Global and local existence for the dissipative critical SQG equation with small oscillations

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Abstract: This article is devoted to the study of the critical dissipative surface quasi-geostrophic (SQG) equation in \( \mathbb{R}^2 \). For any initial data \( \theta_0 \) belonging to the space \( \Lambda^s(H^s_{uloc}(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}^2) \), we show that the critical (SQG) equation has at least one global weak solution in time for all \( \frac{1}{4} \leq s \leq \frac{1}{2} \) and at least one local weak solution in time for all \( 0 < s < \frac{1}{4} \). The proof for the global existence is based on a new energy inequality which improves the one obtain in [21] whereas the local existence uses more refined energy estimates based on Besov space techniques.

Keywords: Quasi-geostrophic equation, fluid mechanics, Riesz transforms, Morrey-Campanato spaces, Besov spaces.

1 Introduction

In this article, we study the following two dimensional dissipative surface quasi-geostrophic equation:

\[
\begin{align*}
\partial_t \theta(x,t) + u \cdot \nabla \theta + \nu \Lambda^\alpha \theta &= 0, \\
u(\theta) = \mathcal{R}^+ \theta, \\
\theta(0, x) &= \theta_0(x),
\end{align*}
\]

where \( \nu > 0 \) is the viscosity, which we will assume to be equal to 1 without loss of generality, and

\[
\Lambda^\alpha \theta \equiv (-\Delta)^{\alpha/2} \theta = C_\alpha P.V. \int_{\mathbb{R}^2} \frac{\theta(x) - \theta(x-y)}{|y|^{2+\alpha}} dy,
\]

where \( C_\alpha \) is a positive constant and \( \theta : \mathbb{R}^+ \times \mathbb{R}^2 \rightarrow \mathbb{R} \) is a scalar function which models the potential temperature of the fluid (typically ocean). Here \( \alpha \in (0,2] \) is a fixed parameter and the velocity \( u = (u_1, u_2) \) is divergence-free and determined by the Riesz transforms of the potential temperature \( \theta \) via the formula:

\[
u = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta) = (-\partial_{zz} (-\Delta)^{-1/2} \theta, \partial_{x_1} (-\Delta)^{-1/2} \theta).
\]

It is well known that this equation is classified into 3 cases depending on the value of the power of the fractional Laplacian. Namely, the cases \( 0 < \alpha < 1 \), \( \alpha = 1 \) and \( 1 < \alpha < 2 \) are respectively called sub-critical, critical, and super-critical case. Before recalling some well known results, let us mention that this equation was introduced in 1994 by Constantin, Majda and Tabak [12] in order to get a better understanding of the 3D Euler equation. More precisely, the authors in [12] have pointed out that, in the inviscid case (i.e \( \nu = 0 \)), the (SQG) equation written in terms of the gradient \( \nabla \theta \) turns out to share the same properties with the 3D Euler equation written in terms of vorticity (i.e the curl of the velocity field). It was conjectured in [12] that the gradient of the active scalar has a fast growth when the geometry of the level sets contain a hyperbolic saddle. In [15], Córdoba showed that a simple hyperbolic saddle breakdown cannot occur in finite time. Beside being mathematically relevant, the inviscid (SQG) equation appears also in some physical models such as the study of strongly rotating fluids. In fact, the inviscid (SQG) equation comes from more general quasi-geostrophic models of atmospheric and ocean fluid (see e.g [4], [22]).

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Even in the viscous case, and more precisely in the critical case $\alpha = 1$, the equation is physically pertinent. In fact, the term $(-\Delta)^{1/2}\theta$ models the so-called Ekman pumping which, roughly speaking, drives geostrophic flows. The existence of global weak $L^2$ solutions for the inviscid and dissipative (SQG)$_\alpha$ equation goes back to Resnick [25]. Subsequently, Marchand [20] has extended Resnick’s result to the class of initial data belonging to $H^{-1/2}$ or $L^p$ with $p > 4/3$.

The sub-critical case ($1 < \alpha < 2$) is well understood, for instance one can cite the works by Resnick [25] and latter Ju [18] in which they showed that for all $\theta_0 \in H^s$, with $s > 2 - 2\alpha$, and $s \geq 0$ there exists a unique global smooth solution. In the critical case, namely $\alpha = 1$, Constantin, Córdoba and Wu [11] showed that there is a unique global solution when $\theta_0$ is in the critical space $H^1$ under a smallness assumption on $\|\theta_0\|_\infty$. The problem consisting in showing the regularity of those weak $L^2$ solutions has been solved by Caffarelli and Vasseur [8] using De Giorgi iteration method. By using a completely different method, Kiselev, Nazarov and Volberg [19] showed that all periodic smooth initial data give rise to a unique smooth solution. Their proof is based on a new non local maximum principle verified by the gradient of the solution which is shown to be bounded from above by the derivative of the modulus of continuity at time 0. Both methods have been successfully applied in several works. Abidi and Hmidi [1] showed that for all $\theta_0 \in B_{\infty,1}^0$ there exist a unique global solution by using a Langrangian approach and the method of [19] for showing that local solutions are global. See also the work by Dong, Li [6] and Dong, Du [7] where same type of results have been obtained.

As for the supercritical case ($0 < \alpha < 1$), only partial results are known. We can cite the result of Córdoba and Córdoba [14] where global solutions were shown to exist for all small $H^m$ initial data with $m < 2$. This latter result was improved by Chae and Lee [10] where global existence and uniqueness are shown for data in critical Besov space $B^2_{2,1,-\alpha}$ under a smallness assumption of the $B^2_{2,1,-\alpha}$ norm. In [18], Ju established the global regularity for small $H^s$ data $s \geq 2 - \alpha$ and therefore improved the two previous results. We can also cite the work of Chen, Miao and Zhang [5] in which they proved the global well-posedness for small initial in critical Besov spaces and local well-posedness for large data. Constantin and Wu studied the Hölder regularity propagation in [13] where they proved that all weak $L^2$ solutions which have the property to be $\delta$-Hölder continuous (with $\delta > 1$) on the time interval $[t_1, t]$, are a classical solutions on $(t_1, t]$. The result obtained in [13] is sharp. Indeed, Sylvestre, Vicol and Zlatos in [26] proved the existence of a solution with time-independent velocity $u \in C^1$ with $\delta \in (0, 1)$ and small $C^\delta$ initial data that becomes discontinuous in finite time. It is an outstanding open problem to prove that any bounded weak solutions become $C^{1-\alpha}$ after a short time or if they blow-up in finite time, uniqueness of these solutions is also open.

In this paper, we present some existence results for the critical dissipative (SQG) equation with data in the non homogeneous Morrey-Campanato spaces $M_{p,q}(\mathbb{R}^d)$ ($1 \leq q \leq p \leq \infty$) defined by:

$$M_{p,q}(\mathbb{R}^d) = \left\{ f \in L^q_{\text{loc}}(\mathbb{R}^2), \sup_{x \in \mathbb{R}^n} \sup_{0 < R < 1} R^{d(\frac{1}{2} - \frac{1}{p})} \|f\|_{L^q(B_R(x))} < \infty \right\}$$

In the same spirit as the work by Lemarié-Rieusset for the Navier-Stokes equation cases, we study the critical (SQG) equation with data in the space $L^2_{\text{uloc}}(\mathbb{R}^2)$ (see [23], [24]). Other authors have obtained existence result for the Navier-Stokes equation with data in homogeneous Morrey-Campanato spaces (see e.g. Basson [3], Federbush [16], or Taylor [27]). Unfortunately, because of the Riesz operator, this space is not well adapted to the critical (SQG) equation. This is due to the fact that the Riesz transforms are not defined in the space $L^2_{\text{uloc}}(\mathbb{R}^2)$ and it is not even defined in the subspace $L^2_{\text{uloc}}(\mathbb{R}^2)$. In fact, the problem comes from the lack of integrability at infinity of the Riesz kernel. In [21], we have overcome the difficulty by putting some integrability on the solution, that is to say, we have worked with $\Lambda^s \theta$ instead of $\theta$, where $0 < s < 1$. In this new setting, we were able to work in a space close to the space $L^2_{\text{uloc}}(\mathbb{R}^2)$, namely, the space $\Lambda^s(\mathcal{H}^{\alpha}_{\text{uloc}}(\mathbb{R}^2))$ (where $f \in \Lambda^s(\mathcal{H}^{\alpha}_{\text{uloc}}(\mathbb{R}^2))$ if there exists $g \in \mathcal{H}^{\alpha}_{\text{uloc}}(\mathbb{R}^2)$ s.t $f = \Lambda^s g$). In the case $1/2 < s < 1$ we have obtained global existence of weak solutions in [21]. In this article, by slightly improving the energy inequality obtained in [21], we are able to show that the solutions are in fact global for all $1/4 \leq s \leq 1/2$. For the case $0 < s < 1/4$, we only have a local existence result. The case $s = 1/4$ turns out to be critical in some sense. It is worth to mention that the smaller the
value of $s$, the closer the singular case i.e $L^2_{uloc}(\mathbb{R}^2)$ data, and the partial result obtained for small $s$ is in part due to this fact.

The paper is organized as follows. In the first part we recall some tools we will use in the proof of our main results. We also briefly recall the equation with truncated and regularized initial data. In the second part, we prove an energy inequality available in the case $1/4 \leq s \leq 1/2$ which allows us to get a global control of the solutions. The third part is devoted to the proof of an energy inequality available in the case $0 < s < 1/4$, this inequality leads to a local existence theorem. The passage to the limit is omitted here since it is the same as the one in [21].

Throughout this article, $C$ stands for any controlled and positive constant, which therefore could be different from line to line. We also denote $A \lesssim B$ if $A$ is less to $B$ up to a positive multiplicative constant which can be different from line to line as well. Those constants depend only on some controlled norms. We denote by $\mathcal{D}(\mathbb{R}^2)$ the space of smooth functions in $\mathbb{R}^2$ that are compactly supported. As usually, we denote by $L^p$ the usual Lebesgue spaces, $H^s$ the classical Sobolev spaces and $\dot{H}^s$ the homogeneous ones. We shall also use the shorter notation $L^p X$ for $L^p([0, T], X)$ where $X$ is a Lebesgue or Sobolev space.

Our main result reads as follows.

**Theorem 1.** Let us denote $X_T$ and $X_T^s$ the spaces

\[ X_T \equiv L^\infty([0, T], L^2_{uloc}) \cap (L^2_t([0, T], \dot{H}^{1/2}))_{uloc}, \]

and

\[ X_T^s \equiv L^\infty([0, T], \dot{H}^s_{uloc}) \cap (L^2_t([0, T], \dot{H}^{s+1/2}))_{uloc}. \]

Assume that $\theta_0 = \Lambda^s w_0 \in \Lambda^s(H^s_{uloc}) \cap L^\infty$, then:

- If $1/4 \leq s \leq 1/2$, the critical (SQG) has at least one global weak solution $\theta$ which satisfies $\theta \in X_T$ and $w \in X_T^s$ for all $T < \infty$. Furthermore, we have the following control

  \[ \|w(x, t)\|_{H^s_{uloc}(\mathbb{R}^2)}^2 \leq c \ e^{CT}, \]

- If $0 < s < 1/4$, the critical (SQG) has at least one local weak solution $\theta$ so that for all

  \[ T < T^* = \frac{C(\|\theta_0\|_\infty)}{1 + \|w_0\|_{H^s_{uloc}}^2} \]

we have

\[ \theta \in X_{T^*} \text{ and } w \in X_{T^*}^s. \]

Moreover, for all $T \leq T^*$, the solution $w$ satisfies the following energy inequality:

\[ \|w(x, T)\|_{H^s_{uloc}}^2 \leq \|w_0\|_{H^s_{uloc}}^2 + C \int_0^T \left( \|w(x, s)\|_{H^s_{uloc}}^2 + \|w(x, s)\|_{H_{uloc}^s}^4 + C \right) \, ds, \]

where $C$ is a positive constant depending only on $\|\theta_0\|_{L^\infty(\mathbb{R}^2)}$ and $\|w_0\|_{H^s_{uloc}(\mathbb{R}^2)}$.

**Remark 2.** Previous results obtained in [21] and the global result of the above theorem imply that we have global existence for $1/4 \leq s \leq 1$.

**Remark 3.** In fact, for the case $0 < s < 1/4$, we prove a more general inequality, namely, for all $t \leq T^*$ and for all $K \in (0, 2)$, solution $w$ satisfies the following energy inequality

\[ \|w(x, T)\|_{H^s_{uloc}}^2 \leq \|w_0\|_{H^s_{uloc}}^2 + C \int_0^T \left( \|w\|_{H^s_{uloc}}^2 + \|w\|_{H^s_{uloc}}^{2(\frac{2}{2-s})} + C \right) \, ds. \]

The best power that we can obtain in the above inequality is close to 3 (roughly speaking, it corresponds to $K \to 0$) and therefore we only have a local control of the solutions. In order to make the statement and the time existence of the solutions simpler, we have considered the case $K = 1$ which gives a power 4.
2 The $L^p_{uloc}(\mathbb{R}^2)$ and $H^s_{uloc}(\mathbb{R}^2)$ spaces

In this section, we recall the definition of the $L^p_{uloc}(\mathbb{R}^2)$ and $H^s_{uloc}(\mathbb{R}^2)$ spaces. To do so, we need to introduce the set of translations of a given test function.

Definition 4. Let us fix a positive test function $\phi_0$ such that $\phi_0 \in \mathcal{D}(\mathbb{R}^2)$ and

$$\begin{cases} 
\phi_0(x) = 1 & \text{if } |x| \leq 2, \\
\phi_0(x) = 0 & \text{if } |x| \geq 3.
\end{cases}$$

We define the set of translations of the function $\phi_0$ as $B_{\phi_0} \equiv \{ \phi_0(x-k), k \in \mathbb{Z}^2 \}$. We are now ready to define both $L^p_{uloc}(\mathbb{R}^2)$ and $H^s_{uloc}(\mathbb{R}^2)$ spaces.

Definition 5. Let $1 \leq p \leq \infty$ then $f \in L^p_{uloc}(\mathbb{R}^2)$ if and only if $f \in L^p_{loc}(\mathbb{R}^2)$ and the following norm is finite

$$\|f\|_{L^p_{uloc}(\mathbb{R}^2)} = \sup_{\phi \in B_{\phi_0}} \|\phi f\|_{L^p(\mathbb{R}^2)}.$$

We will also use the following useful equivalent norms

$$\|f\|_{L^p_{uloc}(\mathbb{R}^2)} \approx \sup_{k \in \mathbb{Z}^2} \left( \int_{k+[0,1]^2} |f(x)|^p \, dx \right)^{1/p} \approx \sup_{k \in \mathbb{Z}^2} \|\phi_0(x-k)f\|_{L^p(\mathbb{R}^2)}.$$

Let us recall the definition of the $H^s_{uloc}(\mathbb{R}^2)$ spaces with $0 < s < 1$.

Definition 6. Let $\phi_0$ be a positive test function $\phi_0 \in \mathcal{D}(\mathbb{R}^2)$ such that $\phi_0(x) = 1$ if $x \in [-1,1]^2$. We say that $f \in H^s_{uloc}(\mathbb{R}^2)$ if and only if $f \in H^s_{loc}(\mathbb{R}^2)$ and the following norm is finite

$$\|f\|_{H^s_{uloc}(\mathbb{R}^2)}^2 = \sup_{\phi \in B_{\phi_0}} \|\phi f\|_{H^s}.$$

We will also use the following equivalent norms

$$\|f\|_{H^s_{uloc}(\mathbb{R}^2)}^2 = \sup_{\phi \in B_{\phi_0}} \int \frac{\|\phi f\|^2}{2} + \frac{|\Lambda^s(\phi f)|^2}{2} \, dx \equiv \sup_{\phi \in B_{\phi_0}} A_\phi f,$$

and,

$$\|f\|_{H^s_{uloc}(\mathbb{R}^2)}^2 = \sup_{k \in \mathbb{Z}^2} \int |\Lambda^s(\phi_0(x-k)f(x,t))|^2 \, dx.$$

Remark 7. It is important to note that the norms do not depend on the choice of the test functions. In [21], we have seen that the $H^s_{uloc}(\mathbb{R}^2)$ can be considered with $\phi$ inside or outside the fractional derivative since the norms are equivalents. Therefore, it does not matter whether the function $\phi$ is inside or outside the brackets in our computations.

The spaces $(L^2_T,H^s_{uloc})$ and $L^p_T H^s_{uloc}$ with $0 < s < 1$ and $1 \leq p < \infty$ will be used throughout the paper. These spaces are endowed with the following norms

$$\|w\|_{(L^2_T,H^s)} = \sup_{\phi \in B_{\phi_0}} \int_0^T \int |\Lambda^s(\phi w(x,s))|^2 \, dx \, ds < \infty,$$

$$\|w\|_{L^p_T H^s_{uloc}} = \sup_{t \in [0,T]} \|\Lambda^s(\phi w(x,t))\|_{L^p}^2 \, dx < \infty.$$

In the next section, we recall some classical tools from the so-called Littlewood-Paley theory (see e.g [9], or [23]).
3 Besov spaces and Bernstein inequalities.

In order to define the Besov spaces, we need to recall the definition of the dyadic blocks.

**Definition 8.** Let $\phi \in D(\mathbb{R}^2)$ be a non negative function such that $\phi(x) = 1$ if $|x| \leq 1/2$ and 0 if $|x| \geq 1$. We also define $\psi \in D(\mathbb{R}^2)$ by $\psi(x) = \phi(x/2) - \phi(x)$ supported on a corona. Then, we define the Fourier multiplier $S_j$ and $\Delta_j$ by

$$\hat{S_j}f(\xi) = \hat{\phi}(\frac{\xi}{2^j})\hat{f}(\xi)$$

and

$$\hat{\Delta_j}f(\xi) = \hat{\psi}(\frac{\xi}{2^j})\hat{f}(\xi).$$

From these operators we deduce the Littlewood decomposition of a distribution $f \in \mathcal{S}'$, that is, for all $N \in \mathbb{Z}$, we have

$$f = S_N f + \sum_{j \geq N} \Delta_j f \text{ in } \mathcal{S}'(\mathbb{R}^2).$$

If moreover,

$$S_N f \xrightarrow{N \to -\infty} 0 \text{ in } \mathcal{S}'(\mathbb{R}^2),$$

we obtain the homogeneous decomposition of $f \in \mathcal{S}'(\mathbb{R}^2)$ (modulo polynomials) :

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f \text{ in } \mathcal{S}'(\mathbb{R}^2).$$

The inhomogeneous Besov spaces are defined as follow

**Definition 9.** For $s \in \mathbb{R}$, $(p,q) \in [1,\infty]^2$ and $N \in \mathbb{Z}$, a distribution $f \in \mathcal{S}'(\mathbb{R}^2)$ belongs to the inhomogeneous Besov space $\hat{B}^s_{p,q}$ if and only if

$$\|f\|_{\hat{B}^s_{p,q}} = \|\Delta_N f\|_{p} + \left( \sum_{j \geq N} 2^{js} \|\Delta_j f\|_p^q \right)^{1/q} < \infty.$$  

We also recall the definition of the homogeneous Besov spaces which are defined only modulo polynomials $\mathcal{P}$.

**Definition 10.** If $s < 0$, or $0 < s < 1$ a distribution $f \in \mathcal{S}'(\mathbb{R}^2)$ belongs to the homogeneous Besov space $\hat{B}^s_{\infty,\infty}$ if and only if (in the case $0 < s < 1$, $\Delta_j f \in L^\infty$, for all $j \in \mathbb{Z}$)

$$\|f\|_{\hat{B}^s_{\infty,\infty}} = \sup_{j \in \mathbb{Z}} 2^{js} \|\Delta_j f\|_{\infty} < \infty,$$

and $f \in \mathcal{S}'(\mathbb{R}^2)$ belongs to the homogeneous Besov space $\hat{B}^0_{\infty,1}$ if and only if

$$\|f\|_{\hat{B}^0_{\infty,1}} = \sum_{j \in \mathbb{Z}} \|\Delta_j f\|_{\infty} < \infty.$$  

A very useful tool when we work with Besov norms is the Bernstein inequalities.

**Lemma 11.** If $f \in \mathcal{S}'(\mathbb{R}^2)$, then for all $(s,j) \in \mathbb{R} \times \mathbb{Z}$, and for all $1 \leq p \leq q \leq \infty$ we have

$$2^{js} \|\Delta_j f\|_{L^p} \lesssim \|\Lambda^s \Delta_j f\|_{p} \lesssim 2^{js} \|\Delta_j f\|_{L^p},$$

$$\|\Lambda^s S_j f\|_{L^p} \lesssim 2^{js} \|S_j f\|_{L^p},$$

and

$$\|\Delta_j f\|_{q} \lesssim 2^{j(\frac{2}{p} - \frac{2}{q})} \|\Delta_j f\|_{p}.$$
4 The equation with truncated and regularized initial data.

This section is devoted to the introduction of the \((SQG)_{R,\epsilon}\) equation which is the critical \((SQG)\) equation with truncated and regularized initial data. As we said before, we put some integrability on \(\theta\) by setting \(\theta = \Lambda^s w\) and we consider the equation with respect to \(w\) (which is more regular than \(\theta\) by construction). We first truncate the initial data \(w_0\) by considering \(w_0\chi_R\), where \(\chi_R\) is a positive test function in \(C^\infty(\mathbb{R}^2)\) constructed as follows. Let \(\chi \in \mathcal{D}(\mathbb{R}^2)\) be a positive smooth function s.t \(\chi(x) = 1\) if \(|x| \leq 1\), and \(0\) if \(|x| \geq 2\). For \(R > 0\), we introduce the function \(\chi_R(x) \equiv \chi(x/R)\). Then we defined the truncated initial data \(\theta_{0,R}\) by

\[
\theta_{0,R} = \Lambda^s w_{0,R} = \Lambda^s (w_0\chi_R).
\]

We will also need to regularize the initial data via a convolution with a standard mollifier \(\rho\) because we need to have at least \(H^1\) solutions. Namely, we consider a positive test function \(\rho \in \mathcal{D}(\mathbb{R}^2)\), s.t. \(\text{supp}(\rho) \subset [-1, 1]^2\) and the integral over \(\mathbb{R}^2\) of \(\rho\) is equal to one. Then we define \(\rho_\epsilon(x) \equiv \epsilon^{-2} \rho(x \epsilon^{-1})\). Therefore, we consider the \((SQG)\) equation associated with the following truncated and regularized initial data

\[
\theta_{0,R,\epsilon} = \Lambda^s (w_0\chi_R) * \rho_\epsilon.
\]

More precisely, we will focus on the following \((SQG)_{R,\epsilon}\) equation

\[
(SQG)_{R,\epsilon} : \begin{cases}
  \partial_t w_{R,\epsilon} = (\Lambda^{-s}\nabla) \cdot (\Lambda^s w_{R,\epsilon} \mathcal{R}^\perp \Lambda^s w_{R,\epsilon}) - \Lambda w_{R,\epsilon}, \\
  \nabla \cdot w_{R,\epsilon} = 0, \\
  \theta_{0,R,\epsilon} = \Lambda^s (w_0\chi_R) * \rho_\epsilon.
\end{cases}
\]

The condition on the truncated initial data is:

\[
\theta_{0,R} = \Lambda^s (w\chi_R) \in \Lambda^s (H^s(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2).
\]

Therefore, the truncated and regularized initial data satisfy

\[
\theta_{0,R,\epsilon} \in \bigcap_{k \geq 0} H^k \subset H^2 \subset B^0_{\infty,1}(\mathbb{R}^2).
\]

From the result of Resnick (see [25]), we can claim that there exists at least one solution \(\theta_{R,\epsilon}\) which is a global weak Leray-Hopf solution, that is

\[
\theta_{R,\epsilon} \in L^\infty((0, T), L^2(\mathbb{R}^2)) \cap L^2((0, T), \dot{H}^{1/2}(\mathbb{R}^2)).
\]

Therefore, since \(\theta_{R,\epsilon} = \Lambda^s w_{R,\epsilon}\), we also have the existence of a solution \(w_{R,\epsilon}\) satisfies

\[
w_{R,\epsilon} \in L^\infty((0, T), \dot{H}^{s}(\mathbb{R}^2)) \cap L^2((0, T), \dot{H}^{s+1/2}(\mathbb{R}^2)).
\]

Moreover, the result of Abidi and Hmidi [1] (for data in \(B^0_{\infty,1}(\mathbb{R}^2)\)) provides us the existence of a global solution \(\theta_{R,\epsilon}\) with the following regularity

\[
\theta_{R,\epsilon} \in C(\mathbb{R}_+; B^0_{\infty,1}) \cap L^1_{loc}(\mathbb{R}_+; \dot{B}^1_{\infty,1}).
\]

The idea of the proof is to obtain an energy estimate with respect to the parameters \(R\) and \(\epsilon\) for the solutions \(w_{R,\epsilon}\). This energy inequality will provides us the desired compactness which allows us to pass to the limit in the \((SQG)_{R,\epsilon}\) equation. We will omit to treat the uniform bound for the initial data since it is already done in [21]. Nevertheless, it is worth to recall the uniform estimates for the initial data as well as a useful bound for the Riesz transforms. The proof of those estimations can be found in [21].
Lemma 12. The truncated and regularized initial data and the Riesz transforms satisfy the following statements.

- $u_{R,\varepsilon}$ is bounded in $(L^2 L^2)_{uloc}$, moreover we have
  \[ \|u_{R,\varepsilon}\|_{(L^2 L^2)_{uloc}} \lesssim \|w_{R,\varepsilon}\|_{L^2 H^s_{uloc}}. \]

- If $w_{0,R,\varepsilon} \in H^s_{uloc}(\mathbb{R}^2)$ and $\theta_{0,R,\varepsilon} = \Lambda^s w_{0,R,\varepsilon} \in L^\infty(\mathbb{R}^2)$ then
  \[ \sup_{R>1, \varepsilon>0} \|w_{0,R,\varepsilon}\|_{L^\infty(\mathbb{R}^2)} < \infty \quad \text{and} \quad \sup_{R>1, \varepsilon>0} \|w_{0,R,\varepsilon}\|_{H^s_{uloc}(\mathbb{R}^2)} < \infty. \]

We shall also use the $L^p$ maximum principle due to Resnick [25] and also Córdoba and Córdoba [14]. Namely, if $\theta$ is a smooth solution to the (SQG)$_\alpha$ equation then all $p \in [2, +\infty)$ and for all $t \geq 0$

\[ \|\theta\|_{L^p_t} + 2k \int_0^t \int |\Lambda^{\alpha/2}(|\theta|^{p/2})|^2 \, dx \, ds \leq \|\theta_0\|_{L^p_t} \quad \text{and} \quad \|\theta(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty}. \]

5 Global existence, the case $1/4 \leq s \leq 1$.

In this section, we improve the following energy balance obtained in [21],

\[ \partial_t A_\phi(w_{R,\varepsilon}) \lesssim \|w_{R,\varepsilon}\|_{H^s_{uloc}(\mathbb{R}^2)}^2 \quad (1) \]

where $w_{R,\varepsilon}$ is a weak solution of (SQG)$_{R,\varepsilon}$. This inequality leads to a global existence result for all $1/2 < s < 1$ (see [21]). By taking advantage of one term, we are able to prove a new energy inequality. This new energy inequality allows us to extend the global existence result to the case $1/4 \leq s \leq 1$. More precisely, we prove the following proposition

Proposition 13. Assume that $1/4 \leq s \leq 1$ and let $w_{R,\varepsilon}$ be a weak solution of the equation (SQG)$_{R,\varepsilon}$. Then, for all $\gamma > 0$, for all $\phi \in B_{q_0}$, we have the following energy inequality

\[ \partial_t A_\phi(w_{R,\varepsilon}) + \int \phi \theta_{R,\varepsilon} \Lambda \theta_{R,\varepsilon} \, dx \leq (C + \frac{1}{2}) \|w_{R,\varepsilon}\|_{H^s_{uloc}(\mathbb{R}^2)}^2 + \frac{2}{\gamma} \|\phi w_{R,\varepsilon}\|_{H^{s+1/2}}^2, \]

where $C$ is a positive constant depending only on $\|\theta_0\|_{L^\infty(\mathbb{R}^2)}$ and $\|w_0\|_{H^s_{uloc}(\mathbb{R}^2)}$.

Proof of proposition 13. Since

\[ \|w_{R,\varepsilon}\|_{H^s_{uloc}}^2 = \sup_{\phi \in B_{q_0}} \int \frac{\phi w_{R,\varepsilon}}{2} + \frac{|\Lambda^s(\phi w_{R,\varepsilon})|^2}{2} \, dx, \]

it is convenient to study the evolution of the quantity

\[ A_\phi w_{R,\varepsilon} \equiv \int \frac{\phi w_{R,\varepsilon}}{2} + \frac{|\Lambda^s(\phi w_{R,\varepsilon})|^2}{2} \, dx. \]

A straighforward computation gives us the following equality

\[ \partial_t A_\phi(w_{R,\varepsilon}) + \int \phi \Lambda^s w_{R,\varepsilon} \Lambda^{s+1} w_{R,\varepsilon} \, dx = -\int w_{R,\varepsilon} \phi \Lambda^{-s} \nabla (u_{R,\varepsilon} \Lambda^s w_{R,\varepsilon}) \, dx \]
\[ -\int \phi \Lambda^s w_{R,\varepsilon} \nabla (u_{R,\varepsilon} \Lambda^s w_{R,\varepsilon}) \, dx - \int w_{R,\varepsilon} \phi \Lambda w_{R,\varepsilon} \, dx. \]

In [21], we proved that the last two terms of the right-hand side of equality (2) are uniformly controlled by $C \|w_{R,\varepsilon}\|_{H^s_{uloc}}^2$. The discussion about low and high oscillations (which, roughly speaking, corresponds respectively to small $s$ and big $s$) comes from the first term of the right hand side of equality (2), namely

\[ -\int \Lambda^{-s} \nabla (w_{R,\varepsilon} \phi) u_{R,\varepsilon} \Lambda^s w_{R,\varepsilon} \, dx. \]
Since $\phi w_{R,\varepsilon} \in L^2_t \dot{H}^{s+1/2}$ then $\Lambda^{-s} \nabla (w_{R,\varepsilon}\phi) \in L^2_t \dot{H}^{2s-1/2}$ we thus need the condition $2s-1/2 \geq 0$ which implies $1/4 \leq s \leq 1$ (in some sense, integration by parts are forbidden since it would amount to control the $L^\infty$ norm of $u_{R,\varepsilon}$ which is hopeless). Thus if $1/4 \leq s \leq 1$, we have by the Hölder inequality and by introducing a positive test function $\psi \in \mathcal{D}$, which is equal to 1 in a neighborhood of $\text{supp}(\phi)$ and 0 far enough from $\text{supp}(\phi)$, and then writting $\psi = \psi^1$ and finally using the Young inequality, we obtain that for all $\gamma > 0$

$$- \int \Lambda^{-s} \nabla (w_{R,\varepsilon}\phi) \psi u_{R,\varepsilon} \Lambda^s w_{R,\varepsilon} \, dx \leq \| \Lambda^{-s} \nabla (w_{R,\varepsilon}\phi) \|_{L^2} \| \psi u_{R,\varepsilon} \|_{L^2} \| \psi \theta_{R,\varepsilon} \|_{L^\infty}$$

$$\lesssim \| \phi w_{R,\varepsilon} \|_{H^{1-s}} \| \psi^2 u_{R,\varepsilon} \|_{H^s} \| \theta_{0,\varepsilon} \|_{L^\infty}$$

$$\lesssim \| \phi w_{R,\varepsilon} \|_{H^{s+1/2}} \| \psi u_{R,\varepsilon} \|_{H^s} \| \theta_{0,\varepsilon} \|_{L^\infty}$$

$$\lesssim \frac{\gamma}{2} \| \phi w_{R,\varepsilon} \|_{H^{s+1/2}} + \frac{1}{2\gamma} \| \psi u_{R,\varepsilon} \|_{H^s}$$

$$\lesssim \frac{\gamma}{2} \| \phi w_{R,\varepsilon} \|_{H^{s+1/2}} + \frac{1}{2\gamma} \sup_{x \in \mathcal{B}_N} \| \psi u_{R,\varepsilon} \|_{H^s} \tag{3}$$

Therefore, from equality (2) we get the following inequality

$$\partial_t A_{\phi}(w_{R,\varepsilon}) + \int \phi \Lambda^s w_{R,\varepsilon} \Lambda^{s+1} w_{R,\varepsilon} \, dx \leq \frac{C\gamma}{2} \| \phi w_{R,\varepsilon} \|_{H^{s+1/2}} + (C + \frac{1}{2\gamma}) \| \phi w_{R,\varepsilon} \|_{H^s}$$

This completes the proof of Prop. 13.

Now, we integrate the energy inequality of Prop. 13 in time $t \in [0, T]$

$$A_{\phi}(w_{R,\varepsilon}(T)) + \int_0^T \int \phi \theta_{R,\varepsilon} \Lambda \theta_{R,\varepsilon} \, dx \, dt \leq A_{\phi}(w_{0,R,\varepsilon}) + \frac{C\gamma}{2} \int_0^T \| \phi w_{R,\varepsilon} \|_{H^{s+1/2}}^2 \, dt$$

$$+ (C + \frac{2}{\gamma}) \int_0^T \| \phi w_{R,\varepsilon} \|_{H^s}^2 \, dt$$

The last term of the left hand side provides us the $(L^2_t \dot{H}^{s+1/2})_{\text{uloc}}$ norm since we have the equality

$$\int_0^T \int \phi \theta_{R,\varepsilon} \Lambda \theta_{R,\varepsilon} \, dx \, dt = \int_0^T \int \Lambda^{1/2} \theta_{R,\varepsilon}[\Lambda^{1/2}, \phi] \theta_{R,\varepsilon} \, dx \, dt + \int_0^T \int \phi \Lambda^{1/2} \theta_{R,\varepsilon} \, dx \, dt$$

Hence, we get

$$A_{\phi}(w_{R,\varepsilon}(T)) + \int_0^T \int \phi \Lambda^{1/2} \theta_{R,\varepsilon}^2 \, dx \, dt \leq A_{\phi}(w_{0,R,\varepsilon}) + \frac{C\gamma}{2} \int_0^T \| \phi w_{R,\varepsilon} \|_{H^{s+1/2}}^2 \, dt$$

$$+ (C + \frac{2}{\gamma}) \int_0^T \| w_{R,\varepsilon} \|_{H^s}^2 \, dt + \int_0^T \int \Lambda^{1/2} \theta_{R,\varepsilon}[\Lambda^{1/2}, \phi] \theta_{R,\varepsilon} \, dx \, dt$$

Then, we use the following estimate, for all $\nu > 0$ and $\eta > 0$ (see [21] for the proof)

$$\left| \int_0^T \int \Lambda^{1/2} \theta_{R,\varepsilon}[\Lambda^{1/2}, \phi] \theta_{R,\varepsilon} \, dx \, ds \right| \lesssim \left( \frac{\nu}{2} + \frac{\eta}{2} \right) \int_0^T \int \psi \Lambda^{1/2} \theta_{R,\varepsilon} \|_{L^2}^2 \, dx \, ds + \left( \frac{1}{2\nu} + \frac{1}{2\eta} \right) \int_0^T \| \theta_{R,\varepsilon} \|_{L^2}^2 \, ds.$$
where \( \psi \) is an arbitrary positive test function which is equal to 1 in a neighborhood of the support of \( \phi \). Finally, we obtain the following inequality

\[
A_\phi(w_{R,\epsilon}(T)) + \int_0^T \int \phi |A^{1/2} \theta_{R,\epsilon}|^2 \, dx \, ds \leq A_\phi(w_{0,R,\epsilon}) + C \left( \frac{\nu}{2} + \frac{\eta}{2} \right) \int_0^T \| \psi A^{1/2} \theta_{R,\epsilon} \|_{L^2}^2 \, ds \\
+ C \frac{\gamma}{\nu} \int_0^T \| \dot{\psi} \|_{H^{s+1/2}}^2 \, ds \\
+ \left( C + \frac{1}{2\gamma} + \frac{1}{2
u} + \frac{1}{2\eta} \right) \int_0^T \| w_{R,\epsilon} \|_{H^s_{uloc}(\mathbb{R}^2)}^2 \, ds.
\]

By Gronwall’s lemma we conclude that

\[
\| w_{R,\epsilon} \|_{H^s_{uloc}(\mathbb{R}^2)}^2 \leq \sup_{R > 0, \epsilon > 0} \| w_{0,R,\epsilon} \|_{H^s_{uloc}(\mathbb{R}^2)}^2 + C \int_0^T \| w_{R,\epsilon} \|_{H^s_{uloc}(\mathbb{R}^2)}^2 \, ds.
\]

Then, we take the supremum over all \( \phi \in B_{\rho_0} \) and we choose \( \eta, \nu, \gamma \) sufficiently small so that the norms \( (L^2 H^{s+1/2})_{uloc} \) of the right hand side of (4) are absorbed by that of the left.

**Remark 15.** It is important to note that \( \sup_{\phi \in B_{\rho_0}} \int_0^T \int \phi |A^{1/2} \theta_{R,\epsilon}|^2 \, dx \, ds = \| w_{R,\epsilon} \|_{(L^2 H^{s+1/2})_{uloc}}^2 \).

We eventually get

\[
\| w_{R,\epsilon}(T) \|_{H^s_{uloc}(\mathbb{R}^2)}^2 + C \| w_{R,\epsilon} \|_{(L^2([0,T],H^{s+1/2}(\mathbb{R}^2)))_{uloc}}^2 \leq \sup_{R > 0, \epsilon > 0} \| w_{0,R,\epsilon} \|_{H^s_{uloc}(\mathbb{R}^2)}^2 + C \int_0^T \| w_{R,\epsilon} \|_{H^s_{uloc}(\mathbb{R}^2)}^2 \, ds.
\]

In particular, one obtain

\[
\| w_{R,\epsilon}(T) \|_{H^s_{uloc}(\mathbb{R}^2)}^2 \leq \sup_{R > 0, \epsilon > 0} \| w_{0,R,\epsilon} \|_{H^s_{uloc}(\mathbb{R}^2)}^2 + C \int_0^T \| w_{R,\epsilon}(x,s) \|_{H^s_{uloc}(\mathbb{R}^2)}^2 \, ds.
\]

By Gronwall’s lemma we conclude that

\[
\| w_{R,\epsilon} \|_{L^\infty([0,T],H^s_{uloc}(\mathbb{R}^2))} \leq e^{CT}.
\]

On the other hand, inequality (5) gives

\[
\| w_{R,\epsilon} \|_{(L^2 H^{s+1/2})_{uloc}(\mathbb{R}^2)} \leq \sup_{R > 0, \epsilon > 0} \| w_{0,R,\epsilon} \|_{H^s_{uloc}(\mathbb{R}^2)} + C \int_0^T \| w_{R,\epsilon} \|_{H^s_{uloc}(\mathbb{R}^2)}^2 \, ds.
\]

Thus, \( w_{R,\epsilon} \in L^2([0,T],H^s_{uloc}(\mathbb{R}^2)) \) for all finite time \( T > 0 \). For the limiting process when \( R \to +\infty \) and \( \epsilon \to 0 \) we refer to the last section of [21]. This completes the proof of the global existence result stated in Theorem 1. The next section is devoted to the proof of the local existence.

6 Local existence, the case \( 0 < s < 1/4 \).

6.1 Towards the energy inequality.

In the previous section (see Prop 13), we have proved that, for all \( 1/4 \leq s \leq 1 \) and \( \gamma > 0 \),

\[
\partial_t A_\phi(w_{R,\epsilon}) + \int \phi \theta_{R,\epsilon} A \theta_{R,\epsilon} \, dx \leq (C + \frac{1}{2\gamma}) \| w_{R,\epsilon} \|_{H^s_{uloc}(\mathbb{R}^2)}^2 + \frac{\gamma}{2} \| \phi w_{R,\epsilon} \|_{H^{s+1/2}}^2.
\]
and this inequality leads to a global existence result. These kind of estimates do not work for small values of \( s \), that is, for \( 0 < s < 1/4 \). Nevertheless, one can still obtain an energy inequality which provides a local existence result. As we have already recalled in the previous section, the main issue comes from the first term of the right hand side of inequality (2). By integrating by parts, we see that this term can be rewritten as

\[- \int w_{R, \epsilon} \phi \Lambda^{-s} \nabla (u_{R, \epsilon} \Lambda^s w_{R, \epsilon}) \, dx = - \int \Lambda^{-s} \nabla (w_{R, \epsilon} \phi) u_{R, \epsilon} \Lambda^s w_{R, \epsilon} \, dx.\]

As before, we introduce a positive test function \( \psi \in \mathcal{D} \) which is equal to 1 in a neighborhood of the support of \( \phi \) and equal to 0 on supp(\( \phi \)) + 2. Using Hölder’s inequality, we get

\[- \int \Lambda^{-s} \nabla (w_{R, \epsilon} \phi) \psi u_{R, \epsilon} \Lambda^s w_{R, \epsilon} \, dx \leq \| w_{R, \epsilon} \|_{H^{1-s}} \| \theta_{R, \epsilon} \|_{L^\infty} \| \psi u_{R, \epsilon} \|_{L^2}.\]

Unfortunately, this inequality fails for small values of \( s \) (i.e. \( 0 < s < 1/4 \)) since the right hand side is not controlled (note that we only have \( w_{R, \epsilon} \in L^2 H^{s+1/2} \)). Basically, one would like to share the derivatives into the two terms so that both are controlled by a desired norm. In our case, if we share the derivatives (that is to say, if we integrate by parts) so that both terms are controlled i.e we give \( s + 1/2 \) derivatives to \( \psi u_{R, \epsilon} \) and \( 1/2 - 2s \) to \( \psi u_{R, \epsilon} \Lambda^s w_{R, \epsilon} \), we obtain

\[- \int w_{R, \epsilon} \phi \Lambda^{-s} \nabla (\psi u_{R, \epsilon} \Lambda^s w_{R, \epsilon}) \, dx = - \int \Lambda^{s+1/2} (w_{R, \epsilon} \phi) \Lambda^{s-1/2} \nabla (\psi u_{R, \epsilon} \Lambda^s w_{R, \epsilon}) \, dx.\]

Then, one would like to use the inequality (see for instance [2]), where we set \( \psi = \psi_1^2 \)

\[ \| \psi u_{R, \epsilon} \theta_{R, \epsilon} \|_{H^{s+1/2}} \lesssim \| \psi_1 u_{R, \epsilon} \|_{L^\infty} \| \psi \theta_{R, \epsilon} \|_{H^{1-s}} + \| \psi_1 \theta_{R, \epsilon} \|_{L^\infty} \| \psi_1 u_{R, \epsilon} \|_{H^{s+1/2}}, \]

but this inequality clearly fails due to the lack of bound of \( \| \psi_1 u_{R, \epsilon} \|_{L^\infty} \).

To overcome the difficulty, we will show that one can still control the product \( \psi u_{R, \epsilon} \theta_{R, \epsilon} \) in the Sobolev space of positive regularity provided that we lose a little bit of regularity on \( \psi u_{R, \epsilon} \). The idea consists in substituting the \( L^\infty \) norm of \( \psi u_{R, \epsilon} \) for the Besov norm \( B_{\infty, \infty}^{-\delta} \) of \( \psi u_{R, \epsilon} \), with \( \delta > 0 \). Indeed, we have the following lemma

**Lemma 16.** For all \( \phi \in B_\phi \), there exists \( 0 < \delta < 3/2 \) such that \( \phi u_{R, \epsilon} \in B_{\infty, \infty}^{-\delta} \).

**Proof.** Since \( \phi u_{R, \epsilon} \in H^{1/2} \), by using Bernstein’s inequality twice (in \( 2D \)), we get \( H^{1/2} = B^{1/2}_{2, 2} \subset B^{3/2}_{2, \infty} \subset B_{\infty, \infty}^{-3/2} \). Moreover, we have that \( \phi u_{R, \epsilon} \in B_\infty^{0, \infty} \). Finally, by interpolation we conclude that \( \phi u_{R, \epsilon} \in B_{\infty, \infty}^{-\delta} \) for all \( 0 < \delta < 3/2 \).

Then, for \( \sigma > 0 \) we write

\[- \int \Lambda^{-s} \nabla (w_{R, \epsilon} \phi) \psi u_{R, \epsilon} \Lambda^s w_{R, \epsilon} \, dx = \int w_{R, \epsilon} \phi \Lambda^{-s} \nabla (1 - \psi) u_{R, \epsilon} \Lambda^s w_{R, \epsilon} \, dx - \int \Lambda^{-s} \nabla (w_{R, \epsilon} \phi) \Lambda^s (\psi u_{R, \epsilon} \psi_1 \Lambda^s w_{R, \epsilon}) \, dx.\]

Observe that (I) can be estimated as in step 3 (see proof of Prop. 13) since in this step we just use the fact that the kernel \( 1/|x|^{3-s} \in L^1(\mathbb{R}^3) \) far from the origin (note that we are outside the support of \( \phi \) thus far from the origin) and this holds for all \( 0 < s < 1 \) and in particular it holds for \( 0 < s \leq 1/4 \) therefore we still have the same estimates as 3 and we get

\[(I) \lesssim \| w_{R, \epsilon} \|_{H^{s}_{\text{loc}}}^2 \]

For (II), we use the following lemma to estimate the product \( \psi u_{R, \epsilon} \theta_{R, \epsilon} \) (recall that \( \theta_{R, \epsilon} = \Lambda^s w_{R, \epsilon} \)).
Lemma 17. There exist $\sigma > 0$ and $\delta > 0$ with $\sigma + \delta < 1/2$ such that $\psi \vartheta_{R,\epsilon} \in \dot{H}^{\sigma+\delta} \cap L^\infty$ and $\psi_1 u_{R,\epsilon} \in \dot{H}^\sigma \cap B^{-\delta}_{\infty,\infty}$. Moreover, we have $\nu u_{R,\epsilon} \vartheta \in \dot{H}^\sigma$ and the estimate
\[
\|\nu u_{R,\epsilon} \vartheta \|_{\dot{H}^\sigma} \lesssim \|\psi_1 u_{R,\epsilon} \|_{B^{-\delta}_{\infty,\infty}} \|\vartheta\|_{\dot{H}^\sigma} + \|\psi_1 \vartheta\|_{L^\infty} \|\psi_1 u_{R,\epsilon}\|_{\dot{H}^\sigma}.
\]

**Proof.** The first part is nothing but Lemma 16 and the regularity condition on $\vartheta_{R,\epsilon}$ and $u_{R,\epsilon}$. The inequality is a consequence of the Littlewood-Paley decomposition for functions of positive Sobolev regularity whose Fourier transform is supported on a ball. The proof is based on the following fact (see e.g. Theorem 4.1 p 34 [23]). If $E$ is a shift invariant Banach space of distribution, then, for all $1 \leq q \leq \infty$, and all $(s, \tau)$ such that $-s < \tau < 0$ and $s > 0$, the pointwise multiplication is a bounded bilinear operator from $(B^s_{q,\infty} \cap B^\tau_{\infty,\infty}) \times (B^s_{q,\infty} \cap B^\tau_{\infty,\infty})$ to $B^{s+\tau}_{q,\infty}$. Where, for all $\kappa \in \mathbb{R}$, such that $\kappa_0 < \kappa < \kappa_1$ and $s = (1 - \eta)\kappa_0 + \eta\kappa_1$ and for all $\eta \in (0,1)$, the space $B_{q,E}^s$ is defined by the following interpolation space
\[
B_{q,E}^s = [H^{\kappa_0}_{E}, H^{\kappa_1}_{E}]_{\eta, q}.
\]
Let us also recall that, for all $\kappa \in \mathbb{R}$, $H^{\kappa}_{E}$ is the space $(Id - \Delta)^{-\kappa} E$ endowed with the norm
\[
\|f\|_{H^{\kappa}_{E}} = \|(Id - \Delta)^{\frac{\kappa}{2}} f\|_{E}.
\]
Therefore, for all functions $(f, g) \in (B^s_{q,E} \cap B^\tau_{\infty,\infty})^2$ and for all $1 \leq q \leq \infty$, we have
\[
\|fg\|_{B^{s+\tau}_{q,E}} \leq \|f\|_{B^{s}_{q,E}} \|g\|_{B^\tau_{\infty,\infty}} + \|f\|_{B^\tau_{\infty,\infty}} \|g\|_{B^s_{q,E}}. \tag{8}
\]
If we choose $E = L^2$ and $q = 2$ and use the fact that $\|\vartheta_{R,\epsilon}\|_{B^{-\delta}_{\infty,\infty}} \leq \|\vartheta_{R,\epsilon}\|_{L^\infty}$ we conclude the proof by applying inequality (8) with $s = \sigma + \delta$ and $\tau = -\delta$.

The next section is devoted to the proof of the energy inequality.

6.2 The energy inequality

In this section, we prove the following proposition

**Proposition 18.** Let $w_{R,\epsilon}$ be a solution of $(SQG)_{R,\epsilon}$ equation. Then, for all $K \in (0,2)$
\[
\|w_{R,\epsilon}(T)\|_{H_{uloc}^1}^2 \leq \|w_{0,R,\epsilon}\|_{H_{uloc}^1}^2 + C \int_0^T \left( \|w_{R,\epsilon}\|_{H_{uloc}^1}^{2(\frac{2}{\sigma} - 2K)} + \|w_{R,\epsilon}\|_{H_{uloc}^1}^{2(\frac{2}{\delta} - 2K)} \right) ds \tag{9}
\]

**Remark 19.** The constant $C > 0$ depends only on the norms $\|\vartheta_0\|_{L^\infty}$ and $\|w_0\|_{H_{uloc}^1}$.

For the sake of clarity, we omit to write the parameter $\epsilon$ in the following.

6.2.1 Proof of Proposition 18

Let us first choose $\sigma > 0$ such that
\[
\frac{1}{2} - 2s < \sigma < \frac{1}{2},
\]
and then choose $\delta > 0$ such that
\[
\delta + \sigma < \frac{1}{2}.
\]
These conditions and Lemma 17 allow us to write
\[
\int A^\sigma \nabla(w_{R,\epsilon} \vartheta \Lambda^\sigma (\psi u_{R,\epsilon} u_{R,\epsilon} w_{R,\epsilon})) \leq \|w_{R,\epsilon}\|_{H^{1-\sigma-s}} \|\psi u_{R,\epsilon} \vartheta\|_{H^s} \leq \|w_{R,\epsilon}\|_{H^{1-\sigma-s}} \|\psi u_{R,\epsilon} \vartheta\|_{B^{-\delta}_{\infty,\infty}} + \|w_{R,\epsilon}\|_{H^{1-\sigma-s}} \|\vartheta\|_{L^\infty} \|\vartheta\|_{H^s},
\]
where $\psi = \psi^2$. Then we write,
\[
\|\psi u_{R,\epsilon}\|_{B^{-\delta}_{\infty,\infty}} \leq \|u_{R,\epsilon}\|_{B^{-\delta}_{\infty,\infty}} \leq \|S_0 u_{R}\|_{L^\infty} + \|(Id - S_0) u_{R}\|_{B^{-\delta}_{\infty,\infty}},
\]

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where we have used that
\[ \| S_0 u_R \|_{B^{-s}_{\infty, \infty}} \leq \| S_0 u_R \|_{L^\infty}. \]

Then, we observe that
\[ \| (S_0 u_R)(x) \|_{L^\infty} \leq \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{|k|^4} \left( \int_{k+x} |u_R(y)|^2 \, dy \right)^{1/2} \leq C \sup_{k \in \mathbb{Z}^2 \setminus \{0\}} \left( \int_{k+x} |u_R(y)|^2 \, dy \right)^{1/2} \lesssim \| u_R \|_{L^2_{\text{loc}}} \lesssim \| w_R \|_{H^s_{\text{loc}}}, \]

where, in the last inequality, we have used the first point of Lemma 12. Therefore,
\[ \| S_0 u_R \|_{L^\infty} \leq \| u_R \|_{L^2_{\text{loc}}} \leq \| w_R \|_{H^s_{\text{loc}}}. \]

For high frequencies we have, using Bernstein’s inequality
\[ \| (Id - S_0) u_R \|_{B^{-s}_{\infty, \infty}} \leq \sum_{j > 0} \| \Delta_j \theta_R \|_{L^\infty} \leq \sum_{j > 0} 2^{-j\delta} \| \theta_R \|_{L^\infty} \leq C \| \theta_R \|_{L^\infty}, \]

and we obtain,
\[ \| \psi u_R \|_{B^{-s}_{\infty, \infty}} \leq C \| w_R \|_{H^s_{\text{loc}}} + \| \theta_R \|_{L^\infty} \leq C \| w_R \|_{H^s_{\text{loc}}} + \| \theta_0, R \|_{L^\infty}. \]

Thus, the inequality
\[ \int \Lambda^{-s} \nabla (w_R \phi) \Lambda^s (\psi u_R \Lambda^s w_R) \lesssim \| w_R \|_{H^{1-s}} \| \psi \|_{H^{s}} \| \psi \|_{H^{s}} \]
\[ + \| w_R \|_{H^{1-s}} \| \psi \|_{H^{s}} \| \theta_0, R \|_{L^\infty}, \]

becomes
\[ \int \Lambda^{-s} \nabla (w_R \phi) \Lambda^s (\psi u_R \Lambda^s w_R) \lesssim \| w_R \|_{H^{1-s}} \| \psi \|_{H^{s}} \| \psi \|_{H^{s}} \]
\[ + \| w_R \|_{H^{1-s}} \| \psi \|_{H^{s}} \| \theta_0, R \|_{L^\infty}, \]

Then, we use the fact that \( 0 < s \leq 1/4 \) and that the constant \( \sigma > 0 \) is chosen so that \( 1/2 - 2s \leq \sigma \) therefore \( s < 1 - s - \sigma < s + 1/2, \) and \( \delta + \sigma < 1/2 \) by definition. These facts allow us to interpolate and write that for all \( (\kappa, \nu, \eta) \in (0, 1)^3 \)
\[ \partial_t A_\phi w_R + \int \phi \partial R \Lambda^s \theta R \, dx \leq C \| w_R \|_{H^s_{\text{loc}}}^2 + C \| w_R \|_{H^s_{\text{loc}}} \| w_R \|_{H^{1/2}} \| \psi \|_{L^\infty} \| \psi \|_{H^{1/2}} \| \theta_0, R \|_{L^\infty} + C \| w_R \|_{H^s_{\text{loc}}} \| w_R \|_{H^{1/2}} \| \psi \|_{H^{1/2}} \| \psi \|_{L^\infty} \]
\[ + C \| w_R \|_{H^s_{\text{loc}}} \| w_R \|_{H^{1/2}} \| \psi \|_{L^\infty} \| \psi \|_{H^{1/2}} \| \theta_0, R \|_{L^\infty} + C \| w_R \|_{H^s_{\text{loc}}} \| w_R \|_{H^{1/2}} \| \psi \|_{H^{1/2}} \| \psi \|_{L^\infty} \]

**Remark 20.** Note that the first term of the right hand side i.e. \( C \| w_R \|_{H^s_{\text{loc}}}^2 \) comes from the control of the other terms (see 7)

Now, we take the supremum over all \( \phi \in B_{2\eta} \) and all \( \psi \in B_{3\eta} \) only on some norms, namely, those
which will give a $L^\infty_t H^\alpha_{\text{uloc}}$ norm of $w_R$ or equivalently a $L^\infty_t L^2_{\text{uloc}}$ norm of $\theta_R$. We obtain
\[
\partial_t A_\phi w_R + \int \phi \theta_R \Lambda^\alpha \theta_R \, dx \leq C \|w_R\|_{2,\text{loc}}^2 + \sup_{\phi \in B_{\psi_0}} \left\{ \|w_R\|_{H^\alpha_{\text{loc}}} \|\psi_1 \theta_R\|_{L^2} \right\} \|w_R\|_{H^\alpha_{\text{loc}}} \|\psi_1 \theta_R\|_{H^1_{\text{loc}}} \|\theta_0, R\|_{L^\infty} \\
+ C \sup_{\phi \in B_{\psi_0}} \left\{ \|w_R\|_{H^\alpha_{\text{loc}}} \|\psi_1 \theta_R\|_{L^2} \right\} \|w_R\|_{H^\alpha_{\text{loc}}} \|\psi_1 \theta_R\|_{H^1_{\text{loc}}} \|\theta_0, R\|_{L^\infty} \\
+ C \sup_{\phi \in B_{\psi_0}} \left\{ \|w_R\|_{H^\alpha_{\text{loc}}} \|\psi_1 \theta_R\|_{L^2} \right\} \|w_R\|_{H^\alpha_{\text{loc}}} \|w_R\|_{H^\alpha_{\text{loc}}} \|\psi_1 \theta_R\|_{H^1_{\text{loc}}}.
\]

Therefore, we obtain that
\[
\partial_t A_\phi w_R + \int \phi \theta_R \Lambda^\alpha \theta_R \, dx \leq C \|w_R\|_{2,\text{loc}}^2 + \|w_R\|_{H^\alpha_{\text{loc}}}^2 \|\psi_1 \theta_R\|_{H^1_{\text{loc}}} \|\theta_0, R\|_{L^\infty} \\
+ C \|w_R\|_{H^\alpha_{\text{loc}}}^2 \|w_R\|_{H^\alpha_{\text{loc}}} \|\psi_1 \theta_R\|_{H^1_{\text{loc}}} \|\theta_0, R\|_{L^\infty} \\
+ C \|w_R\|_{H^\alpha_{\text{loc}}}^2 \|w_R\|_{H^\alpha_{\text{loc}}} \|\psi_1 \theta_R\|_{H^1_{\text{loc}}}.
\]

**Remark 21.** It is important to note that the norms of the type $\|w_R\|_{H^{\alpha+1/2}_{\text{loc}}}$ or $\|\psi_1 \theta_R\|_{H^{1/2}_{\text{loc}}}$ are the same only after having taken the supremum over all $\phi \in B_{\psi_0}$ and $\psi_1 \in B_{\psi_1}$ respectively. In our case, we have to take those suprema as late as we can (at least after having integrated in time) so that we get the desired $(L^2_t H^{\alpha+1/2}_{\text{uloc}})$ norm of $w_R$ or equivalently the $(L^2_t H^{1/2}_{\text{uloc}})$ norm of $\theta_R$.

Let us focus on the three last terms appearing in the right hand side of the previous inequality, that is to say:
\[
T_1 \equiv \|w_R\|_{H^\alpha_{\text{loc}}}^2 \|w_R\|_{H^\alpha_{\text{loc}}} \|\psi_1 \theta_R\|_{H^1_{\text{loc}}} \|\theta_0, R\|_{L^\infty}, \\
T_2 \equiv \|w_R\|_{H^\alpha_{\text{loc}}}^2 \|w_R\|_{H^\alpha_{\text{loc}}} \|\psi_1 \theta_R\|_{H^1_{\text{loc}}} \|\theta_0, R\|_{L^\infty}, \\
\text{and} \\
T_3 \equiv \|w_R\|_{H^\alpha_{\text{loc}}}^2 \|w_R\|_{H^\alpha_{\text{loc}}} \|\psi_1 \theta_R\|_{H^1_{\text{loc}}} \|\theta_0, R\|_{L^\infty}.
\]

The estimations of the terms $T_1$ and $T_2$ are the same. Let us begin with $T_1$
\[
T_1 = \|w_R\|_{H^\alpha_{\text{loc}}}^2 \|\psi_1 \theta_R\|_{H^1_{\text{loc}}} \|\theta_0, R\|_{L^\infty} \|w_R\|_{H^{\alpha+1/2}_{\text{loc}}} \|\psi_1 \theta_R\|_{H^{1/2}_{\text{loc}}}.
\]

Using Young’s inequality (with the exponent $2/(\kappa + \eta) > 1$) we get for all $\rho_1 > 0$
\[
T_1 \leq \rho_1 \frac{2}{\kappa + \eta} \left( \frac{2 - \kappa - \eta}{2} \right) \|w_R\|_{H^\alpha_{\text{loc}}}^2 \|\psi_1 \theta_R\|_{H^1_{\text{loc}}} \|\theta_0, R\|_{L^\infty} \|w_R\|_{H^{\alpha+1/2}_{\text{loc}}} \|\psi_1 \theta_R\|_{H^{1/2}_{\text{loc}}}.
\]

Since, by Young’s inequality (with the exponent $\kappa + \eta > 1$) we have
\[
\|\phi w_R\|_{H^{\alpha+1/2}_{\text{loc}}} \|\psi_1 \theta_R\|_{H^{1/2}_{\text{loc}}} \leq \frac{\kappa}{\kappa + \eta} \|w_R\|_{H^{\alpha+1/2}_{\text{loc}}}^2 + \frac{\eta}{\kappa + \eta} \|\psi_1 \theta_R\|_{H^{1/2}_{\text{loc}}},
\]

and thus, we obtain
\[
T_1 \leq \rho_1 \frac{2}{\kappa + \eta} \left( \frac{2 - \kappa - \eta}{2} \right) \|w_R\|_{H^\alpha_{\text{loc}}}^2 \|\theta_0, R\|_{L^\infty} \|w_R\|_{H^{\alpha+1/2}_{\text{loc}}}^2 + \frac{\kappa \rho_1}{\kappa + \eta} \|w_R\|_{H^{\alpha+1/2}_{\text{loc}}}^2 + \frac{\eta \rho_1}{\kappa + \eta} \|\psi_1 \theta_R\|_{H^{1/2}_{\text{loc}}}^2.
\]
replacing $\nu$ by $\eta$ we infer that

$$T_2 \leq \rho_1 \frac{2}{2} \left( \frac{2 - \kappa - \nu}{2} \right) \| w_R \phi \|^2_{H^2_{uloc}} \| \theta_{0,R} \|_{L^{2\infty}} + \frac{\kappa \rho_1}{2} \| w_R \phi \|^2_{H^{1/2}} + \frac{\nu \rho_1}{2} \| \psi_1 \theta_R \|^2_{H^{1/2}}.$$

Therefore, for some $A > 0$ we have for $i = 1, 2$

$$T_i \lesssim \rho^{-A} \| w_R \phi \|^2_{H^2_{uloc}} + \rho^A \left( \| w_R \phi \|^2_{H^{1/2}} + \| \psi_1 \theta_R \|^2_{H^{1/2}} \right).$$

The term $T_3$ is the one which is responsible for the local existence. Indeed, using Young’s inequality (with the exponent $2/\kappa > 1$), we obtain that for all $\rho_3 > 0$

$$T_3 \leq \frac{\rho_3^{2}}{2} \left( \frac{2 - \kappa}{2} \right) \| w_R \phi \|^2_{H^{1/2}} + \frac{\rho_3}{2} \| \psi_1 \theta_R \|^2_{H^{1/2}} + \frac{\rho_3}{2} \| \psi_1 \theta_R \|^2_{H^{1/2}}.$$

Once again, Young’s inequality (with the exponent $\frac{2}{\kappa} + \frac{\eta}{\gamma} > 1$) gives

$$\| w_R \phi \|^2_{H^{1/2}} + \| \psi_1 \theta_R \|^2_{H^{1/2}} \leq \frac{\kappa}{\kappa + \eta} \| w_R \phi \|^2_{H^{1/2}} + \frac{\eta}{\kappa + \eta} \| \psi_1 \theta_R \|^2_{H^{1/2}}.$$

Therefore,

$$T_3 \leq \frac{\kappa \rho_3}{2} \| w_R \phi \|^2_{H^{1/2}} + \frac{\eta \rho_3}{2} \| \psi_1 \theta_R \|^2_{H^{1/2}} + \rho_3 \left( \frac{2 - \kappa - \eta}{2} \right) \| w_R \|^2_{\mathcal{H}^{2(3 - \kappa - \eta)}_{uloc}}.$$

Hence, for some constant $B > 0$ we have

$$T_3 \lesssim \rho_3 B \| w_R \phi \|^2_{H^{1/2}} + \rho_3 B \| \psi_1 \theta_R \|^2_{H^{1/2}} + \rho_3 \left( \frac{2 - \kappa - \eta}{2} \right) \| w_R \|^2_{\mathcal{H}^{2(3 - \kappa - \eta)}_{uloc}}.$$

Putting all those terms together leads us to the following inequality

$$\partial_t A_\phi w_R + \int \phi \theta_R \Lambda^s \theta_R \; dx \leq \left( C + \rho^{-A} \right) \| w_R \|^2_{\mathcal{H}^{1/2}_{uloc}} + \rho_3 B \| w_R \|^2_{\mathcal{H}^{2(3 - \kappa - \eta)}_{uloc}} + (\rho^A + \rho_3^B) \| \psi_1 \theta_R \|^2_{H^{1/2}}.$$

We integrate in time $s \in [0, T]$, and we conclude that

$$A_\phi w_R(x, T) + \int_0^T \theta_R \Lambda^s \theta_R \; ds \leq A_\phi w_{0,R}(x, T) + \left( C + \rho^{-A} \right) \int_0^T \| w_R \|^2_{H^{1/2}_{uloc}} \; ds \quad (10)$$

$$+ (\rho^A + \rho_3^B) \left( \int_0^T \| w_R \|^2_{H^{1/2}_{uloc}} \; ds + \int_0^T \| \psi_1 \theta_R \|^2_{H^{1/2}} \; ds \right)$$

$$+ \rho_3 B \int_0^T \| w_R \|^2_{\mathcal{H}^{2(3 - \kappa - \eta)}_{uloc}} \; ds.$$

As before, we write the last integral of the left hand side in terms of the commutator as

$$\int_0^T \int \phi \theta_R \Lambda \theta_R \; dx \; ds = \int_0^T \int \Lambda^{1/2} \theta_R [\Lambda^{1/2}, \phi] \theta_R \; dx \; ds + \int_0^T \int \phi \| \Lambda^{1/2} \theta_R \|^2 \; dx \; ds \quad (11)$$

The last integral of the right hand side provides a $(L^2_t \mathcal{H}^{1/2}_{uloc})$ norm of $w_R$ which allows us to absorb the other $(L^2_t \mathcal{H}^{1/2}_{uloc})$ norms of $w_R$, or equivalently, the $(L^2_t \mathcal{H}^{1/2}_{uloc})$ norms of $\theta_R$ appearing in the right hand side of the inequality (this equality of norms is verified only after having taken the supremum over the test functions, see Remark 21). The reminder term of the equality (10) is controlled, since we have seen that for all $\mu_1 > 0$ and $\rho_4 > 0$,

$$\left| \int_0^T \int \Lambda^{1/2} \theta_R [\Lambda^{1/2}, \phi] \theta_R \; dx \; ds \right| \lesssim \left( \frac{\mu_1}{2} + \frac{\rho_4}{2} \right) \int_0^T \| \psi_1 \theta_R \|^2_{L^2} \; ds$$

$$+ \left( \frac{1}{2\mu_1} + \frac{1}{2\rho_4} \right) \int_0^T \| \theta_R \|^2_{L^2_{uloc}} \; ds.$$
Therefore, inequality (10) becomes

\[ A_0 w_R(x, T) + \int_0^T \int \phi |\Lambda^{1/2} \theta_R|^2 dx dt \lesssim A_0 w_{0,R}(x, T) + (C + \rho^{-A}) \int_0^T \| w_R \|_{H^s_{\text{loc}}}^2 ds \]  
\[ + (\rho^A + \rho_3^B) \left( \int_0^T \| w_R \|_{H^{s+1/2}}^2 ds + \int_0^T \| \psi \|_{H^{1/2}}^2 ds \right) \]
\[ + \left( \frac{\mu_1}{2} + \frac{\rho_1}{2} \right) \int_0^T \| w_R \|_{H^{s+1/2}}^2 ds + \frac{1}{2\mu_1} + \frac{1}{2\rho_4} \int_0^T \| \theta_R \|_{L^2_{\text{loc}}}^2 ds \]
\[ + \rho_3^{-B} \int_0^T \| w_R \|_{H^{s+1/2}_{\text{loc}}}^{2(3-\gamma-n)} ds. \]

Now we take the supremum over \( \phi \in B_{\delta_0}, \psi_1 \in B_{\psi_1} \), and \( \psi \in B_{\varphi} \) in the last inequality. Therefore, we obtain

\[ \| w_R(T) \|_{H_{\text{loc}}}^2 + \| w_R \|_{(L^2 H^{s+1/2})_{\text{loc}}}^2 \lesssim \| w_{0,R} \|^2_{H_{\text{loc}}} + (C + \rho^{-A} + \frac{1}{2\mu_1} + \frac{1}{2\rho_4}) \int_0^T \| w_R \|_{H_{\text{loc}}}^2 ds \]
\[ + (\rho^A + \rho_3^B + \frac{\mu_1}{2} + \frac{\rho_1}{2}) \int_0^T \| w_R \|_{H^{s+1/2}}^2 ds \]
\[ + \rho_3^{-B} \int_0^T \| w_R \|_{H^{s+1/2}_{\text{loc}}}^{2(3-\gamma-n)} ds. \]

We choose \( \rho, \rho_3, \rho_4 \) and \( \mu_1 \) sufficiently small so that the \( (L^2 H^{s+1/2})_{\text{loc}} \) norms appearing in the right hand side of the above inequality are absorbed by that of the left, we get

\[ \| w_R(T) \|_{H_{\text{loc}}}^2 + \| w_R \|_{(L^2 H^{s+1/2})_{\text{loc}}}^2 \lesssim \| w_{0,R} \|^2_{H_{\text{loc}}} + C \int_0^T \| w_R \|_{H_{\text{loc}}}^2 + \| w_R \|_{H_{\text{loc}}}^{2(3-\gamma-n)} ds \]
\[ + \int_0^T \| w_R \|_{H_{\text{loc}}}^{2(3-\gamma-n)} ds. \]

Thus, for all \( K \equiv \gamma + \eta \in (0, 2) \) we conclude that

\[ \| w_R(T) \|_{H_{\text{loc}}}^2 \leq \| w_{0,R} \|^2_{H_{\text{loc}}} + C \int_0^T \left( \| w_R \|^2_{H_{\text{loc}}} + \| w_R \|_{H_{\text{loc}}}^{2(3-\gamma-n)} \right) ds. \]  
(14)

In particular, we have

\[ \| w_R(T) \|_{H_{\text{loc}}}^2 \leq \| w_{0,R} \|^2_{H_{\text{loc}}} + C \int_0^T \left( \| w_R \|^2_{H_{\text{loc}}} + \| w_R \|_{H_{\text{loc}}}^{4(3-\gamma-n)} \right) ds. \]

This completes the proof of Proposition 18.

Since inequality (14) is true for all \( K \in (0, 2) \), it is still verified e.g. for \( K = 1 \) and we get

\[ \| w_R(T) \|_{H_{\text{loc}}}^2 \leq \| w_{0,R} \|^2_{H_{\text{loc}}} + \int_0^T \left( \| w_R \|^2_{H_{\text{loc}}} + \| w_R \|_{H_{\text{loc}}}^4 + C \right) ds. \]

From the previous inequality, we can write

\[ \partial_t \| w_R(t) \|_{H_{\text{loc}}}^2 \leq C(\| \theta_0 \|_{\infty}) \| w_R \|^2_{H_{\text{loc}}} + \| w_R \|_{H_{\text{loc}}}^4 + C. \]

Let us set

\[ \Omega(t) \equiv \| w_R(t) \|_{H_{\text{loc}}}^2 + 1. \]

Then,

\[ \partial_t \Omega(t) = \partial_t \| w_R(t) \|_{H_{\text{loc}}}^2 \leq C(\| \theta_0 \|_{\infty})(\| w_R \|^2_{H_{\text{loc}}} + 1)^2 = C(\| \theta_0 \|_{\infty})(\Omega(t))^2. \]
By integrating, we obtain
\[
\frac{1}{\Omega(0)} - \frac{1}{\Omega(T)} \leq C(\|\theta_0\|_\infty)T,
\]
therefore, we get
\[
\|w_R\|^2_{L^2 H^s_{uloc}} \leq \left( \frac{1}{1 + \|w_0,R\|^2_{H^s_{uloc}}} - C(\|\theta_0\|_\infty)T \right)^{-1}.
\] (15)

Thus, the solutions exist in \(L^\infty H^s_{uloc}\) as long as
\[
T < T^* \equiv \frac{C(\|\theta_0\|_\infty)}{1 + \|w_0,R\|^2_{H^s_{uloc}}}.
\]

Now, we have to show that \(w_R \in (L^2 H^{s+1/2})_{uloc}\). In inequality (13), we have seen that
\[
\|w_R\|^2_{(L^2 H^{s+1/2})_{uloc}} \leq \|w_0,R\|^2_{H^s_{uloc}} + C \int_0^T \|w_R\|^2_{H^s_{uloc}} ds + C \int_0^T \|w_R\|^4_{H^s_{uloc}} ds + CT.
\]

Therefore inequality (15) allows us to conclude that the \((L^2 H^{s+1/2})_{uloc}\) norm of \(w_R\) is controlled for all \(T < C(\|\theta_0\|_\infty)(1 + \|w_0,R\|^2_{H^s_{uloc}})^{-1}\).

Concerning the passage to the weak limit with respect to the parameters \(R\) and \(\epsilon\), we refer to the last section of [21] since we have obtained the same uniform bounds.

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