ON CONFIGURATION SPACES AND WHITEHOUSE’S LIFTS OF THE EUCLERIAN REPRESENTATIONS

NICHOLAS EARLY AND VICTOR REINER

Abstract. S. Whitehouse’s lifts of the Eulerian representations of $\mathfrak{S}_n$ to $\mathfrak{S}_{n+1}$ are reinterpreted, topologically and ring-theoretically, building on the first author’s work on A. Ocneanu’s theory of permutohedral blades.

1. Introduction

In their work on Hochschild cohomology [6] and building on work of Barr [1 Prop. 2.5], Gerstenhaber and Schack introduced certain orthogonal idempotents $\{e_n^{(j)}\}_{i=1,2,\ldots,n}$ inside the group algebra $\mathbb{Q}\mathfrak{S}_n$ of the symmetric group $\mathfrak{S}_n$, now called the Eulerian idempotents. To define them, write for $i = 1, 2, \ldots, n - 1$ the element $s_{i,n-i} := \sum_w \text{sgn}(w)w$ in which the sum ranges over all $w = (w_1, w_2, \ldots, w_n)$ in $\mathfrak{S}_n$ with $w_1 < w_2 < \cdots < w_i$ and $w_{i+1} < w_{i+2} < \cdots < w_n$. Their sum $s_n := \sum_{i=1}^{n} s_{i,n-i}$ turns out to act semisimply on $\mathbb{Q}\mathfrak{S}_n$ via left multiplication, with eigenvalues $\{2^j - 2\}_{j=1}^n$. The $j$th Eulerian idempotent $e_n^{(j)}$ is the polynomial in $s_n$ projecting to the $(2^j - 2)$-eigenspace $E_n^{(j)} := e_n^{(j)}\mathbb{Q}\mathfrak{S}_n$. See Whitehouse [19 §1] for a nice discussion of the relation to Hochschild cohomology, and Reutenauer’s book [13], particularly its Section 9.5, for history and combinatorial context.

The dimension of $E_n^{(j)}$ counts the permutations $w$ in $\mathfrak{S}_n$ with $j$ cycles, that is, it is the (signless) Stirling number of the first kind. The (right-) $\mathfrak{S}_n$-representation on $E_n^{(j)}$ is well-studied, by Hanlon [8] and others:

- For $j = n$, the representation $E_n^{(n)}$ carries the sign representation sign.
- For $j = 1$, the 0-eigenspace $E_n^{(1)}$, after tensoring with sign, becomes isomorphic to the well-known $\mathfrak{S}_n$-representation $\text{Lie}_n$ on the multilinear part of the free Lie algebra on $n$ generators.
- In general, $\text{sgn} \otimes E_n^{(j)}$ is the multilinear part in one of the higher Lie characters in the Poincaré-Birkhoff-Witt decomposition for the tensor algebra in terms of the free Lie algebra.

The $E_n^{(j)}$ also have a topological interpretation. For a topological space $X$, its $n$th (ordered) configuration space $\text{Conf}^n X$ is the complement of the thick diagonal within the $n$-fold cartesian product $X^n$:

$$\text{Conf}^n X := \{(p_1, \ldots, p_n) \in X^n : p_i \neq p_j \text{ for } 1 \leq i, j \leq n\}.$$ 

The symmetric group action on $X^n$ permuting coordinates preserves $\text{Conf}^n X$, and hence acts on its cohomology ring $H^* \text{Conf}^n X$ with rational coefficients [4]. It is well-known (see, e.g., Sundaram and Welker [17]) that for Euclidean spaces $X = \mathbb{R}^d$ with $d \geq 1$, this cohomology is concentrated in homological degrees divisible by $d - 1$. Comparing [17] Theorem 4.4(iii) at $k = 2$ with known descriptions of $E_n^{(j)}$ (e.g., from Hanlon [8]; see Sundaram [16 eqn. (2.16)]) shows that for $d \geq 3$ odd, one has an $\mathfrak{S}_n$-module isomorphism

$$E_n^{(j)} \cong \text{sgn} \otimes H^{(n-j)(d-1)} \text{Conf}^n(\mathbb{R}^d).$$

Earlier, F. Cohen (see [2]) had given a presentation for the cohomology algebra, which for $d \geq 3$ odd is this:

$$H^* \text{Conf}^n \mathbb{R}^d \cong U^n := \mathbb{Q}[u_{ij}]_{1 \leq i < j \leq n}/I,$$

in which $I$ is the ideal generated by

- $u_{ij}^2$ and $u_{ij} + u_{ji}$ for $1 \leq i < j \leq n$,
- $u_{ij}u_{jk} + u_{jk}u_{ki} + u_{ki}u_{ij}$ for distinct triples $(i, j, k)$ with $1 \leq i, j, k \leq n$.

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1 All cohomology $H^*(-)$ in this paper is singular cohomology $H^*(-, \mathbb{Q})$ with rational coefficients, and $\dim(-)$ means $\dim_{\mathbb{Q}}(-)$. 

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Here $u_{ij}$ has cohomological degree $d - 1$, and is the pullback of a nonzero class in $H^{d-1} \operatorname{Conf}^n(\mathbb{R}^d) = \mathbb{Q}$ along the surjection $\operatorname{Conf}^n(\mathbb{R}^d) \to \operatorname{Conf}^2(\mathbb{R}^d)$ that maps $(p_1, \ldots, p_n) \mapsto (p_1, p_j)$. Thus $w$ in $\mathcal{S}_n$ acts via $w(u_{ij}) = u_{w(i), w(j)}$. The above algebra $\mathcal{A}$ in (1.2) is the special case for the reflection hyperplane arrangement of type $\Lambda_{d-1}$ of what has been called the Artinian Orlik-Terao algebra; see, e.g., [11] §2.2.

We wish to give analogous interpretations to certain idempotents and $\mathcal{S}_n$-representations defined by Whitehouse in [19]. Letting $c = (1, 2, \ldots, n)$ be an $n$-cycle, she showed that the idempotent $\Lambda_n = \sum_{i=0}^{n-1} \operatorname{sgn}(c^i)c^i$ of $\mathbb{Q}\mathcal{S}_n$ commutes with the element $s_{n-1}$, after embedding $s_{n-1}$ via the inclusion of $\mathcal{S}_{n-1}$ into $\mathcal{S}_n$ as the permutations fixing $n$. Hence $\Lambda_n$ commutes with all Eulerian idempotents $\{e^{(j)}(n-1)\}_{j=1}$, so that the products $\{\Lambda_ne^{(j)}(n-1)\}_{j=1}$ are also idempotent. She then proved several properties of the $\mathcal{S}_n$-representations

$$F_n^{(j)} := \Lambda_ne^{(j)}(n-1)\mathbb{Q}\mathcal{S}_n = e^{(j)}(n-1)\Lambda_n\mathbb{Q}\mathcal{S}_n \quad \text{for } j = 1, 2, \ldots, n-1.$$  

For one, they lift the Eulerian representations of $\mathcal{S}_{n-1}$ to $\mathcal{S}_n$-representations in this sense [19] Prop. 1.4:

$$F_n^{(j)} \uparrow_{\mathcal{S}_{n-1}} \cong E_n^{(j)}.$$  

She also proved that they have the following virtual description [19] Thm. 3.4:

$$F_n^{(j)} = \sum_{i=1}^j \left( E_n^{(i)} \uparrow_{\mathcal{S}_{n-1}} - E_n^{(i)} \right).$$  

Similarly to the Eulerian representations $E_n^{(j)}$ for $j = n$ and $j = 1$, the representation $F_n^{(n-1)}$ is the sign representation of $\mathcal{S}_n$, while $\operatorname{sgn} \otimes F_n^{(1)}$ is known as the Whitehouse representation of $\mathcal{S}_n$. Thus (1.3) implies $F_n^{(1)} \downarrow_{\mathcal{S}_{n-1}} \cong \operatorname{sgn} \otimes \mathcal{L}_{n-1} \cong E_n^{(1)}$. In work on cyclic operads, Getzler and Kapranov [17] showed that

$$E_n^{(1)} \cong \operatorname{sgn} \otimes \mathcal{L}_n \cong V^{(n-1, 1)} \otimes F_n^{(1)}.$$  

Here $V^\lambda$ denotes the irreducible $\mathcal{S}_n$-representation indexed by a partition $\lambda$ of the number $n$; in particular, $V^{(n-1, 1)}$ is the irreducible reflection representation of $\mathcal{S}_n$. Our first goal is to strengthen (1.5) to a graded statement that characterizes Whitehouse’s lifts $F_n^{(j)}$, and which is proven in Section 2 below. Consider the Grothendieck ring $R(\mathcal{S}_n)[[t]]$ of graded $\mathbb{Q}\mathcal{S}_n$-modules, where $V \cdot t^k$ represents the class of the representation $V[k]$ which is $V$ considered as living in degree $k$.

**Proposition 1.** The equations (1.4) defining $F_n^{(j)}$ in terms of $E_n^{(i)}$ are equivalent to this relation in $R(\mathcal{S}_n)[[t]]$:

$$\sum_{j=1}^n E_n^{(j)}t^{n-j} = \left( 1 + t V^{(n-1, 1)} \right) \sum_{j=1}^{n-1} F_n^{(j)}t^{n-1-j}.$$  

In particular, comparing coefficients of $t^{n-1}$ on the two sides of the proposition recovers (1.3); see also [15, 16].

An observation of Moseley, Proudfoot and Young [11] leads to our next proposition, a configuration space interpretation for $F_n^{(j)}$ that parallels the interpretation (1.1) for $E_n^{(j)}$. To state it, recall that the special unitary group $SU_2$ consists of all $2 \times 2$ complex matrices $A$ having $A^TA = I$ and $\det(A) = 1$, and is homeomorphic to the 3-sphere $S^3$. One has a diagonal action of $SU_2$ on the Cartesian product $(SU_2)^n$ and on $\operatorname{Conf}^n SU_2$, so that one may consider the quotient space

$$X_n := \operatorname{Conf}^n SU_2 / SU_2$$

under this diagonal action. In fact, if one includes $\mathbb{R}^3$ into $SU_2$ as the subspace $\mathbb{R}^3 = SU_2 \setminus \{1\} \subset SU_2$, one can check that $X_n$ is $(\mathcal{S}_{n-1}$-equivariantly) homeomorphic to $\operatorname{Conf}^{n-1} \mathbb{R}^3$ via these inverse homeomorphisms:

$$(p_1, \ldots, p_n) \mapsto \begin{cases} 
(p_1, \ldots, p_{n-1}, p_n) \\
(p_1, \ldots, p_{n-1}, 1)
\end{cases}$$

(1.6)

Our next goal is the following proposition, proven in Section 3 below by re-packaging a proof from [11].

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2 It is called the tree representation in work of Robinson and Whitehouse [18], and appears elsewhere, e.g., Sundaram [15].
Proposition 2. For \( j = 1, 2, \ldots, n - 1 \), the isomorphism of \( \mathfrak{S}_{n-1} \)-representations from (1.1) at \( d = 3 \)
\[
F_{n-1}^{(j)} \cong \text{sgn} \otimes H^{2(n-j)} \text{Conf}^{n-1} \mathbb{R}^3 ,
\]
lifts to an isomorphism of \( \mathfrak{S}_n \)-representations
\[
F_n^{(j)} \cong \text{sgn} \otimes H^{2(n-j)} X_n.
\]

Our main result is Theorem 3 below. It identifies the cohomology ring \( H^* X_n \) as a concrete subalgebra of the ring \( \mathcal{U}^n := H^* \text{Conf}^{n} (\mathbb{R}^3) \) from (1.2), specifically, the subalgebra \( \mathcal{V}^n \) generated by the elements
\[
v_{ijk} := u_{ij} + u_{jk} + u_{ki}
\]
for distinct \((i,j,k)\) with \( 1 \leq i, j, k \leq n \). Using the inclusion \( \mathbb{R}^3 \hookrightarrow \mathbb{S}^3 = SU_2 \), define a composite \( h \circ \pi \circ i \): \( (1.7) \)
\[
\text{Conf}^n(\mathbb{R}^3) \xrightarrow{1} \text{Conf}^n SU_2 \xrightarrow{\pi} \text{Conf}^n(SU_2)/SU_2 = X_n \xrightarrow{h} \text{Conf}^{n-1} \mathbb{R}^3
\]

Theorem 3. For all \( n \geq 3 \), the map between polynomial rings given by
\[
(1.8)
\]
has the following properties.
(i) \( \varphi \) induces an \( \mathfrak{S}_{n-1} \)-equivariant injective algebra map \( \varphi : \mathcal{U}^{n-1} \hookrightarrow \mathcal{U}^n \).
(ii) The injection \( \varphi \) maps \( \mathcal{U}^{n-1} \) isomorphically onto the subalgebra \( \mathcal{V}^n \) inside \( \mathcal{U}^n \) generated by all \( \{v_{ijk}\} \).
(iii) One can choose a scalar \( c \) in \( \mathbb{Q}^\times \) so that \( \varphi \) is identified with \( c \cdot (h \circ \pi \circ i)^* \), where \( (h \circ \pi \circ i)^* \) is the \( \mathfrak{S}_{n-1} \)-equivariant composite of the maps on cohomology in the top row here:

\[
\begin{align*}
\mathcal{U}^n & \xrightarrow{(\pi \circ i)^*} \mathcal{V}^n & \mathcal{U}^n & \xrightarrow{h^*} \mathcal{U}^{n-1}
\end{align*}
\]

The middle vertical map is a grade-doubling \( \mathfrak{S}_n \)-equivariant algebra isomorphism, showing that
\[
( F_n^{(n-j)} ) H^2 X_n \cong (\mathcal{V}^n)_j,
\]
where \( (\mathcal{V}^n)_j \) denotes the \( j \)th graded component of the ring \( \mathcal{V}^n \).

After proving Propositions 1, 2 and Theorem 3 in the next three sections, the last section concludes with some remarks on the relation of this paper to results of Mathieu [10] and of d’Antonio and Gaiffi [5], and to a conjecture of Moseley, Proudfoot, and Young [11].

2. Proof of Proposition 1

Taking coefficients of \( t^{n-j} \) on both sides of the equation in the proposition
\[
\sum_{j=1}^{n} F_n^{(j)} t^{n-j} = \left( 1 + t V^{(n-1,1)} \right) \sum_{j=1}^{n-1} F_n^{(j)} t^{n-1-j},
\]
one sees that it is equivalent to assertion for \( 1 \leq j \leq n - 1 \) that
\[
(2.1) \quad F_n^{(j)} = F_n^{(j-1)} \oplus V^{(n-1,1)} \oplus F_n^{(j)},
\]
with conventions \( F_n^{(0)} := 0 \), \( F_n^{(n)} := 0 \). Use the fact [14, Exer. 7.81] that any virtual \( \mathfrak{S}_n \)-module \( U \) satisfies
\[
V^{(n-1,1)} \otimes U = \left( U \uparrow_{\mathfrak{S}_n}^{\mathfrak{S}_n} \right) \uparrow_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} - U,
\]
to rewrite (2.1) in the equivalent form
\[
F_n^{(j)} = F_n^{(j-1)} \oplus \left( F_n^{(j)} \uparrow_{\mathfrak{S}_n}^{\mathfrak{S}_n} \right) \uparrow_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} - F_n^{(j)}
\]
\[
= F_n^{(j-1)} \oplus F_n^{(j)} \uparrow_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} - F_n^{(j)}
\]
where the second equality used (1.3). This last equation can be rewritten as follows
\[ F_n^{(j)} - F_n^{(j-1)} = E_n^{(j)} + E_n^{(j-1)} - E_n^{(j)} \]
which is equivalent to (1.4). This completes the proof of the proposition.

3. Proof of Proposition 2

The proposition asserts \( F_n^{(j)} \cong \text{sgn} \otimes H^{2(n-j)} X_n \) for \( j = 1, 2, \ldots, n - 1 \). However, in light of (1.1) and Proposition 1 one sees that this is equivalent to exhibiting an isomorphism of graded \( S_n \)-representations:
\[ H^* \text{Conf}^n \mathbb{R}^3 \cong \left( 1 \oplus V^{(n-1,1)}[2] \right) \otimes H^* X_n. \]
Recall \( V^{(n-1,1)}[2] \) means the \( S_n \)-irreducible \( V^{(n-1,1)} \) as a graded representation concentrated in degree 2.

In fact, (3.1) was already proven by Moseley, Proudfoot, and Young in [11, Prop. 2.5]; we simply repeat their proof here for self-containment. Forgetting the \((n+1)^{\text{st}}\) coordinate induces \( S_n \)-equivariant maps
\[
\begin{align*}
(SU_2)^{n+1} & \rightarrow (SU_2)^n \\
\text{Conf}^{n+1} (SU_2) & \rightarrow \text{Conf}^n (SU_2), \\
X_{n+1} & \rightarrow X_n.
\end{align*}
\]
This last map gives rise to a fibration sequence \( F \rightarrow X_{n+1} \rightarrow X_n \), in which the fiber \( F := SU_2 - \{ p_i \}_{i=1}^n \) is the 3-sphere \( SU_2 \) punctured at \( n \) points. Note that the cohomology \( H^* F \) has this simple description:
- \( H^0 F = \mathbb{Q} \) with trivial \( S_n \)-action, since \( F \) is connected, and
- \( H^2 F \) is spanned by \( n \) cocycles \( \{ z_i \} \) dual to cycles that go around the \( n \) punctures \( \{ p_i \}_{i=1}^n \), permuted by \( S_n \), and satisfying the relation \( \sum_{i=1}^n z_i = 0 \) as in \( V^{(n-1,1)} \).
Thus as a graded \( S_n \)-representation, \( H^*(F) = 1 \oplus V^{(n-1,1)}[2] \), the first tensor factor on the right of (3.1).

We claim that (3.1) then follows by applying the the Leray-Serre spectral sequence to \( F \rightarrow X_{n+1} \rightarrow X_n \). From (1.6), the base \( X_n \) is homeomorphic to \( \text{Conf}^{n-1} (\mathbb{R}^3) \), and therefore simply connected when \( n \geq 2 \), since \( \text{Conf}^{n-1} (\mathbb{R}^3) \) is the complement of an arrangement of linear subspaces of real codimension three inside \( \mathbb{R}^{3(n-1)} \). Therefore the fibration spectral sequence will have no twisted coefficients. Furthermore, since both the base \( X_n \) and fiber \( F \) have cohomology only in even degrees, the spectral sequence collapses at the first page, giving an \( S_n \)-equivariant isomorphism
\[ H^* X_{n+1} \cong H^* F \otimes H^* X_n. \]
This is exactly the desired \( S_n \)-isomorphism as in (3.1), noting that their left sides agree due to the homeomorphism \( h \) in (1.6) with \( n \) replaced by \( n + 1 \). This completes the proof of the proposition.

4. Proof of Theorem 3

Each of the parts (i),(ii),(iii) in the theorem will be proven in a separate subsection below.

4.1. Proof of Theorem 3(i).
Part (i) of the theorem asserts that, letting \( v_{ijk} := u_{ij} + u_{jk} + u_{ki} \), the map
\[ \mathbb{Q}[u_{ij}]_{1 \leq i \neq j \leq n-1} \xrightarrow{\varphi} \mathbb{Q}[u_{ij}]_{1 \leq i \neq j \leq n} \]
induces a \( S_{n-1} \)-equivariant injective algebra map \( \varphi : U^{n-1} \hookrightarrow U^n \).

The \( S_{n-1} \)-equivariance follows since \( S_{n-1} \) permutes the subscripts \( 1, 2, \ldots, n - 1 \) on the \( u_{ij} \).

To see that \( \varphi \) induces a well-defined map \( \varphi : U^{n-1} \rightarrow U^n \), start with the easy check (left to the reader) that these three identities involving \( v_{ijk} \) hold in \( U^n \):
- (a) \( v_{ijk}^2 = 0 \).
- (b) \( v_{ijk} \) is antisymmetric in \((i,j,k)\), meaning \( v_{\sigma(i), \sigma(j), \sigma(k)} = \text{sgn}(\sigma) v_{ijk} \) for all \( \sigma \) in \( S_3 \).
- (c) \( v_{ij} v_{jk} + v_{jk} v_{ki} + v_{ki} v_{ij} = 0 \).

The special cases of (a),(b) with \( k = n \) and (c) with \( \ell = n \) show that \( \varphi \) descends to a map \( U^{n-1} \rightarrow U^n \), since \( \varphi \) sends each defining relation \( u_{ij}^2 \) and \( u_{ij} + u_{ji} \) and \( u_{ij} u_{jk} + u_{jk} u_{ki} + u_{ki} u_{ij} \) in \( U^{n-1} \) to zero in \( U^n \).
To see $\bar{\varphi} : U^{n-1} \to U^n$ is injective, define a $\mathbb{Q}$-algebra map $\psi : \mathbb{Q}[u_{ij}]_{1 \leq i \neq j \leq n} \to \mathbb{Q}[u_{ij}]_{1 \leq i \neq j \leq n-1}$ by

$$
u_{ij} \mapsto \begin{cases} u_{ij} & \text{if } i, j \leq n - 1, \\ 0 & \text{if either } i = n \text{ or } j = n. \end{cases}$$

This $\psi$ satisfies $\psi(v_{ijn}) = u_{ij}$ and hence $(\psi \circ \varphi)(u_{ij}) = u_{ij}$ for $1 \leq i \neq j \leq n - 1$. On the other hand, it is easy to check that $\psi$ descends to a map $\psi : U^n \to U^{n-1}$, as it sends each defining relation $u_{ij}$ and $u_{ij} + u_{ji}$ and $u_{ij}u_{jk} + u_{jk}u_{ki} + u_{ki}u_{ij}$ from $U^n$ to zero in $U^{n-1}$. Thus $\psi \circ \varphi = 1_{U^{n-1}}$, so $\varphi$ is injective.

4.2. Proof of Theorem \ref{thm:main}(ii). Recall part (ii) of the theorem asserts that the injection $\bar{\varphi} : U^{n-1} \hookrightarrow U^n$ is a ring isomorphism from $U^{n-1}$ onto the subalgebra $\mathcal{V}^n$ inside $U^n$ generated by all $\{v_{ijk}\}$.

By construction, since $\varphi$ sends $u_{ij}$ to $v_{ijn}$, the image $\text{im}(\varphi)$ lies in $\mathcal{V}^n$. To see that $\text{im}(\varphi) = \mathcal{V}^n$, one needs to know that for $1 \leq i, j, k \leq n - 1$ one also has $v_{ijk}$ in $\text{im}(\varphi)$. However, we claim this will follow from the $\ell = n$ case of another identity in $U^n$ whose easy check is left to the reader:

$$v_{ijk} - v_{ij\ell} + v_{ik\ell} - v_{j\ell k} = 0$$

The $\ell = n$ case of (d) shows that $v_{ijk} = v_{ijn} - v_{ikn} + v_{jkn} = \varphi(u_{ij} - u_{ik} + u_{jk})$, lying in $\text{im}(\varphi)$.

4.3. Proof of Theorem \ref{thm:main}(iii). Recall part (iii) of the theorem asserts that the ring isomorphism $\bar{\varphi}$ from $U^{n-1}$ onto the subalgebra $\mathcal{V}^n$ inside $U^n$ generated by $\{v_{ijk}\}$ is, up to an overall scaling by $c$ in $\mathbb{Q}^*$, the same as the composite $(h \circ \pi \circ i)^*$ of these maps on cohomology derived from the maps in (1.7):

$$H^\ast \text{Conf}^\ast({\mathbb{R}^3}) \to H^\ast X_n \overset{h^\ast}{\to} H^\ast \text{Conf}^\ast({\mathbb{R}^3})$$

where the middle vertical map is a grade-doubling $\mathfrak{S}_n$-equivariant algebra isomorphism $H^\ast X_n \cong \mathcal{V}^n$. Our strategy will be to first check this when $n = 3$, and then deduce the general case using functoriality.

The case $n = 3$. Here we claim that it suffices to check $(\pi \circ i)^* : H^2 X_3 \to H^2 \text{Conf}^3({\mathbb{R}^3})$ is not the zero map. To see this claim, note that since $(\pi \circ i)^*$ is $\mathfrak{S}_3$-equivariant, and $\dim H^2 X_3 = \dim U^2 = 1$, nonzeroness would imply $(\pi \circ i)^*(H^2 X_3)$ is a 1-dimensional $\mathfrak{S}_3$-stable subspace of $H^2 \text{Conf}^3({\mathbb{R}^3}) = (U^3)$. One can explicitly decompose $(U^3)$ into $\mathfrak{S}_3$-irreducibles, using its definition as the quotient of the $\mathbb{Q}$-span of the relations that factors through $\pi \circ i : \text{Conf}^3({\mathbb{R}^3}) \to X_3$.

\footnote{The authors thank D. Grinberg for suggesting this argument.}

\footnote{This identification means that we sometimes use both the additive group structure from $\mathbb{R}^3$ to add/subtract vectors from $\mathbb{R}^3$ in the same formula where we use the multiplicative group structure from $SU_2$ to multiply/invert them, as in (14). To understand this more explicitly, one may use the identification of $\mathbb{R}^4$ with the imaginary quaternions.}
One obtains such a map by precomposing this composite

\[
\begin{align*}
\text{Conf}^3(\mathbb{R}^3) & \to X_3 & \to \text{Conf}^2(\mathbb{R}^3) & \to S^2 \\
(p_1, p_2, p_3) & \mapsto (p_1, p_2, p_3) & \mapsto (p_3, p_1, p_3^{-1} p_2) & \mapsto \frac{p_3^{-1} p_1 p_3^{-1} p_2}{|p_3^{-1} p_1 p_3^{-1} p_2|}
\end{align*}
\]

with the map \(S^2 \to \text{Conf}^3(\mathbb{R}^3)\) sending \(v \mapsto (v, 0, p)\) where \(p \in \mathbb{R}^3\) is any particular choice of a vector having length \(|p| > 1\). The result is the map \(\psi_p\) with this formula:

\[
S^2 \xrightarrow{\psi_p} S^2 \quad v \mapsto \frac{p^{-1} v - p^{-1} 0}{|p^{-1} v - p^{-1} 0|}.
\]

This map \(\psi_p\) is homotopic to the identity map on \(S^2\), by sending \(p\) to \(\infty\) in \(\mathbb{R}^3\), which is 1 in \(SU_2\): one can calculate \(\lim_{p \to \infty} \psi_p(v) = \frac{v - 0}{|v - 0|} = \frac{v}{|v|} = v\). Therefore \(\psi_p\) has degree one as a self-map of \(S^2\), as desired.

The general case \(n \geq 3\). We first show that each \(v_{ijn}\) for \(1 \leq i < j \leq n - 1\) lies in \(\text{im}(\pi \circ i)^*\), by reducing to the case \(n = 3\) as follows. Consider the commutative diagram with vertical maps induced by \((p_1, \ldots, p_n) \mapsto (p_i, p_j, p_n)\)

\[
\begin{align*}
\text{Conf}^n(\mathbb{R}^3) & \xrightarrow{i} \text{Conf}^n(SU_2) & \xrightarrow{\pi} X_n \\
\text{Conf}^3(\mathbb{R}^3) & \xrightarrow{i} \text{Conf}^3(SU_2) & \xrightarrow{\pi} X_3
\end{align*}
\]

This gives rise to horizontal composite maps \((\pi \circ i)^*\) on cohomology:

\[
\begin{align*}
U^n & \xrightarrow{(\pi \circ i)^*} H^* X_n \\
U^3 & \xrightarrow{(\pi \circ i)^*} H^* X_3
\end{align*}
\]

From the \(n = 3\) case applied with the triple of indices \((i, j, n)\) replacing \((1, 2, 3)\), the bottom horizontal map \((\pi \circ i)^* : H^* X_3 \to U^3\) has \(v_{ijn}\) in its image. From the fact that the classes \(u_{ij}, u_{jn}, u_{jn}\) in \(U^n\) are pulled back from the corresponding classes in \(U^3\), and commutativity of the diagram, it follows that the top horizontal map \((\pi \circ i)^* : H^* X_n \to U^n\) also has \(v_{ijn}\) in its image.

Once we know that all \(v_{ijn}\) lie in \(\text{im}(\pi \circ i)^*\), as before, applying the \(\ell = n\) case of property \((d)\) above for the \(v_{ijk}\) shows that all \(\{v_{ijk}\}\) lie in \(\text{im}(\pi \circ i)^*\), and hence so does the entire subalgebra \(Y^n\). Since we already know from part \((ii)\) of the theorem that

\[
\dim Y^n = \dim U^{n-1} = \dim H^* \text{Conf}^{n-1} \mathbb{R}^3 = \dim H^* X_n,
\]

this shows via dimension count that \((\pi \circ i)^*\) maps \(H^* X_n\) isomorphically onto the subalgebra \(Y^n\). Since \(h\) is a homeomorphism, this also means that \((h \circ \pi \circ i)^*\) maps \(H^* \text{Conf}^{n-1} \mathbb{R}^3\) isomorphically onto \(Y^n\). Furthermore, the calculation for \(n = 3\), replacing indices \((1, 2, 3)\) with \((i, j, n)\), shows that \((h \circ \pi \circ i)^*\) maps \(u_{ij} \mapsto c_{ijn} \cdot v_{ijn}\) for some scalars \(c_{ijn}\) in \(Q^X\). However, note that \(h \circ \pi \circ i\) is \(S_{n-1}\)-equivariant, and hence so is \((h \circ \pi \circ i)^*\), which forces all of the scalars \(c_{ijn}\) to equal a single scalar \(c\). This completes the proof of Theorem \((\text{iii})\).

5. Remarks

5.1. Relation to work of Mathieu and of d’Antonio and Gaiffi on the “hidden” action of \(S_{n+1}\). Note that replacing \(n - 1\) by \(n\) in Theorem \((\text{iii})\) gives an \(S_n\)-isomorphism \(\varphi : U^n \to Y^{n+1}\), which reveals why there is a “hidden” \(S_{n+1}\)-action on \(U^n\). It also gives an alternate explanation of Whitehouse’s result \((\text{iv})\) that the \(S_{n+1}\) action on \(E_{n}^{(j)}\) restricts to the action of \(S_n\) on \(E_{n-1}^{(j)}\). This hidden action and our results bear a close relation to previous work of Mathieu \([9, 10]\), and of d’Antonio and Gaiffi \([5]\), which we explain here.

In \([9]\) \([6]\), Mathieu denotes by \(A_n\) the ring that we have abstractly presented as \(U^n\) in \((\text{iv})\). He then identifies it as a certain “limit ring”, which he denotes \(Inv_n(\infty)^*\) in \([9]\) Thm. 7.6] and denotes as \((SU^n)^*\) in \([10]\) \([3]\]. In \([10]\) Lem. 3.3, he extends the natural action of \(S_n\) permuting subscripts on the generators \(x_{ij}\) for \(1 \leq i < j \leq n\) in his \((SU^n)^*\) to a “hidden” action of \(S_{n+1}\). One can check that his action is consistent
with our\footnote{\textit{Further translating notations, his generator }\(x_{ij}\) \textit{in }\(SC_n^*\) \textit{corresponds to what we would call the generator }\(v_{i,j,n+1}\) \textit{in }\(V^{n+1}\).} Bearing these identifications in mind, then the part of his result \cite{10} Cor. 4.5] pertaining to \((SU^n)^*\) is equivalent to our Proposition\cite{11} which he had already noted in his \cite{10} Cor. 4.6] strengthens the Getzler and Kapranov result \cite{15}. On the other hand, in \cite{5}, d’Antonio and Gaiffi give another, more direct representation-theoretic construction of the hidden \(\mathfrak{S}_{n+1}\)-action on \(U^n\), along with several results on the \(\mathfrak{S}_{n+1}\)-irreducible decomposition of this action.

The novelty of our results, compared to these previous works, is in naturally identifying the direct sum of Whitehouse’s lifts \(\bigoplus_j F_n^{(j)}\) as both the cohomology \(H^*X_n\), and as the subalgebra \(\mathcal{V}^n\) of \(U^n = H^*\text{Conf}^n\mathbb{R}^3\).

5.2. Relation to the Moseley, Proudfoot, and Young conjecture. They conjecture the following.

Conjecture 4. \cite{11} Conj. 2.10] One has a grade-doubling isomorphism of \(\mathfrak{S}_n\)-modules (but not \(\mathbb{Q}\)-algebras)

\[
M_n := \mathbb{Q}[u_{ij}]_{1 \leq i \neq j \leq n}/K \cong H^*X_n
\]

where \(\deg(u_{ij}) = 1\), and \(K\) is the ideal generated by

- \(u_{ij} + u_{ji}\) for \(1 \leq i < j \leq n\),
- \(u_{ij}u_{jk} + u_{ik}u_{kj}\) for triples \((i, j, k)\) of distinct integers \(1 \leq i, j, k \leq n\),
- \(z_1, \ldots, z_n\) where \(z_i := \sum_{j=1}^n u_{ij}\) (with convention \(u_{ii} := 0\)).

Here \(\mathfrak{S}_n\) permutes subscripts in the \(u_{ij}\).

Proposition\cite{2} shows that this conjecture is equivalent to the assertion that the above algebra \(M_n\) carries the \(\mathfrak{S}_n\)-representation \(\text{sgn} \otimes F_n^{(n-1)}\) in its \(j\)th graded component. Although both our algebra \(\mathcal{V}^n\) and this algebra \(M_n\) are generated in degree one by the images of the elements \(v_{ijk} := u_{ij} + u_{jk} + u_{ki}\), one can check that the ideal of relations satisfied by these generators are different in the two rings, and they are not isomorphic as graded algebras.

5.3. Presentation and bases for \(\mathcal{V}^n\). Since Theorem\cite{3} gives a ring isomorphism \(g : U^{n-1} \rightarrow \mathcal{V}^n\) sending \(u_{ij}\) to \(v_{ij,n}\), one can use this to give a minimal presentation for \(\mathcal{V}^n\), coming from the one in \cite{12} for \(U^{n-1}\). Also, there are known monomial bases in the \(u_{ij}\) for \(U^{n-1}\), such as the \textit{nbc-basis} \cite{12} Cor. 5.3], which can then be mapped forward to exhibit convenient monomial bases in the \(v_{ij,n}\) for \(\mathcal{V}^n\).

On the other hand, since the dimension of either \(j\)th graded component \((U^{n-1})_j\) or \((\mathcal{V}^n)_j\) is the number of permutations \(w\) in \(\mathfrak{S}_{n-1}\) with \(n - 1 - j\) cycles, one might expect basis elements indexed by such \(w\). Indeed, translating a result of the first author \cite{3} Cor. 41] shows the following: writing each cycle \(C\) of \(w\) uniquely as \(C = (c_1c_2\cdots c_{\ell})\) with convention \(c_1 = \min\{c_1, c_2, \ldots, c_{\ell}\}\), then \((U^{n-1})_j\) and \((\mathcal{V}^n)_j\) have basis elements

\[
\prod_{\text{cycles }C\text{ of }w} u_{c_1c_2}u_{c_2c_3}\cdots u_{c_{\ell-1}c_{\ell}} \quad \text{and} \quad \prod_{\text{cycles }C\text{ of }w} v_{c_1c_2n}v_{c_2c_3n}\cdots v_{c_{\ell-1}c_{\ell}n}
\]

where \(w\) runs through all permutations in \(\mathfrak{S}_{n-1}\) with \(n - 1 - j\) cycles. These bases for \(U^n\) and \(\mathcal{V}^n\) were initially conjectured by the first author during his work on A. Ocneanu’s theory of permutohedral blades. They will play an essential role in \cite{4}.

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E-mail address: eaulnick@gmail.com

Massachusetts Institute of Technology, Cambridge MA, 02139

E-mail address: reiner@umn.edu

School of Mathematics, University of Minnesota, Minneapolis MN 55455