On some fundamental problems of the theory of gravitation

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Abstract

Cosmological observations indicate that the Einstein equation may not be entirely correct to describe gravity. However, numerous modifications of these equations usually do not affect foundations of the theory. In this paper two important issue that lead to a substantial revision of the theory are considered:

1. The significance of relativity of space-time geometry with respect to measuring instruments for theory of gravitation.
2. The gauge transformations of the field variables in correct theory of gravitation.

1 Relativity of Space-Time

Einstein’s theory of gravity is a realization of the idea of the relativity of the properties of space-time with respect to the distribution of matter. However, before the advent of Einstein’s theory, Henri Poincaré showed that the properties of space and time are also relative to the properties of the used measuring instruments. Of course now it can be said also about the properties of space-time too. However, these convincing arguments have never been implemented in physical theory.

We can make a step towards the realization of this idea, if we will pay attention that the properties of measuring instruments are one of the characteristics of the reference frame used. We can, therefore, expect that we deal with the manifestation of a fundamental property of physical reality — with space-time relativity with respect to the reference frame used.

The following simple example shows that this rather unexpected statement makes sense. Consider two reference frames, and two observers which proceed from the notion of the relativity of space-time in the sense of Berkley-Leibnitz-Mach-Poincaré (BLMP). Let the reference body of the first, inertial frame (IRF), is associated with the surface of a non-rotating planet, and the
reference body of the second frame formed by a set of material points, falling
freely under the influence of the planet gravity. (It can be named by proper
reference frame of the given force field (PRF)).

The observer, located in the first, inertial, frame of reference, of course
will examine the fall of test bodies as happens under the action of a force
field $\mathcal{F}$ in the Minkowski space-time, the source of which is the planet. He
sees no need to explain the motion of test bodies with curvature.

However, the observer, located in the second reference frame, does not
detect this force field. Instead, he observes rapprochement of points of the
reference body of his frame which for him are points of his physical space.

If he is denied the opportunity to see the planets and stars, it seems
impossible for him to find another explanation of this fact, which is differ-
ent from the generally accepted explanation — of an evidence of space-time
curvature.

Thus, if an observer in a IRF can consider space-time as flat, then the
observer in the PRF of the force field $\mathcal{F}$, who proceeds from relativity of space
and time in the BLMP meaning, is forced to consider it as a non-Euclidean.

Some quantitative results on the metric of space-time in PRFs were ob-
tained earlier by the author [4]. Namely, we postulate that space-time $E$
in inertial frames is the Minkowski one, according to the spacial relativity.
From our point of view, space-time geometry and properties of the reference
frame do not have meaning by themselves. Therefore, this postulate means
that only a complex “Minkowski space-time $E +$ inertial reference system”
makes a physical sense. Starting from this postulate and based on the rela-
tivity of space-time, it is possible to find the line element of space-time $V$
in a PRF of any given in the $E$ force field.

Consider a PRF, the reference body of which formed by materi-
al points with masses $m$ moving under the action of the force field $\mathcal{F}$. If we proceed
from relativity of space and time in the BLMP sense, then the line element
of space-time in PRFs can be expected to have the following form [4]

$$ds = -(mc)^{-1}dS(x, dx),$$
where $dS = \mathcal{L}(x, \dot{x})dt$, and $\mathcal{L}(x, \dot{x})$ is a Lagrange function describing in
Minkowski space-time the motion of the identical point masses $m$.

2 Examples

1. Suppose that in the Minkowski space-time gravitation can be described
as a tensor field $\psi_{\alpha\beta}(x)$ in $E$, and the Lagrangian, describing the motion of
a test particle with the mass $m$ in $E$ is given by the form

$$\mathcal{L} = -mc\left[\psi_{\alpha\beta}(\psi) \dot{x}^\alpha \dot{x}^\beta\right]^{1/2},$$  \hspace{1cm} (2)$$

where $\dot{x}^\alpha = dx^\alpha/dt$ and $g_{\alpha\beta}$ is a symmetric tensor whose components are functions of $\psi_{\alpha\beta}$ \cite{2}.

If particles move under influence of the force field $\psi_{\alpha\beta}(x)$, then according to (1) the space-time line element in PFRs of this field takes the form

$$ds^2 = g_{\alpha\beta}(x) dx^\alpha dx^\beta \hspace{1cm} (3)$$

Consequently, the space-time in such PRFs is Riemannian $V$ with curvature other than zero. The tensor $g_{\alpha\beta}(\psi)$ is a space-time metric tensor in the PRFs.

Viewed by an observer located in the IRF, the motion of the particles, forming the reference body of the PRF, is affected by the force field $\psi_{\alpha\beta}$. Let $x^i(t, \chi)$ be a set of the particles paths, depending on the parameter $\chi$. Then, for the observer located in the IRF the relative motion of a pair of particles from the set is described in non-relativistic limit by the differential equations \cite{3}

$$\frac{\partial^2 n^i}{\partial t^2} + \frac{\partial^2 U}{\partial x^i \partial x^k} n^k = 0,$$  \hspace{1cm} (4)$$

where $n^k = \partial x^k/\partial \chi$ and $U$ is the gravitational potential.

However, the observer in a PRF of this field will not feel the existence of the field. The presence of the field $\psi_{\alpha\beta}$ will be displayed for him differently — as space-time curvature which manifests itself as a deviation of the world lines of nearby points of the reference body.

For a quantitative description of this fact it is natural for him to use the Riemannian normal coordinates. \cite{1} In these coordinates spatial components of the deviation equations of geodesic lines are

$$\frac{\partial n^i}{\partial t^2} + R^i_{0k0} n^k = 0,$$  \hspace{1cm} (5)$$

where $R^i_{0k0}$ are the components of the Riemann tensor. In the Newtonian limit these equations coincide with (4).

Thus, in two frames of reference being used we have two different descriptions of particles motion — as moving under the action of a force field in the Minkowski space-time, and as moving along the geodesic line in a Riemann space-time with the curvature other than zero.

\footnote{This and the above consideration does not depend on the used coordinate system, it can be performed by a covariant method.}
2. Another, rather unexpected example, give the recent results on the motion of small elements of a perfect isentropic fluid \[4\].

Instead of the traditional continuum assumption, the behavior of the fluid flow can be considered as the motion of a finite number of particles under the influence of interparticles forces which mimic effects of pressure, viscosity, etc. \[5\]. Owing to replacement of integration by summation over a number of particles, continual derivatives become simply time derivatives along the particles trajectories. The velocity of the fluid at a given point is the velocity of the particle at this point. The continuity equation is always fulfilled and can consequently be omitted. Owing to such discretization the motion of particles is governed by means of solutions of ordinary differential equations of classical or relativistic dynamics.

In \[4\] it was shown that the following Lagrangian describes the motion of small elements of a perfect isentropic fluid in adiabatic processes is given by

\[
L = -mc \left( G_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right)^{1/2} d\lambda. \tag{6}
\]

In this equation \( G_{\alpha\beta} = \kappa^2 \eta_{\alpha\beta} \eta_{\alpha\beta} \) is the metric tensor of the space-time \( E \),

\[
\kappa = \frac{w}{mc^2} = 1 + \frac{\varepsilon}{\rho c^2} + \frac{P}{\rho c^2}, \tag{7}
\]

\( \varepsilon \) is the fluid density energy, \( m \) is the mass of the fluid particles, \( c \) is speed of light, and \( \rho = mn \), \( n \) is the particles number density, \( P \) is the pressure in the fluid, \( \lambda \) is a parameter along 4-pathes of particles.

In an inertial reference drame (i.e. in Minkowski space-time \( E \)) we can set the parameter \( \lambda = \sigma \) which yields the following Lagrange equations:

\[
\frac{d}{d\sigma} (\kappa u_\alpha) - \frac{\partial \kappa}{\partial x^\alpha} = 0 \tag{8}
\]

where \( u_\alpha = \eta_{\alpha\beta} u^\beta \), and \( u^\alpha = \frac{dx^\alpha}{d\sigma} \). For adiabatic processes \[6\]

\[
\frac{\partial}{\partial x^\alpha} \left( \frac{w}{n} \right) = \frac{1}{n} \frac{\partial P}{\partial x^\alpha}, \tag{9}
\]

and we arrive at the equations of the motion of the set of the particles in the form

\[
w \frac{du_\alpha}{d\sigma} + u_\alpha u^\beta \frac{\partial P}{\partial x^\beta} - \frac{\partial P}{\partial x^\alpha} = 0. \tag{10}
\]

where \( du_\alpha/d\sigma = (\partial u_\alpha/\partial x^\nu) u^\nu \). It is the general accepted relativistic equations of the motion of fluid \[6\].
In a comoving reference frame the space-time the line element is of the form
\[ ds^2 = G_{\alpha\beta} dx^\alpha dx^\beta. \] (11)

In this case the element of the proper time is \( ds \). After the setting \( \lambda = s \), the Lagrangian equation of the motion takes the standard form of a congruence of geodesic lines:
\[ \frac{du^\alpha}{ds} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = 0, \] (12)
where \( du_\alpha/ds = (\partial u_\alpha/\partial x^\epsilon) u^\epsilon \), \( u^\alpha = du^\alpha/ds \), and
\[ \Gamma^\alpha_{\beta\gamma} = \frac{1}{2} G^{\alpha\epsilon} \left( \frac{\partial G_{\epsilon\beta}}{\partial x^\gamma} + \frac{\partial G_{\epsilon\gamma}}{\partial x^\beta} - \frac{\partial G_{\beta\gamma}}{\partial x^\epsilon} \right). \] (13)

In the Cartesian coordinates
\[ \Gamma^\gamma_{\alpha\beta} = \frac{1}{\kappa} \left( \frac{\partial \kappa}{\partial x^\gamma} \delta^\alpha_{\beta} + \frac{\partial \kappa}{\partial x^\beta} \delta^\alpha_{\gamma} - \eta^{\alpha\epsilon} \frac{\partial \kappa}{\partial x^\epsilon} \eta_{\beta\gamma} \right), \] (14)
so that
\[ \Gamma^1_{00} = -\frac{1}{\rho c^2} \frac{\partial P}{\partial x^1} \] (15)

In the spherical coordinates the scalar curvature \( R \) is given by
\[ R = \frac{6}{x^4 r^2} (r^2 x')', \] (16)
where the prime denotes a derivative with respect to \( r \).

Therefore, the motion of small elements of the fluid in a comoving reference frame can be viewed as the motion in a Riemannian space-time with a nonzero curvature.

Of course, (1) refers to any classical field \( F \). For instance, space-time in PRFs of an electromagnetic field is Finslerian. However, since \( ds \), in this case, depends on the mass and charge of the particles forming the reference body, this fact is not of great significance.

Thus any force field can be considered based on the aggregate "IRF + Minkowski space", and based on the aggregate "PRF + non-Euclidean space-time with metric (1)" From this point of view of geometrization of gravity is the second possibility, which was discovered by Einstein’s intuition.

It is important to realize that the relativity of space-time geometry to the frame of reference is the same important and fundamental property of physical relativity as relativity to act of measurement, the physical realization of which is quantum mechanics. Full implementation of these ideas can have far-reaching implications for fundamental physics.
3 Gravity equations and gauge-invariance

In the theory of gravitation, the equations of motion of test particles play a fundamental role. Notion of "gravitational field" emerged as something necessary to correctly describe the motion of bodies. The values that appear in the equations of motion, become the main characteristic of the field. The field equations have emerged as a tool for finding these values for a given distribution of masses.

All this is very similar to classical electrodynamics. It is very important in this case that the equations of motion of test charges are invariant under gauge transformations of 4-potentials. For this reason, all 4-potentials, obtained from a given by a gauge transformation, describe the same field. That is why the field equations of classical electrodynamics are invariant under gauge transformations.

Einstein’s equations of the motions of test particles in gravitational field are also invariant with respect to some class of transformations of the field variables in any given coordinate system — with respect to geodesic transformations of Christoffel symbols (or metric tensor). \[7\] Such transformations for the Christoffel symbols are of the form

\[
\Gamma^\alpha_{\beta\gamma}(x) = \Gamma^\alpha_{\beta\gamma}(x) + \delta^\alpha_\beta \phi_\gamma(x) + \delta^\alpha_\gamma \phi_\beta(x),
\]

where \(\phi_\alpha(x)\) is a continuously differentiable vector field. (The transformations for the metric tensor are solutions of some complicate partial differential equations).

Consequently, all Christoffel symbols obtained from a given by geodesic transformations, describe the same gravitational field. The equations for determining the gravitational field must be invariant under such transformations, and the physical meaning can only have values which are invariant under geodesic transformations.

However, Einstein’s gravitational equations are not consistent completely with the requirements which imposes on them the main hypothesis of the motion of test particles along geodesics, because they are not geodesically invariant \[8\].

Therefore, we can assume that in a fully correct theory of gravity, based on the hypothesis of the motion of test particles along geodesics, geodesic transformations should play the role of gauge transformations, and coordinate transformation should play the same role as in electrodynamics.

Einstein equations are in good agreement with observations in weak and moderately strong fields. Therefore, if there are more correct equation of gravitation, then deriving from them physical results should differ observably from Einstein’s equations only in strong fields.
Simplest vacuum equation of this kind were first proposed (from a different point of view) in [9], and discussed in greater detail in [10] - [12], and the equations in the presence of matter - in [13]. They are some geodesic-invariant modification of Einstein’s equations.

From a theoretical point of view, the most satisfactory are the vacuum equations.

They predict some fundamentally new physical consequences which can be tested experimentally.

Under geodesic transformations the Ricci tensor $R_{\alpha\beta}$ of space-time $V$ in PRFs of gravitational field transforms as follows:

$$ \overline{R}_{\alpha\beta} = R_{\alpha\beta} + (n - 1)\psi_{\alpha\beta}, \quad (18) $$

where

$$ \psi_{\alpha\beta} = \psi_{\alpha;\beta} - \psi_{\alpha} \psi_{\beta}, \quad (19) $$

and a semicolon denotes a covariant differentiation in $V$. Therefore, the simplest generalization of the Einstein equations is of the form

$$ R_{\alpha\beta} + (n - 1)\Gamma_{\alpha\beta} = 0, \quad (20) $$

where $\Gamma_{\alpha\beta}$ is a tensor transformed under geodesic transformations as follows

$$ \Gamma_{\alpha\beta} = \Gamma_{\alpha\beta} - \psi_{\alpha\beta}. \quad (21) $$

Due to the fact that our space-time is a bimetric, there exists a vector field

$$ Q_{\alpha} = \Gamma_{\alpha} - \overline{\Gamma}_{\alpha} \quad (22) $$

where $\Gamma_{\alpha} = \Gamma^{\beta}_{\alpha\beta}$, $\overline{\Gamma}_{\alpha} = \overline{\Gamma}^{\beta}_{\alpha\beta}$, $\Gamma^{\gamma}_{\alpha\beta}$ and $\overline{\Gamma}^{\gamma}_{\alpha\beta}$ are the Christoffel symbols in $V$ and $E$, respectively.

Under geodesic transformations in $V$ the quantities $\Gamma_{\alpha}$ are transformed as follows:

$$ \Gamma_{\alpha} = \Gamma_{\alpha} + (n + 1)\psi_{\alpha} \quad (23) $$

For this reason, a tensor object

$$ A_{\alpha\beta} = Q_{\alpha;\beta} - Q_{\alpha} Q_{\beta}, \quad (24) $$

where $Q_{\alpha;\beta}$ is a covariant derivative of $Q_{\alpha}$ in $V$, has the same transformation properties under geodesic transformations as must have the above vector field $\Gamma_{\alpha\beta}$.
The line element of space-time in PRFs was obtained from the Lagrangian motion of test particles in the Minkowski space-time $E$. If we want to find the equation of gravity in space-time $E$, you must realize that in this space, the Christoffel symbols $\Gamma^\gamma_{\alpha\beta}$ can be regarded as components of the tensor $\Gamma^\gamma_{\alpha\beta}$ in the Cartesian coordinate system, i.e. as components of $\Gamma^\gamma_{\alpha\beta}$, where the ordinary derivatives replaced by covariant in the metric of space-time $E$. (Just as in bimetric Rosen’s theory [14]).

Given this, we arrive at the conclusion that the equation

$$R_{\alpha\beta} - A_{\alpha\beta} = 0 \quad (25)$$

is the simplest geodesic invariant modification of the vacuum Einstein equations, considered from the point of view of flat space-time.

These equations can be written in another form. The simplest geodesic-invariant object in $V$ is a Thomas symbols:

$$\Pi^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} - \frac{1}{n+1} \left( \delta^\gamma_a \Gamma_{\beta} - \delta^\gamma_\beta \Gamma_{\alpha} \right). \quad (26)$$

It is not a tensor. However, from point of view of flat space-time $E$, they can be considered as components of the tensor $B^\gamma_{\alpha\beta} = \Pi^\gamma_{\alpha\beta} - \circ \Pi^\gamma_{\alpha\beta}$, where $\Pi^\gamma_{\alpha\beta}$ is the Thomas symbols in $E$. In another words, $B^\gamma_{\alpha\beta}$ can be considered as the Thomas symbols where derivatives replaced by the covariant ones with respect to the metric $\eta_{\alpha\beta}$. This geodesic-invariant tensor can be named by strength tensor of gravitational field.

The above gravitation equation can be written by tensor $B^\gamma_{\alpha\beta}$ as follows:

$$\nabla_\gamma B^\gamma_{\alpha\beta} - B^\gamma_{\alpha\delta} B^\delta_{\beta\gamma} = 0. \quad (27)$$

where $\nabla$ denotes a covariant derivative in $E$.

The physical consequences following from these equations do not contradict any observational data, however, lead to some unexpected results, which allow to us to test the theory. The first result is that they predict the existence of supermassive compact objects without event horizon which are an alternative to supermassive black holes in the centers of galaxies. The second result is that they provide a simple and natural explanation for the fact of an acceleration of the universe as of a consequence of the gravity properties.

4 Remarks on the equations inside matter

We can not claim that the particles inside the material medium move along geodesics. The exception is the case of dust matter and perfect fluid. Consequently, it is unclear whether the field equations inside the matter to be a
generalization of the geodesic equations of Einstein. However, such equations have been proposed in the work [13]. Comparison of the results obtained from them with observations of the binary pulsar PSR 1913+16 shows good agreement with observations. Despite this, doubts as to their correctness are still remain. The problem is that the writing of generalization of the equations in the matter requires significantly narrow the class of admissible geodesic transformations of the metric tensor of space-time \( V \). It is not clear whether such space-time is Riemannian. It is possible, geodesic invariance is violated in a material medium. For this reason, we do not consider these equations here in more detail, assuming that this is still a subject for further research.

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