S3 General SI Analysis

We now give an example of how the analysis in section S2 can be used to determine the optimal sampling scheme for maximizing the probability of detecting a vector borne disease that follows SI dynamics. In the following, we use system (1) (presented in the main text) to describe the dynamics of a disease. We consider the case where there is one vector population and one host population. Thus we consider the system

\[
\frac{d}{dt} I_V(t) = \beta_{V,H} \frac{I_H(t)}{N_H} (N_V - I_V(t)) \\
\frac{d}{dt} I_H(t) = \beta_{H,V} \frac{I_V(t)}{N_V} (N_H - I_H(t))
\]

where we denote the number of infected vectors by \(I_V(t)\), the number of infected hosts by \(I_H(t)\), the total number of vectors by \(N_V\), the total number of hosts by \(N_H\), the transmission rate from hosts to vectors by \(\beta_{V,H}\), and the transmission rate from vectors to hosts by \(\beta_{H,V}\).

For ease of notation, define

\[
V(t) = \frac{I_V(t)}{N_V} \quad \text{and} \quad H(t) = \frac{I_H(t)}{N_H}
\]

to be the proportion of the infected vector (host) population. Then rewriting the above system, we have that

\[
\frac{d}{dt} V(t) = \beta_{V,H} \frac{I_H(t)}{N_H} S_v(t) \\
\frac{d}{dt} H(t) = \beta_{H,V} \frac{I_V(t)}{N_V} \left( 1 - \frac{I_V(t)}{N_V} \right)
\]

Similarly,

\[
\frac{d}{dt} V(t) = \beta_{V,H} H(t)(1 - V(t))
\]

Lastly, for ease of notation, redefine

\[
\beta_{V,H} = \alpha \quad \text{and} \quad \beta_{H,V} = \gamma.
\]

Then our system becomes

\[
\frac{d}{dt} V(t) = \alpha H(t)(1 - V(t)) \quad (S16a) \\
\frac{d}{dt} H(t) = \gamma V(t)(1 - H(t)) \quad (S16b)
\]

The model variables and parameters are summarized in Table 2 in Text S3.
Table 2 in Text S3. SI model parameters and variables.

| Parameter or Variable | Definition |
|-----------------------|------------|
| $I_H$                 | Number of infected hosts. |
| $S_h$                 | Number of susceptible hosts. |
| $I_V$                 | Number of infected vectors. |
| $S_v$                 | Number of susceptible vectors. |
| $N_H$                 | Total number of hosts. |
| $N_V$                 | Total number of vectors. |
| $H = \frac{I_H}{N_H}$ | Proportion of infected hosts. |
| $V = \frac{I_V}{N_V}$ | Proportion of infected vectors. |
| $\beta_{V,H} = \alpha$ | Transmission rate from hosts to vectors. |
| $\beta_{H,V} = \gamma$ | Transmission rate from vectors to hosts. |

S3.1 Basic analysis of \((S16)\)

It is easy to see that system \((S16)\) has two steady states, \((V, H) = (0, 0)\) and \((1, 1)\). Examining the vector field of \((S16)\), we see that

- If \(V = 0, 0 < H < 1\) \(\Rightarrow \frac{dV}{dt} = \alpha H > 0, \frac{dH}{dt} = 0\)
- If \(0 < V < 1, H = 0\) \(\Rightarrow \frac{dV}{dt} = 0, \frac{dH}{dt} = \gamma V > 0\)
- If \(0 < V < 1, H = 1\) \(\Rightarrow \frac{dV}{dt} = \alpha(1 - V) > 0, \frac{dH}{dt} = 0\)
- If \(V = 1, 0 < H < 1\) \(\Rightarrow \frac{dV}{dt} = 0, \frac{dH}{dt} = \gamma(1 - H) > 0\)
- If \(0 < V < 1, 0 < H < 1\) \(\Rightarrow \frac{dV}{dt} = \alpha H(1 - V) > 0, \frac{dH}{dt} = \gamma V(1 - H) > 0\)

Thus, given an initial condition \((V_0, H_0)\) such that \(V_0, H_0 \in [0, 1], (V_0, H_0) \notin \{(0, 0), (1, 1)\}\), the solution \((V(t), H(t))\) approaches \((1, 1)\) in infinite time. (Figure S1) Though this is not a realistic scenario, we are only concerned with the early-time behavior of the system and the infinite time dynamics are only of academic interest.

S3.2 First integral

Note that if \(V(t) > 0\), then \(\frac{dH}{dt} > 0\) for all \(0 \leq H(t) < 1\). Similarly, if \(H(t) > 0\), then \(\frac{dV}{dt} > 0\) for all \(0 \leq V(t) < 1\). Note that neither \(V\) nor \(H\) may become negative so long as \(V(0) \geq 0\) and \(H(0) \geq 0\). Then we may reparameterize our system \((S16)\) as a function of \(V\) or as a function of \(H\). We will choose to reparameterize our system as a function of \(H\). Dividing \((S16a)\) by \((S16b)\), we have the auxiliary equation

\[
\frac{dV}{dH} = \frac{\alpha H(t)(1 - V(t))}{\gamma V(t)(1 - H(t))} = \frac{\frac{\alpha}{V} - \alpha}{\frac{\gamma}{V} - \gamma}
\]  

\((S17)\)

which we can solve by separation of variables:
\[
dV \left( \frac{V}{\alpha - \alpha V} \right) = dH \left( \frac{H}{\gamma - \gamma H} \right)
\]

\[
\int \left( \frac{V}{\alpha - \alpha V} \right) dV = \int \left( \frac{H}{\gamma - \gamma H} \right) dH
\]

\[
\frac{1}{\alpha} (-V - \ln(1 - V)) = \frac{1}{\gamma} (-H - \ln(1 - H)) + c
\]

where \(c\) is some constant. Then solutions of (S16) lie within the level sets of the function

\[
I(V, H) = \frac{1}{\alpha} (-V - \ln(1 - V)) + \frac{1}{\gamma} (H + \ln(1 - H))
\]

In particular, given an initial condition \((V_0, H_0)\), the solution \((V, H)\) of the initial value problem (S16), \(V(0) = V_0, H(0) = H_0\) satisfies

\[
I(V, H) = I(V_0, H_0)
\]

\[
\frac{1}{\alpha} (-V - \ln(1 - V)) + \frac{1}{\gamma} (H + \ln(1 - H)) = \frac{1}{\alpha} (-V_0 - \ln(1 - V_0)) + \frac{1}{\gamma} (H_0 + \ln(1 - H_0))
\]

that is, the solution \((V, H)\) lies in the level set

\[
\{(V, H) | I(V, H) = I(V_0, H_0)\}
\]

### S3.3 Optimal sampling

Suppose that \(C(s_V, s_H)\) is a strictly increasing cost function where \(s_V\) denotes the number of vectors sampled and \(s_H\) denotes the number of hosts sampled. Our goal is to find possible optimal sampling schemes \(s^* = (s^*_V, s^*_H)\) that maximize the probability of detecting a disease in a single sampling trial at a fixed time \(t\), assuming that the vector and host population dynamics are known. With reference to Table 1 in Text S2, we see that there are three possible sampling schemes. First (Case 2 in Table 1 in Text S2), if there exists some \(s^*\) such that \(s^*_V \geq 0\) and \(s^*_H \geq 0\) and

\[
\frac{V(t)}{C_{s_V}(s^*)} = \frac{H(t)}{C_{s_H}(s^*)}
\]

then we may choose to sample both the vector and the host populations. Second (Case 4 in Table 1 in Text S2), if there exists some \(s^*\) such that \(s^*_V \geq 0\) and \(s^*_H = 0\) and

\[
\frac{V(t)}{C_{s_V}(s^*)} > \frac{H(t)}{C_{s_H}(s^*)}
\]

then we may choose to sample only the vector population. Third (Case 4 in Table 1 in Text S2), if there exists some \(s^*\) such that \(s^*_V = 0\) and \(s^*_H \geq 0\) and

\[
\frac{V(t)}{C_{s_V}(s^*)} < \frac{H(t)}{C_{s_H}(s^*)}
\]

then we may choose to sample only the host population.
then we may choose to sample only the host population. Since each of these cases depends on the ratio \( \frac{V}{H} \), we now characterize this curve. We will first restate two useful equations and make some easy observations. Then, we will give two lemmas that elucidate some properties of the curve \( \frac{V(H)}{H} \).

First, by (S17),

\[
\frac{dV}{dH} = \frac{\alpha H (1 - V)}{\gamma V (1 - H)} \tag{S18}
\]

Then \( \frac{dV}{dH} > 0 \) for all \( (V, H) \in [0, 1] \times [0, 1] \setminus \{(0, 0), (1, 1)\} \). Since \( V(H) \) is increasing and \( V(H) \leq 1 \) for \( H \in (0, 1) \), \( \lim_{H \to 1} V(H) \) exists. We claim that for \( (V_0, H_0) \in [0, 1] \times [0, 1] \setminus \{(0, 0)\} \), \( \lim_{H \to 1} V(H) = 1 \). If not, then there exists some \( 0 < M < 1 \) such that \( V_0, H_0 \) is positive, and let \( (V, H) \) be the solution to this initial value problem. Let \( I(V_0, H_0) = c \). Then recall that \( (V, H) \) solves

\[
\frac{1}{\alpha} (-V - \ln(1 - V)) + \frac{1}{\gamma} (H + \ln(1 - H)) = c. \tag{S19}
\]

Note that

\[
V_0 = 0, \ H_0 > 0 \Rightarrow I(V_0, H_0) < 0
\]

\[
V_0 = 0, \ H_0 = 0 \Rightarrow I(V_0, H_0) = 0
\]

\[
V_0 > 0, \ H_0 = 0 \Rightarrow I(V_0, H_0) > 0
\]

by (S19). Then, since at most one of \( V_0 \) or \( H_0 \) is positive, any initial condition \( (V_0, H_0) \) must satisfy exactly one of the above conditions. Since the sign of \( I(V_0, H_0) \) implied in the above relations is unique for each class of initial condition \( (V_0, H_0) \), we have that

\[
V_0 = 0, \ H_0 > 0 \Leftrightarrow I(V_0, H_0) < 0
\]

\[
V_0 = 0, \ H_0 = 0 \Leftrightarrow I(V_0, H_0) = 0
\]

\[
V_0 > 0, \ H_0 = 0 \Leftrightarrow I(V_0, H_0) > 0.
\]

We now prove two lemmas that are useful in characterizing the curve \( \frac{V(H)}{H} \).

**Lemma 1.** Under the following conditions, there exists some unique \( H^* \in (H_0, 1) \) that solves \( V(H) = H^* \):

1. \( \alpha < \gamma \) and \( I(V_0, H_0) = c > 0 \) or
2. \( \alpha > \gamma \) and \( I(V_0, H_0) = c < 0 \).

Otherwise, there exists no such \( H^* \).

**Proof.** Suppose that

\[
\left( \frac{1}{\alpha} - \frac{1}{\gamma} \right) (-H^* - \ln(1 - H^*)) = c. \tag{S20}
\]

Then by (S19),

\[
-H^* - \ln(1 - H^*) = -V(H^*) - \ln(1 - V(H^*)).
\]

Since the function \( F(x) = -x - \ln(1 - x) \) is strictly increasing for \( x \in (0, 1) \), the above equation implies that \( V(H^*) = H^* \). We now show that there exists some \( H^* \in (0, 1) \) that solves (S20).

Note that the function \( F(x) = -x - \ln(1 - x) \) is positive and strictly increasing for \( x \in (0, 1) \). In addition, \( F(0) = 0 \) and \( \lim_{H \to 1} F(H) = \infty \). Then, since \( c, \alpha \) and \( \gamma \) are constants, there exists
some unique $H^* \in (0, 1)$ that solves (S20) if and only if $c \neq 0$ and the sign of $c$ is the same as the sign of $\left( \frac{1}{\alpha} - \frac{1}{\gamma} \right)$. It remains only to be shown that $H^* \in (H_0, 1)$.

If Condition (1) in Lemma 1 holds, then $H_0 = 0$ and $H^* \in (H_0, 1) = (0, 1)$ trivially. If Condition (2) in Lemma 1 holds, then $H_0 > 0$, $V_0 = 0$,

$$0 > \left( \frac{1}{\alpha} - \frac{1}{\gamma} \right) (-H_0 - \ln(1 - H_0)) > \frac{1}{\gamma} (-H_0 - \ln(1 - H_0)) = c$$

by (S19) and

$$\lim_{H \to 1} \left( \frac{1}{\alpha} - \frac{1}{\gamma} \right) (-H - \ln(1 - H)) = -\infty.$$  

Then there exists some $H^* \in (H_0, 1)$ which solves (S20). 

Lemma 1 gives conditions under which the curve $V(H)$ intersects the horizontal line at 1.

Note that if there exists some $H^* \in (H_0, 1)$ such that $V(H^*) = H^*$, then $\frac{V(H^*)}{H^*} = 1$. If no such $H^*$ exists, then the curve $\frac{V(H)}{H}$ must remain above or below the horizontal line at 1 for all $H \in (H_0, 1)$.

Lemma 2. Suppose that $H < V(H)$ for all $H \in (H_1, H_2) \subseteq (H_0, 1)$, $\alpha > \gamma > 0$. Then there exists at most one $\bar{H} \in (H_1, H_2)$ such that $\left. \frac{dV(H)}{dH} \right|_{H = \bar{H}} = 0$.

Proof. Note that

$$\frac{d}{dH} \frac{V(H)}{H} = \frac{H \frac{dV}{dH} - V}{H^2} = \frac{\alpha H \left( \frac{1}{V} - 1 \right) - \gamma V \left( \frac{1}{H} - 1 \right)}{\gamma H (1 - H)} \tag{S21}$$

and suppose $\bar{H} \in (H_1, H_2)$ such that $\left. \frac{dV(H)}{dH} \right|_{H = \bar{H}} = 0$. Since the denominator of (S21) is strictly positive, it must be true that

$$\alpha \bar{H} \left( \frac{1}{V(\bar{H})} - 1 \right) = \gamma V(\bar{H}) \left( \frac{1}{\bar{H}} - 1 \right) \tag{S22}$$

Now,

$$\frac{d}{dH} \frac{1 - V(\bar{H})}{V^2(\bar{H})} = \frac{1}{V^2} \left[ -V^2 \frac{dV}{dH} - 2V(1 - V) \frac{dV}{dH} \right]$$

by (S18) and

$$\frac{d}{dH} \frac{1 - H}{H^2} = \frac{H - 2}{H^3}.$$  

We claim that

$$\frac{d}{dH} \frac{1 - V(\bar{H})}{V^2(\bar{H})} > \frac{d}{dH} \frac{1 - H}{H^2} \tag{S23}$$

for $H \in (H_1, H_2)$. (S23) holds if and only if
\[
\frac{\alpha(V - 2)H(1 - V)}{\gamma V^4(1 - H)} > \frac{H - 2}{H^3}, \\
\frac{\alpha(V - 2)(1 - V)}{V^4} > \frac{\gamma(H - 2)(1 - H)}{H^4}.
\]
Since \(\frac{(x-2)(1-x)}{x^2}\) is an increasing function for \(x \in [0, 1]\) and since \(H < V\) we have that
\[
\frac{(V - 2)(1 - V)}{V^4} > \frac{(H - 2)(1 - H)}{H^4}
\]
since \(\alpha > \gamma\). Thus (S23) holds. Since \(\alpha > \gamma > 0\),
\[
\frac{d}{dH} \alpha \left( \frac{1 - V(H)}{V^2(H)} \right) > \frac{d}{dH} \gamma \left( \frac{1 - H}{H^2} \right)
\]
for all \(H \in (H_1, H_2)\). Then if there exists some \(\tilde{H} \in (H_1, H_2) \subseteq (0, 1)\) such that (S22) holds, it is unique.

Assuming that the disease starts in either the vector population or the host population (not both), there are six possible characterizations of \(\frac{V(H)}{H}\).

Case 1: \(\alpha > \gamma, V_0 > 0, H_0 = 0\). Since \(V_0 > 0\) and \(H_0 = 0\), we have that \(c > 0\). Note that
\[
\frac{V(H_0)}{H_0} = \frac{V_0}{H_0} = \infty.
\]
Then since \(\alpha > \gamma\), \(\frac{V(H)}{H} > 1\) for all \(H \in (H_0, 1)\) by Lemma 1.

We claim that \(\frac{d}{dH} \frac{V(H)}{H} \leq 0\) for all \(H \in (H_0, 1)\). Indeed, if not, then there exists some \(\tilde{H} \in (H_0, 1)\) such that \(\frac{d}{dH} \frac{V(H)}{H} \big|_{H=\tilde{H}} > 0\). Note that since \(V_0 > 0\) and \(H_0 = 0\), \(\frac{d}{dH} \frac{V(H)}{H} \big|_{H=H_0} = -\infty\).

Then there must exist some \(\bar{H} \in (H_0, \tilde{H})\) such that \(\frac{d}{dH} \frac{V(H)}{H} \big|_{H=\bar{H}} = 0\). Since \(\frac{d}{dH} \frac{V(H)}{H} > 1\) for all \(H \in (H_0, 1)\) by the above argument and since \(\tilde{H} \in (H_0, 1)\), we have that \(\frac{V(H)}{H} \big|_{H=H_0} = 1\). Since \(\lim_{H \to 1} \frac{V(H)}{H} = 1\), it must be true that \(\frac{V(H)}{H}\) is decreasing for some \(H > \bar{H}\) and therefore that there exists some \(\tilde{H}_2 \in (\bar{H}, 1)\) such that \(\frac{d}{dH} \frac{V(H)}{H} \big|_{H=\tilde{H}_2} = 0\). This contradicts the uniqueness of \(\tilde{H}\) by Lemma 2. Thus, the claim holds.

Case 2: \(\alpha > \gamma, V_0 = 0, H_0 > 0\). Since \(V_0 = 0\) and \(H_0 > 0\), we have that \(c < 0\). Note that
\[
\frac{V(H_0)}{H_0} = \frac{V_0}{H_0} = 0.
\]
By Lemma 1 there exists some unique \(H^* \in (H_0, 1)\) such that \(\frac{V(H^*)}{H^*} = 1\). Then \(\frac{V(H)}{H} < 1\) for \(H \in (H_0, H^*)\). We claim that \(\frac{V(H)}{H} > 1\) for \(H \in (H^*, 1)\). If not, then \(V(H) \leq H\) for all \(H \in (H_0, 1)\) and by (S21)
\[
\frac{d}{dH} \frac{V(H)}{H} = \frac{\alpha H \left( \frac{1}{H} - 1 \right) - \gamma V \left( \frac{1}{H} - 1 \right)}{\gamma H(1 - H)} \\
\geq \frac{\alpha H \left( \frac{1}{H} - 1 \right) - \gamma H \left( \frac{1}{H} - 1 \right)}{\gamma H(1 - H)} = \frac{\alpha - \gamma}{\gamma H} > 0
\]
(S24)
since \(\alpha > \gamma > 0\). Then since \(\frac{V(H^*)}{H^*} = 1\), there must exist some \(\bar{H} \in (H^*, 1)\) such that \(\frac{V(H)}{H} > 1\), a contradiction to that \(V(H) \leq H\) for all \(H \in (H_0, 1)\). Thus the claim holds.

Now we examine the sign of \(\frac{d}{dH} \frac{V(H)}{H}\). First, we claim that \(\frac{d}{dH} \frac{V(H)}{H} > 0\) for all \(H \in (H_0, H^*)\). Note that for \(H \in (H_0, H^*)\), \(H \geq V(H)\). Then the claim holds by (S24).

Next we claim that there exists some \(H^* \in (H^*, 1)\) such that \(\frac{d}{dH} \frac{V(H)}{H} > 0\) for all \(H \in (H_0, H^*)\) and \(\frac{d}{dH} \frac{V(H)}{H} < 0\) for all \(H \in (\bar{H}, 1)\). Indeed, by the above argument, \(\frac{d}{dH} \frac{V(H)}{H} \big|_{H=H^*} = 0\). Recall that \(\frac{V(H^*)}{H^*} = 1\). Then there exists some \(H_1 \in (H^*, 1)\) such that \(\frac{V(H_1)}{H_1} > 1\). Then since \(\lim_{H \to 1} \frac{V(H)}{H} = 1\)
there must exist some $H_2 \in (H_1, 1)$ such that $\frac{d}{dH} \frac{V(H)}{H} \bigg|_{H=H_2} < 0$. Then by the continuity of $\frac{d}{dH} \frac{V(H)}{H}$ there exists some $\bar{H} \in (H^*, H_2) \subset (H^*, 1)$ such that $\frac{d}{dH} \frac{V(H)}{H} \bigg|_{H=\bar{H}} = 0$. By Lemma 2 this $\bar{H}$ is unique. Thus the claim holds.

Case 3: $\alpha = \gamma$, $V_0 > 0$, $H_0 = 0$. Since $V_0 > 0$ and $H_0 = 0$, we have that $c > 0$. Note that

$\frac{V(H_0)}{H_0} = \frac{V_0}{H_0} = \infty$. Then since $\alpha = \gamma$, $\frac{V(H)}{H} > 1$ for all $H \in (H_0, 1)$ by Lemma 1.

We claim that $\frac{d}{dH} \frac{V(H)}{H} < 0$ for all $H \in (H_0, 1)$. Since $V(H) > H$ for all $H \in (H_0, 1)$,

$$\alpha H \left( \frac{1}{V} - 1 \right) - \gamma V \left( \frac{1}{H} - 1 \right) < \alpha H \left( \frac{1}{H} - 1 \right) - \gamma H \left( \frac{1}{H} - 1 \right) = \alpha - \gamma = 0.$$ 

then the claim holds by (S21).

Case 4: $\alpha = \gamma$, $V_0 = 0$, $H_0 > 0$. Since $V_0 = 0$ and $H_0 > 0$, we have that $c < 0$. Note that

$\frac{V(H_0)}{H_0} = \frac{V_0}{H_0} = 0$. Then since $\alpha = \gamma$, $\frac{V(H)}{H} < 1$ for all $H \in (H_0, 1)$ by Lemma 1. By an argument similar to that in Case 3, $\frac{d}{dH} \frac{V(H)}{H} > 0$ for all $H \in (H_0, 1)$.

Case 5: $\alpha < \gamma$, $V_0 > 0$, $H_0 = 0$. Since $V_0 > 0$ and $H_0 = 0$, we have that $c > 0$. Note that

$\frac{V(H_0)}{H_0} = \frac{V_0}{H_0} = \infty$. By Lemma 1 there exists some unique $H^* \in (H_0, 1)$ such that $\frac{V(H^*)}{H^*} = 1$. By an argument similar to that in Case 3, we have that $\frac{d}{dH} \frac{V(H)}{H} < 1$ for $H \in (H^*, 1)$ and there exists some $\bar{H} \in (H^*, 1)$ such that $\frac{d}{dH} \frac{V(H)}{H} < 0$ for all $H \in (H_0, \bar{H})$ and $\frac{d}{dH} \frac{V(H)}{H} > 0$ for all $H \in (\bar{H}, 1)$.

Case 6: $\alpha < \gamma$, $V_0 = 0$, $H_0 > 0$. Since $V_0 = 0$ and $H_0 > 0$, we have that $c < 0$. Note that

$\frac{V(H_0)}{H_0} = \frac{V_0}{H_0} = 0$. Then since $\alpha < \gamma$, $\frac{V(H)}{H} < 1$ for all $H \in (H_0, 1)$ by Lemma 1. By an argument similar to that in Case 1, $\frac{d}{dH} \frac{V(H)}{H} \geq 0$ for all $H \in (H_0, 1)$.

We summarize the above cases in Table 3 in Text S3 and illustrate them in Figures S2 and 1.

Table 3 in Text S3. Summary of possible characterizations of $\frac{V(H)}{H}$ with all possible cases listed.
S3.3.1 Linear cost function

Now we assume that our cost function is linear:

\[ C(s_V, s_H) = a_V s_V + a_H s_H. \]

Then

\[ C_{s_V} = b_V \quad \text{and} \quad C_{s_H} = b_H. \]

By the analysis in Section S3.3, it is clear that the optimal sampling design is determined by the relative magnitudes of \( \frac{V(H)}{H} \) and the constant \( \frac{b_V}{b_H} \). We consider only cases 1 and 2 above. The other cases follow similarly.

Case 1: \( \alpha > \gamma, V_0 > 0, H_0 = 0. \) By the above analysis, \( \frac{V(H)}{H} > 1 \) and \( \frac{d}{dH} \frac{V(H)}{H} \leq 0 \) for all \( H \in (H_0, 1) \). If \( \frac{b_V}{b_H} \leq 1 \), then

\[
\frac{V(H)}{H} > 1 \geq \frac{b_V}{b_H} = \frac{C_{s_V}}{C_{s_H}}
\]

for all \( H \in (H_0, 1) \), so by the analysis of Section S3.3, we choose to sample only the vector population. If \( \frac{b_V}{b_H} > 1 \), then since \( \lim_{V \to V_0, H \to H_0} \frac{V(H)}{H} = \infty \), there exists some \( \hat{H} \) such that

\[
\begin{cases}
\frac{V(H)}{V(\hat{H})} > \frac{b_V}{b_H} & H \in (H_0, \hat{H}) \\
\frac{V(H)}{V(\hat{H})} = \frac{b_V}{b_H} & H = \hat{H} \\
\frac{V(H)}{V(\hat{H})} > \frac{b_V}{b_H} & H \in (\hat{H}, 1)
\end{cases}
\]

Note that \( \hat{H} \) is unique by Lemma 2. Then by the analysis of Section S3.3, at early times in the epidemic (when \( H \in (H_0, \hat{H}) \)), we should sample only the vector population and at late times (when \( H \in (\hat{H}, 1) \)) we should sample only the host population. Additionally, there exists some intermediate instant (when \( H = \hat{H} \)) at which we should sample both the vector and host populations.

Case 2: \( \alpha > \gamma, V_0 = 0, H_0 > 0. \) By the above analysis, there exists some unique \( H^* \in (H_0, 1) \) such that

\[
\begin{cases}
\frac{V(H)}{V(H^*)} < 1 & H \in (H_0, H^*) \\
\frac{V(H)}{V(H^*)} = 1 & H = H^* \\
\frac{V(H)}{V(H^*)} > 1 & H \in (H^*, 1)
\end{cases}
\]

and there exists some unique \( \hat{H} \in (H^*, 1) \) such that

\[
\begin{cases}
\frac{d}{dH} \frac{V(H)}{H} > 0 & H \in (H_0, \hat{H}) \\
\frac{d}{dH} \frac{V(H)}{H} = 0 & H = \hat{H} \\
\frac{d}{dH} \frac{V(H)}{H} < 0 & H \in (\hat{H}, 1)
\end{cases}
\]

Then \( \frac{V(H)}{H} \) achieves a unique maximum at \( \hat{H} \in (H_0, 1) \). If

\[
\frac{V(\hat{H})}{\hat{H}} < \frac{b_V}{b_H}
\]

then

\[
\frac{V(H)}{H} < \frac{b_V}{b_H} = \frac{C_{s_V}}{C_{s_H}}
\]

for all \( H \in (H_0, 1) \). By the analysis in Section S3.3, we choose to sample only the host population. If

\[
\frac{V(\hat{H})}{\hat{H}} > \frac{b_V}{b_H}
\]
then there exist some $H_1, H_2 \in (H_0, 1)$, $H_1 < H_2$ such that

$$
\begin{align*}
\frac{V(H)}{V(H_1)} &< \frac{b_V}{b_W} \quad \forall \; H \in (H_0, H_1) \\
\frac{V(H_1)}{H} &= \frac{b_V}{b_W} \\
\frac{V(H)}{V(H_2)} &> \frac{b_V}{b_W} \quad \forall \; H \in (H_1, H_2) \\
\frac{V(H_2)}{H} &= \frac{b_V}{b_W} \\
\frac{V(H)}{H} &< \frac{b_V}{b_W} \quad \forall \; H \in (H_2, 1)
\end{align*}
$$

Then at early stages of the epidemic (while $H \in (H_0, H_1)$), we should sample only the host population, at intermediate times (while $H \in (H_1, H_2)$) we sample only the vector population, and at late times in the epidemic (while $H \in (H_2, 1)$) we return to sampling only host population. As in Case 1, if $H = H_1$ or $H = H_2$, then we should sample both the vector and host populations.

As in the main text, we find that there is a critical time at which we should switch our sampling scheme. We can solve for this critical time numerically.