Stability Indices of Non-Hyperbolic Equilibria

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Abstract

We consider families of systems of two-dimensional ordinary differential equations with the origin 0 as a non-hyperbolic equilibrium. For any number \( a \in (-\infty, +\infty) \) we show that it is possible to choose a parameter in these equations such that the stability index \( \sigma(0) \) is precisely \( \sigma(0) = a \). In contrast to that, for a hyperbolic equilibrium \( x \) it is known that either \( \sigma(x) = -\infty \) or \( \sigma(x) = +\infty \). Furthermore, we discuss a system with an equilibrium that is locally unstable but seems to be globally attracting, highlighting some subtle differences between the local and non-local stability indices.

Keywords: stability, attraction, non-hyperbolic equilibrium

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1 Introduction

Attraction and stability of invariant sets are crucial concepts in the qualitative theory of dynamical systems: the degree to which a set possesses these properties is directly linked to the way it influences the overall (long-term) dynamics of a system. Beyond the classic notion of asymptotic (Lyapunov) stability several levels of so-called non-asymptotic stability have been identified. These include fragmentary asymptotic stability (f.a.s.) \[10\] and essential asymptotic stability (e.a.s.) \[9\] to mention probably the two most frequent ones. Loosely speaking, an f.a.s. set attracts something of positive measure while an e.a.s. set attracts “almost everything” in a small neighbourhood.

In 2011 Podvigina and Ashwin \[11\] introduced a (local) stability index as a means of quantifying stability and attraction of invariant sets in discrete and continuous dynamical systems. It is linked to the stability properties mentioned above: roughly speaking, positive indices correspond to essential asymptotic stability, while negative (but finite) ones are associated with fragmentary asymptotic stability, see \[8\] for a detailed discussion of this. In the last decade, this concept has been used to characterize various types of attractors, e.g. heteroclinic cycles/networks \[3, 4, 5\], invariant graphs in skew product systems \[6\] or attractors with riddled basins \[12\].
For the simple case of a hyperbolic equilibrium the stability index does not reveal significant information, since it turns out to be either $+\infty$ (for a sink) or $-\infty$ (for a saddle or source). In this short paper we discuss two families of ordinary differential equations on $\mathbb{R}^2$ that possess the origin 0 as a non-hyperbolic equilibrium. We show that

(i) for any given real number $a > 0$ we can choose a parameter in the first family such that we obtain $\sigma(0) = a$, and
(ii) that the same is possible for any $a < 0$ in the second family.

This confirms that non-hyperbolic equilibria can indeed be f.a.s. or e.a.s without being asymptotically stable. Moreover, the way we design the systems might serve as a prototype for controlling stability indices in more involved settings, e.g. along heteroclinic connections.

The paper is organized as follows: in section 2 we briefly discuss the (local) stability index from [11]. In section 3 we present our examples and prove that the equilibria possess the desired stability indices. We conclude with some comments in section 4.

### 2 Preliminaries

In this section we recall the stability index that Podvigina and Ashwin [11] introduced to quantify stability and attraction of a compact, invariant set $X \subset \mathbb{R}^n$ of a dynamical system on $\mathbb{R}^n$ given by $\dot{x} = f(x)$.

We write $B_\varepsilon(x)$ for an $\varepsilon$-neighbourhood of a point $x \in \mathbb{R}^n$ and use $\ell(.)$ for Lebesgue measure. The basin of attraction of $X$, i.e. the set of points in $\mathbb{R}^n$ with $\omega$-limit set in $X$, is denoted by $B(X)$. For $\delta > 0$ the $\delta$-local basin of attraction $B_\delta(X)$ is the subset of points in $B(X)$ for which the trajectory never leaves $B_\delta(X)$ in positive time.

With this terminology we reproduce the following definition.

**Definition 2.1** ([11], definition 5). For $x \in X$ and $\varepsilon, \delta > 0$ set

$$
\Sigma_\varepsilon(x) := \frac{\ell(B_\varepsilon(x) \cap B(X))}{\ell(B_\varepsilon(x))}, \quad \Sigma_{\varepsilon,\delta}(x) := \frac{\ell(B_\varepsilon(x) \cap B_\delta(X))}{\ell(B_\varepsilon(x))}.
$$

Then the stability index at $x$ with respect to $X$ is defined as

$$
\sigma(x) := \sigma_+(x) - \sigma_-(x),
$$

with

$$
\sigma_-(x) := \lim_{\varepsilon \to 0} \frac{\ln(\Sigma_\varepsilon(x))}{\ln(\varepsilon)}, \quad \sigma_+(x) := \lim_{\varepsilon \to 0} \frac{\ln(1 - \Sigma_\varepsilon(x))}{\ln(\varepsilon)}.
$$
Figure 1: Positive (left) and negative (right) stability indices.

The convention that $\sigma_-(x) = \infty$ if $\Sigma_\varepsilon(x) = 0$ for some $\varepsilon > 0$, and $\sigma_+(x) = \infty$ if $\Sigma_\varepsilon(x) = 1$ for some $\varepsilon > 0$, implies $\sigma(x) \in [-\infty, \infty]$.

Analogously, the local stability index at $x \in X$ is defined to be

$$\sigma_{\text{loc}}(x) := \sigma_{\text{loc},+}(x) - \sigma_{\text{loc},-}(x),$$

with

$$\sigma_{\text{loc},-}(x) := \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \frac{\ln(\Sigma_{\varepsilon,\delta}(x))}{\ln(\varepsilon)}, \quad \sigma_{\text{loc},+}(x) := \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \frac{\ln(1 - \Sigma_{\varepsilon,\delta}(x))}{\ln(\varepsilon)}.$$

For an invariant set $X \subset \mathbb{R}^n$ and a point $x \in X$ the index $\sigma(x)$ quantifies attraction to $X$ near $x$ in the system. In the same way the local index $\sigma_{\text{loc}}(x)$ characterizes (Lyapunov) stability of $X$ near $x$. While these two properties often go hand in hand (and the local and non-local indices may coincide), it is well-known that they are independent of each other (so local and non-local indices may differ), see examples in [7].

For a geometric intuition consider Figure 1 if $\sigma(x) > 0$, then in a small neighbourhood of $x$ an increasingly large portion of points is contained in the basin of attraction $B(X)$ and therefore attracted to $X$. If on the other hand $\sigma(x) < 0$, then the portion of such points goes to zero as the neighbourhood $B_\varepsilon(x)$ shrinks. The meaning of signs for the local stability index may be illustrated analogously.

Since here we are interested in the stability of equilibria, we typically have $X = \{0\}$ in the following, which prompts us to conveniently shorten our notation to $B(0) = B(\{0\})$ etc.

3 Stability Indices

In this section we discuss several families of systems in $\mathbb{R}^2$, each with a non-hyperbolic equilibrium which, depending on a parameter in the equations,
may possess any given real number as its stability index. Note that we define
the systems only for $x, y \geq 0$, but they can easily be symmetrically extended
to the whole plane. Most of the time local and non-local stability indices
coincide – we therefore only distinguish between the two when this is not the
case.

### 3.1 Positive Stability Indices

We first present a class of systems in $\mathbb{R}^2$ with the origin 0 as an equilibrium
that can have any stability index in $(0, +\infty)$. With a parameter $a > 1$, for
$x, y \geq 0$ our system reads:

$$
\begin{align*}
\dot{x} &= x(x^a - y) \\
\dot{y} &= y \left(\frac{1}{2}x^a - y\right)
\end{align*}
$$

(1)

We remark that the right-hand side is at least $C^1$, but not $C^\infty$ if $a \notin \mathbb{N}$.

It is easy to see that 0 is a non-hyperbolic equilibrium of the system since
the Jacobian is just the zero matrix. Both coordinate axes are invariant: for
$y = 0$ we have $\dot{x} = x^{a+1} > 0$, so the $x$-axis belongs to the unstable set of 0.
Similarly, for $x = 0$ we have $\dot{y} = -y^2 < 0$, so the $y$-axis belongs to the stable
set of 0.

The $x$- and $y$-nullclines off the coordinate axes are given by:

$$
\dot{x} = 0 \iff y = x^a \quad \text{and} \quad \dot{y} = 0 \iff y = \frac{1}{2}x^a
$$

This enables us to sketch the dynamics of system (1) as in Figure 2. We
now proceed to state and prove our result about the stability index.

**Proposition 3.1.** In system (1), for $a > 1$ the stability index of the origin
is $\sigma(0) = a - 1 > 0$.

**Proof.** From Figure 2 it is clear that all points $(x, y)$ with $y < x^a$ do not
belong to the basin of attraction $B(0)$. This enables our first estimate:

$$
\ell(B_\varepsilon(0) \cap B(0)) \leq \varepsilon - \int_0^\varepsilon x^a \, dx = \varepsilon^2 - \frac{1}{1 + a} \varepsilon^{1+a}
$$

and therefore

$$
\Sigma_\varepsilon(0) = \frac{\ell(B_\varepsilon(0) \cap B(0))}{\ell(B_\varepsilon(0))} \leq \frac{1}{\varepsilon^2} \left(\varepsilon^2 - \frac{1}{1 + a} \varepsilon^{1+a}\right) = 1 - \frac{1}{1 + a} \varepsilon^{a-1},
$$
or equivalently
\[ 1 - \Sigma(0) \geq \frac{1}{1 + a} \varepsilon^{a-1}. \]

Hence
\[ \sigma(0) = \lim_{\varepsilon \to 0} \frac{\ln(1 - \Sigma(0))}{\ln(\varepsilon)} \leq \lim_{\varepsilon \to 0} \frac{\ln(\varepsilon^{a-1})}{\ln(\varepsilon)} = a - 1, \]

which finally implies \( \sigma(0) = \sigma_+(0) - \sigma_-(0) \leq a - 1. \)

For the other inequality we show that there is a constant \( k > 1 \) such that all \( (x, y) \) with \( y > kx^a \) belong to \( B(0) \), in fact, even to all \( B_\delta(0) \) with suitable \( \delta > 0 \). We claim that for a given \( a > 1 \) a choice of \( k > \frac{a-1}{a-1} > 1 \) suffices. This we prove by showing that the vector \( (\dot{x}, \dot{y}) \) in this region always points “to the left” of the curve \( (x, kx^a) \), which means the corresponding solution is for all positive times confined between \( (x, kx^a) \) and the y-axis, and thus must limit to 0. To see this, we calculate that the angle \( \alpha \) between \( (\dot{x}, \dot{y}) \) and the normal vector \( (-akx^{a-1}, 1) \) is always in \( (-\frac{\pi}{2}, \frac{\pi}{2}) \) along \( (x, kx^a) \), see Figure 3. To that end, consider the scalar product:
Figure 3: The angle $\alpha$ between the (dotted) normal vector to $(x, kx^a)$ and the flow of system (1).

\[
\langle (\dot{x}, \dot{y}), (-akx^{a-1}, 1) \rangle = -akx^a(x^a - y) + y \left( \frac{1}{2}x^a - y \right)
= -akx^a(x^a - kx^a) + kx^a \left( \frac{1}{2}x^a - kx^a \right)
= kx^{2a} \left( a(k - 1) + \frac{1}{2} - k \right)
= kx^{2a} \left( k(a - 1) - a + \frac{1}{2} \right),
\]

which is positive for all $x > 0$ if and only if $k > \frac{1}{a-1}$ is chosen as above. Such a choice is obviously possible for any $a > 1$. An analogous calculation to that at the beginning of this proof now yields $\sigma_+(0) \geq a - 1$ and therefore $\sigma(0) \geq a - 1$. Therefore, $\sigma(0) = a - 1$ as claimed.

We have thus established that the non-hyperbolic equilibrium in system (1) is e.a.s.

### 3.2 Negative Stability Indices

We now strive for a similar result with negative stability indices. An analogous calculation for system (1) with $a < 1$ does not yield the desired flow, since no suitable $k$ can be found to obtain a positive scalar product as above. This fits the observation that in the proof of Proposition 3.1 we need $k > \frac{a-1}{a-1}$.
which goes to $\infty$ when $a \to 1$ from above. However, the following modification of system (1) does the job:

$$\begin{cases}
\dot{x} &= x(\frac{1}{2}x^a - y) \\
\dot{y} &= y^2 (x^a - y)
\end{cases} \quad (2)$$

Note that the smoothness of system (2) is most severely limited by the $x$-term in the $y$-equation, since $a \in (0, 1)$. It is also worth pointing out that a stronger contraction in the $y$-direction than in system (1) is required to achieve the desired result, as becomes apparent in the calculations below.

As before the coordinate axes are invariant, and for $y = 0$ we have $\dot{x} = \frac{1}{2}x^{a+1} > 0$, so expanding dynamics on the $x$-axis; while for $x = 0$ we have $\dot{y} = -y^3 < 0$, so contracting dynamics on the $y$-axis. Note that the position of the $x$- and $y$-nullclines has been reversed compared to system (1), and we may sketch the phase portrait as in Figure 4.

![Figure 4: Nullclines for system (2) with $a < 1$.](image)

**Proposition 3.2.** In system (2), for $a < 1$ the stability index of the origin is $\sigma(0) = 1 - \frac{1}{a} < 0$.

**Proof.** We argue in the same way as in the proof of Proposition 3.1 but with reversed justifications for the two inequalities: first observe from Figure 4 that all points with $y > x^a$ clearly belong to the (local) basin of attraction of the origin. Thus, we obtain:

$$\ell(B_{\varepsilon}(0) \cap B(0)) \geq \int_{0}^{\varepsilon} x^{\frac{a}{2}}dx = \frac{\varepsilon^{1+\frac{a}{2}}}{1 + \frac{1}{a}}$$
and therefore
\[ \Sigma(0) = \ell(B_\varepsilon(0) \cap B(0)) \geq \frac{1}{1 + a} \frac{a - 1}{\varepsilon^{1/a}} \geq \frac{a - 1}{a + \varepsilon^{1/a}}. \]

hence
\[ \sigma_-(0) = \lim_{\varepsilon \to 0} \frac{\ln(\Sigma(0))}{\ln(\varepsilon)} \leq \lim_{\varepsilon \to 0} \frac{\ln(\varepsilon^{1/a})}{\ln(\varepsilon)} = \frac{1}{a} - 1, \]

which finally implies \( \sigma(0) = \sigma_+(0) - \sigma_-(0) \geq 1 - \frac{1}{a} \).

For the other inequality, we also proceed in a similar way as before, showing that along \((x, kx^a)\) the angle between \((\dot{x}, \dot{y})\) and the normal vector \((akx^{a-1}, -1)\) is in \((-\frac{\pi}{2}, \frac{\pi}{2})\) for suitable \(0 < k < \frac{1}{2}\). This implies that solutions with initial conditions in the region \(y < kx^a\) do not limit to 0 in forward time and thus do not belong to \(B(0)\), which enables our second estimate for the stability index. Again we consider the scalar product:

\[
\langle \dot{x}, \dot{y}, (akx^{a-1}, -1) \rangle = akx^a \left( \frac{1}{2} x^a - y \right) - y^2(x^a - y) = akx^a \left( \frac{1}{2} x^a - kx^a \right) - \left( kx^a \right)^2(x^a - kx^a) = kx^{2a} \left( a \left( \frac{1}{2} - k \right) - kx^a(1 - k) \right),
\]

which, given \(k < \frac{1}{2}\), is positive for all sufficiently small \(x > 0\), since the second term in parentheses goes to zero when \(x \to 0\) while the first one is positive and constant in \(x\). Again, similar calculations as above now yield \(\sigma_-(0) \geq \frac{1}{a} - 1\) and thus finally \(\sigma(0) = 1 - \frac{1}{a} < 0\).

With Propositions 3.1 and 3.2 we have established that in these systems of equations we can obtain any positive or negative number as the (local) stability index of the origin. In the case of system (2) the equilibrium is f.a.s. but not e.a.s.

### 3.3 Infinite Stability Indices

More generally, instead of \(x \mapsto x^a\) one can take any strictly increasing function \(x \mapsto \phi(x)\) with \(\phi(0) = 0\) and consider the following system for \(x, y \geq 0\):

\[
\begin{align*}
\dot{x} &= x \left( y - \frac{1}{2} \phi(x) \right) \\
\dot{y} &= y(y - \phi(x))
\end{align*}
\]

(3)
The smoothness of system (3) is determined by the smoothness of \( \phi \). We draw similar initial conclusions as above: the coordinate axes are invariant with contraction along the \( x \)-axis, where \( \dot{x} = -\frac{1}{2}x\phi(x) > 0 \); and expansion along the \( y \)-axis, where \( \dot{y} = y^2 \). Looking at the nullclines we obtain the sketch of the dynamics in Figure 5.

As a specific choice of \( \phi \), consider now \( \phi(x) = (2x + 1) \exp\left(-\frac{1}{x}\right) \) for \( x > 0 \) and \( \phi(0) = 0 \). In [7] this function is used to illustrate that it is possible to have \( \sigma(0) = -\infty \) even though the basin of attraction \( B(0) \) is of positive measure in any neighbourhood \( B_\epsilon(0) \). This is achieved by confining \( B(0) \) within the region where \( y < \phi(x) \). Indeed, for system (3) it is clear from Figure 5 that all \((x, y)\) with \( y < \frac{1}{2}\phi(x) \) belong to \( B(0) \), even to \( B_\delta(0) \) for suitable \( \delta > 0 \).

Conjecture 3.3. In system (3) all trajectories off the coordinate axes limit to 0 in forward time and are thus homoclinic to the origin.

In order to prove Conjecture 3.3 it suffices to show that all trajectories starting above \( y = \phi(x) \) eventually cross the graph of \( \phi \). To this end, we can check how the angle \( \beta \) between \((\dot{x}, \dot{y})\) and the horizontal axis changes along a trajectory. Since \( \tan \beta = \frac{\dot{y}}{\dot{x}} \) we would like to show that \( \frac{\partial \dot{y}}{\partial \dot{x}} \) is negative for all \( t > 0 \) and bounded away from zero. A straightforward calculation yields:

\[
\frac{\partial \dot{y}}{\partial \dot{x}} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2} = \frac{y}{2\dot{x}^2} \left[ \phi\left(\frac{1}{2}x\phi(x)(y - \phi(x)) - y\left(y - \frac{1}{2}\phi(x)\right)\right) - 2x^2y\left(y - \frac{1}{2}\phi(x)\right)\exp\left(-\frac{1}{x}\right) \right]
\]
While the expression above seems hard to tackle analytically, numerical evaluation suggests that this is indeed negative for \( y > \phi(x) \) as long as \( y \neq 0 \), as required.

This would make system (3) a promising candidate for having \( \sigma(0) = +\infty \) while the local stability index of the origin is \( \sigma_{\text{loc}}(0) = -\infty \). In [7] such an example is discussed, but its right-hand side is only continuous, not differentiable. Note that here \( \phi \) is \( C^\infty \), so if our conjecture can be confirmed, system (3) would provide a smooth example of this kind.

4 Concluding Remarks

We have designed two families of systems of ordinary differential equations on \( \mathbb{R}^2 \) that possess a non-hyperbolic equilibrium with an arbitrary real number as its stability index. Note that while in section 3 we have not considered the case \( \sigma(0) = 0 \), it is straightforward to write down such a system: one simply needs to make sure that \( \Sigma_\varepsilon(0) \) is constant, i.e. independent of \( \varepsilon > 0 \). This is the case if the basin of attraction is linearly bounded, see e.g. the piecewise linear vector field on \( \mathbb{R}^2 \) displayed in Figure 6, where we have \( \Sigma_\varepsilon(0) = \frac{1}{4} \) for all \( \varepsilon > 0 \).

Our work establishes explicit examples for non-asymptotically stable equilibria that are fragmentarily or essentially asymptotically stable. This may prove useful in future endeavors to develop more complicated systems with heteroclinic connections that possess a prescribed level of stability, thus extending previous efforts towards the design of systems with a desired connection structure between equilibria, see e.g. [1, 2].
\[ \dot{x} = -x, \quad \dot{y} = y \]
\[ \dot{x} = x, \quad \dot{y} = y \]

Figure 6: A system with \( \sigma(0) = 0 \).

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