Identification of dynamic errors-in-variables bilinear systems of fractional order

D V Ivanov¹, I L Sandler² and N V Chertykovtseva²

¹Samara National Research University, Moskovskoe Shosse 34A, Samara, Russia, 443086
²Samara State University of Transport, Svobody str. 2B, Samara, Russia, 443066

e-mail: dvi85@list.ru, viruskvam@bk.ru, chertykovtseva@mail.ru

Abstract. An approach for the identification of dynamic single-input single-output bilinear discrete-time fractional order system models within the errors-in-variables framework for the case of white input and output noise sequences is presented. A criterion is obtained that allows obtaining highly consistent estimates of the parameters of the system. The propose algorithm was realized in Matlab. The simulation results show the high efficiency of the propose algorithm.

1. Introduction

In order to describe processes of different nature, equations with derived differences of fractional order are increasingly used. Although there is no simple interpretation, which is possessed by derivatives, integrals and differences of integers, models described by equations of fractional order, used in physics and engineering [1-4], the branch of the theory of management related to the synthesis of regulators of fractional order is actively developing.

In connection with the active development and application of equations with differences and derivatives of fractional order for modeling and forecasting, methods of identification of systems described by equations and differences of fractional order are being actively developed. Most of the papers are devoted to the parametric identification of differential equations of fractional order with noise in the equation or the output signal [5-8].

The problem of identification with the presence of noises in the input and output signals is much more complicated, for example in [9] a method based on the cumulant, involving significant restrictions on signal and noise is proposed.

Papers [10-12] are devoted to identification of systems described by equations, with differences of fractional order.

A nonlinear model with a simple structure is the bilinear system. It is a simple nonlinear extension of a linear system: the evolution of the output of the bilinear system does not only depend on the input and the output, but also on the product between the input and the output. In the field of physical system modelling the bilinear system (BS) models have been exploited extensively in various scientific areas, such as nuclear fission, electric networks, heat transfer, fluid flow, chemical kinetics etc., see e.g. [13-15]. Their popularity and broad applicability stems from the fact that BS models are able to satisfactorily approximate many nonlinear processes.

For the identification of bilinear systems apply methods, such as methods of maximum-likelihood method [16], bias-compensated least square [17], instrumental variables [18], total least square [19] and methods based on higher-order statistics [20]. Recursive identification methods for bilinear
systems can be derived from recursive methods for linear system identification as described by [21]. Several methods were proposed to reduce the bias of the recursive equation-error approach, by modifying the least squares cost function: improved least-squares [22].

Furthermore, building on the theory of linear systems, the state dependent steady-state and dynamic properties of BS models are well understood. Therefore, the BS models are commonly utilised as a stepping stone when analyzing systems exhibiting nonlinear behavior. Consequently, a need to extend the errors-in-variables (EIV) system description to encompass BS models is prompted. This allows a combination of both, i.e. the increased applicability offered by the BS models and the generalized system setup afforded by the EIV framework.

In this paper an approach for the identification of single-input-single-output (SISO) discrete-time fractional EIV BS models is proposed considering the case when the input and output noise sequences are white.

The paper is organized as follows. In the next section, we present the problem statement. In section 3, we define criteria for identifying fractional bilinear systems. The simulation results are presented in section 4. Finally, section 5 concludes this paper.

2. Problem statement
Consider the class of the discrete-time input-output SISO (single-input single-output) bilinear fractional systems that can be represented by the following nonlinear auto-regressive with exogenous input process, i.e.

$$\sum_{m=1}^{r} a_{m}(\Delta) z_{i-f}^{(m)} = \sum_{m=1}^{r} a_{0}(\Delta) x_{i-f}^{(m)} + \sum_{m=0}^{r} c_{m}(\Delta) \Delta x_{i-f}^{(m)} z_{i-f}^{(m)}$$

where $0 < \alpha(1) \ldots < \alpha(r), 0 < \beta(1) \ldots < \beta(r), 0 < \gamma(1) \ldots < \gamma(r), k = 1, r$,

$$\Delta z_{i-f}^{(m)} = \sum_{j=0}^{l} (-1)^{j} \begin{pmatrix} \alpha(m) \\ j \end{pmatrix} z_{i-f-j}^{(m)}, \Delta x_{i-f}^{(m)} = \sum_{j=0}^{l} (-1)^{j} \begin{pmatrix} \beta(m) \\ j \end{pmatrix} x_{i-f-j}^{(m)},$$

$$\Delta x_{i-f-j}^{(m)} y_{j} = \sum_{j=0}^{l} (-1)^{j} \begin{pmatrix} \gamma(m) \\ j \end{pmatrix} x_{i-f-j}^{(m)} y_{j}.$$
A4. Input/output noise sequences \( \{ \xi_i \} \) and \( \{ \zeta_i \} \) are zero mean, ergodic, white signals with known variances, denoted \( \sigma^2_\xi \) and \( \sigma^2_\zeta \).

A5. Input/output noise sequences \( \{ \xi_i \} \) and \( \{ \zeta_i \} \) mutually uncorrelated and uncorrelated with the noise-free signal sequences \( \{ z_i \} \), \( \{ x_i \} \).

The system parameter vector is defined as

\[
\theta_0 = \begin{bmatrix} b_0^T & a_0^r & c_0^r \end{bmatrix}^T, \quad b_o = (b_0^{(2)} \ldots b_0^{(r)})^T, \quad a_o = (a_0^{(1)} \ldots a_0^{(r)})^T, \\
\end{align}
\]

\[
c_o = \begin{bmatrix} c_o^{(11)} \ldots c_o^{(1s)} \\
                      \vdots \\
                      c_o^{(21)} \ldots c_o^{(2s)} \\
                      \vdots \\
                      \end{bmatrix}
\]

\[
\begin{bmatrix}
\end{bmatrix}^T.
\]

The regressor vector for the measured signals is defined as

\[
\varphi_{ij} = \begin{bmatrix} \varphi_{ij}^{(1)} \varphi_{ij}^{(2)} \ldots \varphi_{ij}^{(m)} \end{bmatrix}^T, \quad \varphi_{ij}^{(k)} = \begin{bmatrix} \Delta_{ij}^{(k)} w_{ij}^{(k1)} \ldots \Delta_{ij}^{(k)} w_{ij}^{(k2)} \end{bmatrix}^T.
\]

Given N samples of the measured signals, i.e. \( \{ y_i \} \), \( \{ w_i \} \), determine the vector \( \hat{\theta} \).

3. Criteria for identification

This system can be reformulated in equation error form as

\[
\varphi_{ij} - \varphi_{ij} = \epsilon_i,
\]

where the equation error or residual is given by

\[
\epsilon_i = b_0^T \varphi_{i}^{(1)} - b_0^T \varphi_{i}^{(2)} - c_0^T \varphi_{i}^{(2)} - \varphi_{i}^{(2)} + \varphi_{i}^{(2)} + \varphi_{i}^{(2)} = \begin{bmatrix} \Delta_{ij}^{(1)} \xi_{ij}^{(1)} \ldots \Delta_{ij}^{(2)} \xi_{ij}^{(2)} \end{bmatrix}^T.
\]

From requirement A4, A5 it follows that generalized error \( \epsilon_i \) has zero mean. We obtain that variance of generalized error equal to

\[
\sigma^2_{\epsilon} = \sigma^2_\xi + \sigma^2_\zeta + \sigma^2_\epsilon h_0^T H_\alpha h_0 + \sigma^2_\epsilon^T H_\alpha^T a_o + c_0^T H_\epsilon c_0.
\]

\[
H_\alpha = E \left[ \sum_{i=1}^{N} \varphi_{ij}^{(1)} \left( \varphi_{ij}^{(2)} \right)^T \right] = \begin{bmatrix} b_0^{(1)} & b_0^{(2)} \ldots b_0^{(r)} \\
b_0^{(1)} & b_0^{(2)} \ldots b_0^{(r)} \\
\vdots & \vdots & \ddots \end{bmatrix},
\]

\[
h_0^{(m)} = E \left( \Delta_{ij}^{(m)} \xi_{ij}^{(m)} \cdot \Delta_{ij}^{(m)} \xi_{ij}^{(m)} \right) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N} \left( \begin{array}{c} \alpha_j^{(m)} \\
\tilde{N} - j \end{array} \right) \left( \begin{array}{c} \alpha_k^{(m)} \\
\tilde{N} - j \end{array} \right),
\]

\[
m = 1, r, k = 1, r.
\]
\[ H_\beta = E\left[ \sum_{i=1}^{N} \varphi_{\zeta}^{(i)} \left( \varphi_{\zeta}^{(i)} \right)^{T} \right] = \sigma_\zeta^2 \left( \begin{array}{ccc}
 H_{\beta}^{(11)} & H_{\beta}^{(12)} & \cdots & H_{\beta}^{(1N)} \\
 H_{\beta}^{(21)} & H_{\beta}^{(22)} & \cdots & H_{\beta}^{(2N)} \\
 \vdots & \vdots & \ddots & \vdots \\
 H_{\beta}^{(N1)} & H_{\beta}^{(N2)} & \cdots & H_{\beta}^{(NN)}
\end{array} \right), \]

\[ H_\gamma = E\left[ \sum_{i=1}^{N} \varphi_{\zeta}^{(i)} \left( \varphi_{\zeta}^{(i)} \right)^{T} + \varphi_{\omega}^{(i)} \left( \varphi_{\omega}^{(i)} \right)^{T} - \varphi_{\zeta}^{(i)} \left( \varphi_{\zeta}^{(i)} \right)^{T} \right] = \sigma_\zeta^2 \left( \begin{array}{ccc}
 H_1^{(11)} & H_1^{(12)} & \cdots & H_1^{(1N)} \\
 H_1^{(21)} & H_1^{(22)} & \cdots & H_1^{(2N)} \\
 \vdots & \vdots & \ddots & \vdots \\
 H_1^{(N1)} & H_1^{(N2)} & \cdots & H_1^{(NN)}
\end{array} \right), \]

\[ H_\gamma^{(mmkk')} = \sigma_\zeta^2 \left( \begin{array}{ccc}
 H_{\gamma}^{(11kk')} & H_{\gamma}^{(12kk')} & \cdots & H_{\gamma}^{(1Nkk')} \\
 H_{\gamma}^{(21kk')} & H_{\gamma}^{(22kk')} & \cdots & H_{\gamma}^{(2Nkk')} \\
 \vdots & \vdots & \ddots & \vdots \\
 H_{\gamma}^{(N1kk')} & H_{\gamma}^{(N2kk')} & \cdots & H_{\gamma}^{(NNkk')}
\end{array} \right), \]

\[ H_\gamma^{(mmkk')} = E\left( \Delta_{\zeta_{j-f_{(k)}}}^{(mk)} \cdot \Delta_{\zeta_{j-f_{(k)}}}^{(mm)} \cdot \Delta_{\zeta_{j-f_{(k)}}}^{(mm)} \right) = \sum_{j=0}^{\infty} \frac{1}{N} \left( \sum_{i=1}^{N} \gamma_{j-f_{(k)}}^{(mk)} \sum_{i=1}^{N} \gamma_{j-f_{(k)}}^{(mm)} \right) N - j, \]

\[ h^{(i)}_{\alpha} = \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{\infty} \gamma_{j-f_{(k)}}^{(mk)} \gamma_{j-f_{(k)}}^{(mm)}. \]

\[ h^{(i)}_{\alpha} = E\left( \Delta_{\zeta_{j-f_{(k)}}}^{(mk)} \cdot \Delta_{\zeta_{j-f_{(k)}}}^{(mm)} \cdot \Delta_{\zeta_{j-f_{(k)}}}^{(mm)} \right) = \sigma_\zeta^2 \sum_{j=0}^{\infty} \frac{1}{N} \left( \sum_{i=1}^{N} \gamma_{j-f_{(k)}}^{(mk)} \sum_{i=1}^{N} \gamma_{j-f_{(k)}}^{(mm)} \right) N - j. \]

With the known structure of the model, which means that orders \( r, r_1, r_2, r_3, f^{(m)}, f_1^{(m)}, f_2^{(m)}, f_3^{(m)}, \alpha^{(m)}, \beta^{(m)}, \gamma^{(m)} \) are determined, the following criterion can be applied to get strongly consistent estimates of parameters:

\[ \min_{a,b,c} \sum_{i=1}^{N} \left( a^T \phi_{\alpha}^{(i)} - b^T \phi_{\beta}^{(i)} - c^T \phi_{\gamma}^{(i)} \right)^2. \]
Theorem. Suppose that a random process \( \{y_i, i = \ldots -1,0,1,\ldots\} \) is described by equation (1) with zero initial conditions and the assumptions A1-A5 are met. Then estimate \( \hat{\theta}(N) \), determined by expression (2) with probability 1 when \( N \to \infty \), exists and is a unique and strongly consistent estimate,

\[
\hat{\theta} \xrightarrow{N \to \infty} \theta_0, \text{ a.s.,}
\]

i.e.

\[
\hat{a} \xrightarrow{N \to \infty} a_0, \text{ a.s.,}
\]

\[
\hat{b} \xrightarrow{N \to \infty} b_0, \text{ a.s.,}
\]

\[
\hat{c} \xrightarrow{N \to \infty} c_0, \text{ a.s.}
\]

Proof. The equation (2) is a generalized Rayleigh relation. The proof of solvency is based on the properties of the generalized Rayleigh relation. The proof is analogous to the proof for bilinear systems of order [23].

4. Simulation Results

The algorithm (2) compared to Least Square (LS). The dynamic system is described by the equations

\[
z_i = 0.5 \Delta^{0.9} z_{i-1} - 0.2 \Delta^{1.7} z_{i-1} + \Delta^{0.7} x_i - 0.2 \Delta^{1.4} x_i + 0.3 \Delta^{0.2} x_i z_{i-1},
\]

\[
y_i = z_i + \zeta_i, \quad w_i = x_i + \zeta_i,
\]

(3)

The coefficients of the bilinear system are chosen from the condition of stability of the system. The coefficients of the bilinear system were given by

\[
\theta_0 = (0.7 \quad -0.4 \quad 0.3 \quad 0.7 \quad 0.2 \quad 0.2)^T.
\]

The noise-free input \( x_i \) is modelled as

\[
x_i = -0.8 \cdot x_{i-1} - 0.6 \cdot x_{i-2} + \zeta_{1i} + 1.7 \cdot \zeta_{i-1} + 0.5 \cdot \zeta_{i-2},
\]

where \( \zeta_{1i} \) - is are zero mean, white signal.

The estimates of the parameter vector are given in Table 1, as well as the normalized root mean square error, defined as

\[
\delta \theta = \sqrt{\frac{\|\hat{\theta} - \theta_0\|^2}{\|\theta_0\|^2}} \cdot 100\%.
\]

| Noise-signal ratio | Mean square error ± SD |
|-------------------|------------------------|
| \( \sigma_{\zeta} / \sigma_z \) | \( \sigma_{\zeta} / \sigma_z \) | \( \delta \theta \),% | \( \delta \theta_{1.5} \),% |
| 0.25              | 0.25                   | 2.94±2.27 | 11.57±2.37 |
| 0.50              | 0.50                   | 8.49±5.14 | 25.81±2.20 |
| 0.75              | 0.75                   | 16.58±10.85 | 38.31±2.58 |

The results are based on 50 independent Monte-Carlo simulations. The number of data points Nin each simulation was 2000. For each method we have given the sample mean and sample standard deviation denoted by SD. Output model for validation data is shown in the Figures 1-3.

4. Conclusions

An approach for the identification of dynamic bilinear discrete-time errors-in-variables system models has been developed. A Monte-Carlo simulation study compares two realizations of the proposed approach with least square technique. Simulation results indicate that the LS method gives biased results. The results obtained demonstrate the relatively high accuracy and the robustness against noise of the algorithms proposed.
Future work could encompass potential extensions to handle the case of the colored output noise together with the recursive implementation of the algorithm.

**Figure 1.** Output model for validation data for $\sigma_z / \sigma_x = 0.25$ and $\sigma_y / \sigma_x = 0.25$ (the blue line is the true model, the red line is the criteria (2) model, the green line is the LS model).

**Figure 2.** Output model for validation data for $\sigma_z / \sigma_x = 0.5$ and $\sigma_y / \sigma_x = 0.5$ (the blue line is the true model, the red line is the criteria (2) model, the green line is the LS model).

**Figure 3.** Output model for validation data for $\sigma_z / \sigma_x = 0.75$ and $\sigma_y / \sigma_x = 0.75$ (the blue line is the true model, the red line is the criteria (2) model, the green line is the LS model).
5. References

[1] Stiassnie M 1979 On the application of fractional calculus for the formulation of viscoelastic models Applied Mathematical Modelling 3(4) 300-302

[2] Bagley R L 1983 Fractional calculus – a different approach to the analysis of viscoelastically damped structures AIAA J 21 741-748

[3] Reyes-Melo E, Martinez-Vega J J, Guerrero-Salazar C A and Ortiz-Mendez U 2005 Application of fractional calculus to the modeling of dielectric relaxation phenomena in polymeric materials Journal of Applied Polymer Science 98(2) 923-935

[4] Vinagre B M and Feliu V 2002 Modeling and control of dynamic system using fractional calculus: Application to electrochemical processes and flexible structures Proc 41-st IEEE Conf. Decision Control 214-239

[5] Mathieu B, Le Lay L and Oustaloup A 1996 Identification of non integer order systems in the timedomain IEEE SMC/IMACS Symp Control, Optimization and Supervision 952-956

[6] Cois O, Oustaloup A, Battaglia E and Battaglia J-L 2000 Non-integer model from modal decomposition for time domain system identification IFAC Proceedings 33(15) 989-994

[7] Cois O, Oustaloup A, Poinot T and Battaglia J-L 2001 Fractional state variable filter for systemidentification by fractional model Proc. European Contr. Conf. 2481-2486

[8] Malti R, Aoun M, Sabatier J and Oustaloup A 2006 Tutorial on system identification using fractional differentiation models IFAC Proceedings 39(1) 606-611

[9] Chetoui M, Malti R, Thomassin R, Aoun M, Najar S, Oustaloup A and Abdelkrim M N 2012 EIV methods for system identification with fractional models IFAC Proceedings 45(16) 1641-1646

[10] Ivanov D V 2013 Identification discrete fractional order linear dynamic systems with output-error Proceedings International Siberian Conference on Control and Communications

[11] Ivanov D V and Ivanov A V 2017 Identification Fractional Linear Dynamic Systems with fractional errors-in-variables J. Phys.: Conf. Ser. 803

[12] Ivanov D V 2015 Identification Discrete Fractional Order Hammerstein Systems International Siberian Conference on Control and Communications (SIBCON)

[13] Mohler R R 1991 Nonlinear Systems: Applications to Bilinear Control (Prentice Hall, Englewood Cliffs, NJ) p 192

[14] Mohler R R and Khapalov A Y 2000 Bilinear control and application to flexible transmission systems Journal of Optimization Theory and Applications 105(3) 621-637

[15] Mohler R R and Kolodziej W J 1980 An overview of bilinear system theory and applications IEEE Trans. On Systems, Man, and Cybernetics 10(10) 683-688

[16] Gabr M M and Rao T S 1984 On the identification of bilinear systems from operating records International Journal of Control 40(1) 121-128

[17] Ekman M 2005 Modeling and control of bilinear systems: application to the activated sludge process (PhD thesis, Uppsala University, Sweden)

[18] Ahmed M S 1986 Parameter estimation in bilinear systems by instrumental variable methods International Journal of Control 44(4) 1177-1183

[19] Ivanov D V, Bobkova E U and Zharkova A A 2017 Recursive identification of bilinear dynamical systems with noise in output signal Proceedings of 2017 IEEE East-West Design and Test Symposium 651-655

[20] Tsoulkas V, Koukoulas P and Kalouptsidis N 1999 Identification of input-output bilinear systems using cumulants Proceedings of the 6th IEEE International Conference on Electronics, Circuits and Systems 1105-1108

[21] Fnaiech F and Ljung L 1987 Recursive identification of bilinear systems International Journal of Control 45(2) 453-470

[22] Zhu Z and Leung H 1999 Adaptive identification of bilinear systems Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing 1289-1292
[23] Ivanov D V, Katsyuba O A and Uskov O V 2015 Identification of bilinear dynamic systems with noise in output signal *Journal of Information Technologies and Computing Systems* 3 81-91 (in Russian)

**Acknowledgments**

The authors gratefully acknowledge the contributions of prof. O.A. Katsyuba in improving the paper.