On LP-Sasakian manifold admitting a generalized symmetric metric connection

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Abstract
In this paper we study certain curvature properties of Lorentzian Para-Sasakian manifold (shortly, LPSM) with respect to the generalized symmetric metric connection. Here we discuss $\xi$-conically, $\xi$-conformally and $\xi$-projectively flat LPSM with respect to the generalized symmetric metric connection and obtain various interesting results. Moreover, we study LPSM with $\tilde{Z}(\xi, V).\tilde{S} = 0$, where $\tilde{Z}$ and $\tilde{S}$ are the concircular curvature tensor and Ricci tensor respectively with respect to the generalized symmetric metric connection.

Keywords
LP-Sasakian manifold, quarter-symmetric connection, generalized symmetric connection, $\eta$-Einstein manifold, $\xi$-conically, projectively, conformally flat manifold.

AMS Subject Classification
53C15, 53C25.

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1. Introduction
The Lorentzian Para-Sasakian manifold (shortly, LPSM) has been introduced by Matsumoto [9]. Again, Mihai and Rosca [11] also introduced LPSM and acquired various features. During the last three decades LPSM has been studied by various authors and obtained several results. For this we refer the reader to see [1], [2], [3], [5], [10], [11], [12], [13], [17], [18], [19] and references therein. Among the study of LPSM, most of the research works of this manifold, admitting either semi-symmetric metric connection (see [6], [7], [14], [15], [21]) or quarter-symmetric metric connection (see [7], [8], [16], [20] and also references therein). It should be noted that the quarter-symmetric metric connection is more generalized form that of semi-symmetric metric connection.

In [1], Bahadir and Chaubey introduced a new type of linear connection called generalized symmetric metric connection which is a combination of semi-symmetric metric connection and quarter-symmetric metric connection. A manifold $M$ of dimension $n$ is said to be LPSM if it satisfies

$$\phi\xi = 0, \quad \eta(\phi U) = 0, \quad \phi^2 U = U + \eta(U)\xi,$$

$$g(\phi U, V) = g(U, \phi V), \quad \eta(U) = g(U, \xi), \quad \eta(\xi) = -1,$$

$$g(\phi U, \phi V) = g(U, V) + \eta(U)\eta(V),$$

for all vector fields $U, V$ on $M$, where $\nabla, \phi, \xi, \eta$ and $g$ denote the Levi-Civita connection, a $(1, 1)$ tensor field, a vector field, a $1$-form and a Lorentzian metric respectively.

Relation between generalized symmetric metric connection $\tilde{\nabla}$ and $\nabla$ is given by [1]

$$\tilde{\nabla}_U V = \nabla_U V + \alpha\{\eta(V)U - g(U, V)\xi\} + \beta\{\eta(V)\phi U - g(\phi V, U)\xi\}.$$ 

In section 2 some preliminaries are discussed for further calculations in the subsequent sections. Section 3 is concerned
with $\xi$-concircularity flat LPSM, which gives us an interesting result. Again, $\xi$-conformally flat LPSM is studied in section 4, where we find out a condition for which this manifold can be $\eta$-Einstein. Section 5 deals with projectively flat LPSM. Section 6 is concerned with LPSM satisfying $\tilde{Z}(\xi, V)\tilde{S} = 0$, where we establish a theorem as well as a corollary.

### 2. Preliminaries

Let $R$, $S$ and $r$ be the curvature tensor, Ricci tensor and scalar curvature of $M$ respectively. $\chi(M)$ be the Lie algebra of vector fields of $M$.

The semi-symmetric linear connection have been introduced by Friedmann and Schouten [6].

**Definition 2.1.** [14] $\nabla$ is called semi-symmetric connection [6] if

$$T(U, V) = \eta(V)U - \eta(U)V,$$

holds, where $T$ is the torsion tensor and $U, V \in \chi(M)$.

Again in 1975, Golab [7] introduced quarter-symmetric connection.

**Definition 2.2.** [16] $\nabla$ is said to be quarter-symmetric connection [7] if

$$T(U, V) = \eta(V)\phi U - \eta(U)\phi V,$$

where $T$ is the torsion tensor and $U, V \in \chi(M)$.

**Definition 2.3.** A semi-symmetric and quarter-symmetric connection $\nabla$ satisfying

$$\nabla g = 0,$$

is called a semi-symmetric metric connection (shortly, ssmc) and quarter-symmetric metric connection (shortly, qsmc) respectively. Otherwise it is called non-metric connection.

**Definition 2.4.** [1] A metric connection $\nabla$ is said to be a generalized symmetric metric connection (shortly, gsmc) if $T$ satisfies

$$T(U, V) = \alpha\{\eta(V)U - \eta(U)V\} + \beta\{\eta(V)\phi U - \eta(U)\phi V\},$$

where $\alpha$ and $\beta$ are smooth functions.

We mention that if $\alpha = 1$ and $\beta = 0$, then the gsmc reduces to ssmc. Again if $\alpha = 0$ and $\beta = 1$, then gsmc turns into qsmc. For $M^n(\phi, \xi, \eta, g)$ the following results can be proved easily [2].

$$\phi \xi = 0, \quad \eta(\phi U) = 0, \quad \phi^2 U = U + \eta(U)\xi,$$

$$\phi \xi = 0, \quad \eta(\phi U) = 0, \quad \text{rank}\phi = n - 1.$$

If we write

$$\tilde{\phi} = g(\phi U, V)$$

then we have

$$\nabla e(V) = \tilde{\phi}(U, V), \quad \tilde{\phi}(U, \xi) = 0.$$

$$g(R(U, V)W, \xi) = g(V, W)\eta(U) - g(U, W)\eta(V),$$

$$R(U, V)\xi = \eta(V)U - \eta(U)V,$$

$$S(U, \xi) = (n - 1)\eta(U),$$

$$S(\phi U, \phi V) = S(U, V) + (n - 1)\eta(U)\eta(V)$$

for all vector fields $U, V, W \in \chi(M)$.

**Definition 2.5.** An LPSM is said to be an $\eta$-Einstein if the Ricci sensor $S$ satisfies

$$S(U, V) = a\eta(U)\eta(V) + b\eta(U)\eta(V)$$

for all $U, V \in \chi(M)$, where $a$ and $b$ are scalar functions on $M$.

In LPSM admitting gsmc the following results holds [1].

$$\tilde{V}U\xi = (1 - \beta)\phi U - \alpha U - \alpha\eta(U)\xi,$$

$$\tilde{R}(U, V)\xi = (1 - \beta + \beta^2)\{\eta(V)U - \eta(U)V\} + \alpha(1 - \beta)\{\eta(U)\phi V - \eta(V)\phi U\}$$

$$= k_1\{\eta(V)U - \eta(U)V\} + k_2\{\eta(U)\phi V - \eta(V)\phi U\},$$

(2.1)

$$\tilde{R}(\xi, V)\xi = k_1\{\eta(V)\xi + V\} - k_2\phi U,$$

(2.2)

where $k_1 = (1 - \beta + \beta^2)$, $k_2 = \alpha(1 - \beta)$.

$$\tilde{S}(U, V) = S(U, V) + \{-\alpha\beta + (n - 2)(\alpha\beta - \alpha) + (\beta^2 - 2\beta)\eta(U)\eta(V)$$

$$+ \{(\alpha^2 + \alpha\beta - \beta^2 + \alpha\beta)\eta(U)\eta(V)$$

$$+ \{(\alpha^2 + \alpha\beta - \beta^2 + \alpha\beta)\eta(U)\eta(V)$$

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(2.3)

$$\tilde{S}(U, \xi) = S(U, \xi) + Bg(U, \xi) + C\eta(U)\eta(V),$$

where, $A = -\alpha\beta + (n - 2)(\alpha\beta - \alpha) + (\beta^2 - 2\beta)\eta(U)\eta(V)$

and $C = -2\alpha^2 + n(\alpha^2 + \beta - \beta^2)\eta(U)\eta(V)$. Or,

$$\tilde{S}(U, V) = S(U, V) + A\phi(U, V) + Bg(U, V) + C\eta(U)\eta(V),$$

$$\tilde{S}(U, \xi) = S(U, \xi) + Bg(U, \xi) + C\eta(U)\eta(V),$$

$$\tilde{R}(\xi, V) = \{-\alpha\phi(U) + (1 - \beta)\phi(V) + \beta^2\eta(U)\eta(V)\}$$

$$- k_1\eta(U) + k_2\eta(U)\phi U.$$
Theorem 3.1. The scalar curvature $\check{\kappa}$ of a $\xi$-concircularly flat LPSM with respect to gsme $\tilde{V}$ is given by $\check{\kappa} = n[(n - 1)(1 - \beta + \beta^2) - \alpha(1 - \beta)\text{trace}\tilde{\phi}]$. If we consider the manifold with respect to gsme i.e., $\alpha = 0$ and $\beta = 1$, then the following corollary holds:

Corollary 3.2. The scalar curvature of $\xi$-concircularly flat LPSM with respect to gsme is given by $\check{\kappa} = n(n - 1)$.

Again, if we consider the manifold with respect to ssme i.e., $\alpha = 1$ and $\beta = 0$, then we have

Corollary 3.3. The scalar curvature of $\xi$-concircularly flat LPSM with respect to ssme is given by $\check{\kappa} = n[(n - 1) - \text{trace}\tilde{\phi}]$.

4. $\xi$-conformally flat LPSM

The LPSM is said to be $\xi$-conformally flat w.r.t a gsme $\tilde{V}$ if $\tilde{C}(U, V)\xi = 0$, where $\tilde{C}$ is the conformal curvature tensor. Then

$$\tilde{R}(U, V)\xi = \frac{1}{(n - 2)}[\tilde{S}(V, \xi)V - \tilde{S}(U, \xi)V] + \eta(U)\tilde{Q}V - \eta(U)\tilde{Q}V + \frac{\tilde{\kappa}}{(n - 1)(n - 2)}[\eta(V)U - \eta(U)V] = 0. \quad (4.1)$$

Using (2.1), (2.3) and (2.4) in above we get

$$k_1[\eta(V)U - \eta(U)V] + k_2[\eta(U)\phi V - \eta(\kappa)\phi U]$$

$$- \frac{1}{n - 2}U\{\eta(V)U - \eta(U)V\} = 0.$$
then we have from (4.1)
\[ S(U, V) = [k^2 + B]g(U, V) + [B - C + k^2] \eta(U) \eta(V) \]
i.e., the manifold is \( \eta \) Einstein.

From (4.2) we have
\[ \beta \{ - \alpha + (\beta - 2) \text{trace} \, \phi \} = 0. \tag{4.3} \]

Therefore in view of above the following theorem exists.

**Theorem 4.1.** A \( \xi \)-conformally flat LPSM with respect to gsMc is \( \eta \) Einstein if and only if (4.3) holds.

If we consider the ssMc instead of gsMc i.e., \( \alpha = 1 \) and \( \beta = 0 \), then the following corollary holds:

**Corollary 4.2.** A \( \xi \)-conformally flat LPSM with respect to ssMc is \( \eta \) Einstein.

Again, if we consider the gsMc instead of gsMc i.e., \( \alpha = 0 \) and \( \beta = 1 \), then there is a corollary given as:

**Corollary 4.3.** A \( \xi \)-conformally flat LPSM with respect to gsMc is \( \eta \) Einstein iff trace \( \phi = 0 \).

### 5. \( \xi \)-projectively flat LPSM

We consider an LPSM with gsMc which is \( \xi \)-projectively flat then
\[ \tilde{R}(U, V) \xi = \frac{1}{n - 1} [\tilde{S}(V, \xi)U - \tilde{S}(U, \xi)V]. \]

Putting \( U = \xi \) in above we have
\[ \tilde{R}(\xi, V) \xi = \frac{1}{n - 1} [\tilde{S}(V, \xi)\xi - \tilde{S}(\xi, \xi)V]. \]

Using (2.2) and (2.5) in above
\[ k_1 \{ \eta(V) \xi + V \} - \phi \eta(V) = - \frac{1}{n - 1} \left[ ((n - 1) \eta(V) + B_2 \eta(V) - C \eta(V)) \right] \xi \\
- \left( -(n - 1) - B + C \right) \xi. \]

Or, \( (n - 1) \phi \eta(V) = [(n - 1)k_1 - B]\eta(V) + B_2 \eta(V) - C \eta(V) \xi + V, \)
where \( B = (n - 1) + B - C \), or,
\[ \alpha(1 - \beta)((n - 1) \eta(\xi)U - \text{trace} \, \phi \eta(U) \xi) = 0. \]

If \( \alpha(1 - \beta) \neq 0 \), then
\[ (n - 1) \eta(\xi)U - \text{trace} \, \phi \eta(U) \xi = 0. \]

Hence we come to the following conclusion.

**Theorem 5.1.** If an LPSM is \( \xi \)-projectively flat with respect to gsMc, then \( (n - 1) \eta(\xi)U = \text{trace} \, \phi \eta(U) \xi \), provided \( \alpha(1 - \beta) \neq 0 \).

If we consider ssMc instead of gsMc i.e., \( \alpha = 1 \) and \( \beta = 0 \), then the following corollaries holds:

**Corollary 5.2.** If an LPSM is \( \xi \)-projectively flat with respect to ssMc, then \( (n - 1) \eta(\xi)U = \text{trace} \, \phi \eta(U) \xi \).

**Corollary 5.3.** In an LPSM with respect to ssMc, if \( \text{trace} \, \phi = 0 \) then, it can not be \( \xi \)-projectively flat.

### 6. LPSM with \( \tilde{Z}(\xi, V), \tilde{S} = 0 \)

An LPSM with \( \tilde{Z}(\xi, V), \tilde{S} = 0 \) imply
\[ \tilde{S}(\tilde{Z}(\xi, V)W, U) + \tilde{S}(\tilde{W}, \tilde{Z}(\xi, V)U) = 0. \]

Putting \( W = \xi \) we get
\[ \tilde{S}(\tilde{Z}(\xi, V)\xi, U) + \tilde{S}(\tilde{\xi}, \tilde{Z}(\xi, V)U) = 0 \]

Or, \( \tilde{S}(\tilde{R}(\xi, V)\xi - m[\eta(V)\xi + V], U) \)
\[ + \tilde{S}(\tilde{\xi}, \tilde{R}(\xi, V)U - m[g(V, U)\xi - \eta(U)V]) = 0, \]
where \( m = \frac{r}{n(n - 1)}. \)

Now
\[ \tilde{S}(\tilde{R}(\xi, V)\xi - m[\eta(V)\xi + V], U) = \tilde{S}(\tilde{Z}(\xi, V)W, U) + \tilde{S}(\tilde{W}, \tilde{Z}(\xi, V)U) = 0. \]

Therefore
\[ \tilde{S}(\tilde{Z}(\xi, V)\xi, U) = \tilde{S}(\tilde{W}, \tilde{Z}(\xi, V)U - m[g(V, U)\xi - \eta(U)V]) = 0. \]

(6.2)
Theorem 6.1. Chaubey, S. K. and De, U. C., Lorentzian para-Sasakian metric manifolds admitting a new type of quarter-symmetric non-metric $\xi$-connection, Inter. J. Math. Phys. Astron., 12(2019), 250–259.

Using (6.2) and (6.3) in (6.1) we get
\[
\begin{align*}
D_1S(U, V) &- k_2S(\phi V, U) + \{D_2 + \alpha \} \hat{\phi}(V, U) \\
&+ g(U, V)\{D_3 + \frac{\beta}{n} - (n - 1)(1 - \beta)\} \\
&+ \{D_4 + B^1m + \beta^2 - K_1B^1\} \eta(U)\eta(V) = 0,
\end{align*}
\]

or,
\[
\begin{align*}
D_1S(U, V) &- k_2S(\phi V, U) \\
&= D_2\hat{\phi}(V, U) + D_6g(U, V) + D_7\eta(U)\eta(V),
\end{align*}
\]

where $D_5 = \{-D_2 + (n - 1)\alpha\}$,

$D_6 = \{-D_3 + \frac{\beta}{n} - (n - 1)(1 - \beta)\}$,

$D_7 = \{-D_4 + B^1m + (n - 1)\beta^2 - K_1B^1\}$. Again
\[
\begin{align*}
D_1S(U, V) &- k_2S(\phi V, U) \\
&= D_2\hat{\phi}(V, U) + D_6g(U, V) + D_7\eta(U)\eta(V).
\end{align*}
\]

Putting $V = \phi V$, we get
\[
\begin{align*}
-k_2S(U, V) + D_1S(\phi V, U) \\
&= D_6\hat{\phi}(V, U) + D_6g(U, V) + \{k_2(n - 1) + D_5\} \eta(U)\eta(V).
\end{align*}
\]

Solving (6.4) and (6.5) we get
\[
\begin{align*}
(D_1^2 - k_2^2)S(U, V) &= (D_1D_5 + k_2D_6)\hat{\phi}(V, U) \\
&+ g(U, V)\{D_1D_6 + k_2D_5\} \\
&+ \{D_1D_7 + k_2(n - 1) + D_3\} \eta(U)\eta(V).
\end{align*}
\]

If $(D_1D_5 + k_2D_6) = 0$ then $M$ is of $\eta$ Einstein. Hence the following theorem can be stated.

Theorem 6.1. If an LPSM admitting gsme satisfies
\[
\begin{align*}
(i)\quad &Z(\xi, V)\hat{S} = 0, \\
(ii)\quad &D_1D_5 + k_2D_6 = 0,
\end{align*}
\]

is $\eta$ Einstein.

If we consider gsme instead of gsme i.e., $\alpha = 0$ and $\beta = 1$, then $k_2 = 0$ and hence $(D_1D_5 + k_2D_6) = 0$ reduces to $(1 - \frac{\beta}{n(n - 1)})^2\text{trace } \hat{\phi} = 0$ and hence we can state the corollary as:

Corollary 6.2. If an LPSM with respect to qsmc satisfies
\[
\begin{align*}
(i)\quad &Z(\xi, V)\hat{S} = 0, \\
(ii)\quad &\hat{\beta} \neq n(n - 1),
\end{align*}
\]

is $\eta$ Einstein if and only if trace $\hat{\phi} = 0$. 

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