Some Structure Theory for Cayley Graphs and Associated Hypergraphs

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Abstract

We expand the structure theory of finite Cayley graphs that avoid specific cyclic coset patterns. A focus lies on the exploration of duality in related structures and associated hypergraphs, especially applied to the local analysis of paths and cycles. We present several characterisations of local tree-likeness for these structures and show a close connection to $\alpha$-acyclicity of hypergraphs.

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1 Introduction

Acyclic, discrete structures play a significant role in computer science. Many algorithmic graph problems that are hard in general become tractable for trees. These efficient algorithms can be further adapted to larger classes of graphs, like graphs of bounded tree-width [10], [6], [24]. These are graphs that are
not necessarily trees, but still structurally simple and in some sense tree-like. Generalizing from graphs to hypergraphs, there are several different notions of acyclicity, like \( \gamma \)-, \( \beta \)- and \( \alpha \)-acyclicity and tree decomposability, that admit many different characterisations and find applications in database theory, constraint satisfaction problems and finite model theory [3], [14], [22], [12], [16], [2].

This work focuses on the notion of coset acyclicity for Cayley graphs, a class of graphs that plays an important role in discrete mathematics [5], [11], [20], combinatorics [1], information theory [18], coding theory [21], [15], and network theory [17], [19] among other fields of mathematics. Coset acyclicity was introduced in [23] to construct certain finite hypergraph coverings that have an arbitrarily high degree of \( \alpha \)-acyclicity for a model-theoretic characterisation theorem in the vein of the van Benthem-Rosen theorem [26], [25]. In [8], [7] and [9] coset acyclic Cayley graphs were used to cover transition systems directly in order to prove further model-theoretic characterisation theorems. These results build upon the work in [13] and further develop the model-theoretic techniques that were used there. The current work presents the graph structure theory, which was essentially used in [8], [7], [9], in greater detail and in a self-contained manner that makes it accessible for wider ranging applications beyond model-theory. It is the goal to provide a general toolbox for the analysis of coset patterns in finite Cayley graphs that can be regarded as locally tree-like.

Coset acyclicity generalises the ordinary graph-theoretic notion of a cycle in the context of Cayley graphs. In a Cayley graph, every edge is induced by an element \( e \) from a generating set \( E \) of the associated Cayley group. A single step can therefore be represented by a single generator. Coset cycles generalise this notion by combining several generator steps into a larger step that is represented by a coset that is generated by a subset \( \alpha \subseteq E \) of generators, compared to a single generator \( e \in E \). The formal definition of coset cycles that stipulates precisely which sequences of cosets form a coset cycle leads to a nice structure theory for coset acyclic Cayley graphs and Cayley graphs without short coset cycles [23]. We further investigate the structure theory of coset acyclic Cayley graphs and present several ways in which these structures can be regarded as locally tree-like. The other central concept, besides coset cycles, is the notion of coset paths, which generalise graph-theoretic paths in the same way that coset cycles generalise graph-theoretic cycles. Among other results, we present a qualified uniqueness property for coset paths in coset acyclic Cayley graphs, and establish several close connections between coset acyclic Cayley graphs and \( \alpha \)-acyclic hypergraphs.

Outline

Section 2 introduces the basic notions and definitions: Cayley graphs, cycles, paths, acyclicity, and hypergraphs and \( \alpha \)-acyclicity. Section 3 presents the formal definition of coset cycles, takes a close look at Cayley graphs without coset cycles of length 2, which play a special role, and establishes the first connections between coset acyclicity and \( \alpha \)-acyclicity. Section 4 contains the main results of this work. It introduces coset paths, develops uniqueness properties for coset paths in acyclic Cayley graphs, and deepens the connection between Cayley graphs and hypergraphs with a focus on the equivalence between two
different notions of distance, one in Cayley graphs w.r.t. coset paths and one in hypergraphs.

2 Preliminaries

In this section, we introduce the main objects we want to investigate, Cayley graphs and hypergraphs. We also present some basic and well-known notions of acyclicity for these structures, which will be further developed and investigated in the course of this work. We start with fixing some notation.

For an equivalence relation $R$ on $A$, we denote the equivalence class of an element $a \in A$ by $[a]_R$ and write $A/R = \{[a]_R : a \in A\}$ for the set of all equivalence classes. The set of $R$-successors $\{b \in A : (a,b) \in R\}$ of an element $a$ is denoted by $R[a]$. 

2.1 Cayley graphs and acyclicity

This work further investigates the notion of coset acyclicity of Cayley graphs, which was introduced by Otto in [23], and its connection to $a$-acyclicity of hypergraphs. This section introduces Cayley graphs formally, the usual graph-theoretic notion of acyclicity and associated concepts. Coset acyclicity is a generalisation of the usual notion of graph acyclicity.

A Cayley group is a group $(G, \circ, 1)$ with an associated generator set $E$ that consists of non-trivial involutions, i.e. $e \neq 1$ and $e \circ e = 1$, for all $e \in E$. That $G$ is generated by the set $E$ means that every group element can be represented as a product of generators. In other words, every $g \in G$ can be represented as a word in $E^*$; w.l.o.g. such a representation is reduced in the sense that it does not have any factors $e^2$. We can view a non-empty generator set $E$ as an alphabet and interpret any word $v = e_1 \ldots e_n$ over $E$ as a group element in $G$ via $[v]^G = e_1 \circ \cdots \circ e_n$. We can also think of the letters $e_i$ as the labels of a path from 1 to $[v]^G$ in the Cayley graph of $G$. For $v = e_1 \ldots e_n$, we denote by $v^{-1}$ the word $e_n \ldots e_1$; since all generators are involutions, $[v^{-1}]^G = ([v]^G)^{-1}$.

Definition 2.1. With every Cayley group $(G, \circ, 1)$ generated by $E$ one associates its Cayley graph $(G, (R_e)_{e \in E})$: its vertex set is the set of group elements $G$, and its edge relations are

$$R_e = \{\{v, v \circ e\} \in G \times G : v \in G\}.$$ 

If $G$ is a Cayley group, we denote the group itself, its Cayley graph and its set of group elements with $G$. If $G$ is a Cayley graph, we also write $V[G]$ for its vertex set and $R_e[G]$ for its $e$-labelled edge relation.

In our case, all edge relations are loop-free, undirected and complete matchings on $G$. Since $E$ generates $G$, the graph $(G, (R_e)_{e \in E})$ is connected. Furthermore, it is homogeneous in the sense that every two vertices $v$ and $u$ are related by a graph automorphism that is induced by multiplication from the left with $u^{-1}$.

For a subset $A \subseteq E$ we consider the subgroup $G_A$, which is the subgroup of $G$ generated by the generators from $A$. Its Cayley graph, also denoted $G_A$, is a subgraph of $G$; it is isomorphic to the $a$-component of 1. The $a$-component

3
of an arbitrary group element \(v\) is described by its \(\alpha\)-coset \(vG_\alpha = \{v \circ u \in G : u \in G_\alpha\}\). Every \(\alpha \subseteq E\) induces an equivalence relation on \(G\) through partitioning \(G\) into its \(\alpha\)-cosets. Hence, we usually denote the \(\alpha\)-coset of a group element \(v\) as \([v]_\alpha\).

The main notions that we investigate in this work are paths and cycles. Cayley graphs have multiple edge relations \(R_e\) that are labelled with generators \(e \in E\) of the associate Cayley group. Hence, all paths and cycles will be labelled with generators to differentiate the kind of steps that lead from one vertex to the next.

**Definition 2.2.** An \((E\text{-labelled})\) path of length \(\ell\) in a Cayley graph \(G\) is an alternating sequence \(v_1, e_1, v_2, \ldots, v_\ell, e_\ell, v_{\ell+1}\) of vertices \(v_i \in V[G]\) and labels \(e_i \in E\) such that \(\{v_i, v_{i+1}\} \in R_{e_i}\), for all \(1 \leq i \leq \ell\), end all vertices and edges are distinct, with the possible exception of \(v_1 = v_{\ell+1}\), in which case the path is called a cycle. The vertices \(v_1\) and \(v_{\ell+1}\) are called the endpoints of the path, and we speak of a path from \(v_1\) to \(v_{\ell+1}\). If every edge of a path is labelled with an element from a subset \(\alpha \subseteq E\), we call it an \(\alpha\)-path.

The definition of paths leads to several well-known notions like distance, reachability and connectedness: The distance \(d(v, u)\) between two vertices \(v, u\) in a graph is the minimal length of a path from \(v\) to \(u\); \(d(v, v) = 0\) for all \(v \in V\), and \(d(v, u) = \infty\) if there is no path from \(v\) to \(u\). The \(\ell\)-neighbourhood of a vertex \(v\), denoted \(N^\ell(v)\), is the set of vertices of distance at most \(\ell\) from \(v\), i.e. \(\{u : d(v, u) \leq \ell\}\). For \(\alpha \subseteq E\), a vertex \(u\) is \(\alpha\)-reachable from \(v\) if there is an \(\alpha\)-path from \(v\) to \(u\).

Graphs without cycles or without any short cycles are structurally more simple. This lends itself to be exploited by various applications like efficient algorithms for generally intractable problems, and it is important for model-theoretic constructions. We will compare and generalise properties of graphs without short cycles to graphs without short coset cycles. We present some further notions connected to cycles that will be important throughout.

**Definition 2.3.** Let \(G\) be a Cayley graph.

1. \(G\) is acyclic if it has no cycles.
2. A \(k\)-cycle in \(G\) is a cycle of length \(k\) in \(G\).
3. \(G\) is \(k\)-acyclic if it has no cycles of length \(\leq k\).
4. The girth of \(G\) is the length of a minimal cycle.
5. \(G\) is a tree if it is acyclic.

Usually, trees are defined as acyclic and connected graphs. Since Cayley graphs are always connected, it suffices to require acyclicity. In the case of Cayley graphs, every tree must be infinite. Take as an example the Cayley graph of the free group over \(E\), for a set of involutive generators \(E\). A finite Cayley graph can never be fully acyclic, but finite and \(k\)-acyclic Cayley graphs can be constructed easily.

**Proposition 2.4.** [13] For every finite set \(E\) and every \(k \in \mathbb{N}\) there is a finite, \(k\)-acyclic Cayley graph with generator set \(E\).
If \( G \) is a tree, then two vertices are always connected. If a graph is 2\( k + 1 \)-acyclic, then the subgraphs induced by the \( k \)-neighbourhoods of all vertices are \( k \)-acyclic, i.e. all \( k \)-neighbourhoods look like trees. This implies that paths of length up to \( k \) in 2\( k + 1 \)-acyclic graphs are unique because two vertices at distance \( \leq k \) from each other must share some tree-like \( k \)-neighbourhood. These concepts generalise to coset acyclic graphs in non-trivial ways and will be explored in Section 4.1.

### 2.2 Hypergraphs

This section introduces hypergraphs, \( \alpha \)-acyclicity and other already known related notions like tree decompositions. A hypergraph is a generalisation of a graph in which an edge can contain any number of vertices.

**Definition 2.5.** A **hypergraph** is a structure \( \mathcal{A} = (A, S) \) with a set of vertices \( A \) and a set of hyperedges \( S \subseteq P(A) \).

With a hypergraph \( \mathcal{A} = (A, S) \) we associate its **Gaifman graph** \( G(\mathcal{A}) = (A, G(S)) \) with an undirected edge relation \( G(S) \) that links two vertices \( a \neq a' \) if \( a, a' \in s \), for some \( s \in S \). An \( n \)-cycle in a hypergraph is a cycle of length \( n \) in its Gaifman graph, and an \( n \)-path in a hypergraph is a path of length \( n \) in its Gaifman graph. The distance \( d(X, Y) \) in a hypergraph between two subsets of vertices \( X \) and \( Y \) is the usual graph-theoretic distance between \( X \) and \( Y \) in its Gaifman graph, i.e. the minimal length of a path from \( X \) to \( Y \). A chord of an \( n \)-cycle or \( n \)-path is an edge between vertices that are not next neighbours along the cycle or path.

There are several, non-equivalent ways to define acyclic hypergraphs. However, all the different notions of acyclicity coincide for the usual undirected, loop-free graphs. The following definition of hypergraph acyclicity is the classical one from [4], also known as \( \alpha \)-acyclicity in [3]; \( n \)-acyclicity was introduced in [23].

**Definition 2.6.** A hypergraph \( \mathcal{A} = (A, S) \) is **acyclic** if it is conformal and chordal:

1. conformality requires that every clique in the Gaifman graph \( G(\mathcal{A}) \) is contained in some hyperedge \( s \in S \);  
2. chordality requires that every cycle in the Gaifman graph \( G(\mathcal{A}) \) of length greater than \( 3 \) has a chord.

For \( n \geq 3 \), \( \mathcal{A} = (A, S) \) is **\( n \)-acyclic** if it is **\( n \)-conformal** and **\( n \)-chordal**:

3. \( n \)-conformality requires that every clique in \( G(\mathcal{A}) \) up to size \( n \) is contained in some hyperedge \( s \in S \);  
4. \( n \)-chordality requires that every cycle in \( G(\mathcal{A}) \) of length greater than \( 3 \) and up to \( n \) has a chord.

**Remark 2.7.** If a hypergraph is \( n \)-acyclic, then every induced substructure of size up to \( n \) is acyclic [29].

Conformal and chordal hypergraphs are called acyclic because they are tree-like in the sense that they are **tree decomposable**.
Definition 2.8. A hypergraph \((A, S)\) is tree decomposable if it admits a tree decomposition \(T = (T, \delta)\): \(T\) is a tree and \(\delta: T \rightarrow S\) is a map such that \(\text{image}(\delta) = S\) and, for every node \(a \in A\), the set \(\{v \in T : a \in \delta(v)\}\) is connected in \(T\).

A well-known result from classical hypergraph theory states that a hypergraph is tree decomposable if and only if it is acyclic (see [3], [3]).

3 Acyclicity in Cayley graphs and hypergraphs

This chapter is concerned with a more general notion of cycles called coset cycles; it was introduced by Otto in [23]. Some of the results in this section can also be found in [8].

3.1 Coset acyclicity

We can write a labelled cycle of length \(m\) as a finite sequence \(\left((v_i, e_i)\right)_{i \in \mathbb{Z}_m}\) of pairs from \(G \times E\) with \((v_i, v_{i+1}) \in R_{\alpha_i}\), for all \(i \in \mathbb{Z}_m\). In such an ordinary cycle, every step from \(v_i\) to \(v_{i+1}\) goes along exactly one edge. Coset cycles allow for steps that consist of multiple edges at once, or in other words some group element that is the product of multiple generators from some subset \(\alpha \subseteq E\). To differentiate ordinary cycles from coset cycles, we use the following conventions. A cycle can both be a finite sequence of the form \(\left((v_i, e_i)\right)_{i \in \mathbb{Z}_m}\), \(e_i \in E\), or \(\left((v_i, a_i)\right)_{i \in \mathbb{Z}_m}\), \(a_i \subseteq E\), where \(v_i^{-1}v_{i+1} \in G_{\alpha_i}\) or \(v_i^{-1}v_{i+1} \in G_{\alpha_i}\), respectively. A generator cycle is cycle of the form \(\left((v_i, e_i)\right)_{i \in \mathbb{Z}_m}\), where all \(e_i\) are single generators.

Definition 3.1. Let \(G\) be a Cayley graph with generator set \(E\). A coset cycle of length \(m\) in \(G\) is a finite sequence \(\left((v_i, a_i)\right)_{i \in \mathbb{Z}_m}\) with \(v_i \in G\) and \(a_i \subseteq E\), for all \(i \in \mathbb{Z}_m\), where \(v_i^{-1}v_{i+1} \in G_{a_i}\) and

\[
[v_i]_{a_i-1 \cap a_i} \cap [v_{i+1}]_{a_i \cap a_{i+1}} = \emptyset.
\]

Remark 3.2. For \(\left((v_i, a_i)\right)_{i \in \mathbb{Z}_m}\) we call \([v_i]_{a_i-1 \cap a_i} \cap [v_{i+1}]_{a_i \cap a_{i+1}} = \emptyset\) the coset cycle property. It essentially states that every \(a_i\)-step from \(v_i\) to \(v_{i+1}\) has to count in the sense that it cannot be replaced by the previous \(a_{i-1}\)-step and the subsequent \(a_{i+1}\)-step. Without this property we would admit “too many” cycles and would not obtain a sensible theory for coset cycles.

Definition 3.3. A Cayley graph is acyclic if it does not contain a coset cycle, and \(n\)-acyclic if it does not contain a coset cycle of length up to \(n\).

This definition leads to a theory of coset acyclic Cayley graphs that is interesting in itself and has been shown to be useful for applications in finite model theory in [23] and [3]. The exploration of the structure theory of coset acyclic Cayley graphs is the main topic of this work. For the remainder of this work, if we speak about acyclic or \(n\)-acyclic Cayley graphs, we always mean coset acyclic or coset \(n\)-acyclic. Acyclicity in the usual graph-theoretic sense will be indicated specifically.

Coset acyclicity is of further special interest because every Cayley group can be covered by an acyclic group and every finite Cayley group can be covered by a finite \(n\)-acyclic group, for arbitrary \(n\).
Definition 3.4. A homomorphism $\pi: \hat{G} \to G$ is a covering of $G$ by $\hat{G}$ if it is surjective and for every $v \in V[\hat{G}]$, the restriction of $\pi$ to the 1-neighbourhood of $v$ is an isomorphism onto the 1-neighbourhood of $\pi(v)$. If $\pi: \hat{G} \to G$ is a covering, we also often refer to the structure $\hat{G}$ as a covering of $G$, or say that $\hat{G}$ covers $G$.

If $G$ is a Cayley group that is generated by $E$, we can construct a covering $\pi: \hat{G} \to G$ and give the function rule of $\pi$ based on the representation of a group element $v$ as a word over $E$. However, since an element $v$ can be represented by multiple words, the covering must be compatible with the original group in the following sense.

Definition 3.5. Let $H$ and $G$ be groups with generator set $E$. $H$ is compatible with $G$ if for all words $w$ over $E$ if $[w]^G = 1$ implies $[w]^H = 1$.

If $H$ is compatible with $G$, it is easy to see that $G$ in fact covers $H$.

Remark 3.6. If $H$ is compatible with $G$, then $\pi: G \to H, [w]^G \mapsto [w]^H$ is a well-defined, surjective group homomorphism. In particular, $\pi$ is a covering of $H$ by $G$.

Fully acyclic and infinite coverings can be obtained easily by using the free group over $E$. Constructing finite, fully acyclic coverings is out of the question. But Otto showed in [23] that it is possible to construct finite coverings that have an arbitrarily high degree of acyclicity:

Lemma 3.7. For every finite Cayley group $G$ with finite generator set $E$ and every $n \in \mathbb{N}$, there is a finite, $n$-acyclic Cayley group $\hat{G}$ with generator set $E$ such that $\hat{G}$ is compatible with $G$, and $\pi: \hat{G} \to G, [w]^{\hat{G}} \mapsto [w]^G$ is a covering.

Many concepts for graphs that are acyclic in the usual sense can be generalise to Cayley graphs that are coset acyclic. We establish several close connections between acyclic Cayley graphs and $\alpha$-acyclic hypergraphs, and argue that acyclic Cayley graphs can be considered tree-like in a more general sense. First, we take a closer look at 2-acyclicity because it provides the backbone for most of the forthcoming definitions and all further analysis.

3.2 2-acyclicity

A Cayley graph is 2-acyclic if there are no coset cycles of length 2, i.e. if for all vertices $v, u$ and all sets of generators $\alpha, \beta$ with $[v]_\alpha = [u]_\alpha$ and $[v]_\beta = [u]_\beta$: $[v]_{\alpha \cap \beta} \cap [u]_{\alpha \cap \beta} \neq \emptyset$. 2-acyclicity imposes a high degree of order in Cayley graphs.

Lemma 3.8. A Cayley graph $G$ is 2-acyclic if and only if for all $v \in G, \alpha, \beta \subseteq E$

$$[v]_\alpha \cap [v]_\beta = [v]_{\alpha \cap \beta}.$$ 

Proof. "$\Rightarrow$": If there is a 2-cycle $v, \alpha, u, \beta, v$, then $u \in [v]_\alpha \cap [v]_\beta$ and $[v]_{\alpha \cap \beta} \cap [u]_{\alpha \cap \beta} = \emptyset$. In particular, this means $u \notin [v]_{\alpha \cap \beta}$, which implies $[v]_\alpha \cap [v]_\beta \neq [v]_{\alpha \cap \beta}$.

"$\Leftarrow$": Assume there are $v \in G, \alpha, \beta \subseteq E$ such that $[v]_\alpha \cap [v]_\beta \neq [v]_{\alpha \cap \beta}$. Since by definition always $[v]_{\alpha \cap \beta} \subseteq [v]_\alpha \cap [v]_\beta$, there must be some $u \in ([v]_\alpha \cap [v]_\beta) \setminus [v]_{\alpha \cap \beta}$. In particular, $u \notin [v]_{\alpha \cap \beta}$ implies $[v]_{\alpha \cap \beta} \cap [u]_{\alpha \cap \beta} = \emptyset$. Hence $v, \alpha, u, \beta, v$ forms a 2-cycle.

$\Box$
Example 3.9. A Cayley graph can be of girth 4 without being even coset 2-acyclic: The symmetric group $S_3$ generated by the transpositions $(1,2), (1,3), (2,3)$ has such a Cayley graph. Its shortest cycle has length 4, but it contains the coset 2-cycle $\{(1), (1,2), (2,3)\}, $ $(1,3), $ $(1,3)\}$, $(1)$. This example further illustrates that there is no unique minimal connecting subset of generators between two group elements; both $\{(1,2), (2,3)\}$ and $\{(1,3)\}$ connect $(1)$ and $(1,3)$, but neither is contained in the other. This is not the case in coset 2-acyclic graphs, as Lemma 3.12 shows.

The characterisation of 2-acyclicity in Lemma 3.8 implies that the intersections of cosets with different subsets of generators in 2-acyclic Cayley groups are already far form arbitrary. As mentioned above, 2-acyclicity provides the backbone of our further structural analysis. Lemma 3.12 shows that in 2-acyclic groups two elements $v,u$ are always connected by some unique minimal set of generators $\alpha$, i.e. $[v]_{\beta} = [u]_{\beta}$ if and only if $\beta \supseteq \alpha$. Before we present the lemma, we define the dual hyperedge.

Definition 3.10. In a Cayley graph $G$, define the dual hyperedge induced by an element $v$ to be the set of cosets that contain $v$:

$$[v] := \{ [v]_{\alpha} : \alpha \subseteq E \}$$

Remark 3.11. In a Cayley graph $G$ for all $v,u \in G$ and all $\alpha \subseteq E$:

$$[v]_{\alpha} = [u]_{\alpha} \iff v \in [u]_{\alpha} \iff [u]_{\alpha} \in [v]$$

Lemma 3.12. In a 2-acyclic Cayley group $G$ with elements $v,v_1, \ldots, v_k$ and sets of generators $\alpha_1, \ldots, \alpha_k \subseteq E$:

1. For $\beta := \bigcap_{1 \leq i \leq k} \alpha_i$:

$$v \in \bigcap_{1 \leq i \leq k} [v_i]_{\alpha_i} \Rightarrow \bigcap_{1 \leq i \leq k} [v_i]_{\alpha_i} = [v]_{\beta}$$

2. The set $\bigcap_{1 \leq i \leq k} [v_i]_{\alpha_i}$ has a least element in the sense that there is an $\alpha_0 \subseteq E$ such that $[v_1]_{\alpha_0} \cap \bigcap_{1 \leq i \leq k} [v_i]_{\alpha_i}$ and, for any $\alpha \subseteq E$:

$$[v_1]_{\alpha} \in \bigcap_{1 \leq i \leq k} [v_i]_{\alpha_i} \iff \alpha_0 \subseteq \alpha'$$

Proof. 1. Lemma 3.8 implies $\bigcap_{1 \leq i \leq k} [v_i]_{\alpha_i} = \bigcap_{1 \leq i \leq k} [v_i]_{\alpha_i} = [v]_{\beta}$.

2. 2-acyclicity implies that the collection

$$\{ \alpha \subseteq E : [v_1]_{\alpha} \in \bigcap_{1 \leq i \leq k} [v_i]_{\alpha_i} \}$$

is closed under intersections: otherwise there would be $\alpha, \beta \subseteq E$ with

$$[v_1]_{\alpha}, [v_1]_{\beta} \in \bigcap_{1 \leq i \leq k} [v_i]_{\alpha_i} \quad \text{and} \quad [v_1]_{\alpha \cap \beta} \notin \bigcap_{1 \leq i \leq k} [v_i]_{\alpha_i}. $$

This implies $[v_1]_{\alpha \cap \beta} \notin [v_i]_{\alpha_i}$, but $[v_1]_{\alpha}, [v_1]_{\beta} \in [v_i]_{\alpha_i}$, for some $1 \leq i \leq k$. Hence, there would be a 2-cycle $v_1, \alpha, v_j, \beta, v_1$. □
Lemma 3.12 justifies the following definition.

**Definition 3.13.** In a 2-acyclic Cayley graph we denote the unique minimal set of generators that connects the vertices in a tuple \( v \) by \( \text{gen}(v) \subseteq E \).

Intuitively, \( \text{gen}(v) \) sets the scale for zooming-in on the minimal substructure that connects the vertices \( v \). It behaves in a regular manner.

**Lemma 3.14.** In a 2-acyclic Cayley graph \( G \) for vertices \( v, u \) and every generator \( e \notin \text{gen}(v, u) \):

\[
\text{gen}(v, u \circ e) = \text{gen}(v, u) \cup \{ e \}
\]

**Proof.** Set \( \alpha := \text{gen}(v, u) \), and let \( e \in E \setminus \text{gen}(v, u) \), \( u' := u \circ e \neq u \), and set \( \beta := \text{gen}(v, u') \). The choice of \( u' \) implies an \( (a \cup \{ e \}) \)-path from \( v \) to \( u' \). Hence, \( \beta \subseteq (a \cup \{ e \}) \) because of 2-acyclicity and Lemma 3.12.

Assume \( \beta \subseteq (a \cup \{ e \}) \). First, if \( e \notin \beta \), then \( \beta \subseteq a \), which means there is an \( a \)-path from \( v \) to \( u' \) that can be combined with the \( a \)-path from \( v \) to \( u \) to an \( a \)-path from \( u \) to \( u' \). Furthermore, \( [u]_a \cap [e] = [u]_a = \{ u \} \) and \( [u']_a \cap [e] = [u']_a = \{ u' \} \) since \( e \notin a \). Together with \( u \neq u' \) this implies that \( v, a, u', e, \) \( u \) forms a 2-cycle. Thus, \( a \in \beta \) since \( G \) is 2-acyclic.

Second, assume there is some generator \( e' \in a \) with \( e' \notin \beta \). Additionally, \( e \in \beta \) and \( [u]_e = [u']_e \), imply \( [u]_\beta = [u']_\beta = [v]_\beta \). However, if \( \beta \cap a \neq a \), then a \( \beta \)-path from \( v \) to \( u \) contradicts the minimality property of \( a \).

**Lemma 3.15** gives us some additional useful insight into the structure of 2-acyclic Cayley graphs.

**Lemma 3.15.** Let \( G \) be a 2-acyclic Cayley graph. Then, for all vertices \( v \) and all \( a, \beta \subseteq E, \beta \subseteq a \) if and only if \( [v]_\beta \subseteq [v]_a \).

**Proof.** The direction from left to right is, of course, true in general.

For the converse direction, let \( e \in \beta \), and assume \( e \notin a \). Since \( e \neq 1 \), the element \( v' = v \circ e \in [v]_e \) is different from \( v \). Additionally, \( v' \in [v]_e \subseteq [v]_\beta \subseteq [v]_a \) implies an \( a \)-path from \( v \) to \( v' \). However, this means that \( v, \{ e \}, v', \alpha, v \) is a coset cycle of length 2 since

\[
[v]_e \cap [v']_a \cap [v']_\beta = [v]_\emptyset \cap [v']_\emptyset = [v]_\emptyset \cap [v']_\emptyset = \emptyset,
\]

which contradicts the assumption of 2-acyclicity.

**Lemma 3.16** gives another characterisation of the coset cycle property in 2-acyclic Cayley graphs that provides a helpful tool in dealing with coset cycles; it’s proof is straightforward.

**Lemma 3.16.** If \( G \) is a 2-acyclic Cayley group and \( (v_i, a_i)_{i \in \mathbb{Z}_m} \) a finite sequence with \( [v_i]_{a_i} = [v_{i+1}]_{a_{i+1}} \) for all \( i \in \mathbb{Z}_m \). Then for all \( i \in \mathbb{Z}_m \)

\[
[v_i]_{a_{i-1}} \cap [v_{i+1}]_{a_{i} \cap a_{i+1}} = [v_{i-1}]_{a_{i-1}} \cap [v_i]_{a_i} \cap [v_{i+1}]_{a_{i+1}}.
\]
3.3 Dual hypergraphs

In this section, we define for every Cayley graph $G$ an associated structure $d(G)$, the *dual hypergraph* of $G$, and present the first connections between coset acyclicity for Cayley graphs and $\alpha$-acyclicity for their dual hypergraphs.

**Definition 3.17.** Let $G = (V, (R_\alpha)_{\alpha \subseteq E})$ be a Cayley graph, and define the equivalence relation $R_\alpha := \text{TC}(\bigcup_{e \in \alpha} R_e)$, for all $\alpha \subseteq E$ (TC denotes the transitive closure). The *dual hypergraph* of $G$ is the vertex-coloured hypergraph

$$d(G) := (d(V), S, (Q_\alpha)_{\alpha \subseteq E}) \text{ where }$$

$$d(V) := \bigcup_{\alpha \subseteq E} Q_\alpha \quad \text{for } Q_\alpha := V / R_\alpha,$$

$$S := \{ [v] \subseteq d(V) : v \in V \}.$$

As the name suggests, everything in the dual hypergraph is flipped. The vertices of $G$ are the hyperedges of $d(G)$, the $\alpha$-cosets of $G$ are the $\alpha$-coloured vertices of $d(G)$. Furthermore, Lemma 3.12 implies that every intersection between hyperedges can be described by the unique set of generators $\text{gen}(v)$. This means, for every $v \in V$ and every $\alpha \subseteq E$:

$$[v]_\alpha \in \bigcap_{v \in V} [v] \Leftrightarrow \alpha \supseteq \text{gen}(v).$$

The notions of acyclicity for Cayley graphs and hypergraph acyclicity are directly connected. Otto showed that the dual hypergraph $d(G)$ is $n$-acyclic if $G$ is coset $n$-acyclic, and we show the other direction for 2-acyclic $G$.

**Lemma 3.18.** [23] For $n \geq 3$, if $G$ is an $n$-acyclic Cayley graph, then $d(G)$ is an $n$-acyclic hypergraph.

**Lemma 3.19.** Let $G$ be a 2-acyclic Cayley graph. For $n \geq 3$, if $d(G)$ is an $n$-acyclic hypergraph, then $G$ is $n$-acyclic.

**Proof.** Let $\{(v_i, a_i)\}_{i \in \mathbb{Z}_m}$ be a coset cycle of minimal length in $G$. We need to show that $m > n$. The cycle $\{(v_i, a_i)\}_{i \in \mathbb{Z}_m}$ in $G$ induces an associated cycle $\{(v_{i, a, i}, [v_{i+1}])\}_{i \in \mathbb{Z}_m}$ in the dual hypergraph $d(G)$ because $[v_i]_{a_i} \in [v_{i+1}]$ since $(v_i, v_{i+1}) \in R_{a_i}$. If we show that this cycle is chordless, then $\alpha$-acyclicity of $d(G)$ implies $m > n$.

The length of $\{(v_i, a_i)\}_{i \in \mathbb{Z}_m}$ is at least 3 because $G$ is 2-acyclic. If it is 3, then the induced cycle $\{(v_{i, a, i}, [v_{i+1}])\}_{i \in \mathbb{Z}_m}$ must be contained in some hyperedge $[v]$ because $d(G)$ is, in particular, 3-conformal. However, the definition of $d(G)$ and 2-acyclicity of $G$ together with Lemma 3.18 imply

$$[v_1]_{a_1} \cap [v_2]_{a_2} \cap [v_3]_{a_3} \subseteq [v] \Rightarrow v \in [v_1]_{a_1} \cap [v_2]_{a_2} \cap [v_3]_{a_3} = [v_1]_{a_1 \cap a_2} \cap [v_2]_{a_1 \cap a_2};$$

this violates the coset cycle property $[v_1]_{a_1 \cap a_2} \cap [v_2]_{a_1 \cap a_2} = \emptyset$. Hence, $m$ must be at least 4.

Now, assume that the cycle $\{(v_{i, a, i}, [v_{i+1}])\}_{i \in \mathbb{Z}_m}$ has a chord, i.e. there is some hyperedge $[u]$ and there are $1 \leq i, j \leq m$ with $j > i + 1$ such that $[v_i]_{a_i}, [v_j]_{a_j} \in [u]$. First, we choose $[u]$ such that the distance between $[v_i]_{a_i}$
and \([v_i]_{a_i}\) on the cycle is minimal, i.e. there are no other vertices on the cycle that are connected by a chord and have a shorter distance on the cycle than \([v_i]_{a_i}\) and \([v_j]_{a_j}\). Then
\[
[u]_{a_{i,j}} [v_{i+1}]_{a_{i+1}, \ldots, a_j}, [v_j]_{a_j}, [u]_{a_i}
\]
is a cycle in \(d(G)\) since \([u]_{a_i} = [v_i]_{a_i}\) and \([u]_{a_j} = [v_j]_{a_j}\). This cycle in the dual hypergraph induces a cycle
\[
u_{i,j}, v_{i+1}, a_{i+1}, \ldots, a_j, v_j, a_j, u
\]
in \(G\) of length shorter than \(m\). If we can show that this cycle is also a coset cycle, then the chord \([u]\) could not exists because it would contradict that we chose \(((v_i, a_i))_{i \in \mathbb{Z}_m}\) as a coset cycle of minimal length.

We need to check the coset property at \(u\), i.e. \([u]_{a_i} \cap [v_{i+1}]_{a_i \cap a_{i+1}} = \emptyset\) and \([v_j]_{a_j} \cap [u]_{a_j \cap a_i} = \emptyset\). Assume there is some \(w \in [u]_{a_j} \cap [v_{i+1}]_{a_j \cap a_{i+1}}\). 2-acyclicity of \(G\) and Lemma 3.16 imply \(w \in [u]_{a_j} \cap [v_j]_{a_j} \cap [v_{i+1}]_{a_j \cap a_{i+1}} = [v_j]_{a_j} \cap [v_{i+1}]_{a_j \cap a_{i+1}}\). We assumed \([u]\) to be such that the distance between \([v_i]_{a_i}\) and \([v_j]_{a_j}\) on the cycle is minimal, hence \(j > i + 2\) cannot be the case because \([v_{i+1}]_{a_{i+1}} [v_j]_{a_j} \in [u]\) have shorter distance. This leaves \(j = i + 2\), which implies
\[
\emptyset \neq [v_i]_{a_i} \cap [v_{i+1}]_{a_i} \cap [v_{i+2}]_{a_i} = [v_{i+1}]_{a_i} \cap [v_{i+2}]_{a_i} \cap [v_{i+1}]_{a_i}. 
\]
But this contradicts the coset property of the given coset cycle. Showing \([v_j]_{a_j} \cap [u]_{a_j} = \emptyset\) works analogously.

Thus, we found a coset cycle that is shorter than \(m\). This contradicts the choice of \(((v_i, a_i))_{i \in \mathbb{Z}_m}\) as a coset cycle of minimal length in \(G\). This means that \(((v_i, a_i))_{i \in \mathbb{Z}_m}\) must be chordless, which implies \(m > n\) by \(n\)-acyclicity of \(d(G)\).

Thus, the previous lemmas show that an acyclic Cayley graph is tree-like in the sense that its dual hypergraph is tree-decomposable.

### 4 Analysis of paths and distances

Coset cycles generalise the graph-theoretic notion of a cycle for Cayley graphs. Coset paths generalise the graph-theoretic notion of a path in the same way. These coset paths and their behaviour in \(n\)-acyclic Cayley graphs are the subject of this chapter.

Many of the various definitions and notions that we introduce from now on only make sense in 2-acyclic Cayley graphs, because they are based on the set \(\text{gen}(v)\). Therefore, and because every Cayley graph has a 2-acyclic covering, we make the following assumption for the remainder of this section.

**Proviso 4.1.** Every Cayley graph is assumed to be 2-acyclic.

**Definition 4.2** (Coset path). Let \(G\) be a Cayley graph. A coset path of length \(\ell \geq 1\) is a labelled path \(v_1, a_1, v_2, a_2, \ldots, a_\ell, v_\ell+1\) such that, for \(1 \leq i \leq \ell\),
\[
[v_i]_{a_i} \cap [v_{i+1}]_{a_i} = \emptyset,
\]
with $a_0 = a_{\ell + 1} = \emptyset$. A coset path $v_1, a_1, \ldots, a_{\ell}, v_{\ell + 1}$ of length $\ell \geq 2$ is non-trivial if, for $\alpha = \text{gen}(v_1, v_{\ell + 1})$, for all $1 \leq i \leq \ell$,

$$[v_1]_{\alpha} \not\subseteq [v_i]_{\alpha_i}.$$ 

A coset path $v_1, a_1, \ldots, a_{\ell}, v_{\ell + 1}$ of length $\ell \geq 2$ is an inner path if, for $\alpha = \text{gen}(v_1, v_{\ell + 1})$, for all $1 \leq i \leq \ell$,

$$[v_i]_{\alpha_i} \not\subseteq [v_1]_{\alpha}.$$ 

A non-trivial coset path from $v$ to $u \neq v$ is minimal if there is no shorter non-trivial coset path from $v$ to $u$.

**Remark 4.3.** Non-trivial and inner coset paths are only well-defined in 2-acyclic graphs.

**Observation 4.4.** Inner coset paths are non-trivial.

In other words, a coset path is a path that links two consecutive vertices not via a single edge or generator, but via a coset in a way that respects the coset property of coset cycles in every step. An analogue of Lemma 3.16 is also true for coset paths.

**Lemma 4.5.** If $G$ is a Cayley graph and $v_1, a_1, v_2, \ldots, v_{\ell}, a_{\ell}, v_{\ell + 1}$ a path, then, for all $2 \leq i \leq \ell$,

$$[v_i]_{a_{i-1} \cap a_i} \cap [v_{i+1}]_{a_i \cap a_{i+1}} = [v_{i-1}]_{a_{i-1}} \cap [v_i]_{a_{i+1}} \cap [v_{i+1}]_{a_{i+1}},$$

with $a_{\ell + 1} = \emptyset$.

The following sections develop a theory of coset paths in $n$-acyclic Cayley graphs.

### 4.1 Short coset paths

If a Cayley graph is $2k + 1$-acyclic in the usual sense, then every $k$-neighbourhood $N^k(v)$ induces a substructure that is a tree. This entails that two vertices that have a distance of at most $k$ are connected by a unique path of length at most $k$. This concept generalises to coset acyclic Cayley graphs w.r.t. coset paths.

In an acyclic Cayley graph, two distinct vertices $v$ and $u$ are always uniquely connected by a coset path of the form $v_1, \{e_1\}, \ldots, \{e_{\ell}\}, v_{\ell + 1}$ where all the sets of generators are singletons. But there might be a myriad of different recombinations of sets of these generators that pass as proper coset paths. However, all these paths overlap in some sense, and if the Cayley graph is $2n$-acyclic all paths of length up to $n$ overlap in this way. This is the content of the zipper lemma (Lemma 4.8), the central result of this section. Let us make precise what we mean by short coset paths.

**Definition 4.6.** Let $G$ be a Cayley graph that is $2n$-acyclic. We call a coset path short if its length is $\leq n$.

Often we do not make it explicit to what degree a Cayley graph is acyclic. Instead, we write that a Cayley graph $G$ is sufficiently acyclic, i.e. there is some $n \in \mathbb{N}$ such that $G$ is $n$-acyclic and all the arguments go through.
Essentially, the zipper lemma states that in a sufficiently acyclic Cayley graph two short coset paths that both start at the same vertex \( v \) overlap non-trivially at both ends. Thus, multiple applications of the zipper lemma imply that two short coset paths of this kind behave like a zipper that can be closed from both ends. Furthermore, the zipper lemma implies that, for all pairs of vertices \((v, u)\), there is a unique minimal set of generators \( a_0 \) such that \( a_0 \subseteq a_1 \), for all short coset paths \( v, a_1, \ldots, a_\ell, u \). This set \( a_0 \) can be interpreted as the direction one has to take if one wants to move from \( v \) to \( u \) on a short coset path.

In order to prove the zipper lemma, we begin with considering short coset paths \( v_1, a_1, v_2, \ldots, v_\ell, a_\ell, v_1 \) that start and end at the same vertex \( v_1 \). Such a path may differ from a coset cycle regarding the overlaps at the ends. If \( v_1, a_1, v_2, \ldots, v_\ell, a_\ell, v_1 \) is just a path, we can by definition only assume
\[
[v_1]\cap a_1 \cap [v_2]a_1\cap a_2 = \emptyset \quad \text{and} \quad [v_\ell]a_{\ell-1}\cap a_\ell \cap [v_1]a_\cap a_0 = \emptyset,
\]
i.e. \( v_1 \notin [v_2]a_1\cap a_2 \) and \( v_\ell \notin [v_\ell]a_{\ell-1}\cap a_\ell \), but not that it is a complete coset cycle, i.e. that also
\[
[v_1]a_1\cap a_2 \cap [v_2]a_2\cap a_3 = \emptyset \quad \text{and} \quad [v_\ell]a_{\ell-1}\cap a_\ell \cap [v_1]a_\cap a_0 = \emptyset.
\]
Hence, these cyclic coset paths are not directly ruled out by acyclicity but by the following lemma.

**Lemma 4.7.** Let \( v \) be a vertex in a Cayley graph \( G \). If \( G \) is \( n \)-acyclic, then there is no coset path of length up to \( n \) that starts and ends at \( v \).

**Proof.** The claim is shown by induction on the length \( \ell \) of the coset path, for \( 1 \leq \ell \leq n \).

For \( \ell = 1 \), Definition 4.2 rules out coset loops \( v, a, v \) because it implies
\[
\emptyset = [v]\cap a \cap [v]a\cap a_0 = \{v\}.
\]

For \( \ell = 2 \), coset paths \( v_1, a_1, v_2, a_2, v_1 \) with \( v_1 \notin [v_2]a_1\cap a_2 \) are ruled out because 2-acyclicity implies
\[
[v_1]a_1\cap a_2 = [v_2]a_1\cap a_2,
\]
leading to the contradiction \( v_1 \notin [v_1]a_1\cap a_2 \).

For \( 2 < \ell \leq n \), assume there are no coset paths of length up to \( \ell - 1 \) from any vertex back to itself. Consider a coset path
\[
v_1, a_1, v_2, \ldots, v_\ell, a_\ell, v_\ell+1
\]
of length \( \ell \) with \( v_1 = v_{\ell+1} \). That \( G \) is \( n \)-acyclic implies
\[
[v_1]a_1\cap a_2 \cap [v_2]a_2\cap a_3 \neq \emptyset \quad \text{or} \quad [v_\ell]a_{\ell-1}\cap a_\ell \cap [v_1]a_\cap a_0 \neq \emptyset.
\]
W.l.o.g. we assume there is some \( u \in [v_1]a_1\cap a_2 \cap [v_2]a_2\cap a_3 \). If \( u \notin [v_\ell]a_{\ell-1}\cap a_\ell \), then
\[
u, a_2, v_3, a_3, v_4, \ldots, v_\ell, a_\ell, u
\]
is a coset path of length \( \ell - 1 \) from \( u \) to itself. Otherwise,
\[
u, a_2, v_3, a_3, v_4, \ldots, v_{\ell-1}, a_{\ell-1}, u
\]
is a coset path of length \( \ell - 2 \) from \( u \) to itself. In both cases, such a coset path cannot exist according to the induction hypothesis. \( \square \)
The proof of Lemma 4.7 shows that a short cyclic path cannot exist in a sufficiently acyclic graph because it would collapse onto itself. The zipper lemma follows easily from this.

**Lemma 4.8 (Zipper lemma).** Let $G$ be a $2n$-acyclic Cayley graph, $v, u \in G,$ and

$$v, \alpha_1, l_2, \alpha_2, l_3, \ldots, l_{\ell}, \alpha_\ell, u \quad \text{and} \quad v, \beta_1, r_2, \beta_2, r_3, \ldots, r_k, \beta_k, u$$

be two coset paths from $v$ to $u$ of length up to $n$. Then

1. $[v]_{\beta_1 \cap \alpha_1} \cap [t_2]_{\alpha_1 \cap \alpha_2} \neq \emptyset$ or $[v]_{\alpha_1 \cap \beta_1} \cap [r_2]_{\beta_1 \cap \beta_2} \neq \emptyset$;
2. $[u]_{\beta_1 \cap \alpha_1} \cap [t_\ell]_{\alpha_1 \cap \alpha_{\ell - 1}} \neq \emptyset$ or $[u]_{\alpha_1 \cap \beta_1} \cap [r_k]_{\beta_1 \cap \beta_{k - 1}} \neq \emptyset$.

**Proof.** Both paths are short and share the start vertex $v$ and the end vertex $u$. Both paths fulfill the coset cycle property at every link between $v$ and $u$ by definition. However, the assumptions do not tell us exactly what the situation looks like at $v$ and $u$, the places where the paths overlap. The zipper lemma claims that there is an overlap that violates the coset cycle property at both ends.

Since $G$ is $2n$-acyclic we know that there must be an overlap at one of the edges, i.e.

- $[v]_{\beta_1 \cap \alpha_1} \cap [t_2]_{\alpha_1 \cap \alpha_2} \neq \emptyset$, or
- $[v]_{\alpha_1 \cap \beta_1} \cap [r_2]_{\beta_1 \cap \beta_2} \neq \emptyset$, or
- $[u]_{\beta_1 \cap \alpha_1} \cap [t_\ell]_{\alpha_1 \cap \alpha_{\ell - 1}} \neq \emptyset$, or
- $[u]_{\alpha_1 \cap \beta_1} \cap [r_k]_{\beta_1 \cap \beta_{k - 1}} \neq \emptyset$.

occurs because otherwise the two coset paths would form a coset cycle of length up to $2n$; w.l.o.g. assume $[v]_{\beta_1 \cap \alpha_1} \cap [t_2]_{\alpha_1 \cap \alpha_2} \neq \emptyset$. If we now assume that there is no overlap at $u$, i.e.

$$[u]_{\beta_1 \cap \alpha_1} \cap [t_\ell]_{\alpha_1 \cap \alpha_{\ell - 1}} = \emptyset \quad \text{and} \quad [u]_{\alpha_1 \cap \beta_1} \cap [r_k]_{\beta_1 \cap \beta_{k - 1}} = \emptyset,$$

then there would be a cyclic coset path of length up to $2n$ from $v$ to $v$, contradicting Lemma 4.7.

The zipper lemma states that two short coset paths that start and end at the same vertices can be considered two recombinations of the constituents of a common core path. Short coset paths in acyclic Cayley graphs are unique in the sense that the zipper lemma applies to them. Thus, $n$-acyclic Cayley graphs can be considered locally tree-like. The zipper lemma has several important consequences.

**Corollary 4.9.** Let $G$ be a $2n$-acyclic Cayley graph, $v, u \in G$. If there are two short coset paths

$$v, \alpha_1, l_2, \alpha_2, l_3, \ldots, l_{\ell}, \alpha_\ell, u \quad \text{and} \quad v, \beta_1, r_2, \beta_2, r_3, \ldots, r_k, \beta_k, u$$

from $v$ to $u$ with $\ell, k \leq n$, then there is a short coset path from $v$ to $u$ that starts with an $(\alpha_1 \cap \beta_1)$-edge.
Proof. W.l.o.g. we can assume that there is some \( v_2 \in [v]_{\beta_1 \cap \alpha_1} \cap [t_2]_{\alpha_1 \cap \alpha_3} \) by Lemma 4.8. First, the choice of \( v_2 \) and the coset property of the original path imply
\[
[v_2]_{\alpha_1 \cap \alpha_2} \cap [t_2]_{\alpha_1 \cap \alpha_3} = [t_2]_{\alpha_1 \cap \alpha_2} \cap [t_2]_{\alpha_1 \cap \alpha_3} = \emptyset.
\]
Second,
\[
v \notin [t_2]_{\alpha_1 \cap \alpha_2} = [v_2]_{\alpha_1 \cap \alpha_2} \supseteq [t_2]_{\alpha_1 \cap \beta_1}
\]
implies
\[
[v \cap (\alpha_1 \cap \beta_1)] \cap [v_2]_{(\alpha_1 \cap \beta_1) \cap \alpha_2} = \emptyset.
\]
Thus, \( v, (\alpha_1 \cap \beta_1), v_2, \alpha_2, t_3, \ldots, t_\ell, a_i, u \) is a short coset path. \( \square \)

Let \( G \) be a 2-acyclic Cayley graph and \( v, u \in G \). Based on Corollary 4.9 we define the unique minimal set of generators \( \text{short}(v, u) \subseteq E \).

**Definition 4.10.** A set of generators \( \alpha \) is a **first generator set for** \((v, u)\) if there is a short coset path from \( v \) to \( u \) that starts with an \( \alpha \)-edge. The **minimal first generator set for** \((v, u)\) \( \text{short}(v, u) \) is the intersection of all first generator sets:
\[
\text{short}(v, u) := \bigcap \{\alpha \subseteq E : \alpha \text{ is a first generator set for } (v, u)\}
\]
The unique set \( \text{short}(v, u) \) is well-defined because the intersection of two first generator sets is again a first generator set by Corollary 4.9. In general, \( \text{short}(v, u) \neq \text{short}(u, v) \) but
\[
\text{short}(v, u), \text{short}(u, v) \subseteq \text{gen}(v, u) = \text{gen}(u, v)
\]
because \( \text{gen}(v, u) \) is a first generator set for \((v, u)\) and \((u, v)\). The set \( \text{short}(v, u) \) gives us another perspective on the uniqueness of short coset paths. If one wants to move from one vertex to another on a short coset path, then there might be many possibilities but just one single “direction” to start with.

Furthermore, the zipper lemma implies that all short coset paths of length \( \leq 2 \) can be assumed to be inner paths.

**Corollary 4.11.** Let \( G \) be a 2n-acyclic Cayley graph, \( 2 \leq \ell \leq n \),
\[
v_1, a_1, v_2, a_2, v_3, \ldots, v_\ell, a_\ell, v_{\ell+1}
\]
be a coset path and \( \alpha \supseteq \text{gen}(v_1, v_{\ell+1}) \). Then \( \alpha_i \not\supseteq \alpha \), for \( 1 \leq i \leq \ell \), and there are \( v'_i \in [v_i]_{\alpha_i \cap \alpha} \) for \( 1 \leq i \leq \ell \), such that
\[
v_1, (\alpha_1 \cap \alpha), v'_2, (\alpha_2 \cap \alpha), v'_3, \ldots, v'_\ell, (\alpha_\ell \cap \alpha), v_{\ell+1}
\]
is an inner coset path.

**Proof.** First, \( \alpha \not\supseteq \alpha \) cannot be the case: if \( \ell = 2 \), then \( v_1, a_1, v_2, a_2, v_3 \) would not be a coset path since \( v_3 \in [v_2]_{\alpha_1 \cap \alpha_2} \), and \( \ell > 2 \) would imply a short cyclic coset path from \( v_{\ell+1} \) to itself, contradicting Lemma 4.7. Hence, in both cases \( \alpha_i \not\supseteq \alpha \), and with that \( \alpha_i \cap \alpha \subseteq \alpha \) which implies \( [v_i]_{\alpha_i \cap \alpha} \subseteq [v_i]_{\alpha} \).

Second, analogously to the proof of Corollary 4.9 one can show that there is some \( v'_2 \in [v_2]_{\alpha_1 \cap \alpha_2} \) such that
\[
v_1, (\alpha_1 \cap \alpha), v'_2, a_2, v_3, \ldots, v_\ell, a_\ell, v_{\ell+1}
\]
is a coset path because \( v_1, a_1, v_{\ell+1} \) is also a short coset path from \( v_1 \) to \( v_{\ell+1} \). Applying the same argument iteratively to the paths \( v'_i, a_i, v_{i+1}, \ldots, v_\ell, a_\ell, v_{\ell+1} \) and \( v'_i, a, v_{i+1} \), for \( 2 \leq i \leq \ell \), shows \( \alpha_i \not\supseteq \alpha \) and yields the desired vertices. \( \square \)
Corollary 4.11 illustrates the special role of the subgraph induced by \([v]_{\text{gen}(v,u)}\): all short coset paths between \(v\) and \(u\) essentially move within \([v]_{\text{gen}(v,u)}\). Conversely, if a coset path has a link that is disjoint from \([v]_{\text{gen}(v,u)}\), then it must be long.

**Corollary 4.12.** Let \(G\) be a 2n-acyclic Cayley graph. If \(v_1, \alpha_1, \ldots, \alpha_\ell, v_{\ell+1}\) is a coset path with
\[
[v_1]_{\text{gen}(v_1,v_{\ell+1})} \cap [v_1]_{\alpha_{i-1}\cap\alpha_i} = \emptyset,
\]
for some \(2 \leq i \leq \ell\), then \(\ell > n\).

### 4.2 Distance in Cayley graphs

In a Cayley graph, every pair of vertices \(v, u\) is connected by the coset path \(v, E, u\) of length 1. This makes the definition of a sensible measure of distance w.r.t. coset paths non-obvious. However, we find a solution with the help of 2-acyclicity and its implications. Using 2-acyclicity and the set \(\text{gen}(v,u)\), for vertices \(v, u\), we defined non-trivial coset paths, the paths that remain if one forbids all cosets that connect \(v\) and \(u\) in one step. This leads us to a non-trivial notion of distance in 2-acyclic Cayley graphs.

**Definition 4.13 (Distance in Cayley graphs).** Let \(G\) be a Cayley graph. The distance \(d(v,u)\) between two vertices \(v \neq u\) is defined as the length of a minimal non-trivial coset path from \(v\) to \(u\).

**Remark 4.14.** Definition 4.13 does not allow for \(d(v,u) = 1\). This might seem peculiar compared to other distance measures. However, the measure \(d(v,u)\) is precisely designed to capture the length of the non-trivial coset path connections between two vertices, and their length is always at least 2.

In the previous section, we showed that in sufficiently acyclic structures all short coset paths can be considered inner paths. This has implications for the distance. If we want to know if the distance between \(v\) and \(u\) is long, it suffices to look at the inner paths within the substructure induced by \([v]_{\text{gen}(v,u)}\).

**Lemma 4.15.** Let \(m \in \mathbb{N}\), \(G\) be a sufficiently acyclic Cayley graph and \(v, u\) two vertices. If there are no inner coset paths from \(v\) to \(u\) of length \(\leq m\), then \(d(v,u) > m\).

**Proof.** Let \(\ell \leq m\), and assume there is a non-trivial coset path \(v_1, \alpha_1, \ldots, \alpha_\ell, v_{\ell+1}\) of length \(\ell\) from \(v = v_1\) to \(u = v_{\ell+1}\). First, any non-trivial coset path has at least length 2. Second, we can assume that the path is an inner coset path by Lemma 4.11 since \(G\) is sufficiently acyclic. This contradicts our assumption. Thus, \(d(v,u) > m\). \(\Box\)

The original motivation for this distance stems from [8]. The central problem there is to play Ehrenfeucht-Fraïssé games on Cayley graphs with their complex overlapping edge patterns w.r.t. cosets. To win an Ehrenfeucht-Fraïssé game, one must be able to control distances between multiple vertices of a structure. In the case of Cayley graphs one needs to find a suitable measure of distance first. The one from Definition 4.13 suffices.

Furthermore, this distance for Cayley graphs closely corresponds to a very natural distance in their dual hypergraphs. In dual hypergraphs, the two hyperedges \([v]\) and \([u]\), for vertices \(v\) and \(u\), always intersect. This intersection
is exactly the set of $a$-cosets, for $a \subseteq E$, that contain both $v$ and $u$. At first glance, the distance between two hyperedges seems always trivially 0. But we obtain a meaningful measure of distance in dual hypergraphs between $[v]$ and $[u]$ if we cut out the intersection $[v] \cap [u]$ and consider the remaining paths in the Gaifman graph. Essentially, we look for the non-trivial paths of minimal length between $[v]$ and $[u]$.

Definition 4.16. Let $G$ be a Cayley graph, $v, u \in G$ and $t = [v] \cap [u]$. The distance $d([v], [u])$ between the hyperedges $[v]$ and $[u]$ in the dual hypergraph $d(G)$ is the usual graph-theoretic distance in the Gaifman graph of $d(G) \upharpoonright (d(V[G]) \setminus t)$ between $[v] \setminus t$ and $[u] \setminus t$.

It is the main result of this section that this measure of distance for dual hypergraphs corresponds exactly to the distance defined in (4.13) for Cayley graphs if certain acyclicity conditions are met. If $G$ is a 2-acyclic Cayley graph and $v \neq u$ are vertices, then $d(v, u) = d([v], [u]) + 1$. However, we will prove a more general statement that has a wider range of graph and model-theoretic applications. In order to obtain a meaningful notion of distance, we followed the same idea both in Cayley graphs and their dual hypergraphs: cut out the trivial connections, or more, and look at what remains.

Let $G$ be a Cayley graph and $v \neq u$ vertices. The intersection of the dual hyperedges $t = [v] \cap [u]$ is always a non-empty set of cosets. If $G$ is 2-acyclic, then $t$ is generated by the unique set $\text{gen}(v, u)$ (cf. Lemma 3.12), i.e.

$$t = \{[v]_{\beta} : \beta \supseteq \text{gen}(v, u)\} = \{[u]_{\beta} : \beta \supseteq \text{gen}(v, u)\}.$$  

We can further generalise the distance measure $d([v], [u])$ if we do not forbid $[v] \cap [u]$, but a more general set of cosets that has the same structure as $[v] \cap [u]$. If such a set is a superset of $[v] \cap [u]$, we arrive at a more general measure of distance that still has a correspondent in Cayley graphs. Before we formally define these distances, we introduce some notation to describe the forbidden sets.

Definition 4.17. For a 2-acyclic Cayley graph $G = (V, (R_e)_{e \in E})$ with the dual hypergraph $d(G) = (d(V), S, (Q_a)_{a \subseteq E})$, we define the following mapping:

$$\rho^G : V \times P(E) \to P(d(V)), (v, \gamma) \mapsto \{[v]_{\beta} : \beta \supseteq \gamma\}.$$  

If it is clear from the context, we drop the superscript $G$ and just write $\rho$ instead.

The following lemma characterises the relationship of the sets $[v] \cap [u]$ and $\rho(v, \gamma)$ in $d(G)$ in terms of $\text{gen}(v, u)$ and $\gamma$. We can observe the usual duality in the transition from Cayley graphs to their dual hypergraphs.

Lemma 4.18. Let $G$ be a 2-acyclic Cayley graph, $v, u$ two vertices and $\gamma \subseteq E$ a set of generators, then $[v] \cap [u] \subseteq \rho(v, \gamma)$ if and only if $\gamma \subseteq \text{gen}(v, u)$.

Proof. Put $a := \text{gen}(v, u)$. From right to left: assume $\gamma \subseteq a$. Together with 2-acyclicity this implies

$$[v] \cap [u] = \{[v]_{\beta} : \beta \supseteq a\} \subseteq \{[v]_{\beta} : \beta \supseteq \gamma\} = \rho(v, \gamma).$$

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eral measure of distance that is parametrized by $\rho$.

From left to right: assume $[v] \cap [u] \subseteq \rho(v, \gamma)$. As before, $[v] \cap [u] = \{[v]_\beta : \beta \supseteq \alpha\}$ because of 2-acyclity. Hence, for all $\beta \subseteq E$

$$
\beta \supseteq \alpha \iff [v]_\beta \in [u] \cap [v]
\Rightarrow [v]_\beta \in \rho(v, \gamma)
\iff \beta \supseteq \gamma,
$$

which implies, in particular, $\gamma \subseteq a$. \hfill \square

We will use the mapping $p$ to define generalisations of $d([v], [u])$ and $d(v, u)$. For 2-acyclic Cayley graphs, the sets $[v] \cap [u]$ and $\rho(v, \gamma)$ are generated, in some sense, by the single sets $\text{gen}(v, u)$ and $\gamma$, respectively. If $\gamma$ is a subset of $\text{gen}(v, u)$, then $\rho(v, \gamma)$ is a superset of $[v] \cap [u]$ by Lemma 4.18.

Hence, cutting out $\rho(v, \gamma)$ leaves a bigger hole in the dual hypergraph and fewer paths from $[v] \setminus \rho(v, \gamma)$ to $[u] \setminus \rho(v, \gamma)$, and we can define a more general measure of distance that is parametrized by $\rho(v, \gamma)$.

**Definition 4.19.** Let $A = (A, S)$ be a hypergraph and $t, X, Y \subseteq A$. We denote with $d_t(X, Y)$ the distance between $X \setminus t$ and $Y \setminus t$ in the induced sub-hypergraph $A \setminus t := A \upharpoonright (A \setminus t)$, i.e. the graph-theoretic distance in its Gaifman graph.

Essentially, we measure the length of the minimal paths that go from one set to another and do not go through a third subset $t$; we call such a path a **non-$t$ path**. The next step is to define the suitable analogon of non-$t$ paths in Cayley graphs. In Definition 4.19, we extended the set that is to be avoided to the possibly larger set $t$. Hence, the analogon on the side of Cayley graphs needs to avoid more cosets as links which means that we need to forbid a **smaller** coset and all its supersets.

**Definition 4.20.** Let $G$ be a Cayley graph, $v_1, v_{\ell+1}$ two vertices, $\gamma$ a set of generators and $t = \rho(v_1, \gamma)$. A coset path $v_1, a_1, v_2, a_2, \ldots, a_{\ell}, v_{\ell+1}$ is a **non-$t$ path** if, for all $1 \leq i \leq \ell$,

$$
[v_1]_\gamma \notin [v_i]_{a_i}.
$$

Non-$t$ coset paths are a generalisation of non-trivial coset paths (cf. Definition 4.2) because every non-trivial coset path from $v$ to $u$ is a non-$t$ coset path, for $t = \rho(v, \text{gen}(v, u))$. Based on this generalisation, we can generalise the former notion of distance to a notion that depends on $t$ in a straightforward manner.

**Definition 4.21.** Let $G$ be a 2-acyclic Cayley graph, $v \neq u$ two vertices, $\gamma \subseteq \Gamma$ and $t = \rho(v, \gamma)$. The **$t$-distance** $d_t(v, u)$ between $v$ and $u$ is defined as the length of a minimal non-$t$ coset path from $v$ to $u$.

**Remark 4.22.** $t$-distance generalises the notion of distance from Definition 4.13 in the sense that $d_t(v, u) = d(v, u)$, for $t = \rho(v, \text{gen}(v, u)) = \rho(u, \text{gen}(v, u))$.

**Remark 4.23.** Depending on $t$, $t$-distance allows for distance 1: $d_t(v, u) = 1$ if and only if $[v]_{\text{gen}(v, u)} \notin t$. However, the interesting cases are the ones where $\gamma \subseteq \text{gen}(v, u)$, which implies $[v]_{\text{gen}(v, u)} \in t$, for $t = \rho(v, \gamma)$.
These two parametrized notions of distance, $d_t(v,u)$ for Cayley graphs and $d_t([v],[u])$ for dual hypergraphs, are closely connected in the following sense.

**Proposition 4.24.** For $\ell \geq 1$, let $G$ be a sufficiently acyclic Cayley graph, $v \neq u$ two vertices, $\gamma \subseteq E$ and $t = \rho(v, \gamma)$. Then

$$d_t(v,u) = \ell \iff d_t([v],[u]) = \ell - 1.$$

We give a formal proof in Section 4.2.2. But first, we have a closer look at short non-$t$ coset paths and generalise some concepts from the previous section about short coset paths.

### 4.2.1 Short non-$t$ coset paths

In this section, we combine the notions of the set short$(v,u)$ and non-$t$ coset paths to obtain the parametrize operator short$_t(\cdot,\cdot)$. It describes the direction one has to take if one wants to move on a short non-$t$ coset path from $v$ to $u$, if such a path exists.

The zipper lemma implies the existence of short$(v,u)$. For short non-$t$ coset paths we need a specialized version of this operator. As a reminder: $a \subseteq E$ is a first edge set for the pair of vertices $(v,u)$ if there is a short coset path from $v$ to $u$ that starts with an $a$-edge.

**Definition 4.25.** Let $G$ be a 2-acyclic Cayley graph, $v, u \in G$ and $\gamma \subseteq \text{gen}(v,u)$ a set of generators. For $t = \rho(v,\gamma)$, we define the set of generators short$_t(v,z)$ as the intersection of all the first generator sets of short non-$t$ coset paths from $v$ to $u$.

In the definition of short$_t(v,u)$ we considered a certain subset of all short coset paths from $v$ to $u$. If there are no such paths, then this subset is empty and short$_t(v,u)$ is not defined. However, if there are short non-$t$ coset paths $v, \alpha, \ldots, u$ and $v, \beta, \ldots, u$, then there is a short coset path $v, \alpha \cap \beta, \ldots, u$ by Corollary 4.9 which is also non-$t$ because $[v]\gamma \not\subseteq [v]_a$ and $[v]\gamma \not\subseteq [v]_\beta$ imply $[v]_\gamma \not\subseteq [v]_{\alpha \cap \beta}$. Thus, short$_t(v,u)$ is well-defined if short non-$t$ coset paths from $v$ to $u$ exist.

We continue with investigating the properties of short$_t(v,u)$. This behavior is in a controlled and intuitive manner in sufficiently acyclic graphs. As short$_t(v,u)$ describes the direction of short non-$t$ coset paths from $v$ to $u$, it changes as one would expect if one moves to a neighbour $v'$ of $v$ via some $a$-edge: the direction for short non-$t$ coset paths from $v'$ to $u$ necessarily includes the generator $a$.

**Lemma 4.26.** Let $m \in \mathbb{N}$, $G$ be a Cayley graph, $v, u$ two vertices, $\gamma \subseteq \text{gen}(v,u)$ and $t = \rho(v,\gamma)$. Assume $G$ is $2m+1$-acyclic, $d_t(v,u) \leq m$, and that there is $a \not\in \text{short}_t(v,u)$ such that $d_t(va,u) \leq m$, then $a \in \text{short}_t(va,u)$.

**Proof.** Let $\ell, k \leq m$, and $w_1, a_1, \ldots, a_\ell, w_{\ell+1}$ and $z_1, b_1, \ldots, b_k, z_{k+1}$ be two coset paths that avoid $t$ with

- $w_1 = z_1 = u$, $w_{\ell+1} = v$, $z_{k+1} = v'$, and
- $a_\ell = \text{short}_t(v,u)$, $b_k = \text{short}_t(v',u)$.
chordless. Analogously, one proves between non-
vertex $1$
Since Proof.
In $d$$(G)$
Let $G$ be a 2-acyclic Cayley graph, $v_t$
In Section 4.2, we claimed that $d$(v, u) ≤ $m,
Such paths exist by choice of $v, u$ and $v'$ and Definition 4.25. If we assume $a \notin$ short$(v', u)$, then $a \notin a_t \cup \beta_k$. Together with $w_{\ell+1} \notin [w_{\ell}]_{a_{\ell-1} \cap a_{\ell}}$, $z_{k+1} \notin [z_k]_{\beta_k \cap \beta_k}$ and $w_{\ell+1} \neq z_{k+1}$ this implies
\begin{itemize}
  \item $[w_{\ell}]_{a_{\ell-1} \cap a_{\ell}} \cap [w_{\ell+1}]_{a_{\ell}} \cap (a) = \emptyset$,
  \item $[w_{\ell+1}]_{a_{\ell}} \cap (a) \cap [z_{k+1}]_{\beta_k} = \emptyset$, and
  \item $[z_{k+1}]_{\beta_k} \cap [z_k]_{\beta_k \cap \beta_k} = \emptyset$.
\end{itemize}
Hence,
\begin{equation}
  w_1, a_1, w_2, \ldots, w_{\ell}, a_{\ell}, w_{\ell+1}, a, z_{k+1}, \beta_k, z_k, \ldots, z_2, \beta_1, z_1
\end{equation}
is a coset path of length $\ell + k + 1 \leq 2m + 1$ from $u$ to $u$, which cannot exist by Lemma 4.7 in a $2m + 1$-acyclic Cayley graph.

If we choose $\gamma = \text{gen}(v, u)$ in the lemma above, we obtain this special case:

**Corollary 4.27.** Let $m \in \mathbb{N}$, $G$ be a Cayley graph and $v, u$ two vertices. Assume $G$ is $2m + 1$-acyclic, $d(v, u) \leq m$, and that there is a $\notin$ short$(v, u)$ such that $d(va, u) \leq m$, then $a \in$ short$(va, u)$.

### 4.2.2 Duality of paths

In Section 4.2, we claimed that $d_1(v, u)$ and $d_1([v], [u])$ are equivalent (Proposition 4.24) although they are based on two seemingly very different kinds of paths. In this section, Lemmas 4.28 and 4.29 show a correspondence between non-$t$ coset paths and chordless paths in $d(G) \setminus t$. The former states that minimal paths in $d(G) \setminus t$ induce non-$t$ coset paths.

**Lemma 4.28.** Let $G$ be a 2-acyclic Cayley graph, $v_1 \neq v_{\ell+1}$ two vertices, $\gamma$ a set of generators and $t = p(v_{\ell+1}, \gamma)$. Then a chordless path of length $\ell + 1 \geq 2$
\begin{equation}
  [v_1]_t, [v_2]_{a_1}, [v_2]_{a_1}, [v_3]_{a_2}, \ldots, [v_{\ell+1}]_{a_t}, [v_{\ell+1}]_t
\end{equation}
in $d(G) \setminus t$ from $[v_1]_t$ to $[v_{\ell+1}]_t$ induces a non-$t$ coset path
\begin{equation}
  v_1, a_1, v_2, \ldots, v_{\ell}, a_{\ell}, v_{\ell+1}
\end{equation}
of length $\ell$ in $G$.

**Proof.** Since $[v_{\ell+1}]_{a_t} \in [v_1]_t$ implies $v_i \in [v_{\ell+1}]_{a_t}$ for all $1 \leq i \leq \ell$,
\begin{equation}
  v_1, a_1, v_2, \ldots, v_{\ell}, a_{\ell}, v_{\ell+1}
\end{equation}
is a path in $G$. First, we need to prove that it is also a coset path. If there is a vertex
\begin{equation}
  u \in [v_1]_{a_1} \cap [v_2]_{a_1} \cap [v_3]_{a_2} = [v_1] \cap [v_2]_{a_1} \cap [v_3]_{a_2} = [v_1] \cap [v_1]_{a_1} \cap [v_3]_{a_2},
\end{equation}
then $u = v_1$ and $[v_3]_{a_2} \in [v_1]_t$, which implies that $[v_3]$ is a chord that connects $[v_1]_t$ and $[v_3]_{a_2}$; this cannot be because we assumed that the path is chordless. Analogously, one proves $[v_{\ell}]_{a_{\ell-1} \cap a_{\ell}} \cap [v_{\ell+1}]_{a_t} = \emptyset$. If there is an $1 < i \leq \ell$ and some vertex
\begin{equation}
  u \in [v_i]_{a_{i-1} \cap a_i} \cap [v_{i+1}]_{a_i} \cap [v_{i+1}]_{a_{i+1} = [v_{i-1}]_{a_{i-1}} \cap [v_i]_{a_i} \cap [v_{i+1}]_{a_{i+1}},
\end{equation}
then \([v_{i-1}]_{\alpha_{i-1}}, [v_{i+1}]_{\alpha_{i+1}} \in \gamma\), which makes \([u]\) a chord for the path in \(d(G)\), contradicting chordlessness again. Second, the coset path is also non-t because, for all \(1 \leq i \leq \ell\),

\[[v_{i+1}]_{\alpha} \not\in t \iff [v_{\ell+1}]_{\gamma} \not\subseteq [v_{i+1}]_{\alpha}.

\]

□

Lemma 4.29 states the converse direction: a minimal non-t coset path in a Cayley graph \(G\) induces a chordless path in \(d(G) \setminus t\).

**Lemma 4.29.** Let \(\ell \geq 1\), \(G\) be a sufficiently acyclic Cayley graph \(v_1, v_{\ell+1}\) two vertices, \(\gamma \subseteq \text{gen}(v_1, v_{\ell+1})\) a set of generators and \(t = \rho(v_{\ell+1}, \gamma)\). A non-t coset path of length \(\ell \geq 1\)

\[v_1, \alpha_1, v_2, \ldots, v_{\ell}, \alpha_\ell, v_{\ell+1}\]

induces a chordless path of length \(\ell + 1\)

\[[v_1]_{\alpha}, [v_2]_{\alpha_1}, [v_2]_{\alpha}, [v_3]_{\alpha}, \ldots, [v_{\ell+1}]_{\alpha_{\ell}}, [v_{\ell+1}]_{\alpha}, [v_{\ell+1}]_{\alpha} \in d(G) \setminus t.

Proof. For all \(1 \leq i \leq \ell\), \(v_i \in [v_{i+1}]_{\alpha_i}\) implies \([v_{i+1}]_{\alpha_i} \in \gamma]\), hence

\[[v_1]_{\alpha}, [v_2]_{\alpha_1}, [v_2]_{\alpha}, [v_3]_{\alpha}, \ldots, [v_{\ell+1}]_{\alpha_{\ell}}, [v_{\ell+1}]_{\alpha} \in d(G) \setminus t\]

is indeed a path in \(d(G)\). Furthermore, the coset path is non-t because, for all \(1 \leq i \leq \ell\), \([v_{i+1}]_{\alpha_i} \not\in t\) if and only if \([v_{\ell+1}]_{\gamma} \not\subseteq [v_{i+1}]_{\alpha_i}\). It remains to show that the path is chordless.

Assume there is a chord, i.e. a hyperedge \([v] \subseteq d(G)\) that contains two vertices of the path in \(d(G)\) that have at least distance 2 on the path. Set \(s_0 = \emptyset\). If \([v]\) contains \([v_{\ell+1}]_{\emptyset}\) and some vertex \([v_i]_{\alpha_{i-1}}\), for \(1 \leq i \leq \ell\), then \(v = v_{\ell+1}\) and \(v_{\ell+1} \in [v_i]_{\alpha_{i-1}}\); this implies a short cyclic coset path from \(v_{\ell+1}\) to \(v_{\ell+1}\), which cannot exist in sufficiently acyclic Cayley graphs by Lemma 4.27. Otherwise, \([v]\) contains two vertices \([v_i]_{\alpha_{i-1}}, [v_j]_{\alpha_{j-1}}\), for some \(1 \leq i, j \leq \ell + 1\) with \(j > i + 1\). Then \(v \in [v_i]_{\alpha_{i-1}}\) and \(v \in [v_j]_{\alpha_{j-1}}\). The case \(j = i + 2\) and \(v \in [v_{i+1}]_{\alpha_{i+1}} \cap [v_{j+1}]_{\alpha_{j+1}}\) (keep in mind that \([v_{i+2}]_{\alpha_{i+2}} = [v_{j+1}]_{\alpha_{j+1}}\) violates the coset cycle property. In any other case, we can find again a short cyclic coset path from \(v\) to itself.

□

If a coset path is denoted as \(v_1, \alpha_1, v_2, \ldots, v_{\ell+1}\), as in the lemma above, then

\([v_{i+1}]_{\alpha_i} = [v_i]_{\alpha_i}\) for all \(1 \leq i \leq \ell\). Additionally, removing the first and last edge from a chordless path does not change that it is chordless.

**Corollary 4.30.** Let \(\ell \geq 1\), \(G\) be a sufficiently acyclic Cayley graph \(v_1, v_{\ell+1}\) two vertices, \(\gamma \subseteq \text{gen}(v_1, v_{\ell+1})\) a set of generators and \(t = \rho(v_{\ell+1}, \gamma)\). A non-t coset path of length \(\ell \geq 1\)

\[v_1, \alpha_1, v_2, \ldots, v_{\ell+1}\]

induces a chordless path of length \(\ell - 1\)

\[[v_1]_{\alpha_1}, [v_2]_{\alpha_2}, [v_3]_{\alpha_3}, \ldots, [v_{\ell}]_{\alpha_{\ell}} \in d(G) \setminus t.

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We can combine Proposition 4.24 with the zipper lemma and its implications to obtain a way to verify that the distance between two vertices in a Cayley graph or the distance between two hyperedges in its dual hypergraph is long by looking only at a inner non-t coset paths.

Lemma 4.31. Let \( \ell \geq 1 \), \( G \) be a sufficiently acyclic Cayley graph, \( v \neq u \) two vertices, \( \gamma \subseteq E \) and \( t = \rho(v, \gamma) \). If there is no inner non-t coset path from \( v \) to \( u \) of length \( \leq \ell \), then

\[
d_t(v, u) > \ell \quad \text{and} \quad d_t([v], [u]) > \ell - 1.
\]

Proof. Assume \( d_t(v, u) = k \leq \ell \), and let \( v_1, a_1, v_2, \ldots, v_k, a_k, v_{k+1} \), with \( v_1 = v \) and \( v_{k+1} = u \), be a non-t coset path. Since \( G \) is sufficiently acyclic, this path is short. Hence, Corollary 4.11 implies there are \( v'_i \in [v_i]_{a_i \cap a} \), for \( a = \text{gen}(v, u) \) and \( 1 < i \leq k \), such that

\[
v_1, a'_1, v'_2, a'_2, v'_3, \ldots, v'_k, a'_k, v_{k+1},
\]

for \( a'_i = a_i \cap a, 1 \leq i \leq k \), is a short inner coset path. This inner coset path is also non-t because \( [v_{i+1}]_\gamma \not\subseteq [v_i]_{a_i} \) and \( [v_i]_{a_i \cap a} = [v'_i]_{a_i \cap a} \subseteq [v_i]_{a_i \cap a} \) imply \( [v_{i+1}]_\gamma \not\subseteq [v'_i]_{a_i \cap a} \). However, we assumed that such inner paths do not exist. Thus, \( d_t(v, u) > \ell \) and by Corollary 4.23 also \( d_t([v], [u]) > \ell - 1 \). \( \square \)

Conclusion

This work provides a general toolbox for dealing with the highly intricate overlap patterns of cosets in Cayley graphs without short coset cycles. These patterns are extremely dense, yet their highly regular structure allows us to invoke notions of locality at multiple scales. The overlap patterns are analysed in terms of related structures, with a focus on the duality between Cayley graphs and their associated dual hypergraphs.

We present several characterisations of local tree-likeness in Cayley structures, like the zipper lemma or regarding coset acyclic Cayley graphs as the dual image of \( a \)-acyclic hypergraphs, which are locally tree-decomposable. The zipper lemma gives us further insight into the structure of coset paths on every level of granularity of the coset overlap pattern. The duality between Cayley graphs and associated hypergraphs allows us to translate between coset paths in Cayley graphs and chordless graphs in the dual hypergraph. Thus, we can translate problems in \( n \)-acyclic Cayley graphs to problems in \( n \)-acyclic hypergraphs and use well-known results about \( a \)-acyclicity to solve these problems. Conversely, we know how certain model-theoretic constructions on Cayley graphs impact their dual hypergraphs. So far such techniques were successfully applied in \([8],[7],[9]\) to characterise the expressive power of Common Knowledge logic in certain classes of Kripke structures that are based on Cayley graphs. This work makes these techniques accessible for a wider range of applications.
References

[1] L. Babai. Spectra of Cayley graphs. *Journal of Combinatorial Theory, Series B*, 27(2):180–189, 1979.

[2] G. Bagan, A. Durand, and E. Grandjean. On acyclic conjunctive queries and constant delay enumeration. In *Computer Science Logic*, pages 208–222, 2007.

[3] C. Beeri, R. Fagin, D. Maier, and M. Yannakakis. On the desirability of acyclic database schemes. *Journal of the ACM*, 30:497–513, 1983.

[4] C. Berge. *Graphs and Hypergraphs*. North-Holland, 1973.

[5] N. Biggs, N. L. Biggs, and B. Norman. *Algebraic graph theory*. Number 67. Cambridge university press, 1993.

[6] H. L. Bodlaender. A tourist guide through treewidth. *Acta Cybern.*, 11(1-2):1–21, 1993.

[7] F. Canavoi. *Cayley Structures and the Expressiveness of Common Knowledge Logic*. PhD thesis, Technische Universität Darmstadt, 2018.

[8] F. Canavoi and M. Otto. Common knowledge and multi-scale locality analysis in cayley structures. In *32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS*, pages 1–12, 2017.

[9] F. Canavoi and M. Otto. Cayley structures and common knowledge. *arXiv e-prints*, arXiv:1909.11521, 2021.

[10] B. Courcelle. Graph rewriting: An algebraic and logic approach. In *Handbook of Theoretical Computer Science, Volume B: Formal Models and Semantics (B)*, pages 193–242. 1990.

[11] S. J. Curran and J. A. Gallian. Hamiltonian cycles and paths in Cayley graphs and digraphs – a survey. *Discrete Mathematics*, 156(1):1–18, 1996.

[12] V. Dalmau, P. G. Kolaitis, and M. Y. Vardi. Constraint satisfaction, bounded treewidth, and finite-variable logics. In P. V. Hentenryck, editor, *Principles and Practice of Constraint Programming - CP 2002, 8th International Conference, CP 2002, Ithaca, NY, USA, September 9-13, 2002, Proceedings*, volume 2470 of *Lecture Notes in Computer Science*, pages 310–326. Springer, 2002.

[13] A. Dawar and M. Otto. Modal characterisation theorems over special classes of frames. *Annals of Pure and Applied Logic*, 161:1–42, 2009.

[14] R. Dechter and J. Pearl. Tree clustering for constraint networks. *Artif. Intell.*, 38(3):353–366, 1989.

[15] I. Dinur, S. Evra, R. Livne, A. Lubotzky, and S. Mozes. Locally testable codes with constant rate, distance, and locality. *arXiv e-prints*, arXiv:2111.04808, 2021.
[16] M. Grohe and J. Mariño. Definability and descriptive complexity on databases of bounded tree-width. In C. Beeri and P. Buneman, editors, *Database Theory - ICDT '99, 7th International Conference, Jerusalem, Israel, January 10-12, 1999, Proceedings.*, volume 1540 of *Lecture Notes in Computer Science*, pages 70–82. Springer, 1999.

[17] M.-C. Heydemann. *Cayley graphs and interconnection networks*, pages 167–224. Springer Netherlands, Dordrecht, 1997.

[18] A. Kelarev, J. Ryan, and J. Yearwood. Cayley graphs as classifiers for data mining: The influence of asymmetries. *Discrete Mathematics*, 309(17):5360–5369, 2009. Generalisations of de Bruijn Cycles and Gray Codes/Graph Asymmetries/Hamiltonicity Problem for Vertex-Transitive (Cayley) Graphs.

[19] S. Lakshmivarahan, J.-S. Jwo, and S. Dhall. Symmetry in interconnection networks based on Cayley graphs of permutation groups: A survey. *Parallel Computing*, 19(4):361–407, 1993.

[20] C. H. Li. On isomorphisms of finite Cayley graphs – a survey. *Discrete Mathematics*, 256(1):301–334, 2002.

[21] C. Martinez, R. Beivide, and E. M. Gabidulin. Perfect codes from Cayley graphs over lipschitz integers. *IEEE Transactions on Information Theory*, 55(8):3552–3562, 2009.

[22] S. Ordyniak, D. Paulusma, and S. Szeider. Satisfiability of acyclic and almost acyclic CNF formulas. *Theor. Comput. Sci.*, 481:85–99, 2013.

[23] M. Otto. Highly acyclic groups, hypergraph covers and the guarded fragment. *Journal of the ACM*, 59 (1), 2012.

[24] N. Robertson and P. D. Seymour. Graph minors 1–23. *J. Combinat. Theory, Series B*, 1983–2012.

[25] E. Rosen. Modal logic over finite structures. *Journal of Logic, Language and Information*, 6:427–439, 1997.

[26] J. van Benthem. *Modal Logic and Classical Logic*. Bibliopolis, Napoli, 1983.