phi \) is also smooth. We define the moduli space \( \mathcal{M}(X) \) consisting of smooth forms as in section one. We shall extend this moduli in terms of Sobolev Space. We define \( \mathcal{M}_s(X) = \mathcal{M}_{\mathcal{O},s}(X) \) by

\[
\mathcal{M}_s(X) = \tilde{\mathcal{M}}_s(X)/\operatorname{Diff}^{s+1}_0(X),
\]
where
\[ \widetilde{M}_s(X) = \{ \Phi \in \mathcal{E}_s \mid d\Phi = 0 \}. \]

We denote by \( \pi \) the natural projection
\[ \pi: \widetilde{M}_s(X) \longrightarrow M_s(X). \]

§3-2. In section two, we construct the family of smooth closed forms \( \rho_{g(t)}\Phi^0 \) parametrized by \( t \), where \( g(t) = \exp a(t) \) is written as
\[ a(t) = \sum_{k=1}^{\infty} \frac{1}{k!} a_k t^k, \]
\[ \hat{\rho}_{a_1} \Phi^0 \in \mathbb{H}^1(\#\Phi^0). \]

Substitute \( t = 1 \) into \( \rho_{g(t)}\Phi^0 \), we have a map \( \tilde{\kappa} \),
\[ \tilde{\kappa}: S_{\Phi^0} \rightarrow \widetilde{M}(X), \]
\[ \hat{\rho}_{a_1} \Phi^0 \mapsto \rho_{g(1)}\Phi^0, \]
where \( S(= S_{\Phi^0}) \) is a sufficiently small open set of \( \mathbb{H}^1(\#\Phi^0) \). Let \( \pi \) be the natural projection \( \widetilde{M}(X) \rightarrow M(X) \). We define \( \kappa \) as
\[ \kappa = \pi \circ \tilde{\kappa}: S \rightarrow M(X). \]

Then we shall show

**Theorem 3-2-1.** We assume that \( p^1: H^1(\#\phi^0) \rightarrow \bigoplus_i H^{p_i}(X) \) is injective. Then \( \kappa: S \rightarrow M(X) \) is injective for a sufficiently small open set \( S \subset \mathbb{H}^1(\#\phi^0) \).

**Proof.** We denote by \( S(= S_{\Phi^0}) \) the image \( \tilde{\kappa}(S) \). We call \( S \) a slice, which is a transversal submanifold for the action of \( \text{Diff}_0(X) \) on \( \widetilde{M}(X) \). Since the action of \( \text{Diff}_0(X) \) preserves the de Rham cohomology class, taking the cohomology class, we have the map
\[ P: M(X) \longrightarrow \bigoplus_i H^{p_i}_{dR}(X). \]
The slice $S_{\Phi^0}$ is parametrized by an open set $S$. We denote by $P|_S$ the restricted map to $S$. Then the differential $dP|_S$ at $\Phi^0$ is given by $p^1: H^1(#\Phi^0) \to \bigoplus_i H^p_{dR}(X)$. Since $p^1$ is injective, it follows that $P|_S$ is locally injective. Hence there exists an open set $S \in \mathbb{H}^1(#)$ such that $P \circ \kappa: S \to \bigoplus_i H^p_{dR}(X)$ is injective. We assume that there exists a diffeomorphism $f \in \text{Diff}_0(X)$ such that $f^*\Phi^1 = \Phi^2$, for some $\Phi^1, \Phi^2 \in S_{\Phi^0}$. By taking each cohomology class, we have

$$P(f^*\Phi^1) = P(\Phi^2).$$

Since $\text{Diff}_0(X)$ trivially acts on cohomology groups, we also have

$$P(f^*\Phi^1) = P(\Phi^1).$$

Hence we have

$$P(\Phi^1) = P(\Phi^2).$$

Since $P|_S$ is injective, we see that $\Phi^1 = \Phi^2$. Hence $\kappa$ is injective. \qed

Next we shall show that $\kappa$ is surjective:

**Theorem 3-2-2.** If we take a sufficiently small open set $U_{\Phi^0}$ of $\pi(\Phi^0)$ in $\mathfrak{M}(X)$, the map $\kappa: S \to U_{\Phi^0}$ is surjective for an open set $S$ in $\mathbb{H}^1(#)$.

In order to prove theorem 3-2-2, we must explain the completion with respect to Sobolev norm and the following lemma 3-2-3 and proposition 3-2-4. Let $\xi$ be a vector field on $X$. We assume that $\xi \in C^1(TX) \cap L^2_{s+1}(TX)$. Then there is the diffeomorphism $f_\xi$ corresponding to $\xi$, where $f_\xi \in \text{Diff}_0^{s+1}(X)$ (see 3-1). Since $\mathcal{E}_s(X)$ is invariant under the action of diffeomorphism, we have the action of $f_\xi$ on $\mathcal{E}_s(X)$, i.e., $\rho_{e^a}\Phi^0 \to f_\xi^*\rho_{e^a}\Phi^0$, where $a \in \text{End}(TX)$ and $\rho_{e^a}\Phi^0 \in \mathcal{E}_s(X)$. Hence there exists $b_\xi(= b_{f_\xi}) \in \text{End}(TX)$ such that

$$f_\xi^*\rho_{e^a}\Phi^0 = \rho_{\exp b_\xi}\Phi^0.$$
We assume that $\rho_{e^a} \Phi^0$ is smooth with $d \rho_{e^a} \Phi^0 = 0$, i.e., $\rho_{e^a} \Phi^0 \in \tilde{\mathcal{M}}(X)$. Then we shall show that for a sufficiently small $a$, there exists a vector field $\xi$ satisfying

\[(2) \quad d_0^* \hat{\rho}_{b\xi} = 0\]

Since $\hat{\rho}$ is the differential representation of $\rho$, $\hat{\rho}_{b\xi} \Phi^0$ is written as

\[(3) \quad \hat{\rho}_{b\xi} \Phi^0 = \rho_{\exp b\xi} \Phi^0 - \Phi^0 - \sum_{k \geq 2} \frac{1}{k!} \hat{\rho}_b^k \Phi^0.\]

Since $\rho_{\exp b\xi} \Phi^0 = f^* \rho_{e^a} \Phi^0$ and $f^* \rho_{e^a} \Phi^0$ is closed, by using the identity: $L_\xi = d \circ i_\xi + i_\xi \circ d$, we have

\[(4) \quad \rho_{\exp b\xi} \Phi^0 = f^* \rho_{e^a} \Phi^0 = \rho_{e^a} \Phi^0 + di_\xi \rho_{e^a} \Phi^0 + \cdots , \]

\[= \Phi^0 + \hat{\rho}_a \Phi^0 + di_\xi \Phi^0 + H(\xi, a),\]

where $H(\xi, b)$ denotes the higher order terms with respect to $\xi$ and $a$. Substituting (4) into (3), we have

\[(5) \quad \hat{\rho}_{b\xi} \Phi^0 = \hat{\rho}_a \Phi^0 + di_\xi \Phi^0 + W(\xi, a),\]

where $W(\xi, a)$ denotes the higher order terms. In order to solve the equation (2), we need the following estimate of $W(\xi, a)$ for sufficiently small $\xi_1, \xi_2$ and $a$:

**Lemma 3-2-3.**

\[\|W(\xi_1, a) - W(\xi_2, a)\|_s < \varepsilon \|\xi_1 - \xi_2\|_{s+1},\]

where $\varepsilon < 1$ is a constant.

**Proof.** We have the action of the diffeomorphism $\text{Diff}^s_0(X)$ by

\[(6) \quad \text{Diff}^{s+1}_0(X) \times L^2_s(\oplus i \wedge p_i) \rightarrow L^2_s(\oplus i \wedge p_i).\]
We denote this map by $A$. Then we have the map $A_{\Phi}: \text{Diff}^{s+1}_0(X) \to \mathcal{E}_s$ by $A_{\Phi}(f) = f^*\Phi$ as in section 3-1. Then $A_{\Phi}$ is smooth if $\Phi$ is smooth. From our assumption $\rho_{e^a}\Phi^0 \in \tilde{\mathcal{M}}(X)$, $a$ is smooth. We identify $\xi$ with $f_{\xi} \in \text{Diff}_0(X)$. Hence we have

\begin{equation}
\|A(\xi_1, \rho_{e^a}\Phi^0) - A(\xi_2, \rho_{e^a}\Phi^0)\|_s \leq \|dA\|_0 \|\xi_1 - \xi_2\|_{s+1},
\end{equation}

where $dA$ denotes the differential of $A$, $\|dA\|_0$ is the $C^0$-norm of $dA$. Since $W$ is essentially written in terms of $A$, from (7) we also have

\begin{equation}
\|W(\xi_1, a) - W(\xi_2, a)\|_s \leq \|dW\|_0 \|\xi_1 - \xi_2\|_{s+1}.
\end{equation}

Since the differential of $W$ at the $(\xi, b) = (0, 0)$ vanishes, we have

\begin{equation}
\|W(\xi_1, a) - W(\xi_2, a)\|_s \leq \varepsilon \|\xi_1 - \xi_2\|_{s+1},
\end{equation}

where $\varepsilon < 1$ is a constant. Hence by (9), we have the result.

**Proposition 3-2-4.** There exists a sufficiently small $\varepsilon > 0$ satisfying the following:

For any $\rho_{e^a}\Phi^0 \in \tilde{\mathcal{M}}(X)$ with $\|a\|_s < \varepsilon$, there exists a smooth vector field $\xi$ such that $d_0^*\hat{b}_{\xi}\Phi^0 = 0$.

**Proof.** Let $\xi$ be a vector field and $f_{\xi}$ the diffeomorphism corresponding to $\xi$, where $\xi \in C^1(TX) \cap L^2_{s+1}(TX)$ and $f_{\xi} \in \text{Diff}^{s+1}_0(X)$ as before. We shall construct a vector field $\xi$ satisfying the equation (2). By (5), the equation (2) is written as

\begin{equation}
d_0^*d_\xi \Phi^0 + d_0^*\hat{b}_{\xi}\Phi^0 + d_0^*W(\xi, a) = 0.
\end{equation}

We recall the complex $\#_{\Phi^0}$:

\[
\begin{array}{cccccc}
0 & \rightarrow & E^0_{\Phi^0} & \xrightarrow{d_0} & E^1_{\Phi^0} & \xrightarrow{d_1} & E^2_{\Phi^0} & \xrightarrow{d_2} \cdots,
\end{array}
\]

where $d_0^*$ denotes the adjoint operator of $d_0$, $i_\xi \Phi^0 \in E^0$ and $\hat{b}_{\xi}\Phi^0 \in E^1$. Then by using the Hodge decomposition of $E^0$, we take $\xi$ such that the
harmonic component of \( i_{\xi} \Phi^0 \) vanishes with respect to the complex \( \#_{\Phi^0} \). Then the equation (10) is equivalent to the following:

\[
(11) \quad i_{\xi} \Phi^0 + G_{\#} d_0^* \hat{\rho}_a \Phi^0 + G_{\#} d_0^* W(\xi, a) = 0
\]

where \( G_{\#} \) denotes the Green operator with respect to the complex \( \#_{\Phi^0} \). Then given \( a \) and \( \Phi^0 \), we shall show that there exists a solution \( \xi \) of (11). We denote by \( \text{Ker} \Phi^0 \) the subbundle of \( TX \) given by

\[
\text{Ker} \Phi^0 = \{ \xi \in TX \mid i_{\xi} \Phi^0 = 0 \}.
\]

Let \( \text{Ker}^\perp \) be the orthogonal complement of \( \text{Ker} \Phi^0 \) in \( TX \). At first we define \( \xi_1 \in \text{Ker}^\perp \) by

\[
(12) \quad i_{\xi_1} \Phi^0 = -d_0^* G_{\#} \hat{\rho}_a \Phi^0.
\]

Note that since the image of \( d_0^* \) is in \( E^0 \), there is a unique vector field \( \xi_1 \) satisfying (12). Secondly we define \( \xi_2 \in \text{Ker}^\perp \) by

\[
(13) \quad i_{\xi_2} \Phi^0 = -G_{\#} d_0 \hat{\rho}_a \Phi^0 - G_{\#} d_0^* W(\xi_1, a) \Phi^0.
\]

Inductively we define \( \xi_k \in \text{Ker}^\perp \) by

\[
(14) \quad i_{\xi_k} \Phi^0 = -G_{\#} d_0 \hat{\rho}_a \Phi^0 - G_{\#} d_0^* W(\xi_{k-1}, a) \Phi^0.
\]

Since \( \xi_k \in \text{Ker}^\perp \), we have an estimate

\[
(15) \quad \| \xi_k \|_{s+1} = C \| i_{\xi_k} \Phi^0 \|_{s+1},
\]

where \( C \) denotes a constant. Then by (14) and the elliptic estimate of \( G_{\#} \) and \( d_0^* \), we have

\[
(16) \quad \| \xi_{k+1} - \xi_k \|_{s+1} = \| (G_{\#} d_0^* W(\xi_k, a) - G_{\#} d_0^* W(\xi_{k-1}, a)) \Phi^0 \|_{s+1} \leq C_1 \| (W(\xi_k, a) - W(\xi_{k-1}, a)) \|_s
\]
where $C_1$ is a constant. Hence by lemma 3-2-3, we have

(17) \[ \| \xi_{k+1} - \xi_k \|_{s+1} = \varepsilon \| \xi_k - \xi_{k-1} \|_{s+1}, \]

where $\varepsilon < 1$ is a constant. Hence it follows from (17) that the sequence \{\xi_k\} uniformly converges to some $\xi_{\infty}$ with respect to the norm $\| \cdot \|_{s+1}$. Then by (14) we see that $\xi_{\infty}$ is a solution of the equation (11). Hence we have a vector field satisfying (2). □

**Proof of theorem 3-2-2.** By proposition 3-2-4, if we define an open set $U_{\Phi^0}$ by the image:

\[ U_\varepsilon(\Phi^0) = \pi \left( \{ \rho_\varepsilon \Phi^0 \in \widetilde{\mathcal{M}}(X) \mid \| a \|_s < \varepsilon, \} \right), \]

then there exists a diffeomorphism $f_\xi$ such that

\[
\begin{align*}
    f_\xi^* \rho_\varepsilon a \Phi^0 &= \rho_\varepsilon b_\xi \Phi^0 \\
    d_0^* \rho_\varepsilon b_\xi \Phi^0 &= 0.
\end{align*}
\]

Hence $\rho_\varepsilon b_\xi \Phi^0$ is in the image of $\kappa(S)$. Hence it follows that $\kappa$ is surjective. □

§3-3. Let $\mathcal{O}$ be an orbit in $\oplus_i \Lambda^{p_i}$ as in section one. Then we have the moduli space $\mathcal{M}_\mathcal{O}(X) (= \mathcal{M}(X))$ as the quotient space $\widetilde{\mathcal{M}}(X) / \text{Diff}_0(X)$.

**Proposition 3-3-1.** We assume that the orbit $\mathcal{O}$ is elliptic, metrical and topological. Then the quotient $\mathcal{M}_\mathcal{O}(X)$ is Hausdorff.

**Proof.** Since the orbit $\mathcal{O}$ is metrical, we have the metric $g_\Phi$ for every $\Phi \in \mathcal{E}_s(X)$. Then each tangent space $T_\Phi \mathcal{E}_s(X) \subset L^2_s(\oplus_i \Lambda^{p_i})$ has the $L^2$ metric in terms of $g_\Phi$. Hence it gives $\mathcal{E}_s(X)$ a smooth Riemannian structure (see section 4 in [7]). The fundamental property of the Riemannian structure on $\mathcal{E}_s(X)$ is that it is invariant under the action of $\text{Diff}_{s+1}^0(X)$, so that is, $\text{Diff}_{s+1}^0(X)$ acts on $\mathcal{E}_s(X)$ isometrically. Since $\widetilde{\mathcal{M}}_s(X)$ is the intersection $\mathcal{H} \cap \mathcal{E}_s(X)$, we have the induced distance on $\widetilde{\mathcal{M}}_s(X)$ from
the Riemannian structure on \( \mathcal{E}_s(X) \). We denote by \( d \) the distance on \( \tilde{\mathcal{M}}_s(X) \) and \( \pi \) the natural projection \( \pi : \tilde{\mathcal{M}}_s(X) \to \mathcal{M}_s(X) \). Then we define \( d(\pi(\Phi^1), \pi(\Phi^2)) \) by

\[
(3-3-1) \quad d(\pi(\Phi^1), \pi(\Phi^2)) = \inf_{f, g \in \text{Diff}_0^{s+1}(X)} d(f^*\Phi^1, g^*\Phi^2),
\]

where \( \Phi^1, \Phi^2 \in \tilde{\mathcal{M}}_s(X) \). For simplicity we denote by \( \mathcal{D} \) the group \( \text{Diff}_0^{s+1}(X) \). Since the action of \( \text{Diff}_0^{s+1}(X)(= \mathcal{D}) \) preserves the distance \( d \), we have

\[
(3-3-2) \quad d(\pi(\Phi^1), \pi(\Phi^2)) = \inf_{f \in \mathcal{D}} d(f^*\Phi^1, \Phi^2).
\]

Hence we have the triangle inequality,

\[
(3-3-3) \quad d(\pi(\Phi^1), \pi(\Phi^2)) + d(\pi(\Phi^2), \pi(\Phi^3)) = \inf_{f \in \mathcal{D}} d(f^*\Phi^1, \Phi^2) + \inf_{g \in \mathcal{D}} d(\Phi^2, g^*\Phi^3) \leq \inf_{f, g \in \mathcal{D}} d(f^*\Phi^1, g^*\Phi^3) = d(\pi(\Phi^1), \pi(\Phi^3)).
\]

We shall show that \( d \) induces a distance of \( \mathcal{M}(X) \). We assume that \( d(\pi(\Phi^0), \pi(\Phi)) = 0 \) for smooth elements \( \Phi, \Phi^0 \in \tilde{\mathcal{M}}(X) \). Then by (3-3-1), we have

\[
(3-3-4) \quad \inf_{f \in \text{Diff}_0(X)} d(\Phi^0, f^*\Phi) = 0.
\]

Hence \( f^*\Phi \) is in a small neighborhood \( U_\varepsilon(\Phi^0) \) at \( \Phi^0 \). By theorem 3-2-3 and proposition 3-2-4, there exists a diffeomorphism \( f_\xi \) such that

\[
(3-3-5) \quad f_\xi(f^*\Phi^0) \in S_{\Phi^0},
\]

where \( S_{\Phi^0} \) is the family parametrized by an open set of harmonic forms \( \mathbb{H}^1(#\Phi^0) \). We define the distance \( d_{\mathbb{H}^1(#)} \) on \( \mathbb{H}^1(#\Phi^0) \) by using harmonic
forms in terms of $g_{\Phi^0}$. There is the distance $d_{dR}$ on the direct sum of the de Rham cohomology groups $\oplus_i H^p_i(X)$ by using harmonic representations with respect to $g_{\Phi^0}$. Since $p^1$ is injective, $(H^1(\#\Phi^0), d_{dR}(\#))$ is isometrically embedded into $(\oplus_i H^p_i(X), d_{dR})$. Since $p^1$ is injective, we have the injective map

\[ P|_S : S \to \oplus_i H^p_i(X). \]

Since the differential $dP|_S : T_{\Phi^0}S(= H^1(\#\Phi^0)) \to \oplus_i H^p_i(X)$ is isometric, we have that

\[ d(\Phi^0, f^*_\xi(f^*\Phi)) > Cd_{dR}(P(\Phi^0), P(f^*_\xi(f^*\Phi)), \]

where $C$ is a positive constant. (Note that the distance $d$ restricted to the slice $S$ is (locally) equivalent to the distance $d_{dR}(\#)$.) Since $\text{Diff}_0(X)$ acts on $\oplus_i H^p_i(X)$ trivially, we have

\[ d(\Phi^0, f^*_\xi(f^*\Phi)) > Cd_{dR}(P(\Phi^0), P(\Phi)), \]

where $C$ does not depend on $f$ and $\Phi$. Hence

\[ \inf_{f \in \text{Diff}_0(X)} d(\Phi^0, f^*_\xi(f^*\Phi)) > C d_{dR}(P(\Phi^0), P(\Phi)). \]

Hence from our assumption (3-3-4), we have $P(\Phi) = P(\Phi^0)$. Since $P|_S$ is injective, $\Phi^0 = f^*_\xi(f^*\Phi)$. Hence we have $\pi(\Phi) = \pi(\Phi^0)$. Hence $d$ is a distance on $\mathcal{M}(X)$. □

§3-4 Proof of main theorems.

Proof of theorem 1-8. Since the orbit $\mathcal{O}$ is elliptic and topological, we have the slice $S_{\Phi^0}$ as coordinates of the moduli space $\mathcal{M}(X)$ by theorem 3-2-1 and 3-2-2 in section 3-2. Hence $\mathcal{M}(X)$ is a manifold. Since $\mathcal{O}$ is metrical, $\mathcal{M}(X)$ is Hausdorff by proposition 3-3-1 in section 3-3. The slice $S_{\Phi^0}$ is homeomorphic to an open set of the cohomology group $H^1(\#\Phi^0)$. □
Proof of theorem 1-9. The slice $S_{\Phi^0}$ is local coordinates of $\mathcal{M}_\circ(X)$. The differential $dP$ coincides with the injective map $p^1$. Hence $P$ is locally injective. □

Since $\mathcal{O}$ is metrical, we have the metric $g_\Phi$ for each $\Phi \in \mathcal{E}$. Hence the metric $g_\Phi$ defines the metric on each tangent space $E^1 = T_\Phi \mathcal{E}$. Then $\mathcal{E}$ can be considered as a Riemannian manifold. Then we see that the action of $\text{Diff}_0(X)$ on $\mathcal{E}$ is isometry. Let $I(\Phi)$ be the isotropy group of $\text{Diff}_0(X)$ at $\Phi$,

$I(\Phi) = \{ f \in \text{Diff}_0(X) \mid f^* \Phi = \Phi \}.$

Let $S_{\Phi^0}$ be a slice at $\Phi^0$. Then we shall compare $I(\Phi^0)$ to other isotropy group $I(\Phi)$ for $\Phi \in S_{\Phi^0}$.

**Theorem 1-10.** Let $I(\Phi^0)$ be the isotropy group of $\text{Diff}_0(X)$ at $\Phi^0$ and $S_{\Phi^0}$ the slice at $\Phi^0$. Then $I(\Phi^0)$ is a subgroup of the isotropy group $I(\Phi)$ for each $\Phi \in S_{\Phi^0}$. (We take $S_{\Phi^0}$ sufficiently small for necessary.)

**Proof of theorem 1-10.** From definition of $S_{\Phi^0}$, the slice $S_{\Phi^0}$ is invariant under the action of $I(\Phi^0)$. The restricted map $P|_{S_{\Phi^0}} : S_{\Phi^0} \to \bigoplus_i H^{p_i}(X)$ is locally injective. Since the action of $\text{Diff}_0(X)$ preserves each class of de Rham cohomology group, we see that the action of $I(\Phi^0)$ is trivial on the slice $S_{\Phi^0}$ for sufficiently small $S_{\Phi^0}$. Hence $I(\Phi^0)$ is a subgroup of the isotropy group $I(\Phi)$ for each $\Phi \in S_{\Phi^0}$. □

**Proof of theorem 1-11.** The slice $S_{\Phi^0}$ is local coordinates of $\mathcal{M}_\circ(X)$ and the action of $\text{Diff}_0(X)$ on $\mathcal{E}$ is isometry. Hence the moduli space $\widetilde{\mathcal{M}}_\circ(X)/\text{Diff}_0(X)$ is locally homeomorphic to the quotient space $S_{\Phi^0}/I(\Phi^0)$, where $I(\Phi^0)$ is the isotropy. Hence we see that there is an open set $V$ of $T_{\Phi^0}S_{\Phi^0}$ with the action of $I(\Phi^0)$ such that the quotient $V/I(\Phi^0)$ is homeomorphic to $S_{\Phi^0}/I(\Phi^0)$. $T_{\Phi^0}S_{\Phi^0}$ is isomorphic to $H^1(\#_{\Phi^0})$ and the action of $I(\Phi^0)$ on $H^1(\#_{\Phi^0})$ is a isometry with respect to $g_{\Phi^0}$. The action of $I(\Phi^0)$ preserves the integral cohomology class. It implies that the image of $I(\Phi^0)$ is a subgroup of $O(H^1(\#_{\Phi^0})) \cap \text{End}(\bigoplus_i (H^{p_i}(X,\mathbb{Z})))$, where $O(H^1(\#_{\Phi^0}))$ denotes the orthogonal group. Then we see that $V/I(\Phi^0)$ is the quotient by a finite group. □
§4. CALABI-YAU STRUCTURES

§4.1. $\text{SL}_n(\mathbb{C})$ structures. Let $V$ be a real $2n$ dimensional vector space. We consider the complex vector space $V \otimes \mathbb{C}$ and a complex form $\Omega \in \wedge^n V^* \otimes \mathbb{C}$. The vector space $\ker \Omega$ is defined as

$$\ker \Omega = \{ v \in V \otimes \mathbb{C} \mid i_v \Omega = 0 \},$$

where $i_v$ denotes the interior product.

Definition 4-1-1 ($\text{SL}_n(\mathbb{C})$ structures). A complex $n$ form $\Omega$ is a $\text{SL}_n(\mathbb{C})$ structure on $V$ if $\dim \mathbb{C} \ker \Omega = n$ and $\ker \Omega \cap \overline{\ker \Omega} = \{0\}$, where $\overline{\ker \Omega}$ is the conjugate vector space.

We denote by $\mathcal{A}_{\text{SL}}(V)$ the set of $\text{SL}_n(\mathbb{C})$ structures on $V$. We define the almost complex structure $I_\Omega$ on $V$ by

$$I_\Omega(v) = \begin{cases} -\sqrt{-1}v & \text{if } v \in \ker \Omega, \\ \sqrt{-1}v & \text{if } v \in \overline{\ker \Omega}. \end{cases}$$

So that is, $\ker \Omega = T^{0,1}V$ and $\overline{\ker \Omega} = T^{1,0}V$ and $\Omega$ is a non-zero $(n,0)$ form on $V$ with respect to $I_\Omega$. Let $\mathcal{J}$ be the set of almost complex structures on $V$. Then $\mathcal{A}_{\text{SL}}(V)$ is the $\mathbb{C}^*-$bundle over $\mathcal{J}$. We denote by $\rho$ the action of the real general linear group $G = GL(V) \cong GL(2n, \mathbb{R})$ on the complex $n$ forms,

$$\rho: GL(V) \longrightarrow \text{End} (\wedge^n (V \otimes \mathbb{C})^*).$$

For simplicity we denote by $\wedge^n \mathbb{C}$ complex $n$ forms. Since $G$ is a real group, $\mathcal{A}_{\text{SL}}(V)$ is invariant under the action of $G$. Then we see that the action of $G$ on $\mathcal{A}_{\text{SL}}(V)$ is transitive. The isotropy group $H$ is defined as

$$H = \{ g \in G \mid \rho_g \Omega = \Omega \}.$$

Then we see $H = \text{SL}(n, \mathbb{C})$. Hence the set of $\text{SL}_n(\mathbb{C})$ structures $\mathcal{A}_{\text{SL}}(V)$ is the homogeneous space,

$$\mathcal{A}_{\text{SL}}(V) = G/H = GL(2n, \mathbb{R})/\text{SL}(n, \mathbb{C}).$$
(Note that the set of almost complex structures $\mathcal{J} = \text{GL}(2n, \mathbb{R})/\text{GL}(n, \mathbb{C})$.)

An almost complex structure $I$ defines a complex subspace $T^{1,0}$ of dimension $n$. Hence we have the map $\mathcal{J} \rightarrow \text{Gr}(n, \mathbb{C}^{2n})$. We also have the map from $\mathcal{A}_{SL}(V)$ to the tautological line bundle $L$ over the Grassmanian $\text{Gr}(n, \mathbb{C}^{2n})$ removed $0$–section. Then we have the diagram:

\[
\begin{array}{ccc}
\mathcal{A}_{SL}(V) & \longrightarrow & L\backslash 0 \\
\downarrow \mathbb{C}^* & & \downarrow \\
\mathcal{J} & \longrightarrow & \text{Gr}(n, \mathbb{C}^{2n})
\end{array}
\]

$\mathcal{A}_{SL}(V)$ is embedded as a smooth submanifold in $n$–forms $\wedge^n$. This is Plücker embedding described as follows,

\[
\begin{array}{ccc}
\mathcal{A}_{SL}(V) & \longrightarrow & L\backslash 0 \longrightarrow \wedge^n \backslash \{0\} \\
\downarrow \mathbb{C}^* & & \downarrow \\
\mathcal{J} & \longrightarrow & \text{Gr}(n, \mathbb{C}^{2n}) \longrightarrow \mathbb{C}\mathbb{P}^n.
\end{array}
\]

Hence the orbit $\mathcal{O}_{SL} = \mathcal{A}_{SL}(V)$ is a submanifold in $\wedge^n$ defined by Plücker relations. Let $X$ be a real $2n$ dimensional compact manifold. Then we have the $G/H$ bundle $\mathcal{A}_{SL}(X)$ over $X$ as in section 1. We denote by $\mathcal{E} = \mathcal{E}_{SL}^1$ the set of smooth global sections of $\mathcal{A}_{SL}(X)$. Then we have the almost complex structure $I_\Omega$ corresponding to $\Omega \in \mathcal{E}^1$. Then we have

**Lemma 4-1-2.** If $\Omega \in \mathcal{E}^1$ is closed, then the almost complex structure $I_\Omega$ is integrable.

**Proof.** Let $\{\theta_i\}_{i=1}^n$ be a local basis of $\Gamma(\wedge^{1,0})$ with respect to $\Omega$. From Newlander-Nirenberg’s theorem it is sufficient to show that $d\theta_i \in \Gamma(\wedge^{2,0} \oplus \wedge^{1,1})$ for each $\theta_i$. Since $\Omega$ is of type $\wedge^{n,0}$,

\[
\theta_i \wedge \Omega = 0.
\]

Since $d\Omega = 0$, we have

\[
d\theta_i \wedge \Omega = 0.
\]
Hence \( d\theta_i \in \Gamma(\wedge^{2,0} \oplus \wedge^{1,1}) \). \( \square \)

Then we define the moduli space of \( \text{SL}_n(\mathbb{C}) \) structures on \( X \) by

\[
\mathcal{M}_{\text{SL}}(X) = \{ \Omega \in \mathcal{E}^1_{\text{SL}} | d\Omega = 0 \}/\text{Diff}_0(X).
\]

From lemma 4-1-2 we see that \( \mathcal{M}_{\text{SL}}(X) \) is the \( \mathbb{C}^* \)–bundle over the moduli space of integrable complex structures on \( X \) with trivial canonical line bundles.

**Proposition 4-1-3.** The orbit \( \mathcal{O}_{\text{SL}} \) is elliptic.

**Proof.** Let \( \wedge^{p,q} \) be \((p, q)\)–forms on \( V \) with respect to \( I_{\Omega^0} \in \mathcal{A}_{\text{SL}}(V) \). In this case we see that

\[
\begin{align*}
E^0 &= \wedge^{n-1,0} \\
E^1 &= \wedge^{n,0} \oplus \wedge^{n-1,1} \\
E^2 &= \wedge^{n,1} \oplus \wedge^{n-1,2}.
\end{align*}
\]

Hence we have the complex :

\[
\wedge^{n-1,0} \xrightarrow{\wedge u} \wedge^{n,0} \oplus \wedge^{n-1,1} \xrightarrow{\wedge u} \wedge^{n,1} \oplus \wedge^{n-1,2} \xrightarrow{\wedge u} \cdots,
\]

for \( u \in V \). Since the Dolbeault complex is elliptic, we see that the complex \( 0 \rightarrow E^1 \rightarrow E^2 \rightarrow \cdots \) is exact. \( \square \)

**Proposition 4-1-4.** Let \( I_{\Omega} \) be the complex structure corresponding to \( \Omega \in \mathcal{E} \). If \( \partial \bar{\partial} \) lemma holds for the complex manifold \((X, I_{\Omega})\), then

\[
\begin{align*}
H^0(\#) &\cong H^{n-1,0}(X), & H^1(\#) &\cong H^{n,0}(X) \oplus H^{n-1,1}(X), \\
H^2(\#) &\cong H^{n,1}(X) \oplus H^{n-1,2}(X)
\end{align*}
\]

and \( p^1, p^2 \) are respectively injective,

\[
p^1 : H^1(\#) \rightarrow H^n(X, \mathbb{C}), \quad p^2 : H^2(\#) \rightarrow H^{n+1}(X, \mathbb{C})
\]
In particular, if \((X, I_\Omega)\) is Kählerian, \(p^k\) is injective for \(k = 1, 2\).

**Proof.** As in proof of proposition 4-1-3 the complex \(#_\Omega\) is given as
\[
\begin{array}{c}
\Gamma(\wedge^{n-1,0}) \xrightarrow{d} \Gamma(\wedge^{n,0} \oplus \wedge^{n-1,1}) \xrightarrow{d} \Gamma(\wedge^{n,1} \oplus \wedge^{n-1,2}) \xrightarrow{d} \cdots
\end{array}
\]
Then we have the following double complex:
\[
\begin{array}{cccc}
\Gamma(\wedge^{n,0}) & \xrightarrow{\bar{\partial}} & \Gamma(\wedge^{n,1}) & \xrightarrow{\bar{\partial}} & \Gamma(\wedge^{n,2}) & \xrightarrow{\bar{\partial}} & \cdots \\
\partial & & \partial & & \partial & & \\
\Gamma(\wedge^{n-1,0}) & \xrightarrow{\bar{\partial}} & \Gamma(\wedge^{n-1,1}) & \xrightarrow{\bar{\partial}} & \Gamma(\wedge^{n-1,2}) & \xrightarrow{\bar{\partial}} & \cdots \\
\partial & & \partial & & \partial & & \\
\Gamma(\wedge^{n-2,0}) & \xrightarrow{\bar{\partial}} & \Gamma(\wedge^{n-2,1}) & \xrightarrow{\bar{\partial}} & \Gamma(\wedge^{n-2,2}) & \xrightarrow{\bar{\partial}} & \cdots \\
\partial & & \partial & & \partial & & \\
\end{array}
\]
Let \(a = x + y\) be a closed element of \(\Gamma(\wedge^{n,1}) \oplus \Gamma(\wedge^{n-1,2})\). Then we have the following equations,
\[
\begin{align*}
(1) & \quad \bar{\partial}y = 0, \\
(2) & \quad \bar{\partial}x + \partial y = 0.
\end{align*}
\]
Using the Hodge decomposition, we have
\[
(3) \quad y = \text{Har}(y) + \bar{\partial}(\bar{\partial}^* G_{\bar{\partial}} y),
\]
where \(G_{\bar{\partial}}\) is the Green operator with respect to the \(\bar{\partial}\)-Laplacian and \(\text{Har}(y)\) denotes the harmonic component of \(y\). We also have
\[
(4) \quad x = \text{Har}(x) + \partial(\partial^* G_\partial x),
\]
where \(G_\partial\) is the Green operator with respect to the \(\partial\)-Laplacian and \(\text{Har}(x)\) denotes the harmonic component of \(x\). We put \(s = \partial^* G_\partial x\) and \(t = \bar{\partial}^* G_{\bar{\partial}} y\) respectively. Then we have from (2)
\[
(5) \quad \bar{\partial}\partial s + \partial\bar{\partial}t = \bar{\partial}\partial(s - t) = 0.
\]
Applying $\partial \bar{\partial}$-lemma, we see from (5) that there exists a $\gamma \in \wedge^{n-1,0}$ such that

\[(6) \quad \partial(s - t) = \partial \bar{\partial} \gamma.\]

Hence we have from (4),

\[x = Har(x) + \partial s = Har(x) + \partial t + \bar{\partial}(-\partial \gamma)\]
\[y = Har(y) + \bar{\partial} t.\]

Thus if $Har(x) = 0$ and $Har(y) = 0$, then $a$ is written as $a = x + y = d(t - \bar{\partial} \gamma)$ where $t - \bar{\partial} \gamma \in E^1 \cong \wedge^{n,0} \oplus \wedge^{n-1,1}$. It implies that the map $p^2: H^2(\#) \to H^{n+1}(X, \mathbb{C})$ is injective and $H^2(\#) \cong H^{n,1}(X) \oplus H^{n-1,2}(X)$. \(\square\)

Hence from section 2, we have the smooth deformation space of $SL_n(\mathbb{C})$ structures. However $O_{SL}$ is not metrical and the moduli space $\mathcal{M}_{SL}(X)$ is not Hausdorff in general. In fact it is known that it is not Hausdorff for $K3$ surface. Hence in order to obtain a Hausdorff moduli space, we must introduce extra geometric structures. The most natural structure is a Calabi-Yau structure.

§4-2. Calabi-Yau structures. Let $V$ be a real vector space of $2n$ dimensional. We consider a pair $\Phi = (\Omega, \omega)$ of a $SL_n(\mathbb{C})$ structure $\Omega$ and a real symplectic structure $\omega$ on $V$,

\[\Omega \in \mathcal{A}_{SL}(V),\]
\[\omega \in \wedge^2 V^*, \quad \omega \wedge \cdots \wedge \omega \neq 0.\]

We define $g_{\Omega, \omega}$ by

\[g_{\Omega, \omega}(u, v) = \omega(I_{\Omega}u, v),\]

for $u, v \in V$. 

Definition 4-2-1 (Calabi-Yau structures). A Calabi-Yau structure on $V$ is a pair $\Phi = (\Omega, \omega)$ such that

1. $\Omega \wedge \omega = 0, \quad \overline{\Omega} \wedge \omega = 0$
2. $\Omega \wedge \overline{\Omega} = c_n \omega \wedge \cdots \wedge \omega$
3. $g_{\Omega, \omega}$ is positive definite.

where $c_n$ is a constant depending only on $n$, i.e.,

$$c_n = (-1)^{\frac{n(n-1)}{2}} \frac{2^n}{i^n n!}.$$ 

From the equation (1) we see that $\omega$ is of type $\wedge^{1,1}$ with respect to the almost complex structure $I_\Omega$. The equation (2) is called Monge-Ampère equation.

Lemma 4-2-2. Let $\mathcal{A}_{CY}(V)$ be the set of Calabi-Yau structures on $V$. Then there is the transitive action of $G = GL(2n, \mathbb{R})$ on $\mathcal{A}_{CY}(V)$ and $\mathcal{A}_{CY}(V)$ is the homogeneous space

$$\mathcal{A}_{CY}(V) = GL(2n, \mathbb{R})/SU(n).$$

Proof. Let $g_{\Omega, \omega}$ be the Kähler metric. Then we have a unitary basis of $TX$. Then the result follows from (1) and (2). □

Hence the set of Calabi-Yau structures on $V$ is the orbit $\mathcal{O}_{CY}$,

$$\mathcal{O}_{CY} \subset \wedge^n (V \otimes \mathbb{C})^* \oplus \wedge^2 V^*.$$ 

Let $V$ be a real $2n$ dimensional vector space with a Calabi-Yau structure $\Phi^0 = (\Omega^0, \omega^0)$. We define the complex Hodge star operator $*_{\mathbb{C}}$ by

$$\alpha \wedge *_{\mathbb{C}} \beta = \langle \alpha, \beta \rangle \Omega^0,$$
where $\alpha, \beta \in \wedge^{*,0}$. The complex Hodge star operator $\ast_C$ is a natural generalization of the ordinary Hodge star $\ast$,

$$
\ast_C : \wedge^{i,0} \to \wedge^{n-i,0}.
$$

The vector space $E^0$ is, by definition,

$$
E^0_C(V) = \{ (i_v\Omega^0, i_v\omega^0) \in \wedge^{n-1,0} \oplus \wedge^1 \mid v \in V \}
$$

The map $TX \to \wedge^{n-1,0}$ is given by $v \mapsto i_v\Omega^0$. Then we see that this map is an isomorphism. Hence the projection to the first component defines an isomorphism:

$$
E^0_C \to \wedge^{n-1,0},
$$

$$(i_v\Omega^0, i_v\omega^0) \mapsto i_v\Omega^0
$$

The $E^1_C$ is the tangent space of Calabi-Yau structures $A_C(X)$. Hence by (1) and (2) of definition 4-2-1, the vector space $E^1(V) = E^1_C(V)$ is the set of $(\alpha, \beta) \in \wedge^n_C \oplus \wedge^2$ satisfying equations

$$
\alpha \wedge \omega^0 + \Omega^0 \wedge \beta = 0,
$$

(4)

$$
\alpha \wedge \overline{\Omega^0} + \Omega^0 \wedge \overline{\alpha} = n c_n \beta \wedge (\omega^0)^{n-1}
$$

Let $P^{p,q}$ be the primitive cohomology group with respect to $\omega^0$. Then we have the Lefschetz decomposition,

$$
\alpha = \alpha^{n,0} + \alpha^{n-1,1} + \alpha^{n-2,0} \wedge \omega^0 \in P^{n,0} \oplus P^{n-1,1} \oplus P^{n-2,0} \wedge \omega^0,
$$

(5)

$$
\beta = \beta^{2,0} + \beta^{1,1} + \beta^{0,0} \wedge \omega^0 + \beta^{0,2} \in P^{2,0} \oplus P^{1,1} \oplus P^{0,0} \wedge \omega^0 \oplus P^{0,2},
$$

where $\beta^{2,0} = \overline{\beta^{0,2}}$ and $P^{1,1}_R$ denotes the real primitive forms of type (1,1). Then equation (4) is written as

$$
\alpha^{n-2,0} \wedge \omega \wedge \omega + \Omega \wedge \beta^{0,2} = 0,
$$

(6)

$$
\alpha^{n,0} \wedge \overline{\Omega} = n c_n \beta^{0,0} \omega^n
$$

(7)
Then we see that (6) gives a relation between $\alpha^{n-2,0}$ and $\beta^{2,0}$ and (7) also describes a relation between $\alpha^{n,0}$ and $\beta^{0,0}$. Since there is no relation between the primitive parts $P^{n-1,1}$ and $P_{\mathbb{R}}^{1,1}$, the kernel of the projection $E_{CY}^1 \rightarrow \wedge^{n,0} \oplus \wedge^{n-1,1}$ is given by the primitive forms $P_{\mathbb{R}}^{1,1}$. Hence we have an exact sequence:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & P_{\mathbb{R}}^{1,1} & \longrightarrow & E_{CY}^1 & \longrightarrow & \wedge^{n,0} \oplus \wedge^{n-1,1} & \longrightarrow & 0.
\end{array}
\]

The vector space $E_{CY}^2$ is the subspace of $\wedge^{n,1} \oplus \wedge^{n-1,2} \oplus \wedge^3_{\mathbb{R}}$. We also consider the projection to the first component and we have an exact sequence:

\[
\begin{array}{cccccc}
0 & \longrightarrow & (\wedge^{2,1} \oplus \wedge^{1,2})_{\mathbb{R}} & \longrightarrow & E_{CY}^2 & \longrightarrow & \wedge^{n,1} \oplus \wedge^{n-1,2} & \longrightarrow & 0,
\end{array}
\]

where $(\wedge^{2,1} \oplus \wedge^{1,2})_{\mathbb{R}}$ denotes the real part of $\wedge^{2,1} \oplus \wedge^{1,2}$. Let $X$ be a 2n dimensional compact Kähler manifold. We denote by $\wedge^{i,j}$ (global) differential forms on $X$ of type $(i,j)$. The real primitive forms of type $(i,j)$ is denoted by $P_{\mathbb{R}}^{i,j}$. Then we have a complex of forms on $X$ by using the exterior derivative $d$:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & P_{\mathbb{R}}^{1,1} & \stackrel{d}{\longrightarrow} & (\wedge^{2,1} \oplus \wedge^{1,2})_{\mathbb{R}} & \stackrel{d}{\longrightarrow} & \cdots.
\end{array}
\]

**Proposition 4-2-3.** The cohomology groups of the complex (10) are respectively given by

\[
\begin{align*}
\mathbb{P}_{\mathbb{R}}^{1,1}, & \quad (H^{2,1}(X) \oplus H^{1,2}(X))_{\mathbb{R}},
\end{align*}
\]

where $\mathbb{P}_{\mathbb{R}}^{1,1}$ denotes the harmonic and primitive forms.

**Proof.** By using Kähler identity, we see that a closed primitive form of type $(1, 1)$ is harmonic. Hence the first cohomology group of the complex (10) is $\mathbb{P}_{\mathbb{R}}^{1,1}$. Let $q$ be a real $d$-exact form of type $\wedge^{2,1} \oplus \wedge^{1,2}$. The applying $\partial \bar{\partial}$-lemma, we show that $q$ is written as

\[
q = da,
\]
where \( a = d^* \eta \in \bigwedge^1_R \) and \( \eta \in (\bigwedge^{2,1} \oplus \bigwedge^{1,2})_R \). We shall show that there exists \( k \in \bigwedge^1 \) such that \( d^* \eta + dk \in P^{1,1}_R \). By the Lefschetz decomposition, the three form \( \eta \) is written as

\[
\eta = s + \theta \wedge \omega^0,
\]

where \( s \in (P^{2,1} \oplus P^{1,2})_R \), and \( \theta \in \bigwedge^1_R \). Let \( \wedge_{\omega^0} \) be the contraction with respect to the Kähler form \( \omega^0 \). Since \( \wedge_{\omega^0} \) and \( d^* \) commutes,

\[
\wedge_{\omega^0} d^* \eta = d^* \wedge_{\omega^0} \eta = d^* \wedge_{\omega^0} (s + \theta \wedge \omega^0) = d^* \theta.
\]

On the other hand, applying Kähler identity again, we have

\[
\wedge_{\omega^0} dk = d \wedge_{\omega^0} k + \sqrt{-1} d_c^* k = \sqrt{-1} d_c^* k,
\]

where \( d_c^* = \partial^* - \overline{\partial}^* \). Since \( k \in \bigwedge^1 \),

\[
d_c^* k = (\partial^* - \overline{\partial}^*) k = \partial^* k^{1,0} - \overline{\partial}^* k^{0,1} = (\partial^* + \overline{\partial}^*) (k^{1,0} - k^{0,1}).
\]

Hence if we define \( k \) by

\[
k = \sqrt{-1}(\theta^{1,0} - \theta^{0,1}),
\]

then

\[
\wedge_{\omega^0} (d^* \eta + dk) = d^* \theta + \sqrt{-1} d^* (k^{1,0} - k^{0,1}) = d^* \theta + (-d^* \theta^{1,0} - d^* \theta^{0,1}) = 0.
\]

Hence each exact form \( q \) of type \((\bigwedge^{2,1} \oplus \bigwedge^{1,2})_R\) is given by

\[
q = d(d^* \eta + dk),
\]

where \( d^* \eta + dk \in P^{1,1}_R \). Thus the second cohomology group of the complex (10) is \((H^{2,1}(X) \oplus H^{1,2}(X))_R\). □
Theorem 4-2-4. The cohomology groups of the complex $\#_{CY}$:

$$
\begin{array}{cccccccc}
0 & \longrightarrow & E^0_{CY} & \overset{d}{\longrightarrow} & E^1_{CY} & \overset{d}{\longrightarrow} & E^2_{CY} & \overset{d}{\longrightarrow} & \cdots \\
\end{array}
$$

is respectively given by

\begin{align*}
H^0(\#_{CY}) &= H^{n-1,0}(X), \\
H^1(\#_{CY}) &= H^{n,0}(X) \oplus H^{n-1,1}(X) \oplus P^1_\mathbb{R}, \\
H^2(\#_{CY}) &= H^{n,1}(X) \oplus H^{n-1,2}(X) \oplus (H^{2,1}(X) \oplus H^{1,2}(X))_\mathbb{R},
\end{align*}

In particular, $p^k$ is injective for $k = 0, 1, 2$.

Proof. By (8) and (9), we have the following diagram:

\begin{equation}
\begin{array}{cccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\end{array}
\end{equation}

\begin{equation}
\begin{array}{cccccccc}
0 & \longrightarrow & P^1_\mathbb{R} & \longrightarrow & (\wedge^{2,1} \oplus \wedge^{1,2})_\mathbb{R} & \longrightarrow & \cdots \\
\end{array}
\end{equation}

\begin{equation}
\begin{array}{cccccccc}
0 & \longrightarrow & E^0_{CY} & \longrightarrow & E^1_{CY} & \longrightarrow & E^2_{CY} & \longrightarrow & \cdots \\
\end{array}
\end{equation}

\begin{equation}
\begin{array}{cccccccc}
0 & \longrightarrow & \wedge^{n-1,0} & \longrightarrow & \wedge^{n,0} \oplus \wedge^{n-1,1} & \longrightarrow & \wedge^{n,1} \oplus \wedge^{n-1,2} & \longrightarrow & \cdots \\
\end{array}
\end{equation}

\begin{equation}
\begin{array}{cccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\end{array}
\end{equation}

At first we shall consider $H^2(\#_{CY})$. We assume that $(s, t) \in E^2_{CY}$ is written as an exact form, i.e., $(s, t) = (da, db)$. Let $a$ be an element of $\wedge^{n,0} \oplus \wedge^{n-1,1}$. There is a splitting map $\lambda: \wedge^{n,0} \oplus \wedge^{n-1,1} \rightarrow \wedge^2$ such that $(a, \lambda(a))$ is an element of $E^1_{CY}$. Hence

$$(da, d\lambda(a)) \in E^2_{CY}.$$
By (10), we see that

\[ db - d\lambda(a) \in (\wedge^{2,1} \oplus \wedge^{1,2})_\mathbb{R}. \]

Then by proposition 4-2-3, there exists \( p \in P_{\mathbb{R}}^{1,1} \) such that

\[ db - d\lambda(a) = dp. \]

Hence \((s, t)\) is written as

\[ (s, t) = (da, db) = (da, d(\lambda(a) + p)), \]

where \((a, \lambda(a) + p) \in E_{\text{CY}}^1\). Hence we see that

\[ H^1(\#_{\text{CY}}) = H^{n,1}(X) \oplus H^{n-1,2}(X) \oplus (H^{2,1}(X) \oplus H^{1,2}(X))_\mathbb{R}. \]

Next we shall consider \( H^1(\#_{\text{CY}}) \). Let \((a, b)\) be an element of \( E_{\text{CY}}^1 \) and we assume that \((a, b) = (d\eta, d\gamma)\). Then \( s \) is written as \( s = i_v \Omega^0 \) for some \( v \in TX \). By our definition \( E_{\text{CY}}^0 \), \((i_v \Omega^0, i_v \omega^0)\) is an element of \( E_{\text{CY}}^0 \). Hence \( d\gamma - di_v \omega^0 \in P_{\mathbb{R}}^{1,1} \). By proposition 4-2-3, a \( d \)-exact, primitive form vanishes. Thus \( dt - di_v \omega^0 = 0 \). Hence \((a, b) = (d\eta, d\gamma) = (di_v \Omega^0, di_v \omega^0)\), where \((i_v \Omega^0, i_v \omega^0) \in E_{\text{CY}}^0\). Hence we see that

\[ H^1(\#_{\text{CY}}) = H^{n,0}(X) \oplus H^{n-1,1}(X) \oplus \mathbb{P}_{\mathbb{R}}^{1,1}(X). \]

Similarly we see that \( E_{\text{CY}}^0(X) = H^{n-1,0}(X) \). \( \square \)

Hence we have

**Theorem 4-2-5.** The orbit \( \mathcal{O}_{\text{CY}} \) is metrical, elliptic and topological.

We also have

**Theorem 4-2-6.** The cohomology group \( H^1(\#) \) is the subspace of \( H^n(X, \mathbb{C}) \oplus H^2(X, \mathbb{R}) \) which is defined by equations

\[ \alpha \wedge \omega + \Omega \wedge \beta = 0, \]

\[ \alpha \wedge \overline{\Omega} + \Omega \wedge \overline{\omega} = nc_n \beta \wedge \omega^{n-1}, \]
where \( \alpha \in H^n(X, \mathbb{C}) \), \( \beta \in H^2(X, \mathbb{R}) \).

Let \( P^{p,q}(X) \) be the primitive cohomology group with respect to \( \omega \). Then we have Lefschetz decomposition,

\[
\alpha = \alpha^{n,0} + \alpha^{n-1,1} + \alpha^{n-2,0} \wedge \omega \in P^{n,0}(X) \oplus P^{n-1,0}(X) \oplus P^{n-2,0}(X) \wedge \omega.
\]

\[
\beta = \beta^{2,0} + \beta^{1,1} + \beta^{0,0} \wedge \omega + \beta^{0,2} \in P^{2,0}(X) \oplus P^{1,1}(X) \oplus P^{0,0}(X) \wedge \omega \oplus P^{0,2}(X).
\]

Then the equation in theorem 4-2-6 is written as

\[
\alpha^{n-2,0} \wedge \omega \wedge \omega + \Omega \wedge \beta^{0,2} = 0,
\]

\[
\alpha^{n,0} \wedge \overline{\Omega} = nc_n \beta^{0,0} \omega^n
\]

We see that \( \alpha^{n,0} \in P^{n,0}(X) \) and \( \beta^{0,0} \in P^{0,0}(X) \) are corresponding to the deformation in terms of constant multiplication:

\[
\Omega \rightarrow t\Omega, \quad \omega \rightarrow s\omega
\]

If a Kähler class \([\omega]\) is not invariant under a deformation, such a deformation corresponds to an element of \( \beta^{2,0} \) and \( \alpha^{n-2,0} \). This is in the case of Calabi family of hyperKähler manifolds, i.e., Twistor space gives such a deformation. It must be noted that there is no relation between \( \alpha^{n-1,1} \in P^{n-1,1}(X) \) and \( \beta^{1,1}(X) \in P^{1,1}(X) \). We have from theorem 1-8 in section 1,

**Theorem 4-2-7.** The map \( P \) is locally injective,

\[
P: \mathcal{M}_{CY}(X) \rightarrow H^n(X, \mathbb{C}) \oplus H^2(X, \mathbb{R}).
\]

We also have from theorem 1-9 in section 1,

**Theorem 4-2-8.** Let \( I(\Omega, \omega) \) be the isotropy group of \( (\Omega, \omega) \),

\[
I(\Omega, \omega) = \{ f \in \text{Diff}_0(X) \mid f^*\Omega = \Omega, f^*\omega = \omega \}.
\]
We consider the slice $S_0$ at $\Phi^0 = (\Omega^0, \omega^0)$. Then the isotropy group $I(\Omega^0, \omega^0)$ is a subgroup of $I(\Omega, \omega)$ for each $(\Omega, \omega) \in S_0$.

We define the map $P_{H^2}$ by
\[
P_{H^2} : \mathcal{M}_{CY}(X) \rightarrow \mathbb{P}(H^2(X)),
\]
where
\[
P_{H^2}([\Omega, \omega]) \rightarrow [\omega]_{dR} \in \mathbb{P}(H^2(X)),
\]
$\mathbb{P}(H^2(X))$ denoted the projective space $(H^2(X) - \{0\})/\mathbb{R}^*$. Then we have

**Theorem 4-2-9.** The inverse image $P_{H^2}^{-1}([\omega]_{dR})$ is a smooth manifold.

**Proof.** From theorem 4-2-6 and theorem 4-2-7 the differential of the map $P_{H^2}$ is surjective. Hence from the implicit function theorem $P_{H^2}^{-1}([\omega]_{dR})$ is a smooth manifold.

**Remark.** $P_{H^2}^{-1}([\omega]_{dR})$ is the $\mathbb{C}^*$ bundle over the moduli space of polarized manifolds [8].

§5 HyperKähler structures

Let $V$ be a $4n$ dimensional real vector space. A hyperKähler structure on $V$ consists of a metric $g$ and three complex structures $I, J$ and $K$ which satisfy the followings:

1. $g(u, v) = g(Iu, Iv) = g(JuJv) = g(Ku, Kv)$, for $u, v \in V$,
2. $I^2 = J^2 = K^2 = IJK = -1$.

Then we have the fundamental two forms $\omega_I, \omega_J, \omega_K$ by
\[
\omega_I(u, v) = g(Iu, v), \quad \omega_J(u, v) = g(Ju, v),
\]
\[
\omega_K(u, v) = g(Ku, v).
\]

We denote by $\omega_C$ the complex form $\omega_J + \sqrt{-1} \omega_K$. Let $A_{HK}(V)$ be the set of pairs $(\omega_I, \omega_C)$ corresponding to hyperKähler structures on $V$. As
in section one $A_{HK}(V)$ is the subset of $\wedge^2 \oplus \wedge^2_C$ and the group $GL(4n, \mathbb{R})$ acts on $A_{HK}(V)$. Then we see that $A_{HK}(V)$ is $GL(4n, \mathbb{R})$—orbit with the isotropy group $\text{Sp}(n),$

\[(4) \quad A_{HK}(V) = GL(4n, \mathbb{R})/\text{Sp}(n).\]

We denote by $O_{HK}$ the orbit $A_{HK}(V)$.

**Theorem 5-1.** The orbit $O_{HK}$ is metrical, elliptic and topological.

Let $\Phi^0 = (\omega^0_I, \omega^0_J, \omega^0_K)$ be a hyperKähler structure on a $4n$ dimensional vector space $V$. We denote by $\omega_C^0$ the complex symplectic form $\omega^0_J + \sqrt{-1}\omega^0_K$. Then we consider the pair $(\omega^0_I, \omega^0_C)$. The vector space $E^k_{HK}$ are respectively given by

\[
E^0_{HK} = \{ (i_v \omega^0_I, i_v \omega^0_C) | v \in TX \}
\]
\[
E^1_{HK} = \{ (\rho_a \omega^0_I, \rho_a \omega^0_C) | a \in \text{End}(TX) \}.
\]

Then we consider the projection to the second component and we have the diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & E^0_{HK} & \longrightarrow & E^1_{HK} & \longrightarrow & E^2_{HK} & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \wedge^{1,0} & \longrightarrow & \wedge^{2,0} \oplus \wedge^{1,1} & \longrightarrow & \wedge^{3,0} \oplus \wedge^{2,1} \oplus \wedge^{1,2} & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & &
\end{array}
\]

Let $I, J, K$ be the three almost complex structures on $V$. Then we denote by $\wedge^1_I$ forms of type $(1, 1)$ with respect to $I$. Similarly $\wedge^1_J$ (resp. $\wedge^1_K$) denotes forms of type $(1, 1)$ w.r.t $J$ (resp. $K$). We define $\wedge^2_{HK}$ by the intersection between them,

\[
\wedge^2_{HK} = \wedge^1_I \cap \wedge^1_J \cap \wedge^1_K.
\]
Note that $a \in \wedge^2_{HK}$ is the primitive form with respect to $I, J,$ and $K$. When we identify two forms with $so(4m)$, we have the decomposition:

$$\wedge^2 = sp(4m) \oplus so(4m)/sp(4m).$$

Then $\wedge^2_{HK}$ corresponds to $sp(4m)$. Hence the dimension of $\wedge^2_{HK}$ is $2m^2 + m$. We also see that

$$\dim_{\mathbb{R}} E^1_{HK} = \dim_{\mathbb{R}} gl(4m, \mathbb{R})/sp(4m) = 14m^2 - m,$$

$$\dim_{\mathbb{R}} \wedge^{2,0} \oplus \wedge^{1,1} = 12m^2 - 2m.$$

In fact we see that the kernel of the map $E^1_{HK} \to \wedge^{2,0} \oplus \wedge^{1,1}$ is given by $\wedge^2_{HK}$. We also define $\wedge^3_{HK}$ by real forms of type $(\wedge^{2,1} \oplus \wedge^{2,1})_{\mathbb{R}}$ for each $I, J,$ and $K$. Then we also see that the kernel of the map $E^2_{HK} \to \wedge^{3,0} \oplus \wedge^{2,1} \oplus \wedge^{1,2}$ is $\wedge^3_{HK}$. We consider the following complex:

(\text{HK}) \quad 0 \longrightarrow \wedge^2_{HK} \longrightarrow \wedge^3_{HK} \longrightarrow \cdots

As in proof of Calabi-Yau structures, we see that the cohomology groups of the complex (HK) are respectively given by

$$\mathbb{H}^2_{HK} = \{ \text{real harmonic forms of type } (1, 1) \text{ w.r.t } I, J, K \}$$

$$\mathbb{H}^3_{HK} = \{ \text{real harmonic forms of type } \wedge^{2,1} \oplus \wedge^{1,2} \text{ w.r.t } I, J, K \}.$$

Hence we have the following:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \wedge^2_{HK} & \longrightarrow & \wedge^3_{HK} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
E^0_{HK} & \longrightarrow & E^1_{HK} & \longrightarrow & E^2_{HL} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\wedge^{1,0} & \longrightarrow & \wedge^{2,0} \oplus \wedge^{1,1} & \longrightarrow & \wedge^{3,0} \oplus \wedge^{2,1} \oplus \wedge^{1,2} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 
\end{array}
\]
Theorem 5-2. The cohomology groups of the complex $\#_{HK}$ are given by

\begin{align*}
H^0(\#_{HK}) &= H^{1,0}(X) \\
H^1(\#_{HK}) &= H^{2,0}(X) \oplus H^{1,1}(X) \oplus \mathbb{H}^2_{HK}, \\
H^3(\#_{HK}) &= H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \oplus \mathbb{H}^3_{HK}.
\end{align*}

In particular, the map $p^k$ is injective for $k = 0, 1, 2$.

Proof. The proof is essentially same as in the case of Calabi-Yau structures. Let $\lambda$ be the splitting map $\wedge^{2,0} \oplus \wedge^{1,1} \to E^1_{HK}$. Let $(s, t)$ by an element of $E^2_{HK}$. We assume that $(s, t) = (da, db)$ for $b \in \wedge^{2,0} \oplus \wedge^{1,1}$ and $a \in \wedge^2$. By using the splitting map $\lambda$, we have $(\lambda(b), b) \in E^1_{HK}$. Then $(d\lambda(b), db) \in E^2_{HK}$. Hence $da - d\lambda(b) \in \wedge^3$. Then there is an element $\gamma \in \wedge^2_{HK}$ such that

$$da - d\lambda(b) = d\gamma.$$ 

Hence $(s, t) = (da, db) = (d(\lambda(b) + \gamma), db)$, where $((\lambda(b) + \gamma, b) \in E^1_{HK}$. Thus we have

$$H^2(\#_{HK}) = H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \oplus \mathbb{H}^3_{HK}.$$ 

Similarly we see that

\begin{align*}
H^1(\#_{HK}) &= H^{2,0}(X) \oplus H^{1,1}(X) \oplus \mathbb{H}^2_{HK} \\
H^0(\#_{HK}) &= H^{1,0}(X)
\end{align*}

$\square$

Proof of theorem 5-1. This follows from theorem 5-2. $\square$

§6. $G_2$ structures

Let $V$ be a real 7 dimensional vector space with a positive definite metric. We denote by $S$ the spinors on $V$. Let $\sigma^0$ be an element of $S$
with $\|\sigma^0\| = 1$. By using the natural inclusion $S \otimes S \subset \wedge^* V^*$, we have a calibration by a square of spinors,

$$\sigma^0 \otimes \sigma^0 = 1 + \phi^0 + \psi^0 + \text{vol},$$

where vol denotes the volume form on $V$ and $\phi^0$ (resp. $\psi^0$) is called the associative 3 form (resp. coassociative 4 form). Our construction of these forms in terms of spinors is written in chapter IV §10 of [20] and in section 14 of [11]. Background materials of $G_2$ geometry are found in [13],[15] and [24]. We also have an another description of $\phi^0$ and $\psi^0$. We decompose $V$ into a real 6 dimensional vector space $W$ and the one dimensional vector space $\mathbb{R}$. Let $(\Omega^0, \omega^0)$ be an element of Calabi-Yau structure on $W$ and $t$ a nonzero 1 form on $\mathbb{R}$. Then the 3 form $\phi^0$ and the 4 form $\psi^0$ are respectively written as

$$\phi^0 = \omega^0 \wedge t + \text{Im} \Omega^0, \quad \psi^0 = \frac{1}{2} \omega^0 \wedge \omega^0 - \text{Re} \Omega^0 \wedge t.$$

Then as in section 1, we define $G_2$ orbit $\mathcal{O} = \mathcal{O}_{G_2}$ as

$$\mathcal{O}_{G_2} = \{ (\phi, \psi) = (\rho_g \phi^0, \rho_g \psi^0) \mid g \in \text{GL}(V) \}.$$  

Note that the isotropy group is the exceptional Lie group $G_2$. We denote by $\mathcal{A}_{G_2}(V)$ the orbit $\mathcal{O}_{G_2}$. Let $X$ be a real 7 dimensional compact manifold. Then we define a $GL(7, \mathbb{R})/G_2$ bundle $\mathcal{A}_{G_2}(X)$ by

$$\mathcal{A}_{G_2}(X) = \bigcup_{x \in X} \mathcal{A}_{G_2}(T_x X).$$

Let $\mathcal{E}^1_{G_2}$ be the set of smooth global sections of $\mathcal{A}_{G_2}(X)$,

$$\mathcal{E}^1_{G_2}(X) = \Gamma(X, \mathcal{A}_{G_2}(X)).$$

Then the moduli space of $G_2$ structures over $X$ is given as

$$\mathcal{M}_{G_2}(X) = \{ (\phi, \psi) \in \mathcal{E}^1_{G_2} \mid d\phi = 0, d\psi = 0 \} / \text{Diff}_0(X).$$

We shall prove unobstructedness of $G_2$ structures.
Theorem 6-1. The orbit \( \mathcal{O}_{G_2} \) is metrical, elliptic and topological.

The rest of this section is devoted to prove theorem 6-1. In the case of \( G_2 \), each \( E^i \) is written as

\[
E^0 = E^0_{G_2} = \{ (i_v\phi^0, i_v\psi^0) \in \Lambda^2 \oplus \Lambda^3 | v \in V \} \\
E^1 = E^1_{G_2} = \{ (\rho_\xi \phi^0, \rho_\xi \psi^0) \in \Lambda^3 \oplus \Lambda^4 | \xi \in \mathfrak{gl}(V) \} \\
E^2 = E^2_{G_2} = \{ (\theta \wedge \phi, \theta \wedge \psi) \in \Lambda^4 \oplus \Lambda^5 | \theta \in \Lambda^1, (\phi, \psi) \in E^1_{G_2} \}.
\]

The Lie group \( G_2 \) is a subgroup of \( \text{SO}(7) \) and we see that \( G_2 = \{ g \in \text{GL}(V) | \rho_g \phi^0 = \phi^0 \} \). Hence we have the metric \( g_\phi \) corresponding to each 3 form \( \phi \). Let \( *_\phi \) be the Hodge star operator with respect to the metric \( g_\phi \). Then a non linear operator \( \Theta(\phi) \) is defined as

\[
(1) \quad \Theta(\phi) = *_\phi \phi.
\]

According to [13], the differential of \( \Theta \) at \( \phi \) is described as

\[
(2) \quad J(\phi) = d\Theta(a)_{\phi} = \frac{4}{3} *_1(a) + *_{\pi_7}(a) - *_{\pi_{27}}(a),
\]

for each \( a \in \Lambda^3 \), where we use the irreducible decomposition of 3 forms on \( V \) under the action of \( G_2 \),

\[
(3) \quad \Lambda^3 = \Lambda^3_1 + \Lambda^3_7 + \Lambda^3_{27},
\]

and each \( \pi_i \) is the projection to each component for \( i = 1, 7, 27 \), ( see also [12] for the operator \( J \) ). From (1) the orbit \( \mathcal{O}_{G_2} \) is written as

\[
(4) \quad \mathcal{O}_{G_2} = \{ (\phi, \Theta(\phi)) | \phi \in \Lambda^3 \}.
\]

Since \( E^1_{G_2}(V) \) is the tangent space of the orbit \( \mathcal{O}_{G_2} \) at \( (\phi^0, \psi^0) \), from (2) the vector space \( E^1_{G_2}(V) \) is also written as

\[
(5) \quad E^1_{G_2}(V) = \{ (a, Ja) \in \Lambda^3 \oplus \Lambda^4 | a \in \Lambda^3 \}.
\]
Let $X$ be a real 7 dimensional compact manifold and $(\phi^0, \psi^0)$ a closed element of $E^1_{G_2}(X)$. Then we have a vector bundle $E^i_{G_2}(X) \to X$ by

$$E^i_{G_2}(X) = \bigcup_{x \in X} E^i_{G_2}(T_x X),$$

for each $i = 0, 1, 2$. Then we have the complex $\#G_2$,

$$0 \longrightarrow \Gamma(E^0_{G_2}) \xrightarrow{d_0} \Gamma(E^1_{G_2}) \xrightarrow{d_1} \Gamma(E^2_{G_2}) \longrightarrow \cdots.$$

The complex $\#G_2$ is a subcomplex of the de Rham complex,

$$0 \longrightarrow \Gamma(E^0_{G_2}) \xrightarrow{d_0} \Gamma(E^1_{G_2}) \xrightarrow{d_1} \Gamma(E^2_{G_2}) \xrightarrow{d_2} \cdots \xrightarrow{d} \Gamma(\wedge^2 \oplus \wedge^3) \xrightarrow{d} \Gamma(\wedge^3 \oplus \wedge^4) \xrightarrow{d} \Gamma(\wedge^4 \oplus \wedge^5) \xrightarrow{d} \cdots.$$

Then we have the map $p^1: H^1(\#G_2) \to H^3(X) \oplus H^4(X)$ and $p^2: H^2(\#G_2) \to H^4(X) \oplus H^5(X)$. The following lemma is shown in [12].

**Lemma 6-2.** Let $a^3 = db^2$ be an exact 3 form, where $b^2 \in \Gamma(\wedge^2)$. If $dJdb^2 = 0$, then there exists $\gamma^2 \in \Gamma(\wedge^2)$ such that $db^2 = d\gamma^2$.

We shall show that $p^1$ is injective by using lemma 6-2.

**Proposition 6-3.** Let $\alpha = (a^3, a^4)$ be an element of $\Gamma(E^1_{G_2})$. We assume that there exists $(b^2, b^3) \in \Gamma(\wedge^2 \oplus \wedge^3)$ such that

$$a^3 = db^2.$$

Then there exists $\gamma = (\gamma^2, \gamma^3) \in \Gamma(E^0_{G_2})$ satisfying

$$(db^2, db^3) = (d\gamma^2, d\gamma^3).$$

**Proof.** From (5) an element of $\Gamma(E^1_{G_2})$ is written as

$$(a^3, a^4) = (a^3, Ja^3).$$
From (7) we have

\[ dJdb^2 = da^4 = ddb^3 = 0 \]  

From lemma 6-2 we have \( \gamma^2 \in \Gamma(\wedge^2) \) such that

\[ db^2 = d\gamma^2. \]  

Since \( \gamma \in \Gamma(\wedge^2) \), \( \gamma \) is written as

\[ \gamma = i_v \phi^0, \]  

where \( v \) is a vector field. Since \( \phi^0 \) is closed, \( d\gamma \) is given by the Lie derivative,

\[ d\gamma = di_v \phi^0 = L_v \phi^0. \]  

Then since \( \text{Diff}_0 \) acts on \( E^1_{G_2} \), \( (L_v \phi^0, L_v \psi^0) = (di_v \phi^0, di_v \psi^0) \) is an element of \( \Gamma(E^1_{G_2}) \). Hence from (5), we see

\[ di_v \psi^0 = Jdi_v \phi^0 = Jd\gamma^2. \]  

From (12) we have

\[ (db^2, db^3) = (db^2, Jdb^2) = (di_v \phi, di_v \psi), \]  

where \( (i_v \phi^0, i_v \psi^0) \in \Gamma(E^0_{G_2}) \). \( \Box \)

Next we shall show that \( p^2 \) is injective.

**Lemma 6-4.** Let \( V \) be a real 7 dimensional vector space with a \( G_2 \) structure \( \Phi^0_V \). Let \( u \) be a non-zero one form on \( V \). Then for any two form \( \eta \) there exists \( \gamma \in \wedge^2_{14} \) such that

\[ u \wedge J(u \wedge \eta) = u \wedge J(u \wedge \gamma) = -2 \ast \|u\| \gamma, \]

\[ i_v \gamma = 0, \]
where \( v \) is the vector which is metrical dual of the one form \( u \) and \( * \) is the Hodge star operator.

**Proof.** The two forms \( \wedge^2 \) is decomposed into the irreducible representations of \( G_2 \),

\[
\wedge^2 = \wedge^2_7 \oplus \wedge^2_{14}.
\]

We denote by \( \eta_7 \) the \( \wedge^2_7 \)-component of \( \eta \in \wedge^2 \). The subspace \( u \wedge \wedge^2 \) is defined by \( \{ u \wedge \eta \in \wedge^3 | \eta \in \wedge^2 \} \). we also denote by \( u \wedge \wedge^2_7 \) the subspace \( \{ u \wedge \eta_7 \in \wedge^3 | \eta \in \wedge^2 \} \). Then we have the orthogonal decomposition,

\[
(6-4-1) \quad u \wedge \wedge^2 = u \wedge \wedge^2_7 \oplus \left( u \wedge \wedge^2_7 \right)^\perp,
\]

where \( \left( u \wedge \wedge^2_7 \right)^\perp \) is the orthogonal complement. By the decomposition 6-4-1, \( u \wedge \eta \) is written as

\[
(6-4-2) \quad u \wedge \eta = u \wedge \eta_7 + u \wedge \hat{\eta}.
\]

for \( \hat{\eta} \in \wedge^2 \). Then we see that

\[
(6-4-3) \quad i_v(u \wedge \hat{\eta}) \in \wedge^2_{14}.
\]

Since \( \eta_7 \) is expressed as \( i_w \phi^0 \) for \( w \in V \), we have

\[
\begin{align*}
    u \wedge J(u \wedge \eta_7) &= u \wedge J(u \wedge i_w \phi^0) \\
    &= u \wedge J\hat{\rho}_a \phi^0,
\end{align*}
\]

where \( a = w \otimes u \in V \otimes V^* \cong \text{End}(V) \). Since \( J\hat{\rho}_a \phi^0 = \hat{\rho}_a \psi^0 \),

\[
    u \wedge J\hat{\rho}_a \phi^0 = u \wedge \hat{\rho}_a \psi^0 = u \wedge (u \wedge i_w \psi^0) = 0.
\]

Hence

\[
(6-4-4) \quad u \wedge J(u \wedge \eta_7) = 0.
\]
Then by 6-4-2 we have

\[(6-4-5) \quad u \wedge J(u \wedge \eta) = u \wedge J(u \wedge \hat{\eta}).\]

\(\hat{\eta}\) is written as

\[(6-4-6) \quad \hat{\eta} = \frac{1}{2\|u\|^2} (i_v(u \wedge \hat{\eta}) + u \wedge i_v\hat{\eta}).\]

We define \(\gamma\) by

\[\gamma = \frac{1}{2\|u\|^2} i_v(u \wedge \hat{\eta}).\]

By 6-4-3, \(\gamma \in \wedge^2_{14}\). By 6-4-5,6 we have

\[(6-4-7) \quad u \wedge J(u \wedge \eta) = u \wedge J(u \wedge \gamma).\]

Since \(\gamma \in \wedge^2_{14}\), \(\gamma \wedge \psi^0 = 0\). Then it follows that

\[(6-4-8) \quad \psi^0 \wedge u \wedge \gamma = 0.\]

We also have \(*\gamma = -\gamma \wedge \phi^0\) from \(\gamma \in \wedge^2_{14}\). Since \(i_v\gamma = 0\), we have

\[(6-4-9) \quad \phi^0 \wedge u \wedge \gamma = 0.\]

By 6-4-8 and 6-4-9, we have

\[(6-4-10) \quad u \wedge \gamma \in \wedge^3_{27}.\]

Then by 6-4-7,

\[u \wedge J(u \wedge \eta) = u \wedge J(u \wedge \gamma)\]

\[= -u \wedge *(u \wedge \gamma) = -*i_vu \wedge \gamma\]

\[= -2\|u\|^2(*\gamma)\]

\(\square\)
Proposition 6-5. Let $E^2_{G_2}(V)$ be the vector space as in before. Then we have an exact sequence,

$$\begin{array}{cccc}
0 & \longrightarrow & \wedge^5_{14} & \longrightarrow & E^2_{G_2}(V) & \longrightarrow & \wedge^4 & \longrightarrow & 0
\end{array}$$

Proof. The map $E^2_{G_2} \rightarrow \wedge^4$ is the projection to the first component. We denote by Ker the Kernel of the map $E^2_{G_2} \rightarrow \wedge^4$. We shall show that Ker $\cong \wedge^5_{14}$. Let $\{v_1, v_2, \cdots, v_7\}$ be an orthonormal basis of $V$. We denote by $\{u^1, u^2, \cdots, u^7\}$ the dual basis of $V^*$. Let $(s, t)$ be an element of $E^2_{G_2}(V)$, where $s \in \wedge^4$ and $t \in \wedge^5$. Then we have the following description:

(6-5-1) \quad s = u^1 \wedge a_1 + u^2 \wedge a_2 + \cdots + u^7 \wedge a_7,

(6-5-2) \quad t = u^1 \wedge Ja_1 + u^2 \wedge Ja_2 + \cdots + u^7 \wedge Ja_7.

where $a_1, a_2, \cdots, a_7 \in \wedge^3$ satisfying

$$i_{v_l}a_m = 0, \forall l < m.$$ 

We assume that $(s, t) \in$ Ker. Then $s = 0$. By 6-5-1, we see that $u^l \wedge a_l = 0$, for all $l$. Hence each $a_l$ is written as

(6-5-3) \quad a_l = u^l \wedge \eta_l

where $\eta_l \in \wedge^2$. By (6-5-2) we have

$$t = \sum_{l=1}^7 u^l \wedge J(u^l \wedge \eta_l).$$

Then it follows from lemma 6-4 there exists $\gamma_l$ such that

$$t = \sum_{l=1}^7 u^l \wedge J(u^l \wedge \gamma_l) = -2 \sum_{l=1}^7 ||u^l||^2(*\gamma_l),$$

where $\gamma_l \in \wedge^2_{14}$. Hence $t \in \wedge^5_{14}$. Therefore we see that Ker $= \wedge^5_{14}$. □
Lemma 6-6. Let $X$ be a compact 7 dimensional manifold with $G_2$ structure $\Phi^0$, (i.e., $d\Phi^0 = 0$). Then for any two form $\eta$ there exists $\gamma \in \bigwedge_{14}^2$ such that
\[
  dJd\eta = dJd\gamma = -*\triangle\gamma, \\
  d^*\gamma = 0.
\]

Proof. We denote by $d\wedge^2$ the closed subspace $\{d\eta | \eta \in \bigwedge^2\}$. Since $d\wedge_7^2 = \{d\eta_7 | \eta_7 \in \bigwedge_7^2\}$ is the closed subspace of $d\wedge^2$, we have the decomposition,
\[
  (6-6-1) \quad d\wedge^2 = d\wedge_7^2 \oplus (d\wedge_7^2)^\perp
\]
where $(d\wedge_7^2)^\perp$ denotes the orthogonal subspace of $d\wedge_7^2$. By 6-6-1, $d\eta$ is written as
\[
  d\eta = d\eta_7 + d\hat{\eta},
\]
where $d\hat{\eta} \in (d\wedge_7^2)^\perp$. Hence we have
\[
  (6-6-2) \quad d^*d\hat{\eta} \in \bigwedge_{14}^2
\]
As in the proof of lemma 6-4, $\eta_7$ is written as $i_w\phi^0$ for some $w \in TX$. Hence
\[
  (6-6-3) \quad dJd\eta_7 = dJdi_w\phi^0 = dJL_w\phi^0 = dL_w\psi^0 = ddi_w\psi^0 = 0.
\]
Thus $dJd\eta = dJd\hat{\eta}$. By the Hodge decomposition, we have
\[
  \hat{\eta} = Harm(\hat{\eta}) + dd^*G\hat{\eta} + d^*dG\hat{\eta},
\]
where $Harm(\hat{\eta})$ is the harmonic part of $\hat{\eta}$ and $G$ denotes the Green operator. We define $\gamma$ by
\[
  \gamma = d^*dG\hat{\eta}.
\]
Then by Chern’s theorem \((\pi_7 G = G \pi_7)\) and 6-6-2, we see that \(\gamma \in \wedge_{14}^2\).
Then \(d\hat{\eta} = d\gamma\) and \(d^*\gamma = 0\). Since \(\gamma \in \wedge_{14}^2\), we have \(\gamma \wedge \psi^0 = 0\) and \(*\gamma = -\gamma \wedge \phi^0\). Hence we have

\[
\begin{align*}
(6-6-4) & \quad d\gamma \wedge \phi^0 = 0, \\
(6-6-5) & \quad d\gamma \wedge \psi^0 = 0.
\end{align*}
\]

Hence it follows from 6-6-4,5 that

\[
\begin{align*}
(6-6-6) & \quad d\gamma \in \wedge_{27}^3.
\end{align*}
\]

Then by 6-6-6,

\[
dJd\gamma = -d^*d\gamma = -*\Delta\gamma.
\]

By 6-6-3,

\[
dJd\eta = -*\Delta\gamma.
\]

\[\square\]

**Proposition 6-7.**

\[
H^2(\#G_2) = H^4(X) \oplus H^5_{14}(X).
\]

In particular,

\[
p^2: H^2(\#G_2) \longrightarrow H^4(X) \oplus H^5(X)
\]

is injective.

**Proof.** Let \((s, t)\) be an element of \(E^2_{G_2}(X)\). We assume that \(s, t\) are exact forms respectively, i.e.,

\[
(6-7-1) \quad s = da, \ t = db,
\]

for some \(a \in \wedge^3\) and \(b \in \wedge^4\). Then we shall show that there exists \(\tilde{a} \in \wedge^3\) such that \(da = d\tilde{a}\) and \(db = dJ\tilde{a}\). Since \((da, dJa)\) is an element of \(E^2_{G_2}\), it follows from proposition 6-5 that

\[
(6-7-2) \quad db - dJa \in \wedge_{14}^5.
\]
We shall show that there exists $\eta \in \wedge^2$ satisfying,

$$(6-7-3)\quad db = dJ(a + d\eta)$$

In order to solve the equation (6-7-3), we apply lemma 6-6. Then there exists $\gamma \in \wedge^2_{14}$ such that

$$(6-7-4)\quad dJd\eta = -* \triangle \gamma$$

$$d^*\gamma = 0.$$ 

Substituting 6-7-4 to the equation (6-7-3), we have

$$(6-7-5)\quad -* \triangle \gamma = db - dJa$$

Then by (6-7-2), there exists a solution $\gamma$ of the equation (6-7-5),

$$\gamma = -G * (db - dJa) \in \wedge^2_{14}.$$ 

Hence if we set $\tilde{a} = a + d\gamma$, $(s, t)$ is written as

\begin{align*}
    s &= d\tilde{a} = d(a + d\gamma), \\
    t &= dJ\tilde{a} = dJ(a + d\gamma)
\end{align*}

Therefore $p^2: H^2(#_{G_2}) \rightarrow H^4(X) \oplus H^5(X)$ is injective. Furthermore we consider harmonic forms $\mathbb{H}^4(X)$ and $\mathbb{H}^5_{14}(X)$. By Chern’s theorem $H^4(X) \oplus H^5_{14}(X) \cong \mathbb{H}^4(X) \oplus \mathbb{H}^5_{14}(X)$. Since the complex $#_{G_2}$ is elliptic, $H^2(#_{G_2})$ is represented by harmonic forms of the complex $#_{G_2}$, i.e.,

$$H^2(#_{G_2}) \cong \mathbb{H}^2(#_{G_2}).$$

Then we see that there is the injective map

$$\mathbb{H}^4(X) \oplus \mathbb{H}^5_{14}(X) \rightarrow \mathbb{H}^2(#_{G_2}).$$
Since \( p^2 \) is injective, we have
\[
H^2(\#G_2) \cong H^4(X) \oplus H^5_{14}(X).
\]

\( \square \)

proof of theorem 6-1. By proposition 6-3 and proposition 6-7, we have
\[
H^0(\#G_2) \cong H^2_7(X) \cong H^3_7(X)
\]
\[
H^1(\#G_2) \cong H^3(X) \cong H^4(X)
\]
\[
H^2(\#G_2) \cong H^4(X) \oplus H^5_{14}(X).
\]

Hence we have the result.

§7. Spin(7) structures

Let \( V \) be a real 8 dimensional vector space with a positive definite metric. We denote by \( S \) the spinors of \( V \). Then \( S \) is decomposed into the positive spinor \( S^+ \) and the negative spinor \( S^- \). Let \( \sigma_0^+ \) be a positive spinor with \( \| \sigma_0^+ \| = 1 \). Then under the identification \( S \otimes S \cong \wedge^* V \), we have a calibration by the square of the spinor,
\[
\sigma_0^+ \otimes \sigma_0^+ = 1 + \Phi^0 + vol,
\]
where \( vol \) denotes the volume form on \( V \) and \( \Phi^0 \) is called the Cayley 4 form on \( V \) (see [11], [20] for our construction in terms of spinors). Background materials of Spin(7) geometry are found in [14],[15] and [24]. we decompose \( V \) into a real 7 dimensional vector space \( W \) and the one dimensional vector space \( \mathbb{R} \),

\[
V = W \oplus \mathbb{R}.
\]

Then a Cayley 4 form \( \Phi^0 \) is defined as
\[
\phi^0 \wedge \theta + \psi^0 \in \wedge^4 V^*,
\]
where \((\phi^0, \psi^0) \in \mathcal{O}_{G_2}(W)\) and \(\theta\) is non zero one form on \(\mathbb{R}\). We define an orbit \(\mathcal{O}_{\text{Spin}(7)} = \mathcal{A}_{\text{Spin}(7)}(V)\) by
\[
\mathcal{O}_{\text{Spin}(7)} = \{ \rho_g \Phi^0 \mid g \in \text{GL}(V) \}.
\]
Since the isotropy is \(\text{Spin}(7)\), the orbit \(\mathcal{O}_{\text{Spin}(7)}\) is written as
\[
\mathcal{O}_{\text{Spin}(7)} = \text{GL}(V)/\text{Spin}(7).
\]
Let \(X\) be a real 8 dimensional compact manifold. Then we define \(\mathcal{A}_{\text{Spin}(7)}(X)\) by
\[
\mathcal{A}_{\text{Spin}(7)}(X) = \bigcup_{x \in X} \mathcal{A}_{\text{Spin}(7)}(T_x X) \rightarrow X.
\]
We denote by \(\mathcal{E}^1_{\text{Spin}(7)}(X)\) the set of global section of \(\mathcal{A}_{\text{Spin}(7)}(X)\),
\[
\mathcal{E}^1_{\text{Spin}(7)}(X) = \Gamma(X, \mathcal{A}_{\text{Spin}(7)}(X)).
\]
Then we define the moduli space of \(\text{Spin}(7)\) structures over \(X\) as
\[
\mathcal{M}_{\text{Spin}(7)}(X) = \{ \Phi \in \mathcal{E}^1_{\text{Spin}(7)} \mid d\Phi = 0 \} / \text{Diff}_0(X).
\]
The following theorem is shown in [15]

**Theorem 7-1.** [15] The moduli space \(\mathcal{M}_{\text{Spin}(7)}(X)\) is a smooth manifold with
\[
\dim \mathcal{M}_{\text{Spin}(7)}(X) = b_1^4 + b_7^4 + b_{35}^4,
\]
where Harmonic 4 forms on \(X\) is decomposed into irreducible representations of \(\text{Spin}(7)\),
\[
\mathbb{H}^4(X) = \mathbb{H}_1^4 \oplus \mathbb{H}_7^4 \oplus \mathbb{H}_{27}^4 \oplus \mathbb{H}_{35}^4,
\]
each \(b_i^4\) denoted \(\dim \mathbb{H}_i^4\), for \(i = 1, 7, 27\) and \(35\).

Note that \(\mathbb{H}^4(X)\) is decomposed into self dual forms and anti-self dual forms,
\[
\mathbb{H}^4(X) = \mathbb{H}^+ \oplus \mathbb{H}^-,
\]
where
\[
\mathbb{H}^+(X) = \mathbb{H}_1^4 \oplus \mathbb{H}_7^4 \oplus \mathbb{H}_{27}^4, \quad \mathbb{H}^- = \mathbb{H}_{35}^4.
\]
We shall show theorem 7-1 by using our method in section one.
**Theorem 7-2.** The orbit \( O_{\text{Spin}(7)} \) is metrical, elliptic and topological.

Since \( \text{Spin}(7) \) is a subgroup of \( \text{SO}(8) \), we have the metric \( g_{\Phi^0} \) for each \( \Phi^0 \in O_{\text{Spin}(7)} \). For each \( \Phi^0 \in O_{\text{Spin}(7)}(V) \), \( \wedge^3 \) and \( \wedge^4 \) are orthogonally decomposed into the irreducible representations of \( \text{Spin}(7) \),

\[
\wedge^3 = \wedge^3_8 \oplus \wedge^3_{48},
\]

\[
\wedge^4 = \wedge^+ \oplus \wedge^- = (\wedge^4_1 \oplus \wedge^4_7 \oplus \wedge^4_{27}) \oplus \wedge^4_{35},
\]

where \( \wedge^p_i \) denotes the irreducible representation of \( \text{Spin}(7) \) of \( i \) dimensional. We denote by \( \pi_i \) the orthogonal projection to each component. Let \( X \) be a real 8 dimensional compact manifold with a closed form \( \Phi^0 \in \mathcal{E}^1_{\text{Spin}(7)}(X) \). Let \( g_{\Phi^0} \) be the metric corresponding to \( \Phi^0 \). Then there is a unique parallel positive spinor \( \sigma^+_0 \in \Gamma(S^+) \) with

\[
\sigma^+_0 \otimes \sigma^+_0 = 1 + \Phi^0 + \text{vol},
\]

where \( S^+ \otimes S^+ \) is identified with the subset of Clifford algebra \( \text{Cliff} \cong \wedge^* \) (see [16]). By using the parallel spinor \( \sigma^+_0 \), the positive and negative spinors are respectively identified with following representations,

\[
\Gamma(S^+) \cong \Gamma(\wedge^4_1 \oplus \wedge^4_7),
\]

\[
\sigma^+ \rightarrow \sigma^+ \otimes \sigma_0^+,
\]

(1)

\[
\Gamma(S^-) \cong \Gamma(\wedge^3_8),
\]

\[
\sigma^- \rightarrow \sigma^- \otimes \sigma_0^+,
\]

(2)

where \( \sigma^\pm \in \Gamma(S^\pm) \). Under the identification (1) and (2), The Dirac operator \( D^+: \Gamma(S^+) \rightarrow \Gamma(S^-) \) is written as

\[
\pi_8 \circ d^*: \Gamma(\wedge^4_1 \oplus \wedge^4_7) \rightarrow \Gamma(\wedge^3_8).
\]

In particular \( \text{Ker} \; \pi_8 \circ d^* \) are Harmonic forms in \( \Gamma(\wedge^4_1 \oplus \wedge^4_7) \). Hence we have
Lemma 7-3. 

\[ \text{Ker } \pi_8 \circ d^* = \mathbb{H}^4_1(X) \oplus \mathbb{H}^4_7(X). \]

In the case of Spin(7), each \( E^i = E^i_{\text{Spin}(7)} \) is given by

\[ E^0_{\text{Spin}(7)} = \wedge^3_8, \quad E^1_{\text{Spin}(7)} = \wedge^1_1 \oplus \wedge^4_7 \oplus \wedge^4_. \]

Let \( \alpha \) be an element of \( \Gamma(E^1_{\text{Spin}(7)}(X)) \). We assume that

\( (3) \quad d\alpha = 0, \quad \pi_8 d^* \alpha = 0, \)

So that is, \( \alpha \) is an element of \( \mathbb{H}^1(\#) \), where \( \# \) is the complex

\[
\begin{array}{cccccc}
0 & \longrightarrow & \Gamma(E^0_{\text{Spin}(7)}) & \longrightarrow & \Gamma(E^1_{\text{Spin}(7)}) & \longrightarrow & \Gamma(E^2_{\text{Spin}(7)}) & \longrightarrow & \cdots \\
& & d_0 & & d_1 & & d_1 & & d_1 \\
& & \| & & \| & & \| & & \\
\cdots & \longrightarrow & \Gamma(\wedge^3_8) & \longrightarrow & \Gamma(\wedge^4_1 \oplus \wedge^4_7 \oplus \wedge^4_) & \longrightarrow & \Gamma(\wedge^5) & \longrightarrow & \cdots \\
\end{array}
\]

(Note that \( d_0^* = \pi_8 d^* \).) We decompose \( \alpha \) into the self-dual form and the anti-self-dual form,

\[ \alpha = \alpha^+ + \alpha^- \in \Gamma(\wedge^+) \oplus \Gamma(\wedge^-). \]

From (3) we have

\[ d\alpha^+ + da^- = 0 \]
\[ \pi_8 * d\alpha^+ - \pi_8 * d\alpha^- = 0. \]

Hence we have \( \pi_8 d^* \alpha^+ = 0 \). From lemma 7-3, we see that \( d\alpha^+ = 0 \). Hence we also have \( d\alpha^- = 0 \) and it implies that \( \alpha \) is a harmonic form with respect to the metric \( g_{\Phi_0} \). Hence the map \( p: H^1(\#) \cong \mathbb{H}^1(\#) \longrightarrow H^4(X) \cong \mathbb{H}^4(X) \) is injective.
Theorem 7-4. The cohomology groups of the complex $\#_{\text{Spin}(7)}$ are respectively given by

\[
\begin{align*}
H^0(\#_{\text{Spin}(7)}) & \cong H^3_8(X), \\
H^1(\#_{\text{Spin}(7)}) & \cong H^4_1(X) \oplus H^4_7(X) \oplus H^4_\perp(X), \\
H^2(\#_{\text{Spin}(7)}) & = H^5(X),
\end{align*}
\]

In particular $p^1$ and $p^2$ are respectively injective.

Proof. It is sufficient to show that $H^2(\#_{\text{Spin}(7)}) = H^5(X)$. Since anti-self dual forms $\wedge^4_\perp$ is the subset of $E^1_{\text{Spin}(7)}$, we see that our result.

Proof of theorem 7-2. This follows from theorem 7-4.

REFERENCES

[1] A.L.Besse, Einstein manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete 10, Springer-Verlag, Berlin-New York, 1987.
[2] F. A. Bogomolov, Hermitian Kähler manifolds, Dokl. Akad. Nauk SSSR 243 no. 5 (1978), 1101–1104.
[3] R. Bryant, Metrics with exceptional holonomy, Ann of Math 126 (1987), 525-576.
[4] P. Candelas and X.C. de la Ossa, Moduli space of Calabi-Yau manifolds, Nuclear Phys. B 355 (1991), 455–481.
[5] S. S. Chern, On a generalization of Kähler geometry, Algebraic geometry and Topology, A symposium in honor of S. Lefshetz (1957), Princeton, 103-121.
[6] S.K. Donaldson and P.B. Kronheimer, The Geometry of Four-Manifolds, Oxford Mathematical Monographs, Oxford Science publications, 1990.
[7] D.G. Ebin, The moduli space of riemannian metrics, Global Analysis, Proc. Symp. Pure Math. AMS 15 (1968), 11-40.
[8] A. Fujiki and G. Schumacher, The moduli space of Extremal compact Kähler manifolds and Generalized Weil-Perterson Metrics, Publ. RIMS, Kyoto Univ 26. No.1 (1990), 101-183.
[9] P.B. Gilky, Invariance Theory, The Heat Equation, and the Atiyah-Singer Index Theorem, Mathematical Lecture Series, vol. 11, Publish or Perish, Inc, 1984.
[10] G.B. Gurevich, Foundations of the Theory of Algebraic Invariants, P.Noordhoff LTD-Groningen, The Netherlands, 1964.
[11] F.R. Harvey, Spinors and Calibrations, Perspectives in Mathematics, vol. 9, Academic Press, Inc, 1990.
[12] N. Hitchin, *The geometry of three-forms in six and seven dimensions*, math.DG/0010054 (2000).
[13] D.D. Joyce, *Compact Riemannian 7-manifolds with holonomy G_2*, I, II, J. Differential Geometry 43 (1996), 291-328, 329-375.
[14] D.D. Joyce, *Compact 8—manifolds with holonomy Spin(7)*, Inventiones mathematicae 128 (1996), 507-552.
[15] D.D. Joyce, *Compact Manifolds with Special Holonomy*, Oxford mathematical Monographs, Oxford Science Publication, 2000.
[16] Y. Kawamata, *Unobstructed deformations. A remark on a paper of Z. Ran: ”Deformations of manifolds with torsion or negative canonical bundle”,* J. Algebraic Geom. 1 no. 2 (1992), 183–190.
[17] S. Kobayashi, *Differential Geometry of complex vector bundles*, Iwanami Shoten and Princeton University Press, 1987.
[18] K. Kodaira, *Complex manifolds and deformation of complex structures*, Grundlehren der Mathematischen Wissenschaften, vol. 283, Springer-Verlag, New York-Berlin, 1986.
[19] S. Lang, *Differential manifolds*, Springer-Verlag.
[20] H.B. Lawson, Jr and M. Michelsohn, *Spin Geometry*, Princeton University press, 1989.
[21] H. Omori, *Infinite dimensional Lie group*, Translations of Mathematical Monograph, vol. 158, American Mathematical Society.
[22] R.S. Palais, *Foundations of non-linear functional analysis*, Benjamin, New york, 1968.
[23] Z. Ran, *Essays on mirror manifolds*, Internat. Press, 1992, pp. 451-457.
[24] S. Salamon, *Riemannian geometry and holonomy groups*, Pitman Research Notes in Mathematics Series, vol. 201, Longman, Harlow, 1989.
[25] G. Tian, *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Peterson-Weil metric*, Mathematical aspects of string theory (ed. S.-T. Yau), 10 (1987), Advanced Series in Mathematical Physics, World Scientific Publishing Co., Singapore, 629–646.
[26] G. Tian, *Smoothing 3—folds with trivial canonical bundle and ordinary double points*, Essays on mirror manifolds (1992), Internat. Press, 458-479.
[27] A.N. Todorov, *The Weil-Peterson geometry of the moduli space of SU(n ≥ 3) (Calabi-Yau) manifolds. I*, Comm. Math. Phys. 126 (1989), 325–346.

TOYONAKA, OSAKA, 560, JAPAN

E-mail address: goto@math.sci.osaka-u.ac.jp