The Circular, Elliptic Three-spin String from the $SU(3)$ Spin Chain

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Abstract

We complete the description of the circular, elliptic three spin string on $AdS_5 \times S^5$ having three large angular momenta $(J_1, J_2, J_3)$ on $S^5$ in the language of the integrable $SU(3)$ spin chain. First, we recover the string solution directly from the spin chain sigma model and secondly, we identify the appropriate Bethe root configuration in the so far unexplored region of parameter space.

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1 Introduction

Semi-classical analysis of strings propagating on $AdS_5 \times S^5$ has provided a novel approach to investigating the AdS/CFT correspondence, the prime example being the study of strings with several large angular momenta on $S^5$. For such strings the classical string energy has an analytical dependence on the parameter $\frac{\lambda}{L^2}$ where $\lambda$ is the squared string tension and $L$ the total angular momentum. In addition quantum corrections to the string energy are suppressed as $\frac{1}{L}$ when $L \to \infty$ \cite{1, 2}. The AdS/CFT correspondence \cite{3} relates the energy of a IIB string state with given quantum numbers to the conformal dimension of a single trace operator of planar $\mathcal{N} = 4$ super Yang-Mills theory with corresponding representation labels, mapping $\lambda$ to the 't Hooft coupling and $L$ to the number of constituent fields of the operator. This led to the suggestion that the result of the semi-classical string analysis should be reproduced on the gauge theory side by a perturbative calculation of the anomalous dimension followed by the limit $L \to \infty$, $\frac{\lambda}{L^2}$ fixed — a generalization of the BMN idea. The BMN idea \cite{4} had triggered the development of efficient techniques based on the use of effective vertices for the perturbative calculation of anomalous dimensions of operators of $\mathcal{N} = 4$ SYM \cite{5}. These techniques were later substantially improved by focusing on the dilatation generator of the gauge theory \cite{6, 7} but their applicability were in practice limited to short operators or operators carrying at most one large representation label such as BMN-like operators. This limitation was overcome with the discovery that the one loop dilatation generator of $\mathcal{N} = 4$ SYM could be identified as the Hamiltonian of an integrable spin chain \cite{8, 9, 10}. A connection between gauge theories and spin chains was observed earlier in the context of QCD \cite{11} and recently further integrable structures in QCD were revealed \cite{12}. In the spin chain formulation considering large representation labels translates into going to the thermodynamical limit. When the number of large representation labels exceeds one the spin chain Bethe equations \cite{13} turn into a set of integral equations involving a number of continuum Bethe root densities. In certain cases corresponding to certain sub-sectors of $\mathcal{N} = 4$ SYM it has been possible to solve these equations exactly. The simplest possible closed sub-sector of $\mathcal{N} = 4$ SYM is the $SU(2)$ sub-sector consisting of operators composed of two out of the three complex scalar fields. In the $SU(2)$ sub-sector at one loop level, assuming both of the possible representation labels to be large, two types of solutions of the Bethe equations were found and these were identified as the gauge theory duals of respectively a folded and a circular string in $AdS_5 \times S^5$ having two large angular momenta on $S^5$ \cite{14, 15}. The $SU(2)$ sector remains closed to all loop orders \cite{7} and an extension of the spin chain picture including an appropriate Bethe ansatz was proposed in \cite{16} to three loops, see also \cite{17}. Furthermore, at one and two-loop order there exists a general proof of the equivalence between solutions of the Bethe equations in the thermodynamical limit and solutions of the string sigma model for large conserved charges \cite{18}. Equivalence between semi-classical strings and long operators has also been proved at the level of actions at one as well as at two loop order by matching continuum sigma models derived from respectively the spin chain and the string theory \cite{19, 20}.
The study of the relation between gauge theory operators and semi-classical strings is less developed in other sub-sectors of $\mathcal{N} = 4$ SYM. The $SU(3)$ sub-sector, consisting of operators built from the three complex scalars of $\mathcal{N} = 4$ SYM is a natural place to start extending the analysis. At one-loop order the dilatation operator restricted to this sub-sector is identical to the Hamiltonian of an integrable $SU(3)$ spin chain, the length $L$ of the spin chain being given by the number of constituent fields of the operators considered. The $SU(3)$ sub-sector is, however, only a closed sub-sector at this order. Beyond one loop one has to consider the larger $SU(2|3)$ sub-sector in order to have a strictly closed set of operators \[9, 21\]. Recently, arguments were given, though, that the $SU(3)$ sector can be considered as closed in the thermodynamical limit \[22\]. Generic operators in the $SU(3)$ sub-sector are expected to be dual to strings carrying three non-vanishing angular momenta $(J_1, J_2, J_3)$ on $S^5$. The first classical solution of the string sigma model describing such a three-spin situation was provided by Frolov and Tseytlin and had two out of the three spins identical, i.e. $(J_1, J_2, J_3) = (J, J', J')$ \[1, 2\]. The corresponding Bethe root configuration of the $SU(3)$ spin chain was identified in \[23\]. Also fluctuations around the classical solution has been understood from the spin chain perspective \[24\]. Later numerous other three-spin string solutions were found and classified \[25, 26\]. Briefly stated, three spin string solutions can be classified as being either rational \[26\], elliptic or hyper-elliptic \[25\]. The case $(J_1, J_2, J_3) = (J, J', J')$ can be reached as a limiting case of the rational as well as of the elliptic situation. In reference \[27\] the Bethe root configuration corresponding to an elliptic three spin string of circular type was identified in the region of parameter space where $J_2 \approx J_3$, $J_1 > J_2, J_3$. In the present paper we identify the Bethe root configuration in the opposite limit, i.e. $J_1 \approx J_2$, $J_3 < J_1, J_2$. Furthermore, we show how to recover the circular, elliptic three spin string directly from the continuum $SU(3)$ spin chain sigma model, derived in \[28, 29\].

2 The continuum $SU(3)$ spin chain sigma model

Imposing the thermodynamical limit $L \to \infty$ and considering long wavelength excitations, the $SU(3)$ spin chain can be described in terms of the following continuum sigma model action \[28, 29\]

$$S = \frac{L}{2\pi} \int d\sigma dt \left( \dot{\alpha} + \sin^2 \theta \dot{\phi} + \cos^2 \theta \cos(2\psi) \dot{\varphi} \right) - \frac{\lambda}{4\pi L} \int d\sigma dt \left[ \theta'^2 + \cos^2 \theta (\psi'^2 + \sin^2(2\psi) \varphi'^2) + \frac{1}{4} \sin^2(2\theta)(\phi' - \cos(2\psi) \varphi')^2 \right],$$

with $\sigma \in [0, 2\pi]$. Here the four variables $\theta, \psi, \phi, \varphi$ are the four angles needed to specify a coherent $SU(3)$ spin state and $\alpha$ is an additional overall phase\(^1\). The variable $\alpha$ is redundant as regards the dynamics of the spin chain but is useful for establishing the connection to the string sigma model where in particular it may

\(^{1}\)By introducing the variable $\alpha$ we have effectively, in a trivial way, extended the symmetry of the spin chain to $U(3)$.  

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play a role when it comes to constraints. The model in eqn. (1) has the conserved angular momenta

\[ P_\phi = \frac{L}{2\pi} \int d\sigma \sin^2 \theta, \quad P_\varphi = \frac{L}{2\pi} \int d\sigma \cos^2 \theta \cos(2\psi), \quad P_\alpha = \frac{L}{2\pi} \int d\sigma = L, \] (2)

where we notice that \( P_\alpha \) is simply the length of the spin chain. The angular variables in eqn. (1) are conveniently chosen so that starting from the string metric involving \( S^5 \) and the decoupled time coordinate \( t \)

\[ ds^2 = -dt^2 + d\theta^2 + \sin^2 \theta d\phi_3^2 + \cos^2 \theta \left( d\psi^2 + \cos^2 \psi d\phi_1^2 + \sin^2 \psi d\phi_2^2 \right), \] (3)

with

\[ \phi_1 = \alpha + \varphi, \quad \phi_2 = \alpha - \varphi, \quad \phi_3 = \alpha + \phi, \] (4)

the same sigma model is obtained once the appropriate large angular momentum limit is taken. One can thus make the following identification \[28\]

\[ P_\phi = J_3, \quad P_\varphi = J_1 - J_2. \] (5)

The Hamiltonian of the model in eqn. (1) is \[28\]

\[ H = \frac{\lambda}{4\pi L} \int d\sigma \left[ \theta'^2 + \cos^2 \theta (\psi'^2 + \sin^2 (2\psi) \varphi'^2) + \frac{1}{4} \sin^2 (2\theta) (\varphi' - \cos (2\psi) \varphi')^2 \right]. \] (6)

In order that the solutions of the sigma model capture the cyclicity of the trace appearing in the gauge theory operators all variables must be periodic in \( \sigma \) with period \( 2\pi \) and the momentum along the \( \sigma \)-direction should vanish. This momentum is given by

\[ P_\sigma = -\frac{L}{2\pi} \int d\sigma \left( \sin^2 \theta \partial_\sigma \phi + \cos^2 \theta \cos(2\psi) \partial_\sigma \varphi + \partial_\sigma \alpha \right). \] (7)

For \( \theta = \phi = 0 \) we recover the sigma model describing the continuum limit of the integrable \( SU(2) \) spin chain. From this sigma model, one reproduces the two-spin folded and circular string solution when \( \psi = 0, \varphi = a \) and \( \varphi' = \alpha' = 0 \) where \( a \) is a constant \[19\]. In reference \[28\] it was shown how to recover the circular, rational three spin string solutions of \[26\] from the continuum \( SU(3) \) spin chain sigma model. These solutions follow from the ansatz \( \theta = \theta_0, \psi = \psi_0 \) with \( \theta_0 \) and \( \psi_0 \) constant and \( \varphi' = m, \varphi' = n \) and \( \alpha' = p \) with \( m, n \) and \( p \) integer.\(^2\) The energy as a function of the spins reads

\[ E = \frac{\lambda}{2L \Lambda^2} \left[ (2m)^2 J_1 J_2 + (n - m)^2 J_1 J_3 + (n + m)^2 J_2 J_3 \right], \] (8)

and the condition \( P_\sigma = 0 \) turns into

\[ (p + m)J_1 + (p - m)J_2 + (n + p)J_3 = 0. \] (9)

\(^2\)In \[28\] the variable \( \alpha \) was left out from the analysis.
In the present paper we are interested in elliptic three spin solutions. Such solutions follow from the ansatz \( \dot{\theta} = \dot{\psi} = 0, \varphi' = \dot{\phi} = \alpha' = 0 \) and \( \dot{\phi} = a, \dot{\varphi} = b \) where \( a \) and \( b \) are constants. With this ansatz the momentum along the \( \sigma \) direction vanishes (cf. eqn. (7)) and the equations of motion take the form

\[
 b \cos^2 \theta \sin(2\psi) - \frac{\lambda}{2L^2} (\cos^2 \theta \psi')' = 0, \tag{10}
\]

\[
 \frac{\lambda}{L^2} \theta'' + \sin(2\theta)(a - b + 2b \sin^2 \psi + \frac{\lambda}{2L^2} \psi'^2) = 0. \tag{11}
\]

One simple solution to the equations is to have \( \psi \) constant and \( b = 0 \). In this case \( \sin \theta = \text{dn} v \sigma \). The solution which has our interest can be obtained by replacing this relation by the more general ansatz

\[
 \theta = \arcsin \left( \frac{\gamma}{\text{dn}(v \sigma, k)} \right), \tag{12}
\]

\[
 \psi = \arcsin \frac{\beta \text{cn}(v \sigma, k)}{\sqrt{1 - \delta^2 \text{dn}^2(v \sigma, k)}}, \tag{13}
\]

where \( \gamma, \beta \) and \( \delta \) are constants. We then notice that the equations (10) and (11) simplify if \( \delta = \gamma \) and if \( \beta \) and \( \gamma \) are related to each other as \( \beta^2 = 1 - \gamma^2 \). In particular, the derivative of \( \psi \) takes a very simple form

\[
 \psi' = -v \sqrt{1 - \gamma^2} \sqrt{1 - \gamma^2(1 - k^2)} \frac{\text{dn} v \sigma}{1 - \gamma^2 \text{dn}^2 v \sigma}. \tag{14}
\]

The first equation (10) now relates \( b \) with \( k \) as

\[
 b = \frac{v^2 \lambda k^2}{4L^2}, \tag{15}
\]

and the second equation is fulfilled if

\[
 b - a = \frac{v^2 \lambda}{2L^2}. \tag{16}
\]

Furthermore, the requirement that the angles are invariant under a shift \( \sigma \) by \( 2\pi \) forces \( v = 2K/\pi \).

Making use of the relations (2) and (5) we can now determine the normalized spin \( j_3 = J_3/L \)

\[
 j_3 = \frac{1}{2 \pi} \int_0^{2\pi} \gamma^2 \text{dn}^2 v \sigma = \gamma^2 \frac{E(k)}{K(k)}, \tag{17}
\]

where it has been used that \( v = 2K/\pi \). Let us furthermore define \( 2\epsilon = (J_1 - J_2)/L \). Then according to eqns. (2) and (5) we have

\[
 2\epsilon = \frac{1}{2 \pi} \int_0^{2\pi} d\sigma \left( 1 - \gamma^2 \text{dn}^2 v \sigma - 2\beta^2 \text{cn}^2 v \sigma \right) \\
 = \frac{2 - 2\gamma^2 + k^2 \gamma^2}{k^2} \left[ 1 - \frac{E(k)}{K(k)} \right] - (1 - \gamma^2). \tag{18}
\]
Using that $\gamma^2 = j_3 K/E$ we get a relation between $\epsilon$ and $j_3$
\begin{equation}
\epsilon = \frac{1}{k^2} \left[ 1 - \frac{E(k)}{K(k)} \right] - \frac{1}{2} + j_3 \left[ \frac{1 - \frac{1}{k^2}}{E(k)} \right] - \frac{1}{2} + \frac{1}{k^2}.
\end{equation}

Finally, from eqn. (6) we obtain an expression for the energy as a function of the spins
\begin{equation}
H = \frac{\lambda}{4\pi L} \int_0^{2\pi} d\sigma \left( \frac{\gamma^2 \epsilon^2 k^4 \sin^2 \sigma \cos^2 \sigma + \epsilon^2 \beta^2 (1 - \gamma^2 + \gamma^2 k^2) \sin^2 \sigma}{1 - \gamma^2 \sin^2 \sigma} \right) \left( \frac{\sin \lambda}{\sin \lambda} \right)
\end{equation}

\begin{equation}
= \frac{\epsilon^2}{2L} K(k) - \gamma^2 (1 - k^2)
\end{equation}

\begin{equation}
= \frac{2\epsilon}{\pi^2 L} [K(k) + j_3 (k^2 - 1)] \frac{K^3(k)}{E(k)}.
\end{equation}

where $k$ is supposed to be expressed via $j_3$ and $\epsilon$ using eqn. (19). The relations (19) and (20) are exactly the relations defining the circular, elliptic three spin string [25, 30, 27]. For later convenience we note that solving eqn. (19) for $k$ in terms of $j_3$ to leading order in $\epsilon$ and inserting the solution in eqn. (20) we get
\begin{equation}
H = \frac{\lambda}{2L} \left( 1 - j_3 + 8\epsilon^2 \frac{1}{1 + 3j_3} + O(\epsilon^4) \right).
\end{equation}

3 The discrete $SU(3)$ spin chain.

At the discrete level, finding an eigenstate and an eigenvalue of the $SU(3)$ spin chain amounts to solving a set of algebraic equations for the Bethe roots. The Bethe roots come in two different types, reflecting the fact that the Lie algebra $SU(3)$ has two simple roots. Denoting the number of roots of the two types as $n_1$ and $n_2$ and the roots themselves as $\{u_{1,j}\}_{j=1}^{n_1}$ and $\{u_{2,j}\}_{j=1}^{n_2}$ the Bethe equations read
\begin{equation}
\left( \frac{u_{1,j} + i/2}{u_{1,j} - i/2} \right)^L = \prod_{k \neq j}^{n_1} \frac{u_{1,j} - u_{1,k} + i}{u_{1,j} - u_{1,k} - i} \prod_{k=1}^{n_2} \frac{u_{1,j} - u_{2,k} - i/2}{u_{1,j} - u_{2,k} + i/2}.
\end{equation}

\begin{equation}
1 = \prod_{k \neq j}^{n_1} \frac{u_{2,j} - u_{2,k} + i}{u_{2,j} - u_{2,k} - i} \prod_{k=1}^{n_2} \frac{u_{2,j} - u_{1,k} - i/2}{u_{2,j} - u_{1,k} + i/2}.
\end{equation}

We shall assume that $n_1 \leq \frac{L}{2}$, $n_2 \leq \frac{n_1}{2}$. The $SO(6)$ representation implied by this choice of Bethe roots is given by the Dynkin labels $[n_1 - 2n_2, L - 2n_1 + n_2, n_1]$. In terms of the spin quantum numbers, assuming $J_1 \geq J_2 \geq J_3$ this corresponds to $[J_2 - J_3, J_1 - J_2, J_2 + J_3]$ or $J_1 = L - n_1$, $J_2 = n_1 - n_2$, $J_3 = n_2$. A given solution of the Bethe equations gives rise to an eigenvalue of the spin chain Hamiltonian i.e. a one loop anomalous dimension which is
\begin{equation}
\gamma = \frac{\lambda}{8\pi^2} \sum_{j=1}^{n_1} \frac{1}{(u_{1,j})^2 + 1/4}.
\end{equation}
The cyclicity of the trace is ensured by imposing the following constraint

\[ 1 = \prod_{j=1}^{n_1} \left( \frac{u_{1,j} + i/2}{u_{1,j} - i/2} \right). \]

(25)

Let us define

\[ \alpha = \frac{n_1}{L}, \quad \beta = \frac{n_2}{L}. \]

(26)

Then the spin quantum numbers are given by \((J_1, J_2, J_3) = ((1 - \alpha)L, (\alpha - \beta)L, \beta L)\). In references [23, 27] the above Bethe equations were studied under the assumption that the roots \(\{u_{2,j}\}_{j=1}^{n_2}\) were confined to an interval \([-ic, ic]\) on the imaginary axis and the roots \(\{u_{1,j}\}_{j=1}^{n_1}\) were living on two arches \(C_+\) and \(C_-\), each others mirror images with respect to zero, each symmetric around the real axis and not intersecting the imaginary axis. For \(c = 0\) the corresponding gauge theory operator is the dual of the folded string with two large angular momenta on \(S^5\) [14] and for \(c \to \infty\) the operator could be identified as the dual of the circular string with three large angular momenta, \((J, J', J')\), \(J > J'\) on \(S^5\) [23]. At an intermediate value of \(c\) a critical line \(\beta = \beta_{\text{crit}}(\alpha)\) was located [27] and it was proposed that above the critical line the operator was the dual of the circular, elliptic three spin string of references [25, 30]. The proposal was supported by a perturbative calculation in the region \(\beta \approx \frac{\alpha}{2}\), i.e. \(J_2 \approx J_3, J_1 > J_2, J_3\). Now, it is known that the three spin string with angular momentum assignment \((J', J', J)\) where \(J < J'\) is characterized by the Bethe roots \(\{u_{1,j}\}_{j=1}^{n_1}\) and \(\{u_{2,j}\}_{j=1}^{n_2}\) being all imaginary [23]. It is therefore natural to expect that something similar should characterize the circular, elliptic three spin string with \(J_1 \approx J_2, J_3 < J_1, J_2\), i.e. \(1 - 2\alpha + \beta \approx 0\). Below, we shall show that this is indeed the case.

4 The imaginary root solution

We assume that the Bethe roots \(\{u_{1,j}\}_{j=1}^{n_1}\) are all imaginary and distributed symmetrically around zero. Furthermore, in an interval of length of \(\mathcal{O}(L)\) around zero the roots are equidistant, placed at the half-integer imaginary numbers. This sub-set of the root configuration is denoted as the condensate. Outside the condensate the roots are more distant. This distribution of the roots \(\{u_{1,j}\}_{j=1}^{n_1}\) is the one characteristic of the two spin circular string [14]. It ensures that the condition (25) is fulfilled (provided \(n_1\) is odd and \(L\) is even — a constraint which should not affect quantities extracted in the thermodynamical limit). The roots \(\{u_{2,j}\}_{j=1}^{n_2}\) are likewise assumed to be imaginary and symmetrically distributed around zero. They are furthermore assumed to be confined to the interval defined by the above mentioned condensate. The possibility of this configuration for the roots \(\{u_{2,j}\}_{j=1}^{n_2}\) was pointed out in [23]. Rewriting the roots as \(u_{1,k} = iq_{1,k}L\) and \(u_{2,k} = iq_{2,k}L\), taking the logarithm of the Bethe equations and imposing the limit \(L \to \infty\) one is left with the following set of integral equations [23]

\[ 2 \int_s^t dq' \frac{\sigma(q')}{q - q'} + 2 \int_s^t dq' \frac{\sigma(q')}{q + q'} = \frac{2}{q} - 8 \log \frac{q - s}{q + s} + \int_{-v}^v dq' \frac{\rho(q')}{q - q'}, \quad s < q < t, \]

(27)
\[
\int_{-v}^{v} dq' \frac{\rho(q')}{q - q'} = 2 \log \frac{s + q}{s - q} + q \int_{s}^{t} dq' \frac{\sigma(q')}{q^2 - q'^2}, \quad -v < q < v, \tag{28}
\]

where \(v < s\) and where \(\rho(q)\) and \(\sigma(q)\) are root densities describing respectively the continuum distribution of \(\{q_{2,k}\}_{k=1}^{n_2}\) and the subset of \(\{q_{1,k}\}_{k=1}^{n_1}\) which are positive and lie outside the condensate. The presence of the condensate, located at \([-s, s]\), is reflected by the appearance of the logarithmic terms in the two equations. The densities are normalized as

\[
\int_{-v}^{v} \rho(q) dq = 2 \beta, \tag{29}
\]
\[
\int_{s}^{t} \sigma(q) dq = \alpha - 4s. \tag{30}
\]

Furthermore, the anomalous dimension can be expressed as \[14\]
\[
\gamma = \frac{\lambda}{8 \pi^2 L} \left( \frac{4}{s} - \int_{s}^{t} dq \frac{\sigma(q)}{q^2} \right). \tag{31}
\]

In order to solve the coupled integral equations \[27\] and \[28\] we shall follow the strategy of \[23\], i.e. we express \(\sigma(q)\) in terms of \(\rho(q)\) by means of eqn. \[27\] and use the resulting expression to eliminate \(\sigma(q)\) from eqn. \[28\]. First of all, let us introduce the resolvent corresponding to the root density \(\sigma(q)\)
\[
W(q) = \int_{s}^{t} dq' \frac{\sigma(q')}{q - q'} \equiv W_{+}(q) + qW_{-}(q), \tag{32}
\]
with \(W_{\pm}(q) = W_{\pm}(-q)\). The resolvent is analytic in the complex plane except for a cut along the interval \([s, t]\). We notice that \(\sigma(q)\) only enters the equation \[28\] via the function \(qW_{-}(q)\) and the expression for \(\gamma\) via \(W_{-}(0)\). Thus, we do not need to determine neither \(\sigma(q)\) nor \(W(q)\). We recognize the integral equation \[27\] as the saddle point equation of the \(O(n)\) model on a random lattice \[31\] for \(n = -2\) with the terms on the right hand side playing the role of the derivative of the potential \(V(q)\), i.e.
\[
V'(q) = \frac{2}{q} + \int_{-v}^{v} dq' \rho(q') \frac{1}{q - q'} - 8 \log \frac{q - s}{q + s}. \tag{33}
\]

Therefore, we can immediately, following \[32\], write down a contour integral expression for \(W_{-}(q)\)
\[
W_{-}(q) = \frac{1}{2} \oint_{C} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{\omega^2 - \omega^2} \left\{ \frac{(q^2 - s^2)(q^2 - t^2)}{(\omega^2 - s^2)(\omega^2 - t^2)} \right\}^{1/2}, \tag{34}
\]
where \(C\) is a contour which encircles the cut \([s, t]\) but not the other singularities of the integrand and where the endpoints \(s\) and \(t\) are determined by the boundary conditions
\[
\oint_{C} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{(\omega^2 - s^2)^{1/2}(\omega^2 - t^2)^{1/2}} = 0, \tag{35}
\]
\[
\oint_{C} \frac{d\omega}{2\pi i} \frac{V'(\omega) \omega^2}{(\omega^2 - s^2)^{1/2}(\omega^2 - t^2)^{1/2}} = 2\alpha - 8s. \tag{36}
\]
Here, the latter relation is equivalent to the normalization condition (30). Inserting the expression (33) into (34), (35) and (36) we find

\[ qW_-(q) = -\frac{1}{2qst} \left( \sqrt{(q^2 - s^2)(q^2 - t^2)} - st \right) \]

\[ -q \int_{-v}^{v} d\omega \frac{\rho(\omega)}{q^2 - \omega^2} \left\{ \frac{(s^2 - q^2)(t^2 - q^2)}{(s^2 - \omega^2)(t^2 - \omega^2)} - 1 \right\} \]

\[ + 2q \int_{-s}^{s} d\omega \frac{1}{s^2 - \omega^2} \left\{ \frac{(s^2 - q^2)(t^2 - q^2)}{(s^2 - \omega^2)(t^2 - \omega^2)} - 1 \right\}, \quad (37) \]

with the boundary conditions

\[ \frac{2}{st} + \int_{-v}^{v} d\omega \frac{\rho(\omega)}{\sqrt{(s^2 - \omega^2)(t^2 - \omega^2)}} - 8 \int_{-s}^{s} d\omega \frac{1}{\sqrt{(s^2 - \omega^2)(t^2 - \omega^2)}} = 0, \quad (38) \]

\[ -\frac{1}{2} \int_{-v}^{v} d\omega \frac{\rho(\omega)\omega^2}{\sqrt{(s^2 - \omega^2)(t^2 - \omega^2)}} + 4 \int_{-s}^{s} d\omega \frac{\omega^2}{\sqrt{(s^2 - \omega^2)(t^2 - \omega^2)}} = 1 - 2\alpha - \beta. \quad (39) \]

Furthermore, we find for \( \gamma \)

\[ \gamma = \frac{\lambda}{8\pi^2 L} \left( \frac{4}{s} + W_-(0) \right) \]

\[ = \frac{\lambda}{32\pi^2 L} \left\{ \frac{16}{s} + \frac{1}{s^2} + \frac{1}{t^2} + \int_{-v}^{v} d\omega \frac{\rho(\omega)}{\omega^2} \left[ \frac{st}{\sqrt{(s^2 - \omega^2)(t^2 - \omega^2)}} - 1 \right] \right\} \]

\[ -8 \int_{-s}^{s} d\omega \left[ \frac{st}{\sqrt{(s^2 - \omega^2)(t^2 - \omega^2)}} - 1 \right] \right\} \quad (40) \]

Finally, the integral equation for \( \rho(q) \) takes the form

\[ \int_{-v}^{v} dx \frac{\rho(x)}{q - x} \left( 3 + \sqrt{\frac{(s^2 - q^2)(t^2 - q^2)}{(s^2 - x^2)(t^2 - x^2)}} \right) \]

\[ = \frac{2}{qst} \left( st - \sqrt{(s^2 - q^2)(t^2 - q^2)} \right) \]

\[ + 8 \int_{-s}^{s} dx \frac{1}{q^2 - x^2} \sqrt{\frac{s^2 - q^2}{s^2 - x^2}} \left( \sqrt{\frac{t^2 - q^2}{t^2 - x^2}} - 1 \right), \quad -v < q < v. \quad (41) \]

### 5 Perturbative solution for \( 1 - 2\alpha + \beta \approx 0 \)

As mentioned earlier for \( 1 - 2\alpha + \beta = 0 \) the gauge theory operator in question is known to be the dual of the circular three-spin string of [11] which has angular momenta \((J', J', J), J < J'\) [23]. In the following we shall show that as we perturb away from \( 1 - 2\alpha + \beta = 0 \), the operator becomes the gauge theory dual of the circular, elliptic three-spin string given by eqns. [19] and [20].
Let us define
\[ 2\epsilon = 1 - 2\alpha + \beta, \] (42)
and let us consider \( \epsilon \ll \alpha, \beta \). In terms of angular momenta we have \((J_1, J_2, J_3) = (\frac{1}{2}(1 - \beta + 2\epsilon)L, \frac{1}{2}(1 - \beta - 2\epsilon)L, \beta L)\) or
\[ \epsilon = \frac{1}{2L}(J_1 - J_2) \equiv \frac{1}{2}(j_1 - j_2), \quad \beta = \frac{J_3}{L} \equiv j_3, \quad j_3 < j_1, j_2. \] (43)

As pointed out in [23], for \( \epsilon = 0 \), the boundary equation (36) is solved by setting \( t = \infty \). For a small, non-zero value of \( \epsilon \) consistency of the boundary equations requires that \( t \sim \frac{1}{\epsilon} \). Expanding the two boundary conditions to leading order in \( \epsilon \) we get
\[ 2s^2\pi - \frac{1}{2} \int_{-v}^{v} dx \frac{\rho(x)x^2}{\sqrt{s^2 - x^2}} = 2\epsilon t, \] (44)
\[ 4\pi = \frac{1}{s} + \frac{1}{2} \int_{-v}^{v} dx \frac{\rho(x)}{\sqrt{s^2 - x^2}} - \frac{\epsilon}{t}. \] (45)

Working at leading order in \( \epsilon \), the first of these two equations gives us \( t \) as a function of \( \epsilon \) and the second tells us how \( s \) (and \( v \)) depend on \( \epsilon \). As we shall see we do not need to know the explicit form of these corrections. We furthermore notice that for symmetry reasons, corrections to the integral equation (41) and to the expression for \( \gamma \), i.e. eqn. (40) can involve only even powers of \( \epsilon \). Now, expanding (41) for large \( t \) we find that the corrections of order \( \epsilon^2 \) cancel out due to the boundary conditions and we are left with
\[ \int_{-v}^{v} dx \rho(x) q - x \left( 3 + \sqrt{\frac{s^2 - q^2}{s^2 - x^2}} \right) = \frac{2}{q} \left( 1 - \sqrt{1 - \frac{q^2}{s^2}} \right) + \mathcal{O}(\epsilon^4). \] (46)

A similar cancellation of order \( \epsilon^2 \) terms takes place in the expression for \( \gamma \) and we get\(^3\).
\[ \gamma = \frac{\lambda}{32\pi^2 L} \left[ \frac{1}{s^2} + \int_{-v}^{v} dq \rho(q) \frac{1}{q^2} \left( \frac{s}{\sqrt{s^2 - q^2}} - 1 \right) \right] + \mathcal{O}(\epsilon^4). \] (47)

The two equations (46) and (47) thus to the given order in \( \epsilon \) take the same form as for \( t = \infty \) and we can proceed using a solution strategy similar to the one employed in that case. The new element then consists in correctly taking into account the modified boundary conditions. Following [23] we introduce the new variables
\[ q = \frac{2s\eta}{1 + \eta^2}, \quad x = \frac{2s\xi}{1 + \xi^2}, \] (48)
with \( dx\rho(x) \equiv d\xi\rho(\xi) \). In these variables the integral equation (46) takes the form
\[ \int_{-v}^{v} d\xi\rho(\xi) \frac{1 + \xi^2}{1 - \xi^2} \left( \frac{2}{\eta - \xi} + \frac{\eta + \xi}{1 - \eta\xi} \right) = 2\eta, \] (49)
\(^3\)Notice that \( s \) and \( v \) might still get \( \epsilon \) corrections. However, as already mentioned it is not necessary to know the explicit form of these corrections.
where \( \nu \) is related to \( v \) by

\[
v = \frac{2s\nu}{1 + \nu^2}.
\] (50)

The integral equation (49) is of the type characteristic of the \( O(n) \) plaquette matrix model studied in [33] and an explicit expression for \( \rho(\xi) \) valid for any \( n \) can be written down by contour integral techniques. However, since we do not need all the information stored in \( \rho(\xi) \) and since the present case corresponds to \( n = 1 \) which is one of the so-called rational points of the \( O(n) \) model \([31, 34, 35]\) we shall proceed along the lines of [23], using a method developed in [35]. We introduce a resolvent \( F(z) \) by

\[
F(z) = \int_{-\nu}^{\nu} d\xi \rho(\xi) \frac{1 + \xi^2}{1 - \frac{z}{\xi} z - \xi}.
\] (51)

This object is analytic in the complex plane except for a cut along the interval \([-\nu, \nu]\) and it has the following asymptotic behaviour as \( z \to \infty \)

\[
F(z) \sim \frac{p}{z}, \quad \text{as} \quad z \to \infty,
\] (52)

with

\[
p = \int_{-\nu}^{\nu} d\xi \rho(\xi) \frac{(1 + \xi^2)^2}{1 - \xi^2}.
\] (53)

The constant \( p \) plays a very central role since \( \gamma \) can be expressed as

\[
\gamma = \frac{\lambda}{32\pi^2 s^2} \left( 1 + \frac{p}{2} \right).
\] (54)

Using the definition of \( F(z) \) one can now write the boundary conditions (29), (30) and (35) as

\[
F(i) = -8\pi is \left( 1 + \frac{\epsilon}{4\pi t} \right) + 2i,
\] (55)

\[
F''(i) = 2\beta,
\] (56)

\[
F'''(i) = -2i(1 - \beta) + 8i \frac{t}{s} \epsilon.
\] (57)

Furthermore, by using analyticity arguments as in [35, 23] one can show that the function \( \omega(z) \), defined by

\[
\omega(z) = F(z) - \frac{4z}{3} + \frac{2}{3z},
\] (58)

fulfills the following cubic equation

\[
\omega^3(z) - R(z)\omega(z) = S(z),
\] (59)

where

\[
R(z) = \frac{4}{3} \left( z + \frac{1}{z} \right)^2 - 64\pi^2 s^2 \left( 1 + \frac{\epsilon}{2\pi t} \right),
\] (60)

\[
S(z) = -\frac{16}{27} \left( z + \frac{1}{z} \right)^3 + \frac{4}{3} \left( 6 + 3p - 64\pi^2 s^2 \left( 1 + \frac{\epsilon}{2\pi t} \right) \right) \left( z + \frac{1}{z} \right).
\] (61)
Now, by considering the first derivative of eqn. (59) we get the following expression for \( p \) in terms of \( s, t, \epsilon \) and \( \beta \)

\[
p = 32\pi^2 s^2 (1 - \beta) \left( 1 + \frac{\epsilon}{2\pi t} \right) - 2.
\]  

(62)

Furthermore, from the second derivative of eqn. (59) we get an expression for \( t \) as a function of \( \epsilon \) and \( \beta \)

\[
t = \frac{1}{16\pi \epsilon} (1 - \beta) (1 + 3\beta).
\]  

(63)

Finally, inserting eqns. (62) and (63) in the expression (54) for \( \gamma \) we see that the \( s \)-dependence very neatly cancels out and we are left with

\[
\gamma = \frac{\lambda}{2L} \left( 1 - j_3 + 8\epsilon^2 \frac{1}{1 + 3j_3} + O(\epsilon^4) \right),
\]  

(64)

where we have replaced \( \beta \) by \( j_3 \), cf. eqn. (13). This is precisely the result expected for the circular, elliptic three spin string, cf. eqn. (21). It would of course be interesting to reproduce the equations (19) and (20) from an exact solution of eqn. (11).

6 Conclusion

The continuum \( SU(3) \) spin chain sigma model in principle contains all information about the \( \mathcal{O}(\lambda') \) classical energy of strings with three angular momenta \( (J_1, J_2, J_3) \) on \( S^5 \) in the limit \( L = J_1 + J_2 + J_3 \to \infty, \lambda' = \frac{1}{L} \) fixed. Its most general equations of motion are, however, rather involved, cf. [28, 29]. It is therefore of interest to put forward possible simplifying ansätze which lead to non trivial solutions. Previously, it was shown how to recover from the spin chain sigma model the simple rational three spin string of [26]. In the present paper we have presented an ansatz which leads to the circular, elliptic three spin string of [25, 30, 27]. The most generic three spin string solutions are parametrized in terms of hyper-elliptic integrals. It would be interesting to understand how these solutions are encoded in the spin chain sigma model. Furthermore, it might be that the continuum spin chain sigma model could reveal solutions overlooked in the string theory analysis so far.

In the language of the discrete \( SU(3) \) spin chain a given three spin string solution is characterized by a certain Bethe root configuration. For the circular, elliptic three spin string with angular momentum assignment \( (J_1, J_2, J_3) = ((1 - \alpha)L, (\alpha - \beta)L, \beta L) \) it follows from the analysis of [27] that the Bethe root configuration has to be of a different type for \( \beta < \beta_c(\alpha) \) and \( \beta > \beta_c(\alpha) \) where \( \beta = \beta_c(\alpha) \) denotes a line of critical points in parameter space. In [27] the appropriate Bethe root configuration for \( \beta > \beta_c(\alpha) \) was identified. We propose that the imaginary root configuration of section 4 constitutes the appropriate Bethe root configuration for \( \beta < \beta_c(\alpha) \). Clearly the expression (14) for the one loop anomalous dimension as a function of the spins supports this proposal. In particular, we thus expect that the imaginary root solution should cease to exist for \( \beta \to (\beta_c(\alpha))_- \). Certainly, it would be interesting to understand the mechanism behind this phenomenon in the spirit
of the understanding of the singular limit $\beta \to (\beta_c(\alpha))^\pm \ [27]$. Likewise it would be interesting to determine the exact location of the critical line. This would require an exact solution of the integral equation (41) or of the corresponding integral equation of [27]. We note in passing that neither for the rational three spin string, nor for the hyper-elliptic one the relevant Bethe root configuration is known.

A recently initiated line of investigation, relying on the observation that the $SU(3)$ sub-sector may be considered as closed in the thermodynamical limit, is the generalization of the $SU(3)$ spin chain picture to include higher gauge theory loop orders [22]. A spin chain description going beyond one loop order was proposed for the $SU(2)$ sub-sector in [16]. The corresponding Bethe ansatz implied that inclusion of higher loop orders required only a rather simple modification of the one loop integral equation. In [22] it was assumed that inclusion of higher loop corrections in the $SU(3)$ sub-sector lead to a similar modification of the one loop Bethe equations and the evaluation of higher loop corrections was carried out for the gauge theory dual of a circular three spin string with angular momentum assignment $(J, J', J')$, $J' < J$. An exact solution of either of the earlier mentioned integral equations would allow an extension of the analysis to the case of the more general circular, elliptic three spin string. The study of higher loop corrections has so far revealed a disagreement between semi-classical string analysis and perturbative gauge theory at three loop order for all cases treated, i.e. for folded and circular two spin strings [16], a certain class of so-called pulsating strings as well as for the above mentioned special three spin string [22]. A possible explanation for this discrepancy was proposed in [16] and elaborated in [17]. Whereas the analysis of the circular, elliptic three spin string is not expected to change the picture as regards the presence of the discrepancy it will provide additional data that might help in ultimately resolving it.

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