A Minimax Converse for Quantum Channel Coding

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Abstract

We prove a one-shot “minimax” converse bound for quantum channel coding assisted by positive partial transpose channels between sender and receiver. The bound is similar in spirit to the converse by Polyanskiy, Poor, and Verdú [IEEE Trans. Info. Theory 56, 2307–2359 (2010)] for classical channel coding, and also enjoys the saddle point property enabling the order of optimizations to be interchanged. Equivalently, the bound can be formulated as a semidefinite program satisfying strong duality. The convex nature of the bound implies channel symmetries can substantially simplify the optimization, enabling us to explicitly compute the finite blocklength behavior for several simple qubit channels. In particular, we find that finite blocklength converse statements for the classical erasure channel apply to the assisted quantum erasure channel, while bounds for the classical binary symmetric channel apply to both the assisted dephasing and depolarizing channels. This implies that these qubit channels inherit statements regarding the asymptotic limit of large blocklength, such as the strong converse or second-order converse rates, from their classical counterparts. Moreover, for the dephasing channel, the finite blocklength bounds are as tight as those for the classical binary symmetric channel, since coding for classical phase errors yields equivalently-performing unassisted quantum codes.

1 Introduction

The capacity of a noisy channel is the ultimate, in-principle limit on its capability for reliable communication, and therefore studying channel capacity is an important goal in information theory. By its nature, the capacity is not of immediate practical concern, as it ignores the resource requirements that would be needed to achieve the limit. Approaching capacity might, in principle, require coding operations and blocklengths too cumbersome or large to be implementable. Nevertheless, several classical coding techniques developed in recent years have narrowed the gap between in-principle and in-practice for classical communication over classical channel, in particular polar codes [1] and spatially-coupled low-density parity-check codes [2]. Coding and decoding operations can be performed efficiently (quasilinearly) in the blocklengths of these codes, though the blocklengths themselves must still be rather large to approach capacity. The situation is dramatically different for quantum coding, where accurate control of quantum systems is a major experimental challenge, while manipulation and storage of classical bits is obscenely easy by comparison.

Thus, it is of interest to better understand the possible performance of codes operating with fixed resources, in particular at finite blocklength. A bound which limits the performance of a coding scheme given fixed resources is known as a converse bound. For classical channels, the first truly systematic results on converse bounds limiting the size (blocklength) of codes with a given error probability were given by Polyanskiy, Poor, and Verdú [3]. They formulated the converse bound in terms of a minimax optimization, and showed by numerical examples that the bound is quite tight for several channels of interest even at small blocklengths, by comparing to existing and novel achievability bounds. Subsequently, Matthews [4] and Polyanskiy [5] demonstrated concavity and convexity properties of the bound which enable it to be formulated as a linear program (Matthews) or equivalently that the order of minimization and maximization can be interchanged (Polyanskiy). Their results imply that channel symmetries can be used to simplify the optimization.

For quantum channels, Matthews and Wehner extended the minimax approach to the task of transmitting classical information over quantum channels and formulated a bound in terms of a semidefinite
shows that the minimax bound is a somewhat elaborate discussion of how symmetry can be used to simply the bound, which ultimately rests on its concavity and convexity properties. The particular qubit channel examples are detailed in Sec. 5 and the paper finishes with a discussion of related bounds.

2 Mathematical Setup

2.1 States and Channels

In this paper we consider finite-dimensional quantum systems, labelled by capital letters A, B, and so forth. The state space of system A is denoted by \( \mathcal{H}_A \), and the dimension of this space |A|. The set of bounded operators on \( \mathcal{H}_A \) is denoted by \( \mathcal{B}(A) \), while the set of bounded operators with unit trace, i.e. the states of A, is denoted by \( \mathcal{S}(A) \). The maximally mixed state on \( \mathcal{H}_A \) is denoted by \( \tau_A = \frac{1}{|A|} \mathbf{1} \).

A channel \( \mathcal{N}_{B|A} \) is a linear map from \( \mathcal{B}(A) \) to \( \mathcal{B}(B) \) which is both trace preserving and completely positive. Adjoints of channels and operators are denoted by \( * \). The Choi representative (or Choi operator) of the channel is the bipartite operator \( \hat{N}_{B|A} = (\mathcal{N}_{B|A'} \otimes I_{A'})[\Omega_{AA'}] \). The Choi representative of a channel satisfies \( N_{B|A} \geq 0 \) and \( Tr_B[N_{B|A}] = \mathbf{1}_A \), and any bipartite operator \( \tilde{N}_{B|A} \) satisfying these conditions defines a valid channel via \( \rho_A \rightarrow Tr_A[\tilde{N}_{B|A}\rho_A^{T_A}] \), a statement known as the Choi isomorphism [8].

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In contrast, the Jamiołkowski representative of \( \mathcal{N}_{B|A} \) is the operator \( \hat{N}_{B|A} = \rho_A^{T_A} \) where \( T_A \) denotes the transpose of system A. Put differently, \( \hat{N}_{B|A} = (\mathcal{N}_{A|A'} \otimes \mathcal{N}_{B|A'})[\Omega_{AA'}^{T_{A'}}] \) [9]. An appealing property of the Jamiołkowski representative is that the action of the channel no longer involves the transpose and is just \( \rho_A \rightarrow Tr_A[\hat{N}_{B|A}\rho_{A'}^{T_{A'}}] \). This formulation makes channel action look quite similar to marginalizing random variables in a classical probability distribution; for more on this point, see [10]. Further, the Jamiołkowski isomorphism is natural, in the sense that it does not require a choice of basis, since \( \Omega_{AA'}^{T_{A'}} = \Pi_{AA'}^{sym} - \Pi_{AA'}^{antisym} \) and the projectors onto the symmetric and antisymmetric spaces of \( \mathcal{H}_A \otimes \mathcal{H}_{A'} \) are basis-independent.

However, we shall not make particular use of this fact here.

Bipartite states \( \rho_{AB} \) for which \( \rho_{AB}^{T_B} \geq 0 \) (equivalently, \( \rho_{AB}^{T_A} \geq 0 \)) will be called positive partial transpose states, or PPT for short. A channel is PPT-preserving if a PPT input necessarily results in a PPT output. In [11], it is shown that PPT-preserving channels have PPT Choi states (see the discussion after Eq. 4.13). More directly, suppose that \( \varphi_{AB} \) is a state with \( \varphi_{AB}^{T_A} \geq 0 \) and \( \mathcal{D}_{MM'|AB} \) is a channel with \( \mathcal{D}_{MM'|AB}^{T_{M'M|AB}} \geq 0 \). Let \( \sigma_{MM'} = \mathcal{D}_{MM'|AB}(\varphi_{AB}) \) and compute \( \sigma_{MM'}^{T_{M'M|AB}} \):
Although the trace of positive operators is a positive number, the partial trace need not be a positive operator (consider $\text{Tr}_B[\varphi_{AB} \Phi_{BC}]$), so we cannot conclude that $\varphi_{MM'}^{T_M} \geq 0$ on this basis alone. However, the fact that the $\varphi_{AB}$ factor is completely traced out, along with positivity of it and $D_{MM'|AB^T}^{T_{MM'}}$, together imply that all expectation values of $\varphi_{MM'}^{T_M}$ are positive.

2.2 Semidefinite programming

A semidefinite program (SDP) is simply an optimization of a linear function of a matrix or operator over a feasible set of inputs defined by positive semidefinite constraints. We give only the bare essentials here, for more detail see [12, 13].

The maximization form of an SDP is defined by a Hermiticity-preserving superoperator $\mathcal{E}_{BA}$ taking $\mathcal{B}(A)$ to $\mathcal{B}(B)$, a constraint operator $C \in \mathcal{B}(B)$, and an operator $K \in \mathcal{B}(A)$ which defines the objective function. The SDP is the following optimization, which we will also refer to as the primal form,

$$\begin{align*}
\alpha = \sup X \in \mathbb{R}^+ \text{ subject to } & \mathcal{E}(X) \leq C, \\
& X \geq 0.
\end{align*}$$

When the feasible set is empty, i.e. no $X$ satisfy the constraints, we set $\alpha = -\infty$.

The dual form arises as the optimal upper bound to the primal form, and takes the form

$$\begin{align*}
\beta = \inf Y \in \mathbb{R}^+ \text{ subject to } & \mathcal{E}^*(Y) \geq K, \\
& Y \geq 0.
\end{align*}$$

Again, when the set of feasible $Y$ is empty, $\beta = \infty$. Weak duality is the statement that $\alpha \leq \beta$, that indeed the dual form gives upper bounds to the primal (or that the primal lower bounds the dual).

Strong duality is the statement that the optimal upper bound equals the value of the primal problem, $\alpha = \beta$. This state of affairs often holds in problems of interest, and can be established by either of the following Slater conditions. In the first, called strict primal feasibility, strong duality holds if $D_{MM'|AB^T}$ is finite and there exists an $X > 0$ such that $\mathcal{E}(X) < C$. Contrariwise, under strict dual feasibility strong duality holds when $\alpha$ is finite and there exists a $Y > 0$ such that $\mathcal{E}^*(Y) > K$. For strongly dual SDPs we also have the so-called complementary slackness conditions $\mathcal{E}^*(Y)X = KX$ and $\mathcal{E}(X)Y = CY$ that relate the primal and dual optimizers.

Neyman-Pearson hypothesis testing is an SDP particularly useful in the study of information processing protocols. Given one of two states $\sigma$ or $\rho$, called the null and alternate hypothesis, an asymmetric hypothesis test is a two-outcome POVM $\{\Gamma, \Gamma - \Gamma\}$ that indicates which of the two hypotheses (states) is actually present. Here $\Gamma$ indicates $\rho$. Any such test makes two kinds of errors, the type-I error in which the null hypothesis is erroneously rejected, and the type-II error in which the alternate hypothesis is erroneously rejected. The probabilities of type-I and type-II errors are just $\text{Tr}[\rho (1 - \Gamma)]$ and $\text{Tr}[\sigma \Gamma]$, respectively. Fixing the probability of type-I error to $1 - \epsilon$, we may ask for the test with the optimal (minimal) type-II error, called the Neyman-Pearson test. The optimal POVM is specified by $\Gamma$ with the smallest value of $\text{Tr}[\sigma \Gamma]$ such that $0 \leq \Gamma \leq 1$ and $\text{Tr}[\rho \Gamma] \geq \epsilon$. This is a dual-form SDP with $Y = \Gamma$, $C = \sigma$, $K = (1 - \epsilon)$ and $\mathcal{E}^*(Y) = (\epsilon, \text{Tr}[\rho Y])$, and it satisfies strong duality. For future use, let us denote this optimal value by $\beta_\epsilon(\rho, \sigma)$:

$$\begin{align*}
\beta_\epsilon(\rho, \sigma) := \min Y \in \mathbb{R}^+ \text{ subject to } & \text{Tr}[\Gamma \sigma] \\
& \text{Tr}[\Gamma \rho] \geq \epsilon, \\
& 0 \leq \Gamma \leq 1.
\end{align*}$$

The slackness conditions can be used to infer the form of the optimal test, recovering the Neyman-Pearson lemma of classical statistics.
3 Converse bound

3.1 Coding scenario

In this section we define the coding scenario precisely. Here we consider coding schemes for using a noisy quantum channel $\mathcal{N}_{B|A}$ together with some auxiliary assistance channels to create a high-fidelity entangled state between sender Alice and receiver Bob. In particular, a PPT-assisted code denoted by $\mathcal{C} = (\varphi_{AA'B'}, \mathcal{D}_{MM'|BA'B'})$ consists of a PPT state $\varphi_{AA'B'}$ with Alice in possession of systems $AA'$ and Bob $B'$ as well as a PPT-preserving decoding operation $\mathcal{D}_{MM'|BA'B'}$ which outputs system $M$ to Alice and $M' \simeq M$ to Bob. By a slight abuse of notation, we also denote the dimension of system $M$ by $M$. Note that the decoding operation might consist of several rounds of forward and backward communication between sender and receiver.

The coding scheme proceeds by Alice and Bob creating the state $\varphi_{AA'B'}$ by means of a PPT channel, and Alice subsequently transmitting the $A$ system to Bob via $\mathcal{N}_{B|A}$. Finally, they perform the operation $\mathcal{D}_{MM'|BA'B'}$, leaving them with a bipartite state in $MM'$. The resulting entanglement fidelity is defined by

$$F(\mathcal{N}; \mathcal{C}) := \text{Tr}[\Phi_{MM'}^+\mathcal{D}_{MM'|BA'B'} \circ \mathcal{N}_{B|A}(\varphi_{AA'B'})].$$

Finally, an $(M, \epsilon)$ code for $\mathcal{N}_{B|A}$ is a pair $\mathcal{C} = (\mathcal{E}_{AA'B'}, \mathcal{D}_{MM'|BA'B'})$ with output dimension $M$ such that $F(\mathcal{N}; \mathcal{C}) \geq 1 - \epsilon$.

As a side remark, when the initial state $\varphi_{AA'B'}$ is separable and not only PPT, the coding scheme can be simplified without loss of generality. In particular, one can dispense with system $B'$ and restrict to pure states $\varphi_{AA'}$. To see this, note that the fidelity is a convex combination of entanglement fidelities for the constituent states $\sigma_{AA'}^j \otimes \eta_{B'}^j$ in $\varphi_{AA'B'} = \sum_j \sigma_{AA'}^j \otimes \eta_{B'}^j$, so it will certainly not decrease when shifting to the code $\mathcal{C}_j = (\sigma_{AA'}^j \otimes \eta_{B'}^j, \mathcal{D}_{MM'|BA'B'})$ with the largest entanglement fidelity. Further, as $\eta_{B'}$ is fixed, it can be absorbed into the decoding operation. Finally, the same argument implies that one can further modify $\sigma_{AA'}^j$ to be a pure state. However, this argument does not necessarily go through in the general case when $\varphi_{AA'B'}$ is PPT.

3.2 PPT-preserving channels

Instead of the actual channel $\mathcal{N}_{B|A}$, consider an arbitrary PPT-preserving channel $\mathcal{M}_{B|A}$. This produces a PPT state $\gamma_{MM'} = \mathcal{D}_{MM'|AA'B'} \circ \mathcal{M}_{B|A}(\varphi_{AA'B'})$. The following argument, due to Rains, then immediately implies

$$F(\mathcal{M}; \mathcal{C}) \leq \frac{1}{M}. \tag{7}$$

Lemma 1 (Rains [14]). For any subnormalized state $\gamma_{MM'}$ with $\gamma_{MM'}^T \geq 0$, $\text{Tr}[\Phi_{MM'}^+\gamma_{MM'}] \leq \frac{1}{M}$.

Proof. The proof proceeds by observing that the partial transpose of the maximally entangled state is the swap operator, normalized by $M$.

$$\text{Tr}[\Phi_{MM'}^+\gamma_{MM'}^T] = \text{Tr}[\Phi_{MM'}^T\gamma_{MM'}^T] \leq \|\Phi_{MM'}^T\|_\infty \text{Tr}[\gamma_{MM'}^T] \leq \frac{1}{M}. \tag{10}$$

The first inequality follows from the fact that $\Phi_{MM'}^T = \frac{1}{M} U^\text{SWAP}_{MM'}$ and therefore $\Phi_{MM'}^T \leq \|\Phi_{MM'}^T\|_\infty \mathbb{I}_{MM'} = \frac{1}{M} \mathbb{I}_{MM'}$. The second inequality is the subnormalization of $\gamma_{MM'}$. \hfill \Box

3.3 Fidelity in terms of the Choi operator

Now we show that the entanglement fidelity (6) can be expressed directly in terms of its Choi operator of the channel and without explicit reference to the systems $MM'A'B'$. To do so, it is actually convenient
to start with the Jamiołkowski representation of channel action. In this representation we can write the fidelity as

\[
F(\mathcal{N}; \varrho) = \text{Tr}_{MM'}[\Phi_{MM'} D_{MM'|AB} \varphi_{AB}],
\]

(11)

\[
= \text{Tr}_{MM'}[\Phi_{MM'} D_{MM'|AB} \mathcal{N}_{B|A}(\varphi_{AB})],
\]

(12)

\[
= \text{Tr}_{MM'|AB} [\Phi_{MM'} D_{MM'|AB} \mathcal{N}_{B|A}(\varphi_{AB})],
\]

(13)

\[
= \text{Tr}_{MM'|AB} [\Phi_{MM'} D_{MM'|AB} \varphi_{AB} \mathcal{N}_{B|A}],
\]

(14)

where, in the last equation, the order of \(\varphi\) and \(N\) is interchanged in accordance with the usual rules of transposition.

Define the operator

\[
\Lambda_{AB} = \text{Tr}_{MM'|AB} [\Phi_{MM'} D_{MM'|AB} \varphi_{AB} \mathcal{N}_{B|A}],
\]

(15)

Since \(N_{B|A} = \mathcal{N}_{B|A}\), the fidelity can be expressed as a linear function of the Choi operator of the channel,

\[
F(\mathcal{N}; \varrho) = \text{Tr}[\Lambda_{AB} N_{B|A}],
\]

(16)

for the particular \(\Lambda_{AB}\) defined by the code. An \((M, \epsilon)\) code will have a \(\Lambda_{AB}\) which satisfies

\[
\text{Tr}[\Lambda_{AB} N_{B|A}] \geq 1 - \epsilon.
\]

(17)

Any such operator \(\Lambda_{AB}\) satisfies the following two simple properties

**Proposition 1.** Any operator \(\Lambda_{AB}\) defined as in (15) satisfies

\[
0 \leq \Lambda_{AB} \leq \varphi_A^T \otimes \mathbb{1}_B,
\]

(18)

**Proof.** First regard \(\varphi_{AB}^{T_A}\) as the Jamiołkowski representative \(\hat{\mathcal{R}}_{AB}'|A = \varphi_{AB}^{T_A}\) of a channel \(\mathcal{R}_{AB}'|A\). Then

\[
\Lambda_{AB} = \mathcal{R}_{AB}'|A \circ \varphi_{MM'|BA}^T (\Phi_{MM'}).\n\]

(19)

Now observe that \(\mathcal{R}_{AB}'|A\) is completely positive and trace decreasing:

\[
\hat{\mathcal{R}}_{AB}'|A \geq 0,
\]

(20)

\[
\text{Tr}_{AB} [\hat{\mathcal{R}}_{AB}'|A] = \varphi_A^T \leq \mathbb{1}_A,
\]

(21)

meaning its adjoint action is completely positive and subunital. Therefore, (18) follows from the fact that

\[
0 \leq \Phi_{MM'} \leq \mathbb{1}_{MM'}.
\]

\[\Box\]

These constraints hold for more than just PPT-assisted codes, as is easily demonstrated. Let \(\varphi_{MM'}^{T_A}\) be the channel which creates \(\Phi_{MM'}\) and ignores (traces out) the input systems \(BA'\). Then \(\hat{D}_{MM'|BA'} = \Phi_{MM'} \mathbb{1}_{BA'}\). This leads to \(\Lambda_{AB} = \varphi_A^T \mathbb{1}_B\), which satisfies both constraints.

### 3.4 Minimax converse bound

It is now straightforward to derive the minimax converse bound. The idea is simple: Maximizing over PPT channels \(\mathcal{M}_{B|A}\) in the fidelity expression (16) and using (7) gives a lower bound on \(1/M\), i.e. an upper bound on \(M\). However, the result depends on the details of the code via \(\Lambda_{AB}\). This dependence can be removed by minimizing \(\Lambda_{AB}\) over the smallest conveniently-described set which certainly contains the \(\Lambda_{AB}\) associated with the code. Here, this is the set defined by the constraints in (17) and (18), for arbitrary subnormalized \(\varphi_{AB}\) as these do not depend on the precise details of the coding operations.

To state the minimax bound formally, first define the following sets

\[
ppt := \{M_{B|A} : M_{B|A} \geq 0, M_{B|A}^{T_A} \geq 0, \text{Tr}_B [M_{B|A}] \leq \mathbb{1}_A\}
\]

(22)

\[
F(\mathcal{N}, \epsilon) := \{(\varphi_A, \Lambda_{AB}) : \varphi_A \geq 0, \text{Tr}[\varphi_A] \leq 1, 0 \leq \Lambda_{AB} \leq \varphi_A^T \mathbb{1}_B, \text{Tr}[\Lambda_{AB} N_{B|A}] \geq 1 - \epsilon\}.
\]

(23)

Then we have
Corollary 1. Any \((M, \varepsilon)\) PPT-assisted code satisfies
\[
\min_{(\varphi_A, \Lambda_{AB}) \in F(\mathcal{N}, \varepsilon) M_{B|A} \in \text{PPT}} \max \quad \text{Tr}[\Lambda_{AB} M_{B|A}] \leq \frac{1}{M}. \tag{24}
\]

For later convenience, let us define
\[
f(\mathcal{N}, \varepsilon) := \min_{(\varphi_A, \Lambda_{AB}) \in F(\mathcal{N}, \varepsilon) M_{B|A} \in \text{PPT}} \max \quad \text{Tr}[\Lambda_{AB} M_{B|A}]. \tag{25}
\]

Before proceeding with the proof, observe that we can interchange the order of optimization in \(f\), due to von Neumann’s minimax theorem \([15]\), as the objective function is linear and both \(F(\mathcal{N}, \varepsilon)\) and \(\text{PPT}\) are compact, convex sets. This gives

Corollary 1. Any \((M, \varepsilon)\) PPT-assisted code satisfies
\[
\max_{M_{B|A} \in \text{PPT}} \min_{(\varphi_A, \Lambda_{AB}) \in F(\mathcal{N}, \varepsilon)} \quad \text{Tr}[\Lambda_{AB} M_{B|A}] \leq \frac{1}{M}. \tag{26}
\]

Proof of Theorem 1. Consider then the following function,
\[
f_0(O_{AB}) := \sup_{M_{B|A} \in \text{PPT}} \text{Tr}[O_{AB} M_{B|A}], \tag{27}
\]
defined on all bipartite operators \(\{O_{AB} : 0 \leq O_{AB} \leq I_{AB}\}\). First note that the supremum is attained, since the objective function is a continuous function \(M \mapsto \text{Tr}[OM]\) on a compact set, \(\text{PPT}\). Moreover, \(f_0\) is continuous; specifically, it obeys
\[
|f_0(O'_{AB}) - f_0(O_{AB})| \leq |A||O'_{AB} - O_{AB}|_1. \tag{28}
\]
To see this, suppose that \(f_0(O_{AB}) \leq f_0(O'_{AB})\), otherwise swap the two. Then \(|f_0(O'_{AB}) - f_0(O_{AB})| = f_0(O'_{AB}) - f_0(O_{AB})\). Let \(M'_{B|A}\) be the optimizer in \(f_0(O'_{AB})\). By the variational characterization of the trace norm,
\[
f_0(O_{AB}) \geq \text{Tr}[M'_{B|A} O_{AB}] \geq \text{Tr}[M'_{B|A} O'_{AB}] - |A||O'_{AB} - O_{AB}|_1 \tag{29}
\]
\[= f_0(O'_{AB}) - |A||O'_{AB} - O_{AB}|_1. \tag{30}
\]

For \(\Lambda_{AB}\) defined from an \((M, \varepsilon)\) code as in \((15)\), \((7)\) implies \(f_0(\Lambda_{AB}) \leq \frac{1}{M}\). Taking the infimum over \((\varphi_A, \Lambda_{AB}) \in F(\mathcal{N}, \varepsilon)\) gives a bound independent of the precise details of the code. Finally, again since \(F(\mathcal{N}, \varepsilon)\) is convex and compact and \(f_0\) is continuous, the infimum is attained. \(\square\)

4 The minimax bound as a semidefinite program

In this section we describe how to formulate the minimax bound as a semidefinite program satisfying strong duality. Doing so is straightforward: We simply use the dual of the inner optimization in \((24)\) to obtain a minimization problem, or the dual of the inner optimization in \((26)\) to obtain a maximization problem. Ultimately we find the following

Proposition 2. For any channel \(\mathcal{N}_{B|A}\) and \(0 \leq \varepsilon \leq 1\),
\[
f(\mathcal{N}, \varepsilon) = \min_{\xi_A} \quad \text{Tr}[\xi_A] \quad \text{subject to} \quad (\varphi_A, \Lambda_{AB}) \in F(\mathcal{N}, \varepsilon), \varphi_A, \Lambda_{AB}, \Gamma_{AB}, \xi_A \geq 0
\]
\[= \max_{M_{B|A} \in \text{PPT}, mN_{B|A} \leq M_{B|A} + R_{AB}} \quad m(1 - \varepsilon) - n \quad \text{subject to} \quad \text{Tr}_{B} [R_{AB}] \leq nI_A, m, n, R_{AB} \geq 0. \tag{32}
\]
Proof. Let us take the former approach, dualize the inner optimization in (26), and then show that strong duality holds. Of course, strong duality must hold, as it is equivalent to the saddle point property, but we shall give a simple independent argument for strong duality based on Slater’s condition.

Observe that $f_0$ is a semidefinite program, in particular, a primal problem as we have defined it, with $X = M_{B|A}$, $K = O_{AB}$, $C = (0, 1_A)$, and $\mathcal{E}(X) = (-X^T, \text{Tr}_B[X])$. Choosing for the dual variables $Y = (\Gamma_{AB}, \xi_A)$, the dual of $f_0$ is

$$f_0(O_{AB}) := \min_{\Gamma_{AB}, \xi_A} \text{Tr}[\xi_A]$$

subject to

$$\xi_A 1_B \geq O_{AB} + \Gamma_{AB}^T,$$

$$\Gamma_{AB}, \xi_A \geq 0.$$  \hspace{1cm} (33)

Combining this with the outer optimization over $F(\mathcal{N}, \epsilon)$ gives the minimization program in (32).

The equality statement is precisely strong duality of the primal and dual forms of the inner optimization. By Slater’s condition, strong duality holds if $f_0$ is finite and there exists a strictly feasible set of dual variables. Observe that $f_0(O_{AB}) \leq |\mathcal{A}|$, since for the optimal $M_{B|A}$ we have $f_0(O_{AB}) = \text{Tr}[M_{B|A} O_{AB}] \leq \text{Tr}[M_{B|A}] \leq \text{Tr}_A 1_A = |\mathcal{A}|$. Here we have used the upper bounds $O_{AB} \leq 1_{AB}$ and $\text{Tr}_B[M_{B|A}] \leq 1_A$. Thus, the first condition is fulfilled. Meanwhile, $\Gamma_{AB} = 1_{AB}$ and $\xi_A = 3 1_A$ are a strictly feasible pair. Thus, $f_0 = f_0$ over the domain of interest.

To construct the maximization program, we simply dualize the minimization program. In particular, $f(\mathcal{N}, \epsilon)$ is a dual-form semidefinite program in the variable $Y = (\varphi_A, \Lambda_{AB}, \Gamma_{AB}, \xi_A)$ with $C = (0, 0, 0, 1_A)$, $K = (1 - \epsilon, -1, 0, 0)$, and

$$\mathcal{E}^+(Y) = (\text{Tr}[N_{B|A} \Lambda_{AB}], -\text{Tr}[\varphi_A], \varphi_A^T 1_B - \Lambda_{AB}, \xi_A 1_B - \Lambda_{AB} - \Gamma_{AB}^T).$$ \hspace{1cm} (34)

Choosing primal variables $X = (m, n, R_{AB}, M_{AB})$ leads to the maximization in (32).

Equality again follows from Slater’s condition: $f$ is finite by the minimax formulation (in particular the bound on $f_0$ used above), while a feasible choice of dual variables is given by $M_{AB} = R_{AB} = \frac{1}{2|\mathcal{A}|} 1_{AB}$, $n = 1$, and $m = \frac{1}{2|\mathcal{A}|}$. The choice of $m$ ensures the first constraint holds strictly, since any Choi operator of a trace-preserving map satisfies $||N_{B|A}||_\infty = |\mathcal{A}|$. \hspace{1cm} \square

No discussion of strong duality of semidefinite programs is complete until the complementary slackness conditions have been formulated. Often, these give considerable insight into the form and properties of the optimizing variables. First observe that

$$\mathcal{E}(X) = (-n 1_A + \text{Tr}_B[R_{AB}^T], mN_{B|A} - M_{B|A} - R_{AB}, -M_{AB}^T, \text{Tr}_B[M_{B|A}]).$$ \hspace{1cm} (35)

Then the conditions are easy to read off from the form of $C$ and $K$. They are

$$\text{Tr}[\varphi_A] = 1$$ \hspace{1cm} (36)

$$\text{Tr}[\Lambda_{AB} N_{B|A}] = 1 - \epsilon,$$ \hspace{1cm} (37)

$$\varphi_A^T R_{AB} = \Lambda_{AB} R_{AB},$$ \hspace{1cm} (38)

$$\xi_A M_{B|A} = (\Lambda_{AB} + \Gamma_{AB}^T) M_{B|A},$$ \hspace{1cm} (39)

$$n \varphi_A = \text{Tr}_B[R_{AB}^T] \varphi_A,$$ \hspace{1cm} (40)

$$M_{B|A} \Gamma_{AB} = 0$$ \hspace{1cm} (41)

$$\text{Tr}_B[M_{B|A}] \xi_A = \xi_A,$$ \hspace{1cm} (42)

$$m N_{B|A} \Lambda_{AB} = (M_{B|A} + R_{AB}) \Lambda_{AB},$$ \hspace{1cm} (43)

5 Channel symmetry

Symmetries of the channel can greatly simplify the calculation of the minimax bound. First let us state precisely what we mean by channel symmetries. Suppose $G$ is a group, possibly a topological group, represented by operators $U_g$ on $A$ and $V_g$ on $B$. A channel $\mathcal{N}_{B|A}$ is covariant with respect to $G$ when

$$V_g \mathcal{N}(\cdot)V_g^* = \mathcal{N}(U_g^* \cdot U_g) \hspace{1cm} \forall g \in G.$$ \hspace{1cm} (44)
We can write this as an invariance of the channel:

\[ \mathcal{N}(\cdot) = V_g^* \mathcal{N}(U_g \cdot U^*_g) V_g \quad \forall g \in G. \]  

(45)

In terms of the Choi operator, the condition is simply

\[ (U^T_g)_A \otimes (V_g^*)_B N_{B|A}(U^T_g)_A \otimes (V_g)_B = N_{B|A} \quad \forall g \in G. \]  

(46)

Thus, the Choi state is a fixed point when averaging over the action of the group. To enforce such averaging, introduce the superoperator \( \mathcal{G}_{AB} \):

\[ \mathcal{G}(O_{AB}) = \int d\mu(g) (U^T_g)_A \otimes (V^*_g)_B O_{AB}(U^T_g)_A \otimes (V_g)_B, \]  

(47)

where \( \mu \) is the Haar measure of the group. Observe that \( \mathcal{G}^* = \mathcal{G} \), since taking the adjoint of the group elements just reparameterizes the group, sending \( g \) to \( g^{-1} \).

Due to the structure of the symmetrization \( \mathcal{G} \), we have the following

**Proposition 3.** Suppose \((\varphi_A, \Lambda_{AB}) \in F(\mathcal{N}, \varepsilon)\). Then \((\mathcal{G}(\varphi_A), \mathcal{G}(\Lambda_{AB})) \in F(\mathcal{N}, \varepsilon)\). Similarly, \( \mathcal{G}(M_{B|A}) \in \text{PPT} \) for any \( M_{B|A} \in \text{PPT} \).

**Proof.** Start with the latter claim, and let \( \tilde{M}_{B|A} = \mathcal{G}(M_{B|A}) \). The positivity condition in (22) holds for \( \tilde{M}_{B|A} \) since \( \mathcal{G} \) is completely positive. For the trace condition, we have

\[ \text{Tr}_B[\tilde{M}_{B|A}] = \int d\mu(g) (U^T_g)_A \text{Tr}_B[(V^*)_B M_{B|A}(V_g)_B](U^T_g)_A \]  

(48)

\[ \leq \int d\mu(g) (U^T_g)_A \mathbb{1}_A (U^T_g)_A^* \]  

(49)

\[ = \mathbb{1}_A \]  

(50)

For the partial transpose condition, note that for any operator \( O_{AB} \),

\[ \tilde{O}^T_{AB} = \int d\mu(g) (U_g)_A \otimes (V^*_g)_B O_{AB}(U^T_g)_A \otimes (V_g)_B, \]  

(51)

which is the action of a slightly different, yet still-completely-positive, version of \( \mathcal{G} \) on \( O_{AB} \). Thus, \( \tilde{M}^T_{B|A} \) is positive if \( M^T_{B|A} \) is.

For the former claim, again let \( \tilde{\varphi}_A = \mathcal{G}(\varphi_A) \) and \( \tilde{\Lambda}_{AB} = \mathcal{G}(\Lambda_{AB}) \). Returning to (23), the positivity conditions \( \tilde{\varphi}_A, \tilde{\Lambda}_{AB} \geq 0 \) hold because \( \mathcal{G} \) is completely positive. The two trace conditions still hold because \( \mathcal{N}_{B|A} \) is \( G \)-covariant. It remains to show the upper bound on \( \tilde{\Lambda}_{AB} \):

\[ \tilde{\Lambda}_{AB} \leq \int d\mu(g) (U^T_g)_A \otimes (V^*_g)_B (\varphi_A^T \otimes \mathbb{1}_B)(U^T_g)_A^* \otimes (V_g)_B \]  

(52)

\[ = \int d\mu(g) (U^T_g \varphi^T U_g)_A \otimes \mathbb{1}_B \]  

(53)

\[ = \left( \int d\mu(g) U^*_g \varphi U_g \right)_A \otimes \mathbb{1}_B \]  

(54)

\[ = \varphi^T_\Lambda \mathbb{1}_B. \]  

(55)

\[ \square \]

As shown by Polyanskiy for the classical metaconverse [5], we can now show that \( G \)-covariant quantum channels have \( G \)-invariant optimizers. Letting \( \text{PPT}^G \) be the set of Choi operators of PPT-preserving channels which are invariant under \( \mathcal{G} \), and similarly \( F^G(\mathcal{N}, \varepsilon) \) the intersection of \( F(\mathcal{N}, \varepsilon) \) with \( G \)-invariant operators, we have
Theorem 2. For G-covariant channels \( \mathcal{N}_{B|A} \) we can restrict the optimizations in (24) to \( \text{PPT}^G \) and \( F^G(\mathcal{N}_{B|A}, \varepsilon) \).

Proof. The proof proceeds similarly to that of [5, Theorem 20]. To simplify notation, define

\[
g(O_{AB}) = (U_g^T)_A \otimes (V_g^T)_B O_{AB} (U_g^T)_A \otimes (V_g^T)_B.
\]

(56)

Now consider the outer optimization in (24). First note that the function \( f_0 \) from (27) is convex, as it is the pointwise maximum of linear functions. Furthermore, \( f_0 \) must be constant on orbits of \( G \). Suppose \( M_{B|A}^* \) is the optimizer for \( g(O_{AB}) \) for some arbitrary \( g \in G \), so that \( f_0(O_{AB}) = \text{Tr}[M_{B|A}^* (O_{AB})] \). It follows that \( g^{-1}(M_{B|A}^*) \) is feasible for \( f_0(O_{AB}) \), since independent unitary operations on \( A \) and \( B \) are PPT-preserving. Hence, \( f_0(O_{AB}) \geq f_0(g(O_{AB})) \). But the same argument implies \( f_0(g(O_{AB})) \geq f_0(g^{-1} \circ g(O_{AB})) = f_0(O_{AB}) \).

Applying Jensen’s inequality and taking the minimum over \( (\Phi, \Lambda_{AB}) \in F(\mathcal{N}, \varepsilon) \) gives

\[
\min_{(\Phi, \Lambda_{AB}) \in F(\mathcal{N}, \varepsilon)} f_0(\mathcal{G}(\Lambda_{AB})) \leq \min_{(\Phi, \Lambda_{AB}) \in F(\mathcal{N}, \varepsilon)} f_0(\Lambda_{AB}).
\]

(57)

By Proposition 3, we can restrict the optimization on the left-hand side to \( F^G(\mathcal{N}, \varepsilon) \) and obtain

\[
\min_{(\Phi, \Lambda_{AB}) \in F^G(\mathcal{N}, \varepsilon)} f_0(\Lambda_{AB}) \leq \min_{(\Phi, \Lambda_{AB}) \in F^G(\mathcal{N}, \varepsilon)} f_0(\Lambda_{AB}).
\]

(58)

On the other hand, since we are now minimizing over a smaller set, the left-hand side of this expression cannot be smaller than the right, so equality holds.

Next consider the inner optimization in (24), with the additional restriction to \( F^G(\mathcal{N}, \varepsilon) \) in the outer optimization. For \( f_0(\Lambda_{AB}) \) with \( G \)-invariant \( \Lambda_{AB} \), the objective function does not change upon replacing \( M_{B|A} \) by \( \mathcal{G}(M_{B|A}) \). Proposition 3 then implies that we can safely restrict the optimization to \( \text{PPT}^G \).

6 Examples

6.1 Qubit dephasing channel

Here we show that both finite blocklength converse and achievability bounds for the qubit dephasing channel can be inherited from the corresponding bounds in [3] for the classical binary symmetric channel (BSC). The dephasing channel \( \mathcal{N}_{B|A} \) has Kraus operators \( \sqrt{1-p} \mathbb{1} \) and \( \sqrt{p} \sigma_z \), and the Choi operator is \( N_{B|A} = 2(1-p)\Phi_{AB}^+ + 2p\Phi_{AB}^- \). Here \( \Phi^+ \) is the canonical maximally entangled state, i.e. \( \Phi^+ = \Phi \), and \( \Phi^- = (\mathbb{1} \otimes \sigma_z)\Phi(\mathbb{1} \otimes \sigma_z) \). Since the Bell states are orthogonal, dephasing is essentially the BSC for phase errors.

Clearly \( \mathcal{N}_{B|A} \) is covariant under the action of \( \sigma_z \); indeed, it is covariant under any unitary operator diagonal in the dephasing basis. Since \( \sigma_x \sigma_z \sigma_x = -\sigma_z \), it is covariant under the action of \( \sigma_x \) as well. For a single qubit the relevant symmetry group is \( G_1 = \{1, \sigma_x, \sigma_y, \sigma_z\} \) (up to phases, which are irrelevant since the group acts by conjugation by the Pauli operators). The corresponding \( \mathcal{G}_1 \) on the input space has the action \( \mathcal{G}_1(\rho) = \pi \) for all \( \rho \), where \( \pi \) is the maximally mixed state.

The memoryless extension \( \mathcal{N}^{(n)}_{B|A} \) inherits these symmetries in each input space, so \( G_n = G_1 \times G_1 \times \cdots \times G_1 \). Similarly, \( \mathcal{G}_n(\rho_n) = \pi_n \) has the effect of completely depolarizing the input state. That is, the optimal input state \( \varphi_{A^*} \) for the bound is maximally mixed. Thus,

\[
F^G(\mathcal{N}^{(n)}_{B|A}, \varepsilon) = \{ \Lambda_{A^nB^n} : 0 \leq \Lambda_{A^nB^n} \leq \frac{1}{M} \mathbb{1}_{A^nB^n}, \text{Tr}[\Lambda_{A^nB^n} N^{(n)}_{B|A}] \geq 1 - \varepsilon \}. \]

(59)

Furthermore, we can restrict \( \Lambda_{A^nB^n} \) to be in the support of \( N^{(n)}_{B|A} \) without loss of generality, since feasibility will not be affected and the objective function in the inner minimization of the maximin bound (26) can only decrease. Staying with the maximin bound, we may choose \( \mathcal{N}_{B^n|A^n} \) to have Choi state \( M^{\otimes n}_{B|A} \) with \( M_{B|A} = \Phi_{AB}^+ + \Phi_{AB}^- \), i.e. the fully dephasing channel. Defining \( L_{A^nB^n} = 2^n \Lambda_{A^nB^n} \), (26) then yields

\[
\frac{1}{M} \geq \min_{L_{A^nB^n}} \text{Tr}[L_{A^nB^n} \cdot \mathcal{M}_{B^n|A^n}(\Phi_{AB})^{\otimes n}] \\
\text{subject to } \text{Tr}[L_{A^nB^n} \cdot \mathcal{N}_{B^n|A^n}(\Phi_{AB})^{\otimes n}] \geq 1 - \varepsilon, \\
0 \leq L_{A^nB^n} \leq \frac{1}{M} \mathbb{1}_{A^nB^n}.
\]

(60)
Letting $\omega_{AB} = N_{B|A}(\Phi_{AB}) = (1 - p)\Phi_{AB}^+ + p\Phi_{AB}^-$ and $\sigma_{AB} = M_{B|A}(\Phi_{AB}) = \frac{1}{2}(\Phi_{AB}^+ + \Phi_{AB}^-)$, the righthand side is just the minimal type-II error of distinguishing $\omega_{AB}^{\otimes n}$ from $\sigma_{AB}$, for type-I error constrained to be no larger than $\epsilon$. That is,

$$\frac{1}{M} \geq \beta_{1-\epsilon}(\omega_{AB}^{\otimes n}, \sigma_{AB}^{\otimes n}).$$

(61)

Since both states are diagonal in the same basis, the hypothesis test between $\omega_{AB}$ and $\sigma_{AB}$ can be recast as a test between classical distributions. Observe that measuring each system of $\omega_{AB}$ and $\sigma_{AB}$ in the basis of $\sigma_x$ produces the probability distributions $P_{XY}$ and $P_XQ_Y$, respectively, with $P_X$ and $Q_Y$ uniformly distributed and $P[X = Y] = 1 - p$. Moreover, we can reconstruct the original states $\omega_{AB}$ and $\sigma_{AB}$ from $P_XY$ and $P_XQ_Y$ with the map that sends $(X,Y)$ to $\Phi_{AB}^+$ when $X = Y$ and to $\Phi_{AB}^-$ otherwise. Therefore we have

$$\frac{1}{M} \geq \beta_{1-\epsilon}(P_{XY}^{\otimes n}, P_X^{\otimes n}Q_Y^{\otimes n}).$$

(62)

This bound is precisely the expression obtained for the binary symmetric channel by Polyanskiy, Poor, and Verdú [3, Theorem 26] (see also [5, Theorem 22]). Thus, the dephasing channel inherits the finite blocklength converse of the BSC, and the bounds are identical if our choice for $M_{B|A}$ is optimal.

Regardless of the optimality of $M_{B|A}$, asymptotic results such as the strong converse and second order coding rate for the dephasing channel follow directly from the classical problem, in particular Eq. 160 and Theorem 52 in [3], respectively. Alternately, one can deduce the strong converse property more immediately by simply invoking Stein’s lemma on (61) or (62).

Finally, the achievable bounds for dephasing are also at least as good as those of the classical BSC, simply because any classical code for the BSC can be regarded as correcting phase errors and applied to the dephasing channel. More specifically, the classical code can be used as part of a CSS-like quantum code, as described in [16]. The ”error correction” part of the code (see the Remark prior to §V) is just the code for the BSC, applied to the $\sigma_x$ basis (i.e. with inputs diagonal in this basis). We can dispense with the “privacy amplification” part of the code, since it may be easily verified that the complement of the dephasing channel has a constant output on inputs diagonal in the $\sigma_x$ basis. We require a CSS-like code and not a proper CSS code because the classical BSC code need not be a linear code. Note also that, importantly, the guessing probability of the classical code is equal to the fidelity of the quantum code for this channel.

### 6.2 Erasure channel

For the qubit erasure channel we can inherit a converse bound from the metaconverse of the classical binary erasure channel (BEC). The qubit erasure channel has qubit input and output $B$ of dimension three, namely $B = A \oplus \mathbb{C}$. The extra dimension indicates to the receiver that the input was erased. The Choi state of the erasure channel with probability $p$ is simply $N_{B|A} = 2(1 - p)\Phi_{AB} + 2p\pi_A \otimes |e\rangle\langle e|_B$, where $|e\rangle$ is the additional vector in $B$. The channel is covariant with respect to action by any unitary on the input, with corresponding inverse on the output, plus dephasing of the output into the $A$ and $|e\rangle$ subspaces. Therefore, the optimal input state is the maximally mixed state. Let $\omega_{AB}^{\otimes n}$ be the output of the PPT map $N_{B|A}^{\otimes n}$.

As with the dephasing channel, consider a measurement of $A^n$ and $B^n$ in the standard basis and call the output random variables $X^n$ and $Y^n$, respectively. From $\omega_{AB}^{\otimes n}$ we obtain the distribution $P_{XY}^{\otimes n}$, with $P_X$ uniform and $Y = X$ with probability $1 - p$ and equal to $e$ with probability $p$. Note that we can recover $\omega_{AB}$ from $P_{XY}$ by employing the map which produces $\pi_A \otimes |e\rangle\langle e|_B$ when $Y = e$ and otherwise $\Phi_{AB}^+$. Now let $\sigma_{A^nB^n}$ be the state obtained by this map for the distribution $P_{X^nY^n}$ with $Q_{Y^n}$ the optimal choice for the metaconverse of the classical BEC, Eq. 168 of [5]. Due to the product form of the classical distribution, $\sigma_{A^nB^n}$ can be obtained from a PPT channel acting on the maximally entangled state. Thus, we have

$$\beta_{1-\epsilon}(\omega_{AB}^{\otimes n}, \sigma_{A^nB^n}) = \beta_{1-\epsilon}(P_{XY}^{\otimes n}, P_X^{\otimes n}Q_{Y^n}),$$

where the latter quantity appears in the converse bound for the classical BEC given in [5]. Therefore, by the minimax bound we obtain

$$\frac{1}{M} \geq \beta_{1-\epsilon}(P_{XY}^{\otimes n}, P_X^{\otimes n}Q_{Y^n}).$$

(63)
meaning the minimax bound for the BEC also applies to the qubit erasure channel with PPT assistance. Again, we may infer asymptotic statements such as the strong converse and second order coding rates from this bound.

6.3 Depolarization

The depolarizing channel has Choi state \( N_{B|A} = 2(1-p)\Phi_{AB}^+ + \frac{2p}{3}(\Phi_{AB}^+ + \Psi_{AB}^+ + \Psi_{AB}^-) \), where \( p \) is the probability of depolarization, and \( \Psi^\pm \) are obtained from \( \Phi^+ \) by conjugation with \( \sigma_x \) and \( \sigma_y \), respectively. The symmetry group of this channel includes all unitary operations, meaning that the optimal \( \psi_A \) is again the maximally mixed state. If we choose \( M_{B'|A'} = M_{B|A}' \) with \( M_{B|A} = \Phi_{AB}^+ + \frac{1}{3}(\Phi_{AB}^+ + \Psi_{AB}^+ + \Psi_{AB}^-) \), the minimax bound involves the optimal hypothesis test between \( n \) copies of \( \omega_{AB} = (1-p)\Phi_{AB}^+ + \frac{2}{3}(\Phi_{AB}^+ + \Psi_{AB}^+ + \Psi_{AB}^-) \) and \( n \) copies of \( \sigma_{AB} = \frac{1}{2}\Phi_{AB}^+ + \frac{1}{3}(\Phi_{AB}^+ + \Psi_{AB}^+ + \Psi_{AB}^-) \): As in the case of dephasing, we can convert the hypothesis test between \( \omega_{AB} \) and \( \sigma_{AB} \) into a test between classical distributions, in fact precisely those distributions which were used in the dephasing example. This follows by considering the map which generates \( \Phi^+ \) when \( X = Y \) and otherwise randomly generates one of the other Bell states when \( X \neq Y \). Therefore, we obtain the same bound, (62), for depolarization as for dephasing.

This raises the question of whether PPT assistance can turn the depolarizing channel into the dephasing channel. To investigate this further, a sensible first step would be to establish optimality of the two bounds, to ensure they are truly equivalent.

7 Discussion

We have derived a minimax bound for the size of a PPT-assisted quantum code given a target entanglement fidelity, very much along the lines of the classical bound by Polyanskiy, Poor, and Verdú [3]. The restriction to PPT-assistance comes from the use of Rains’s bound, Lemma 1, though in principle, the bound applies to all channels \( \mathcal{M}_{B|A} \) which deliver a state having overlap \( 1/M \) with the maximally entangled state. Focussing on PPT has the advantage that the PPT conditions can be phrased as linear constraints, and lead to a semidefinite program formulation of the bound.

It would be desirable to incorporate the PPT constraint into the feasible set \( F(\mathcal{N}, \epsilon) \) itself, so as to tighten the bound. Additional constraints can indeed be found along the lines of Leung and Matthews [7]. However, it appears to be impossible to incorporate such additional constraints on \( F(\mathcal{N}, \epsilon) \) and obtain a bound, like that of Theorem 1, in which we optimize the code size for fixed target fidelity. The difficulty is that the further constraints on \( F(\mathcal{N}, \epsilon) \) directly involve \( M \), the size of the code. Leung and Matthews avoid this problem by optimizing the target fidelity for fixed code size, rather than the other way around.

The analog of this issue in the classical case is that the metaconverse in [3] also applies to non-signalling assisted codes [4, 5], not just unassisted codes. There, however, no great gain is to be had by adding further constraints to the analog of \( F(\mathcal{N}, \epsilon) \), potentially tightening the bound: The \( 1/M \) bound is already quite tight at moderate blocklengths for channels of interest, as shown in [3].

One might also hope to obtain a useful bound for unassisted codes by substituting \( 1_A \otimes \sigma_B \) for the output of the PPT channel \( \mathcal{M}_{B|A} \). Despite its normalization, this operator also has the correct \( 1/M \) overlap with the maximally entangled state. Moreover, it would lead to a hypothesis-testing quantity reminiscent of the coherent information, whose regularized version is part of the formula for quantum channel capacity.

However, the resulting optimization is not a semidefinite program, as now we will have to set \( M_{B|A} = \varphi_A^{-1} \otimes \sigma_B \), leading to \( \varphi_A^{-1} \) appearing in the objective function along with \( \Lambda_{AB} \). Concretely, we obtain

\[
\frac{1}{M} \geq \min_{\varphi_A, \Lambda_{AB} \in F(\mathcal{N}, \epsilon)} \max_{\sigma_B} \text{Tr}[\Lambda_{AB} \varphi_A^{-1} \otimes \sigma_B] \\
= \min_{(\varphi_A, \Gamma_{AB}) \in F(\mathcal{N}, \epsilon)} \max_{\sigma_B} \text{Tr}[\Gamma_{AB} \varphi_A^{1/2} \Lambda_{AB} \varphi_A^{1/2} \otimes \sigma_B],
\]

where \( F(\mathcal{N}, \epsilon) \) consists of normalized states \( \varphi_A \) and \( \Gamma_{AB} \) with \( 0 \leq \Gamma_{AB} \leq 1_{AB} \) such that \( \text{Tr}[\Gamma_{AB} \varphi_A^{1/2} N_{B|A} \varphi_A^{1/2}] \geq 1 - \epsilon \). Furthermore, the symmetry arguments employed in Proposition 3 no longer go through, making
the resulting bound difficult to work with. This difficulty is perhaps to be expected, since otherwise sym-
metrization might lead to a single-letter bound in terms of the coherent information for general channels,
which is known to be false.

The same difficulty applies to formulating an SDP-based bound using any PPT state, not just the output
of a PPT channel $\mathcal{M}_{\beta|\alpha}$ on the input $\varphi_A$ as done in Theorem 1. This is the approach taken by Tomamichel
and Berta to obtain second-order coding rates for simple channels in [17]. In the notation of this paper,
their bound can be expressed as

$$\min_{(\varphi_A, \Gamma_{AB}) \in F(\mathcal{M}_{\beta|\alpha}, \epsilon)} \max_{\sigma_{AB} \in \text{PPT}} \text{Tr}[\Gamma_{AB}\sigma_{AB}] \leq \frac{1}{M},$$  \hspace{1cm} (66)

with the same $F'$ as in the previous paragraph. (Actually, we have interchanged a minimization over $\Gamma$ with
the maximization over $\sigma$, but this is permissible by the minimax theorem.) Nevertheless, symmetry
arguments do go through in this case, and can be employed to infer that optimal input states $\varphi_A$ can
be chosen to be invariant under the channel symmetry group. For highly symmetric channels such qubit
dehasing, depolarization, and erasure, this fixes $\varphi_A$ to the mixed state. Then the nonlinearities of the
optimization disappear and the resulting bound is identical to the minimax bound (24). Indeed, for these
channels, one could use the results of §6 to more quickly obtain their results regarding second-order
coding rates. The form of the bound also implies that the minimax bound can be obtained by loosening
(66), restricting the optimization of $\sigma_{AB}$ to states of the form $\sigma_{AB} = \varphi_{A}^{1/2} \mathcal{M}_{\beta|\alpha} \varphi_{A}^{1/2}$ for some PPT channel
$\mathcal{M}_{\beta|\alpha}$. Equivalently, $\sigma_{AB}$ must be PPT and satisfy $\text{Tr}_B[\sigma_{AB}] = \varphi_A$.

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