OCTAHEDRAL NORMS IN SPACES OF OPERATORS

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Abstract. We study octahedral norms in the space of bounded linear operators between Banach spaces. In fact, we prove that \( L(X, Y) \) has octahedral norm whenever \( X^* \) and \( Y \) have octahedral norm. As a consequence the space of operators from \( L(\ell_1, X) \) has octahedral norm if, and only if, \( X \) has octahedral norm. These results also allows us to get the stability of strong diameter 2 property for projective tensor products of Banach spaces, which is an improvement of the known results about the size of nonempty relatively weakly open subsets in the unit ball of the projective tensor product of Banach spaces.

1. Introduction.

Octahedral norms were introduced by G. Godefroy in \([11]\) and they have been used in order to characterize when a Banach space contains an isomorphic copy of \( \ell_1 \). In fact, G. Godefroy shows in \([11]\) that a Banach space \( X \) contains an isomorphic copy of \( \ell_1 \) if, and only if, \( X \) can be equivalently renormed so that the new norm in \( X \) is octahedral. Also, it is proved in \([5]\) that the norm of a Banach space \( X \) is octahedral if, and only if, every convex combination of \( w^* \)-slices in the unit ball of \( X^* \) has diameter 2. As a consequence, the norm of \( X^* \) is octahedral if, and only if, every convex combination of slices in the unit ball of \( X \) has diameter 2.

In the last years intensive efforts have been done in order to discover new families of Banach spaces satisfying that every slice or every nonempty relatively weakly open subset in its ball has diameter 2, (see \([1]\), \([4]\), \([14]\), \([15]\)). It is known that every nonempty relatively weakly open subset in the unit ball of a Banach space contains a convex combination of slices. Then having every convex combination of slices with diameter 2 implies having every slice or every nonempty relatively weakly open subsets with diameter two. We say that a Banach space has the strong diameter two property (SD2P) if every convex combination of slices in its unit ball has diameter 2.

Let us observe that having SD2P implies failing in an extreme way the well known Radon-Nikodym property, since this property is characterized

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in terms of the existence of slices with diameter arbitrarily small. Also it is
known that the projective tensor product of Banach spaces with the RNP
is not necessarily a RNP space [7]. However, the unit ball of a such product
has slices with diameter arbitrarily small. It is proved in [3] that every
nonempty relatively weakly open subset in the unit ball of the projective
tensor product of two Banach spaces with infinite-dimensional centralizer
has diameter 2, a property weaker that SD2P. Then it is a natural open
question if the SD2P is stable for projective tensor products [1].

The aim of this note is to study the existence of octahedral no-
mrs in
spaces of operators. In this setting we prove that the space of operators
$L(X,Y)$ has octahedral norm whenever $X$ and $Y^*$ have it. Also we get
some necessary conditions in order to $L(X,Y)$ has octahedral norm. Now,
taking into account the dual relation between having octahedral norm and
satisfying SD2P we show, as a consequence, that the SD2P is stable for
projective tensor products of Banach spaces, which improves the stability
known results in [3]. Also, we get necessary conditions on Banach spaces in
order that the projective tensor product of these spaces has the SD2P.

We pass now to introduce some notation. We consider real Banach spaces.
$B_X$ and $S_X$ stand for the closed unit ball and the unit sphere of the Banac
space $X$. $X^*$ is the topological dual space of $X$. A slice in the unit ball of
$X$ is the set

$$S(B_X, x^*, \alpha) := \{ x \in B_X : x^*(x) > 1 - \alpha \},$$

where $x^* \in S_{X^*}$ and $0 < \alpha < 1$.

The norm of a Banach space $X$ is said to be octahedral if, for every finite-
dimensional subspace $E$ of $X$ and for every $\varepsilon > 0$ there is $y \in S_X$ such
that

$$\|x + \lambda y\| \geq (1 - \varepsilon)(\|x\| + |\lambda|)$$

for every $x \in E$ and scalar $\lambda$.

Let $X, Y$ be Banach spaces. According to [8],we recall that the projective
tensor product of $X$ and $Y$, denoted by $X \hat{\otimes}_\pi Y$, is the completion of $X \otimes Y$
under the norm given by

$$\|u\| := \inf \left\{ \sum_{i=1}^n \|x_i\|\|y_i\| / n \in \mathbb{N}, x_i \in X, y_i \in Y \forall \{1, \ldots, n\}, u = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

We recall that the space $B(X \times Y)$ of bounded bilinear forms defined on
$X \times Y$ is linearly isometric to the topological dual of $X \hat{\otimes}_\pi Y$.

Let $X$ be a Banach space and $L(X)$ the space of all bounded and linear
operators on $X$. By a multiplier on $X$ we mean an element $T \in L(X)$ such
that every extreme point of $B_{X^*}$ becomes an eigenvector for $T^t$. Thus, given
a multiplier $T$ on $X$, and an extreme point $p$ of $B_{X^*}$, there exists a unique
scalar $a_T(p)$ satisfying $T^t(p) = a_T(p)p$. The centralizer of $X$ (denoted by
$Z(X)$) is defined as the set of those multipliers $T$ on $X$ such that there exists
a multiplier $S$ on $X$ satisfying $a_S(p) = a_T(p)$ for every extreme point $p$ of
Thus, if $X$ a real Banach space, then $Z(X)$ coincides with the set of all multipliers on $X$. In any case, $Z(X)$ is a closed subalgebra of $L(X)$ isometrically isomorphic to $C(K_X)$, for some compact Hausdorff topological space $K_X$ (see [6, Proposition 3.10]).

Given a Banach space $X$, we consider the increasing sequence of its even duals

$$X \subseteq X^{**} \subseteq X^{(4)} \subseteq \cdots \subseteq X^{(2n)} \subseteq \cdots.$$ 

Since every Banach space is isometrically embedded into its second dual, we can define $X^{(\infty)}$ as the completion of the normed space $\bigcup_{n=0}^{\infty} X^{(2n)}$.

For a Banach space $X$, an $L$-projection on $X$ is a (linear) projection $P : X \to X$ satisfying $\|x\| = \|P(x)\| + \|x - P(x)\|$ for every $x \in X$. In such a case, we will say that the subspace $P(X)$ is an $L$-summand of $X$. Let us notice that the composition of two $L$-projections on $X$ is an $L$-projection [6, Proposition 1.7], so the closed linear subspace of $L(X)$ generated by all $L$-projections on $X$ is a subalgebra of $L(X^*), the space of all bounded and linear operators on $X$. This algebra, denoted by $C(X)$, is called the Cunningham algebra of $X$. It is known that $C(X)$ is linearly isometric to $Z(X^*)$ ([6, Theorems 5.7 and 5.9]). For instance, any infinite-dimensional space $L_1(\mu)$ satisfies that its Cunningham algebra is infinite-dimensional.

By [13, Lemma VI.1.1], the centralizer of $L(X,Y)$, the space of all bounded and linear operators from $X$ to $Y$, is infinite dimensional whenever either $C(X)$ is infinite dimensional or $Z(Y)$ is infinite dimensional.

2. Main results.

Our start point is the next proposition, which will be used in order to deal with spaces satisfying the strong diameter two property.

**Proposition 2.1.** Let $X$ be a Banach space and $C := \sum_{i=1}^{n} \lambda_i S_i$ a convex combination of slices of $B_X$ such that

$$\text{diam}(C) = 2.$$ 

Then for every $\varepsilon > 0$ there exist $x_i, y_i \in S_i \forall i \in \{1, \ldots, n\}$ and $f \in S_{X^*}$ such that

$$f(x_i - y_i) > 2 - \varepsilon \ \forall i \in \{1, \ldots, n\}.$$ 

Thus

$$f(x_i), \ f(-y_i) > 1 - \varepsilon \ \forall i \in \{1, \ldots, n\}.$$ 

**Proof.** Fix an arbitrary $\varepsilon > 0$. Choose $\delta > 0$ such that $\delta < m\varepsilon$, for $m := \min_{1 \leq i \leq n} \lambda_i$ (notice that we can assume that $\lambda_i \neq 0 \forall i \in \{1, \ldots, n\}$).

Since $\text{diam}(C) = 2$ then for every $1 \leq i \leq n$ there exist $x_i, y_i \in S_i$ such that

$$\left\| \sum_{i=1}^{n} \lambda_i x_i - \sum_{i=1}^{n} \lambda_i y_i \right\| > 2 - \delta.$$
Hence there exists \( f \in S_{X^*} \) satisfying
\[
f \left( \sum_{i=1}^{n} \lambda_i x_i - \sum_{i=1}^{n} \lambda_i y_i \right) = \sum_{i=1}^{n} \lambda_i f(x_i - y_i) > 2 - \delta.\]

As a consequence we have
\[
(2.1) \quad f(x_i - y_i) > 2 - \varepsilon \quad \forall i \in \{1, \ldots, n\}.
\]
Indeed, assume that there exists \( i \in \{1, \ldots, n\} \) such that \( f(x_i - y_i) \leq 2 - \varepsilon \) then
\[
2 - \delta < \sum_{j=1}^{n} \lambda_j f(x_j - y_j) = \lambda_i f(x_i - y_i) + \sum_{j \neq i} \lambda_j f(x_j - y_j) \leq \lambda_i (2 - \varepsilon) + 2(1 - \lambda_i) = 2 - \lambda_i \varepsilon < 2 - \delta,
\]
a contradiction. So \((2.1)\) holds.

We omit the proof of next proposition which is the dual version of the above one.

**Proposition 2.2.** Let \( X \) be a Banach space and \( C := \sum_{i=1}^{n} \lambda_i S_i \) a convex combination of \( w^* \)-slices of \( B_{X^*} \) such that \( \text{diam}(C) = 2 \).

Then for every \( \varepsilon > 0 \) there exist \( f_i, g_i \in S_i \) \( \forall i \in \{1, \ldots, n\} \) and \( x \in S_X \) such that
\[
(f_i - g_i)(x) > 2 - \varepsilon \quad \forall i \in \{1, \ldots, n\}.
\]
Thus
\[
f_i(x), \ g_i(-x) > 1 - \varepsilon \quad \forall i \in \{1, \ldots, n\}.
\]

Our last preliminary lemma allows us writing the unit ball of the dual of a space of operators in terms of elements in a tensor product of spheres.

**Lemma 2.3.** Let \( X, Y \) be Banach spaces. We consider \( H \) a closed subspace of \( L(X, Y) \) such that \( X^* \otimes Y \subseteq H \). Then we have that \( B_{H^*} = \overline{\text{co}}^{w^*}(S_X \otimes S_{Y^*}) \).

**Proof.** Given \( T \in H \), it is clear that
\[
\|T\| = \sup\{y^*(T(x)) : x \in S_X, \ y^* \in S_{Y^*}\}.
\]
This implies that the set of continuous linear functional \( x \otimes y^* \in H^* \) given by \( (x \otimes y^*)(T) := y^*(T(x)), \) for \( x \in S_X \) and \( y^* \in S_{Y^*} \), is a norming subset of \( H^* \). Since \( X^* \otimes Y \subseteq H \), we have that \( \|x \otimes y^*\| = 1 \) for every \( x \in S_X \) and \( y^* \in S_{Y^*} \). By a separation argument, we get that \( B_{H^*} = \overline{\text{co}}^{w^*}(S_X \otimes S_{Y^*}) \).
Note that as an easy consequence of the above lemma we get that every $w^*$-slice of the unit ball in $L(X,Y)^*$ has diameter 2, whenever every slice of the unit ball in $X$ has diameter 2.

Let $X$ be a Banach space, and let $u$ be a norm-one element in $X$. We put

$$D(X, u) := \{ f \in B_{X^*} : f(u) = 1 \}.$$ 

Now, assume that $X$ has a (complete) predual $X_*$, and put

$$D^w(X, u) := D(X, u) \cap X_*.$$ 

If $D^w(X, u) = \emptyset$, then we define $n^w(X, u) := 0$. Otherwise, we define $n^w(X, u)$ as the largest non-negative real number $k$ satisfying

$$k\|x\| \leq v^w(x) := \sup\{ \|f(x)\| : f \in D^w(X, u) \}$$

for every $x \in X$. We say that $u$ is a $w^*$-unitary element of $X$ if the linear hull of $D^w(X, u)$ equals the whole space $X^*$.

Now, let $X$ be an arbitrary Banach space, and let $u$ be a norm-one element in $X$. We define $n(X, u)$ as the largest non-negative real number $k$ satisfying

$$k\|x\| \leq v(x) := \sup\{ \|f(x)\| : f \in D(X, u) \}$$

for every $x \in X$, and we say that $u$ is a unitary element of $X$ if the linear hull of $D(X, u)$ equals the whole space $X^*$. Noticing that $D(X, u) = D^w(X^{**}, u)$, it is clear that $u$ is unitary in $X$ if and only if it is $w^*$-unitary in $X^{**}$, and that $n(X, u) = n^w(X^{**}, u)$.

The next result is a sufficient condition in order to have octahedral norm in a space of operators.

**Theorem 2.4.** Let $X, Y$ be Banach spaces. Assume that $Y$ has octahedral norm and that there exists $f \in S_{X^*}$ such that $n(X^*, f) = 1$. Let be $H$ a closed subspace of $L(X, Y)$ such that $X^* \otimes Y \subseteq H$. Then $H$ has octahedral norm.

**Proof.** By [3, Theorem 2.1], $H$ has octahedral norm if and only if every convex combination of $w^*$-slices of $H^*$ has diameter 2. Let $C := \sum_{i=1}^{n} \lambda_i S_i$ be a convex combination of $w^*$-slices in $B_{H^*}$. We can assume that there exist $\varepsilon \in \mathbb{R}^+$ and $A_i \in S_H$ such that $S_i = S(B_{H^*}, A_i, \varepsilon)$ for every $i \in \{1, \ldots, n\}$.

Fix $\delta \in \mathbb{R}^+$. By lemma 2.3 we can assume that there exists $x_i \in S_X$, $y_i^* \in S_{Y^*}$ such that $x_i \otimes y_i^* \in S_i$ for all $i \in \{1, \ldots, n\}$. Thus $\sum_{i=1}^{n} \lambda_i x_i \otimes y_i^* \in C$.

As $B_X = [\text{co}(D^w(X^*, f))] \overline{3}$ Corollary 3.5 we can assume that $x_i \in D^w(X^*, f) \forall i \in \{1, \ldots, n\}$.

Fix $i \in \{1, \ldots, n\}$. For $y^* \in B_{Y^*}$ we have $x_i \otimes y^* \in S_{H^*}$. Hence

$$x_i \otimes y^* \in S_i \iff A_i(x_i \otimes y^*) > 1 - \varepsilon \iff A_i(x_i)(y^*) > 1 - \varepsilon.$$ 

Thus

$$x_i \otimes y^* \in S_i \iff y^* \in S(B_{Y^*}, A_i(x_i), \varepsilon).$$
Since $\sum_{i=1}^{n} \lambda_i S(B_{Y^*}, A_i(x_i), \varepsilon)$ is a convex combination of $u^*$-slices in $B_{Y^*}$ we deduce by corollary 2.2 and since $Y$ has octahedral norm the existence of $u_i^*, v_i^* \in B_{Y^*}$ such that $x_i \otimes u_i^*, x_i \otimes v_i^* \in S_i$ for $i \in \{1, \ldots, n\}$, and $y \in S_Y$ such that

$$y(u_i^* - v_i^*) > 2 - \delta \quad \forall i \in \{1, \ldots, n\}.$$ 

Then $\sum_{i=1}^{n} \lambda_i x_i \otimes u_i^*, \sum_{i=1}^{n} \lambda_i x_i \otimes v_i^* \in C$.

Define $T : X \rightarrow Y$ by $T(x) := f(x)y$ for $x \in X$. Clearly $\|T\| = 1$ and $T \in H$. Thus

$$diam(C) \geq \left\| \sum_{i=1}^{n} \lambda_i x_i \otimes u_i^* - \sum_{i=1}^{n} \lambda_i x_i \otimes v_i^* \right\| = \left\| \sum_{i=1}^{n} \lambda_i x_i \otimes (u_i^* - v_i^*) \right\| \geq \sum_{i=1}^{n} \lambda_i T(x_i)(u_i^* - v_i^*) = \sum_{i=1}^{n} \lambda_i f(x_i)y(u_i^* - v_i^*) > (2 - \delta) \sum_{i=1}^{n} \lambda_i = 2 - \delta.$$

From the arbitrariness of $\delta$ we deduce that $diam(C) = 2$ and the theorem is proved.

From the symmetry in the proof of the above theorem we get the following

**Theorem 2.5.** Let $X, Y$ be Banach spaces. Assume that $X^*$ has octahedral norm and that there exists $f \in S_Y$ such that $n(Y, f) = 1$. Let be $H$ a closed subspace of $L(X, Y)$ such that $X^* \otimes Y \subseteq H$. Then $H$ has octahedral norm.

The sufficient condition obtained in the above theorem is satisfied when one assume an infinite-dimensional condition on the Cunningham algebra.

**Corollary 2.6.** Let $X, Y$ be Banach spaces. Assume that $C(Y^{(\infty)})$ is infinite-dimensional and that there exists $f \in S_X$ such that $n(X, f) = 1$. Let be $H$ a closed subspace of $L(X, Y)$ such that $X^* \otimes Y \subseteq H$. Then $H$ has octahedral norm.

**Proof.** Since $C(Y^{(\infty)})$ is infinite-dimensional, we have that $Z((Y^{(\infty)})^*)$ is infinite-dimensional. By [2, Theorem 3.4], $Y^*$ has the strong diameter two property, so theorem 2.4 applies.

Keeping in mind the projective tensor product, we improve the thesis of [3, Theorem 3.6] invoking theorem 2.4

**Corollary 2.7.** Let $X, Y$ be Banach spaces. Assume that $Z(Y^{(\infty)})$ is infinite-dimensional and that there exists $f \in S_X$ such that $n(X, f) = 1$. Then $X \otimes \pi Y$ has the strong diameter two property.

The next result follows the lines of the theorem 2.4, but for another kind of spaces and we get, as a consequence, the complete stability of octahedral norms in spaces of operators.
Theorem 2.8. Let $X, Y$ be Banach spaces. Assume that $X^*$ and $Y$ have octahedral norm. Let be $H$ a closed subspace of $L(X,Y)$ such that $X^* \otimes Y \subseteq H$. Then $H$ has octahedral norm.

Proof. Again by [5, Theorem 2.1], $H$ has octahedral norm if and only if every convex combination of $w^*$-slices of $H^*$ has diameter 2. Let $C := \sum_{i=1}^{n} \lambda_i S_i$ a convex combination of $w^*$-slices in $B_{H^*}$. Hence there exist $\varepsilon \in \mathbb{R}^+$ and $A_i \in S_H$ such that $S_i = S(B_{H^*}, A_i, \varepsilon)$ for every $i \in \{1, \ldots, n\}$. Fix $\delta \in \mathbb{R}^+$.

By lemma 2.3 we can ensure the existence of $x_i \in S_X, y_i^* \in S_{Y^*}$ such that $S_i = S(B_X, A_i^*(y_i^*), \varepsilon)$ for every $i \in \{1, \ldots, n\}$. For every $i \in \{1, \ldots, n\}$ we consider $A_i^*(y_i^*): X \to \mathbb{R}$ defined by $A_i^*(y_i^*)(x) = y_i^*(A_i(x))$ for every $x \in X$. Then we have that

$$x \otimes y_i^* \in S_i \iff x \in S(B_X, A_i^*(y_i^*), \varepsilon).$$

Now, $X^*$ has octahedral norm and by [5, Theorem 2.1], we have that every convex combination of slices of $X$ has diameter 2.

Applying Proposition 2.1 there exists $w_i \in S(B_X, A_i^*(y_i^*), \varepsilon)$ and $f \in S_{X^*}$ such that for all $i \in \{1, \ldots, n\}$

$$f(w_i) > 1 - \delta,$$

and

$$\sum_{i=1}^{n} \lambda_i w_i \otimes y_i \in C.$$ 

Following as in proof of theorem 2.4, we deduce that $\text{diam}(C) > 2 - \delta$. Due to the arbitrariness of $\delta$ we deduce that $\text{diam}(C) = 2$ and the theorem is proved.

The first consequence of the above theorem is the stability of SD2P for projective tensor products of Banach spaces. Indeed, it is enough taking into account that the dual of the projective tensor product of Banach spaces $X$ and $Y$ is the space of operators $L(X,Y^*)$ joint to the relation between having octahedral norm and the SD2P, as explained in the introduction, and apply the above theorem to get the following

Corollary 2.9. The projective tensor product, $X \hat{\otimes}_\pi Y$, of two Banach spaces $X$ and $Y$, verifies the SD2P whenever $X$ and $Y$ have the SD2P.

This last corollary gives the stability of SD2P for projective tensor products. However, in order to complete the study of the behavior of SD2P for projective tensor products it would be interesting to know if the projective tensor product of two Banach spaces has SD2P when one assumes that only one of the spaces has the SD2P. We think that a possible candidate to answer by the negative this question could be $X \hat{\otimes}_\pi \ell^2_2$, for some Banach space $X$ with the SD2P, but we don’t know the answer.
We recall, from proof of corollary 2.6, that given Banach spaces $X$ and $Y$ such that $Z(X^{(\infty)})$ and $C(Y^{(\infty)})$ are infinite-dimensional, then both $X^*$ and $Y$ have octahedral norm.

**Corollary 2.10.** Let $X, Y$ be Banach spaces such that $Z(X^{(\infty)})$ and $C(Y^{(\infty)})$ are infinite-dimensional. Let be $H$ a closed subspace of $L(X, Y)$ such that $X^* \otimes Y \subseteq H$. Then $H$ has octahedral norm.

As a new consequence, we get a result improving [3, Theorem 2.6], where from the same hypotheses is obtained that every nonempty relatively weakly open subset of the unit ball in $X \hat{\otimes}_\pi Y$ has diameter 2.

**Corollary 2.11.** Let $X, Y$ be Banach spaces such that $Z(X^{(\infty)})$ and $Z(Y^{(\infty)})$ are infinite-dimensional. Then the space $X \hat{\otimes}_\pi Y$ has the strong diameter two property.

Let $X$ be a Banach space. For $u \in S_X$ we define the roughness of $X$ at $u$, denoted by $\eta(X, u)$, by the equality

$$\eta(X, u) = \limsup_{\|h\| \to 0} \frac{\|u + h\| + \|u - h\| - 2}{\|h\|}$$

Let us remark that, for $u \in S_X$, we have that $\eta(X, u) = 0$ if, and only if, the norm is Fréchet differentiable at $u$ [9, Lemma 1.3].

For $\varepsilon \in \mathbb{R}^+$, the Banach space is said to be $\varepsilon$-rough if, for every $u \in S_X$ we have $\eta(X, u) \geq \varepsilon$. We say that $X$ is rough whenever it is $\varepsilon$-rough for some $\varepsilon \in \mathbb{R}^+$, and we say that $X$ is non-rough otherwise.

Our next result is a necessary condition in order to a space of operators has an octahedral norm.

**Proposition 2.12.** Let $X, Y$ be Banach spaces and let be $H$ a closed subspace of $L(X, Y)$ with octahedral norm such that $X^* \otimes Y \subseteq H$. Assume that $X^*$ has non-rough norm. Then $Y$ has octahedral norm.

**Proof.** We prove that every convex combination of $w^*$-slices of $B_Y^*$ has diameter 2. We put $y_1, \ldots, y_n \in S_Y$, $\delta > 0$ and $\lambda_1, \ldots, \lambda_n \in (0, 1)$ with $\sum_{i=1}^n \lambda_i = 1$, and consider the convex combination of $w^*$-slices

$$\sum_{i=1}^n \lambda_i S(B_Y^*, y_i, \delta).$$

Let be $\varepsilon > 0$. Since the norm of $X^*$ is non-rough, from [3, Proposition 1.11], we have that there exist $x^* \in S_{X^*}$ and $\alpha > 0$ such that $diam(S(B_X, x^*, \alpha)) < \varepsilon$. Put $\rho := \min\{\delta, \alpha\}$ and $x_0 \in S_X \cap S(B_X, x^*, \alpha)$. Consider the convex combination of $w^*$-slices of $B_{H^*}$ given by

$$\sum_{i=1}^n \lambda_i S(B_{H^*}, x^* \otimes y_i, \rho^2).$$

Now, $H$ has octahedral norm, then for $i \in \{1, \ldots, n\}$ there exist $f_i, g_i \in S_{H^*} \cap S(B_{H^*}, x^* \otimes y_i, \rho^2)$ such that
\[ \left\| \sum_{i=1}^{n} \lambda_i f_i - \sum_{i=1}^{n} \lambda_i g_i \right\| > 2 - \varepsilon. \]

By lemma 2.3, we can assume that \( f_i = \sum_{k=1}^{m_i} \gamma(k,i)x(k,i) \otimes y^*_k \) and \( g_i = \sum_{k=1}^{m_i} \gamma'(k,i)u(k,i) \otimes v^*_k \), where \( x(k,i), u(k,i) \in S_X, \ y^*_k, v^*_k \in S_Y \), and \( \sum_{k=1}^{m_i} \gamma(k,i) = 1 = \sum_{k=1}^{m_i} \gamma'(k,i) \) which \( \gamma(k,i), \gamma'(k,i) \in [0,1] \) for all \((k,i), k \in \{1, \ldots, m_i\} \text{ and } i \in \{1, \ldots, n\} \). For \( i \in \{1, \ldots, n\} \), we consider the sets \( P_i := \{(k,i) \in \{1, \ldots, m_i\} \times \{i\} : (x^* \otimes y_i)(x(k,i) \otimes y^*_k) > 1 - \rho \} \) and \( Q_i := \{(k,i) \in \{1, \ldots, m_i\} \times \{i\} : (x^* \otimes y_i)(u(k,i) \otimes v^*_k) > 1 - \rho \} \). Then we have that

\[ 1 - \rho^2 < (x^* \otimes y_i) \left( \sum_{k=1}^{m_i} \gamma(k,i)x(k,i) \otimes y^*_k \right) = \]

\[ \sum_{i \in P_i} \gamma(k,i)(x^* \otimes y_i)(x(k,i) \otimes y^*_k) + \sum_{i \notin P_i} \gamma(k,i)(x^* \otimes y_i)(x(k,i) \otimes y^*_k) \leq \]

\[ \sum_{i \in P_i} \gamma(k,i) + (1 - \rho) \sum_{i \notin P_i} \gamma(k,i) = 1 - \sum_{i \notin P_i} \gamma(k,i) + (1 - \rho) \sum_{i \notin P_i} \gamma(k,i). \]

We conclude that \( \sum_{i \notin P_i} \gamma(k,i) < \rho \), and hence we have that

\[ (x^* \otimes y_i) \left( \sum_{i \in P_i} \gamma(k,i)x(k,i) \otimes y^*_k \right) > 1 - \rho. \]

It follows that

\[ y_i \left( \sum_{i \in P_i} \gamma(k,i) y^*_k \right) \geq (x^* \otimes y_i) \left( \sum_{i \in P_i} \gamma(k,i)x(k,i) \otimes y^*_k \right) > 1 - \rho, \]

and \( \sum_{i \in P_i} \gamma(k,i) y^*_k \in B_{Y^*} \), so \( \varphi_i := \sum_{i \in P_i} \gamma(k,i) y^*_k \in S(B_{Y^*}, y_i, \delta). \)

For \((k,i) \in P_i\), we have that \( (x^* \otimes y_i)(x(k,i) \otimes y^*_k) > 1 - \rho \). This implies that \( x^*(x(k,i)) > 1 - \rho \), and as a consequence \( \|x(k,i) - x_0\| < \varepsilon \). In a similar way, we have that

\[ y_i \left( \sum_{i \in Q_i} \gamma(k,i) y^*_k \right) \geq (u^* \otimes v_i) \left( \sum_{i \in Q_i} \gamma(k,i) u(k,i) \otimes v^*_k \right) > 1 - \rho, \]

and \( \sum_{i \in Q_i} \gamma'(k,i) v^*_k \in B_{Y^*} \), so \( \psi_i := \sum_{i \in Q_i} \gamma(k,i) v^*_k \in S(B_{Y^*}, y_i, \delta). \)

For \((k,i) \in Q_i\), we have that \( (x^* \otimes y_i)(u(k,i) \otimes v^*_k) > 1 - \rho \). This implies that \( x^*(u(k,i)) > 1 - \rho \), and as a consequence \( \|u(k,i) - x_0\| < \varepsilon \). It follows that
In a similar way, we have that

$$\|f_i - x_0 \otimes \varphi_i\| \leq \left\| f_i - \sum_{i \in P_i} \gamma(k,i)x(k,i) \otimes y^*_i(k,i) \right\| + \left\| \sum_{i \in P_i} \gamma(k,i)x(k,i) \otimes y^*_i(k,i) - x_0 \otimes \varphi_i \right\| =$$

$$\left\| \sum_{i \in P_i} \gamma(k,i)x(k,i) \otimes y^*_i(k,i) \right\| + \left\| \sum_{i \in P_i} \gamma(k,i)(x(k,i) - x_0) \otimes y^*_i(k,i) \right\| \leq$$

$$\sum_{i \notin P_i} \gamma(k,i) + \sum_{i \in P_i} \gamma(k,i)\|x(k,i) - x_0\|\|y^*_i(k,i)\| \leq \rho + \varepsilon.$$

In a similar way, we have that

$$\|g_i - x_0 \otimes \psi_i\| \leq \rho + \varepsilon.$$

As a consequence we have that

$$2 - \varepsilon < \left\| \sum_{i=1}^{n} \lambda_i f_i - \sum_{i=1}^{n} \lambda_i g_i \right\| \leq$$

$$\left\| \sum_{i=1}^{n} \lambda_i (f_i - x_0 \otimes \varphi_i) \right\| + \left\| \sum_{i=1}^{n} \lambda_i x_0 \otimes (\varphi_i - \psi_i) \right\| + \left\| \sum_{i=1}^{n} \lambda_i (g_i - x_0 \otimes \psi_i) \right\| \leq$$

$$2(\rho + \varepsilon) + \left\| \sum_{i=1}^{n} \lambda_i x_0 \otimes (\varphi_i - \psi_i) \right\| = 2(\rho + \varepsilon) + \left\| x_0 \otimes \sum_{i=1}^{n} \lambda_i (\varphi_i - \psi_i) \right\| \leq$$

$$2(\rho + \varepsilon) + \|x_0\| \left\| \sum_{i=1}^{n} \lambda_i (\varphi_i - \psi_i) \right\| \leq 2(\rho + \varepsilon) + \left\| \sum_{i=1}^{n} \lambda_i (\varphi_i - \psi_i) \right\|.$$

It follows that

$$\left\| \sum_{i=1}^{n} \lambda_i \varphi_i - \sum_{i=1}^{n} \lambda_i \psi_i \right\| > 2 - 2\rho - 3\varepsilon.$$

We recall that $\varphi_i, \psi_i \in S(B_{Y^*}, y_i, \delta)$, and hence

$$\text{diam}\left(\sum_{i=1}^{n} \lambda_i S(B_{Y^*}, y_i, \delta)\right) \geq 2 - 2\rho - 3\varepsilon \geq 2 - 2\delta - 3\varepsilon.$$

Since $\varepsilon$ is arbitrary, we conclude that

$$\text{diam}\left(\sum_{i=1}^{n} \lambda_i S(B_{Y^*}, y_i, \delta)\right) \geq 2 - 2\delta.$$

Hence, for $0 < \eta < \delta$ we have

$$\sum_{i=1}^{n} \lambda_i S(B_{Y^*}, y_i, \eta) \subseteq \sum_{i=1}^{n} \lambda_i S(B_{Y^*}, y_i, \delta),$$
and \( \text{diam} \left( \sum_{i=1}^{n} \lambda_i S(B_{Y^*}, y_i, \eta) \right) \geq 2 - 2\eta \) by using a similar argument. Hence
\[
\text{diam} \left( \sum_{i=1}^{n} \lambda_i S(B_{Y^*}, y_i, \delta) \right) \geq 2 - 2\eta.
\]

Since \( \eta \in (0, \delta) \) is arbitrary we deduce that
\[
\text{diam} \left( \sum_{i=1}^{n} \lambda_i S(B_{Y^*}, y_i, \delta) \right) = 2,
\]
and we are done. \( \square \)

From the symmetry of the spaces \( X \) and \( Y \) in the proof of the above result, this one can be written also in the following way.

**Corollary 2.13.** Let \( X, Y \) be Banach spaces and let be \( H \) a closed subspace of \( L(X,Y) \) with octahedral norm such that \( X^* \otimes Y \subseteq H \). Assume that \( Y \) has non-rough norm. Then \( X^* \) has octahedral norm.

As a consequence of the above proposition and the Theorem 2.4 one gets the following equivalence.

**Corollary 2.14.** Let \( X, Y \) be Banach spaces. Assume that the norm of \( X^* \) is non-rough and that there exists \( f \in S_{X^*} \) such that \( n(X^*, f) = 1 \). Then for every closed subspace \( H \) of \( L(X,Y) \) such that \( X^* \otimes Y \subseteq H \), the following assertion are equivalent:

i) \( H \) has octahedral norm.

ii) \( Y \) has octahedral norm.

Taking \( X = \ell_1 \) in the above corollary one has

**Corollary 2.15.** Let \( Y \) be Banach space. Then \( L(\ell_1,Y) \) has octahedral norm if and only if \( Y \) has octahedral norm.

Again, using the duality between having octahedral norm and the SD2P joint to the duality \( (X \widehat{\otimes}_\pi Y)^* = L(X,Y^*) \), we get from Proposition 2.12, a necessary condition in order to the projective tensor product of Banach spaces has the SD2P.

**Corollary 2.16.** Assume that \( X \) and \( Y \) are Banach spaces such that \( X \widehat{\otimes}_\pi Y \) has SD2P and \( X^* \) has non-rough norm. Then \( Y \) has SD2P.

Let \( X \) be a Banach space. According to [12] we say that (the norm on) \( X \) is **weakly octahedral** if, for every finite-dimensional subspace \( Y \) of \( X \), every \( x^* \in B_{X^*} \), and every \( \varepsilon \in \mathbb{R}^+ \) there exists \( y \in S_X \) satisfying
\[
\|x + y\| \geq (1 - \varepsilon)(|x^*(x)| + \|y\|) \quad \forall x \in Y.
\]

It is clear that octahedral norm implies weakly octahedral norm, but the converse is not true. In fact, by [5, Theorem 2.1] and [12, Theorem 3.4],
(\(c_0 \oplus p\ c_0\))^* has weakly octahedral norm but does not have octahedral norm \([2\, \text{Theorem 3.2}]\) for every \(1 < p < \infty\). From \([12]\), we know that a Banach space \(X\) has a weakly octahedral norm if, and only if, every nonempty relatively weak-star open subset of the unit ball in \(X^*\) has diameter 2.

Now our aim is to enlarge the family of examples of spaces enjoying weakly octahedral norm but failing to have octahedral norm.

**Proposition 2.17.**

Let \(p \geq 1\) and \(X\) a non-null Banach space. Then

\[Y := L(c_0 \oplus p\ c_0, X),\]

has weakly octahedral norm.

**Proof.** By \([12, \text{Theorem 3.4}]\) it is equivalent to show that \(Y^*\) has the weak\(^*\) diameter 2 property, that is, every nonempty relatively weak-star open subset of the unit ball in \(Y^*\) has diameter 2.

So let be \(\varepsilon \in \mathbb{R}^+\) and \(W\) be a non-empty relatively \(w^*\)-open subset of \(B_Y\). As \(X \neq \{0\}\) and \(c_0 \oplus p\ c_0\) is infinite-dimensional, \(Y\) is also infinite-dimensional. Then

\[W \cap S_Y \neq \emptyset.\]

By lemma 2.6 there are \(m \in \mathbb{N}\), \((f_1, g_1), \ldots, (f_m, g_m) \in S_{c_0 \oplus p\ c_0}, x_1^*, \ldots, x_n^* \in S_{X^*}, \lambda_1, \ldots, \lambda_n \in [0, 1], \sum_{i=1}^m \lambda_i = 1\) such that

\[
\sum_{i=1}^m \lambda_i (f_i, g_i) \otimes x_i^* \in W;
\]

and \(\|\sum_{i=1}^m \lambda_i (f_i, g_i) \otimes x_i^*\| > 1 - \varepsilon^2\).

As \(f_1, \ldots, f_m, g_1, \ldots, g_m \in c_0\) we can assume that \(f_i, g_i\) have finite support for every \(i \in \{1, \ldots, n\}\) because \(W\) is a norm open set. Hence there exists \(k \in \mathbb{N}\) such that for every \(n \geq k\) one has

\(f_i(n) = g_i(n) = 0 \ \forall i \in \{1, \ldots, m\}\).

In other words

\[
(f_i, g_i) = \sum_{j=1}^k f_i(j)(e_j, 0) + \sum_{j=1}^k g_i(j)(0, e_j) \ \forall i \in \{1, \ldots, m\}.
\]

As \(\|\sum_{i=1}^m \lambda_i (f_i, g_i) \otimes x_i^*\| > 1 - \varepsilon^2\) we can find \(T \in S_{L(c_0 \oplus p\ c_0, X)}\) such that

\[
\sum_{i=1}^m \lambda_i x_i^*(T(f_i, g_i)) > 1 - \varepsilon.
\]

Fix \(i \in \{1, \ldots, n\}\). Define
Octahedral norms in spaces of operators

\[ u_{i,n} := (f_i, g_i) + \sum_{j=1}^{k} f_i(j)(e_{nk+j}, 0) + \sum_{j=1}^{k} g_i(j)(0, e_{nk+j}), \]

\[ v_{i,n} := (f_i, g_i) - \sum_{j=1}^{k} f_i(j)(e_{nk+j}, 0) - \sum_{j=1}^{k} g_i(j)(0, e_{nk+j}). \]

We claim that \( \| (f_i, g_i) \| = \| u_{i,n} \| = \| v_{i,n} \| \geq 2 \) \( \forall i \in \{1, \ldots, n\} \). Let check only the first equality because the second one is similar. From (2.2) we deduce that

\[ \| u_{i,n} \| = \left( \left( \max_{1 \leq j \leq k} |f(j)| \right)^p + \left( \max_{1 \leq j \leq k} |g(j)| \right)^p \right)^{\frac{1}{p}} = \| (f_i, g_i) \|. \]

On the other hand, since \( \{ e_n \} \rightarrow 0 \) in the weak topology of \( c_0 \), it is clear that

\[ \{ u_{i,n} \}_{n \in \mathbb{N}} \rightarrow (f_i, g_i) \ \forall i \in \{1, \ldots, m\}, \]

where the last convergence is in the weak topology of \( c_0 \oplus_p c_0 \).

In a similar way \( \{ v_{i,n} \}_{n \in \mathbb{N}} \rightarrow (f_i, g_i) \ \forall i \in \{1, \ldots, m\} \) in the weak topology of \( c_0 \oplus_p c_0 \). Hence

\[ \{ u_{i,n} \otimes x_i^* \}_{n \in \mathbb{N}} \rightarrow (f_i, g_i) \otimes x_i^* \ \forall i \in \{1, \ldots, n\}, \]

in the weak* topology of \( Y^* \).

In order to check the last assertion pick \( G \in L(c_0 \oplus_p c_0, X) \). As \( G \) is norm-norm continuous then \( G \) is weak-weak continuous. Hence

\[ \{ G(u_{i,n}) \} \rightarrow^w G(f_i, g_i). \]

By definition of weak topology in \( X \) then

\[ \{ u_{i,n} \otimes x_i^*(T) \} = \{ x_i^*(T(u_{i,n})) \} \rightarrow x_i^*(T(f_i, g_i)) = (f_i, g_i) \otimes x_i^*(T). \]

Last equality proves that \( \{ u_{i,n} \otimes x_i^* \} \rightarrow^{w^*} (f_i, g_i) \otimes x_i^* \).

Similarly

\[ \{ v_{i,n} \otimes x_i \}_{n \in \mathbb{N}} \rightarrow (f_i, g_i) \otimes x_i \ \forall i \in \{1, \ldots, n\}, \]

in the weak* topology of \( Y \). Thus there exist \( n \in \mathbb{N} \) big enough such that

\[ u := \sum_{i=1}^{m} \lambda_i u_{i,n} \otimes x_i^*, \]

\[ v := \sum_{i=1}^{m} \lambda_i v_{i,n} \otimes x_i^*, \]
are elements of \( W \) (Notice that \( \| (f_i, g_i) \| = \| u_{i,n} \| = \| v_{i,n} \| n \geq 2 \ \forall i \in \{1, \ldots, n\} \) implies that \( u, v \) are elements in the unit ball of \( Y \)).

Now our aim is to estimate the norm of the element

\[
u - v = 2 \sum_{i=1}^{n} \lambda_i \left( \sum_{j=1}^{k} f_i(j)(e_{nk+j}, 0) + \sum_{j=1}^{k} g_i(j)(0, e_{nk+j}) \right) \otimes x_i^*.
\]

Define

\[
M := \text{span} \{(e_{nk+j}, 0), (0, e_{nk+j}) : j \in \{1, \ldots, k\}\},
\]

\[
Z := \text{span} \{(e_j, 0), (0, e_j) : j \in \{1, \ldots, k\}\}.
\]

Then \( \Phi : M \rightarrow Z \) given by

\[
\Phi(e_{nk+j}, 0) = (e_j, 0) \quad \Phi(0, e_{nk+j}) = (0, e_j),
\]

defines a linear isometry. Moreover it is clear that

\[
\Phi \left( \sum_{j=1}^{k} f_i(j)(e_{nk+j}, 0) + \sum_{j=1}^{k} g_i(j)(0, e_{nk+j}) \right) = (f_i, g_i) \ \forall i \in \{1, \ldots, m\}.
\]

Define \( P : c_0 \oplus_p c_0 \rightarrow c_0 \oplus_p c_0 \) by the equation

\[
P(f, g) = \sum_{s=nk+1}^{(n+1)k} f(s)(e_s, 0) + \sum_{s=nk+1}^{(n+1)k} g(s)(0, e_s).
\]

\( P \) is a linear projection and clearly \( \|P\| = 1 \). In addition, for every \( i \in \{1, \ldots, n\} \) we have

\[
\Phi \left( P \left( \sum_{j=1}^{k} f_i(j)(e_{nk+j}, 0) + \sum_{j=1}^{k} g_i(j)(0, e_{nk+j}) \right) \right) =
\]

\[
= \sum_{j=1}^{k} f_i(j)(e_j, 0) + \sum_{j=1}^{k} g_i(j)(0, e_j) = (f_i, g_i).
\]

Define

\[
S : c_0 \oplus_p c_0 \rightarrow X \quad S = T \circ \Phi \circ P.
\]

\( S \in L(c_0 \oplus_p c_0, X) \) and \( \|S\| \leq 1 \). If we compute \((u - v)(S)\) we have

\[
(u-v)(S) = 2 \sum_{i=1}^{n} \lambda_i x_i^* \left( T \left( \Phi \left( P \left( \sum_{j=1}^{k} f_i(j)(e_{nk+j}, 0) + \sum_{j=1}^{k} g_i(j)(0, e_{nk+j}) \right) \right) \right) \right).
\]
\[= 2 \sum_{i=1}^{n} \lambda_i x_i^*(T(f_i, g_i)).\]

Hence

\[\text{diam}(W) \geq \|u - v\| \geq (u - v)(S) = 2 \sum_{i=1}^{m} \lambda_i x_i^*(T(f_i, g_i)) \geq 2(1 - \varepsilon).\]

From the last estimation we deduce that \(\text{diam}(W) = 2\) from the arbitrariness of \(\varepsilon\).

By the arbitrariness of \(W\) we deduce that \(Y\) has weakly octahedral norm, so we are done.

\[\blacksquare\]

Applying the above result joint to Corollary 2.13 and [2, Theorem 3.2] we obtain the desired example.

**Corollary 2.18.**

Let \(X\) be a non-null Banach space such that \(X\) has non-rough norm and \(p > 1\). Then the norm of the space

\[Y = L(c_0 \oplus_p c_0, X),\]

is weakly octahedral but not octahedral.

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