On the Ground Level of Purely Magnetic Algebro-Geometric 2D Pauli Operator (spin 1/2)

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Abstract

Full manifold of the complex Bloch-Floquet eigenfunctions is investigated for the ground level of the purely magnetic 2D Pauli operators (equal to zero because of supersymmetry). Deep connection of it with the 2D analog of the "Burgers Nonlinear Hierarchy" plays fundamental role here. Everything is completely calculated for the broad class of Algebro-Geometric operators found in this work for this case. For the case of nonzero flux the ground states were found by Aharonov-Casher (1979) for the rapidly decreasing fields, and by Dubrovin-Novikov (1980) for the periodic fields. No Algebro-Geometric operators where known in the case of nonzero flux. For genus $g = 1$ we found periodic operators with zero flux, singular magnetic fields and Bohm-Aharonov

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4All papers of S.P.Novikov cited in the References are available on the web page http://www.mi.ras.ru/~snovikov click Publications.
Our arguments imply that the delta-term really does not affect seriously the spectrum nearby of the ground state. For \( g > 1 \) our theory requires to use only algebraic curves with selected point leading to the solutions elliptic in the variable \( x \) for KdV and KP in order to get periodic magnetic fields. The algebro-geometric case of genus zero leads, in particular, to the slowly decreasing lump-like magnetic fields with especially interesting variety of ground states in the Hilbert Space \( L_2(\mathbb{R}^2) \).

0 Introduction. Magnetic Pauli Operator and factorizable Schrodinger Operators

A nonrelativistic 2D Pauli operator for the charged particles with spin=1/2 moving in electric and magnetic fields \( E_\alpha = \partial_\alpha U, A_\alpha \) (under the Lorentz gauge condition) has a form (see [1], let \( e = 1, \frac{m}{c} = 1/2 \), we neglect the universal constants \( c, \hbar \) whose values are inessential here)

\[
L^P = \sum_{\alpha=1,2} (p_\alpha)^2 + B \sigma_3 + U, \ i p_\alpha = \partial_\alpha + i A_\alpha, \ \sum_{\alpha=1}^2 \partial_\alpha A_\alpha = 0, \quad (1)
\]

\( \sigma_\alpha \) are the Pauli matrices

\[
\sigma_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma_2 = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \sigma_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad \sigma_0 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right).
\]

Let \( U = 0 \) (i.e. electric field is equal to zero). The operator \( L^P \) is reduced to the direct sum of two scalar Schrodinger operators. They are written in the “factorized” form (see [2] [3] [4]):

\[
L^P = QQ^+ \oplus Q^+ Q = L_+ \oplus L_-
\]

(we neglect all constants unimportant for our goals). Here \( Q = \partial + A, \ Q^+ = -(\bar{\partial} - \bar{A}), \ -\bar{A} = A^{(z)}, \ \bar{\partial} A + A \bar{\partial} = 2B \) is magnetic field, \( \bar{\partial} = \partial_x - i \partial_y, \ \bar{\partial}\bar{\partial} = \Delta = \partial_x^2 + \partial_y^2. \)

Magnetic field \( B \) here is perpendicular to the oriented plane \( (x, y) \). Therefore, the magnetic field has a sign. For the rapidly decreasing field \( B \) at \( x^2 + y^2 \rightarrow \infty \), the magnetic flux \( \{ B \} \) is finite by definition. The Operator \( L^P \) is nonnegative. Therefore, the ground state energy is equal to zero \( \varepsilon_0 = 0 \).
or positive $\varepsilon_0 > 0$. Let $|\{B\}| \geq 1$ (in natural quantum units). Then the ground state subspace in the Hilbert Space is a linear space of the dimension $|\{B\}| = m$ (see [2]). For the periodic case and integer flux $\{B\} \in \mathbb{Z}$, the linear subspace of ground states is infinite-dimensional. It is isomorphic to the Landau level in a homogeneous magnetic field (see [3, 4]). The higher levels are separated from the ground state by the non-zero gap $\Delta_B > 0$. According to the literature of the late 80-s, this operator admits a "supersymmetry" transformation $P : L_+ \rightarrow L_- \rightarrow 0$, $P^2 = 0$, $P : \Psi \rightarrow Q_+ \Psi$ for $\{B\} > 0$, $\Psi \in L_+$. All positive energy levels $\varepsilon > 0$ are degenerate since $(\Psi, P \Psi)$ both are eigenfunctions. For the zero energy we have $P \Psi = 0$, $\varepsilon_0 = 0$, if function $\Psi$ belongs to the Hilbert space $L_2(\mathbb{R}^2)$. Using the ancient language of XIX (or even XVIII) century, there exists a "Laplace transformation" of the

2D second order scalar operators (see [11])

$$L = (\partial_x + A)(\partial_y + D) + U, \quad U = e^f$$

$$L \rightarrow \tilde{L} = e^f(\partial_y + D)e^{-f}(\partial_x + A) + U$$

$$\Psi \rightarrow \tilde{\Psi} = (\partial_y + D)\Psi$$

(3)

The equality $L \Psi = 0$ implies $\tilde{L} \tilde{\Psi} = 0$. We have here $L = Q_1Q_2 + U$.

In the selfadjoint elliptic case, studied in detail in [11] from the point of view of spectral properties of operators, we have

$$L = QQ^+ + U, \quad U = e^f$$

$$Q = (\partial + A), \quad Q^+ = -(\bar{\partial} - \bar{A}).$$

(4)

For the "purely factorizable" operators $U = \text{const}$, we have $\tilde{L} = Q^+Q + U$ (let $U = 0$). The Laplace Transformation $\Psi \rightarrow \tilde{\Psi} = Q^+\Psi$ coincides with the "Supersymmetry" with $P = Q^+$ in the sector $L^+$ and $P = 0$ in the second sector $L^-$. It acts on the whole spectrum $L \Psi = \varepsilon \Psi$, $\tilde{L} \tilde{\Psi} = \varepsilon \tilde{\Psi}$. The Ground States of $L^p$ are all "instantons", i.e. they satisfy to the equation $Q^+ \Psi = 0$ if $\Psi \in \mathcal{L}_2(\mathbb{R}^2)$. Let zero mode $L \Psi = 0$ does not belong to the Hilbert space but "belongs to the spectrum". It simply means that its growth rate is slower than some polynomial for $x^2 + y^2 \rightarrow \infty$. The instanton argument disappears. In the last case the point $\varepsilon_0 = 0$ is the bottom of continuous spectrum. Even if no "instanton" type solutions $\Psi$ of that kind exist, it is impossible to conclude immediately that the true ground state for the operator $L$ is positive $\varepsilon_0 > 0$ (but it is highly probable).
In the case of nonzero flux \( \{B\} \neq 0 \) the ground energy of the Pauli operator is equal to zero \( \varepsilon_0 = 0 \). For \( L^p = L_+ \oplus L_- \) it is realized inside of the sector \( L = L_+ \) (if \( \{B\} > 0 \)) or \( L = L_- \) (if \( \{B\} < 0 \)) (see [2, 3, 4]), in the rapidly decreasing and periodic case (see also [5], where other functional classes of magnetic fields were considered). The rest of the spectrum is twofold and separated from zero by a positive gap \( \Delta_B \). Interesting classes of the “factorizable” operators \( L \), having one more infinitely degenerate level except of the ground one, were found in [1]. These works have a “soliton” origin. Let us point out that the connection between Laplace transformations and 2D Toda chain found in the soliton theory, was in fact known in the XIX century to Darboux and his school. However, all calculations at the end of XIX — beginning of XX centuries were purely formal, and the elliptic case was completely missing.

In the present paper we investigate the algebro-geometric case. For the smooth periodic operators (i.e vector-potentials are periodic) we have magnetic flux equal to zero, but for the degenerate soliton-type case it might be not so. In our case the whole complex variety \( \Gamma \) of the Bloch-Floquet zero level eigenfunctions \( L \Psi = 0 \) appears ("The Complex Fermi Curve"). This operator is called “algebro-geometric” if genus is finite.

A Purely Factorizable Reduction of the self-adjoint Schrödinger operator \( L = -(\partial + A)(\bar{\partial} - A) \) is studied here from the point of view of algebro-geometric operators. It was recently found by the present authors [12] using the 2D Soliton-Type Completely Integrable System. The operator \( L = \partial_x \partial_y + G \partial_y + S \) is by definition hyperbolic in the work [12]. Its reduction \( S = 0 \) leads to very interesting "2D Burgers Hierarchy" which is linearizable similar to the classical Burgers Equation. The spectral meaning of this linearization is revealed in [12]. For the nonreduced system \( L_t = (LH - HL) + fL \) the second operator \( H = \Delta + F \partial_y + A \) may have an interesting spectral theory in the stationary "finite-gap" case \( LH - HL = -f(x,y)L \). The operators \( L \) and \( H \) form "The Algebrogeometric Pair of PDE’s Commuting Relative to the Level of Operator \( L = 0 \)" according to the terminology used by Kriecher and Novikov in the late 1970s-early 1980s. The operator \( H \) is elliptic here. Its study is the second main goal of [12]. It is easy to make \( H \) smooth, periodic and real. However, we failed to find nontrivial self-adjoint operators \( H \) within this approach. So the Conjecture is formulated in [12]: For the smooth periodic self-adjoint 2D Schrödinger operator in \( \mathbb{R}^2 \) the Full Complex Manifold of Bloch-Floquet Eigenfunctions might contain Complex Algebraic Subman-
ifolds only belonging to one energy level (except some trivial cases which essentially can be reduced to one-dimensional equation).

1 Algebro-Geometric self-adjoint factorizable operators. The inverse spectral data

As it was demonstrated above, the purely magnetic 2D Pauli Operator $L^P = L_+ \oplus L_-$ is a direct sum of two "Strongly Factorizable" Schrodinger operators

$$L_+ = QQ^+, \quad L_- = Q^+Q$$

$$Q = (\partial + A), \quad Q^+ = - (\bar{\partial} - \bar{A}).$$

Following [13], let us recall what is the algebro-geometric operator with period coefficients $A, U$

$$L = -(\partial + A)(\bar{\partial} + D) + U,$$

where $A, D, U$ are periodic in $x, y$.

Let us describe "The Inverse Spectral Data" for the Operator $L$: we take nonsingular Riemann surface $\Gamma$ of genus $g > 0$, two marked "infinite" points $\infty_1, \infty_2$ with local parameters $k^{l-1}(\infty_1) = 0, k^{m-1}(\infty_2) = 0$, and set of $g$ points ( a "divisor" of degree $g$) $D = (P_1, \ldots, P_g)$. We write it as a formal sum $D = P_1 + \ldots + P_g$. In the work [13] the "Two-point Baker-Akhiezer function" was introduced, $\Psi(P, x, y)$, $P \in \Gamma$, with the following properties:

a) It is meromorphic in the variable $P$ outside of infinities;

b) It has following asymptotic near infinities $\infty_1, \infty_2$:

$$\infty_1 : \Psi = c(x, y)e^{k'z} \left(1 + O \left(\frac{1}{k'}\right)\right), \quad \frac{1}{k'}(\infty_1) = 0$$

$$\infty_2 : \Psi = e^{k''z} \left(1 + O \left(\frac{1}{k''}\right)\right), \quad \frac{1}{k''}(\infty_2) = 0.$$  

c) It has poles of the first order in the points of divisor $D$ which are independent on the space variables $x, y$.

d) $\Psi \equiv 1$ at $x = 0, y = 0$. 

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Such function satisfies to the equation

\[ L\Psi = 0, \quad L = \partial \bar{\partial} - 2(\ln c)_z \bar{\partial} + U(x, y). \]

Our function \( \Psi \) is a Bloch-Floquet eigenfunction if \( c(x, y), U(x, y) \) are periodic. In general, they are quasiperiodic but for a dense set of Riemann surfaces \( \Gamma \) these functions are periodic. It is possible to write down \( \Psi, c, U \) through the \( \Theta \)-functions in a standard way of the periodic soliton theory (see [13] and surveys [16], [17]). A Purely Potential Reduction \( c \equiv 1 \) was found for this data in [14], [15]. The Self-adjoint reduction in the presence of magnetic field \( B = -\Delta (\ln c)/2 = -\partial \bar{\partial} (\ln c)/2 \neq 0 \) was found in [18]:

\begin{equation}
\sigma(k') = -k', \quad \sigma(D) + D \sim (K) + \infty_1 + \infty_2,
\end{equation}

where \( (K) \) is divisor of zeros and poles of holomorphic 1-forms, and the symbol \( \sim \) means “linear equivalence” of divisors, i.e. every divisor of zeros and poles of meromorphic function is equal to zero. These conditions are sufficient and (probably) necessary, but no rigorous proof of necessity was obtained in the literature yet.

Let us describe “The inverse spectral data” for our ”Factorizable Self-adjoint Reduction”. It is result of the present work.

Riemann surface \( \Gamma \) is degenerate

\[ \Gamma = \Gamma' \cup \Gamma'', \infty_1 \in \Gamma', \infty_2 \in \Gamma'', \]

and the intersection \( \Gamma' \cap \Gamma'' \) is a set of \( k + 1 \) points \( Q'_0, \ldots, Q'_k \).

![Diagram](image-url)
An antiholomorphic involution should exist $\sigma : \Gamma' \to \Gamma''$, $\Gamma'' \to \Gamma'$, which permutes points $\infty_1 \to \infty_2$, $\infty_2 \to \infty_1$, $(Q', 0, \ldots, Q'_k) \to (Q''_{i_0}, \ldots, Q''_{i_k})$. Let $g = \text{genus } \Gamma' = \text{genus } \Gamma''$. Let us specify $g + k$ points $D' = (P'_1, \ldots, P'_{g+k})$ on $\Gamma'$ and $g$ points $D'' = (P''_1, \ldots, P''_g)$ on $\Gamma''$ satisfying to the linear equivalence:

$$\sigma(D' + D'') + D' + D'' \sim (K) + \infty_1 + \infty_2 \text{ on } \Gamma,$$

where $(K) = (K') + (K'')$ is a divisor of 1-form $\omega$ with conditions on residues:

$$\omega = \omega' \text{ ( } \Gamma' \text{)}$$
$$\omega = \omega'' \text{ ( } \Gamma'' \text{)}$$
$$\text{Res}_{Q_j} \omega' = \text{Res}_{R_j=\sigma(Q_j)} \omega'', \sigma(R) = Q.$$

Let us reformulate it in terms of one curve $\Gamma'$ with local parameter $1/k'$, set of points $Q_0, \ldots, Q_n$ and $\sigma(Q_j) = Q_{\sigma(j)}$.

Find $\Psi$-function on $\Gamma'$ such that:

1) It has poles of first order in points $D'$, $(g + k$ points).

2) It has asymptotic near $\infty_1$:

$$\Psi = c(x, y)e^{k'z} \left( 1 + O \left( \frac{1}{k'} \right) \right)$$

3) $\Psi|_{Q_j} = \varphi_j(z)$ holomorphic in $z$.

$(k + 1)$ points $Q''_0 = Q'_{i_0}, \ldots, Q''_k = Q'_{i_k}$ are fixed on the surface $\Gamma'$. Let us construct an antiholomorphic (one-point) Baker–Akhiezer function $\varphi$ with properties: $\varphi(P, z)$ is antiholomorphic in $P \in \Gamma'$; it has a fixed divisor of first order poles $\sigma(D'')$ and asymptotic

$$\varphi \sim e^{-k'z} \left( 1 + O \left( \frac{1}{k'} \right) \right), \varphi \equiv 1 \text{ when } x = 0, y = 0.$$

It is necessary to satisfy the condition

$$\sigma(D'') + D' = (Q) + (K')_{\Gamma'} + \infty_1$$

with restriction on residues (above) in the points $Q_j \sim Q_{\sigma(j)}$ for the form defining $(K')$.  

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Let
\[ \varphi_j(z) = \varphi(Q_j'', z), \quad Q_j'' = Q_{\sigma(j)} \]
We described everything in terms of one nonsingular surface \( \Gamma' \) because \( \Gamma'' = \sigma(\Gamma') \).

Under these conditions we prove that our function \( \Psi(P, z, \bar{z}) \) on the surface \( \Gamma' \ni P \) satisfies to the equation below in the space variables \( z, \bar{z} \):
\[ \tilde{L}_+ \Psi = 0, \quad \tilde{L}_+ = (\partial + \tilde{A})\bar{\partial}, \]
where \( \tilde{A} = -\partial \ln c \). There is a constant \( \alpha \neq 0 \) such that the function \( \alpha c \) is real. It generates symmetric operator by the following rule: let
\[ c = e^{2\Phi}, \quad L_+ = -e^{-\Phi}(\partial + A)\bar{\partial}e^{\Phi} = -(\partial + A)(\bar{\partial} - \tilde{A}) = QQ^+, \quad A = \tilde{A}/2 \]
\[ \Psi \Rightarrow e^{-\Phi}\Psi = \frac{1}{\sqrt{c}}\Psi, \quad 1 \Rightarrow \frac{1}{\sqrt{c}}. \]
The operator \( L_+ \) is nonsingular, if there exists a constant \( \alpha \) such that \( \alpha c(x, y) > 0 \). Let \( \varepsilon_0 = 0 \) be a point of the spectrum for \( L_+ \). A natural candidate for the ground state is a function \( 1/\sqrt{c} \) satisfying to the equation
\[ Q^+ \left( 1/\sqrt{c} \right) = (-\bar{\partial} + \tilde{A})e^{-\Phi} = 0. \]
As \( (\partial + A)e^{\Phi} = 0 \), the function \( e^{\Phi} \) is also a candidate for the ground state for the operator \( L_- = Q^+Q \). However it is true only if these functions belong to the spectrum in \( L_2(\mathbb{R}^2) \). Let us formulate sufficient conditions for that:

a) The coefficients are periodic (quasiperiodic) and nonsingular \( \alpha c > 0 \). Both functions \( \sqrt{\alpha c}, \sqrt{1/\alpha c} \) are positive and belong to the spectrum. We know that \( A = -(\partial \ln c)/2, \quad B = -(\Delta \ln c)/2 \), and the magnetic flux is equal to zero:
\[ 0 = \int \int \Box Bdz \wedge d\bar{z} = 0, \quad \text{where} \quad \Box \quad \text{is an elementary cell}. \]
It is true because \( B = -(\partial \bar{\partial} \ln c)/2, \quad Bdz \wedge d\bar{z} = -\Phi_{zz}dz \wedge d\bar{z} = -d(\Phi_zdz) = 1/2 \cdot d(Adz) \), so our flux is an integral of the exact form.
b) Let us consider an exponential case now.

**Ground States of the Nonrelativistic Pauli operator:**

Let \( L = L_+ \oplus L_- \), \( L_+ = QQ^+ \), \( L_- = Q^+ Q \), where \( Q = (\partial + A) \), \( Q^+ = (-\bar{\partial} + \bar{A}) \), and \( A = -2\Phi_z \), \( \bar{A} = -2\Phi_{\bar{z}} \), where \( \Phi \) is real.

In the whole class \( c \to e^W c = c' \), where \( W = \alpha x + \beta y \) for real \( \alpha \), \( \beta \), we have the same magnetic field \( B = -\Delta \Phi = -(\Delta \ln c)/2 = -(\Delta \ln (ce^W))/2 \).

The operators \( L_+ \) and \( L_+' \) are unitary equivalent. Indeed, we have:

\[
\begin{align*}
L_\pm &= (\partial_x - i\Phi_y)^2 - (\partial_y + i\Phi_x)^2 \pm \Delta \Phi \\
L_+'\pm &= (\partial_x - i\Phi_y - i\beta/2)^2 - (\partial_y + i\Phi_x + i\alpha/2)^2 \pm \Delta \Phi
\end{align*}
\]

The Unitary Gauge Transformation

\[
\Psi \to \Psi e^{i(-\beta x + \alpha y)/2} = \Psi' \\
L_+ \to L_+' \quad L_- \to L_-'
\]

realizes this equivalence. If any one of the functions \( \sqrt{c} \) or \( 1/\sqrt{c} \) is bounded, then \( \varepsilon_0 = 0 \), and this function serves the spectrum in \( L_2(\mathbb{R}^2) \).

We have \( \Psi_+ = e^{-i(\alpha x - \beta y)/\sqrt{c}} \), for \( L_+ \), \( \Psi_- = e^{i(\alpha x - \beta y)/2}/\sqrt{c} \) for \( L_- \).

Thus we constructed as many different ground state vectors \( \varepsilon_0 = 0 \) for \( L_+ \) or \( L_- \) as there are bounded functions \( \{e^W c\} \) or \( \{e^{-W c-1}\} \) in this class.

**Example 1.** Consider \( c = 1 + e^y \). It does not depend on \( x \). Here \( c^{-1} \) is bounded, and \( c \) is unbounded. Let \( c'_0 = c^{\alpha x + \beta y} \). To have bounded function \( 1/c' \) in the class \( c' = e^W c, W = \alpha x + \beta y \), we need to satisfy the conditions: \( \alpha = 0, -1 \leq \beta \leq 0 \). So we obtain continuum of eigenfunctions for the ground energy level \( \varepsilon_0 = 0 \) parametrized by the index \( \beta \):

\[
\Psi_\beta = e^{i\beta x/2} \cdot 1/\sqrt{c}, \quad \beta \in [-1, 0]
\]

Here \( c_{0,0} = c \), \( \Psi_0 = 1/\sqrt{c} \). The unbounded functions \( (\sqrt{c}, 1/\sqrt{c}) \) satisfy to the equation \( L_+(\sqrt{c}) = 0, L_-(1/\sqrt{c}) = 0 \) but do not belong to the spectrum.

So we are coming to the following conclusion:

For the purely exponential generating function \( c > 0 \), the level \( \varepsilon_0 = 0 \) is the lowest point of the spectrum if we can find in the class \( \{e^{\alpha x + \beta y} c\} \) for real \( \alpha, \beta \) a bounded function \( c' = e^{\alpha x + \beta y} c \) or \( 1/c' = e^{-\alpha x - \beta y} c \), \( L_+^{'} = \)
\( L_+ \oplus L_- \). But it is always true for all \( k > 0 \) where \( k + 1 \) is the number of intersection points if all coefficients \( \kappa_j \) are positive.

In the smooth periodic case we know that the smooth periodic functions \( \sqrt{c} \) and \( 1/\sqrt{c} \) both are the ground states if \( c \) is positive. They are periodic, and the zero energy level is always a twice degenerate point of the spectrum (not like in the case on nonzero flux).

In the next paragraphs we present calculations for the genuses \( g = 0, 1 \).

2 The Algebogeometric self-adjoint factorizable operators

2.1 Solutions of genus \( g = 0 \).

As we can see below, all algebrogeometric purely magnetic Pauli operators with Complex Fermi Surface of genus zero correspond to the functions \( c \) of the form

\[
    c = \sum_j \kappa_j e^{W_j}
\]

Here \( \kappa_j \) are constants, and \( W_j = a_j z + b_j \bar{z} \) are the linear forms with constant coefficients. In general, all coefficients here are complex. However, for the selection of physically meaningful self-adjoint operators and real magnetic fields \( B = -(\Delta \ln c)/2 \) we are going to formulate proper restrictions below. These "Degenerate Algebogeometric Purely Magnetic Pauli Operators" are the n Natural Analogs of the "Multisoliton Potentials" for the 1D Schrodinger Operators in the case of KdV. It deserves to point out that our generating functions \( c \) are linear combinations of the elementary exponents with constant coefficients. Indeed, the magnetic field \( B \) is equal to \(-1/2(\Delta \ln c)\), i.e. it is a strongly nonlinear object. Such linear behavior of the quantity \( c \) reflects the main property of the "2D Burgers Hierarhy" discovered in the work [12]. For KdV the Multisoliton functions are also constructed as the second logarithmic derivatives of something which is indeed a nonlinear expression like some determinant made out of the one-soliton functions. In spite of linearity of \( c \), the Spectral Theory
is quite nontrivial for these 2D Purely Magnetic Pauli Analogs of the Multisoliton Operators.

In the case of genus zero our $\Psi$-function is written in the form ($k = k' \in \Gamma'$):

$$c \equiv u_0, \Psi = e^{k\bar{z}} \frac{u_0 k^n + \cdots + u_n}{(k - a_1) \cdots (k - a_n)}, D' = (a_1, \ldots, a_n)$$

(11)

(Here $n + 1$ is the number of intersection points)

$$\Gamma = \Gamma' \cup \Gamma'', \Gamma' = \Gamma'' = S^2$$

with local parameters $k = k' (\Gamma')$, $p = k'' (\Gamma'')$; points $\infty_1, \infty_2$ have the form $k = \infty (\infty_1), p = \infty (\infty_2)$. We have $\varphi (z, p) = e^{pz}$. The intersection points are $k_0, \ldots, k_n$ for $\Gamma'$ and $p_0, \ldots, p_n$ for $\Gamma''$. They lead to the set of equations for $\Psi$ in these points:

$$\Psi|_{k=k_j} = e^{p_j z}, j = 0, \ldots, n.$$  

(12)

So our solution has a form:

$$c = \sum_{j=0}^{n} (-1)^j e^{W_j(z, \bar{z})} \theta_j \frac{\Delta^{(n-1)}}{\Delta^{(n)}}, W_j = p_j z - k_j \bar{z},$$

(13)

as it follows from the system of linear equations

$$\left\{ \begin{array}{l}
u_0 k_0^n + \cdots + u_n = (k_0 - a_1) \cdots (k_0 - a_n)e^{p_0 z - k_0 \bar{z}} \\
u_0 k_1^n + \cdots + u_n = (k_1 - a_1) \cdots (k_1 - a_n)e^{p_1 z - k_1 \bar{z}} \\
u_0 k_n^n + \cdots + u_n = (k_n - a_1) \cdots (k_n - a_n)e^{p_n z - k_n \bar{z}}. \end{array} \right.$$  

(14)

Here

$$\Delta^{(n)} = \left| \begin{array}{cccc}
k_0^n & \cdots & 1 \\
\cdots & \cdots & \cdots \\
k_n^n & \cdots & 1 \end{array} \right| = \prod_{i < j} (k_i - k_j)$$

and $\Delta_j^{(n-1)}$ are similar Vandermonde determinants with the set of generating numbers $(k_0, \ldots, k_j, \ldots, k_n)$, where $k_j$ is erased, $\theta_j = (k_j - a_1) \cdots (k_j - a_n)$; $u_0 = c$.

The quantity $c(x, y) = u_0$ is determined by the field $B$ up to the transformation $c \rightarrow \alpha e^{W} c = c'$, where $\alpha = const, W = \gamma z + \delta \bar{z}$ because
\[ B = -(\Delta \ln c)/2 \] and \[-(\Delta \ln c')/2 = -(\Delta \ln c)/2, \Delta = \partial \bar{\partial}. \] Therefore we have exactly \( n \) unknown coefficients in the formula (15):

\[ j = 0, \ldots, n, \quad c = \sum_{q=0}^{n} \kappa_q e^{W_q(z, \bar{z})}, \quad (15) \]

For the given \( k_j, p_j \) all coefficients \( \kappa_q \) are determined up to the common multiplier.

For the differentials below the conditions on residues should also be satisfied:

\[ \Omega_1 = \frac{(k - a_1) \cdots (k - a_n) dk}{(k - k_0) \cdots (k - k_n)}, \quad \Omega_2 = \frac{s(p + a_1) \cdots (p + a_n) dp}{(p - p_0) \cdots (p - p_n)}, \]

where \( s \) is a constant.

\[ \text{Res}_{k_j} \Omega_1 + \text{Res}_{p_j} \Omega_2 = 0. \]

Choosing appropriate divisors \( D' = (a_1, \ldots, a_n) \), we obtain all complex coefficients \( \kappa_j \in \mathbb{C} \).

We need to classify such divisors \( D' \) and linear forms \( W_j = p_j z - k_j \bar{z} \) that \( c \) is real and positive in the equivalence class \( c \to \alpha c = c', \alpha = \text{const}. \)

For the reality of \( c \), \((x, y) \in R, z = x + iy, \bar{z} = x - iy, \) our expression must consists of the following terms:

1. The "exponential type" term for some index \( j \):

\[ \tilde{p}_j = -k_j, W_j = p_j z + \bar{p}_j \bar{z}, \]

\( \kappa_j \) is real and \( \kappa_j e^{W_j} \) is also real (a purely real exponent)

2. The "mixed type" term for the pair of indices \((j, l)\):

\[ p_l = -\bar{k}_l, k_l = -\bar{p}_l, \]

\[ \kappa_j = \kappa_l, \quad \kappa_j e^{W_j} + \kappa_l e^{W_l} = \kappa_j e^{W_j} + \bar{\kappa}_l e^{\bar{W}_l} \]

is real. We assume that all points \( k_j \neq k_q, j \neq q \) and \( p_j \neq p_q, j \neq q \) are distinct.

For the case 1: We obtain terms like real exponent \( \kappa_j e^{(\alpha_j x + \beta_j y)} \), where \( p_j = \alpha_j + i\beta_j, k_j = -\bar{p}_j, \kappa_j - \bar{\kappa}_j = 0. \)
For the case 2: We obtain terms of the form
\[ e^{W_{R,j}} (\kappa'_j \cos \theta_{I,j} - \kappa''_j \sin \theta_{I,j}), \kappa_j = \kappa'_j + i\kappa''_j, \]
\[ W_j = W_{R,j} + iW_{I,j} = [(\alpha_j - \gamma_j)x - (\beta_j - \delta_j)y] + i[(\beta_j - \delta_j)x + (\alpha_j + \gamma_j)y], \]
where \( p_j = \bar{k}_q = \alpha_j + i\beta_j, k_j = -\bar{p}_q = \gamma_j + i\delta_j. \)
For \( W_{I,j} = 0, \) we have the case 1: \( p_j = -\bar{k}. \)

3. The "purely trigonometric type" appears as another special subcase of the case 2 if \( W_{R,j} = 0 \) or \( \alpha_j = \gamma_j, \beta_j = -\delta_j, \) i.e. \( k_j = \bar{p}_j, W_{I,j} = (\beta_j x + \alpha_j y), e^{W_j} = e^{p_j z - p_j \bar{z}}. \)
In all these cases \( c \) is real. The mixed case 2 leads to the zeros of \( c \) and singularities of magnetic field if they are not "blocked" by other stronger terms.
\[ c = \sum_j \kappa_j e^{W_j(x,y)}, \ c \rightarrow \kappa e^W c = c', \]
where all \( \kappa_j \) are real. Let \( c = c^+ + c^- \). Here \( \kappa_j > 0 \) for \( j \in I, \kappa_j < 0 \) \( j \in \Pi. \)
Consider at first the case \( c = c^+, \) i.e. \( \kappa_j > 0 \) for all \( j. \) So we have \( c > 0. \)
As one can see (below), the magnetic field \( B = -\frac{\Delta \ln c}{2} \) is bounded in \( \mathbb{R}^2. \) In the class \( \{\kappa e^W c\} \) both \( \sqrt{c'}, \frac{1}{\sqrt{c'}} \) never can be bounded. Either they both exponentially increase along some directions at \( x^2 + y^2 \rightarrow \infty \) or one of them (i.e. \( 1/\sqrt{c'} \)) is bounded. In the last case the pair \( \{c, W\} \)
or simply a function \( e^W c = c' \) defines the ground state. As we can see below, such functions \( c' \) form a domain inside of the convex polygon \( T \) in \( \mathbb{R}^2 \) which is always nonempty. This domain is completely determined by the set of linear forms \( W_j \) in the class \( \{W_j\}: \) it is a convex hull of the set of points \( (\alpha_j, \beta_j) \in \mathbb{R}^2, W_j = \alpha_j x + \beta_j y \) (see below).

2. Let us consider the purely trigonometric case. Here we have the cases of odd and even numbers of intersection points \( n + 1. \) They are drastically different.
a) The number of intersection points \( n + 1 \) is even.
b) The number of intersection points \( n + 1 \) is odd.
\[ a) \quad c = \sum_{j=0}^{(1+n)/2} \kappa_j' \cos W_{I,j} + \kappa_j'' \sin W_{I,j}, \]

where \( \kappa_j = \kappa_j' + i\kappa_j'' \), \( W_j = iW_{I,j} = -k_j\bar{z} - \bar{k}_jz \), \( k_j = \alpha_j + i\beta_j \). Here function \( c \) always has zeros.

\[ b) \quad c = 1 + \sum_{j=1}^{\frac{n}{2}} \kappa_j' \cos W_{I,j} + \kappa_j'' \sin W_{I,j}, \]

For the appropriate constants \( \kappa', \kappa'' \in \mathbb{R} \) we have \( c > 0 \) and magnetic field \( B = -\left( \Delta \ln c \right)/2 \) is smooth, periodic and has zero flux through the elementary cell of periodic lattice (or the quasiperiodic mean value if \( c \) is quasiperiodic). It would be interesting to describe corresponding domains in the space of constants. The set of lines \( W_{I,j} = \alpha_jx + \beta_jy \), should pass through the integer vectors of the lattice in \( \mathbb{R}^2 \). Otherwise, the fields are quasiperiodic. Our conclusion is that in the regular trigonometric case both functions \( \sqrt{c}, 1/\sqrt{c} \) are periodic and positive; they are the ground states in both sectors \( L_+, L_- \) of the Operator \( L^P \). In the general quasiperiodic case the situation is the same.

**A Curious Remark.** There are “critical” values of constants \( \kappa', \kappa'' \) such that \( c \) has isolated zeroes \( c = 0 \) (repeated periodically). It is possible to choose parameters such that we have in this critical point an isotropic hessian \( d^2c = \pm a^2(dx^2 + dy^2) \). Then the magnetic field has a \( \delta \)-shaped singularity \( B = -\left( \Delta \ln c \right)/2 \sim \delta(x - x_0, y - y_0) \).

**Example 2.** Let \( n = 4 \). We demonstrate here a simplest nonsingular purely trigonometric (i.e degenerate algebrogeometric) operator, essentially dependent on both variables \( x, y \): we write \( \Psi \)-function in the form

\[ \Psi = e^{k \bar{z}} \frac{u_0k^4 + u_1k^3 + u_2k^2 + u_3k + u_4}{(k^2 - a_1^2)(k^2 - a_2^2)}, \quad D' = (a_1, a_2, -a_1, -a_2), \]

and \( \varphi = e^{pz} \).

Take the intersection points of \( \Gamma' \) and \( \Gamma'' \) in the form \( 0, k_1, k_2, -k_1, -k_2 \) for \( \Gamma' \) and \( 0, p_1, p_2, -p_1, -p_2 \) for \( \Gamma'' \). Let

\[ p_1 = k_1 \in \mathbb{R}, \quad p_2 = -k_2 = iK \in i\mathbb{R}. \quad (16) \]
In this case the antiinvolution $\sigma : k \to -\mathring{p}$ is correctly defined on $\Gamma$. The differentials look like

$$\Omega_1 = \frac{(k^2 - a_1^2)(k^2 - a_2^2)dk}{(k^2 - k_1^2)(k^2 - k_2^2)}$$

$$\Omega_2 = \frac{s(p^2 - \mathring{a}_1^2)(p^2 - \mathring{a}_2^2)dp}{(p^2 - p_1^2)(p^2 - p_2^2)}$$

where $s$ is some number. The condition on the residues

$$\text{Res}_0 \Omega_1 + \text{Res}_0 \Omega_2 = 0, \text{Res}_{xk_j} \Omega_1 + \text{Res}_{xp_j} \Omega_2 = 0.$$

must be valid. In the points of intersection we have

$$\Psi(0) = 1, \Psi(k_1) = e^{p_1z}, \Psi(k_2) = e^{p_2z}, \Psi(-k_1) = e^{-p_1z}, \Psi(-k_2) = e^{-p_2z}.$$ 

So the equalities follow:

$$u_4 = a_1^2a_2^2,$$

$$u_0k_1^4 + u_1k_1^3 + u_2k_1^2 + u_3k_1 = -a_1^2a_2^2 + (k_1^2 - a_1^2)(k_1^2 - a_2^2)e^{p_1z-k_1\mathring{z}},$$

$$u_0k_2^4 + u_1k_2^3 + u_2k_2^2 + u_3k_2 = -a_1^2a_2^2 + (k_2^2 - a_1^2)(k_2^2 - a_2^2)e^{p_2z-k_2\mathring{z}},$$

$$u_0k_1^4 - u_1k_1^3 + u_2k_1^2 - u_3k_1 = -a_1^2a_2^2 + (k_1^2 - a_1^2)(k_1^2 - a_2^2)e^{-p_1z+k_1\mathring{z}},$$

$$u_0k_2^4 - u_1k_2^3 + u_2k_2^2 - u_3k_2 = -a_1^2a_2^2 + (k_2^2 - a_1^2)(k_2^2 - a_2^2)e^{-p_2z+k_2\mathring{z}}.$$ 

Sum of the second equality with the fourth equality, and of the third equality with the fifth one are written below:

$$2u_0k_1^4 + 2u_2k_1^2 = -2a_1^2a_2^2 + (k_1^2 - a_1^2)(k_1^2 - a_2^2)2\cos(2k_1y),$$

$$2u_0K^4 - 2u_2K^2 = -2a_1^2a_2^2 + (K^2 + a_1^2)(K^2 + a_2^2)2\cos(2Kx).$$

Take

$$a_1 \in \mathbb{R}, a_2 = ia \in i\mathbb{R}.$$ 

Using (16) and taking $s = -1$, we can see that the conditions on residues of the differentials $\Omega_1$ and $\Omega_2$ are satisfied. We have

$$c = u_0 = \frac{a_1^2a_2^2}{k_1^2K^2}(1 - A\cos(2k_1y) - B\cos(2Kx)).$$
where

\[ A = \frac{(k_1^2 - a_1^2)(k_1^2 + a^2)}{k_1^2(k_1^2 + K^2)}, \quad B = \frac{(K^2 + a_1^2)(K^2 - a^2)}{K^2(k_1^2 + K^2)}. \]

For \( K = 10, \ k_1 = 5, a_1 = 2, \ a = 1 \) we obtain

\[ A = \frac{546}{3125}, \quad B = \frac{2574}{3125}, \quad A + B = \frac{624}{625}, \]

So the magnetic field is smooth and periodic.

**Example 3.** Consider now the Full Class of the Purely Exponential Real Type functions \( C \). We introduce below an important notion of “the Indicator of Growth” for the set \( \{W_j\} \) of all real exponents entering formula for \( c \) with positive coefficients. Suppose this function \( c \) and therefore \( \sqrt{c} \) grows exponentially in all directions. This is a Stable Property. The function \( \frac{1}{\sqrt{c}} \) has exponential decay in \( \mathbb{R}^2 \). Many other functions \( \frac{1}{\sqrt{c'}} \) in the same class \( c' = e^{Wc} \) are such that \( \frac{1}{\sqrt{c'}} \) automatically have similar decay (for example, it is true for all “small” linear forms \( W = \varepsilon(ax + by), \varepsilon \to 0 \)).

Consider first the Unstable Case \( n = 1 \), i.e. with 2 intersection points. Let us take absolutely typical example \( c = 1 + e^y \) following notations in the Example 1 above. The polygon \( T \) numerating all bounded functions \( 1/c' \) in the class \( c' = e^{Wc} \), coincides with a segment \( \beta \in [-1, 0], W = \alpha x + \beta y \). It does not have inner points. We never have \( c' \) in this class which has exponential growth in all directions. Magnetic field here depends on one variable. **The same result is true for all cases with** \( n + 1 = 2 \) **where** \( c = \kappa_1e^{W_1} + \kappa_2e^{W_2} \). The Unstable cases for all \( n > 1 \) are given by the sets linear forms \( \{W_j\} \) such that all differences are proportional to one linear form with constant coefficients. Magnetic field here depends on one variable only.
\[ \mathbb{R}^2 = (\alpha, \beta) \]

\[ W_j = \alpha_j x + \beta_j y \]

\[ -W_1 = (0,0) \]

\[ -W_2 = (0,-1) \]

Let us consider examples of the indicators of growth in this class. Choose \( W = e^{-y/2} \). We have \( c' = e^{y/2} + c^{-y/2} \). The indicator of growth has zeroes \( I_{W'}(\varphi) = 0 \) exactly in two points \( \varphi = 0, \pi \). Put \( W = e^{(x-y)/2} \). We get \( c'' = e^{(x+y)/2} + e^{(x-y)/2} \). For this case \( I_{W''}(\varphi) = 0 \) on the connected segment. However, the zero set of the indicator of growth is never empty for \( n = 1 \). It is not surprising because magnetic field always depends on one variable for \( n = 1 \).
(the growth directions for $W_j$ are shown by the rows; every ray from the zero point belongs to at least one sector of exponential growth provided by the 3 exponents entering $c$.)

The Stable Cases start with $n = 3$ (like in Fig 4,5). Let, for example,

$$c = e^{W_1} + e^{W_2} + e^{W_3} = e^x + e^y + e^{-x-y}$$

a) $\mathbb{R}^2 = (x, y)$

b) $\mathbb{R}^2 = (\alpha, \beta)$

$W_j = \alpha_j x + \beta_j y$

In this example the function \( \frac{1}{c'} = \frac{1}{e^{Wc}} \) is bounded if and only if $W \in T$ where $T$ is a triangle with vertices ($-W_1, -W_2, -W_3$) $\subset \mathbb{R}^2$ with coordinates $(\alpha, \beta)$. It belongs to the Hilbert Space (i.e. it is square integrable function on the $x, y$-plane $\mathbb{R}^2$) if and only if $c' = e^{Wc}$ where $W$ belongs to the interior of $T$.

**We describe below all nonsingular cases for the exponential generating functions $c$:**

**Definition.** Let $W = \alpha x + \beta y$ be a real linear form. We call the function on the circle $I_W \geq 0$, $I_W(\varphi) = \max(\alpha \cos \varphi + \beta \sin \varphi, 0)$ “the Indicator of Growth“ for the linear form $W = \alpha x + \beta y$. For the set of real linear forms \( \{W_j\} \) we call function $I_{\{W_j\}}(\varphi) = \max(I_{W_j}) \geq 0$ the “the Indicator of Growth of this set“. It is what some people call ”a Tropical Sum“.
I. Consider the case
\[ c_I = \sum_j e^{W_j \kappa_j}, \kappa_j > 0 \text{ or } c_{II} = \sum_j e^{W_j \kappa_j}, \kappa_j < 0. \]

Let us note that the indicators of growth are different for the different sets within the same class \( \{W_j\} \) and \( \{W_j + W\} \). It is an invariant of the set of exponents entering the function \( c \), not of magnetic field. It does not depend also on the coefficients \( \kappa_j \) entering the function \( c \).

There are following possibilities:
1. \( I_{\{W_j\}}(\varphi) > 0 \) for all \( \varphi \in S^1 \) (see Fig. 6 a).
2. \( I_{\{W_j\}}(\varphi) = 0 \) on the connected closed segment on the circle \( \varphi \in S^1 \) (see Fig. 6 b).
3. \( I_{\{W_j\}}(\varphi) = 0 \) in the isolated points \( \varphi = \varphi_1, \ldots, \varphi_s \in S^1 \). Then either \( s = 1 \) or \( s = 2 \) (see. Fig. 4 a to the Example 2).

We choose in the class \( \{ce^W\} \) a representative \( c' \) with minimal set of zeroes of the indicator of growth \( W_j \rightarrow W_j + W = W_j' \) for \( c' = e^W c \).
Possibility 3 with 2 opposite zeroes $I_{W_j}(\varphi_1) = I_{W_j}(\varphi + \pi)$ is realized for $c = e^{W_1} + e^{-W_1}$. It is impossible to reduce it to the case 1. If $s = 1$, then it is possible to reduce it to the “Stable Positive Case” case $I_{W_j+W}(\varphi) > 0$ by choosing $W$. The last case we call “stable”. The function $c'$ in this class $\{W_j + W = W'_j\}$ where $I_{W'_j}(\varphi_1) = 0$ has an isolated zero, we call “a boundary function”. It is easy to see that in the stable case where the function $c > 0$ grows exponentially in all directions, its opposite $c^{-1}$ has exponential decay in all directions.

Let $c' = c_0 \in \{ce^W\}$ be stable (i.e has exponential growth in all directions). All such functions $c_0^{-1/2}$ are the ground states for $L_-$ where $L^p = L_+ \oplus L_-$. All stable functions $c'' = c_W$ from the class $\{c_0e^W\}$ define the ground states for $L^p$ located in the sector $L_-$.)

$$\Psi_W = \frac{e^{-i(\alpha y - \beta x)/2}}{\sqrt{c_0e^{(\alpha x + \beta y)/2}}} = \frac{e^{-i(\alpha y - \beta x)/2}}{\sqrt{c_W}}.$$  

**Conclusion.** The operator $L_-$ has the family of the square integrable ground states $\Psi_W$, parametrized by the linear forms $W = \alpha x + \beta y$ (or by the pair $\alpha, \beta \in \mathbb{R}^2$ such that the set $\{W_j + W\}$ has strictly positive indicator of growth $I_{W_j+W} > 0$ everywhere on the circle. This domain $T$ on the plane $\mathbb{R}^2$ with coordinates $(\alpha, \beta)$ is a convex polygon $T$. The interior of $T$ is nonempty for all stable cases (i.e. if the set of linear forms $W_j - W_s$ generates linear space of dimension more that one). Inside $T$ the ground states $\Psi_W$ belong to $L_2(\mathbb{R}^2)$, $W \in \text{Int } T$. On the boundary of $T$ the states $\Psi_W$ do not belong to $L_2(\mathbb{R}^2)$. Therefore they represent the bottom of continuous spectrum for the operators $L_-$, and $L^p = L_+ \oplus L_-$. Apparently, the lowest level for the another sector $L_+$, is strictly positive.

Every state $\Psi_W$ for $L_-$ except $c_0$ generate a nonzero current $J_W$ defined by the phase of the complex function $\psi_W$. The Current vectors $J_W$ cover some convex bounded domain on the plane.

**Important fact.** The magnetic flux is divergent on the plane $\mathbb{R}^2$:  

$$\int \int_{x^2 + y^2 \leq R^2} B dx dy = -\frac{1}{2} R \cdot \int_0^{2\pi} I_{W_j}(\varphi) d\varphi + o(R), \quad R \to \infty,$$  

20
where \( I_{\{W\}} \) is the indicator of growth (see Appendix).

II. For the generating functions of the form:

\[
c = \sum_j \kappa_j e^{W_j} + \sum_q e^{W_{R,q}} (\kappa'_q \cos W_{I,q} + \kappa''_q \sin W_{I,q})
\]

we define the indicator of growth using only its part: the real subset should be chosen \( \{W_j\}_+ \), such that \( \kappa_j > 0 \). It defines a domain \( T^+ \) as above. (Possibly, it is necessary to change the sign \( c \to -c \), if this operation leads to the bigger domain). Anyway, all other linear forms \( W_j, W_{R,q} \) should belong to the domain \( T^+ \) defined by the main positive part of linear forms selected above. Under these conditions there exists a nonempty set of coefficients such that \( c \neq 0 \). It is possible to choose remaining coefficients such that zeroes of \( c \) appear.

2.2 Solutions of genus \( g=1 \). Operators with Bohm-Aharonov Singularity and the Magnetic Bloch functions

In the current section we study the case of elliptic curves \( (g = 1) \). As \( w \) are going to show, in this case we are facing a new very interesting phenomena, connecting out paper with the case of non-zero magnetic flux \( [2, 3, 4] \) (magnetic translations and topological phenomena are also discussed in \( [6, 7, 8, 9, 10] \)).

Let

\[
\Gamma' = \Gamma'' = \mathbb{C}/\Lambda,
\]

where

\[
\Lambda = \{2m_1\omega_1 + 2m_2\omega_2, \ m_1, m_2 \in \mathbb{Z}\} \subset \mathbb{C}
\]

denotes a lattice. We assume that \( \omega_1 = 1 \) and the lattice is invariant with respect to the complex conjugation

\[
\bar{\Lambda} = \Lambda
\]

(it is true, in particular, if \( \omega_2 \in i\mathbb{R} \)).

Let \( \infty_1 = 0 \in \Gamma' \), \( \infty_2 = 0 \in \Gamma'' \). Assume, that \( Q_0, \ldots, Q_n \in \Gamma' \) and \( R_0, \ldots, R_n \in \Gamma'' \) correspond to the intersection points \( \Gamma' \cap \Gamma'' \).

Let us express the \( \psi \)-function through the Weierstrass \( \sigma \)-function and \( \zeta \)-function.
We know that $\zeta(w)$ is meromorphic in $\mathbb{C}$ with first order poles at the points of $\Lambda$ and
\begin{equation}
\zeta(w + 2\omega_s) = \zeta(w) + 2\eta_s,
\end{equation}
where $\eta_s = \zeta(\omega_s)$.

The lattice is invariant with respect to the complex conjugation, therefore
\[ \zeta(w) = \overline{\zeta(w)}. \]

The function $\sigma(w)$ is analytic in $\mathbb{C}$ with first-order zeroes at the points of $\Lambda$ and
\begin{align*}
\sigma(w + 2\omega_s) &= -e^{2\eta_s(w+\omega_s)}\sigma(w), \\
\sigma(w - 2\omega_s) &= -e^{-2\eta_s(w-\omega_s)}\sigma(w).
\end{align*}

The invariance of the lattice with respect to the complex conjugation implies that
\[ \sigma(w) = \overline{\sigma(w)}. \]

The function $\psi'' = \psi|_{\Gamma''}$ has the following form ($z, \bar{z} \in \mathbb{C}, p \in \Gamma'', P = D''):
\begin{equation}
\psi''(p, z) = e^{-z\zeta(p)} \frac{\sigma(p + z + P)}{\sigma(z + P)\sigma(p + P)}.
\end{equation}

The function $\psi' = \psi|_{\Gamma'}$ has the form
\begin{align*}
\psi'(k, z, \bar{z}) &= e^{-z\zeta(k)} \left( \frac{\sigma(k + \bar{z} + A_0)\sigma(k - Q_1)\ldots\sigma(k - Q_n)}{\sigma(k + P_1)\ldots\sigma(k + P_{n+1})} f_0(z, \bar{z}) + \ldots + \frac{\sigma(k + \bar{z} + A_n)\sigma(k - Q_0)\ldots\sigma(k - Q_{n-1})}{\sigma(k + P_1)\ldots\sigma(k + P_{n+1})} f_n(z, \bar{z}) \right),
\end{align*}
where
\begin{align*}
A_0 &= Q_1 + \cdots + Q_n + P_1 + \cdots + P_{n+1}, \\
A_1 &= Q_0 + Q_2 + \cdots + Q_n + P_1 + \cdots + P_{n+1}, \\
\cdots \\
A_n &= Q_0 + \cdots + Q_{n-1} + P_1 + \cdots + P_{n+1}, \\
D' &= P_1 + \cdots + P_{n+1}.
\end{align*}

The compatibility conditions
\[ \psi'(Q_s) = \psi''(R_s) \]
imply
\[
e^{-\bar{z}(Q_0)} \frac{\sigma(Q_0 + \bar{z} + A_0)\sigma(Q_0 - Q_1) \cdots \sigma(Q_0 - Q_n)}{\sigma(Q_0 + P_1) \cdots \sigma(Q_0 + P_{n+1})} f_0(z, \bar{z}) = e^{-z(Q_0)} \frac{\sigma(R_0 + z + P)}{\sigma(z + P)\sigma(R_0 + P)},
\]
\[
\quad \cdots \cdots \cdots 
\]
\[
e^{-\bar{z}(Q_n)} \frac{\sigma(Q_n + \bar{z} + A_n)\sigma(Q_n - Q_0) \cdots \sigma(Q_n - Q_{n-1})}{\sigma(Q_n + P_1) \cdots \sigma(Q_n + P_{n+1})} f_n(z, \bar{z}) = e^{-z(Q_n)} \frac{\sigma(R_n + z + P)}{\sigma(z + P)\sigma(R_n + P)},
\]
therefore \( f_s(z, \bar{z}) \):
\[
f_0 = e^{-z(Q_0)} \frac{\sigma(R_0 + z + P)}{\sigma(z + Q_0 + \cdots + Q_n + P_1 + \cdots + P_{n+1})\sigma(z + P)} S_0,
\]
\[
S_0 = \frac{\sigma(Q_0 + P_1) \cdots \sigma(Q_0 + P_{n+1})}{\sigma(R_0 + P)\sigma(Q_0 - Q_1) \cdots \sigma(Q_0 - Q_n)},
\]
\[
f_n = e^{-z(Q_n)} \frac{\sigma(R_n + z + P)}{\sigma(z + Q_0 + \cdots + Q_n + P_1 + \cdots + P_{n+1})\sigma(z + P)} S_n,
\]
\[
S_n = \frac{\sigma(Q_n + P_1) \cdots \sigma(Q_n + P_{n+1})}{\sigma(R_n + P)\sigma(Q_n - Q_0) \cdots \sigma(Q_n - Q_{n-1})}.
\]
We have
\[
c(z, \bar{z}) = \left( \frac{\sigma(Q_0 + \bar{z})\sigma(-Q_1) \cdots \sigma(-Q_n)}{\sigma(P_1) \cdots \sigma(P_{n+1})} f_0(z, \bar{z}) + \cdots + \frac{\sigma(Q_n + \bar{z})\sigma(-Q_0) \cdots \sigma(-Q_{n-1})}{\sigma(P_1) \cdots \sigma(P_{k+1})} f_n(z, \bar{z}) \right).
\]
Let us point out that all \( f_s \) have the same factor at the denominator
\[
\sigma' = \sigma(z + Q_0 + \cdots + Q_n + P_1 + \cdots + P_{n+1})\sigma(z + P).
\]
Multiplying all \( f_s \) to \( \sigma' \) we obtain nonsingular functions \( \tilde{c} = c\sigma(z + Q + D')\sigma(z + D'') \). The corresponding functions \( \tilde{\psi} = \psi\sigma(z + Q + D')\sigma(z + D'') \) do not have Bloch-Floquet properties. They associated with the ground state of
Pauli operator with magnetic field $\tilde{B} = 1/2\Delta(\ln \tilde{c})$. We are going to discuss their analytic properties and relationship to the Dubrovin-Novikov states (1980) below.

Easy to prove following statements.

**Proposition 1:** All solutions $\tilde{c}$ can be presented in the form

$$\tilde{c} = \sum_q \alpha_q \exp\{-z\zeta(R_q) + \bar{z}\zeta(Q_q)\}\sigma(z + R_q)\sigma(z - Q_q)$$

Here $\alpha_q, R_q, Q_q$ can be any generic set of numbers, no restrictions.

**Proposition 2:** Every real solution can be presented as a sum of two type of terms like in Proposition 1: The Type 1 where $\alpha_q \in \mathbb{R}$ and $R_q = -\bar{Q}_q$, and The Type 2 where we have pair of indices $j, l$ satisfying to relations $\alpha_j = \bar{\alpha}_l, R_j = -\bar{Q}_l, R_l = -\bar{Q}_j$.

**Definition.** We say that real solution $\tilde{c}$ has Type $(k, l)$ if it is realized as a sum of $k$ Type 1 terms and $l$ Type 2 terms (total number of intersection points is $k + 2l$).

No problem to choose parameters leading to the nonzero real function $\tilde{c} \neq 0, \tilde{c} \in \mathbb{R}$.

**How to choose $\tilde{c}$ with periodic magnetic field $\tilde{B}$ with same periods as our lattice?**

**Example.** Let us take $n = 2$ (3 intersection points) and real function $\tilde{c}$ of the type $(1, 1)$:

$$\tilde{c} = \alpha \exp\{z\eta + \bar{z}\bar{\eta}\}|\sigma(z - \omega)|^2 +$$

$$+ \beta \exp\{z\zeta(Q_1) - z\zeta(R_1)\}\sigma(z - Q_1)\sigma(z + R_1) + CC$$

Here CC means ”complex conjugate”. We can choose parameters such that this expression is positive everywhere.

**Proposition 3.** Fix $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$. There exists a countable number of data $R_2 = -R_1 = -\bar{Q}_1 = Q_2, R_0 = -\omega = -Q_0$ such that magnetic field $\tilde{B} = 1/2\Delta \log \tilde{c}$ is periodic with the same periods as lattice. This set is determined by one number which satisfies to the equations

$$U - \bar{U} = -i\pi n, V + \bar{V} = -\pi m, m, n \in \mathbb{Z}$$

$$U(\lambda) = \lambda\eta - \omega\zeta(\lambda), V(\lambda) = \lambda\eta' - \omega'\zeta(\lambda)$$

Sum of such positive expressions $\tilde{c} = \sum_s \tilde{c}_s$ also leads to the Algebro-Geometric operator with nonsingular real magnetic field $\tilde{B}$ with flux equal to one quantum unit.
Conjecture: Every periodic nonsingular magnetic field $\tilde{B}$ with flux equal to one quantum unit through the elementary cell, can be approximated by the Algebro-Geometric Fields described above.

Let us make comparison with the Dubrovin-Novikov bases [3, 4].

Our function $\tilde{\psi}$ is an eigenfunction for the Pauli Operator $L^P$ with the magnetic field $\tilde{B}$. Is it a magnetic analog of the Bloch functions? A complete family of the Magnetic Bloch Functions, parametrized by the points of the torus $T^2 = \Gamma'$ was constructed in [3, 4] for all non-zero values of magnetic flux for an arbitrary nonsingular periodic field $B(x, y)$. We consider at the moment the case of flux equal to one. The whole family determines a direct summand in the Hilbert space $L_2(\mathbb{R}^2)$. These functions were written in the form:

$$\tilde{\psi} = (const) \exp\{S\} \sigma(z + a')$$

where $-1/2\Delta S = \tilde{B}$, and inversion of the Laplacian was especially normalized. Here $S = \tilde{S} + (const)z$, where $\tilde{S}$ is a real functions, independent on $a'$ and $(const)$ depends on $a'$ and the lattice. In our case we start with function $\psi''$ holomorphic in $z$ because the actual ground states (corresponding to the spectrum in the Hilbert Space) belong to the spin-sector described by the curve $\Gamma'$. We have to reduce the operator $L^P = L^+ \bigoplus L^-$ to the self-adjoint form by the "Gauge Transformation". We choose the proper spin-sector where the ground states are located, i.e. eigenfunctions decay at infinity (as for genus zero). It corresponds to the curve $\Gamma''$. For that sector we need to divide $\psi = \psi''$ by the $\sqrt{c}$. Use here notations $a = P, b = A, d = p - P$. So we have finally

$$\psi(p, z) = (1/\sqrt{c})\psi''(p, z) = (\sqrt{\sigma(z - a)\sigma(z - b)})/(\sqrt{c})e^{-\zeta(d)z}\frac{\sigma(z + d - a)}{\sigma(z - a)\sigma(d - a)} =$$

$$= (const)\frac{\sigma(\bar{z} - b)}{\sigma(z - a)}(1/\sqrt{c})e^{-\zeta(d)z}\sigma(z + d - a)$$

where $p \in \Gamma'' = C/\Lambda$. In order to remove a singular Bohm-Aharonov $\delta$-term we apply a unitary singular gauge transformation and multiply the result to a linear exponent

$$\psi \rightarrow \tilde{\psi} = \sqrt{\frac{\sigma(z - a)}{\sigma(\bar{z} - b)}}e^{(const)z}\psi$$
where \((\text{const})\) depending on \(p\) is chosen to obtain bounded magnetic Bloch eigenfunctions.

The last function \(\tilde{\psi}\) leads exactly to the Dubrovin-Novikov magnetic Bloch family of eigenfunctions for the smooth field \(\tilde{B}\):

\[
\tilde{\psi} = (\text{const}) \frac{1}{\sqrt{\tilde{c}}} e^{-\zeta(d)z} e^{(\text{const})z} \sigma(z + d - a) = \exp\{S\} \sigma(z + a')
\]

For smooth fields this family is uniquely defined.

Another form of this argument is following: we can always multiply \(\psi''\) by any holomorphic function, which we choose as \(\sigma(z - a)e^{(\text{const})z}\) with cons depending on \(p\). The product also satisfies to the Cauchy-Riemann (="self-duality") equation as an "Instanton" in the first nonself-adjoint form of the Pauli operator. After that we multiply result by the \(1/\sqrt{\tilde{c}}\) realizing a non unitary "Gauge Transformation" leading to the self-adjoint form of \(L^P\) in the sector where ground states are located. Finally we get exactly the Dubrovin-Novikov Magnetic Bloch function (see\[3\]).

These arguments work because we are dealing with the "Instanton Family" (i.e. satisfying to the first order Cauchy-Riemann equation \(\tilde{\partial}\psi'' = 0\)) for the operator \(L^P\) written in the first non-selfadjoint form. It satisfies to the covariant first order equation \([\tilde{\partial} + 1/2\partial(ln \tilde{c})][(1/\sqrt{\tilde{c}})\psi''] = 0\) after nonunitary gauge transformation realized as a division of all eigenfunctions including ground-states by the factor \(\sqrt{\tilde{c}}\). Here \(a, b, d\) are constants with the proper reality restrictions replacing the points \(a = P, b = A = \sum_s P_s - \sum_s Q_s, d = p - P\) correspondingly on the elliptic curve. In particular \(b = \bar{a}\).

For \(g > 1\) we always have magnetic flux with more than one quantum unit. We have to use Riemann surfaces \(\Gamma'\) with selected point \(\infty_1\) such that solutions of the corresponding KP hierarchy is Elliptic in the variable \(x\) in order to get magnetic fields \(B\) and \(\tilde{B}\) periodic in both directions in the \(z\)-plane. The theory of elliptic solutions to the KdV equation was started by Dubrovin and Novikov in 1974. For the KP hierarchy it was developed by Krichever since 1979 . A number of works were dedicated to it in the later literature. Details will be presented in the next work of the present authors.

Let us discuss here an extremely important physical question: What is a quantum Bohm-Aharonov Phenomenon? How the \(\delta\)-term in magnetic field affects the spectrum?

In our case the singular magnetic field \(B\) with the \(\delta\)-type singularity has an algebro-geometric realization. It has a zero magnetic flux. The family of
the complex Bloch-Floquet eigenfunctions is found for it which has very specific analytical properties valid only in the case of zero total flux through the elementary cell. It was explicitly calculated in the appropriate spin-sector. Its "instanton part" is simply the Baker-Akhiezer function $\psi''$. After reduction to the self-adjoint form of operators $L^P$ it became $\psi''/\sqrt{c}$ with same quasimomentum. Other part $\psi'/\sqrt{c}$ certainly does not have the instanton form outside of the intersection points. Is everything correctly and uniquely defined for the singular operators of this type? Are the Bloch-Floquet multipliers (whose logarithms divided by periods define the components of quasimomentum) canonically well-defined for such singular operators?

This question needs clarification. **Our statement is following: This family is correctly defined as a limit of corresponding families for the smooth operators with zero magnetic flux.** Such a procedure to define spectrum and the whole complex family of Bloch-Floquet functions can be realized for the zero level by the family of Riemann surfaces $\Gamma_\tau$ degenerating to $\Gamma_0 = \Gamma' \cap \Gamma''$. As we can see, only some special isolated state in this family might serve the standard Hilbert Space. In the Hilbert space $\mathcal{L}_2(\mathbb{R}^2)$ such state corresponds to the bottom of the continuous spectrum for the field $B$. So either the spectrum near 0 is continuous or the dispersion relation near the zero point is identically trivial, and we have in fact more Bloch functions on the level $\epsilon = 0$. It is exactly the case here. We will clarify this question below.

The Bloch-Floquet multipliers of the family $\psi''/\sqrt{c}$ are $\kappa_x, \kappa_y$. They are equal to

$$\kappa_x = \exp\{-2\omega\zeta(p) - 2\eta p\}, \kappa_y = \exp\{-2i\zeta(p)\omega' - 2i\eta' p\}$$

The equations $|\kappa_x| = 1, |\kappa_y| = 1$ can be easily solved

$$p^R = -\zeta(p) R\omega/\eta, p^I = -\zeta^I \omega'/\eta'$$

or

$$-\zeta(p) = p^R \eta/\omega + ip^I \eta'/\omega'$$

So they are non-unitary $|\kappa| \neq 1$ for other points $p \in \Gamma''$.

**Removing singular part from $B$ we are coming to the Magnetic-Bloch Functions found in 1980: the multipliers (i.e.the eigenvalues of the Magnetic Translations) became unimodular $|\tilde{\kappa}| = 1$. Remaining spectrum will be separated from 0 by the finite gap for the field $B$.**
The formal procedure is following: We introduce function

$$
\psi_{\text{new}}(p, z) = \left(\psi'' / \sqrt{c}\right) \exp\{u(p)z\}
$$

choosing $u(p)$ such that all multiplicators became unitary $\kappa \to \tilde{\kappa}$:

$$
|\kappa_x e^{2u(p)\omega}| = 1 = |\kappa_y e^{2iu(p)\omega'}|
$$

For the new multiplicators (after this renormalization of eigenfunction) we have

$$
\tilde{\kappa}_x = \exp\{2ip[\eta'\omega - \eta\omega']/\omega\}, \tilde{\kappa}_y = \exp\{2ipR[\eta\omega' - \eta'\omega]/\omega\}
$$

Finally we remove singularity by multiplication $\psi_{\text{new}} \to |\sigma'|\psi_{\text{new}}$ replacing $c$ by the smooth $\bar{c}$ in the formula above. So we are coming exactly to the magnetic Bloch eigenfunctions of Dubrovin and Novikov(1980) with multiplicants $\tilde{\kappa}_x, \tilde{\kappa}_y$ (may be shifted by constant which is inessential).

The multiplicators $\tilde{\kappa}$ form together a point of the 2-torus $\tilde{\kappa} \in T^2$ forming the whole component space of the real quasimomentum at the zero level and nearby. We have a big complex 2-dimensional manifold $M^2$ of the Bloch-Floquet eigenfunctions for the singular operator $L^\tau$ with zero flux of the form

$$
M^2 = \Gamma'' \times CP^1
$$

compactified at the infinities $u \in CP^1 = C \cup \infty$. It presents an irreducible component of the whole Bloch-Floquet manifold for this operator. The dispersion relation $\epsilon \to C$ degenerates at this component, i.e. $\epsilon = 0$ identically. The Bloch eigenfunction already written above has a form

$$
\Psi(p, u, z) = \psi'' / \sqrt{c} \times \exp\{uz\}
$$

where $(p, u) \in M^2$. This manifold presents exactly one component of the limit of the whole Bloch-Floquet manifolds $M^2_\tau$ for $\tau \to 0$ and $M^2_0 = M^2 \cup M'$. Every small purely magnetic perturbation leading to the smooth magnetic field $B_\tau$ close to our singular field $B$, has quite similar Bloch-Floquet eigenfunction: The Bloch manifold here is $\Gamma'' \times CP^1$ with the instanton-type Bloch eigenfunction like $\Psi_\tau = \psi''(p, z) / \sqrt{c_\tau} \times \exp\{uz\}$ where the curve $\Gamma''_\tau$ might have an infinite genus. So the small electric perturbation of operator $L^\tau$ is needed. We perturb by the small periodic potential $\tau U(x, y)$. Already the first order in coupling parameter $\tau$ probably leads to a nontrivial dispersion
relation as a complex meromorphis function on the same manifold. It has the order \(\tau\), so we have

\[ \epsilon_\tau : M^2_\tau \to \mathbb{C} \]

The real levels \(\epsilon_\tau = \text{const} \in \mathbb{R}\) give a function on the real "quasimomentum" torus \(T^2\) whose levels are the real Fermi-curves. Its minimum lies nearby of the initial point on the curve \(\Gamma''\) found above where \(\kappa_x, \kappa_y\) are imaginary.

We are going to calculate this perturbation in the next work. The complex level curves should have analytical properties at infinity typical for the Baker-Akhiezer functions ( maybe of the infinite genus where small handles appear from the "resonance points").

So it looks like the delta-term does not affect deeply the spectrum near the ground state.

3 Appendix: The Asymptotic of Magnetic Flux

Let us calculate the Asymptotic of Magnetic Flux through the round ball of radius \(R\) for the purely exponential case

\[ e^{2\Phi} = c = \sum_j \kappa_j e^{W_j}, \kappa_j > 0, W_j = R(\alpha_j \cos(\phi) + \beta_j \sin(\phi)) \]

We have for the magnetic field \(B = -(\Delta(\ln c))/2\). For the vector-potential restricted on the circle \(r = R\) in polar coordinates, we obtain

\[ A = \Phi_y dx - \Phi_x dy = -\frac{1}{2} R[\sum_j \kappa_j e^{W_j}(\alpha_j \cos(\phi) + \beta_j \sin(\phi))]d\phi/c. \]

Our assumption is that there exist exactly \(N\) indices \(j = 1, 2, ..., N\) such that the Indicator of our family \(I_{(W_j)}(\phi) = \max_j I_{W_j}(\phi)\) where \(I_{W_j} = \max[\alpha_j \cos(\phi) + \beta_j \sin(\phi), 0]\) is strictly positive, and all other indices \(p \neq 1, 2, ...N\) are inessential (i.e. corresponding linear forms \(W_p\) are located strictly inside of the convex domain \(T \subset \mathbb{R}^2\) with coordinates \(\alpha, \beta\) numerating the rapidly decreasing ground state vectors of our operator).

There are domains \(\Delta_j\) on the circle \(S^1\), where \(I_{(W_j)} = I_{W_j}(\phi)\) with end points \(\Delta_j = [\phi^0_j, \phi^0_{j-1}]\), and for \(j = N, 1\) we have \(\phi^0_N = \phi^0_N\). So \(\Delta_N\) is a neighbor of \(\Delta_{N-1}\) and \(\Delta_1\) (i.e. our numeration is circle contr-clockwise).
Our claim is following: **Following Asymptotic Formula is true:**

\[
\int \int_{D_R^2} B(x,y)dxdy + \frac{1}{2}R \oint_{S_1} I_{\{W_k\}}(\phi)d\phi =
\]

\[
= \sum_{s \geq 1} R^{-s} \sum_{j=1}^{N} \lambda_j^{(s)} \{Q_s(a_j) + Q_s(a_j^{-1})(-1)^s\} + \text{(Remainder)}.
\]

Apparently, this series is nonconvergent, since the coefficients \(Q_s\) grow, as we think, as \(s!\). About the remainder we claim now that its decay is more rapid than any negative degree of \(R\). We claim only that the "Regularized Flux"

\[
\int \int_{D_R^2} Bdx dy + \frac{1}{2}R \oint_{S_R^1} I_{\{W_j\}}(\phi)d\phi = O\left(\frac{1}{R}\right)
\]

is tend to zero in this sum, for \(R \to \infty\). Performing this calculation near the critical points \(\phi_0^j\), we use following functions

\[
(W_{j+1} - W_j)/R = (\alpha_{j+1} - \alpha_k) \cos(\phi) + (\beta_{j+1} - \beta_j) \sin(\phi) = t_j(z).
\]

Here \(\phi = (\phi_0^j + z), |z| < \epsilon\). It is located near the points \(\phi_0^j\) or \(z = 0\): in this point \(W_j = W_{j+1}, t_j = 0, z = 0\), and the inverse function \(z(t_j)\) is given by the inverse series with a finite radius:

\[
z = \sum_{k \geq 1} \lambda_j^{(k)} t_j^{k+1} / (k + 1)
\]

\[
d\phi = dz = \sum_{k \geq 0} \lambda_j^{(k)} t_j^k dt_j.
\]

We define numbers

\[
Q_k(a) = \int_0^\infty [aw^k e^{-w}/(1 + ae^{-w})]dw
\]

useful for the investigation of the difference

\[
\oint_{S_R^1} A + \frac{1}{2}R \oint_{S_1} I_{\{W_q\}}(\phi)d\phi.
\]

Probably, \(Q_k \sim k!\). Our function \(c\) has exponential growth everywhere, but magnetic field has decay only outside of the small domains surrounding the
"critical" points $\phi_0^j$. It is easy to see that our vector-potential $A$ after extracting the Indicator of Growth $RI_{(W_q)}(\phi)(d\phi)$, became exponentially small outside these small domains. Only two exponential terms $\kappa_j e^{W_j}$, $\kappa_{j+1} e^{W_{j+1}}$ in $c$ are essential in every such small domain, between $\Delta_j$ and $\Delta_{j+1}$. Dropping all other terms in the sum for $c = \sum_q \kappa_q e^{W_q}$ and for $A$ in every such small area costs us exponentially small. In the area $\phi \in \Delta_q$ we multiply both numerator and denominator in the expression for $A$-by the exponent $\kappa_q^{-1} e^{-W_q}$. The exponent $e^{-W_j}$ is the vertex of the convex polygon $T$ containing all functions $c' = ce^W \in T$ such that $(ce^{W_q})^{-1/2}$ are the ground states of the Pauli Operator. We need $q = j$ for $\phi \leq \phi_0^j$ (or $\phi \in \Delta_j$), and $q = j + 1$ for $\phi \in \Delta_{j+1}$. So only two terms remain in the numerator and denominator.

Similar result we obtain in the domain $\Delta_j$ just below the point $\phi_0^j$ with inverse constant $\kappa_{j+1}/\kappa_j$ and exponent $e^{(W_{j+1} - W_j)}$, plus we have to turn back the direction of integration. Taking $\epsilon$ such that $Re = O(R^\delta), \delta > 0$, we see following: The integration between the local limits $[\phi_0^j - \epsilon, \phi_0^j + \epsilon]$ of such expressions with $w = Rt_j$, which appear in our calculation of the regularized magnetic flux, can be extended to the limits $[-\infty, +\infty]$. It is true because the remaining terms have order $O(e^{-R^\delta})$: more precisely their decay is more rapid than any polynomial.

Expressing the variable $z = \phi - \phi_0^j$ by the variable $t_j = (W_j - W_{j+1})/R$, we are easily coming to our result. In the final integration we have a sum of integrals looking like

$$Q_s(a) = R^{-s-1} \int_0^\infty ae^{-w}/(1 + ae^{-w})w^s dw,$$

where $w = \pm Rt_j$. The sign is + and $a = \kappa_{j+1}/\kappa_j$ for $z \leq 0$, and sign − and $a$ replaced by $a^{-1}$ for $z \geq 0$. So we are coming to our result.

Note, it is easy to show that the expressions

$$Q'_k = \int_0^\infty e^{-w}w^k dw$$

grow as $(k!)$. Probably, it is true for our expressions $Q_k$.

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$$c = e^y + e^{y-2x} + e^{-y-2x}$$