ON STATE-CLOSED REPRESENTATIONS OF RESTRICTED
WREATH PRODUCT OF GROUPS OF TYPE $G_{p,d} = C_p \wr C^d$

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ABSTRACT. Let $G_{p,d}$ be the restricted wreath product $C_p \wr C^d$ where $C_p$ is a cyclic group of order a prime $p$ and $C^d$ a free abelian group of finite rank $d$. We study the existence of faithful state-closed (fsc) representations of $G_{p,d}$ on the 1-rooted $m$-ary tree for some finite $m$. The group $G_{2,1}$, known as the lamplighter group, admits a fsc representation on the binary tree. We prove that for $d \geq 2$ there are no fsc representations of $G_{p,d}$ on the $p$-adic tree. We characterize all fsc representations of $G = G_{p,1}$ on the $p$-adic tree where the first level stabilizer of the image of $G$ contains its commutator subgroup. Furthermore, for $d \geq 2$, we construct uniformly fsc representations of $G_{p,d}$ on the $p^2$-adic tree and exhibit concretely the representation of $G_{2,2}$ on the 4-tree as a finite-state automaton group.

1. INTRODUCTION

This paper is a study of representations of the restricted wreath product of groups of type $G_{p,d} = C_p \wr X$ where $C_p$ is a cyclic group of prime order $p$ and $X$ is a free abelian group of finite rank $d \geq 1$, as groups of automorphisms of 1-rooted regular $m$-trees satisfying the state-closed property (or, self-similarity in dynamics language); the representations are said to be of degree $m$. We let $C_p$ be generated by $a$, denote its the normal closure by $A$ and let $X$ be generated by $\{x_1, x_2, ..., x_d\}$.

The groups $G_{2,d}$ appeared as examples in the study of probabilistic properties of random walks on groups ([1], page 480). Among these examples, the group $G_{2,1}$ which goes by the picturesque name of lamplighter, admits a (classical) faithful self-similar representation on the 2-tree, as the state-closure of the tree automorphism $\xi = (\xi, \xi \alpha)$ where $\alpha$ is the transposition automorphism. Further interest in the lamplighter group arose from the calculation of its spectrum in [2] which was then used to disprove a conjecture about the range of $L_2$-Betti numbers of closed manifolds (see, [3]). Since then, several articles have appeared on generalizations of the lamplighter group ([4], [5], [6], [8], [7]).

An important type of representation of groups as automorphisms of trees is that of finite-state; or equivalently, representations by finite automata. It follows from a general technique called tree-wreathing introduced in [9] that the groups $G_{p,d}$ admit faithful finite-state representations of degree $p$, independently of $d$. In

Date: May11th, 2015.

2000 Mathematics Subject Classification. Primary20E08, 20F18.
Key words and phrases. Tree automorphisms, state-closed representations, wreath products, lamplighter group.

The first author acknowledges a doctoral scholarship from CNPq.
The second author thanks Tatiana Smirnova-Nagnibeda for a visit to Université de Genève and thanks Dmytro Savchuk for a visit to Southern Florida University both of which took place in 2014 and stimulated this work.
contrast, as we will prove, if $d \geq 2$, a necessary condition for the existence of faithful state-closed representations of $G_{p,d}$ is that the degree of the representation be a composite number.

State-closed representations of a general group $G$, are constructible from similarity pairs $(H, f)$ where $H$ is a subgroup of $G$ of finite index $m$ and $f$ is a homomorphism $H \to G$ called a virtual endomorphism of $G$. A similarity pair $(H, f)$ leads to a recursively defined representation $\varphi$ of $G$ as a group of automorphisms of the 1-rooted regular $m$-tree. The image $G^r$ is a state-closed group of automorphisms of the tree. The kernel of $\varphi$, called the $f$-core of $H$, is the largest subgroup $K$ of $H$ which is normal in $G$ and $f$-invariant (in the sense $Kf \leq K$). When the kernel of $\varphi$ is trivial, $f$ and the similarity pair $(H, f)$ are said to be simple. A typical example of a group with a simple similarity pair is that of the free abelian group $X = \langle x_1, x_2, \ldots, x_d \rangle$ of rank $d$ and pair $(Y, f)$ with $Y = \langle x_1^p, x_2, \ldots, x_d \rangle$ and homomorphism $f$ which is an extension of $x_1^p \to x_2, x_j \to x_{1+j}$ ($2 \leq j \leq d-1$), $x_d \to x_1$ (see, [10]).

State-closed representations are known for many finitely generated groups ranging from the torsion groups of Grigorchuk and Gupta-Sidki to free groups [11]. Furthermore, such representations have been studied for the family of abelian groups [12], of finitely generated nilpotent groups [13], as well as for arithmetic groups [14]. A useful software for computation in self similar groups is available in [15].

Section 2 of this paper is a preliminary analysis of similarity pairs for groups which are semidirect products $G = AX$ where $A$ is a self-centralizing abelian normal subgroup. We show how to replace a simple similarity pair $(H, f)$ where $f : A_0 (= A \cap H) \to A$ by a simple $(\tilde{H}, \tilde{f})$ satisfying

$$[G : \tilde{H}] = [G : H], \quad \tilde{H} = A_0 Y, \quad Y = AH \cap X,$$

$$\tilde{f} : A_0 \to A, \quad Y \to \infty.$$

In addition, we provide a module theoretic formulation of state-closed representations of $G$.

In Section 3 we prove that a state-closed representations of $G_{p,d}$, where $H = A_0 X$, is faithful only if $d = 1$. The exceptional case occurs in the classical representation of the lamplighter group $G = G_{2,1}$ where, in addition, $H$ contains the commutator subgroup $G'$.

**Theorem 1.** (Nonexistence Result) Let $G_{p,d} = C_p \wr X$ where $C_p = \langle a \rangle$ of prime order $p$ and $X$ is a free abelian group of finite rank $d \geq 2$. Let $A$ be the normal closure of $\langle a \rangle$, let $H$ be a subgroup of finite index in $G_{p,d} (= AX)$ and $f : H \to G$ a homomorphism. Suppose $H$ projects onto $X$ modulo $A$. Then $f$ is not simple.

An application of this result is

**Corollary 1.** If $d \geq 2$, then $G_{p,d}$ does not admit faithful state-closed representations of degree $p$.

Previously, it was shown that finitely generated torsion-free nilpotent groups of nilpotency class $c > 1$ do not admit faithful state-closed representations of degree $p$ [13].

In Section 4, we study representations of $G_{p,1}$ on the $p$-adic tree. First, we characterize the faithful ones obtained with respect to normal subgroups of index $p$. 
Theorem 2. (Degree 1 Representations) Suppose $H$ is a normal subgroup of $G_{p,1}$ of index $p$. Then every faithful state-closed representations of $G$ on the $p$-adic tree obtained with respect to $H$ is reducible to

$$
\varphi : \quad a \rightarrow \alpha = (0,1,\ldots,p-1),
$$

$$
x \rightarrow \xi = \left(\xi^n, \xi^n \alpha u(\xi), \ldots, \xi^n \alpha u(\xi)(p-1)\right)
$$

for some integer $n$ and $u(x) \in k \langle x \rangle$ such that $\gcd(p,n) = 1$ and $u(1) \neq 0$.

The methods of reduction use the replacement arguments discussed in Section 2. Next, we produce faithful representations from those subgroups $H$ of index $p$ which are not necessarily normal.

Theorem 3. Let $G_{p,1} = C_{p,wr}C$, $C_p = \langle a \rangle$, $C = \langle x \rangle$. Also, let $j \in \{1,\ldots,p-1\}$ and $\beta$ be the permutation of $\{0,1,\ldots,p-1\}$ defined by $\beta : i \mapsto ij$ modulo $p$. Then

$$
\varphi : \quad a \rightarrow \alpha = (0,1,\ldots,p-1),
$$

$$
x \rightarrow \xi = \left(\xi, \ldots, \xi \alpha^i, \ldots, \xi \alpha^{(p-1)}\right) \beta
$$

extends to a faithful state-closed representation of $G_{p,1}$ on the $p$-adic tree.

We note that this representation is finite state; indeed the group $G^n$ is an automaton group generated by the $p$ states of $\xi$. For $p = 2$, the representation is defined by $\varphi : a \rightarrow \alpha = (0,1), x \rightarrow \xi = (\xi, \xi \alpha)$ and is the classical representation of $G_{2,1}$. Also, for $p = 3$, $\varphi : a \rightarrow \alpha = (0,1,2), x \rightarrow \xi = (\xi, \xi \alpha, \xi \alpha^2)$ $(1,2)$ and $\xi$ is equivalent to the automaton in [17].

In Section 4, we provide uniformly faithful state-closed representations of $G = G_{p,d}$ on the $p^2$-tree, for all primes $p$ and for all $d \geq 2$.

Theorem 4. (Degree 2 Representations) Let $d \geq 2, G = G_{p,d}$ and $G'$ be its commutator subgroup. Furthermore, let $H = G'Y$ where $Y = \langle x_1^p, x_2, \ldots, x_d \rangle$. Then the map

$$
f : \quad a^{z-1} \rightarrow a^i \quad (1 \leq i \leq p-1), \quad a^z \rightarrow e \quad \text{for all } z \in Y,
$$

$$
x_1^p \rightarrow x_2, \quad x_j \rightarrow x_{1+j} \quad (2 \leq j \leq d-1), \quad x_d \rightarrow x_1
$$

extends to a simple homomorphism $f : H \rightarrow G_{p,d}$.

Finally, we write down concretely the above representation for $G_{2,2}$ in its action on the 4-tree which is indexed by sequences from $\{0,1,2,3\}$.

Theorem 5. The following automorphisms of the 4-tree

$$
\alpha = (0,1) \langle 2,3 \rangle,
$$

$$
\xi_1 = \left(1, a^{\xi_2^{-1}}, \xi_2, a^{\xi_2^{-1}} \xi_2\right) \langle 0,2 \rangle \langle 1,3 \rangle,
$$

$$
\xi_2 = \left(\xi_1, \xi_3, \xi_4, a^{(1+\xi_1^{-1}) \xi_2^{-1}} \xi_1\right).
$$

generate a group $G$ isomorphic to $G_{2,2}$. In addition, $G$ is the state-closure of $\xi_1$ and is finite-state; indeed, $\xi_1$ has 12 states.
2. Virtual endomorphisms of semidirect products

2.1. Replacement arguments. We consider in this section groups $G$ which are semidirect products $G = AX$ where $A$ is abelian, $C_X(A) = 1$, $H$ a subgroup of $G$ of index $m$ and homomorphism $f : H \to G$ such that $f : A_0 (= H \cap A) \to A$; the last condition is necessary in case $G = G_{p,d}$.

We recall that a subgroup $K$ of $G$ is said to be $f$-invariant provided $K^f \leq K$.

Proposition 1. Let $(G, H, f)$ be as above, $Y = AH \cap X$ and let $\gamma$ be an automorphism of $X$. Define the subgroup $\hat{H} = A_0 Y$ of $G$. Then: (1) $[G : \hat{H}] = [G : H]$; (2) $f$ induces a homomorphism $\alpha : Y \to X$ which together with $f|_{A_0} : A_0 \to A$ defines a homomorphism $\hat{f} : \hat{H} \to G$; (3) $(H, f)$ may be replaced by $(\hat{H}, \hat{f})$ and by $(H^*, f^*)$; (3) both replacements preserve normality of $H$ and simplicity of $f$.

Proof. (1) The subgroup $A_0$ is normal in $H$ and $[A : A_0] = m_1$ a divisor of $m$. Also, $Y$ is a subgroup of $X$ and $|X : Y| = m_2$ a divisor of $m$. Let $S$ be a right transversal of $A_0$ in $A$ and let $T$ be a right transversal of $Y$ in $X$. For every $y \in Y$, there exists $v(y) \in A$ such that $H = A_0 \langle v(y)y \mid y \in Y \rangle$. The set $ST$ is a right transversal of $H$ in $G$ and $m = m_1 m_2$:



\[
HST = A_0 \langle v(y)y \mid y \in Y \rangle ST = \langle v(y)y \mid y \in Y \rangle (A_0 S) T = \langle v(y)y \mid y \in Y \rangle A T = A \langle v(y)y \mid y \in Y \rangle T = A (YT) = AX = G.
\]

As $Y$ normalizes $A_0$ the set $\hat{H} = A_0 Y$ is a subgroup of $G$ with the same transversal $ST$.

(2) For every $y \in Y$ there exist a unique pair $w(y) \in A, y' \in X$ such that $f : v(y)y \to w(y)y'$. Define $\alpha : Y \to X$ by $y \to y'$; then $\alpha$ is a homomorphism.

Now let $\hat{f} : \hat{H} \to G$ be an extension of $\mu := f|_{A_0} : A_0 \to A$ and $\alpha : Y \to X$, by $(a_0 y)^\hat{f} = (a_0^\mu y^\alpha)$ for all $a_0 \in A_0, y \in Y$. To prove that $f$ is a homomorphism it suffices to prove $((a_0^\mu y)^\alpha) = ((a_0 y)^\mu)$ for all $a_0 \in A_0, y \in Y$:

\[
((a_0 y)^\mu)^\alpha = \left((a_0) v(y)y\right)^\hat{f} = \left((a_0)^\hat{f} \right) (v(y)y)^\hat{f} = a_0^\mu w(y)y^\alpha = \left((a_0^\mu) y^\alpha\right) = ((a_0^\mu) y^\alpha).
\]

(3.1) Suppose $H$ is a normal subgroup of $G$; then, $A_0 = A \cap H$ is normal in $G$. Since $Y \leq X$ and $Y$ is normal in $X$ modulo $A$, it follows that $Y$ is normal in $X$. Therefore, for $a_0 \in A_0, a \in A, y \in Y$ we have

\[
(a_0 y)^x = (a_0)^x y^x, \quad [y, a] = [v(y)y, a]
\]

from which it follows that $\hat{H}$ is normal in $G$.

(3.2) Suppose $f$ is simple. To prove that $\hat{f} : \hat{H} \to G$ is simple, we consider a subgroup $K \leq \hat{H}$ such that $K$ is normal in $G$ and $K^f \leq K$. Then $K_0 = K \cap A$ is a normal subgroup of $G$ and is $\hat{f}$-invariant:

\[
(K_0)^\hat{f} = (K_0)^f = (K \cap A_0)^f \leq K^f \cap A \leq K \cap A = \left(K \cap \hat{H}\right) \cap A = K \cap \left(\hat{H} \cap A\right) = K \cap A_0 = K_0.
\]
Thus, \( K_0 = (K \cap A) = 1 \) and from \( C_X(A) = 1 \), we conclude \( K = 1 \).

(3.3) The assertions about the replacement \((H, f)\) by \((H^\ast, f^\ast)\) are easily verified.

\[ \square \]

**Remark 1.** As \( H \) is obtained from \( \hat{H} = A_0 Y \) by multiplying the elements \( y \in Y \) by certain elements \( v(y) \in (A_0 \setminus A_0) \cup \{ e \} \), the subgroup \( H \) may be viewed as a deformation of \( \hat{H} \) (maintaining the subgroup \( A_0 \) unchanged). Suppose \( Y \) is finitely presented, say with generators \( y_i \) and relators \( r_j \). Then the process of obtaining all deformations of \( \hat{H} \) is algorithmic: let \( T \) be a transversal of \( A_0 \) in \( A \) then for every choice \( v_i \in T \ (1 \leq i \leq s) \) we need that \( r_j (v_i y_i) \in A_0 \ (1 \leq i \leq s) \) be satisfied. In the case of \( G_{p, d} \), the subgroup \( Y \) is freely generated by \( \{ y_1, \ldots, y_d \} \), satisfying \( [y_i, y_j] = e \) for \( 1 \leq i < j \leq d \). Then, since

\[ [v_i y_i, v_j y_j] = [w_i, y_j] [y_i, w_j] \]

where \( w_i = v_i^{y_i} \ (1 \leq i \leq d) \), the set \( \{ w_1, \ldots, w_s \} \) should satisfy

\[ [y_i, w_j] \in [y_j, w_i] A_0 \text{ for all } i, j. \]

**Remark 2.** For the groups \( G = G_{p, d} \), the ring \( A \) is the commutative group algebra \( k \langle X \rangle \) where \( k = GF(p) \) and \( B = k \langle Y \rangle \). Suppose \( A_0 \) is a normal subgroup of \( G_{p, d} \). Then \( A_0 \) corresponds to an ideal \( I \) of \( A \). If \( v \) is the exponent of the group of units of the quotient ring \( \frac{A}{I} \) then \( I \) contains the ideal \( \mathcal{A}(v) = \sum_{1 \leq i \leq d} A(x_i^v - 1) \). Moreover, the skew action reads as

\[ (w, v)^\mu = (w^\alpha (v))^\mu \]

for all \( v \in I \) and \( w \in B = \mathbb{Z}[Y] \). On denoting \( K = \ker_B(\alpha) \), it follows that \( IK \) is an ideal of \( A \) contained in \( \ker_B(\mu) \). If \( f \) is simple then the only \( \mu \)-invariant ideal of \( A \) is the zero ideal and therefore, \( \alpha : Y \to X \) is a monomorphism.
3. Nonexistence of Simple Virtual Endomorphisms

The following result on solving certain polynomial equations will be used in the proof of Theorem 1.

Proof.

Proposition 2. Let $M = (m_{ij})$ be an $n \times n$ integral matrix where $n \geq 2$ and $v$ a nonzero integer; denote $\det (M) = t$. Let $K$ be a field and let $u_i \in K \langle x_1, x_2, \ldots, x_n \rangle$ satisfy the equations

$$(x_1^{vm_{11}}x_2^{vm_{12}}\ldots x_n^{vm_{jn}} - 1)u_j = (x_1^{vm_{j1}}x_2^{vm_{j2}}\ldots x_n^{vm_{jn}} - 1)u_i$$

for all $1 \leq i < j \leq n$. Then, either $t = 0$ or $u_i \in \mathcal{I} = \langle x_1^v - 1, x_2^v - 1, \ldots, x_n^v - 1 \rangle_{\text{ideal}}$ for all $i$.

Proof. (1) Let $n = 2$. Suppose $g \in k \langle x_1, x_2 \rangle$ is a nonconstant common factor of $x_1^{vm_{11}}x_2^{vm_{12}} - 1, x_1^{vm_{21}}x_2^{vm_{22}} - 1$. Then, $x_1^{vm_{11}}x_2^{vm_{12}} \equiv 1, x_1^{vm_{21}}x_2^{vm_{22}} \equiv 1 \mod g$. Therefore,

$$(x_1^{vm_{11}}x_2^{vm_{12}})^{m_{22}}(x_1^{vm_{21}}x_2^{vm_{22}})^{-m_{22}} = 1$$

$$\equiv x_1^{vm_{11}m_{22} - vm_{21}m_{22}}x_2^{vm_{12}m_{22} - vm_{12}m_{22}}$$

$$\equiv x_1^{vm_{11}m_{22} - vm_{21}m_{22}} \equiv x_1^v \mod g;$$

similarly,

$$x_2^v \equiv 1 \mod g.$$

That is, there exist $h, h' \in k \langle x_1, x_2 \rangle$ such that

$$x_1^v - 1 = gh, \; x_2^v - 1 = gh';$$

the left hand of each equation is non-zero. Since $K [x_1, x_2]$ is a unique factorization domain, $g, h, h' \in k \langle x_1 \rangle$ and $g, h, h' \in k \langle x_2 \rangle$; therefore, $g \in k$; contradiction.

We conclude $\langle x_1^{vm_{21}}, x_2^{vm_{22}} - 1 \rangle u_1, \langle x_1^{vm_{11}}, x_2^{vm_{12}} - 1 \rangle u_2$ and thus, there exists $l (x_1, x_2) \in k [x_1, x_2]$ such that

$$u_1 = (x_1^{vm_{21}}x_2^{vm_{22}} - 1)l (x_1, x_2), \; u_2 = (x_1^{vm_{11}}x_2^{vm_{12}} - 1)l (x_1, x_2).$$

Hence $u_1, u_2 \in \langle x_1^v - 1, x_2^v - 1 \rangle_{\text{ideal}}$.

(2) Let $n \geq 3$ and suppose by induction the assertion true for $n - 1$. Suppose furthermore $u_s \notin \mathcal{I}$ for some $s$; without loss, $s = 1$. Let $M_{i,j}$ denote the $(i, j)$th minor of $M$. On letting $x_i = 1 (1 \leq i \leq n)$ in

$$(x_1^{vm_{11}}x_2^{vm_{12}}\ldots x_n^{vm_{jn}} - 1)u_i = (x_1^{vm_{11}}x_2^{vm_{12}}\ldots x_n^{vm_{jn}} - 1)u_1 \; (i \neq 1)$$

we produce by induction $M_{i,j} = 0$ for all $(i, j)$ where $i \neq 1$. Therefore

$$\det (M) = \sum_{1 \leq i \leq n} (-1)^{i+j} m_{i,j} M_{i,j} = (-1)^{1+j} m_{1,j} M_{1,j} \text{ for } (1 \leq j \leq n),$$

$$\sum_{1 \leq j \leq n} \det (M) = \sum_{1 \leq j \leq n} (-1)^{1+j} m_{1,j} M_{1,j} = \det (M),$$

$$\det (M) = 0, \; \det (M) = 0;$$

contradiction. □
3.1. Proof of Theorem 1. The subgroup $A_0$ is normal in $G$, since it is centralized by $A$ and is normalized by $X$. In additive notation, the subgroup $A_0$ is an ideal $I$ of $A = k[X]$. To prove Theorem 1, the following proposition suffices.

**Proposition 3.** Let $X = \langle x_1, x_2, \ldots, x_d \rangle$ be a free abelian group with $d \geq 2$. Furthermore let $A$ be the group algebra $k[X]$ and let $I$ be an ideal in $A$ of finite index $m > 1$. Let $\mu : I \rightarrow A$, be a $k$-homomorphism and $\alpha : X \rightarrow X$ a multiplicative homomorphism such that the skew action

\[(**): (r\nu)^\mu = r^\alpha \nu^\mu\]

is satisfied for all $r \in A$, $\nu \in I$. Then the ideal $I$ contains a nontrivial $\mu$-invariant ideal of $A$.

**Proof.** We suppose by contradiction that there exists a simple homomorphism $\mu : I \rightarrow A$. Then, by Remark 2, $\alpha$ is a monomorphism.

Let the order of the group of units of $\frac{1}{\alpha}$ be $\nu$. Then $J = \langle X^\nu - 1 \rangle$ ideal is contained in $I$ and so, we may assume $I = J$.

Denote

\[(x_i^\nu - 1)^\mu = u_i \quad (1 \leq i \leq d).\]

Then, conditions (**) imply that for $i \neq j$,

\[\mu : (x_i^\nu - 1) (x_j^\nu - 1) \rightarrow (x_i^\nu - 1)^\alpha u_j\]

\[\mu : (x_j^\nu - 1) (x_i^\nu - 1) \rightarrow (x_j^\nu - 1)^\alpha u_i\]

and therefore,

\[(x_i^{\nu^\alpha} - 1) u_j = (x_j^{\nu^\alpha} - 1) u_i.\]

Since $\alpha : X \rightarrow X$ is a monomorphism, by the above proposition, $u_i = (x_i^\nu - 1)^\mu \in I$ holds for all $i$. As, $\mu : r (x_i^\nu - 1) \rightarrow r^\alpha u_i$ for all $i$, we conclude that $I$ is $\mu$-invariant. \(\square\)

**Proof of Corollary 1.**

**Proof.** Suppose $H$ is a subgroup of $G$ of prime index $q$ and $f : H \rightarrow G$ a simple homomorphism. Then, as $f : A_0 \rightarrow A$, it follows that $A_0$ is a proper subgroup of $A$ and so, $[A : A_0] = q$. Therefore, $H$ projects onto $X$ modulo $A$ and $A_0$ is normal in $G$.

We apply the previous theorem to conclude the existence of a nontrivial subgroup of $A_0$ which is normal in $G$ and $f$-invariant and so reach a contradiction. \(\square\)

4. Representations of $G_{p,1}$ of Degree $p$

Given a general group $G$ and triple $(G, H, f)$ with $[G; H] = m$, we recall the representation $\varphi$ of $G$ on the $m$-ary tree indexed by sequences from $N = \{0, 1, \ldots, m - 1\}$. Let $T = \{e, t_2, \ldots, t_{m-1}\}$ be a right transversal $T$ of $H$ in $G$ and $\sigma$ be the permutational representation of $G$ on $T$. Then

\[g^\varphi = \left(\left((h_i)^f\right)^\varphi \mid 0 \leq i \leq m - 1\right) g^\varphi,\]

where $h_i$ are the Schreier elements of $H$ defined by

\[h_i = (t_i g) (t_j)^{-1}, \quad t_j = (t_i)^{g^\varphi};\]

see [10].
Let \( G = G_{p,1} \). We observe

\[
\frac{G}{G'} = C_p \times C, \quad G = A \langle x \rangle, \quad A = G' \langle u \rangle.
\]

The following items deal with general state-closed representations of \( G_{p,1} \) of degree \( p \).

(i) Let \( H \) be a subgroup of index \( p \) in \( G \). Then from the representation of \( G \) into \( Sym(\pi) \) we conclude that \( \frac{G}{\text{Core}(H)} \) is isomorphic to a metabelian transitive subgroup of \( Sym(p) \) and therefore is of order multiple of \( p \) and is isomorphic to a subgroup of the semidirect product \( C_pC_{p-1} \) where \( C_p \) is normal and \( C_{p-1} \) acts on it as the full automorphism group \( \text{Aut}(C_p) \).

As \( A_0 = A \cap H \) and \( [G:H] = p \), we conclude that \( A_0 = A \cap \text{Core}(H) \) and so is normal in \( G \). As we have argued in Section 2, we may assume \( H = A_0X \).

(ii) Additively, \( A \) corresponds to the \( k \)-algebra \( A = k \langle x \rangle \) and \( A_0 \) corresponds to a maximal ideal \( M \) of \( A \). As \( \frac{k[x]}{(x)} \cong k \), it follows that \( M+x = M+c \) where \( (1 \leq c \leq p-1) \). Therefore, \( M = \langle x-c \rangle_{\text{ideal}} \) and so we have \( p-1 \) distinct maximal ideals

\[
M = \langle x-c \rangle_{\text{ideal}} \quad (1 \leq c \leq p-1).
\]

(iii) Let \( A_0 \) correspond to \( M \) and let \( f : H \to G \) be simple. Then, as before, \( f \) corresponds to a pair of homomorphisms \( (\mu, \alpha) \) where \( \mu : M \to A \) is an additive homomorphism and \( \alpha : x \to x^n \) is a multiplicative monomorphism. Let \( \mu : x-c \to u(x) \) for some \( u(x) \in A \), \( \alpha : x \to x^n \) for some integer \( n \). Then \( \mu : M \to A \) is defined by

\[
\mu : r(x)(x-c) \to r(x^n)u(x).
\]

We derive below some restrictions on \( n, u(x) \).

(iii.1) **Assertion.** \( u(c^n) \neq 0 \) for all \( i \).

**Proof.** If \( (x-c)|u \) then \( \mu : M \to M \); contradiction.

Suppose \( u(x^n) \) is not a multiple of \( x-c \) for \( 0 \leq t \leq i-1 \), yet \( u(x^n) = u'(x)(x-c) \). Then

\[
\mu : u(x)u(x^n) \cdots u(x^{n-1})(x-c) \to u(x^n) = u'(x)(x-c);
\]

that is, the ideal \( Mu(x)u(x^n) \cdots u(x^{n-1}) \) is \( \mu \)-invariant; contradiction.

(iii.2) Let \( n = p^sn' \), \( \gcd(p,n') = 1 \), \( n'\equiv 1 \mod p \). Then, \( s = 0 \) or \( o(c) \mid n' - 1 \).

**Proof.** Suppose \( s \neq 0 \) and \( o(c) \mid n' - 1 \). Then,

\[
\mu : r(x)(x-c)^2 = (r(x)(x-c))(x-c) \to r(x^n)(x-c)^\alpha (x-c)^\mu \equiv r(x^n)(x-c) + \nu(x) \equiv r(x^n)(x-c) + \nu(x) ;
\]

thus \( M(x-c)^2 \) is \( \mu \)-invariant; a contradiction.
4.1. **Proof of Theorem 2.** (1) Since $H$ contains $G'$, it follows that $G' = A_0$. By the replacement argument in Section 2, we may assume $H = A_0 \langle x \rangle$ and $f : x \to x^n$. for some nonzero integer. Furthermore, additively, $A_0$ corresponds to the ideal $I = (x - 1)_{\text{ideal}}$ in the group algebra $A = k \langle x \rangle$. Therefore, $f$ induces on $I$ the additive homomorphism $\mu : I \to A$ defined by

$$\mu : r(x) (x - 1) \to r(x^n) u(x)$$

for all $r(x) \in A$, where $u(x)$ is a fixed non-zero element of $A$. Since $I$ is not $\mu$-invariant, $\gcd(−1 + x, u) = 1$; that is, $u(1) ≠ 0$.

(2) **Assertion:** $\gcd(p, n) = 1$.

**Proof.** Suppose $n = pm′$. Then, as in the previous (iii.2),

$$\mu : r(x) (x - 1) \cdot (x - 1) \to r((x - 1))^{m′} \cdot (x - 1)^m$$

$$= (r(x^n) (x^n - 1)). u(x)$$

$$= r(x^n) (x^n - 1)^p u(x) = t(x) (x - 1)^2$$

which proves that $K = I (x - 1)^2$ is $\mu$-invariant; a contradiction.

(3) **Assertion:** $\mu$ is simple.

**Proof.** Denote $x^{n−1}$ by $\Phi_n(x)$. Let $v = (x - 1)^j r(x)$ be a non-zero polynomial in $I$ such that $j ≥ 1$, $r(1) ≠ 0$. We have

$$\mu : v = (x - 1)^j r(x) = (x - 1)^j \cdot r(x - 1) \to \nu_1 = (x - 1)^j \cdot r(x^n) \cdot u(x),$$

$$= (x - 1)^j \cdot (\Phi_n(x) u(x) r(x^n)),$$

where

$$\Phi_n(1)^{j−1} u(1) r(1) ≠ 0.$$

Thus, $\mu^j : v → v'$ and $v'(1) ≠ 0$; that is, $(v)^{\mu^j} \notin I$.

(4) Choose the transversal $\{a^i | i = 0, ..., p - 1\}$ for $H$ in $G$. Then on identifying $g$ with $g^p$, the image of $G$ on the $p$-adic tree takes the form

$$a = (0, 1, ..., p - 1), x = (x^n, x^na(u(x)), ..., x^na(u(x)(p-1))).$$

**Proof.** Since $Ha^ix = Ha^{ic}$ ($0 ≤ i ≤ p - 1$), the cofactor is $a^i x a^{-ic} = xa^ix a^{-ic} = xa^{i(x-c)} ∈ H$. Thus,

$$x^σ : i → ic, f : xa_{(x-c)} → x^n a^{iu(x)}.$$

4.2. **Proof of Theorem 3.** Recall Let $u(x) = 1, n = 1$.

(1) **Assertion** The homomorphism $f : H → G$ is simple.

**Proof.** It is sufficient to prove the induced $\mu : I → A$ is simple. A non-zero polynomial in $I$ can be written as $v = r(x) (x - c)^j$ with $j ≥ 1$, $r(c) ≠ 0$.

We have

$$\mu : v = r(x) (x - c)^j = r(x) (x - c)^{j−1} \cdot (x - c) \to \nu_1(x) = r(x) (x - c)^{j−1}.$$ 

Thus, $\mu^j : v → v'$ where $v'(1) ≠ 0$. 


(2) Choose the transversal \( \{a_i \mid i = 0, \ldots, p-1\} \) for \( H \) in \( G \). Then on identifying \( g \) with \( g^\sigma \), the representation \( \varphi \) of \( G \) on the \( p \)-adic tree takes the form

\[
a = (0, 1, \ldots, p-1), \quad x = \left( x, \ldots, xa^i, \ldots, xa^{(p-1)} \right) x^\sigma,
\]

where \( x^\sigma : i \to ic. \)

5. Representations of \( G_{p,d} \) (\( d \geq 2 \)) of Degree \( p^2 \)

If \( C \) is a group algebra, we denote its augmentation ideal by \( C' \).

Let

\[
G = G_{p,d}, \quad H = G'Y,
\]

\[
Y = \langle x_1, x_2, \ldots, x_d \rangle,
\]

\[
X_2 = \{ x_2, \ldots, x_d \}, \quad Z = \langle X_2 \rangle.
\]

Then, \([G; H] = p^2\) and \( A_0 = A \cap H = G' \). Also let \( \alpha : Y \to X \) be the monomorphism defined by

\[
x_p^p \to x_2, \quad x_j \to x_{1+j} \quad (2 \leq j \leq d-1), \quad x_d \to x_1.
\]

Let \( A = k \langle X \rangle, B = k \langle Y \rangle \); then, \( A = \sum_{0 \leq i \leq p-1} B x^i \). Also, \( A' \) is the same as the ideal \( \mathcal{I} \) generated by \( \{ x_1 - 1, x_2 - 1, \ldots, x_d - 1 \} \). We continue with: \( A \) corresponds to \( A \) and \( A_0 \) to \( \mathcal{I} \).

Given an integer \( j \), write \( j = j_0 + j_1p, \quad 0 \leq j_0 \leq p - 1 \). Then,

\[
k \langle x \rangle' = k \langle x^p \rangle' + \sum_{1 \leq i \leq p-1} k \langle x^p \rangle (x^i - 1).
\]

(1) **Decomposition of \( \mathcal{I} \).**

The ideal \( \mathcal{I} \) decomposes as

\[
\mathcal{I} = \sum_{x \in (x_1)} \{ k (xz - 1) \mid x \in (x_1), \, z \in Z \}
\]

\[
= \sum_{x \in (x_1)} k (x - 1) \oplus \sum_{z \in Z} k (z - 1) \oplus \sum_{x \in (x_1), z \in Z} k (x - 1) (z - 1)
\]

and on substituting

\[
\sum_{x \in (x_1)} k (x - 1) = k \langle x_1^p \rangle' + \sum_{1 \leq i \leq p-1} k \langle x_1^p \rangle (x_1^i - 1),
\]

the decomposition is refined to

\[
\mathcal{I} = k \langle x_1^p \rangle' + \sum_{1 \leq i \leq p-1} k \langle x_1^p \rangle (x_1^i - 1)
\]

\[
\oplus \sum_{z \in Z} k (z - 1) \oplus \sum_{z \in Z} k \langle x_1^p \rangle' (z - 1)
\]

\[
\oplus \sum_{z \in Z} \sum_{1 \leq i \leq p-1} k \langle x_1^p \rangle (x_1^i - 1) (z - 1).
\]

Therefore, an element \( v \) in \( \mathcal{I} \) has the unique form

\[
v = b_0 + \sum_{1 \leq i \leq p-1} b_i (x_1^i - 1) \oplus \sum_{z \in Z} a_z (z - 1) \oplus \sum_{z \in Z} \sum_{0 \leq i \leq p-1} b_{i,z} (z - 1) (x_1^i - 1),
\]

where

\[
b_0 \in k \langle x_1^p \rangle', \quad a_z, b_i, b_{i,z} \in k \langle x_1^p \rangle.
\]
(2) Definition of $\mu$ on $I$
Choose $\mu : B' \to 0$ and

$$\mu : x^i_1 - 1 \to i \text{ for } 1 \leq i \leq p - 1.$$ 

Given the algebra homomorphism $\alpha : B \to A$, we extend $\mu$ to an additive homomorphism $I \to A$ satisfying the skew condition as follows:

$$\nu = b_0 + \sum_{0 \leq i \leq p-1} b_i (x^i_1 - 1) + \sum_{z \in Z} \sum_{0 \leq i \leq p-1} b_{i,z} (z-1) (x^i_1 - 1) \to$$

$$(b_0)^\alpha + \sum_{0 \leq i \leq p-1} (b_i)^\alpha (x^i_1 - 1)^\alpha + \sum_{z \in Z} \sum_{0 \leq i \leq p-1} (b_{i,z})^\alpha (z-1)^\alpha (x^i_1 - 1)^\alpha$$

$$= \sum_{0 \leq i \leq p-1} \sum_{z \in Z} \sum_{0 \leq i \leq p-1} i (b_{i,z})^\alpha (z-1)^\alpha.$$

5.1. Proof of Theorem 4.

Lemma 2. Let $u, i \geq 1$ and write $u = u_0 + u_1 p, i = i_0 + i_1 p$ where $0 \leq u_0, i_0 \leq p - 1$ and let $2 \leq j \leq d$. Then,

$$\mu : x^u_1 (x^i_1 - 1) \to (u_0 + i_0) x^i_2 - u_0 x^u_2,$$

and

$$x^u_1 (x^j_2 - 1) \to u_0 x^u_2 (x^{i+j}_1 - 1),$$

$$x^u_2 (x^i_1 - 1) \to i_0 x^u_1 x^i_2,$$

where $1 + j$ is computed modulo $d$.

Proof. Then,

$$\mu : x^u_1 (x^i_1 - 1) \to x^{u+i}_2 = x^u_2;$$

$$\mu : x^u_1 (x^j_2 - 1) = x^{u+i+1}_1 (x^i_1 - 1) = x^{u+i+1}_1 - x^{u+i}_1$$

$$\to (x^{u+i+1}_1 - 1) u_0 + 0 + u_0 = u_0 x^{u+i}_2;$$

$$\mu : x^{u+i}_1 (x^i_1 - 1) = x^{u+i+1}_1 - x^{u+i}_1$$

$$= \left( x^{u+i+1}_1 - 1 \right) (x^{u+i}_1 - 1)$$

$$\to (u_0 + 1) x^{u+i}_2 - u_0 x^{u+i}_2 = x^{u+i}_2 \text{ if } u_0 + 1 \leq p - 1;$$

$$\mu : x^{u+i+1}_1 (x^i_1 - 1) = x^{u+i+1}_1 - x^{u+i}_1$$

$$\to (x^{u+i+1}_1 - 1) (x^{u+i+1}_1 - 1)$$

$$= (u_0 + i) x^{u+i+1}_2 - u_0 x^{u+i}_2 \text{ if } (u+i)_0 \leq p - 1;$$

$$\mu : x^{u+i+1}_1 (x^i_1 - 1) \to -u_0 x^{u+i}_2 \text{ if } (u+i)_0 = p.$$

More generally,

$$\mu : x^u_1 (x^i_1 - 1) = x^{u+i}_1 - x^u_1 = (x^{u+i}_1 - 1) - (x^u_1 - 1)$$

$$\to (u+i)_0 x^{(u+i+1)}_2 - u_0 x^u_2$$

$$= (u_0 + i) x^{u+i+1}_2 - u_0 x^{u+i}_2 \text{ if } (u+i)_0 \leq p - 1;$$

$$\mu : x^u_1 (x^i_1 - 1) \to -u_0 x^{u+i}_2 \text{ if } (u+i)_0 = p.$$
In all cases, 
\[ x_i^u (x_i^1 - 1) \rightarrow (u_0 + i_0) x_2^{u_i + i} - u_0 x_2^{u_i}. \]
Also, for \( 2 \leq j \leq d \),
\[ x_i^u (x_j^i - 1) = (x_i^u - 1)(x_j^i - 1) + (x_j^i - 1)^\alpha (x_i^u - 1)^\mu = (x_{j+1}^i - 1) u_0 x_2^{u_i}; \]
\[ x_j^i (x_1^1 - 1) \rightarrow (x_{j+1}^i) i_0 x_2^{i_1} = i_0 x_2^{u_i} x_2^{i_1}. \]

**Lemma 3.** Let \( q(x) = c_0 + c_1 x + \ldots + c_s x^s \in k[x] \), and let \( 0 \leq u = u_0 + u_1 p \) where \( 0 \leq u_0 \leq p - 1 \). Suppose \( q(x_1) \in \mathcal{I} \). Then, \( \sum_{0 \leq i \leq s} c_i = 0 \) and
\[ \mu : q(x_1) \rightarrow \sum_{1 \leq i \leq s} c_i i_0 x_2^{i_1}. \]
Furthermore,
\[ (x_i^u q(x_1))^\mu - x_2^{u_i} \cdot q(x_1)^\mu = u_0 \left( \sum_{0 \leq i \leq s} c_i (x_2^{i_1} - 1) \right) x_2^{u_i}. \]

**Proof.** Let \( 1 \leq j \leq d \). Then,
\[ q(x) = c_0 + c_1 x + \ldots + c_s x^s = \sum_{0 \leq i \leq s} c_i + \sum_{1 \leq i \leq s} c_i (x^i - 1), \]
and clearly, \( q(x_1) \in \mathcal{I} \) if and only if \( \sum_{0 \leq i \leq s} c_i = 0 \). Therefore, \( q(x_1) \in \mathcal{I} \) implies
\[ \mu : q(x_1) = \sum_{1 \leq i \leq s} c_i (x_1^i - 1) \rightarrow \sum_{1 \leq i \leq s} c_i i_0 x_2^{i_1}. \]
Next,
\[ \mu : x_1^u q(x_1) = \sum_{1 \leq i \leq s} c_i (x_1^u (x_1^i - 1)) \rightarrow \sum_{1 \leq i \leq s} c_i ((u_0 + i_0) x_2^{i_1} - u_0) x_2^{u_i}; \]
\[ = \left( \sum_{1 \leq i \leq s} c_i u_0 x_2^{i_1} + c_i i_0 x_2^{i_1} - c_i u_0 \right) x_2^{u_i}; \]
\[ = \left( u_0 \sum_{1 \leq i \leq s} c_i (x_2^{i_1} - 1) + \sum_{1 \leq i \leq s} c_i i_0 x_2^{i_1} \right) x_2^{u_i}; \]
\[ = \left( u_0 \sum_{1 \leq i \leq s} c_i (x_2^{i_1} - 1) + q(x_1)^\mu \right) x_2^{u_i}; \]
therefore,
\[ (x_i^u q(x_1))^\mu - x_2^{u_i} \cdot q(x_1)^\mu = u_0 \left( \sum_{1 \leq i \leq s} c_i (x_2^{i_1} - 1) \right) x_2^{u_i}. \]

**Lemma 4.** Let \( \mathcal{K} \) be an invariant \( \mu \)-ideal. If \( q(x_j) \in \mathcal{K} \) for some \( j \) then \( q(x_j) = 0 \).
Proof. Given \( q(x) = c_0 + c_1 x + \ldots + c_s x^s \neq 0 \), define \( e(q) = \{ i \mid c_i \neq 0 \} \) and \( \lambda(q(x)) = \sum \{ i \mid i \in e(q) \} \). On writing \( 0 \leq i = i_0 + i_1 p \) \((0 \leq i_0 \leq p - 1)\),

\[
\lambda(q(x)) = \sum \{ i_0 \mid i \in e(q) \} + \left( \sum \{ i_1 \mid i \in e(q) \} \right) p.
\]

Suppose there exists a nonzero polynomial \( q(x) \) such that \( q(x_j) \in K \) for some \( j \) and choose one with minimum \( \lambda(q(x)) \). We may assume \( c_0 \neq 0 \).

(1) Suppose \( j = 1 \). \( q(x_1) \in K \). Then on choosing \( u_0 \neq 0 \), by the previous lemma,

\[
(x^n q(x_1))^\mu - x_2^{u_1} q(x_1)^\mu = u_0 \left( \sum_{1 \leq i \leq s} c_i (x_2^{i_1} - 1) \right) x_2^{u_1} \in K
\]

and therefore, \( l(x_2) = \sum_{1 \leq i \leq s} c_i (x_2^{i_1} - 1) \in K \) and \( \lambda(l(x_2)) \leq \sum \{ i_1 \mid i \in e(q) \} \).

Thus, either \( \lambda(q(x)) = \lambda(l(x)) \) or \( l(x_2) = 0 \). In the first case,

\[
\sum \{ i_0 \mid i \in e(q) \} + \left( \sum \{ i_1 \mid i \in e(q) \} \right) p \leq \sum \{ i_1 \mid i \in e(q) \},
\]

\[
\sum \{ i_0 \mid i \in e(q) \} \leq \left( \sum \{ i_1 \mid i \in e(q) \} \right) (1 - q),
\]

\[
\sum \{ i_0 \mid i \in e(q) \} = \sum \{ i_1 \mid i \in e(q) \} = 0,
\]

\[
q(x_1) = c_0 \in K.
\]

Thus, \( c_0 = 0 \); contradiction.

In the second case,

\[
l(x_2) = \sum_{1 \leq i \leq s} c_i (x_2^{i_1} - 1) = 0,
\]

\[
l(x_2) = l(x_1)^\alpha = 0.
\]

and as \( \alpha \) is a monomorphism, it follows that \( l(x_1) = 0 \).

Define \( L_j = \{ i \in e(q) \mid i_1 = j \} \) and let \( t \) be such that \( p^t \leq s < p^{t+1} \). Then,

\[
l(x_1) = \sum_{1 \leq i \leq s} c_i (x_1^{i_1} - 1) = \sum_{i \in L_0} c_i (x_1^{i_1} - 1) + \sum_{i \in L_1} c_i (x_1^{i_1} - 1) + \ldots + \sum_{i \in L_t} c_i (x_1^{i_1} - 1)
\]

\[
= \sum_{i \in L_1} c_i (x_1 - 1) + \sum_{i \in L_2} c_i (x_1^2 - 1) + \ldots + \sum_{i \in L_t} c_i (x_1^t - 1)
\]

\[
= \left( \sum_{1 \leq i \leq p - 1} c_i \right) (x_1 - 1) + \left( \sum_{0 \leq t < p - 1} c_{p+t} \right) (x_1^2 - 1) + \ldots + \left( \sum_{0 \leq t < p - 1} c_{p^t+t} \right) (x_1^t - 1) = 0
\]

and

\[
\left( \sum_{1 \leq i \leq p - 1} c_i \right) = \left( \sum_{0 \leq t < p - 1} c_{p+t} \right) = \ldots = \left( \sum_{0 \leq t < p - 1} c_{p^t+t} \right) = 0,
\]

\[
\sum_{1 \leq i \leq s} c_i = 0.
\]

Since \( \sum_{0 \leq t < p - 1} c_i = 0 \), we reach \( c_0 = 0 \); a contradiction.
(2) Suppose \( q(x_j) \in \mathcal{K} \) for some \( 2 \leq j \leq d \). Then, \( q(x_j) (x_i^s - 1) \in \mathcal{K} \) and \( \mu : q(x_j) (x_i^s - 1) \to q(x_{j+1}) u \alpha x_2^{\mu 1} \); therefore \( q(x_{j+1}) \in \mathcal{K} \) which leads us back to \( q(x_1) \in \mathcal{K} \). \( \square \)

**Lemma 5.** \( \mathcal{K} = \{0\} \).

**Proof.** Suppose \( \mathcal{K} \neq \{0\} \). Choose a polynomial \( w \neq 0 \) in \( \mathcal{K} \) having a minimum number of variables. Using the argument in (2) above, we may assume one of the variables to be \( x_1 \). Let \( \delta_i(w) \) be the \( x_i \)th degree of \( w \) and

\[
\delta(w) = \sum_{1 \leq i \leq d} \delta_i(w)
\]

the total degree of \( w \). Then \( \delta(w) \neq 0 \). Choose \( w \) having minimum \( \delta(w) \).

(1) Write \( w = w(x_1, x_j, ..., x_t) = \sum_{i=0, ..., s} w_i(x_1, ..., x_j) x_1^i \) where \( 2 \leq j_1 < ... < j_t \leq d \). Then,

\[
\delta(w) = \sum_{j \neq 1} \delta_j(w) + s.
\]

We note that in

\[
w = \sum_{i=0, ..., s} w_i(x_1, ..., x_j) + \sum_{i=1, ..., s} w_i(x_1, ..., x_j) (x_1^i - 1)
\]

\( w \in \mathcal{K} \) and second term on the right hand side is in \( \mathcal{I}_t \); it follows that the first term is in \( \mathcal{B}' \).

Thus,

\[
\mu : w \to 0 + \sum_{i=1, ..., s} w_i(x_{j+1}, ..., x_{j+1}) (i_0 x_2^{i_0})
\]

\[
= \sum_{i=1, ..., s} w_i(x_{j+1}, ..., x_{j+1}) i x_2^{i_1}
\]

and

\[
\delta(w^\mu) \leq \sum_{j \neq 1} \delta_j(w) + \left[ \frac{s}{p} \right] \leq \delta(w) = \sum_{j \neq 1} \delta_j(w) + s.
\]

Since \( w^\mu \in \mathcal{K} \), by the minimality of \( \delta(w) \), we conclude

\[ w^\mu = 0 \text{ or } 1 \leq s \leq p - 1. \]

(1.1) Suppose \( s = 1 \). Then, \( w = w_0(x_1, ..., x_j) + w_1(x_j, ..., x_j) x_1 \) and \( w_0 \neq 0 \neq w_1 \). Since \( w^\mu = w_1(x_{j+1}, ..., x_{j+1}) \in \mathcal{K} \), by the minimality of \( w \), we deduce \( w_1(x_{j+1}, ..., x_{j+1}) = 0 \). However, as \( w^\mu = w_1(x_{j+1}, ..., x_{j+1}) = \alpha \) and \( \alpha \) monomorphism, we have \( w_1(x_1, ..., x_j) = 0 \); contradiction.

(2) Define \( W_j = w_0 \left( x_j^1 - 1 \right) \) for \( 1 \leq j \leq p - 1 \). We apply the above analysis to \( W_j \).
Here we have

\[ W_j = w \cdot (x_1^j - 1) = \left( \sum_{i=0} w_i(x_j, \ldots, x_j) + \sum_{i=1, \ldots, s} w_i(x_j, \ldots, x_j) (x_1^j - 1) \right) (x_1^j - 1) \]

\[ = \sum_{i=0, \ldots, s} w_i(x_j, \ldots, x_j) (x_1^j - 1) + \sum_{i=1, \ldots, s} w_i(x_j, \ldots, x_j) (x_1^j - 1) \]

\[ = \sum_{i=0, \ldots, s} w_i(x_j, \ldots, x_j) (x_1^j - 1) + \sum_{i=1, \ldots, s} w_i(x_j, \ldots, x_j) \left( (x_1^j - 1) - (x_1^j - 1) \right). \]

Since \([\frac{p}{p}] = 0\), we have

\[ \mu : W_j \rightarrow \sum_{i=0, \ldots, s} w_i(x_j, \ldots, x_j+1) j_0 x_2 \frac{[i]}{p} + \sum_{i=1, \ldots, s} w_i(x_j, \ldots, x_j+1) \left( (i + j) \right)_0 x_2 \frac{[i+j]}{p} - j_0 x_2 \frac{[i]}{p} \]

\[ = \sum_{i=0, \ldots, s} w_i(x_j, \ldots, x_j+1) - \sum_{i=1, \ldots, s} w_i(x_j, \ldots, x_j+1) \]

\[ + \sum_{i=1, \ldots, s} w_i(x_j, \ldots, x_j+1) (i + j) \left( (i + j)_0 x_2 \frac{[i+j]}{p} - \sum_{i=1, \ldots, s} w_i(x_j, \ldots, x_j+1) i_0 x_2 \frac{[i]}{p} \right) \]

\[ = w_0(x_j, \ldots, x_j+1) j + \sum_{i=1, \ldots, s} w_i(x_j, \ldots, x_j+1) (i + j) x_2 \frac{[i+j]}{p} \]

\[ - \sum_{i=1, \ldots, s} w_i(x_j, \ldots, x_j+1) i_0 x_2 \frac{[i]}{p}. \]

As \(w^\mu = \sum_{i=1, \ldots, s} w_i(x_j, \ldots, x_j+1) i x_2 \frac{[i]}{p} \in K\), we conclude that for \(1 \leq j \leq p-1\),

\[ V_j = w_0(x_j, \ldots, x_j+1) j + \sum_{i=1, \ldots, s} w_i(x_j, \ldots, x_j+1) (i + j) x_2 \frac{[i+j]}{p} \in K. \]

For each \(j\), either \(V_j = 0\) or \(\delta(V_j) \geq 1\).

\(2.1\) Suppose for some \(j\), \(V_j \neq 0\). Then

\[ \delta(w) = \sum_{j \neq 1} \delta_j(w) + s \leq \delta(V_j) \leq \sum_{j \neq 1} \delta_j(w) + \left[ \frac{s + j}{p} \right], \]

\[ s \leq \left[ \frac{s + j}{p} \right]. \]

On writing \(s = s_0 + s_1 p\) where \(0 \leq s_0 \leq p - 1\), we conclude

\[ s_0 + s_1 p \leq \left[ \frac{s_0 + s_1 p + j}{p} \right] \leq \left[ \frac{s_0 - 1 + (s_1 + 1) p}{p} \right]; \]

if \(s_0 = 0\) then \(s_1 p \leq s_1\) and \(s_1 = 0\) (impossible),
if \( s_0 \neq 0 \) then \( s_0 + s_1 p \leq s_1 + 1 \),
\( s_1 (p - 1) \leq 1 - s_0 \leq 0 \) and \( s_1 = 0, s_0 = 1 \).

Hence, \( s = 1 \); by (1.1), we have a contradiction.

(2.2) Suppose

\[
V_j = w_0 (x_{j_1}, \ldots, x_{j_{s-1}+1}) j + \sum_{i=1,\ldots,s} w_i (x_{j_1}, \ldots, x_{j_{s-1}+1}) (i + j) x_1^{i+j} = 0
\]

for all \( j \). Then

\[
(V_j)^{-1} = w_0 (x_{j_1}, \ldots, x_{j_{s-1}+1}) j + \sum_{i=1,\ldots,s} w_i (x_{j_1}, \ldots, x_{j_{s-1}+1}) (i + p - 1) x_1^{i+p-1} = 0
\]

for all \( j \).

In particular,

\[
(V_{p-1})^{-1} = w_0 (x_{j_1}, \ldots, x_{j_{s-1}+1}) (p - 1) + \sum_{i=1,\ldots,s} w_i (x_{j_1}, \ldots, x_{j_{s-1}+1}) (i + p - 1) x_1^{i+p-1} = 0
\]

and therefore, \( w_0 (x_{j_1}, \ldots, x_{j_{s-1}+1}) = 0 \); a contradiction. \( \square \)

5.1.1. Representation on the \( p^2 \)-tree. We index the \( p^2 \)-tree by sequences from the set of strings \( ij \) where \( 0 \leq i, j \leq p - 1 \) and choose the following transversal for \( H \):

\[
T = \{ x_1^i a^j | \ 0 \leq i, j \leq p - 1 \};
\]

we will indicate \( x_1^i a^j \) by \( ij \).

(1) Permutation representation. Given \( g \in G \), we denote the permutation induced by \( g \) on the transversal \( T \) by \( g^\sigma \). Then,

\[
\begin{align*}
\sigma : & \quad x_1^i a^j \rightarrow x_1^i a^{j+1} \text{ if } 0 \leq j \leq p - 2, \\
x_1^i a^{p-1} & \rightarrow x_1^i, \\
\sigma : & \quad ij \rightarrow i (j + 1) \text{ if } 0 \leq j \leq p - 2, \\
i (p - 1) & \rightarrow i 0; \\
(x_1)^\sigma : & \quad x_1^i a^j \rightarrow x_1^{i+1} a^j \text{ if } 0 \leq i \leq p - 2, \\
x_1^{p-1} a^j & \rightarrow a^j, \\
ij & \rightarrow (i + 1) j \text{ if } 0 \leq i \leq p - 2, \\
(p - 1) j & \rightarrow 0 j; \\
(x_1)^\sigma : & \quad x_1^i a^j \rightarrow x_1^j a^i \text{ for } 0 \leq i \leq p - 1, 2 \leq l \leq d, \\
ij & \rightarrow ij.
\end{align*}
\]

(2) Cofactors (or Schreir elements) of actions on \( T \).
We calculate

\[
\begin{align*}
\text{cofactors of } a : & \quad (x_1^i a^j, a) (x_1^i a^{j+1})^{-1} = 1 \text{ if } 0 \leq j \leq p - 2, \\
(x_1^i a^{p-1}, a) (x_1^i)^{-1} & = 1;
\end{align*}
\]
cofactors of $x_1$:

$$(x_1 a_1 x_1)(x_1^{-1} a_1^{-1})^{-1} = x_1^i a_j x_1 a_j x_1^{-i-1} = (a^j)^{-x_1^{-i-1}} (a^{-j}) x_1^{-1} = (a^j)(x_1^{i-1} x_1^{-1/(i+1)})$$

$$= (a^j)^{-x_1^{-1}(1-x_1^{-1})} \text{ if } 0 \leq i \leq p-2,$$

$$(x_1^p a_1 x_1) a_j^{-1} = x_1^{p-1} a_1 a_j^{-1} = x_1^p (a^j)^{x_1^{-1}} = (a^j)(x_1^{-1} x_1^p) x_1^p;$$

cofactors of $x_l$ ($2 \leq l \leq d$):

$$(x_1^j a_j x_l)(x_1^j a_j^{-1})^{-1} = x_1^j a_j x_l a_j^{-1} = (a^j)(x_1^{-1} x_l^{-1}) x_l.$$  

(3) Images of cofactors under $f$.

We calculate

$$f : (a^j)^{-x_1^{-1}(1-x_1^{-1})} \rightarrow (a^j)^{x_2^{-1}},$$

$$f : (a^j)^{(x-1)x_l^{-p}} x_l^p \rightarrow (a^j)^{x_2^{-1}} x_2;$$

$$f : (a^j)^{-x_1^{-1}(x_1^{-1})} x_l \rightarrow (a^j)^{-x_2^{-1} x_l^{-1}(x_1^{-1})} x_l.$$  

Therefore,

$$(x_1^j)_{ij} = \left( (a^j)^{x_2^{-1}} \right)^{x_l} \text{ for } 0 \leq i \leq p-2$$

$$= \left( (a^j)^{x_2^{-1}} x_2 \right)^{x_l} \text{ for } i = p-1$$

$$(x_1^j)_{ij} = \left( (a^j)^{-x_2^{-1} x_l^{-1}(x_1^{-1})} x_l \right)^{x_l} \text{ for } 0 \leq i \leq p-1.$$  

On identifying $a^x$ with $a$, $x_1^x$ with $x_1$ and $x_l^x$ with $x_l$ we obtain

$$(x_1)_{ij} = \left( (a^j)^{x_2^{-1}} \right)^{x_l} \text{ for } 0 \leq i \leq p-2$$

$$= \left( (a^j)^{x_2^{-1}} x_2 \right)^{x_l} \text{ for } i = p-1,$$

$$(x_1)_{ij} = \left( a^{-x_2^{-1} x_l^{-1}(x_1^{-1})} x_l \right)^{x_l} \text{ for } 0 \leq i \leq p-1.$$  

5.2. Proof of Theorem 5. Let $p = 2$ and re-index the 4-tree by sequences from \{0, 1, 2, 3\}. Then the above representation becomes

$$a = (0, 1)(2, 3),$$

$$x_1 = \left( 1, a x_2^{-1}, x_2, a x_2^{-1} x_2 \right)(0, 2)(1, 3),$$

$$x_2 = \left( x_1, x_1, x_1, a x_2^{-1} x_1 x_1^{-1} x_2^{-1} \right) = \left( 1, 1, 1, a x_2^{-1} x_1^{-1} x_2^{-1} \right) (x_1)_{(1)}.$$

(1) Powers of $x_1, x_2$:

$$x_1^{2^n} = \left( x_2^n, x_2^n, x_2^n, a x_2^{-1} x_2^{-1} x_2^{-1(n+1)} x_2^n \right),$$

$$x_1^{2n+1} = \left( x_2^n, a x_2^{-1} x_2^{-1} x_2^{-1} x_2^n \right) (0, 2)(1, 3),$$

$$x_2^n = \left( x_1^n, x_1^n, x_1^n, a x_1^{-1} x_1^{-1} x_1^{-1} x_2^n \right)$$

for all $n$.  

(2) Conjugates of $a$:

\[
\begin{align*}
\alpha^{2n}_1 &= (1, 1, a^{x_2^{-1}} , a^{x_2^{-1} + x_2^{-1}}) (0, 1) (2, 3), \\
\alpha^{2n+1}_1 &= (a^{x_2^{-1}} , a^{x_2^{-1}} , a^{x_2^{-1}}) (0, 1) (2, 3), \\
\alpha^n_2 &= (1, 1, a^{x_2^{-1} + x_2^{-1}} , a^{x_2^{-1} + x_2^{-1}}) (0, 1) (2, 3) \\
\alpha^{2n}_1 \alpha^{x_2^{-1}} &= (1, 1, a^{x_2^{-1} + x_2^{-1}} , a^{x_2^{-1} + x_2^{-1}}) (0, 1) (2, 3), \\
\alpha^{2n+1}_1 \alpha^{x_2^{-1}} &= (a^{x_2^{-1}} , a^{x_2^{-1}} , a^{x_2^{-1}}) (0, 1) (2, 3)
\end{align*}
\]

for all $n, l$.

(3) Products of Conjugates of $a$

Let $0 < n_1 < n_2 < \ldots < n_2s$. Then

\[
\begin{align*}
\alpha^{2n_1+1}_1 + \alpha^{2n_2+1}_1 + \cdots + \alpha^{2n_s+1}_1 &= (1, 1, a^{x_2^{-1} + x_2^{-1} + \cdots + x_2^{-1}} , a^{x_2^{-1} + x_2^{-1} + \cdots + x_2^{-1}}), \\
\alpha^{2n_1+1}_1 + \alpha^{2n_2+1}_2 + \cdots + \alpha^{2n_s+1}_2 &= (1, 1, a^{x_2^{-1} + x_2^{-1} + \cdots + x_2^{-1} + x_2^{-1}} , a^{x_2^{-1} + x_2^{-1} + \cdots + x_2^{-1} + x_2^{-1}}), \\
\alpha^{1+1}_1 + \alpha^{2n_1+1}_2 + \cdots + \alpha^{2n_s+1}_2 &= (1, 1, a^{x_2^{-1} + x_2^{-1} + \cdots + x_2^{-1} + x_2^{-1}} , a^{x_2^{-1} + x_2^{-1} + \cdots + x_2^{-1} + x_2^{-1}}).
\end{align*}
\]

(4) States of the automaton $x_1 = (1, a^{x_2^{-1}} , x_2 , a^{x_2^{-1}} x_2) (0, 2) (1, 3)$

Using the above equations together with

\[
\begin{align*}
\alpha^{2n_1+1}_1 \alpha^{x_2^{-1}} x_1 &= (a^{x_2^{-1}}, 1, a^{x_1^{-1}} x_2, a^{x_2^{-1}} x_1 x_2) (0, 3) (1, 2), \\
\alpha^{2n_1+1}_1 \alpha^{x_2^{-1}} x_2 &= (a^{x_2^{-1}}, a^{x_2^{-1}}, a^{x_2^{-1}}, x_2, a^{x_2^{-1}}) (0, 3) (1, 2),
\end{align*}
\]

we calculate the states of $x_1$:

\[
\begin{align*}
\alpha^{x_2^{-1}} x_1 &= (1, 1, a^{x_2^{-1} + x_2^{-1} + x_2^{-1}} , a^{x_2^{-1} + x_2^{-1} + x_2^{-1}}) (0, 1) (2, 3), \\
\alpha^{x_2^{-1}} x_2 &= (x_1, x_1, x_1, a^{x_2^{-1} + x_2^{-1} + x_2^{-1}}) (0, 1) (2, 3), \\
\alpha^{x_2^{-1} + x_1^{-1}} x_1 &= (a^{x_2^{-1}}, a^{x_2^{-1}}, a^{x_2^{-1}}, a^{x_2^{-1}}) (0, 1) (2, 3), \\
\alpha^{x_2^{-1} + x_1^{-1}} x_2 &= (a^{x_2^{-1}}, a^{x_2^{-1}}, a^{x_2^{-1}}, a^{x_2^{-1}}) (0, 1) (2, 3),
\end{align*}
\]

\[
\begin{align*}
\alpha^{x_1^{-1} x_1^{-1}} x_1 &= (a^{x_2^{-1}}, a^{x_2^{-1}}, a^{x_2^{-1}}, a^{x_2^{-1}}) (0, 1) (2, 3), \\
\alpha^{x_1^{-1} x_2^{-1}} x_2 &= (a^{x_2^{-1}}, a^{x_2^{-1}}, a^{x_2^{-1}}, a^{x_2^{-1}}) (0, 1) (2, 3), \\
\alpha^{x_2^{-1} + x_1^{-1}} x_2 &= (a^{x_2^{-1}}, a^{x_2^{-1}}, a^{x_2^{-1}}, a^{x_2^{-1}}) (0, 1) (2, 3), \\
\alpha^{x_2^{-1} + x_1^{-1}} x_1 &= (a^{x_2^{-1}}, a^{x_2^{-1}}, a^{x_2^{-1}}, a^{x_2^{-1}}) (0, 1) (2, 3),
\end{align*}
\]
Therefore $x_1$ has in total 12 states:

$$\begin{align*}
\{ & e, x_1, x_2, \\
& a^{-2} x_2, a x_2^{-1} x_1, a x_2^{-1} x_2, \\
& a x_1^{-1} x_2, a x_1^{-1} x_2^{-1} x_1, a x_1^{-1} x_2^{-1} x_2, \\
& a x_1^{-1} x_2^{-1} x_2, a x_2^{-1} x_2^{-1} x_1, a x_2^{-1} x_2^{-1} x_2 \}.
\end{align*}$$

Since $x_2$ is a state of $x_1$ it follows that $G$ is a finite state group.

(5) Incidence matrix for the graph of states of $x_1$.

Rewrite the states of $x_1$ as

$$
\begin{align*}
& s_1 = x_1, s_2 = x_2, \\
& s_3 = a x_2^{-1}, s_4 = a x_2^{-1} x_1, s_5 = a x_2^{-1} x_2, \\
& s_6 = a x_1^{-1} x_2, s_7 = a x_1^{-1} x_2^{-1} x_1, s_8 = a x_1^{-1} x_2^{-1} x_2, \\
& s_9 = a x_1^{-1} x_2^{-1} x_2, s_{10} = a x_1^{-1} x_2^{-1} x_2 x_1, s_{11} = a x_1^{-1} x_2^{-1} x_2 x_2.
\end{align*}
$$

The incidence matrix of the automaton $x_1$ reads as follows:

$$
\begin{pmatrix}
& e & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} \\
e & 4 & & & & & & & & & & \\
s_1 & 1 & 1 & 1 & 1 & & & & & & & & \\
s_2 & 3 & 1 & & & & & & & & & & \\
s_3 & 2 & & 2 & & & & & & & & & & \\
s_4 & 1 & & 1 & & & & & & & & & & \\
s_5 & 3 & & & 1 & & & & & & & & & \\
s_6 & 2 & & & 2 & & & & & & & & & \\
s_7 & 1 & & 1 & & & & & 1 & & & & & \\
s_8 & 1 & & 1 & & & & & & & & & & \\
s_9 & 4 & & & & & & & & & & & & \\
s_{10} & 1 & & 1 & & & & & & & & & & \\
s_{11} & 1 & & & 3 & & & & & & & & &
\end{pmatrix}.
$$

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