A general perturbation theorem with applications to nonhomogeneous critical growth elliptic problems

Kanishka Perera
Department of Mathematical Sciences
Florida Institute of Technology
Melbourne, FL 32901, USA
kperera@fit.edu

Abstract

We prove a general perturbation theorem that can be used to obtain pairs of nontrivial solutions of a wide range of local and nonlocal nonhomogeneous elliptic problems. Applications to critical $p$-Laplacian problems, $p$-Laplacian problems with critical Hardy-Sobolev exponents, critical fractional $p$-Laplacian problems, and critical $(p, q)$-Laplacian problems are given. Our results are new even in the semilinear case $p = 2$.

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1 Introduction

In the pioneering paper [24], Tarantello showed that the problem
\[
\begin{aligned}
-\Delta u &= |u|^{2^* - 2} u + h(x) \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\] (1.1)
where \( \Omega \) is a bounded domain in \( \mathbb{R}^N, \ N \geq 3 \), and \( 2^* = 2N/(N - 2) \) is the critical Sobolev exponent, has two nontrivial solutions if \( h \in H^{-1}\setminus\{0\} \) satisfies
\[
\int_{\Omega} hu \, dx < \frac{4}{N - 2} \left( \frac{N - 2}{N + 2} \right)^{(N+2)/4} |\nabla u|^{(N+2)/2} \quad \text{for all } u \in H^1_0(\Omega) \quad \text{with } |u|^{2^*} = 1,
\]
where \( |\cdot| \) denotes the norm in \( L^p(\Omega) \). In particular, problem (1.1) has two nontrivial solutions for all \( h \in L^{2N/(N+2)}(\Omega) \setminus \{0\} \) sufficiently small. In [5], Cao and Zhou extended this result to the problem
\[
\begin{aligned}
-\Delta u &= \lambda u + |u|^{2^* - 2} u + h(x) \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\] (1.2)
for \( 0 < \lambda < \lambda_1 \), where \( \lambda_1 > 0 \) is the first Dirichlet eigenvalue of \( -\Delta \) in \( \Omega \). There is now a large literature generalizing these results (see, e.g., [2, 6, 7, 11, 15, 16, 23] and their references). However, the question of whether problem (1.2) still has two nontrivial solutions for all \( h \in L^{2N/(N+2)}(\Omega) \setminus \{0\} \) with \( |h|_{2N/(N+2)} \) sufficiently small when \( \lambda \geq \lambda_1 \) has remained open over the years. In the present paper we show that this is indeed the case when \( N = 4 \) and \( \lambda > \lambda_1 \) is not an eigenvalue, and when \( N \geq 5 \) and \( \lambda \geq \lambda_1 \). More specifically, we have the following theorem.

Theorem 1.1. There exists \( \mu_0 > 0 \) such that problem (1.2) has two nontrivial solutions for all \( h \in L^{2N/(N+2)}(\Omega) \setminus \{0\} \) with \( |h|_{2N/(N+2)} < \mu_0 \) in each of the following cases:

(i) \( N = 4 \) and \( \lambda > 0 \) is not an eigenvalue,

(ii) \( N \geq 5 \) and \( \lambda > 0 \).

We will in fact prove the corresponding result for the \( p \)-Laplacian. Consider the problem
\[
\begin{aligned}
-\Delta_p u &= \lambda |u|^{p^* - 2} u + |u|^{p^* - 2} u + h(x) \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\] (1.3)
where \( \Omega \) is a bounded domain in \( \mathbb{R}^N, \ N \geq 2 \), \( \Delta_p u = \text{div}(\nabla |u|^{p^* - 2} \nabla u) \) is the \( p \)-Laplacian of \( u, \ 1 < p < N, \ p^* = Np/(N - p) \) is the critical Sobolev exponent, \( \lambda > 0, \ h \in L^{p^*}(\Omega) \setminus \{0\} \), and \( p^* = p^*/(p^* - 1) \) is the Hölder conjugate of \( p^* \). We have the following theorem.

Theorem 1.2. There exists \( \mu_0 > 0 \) such that problem (1.3) has two nontrivial solutions for all \( h \in L^{p^*}(\Omega) \setminus \{0\} \) with \( |h|_{p^*} < \mu_0 \) in each of the following cases:
(i) \( N \geq p^2 \) and \( \lambda > 0 \) is not a Dirichlet eigenvalue of \(-\Delta_p\) in \( \Omega \),

(ii) \( N^2/(N+1) > p^2 \) and \( \lambda > 0 \).

Remark 1.3. When \( h = 0 \), one nontrivial solution of problem (1.3) was obtained by García Azorero and Peral Alonso [12], Egnell [9], Guedda and Véron [14], Arioli and Gazzola [1], and Degiovanni and Lancelotti [8].

We will first prove a general perturbation theorem that can be used to obtain pairs of nontrivial solutions of a wide range of local and nonlocal nonhomogeneous elliptic problems (see Theorem 2.1). We will then apply this result to prove Theorem 1.2 for a more general class of critical \( p \)-Laplacian problems (see Theorem 2.2). We also present applications of our general result to \( p \)-Laplacian problems with critical Hardy-Sobolev exponents (Theorem 2.4), critical fractional \( p \)-Laplacian problems (Theorem 2.7), and critical \((p, q)\)-Laplacian problems (Theorem 2.10).

Proof of Theorem 2.1 makes use of a certain linking structure associated with a sequence of eigenvalues introduced by the author in [18]. This sequence is defined using the genus does not provide this linking structure and therefore cannot be used to prove Theorem 2.1.

2 Statement of results

2.1 A general perturbation theorem

Let \((W, \| \cdot \|)\) be a uniformly convex Banach space with dual \((W^*, \| \cdot \|^*)\) and duality pairing \((\cdot, \cdot)\). Recall that \( f \in C(W, W^*) \) is a potential operator if there is a functional \( F \in C^1(W, \mathbb{R}) \), called a potential for \( f \), such that \( F' = f \). We consider the nonlinear operator equation

\[
A_p u = \lambda B_p u + f(u) + \mu g(u) + h
\]

in \( W^* \), where \( A_p, B_p, f, g \in C(W, W^*) \) are potential operators satisfying the following assumptions, \( \lambda > 0 \) and \( \mu \in \mathbb{R} \) are parameters, and \( h \in W^* \setminus \{0\} \): 

(A1) \( A_p \) is \((p - 1)\)-homogeneous and odd for some \( p \in (1, \infty) \): \( A_p \, (tu) = |t|^{p-2} t A_p \, u \) for all \( u \in W \) and \( t \in \mathbb{R} \),

(A2) \( (A_p \, u, v) \leq \|u\|^{p-1} \|v\| \) for all \( u, v \in W \), and equality holds if and only if \( \alpha u = \beta v \) for some \( \alpha, \beta \geq 0 \), not both zero (in particular, \( (A_p \, u, u) = \|u\|^p \) for all \( u \in W \)),

(B1) \( B_p \) is \((p - 1)\)-homogeneous and odd: \( B_p \, (tu) = |t|^{p-2} t B_p \, u \) for all \( u \in W \) and \( t \in \mathbb{R} \),

(B2) \( (B_p \, u, u) > 0 \) for all \( u \in W \setminus \{0\} \), and \( (B_p \, u, v) \leq (B_p \, u, u)^{(p-1)/p} (B_p \, v, v)^{1/p} \) for all \( u, v \in W \),

(B3) \( B_p \) is a compact operator,

(F1) the potential \( F \) of \( f \) with \( F(0) = 0 \) satisfies \( F(u) = o(\|u\|^p) \) as \( u \to 0 \),
(F_2) $F(u) \geq 0$ for all $u \in W$,

(F_3) $F$ is bounded on bounded subsets of $W$,

(G) the potential $G$ of $g$ with $G(0) = 0$ is bounded on bounded subsets of $W$.

Solutions of equation (2.1) coincide with critical points of the $C^1$-functional

$$E(u) = I_p(u) - \lambda J_p(u) - F(u) - \mu G(u) - (h, u), \quad u \in W,$$

where

$$I_p(u) = \frac{1}{p}(A_p u, u), \quad J_p(u) = \frac{1}{p}(B_p u, u)$$

are the potentials of $A_p$ and $B_p$ satisfying $I_p(0) = 0$ and $J_p(0) = 0$, respectively (see Perera [19, Proposition 3.1]). The nonlinear eigenvalue problem

$$A_p u = \lambda B_p u$$

will play a role in our result. Let $M = \{u \in W : I_p(u) = 1\}$. Then $M \subset W\setminus\{0\}$ is a bounded complete symmetric $C^1$-Finsler manifold radially homeomorphic to the unit sphere in $W$, and eigenvalues of problem (2.4) coincide with critical values of the $C^1$-functional

$$\Psi(u) = \frac{1}{J_p(u)}, \quad u \in M.$$

Denote by $\mathcal{F}$ the class of symmetric subsets of $M$ and by $i(M)$ the $\mathbb{Z}_2$-cohomological index of $M \in \mathcal{F}$ (see Fadell and Rabinowitz [10]), let $\mathcal{F}_k = \{M \in \mathcal{F} : i(M) \geq k\}$, and set

$$\lambda_k := \inf_{M \in \mathcal{F}_k} \sup_{u \in M} \Psi(u), \quad k \in \mathbb{N}.$$

Then $\lambda_1 > 0$ is the first eigenvalue and $\lambda_1 \leq \lambda_2 \leq \cdots$ is an unbounded sequence of eigenvalues. Moreover, denoting by $\Psi^a = \{u \in M : \Psi(u) \leq a\}$ (resp. $\Psi_a = \{u \in M : \Psi(u) \geq a\}$) the sublevel (resp. superlevel) sets of $\Psi$, if $\lambda_k < \lambda_{k+1}$, then

$$i(\Psi^{\lambda_k}) = i(M \setminus \Psi^{\lambda_{k+1}}) = k$$

and $\Psi^{\lambda_k}$ has a compact symmetric subset of index $k$ (see Perera et al. [20, Theorem 4.6] and Perera [19, Theorem 1.3]).

We assume that there is a threshold level $c^*_{\mu, h} > 0$ such that $E$ satisfies the $(PS)_c$ condition at all levels $c < c^*_{\mu, h}$. Set

$$c^* = \liminf_{\mu, \|h\| \to 0} c^*_{\mu, h}$$

and

$$E_0(u) = I_p(u) - \lambda J_p(u) - F(u), \quad u \in W,$$

Let $\pi_M : W \setminus \{0\} \to M$, $u \mapsto u/I_p(u)^{1/p}$ be the radial projection onto $M$. We will prove the following theorem.
Theorem 2.1. Let $\lambda_k \leq \lambda < \lambda_{k+1}$. Assume that there exist $R > 0$ and, for all sufficiently small $\delta > 0$, a compact symmetric subset $C_\delta$ of $\Psi^{k+\delta}$ with $i(C_\delta) = k$ and $w_\delta \in \mathcal{M} \setminus C_\delta$ such that, setting $A_\delta = \{ \pi_{\mathcal{M}}((1 - \tau)v + \tau w_\delta) : v \in C_\delta, 0 \leq \tau \leq 1 \}$, we have
\[
\sup_{u \in A_\delta} E_0(Ru) \leq 0 \tag{2.8}
\]
and
\[
\sup_{u \in A_\delta, 0 \leq t \leq R} E_0(tu) < c^*. \tag{2.9}
\]
Then $\exists \mu_0 > 0$ such that equation (2.1) has two nontrivial solutions $u_1$ and $u_2$ satisfying
\[
E(u_1) < E(u_2), \quad 0 < E(u_2) < c^*_{\mu, h} \tag{2.10}
\]
for all $\mu \in \mathbb{R}$ and $h \in W^* \setminus \{0\}$ with $|\mu| + \|h\|^* < \mu_0$.

Proof of this theorem and those of its applications that follow are given in Section 3.

2.2 Critical $p$-Laplacian problems

Consider the critical $p$-Laplacian problem
\[
\begin{aligned}
-\Delta_p u &= \lambda |u|^{p-2} u + \mu |u|^{q-2} u + |u|^{p^*-2} u + h(x) \quad \text{in } \Omega \\
0 &= \text{on } \partial \Omega,
\end{aligned} \tag{2.11}
\]
where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $1 < p < N$, $1 < q < p^*$, $p^* = Np/(N - p)$ is the critical Sobolev exponent, $\lambda > 0$, $\mu \in \mathbb{R}$, $h \in L^{p^*}(\Omega) \setminus \{0\}$, and $p^* = p^*/(p^* - 1)$ is the Hölder conjugate of $p^*$. Let
\[
E(u) = \int_\Omega \left( \frac{1}{p} |\nabla u|^p - \frac{\lambda}{p} |u|^p - \frac{\mu}{q} |u|^q - \frac{1}{p^*} |u|^{p^*} - h(x) u \right) dx, \quad u \in W_0^{1, p}(\Omega) \tag{2.12}
\]
be the associated variational functional and let
\[
S_{N, p} = \inf_{u \in W_0^{1, p}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p dx}{\left( \int_\Omega |u|^{p^*} dx \right)^{p/p^*}} \tag{2.13}
\]
be the best Sobolev constant. We have the following theorem.

Theorem 2.2. There exists $\mu_0 > 0$ such that problem (2.11) has two nontrivial solutions $u_1$ and $u_2$ satisfying
\[
E(u_1) < E(u_2), \quad 0 < E(u_2) < \frac{1}{N} S_{N, p}^{N/p} \tag{2.14}
\]
for all $\mu \in \mathbb{R}$ and $h \in L^{p^*}(\Omega) \setminus \{0\}$ with $|\mu| + \|h\| < \mu_0$ in each of the following cases:

(i) $N \geq p^2$ and $\lambda > 0$ is not a Dirichlet eigenvalue of $-\Delta_p$ in $\Omega$,

(ii) $N (N - p^2) > p^2$ and $\lambda > 0$.

We note that Theorem 2.2 allows the full subcritical range $1 < q < p^*$ for $q$ and makes no assumptions on the sign of $\mu$.

Remark 2.3. Theorem 1.2 is the special case $\mu = 0$ of Theorem 2.2.
2.3 $p$-Laplacian problems with critical Hardy-Sobolev exponents

Consider the problem

\[
\begin{cases}
-\Delta_p u = \lambda |u|^{p-2} u + \frac{|u|^{p^*(\sigma) - 2}}{|x|^\sigma} u + h(x) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ containing the origin, $1 < p < N$, $0 < \sigma < p$, $p^*(\sigma) = (N - \sigma)p/(N - p)$ is the critical Hardy-Sobolev exponent, $\lambda > 0$, $h \in L^{p^*(\sigma)'(\Omega)\setminus\{0\}}$, and $p^*(\sigma)' = p^*(\sigma)/(p^*(\sigma) - 1)$ is the Hölder conjugate of $p^*(\sigma)$. Let

\[
E(u) = \int_\Omega \left( \frac{1}{p} |\nabla u|^p - \frac{\lambda}{p} |u|^p - \frac{1}{p^*(\sigma)} \frac{|u|^{p^*(\sigma)}}{|x|^\sigma} - h(x) u \right) dx, \quad u \in W^{1,p}_0(\Omega)
\]

be the associated variational functional and let

\[
S_{N,p,\sigma} = \inf_{u \in W^{1,p}_0(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p dx}{\left( \int_\Omega \frac{|u|^{p^*(\sigma)}}{|x|^\sigma} dx \right)^{p/p^*(\sigma)}}
\]

be the best constant in the Hardy-Sobolev inequality. We have the following theorem.

**Theorem 2.4.** There exists $\mu_0 > 0$ such that problem (2.14) has two nontrivial solutions $u_1$ and $u_2$ satisfying

\[
E(u_1) < E(u_2), \quad 0 < E(u_2) < \frac{p - \sigma}{(N - \sigma) p} S_{N,p,\sigma}^{(N-\sigma)/(p-\sigma)}
\]

for all $h \in L^{p^*(\sigma)'}(\Omega)\setminus\{0\}$ with $|h|_{p^*(\sigma)'} < \mu_0$ in each of the following cases:

(i) $N \geq p^2$ and $\lambda > 0$ is not a Dirichlet eigenvalue of $-\Delta_p$ in $\Omega$,

(ii) $(N - \sigma)(N - p^2) > (p - \sigma) p$ and $\lambda > 0$.

**Remark 2.5.** When $h = 0$, one nontrivial solution was obtained by Ghoussoub and Yuan [13] and Perera and Zou [22].

**Remark 2.6.** Theorem 1.2 is the special case $\sigma = 0$ of Theorem 2.4.

2.4 Critical fractional $p$-Laplacian problems

Consider the critical fractional $p$-Laplacian problem

\[
\begin{cases}
(-\Delta)^s_p u = \lambda |u|^{p-2} u + |u|^{p^*-2} u + h(x) & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

(2.17)
where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, $(-\Delta)^s_p$ is the fractional $p$-Laplacian operator defined on smooth functions by

$$
(-\Delta)^s_p u(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} \, dy, \quad x \in \mathbb{R}^N,
$$

$s \in (0, 1)$, $1 < p < N/s$, $p^*_s = Np/(N - sp)$ is the fractional critical Sobolev exponent, $\lambda > 0$, $h \in L^{p^*_s'}(\Omega) \setminus \{0\}$, and $p''_s = p^*_s/(p^*_s - 1)$ is the Hölder conjugate of $p^*_s$. Let

$$
[u]_{s,p} = \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dxdy \right)^{1/p}
$$

be the Gagliardo seminorm of a measurable function $u : \mathbb{R}^N \to \mathbb{R}$ and let

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty \right\}
$$

be the fractional Sobolev space endowed with the norm

$$
\|u\|_{s,p} = \left( |u|^p_p + [u]_{s,p}^p \right)^{1/p}.
$$

We work in the closed linear subspace

$$W^{s,p}_0(\Omega) = \left\{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\},
$$

equivalently renormed by setting $\|\cdot\| = [\cdot]_{s,p}$. Let

$$
E(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dxdy - \int_{\Omega} \left( \frac{\lambda}{p} |u|^p + \frac{1}{p^*_s} |u|^{p^*_s} + h(x) u \right) \, dx,
$$

$u \in W^{s,p}_0(\Omega)$ \hspace{1cm} (2.18)

be the associated variational functional and let

$$
S_{N,p,s} = \inf_{u \in W^{s,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dxdy}{\left( \int_{\mathbb{R}^N} |u|^{p^*_s} \, dx \right)^{p/p^*_s}}
$$

(2.19)

be the best fractional Sobolev constant, where

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^{p^*_s}(\mathbb{R}^N) : [u]_{s,p} < \infty \right\}
$$

endowed with the norm $\|\cdot\|$. We have the following theorem.

**Theorem 2.7.** There exists $\mu_0 > 0$ such that problem \hspace{1cm} (2.17) \hspace{1cm} has two nontrivial solutions $u_1$ and $u_2$ satisfying

$$
E(u_1) < E(u_2), \quad 0 < E(u_2) < \frac{S}{N} S_{N,p,s}^{N/sp}
$$

for all $h \in L^{p^*_s}(\Omega) \setminus \{0\}$ with $|h|_{p^*_s} < \mu_0$ in each of the following cases:
(i) \( N \geq sp^2 \) and \( \lambda > 0 \) is not a Dirichlet eigenvalue of \((-\Delta)^s\) in \( \Omega \),

(ii) \( N \) \((N - sp^2) > s^2p^2 \) and \( \lambda > 0 \).

**Remark 2.8.** When \( h = 0 \), one nontrivial solution was obtained in Mosconi et al. [17] except when \( N = sp^2 \) and \( \lambda > \lambda_1 \) is not an eigenvalue in (i), where \( \lambda_1 > 0 \) is the first eigenvalue.

Theorem 2.7 is new even in the semilinear case \( p = 2 \) when \( \lambda \geq \lambda_1 \), which we state as the following corollary.

**Corollary 2.9.** There exists \( \mu_0 > 0 \) such that the problem

\[
\begin{cases}
(-\Delta)^s u = \lambda u + |u|^{2^* - 2} u + h(x) \quad &\text{in } \Omega \\
u = 0 \quad &\text{in } \mathbb{R}^N \setminus \Omega
\end{cases}
\]

has two nontrivial solutions for all \( h \in L^{2N/(N+2s)}(\Omega) \setminus \{0\} \) with \( |h|_{2N/(N+2s)} < \mu_0 \) for each of the following cases:

(i) \( N \geq 4s \) and \( \lambda > 0 \) is not a Dirichlet eigenvalue of \((-\Delta)^s\) in \( \Omega \),

(ii) \( N \) \((N - 4s) > 4s^2 \) and \( \lambda > 0 \).

**2.5 Critical \((p, q)\)-Laplacian problems**

Consider the critical \((p, q)\)-Laplacian problem

\[
\begin{cases}
-\Delta_p u - \mu \Delta_q u = \lambda |u|^{p^* - 2} u + |u|^{q^* - 2} u + h(x) \quad &\text{in } \Omega \\
u = 0 \quad &\text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( 1 < q < p < N \), \( p^* = Np/(N - p) \), \( \lambda, \mu > 0 \), \( h \in L^{p^*}(\Omega) \setminus \{0\} \), and \( q^* = p^*/(p^* - 1) \). Let

\[
E(u) = \int_\Omega \left( \frac{1}{p} |\nabla u|^p + \frac{\mu}{q} |\nabla u|^q - \frac{\lambda}{p} |u|^p - \frac{1}{p^*} |u|^{p^*} - h(x) u \right) \, dx, \quad u \in W^{1,p}(\Omega)
\]

be the associated variational functional and let \( S_{N,p} \) be as in (2.13). We have the following theorem.

**Theorem 2.10.** There exists \( \mu_0 > 0 \) such that problem (2.20) has two nontrivial solutions \( u_1 \) and \( u_2 \) satisfying

\[
E(u_1) < E(u_2), \quad 0 < E(u_2) < \frac{1}{N} S_{N,p}^{N/p}
\]

for all \( \mu > 0 \) and \( h \in L^{p^*}(\Omega) \setminus \{0\} \) with \( \mu + |h|_{p^*} < \mu_0 \) in each of the following cases:

(i) \( N \geq p^2 \) and \( \lambda > 0 \) is not a Dirichlet eigenvalue of \(-\Delta_p\) in \( \Omega \),

(ii) \( N \) \((N - p^2) > p^2 \) and \( \lambda > 0 \).

**Remark 2.11.** Theorem 1.2 is the special case \( \mu = 0 \) of Theorem 2.10.
3 Proofs

3.1 Proof of Theorem 2.1

Proof of Theorem 2.1 will be based on a special case of an abstract critical point theorem proved in Perera [19]. Let \( W \) be a Banach space and let \( M \) be a bounded symmetric subset of \( W \setminus \{0\} \) radially homeomorphic to the unit sphere \( S = \{u \in W : \|u\| = 1\} \), i.e., the restriction to \( M \) of the radial projection \( \pi : W \setminus \{0\} \to S, u \mapsto u/\|u\| \) is a homeomorphism. Then the radial projection from \( W \setminus \{0\} \) onto \( M \) is given by \( \pi_M = (\pi|_M)^{-1} \circ \pi \). For a symmetric set \( A \subset W \setminus \{0\} \), we denote by \( i(A) \) its \( \mathbb{Z}_2 \)-cohomological index (see Fadell and Rabinowitz [10]). The following theorem is the special case \( r = 0 \) of [19, Theorem 1.1].

**Theorem 3.1.** Let \( E \) be a \( C^1 \)-functional on \( W \) and let \( A_0 \) and \( B_0 \) be disjoint closed symmetric subsets of \( M \) such that

\[
i(A_0) = i(M \setminus B_0) = k < \infty. \tag{3.1}
\]

Assume that there exist \( w_0 \in M \setminus A_0 \), \( 0 < \rho < R \), and \( a < b \) such that, setting

\[
A_1 = \{\pi_M((1 - \tau)v + \tau w_0) : v \in A_0, 0 \leq \tau \leq 1\}, \quad A = \{tv : v \in A_0, 0 \leq t \leq R\} \cup \{Ru : u \in A_1\}, \quad B = \{\rho w : w \in B_0\}, \tag{3.2}
\]

\[
A^* = \{tu : u \in A_1, 0 \leq t \leq R\}, \quad B^* = \{tw : w \in B_0, 0 \leq t \leq \rho\}, \tag{3.3}
\]

we have

\[
a < \inf_{B^*} E, \quad \sup_{A} E < \inf_{B} E, \quad \sup_{A^*} E < b.
\]

If \( E \) satisfies the \((PS)_c\) condition for all \( c \in (a, b) \), then \( E \) has two critical points \( u_1 \) and \( u_2 \) with

\[
\inf_{B^*} E \leq E(u_1) \leq \sup_{A} E, \quad \inf_{B} E \leq E(u_2) \leq \sup_{A^*} E.
\]

We are now ready to prove Theorem 2.1.

**Proof of Theorem 2.1.** We apply Theorem 3.1 to the functional \( E \) defined in (2.2), taking \( A_0 = C_\delta, B_0 = \Psi_{\lambda_{k+1}}, w_0 = w_\delta \), and \( b = c_{\mu, k}^* \), where \( \delta \in (0, \lambda_{k+1} - \lambda) \) is to be chosen. Since \( A_0 \subset \Psi^{\lambda + \delta} \) and \( \lambda + \delta < \lambda_{k+1} \), \( A_0 \) and \( B_0 \) are disjoint. We have \( i(A_0) = k \) by assumption and \( i(M \setminus B_0) = k \) by (2.5), so (3.1) holds.

For \( u \in M \) and \( t > 0 \),

\[
E_0(tu) = t^p \left( 1 - \frac{\lambda}{\Psi(u)} \right) - F(tu). \tag{3.4}
\]

For \( w \in B_0 \), this together with \((F_1)\) gives

\[
E_0(tw) \geq t^p \left( 1 - \frac{\lambda}{\lambda_{k+1}} + o(1) \right) \quad \text{as } t \to 0.
\]
Since $\lambda < \lambda_{k+1}$, it follows from this that $\exists \rho \in (0, R)$ such that
\[
\inf_B E_0 > 0,
\tag{3.5}
\]
where $B$ is as in (3.2). For $v \in A_0$ and $0 \leq t \leq R$, (3.4) together with $(F_2)$ gives
\[
E_0(tv) \leq \frac{\delta R^p}{\lambda + \delta},
\tag{3.6}
\]
since $A_0 \subset \Psi^{\lambda+\delta}$. Fix $\delta$ so small that the right-hand side is less than $\inf_B E_0$. Then it follows from (3.6) and (2.8) that
\[
\sup_A E_0 \leq \frac{\delta R^p}{\lambda + \delta} < \inf_B E_0,
\tag{3.7}
\]
where $A$ is as in (3.2).

We have
\[
|E(u) - E_0(u)| \leq |\mu||G(u)| + \|h\|^* \|u\| \quad \forall u \in W.
\tag{3.8}
\]
Let $A^*$ and $B^*$ be as in (3.3). Since $A$, $B$, and $A^*$ are bounded and $G$ is bounded on bounded sets, it follows from (3.8), (3.5), (3.7), (2.6), and (2.9) that $\exists \mu_0 > 0$ such that
\[
\inf_B E > 0, \quad \sup_A E < \inf_B E, \quad \sup A^* E < c^*_{\mu, h},
\tag{3.9}
\]
for all $\mu \in \mathbb{R}$ and $h \in W^* \setminus \{0\}$ with $|\mu| + \|h\|^* < \mu_0$. Since $B^*$ is bounded and $F$ is also bounded on bounded sets,
\[
\inf_{B^*} E > -\infty.
\]

So we can apply Theorem 3.1 with $a < \inf E(B^*)$ to get two critical points $u_1$ and $u_2$ with
\[
\inf_{B^*} E \leq E(u_1) \leq \sup_A E, \quad \inf_{B^*} E \leq E(u_2) \leq \sup_{A^*} E.
\tag{3.10}
\]
The inequalities in (2.10) follow from (3.9) and (3.10).

### 3.2 Proof of Theorem 2.2

We prove Theorem 2.2 by applying Theorem 2.1 with $W = W^{1,p}_0(\Omega)$ and the operators $A_p, B_p, f, g \in C(W^{1,p}_0(\Omega), W^{-1,p'}(\Omega))$ and $h \in W^{-1,p'}(\Omega)$ given by
\[
(A_p u, v) = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx, \quad (B_p u, v) = \int_\Omega |u|^{p-2} uv \, dx,
\]
\[
(f(u), v) = \int_\Omega |u|^{p-2} uv \, dx, \quad (g(u), v) = \int_\Omega |u|^{q-2} uv \, dx, \quad u, v \in W^{1,p}_0(\Omega)
\]
and
\[
(h, v) = \int_\Omega h(x) \, v \, dx, \quad v \in W^{1,p}_0(\Omega).
\]

We begin by determining a threshold level below which the functional $E$ in (2.12) satisfies the (PS) condition.
Lemma 3.2. There exists \( \kappa > 0 \) such that \( E \) satisfies the \((PS)_c\) condition for all \( c < \frac{1}{N} S_{N,p}^{N/p} - \kappa \left( |\mu|^{p'/q'} + |h|^{|p'\star}| \right) \). (3.11)

Proof. Let \( c \in \mathbb{R} \) and let \((u_j)\) be a sequence in \( W^{1,p}_0(\Omega) \) such that

\[
E(u_j) = \int_{\Omega} \left( \frac{1}{p} |\nabla u_j|^p - \frac{\lambda}{p} |u_j|^p - \frac{\mu}{q} |u_j|^q - \frac{1}{p^\star} |u_j|^{p^\star} - h(x) u_j \right) \, dx = c + o(1) \quad (3.12)
\]

and

\[
(E'(u_j), v) = \int_{\Omega} \left( |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla v - \lambda |u_j|^{p-2} u_j v - \mu |u_j|^{q-2} u_j v - |u_j|^{p^\star-2} u_j v - h(x) v \right) \, dx = o(\|v\|) \quad \forall v \in W^{1,p}_0(\Omega). \quad (3.13)
\]

Taking \( v = u_j \) in (3.13) gives

\[
\int_{\Omega} \left( |\nabla u_j|^p - \lambda |u_j|^p - \mu |u_j|^q - |u_j|^{p^\star} - h(x) u_j \right) \, dx = o(\|u_j\|). \quad (3.14)
\]

Let \( r \in (p, p^\star) \). Dividing (3.14) by \( r \) and subtracting from (3.12) gives

\[
\int_{\Omega} \left[ \left( \frac{1}{p} - \frac{1}{r} \right) |\nabla u_j|^p - \lambda \left( \frac{1}{p} - \frac{1}{r} \right) |u_j|^p - \mu \left( \frac{1}{q} - \frac{1}{r} \right) |u_j|^q + \left( \frac{1}{r} - \frac{1}{p^\star} \right) |u_j|^{p^\star} \right. \\
- \left. \left( 1 - \frac{1}{r} \right) h(x) u_j \right] \, dx = c + o(1) + o(\|u_j\|),
\]

and it follows from this that \((u_j)\) is bounded. So a renamed subsequence converges to some \( u \) weakly in \( W^{1,p}_0(\Omega) \), strongly in \( L^t(\Omega) \) for all \( t \in [1, p^\star) \), and a.e. in \( \Omega \). Setting \( \tilde{u}_j = u_j - u \), we will show that \( \tilde{u}_j \to 0 \) in \( W^{1,p}_0(\Omega) \).

Equation (3.14) gives

\[
\|u_j\|^p = |u_j|^{p^\star}_p + \int_{\Omega} \left( \lambda |u|^p + \mu |u|^q + h(x) u \right) \, dx + o(1). \quad (3.15)
\]

Taking \( v = u \) in (3.13) and passing to the limit gives

\[
\|u\|^p = |u|^{p^\star}_p + \int_{\Omega} \left( \lambda |u|^p + \mu |u|^q + h(x) u \right) \, dx. \quad (3.16)
\]

Since

\[
\|\tilde{u}_j\|^p = \|u_j\|^p - \|u\|^p + o(1) \quad (3.17)
\]

and

\[
|\tilde{u}_j|^{p^\star}_p = |u_j|^{p^\star}_p - |u|^{p^\star}_p + o(1)
\]
by the Brézis-Lieb lemma \[4\], Theorem 1, \((3.15)\) and \((3.16)\) imply

\[
\|\tilde{u}_j\|^p = |\tilde{u}_j|_{p^*}^{p^*} + o(1) \leq \frac{\|\tilde{u}_j\|^{p^*}}{S_{N,p}^{p^*/p}} + o(1),
\]

so

\[
\|\tilde{u}_j\|^p \left( S_{N,p}^{N/(N-p)} - \|\tilde{u}_j\|^{p^2/(N-p)} \right) \leq o(1).
\]  \((3.18)\)

On the other hand, \((3.12)\) gives

\[
c = \frac{1}{p} \|u_j\|^p - \frac{1}{p^*} |u_j|_{p^*}^{p^*} - \int_{\Omega} \left( \frac{\lambda}{p} |u|^p + \frac{\mu}{q} |u|^q + h(x) u \right) \, dx + o(1),
\]

and a straightforward calculation combining this with \((3.13)\)–\((3.17)\) gives

\[
c = \frac{1}{N} \|\tilde{u}_j\|^p + \int_{\Omega} \left[ \frac{1}{N} |u|^{p^*} - \mu \left( \frac{1}{q} - \frac{1}{p} \right) |u|^q - \left( 1 - \frac{1}{p} \right) h(x) u \right] \, dx + o(1).
\]

The integral on the right-hand side is greater than or equal to

\[
\frac{1}{N} |u|^{p^*} - |\mu| \left( \frac{1}{q} - \frac{1}{p} \right) |\Omega|^{1-\frac{q}{p^*}} |u|^{p^*} - \left( 1 - \frac{1}{p} \right) |h|_{p^*} |u|_{p^*} \geq -\kappa \left( |\mu|^{(p^*/q)'} + |h|_{p^{'*}}^{p^{'*}} \right)
\]

for some $\kappa > 0$ by the Hölder and Young's inequalities, so

\[
\|\tilde{u}_j\|^p \leq N \left[ c + \kappa \left( |\mu|^{(p^*/q)'} + |h|_{p^{'*}}^{p^{'*}} \right) \right] + o(1).
\]

Combining this with \((3.18)\) shows that $\tilde{u}_j \to 0$ when \((3.11)\) holds.

We will apply Theorem \[2.1\] with

\[
c_{\mu,h}^* = \frac{1}{N} S_{N,p}^{N/p} - \kappa \left( |\mu|^{(p^*/q)'} + |h|_{p^{'*}}^{p^{'*}} \right),
\]

where $\kappa > 0$ is as in Lemma \[3.2\]. Note that

\[
\lim_{\mu,|h|_{p^{'*}} \to 0} c_{\mu,h}^* = \frac{1}{N} S_{N,p}^{N/p}.
\]

We have

\[
\mathcal{M} = \{ u \in W_0^{1,p}(\Omega) : \|u\|^p = p \},
\]

\[
\Psi(u) = \frac{p}{|u|_p^p}, \quad u \in \mathcal{M},
\]

\[
\pi_{\mathcal{M}}(u) = \frac{p^{1/p}}{\|u\|} \quad (u \in W_0^{1,p}(\Omega) \setminus \{0\}).
\]
and

\[ E_0(u) = \int_{\Omega} \left( \frac{1}{p} |\nabla u|^p - \frac{\lambda}{p} |u|^p - \frac{1}{p^*} |u|^{p^*} \right) \, dx, \quad u \in W_0^{1,p}(\Omega). \]

Let \( \lambda_k \leq \lambda < \lambda_{k+1} \). We need to show that there exist \( R > 0 \) and, for all sufficiently small \( \delta > 0 \), a compact symmetric subset \( C_\delta \) of \( \Psi^{\lambda+\delta} \) with \( i(C_\delta) = k \) and \( w_\delta \in M \setminus C_\delta \) such that, setting \( A_\delta = \{ \pi_M((1-\tau) v + \tau w_\delta) : v \in C_\delta, \ 0 \leq \tau \leq 1 \} \), we have

\[
\sup_{u \in A_\delta} E_0(Ru) \leq 0, \quad \sup_{u \in A_\delta, 0 \leq t \leq R} E_0(tu) < \frac{1}{N} S_{N,p}^{N/p}.
\]

(3.19)

Since \( \lambda_k < \lambda_{k+1} \), \( \Psi^{\lambda_k} \) has a compact symmetric subset \( C_0 \) of index \( k \) that is bounded in \( L^\infty(\Omega) \cap C^1_{loc}(\Omega) \) (see Degiovanni and Lancelotti [8, Theorem 2.3]). We may assume without loss of generality that \( 0 \in \Omega \). Let \( \rho_0 = \text{dist} (0, \partial \Omega) \), let \( \eta : [0, \infty) \to [0, 1] \) be a smooth function such that \( \eta(t) = 0 \) for \( t \leq 3/4 \) and \( \eta(t) = 1 \) for \( t \geq 1 \), let

\[ u_\rho(x) = \eta \left( \frac{|x|}{\rho} \right) u(x), \quad u \in C_0, \ 0 < \rho \leq \rho_0/2, \]

and let

\[ C = \{ \pi_M(u_\rho) : u \in C_0 \}. \]

**Lemma 3.3.** The set \( C \) is a compact symmetric subset of \( \Psi^{\lambda_k + c_1 \rho^{N-p}} \) for some constant \( c_1 > 0 \). If \( \lambda_k + c_1 \rho^{N-p} < \lambda_{k+1} \), then \( i(C) = k \).

**Proof.** Let \( u \in C_0 \). Since functions in \( C_0 \) are bounded in \( C^1(B_{\rho_0/2}(0)) \) and belong to \( \Psi^{\lambda_k} \),

\[ \int_{\Omega} |\nabla u_\rho|^p \, dx \leq \int_{\Omega \setminus B_\rho(0)} |\nabla u|^p \, dx + \int_{B_\rho(0)} \left( |\nabla u| + \frac{|\eta'| |u|}{\rho} \right)^p \, dx \leq p + c_2 \rho^{N-p} \]

and

\[ \int_{\Omega} |u_\rho|^p \, dx \geq \int_{\Omega \setminus B_\rho(0)} |u|^p \, dx = \int_{\Omega} |u|^p \, dx - \int_{B_\rho(0)} |u|^p \, dx \geq \frac{p}{\lambda_k} - c_3 \rho^N \]

for some constants \( c_2, c_3 > 0 \). So

\[ \Psi(\pi_M(u_\rho)) = \frac{\int_{\Omega} |\nabla u_\rho|^p \, dx}{\int_{\Omega} |u_\rho|^p \, dx} \leq \lambda_k + c_1 \rho^{N-p} \]

for some constant \( c_1 > 0 \). Then \( C \subset \Psi^{\lambda_k + c_1 \rho^{N-p}} \). Since \( C_0 \) is a compact symmetric set and \( u \mapsto \pi_M(u_\rho) \) is an odd continuous map of \( C_0 \) onto \( C \), \( C \) is also a compact symmetric set and

\[ i(C) \geq i(C_0) = k \]

by the monotonicity of the index. If \( \lambda_k + c_1 \rho^{N-p} < \lambda_{k+1} \), then \( C \subset M \setminus \Psi^{\lambda_{k+1}} \) and hence

\[ i(C) \leq i(M \setminus \Psi^{\lambda_{k+1}}) = k \]

by [25], so \( i(C) = k \). \( \square \)
Lemma 3.4. For any \( w \in \mathcal{M} \setminus C \) with support in \( \overline{B_{\rho/2}(0)} \), \( \exists R > 0 \) such that, setting \( A = \{ \pi_\mathcal{M}((1 - \tau) v + \tau w) : v \in C, 0 \leq \tau \leq 1 \} \), we have

\[
\sup_{u \in A} E_0(Ru) \leq 0.
\]

Proof. Let \( u = \pi_\mathcal{M}((1 - \tau) v + \tau w) \in A \). For \( R > 0 \),

\[
E_0(Ru) \leq \int_\Omega \left( \frac{R^p}{p} |\nabla u|^p - \frac{R^p}{p^*} |u|^{p^*} \right) dx = R^p - \frac{R^p}{p^*} |u|^{p^*},
\]

so it suffices to show that \( |u|^{p^*} \) is bounded away from zero on \( A \). By the Hölder inequality, it is enough to show that \( |u|_p \) is bounded away from zero. Since \( v, w \in \mathcal{M} \) have disjoint supports,

\[
|u|_p^p = \frac{p}{p} \frac{1}{(1 - \tau) v + \tau w} \left( \frac{p}{p} \frac{1}{(1 - \tau) v + \tau w} \right) = \frac{p}{p} \frac{1}{(1 - \tau) v + \tau w} \left( \frac{p}{p} \frac{1}{(1 - \tau) v + \tau w} \right) = \frac{p}{p} \frac{1}{(1 - \tau) v + \tau w} \left( \frac{p}{p} \frac{1}{(1 - \tau) v + \tau w} \right) \geq \min \left\{ |v|_p^p, |w|_p^p \right\},
\]

so it suffices to show that \( |v|_p \) is bounded away from zero on \( C \). Since \( C \subset \Psi^{\lambda + c_1 \rho N - p} \) by Lemma 3.3, we have

\[
|v|_p^p = \frac{p}{\Psi(v)} \geq \frac{p}{\lambda_k + c_1 \rho N - p}. \quad \square
\]

Let \( \delta \in (0, \lambda_{k+1} - \lambda) \), let \( \rho \in (0, \rho_0/2] \) be so small that \( \lambda_k + c_1 \rho N - p < \lambda + \delta \), and let \( C_\delta = C \). Then \( C_\delta \) is a compact symmetric subset of \( \Psi^{\lambda + \delta} \) with \( i(C_\delta) = k \) by Lemma 3.3. We will show that if \( \delta > 0 \) is sufficiently small, then \( \exists w_\delta \in \mathcal{M} \setminus C_\delta \) with support in \( \overline{B_{\rho/2}(0)} \) such that, setting \( A_\delta = \{ \pi_\mathcal{M}((1 - \tau) v + \tau w_\delta) : v \in C_\delta, 0 \leq \tau \leq 1 \} \), we have

\[
\sup_{u \in A_\delta, t \geq 0} E_0(tu) < \frac{1}{N} S^{N/p}_{N,p}. \tag{3.20}
\]

Then Lemma 3.3 will give an \( R > 0 \) such that (3.19) holds and complete the proof. We note that (3.20) is equivalent to

\[
\sup_{u \in C_\delta, t \geq 0} E_0(tu + \tau w_\delta) < \frac{1}{N} S^{N/p}_{N,p}. \tag{3.21}
\]

To choose \( w_\delta \), recall that the infimum in (2.13) is attained by the Aubin-Talenti functions

\[
u_\varepsilon(x) = \frac{c_{N,p} \varepsilon(N-p)/p^2}{(\varepsilon + |x|p/(p-1))^{(N-p)/p}}, \quad \varepsilon > 0
\]

when \( \Omega = \mathbb{R}^N \), where the constant \( c_{N,p} > 0 \) is chosen so that

\[
\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^p dx = \int_{\mathbb{R}^N} u_\varepsilon^{p^*} dx = S^{N/p}_{N,p}.
\]
Let \( \zeta : [0, \infty) \to [0,1] \) be a smooth function such that \( \zeta(t) = 1 \) for \( t \leq 1/4 \) and \( \zeta(t) = 0 \) for \( t \geq 1/2 \), and let

\[
u_{\varepsilon, \rho}(x) = \zeta \left( \frac{|x|}{\rho} \right) u_{\varepsilon}(x), \quad w_{\varepsilon, \rho}(x) = \frac{u_{\varepsilon, \rho}(x)}{\left( \int_{\mathbb{R}^N} w_{\varepsilon, \rho}^{\ast} \, dx \right)^{1/p}}, \quad 0 < \rho \leq \rho_0/2.
\]

Then

\[
\int_{\mathbb{R}^N} w_{\varepsilon, \rho}^{\ast} \, dx = 1 \tag{3.22}
\]

and we have

\[
\int_{\mathbb{R}^N} |\nabla w_{\varepsilon, \rho}|^p \, dx \leq S_{N,p} c_4 \varepsilon^{(N-p)/p} \rho^{-(N-p)/(p-1)}, \tag{3.23}
\]

\[
\int_{\mathbb{R}^N} w_{\varepsilon, \rho}^{\ast} \, dx \geq \begin{cases} c_5 \varepsilon^{p-1} & \text{if } N > p^2 \\ c_5 \varepsilon^{p-1} \log \left( \varepsilon \rho^{p/(p-1)} \right) & \text{if } N = p^2 \end{cases} \tag{3.24}
\]

for some constants \( c_4, c_5 > 0 \) (see, e.g., Perera and Zou [22]). Let

\[w_\delta = \pi_{M}(w_{\varepsilon, \rho}).\]

Since functions in \( C_\delta \) have their supports in \( \Omega \setminus B_{3\rho/4}(0) \), while the support of \( w_\delta \) is in \( \overline{B_{\rho/2}(0)} \), \( w_\delta \in M \setminus C_\delta \). We will show that (3.21) holds if \( \varepsilon, \rho > 0 \) are sufficiently small.

Inequality (3.21) is equivalent to

\[
\sup_{v \in C_\delta, t, \tau \geq 0} E_0(tv + \tau w_{\varepsilon, \rho}) < \frac{1}{N} S_{N,p}^{N/p}. \tag{3.25}
\]

For \( v \in C_\delta \) and \( t, \tau \geq 0 \),

\[
E_0(tv + \tau w_{\varepsilon, \rho}) = E_0(tv) + E_0(\tau w_{\varepsilon, \rho})
\]

since \( v \) and \( w_{\varepsilon, \rho} \) have disjoint supports. So

\[
\sup_{v \in C_\delta, t, \tau \geq 0} E_0(tv + \tau w_{\varepsilon, \rho}) = \sup_{v \in C_\delta, t \geq 0} E_0(tv) + \sup_{\tau \geq 0} E_0(\tau w_{\varepsilon, \rho}). \tag{3.26}
\]

**Lemma 3.5.** We have

\[
\sup_{v \in C_\delta, t \geq 0} E_0(tv) \leq \begin{cases} 0 & \text{if } \lambda_k + c_1 \rho^{N-p} \leq \lambda < \lambda_{k+1} \\ c_6 \rho^{N(N-p)/p} & \text{if } \lambda = \lambda_k, \end{cases}
\]

where \( c_1 \) is as in Lemma 3.3 and \( c_6 > 0 \) is a constant.
Proof. For \( v \in C_\delta \) and \( t \geq 0 \),
\[
E_0(tv) = \frac{t^p}{p} \int_{\Omega} (|\nabla v|^p - \lambda |v|^p) \, dx - \frac{t^{p^*}}{p^*} \int_{\Omega} |v|^{p^*} \, dx,
\]
and
\[
\frac{1}{p} \int_{\Omega} (|\nabla v|^p - \lambda |v|^p) \, dx = 1 - \frac{\lambda}{\Psi(v)} \leq 1 - \frac{\lambda}{\lambda_k + c_1 \rho^{N-p}}\tag{3.27}
\]
since \( C_\delta \subset \Psi^{\lambda_k + c_1 \rho^{N-p}} \) by Lemma 3.3. So \( E_0(tv) \leq 0 \) if \( \lambda_k + c_1 \rho^{N-p} \leq \lambda < \lambda_{k+1} \). If \( \lambda = \lambda_k \), then
\[
\frac{1}{p} \int_{\Omega} (|\nabla v|^p - \lambda |v|^p) \, dx \leq \frac{c_1 \rho^{N-p}}{\lambda_k + c_1 \rho^{N-p}} \leq c_7 \rho^{N-p},\tag{3.28}
\]
where \( c_7 = c_1/\lambda_k > 0 \), and
\[
\frac{1}{p^*} \int_{\Omega} |v|^{p^*} \, dx \geq c_8
\]
for some constant \( c_8 > 0 \) as in the proof of Lemma 3.4, so
\[
E_0(tv) \leq c_7 \rho^{N-p} t^p - c_8 t^{p^*}
\]
and maximizing the right-hand side over all \( t \geq 0 \) gives the desired conclusion. \( \square \)

**Lemma 3.6.** We have
\[
\sup_{\tau \geq 0} E_0(\tau w_{\varepsilon,\rho}) \leq \begin{cases} 
\frac{1}{N} \left[ S_{N,p} + c_4 \varepsilon^{(N-p)/p} \rho^{-(N-p)/(p-1)} - \lambda c_5 \varepsilon^{p-1} \right]^{N/p} & \text{if } N > p^2 \\
\frac{1}{N} \left[ S_{N,p} + c_4 \varepsilon^{p-1} \rho^{-p} - \lambda c_5 \varepsilon^{p-1} \left| \log \left( \varepsilon \rho^{-p}/(p-1) \right) \right| \right]^{N/p} & \text{if } N = p^2.
\end{cases}
\]

**Proof.** We have
\[
E_0(\tau w_{\varepsilon,\rho}) = \frac{\tau^p}{p} \int_{\Omega} (|\nabla w|_{\varepsilon,\rho}|^p - \lambda w_{\varepsilon,\rho}^p) \, dx - \frac{\tau^{p^*}}{p^*}
\]
by (3.22), and maximizing the right-hand side over all \( \tau \geq 0 \) gives
\[
\sup_{\tau \geq 0} E_0(\tau w_{\varepsilon,\rho}) = \frac{1}{N} \left[ \int_{\Omega} (|\nabla w_{\varepsilon,\rho}|^p - \lambda w_{\varepsilon,\rho}^p) \, dx \right]^{N/p},
\]
so the desired conclusion follows from (3.23) and (3.24). \( \square \)

We can now complete the proof of Theorem 2.2. First suppose \( N \geq p^2 \) and \( \lambda > \lambda_1 \) is not an eigenvalue. Then \( \lambda_k < \lambda < \lambda_{k+1} \) for some \( k \in \mathbb{N} \). Let \( \rho \in (0, \rho_0/2] \) be so small that \( \lambda_k + c_1 \rho^{N-p} \leq \lambda \). Then (3.25) follows from (3.26), Lemma 3.5 and Lemma 3.6 for sufficiently small \( \varepsilon > 0 \).
Now suppose $N(N-p^2) > p^2$ and $\lambda \geq \lambda_1$. Then $\lambda_k \leq \lambda < \lambda_{k+1}$ for some $k \in \mathbb{N}$. We have already considered the case where $N > p^2$ and $\lambda_k < \lambda < \lambda_{k+1}$, so suppose $\lambda = \lambda_k$. Then

$$\sup_{v \in C_k, t, \tau \geq 0} E_0(tv + \tau w_{\varepsilon, \rho}) \leq \frac{1}{N} \left[ S_{N, p} + c_4 \varepsilon^{(N-p)/p} \rho^{-(N-p)/(p-1)} - \lambda c_5 \varepsilon^{p-1} \right]^{N/p} + c_6 \rho^{N(N-p)/p}$$

by (3.26), Lemma 3.5, and Lemma 3.6. Set $\rho = \varepsilon^\alpha$, where $\alpha > 0$ is to be chosen. Then the right-hand side is less than or equal to

$$\frac{1}{N} S_{N, p}^{N/p} \left[ 1 + c_9 \varepsilon^{(N-p)[1/p - \alpha/(p-1)]} - c_{10} \varepsilon^{p-1} \right]^{N/p} + c_6 \varepsilon^\alpha N(N-p)/p$$

for some constants $c_9, c_{10} > 0$, so (3.25) will follow for sufficiently small $\varepsilon > 0$ if $\alpha$ can be found so that

$$(N-p)[1/p - \alpha/(p-1)] > p - 1$$

and

$$\alpha N(N-p)/p > p - 1.$$ 

This is possible if and only if

$$(p-1)p/N(N-p) < (p-1)[1/p - (p-1)/(N-p)],$$

i.e.,

$$N(N-p^2) > p^2. \quad \Box$$

### 3.3 Proof of Theorem 2.4

We prove Theorem 2.4 by applying Theorem 2.1 with $W = W_{0}^{1,p}(\Omega)$, the operators $A_p, B_p, f \in C(W_{0}^{1,p}(\Omega), W^{-1,p}(\Omega))$ and $h \in W^{-1,p}(\Omega)$ given by

$$(A_p u, v) = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx, \quad (B_p u, v) = \int_{\Omega} |u|^{p-2} uv \, dx,$$

$$(f(u), v) = \int_{\Omega} \frac{|u|^{p\sigma}\gamma-2}{|x|^{\sigma}} \, uv \, dx, \quad u, v \in W_{0}^{1,p}(\Omega)$$

and

$$(h, v) = \int_{\Omega} h(x) v \, dx, \quad v \in W_{0}^{1,p}(\Omega),$$

and $g = 0$. The proof is similar to that of Theorem 2.2 so we will be sketchy.

**Lemma 3.7.** There exists $\kappa > 0$ such that the functional $E$ in (2.15) satisfies the $(\text{PS})_c$ condition for all

$$c < \frac{p - \sigma}{(N - \sigma)p} S_{N, p, \sigma}^{(N-\sigma)/(p-\sigma)} - \kappa |h|_{p^\sigma}^\gamma.$$  \hspace{1cm} (3.29)
Proof. Let \( c \in \mathbb{R} \) and let \( (u_j) \) be a sequence in \( W_0^{1,p}(\Omega) \) such that
\[
E(u_j) = \int_{\Omega} \left( \frac{1}{p} |\nabla u_j|^p - \frac{\lambda}{p} |u_j|^p - \frac{1}{p^*(\sigma)} \frac{|u_j|^{p^*(\sigma)}}{|x|^\sigma} - h(x) u_j \right) \, dx = c + o(1) \tag{3.30}
\]
and
\[
(E'(u_j), v) = \int_{\Omega} \left( |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla v - \lambda |u_j|^{p-2} u_j v - \frac{|u_j|^{p^*(\sigma)-2}}{|x|^\sigma} u_j v - h(x) v \right) \, dx
= o(\|v\|) \quad \forall v \in W_0^{1,p}(\Omega). \tag{3.31}
\]
Taking \( v = u_j \) in (3.31) gives
\[
\int_{\Omega} \left( |\nabla u_j|^p - \lambda |u_j|^p - \frac{|u_j|^{p^*(\sigma)}}{|x|^\sigma} - h(x) u_j \right) \, dx = o(\|u_j\|). \tag{3.32}
\]
Let \( r \in (p, p^*(\sigma)) \). Dividing (3.32) by \( r \) and subtracting from (3.30) gives
\[
\int_{\Omega} \left[ \left( 1 - \frac{1}{p} \right) |\nabla u_j|^p - \lambda \left( 1 - \frac{1}{p} \right) |u_j|^p + \left( 1 - \frac{1}{p^*(\sigma)} \right) \frac{|u_j|^{p^*(\sigma)}}{|x|^\sigma} - \left( 1 - \frac{1}{r} \right) h(x) u_j \right] \, dx
= c + o(1) + o(\|u_j\|),
\]
and it follows from this that \( (u_j) \) is bounded. So a renamed subsequence converges to some \( u \) weakly in \( W_0^{1,p}(\Omega) \), strongly in \( L^p(\Omega) \) for all \( t \in [1, p^*) \), and a.e. in \( \Omega \). Setting \( \tilde{u}_j = u_j - u \), we will show that \( \tilde{u}_j \to 0 \) in \( W_0^{1,p}(\Omega) \).

Equation (3.32) gives
\[
\|u_j\|^p = \int_{\Omega} \frac{|u_j|^{p^*(\sigma)}}{|x|^\sigma} \, dx + \int_{\Omega} (\lambda |u|^p + h(x) u) \, dx + o(1). \tag{3.33}
\]
Taking \( v = u \) in (3.31) and passing to the limit gives
\[
\|u\|^p = \int_{\Omega} \frac{|u|^{p^*(\sigma)}}{|x|^\sigma} \, dx + \int_{\Omega} (\lambda |u|^p + h(x) u) \, dx. \tag{3.34}
\]
Since
\[
\|\tilde{u}_j\|^p = \|u_j\|^p - \|u\|^p + o(1) \tag{3.35}
\]
by the Brézis-Lieb lemma [4, Theorem 1] and
\[
\int_{\Omega} |\nabla \tilde{u}_j|^{p^*(\sigma)} \, dx = \int_{\Omega} |u_j|^{p^*(\sigma)} \, dx - \int_{\Omega} |u|^{p^*(\sigma)} \, dx + o(1)
\]
by Ghoussoub and Yuan [13, Lemma 4.3], (3.33) and (3.34) imply
\[
\|\tilde{u}_j\|^p = \int_{\Omega} \frac{|\tilde{u}_j|^{p^*(\sigma)}}{|x|^\sigma} \, dx + o(1) \leq \frac{\|\tilde{u}_j\|^{p^*(\sigma)}}{S_{N,p,\sigma}^{p^*(\sigma)/p}} + o(1),
\]

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so
\[
\| \tilde{u}_j \|^p \left( S_{N, p, \sigma}^{(N-\sigma)/(N-p)} - \| \tilde{u}_j \|^{(p-\sigma)p/(N-p)} \right) \leq o(1).
\]

(3.36)

On the other hand, (3.30) gives
\[
c = 1 \frac{\| u_j \|^p - 1}{p^*(\sigma)} \int_{\Omega} \frac{|u_j| p^*(\sigma)}{|x|^\sigma} \, dx - \int_{\Omega} \left( \frac{\lambda}{p} |u|^p + h(x) u \right) \, dx + o(1),
\]
and a straightforward calculation combining this with (3.33)–(3.35) gives
\[
c = \frac{p - \sigma}{(N - \sigma)p} \| \tilde{u}_j \|^p + \int_{\Omega} \left[ \frac{p - \sigma}{(N - \sigma)p} \frac{|u| p^*(\sigma)}{|x|^\sigma} \right. - \left. \left( 1 - \frac{1}{p} \right) h(x) u \right] \, dx + o(1).
\]

The integral on the right-hand side is greater than or equal to
\[
\frac{p - \sigma}{(N - \sigma)p} \int_{\Omega} \frac{|u| p^*(\sigma)}{|x|^\sigma} \, dx - \left( 1 - \frac{1}{p} \right) \left( \int_{\Omega} \frac{|u| p^*(\sigma)}{|x|^\sigma} \, dx \right)^{1/p^*(\sigma)} \geq -\kappa |h|_{p^*(\sigma)'}^{p^*(\sigma)'}
\]
for some \( \kappa > 0 \) by the Hölder and Young’s inequalities, so
\[
\| \tilde{u}_j \|^p \leq \frac{(N - \sigma)p}{p - \sigma} \left( c + \kappa |h|_{p^*(\sigma)'}^{p^*(\sigma)'} \right) + o(1).
\]

Combining this with (3.36) shows that \( \tilde{u}_j \to 0 \) when (3.29) holds.

We will apply Theorem 2.1 with
\[
c^*_{\mu, h} = \frac{p - \sigma}{(N - \sigma)p} S_{N, p, \sigma}^{(N-\sigma)/(p-\sigma)} - \kappa |h|_{p^*(\sigma)'}^{p^*(\sigma)'}
\]
where \( \kappa > 0 \) is as in Lemma 3.7, noting that
\[
\lim_{|h|_{p^*(\sigma)'} \to 0} c^*_{\mu, h} = \frac{p - \sigma}{(N - \sigma)p} S_{N, p, \sigma}^{(N-\sigma)/(p-\sigma)}.
\]

We have
\[
\mathcal{M} = \{ u \in W_0^{1,p}(\Omega) : \| u \|^p = p \},
\]
\[
\Psi(u) = \frac{p}{\| u \|^p}, \quad u \in \mathcal{M},
\]
\[
\pi_{\mathcal{M}}(u) = \frac{p^{1/p} u}{\| u \|}, \quad u \in W_0^{1,p}(\Omega) \setminus \{0\}
\]
Let \( \rho \leq \lambda < \lambda_{k+1} \). We need to show that there exist \( R > 0 \) and, for all sufficiently small \( \delta > 0 \), a compact symmetric subset \( C_\delta \) of \( \Psi^{1+\delta} \) with \( i(C_\delta) = k \) and \( w_\delta \in \mathcal{M} \setminus C_\delta \) such that, setting \( A_\delta = \{ \pi_M((1 - \tau) v + \tau w_\delta) : v \in C_\delta, 0 \leq \tau \leq 1 \} \), we have

\[
\sup_{u \in A_\delta} E_0(Ru) \leq 0, \quad \sup_{u \in A_\delta, 0 \leq \tau \leq R} E_0(tu) < \frac{p - \sigma}{(N - \sigma) p} S_{N, p, \sigma}^{(N - \sigma)/(p - \sigma)}. \tag{3.37}
\]

Let \( \rho_0 = \text{dist } (0, \partial \Omega) \), let \( 0 < \rho \leq \rho_0/2 \), and let \( C \) be as in the proof of Theorem 2.2.

**Lemma 3.8.** For any \( w \in \mathcal{M} \setminus C \) with support in \( \overline{B_{\rho/2}(0)} \), \( \exists R > 0 \) such that, setting \( A = \{ \pi_M((1 - \tau) v + \tau w) : v \in C, 0 \leq \tau \leq 1 \} \), we have

\[
\sup_{u \in A} E_0(Ru) \leq 0.
\]

**Proof.** Let \( u = \pi_M((1 - \tau) v + \tau w) \in A \). For \( R > 0 \),

\[
E_0(Ru) \leq \int_\Omega \left( \frac{R^p}{p} |\nabla u|^p - \frac{R^{p^*}(\sigma)}{p^*(\sigma)} \left| \frac{|u|^{p^*(\sigma)}}{|x|^\sigma} \right| \right) dx = R^p - \frac{R^{p^*(\sigma)}}{p^*(\sigma)} \int_\Omega \left| \frac{|u|^{p^*(\sigma)}}{|x|^\sigma} \right| dx,
\]

so it suffices to show that the last integral is bounded away from zero on \( A \). By the Hölder inequality,

\[
\int_\Omega |u|^p dx \leq \left( \int_\Omega |x|^{p/(p^*(\sigma) - p)} dx \right)^{1 - p/p^*(\sigma)} \left( \int_\Omega \left| \frac{|u|^{p^*(\sigma)}}{|x|^\sigma} \right| dx \right)^{p/p^*(\sigma)},
\]

and \( |u|_p \) is bounded away from zero as in the proof of Lemma 3.4, so the desired conclusion follows.

Let \( \delta \in (0, \lambda_{k+1} - \lambda) \), let \( \rho \in (0, \rho_0/2) \) be so small that \( \lambda_k + c_1 \rho^{N-p} < \lambda + \delta \), and let \( C_\delta = C \). Then \( C_\delta \) is a compact symmetric subset of \( \Psi^{1+\delta} \) with \( i(C_\delta) = k \) by Lemma 3.3. We will show that if \( \delta > 0 \) is sufficiently small, then \( \exists w_\delta \in \mathcal{M} \setminus C_\delta \) with support in \( \overline{B_{\rho/2}(0)} \) such that, setting \( A_\delta = \{ \pi_M((1 - \tau) v + \tau w_\delta) : v \in C_\delta, 0 \leq \tau \leq 1 \} \), we have

\[
\sup_{u \in A_\delta, t \geq 0} E_0(tu) < \frac{p - \sigma}{(N - \sigma) p} S_{N, p, \sigma}^{(N - \sigma)/(p - \sigma)}. \tag{3.38}
\]

Then Lemma 3.8 will give an \( R > 0 \) such that (3.37) holds and complete the proof. We note that (3.38) is equivalent to

\[
\sup_{u \in C_\delta, t \geq 0} E_0(tu + \tau w_\delta) < \frac{p - \sigma}{(N - \sigma) p} S_{N, p, \sigma}^{(N - \sigma)/(p - \sigma)}. \tag{3.39}
\]

To choose \( w_\delta \), recall that the infimum in (2.16) is attained by the family of functions

\[
u_\varepsilon(x) = \frac{c_{N, p, \sigma} \varepsilon^{(N-p)/(p-\sigma)}}{\left( \varepsilon + |x|^{(p-\sigma)/(p-1)} \right)^{(N-p)/(p-\sigma)}}, \quad \varepsilon > 0
\]
when $\Omega = \mathbb{R}^N$, where the constant $c_{N,p,\sigma} > 0$ is chosen so that

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^p \, dx = \int_{\mathbb{R}^N} \frac{u_\varepsilon^{p^*(\sigma)}}{|x|^{\sigma}} \, dx = S_{N,p,\sigma}^{(N-\sigma)/(p-\sigma)}$$

(see [13, Theorem 3.1.(2)]). Let $\zeta : [0, \infty) \to [0, 1]$ be a smooth function such that $\zeta(t) = 1$ for $t \leq 1/4$ and $\zeta(t) = 0$ for $t \geq 1/2$, and let

$$u_{\varepsilon, \rho}(x) = \zeta \left( \frac{|x|}{\rho} \right) u_\varepsilon(x), \quad w_{\varepsilon, \rho}(x) = \frac{u_{\varepsilon, \rho}(x)}{\left( \int_{\mathbb{R}^N} \frac{u_{\varepsilon, \rho}^{p^*(\sigma)}}{|x|^{\sigma}} \, dx \right)^{1/p^*(\sigma)}}, \quad 0 < \rho \leq \rho_0/2.$$

Then

$$\int_{\mathbb{R}^N} \frac{w_{\varepsilon, \rho}^{p^*(\sigma)}}{|x|^{\sigma}} \, dx = 1 \quad (3.40)$$

and we have

$$\int_{\mathbb{R}^N} |\nabla w_{\varepsilon, \rho}|^p \, dx \leq S_{N,p,\sigma} + c_{11} \varepsilon^{(N-p)/(p-\sigma)} \rho^{-(N-p)/(p-1)}, \quad (3.41)$$

$$\int_{\mathbb{R}^N} w_{\varepsilon, \rho}^p \, dx \geq \begin{cases} c_{12} \varepsilon^{(p-1)p/(p-\sigma)} & \text{if } N > p^2 \\ c_{12} \varepsilon^{(p-1)p/(p-\sigma)} \left| \log \left( \varepsilon \rho^{-(p-\sigma)/(p-1)} \right) \right| & \text{if } N = p^2 \end{cases} \quad (3.42)$$

for some constants $c_{11}, c_{12} > 0$ (see Perera and Zou [22]). Let

$$w_\delta = \pi_M(w_{\varepsilon, \rho}).$$

Since functions in $C_\delta$ have their supports in $\Omega \setminus B_{3\rho/4}(0)$, while the support of $w_\delta$ is in $B_{\rho/2}(0)$, $w_\delta \in M \setminus C_\delta$. We will show that (3.39) holds if $\varepsilon, \rho > 0$ are sufficiently small.

Inequality (3.39) is equivalent to

$$\sup_{v \in C_\delta, t, \tau \geq 0} E_0(tv + \tau w_{\varepsilon, \rho}) \leq \frac{p - \sigma}{(N - \sigma) p} S_{N,p,\sigma}^{(N-\sigma)/(p-\sigma)}. \quad (3.43)$$

For $v \in C_\delta$ and $t, \tau \geq 0$,

$$E_0(tv + \tau w_{\varepsilon, \rho}) = E_0(tv) + E_0(\tau w_{\varepsilon, \rho})$$

since $v$ and $w_{\varepsilon, \rho}$ have disjoint supports. So

$$\sup_{v \in C_\delta, t, \tau \geq 0} E_0(tv + \tau w_{\varepsilon, \rho}) = \sup_{v \in C_\delta, t \geq 0} E_0(tv) + \sup_{\tau \geq 0} E_0(\tau w_{\varepsilon, \rho}). \quad (3.44)$$

**Lemma 3.9.** We have

$$\sup_{v \in C_\delta, t \geq 0} E_0(tv) \leq \begin{cases} 0 & \text{if } \lambda_k + c_1 \rho^{N-p} \leq \lambda < \lambda_{k+1} \\ c_{13} \rho^{(N-\sigma)(N-p)/(p-\sigma)} & \text{if } \lambda = \lambda_k, \end{cases}$$

where $c_1$ is as in Lemma 3.3 and $c_{13} > 0$ is a constant.
Proof. For \( v \in C_\delta \) and \( t \geq 0 \),
\[
E_0(tv) = \frac{t^p}{p} \int_\Omega (|\nabla v|^p - \lambda |v|^p) \, dx - \frac{t^{p^*}(\sigma)}{p^*(\sigma)} \int_\Omega \frac{|v|^{p^*(\sigma)}}{|x|^\sigma} \, dx
\]
and (3.27) holds. So \( E_0(tv) \leq 0 \) if \( \lambda_k + c_1 \rho^{N-p} \leq \lambda < \lambda_{k+1} \). If \( \lambda = \lambda_k \), then (3.28) holds and
\[
\frac{1}{p^*(\sigma)} \int_\Omega \frac{|v|^{p^*(\sigma)}}{|x|^\sigma} \, dx \geq c_{14}
\]
for some constant \( c_{14} > 0 \) as in the proof of Lemma 3.8 so
\[
E_0(tv) \leq c_7 \rho^{N-p} t^p - c_{14} t^{p^*(\sigma)}
\]
and maximizing the right-hand side over all \( t \geq 0 \) gives the desired conclusion.

\[ \square \]

**Lemma 3.10.** We have
\[
\sup_{\tau \geq 0} E_0(\tau w_{\epsilon, \rho}) \leq \begin{cases} 
\frac{p - \sigma}{(N - \sigma) p} \left[ S_{N, p, \sigma} + c_{11} \epsilon^{(N-p)/(p-\sigma)} \rho^{-(N-p)/(p-1)} \right] 
- \lambda c_{12} \epsilon^{(p-1)/p} \left[ (N-\sigma)/(p-\sigma) \right] 
& \text{if } N > p^2 \\
\frac{p - \sigma}{(N - \sigma) p} \left[ S_{N, p, \sigma} + c_{11} \epsilon^{(p-1)/p} \rho^{p} \right] 
- \lambda c_{12} \epsilon^{(p-1)/p} \left[ \log \left( \epsilon \rho^{-(p-\sigma)/(p-1)} \right) \right]^{(N-\sigma)/(p-\sigma)} 
& \text{if } N = p^2.
\end{cases}
\]

Proof. We have
\[
E_0(\tau w_{\epsilon, \rho}) = \frac{\tau^p}{p} \int_\Omega (|\nabla w_{\epsilon, \rho}|^p - \lambda w_{\epsilon, \rho}^p) \, dx - \frac{\tau^{p^*(\sigma)}}{p^*(\sigma)}
\]
by (3.40), and maximizing the right-hand side over all \( \tau \geq 0 \) gives
\[
\sup_{\tau \geq 0} E_0(\tau w_{\epsilon, \rho}) \leq \begin{cases} 
\frac{p - \sigma}{(N - \sigma) p} \int_\Omega (|\nabla w_{\epsilon, \rho}|^p - \lambda w_{\epsilon, \rho}^p) \, dx \left[ (N-\sigma)/(p-\sigma) \right] 
& \text{if } N > p^2 \\
\frac{p - \sigma}{(N - \sigma) p} \left[ S_{N, p, \sigma} + c_{11} \epsilon^{(p-1)/p} \rho^{p} \right] 
& \text{if } N = p^2.
\end{cases}
\]

so the desired conclusion follows from (3.41) and (3.42).

\[ \square \]

We can now complete the proof of Theorem 2.14. First suppose \( N \geq p^2 \) and \( \lambda > \lambda_1 \) is not an eigenvalue. Then \( \lambda_k < \lambda < \lambda_{k+1} \) for some \( k \in \mathbb{N} \). Let \( \rho \in (0, \rho_0/2] \) be so small that \( \lambda_k + c_1 \rho^{N-p} \leq \lambda \). Then (3.43) follows from (3.44), Lemma 3.9, and Lemma 3.10 for sufficiently small \( \epsilon > 0 \).

Now suppose \( (N-\sigma)(N-p^2) > (p-\sigma)p \) and \( \lambda \geq \lambda_1 \). Then \( \lambda_k \leq \lambda < \lambda_{k+1} \) for some \( k \in \mathbb{N} \). We have already considered the case where \( N > p^2 \) and \( \lambda_k < \lambda < \lambda_{k+1} \), so suppose \( \lambda = \lambda_k \). Then
\[
\sup_{v \in C_\delta, t, \tau \geq 0} E_0(tv + \tau w_{\epsilon, \rho}) \leq \frac{p - \sigma}{(N - \sigma) p} \left[ S_{N, p, \sigma} + c_{11} \epsilon^{(N-p)/(p-\sigma)} \rho^{-(N-p)/(p-1)} \right] 
- \lambda c_{12} \epsilon^{(p-1)/p} \left[ (N-\sigma)/(p-\sigma) \right] 
+ c_{13} \rho^{(N-\sigma)(N-p)/(p-\sigma)}
\]

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by (3.44), Lemma 3.9, and Lemma 3.10. Set \( \rho = \varepsilon^\alpha \), where \( \alpha > 0 \) is to be chosen. Then the right-hand side is less than or equal to

\[
\frac{p - \sigma}{(N - \sigma)p} S_{N, p, \sigma}^{(N-\sigma)/(p-\sigma)} \left[ 1 + C_{15} \varepsilon^{(N-p)[1/(p-\sigma) - \alpha/(p-1)]} - C_{16} \varepsilon^{(p-1)p/(p-\sigma)} \right]^{(N-\sigma)/(p-\sigma)} + C_{13} \varepsilon^{\alpha(N-p)/(p-\sigma)}
\]

for some constants \( C_{15}, C_{16} > 0 \), so (3.43) will follow for sufficiently small \( \varepsilon > 0 \) if \( \alpha \) can be found so that

\[
(N - p)[1/(p - \sigma) - \alpha/(p - 1)] > (p - 1)p/(p - \sigma)
\]

and

\[
\alpha (N - \sigma)(N - p)/(p - \sigma) > (p - 1)p/(p - \sigma).
\]

This is possible if and only if

\[
(p - 1)p/(N - \sigma)(N - p) < (p - 1)[1 - (p - 1)p/(N - p)]/(p - \sigma),
\]

i.e.,

\[
(N - \sigma)(N - p^2) > (p - \sigma)p.
\]

### 3.4 Proof of Theorem 2.7

We prove Theorem 2.7 by applying Theorem 2.1 with \( W = W_0^{s, p}(\Omega) \), the operators \( A_p, B_p, f \in C(W_0^{s, p}(\Omega), W_0^{s, p}(\Omega)^*) \) and \( h \in W_0^{s, p}(\Omega)^* \) given by

\[
(A_p u, v) = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} dxdy,
\]

\[
(B_p u, v) = \int_{\Omega} |u|^{p-2}uv dx, \quad (f(u), v) = \int_{\Omega} |u|^{p^*-2}uv dx, \quad u, v \in W_0^{s, p}(\Omega)
\]

and

\[
(h, v) = \int_{\Omega} h(x)v dx, \quad v \in W_0^{s, p}(\Omega),
\]

and \( g = 0 \).

**Lemma 3.11.** There exists \( \kappa > 0 \) such that the functional \( E \) in (2.18) satisfies the \((PS)_c\) condition for all

\[
c < \frac{s}{N} S_{N, p, s}^{N/sp} - \kappa |h|_{p^*_s}^{p^*_s}.
\]

(3.45)
Proof. Let $c \in \mathbb{R}$ and let $(u_j)$ be a sequence in $W_0^{s,p}(\Omega)$ such that

$$E(u_j) = \frac{1}{p} \int_{\mathbb{R}^N} \frac{|u_j(x) - u_j(y)|^p}{|x - y|^{N+sp}} \, dx \, dy - \int_{\Omega} \left( \frac{\lambda}{p} |u_j|^p + \frac{1}{p^*_s} |u_j|^{p^*_s} + h(x) u_j \right) \, dx$$

$$= c + o(1) \quad (3.46)$$

and

$$(E'(u_j), v) = \int_{\mathbb{R}^N} \frac{|u_j(x) - u_j(y)|^p}{|x - y|^{N+sp}} \, dx \, dy - \int_{\Omega} \left( \lambda |u_j|^p + |u_j|^{p^*_s} + h(x) u_j \right) \, dx = o(\|v\|) \quad \forall v \in W_0^{s,p}(\Omega). \quad (3.47)$$

Taking $v = u_j$ in (3.47) gives

$$\int_{\mathbb{R}^N} \frac{|u_j(x) - u_j(y)|^p}{|x - y|^{N+sp}} \, dx \, dy - \int_{\Omega} \left( \lambda |u_j|^p + |u_j|^{p^*_s} + h(x) u_j \right) \, dx = o(\|u_j\|). \quad (3.48)$$

Let $r \in (p, p^*_s)$. Dividing (3.48) by $r$ and subtracting from (3.46) gives

$$\left( \frac{1}{p} - \frac{1}{r} \right) \int_{\mathbb{R}^N} \frac{|u_j(x) - u_j(y)|^p}{|x - y|^{N+sp}} \, dx \, dy - \int_{\Omega} \left[ \lambda \left( \frac{1}{p} - \frac{1}{r} \right) |u_j|^p - \left( \frac{1}{r} - \frac{1}{p^*_s} \right) |u_j|^{p^*_s} \right. \left. + \left( 1 - \frac{1}{r} \right) h(x) u_j \right] \, dx = c + o(1) + o(\|u_j\|),$$

and it follows from this that $(u_j)$ is bounded. So a renamed subsequence converges to some $u$ weakly in $W_0^{s,p}(\Omega)$, strongly in $L^1(\Omega)$ for all $t \in [1, p^*_s)$, and a.e. in $\Omega$. Setting $\tilde{u}_j = u_j - u$, we will show that $\tilde{u}_j \to 0$ in $W_0^{s,p}(\Omega)$.

Equation (3.48) gives

$$\|u_j\| = |u_j|^{p^*_s} + \int_{\Omega} (\lambda |u|^p + h(x) u) \, dx + o(1). \quad (3.49)$$

Taking $v = u$ in (3.47) and passing to the limit gives

$$\|u\|^p = |u|^{p^*_s} + \int_{\Omega} (\lambda |u|^p + h(x) u) \, dx. \quad (3.50)$$

Since

$$\|\tilde{u}_j\| = \|u_j\| - \|u\|^p + o(1) \quad (3.51)$$

by Perera et al. [21] Lemma 3.2] and

$$|\tilde{u}_j|^{p^*_s}_{p^*_s} = |u_j|^{p^*_s} - |u|^{p^*_s} + o(1)$$
by the Brézis-Lieb lemma [4, Theorem 1], (3.49) and (3.50) imply
\[ \| \tilde{u}_j \|^p = |\tilde{u}_j|_{P_s^*}^p + o(1) \leq \frac{\| \tilde{u}_j \|_{P_s^*/p}^p}{S_{N,p,s}^{N/(N-sp)}} + o(1), \]
so
\[ \| \tilde{u}_j \|^p \left( S_{N,p,s}^{N/(N-sp)} - \| \tilde{u}_j \|_{sp^2/(N-sp)} \right) \leq o(1). \] (3.52)

On the other hand, (3.46) gives
\[ c = \frac{1}{p} \| u_j \|^p - \frac{1}{p^*_s} |u_j|_{p^*_s}^p - \int_{\Omega} \left( \frac{\lambda}{p} |u|^p + h(x) u \right) dx + o(1), \]
and a straightforward calculation combining this with (3.49) – (3.51) gives
\[ c = \frac{s}{N} \| \tilde{u}_j \|^p + \int_{\Omega} \left[ \frac{s}{N} |u|_{p^*_s}^p - \left( 1 - \frac{1}{p} \right) h(x) u \right] dx + o(1). \]

The integral on the right-hand side is greater than or equal to
\[ \frac{s}{N} |u|_{p^*_s}^p - \left( 1 - \frac{1}{p} \right) |h|_{p^*_s}^p \geq -\kappa |h|_{p^*_s}^p, \]
for some \( \kappa > 0 \) by the Hölder and Young’s inequalities, so
\[ \| \tilde{u}_j \|^p \leq \frac{N}{s} \left( c + \kappa |h|_{p^*_s}^p \right) + o(1). \]
Combining this with (3.52) shows that \( \tilde{u}_j \to 0 \) when (3.45) holds. \( \square \)

We will apply Theorem 2.1 with
\[ c_{\mu,h}^* = \frac{s}{N} S_{N,p,s}^{N/(sp)} - \kappa |h|_{p^*_s}^p, \]
where \( \kappa > 0 \) is as in Lemma 3.11, noting that
\[ \lim_{|h|_{p^*_s}^p \to 0} c_{\mu,h}^* = \frac{s}{N} S_{N,p,s}^{N/(sp)}. \]
We have
\[ \mathcal{M} = \{ u \in W_{0}^{s,p}(\Omega) : \| u \|^p = p \}, \]
\[ \Psi(u) = \frac{p}{|u|_p^p}, \quad u \in \mathcal{M}, \]
\[ \pi_{\mathcal{M}}(u) = \frac{p^{1/p}}{\| u \|}, \quad u \in W_{0}^{s,p}(\Omega) \setminus \{0\}, \]

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and
\[ E_0(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy - \int_{\Omega} \left( \frac{\lambda}{p} |u|^p + \frac{1}{p_s^*} |u|^{p_s^*} \right) \, dx, \quad u \in W_0^{s,p}(\Omega). \]

Let \( \lambda_k \leq \lambda < \lambda_{k+1} \). We need to show that there exist \( R > 0 \) and, for all sufficiently small \( \delta > 0 \), a compact symmetric subset \( C_\delta \) of \( \Psi^{\lambda+\delta} \) with \( i(C_\delta) = k \) and \( w_\delta \in M \setminus C_\delta \) such that, setting \( A_\delta = \{ \pi_M((1 - \tau) v + \tau w_\delta) : v \in C_\delta, \ 0 \leq \tau \leq 1 \} \), we have

\[ \sup_{u \in A_\delta} E_0(Ru) \leq 0, \quad \sup_{u \in A_\delta, 0 \leq t \leq R} E_0(tu) < \frac{s}{N} S_{N,p,s}^{N/sp}. \quad (3.53) \]

Since \( \lambda_k < \lambda_{k+1} \), \( \Psi^{\lambda_k} \) has a compact symmetric subset \( C_0 \) of index \( k \) that is bounded in \( L^\infty(\Omega) \) (see Mosconi et al. [17, Proposition 3.1]). We may assume without loss of generality that \( 0 \in \Omega \). Let \( \rho_0 = \text{dist}(0, \partial \Omega) \), let \( \eta : [0, \infty) \to [0, 1] \) be a smooth function such that \( \eta(t) = 0 \) for \( t \leq 3/4 \) and \( \eta(t) = 1 \) for \( t \geq 1 \), let

\[ u_\rho(x) = \eta \left( \frac{|x|}{\rho} \right) u(x), \quad u \in C_0, \ 0 < \rho \leq \rho_0/2, \]

and let

\[ C = \{ \pi_M(u_\rho) : u \in C_0 \}. \]

The following lemma was proved in Mosconi et al. [17].

**Lemma 3.12** (Mosconi et al. [17, Proposition 3.2]). The set \( C \) is a compact symmetric subset of \( \Psi^{\lambda_k + c_{17} \rho^{N-sp}} \) for some constant \( c_{17} > 0 \) and is bounded in \( L^\infty(\Omega) \). If \( \lambda_k + c_{17} \rho^{N-sp} < \lambda_{k+1} \), then \( i(C) = k \).

**Lemma 3.13.** For any \( w \in M \setminus C \) with support in \( B_{\rho/2}(0) \), \( \exists R > 0 \) such that, setting \( A = \{ \pi_M((1 - \tau) v + \tau w) : v \in C, \ 0 \leq \tau \leq 1 \} \), we have

\[ \sup_{u \in A} E_0(Ru) \leq 0. \]

**Proof.** Let \( u = \pi_M((1 - \tau) v + \tau w) \in A \). For \( R > 0 \),

\[ E_0(Ru) \leq \frac{R^p}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy - \frac{R^{p_s^*}}{p_s^*} \int_{\Omega} |u|^{p_s^*} \, dx = R^p - \frac{R^{p_s^*}}{p_s^*} |u|_{p_s^*}^{p_s^*}, \]

so it suffices to show that \( |u|_{p_s^*} \) is bounded away from zero on \( A \). By the Hölder inequality, it is enough to show that \( |u|_p \) is bounded away from zero. As in the proof of Lemma 3.4, it suffices to show that \( |v|_p \) is bounded away from zero on \( C \). Since \( C \subset \Psi^{\lambda_k + c_{17} \rho^{N-sp}} \) by Lemma 3.12, we have

\[ |v|_p = \frac{p}{\Psi(v)} \geq \frac{p}{\lambda_k + c_{17} \rho^{N-sp}}. \]
Let $\delta \in (0, \lambda_{k+1} - \lambda)$, let $\rho \in (0, \rho_0/2]$ be so small that $\lambda_k + c_{17} \rho^{N-sp} < \lambda + \delta$, and let $C_\delta = C$. Then $C_\delta$ is a compact symmetric subset of $\Psi^{\lambda+\delta}$ with $i(C_\delta) = k$ that is bounded in $L^\infty(\Omega)$ by Lemma 3.12. We will show that if $\delta > 0$ is sufficiently small, then $\exists w_\delta \in \mathcal{M} \setminus C_\delta$ with support in $B_{\rho/2}(0)$ such that, setting $A_\delta = \{ \pi_\mathcal{M}((1 - \tau)v + \tau w_\delta) : v \in C_\delta, 0 \leq \tau \leq 1 \}$, we have

$$ \sup_{u \in A_\delta, t \geq 0} E_0(tu) < \frac{S}{N} S^{N/sp}_{N,p,s}. \quad (3.54) $$

Then Lemma 3.13 will give an $R > 0$ such that (3.53) holds and complete the proof. We note that (3.54) is equivalent to

$$ \sup_{v \in C_\delta, t, \tau \geq 0} E_0(tv + \tau w_\delta) < \frac{S}{N} S^{N/sp}_{N,p,s}. \quad (3.55) $$

In the absence of an explicit formula for a minimizer for $S_{N,p,s}$ in (2.19), we will use certain asymptotic estimates for minimizers obtained in Brasco et al. [3] to choose $w_\delta$. It was shown in [3] that there exists a nonnegative, radially symmetric, and decreasing minimizer $U(x) = U(r), r = |x|$ satisfying

$$ \int_{\mathbb{R}^N} \frac{|U(x) - U(y)|^p}{|x - y|^{N+sp}} \, dx \, dy = \int_{\mathbb{R}^N} U(x)^p \, dx = S^{N/sp}_{N,p,s} \quad (3.56) $$

and

$$ c_{18} r^{-(N-sp)/(p-1)} \leq U(r) \leq c_{19} r^{-(N-sp)/(p-1)} \quad \forall r \geq 1 $$

for some constants $c_{18}, c_{19} > 0$. Then the functions

$$ u_\varepsilon(x) = \varepsilon^{-(N-sp)/p} U \left( \frac{|x|}{\varepsilon} \right), \quad \varepsilon > 0 $$

are also minimizers for $S_{N,p,s}$ satisfying

$$ \int_{\mathbb{R}^N} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^p}{|x - y|^{N+sp}} \, dx \, dy = \int_{\mathbb{R}^N} u_\varepsilon(x)^p \, dx = S^{N/sp}_{N,p,s} \quad (3.57) $$

and

$$ \frac{U(\theta r)}{U(r)} \leq \frac{c_{19}}{c_{18}} \theta^{-(N-sp)/(p-1)} \leq \frac{1}{2} \quad \forall r \geq 1 $$

if $\theta > 1$ is a sufficiently large constant. Let

$$ u_{\varepsilon, \rho}(x) = \begin{cases} 
    u_\varepsilon(x) & \text{if } |x| \leq \rho \\
    u_\varepsilon(\rho) \frac{(u_\varepsilon(x) - u_\varepsilon(\theta \rho))}{u_\varepsilon(\rho) - u_\varepsilon(\theta \rho)} & \text{if } \rho < |x| < \theta \rho \\
    0 & \text{if } |x| \geq \theta \rho 
\end{cases} $$

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and
\[ w_{\varepsilon, \rho}(x) = \frac{u_{\varepsilon, \rho}(x)}{\left(\int_{\mathbb{R}^N} u_{\varepsilon, \rho}(x)^{p_s^*} \, dx\right)^{1/p_s^*}} \]
for \(0 < \rho \leq \rho_0/2\). Then
\[ \int_{\mathbb{R}^N} w_{\varepsilon, \rho}(x)^{p_s^*} \, dx = 1 \tag{3.57} \]
and for \(\varepsilon \leq \rho/2\) we have the estimates
\[ \int_{\mathbb{R}^{2N}} \frac{|w_{\varepsilon, \rho}(x) - w_{\varepsilon, \rho}(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \leq S_{N, p, s} + c_{20} \left(\varepsilon/\rho\right)^{(N-sp)/(p-1)}, \tag{3.58} \]
\[ \int_{\mathbb{R}^N} w_{\varepsilon, \rho}^p(x) \, dx \geq \begin{cases} c_{21} \varepsilon^{sp} & \text{if } N > sp^2 \\ c_{21} \varepsilon^{sp} \log(\varepsilon/\rho) & \text{if } N = sp^2 \end{cases} \tag{3.59} \]
for some constants \(c_{20}, c_{21} > 0\) (see Mosconi et al. [17, Lemma 2.7]). Let
\[ w_\delta = \pi_M(w_{\varepsilon, \rho}/2\theta). \]
Since functions in \(C_\delta\) have their supports in \(\Omega \setminus B_{3\rho/4}(0)\), while the support of \(w_\delta\) is in \(B_{\rho/2}(0)\), \(w_\delta \in M \setminus C_\delta\). We will show that (3.55) holds if \(\varepsilon, \rho > 0\) are sufficiently small.

Inequality (3.55) is equivalent to
\[ \sup_{v \in C_\delta, t, \tau \geq 0} E_0(tv + \tau w_{\varepsilon, \rho}/2\theta) < \frac{S}{N} S_{N, p, s}^{N/sp}. \tag{3.60} \]

**Lemma 3.14.** For \(v \in C_\delta\) and \(t, \tau \geq 0\),
\[ E_0(tv + \tau w_{\varepsilon, \rho}/2\theta) \leq \left[ E_0(tv) + c_{22} \rho^{N-sp} t^p \right] + \left[ E_0(\tau w_{\varepsilon, \rho}/2\theta) + c_{23} \left(\varepsilon/\rho\right)^{(N-sp)/(p-1)} \tau^p \right] \]
for some constants \(c_{22}, c_{23} > 0\).

**Proof.** Since \(v\) and \(w_{\varepsilon, \rho}/2\theta\) have disjoint supports,
\[ E_0(tv + \tau w_{\varepsilon, \rho}/2\theta) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|(tv(x) + \tau w_{\varepsilon, \rho}/2\theta(x)) - (tv(y) + \tau w_{\varepsilon, \rho}/2\theta(y))|^p}{|x - y|^{N+sp}} \, dxdy \]
\[ - \int_{\Omega} \left( \frac{\lambda^p}{p} |v|^p + \frac{\lambda^{p_s^*}}{p_s^*} |v|^{p_s^*} \right) dx - \int_{\Omega} \left( \frac{\lambda^p}{p} w_{\varepsilon, \rho}/2\theta + \frac{\tau^p}{p} w_{\varepsilon, \rho}/2\theta \right)^p \, dx. \tag{3.61} \]
Denote by \(I_1\) the first integral on the right-hand side. Since \(\text{supp } v \subset B_{3\rho/4}^c\) and \(\text{supp } w_{\varepsilon, \rho}/2 \subset B_{\rho/2}^c\),
\[ I_1 \leq t^p \int_{B_{\rho/2}^c \times B_{\rho/2}^c} \frac{|v(x) - v(y)|^p}{|x - y|^{N+sp}} \, dxdy + \tau^p \int_{B_{3\rho/4}^c \times B_{3\rho/4}^c} \frac{|w_{\varepsilon, \rho}/2\theta(x) - w_{\varepsilon, \rho}/2\theta(y)|^p}{|x - y|^{N+sp}} \, dxdy \]
\[ + 2 \int_{B_{3\rho/4}^c \times B_{\rho/2}^c} \frac{|tv(x) - \tau w_{\varepsilon, \rho}/2\theta(y)|^p}{|x - y|^{N+sp}} \, dxdy =: t^p I_2 + \tau^p I_3 + 2I_4. \tag{3.62} \]
First suppose \( p \geq 2 \). To estimate \( I_4 \), we use the elementary inequality

\[
|a + b|^p \leq |a|^p + |b|^p + C_p \left( |a|^{p-1}|b| + |a||b|^{p-1} \right) \quad \forall a, b \in \mathbb{R}
\]

for some constant \( C_p > 0 \). Since \( v(y) = 0 \) for \( y \in B_{p/2} \) and \( w_{\varepsilon,p/2\theta}(x) = 0 \) for \( x \in B_{3p/4} \), we get

\[
I_4 \leq t^p \int_{B_{3p/4} \times B_{p/2}} \frac{|v(x) - v(y)|^p}{|x - y|^{N + sp}} \, dx \, dy + \tau^p \int_{B_{3p/4} \times B_{p/2}} \frac{|w_{\varepsilon,p/2\theta}(x) - w_{\varepsilon,p/2\theta}(y)|^p}{|x - y|^{N + sp}} \, dx \, dy
\]

\[
+ C_p \left( t^{p-1} \tau \int_{B_{3p/4} \times B_{p/2}} \frac{|v(x)|^{p-1} w_{\varepsilon,p/2\theta}(y)}{|x - y|^{N + sp}} \, dx \, dy + \tau t^{p-1} \int_{B_{3p/4} \times B_{p/2}} \frac{|v(x)| w_{\varepsilon,p/2\theta}(y)^{p-1}}{|x - y|^{N + sp}} \, dx \, dy \right)
\]

\[
=: t^p I_5 + \tau^p I_6 + C_p \left( t^{p-1} \tau J_1 + \tau t^{p-1} J_{p-1} \right), \quad (3.63)
\]

where

\[
J_q = \int_{B_{3p/4} \times B_{p/2}} \frac{|v(x)|^{p-q} w_{\varepsilon,p/2\theta}(y)^q}{|x - y|^{N + sp}} \, dx \, dy, \quad q = 1, p - 1.
\]

Since \( C_q \) is bounded in \( L^\infty(\Omega) \) and

\[
|x - y| \geq |x| - |y| > |x| - \frac{p}{2} \geq |x| - \frac{2}{3}|x| = \frac{|x|}{3} \quad \forall (x, y) \in B_{3p/4} \times B_{p/2},
\]

we have

\[
J_q \leq c_{24} \int_{B_{3p/4} \times B_{p/2}} \frac{w_{\varepsilon,p/2\theta}(y)^q}{|x|^{N + sp}} \, dx \, dy = c_{25} \rho^{-sp} \int_{B_{p/2}} w_{\varepsilon,p/2\theta}(y)^q \, dy \quad (3.64)
\]

for some constants \( c_{24}, c_{25} > 0 \). By Mosconi et al. [17] Lemma 2.7, \( |u_{\varepsilon,p/2\theta}|_{p^*_\infty} \) is bounded away from zero and hence

\[
\int_{B_{p/2}} w_{\varepsilon,p/2\theta}(y)^q \, dy \leq c_{26} \int_{B_{p/2}} u_{\varepsilon,p/2\theta}(y)^q \, dy \quad (3.65)
\]

for some constant \( c_{26} > 0 \). Noting that \( u_{\varepsilon,p/2\theta} \leq u_\varepsilon \), we have

\[
\int_{B_{p/2}} u_{\varepsilon,p/2\theta}(y)^q \, dy \leq \int_{B_{p/2}} u_\varepsilon(y)^q \, dy = \varepsilon^{-(N - sp)/p} \int_{B_{p/2}} U \left( \frac{|y|}{\varepsilon} \right)^q \, dy
\]

\[
= \varepsilon^{N - (N - sp)/q/p} \int_{B_{p/2\varepsilon}} U(|y|)^q \, dy. \quad (3.66)
\]

When \( q < N(p - 1)/(N - sp) \), (3.66) gives

\[
\int_{B_{p/2\varepsilon}} U(|y|)^q \, dy \leq c_{27} \left( \frac{p}{\varepsilon} \right)^{N - (N - sp)/q(p - 1)} \quad (3.67)
\]
for some constant $c_{27} > 0$, in particular, \(3.67\) holds for $q = 1$ when $p > 2N/(N + s)$ and for $q = p - 1$. Combining \(3.64\)--\(3.67\) gives

$$J_q \leq c_{28} \rho^{(N-sp)(p-q-1)/(p-1)} \varepsilon^{(N-sp)q/p(p-1)}$$

for some constant $c_{28} > 0$, so

$$t^{p-q-1} J_q \leq c_{28} \left( \rho^{N-sp} t^p \right)^{1-q/p} \left( (\varepsilon/\rho)^{(N-sp)/(p-1)} \tau^p \right)^{q/p} \leq c_{29} \rho^{N-sp} t^p$$

for some constants $c_{29}, c_{30} > 0$ by Young’s inequality. Combining this with \(3.61\)--\(3.63\), and noting that

$$1 \leq \frac{|v(x) - v(y)|^p}{|x - y|^{N+sp}}$$

we get the desired conclusion in this case.

If $1 < p < 2$, we use the elementary inequality

$$|a + b|^p \leq |a|^p + |b|^p + p|a||b|^{p-1} \quad \forall a, b \in \mathbb{R}$$

to get

$$I_4 \leq t^p I_5 + \tau^p I_6 + p t \tau^{p-1} J_{p-1}$$

and proceed as above. \(\square\)

By Lemma \(3.14\)

$$\sup_{v \in C_\delta, t, \tau \geq 0} E_0(tv + \tau w_{\varepsilon, \rho/29}) \leq \sup_{v \in C_\delta, t, \tau \geq 0} \left[ E_0(tv) + c_{22} \rho^{N-sp} t^p \right] + \sup_{\tau \geq 0} \left[ E_0(\tau w_{\varepsilon, \rho/29}) + c_{23} (\varepsilon/\rho)^{(N-sp)/(p-1)} \tau^p \right] =: K_1 + K_2. \quad \text{(3.68)}$$

**Lemma 3.15.** We have

$$K_1 \leq \begin{cases} 0 & \text{if } (\lambda_k + c_{17} \rho^{N-sp})(1 + c_{22} \rho^{N-sp}) \leq \lambda < \lambda_{k+1} \\ c_{31} \rho^{N(N-sp)/sp} & \text{if } \lambda = \lambda_k, \end{cases}$$

where $c_{17}$ is as in Lemma \(3.12\) and $c_{31} > 0$ is a constant.

**Proof.** For $v \in C_\delta$ and $t \geq 0$,

$$E_0(tv) + c_{22} \rho^{N-sp} t^p = t^p \left( \frac{1}{p} \int_{\mathbb{R}^2} \frac{|v(x) - v(y)|^p}{|x - y|^{N+sp}} \, dxdy - \frac{\lambda}{p} \int_{\Omega} |v|^p \, dx + c_{22} \rho^{N-sp} \right)$$

$$- \frac{t^{p^*_s}}{p^*_s} \int_{\Omega} |v|^{p^*_s} \, dx =: K_3 t^p - K_4 t^{p^*_s},$$

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and
\[ K_3 = 1 - \frac{\lambda}{\Psi(v)} + c_{22} \rho^{N-sp} \leq 1 - \frac{\lambda}{\lambda_k + c_{17} \rho^{N-sp}} + c_{22} \rho^{N-sp} \]

since \( C_6 \subset \Psi^{\lambda_k + c_{17} \rho^{N-sp}} \) by Lemma 3.12. So \( E_0(tv) + c_{22} \rho^{N-sp} t^p \leq 0 \) if \((\lambda_k + c_{17} \rho^{N-sp})(1 + c_{22} \rho^{N-sp}) \leq \lambda < \lambda_{k+1}\). If \( \lambda = \lambda_k \), then

\[ K_3 \leq \frac{c_{17} \rho^{N-sp}}{\lambda_k + c_{17} \rho^{N-sp}} + c_{22} \rho^{N-sp} \leq c_{32} \rho^{N-sp}, \]

where \( c_{32} = c_{17}/\lambda_k + c_{22} > 0 \), and \( K_4 \geq c_{33} \) for some constant \( c_{33} > 0 \) as in the proof of Lemma 3.13, so

\[ E_0(tv) + c_{22} \rho^{N-sp} t^p \leq c_{32} \rho^{N-sp} t^p - c_{33} t^p \]

and maximizing the right-hand side over all \( t \geq 0 \) gives the desired conclusion.

\[ \square \]

**Lemma 3.16.** We have

\[ K_2 \leq \begin{cases} 
\frac{s}{N} \left[ S_{N,p,s} + c_{34} (\varepsilon/\rho)^{(N-sp)/(p-1)} - \lambda c_{35} \varepsilon^{sp} \right]^{N/sp} & \text{if } N > sp^2 \\
\frac{s}{N} \left[ S_{N,p,s} + c_{34} (\varepsilon/\rho)^{sp} - \lambda c_{35} \varepsilon^{sp} \log (\varepsilon/\rho) \right]^{N/sp} & \text{if } N = sp^2 
\end{cases} \]

for some constants \( c_{34}, c_{35} > 0 \).

**Proof.** We have

\[ E_0(\tau w_{\varepsilon,\rho/2\theta}) + c_{23} (\varepsilon/\rho)^{(N-sp)/(p-1)} \tau^p = \frac{\tau^p}{p} \left( \int_{\mathbb{R}^{2N}} \frac{|w_{\varepsilon,\rho/2\theta}(x) - w_{\varepsilon,\rho/2\theta}(y)|^p}{|x - y|^{N+sp}} dxdy \right) - \lambda \int_{\Omega} w_{\varepsilon,\rho/2\theta} dx + p c_{23} (\varepsilon/\rho)^{(N-sp)/(p-1)} - \frac{\tau^p s}{p^s} \]

by (3.57), and maximizing the right-hand side over all \( \tau \geq 0 \) gives

\[ K_2 = \frac{s}{N} \left( \int_{\mathbb{R}^{2N}} \frac{|w_{\varepsilon,\rho/2\theta}(x) - w_{\varepsilon,\rho/2\theta}(y)|^p}{|x - y|^{N+sp}} dxdy - \lambda \int_{\Omega} w_{\varepsilon,\rho/2\theta} dx + p c_{23} (\varepsilon/\rho)^{(N-sp)/(p-1)} \right)^{N/sp} \]

so the desired conclusion follows from (3.58) and (3.59).

\[ \square \]

We can now complete the proof of Theorem 2.7. First suppose \( N \geq sp^2 \) and \( \lambda > \lambda_1 \) is not an eigenvalue. Then \( \lambda_k < \lambda < \lambda_{k+1} \) for some \( k \in \mathbb{N} \). Let \( \rho \in (0, \rho_0/2] \) be so small that \((\lambda_k + c_{17} \rho^{N-sp})(1 + c_{22} \rho^{N-sp}) \leq \lambda \). Then (3.60) follows from (3.68), Lemma 3.15, and Lemma 3.16 for sufficiently small \( \varepsilon > 0 \).

Now suppose \( N(N-sp^2) > sp^2 \) and \( \lambda \geq \lambda_1 \). Then \( \lambda_k \leq \lambda < \lambda_{k+1} \) for some \( k \in \mathbb{N} \). We have already considered the case where \( N > sp^2 \) and \( \lambda_k < \lambda < \lambda_{k+1} \), so suppose \( \lambda = \lambda_k \). Then

\[ \sup_{v \in C_6, t, \tau \geq 0} E_0(tv + \tau w_{\varepsilon,\rho/2\theta}) \leq \frac{s}{N} \left[ S_{N,p,s} + c_{34} (\varepsilon/\rho)^{(N-sp)/(p-1)} - \lambda c_{35} \varepsilon^{sp} \right]^{N/sp} + c_{31} \rho^{N(N-sp)/sp} \]
by \([3.68]\), Lemma \([3.15]\) and Lemma \([3.16]\). Set \(\rho = \varepsilon^\alpha\), where \(\alpha > 0\) is to be chosen. Then the right-hand side is less than or equal to

\[
\frac{S}{N} S_{N,p,s}^{N/sp} \left[ 1 + c_{36} \varepsilon^{(1-\alpha)(N-sp)/(p-1)} - c_{37} \varepsilon^{sp} \right]^{N/sp} + c_{31} \varepsilon^{\alpha N(N-sp)/sp}
\]

for some constants \(c_{36}, c_{37} > 0\), so \([3.60]\) will follow for sufficiently small \(\varepsilon > 0\) if \(\alpha\) can be found so that

\[(1-\alpha)(N-sp)/(p-1) > sp\]

and

\[\alpha N(N-sp)/sp > sp.\]

This is possible if and only if

\[s^2 p^2 / N(N-sp) < (N-sp^2)/(N-sp),\]

i.e.,

\[N(N-sp^2) > s^2 p^2.\]

\[\square\]

### 3.5 Proof of Theorem 2.10

We prove Theorem 2.10 by applying Theorem 2.1 with \(W = W_0^{1,p}(\Omega)\) and the operators \(A_p, B_p, f, g \in C(W_0^{1,p}(\Omega), W^{-1,p'}(\Omega))\) and \(h \in W^{-1,p'}(\Omega)\) given by

\[
(A_p u, v) = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx, \quad (B_p u, v) = \int_\Omega |u|^{p-2} uv \, dx,
\]

\[
(f(u), v) = \int_\Omega |u|^{p^*-2} uv \, dx, \quad (g(u), v) = -\int_\Omega |\nabla u|^{q-2} \nabla u \cdot \nabla v \, dx, \quad u, v \in W_0^{1,p}(\Omega)
\]

and

\[
(h, v) = \int_\Omega h(x) v \, dx, \quad v \in W_0^{1,p}(\Omega).
\]

**Lemma 3.17.** There exists \(\kappa > 0\) such that the functional \(E\) in \([2.21]\) satisfies the \((PS)_c\) condition for all

\[
c < \frac{1}{N} S_{N,p,s}^{N/p} - \kappa |h|^{p^*}. \tag{3.69}
\]

**Proof.** Let \(c \in \mathbb{R}\) and let \((u_j)\) be a sequence in \(W_0^{1,p}(\Omega)\) such that

\[
E(u_j) = \int_\Omega \left( \frac{1}{p} |\nabla u_j|^p + \frac{\mu}{q} |\nabla u_j|^q - \frac{\lambda}{p} |u_j|^p - \frac{1}{p^*} |u_j|^{p^*} - h(x) u_j \right) \, dx = c + o(1) \tag{3.70}
\]
and

\[
(E'(u_j), v) = \int_\Omega (|\nabla u_j|^{p-2} \nabla u_j \cdot \nabla v + \mu |\nabla u_j|^{q-2} \nabla u_j \cdot \nabla v - \lambda |u_j|^{p-2} u_j v
- |u_j|^{p'-2} u_j v - h(x) v) \, dx = o(\|v\|) \quad \forall v \in W^{1,p}_0(\Omega). \tag{3.71}
\]

Taking \( v = u_j \) in (3.71) gives

\[
\int_\Omega (|\nabla u_j|^p + \mu |\nabla u_j|^q - \lambda |u_j|^p - |u_j|^p - h(x) u_j) \, dx = o(\|u_j\|). \tag{3.72}
\]

Let \( r \in (p, p^*) \). Dividing (3.72) by \( r \) and subtracting from (3.70) gives

\[
\int_\Omega \left[ \left( \frac{1}{p} - \frac{1}{r} \right) |\nabla u_j|^p + \mu \left( \frac{1}{q} - \frac{1}{r} \right) |\nabla u_j|^q - \lambda \left( \frac{1}{p} - \frac{1}{r} \right) |u_j|^p + \left( \frac{1}{r} - \frac{1}{p^*} \right) |u_j|^{p^*}
- \left( 1 - \frac{1}{r} \right) h(x) u_j \right] \, dx = c + o(1) + o(\|u_j\|),
\]

and it follows from this that \((u_j)\) is bounded. So a renamed subsequence converges to some \( u \) weakly in \( W^{1,p}_0(\Omega) \), strongly in \( L^t(\Omega) \) for all \( t \in [1, p^*) \), and a.e. in \( \Omega \). Setting \( \tilde{u}_j = u_j - u \), we will show that \( \tilde{u}_j \to 0 \) in \( W^{1,p}_0(\Omega) \).

Equation (3.72) gives

\[
\|u_j\|^p + \mu |\nabla u_j|^q = |u_j|^{p^*} + \int_\Omega (\lambda |u|^p + h(x) u) \, dx + o(1). \tag{3.73}
\]

Taking \( v = u \) in (3.71) and passing to the limit gives

\[
\|u\|^p + \mu |\nabla u|^q = |u|^{p^*} + \int_\Omega (\lambda |u|^p + h(x) u) \, dx. \tag{3.74}
\]

Since

\[
\|\tilde{u}_j\|^p = \|u_j\|^p - \|u\|^p + o(1) \tag{3.75}
\]

and

\[
|\tilde{u}_j|^{p^*} = |u_j|^{p^*} - |u|^{p^*} + o(1)
\]

by the Brézis-Lieb lemma [4, Theorem 1], and

\[
\liminf |\nabla u_j|^q \geq |\nabla u|^q, \tag{3.76}
\]

(3.73) and (3.74) imply

\[
\|\tilde{u}_j\|^p \leq \|\tilde{u}_j\|^{p^*} + o(1) \leq \frac{||\tilde{u}_j||^{p^*}}{S_{p^*/p}^{p^*/p}} + o(1),
\]

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so
\[ \| \tilde{u}_j \|^p \left( S^{N/(N-p)}_{N,p} - \| \tilde{u}_j \|^{p^*/(N-p)} \right) \leq o(1). \] (3.77)

On the other hand, (3.70) gives
\[ c = \frac{1}{p} \| u_j \|^p + \frac{\mu}{q} |\nabla u_j|_q^q - \frac{1}{p^*} |u_j|^{p^*} - \int_{\Omega} \left( \frac{\lambda}{p} |u|^p + h(x) u \right) dx + o(1), \]
and a straightforward calculation combining this with (3.73)–(3.75) gives
\[ c = \frac{1}{N} \| \tilde{u}_j \|^p + \mu \left[ \left( \frac{1}{q} - \frac{1}{p^*} \right) |\nabla u_j|_q^q - \frac{1}{N} |\nabla u|_q^q \right] + \int_{\Omega} \left[ \frac{1}{N} |u|^{p^*} - \left( 1 - \frac{1}{p} \right) h(x) u \right] dx + o(1). \]

The second term on the right-hand side is greater than or equal to \( o(1) \) by (3.76) and the integral is greater than or equal to
\[ \frac{1}{N} |u|^{p^*} - \left( 1 - \frac{1}{p} \right) |h|_{p^*} |u|_{p^*} \geq -\kappa |h|^{p^*} \]
for some \( \kappa > 0 \) by the Hölder and Young’s inequalities, so
\[ \| \tilde{u}_j \|^p \leq N \left( c + \kappa |h|^{p^*} \right) + o(1). \]

Combining this with (3.77) shows that \( \tilde{u}_j \to 0 \) when (3.69) holds.

We apply Theorem 2.1 with
\[ c^*_{\mu, h} = \frac{1}{N} S^{N/p}_{N,p} - \kappa |h|^{p^*}, \]
where \( \kappa > 0 \) is as in Lemma 3.17 noting that
\[ \lim_{\mu, |h|^{p^*} \to 0} c^*_{\mu, h} = \frac{1}{N} S^{N/p}_{N,p}. \]

The set \( \mathcal{M} \) and the functions \( \Psi, \pi_M \), and \( E_0 \) are the same as in the proof of Theorem 2.2. Let \( \lambda_k \leq \lambda < \lambda_{k+1} \). Exactly as in that proof, there exist \( R > 0 \) and, for all sufficiently small \( \delta > 0 \), a compact symmetric subset \( C_\delta \) of \( \Psi^{\lambda+\delta} \) with \( i(C_\delta) = k \) and \( w_\delta \in \mathcal{M} \setminus C_\delta \) such that, setting \( A_\delta = \{ \pi_M((1-\tau) v + \tau w_\delta) : v \in C_\delta, 0 \leq \tau \leq 1 \} \), we have
\[ \sup_{u \in A_\delta} E_0(Ru) \leq 0, \quad \sup_{u \in A_\delta, 0 \leq t \leq R} E_0(tu) < \frac{1}{N} S^{N/p}_{N,p}. \]

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