MAXIMAL VARIATION OF CURVES ON K3 SURFACES

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Abstract. We prove that curves in a non-primitive, base point free, ample linear system on a K3 surface have maximal variation. The result is deduced from general restriction theorems applied to the tangent bundle. We also show how to use specialisation to spectral curves to deduce information about the variation of curves contained in a K3 surface more directly. The situation for primitive linear systems is not clear at the moment. However, the maximal variation holds in genus two and can, in many cases, be deduced from a recent result of van Geemen and Voisin [vGV16] confirming a conjecture due to Matsushita.

1. Introduction

Smooth curves contained in K3 surfaces are special yet general. This has been a guiding principle for important work over the last decades, cf. [Be05, La86, Vo08]. For dimension reasons, a general curve of high genus cannot be contained in any smooth K3 surface, although those that are behave in many respects like a general curve. It is a basic question to ask in how many ways, if at all, a curve can be embedded into a K3 surface. In other words, how much do curves contained in a K3 surface vary, either within the given K3 surface or together with it?

1.1. Our main result is concerned with the variation of curves in an arbitrary but fixed K3 surface.

Theorem 1.1. Let $|H|$ be a base point free, ample linear system on a K3 surface such that the locus of non-reduced curves in $|H|$ has codimension at least three. Then, for all $m \geq 2$, the family of smooth curves in the linear system $|mH|$ has maximal variation.

In other words, a generic curve in a non-primitive, base point free ample linear system occurs only finitely many times in its linear system. The result can also be expressed by saying that the rational map $|mH| \dashrightarrow M_{g_m}$ that associates with a smooth curve $C \subset S$ its isomorphism class $[C] \in M_{g_m}$ in the moduli space of smooth curves of genus $g_m := \dim |mH|$ is generically quasi-finite as soon as $m \geq 2$.

For $m \geq 3$ the assertion holds without any assumption on the non-reduced locus and we expect this to be true also for $m = 2$. For $m = 1$, we can currently prove maximal variation only for $(H.H) = 2$. It would follow in this generality from a conjecture of Matsushita, which has been proved for generic K3 surfaces by van Geemen and Voisin [vGV16].

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1.2. The problem of deforming curves in varying K3 surfaces was already studied by Mukai [Mu92] who showed that the generic fibre of the map \((C \subset S) \rightarrow C\) is finite if the genus satisfies \(g = 11\) or \(g \geq 13\). Here, \(C\) is a primitive smooth curve contained in a varying K3 surface \(S\).

Results due to Arbarello–Bruno–Sernesi [ABS14] and to Feyzbaklsh [Fe20] showed that for a polarized K3 surface \((S, H)\) with the same hypotheses on the genus and with \(\text{Pic}(S) = \mathbb{Z} \cdot H\) the K3 surface \(S\) can be uniquely reconstructed from any smooth curve \(C \in |H|\). In particular, the map \((C \subset S) \rightarrow C\) is injective at such a point and then, of course, for fixed \(S\) the rational map \(|H| \rightarrow M_g\) has finite fibres. Results without restrictions on the Picard number have been proved by Arbarello–Bruno–Sernesi [ABS17], Ciliberto–Dedieu–Sernesi [CDS20], and Ciliberto–Dedieu [CD20]. Their results cover most of the cases dealt with by Theorem 1.1 using completely different methods. In fact, they can handle also the case \(m = 1\) albeit with the additional assumption on the Clifford index to be at least three. Another advantage of their approach is that they prove results for all smooth curves in a linear system, while our approach is mostly restricted to generic curves.

An infinitesimal analogue, proving the injectivity of the tangent map under similar conditions, was worked out by Totaro [To20]. These results imply maximal variation for primitive and therefore, also non-primitive ample linear systems of high enough genus on a generic polarized K3 surfaces. We also wish to draw attention to the recent paper [CG20] where maximal variation of the normalization of singular curves of unbounded degree is studied.

1.3. In this note, we study curves contained in an arbitrary but fixed K3 surface \(S\). Without any condition on the Picard group of \(S\) or on the genus of the curves, our result says that curves in a non-primitive, base point free, ample linear system have maximal variation, i.e. up to finite ambiguity a generic curve \(C\) is embedded uniquely in its ambient K3 surface.

Under additional assumptions on \(m\) or \((H.H)\) the result follows directly from existing results. For example, if \((H.H) \geq 48\), it is an immediate consequence of the work of Hein [He06]. For \(m \geq 6\) it follows, at least for non-hyperelliptic curves, from a general restriction result due to Balaji–Kollár [BK12] and for \(m \geq 5\) from Totaro’s Bott vanishing theorems [To20]. Our approach is logically independent of these results and uses as the main input another result of Hein [He96] which we state and prove in Section 2.1.

For K3 surfaces with maximal Mumford–Tate group, maximal variation as claimed here is related to a recent result of van Geemen and Voisin [vGV16]. Establishing the result in the non-generic case lends further strong evidence for a conjecture of Matsushita.

1.4. The original idea of our approach was to deduce maximal variation for curves contained in a K3 surface from a similar statement for spectral curves. More precisely, instead of studying curves in a linear system \(|mH|\) on a K3 surface one can look at spectral covers \(D \rightarrow C\) of degree \(m\) of a distinguished primitive curve \(C \in |H|\).
Here, one thinks of the spectral curve $D$ as being contained in the cotangent bundle $\mathcal{V}(\omega_C)$ of $C$. So the projective K3 surface $S$ is replaced by the quasi-projective symplectic surface $\mathcal{V}(\omega_C)$ and curves in $|mH|$ specialise to spectral covers $\mathcal{V}(\omega_C) \supset D \to C$. This degeneration technique was first studied by Donagi–Ein–Lazarsfeld [DEL96] and exploited in a recent paper of de Cataldo–Maulik–Shen [dCMS19].

It turns out that the variation of spectral curves is not maximal, so Theorem 1.1 fails when K3 surfaces are replaced by $\mathcal{V}(\omega_C)$. However, due to a result of Hodge and Mulase [HM10], the isotrivial families of spectral curves are well understood, which allows one to bound the dimension of isotrivial families in $|mH|$ for $m \geq 2$. Assuming Matsushita’s conjecture that Lagrangian fibrations of hyperkähler manifolds have zero or maximal variation, would then immediately yield maximal variation in $|mH|$ for $m \geq 2$. Thus, the verification of the conjecture [vGV16] for K3 surfaces with maximal Mumford–Tate group and Picard number $\rho \leq 17$ gives an alternative proof for the maximal variation in these cases.

1.5. Outline of content. In Section 2 we recall and prove a result due to Hein, see Theorem 2.1, from which we conclude that $h^0(T_S|_C) \leq 2$ for a generic ample curve $C$ on $S$, see Corollary 2.4. Next, we explain how to use a result of Balaji–Kollár to prove Theorem 1.1 for $|H|$ non-hyperelliptic and $m \geq 6$, see Corollary 2.6. The main result of Section 2 and the key technical result of the paper is Theorem 2.9 which shows $h^0(T_S|_C) = 0$ for a generic $C \in |mH|$, $m \geq 2$. The section concludes with Remark 2.11 addressing the case $m \geq 3$ without additional assumptions on the non-reduced locus.

In Section 3 we study the rational map $|H| \dashrightarrow M_g$, describe its derivative and show that its generic fibre is of dimension at most two, see Corollary 3.2. Theorem 1.1 is then an immediate consequence, its proof is presented in Section 3.2. It is here where the technical assumption on the non-reduced locus is used. In this section, we also provide examples to show that neither ampleness nor base point freeness can be dropped from the hypothesis of our main theorem. In the rest of this section, we explain the relation to Totaro’s Bott vanishing results.

In Section 4 we study the analogous situation provided by spectral curves and the Hitchin system. After briefly recalling the basic setup we give a short proof of a result due to Hodge and Mulase showing that every spectral curve of degree at least two is contained in a $(g+1)$-dimensional isotrivial family, see Corollary 4.2. We also explain how to use specialisation to the normal cone to deduce from the spectral curves situation information about variation of curves on K3 surfaces, see Corollary 4.4. The result is then combined with the verification of Matsushita’s conjecture by van Geemen and Voisin for very general hyperkähler manifolds to give an alternative proof of Theorem 1.1 for very general K3 surfaces without additional conditions on $|H|$ and without using any information about the restriction of the tangent bundle.

The final Section 5 contains more on the relation to Matsushita’s conjecture and the proof of Theorem 1.1 for $g = 2$ and $m \geq 1$. 
1.6. **Conventions.** The main result is proved for smooth projective K3 surfaces over an algebraically closed field $k$ of characteristic zero. In Sections 4 and 5 we mention the Hodge theoretic result [vGV16] for which $k = \mathbb{C}$ is needed.

By $H$ we denote a base point free, ample line bundle on $S$, not necessarily primitive. The linear system $|H|$ is called non-hyperelliptic if it contains a smooth non-hyperelliptic curve. Any very ample linear system is non-hyperelliptic. The degree of $S$ with respect to $H$ is $d := (H.H)$ and the genus of $H$ is the genus of a smooth curve in $|H|$, i.e. $2g - 2 = d$ or, in other words, $g = \dim |H|$.

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2. Restricting the tangent bundle

As we will recall in Section 3, the infinitesimal nature of the rational map $|H| \to Mg$ at a smooth curve $C \in |H|$ is determined by the restriction $T_{S|C}$ of the tangent bundle $T_S$ of the K3 surface $S$, see Lemma 3.1. Ideally one would like $T_{S|C}$ to be stable, at least for the generic curve $C \in |H|$, since stability ensures the vanishing of global sections of $T_{S|C}$ which is crucial for our main theorem. However, semistability can also be exploited at the expense of significantly more work. This is the core of our main technical result. All three aspects will be discussed in this section.

2.1. The following result is originally due to Hein [He96, Kor. 3.11]. Since the result was never published and is crucial for what follows, we include a complete proof for the convenience of the reader. See Remark 2.3 for a weaker assertion in the case of hyperelliptic linear systems.

**Theorem 2.1** (Hein). Let $E$ be a bundle of rank two on a K3 surface $S$. Assume $E$ is $\mu$-semistable of degree zero with respect to a base point free, ample line bundle $H$. Then the restriction $E|_{C}$ to a generic, non-hyperelliptic curve $C \in |H|$ is again semistable.

**Proof.** Consider the universal family

$$
\begin{array}{ccc}
|H| & \xrightarrow{p} & \mathcal{C} \\
& & \xrightarrow{q} \quad S
\end{array}
$$

of curves in the linear system $|H|$. If the restriction $E|_{C}$ to the generic curve $C \in |H|$ is not semistable, then there exists a relative Harder–Narasimhan filtration $0 \subset F \subset q^*E$, i.e. $F$ and $F' := q^*E/F$ are flat over some dense open subset $U \subset |H|$ and the restriction $F|_{C}$ to a fibre $C := \mathcal{C}_s \subset \mathcal{C}$, $s \in U$, is a line bundle of positive degree, see [HL10, Sec. 2.3] for details and references.
Now, unless $F \subset q^*E|_U$ is the pull-back of a sub-line bundle of $E$ on $S$, which would contradict the semistability of $E$, the differential of the induced morphism $C|_U \rightarrow \Grass_S(E,1)$ yields a non-zero map $T_q|_U \rightarrow F^* \otimes F'$. This is the standard argument in the proof of the Grauert–Mülich theorem, cf. [HL10, Thm. 3.1.2]. Here, $T_q$ is the relative tangent bundle of the projection $q$. Note that $T_q$ restricted to a fibre $C = C_s$, $s \in U$, is isomorphic to the kernel $M_C$ of the evaluation map $H^0(C,\omega_C) \otimes O_C \rightarrow \omega_C$. Indeed, the relative Euler sequence

$$0 \rightarrow O_C \rightarrow q^*K \otimes p^*O(1) \rightarrow T_q \rightarrow 0$$

of the projective bundle $q: C \simeq \mathbb{P}(K) \rightarrow S$, where $K := \text{Ker} \left( H^0(S,H) \otimes O_S \rightarrow H \right)$, restricted to $C$ is of the form $0 \rightarrow O_C \rightarrow K|_C \rightarrow T_q|_C \rightarrow 0$ and, therefore, $M_C \simeq T_q|_C$.

Hence, the image of any non-trivial morphism $T_q \rightarrow F^* \otimes F'$ restricted to $C$ would yield a line bundle quotient $M_C \rightarrow L$ of degree $\deg(L) \leq \deg(F^*) + \deg(F') = -2 \deg(F) \leq -2$ on the generic curve $C \in |H|$.

The latter is excluded by a result of Paranjape and Ramanan [PR88, Lem. 3.41] saying that on a smooth projective curve $C$ the bundle $M_C$ is stable if $C$ is not hyperelliptic, see also [Pa92]. Indeed, stability in particular implies that any quotient line bundle $M_C \rightarrow L$ satisfies

$$-2 = \frac{2 - 2g(C)}{g(C) - 1} = \mu(M_C) < \deg(L),$$

which concludes the proof.\footnote{The last part of the argument replaces the proof of [He96, Kor. 3.11], which seems incomplete. An argument is missing to ensure that the line bundle $L \otimes \omega_C \otimes A^*$ is special and, more importantly, effective or weaker that it has non-negative degree.}

\[\square\]

**Remark 2.2.** The classical restriction result due to Flenner asserts the semistability of the generic restriction $E|_C$ for $C \in |mH|$ as soon as $m \geq 2(H,H)$ and $H$ is very ample, cf. [HL10, Thm. 7.1.1]. So not only that one has to pass to multiples of the linear system, but the bound in Flenner’s theorem depends on the degree of the surface.

**Remark 2.3.** When the generic curve $C \in |H|$ is hyperelliptic, the bundle $M_C$ is the pull-back of the kernel of the evaluation map $\eta: H^0(\mathbb{P}^1, O(g-1)) \otimes O \rightarrow O(g-1)$ on $\mathbb{P}^1$ under the hyperelliptic quotient $\pi: C \rightarrow \mathbb{P}^1$. The kernel of $\eta$ is readily identified as $O(-1)^{\oplus g-1}$ and hence $M_C \simeq \pi^*O(-1)^{\oplus g-1}$. In particular, $M_C$ is really not stable in this case, but it is still polystable. Therefore, any line bundle quotient $M_C \rightarrow L$ as in the proof above satisfies the weaker inequality $\deg(L) \geq -2$. Furthermore, if $\deg(L) = -2$, then $L \simeq \pi^*O(-1)$.

We apply the theorem to the tangent bundle $T_S$ of a K3 surface $S$ which is $\mu$-stable with respect to any ample line bundle on $S$, see [Hu16, Sec. 9.4] for a proof and references.
Corollary 2.4. Let $H$ be a base point free, ample line bundle on a K3 surface $S$. Then for a generic curve $C ∈ |H|$ either $H^0(C, T_S|C) = 0$ or there exists a short exact sequence

$0 \rightarrow \mathcal{O}_C \rightarrow T_S|C \rightarrow \mathcal{O}_C \rightarrow 0$. In particular,

$$\dim H^0(C, T_S|C) ≤ 2 \text{ and } H^0(C, T_S|C(-H)) = 0.$$  

Proof. Since $H^0(C, T_S|C) \simeq \text{Hom}(\mathcal{O}_C, T_S|C)$ and $\deg(T_S|C) = 0$, stability of the restriction $T_S|C$ would yield $h^0(C, T_S|C) = 0$.

If $T_S|C$ is only semistable, then we have an exact sequence

$0 \rightarrow L_1 \rightarrow T_S|C \rightarrow L_2 \rightarrow 0$

with $L_1, L_2$ both line bundles of degree zero. In fact, since $\det(T_S|C) \simeq \mathcal{O}_C$, we have $L_1 \simeq L^*_2$. Thus, only two cases can occur. Either, $L_1 \simeq L_2 \simeq \mathcal{O}_C$, in which case $h^0(C, T_S|C) ≤ 2$, or both line bundles, $L_1$ and $L_2$, are non-trivial and hence $h^0(C, T_S|C) = 0$. By Theorem 2.1, this concludes the proof in the case of a non-hyperelliptic linear system.

If the generic curve $C ∈ |H|$ is hyperelliptic and $T_S|C$ is not semistable, then by Remark 2.3 the Harder–Narasimhan filtration

$0 \rightarrow L_1 \rightarrow T_S|C \rightarrow L_2 \rightarrow 0$

satisfies $L_2 \simeq L^*_1$ and $\deg(L) = −2$ for the invertible sheaf $L := L^*_1 \otimes L_2 \simeq L^{-2}$, which is in fact the pull-back $\pi^*\mathcal{O}(−1)$ under the hyperelliptic quotient $\pi: C \rightarrow \mathbb{P}^1$. Therefore, $\deg(L_1) = 1$, $\deg(L_2) = −1$, and $H^0(C, L_1) \simeq H^0(C, T_S|C)$. Hence, either $L_1$ is effective, and then $h^0(C, T_S|C) = h^0(C, L_1) = 1$, or it is not, in which case $h^0(C, T_S|C) = 0$.

If $L_1$ is effective, then the Harder–Narasimhan filtration of the restriction $T_S|C$ to the generic curve is of the form

$0 \rightarrow \mathcal{O}(x_C) \rightarrow T_S|C \rightarrow \mathcal{O}(-x_C) \rightarrow 0$ for a distinguished point $x_C ∈ C$. Note that $\mathcal{O}(-2x_C) \simeq L \simeq \pi^*\mathcal{O}(−1)$ implies that $x_C$ is a fixed point of the hyperelliptic involution or, in other words, $x_C$ is contained in the ramification curve $C_0$ of the hyperelliptic quotient $S \rightarrow \tilde{S}$. Consider the rational section of the universal curve $C → |H|$ given by mapping $C$ to the distinguished point $x_C$. Then the image of the induced rational map $|H| → S$ is $C_0$. However, it is known that $\tilde{S}$ is either $\mathbb{P}^2$ or one of the Hirzebruch surfaces $\mathbb{F}_i$, $i = 0, \ldots, 4$, see [Do73, Re76]. Since $C_0$ considered as a curve on $\tilde{S}$ must be contained in $|ω_{\tilde{S}}^{-2}|$, it cannot be rational. This yields a contradiction.

The claimed vanishing is a consequence of the above.

□

Remark 2.5. (i) Note that the vanishing $H^0(C, T_S|C \otimes H^*) = 0$ for generic $C$ can alternatively be deduced from a result due to Beauville–Mérindol [BM87, Prop. 3], at least for hyperplane sections: The normal bundle sequence of any smooth hyperplane section $C \subset S$ is not split unless $C$ is the fixed locus of an involution. In particular, this does not happen for the generic smooth hyperplane sections and, in fact, can only occur for small genus. See Section 4.1 for a comparison with the situation for spectral curves.

(ii) It turns out that the situation is even better for singular curves. More precisely, we have $H^0(C, T_S|C(-C)) = 0$ for every singular, but reduced curve $C ∈ |H|$ as soon as $g > 2$. To prove
this, we use the conormal bundle sequence
\[
0 \longrightarrow \mathcal{O}_C(-C) \longrightarrow \Omega_S|_C \longrightarrow \Omega_C \longrightarrow 0
\]
and the natural map \(\varphi: \Omega_C \longrightarrow \omega_C\) obtained by tensoring \(\mathcal{O}_C(-C) \otimes \Omega_C \longrightarrow \omega_S|_C \simeq \mathcal{O}_C, f \otimes \omega \longrightarrow df \wedge \omega\), with the line bundle \(\mathcal{O}_C(C)\). Clearly, \(\ker(\varphi)\) and \(\coker(\varphi)\) are concentrated in the singular points of \(C\) of which there are at most \(g\). In fact, it is known classically that \(h^0(\ker(\varphi)) = h^0(\coker(\varphi)) \leq 2(g - g(\tilde{C}))\), where \(\tilde{C} \longrightarrow C\) is the normalization, cf. [DG18, BG80]. In particular, \(0 < h^0(\ker(\varphi)) \leq 2g\).

Consider \(0 \neq s \in H^0(C, T_S|_C(-C))\) interpreted as a homomorphism \(s: \mathcal{O}_C(C) \longrightarrow T_S|_C \simeq \Omega_S|_C\). Since \(\omega_C \simeq \mathcal{O}_C(C)\) and \(\coker(\varphi) \neq 0\), the composition of \(s\) with \(\Omega_S|_C \longrightarrow \Omega_C \longrightarrow \omega_C\) has to be trivial. In other words, the composition \(\mathcal{O}_C(C) \longrightarrow \Omega_S|_C \longrightarrow \Omega_C\) takes image in the torsion subsheaf \(\ker(\varphi) \subset \Omega_C\).

Hence, \(s\) induces a non-trivial homomorphism \(\mathcal{O}_C(C) \otimes I_Z \longrightarrow \mathcal{O}_C(-C)\):
\[
\begin{array}{ccc}
\mathcal{O}_C(C) \otimes I_Z & \longrightarrow & \mathcal{O}_C(C) \\
\downarrow & & \downarrow \\
\mathcal{O}_C(-C) & \longrightarrow & \Omega_S|_C \\
\downarrow \quad s & & \downarrow \\
\mathcal{O}_C(-C) & \longrightarrow & \Omega_C
\end{array}
\]

Here, \(Z \subset C\) is contained in the singular locus of \(C\) with \(h^0(\mathcal{O}_Z) \leq h^0(\ker(\varphi))\). Together with \((2C.C) = 4g - 4\) this contradicts \(h^0(\ker(\varphi)) \leq 2g\) for \(g > 2\).

2.2. A related result of Balaji and Kollár [BK12, Thm. 2] translated to our situations gives the stability of \(T_S|_C\) for a generic \(C \in |mH|, m \geq 6\), and \(|H|\) a base point free, ample, non-hyperelliptic linear system on \(S\). As observed in the proof of Corollary 2.4, stability has the following direct consequence.

**Corollary 2.6.** Assume \(|H|\) is a base point free, ample, non-hyperelliptic linear system on a K3 surface \(S\). Then for a generic curve \(D \in |mH|, m \geq 6\), the restriction \(T_S|_D\) is stable and, in particular, \(H^0(D, T_S|_D) = 0\). \(\square\)

Neither [BK12] nor the above consequence will be used in what follows. Nonetheless it emphasises our point that multiple linear systems behave better than primitive ones.

**Remark 2.7.** (i) One should compare their stability result to other ones. For instance, a result of Bogomolov, cf. [HIL10, Thm. 7.3.5], implies that \(T_S|_C\) is stable for any smooth curve \(C \in |mH|\) for \(m \geq 96\) and \(|H|\) a very ample linear system on \(S\). The bound is significantly bigger than that of [BK12]. A better bound is due to Hein [He06, Thm. 2.8]; namely the restriction \(T_S|_C\) to any smooth curve \(C \in |mH|\) is stable if \(2m(H.H) > 96\), which holds for all \(m \geq 1\) as soon as \((H.H) > 48\).

(ii) For smaller degree there exist examples of positive-dimensional isotrivial subfamilies of curves \(C \in |H|\) with non-stable restriction \(T_S|_C\), see [GO20] and Example 3.7.
Remark 2.8. The vanishing \( H^0(D, \mathcal{T}_S|_D) = 0 \) for \( D \in |\mathcal{O}(m)|, m \geq 5, \) also follows from \cite[Thm. 5.1]{To20}, see Remark 3.9.

2.3. The two results, Theorem 2.1 and Corollary 2.6, are complemented by the following, which is the key to the proof of the main result Theorem 1.1.

Theorem 2.9. Assume \( |H| \) is a base point free, ample linear system on a K3 surface \( S \) such that the locus of non-reduced curves in \( |H| \) has codimension at least three. Then, for a generic curve \( D \in |mH|, m \geq 2, \) one has \( H^0(D, \mathcal{T}_S|_D) = 0. \)

Proof. For the case \( g = 2 \) see Proposition 5.4, so we restrict to the case \( g > 2 \) and hence can make use of Remark 2.5, (ii). We first show that it suffices to prove the assertion for \( m = 2. \)

Indeed, assume \( H^0(D, \mathcal{T}_S|_D) = 0 \) for the generic curve \( D \in |2H| \). Then, if \( m > 2 \), pick a generic curve \( D' \in |(m-2)H| \) and tensor the exact sequence

\[
0 \longrightarrow \mathcal{O}_{D'}(-D) \longrightarrow \mathcal{O}_{D\cup D'} \longrightarrow \mathcal{O}_D \longrightarrow 0
\]

with \( \mathcal{T}_S \). The resulting long exact cohomology sequence is of the form

\[
0 \longrightarrow H^0(D', \mathcal{T}_S|_{D'}(-D)) \longrightarrow H^0(D \cup D', \mathcal{T}_S|_{D\cup D'}) \longrightarrow H^0(D, \mathcal{T}_S|_D) \longrightarrow \cdots.
\]

By Corollary 2.4, the restriction \( \mathcal{T}_S|_{D'} \) satisfies \( H^0(D', \mathcal{T}_S|_{D'}(-D)) = 0 \). Combined with the assertion for \( m = 2 \), this yields the vanishing \( H^0(D \cup D', \mathcal{T}_S|_{D\cup D'}) = 0 \) and, by semi-continuity, this proves the assertion for the generic curve in \( |mH| \).

To prove the assertion for \( m = 2 \), let \( C, C' \in |H| \) be generic curves. Again by semi-continuity, it suffices to show \( H^0(C \cup C', \mathcal{T}_S|_{C \cup C'}) = 0 \).

If the restriction \( \mathcal{T}_S|_C \) to a generic \( C \in |H| \) satisfies \( H^0(C, \mathcal{T}_S|_C) = 0 \), then also \( H^0(C \cup C', \mathcal{T}_S|_{C \cup C'}) = 0 \), by the same arguments as above. Otherwise, by Corollary 2.4, there exists a short exact sequence

\[
0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{T}_S|_C \longrightarrow \mathcal{O}_C \longrightarrow 0.
\]

This leaves us with the two cases: (i) \( h^0(C, \mathcal{T}_S|_C) = 1 \), or equivalently (2.1) does not split, or (ii) \( h^0(C, \mathcal{T}_S|_C) = 2 \), i.e. \( \mathcal{T}_S|_C \simeq \mathcal{O}_C \oplus \mathcal{O}_C \), for the generic curve \( C \in |H| \).

(i) We assume \( h^0(C, \mathcal{T}_S|_C) = 1 \) for a generic \( C \in |H| \) and suppose that also \( h^0(C \cup C', \mathcal{T}_S|_{C \cup C'}) = 1 \) for two generic curves \( C, C' \in |H| \). To derive a contradiction we pick a generic pencil \( p: \mathbb{C} \longrightarrow \mathbb{P}^1 \) in the base point free linear system \( |H| \) and fix a generic curve \( C' \in |H| \). By our assumption on the non-reduced locus in \( |H| \), all fibres of \( p \) are reduced. The other projection \( q: \mathbb{C} \longrightarrow S \) is the blow-up in the base locus of the pencil. By Remark 2.5, (ii), the natural restriction map \( H^0(C \cup C', \mathcal{T}_S|_{C \cup C'}) \longrightarrow H^0(C', \mathcal{T}_S|_{C'}) \) is injective for all fibres \( C = C_s, s \in \mathbb{P}^1 \), and under our assumption in fact bijective. Similarly, also the restriction maps \( H^0(C \cup C', \mathcal{T}_S|_{C \cup C'}) \longrightarrow H^0(C, \mathcal{T}_S|_C) \) are injective. Thus, the bundle with fibres \( H^0(C \cup C', \mathcal{T}_S|_{C \cup C'}) \) is isomorphic to the trivial line bundle with constant fibre \( H^0(C', \mathcal{T}_S|_{C'}) \).
and, at the same time, it is contained in \( p_*q^*T_S \), which generically has fibre \( H^0(C, T_S|_C) \). Thus, \( O_{\mathbb{P}^1} \subset p_*q^*T_S \), which yields the contradiction \( k = H^0(\mathbb{P}^1, O_{\mathbb{P}^1}) \subset H^0(\mathbb{P}^1, p_*q^*T_S) \simeq H^0(C, q^*T_S) \simeq H^0(S, T_S) = 0 \).

To formalize the argument, let \( \tilde{p} : \tilde{C} \longrightarrow \mathbb{P}^1 \) be the family obtained by gluing \( C \) and the constant family \( C' := C' \times \mathbb{P}^1 \) along \( C' \). Here, \( C' \) is naturally embedded into \( C \) by viewing \( C \) as the blow-up of \( S \) and \( C' = C' \times \mathbb{P}^1 \) as the graph of the induced projection \( C' \subset C \longrightarrow \mathbb{P}^1 \). So, if \( C \) is the fibre \( C_s = p^{-1}(s), s \in \mathbb{P}^1 \), then \( p^{-1}(s) \simeq C \cup C' \). We denote by \( \tilde{q} : \tilde{C} \longrightarrow S \) the second projection. Then the fibres of \( E := \tilde{p}^*q^*T_S \) are the lines \( H^0(C \cup C', T_S|_{C \cup C'}) \). Moreover, we have a natural inclusion \( E \subset p_*q^*T_S \), which is the direct image of the restriction \( \tilde{q}^*T_S \longrightarrow q^*T_S \) under the inclusion \( C \subset \tilde{C} \). Via the restriction to \( C' \subset \tilde{C} \), the sheaf \( E \) can also be identified with the trivial bundle \( H^0(C', T_S|_{C'}) \otimes O_{\mathbb{P}^1} \simeq O_{\mathbb{P}^1} \).

(ii) Let us now assume that \( T_S|_C \simeq O_C \oplus O_C \) for the generic curve \( C \in |H| \). As above, we choose a generic pencil \( C \longrightarrow \mathbb{P}^1 \) and let \( C' \in |H| \) be a fixed generic curve. As before, it is enough to show that \( h^0(C \cup C', T_S|_{C \cup C'}) = 0 \) for the generic fibre \( C = C_s, s \in \mathbb{P}^1 \). Due to Remark 2.5, (ii), for every fibre \( C = C_s \) we have natural inclusions
\[
H^0(C \cup C', T_S|_{C \cup C'}) \subset H^0(C, T_S|_C) \quad \text{and} \quad H^0(C \cup C', T_S|_{C \cup C'}) \subset H^0(C', T_S|_{C'}) \simeq k^2.
\]

Hence, \( h^0(C \cup C', T_S|_{C \cup C'}) \leq 2 \). Furthermore, if \( h^0(C \cup C', T_S|_{C \cup C'}) = 2 \) holds generically, then it holds for all fibres \( C \). This, however, would result in the inclusion of the trivial bundle \( O^{\mathbb{P}^2}_{\mathbb{P}^1} \) with fibre at the point \( s \in \mathbb{P}^1 \) naturally identified with \( H^0(C_s \cup C', T_S|_{C_s \cup C'}) \simeq H^0(C', T_S|_{C'}) \simeq k^2 \) into the sheaf \( p_*q^*T_S \) and eventually lead to a similar contradiction \( k^2 \simeq H^0(\mathbb{P}^1, O^{\mathbb{P}^2}_{\mathbb{P}^1}) \subset H^0(\mathbb{P}^1, p_*q^*T_S) \simeq H^0(S, T_S) = 0 \).

(iii) It remains to deal with the case \( h^0(C, T_S|_C) = 2 \) and \( h^0(C \cup C', T_S|_{C \cup C'}) = 1 \) for generic curves \( C, C' \in |H| \). Here, we use the assumption that the non-reduced curves in \( |H| \) form a closed set of codimension at least three. We denote its open complement of all reduced curves by \( U := |H|_{\text{red}} \subset |H| \).

Now, under our assumptions, we can consider the rational map
\[
\psi : |H| - \longrightarrow \mathbb{P}(H^0(C', T_S|_{C'})), \quad C \longmapsto H^0(C \cup C', T_S|_{C \cup C'}).
\]

The map is regular on the complement \( U \setminus Z_{C'} \) of the proper, closed subset \( Z_{C'} \subset U \) of all curves \( C \in U \) with \( H^0(C \cup C', T_S|_{C \cup C'}) \longrightarrow H^0(C', T_S|_{C'}) \) surjective (and hence bijective by Remark 2.5, (ii)). We proceed by distinguishing two cases: Either, \( \text{codim}(Z_{C'}) \geq 2 \) or \( \text{codim}(Z_{C'}) = 1 \).

In the first case, the generic fibre \( \psi^{-1}(\ell) \) of \( \psi : U \setminus Z_{C'} \longrightarrow \mathbb{P}(H^0(C', T_S|_{C'})) \) contains a complete, integral curve, say \( T \subset \psi^{-1}(\ell) \). We then consider the restriction \( C_T \) of the universal curve to \( T \) and its two projections \( T \xleftarrow{p} C_T \xrightarrow{q} S \). As in step (ii), \( E \) on \( T \) shall denote the bundle with fibre \( H^0(C_t \cup C', T_S|_{C_t \cup C'}) \) over the point \( t \in T \). By our choice of \( T \), this bundle is isomorphic to \( \ell \otimes O_T \) and hence trivial. On the other hand, it is a subsheaf of \( p_*q^*T_S \) and, therefore, \( H^0(C_T, q^*T_S) \neq 0 \). In (ii), the total space of the one-dimensional family was the blow-up of \( S \).
which was enough to derive the contradiction \( H^0(S, \mathcal{T}_S) \neq 0 \). The same argument works here due to Lemma 2.10 below.

To conclude we have to deal with the case that \( \text{codim}(\mathcal{Z}_{C'}) = 1 \). In this case, \( \mathcal{Z}_{C'} \) itself contains a complete, integral curve \( T \subset \mathcal{Z}_{C'} \), simply because the boundary \( \mathcal{Z}_{C'} \setminus \mathcal{Z}_{C'} \) of the closure \( \mathcal{Z}_{C'} \subset |H| \) is contained in \( |H| \setminus |H|_{\text{red}} \), which by assumption is of dimension at most \( g - 3 \) and hence \( \text{codim}(\mathcal{Z}_{C'} \setminus \mathcal{Z}_{C'}) \geq 2 \). For the corresponding family \( \mathcal{C}_T \) as above, the bundle with fibres \( H^0(C_t \cup C', \mathcal{T}_S|_{C_t \cup C'}) \simeq H^0(C', \mathcal{T}_S|_{C'}) \) is trivial. Hence, \( p_*q^*\mathcal{T}_S \) contains \( \mathcal{O}_T^{\oplus 2} \) which again yields a contradiction to \( H^0(S, \mathcal{T}_S) = 0 \).

\[ \square \]

**Lemma 2.10.** Assume \( q: S' \longrightarrow S \) is a generically finite morphism from a normal projective irreducible surface \( S' \) onto a K3 surface \( S \). Then \( H^0(S', q^*\mathcal{T}_S) = 0 \).

*Proof.* Using Stein factorization, we can reduce to the case that \( q \) is finite. Then use that for a finite morphism \( q: S' \longrightarrow S \) the pull-back \( q^*\mathcal{T}_S \) is \( \mu \)-polystable, cf. [HL10, Lem. 3.2.3]. Clearly, if \( q^*\mathcal{T}_S \) is \( \mu \)-stable, then \( H^0(S', q^*\mathcal{T}_S) = 0 \). Otherwise, \( q^*\mathcal{T}_S \) is isomorphic to a sum of invertible sheaves, say \( q^*\mathcal{T}_S \simeq L_1 \oplus L_2 \), which satisfy \( L_1 \simeq L_2^2 \). There are two cases: Either, \( L_1 \) and \( L_2 \) are not trivial, in which case \( H^0(S', L_i) = 0 \) and hence \( H^0(S', q^*\mathcal{T}_S) = 0 \), or \( L_1 \simeq L_2 \simeq \mathcal{O}_{S'} \), which contradicts \( c_2(q^*\mathcal{T}_S) = q^*c_2(\mathcal{T}_S) \neq 0 \).

\[ \square \]

**Remark 2.11.** Without the assumption on the reduced locus \( |H|_{\text{red}} \subset |H| \) the arguments in the proof show that \( h^0(C, \mathcal{T}_S|_C) \leq 1 \) for generic \( C \in |2H| \). This renders case (iii), which is where we used the assumption, in the proof unnecessary. Running the other two cases again for \( C \in |2H| \) and \( C' \in |H| \), we obtain \( H^0(D, \mathcal{T}_S|_D) = 0 \) for \( D = C \cup C' \) and hence also for generic curves \( D \in |3H| \).

### 3. Maximal variation

We study the family \( \mathcal{C} \longrightarrow |H| \) given by the universal curve over a base point free linear system \( |H| \) on a K3 surface \( S \). We denote by \( |H|_{\text{sm}} \subset |H| \) the subset of all smooth curves, which is Zariski open and dense.

Mapping a point \( s \in |H| \) corresponding to a smooth curve \( C := \mathcal{C}_s \) to the corresponding point \( [C] \in M_g \) in the Deligne–Mumford stack of smooth curves of genus \( g \), where \( 2g - 2 = (H.H)^2 = d \), describes a rational map

\[ \Phi: |H| -\longrightarrow M_g \ . \]

The map is regular on the open subset \( |H|_{\text{sm}} \subset |H| \) of all smooth curves. The curves in \( |H| \) have **maximal variation** if this map is generically quasi-finite or, equivalently, if its image is of dimension \( g = \dim |H| \).

#### 3.1. Consider a smooth fibre \( C := \mathcal{C}_s, s \in |H|_{\text{sm}} \). The tangent spaces of \( |H| \) at \( s \) and of \( M_g \) at \( [C] = \Phi(s) \) are naturally isomorphic to \( H^0(C, \omega_C) \simeq H^0(C, H|_C) \) and \( H^1(C, \mathcal{T}_C) \). The
differential $d\Phi$ at the point $s$ is the boundary map

$$
d\Phi: H^0(C, H|_C) \longrightarrow H^1(C, \mathcal{T}_C)
$$

of the long exact cohomology sequence associated with the normal bundle sequence

$$(3.1) \quad 0 \rightarrow \mathcal{T}_C \rightarrow \mathcal{T}_S|_C \rightarrow \omega_C \simeq H|_C \rightarrow 0.$$  

In particular, using $H^0(C, \mathcal{T}_C) = 0$, the relative Zariski tangent space $\mathcal{T}_\Phi(s)$ of $\Phi$ at the point $s \in |H|_{\text{sm}}$ is the vector space:

$$\mathcal{T}_\Phi(s) \simeq H^0(C, \mathcal{T}_S|_C).$$

**Lemma 3.1.** The rational map $\Phi: |H| \dashrightarrow M_g$ is generically quasi-finite if and only if $H^0(C, \mathcal{T}_S|_C) = 0$ for the generic curve $C \in |H|$. More generally, the dimension of the generic fibre of $\Phi$ is bounded from above by $h^0(C, \mathcal{T}_S|_C)$ for any (generic) smooth curve $C \in |H|$.

**Proof.** For a smooth curve $C = C_s$, $s \in |H|$, the differential $d\Phi: H^0(C, H|_C) \longrightarrow H^1(C, \mathcal{T}_C)$, is injective if and only if $H^0(C, \mathcal{T}_S|_C) = 0$. More generally, as we work in characteristic is zero, the dimension of a generic fibre $\Phi^{-1}([C])$ is the dimension of $\text{Ker}(d\Phi)$ at the generic point of that fibre. In particular, the generic fibre is zero-dimensional if and only if $H^0(C, \mathcal{T}_S|_C) = 0$ for the generic curve $C = C_s$. □

**Corollary 3.2.** Assume $|H|$ is a base point free, ample linear system on a K3 surface $S$. Then the generic fibre of the rational map $\Phi: |H| \dashrightarrow M_g$ is of dimension at most two. Equivalently,

$$\dim \text{Im}(\Phi) \geq g - 2.$$  

**Proof.** This is a direct consequence of Corollary 2.4 and Lemma 3.1. □

In particular, the family of smooth curves in $|H|$ can never be isotrivial for $g > 2$. For $7 \leq g \leq 23$ this can also be deduced from [Be14].

As another immediate consequence of the description of the relative Zariski tangent space, we state the following result which is strengthened by the main result Theorem 1.1.

**Corollary 3.3.** Assume $H$ is a base point free, ample, non-hyperelliptic linear system on a K3 surface. Then the rational map

$$\Phi_m: |mH|_{\text{sm}} \longrightarrow M_{g_m},$$

$g_m := \dim |mH|$, is generically quasi-finite for all $m \geq 6$. Equivalently,

$$\dim \text{Im}(\Phi_m) = g_m = \frac{m^2}{2} \cdot (H.H) + 1.$$  

**Proof.** For $m \geq 6$ this follows from the result of Balaji–Kollár, see Corollary 2.6. □
The result holds more generally for $m \geq 5$ and without the non-hyperelliptic assumption due to Bott vanishing [To20, Lem. 3.7 & Rem. 3.8]. We discuss the connection of this vanishing to the vanishing of $H^0(C, \mathcal{T}_S|_C)$ in Section 3.3.

The next result is proved by techniques similar to the ones in the proof of Theorem 2.9. It can be seen as the analogue of [To20, Lem. 3.5], see Remark 3.9.

**Lemma 3.4.** Assume $|H|$ is a base point free, ample linear system such that $H^0(C, \mathcal{T}_S|_C) = 0$ for the generic curve $C \in |H|$. Then $H^0(D, \mathcal{T}_S|_D) = 0$ for the generic curve $D \in |mH|$ for every $m \geq 1$.

**Proof.** We explain the case $m = 2$. The general case is similar. Pick a curve $C \in |H|$ with $H^0(C, \mathcal{T}_S|_C) = 0$ and consider the non-reduced curve $D = 2C \in |2H|$. Twisting the short exact sequence $0 \longrightarrow \mathcal{O}_C(-C) \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_C \longrightarrow 0$ with the locally free sheaf $\mathcal{T}_S$, we obtain an exact sequence $0 \longrightarrow H^0(C, \mathcal{T}_S|_C(-C)) \longrightarrow H^0(D, \mathcal{T}_S|_D) \longrightarrow H^0(C, \mathcal{T}_S|_C) \longrightarrow \cdots$. Since $H^0(C, \mathcal{T}_S|_C)$ and hence $H^0(C, \mathcal{T}_S|_C(-C))$ are both trivial, the vanishing of $H^0(D, \mathcal{T}_S|_D)$ follows. By semi-continuity, the vanishing then holds for the generic $C \in |2H|$. \hfill $\square$

3.2. We now come to the proof of Theorem 1.1, which, having set up all the necessary machinery, is now almost immediate.

Assume $|H|$ is a base point free, ample linear system with a non-reduced locus of codimension at least three. Then, by virtue of Theorem 2.9, we know that $H^0(D, \mathcal{T}_S|_D) = 0$ for a generic curve $D \in |mH|$, $m \geq 2$. Applying Lemma 3.1 concludes the proof. \hfill $\square$

**Remark 3.5.** Note that by virtue of Remark 2.11, maximal variation holds for $|mH|$, $m \geq 3$, without any assumption on the non-reduced locus. Also note that the assumption on the non-reduced locus in $|H|$ is a Zariski open condition for polarized K3 surfaces $(S, H)$. It always holds for polarized K3 surfaces with $\text{Pic}(S) = \mathbb{Z} \cdot H$.

**Example 3.6.** The two assumptions on the linear system $|H|$ to be base point free and ample are both necessary.

(i) Let $\pi: S \longrightarrow \mathbb{P}^1$ be an isotrivial elliptic K3 surface, then $H = \pi^* \mathcal{O}(1)$ is base point free but not ample. In this case, none of the linear systems $|mH|$, $m \geq 1$, has maximal variations.

(ii) If $|H|$ is ample but not base point free, then $H \simeq \mathcal{O}((g - 1)E + C)$. Here, $E$ is a smooth elliptic curve and $C \simeq \mathbb{P}^1$ with $(C.E) = 1$, see [Hu16, Cor. 2.3.15]. The general member in $|H|$ is a curve of the form $E_1 + \cdots + E_{g-1} + C$. For example, for $g = 4$ this provides a family of dimension at most three of abstract curves in the four-dimensional linear system $|H|$.

**Example 3.7.** Similarly, the conclusion of Theorem 1.1 cannot be strengthened. In general, neither is the map $|H|_{\text{sm}} \longrightarrow M_g$ quasi-finite nor generically injective. For a concrete example, consider the Fermat quartic $S \subset \mathbb{P}^3$ given by the equation $x_0^4 + \cdots + x_3^4 = 0$. Then the hyperplane sections $x_0 = tx_1$ describe a one-dimensional isotrivial family of smooth curves in $|\mathcal{O}(1)|$. Due
to the non-trivial automorphism group of \((S, \mathcal{O}_S(1))\), namely \(\text{Aut}(S, \mathcal{O}_S(1)) = \mu_4^4 / \mu_4 \times \mathfrak{S}_4\), the map \(|\mathcal{O}_S(1)|_{\text{sm}} \rightarrow M_3\) cannot be generically injective.

3.3. Let us briefly spell out the relation between the vanishing of \(H^0(C, \mathcal{T}_S|_C)\) and that of \(H^1(S, \mathcal{T}_S \otimes H^*)\). Note that while the former is the relative tangent space of the map \(\Phi: |H| \rightarrow M_g\) at \(C \in |H|\) the latter is the relative tangent space of the map

\[ \Phi: (C \subset S) \rightarrow C \]

with varying K3 surface \(S\), cf. [Be04, Sec. 5.2].

From \(0 \rightarrow \mathcal{T}_S \otimes H^* \rightarrow \mathcal{T}_S \rightarrow \mathcal{T}_S|_C \rightarrow 0\) and \(H^0(S, \mathcal{T}_S) = 0 = H^2(S, \mathcal{T}_S)\) we obtain the long exact cohomology sequence

\[ 0 \rightarrow H^0(\mathcal{T}_S|_C) \rightarrow H^1(\mathcal{T}_S \otimes H^*) \rightarrow H^1(\mathcal{T}_S) \rightarrow H^1(\mathcal{T}_S|_C) \rightarrow H^2(\mathcal{T}_S \otimes H^*) \rightarrow 0. \]

Hence, if \(H^1(S, \mathcal{T}_S \otimes H^*) = 0\), then also \(H^0(C, \mathcal{T}_S|_C) = 0\), but the converse need not hold. This immediately leads to the following observation.

**Lemma 3.8.** Assume \(H^1(S, \mathcal{T}_S \otimes H^*) = 0\). Then the morphism \(|H|_{\text{sm}} \rightarrow M_g\) is quasi-finite.

**Proof.** Note that the vanishing only depends on the linear system and not on the individual curve \(C \in |H|\). It implies \(H^0(C, \mathcal{T}_S|_C) = 0\) for all curves \(C \in |H|_{\text{sm}}\). Therefore, \(\Phi: |H|_{\text{sm}} \rightarrow M_g\) is immersive and, hence, quasi-finite.

**Remark 3.9.** (i) Totaro [To20] calls \(H^1(S, \mathcal{T}_S \otimes H^*) = 0\), or dually \(H^1(S, \Omega_S \otimes H) = 0\), Bott vanishing. If \(|H|\) is base point free and ample, then Bott vanishing for \(H\) implies Bott vanishing for all multiples \(mH\), \(m \geq 1\), which in particular implies \(H^0(D, \mathcal{T}_S|_D) = 0\) for every curve \(D \in |mH|\). In this sense, Lemma 3.4 really is the analogue of [To20, Lem. 3.5].

Bott vanishing holds if the following two conditions are satisfied: \((H.H) \geq 74\) and there is no curve \(E\) with \((E.E) = 0\) and \(1 \leq (H.E) \leq 4\), see [To20, Thm. 5.1]. For example, these assumptions are met for the linear system \(|mH|\) if \(m \geq 5\) and \((H.H) \geq 4\). Note that there exist examples where Bott vanishing fails for arbitrary large \((H.H)\), see [To20, Sec. 6].

(ii) Most of the results in [To20] concern K3 surfaces of Picard number one. For example, under this assumption, [To20, Thm. 3.3] asserts Bott vanishing \(H^1(S, \mathcal{T}_S \otimes H^*) = 0\) for \((H.H) = 20\) or \((H.H) \geq 24\). In particular, under these assumptions also the primitive linear system \(|H|\) and hence all its multiples \(|mH|\) have maximal variation, see [To20, Lem. 3.5]. On the other hand, Bott vanishing definitely fails for \((H.H) = 22\), see [To20, Thm. 3.2], but for the generic K3 surface \(|H|\) nevertheless has maximal variation.

(iii) The dimension \(h^1(S, \mathcal{T}_S \otimes H^*)\) is linked to the corank of the Wahl map

\[ \bigwedge^2 H^0(C, \omega_C) \rightarrow H^0(C, \omega_C^3), \]

see [CDS20] for details. Recall that for a smooth curve \(C \subset S\) in a K3 surface the Wahl map is not surjective, cf. [BM87].
4. Comparison with the Hitchin system

Instead of passing from a (smooth) curve $C \in |H|$ to non-primitive curves $D \in |mH|$ we now look at spectral curves $\dashrightarrow$ of degree $m$. There are many similarities between the two situations. To some extent the space $V(\omega_C)$, where the spectral curves live, is in a way easier and more uniform that K3 surfaces, even though the stability of the restriction of the tangent bundle fails. We will also explain how to use the information for spectral curves to deduce partial results for K3 surfaces more directly.

4.1. Let us begin by recalling some basic facts. A spectral curve of degree $m$ is naturally associated to $(s_i) \in \bigoplus_{i=1}^m H^0(C, \omega_C^i)$. More precisely, given $(s_i)$ one obtains a curve

$$D := D_{(s_i)} \subset V := \mathcal{V}(\omega_C) \subset T := \mathbb{P}(\mathcal{O}_C \oplus \omega_C)$$

in the linear system $|\pi^* \omega_C^m \otimes \mathcal{O}_\pi(m)|$, where $\pi: T \longrightarrow C$ is the natural projection. Indeed, $H^0(T, \pi^* \omega_C^m \otimes \mathcal{O}_\pi(m)) \cong H^0(C, \omega_C^m \otimes S^m(\mathcal{O}_C \oplus \omega_C)) \cong \bigoplus_{i=0}^m H^0(C, \omega_C^i)$. More explicitly, using the canonical sections $s$ and $t$ of $\pi_* \mathcal{O}_\pi(1) \cong \mathcal{O}_C \oplus \omega_C$ and $\pi_*(\pi^* \omega_C \otimes \mathcal{O}_\pi(1)) \cong \omega \oplus \mathcal{O}_C$ allows one to write an explicit equation for $D = D_{(s_i)}$:

$$t^m + \pi^* s_1 \cdot t^{m-1} \cdot s + \cdots + \pi^* s_m \cdot t^m \in H^0(T, \pi^* \omega_C^m \otimes \mathcal{O}_\pi(m)),$$

cf. [BNR89, Sec. 3]. For example, the zero section $C \subset V \subset T$ corresponds to $(s_1 = 0) \in H^0(C, \omega_C)$. Also, recall that $\pi_* \mathcal{O}_D \cong \mathcal{O}_C \oplus \cdots \oplus \omega_C^{m+1}$ for a spectral curve $\pi: D \longrightarrow C$ of degree $m$ and observe that the restriction of $\mathcal{O}_\pi(1)$ to $V \subset T$ is trivial.

The tangent bundle $\mathcal{T}_V$ of $V$ can be described naturally as an extension of pull-back line bundles

$$0 \longrightarrow \pi^* \omega_C \longrightarrow \mathcal{T}_V \longrightarrow \pi^* \mathcal{T}_C \cong \pi^* \omega_C^* \longrightarrow 0.$$  

The sequence splits when restricted to the zero-section $C \subset V$. Contrary to the case of K3 surfaces, the symplectic surface $V$ admits global vector fields. More precisely, $H^0(V, \mathcal{T}_V) \cong H^0(C, \pi_* \mathcal{T}_V)$ which can be computed via the direct image

$$0 \longrightarrow \pi_* \omega_C \otimes \bigoplus_{i \leq 0} \omega^i \longrightarrow \pi_* \mathcal{T}_V \longrightarrow \pi_* \pi^* \mathcal{T}_C \cong \omega_C^* \otimes \bigoplus_{i \leq 0} \omega^i \longrightarrow 0$$

of (4.2) as

$$H^0(V, \mathcal{T}_V) \cong H^0(C, \mathcal{O}_C) \oplus H^0(C, \omega_C).$$

In fact, thinking of $H^0(V, \mathcal{T}_V)$ as the tangent space of $\text{Aut}(V)$ both parts can be explained geometrically: (i) The vector bundle $V$ comes with a natural $k^*$-action; (ii) Any section $u \in H^0(C, \omega_C)$ acts by translation on $V$. Combining the two, one finds that $(\lambda, u) \in k^* \times H^0(C, \omega_C)$
acts on $V$ by $v \mapsto \lambda v + u$. From this, one also obtains an action of $k^* \times H^0(C, \omega_C)$ on the space of spectral covers or, equivalently, on the parameter space $\bigoplus_{i=1}^m H^0(C, \omega_C^i)$. Concretely, $k^*$ acts by $\lambda \cdot (s_i) = (\lambda^i \cdot s_i)$ and $H^0(C, \omega_C)$ by applying Tschirnhaus transformation to (4.1).

As in the case of K3 surfaces, we are interested in the restriction $\mathcal{T}_V|_D$ to a smooth spectral curve $D \to C$. The restriction of vector fields on $V$ to $D$ can be understood in terms of the exact sequence

$$0 \to H^0(V, \mathcal{T}_V(-D)) \to H^0(V, \mathcal{T}_V) \xrightarrow{r} H^0(D, \mathcal{T}_V|_D) \to H^1(V, \mathcal{T}_V(-D)) \to \cdots,$$

which implies the injectivity of $r$, as $H^0(V, \mathcal{T}_V(-D)) \simeq H^0(V, \mathcal{T}_V \otimes \pi^* \omega_C^{-m}) = 0$ for $m > 1$. In fact, $r$ is bijective according to the next result.

**Lemma 4.1.** Let $D \to C$ be a spectral curve of degree $m \geq 2$. Then the restriction map is an isomorphism

$$H^0(C, \mathcal{O}_C) \otimes H^0(C, \omega_C) \simeq H^0(V, \mathcal{T}_V) \xrightarrow{\sim} H^0(D, \mathcal{T}_V|_D).$$

In particular, the restriction $\mathcal{T}_V|_D$ is not semistable and $h^0(D, \mathcal{T}_V|_D) = g(C) + 1$.

**Proof.** We give an alternative argument to [HM10, Sec. 3]. Restricting the exact sequence (4.2) to $D$ and then taking direct image under $\pi: D \to C$, we obtain the exact sequence

$$0 \to \omega_C \otimes \bigoplus_{i=-m+1}^0 \omega_C^i \to \pi_* (\mathcal{T}_V|_D) \to \omega_C^* \otimes \bigoplus_{i=-m+1}^0 \omega_C^i \to 0.$$

Taking cohomology yields the result. \qed

Despite not giving the desired vanishing even for large $m$ and generic $D$, compare with Corollary 2.4 and Theorem 2.9, the observation is useful as the bound is independent of $m$ and, for example, stronger than the obvious one $h^0(D, \mathcal{T}_V|_D) \leq g_m := g(D)$ for $m \geq 2$, cf. Remark 2.5.

The analogue of Theorem 1.1 in the case of spectral curves is the following consequence, cf. [HM10, Thm. 3.1].

**Corollary 4.2** (Hodge–Mulase). Let $C$ be a smooth curve of genus $g$. We view the space $\bigoplus_{i=1}^m H^0(C, \omega_C^i)$ as the variety of spectral covers $D \to C$ of degree $m$ and consider the rational map

$$\Phi: \bigoplus_{i=1}^m H^0(C, \omega_C^i) - \to M_{g_m}, (s_i) \longmapsto [D_{(s_i)}].$$

Then every fibre of $\Phi$ through a smooth spectral curve is of dimension $g + 1$. In particular, $\Phi$ is nowhere quasi-finite.

**Proof.** The arguments are identical to the ones in Sections 3.1 and 3.2. \qed
Remark 4.3. It turns out that Bott vanishing holds for the surface $T$ with respect to spectral curves, i.e. $H^1(T, \mathcal{T}_T \otimes \mathcal{O}(-D)) = 0$ for $D \in |\pi^*\omega_C^m \otimes \mathcal{O}_\pi(m)|$. To prove this, one uses the exact sequence $0 \longrightarrow \mathcal{T}_\pi \simeq \pi^*\omega_C \otimes \mathcal{O}_\pi(2) \longrightarrow \mathcal{T}_T \longrightarrow \pi^*\mathcal{T}_C \longrightarrow 0$ for the projective bundle $\pi: T = \mathbb{P}(\mathcal{O}_C \oplus \omega_C) \longrightarrow C$.

Geometrically this means that the map $\tilde{\Phi}: (D \subset T) \longrightarrow D$ is infinitesimally injective. Here, $T$ can vary as well, either as a general surface, i.e. letting both $C$ and the bundle $\mathcal{O}_C \oplus \omega_C$ vary, or only as the completion of the cotangent bundle of deformations of $C$. So, the relative tangent spaces of $\Phi$ and $\tilde{\Phi}$ at a spectral curve $D$ are

$$T_\Phi([D]) \simeq H^0(C, \omega_C) \oplus H^0(C, \mathcal{O}_C) \text{ and } T_{\tilde{\Phi}}([D]) \simeq H^1(T, \mathcal{T}_T(-D)) = 0.$$  

Note this does not contradict the idea of Lemma 3.8, as for the map $\tilde{\Phi}$ one divides by the action of the group $\text{Aut}(T)$ whose tangent space is exactly $T_\Phi([D])$.

4.2. Specialisation from curves $D \in |mH|$ on a K3 surface to spectral curves $D \longrightarrow C$ of degree $m$ immediately yields a version of Corollary 3.2 and Theorem 1.1. The bound here is worse but below we will see that combined with a conjecture of Matsushita it would in fact suffice to prove Theorem 1.1, thereby giving an alternative approach to our main result.

Corollary 4.4. Assume $|H|$ is a base point free, ample linear system on a K3 surface $S$ of genus $g$. Then for all $m \geq 1$ the generic fibre of the rational map $\Phi: |mH| \longrightarrow M_{g_m}$ is of dimension at most $g + 1$. In other words, a generic curve $D \in |mH|$ satisfies $h^0(D, \mathcal{T}_S|D) \leq g + 1$ and $\dim \Phi^{-1}([D]) \leq g + 1$.

Proof. The assertion is certainly true for $m = 1$, so let us restrict to the case $m \geq 2$. Fix a smooth curve $C \in |H|$ and consider the degeneration of $C \subset S$ to its normal cone $C \subset V = V(\omega_C)$ or, alternatively, to $C \subset \bar{T}$, where $\bar{T}$ is the contraction of the section at infinity $\mathbb{P}(\mathcal{O}_C) \subset T = \mathbb{P}(\mathcal{O}_C \oplus \omega_C)$.

In the process, the linear system $|mH| = |mC|$ on $S$ specialises to the linear system $|\pi^*\omega_C^m \otimes \mathcal{O}_\pi(m)| = |mC|$ on $V$ (or on $\bar{T}$). Thus, spectral curves $D \longrightarrow C$ can be seen as specialisations of curves $D_t \in |mH|$, $t \neq 0$, on the constant family of K3 surfaces $S$. By semi-continuity, $h^0(D, \mathcal{T}_S|D) \leq g + 1$ follows from the corresponding statement $h^0(D, \mathcal{T}_V|D) = g + 1$ for spectral curves, see Lemma 4.1.

Alternatively and more geometrically, one can view (4.3) as the specialisation of the rational map $\Phi: |H| \longrightarrow M_{g_m}$, which also proves the claim. 

4.3. Corollary 4.4 can be combined with a result of van Geemen–Voisin [vGV16] to reprove Theorem 1.1 for the generic polarized K3 surface $(S, H)$. To make the connection, we need to recall how to associate a Lagrangian fibration to the linear system $|mH|$ on a K3 surface. For this consider the Mukai vector $v_m = (0, mh, s)$ with $s$ an integer prime to $m$. Then the moduli space $M(v_m)$ of sheaves $E$ with Mukai vector $v(E) = v_m$ which are stable with respect to a
generic polarization is a projective hyperkähler manifold deformation equivalent to the Hilbert scheme $S^{[m]}$, cf. [HL10, Sec. 6.2] or [Hu16, Sec. 10.2] for the precise statement and references. The map that sends $E \in M(v_m)$ to its scheme-theoretic support defines a Lagrangian fibration
\[ f : M(v_m) \rightarrow |mH|. \]

The fibre over a generic curve $D \in |mH|$ is isomorphic to the Jacobian $J(D)$. The infinitesimal Torelli theorem for curves ensures that the family of curves $D \in |mH|$ has maximal variation if and only if the same holds for the abelian fibres of the Lagrangian fibration $f$. More generally, the amount of variation is the same for $M(v_m) \rightarrow |mH|$ and the universal curve $D \rightarrow |mH|$.

**Corollary 4.5.** For each degree $d$, there exists a Zariski dense open subset $U \subset N_d$ in the moduli space of polarized K3 surfaces $(S, H)$ of degree $(H.H) = d$ such that for every $(S, H) \in U$ and $m \geq 2$ the rational map $|mH| \dashrightarrow M_{g_m}$ is generically quasi-finite.

**Proof.** Quasi-finiteness is an open condition. Therefore, it is enough to find one polarized K3 surface $(S, H)$ for which the assertion holds. It is known that the very general polarized K3 surface of degree $d$ has Picard number one and a trivial endomorphism field $\mathbb{Q} \simeq \text{End}_{\text{Hdg}}(T(S))$, cf. [vGV16, Lem. 9]. In particular, its Mumford–Tate group is maximal and, thus, the assumptions of [vGV16, Thm. 5] are satisfied. Hence, either the curves in the linear system $|mH|$, $m \geq 2$ have maximal or trivial variation. The latter is excluded by Corollary 4.4. \[ \square \]

**Remark 4.6.** (i) The technique actually shows more. It shows that in any irreducible closed subset $Z \subset N_d$ of the moduli space of polarized K3 surface containing at least one K3 surface $(S, H) \in Z$ with maximal Mumford–Tate group, the set of K3 surfaces $(S, H) \in Z$ with maximal variation in $|mH|$, $m \geq 2$, is Zariski open and dense.

(ii) Note that in the above proof, instead of Corollary 4.4 one could have evoked Corollaries 2.4 and 3.2, which would have allowed us to prove the result for $m = 1$ and $g > 2$ as well. However, the approach via Hitchin systems is easier for it avoids any restriction result as Theorem 2.1.

5. PRIMITIVE LINEAR SYSTEMS AND MATSUSHITA’S CONJECTURE

In the proof of Corollary 4.5 we used [vGV16, Thm. 5] which proves a conjecture of Matsushita under additional assumptions on the rank of the transcendental lattice $T(X) \subset H^2(X, \mathbb{Z})$ and its Mumford–Tate group. More precisely, van Geemen–Voisin answer affirmatively the following conjecture for very general hyperkähler manifolds.

**Conjecture 5.1** (Matsushita). Let $f : X \dashrightarrow \mathbb{P}^m$ be a Lagrangian fibration of a projective hyperkähler manifold. Then, either the family of smooth fibres, which are abelian varieties, has maximal variation or it is isotrivial.
5.1. Our main result Theorem 1.1 can be seen as further evidence for the conjecture, because it immediately confirms it for essentially all Lagrangian fibrations $f: M(v_m) \hookrightarrow \mathbb{P}^{2m}$ associated with the non-primitive, base point free ample linear system $|mH|$, $m \geq 2$, see Section 4.3 for the notation and explanations. As it is in principle possible that Conjecture 5.1 holds generically but fails for special Lagrangian fibrations, the fact that Theorem 1.1 holds for essentially all K3 surfaces is seen as strong evidence in favour of the conjecture.

Conversely, Matsushita’s conjecture combined with Corollary 2.4 would provide a quick way to prove Theorem 1.1 and not only for $m \geq 2$, but in fact for all $m \geq 1$.

**Proposition 5.2.** Let $H$ be a base point free, ample linear system on a K3 surface $S$ of genus $g \geq 3$. Assume Conjecture 5.1 holds for the moduli space $M(v)$ of stable sheaves with Mukai vector $v = (0, H, s)$ and the natural Lagrangian fibration $f: M(v) \hookrightarrow \mathbb{P}^g$. Then the linear system $|H|$ has maximal variation.

**Proof.** Due to Corollaries 2.4 and 3.2, we know that an isotrivial family of curves through the generic point in $|H|$ is of dimension at most two. In particular, for $g \geq 3$ the Lagrangian fibration $M(v) \hookrightarrow \mathbb{P}^g$ cannot be isotrivial. Hence, by Conjecture 5.1, it has to have maximal variation. □

As [vGV16, Thm. 5] verifies Conjecture 5.1 under additional hypotheses, one immediately obtains the following consequence.

**Corollary 5.3.** Assume $S$ is a K3 surface with Picard number $\rho(S) \leq 17$ and Hodge endomorphism field $\text{End}(T(S) \otimes \mathbb{Q}) \simeq \mathbb{Q}$. Then any ample, base point free linear system $|H|$ has maximal variation. □

5.2. To conclude, let us look at the case of primitive linear systems of genus two which are not covered by Proposition 5.2 and for which maximal variation can in fact be shown directly and without relying on Conjecture 5.1.

Recall that a base point free, ample linear system $|H|$ of genus two, i.e. $(H.H) = 2$, induces a degree two morphism $\pi: S \hookrightarrow \mathbb{P}^2$ which is branched along a smooth sextic curve $D_0 \in |\mathcal{O}_{\mathbb{P}^2}(1)|$.

**Proposition 5.4.** A base point free, ample linear system $|H|$ of degree $(H.H) = 2$ on a K3 surface has maximal variation.

**Proof.** According to [EV92, Lem. 3.16(d)], we have $\pi_* \Omega_S \simeq \Omega_{\mathbb{P}^2} \oplus \Omega_{\mathbb{P}^2}(\log D_0)(-3)$ and, therefore, for any line $\ell \subset \mathbb{P}^2$

$$H^0(\pi^{-1}(\ell), \Omega_S|_{\pi^{-1}(\ell)}) \simeq H^0(\ell, (\Omega_{\mathbb{P}^2} \oplus \Omega_{\mathbb{P}^2}(\log D_0)(-3))|_{\ell}).$$

Now, both bundles, $\Omega_{\mathbb{P}^2}$ and $\Omega_{\mathbb{P}^2}(\log D_0)(-3)$, are stable of rank two with determinant $\mathcal{O}(-3)$. Indeed, the stability of $\Omega_{\mathbb{P}^2}$ is well known and for $\Omega_{\mathbb{P}^2}(\log D_0)$ it is deduced by a standard
argument from the exact residue sequence

\[ 0 \longrightarrow \Omega_{\mathbb{P}^2} \longrightarrow \Omega_{\mathbb{P}^2}(\log D_0) \longrightarrow \mathcal{O}_{D_0} \longrightarrow 0. \]

The Grauert–Mülich theorem, cf. [HL10, Thm. 3.0.1], then proves that the restriction of both bundles, \( \Omega_{\mathbb{P}^2} \) and \( \Omega_{\mathbb{P}^2}(\log D_0)(-3) \), to the generic line \( \ell \subset \mathbb{P}^2 \) is isomorphic to \( \mathcal{O}(-1) \oplus \mathcal{O}(-2) \). Therefore, \( H^0(C, \mathcal{T}_S|_C) \simeq H^0(\pi^{-1}(\ell), \Omega_S|_{\pi^{-1}(\ell)}) = 0 \) for the generic curve \( C = \pi^{-1}(\ell) \in |H| \simeq |\mathcal{O}(1)|. \)

\[ \square \]

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