Time evolution of observable properties of reparametrization-invariant systems

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Abstract

A short review of some recent work on the problem of time and of observables for the reparametrization-invariant systems is given. A talk presented at the Second Meeting on Constrained Dynamics and Quantum Gravity at Santa Marguerita Ligure, September 17–21 1996.
1 Introduction

In the quest for quantum gravity, a number of problems have been recognized. This paper concerns mainly the problem of time and the problem of observables within the canonical approach.

The problem of time has many aspects; we shall focus on the time evolution. In the quantum theory, the evolution is defined by the Schrödinger equation (or Heisenberg equations) and the basic structure is the Hamiltonian operator. The classical counterpart thereof is a Hamiltonian dynamical system and there will be a Hamiltonian function. In the classical theory of gravity—the general relativity—we do not know what could play the role of Hamiltonian: the general relativity is not a Hamiltonian dynamical, but rather a reparametrization invariant system (RIS). In order to capture the structure which underlies the Hamiltonian in general, we give a definition of RIS such that all Hamiltonian dynamical systems are included. Any Hamiltonian dynamical system can be rewritten in the form of RIS by the well-known process of parametrization. It turns out that the Hamiltonian dynamics is equivalent to a kind of reference system in the phase space, the so-called auxiliary rest frame (ARF). Each parametrization defines a unique ARF in the resulting RIS and each choice of ARF in a RIS enables one to reformulate the RIS uniquely as a Hamiltonian dynamical system.

The notion of ARF allows broad generalization such that RIS’s with a generalized ARF can still be brought to the form of a (generalized) Hamiltonian dynamical system. Many apparently different approaches to the quantum gravity can then be recognized as just different choices of ARF. The time evolution between two non-global transversal surfaces can also be incorporated, see (Hawking effect), and .

However, the general relativity does not seem to possess any natural (generalized) ARF. Thus, the question could arise: are there any alternatives to the Schrödinger equation form of dynamics in the quantum theory? The answer is yes; an example of such a quantum theory is the quantum field theory (QFT) on fixed curved background spacetime, if this background does not admit any timelike Killing vector. We analyze this model and describe the two strategies which were invented long ago to cope with the problem: the scattering and the algebraic methods. The dynamical equation that replaces the Schrödinger equation in this case is the quantum field equation.

The problem of observables is closely related to that of time evolution. An observable in a Hamiltonian dynamical system represents a whole set of measurements, each performable at a particular time; these measurements are considered to be “the same” (for example, the same apparatus measures the position at different times). It turns out that a precise definition of such a “standard” observable requires
the same reference frame as the definition of time evolution. Thus, only if an ARF is chosen, standard observables are well-defined.

In general, the observable properties of RIS’s can be represented by reparametrization invariant functions, the so-called perennials. Originally, perennials were called “first-class quantities” or “observables” by Bergmann and Dirac [9], [10]. Bergmann recognized the two major problems:

1. Are there enough perennials available? On one hand, even if they exist, they are difficult to find in practice. On the other, the very existence of perennials has been questioned recently (cf. [11] or [12]).

2. Reparametrizations can be identified with diffeomorphisms that act transitively along dynamical trajectories of RIS’s. This implies that perennials must be functions that are constant along classical solutions (this is a justification for the name, which has been introduced by Kuchař [11] to distinguish them from standard observables). In a theory, in which the perennials are used to describe observable properties, no time evolution seems possible (the “frozen dynamics” of Bergmann).

A new impetus for research in this area came from Ashtekar’s [13] work on canonical quantization. The papers [14], [15], [16] by Rovelli and [11] by Kuchař contain crucial ideas. These ideas have been worked out in a rigorous and systematic way in [17], [4], [8], [18], [7] and [5]. Bergmann’s problems have been solved or circumvented; in particular, the existence of a sufficient number of perennials has been shown [3] for physically reasonable RIS’s.

It turns out, then that a standard observable can be identified with a certain class of perennials. The class can be defined by one perennial and the ARF. The time evolution of the corresponding measurable property is determined by the Schrödinger or Heisenberg equations. In the case that there is no ARF, one can still describe the observable properties by a special kind of perennials. They have been introduced by Rovelli, and we call them “labelled perennials.”

The plan of the paper is as follows. In Sec. 2, we give a definition of RIS that is sufficiently general for our purposes. The rest of the section is devoted to the first problem of Bergmann. We restrict ourselves to the RIS’s that possess a manifold-like space of physical degrees of freedom and show that they admit a complete system of perennials. Then, we explain the general structure and physical meaning of perennials. In Sec. 3, we describe the relation between the Hamiltonian dynamics and the ARF. The physical meaning of the ARF is analyzed; the definition of the standard observable follows immediately from this analysis. The (classical) dynamical equations of Schrödinger and Heisenberg are then derived from the ARF. The
second problem of Bergmann disappears. In Sec. 4, we describe examples of generalized ARF’s. We show how some apparently qualitatively different approaches to quantum gravity result just by different choices of ARF. In Sec. 5, we study the parametrized QFT on fixed classical background spacetime as an example of RIS that, on one hand, is relatively well-understood and that, on the other, is afflicted with problems associated with a choice of ARF. We review the well-known scattering and algebraic approaches to the dynamics of this model. After everything is translated into the language of RIS and perennials [18], [7], these approaches can be seen as ARF-free methods of describing dynamics; they are, in fact, using a particular kind of labelled perennials.

2 Existence and nature of perennials

The existence is mainly a mathematical problem, so we have to be relatively precise. Let \((\tilde{\Gamma}, \tilde{\Omega})\) be a symplectic manifold; it will play the role of the extended phase space. Let \(\Gamma\) be a submanifold of \(\tilde{\Gamma}\). Let us define, for any \(x \in \Gamma\), \(\tilde{L}_x := \{X \in T_x \tilde{\Gamma} | \tilde{\Omega}(X, Y) = 0 \ \forall Y \in T_x \Gamma\}\) and the space \(L_x\) of longitudinal vectors at \(x\) by \(L_x := \tilde{L}_x \cap T_x \Gamma\). Suppose that \(L_x \neq \{0\}\) and that \(L_x\) for all \(x \in \Gamma\) defines a subbundle \(L\Gamma\) of the tangent bundle \(T\Gamma\). Then, the triplet \((\tilde{\Gamma}, \tilde{\Omega}, \Gamma)\) is called reparametrization-invariant (constraint) system (RIS) and \(\Gamma\) is called the constraint surface; the points of \(\Gamma\) are interpreted as physical states. \(L_x\) is also the subspace of degeneracy of the pull-back \(\Omega\) of \(\tilde{\Omega}\) to \(\Gamma\). \(L\Gamma\) is an integrable subbundle, because \(\Omega\) is closed. Thus, there are integral submanifolds of \(L\Gamma\) through any point \(x\) of \(\Gamma\). Let us call maximal integral submanifolds of \(L\Gamma\) c-orbits (orbits of first-class constraints). For any RIS, there is one-to-one correspondence between c-orbits and maximal dynamical trajectories [11]. That means that each classical solution is just a c-orbit rather that a set of c-orbits like, e.g., in gauge theories. The motivation for choosing the name “reparametrization-invariant” is that the dynamical trajectories of the system are submanifolds (sets) rather than maps (like curves) and the coordinates along such submanifolds (parametrization) is arbitrary.

Non-geometrical formulations of RIS’s usually focus on the Hamiltonian action \(S = \sum p\dot{q} - \mathcal{H}\), where the second class constraints have been removed and \(\mathcal{H}\) is a linear combination of the first class ones (see [11]). Let us call \(\mathcal{H}\) extended Hamiltonian to distinguish it from the genuine (Schrödinger or Heisenberg) Hamiltonian (see Sec. 3), which will be called simply Hamiltonian.

The reparametrization-invariant systems form a very general class: they include all (constrained) Hamiltonian dynamical systems, because such systems can be transformed to equivalent RIS’s by parametrization (see the example at the end of this section). This holds also for the asymptotically flat sector of general relativ-
ity, which can be parametrized at infinity (for an example see [19]). A particular
class of RIS’s is formed by the so-called first-class systems; they are defined by
\( L_x = \tilde{L}_x \) for all \( x \in \Gamma \); that means that \( \Omega \) is maximally degenerate.

Consider the space \( \bar{\Gamma} \) defined by \( \bar{\Gamma} := \Gamma/\gamma \), where \( \gamma \) denotes the c-orbits. With
the quotient topology, \( \bar{\Gamma} \) is a topological space; it is interpreted as the space of
physical degrees of freedom (the number of physical degrees of freedom is half of the
dimension of \( \bar{\Gamma} \)), or the space of classical solutions, or the physical phase space. An
important notion in the theory of RIS’s is that of transversal surface (a section of
\( \Gamma/\gamma \)). It is any submanifold of \( \Gamma \) which has no common tangent vectors with any
c-orbit (except for zero vectors) and which intersects each c-orbit in at most one
point. A global transversal surface must intersect every c-orbit.

Our formal mathematical definition implies little restriction on the topology of
\( \bar{\Gamma} \). Consider the following example (an ergodic system, cf. [20]). Let \((M, g)\) be
a compact Riemannian (positive-definite metric) manifold with negative constant
curvature. Let \( \tilde{\Gamma} \) be \( T^*M \) with coordinates \( x^\mu, p_\mu \) and \( \tilde{\Omega} \) be the standard symplectic
form of cotangent bundles. Let \( \Gamma \) be given by the equation \( g^{\mu\nu} p_\mu p_\nu = 1 \). The
dynamical trajectories of this RIS are geodesic arcs of \((M, g)\). One can show that \( \bar{\Gamma} \)
has the coarsest possible topology: just the empty set and \( \bar{\Gamma} \) itself are open; there
is no transversal surface through any point of \( \Gamma \).

However, for a RIS to be physically sensible, its physical degrees of freedom must
form, at least locally, a manifold. For general relativity, the structure of this space
has been studied for some time. Large portions of it have been proved to possess
manifold structure; points, at which the manifold structure breaks down have also
been found, but they form a small subset (at least as yet, a recent paper is [21]).
That motivates the following definition [5]:

**Definition 1** A RIS \((\tilde{\Gamma}, \tilde{\Omega}, \Gamma)\) is called locally reducible, if an open dense subset of
\( \bar{\Gamma} \) is a quotient manifold.

For the sake of simplicity, we assume that \( \bar{\Gamma} \) is a quotient manifold; the general
case is dealt with in [19]. Then, the natural projection \( \bar{\pi} : \Gamma \mapsto \bar{\Gamma} \) is a submersion
and it determines a two-form \( \bar{\Omega} \) by \( \bar{\pi}^* \bar{\Omega} = \Omega \). The symplectic manifold \((\bar{\Gamma}, \bar{\Omega})\) is the
physical phase space. \( \bar{\pi}^* \) also gives a one-to-one relation between functions on \( \bar{\Gamma} \) and
functions on \( \Gamma \) that are constant along all c-orbits. We define: a function \( o : \bar{\Gamma} \mapsto \mathbb{R} \)
that is constant along all c-orbits is a perennial. We call a set of perennials complete,
if it separates points in an open dense subset of \( \bar{\Gamma} \) (cf. [3]).

Now, the example above does not admit any perennial. There are some results
on an analogous problem from the theory of Hamiltonian systems suggesting that
the existence of perennials for a generic RIS is very unlikely: a theorem by Robinson
[22]. It states, roughly, that a generic Hamiltonian system possesses no integral (of
motion) that is independent from the Hamiltonian. However, we have the following theorem (for finite-dimensional systems)

**Theorem 1** Let a RIS \((\tilde{\Gamma}, \tilde{\Omega}, \Gamma)\) be locally reducible. Then, \((\tilde{\Gamma}, \tilde{\Omega}, \Gamma)\) admits a complete system of perennials.

A proof using Whitney embedding theorem was given in [5].

Two questions still arise.

1. It might be desirable that physically interesting perennials satisfy some further conditions. For example, in field theories, one is tempted to consider only local functions. Are there perennials that are local? The answer to this question, at least for general relativity, is negative [12].

2. Perennials, even if they exist, are likely to be complicated functions on \(\Gamma\), difficult to construct explicitly and to work with. This may be more than just a technical problem, if we are going to quantize the system.

Let us address these two problems by studying a simple example (a non-relativistic conservative system). Let the physical phase space be \(\mathbb{R}^{2n}\) with canonical coordinates \(q^k\) and \(p_k\), \(k = 1, \ldots, n\), and let the Hamiltonian be some function \(H(q, p)\) on the phase space. We assume that the model describes some physically interesting system so that both \(q^k\) and \(p_k\) are directly measurable (coordinate and momentum). The dynamical equations are

\[
\dot{q}^k = \{q^k, H\}, \quad \dot{p}_k = \{p_k, H\}.
\]

We construct an equivalent RIS by the so-called parametrization. This can be done in many ways; we choose \(\tilde{\Gamma} = \mathbb{R}^{2n+2}\) with canonical coordinates \(T, P, q^k\) and \(p_k\), \(k = 1, \ldots, n\), so that \(\tilde{\Omega} = dP \land dT + dp_k \land dq^k\). Then \(\Gamma\) is defined by the equation \(P + H(q, p) = 0\) and \(\tilde{L}_x = L_x\) is one-dimensional, spanned by the vector

\[
\left(1, 0, \frac{\partial H}{\partial p_k}, -\frac{\partial H}{\partial q^k}\right).
\]

This form of \(L_x\) implies that 1) the orbits parametrized by \(T\) yield solutions to Eqs. (1), and 2) each \(\Gamma_t\) defined by \(T = t\) is a global transversal surface.

Using global transversal surfaces, one can define a complete set of perennials by their “initial data”: let the perennials \(Q_t^k\) and \(P_{tk}\) satisfy \(Q_t^k|_{\Gamma_t} = q^k|_{\Gamma_t}\) and \(P_{tk}|_{\Gamma_t} = p_k|_{\Gamma_t}\). Then the functions \(Q^k_t(T, P, q, p)\) and \(P_{tk}(T, P, q, p)\) are well-defined on \(\Gamma\), because they must be constant along the c-orbits. This particular kind of perennials were well-known to Bergmann, who called them “canonical constants of motion of the Hamilton-Jacobi theory” [9]. As functions of \(t\), they are special cases of Rovelli’s “evolving constants of motion” [16].
Let us list some properties of these perennials. First, Kuchař has given an important argument \([11]\) that these perennials are not directly observable (that is, they are in principle not measurable in the quantum mechanics) at a time different from \(T = t\). Second, at \(T = t\), the values of these perennials coincide with values of observable quantities, and so they are measurable. Third, the values of these perennials are given by complicated (non-polynomial, in a non-linear field theory surely non-local) functions on the phase space outside of \(\Gamma_t\), but their restrictions to \(\Gamma_t\) are simple (polynomial, local) functions. Observe that these perennials carry a label that specifies the transversal surface, where they have a direct physical meaning and are given by simple functions. Thus, for a perennial to be physically reasonable, it seems to be sufficient to require that 1) it carries a label specifying some position, 2) the values of the perennial are measurable at the position and 3) the functional form of the perennial at the position is simple. This may answer the first question \([5]\).

The second question can be dealt with as follows (for detail see \([4]\) and \([3]\)). The basic knowledge necessary for any calculation with perennials is the Poisson algebra of the perennials (all three operations of linear combination, functional multiplication and Poisson brackets). This algebra (e.g. the Poisson brackets) can be calculated, if the functional form of all perennials in at least an open neighbourhood in \(\tilde{\Gamma}\) is known. By initial data, the perennials are not even defined in such a neighbourhood (only at \(\Gamma\)), and their form is not explicitly known outside of \(\Gamma_t\). However, the Poisson algebra of perennials has been shown in \([4]\) to coincide with the Poisson algebra of their restrictions to a transversal surface. Thus, the values of the perennials elsewhere than at \(\Gamma_t\) are not needed even for calculations.

3 Hamiltonian dynamics

The standard kind of time evolution of dynamical systems is based on some additional assumptions about the structure of the corresponding RIS. We are going to reveal these assumptions and describe them in symplectic-geometrical terms using the example of non-relativistic conservative system (see Sec. \([4]\)). For this system, we can of course define the surfaces \(\Gamma_t\) of constant time unambiguously: \(\Gamma_t := \{x \in \Gamma | T(x) = t\}\); they are global transversal surfaces. On each \(\Gamma_t\), we can choose the coordinates \(x^k_t\) and \(y^k_t\) defined by \(x^k_t := q^k|\Gamma_t\) and \(y^k_t := p^k|\Gamma_t\). The pull-back \(\omega_t\) of \(\Omega\) to \(\Gamma_t\) is, then, \(\omega_t = dy^k_t \wedge dx^k_t\) and \((\Gamma_t, \omega_t)\) is a symplectic manifold. Observe that \((\Gamma_t, \omega_t)\) can be identified with \((\tilde{\Gamma}, \tilde{\Omega})\). We call these surfaces time levels. This is the interpretation of transversal surfaces in general \([3]\). Indeed, for example the following property of general relativity can easily be shown. Let \(\Gamma_1\) be an arbitrary transversal surface and let \(\tilde{\Gamma}_1\) be the set of c-orbits that intersect \(\Gamma_1\).
Let $\gamma \in \bar{\Gamma}_1$ be associated with the spacetime $(M, g)$ that does not admit any isometry (an open dense subset of $\bar{\Gamma}_1$). Then the intersection point $\gamma \cap \Gamma_1$ determines a unique Cauchy surface in $(M, g)$. Thus, roughly speaking, each transversal surface defines a particular instant of time in each solution to Einstein’s equations.

The coordinates $x^k_t$ and $y^k_t$ represent results of measurements: $x^k_t$ that of coordinate and $y^k_t$ that of momentum. Given the value of all these coordinates, a state of the system is determined; two states at different times $T = t$ and $T = s$ are considered to be the same, if the corresponding coordinates coincide: $x^k_t = x^k_s$ and $y^k_t = y^k_s$. This is because the coordinate and momentum measurements are defined for all times; each of them specifies a class of the “same measurements at different times”. This leads to a further structure in the phase space: the system of maps $\theta_{ts} : \Gamma_s \mapsto \Gamma_t$ such that the values of these coordinates are preserved:

$$x^k_t(\theta_{ts}(x_s, y_s)) = x^k_s, \quad y^k_t(\theta_{ts}(x_s, y_s)) = y^k_s. \quad (2)$$

We call the maps $\theta_{ts}$ time shifts. Time shifts define an equivalence relation of states: if $u \in \Gamma_t$, $v \in \Gamma_s$ and $v = \theta_{st}(u)$, then $u$ and $v$ are equivalent—they are two states with the same measurable properties. Thus, the maps $\theta_{st}$ satisfy the axioms:

1. $\theta_{ts}$ is a symplectic diffeomorphism of $\Gamma_s$ to $\Gamma_t$ for all $t$ and $s$,
2. $\theta_{ts} \circ \theta_{sr} = \theta_{tr}$ for all $t$ and $s$ and $r$,
3. $\theta_{tt} = \text{id}$ for all $t$.

We call the set $(\Gamma_t, \theta_{ts})$ an auxiliary rest frame (ARF) (in general, the time index $t$ runs through a more general index set $I$).

On one hand, the above construction can be generalized to show that a RIS that is obtained by parametrizing a Hamiltonian dynamical system possesses a unique ARF’s. On the other, if a RIS possesses an ARF, then we can reduce it to a Hamiltonian dynamical system. Indeed, the motion of the system can then be defined as the motion with respect of the ARF. Let us consider a particular c-orbit $\gamma$. Define $\eta_{\gamma}(t)$ to be the point of intersection of $\gamma$ with $\Gamma_t$; for each $t$, and for each $\gamma$, there is an $\eta_{\gamma}(t)$ and it is unique. Project $\eta_{\gamma}(t)$ to a fixed but arbitrary time level $\Gamma_0$ by $\xi_{\gamma}(t) := \theta_{0t}(\eta_{\gamma}(t))$. We call the curve $\xi_{\gamma} : \mathbb{R} \mapsto \Gamma_0$ trajectory of the system in the physical phase space $(\Gamma_0, \omega_0)$. The functions (we drop the index 0)

$$x^k(t) := x^k(\xi_{\gamma}(t)), \quad y^k(t) := y^k(\xi_{\gamma}(t)),$$

satisfy the equations

$$\dot{x}^k = \{x^k, H(x, y)\}, \quad (3)$$

$$\dot{y}^k = \{y^k, H(x, y)\}. \quad (4)$$

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where \( H(x, y) \) is independent of \( \gamma \) and can be constructed directly from the set of time shifts. In our case, it coincides with the original Hamiltonian of the system; \( \{\cdot, \cdot\} \) is the Poisson bracket of the symplectic space \((\Gamma_0, \omega_0)\). The system (3)–(4) is called Schrödinger dynamical equation. The coordinates \( x_k \) and \( y_k \) are called Schrödinger observables associated with the ARF \((\Gamma_t, \theta_{ts})\). This is a very important point: Schrödinger observables are always associated with some ARF [5].

One can also describe Heisenberg evolution. Let \( o_t \) be a perennial that represents a measurement at the time (level) \( \Gamma_t \). Then the perennial \( o_s \) defined by \( o_s|_{\Gamma_s} = o_t|_{\Gamma_t} \circ \theta_{ts} \) represents the same measurement at the time \( \Gamma_s \). The set \( \{o_t|t \in \mathbf{R}\} \) is called Heisenberg observable associated with the ARF \((\Gamma_t, \theta_{ts})\). We stress again: there are no more general Heisenberg observables; to construct or define a Heisenberg observable, an ARF is necessary [5]. Let us project a Heisenberg observable \( \{o_t|t \in \mathbf{R}\} \) to a particular time level \((\Gamma_0, \omega_0)\) by \( \bar{o}_t := o_t|_{\Gamma_t} \). The projection \( \bar{o}_t(x, y) \) is a function of \( 2n + 1 \) variables \( x_k, y_k \) and \( t \). Its \( t \)-dependence satisfies the equation (5)

\[
\frac{\partial}{\partial t} \bar{o}_t(x, y) + \{\bar{o}_t, H\} = 0.
\]

This is the Heisenberg dynamical equation.

The construction of the Hamiltonian from an ARF \((\Gamma_t, \theta_{ts})\) starts by the observation that the c-orbits define another map \( \rho_{st} : \Gamma_t \mapsto \Gamma_s \) by \( \{\rho_{st}(p)\} := \gamma_p \cap \Gamma_s \), where \( \gamma_p \) is the c-orbit through \( p \in \Gamma_t \). \( \rho_{st} \) is a symplectic diffeomorphism for all \( t \) and \( s \) [4]. Then the map \( \chi_t : \Gamma_0 \mapsto \Gamma_0 \) defined by \( \chi_t := \theta_{0t} \circ \rho_{0t} \) is also a symplectic diffeomorphism for each \( t \), and its derivative \( d_t \chi_t \) defines a locally Hamiltonian vector field on \( \Gamma_0 \). The corresponding Hamiltonian function \( H_t^S \) is called Schrödinger Hamiltonian associated with the ARF \((\Gamma_t, \theta_{ts})\). \( H_t^S \) is defined up to a constant, and only locally in general. In general, it also depends on \( t \) in a non-trivial way (time-dependent Hamiltonian). In this case, the Heisenberg Hamiltonian \( H_t^H \) (which features in Eq. (5)) is given by \( H_t^H = H_t^S \circ \chi_t \) and \( H_t^H \neq H_t^S \). The detail has been given in [3].

Some perennials can be exceptional in that they are associated with measurements at more than one time level. Let \( o \) be associated with some measurement at \( \Gamma_t \) and suppose that there is \( s \neq t \) such that \( o|_{\Gamma_s} = o|_{\Gamma_t} \circ \theta_{ts} \). Then \( o \) is also associated with the same measurement at \( \Gamma_s \). A perennial that is associated with one and the same measurement at all time levels is called integral of motion. For example, the Hamiltonian of a parametrized Hamiltonian system together with all functions which have vanishing Poisson brackets with the Hamiltonian are integrals of motion. We can understand why integrals of motion are exceptional. Suppose that there is a complete system of integrals of motion. Then it must hold \( \theta_{st}(\gamma \cap \Gamma_t) = \gamma \cap \Gamma_s \) for all \( \gamma, s \) and \( t \). Such an ARF is trivial in that the time shifts do not move the c-orbits; the dynamics is frozen.

We conclude that the perennials can represent measurements and, unlike the
standard observables, they do not need any additional structure like ARF in order to be well-defined.

4 Examples of ARF’s

4.1 Reduction of 2+1 gravity

This will be very brief, for detail see [23]. Let us denote the three-dimensional spacetime of the 2+1 gravity model by \((M, g)\) and the two-dimensional Cauchy manifold by \(\Sigma\). Then the points of \(\tilde{\Gamma}\) are described by pairs of fields \(q_{kl}(x)\) (the two-metric) and \(\pi^{kl}(x)\) (the two-dimensional ADM momentum) on \(\Sigma\). \(\Gamma\) is determined by the three-dimensional analogon of ADM constraints.

As time levels, Moncrief has chosen the surfaces of constant mean curvature given by \(\Gamma_{\tau} := \{q, \pi \in \tilde{\Gamma} | (2\sqrt{q})^{-1} q^{kl} \pi_{kl} = \tau\}\). “The same states” are defined as those which have the same Teichmüller parameters \(q^a\) and the same values of the conjugate momenta \(p_a\). The knowledge of these \(12h - 12\) values (\(h > 1\) is the genus of \(\Sigma\)) determines uniquely a point at \(\Gamma_{\tau}\) (because the constraints are satisfied).

The Hamiltonian associated with the corresponding ARF coincides with Moncrief’s Hamiltonian. The time is one-dimensional in this case: \(\tau\) runs over some interval.

4.2 Three forms of relativistic mechanics

Dirac [24] considered a system of massive particles in Minkowski spacetime \(M\). Let us restrict ourselves just to one particle of mass \(m\). \(\tilde{\Gamma} = T^* M\) with coordinates \(x^\mu, p_\mu\) and \(\tilde{\Omega} = dp_\mu \wedge dx^\mu\). The constraint is \(p^2 = -m^2\).

Let \(G\) be the Poincaré group. The action of \(G\) on \(\tilde{\Gamma}\) is defined by its 10 generator functions (via Poisson brackets) \(p_\mu\) (momentum), \(J_k\) (angular momentum), and \(K_k\) (boost momentum). \((p_\mu, J_k, K_k)\) forms a basis of a complete Lie algebra of perennials. Next, Dirac chose three non-timelike surfaces in \(M\) with the maximal symmetry with respect to \(G\) (“three forms”): let us take just one, the spacelike plane \(x^0 = 0\). The associated transversal surface is \(\Gamma_0 := \{(x^\mu, p_\mu) \in \tilde{\Gamma} | x^0 = 0, p^2 = -m^2\}; define \(\Gamma_g := g\Gamma_0\) for all \(g \in G\). Thus, the time is ten-dimensional. Let \(\theta_{gh} := gh^{-1}|_{\Gamma_0}\).

The Hamiltonian associated with this ARF is \(H_X(x, p) = (X^\mu p_\mu + Y^k J_k + Z^k K_k)|_{\Gamma_0}\), where \(X = (X^\mu, Y^k, Z^k)\) is an element of the Lie algebra of \(G\). For more detail, see [4] and [17].

This is an elegant construction of dynamics, where the complete system of perennials generates a group and contains all Hamiltonians. Observe that the maps \(\theta_{gh}\) can be extended to symplectic diffeomorphisms on the whole of \(\tilde{\Gamma}\). The method is applicable to less trivial systems than the relativistic particle. An example seems to be the torus sector of the 2+1 gravity, for which one might choose the group.
ISO(2,1) to play the role of the Poincaré group ([25]). Its generators that form a complete Lie algebra of perennials have been found by Moncrief [26].

Being confronted with a whole family of Hamiltonians, one can wonder if the condition of positivity applies to them, and if not, which form will this condition take. Indeed, there is no reason for boosts, space shifts or rotations to be positive, and they are not. However, in constructing the quantum theory by the so-called group quantization method (see [27]), one looks for some suitable representation of the group generated by the Lie algebra of perennials (Poincaré group) [4], and one can require that in this representation the spectrum of \( p_\mu \) lies in the future light cone. Thus, the positivity of Hamiltonian becomes a condition on the space of states.

### 4.3 Functional Schrödinger equation

In several papers [28], [19] and [29], Kuchař and his co-workers have studied models of gravity with (effectively) two-dimensional spacetime \( M \) and one-dimensional Cauchy manifold \( \Sigma \). In all cases, they managed to find canonical coordinates \( X^\mu(x), P_\mu(x), q^k(x), \pi_k(x) \), \( x \in \Sigma \), in \( \tilde{\Gamma} \) such that \( X^\mu(x) \) are spacelike embeddings, \( X: \Sigma \mapsto M \). The constraints can then be rewritten in the form \( P_\mu(x) + H_\mu[X, q, \pi; x] = 0 \). If we choose the time levels \( \Gamma_X \) by fixing the embedding \( X \), then \( q^k(x) \) and \( \pi_k(x) \) are canonical coordinates on \( \Gamma_X \), and we can define \( \theta_{X':X} \) by the requirement that the values of these coordinates be preserved similarly to Eq. (2). The associated Hamiltonian can then be shown to coincide with Kuchař Hamiltonian \( H_\mu([X, q, \pi; x]) \) ([4]).

Observe that the time is now \( \infty \)-dimensional, as \( X \) is an arbitrary spacelike embedding. Still, the set of all resulting \( \Gamma_X \) comprises only a very small subset of the set of all transversal surfaces (this has to do with the multiple choice problem [3]).

### 5 ARF-free dynamics

There are RIS’s that do not admit any ARF or that seem to have no natural, unique, physically distinguished ARF (like general relativity). One can be tempted to surrender the tiring search for an ARF. Then one should attempt to formulate the dynamics without the help of ARF’s. Such attempts may be classified as “timeless interpretations of quantum gravity” according to [1]. Ours is based on the fact that not only observable properties, but also the dynamics can be represented by perennials. Indeed, a complete system of perennials determines all c-orbits completely. Thus, it must be possible to extract, directly from the definition of perennials, some small-step (local) dynamical principle. For QFT on a background spacetime, this
turns out to be the field equation. Let us explain how it works.

Consider a scalar field $\phi$ of mass $m$ on a fixed globally hyperbolic spacetime $(M, g)$ with a Cauchy manifold $\Sigma$; the field equation reads

$$\frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu\nu} \partial_\nu \phi) - m^2 \phi = 0.$$  \hspace{1cm} (6)

Such a system can be reformulated as a RIF [31]: the points of the extended phase space $\tilde{\Gamma}$ are quadruples of fields, $X^\mu(x), P_\mu(x), \varphi(x)$ and $\pi(x)$, on $\Sigma$; $X: \Sigma \mapsto M$ is a spacelike embedding, $P$ is the conjugate momentum, $\varphi := \phi|_{X(\Sigma)}$ and $\pi := (\sqrt{\gamma} n^\mu \partial_\mu \phi)|_{X(\Sigma)}$, where $\gamma_{kl}$ is the metric induced on $X(\Sigma)$ by $g_{\mu\nu}$ and $n^\mu$ is the unite normal vector to $X(\Sigma)$ in $M$. The constaint surface $\Gamma$ is determined by an equation of the form (see [31])

$$P_\mu(x) + H_\mu[X, \varphi, \pi; x] = 0.$$  \hspace{1cm} (7)

Thus, points of $\Gamma$ can be labelled by the triples $(X, \varphi, \pi)$.

A fixed embedding $Y$ defines a transversal surface $\Gamma_Y$ by $\Gamma_Y := \{(X, \varphi, \pi) \in \Gamma | X = Y\}$; $\varphi, \pi$ are coordinates on the physical space $\Gamma_Y$ and there is an ARF ($\Gamma_X, \theta_{XY}$) analogous to that in Sec. [43]. One can try to quantize the system by the functional Schrödinger equation method; the main obstacle are anomalies. A thorough study of a QFT model on a 2-dimensional spacetime in [32] shows how the anomalies can be neutralized; the method has not yet been extended to higher dimensional spacetimes, however. Anomalies may set limits to enlargements of ARF’s in general.

QFT on a general background (in the non-parametrized version) has been intensively studied in the seventies and eighties, and it seems to be well-understood today (for a review, see [33]). In particular, if the background metric does not admit any timelike Killing vector, then the quantum field algebra has no unique physical (Hilbert space) representation and there is no unique Hamiltonian. Thus, it may be futile to look for some Hilbert space of states and for a Schrödinger-like evolution in this Hilbert space. Two methods of how this problem can be met have been worked out: the scattering and the algebraic methods [33]. These methods find a very natural reformulation in terms of perennials [7].

5.1 Scattering method

Suppose that there are two Cauchy surfaces, $X_1(\Sigma)$ and $X_2(\Sigma)$ in $M$ such that $X_2(\Sigma)$ lies in the future of $X_1(\Sigma)$ and each $X_i(\Sigma)$ possesses a static neighbourhood $U_i$ in $M, i = 1, 2$ (this can be generalized to include also the scattering between two asymptotic regions). Then, one can choose a positive-frequency basis $\{\Psi_\omega\}$ for the field in $U_i$, the frequency $\omega$ being defined with respect to the Killing vector.
The functions $\Psi_{i\omega}$ are interpreted as wave functions of particles of $(i, \omega)$-kind; the Fock spaces $F_i$ constructed from them are called \textit{in} and \textit{out Hilbert spaces}. As the $\omega$-spectra of in and out particles may be different, the two Fock spaces cannot in general be identified, and we have no unique physical Hilbert space.

One possible choice of perennials is the following \cite{7}. Any point $(\varphi, \pi)$ of $\Gamma_X$ can be considered as an initial datum for the field equation (6) and the solution can be expanded in the basis $\{\Psi_{i\omega}\}$; the coefficients $a_{i\omega}$ and $a_{i\omega}^*$ of this expansion are, therefore, functions $a_{i\omega}(\varphi, \pi)$ on the physical phase space $\Gamma_X$. $a_{i\omega}(\varphi, \pi)$ can then be considered as the initial datum for a perennial, $A_{1\omega}$, on $\Gamma$. The value of $A_{1\omega}$ at $(X, \varphi, \pi) \in \Gamma$ can be calculated as follows. The pair $(\varphi, \pi)$ defines an initial datum along $X(\Sigma)$ for the field equation; let us denote the corresponding solution by $\phi$. $\phi$ defines, in turn, an initial datum $(\varphi_1, \pi_1)$ at $X_1(\Sigma)$. Then, $A_{1\omega}(X, \varphi, \pi) = a_{1\omega}(\varphi_1, \pi_1)$. Analogously, the perennial $A_{2\omega}$ is defined. The Lie algebra of the complete system $\{A_{1\omega}\}$ ($\{A_{2\omega}\}$) of perennials is called \textit{in} (\textit{out}) algebra.

As the field equation is linear, the relation between the in and out algebras is linear:

$$A_{1\omega} = \sum_{\omega'}(\alpha_{\omega\omega'}A_{2\omega'} + \beta_{\omega\omega'}A_{2\omega'}^*); \quad (8)$$

from the Bogoliubov coefficients $\alpha$ and $\beta$, the scattering matrix can be calculated \cite{33}.

The construction above shows the role of the field equation in the definition of perennials. In a linear theory like QFT on background, the classical field equation can be directly promoted to a \textit{quantum field equation}: just replace the real field $\phi$ by a quantum field $\hat{\phi}$ (an operator valued distribution). The relation between the quantum in and out algebras $\{\hat{A}_{1\omega}\}$ ($\{\hat{A}_{2\omega}\}$) obtained from the quantum field equation has then the same form as (8). Observe that the quantum field equation is less awkward and more 4-covariant than the Wheeler-DeWitt equation based on (7) (functional Schrödinger equation).

Of course, interesting models will include interaction and non-abelian gauge symmetry, and the quantum field equation may be difficult to write down (the products of quantum fields are not well-defined). If it can be written down, and if it is a reasonable dynamical equation, then the relation between the quantum in and out algebras can be calculated from it. This relation need not have the same form as the corresponding classical one.

The scattering approach can be extended to a pair of transversal surfaces $\Gamma_1$ and $\Gamma_2$ that are not global. Three different models of this kind have been studied: two finite dimensional systems in \cite{8} and \cite{5} and the massless scalar field on a collapse background spacetime (Hawking effect) in \cite{7}. In all cases, some c-orbit may intersect $\Gamma_1$ ($\Gamma_2$) but miss $\Gamma_2$ ($\Gamma_1$). Still, the relations between the in and out algebras are well defined and have been calculated (for the Hawking effect, this has in fact been done.
These relations cannot be, however, implemented by a unitary map between the in and out Hilbert spaces. This happens even in the finite-dimensional models; thus, the problem is not only an infinite number of out particles being created from in vacuum.

5.2 Algebraic method

The algebraic method of the (non-parametrized) QFT on background (see [33]) is based on a particular class of observables, for example those called smeared fields $\phi_f$: each $C^\infty$ compact-support function $f : M \mapsto \mathbb{R}$ defines $\phi_f$ by $\phi_f := \int_M d^4x \, f \phi$. Polynomials of smeared fields generate the basic space of the algebraic approach, the so-called $C^*$ algebra; the states are certain linear functionals on this algebra, which are interpreted as expected values; they do not form a Hilbert space. A Hilbert space can still be obtained as a representation space of the algebra. However, if there is no timelike Killing vector, the algebra has no unique, physically distinguished representation. The physical interpretation of the algebra is then based on a sufficient amount of physically interesting states (the Hadamard quasi-free states), the associated Hilbert spaces (GNS construction) and well-defined expected stress-energy tensor in suitable states. For an explanation of how such an approach to a quantum field theory works, see [33], Section 4.5.

In the parametrized QFT on background [7], the smeared fields can be expressed as functions $\kappa_f$ on the extended phase space $\tilde{\Gamma}$. Such functions are automatically perennials, because they depend only on classical solutions $\phi$. The perennials $\kappa_f$'s form a Lie algebra with the operations of linear combination and Poisson bracket; the Poisson bracket is determined by the field equation (6). In [7], the group quantization method is applied to this system of perennials [27], [14]. The resulting quantum theory coincides with the $C^*$ algebra described above.

The constructions described briefly in this and the foregoing subsections lead to manifestly reparametrization invariant quantum theories. The interpretation of these theories is based on what may be called labelled perennials. The labels of such perennials carry information on the time and space position (not necessarily a point), where the corresponding measurement is done. If we are given a sufficient amount of labelled perennials (like all smeared fields, for example), then we can calculate what happens at any position. It might, therefore, better be called “Hamiltonian-less” rather than “timeless” approach. The “sufficient amount” is a highly overcomplete system of perennials. For instance, all perennials that form a complete set of Heisenberg observables is such a system. In general, however, the labels do not need to refer to global transversal surfaces and the perennials need not form classes whose elements are related by time shifts (cf. [16]).
6 Outlook

There are some hopes that the notion of ARF will help to better understand and to compare all current attempts at quantizing RIS’s, especially the general relativity. This requires systematic and extended work which is currently being done; for example, a question of how much use the different path integral methods make of ARF is interesting. The ARF free methods may deserve further investigation.

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