Friedmann-Robertson-Walker brane cosmological equations from the five-dimensional bulk (A)dS black hole

Shin’ichi NOJIRI, Sergei D. ODINTSOV, and Sachiko OGUSHI

Department of Applied Physics
National Defence Academy, Hashirimizu Yokosuka 239-8686, JAPAN

Lab. for Fundamental Study, Tomsk State Pedagogical University, 634041 Tomsk, RUSSIA

Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, JAPAN

ABSTRACT

In the first part of this work we review the equations of motion for the brane presented in Friedmann-Robertson-Walker (FRW) form, when bulk is 5-dimensional (A)dS Black Hole. The spacelike (timelike) FRW brane equations are considered from the point of view of their representation in the form similar to 2-dimensional CFT entropy, so-called Cardy-Verlinde (CV) formula. The following 5-dimensional gravities are reviewed: Einstein, Einstein-Maxwell and Einstein with brane quantum corrections. The second part of the work is devoted to study FRW brane equations and their representation in CV form, brane induced matter and brane cosmology in Einstein-Gauss-Bonnet (GB) gravity. In particular, we focus on the inflationary brane cosmology. The energy conditions for brane matter are also analyzed. We show that for some values of GB coupling constant (bulk is AdS BH) the brane matter is not CFT. Its energy density and pressure are not always positive. The appearance of logarithmic corrections in brane cosmology is discussed.

PACS: 98.80.Hw,04.50.+h,11.10.Kk,11.10.Wx

1nojiri@cc.nda.ac.jp
2 odintsov@mail.tomsknet.ru
3 JSPS fellow, ogushi@yukawa.kyoto-u.ac.jp
1 Introduction

The recent astronomical data indicate that observable universe is currently-accelerating [1]. This observation, in turn, indicates that the universe has a positive cosmological constant. As a result it is likely that universe evolves into the future (asymptotically) de Sitter phase.

The recent brane-world approach (as manifestation of holographical principle) to the description of the observable universe as a brane embedded in
higher dimensional bulk space has brought many interesting ideas to the realization of (asymptotically) de Sitter brane universe. First of all, it became clear that brane matter may be induced by bulk space. Such brane matter which may play the role of dark matter may even violate the Dominant Energy Condition. Moreover, as a result of acceleration it looks that the cosmology with negative density energy and negative pressure does not seem to contradict to the astronomical data (for recent initial steps to study such cosmology, see [2]). The brane-world approach may be also related with Cosmic Censorship via holographic principle (see recent discussion in [3]).

It is even more important that holographic principle is somehow encoded in the usual gravitational field equations. Indeed, as it has been shown by E. Verlinde [4] the usual 4-dimensional Friedmann-Robertson-Walker (FRW) cosmological equations may be re-written in the form reminding about the entropy of 2-dimensional conformal field theory (CFT). It again appears the connection between 4-dimensional classical gravitational physics and quantum 2-dimensional CFT. The corresponding 2-dimensional CFT entropy has been extensively studied sometime ago in [5]. That is why, the 2-dimensional CFT entropy representation of FRW equations is sometimes called generalized Cardy or Cardy-Verlinde formula.

Finally, holographic principle may suggest new interpretation of the evolution of the observable universe. Indeed, using dual description one can think about de Sitter phase as preferrable solution of cosmological equations. One possibility (which is not successfully worked out so far) could be provided by fixed points of the coupling constants for corresponding dual CFT where de Sitter space is realized.

The purpose of this work is two-fold. From one side we review the FRW brane cosmology where brane is embedded in AdS or dS Black Hole (BH) and induced brane matter in various higher dimensional gravitational theories from the point of view of the representation of corresponding field equations in Cardy-Verlinde form. From another side we study brane-world cosmology in Einstein-Gauss-Bonnet (Einstein-GB) gravity where brane description is extremely complicated. It is shown that for various values of GB coupling constant and AdS or dS bulk BHs the induced brane matter may have negative energy and pressure. It is demonstrated that de Sitter brane is not preferrable solution of Einstein-GB brane-world cosmology.

The paper is organized as follows. In the next section we give simple and pedagogical introduction to the Cardy-Verlinde (CV) formula in $n + 1$-
dimensional gravity with general state matter. In other words, FRW cosmological equations are rewritten (using several definitions of cosmological entropy) in the form similar to quantum 2-dimensional CFT entropy. In the second appearance of CV formula the calculation of the universe entropy (supposing the valid first law of thermodynamics) gives it in generalized CV form. For radiation-dominated universe the cosmological entropy reduces to the standard CV form. In section three we give the introduction to brane-world program on the example of 5-dimensional Einstein gravity. Of course, the study of brane cosmology when bulk space is AdS includes huge number of works (see [6] and references therein). We consider both types of bulks: AdS and dS BHs and discuss the brane equations of motion presented in FRW form. Moreover, for each 5-dimensional bulk the FRW equations for space-like as well as time-like branes are reviewed. The induced brane matter is introduced. The presentation of such FRW brane equations in CV form is discussed as well as cosmological entropy bounds and the relation of AdS BH entropy with cosmological entropy (second way appearance of CV formula).

Section four is devoted to the review of FRW brane equations from 5-dimensional Reissner-Nordstrom-de Sitter (RNdS) BH and their relation with CV formula. It is shown that while some contributions due to Maxwell field appear in the intermediate identifications the corresponding FRW equations may still be presented in the form similar to 2-dimensional CFT entropy. In section five we review the role of quantum brane matter to above FRW equations. Using conformal anomaly induced effective action on the brane, the quantum-corrected FRW brane equations are written. Bulk space is again AdS BH. The values (signs) of induced brane matter pressure and energy are considered in connection with Dominant and Weak Energy Conditions (DEC and WEC). The quantum-corrected de Sitter brane solution is discussed.

Section six is devoted to the study of induced brane matter from bulk AdS BH in Einstein-GB gravity. GB combination naturally appears in the next-to-leading order term of the heterotic string effective action [7]. Despite the presence of higher derivative terms, the Einstein-GB field equations (being much more complicated) include only second derivatives like in ordinary Einstein gravity. That was the reason why there was much activity in the study of brane-world aspects of Einstein-GB theory [8, 9, 10]. In particularly, we show that FRW brane equations being very complicated may be still presented in CV form. The analysis of induced brane matter shows that for some values of GB coupling constant and in different limits on scale
factor the dual brane matter is not always CFT. Moreover, there are cases where matter pressure and energy are not positive. The careful analysis of DEC and WEC is presented for various types of brane matter. These two conditions are violated for some choices of GB coupling constant.

In section seven the FRW brane cosmology in Einstein-GB theory is discussed using the effective potential approach. The same cases of induced brane matter (for the same values of GB coupling) as in section 6 are considered. As is shown explicitly, there are various types of brane cosmology: de Sitter, hyperbolic and flat. Such brane universes may expand or contract, they may be singular or non-singular. De Sitter (inflationary) brane cosmology does not seem to be the preferrable solution of FRW cosmological equations. Some summary and outlook are given in the last section. In Appendix A, a brief review of the AdS/CFT is given. In Appendix B, the logarithmic corrections to CV formula (or FRW equations) are found.

2 Brief look to Cardy-Verlinde formula

In the seminal work, the very interesting approach to rewrite FRW cosmological equations in the form reminding about 2-dimensional quantum field theory has been suggested. In fact, E. Verlinde drew an interesting analogy between the FRW equations of a standard, closed, radiation-dominated universe and the 2-dimensional entropy formula due to Cardy. The physical origin of this analogy between classical gravity theory and 2-dimensional QFT remains completely hidden.

Before discussing the cosmological Cardy-Verlinde formula, we briefly explain how the 2-dimensional Cardy formula can be derived. As conformal fields we consider Majorana spinors. The Majorana spinor in 2 dimensions has the central charge \( c = \frac{1}{2} \). We now consider the system given by \( N \) Majorana spinors. The total central charge is

\[
    c = \frac{N}{2} .
\]

The partition function \( Z(\beta) \) of the system is given by

\[
    \ln Z(\beta) = N \sum_{n=0}^{\infty} \ln \left( 1 + e^{-\alpha \beta (n+\frac{1}{2})} \right) .
\]
Here the left-moving sector of the Neveu-Schwarz fermions is considered. \( \beta \) is the inverse of the temperature \( T \)

\[
\beta = \frac{1}{T} .
\]  

(2.3)

If the spinors live on the circle with the radius \( R \), the dimensional parameter \( \alpha \) is given by

\[
\alpha = \frac{1}{2\pi R}.
\]  

(2.4)

We now approximate the sum in (2.2) by the integration

\[
\ln Z(\beta) = N \int_0^\infty dx \ln \left(1 + e^{-\alpha \beta x}\right). \tag{2.5}
\]

By changing the variable \( x \) to \( y = \alpha \beta x \), one has

\[
\ln Z(\beta) = \frac{N}{\alpha \beta} \int_0^\infty dy \ln \left(1 + e^{-y}\right) = \frac{N \pi^2}{12\alpha \beta} . \tag{2.6}
\]

Here the formula

\[
\int_0^\infty dy \ln \left(1 + e^{-y}\right) = \frac{\pi^2}{12} . \tag{2.7}
\]

is used. Since the free energy \( F \) is given by

\[
- \beta F = \ln Z(\beta) , \tag{2.8}
\]

we have

\[
F = \frac{N \pi^2}{12\alpha \beta^2} = \frac{N \pi^2 T^2}{12\alpha} . \tag{2.9}
\]

Then the following expressions of the entropy and the (thermal average of) the energy are:

\[
S = \frac{\partial F}{\partial T} = \frac{N \pi^2 T}{6\alpha} , \tag{2.10}
\]

\[
E = - \frac{\partial}{\partial \beta} \left( \ln Z(\beta) \right) = \frac{N \pi^2}{12\alpha \beta^2} = \frac{N \pi^2 T^2}{12\alpha} . \tag{2.11}
\]

Then

\[
S^2 = \frac{N \pi^2}{3\alpha} E . \tag{2.12}
\]
Now the energy $E$ is related with the Virasoro operator $L_0$ by
\[
\frac{E}{\alpha} = L_0 - \frac{c}{24} .
\] (2.13)

Then we obtain
\[
S^2 = \frac{N\pi^2}{3} \left( L_0 - \frac{c}{24} \right) .
\] (2.14)

Finally using (2.1) we have
\[
S^2 = \frac{2c\pi^2}{3} \left( L_0 - \frac{c}{24} \right) ,
\] (2.15)

which is nothing but the Cardy formula
\[
S = 2\pi \sqrt{\frac{c}{6} \left( L_0 - \frac{c}{24} \right)} ,
\] (2.16)

It is not so difficult to calculate the entropy for massless fields even in higher dimensions although the obtained formula is not so simple. If one can, however, generalize the Cardy forumula to higher dimensions, the Virasoro operator $L_0$ should be related with the energy and the central charge with the Casimir energy. What E. Verlinde has shown is that such total energy and Casimir one are related with the cosmological entropies in a form similar to the Cardy formula via the FRW equation of the radiation-dominated universe. Since the radiation is, of course, the conformal matter, that is, the trace of the energy-momentum tensor vanishes, it would be natural if we expect that such a generalized Cardy formula has some physical meaning.

As one sees in the next section, the meaning becomes more clear from the AdS/CFT correspondence (see Appendix A) if we consider the brane universe in the bulk spacetime, which expresses the AdS black hole [18]. The entropy of the CFT is related with the black hole entropy via AdS/CFT.

Let us review this analogy in Einstein gravity for the usual $(n + 1)$-dimensional FRW Universe with a metric
\[
ds^2 = -d\tau^2 + a^2(\tau)g_{ij}dx^i dx^j ,
\] (2.17)
where the \( n \)-dimensional spatial hypersurfaces with negative, zero or positive curvature are parametrized by \( K = -1, 0, 1 \), respectively. For example, \( K = -1 \) corresponds to hyperboloid (of one sheet), \( K = 0 \) to flat surface, and \( K = 1 \) to sphere, whose metric is given by
\[
g_{ij}dx^i dx^j = \frac{dr^2}{1 - Kr^2} + r^2dΩ^2_{n-1}. \tag{2.18}
\]
Here \( dΩ^2_{n-1} \) is the metric of \( n-1 \)-dimensional sphere with unit radius. If \( R_{ij} \) is the Ricci tensor given by \( g_{ij} \), we have
\[
R_{ij} = (n-1)Kg_{ij}. \tag{2.19}
\]
In our discussion of cosmology based on Einstein gravity, we parametrize the curvature of the spatial hypersurfaces in terms of \( K = -1, 0, 1 \) since direct comparison with the standard cosmological equations is then straightforward. In other sections, we often use the lower letter \( k \) defined by \( k \equiv (n-1)K \), instead of the capital letter \( K \). We limit our discussion mainly to that of the closed universe \( (K = 1) \), with a spatial volume defined by \( V = a^n \int d^n x \sqrt{g} \).

The standard FRW equations, which follow from the Einstein equations may then be written as
\[
H^2 = \frac{16\pi G}{n(n-1)}\rho - \frac{K}{a^2},
\]
\[
\dot{H} = -\frac{8\pi G}{(n-1)}(\rho + p) + \frac{K}{a^2}, \tag{2.20}
\]
where \( \rho = \rho_m + \frac{\Lambda}{8\pi G} \), \( p = p_m - \frac{\Lambda}{8\pi G} \). \( \Lambda \) is a cosmological constant and \( \rho_m \) and \( p_m \) are the energy density and pressure of the matter contributions. The energy conservation equation is
\[
\dot{\rho} + n(\rho + p)\frac{\dot{a}}{a} = 0 \tag{2.21}
\]
and for a perfect fluid matter source with equation of state \( p_m = \omega \rho_m \) (\( \omega = \) constant) Eq. \((2.21)\) is solved as:
\[
\rho = \rho_0 a^{-n(1+\omega)} + \frac{\Lambda}{8\pi G}. \tag{2.22}
\]
When $\omega = 0$ the pressure vanishes $p = 0$, which corresponds to dust and $\rho$ behaves as $\rho \propto a^{-n}$, on the other hand, if $\omega = \frac{1}{n}$, the trace of the energy momentum tensor vanishes: $T_{\mu}^{\mu} = -\rho + np = 0$ which corresponds to CFT or radiation:

$$
\begin{align*}
\omega &= 0 \quad \text{dust} \quad p_m = 0 \quad , \quad \rho_m \propto a^{-n} \\
\omega &= \frac{1}{n} \quad \text{radiation} \quad -\rho_m + np_m = 0 \quad , \quad p_m \propto \rho_m \propto a^{-(n+1)} .
\end{align*}
$$

We should note that if we identify

$$
\begin{align*}
\frac{2\pi}{n} V \rho a &\Rightarrow 2\pi L_0 , \\
\frac{(n-1)V}{8\pi G a} &\Rightarrow \frac{c}{12} , \\
\frac{(n-1)HV}{4G} &\Rightarrow S ,
\end{align*}
$$

when $K = 1$, we can rewrite the first FRW equation (2.20) in the form of Cardy formula Eq.(2.15). Note that in (2.24), $\frac{(n-1)HV}{4G}$ is nothing but the Hubble entropy.

The definitions for the Hubble, Bekenstein [11] and Bekenstein-Hawking entropies are given as following [4]:

$$
S_H = (n-1) \frac{HV}{4G} , \quad S_{BH} = (n-1) \frac{V}{4Ga} , \quad S_B = \frac{2\pi a}{n} E ,
$$

(2.25)

where the total energy, $E$, is defined as $E = \rho V$ and contains the contribution from the cosmological constant term. This differs from that of the standard case, where the definitions of the entropies $S_{BH}$ and $S_B$ may differ slightly in their coefficients. This is specific to the presence of a cosmological constant [13, 14].

The Bekenstein entropy $S_B$ [11] gives the bound for the total entropy $S \leq S_{BH}$ for the system with limited gravity. The bound is useful for relatively low energy density or small volumes. Then the bound is not appropriate for strongly-gravitating universe, where $Ha \geq 1$. In this case $S_B \geq S_{BH}$. In the strongly-gravitating universe, the black hole production should be accounted for. $S_{BH}$ grows like an area rather than the volume and for the closed universe $S_{BH}$ reduces to the well-known expression of $A/4G$, where $A$ expresses the area. (This is, of course, typical for Einstein gravity as for
higher derivative gravity the area law may not hold.) In [12], however, it has been argued that, when $Ha > 1$, the total entropy should be bounded by the Hubble entropy $S_H$, which is the entropy of the black hole with the radius of the Hubble size.

By employing the definitions (2.25), one can easily rewrite the FRW equations (2.20) as a cosmological Cardy-Verlinde (CV) formula:

$$ S_H = \frac{2\pi}{n} a \sqrt{E_{BH}(2E - KE_{BH})}, $$

$$ KE_{BH} = n(E + pV - T_H S_H), $$

(2.26)

where the energy and Hawking temperature of the black hole are defined as

$$ E_{BH} = n(n - 1)\frac{V}{8\pi G a^2}, \quad T_H = -\frac{\dot{H}}{2\pi H}. $$

and we have separated the energy into a matter part and a cosmological constant part, i.e., $E = E_m + E_{\text{cosm}}$, where $E_{\text{cosm}} = \frac{\Lambda}{8\pi G} V$. This is simply a way to rewrite the FRW equations in a form that resembles the equation defining the entropy of a 2-dimensional CFT. However, the following remark is in order: the presence of cosmological constant may change some of the coefficients in Eq. (2.26) and this depends on precisely how the separation between the strongly and weakly interacting gravitational phases is made (compare with [13, 14]). In any case, the energy associated with the cosmological constant term is hidden in the expression for $E$, Eq. (2.26).

Later, we discuss the motion of the brane in Schwarzschild-(A)dS bulk space. The motion is again described by the FRW-like equation from which Cardy-Verlinde formula follows. The Hawking temperature $T_H$ (2.27) coincides with that of the black hole, when the radius $a$ of the universe is equal to the horizon radius, that is, the brane crosses the horizon.

Eq. (2.26) may also be rewritten in another form:

$$ S_H^2 = S_{BH} (2S_B - KS_{BH}) . $$

(2.28)

Since the definition of $S_B$ normally contains only matter contributions, it is reasonable to define $S_B \equiv S_B^{m} + S_B^{\text{cosm}}$, where the entropy associated with the cosmological constant is given by

$$ S_B^{\text{cosm}} = \frac{a V \Lambda}{4nG}. $$

(2.29)
The appearance of such a new “cosmological constant” contribution to the entropy in the CV formula is quite remarkable.

Thus far, we have discussed the appearance of the CV formula as a way to rewrite the FRW equations. However, the CV formula appears in a second formulation when one calculates the entropy $S$, of the universe. Indeed, following Ref. [4], one can represent the total energy $E = \rho V$ of the universe as the sum of the extensive energy, $E_E$, and the subextensive (Casimir) energy $E_C$:

$$E(S, V) = E_E(S, V) + \frac{1}{2} E_C(S, V) .$$

(2.30)

Note that unlike the case considered by Verlinde [4], the cosmological constant contribution appears in $E_E$. Nevertheless, the constant rescaling of the energy is given by

$$E_E(\lambda S, \lambda V) = \lambda E_E(S, V) ,$$

$$E_C(\lambda S, \lambda V) = \lambda^{1 - \frac{2}{n}} E_C(S, V) .$$

(2.31)

Now, if one assumes that the first law of thermodynamics is valid and that the expansion is adiabatic, one deduces that

$$dS = 0 , \quad s = \frac{a^n}{T}(\rho + p) + s_0 ,$$

(2.32)

where the entropy $S \equiv s \int d^n x \sqrt{g}$, $s_0$ is an integration constant and $T$ is the temperature of the universe. It then follows that the Casimir energy is given by [15]

$$E_C = n (E + pV - TS) = -n Ts_0 \int d^n x \sqrt{g}$$

(2.33)

and, consequently, that $E_C \sim a^{-n\omega}$ and $E_E - E_{\text{cosm}} \sim a^{-n\omega}$. This further implies that the products $E_C a^{n\omega}$ and $(E_E - E_{\text{cosm}}) a^{n\omega}$ are independent of the spatial volume of the universe, $V$. By employing the scaling relations (2.31) one then concludes that [15]:

$$E_E - E_{\text{cosm}} = \frac{\alpha}{4\pi a^{n\omega}} s^{\omega+1} , \quad E_C = \frac{\beta}{2\pi a^{n\omega}} s^{\omega+1 - \frac{2}{n}} ,$$

(2.34)

where $\alpha$ and $\beta$ are some unknown constants. (They are known for CFT in 4 dimensions). Hence, the entropy is given by

$$S = \left[ \frac{2\pi a^{n\omega}}{\sqrt{\alpha \beta}} \sqrt{E_C(E_E - E_{\text{cosm}})} \right] \frac{a^n}{(\omega+1)n-1} .$$

(2.35)
Eq. (2.35) represents the generalization of the Cardy-Verlinde formula found by Youm [15] in the absence of a contribution from the cosmological constant. The negative term associated with such a cosmological entropy is quite remarkable. In the case of a radiation-dominated universe, Eq. (2.35) reduces to the standard CV formula with the familiar square root term [5]. This formulation will be used below when one writes the equation of brane motion as FRW equation.

3 FRW brane equations in the background of AdS and dS Schwarzschild black hole

As shown in the previous section, Verlinde [4] has found that the FRW equation describing the radiation dominated universe has a structure similar to the Cardy formula which gives a relation between the entropy and the Virasoro operator. Since the Cardy-Verlinde formula follows from the comparison between the FRW equation and the Cardy formula, the physical origin or meaning of the correspondence was not clear. Savonije and Verlinde [18] have shown that the origin of the formula becomes more clear in terms of AdS/CFT set-up (Appendix A). If we consider the AdS black hole spacetime instead of the pure AdS, we can introduce the temperature and the entropy via the Hawking temperature and the Bekenstein-Hawking entropy. By the AdS/CFT correspondence, the temperature and the entropy can be related with those of the corresponding CFT. In the AdS/CFT, the CFT can be regarded to exist on the boundary which lies at infinity in the AdS. One can put the boundary at finite distance. Then the boundary can be regarded as a brane where CFT exists. The dynamics of the brane is described by FRW-like equations. If the brane is our universe, the entropy of the universe can be related with the black hole entropy. Then the Cardy-Verlinde formula would tell that the relation of the black hole entropy with the black hole energy (mass) corresponds to the relation of the entropy and the energy of the CFT at the finite temperature.

In this section, we review briefly the relationship between the entropy of AdS and dS Schwarzschild space and those of the dual CFT which lives on the brane by using Friedmann-Robertson-Walker (FRW) equations and Cardy-Verlinde formula. The holographic principle between the radiation
dominated FRW universe in $d$-dimensions and same dimensional CFT with a dual $d + 1$-dimensional AdS description was studied by E. Verlinde \[4\]. Especially, one can see the correspondence between black hole entropy and the entropy of the CFT which is derived by making the appropriate identifications for FRW equation with the generalized Cardy formula. The Cardy formula is originally the entropy formula of the CFT only for 2 dimensions \[5\], while the generalized Cardy formula expresses that of the CFT for any dimensions \[4\]. From the point of brane-world physics \[16\], the CFT/FRW relation sheds further light on the study of the brane CFT in the background of AdS Schwarzschild black hole \[18\]. There was much activity on the studies of related questions \[9,10,19,20,22,23,24,25,26\] making use the connection with Cardy-Verlinde formula.

We will describe 4 kinds of FRW Eqs. following from AdS and dS Schwarzschild black hole, i.e., time(space)-like FRW Eqs. from AdS(dS) Schwarzschild background. If all the vectors tangential to the brane are space-like, we call the brane space-like one. If there is any time-like tangential vector, we call the brane time-like one. We consider the FRW Eqs. for both cases.

One first considers a 4-dimensional time-like brane in 5-dimensional AdS Schwarzschild background. From the analogy with the AdS/CFT correspondence, one can regard that 4-dimensional CFT exists on the brane which is the boundary of the 5-dimensional AdS Schwarzschild background. The bulk action is given by the 5-dimensional Einstein action with cosmological term. The dynamics of the brane is described by the boundary action:\footnote{In this paper, lower case Latin indices span the world-volume, $(i, j) = (0, 1, 2, 3)$, lower case Greek indices span the bulk coordinates and a comma denotes partial differentiation.}

\[
\mathcal{L}_b = -\frac{1}{8\pi G_5} \int_{\partial M} \sqrt{-g} K + \frac{\kappa}{8\pi G_5} \int_{\partial M} \sqrt{-g} \ , \quad K = K^i_i
\]

(3.1)

Here $G_5$ is 5-dimensional bulk Newton constant, $\partial M$ denotes the surface of the brane, $g$ is the determinant of the induced metric on $\partial M$, $K_{ij}$ is the extrinsic curvature, $\kappa$ is a parameter related to tension of the brane. From this Lagrangian, we can get the equation of motion of the brane as \[18\]:

\[
K_{ij} = \frac{\kappa}{2} g_{ij} \ ,
\]

(3.2)

which implies that $\partial M$ is a brane of constant extrinsic curvature. The bulk action is given by 5-dimensional Einstein action with cosmological constant.
The AdS Schwarzschild space is one of the exact solutions of bulk equations of motion and can be written in the following form,

\[ ds_5^2 = \hat{G}_{\mu\nu} dx^\mu dx^\nu = -e^{2\rho} dt^2 + e^{-2\rho} da^2 + a^2 d\Omega_3^2, \]

\[ e^{2\rho} = \frac{1}{a^2} \left( -\mu + a^2 + \frac{a^4}{l_{AdS}^2} \right). \]  

(3.3)

Here \( l_{AdS} \) is the curvature radius of AdS and \( \mu \) is the black hole mass. Following the method of the work [18], one rewrites AdS Schwarzschild metric (3.3) in the form of FRW metric by using a new time parameter \( \tau \). Note that the parameters \( t \) and \( a \) in (3.3) are the functions of \( \tau \), namely \( a = a(\tau), t = t(\tau) \).

For the purpose of getting the 4-dimensional FRW metric, we impose the following condition,

\[ -e^{2\rho} \left( \frac{\partial t}{\partial \tau} \right)^2 + e^{-2\rho} \left( \frac{\partial a}{\partial \tau} \right)^2 = -1. \]  

(3.4)

Thus one obtains time-like FRW metric:

\[ ds_4^2 = g_{ij} dx^i dx^j = -d\tau^2 + a^2 d\Omega_3^2. \]  

(3.5)

The extrinsic curvature, \( K_{ij} \), of the brane can be calculated and expressed in terms of the function \( a(\tau) \) and \( t(\tau) \). Thus one rewrites the equations of motion (3.2) as

\[ \frac{dt}{d\tau} = -\frac{\kappa a}{2} e^{-2\rho}. \]  

(3.6)

Using (3.4) and (3.6), we can derive FRW equation for a radiation dominated universe where Hubble parameter \( H \) which is defined by \( H = \frac{1}{a} \frac{da}{d\tau} \) is given by

\[ H^2 = -\frac{1}{l_{AdS}^2} - \frac{1}{a^2} + \frac{\mu}{a^4} + \frac{\kappa^2}{4}. \]  

(3.7)

From the point of view of brane-world physics [16], the tension of brane should be determined without ambiguity as \( \kappa = 2/l_{AdS} \), so we take it as above from now on. In fact, one can calculate \( \kappa \) requiring to cancel the leading divergence of bulk AdS Schwarzschild.
This equation can be rewritten by using 4-dimensional energy density $\rho$ and volume $V$ in the form of the standard FRW equation with the cosmological constant $\Lambda$:

$$H^2 = -\frac{1}{a^2} + \frac{8\pi G_4}{3} \rho + \frac{\Lambda}{3} ,$$

$$\rho = \frac{3\mu}{8\pi G_4 a^4} , \quad A = 0 . \quad (3.8)$$

Here $G_4$ is the 4-dimensional gravitational coupling, which is defined by

$$G_4 = \frac{2G_5}{l_{\text{AdS}}} . \quad (3.9)$$

$\rho$ can be regarded as 4-dimensional energy density on the brane in AdS Schwarzschild background. We should note that when the bulk is pure AdS with $\mu = 0$ and therefore $\rho = 0$, the FRW equation (3.8) reduces to

$$H^2 = -\frac{1}{a^2} . \quad (3.10)$$

The equation (3.10) has no solution since l.h.s. is positive but r.h.s. is negative.

By differentiating Eq.(3.8) with respect to $\tau$, we obtain the second FRW equation:

$$\dot{H} = -4\pi G_4 (\rho + p) + \frac{1}{a^2} ,$$

$$p = \frac{\mu}{8\pi G_4 a^4} . \quad (3.11)$$

Here $p$ is 4-dimensional pressure of the matter on the boundary.

From Eqs.(3.8) and (3.11), one finds that the energy-momentum tensor is traceless:

$$T_{\text{matter}}^{\mu\nu} = -\rho + 3p = 0 . \quad (3.12)$$

Therefore the matter on the brane can be regarded as the radiation. This result means the field theory on the brane should be CFT as in case of AdS Schwarzschild background [17, 18].
Next, one considers space-like brane in 5-dimensional AdS Schwarzschild background. Similarly, we impose the following condition to obtain space-like brane metric instead of Eq. (3.4):

\[-e^{2\rho} \left( \frac{\partial t}{\partial \tau} \right)^2 + e^{-2\rho} \left( \frac{\partial a}{\partial \tau} \right)^2 = 1. \tag{3.13}\]

Thus the following FRW-like metric is obtained:

\[ds_4^2 = g_{ij} dx^i dx^j = d\tau^2 + a^2 d\Omega_3^2. \tag{3.14}\]

Note that this metric is also derived by Wick-rotation \( \tau \rightarrow i\tau \) in Eq. (3.5). We again calculate the equations of motion and the extrinsic curvature of space-like brane instead of (3.2) and (3.6). These equations lead to FRW-like equation as follows:

\[H^2 = \frac{1}{l_{\text{AdS}}^2} + \frac{1}{a^2} - \frac{\mu}{a^4} + \frac{\kappa^2}{4}. \tag{3.15}\]

One assumes this equation can be rewritten by using 4-dimensional energy density \( \rho \) in the form analogous to the standard FRW equations:

\[
H^2 = \frac{1}{a^2} - \frac{8\pi G_4}{3} \rho - \frac{\Lambda}{3}, \tag{3.16}
\]

\[
\rho = \frac{3\mu}{8\pi G_4 a^4}, \quad \Lambda = -\frac{6}{l_{\text{AdS}}^2}.
\]

\[
\dot{H} = 4\pi G_4 (\rho + p) - \frac{1}{a^2}, \quad p = \frac{\mu}{8\pi G_4 a^4}. \tag{3.17}
\]

The reason why the sign of FRW equations is different from the standard FRW equations (3.8) results from the condition (3.13), namely \( \tau \rightarrow i\tau \) in Eq. (3.5). From Eqs. (3.16) and (3.17), it follows the energy-momentum tensor is traceless again. It is interesting that the cosmological constant on the brane doesn’t appear for time-like FRW metric, while it appears for space-like FRW metric in AdS Schwarzschild background. We will see what happens in dS Schwarzschild case next.

When the bulk is pure AdS, \( \mu = 0 \) and therefore \( \rho = 0 \). The solution of (3.18) is given by

\[a = \frac{l_{\text{AdS}}}{\sqrt{2}} \sinh \left( \frac{\tau \sqrt{2}}{l_{\text{AdS}}} \right), \tag{3.18}\]
which is one of two sheet hyperboloid.

One assumes that there is some holographic relation between FRW universe which is reduction from dS Schwarzschild background and boundary CFT. The dS Schwarzschild space is also one of the exact solutions of bulk equations (with proper sign of bulk cosmological constant)

\[
\begin{align*}
    ds_5^2 &= \hat{G}_{\mu\nu}dx^\mu dx^\nu \\
    &= -e^{2\rho}dt^2 + e^{-2\rho}da^2 + a^2d\Omega_3^2 , \\
    e^{2\rho} &= \frac{1}{a^2} \left( -\mu + a^2 - \frac{a^4}{l_{\text{dS}}^2} \right) .
\end{align*}
\] (3.19)

Here \( l_{\text{dS}} \) is the curvature radius of dS and \( \mu \) is the black hole mass. This metric is very similar to the AdS Schwarzschild metric (3.3). The difference of Eq.(3.3) and Eq.(3.19) is in the sign in the third term of \( e^{2\rho} \), which corresponds to cosmological term in the bulk.

We first consider time-like brane in 5-dimensional dS Schwarzschild background. Similarly to AdS case, one imposes the same condition (3.4) in order to obtain time-like FRW metric (3.5). Then using equation of motion which has the same form as Eq.(3.6), we obtain FRW like equation as follows:

\[
H^2 = \frac{1}{l_{\text{dS}}^2} - \frac{1}{a^2} + \frac{\mu}{a^4} + \frac{\kappa^2}{4} .
\] (3.20)

From the standard FRW equations (3.8),(3.11), one defines \( E_4, \Lambda \) and \( p \) as

\[
\begin{align*}
    \rho &= \frac{3\mu}{8\pi G_4a^4}; \quad \Lambda = \frac{6}{l_{\text{dS}}^2} , \\
    p &= \frac{\mu}{8\pi G_4a^4} .
\end{align*}
\] (3.21)

The cosmological constant \( \Lambda \) has the opposite sign to AdS Schwarzschild case in Eq.(3.16), \( G_4 \) is the 4-dimensional gravitational coupling, which is defined by

\[
G_4 = \frac{2G_5}{l_{\text{dS}}} .
\] (3.22)

We now choose \( \kappa^2 = 4l_{\text{dS}}^2 \). When the bulk is pure de Sitter with \( \mu = 0 \) and therefore \( \rho = 0 \), the solution of (3.21) is given by

\[
a = \frac{l_{\text{dS}}}{\sqrt{2}} \cosh \left( \frac{\tau \sqrt{2}}{l_{\text{dS}}} \right) ,
\] (3.23)

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which is so-called one-sheet hyperboloid.

On the other hand, by using the same method for AdS Schwarzschild background with space-like FRW metric, from FRW-like equation,

\[ H^2 = -\frac{1}{l_{dS}^2} + \frac{1}{a^2} - \frac{\mu}{a^4} + \frac{\kappa^2}{4}, \quad (3.24) \]
one can derive the \( E_4, \Lambda \) and \( p \) with \( \kappa^2 = 4l_{dS}^2 \), as

\[ \rho = \frac{3\mu}{8\pi G_4 a^4}, \quad \Lambda = 0, \]

\[ p = \frac{\mu}{8\pi G_4 a^4}. \quad (3.25) \]

Therefore we find the energy-momentum tensor is traceless (dual QFT is CFT) for the dS Schwarzschild background too. When the bulk is pure de Sitter, the solution of (3.24) is given by

\[ a = \tau, \quad (3.26) \]

which is the cone.

We point out that\[20\] the cosmological constant on the brane appears in AdS Schwarzschild background with space-like brane, while it appears in dS Schwarzschild background with time-like brane. The energy-momentum tensor is traceless for all cases.

Moreover, for all cases Hubble parameter \( H \) takes the same form as \( \pm 1/l_{AdS} \) or \( \pm 1/l_{dS} \) when brane crosses the horizon. Here the plus sign corresponds to the expanding brane universe and the minus one to the contracting universe. Let us choose the expanding case below. The 4-dimensional Hubble entropy is defined as\[4\]

\[ S = \frac{HV}{2G_4} \quad (3.27) \]

It takes the following forms

\[ S_4 = \frac{V}{2l_{AdS} G_4}, \quad \frac{V}{2l_{dS} G_4} = \frac{V}{4G_5}. \quad (3.28) \]

when brane crosses the horizon. Here Eqs.(3.9), (3.22) are used. The entropy (3.28) is nothing but the Bekenstein-Hawking entropy of 5-dimensional AdS (dS) black hole similarly to ref.\[18, 19\].
Coming back to the discussion of previous section where it was shown that the \(d\)-dimensional FRW equation can be regarded as an analogue of the Cardy formula of 2-dimensional CFT \([4]\) one gets

\[
S_4 = 2\pi \sqrt{\frac{c}{6}} \left( L_0 - \frac{c}{24} \right). \tag{3.29}
\]

Let us use the Cardy formula for AdS(dS) Schwarzschild background with time-like and space-like branes.

For time-like brane of AdS Schwarzschild background, identifying

\[
\begin{align*}
&\frac{2\pi}{3} V \rho a \Rightarrow 2\pi L_0, \\
&\frac{V}{8\pi G_4 a} \Rightarrow \frac{c}{24}, \\
&\frac{HV}{2G_4} \Rightarrow S_4, \tag{3.30}
\end{align*}
\]

one can rewrite FRW equation (3.8) in the form of Cardy formula Eq.(3.29). Note that the identification of Eq.(3.30) is identical with original one \([4]\) exactly.

For space-like brane of AdS Schwarzschild background, identifying

\[
\begin{align*}
&\frac{2\pi}{3} V \rho a + \frac{\Lambda V a}{8\pi G_4} \Rightarrow 2\pi L_0, \\
&\frac{V}{8\pi G_4 a} \Rightarrow \frac{c}{24}, \\
&\frac{-iHV}{2G_4} \Rightarrow S_4, \tag{3.31}
\end{align*}
\]

one rewrites FRW equation (3.11). Here \(H\) changes as \(H \to -iH\) since \(H\) is defined by \(H = \frac{d a}{a d\tau}\) and \(\frac{d}{d\tau}\) change as \(-i\frac{d}{d\tau}\) by the Wick-rotation, \(\tau \to i\tau\).

For time-like brane of dS Schwarzschild background, we assume the identification as

\[
\begin{align*}
&\frac{2\pi}{3} \left( V \rho a + \frac{\Lambda V a}{8\pi G_4} \right) \Rightarrow 2\pi L_0, \\
&\frac{V}{8\pi G_4 a} \Rightarrow \frac{c}{24}, \\
&\frac{HV}{2G_4} \Rightarrow S_4, \tag{3.32}
\end{align*}
\]
In above two cases, space-like brane of AdS and time-like brane of dS Schwarzschild background, the effect of the cosmological constant appears in Cardy formula. We included contribution of the cosmological constant in $L_0$ because it shifts the vacuum energy. This means the cosmological entropy bound [4] should be changed. The Bekenstein bound in 4-dimensions is

$$S \leq S_B, \quad S_B \equiv \frac{2\pi}{3} \rho V a.$$  \hfill (3.33)$$

Using Eq.(3.32), the Bekenstein entropy bound should be changed as follows:

$$S \leq S_B, \quad S_B \equiv \frac{2\pi}{3} V a \left( \rho + \frac{\Lambda}{8\pi G_4} \right).$$  \hfill (3.34)$$

Thus, the effect of the cosmological constant appears in the change of the Bekenstein entropy bound.

For space-like brane of dS Schwarzschild background, the identification looks as follows:

$$\frac{2\pi}{3} V \rho a \Rightarrow 2\pi L_0, \quad \frac{V}{8\pi G_4 a} \Rightarrow \frac{c}{24}, \quad -\frac{HV}{2G_4} \Rightarrow S_4,$$  \hfill (3.35)$$

which represents the FRW-like equation (3.24) in the form of Cardy formula Eq.(3.29) again. Thus, we presented the review of brane motion as FRW equation (or Cardy formula) in AdS (dS) Schwarzschild black hole background when the theory is Einstein gravity with cosmological constant.

In order to discuss the second way of getting the Cardy formula by using Casimir energy like in Eq.(2.35) one should use the holographic principle. For simplicity, we examine only the 5-dimensional AdS Schwarzschild black hole case here.

The horizon radius, $a_H$, is deduced by solving the equation $e^{2\rho(a_H)} = 0$ in (3.3), i.e.,

$$a_H^2 = -\frac{l_{AdS}^2}{2} + \frac{1}{2} \sqrt{l_{AdS}^4 + 4\mu l_{AdS}^2}.$$  \hfill (3.36)$$
The Hawking temperature, \( T_H \), is then given by

\[
T_H = \frac{(e^{2\rho})|_{a=a_H}}{4\pi} = \frac{1}{2\pi a_H} + \frac{a_H}{\pi l^2_{\text{AdS}}},
\]

where a prime denotes differentiation with respect to \( r \). One can also rewrite the mass parameter, \( \mu \), using \( a_H \) or \( T_H \) from Eq. (3.36) as follows:

\[
\mu = \frac{a_H^4}{l^2_{\text{AdS}}} + a_H^2 = a_H^2 \left( \frac{a_H^2}{l^2_{\text{AdS}}} + 1 \right)
\]

\[
= \frac{1}{4} \left( \pi l^2_{\text{AdS}} T_H \pm \sqrt{(\pi l^2_{\text{AdS}} T_H)^2 - 2l^2_{\text{AdS}}} \right)^2
\]

\[
\times \left( \frac{1}{4l^2_{\text{AdS}}} \left( \pi l^2_{\text{AdS}} T_H \pm \sqrt{(\pi l^2_{\text{AdS}} T_H)^2 - 2l^2_{\text{AdS}}} \right)^2 + 1 \right). \quad (3.38)
\]

The entropy \( S \) and the thermodynamical energy \( E \) of the black hole are given in [18, 19]

\[
S = \frac{V_3 \pi a_H^3}{2} \frac{8}{16\pi G_5}
\]

\[
= \frac{V_3 \pi}{32\pi G_5} \left( \pi l^2_{\text{AdS}} T_H \pm \sqrt{(\pi l^2_{\text{AdS}} T_H)^2 - kl^2} \right)^3. \quad (3.39)
\]

\[
E = \frac{3V_3 \mu}{16\pi G_5}. \quad (3.40)
\]

On the other hand, the 4-dimensional energy can be derived from the FRW equations of the brane universe in the SAdS background. It is given by Eq. (3.8)

\[
E_4 = \frac{3V_3 l_{\text{AdS}} \mu}{16\pi G_5 a}. \quad (3.41)
\]

Then the relation between 4-dimensional energy \( E_4 \) on the brane and 5-dimensional energy in Eq. (3.40) is as follows [18]

\[
E_4 = \frac{l_{\text{AdS}}}{a} E. \quad (3.42)
\]

It is assumed that the total entropy \( S \) of the dual CFT on the brane is given by Eq. (3.39). If this entropy is constant during the cosmological evolution,
the entropy density $s$ is given by

$$s = \frac{S}{a^3 V_3} = \frac{a_H^3}{2a^3 G_4 l_{\text{AdS}}}$$  \hspace{1cm} (3.43)

If one further assumes that the temperature $T$ on the brane differs from the Hawking temperature $T_H$ by the factor $l_{\text{AdS}}/a$ like energy relation, it follows that

$$T = \frac{l_{\text{AdS}}}{a} T_H = \frac{r_H}{\pi a l_{\text{AdS}}} + \frac{l_{\text{AdS}}}{2\pi a a_H}$$  \hspace{1cm} (3.44)

and, when $a = a_H$, this implies that

$$T = \frac{1}{\pi l_{\text{AdS}}} + \frac{l_{\text{AdS}}}{2\pi a^2_H}.$$  \hspace{1cm} (3.45)

If the energy and entropy are purely extensive, the quantity $E_4 + pV - TS$ vanishes. In general, this condition does not hold and one can define the Casimir energy $E_C$ as in case for Einstein gravity (section two):

$$E_C = 3 (E_4 + pV - TS).$$  \hspace{1cm} (3.46)

Then, by using Eqs. (3.39), (3.41), and (3.44), and the relation $3p = E_4/V$, we find that

$$E_C = \frac{3l_{\text{AdS}} a_H^2 V_3}{8\pi G_5 a}.$$  \hspace{1cm} (3.47)

Finally, by combining Eqs. (3.39), (3.41), and (3.47) one gets

$$S = \frac{4\pi a}{3\sqrt{2}} \sqrt{E_C \left( E_4 - \frac{1}{2} E_C \right)}.$$  \hspace{1cm} (3.48)

Thus, we have demonstrated how the FRW equation which is written in CV form (second way) can be related to the thermodynamics of the bulk black hole. Similarly, one can consider bulk dS black hole thermodynamics.

### 4 FRW brane equations from 5-dimensional Einstein-Maxwell gravity

In this section, we discuss the Cardy-Verlinde formula from the 5-dimensional Einstein-Maxwell gravity. The example of such sort is necessary in order to
understand the role of non-gravitational fields in rewriting of FRW equations in the Cardy form.

Let us consider 4-dimensional brane in 5-dimensional Reissner-Nordstrom-de Sitter (RNdS) background following ref. [22]. The bulk solution is given as:

\[
ds^2 = -h(a)dt^2 + \frac{1}{h(a)}da^2 + a^2d\Omega_3^2,
\]

\[
h(a) = e^{2\rho} = 1 - \frac{a^2}{l_{dS}^2} - \frac{\omega_4 M}{a^2} + \frac{3\omega_4^2 Q^2}{16a^4},
\]

\[
\omega_4 = \frac{16\pi G_5 V_3}{3a^2}, \quad \phi_{dS}(a) = \frac{3\omega_4 Q}{8a^2}.
\]

Here \(l_{dS}\) is the curvature radius of dS background, \(d\Omega_3^2\) is a unit 3-dimensional sphere with volume \(V_3\), \(G_5\) is the 5-dimensional Newton constant, \(M\) and \(Q\) are the conserved quantities of black hole mass and charge respectively. \(\phi_{dS}(a)\) is a measure of the electrostatic potential at \(a\).

To derive FRW equations, one adopts the same method of Section 3. The dynamics of the brane is assumed to be described by the boundary action (3.1) even in RNdS bulk. Then one can get the equation of motion of the brane from this Lagrangian as in (3.2). Following the method of Section 3, we rewrite RNdS metric (4.1) in the form of FRW metric by using a new time parameter \(\tau\). On the brane the coordinates \(t\) and \(a\) in (4.1) are the functions of \(\tau\), namely \(a = a(\tau)\), \(t = t(\tau)\) as in (3.5). By imposing the condition (3.4), we can rewrite the equations of motion (3.2) as (3.6), again. From Eqs. (3.4) and (3.6), one can derive FRW equation for a radiation dominated universe (Hubble parameter \(H\) is defined by \(H = \frac{1}{a} \frac{da}{d\tau}\)

\[
H^2 = \frac{1}{l_{dS}^2} - \frac{1}{a^2} + \frac{\omega_4 M}{a^4} - \frac{3\omega_4^2 Q^2}{16a^6} + \frac{\kappa^2}{4}.
\]

The tension of brane is chosen as \(\kappa = 2/l_{dS}\). The above equation can be written in the form of the standard FRW equation with \(Q\) and the cosmological

---

\footnote{In our conventions, black hole mass \(\mu\) represents the \(\omega_4 M\) in Eq.(4.2). Hereafter we adopt \(M\) as black hole mass instead of \(\mu\) for later convenience.}
constant $\Lambda$:

$$H^2 = \frac{1}{a^2} + \frac{8\pi G_4}{3 \rho} \left( \rho - \frac{1}{2} \phi \rho_Q \right) + \frac{\Lambda}{3},$$

$$\rho = \frac{E_4}{V} = \frac{3 \omega_4 M}{8\pi G_4 a^4}, \quad V = a^3 V_3, \quad \Lambda = \frac{6}{l_{dS}^2},$$

$$\rho_Q = \frac{Q}{V}, \quad \phi = \frac{l_{dS}}{a} \phi_{dS}, \quad G_4 = \frac{2G_5}{l_{dS}}. \quad (4.5)$$

Here $G_4$ is the 4-dimensional gravitational coupling again and $\rho$ and $\rho_Q$ can be regarded as 4-dimensional energy density and charge density on the brane in RNdS background respectively.

By differentiating Eq.(4.5) with respect to $\tau$, we obtain the second FRW equation:

$$\dot{H} = -4\pi G_4 (\rho + p) - \frac{1}{a^2},$$

$$p = \frac{\omega_4 M}{8\pi G_4 a^4}. \quad (4.6)$$

Here $p$ is 4-dimensional pressure of the matter on the boundary.

Next, we recall the Cardy formula of 2-dimensional CFT:

$$S_4 = 2\pi \sqrt{\frac{c}{6} \left( L_0 - \frac{c}{24} \right)}. \quad (4.7)$$

One can get now the Cardy formula for RNdS background with time-like branes. For time-like brane on RNdS background, the following identifications may be done

$$\frac{2\pi}{3} \left( E_4 a - \frac{1}{2} \phi Q a + \frac{\Lambda V a}{8\pi G_4} \right) \Rightarrow 2\pi L_0,$$

$$\frac{V}{8\pi G_4 a} \Rightarrow \frac{c}{24},$$

$$\frac{HV}{2G_4} \Rightarrow S_4. \quad (4.8)$$

\[For space-like brane, this may be easily repeated like for AdS(dS) Schwarzschild background.\]
It is clear that with above identifications FRW equations take the form of Cardy formula. Note that the effect of $Q$ and the cosmological constant appears in Cardy formula (in the shift of energy for Hamiltonian). We included the contribution of the cosmological constant in $L_0$ because it shifts the vacuum energy. This means the cosmological entropy bound\footnote{4} should be changed. The Bekenstein bound in 4-dimensions is

$$S \leq S_B, \quad S_B \equiv \frac{2\pi}{3} E_4 a. \quad (4.9)$$

Using Eq.(4.8), the Bekenstein entropy bound should be changed as follows:

$$S \leq S_B, \quad S_B \equiv \frac{2\pi}{3} a \left( E_4 - \frac{1}{2} \phi Q + \frac{\Lambda V}{8\pi G_4} \right). \quad (4.10)$$

Thus, the matter fields (vectors in the example under consideration) modify some quantities which appear in Cardy formula via the contribution of the electric potential in the operator of Virasoro algebra of zero level. Moreover, the Bekenstein entropy bound is also modified. In the same way, the other fields (fermions, tensor fields, etc) will influence to Cardy representation of FRW equations. Of course, the explicit equations may be quite complicated. Note also that similarly to the discussion of the previous section one can relate thermodynamic entropy of RNdS BH with dual CFT entropy and to get the CV formula from such relation. AdS/CFT correspondence is again used in such calculation.

5 Brane New World from 5-dimensional AdS-Schwarzschild Black Hole

The interesting question now is: what is the role of quantum brane effects to FRW brane cosmology and to representation of FRW equations in the form of 2-dimensional entropy equation? In this section based on \cite{26}, we review the appearance of quantum matter effects in the brane equations of motion.

We assume the brane connects two bulk spaces and we may also identify the two bulk spaces as in \cite{16} by imposing $Z_2$ symmetry. One starts with the Minkowski signature action $S$ which is the sum of the Einstein-Hilbert action $S_{EH}$ with the cosmological term, the Gibbons-Hawking surface term
$S_{\text{GH}}$, the surface counter term $S_1$ and the trace anomaly induced action $\mathcal{W}$:

\begin{align}
S &= S_{\text{EH}} + S_{\text{GH}} + 2S_1 + \mathcal{W}, \\
S_{\text{EH}} &= \frac{1}{16\pi G_5} \int d^5x \sqrt{-g_{(5)}} \left( R_{(5)} + \frac{12}{l^2} \right), \\
S_{\text{GH}} &= \frac{1}{8\pi G_5} \int d^4x \sqrt{-g_{(4)}} \nabla_\mu n^\mu, \\
S_1 &= -\frac{6}{16\pi G_5 l_{\text{AdS}}} \int d^4x \sqrt{g_{(4)}}, \\
\mathcal{W} &= b \int d^4x \sqrt{-\tilde{g}} \tilde{F} A + b' \int d^4x \sqrt{\tilde{g}} \left\{ A \left[ 2 \tilde{\Box}^2 + \tilde{R}_{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \right. \\
& \quad - \frac{4}{3} \tilde{R} \tilde{\Box} - \frac{2}{3} \left( \tilde{\nabla}^\mu \tilde{R} \right) \tilde{\nabla}_\mu \left\} A + \left( \tilde{G} - \frac{2}{3} \tilde{\Box} \tilde{R} \right) A \right\} + \frac{1}{12} \left\{ b'' + \frac{2}{3} (b + b') \right\} \int d^4x \sqrt{\tilde{g}} \left[ \tilde{R} - 6 \tilde{\Box} A - 6(\tilde{\nabla}_\mu A)(\tilde{\nabla}^\mu A) \right]^2.
\end{align}

Here the quantities in the 5-dimensional bulk spacetime are specified by the suffices (5) and those in the boundary 4-dimensional spacetime are specified by (4) (for details, see [27]). In (5.3), $n^\mu$ is the unit vector normal to the boundary. The Gibbons-Hawking term $S_{\text{GH}}$ is necessary in order to make the variational method well-defined when there is boundary in the spacetime. In (5.4), the coefficient of $S_1$ is determined from AdS/CFT [28]. The factor 2 in front of $S_1$ is coming from that we have two bulk regions which are connected with each other by the brane. In (5.5), one chooses the 4-dimensional boundary metric as $g_{(4)\mu\nu} = e^{2A} \tilde{g}_{\mu\nu}$, where $\tilde{g}_{\mu\nu}$ is a reference metric. $G (\tilde{G})$ and $F (\tilde{F})$ are the Gauss-Bonnet invariant and the square of the Weyl tensor. $\mathcal{W}$ can be obtained by integrating the conformal anomaly with respect to the scale factor $A$ of the metric tensor since the conformal anomaly should be given by the variation of the quantum effective action with respect to $A$. Note that quantum effects of brane CFT are taken into account via Eq.(5.5).

In the effective action (5.5) induced by brane quantum conformal matter, in general, with $N$ scalar, $N_{1/2}$ spinor, $N_1$ vector fields, $N_2$ ($=0$ or 1) gravitons and $N_{\text{HD}}$ higher derivative conformal scalars, $b$, $b'$ and $b''$ are [27]

\[ b = \frac{N + 6N_{1/2} + 12N_1 + 611N_2 - 8N_{\text{HD}}}{120(4\pi)^2} \]
\[ b' = -\frac{N + 11N_{1/2} + 62N_1 + 1411N_2 - 28N_{HD}}{360(4\pi)^2}, \quad b'' = 0. \quad (5.6) \]

For typical examples motivated by AdS/CFT correspondence one has: a) \( \mathcal{N} = 4 \) \( SU(N) \) SYM theory : \( b = -b' = \frac{N^2 - 1}{4(4\pi)^2} \), b) \( \mathcal{N} = 2 \) \( Sp(N) \) theory : \( b = \frac{12N^2 + 18N - 2}{24(4\pi)^2} \) and \( b' = -\frac{12N^2 + 12N - 1}{24(4\pi)^2} \). Note that \( b' \) is negative in the above cases. It is important to note that brane quantum gravity may be taken into account via the contribution to correspondent parameters \( b, b' \).

Then on the brane, we have the following equation which generalizes the classical brane equation of the motion:

\[ 0 = \frac{48l_{AdS}^4}{16\pi G_5} \left(A_{xz} - \frac{1}{l_{AdS}^4}\right) e^{4A} + b' \left(4\partial_r^4 A + 16\partial_r^2 A\right) \\
- 4(b + b') \left(\partial_r^4 A - 2\partial_r^2 A - 6(\partial_r A)^2\partial_r^2 A\right). \quad (5.7) \]

This equation is derived from the condition that the variation of the action on the brane, or the boundary of the bulk spacetime, vanishes under the variation over \( A \). The first term proportional to \( A_{xz} \) expresses the bulk gravity force acting on the brane and the term proportional to \( \frac{1}{l_{AdS}^4} \) comes from the brane tension. The terms containing \( b \) or \( b' \) express the contribution from the conformal anomaly induced effective action (quantum effects). In (5.7), one uses the form of the metric as

\[ ds^2 = dz^2 + e^{2A(z, \tau)} \tilde{g}_{\mu\nu} dx^\mu dx^\nu, \quad \tilde{g}_{\mu\nu} dx^\mu dx^\nu \equiv l^2 \left(-d\tau^2 + d\Omega_3^2\right). \quad (5.8) \]

Here \( d\Omega_3^2 \) corresponds to the metric of 3-dimensional unit sphere.

As a bulk space, one considers 5d AdS-Schwarzschild black hole spacetime (1.3). By putting \( a = r, \ h = e^{2\rho}, \) and \( \mu = \frac{16\pi G_5 M}{3V_3} \) (\( V_3 \) is the volume of the unit 3 sphere) and by choosing new coordinates \( (z, \tau) \) as

\[ \frac{e^{2A}}{h(a)} A_{xz}^2 - h(a)t_z^2 = 1, \quad \frac{e^{2A}}{h(a)} A_{xz} A_{z\tau} - h(a)t_z t_{z\tau} = 0 \]
\[ \frac{e^{2A}}{h(a)} A_{x\tau}^2 - h(a)t_{x\tau}^2 = -e^{2A}. \quad (5.9) \]

the metric takes the warped form (5.8). Here \( a = l_{AdS} e^A \). Further choosing a coordinate \( \tilde{t} \) by \( d\tilde{t} = le^A d\tau \), the metric on the brane takes FRW form:

\[ e^{2A} \tilde{g}_{\mu\nu} dx^\mu dx^\nu = -d\tilde{t}^2 + l_{AdS}^2 e^{2A} d\Omega_3^2. \quad (5.10) \]
By solving Eqs. (5.9), we have

\[ H^2 = A_z^2 - e^{-2A} = A_z^2 - \frac{1}{l_{\text{AdS}}^2} - \frac{1}{a^2} + \frac{16\pi G_5 M}{3 V_3 a^4}. \]  

(5.11)

Here the Hubble constant \( H \) is introduced: \( H = \frac{1}{a} \frac{da}{d\tilde{t}} = \frac{dA}{d\tilde{t}} \). On the other hand, from (5.7) one gets

\[ A_z = \frac{1}{l_{\text{AdS}}} + \frac{\pi G_5}{3} \left\{ -4b' \left( H_{\tilde{t}\tilde{t}} + 4H^2 + 7HH_{\tilde{t}} + 18H^2 H_{\tilde{t}} + 6H^4 \right) \\
+ \frac{4}{a^2} \left( H_{\tilde{t}} + H^2 \right) \right\} + 4 \left( b + b' \right) \left( H_{\tilde{t}\tilde{t}} + 4H^2 \right) + 7HH_{\tilde{t}} + 12H^2 H_{\tilde{t}} \right) - \frac{2}{a^2} \left( H_{\tilde{t}} + H^2 \right) \} \right\}. \]  

(5.12)

Then combining (5.11) and (5.12), we find

\[ H^2 = -\frac{1}{l_{\text{AdS}}^2} - \frac{1}{a^2} + \frac{16\pi G_5 M}{3 V_3 a^4} + \left[ \frac{1}{l_{\text{AdS}}} + \frac{\pi G_5}{3} \left\{ -4b' \left( H_{\tilde{t}\tilde{t}} + 4H^2 \right) \\
+ 7HH_{\tilde{t}} + 18H^2 H_{\tilde{t}} + 6H^4 \right) \right] + \frac{4}{a^2} \left( H_{\tilde{t}} + H^2 \right) \right\} \right\}^2. \]  

(5.13)

This expresses the quantum correction to the corresponding brane equation in (18). In fact, if we put \( b = b' = 0 \), Eq. (5.13) reduces to the classical FRW equation

\[ H^2 = -\frac{1}{a^2} + \frac{16\pi G_5 M}{3 V_3 a^4}. \]  

(5.14)

Further by differentiating Eq. (5.13) with respect to \( \tilde{t} \), one arrives to second FRW equation. One can rewrite FRW equations in more familiar form

\[ H^2 = -\frac{1}{a^2} + \frac{8\pi G_4 \rho}{3} \]  

(5.15)

\[ \rho = \frac{l_{\text{AdS}}}{a} \left[ \frac{M}{V_3 a^3} + \frac{3a}{16\pi G_5} \left[ \frac{1}{l_{\text{AdS}}} + \frac{\pi G_5}{3} \left\{ -4b' \left( H_{\tilde{t}\tilde{t}} + 4H^2 + 7HH_{\tilde{t}} \\
+ 18H^2 H_{\tilde{t}} + 6H^4 \right) + \frac{4}{a^2} \left( H_{\tilde{t}} + H^2 \right) \right\} + 4 \left( b + b' \right) \left( H_{\tilde{t}\tilde{t}} + 4H^2 \right) \right\} \right], \]  

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Here 4d Newton constant $G_4$ is given by (3.9) and quantum corrections from CFT are included into the definition of energy (pressure). These quantum corrected FRW equations are written from quantum-induced brane-world perspective. As the correction terms include higher derivatives, these terms become relevant when the universe changes its size very rapidly as in the very early universe. It is also very important to note that doing the same identification as in the previous section one easily rewrites above FRW equations in the Cardy formula form. This is caused by the fact that quantum effects are included into the definition of energy and pressure. In other words, formally these equations look like classical FRW equations.

It is not so clear if the energy density $\rho$ and the pressure $p$ satisfy the energy conditions because quantum effects generally may violate the energy conditions. For the solution of (5.15), however, $\rho$ is always positive since (5.19) can be rewritten

$$\rho = \frac{3}{8\pi G_4} \left( H^2 + \frac{1}{a^2} \right) > 0. \quad (5.19)$$

We also have from (5.17)

$$\rho + p = \frac{1}{4\pi G_4} \left( \frac{1}{a^2} - H_i \right). \quad (5.20)$$
Therefore the weak energy condition should be satisfied if $\frac{1}{a^2} - H_j > 0$ in the solution. In order to clarify the situation, we consider the specific case of $b + b' = 0$ as in $\mathcal{N} = 4$ theory and we assume that $b'$ is small. Then from (5.13) and (5.17) and by differentiating (5.17) with respect $\tilde{t}$, one gets

$$H^2 = -\frac{1}{a^2} + \frac{8\pi G_4 M_{\text{AdS}}}{3V_3 a^4} + \mathcal{O}(b') \ , \quad H_{,\tilde{t}} = \frac{1}{a^2} - \frac{16\pi G_4 M_{\text{AdS}}}{3V_3 a^4} + \mathcal{O}(b') \ ,$$

$$H_{,\tilde{t}} = -\frac{2}{a^2} H + \frac{64\pi G_4 M_{\text{AdS}}}{3V_3 a^4} H + \mathcal{O}(b') \ , \quad \text{etc.} \quad (5.21)$$

Then by using (5.16) and (5.18), we find

$$\rho = \frac{M l_{\text{AdS}}}{V_3 a^4} - \frac{b'}{2} \left( \frac{8\pi G_4 M_{\text{AdS}}}{V_3 a^6} - \frac{128\pi^2 G_4^2 M^2 l_{\text{AdS}}}{3V_3^2 a^8} \right) + \mathcal{O}(b'^2) \ ,$$

$$p = \frac{M l_{\text{AdS}}}{3V_3 a^4} - \frac{b'}{2} \left( \frac{8\pi G_4 M_{\text{AdS}}}{V_3 a^6} - \frac{640\pi^2 G_4^2 M^2 l_{\text{AdS}}}{9V_3^2 a^8} \right) + \mathcal{O}(b'^2) \quad (5.22)$$

The correction part of $\rho$ is not always positive but $\rho$ itself should be positive, which is clear from (5.19). One also gets

$$\rho + p = \frac{4M l_{\text{AdS}}}{3V_3 a^4} - \frac{b'}{2} \left( \frac{16\pi G_4 M_{\text{AdS}}}{V_3 a^6} - \frac{1024\pi^2 G_4^2 M^2 l_{\text{AdS}}}{9V_3^2 a^8} \right) + \mathcal{O}(b'^2) \ . \quad (5.23)$$

Then the correction part seems to be not always positive and the weak energy condition might be broken. As the above discussion is based on the perturbation theory, we will discuss the weak energy condition later using the de Sitter type brane universe solution.

Let us consider the solution of quantum-corrected FRW equation (5.13). Assume the de Sitter type solution

$$a = A \cosh B\tilde{t} . \quad (5.24)$$

Substituting (5.24) into (5.13), one finds the following equations should be satisfied:

$$0 = -\frac{1}{B^2} - \frac{1}{l_{\text{AdS}}^2} + \left( \frac{1}{l_{\text{AdS}}} - 8\pi G_5 b'B^4 \right)^2 \quad (5.25)$$

$$0 = B^2 - \frac{1}{A^2}$$
\[ 0 = 1 + 2 \left( \frac{1}{l_{AdS}^2} - 8\pi G_5 b' B^4 \right) \frac{\pi G_5}{3} \left( 24b' + 8b \right) \left( B^4 - \frac{B^2}{A^2} \right) \]  

(5.26)

\[ 0 = \frac{16\pi G_5 M}{3V_3} + \left( \frac{\pi G_5}{3} \right)^2 \left( 24b' + 8b \right)^2 \left( 24b' + 8b \right) \left( B^4 - \frac{B^2}{A^2} \right)^2 . \]  

(5.27)

Eq. (5.25) tells that there is no de Sitter type solution if there is no quantum correction, or if \( b' = 0 \). Eq. (5.27) tells that if the black hole mass \( M \) is non-vanishing and positive, there is no any solution of the de Sitter-like brane. When \( M = 0 \), Eqs. (5.26) and (5.27) are trivially satisfied if \( A^2 = \frac{1}{b^2} \). Actually this case corresponds to well-known anomaly-driven inflation (for recent discussion, see [30]). Eq. (5.26) has unique non-trivial solution for \( B^2 \), which corresponds to the de Sitter brane universe in [28, 27]. This brane-world is called Brane New World.

When \( M < 0 \), there is no horizon and the curvature singularity becomes naked. We will, however, formally consider the case since there is no de Sitter-like brane solution in the classical case (\( b' = 0 \)) even if \( M \) is negative. If \( M \neq 0 \) or \( A^2 \neq \frac{1}{b^2} \), Eq. (5.26) has the following form:

\[ 0 = 1 + 2 \left( \frac{1}{l_{AdS}^2} - \frac{8\pi G_5 b'}{l_{AdS}^4} B^4 \right) \frac{\pi G_5}{3l_{AdS}^4} \left( 24b' + 8b \right) B^2 . \]  

(5.28)

Eq. (5.28) is not always compatible with Eq. (5.25) and gives a non-trivial constraint on \( G_5, l_{AdS}, b \) and \( b' \). If the constraint is satisfied, \( B^2 \) can be uniquely determined by (5.25) or (5.28). Then (5.27) can be solved with respect to \( A^2 \).

Now we consider the above constraint and solution for \( B^2 \). By combining (5.25) and (5.28), one obtains

\[ 0 = B^6 + \frac{1}{l_{AdS}^2} B^4 - \frac{1}{\eta} , \quad \eta \equiv 4 \left( 24b' + 8b \right) \left( \frac{\pi G_5}{3} \right)^2 \]  

(5.29)

\[ 0 = \left( \frac{1}{l_{AdS}^2} + B^2 \right)^3 - \left\{ \frac{1}{l_{AdS}^2} \left( \frac{1}{l_{AdS}^2} + B^2 \right) - \zeta \right\} , \]

\[ \zeta \equiv \frac{6b'}{24b' + 8b} \left( \frac{3}{\pi G_5} \right) . \]  

(5.30)

In most of cases, \( \eta \) is negative and \( \zeta \) is positive. The explicit solution of
(5.29) is given by
\[
B^2 = -\frac{1}{3l_{AdS}^2} + \left(\frac{1}{27l_{AdS}^6} - \frac{1}{2\eta} + \sqrt{\frac{1}{4\eta^2} - \frac{1}{27l_{AdS}^6\eta}}\right)^\frac{1}{3} + \left(\frac{1}{27l_{AdS}^6} - \frac{1}{2\eta} - \sqrt{\frac{1}{4\eta^2} - \frac{1}{27l_{AdS}^6\eta}}\right)^\frac{1}{3}.
\]

On the other hand, if \(\frac{\zeta^4}{4} - \frac{\zeta^3}{27l_{AdS}^4} > 0\), the solution of (5.30) is given by
\[
B^2 = -\frac{2}{3l_{AdS}^2} + \left(-\frac{1}{27l_{AdS}^6} + \frac{\zeta}{3l_{AdS}^3} - \frac{\zeta^2}{2} + \sqrt{\frac{\zeta^4}{4} - \frac{\zeta^3}{27l_{AdS}^6}}\right)^\frac{1}{3} + \left(-\frac{1}{27l_{AdS}^6} + \frac{\zeta}{3l_{AdS}^3} - \frac{\zeta^2}{2} - \sqrt{\frac{\zeta^4}{4} - \frac{\zeta^3}{27l_{AdS}^6}}\right)^\frac{1}{3}.
\]

or if \(\frac{\zeta^4}{4} - \frac{\zeta^3}{27l_{AdS}^4} < 0\), the solutions are
\[
B^2 + \frac{2}{3l_{AdS}^2} = \xi + \xi^*, \quad \zeta\omega + \xi^*\omega^2, \quad \xi\omega^2 + \xi^*\omega.
\]

Here
\[
\xi = \left(-\frac{1}{27l_{AdS}^6} + \frac{\zeta}{3l_{AdS}^3} - \frac{\zeta^2}{2} + i\sqrt{\frac{\zeta^4}{27l_{AdS}^6} - \frac{\zeta^3}{4}}\right)^\frac{1}{3}, \quad \omega = e^{\frac{2\pi}{3}}.
\]

Then if the solution (5.31) coincides with any of the solutions (5.32) or (5.33), there occurs quantum-induced de Sitter-like brane realized in d5 AdS BH. In a sense, we got the extension of scenario of refs. [28, 27] for quantum-induced brane-worlds within AdS/CFT set-up when bulk is given by d5 AdS BH.

For the de Sitter type solution (5.24), Eq. (5.20) has the following form:
\[
\rho + p = \frac{1}{4\pi G_4} \left(\frac{1}{A^2} - B^2\right) \frac{1}{\cosh^2 Bt}.
\]

Then the weak energy condition can be satisfied if
\[
\frac{1}{A^2} \geq B^2.
\]
For the exact de Sitter solution corresponding to $M = 0$, we have $\frac{1}{A^2} = B^2$ and Eq. (5.36) is satisfied. For more general solution in (5.31) or (5.33), $B$ and $A$ non-trivially depend on the parameters $G_5$, $M$, $b$ and $b'$ and it is not so clear if Eq. (5.36) is always satisfied.

The more detailed analysis shows that quantum corrections induce de Sitter brane not only in the case of zero black hole mass but also in the case of negative black hole mass. Of course, the specific details of such brane-world inflation depend on the fields content on the brane. As a final remark one can note that above picture may be considered also in the case when bulk space is de Sitter black hole (see first work in ref. [25]). The presentation of brane equations in Cardy form is again possible.

6 Brane matter induced by 5-dimensional Einstein-Gauss-Bonnet gravity

In the present section we study more complicated theory, i.e. Einstein-GB gravity. As the bulk space, Anti-de Sitter black hole is considered. The question is: how looks the brane matter (which may be considered as dark matter) induced in such brane-world theory? As it is shown explicitly such brane matter is quite complicated which is caused by higher derivatives terms.

The action of the $(d + 1)$-dimensional Einstein–GB bulk action is given by

$$S = \int d^{d+1}x \sqrt{-g} \left\{ \frac{1}{\kappa_g^2} R - \Lambda_{\text{bulk}} + c \left( R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\xi\sigma} R^{\mu\nu\xi\sigma} \right) \right\}, \quad (6.1)$$

where $c$ is an arbitrary coupling constant, $\kappa_g^2 = 16\pi G_5$ parametrizes the $(d + 1)$-dimensional Planck mass, the Riemann tensor, $R_{\mu\nu\xi\sigma}$, and its contractions are constructed from the metric, $g_{\mu\nu}$, and its derivatives, $g \equiv \det g_{\mu\nu}$ and $\Lambda_{\text{bulk}}$ represents the bulk cosmological constant.

By extremising the variations of the action (6.1) with respect to the metric

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$^7$In this section, upper case Latin indices run from $(A, B) = (1, 2, 3)$ over the spatial sections of the world–volume of the brane and $y$ is the coordinate associated with the 5-dimension.
tensor we obtain the field equations

\[ 0 = \frac{1}{2} g_{\mu\nu} \left\{ c \left( R^2 - 4 R_{\rho\sigma} R^{\rho\sigma} + R_{\rho\lambda\xi\sigma} R^{\rho\lambda\xi\sigma} \right) + \frac{1}{\kappa_g^2} R - \Lambda_{\text{bulk}} \right\} \] (6.2)

\[ + c \left( -2 R R^{\mu\nu} + 4 R^{\mu}_{\rho} R^{\nu\rho} + 4 R^{\mu\nu\sigma\tau} R_{\rho\sigma} - 2 R^{\mu\rho\sigma\tau} R_{\rho\sigma} \right) - \frac{1}{\kappa_g^2} R^{\mu\nu} . \]

One considers the case where the bulk spacetime corresponds to a static, hyper-spherically symmetric geometry with a line element given by

\[ ds^2 = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 \sum_{A,B=1}^{d-1} \tilde{g}_{AB} dx^A dx^B , \] (6.3)

where \( \{\nu(r), \lambda(r)\} \) are functions of the radial coordinate, \( r \), and the metric \( \tilde{g}_{ij} \) is the metric of the \((d-1)\)-dimensional Einstein manifold with a Ricci tensor defined by \( \tilde{R}_{ij} = k g_{ij} \). The constant \( k \) has values \( k = \{d - 2, 0, -(d - 2)\} \) for a \((d-1)\)-dimensional unit sphere, a flat Euclidean space and a \((d-1)\)-dimensional unit hyperboloid, respectively.

Restricting to the 5-dimensional case \((d = 4)\), Eq. (6.2) admits the black hole solution [2, 9]:

\[ e^{2\nu} = e^{-2\lambda} = \frac{1}{2c} \left\{ c k + \frac{r^2}{2\kappa_g^2} \right\} \]

\[ \pm \sqrt{\frac{r^4}{4\kappa_g^4} \left( \frac{4c\kappa_g^2}{l^2} - 1 \right)^2 - \frac{2c\mu}{\kappa_g^2} \left( \frac{4c\kappa_g^2}{l^2} - 1 \right)} \] , (6.4)

where the constant \( \mu \) is related to the gravitational mass of the black hole and

\[ \frac{1}{l^2} \equiv \frac{1}{4c\kappa_g^2} \left( 1 \pm \sqrt{1 + \frac{2c\Lambda_{\text{bulk}}\kappa_g^4}{3}} \right) . \] (6.5)

The constant, \( l^2 \), is determined by the Gauss–Bonnet coupling parameter, \( c \), and the bulk cosmological constant, \( \Lambda_{\text{bulk}} \). It corresponds to the length parameter of the asymptotically AdS space when \( r \) is large. If \( c\Lambda_{\text{bulk}} > 0 \), \( l^2 \) can be formally negative and the spacetime then becomes asymptotically AdS.
de Sitter. In principle, the above solution (6.4) for positive $l^2$ generalizes the well-known Schwarzschild-AdS black hole solution to Einstein–GB gravity.

We now proceed to consider the motion of a domain wall (three-brane) along a timelike geodesic of the 5-dimensional, static background defined by Eqs.(6.3) and (6.4). The equation of motion of the brane is interpreted by an observer confined to the brane as an effective Friedmann equation describing the expansion or contraction of the universe. From this Friedmann equation, we can deduce the energy and entropy of the matter in the brane universe. Specifically, we consider a brane action of the form:

$$S_{br} = -\eta \int d^4x \sqrt{-h}, \quad (6.6)$$

where $\eta$ is a positive constant representing the tension associated with the brane and $h$ is the determinant of the boundary metric, $h_{ij}$, induced by the bulk metric, $g_{\mu\nu}$.

We employ the method developed in Ref.[19] to derive the Friedmann equation. (Note that it is more complicated than Einstein gravity case considered in section 3.) The metric (6.3) is rewritten by introducing new coordinates $(y, \tau)$ and a scalar function $A = A(y, \tau)$ that satisfies the set of constraint equations:

$$l^2 e^{2A+2\lambda} A_y^2 - e^{-2\lambda} t_y^2 = 1, \quad (6.7)$$

where a comma denotes partial differentiation. When $\lambda = -\nu$, as in Eq. (B.4), the metric (6.3) may then be written in the form

$$ds^2 = dy^2 + e^{2A(y, \tau)} \sum_{i,j=1}^{4} \tilde{g}_{ij} dx^i dx^j, \quad (6.8)$$

where $r = l \exp(A)$. Since we are interested in the cosmological implications, we assume that the metric, $\tilde{g}_{ij}$, respects the same symmetries as the metric of the Friedmann–Robertson–Walker (FRW) models, i.e., we assume that

$$\tilde{g}_{ij} dx^i dx^j \equiv l^2 \left(-d\tau^2 + d\Omega_{k,3}^2 \right), \quad (6.9)$$
where \( d\Omega^2_{k,3} \) is the metric of unit three–sphere for \( k > 0 \), 3-dimensional Euclidean space for \( k = 0 \) and the unit three–hyperboloid for \( k < 0 \). Thus, choosing a timelike coordinate, \( \tilde{t} \), such that \( d\tilde{t} \equiv le^A d\tau \), implies that the induced metric on the brane takes the FRW form:

\[
 ds^2_{\text{brane}} = -d\tilde{t}^2 + l^2 e^{2A} d\Omega^2_{k,3} .
\]  

(6.10)

It follows by solving Eqs. (6.7) that

\[
 H^2 = A^2_{,y} - \frac{e^{-2A} e^{-2A}}{l^2} ,
\]

(6.11)

where the Hubble parameter on the brane is defined by \( H \equiv dA/d\tilde{t} \). Thus, for a vacuum brane that has no matter confined to it, the cosmic expansion (contraction) is determined once the functional forms of \( \{A, \lambda\} \) have been determined. Hereafter we use the scale factor of the metric on the brane as \( a \equiv e^{A(\tilde{t})} \), then the Hubble parameter is rewritten in the standard form \( H = \frac{1}{a} \frac{da}{d\tilde{t}} \).

Then the Friedmann equation describing the motion of the 3-dimensional brane in the AdS BH bulk space-time is given by the following equation (see work by Lidsey et al from ref. [10]) where the function \( f(a) \) is introduced for later convenience:

\[
 H^2 = \frac{G^2}{H^2} - \frac{X(a)}{a^2} \equiv f(a) ,
\]

(6.12)

where

\[
 G = 4\eta \pm \frac{12X^{1/2}}{Y^{3/2}} \left\{ 16e^2 \tilde{\mu}^2 (4\epsilon - 1)^2 a^{-3} \right\}
\]

\[
 \eta \equiv \frac{6}{\kappa_g^2} (12\epsilon - 1) .
\]

(6.13)

and

\[
 H = -\frac{48}{\kappa_g^2} \pm \frac{24}{Y^{3/2}} \left( 5 (2\epsilon \tilde{\mu} (4\epsilon - 1))^2 a^{-2} + 6 \left( \frac{(4\epsilon - 1)^2}{4\kappa_g^2} \right)^2 a^6 - \frac{9 (4\epsilon - 1)^3 \epsilon \tilde{\mu}}{2\kappa_g^4} a^2 \right) ,
\]

(6.14)
\[
X \equiv \frac{k}{2} + \frac{a^2}{4\ell l^2} \pm \frac{Y^{1/2} \kappa_g^2}{2\ell l^2}, \quad Y \equiv -2\epsilon \bar{\mu}(4\epsilon - 1) + \frac{(4\epsilon - 1)^2}{4\kappa_g^4} a^4. \quad (6.15)
\]

Here we have defined the rescaled parameters:
\[
\epsilon \equiv \frac{c \kappa_g^2}{l^2}, \quad \bar{\mu} \equiv \frac{l^2 \mu}{\kappa_g^4}. \quad (6.16)
\]

The standard FRW equations for 4-dimensions can be written as:
\[
H^2 = \frac{8\pi G}{3} \rho - \frac{k}{2a^2}, \quad \dot{H} = -4\pi G (\rho + p) + \frac{k}{2a^2}. \quad (6.17)
\]

here, \(\dot{}\) is the derivative with respect to cosmological time \(\tilde{t}\). Then \(\rho\) and \(p\) are
\[
\rho = \frac{3}{8\pi G} \left( f(a) + \frac{k}{2a^2} \right), \quad (6.18)
\]
\[
p = -\frac{1}{8\pi G} \left( \frac{k}{2a^2} + a f'(a) + 3f(a) \right), \quad (6.19)
\]

where \(\dot{}\) denotes the derivative with respect to \(a\). Similarly, FRW equations from the bulk dS BH may be constructed (see work by Lidsey et al in ref.[10].

In [4], it was shown that the standard FRW equation in \(d\) dimensions can be regarded as a \(d\)-dimensional analogue of the Cardy formula for a 2-dimensional CFT [5]:
\[
\tilde{S} = 2\pi \sqrt{\frac{c}{6} \left( L_0 - \frac{k}{d-2} \frac{c}{24} \right)}, \quad (6.20)
\]

where \(c\) is the analogue of the 2-dimensional central charge and \(L_0\) is the analogue of the 2-dimensional Hamiltonian. In the present case \((d = 4)\), we make the following identifications (similarly to the case of Einstein-Maxwell gravity in section 4):
\[
\frac{2\pi \rho V a}{3} \rightarrow 2\pi L_0,
\]

---

\(8\) In our conventions, \(k\) takes \(-2, 0, 2\).
\[
\frac{2V}{16\pi Ga} \rightarrow \frac{c}{24},
\]
\[
\frac{8\pi HV}{16\pi G} \rightarrow \tilde{S},
\]
\[
V = a^3V_3,
\]  \hspace{1cm} (6.21)

where \(V_3\) is the volume of the 3-dimensional sphere with unit radius. Then one finds the first FRW-like equation (6.17) has the same form as Eq. (6.20).

In order to get explicit form of \(p\), one calculates \(f'(a)\) as
\[
f'(a) = \frac{2G^2}{H^2} \left( \frac{G'}{G} - \frac{H'}{H} \right) - \frac{X'}{a^2} + \frac{2X}{a^3}.
\]  \hspace{1cm} (6.22)

From Eqs.(6.15), we get
\[
X' = \frac{a}{2\epsilon l^2} \pm \frac{Y^{-1/2}\kappa_2^2 (4\epsilon - 1)^2}{4\epsilon l^2} a^3, \quad Y' = \frac{(4\epsilon - 1)^2}{\kappa_4^4 a^3}.
\]  \hspace{1cm} (6.23)

Then, the last two terms of Eq.(6.22) are written as
\[
- \frac{X'}{a^2} + \frac{2X}{a^3} = \pm \frac{\kappa_2^2 (4\epsilon - 1)^2 a^3}{4\epsilon l^2} \pm \frac{k}{a^3} \pm \frac{\kappa_2^2}{a^3 l^2} Y^{1/2}.
\]  \hspace{1cm} (6.24)

To calculate the first two terms of Eq.(6.22), the derivatives of \(G\) and \(H\) are needed
\[
G' = \pm \frac{12X^{1/2}}{Y^{3/2}} \left\{ 16\epsilon^2 \tilde{\mu}^2 (4\epsilon - 1)^2 a^{-3} \right\} \left( \frac{1}{2 X} - \frac{3 Y'}{2 Y} - \frac{3}{a} \right)
\]
\[
H' = \pm \frac{24}{Y^{3/2}} \left\{ -3 \frac{Y'}{2 Y} \left( 5 (2\epsilon \tilde{\mu} (4\epsilon - 1))^2 a^{-2} + 6 \left( \frac{(4\epsilon - 1)^2}{4\kappa_4^4} \right)^2 a^6 
- \frac{9 (4\epsilon - 1)^3 \epsilon \tilde{\mu}}{2\kappa_4^4 a^2} + \left( -10 (2\epsilon \tilde{\mu} (4\epsilon - 1))^2 a^{-3}
+ 36 \left( \frac{(4\epsilon - 1)^2}{4\kappa_4^4} \right)^2 a^5 - \frac{18 (4\epsilon - 1)^3 \epsilon \tilde{\mu}}{2\kappa_4^4 a} \right) \right\}
\]  \hspace{1cm} (6.25)

Substituting above equations into Eq.(6.22), one obtains
\[
f'(a) = \frac{2G^2}{H^2} \times \left( \pm \frac{1}{G} \frac{12X^{1/2}}{Y^{3/2}} \left\{ 16\epsilon^2 \tilde{\mu}^2 (4\epsilon - 1)^2 a^{-3} \right\} \left( \frac{1}{2 X} - \frac{3 Y'}{2 Y} - \frac{3}{a} \right) \right)
\]
\[ \pm \frac{1}{g} \frac{24}{Y^{3/2}} \left\{ -3 \frac{Y'}{2} \left( 5 (2\epsilon \bar{\mu} (4\epsilon - 1))^2 a^{-2} + 6 \left( \frac{(4\epsilon - 1)^2}{4\kappa_g^4} \right)^2 a^6 \right) - \frac{9 (4\epsilon - 1)^3 \epsilon \bar{\mu}}{2\kappa_g^4 a^2} \right\} + \left( -10 (2\epsilon \bar{\mu} (4\epsilon - 1))^2 a^{-3} + 36 \left( \frac{(4\epsilon - 1)^2}{4\kappa_g^4} \right)^2 a^5 \right) - \frac{18 (4\epsilon - 1)^3 \epsilon \bar{\mu}}{2\kappa_g^4 a} \right\} \right) \right) + \frac{\kappa_g^2 (4\epsilon - 1)^2}{4\epsilon l^2} a^{-1/2} + \frac{k}{a^3} \pm \frac{\kappa_g^2}{\epsilon l^2 a^3} Y^{1/2} \right) \right). \] (6.26)

This equation has very complicated form, so we consider mainly the limit \( a \to \infty \) or \( a \to 0 \).

One first considers the \( \rho \) and \( \rho \) until the order of \( a^{-4} \), in the limit of \( a \to \infty \). Then

\[ X = \frac{k}{2} + \frac{a^2}{4\epsilon l^2} \pm \frac{\kappa_g^2}{2\epsilon l^2} Y^{1/2}, \]
\[ = \frac{k}{2} + \frac{a^2}{4\epsilon l^2} \pm \frac{4\epsilon - 1}{4\epsilon l^2} a^2 \left( 1 - \epsilon \bar{\mu} \frac{4\kappa_g^4}{(4\epsilon - 1)a^4} \right), \]
\[ Y = \frac{(4\epsilon - 1)^2}{4\kappa_g^4 a^4} \left( 1 - \epsilon \bar{\mu} \frac{8\kappa_g^4}{(4\epsilon - 1)a^4} \right), \]
\[ G \to 4\eta, \]
\[ \mathcal{H} \to -\frac{48}{\kappa_g^2} + 24 \frac{8\kappa_g^6}{(4\epsilon - 1)^3} a^{-6} \left( 1 - \epsilon \bar{\mu} \frac{8\kappa_g^4}{(4\epsilon - 1)a^4} \right)^{-3/2} \]
\[ \times \left( \frac{(4\epsilon - 1)^2}{4\kappa_g^4} \right)^2 a^6 - \frac{9(4\epsilon - 1)^3 \epsilon \bar{\mu}}{2\kappa_g^4 a^2} \right) \right) \right) \right), \]
\[ \approx -\frac{48}{\kappa_g^2} + 24 \frac{8\kappa_g^6}{(4\epsilon - 1)^3} \left( 1 + \epsilon \bar{\mu} \frac{12\kappa_g^4 a^{-4}}{(4\epsilon - 1)} \right) \]
\[ \times \left( \frac{(4\epsilon - 1)^2}{4\kappa_g^4} \right)^2 - \frac{9(4\epsilon - 1)^3 \epsilon \bar{\mu}}{2\kappa_g^4 a^4} \right) \right) \right) \right) \right) \right) \right) \right) \right) \right), \] (6.27)
Thus,

\[
f(a) = \frac{1}{l^2} (12\epsilon - 1)^2 (2 \pm 3(4\epsilon - 1))^{-2} - \frac{k}{2a^2} - \frac{1}{4\epsilon l^2} \mp \frac{4\epsilon - 1}{4\epsilon l^2} \left(1 - \epsilon \tilde{\mu} \frac{4\kappa_g^4}{(4\epsilon - 1)a^4}\right)\quad (6.28)
\]

\[
\rho = \frac{3}{8\pi G} \left(\frac{1}{l^2} (12\epsilon - 1)^2 (2 \pm 3(4\epsilon - 1))^{-2} - \frac{1}{4\epsilon l^2} \mp \frac{4\epsilon - 1}{4\epsilon l^2} \left(1 - \epsilon \tilde{\mu} \frac{4\kappa_g^4}{(4\epsilon - 1)a^4}\right)\right)\quad (6.29)
\]

\[
p = -\frac{1}{8\pi G} \left(\frac{3}{l^2} (12\epsilon - 1)^2 (2 \pm 3(4\epsilon - 1))^{-2} - \frac{3}{4\epsilon l^2} \pm \frac{3(4\epsilon - 1)}{4\epsilon l^2} \pm \frac{1}{l^2} \tilde{\mu} \kappa_g^4 a^{-4}\right)\quad (6.30)
\]

If we choose the upper sign, that is, \(+\) of \(\pm\) and \(-\) of \(\mp\) in Eqs.\((6.28)\), \((6.29)\), \((6.30)\), then

\[
f(a) = -\frac{k}{2a^2} + \frac{\tilde{\mu} \kappa_g^4}{l^2 a^4},\quad (6.31)
\]

\[
\rho = \frac{3}{8\pi G} \frac{\tilde{\mu} \kappa_g^4}{l^2 a^4},\quad (6.32)
\]

\[
p = \frac{1}{8\pi G} \frac{\tilde{\mu} \kappa_g^4}{l^2 a^4},\quad (6.33)
\]

This shows that energy-momentum tensor is traceless, \(T^\mu_\mu = \rho - 3p = 0\). This means the theory on the brane is CFT.

However, if we take the lower sign, that is \(-\) of \(\pm\) and \(+\) of \(\mp\) in Eqs.\((6.28)\), \((6.29)\), \((6.30)\), there is a constant term and \(a^{-4}\) term which comes from conformal matter. Since the original Friedmann equation which includes cosmological constant \(\Lambda\) has the following form:

\[
H^2 = \frac{8\pi G}{3} \rho_m - \frac{k}{2a^2} + \frac{\Lambda}{3},
\]

\[
\dot{H} = -4\pi G (\rho + p) + \frac{k}{2a^2},\quad (6.34)
\]

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one can divide $\rho$ and $p$ into the sum of the contributions from matter fields $\rho_m$ and $p_m$ and those from the cosmological constant:

$$
\rho = \rho_m + \rho_0, \quad p = p_m - \rho_0, \quad \rho_0 = \frac{\Lambda}{8\pi G}.
$$

(6.35)

Then the constant term in (6.29) corresponds to the effective cosmological constant on the brane:

$$
\rho_0 = \frac{\Lambda}{8\pi G} = \frac{3}{8\pi G} \left\{ \frac{1}{l^2} (12\epsilon - 1)^2 (5 - 12\epsilon)^{-2} - \frac{1 - 2\epsilon}{2\epsilon l^2} \right\}
$$

(6.36)

The matter parts $\rho_m$ and $p_m$ in $\rho$ and $p$ are given by

$$
\rho_m = -\frac{3}{8\pi G l^2 a^4},
$$

(6.37)

$$
p_m = -\frac{1}{8\pi G l^2 a^4},
$$

(6.38)

and the matter energy-momentum tensor $T^{\mu\nu}_m$ is traceless, $T^{\mu\nu}_m = \rho_m - 3p_m = 0$. Thus, having the effective cosmological term in FRW equations, the brane matter is again the conformal one.

Next, one takes $a \to 0$ limit in case $-2\epsilon \tilde{\mu}(4\epsilon - 1) > 0$. Then

$$
X \to \frac{k}{2} \pm \frac{\kappa^2}{2\epsilon l^2} \sqrt{-2\epsilon \tilde{\mu}(4\epsilon - 1)}, \quad Y \to -2\epsilon \tilde{\mu}(4\epsilon - 1)
$$

$$
G \to \pm 48(-2\epsilon \tilde{\mu}(4\epsilon - 1))^{1/2} \left\{ \frac{k}{2} \pm \frac{\kappa^2}{2\epsilon l^2} (-2\epsilon \tilde{\mu}(4\epsilon - 1))^{1/2} \right\}^{1/2} a^{-3},
$$

$$
H \to \mp 120(-2\epsilon \tilde{\mu}(4\epsilon - 1))^{1/2} a^{-2}.
$$

(6.39)

These equations give $\rho$ and $p$ as

$$
\rho = \frac{3}{8\pi G} \left\{ \frac{2k}{25} \pm \frac{21}{25} \frac{\kappa^2}{2\epsilon l^2} (-2\epsilon \tilde{\mu}(4\epsilon - 1))^{1/2} \right\} a^{-2}
$$

$$
p = -\frac{1}{8\pi G} \left\{ \frac{2k}{25} \pm \frac{21}{25} \frac{\kappa^2}{2\epsilon l^2} (-2\epsilon \tilde{\mu}(4\epsilon - 1))^{1/2} \right\} a^{-2}.
$$

(6.40)

The energy-momentum tensor is not traceless. The case that $\rho$ and $p$ is proportional to $a^{-2}$ is known as curvature dominant case. The original Friedmann equation (6.34) can be rewritten in the following form:

$$
H^2 = \frac{8\pi G}{3} \left( \rho_m - \frac{3}{8\pi G 2a^2} \right) + \frac{\Lambda}{3},
$$

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Here \( \tilde{\rho} \) and \( \tilde{k} \) are effective energy density and effective \( k \) respectively defined by

\[
\tilde{\rho} = \rho_m - \frac{3}{8\pi G} \frac{k}{2a^2}, \quad \tilde{k} = 0. \tag{6.42}
\]

Such effective energy density is proportional to \( a^{-2} \) when \( k \) is dominant, that is \( a \to \infty \), because the density \( \rho_m \) must decrease with increasing \( a \) at least as fast as \( a^{-3} \). It is interesting that the original behavior of \( \tilde{\rho} \) is proportional to \( a^{-2} \) in the limit \( a \to \infty \), while our \( \rho \) (6.40) behaves like \( a^{-2} \) in the limit of \( a \to 0 \).

Note that if \( -2\epsilon\tilde{\mu}(4\epsilon - 1) \) is less than zero, \( X, \mathcal{H}, \mathcal{G} \) become imaginary when \( a = 0 \). Then we consider \( Y = 0 \) case instead of \( a \to 0 \) limit. When \( Y = 0, a \) and \( X \) are

\[
a = \left( \frac{8\epsilon\tilde{\mu}}{(4\epsilon - 1)} \right)^{\frac{1}{4}} \kappa_g, \\
X = \frac{k}{2} + \frac{a^2}{4\epsilon l^2} \\
= \frac{k}{2} + \frac{1}{4\epsilon l^2} \left( \frac{2\epsilon\tilde{\mu}}{(4\epsilon - 1)} \right)^{\frac{1}{2}} (2\kappa_g^2) \tag{6.43}
\]

\[
\frac{\mathcal{G}^2}{\mathcal{H}^2} \quad \text{is}
\]

\[
\frac{\mathcal{G}^2}{\mathcal{H}^2} = \frac{4}{81} \left\{ \frac{k}{2} \left( \frac{2\epsilon\tilde{\mu}}{(4\epsilon - 1)} \right)^{\frac{1}{2}} (2\kappa_g^2)^{-1} + \frac{1}{4\epsilon l^2} \right\}, \tag{6.44}
\]

which leads to

\[
\rho = \frac{3}{8\pi G} \left\{ \frac{2k}{81} \left( \frac{2\epsilon\tilde{\mu}}{(4\epsilon - 1)} \right)^{-\frac{1}{2}} (2\kappa_g^2)^{-1} - \frac{77}{81} \frac{1}{4\epsilon l^2} \right\}, \tag{6.45}
\]

\[
p = -\frac{1}{8\pi G} \left\{ \frac{2k}{81} \left( \frac{2\epsilon\tilde{\mu}}{(4\epsilon - 1)} \right)^{-\frac{1}{2}} (2\kappa_g^2)^{-1} - \frac{77}{81} \frac{3}{4\epsilon l^2} \right\}. \tag{6.45}
\]

\(^9\)When curvature \( k \) becomes large, \( k \) can be divided as \( k = k + \tilde{k} \). The \( \tilde{k} \) takes the original value, namely 0, ±2.
One can divide $\rho$ and $p$ in (6.45) into the sum of the contributions from matter fields and the cosmological constant as in (6.35). Then one arrives at

$$
\rho_0 = -\frac{3}{8\pi G} \frac{77}{81} \frac{1}{4\epsilon \bar{\mu}^2}, \\
\rho_m = \frac{3}{8\pi G} \frac{2k}{81} \left( \frac{2\epsilon \bar{\mu}}{4\epsilon - 1} \right)^{-\frac{1}{2}} (2\kappa_g^2)^{-1}, \\
p_m = -\frac{1}{8\pi G} \frac{2k}{81} \left( \frac{2\epsilon \bar{\mu}}{4\epsilon - 1} \right)^{-\frac{1}{2}} (2\kappa_g^2)^{-1}.
$$

(6.46)

As a special case, we consider $\epsilon = 1/4$ case where $\mathcal{G}, \mathcal{H}$ become

$$
\mathcal{G} = \frac{48}{\kappa_g^2}, \quad \mathcal{H} = -\frac{48}{\kappa_g^2}.
$$

(6.47)

$f(a), f'(a)$ take simple forms

$$
f(a) = -\frac{k}{2a^2}, \quad f'(a) = \frac{k}{a^3}.
$$

(6.48)

Note that $k$ should be negative, i.e. $k = -2$ since $f(a) = H^2$ is always positive. Thus $\rho$ and $p$ are

$$
p = 0, \quad \rho = 0.
$$

(6.49)

Therefore the energy-momentum tensor is zero. Next, we consider $\epsilon = 1/4 - \delta^2$ case. Here $\delta^2 > 0$ and $|\delta| < 1$. In this case, $X, Y$ are

$$
X = \frac{k}{2} + \frac{a^2}{l^2} \pm \frac{2\sqrt{2\bar{\mu}\kappa_g^2}\delta}{l^2} + \mathcal{O}(\delta^4), \quad Y \sim 2\bar{\mu}\delta^2 + \mathcal{O}(\delta^4)
$$

(6.50)

Using Eqs.(6.50), one obtains $\mathcal{G}, \mathcal{H}$ as

$$
\mathcal{G} = \frac{48}{\kappa_g^2} \pm 48 \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{1/2} (2\bar{\mu})^{1/2} a^{-3} \delta, \\
\mathcal{H} = -\frac{48}{\kappa_g^2} + 120(2\bar{\mu})^{1/2} a^{-2} \delta.
$$

(6.51)
until the order of $\delta$. Then $f(a)$ is

$$f(a) = -\frac{k}{2a^2} \pm \frac{1}{l^2} \left\{ \frac{2\kappa_g^2 l}{a^3} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{1/2} - \frac{7\kappa_g^2}{a^2} \right\} (2\tilde{\mu})^{1/2} \delta \, . \quad (6.52)$$

This leads to the following $\rho$

$$\rho = \pm \frac{3}{8\pi G l^2} \left\{ \frac{2l}{a^3} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{1/2} - \frac{7}{a^2} \right\} (2\tilde{\mu})^{1/2} \delta \, . \quad (6.53)$$

$f'(a)$ is

$$f'(a) = \frac{k}{a^3} \pm \frac{\kappa_g^2}{l^2} \left\{ -\frac{6l}{a^4} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{1/2} + \frac{2}{a^2 l} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{-1/2} + \frac{14}{a^3} \right\} (2\tilde{\mu})^{1/2} \delta,$$

which leads to the following $p$

$$p = \pm \frac{1}{8\pi G l^2} \frac{\kappa_g^2}{l^2} \left\{ \frac{2}{a} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{-1/2} - \frac{7}{a^2} \right\} (2\tilde{\mu})^{1/2} \delta \, . \quad (6.54)$$

In the limit of $a \to 0$, $\rho$ which is much larger than $p$ is proportional to $a^{-3}$. This means there is “dust” on the brane. We should note that when $k = -2$, there is a minimum of $a$ at $a = l$ since $f(a)$ becomes complex values if $a < l$.

In the limit of $a \to \infty$ for $k = 2$ or $k = 0$, $\rho$ and $p$ are proportional to $a^{-2}$ like in Eq. (6.40), which agrees with the behavior of the original effective energy density $\tilde{\rho}$ as it was mentioned.

The trace of the energy-momentum tensor is

$$T_{\mu}^{\mu} = -\rho + 3p \quad (6.55)$$

$$= \mp \frac{3}{8\pi G l^2} \frac{\kappa_g^2}{l^2} \left\{ \frac{2l}{a^3} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{1/2} - \frac{7}{a^2} \right\} (2\tilde{\mu})^{1/2} \delta$$

$$\mp \frac{3}{8\pi G l^2} \frac{\kappa_g^2}{l^2} \left\{ \frac{2}{a} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{-1/2} - \frac{7}{a^2} \right\} (2\tilde{\mu})^{1/2} \delta$$

$$= \mp \frac{3}{8\pi G l^2} \frac{\kappa_g^2}{l^2} \left\{ \frac{2l}{a^3} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{1/2} + \frac{2}{a} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{-1/2} - \frac{14}{a^2} \right\} (2\tilde{\mu})^{1/2} \delta,$$
which is not zero. In the limit \( a \to \infty \), the energy-momentum tensor is traceless. This indicates that dual CFT description is valid only in such a limit.

Another special case is \( \epsilon = 1/12 \), which gives \( \eta = 0 \). Then

\[
X = \frac{k}{2} + \frac{3a^2}{l^2} \pm \frac{2a^2}{l^2} \left( \frac{\bar{\mu} \kappa^4}{a^4} + 1 \right)^{1/2}, \quad Y = \frac{a^4}{9\kappa^4} \left( \frac{\bar{\mu} \kappa^4}{a^4} + 1 \right). \quad (6.56)
\]

\[
G = \pm 16\bar{\mu}^2 a^{-9} \kappa^6 g \left( \frac{\bar{\mu} \kappa^4}{a^4} + 1 \right)^{-3/2} \left( \frac{k}{2} + \frac{3a^2}{l^2} \pm \frac{2a^2}{l^2} \left( \frac{\bar{\mu} \kappa^4}{a^4} + 1 \right)^{1/2} \right)^{1/2}
\]

\[
H = -\frac{48}{k^2} + a^{-6} \kappa^6 g \left( \frac{\bar{\mu} \kappa^4}{a^4} + 1 \right)^{-3/2} \left( 40\bar{\mu} a^{-2} + 48 \frac{a^6}{\kappa^6 g} + 72 \frac{\bar{\mu}}{\kappa^4} a^2 \right). \quad (6.57)
\]

Above equations lead to the following \( f(a) \)

\[
f(a) = 256\bar{\mu}^4 a^{-18} \kappa^12 g \left( \frac{\bar{\mu} \kappa^4}{a^4} + 1 \right)^{-3} \left( \frac{k}{2} + \frac{3a^2}{l^2} \pm \frac{2a^2}{l^2} \left( \frac{\bar{\mu} \kappa^4}{a^4} + 1 \right)^{1/2} \right)
\]

\[
\times \left\{ -\frac{48}{k^2} \mp a^{-6} \kappa^6 g \left( \frac{\bar{\mu} \kappa^4}{a^4} + 1 \right)^{-3/2} \left( 40\bar{\mu} a^{-2} + 48 \frac{a^6}{\kappa^6 g} + 72 \frac{\bar{\mu}}{\kappa^4} a^2 \right) \right\}^{-2}
\]

\[-\frac{k}{2a^2} - \frac{3}{l^2} \mp \frac{2}{l^2} \left( \frac{\bar{\mu} \kappa^4}{a^4} + 1 \right)^{1/2}. \quad (6.58)
\]

The structure of \( \rho, p \) is very complicated, so we consider them in the limit \( a \to \infty \) or \( a \to 0 \). Taking \( a \to \infty \), one gets

\[
f(a) \to -\frac{k}{2a^2} - \frac{3}{l^2} \mp \frac{2}{l^2} \left( \frac{\bar{\mu} \kappa^4}{a^4} + 1 \right)^{1/2},
\]

\[
\rho \to \frac{3}{8\pi G} \left( -\frac{3}{l^2} \mp \frac{2}{l^2} \pm \frac{\bar{\mu} \kappa^4}{l^2 a^4} \right) \quad (6.59)
\]

\[
p \to -\frac{1}{8\pi G} \left( -\frac{9}{l^2} \mp \frac{6}{l^2} \pm \frac{\bar{\mu} \kappa^4}{l^2 a^4} \right). \quad (6.60)
\]

Similarly to Eqs. (6.29) and (6.30), there are constant and \( a^{-4} \) terms which correspond to effective cosmological constant on the brane and the effect of
conformal matter, respectively. Then \( \rho \) and \( p \) in (6.59), (6.60) can be divided into the sum of the contributions from matter and from the cosmological constant as in (6.35) again. Then they look as

\[
\rho_0 = \frac{1}{8\pi G} \left( \frac{-9}{l^2} \mp \frac{6}{l^2} \right), \\
\rho_m = \mp \frac{1}{8\pi G l^2} \frac{\tilde{\mu}}{a^4}, \\
p_m = \mp \frac{1}{8\pi G l^2} \frac{\kappa_g^4}{a^4}.
\]  

(6.61)

and the matter energy-momentum tensor \( T^{\mu\nu}_m \) is traceless, \( T^{\mu\mu}_m = \rho_m - 3p_m = 0 \). On the other hand, when \( a \) is small, we find

\[
\rho \to \pm \frac{6}{8\pi G} \frac{21}{25} \frac{\kappa_g^2}{l^2} \frac{\tilde{\mu}}{a^2}, \quad p \to \pm \frac{2}{8\pi G} \frac{21}{25} \frac{\kappa_g^2}{l^2} \frac{\tilde{\mu}}{a^2}.
\]  

(6.62)

This corresponds to the curvature dominant case as in (5.40).

It is quite interesting now to check the Weak Energy Condition (WEC) and the Dominant Energy Condition (DEC) for above 4-dimensional cases. The WEC is defined as the condition where

\[
\rho_m + p_m \geq 0, \quad \rho_m \geq 0,
\]  

(6.63)

and the DEC is given by

\[
\rho_m + p_m \geq 0, \quad \rho_m + 3p_m \geq 0,
\]  

(6.64)

In the general case it follows from Eqs.(6.18),(6.19):

\[
\text{WEC} \quad \rho + p = \rho_m + p_m \geq 0 \Leftrightarrow k \geq a^3 f'(a), \quad \rho_m \geq 0 \Leftrightarrow \frac{k}{2a^2} \geq -f(a) - \rho_0, \\
\text{DEC} \quad \rho + p = \rho_m + p_m \geq 0 \Leftrightarrow k \geq a^3 f'(a), \quad \rho_m + 3p_m \geq 0 \Leftrightarrow -2f(a) \geq af'(a) + 2\rho_0.
\]  

(6.65)

Note that \( f(a) \) is always positive , \( f(a) \geq 0 \). We will check some limits of \( a \) and specific \( \epsilon \) cases mentioned above.

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1. In the limit $a \to \infty$, including the order of $a^{-4}$, one finds

$$\rho_m + p_m = \pm \frac{4}{8\pi G} \frac{\tilde{\mu} \kappa_{g}^4}{l^2 a^4}, \quad \rho_m = \pm \frac{3}{8\pi G} \frac{\tilde{\mu} \kappa_{g}^4}{l^2 a^4}$$

$$\rho_m + 3p_m = \pm \frac{6}{8\pi G} \frac{\tilde{\mu} \kappa_{g}^4}{l^2 a^4}. \quad (6.67)$$

Then for the upper signs both of the DEC and WEC are satisfied but for the lower signs, both of the conditions are not satisfied.

2. The limit $a \to 0$: If $\frac{2k}{25} \mp \frac{21}{25} \frac{\kappa_{g}^2}{26l^2}(-2\epsilon \tilde{\mu}(4\epsilon - 1))^{1/2} \geq 0$,

$$\rho + p = \frac{2}{8\pi G} \left\{ \frac{2k}{25} \mp \frac{21}{25} \frac{\kappa_{g}^2}{26l^2}(-2\epsilon \tilde{\mu}(4\epsilon - 1))^{1/2} \right\} a^{-2} \geq 0$$

$$\rho = \frac{3}{8\pi G} \left\{ \frac{2k}{25} \mp \frac{21}{25} \frac{\kappa_{g}^2}{26l^2}(-2\epsilon \tilde{\mu}(4\epsilon - 1))^{1/2} \right\} a^{-2} \geq 0$$

$$\rho + 3p = 0. \quad (6.68)$$

Then both of the DEC and WEC are satisfied. If $\frac{2k}{25} \mp \frac{21}{25} \frac{\kappa_{g}^2}{26l^2}(-2\epsilon \tilde{\mu}(4\epsilon - 1))^{1/2} \leq 0$, both WEC and DEC do not hold.

3. $Y = 0$ case. If $k \geq 0$,

$$\rho_m + p_m = \frac{2}{8\pi G} \left\{ \frac{2k}{81} \left( \frac{2\epsilon \tilde{\mu}}{(4\epsilon - 1)} \right)^{1/3} (2\kappa_{g}^2)^{-1} \right\} \geq 0, \quad (6.69)$$

$$\rho_m = \frac{3}{8\pi G} \frac{2k}{81} \left( \frac{2\epsilon \tilde{\mu}}{(4\epsilon - 1)} \right)^{1/3} (2\kappa_{g}^2)^{-1} \geq 0$$

$$\rho_m + 3p_m = 0. \quad (6.70)$$

Then both of the DEC and WEC are satisfied. If $k \leq 0$ both WEC and DEC do not hold.

4. When $\epsilon = 1/4$, we find

$$\rho + p = 0, \quad \rho = 0, \quad \rho + 3p = 0. \quad (6.71)$$

Then both of the DEC and WEC are trivially satisfied.
5. When \( \epsilon = 1/4 - \delta^2 \)

\[
\rho + p = \pm \frac{1}{8\pi G l^2} \left[ \frac{6l}{a^3} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{1/2} \right.
- \frac{2}{al} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{-1/2} - \frac{14}{a^2} \left(2\tilde{\mu}\right)^{1/2} \delta
\]

\[
\rho + 3p = \pm \frac{1}{8\pi G l^2} \left[ \frac{6l}{a^3} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{1/2} \right.
- \frac{6}{al} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{-1/2} \left(2\tilde{\mu}\right)^{1/2} \delta
\]

(6.72)

If the upper signs are chosen, that is, + of \(\pm\) in above equations and in the limit of \(a \to \infty\) for \(k = 2, 0\) or in the case of \(a = l + 0\) for \(k = -2\), then

\[
\rho + p \leq 0, \quad \rho + 3p \leq 0.
\]

(6.73)

Then both of the DEC and WEC are not satisfied. In the limit of \(a \to 0\),

\[
\rho + p \geq 0, \quad \rho \geq 0 \quad \rho + 3p \geq 0.
\]

(6.74)

Then both of the DEC and WEC are satisfied. On the other hand if we choose the lower signs, that is, \(\pm\) of \(\pm\) in Eqs.(6.72), in the limit of \(a \to \infty\) for \(k = 2, 0\) or in the case of \(a = l + 0\) for \(k = -2\), both of WEC and DEC hold, while in the limit of \(a \to 0\), both of WEC and DEC do not hold.

6. When \( \epsilon = 1/12 \) and \( a \) is large

\[
\rho_m + p_m = \mp \frac{1}{8\pi G l^2} \frac{4\tilde{\mu} \kappa_g^4}{a^4}
\]

(6.75)

\[
\rho_m + 3p_m = \mp \frac{1}{8\pi G l^2} \frac{6\tilde{\mu} \kappa_g^4}{a^4}
\]

(6.76)

If we choose + of \(\mp\) in above equations, both of the DEC and WEC are satisfied. If we choose - of \(\mp\) in Eq.(6.73), however, both of the
DEC and WEC are not satisfied. On the other hand when \( a \) is small one gets

\[
\rho_m + p_m = \mp \frac{4}{8\pi G l^2} \frac{21}{25} \sqrt{\mu \kappa_g^2} \, \frac{a}{a^2}
\]

(6.77)

\[
\rho_m + 3p_m = 0.
\]

(6.78)

Then for + of \( \mp \) in above equations both of the DEC and WEC are satisfied again but for − both of the DEC and the WEC are not satisfied.

In the above analysis one finds that the matter on the brane shows the singular behavior when \( \epsilon = \frac{1}{4} \) or \( \epsilon = \frac{1}{12} \). In [1] it has been shown that the black hole entropy is given by

\[
S = V_3 \left( \frac{1 - 12 \epsilon}{1 - 4 \epsilon} \right) \left( 4\pi r_H^3 + 24\pi \kappa \pi r_H \right) + S_0.
\]

(6.79)

Here \( r_H \) is the horizon radius and \( V_3 \) is the volume of the Einstein manifold with unit radius. \( S_0 \) is a constant of the integration and if we assume \( S = 0 \) when \( r_H = 0 \), we have \( S_0 = 0 \). Then the entropy vanishes when \( \epsilon = \frac{1}{12} \) and diverges when \( \epsilon = \frac{1}{4} \). Note that above entropy should be identified with cosmological entropy of dual QFT (second way appearance of CV formula).

If we take \( \epsilon \) between \( 1/12 \) and \( 1/4 \), that is, \( 1/12 < \epsilon < 1/4 \) then the entropy (6.79) takes negative value. (The appearance of negative entropy black holes in HD gravity has been discussed in [2]. As it was shown in last work of ref.[10] black hole with negative entropy is very unstable and quickly decays.) Let us check what happens in the region around \( 1/4 \) and \( 1/12 \).

How looks the behavior around \( \epsilon = 1/4 \). For this purpose, we extend \( \epsilon \) to the complex value. Since Eqs.(6.13), (6.14) and (6.15) contain the half-integer power of \( Y \), the expressions have branch points in the complex \( \epsilon \)-plane when \( Y = 0 \), that is, at

\[
\epsilon = \epsilon_1 \equiv \frac{1}{4}, \quad \epsilon = \epsilon_2 \equiv \frac{1}{4} \left( 1 - \frac{2\tilde{\mu} \kappa_g^2}{a^4} \right).
\]

(6.80)

and there is a cut connecting two branch points as in Fig.6. In the limit \( a \to \infty \), the two branch points coincide with each other, \( \epsilon_2 \to \epsilon_1 \). Then if we consider the limit \( a \to \infty \) first, the cut does not appear as in (6.29) and
When $\epsilon$ is real, Eqs. (6.53) and (6.54) tell that $\rho$ and $p$ becomes complex on the cut, that is, when $\epsilon_1 < \epsilon < \epsilon_2$ since $\delta$ should be pure imaginary. If we choose the path $A$ in Fig. 1 which path the cut the sign of $\rho$ and $p$ is changed, that is, the value $\rho$ and $p$ in (6.29) and (6.30) are changed into those in (6.37) and (6.38) when $a$ is large. On the other hand, if we choose the path $B$ in Fig. 1 the sign is not changed.

We now consider the behavior near $\epsilon = 1/12$. Eq. (6.61) shows that the signs of $\rho$ and $p$ at $\epsilon = 1/12$ are not changed since $\rho$ and $p$ have finite values there. By comparing the contribution $\rho_0$ which comes from the effective cosmological constant in the effective gravity on the brane with (6.29) where $\rho_0 = 0$, we find that there is a jump in the value of $\rho_0$ at $\epsilon = 1/12$ (there is no jump for (6.36) which corresponds to the lower $-$ sign in (6.29)). The jump might make a potential barrier at $\epsilon = 1/12$ since $\rho$ corresponds the energy density on the brane.

In the similar way one can discuss the other values of Gauss-Bonnet coupling constant and to find the brane matter energy and pressure for such values. As it follows from the discussion in this section this is straightforward while technically a little bit complicated. In the next section we apply the found matter energy and pressure in the consideration of the FRW brane cosmology.

## 7 FRW Brane Cosmology from Einstein-Gauss-Bonnet gravity

In this section we discuss FRW brane equations for various examples of matter induced by Einstein-GB gravity (see previous section) by using effective...
potential technique. It is assumed that the bulk spacetime is asymptotically anti-de Sitter. First, we review FRW brane cosmology in 5-dimensional Einstein gravity, namely $\epsilon = 0$ (no higher derivative term). If one uses effective potential technique for FRW brane equation (see section 3):

$$H^2 = -\frac{k}{2a^2} + \frac{8\pi G}{3}\rho ,$$

one has to rewrite this as

$$\left(\frac{da}{dt}\right)^2 = -\frac{k}{2} - V(a) .$$

Here $V(a)$ is the effective potential: $V(a) = -\frac{8\pi Ga^2}{3}\rho$ which is proportional to $-1/a^2$. $V(a)$ is plotted in Fig.2. Then the universe can only exist in regions where the line $V(a) = -k/2$ exceeds $V(a)$, so that $H^2 > 0$. For the case of $k = 2$ the spherical (inflationary) brane starts at $a = 0$ and reaches its maximal size at $a_{\text{max}}$ and then it re-collapses. For $k = 0$ or $k = -2$, the brane starts at $a = 0$ and expands to infinity.
Similarly, we consider the cosmology for the higher derivative case. From the analysis of the previous section, one can easily obtain the effective potential $V(a)$ by using energy density $\rho$. We consider several limits of $a$ and particular values of $\epsilon$ which were discussed in previous section.

1. First, we consider large $a$ limit. From Eqs. (6.31), (6.36), (6.37), there appear two types of effective potential:

$$V(a) = -\frac{8\pi G a^2}{3} \rho \xrightarrow{a \to \infty} -\frac{\bar{\mu} \kappa_4^4}{l^2 a^2}.$$ 

(7.3)

and

$$V(a) = \frac{\bar{\mu} \kappa_4^4}{l^2 a^2} - \frac{\Lambda}{3} a^2 \xrightarrow{a \to \infty} -\frac{\Lambda}{3} a^2.$$ 

(7.4)

Here

$$\Lambda = 3 \left\{ \frac{1}{l^2} (12\epsilon - 1)^2 (5 - 12\epsilon)^{-2} - \frac{1 - 2\epsilon}{2cl^2} \right\}.\tag{7.5}$$

The former case is obtained by taking the upper sign in Eq.(6.29) and the latter is obtained by taking the the lower sign. The former case is similar to the original FRW cosmology in Einstein gravity as we mentioned. From the point of view of the brane:

- $k = 2$, the brane which is sphere reaches its maximum size at $a_{\text{max}}$ and then it re-collapses. Note that this case is the reverse version of “bounce” (see work by Medved in ref.[24]). It can be called the bounce universe when the brane starts at $a = \infty$ and reaches its minimum size at $a = a_{\text{min}}$ and then it re-expands.

- $k = 0$ or $k = -2$, the brane which is flat or hyperbolic expands to infinity.

In the latter case of Eq.(7.4) there are three kinds of potential which are shown in Figure 3 for $\Lambda \neq 0$ cases. For the case of $\Lambda > 0$:

- $k = 2$, the spherical brane starts at $a = \infty$ and reaches its minimum size at $a = a_{\text{min}}$ and then it re-expands. It can be called the bounce universe as we mentioned above.
Figure 3: The effective potential for the evolution of FRW universe corresponding to Eq.(7.4) with \( \Lambda \neq 0 \).

- \( k = 0 \) or \( -2 \), the flat or hyperbolic brane expands to infinity.

When \( \Lambda < 0 \) one can take \( k = -2 \) because the universe can only exist in the regions where the line \( -k/2 \) exceeds \( V(a) \). Since FRW equation becomes inconsistent when \( k = 0 \) or \( 2 \) in the limit \( a \to \infty \).

- \( k = -2 \), the hyperbolic brane reaches its maximum size at \( a = a_{\text{max}} \) and then it re-collapses.

When \( \Lambda = 0 \), the effective potential is given in Figure 4 and one can take only \( k = -2 \).

- For \( k = -2 \) case the brane starts at \( a = \infty \) and reaches its minimum size at \( a = a_{\text{min}} \) and then it re-expands.

2. Next case corresponds to \( a \to 0 \) limit for \(-2\epsilon\tilde{\mu}(4\epsilon - 1) > 0 \). From Eq.(6.40) the effective potential is

\[
V(a) \xrightarrow{a \to 0} - \left\{ \frac{2k}{25} \mp \frac{21}{25} \frac{\kappa^2_g}{2\epsilon l^2} (-2\epsilon\tilde{\mu}(4\epsilon - 1))^{1/2} \right\}
\]  

(7.6)

In this case the brane exists at \( a = 0 \). Namely, the brane universe may have a cosmological singularity under the condition \(-k/2 \geq \lim_{a \to 0} V(a)\),
Figure 4: The effective potential for the evolution of FRW universe corresponding to Eq.(7.4) with \( \Lambda = 0 \). For \( k = -2 \) case, the brane starts from \( a = \infty \) and reaches its minimum size at \( a = a_{\text{min}} \) and then it re-expands.

that is

\[
k \leq \pm 2 Z_1(\epsilon) , \quad Z_1(\epsilon) \equiv \frac{\kappa_g^2}{2\epsilon l^2} (-2\epsilon \bar{\mu}(4\epsilon - 1))^{1/2} . \tag{7.7}
\]

Otherwise the brane universe has no cosmological singularity. The conditions for \( Z \) that the brane exists at \( a = 0 \) are

- For \( k = 2 \) (the spherical brane) it is \( Z_1(\epsilon) \geq 1 \).
- For \( k = 0 \) (the flat brane) it is \( Z_1(\epsilon) \geq 0 \).
- For \( k = -2 \) (the hyperbolic brane) it is \( Z_1(\epsilon) \geq -1 \).

For \( Z_1(\epsilon) \geq 1 \), all three kinds of brane reach the point \( a = 0 \).

3. If \(-2\epsilon \bar{\mu}(4\epsilon - 1) < 0\), \( X, H, G \) become imaginary when \( a = 0 \). Let us consider \( Y = 0 \) case instead of \( a \to 0 \) limit. When \( Y = 0 \) Eq.(6.46) leads to the following effective potential

\[
V(a) \overset{Y=0}{=} - \frac{2k}{81} + \frac{77}{81} \frac{1}{4\epsilon l^2} \left( \frac{8\epsilon \bar{\mu}}{4\epsilon - 1} \right)^{\frac{1}{2}} \kappa_g^2
\tag{7.8}
\]
Similarly to the case 2 the brane reaches the singularity at \( Y = 0 \), only if the condition \(-k/2 \geq V(a)|_{Y=0}\) is fulfilled, i.e.

\[
k \leq -Z_2(\epsilon), \quad Z_2(\epsilon) \equiv \frac{1}{2\epsilon l^2} \left( \frac{8\epsilon \tilde{\mu}}{4\epsilon - 1} \right)^{1/2} \kappa_g^2 = -Z_2(\epsilon). \tag{7.9}
\]

is satisfied. The conditions for \( Z_2(\epsilon) \) that the brane reaches the singularity at \( Y = 0 \) are

- For \( k = 2 \) (the spherical brane) it is \( Z_2(\epsilon) \leq -2 \).
- For \( k = 0 \) (the flat brane) it is \( Z_2(\epsilon) \leq 0 \).
- For \( k = -2 \) (the hyperbolic brane) it is \( Z_2(\epsilon) \leq 2 \).

Then, if \( Z_2(\epsilon) \leq -2 \), all three kinds of brane reach the singularity at \( Y = 0 \).

4. The most special case is \( \epsilon = 1/4 \) case, where the effective potential is zero. Then one can take \( k = -2 \), \( k = 0 \) only. The hyperbolic brane starts at \( a = 0 \) and expands to infinity and the flat brane cannot move.

5. Next, we consider \( \epsilon = 1/4 - \delta^2 \) case where \( \delta^2 > 0 \) and \(|\delta| \ll 1\). From Eq.(6.53), the effective potential looks as

\[
V(a) = \mp \kappa_g^2 \left\{ 2l \left( \frac{k}{2a^2} + \frac{1}{l^2} \right)^{1/2} - 7 \right\} (2\tilde{\mu})^{1/2} \delta. \tag{7.10}
\]

Under the condition \( \frac{k}{2a^2} + \frac{1}{l^2} \geq 0 \), if we take the upper sign of \( \mp \), the situation is same as in Fig.2.

- \( k = 2 \), the spherical brane starts at \( a = 0 \) and reaches its maximum size at \( a_{\text{max}} \) and then it re-collapses. That is the reverse version of “bounce” universe.
- \( k = 0 \), the flat brane starts at \( a = 0 \) and expands to infinity.
- \( k = -2 \), the hyperbolic brane might reach the singularity at \( a = l \).

If the brane starts at \( a = l + 0 \), the brane expands to infinity.

Taking the lower sign of \( \mp \), the situation is shown in Fig.4. In this case only hyperbolic brane exists. If the hyperbolic brane starts at \( a = \infty \), the brane reaches its minimum size at \( a = l \).
6. Finally, we consider the case of $\epsilon = 1/12$. One can get the effective potential from Eq.(6.59) but it has very complicated form. It is easier to consider only the limits of $a \to \infty$ or $a \to 0$. In the limit of $a \to \infty$ the effective potential follows from Eq.(6.59) as

$$V(a) \xrightarrow{a \to \infty} \left( \frac{3}{l^2} \pm \frac{2}{l^2} \right) a^2.$$  

(7.11)

Since $\frac{3}{l^2} \pm \frac{2}{l^2}$ is always positive, one should take only $k = -2$ or $k = 0$.

- $k = -2$, the hyperbolic brane reaches maximum at $a_{\text{max}}$ and then it re-collapses.
- $k = 0$, the flat brane cannot move.

In the limit of $a \to 0$, the effective potential can be derived from Eq.(6.62) as

$$V(a) \xrightarrow{a \to 0} \pm \frac{42}{25} \sqrt{\mu} \frac{l^2}{l^2} \kappa_g^2 .$$  

(7.12)

Under the condition of $-k/2 \geq V(a)$, the brane reaches the point $a = 0$, namely, the brane universe has the cosmological singularity. The conditions for $\epsilon$ that the brane reaches the point $a = 0$ are

- For $k = 2$ (the spherical brane) it is $\frac{25}{42} \leq \frac{\sqrt{\mu}}{l^2} \kappa_g^2$.
- For flat brane it is $0 \leq \frac{\sqrt{\mu}}{l^2} \kappa_g^2$.
- For $k = -2$ (the hyperbolic brane) it is $\frac{25}{42} \geq \frac{\sqrt{\mu}}{l^2} \kappa_g^2$.

Otherwise the brane universe has no cosmological singularity.

In the same way, one can study FRW brane cosmology from Einstein-GB gravity for other values of GB coupling constant.

There is an important lesson which follows from the results of this section. It is indicated by recent astrophysical data that currently our observable universe has small and positive cosmological constant. In other words, the universe is in de Sitter phase. It would be nice to have some mechanism which would predicted the de Sitter universe as some preferrable state of FRW brane cosmology. Unfortunately, despite the number of attempts such mechanism was not found in brane-world approach to 5-dimensional Einstein gravity. As it follows from our analysis it is unlikely that such mechanism exists in brane-world approach to higher derivative gravity too.
8 Discussion

In summary, we discussed FRW brane equations in the situation when brane is embedded in the 5-dimensional (A)dS black hole. One of the main points of such discussion is the possibility in all cases under consideration (Einstein, Einstein-Maxwell, quantum-corrected or Einstein-GB gravity) to rewrite the FRW equations in the form similar to 2-dimensional CFT entropy. In the first part of the paper (sections 2,3,4,5) we review the analogy between FRW cosmological equations and 2-dimensional CFT entropy (so-called Cardy-Verlinde formula), two ways where Cardy-Verlinde formula appears in gravity theory, the presentation of brane equations of motion in 5-dimensional (A)dS BH in the FRW form and then in Cardy-Verlinde form. The modification of such FRW presentation in case when 5-dimensional Maxwell field or 4-dimensional (brane) quantum fields are present is also reviewed. Mainly, the first way (formal re-writing of FRW equation) in Cardy-Verlinde formula appearance is reviewed. However, using holographic duality between bulk BH and dual CFT entropies, the second way of Cardy-Verlinde formula appearance is given in brane-world too.

In the second part of this work (sections 6,7) we investigate the brane matter induced by 5-dimensional AdS BH in Einstein-GB gravity. The corresponding FRW brane equations are written. The novelty of this study consists in the fact that brane matter energy and pressure significally depend on the choice of GB coupling constant. In particular, in some cases dual brane matter is not CFT (the possibility of (A)dS/non-CFT correspondence?). Moreover, DEC and WEC are not always satisfied because for some values of GB coupling constant the brane matter energy and pressure are negative. The presentation of FRW brane equations in the form similar to 2-dimensional CFT entropy formula is again possible. Finally, FRW brane cosmology is studied using the effective potential technique. The number of brane universes: spherical, hyperbolic or flat with various dynamics (expanding, contracting, first expanding then contracting, etc) is explicitly constructed. The brief analysis of singularity (when it appears) is also presented.

It is clear that CV representation of FRW (brane) cosmology has some holographic origin. However, the details of such holography remain to be found. Moreover, as quite much is known about 2-dimensional CFT, it is possible that FRW equation representation as 2-dimensional CFT entropy
may have various applications in the modern cosmology. In particular, it may give an idea about why dS brane cosmology seems to be so fundamental in our universe? In this respect it is quite important to search for various modifications of CV formulation.

The questions discussed in this work may be also important for the establishment of bulk/non-CFT correspondence. For example, in section three it has been shown how thermodynamic entropy of bulk AdS BH may be used to get the holographic CV description of dual 4-dimensional CFT cosmology. It turns out that similar procedure does not work when it is applied to bulk (A)dS BH in higher derivative gravity (if Riemann tensor squared term presents) [19, 4]. Taking into account that still dual (non-CFT) description of induced brane matter is possible (see section six) this looks quite promising.

Acknowledgments

The research by S.N. is supported in part by the Ministry of Education, Science, Sports and Culture of Japan under the grant n. 13135208. S.O. thanks N. Sasakura, M. Fukuma, S. Matsuura and O. Seto for useful discussions. The research by S.O. is supported in part by the Japan Society for the Promotion of Science under the Postdoctoral Research Program and that by S.D.O. is supported in part by FC grant E00-3.3-461 and in part by INFN (Gruppo Collegato di Trento).
Appendix

A Brief review of AdS/CFT correspondence

AdS/CFT (or bulk/boundary) correspondence is one of bright examples of manifestation of holographic principle. It is conjectured in ref. \[31\] (see also \[32, 33\]). After that several thousand papers were devoted to the discussion of related questions. Various aspects of AdS/CFT correspondence are reviewed in \[34\]). Clearly, it is impossible to give its good review in this Appendix so we just briefly mention several main ideas (and problems) related with the directions discussed in this work.

When \( N \) p-branes in superstring theory (coming from so-called M-theory) coincide with each other and the coupling constant is small, the classical supergravity on \( \text{AdS}_{D+d+1=p+2} \), which is the low energy effective theory of superstring, is, in some sense, dual to large \( N \) conformal field theory on \( M^d \), which is the boundary of the AdS. For example, \( d = 2 \) case corresponds to \((4, 4)\) superconformal field theory, \( d = 4 \) case corresponds to \( U(N) \) or \( SU(N) \) \( \mathcal{N} = 4 \) super Yang Mills theory and \( d = 6 \) case to \((0, 2)\) superconformal field theory.

AdS/CFT correspondence may provide new insights to the understanding of non-perturbative (supersymmetric) QCD. For example, in frames of Type 0 String Theory the attempts have been done to reproduce such well-known QCD effects as running gauge coupling and possibly confinement. It is among the first problems to get the description of well-known QCD phenomena from bulk/boundary correspondence.

In another approach one can consider IIB supergravity (SG) vacuum which describes the strong coupling regime of a non-supersymmetric gauge theory. This can be achieved by the consideration of deformed IIB SG vacuum, for example, with non-constant dilaton which breaks conformal invariance and supersymmetry (SUSY) of boundary supersymmetric Yang-Mills (YM) theory. Such a background will be the perturbation of \( \text{AdS}_5 \times S_5 \) vacuum. The background of such a sort (with non-trivial dilaton) which interpolates between AdS (UV) and flat space with singular dilaton (IR) may be constructed.
Such solution of IIB SG is used with the interpretation of it as the one describing the running gauge coupling (via exponent of dilaton). It is shown that running gauge coupling has a power law behavior with ultraviolet (UV) stable fixed point and quark-antiquark potential can be calculated. Unfortunately, situation is very complicated here due to the double role of IIB SG background. From one side it may indeed correspond to IR gauge theory (deformation of initial SUSY YM theory). On the same time such a background may simply describe another vacuum of the same maximally supersymmetric YM theory with non-zero vacuum expectation value (VEV) of some operator. Due to the fact that operators corresponding to deformation to another gauge theory are not known, it is unclear what is the case under discussion (interpretation of SG background). Only some indirect arguments may be given. It seems that IIB SG background with running dilaton most probably correspond to another vacuum of super YM theory under consideration. Then renormalization group (RG) flow is induced in the theory via giving a non-zero VEV to some operator.

In this appendix, we briefly review the AdS/CFT correspondence. This appendix is mainly based on [34, 35].

Conformal field theories (CFTs) are field theories invariant under the conformal transformation, which is the coordinate transformation preserving the angle:

\[
x^\mu \rightarrow x'^\mu = x^\mu + f^\mu(x) \implies g_{\mu\nu} \rightarrow g'_{\mu\nu} \propto g_{\mu\nu}.
\]

Especially if \( g_{\mu\nu} \propto \delta_{\mu\nu} \) in \( d \)-dimensional (Euclidean) space \( g_{\mu\nu} \propto \delta_{\mu\nu} \), we obtain

\[
\partial_i f_j + \partial_j f_i = \frac{2}{d} \delta_{ij} \sum_k \partial_k f_k.
\]  (A.1)

For \( d = 2 \), the solutions of (A.1) are given by an analytic function \( f = f(z) \) if one defines \( f = f_1 + i f_2, \ z = x^1 + i x^2 \) (\( i^2 = -1 \)). When \( d \geq 3 \), the general solution is given by

\[
f_k = \sum_{i=1}^{d} \left\{ 2 \left( c_i x^i \right) x^k - c_i \left( x^k \right)^2 + \omega_{ki} x^i \right\} + \epsilon x^k + a_k.
\]  (A.2)

Here \( a_k \) corresponds to the translation, \( \omega_{ij} \) to rotation (including Lorentz transformation in the Minkowski signature), \( \epsilon \) to the scale transformation (dilatation) and \( c_k \) to conformal boost (special conformal). Under the conformal boost, we find \( \frac{x^k}{x^\nu} \rightarrow \frac{x^k}{x^\nu} + c^k \), \( (x \cdot x \equiv \sum_j (x^j)^2) \). Therefore the conformal boost is given by reflecting the coordinate with respect to the unit
sphere \((x^k \to \frac{x^k}{x^2})\) and after that, translating the reflected coordinate and finally reflecting the coordinate with respect to the sphere again.

Let the generators corresponding to the transformations be \(P_i, M_{ij}, D,\) and \(K_i\) (here we choose them to be hermitian operators), then one obtains the following commutation relations:

\[
[M_{ij}, P_k] = -i(\delta_{ik}P_j - \delta_{jk}P_i) \quad [M_{ij}, K_k] = -i(\delta_{ik}K_j - \delta_{jk}K_i)
\]

\[
[M_{ij}, M_{kl}] = -i\delta_{ik}M_{jl} + i\delta_{il}M_{jk} - i\delta_{jl}M_{ik} + i\delta_{jk}M_{il}
\]

\[
[D, K_i] = iK_i \quad [D, P_i] = -iP_i \quad [P_i, K_j] = 2iM_{ij} - 2i\delta_{ij}D .
\]

(A.3)

Other commutation relations vanish. If we redefine the generators by \(J_{ij} = M_{ij}, J_{i d+1} = \frac{1}{2}(K_i - P_i), J_{d+2} = \frac{1}{2}(K_i + P_i),\) and \(J_{d+2 d+1} = D,\) we find that the generators \(J_{ab}\) \((a, b = 1, 2, \cdots, d + 1, d + 2)\) satisfy the algebra of \(SO(d + 1, 1)\) (\(SO(d, 2)\) in the Lorentz signature).

The anti-de Sitter space (AdS) is vacuum (constant curvature) solution of the Einstein equation with negative cosmological constant. The \(D\)-dimensional AdS can be constructed by embedding it in the flat \(D + 1\)-dimensional space:

\[-L^2 = \sum_{\mu=1}^{D} (X^\mu)^2 - (X^{D+1})^2, \quad ds^2 = \sum_{\mu=1}^{D} (dX^\mu)^2 - (dX^{D+1})^2 .\]  

(A.4)

(For Minkowski signature, we use analytic continuation: \(x^d = -ix^{0}\)). From the construction in (A.4), it is manifest that the AdS has a symmetry of \(SO(D, 1)\), which is the algebra of the conformal transformation \((D = d + 1)\). The transformation of the \(SO(D, 1)\) is linear for the coordinates \(X^M\) \((M = 1, \cdots, D + 1)\). If one changes the coordinates by

\[
U = X^D + X^{D+1}, \quad V = X^D - X^{D+1}, \quad x^i = \frac{X^i L}{U} \quad (i = 1, \cdots, d = D - 1),
\]

(A.5)

and deletes \(V\) by using the constraint equation in (A.4), the metric follows

\[
ds^2 = \frac{L^2}{U^2}dU^2 + \frac{U^2}{L^2} \sum_{i=1}^{d} (dx^i)^2 .
\]

(A.6)

By the previous identification with the generators in the conformal transformation and those in \(SO(D, 1)\), the following transformation laws for the new
coordinates in (A.5) are obtained

\begin{align*}
\text{dilatation} & \quad \delta U = -\epsilon U, \quad \delta V = \epsilon V, \quad \delta x^i = \epsilon x^i \\
\text{translation} & \quad \delta U = 0, \quad \delta V = \frac{2a^i X_i}{L} = \frac{a^i x_i}{U}, \quad \delta x^i = a^i \\
\text{conformal boost} & \quad \delta U = -2c^i x_i U, \quad \delta V = 0, \\
& \quad \delta x^i = 2a^j x_j x^i - a^i x^i x_j - a^i L^4 U^2. \tag{A.7}
\end{align*}

(Rotation is the same as the usual one for $x^i$.) Then we find that $x^i$ transforms as if they are coordinates in the $d$-dimensional space (A.2) except for the conformal boost. Even for the conformal boost, the transformation law of $x^i$ coincides with (A.2) in the limit of $U \to \infty$, which tells that the conformal invariance is realized at the boundary ($U \to \infty$) of AdS.

The AdS/CFT correspondence can be found by identifying the extremal limit of the black $p$-brane, which appeared in the classical solution of the type IIB supergravity and D$p$-brane (see [36]).

Type IIB superstring theory contains $2n$-rank ($n = 0, \cdots, 5$) anti-symmetric tensor. One is interested in the classical solution of the type IIB supergravity, which is the low energy effective theory of type IIB superstring. We can assume the 5-dimensional part in 10-dimensional spacetime, where the superstring lives, to be $S_5$ (5-dimensional sphere) and rank 4 antisymmetric tensor $A_{\mu\nu\rho\sigma}$ is self-dual:

\[
F_{\mu\nu\rho\sigma} \equiv \partial_{[\mu} A_{\nu\rho\sigma]}, \quad F^*_{\mu\nu\rho\sigma} \equiv \frac{1}{5!} \epsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta\eta} F_{\alpha\beta\gamma\delta\eta}. \tag{A.8}
\]

Here $\epsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta\eta}$ is a (constant) rank 10 anti-symmetric tensor. One more assumption is

\[
\int_{S_5} F^* = N \quad (N \text{ is an integer}), \quad F^* \equiv F^*_{\mu\nu\rho\sigma} dx^\mu dx^\nu dx^\rho dx^\sigma dx^\tau. \tag{A.9}
\]

There exists black 3-brane solution, which is spatially 3-dimensional extended black hole like object. This solution has two horizons. We can consider an extremal limit, where two horizon coincides. The extremal solution has the following form:

\[
ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-\frac{1}{2}} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \left(1 + \frac{R^4}{r^4}\right)^{-\frac{1}{2}} (dr^2 + r^2 d\Omega_5^2). \tag{A.10}
\]
Here $d\Omega_5^2$ expresses the metric tensor of the 5-dimensional unit sphere and $R^4 \sim g_s N l_s^4$ ($g_s$ is string coupling and $l_s$ is string length). In the coordinate system in (A.10), the horizon lies at $r = 0$. Since the solution is extremal, the mass density $M$ is equal to the charge density $N$ of rank 4 antisymmetric field like charged black hole, which tells that the extremal black $p$-brane is BPS saturated state. Therefore the relation $M = N$ does not suffer the quantum correction. One can also show that $N$ is quantized to be an integer, therefore both of $N$ and $M$ do not suffer the quantum correction. We should note that there is no any object which has the charge of the rank 4 anti-symmetric tensor at the level of the perturbation of the string theory. The D-brane is believed to correspond to the extremal limit of the black $p$-brane.

D-brane first appeared through T-duality. The T-duality is the symmetry where left-mover and right-mover in the string are exchanged. By exchanging left-mover and right-mover, the Neumann boundary condition at the ends of the open string is changed to the Dirichlet one. This suggests that there is an object where the ends of open string are attached. This object is called D-brane. If the object is the (spatially) $p$-dimensional extended object, we call the object D$p$-brane.

A conjecture that D$p$-brane is nothing but the extremal limit of black $p$-brane can be supported by the following considerations:

1. D$p$-brane can naturally couple with rank $p + 1$ antisymmetric tensor field by considering the following interaction :

$$ S_{DA} = q \int_{Dp\text{-brane}} A. \quad \text{(A.11)} $$

2. Consider a situation that two D$p$-branes interact with each other by the closed string. This can be regarded as one loop vacuum amplitude of the open string (with Chan-Paton factor) by the T-duality. Due to the supersymmetry, the amplitude of the open string vanishes. This tells (especially when the distance between the 2 D$p$-brane is large) that the interaction due to the graviton exactly cancels with that due to rank $p + 1$ antisymmetric tensor. That is, the D$p$-brane should be BPS saturated state.
3. We can also show that the charge density of $p$-brane for rank $p + 1$ antisymmetric tensor is quantized by the argument similar to Dirac’s quantization condition for the electric and magnetic charges. Since D$p$-brane is BPS saturated state, this tells the mass density is also quantized.

We now consider the dynamics on D$p$-brane especially for $p = 3$. The D3-brane with charge and mass densities which are $N$ times compared to the minimal ones can be regarded as the object composed of $N$ D3-branes with minimal charge and mass densities. Then there exists an open string which connects $i$-th D3-brane and $j$-th D3-brane. By the T-duality, the open string can be regarded as an open string with Chan-Paton factor $i$ and $j$. If the distance between the branes is very small, the open string becomes massless and corresponds to the vector fields $A_{\mu}^{ij}$. If we consider the orientable string theory, we find $A_{\mu}^{ij} = (A_{\mu}^{ji})^*$, that is $A_{\mu}^{ij}$ is a hermitian matrix and can be regarded as the $U(N)$ or $SU(N)$ gauge fields. These gauge fields can be described by a non-linear action called the Born-Infeld action, which contains $\alpha'$ corrections but the leading order term is nothing but the usual (super) Yang-Mills action.

We now assume that the string theory with black $p$-brane is equivalent to that with D$p$-brane and compare the low energy limits of the two theories.

In the low energy limit of the string theory with black $p$-brane, the theory is decoupled into two theories. The gravity theory becomes almost free in the region (called bulk) far from the brane. On the other hand, due to the red-shift, there can be any kind of configurations near the horizon of black $p$-brane ($r \sim 0$). The gravity theory in the bulk has long wave length and cannot observe the horizon. On the other hand, due to infinitely large red-shift, nothing can come from the horizon to the bulk. Therefore one has two decoupled theories, the free gravity theory in the bulk and the (full) gravity theory near the horizon.

On the other hand, in the string theory with D$p$-brane, the gravity becomes free in the bulk and the gravitational wave does not observe the brane, again. On the brane, the $\alpha'$ corrections in the Born-Infeld action vanish and there remains (super) Yang-Mills action. In the low energy limit, the long string which has large energy cannot appear and any interaction between the
bulk and the brane vanishes.

In the low energy limit of two string theories, which is believed to be equivalent with each other, we have common free gravity theories in the bulk. Therefore the remaining decoupled two theories, the gravity theory near the horizon and (super) Yang-Mills theory on the brane, should be equivalent with each other. The Yang-Mills theory on the D3-brane is the $\mathcal{N} = 4$ supersymmetric $U(N)$ or $SU(N)$ gauge theory, which is only one known interacting theory with conformal symmetry in 4-dimensions. On the other hand, near the horizon of the black $p$-brane, the metric has the following form:

$$ds^2 = \frac{r^2}{R^2} \left( -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \right) + \frac{R^2}{r^4} \left( dr^2 + r^2 d\Omega_5^2 \right). \quad (A.12)$$

Except the $S_5$ part, the metric is nothing but the one of AdS$_5$ ($r = U, R = L$). Therefore one finds AdS$_5$/CFT$_4$ correspondence; the (super)gravity theory in AdS$_5$ is equivalent to $\mathcal{N} = 4$ supersymmetric $U(N)$ or $SU(N)$ gauge theory.

In the further low energy limit, we can treat the AdS$_5$ (super)gravity theory classically. The low energy approximation would be valid when the (curvature) radius of the AdS$_5$ is large, which requires $R \gg \ell_s$. Since $R^4 \sim g_s N \ell_s^4$, the large $R$ corresponds to the strong coupling region in $\mathcal{N} = 4$ gauge theory.

Let the bulk space is not pure AdS space but the AdS black hole:

$$ds^2 = \hat{G}_{\mu\nu} dx^\mu dx^\nu$$

$$= -e^{2\rho_0} dt^2 + e^{-2\rho_0} dr^2 + r^2 \sum_{i,j} g_{ij} dx^i dx^j,$$

$$e^{2\rho_0} = \frac{1}{r^{d-2}} \left( -\mu + \frac{k_{d-2}}{d-2} + \frac{r^d}{\ell^2} \right). \quad (A.13)$$

Here $g_{ij}$ expresses the Einstein manifold, defined by $r_{ij} = kg_{ij}$, where $r_{ij}$ is the Ricci tensor defined by $g_{ij}$ and $k$ is the constant. For example, if $k > 0$ the boundary can be 4-dimesional de Sitter space (sphere when Wick-rotated), if $k < 0$, anti-de Sitter space or hyperboloid, or if $k = 0$, flat space. In the following, $k = 0$ is considered for simplicity. Then the radius $r_h$ of the horizon and the temperature $T$ are given by

$$r_h \equiv \mu^{\frac{1}{d-2}}, \quad T = \frac{\mu^{\frac{1}{d-2}}}{\pi}. \quad (A.14)$$
Since the black hole has the Hawking temperature, it is natural if we expect that the corresponding field theory should be CFT at finite temperature \[\text{(17)}\]. As an explicit check, we compare the entropy from the 5-dimensional gravity side and the 4-dimensional CFT side. The black hole spacetime in \[\text{(A.13)}\] is a solution of the Einstein equation which can be derived from the following Einstein-Hilbert action with negative cosmological constant \(-\frac{12}{l^2}\):

\[
S = \frac{1}{\kappa^2} \int d^5x \sqrt{-g} \left( R + \frac{12}{l^2} \right).
\]

(A.15)

In the previous AdS$_5 \times$S$_5$ case \[\text{(A.12)}\], which is equivalent to $\mathcal{N} = 4$ supersymmetric $U(N)$ or $SU(N)$ gauge theory

\[
\frac{1}{\kappa^2} = \frac{N^2}{8\pi}.
\]

(A.16)

Instead of the type IIB string theory on AdS$_5 \times$S$_5$, one may consider the string theory on AdS$_5 \times X_5$, where $X_5 = S_5/Z_2$, which corresponds to $\mathcal{N} = 2$ supersymmetric $Sp(N)$ gauge theory. In this case, we have

\[
\frac{1}{\kappa^2} = \frac{N^2}{4\pi}.
\]

(A.17)

We now consider the thermodynamical quantities like free energy. After Wick-rotating the time variables by $t \rightarrow i\tau$, the free energy $F$ can be obtained from the action $S$ where the classical solution is substituted: $F = \frac{1}{T} S$. Then

\[
F = -\frac{V_3}{\kappa^2 T} \frac{8}{\pi T} \left( \int_{r_h}^{\infty} dr r^3 \right).
\]

(A.20)

Here $V_3$ is the volume of 3d flat space and we assume $\tau$ has a period of $\frac{1}{T}$. The expression of $S$ contains the divergence coming from large $r$. In order to subtract the divergence, one regularizes $S$ in \[\text{(A.18)}\] by cutting off the integral at a large radius $r_{\text{max}}$ and subtracting the solution with $\mu = 0$:

\[
S_{\text{reg}} = \frac{8V_3}{\kappa^2 T} \left( \int_{r_h}^{\infty} dr r^3 - e^{\rho(r=r_{\text{max}})-\rho(r=r_{\text{max}};\mu=0)} \int_0^{r_{\text{max}}} dr r^3 \right).
\]

(A.19)

The factor $e^{\rho(r=r_{\text{max}})-\rho(r=r_{\text{max}};\mu=0)}$ is chosen so that the proper length of the circle which corresponds to the period $\frac{1}{T}$ in the Euclidean time at $r = r_{\text{max}}$ coincides with each other in the two solutions. Then one gets

\[
F = -\frac{V_3}{\kappa^2 T} \left( \pi T \right)^4.
\]

(A.20)
The entropy $S$ and the mass (energy) $E$ are given by

$$S = -\frac{dF}{dT} = \frac{4V_3(\pi T)^4}{\kappa^2 T}, \quad E = F + TS = \frac{3V_3(\pi T)^4}{\kappa^2 T}. \quad (A.21)$$

Then in case of the string theory on $\text{AdS}_5 \times S_5$ (A.12), we find

$$S_{\text{AdS}_5 \times S_5} = \frac{N^2 V_3(\pi T)^4}{2\pi^2 T} \quad (A.22)$$

and in case of the string theory on $\text{AdS}_5 \times X_5$,

$$S_{\text{AdS}_5 \times X_5} = \frac{N^2 V_3(\pi T)^4}{\pi^2 T}. \quad (A.23)$$

It is useful to compare the above results with those of field theories. In case of $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $U(N)$, there are $8N^2$ set of the bosonic and fermionic degrees of freedom on-shell. With $SU(N)$, $8(N^2 - 1)$ is corresponding number. Then from the perturbative QFT its free energy is given by,

$$F = \begin{cases} 
-\frac{\pi^2 V_3 N^2 T^4}{6} & \text{U(N) case} \\
-\frac{\pi^2 V_3 N^2 T^4}{6} \left(1 - \frac{1}{N^2}\right) & \text{SU(N) case}
\end{cases} \quad (A.24)$$

Then the entropy is given by (in the leading order of the $1/N$ expansion)

$$S_{\mathcal{N}=4} = \frac{2\pi^2 V_3 N^2 T^3}{3} \quad (A.25)$$

On the other hand, $\mathcal{N} = 2$ $Sp(N)$ gauge theory contains $n_V = 2N^2 + N$ vector multiplet and $n_H = 2N^2 + 7N - 1$ hypermultiplet. Vector multiplet consists of two Weyl fermions, one complex scalar and one real vector which gives 4 bosonic (fermionic) degrees of freedom on shell and hypermultiplet contains two complex scalars and two Weyl fermions, which also gives 4 bosonic (fermionic) degrees of freedom on shell. Therefore there appear $4 \times (n_V + n_H) = 16 \left(N^2 + 2N - \frac{1}{4}\right)$ boson-fermion pairs. In the limit which we consider, the interaction between the particles can be neglected. The contribution to the free energy from one boson-fermion pair in the space with
the volume $V_3$ can be easily estimated [37, 38]. Each pair gives a contribution to the free energy of $\frac{\pi^2 V_3 T^4}{48}$. Therefore the total free energy $F$ should be

$$ F = -\frac{\pi^2 V_3 N^2 T^4}{3} \left( 1 + \frac{2}{N} - \frac{1}{4N^2} \right). \quad (A.26) $$

Then the entropy is given by, in the leading order of the $1/N$ expansion,

$$ S_{N=2} = \frac{4\pi^2 V_3 N^2 T^3}{3} \quad (A.27) $$

Comparing (A.22) with (A.25) or (A.23) with (A.27), there is the difference of factor $\frac{4}{3}$ in the leading order of $1/N$ as observed in [37, 38].

This difference of the factor $\frac{4}{3}$ is presumably disappear when all orders of perturbation theory are taken into account from dual QFT side. Similarly, one can consider other phenomena in AdS/CFT duality. In fact, as we saw in Section 2, the entropy of the black hole is nothing but the entropy of the matter on the brane universe. As explained in detail in Section 2, the first clue was the analogy between the FRW equation of the radiation dominant universe and the Cardy formula [4]. After that, the analogy was shown to be natural from the AdS/CFT viewpoint if the FRW equation of the 4-dimensional spacetime is the equation describing the motion of the 3-brane in the 5-dimensional AdS-Schwarzschild bulk spacetime.

The (first) FRW equation for the 4-dimensional universe, given by

$$ H^2 = \frac{8\pi G}{3} \rho - \frac{1}{a^2}, \quad (A.28) $$

can be rewritten in the form of the Cardy formula

$$ S_H = 2\pi \sqrt{\frac{c}{6} \left( L_0 - \frac{c}{24} \right)}. \quad (A.29) $$

by identifying

$$ -\frac{2\pi}{n} V \rho a \Rightarrow 2\pi L_0, $$

$$ \frac{(n-1)V}{8\pi Ga} \Rightarrow \frac{c}{12}, $$

$$ \frac{(n-1)HV}{4G} \Rightarrow S_H. \quad (A.30) $$
In fact, $S_H$ is called the Hubble entropy, which give the upper bound of the whole entropy of the universe when $Ha > 1$.

On the other hand, the motion of the 3-brane in the 5-dimensional AdS-Schwarzschild spacetime is given by

$$H^2 = -\frac{1}{a^2} + \frac{\mu}{a^4}, \quad (A.31)$$

which can be rewritten in the form of the standard FRW equation in (A.28), by defining the energy density on the brane as

$$\rho = \frac{3\mu}{8\pi G_4 a^4}, \quad G_4 = \frac{2}{l_{AdS}}. \quad (A.32)$$

When the brane cross the black hole, the Hubble entropy $S_H$ in (A.30) is given by

$$S_H = \frac{V}{2l_{AdS}G_4} = \frac{V}{4G_5}, \quad (A.33)$$

which is nothing but the Bekenstein-Hawking entropy of the 5-dimensional black hole. If the whole entropy of the brane universe is constant during the time development of the universe, the whole entropy of the universe is equal to that of the bulk black hole. Then the Cardy-Verlinde formula (A.29) expresses the duality between the entropies of the brane universe and the bulk black hole.

### B Logarithmic Corrections to Cardy-Verlinde formula

In this Appendix we take into account thermal fluctuations of 5-dimensional AdS BH. As a result, it is shown the logarithmic corrections to brane FRW equations and CV formula appear.

It has been noted sometime ago that thermal fluctuations produce the logarithmic corrections [39] to BH entropy. This also occurs for AdS BHs.

One can get CV formula starting from the thermodynamics of the bulk black hole. The horizon radius $a_H$ is deduced by solving the equation $e^{2\rho(a_H)} = 0$ in (3.3), i.e.,

$$a_H^2 = -\frac{l^2}{2} + \frac{1}{2}\sqrt{l^4 + 4\mu l^2}. \quad (B.1)$$

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The Hawking temperature, \(T_H\), is then given by
\[
T_H = \left. \left( \frac{e^{2\rho}}{4\pi} \right) \right|_{a = a_H} = \frac{1}{2\pi a_H} + \frac{a_H}{\pi l^2},
\]
where a prime denotes differentiation with respect to \(r\). One can also rewrite the mass parameter, \(\mu\), using \(a_H\) or \(T_H\) from Eq. (B.1) as follows:
\[
\mu = \frac{a_H^4}{l^2} + a_H^2 = a_H^2 \left( \frac{a_H^2}{l^2} + 1 \right).
\]

The free energy \(F\), the entropy \(S\) and the thermodynamical energy \(E\) of the black hole are given as
\[
F = -\frac{V_3}{16\pi G_5} a_H^2 \left( \frac{a_H^2}{l^2} - 1 \right), \quad S = \frac{V_3 a_H^3}{4G_5}, \quad E = F + T_H S = \frac{3V_3\mu}{16\pi G_5}.
\]

Now we consider the logarithmic corrections to the above entropy. The corrected entropy has the form:
\[
S \equiv S_0 + c \ln S_0
\]
where, \(S_0\) is identical with the entropy in Eq.(B.4), and \(c\) is the constant determined later. The mechanism by which the logarithmic corrections appear is the following.\(^\ddagger\) We first describe how to calculate entropy based on the grand canonical ensemble. The partition function in the grand canonical ensemble is given by
\[
Z(\alpha, \beta) = \int_0^\infty \int_0^\infty \rho(n, E)e^{\alpha n - \beta E} dndE,
\]
where \(\alpha = \beta \mu\), \(\mu\) is the chemical potential and \(\beta = \frac{1}{T}\), \(T\) is the temperature.

In order that the temperature has the dimension of energy, \(k_B = 1\). The density of state \(\rho(n, E)\) can be obtained from the above equation by inverse Laplace transformation
\[
\rho(n, E) = \left( \frac{1}{2\pi i} \right)^2 \int_{c-i\infty}^{c+i\infty} \int_{c-i\infty}^{c+i\infty} Z(\alpha, \beta) e^{-\alpha n + \beta E} d\alpha d\beta,
\]
\[
= \left( \frac{1}{2\pi i} \right)^2 \int_{c-i\infty}^{c+i\infty} \int_{c-i\infty}^{c+i\infty} e^{S(n, E, \alpha, \beta)} d\alpha d\beta.
\]
\(^\ddagger\)The discussion here is based on last work from Ref.[39].
Here the function $S(n, E, \alpha, \beta)$ is defined as

$$S(n, E, \alpha, \beta) \equiv \ln Z(\alpha, \beta) - \alpha n + \beta E.$$  

To calculate the integral in (B.8), we take the saddle-point approximation, namely the main contribution can be evaluated around the equilibrium point $(\alpha_0, \beta_0)$ where the integral is stationary. Evaluating the integral of (B.8) to second order around the point $(\alpha_0, \beta_0)$, one can obtain the form which needs the Gaussian integration. After the integration, $\rho(n, E)$ becomes

$$\rho(n, E) \simeq \frac{e^{S(\alpha_0, \beta_0)}}{2\pi \sqrt{\frac{\partial^2 \ln Z}{\partial \alpha^2} \bigg|_{(\alpha_0, \beta_0)} \times \frac{\partial^2 \ln Z}{\partial \beta^2} \bigg|_{(\alpha_0, \beta_0)}}}.$$  

Therefore the entropy is

$$S \equiv \ln(\rho) = S(\alpha_0, \beta_0) + \ln \frac{\epsilon}{\sqrt{\frac{\partial^2 \ln Z}{\partial \beta^2} \bigg|_{(\alpha_0, \beta_0)}}} + \text{higher order terms.}$$  

Here we choose the scale factor $\epsilon$ to have the dimension of energy. From the definition of the specific heat, $\left.\frac{\partial^2 \ln Z}{\partial \beta^2}\right|_{(\alpha_0, \beta_0)} = C_v T^2$. Furthermore, one can set the scale $\epsilon \propto T$ since the temperature is the only available scale in canonical ensemble. Hence, the entropy is

$$S = S(\alpha_0, \beta_0) - \frac{1}{2} \ln C_v A + \cdots.$$  

where the constant $A$ is determined later. From the metric Eq.(3.3), and the entropy Eq.(B.4) one can calculate the specific heat $C_v$ of the black hole:

$$C_v \equiv \left(\frac{3V_3}{16\pi G_5}\right) \frac{d\mu}{dT_H} = 3 \frac{2a_H^2 + l^2}{2a_H^2 - l^2} S_0$$  

In the limit $a_H > \frac{l^2}{2}$, which gives $C_v > 0$, the specific heat $C_v$ can be approximated as

$$C_v \sim 3S_0.$$  

\footnote{It is assumed that $\left.\frac{\partial^2 \ln Z}{\partial \alpha \partial \beta}\right|_{(\alpha_0, \beta_0)}$ is smaller than $\left.\frac{\partial^2 \ln Z}{\partial \alpha^2}\right|_{(\alpha_0, \beta_0)}$ or $\left.\frac{\partial^2 \ln Z}{\partial \beta^2}\right|_{(\alpha_0, \beta_0)}$.}
Hence, taking the constant $A$ as $\frac{1}{3}$, we obtain

$$S = S_0 - \frac{1}{2} \ln S_0 + \cdots.$$  \hfill (B.15)

With above set-up one can find the logarithmic corrections to Cardy-Verlinde formula. Let us recall the 4-dimensional energy which can be derived from the FRW equations (B.8) of the brane universe in the SAdS background

$$E_4 = \frac{3V_3 l_\mu}{16\pi G_5 a}.$$ \hfill (B.16)

Then the relation between 4-dimensional energy $E_4$ on the brane and 5-dimensional energy $E$ in Eq.(B.5) is

$$E_4 = \frac{l}{a} E.$$ \hfill (B.17)

Note that $E$ doesn’t have logarithmic corrections since it agrees with the energy given by Eq.(B.9). If one further assumes that the temperature $T$ on the brane differs from the Hawking temperature $T_H$ by the factor $l/a$ like energy relation, it follows that

$$T = \frac{l}{a} T_H = \frac{a_H}{\pi a l} + \frac{l}{2\pi a a_H}$$ \hfill (B.18)

and, when $a = a_H$, this implies that

$$T = \frac{1}{\pi l} + \frac{l}{2\pi a_H^2}.$$ \hfill (B.19)

If the energy and entropy are purely extensive, the quantity $E_4 + pV - TS$ vanishes. In general, this condition does not hold and one can define the Casimir energy $E_C$.

$$E_C = 3 (E_4 + pV - TS).$$ \hfill (B.20)

Then, by using Eqs. (B.4), (B.10), and (B.18), and the relation $3p = E_4/V$, we find that

$$E_C = \frac{3l a_H^2 V_3}{8\pi G_5 a} + \frac{3}{2} T \ln S_0.$$ \hfill (B.21)
By combining Eqs. (B.4), (B.16), and (B.21) one gets

\[ S_0 + \frac{\pi al}{2a_H^2} T \left( \frac{a_H^4}{l^2} - a_H^2 \right) \ln S_0 \sim \frac{4\pi a}{3\sqrt{2}} \left| \frac{E_C \left( E_4 - \frac{1}{2} E_C \right)}{E_C} \right| . \]  

(B.22)

Here we assume the ln-correction term is small. Note that the coefficient of ln-correction in the l.h.s. of Eq. (B.22) is a constant, i.e., this quantity does not depend on \( a \).

Assuming the ln-correction term is small, the following relation appears

\[ \frac{E_4 - \frac{1}{2} E_C}{E_C} = \frac{a_H^2}{2l^2} . \]  

(B.23)

By using (B.18) and (B.23), the coefficient of the second term in (B.22) can be rewritten as follows,

\[ -\frac{\pi al}{2a_H^2} T \left( \frac{a_H^4}{l^2} - a_H^2 \right) = -\frac{\pi al}{2a_H^2} \left( \frac{a_H}{\pi al} + \frac{l}{2\pi a a_H} \right) \left( \frac{a_H^4}{l^2} - a_H^2 \right) = -\frac{2E_4 \left( E_4 - E_C \right)}{(2E_4 - E_C) E_C} . \]  

(B.24)

Therefore when the ln-correction is small, Eq. (B.22) can be rewritten in the following form:

\[ S_0 = \frac{4\pi a}{3\sqrt{2}} \left| \frac{E_C \left( E_4 - \frac{1}{2} E_C \right)}{E_C} \right| - \frac{2E_4 \left( E_4 - E_C \right)}{(2E_4 - E_C) E_C} \ln \left( \frac{4\pi a}{3\sqrt{2}} \left| \frac{E_C \left( E_4 - \frac{1}{2} E_C \right)}{E_C} \right| \right) . \]  

(B.25)

Then the total entropy Eq. (B.15) can be written as

\[ S = S_0 - \frac{1}{2} \ln S_0, \]

\[ = \frac{4\pi a}{3\sqrt{2}} \left| \frac{E_C \left( E_4 - \frac{1}{2} E_C \right)}{E_C} \right| - \frac{2E_4 \left( E_4 - E_C \right)}{(2E_4 - E_C) E_C} \ln \left( \frac{4\pi a}{3\sqrt{2}} \left| \frac{E_C \left( E_4 - \frac{1}{2} E_C \right)}{E_C} \right| \right). \]
\[
-\frac{1}{2} \ln \left( \frac{4\pi a}{3\sqrt{2}} \sqrt{\frac{E_C (E_4 - \frac{1}{2} E_C)}{}} \right) + \cdots \\
\sim \frac{4\pi a}{3\sqrt{2}} \sqrt{\frac{E_C (E_4 - \frac{1}{2} E_C)}{}} + \frac{4E_4^2 - 2E_4 E_C - E_C^2}{2 (2E_4 - E_C) E_C} \ln \left( \frac{4\pi a}{3\sqrt{2}} \sqrt{\frac{E_C (E_4 - \frac{1}{2} E_C)}{}} \right) . \quad (B.26)
\]

up to the first order of ln term. Then the logarithmic corrections to Cardy-Verlinde formula, are given by the second term in right hand side of Eq. (B.26), which can be found by the ln of the original Cardy-Verlinde formula.

Moreover, the 4-dimensional FRW equations are also deformed by the logarithmic corrections. The Hubble parameter \( H \) for 4-dimensions is related to the 4-dimensional entropy (Hubble entropy), as \( S = \frac{H V}{2\pi a^4} \). Hence, FRW equation is calculated by

\[
H^2 = \left( \frac{2G_4}{V} \right)^2 S^2 . \quad (B.27)
\]

Here \( G_4 = \frac{2G_5}{l} \). Using Eqs. (B.3), (B.4), (B.18), (B.21), (B.25), (B.26), the 4-dimensional FRW equation with the logarithmic corrections, up to the first order of ln term, is obtained by

\[
H^2 = \left( \frac{2G_4}{V} \right)^2 \left[ \frac{4\pi a}{3\sqrt{2}} \sqrt{\frac{E_C (E_4 - \frac{1}{2} E_C)}{}} - \frac{4\pi a}{3\sqrt{2}} \frac{4E_4^2 - 2E_4 E_C - E_C^2}{2 (2E_4 - E_C) E_C} \ln \left( \frac{4\pi a}{3\sqrt{2}} \sqrt{\frac{E_C (E_4 - \frac{1}{2} E_C)}{}} \right) \right] ,
\]

\[
= -\frac{1}{a_H^2} + \frac{8\pi G_4}{3} \rho - \frac{2G_4}{V} \ln S_0 . \quad (B.28)
\]

Here \( \rho \) is the energy density defined by \( \rho = \frac{E_4}{V} \), and \( V \) is the volume given by \( V = a_H^3 V_3 \). Since the first term in Eq. (B.28) is identical to the standard FRW equation: \( H^2 = -\frac{1}{a^2} + \frac{8\pi G_4}{3} \rho \) at the horizon \( a = a_H \), the logarithmic corrections for FRW equation are given by ln \( S_0 \) terms in Eq. (B.28). It remains to study the role of logarithmic corrections to the explicit examples of brane cosmology.
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