A LINEAR-CONFINED PARTICLE AND THE DIRAC EQUATION

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The model of a classical particle with the weak linear AAD potential is subjected to path integral quantization. The light cone constraints and peculiar properties of its internal variables permit to use in calculations commutative dynamics and apply path integrals for a matrix form of the transition amplitude. Quantization leads to description of a Dirac particle.

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1. Introduction

Motivated by the Wheeler-Feynman action-at-a-distance (AAD) electrodynamics, a model of particle with the weak linear potential was developed [1],[3], which allows to make reasonable physical implications linking elements of its commutative dynamics with the light cone constraints. Due to these constraints, there occurs the possibility to perform relatively easily quantization of the model in the intention of Feynman. The quantization requires to fulfil the important condition: the internal variables of the model, inputted by AAD dynamics, have to be regarded as a canonical pair. In this case they become a specimen element of path integrals, which admits to introduce into the Feynman formalism chains of Dirac δ functions and thereby to facilitate calculations. Beside the use of the internal variables as the tools of pure Hamiltonian dynamics, it will be emphasized in this paper even the role of the Routh function in the dynamical considerations. Its introduction consists in the effort to give a more general meaning to these variables in the process of quantization. They could express, in our opinion, the connection between their light-cone constraint content and spinning of particle; at least to that extent as it was anticipated by Feynman [2], when he tried to establish a compromise solution of spin 1/2 particle dynamics.

In the paper [3] we have formulated major features of relativistic classical dynamics of the point particle with the internal degrees of freedom, defined according to the AAD theory of linear interaction taken for weak coupling. Now we want to show that this model leads to quantum dynamics of a particle with the fixed mass and the spin 1/2. Dynamics of this particle must be supplemented by subsidiary conditions, which are, however, consistent with the equations of motions. Moreover, they are sufficient to construct the Feynman integrals along the paths. The appropriate quantum equation, obtained in this formalism, is the equation for the two-component spinor, proposed by Feynman and Gell-Mann [4], which is equivalent to the Dirac equation. The nonrelativistic Feynmanian quantization of the model is given in [7].

In Sec. 2 a brief summary of main properties of classical model is given. Sec. 3 explains that if the constraints are considered to involve even virtual motions and the presence of weak electromagnetic action, they are more flexible, when the Routh function is introduced. In Sec. 4 it is argued that the Feynman formalism demands to adopt four transition amplitudes and to add some subsidiary conditions in close connection with the constraints and the equations of motion. The form, in which the internal variables must be represented to correspond to the complete structure of transition amplitude, appears to have the spinor behaviour.

2. Glossary of the classical model

Before discussing the quantum version of the model let us summarize major properties of its classical picture. The examined particle was described [1] as a point object with internal degrees of freedom, dynamics of which is related to the transitive
realizations of Poincaré’s group with the generators
\[ M^\mu{}\nu = x^\mu p^\nu - x^\nu p^\mu + \xi^\mu \eta^\nu - \xi^\nu \eta^\mu, \]  
where \( \xi \) and \( \eta \) were considered as the canonically conjugate internal variables (properly \( \xi \) and \( b\eta \) evaluated with \( b = 1 \)) obeying the constraints
\[ \xi^2 = 0 \quad \eta^2 = 0. \]  
The physically most interesting are realizations that correspond to the case \( p^2 \geq 0 \). In this case the irreducible unitary representation of small Lorentz’s group is finitely dimensional. Then one can avoid calculations with \( \xi \) and \( \eta \) as the continual variables and pass straightforwardly to matrices, which is a standard approach resulting into conventional matrix description of spin [2].

Hence, we utilize the variables \( \xi \) and \( \eta \) for all region \( p^2 \geq 0 \). In order that the realization of the Poincaré group remains transitive, we subject these variables beside (2.2) the following conditions
\[ (\xi, p) = \kappa \quad (\eta, p) = \kappa' \]  
\[ (\xi, \eta)p^2 = 2(p, \xi)(p, \eta) = 2\kappa\kappa', \]  
\( \kappa \) and \( \kappa' \) being some constants. Such a particle is possible then to be conceived as an elementary object, deduced from the form of linear AAD interaction, having all necessary attributes of conventional particles, \textit{i.e.} \((m, \vec{p}, \vec{x})\). Moreover, its state can be characterized by one unity vector \( \vec{n} = \xi^0 / \xi_0 \). The two degrees of freedom, peculiar to this vector, give the direct physical content for the quantum picture of the particle.

The particle fulfills the variation principle
\[ \delta \left\{ - \int_{\tau_1}^{\tau_2} d\tau \left[ (\eta, \dot{\xi}) + R \right] + (\xi, \eta)_{\tau=\tau_2} \right\} = 0, \]  
where \( R \) is the Routh function and the constraints (2.2) and (2.3) are valid. The equations of motion, derived from (2.4), are
\[ \frac{d}{d\tau} \frac{\partial R}{\partial \dot{w}^\mu} - \frac{\partial R}{\partial w^\mu} = 0 \]  
\[ \frac{d\dot{\xi}}{d\tau} = \frac{\partial R}{\partial \dot{\eta}} \left( \frac{\xi}{(\xi, \eta)} \right) \quad \frac{d\dot{\eta}}{d\tau} = -\left[ \frac{\partial R}{\partial \xi} + \frac{\xi}{(\xi, \eta)} \frac{\partial R}{\partial \xi} \right] \]  
\[ \frac{d\xi^0}{d\tau} = \frac{\eta^0}{(\xi, \eta)} \left( \frac{\xi}{\eta} \right) \quad \frac{d\eta^0}{d\tau} = -\frac{\xi^0}{(\xi, \eta)} \left( \frac{\eta}{\xi} \right). \]  

One expects naturally that dynamics of the free particle will be sufficiently simple. We can also believe that on the classical level the fourvectors \( \xi \) and \( \eta \) will refer to a "residuum" of particle spin according to intuition of Feynman. This spin relic, of course, must be a constant in time without the presence of electromagnetic field [2]. Therefore it is natural to assume for the free particle to be
\[ \frac{d\xi^\mu}{d\tau} = 0; \quad \frac{d\eta^\mu}{d\tau} = 0. \]
The Routh function, independent in this case on $x$, $\xi$ and $\eta$, and expressed in terms of the fourvelocity $u$, is given by

$$R = \frac{1}{2} \left( \tilde{\mu} u^2 + \frac{m^2}{\tilde{\mu}} \right)$$  \hspace{1cm} (2.7)$$

where $\tilde{\mu}$ is an auxiliary variable introduced for the action $S$ to be invariant under the reparametrisation. It is obvious that for the momentum we have $p^\mu = \frac{\partial R}{\partial u^\mu}$ and $p^\mu = \tilde{\mu} u^\mu$. It is also evident that the constraints (2.3) are consistent with the equations of motion of this free particle.

Note that the conditions (2.3), considered either for $p$ or $u$, admit in fact only two degrees of freedom, associated e.g. with the orientation of $\vec{\xi}$. Furthermore, for $\xi^\mu$ it holds $\xi^2 = 0$ and $(\xi, p) = \kappa$, which admits any direction for $\vec{\xi}/\xi^0$.

Let us give the form of the action functional that leads directly to the Hamilton canonical equations. It is

$$S = - \int_{\tau_1}^{\tau_2} d\tau \left[ (p, u) + (\eta, \dot{\xi}) + H \right] + (\xi, \eta)_{\tau=\tau_2},$$  \hspace{1cm} (2.8)$$

where

$$H = \frac{1}{2\tilde{\mu}} (m^2 - p^2).$$  \hspace{1cm} (2.9)$$

The Hamilton-Jacobi function $S_{21}$ can be evaluated as $S$ taken for real motion. We have

$$S_{21} = - \frac{1}{2} \left[ \frac{(x_2 - x_1)^2}{\rho_{21}} + m^2 \rho_{21} \right],$$  \hspace{1cm} (2.10)$$

$\rho$ being defined as $d\rho = \frac{d\tau}{\tilde{\mu}}$. In (2.10) the dependence of $S_{21}$ on $\xi$ and $\eta$ is not present because $(\eta, \dot{\xi}) = 0$ and the term $(\xi, \eta)_{\tau=\tau_2}$ yields a constant factor due to (2.3). The form of the action $S_{21}$ is now

$$S_{21} = - \text{Extr} \left\{ (p_{21}, (x_2 - x_1)) + H_{21} \rho_{21} \right\},$$  \hspace{1cm} (2.11)$$

where now

$$H_{21} = \frac{1}{2} (m^2 - p_{21}^2).$$

The behaviour of the model if the electromagnetic field is present is described by the Routh function

$$R = \frac{1}{2} \left( \tilde{\mu} u^2 + \frac{m^2}{\tilde{\mu}} \right) + e \tilde{A}^\mu u^\mu + \lambda F_{\mu\nu} \xi^\mu \eta^\nu,$$  \hspace{1cm} (2.12)$$

where $\lambda$ is a coupling constant and $\tilde{A}^\mu$ and $F_{\mu\nu}$ are the fourpotential and strenght of electromagnetic field, respectively. $\lambda$ can be determined by the requirement for the constraints $\tilde{\mu}(\xi, u) = \kappa$ and $\tilde{\mu}(\eta, u) = \kappa'$ to be in agreement with the equations of motion. Of course, they are

$$\frac{d}{d\tau} (\tilde{\mu} u^\mu) = e F^{\mu\nu} u^\nu,$$  \hspace{1cm} (2.13)$$
with \( F^{\mu\nu} = \tilde{A}_{\nu,\mu} - \tilde{A}_{\mu,\nu} \). We assume, again as in the case of linear field, that action of electromagnetic field is weak, omitting the term \( \partial_\mu (F_{\rho\nu} \xi^\rho \eta^\nu) \).

The equations of motion for \( \xi \) and \( \eta \) are the consequences of the variation principle. We must here evaluate the corresponding derivatives of \( R \) over \( \vec{\xi} \) and \( \vec{\eta} \). The result is simple:

\[
\frac{d\xi^\mu}{d\tau} = -\lambda F^{\mu\nu} \xi^\nu ; \quad \frac{d\eta^\mu}{d\tau} = \lambda F^{\mu\nu} \eta^\nu,
\]

with \( \lambda = e/\bar{\mu} \) deduced from the condition \( \bar{\mu}(\xi, u) = \text{const} \).

It can be shown that the conditions (2.3) are consistent with the condition

\[
\frac{\xi^\mu}{\bar{\mu}(u, \xi)} + \frac{\eta^\mu}{\bar{\mu}(u, \eta)} - 2 \frac{u^\mu}{\bar{\mu} u^2} = 0, \quad (2.14)
\]

and thus with the equations of motion. Eq.(2.14) may be viewed physically as the fact that the external field is not able to cause spin excitations of the particle.

The Hamilton formalism requires now for \( p \) and \( H \) the following forms

\[
p^\mu = \bar{\mu} u^\mu + e \bar{A}^\mu \quad (2.15)
\]

\[
H = \frac{1}{2\bar{\mu}} \left[ m^2 - (p^\mu - e \bar{A}^\mu)^2 \right] + \frac{e}{\bar{\mu}} F_{\mu\nu} \xi^\mu \eta^\nu, \quad (2.16)
\]

respectively, or using the variable \( \rho \), equivalently

\[
H = \frac{1}{2} \left[ m^2 - (p^\mu - e \bar{A}^\mu)^2 \right] + e F_{\mu\nu} \xi^\mu \eta^\nu. \quad (2.17)
\]

It can be easily verified that the Hamilton-Jacobi function depends only on \( x \) and \( x' \), but not on \( \xi \) and \( \eta \). As a result the action integral is identical with that defined for the scalar particle.

Dynamics for the variables \( \xi \) and \( \eta \) is thus determined, if \( F_{\mu\nu} \neq 0 \), by the same subsidiary conditions as that for the free particle. This means, for the same times \( \rho \) the condition (2.14) will be identical with that for the free particle. For the different \( \rho \) it will differ.

3. Quantum dynamics of the model

In the Feynman formalism dynamics of a particle is determined by the transition amplitude from the state at the ”time” \( \rho_1 \), to the state at the ”time” \( \rho_2 \). Our model is accommodated to have four amplitudes

\[
A_{21} = A_{21}(x_2, x_1, \xi_2, \xi_1, \rho_{21}); \quad B_{21} = B_{21}(x_2, x_1, \eta_2, \eta_1, \rho_{21})
\]

\[
C_{21} = C_{21}(x_2, x_1, \eta_2, \xi_1, \rho_{21}); \quad D_{21} = D_{21}(x_2, x_1, \xi_2, \eta_1, \rho_{21}). \quad (3.1)
\]
The structure of these amplitudes demands their dependence on $x_2$ and $x_1$ to be defined by the standard exponential of the type

$$\rho_{21}^{-2} e^{-\frac{i}{\hbar} [\rho_{21}^{-1}(x_2-x_1)^2 + m^2 \rho_{21}]} . \quad (3.2)$$

On the other hand, the dependence on $\xi$ and $\eta$ have to be characterized by Dirac’s delta functions in a way by which the subsidiary conditions for $\xi$ and $\eta$ are suitably expressed (relations (2.3) or (2.14)).

The integrals over trajectories are usually taken in the form involving the integration over momenta. Therefore we shall use the following integrals

$$\int e^{\frac{i}{\hbar} \{-[p_{21}.(x_2-x_1)] + \frac{1}{2} (p_{21}^2 - m^2) \rho_{21} \}} \frac{d^4 p_{21}}{(2\pi \hbar)^4} =$$

$$= \frac{1}{i(2\pi \hbar)^2 \rho_{21}^2} e^{-\frac{i}{2\hbar} [\rho_{21}^{-1}(x_2-x_1)^2 + m^2 \rho_{21}]} . \quad (3.3)$$

The continual integration over $\xi$ and $\eta$ will be represented by the differentials

$$\frac{d^3 \eta_2}{\eta_2^0} \frac{d^3 \xi_2}{\xi_2^0} \frac{d^3 \eta_3}{\eta_3^0} \frac{d^3 \xi_3}{\xi_3^0} \ldots$$

which are the Lorentz invariant quantities. One of kinds of auxiliary conditions will be expressed by the combinations of $\delta$ functions:

$$\delta[(\xi_1.p_{21}) - \kappa] \delta[(\eta_2.p_{21}) - \kappa'] \delta[(\xi_2.p_{32}) - \kappa] \delta[(\xi_3.p_{32}) - \kappa'] \ldots . \quad (3.4)$$

If one introduces into (2.3b)

$$\tilde{\xi}^\mu = \xi^\mu - 2p^{-2} p^\mu (p.\xi) ; \quad \tilde{\eta}^\mu = \eta^\mu - 2p^{-2} p^\mu (p.\eta) , \quad (3.5)$$

then the subsidiary conditions may be written, respectively

$$\langle \tilde{\xi} . \eta \rangle = 0 ; \quad \langle \tilde{\eta} . \xi \rangle = 0 . \quad (3.6)$$

So we find straightforwardly the analogon of conditions (2.2)

$$\tilde{\xi}^2 = 0 ; \quad \tilde{\eta}^2 = 0 . \quad (3.7)$$

It is apparent that the relations (3.5) lead to the new conditions

$$\delta[(\tilde{\xi} . \eta)] = \delta[(\xi . \eta)] , \quad (3.8)$$

expressed in terms of $\delta$ functions. As a result, in the functional integral there appears a $\delta$ function chain of the form

$$\delta[(\tilde{\xi_1} . \eta_2)] \delta[(\eta_2 . \tilde{\xi_2})] \delta[(\tilde{\xi_2} . \eta_3)] \delta[(\eta_3 . \tilde{\xi_3})] \delta[(\tilde{\xi_3} . \eta_4)] \delta[(\eta_4 . \tilde{\xi_4})] \ldots . \quad (3.9)$$
Let us calculate, for instance, the integral
\[
\int \delta((\xi_1, \eta_2)) \delta((\eta_2, \xi_2)) \kappa' \delta((\eta_2, p_{21}) - \kappa') \frac{d^3\eta_2}{\eta_2^0},
\] (3.9)
to find the way for verification of composition law. Due to relations
\[
\kappa' \delta((\eta_2, p_{21}) - \kappa') = \frac{\kappa'}{p_{21}^0 - \tilde{n}'_{21} \tilde{p}_{21}} \delta(\eta_2^0 - \frac{\kappa'}{p_{21}^0 - \tilde{n}'_{21} \tilde{p}_{21}}),
\] (3.10)
where
\[
\delta((\eta_2, \xi_2)) = (\eta_2^0 \xi_2^0)^{-1} \delta(1 - \tilde{n}_2 \tilde{p}_2);
\]
and we put \( \tilde{n}_1 = \tilde{\xi}_1 / \tilde{\xi}_1; \tilde{n}_2 = \tilde{\xi}_2 / \tilde{\xi}_2; \tilde{n}_1 = 1; \tilde{n}_2 = 1 \). Then the integral (3.9) yields
\[
\int d\Omega_{\tilde{n}_2} (\tilde{\xi}_1 \tilde{\xi}_2)^{-1} \delta(1 - \tilde{n}_1 \tilde{p}_2) \delta(1 - \tilde{n}_1 \tilde{n}_2) = 2\pi \delta((\tilde{\xi}_1, \tilde{\xi}_2)).
\] (3.11)
However, since \((\tilde{\xi}_1, \tilde{\xi}_2) = (\xi_1, \xi_2)\), one has finally
\[
\int \frac{1}{2\pi} \delta((\tilde{\xi}_1, \eta_2)) \frac{1}{2\pi} \delta((\eta_2, \tilde{\xi}_2)) \frac{\kappa' \delta((\eta_2, p_{21}) - \kappa')}{\eta_2^0} = \frac{1}{2\pi} \delta((\xi_1, \xi_2)).
\] (3.12)
So we see that the validity of the composition law requires to accept in the role of coefficients staying before the \( \delta \) functions the factors \( \frac{1}{2\pi} \), \( \kappa \) and \( \kappa' \), respectively. In a similar way we can compute integrals of other combinations for arguments of \( \delta \).

Introduce the symbolical denotation
\[
\mathcal{D}x(\rho) = d^4x_2 d^4x_3 d^4x_4 \ldots d^4x_{N-1}; \quad \mathcal{D}p(\rho) = \frac{d^4p_{21}}{(2\pi \hbar)^4} \frac{d^4p_{32}}{(2\pi \hbar)^4} \ldots \frac{d^4p_{N,N-1}}{(2\pi \hbar)^4}
\]
\[
\mathcal{D}\xi(\rho) = \frac{d^3\xi_2}{\xi_2^0} \frac{d^3\xi_3}{\xi_3^0} \ldots \frac{d^3\xi_{N-1}}{\xi_{N-1}^0}; \quad \mathcal{D}\eta(\rho) = \frac{d^3\eta_2}{\eta_2^0} \frac{d^3\eta_3}{\eta_3^0} \ldots \frac{d^3\eta_{N-1}}{\eta_{N-1}^0}.
\] (3.13)
The amplitudes acquire then to have the form
\[
A_{N1} = \int e^{\frac{i}{\hbar} \sum k S_{k+1,k} \delta[\xi(\rho), \xi(\rho)]} \mathcal{D}p(\rho) \mathcal{D}x(\rho) \mathcal{D}\xi(\rho) \mathcal{D}\eta(\rho)
\] (3.14)
and analogically for \( B_{N1}, C_{N1} \) and \( D_{N1} \).

The integration over \( x \) and \( p \) in the composition law can be performed in the standard way and reads
\[
\int \frac{d^4p_{21}}{(2\pi \hbar)^4} \int \frac{d^4p_{32}}{(2\pi \hbar)^4} \left[ e^{\frac{i}{\hbar} \left( -[p_{21}.(x_2-x_1) + p_{32}.(x_3-x_2)] + \frac{1}{2}(p_{21}^2 - m^2)p_{21} + (p_{32}^2 - m^2)p_{32}) \right)} \right] \times
\]
\[ d^4x_2 = \int \frac{d^4p_{31}}{(2\pi\hbar)^4} e^{\frac{i}{\hbar}} \left\{ -[p_{31} \cdot (x_3 - x_1)] + \frac{i}{2}(p^2_{31} - m^2)p_{31} \right\}, \] (3.15)

where \( p_{31} = p_{32} + p_{21} \). We see that due to the integration over \( x \) the neighbouring momenta \( p_{21} \) and \( p_{32} \) are always equal. Therefore it is irrelevant what momentum was used to define \( \tilde{\xi} \) and \( \tilde{\eta} \). However the ambiguity of selecting \( p \) plays no role.

4. Quantum model and spinors

The previous analysis has shown that we can make a selection of four amplitudes and also that we have the possibility to use a matrix form of the amplitude with the 2x2 dimensions. The complete structure of the propagator must of course be of the type consisting of 4x4 Dirac’s \( \gamma \) matrices. Therefore we adjoint the undot spinor \( \zeta \) to each \( \xi^\mu \), namely

\[ (\xi^0 - \vec{\sigma}.\vec{\zeta})\zeta = 0, \] (4.1)

where \( \vec{\sigma} \) are the Pauli matrices. The solution of Eq.(4.1) is

\[ \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = c \begin{pmatrix} e^{-i\frac{\varphi}{2}} \cos \frac{\vartheta}{2} \\ e^{i\frac{\varphi}{2}} \sin \frac{\vartheta}{2} \end{pmatrix}, \] (4.2)

where \( \vartheta \) and \( \varphi \) are the spherical angles of the unity vector \( \vec{n} \), and \( c \) a normalization factor. The spinor \( \zeta \), normalized so that

\[ \xi^\mu = (\zeta^+ \zeta, \zeta^+ \vec{\sigma} \zeta), \] (4.3)

yields \( |c| = \sqrt{\xi^0} \). The phase may be choosen arbitrarily and so we chose \( c = |c| \). Note that we can refer \( \zeta \) to \( \xi^\mu \) unambiguosly only if the condition (2.2) is obeyed.

Likewise we can introduce for each \( \eta \) the dot spinor \( \chi \), using the equation

\[ (\eta^0 + \vec{\sigma}.\vec{\eta})\chi = 0, \] (4.4)

the solution of which (now \( \eta^2 = 0 \)) is

\[ \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = c' \begin{pmatrix} e^{-i\frac{\varphi}{2}} \sin \frac{\vartheta}{2} \\ e^{i\frac{\varphi}{2}} \cos \frac{\vartheta}{2} \end{pmatrix}, \] (4.5)

this time with the spherical angles of the vector \( \vec{n} = \vec{\eta}/\eta^0 \). Normalizing \( \chi \) according to

\[ \eta^\mu = (\chi^+ \chi, -\chi^+ \vec{\sigma} \chi), \] (4.6)

we obtain \( |c'| = \sqrt{\eta^0} \).

The relative phase of \( \chi \) under \( \zeta \) is deduced from the requirement to assure the transition \( \zeta \to \xi \) using the space inversions \( \vec{\xi} \to -\vec{\xi} \) and \( \vec{\eta} \to -\vec{\eta} \), because in this
case Eq.(4.1) passes to Eq.(4.4) and vice versa. Due to the inversions \( \theta \to \pi - \theta \) and \( \phi \to \phi + \pi \), Eq.(4.2) acquires the fashion
\[
\zeta = i \sqrt{\xi^0} \begin{pmatrix} e^{-i \frac{\phi}{2}} \sin \frac{\phi}{2} \\ e^{i \frac{\phi}{2}} \cos \frac{\phi}{2} \end{pmatrix}.
\] (4.7)

Next we adopt \( \chi \) as follows
\[
\chi = i \sqrt{\eta^0} \begin{pmatrix} -e^{-i \frac{\phi}{2}} \sin \frac{\phi}{2} \\ e^{i \frac{\phi}{2}} \cos \frac{\phi}{2} \end{pmatrix}.
\] (4.8)

and at the same time we secure the validity
\[
\frac{\xi}{\xi^0} = -\frac{\eta}{\eta^0}; \quad (\xi^0)^{-1/2} \zeta = (\eta^0)^{-1/2} \chi.
\] (4.9)

This simple equation suits to the situation, when \( \vec{p} = 0 \), since in this case from (2.3) it follows (4.9), as well \( p_0 \xi^0 = \kappa; \ p_0 \eta^0 = \kappa' \). It means we have for \( \vec{p} = 0 \): \( (\kappa)^{-1/2} \zeta = (\kappa')^{-1/2} \chi \). For \( \vec{p} \neq 0 \) we again use (2.3) and derive
\[
p^{-2}(p^0 - \vec{\sigma} \cdot \vec{p}) \zeta = \chi' \zeta; \quad p^{-2}(p^0 + \vec{\sigma} \cdot \vec{p}) \chi = \zeta' \chi,
\] (4.10)

where
\[
\chi' = \frac{p^2}{2(p_0, \eta)}(\eta^0 - \vec{\sigma} \cdot \vec{\eta}) \zeta; \quad \zeta' = \frac{p^2}{2(p, \xi)}(\xi^0 + \vec{\sigma} \cdot \vec{\xi}) \chi.
\]

The solutions of Eqs.(4.10) are unique and equal \( \xi' = c' \chi \) and \( \zeta' = c \zeta \) up to the two Lorentz invariant factors \( c \) and \( c' \). It is easily verified for both the factors to be
\[
c = \sqrt{\frac{p^2 \sqrt{\kappa'}}{\sqrt{\kappa}}}; \quad c' = \sqrt{\frac{p^2 \sqrt{\kappa}}{\sqrt{\kappa'}}}.
\] (4.11)

The ultimate form of Eqs.(4.10) is hence
\[
(p^0 - \vec{\sigma} \cdot \vec{p}) \zeta = \frac{\sqrt{p^2 \kappa}}{\sqrt{\kappa'}} \chi; \quad (p^0 + \vec{\sigma} \cdot \vec{p}) \chi = \frac{\sqrt{p^2 \kappa'}}{\sqrt{\kappa}} \zeta,
\] (4.12)

being
\[
(\eta^0 - \vec{\sigma} \cdot \vec{\eta}) \zeta = 2 \frac{\sqrt{\kappa \kappa'}}{\sqrt{p^2}} \chi; \quad (\xi^0 + \vec{\sigma} \cdot \vec{\xi}) \chi = 2 \frac{\sqrt{\kappa \kappa'}}{\sqrt{p^2}} \zeta.
\]

Continual integrals over the variables \( \xi \) and \( \eta \) are possible to be carried out just as in case of integration over \( p \) in the previous section. Now, there remains in the transition amplitude only the \( \delta \) function chain of types \( \delta[(\xi_1, \xi_2)], \delta[(\eta_1, \eta_2)] \), or possibly \( \delta[(\xi, \eta)] \).

The integrals with the spinor products are coupled with the following form, which gives for \( p^2 > 0 \)
\[
\int \delta[(\xi \cdot p) - \kappa] \frac{d^3 \xi}{\xi^0} = \int \frac{\kappa d\Omega_{\vec{p}}}{(p^0 - \vec{n} \cdot \vec{p})^2} = \frac{4\pi \kappa}{p^2}.
\] (4.13)
Using (4.13) these integrals read

\[ \int \zeta + \delta((\xi \cdot p) - \kappa) \frac{d^3 \xi}{\xi^0} = 2 \pi \frac{\kappa^2}{p^4} (p^0 + \vec{\sigma} \cdot \vec{p}) \]  
(4.14a)

\[ \int \chi + \delta((\eta \cdot p) - \kappa') \frac{d^3 \eta}{\eta^0} = 2 \pi \frac{\kappa'^2}{p^4} (p^0 - \vec{\sigma} \cdot \vec{p}) . \]  
(4.14b)

We exploit now the results (4.14) and derive the appropriate amplitude

\[ \int \frac{d^4 p}{(2\pi \hbar)^4} \int \frac{d^3 \xi_1}{\xi_1^0} \int \frac{d^3 \xi_2}{\xi_2^0} e^{i \hat{p} S_{21}} \zeta_1 \zeta_2^+ \kappa \delta((\xi_1 \cdot p) - \kappa) . \]  
(4.15)

where \( \zeta_1 \) refers to \( \xi_1 \) and \( \zeta_2 \) to \( \xi_2 \), respectively. The amplitude (4.15) can be modified as follows

\[ \int \frac{d^4 p}{(2\pi \hbar)^4} \int \frac{d^3 \xi_1}{\xi_1^0} e^{i \hat{p} S_{21}} \zeta_1 \zeta_2^+ \kappa \delta((\xi_1 \cdot p) - \kappa) , \]  
(4.16)

because due to \( \xi_1^0 = \kappa(p^0 - \vec{n}_1 \cdot \vec{p})^{-1} \) we have

\[ \int \frac{d^3 \xi_2}{\xi_2^0} \zeta_2^+ \delta((\xi_1 \cdot \xi_2)) \delta((\xi_2 \cdot p) - \kappa) = \frac{2\pi}{\kappa} \zeta_1^+ . \]  

Next, with help of (4.14a) the integral (4.16) yields

\[ 2\pi \int \frac{d^4 p}{(2\pi \hbar)^4} e^{i \hat{p} S_{21}} \kappa^3 p^{-4} (p^0 + \vec{\sigma} \cdot \vec{p}) . \]  
(4.17)

Note that a correct propagator should not have \( p^4 \) in the denominator. We must thus eliminate in (4.17) the quantity \( p^{-4} \). This may be made easily by the direct applying the operator \(-\hbar \partial_{\mu}^2 \partial_{\nu}^2\) on the amplitude deduced. So we have

\[ 2\pi \kappa^3 i \hbar \left( \frac{\partial}{\partial x_0^2} - \vec{\sigma} \cdot \frac{\partial}{\partial x_2^2} \right) \int \frac{d^4 p}{(2\pi \hbar)^4} e^{i \hat{p} S_{21}} . \]  
(4.18)

By analogy with (4.15) one can create the amplitude associated with the variable \( \eta \)

\[ \int \frac{d^4 p}{(2\pi \hbar)^4} \int \frac{d^3 \eta_1}{\eta_1^0} \int \frac{d^3 \eta_2}{\eta_2^0} e^{i \hat{p} S_{21}} \chi_1 \chi_2^+ \kappa' \delta((\eta_1 \cdot p) - \kappa') . \]  
(4.19)

Likewise as in the previous case we obtain the formula analogical with (4.18)

\[ 2\pi \kappa^3 i \hbar \left( \frac{\partial}{\partial x_0^2} + \vec{\sigma} \cdot \frac{\partial}{\partial x_2^2} \right) \int \frac{d^4 p}{(2\pi \hbar)^4} e^{i \hat{p} S_{21}} . \]  
(4.20)
Finally we sketch the derived amplitudes in terms of Dirac’s matrices taken in the spinor representation. Both the amplitudes can be written up in terms of the Dirac matrices \( \gamma \) in the following way

\[
i \hbar \left( \gamma^0 \frac{\partial}{\partial x^0} + \gamma^i \frac{\partial}{\partial x^i} \right) \int \frac{d^4p}{(2\pi\hbar)^4} e^{iS_{21}}
\]

\[
i \hbar \left( \frac{\partial}{\partial x^0} + \tilde{\sigma} \frac{\partial}{\partial x^0} - \sigma \frac{\partial}{\partial x^0} \right) \int \frac{d^4p}{(2\pi\hbar)^4} e^{iS_{21}}
\]

\[
\gamma^\mu p_\mu \int \frac{d^4p}{(2\pi\hbar)^4} e^{iS_{21}},
\]

with the operator \( p_\mu = i\hbar \frac{\partial}{\partial x_\mu} \). We see that both the amplitudes are defined as the out-diagonal ones.

The diagonal amplitudes are determined straightforwardly. Let us compute the following integral

\[
\int \frac{d^4p}{(2\pi\hbar)^4} \int \frac{d^4\xi_1^0}{\xi_1^0} \int \frac{d^4\eta_2^0}{\eta_2^0} \zeta_1 \chi_2^+ e^{iS_{21}} \delta[(\xi_1, p) - \kappa] \cdot \frac{1}{2\pi} \delta[(\xi_1, \eta_2) - \kappa'].
\]

Since it holds \( \delta[(\xi_1, \eta_2) = \delta[(\tilde{\xi}, \eta_2)] \), we have then

\[
\int \frac{d^4\eta_2^0}{\eta_2^0} \chi_2^+ \delta[(\tilde{\xi}, \eta_2)] \delta[(\eta_2, p) - \kappa'] = \frac{2\pi\chi_2^+}{(\tilde{\xi}, p)},
\]

being \( \eta_2^0 = \kappa'(p^0 - \tilde{n}_1 \tilde{p})^{-1} \) and \( \tilde{n}_2 = \tilde{n}_1 \). It can be also easily found that both the equations are equivalent to the equations

\[
\frac{\xi_1^\mu}{\kappa} + \frac{\eta_2^\mu}{\kappa'} = \frac{2}{p^2}.
\]

However, Eq.(4.25) is the equation from which, as it was seen, Eqs. (4.12) have been derived. Thus if we take the first equation of (4.12) in the form

\[
\zeta_1^+(p^0 - \tilde{\sigma}, \tilde{p}) = \sqrt{p^2} \sqrt{\frac{\kappa}{\kappa'}} \chi_2^+
\]

and we multiply this equation by \( \zeta_1 \), we obtain as a result

\[
\zeta_1 \zeta_1^+(p^0 - \tilde{\sigma}, \tilde{p}) = \sqrt{p^2} \sqrt{\frac{\kappa}{\kappa'}} \zeta_1 \chi_2^+.
\]

We see thus that the total amplitudes \( < \phi_f | \phi_i > \) given by Eqs.(4.14) and (4.23) assert that the equations of motion linking to (2.12) plus the constraints (2.2) and (3.4) are sufficient to construct the Feynman continual integrals (in the \( p, \xi, \eta \) space) for the relativistic particle of 1/2 spin in the spinor form. Eq.(4.18) is the equation equivalent...
to Dirac’s equation, defined for the two-component spinor with two prescribed initial constants. Such a form of the equation was suggested in [4] by Feynman and Gell-Mann to characterize weak interaction decays within the V-A Lagrangian (see also [5], [6]). It is not unexpectable that in a similar quantization scheme one is able to yield also the Pauli equation [7].

5. Conclusion

The original Feynman approach [4] meant a compromise between the standard operator formalism for the spin degrees of freedom and the scheme of continual integrals for the orbital ones. The infirmity of such hybrid theory was conceded by Feynman himself (see his Nobel lecture in [8]). One way or another, the formulation of any relativistic dynamics for Dirac particles in terms of the continual integrals deserves to pay attention in many regards, without excepting quantization of AAD models others than merely the electrodynamical ones. Our model, operating with the linear interaction and having the degree of intrinsic relationship with the electrodynamic interaction (at least in sense of mathematical structures [10]), seems to suggest through the Feynmanian quantization at the same time also a way how to approach to the delicate question of spin by Feynman, using directly the concept of linear interaction, taken in the traditionally well understand region of quantum theory, i.e. in the domain of weak coupling. We have shown that dynamics of the point particle with the internal degrees of freedom, created by AAD theory, with the canonical pair of variables $\xi$ and $\eta$, is just able to realize quantization and to yield the standard equation of the particle with the fixed mass and 1/2 spin value.

Naturally, here one can easily anticipate difficulties, if one has to consider states with strong coupling. In a way, the models of quantum AAD theory can have in this case a predictive value for current methods of the light-cone quantization [9].

Finally we note that the discussed model provides its verifiable force and common sense, furthermore, in view of it means a natural generalization of the nonrelativistic version of Pauli’s equation and brings also new considerations on the problem of particle structure [7], close to trends, for which today the new idea of duality [11], [12] paves the way.

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A LINEAR-CONFINED PARTICLE AND THE DIRAC EQUATION

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The model of a classical particle with the weak linear AAD potential is subjected to path integral quantization. The light cone constraints and peculiar properties of its internal variables permit to use in calculations commutative dynamics and apply path integrals for a matrix form of the transition amplitude. Quantization leads to description of a Dirac particle.

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1. Introduction

Motivated by the Wheeler-Feynman action-at-a-distance (AAD) electrodynamics, a model of particle with the weak linear potential was developed [1],[3], which allows to make reasonable physical implications linking elements of its commutative dynamics with the light cone constraints. Due to these constraints, there occurs the possibility to perform relatively easily quantization of the model in the intention of Feynman. The quantization requires to fulfil the important condition: the internal variables of the model, inputted by AAD dynamics, have to be regarded as a canonical pair. In this case they become a specimen element of path integrals, which admits to introduce into the Feynman formalism chains of Dirac $\delta$ functions and thereby to facilitate calculations. Beside the use of the internal variables as the tools of pure Hamiltonian dynamics, it will be emphasized in this paper even the role of the Routh function in the dynamical considerations. Its introduction consists in the effort to give a more general meaning to these variables in the process of quantization. They could express, in our opinion, the connection between their light-cone constraint content and spinning of particle; at least to that extent as it was anticipated by Feynman [2], when he tried to establish a compromise solution of spin 1/2 particle dynamics.

In the paper [3] we have formulated major features of relativistic classical dynamics of the point particle with the internal degrees of freedom, defined according to the AAD theory of linear interaction taken for weak coupling. Now we want to show that this model leads to quantum dynamics of a particle with the fixed mass and the spin 1/2. Dynamics of this particle must be supplemented by subsidiary conditions, which are, however, consistent with the equations of motions. Moreover, they are sufficient to construct the Feynman integrals along the paths. The appropriate quantum equation, obtained in this formalism, is the equation for the two-component spinor, proposed by Feynman and Gell-Mann [4], which is equivalent to the Dirac equation. The nonrelativistic Feynmanian quantization of the model is given in [7].

In Sec. 2 a brief summary of main properties of classical model is given. Sec. 3 explains that if the constraints are considered to involve even virtual motions and the presence of weak electromagnetic action, they are more flexible, when the Routh function is introduced. In Sec. 4 it is argued that the Feynman formalism demands to adopt four transition amplitudes and to add some subsidiary conditions in close connection with the constraints and the equations of motion. The form, in which the internal variables must be represented to correspond to the complete structure of transition amplitude, appears to have the spinor behaviour.

2. Glossary of the classical model

Before discussing the quantum version of the model let us summarize major properties of its classical picture. The examined particle was described [1] as a point object with internal degrees of freedom, dynamics of which is related to the transitive realizations of Poincaré’s group with the generators

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu + \xi^\mu \eta^\nu - \xi^\nu \eta^\mu,$$  

(2.1)
where $\xi$ and $\eta$ were considered as the canonically conjugate internal variables (properly $\xi$ and $b\eta$ evaluated with $b = 1$) obeying the constraints

$$\xi^2 = 0 \quad \eta^2 = 0.$$ (2.2)

The physically most interesting are realizations that correspond to the case $p^2 \geq 0$. In this case the irreducible unitary representation of small Lorentz's group is finitely dimensional. Then one can avoid calculations with $\xi$ and $\eta$ as the continual variables and pass straightforwardly to matrices, which is a standard approach resulting into conventional matrix description of spin [2].

Hence, we utilize the variables $\xi$ and $\eta$ for all region $p^2 \geq 0$. In order that the realization of the Poincaré group remains transitive, we subject these variables beside (2.2) the following conditions

$$(\xi, p) = \kappa \quad (\eta, p) = \kappa' \quad (\xi, \eta) p^2 = 2(p, \xi) (p, \eta) = 2\kappa \kappa',$$ (2.3a)

$$(\xi, \eta) p^2 = 2(p, \xi) (p, \eta) = 2\kappa \kappa',$$ (2.3b)

$\kappa$ and $\kappa'$ being some constants. Such a particle is possible then to be conceived as an elementary object, deduced from the form of linear AAD interaction, having all necessary attributes of conventional particles, $(m, \vec{p}, \vec{x})$. Moreover, its state can be characterized by one unity vector $\vec{n} = \vec{\xi}/\xi^0$. The two degrees of freedom, peculiar to this vector, give the direct physical content for the quantum picture of the particle.

The particle fulfills the variation principle

$$\delta \left\{ - \int_{\tau_1}^{\tau_2} d\tau [ (\eta, \dot{\xi}) + R ] + (\xi, \eta) \tau = \tau_2 \right\} = 0,$$ (2.4)

where $R$ is the Routh function and the constraints (2.2) and (2.3) are valid. The equations of motion, derived from (2.4), are

$$\frac{d}{d\tau} \frac{\partial R}{\partial u^\mu} - \frac{\partial R}{\partial x^\mu} = 0$$

$$\frac{d\vec{\xi}}{d\tau} = \frac{\partial R}{\partial \eta} + \frac{\eta}{(\xi, \eta)} (\vec{\xi} \cdot \frac{\partial R}{\partial \eta})$$;

$$\frac{d\dot{\eta}}{d\tau} = - \left[ \frac{\partial R}{\partial \xi} + \frac{\xi}{(\xi, \eta)} (\vec{\eta} \cdot \frac{\partial R}{\partial \xi}) \right]$$;

$$\frac{d\xi^0}{d\tau} = \frac{\eta^0}{(\xi, \eta)} (\vec{\xi} \cdot \frac{\partial R}{\partial \eta})$$;

$$\frac{d\eta^0}{d\tau} = - \frac{\xi^0}{(\xi, \eta)} (\vec{\eta} \cdot \frac{\partial R}{\partial \xi}).$$ (2.5)

One expects naturally that dynamics of the free particle will be sufficiently simple. We can also believe that on the classical level the fourvectors $\xi$ and $\eta$ will refer to a "residuum" of particle spin according to intuition of Feynman. This spin relic, of course, must be a constant in time without the presence of electromagnetic field [2]. Therefore it is natural to assume for the free particle to be

$$\frac{d\xi^\mu}{d\tau} = 0; \quad \frac{d\eta^\mu}{d\tau} = 0.$$ (2.6)

The Routh function, independent in this case on $x$, $\xi$ and $\eta$, and expressed in terms of the fourvelocity $u$, is given by

$$R = \frac{1}{2} \left( \mu u^2 + \frac{m^2}{\mu} \right)$$ (2.7)
where $\bar{\mu}$ is an auxiliary variable introduced for the action $S$ to be invariant under the reparametrization. It is obvious that for the momentum we have $p^\mu = \frac{\partial R}{\partial \dot{u}^\mu} : \frac{dp^\mu}{d\tau} = 0$ and $p^\mu = \bar{\mu} u^\mu$. It is also evident that the constraints (2.3) are consistent with the equations of motion of this free particle.

Note that the conditions (2.3), considered either for $p$ or $u$, admit in fact only two degrees of freedom, associated with the orientation of $\tilde{\xi}$. Furthermore, for $\xi^\mu$ it holds $\xi^2 = 0$ and $(\xi.p) = \kappa$, which admits any direction for $\tilde{\xi}/\xi^0$.

Let us give the form of the action functional that leads directly to the Hamilton canonical equations. It is

$$ S = -\int_{\tau_1}^{\tau_2} d\tau \left[ (p.u) + (\eta.\dot{\xi}) + H \right] + (\xi.\eta)_{\tau=\tau_2}, \quad (2.8) $$

where

$$ H = \frac{1}{2\bar{\mu}} (m^2 - p^2). \quad (2.9) $$

The Hamilton-Jacobi function $S_{21}$ can be evaluated as $S$ taken for real motion. We have

$$ S_{21} = -\frac{1}{2} \left[ \frac{(x_2 - x_1)^2}{\rho_{21}} + m^2 \rho_{21} \right], \quad (2.10) $$

$\rho$ being defined as $d\rho = \frac{d\tau}{\bar{\mu}}$. In (2.10) the dependence of $S_{21}$ on $\xi$ and $\eta$ is not present because $(\eta.\dot{\xi}) = 0$ and the term $(\xi.\eta)_{\tau=\tau_2}$ yields a constant factor due to (2.3). The form of the action $S_{21}$ is now

$$ S_{21} = -\text{Extr} \left\{ [p_{21}.(x_2 - x_1)] + H_{21}\rho_{21} \right\}, \quad (2.11) $$

where now

$$ H_{21} = \frac{1}{2} (m^2 - \rho_{21}^2). $$

The behaviour of the model if the electromagnetic field is present is described by the Routh function

$$ R = \frac{1}{2} \left( \bar{\mu} u^2 + \frac{m^2}{\bar{\mu}} \right) + e\tilde{A}_\mu u^\mu + \lambda F_{\mu\nu} \xi^\mu \eta^\nu, \quad (2.12) $$

where $\lambda$ is a coupling constant and $\tilde{A}_\mu$ and $F_{\mu\nu}$ are the fourpotential and strenght of electromagnetic field, respectively. $\lambda$ can be determined by the requirement for the constraints $\tilde{\mu}(\xi,u) = \kappa$ and $\tilde{\mu}(\eta,u) = \kappa'$ to be in agreement with the equations of motion. Of course, they are

$$ \frac{d}{d\tau} (\tilde{\mu} u^\mu) = eF^{\mu\nu} u^\nu, \quad (2.13) $$

with $F^{\mu\nu} = \tilde{A}_{\nu,\mu} - \tilde{A}_{\mu,\nu}$. We assume, again as in the case of linear field, that action of electromagnetic field is weak, omitting the term $\partial_\mu (F_{\rho\nu} \xi^\rho \eta^\nu)$.

The equations of motion for $\xi$ and $\eta$ are the consequences of the variation principle. We must here evaluate the corresponding derivates of $R$ over $\tilde{\xi}$ and $\tilde{\eta}$. The result is simple:

$$ \frac{d\xi^\mu}{d\tau} = -\lambda F^{\mu\nu} \dot{\xi}_\nu; \quad \frac{d\eta^\mu}{d\tau} = \lambda F^{\mu\nu} \dot{\eta}_\nu. $$
with \( \lambda = e/\mu \) deduced from the condition \( \mu(\xi, u) = \text{const.} \)

It can be shown that the conditions (2.3) are consistent with the condition

\[
\frac{\xi^\mu}{\mu(u, \xi)} + \frac{\eta^\mu}{\mu(u, \eta)} - 2 \frac{u^\mu}{\mu u^2} = 0 ,
\]

and thus with the equations of motion. Eq (2.14) may be viewed physically as the fact that the external field is not able to cause spin excitations of the particle.

The Hamilton formalism requires now for \( p \) and \( H \) the following forms

\[
p^\mu = \mu u^\mu + e \tilde{A}^\mu
\]

\[
H = \frac{1}{2\mu} \left[ m^2 - (p^\mu - e \tilde{A}^\mu)^2 \right] + e \frac{F_{\mu \nu} \xi^\mu \eta^\nu}{\mu},
\]

respectively, or using the variable \( \rho \), equivalently

\[
H = \frac{1}{2} \left[ m^2 - (p^\mu - e \tilde{A}^\mu)^2 \right] + e F_{\mu \nu} \xi^\mu \eta^\nu.
\]

It can be easily verified that the Hamilton-Jacobi function depends only on \( x \) and \( x' \), but not on \( \xi \) and \( \eta \). As a result the action integral is identical with that defined for the scalar particle.

Dynamics for the variables \( \xi \) and \( \eta \) is thus determined, if \( F_{\mu \nu} \neq 0 \), by the same subsidiary conditions as that for the free particle. This means, for the same times \( \rho \) the condition (2.14) will be identical with that for the free particle. For the different \( \rho \) it will differ.

3. Quantum dynamics of the model

In the Feynman formalism dynamics of a particle is determined by the transition amplitude from the state at the "time" \( \rho_1 \), to the state at the "time" \( \rho_2 \). Our model is accomodated to have four amplitudes

\[
A_{21} = A_{21}(x_2, x_1, \xi_2, \xi_1, \rho_{21}); \quad B_{21} = B_{21}(x_2, x_1, \eta_2, \eta_1, \rho_{21})
\]

\[
C_{21} = C_{21}(x_2, x_1, \eta_2, \xi_1, \rho_{21}); \quad D_{21} = D_{21}(x_2, x_1, \xi_2, \eta_1, \rho_{21}).
\]

The structure of these amplitudes demands their dependence on \( x_2 \) and \( x_1 \) to be defined by the standard exponential of the type

\[
\rho_{21}^{-2} e^{-\frac{1}{2\mu} |p_2^{-1}(x_2 - x_1)|^2 + m^2 \rho_{21}}.
\]

On the other hand, the dependence on \( \xi \) and \( \eta \) have to be characterized by Dirac’s delta functions in a way by which the subsidiary conditions for \( \xi \) and \( \eta \) are suitably expressed (relations (2.3) or (2.14)).
The integrals over trajectories are usually taken in the form involving the integration over momenta. Therefore we shall use the following integrals
\[
\int e^{i\int \left\{-[p_{21} \cdot (x_2 - x_1)] + \frac{i}{2} (p_{21}^2 - m^2) \rho_{21}\right\}} \frac{d^4 p_{21}}{(2\pi \hbar)^4} = \frac{1}{i(2\pi \hbar)^2} e^{-\frac{i}{\hbar} \rho^{-1}_{21} (x_2 - x_1)^2 + m^2 \rho_{21}}.
\]
(3.3)

The continual integration over \( \xi \) and \( \eta \) will be represented by the differentials
\[
\frac{d^3 \eta_2}{\eta_2^0} \frac{d^3 \xi_2}{\xi_2^0} \frac{d^3 \eta_3}{\eta_3^0} \frac{d^3 \xi_3}{\xi_3^0} \ldots
\]
which are the Lorentz invariant quantities. One of kinds of auxiliary conditions will be expressed by the combinations of \( \delta \) functions:
\[
\delta[(\xi_1, p_{21}) - \kappa]\delta[(\eta_2, p_{21}) - \kappa']\delta[(\xi_2, p_{32}) - \kappa]\delta[(\xi_3, p_{32}) - \kappa']\ldots.
\]

If one introduces into (2.3b)
\[
\tilde{\xi}^\mu = \xi^\mu - 2p^\mu p^\mu(p, \xi); \quad \tilde{\eta}^\mu = \eta^\mu - 2p^\mu p^\mu(p, \eta),
\]
then the subsidiary conditions may be written, respectively
\[
(\tilde{\xi}, \eta) = 0; \quad (\tilde{\eta}, \xi) = 0.
\]
(3.5)

So we find straightforwardly the analogon of conditions (2.2)
\[
\tilde{\xi}^2 = 0; \quad \tilde{\eta}^2 = 0.
\]
(3.6)

It is apparent that the relations (3.5) lead to the new conditions
\[
\delta[(\tilde{\xi}, \eta)] = \delta[(\xi, \eta)],
\]
(3.7)
expressed in terms of \( \delta \) functions. As a result, in the functional integral there appears a \( \delta \) function chain of the form
\[
\delta[(\tilde{\xi}_1, \eta_2)]\delta[(\eta_2, \xi_2)]\delta[(\xi_2, \eta_3)]\delta[(\eta_3, \xi_3)]\delta[(\xi_3, \eta_4)]\delta[(\eta_4, \xi_4)]\ldots
\]
(3.8)

Let us calculate, for instance, the integral
\[
\int \delta[(\tilde{\xi}_1, \eta_2)]\delta[(\eta_2, \xi_2)]\kappa'\delta[(\eta_2, p_{21}) - \kappa']\frac{d^3 \eta_2}{\eta_2^0},
\]
(3.9)
to find the way for verification of composition law. Due to relations
\[
\kappa'\delta[(\eta_2, p_{21}) - \kappa'] = \frac{\kappa'}{p_{21}^0 - \vec{n}_{21}^0} \delta\left(\frac{\eta_2^0}{p_{21}^0 - \vec{n}_{21}^0} - \frac{\kappa'}{p_{21}^0 - \vec{n}_{21}^0}\right),
\]
(3.10)
where
\[
\delta[(\eta_2, \tilde{\xi}_2)] = (\eta_2^0 \tilde{\xi}_2^0)^{-1} \delta(1 - \tilde{n}_2', \tilde{n}_2), \quad \delta[(\tilde{\xi}_1, \eta_2)] = (\eta_1^0 \eta_2^0)^{-1} \delta(1 - \tilde{n}_1', \tilde{n}_1'),
\]
and we put \(\tilde{n}_1 = \tilde{\xi}_1 / \tilde{\xi}_1^0 ; \tilde{n}_2 = \tilde{\xi}_2 / \tilde{\xi}_2^0 ; \tilde{n}_1^0 = 1 ; \tilde{n}_2^0 = 1\). Then the integral (3.9) yields
\[
\int d\Omega_n \tilde{\xi}_1 \tilde{\xi}_2 (\tilde{\xi}_1^0 \tilde{\xi}_2^0)^{-1} \delta(1 - \tilde{n}_1', \tilde{n}_2) \delta(1 - \tilde{n}_1^0, \tilde{n}_2^0) = 2\pi \delta[(\tilde{\xi}_1, \tilde{\xi}_2)].
\]
(3.11)

However, since \((\tilde{\xi}_1, \tilde{\xi}_2) = (\xi_1, \xi_2)\), one has finally
\[
\int \frac{1}{2\pi} \delta[(\xi_1, \eta_2)] \frac{1}{2\pi} \delta[(\eta_2, \xi_2)] \kappa' \delta[(\eta_2, \eta_2)] = \frac{1}{2\pi} \delta[(\xi_1, \xi_2)].
\]
(3.12)

So we see that the validity of the composition law requires to accept in the role of coefficients staying before the \(\delta\) functions the factors \(\frac{1}{2\pi}\), \(\kappa\) and \(\kappa'\), respectively. In a similar way we can compute integrals of other combinations for arguments of \(\delta\).

Introduce the symbolical denotation
\[
\mathcal{D} x(\rho) = d^4 x_2 d^4 x_3 d^4 x_4 \ldots d^4 x_{N-1}; \quad \mathcal{D} p(\rho) = \frac{d^4 p_{21}}{(2\pi h)^4} \frac{d^4 p_{32}}{(2\pi h)^4} \ldots \frac{d^4 p_{N,N-1}}{(2\pi h)^4}.
\]

The amplitudes acquire then to have the form
\[
A_{N1} = \int \left[ e^{\frac{i}{\hbar} \sum k \cdot s_{k+1,k} \delta[(\xi(\rho), \xi(\rho))] \mathcal{D} p(\rho) \mathcal{D} x(\rho) \mathcal{D} \xi(\rho) \mathcal{D} \eta(\rho)} \right] (3.14)
\]
and analogically for \(B_{N1}, C_{N1}\) and \(D_{N1}\).

The integration over \(x\) and \(p\) in the composition law can be performed in the standard way and reads
\[
\int \frac{d^4 p_{21}}{(2\pi h)^4} \int \frac{d^4 p_{32}}{(2\pi h)^4} \int \left[ e^{\frac{i}{\hbar} \{ -[p_{21} \cdot (x_3 - x_1) + p_{32} \cdot (x_3 - x_2)] + \frac{1}{2} \left[ (p_{21}^2 - m^2) \rho_{21} + (p_{32}^2 - m^2) \rho_{32} \right] \} \right] \times
\]
\[
d^4 x_2 = \int \frac{d^4 p_{31}}{(2\pi h)^4} e^{\frac{i}{\hbar} \{ -[p_{31} \cdot (x_3 - x_1)] + \frac{1}{2} (p_{31}^2 - m^2) \rho_{31} \},}
\]
(3.15)

where \(\rho_{31} = \rho_{32} + \rho_{21}\). We see that due to the integration over \(x\) the neighbouring momenta \(p_{21}\) and \(p_{32}\) are always equal. Therefore it is irrelevant what momentum was used to define \(\tilde{\xi}\) and \(\tilde{\eta}\). However the ambiguity of selecting \(p\) plays no role.

4. Quantum model and spinors

The previous analysis has shown that we can make a selection of four amplitudes and also that we have the possibility to use a matrix form of the amplitude with the 2x2
dimensions. The complete structure of the propagator must of course be of the type consisting of 4x4 Dirac's γ matrices. Therefore we adjoint the undot spinor \( \zeta \) to each \( \xi^\mu \), namely

\[
(\xi^0 - \vec{\sigma} \cdot \vec{\xi}) \zeta = 0,
\]

where \( \vec{\sigma} \) are the Pauli matrices. The solution of Eq.(4.1) is

\[
\begin{pmatrix}
\zeta_1 \\
\zeta_2
\end{pmatrix} = c \begin{pmatrix}
e^{-i\frac{\vartheta}{2}} \cos \frac{\varphi}{2} \\
e^{i\frac{\vartheta}{2}} \sin \frac{\varphi}{2}
\end{pmatrix}
\]

(4.2)

where \( \vartheta \) and \( \varphi \) are the spherical angles of the unity vector \( \vec{n} \), and \( c \) a normalization factor. The spinor \( \zeta \), normalized so that

\[
\zeta^\mu = (\zeta^+ \zeta, \zeta^+ \vec{\sigma} \zeta),
\]

yields \( |c| = \sqrt{\xi^0} \). The phase may be chosen arbitrarily and so we chose \( c = |c| \). Note that we can refer \( \zeta \) to \( \xi^\mu \) unambiguously only if the condition (2.2) is obeyed.

Likewise we can introduce for each \( \eta \) the dot spinor \( \chi \), using the equation

\[
(\eta^0 + \vec{\sigma} \cdot \vec{\eta}) \chi = 0,
\]

the solution of which (now \( \eta^2 = 0 \)) is

\[
\chi = \begin{pmatrix}
\chi_1 \\
\chi_2
\end{pmatrix} = c' \begin{pmatrix}
e^{-i\frac{\vartheta}{2}} \sin \frac{\varphi}{2} \\
e^{i\frac{\vartheta}{2}} \cos \frac{\varphi}{2}
\end{pmatrix}
\]

(4.5)

this time with the spherical angles of the vector \( \vec{n} = \frac{\vec{\eta}}{\eta^0} \). Normalizing \( \chi \) according to

\[
\eta^\mu = (\chi^+ \chi, -\chi^+ \vec{\sigma} \chi),
\]

we obtain \( |c'| = \sqrt{\eta^0} \).

The relative phase of \( \chi \) under \( \zeta \) is deduced from the requirement to assure the transition \( \zeta \to \xi \) using the space inversions \( \vec{\xi} \to -\vec{\xi} \) and \( \vec{\eta} \to -\vec{\eta} \), because in this case Eq.(4.1) passes to Eq.(4.4) and vice versa. Due to the inversions \( \theta \to \pi - \theta \) and \( \phi \to \phi + \pi \), Eq.(4.2) acquires the fashion

\[
\zeta = i \sqrt{\xi^0} \begin{pmatrix}
e^{-i\frac{\vartheta}{2}} \sin \frac{\varphi}{2} \\
e^{i\frac{\vartheta}{2}} \cos \frac{\varphi}{2}
\end{pmatrix}.
\]

(4.7)

Next we adopt \( \chi \) as follows

\[
\chi = i \sqrt{\eta^0} \begin{pmatrix}
-e^{-i\frac{\vartheta}{2}} \sin \frac{\varphi}{2} \\
e^{i\frac{\vartheta}{2}} \cos \frac{\varphi}{2}
\end{pmatrix}
\]

(4.8)

and at the same time we secure the validity

\[
\frac{\vec{\xi}}{\xi^0} = -\frac{\vec{\eta}}{\eta^0}; \quad (\xi^0)^{-1/2} \zeta = (\eta^0)^{-1/2} \chi.
\]

(4.9)
This simple equation suits to the situation, when $\vec{p} = 0$, since in this case from (2.3) it follows (4.9), as well $p_0\xi^0 = \kappa$; $p_0\eta^0 = \kappa'$. It means we have for $\vec{p} = 0$: $(\kappa)^{-1/2}\zeta = (\kappa')^{-1/2}\chi$. For $\vec{p} \neq 0$ we again use (2.3) and derive
\[
p^{-2}(p^0 - \vec{\sigma} \cdot \vec{p})\zeta = \chi'\zeta; \quad p^{-2}(p^0 + \vec{\sigma} \cdot \vec{p})\chi = \zeta'\chi, \quad (4.10)
\]
where
\[
\chi' = \frac{p^2}{2(p^2 - \vec{\eta} \cdot \vec{\eta})}\zeta; \quad \zeta' = \frac{p^2}{2(p,\xi)}(\xi^0 + \vec{\sigma} \cdot \vec{\xi})\chi.
\]
The solutions of Eqs.(4.10) are unique and equal $\xi' = c'\chi$ and $\zeta' = c\zeta$ up to the two Lorentz invariant factors $c$ and $c'$. It is easily verified for both the factors to be
\[
c = \sqrt{p^2 \kappa \kappa'}; \quad c' = \sqrt{\frac{p^2}{\kappa'}.} \quad (4.11)
\]
The ultimate form of Eqs.(4.10) is hence
\[
(p^0 - \vec{\sigma} \cdot \vec{p})\zeta = \frac{\sqrt{p^2 \kappa \kappa'}}{\sqrt{\kappa'}}\chi; \quad (p^0 + \vec{\sigma} \cdot \vec{p})\chi = \frac{\sqrt{p^2 \kappa \kappa'}}{\sqrt{\kappa}}\zeta, \quad (4.12)
\]
being
\[
(\eta^0 - \vec{\sigma} \cdot \vec{\eta})\zeta = 2\frac{\sqrt{\kappa \kappa'}}{\sqrt{p^2}}\chi; \quad (\xi^0 + \vec{\sigma} \cdot \vec{\xi})\chi = 2\frac{\sqrt{\kappa \kappa'}}{\sqrt{p^2}}\zeta.
\]
Continual integrals over the variables $\xi$ and $\eta$ are possible to be carried out just as in case of integration over $p$ in the previous section. Now, there remains in the transition amplitude only the $\delta$ function chain of types $\delta[(\xi_1,\xi_2)], \delta[(\eta_1,\eta_2)],$ or possibly $\delta[(\xi,\eta)]$.

The integrals with the spinor products are coupled with the following form, which gives for $p^2 > 0$
\[
\int \delta[(\xi,\eta) - \kappa] \frac{d^3\xi}{\xi^0} \int \frac{\kappa d\Omega_\vec{\eta}}{(p^0 - \vec{n},\vec{p})^2} = \frac{4\pi \kappa}{p^2}. \quad (4.13)
\]
Using (4.13) these integrals read
\[
\int \zeta\zeta^+ \delta[(\xi,\eta) - \kappa] \frac{d^3\xi}{\xi^0} = 2\pi \frac{\kappa^2}{p^4}(p^0 + \vec{\sigma} \cdot \vec{p}) \quad (4.14a)
\]
\[
\int \chi\chi^+ \delta[(\eta,\eta) - \kappa] \frac{d^3\eta}{\eta^0} = 2\pi \frac{\kappa^2}{p^4}(p^0 - \vec{\sigma} \cdot \vec{p}). \quad (4.14b)
\]
We exploit now the results (4.14) and derive the appropriate amplitude
\[
\int \frac{d^4p}{(2\pi \hbar)^4} \int \frac{d^3\xi_1}{\xi_1^0} \int \frac{d^3\xi_2}{\xi_2^0} e^{i\frac{S_{21}}{\hbar} \xi_1^+ \kappa \delta[(\xi_1,\eta) - \kappa]} \cdot \frac{1}{2\pi} \delta[(\xi_1,\xi_2)] \kappa \delta[(\xi_2,\eta) - \kappa], \quad (4.15)
\]
where $\xi_1$ refers to $\xi_1$ and $\xi_2$ to $\xi_2$, respectively. The amplitude (4.15) can be modified as follows
\[
\int \frac{d^4p}{(2\pi \hbar)^4} \int \frac{d^3\xi_1}{\xi_1^0} e^{i\frac{S_{21}}{\hbar} \xi_1^+ \kappa \delta[(\xi_1,\eta) - \kappa]}, \quad (4.16)
\]
because due to $\xi_1^0 = \kappa(p^0 - \vec{n}_1 \cdot \vec{p})^{-1}$ we have

$$
\int \frac{d^3 \xi_2}{\xi_2^0} \zeta_2^+ \delta[(\xi_1 \cdot \xi_2)] \delta[(\xi_2, p) - \kappa] = \frac{2\pi}{\kappa} \zeta_1^+.
$$

Next, with help of (4.14a) the integral (4.16) yields

$$
2\pi \int \frac{d^4 p}{(2\pi \hbar)^4} e^{i S_{21}} \kappa' p^{-4} (p^0 + \vec{p} \cdot \vec{x}) . \tag{4.17}
$$

Note that a correct propagator should not have $p^4$ in the denominator. We must thus eliminate in (4.17) the quantity $p^{-4}$. This may be made easily by the direct applying the operator $-\hbar \partial_{\mu}^2 \partial_{\mu}^2$ on the amplitude deduced. So we have

$$
2\pi \kappa^3 i\hbar \left( \frac{\partial}{\partial \xi_0^2} - \vec{\sigma} \cdot \frac{\partial}{\partial \vec{x}_2} \right) \int \frac{d^4 p}{(2\pi \hbar)^4} e^{i S_{21}} . \tag{4.18}
$$

By analogy with (4.15) one can create the amplitude associated with the variable $\eta$

$$
\int \frac{d^4 \eta_1}{(2\pi \hbar)^4} \int \frac{d^3 \eta_2}{\eta_2^0} \int \frac{d^3 \eta_2}{\eta_2^0} e^{i S_{21} \chi_1^+ \chi_2^+ \kappa' \delta[(\eta_1, p) - \kappa']}

\cdot \frac{1}{2\pi} \delta[(\eta_1, \eta_2)] \kappa' \delta[(\eta_2, p) - \kappa'] . \tag{4.19}
$$

Likewise as in the previous case we obtain the formula analogical with (4.18)

$$
2\pi \kappa^3 i\hbar \left( \frac{\partial}{\partial \xi_0^2} + \vec{\sigma} \cdot \frac{\partial}{\partial \vec{x}_2} \right) \int \frac{d^4 p}{(2\pi \hbar)^4} e^{i S_{21}} . \tag{4.20}
$$

Finally we sketch the derived amplitudes in terms of Dirac’s matrices taken in the spinor representation. Both the amplitudes can be written up in terms of the Dirac matrices $\gamma$ in the following way

$$
i\hbar \left( \gamma^0 \frac{\partial}{\partial \xi_0^2} + \gamma^3 \frac{\partial}{\partial \vec{x}_2} \right) \int \frac{d^4 p}{(2\pi \hbar)^4} e^{i S_{21}}

i\hbar \left( \frac{\partial}{\partial \xi_0^0} + \vec{\sigma} \cdot \frac{\partial}{\partial \vec{x}_2} \right) \int \frac{d^4 p}{(2\pi \hbar)^4} e^{i S_{21}}

\gamma^\mu p_\mu \int \frac{d^4 p}{(2\pi \hbar)^4} e^{i S_{21}} , \tag{4.21}
$$

with the operator $p_\mu = i\hbar \frac{\partial}{\partial x_\mu}$. We see that both the amplitudes are defined as the out-diagonal ones.

The diagonal amplitudes are determined straightforwardly. Let us compute the following integral

$$
\int \frac{d^4 p}{(2\pi \hbar)^4} \int \frac{d^3 \xi_1}{\xi_1^0} \int \frac{d^3 \eta_2}{\eta_2^0} \zeta_1^+ \zeta_2^+ e^{i S_{21} \kappa \delta[(\xi_1, p) - \kappa']}.
$$




\[ \frac{1}{2\pi} \delta[(\xi_1, \eta_2) - \kappa'] \cdot \delta[(\xi_2, p) - \kappa] \cdot (4.22) \]

Since it holds \( \delta[(\xi_1, \eta_2)] = \delta[(\xi_2, \eta_2)] \), we have then

\[ \int \frac{d^3}{\eta_2^2} \chi_2^+ \delta[(\xi_1, \eta_2)] \delta[(\eta_2, p)] = \frac{2\pi \chi_2^+}{(\xi_1, p)} \cdot (4.23) \]

being \( \eta_2^0 = \kappa'(p^0 - \vec{n}_1, \vec{p})^{-1} \) and \( \vec{n}_2 = \vec{n}_1 \). It can be also easily found that both the equations are equivalent to the equations

\[ \frac{\xi_1}{\kappa} + \frac{\eta_2^\mu}{\kappa'} = \frac{2p^\mu}{p^2} \cdot (4.24) \]

However, Eq.(4.25) is the equation from which, as it was seen, Eqs. (4.12) have been derived. Thus if we take the first equation of (4.12) in the form

\[ \zeta^+_1 (p^0 - \vec{\sigma}, \vec{p}) = \sqrt{p^2} \sqrt{\frac{\kappa}{\kappa'}} \chi_2^+ \]

and we multiply this equation by \( \zeta_1 \), we obtain as a result

\[ \zeta_1 \zeta^+_1 (p^0 - \vec{\sigma}, \vec{p}) = \sqrt{p^2} \sqrt{\frac{\kappa}{\kappa'}} \zeta_1 \chi_2^+ \cdot (4.25) \]

We see thus that the total amplitudes \( < \phi_f | \phi_i > \) given by Eqs.(4.14) and (4.23) assert that the equations of motion linking to (2.12) plus the constraints (2.2) and (3.4) are sufficient to construct the Feynman continual integrals (in the \( p, \xi, \eta \) space) for the relativistic particle of 1/2 spin in the spinor form. Eq.(4.18) is the equation equivalent to Dirac’s equation, defined for the two-component spinor with two prescribed initial constants. Such a form of the equation was suggested in [4] by Feynman and Gell-Mann to characterize weak interaction decays within the V-A Lagrangian (see also [5],[6]). It is not unexpectable that in a similar quantization scheme one is able to yield also the Pauli equation [7].

5. Conclusion

The original Feynman approach [4] meant a compromise between the standard operator formalism for the spin degrees of freedom and the scheme of continual integrals for the orbital ones. The infirmity of such hybrid theory was conceded by Feynman himself (see his Nobel lecture in [8]). One way or another, the formulation of any relativistic dynamics for Dirac particles in terms of the continual integrals deserves to pay attention in many regards, without excepting quantization of AAD models others than merely the electrodynamical ones. Our model, operating with the linear interaction and having the degree of intrinsic relationship with the electrodynamic interaction (at least in sense of mathematical structures [10]), seems to suggest through the Feynmanian quantization at the same time also a way how to approach to the delicate question of spin by Feynman, using directly the concept of linear interaction, taken in the traditionally well understand region of quantum theory, i.e. in the domain of weak coupling. We have
shown that dynamics of the point particle with the internal degrees of freedom, created by AAD theory, with the canonical pair of variables $\xi$ and $\eta$, is just able to realize quantization and to yield the standard equation of the particle with the fixed mass and $1/2$ spin value.

Naturally, here one can easily anticipate difficulties, if one has to consider states with strong coupling. In a way, the models of quantum AAD theory can have in this case a predictive value for current methods of the light-cone quantization [9].

Finally we note that the discussed model provides its verifiable force and common sense, furthermore, in view of it means a natural generalization of the nonrelativistic version of Pauli's equation and brings also new considerations on the problem of particle structure [7], close to trends, for which today the new idea of duality [11], [12] paves the way.

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