Decoupled Modified Characteristic Finite Element Method with Different Subdomain Time Steps for Nonstationary Dual-Porosity-Navier-Stokes Model

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Abstract

In this paper, we develop the numerical theory of decoupled modified characteristic finite element method with different subdomain time steps for the mixed stabilized formulation of nonstationary dual-porosity-Navier-Stokes model. Based on partitioned time-stepping methods, the system is decoupled, which means that the Navier-Stokes equations and two different Darcy equations are solved independently at each time step of subdomain. In particular, the Navier-Stokes equations are solved by the modified characteristic finite element method, which overcome the computational difficulties caused by the nonlinear term. In order to increase the efficiency, different time steps are used to different subdomains. The stability of this method is proved. In addition, we verify the optimal $L^2$-norm error convergence order of the solutions by mathematical induction, whose proof implies the uniform $L^\infty$-boundedness of the fully discrete velocity solution. Finally, some numerical tests are presented to show efficiency of the proposed method.

Keywords: nonstationary dual-porosity-Navier-Stokes model; decoupled method; different subdomain time steps; mixed finite element method; modified characteristic finite element method; stability; convergence analysis

1. Introduction

Coupled free flow and porous medium flow systems play an important role in many fields. For example, the flood simulation of arid areas in geological science \cite{1}, filtration treatment in industrial production \cite{2, 3}, petroleum exploitation in mining and blood penetration between vessels and organs in life science \cite{4}.

Usually, the system can be described by a Stokes (Navier-Stokes) coupled Darcy equation. There are a great deal of achievements \cite{5, 6, 7, 8, 9, 10, 11}. However, the Darcy equation is

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a single porosity model, which is not accurate to deal with the complicated multiple porous media similar to naturally fractured reservoir. Actually, the naturally fractured reservoir is comprised of low permeable rock matrix blocks surrounded by an irregular network of natural microfractures. And they have different fluid storage and conductivity properties [12, 13, 14]. In 2016, Hou et al. proposed and numerically solved a coupled dual-porosity-Stokes multi-physics interface system [15] where dual-porosity equations were used to describe the multiple porous media flow. At present, the research on this model can be found in the literature [16, 17, 18, 19]. To our best knowledge, up till now, there has been no research on the dual-porosity-Navier-Stokes model.

For the nonstationary dual-porosity-Navier-Stokes model, it has some features in physical and some difficulties in numerical analysis. As we all know, the fluid velocity in the free flow domain is usually much higher than that in the porous medium. Therefore, it is reasonable to apply different time steps in different subdomains. For coupled free flow and porous media flow with different subdomain time steps, see [20, 21, 22, 23, 24] and the references therein. And for the difficulties, it is a coupling problem. It will be considered to take direct method or decoupled method. In order to reduce the scale of solving problems and increase the reusability of software packages, we choose the decoupled method. When using partitioned time-stepping method, as a simple decoupled strategy, we have to face a difficulty that the artificial energy transfers generated by the interface time-splitting. This results in numerical instabilities [25]. To this point, Nitsches interface method is used to control the artificial energy transfers [26, 27, 28, 29, 17]. On the other hand, we need to consider how to solve the nonlinear term in Navier-Stokes equation. Since the modified characteristic finite element methods could link the nonlinear term to the time term, it can greatly improve efficiency [30, 31, 32]. The method is preferred.

In this paper, we propose and develop the numerical theory of decoupled modified characteristic finite element method with different subdomain time steps for the mixed stabilized formulation of this model. In order to hold the numerical stability, a mesh dependent stabilization term is introduced. The partitioned time-stepping method decomposes the original problem into Navier-Stokes equations and two different Darcy equations. For Navier-Stokes equations, the modified characteristic finite element method is employed to deal with the time and nonlinear terms. And the other Darcy equations are used by mixed finite element method. The stability of this method is proved. In the error analysis framework, proposed in [31, 32], we prove the optimal $L^2$-norm error convergence order by mathematical induction, whose proof implies the uniform $L^\infty$-boundedness of the fully discrete velocity solution. Finally, some numerical tests are presented to show the validity of our theoretical results, especially high efficiency of the proposed method.

The rest of this paper is organized as follows: In Section 2, we introduce the nonstationary dual-porosity-Navier-Stokes model, and construct the fully discrete mixed stabilized decoupled modified characteristic scheme with different subdomain time steps. Stability of the method is presented in Section 3. In Section 4, some preliminaries and convergence analysis are shown. Section 5 reports some numerical examples, and the conclusions are given in Section 6.

2. Modified characteristic finite element method for the model problem

2.1. The nonstationary dual-porosity-Navier-Stokes model

We consider a coupled dual-porosity-Navier-Stokes system on a bounded domain $\Omega = \Omega_c \cup \Omega_d \subset \mathbb{R}^D$, $D = 2, 3$, where $\Omega_c$ and $\Omega_d$ denote disjoint nonoverlapping bounded open
convex regions with common boundary $\mathbb{I} = \overline{\Omega_c} \cap \overline{\Omega_d}$, i.e. $\Omega_c \cap \Omega_d = \emptyset$. See Figure 1.

In the conduit region $\Omega_c$, let $u_c$ denote the fluid velocity, $p_c$ denote the kinematic pressure, $f_c$ denote the external body force density, and $\nu > 0$ is the kinematic viscosity of the fluid. The conduit flow in $\Omega_c$ is assumed to satisfy, for $t \in (0, T]$, the Navier-Stokes system

$$\frac{\partial u_c}{\partial t} - \nu \Delta u_c + \nabla p_c + (u_c \cdot \nabla)u_c = f_c, \quad \text{in } \Omega_c \times (0, T],$$

$$\nabla \cdot u_c = 0, \quad \text{in } \Omega_c \times (0, T].$$

In the dual-porosity region $\Omega_d$, let $u_f$ denote the velocity of microfracture flow, $\phi_f$ denotes the pressure. Accordingly, $u_m$ denotes the velocity in matrix flow and $\phi_m$ denotes the pressure. Then, the flow in the dual-porosity region is assumed to satisfy, for $t \in (0, T]$, the dual-porosity system

$$\eta_f C_{ft} \frac{\partial \phi_f}{\partial t} + \nabla \cdot u_f + \frac{\sigma k_f}{\mu} (\phi_f - \phi_m) = f_d, \quad \text{in } \Omega_d \times (0, T],$$

$$u_f = -\frac{k_f}{\mu} \nabla \phi_f, \quad \text{in } \Omega_d \times (0, T],$$

$$\eta_m C_{mt} \frac{\partial \phi_m}{\partial t} + \nabla \cdot u_m + \frac{\sigma k_m}{\mu} (\phi_m - \phi_f) = 0, \quad \text{in } \Omega_d \times (0, T],$$

$$u_m = -\frac{k_m}{\mu} \nabla \phi_m, \quad \text{in } \Omega_d \times (0, T],$$

where the porosity of the microfracture and matrix region are denoted by $\eta_f$ and $\eta_m$, $C_{ft}$ and $C_{mt}$ are the total compressibility for the matrix and microfractures system, $k_f$ and $k_m$ denote the intrinsic permeability. Additionally, $\sigma$ is the shape factor characterizing the morphology and dimension of the microfractures, $\mu$ is the dynamic viscosity and $f_d$ is a source/sink term. The term $\frac{\sigma k_m}{\mu} (\phi_m - \phi_f)$ describes the mass exchange between the matrix and the microfractures.

Along the interface $\mathbb{I}$, there is a no-exchange situation between the matrix and conduits/microfractures:

$$u_m \cdot n_d = 0,$$
where \( n_d \) denotes the unit outer normal on the interface edges from \( \Omega_d \) to \( \Omega_c \) and \( n_c = -n_d \).

Similar to the Navier-Stokes-Darcy model, the three well-accepted interface conditions are imposed:

- The mass conservation between conduit flow and microfracture flow,
  \[ u_c \cdot n_c + u_f \cdot n_d = 0. \]
- The balance of forces normal,
  \[ \frac{\phi_f}{\rho} = p_c - \nu n_c \cdot \nabla u_c \cdot n_c. \]
- Beavers-Joseph-Saffman (BJS) interface condition,
  \[ -\nu \tau_i \cdot \nabla u_c \cdot n_c = \frac{\alpha \nu \sqrt{D}}{\sqrt{\text{trace}(\Pi)}} (u_c \cdot \tau_i). \]

Here \( \tau_i (i = 1, 2, 3, ..., D - 1) \) denote mutually orthogonal unit tangential vectors along the interface. In addition, \( \rho \) is the density of fluid, \( \alpha \) is constant parameters and \( D \) is the spatial dimension. \( \Pi = k_f \mathbb{I} \) is the intrinsic permeability of microfractures.

For simplicity, except on \( I \), homogeneous Dirichlet boundary conditions are imposed on the boundaries \( \partial \Omega_c \) and \( \partial \Omega_d \), respectively. We have

\[ u_c = 0, \quad \text{on } \partial \Omega_c \setminus \mathbb{I}, \]
\[ u_f \cdot n_d = 0, \quad \text{on } \partial \Omega_d \setminus \mathbb{I}, \]
\[ u_m \cdot n_d = 0, \quad \text{on } \partial \Omega_d \setminus \mathbb{I}, \]

where boundaries \( \partial \Omega_c \setminus \mathbb{I} \) and \( \partial \Omega_d \setminus \mathbb{I} \) are smooth enough and Lipschitzian continuous.

Finally, initial conditions are imposed

\[ u_c(0, x) = u_{c0}(x), \quad \text{in } \Omega_c, \]
\[ \phi_f(0, x) = \phi_{f0}(x), \quad \text{in } \Omega_d, \]
\[ \phi_m(0, x) = \phi_{m0}(x), \quad \text{in } \Omega_d. \]

2.2. The weak formulation of model problem

To begin with, we introduce some notations. For the Sobolev space \( W^{k,p}(\Omega_\Lambda), \Lambda = c \) or \( d \), the integer \( k \geq 0 \) and \( 1 \leq p \leq \infty \). The scalar value function \( \psi \in W^{k,p}(\Omega_\Lambda) \) is equipped with the following norms:

\[ \| \psi \|_{W^{k,p}} = \begin{cases} \left( \sum_{|\beta| \leq k} \int_{\Omega_\Lambda} |D^\beta \psi(x)|^p dx \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \sum_{|\beta| \leq k} \sup_{x \in \Omega_\Lambda} |D^\beta \psi(x)|, & p = \infty, \end{cases} \]

where

\[ D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_D^{\beta_D}}, \]

For the multi-index \( \beta = (\beta_1, ..., \beta_D), \beta_i \geq 0, i = 1, ..., D \) and \( |\beta| = \beta_1 + ... + \beta_D \). The vector value function \( \mathbf{v} = \{v_j\} \in W^{k,p}(\Omega_\Lambda)^D \) is equipped with the norm

\[ \| \mathbf{v} \|_{W^{k,p}} = \left( \sum_{j=1}^D \| v_j \|_{W^{k,p}(\Omega_\Lambda)}^2 \right)^{\frac{1}{2}}. \]
When $p = 2$, $W^{k,2}$ is denoted by $H^k$. Define
\[ \|\psi\|_k := \|\psi\|_{H^k}, \quad \|v\|_k := \|v\|_{H^k}. \]
When $k = 0$, $W^{0,p}$ is denoted by $L^p$. Especially, when $p = 2$,
\[ \|\psi\|_0 := \|\psi\|_{L^2}, \quad \|v\|_0 := \|v\|_{L^2}. \]
In addition, $(\cdot, \cdot)_{\Omega}$ denotes the inner product of $L^2(\Omega)$. Define
\[ H(\text{div}, \Omega_d) := \{ v \in L^2(\Omega_d)^D, \nabla \cdot v \in L^2(\Omega_d) \}, \]
and equip the space with norm
\[ \|v\|_{H(\text{div})} = (\|v\|_0^2 + \|\nabla \cdot v\|_0^2)^{1/2}. \]

Next, we recall some inequalities [33] that are useful in the analysis.

A$_1$. (Poincaré-Friedrichs inequality): For all $v_c \in Y_c$, there exists a positive constant $C_{PF}$ which only depends on the area of $\Omega_c$ such that
\[ \|v_c\|_0 \leq C_{PF} \|\nabla v_c\|_0. \]

A$_2$. (Trace inequality): For all $v_c \in Y_c$, there exists a positive constant $C_T$ which only depends on the area of $\Omega_c$ such that
\[ \|v_c\|_{L^2(\Omega)} \leq C_T \|v_c\|_0^{1/2} \|\nabla v_c\|_0^{1/2}. \]

(Generic trace inequality): Assume that $\Omega_c$ is a bounded area with Lipschitz boundary and $v_c \in H^m(\Omega_c)$. Define the trace function $\gamma_0 v_c, \gamma_1 v_c, \ldots, \gamma_{m-1} v_c$ on $\partial \Omega_c$, where $\gamma_j(0 \leq j \leq m - 1)$ is a linear continuous map from $H^m(\Omega_c)$ to $H^{m-j-\frac{1}{2}}(\partial \Omega_c)$. There exists a constant $C_T$ which only depends on $\Omega_c$ such that
\[ \|v_c\|_{H^{m-j-\frac{1}{2}}(\partial \Omega_c)} \leq C_T \|v_c\|_m. \]

A$_3$. (Properties of $H(\text{div})$ space) For all $v_c \in H(\text{div}, \Omega_c)$ and $v_c \cdot n_d \in H^{-1/2}(\partial \Omega_c)$, there exists a positive constant $C_{div}$ satisfying
\[ \|v_c \cdot n_d\|_{H^{-1/2}(\partial \Omega_c)} \leq C_{div} \|v_c\|_{H(\text{div})}. \]

A$_4$. (Sobolev interpolation inequality)
\[
\begin{align*}
\|\psi\|_{L^q} &\leq C_S \|\psi\|_{1}, \quad q \leq 6, \\
\|\psi\|_{L^\infty} &\leq C_s \|\psi\|_{W^{1,q}}, \quad q > d^*, \\
\|\psi\|_{L^q} &\leq C_q \|\psi\|^\beta \|\psi\|^{1-\beta}_{1}, \quad 2 \leq q \leq 6, \beta = \frac{6-q}{2q}.
\end{align*}
\]

In order to deduce the weak formulation of dual-porosity-Navier-Stokes equations, we introduce the following spaces $(Y_c, Q_c, Y_f, Q_f; Y_m, Q_m)$:
\[
\begin{align*}
Y_c &:= \{ v_c \in H^1(\Omega_c)^D : v_c = 0 \text{ on } \partial \Omega_c \setminus \mathbb{I} \}, \\
Q_c &:= L^2_0(\Omega_c) := \{ q \in L^2(\Omega_c) : \int_{\Omega_c} q dx = 0 \}, \\
Y_f &:= \{ v_f \in H(\text{div}, \Omega_d) : v_f \cdot n_d = 0 \text{ on } \partial \Omega_d \setminus \mathbb{I} \}, \\
Q_f &:= L^2_0(\Omega_d) := \{ \psi_f \in L^2(\Omega_d) : \int_{\Omega_d} \psi_f dx = 0 \}, \\
Y_m &:= \{ v_m \in H(\text{div}, \Omega_d) : v_m \cdot n_d = 0 \text{ on } \partial \Omega_d \}, \\
Q_m &:= L^2_0(\Omega_d) := \{ \psi_m \in L^2(\Omega_d) : \int_{\Omega_d} \psi_m dx = 0 \}.
\end{align*}
\]
For simplicity, define the product space as follows:

\[ X := Y_c \times Q_c \times Y_f \times Q_f \times Y_m \times Q_m. \]

Furthermore, define the involving time Sobolev space

\[ X_T := L^2(0, T; X). \]

In weak formulation, the interface term \( \frac{1}{\rho} \int_{\Omega} \phi_f (v_c - v_f) \cdot n_d ds \) is difficult to control in numerical calculation. According to \([28, 17]\), in order to overcome this difficulty of numerical instability which generated by the interface time-splitting, take the Nitsche’s interface numerical calculation. According to \([28, 17]\), in order to overcome this difficulty of numerical instability which generated by the interface time-splitting, take the Nitsche’s interface numerical calculation. According to \([28, 17]\), in order to overcome this difficulty of numerical instability which generated by the interface time-splitting, take the Nitsche’s interface numerical calculation. According to \([28, 17]\), in order to overcome this difficulty of numerical instability which generated by the interface time-splitting, take the Nitsche’s interface numerical calculation. According to \([28, 17]\), in order to overcome this difficulty of numerical instability which generated by the interface time-splitting, take the Nitsche’s interface numerical calculation. According to \([28, 17]\), in order to overcome this difficulty of numerical instability which generated by the interface time-splitting, take the Nitsche’s interface numerical calculation. According to \([28, 17]\), in order to overcome this difficulty of numerical instability which generated by the interface time-splitting, take the Nitsche’s interface numerical calculation. According to \([28, 17]\), in order to overcome this difficulty of numerical instability which generated by the interface time-splitting, take the Nitsche’s interface numerical calculation. According to \([28, 17]\), in order to overcome this difficulty of numerical instability which generated by the interface time-splitting, take the Nitsche’s interface numerical calculation. According to \([28, 17]\), in order to overcome this difficulty of numerical instability which generated by the interface time-splitting, take the Nitsche’s interface numerical calculation.

The dual-porosity-Navier-Stokes weak formulation is as follows. Assuming that \( f_e \in L^2(0, T; H^{-1}(\Omega_c)^D) \), \( f_d \in L^2(0, T; L^2(\Omega_d)) \), for all \((v_c, q; v_f, \psi_f; v_m, \psi_m) \in X\), find \((u_c, p_c; u_f, \phi_f; u_m, \phi_m) \in X_T\) satisfying

\[
\left( \frac{\partial u_c}{\partial t}, v_c \right)_{\Omega_c} + \eta_d C_{\text{dt}} \left( \frac{\partial \phi_d}{\partial t}, \psi_d \right)_{\Omega_d} + a_{\Omega_c}(u_c, v_c) + ((u_c \cdot \nabla)u_c, v_c)_{\Omega_c} - b(v_c, p_c) + b(u_c, q) \\
+ a_{\phi_d}(\phi_d, \psi_d) + a_{ud}(u_d, v_d) + b_{\phi_d}(u_d, \psi_d) - b_{\phi_d}(v_d, \phi_d) \\
- \frac{1}{\rho} \int_{\Omega} \phi_f (v_c - v_f) \cdot n_d ds + \frac{\gamma}{\rho h} \int_{\Omega} (u_c - u_f) \cdot n_d ds \\
= (f_c, v_c)_{\Omega_c} + \frac{1}{\rho}(f_d, \psi_f)_{\Omega_d},
\]

(2.1)

where

\[
[u_f, u_m]^T = u_d, \quad [v_f, v_m]^T = v_d, \quad [\phi_f, \phi_m]^T = \phi_d, \quad [\psi_f, \psi_m]^T = \psi_d;
\]

\[
[\eta_f, \eta_m]^T = \eta_d, \quad [C_{ft}, C_{mt}]^T = C_{dt};
\]

\[
\frac{\eta_d C_{dt}}{\rho} (\frac{\partial \phi_d}{\partial t}, \psi_d)_{\Omega_d} = \frac{\eta_d C_{dt}}{\rho} (\frac{\partial \phi_f}{\partial t}, \psi_f)_{\Omega_d} + \frac{\eta_d C_{mt}}{\rho} (\frac{\partial \phi_m}{\partial t}, \psi_m)_{\Omega_d},
\]

\[
a_{\Omega_c}(u_c, v_c) = \nu(\nabla u_c, \nabla v_c)_{\Omega_c} + \sum_{i=1}^{D-1} \int \frac{\alpha_{\nu} \sqrt{D}}{\text{trace}(\Pi)} (u_c \cdot \tau_i) (v_c \cdot \tau_i) ds,
\]

\[
a_{\phi_d}(\phi_d, \psi_d) = \frac{\sigma k_m}{\rho \mu} (\phi_d - \phi_m, \psi_f)_{\Omega_d} + \frac{\sigma k_m}{\rho \mu} (\phi_f - \phi_m, \psi_f)_{\Omega_d},
\]

\[
a_{ud}(u_d, v_d) = \frac{1}{\rho} (\mu k_f^{-1} u_f, \psi_f)_{\Omega_d} + \frac{1}{\rho} (\mu k_m^{-1} u_m, v_m)_{\Omega_d},
\]

\[
b_{\phi_d}(u_d, \psi_d) = \frac{1}{\rho} (\nabla \cdot u_f, \psi_f)_{\Omega_d} + \frac{1}{\rho} (\nabla \cdot u_m, \psi_m)_{\Omega_d},
\]

\[
b(v_c, p_c) = (p_c, \nabla \cdot v_c)_{\Omega_c}.
\]

As we all know, the velocity in the conduit flow is faster than that in the dual-porosity flow, therefore it is reasonable to apply different time steps in different subdomain. For \( \Omega_c \),
divide the time interval $[0, T]$ into $N > 0$ averagely. For the segments $[t_n, t_{n+1}](n = 0, 1, ..., N − 1)$ satisfying

$$0 = t_0 ≤ t_1 ≤ ... ≤ t_{N−1} ≤ t_N = T, \quad t_n = nΔt,$$

and the time step is $Δt = \frac{T}{N}$.

![Figure 2: Different time steps in different subdomain](image)

For $\Omega_d$, divide the time interval $[0, T]$ into $M > 0$ averagely. For the segments $[t_{nk}, t_{nk+1}](k = 0, 1, ..., M − 1)$ satisfying

$$0 = t_{n0} ≤ t_{n1} ≤ ... ≤ t_{nM−1} ≤ t_n = kΔs,$$

and the time step is $Δs = \frac{T}{M}$. Note that time level in $\Omega_c$ and that in $\Omega_d$ are nested, i.e. $N = rM$, where integer time step ratio $r = \frac{N}{M}$.

Following [31, 32], we have

$$\left(\frac{∂u_c}{∂t} + u_c \cdot \nabla u_c\right)\bigg|_{t=n+1} ≈ \frac{u_c(t_{n+1}) - \bar{u}_c(t_n)}{Δt},$$

where $\bar{u}_c(t_n) = u_c(\bar{x}, t_n), \bar{x} = x - u_c(t_n)Δt$.

Therefore, we obtain the characteristic version of variational formulation.

### A. The characteristic version of variational formulation:

$$
(du_c(t_{n+1}), v_c)_{Ω_c} + \frac{η_dC_d}{ρ}(du_d(t_{n+1}), ψ_d)_{Ω_d} + a_{Ω_c}(u_c(t_{n+1}), v_c) - b(v_c, p_c(t_{n+1}))
+b(u_c(t_{n+1}), q) + a_{φ_d}(φ_d(t_{n+1}), ψ_d) + a_{u_d}(u_d(t_{n+1}), v_d) + b_{φ_d}(u_d(t_{n+1}), ψ_d)
− b_{φ_d}(v_d, φ_d(t_{n+1})) − \left(\frac{γ}{ρh}\int(\mathbf{u}_c(t_{n+1}) - \mathbf{u}_f(t_{n+1})) \cdot \mathbf{n}_d ds\right)
\cdot \mathbf{n}_d =
\left(f_c(t_{n+1}), v_c\right)_{Ω_c} \quad \text{for} \quad (\mathbf{p}_{tr}^{n+1}, \mathbf{v}_c)_{Ω_c}
\quad \text{and}
\quad \left(\frac{η_dC_d}{ρ}(du_d(t_{n+1}), \frac{∂φ_d}{∂t}(t_{n+1}), ψ_d)\right)_{Ω_d},
$$

where $d_t \mathbf{u}_c(t_{n+1}) = \frac{u_c(t_{n+1}) - u_c(t_n)}{Δt}, \mathbf{p}_{tr}^{n+1} = \frac{u_c(t_{n+1}) - \bar{u}_c(t_n)}{Δt} - \frac{∂u_c}{∂t}(t_{n+1}) − (u_c(t_{n+1}) \cdot \nabla)u_c(t_{n+1})$.

We also introduce the discrete Gronwall lemma [34].
Lemma 2.1. Assume that $E \geq 0$, for any integer $M \geq 0$, $\kappa_m, A_m, B_m, C_m \geq 0$ satisfying
\[
A_M + \Delta t \sum_{m=0}^{M} B_m \leq \Delta t \sum_{m=0}^{M} \kappa_m A_m + \Delta t \sum_{m=0}^{M} C_m + E.
\]
For all $m$, assume that
\[
\kappa_m \Delta t < 1,
\]
and set $g_m = (1 - \kappa_m \Delta t)^{-1}$, then
\[
A_M + \Delta t \sum_{m=0}^{M} B_m \leq \exp(\Delta t \sum_{m=0}^{M} g_m \kappa_m)(\Delta t \sum_{m=0}^{M} C_m + E).
\]

2.3. The discrete formulation of model problem

Let $T_h$ be a uniform simplex partition of $\overline{\Omega}_c \cup \overline{\Omega}_d$, and $h := \{\max_{K \in T_h} h_K : h_K = \text{diam}(K)\}$. $T_h^e$ and $T_h^d$ denote the partition of subdomain $\Omega_c$ and $\Omega_d$, respectively. Furthermore, the partition matches along the interface $\Gamma$, that is to say, there is no hanging point on $\Gamma$. If $D = 2$, adjacent elements share the same edge; If $D = 3$, adjacent elements share the same face.

Choose the finite element space $(Y_c^h, Q_c^h)$ for Navier-Stokes model satisfying the velocity-pressure inf-sup condition: there exists $\chi_c$ independent of $h$, such that
\[
Y_c^h \subset Y_c, Q_c^h \subset Q_c, \quad \inf_{0 \neq q_c \in Q_c^h} \sup_{0 \neq v_c \in Y_c^h} \frac{\langle q_c, \nabla \cdot v_c \rangle_{\Omega_c}}{\|\nabla v_c^h\|_0} \geq \chi_c.
\]

Define a discrete divergence-free velocity space
\[
V_c^h = \{v_c \in Y_c^h : \langle q, \nabla \cdot v_c \rangle_{\Omega_c} = 0, \forall q \in Q_c^h\}.
\]

For dual-porosity model, choose the finite element space $(Y_f^h, Q_f^h)$ also satisfying the velocity-pressure inf-sup condition: there exists a positive constant $\chi_f$, for all $\psi_f^h \in Q_f^h$, we have
\[
Y_f^h \subset Y_f, Q_f^h \subset Q_f, \quad \sup_{0 \neq \psi_f \in Y_f^h} \frac{\langle \psi_f^h, \nabla \cdot v_f^h \rangle_{\Omega_d}}{\|v_f^h\|_{H(\text{div},\Omega_d)}} \geq \chi_f \|\psi_f^h\|_0.
\]

Similarly, for all $\psi_m^h \in Q_m^h$, there exists a positive constant $\chi_m$, we have
\[
Y_m^h \subset Y_m, Q_m^h \subset Q_m, \quad \sup_{0 \neq \psi_m \in Y_m^h} \frac{\langle \psi_m^h, \nabla \cdot v_m^h \rangle_{\Omega_d}}{\|v_m^h\|_{H(\text{div},\Omega_d)}} \geq \chi_m \|\psi_m^h\|_0.
\]

In the next place, we introduce some inequalities [33] and lemmas [35] that may be used in discrete spaces.

$B_1$. (The inverse inequality) When $1 \leq p, q \leq \infty, 0 \leq l \leq k$,
\[
\|u_c^h\|_{W^{k,p}} \leq C_{inv} h^{-\max\{0, \frac{k-l}{2}\}} h^{l-k} \|u_c^h\|_{W^{l,q}}, \quad \forall u_c^h \in Y_c^h.
\]
B2. (The trace inverse inequality) For all $\psi_f^h \in Q_f^h$, we have
$$\|\psi_f^h\|_{L^2(\Omega)} \leq \hat{C}_{inv} h^{-1/2}\|\psi_f^h\|_0, \quad \forall \psi_f^h \in Q_f^h.$$  
\hfill (2.4)

B3. (The discrete Sobolev inequality [36, 33]) There exists $C_{DS} > 0$ such that for all $u_c^h \in Y_c^h$, the following inequalities hold:
$$\|u_c^h\|_{0,\infty} \leq C_{DS}(1 + |\ln(h)|)^{1/2}\|u_c^h\|_1, \quad \text{in } d = 2,$$
$$\|u_c^h\|_{0,\infty} \leq C_{DS} h^{-\frac{1}{2}}\|u_c^h\|_1, \quad \text{in } d = 3.$$  
\hfill (2.5)

**Lemma 2.2 ([35, 37]).** Assume that $u_c^h \in W^{1,\infty}$, for all $u_c^h \in H_0^1(\Omega_c)$, we have
$$(\hat{u}_c^h, u_c^h)_{\Omega_c} - (u_c^h, \hat{u}_c^h)_{\Omega_c} \leq \hat{C}\Delta t (u_c^h, u_c^h)_{\Omega_c},$$
where $\hat{u}_c^h = u_c^h(x - u_c^h\Delta t)$ and $\hat{C}$ is a constant which is independent of the spatial and temporal grid sizes $h$ and $\Delta t$.

For simplicity, for $t_{n+1}, t_{n+k+1} \in [0, T]$, $n_{k+1} = kr$, we use $(u_c^{n+1}, p^{n+1}; u_f^{n+1}, \phi_f^{n+1}; u_m^{n+1}, \phi_m^{n+1})$ denotes $(u_c^{h,n+1}, p^{h,n+1}; u_f^{h,n+1}, \phi_f^{h,n+1}; u_m^{h,n+1}, \phi_m^{h,n+1})$. Define the following product space
$$\mathcal{X}^h := Y_c^h \times Q_f^h \times Y_f^h \times Q_f^h \times Y_m^h \times Q_m^h \subset \mathcal{X}.$$  
And define the $L^2$ bounded linear projection operator $P^h : \mathcal{X} \to \mathcal{X}^h$.

For the formulation A, we take the mixed finite element method for space, the backward-Euler discretization in time and the decoupled approach by partition time-stepping method. These yield that

**B. The fully discrete decoupled modified characteristic scheme**

Given $u_c^0 = P_c^h u_c^0, p_m^0 = P_m^h \phi_m^0, \phi_f^0 = P_f^h \phi_f^0$, for all $(v_c^h, \psi^h; v_f^h, \psi_f^h; v_m^h, \psi_m^h) \in \mathcal{X}^h$, find $(u_c^{n+1}, p_c^{n+1}; u_f^{n+1}, \phi_f^{n+1}; u_m^{n+1}, \phi_m^{n+1}) \in \mathcal{X}^h$, $n = n_k, n_k + 1, ..., n_{k+1} - 1$ such that the following formulations are established.

- **Step 1**

\begin{align*}
&\left( d_t u_c^{n+1}, v_c^h \right)_{\Omega_c} + a_{\Omega_c}(u_c^{n+1}, v_c^h) = b(v_c^h, p_c^{n+1}) + b(u_c^{n+1}, \psi^h) - \frac{1}{\rho} \int f_c^{n+1}(v_c^h, \psi_c^h)(n_d) ds \\
&+ \frac{\mu}{\rho h} \int (u_c^{n+1} - u_f^{n+1}) \cdot n_d) (v_c^h \cdot n_d) ds = (f_c(t_{n+1}), v_c^h)_{\Omega_c} - \left( \frac{u_c^n - u_c^n}{\Delta t}, v_c^h \right)_{\Omega_c} ,
\end{align*}
\hfill (2.6)

where $d_t u_c^{n+1} = \frac{u_c^{n+1} - u_c^n}{\Delta t}$, $\hat{u}_c^n = u_c^n(\hat{x})$, $\hat{x} = x - u_c^n\Delta t$, and $\Delta t$ is a small step in $\Omega_c$.

- **Step 2**

\begin{align*}
&\frac{\eta_m C_{mt}}{\rho} \left( d_s \phi_m^{n+1}, \psi_m^h \right)_{\Omega_d} + \frac{1}{\rho} \left( \nabla \cdot u_m^{n+1}, \psi_m^h \right)_{\Omega_d} + \frac{1}{\rho} (\mu k_m^{n-1} u_m^{n+1}, v_m^h)_{\Omega_d} \\
&- \frac{1}{\rho} \left( \phi_m^{n+1}, \nabla \cdot v_m^h \right)_{\Omega_d} + \frac{\sigma k_m}{\rho \mu} (\phi_m^{n+1} - \phi_f^{n_k}, \psi_m^h)_{\Omega_d} = 0,
\end{align*}
\hfill (2.7)

where $d_s \phi_m^{n+1} = \frac{\phi_m^{n+1} - \phi_m^n}{\Delta s}$ and $\Delta s$ is a large step in $\Omega_d$. 

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Step 3

\[ \frac{\eta_f C_{ft}}{\rho} (d_s \phi_f^{n_{k+1}}, \psi_f^h)_{\Omega_d} + \frac{1}{\rho} (\nabla \cdot u_f^{n_{k+1}}, \psi_f^h)_{\Omega_d} + \frac{1}{\rho} (\mu k_f^{-1} u_f^{n_{k+1}}, v_f^h)_{\Omega_d} \]

\[ - \frac{1}{\rho} (\phi_f^{n_k+1}, \nabla \cdot v_f^h)_{\Omega_d} + \frac{\sigma k_m (\phi_f^{n_{k+1}} - \phi_m^{n_k}, v_f^h)_{\Omega_d} + \frac{1}{\rho} \int_0^1 \phi_f^{n_{k+1}} v_f^h \cdot n_d ds \]

\[ - \frac{\gamma}{\rho h} \int_0^1 ((\mathbf{S}^{n_{k+1}} - u_f^{n_{k+1}}) \cdot n_d) (v_f^h \cdot n_d) ds = \frac{1}{\rho} (f_d(t_{n_{k+1}}), \psi_f^h)_{\Omega_d}. \quad (2.8) \]

where \( d_s \phi_f^{n_{k+1}} = \frac{\phi_f^{n_{k+1}} - \phi_f^{n_k}}{\Delta s} \) and \( S^{n_{k+1}} = \frac{1}{r} \sum_{n_{k+1}} u_c^{n_{k+1}}. \)

**Remark 2.1.** Note that Step 1 and Step 2 can be calculated at the same time. In the following numerical experiments, we take parallel algorithm for Step 1 and Step 2. By this means, we can improve the computing efficiency.

### 3. Stability of the method

Hereafter, \( C > 0 \) denotes a generic constant whose value may be different from place to place, but which is independent of the spatial and temporal grid sizes \( h \) and \( \Delta t \), respectively.

**Theorem 3.1.** If \( f_c \in L^2(0, T; H^{-1}(\Omega_c)), f_d \in L^2(0, T; L^2(\Omega_d)) \). Assume that \( \bar{C} \Delta t < 1 \), we get the stability for the fluid velocity in the first large time interval \([0, t_{n_1}]\), for any \( 0 < J \leq r - 1 \)

\[ \| u_c^{J+1} \|_0^2 + \frac{\nu \Delta t}{2} \sum_{n=0}^J \| \nabla u_c^{n+1} \|_{\Omega_d}^2 + \frac{2 \alpha \nu \Delta t}{\sqrt{k}} \sum_{n=0}^J \| P_r(u_c^{n+1}) \|_{L^2(\Omega_d)}^2 \]

\[ + \frac{\gamma \Delta t}{\rho h} \sum_{n=0}^J \| (u_c^{n+1} - u_c^{n_k}) \cdot n_d \|_{L^2(\Omega_d)}^2 + \frac{\gamma \Delta t}{\rho h} \sum_{n=0}^J \| u_c^{n+1} \cdot n_d \|_{L^2(\Omega_d)}^2 \]

\[ \leq C \left( \frac{2 C_P^2 \rho \Delta t}{\nu} \sum_{n=0}^J \| f_c(t_{n+1}) \|_{H^{-1}(\Omega_c)}^2 + \frac{C_P^2 C_m^2 \bar{C} \rho \Delta t}{\nu} \sum_{n=0}^J \| \phi_f^0 \|_0^2 + \frac{\gamma \Delta t}{\rho h} \sum_{n=0}^J \| u_f \cdot n_d \|_{L^2(\Omega_d)}^2 + \| u_c^0 \|_{0}^2 \right). \]

On the other hand, under the condition

\[ \gamma \geq \frac{4C^2_{t_{in}}}{\eta_f C_{ft}} \left( \frac{\Delta t}{1 - \Delta t - 2\bar{C} \Delta t} \right), \]

we obtain the stability for the time interval \([0, t_{n_M}]\)

\[ \| u_c^{n_M} \|_0^2 + \nu \Delta t \sum_{k=0}^{M-1} \sum_{n=n_k}^{n_{k+1}-1} \| \nabla u_c^{n+1} \|_{\Omega_d}^2 + \frac{2 \alpha \nu \Delta t}{\sqrt{k}} \sum_{k=0}^{M-1} \sum_{n=n_k}^{n_{k+1}-1} \| P_r(u_c^{n+1}) \|_{L^2(\Omega_d)}^2 \]

\[ + \frac{\gamma \Delta t}{\rho h} \sum_{k=0}^{M-1} \sum_{n=n_k}^{n_{k+1}-1} \| (u_c^{n+1} - u_c^{n_k}) \cdot n_d \|_{L^2(\Omega_d)}^2 + \frac{\gamma \Delta s}{\rho k_f} \sum_{k=0}^{M-1} \| u_f^{n_{k+1}} \|_{0}^2 \]

\[ + \frac{\eta_m C_{mt}}{\rho} \left( \| \phi_m^{n_{M}} \|_0^2 + \sum_{k=0}^{M-1} \| \phi_m^{n_{k+1}} - \phi_m^{n_k} \|_0^2 \right) + \frac{\eta_f C_{ft}}{\rho} \left( \| \phi_f^{n_{M}} \|_0^2 + \sum_{k=0}^{M-1} \| \phi_f^{n_{k+1}} - \phi_f^{n_k} \|_0^2 \right). \]
Combining (3.2) (3.3) and (3.4), we obtain

\[ v \]

Taking \( v^h = 2\Delta t u_c^{n+1} \) and \( q^h = 2\Delta t p_c^{n+1} \) in (2.6), sum over \( n = n_k, \ldots, n_{k+1} - 1, \)

\[
2\Delta t \sum_{n=n_k}^{n_{k+1}-1} \| \nabla u_c^{n+1} \|^2_0 + \frac{2\alpha \Delta t}{\sqrt{k}} \| P_r(u_c^{n+1}) \|^2_{L^2(\Omega)} = 2\Delta t \sum_{n=n_k}^{n_{k+1}-1} (f_c(t_{n+1}), \nabla u_c^{n+1})_\Omega_c - 2\Delta t \sum_{n=n_k}^{n_{k+1}-1} \left( \frac{u_c^{n+1} - \hat{u}_c^n}{\Delta t} \right) \cdot (\nabla u_c^{n+1})_\Omega_c
\]

Taking \( v^m = 2\Delta s u_m^{n+1} \) and \( \psi^m = 2\Delta s \phi_m^{n+1} \) in (2.7), we obtain

\[
\sum_{n=n_k}^{n_{k+1}-1} \eta_m C_{mt} (\| \phi_m^{n+1} \|^2_0 - \| \phi_m^n \|^2_0) + \| \phi_m^{n+1} - \phi_m^n \|^2_0 = 2\mu \Delta s \| u_m^{n+1} \|^2_0
\]

Taking \( v^f = 2\Delta s u_f^{n+1} \) and \( \psi^f = 2\Delta s \phi_f^{n+1} \) in (2.8), we have

\[
\sum_{n=n_k}^{n_{k+1}-1} \eta_f C_{ft} (\| \phi_f^{n+1} \|^2_0 - \| \phi_f^n \|^2_0) + \| \phi_f^{n+1} - \phi_f^n \|^2_0 = \frac{2\mu \Delta s}{\rho k_f} \| u_f^{n+1} \|^2_0
\]

Combining (3.2) (3.3) and (3.4), we obtain

\[
2\Delta t \sum_{n=n_k}^{n_{k+1}-1} \| \nabla u_c^{n+1} \|^2_0 + \frac{2\alpha \Delta t}{\sqrt{k}} \| P_r(u_c^{n+1}) \|^2_{L^2(\Omega)} = 2\Delta t \sum_{n=n_k}^{n_{k+1}-1} (f_c(t_{n+1}), \nabla u_c^{n+1})_\Omega_c + \frac{\eta_m C_{mt}}{\rho} (\| \phi_m^{n+1} \|^2_0 - \| \phi_m^n \|^2_0) + \| \phi_m^{n+1} - \phi_m^n \|^2_0)
\]

\[
+ \frac{2\mu \Delta s}{\rho k_f} \| u_f^{n+1} \|^2_0 + \frac{2\mu \Delta s}{\rho k_m} \| u_m^{n+1} \|^2_0
\]

\[
= [2\Delta t \sum_{n=n_k}^{n_{k+1}-1} (f_c(t_{n+1}), \nabla u_c^{n+1})_\Omega_c + \frac{2\Delta s}{\rho} (f_d(t_{n+1}), \phi_f^{n+1})_\Omega_d] - 2\Delta t \sum_{n=n_k}^{n_{k+1}-1} \left( \frac{u_c^{n+1} - \hat{u}_c^n}{\Delta t} \right) \cdot (\nabla u_c^{n+1})_\Omega_c
\]
\[ + \frac{2\Delta t}{\rho} \sum_{n=n_k}^{n_{k+1}-1} \int \phi_f^n (u_c^{n+1} - u_f^{n+1}) \cdot n_d ds + \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_{k+1}-1} \int ((u_c^{n+1} - u_f^{n+1}) \cdot n_d) (u_f^{n+1} \cdot n_d) ds \]

\[ - \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_{k+1}-1} \int ((u_c^{n+1} - u_f^{n_k}) \cdot n_d) (u_c^{n+1} \cdot n_d) ds \]

\[ := \sum_{i=1}^{4} I_i. \tag{3.5} \]

For \( I_1 \), we can split it into the following form.

\[ I_1 = 2\Delta t \sum_{n=n_k}^{n_{k+1}-1} (f_c(t_{n+1}), u_c^{n+1}) \Omega_c + \frac{2\Delta s}{\rho} (f_d(t_{n+1}), \phi_f^{n+1}) \Omega_d \]

\[ := I_{11} + I_{12}. \]

The two terms are bounded by Hölder inequality, Poincaré-Friedriches inequality and Young inequality,

\[ I_{11} \leq 2CP_F \Delta t \sum_{n=n_k}^{n_{k+1}-1} \| f_c(t_{n+1}) \|_{H^{-1}} \| \nabla u_c^{n+1} \|_0 \]

\[ \leq \frac{C^2_P F \Delta t}{\nu} \sum_{n=n_k}^{n_{k+1}-1} \| f_c(t_{n+1}) \|_{H^{-1}}^2 + \nu \Delta t \sum_{n=n_k}^{n_{k+1}-1} \| \nabla u_c^{n+1} \|_0^2, \]

and

\[ I_{12} \leq \frac{2\Delta s}{\rho n_f C_f t} \| f_d(t_{n+1}) \|_0^2 + \frac{\eta_f C_f t \Delta s}{2\rho} \| \phi_f^{n+1} \|_0^2. \]

For the first term on the right-hand side in (3.5),

\[ I_1 \leq \nu \Delta t \sum_{n=n_k}^{n_{k+1}-1} \| \nabla u_c^{n+1} \|_0 + \frac{C^2_P F \Delta t}{\nu} \sum_{n=n_k}^{n_{k+1}-1} \| f_c(t_{n+1}) \|_{H^{-1}}^2 + \frac{2\Delta s}{\rho n_f C_f t} \| f_d(t_{n+1}) \|_0^2 + \frac{\eta_f C_f t \Delta s}{2\rho} \| \phi_f^{n+1} \|_0^2. \]

Applying the lemma 2.2, the term

\[ -2\Delta t \left( \frac{u_c^{n+1} - \hat{u}_c^n}{\Delta t}, u_c^{n+1} \right) \Omega_c \]

\[ = - \left( u_c^{n+1} - \hat{u}_c^n, u_c^{n+1} + \hat{u}_c^n - \hat{u}_c^n \right) \Omega_c \]

\[ \leq - \left( u_c^{n+1}, u_c^{n+1} \right) \Omega_c - \left( \hat{u}_c^n, \hat{u}_c^n \right) \Omega_c \]

\[ = - \left( \| u_c^{n+1} \|_0^2 - \| u_c^n \|_0^2 + \| u_c^n \|_0^2 - \| u_c^n \|_0^2 \right) \Omega_c - \left( \hat{u}_c^n, \hat{u}_c^n \right) \Omega_c \]

\[ \leq - \left( \| u_c^{n+1} \|_0^2 - \| u_c^n \|_0^2 \right) + \hat{C} \Delta t \| u_c^n \|_0^2. \]

The second term on the right-hand in (3.5) is bounded by

\[ I_2 \leq - \sum_{n=n_k}^{n_{k+1}-1} \left[ \| u_c^{n+1} \|_0^2 - \| u_c^n \|_0^2 \right] + \hat{C} \Delta t \sum_{n=n_k}^{n_{k+1}-1} \| u_c^n \|_0^2. \]
Using the Cauchy-Schwarz inequality, trace inverse inequality and the Young inequality, we show that

\[
I_3 \leq \frac{2C_m^2}{\rho \gamma} \| \phi_f^{n_k} \|^2_0 + \frac{\gamma \Delta t}{2 \rho h} \sum_{n=n_k}^{n_{k+1}-1} \| (u_c^{n+1} - u_f^{n+1}) \cdot n_d \|^2_{L^2(\Omega)}.
\]

For \( I_4 \), the identity transformation is carried out.

\[
I_4 = \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_{k+1}-1} \int_{\Omega} ((u_c^{n+1} - u_f^{n+1}) \cdot n_d)(u_f^{n+1} \cdot n_d)ds
- \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_{k+1}-1} \int_{\Omega} ((u_c^{n+1} - u_f^{n+1}) \cdot n_d)^2 ds
- \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_{k+1}-1} \int_{\Omega} ((u_f^{n+1} - u_c^{n+1}) \cdot n_d)(u_f^{n+1} \cdot n_d)ds
- \int_{\Omega} ((u_f^{n+1} - u_c^{n+1}) \cdot n_d)((u_f^{n+1} - u_c^{n+1}) \cdot n_d)ds
\]

Since

\[
\int_{\Omega} ((u_f^{n+1} - u_c^{n+1}) \cdot n_d)((u_f^{n+1} - u_c^{n+1}) \cdot n_d)ds
\]

\[
\leq \frac{1}{2} \| (u_f^{n+1} - u_c^{n+1}) \cdot n_d \|^2_{L^2(\Omega)} + \frac{1}{2} \| (u_f^{n+1} - u_c^{n+1}) \cdot n_d \|^2_{L^2(\Omega)}
\]

\[
\leq \frac{1}{2} \| u_f^{n+1} \cdot n_d \|^2_{L^2(\Omega)} + \frac{1}{2} \| u_f^{n+1} \cdot n_d \|^2_{L^2(\Omega)} - \int_{\Omega} u_f^{n+1} \cdot n_d u_f^{n+1} \cdot n_d ds + \frac{1}{2} \| (u_f^{n+1} - u_c^{n+1}) \cdot n_d \|^2_{L^2(\Omega)},
\]

then the fourth term \( I_4 \) on the right-hand in (3.5) is yielded that

\[
I_4 \leq \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_{k+1}-1} \| (u_c^{n+1} - u_f^{n+1}) \cdot n_d \|^2_{L^2(\Omega)} - \gamma \Delta t \sum_{n=n_k}^{n_{k+1}-1} (\| u_f^{n+1} \cdot n_d \|^2_{L^2(\Omega)}
- \| u_f^{n+1} \cdot n_d \|^2_{L^2(\Omega)}).
\]

Combining above inequalities, summing over \( k = 0, \ldots, M-1 \) and using the discrete Growall lemma, we get the stability in the time interval \([0, t_{nM}]\)

\[
\| u_c^{nM} \|^2_0 + \nu \Delta t \sum_{k=0}^{M-1} \sum_{n_n_k} \| \nabla u_c^{n+1} \|^2_0 + \frac{2\Delta t}{\sqrt{k}} \sum_{k=0}^{M-1} \sum_{n=n_k} \| P u_c^{n+1} \|^2_{L^2(\Omega)} + \frac{2\mu \Delta s}{\rho k_m} \sum_{k=0}^{M-1} \sum_{n=n_k} \| u^{n+1} \|^2_{L^2(\Omega)}
+ \frac{\gamma \Delta t}{2 \rho h} \sum_{k=0}^{M-1} \sum_{n=n_k} \| (u_c^{n+1} - u_f^{n+1}) \cdot n_d \|^2_{L^2(\Omega)} + \frac{\gamma \Delta s}{\rho h} \sum_{k=0}^{M-1} \sum_{n=n_k} \| u_f^{n+1} \cdot n_d \|^2_{L^2(\Omega)} + \frac{2\mu \Delta s}{\rho k_f} \sum_{k=0}^{M-1} \sum_{n=n_k} \| u^{n+1} \|^2_{L^2(\Omega)}
\]

\[
+ \frac{\eta M C_m}{\rho} \left( \| \phi_m^{nM} \|^2_0 + \sum_{k=0}^{M-1} \| \phi_m^{n+1} - \phi_m^{n+k} \|^2_0 \right) + \frac{\eta f C_f}{\rho} \left( \| \phi_f^{nM} \|^2_0 + \sum_{k=0}^{M-1} \| \phi_f^{n+1} - \phi_f^{n+k} \|^2_0 \right)
\]

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\[ + \frac{\sigma_k \Delta t}{\rho \mu} \left( \| \phi_m^{n+1} \|^2 + \sum_{k=0}^{M-1} \| \phi_m^{n+k+1} - \phi_m^{n} \|^2 + \| \phi_f^{n+k+1} \|^2 + \sum_{k=0}^{M-1} \| \phi_f^{n+k+1} - \phi_f^{n} \|^2 \right) \]

\[ \leq C \left( \frac{C_P^2 \Delta t}{\nu} \sum_{k=0}^{M-1} \sum_{n=n_k}^{n_{k+1}-1} \| f_c(t_{n+1}) \|^2_{H^{-1}} + \frac{2\Delta t}{\rho \eta_f C_{ft}} \sum_{k=0}^{M-1} \sum_{n=n_k}^{n_{k+1}-1} \| f_d(t_{n+1}) \|^2_{H^{-1}} + \frac{\gamma \Delta t}{\rho h} \| u_f^0 \cdot n_d \|^2_{L^2(I)} \right) + \| u_0^0 \|^2 + \frac{\eta_m \rho C_m}{\rho} \| \phi_m^{n+1} \|^2 + \frac{\eta_f C_{ft}}{\rho} \| \phi_f^{n+1} \|^2 + \| \sigma_k \Delta t \|_{\Omega} \left( \| \phi_m^{n+1} \|^2 + \| \phi_f^{n+1} \|^2 \right). \]

with the condition \( \left( \frac{\eta_f C_{ft}}{\rho} + \frac{2C_{inv}^2 \rho}{\eta_f C_{ft}} \right) + \hat{C} \Delta t \leq \frac{1}{2} \), which leads to

\[ \gamma \geq \frac{4C_{inv}^2}{\eta_f C_{ft} \left( 1 - \Delta t - 2\hat{C} \Delta t \right)}. \]

Taking \( v_c^h = 2\Delta t u_c^{n+1} \) and \( q^h = 2\Delta t p_c^{n+1} \) in (2.6), sum over \( n = n_k, \ldots, n_k + J \) \((0 \leq J \leq r - 1)\),

\[ 2\nu \Delta t \sum_{n=n_k}^{n_k+J} \| \nabla u_c^{n+1} \|^2_0 + \frac{2\alpha \Delta t}{\sqrt{\kappa}} \sum_{n=n_k}^{n_k+J} \| P_\gamma(u_c^{n+1}) \|^2_{L^2(I)} \]

\[ = 2\Delta t \sum_{n=n_k}^{n_k+J} (f_c(t_{n+1}), u_c^{n+1})_{\Omega_c} - 2\Delta t \sum_{n=n_k}^{n_k+J} \left( \frac{u_c^{n+1} - u_c^n}{\Delta t}, u_c^{n+1} \right)_{\Omega_c} \]

\[ + \frac{2\Delta t}{\rho} \sum_{n=n_k}^{n_k+J} \int_\Omega \phi_n u_c^{n+1} \cdot n_d \, ds - \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_k+J} \int_\Omega ((u_c^{n+1} - u_c^n) \cdot n_d) (u_c^{n+1} \cdot n_d) \, ds \]

\[ := 4 II_i. \]

Similarly, we obtain

\[ II_1 \leq \frac{2C_P^2 \Delta t}{\nu} \sum_{n=n_k}^{n_k+J} \| f_c(t_{n+1}) \|^2_{H^{-1}} + \frac{\nu \Delta t}{2} \sum_{n=n_k}^{n_k+J} \| \nabla u_c^{n+1} \|^2_0, \]

\[ II_2 \leq - \sum_{n=n_k}^{n_k+J} \left( \| u_c^{n+1} \|^2_0 - \| u_c^n \|^2_0 \right) + \hat{C} \Delta t \sum_{n=n_k}^{n_k+J} \| u_c^n \|^2_0, \]

\[ II_3 \leq \frac{C_P^2 \Delta t}{\rho^2 \nu h} \sum_{n=n_k}^{n_k+J} \| \phi_f^n \|^2_0 + \nu \Delta t \sum_{n=n_k}^{n_k+J} \| \nabla u_c^{n+1} \|^2_0, \]

\[ II_4 = -\frac{\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_k+J} \| (u_c^{n+1} - u_f^n) \cdot n_d \|^2_{L^2(I)} + \frac{\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_k+J} \| u_f^n \cdot n_d \|^2_{L^2(I)} - \| u_c^{n+1} \cdot n_d \|^2_{L^2(I)}. \]

Using the discrete Growall lemma, when \( \hat{C} \Delta t < 1 \), then

\[ \| u_c^{n_k+J+1} \|^2_0 + \frac{\nu \Delta t}{2} \sum_{n=n_k}^{n_k+J} \| \nabla u_c^{n+1} \|^2_0 + \frac{2\alpha \Delta t}{\sqrt{\kappa}} \sum_{n=n_k}^{n_k+J} \| P_\gamma(u_c^{n+1}) \|^2_{L^2(I)} \]

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\[ + \frac{\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_k+J} \|(u_{c,n}^{n+1} - u_f^{n_k}) \cdot n_d\|^2_{L^2(\Omega)} + \frac{\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_k+J} \|u_{f,n}^{n+1} \cdot n_d\|^2_{L^2(\Omega)} \]

\[ \leq C \left( \frac{2C_{PF}^2 \Delta t}{\nu} \sum_{n=n_k}^{n_k+J} \|f_c(t_{n+1})\|^2_{H^{-1}} + \frac{C_{PF}^2 \Delta t}{\rho^2 \nu h} \sum_{n=n_k}^{n_k+J} \|f_{\phi,n}^{n+1}\|^2_0 + \frac{\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_k+J} \|u_{f,n}^{n_k} \cdot n_d\|^2_{L^2(\Omega)} + \|u_{f,n}^{n_k}\|^2_0 \right). \]

In particular, taking \( k = 0 \), we have the stability in first large time \([0, t_{n_1}]\),

\[ \|u_{c,j+1}\|^2_0 + \frac{\nu \Delta t}{2} \sum_{n=0}^{J} \|\nabla u_{c,n+1}\|^2_0 + \frac{2\alpha \nu \Delta t}{k} \sum_{n=0}^{J} \|P_f(u_{c,n+1})\|^2_{L^2(\Omega)} \]

\[ + \frac{\gamma \Delta t}{\rho h} \sum_{n=0}^{J} \|(u_{c,n+1} - u_{f,n}) \cdot n_d\|^2_{L^2(\Omega)} + \frac{\Delta t}{\rho h} \sum_{n=0}^{J} \|u_{c,n+1} \cdot n_d\|^2_{L^2(\Omega)} \]

\[ \leq C \left( \frac{2C_{PF}^2 \Delta t}{\nu} \sum_{n=0}^{J} \|f_c(t_{n+1})\|^2_{H^{-1}} + \frac{C_{PF}^2 \Delta t}{\rho^2 \nu h} \sum_{n=0}^{J} \|f_{\phi,n+1}\|^2_0 + \frac{\Delta t}{\rho h} \sum_{n=0}^{J} \|u_{f,n}^{0} \cdot n_d\|^2_{L^2(\Omega)} + \|u_{c,n}^{0}\|^2_0 \right). \]

4. Error analysis

4.1. Preliminaries

In this subsection, we will introduce a projection operator, some lemmas and techniques that need to be used.

A. Some lemmas

Define a bounded linear projection operator \( P^h \), for all \( t \in [0, T] \),

\[ P^h = (P_{c_1}^h, P_{c_2}^h, P_f^h, P_m^h, P_m^h) : (u_c(t), p_c(t), u_f(t), \phi_f(t), u_m(t), \phi_m(t)) \in X \mapsto \]

\[ (P_{c_1}^h u_c(t), P_{c_2}^h p_c(t), P_f^h u_f(t), P_f^h \phi_f(t), P_m^h u_m(t), P_m^h \phi_m(t)) \in X_h, \]

satisfying

\[ a_{\Omega_c}(u_c - P^h u_c, v_c^h) - b(v_h, p_c - P^h p_c) + b(u_c - P^h u_c, q^h) + a_{\phi_c}(\phi_d, P^h \phi_d, \psi_d^h) \]

\[ + a_{u_d}(u_d - P^h u_d, v_d^h) + a_{\phi_d}(v_d^h, \phi_d - P^h \phi_d) + b_{\phi_d}(u_d - P^h u_d, v_d^h) \]

\[ - \frac{1}{\rho} \int_{1}^{0} (\phi_j - P^h \phi_j)(v^h_j - v^h_j) \cdot n_d ds + \frac{\gamma}{\rho h} \int_{1}^{0} ((u_c - P^h u_c) - (u_f - P^h u_f)) \cdot n_d (v^h_c - v^h_f) \cdot n_d ds \]

\[ = 0, \forall (v^h_c, q^h, v^h_f, \psi_f^h, v^h_m, \psi_m^h) \in (Y_c^h, Q_c^h, Y_f^h, Q_f^h, Y_m^h, Q_m^h). \]

(4.1)

Suppose that the solution \((u_c(t), p_c(t); u_f(t), \phi_f(t); u_m(t), \phi_m(t))\) to the variational formulation (2.1) satisfy

\[ \|u_c\|_{L^\infty(0,T;H^2)} + \|u_c\|_{L^\infty(0,T;H^2;\sigma^*)} + \|P_c\|_{L^\infty(0,T;H^2)} + \|u_f\|_{L^\infty(0,T;H^2)} + \|\phi_f\|_{L^\infty(0,T;H^1)} \]

\[ + \|u_m\|_{L^\infty(0,T;H^2)} + \|\phi_m\|_{L^\infty(0,T;H^1)} + \|u_c\|_{L^2(0,T;H^1)} + \|u_f\|_{L^2(0,T;H^1)} + \|\phi_f\|_{L^2(0,T;L^2)} \]

\[ + \|\phi_f t\|_{L^2(0,T;L^2)} + \|\phi_m t\|_{L^2(0,T;L^2)} + \|\phi_t m\|_{L^2(0,T;L^2)} \leq C_H, \]

(4.2)

where \( d^* > D \).
And assume that the solutions satisfy the following approximation properties \[38, 39]\n
\[
\|u_c - P^h u_c\|_0 + h\|
abla(u_c - P^h u_c)\|_0 + h\|p_c - P^h p_c\|_0 \leq C_{pc} h^2, \tag{4.3}
\]
\[
\|u_f - P^h u_f\|_0 + h\|
abla \cdot (u_f - P^h u_f)\|_0 + h\|\phi_f - P^h \phi_f\|_0 \leq C_{pf} h^2, \tag{4.4}
\]
\[
\|u_m - P^h u_m\|_0 + h\|
abla \cdot (u_m - P^h u_m)\|_0 + h\|\phi_m - P^h \phi_m\|_0 \leq C_{pm} h^2. \tag{4.5}
\]

Furthermore, we suppose the projection operator \(P^h\) satisfying
\[
\|P^h u_c(t_n)\|_{L^\infty} \leq C\|u_c(t_n)\|_2, \quad \|P^h u_c(t_n)\|_{W^{1,\infty}} \leq C\|u_c(t_n)\|_{W^{1,\infty}}. \tag{4.6}
\]

**Lemma 4.1** ([40]). Let \(R^{n+1}_{tr} = \frac{u_c(t_{n+1}) - u_c(t_n)}{\Delta t} - \frac{\partial u_c}{\partial t}(t_{n+1}) - (u_c(t_{n+1}) \cdot \nabla)u_c(t_{n+1})\). It holds that
\[
\Delta t \sum_{n=0}^{N-1} \|R^{n+1}_{tr}\|_0^2 \leq C \Delta t^2.
\]

**Lemma 4.2** ([31]). Assume that \(g_1, g_2\) and \(\rho\) are three functions defined in \(\Omega\) and vanish on \(\partial \Omega\). If
\[
\Delta t(\|g_1\|_{W^{1,\infty}} + \|g_2\|_{W^{1,\infty}}) \leq \frac{1}{2},
\]
then
\[
\|\rho(x - g_1(x)\Delta t) - \rho(x - g_2(x)\Delta t)\|_{L^q} \leq C N \Delta t \|\rho\|_{W^{1,q}} \|g_1 - g_2\|_{L^2},
\]
\[
\|\rho(x) - \rho(x - g_2(x)\Delta t)\|_{-1} \leq C N \Delta t \|\rho\|_0 \|g_2\|_{W^{1,4}},
\]
where \(1/q_1 + 1/q_2 = 1/q\) and \(1 \leq q < \infty\).

For \((v_c, q; v_f, \psi_f; v_m, \psi_m) \in \mathcal{X}^h\), using the characteristic version of the variational formulation and the definition of projection operator \(P^h\), we have
\[
(d_t u_c(t_{n+1}), v_c)_{\Omega_c} + a_{\Omega_c}(P^h u_c(t_{n+1}), v_c) - b(v_c, P^h p_c(t_{n+1})) + b(P^h u_c(t_{n+1}), q) = 0,
\]
\[
- \frac{1}{\rho} \int 1 \left( \rho h \right) \int (P^h u_c(t_{n+1}) - P^h u_f(t_{n+1})) \cdot n_d v_c \cdot n_d ds
\]
\[
= (f_c(t_{n+1}), v_c)_{\Omega_c} - \left( \frac{u_c(t_n) - \bar{u}_c(t_n)}{\Delta t}, v_c \right)_{\Omega_c} + (R_{tr}^{n+1}, v_c)_{\Omega_c}, \tag{4.7}
\]
\[
\frac{\eta_m C_{mt}}{\rho} (d_s \phi_m(t_{n+1}), \psi_m)_{\Omega_d} + \frac{\sigma k_m}{\rho \mu} (P^h \phi_m(t_{n+1}) - P^h \phi_f(t_{n+1}), \psi_m)_{\Omega_d}
\]
\[
+ \frac{1}{\rho} (\mu k_m^{-1} P^h u_m(t_{n+1}), v_m)_{\Omega_d} + \frac{1}{\rho} (\nabla \cdot P^h u_m(t_{n+1}), \psi_m)_{\Omega_d} - \frac{1}{\rho} (P^h \phi_m(t_{n+1}), \nabla \cdot v_m)_{\Omega_d}
\]
\[
= \frac{\eta_m C_{mt}}{\rho} (d_s \phi_m(t_{n+1}) - \frac{\partial \phi_m}{\partial t}(t_{n+1}), \psi_m)_{\Omega_d}, \tag{4.8}
\]
and
\[
\frac{\eta f C_{ft}}{\rho} (d_s \phi_f(t_{n+1}), \psi_f)_{\Omega_d} + \frac{\sigma k_m}{\rho \mu} (P^h \phi_f(t_{n+1}) - P^h \phi_m(t_{n+1}), \psi_f)_{\Omega_d}
\]
\[
+ \frac{1}{\rho} (\mu k_f^{-1} P^h u_f(t_{n+1}), v_f)_{\Omega_d} + \frac{1}{\rho} (\nabla \cdot P^h u_f(t_{n+1}), \psi_f)_{\Omega_d} - \frac{1}{\rho} (P^h \phi_f(t_{n+1}), \nabla \cdot v_f)_{\Omega_d}
\]
\[
+ \frac{1}{\rho} \int 1 \left( \rho h \right) \int (P^h u_c(t_{n+1}) - P^h u_f(t_{n+1})) \cdot n_d v_f \cdot n_d ds
\]
\[
= \frac{1}{\rho} (f_d(t_{n+1}), \psi_f)_{\Omega_d} + \frac{\eta f C_{ft}}{\rho} (d_s \phi_f(t_{n+1}) - \frac{\partial \phi_f}{\partial t}(t_{n+1}), \psi_f)_{\Omega_d}. \tag{4.9}
\]
B. Some techniques

For simplicity, we show some techniques in estimation. Let $\Psi_{d,t}^{n+1} = \Psi_{d,t,1}^{n+1} + \Psi_{d,t,2}^{n+1}$, where $\Psi_{d,t,1}^{n+1} = d_t \phi_d(t_{n+1}) - \frac{\partial \phi_d}{\partial t}(t_{n+1})$ and $\Psi_{d,t,2}^{n+1} = d_t(P^h \phi_d(t_{n+1}) - \phi_d(t_{n+1}))$, $d = f$ and $m$. Let $\Psi_{c,t}^{n+1} = d_t(P^h u_c(t_{n+1}) - u_c(t_{n+1}))$. For

$$\Delta t \Psi_{f,t,1}^{n+1} = \phi_f(t_{n+1}) - \phi_f(t_n) - \Delta t \frac{\partial \phi_f}{\partial t}(t_{n+1}) = \int_{t_n}^{t_{n+1}} (t_{n+1} - t) \frac{\partial^2 \phi_f}{\partial t^2}(t) dt,$$

then

$$\|\Psi_{f,t,1}^{n+1}\|_0^2 \leq \frac{1}{\Delta t^2} \int_{t_n}^{t_{n+1}} \left\{ \int_{t_n}^{t_{n+1}} \left( \frac{\partial^2 \phi_f}{\partial t^2}(t) \right)^2 dt \right\} dt \int_{t_n}^{t_{n+1}} (t_{n+1} - t)^2 dt dx$$

$$\leq \frac{\Delta t}{3} \int_{t_n}^{t_{n+1}} \left\| \frac{\partial^2 \phi_f}{\partial t^2} \right\|^2_0 dt.$$

For

$$\Psi_{f,t,2}^{n+1} = \frac{P^h \phi_f(t_{n+1}) - P^h \phi_f(t_n)}{\Delta t} - \frac{\phi_f(t_{n+1}) - \phi_f(t_n)}{\Delta t}$$

$$= (P^h - I) \frac{\phi_f(t_{n+1}) - \phi_f(t_n)}{\Delta t}$$

$$= \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} (P^h - I) \frac{\partial \phi_f}{\partial t}(t) dt,$$

then

$$\|\Psi_{f,t,2}^{n+1}\|_0^2 = \frac{1}{\Delta t^2} \int_{\Omega_d} \left( \int_{t_n}^{t_{n+1}} (P^h - I) \frac{\partial \phi_f}{\partial t}(t) dt \right)^2 dx$$

$$\leq \frac{1}{\Delta t^2} \int_{\Omega_d} \left\{ \int_{t_n}^{t_{n+1}} ((P^h - I) \frac{\partial \phi_f}{\partial t}(t))^2 dt \int_{t_n}^{t_{n+1}} 1^2 dt \right\} dx$$

$$\leq \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \left\| (P^h - I) \frac{\partial \phi_f}{\partial t}(t) \right\|^2_0 dt.$$

Similarly,

$$\|\Psi_{m,t,1}^{n+1}\|_0^2 \leq \frac{\Delta t}{3} \int_{t_n}^{t_{n+1}} \left\| \frac{\partial^2 \phi_m}{\partial t^2} \right\|^2_0 dt,$$

$$\|\Psi_{m,t,2}^{n+1}\|_0^2 \leq \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \left\| (P^h - I) \frac{\partial \phi_m}{\partial t}(t) \right\|^2_0 dt,$$

$$\|\Psi_{c,t}^{n+1}\|_0^2 \leq \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \left\| (P^h - I) \frac{\partial u_c}{\partial t}(t) \right\|^2_0 dt.$$

4.2. Convergence analysis

Some notations are defined by

$$e_c^n = P^h u_c(t_n) - u_c^n, \quad \delta_c^n = P^h p_c(t_n) - p_c^n, \quad e_d^n = P^h u_d(t_n) - u_d^n, \quad \theta_d^n = P^h \phi_d(t_n) - \phi_d^n.$$

Then

$$u_c(t_n) - u_c^n := u_c(t_n) - P^h u_c(t_n) + e_c^n,$$

$$p_c(t_n) - p_c^n := p_c(t_n) - P^h p_c(t_n) + \delta_c^n,$$

$$u_d(t_n) - u_d^n := u_d(t_n) - P^h u_d(t_n) + e_d^n,$$

$$\phi_d(t_n) - \phi_d^n := \phi_d(t_n) - P^h \phi_d(t_n) + \theta_d^n.$$
where \( d = f \) or \( m \).

Taking \((v_c; q; v_f; \psi_f; v_m; \psi_m) = (v_c^h; q^h; v_f^h; \psi_f^h; v_m^h; \psi_m^h)\) in (4.7), (4.8) and (4.9). (2.6) (2.7) and (2.8) are subtracted from the corresponding term, respectively. Which leads to

\[
\frac{\left(e_c^{n+1} - e_c^n\right)}{\Delta t} + a_t(e_c^{n+1}, v_c^h) - b(v_h, \delta_c^{n+1}) + b(e_c^{n+1}, q^h) - \frac{1}{\rho} \int_{\Omega} \left( P^h \phi_f(t_{n+1}) - \phi_f^n \right) v_c^h \cdot n_d d\Omega
\]

\[
+ \frac{\gamma}{\rho h} \int_{\Omega} \left( e_c^{n+1} - (P^h u_f(t_{n+1}) - u_f^n) \right) \cdot n_d v_c^h \cdot n_d d\Omega
\]

\[
= \left( d_t(P^h u_c(t_{n+1}) - u_c(t_{n+1})) + \frac{u_c^{n} - \bar{u}_c^{n}}{\Delta t} - \bar{u}_c(t_{n}) + \bar{u}_c(t_{n}) + P_{tr+1}, v_c^h \right)_{\Omega_c},
\]

(4.10)

\[
\frac{\eta_m C_{mt}}{\rho} \left( \frac{\theta_m^{n+1} - \theta_m^n}{\Delta s}, \psi_f^h \right)_{\Omega_d} + \frac{1}{\rho} \left( \nabla \cdot \epsilon_m^{n+1}, \psi_m^h \right)_{\Omega_d} + \frac{1}{\rho} \left( \mu k_m^{-1} e_m^{n+1}, v_m^h \right)_{\Omega_d}
\]

\[
- \frac{1}{\rho} \left( \theta_m^{n+1}, \nabla \cdot v_m^h \right)_{\Omega_d} + \frac{\sigma k_m}{\rho \mu} \left( \phi_m(t_{n+1}) - (P^h \phi_f(t_{n+1}) - \phi_f^n), \psi_f^h \right)_{\Omega_d}
\]

\[
\frac{\eta_f C_{ft}}{\rho} \left( \frac{\theta_f^{n+1} - \theta_f^n}{\Delta s}, \psi_f^h \right)_{\Omega_d} + \frac{1}{\rho} \left( \nabla \cdot e_f^{n+1}, \psi_f^h \right)_{\Omega_d} + \frac{1}{\rho} \left( \mu k_f^{-1} e_f^{n+1}, v_f^h \right)_{\Omega_d}
\]

\[
- \frac{1}{\rho} \left( \theta_f^{n+1}, \nabla \cdot v_f^h \right)_{\Omega_d} + \frac{\sigma k_m}{\rho \mu} \left( \phi_f(t_{n+1}) - (P^h \phi_m(t_{n+1}) - \phi_m^n), \psi_f^h \right)_{\Omega_d}
\]

\[
+ \frac{1}{\rho} \int_{\Omega} \left( P^h \phi_f(t_{n+1}) - \phi_f^n \right) v_f^h \cdot n_d d\Omega - \frac{\gamma}{\rho h} \int_{\Omega} \left[ \left( P^h u_c(t_{n+1}) - S_{n+1} - e_f^n \right) \cdot n_d v_f^h \cdot n_d d\Omega
\]

\[
= \left( \frac{P^h \phi_f(t_{n+1}) - P_h \phi_f(t_{n}),}{\Delta s} - \frac{\partial \phi_f}{\partial s} (t_{n+1}), \psi_f^h \right)_{\Omega_d}.
\]

(4.11)

We below prove error convergence of solutions in sense of \( L^2 \)-norm and \( H^1 \)-seminorm for different time steps in different region. The key of successful proof is to obtain the uniform \( L^\infty \)-boundedness of \( u_h^m \) at the assumption step.

**Theorem 4.3.** Assume that (2.2) has a unique solution \((u_c(t_n), p_e(t_n); u_f(t_n), \phi_f(t_n); u_m(t_n), \phi_m(t_n))\) satisfying the boundedness of assumption. There exists some positive constants \( \tau_1 \) and \( h_1 \) such that when \( \Delta t < \tau_1, h < h_1 \) and \( \Delta t = \mathcal{O}(h^2) \), the solution of fully discrete decoupled modified characteristic scheme (2.6),(2.7) and (2.8) in the first large time interval \([0, t_{n_1}]\), for any \( 0 \leq J \leq r \) satisfies

\[
\max_{0 \leq J \leq r} \| e_f^J \|_0 + \frac{\mu \Delta t}{2} \sum_{n=0}^J \| \nabla e_c^n \|_0^2 \leq C(h^4 + \Delta t^2 + \Delta t^2 h^{-1}).
\]

(4.13)
On the other hand, for \(0 \leq t \leq M - 1\), we obtain
\[
\max_{0 \leq t \leq M - 1} \left( \|e_c^{n+1}\|^2_0 + \frac{\eta C_m}{\rho} \|\theta_f^{n+1}\|^2_0 + \frac{\eta_m C_m t}{\rho} \|\theta_m^{n+1}\|^2_0 \right) + \nu \Delta t \sum_{k=0}^l \sum_{n=n_k}^{n_{k+1}-1} \|\nabla e_c^{n+1}\|^2_0
+ \frac{2 \mu \Delta s}{\rho k_f} \sum_{k=0}^l \|e_f^{n+1}\|^2_0 + \frac{2 \mu \Delta s}{\rho k_m} \sum_{k=0}^l \|e_m^{n+1}\|^2_0 \leq C(h^4 + \Delta t^2 + \Delta t^2 h^{-1}),
\] (4.14)

and
\[
\Delta t \sum_{k=0}^{M-1} \sum_{n=n_k+1}^{n_{k+1}} \|u_c^n\|^2_{W^{1,\infty}} + \max_{n \in \{n_k+1, \ldots, n_{k+1}\}, 0 \leq k \leq M-1} \|u_c^n\|_{L^\infty} \leq C_B.
\] (4.15)

**Proof.** We give the proof of (4.13) by mathematical induction. First of all, when \(J = 0\), \(e_c^0 = 0, \nabla e_c^0 = 0\). It’s obvious that (4.13) holds at the initial time step. When \(J = m, 1 \leq m \leq r - 1\), assume that (4.13) holds, i.e.,
\[
\|e_c^n\|^2_0 + \frac{\nu \Delta t}{2} \sum_{n=0}^m \|\nabla e_c^n\|^2_0 \leq C(h^4 + \Delta t^2 + \Delta t^2 h^{-1}).
\] (4.16)

Using the discrete Sobolev inequality (2.5), the properties (4.6) and the regularity assumption in (??), when \(\Delta t < \tau_1, h < h_1\) and \(\Delta t = O(h^2)\), if \(d = 2\), we have
\[
\|u_c^n\|_{L^\infty} \leq \|e_c^n\|_{L^\infty} + \|P^h u_c(t_m)\|_{L^\infty} \leq C_D S (1 + |\ln(h)|)^{1/2} \|e_c^n\|_1 + C \|u_c(t_m)\|_2
\leq C (1 + |\ln(h)|)^{1/2} (\Delta t^{-1/2} h^2 + \Delta t^{1/2} + \Delta t^{1/2} h^{-1/2}) + C
\leq C_B.
\] (4.17)

Actually, \(h|\ln(h)| \to 0(h \to 0)\). If \(d = 3\), we obtain
\[
\|u_c^n\|_{L^\infty} \leq \|e_c^n\|_{L^\infty} + \|P^h u_c(t_m)\|_{L^\infty} \leq C_D S h^{-1/2} \|e_c^n\|_1 + C \|u_c(t_m)\|_2
\leq C h^{-1/2} (\Delta t^{-1/2} h^2 + \Delta t^{1/2} + \Delta t^{1/2} h^{-1/2}) + C \leq C_B.
\] (4.18)

In addition, by the properties of (4.6), inverse inequality (2.3), imbedding theorem and the regularity assumption in (4.2), we obtain
\[
\Delta t \|u_c^n\|_{W^{1,\infty}} \leq \Delta t \left( \|e_c^n\|_{W^{1,\infty}} + \|P^h u_c(t_m)\|_{W^{1,\infty}} \right)
\leq \Delta t \left( C h^{-3/2} \|\nabla e_c^n\|_0 + C \|u_c(t_m)\|_{W^{2,d^*}} \right)
\leq C \left( \Delta t^{1/2} h^{1/2} + \Delta t^{3/2} h^{-3/2} + \Delta t^{2/2} h^{-2} + \Delta t \right)
\leq \frac{1}{4},
\] (4.19)

where \(d^* > D\).

When \(J = m + 1\), taking \(\nu_c = 2\Delta t e_c^{n+1}\) and \(q_h = 2\Delta t e_c^{n+1}\) in (4.10), sum over \(n = 0, 1, \ldots, m(1 \leq m \leq r - 1)\), we have
\[
\|e_c^{n+1}\|^2_0 - \|e_c^n\|^2_0 + \sum_{n=0}^m \|e_c^{n+1} - e_c^n\|^2_0 + 2 \nu \Delta t \sum_{n=0}^m \|\nabla e_c^{n+1}\|^2_0 + \frac{2 \alpha \nu \Delta t}{\sqrt{k}} \sum_{n=0}^m \|P_r(e_c^{n+1})\|^2_{L^2(I)}
= \sum_{n=0}^m \left( \psi_{c,t}^{n+1} + \frac{u_c^n - u_c(t_n) + \bar{u}_c(t_n)}{\Delta t} + R_{tr}^{n+1} + 2\Delta t e_c^{n+1} \right)_{\Omega_c}
\]
\[ + \frac{2\Delta t}{\rho} \sum_{n=0}^{m} \int (P^h \phi_f(t_{n+1}) - \phi^0) e_c^{n+1} \cdot n_d ds \]

\[ - \frac{2\gamma \Delta t}{\rho h} \sum_{n=0}^{m} \int (e_c^{n+1} - (P^h u_f(t_{n+1}) - u^0_f)) \cdot n_d e_c^{n+1} \cdot n_d ds \]

\[ := A_1 + A_2 + A_3. \]

Note that in the first large time interval, \( n_k = n_0 = 0 \). The first term

\[ A_1 = 2\Delta t \sum_{n=0}^{m} (\Psi_{c,t}^{n+1}, e_c^{n+1})_{\Omega_c} + 2\Delta t \sum_{n=0}^{m} \left( \frac{u_c^n - \bar{u}_c^n - u_c(t_n) + \bar{u}_c(t_n)}{\Delta t}, e_c^{n+1} \right)_{\Omega_c} \]

\[ + 2\Delta t \sum_{n=0}^{m} (P_{tr}^{n+1}, e_c^{n+1})_{\Omega_c} \]

\[ := A_{11} + A_{12} + A_{13}. \]

By the Cauchy-Schwarz inequality, Poincaré-Friedrichs inequality, Young inequality and some techniques in part 4.1, we estimate

\[ A_{11} \leq 2C_{PF} \Delta t \sum_{n=0}^{m} \| \Psi_{c,t}^{n+1} \|_0 \| \nabla e_c^{n+1} \|_0 \]

\[ \leq \frac{6C_{PF}^2}{\nu} \int_0^{t_{n+1}} \left\| (P^h - I) \frac{\partial u_c}{\partial t}(t) \right\|_0^2 dt + \frac{\nu \Delta t}{6} \sum_{n=0}^{m} \| \nabla e_c^{n+1} \|_0^2. \]

At the same time, using the Hölder inequality, Poincaré-Friedrichs inequality, Lemma 4.2, the trace inequality, inverse inequality, Sobolev interpolation formulas, imbedding theorem and the boundedness of (4.17) (4.18) (4.19), we show that

\[ 2\Delta t \left( \frac{u_c^n - \bar{u}_c^n - u_c(t_n) + \bar{u}_c(t_n)}{\Delta t}, e_c^{n+1} \right)_{\Omega_c} \]

\[ \leq 2C_{PF} \| P^h u_c(t_n) - u_c(t_n) - (P^h \bar{u}_c(t_n) - \bar{u}_c(t_n)) \|_1 \| \nabla e_c^{n+1} \|_0 \]

\[ + 2\| P^h \bar{u}_c(t_n) - u_c(t_n) - (P^h \bar{u}_c(t_n) - \bar{u}_c(t_n)) \|_1 \| \nabla e_c^{n+1} \|_0 \]

\[ + 2\| \bar{u}_c^n - u_c^n \|_0 \| \nabla e_c^{n+1} \|_0 \]

\[ \leq 2C_{PF} C_{\Delta t} \| P^h u_c(t_n) - u_c(t_n) \|_0 \| \nabla e_c^{n+1} \|_0 \]

\[ + 2C_{N} C_{\Delta t} \| P^h u_c(t_n) - u_c(t_n) \|_1 \| \nabla e_c^{n+1} \|_0 \]

\[ \leq 2C_{PF} C_{\Delta t} \| \nabla e_c^n \|_0 \| \nabla e_c^{n+1} \|_0 \]

\[ \leq 2C_{PF} C_{\Delta t} \| \nabla e_c^n \|_0 \| \nabla e_c^{n+1} \|_0 \]

\[ \leq \frac{\nu \Delta t}{6} \| \nabla e_c^{n+1} \|_0^2 + \frac{\nu \Delta t}{2} \| \nabla e_c^n \|_0^2 + \frac{2C_{PF}^2 C_{\Delta t}}{\nu} \sum_{n=0}^{m} \| e_c^{n+1} \|_0^2 \]

\[ + Ch^h \Delta t. \]

Therefore

\[ A_{12} \leq \frac{\nu \Delta t}{6} \sum_{n=0}^{m} \| \nabla e_c^{n+1} \|_0^2 + \frac{\nu \Delta t}{2} \sum_{n=0}^{m} \| \nabla e_c^n \|_0^2 + \frac{2C_{PF}^2 C_{\Delta t}}{\nu} \sum_{n=0}^{m} \| e_c^{n+1} \|_0^2. \]
\[ + \frac{30C_N^2 C_S^2 C_{PF}^2 \Delta t}{\nu} \sum_{n=0}^{m} \|e_c^n\|_0^2 + Ch^4 \Delta t. \]

Analogous to \( A_{11} \), using Lemma 4.1, we deduce that
\[ A_{13} \leq \frac{6C_{PF}^2 \Delta t}{\nu} \sum_{n=0}^{m} \|R_{tr}^{n+1}\|_0^2 + \frac{\nu \Delta t}{6} \sum_{n=0}^{m} \|\nabla e_c^{n+1}\|_0^2. \]

For the first term
\[
A_1 \leq \frac{\nu \Delta t}{2} \sum_{n=0}^{m} (\|\nabla e_c^{n+1}\|_0^2 + \|\nabla e_c^n\|_0^2) + \frac{2C_N^2 C_S^2 C_{PF}^2 \Delta t}{\nu} \sum_{n=0}^{m} \|e_c^{n+1}\|_0^2
+ \frac{30C_N^2 C_S^2 C_{PF}^2 \Delta t}{\nu} \sum_{n=0}^{m} \|e_c^n\|_0^2 + \frac{6C_{PF}^2 \Delta t}{\nu} \sum_{n=0}^{m} \|R_{tr}^{n+1}\|_0^2
+ \frac{6C_{PF}^2}{\nu} \int_{t_0}^{t_1} \left\| (\mathbf{p}h - I) \frac{\partial \mathbf{u}_c}{\partial t} (t) \right\|^2_0 dt + Ch^4 \Delta t.
\]

For the second term, \( A_2 \) is bounded by the trace inequality and trace inverse inequality,
\[
A_2 = \frac{2 \Delta t}{\rho} \sum_{n=0}^{m} \int_0^1 (P^h \phi_f(t_{n+1}) - \phi_f^0) e_c^{n+1} \cdot \mathbf{n}_d ds
\leq \frac{2 \Delta t}{\rho} \sum_{n=0}^{m} \|P^h \phi_f(t_{n+1}) - P^h \phi_f(0)\|_{L^2(\Omega)} \|e_c^{n+1}\|_0 \cdot \mathbf{n}_d \|_{L^2(\Omega)} + \frac{2 \Delta t}{\rho} \sum_{n=0}^{m} \|\phi_f\|_{L^2(\Omega)} \|e_c^{n+1}\|_0 \cdot \mathbf{n}_d \|_{L^2(\Omega)}
\leq \frac{2 \tau C_T \tilde{C}_{inv} C_{PF}^{1/2} \Delta t}{\rho h^2} \sum_{n=0}^{m} \|P^h \phi_f(t_{n+1}) - P^h \phi_f(t_n)\|_0 \|\nabla e_c^{n+1}\|_0 + \frac{2 \tau C_T \tilde{C}_{inv} C_{PF}^{1/2} \Delta t}{\rho h^2} \sum_{n=0}^{m} \|\phi_f^0\|_0 \|\nabla e_c^{n+1}\|_0
\leq \frac{4\tau C_T^2 \tilde{C}_{inv} C_{PF}^2 \Delta t^2}{\rho^2 \nu h} \int_0^{t_1} \left\| \frac{\partial P^h \phi_f}{\partial t} (t) \right\|^2_0 dt + \frac{\nu \Delta t}{2} \sum_{n=0}^{m} \|\nabla e_c^{n+1}\|_0^2 + \frac{4\tau C_T^2 \tilde{C}_{inv} C_{PF} \Delta t}{\rho^2 \nu h} \sum_{n=0}^{m} \|\phi_f^0\|^2_0.
\]

For the third term, \( A_3 \) does the identity transformation,
\[
A_3 = -\frac{2 \gamma \Delta t}{\rho h} \sum_{n=0}^{m} \int_0^1 (e_c^{n+1} - (P^h \mathbf{u}_f(t_{n+1}) - \mathbf{u}_f^0)) \cdot \mathbf{n}_d e_c^{n+1} \cdot \mathbf{n}_d ds
= -\frac{2 \gamma \Delta t}{\rho h} \sum_{n=0}^{m} \int_0^1 (e_c^{n+1} - e_f^0 - (P^h \mathbf{u}_f(t_{n+1}) - P^h \mathbf{u}_f(0))) \cdot \mathbf{n}_d (e_c^{n+1} - e_f^0 + e_f^0) ds
= -\frac{2 \gamma \Delta t}{\rho h} \sum_{n=0}^{m} \|e_c^{n+1} - e_f^0\|_{L^2(\Omega)}^2 + \frac{2 \gamma \Delta t}{\rho h} \sum_{n=0}^{m} \int_0^1 (P^h \mathbf{u}_f(t_{n+1}) - P^h \mathbf{u}_f(0)) \cdot \mathbf{n}_d e_c^{n+1} \cdot \mathbf{n}_d ds
+ \frac{2 \gamma \Delta t}{\rho h} \sum_{n=0}^{m} \int_0^1 (e_f^0 - e_c^{n+1}) \cdot \mathbf{n}_d e_f^0 \cdot \mathbf{n}_d ds,
\]
where using Hölder inequality, the general trace inequality, Young inequality, the properties of \( H(\text{div}) \) space and the divergence free condition, we arrive at
\[
\frac{2 \gamma \Delta t}{\rho h} \sum_{n=0}^{m} \int_0^1 (P^h \mathbf{u}_f(t_{n+1}) - P^h \mathbf{u}_f(0)) \cdot \mathbf{n}_d e_c^{n+1} \cdot \mathbf{n}_d ds.
\]
In conclusion, in the first large time interval \([0, t_{n+1}]\),

Therefore,

\[
\begin{align*}
A_3 &\leq -\frac{2\gamma \Delta t}{\rho h} \sum_{n=0}^{m} \|e^{n+1}_c - e^n_c\|_2^2 + \frac{\nu \Delta t}{2} \sum_{n=0}^{m} \|\nabla e^{n+1}_c\|_0^2 + \frac{2\alpha \nu \Delta t}{\sqrt{k}} \sum_{n=0}^{m} \|P_\gamma(e^{n+1}_c)\|_2^2 \\
&\quad + \frac{\gamma \Delta t}{\rho h} \sum_{n=0}^{m} \|e^{n+1}_c - e^0_f\|_2^2 + \frac{\gamma \Delta t}{\rho h} \sum_{n=0}^{m} \|e^{n+1}_c \cdot n_d\|_2^2 \\
&\leq \Delta t \sum_{n=0}^{m} \left( \frac{2c^2 N C^2_{PF} C_B^2}{\nu} \|e^{n+1}_c\|_2^2 + \frac{30c^2 N C^2_S C^2_{PF} C_H^2}{\nu} \|e^{n}_c\|_2^2 + \frac{C^2_{div}}{h} \|e^{n+1}_c\|_2^2 \right) \\
&\quad + \frac{6C^2_{PF} \Delta t}{\nu} \sum_{n=0}^{m} \|R^{n+1}_t\|_2^2 + \frac{6C^2_{PF}}{\nu} \int_{t^n}^{t_{n+1}} \|\frac{\partial P^h \phi_f(t)}{\partial t}\|_0^2 dt + Ch^4 \Delta t \\
&\quad + \frac{4r^2 C^2_{inv} C^2_{PF} \Delta t^2}{\rho^2 \nu h} \int_{t^n}^{t_{n+1}} \|\frac{\partial P^h u_f(t)}{\partial t}\|_0^2 dt + \frac{\gamma \Delta t}{\rho h} \sum_{n=0}^{m} \|e^0_f\|_2^2 + \|e^{n+1}_c\|_0^2 \\
&\quad + \frac{\nu \Delta t}{2} \sum_{n=0}^{m} \|\nabla e^{n+1}_c\|_0^2.
\end{align*}
\]

By the discrete Growall lemma, Lemma 4.1 and \(\Delta t = O(h^2)\), when \(\kappa \Delta t \leq \frac{1}{2}\), where \(\kappa = \frac{2c^2 N C^2_{PF} C_B^2}{\nu} + \frac{30c^2 N C^2_S C^2_{PF} C_H^2}{\nu} + \frac{C^2_{div}}{h}\), then

\[
\|e^{n+1}_c\|_0^2 + \frac{\nu \Delta t}{2} \sum_{n=0}^{m} \|\nabla e^{n+1}_c\|_0^2 \leq C(h^4 + \Delta t^2 + \Delta t^2 h^{-1}).
\]

In conclusion, in the first large time interval \([0, t_{n+1}]\), when \(0 \leq J \leq r\), it yields that

\[
\max_{0 \leq J \leq r} \|e^J\|_0^2 + \frac{\nu \Delta t}{2} \sum_{n=0}^{J} \|\nabla e^n_c\|_0^2 \leq C(h^4 + \Delta t^2 + \Delta t^2 h^{-1}).
\]

Taking \(v^h = 2\Delta s e^{n+1}_m, \psi^h = 2\Delta s \theta^{n+1}_m, v^f = 2\Delta s e^{n+1}_f\) and \(\psi^f = 2\Delta s \theta^{n+1}_f\) in (4.11) and (4.12), sum over two equations

\[
\frac{\eta d C_{dt}}{\rho} (\|\theta^{n+1}_d\|_0^2 - \|\theta^n_d\|_0^2 + \|\theta^{n+1}_d - \theta^n_d\|_0^2) + \frac{2\mu \Delta s}{\rho k_m} \|e^{n+1}_m\|_0^2 + \frac{2\mu \Delta s}{\rho k_f} \|e^{n+1}_f\|_0^2
\]
Using Cauchy-Schwarz inequality and trace inequality,

\[
\frac{\sigma_m \Delta_s}{\rho \mu} (\|\theta_m^{n+1}\|_0^2 - \|\theta_f^{n_k}\|_0^2) + \frac{\sigma_m \Delta_s}{\rho \mu} (\|\theta_m^{n_k+1}\|_0^2 - \|\theta_m^n\|_0^2 + \|\theta_f^{n_k+1}\|_0^2)
\]

\[
= \eta_d \Delta_s (\psi_{d,s}^{n_k+1}, 2 \Delta \theta_d^{n_k+1})_{\Omega_d} + \frac{2 \gamma \Delta_s}{\rho \mu} \int_{\Omega_d} ((P^h u_c(t_{n_k+1}) - S^{n_k+1}) - e_f^{n_k+1}) \cdot n_d e_f^{n_k+1} \cdot n_d ds
\]

\[+ \left[ 2 \sigma_k \Delta_s (P^h \phi_m(t_{n_k+1}) - P^h \phi_m(t_{n_k}), \theta_f^{n_k+1})_{\Omega_d} + \frac{2 \sigma_k \Delta_s}{\rho \mu} (P^h \phi_f(t_{n_k+1}) - P^h \phi_f(t_{n_k}), \theta_m^{n_k+1})_{\Omega_d} \right]
\]

\[- 2 \Delta_s \int_{\Omega_d} (P^h \phi_f(t_{n_k+1}) - \phi_f^{n_k}) e_f^{n_k+1} \cdot n_d ds
\]

\[
:= \sum_{i=1}^4 B_i,
\]

(4.20)

where \(\psi_{d,s}^{n_k+1} = \psi_{d,s,1}^{n_k+1} + \psi_{d,s,2}^{n_k+1}\).

For the first term, use the previous techniques

\[
B_1 = \eta_d \Delta_s (\psi_{d,s}^{n_k+1}, 2 \Delta \theta_d^{n_k+1})_{\Omega_d}
\]

\[
= \eta_d \Delta_s \left( \Psi_{f,s}^{n_k+1}, 2 \Delta \theta_f^{n_k+1} \right)_{\Omega_d}
\]

\[
\leq 2 \eta_d \Delta_s \left( \||\Psi_{f,s}^{n_k+1}||_0^2 + \||\Psi_{m,s}^{n_k+1}||_0^2 \right) + \frac{2 \eta_d \Delta_s}{\rho \mu} \||\theta_f^{n_k+1}||^2_0 + \frac{2 \eta_d \Delta_s}{\rho \mu} \||\theta_m^{n_k+1}||^2_0
\]

\[
\leq 2 \eta_d \Delta_s \left( \||\Psi_{f,s,1}^{n_k+1}||_0^2 + \||\Psi_{m,s,2}^{n_k+1}||_0^2 \right) + \frac{2 \eta_d \Delta_s}{\rho \mu} \||\theta_f^{n_k+1}||^2_0 + \frac{2 \eta_d \Delta_s}{\rho \mu} \||\theta_m^{n_k+1}||^2_0
\]

For the second term

\[
B_2 = \frac{2 \gamma \Delta_s}{\rho \mu} \int_{\Omega_d} (P^h u_c(t_{n_k+1}) - S^{n_k+1}) - e_f^{n_k+1}) \cdot n_d e_f^{n_k+1} \cdot n_d ds
\]

\[
= \frac{2 \gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_k+1} \int_{\Omega_d} (P^h u_c(t_{n_k+1}) - P^h u_c(t_{n+1})) \cdot n_d e_f^{n_k+1} \cdot n_d ds
\]

\[
+ \frac{2 \gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_k+1} \int_{\Omega_d} (e_c^{n_k+1} - e_f^{n_k+1}) \cdot n_d e_f^{n_k+1} \cdot n_d ds
\]

\[:= B_{21} + B_{22}.
\]

Using Cauchy-Schwarz inequality and trace inequality,

\[
B_{21} = \frac{2 \gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_k+1} \int_{\Omega_d} (P^h u_c(t_{n_k+1}) - P^h u_c(t_{n+1})) \cdot n_d e_f^{n_k+1} \cdot n_d ds
\]
For the fourth term
We can conclude that

\[
B_4 = -\frac{2\Delta s}{\rho} \int_{t_n}^{t_{n+1}} \left( P^h \phi_f(t_{n+1}) - \phi_f^{n+1} \right) e_f^{n+1} \cdot n_d ds
= -\frac{2\Delta s}{\rho} \int_{t_n}^{t_{n+1}} \left( P^h \phi_f(t_{n+1}) - P^h \phi_f(t_n) \right) e_f^{n+1} \cdot n_d ds
- \frac{2\Delta s}{\rho} \int_{t_n}^{t_{n+1}} \theta_f^{n+1} e_f^{n+1} \cdot n_d ds
:= B_{41} + B_{42}.
\]

It is easy to see \(2(a - b, b) \leq (a, a) - (b, b)\), we have

\[
B_{22} = \frac{2\gamma t^2 C^2}{\rho} \int_{t_n}^{t_{n+1}} \left\| P^h u_e \right\|^2 dt + \frac{\gamma t^2}{\rho} \sum_{n = n_k}^{n_{n+1}} || e_f^{n+1} ||^2_{L^2(\Omega)}
\]

Therefore,

\[
B_2 \leq \frac{4\gamma t^2 C^2}{\rho} \int_{t_n}^{t_{n+1}} \left\| P^h u_e \right\|^2 dt + \frac{4\gamma t^2}{\rho} \sum_{n = n_k}^{n_{n+1}} || e_f^{n+1} ||^2_{L^2(\Omega)}
\]

For the third term

\[
B_3 = \frac{2\sigma k_m \Delta s}{\rho \mu} (P^h \phi_m(t_{n+1}) - P^h \phi_m(t_n), \theta_f^{n+1}) + \frac{2\sigma k_m \Delta s}{\rho \mu} \int_{t_n}^{t_{n+1}} \left\| \frac{\partial P^h \phi_m}{\partial t} \right\|^2 dt + \frac{\eta_f C_{fi} \Delta s}{2 \rho} \theta_f^{n+1}
\]

By Cauchy-Schwarz inequality and the Young inequality, we have

\[
B_{31} = \frac{2\sigma k_m \Delta s}{\rho \mu} \int_{t_n}^{t_{n+1}} \left\| \frac{\partial P^h \phi_m}{\partial t} \right\|^2 dt + \frac{\eta_f C_{fi} \Delta s}{2 \rho} \theta_f^{n+1}
\]

Similarly available,

\[
B_{32} = \frac{2\sigma k_m \Delta s}{\rho \mu} \int_{t_n}^{t_{n+1}} \left\| \frac{\partial P^h \phi_f}{\partial t} \right\|^2 dt + \frac{\eta_m C_{mt} \Delta s}{2 \rho} \theta_f^{n+1}
\]

We can conclude that

\[
B_3 \leq \frac{2\sigma k_m \Delta s}{\rho \mu} \int_{t_n}^{t_{n+1}} \left\| \frac{\partial P^h \phi_m}{\partial t} \right\|^2 dt + \frac{\eta_f C_{fi} \Delta s}{2 \rho} \theta_f^{n+1}
\]

For the fourth term

\[
B_4 = -\frac{2\Delta s}{\rho} \int_{t_n}^{t_{n+1}} \left( P^h \phi_f(t_{n+1}) - \phi_f^{n+1} \right) e_f^{n+1} \cdot n_d ds
= -\frac{2\Delta s}{\rho} \int_{t_n}^{t_{n+1}} \left( P^h \phi_f(t_{n+1}) - P^h \phi_f(t_n) \right) e_f^{n+1} \cdot n_d ds
- \frac{2\Delta s}{\rho} \int_{t_n}^{t_{n+1}} \theta_f^{n+1} e_f^{n+1} \cdot n_d ds
:= B_{41} + B_{42}.
\]
$B_{41}$ is bounded by the trace inequality and the Young inequality,

$$B_{41} = -\frac{2\Delta s}{\rho} \int_{n_k}^{n_{k+1}} \left( P^h \phi_f(t_{n_k}) - P^h \phi_f(t_{n_k}) \right) e^{n_{k+1}}_f \cdot n_d \, ds$$

$$\leq \frac{8C^2 h^3 \Delta s^2}{\rho \gamma} \int_{n_k}^{n_{k+1}} \left\| \partial P^h \phi_f \right\|_1 ^2 \, dt + \frac{\gamma \Delta s}{8\rho} \sum_{n=n_k}^{n_{k+1}-1} \left\| e^{n_{k+1}}_f \cdot n_d \right\|_{L^2(I)}^2.$$

And $B_{42}$ is bounded by the trace inverse inequality and the Young inequality,

$$B_{42} = -\frac{2\Delta s}{\rho} \int_{t_k}^{t_{k+1}} \left( \theta_f e^{n_{k+1}}_f \cdot n_d \right) ds$$

$$\leq \frac{\tilde{C}_{inv} h^{1/2} \Delta s}{\rho \gamma} \sum_{n=n_k}^{n_{k+1}} \left\| \theta_f \right\|_0 ^2 + \frac{\gamma \Delta s}{8\rho} \sum_{n=n_k}^{n_{k+1}-1} \left\| e^{n_{k+1}}_f \cdot n_d \right\|_{L^2(I)}^2.$$

Therefore,

$$B_4 \leq \frac{8C^2 h^3 \Delta s^2}{\rho \gamma} \int_{n_k}^{n_{k+1}} \left\| \partial P^h \phi_f \right\|_1 ^2 \, dt + \frac{\gamma \Delta s}{4\rho} \sum_{n=n_k}^{n_{k+1}-1} \left\| e^{n_{k+1}}_f \cdot n_d \right\|_{L^2(I)}^2 + \frac{8\tilde{C}_{inv} \Delta s}{\rho \gamma} \sum_{n=n_k}^{n_{k+1}-1} \left\| \theta_f \right\|_0 ^2.$$

Substitute $B_1 - B_4$ into (4.20) and sum over $k = 0, ..., M-1$, 

$$\frac{\eta_C}{\rho} \sum_{k=0}^{M-1} \left( \left\| \theta_d^{n_{k+1}} \right\|_0 ^2 - \left\| \theta_d^{n_{k}} \right\|_0 ^2 + \left\| \theta_d^{n_{k+1}} - \theta_d^{n_{k}} \right\|_0 ^2 \right) + \frac{2\mu \Delta s}{\rho k_m} \sum_{k=0}^{M-1} \left\| e^{n_{k+1}}_m \right\|_0 ^2 + \frac{2\mu \Delta s}{\rho k_f} \sum_{k=0}^{M-1} \left\| e^{n_{k+1}}_f \right\|_0 ^2$$

$$+ \frac{\sigma k_m \Delta s}{\rho \mu} \sum_{k=0}^{M-1} \left( \left\| \theta_m^{n_{k+1}} \right\|_0 ^2 - \left\| \theta_m^{n_{k}} \right\|_0 ^2 + \left\| \theta_m^{n_{k+1}} - \theta_m^{n_{k}} \right\|_0 ^2 \right) + \frac{\sigma k_m \Delta s}{\rho \mu} \sum_{k=0}^{M-1} \left( \left\| \theta_f^{n_{k+1}} \right\|_0 ^2 - \left\| \theta_f^{n_{k}} \right\|_0 ^2 + \left\| \theta_f^{n_{k+1}} - \theta_f^{n_{k}} \right\|_0 ^2 \right)$$

$$+ \frac{\gamma \Delta t}{2\rho h} \sum_{k=0}^{M-1} \sum_{n=n_k}^{n_{k+1}-1} \left\| e^{n_{k+1}}_f \cdot n_d \right\|_{L^2(I)}^2$$

$$\leq \Delta s \sum_{k=0}^{M-1} \left( \frac{\eta_f C_{C_f} \Delta s^2}{\rho} \left\| \theta_f^{n_{k+1}} \right\|_0 ^2 + \frac{\eta_m C_{C_m}}{\rho} \left\| \theta_m^{n_{k+1}} \right\|_0 ^2 + \frac{8\tilde{C}_{inv} \Delta s}{\rho \gamma} \left\| \theta_f^{n_{k+1}} \right\|_0 ^2 \right) + \frac{\Delta t}{\rho h} \sum_{k=0}^{M-1} \sum_{n=n_k}^{n_{k+1}-1} \left\| e^{n_{k+1}}_f \cdot n_d \right\|_{L^2(I)}^2$$

$$+ \frac{2\eta_f C_{C_f} \Delta s^2}{\rho} \int_0^T \left\| \partial^2 \phi_f \right\|_1 ^2 \, dt + \frac{2\eta_f C_{C_f}}{\rho} \int_0^T \left\| \partial P^h \phi_f \right\|_1 ^2 \, dt$$

$$+ \frac{2\eta_m C_{C_m} \Delta s^2}{\rho} \int_0^T \left\| \partial^2 \phi_m \right\|_1 ^2 \, dt + \frac{2\eta_m C_{C_m} \Delta s}{\rho} \int_0^T \left\| \partial P^h \phi_m \right\|_1 ^2 \, dt$$

$$+ \frac{4\gamma^2 C^2 \Delta t^2}{\rho h} \int_0^T \left\| \partial P^h u_c \right\|_1 ^2 \, dt + \frac{2\gamma \Delta t}{\eta_f C_{C_f} \rho \mu} \int_0^T \left\| \partial P^h \phi_f \right\|_1 ^2 \, dt + \frac{\Delta t}{\eta_m C_{C_m} \rho \mu} \int_0^T \left\| \partial P^h \phi_m \right\|_1 ^2 \, dt$$

By the discrete Gronwall lemma, when $\kappa_n \Delta t \leq \frac{1}{2}$, $\kappa_n = \kappa_1 + \kappa_2 + \kappa_3, \kappa_1 = \frac{\gamma}{\rho h}, \kappa_2 = \frac{\eta_f C_{C_f}}{\rho} + \frac{2\eta_m C_{C_m}}{\rho} \frac{\Delta s}{\rho}$, and \( \kappa_3 = \frac{2\gamma \Delta t}{\eta_f C_{C_f} \rho \mu} \frac{\Delta s^2}{\rho} \), then 

The inequality holds true.
\[
\frac{8C_{\text{cn}}^2}{\rho \kappa d} = \frac{\eta_m C_{\text{mt}}}{\rho}, \text{ we have}
\]
\[
\frac{\eta_d C_{dt}}{\rho} \sum_{k=0}^{M-1} (\|\theta_d^{n_k+1}\|_0^2 - \|\theta_d^{n_k}\|_0^2 + \|\theta_d^{n_k+1} - \theta_d^{n_k}\|_0^2) + \frac{2\mu \Delta s}{\rho k_m} \sum_{k=0}^{M-1} \|e_m^{n_k+1}\|_0^2 + \frac{2\mu \Delta s}{\rho k_f} \sum_{k=0}^{M-1} \|e_f^{n_k+1}\|_0^2
\]
\[
\frac{\sigma k_m \Delta s}{\rho \mu} \sum_{k=0}^{M-1} (\|\theta_m^{n_k+1}\|_0^2 - \|\theta_m^{n_k}\|_0^2 + \|\theta_m^{n_k+1} - \theta_m^{n_k}\|_0^2) + \frac{\sigma k_m \Delta s}{\rho \mu} \sum_{k=0}^{M-1} (\|\theta_f^{n_k+1}\|_0^2 - \|\theta_f^{n_k}\|_0^2 + \|\theta_f^{n_k+1} - \theta_f^{n_k}\|_0^2)
\]
\[
+ \frac{\gamma \Delta t}{2 \rho h} \sum_{k=0}^{M-1} \sum_{n=0}^{n_k} \|e_{f,\infty}^{n_k+1} \cdot n_d\|_{L^2(\Omega)}^2 \leq C(h^4 + \Delta t^2 + \Delta t^2 h^{-1}).
\]

Especially, when \(k = 0\), \(\phi_0^0 = P_h \phi_{f0}, \phi_0^m = P_h \phi_{m0},\)
\[
\frac{2\eta_d C_{dt}}{\rho} \|\theta_d^{n_1}\|_0^2 + \frac{2\mu \Delta s}{\rho k_m} \|e_m^{n_1}\|_0^2 + \frac{2\mu \Delta s}{\rho k_f} \|e_f^{n_1}\|_0^2 + \frac{\gamma \Delta t}{2 \rho h} \|e_{f,\infty}^{n_1}\|_{L^2(\Omega)}^2 + \frac{2\sigma k_m \Delta s}{\rho \mu} \|\theta_m^{n_1}\|_0^2 + \|\theta_f^{n_1}\|_0^2)
\]
\[
\leq C(h^4 + \Delta t^2 + \Delta t^2 h^{-1}).
\]

Next, we give the proof of (4.14) by mathematical induction in time interval \([0, T]\).
When \(l = 0\), by (4.21) and (4.13) to know
\[
\|e_{c,\infty}^{n_1}\|_0^2 + \frac{\eta_f C_{ft}}{\rho} \|\theta_f^{n_1}\|_0^2 + \frac{\eta_m C_{mt}}{\rho} \|\theta_m^{n_1}\|_0^2 + \nu \Delta t \sum_{n=1}^{n_1} \|\nabla e_{c,\infty}^n\|_0^2 + \frac{2\mu \Delta s}{\rho k_f} \|e_f^{n_1}\|_0^2 + \frac{2\mu \Delta s}{\rho k_m} \|e_m^{n_1}\|_0^2
\]
\[
\leq C(h^4 + \Delta t^2 + \Delta t^2 h^{-1}).
\]

Assume that (4.14) holds for \(l = j(1 \leq j \leq M - 2)\), i.e.,
\[
\max_{1 \leq j \leq M - 2} \left\{ \|e_{c,\infty}^{n_j+1}\|_0^2 + \frac{\eta_f C_{ft}}{\rho} \|\theta_f^{n_j+1}\|_0^2 + \frac{\eta_m C_{mt}}{\rho} \|\theta_m^{n_j+1}\|_0^2 \right\} + \nu \Delta t \sum_{j=0}^{n_{j+1}} \|\nabla e_{c,\infty}^{n_{j+1}}\|_0^2 + \frac{2\mu \Delta s}{\rho k_f} \|e_f^{n_{j+1}}\|_0^2 + \frac{2\mu \Delta s}{\rho k_m} \|e_m^{n_{j+1}}\|_0^2 \leq C(h^4 + \Delta t^2 + \Delta t^2 h^{-1}).
\]

Using inverse inequality (2.3), the properties of (4.6), (4.13), the regularity of (??) and imbedding theorem, when \(\Delta t < \tau_1, h < h_1\) and \(\Delta t = O(h^2)\), we have \(n = n_k + 1, \ldots, n_{k+1}, 1 \leq k \leq j,\)
\[
\|u_{c,\infty}^n\|_{L^\infty} \leq \|e_{c,\infty}^n\|_{L^\infty} + \|P_h u_c(t_n)\|_{L^\infty} \leq Ch^{-3/2} \|e_{c,\infty}^n\|_0 + C \|u_c(t_n)\|_2 \leq C(h^{1/2} + h^{-3/2} \Delta t + h^{-2} \Delta t) + C \leq C_B.
\]

and
\[
\Delta t \|u_{c,\infty}^n\|_{W^{1,\infty}} \leq \Delta t (\|e_{c,\infty}^n\|_{W^{1,\infty}} + \|P_h u_c(t_n)\|_{W^{1,\infty}}) \leq \Delta t (Ch^{-3/2} \|\nabla e_{c,\infty}^n\|_0 + C \|u_c(t_n)\|_{W^{2,4}}) \leq C(\Delta t^{1/2} C_B + \Delta t^{3/2} h^{-3/2} + \Delta t^{3/2} h^{-2} + \Delta t) \leq \frac{1}{4}.
\]
When \( l = j + 1 \), taking \((\nu^{k}, q^{k}, \psi^{k}, \psi^{k}, \psi^{k}, \psi^{k}) = (2 \Delta t \varepsilon_{c}^{n+1}, 2 \Delta t \delta_{c}^{n+1}, 2 \Delta s \varepsilon_{f}^{n+1}, 2 \Delta s \theta_{f}^{n+1}, 2 \Delta s \epsilon_{m}^{n+1}, 2 \Delta s \delta_{m}^{n+1})\), combining (4.10), (4.11) and (4.12), and summing over \( n = n_{k}, n_{k} + 1, ..., n_{k+1} - 1 \), we have

\[
\begin{align*}
\|e_{c}^{n+1}\|^{2} - \|e_{c}^{n}\|^{2} & = \sum_{n=n_{k}}^{n_{k+1}-1} \|e_{c}^{n+1} - e_{c}^{n}\|^{2} + \frac{\eta_{d} C_{d}}{\rho} (\|\theta_{d}^{n+1}\|^{2} - \|\theta_{d}^{n}\|^{2} + \|\theta_{d}^{n+1} - \theta_{d}^{n}\|^{2}) \\
& + \frac{\sigma_{k_m} \Delta s}{\rho \mu} (\|\theta_{m}^{n+1}\|^{2} - \|\theta_{m}^{n}\|^{2} + \|\theta_{m}^{n+1} - \theta_{m}^{n}\|^{2}) + \sum_{n=n_{k}}^{n_{k+1}-1} \|\nabla e_{c}^{n+1}\|^{2} + \frac{2 \nu \Delta t}{\sqrt{k}} \sum_{n=n_{k}}^{n_{k+1}-1} \|P_{e} (e_{c}^{n+1})\|^{2}_{L^{2}(\Omega)} + \frac{2 \mu \Delta s}{\rho \kappa_{f}} \|e_{f}^{n+1}\|^{2} + \frac{2 \mu \Delta s}{\rho \kappa_{m}} \|e_{m}^{n+1}\|^{2} \\
& = 2 \nu \Delta t \sum_{n=n_{k}}^{n_{k+1}-1} \left( \frac{\psi_{c}^{n+1} + \tilde{u}_{c}^{n} - \tilde{u}_{c}^{n}(t_{n}) + \tilde{u}_{c}(t_{n})}{\Delta t} + \frac{\eta_{d} C_{d}}{\rho} \left( \frac{\psi_{d}^{n+1} + \tilde{u}_{d}^{n} - \tilde{u}_{d}^{n}(t_{n}) + \tilde{u}_{d}(t_{n})}{\Delta t} \right) \right) + \frac{\eta_{d} C_{d}}{\rho} \left( \frac{\psi_{d}^{n+1} + \tilde{u}_{d}^{n} - \tilde{u}_{d}^{n}(t_{n}) + \tilde{u}_{d}(t_{n})}{\Delta t} \right) \\
& + \left[ \frac{2 \Delta t}{\rho} \sum_{n=n_{k}}^{n_{k+1}-1} \int_{\Omega} (P h \phi_{f}(t_{n+1}) - \phi_{f}^{n+1}) \cdot n_{d} + \frac{2 \Delta s}{\rho} \int_{\Omega} (P h \phi_{f}(t_{n+1}) - \phi_{f}^{n+1}) \cdot n_{d} \\
& + \left[ \frac{2 \gamma \Delta s}{\rho h} \sum_{n=n_{k}}^{n_{k+1}-1} \int_{\Omega} (P h u(t_{n+1}) - S^{n+1}) \cdot n_{d} e_{f}^{n+1} \cdot n_{d} + \frac{2 \gamma \Delta t}{\rho h} \sum_{n=n_{k}}^{n_{k+1}-1} \int_{\Omega} (P h u(t_{n+1}) - S^{n+1}) \cdot n_{d} e_{f}^{n+1} \cdot n_{d} \\
& = \sum_{i=1}^{5} T_{i}. \tag{4.25}
\end{align*}
\]

For the first term

\[
T_{1} = 2 \nu \Delta t \sum_{n=n_{k}}^{n_{k+1}-1} (\psi_{c}^{n+1}, e_{c}^{n+1})_{\Omega_{c}} + 2 \nu \Delta t \sum_{n=n_{k}}^{n_{k+1}-1} \left( \frac{\psi_{c}^{n+1} + \tilde{u}_{c}^{n} - \tilde{u}_{c}^{n}(t_{n}) + \tilde{u}_{c}(t_{n})}{\Delta t} \right)_{\Omega_{c}} \\
+ 2 \nu \Delta t \sum_{n=n_{k}}^{n_{k+1}-1} (P h_{r}^{n+1}, e_{c}^{n+1})_{\Omega_{c}} \\
:= T_{11} + T_{12} + T_{13}.
\]

We repeat the same procedure in A1, by the boundedness of (4.23) and (4.24),

\[
T_{1} \leq \frac{\nu \Delta t}{2} \sum_{n=n_{k}}^{n_{k+1}-1} (\|\nabla e_{c}^{n+1}\|^{2} + \|\nabla e_{c}^{n}\|^{2}) + \frac{2 C_{s} C_{s} C_{c}^{2} \Delta t}{\nu} \sum_{n=n_{k}}^{n_{k+1}-1} \|e_{c}^{n+1}\|^{2} \\
+ \frac{30 C_{s}^{2} C_{s}^{2} C_{c}^{2} \Delta t}{\nu} \sum_{n=n_{k}}^{n_{k+1}-1} \|e_{c}^{n+1}\|^{2} + \frac{6 C_{s} \Delta t}{\nu} \sum_{n=n_{k}}^{n_{k+1}-1} \|R_{r}^{n+1}\|^{2} \\
+ \frac{6 C_{s}^{2} \nu \int_{t_{n_{k}}}^{t_{n_{k+1}}} \left( (P h_{r} - I) \frac{\partial u_{c}}{\partial t}(t) \right)^{2}_{\Omega} dt + C h^{3} \Delta t.}
\]
For the second term, similar with the deduce of $B_1$, by Cauchy-Schwarz inequality and Young inequality

\[
T_2 \leq \frac{2\eta f C_{ft} \Delta s^2}{3\rho} \int_{t_{n_k}}^{t_{n_k+1}} \left( \frac{\partial^2 \phi_f(t)}{\partial t^2} \right)^2 dt + \frac{2\eta f C_{ft}}{\rho} \int_{t_{n_k}}^{t_{n_k+1}} \left( P^h - I \right) \frac{\partial \phi_f(t)}{\partial t} \left( t \right) \bigg|_0^2 dt \\
+ \frac{2\eta m C_{mt} \Delta s^2}{3\rho} \int_{t_{n_k}}^{t_{n_k+1}} \left( \frac{\partial^2 \phi_m(t)}{\partial t^2} \right)^2 dt + \frac{2\eta m C_{mt}}{\rho} \int_{t_{n_k}}^{t_{n_k+1}} \left( P^h - I \right) \frac{\partial \phi_m(t)}{\partial t} \left( t \right) \bigg|_0^2 dt \\
+ \frac{\eta f C_{ft} \Delta s}{2\rho} \left\| \theta_f^{n_{k+1}} \right\|_0^2 + \frac{\eta m C_{mt} \Delta s}{2\rho} \left\| \theta_m^{n_{k+1}} \right\|_0^2.
\]

For the third term

\[
T_3 = \frac{2\Delta t}{\rho} \sum_{n=n_k}^{n_k+1} \int \left( P^h \phi_f(t_{n+1}) - \phi_f^{n+1} \right) \cdot n_{d} ds - \frac{2\Delta s}{\rho} \int \left( P^h \phi_f(t_{n_k+1}) - \phi_f^{n_k+1} \right) \cdot n_{d} ds
\]

\[
= \frac{2\Delta t}{\rho} \sum_{n=n_k}^{n_k+1} \int \left( P^h \phi_f(t_{n+1}) - P^h \phi_f(t_{n_k+1}) \right) e_c^{n+1} \cdot n_{d} ds
\]

\[
+ \frac{2\Delta t}{\rho} \sum_{n=n_k}^{n_k+1} \int \theta_f^{n_k} \left( e_c^{n+1} - e_f^{n_k+1} \right) \cdot n_{d} ds
\]

\[
+ \frac{2\Delta t}{\rho} \sum_{n=n_k}^{n_k+1} \int \left( P^h \phi_f(t_{n_k+1}) - P^h \phi_f(t_{n_k}) \right) \left( e_c^{n+1} - e_f^{n_k+1} \right) \cdot n_{d} ds
\]

\[
:= T_{31} + T_{32} + T_{33}.
\]

Using Cauchy-Schwarz inequality, the trace inverse inequality and the Young inequality, we arrive at

\[
T_{31} = \frac{2\Delta t}{\rho} \sum_{n=n_k}^{n_k+1} \int \left( P^h \phi_f(t_{n+1}) - P^h \phi_f(t_{n_k+1}) \right) e_c^{n+1} \cdot n_{d} ds
\]

\[
\leq \frac{2\tilde{C}_{inv} \rho^{-1} \Delta t}{\rho} \sum_{n=n_k}^{n_k+1} \left\| P^h \phi_f(t_{n+1}) - P^h \phi_f(t_{n_k+1}) \right\| e_c^{n+1} \cdot n_{d} ds
\]

\[
\leq \frac{\tilde{C}_{inv} \rho^{-1} \Delta t}{\rho} \sum_{n=n_k}^{n_k+1} \left\| e_c^{n+1} \right\|_0^2 + \frac{\tilde{C}_{inv} \rho^{-1} \Delta t}{\rho} \int_{t_{n_k}}^{t_{n_k+1}} \left\| \frac{\partial P^h \phi_f(t)}{\partial t} \right\|_0^2 dt,
\]

\[
T_{32} = \frac{2\Delta t}{\rho} \sum_{n=n_k}^{n_k+1} \int \theta_f^{n_k} \left( e_c^{n+1} - e_f^{n_k+1} \right) \cdot n_{d} ds
\]

\[
\leq \frac{2\tilde{C}_{inv} \rho^{-1/2} \Delta t}{\rho} \sum_{n=n_k}^{n_k+1} \left\| \theta_f^{n_k} \right\|_0 \left( e_c^{n+1} - e_f^{n_k+1} \right) \cdot n_{d} \|L^2(I)
\]

\[
\leq \frac{4\tilde{C}_{inv} \rho^{-2} \Delta s}{\rho \gamma} \left\| \theta_f^{n_k+1} \right\|_0^2 + \frac{\gamma \Delta t}{4 \rho \Delta s} \sum_{n=n_k}^{n_k+1} \left( e_c^{n+1} - e_f^{n_k+1} \right) \cdot n_{d} \|L^2(I),
\]

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Therefore,

\[ T_3 \leq \frac{\tilde{C}_{\text{inv}}}{\rho} \Delta t^{1/2} \sum_{n=k}^{n+k-1} \left\| (P^h \phi_f(t_{n+k}) - P^h \phi_f(t_n)) \right\|_0 \| (e_c^{n+1} - e_f^{n+1}) \cdot n_d \|_{L^2(\Omega)} \]

\[ + \frac{4r \tilde{C}_{\text{inv}}^2 \Delta t^2}{\rho \gamma} \int_{t_{n+k}}^{t_{n+1}} \left\| \frac{\partial P^h \phi_f(t)}{\partial t} \right\|_0^2 \, dt + \frac{\tilde{C}_{\text{inv}}}{2\rho h} \sum_{n=n_k}^{n+k-1} \left\| (e_c^n - e_f^n) \cdot n_d \right\|_{L^2(\Omega)}^2. \]

For the fourth term, since \( S^{n+1}_n = \frac{1}{r} \sum_{n=n_k}^{n+k-1} u_c^{n+1} \), then

\[ T_4 = \frac{2\gamma \Delta s}{\rho h} \int_{t_{n+k}}^{t_{n+1}} \left( (P^h u_c(t_{n+1}) - S^{n+1}_n) - e_f^{n+1} \right) \cdot n_d e_f^{n+1} \cdot n_d \, ds \]

\[ - \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n+k-1} \int_{t_{n+k}}^{t_{n+1}} \left( e_c^{n+1} - (P^h u_c(t_{n+1}) - u_f^n) \right) \cdot n_d e_f^{n+1} \cdot n_d \, ds \]

\[ = \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n+k-1} \int_{t_{n+k}}^{t_{n+1}} \left( (P^h u_c(t_{n+1}) - P^h u_c(t_{n+1}) + e_c^{n+1} - e_f^{n+1}) \right) \cdot n_d e_f^{n+1} \cdot n_d \, ds \]

\[ - \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n+k-1} \int_{t_{n+k}}^{t_{n+1}} \left( e_f^{n+1} - (P^h u_f(t_{n+1}) - u_f^n) \right) \cdot n_d e_f^{n+1} \cdot n_d \, ds \]

\[ = \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n+k-1} \int_{t_{n+k}}^{t_{n+1}} \left( (P^h u_c(t_{n+1}) - P^h u_c(t_{n+1})) \right) \cdot n_d e_f^{n+1} \cdot n_d \, ds \]

\[ + \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n+k-1} \int_{t_{n+k}}^{t_{n+1}} \left( (P^h u_f(t_{n+1}) - P^h u_f(t_{n+1})) \right) \cdot n_d e_f^{n+1} \cdot n_d \, ds \]

\[ - \left[ \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n+k-1} \left\| (e_c^{n+1} - e_f^{n+1}) \cdot n_d \right\|_{L^2(\Omega)}^2 + \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n+k-1} \left\| (e_f^{n+1} - e_f^n) \right\|_{L^2(\Omega)}^2 \right] \]

\[ := T_{41} + T_{42} + T_{43}. \]

Applying Cauchy-Schwarz inequality and Young inequality, we show that

\[ T_{41} = \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n+k-1} \int_{t_{n+k}}^{t_{n+1}} \left( (P^h u_c(t_{n+1}) - P^h u_c(t_{n+1})) \right) \cdot n_d e_f^{n+1} \cdot n_d \, ds \]

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\[
\begin{align*}
&\leq \frac{2\gamma C_T \Delta t}{\rho h} \sum_{n=n_k}^{n_{k+1}-1} \|P^h u_c(t_{n+1}) - P^h u_c(t_{n+1})\|_1 \|e_{c_{n+1}} - e_{c_{n+1}}\| \|n_d\|_{L^2(\Omega)} \\
&\quad + \frac{r^2 \gamma C_T^2 \Delta t}{\rho^2 h} \int_{t_{n_k}}^{t_{n_k+1}} \left\| \frac{\partial P^h u_c}{\partial t}(t) \right\|_1^2 dt + \frac{C^2_{\text{div}} \Delta t}{h} \sum_{n=n_k}^{n_{k+1}-1} \|e_{c_{n+1}}\|_0^2 \\
&\leq \frac{4\gamma C_T^2 \Delta t}{\rho h} \int_{t_{n_k}}^{t_{n_k+1}} \left\| \frac{\partial P^h u_c}{\partial t}(t) \right\|_1^2 dt + \frac{\gamma \Delta t}{4 \rho h} \sum_{n=n_k}^{n_{k+1}-1} \|e_{c_{n+1}} - e_{c_{n+1}}\|_{L^2(\Omega)}^2 \\
&\quad + \frac{r^2 \gamma C_T^2 \Delta t}{\rho^2 h} \int_{t_{n_k}}^{t_{n_k+1}} \left\| \frac{\partial P^h u_c}{\partial t}(t) \right\|_1^2 dt + \frac{C^2_{\text{div}} \Delta t}{h} \sum_{n=n_k}^{n_{k+1}-1} \|e_{c_{n+1}}\|_0^2.
\end{align*}
\]

By Hölder inequality, the general trace inequality, the Young inequality, the property of \(H(\text{div})\) space and the divergence free condition,

\[
T_{42} = \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_{k+1}-1} \int_{1}^{\Omega} \left( P^h u_f(t_{n+1}) - P^h u_f(t_{n_k}) \right) \cdot n_d e_{c_{n+1}} \cdot n_d ds
\]

\[
= \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_{k+1}-1} \int_{\partial \Omega_c} \left( P^h u_f(t_{n+1}) - P^h u_f(t_{n_k}) \right) \cdot n_d e_{c_{n+1}} \cdot n_d ds
\]

\[
\leq \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_{k+1}-1} \left\| \left( P^h u_f(t_{n+1}) - P^h u_f(t_{n_k}) \right) \cdot n_d \right\|_{H^{1/2}(\partial \Omega_c)} \|e_{c_{n+1}} \|_{H^{-1/2}(\partial \Omega_c)}
\]

\[
\leq \frac{2r \gamma C_T C_{\text{div}} \Delta t}{\rho h} \sum_{n=n_k}^{n_{k+1}-1} \|P^h u_f(t_{n+1}) - P^h u_f(t_{n_k})\|_1 \|e_{c_{n+1}}\|_{H(\text{div})}
\]

\[
\leq \frac{r^2 \gamma C_T^2 \Delta t}{\rho^2 h} \int_{t_{n_k}}^{t_{n_k+1}} \left\| \frac{\partial P^h u_f}{\partial t}(t) \right\|_1^2 dt + \frac{C^2_{\text{div}} \Delta t}{h} \sum_{n=n_k}^{n_{k+1}-1} \|e_{c_{n+1}}\|_0^2.
\]

And

\[
T_{43} = -\frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_{k+1}-1} \left\| (e_{c_{n+1}} - e_{c_{n+1}}) \cdot n_d \right\|_{L^2(\Omega)}^2 - \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_{k+1}-1} \int_{1}^{\Omega} (e_{c_{n+1}} - e_{c_{n+1}}) \cdot n_d e_{c_{n+1}} \cdot n_d ds
\]

\[
= -\frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_{k+1}-1} \left\| (e_{c_{n+1}} - e_{c_{n+1}}) \cdot n_d \right\|_{L^2(\Omega)}^2 - \frac{2\gamma \Delta s}{\rho h} \|e_{c_{n+1}}\|_{L^2(\Omega)}^2
\]

\[
+ \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_{k+1}-1} \int_{1}^{\Omega} e_{c_{n+1}} \cdot n_d e_{c_{n+1}} \cdot n_d ds + \frac{2\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_{k+1}-1} \int_{1}^{\Omega} (e_{c_{n+1}} - e_{c_{n+1}}) \cdot n_d (e_{c_{n+1}} - e_{c_{n+1}}) \cdot n_d ds
\]

\[
\leq -\frac{\gamma \Delta t}{\rho h} \sum_{n=n_k}^{n_{k+1}-1} \left\| (e_{c_{n+1}} - e_{c_{n+1}}) \cdot n_d \right\|_{L^2(\Omega)}^2 - \frac{\gamma \Delta s}{\rho h} \|e_{c_{n+1}}\|_{L^2(\Omega)}^2
\]

Therefore,

\[
T_4 \leq -\frac{\gamma \Delta s}{\rho h} \left[ \|e_{c_{n+1}}\|_{L^2(\Omega)}^2 - \|e_{c_{n+1}}\|_{L^2(\Omega)}^2 \right] - \frac{3\gamma \Delta t}{4 \rho h} \sum_{n=n_k}^{n_{k+1}-1} \left\| (e_{c_{n+1}} - e_{c_{n+1}}) \cdot n_d \right\|_{L^2(\Omega)}^2
\]
\[
+ \frac{4 \gamma C_{f}^{2} \varDelta t^{2}}{\rho h} \int_{t_{n_{k}}}^{t_{n_{k}+1}} \left\| \frac{\partial P h u_{c}^{e}}{\partial t} (t) \right\|_{1}^{2} dt + \frac{r^{2} \gamma C_{f}^{2} \varDelta t^{2}}{\rho^{2} h} \int_{t_{n_{k}}}^{t_{n_{k}+1}} \left\| \frac{\partial P h u_{c}^{e}}{\partial t} (t) \right\|_{1}^{2} dt \\
+ \frac{r^{2} \gamma C_{f}^{2} \varDelta t^{2}}{\rho^{2} h} \int_{t_{n_{k}}}^{t_{n_{k}+1}} \left\| \frac{\partial P h u_{c}^{l}}{\partial t} (t) \right\|_{1}^{2} dt + \frac{2 \gamma^{2} \varDelta t^{2}}{h} \sum_{n=n_{k}}^{n_{k+1}} \left\| e_{c}^{n+1} \right\|_{0}^{2}.
\]

For the fifth term
\[
T_{5} = \frac{2 \sigma k_{m} \varDelta s}{\rho \mu} (P h \phi_{m}(t_{n_{k}+1}) - P h \phi_{m}(t_{n_{k}}), \theta_{f}^{n_{k}+1})_{\Omega_{d}} + \frac{2 \sigma k_{m} \varDelta s}{\rho \mu} (P h \phi_{f}(t_{n_{k}+1}) - P h \phi_{f}(t_{n_{k}}), \theta_{m}^{n_{k}+1})_{\Omega_{d}}
\]

\[
:= T_{51} + T_{52}.
\]

\(T_{51}\) is bounded by Cauchy-Schwarz inequality and the Young inequality,
\[
T_{51} = \frac{2 \sigma k_{m} \varDelta s}{\rho \mu} (P h \phi_{m}(t_{n_{k}+1}) - P h \phi_{m}(t_{n_{k}}), \theta_{f}^{n_{k}+1})_{\Omega_{d}}
\]

\[
\leq \frac{2 \sigma k_{m} \varDelta s}{\eta_{f} C_{f} \rho \mu^{2}} \int_{t_{n_{k}}}^{t_{n_{k}+1}} \left\| \frac{\partial P h \phi_{m}}{\partial t} (t) \right\|_{0}^{2} dt + \frac{\eta_{f} C_{f} \varDelta s}{2 \rho} \left\| \theta_{f}^{n_{k}+1} \right\|_{0}^{2}.
\]

In the same spirit,
\[
T_{52} = \frac{2 \sigma k_{m} \varDelta s}{\rho \mu} (P h \phi_{f}(t_{n_{k}+1}) - P h \phi_{f}(t_{n_{k}}), \theta_{m}^{n_{k}+1})_{\Omega_{d}}
\]

\[
\leq \frac{2 \sigma k_{m} \varDelta s}{\eta_{m} C_{m} \rho \mu^{2}} \int_{t_{n_{k}}}^{t_{n_{k}+1}} \left\| \frac{\partial P h \phi_{f}}{\partial t} (t) \right\|_{0}^{2} dt + \frac{\eta_{m} C_{m} \varDelta s}{2 \rho} \left\| \theta_{m}^{n_{k}+1} \right\|_{0}^{2}.
\]

Therefore,
\[
T_{5} \leq \frac{2 \sigma k_{m} \varDelta s}{\eta_{f} C_{f} \rho \mu^{2}} \int_{t_{n_{k}}}^{t_{n_{k}+1}} \left\| \frac{\partial P h \phi_{m}}{\partial t} (t) \right\|_{0}^{2} dt + \frac{\eta_{f} C_{f} \varDelta s}{2 \rho} \left\| \theta_{f}^{n_{k}+1} \right\|_{0}^{2}
\]

\[
+ \frac{2 \sigma k_{m} \varDelta s}{\eta_{m} C_{m} \rho \mu^{2}} \int_{t_{n_{k}}}^{t_{n_{k}+1}} \left\| \frac{\partial P h \phi_{f}}{\partial t} (t) \right\|_{0}^{2} dt + \frac{\eta_{m} C_{m} \varDelta s}{2 \rho} \left\| \theta_{m}^{n_{k}+1} \right\|_{0}^{2}.
\]

Combining \(T_{1}, T_{2}, T_{3}, T_{4}\) and \(T_{5}\), summing over \(k = 0, ..., j + 1\) \((1 \leq j \leq M - 2)\), we arrive at
\[
\left\| e_{c}^{n+1} \right\|_{0}^{2} + \sum_{k=0}^{j+1} \sum_{n=n_{k}}^{n_{k+1}} \left\| e_{c}^{n+1} - e_{c}^{n} \right\|_{0}^{2} + \frac{\eta_{f} C_{f} \varDelta t}{\rho} \left\| \theta_{f}^{n_{k}+1} \right\|_{0}^{2} + \frac{\eta_{m} C_{m} \varDelta t}{\rho} \left\| \theta_{m}^{n_{k}+1} \right\|_{0}^{2} + \frac{\eta_{f} C_{f} \varDelta t}{\rho} \sum_{k=0}^{j+1} \left\| \theta_{f}^{n_{k}+1} - \theta_{f}^{n_{k}} \right\|_{0}^{2}
\]

\[
+ \frac{\eta_{m} C_{m} \varDelta t}{\rho} \sum_{k=0}^{j+1} \left\| \theta_{m}^{n_{k}+1} - \theta_{m}^{n_{k}} \right\|_{0}^{2} + \frac{\sigma k_{m} \varDelta s}{\rho \mu} \sum_{k=0}^{j+1} \left\| \theta_{m}^{n_{k}+1} - \theta_{m}^{n_{k}} \right\|_{0}^{2} + \frac{\gamma \varDelta t}{\rho h} \sum_{k=0}^{j+1} \left\| \left( e_{c}^{n_{k}+1} - e_{c}^{n_{k}} \right) \cdot n_{c} \right\|_{0}^{2} + \frac{\gamma \varDelta t}{\rho h} \sum_{k=0}^{j+1} \left\| e_{c}^{n_{k}+1} \right\|_{0}^{2}
\]

\[
+ \frac{2 \mu \varDelta s}{\rho k_{m}} \sum_{k=0}^{j+1} \left\| e_{m}^{n_{k}+1} \right\|_{0}^{2} + \frac{\gamma \varDelta t}{4 \rho h} \sum_{k=0}^{j+1} \left\| \left( e_{c}^{n_{k}+1} - e_{c}^{n_{k}} \right) \cdot n_{d} \right\|_{0}^{2} + \frac{\gamma \varDelta s}{\rho h} \sum_{k=0}^{j+1} \left\| e_{m}^{n_{k}+1} \right\|_{0}^{2} + \frac{2 \mu \varDelta s}{\rho k_{m}} \sum_{k=0}^{j+1} \left\| e_{m}^{n_{k}+1} \right\|_{0}^{2}
\]

\[
\leq \frac{6 C_{f}^{2} \varDelta t}{\nu} \sum_{k=0}^{j+1} \sum_{n=n_{k}}^{n_{k+1}} \left\| R_{\nu}^{n_{k}+1} \right\|_{0}^{2} + \frac{6 C_{f}^{2} \varDelta t}{\nu} \int_{0}^{T} \left( P h - I \right) \left\| \frac{\partial u_{c}^{e}}{\partial t} (t) \right\|_{0}^{2} dt + \frac{r^{2} \gamma C_{f}^{2} \varDelta t^{2}}{\rho^{2} h} \int_{t_{n_{k}}}^{t_{n_{k}+1}} \left\| \frac{\partial P h u_{c}^{e}}{\partial t} (t) \right\|_{1}^{2} dt
\]

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the following corollary.

Further, by imbedding theorem and inverse inequality, \( \kappa_n \Delta t \leq \frac{1}{2} \), where \( \kappa_\nu = \kappa_1 + \kappa_2 + \kappa_3 \), \( \kappa_1 = \frac{2C^2_{\mu \nu}C_{\mu \nu}}{\nu} + \frac{C_{\mu \nu} + 2C^2_{\mu \nu}}{\nu} \), \( \kappa_2 = \frac{\eta C_{f t}}{\rho} + \frac{4C^2_{\mu \nu}}{\rho \gamma} \), \( \kappa_3 = \frac{\eta m C_{m t}}{\rho} \) and \( \Delta t = O(h^2) \), we conclude that

\[
\frac{\max_{1 \leq j \leq M-2} \left\{ \|e_f^{n+j+1}\|_\infty^2 + \frac{\eta C_{f t}}{\rho} \|\theta_f^{n+j+1}\|_\infty^2 + \frac{\eta m C_{m t}}{\rho} \|\theta_m^{n+j+1}\|_\infty^2 \right\} + \nu \Delta t \sum_{k=0}^{j+1} \sum_{n=n_k}^{n_{k+1}} \|\nabla e_{c}^{n+1}\|_\infty^2 + 2 \mu \Delta s \sum_{k=0}^{j+1} \sum_{n=n_k}^{n_{k+1}} \|e_{c}^{n+1}\|_\infty^2 \leq C(h^4 + \Delta t^2 + \Delta t^2 h^{-1}).
\]

Further more, by imbedding theorem and inverse inequality, \( n = n_k + 1, ..., n_{k+1}, 0 \leq k \leq M - 1 \),

\[
\max_{0 \leq k \leq M-1 \atop n \in \{n_k+1, ..., n_{k+1}\}} \|u_{c}^n\|_{L^\infty} \leq \max_{0 \leq k \leq M-1 \atop n \in \{n_k+1, ..., n_{k+1}\}} \left( \|e_{c}^n\|_{L^\infty} + \|P^h u_c(t_n)\|_{L^\infty} \right)
\]

\[
\leq C h^{-3/2} \max_{0 \leq k \leq M-1 \atop n \in \{n_k+1, ..., n_{k+1}\}} \|e_{c}^n\|_\infty + C \max_{0 \leq k \leq M-1 \atop n \in \{n_k+1, ..., n_{k+1}\}} \|u_c(t_n)\|_2
\]

\[
\leq C_B,
\]

and

\[
\Delta t \sum_{k=0}^{M-1} \sum_{n=n_k+1}^{n_{k+1}} \|u_{c}^n\|_{W^{1,\infty}}^2 \leq \Delta t \sum_{k=0}^{M-1} \sum_{n=n_k+1}^{n_{k+1}} \left( \|e_{c}^n\|_{W^{1,\infty}}^2 + \|P^h u_c(t_n)\|_{W^{1,\infty}}^2 \right)
\]

\[
\leq C h^{-3} \Delta t \sum_{k=0}^{M-1} \sum_{n=n_k+1}^{n_{k+1}} \|\nabla e_{c}^n\|_0^2 + C \Delta t \sum_{k=0}^{M-1} \sum_{n=n_k+1}^{n_{k+1}} \|u_c(t_n)\|_{W^{2,d^*}}^2
\]

\[
\leq C_B,
\]

where \( d^* > D \).

From the triangle inequality and the approximation properties (4.3)(4.4)(4.5), we show the following corollary.
Corollary 4.4 (Error convergence). Let assumptions of Theorem 4.3 hold, we have the following error convergence

\[
\max_{0 \leq m \leq M} \left( \| u_c(t_m) - u_c^m \|_0^2 + \| \phi_d(t_m) - \phi_d^m \|_0^2 \right) \leq C(h^4 + \Delta t^2 + \Delta t^2 h^{-1}),
\]

(4.26)

where

\[
\| \phi_d(t_m) - \phi_d^m \|_0^2 := \| \phi_f(t_m) - \phi_f^m \|_0^2 + \| \phi_m(t_m) - \phi_m^m \|_0^2.
\]

5. Numerical examples

In this section, some numerical examples are used to show the convergence performance and efficiency of decoupled modified characteristic finite element method with different sub-domain time steps for the mixed stabilized formulation. The first example’s results validate the optimal convergence order with different time steps. Furthermore, by changing the penalty parameter to search the impact for the convergence performance. Finally, we use 2-cores CPU to solve this problem. The second example is used to investigate the performance of practical issue by adjusting some different physical parameters. Such as the velocity of injection well, the matrix velocity on the boundaries and the deep relationship between injection wellborn and horizontal open-hole completion wellborn. In this way, we can get some comparative experimental phenomena. For the finite element space constructed in Section 2, we use MINI elements (P1b-P1) for the Navier-Stokes equations and the Brezzi-Douglas-Marini (BMD1) for the microfracture flow velocity \( u_f \) and matrix flow velocity \( u_m \) in two different Darcy equations. The corresponding pressure \( \phi_f \) and \( \phi_m \) use piecewise constant elements P0. The FreeFEM++ package [41] is used to carry out experiments and all results are generated by the same computer.

5.1. Example 1

The solving area of model problem is \( \Omega = \Omega_c \cup \Omega_d \), where the conduct domain is \( \Omega_c = [0, 1] \times [1, 2] \) and the dual-porosity domain is \( \Omega_d = [0, 1] \times [0, 1] \). The non-slip interface is \( \mathbb{I} = (0, 1) \times \{1\} \). The analytical solutions satisfying the dual-porosity-Navier-Stokes model is as follows:

\[
\begin{align*}
 u_c &= \begin{bmatrix} (x^2(y-1)^2+y) \cos(t), & -\frac{2}{3}x(y-1)^3 \cos(t) + (2 - \pi \sin(\pi x)) \cos(t) \end{bmatrix}^T, \\
p_c &= (2 - \pi \sin(\pi x)) \sin(0.5\pi y) \cos(t), \\
\phi_f &= (2 - \pi \sin(\pi x))(1 - y - \cos(\pi y)) \cos(t), \\
\phi_m &= (2 - \pi \sin(\pi x))(\cos(\pi(1 - y))) \cos(t), \\
 u_f &= -\frac{k_f}{\mu} \nabla \phi_f, \\
 u_m &= -\frac{k_m}{\mu} \nabla \phi_m.
\end{align*}
\]

In addition, the initial condition, boundary conditions and the forcing terms follow the analytical solutions. For simplicity, take the parameters \( \nu, \mu, \sigma, \alpha, \rho, \eta_f, \eta_m, C_{ft}, C_{mt}, k_f, k_m \) are all 1.0 in numerical example, penalty parameter \( \gamma = 0.1 \), and the final time \( T = 0.5 \).
5.1.1. Convergence performance and CPU comparison with different subdomain time steps

We take varying space steps \( h = 1/4, 1/8, 1/16, 1/32, 1/64 \), different time step ratios \( r = 1, 2, 4, 8 \) and choose the corresponding time step \( \Delta t = h^2 \). Calculate the errors between the exact solution and numerical solution of \( L^2 \)-norm and \( H^1 \)-seminorm for the velocity separately. And we compute the pressure and its \( L^2 \)-norm. In addition, the corresponding rate of convergence are obtained.

From Table 1-4, we see that \( L^2 \)-norm convergence rate of \( u_c, u_f \) and \( u_m \) is 2.0. The \( H^1 \)-seminorm convergence rate of \( u_c \), the \( L^2 \)-norm convergence rate of \( \phi_f \) and \( \phi_m \) is 1.0. In other words, the error in sense of \( L^2 \)-norm obtains the optimal convergence rate \( O(h^2) \), and the error in sense of semi-\( H^1 \) norm obtains the optimal rate \( O(h) \).

### Table 1: The convergence performance for \( \Delta t = h^2 \) at \( r = 1 \)

| \( h \) | \( \| u_c - u^h_c \|_0 \) | Rate | \( \| u_f - u^h_f \|_0 \) | Rate | \( \| u_m - u^h_m \|_0 \) | Rate |
|---|---|---|---|---|---|---|
| \( 1/4 \) | 0.129317 | – | 0.536254 | – | 0.482127 | – |
| \( 1/8 \) | 0.029965 | 2.11 | 0.115880 | 2.21 | 0.111867 | 2.11 |
| \( 1/16 \) | 0.007332 | 2.03 | 0.023838 | 2.28 | 0.024654 | 2.18 |
| \( 1/32 \) | 0.001793 | 2.03 | 0.005283 | 2.17 | 0.005993 | 2.04 |
| \( 1/64 \) | 0.000447 | 2.00 | 0.001336 | 1.98 | 0.001532 | 1.97 |
| \( h \) | \( \| \nabla (u_c - u^h_c) \|_0 \) | Rate | \( \| \phi_f - \phi^h_f \|_0 \) | Rate | \( \| \phi_m - \phi^h_m \|_0 \) | Rate |
|---|---|---|---|---|---|---|
| \( 1/4 \) | 1.660150 | – | 0.240974 | – | 0.264592 | – |
| \( 1/8 \) | 0.703788 | 1.24 | 0.122430 | 0.98 | 0.135008 | 0.97 |
| \( 1/16 \) | 0.333161 | 1.08 | 0.054168 | 1.18 | 0.061704 | 1.13 |
| \( 1/32 \) | 0.158962 | 1.07 | 0.027427 | 0.98 | 0.031148 | 0.99 |
| \( 1/64 \) | 0.078633 | 1.02 | 0.014070 | 0.96 | 0.015801 | 0.98 |

### Table 2: The convergence performance for \( \Delta t = h^2 \) at \( r = 2 \)

| \( h \) | \( \| u_c - u^h_c \|_0 \) | Rate | \( \| u_f - u^h_f \|_0 \) | Rate | \( \| u_m - u^h_m \|_0 \) | Rate |
|---|---|---|---|---|---|---|
| \( 1/4 \) | 0.129303 | – | 0.530867 | – | 0.480194 | – |
| \( 1/8 \) | 0.029945 | 2.11 | 0.115879 | 2.20 | 0.111907 | 2.10 |
| \( 1/16 \) | 0.007323 | 2.03 | 0.023828 | 2.28 | 0.024687 | 2.18 |
| \( 1/32 \) | 0.001790 | 2.03 | 0.005282 | 2.17 | 0.005993 | 2.04 |
| \( 1/64 \) | 0.000446 | 2.01 | 0.001336 | 1.98 | 0.001535 | 1.97 |
| \( h \) | \( \| \nabla (u_c - u^h_c) \|_0 \) | Rate | \( \| \phi_f - \phi^h_f \|_0 \) | Rate | \( \| \phi_m - \phi^h_m \|_0 \) | Rate |
|---|---|---|---|---|---|---|
| \( 1/4 \) | 1.660140 | – | 0.240974 | – | 0.264592 | – |
| \( 1/8 \) | 0.703787 | 1.24 | 0.122430 | 0.98 | 0.135008 | 0.97 |
| \( 1/16 \) | 0.333162 | 1.08 | 0.054168 | 1.18 | 0.061704 | 1.13 |
| \( 1/32 \) | 0.158962 | 1.07 | 0.027427 | 0.98 | 0.031148 | 0.99 |
| \( 1/64 \) | 0.078633 | 1.02 | 0.014070 | 0.96 | 0.015801 | 0.98 |
Table 3: The convergence performance for $\Delta t = h^2$ at $r = 4$

| $h$ | $\|u_c - u_h^h\|_0$ | Rate | $\|u_f - u_f^h\|_0$ | Rate | $\|u_m - u_m^h\|_0$ | Rate |
|-----|------------------|------|------------------|------|------------------|------|
| $\frac{1}{4}$ | 0.129279 | – | 0.522902 | – | 0.475665 | – |
| $\frac{1}{8}$ | 0.029894 | 2.11 | 0.115875 | 2.17 | 0.111991 | 2.09 |
| $\frac{1}{16}$ | 0.007305 | 2.03 | 0.023810 | 2.28 | 0.024757 | 2.18 |
| $\frac{1}{32}$ | 0.001783 | 2.01 | 0.005282 | 2.17 | 0.006024 | 2.04 |

Also, we record the corresponding time cost in different time steps.

Table 4: The convergence performance for $\Delta t = h^2$ at $r = 8$

| $h$ | $\|u_c - u_h^h\|_0$ | Rate | $\|u_f - u_f^h\|_0$ | Rate | $\|u_m - u_m^h\|_0$ | Rate |
|-----|------------------|------|------------------|------|------------------|------|
| $\frac{1}{4}$ | 0.129111 | – | 0.517255 | – | 0.463726 | – |
| $\frac{1}{8}$ | 0.029782 | 2.12 | 0.115885 | 2.16 | 0.112212 | 2.05 |
| $\frac{1}{16}$ | 0.007305 | 2.03 | 0.023810 | 2.28 | 0.024910 | 2.17 |
| $\frac{1}{32}$ | 0.001783 | 2.01 | 0.005282 | 2.17 | 0.006024 | 2.04 |

Also, we record the corresponding time cost in different time steps.

Table 5: The CPU cost performance with different time step

| CPU Time(s) | $h$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ | $\frac{1}{32}$ | $\frac{1}{64}$ |
|-------------|-----|-------------|-------------|-------------|-------------|-------------|
| $r$        |     | 1.04       | 1.29       | 1.35       | 2.08       | 3.89       |
| 1          |     | 1.10       | 1.32       | 2.08       | 3.89       | 3.04       |
| 2          |     | 0.06       | 0.60       | 8.43       | 13.49      | 2356.03    |
| 3          |     | 0.05       | 0.49       | 6.76       | 107.78     | 1885.20    |
| 4          |     | 0.04       | 0.44       | 5.89       | 94.31      | 1649.38    |

When taking different $r$, we obtain similar errors. From Table 5, as time step ratio $r$ grows, the CPU time decreases. When $h$ is smaller, the CPU time is so much less. Especially, $r = 2$ is obvious.

Next, fixing the time step $\Delta t = 0.001$, we give the relative errors and the CPU time at $r = 1$ as follows.
Table 6: The relative errors and time cost of the characteristic FEMs

| h  | $||u_c - u_h||_0$ | $||\nabla(u_c - u_h)||_0$ | $||u_f - u_h^f||_0$ | $||\phi_f - \phi_h^f||_0$ | $||u_m - u_h^m||_0$ | $||\phi_m - \phi_h^m||_0$ | CPU(s) |
|----|------------------|------------------|------------------|------------------|------------------|------------------|-------|
| 1/4| 0.075926         | 0.264307         | 0.129379         | 0.448490         | 0.102381         | 0.442385         | 8.50  |
| 1/8| 0.017523         | 0.112033         | 0.027754         | 0.220726         | 0.023679         | 0.225042         | 17.96 |
| 1/16| 0.004300      | 0.053037         | 0.007509         | 0.097581         | 0.005215         | 0.0102845        | 47.70 |
| 1/32| 0.001060        | 0.025306         | 0.001265         | 0.049407         | 0.001269         | 0.051915         | 184.04|
| 1/64| 0.000275        | 0.012519         | 0.000319         | 0.025345         | 0.000326         | 0.026335         | 835.67|

At the same time, we apply the Newton iteration method to solve the Navier-Stokes equations as for a contrast of using modified characteristic finite element method. Also, we solve the problem in the same mesh.

Table 7: The relative errors and time cost of the Newton iteration method

| h  | $||u_c - u_h||_0$ | $||\nabla(u_c - u_h)||_0$ | $||u_f - u_h^f||_0$ | $||\phi_f - \phi_h^f||_0$ | $||u_m - u_h^m||_0$ | $||\phi_m - \phi_h^m||_0$ | CPU(s) |
|----|------------------|------------------|------------------|------------------|------------------|------------------|-------|
| 1/4| 0.075927         | 0.264311         | 0.129379         | 0.448490         | 0.102381         | 0.442385         | 20.72 |
| 1/8| 0.017515         | 0.112035         | 0.027753         | 0.220726         | 0.023679         | 0.225042         | 32.47 |
| 1/16| 0.004293         | 0.053038         | 0.007510         | 0.097581         | 0.005215         | 0.0102845        | 110.48|
| 1/32| 0.001051         | 0.025306         | 0.001266         | 0.049407         | 0.001269         | 0.051915         | 433.19|
| 1/64| 0.000260         | 0.012518         | 0.000319         | 0.025345         | 0.000326         | 0.026335         | 1442.66|

Obviously, the two method obtain similar accuracy. Note that the time to solve is reduced greatly with modified characteristic finite element method. Therefore, when we increase the mesh size, the modified characteristic finite element method performances efficiently in same accuracy. Also, when take different $r$, we can get the same conclusion.

5.1.2. The stability performance with different penalty parameters

Next, we use the method to test the convergence performance in different penalty parameter $\gamma = 0, 0.0001, 0.001, 0.01, 1$. Show the log-log plot of the errors as follows.

Figure 3: The effect of the different values of the penalty parameter on the order of convergence for velocity.
The dual-porosity area $\Omega_d$ is made up microfracture area and matrix area. The boundary is $\partial \Omega_d^1 = \{(x,y) : y = 0, 0 \leq x \leq 6\}$, $\partial \Omega_d^2 = \{(x,y) : x = 6, 0 \leq y \leq 1.38\}$, $\partial \Omega_d^3 = \{(x,y) : x = 5.75, 1.63 \leq y \leq 3\}$, $\partial \Omega_d^4 = \{(x,y) : y = 3, 0.25 \leq x \leq 5.75\}$, $\partial \Omega_d^5 = \{(x,y) : x = 0, 0 \leq y \leq 3\}$. In $\partial \Omega_d^1, \partial \Omega_d^2, \partial \Omega_d^3, \partial \Omega_d^4, \partial \Omega_d^5$, the velocity in microfracture area is $\mathbf{u}_f$, and the velocity in matrix area is $\mathbf{u}_m$. Here, The velocity in the two domains are supposed in $(0,0.5),(-0.5,0),(0,-0.5),(0.5,0)$ and $(0,0.001),(-0.001,0)$ separately. Furthermore, in contrast, we will increase the velocity value $(0,\Theta),(-\Theta,0),(\Theta,0)$ on the boundary of matrix $\mathbf{u}_m$, where the $\Theta = 0.05,0.1$. At the same time, other parameters are not changed.
Figure 5: A sketch of the conduit region $\Omega_c$, the dual-porosity region $\Omega_d$ and the interface $I$.

The domain of free flow is made up with a vertical injection and production wellbore with horizontal open-hole completion. The inflow boundary condition is applied on the top of the left vertical well $\partial \Omega_{c,in} = \{(x, y) : y = 7, 0 \leq x \leq 0.25\}$, $U_{c1} = 0, U_{c2} = -64x(0.25 - x)$. Accordingly, on the top of the right vertical wellbore $\partial \Omega_{c,out} = \{(x, y) : y = 7, 5.75 \leq x \leq 6\}$, we apply the Neumann boundary condition $(-p_cI + \nu \nabla u_c) \cdot n_c = 0$. The height of injection wellbore and production wellbore is $\partial \Omega_c = \{(x, y) : x = 0, 3 \leq y \leq 7\} \cup \{(x, y) : x = 0.25, 3 \leq y \leq 7\} \cup \{(x, y) : x = 6, 1.38 \leq y \leq 7\} \cup \{(x, y) : x = 5.75, 1.63 \leq y \leq 7\}$. In here, we impose the non-slip boundary condition $u_c = (0, 0)$.

The model parameters are chosen as follows, $\eta_f = 10^{-4}, \eta_m = 10^{-2}, C_{ft} = 10^{-5}, C_{mt} = 10^{-5}, k_f = 10^{-4}, k_m = 10^{-8}, \mu = 10^{-2}, \nu = 10^{-2}, \sigma = 0.9, \alpha = 1.0, \rho = 1.0, f_c = 0, f_d = 0$, and $\gamma = 10$. The mesh size and time step are taken $h = 1/32, \Delta t = 0.01$. And the final time $T = 5.0$.

Figure 6: The flow speed and streamlines around a injection well and horizontal open-hole attached with vertical production wellbore completion. Left: the flow in the microfractures and conduits; Right: the flow in the matrix.
6. Conclusions

In this paper, we develop and analyze the decoupled modified characteristic finite element method with different subdomain time steps for the mixed stabilized formulation of nonstationary dual-porosity-Navier-Stokes model. Its main feature is a combination of characteristic methods and finite element methods, which leads to a decoupled and fully discrete scheme without nonlinear terms. Under certain assumptions the $L^\infty$-norm of the fully discrete velocity solution is uniformly bounded, and then we prove the error convergence of the velocity and the corresponding pressures in sense of the $L^2$-norm and the $H^1$-seminorm. Further the numerical tests show the validity of the methods we develop. In order to improve our methods, several open problems remain to be solved, e.g., how to relax assumptions but still keep the uniform $L^\infty$ boundedness of $\mathbf{u}_c^n$; whether there exist other solutions of dealing with the interface terms in the fully discrete scheme for better convergence results with $H^1$-seminorm; the modified characteristic FEM mixed with other numerical methods such as stabilization methods and high-order time discretization methods are worthy of research items.

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