The resistivity of high-Tc cuprates

R Arouca\textsuperscript{1,2} and E C Marino\textsuperscript{1}\textsuperscript{\circledR}

\textsuperscript{1} Instituto de Física, Universidade Federal do Rio de Janeiro, C.P. 68528, Rio de Janeiro, RJ 21941-972, Brazil
\textsuperscript{2} Institute for Theoretical Physics, Center for Extreme Matter and Emergent Phenomena, Utrecht University, Princetonplein 5, 3584 CC Utrecht, The Netherlands

E-mail: arouca@pos.if.ufrj.br and marino@if.ufrj.br

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Abstract
We show that the resistivity in each phase of the high-Tc cuprates is a special case of a general expression derived from the Kubo formula. We obtain, in particular, the T-linear behavior in the strange metal and upper pseudogap phases, the pure $T^2$ Fermi liquid behavior observed in the strongly overdoped regime as well as in the mid-pseudogap phase. We also describe the $T^{1+\delta}$ behavior that interpolates the linear and quadratic behaviors in the crossover regime. We calculate the coefficients: (a) of $T$ in the linear regime and show that it is proportional to the pseudogap transition temperature $T^*(x)$; (b) of the $T^2$-term in the Fermi Liquid regime, without adjusting any parameter; and (c) of the $T^{1.6}$ term in the crossover regime, all in excellent agreement with the experimental data. From our model, we are able to infer that the resistivity in cuprates is caused by the scattering of holes by excitons, which naturally form as holes are doped into the electron background.

Keywords: high-Tc cuprates, resistivity, superconductivity

(Some figures may appear in colour only in the online journal)

1. Introduction

High-$T_c$ superconductivity in the cuprates [1] is, at the same time, one of the most interesting and complex problems in contemporary physics. Although some features of these systems, such as the relevance of magnetic interactions in the CuO\textsubscript{2} planes [2, 3] and the non-s-wave character of the superconducting order parameter [4, 5], are consensual in the community, there are important issues that, so far, remain unsolved.

Among the fundamental unanswered questions related to the cuprates, there are some that concern the normal state of these materials. A particularly intriguing one is: how to explain the perfectly linear dependence of the resistivity with the temperature [6, 7], which is observed in all cuprate materials? This deviates from the well-known $T^2$ behavior, typical of Fermi liquids, which are usually associated with conventional metals [8–10]. Nevertheless, the metallic nature of this state is suggested by the resistivity increase with the temperature, hence justifying the name strange metal (SM), by which this phase is known [11, 12]. In the attempt to explain the linear behavior of the resistivity in this phase, different mechanisms have been proposed [13–17]. Among these, we find the ‘Planckian dissipation hypothesis’ [7, 18, 19] that associates the scattering rate of the charge carriers, $1/\tau$, with the inverse characteristic time of thermal fluctuations: $k_B T/h$, which ultimately follows from the uncertainty principle. Also the state responsible for the linear resistivity in cuprates has been associated to a regime of quantum criticality, namely, a quantum critical point (QCP) would exist, producing a phase consisting in a quantum critical fluid whose properties would be universally determined [20, 21].

Nevertheless, as discussed in [22], the Planckian dissipation hypothesis combined with scaling arguments would be, in principle, not compatible with a T-linear dependence of the resistivity. Yet, it is argued in [22] that in case resistivity could be ascribed to the scattering of holes out of a bosonic field, then the Planckian dissipation hypothesis in a quantum critical regime could be reconciled with the linear behavior of resistivity.

The QCP supposedly responsible for this universal quantum critical behavior has been associated to a metal–insulator quantum phase transition shown to exist at a doping value $x_p$ and $T = 0$, located inside the SC dome of these materials under the effect of strong magnetic fields that would...
destroy the SC state [12, 23, 24]. The assumption that the PG temperature transition line \( T^*(x) \) ends at a point \( x_p \), which is inside the SC dome however, does not seem to have experimental support, since no other \( T^*(x) \) points are observed inside the dome [24]. It seems more plausible, therefore, that the quantum phase transition at \( x = x_p \) is not related to the pseudogap.

In a recent publication, we proposed a model which provides a general and accurate description of the high-Tc superconductivity in cuprates. This model allowed for the obtainment of analytical expressions for the SC and PG temperatures: \( T_c(x) \) and \( T^*(x) \), showing excellent agreement with the experimental data for different compounds [25]. This analytical solution clearly shows that the PG temperature line meets the SC temperature line at \( T = 0 \), on a QCP located at the right extremity of the SC dome.

The model we propose for understanding the cuprates [25], exhibits two quartic interaction terms: (a) one which is hole-attractive and derives from the magnetic Kondo interaction between the itinerant holes and localized copper spins and (b) another, which is hole-repulsive and stems from the Hubbard electric repulsion between the holes. Performing a Hubbard-Stratonovich transformation in both terms, we introduce two scalar fields, respectively, \( \Phi \) and \( \chi \). \( \Phi \) is the creation operator of Cooper pairs, that condense on the superconducting (SC) phase, \( \chi \), on its turn, is the creation operator of excitons (electron–hole bound-states) that, upon condensation, gives rise to the pseudogap (PG) phase [25]. With the aid of this picture, we are able to conclude that the main mechanism responsible for resistivity in the normal phases of cuprates is the scattering of charged holes by excitons. Such exciton states should be observable in the insulating regime of the cuprates, namely, in the low-temperature region of the strongly underdoped PG phase.

In the present study, we investigate this matter inserting the current correlators derived from our model into the Kubo formula at a finite temperature, in order to obtain a general expression for the resistivity, which reduces to the ones occurring in the different normal states of the high-Tc cuprates.

The paper is organized as follow: in section 2, we briefly review our model, focusing on the phase diagram obtained from the thermodynamic potential, calculate the resistivity using Kubo formula and show how one can interpret the general expression obtained in terms of a scaling function. In section 3, we describe how this general expression manifests in each phase and show how one can describe the different resistivity behaviors by taking suitable limits of \( M \) and \( \mu \). In section 4, we compare our results to the experimental data, showing that we can describe the data consistently. Furthermore, we show that using the values of \( T^* \) and the value of the chemical potential in the quantum critical region, we can obtain the coefficient of both the SM, FL and crossover phases with excellent agreement with the experimental data.

2. The model and the resistivity

2.1. The model

We take as a starting point the Spin-Fermion-Hubbard model in [25]. This model describes the interaction between the itinerant holes that are located on the oxygen sites and the localized spins, located in the copper sites. The oxygen square lattice is bipartite and breaks down into two sublattices, \( A \) and \( B \), according to whether the \( p_z \) or \( p_x \) orbitals hybridize with the copper d-orbital. We denote by \( R \) the sites of sublattice \( A \) and by \( d_i \), \( i = 1 \ldots 4 \), the vectors connecting site \( R \) with its four nearest neighbors in sublattice \( B \) [25].

This Hamiltonian is composed by four terms. A fermionic kinetic term:

\[
H_0 = -t_p \sum_{R,\alpha, \sigma} \psi_{A, \sigma}^\dagger(R) \psi_{B, \sigma}(R + d_i) + hc,
\]

where \( \psi_{c, \sigma}(r) \) is a hole operator on sublattice \( c = A, B \), with spin \( \sigma = \uparrow, \downarrow \) and position \( r \), and \( t_p \) is the hopping parameter. Then, we have a Hubbard term describing the Coulomb repulsion between the holes:

\[
H_U = U_p \sum_R n_{\uparrow}^A(R) n_{\downarrow}^A(R) + U_p \sum_R n_{\uparrow}^B(R) n_{\downarrow}^B(R),
\]

where \( n_{\sigma} \) denotes the number operators in sublattice \( c \) with spin \( \sigma \), and \( U_p \) is the intensity of the Hubbard interaction. An antiferromagnetic Heisenberg interaction term describes the magnetic interaction among the localized copper spins, namely

\[
H_{AF} = J_{AF} \sum_{\langle R \rangle} S_I \cdot S_J,
\]

where \( S_I \) is a Cu spin operator on site \( I \) and \( J_{AF} \) is the antiferromagnetic coupling. Finally a Kondo term couples the holes located at the \( O \) orbitals with the localized spins placed at the Cu orbitals:

\[
H_K = J_K \sum_{I \in \{1, 2\}} \sum_{\langle R, R + d_i \rangle} S_I \cdot \left[ S_R(R) + S_d(R + d_i) \right],
\]

where \( J_K \) is the Kondo coupling and

\[
S_I(r) = \frac{1}{2} \psi_{c, \sigma}^\dagger(r) \psi_{c, \sigma}(r),
\]

is the spin operator of the hole in the sublattice \( c = A, B \).

Integrating out the \( \mu \) degree of freedom and treating the kinetic term as a perturbation for the Coulomb repulsion (see [25]) we arrive at an effective electronic Hamiltonian \( H_{\text{eff}} \) that involves only holes degrees of freedom

\[
H_{\text{eff}}[\psi] = -t_p \sum_{R, d_i} \psi_{A, \sigma}^\dagger(R) \psi_{B, \sigma}(R + d_i) + hc
\]

\[
- g_s \sum_{R, d_i} \left[ \psi_{A, \sigma}^\dagger(R) \psi_{B, \sigma}(R + d_i) + \psi_{B, \sigma}^\dagger(R + d_i) \psi_{A, \sigma}(R) \right] - g_p \sum_{R, d_i} \left[ \psi_{B, \sigma}^\dagger(R) \psi_{A, \sigma}(R + d_i) + \psi_{A, \sigma}^\dagger(R + d_i) \psi_{B, \sigma}(R) \right] \times \left[ \psi_{B, \sigma}^\dagger(R + d_i) \psi_{A, \sigma}(R) + \psi_{A, \sigma}^\dagger(R) \psi_{B, \sigma}(R + d_i) \right],
\]
where the superconducting and pseudogap coupling parameters are given, respectively, by $g_s = \frac{i}{\hbar v_F}$ and $g_p = \frac{2\hbar v_F}{\pi}$ in terms of the original parameters of the three bands Hubbard model. These show an excellent agreement with the values obtained from the experimental data, thus indicating the success of our first principles approach [25].

Notice that the above Hamiltonian contains two interaction terms, one that is responsible for the formation of Cooper pairs (and the SC phase) and another one, which leads to the formation of excitons (and the PG phase). In this way, it is natural to perform a Hubbard-Stratonovich transformation using the scalar fields $\Phi$,

$$\Phi = g_s \left[ \psi_\uparrow \downarrow \psi_\uparrow \downarrow + \psi_\downarrow \uparrow \psi_\downarrow \uparrow \right], \quad (7)$$

that is related to the Cooper pairs and the SC phase, and $\chi$,

$$\chi = g_p \left[ \psi_\uparrow \downarrow \psi_\uparrow \downarrow + \psi_\downarrow \uparrow \psi_\downarrow \uparrow \right], \quad (8)$$

that describe excitons and the PG phase.

After doing a Fourier transform, $H_{\text{eff}}$ is expressed in terms of these Hubbard-Stratonovich fields as:

$$H_{\text{eff}} = \sum_{k,\sigma} \varepsilon(k) \left[ \psi_{A\sigma}^\dagger(k) \psi_{b\sigma}^\dagger(k) + \hbar c \right] + \sum_k \Phi(k) \left[ \psi_{A\uparrow}(k) \psi_{b\downarrow}(k) + \psi_{b\uparrow}(k) \psi_{A\downarrow}(k) \right] - \mu \sum_k \psi_{A\sigma}^\dagger(k) \psi_{A\sigma}(k) + \frac{1}{g_s} \sum_k \psi_{A\uparrow}^\dagger(k) \psi_{A\downarrow}^\dagger(k) + \frac{1}{g_p} \sum_k \chi^\dagger(k) \chi(k), \quad (9)$$

where $\varepsilon(k) = 2\hbar v_F [\cos k_x a + \cos k_y a]$ is the usual tight-binding kinetic energy of the free-holes on a square lattice. Notice that the PG field $\chi$ acts as a scattering potential for the holes, being therefore responsible for their resistivity.

By integrating on the fermions, we arrive at an effective expression for the thermodynamic grand-canonical potential that depends on the SC and PG order parameters, respectively, $\Delta = \langle \Phi \rangle$ and $M = \langle \chi \rangle$, as well as on the chemical potential $\mu$: $\Omega(\Delta, M, \mu, T)$

$$\exp \left\{ - \beta \Omega(\Delta, M, \mu) \right\} = \text{Tr}_{\psi} e^{-\beta[H-M\mu]},$$

Minimizing the grand-canonical potential $\Omega$, respectively, with respect to $\Delta, M, \mu$, we obtain the equations that determine the behavior of $\Delta, M$ and $\mu$ in the thermodynamic equilibrium state, namely [25]:

$$2\Delta \left[ \frac{\partial}{\partial \mu} F(\Delta, M, \mu) + \frac{\eta(N g_s)}{g_s} \right] = 0 \quad (10)$$

$$2M \left[ \frac{\partial}{\partial \mu} F(\Delta, M, \mu) + \frac{\eta(N g_s)}{g_s} \right] = 0 \quad (11)$$

$$d(x) = \mu \frac{\partial}{\partial \mu} F(\Delta, M, \mu), \quad (12)$$

where $F$ is given (close to the critical curves $T_c(x)$ and $T^*(x)$) by:

$$\frac{\partial}{\partial \mu} F(\Delta, M, \mu) \bigg|_{\Delta=0, M=0} = \ln 2 + \frac{1}{2} \ln \cosh \left[ \frac{\sqrt{\Delta^2 + (M + \mu(x))^2}}{2T} \right]$$

$$+ \frac{1}{2} \ln \cosh \left[ \frac{\sqrt{\Delta^2 + (M - \mu(x))^2}}{2T} \right], \quad (13)$$

and $\eta(g) = 1 - \frac{g}{g_c}$ is a threshold coupling: $g > g_c$ [25].

Notice that our approach is not an overall mean-field approximation, since, by integrating over the fermion (holes) field, we take into account the full-fledged quantum fluctuations of the holes’ degrees of freedom. Concerning the Hubbard-Stratonovich fields, $\Phi$ and $\chi$, we are, rather, performing a mean-field semi-classical expansion where we take the classical, vacuum expectation values thereof, respectively, as the SC and PG order parameters. This semi-classical mean-field approximation is non-perturbative, what is testified, for instance, by the fact that our expression for the optimal transition temperature $T_{\text{max}}$ depends on the coupling $g_s$ through $\eta(g_s)$, being therefore non-analytic in $g_s$. This kind of approximation, which is fully controlled, resembles the $1/N$ expansion, that has been extensively used in order to successfully describe the non-perturbative regime of QCD [26].

In the superconducting phase, $\Delta \neq 0$ and $M = 0$, so that equation (11) is trivially satisfied and from equations (10) and (12) we arrive at the expressions for the superconducting transition temperature $T_c(x)$ [25]:

$$T_c(x) = \frac{\ln 2 + \ln(\frac{\eta(N g_s)}{g_s}) + e^{\frac{\ln(\frac{\eta(N g_s)}{g_s})}{T_{\text{max}}}} - 1}{\ln 2 + \ln(\frac{\eta(N g_s)^2}{g_s}) + e^{\frac{\ln(\frac{\eta(N g_s)^2}{g_s})}{T_{\text{max}}}} - 1}, \quad x < x_0$$

$$T_c(x) = \frac{\ln 2 + \ln(\frac{\eta(N g_s)}{g_s}) + e^{\frac{\ln(\frac{\eta(N g_s)}{g_s})}{T_{\text{max}}}} - 1}{\ln 2 + \ln(\frac{\eta(N g_s)^2}{g_s}) + e^{\frac{\ln(\frac{\eta(N g_s)^2}{g_s})}{T_{\text{max}}}} - 1}, \quad x > x_0 \quad (14)$$

$$T_c(x) = \frac{\ln 2 + \ln(\frac{\eta(N g_s)}{g_s}) + e^{\frac{\ln(\frac{\eta(N g_s)^2}{g_s})}{T_{\text{max}}}} - 1}{\ln 2 + \ln(\frac{\eta(N g_s)^2}{g_s}) + e^{\frac{\ln(\frac{\eta(N g_s)^2}{g_s})}{T_{\text{max}}}} - 1}, \quad \text{LSCO,} \forall x$$

where $\mu_0(x) = 2\gamma(x_0 - x)$.

$\gamma$ being a parameter which has been determined for each compound. $T_{\text{max}}$ is the transition temperature at optimal doping, $x_0$, which is given by $T_{\text{max}} = \frac{\Lambda}{2\pi} \eta(N g_s)/\eta(N g_s)$, where $\Lambda = 0.018$ eV is a characteristic energy scale. Notice that for LSCO we use a symmetrized version of the equations to comply with the experimental observation of a symmetrical SC dome.

For the pseudogap phase, $\Delta = 0$ and $M \neq 0$ so that now equation (10) is trivially satisfied and equations (11) and (12) imply for the pseudogap transition temperature $T^*(x)$:

$$T^*(x) = \frac{\Lambda \eta(g N)}{2 \pi} \ln \left[ 1 + \exp \left( - \frac{2\gamma(x_0 - x)}{F(x)} \right) \right], \quad (16)$$

$\Lambda = 0.018$ eV is a characteristic energy scale. Notice that for LSCO we use a symmetrized version of the equations to comply with the experimental observation of a symmetrical SC dome.

For the pseudogap phase, $\Delta = 0$ and $M \neq 0$ so that now equation (10) is trivially satisfied and equations (11) and (12) imply for the pseudogap transition temperature $T^*(x)$:
with \( \eta (g_p N) = 1 - \frac{g_p N}{\delta} \), \( x_0 = x_0^+ \) and \( \tilde{\gamma} \) has been determined for each compound [25].

Using a systematic procedure, discussed extensively in [25], we were able to obtain the parameters \( \gamma \) and \( \tilde{\gamma} \) for each material and characterize their phase diagram with an excellent agreement with experimental data. Using that potential, we could explain the SC phase diagram of different cuprate compounds, obtaining in particular, the left and right limiting points of the SC dome, namely \( x_0^c SC \) and \( x_0^+ \) [25]. For doping larger than \( x_0 \) we have the FL phase, at sufficiently low temperatures, the SM phase at high temperatures and a crossover between both [27, 28]. The LSCO phase diagram, displaying the analytical expressions for the SC and PG temperatures: \( T_c (x) \) and \( T^* (x) \), as well as the crossover temperatures: \( T^* (x) \equiv T^* (2x_0 - x) \) and \( T_{crossover} = C_0 T^* (x) \) (with \( C_0 \) defined below) is presented in figure 1. As discussed in the following section, we can understand the normal phase in terms of the critical variables \( K_1 \equiv M/k_B T \) and \( K_2 \equiv \mu/k_B T \).

### 2.2. General expression for \( \rho (T, x) \)

In order to obtain the resistivity in our model, we shall use the Kubo formula at a finite temperature [9], which is given by

\[
\sigma^R_{DC} = \lim_{\omega \to 0} \frac{i}{\omega} \left[ 1 - e^{-\beta \hbar \omega} \right] \lim_{\delta \to 0} \Pi^R (\omega + i\delta, \mathbf{k}), \tag{17}
\]

where \( \Pi^R \) is the retarded, connected current–current correlation function:

\[
\Pi^R = \langle \{ j^i j^j \} \rangle_C,
\]

which is given by,

\[
\langle \{ j^i j^j \} \rangle_C (\omega, \mathbf{k}) = \frac{\delta^2 \Omega [\mathbf{A}]}{\delta \mathbf{A}^i (\omega, \mathbf{k}) \delta \mathbf{A}^j (\omega, \mathbf{k})}, \tag{19}
\]

where \( \Omega [\mathbf{A}] \) is the grand-canonical potential in the presence of an applied electromagnetic vector potential \( \mathbf{A} (\omega, \mathbf{k}) \). This relates to the grand-partition functional \( Z[\mathbf{A}] \) as:

\[
\Omega [\mathbf{A}] = -\frac{1}{\beta} \ln Z[\mathbf{A}], \tag{20}
\]

which is given by:

\[
Z[\mathbf{A}] = T_{Total} e^{-\beta [U(\mathbf{A}) - \mu N]}, \tag{21}
\]

The electromagnetic field \( \mathbf{A} \) is introduced through the usual minimal coupling prescription:

\[
\epsilon (\hbar \mathbf{k}) \rightarrow \epsilon (\hbar \mathbf{k} + e \mathbf{A}), \tag{22}
\]

which yields the grand-canonical potential in the presence of an applied electromagnetic vector potential \( \mathbf{A} \), namely, \( \Omega [\mathbf{A}] \).

We can write the eigenvalues of \( H - \mu N \), in terms of the stationary values \( \Delta, M; \mu \) as

\[
E_{\pm} (\mathbf{k}) = \pm \sqrt{\left( \sqrt{\epsilon^2 (\mathbf{k}) + M^2} \pm \mu \right)^2 + \Delta^2}. \tag{23}
\]

The grand-partition functional \( Z[\mathbf{A}] \) follows from equations (21) and (22), namely,

\[
Z[\mathbf{A}] = \exp \left\{ -\beta \left[ \frac{|\Delta|^2}{8\epsilon} + \frac{|M|^2}{8\epsilon} + N \mu (x) \right] - NTA \sum_{\alpha = -\infty}^{\infty} \sum_{\beta = -\infty}^{\infty} \int \frac{d^2k}{4\pi^2} \ln \left[ \omega^2 + E_j^2 [\mathbf{A}] \right] \right\}
\]

\[
= Z[0] \exp \left\{ -\beta T \sum_{\omega_n} \sum_{\beta = \pm 1} \int \frac{d^2k}{(2\pi)^2} \ln \left[ \omega^2_n + E_j^2 [\mathbf{A}] \right] \right\}, \tag{24}
\]

where

\[
E_j^2 [\mathbf{A}] = \Delta^2 + \left( \sqrt{\epsilon^2 (\mathbf{k}) + M^2} + l \mu \right)^2. \tag{25}
\]

Using (19), (22) and (25), we obtain the average current:

\[
\langle \{ j^i \} \rangle (k = 0, \omega = 0) = N \sum_{j = \pm 1} 2 T E_j [\mathbf{A}] \frac{\partial E_j [\mathbf{A}]}{\partial A^j} \sum_{\omega_n} \frac{1}{\omega_n^2 + E_j^2 [\mathbf{A}]} \tag{26}
\]

\[
= N \sum_{j = \pm 1} \frac{\partial E_j [\mathbf{A}]}{\partial A^j} \tanh \left[ \frac{E_j [\mathbf{A}]}{2k_B T} \right].
\]
To calculate the conductivity matrix, \( \sigma^{ij} \) we must take the derivative of \( \langle f^i \rangle \) with respect to \( A^j \), at \( A = k = 0 \). Considering that, in this case

\[
\frac{\partial \mathcal{E}[A]}{\partial A^j} \bigg|_{A=k=0} = 0, \quad (27)
\]

we have

\[
\langle f^i \rangle (k = 0, \omega = 0) = N \sum_{j=1}^{3} \frac{\partial^2 \mathcal{E}[A]}{\partial A^i \partial A^j} \tanh \left( \frac{\mathcal{E}[A]}{2k_B T} \right). \quad (28)
\]

Under the latter conditions, only the diagonal (\( \delta^{ij} \)) terms survive, namely:

\[
\langle f^i \rangle (k = 0, \omega = 0) = N \sum_{j=1}^{3} \frac{\partial^2 \mathcal{E}[A]}{\partial A^i \partial A^j} \tanh \left( \frac{\mathcal{E}[A]}{2k_B T} \right). \quad (29)
\]

In order to obtain the DC conductivity per CuO\(_2\) plane, we just divide by \( N \). The corresponding DC resistivity per CuO\(_2\) plane, then, will be given by:

\[
\rho^{ij} = \left( \frac{\sigma^{ij}_{\text{DC}}}{N} \right)^{-1} = \frac{\delta^{ij} M}{h \beta V^{-1} e^2 v^2 \left( \frac{|M+\mu|}{\sqrt{\Delta^2 + (M+\mu)^2}} \tanh \left[ \frac{\sqrt{\Delta^2 + (M+\mu)^2}}{2k_B T} \right] + |M-\mu| \tanh \left[ \frac{\sqrt{\Delta^2 + (M-\mu)^2}}{2k_B T} \right] \right)}. \quad (30)
\]

where \( V = da^2 \) is the volume of the primitive unit cell, per CuO\(_2\) plane, with \( d \) being the distance between planes, \( a \) the lattice parameter and \( v \), the characteristic velocity of the holes, such that \( (h/\alpha) \approx 2.86 \times 10^{-3} \text{ eV} \) [25].

In the SC phase, we have \( \Delta \neq 0 \) and \( M = 0 \), implying that the resistivity vanishes as it should:

\[
\rho^{ij}_{\text{SC}} = \frac{\delta^{ij} M}{h \beta V^{-1} e^2 v^2 \left( \frac{|M+\mu|}{\sqrt{\Delta^2 + (M+\mu)^2}} \tanh \left[ \frac{\sqrt{\Delta^2 + (M+\mu)^2}}{2k_B T} \right] + |M-\mu| \tanh \left[ \frac{\sqrt{\Delta^2 + (M-\mu)^2}}{2k_B T} \right] \right)} \quad (31)
\]

In the non-superconducting phases, conversely, we have \( \Delta = 0 \), which leads to the following expression for the resistivity:

\[
\rho^{ij} = \frac{\delta^{ij} M}{h \beta V^{-1} e^2 v^2 \left( \frac{|M+\mu|}{\sqrt{\Delta^2 + (M+\mu)^2}} \tanh \left[ \frac{\sqrt{\Delta^2 + (M+\mu)^2}}{2k_B T} \right] + |M-\mu| \tanh \left[ \frac{\sqrt{\Delta^2 + (M-\mu)^2}}{2k_B T} \right] \right)} \quad (32)
\]

This can be rewritten as:

\[
\rho^{ij} = \frac{\delta^{ij} \mathcal{V} k_B}{h \beta v^2 e^2} \left( \frac{\tanh \left( \frac{M}{2k_B T} \right) + \tanh \left( \frac{\mu}{2k_B T} \right)}{\tanh \left( \frac{M}{2k_B T} \right)} \right) \quad (33)
\]

We can express the above equation in terms of the scaling function \( G \) of the critical variables \( K_1 = M_0/k_B T \) and \( K_2 = \mu/k_B T \), given by:

\[
G(K_1, K_2) = K_1 \cosh K_1 + \cosh K_2 \quad (34)
\]

such that the resistivity becomes (from now on we will drop the \( ij \) index):

\[
\rho(\chi, T) = BT^2 G \left( \frac{M}{k_B T}, \frac{\mu}{k_B T} \right), \quad (35)
\]

where the (almost universal) constant \( B \) is:

\[
B = \frac{\hbar}{e^2} \frac{d}{2\pi} \left( \frac{a}{\hbar v} \right)^2 k_B^2 \approx 0.37 \times d n \Omega \text{ cm K}^{-2}, \quad (36)
\]

where \( \hbar e^2 \approx 25812.807 \Omega \) is the resistance quantum and \( d \) is given in Å units.

This general form of the resistivity, whose dependence on the temperature (\( T \)) and on the doping parameter (\( \chi \)) has been made explicit, holds in all phases of the phase diagram of cuprates, except the SC one. The peculiar form of the resistivity in each of the different phases will be determined by the form the function \( G(K_1, K_2) \) assumes in each phase.
Figure 2. Evolution of the resistivity behavior as a function of doping for LSCO. (a) Strongly underdoped regime ($x < x_{SC}$): there is no SC phase and $T_0 < T^*$. The resistivity diverges for temperatures below $T_0$, (b) Underdoped regime ($x_{SC} < x \lesssim x_0$): $T_0$ diminishes and the upturn of resistivity occurs for lower temperatures and appears as a kink close to $T_c$. (c) Overdoped regime ($x_0 \lesssim x < x_{SC}$): $T_*$ gets higher than $T_0$ and the resistivity becomes completely linear in the normal phase. (d) Strongly overdoped regime ($x > x_{SC}$): for low enough temperatures, the resistivity has the quadratic behavior typical of the FL phase with the coefficient given by equation (36), while at higher temperature it has the power law behavior of equation (55). The experimental data shown above are for LSCO compounds with doping levels of $x = 0.02$ (strongly underdoped), $x = 0.13$ (underdoped), $x = 0.22$ (overdoped) and $x = 0.3$ (strongly overdoped). The red circles are data extracted from [30] and [27]. The values of the coefficient of the FL and crossover phases are obtained by equations (36) and (55) and not by a fitting process.

3. The different resistivity behaviors

3.1 Resistivity in the PG phase: low temperature regime

In the PG phase, both $K_1$ and $K_2$ are different from zero. We subdivide the PG phase in three regions according to the values of $x$, namely: (a) strongly underdoped, for $x < x_{SC}$; (b) underdoped, for $x_{SC} < x \lesssim x_0$, where $x_0$ is the optimal doping; (c) overdoped, for $x_0 \lesssim x < x_{SC}$.

In all subregions, except for the strongly underdoped region, the low-temperature regime is dominated by the SC phase, where the resistivity vanishes. In order to explore the resistivity in the strongly underdoped region, we take the low-temperature limit of (34). As $M$ and $\mu$ are finite for zero temperature in this phase we can approximate

$$\cosh \left( \frac{M}{k_B T} \right) + \cosh \left( \frac{\mu}{k_B T} \right) \approx \exp \left( \frac{M}{k_B T} \right) + \exp \left( \frac{\mu}{k_B T} \right) = \exp \left( \frac{\mu - M}{k_B T} \right),$$

such that

$$G_{PG}(T \to 0) \approx \frac{M}{2k_B T} \exp \left( \frac{\mu - M}{k_B T} \right) = \frac{M}{2k_B T} \exp \left( \frac{T_0}{T} \right),$$

where we define the temperature scale $T_0 \equiv \frac{\mu - M}{k_B}$. This makes that the resistivity in this limit to be given by

$$\rho_{PG}(x, T \to 0) = \frac{BM}{2k_B T} \exp \left( \frac{T_0}{T} \right),$$

(39)

such that it presents an insulating behavior with an activation temperature $T_0$ given by $\left[ \mu(T, x) - M(x, T) \right] / k_B$. This kind of behavior is observed in figures 2(a)–(c) at low temperatures.

3.2 Resistivity in the PG phase: high temperature regime

For $T \to T^*$, $M \to 0$. We can separately inspect the numerator and denominator of $G$ in this limit. First, the sum of two cosh in the numerator,

$$\lim_{T \to T^*} \cosh \left( \frac{M}{k_B T} \right) + \cosh \left( \frac{\mu}{k_B T} \right) = 1 + \cosh \left( \frac{\mu - M}{2k_B T} \right),$$

(40)

then the fraction

$$\lim_{T \to T^*} \frac{M/k_B}{\sinh \left( \frac{M}{k_B T} \right)} = T^*,$$

(41)
such that
\[ G_{\text{PG}}(T \rightarrow T^*) = \frac{T^*}{T} \cosh^2 \left( \frac{\mu_{\text{PG}}(T \rightarrow T^*)}{2k_B T^*} \right). \]  
(42)

Inserting this in (35) we obtain for the resistivity
\[ \rho_{\text{PG}}(T \rightarrow T^*) = BT^* T \cosh^2 \left( \frac{\mu_{\text{PG}}(T \rightarrow T^*)}{2k_B T} \right). \]  
(43)

3.3. Resistivity in the PG phase: intermediate temperature regime

An intermediate regime of the PG phase occurs for temperatures far enough from \( T^* \) but yet, such that \( K_1 \ll 1, K_2 \approx \text{constant} \). In this regime, we have

\[ G(K_1, K_2) \rightarrow 1 + \cosh K_2 \approx \text{constant}. \]  
(44)

Consequently, from (35), we conclude that the resistivity, in this region, is given by

\[ \rho(T) \approx B'T^2, \]  
(45)

which befits a Fermi Liquid. This is in agreement with experimental observations that locate a Fermi Liquid behavior inside the PG phase [31–33].

3.4. Resistivity in the SM phase

In the SM phase, \( M = 0 \) and \( K_1 = 0 \) for all temperatures, so that the function \( G \) is given by

\[ G_{\text{SM}} = \lim_{M \rightarrow 0} M \frac{\cosh \left( \frac{\mu_{\text{SM}}}{2k_B T} \right)}{M \sinh \left( \frac{M}{2k_B T} \right)}. \]  
(46)

The SM phase corresponds to the quantum critical region associated to the QCP located at the right end of the Sc dome, namely \( x_{\text{Sc}}^x \equiv \tilde{x}_0 \). It follows that, in that region, all quantities with dimension of energy should scale with \( T \) [34, 35]. In particular, the chemical potential must be given by

\[ \mu_{\text{SM}}(T, x) = D k_B T, \]  
(47)

where \( D \) is a constant [21, 34]. In this way, we see that in the SM phase \( K_2 = D \) is constant. Imposing the continuity of the resistivity and its derivative across the border between the PG and SM phases, at \( T = T^* \) implies that the scaling function of the SM phase is given by:

\[ G_{\text{SM}} = G_{\text{PG}}(T \rightarrow T^*) = \frac{CT^*}{T}, \]  
(48)

where \( C = \cosh^2 \left( \frac{D}{2T} \right) \). This immediately yields the celebrated linear resistivity

\[ \rho_{\text{SM}} = A_1 T = \left( CBT^* \right) T, \]  
(49)

which ranges from the upper PG phase all the way into the SM phase. For regions of the SM phase where \( x > x_{\text{Sc}}^x \equiv \tilde{x}_0 \), the temperature \( T^*(x) \) is smoothly replaced by \( T^*(x) \equiv T^* (2\tilde{x}_0 - x) \). This kind of behavior is observed in figures 2(a)–(c) at high temperatures.

3.5. Resistivity in the FL phase

In the FL phase, which corresponds to the strongly overdoped regime, \( x > x_{\text{Sc}}^x \equiv \tilde{x}_0 \), at low temperatures, both \( K_1 \) and \( K_2 \) are equal to zero.

For the Fermi Liquid phase, the limit \( M \rightarrow 0 \) is taken without the constraint imposed by the continuity of \( \rho \) at \( T^* \). On the other hand, for \( \Delta = M = 0 \), equations (10) and (11) are trivially satisfied, and equation (12) implies

\[ \frac{\mu_{\text{FL}}}{2T} \left[ \mu_{\text{FL}}^2 - \Lambda \bar{\rho}(x) \right] = 0, \]  
(45)

where \( \bar{\rho}(x) = 2\gamma (\tilde{x}_0 - x) \).

Observe that, consequently, for \( x \rightarrow \tilde{x}_0 \), we have the trivial solution of the above equation: \( \mu_{\text{FL}}(x) = 0 \).

Now, for \( x > \tilde{x}_0 \), we see that \( \bar{\rho}(x) \) becomes negative, hence the only solution of (45) is \( \mu_{\text{FL}}(x) = 0 \) as well.

Hence, in the FL phase, we have

\[ G_{\text{FL}} = 1. \]  
(50)

\[ G_{\text{FL}} = G(K_1 = 0, K_2 = 0) = 1, \]  
(51)

and then, in this phase, the resistivity is given by the quadratic behavior, typical of a Fermi liquid:

\[ \rho_{\text{FL}} = B'T^2, \]  
(52)

with the coefficient \( B \) given by (36). This kind of behavior is observed in figure 2(d) at low temperatures.

3.6. Crossover

The transition between FL and the SM phases is not really a phase transition, but rather a crossover with an intermediate power-law behavior \( \rho \sim T^{1+\delta}, 0 \leq \delta \leq 1 \), that interpolates the resistivity behaviors, namely, linear and quadratic, of the SM and FL phases [27, 28]. This can be accounted for assuming that \( G \) has the power-law behavior

\[ G_{\text{cross}} = \left[ \frac{C_0 T^*(x)}{T} \right]^{1-\delta}, \]  
(53)

which is obtained from the previous one (48) by a scale transformation. The resistivity, then, will be given by

\[ \rho_{\text{cross}} = B \left[ C_0 T^*(x) \right]^{1-\delta} T^{1+\delta} \equiv B C T^{1+\delta}, \]  
(54)

for \( \delta \in [0, 1] \). Notice that the values \( \delta = 0 \) and \( \delta = 1 \) correspond to the SM and FL phases, respectively, which are thereby interpolated by the above expression. The crossover behavior occurs in the region of the phase diagram located between \( T^*(x) \) and the temperature \( T_{\text{cross}} \equiv C_0 T^*(x) \).
has been associated to the PG temperature. In the literature in \[48x81\]

temperature dependence of \[x\] behavior of \[value\] towards the quantum critical point, this stabilizes at a constant parameter the ratio \[\text{figures}\] phase. The results are the ones presented in figure\[48x225\], without any fitting parameter, once we determine the experimental data and we used equations (\[48x249\]) for many compounds used to obtain the values of \[C\] that were used in figure\[48x345\]. The references where the values of \[d\] were obtained are listed.

| Compound   | \(d\) (Å) | \(C_0\) |
|------------|-----------|--------|
| LSCO       | 6.61 [40] | 5.35   |
| Bi2201     | 12.15 [40]| 5.40   |
| Bi2212     | 7.74 [40] | 8.49   |
| Hg1212     | 6.32 [41] | 11.41  |

4. Comparison with experimental data

4.1. General considerations

We have successfully applied the above theoretical framework in the description of the resistivity of cuprates. Our results accurately explain the experimental data of several compounds, namely: LSCO [27, 28, 30], Bi-based (Bi2201 [36], Bi2212 [37]) and Hg-based (Hg1212 [38]) families of cuprates.

To reproduce the experimental data, we have added a constant value \(\rho_0\) to the theoretical expression of equations (39) and (49), in order to account for any effect of material imperfections in the resistivity. So, in summary, we have fitted \(A_1\) and \(\rho_0\) in the strange metal phase and \(T_0\) (assuming that it does not dependent on \(T\)) in the low-T regime of the PG phase. For the FL phase and crossover we can determine directly \(\rho_0\) from the experimental data and we used equations (36) and (55) without any fitting parameter, once we determine \(C\) for the SM phase. The results are the ones presented in figure 2 as well as figures B1, B2, B3, B4, B5 and B6 in appendix B.

In order to determine the value of \(C\) we must consider the ratio \(A_1(x)/[BT^*(x)]\), for different values of the doping parameter \(x\). As it turns out, as we increase \(x\) thus moving towards the quantum critical point, this stabilizes at a constant value \(C_0\). For LSCO, this occurs for \(0.10 \lesssim x\). The constant behavior of \(C\) implies that in the quantum critical region, the \(x\)-dependence of the resistivity slope comes through the dependence of \(T^*\) on \(x\). In other words, \(A_1\) scales with the PG temperature \(T^*(x)\). A similar behavior of \(A_1\) has been reported in the literature in [23], even though, in that case, it has not been associated to the PG temperature.

![Figure 3](image-url)

**Figure 3.** Scaling of \(A_1/(BC_0)\) (red circles) with \(T^*\), both with our theoretical expression (black solid line) as well as experimental values (blue squares), for different compounds. The experimental data was extracted from: LSCO [30], Bi2201 [36], Bi2212 [37] and Hg1212 [38]. The values of \(C_0\) and \(d\) used are in table 1.

**Figure 4.** (a) Evolution of \(A_1/(BT^*)\) with doping, showing that it stabilizes in the value \(C_0\) (black dashed line) displayed in table 1: (i) using our fits for \(A_1\) (red disks) and (ii) using results found in the literature (green and blue disks). (b) Value of \(B\) obtained by: (i) evolution of \(A_1/(CT^*)\) with doping, showing that it stabilizes in the value of \(B\) given by equation (36) using our fits for \(A_1\) (red disks) as well as using results found in the literature (green and blue disks); (ii) the resistivity coefficient of the FL phase; and (iii) the theoretical expression given in equation (36). These show a remarkable agreement close to the critical point \(x = x_0 \approx 0.267\). The experimental data points were extracted from the values of the coefficient, available in the literature [27, 28, 30, 39], with some of them being directly extracted from [6]. The dashed line corresponds to our theoretical calculation.

This analysis can be consistently repeated for different families of cuprates, as shown in figure 3, and the values of \(C_0\) thereby obtained are presented in table 1. The tendency of \(C\)
Quantum criticality results from the loss of the energy scale $M$, related to the exciton scattering as $M$ goes to zero when we approach the transition line $T^*(x)$ separating the PG from the SM phase. The scaling of every energy with $T$ in the quantum critical phase, makes the resistivity to become linear.

The possibility of expressing the resistivity in the SM phase in terms of a scaling function was already pointed out in the literature [20–22, 34]. In this work, however, this fact has been derived from the proposed Hamiltonian, and the explicit form of the scaling function was, thereby, determined. Also, the resistivity being the consequence of hole scattering by a scalar field, our scaling approach does not suffer from the problems pointed out in [22, 34]. We, therefore, reconcile the existence of a quantum critical regime with the Planckian diffusion hypothesis.

The scaling allows us to predict that the slope of the linear resistivity is proportional to the PG temperature $T^*(x)$, where the excitons condense. We, thereby, can infer the connection between the resistivity and the scattering by excitons.

A natural extension of this work would be to consider the inclusion of a third scaling variable, representing the effect of external agents such as pressure or magnetic field on the resistivity of cuprates. That would add a third dimension to the phase diagram of figure 1. This could be easily done in our theoretical framework, by introducing an external magnetic field in the Kubo formula or by considering the effect of an applied pressure in the chemical potential [25]. It would be interesting, then, to compare the new results with the data available in the literature for external magnetic field and with an AC field.

5. Conclusions and future perspectives

Starting from the model introduced in [25], and using the Kubo formula at finite temperature, we have derived a general expression for the resistivity of high-Tc cuprates, equation (35) whose particular forms in the PG, SM, FL phases, as well as the crossover between the two latter, agree very well with the experimentally observed resistivity. This agreement is not only qualitative, but also quantitative. In particular, we obtain $B$ and $B_C$ without adjusting any parameter, what attests the accuracy of our model for the description of high-Tc cuprates.

Based on this model, we conclude that the main cause of resistivity in cuprates is the scattering of the charged holes by excitons, which are associated with a scalar field. The presence of excitons should be expected in a system containing electrons and holes. Being confined to the CuO$_2$ planes, they should present similar properties as, for instance, the exciton states observed in transition metal dichalcogenides by photoluminescence techniques, namely, high binding energy and very short lifetimes [42].

Our expression for the resistivity in cuprates involves a two-variable scaling function. The SM phase, where the linear resistivity is seen, appears as a quantum critical region associated to a quantum critical point located at the right extremity of the SC dome, precisely where the $T_c$, $T^*$, $T^{**}$ and $T_{\text{cross}}$ lines meet.
Table A1. Obtained doping levels using equation (14) for many analyzed compounds, as well as the references where they were experimentally observed.

| Compound | $T_c$ | $x$ | Reference | Compound | $T_c$ | $x$ | Reference |
|----------|-------|-----|-----------|----------|-------|-----|-----------|
| Bi2201   | 14.5  | 0.14| [36]      | Bi2212   | 80    | 0.18| [37]      |
| Bi2201   | 23.7  | 0.17| [36]      | Bi2212   | 87    | 0.20| [37]      |
| Bi2201   | 30.2  | 0.21| [36]      | Bi2212   | 86    | 0.25| [37]      |
| Bi2201   | 34.2  | 0.26| [36]      | Bi2212   | 84    | 0.26| [37]      |
| Bi2201   | 33.1  | 0.30| [36]      | Hg1212   | 86    | 0.12| [38]      |
| Bi2201   | 30.1  | 0.31| [36]      | Hg1212   | 104   | 0.15| [38]      |
| Bi2212   | 35    | 0.13| [37]      | Hg1212   | 124   | 0.22| [38]      |
| Bi2212   | 50    | 0.14| [37]      | Hg1212   | 120   | 0.25| [38]      |
| Bi2212   | 65    | 0.16| [37]      | Hg1212   | 89    | 0.28| [38]      |

experimental data. That is how the curves $T_c(x)$ and $T^*(x)$ were obtained, with excellent agreement to the experiments for several cuprate compounds. Nevertheless, for some of the experimental data we have analyzed, a uniform database of doping parameters for the whole set of samples was needed. This is so because the reported doping levels were either based on different analyses or simply were not provided for each sample, but only its critical temperature. To circumvent this problem, we used our theoretical expression for $T_c(x)$, derived in [25] in order to obtain the doping level of each sample by using the corresponding value of $T_c$. The resulting $x$ values are presented on table A1.

Appendix B. Complete resistivity analysis

The fitted data for LSCO, Bi2201, Bi2212 and Hg1212 are in figures B1, B2, B3, B4, B5 and B6. Observe how these expressions describe the resistivity for a huge range of experimental data. The input values of the latter were obtained by digitalization of the published data.
Figure B1. Comparison between the experimental data extracted from [30] and the fitted equations (39) and (49) (with addition of $\rho_0$) for LSCO in the strongly underdoped regime.
Figure B2. Comparison between the experimental data extracted from [30] and the fitted equations (39) and (49) (with addition of $\rho_0$) for LSCO in the underdoped regime.
Figure B3. Comparison between the experimental data extracted from [30] and the fitted equations (39) and (49) (with addition of $\rho_0$) for LSCO in the overdoped regime.
Figure B4. Comparison between the experimental data extracted from [36] and the fitted equation (49) (with addition of $\rho_0$) for Bi2201.
Figure B5. Comparison between the experimental data extracted from [37] and the fitted equation (49) (with addition of $\rho_0$) for Bi2212.
Figure B6. Comparison between the experimental data extracted from [38] and the fitted equation (49) (with addition of $\rho_0$) for Hg1212.

ORCID iDs
R Arouca https://orcid.org/0000-0003-4214-1437
E C Marino https://orcid.org/0000-0002-8861-5135

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