CLASSIFICATION OF SIMPLE WEIGHT MODULES FOR THE $N = 2$ SUPERCONFORMAL ALGEBRA

DONG LIU, YUFENG PEI, AND LIMENG XIA

Abstract. In this paper, we classify all simple weight modules with finite dimensional weight spaces over the $N = 2$ superconformal algebra. As an application, we give a new proof of the classification of such modules for the $N = 1$ superconformal algebra, which was originally published in [25].

Keywords: Virasoro algebra, superconformal algebra, cuspidal module

Mathematics Subject Classification (2000): 17B10, 17B65, 17B68, 17B70.

1. Introduction

The representation theory of the Virasoro algebra, which is the universal central extension of the Witt algebra, plays a crucial role in many areas of mathematics and physics. It is well-known that the classification of simple weight modules with finite dimensional weight spaces over the Virasoro algebra, conjectured by V. Kac in [13], was completed by Mathieu in 1992 (see [20], also see [15, 26]). Based on this result, such modules for many Lie algebras related to the Virasoro algebra were classified (see [27] for the Weyl algebra, [18, 28] for high rank Virasoro algebras, [19] for the twisted Heisenberg-Virasoro algebra, [16] for the Schrödinger-Virasoro algebra, etc.). Recently, Billig and Futorny extended Mathieu’s classification result to the Lie algebra $W_n$ of vector fields on an $n$-dimensional torus (see [3] and the references therein). The classification theorem in [3] can be roughly stated as: every nontrivial simple weight module with finite dimensional weight spaces for the Lie algebra $W_n$ is either a submodule of a tensor module or a module of highest weight type.

Superconformal algebras may be viewed as natural super-extensions of the Virasoro algebra and have been playing a fundamental role in string theory and conformal field theory. In [14] Kac and van de Leur gave a mathematically rigorous definition of a superconformal algebra as follows: (a) it is a $\mathbb{Z}$-graded complex Lie superalgebra, (b) it is graded simple, (c) the dimensions of its graded spaces are uniformly bounded. Furthermore, it contains the Virasoro algebra as a graded subalgebra. The main examples of superconformal algebras are Cartan families of linearly compact Lie superalgebras. In [13], Kac classified all physical superconformal algebras:
namely, the $N = 0$ (Virasoro), $N = 1$ (Neveu-Schwarz), $N = 2, 3$ and $4$ superconformal algebras, the superalgebra $W(2)$ of all vector fields on the $N = 2$ supercircle, and a new superalgebra $CK(6)$.

Representation theory for superconformal algebras has been the subject of intensive study (see [7, 10, 14], etc.). It is an important and challenging problem to give complete classifications of simple weight modules with finite dimensional weight spaces for superconformal algebras. In [5], all simple unitary weight modules with finite dimensional weight spaces over the $N = 1$ superconformal algebra were classified, which includes highest and lowest weight modules. A complete classification for the $N = 1$ superconformal algebra was given by Su in [25]. However, the complicated computations in the proofs make it extremely difficult to follow. Such modules (also named graded modules here) for superconformal algebras were also studied in [22], where some cuspidal (also named uniformly bounded) modules were constructed. The problem of classifying such simple modules over superconformal algebras turns out to be more subtle.

It is well-known that the $N = 2$ superconformal algebras fall into four sectors: the Ramond sector, the Neveu-Schwarz sector, the topological sector and the twisted sector. The main theme of the present paper is to study simple weight modules with finite dimensional weight spaces for the Ramond sector and the Neveu-Schwarz sector of $N = 2$ superconformal algebras (see [9, 24]). Our methods in the whole paper are suitable for both two sectors, so we just write them for the Ramond sector. The main theorem can be stated as follows.

**Theorem 1.1. (Main Theorem, see Theorem 5.3)** Let $V$ be a simple weight module with finite dimensional weight spaces over the $N = 2$ Ramond algebra. Then $V$ is a highest weight module, a lowest weight module, or a cuspidal module of the length 1 or 2 (see (2.6) below).

In order to achieve the main theorem, we first consider the subalgebras $q^\pm = \text{span}_C\{L_m, H_m, G^\pm_m, C \mid m \in \mathbb{Z}\}$ of the $N = 2$ Ramond algebra $\mathfrak{g}$ (see Section 2.2 below), which also includes the twisted Heisenberg-Virasoro algebra $\mathfrak{t} = \text{span}_C\{L_m, H_m, C \mid m \in \mathbb{Z}\}$ as a subalgebra. Based on the classification result about the twisted Heisenberg-Virasoro algebra $\mathfrak{t}$ in [19] and some facts of Grassmann algebras we classify all simple cuspidal $q^\pm$-modules. With this classification (Proposition 3.4), we can reduce the classification of simple cuspidal $\mathfrak{g}$-modules (uniformly bounded modules) to the combinations of the two pieces of simple $q^\pm$-modules $U^\pm$ as the following diagram (see Proposition 4.1 for details).

\[
\begin{array}{ccc}
U^+ & \longrightarrow & q^- U^+ & \longrightarrow & \cdots & \longrightarrow & (q^-)^n U^+ \\
\uparrow & & \uparrow & & & & \uparrow \\
(q^+) n U^- & \longrightarrow & (q^+) n^{-1} U^- & \longrightarrow & \cdots & \longrightarrow & U^-
\end{array}
\]
By direct calculations we get a useful identity \( G_{r_1}^- G_{r_2}^- G_{s_1}^+ G_{s_2}^+ G_{s_3}^+ U^- = 0, \forall r_i, s_i \in \mathbb{Z}, i = 1, 2, 3 \) for the \( q^\pm \)-modules \( U^\pm \), and then approach our main result for the \( N = 2 \) Ramond algebra \( g \).

As an application of Theorem 1.1, we recover the classification of simple weight modules with finite dimensional weight spaces for the \( N = 1 \) Ramond algebra \( s \), which can be realized as a graded subalgebra of the \( N = 2 \) Ramond algebra \( g \).

**Theorem 1.2.** ([25], see Theorem 6.7) Let \( V \) be a simple weight module with finite dimensional weight spaces over the \( N = 1 \) Ramond algebra. Then \( V \) is a highest weight module, a lowest weight module, or a module of the intermediate series.

To classify all simple cuspidal modules over the \( N = 1 \) Ramond algebra \( s \), we shall use the theory of the \( A \)-cover in [3] for the Virasoro algebra, which can be generalized to the super Virasoro algebra (see Section 6 for detail). Then a cuspidal \( s \)-module can be extended to cuspidal \( g \)-module as the following diagram:

\[
\begin{array}{ccc}
\text{simple cuspidal } s \text{-module } V & \xrightarrow{A\text{-cover}} & \text{cuspidal } (s, t) \text{-module } \hat{V} \\
\downarrow \text{projection} & & \downarrow \\
\text{simple } g \text{-module } \mathcal{R}'_{a,b,c} & \xleftarrow{\text{Theorem 1.1}} & \text{cuspidal } g \text{-module } \hat{V}
\end{array}
\]

It guarantees that we can get a new proof of the classification of simple cuspidal modules for the \( N = 1 \) Ramond algebra.

This paper is arranged as follows. In Section 2, we recall some notations and collect known facts about the \( N = 1, N = 2 \) Ramond algebras. In Section 3, we classify all simple cuspidal modules for the subalgebra \( q^\pm \) of the \( N = 2 \) Ramond algebra. With this classification we do such researches for the \( N = 2 \) Ramond algebra in Section 4. In Section 5, we determine all simple non-cuspidal weight modules over the \( N = 2 \) Ramond algebra which turns out to be highest (or lowest) weight modules, and then get the main result of this paper. As an application of the main result, we get a classification of simple quasi-finite modules for the \( N = 1 \) Ramond algebra in Section 6 instead of calculations in [25].

Throughout this paper, we shall use \( \mathbb{C}, \mathbb{Q}, \mathbb{Z}, \mathbb{Z}^*, \mathbb{Z}_+ \) and \( \mathbb{N} \) to denote the sets of the complex numbers, the rational numbers, the integers, the non-zero integers, the positive integers and the nonnegative integers, respectively. For convenience, all algebras (vector spaces) are based on the field \( \mathbb{C} \), all modules considered in this paper are nontrivial and all elements in superalgebras and modules are homogenous unless specified. We always denote by \( U(L) \) the universal enveloping algebra of a given Lie (super)algebra \( L \), and denote by \( L^nV := \{x_1x_2\cdots x_n \cdot v \mid v \in V, x_i \in L, i = 1, 2, \cdots n\} \) for any positive integer \( n \) and \( L \)-module \( V \). A nonzero \( v \in V \) is called a trivial \( L \)-vector if \( Lv = 0 \).
2. Basics

In this section, we collect the definitions about the $N = 1$ and $N = 2$ superconformal algebras and some known facts for later use.

Let $L$ be a Lie superalgebra. An $L$-module is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ together with a bilinear map, $L \times V \to V$, denoted $(x, v) \mapsto xv$ such that

$$x(yv) - (-1)^{|x||y|}y(xv) = [x, y]v$$

for all $x, y \in L, v \in V$, and $L_\sigma V_\tau \subseteq V_{\sigma + \tau}$ for all $\sigma, \tau \in \mathbb{Z}_2$. Thus there is a parity change functor $\Pi$ on the category of $L$-modules, which interchanges the $\mathbb{Z}_2$-grading of a module.

2.1. The $N = 2$ Ramond algebra. By definition, the $N = 2$ Ramond algebra $g$ is a Lie superalgebra over $\mathbb{C}$ with a basis $\{L_m, H_m, G^\pm_m, C \mid m \in \mathbb{Z}\}$ and the following relations:

$$[L_m, L_n] = (n - m)L_{n+m} + \frac{1}{12}(n^3 - n)C,$$
$$[H_m, H_n] = \frac{1}{3}m\delta_{m+n,0}C, \quad [L_m, H_n] = nH_{m+n},$$
$$[L_m, G^\pm_p] = (p - \frac{m}{2})G^\pm_{p+m}, \quad [H_m, G^\pm_p] = \pm G^\pm_{m+p},$$
$$[G^+_p, G^-_q] = -2L_{p+q} + (p - q)H_{p+q} + \frac{1}{3}(p^2 - \frac{1}{4})\delta_{p+q,0}C,$$
$$[G^\pm_p, G^\pm_q] = 0$$

for $m, n, p, q \in \mathbb{Z}$.

The Lie superalgebra $g$ is $\mathbb{Z}$-graded by $(g)_0 = \text{span}_\mathbb{C}\{L_i, H_i \mid i \in \mathbb{Z}\}$ and $(g)_i = \text{span}_\mathbb{C}\{G^+_i, G^-_i\}$ for any $i \in \mathbb{Z}$. Now we introduce some subalgebras of $g$ as follows:

$$t := g_0 = \text{span}_\mathbb{C}\{L_n, H_n, C \mid n \in \mathbb{Z}\}, \quad \text{(2.1)}$$

which is called the twisted Heisenberg-Virasoro algebra (see [1, 17, 19], etc.).

$$q^\pm := \text{span}_\mathbb{C}\{L_n, H_n, G^\pm_n, C \mid n \in \mathbb{Z}\}, \quad \text{(2.2)}$$

which play a key role in our studies, and will be mainly considered in Section 3.

The $N = 1$ Ramond algebra $s$ can be realized as a subalgebra of $g$ by $G_m = \frac{1}{\sqrt{2}}(G^+_m + G^-_m)$. By definition, the $N = 1$ Ramond algebra

$$s := \text{span}_\mathbb{C}\{L_n, G_n, C \mid n \in \mathbb{Z}\} \quad \text{(2.3)}$$
with the following relations:

\[ [L_m, L_n] = (n - m)L_{n+m} + \frac{1}{12}(n^3 - n)C \]
\[ [L_m, G_p] = (p - \frac{m}{2})G_{p+m}, \]
\[ [G_p, G_q] = -2L_{p+q} + \frac{1}{3}\delta_{p+q,0} \left( p^2 - \frac{1}{4} \right) C \]

for \( m, n, p, q \in \mathbb{Z} \).

Clearly

\[ \text{Vir} := \mathfrak{so}_0 = \text{span}_\mathbb{C}\{L_n, C \mid n \in \mathbb{Z}\} \tag{2.4} \]

is just the Virasoro algebra.

Replaced \( \{G_r^\pm, r \in \mathbb{Z}\} \) (resp. \( \{G_r, r \in \mathbb{Z}\} \)) by \( \{G_r^\pm, r \in \mathbb{Z} + \frac{1}{2}\} \) (resp. \( \{G_r, r \in \mathbb{Z} + \frac{1}{2}\} \)), the superconformal algebra is called the \( N = 2 \) Neveu-Schwarz algebra (resp. the \( N = 1 \) Neveu-Schwarz algebra).

The \( N = 1 \) superconformal algebra (Ramond or Neveu-Schwarz) is also named the super Virasoro algebra.

2.2. Quasi-finite modules. Set \( \mathcal{L} := \mathfrak{so} \) or \( \mathfrak{g} \). For any \( \mathcal{L} \)-module \( V \) and \( \lambda \in \mathbb{C} \), set \( V_\lambda := \{v \in V \mid L_0v = \lambda v\} \), which is generally called the weight space of \( V \) corresponding to the weight \( \lambda \). An \( \mathcal{L} \)-module \( V \) is called a weight module if \( V \) is the sum of all its weight spaces.

For a weight module \( V = V_\lambda + V_i \), we define

\[ \text{Supp}(V) := \{\lambda \in \mathbb{C} \mid V_\lambda \neq 0\}. \tag{2.5} \]

Obviously, if \( V \) is a weight \( \mathcal{L} \)-module, then there exists \( \lambda \in \mathbb{C} \) such that \( \text{Supp}(V) \subset \lambda + \mathbb{Z} \). So \( V = \sum_{i \in \mathbb{Z}} V_i \) is \( \mathbb{Z} \)-graded, where \( V_i = V_{\lambda+i} \). A weight \( \mathcal{L} \)-module \( V = \sum V_i \) is called quasi-finite if all \( V_i \) are finite-dimensional. If, in addition, there exists a positive integer \( p \) such that

\[ \dim(V_i)_\tau \leq p, \forall i \in \mathbb{Z}, \forall \tau \in \mathbb{Z}_2, \tag{2.6} \]

the module \( V \) is called cuspidal. In this case the minimal \( p \) in (2.6) is called the length of \( V \) (in fact, it is just the length of composition series of the Vir-module \( V_\tau \) for \( \tau = 0 \) or \( 1 \)). A cuspidal module \( V \) of the length 1 is called a module of the intermediate series.

An \( \mathcal{L} \)-module \( V \) is called a highest (resp. lowest) weight module, if there exists a nonzero \( v \in V_\lambda \) such that

1) \( V \) is generated by \( v \) as \( \mathcal{L} \)-module with \( L_0v = hv \) and \( Cv = cv \) for some \( h, c \in \mathbb{C} \).
2) $L_+ v = 0$ (resp. $L_- v = 0$), where $L_+ = \sum_{i>0} L_i$, $L_- = \sum_{i<0} L_i$ (in the case of the $N = 2$ Ramond algebra $\mathfrak{g}$, we also need that $G_0^+ v = 0$ and $H_0 v = h' v$ for some $h' \in \mathbb{C}$).

Clearly highest or lowest weight modules are quasi-finite.

For the Virasoro algebra, the module of the intermediate series was given by as follows (see [15]):

$$A_{a, b} = \sum_{i \in \mathbb{Z}} C v_i : L_m v_i = (a + i + bm)v_{m+i}, \forall i, m \in \mathbb{Z},$$

where $a, b \in \mathbb{C}$

It is well-known that $A_{a, b} \cong A_{a+1, b}$ for all $a, b \in \mathbb{C}$, then we can always suppose that $a \notin \mathbb{Z}$ or $a = 0$ in $A_{a, b}$. Moreover, the module $A_{a, b}$ is simple if $a \notin \mathbb{Z}$ or $b \neq 0, 1$. In the opposite case the module contains two simple sub-quotients namely the trivial module and $\mathbb{C}[t, t^{-1}]/\mathbb{C}$. It is also clear that $A_{0,0}$ has $C v_0$ as a submodule, and its corresponding quotient is denoted by $A'_{0,0}$. Dually, $A_{0,1}$ has $C v_0$ as a quotient module, and its corresponding submodule is isomorphic to $A'_{0,0}$. For convenience, we simply write $A'_{a, b} = A_{a, b}$ when $A_{a, b}$ is simple.

All simple weight modules with finite dimensional weight spaces over the Virasoro algebra were mainly classified in [20].

**Theorem 2.1.** ([20]) Let $V$ be a simple weight Vir-module with finite dimensional weight spaces. Then $V$ is a highest weight module, lowest weight module, or a module of the intermediate series.

Based on this classification, all simple weight modules with finite dimensional weight spaces over the twisted Heisenberg-Virasoro algebra $t$ were also classified.

**Theorem 2.2.** ([19]) Let $V$ be a simple weight $t$-module with finite dimensional weight spaces. Then $V$ is a highest weight module, a lowest weight module, or a module of the intermediate series.

**Remark 2.3.** The $t$-module of the intermediate series, denoted by $A_{a, b, c}$ for some $a, b, c \in \mathbb{C}$, was given in [19] (also see [17]) as follows:

$$A_{a, b, c} = \sum_{i \in \mathbb{Z}} C v_i : L_m v_i = (a + i + bm)v_{m+i}, H_m v_i = cv_{m+i}, \forall i, m \in \mathbb{Z}.$$

Moreover, the module $A_{a, b, c}$ is simple if $a \notin \mathbb{Z}$ or $b \neq 0, 1$ or $c \neq 0$. For convenience, we also use $A'_{a, b, c}$ to denote by the simple sub-quotient of $A_{a, b, c}$.

### 3. Cuspidal modules for the Lie sub-superalgebra $q^\pm$

In order to achieve our main result, we shall do such researches for the subalgebra $q^\pm$. Due to that $q^\pm$ are isomorphic each other, we just study $q^+$ in this section,
and then short write \( q := q^+ \), \( Q_j := G^+_j \) for all \( j \in \mathbb{Z} \) and \( q_1 = \text{span}_C \{ Q_j \mid j \in \mathbb{Z} \} \) the subalgebra of \( q \).

First we introduce a result about the twisted Heisenberg-Virasoro algebra \( t \).

**Lemma 3.1.** Let \( V = \sum V_i \) be a cuspidal \( t \)-module including no any trivial \( t \)-vector. There exists \( m \in \mathbb{Z}^+ \) such that for any nonzero \( v \in V_i \), \( i \in \mathbb{Z}^* \), simple \( t \)-module \( V' \subset U(t)v \) satisfy

\[
V' \subset \sum_{1 \leq k \leq m} t^k v. \tag{3.1}
\]

**Proof.** From representation theory of the Virasoro algebra, \( \dim V_j = N \) for some positive integer \( N \) holds for almost all \( j \in \mathbb{Z} \) and \( C \) acts on \( V \) as zero (see [15, 21]). Consider a composition series of \( t \)-submodules of \( U(t)v \):

\[
0 \subset V' = V^{(1)} \subset V^{(2)} \subset \ldots \subset V^{(p)} = U(t)v, \tag{3.2}
\]

where \( V^{(1)}, \ldots, V^{(p)} \) are \( t \)-submodules of \( V \), \( 1 \leq p \leq N \), and the quotient modules \( W^{(j)} = V^{(j)}/V^{(j-1)} = \sum \mathbb{C}e_i^{(j)} \) are isomorphic to \( \mathcal{A}'_{a,b_j,c_j} \) for some \( a, b_j, c_j \in \mathbb{C} \) by Theorem 2.2.

Assume that \( v = a_1v_i^{(1)} + \cdots + a_pv_i^{(p)} \in V_i^{(p)} := V_i \cap V^{(p)} \) for some \( a_1, \ldots, a_p \in \mathbb{C} \). Then we have

\[
L_j v \equiv a_p(a + b_p j + i)v_i^{(p)} \mod V_i^{(p-1)}, \tag{3.3}
\]

\[
H_j v \equiv c_p v_i^{(p)} \mod V_i^{(p-1)} \tag{3.4}
\]

for any \( j \in \mathbb{Z} \).

Now we shall prove that there exists \( 0 \neq u \in V_i^{(p-1)} \) for some \( k \in \mathbb{Z} \) such that

\[
u \in \sum_{i=1}^2 t^j v \tag{3.5}
\]
case by case.

**Case 1.** \( c_p \neq 0 \). In this case, for any \( k, k' \in \mathbb{Z} \), \( X, Y \in \{L, H\} \), there exists \( d_{k,k'} \in \mathbb{C} \) such that \( u = X_kY_{k-k'}v - d_{k,k'}H_kv \in V_i^{(p-1)} \) if \( X_kY_{k-k'}v \neq 0 \) by (3.3) and (3.4). If \( u \) is always zero, then

\[
X_kY_{k-k'}v = d_{k,k'}H_kv, \forall X, Y \in \{L, H\}, \ k, k' \in \mathbb{Z}
\]

for some \( d_{k,k'} \in \mathbb{C} \). Then span \( \mathbb{C} \{ H_kv \mid k \in \mathbb{Z} \} \) becomes a \( t \)-submodule and \( p = 1 \). So (3.5) holds.

**Case 2.** \( c_p = 0 \). In this case, \( H_j v \in V_{i+j}^{(p-1)} \) for any \( j \in \mathbb{Z} \). So there exists \( u = H_kv \in V_{i+k}^{(p-1)} \) such that \( 0 \neq u \in tv \) by (3.4) if \( H_kv \neq 0 \) for some \( k \in \mathbb{Z} \). In this case (3.5) also holds.

**Case 3.** Now we can suppose that \( H_j v = 0 \) for any \( j \in \mathbb{Z} \), then \( U(t)v = U(\text{Vir})v \).
Subcase 3.1. $L_j v \not\in V_{i+j}^{(p-1)}$, $\forall j \in \mathbb{Z}$. For any $k' \in \mathbb{Z}$, there exists $d \in \mathbb{C}$ such that $u = L_{k'} L_{k-k'} v - dL_k v \in V_{i+k}^{(p-1)}$ if $L_{k'} L_{k-k'} v \neq 0$ by (3.3). Similarly, if the $u$ is always zero, then span $c \{ L_k v \mid k \in \mathbb{Z} \}$ becomes a Vir-submodule and then $p = 1$. So (3.5) still holds.

Subcase 3.2. $u = L_k v \in V_{i+k}^{(p-1)}$ for some $k \in \mathbb{Z}$. In this case (3.5) holds if $L_k v \neq 0$.

Subcase 3.3. The only left case is that $L_{k_0} v = 0$ for some $k_0 \in \mathbb{Z}$. In this case $L_j v \not\in V_{i+j}^{(p-1)}$ for any $j \neq k_0$ by (3.3). As in Subcase 3.1, for any $k \neq k_0$, there exists $d_{k,k'} \in \mathbb{C}$ such that $u = L_{k'} L_{k-k'} v - d_{k,k'} L_k v \in V_{i+k}^{(p-1)}$ if $L_{k'} L_{k-k'} v \neq 0$.

Suppose that the above $u$ is always zero, then

$$L_{k'} L_{k-k'} v = d_{k,k'} L_k v$$

for some $d_{k,k'} \in \mathbb{C}$ if $k \neq k_0$. Moreover, for any $k \neq k_0, k \neq 0, j \neq 0$, $(k-2j)L_k L_{k-k'} v = L_j L_{k-j} L_{k_0-k} v - L_{k-j} L_k L_{k_0-k} v - d_{k,j,k_0-k} L_j L_{k_0-j} v - d_{k-j,j} L_k L_{k_0-j} v$. So we have

$$L_j L_{k_0-j} v = \frac{(k-2j) + d_{k,j,k_0-k}}{d_{k-j,k_0-k}} L_k L_{k_0-k} v,$$

where $d_{k-j,k_0-k} \neq 0$ since $L_j v \not\in V_{i+j}^{(p-1)}$ for any $j \neq k_0$. In this case $\sum_{j \in \mathbb{Z}, j \neq k_0} \mathbb{C} L_j v + \mathbb{C} u_{k_0}$ becomes a Vir-submodule, where $u_{k_0} = L_k L_{k_0-k} v \neq 0$ for some $k \in \mathbb{Z}^*$. Then $p = 1$.

Now (3.5) holds in all cases. If $k + i = 0$, we can replace $u$ by $L_j u$ or $H_j u$ for some $j \neq 0$ and get that there exists $0 \neq u \in V_{i+k}^{(p-1)}$ for some $k \in \mathbb{Z}$ with $k + i \neq 0$ such that

$$u \in \sum_{j=1}^{3} t^j v.$$

(3.6)

Now by the induction we can find $v_j^{(1)} \in \sum_{1 \leq k \leq 3p} t^k v$ for some $j \in \mathbb{Z}$ and then

$$V' \subset \sum_{1 \leq k \leq 3p+2} t^k v$$

(3.7)

since $L_{k-j} v_j^{(1)} = (a + b_1(k - j) + j)v_k^{(1)}$ and $H_{k-j} v_j^{(1)} = c_1 v_k^{(1)}$ for any $k \in \mathbb{Z}$.

Set $m = 3N + 2 \geq 3p + 2$, the lemma follows. \hfill $\square$

Remark 3.2. From the proof in Case 3, we see that the lemma also holds for the Virasoro algebra.

Now we return to the subalgebra $\mathfrak{q}$. Set $\mathcal{O}_n := \{0, \pm 1, \pm 2, \ldots, \pm n\}$ for $n \in \mathbb{Z}_+$ in this section.

Lemma 3.3. Let $V$ be a $\mathfrak{q}$-module. Suppose that there exist $n, m \in \mathbb{Z}_+$ and $n > m$ such that

$$Q_{r_1} Q_{r_2} \cdots Q_{r_m} V = 0$$

(3.8)
for all \( r_1, r_2, \ldots, r_m \in \mathcal{O}_n \), then (3.8) holds for all \( r_1, r_2, \ldots, r_m \in \mathbb{Z} \).

**Proof.** We shall use the induction on \( n \) and first prove that

\[
Q_{r_1} \cdots Q_{r_{m-1}} Q_{n+1} V = 0
\]

for all \( r_1, r_2, \ldots, r_{m-1} \in \mathcal{O}_n \).

By action of \( L_{n+1-r_m} \) on (3.8) (where \( r_1 < r_2 < \cdots < r_m \)) we get

\[
Q_{r_1} Q_{r_2} \cdots Q_{r_{m-1}} Q_{n+1} V = 0
\]

(3.9)

for all \(-n \leq r_1 < r_2 < \cdots < r_{m-1} \leq n - 1\). The left is to prove that

\[
Q_{r_1} Q_{r_2} \cdots Q_{r_{m-2}} Q_n Q_{n+1} V = 0
\]

for all \( r_1, r_2, \ldots, r_{m-2} \in \mathcal{O}_n \).

By actions of \( L_1 \) and \( H_1 \) on (3.9) with \( r_{m-1} = n - 1 \), we get

\[
(n - \frac{3}{2})Q_{r_1} \cdots Q_{r_{m-2}} Q_n Q_{n+1} + (n + \frac{1}{2})Q_{r_1} \cdots Q_{r_{m-2}} Q_{n+1} V = 0
\]

(3.10)

\[
(Q_{r_1} \cdots Q_{r_{m-2}} Q_n Q_{n+1} + Q_{r_1} \cdots Q_{r_{m-2}} Q_{n+2}) V = 0.
\]

(3.11)

Combining with (3.10) and (3.11), we get

\[
Q_{r_1} \cdots Q_{r_{m-2}} Q_n Q_{n+1} V = 0
\]

(3.12)

for all \(-n \leq r_1 < \cdots < r_{m-2} \leq n - 2\).

In the same way, by the actions of \( L_1 \) and \( H_1 \) on (3.12) with \( r_{m-2} = n - 2 \), we get

\[
Q_{r_1} \cdots Q_{r_{m-3}} Q_{n-1} Q_n Q_{n+1} V = 0
\]

(3.13)

for all \(-n \leq r_1 < r_2 < \cdots < r_{m-3} \leq n - 3\).

Repeating the above steps by setting \( r_{m-3} = n - 3 \), \( r_1 = n - m + 1 \), respectively, we can get \( Q_{r_1} Q_{r_2} \cdots Q_{r_{m-1}} Q_{n+1} V = 0 \) for all \( r_1, r_2, \ldots, r_{m-1} \in \mathcal{O}_n \).

Similarly we can get \( Q_{n-1} Q_{r_2} Q_{r_3} \cdots Q_{r_m} V = 0 \) for all \( r_2, r_3, \ldots r_m \in \mathcal{O}_n \). Then (3.8) holds for all \( r_1, r_2, \ldots, r_m \in \mathcal{O}_{n+1} \). The lemma follows by induction on \( n \). \( \square \)

**Proposition 3.4.** Let \( V = \sum V_i \) be a simple cuspidal \( q \)-module. Then \( V \) is a \( q \)-module of the intermediate series: \( V = \sum v_i \cong A_{a,b,c}' \) for some \( a, b, c \in \mathbb{C} \) with \( H_j v_i = c v_{i+j} \), \( Q_j V = 0 \) for all \( j \in \mathbb{Z} \).

**Proof.** Clearly \( \dim V_i \leq N \) for some positive integer \( N \) holds for almost \( i \in \mathbb{Z} \) and \( C \) acts on \( V \) as zero (see [15, 21]).

Now \( q_i V \) is \( q \)-submodule since \( q_i^{i+1} V \subset q_i V \) for all \( i \in \mathbb{N} \). So \( q_1 V = V \) or \( q_1 V = 0 \).

If there exists a trivial \( t \)-vector \( v \in V \), then \( V = U(q_1)v \). Clearly \( V = \mathbb{C}v + \sum_{i \geq 1} q_1^i v \). Moreover we have \( q_1^i v = \{ v \in V \mid H_0 v = iv \} \) for any \( i \in \mathbb{Z}_+ \). It is easy
to see that $q_1v = 0$. Otherwise $\bigoplus_{i \geq 1} q_1^i v$ is a proper $q$-submodule of $V$. So we can suppose that $V$ does not include any trivial $t$-vector.

**Claim** For any $n \in \mathbb{Z}_+$, there exists $v \in V_i$ for some $i \in \mathbb{Z}^*$ such that $Q_r v = 0$ for all $r \in O_n := \{0, \pm 1, \pm 2, \cdots, \pm n\}$.

In fact, if $Q_r v \neq 0$ for some $r \in O_n$, we may replace $v$ by $Q_r v \in V_{r+i}$. So this claim holds for any $n \in \mathbb{Z}_+$.

Now we choose $n \gg N$ for the $v,n$ in Claim. Clearly $TV \neq 0$ (Otherwise, $Cv$ is a trivial $p$-module). By Theorem 2.2, we can choose a simple $t$-module $V' = \sum Cv_i \cong \mathcal{A}'_{a,b,c} \subset U(t)v$ for some $a,b,c \in \mathbb{C}$, and then $V = U(q_1)V'$.

By Lemma 3.1, there exists $m \in \mathbb{Z}^+$ with $m < n$ such that

$$V' \subset \sum_{k \leq m} t^k v.$$

(3.14)

Now for all $r_1, r_2, \cdots r_m \in O_n$, we have

$$Q_{r_1}Q_{r_2}\cdots Q_{r_m}V' \subset \bigoplus_{j=1}^m x_j Q_{r_j}v = 0,$$

where the $x_j$’s are some elements in $U(q)$.

So $Q_{r_1}Q_{r_2}\cdots Q_{r_m}V = 0$ for all $r_1, r_2, \cdots r_m \in O_n$ since $V = U(q_1)V'$.

By Lemma 3.3, we get

$$q_1^m V = 0.$$ 

(3.15)

If $q_1 V = V$ then $q_1^m V = V = 0$. It gets a contradiction. So $q_1 V = 0$ and the proposition follows from Theorem 2.2. \hfill \Box

4. cuspidal modules over the $N = 2$ Ramond algebra

In this section we shall classify all simple cuspidal modules for the $N = 2$ Ramond algebra $g$ with Proposition 3.4 in Section 3.

4.1. Classification of simple cuspidal modules.

**Proposition 4.1.** Let $V$ be a simple cuspidal $g$-module. Then $\dim (V_i)_\tau \leq 2$ for any $i \in \mathbb{Z}$ and $\tau \in \mathbb{Z}_2$.

**Proof.** Clearly $C$ acts on $V$ as zero.

Now the subalgebras $q^\pm = \text{span}_\mathbb{C} \{L_m, H_m, G_m^\pm, C \mid m \in \mathbb{Z}\}$ are all isomorphic to the Lie superalgebra $q$ in Section 3. The even part $g_0$ is just the twisted Heisenberg-Virasoro algebra $t$ in Section 2.2.
By Proposition 3.4, we can choose a simple \( \mathfrak{g}^+ \)-module \( U^+ = \sum C u_i^+ \) of \( V \) such that \( G_u^+ U^+ = 0 \) for all \( r \in \mathbb{Z} \). In this case \( V = U(G^-)U^+ \), where \( G^- = \text{span}_C \{ G^-_r \mid r \in \mathbb{Z} \} \) is the subalgebra of \( \mathfrak{g} \).

Now we can suppose that \( G^- U^+ \neq 0 \) (otherwise \( V \) is a trivial \( \mathfrak{g} \)-module). Set \( (G^-)^0 U^+ = U^+ \) and \( (G^-)^{i+1} U^+ = G^- (G^-)^i U^+ \) for all \( i \geq 0 \). Then we have \( G^+ (G^-)^i U^+ \subset (G^-)^i U^+ \). Moreover

\[
V = \sum_{i \geq 0} (G^-)^i U^+. \tag{4.1}
\]

By Proposition 3.4 we can choose a simple \( \mathfrak{g}^- \)-module \( U^- \) of \( V \) such that \( U^- = \sum C u^-_j \cong \mathcal{A}_{\alpha,b,c} \) with \( L_m u^-_j = (a + bm + j) u^-_{m+i}, H_m u^-_j = cu^-_{m+j} \) and \( G^-_m u^-_j = 0 \) for all \( m, j \in \mathbb{Z} \), for some \( a, b, c \in \mathbb{C} \).

Suppose that

\[
u_i^- \in (G^-)^n U^+ \tag{4.2}
\]

for some \( n \in \mathbb{Z}_+ \). By the actions of suitable \( L_m \)'s or \( H_m \)'s on (4.2) we get \( U^- \in (G^-)^n U^+ \).

In this case,

\[
V = \sum_{i=0}^n (G^+)i U^- = \sum_{j=0}^n (G^-)^j U^+ \tag{4.3}
\]

and \( (G^+)^n U^- \subset (G^+)n (G^-)^n U^- \subset U^+ \) (see the diagram below). Moreover \( (G^+)^n U^- \neq 0 \) (in fact, if \((G^+)^n U^- = 0\), replaced \( U^+ \) by \( U^- \) and repeated the above step, one has \((G^-)^n U^+ = 0\)). By the irreducibility of \( U^+ \) as \( t \)-module, we get

\[
(G^+)^n U^- = U^+. \tag{4.4}
\]

So get the following diagram:

\[
\begin{array}{cccccccc}
U^+ & \rightarrow & G^- U^+ & \rightarrow & \cdots & \rightarrow & (G^-)^{n-1} U^+ & \rightarrow & (G^-)^n U^+\\
\| & \| & \| & \| & \| & \| & \| & \| & \|
\end{array}
\]

\[
(G^+)^n U^- \leftarrow (G^+)^{n-1} U^- \leftarrow \cdots \leftarrow G^+ U^- \leftarrow U^-
\]

Moreover from the diagram we see that \((G^-)^n U^+ = U^- \) and \((G^-)^i U^+ = (G^+)^{n-i} U^- \) for all \( 0 \leq i \leq n \).

Now by Lemma 4.3 below we see that \( n \leq 2 \) and get the proposition.

If the length \( n \) of \( V \) is 1, then \( V = U^+ + U^- \) and \( G^- U^+ = U^- \) by (4.4).

If the length \( n \) of \( V \) is 2, then \( V = U^+ + G^- U^+ + U^- \) and \( G^- U^+ = G^+ U^- \) by (4.4).

Now we shall prove Lemma 4.3 by direct calculations.
For convenience, set $K_{r,s} = [G_r^+, G_s^-] = -2L_{r+s} + (r-s)H_{r+s}$, then we have the following equalities:

$$K_{r,s} u_i^- = (-2(a + b(r + s) + i) + (r - s)c)u_{r+s+i}^-, \quad \text{(4.5)}$$

$$[K_{r,s}, G_t^+] = 2(r - t)G_{r+s+t}. \quad \text{(4.6)}$$

**Lemma 4.2.** For any $r_1, r_2, s_1, s_2 \in \mathbb{Z}$,

$$G_{r_1}^- G_{r_2}^- G_{s_1}^+ G_{s_2}^+ u_i^- = d(s_2 - s_1)(r_1 - r_2)u_{i+r+s+s_1+s_2}^- \quad \text{(4.7)}$$

where $d = (2b + c)(2 + c - 2b)$.

**Proof.** By (4.5) and (4.6) we have

$$G_{r_1}^- G_{r_2}^- G_{s_1}^+ G_{s_2}^+ u_i^- = G_{r_1}^- K_{s_1, r_2} G_{s_2}^+ u_i^- - G_{r_1}^- G_{s_1}^+ G_{r_2}^+ G_{s_2}^+ u_i^-$$

$$= G_{r_1}^- G_{s_1}^+ K_{s_1, r_2} G_{s_2}^+ u_i^- + 2(s_1 - s_2)G_{r_1}^- G_{s_2}^+ G_{s_2+s_1+r_2}^+ u_i^- - G_{r_1}^- G_{s_1}^+ K_{s_1, r_2} G_{s_2}^+ u_i^-$$

$$= K_{s_2, r_1} K_{s_1, r_1} K_{s_1, r_2} K_{s_2, r_2} u_i^- + 2(s_1 - s_2)K_{s_1+s_2+r_1+r_2} u_i^-$$

$$= (-2(a + b(s_1 + r_2) + i) + (s_1 - r_2)c)(-2(a + b(s_2 + r_1) + i + s_1 + r_2)+(s_2 - r_1)c)u_i^-$$

$$- (-2(a + b(s_2 + r_2) + i) + (s_2 - r_2)c)(-2(a + b(s_1 + r_1) + i + s_2 + r_2)+(s_1 - r_1)c)u_i^-$$

$$+ 2(s_1 - s_2)(-2(a + b(r_1 + r_2 + s_2 + s_2) + i) + (s_1 + s_2 + r_2 - r_1)c)u_i^-$$

$$= (2 + c - 2b)(2b + c)(s_2 - s_1)(r_1 - r_2)u_i^-,$$

where $u_i^- = u_{i+r_1+r_2+s_1+s_2}^-$. The lemma is obtained. \[\Box\]

Now we shall get a key identity for our proofs by Lemma 4.2.

**Lemma 4.3.** For any $r_1, r_2, r_3, s_1, s_2, s_3 \in \mathbb{Z}$,

$$G_{r_1}^- G_{r_2}^- G_{r_3}^- G_{s_1}^+ G_{s_2}^+ G_{s_3}^+ U^- = 0. \quad \text{(4.8)}$$
Proof.

\[ G_r^+ G_r^- G_r^+ G_r^- G_{s_1}^+ s_{s_2}^+ s_{s_3}^- u_i^- \]
\[ = G_r^+ G_r^- s_{s_1}^+ s_{s_2}^+ s_{s_3}^- u_i^- - G_r^- G_r^+ s_{s_1}^+ G_r^+ G_r^- s_{s_2}^+ s_{s_3}^- u_i^- \]
\[ = 2(s_1 - s_2)G_r^+ G_r^- s_{s_1}^+ s_{s_2}^+ s_{s_3}^- u_i^- + 2(s_1 - s_3)G_r^+ G_r^- s_{s_1}^+ G_r^- s_{s_2}^+ s_{s_3}^- u_i^- \]
\[ + G_r^- G_r^+ G_r^- s_{s_1}^+ G_r^+ s_{s_2}^+ s_{s_3}^- u_i^- - G_r^+ G_r^- G_r^+ G_r^- s_{s_1}^+ s_{s_2}^+ s_{s_3}^- u_i^- \]
\[ = 2(s_1 - s_2)G_r^+ G_r^- s_{s_1}^+ s_{s_2}^+ s_{s_3}^- u_i^- + 2(s_1 - s_3)G_r^+ G_r^- s_{s_1}^+ G_r^- s_{s_2}^+ s_{s_3}^- u_i^- \]
\[ + G_r^- G_r^+ G_r^- G_r^+ s_{s_1}^+ G_r^+ G_r^- G_r^- s_{s_2}^+ s_{s_3}^- u_i^- - G_r^- G_r^- G_r^+ s_{s_1}^+ s_{s_2}^+ s_{s_3}^- u_i^- \]
\[ = 2d(s_1 - s_2)(r_1 - r_2)(s_3 - s_1 - s_2 - s_3)u_i^- + 2d(s_1 - s_3)(r_1 - r_2)(s_1 + s_3 + r_3 - s_2)u_i^- \]
\[ - d(2a + (2b - c)s_1 + (2b + c)r_3 + 2i)(r_1 - r_2)(s_3 - s_2)u_i^- \]
\[ - 2d(s_2 - s_3)(r_1 - r_2)(s_2 + s_3 + r_3 - s_1)u_i^- \]
\[ + d(2a + (2b - c)s_2 + (2b + c)r_3 + 2i)(r_1 - r_2)(s_3 - s_1)u_i^- \]
\[ - d(2a + (2b - c)s_3 + (2b + c)r_3 + 2i)(r_1 - r_2)(s_2 - s_1)u_i^- = 0 \]

for any \( i \in \mathbb{Z} \), where \( u_i^- = u_{i+r_1+r_2+r_3+s_1+s_2+s_3}^- \). The lemma is obtained. \( \square \)

4.2. Simple cuspidal module structure. In this subsection we shall determine the precise module structure of the simple cuspidal modules classified in Section 4.1.

Proposition 4.4. Any simple module of the intermediate series is isomorphic to the simple sub-quotient of \( R_{a,b} \) (up to parity-change) for some \( a, b \in \mathbb{C} \), where \( V = R_{a,b} := \sum \mathbb{C}v_i^+ + \sum \mathbb{C}v_i^- \) with

\[ L_m v_i^+ = (a + i + bm)v_{m+i}^+, \quad L_m v_i^- = (a + i + (b - \frac{1}{2})m)v_{m+i}^- \]
\[ H_m v_i^+ = (2 - 2b)v_{m+i}^+, \quad H_m v_i^- = (1 - 2b)v_{m+i}^- \]
\[ G_m v_i^+ = v_{m+i}^+, \quad G_m v_i^- = -(2a + (4b - 2)m + 2i)v_{m+i}^- \]

for all \( m, i \in \mathbb{Z} \).

Proof. In this case \( V = U^+ + U^- \) and \( G^{-} U^+ = U^- \) by Proposition 4.1, where \( U^\pm \) are simple \( t \)-modules.

Set that \( U^+ = \sum \mathbb{C}v_i^+ \cong \mathcal{A}_{a,b,c}^+ \), for some \( a, b, c \in \mathbb{C} \). Due to \( G^- U^+ = U^- \) we can suppose that \( U^- = \sum \mathbb{C}v_i^- \cong \mathcal{A}_{a,b,c}^- \) for some \( b, c, e \in \mathbb{C} \) and \( G_r^- v_i^+ = g(r, i)v_{r+i}^- \) for almost all \( 0 \neq g(r, i) \in \mathbb{C} \). By \( [L_m, G_n^-]v_i^+ = (n - \frac{m}{2})G_{m+n}^- v_i^+ \) and \([H_m, G_n^-]v_i^+ = -G_{m+n}^- v_i^+ \) we have

\[ (a + b - m + n + i)g(n, i) - (a + bm + i)g(n, m + i) = (n - \frac{m}{2})g(m + n, i), \quad (4.9) \]
\[ c^- g(n, i) - cg(n, m + i) = -g(m + n, i), \quad \forall m, n, i \in \mathbb{Z}. \quad (4.10) \]

Setting \( m = 0 \) in (4.10) we get \( c^- = c - 1 \).

Case 1. \( c = 0 \).
In fact if $c = 0$, then $c^- = -1$. By (4.10) we get $g(n, i) = g(i)$ for some $g(i) \in \mathbb{C}$ for all $n, i \in \mathbb{Z}$.

For any $n, i \in \mathbb{Z}$, we can suppose that $G^+_n v^-_i = h(n, i)v^+_n$ for some $h(n, i)$. By $[H_m, G^+_n] v^-_i = G^+_m v^-_i$, we get $h(n, i) = h(n + i)$ for some $h(n) \in \mathbb{C}$ for all $n, i \in \mathbb{Z}$.

By $[G^+_n, G^-_m] v^-_i = -2L_{m+n} v^+_i + (n - m)H_{m+n} v^-_i$, we get $h(n + i)g(n + i) = -2(a + b^-(m + n) + i) - (n - m)$ for any $m, n, i \in \mathbb{Z}$. So $b^- = \frac{1}{2}$.

Similarly, by $[G^+_n, G^-_m] v^+_i = -2L_{m+n} v^+_i + (n - m)H_{m+n} v^+_i$, we get $b = 1$.

Then $g(n, i) = g(0, 0)$ for all $n, i \in \mathbb{Z}$ follows by (4.9).

**Case 2.** $c = 1$.

It is similar to prove that $g(n, i) = g(0, 0)$ for all $n, i \in \mathbb{Z}$ as in Case 1.

**Case 3.** $c \neq 0, 1$.

Setting $i = 0$ in (4.10) we get

$$g(n, m) = \frac{c - 1}{c} g(n, 0) + \frac{1}{c} g(m, n, 0), \quad \forall m, n \in \mathbb{Z}. \quad (4.11)$$

Substituting (4.11) into (4.10) we get

$$g(m + n + i, 0) = g(n + i, 0) + g(n + m, 0) - g(n, 0), \quad \forall n, m, i \in \mathbb{Z}. \quad (4.12)$$

By (4.12) we get $g(m, 0) = g(0, 0)$ and then $g(m, n) = g(0, 0)$ for all $m, n \in \mathbb{Z}$ by (4.10) again.

So in all case we have $g(m, n) = g(0, 0)$ for all $m, n \in \mathbb{Z}$. Moreover $b^- = b - \frac{1}{2}$ follows by (4.9).

Now we can get the precise module structure on $V = \mathcal{R}_{a,b} := \sum \mathbb{C} v^+_i + \sum \mathbb{C} v^-_i$ as follows (up to parity-change) by some easy calculations:

$$L_m v^+_i = (a + i + bm)v^+_{m+i}, \quad L_m v^-_i = (a + i + (b - \frac{1}{2}) m)v^-_{m+i},$$

$$H_m v^+_i = (2 - 2b)v^+_{m+i}, \quad H_m v^-_i = (1 - 2b)v^-_{m+i},$$

$$G^-_m v^+_i = v^-_{m+i}, \quad G^+_m v^-_i = -(2a + (4b - 2)m + 2i)v^+_{m+i}$$

for all $m, i \in \mathbb{Z}$. \hfill \Box

**Remark 4.5.** $\mathcal{R}_{a,b}$ is not simple if and only if $a = 0, b = 1$ or $a = 0, b = \frac{1}{2}$. All indecomposable modules of the length 1 were given in [8].

**Proposition 4.6.** Any simple module of the length 2 is isomorphic to $\mathcal{R}_{a,b,c}$ (up to parity-change) for some $a, b, c \in \mathbb{C}$ and $2b \pm c \neq 2$, where $\mathcal{R}_{a,b,c} := (U + U^{+\pm}) \oplus$
\[(U^+ + U^-) \text{ with}
\]
\[
L_m v_i = (a + i + bm)v_{m+i}, \quad L_m v^+_k = (a + k + (b - \frac{1}{2})m)v^+_m, \quad (4.13)
\]
\[
H_m v_i = cv_{m+i}, \quad H_m v^+_k = (c \pm 1)v^+_m, \quad (4.14)
\]
\[
G^+_r v_i = v^+_i, \quad G^+_r v^+_i = 0
\]
\[
L_m v^+_i = (a + i + (b - 1)m)v^+_m + \frac{1}{2}(2b - c - 2)m^2 v_i + m, \quad (4.15)
\]
\[
H_m v^+_i = cv^+_m - m(2b - c - 2)v_{m+i}, \quad (4.16)
\]
\[
G^+_r v^+_k = v^+_k + (c + 2 - 2b)rv_k, \quad (4.17)
\]
\[
G^+_r v^+_k = (2a + 2k + (2b + c)r)v_k, \quad (4.18)
\]
\[
G^+_r v^+_i = -(2a + 2i + (2b + c - 2)r)v_{i+r}, \quad (4.19)
\]
\[
G^+_r v^+_i = (2b - c - 2)rv_{i+r}, \quad \forall m, r, i \in \mathbb{Z}.
\]

Proof. In this case \(n = 2\), \(V = U^+ + G^- U^+ + U^- \) and \(G^- U^+ = G^+ U^- \) by Proposition 4.1.

First we can choose a simple \(t\)-module \(U = \sum \mathbb{C}v_i \cong A'_{a,b,c} \) of \(G^- U^+ \) for some \(a, b, c \in \mathbb{C}\). Due to \(U^\pm = G^\pm U \) are all simple \(t\)-modules, we can get \(U^\pm = \sum \mathbb{C}v^\pm_i \cong A'_{a,b-\frac{1}{2},c\pm 1} \) as in Case 1. So we can obtain (4.13), (4.14), (4.15) and the coefficients of \(v^\pm_{m+i} \) in (4.16) and (4.17).

Now the quotient module \(G^- U^+/U = \sum \mathbb{C}v^+_i \) (or \(G^+ U^-/U \)) is also a simple \(t\)-module, and isomorphic to \(A'_{a,b-1,c} \) as in Case 1. So we can suppose that
\[
G^+_r v^+_k = v^+_r + g(r, k)v_k
\]
for some \(g(r, k) \in \mathbb{C} \) and for any \(r, k \in \mathbb{Z} \). Moreover, replaced \(v^+_k + g(0, k)v_k \) by \(v^+_k \) one has \(G^+_0 v^+_k = v^+_k \). It is
\[
g(0, k) = 0, \quad \forall r, k \in \mathbb{Z}.
\]

By the action \(G^- r \) on \(G^+_r v_i = v^+_r \) for any \(r, s, i \in \mathbb{Z} \), we can get
\[
G^- v^+_k = -v^+_{r+k} + h(r, k)v_k
\]
for some \(h(r, k) \in \mathbb{C} \).

By \([G^+_r, G^- v_k] = -2L_{2r}v_k \), one has
\[
g(r, k) = h(r, k) = -2(a + (2b - 1)r + k), \quad \forall r, k \in \mathbb{Z}.
\]

By \([H, G^-] v^+_k = -G^- v^+_k \) and \([H, G^-] v^-_k = G^+ v^-_k \), one has
\[
f_1(m, r + k) + ch(r, k) - (c + 1)h(r, m + k) = -h(m + r, k), \quad (4.23)
\]
Combining with (4.21),(4.22) and (4.23) we get
\[
g(m + r, k) - g(r, m + k) = (c + 2 - 2b)m.
\]
Setting \( r = 0 \) and using (4.20) and (4.21), we get
\[
\begin{align*}
g(m, k) &= (c + 2 - 2b)m, \\
h(m, k) &= -(2a + 2k + (2b + c)m), \quad \forall m, k \in \mathbb{Z}.
\end{align*}
\]
So (4.18) and (4.19) hold.

By actions of \( L_m, H_m, G^\pm_r \) on (4.18), we can easily get the left relations in the proposition. Moreover \( \mathcal{R}_{a,b,c} \) is simple if and only if \( 2b \pm c \neq 2 \).

**Remark 4.7.** If \( 2b - c = 2 \), then \( V = \text{span}_\mathbb{C}\{v_i, v_i^+ | i \in \mathbb{Z}\} \) is a simple module of the intermediate series, which is just isomorphic to the module \( \mathcal{R}_{a,b} \) listed in Case 1 (up to parity-change).

If \( 2b + c = 2 \), then \( V = \text{span}_\mathbb{C}\{v_i, v_i^+ + 2(a + i)v_i | i \in \mathbb{Z}\} \) is a simple module of the intermediate series, which is also isomorphic to \( \mathcal{R}_{a,b} \) (up to parity-change).

If \( 2b - c = 2b + c = 2 \) \( (b = 1, c = 0) \) and \( a = 0, \frac{1}{2} \), then \( V \) is a trivial module.

So we can use \( \mathcal{R}'_{a,b,c} \) to denote by the simple sub-quotient of \( \mathcal{R}_{a,b,c} \) in all cases (including the module \( \mathcal{R}_{a,b} \) in the case of \( n = 1 \)).

\( \mathcal{R}_{a,b,c} \) is just corresponding to the finite conformal module given in [6].

5. Quasi-finite modules over the \( N = 2 \) Ramond algebra

In this section we mainly determine simple quasi-finite modules without highest or lowest weight over the \( N = 2 \) Ramond algebra \( \mathfrak{g} \), and then get the main theorem.

**Theorem 5.1.** Let \( V \) be a simple weight module with finite dimensional weight spaces over \( \mathfrak{g} \). If \( V \) is not a highest and lowest module, then \( V \) is cuspidal.

**Proof.** Suppose that \( V = \sum_{k \in \mathbb{Z}} V_k \) is an simple quasi-finite \( \mathfrak{g} \)-module without highest and lowest weights. We shall prove that for any \( n \in \mathbb{Z}^*, k \in \mathbb{Z} \),
\[
L_n|v_k \oplus L_{n+1}|v_k \oplus H_n|v_k \oplus G^+_n|v_k \oplus G^-_n|v_k : V_k \to V_{k+n} \oplus V_{k+n} \oplus V_{k+n+1}
\]
is injective. In particular, by taking \( n = -k \), we obtain that \( \dim V_k \) is cuspidal.

In fact, suppose there exists some \( v_0 \in V_k \) such that
\[
L_n v_0 = L_{n+1} v_0 = G^+_n v_0 = H_n v_0 = 0.
\]
Without loss of generality, we can suppose \( n > 0 \). Note that when \( \ell \gg 0 \), we have
\[
\ell = n_1n + n_2(n + 1)
\]
for some \( n_1, n_2 \in \mathbb{N} \). From this and the relations in the definition, one can easily deduce that \( L_\ell, G^+_\ell, H_\ell \) can be generated by \( L_n, L_{n+1}, G^+_n, H_n \). Therefore there exists some \( N_0 > 0 \) such that
\[
L_\ell v_0 = G^+_\ell v_0 = H_\ell v_0 = 0, \forall \ell \geq N_0.
\]
This means
\[
\mathfrak{g}_{[N_0, +\infty)} v_0 = 0,
\]
where $\mathfrak{g}_{(N_0, +\infty)} = \bigoplus_{l \geq N_0} \mathfrak{g}_l$.

Clearly $V = U(\mathfrak{g})v_0$ and $V_k = U(\mathfrak{g})_0 v_0$. For any $v \in V_k$, there exists $u \in U(\mathfrak{g})_0$ such that $v = uv_0$. Suppose that
$$u = \sum a_{-i_1, \ldots, -i_m, i_{m+1}, \ldots, i_n}X_{-i_1} \cdots Y_{-i_m}Z_{i_{m+1}} \cdots W_{i_n},$$
where $X, \ldots, Y, Z \cdots, W \in \{L, H, G^\pm\}, i_1, \ldots, i_n > 0$. Define the negative degree
$$n(v) := \max\{i_1 + \cdots + i_m \mid a_{-i_1, \ldots, -i_m, i_{m+1}, \ldots, i_n} \neq 0\}.$$
It is rational since the sum is finite. Choose a basis $\{v_1, v_2, \ldots, v_{d_k}\}$ of the finite dimensional space $V_k$, and set $n_k := \max\{n(v_i) \mid 1 \leq i \leq d_k\}$. Then we have
$$\mathfrak{g}_{(N, +\infty)} V_k = 0, \text{ where } N = N_0 + n_k. \quad (5.4)$$

Clearly $\mathfrak{g}_+$ is generated by $S := \{L_1, H_1, G^+_1, G^-_1\}$.

For any $i \geq 1$, set $U_{k+1} = SV_k = \sum_{x \in S} x V_k \subset V_{k+1}$. If $U_{k+1} = 0$, then there exists a nonzero element $v \in U_k$ such that $Sv = 0$ and then $\mathfrak{g}_+ v = 0$. So $v$ is a highest weight vector, and $V$ is a highest weight module.

Now we can suppose that $U_{k+1} \neq 0$. For any $l \geq N$, we have $\mathfrak{g}_l U_{k+1} = \sum_{x \in S} [\mathfrak{g}_l, x] V_k + \sum_{x \in S} x \mathfrak{g}_l V_k \subset \mathfrak{g}_{l+1}V_k + S\mathfrak{g}_lV_k = 0$.

Then we get
$$\mathfrak{g}_{(N, +\infty)} U_{k+1} = 0. \quad (5.5)$$

Repeat the above step, we get $U_{k+i+1} = SU_{k+i}$ for any $i \geq 0$, and
$$\mathfrak{g}_{(N, +\infty)} U_{k+i} = 0, \forall i \geq 0. \quad (5.6)$$

For any $1 \leq j < N$, $z \in \mathfrak{g}_{N-j}$, there exists $y \in \mathfrak{g}_N$ such that $z = [y, L_{-j}]$. Then $z U_{k+j} = [y, L_{-j}] U_{k+j} = y L_{-j} U_{k+j} - L_{-j} y U_{k+j} \subset y V_k - L_{-j} y U_{k+j} = 0$.

So
$$\mathfrak{g}_{(N-j, +\infty)} U_{k+j} = 0, \forall 1 \leq j < N. \quad (5.7)$$

Then $\mathfrak{g}_+ U_{k+N} = 0$ for some $U_{k+N} \neq 0$.

Since the finite-dimensional subalgebra $\mathfrak{a} := \text{span}_C \{L_0, H_0, C\}$ is commutative, there exists a common eigenvector $w$ of $\mathfrak{a}$ in $U_{k+N}$. Replaced $w$ by $G^+_0 w$ if $G^+_0 w \neq 0$, $w$ becomes a highest weight vector of $\mathfrak{g}$. This contradicts the assumption of the theorem. \hfill \Box

Remark 5.2. The above proof is also suitable for the $N = 1$ Ramond algebra $\mathfrak{s}$.

Now we get the main result of this paper by combining with Theorem 5.1 and Proposition 4.1.

Theorem 5.3. (Main Theorem) Let $V$ be a simple weight $\mathfrak{g}$-module with finite dimensional weight spaces. Then $V$ is a highest weight module, a lowest weight module, or $\mathcal{R}'_{a,b,c}$ for some $a, b, c \in \mathbb{C}$ (up to parity-change).
6. Application to the $N = 1$ superconformal algebra

As an application of the main results, we shall classify all simple quasi-finite modules over the $N = 1$ superconformal algebra. The main method is to regard the $N = 1$ superconformal algebra as a subalgebra of the corresponding $N = 2$ superconformal algebra. The key point is that the $A$-cover for the Virasoro algebra in [3] can be extended to the super Virasoro algebra.

6.1. $A$-cover of $\mathfrak{s}$-modules. Let $A = \mathbb{C}[t, t^{-1}]$ be the algebra of Laurent polynomials. Then the $N = 1$ Ramond algebra $\mathfrak{s}$ has a natural structure of an $A$-module by $t^n L_m = L_{m+n}$ and $t^n G_r = G_{n+r}$ for all $m, n, r \in \mathbb{Z}$.

Definition 6.1. Let $M$ be an $\mathfrak{s}$-module. An $A$-cover of $M$ is the subspace $\hat{M} \subset \text{Hom} (A, M)$ spanned by the set $\{ \psi(x, u) \mid x \in \mathfrak{s}, u \in M \}$, where $\psi(x, u) \in \text{Hom} (A, M)$ is given by

$$\psi(x, u)(f) = (fx)u, \quad \text{for } f \in A.$$  

Proposition 6.2. (1) $\hat{M}$ is an $\mathfrak{s}$-module and an $A$-module by the following actions of $\mathfrak{s}$ and $A$:

$$y \psi(x, u) = \psi([y, x], u) + (-1)^{|x||y|} \psi(x, yu), \quad (6.1)$$

$$g \psi(x, u) = \psi(gx, u), \quad \text{for } x, y \in \mathfrak{s}, u \in M, g \in A. \quad (6.2)$$

(2) If $M$ is a weight module, then so is $\hat{M}$.

(3) $\pi(\hat{M}) = \mathfrak{s} M$.

Proof. For any $x, y, z \in \mathfrak{s}$, $f, g \in A$ and $u \in M$,

$$(yz - (-1)^{|y||z|}zy) \psi(x, u)) = y(z \psi(x, u)) - (-1)^{|y||z|}z(y \psi(x, u))$$

$$= y(\psi([z, x], u) + (-1)^{|x||z|} \psi(x, zu)) - (-1)^{|y||z|}z(\psi([y, x], u) + (-1)^{|x||y|} \psi(x, yu))$$

$$= \psi([y, [z, x]], u) + (-1)^{(|z|+|x|)|y|} \psi([z, x], yu)$$

$$+ (-1)^{|x||z|} \psi([y, x], z u) + (-1)^{|x||z|+|u|} \psi(x, y z u)$$

$$- (-1)^{|y||z|} \left( \psi([z, [y, x]], u) + (-1)^{(|y|+|x|)|z|} \psi([y, x], z u) \right)$$

$$+ (-1)^{|x||y|} \psi([z, x], y u) + (-1)^{|x||y||z|} \psi(x, z y u) \right)$$

$$= \psi([y, z, x], u) + (-1)^{|x||y||z|} \psi(x, [y, z]u)$$

$$= [y, z] \psi(x, u).$$

For the action of $A$ we have

$$f(\psi(x, u)) = f \psi(gx, u) = \psi(fgx, u).$$

Let us prove the second part of the proposition. If $M$ is a weight module, then $\hat{M}$ is spanned by $\psi(x, u)$ with $x$ and $u$ being eigenvectors with respect to the action of
the Cartan subalgebra of $\mathfrak{s}$. Then (6.1) shows that $\psi(x, u)$ is also an eigenvector for the action of $L_0$. Since $\hat{M}$ is spanned by its weight vectors, it is a weight module.

Finally, for part (3) we define $\pi : \hat{M} \to M$ by $\pi(\psi(x, u)) := \psi(x, u)(1) = xu$, which implies the claim. □

Remark 6.3. From Proposition 6.2 we see that the $\mathfrak{A}$-cover $\hat{M}$ can be viewed as a module over the twisted Heisenberg-Virasoro algebra $\mathfrak{t}$ (here $C = 0$) if we identify the action of $H_m$ on $\hat{V}$ and $\mathfrak{s}$ with that of $\mathfrak{t}_m$. In fact,

$$(L_m H_n - H_n L_m) \psi(x, u)$$

$$= L_m \psi(H_n x, u) - H_n \psi([L_m, x], u) + H_n \psi(x, L_m u)$$

$$= \psi([L_m, H_n x], u) - \psi(H_n x, L_m u) - \psi(H_n [L_m, x], u) + \psi(H_n x, L_m u)$$

$$= \psi([L_m, H_n x] - H_n [L_m, x], u)$$

$$= [L_m, H_n] \psi(x, u).$$

The last equality is due to that $[L_m, H_n x] - H_n [L_m, x] = [L_m, H_n] x$ for $x = L_k$ or $G_k$ for any $k \in \mathbb{Z}$.

Theorem 6.4. Let $M$ be a cuspidal $\mathfrak{s}$-module satisfying $\mathfrak{s} M = M$. Then there exists a cuspidal $\mathfrak{s}$-module and $\mathfrak{t}$-module $\hat{M}$ and a surjective homomorphism of $\mathfrak{s}$-modules:

$$\hat{M} \to M.$$

Proof. Let $\hat{M}$ be the $\mathfrak{A}$-cover constructed as above, then $\hat{M}$ is both of an $\mathfrak{s}$-module, and of a $\mathfrak{t}$-module by Remark 6.3. Clearly

$$\hat{M} = M_1 + M_2,$$

where $M_1 = \{ \psi(x, u) \mid x \in \text{Vir}, u \in M \}$ is the $\mathfrak{A}$-cover constructed in [3], and $M_2 = \{ \psi(y, u) \mid y \in G, u \in M \}$, where $G = \text{span} \mathbb{C} \{ G_r \mid r \in \mathbb{Z} \}$ is a vector space over $\mathbb{C}$. By Theorem 4.9 in [3], $M_1$ is cuspidal. For any $r \in \mathbb{Z}^*$, $u \in M$, by (6.1) we see that

$$\psi(G_r, u) = \frac{2}{r} \psi([G_0, L_r], u) = \frac{2}{r} (G_0 \psi(L_r, u) - \psi(L_r, G_0 u)).$$

(6.4)

So $M_2$ is also cuspidal. The left follows from Proposition 6.2. □

6.2. Cuspidal $\mathfrak{s}$-modules. The centerless $N = 2$ Ramond algebra $\mathfrak{g}$ (in the case of $C = 0$) is isomorphic to the following Lie superalgebra $\text{span} \mathbb{C} \{ L_m, J_m, G_m, G_m^c \mid
Suppose that $n \in \mathbb{Z}$, with the following relations:

\[
[L_m, L_n] = (n - m)L_{m+n}, \quad [L_m, J_n] = nJ_{m+n}, \quad (6.5)
\]

\[
[L_m, G_r] = (r - \frac{m}{2})G_{r+m}, \quad [G_r, G_s] = 2L_{r+s}, \quad (6.6)
\]

\[
[L_m, G_r^c] = (r - \frac{m}{2})G_{r+m}^c, \quad (6.7)
\]

\[
[G_r^c, G_s^c] = -2L_{r+s}, \quad (6.8)
\]

\[
[J_m, G_r] = -G_{r+m}^c, \quad (6.9)
\]

\[
[J_m, G_r^c] = G_{m+r}, \quad (6.10)
\]

\[
[G_r, G_s^c] = (r - s)J_{r+s}, \quad (6.11)
\]

for $m, n, r, s \in \mathbb{Z}$. The isomorphism is given by $H_m = -\sqrt{-1}J_m$, $G^+_r = \frac{1}{\sqrt{2}}(G_r + \sqrt{-1}G^c_r)$ and $G^-_r = \frac{1}{\sqrt{2}}(G_r - \sqrt{-1}G^c_r)$ for all $m, r \in \mathbb{Z}$.

Lemma 6.5. Let $V$ be a simple cuspidal $\mathfrak{s}$-module. Then $V$ can induces a cuspidal $\mathfrak{g}$-module $\hat{V}$ and a surjective homomorphism of $\mathfrak{s}$-modules: $\hat{V} \to V$.

Proof. Suppose that $V$ is a simple $\mathfrak{s}$-module. Clearly $C = 0$ and $\mathfrak{s}V = V$.

Clearly, the subalgebra $\text{span}_\mathbb{C}\{L_n, J_n \mid n \in \mathbb{Z}\}$ is isomorphic to the centerless twisted Heisenberg-Virasoro algebra $\mathfrak{t}$. By Theorem 6.4, there exists a cuspidal $\mathfrak{s}$-module and $\mathfrak{t}$-module $\hat{V}$ and a surjective homomorphism of $\mathfrak{s}$-modules $\pi : \hat{V} \to V$.

Set

\[
G_r^c v := [G_{r-1}, J_1] v \quad (6.12)
\]

for any $r \in \mathbb{Z}, v \in \hat{V}$.

For any $r, s \in \mathbb{Z}, m, n \in \mathbb{Z}$, we shall show that (6.7)-(6.11) hold on $\hat{V}$:

Due to $\left[L_m, G_r^c\right] = [L_m, [G_{r-1}, J_1]] = [[L_m, G_{r-1}], J_1] + [G_{r-1}, [L_m, J_1]] = (r - \frac{m}{2})G_{r+m}^c$, (6.7) holds.

Similarly,

\[
\left[G_r, J_m\right] = [G_r, [L_{m-1}, J_1]] = [[G_r, L_{m-1}], J_1] + [L_{m-1}, [G_r, J_1]]
= \left(\frac{m-1}{2} - r\right)[G_{r+m-1}, J_1] + [L_{m-1}, G_{r+1}^c] = G_{m+r}^c.
\]

Then (6.9) is obtained.

\[
\left[G_r, G_s^c\right] = [G_r, [G_{s-n}, J_n]] = [[G_r, G_{s-n}], J_n] - [G_{s-n}, [G_r, J_n]]
= (-2)[L_{r+s-n}, J_n] - [G_{s-n}, G_{r+n}^c].
\]

Setting $n = s - r$, we get (6.11).

Suppose that $[G_r^c, G_s^c] = a_{r,s}L_{r+s} + b_{r,s}J_{r+s}$, then by the action of $L_n$ we get

\[
(r + s - n)a_{r,s} = (r - \frac{n}{2})a_{r+n,s} + (s - \frac{n}{2})a_{r,s+n}, \quad (6.13)
\]

\[
(r + s)b_{r,s} = (r - \frac{n}{2})b_{r+n,s} + (s - \frac{n}{2})b_{r,s+n}. \quad (6.14)
\]
By (6.13) and (6.14) we get \( a_{r,s} = -2a \) and \( b_{s,s} = 0 \) for some \( a \in \mathbb{C} \) and for all \( r, s \in \mathbb{Z} \). Then

\[
[J_m, G_s^c] = \frac{1}{m - 2s}[[G_m - s, G_s^c], G_s^c]
\]

\[
= \frac{1}{m - 2s}[G_m - s, [G_s^c, G_s^c]] + \frac{1}{m - 2s}[G_s^c, [G_m - s, G_s^c]]
\]

\[
= \frac{1}{m - 2s}[G_m - s, -2aL_{2s}] + \frac{1}{m - 2s}[G_s^c, (m - 2s)J_m]
\]

\[
= 2aG_{m+s} - [J_m, G_s^c].
\]

Then

\[
[J_m, G_s^c] = aG_{m+s}.
\] (6.15)

By (6.15), we have

\[
[G_r^c, G_s^c] = [G_r^c, [G_s^{-1}, J_1]]
\]

\[
= [[G_r^c, G_s^{-1}], J_1] - [G_s^{-1}, [G_r^c, J_1]]
\]

\[
= [(s - 1 - r)J_{r+s-1}, J_1] + a[G_{s-1}, G_{r+1}]
\]

\[
= -2aL_{r+s}.
\]

Set \( a = 1 \), (6.10) and (6.8) also hold. Then \( \hat{V} \) becomes a \( g \)-module. \( \square \)

**Proposition 6.6.** Let \( V \) be a simple cuspidal \( \mathfrak{s} \)-module. Then \( V \) is a simple weight module of the intermediate series.

**Proof.** We use the notation as above. Suppose that \( V \) is a simple \( \mathfrak{s} \)-module, then \( C = 0 \). By Lemma 6.5, there exist a cuspidal \( g \)-module \( \hat{V} \) and a surjective homomorphism of \( \mathfrak{s} \)-modules: \( \hat{V} \to V \).

Consider a composition series of \( g \)-submodules of \( \hat{V} \):

\[
0 = \hat{V}^{(0)} \subset \hat{V}^{(1)} \subset \cdots \subset \hat{V}^{(p)} = \hat{V},
\] (6.16)

with the quotients \( \hat{V}^{(i)}/\hat{V}^{(i)} \) being simple \( g \)-modules. Let \( s \) be the smallest integer such that \( \pi(\hat{V}^{(s)}) \neq 0 \). Since \( V \) is a simple \( \mathfrak{s} \)-module we have \( \pi(\hat{V}^{(s)}) = V \) and \( \pi(\hat{V}^{(s-1)}) = 0 \). This gives a surjective homomorphism of \( \mathfrak{s} \)-modules

\[
\pi : \hat{V}^{(s)}/\hat{V}^{(s-1)} \to V.
\] (6.17)

So \( \hat{V}^{(s)}/\hat{V}^{(s-1)} \cong \mathcal{R}_{a,b,c} \) for some \( a, b, c \in \mathbb{C} \) in (??) by Theorem 5.3, and then \( V \) is isomorphic to a simple quotient of \( \mathcal{R}_{a,b,c} \) (up to parity-change).

Now the proposition follows by the fact that \( W := \sum_{i \in \mathbb{Z}} \mathbb{C}v_i + \sum_{k \in \mathbb{Z}} \mathbb{C}u_k \) is an \( \mathfrak{s} \)-submodule of \( \mathcal{R}_{a,b,c} \), where \( u_k := v_k^+ + v_k^- \).

Combining with Proposition 6.6 and Remark 5.2, we can get the following result, which was given in [25] by much complicated calculations.
\textbf{Theorem 6.7.} \cite{25} Let $V$ be a simple $\mathfrak{s}$-module with finite dimensional weight spaces. Then $V$ is a highest weight module, a lowest weight module, or a module of the intermediate series.

The module of the intermediate series over the $N = 1$ Ramond algebra $\mathfrak{s}$ was given in \cite{25} (up to parity-change):

$$S_{a,b} := \sum_{i \in \mathbb{Z}} \mathbb{C}x_i + \sum_{k \in \mathbb{Z}} \mathbb{C}y_k$$

with

$$L_n x_i = (a + bn + i)x_{i+n}, \quad L_n y_k = (a + (b - \frac{1}{2})n + k)y_{k+n},$$

$$G_r x_i = y_{r+i}, \quad G_r y_k = (a + k + 2r(b - \frac{1}{2}))x_{r+k}$$

for all $n, i, r, k \in \mathbb{Z}$. Moreover $S_{a,b}$ is not simple if and only if $a = 0, b = 1$ or $a = b = \frac{1}{2}$.

\textbf{ACKNOWLEDGMENTS}

We gratefully acknowledge the partial financial support from the NNSF (Nos. 11871249, 11771142), the ZJNSF (No. LZ14A010001), the Shanghai Natural Science Foundation (No. 16ZR1425000) and the Jiangsu Natural Science Foundation (No. BK20171294). Part of this work was done during the author’s visiting the Chern Institute of Mathematics, Tianjin, China. The authors would like to thank the institute and Prof. Chengming Bai for their warm hospitality and support, and also deeply indebted to Prof. Rencai Lv for helpful discussions.

\textbf{REFERENCES}

[1] E. Arbarello, C. De Concini, V.G. Kac, C. Procesi, Moduli spaces of curves and representation theory, Commun. Math. Phys. 117 (1988), 1-36.
[2] T. Arakawa, Representation theory of superconformal algebras and the Kac-Roan-Wakimoto conjecture, Duke Math. J. 130 (2005), no. 3, 435-478.
[3] Y. Billig, V. Futorny, Classification of irreducible representations of Lie algebra of vector fields on a torus, J. Reine Angew. Math. 720 (2016), 199-216.
[4] W. Boucher, D. Friedan, and A. Kent. Determinant formulae and unitarity for the $N = 2$ superconformal algebras in two dimensions or exact results on string compactification. Phys. Lett. B, 172(3), (1986), 316-322.
[5] V. Chari and A. Pressley, Unitary representations of the Virasoro algebra and a conjecture of Kac, Compositio Mathematica, 67(1988), 315-342
[6] S. Cheng, N. Lam, Finite conformal modules over the $N = 2, 3, 4$ superconformal algebras. J. Math. Phys. 42(2)(2001), 906-933.
[7] V.K. Dobrev, Characters of the unitarizable highest weight modules over the $N = 2$ superconformal algebras, Phys. Lett. B, 186 (1987), 43-51.
[8] J. Fu, Q. Jiang, Y. Su, Classification of modules of the intermediate series over Ramond $N = 2$ superconformal algebras. J. Math. Phys. 48 (2007), no. 4, 043508, 15 pp.
[9] K. Iohara, Unitarizable highest weight modules of the $N = 2$ super Virasoro algebras: untwisted sectors. Lett. Math. Phys. 91(3) (2010), 289-305.
[10] K. Iohara, Y. Koga, Representation theory of Neveu-Schwarz and Ramond algebras I: Verma modules, Adv. Math. 177 (2003), 61-69.
[11] M. Gorelik, V. Kac, On simplicity of vacuum modules, Adv. Math. 211(2)(2017), 621-677.
[12] V. Kac, Some problems of infinite-dimensional Lie algebras and their representations, Lecture Notes in Mathematics, 933 (1982), 117-126. Berlin, Heidelberg, New York: Springer.
[13] V. Kac, Superconformal algebras and transitive group actions on quadrics, Commun. Math. Phys. 186, (1997) 233-252.
[14] V. Kac, J. van de Leuer, On classification of superconformal algebras, Strings 88, Singapore: World Scientific, 1988.
[15] I. Kaplansky, L. J. Santharoubane, Harish-Chandra modules over the Virasoro algebra, Infinite-dimensional groups with applications (Berkeley, Calif. 1984), 217–231, Math. Sci. Res. Inst. Publ., 4, Springer, New York, 1985.
[16] D. Liu, Classification of Harish-Chandra modules over some Lie algebras related to the Virasoro algebra, J. Algebra 447(2016), 548-559.
[17] D. Liu, C. Jiang, Harish-Chandra modules over the twisted Heisenberg-Virasoro algebra, J. Math. Phys. 49(1)(2008), 012901.
[18] R. Lv, K. Zhao, Classification of irreducible weight modules over higher rank Virasoro algebras, Adv. Math. 201(2)(2006), 630-656.
[19] R. Lu, K. Zhao, Classification of irreducible weight modules over the twisted Heisenberg-Virasoro algebra, Commun. Contemp. Math. 12 (2010), no. 2, 183-205.
[20] O. Mathieu, Classification of Harish-Chandra modules over the Virasoro Lie algebra, Invent. Math. 107(1992), 225-234.
[21] C. Martin, A. Piard, Nonbounded indecomposable admissible modules over the Virasoro algebra, Lett. Math. Phys. 23(1991), 319-324.
[22] C. Martinez and E. Zelmanov, Graded modules over superconformal algebras, Non-Associative and Non-Commutative Algebra and Operator Theory. Springer International Publishing, 2016, 41-53.
[23] V. Mazorchuk, K. Zhao, Classification of simple weight Virasoro modules with a finite-dimensional weight space, J. Algebra, 307(2007), 209-214.
[24] A. Schwimmer, N. Seiberg, Comments on the $N =2$, 3, 4 superconformal algebras in two dimensions, Phys. Lett. B, 184 (1987), 191-196.
[25] Y. Su, Classification of Harish-Chandra modules over the super-Virasoro algebras, Commun. Alg. 23(10)(1995), 3653-3675.
[26] Y. Su, A classification of indecomposable $sl_2(\mathbb{C})$-modules and a conjecture of Kac on irreducible modules over the Virasoro algebra, J. Alg. 161(1993), 33-46.
[27] Y. Su, Classification of quasifinite modules over the Lie algebras of Weyl type, Adv. Math. 174 (2003), 57-68.
[28] Y. Su, Classification of Harish-Chandra modules over the higher rank Virasoro algebras, Commun. Math. Phys. 240 (2003), 539-551.
Department of Mathematics, Huzhou University, Zhejiang Huzhou, 313000, China

E-mail address: liudong@zjhu.edu.cn

Department of Mathematics, Shanghai Normal University, Shanghai, 200234, China

E-mail address: pei@shnu.edu.cn

Institute of Applied System Analysis, Jiangsu University, Jiangsu Zhenjiang, 212013, China

E-mail address: xialimeng@ujs.edu.cn