Real Multiplication Revisited

Nikolaev IV*

Department of Mathematical Sciences, The Fields Institute for Research in Mathematical Sciences, Toronto, ON, Canada

Abstract
It is proved that the Hilbert class field of a real quadratic field \( \mathcal{Q}(fD) \) modulo a power \( m \) of the conductor \( f \) is generated by the Fourier coefficients of the Hecke eigenform for a congruence subgroup of level \( fD \).

Keywords: Class field; Real multiplication

Introduction

The Kronecker’s *Jugendtraum* is a conjecture that the maximal unramified abelian extension (The Hilbert class field) of any algebraic number field is generated by the special values of modular functions attached to an abelian variety. The conjecture is true for the rational and imaginary quadratic fields with the modular functions being an exponent and the \( j \)-invariant, respectively. In the case of an arbitrary number field, a description of the abelian extensions is given by class field theory, but an explicit formula for the generators of these abelian extensions, in the sense sought by Kronecker, is unknown even for the real quadratic fields.

The problem was first studied by Hecke [1]. A description of abelian extensions of real quadratic number fields in terms of coordinates of points of finite order on abelian varieties associated with certain modular curves was obtained in studies of Shimura [2]. Stark formulated a number of conjectures on abelian extension of arbitrary number fields, which in the real quadratic case amount to specifying generators of these extensions using special values of Artin L-functions [3]. Based on an analogy with complex multiplication, Manin suggested to use the so-called “pseudo-lattices” \( \mathbb{Z} + \mathcal{M} \mathbb{Z} \) in \( \mathbb{R} \) having non-trivial real extensions to produce abelian extensions of real quadratic fields [4]. Similar to the case of complex multiplication, the endomorphism ring \( \mathcal{M} = \mathbb{Z} + \mathcal{M} \mathbb{Z} \) of pseudo-lattice \( \mathbb{Z} + \mathcal{M} \mathbb{Z} \) is an order in the real quadratic field \( \mathcal{M} = \mathbb{Q}(\theta) \), where \( \theta \) is a ring of integers of \( \mathcal{M} \) and \( f \) is the conductor of \( \mathcal{M} \). Manin calls these pseudo-lattices with real multiplication.

The aim of our note is a formula for generators of the Hilbert class field of real quadratic fields based on a modularity and a symmetry of complex and real multiplication. To give an idea, let

\[
\Gamma_f(N) = \left\{ \begin{array}{c} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \mod N, c \equiv 0 \mod N \end{array} \right\}
\]

be a congruence subgroup of level \( N \geq 1 \) and \( \mathbb{H} \) be the Lobachevsky half-plane; let \( \Gamma_f(N) = \mathbb{H} / \Gamma_f(N) \) be the corresponding modular curve and \( S_f(\Gamma_f(N)) \) the space of all cusp forms on \( \Gamma_f(N) \) of weight \( 2 \). Let \( \mathcal{E}_f^0(\mathcal{M}) \) be elliptic curve with complex multiplication by an order \( \mathcal{M} = \mathbb{Z} + \mathcal{M} \mathbb{Z} \) in the field \( \mathcal{M} = \mathbb{Q}(\sqrt{D}) \) [5]. Denote by \( \mathcal{K}^0(k) = k(j(\mathcal{E}_f^0(\mathcal{M}))) \) the Hilbert class field of \( f \) modulo conductor \( f + 1 \) and let \( \text{Jac}(X_f(fD)) \) be the Jacobian of modular curve \( X_f(fD) \). There exists an abelian subvariety \( A_f \subset \text{Jac}(X_f(fD)) \), such that its points of finite order generate \( \mathcal{K}^0(k) \), [6,7], Section 8. The \( \mathcal{K}^0(k) \) is a CM-field, i.e. a totally imaginary quadratic extension of the totally real field \( \mathcal{K} \), generated by the Fourier coefficients of the Hecke eigenform \( \phi(z) \in S_f(\Gamma_f(fD)) \) [2]. In particular, there exists a holomorphic map \( X_f^0(fD) \to \mathcal{E}_f^0(\mathcal{M}) \), where \( X_f^0(fD) \) is a Riemann surface such that \( \text{Jac}(X_f^0(fD)) \cong A_f \); we refer to the above as a modularity of complex multiplication.

Recall that (twisted homogeneous) coordinate ring of an elliptic curve \( \epsilon(\mathbb{C}) \) is isomorphic to a Sklyanin algebra, [8]; the norm-closure of a self-adjoint representation of the Sklyanin algebra by the linear operators on a Hilbert space \( \mathcal{H} \) is isomorphic to a noncommutative torus \( \mathcal{A}_{\mathbb{C}} \) [9] for the definition.

Whenever elliptic curve \( \epsilon(\mathbb{C}) \) has complex multiplication, the noncommutative torus \( \mathcal{A}_{\mathbb{C}} \) has real multiplication by an order \( \mathcal{M} = \mathbb{Z} + \mathcal{M} \mathbb{Z} \) in the field \( t = \sqrt{-1} \); moreover, it is known that \( f = m \) for the minimal power \( m \) satisfying an isomorphism:

\[
\mathcal{C}(\mathcal{M}) \cong \mathcal{C}(R),
\]

where \( \mathcal{C}(R) \) and \( \mathcal{C}(\mathcal{M}) \) are the ideal class groups of orders \( R \) and \( \mathcal{M} \), respectively. We shall refer to (2) as a symmetry of complex and real multiplication. The noncommutative torus with real multiplication by \( \mathcal{M} \) will be denoted by \( \mathcal{A}_{\mathbb{R}}^{(\mathcal{M})} \).

**Remark 1:** The isomorphism (2) can be calculated using the well-known formula for the class number of a non-maximal order \( \mathcal{O}_p \) of a quadratic field \( k = \mathcal{Q}(\sqrt{D}) \):

\[
\mathcal{K}_{\mathcal{M},\mathcal{O}_p} = \frac{1}{\epsilon_j} \prod_{p \mid D} \left( 1 - \left( \frac{p}{\epsilon} \right) \right)
\]

where \( \epsilon_j \) is the class number of the maximal order \( \mathcal{O}_p \), \( \epsilon_j \) is the index of the group of units of \( \mathcal{O}_p \), \( \epsilon_j \) is a prime number and \( \prod p \) is the Legendre symbol [10,11].

The (twisted homogeneous) coordinate ring of the Riemann surface \( X_f^0(fD) \) is an AF-algebra \( \mathcal{A}_{\mathbb{R}}^{(\mathcal{M})} \) linked to a holomorphic differential \( \phi(z)dz \) on \( X_f^0(fD) \), see Section 2.2, Definition 1 and Remark 5 for the details; the Grothendieck semigroup \( \mathcal{K}_f^{(\mathcal{M})} \) is a pseudo-lattice \( \mathbb{Z} + \mathbb{Z} \theta_1 + \ldots + \mathbb{Z} \theta_n \) in the number field \( \mathcal{K} \), where \( n \) equals the genus of \( X_f^0(fD) \). Moreover, a holomorphic map \( X_f^0(fD) \to \mathcal{E}_f^0(\mathcal{M}) \) induces the \( \mathbb{C} \)-algebra homomorphism \( \mathcal{A}_{\mathbb{R}}^{(\mathcal{M})} \to \mathcal{K}_f^{(\mathcal{M})} \) between the corresponding coordinate rings, so that the following diagram commutes:

*Corresponding authors: Nikolaev IV, Professor, Department of Mathematical Sciences, The Fields Institute for Research in Mathematical Sciences, 222 College Street, Toronto, Ontario, Canada, Tel: 416-348-9710; E-mail: igor.v.nikolaev@gmail.com

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The Sklyanin algebra $S_{\beta,C}(\mathcal{C})$ is a free $C$-algebra on four generators and six relations:

$$
\begin{align*}
& x_1 x_2 - x_2 x_1 = \alpha(x_1 x_3 + x_2 x_4), \\
& x_3 x_4 + x_4 x_3 = x_3 x_1 - x_4 x_2,
\end{align*}
$$

(5)

where $\alpha + \beta + \gamma + a \beta \gamma = 0$; such an algebra corresponds to a twisted homogeneous coordinate ring of an elliptic curve in the complex projective space $\mathbb{P}^2$ given by the intersection of two quadric surfaces of the form $E_{\alpha,\beta}(\mathcal{C}) = \{(w^2 + v^2 + z^2) = 0\}$. Being such a ring means that the algebra $S_{\beta,C}$ satisfies an isomorphism

$$
\text{Mod} (\, \theta_{\beta,C}(\mathcal{C})) \cong \text{Tors} (\, \theta_{\beta,C}(\mathcal{C})),
$$

(6)

where $\text{Coh}$ is the category of quasi-coherent sheaves on $\text{Mod} (\, \theta_{\beta,C}(\mathcal{C}))$, $\text{Tors}$ the category of graded left modules over the graded ring $\theta_{\beta,C}(\mathcal{C})$ and $\text{Tors}$ the full sub-category of $\text{Mod}$ consisting of the torsion modules, [8].

If one sets $x_i = u_i, x_i = u_i, x_i = v_i, x_i = v_i$, then there exists a self-adjoint representation of the Sklyanin $*$-algebra $S_{\beta,C}(\mathcal{C})$ by linear operators on a Hilbert space $\mathcal{H}$, such that its norm-closure is isomorphic to $\theta_{\beta,C}$, namely, $\theta_{\beta,C} \cong S_{\beta,C}(\mathcal{C})$. For an $\alpha$-subalgebra of $\theta_{\beta,C}$ and $\theta_{\beta,C}$ is an ideal generated by the "scaled unit" relations $x_i x_j = x_j x_i = \frac{1}{\mu}$, where $\mu > 0$ is a constant. Thus the algebra $\theta_{\beta,C}$ is a coordinate ring of elliptic curve $E(\mathcal{C})$, such that isomorphic elliptic curves correspond to the $\theta_{\beta,C}$ satisfies an isomorphism of the Morita equivalent noncommutative tori; this fact explains the modular transformation law in (4). In particular, if $\mathcal{C}$ has complex multiplication by an order $\mathcal{O}_K = \mathbb{Z} + i \mathbb{Z}$, in a quadratic field $t = \mathbb{Q}(\sqrt{-D})$, then $\theta_{\beta,C}$ has real multiplication by an order $\mathcal{O}_K$, $\mathcal{O}_K = \mathbb{Z} + i \mathbb{Z}$ in the quadratic field $t = \mathbb{Q}(\sqrt{-D})$, where $i$ is the smallest integer satisfying $x_i = x_j = \frac{1}{\mu}$, for the constraint $f = \alpha$, see remark 6.

**AF-algebra of the Hecke eigenform**

An AF-algebra (Approximately Finite $C^*$-algebra) is defined to be the norm closure of an ascending sequence of finite dimensional $C^*$-algebras $M_n$, where $M_n$ is the $C^*$-algebra of the $n \times n$ matrices with entries in $\mathbb{C}$. Here the index $n = (n_1, \ldots, n_k)$ represents the semi-simple matrix algebra $M_{n_1} \otimes \cdots \otimes M_{n_k}$. The ascending sequence mentioned above can be written as $M_n = M_n \otimes M_{n-1} \otimes \cdots$ and for $M_n$ the finite dimensional $C^*$-algebras and $\phi:M \to M_n$, the homomorphism. One has two sets of vertices $v_1, \ldots, v_m$ joined by $b_i$ edges whenever the summand $M_{n_i}$ contains $b_i$ copies of the summand $M_{n_i}$ under the embedding $\phi_i$. As $i$ varies, one obtains an infinite graph called the Bratteli diagram of the AF-algebra. The matrix $B = (b_{ij})$ is known as a partial multiplicity matrix; an infinite sequence of $B$ defines a unique AF-algebra. An AF-algebra is called stationary if $B = \text{Cont} = B = B' = B''$, when two non-similar matrices $B$ and $B''$ have the same characteristic polynomial, the corresponding stationary AF-algebras will be called companion AF-algebras.
Let $N \geq 1$ be a natural number and consider a (finite index) subgroup of the modular group given by the formula:

$$
\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid a = d \equiv 1 \mod N, c \equiv 0 \mod N \right\}.
$$

Let $\mathbb{H} = \mathbb{C} \setminus \{x + iy \in \mathbb{C} \mid y > 0\}$ be the upper half-plane and let $\Gamma(N)$ act on $\mathbb{H}$ by the linear fractional transformations; consider an orbifold $\mathbb{H}/\Gamma(N)$. To compactify the orbifold at the cusps, one adds a boundary to $\mathbb{H}$, so that $\mathbb{H} = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ and the compact Riemann surface $X = \mathbb{H}/\Gamma(N)$ is called a modular curve. The meromorphic functions $\sigma \in \mathbb{H}$ that vanish at the cusps and such that

$$
\phi \left( \frac{az + b}{cz + d} \right) = (cz + d)^\sigma(\phi), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N), \quad (8)
$$

are called {cusp forms} of weight two; the (complex linear) space of such forms will be denoted by $S_2(\Gamma(N))$. The function $\psi(z) \mapsto \omega(\psi(z))$ defines an isomorphism $S_2(\Gamma(N)) \cong \Omega_0(2,X(N))$, where $\Omega_0(2,X(N))$ is the space of all holomorphic differentials on the Riemann surface $X(N)$. Note that $\dim S_2(\Gamma(N)) = \dim \Omega_0(2,X(N)) = g$, where $g = \text{genus}(X)$. A Hecke operator, $T_n$, acts on $S_2(\Gamma(N))$ by the formula $T_n \psi = \sum d \psi \left( \frac{az + b}{cz + d} \right)$, and $\psi(z)$ is a modular form of weight $k$; since $\psi$ is an eigenvector for one (and hence all) of $T_n$, hence the lemma.

By $\psi^{(n)}(z) \in \Omega_{0,1}(X(1)(\mathbb{H}))$ we denote the image of the Hecke eigenform $\phi(z) \in \Omega_{0,1}(X(1)(\mathbb{H}))$ under the holomorphic map $X(1)(\mathbb{H}) \rightarrow X(1)(\mathbb{H})$.

**Remark 3.** The surface $X^0(1)(\mathbb{H})$ is correctly denoted. Indeed, since the abelian variety $A_g$ is the product of $g$ copies of an elliptic curve with the complex multiplication, there exists a holomorphic map from $A_g$ to the elliptic curve $E$. For a Riemann surface $X$ of genus $g$ covering the elliptic curve $E$, by the holomorphic map (such a surface and a map always exist), one gets a period map $X \rightarrow A_g$ by closing the arrows of a commutative diagram $A_g \rightarrow E_{CM} \hookrightarrow E$. It is easy to see, that the Jacobian of $X$ coincides with $A_g$ and we set $X^0(1)(\mathbb{H}) = X$.

**Lemma 1.** $g(X^0(1)(\mathbb{H})) = \deg(\mathbb{C}^n(\mathbb{H}) / \mathbb{H})$.

**Proof.** By definition, abelian variety $A_g$ is the quotient of $\mathbb{C}^n$ by a lattice of periods of the Hecke eigenform $\phi(z) \in S_2(\Gamma(N))$ and all its conjugates $\phi^{(n)}(z)$ on the Riemann surface $X(1)(\mathbb{H})$. These periods are complex algebraic numbers generating the Hilbert class field $\mathbb{C}^n$ over imaginary quadratic field $K = \mathbb{Q}((D))$ modulo conductor $f(2,6,7)$.

**Corollary 1.** $g(X^0(1)(\mathbb{H})) = \deg(\text{Cl}(R))$.

**Proof.** Because $\mathbb{C}^n(\mathbb{H})$ is the Hilbert class field over $\mathbb{Q}$ modulo conductor $f$, we have

$$
\text{Gal}(\mathbb{C}^n(\mathbb{H}) / \mathbb{Q}) = \text{Cl}(R),
$$

where $\text{Gal}(\mathbb{C}^n(\mathbb{H}) / \mathbb{Q})$ is the Galois group of the extension $\mathbb{C}^n(\mathbb{H}) / \mathbb{Q}$ and $\text{Cl}(R)$ is the class group of ring $R$. But $\text{Gal}(\mathbb{C}^n(\mathbb{H}) / \mathbb{Q}) = \deg(\mathbb{C}^n(\mathbb{H}) / \mathbb{Q})$ and by lemma 1 we have $\deg(\mathbb{C}^n(\mathbb{H}) / \mathbb{Q}) = g(X^0(1)(\mathbb{H})).$ In view of this and isomorphism (11), one gets $\deg(\mathbb{C}^n(\mathbb{H}) / \mathbb{Q}) = g(X^0(1)(\mathbb{H}))$. Corollary 1 follows.

**Lemma 2.** $g(X^0(1)(\mathbb{H})) = \deg(\text{Cl}(\mathbb{Q}))$.

**Proof.** It is known that $\dim(\text{Cl}(\mathbb{Q})) = \deg(\text{Cl}(\mathbb{Q}))$ [15], Proposition 6.6.4. But abelian variety $A_g$ is Jac($X(1)(\mathbb{H})$), and, therefore, $\dim(\text{Cl}(\mathbb{Q})) = g(X^0(1)(\mathbb{H}))$, hence the lemma.

**Corollary 2.** $\deg(\text{Cl}(\mathbb{Q})) = \deg(\text{Cl}(\mathbb{Q}))$.

**Proof.** From lemma 2 and corollary 1 one gets $\deg(\text{Cl}(\mathbb{Q})) = \deg(\text{Cl}(\mathbb{Q}))$. In view of this and equality (2), one gets the conclusion of corollary 2.

**Lemma 3. (Basic lemma)** $\text{Gal}(\mathbb{Q}(\zeta_g) / \mathbb{Q}) = \text{Cl}(\mathbb{Q})$.

**Proof.** Let us outline the proof. In view of lemma 2 and corollaries 1-2, we denote by $h$ the single integer $g(X^0(1)(\mathbb{H})) = \deg(\text{Cl}(\mathbb{Q})) = \deg(\text{Cl}(\mathbb{Q}))$. Since $\deg(\text{Cl}(\mathbb{Q})) = \deg(\text{Cl}(\mathbb{Q}))$ there exist $\phi_1, \ldots, \phi_h$ conjugate Hecke eigenforms $\phi(z) \in S_2(\Gamma(N))$ [15], Theorem 6.5.4; thus one gets $h$ holomorphic forms $\phi_i \mapsto \phi_i$ on the Riemann surface $X^0(1)(\mathbb{H})$. Let $\phi_1, \ldots, \phi_h$ be the corresponding stationary AF-algebras; the $\phi_j$ are companion AF-algebras, see Section 1.2. Recall that the characteristic polynomial for the partial multiplicity matrices $B_j$ of companion AF-algebras $\phi_j$ is the same; it is a minimal polynomial of degree $h$ and let $\{\lambda_1, \ldots, \lambda_m\}$ be the roots of such a polynomial, compare with studies of Effros [14], Corollary 6.3. Since $\deg(B_j) = 1$, the numbers $\lambda_1, \ldots, \lambda_m$ are algebraic units of the field $\mathbb{Q}(\zeta_g)$. Moreover, $\lambda_j$ are algebraically conjugate and can be taken for generators of the extension $\mathbb{Q}(\zeta_g) / \mathbb{Q}$; since $\deg(\mathbb{Q}(\zeta_g) / \mathbb{Q}) = \deg(\text{Cl}(\mathbb{Q}))$, there exists a natural action of group $\text{Cl}(\mathbb{Q})$ on these generators. The
action extends to automorphisms of the entire field $K_\rho$ preserving $Q_\rho$; thus one gets the Galois group of extension $K_\rho/Q_\rho$ and an isomorphism $Gal(K_\rho/Q_\rho) \cong Cl(\Omega)$. Let us pass to a step-by-step argument.

(i) Let $h = \sigma(x_1^2(d)) = Cl(R_0) = Cl(\Omega_0)$, and let $\phi(z) = S(\Gamma, (d)).$ The Hecke eigenform. It is known that there exists a conjugate Hecke eigenforms, so that $\phi(z)$ is one of them [15], Theorem 6.5.4. Let $\{\phi_1, \ldots, \phi_r\}$ be the corresponding forms on the Riemann surface $X_1(f(D))$. \n
**Remark 4.** The forms $\{\phi_1, \ldots, \phi_r\}$ can be taken for a basis in the space $\Omega_\omega(x_1^2(f(D))).$ We leave it to the reader to verify, that abelian varieties $A_{\phi}$ are isomorphic to the quotient of $C_\omega$ by the lattice of periods of holomorphic differentials $\omega(z)dz$ on $X_1(f(D))$. \n
(ii) Let $A_{\phi}$ be the AF-algebra corresponding to holomorphic differential $\omega(z)dz$ on $X_1(f(D))$, see Section 2.2; the set $\{A_{\phi}, \ldots, A_{\phi_1}\}$ is a basis of the companion AF-algebras. It is known that each $A_{\phi}$ is a stationary AF-algebra, i.e. its partial multiplicity matrix is a constant; therefore corollary 3 is an implication of lemma 3 and isomorphism $Gal(K_\rho/Q_\rho)$ generated by the Fourier coefficients of the Hecke eigenform $\phi(z) = S(\Gamma, (d))$. \n
**Corollary 3.** The Hilbert class field of real quadratic field $t = Q(\sqrt{D})$ modulo conductor $|t|$ is isomorphic to the field $k|t| = 15$. The class field theory says that \n
$$Cl(\mathbb{Q}(\sqrt{D})) \cong Cl(R) \cong \mathbb{Z}/2\mathbb{Z}$$

and isomorphism (2) is trivially satisfied for each power $m$, i.e. one obtains an unramified extension. By theorem 1, the Hilbert class field of $k$ is generated by the Fourier coefficients of the Hecke eigenform $\phi(z) = S(\Gamma, (15))$. Using the computer programme SAGE created by William A. Stein, one finds an irreducible factor of degree $4 + 3 = 5$ of the characteristic polynomial of the Hecke operator $T_p$ acting on the space $S_1(\Gamma, (15))$. Therefore, the Fourier coefficient $c(2)$ coincides with a root of equation $p(x) = 0$; in other words, we arrive at an extension of $k$ by the polynomial $p(x)$. The generator $\alpha$ of the field $K_{\alpha} = \mathbb{Q}(c(2))$ is a root of the bi-quadratic equation $x^4 + 3x^2 + 15 = 0$; it is easy to see that $Z = 2 + \sqrt{1 - 15}$. One concludes, that the field $K_{\alpha} = \mathbb{Q}(c(2))$ is the Hilbert class field of quadratic field $t = Q(\sqrt{5})$. \n
**Example 1.** Let $D = 15$. The class number of quadratic field $k = Q(\sqrt{15})$ is known to be 2; such a number for quadratic field $t = Q(\sqrt{5})$ is also equal to 2. Thus $Cl(\mathbb{Q}(\sqrt{5})) \cong Cl(R) \cong \mathbb{Z}/2\mathbb{Z}$. \n
**Remark 6.** The class field theory says that $f = f_\alpha$, i.e. the extensions of old $k$ and $t$ must ramify over the same set of prime ideals. Indeed, consider the commutative diagram below, where $I_1$ and $I_2$ are groups of all ideals of $k$ and $t$, which are relatively prime to the principal ideals $(f)$ and $(l)$, respectively. Since $Gal(K^\infty(k)/Q_\rho) \cong Gal(K_\rho/Q_\rho)$ one gets an isomorphism $I_1 \cong I_2$, i.e. $f = f_\alpha$ for some positive integer $m$. \n
$$
\begin{array}{ccc}
I_f & \text{Artin homomorphism} & Gal \left( K^\infty(k)/Q_\rho \right) \\
I_l & \text{Artin homomorphism} & Gal \left( K_\rho/Q_\rho \right)
\end{array}
$$

**Examples.** Along with the method of Stark’s units [19], theorem 1 can be used in the computational number theory. For the sake of clarity, we shall consider the simplest examples, the rest can be found in Table 1. \n
**Example 2.** Let $D = 14$. It is known, that for the quadratic field $k = Q(\sqrt{14})$ we have $Cl(R) = 4$, while for the quadratic field $t = Q(\sqrt{5})$ we have $Cl(R) = 1$. However, for the ramified extensions one obtains the following isomorphism: $Cl(\mathbb{Q}(\sqrt{5})) \cong Cl(R) \cong \mathbb{Z}/2\mathbb{Z}$. \n
$$
\begin{array}{ccc}
Cl(R_{\alpha}) & \cong Cl(R) \cong \mathbb{Z}/2\mathbb{Z}
\end{array}
$$
and one finds a generator of \( f/2 \). Thus the field \( \mathbb{Q}(\sqrt{D}) \) modulo conductor \( f \) is maintained by Keith Matthews. We focused on small conductors; the orders in quadratic fields posted at www.numbertheory.org; the site is handled by Keith Matthews. We focused on small conductors; the orders in quadratic fields posted at www.numbertheory.org; the site is handled by Keith Matthews.

Remark 7. Table 1 above lists quadratic fields for some square-free \( p \). Clearly, the extension is ramified over the prime \( p \). The bi-quadratic equation \( x^4 - 2x^3 + 4x^2 - 8x + 16 \) is satisfied by \( \omega \). Clearly, the extension is ramified over the prime ideal \( p = 2 \).

### References

1. Hecke E (1910) Concerning the construction of class field of real quadratic fields by means of automorphic functions, messages from the Gessellschaft of Sciences in Göttingen, Mathematics and Physical Kl E: 619-623.
2. Shimura G (1972) Class fields over real quadratic fields and Hecke operators. Annals of Math 96: 130-190.
3. Stark HM (1976) L-functions at \( s = 1 \). Ill. Totally Real Fields and Hilbert's Twelfth Problem, Advances in Math 22: 64-84.
4. Manin YI (2004) Real multiplication and noncommutative geometry. In: The legacy of Niels Hendrik Abel, Springer, Berlin pp: 685-727.
5. Silverman JH (1994) Advanced Topics in the Arithmetic of Elliptic Curves. Graduate Texts in Mathematics, Springer.
6. Hecke E (1928) Determination of the periods of certain integrals by the theory of class field. Math Z 28: 708-727.
7. Shimura G (1971) On elliptic curves with complex multiplication as factors of class number. Math J 43: 199-208.
8. Stafford JT, van den Bergh M (2001) Noncommutative curves and noncommutative surfaces. Bull Amer Math Soc 38: 171-216.
9. Rieffel MA (1990) Non-commutative tori – A case study of non-commutative differentiable manifolds. Contemp Math 105: 191-211.
10. Boyarchuk ZI, Shafarevich IR (1966) Number Theory, Acad Press, New York, London.
11. Hasse H (1950) Lectures on Number Theory. Springer.
12. Murphy GJ (1990) C*-Algebras and Operator Theory. Academic Press.
13. Blackadar B (1986) K-Theory for Operator Algebras. MSRI Publications, Springer.
14. Effros EG (1981) Dimensions and C*-Algebras. In: Conf. Board of the Math Sciences, Amer Math Soc., Providence, RI.
15. Diamond F, Shurman J (2005) A First Course in Modular Forms. Number Theory and Discrete Mathematics, Springer.
16. Nikolaev I (2016) On a symmetry of complex and real multiplications. Hokkaido Math J 45: 43-51.
17. Bernstein L (1971) The Jacobi-Perron Algorithm, its Theory and Applications. Lect Notes in Math, Springer.
18. Nikolaev I (2012) On the AF-algebra of a Hecke eigenform. Proc Edinburgh Math Soc 55: 207-213.
19. Cohen H, Roblot XF (2000) Computing the Hilbert class field of real quadratic fields. Math Comp 69: 1229-1244.

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**Table 1: Square-free discriminants \( 2 \leq D \leq 101 \).**

| D   | \( f \) | \( Cl(R_f) \) | \( f \) | Hilbert class field of \( \mathbb{Q}(\sqrt{D}) \) modulo conductor \( f \) |
|-----|-------|--------------|-------|----------------------------------|
| 2   | 1     | trivial      | 1     | \( \mathbb{Q}(\sqrt{2}) \)      |
| 3   | 1     | trivial      | 1     | \( \mathbb{Q}(\sqrt{3}) \)      |
| 7   | 1     | trivial      | 1     | \( \mathbb{Q}(\sqrt{7}) \)      |
| 11  | 1     | trivial      | 1     | \( \mathbb{Q}(\sqrt{11}) \)     |
| 14  | 2     | \( \mathbb{Z}/4\mathbb{Z} \) | 8     | \( \mathbb{Q}(\sqrt{-27 + 8\sqrt{14}}) \) |
| 15  | 1     | \( \mathbb{Z}/2\mathbb{Z} \) | 1     | \( \mathbb{Q}(\sqrt{-1 + \sqrt{15}}) \) |
| 19  | 1     | trivial      | 1     | \( \mathbb{Q}(\sqrt{19}) \)     |
| 21  | 2     | \( \mathbb{Z}/4\mathbb{Z} \) | 8     | \( \mathbb{Q}(\sqrt{-3 + 2\sqrt{11}}) \) |
| 35  | 1     | \( \mathbb{Z}/2\mathbb{Z} \) | 1     | \( \mathbb{Q}(\sqrt{7 + \sqrt{35}}) \) |
| 43  | 1     | trivial      | 1     | \( \mathbb{Q}(\sqrt{43}) \)     |
| 51  | 1     | \( \mathbb{Z}/2\mathbb{Z} \) | 1     | \( \mathbb{Q}(\sqrt{17 + \sqrt{51}}) \) |
| 58  | 1     | \( \mathbb{Z}/2\mathbb{Z} \) | 1     | \( \mathbb{Q}(\sqrt{-1 + \sqrt{58}}) \) |
| 67  | 1     | trivial      | 1     | \( \mathbb{Q}(\sqrt{67}) \)     |
| 82  | 1     | -           | -     | \( x^4 - 2x^3 + 4x^2 - 8x + 16 \) |
| 91  | 1     | \( \mathbb{Z}/2\mathbb{Z} \) | 1     | \( \mathbb{Q}(\sqrt{-3 + \sqrt{91}}) \) |

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