Ding injective envelopes in the category of complexes

James Gillespie1 · Alina Iacob2

Received: 22 October 2021 / Accepted: 3 December 2021 / Published online: 27 January 2022 © The Author(s), under exclusive licence to Springer-Verlag Italia S.r.l., part of Springer Nature 2021

Abstract
A complex $X$ is called Ding injective if there exists an exact sequence of injective complexes $\ldots \to E_1 \to E_0 \to E_{-1} \to \ldots$ such that $X = \text{Ker}(E_0 \to E_{-1})$, and the sequence remains exact when the functor $\text{Hom}(A, \cdot)$ is applied to it, for any $FP$-injective complex $A$. We prove that, over any ring $R$, a complex is Ding injective if and only if it is a complex of Ding injective modules. We use this to show that the class of Ding injective complexes is enveloping over any ring.

Keywords Ding injective envelope · Ding injective modules · Ding injective complexes

Mathematics Subject Classification 16E05 · 16E10

1 Introduction

It is an important question to establish relationships between a complex $X$ and the $R$-modules $X_n$, $n \in \mathbb{Z}$, for a given ring $R$. For example, it is well known that a complex $I$ is injective (projective, flat respectively) if and only if it is an exact complex and all its cycles are injective (projective, flat respectively) modules; and a complex is finitely generated if and only if it is a bounded complex of finitely generated modules [3]. By [15], a complex is Gorenstein injective (respectively, projective) if it is a complex of Gorenstein injective (respectively, projective) modules. We consider here the relationship between the Ding injectivity of a complex $X$ and the Ding injectivity of the modules $X_n$, $n \in \mathbb{Z}$.

The Ding injective modules were introduced by Ding and Mao, in [12], where they were called Gorenstein $FP$-injective modules. Later, Gillespie renamed these modules, calling them Ding injective modules (in [10]). The definition uses $FP$-injective modules, so we
recall first that a module $A$ is called $FP$-injective (or absolutely pure) if $\text{Ext}^1(F, A) = 0$ for all finitely presented $R$-modules $F$. We use $\mathcal{FI}$ to denote this class of modules. The Ding injective modules are the cycles of the exact complexes of injective modules that remain exact when applying a functor $\text{Hom}(A, -)$, with $A$ any $FP$-injective module.

The Ding injective complexes are defined in a similar manner: a complex $X$ is Ding injective if it is a cycle of an exact complex of injective complexes $E = \ldots \to E_1 \to E_0 \to E_{-1} \to \ldots$ such that $\text{Hom}(A, E)$ is exact for any $FP$-injective complex $A$. This $\text{Hom}$ too is just the abelian group of chain maps from $A$ to $E$.

This class of complexes has been studied before—see for example [8, 10, 11, 16, 18]. In [18] (Theorem 3.20), the authors prove that a complex $X$ is Ding injective if and only if each term $X_i$ is Ding injective and any chain map $A \to X$ from any $FP$-injective complex $A$ is null homotopic (i.e. the Hom complex $\mathcal{H}\text{om}(A, X)$ is exact, for any $FP$-injective complex $A$). In [10], Corollary 4.4, it is proved that over a Ding-Chen ring (i.e. a two sided coherent ring having finite $FP$-injective dimension on both sides), the Ding injective complexes are precisely the complexes of Ding injective modules. In [16] (Theorem 4.1) this result was extended to all coherent rings. Here we prove that this result holds over any ring: a complex is Ding injective if and only if it is a complex of Ding injective modules.

It was recently proved [16], Theorem 5.4) that over any ring $R$, the class of Ding injective modules, $\mathcal{DI}$, is the right half of a perfect cotorsion pair in $R-\text{Mod}$. Consequently, the class of Ding injective modules is enveloping over any ring. We show that a similar result holds for the class of Ding injective complexes: over any ring $R$, the class of Ding injective complexes is an enveloping class in $Ch(R)$, the category of chain complexes of $R$-modules.

We prove first (Theorem 2) that, over any ring $R$, a complex is Ding injective if and only if it is a complex of Ding injective modules. Then we show (Proposition 3) that the class of Ding injective complexes is special preenveloping in $Ch(R)$. Theorem 3 shows that in fact the class of Ding injective complexes is enveloping in $Ch(R)$, for any ring $R$.

2 Preliminaries

In this paper, $R$ denotes an associative ring with unity, $R-\text{Mod}$ denotes the category of left $R$-modules and $Ch(R)$ denotes the abelian category of complexes of left $R$-modules. A complex

$$\ldots \to C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \to \ldots$$

will be denoted by $(C, d)$, or simply by $C$. The $n$th cycle of $C$, $\text{Ker}(d_n)$, is denoted $Z_n(C)$. The $n$th boundary of $C$, $\text{Im}(d_n)$, is denoted $B_n(C)$. The $n$th homology module of $C$ is $H_n(C) = Z_n(C)/B_{n+1}(C)$. The complex $C$ is exact (or acyclic) if $H_n(C) = 0$ for all $n$.

For two complexes $C, D \in Ch(R)$, we let $\text{Hom}(C, D)$ denote the abelian group of chain maps from $C$ to $D$ in $Ch(R)$. We use the notation $\text{Ext}^i(C, D)$ where $i \geq 1$ for the groups that arise from the right derived functor of $\text{Hom}$.

We use the notation suggested by Brown in [1] and denote by $\mathcal{H}\text{om}(C, D)$ the usual complex formed from two complexes $C$ and $D$. Then $Z_0\mathcal{H}\text{om}(C, D)$ is the group $\text{Hom}(C, D)$ of morphisms from $C$ to $D$.

As already mentioned, the Ding injective complexes are defined in a similar manner with the Ding injective modules. The definition uses $FP$-injective complexes, so we recall that a complex $C$ is $FP$-injective if $\text{Ext}^1(F, C) = 0$ for every finitely presented complex $F$. 

$$\mathcal{F}$$ Springer
A complex $X$ is called Ding injective if it is a cycle of an exact complex of injective complexes $E = \ldots \to E_1 \to E_0 \to E_{-1} \to \ldots$ such that $\text{Hom}(A, E)$ is exact for any FP-injective complex $A$. It is known (\cite{18}, Theorem 3.20), that a complex $X$ is Ding injective if and only if each term $X_i$ is Ding injective and any chain map $A \to X$ from any FP-injective complex $A$ is null homotopic (or equivalently, the Hom complex $\text{Hom}(A, X)$ is exact, for any FP-injective complex $A$).

We recall that a complex $M$ has a Ding injective preenvelope if there exists a chain map $l : M \to A$ with $A$ a Ding injective complex and such that for any Ding injective complex $A'$, any chain map $h : M \to A'$ factors through $l$ ($h = vl$ for some $v \in \text{Hom}(A, A')$).

$$
\begin{array}{ccc}
M & \xrightarrow{l} & A \\
\downarrow{h} & & \downarrow{v} \\
A' & & \\
\end{array}
$$

Such a preenvelope $l$ is said to be an envelope if it has one more property: any $v \in \text{Hom}(A, A)$ such that $vl = l$ is an automorphism of $A$.

Any Ding injective preenvelope $l : M \to A$ is called special if we have $\text{Ext}^1(\text{cok}(l), X) = 0$ for all Ding injective complexes $X$.

One way to prove that a class $\mathcal{L}$ is special preenveloping is showing that $\mathcal{L}$ is the right half of a complete cotorsion pair in $Ch(R)$.

Given a class of objects $C$ in a Grothendieck category $A$, we denote by $C^\perp$ its right orthogonal class, i.e. the class of objects $X$ such that $\text{Ext}^1(C, X) = 0$ for any $C \in C$. The left orthogonal class of $C$ is defined dually. We recall that a pair of classes of objects $(C, \mathcal{L})$, is a cotorsion pair if $C^\perp = \mathcal{L}$ and $\mathcal{L}^\perp = C$. A cotorsion pair $(C, \mathcal{L})$ is complete if for any object $M$ there are exact sequences $0 \to L \to C \to M \to 0$ and respectively $0 \to M \to L' \to C' \to 0$ with $C, C' \in C$ and $L, L' \in \mathcal{L}$. We also recall that a cotorsion pair is said to be hereditary if $\text{Ext}^i(C, L) = 0$ for all $i \geq 1$, all $C \in C$, all $L \in \mathcal{L}$. It is known that this is equivalent with the class $C$ being closed under kernels of epimorphisms, and it is also equivalent with the condition that $\mathcal{L}$ is closed under cokernels of monomorphisms.

A cotorsion pair $(C, \mathcal{L})$ is said to be perfect if $C$ is covering and $\mathcal{L}$ is enveloping.

Given a cotorsion pair $(A, B)$ in the category of $R$-modules, Gillespie introduced (\cite{7}) four classes of complexes in $Ch(R)$ that are associated with it:

1. An acyclic complex $X$ is an $A$-complex if $Z_j(X) \in A$ for all integers $j$. We denote by $\widetilde{A}$ the class of all acyclic $A$-complexes.
2. An acyclic complex $U$ is a $B$-complex if $Z_j(X) \in B$ for all integers $j$. We denote by $\widetilde{B}$ the class of all acyclic $B$-complexes.
3. A complex $Y$ is a dg-$A$ complex if each $Y_j \in A$ and each map $Y \to U$ is null-homotopic, for each complex $U \in \widetilde{B}$. We denote by $\text{dg}(A)$ the class of all dg-$A$ complexes.
4. A complex $W$ is a dg-$B$ complex if each $W_i \in B$ and each map $V \to W$ is null-homotopic, for each complex $V \in \widetilde{A}$. We denote by $\text{dg}(B)$ the class of all dg-$B$ complexes.

Yang and Liu showed in \cite{17}, Theorem 3.5, that when $(A, B)$ is a complete hereditary cotorsion pair in $R$-Mod, the pairs $(\text{dg}(A), \widetilde{B})$ and $(\widetilde{A}, \text{dg}(B))$ are complete and hereditary cotorsion pairs in the category of complexes $Ch(R)$. Moreover, by Gillespie \cite{7}, we...
have that \( \widetilde{A} = \text{dg}(A) \cap \mathcal{E} \) and \( \widetilde{B} = \text{dg}(B) \cap \mathcal{E} \) (where \( \mathcal{E} \) is the class of all acyclic complexes). For example, from the (complete and hereditary) cotorsion pairs \((\text{Proj}, \text{R-Mod})\) and \((\text{R-Mod}, \text{Inj})\), one obtains the standard (complete and hereditary) cotorsion pairs \((\mathcal{E}, \text{dg(Inj)})\) and \((\text{dg(Proj)}, \mathcal{E})\).

We use \( \mathcal{D} \) to denote the class of Ding injective modules. It was recently proved ([6], Theorem 5.4), that \((\perp \mathcal{D}, \mathcal{D})\) is a complete hereditary cotorsion pair over any ring \( R \) (in fact, this is a perfect cotorsion pair). Then by [17], Theorem 3.5, \((\perp \mathcal{D}, \text{dg(\mathcal{D})})\) is a complete hereditary cotorsion pair in the category of complexes, \( \text{Ch}(R) \).

We recall below some of the results that we use.

**Theorem 1** (this is part of [14], Theorem 2.10) Let \( C \) be a complex. Then the following statements are equivalent.

1. \( C \) is FP-injective.
2. \( C \) is exact and \( \mathbb{Z}_n(C) \) is FP-injective in \( \text{R-Mod} \) for all \( n \in \mathbb{Z} \).

We also recall that a short exact sequence of modules \( 0 \to M' \to M \to M'' \to 0 \) is pure exact if and only if the induced sequence of abelian groups \( 0 \to \text{Hom}(Y, M') \to \text{Hom}(Y, M) \to \text{Hom}(Y, M'') \to 0 \) is exact for any finitely presented module \( Y \).

By definition, a complex \( F \) is pure acyclic if it is acyclic and such that the short exact sequences of modules \( 0 \to \mathbb{Z}_n(F) \to F_n \to \mathbb{Z}_{n-1}(F) \to 0 \) are pure for all \( n \).

By [13], Definition 6.7, a complex \( X \) is called coacyclic if \( \text{Ext}^1(X, I) = 0 \) for every complex of injective modules \( I \).

**Lemma 1** ([13], Lemma 6.10) Let \( \mathcal{G} \) be a locally finitely presentable Grothendieck category. Then \( C_{\text{pac}}(\mathcal{F}\mathcal{L}) \subseteq C_{\text{coac}}(\mathcal{G}) \cap C(\mathcal{F}\mathcal{L}) \), where \( C_{\text{pac}}(\mathcal{F}\mathcal{L}) \) is the class of pure acyclic complexes of FP-injective objects, \( C_{\text{coac}}(\mathcal{G}) \) are the coacyclic complexes, and \( C(\mathcal{F}\mathcal{L}) \) are the complexes of FP-injective objects.

This implies the following:

**Lemma 2** Every pure acyclic complex of FP-injective modules is coacyclic.

In fact a stronger version of Lemma 2 holds:

**Lemma 3** Any pure acyclic complex is coacyclic.

**Proof** Let \( X \) be a pure acyclic complex and consider an exact sequence \( 0 \to I \to J \to X \to 0 \) with \( I \) a complex of injective modules. This gives an exact sequence of modules in each degree \( 0 \to I_n \to J_n \to X_n \to 0 \). Since each \( I_n \) is injective, the sequence is split exact. Thus the sequence \( 0 \to I \to J \to X \to 0 \) with \( I \) is isomorphic to a sequence \( 0 \to I \to M(f) \to X \to 0 \) with \( f : X \to I[1] \), and \( M(f) \) its mapping cone. Since injective modules are pure injective, \( f \) is null homotopic (by [13], Theorem 5.4). By [4], Lemma 3.2, \( \text{Ext}^1(X, I) = 0 \). \( \square \)
Corollary 1  Any FP-injective complex is pure acyclic, and hence coacyclic.

Proof  Let C be an FP-injective complex. By Theorem 1 and [2], Proposition 2.2, the complex C is pure acyclic. By Lemma 3 it is coacyclic. □

We also recall that by [9], Definition 3.4, a complete cotorsion pair (W, F) is called an injective cotorsion pair if the class W is thick and if W \cap F coincides with the class of injective objects.

3 Main results

We start by proving that over any ring R, a complex is Ding injective if and only if it is a complex of Ding injective modules.

Theorem 2  Let R be any ring. A complex X is Ding injective if and only if it is a complex of Ding injective R-modules.

Proof  The condition is necessary by [18] Theorem 3.20.

We show that it is also a sufficient condition. Let X be a complex of Ding injective modules. Since (\overset{1}{\underset{D}{\perp}}, \overset{dg}{\perp}) is a complete cotorsion pair in Ch(R), there is an exact sequence 0 \to J \to I \to X \to 0 with I \in \overset{1}{\underset{D}{\perp}} and with J \in \overset{dg}{\perp}. This gives a short exact sequence of modules, 0 \to J_n \to I_n \to X_n \to 0, for any n \in \mathbb{Z}. Since both J_n and X_n are Ding injective modules, it follows that for each n we have I_n \in \overset{1}{\underset{D}{\perp}} \cap \overset{dg}{D} = \text{Inj}. So I is an exact complex of injective R-modules.

Let C be an FP-injective complex, and let f : C \to X be a chain map. Since for each n, Z_n(C) is FP-injective (by Theorem 1), so in \overset{1}{\underset{D}{\perp}}, it follows that C \in \overset{1}{\underset{D}{\perp}}, and therefore Ext_1(C, J) = 0. Thus Hom(C, I) \to Hom(C, X) \to 0 is exact. It follows that f factors through π, f = πg, for some g : C \to I. So it suffices to show that g is null homotopic.

By Corollary 1, any FP-injective complex is coacyclic. It follows that

\[ \text{Ext}_1(C, I) = 0. \]

By [4], Corollary 3.3, this is equivalent to Hom(C, I) being exact. Thus any map from C to I, in particular g, is homotopic to zero. It follows that f = πg is null homotopic. So X is a Ding injective complex by [18, Theorem 3.20]. □

We note that Theorem 2 is proved for coherent rings in [16]. However, it holds for all rings and the above gives a direct proof of this.
We can prove now that the class of Ding injective complexes is the right half of a complete cotorsion pair in \(Ch(R)\) (for any ring \(R\)).

We use \(dw(DI)\) to denote the class of complexes of Ding injective modules. By Theorem 2, these are precisely the Ding injective complexes.

**Proposition 1** Let \(R\) be any ring. The class of Ding injective complexes is special preenveloping in \(Ch(R)\).

**Proof** Since \((\perp DI, DI)\) is an injective cotorsion pair cogenerated by a set ([6], Lemma 3.7, Theorem 3.11, and Theorem 5.4), it follows that

\[
(\perp dw(DI), dw(DI))
\]

is a complete cotorsion pair in \(Ch(R)\) (by [9], Proposition 7.2(1)). So \(dw(DI)\) is special preenveloping. By Theorem 2 above, \(dw(DI)\) is the class of Ding injective complexes. \(\square\)

We show that, in fact, the class of Ding injective complexes, \(dw(DI)\), is enveloping in \(Ch(R)\), for any ring \(R\).

We show first that \(\perp dw(DI)\) is a covering class in \(Ch(R)\). By the proof of Proposition 1, \(\perp dw(DI)\) is a special precovering class. It is known that a precovering class that is also closed under direct limits is covering. So it suffices to show that \(\perp dw(DI)\) is closed under direct limits. The proof uses the following remarks:

**Lemma 4** \((\perp dw(DI), dw(DI))\) is a hereditary cotorsion pair.

**Proof** Let \(0 \to A' \to A \to A'' \to 0\) be an exact sequence with both \(A'\) and \(A \in dw(DI)\). Then for each \(n\) we have an exact sequence of modules \(0 \to A'_n \to A_n \to A''_n \to 0\), with \(A'_n, A''_n \in DI\). Since \((\perp DI, DI)\) is a hereditary cotorsion pair, it follows that \(A'_n \in DI\). Thus \(A'' \in dw(DI)\). \(\square\)

**Lemma 5** \(\perp dw(DI)\) is a thick class.

**Proof** As a left orthogonal class, \(\perp dw(DI)\) is closed under extensions, and by Lemma 4, it is also closed under kernels of epimorphisms.

We check that \(\perp dw(DI)\) is closed under cokernels of monomorphisms. Let \(0 \to D' \to D \to D'' \to 0\) be a short exact sequence with both \(D'\) and \(D\) in \(\perp dw(DI)\). Let \(A \in dw(DI)\). The long exact sequence \(0 = \text{Ext}^1(D', A) \to \text{Ext}^2(D'', A) \to \text{Ext}^2(D, A) = 0\) (by Lemma 4) gives that \(\text{Ext}^2(D'', A) = 0\) for all Ding injective complexes \(A\). By the definition of the Ding injective complexes, there is an exact sequence \(0 \to A' \to I \to A \to 0\) with \(A'\) a Ding injective complex and with \(I\) an injective complex. By the above, \(\text{Ext}^2(D'', A') = 0\). The associated exact sequence

\[
0 = \text{Ext}^1(D'', I) \to \text{Ext}^1(D'', A) \to \text{Ext}^2(D'', A') = 0
\]

gives that \(\text{Ext}^1(D'', A) = 0\) for all \(A \in dw(DI)\). Thus \(D'' \in \perp dw(DI)\).

As a left orthogonal class, \(\perp dw(DI)\) is closed under direct summands. \(\square\)

**Proposition 2** The class \(\perp dw(DI)\) is covering in \(Ch(R)\)
Proof Since $(⊥, d(\mathbb{D}))$ is a cotorsion pair with $d(\mathbb{D})$ thick, it follows that the class $d(\mathbb{D})$ is closed under direct limits (by [10], Proposition 3.2). Thus $d(\mathbb{D})$ is a precovering class that is also closed under direct limits. So $d(\mathbb{D})$ is a covering class.

We recall that a cotorsion pair $(\mathcal{C}, \mathcal{L})$ is called perfect if $\mathcal{C}$ is covering and $\mathcal{L}$ is enveloping.

In order to prove that the class of Ding injective complexes is enveloping in $Ch(R)$, for any ring $R$, we use the following result (showing that $(⊥, d(\mathbb{D}))$ is a perfect cotorsion pair if and only if $⊥, d(\mathbb{D})$ is covering).

Proposition 3 Let $R$ be any ring. The following are equivalent:

1. The class of Ding injective complexes is enveloping.
2. $(⊥, d(\mathbb{D}))$ is a perfect cotorsion pair.
3. The left orthogonal class of that of Ding injective complexes, $⊥, d(\mathbb{D})$, is covering.

Proof

1 $\Rightarrow$ 2. By Proposition 2, every complex has a $⊥, d(\mathbb{D})$ cover. It follows that, assuming (1), the cotorsion pair $(⊥, d(\mathbb{D}))$ is perfect (from the definition).

3 $\Rightarrow$ 2. By Lemma 4, $(⊥, d(\mathbb{D}))$ is a hereditary cotorsion pair. By [5], Theorem 1.4, this cotorsion pair $(⊥, d(\mathbb{D}))$ is perfect if and only if $⊥, d(\mathbb{D})$ is covering and every $X \in ⊥, d(\mathbb{D})$ has a Ding injective envelope.

Let $X \in ⊥, d(\mathbb{D})$. Consider the exact sequence $0 \to X \to I \to Y \to 0$ with $X \to I$ the injective envelope of $X$. By the definition of a Ding injective complex, $\text{Ext}^1(I, L) = 0$, for any Ding injective complex $L$, so $I \in ⊥, d(\mathbb{D})$. Since both $X$ and $I$ are in $⊥, d(\mathbb{D})$, it follows (by Lemma 5) that $Y \in ⊥, d(\mathbb{D})$. So the sequence is $\text{Hom}(−, d(\mathbb{D}))$ exact. Thus $X \to I$ is a special Ding injective preenvelope of $X$. Since any $u : I \to I$ that is the identity on $X$ is an automorphism of $I$, it follows that $X \to I$ is a Ding injective envelope.

2 $\Rightarrow$ 1 and 2 $\Rightarrow$ 3 follow from the definition of a perfect cotorsion pair.

Theorem 3 Let $R$ be any ring. The class of Ding injective complexes is enveloping in $Ch(R)$.

Proof By Proposition 3 above the class of Ding injective complexes is enveloping if and only if $⊥, d(\mathbb{D})$ is covering. By Proposition 2, the class $⊥, d(\mathbb{D})$ is covering.

References

1. Brown, K.: Cohomology of Groups. Springer, New York (1982)
2. Emmanouil, I.: On pure acyclic complexes. J. Algebra 405, 190–213 (2016)
3. Enochs, E.E., Garcia Rozas, J.R.: Flat covers of complexes. J. Algebra 210, 86–102 (1998)
4. Enochs, E.E., Jenda, O.M.G., Xu, J.: Orthogonality in the category of complexes. Math. J. Okayama Univ. 38, 25–46 (1996)
5. Enochs, E.E., Jenda, O.M.G., Lopez-Ramos, J.A.: The existence of Gorenstein flat covers. Math. Scand. 94(1), 46–62 (2004)
6. Gillespie, J., Iacob, A.: Duality pairs, generalized Gorenstein modules and Ding injective envelopes (submitted)
7. Gillespie, J.: The flat model structure on $\mathrm{Ch}(R)$. Trans. Am. Math. Soc. 356, 3369–3390 (2004)
8. Gillespie, J.: Model structures on modules over Ding–Chen rings. Homol. Homotopy Appl. 12(1), 61–73 (2010)
9. Gillespie, J.: Gorenstein complexes and recollements from cotorsion pairs. Adv. Math. 291, 859–911 (2016)
10. Gillespie, J.: On Ding injective, Ding projective and Ding flat modules and complexes. Rocky Mt. J. Math. 47, 2641–2673 (2017)
11. Hu, J.S., Geng, Y.X., Xie, Z.W., Zhang, D.D.: Gorenstein FP-injective dimension for complexes. Commun. Algebra 43(8), 3515–3533 (2015)
12. Mao, L., Ding, N.: Gorenstein FP-injective and Gorenstein flat modules. J. Algebra Appl. 07(04), 491–506 (2008)
13. Stovicek, J.: On purity and applications to coderived and singularity categories. preprint, available at [arXiv:1412.1615]
14. Wang, Z., Liu, Z.: FP-injective complexes and FP-injective dimension of complexes. J. Aust. Math. Soc. 91, 163–187 (2011)
15. Yang, G.: Gorenstein projective, injective and flat complexes. Acta Math. Sinica (Chin. Ser.) 54, 451–460 (2011)
16. Yang, G., Estrada, S.: Characterizations of Ding injective complexes. Bull. Malays. Math. Sci. Soc. 43, 2385–2398 (2020)
17. Yang, G., Liu, Z.: Cotorsion pairs and model structures on $\mathrm{Ch}(R)$. Proc. Edinb. Math. Soc. 54, 783–797 (2011)
18. Yang, G., Liu, Z.K., Liang, L.: Model structures on categories of complexes over Ding-Chen rings. Commun. Alg. 41, 50–69 (2013)

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.