Intuitionistic Fuzzy Ideals of Ternary Near-Rings

Warud Nakkhasen
Department of Mathematics, Faculty of Science, Mahasarakham University, Mahasarakham, Thailand

Abstract
We define the concept of intuitionistic fuzzy ideals of ternary near-rings as a generalization of fuzzy ideals, and we investigate some of their properties. Moreover, we characterize the notions of Noetherian and Artinian ternary near-rings using their intuitionistic fuzzy ideals.

Keywords: Ternary near-ring, Fuzzy ideal, Intuitionistic fuzzy ideal, Intuitionistic fuzzy set

1. Introduction
The concept of fuzzy sets was first introduced by Zadeh [1] as a function of a nonempty set \( X \) on the unit interval \([0, 1]\). The first inspired application to many algebraic structures was the concept of fuzzy groups, introduced by Rosenfeld [2]. Lui [3] studied the fuzzy ideals of rings, and many researchers [4–6] have extended these concepts. The notions of fuzzy subnear-rings and fuzzy left (resp. right) ideals in near-rings were introduced by Abou-Zaid in [7]. They have been studied by many authors [8–10].

The concept of intuitionistic fuzzy sets was introduced by Atanassov [11–13] as a generalization of the concept of fuzzy sets. Fuzzy sets give the degree of membership of an element in a given set. Intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership. This theory has been studied by many mathematicians [14–19]. Biswas [20] considered the notion of intuitionistic fuzzy subgroups of groups. In [21], the authors presented the concept of intuitionistic fuzzy ideals of semi-rings. Later, the concept of intuitionistic fuzzy ideals of near-rings was introduced and studied by Zhan and Ma [22].

In 2012, Nakkhasen and Pibaljomme [23] introduced the concept of left ternary near-rings and investigated some properties of \( L \)-fuzzy ideals of ternary near-rings, where \( L \) is a complete lattice with the greatest element 1 and the least element 0. Later, Uma Maheswari and Meera [24, 25] studied the concepts of fuzzy soft ideals and fuzzy soft prime ideals over right ternary near-rings. In this paper, we introduce the notion of intuitionistic fuzzy ideals of ternary near-rings and investigate some of their properties. Also, we introduce and characterize the notions of Noetherian and Artinian ternary near-rings using their intuitionistic fuzzy ideals.

2. Preliminaries
In this section, we present the basic definitions that are used in the following sections of this paper.

**Definition 1** [26]. A ternary semi-group is an algebraic structure \((N, +, [\ ] )\) such that \( N \) is
a nonempty set, and \([\ ] : N^3 \to N\) is a ternary operation satisfying the following associative law: \([abc]de = [a[bc]d]e = [ab]cde\), for all \(a, b, c, d, e \in N\).

**Definition 2** \([26]\). Let \(A, B,\) and \(C\) be nonempty subsets of a ternary near-ring \(N\). Then, \([ABC] = \{[abc] \in N \mid a \in A, b \in B, c \in C\}\).

**Definition 3** \([23]\). Let \(N\) be a nonempty set together with a binary operation \(+\) and a ternary operation \([\ ] : N^3 \to N\). Then, \((N, +, [\ ]\)) is called a **left ternary near-ring** if it satisfies the following conditions:

(i) \((N, +)\) is a group (not necessarily abelian);

(ii) \((N, [\ ]\)) is a ternary semi-group;

(iii) \([ab(c + d)] = [abc] + [abd]\), for every \(a, b, c, d \in N\).

**Right ternary near-rings** and **lateral near-rings** are defined in a similar manner. In this paper, we focus on left ternary near-rings, and we will use the word “ternary near-rings” to mean “left ternary near-rings.”

**Definition 4** \([23]\). A nonempty subset \(T\) of a ternary near-ring \(N\) is said to be a **ternary subnear-ring** of \(N\) if \(a - b \in T\) and \([abc] \in T\), for all \(a, b, c \in T\).

**Definition 5** \([24]\). Let \(N\) be a nonempty near-ring. Let \(I\) be a normal subgroup of \((N, +)\). Then, for every \(a, b, c \in N\) and \(i \in I\),

(i) \(I\) is called a **left ideal** of \(N\) if \([NNI] \subseteq I\);

(ii) \(I\) is called a **right ideal** of \(N\) if \([a + ibc] - [abc] \in I\);

(iii) \(I\) is called a **lateral ideal** of \(N\) if \([a(b + ic)] - [abc] \in I\).

We call \(I\) an ideal of \(N\) if it is a left ideal, a right ideal, and a lateral ideal of \(N\).

**Example 1** \([23]\). Let \(N = \{a, b, c, d\}\) be a set with a binary operation \(+\) on \(N\) as follows:

| +   | a | b | c | d |
|-----|---|---|---|---|
| a   | a | b | c | d |
| b   | b | a | d | c |
| c   | c | d | b | a |
| d   | d | c | a | b |

The ternary operation \([\ ]\) on \(N\) is defined by \([xyz] = z\) for all \(x, y, z \in N\). Then, we have that \((N, +, [\ ]\)) is a ternary near-ring. Let \(I = \{a, b\}\). It follows that \(I\) is a ternary subnear-ring of \(N\). Next, we show that \(I\) is a left ideal, a right ideal, and a lateral ideal of \(N\), that is, \(I\) is an ideal of \(N\).

**Definition 6** \([23]\). Let \(N\) and \(R\) be ternary near-rings. A mapping \(\varphi : N \to R\) is called a **homomorphism** if \(\varphi(a + b) = \varphi(a) + \varphi(b)\) and \(\varphi([abc]) = [\varphi(a)\varphi(b)\varphi(c)]\), for all \(a, b, c \in N\).

Let \(X\) be a nonempty set. A **fuzzy set** \([1]\) of \(X\) is a mapping \(\mu : X \to [0, 1]\). Let \(\mu\) be a fuzzy set of \(X\). The set \(U(\mu; t) = \{x \in X \mid \mu(x) \geq t\}\) is called an upper-level set of \(\mu\), and the set \(L(\mu; t) = \{x \in X \mid \mu(x) \leq t\}\) is called a lower-level set of \(\mu\), where \(t \in [0, 1]\). The complement of \(\mu\) denoted by \(\mu^c\) is the fuzzy set of \(X\) defined by \(\mu^c(x) = 1 - \mu(x)\), for all \(x \in X\). The intersection and union of two fuzzy sets \(\mu\) and \(\lambda\) of \(X\), denoted by \(\mu \cap \lambda\) and \(\mu \cup \lambda\), respectively, are defined by letting \(x \in X\), \((\mu \cap \lambda)(x) = \min\{\mu(x), \lambda(x)\}\) and \((\mu \cup \lambda)(x) = \max\{\mu(x), \lambda(x)\}\), respectively.

The concept of intuitionistic fuzzy sets was introduced by Atanassov \([11–13]\) as an important generalization of the concept of fuzzy sets. An **intuitionistic fuzzy set** \(A\) in a nonempty set \(X\) is defined by the form

\[A = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in X\},\]

where \(\mu_A : X \to [0, 1]\) and \(\lambda_A : X \to [0, 1]\) denote the **degree of membership** and the **degree of non-membership** of each \(x \in X\) in the set \(A\), and also \(0 \leq \mu_A(x) + \lambda_A(x) \leq 1\), for all \(x \in X\). For the sake of convenience, we will use the symbol \(A = (\mu_A, \lambda_A)\) instead of the intuitionistic fuzzy set \(A = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in X\}\).

### 3. Intuitionistic Fuzzy Ideals

In this section, we introduce the concept of intuitionistic fuzzy ideals of ternary near-rings and investigate some of their properties.

**Definition 7**. An intuitionistic fuzzy set \(A = (\mu_A, \lambda_A)\) of a ternary near-ring \((N, +, [\ ]\)) is called an **intuitionistic fuzzy ideal** of \(N\) if it satisfies for every \(i, x, y, z \in N\),

- (IF1) \(\mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\}\);
- (IF2) \(\mu_A([xyz]) \geq \min\{\mu_A(x), \mu_A(y), \mu_A(z)\}\);
- (IF3) \(\mu_A(y + x - y) \geq \mu_A(x)\);
- (IF4) \(\mu_A([xyz]) \geq \mu_A(z)\);
- (IF5) \(\mu_A([(x + i)yz] - [xyz]) \geq \mu_A(i)\);
- (IF6) \(\mu_A([(x + i)yz] - [xyz]) \geq \mu_A(i)\);
- (AF1) \(\lambda_A(x - y) \leq \max\{\lambda_A(x), \lambda_A(y)\}\);
(AF2) \( \lambda_A([xyz]) \leq \max(\lambda_A(x), \lambda_A(y), \lambda_A(z)) \);

(AF3) \( \lambda_A(y + x - y) \leq \lambda_A(x) \);

(AF4) \( \lambda_A([x + y]z - [xyz]) \leq \lambda_A(i) \);

(AF5) \( \lambda_A([x + y]z - [xyz]) \leq \lambda_A(i) \).

Example 2. In Example 1 we define an intuitionistic fuzzy set \( A = (\mu_A, \lambda_A) \) of a ternary near-ring \( N \) by \( \mu_A(c) = \mu_A(d) < \mu_A(b) < \mu_A(a) \) and \( \lambda_A(a) < \lambda_A(b) < \lambda_A(c) = \lambda_A(d) \). By routine calculations, it is clear that \( A = (\mu_A, \lambda_A) \) is an intuitionistic fuzzy ideal of \( N \).

Proposition 1. If \( A = (\mu_A, \lambda_A) \) is an intuitionistic fuzzy ideal of a ternary near-ring \( N \), then \( \mu_A(0) \geq \mu_A(x) \) and \( \lambda_A(0) \leq \lambda_A(x) \), for all \( x \in N \).

Proof. Assume that \( A = (\mu_A, \lambda_A) \) is an intuitionistic fuzzy ideal of a ternary near-ring \( N \). Let \( x \in N \). Then, \( \mu_A(0) = \mu_A(x - x) \geq \min(\mu_A(x), \mu_A(x)) = \mu_A(x) \) and \( \lambda_A(0) = \lambda_A(x - x) \leq \max(\lambda_A(x), \lambda_A(x)) = \lambda_A(x) \).

Theorem 1. Let \( N \) be a ternary near-ring. Then \( A = (\mu_A, \lambda_A) \) is an intuitionistic fuzzy ideal of \( N \) if and only if for any \( t, s \in [0, 1] \), the nonempty sets \( U(\mu_A; t) \) and \( L(\lambda_A; s) \) are ideals of \( N \).

Proof. Assume that \( A = (\mu_A, \lambda_A) \) is an intuitionistic fuzzy ideal of \( N \). Let \( t, s \in [0, 1] \). First, let \( x, y \in U(\mu_A; t) \). Then, \( \mu_A(x) \geq t \) and \( \mu_A(y) \geq t \). Thus, \( \mu_A(x - y) \geq \min(\mu_A(x), \mu_A(y)) \geq t \) and so \( x - y \in U(\mu_A; t) \). Second, for any \( x \in U(\mu_A; t) \) and \( n \in N \), we have \( \mu_A(x + n - n) \geq \mu_A(x) \geq t \) and then \( n + x - n \in U(\mu_A; t) \). Third, for any \( n, m \in N \) and \( x \in U(\mu_A; t) \), we have \( \mu_A(nmx) \geq \mu_A(x) \geq t \), that is, \( [nmx] \in U(\mu_A; t) \). Finally, for any \( i \in U(\mu_A; t) \) and \( x, y, z \in N \), then \( \mu_A([(x + i)y]z - [x+y]z) \geq \mu_A(i) \geq t \) and \( \mu_A([(x + i)y]z - [x+y]z) \geq \mu_A(i) \geq t \). It follows that \( [(x + i)y]z - [x+y]z \in U(\mu_A; t) \) and \( [x+y]z - [xyz] \in U(\mu_A; t) \). Hence, \( U(\mu_A; t) \) is an ideal of \( N \). Similarly, we can show that \( L(\lambda_A; s) \) is also an ideal of \( N \).

Conversely, assume that for any for any \( s, t \in [0, 1] \), the nonempty sets \( U(\mu_A; t) \) and \( L(\lambda_A; s) \) are ideals of \( N \). We want to show that \( A = (\mu_A, \lambda_A) \) is an intuitionistic fuzzy ideal of \( N \).

In Example 2, it is clear that \( A = (\mu_A, \lambda_A) \) is an intuitionistic fuzzy ideal of \( N \). Then, we can show that the upper-level sets of \( N \) are \( U(\mu_A; \lambda_A(a)) = \{a\} \), \( U(\mu_A; \lambda_A(b)) = \{a, b\} \), \( U(\mu_A; \lambda_A(c)) = N \), and \( U(\mu_A; \lambda_A(d)) = N \). Also, the lower-level sets of \( N \) are \( L(\lambda_A; \lambda_A(a)) = \{a\} \), \( L(\lambda_A; \lambda_A(b)) = \{a, b\} \), \( L(\lambda_A; \lambda_A(c)) = N \), and \( L(\lambda_A; \lambda_A(d)) = N \). By Theorem 1 it follows that \( \{a\}, \{a, b\}, \{a\} \), and \( N \) are the ideals of \( N \).

Theorem 2. Let \( N \) be a ternary near-ring. Then, \( A = (\mu_A, \lambda_A) \) is an intuitionistic fuzzy ideal of \( N \) if and only if \( A^c = (\lambda_A^c, \mu_A^c) \) is an intuitionistic fuzzy ideal of \( N \).

Proof. Assume that \( A = (\mu_A, \lambda_A) \) is an intuitionistic fuzzy ideal of \( N \). Let \( i, x, y, z \in N \). (IF1) \( \lambda_A^c(x - y) = 1 - \lambda_A(x - y) \geq 1 - \max(\lambda_A(x), \lambda_A(y)) = 1 - \lambda_A(x), 1 - \lambda_A(y) \geq \min(\lambda_A^c(x), \lambda_A^c(y)) \).

Similarly, we can show that \( \lambda_A \) satisfies (IF2)-(IF6). Next, we consider (AF1) \( \mu_A^c(x - y) = 1 - \mu_A(x - y) \leq 1 - \min(\mu_A(x), \mu_A(y)) = \max(1 - \mu_A(x), 1 - \mu_A(y)) \).

Similarly, we can show that (AF2)-(AF6). Hence, \( A^c = (\lambda_A^c, \mu_A^c) \) is an intuitionistic fuzzy ideal of \( N \).

Conversely, assume that \( A^c = (\lambda_A^c, \mu_A^c) \) is an intuitionistic fuzzy ideal of \( N \). Let \( i, x, y, z \in N \). (IF1) \( 1 - \mu_A(x - y) \leq \mu_A^c(x - y) = 1 - \lambda_A^c(x - y) \)

This implies that \( \lambda_A(x - y) \geq \min(\lambda_A^c(x), \lambda_A^c(y)) \).

Similarly, we can show that \( A^c \) satisfies (IF2)-(IF6). Next, we consider (AF1) \( 1 - \lambda_A(x - y) \leq \lambda_A^c(x - y) \)

This implies that \( \mu_A(x - y) \geq \min(\mu^c_A(x), \mu^c_A(y)) \).

It follows that \( \lambda_A(x - y) \leq \max(\lambda^c_A(x), \lambda^c_A(y)) \).

Similarly, we can show that \( \lambda^c_A \) satisfies (AF1)-(AF6). This completes the proof. \( \square \)
satisfies (AF2)-(AF6). Therefore, \(A = (\mu_A, \lambda_A)\) is an intuitionistic fuzzy ideal of \(N\).

The following theorem immediately follows from Theorem 2.

**Theorem 3.** Let \(A = (\mu_A, \lambda_A)\) be an intuitionistic fuzzy set of a ternary near-ring \(N\). Then, \(A = (\mu_A, \lambda_A)\) is an intuitionistic fuzzy ideal of \(N\) if and only if \(A^T = (\mu_A, \lambda_A^T)\) and \(A^L = (\lambda_A, \mu_A)\) are intuitionistic fuzzy ideals of \(N\).

**Proof.** Assume that \(A = (\mu_A, \lambda_A)\) is an intuitionistic fuzzy ideal of \(R\). Let \(i, x, y, z \in N\). Then, \(\lambda^T_A(x) = \mu_A(f(x) + f(y)) + \lambda_A(f(x))\geq \lambda_A(f(x))\). Similarly, we can prove that \(\mu^T_A\) satisfies (IF2)-(IF6). Now, (AF1) \(\lambda^T_A(x) = \mu_A(f(x) - f(y))\leq \lambda_A(f(x))\). Similarly, we can prove that \(\lambda_A^T\) satisfies (AF2)-(AF6).

Therefore, \(A^T = (\mu^T_A, \lambda_A^T)\) is an intuitionistic fuzzy ideal of \(N\).

If we strengthen the condition of \(f\), we can construct the converse of Theorem 6 as follows.

**Theorem 7.** Let \(f : N \to R\) be an epimorphism of ternary near-rings, and let \(A = (\mu_A, \lambda_A)\) be an intuitionistic fuzzy set of \(R\). If \(A^T = (\mu^T_A, \lambda_A^T)\) is an intuitionistic fuzzy ideal of \(N\), then \(A = (\mu_A, \lambda_A)\) is an intuitionistic fuzzy ideal of \(R\).

**Proof.** Assume that \(A = (\mu^T_A, \lambda_A^T)\) is an intuitionistic fuzzy ideal of \(N\). Let \(i, x, y, z \in R\). Then, there exist \(n, a, b, c \in N\) such that \(f(i) = n, f(x) = a, f(y) = b, f(z) = c\). Then, \(\mu_A(x) = \mu_A(f(a) - f(b))\). Similarly, we can show that \(\lambda_A^T\) satisfies (IF2)-(IF6). Now, we consider (AF1) \(\lambda_A(x) = \mu_A(f(a) - f(b))\). Similarly, we can show that \(A\) satisfies (AF2)-(AF6). Hence, \(A = (\mu_A, \lambda_A)\) is an intuitionistic fuzzy ideal of \(R\).

Let \(\mu\) be a fuzzy set of a nonempty set \(X\) and \(t \in [0, 1] - \sup_{x \in X} \mu(x)\). The mapping \(\mu^T : X \to [0, 1]\) is called a fuzzy translation of \(\mu\) if \(\mu^T(x) = \mu(x) + t\), for all \(x \in X\).

**Theorem 8.** Let \(N\) be a ternary near-ring, \(A = (\mu_A, \lambda_A)\) be an intuitionistic fuzzy ideal of \(N\), and \(t \in [0, \frac{1}{2}(\sup_{x \in N} \lambda_A(x) + \lambda_A(x))]\). Suppose that \(\mu_A^T\) and \(\lambda_A^T\) are fuzzy translations of \(\mu_A\) and \(\lambda_A\) with respect to \(t\), respectively. Then, \(A = (\mu_A, \lambda_A)\) is an intuitionistic fuzzy ideal of \(N\) if and only if \(A^T = (\mu_A^T, \lambda_A^T)\) is an intuitionistic fuzzy ideal of \(N\).

**Proof.** Assume that \(A = (\mu_A, \lambda_A)\) is an intuitionistic fuzzy ideal of \(N\). Let \(a \in N\) and \(t = \frac{1}{2}(\sup_{x \in N} \lambda_A(x) + \lambda_A(x))\). Then, \(\mu_A^T(a) + \lambda_A^T(a) = \mu_A(a) + \lambda_A(a) + 2t = \mu_A(a) + \lambda_A(a) + 1 - (\mu_A(a) + \lambda_A(a)) = 1\). Thus, \(A^T = (\mu_A^T, \lambda_A^T)\) is an intuitionistic fuzzy ideal of \(N\).

Let \(f : N \to R\) be a homomorphism of ternary near-rings. For any \(A = (\mu_A, \lambda_A)\) of \(R\), we define a new \(A^T = (\mu_A^T, \lambda_A^T)\) of \(N\) by \(\mu_A^T(x) = \mu_A(f(x))\) and \(\lambda_A^T(x) = \lambda_A(f(x))\), for all \(x \in N\).

**Theorem 6.** Let \(f : N \to R\) be a homomorphism of ternary near-rings. If \(A = (\mu_A, \lambda_A)\) is an intuitionistic fuzzy ideal of \(R\), then \(A^T = (\mu_A^T, \lambda_A^T)\) is an intuitionistic fuzzy ideal of \(N\).
Let μ be a fuzzy set of a nonempty set X and m ∈ [0, 1]. The mapping μ^M : X → [0, 1] is called a fuzzy multiplication [27] of μ if μ^M(x) = μ(x), for all x ∈ X.

**Theorem 9.** Let N be a ternary near-ring, A = (μ_A, λ_A) be an intuitionistic fuzzy set of N, and m ∈ [0, 1]. Suppose that μ^M_A and λ^M_A are fuzzy multiplications of μ_A and λ_A, respectively, where μ^M_A(x) = mμ_A(x) and λ^M_A(x) = mλ_A(x), for all x ∈ N. Then, A = (μ_A, λ_A) is an intuitionistic fuzzy ideal of N if and only if A^M = (μ^M_A, λ^M_A) is an intuitionistic fuzzy ideal of N.

**Proof.** Assume that A = (μ_A, λ_A) is an intuitionistic fuzzy ideal of N. Obviously, A^M = (μ^M_A, λ^M_A) is an intuitionistic fuzzy set of N. Let i, x, y, z ∈ N. Then, (IF1) μ^M_A(x − y) = mμ_A(x) ≥ m min{μ_A(x), μ_A(y)} = min{μ^M_A(x), μ^M_A(y)}. The proofs of (IF2)-(IF6) are similar to that of (IF1). Next, we have λ^M_A(x − y) = mλ_A(x − y) ≤ m max{λ_A(x), λ_A(y)} = max{mλ_A(x), mλ_A(y)} = max{λ^M_A(x), λ^M_A(y)}. In the same way, we have λ^M_A satisfying (AF2)-(AF6). Hence, A^M = (μ^M_A, λ^M_A) is an intuitionistic fuzzy ideal of N.

Conversely, assume that A^M = (μ^M_A, λ^M_A) is an intuitionistic fuzzy ideal of N. Let i, x, y, z ∈ N. Then, (IF1) μ^M_A(x − y) = mμ_A(x − y) ≥ m min{μ_A(x − y)} = min{μ^M_A(x − y)} = min{μ^M_A(x), μ^M_A(y)}. Because m > 0, μ_A(x − y) ≥ min{μ_A(x), μ_A(y)}. Uniformly, we have that μ_A satisfies (IF1)-(IF6). Now, consider (AF1) mλ_A(x − y) = mλ_A(x − y) ≤ max{λ^M_A(x − y)} = m max{λ_A(x − y)}.

4. Noetherian and Artinian Ternary Near-Rings

In this section, we define the notions of Noetherian and Artinian ternary near-rings and characterize Noetherian and Artinian ternary near-rings using their intuitionistic fuzzy ideals.

**Definition 8.** A ternary near-ring N is called Noetherian (resp. Artinian) if N satisfies the ascending (resp. descending) chain condition on ideals of N, that is, for any ideals I_1, I_2, I_3, ... of N, with

\[ I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I_i \subseteq \cdots \]

there exists n ∈ N such that I_i = I_{i+1} for all i ≥ n.

**Theorem 11.** If every intuitionistic fuzzy ideal of a ternary near-ring N has a finite image of values, then N is Noetherian.

**Proof.** Assume that every intuitionistic fuzzy ideal of a ternary near-ring N has the finite image of values. Suppose that N is not Noetherian, so there exists an ascending chain condition on ideals of N, that is, I_0 ⊆ I_1 ⊆ I_2 ⊆ \cdots. We define the intuitionistic fuzzy set A = (μ_A, λ_A) of N by

\[
\mu_A(x) = \begin{cases} 
\frac{1}{n + 1} & \text{if } x \in I_{n+1} - I_n; \\
1 & \text{if } x \in I_0; \\
0 & \text{if } x \in S - \bigcup_{n=0}^{\infty} I_n,
\end{cases}
\]
\[
\lambda_A(x) = \begin{cases} 
\frac{n}{n+1} & \text{if } x \in I_{n+1} - I_n; \\
\frac{n}{n} & \text{if } x \in I_0; \\
1 & \text{if } x \in S - \bigcup_{n=0}^{\infty} I_n,
\end{cases}
\]
for all \( x \in N \). It is not difficult to show that \( A = (\mu_A, \lambda_A) \) is an intuitionistic fuzzy ideal of \( N \). We have a contradiction because \( I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \) is an infinitely ascending chain of ideals of \( N \).

The proof of the following theorem is similar to that of Theorem [11]

**Theorem 12.** If every intuitionistic fuzzy ideal of a ternary near-ring \( N \) has a finite image of values, then \( N \) is Artinian.

**Proof.** Assume that \( N \) is Artinian. Suppose that \( A = (\mu_A, \lambda_A) \) is an intuitionistic fuzzy ideal of \( N \), which is not a well-ordered subset of \( [0, 1] \). Then, there exists an infinite descending sequence \( \{t_n\}_{n=1}^{\infty} \) such that \( \mu_A(x) = t_n \) and \( \lambda_A(x) \leq 1 - t_n \), for some \( x \in N \). We define \( I_n = \{ x \in N \mid \mu_A(x) \geq t_n \} \) and \( J_n = \{ x \in N \mid \lambda_A(x) \leq 1 - t_n \} \). By Theorem [11] \( I_n \) and \( J_n \) are ideals of \( N \), for all \( n \in \mathbb{N} \). It follows that \( I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \) and \( J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots \) are strictly infinite ascending chains of ideals of \( N \). This contradicts our hypothesis.

Conversely, suppose that \( N \) is not Noetherian. Then, there exists a strictly infinite ascending chain of ideals of \( N \), namely, \( I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \). Let \( I = \bigcup_{n \in \mathbb{N}} I_n \). It easy to show that \( I \) is ideal for \( N \). Next, we define the intuitionistic fuzzy set of \( A = (\mu_A, \lambda_A) \) of \( N \) by

\[
\mu_A(x) = \begin{cases} 
m & \text{if } m = \min\{n \in \mathbb{N} \mid x \in I_n\}; \\
0 & \text{if } x \not\in I;
\end{cases}
\]

\[
\lambda_A(x) = \begin{cases} 
\frac{m-1}{m+1} & \text{if } m = \min\{n \in \mathbb{N} \mid x \in I_n\}; \\
1 & \text{if } x \not\in I,
\end{cases}
\]

for all \( x \in N \). Obviously, \( A = (\mu_A, \lambda_A) \) is an intuitionistic fuzzy ideal of \( N \). Because the chain is not finite, \( A = (\mu_A, \lambda_A) \) has an infinite ascending sequence of values. This is a contradiction with the idea that the set of values of the intuitionistic fuzzy ideal is not a well-ordered subset of \( [0, 1] \).

**Theorem 14.** A ternary near-ring \( N \) is both Noetherian and Artinian if and only if every intuitionistic fuzzy ideal of \( N \) has a finite image of values.

**Proof.** Suppose that \( A = (\mu_A, \lambda_A) \) is an intuitionistic fuzzy ideal of \( N \) such that \( \text{Im}(\mu_A) \) and \( \text{Im}(\lambda_A) \) are finite. By Theorem [11] \( U(\mu_A; t_n) \) and \( L(\lambda_A; s_m) \) are ideals of \( N \), for all \( m, n \in \mathbb{N} \). Because \( N \) is Noetherian and by Theorem [13] \( \text{Im}(\mu_A) \) and \( \text{Im}(\lambda_A) \) are well-ordered subsets of \( [0, 1] \). Then, we can separate them into two cases, as follows.

**Case 1:** Let \( t_1 < t_2 < t_3 < \cdots \) be an increasing sequence in \( \text{Im}(\mu_A) \) and \( s_1 > s_2 > s_3 > \cdots \) be a decreasing sequence in \( \text{Im}(\lambda_A) \). It follows that \( U(\mu_A; t_1) \supset U(\mu_A; t_2) \supset U(\mu_A; t_3) \supset \cdots \) and \( L(\lambda_A; s_1) \supset L(\lambda_A; s_2) \supset L(\lambda_A; s_3) \supset \cdots \) are exactly descending chains of ideals of \( N \). Because \( N \) is Artinian, there exist \( i, j \in \mathbb{N} \) such that \( U(\mu_A; t_i) = U(\mu_A; t_{i+k}) \) and \( L(\lambda_A; s_j) = L(\lambda_A; s_{j+l}) \), for all \( k, l \in \mathbb{N} \). It turns out that \( t_i = t_{i+k} \) and \( s_j = s_{j+l} \), for all \( k, l \in \mathbb{N} \). This is a contradiction.

**Case 2:** Let \( t_1 > t_2 > t_3 > \cdots \) be a decreasing sequence, in \( \text{Im}(\mu_A) \) and \( s_1 < s_2 < s_3 < \cdots \) be an increasing sequence in \( \text{Im}(\lambda_A) \). It follows that \( U(\mu_A; t_1) \subset U(\mu_A; t_2) \subset U(\mu_A; t_3) \subset \cdots \) and \( L(\lambda_A; s_1) \subset L(\lambda_A; s_2) \subset L(\lambda_A; s_3) \subset \cdots \) are absolutely ascending chains of ideals of \( N \). Because \( N \) is Noetherian, there exist \( i, j \in \mathbb{N} \) such that \( U(\mu_A; t_i) = U(\mu_A; t_{i+k}) \) and \( L(\lambda_A; s_j) = L(\lambda_A; s_{j+l}) \), for all \( k, l \in \mathbb{N} \). It follows that \( t_i = t_{i+k} \) and \( s_j = s_{j+l} \), for all \( k, l \in \mathbb{N} \). We have a contradiction.

Conversely, it follows by Theorem [11] and Theorem [12] □

5. Conclusion

We introduced the concept of the intuitionistic fuzzy ideal in ternary near-rings as a generalization of their fuzzy ideals and studied some of their properties. We also presented the notions of Noetherian and Artinian ternary near-rings and characterized some of their properties using their intuitionistic fuzzy ideals. In the future, we would like to investigate some of the basic properties of the concepts of fuzzy quasi-ideals and fuzzy bi-ideals in ternary near-rings. Next, we will study the concepts of intuitionistic fuzzy quasi-ideals and intuitionistic fuzzy bi-ideals in ternary near-rings as generalizations of their fuzzy quasi-ideals and fuzzy bi-ideals, respectively.

**Conflict of Interest**

No potential conflict of interest relevant to this article is reported.
Acknowledgements

This research was financially supported by the Faculty of Science, Mahasarakham University.

References

[1] L. A. Zadeh, “Fuzzy sets,” Information and Control, vol. 8, no. 3, pp. 338-353, 1965. https://doi.org/10.1016/S0019-9958(65)90241-X

[2] A. Rosenfeld, “Fuzzy groups,” Journal of Mathematical Analysis and Applications, vol. 35, no. 3, pp. 512-517, 1971. https://doi.org/10.1016/0022-247x(71)90199-5

[3] W. J. Lui, “Fuzzy invariant subgroups and fuzzy ideals,” Fuzzy Sets and Systems, vol. 8, no. 2, pp. 133-139, 1982. https://doi.org/10.1016/0165-0114(82)90039-3

[4] V. N. Dixit, R. Kumar, and N. Ajal, “On fuzzy ring,” Fuzzy Sets and Systems, vol. 49, no. 2, pp. 205-213, 1992. https://doi.org/10.1016/0165-0114(92)90325-X

[5] G. Gratzer, General Lattice Theory. New York, NY: Academic Press, 1978.

[6] R. Kumar, “Fuzzy irreducible ideals in rings,” Fuzzy Sets and Systems, vol. 42, no. 3, pp. 369-379, 1991. https://doi.org/10.1016/0165-0114(91)90116-8

[7] S. Abou-Zaid, “On fuzzy subnear-rings,” Fuzzy Sets and Systems, vol. 44, no. 1, pp. 139-146, 1991. https://doi.org/10.1016/0165-0114(91)90039-s

[8] S. M. Hong and Y. B. Jun, “A note on fuzzy ideals in gamma-ring,” Bulletin of Honam Mathematical Society, vol. 12, pp. 39-48, 1995.

[9] Y. B. Jun, M. Sapanci, and M. A. Ozturk, “Fuzzy ideals in gamma near-rings,” Turkish Journal of Mathematics, vol. 22, no. 4, pp. 449-459, 1998.

[10] S. D. Kim and H. S. Kim, “On fuzzy ideals of near-rings,” Bulletin of the Korean Mathematical Society, vol. 33, no. 4, pp. 593-601, 1996.

[11] K. T. Atanassov, Intuitionistic Fuzzy Sets. Sofia, Bulgaria: Central Technical Library, Bulgarian Academy Science, 1983.

[12] K. T. Atanassov, “Intuitionistic fuzzy sets,” Fuzzy Sets and Systems, vol. 20, no. 1, pp. 87-96, 1986. https://doi.org/10.1016/S0165-0114(86)80034-3

[13] K. T. Atanassov, Intuitionistic Fuzzy Sets: Theory and Applications. Heidelberg, Germany: Physica-Verlag, 1999.

[14] T. C. Ahn, K. Hur, and K. W. Jang, “Intuitionistic fuzzy subgroups and level subgroups,” International Journal of Fuzzy Logic and Intelligent Systems, vol. 6, no. 3, pp. 240-246, 2006. https://doi.org/10.5391/IJFIS.2006.6.3.240

[15] K. Hur, S. Y. Jang, and H. W. Kang, “Intuitionistic fuzzy subgroupoids,” International Journal of Fuzzy Logic and Intelligent Systems, vol. 3, no. 1, pp. 72-77, 2003. https://doi.org/10.5391/IJFIS.2003.3.1.072

[16] K. Hur, S. Y. Jang, and P. K. Lim, “Intuitionistic fuzzy semigroups,” International Journal of Fuzzy Logic and Intelligent Systems, vol. 8, no. 3, pp. 207-219, 2008. https://doi.org/10.5391/IJFIS.2008.8.3.207

[17] K. Hur, H. W. Kang, and J. H. Ryou, “Union of intuitionistic fuzzy subgroups,” International Journal of Fuzzy Logic and Intelligent Systems, vol. 6, no. 1, pp. 85-93, 2006. https://doi.org/10.5391/IJFIS.2006.6.1.085

[18] K. Hur, S. R. Kim, and P. K. Lim, “Intuitionistic fuzzy k-ideals of a semiring,” International Journal of Fuzzy Logic and Intelligent Systems, vol. 9, no. 2, pp. 110-114, 2009. https://doi.org/10.5391/IJFIS.2009.9.2.110

[19] J. H. Park, “Operations on generalized intuitionistic fuzzy soft sets,” International Journal of Fuzzy Logic and Intelligent Systems, vol. 11, no. 3, pp. 184-189, 2011. https://doi.org/10.5391/IJFIS.2011.11.3.184

[20] R. Biswas, “On fuzzy sets and intuitionistic fuzzy sets,” Notes on Intuitionistic Fuzzy Sets, vol. 3, no. 1, pp. 3-11, 1997.

[21] K. H. Kim and Y. B. Jun, “Intuitionistic fuzzy ideals of semigroups,” Indian Journal of Pure and Applied Mathematics, vol. 33, no. 4, pp. 443-449, 2002.

[22] J. Zhan and X. Ma, “Intuitionistic fuzzy ideals of near-rings,” Scientiae Mathematicae Japonicae Online, vol. 2004, pp. 289-293, 2004.

[23] W. Nakhashen and B. Pibaljommee, “L-fuzzy ternary subnear-rings,” International Mathematical Forum, vol. 7, no. 41, pp. 2045-2059, 2012.
[24] A. Uma Maheswari and C. Meera, “On fuzzy soft right ternary near-rings,” *International Journal of Computer Applications*, vol. 57, no. 6, pp. 26-33, 2012.

[25] A. Uma Maheswari and C. Meera, “Fuzzy soft prime ideals over right ternary near-rings,” *International Journal of Pure and Applied Mathematics*, vol. 85, no. 3, pp. 507-529, 2013. [https://doi.org/10.12732/ijpam.v85i3.7](https://doi.org/10.12732/ijpam.v85i3.7)

[26] F. M. Sioson, “Ideal theory in ternary semigroups,” *Mathematica Japonica*, vol. 10, pp. 63-84, 1965.

[27] W. B. Vasantha Kandasamy, *Smarandache Fuzzy Algebra*. Rehoboth, NM: Bookman Pub, 2003.

[28] S. K. Sardar and S. K. Majumder, “Fuzzy magnified translation on groups,” *Journal of Mathematics*, vol. 1, no. 2, pp. 117-124, 2008.

**Warud Nakkhasen** received the Ph.D. in Mathematics from Khon Kaen University, Thailand in 2019. Currently, he is a lecturer at the Department of Mathematics, Faculty of Science, Mahasarakham University, Thailand. He research fields focus on algebraic structure, algebraic hyperstructure, fuzzy set theory, and intuitionistic fuzzy set theory.

E-mail: warud.n@msu.ac.th