Defining an SU(3)-Casson/U(2)-Seiberg-Witten integer invariant for integral homology 3-spheres

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Abstract

The SU(3)-Casson invariant for integral homology 3-spheres as studied by Boden-Herald possesses a ‘spectral flow obstruction’ to being an integer valued invariant which depends only on the non-degenerate (perturbed) moduli space of flat SU(3)-connections. This obstruction is the non-trivial spectral flow of a family of twisted signature operators in 3-dimensions. The parallel U(2)-Seiberg-Witten construction also has an obstruction but from the non-trivial spectral flow of a family of twisted Dirac operators. By taking the SU(3)-flat and U(2)-Seiberg-Witten equations simultaneously the obstructions can be made to cancel and an integer invariant is obtained.

1 Introduction

In 1985 Casson [2] introduced his now well-known integer invariant for (oriented) integral homology 3-spheres (ZHS). This beautiful invariant is a lift of the mod2 \( \mu \)-invariant and with it Casson showed how to prove a number of remarkable theorems in low-dimensional topology. Roughly speaking Casson’s invariant algebraically counts, up to conjugacy the number of representations of the fundamental group into \( SU(2) \). Shortly after Taubes [15] showed how to interpret this as an Euler characteristic/Hopf index in the infinite dimensional setting of gauge theory, the bridge being the correspondence between flat connections on an \( SU(2) \)-bundle and representations into \( SU(2) \) of the fundamental group. Meanwhile Floer [8] defined his homology groups based on Witten’s Morse theory ideas applied to gauge theory – the Casson invariant appears as (half) the Euler characteristic of the Floer groups.
A natural idea is to extend Casson’s invariant by utilizing gauge groups different from $SU(2)$, especially to higher $SU(n)$. A proposal along Casson’s original approach involving Heegaard splittings and representation varieties was announced by Cappell-Lee-Miller [6] in 1990. Then in 1998 Boden-Herald [4] presented the first detailed account of an $SU(3)$-Casson invariant based on Taubes’ interpretation of Casson’s invariant.

1.1 Spectral flow obstruction in the $SU(3)$-Casson invariant

Let $Y$ denote an oriented ZHS and $F \to Y$ an $SU(3)$-bundle. In the $SU(3)$-Casson invariant consider $\mathcal{A}_F$ the space of $SU(3)$-connections on $F$ and $\mathcal{G}_F$ the group of gauge transformations of $F$. (For the purpose of this introduction we omit the Sobolev completions of these $C^\infty$ objects. Details are in the main body of the article.) The flat connections on $F$ are the critical points of the Chern-Simons functional $cs$ on $\mathcal{A}_F$. The quotient space $\mathcal{A}_F/\mathcal{G}_F$ is stratified; the highest comes from the irreducibles $\mathcal{A}_F^*$ (finite stabilizer under $\mathcal{G}_F$) and the lowest the trivial connections $\mathcal{A}_F^0$. The only intermediate strata relevant for us comes from the $U(2)$-reducibles $\mathcal{A}_F^I$ (those with $U(1)$-stabilizer). For a ZHS the moduli space of flat connections splits as the stratas above:

$$\mathcal{M}^{su} = \mathcal{M}^{su*} \cup \mathcal{M}^{su,I} \cup \{[\Theta]\}$$

where $[\Theta]$ is the orbit of any trivial connection $\Theta$. $\mathcal{M}^{su,I}$ is exactly the moduli space of flat $U(2)$-connections.

A central ingredient is the necessity of perturbing the Chern-Simons function in order to make the critical points $\mathcal{M}^{su}$ a finite set of non-degenerate points. It is a theorem that this is always possible. Let us denote the perturbed moduli space as $\mathcal{M}^{su}_{\pi}$. Then in the same manner as the splitting above we have

$$\mathcal{M}^{su}_{\pi} = \mathcal{M}^{su*}_{\pi} \cup \mathcal{M}^{su,I}_{\pi} \cup \{[\Theta]\}.$$  

Following Taubes, every point $x$ in $\mathcal{M}^{su}_{\pi}$ can be assigned an orientation $\varepsilon(x) = \pm 1$ (take by convention $\varepsilon([\Theta]) = +1$) by considering the parity of the spectral flow of the Floer-Taubes operator $L^{sw}$ from $[\Theta]$ to $x$. Fundamentally one would like to create an topological invariant by taking the algebraic sum

$$\sum_{x \in \mathcal{M}^{su}_{\pi}} \varepsilon(x)$$
as the Euler characteristic. However this sum can change with different choices of perturbation $\pi$ – the phenomena of bifurcation of $\mathcal{M}_{\pi}^{su}$ along $\mathcal{M}_{\pi}^{su,I}$ (i.e. birth or death of points in $\mathcal{M}_{\pi}^{su}$). This is corrected by the addition of a counter-term associated to each point $x'$ in $\mathcal{M}_{\pi}^{su,I}$. Along the strata $\mathcal{A}_{\pi}^{I}$ the Floer-Taubes operator splits orthogonally as normal and tangential components. (With some care this splitting can be made to extend over the trivial strata.) The normal operator $N_{\pi}^{su,I}$ is complex, the complex structure coming from the stabilizer of $\mathcal{G}_F$ along $\mathcal{A}_{\pi}^{I}$. $N_{\pi}^{su,I}$ is essentially a twisted version of the signature operator in 3-dimensions. Denote by $SF_{\nu}^{su,I}([\Theta],x)$ the complex spectral flow of $N_{\pi}^{su,I}$ from $[\Theta]$ to $x$ in $\mathcal{A}_{\pi}^{0,I}/\mathcal{G}_F$. If this did not depend on the path chosen then the expression

$$\sum_{x' \in \mathcal{M}_{\pi}^{su,I}} \varepsilon(x')SF_{\nu}^{su,I}([\Theta],x')$$

(1.4)

would be the sum total of the required counter-terms. Unfortunately the proposed counter-term does depend on the path chosen. This dependency is traced back to the fact that $\pi_1(\mathcal{A}_{\pi}^{0,I}/\mathcal{G}_F) \cong \mathbb{Z}$ and the value of $SF_{\nu}^{su,I}$ around the generator is $\pm 2$. This is the (spectral flow) obstruction to defining an integer $SU(3)$-Casson invariant utilizing only the non-degenerate (perturbed) moduli space.

Boden-Herald [4] solve this by firstly allowing only very small perturbations so as to compare the perturbed and unperturbed moduli space. This in turn enables using the Chern-Simons function on the unperturbed moduli space to cancel the obstruction on the counter-terms. In this way a topological invariant is obtained, but it is no longer obviously an integer. Boden-Herald-Kirk [5] extract an integer invariant by a slight modification of the preceding. However the definition still relies heavily on the unperturbed moduli space and is not very natural. Cappell-Lee-Miller [7] found an ingenious solution involving a certain differential in the Floer homology chain complex. Their solution has the advantage of allowing large perturbations, but the definition seems unnatural and invokes a much more complicated object, Floer theory.

### 1.2 $U(2)$-Seiberg-Witten and $SU(3)$-Casson

Given the mentioned difficulties in establishing a satisfactory integral valued $SU(3)$-Casson invariant we propose an entirely different approach using the moduli space for the parallel $U(2)$-Seiberg-Witten (SW) equations. Our main thesis may be summarized as follows:
The obstruction in $SU(3)$-Casson can be made to cancel against the obstruction in (twice) $U(2)$-Seiberg-Witten. The $SU(3)$-Casson and $U(2)$-Seiberg-Witten equations when taken in conjunction yield an integral invariant of integral homology spheres involving only the non-degenerate (perturbed) moduli spaces, path independent spectral flow counter-terms and Atiyah-Patodi-Singer spectral invariants and is not limited by small perturbations.

Let us briefly explain the idea which is detailed in the main body of the article. (A preliminary study of this was treated in [14].) Let $E \to Y$ be a $U(2)$-bundle and $S \to Y$ the complex spinor bundle. Denote by $\mathcal{C}_E$ the space of pairs consisting of connection and section of $S \otimes E$ (the tensor product is taken over $\mathbb{C}$). The $U(2)$-Seiberg-Witten solutions are the critical points of the Chern-Simons-Dirac function $csd : \mathcal{C}_E \to \mathbb{R}$. The quotient space $\mathcal{C}_E / \mathcal{G}_E$ by the gauge transformations $\mathcal{G}_E$ has highest strata the irreducibles $\mathcal{C}^*_E / \mathcal{G}_E$, and the lowest the trivial connections $\mathcal{C}^0_E / \mathcal{G}_E$.

There are two intermediate strata relevant to us, we denote them as Type I and Type II.

Type I reducibles are those where the spinor component is zero. Type II reducibles are essentially configurations where the connection gives a parallel reduction $E = L_0 \oplus L_1$ and one of the components of the spinor in $S \otimes E = (S \otimes L_0) \oplus (S \otimes L_1)$ is identically zero.

We have a corresponding decomposition of the Seiberg-Witten moduli space (as before we need to introduce non-degenerate perturbations $\pi'$):

$$\mathcal{M}_\pi^{sw} = \mathcal{M}_\pi^{sw} \cup \mathcal{M}_\pi^{sw, I} \cup \mathcal{M}_\pi^{sw, II} \cup \{[\Theta]\}. \quad (1.5)$$

As the metric and/or perturbation is varied bifurcation phenomena again happens along the lower stratas. Since Type I reducibles are configurations where the spinor component is zero, we can identify $\mathcal{M}_\pi^{sw, I}$ as the moduli space of (perturbed) flat $U(2)$-connections. Notice that $\mathcal{A}^{0,1}_F / \mathcal{G}_F = \mathcal{C}^{0,1}_E / \mathcal{G}_E$. It is straightforward to arrange the perturbations so that we have the identification

$$\mathcal{M}_\pi^{su, I} = \mathcal{M}_\pi^{sw, I} \quad (1.6)$$

In the SW portion of the theory we again need to consider counter-terms associated to a point $x$ in $\mathcal{M}_\pi^{sw, I}$. This involves the complex spectral flow $SF^{sw, I}_\nu([\Theta], x)$ from $[\Theta]$ to $x$ of a corresponding normal operator $N^{sw, I}$ which is a twisted Dirac operator. Again we have a spectral flow obstruction – around a generator of $\pi_1(\mathcal{C}^{0,1}_E / \mathcal{G}_E) \cong \mathbb{Z}$
the value of $\text{SF}^{\text{sw},I}_\nu$ is $\pm 1$. Now following a well established tradition in index theory we play-off $\text{SF}^{\text{su},I}_\nu([\Theta], x)$ (essentially a signature operator) against $\text{SF}^{\text{sw},I}_\nu([\Theta], x)$ (a Dirac operator) by working with the sum (after taking signs into account)

$$
\varepsilon(x) \left\{ (\text{SF}^{\text{su},I}_\nu + 2\text{SF}^{\text{sw},I}_\nu)([\Theta], x) + 4c(g, \pi') \right\}.
$$

(1.7)

The extra $4c(g, \pi')$ is an Atiyah-Patodi-Singer (APS) [3] spectral invariant term inserted to suppress spectral-flow at $[\Theta]$ under variation of the metric. When $\pi' = 0$ this takes the form

$$
c(g, 0) = \xi + \frac{1}{8} \eta(B)
$$

(1.8)

where $\xi$ is the APS spectral invariant for the (untwisted) Dirac operator on $Y$ and $\eta(B)$ that for the (untwisted) signature operator. By index theorems this sum is always an integer and reduces mod2 to the $\mu$-invariant for $Y$. The sum (1.7) is now independent of the path chosen and should appear as the correct counter-term in a proposed invariant. It is then clear that in the highest strata we should base a topological invariant on the following combination of SW and Casson theories:

$$
\sum_{x \in M^{\text{sw},s}} \varepsilon(x) + 2 \sum_{x' \in M^{\text{sw},s}_{\pi'}} \varepsilon(x').
$$

(1.9)

This does not quite complete the invariant for as we vary the metric and/or perturbation, in the SW-case there is also bifurcation along (i) the trivial strata (ii) the Type II strata (this is essentially the $U(1)$-SW moduli space). Case (ii) can be straightforwardly handled by spectral flow terms which do not suffer from anomalies.

Case (i) is different in that bifurcations along the trivial strata give birth or death into the Type II strata. Thus what is needed is a counter-term for the counter-term associated to $M^{\text{sw},II}_{\pi'}$. We identify such an expression which turns out to be

$$
c(g, \pi')(c(g, \pi') - 1).
$$

(1.10)

This completes all the counter-terms required to create a topological invariant.

We finish this introduction with a technical remark. The traditional approach to perturbations is to perturb the Chern-Simons(-Dirac) function. In the SW-case there are problems to getting an adequate many for transversality – the author has yet to find a satisfactory solution this way. Instead we perturb the gradient of the Chern-Simons(-Dirac) function directly (which is a much simpler procedure) in a way which preserves most of the features we would expect from gradient perturbations.
In §2 we discuss the basics of $U(2)$-Seiberg-Witten and $SU(3)$-Casson theory. This includes introducing the class of admissible perturbations, compactness of the moduli space and the splitting of slice spaces along the reducible stata. We also analyze the behaviour of an admissible perturbation near a reducible stata, especially its *normal linearization*. We conclude with showing how to construct adequately many admissible perturbations so as to obtain a non-degenerate moduli space.

§3 begins with a discussion of the Floer-Taubes operator which is the operator which allows us to define orientations as well as counter-terms. We show that along the reducibles the normal component of this operator is self-adjoint and thus the notion of spectral-flow is defined. The definition of the invariant now proceeds after a discussion of the counter-terms. §4: Proof of the main theorem. §5 Treats orientation issues which are crucial to obtaining the correct signs for all the terms in the definition of the invariant.

## 2 $U(2)$-Seiberg-Witten and $SU(3)$-Casson theory

**Standing Convention** *Throughout this article $Y$ will denote an oriented closed integral homology 3-sphere (ZHS). $Y$ will also be assumed to be have a fixed Riemannian metric $g$.*

### 2.1 The Equations

Let $P \to Y$ be the unique spin-structure on $Y$ (up to equivalence). In the (real) Clifford bundle $CL(T^*Y) \cong CL(Y)$ the volume form $\omega_Y$ has the property that $\omega_Y^2 = 1$. The action of $\omega_Y$ on $CL(Y)$ induces a splitting into $\pm 1$ eigenbundles $CL^+ \oplus CL^-$. Both $CL^+$ and $CL^-$ are bundles of algebras over $Y$ with each fibre isomorphic, as an algebra, to the quaternions $\mathbb{H}$ (see for instance [1]). Let $S \to Y$ be the complex spinor bundle on which $CL^+$ acts non-trivially. This is a rank 2 complex Hermitian vector bundle.

Fix $E \to Y$ a (trivial) $U(2)$-vector bundle, i.e. a rank 2 Hermitian complex vector bundle. Twist $S$ by forming the tensor product (over $\mathbb{C}$) $S \otimes E$, a rank 4 complex vector bundle. The Clifford action on $S$ naturally extends to $S \otimes E$ by the rule $\alpha \cdot (\phi \otimes e) = (\alpha \cdot \phi) \otimes e$. The action of $\Lambda^2 \otimes \text{ad}E$ on $S \otimes E$ defines a fibrewise bilinear...
form \{\cdot\}_0^* on \ S \otimes E \ by \ the \ rule

\langle \alpha \cdot \phi, \psi \rangle = \langle \alpha, \{\psi \cdot \phi\}_0^* \rangle. \quad (2.1)

The \textit{U}(2)-Seiberg-Witten Equation (in 3-dimensions)\ is the equation defined for a pair \((A, \Phi)\) consisting of a connection on \(E\) and a spinor \(\Phi\) (i.e. section of \(S \otimes E\)). The equation reads:

\[ F_A - \{\Phi \cdot \Phi\}_0^* = 0, \quad D_A \Phi = 0, \quad (2.2) \]

where \(F_A\) is the curvature of \(A\), and since \(A\) is an \(U(2)\)-connection, \(F_A\) is a section of \(\Lambda^2 \otimes \text{ad}E\). \(D_A\) is the twisted Dirac operator on \(S \otimes E\) and \(\{\cdot\}_0^*\) the quadratic form above.

Let \(F \rightarrow Y\) be a fixed (trivial) \(SU(3)\)-vector bundle over \(Y\), that is a rank 3 Hermitian complex vector bundle with trivialized determinant. In \(SU(3)\)-Casson theory we are concerned with the flat connections on \(F\), i.e. the solutions of the flat equation

\[ F_A = 0 \quad (2.3) \]

where \(A\) is an \(SU(3)\)-connection on \(F\).

2.2 Configuration Spaces

\(C_E\) will denote the \textit{configuration space} of pairs \((A, \Phi)\) where \(A\) is unitary connection on \(E\) and \(\Phi\) a twisted spinor, i.e. a section of \(S \otimes E\). In this article we shall be working in an \(L^2\)-Sobolev gauge theory. This means both \(A\) and \(\Phi\) shall be of class \(L^2\). The \textit{gauge automorphism group} \(G_E\) will be the \(L^2\)-sections of \(\text{Ad}E\), the unitary bundle automorphisms of \(E\). The action of \(G_E\) on \(C_E\) is from the right as \(g \cdot (A, \Phi) = (g(A), g^{-1}\Phi)\) where the convention is that \(g(A)\) is the pull-back connection.

The \textit{U}(2)-SW moduli space \(M^\text{sw}\) is the solutions of \(\text{(2.2)}\) modulo gauge equivalence.

In a likewise manner \(A_F\) is the space of \(L^2\)-Sobolev \(SU(3)\)-connections on \(F\). The gauge group of \(L^2\) automorphisms we denote by \(G_F\). The \textit{SU(3)-flat moduli space} is denoted by \(M^\text{fl}\).

\(C_E\) is an affine space modelled on \(L^2_2(\Lambda^1 \otimes \text{ad}E) \times L^2_2(S \otimes E)\). Here \(\text{ad}E\) denotes the bundle of Hermitian skew endomorphisms of \(E\). The tangent space to the identity of
\( \mathcal{G}_E \) is \( L_3^2(\text{ad}E) \) and the derivative at the identity of the gauge orbit map \( \mathcal{G}_E \to C_E, \ g \mapsto g \cdot (A, \Phi) \) is given by the operator

\[
\delta^0_{A, \Phi} : L_3^2(\text{ad}E) \to L_2^2(\Lambda^1 \otimes \text{ad}E) \oplus L_2^2(S \otimes E), \\
\delta^0_{A, \Phi}(\gamma) = (d_A \gamma, -\gamma(\Phi)).
\] (2.4)

The slice space at \((A, \Phi)\) is the \( L^2 \)-orthogonal to the image of \( \delta^0_{A, \Phi} \) and is denoted by \( X_{A, \Phi} \).

\( \mathcal{A}_F \) is an affine space modelled on \( L_2^2(A^1 \otimes \text{ad}F) \). The tangent space at the identity to \( \mathcal{G}_F \) is \( L_3^2(\text{ad}F) \) and the derivative at the identity of the gauge orbit map \( \mathcal{G}_F \to \mathcal{A}_F, \ g \mapsto g \cdot A \) is given by the operator

\[
d^0_A : L_3^2(\text{ad}F) \to L_2^2(A^1 \otimes \text{ad}F). \] (2.5)

The slice space at \( A \) is the \( L^2 \)-orthogonal to the image of \( d^0_A \) and is denoted by \( X_A \).

**Remark** In \( L_2^2 \)-gauge theory in 3-dimensions the connections and spinors are continuous objects. Since we are in the continuous range for Sobolev theory, the SW and flat equations are well-defined as equations in \( L_1^2 \). Details of Sobolev gauge theory can be found in for instance Freed-Uhlenbeck \[9\].

### 2.3 Reducibles

We shall call \((A, \Phi) \in C_E\) reducible if the stabilizer of \((A, \Phi)\) is non-trivial, otherwise we call \((A, \Phi)\) irreducible. Geometrically a reduction happens in two ways. In the first case \( \Phi = 0 \); then \( \text{stab}(A, \Phi) \) is at least \( U(1) \) (the gauge transformations which are multiplication by a complex unit). The second is when \( A \) is reducible as \( A_0 \oplus A_1 \) in a parallel splitting \( E = L_0 \oplus L_1 \) and \( \Phi \) is \( A \)-reducible in the sense that \( \Phi = (\phi_0, \phi_1) \in L_2^2(S \otimes L_0) \oplus L_2^2(S \otimes L_1) \) with at least one of \( \phi_{0,1} = 0 \). There are various reducible strata with stabilizers \( U(1), U(1) \times U(1) \) and \( U(2) \) however we shall only be concerned with the following ones:

- **Type I**: \( \Phi = 0 \) and \( A \) is irreducible as a connection on \( E \). In this case \( \text{stab}(A, \Phi) = \text{stab}(A) \) under the action of \( \mathcal{G}_E \) and this is easily seen to be just those which are multiplication by a complex unit. Thus \( \text{stab}(A) \cong U(1) \).
- Type II: $A$ is reducible as $A_0 \oplus A_1$ and $\Phi = (\phi_0, 0) \neq 0$. The stabilizer consists of the gauge transformations which have block diagonal form

$$
\begin{pmatrix}
1 & 0 \\
0 & g
\end{pmatrix}.
$$

Since the gauge automorphisms are sections of $\text{Ad}E$ (which is fibrewise $\cong U(2)$) we see that $g \in U(1)$. Thus in this case $\text{stab}(A, \Phi) \cong U(1)$ again.

- Trivial: $A$ is a trivial connection $\Theta$ and $\Phi = 0$. Here $\text{stab}(A, 0) = \text{stab}(A) \cong U(2)$.

As general notation the irreducible portion of $C_E$ shall be denoted by $C_E^*$ and the reducible portion $C_E$. The Trivial, Type I and II reducible stratas shall be denoted by $C_E^0$, $C_E^I$ and $C_E^{II}$ respectively. Note that within our definition $C_E^0$, $C_E^I$ and $C_E^{II}$ are mutually disjoint.

Occasionally it will be useful to specify the splitting $E = L_0 \oplus L_1$ in a Type II reducible. We denote $C^{II}(L_0, L_1) \subset C^{II}$ the subset with the given splitting and with the spinor component in $S \otimes L_1$ vanishing. Under the action of $G_E$, $C^{II}(L_0, L_1)$ sweeps out $C^{II}$.

Finally let $M^{sw*} = M^{sw} \cap C_E^*/G_E$ and $M^{sw,r} = M^{sw} \setminus M^{sw*}$ denote the irreducible and reducible portions of the moduli space.

**Lemma 2.1** The only possible reducible SW-solutions on the ZHS $Y$ are of Type I, II or Trivial. The Type I reducibles correspond to the irreducible solutions of the flat equation $F_A = 0$ on $E$. The Type II reducibles correspond to the solutions of the $U(1)$-SW-equation.

**Proof** (sketch) If $\Phi = 0$ then the SW-equation reduces to the flat equation $F_A = 0$ and the only reducible solutions on a ZHS are trivial ones. The stabilizer of an irreducible solutions is clearly $U(1)$. On the other hand if $A = A_0 \oplus A_1$ is reducible and $\Phi = (\phi_0, \phi_1) \in L_2^2(S \otimes L_0) \oplus L_2^2(S \otimes L_1)$ is $A$-reducible with say $\phi_1 = 0$ then the $U(2)$-SW-equation reduces to the two sets of equations: (i) $F_{A_0} = \{\phi_0 \cdot \phi_0\}_0$, $D_{A_0} \phi_0 = 0$. This is the $U(1)$-SW-equation. Assume $\phi_0 \neq 0$, otherwise as before on a ZHS we are back in the trivial solution (ii) $F_{A_1} = 0$, this clearly has only the trivial solution. Thus, apart from Type I and trivial reducibles, we only get Type II. qed.
A reducible $SU(3)$-connection admits a parallel splitting $A = A_0 \oplus A_1$ corresponding to $F = F_0 \oplus F_1$. We refer to a $U(2)$-reducible or Type I as one for which $F_0$ is an $U(2)$-bundle and $A_0$ is irreducible. Since $F$ is an $SU(3)$-bundle, this forces $F_1$ to be $\cong \det F_0$ and $A_1$ the connection induced by $A_0$. The stabilizer of a $U(2)$-reducible can be verified to be $\cong U(1)$. If additionally $A_1$ is actually trivial we term $A$ to be $SU(2)$-reducible.

The strata of irreducibles is denoted $A^*_F$, the Type I reducibles by $A^I_F$ and the strata of trivial connections by $A^0_F$. Set $M^{su,*} = M^{su} \cap A^*_F / G_F$ and $M^{su,r} = M^{su} \setminus M^{su,*}$.

The following is clear:

**Lemma 2.2** On the ZHS $Y$ the only reducible solutions to the flat $SU(3)$-equations are flat $U(2)$-reducibles (in fact $SU(2)$-reducible) and trivial connections.

**Convention** It will often be convenient to simultaneously treat both the SW and $SU(3)$ theories. To this end we shall employ the notation $Z$ for either $C_E$ or $A_F$, and $G$ the corresponding group of gauge transformations.

### 2.4 Admissible Perturbations

The gauge group $G_E$ acts naturally on the tangent space $L^2_2(A^1 \otimes \text{ad}E) \times L^2_2(S \otimes E)$ by conjugation in the fibres of $\text{ad}E$ and directly on $E$. Thus the notion of a $G_E$-equivariant map $C_E \rightarrow L^2_2(A^1 \otimes \text{ad}E) \times L^2_2(S \otimes E)$ makes sense. Define an *admissible perturbation* $\pi$ on $C_E$ to be a $C^3$ $G_E$-equivariant map $\pi = (k,l): C_E \rightarrow L^2_2(A^1 \otimes \text{ad}E) \times L^2_2(S \otimes E)$ satisfying

(i) $\pi_{A,\Phi} \in X_{A,\Phi}$

(ii) the linearization (i.e. derivative) $(L\pi)_{A,\Phi}$ at $(A,\Phi)$ is a bounded linear operator from $L^2_2(A^1 \otimes \text{ad}E) \oplus L^2_2(S \otimes E)$ back to itself

(iii) there is a uniform bound

$$\|\pi_{A,\Phi}\|_{L^2_{2,A}} = \sum_{i=0}^2 \|(\nabla^A)^i k_{A,\Phi}\|_{L^2} + \|(\nabla^A)^i l_{A,\Phi}\|_{L^2} \leq C$$

(iv) $\pi$ has support contained in $C^*_E \cup C^{0,1,11}_E$. 


(v) \( \pi \) depends only on the spinor component \( \Phi \) in a neighbourhood of the trivial orbit.

If \( \pi = (*k, l) \) as above then we perturb the \( U(2) \)-Seiberg-Witten equation by setting

\[
F_A - \{\Phi \cdot \Phi\}_0 + k_{A,\Phi} = 0, \quad D_A \Phi + l_{A,\Phi} = 0. \tag{2.6}
\]

The corresponding moduli space is denoted \( \mathcal{M}^{sw}_\pi \), the irreducible portion \( \mathcal{M}^{sw\ast}_\pi \), the reducible portion \( \mathcal{M}^{sw, r}_\pi \) etc.

**Lemma 2.3** The only possible reducible perturbed \( U(2) \)-SW-solutions on the ZHS \( Y \) are of Type I, II or Trivial. The Type I reducibles correspond to the irreducible solutions of the perturbed flat equation \( F_A + k_A = 0 \) on \( E \). The Type II reducibles correspond to the solutions of a perturbed \( U(1) \)-SW-equation.

**Proof** The admissible perturbations have by definition support in \( C^*_E \cup C^0_{E,I,II} \) and therefore no new kinds of reductions are introduced. qed.

An admissible perturbation \( \pi' \) on \( A_F \) consists of a \( C^3 \) \( \mathcal{G} \)-equivariant map \( \pi': A_F \to L^2_2(A^1 \otimes \text{ad}F) \) with \( \pi'_A \in X_A \) and satisfying the parallel conditions stated above, i.e. drop the spinor component \( S \otimes E \). In particular \( \pi' \) should be zero in a neighbourhood of the trivial orbit. The perturbed flat \( SU(3) \)-equation now reads as

\[
F_A + *\pi'_A = 0 \tag{2.7}
\]

and the corresponding perturbed moduli spaces \( \mathcal{M}^{su}_\pi, \mathcal{M}^{su\ast}_\pi, \mathcal{M}^{su, r}_\pi \) etc.

**Lemma 2.4** On the ZHS \( Y \) the only reducible solutions to the perturbed flat \( SU(3) \)-equations are perturbed flat \( U(2) \)-reducibles and trivial connections.

**Remark 2.5** The \( SU(2) \)-reducibles which happen in the unperturbed case in general cease to remain so in the perturbed case. (i.e. become \( U(2) \)-reducible but not \( SU(2) \)-reducible.)
2.5 Compactness

Fix a smooth connection $\nabla^0$. A metric on $C_E/G_E$ which induces the (quotient) topology is defined by the rule

$$d([A,\Phi],[A',\Phi']) = \inf_{g \in G_E} \left\{ \sum_{i=0}^{2} \| ((\nabla^0)^i (A-g(A')), \Phi-g^{-1}\Phi') \|_{L^2} \right\}. \quad (2.8)$$

Thus a subset $\mathcal{N} \subset C_E/G_E$ is compact if and only if given any sequence $(A_i,\Phi_i)$ such that the orbits $[A_i,\Phi_i] \in \mathcal{N}$ there exists a subsequence $\{i'\} \subset \{i\}$ and gauge transformations $g_{i'}$ such that $g_{i'}(A_{i'},\Phi_{i'})$ converges in $L^2$. In a similar way a metric is defined on $A_F/G_F$.

**Proposition 2.6** For any admissible perturbation $\mathcal{M}^{\text{sw}}_\pi$ and $\mathcal{M}^{\text{su}}_\pi$ are compact subspaces.

We shall not go through this in detail but refer to [14] where the proof applies in this context. (Property (iii) in the definition of an admissible perturbation plays the crucial role.)

2.6 The Fundamental Elliptic Complex

Following Taubes we should interpret the $U(2)$-SW-equation as the zeros of the gauge equivariant ‘$L^2$-vector field’ on $C_E$

$$\mathcal{X}^{\text{sw}}(A,\Phi) \overset{\text{def}}{=} (\ast F_A - \ast \{\Phi \cdot \Phi\}_0, D_A\Phi). \quad (2.9)$$

This descends to the vector field $\hat{\mathcal{X}}^{\text{sw}}$ on $C_E^*/G_E$ and the zeros are exactly $\mathcal{M}^{\text{sw}*}_\pi$. Let $\mathcal{X}^{\text{sw}}_{\pi} = \mathcal{X}^{\text{sw}} + \pi$, the perturbation of $\mathcal{X}^{\text{sw}}$. The linearization of $\mathcal{X}^{\text{sw}}_\pi$ at $x = (A,\Phi)$ is

$$\delta^1_\pi: L^2_2(A^1 \otimes \text{ad}E) \oplus L^2_2(S \otimes E) \to L^1_2(A^1 \otimes \text{ad}E) \oplus L^1_2(S \otimes E) \quad (2.10)$$

$$(a,\phi) \mapsto (\ast d_Aa - \{\Phi \cdot \Phi\}_0, D_A\phi + a \cdot \Phi) + (L\pi)_{A,\Phi}(a,\phi).$$

At a solution $x = (A,\Phi)$ this fits into an (partial) elliptic complex:

$$L^2_2(\text{ad}E) \xrightarrow{\delta^0_\pi} L^2_2(A^1 \otimes \text{ad}E) \oplus L^2_2(S \otimes E) \xrightarrow{\delta^1_\pi} X_x \cap L^2_1. \quad (2.11)$$

The (harmonic) cohomologies we denote by $H^{\text{sw},i}_x, i = 0, 1, 2$. The non-degeneracy (or regularity) of $x$ is the condition $H^{\text{sw},2}_x = \{0\}$. By equivariance if $x$ is non-degenerate
then so are all points in the orbit of \( x \) and so non-degeneracy of the orbit \([x]\) makes sense.

Let \( \mathcal{X}^{\text{su}} \) denote the map \( \mathcal{A}_F \to L^2(\Lambda^1 \otimes \text{ad}F), \ A \mapsto \star F_A \). This is perturbed as \( \mathcal{X}^{\text{su}} = \mathcal{X}^{\text{su}} + \pi' \). This has the linearization

\[ \star d_A^{1,\pi'}: L^2(\Lambda^1 \otimes \text{ad}F) \to L^2(\Lambda^1 \otimes \text{ad}F), \quad a \mapsto \star d_A a + (L\pi')_A(a). \]  

At a solution the elliptic complex in this instance is

\[ L^2(\text{ad}F) \xrightarrow{\delta^0_x} L^2(\Lambda^1 \otimes \text{ad}F) \xrightarrow{\star d_A^{1,\pi'}} L^2(\Lambda^1 \otimes \text{ad}F) \xrightarrow{\delta^0_x} L^2(\text{ad}F) \]  

with (harmonic) cohomologies \( H^{\text{su},i}_A, \ i = 0,1,2 \). Non-degeneracy is defined just as above.

**Lemma 2.7** \( \dim H^{\text{sw},1}_x = \dim H^{\text{sw},2}_x \) and \( \dim H^{\text{su},1}_x = \dim H^{\text{su},2}_x \).

**Proof** In the SW-case: the partial elliptic complex \( (2.11) \) can be extended to a full one by replacing the last term with

\[ \ldots \xrightarrow{\delta^1_x} L^2(\Lambda^1 \otimes \text{ad}E) \oplus L^2(S \otimes E) \xrightarrow{\delta^0_x} L^2(\text{ad}E) \]  

where \( \delta^0_x \) is the formal \( L^2 \)-adjoint of \( \delta^0_x \). This is an elliptic complex on an odd dimensional manifold and thus has zero index. Finally note that \( H^{\text{sw},0}_x \) is the same as the 3rd-cohomology of the full complex. This proves the lemma in the SW-case. The same argument holds in the \( SU(3) \)-Casson case; this time the full version of \( (2.13) \) is extended by

\[ \ldots \xrightarrow{d^1_x} L^2(\Lambda^1 \otimes \text{ad}F) \xrightarrow{\delta^0_x} L^2(\text{ad}F). \]  

qed.

It follows from the Kuranishi local model that if \([x]\) is a non-degenerate point in \( \mathcal{M}^{\text{sw}}_\pi \) or \( \mathcal{M}^{\text{su}}_\pi \), then \([x]\) is an isolated point in the moduli space.

### 2.7 Slice Splittings along Reducible Stratas

Let \( x \) be a point in a reducible strata \( \mathcal{R} \) in \( \mathcal{Z} = \mathcal{C}_E \) or \( \mathcal{A}_F \). Then there is an \( L^2 \)-splitting of the slice space at \( x \),

\[ X_x = X^\tau_x \oplus X^\nu_x \]  

(2.16)
into a tangential component (superscripted $\tau$) and a normal component (superscripted $\nu$). The tangential component is essentially the slice space for the gauge action on $\mathcal{R}$ and this determines the normal component by taking the $L^2$-orthogonal. A precise way of describing this is as follows. At $x$, $\text{stab}(x)$ acts on $X_x$; the latter can be decomposed into an invariant subspace on which the action is trivial and an invariant subspace on which the action is non-trivial. The first subspace is $X^\tau_x$ and the second $X^\nu_x$.

The splitting is actually induced at the level of the fibers of the various vector bundles involved. In the $U(2)$-SW case the tangent space to the configuration space are the $L^2$-sections of $V = (\Lambda^1 \otimes \text{ad}E) \oplus (S \otimes E)$. $\mathcal{G}_E$ acts on each fiber of $V$ by conjugation $v \mapsto gvg^{-1}$ on the first factor and $\phi \mapsto g^{-1}\phi$ in the second. Then in the manner above, the invariant factors of the action of $\text{stab}(x)$ give rise to a parallel splitting (with respect to the connection component of $x$) of the form

$$V = V^\tau \oplus V^\nu$$

(2.17)

where $V^\tau$ is the factor on which $\text{stab}(x)$ acts trivially. Then $X^\tau_x = X_x \cap L^2_2(V^\tau)$ and $X^\nu_x = X_x \cap L^2_2(V^\nu)$. In the $SU(3)$ case the relevant bundle $V = \Lambda^1 \otimes \text{ad}F$ and $\mathcal{G}_F$ acts on this by conjugation in each fiber.

Let $W$ be $\text{ad}E$ in the SW-case and $\text{ad}F$ in the $SU(3)$-case. The action of $\text{stab}(x)$ on $W$ in the same way as above also gives an $L^2$-decomposition

$$W = W^\tau \oplus W^\nu.$$  

(2.18)

We may identify the components of the splittings in terms of the parallel splitting of $E$ or $F$ determined by the connection component of $x$. For future reference we determine explicitly $W^\tau$, $W^\nu$, $V^\tau$ and $V^\nu$ in the cases that interest us. We precede this by a standard lemma.

**Lemma 2.8** (a) Suppose that $A$ is a reducible $U(2)$-connection on $E$ in a parallel splitting $E = L_0 \oplus L_1$. Then this induces a parallel splitting $\text{ad}E = i\mathbf{R}_0 \oplus i\mathbf{R}_1 \oplus (L_0 \otimes \overline{L}_1)$. Here the $i\mathbf{R}_j$ factor is the subbundle $\text{ad}E$ which is multiplication by pure imaginary constants on $L_j$. (b) Suppose that $A$ is a reducible $SU(3)$-connection on $F$ in a parallel splitting $E \oplus L$ where $E$ is a $U(2)$-bundle. Then $\text{ad}F$ has a parallel splitting with respect to $A$ as $\text{ad}E \oplus (E \otimes \overline{L})$ where the $\text{ad}E$ factor is the natural subbundle induced by the inclusion $E \subset F$. 

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In the following table a vector space denotes the trivialized bundle over $Y$ with fibre that vector space.

|     | Reducible | Parallel splitting | Adjoint bundle | $W^r$ | $W^\nu$ |
|-----|-----------|--------------------|---------------|-------|--------|
| SW  | Trivial   | $E = \mathbb{C}^2$ | $\mathfrak{u}(2)$ | $\{0\}$ | $\mathfrak{u}(2)$ |
|     | Type I    | none               | $\text{ad}E$   | $\text{ad}E$ | $\{0\}$ |
|     | Type II   | $E = L_0 \oplus L_1$ | $i\mathbb{R}_0 \oplus i\mathbb{R}_1 \oplus (L_0 \otimes \overline{T}_1)$ | $i\mathbb{R}_0 \oplus i\mathbb{R}_1 \oplus L_0 \otimes \overline{T}_1$ |
| $SU(3)$ | Trivial | $F = \mathbb{C}^3$ | $\mathfrak{su}(3)$ | $\{0\}$ | $\mathfrak{su}(3)$ |
|     | Type I    | $F = E \oplus L$  | $\text{ad}E \oplus (E_0 \otimes \overline{L})$ | $\text{ad}E \oplus E \otimes \overline{L}$ |

|     | Reduced   | $V^r$               | $V^\nu$      |
|-----|-----------|---------------------|--------------|
| SW  | Trivial   | $\{0\}$            | $A^1 \otimes \mathfrak{u}(2) \oplus (S \otimes \mathbb{C}^2)$ |
|     | Type I    | $A^1 \otimes \text{ad}E$ | $S \otimes E$ |
|     | Type II   | $(A^1 \otimes i\mathbb{R}_0) \oplus (S \otimes L_0) \oplus (A^1 \otimes (L_0 \otimes \overline{L}_1)) \oplus (S \otimes L_1)$ | $(A^1 \otimes (L_0 \otimes \overline{L}_1)) \oplus (S \otimes L_1)$ |
| $SU(3)$ | Trivial | $\{0\}$            | $A^1 \otimes \mathfrak{su}(3)$ |
|     | Type I    | $A^1 \otimes \text{ad}E$ | $A^1 \otimes (E \otimes \overline{L})$ |

### 2.8 Normal Linearization of Perturbations

Now that we have a splitting of the slice spaces along a reducible strata, we wish to establish a corresponding splitting for the linearization

$$(L\pi)_x = (L^\nu\pi)_x \oplus (L^r\pi)_x$$  \hspace{1cm} (2.19)

where the first factor maps into $L^\nu_2(V^\nu)$ and the latter $L^r_2(V^r)$. This and the crucial fact that $(L\pi)_x$ is symmetric are proven below, see Proposition 2.11.

**Lemma 2.9** Let $x \in Z^r$, $g \in \text{stab}(x)$ and $\gamma \in \text{stab}(x)$, the Lie Algebra. Then for all $v$,

$$(L\pi)_x(g \cdot v) = g \cdot (L\pi)_x(v),$$

$$(L\pi)_x(\gamma \cdot v) = \gamma \cdot (L\pi)_x(v).$$

**Proof** The condition that $\pi$ is equivariant with respect to the gauge action means $\pi_{g(x+sv)} = g \cdot \pi_{x+sv}$, $g \in \text{stab}(x)$, $s \in \mathbb{R}$. Note that $g(x + sv) = x + sg \cdot v$ since $g \in \text{stab}(x)$. Differentiating with respect to $s$ and evaluating at $s = 0$ gives

$$(L\pi)_x(g \cdot v) = g \cdot (L\pi)_x(v).$$ \hspace{1cm} (2.20)

The second relation of the lemma is obtained by varying $g$ in (2.20). qed.
Lemma 2.10 \ Let \( x \in \mathcal{Z}^r \). Then \((L\pi)_x\) is a symmetric operator on \( L^2_2(V) \), i.e.

\[
\langle (L\pi)_x(v), w \rangle_{L^2} = \langle v, (L\pi)_x(w) \rangle_{L^2}, \quad v, w \in L^2_2(V).
\] (2.21)

**Proof** \ Let \( \delta^0_x \) denote the linearized gauge action map (2.4) or (2.5) at \( x \). Let \( v \in L^2_2(V) \). The definition of an admissible perturbation implies that

\[
\langle \pi_{x+sv}, \delta^0_{x+sv}(\gamma) \rangle_{L^2} = 0
\] (2.22)

for all \( \gamma \in L^2_3(W) \). Observe that \( \delta^0_{x+sv}(\gamma) = \delta^0_x(\gamma) + s \gamma \cdot v \). Now let \( \gamma \in \text{stab}(x) \) so that \( \delta^0_x(\gamma) = 0 \). Differentiating (2.22) twice with respect to \( s \) and putting \( s = 0 \) gives the relation

\[
\langle (L\pi)_x(v), \gamma \cdot v \rangle_{L^2} = 0.
\] (2.23)

If \( x \in \mathcal{C}^L_{E,F} \) or \( \mathcal{A}^L_F \), then \( \text{stab}(x) \cong U(1) \) induces a complex structure on \( V^\nu \); thus we can find find a \( \gamma \) such that \( \gamma^2 = -1 \). Let \( L = (L\pi)_x \). Applying (2.23) to \( \gamma(v) + w \) and invoking the skew-symmetry of \( \gamma \) and Lemma 2.9 gives

\[
0 = \langle L(\gamma(v) + w), \gamma(\gamma(v) + w) \rangle_{L^2}
= \langle L(v), w \rangle_{L^2} - \langle L(w), v \rangle_{L^2}.
\]

If \( x = \Theta \in \mathcal{A}^L_F \), the definition of \( \pi \) admissible requires \( \pi \) to vanish near \( \Theta \) thus \((L\pi)_\Theta = 0\). If \( x = (\Theta, 0) \in \mathcal{C}^L_E \), the condition of being admissible requires \( \pi_{A,\Phi} \) near \((\Theta, 0)\) to depend only on \( \Phi \). Thus \((L\pi)_\Theta \) acts only on the spinor factor \( L^2_2(S \otimes E) \) of \( V^\nu \) and this as above gets a complex struture from the action of the stabilizer.

Repeating the argument we see that \((L\pi)_\Theta \) is also symmetric. qed.

**Proposition 2.11** \ Let \( x \in \mathcal{Z}^r \). Then there exists an \( L^2 \)-orthogonal splitting

\[
(L\pi)_x = (L^r\pi)_x \oplus (L^\nu\pi)_x : L^2_2(V^r) \oplus L^2_2(V^\nu) \rightarrow L^2_2(V^r) \oplus L^2_2(V^\nu).
\]

Furthermore the normal linearization \((L^r\pi)_x\) has the following properties (i) it commutes with the action of \( \text{stab}(x) \) (ii) it is symmetric with respect to the \( L^2 \)-inner product (iii) if additionally \( \pi = 0 \) on \( \mathcal{Z}^r \) then \((L^r\pi)_x = 0\).

Thus the normal linearization of a perturbation is symmetric and complex linear in the case \( \text{stab}(x) \cong U(1) \) and symmetric and quaternionic linear when \( \text{stab}(x) \cong U(2) \) corresponding to \( x = \Theta \) in the SW-case (and zero for \( x = \Theta \) in the \( SU(3) \)-case).
Firstly if \( x \in \mathbb{Z}^r \) and \( g \in \text{stab}(x) \) then \( \pi_x = \pi_{g(x)} = g \cdot \pi_x \) implies that \( \pi_x \in L^2_2(V^\tau) \). Therefore for \( w \in L^2_2(V^\tau) \) we have \( (L\pi)_x(w) \in L^2_2(V^\tau) \) as well. Now suppose \( v \in L^2_2(V^\nu) \). Since this holds for all \( w \in L^2_2(V^\tau) \) we deduce that \( (L\pi)_x(v) \in L^2_2(V^\nu) \).

This proves the splitting. Items (i) and (ii) follow immediately from Lemmas \ref{lem:splitting} and \ref{lem:splitting}. Item (iii) is clear. \textit{qed.}

**Proposition 2.12** The complexes \((\ref{eq:complex1})\) and \((\ref{eq:complex2})\) at a solution \( x \) decompose orthogonally into tangential and normal complexes:

\[
\begin{align*}
L^2_2(W^\tau) & \xrightarrow{\delta^0_0} L^2_2(V^\tau) \xrightarrow{\delta^1_0} X^\tau_x \cap L^2_1, \\
L^2_2(W^\nu) & \xrightarrow{\delta^0_0} L^2_2(V^\nu) \xrightarrow{\delta^1_0} X^\nu_x \cap L^2_1.
\end{align*}
\]

We have corresponding \( L^2 \)-orthogonal splittings of cohomologies as \( H^i_x = H^i_2^\tau \oplus H^i_2^\nu \).

**Proof** Follows directly from the preceding. \textit{qed.}

### 2.9 Abundance of Perturbations

**Proposition 2.13** There exists non-degenerate admissible perturbations, i.e. \( \pi (\pi') \) such that \( \mathcal{M}^w \) (\( \mathcal{M}^w_\pi \)) consists entirely of non-degenerate points. Furthermore \( \pi (\pi') \) may be chosen to have support in any arbitrarily small gauge invariant neighbourhood of the subspace of unperturbed SW (\( SU(3) \)-flat) solutions.

As before \( Z \) denotes either \( C_E \) or \( A_F \), and \( G \) the gauge group. The strategy is to construct perturbations locally in \( Z/G \). Done correctly these will be admissible. To do so we require some preliminary technical lemmas. Introduce the notation \( B(\varepsilon) \) for the \( \varepsilon \)-ball in the slice space \( X_x \). Denote by \( \beta: X_x \to [0,1] \) a smooth cut-off function with support in \( B(\varepsilon) \). Let \( \delta^0 \) the zeroth differential in the elliptic complex and \( W \to Y \) the adjoint bundle.

**Lemma 2.14** Fix \( x \in \mathcal{Z} \). For all \( \varepsilon > 0 \) sufficiently small there is a differentiable function \( \xi: B(\varepsilon) \times X_x \to (\ker \delta^0)^\perp \subset L^2_3(W) \) such that given any \( (\alpha, v) \in B(\varepsilon) \times X_x \), the equation

\[
v + \delta^0_x \circ \xi(\alpha, v) \in X_{x+\alpha}
\]  

holds. Here \( (\ker \delta^0)^\perp \) denotes the \( L^2 \)-orthogonal complement.
Apply the Implicit Function theorem to the map

\[ H(\xi, \alpha, v) = \delta_{x+\alpha}^0(\delta_x^0(\xi) + v) \]

from \((\ker \delta_x^0)^\perp \times B(\varepsilon) \times X_x \to (\ker \delta_x^0)^\perp \cap L^2_1\). The linearization of \(H\) at the origin restricted to \((\ker \delta_x^0)^\perp\) is an isomorphism. This establishes the existence of the function \(\xi = \xi(\alpha, v)\) as claimed but only for \(\alpha\) and \(v\) defined in sufficiently small neighbourhoods of zero. However notice that if \(v\) satisfies (2.24) then for any real constant \(c\), \(cv\) satisfies the same equation but with \(\xi\) replaced by \(c\xi\). That is we can allow the \(v\) in \(\xi\) to be defined for all \(X_x\) by extending \(\xi\) linearly in that factor. qed.

**Lemma 2.15** Assume the hypotheses of Lemma 2.14. Then for all \(\varepsilon > 0\) sufficiently small there is a constant \(c\) (independent of \(\alpha, v\) and \(\varepsilon\)) such that \(\|\xi(\alpha, v)\|_{L^2_3} \leq c\|v\|_{L^2_2}\).

**Proof** \(\xi\) satisfies \(H(\xi, \alpha, v) = 0\). Thus

\[ \Delta_x \xi + N_1(\alpha, \xi) + N_2(\alpha, v) + \delta_{x}^0(\xi) = 0 \tag{2.25} \]

where \(\Delta_x\) is the Laplacian \(\delta_x^0\delta_x^0\) and \(N_1\) and \(N_2\) are lower order terms. \(N_1\) is a bilinear expression in \(\alpha\) and \(\delta_x^0(\xi)\). \(N_2\) is a bilinear expression in \(\alpha\) and \(v\). After some calculation it is seen that \(N_1, N_2\) satisfy, by Sobolev theorems

\[ \|N_1(\alpha, \xi)\|_{L^2_4} \leq \text{const.} \|\alpha\|_{L^2_2} \|\xi\|_{L^2_3} \tag{2.26} \]

\[ \|N_2(\alpha, v)\|_{L^2_4} \leq \text{const.} \|\alpha\|_{L^2_2} \|v\|_{L^2_2}. \]

On the other hand since \(\Delta_x\) is invertible on \((\ker \delta_x^0)^\perp\),

\[ \|\xi\|_{L^2_3} \leq \text{const.} \|\Delta_x \xi\|_{L^2_3}. \tag{2.27} \]

Now make \(\varepsilon > 0\) sufficiently small so that \(\|\alpha\|_{L^2_2}\) is correspondingly small. Then (2.25), (2.26) and (2.27) give \(\|\xi\|_{L^2_3} \leq \text{const.} \|v\|_{L^2_2}\). qed.

**Proposition 2.16** Assume \(x \in Z^*\). Given any \(v \in X_x\) there is an admissible perturbation \(\pi\) such that \(\pi_x = v\). Furthermore the support of \(\pi\) may be chosen to be contained in an arbitrarily small \(G\)-invariant neighbourhood of the orbit \(G \cdot x\).
Proof Identify the slice space $X_x$ with the actual slice $x + X_x$. Let $\varepsilon > 0$ be sufficiently small so that $B(\varepsilon)$ injects into the quotient space $\mathcal{Z}^*/\mathcal{G}$ and the conclusions of Lemmas \ref{lem:2.11}, \ref{lem:2.12} hold. Construct a perturbation $\pi$ in $X_x$ by the rule that

$$
\pi_{x+\alpha} = \beta(\alpha)v + \delta^{0}_{x} \circ \xi(\alpha, \beta(\alpha)v), \quad \alpha \in B(\varepsilon) \subset X_x.
$$

(2.28)

By construction $\pi$ has support in $B(\varepsilon)$ and $\pi_{x+\alpha} \in X_{x+\alpha}$. Extend $\pi$ to $\mathcal{Z}$ by $\mathcal{G}$-equivariance. What remains is to show that $\pi$ is admissible provided $\varepsilon$ is sufficiently small. Equation (2.28) and Lemmas \ref{lem:2.11}, \ref{lem:2.12} give a uniform bound

$$
\|\pi_{x+\alpha}\|_{L^2} \leq \text{const.} \|v\|_{L^2} \leq C.
$$

Here the Sobolev norm is taken with respect to some fixed connection $A_0$, which is commensurate to the Sobolev norm taken to the connection component $A$ of $x$. Let $a$ denote the non-spinor component of $\alpha$. If $\|\alpha\|_{L^2}$ is sufficiently small then

$$
\|\nabla^{A+a}\pi_{x+\alpha}\|_{L^2} \leq \text{const.} \|\nabla^A\pi_{x+\alpha}\|_{L^2},
$$

$$
\|\nabla^{A+a}\nabla^{A+\alpha}\pi_{x+\alpha}\|_{L^2} \leq \text{const.} \|\nabla^A\nabla^A\pi_{x+\alpha}\|_{L^2}
$$

uniformly. By reducing $\varepsilon$ again if necessary, the uniform bound $\|\pi'_{x'}\|_{L^2_{2,A'}} \leq C$ where $A'$ is the connection component of $x'$, is established. qed.

Proposition 2.17 Assume $x \in \mathcal{Z}^{\alpha}$, $\alpha \in \{I, II\}$. Given any $v \in X^\tau_x$ there is an admissible perturbation $\pi$ such that $\pi_x = v$. Furthermore the support of $\pi$ may be chosen to be contained in an arbitrarily small $\mathcal{G}$-invariant neighbourhood of the orbit $\mathcal{G} \cdot x$.

Proof same argument as Proposition \ref{prop:2.16} qed.

Let $V$ be the vector underlying the configuration space $\mathcal{Z}$ (\S2.7). We saw in Proposition \ref{prop:2.11} that the normal linearization of any admissible perturbation defines a $\text{stab}(x)$-equivariant symmetric bounded linear map $T: X^\nu_x \to X^\nu_x \subset L^2(V^\nu)$. A certain assumption was placed on the definition of an admissible perturbation in order for Proposition \ref{prop:2.11} to be valid: namely that $\pi$ did not depend on the connection component in a neighbourhood of the trivial connection orbit. This forces, when $x = \Theta$ that $T$ is trivial on the first factor of $X^\nu_\Theta = L^2(A^1 \otimes \text{ad}E) \oplus L^2(S \otimes E)$ in the SW-case, and completely trivial on $X^g_\Theta$ in the SU(3)-case. We shall call such $\text{stab}(x)$-equivariant symmetric bounded linear maps $T: X^\nu_x \to X^\nu_x$ admissible.
Proposition 2.18 Let \( x \) be in a reducible strata \( Z^r \). Given any admissible map \( T: X'_x \to X'_x \) there exists an admissible perturbation \( \pi \) such that \( \pi = 0 \) on the reducibles and \( (L\pi)_x = (L'_\rho\pi)_x = T \). Furthermore \( \pi \) may be assumed to be supported in an arbitrarily small \( \mathcal{G} \)-invariant neighbourhood of \( x \).

Proof In the slice space \( X_x \) define

\[
\pi_{x+\alpha} = \beta(\alpha)T(\alpha) + \delta_x^{\alpha} \circ \xi(\alpha, \beta(\alpha)T(\alpha)), \quad \alpha \in B(\varepsilon) \subset X_x.
\] (2.29)

As in the proof of Proposition 2.16 this defines an admissible perturbation with the desired properties provided \( \varepsilon \) is sufficiently small. qed.

Proof of Proposition 2.13 Consider first the SW-case. We proceed inductively up the various strata beginning with the trivial strata \( \{[\Theta]\} \). Here the normal operator \( N_{\Theta} = D \oplus D \) acts on \( X'_\Theta = S \oplus S \). Let \( \Sigma \) be the unit \( L^2 \)-sphere within \( L^2(S) \). Denote by \( \Pi_0 \) the vector space of admissible perturbations which vanish on \( C_{E}^r \). If we introduce a perturbation \( \pi \in \Pi_0 \) then the corresponding normal operator \( N_{\Theta} = D^\pi \oplus D^\pi \) where \( D^\pi = D + (L'_\rho\pi)_{\Theta} \). Denote by \( V \to \Sigma \times \Pi_0 \) the vector bundle whose fibre at \( (v, \pi) \) is the real \( L^2 \)-orthogonal \( \langle iv \rangle^{\perp} \subset L^2_1(S) \). Then \( f(v, \pi) = D^\pi(v) \) is a section of this vector bundle. We claim that \( f \) is a submersion along \( f^{-1}(0, 0) \cap (\Sigma \times \{0\}) \). To see this, consider the derivative of \( f \) at \( (v, 0) \in f^{-1}(0, 0) \) in the direction \( \pi \in \Pi_0 \). We have \( (Lf)_{v,0}(\pi) = (L'_\rho\pi)_{\Theta} \). By varying \( \pi \) and invoking Proposition 2.18 we see that \( (Lf)_{v,0} \) must be surjective. Furthermore we can make \( f \) a submersion by restricting \( \pi \) to some finite dimensional subspace \( \mathcal{H} \subset \Pi_0 \). By the Sard-Smale theorem there must exist a \( \pi' \in H \) (which we can assume arbitrarily small) such that \( f^{-1}(0, \pi') \) is cut out transversely in \( \Sigma \). However if \( (v, \pi') \in f^{-1}(0, \pi') \) then the symmetry and complex linearity of \( D_{\pi'} \) forces both \( v \) and \( iv \) to be orthogonal to its image. This means \( L(f|_\Sigma) \) is not a submersion along \( f^{-1}(0, \pi') \) which is a contradiction. Therefore \( f^{-1}(0, \pi') \) is empty and \( D_{\pi'} \) is invertible. So for \( \pi' \), \( H^1_{\Theta} = H^1_{\Theta} = \ker N^r_{\Theta} = \{0\} \) and \( \{[\Theta]\} \) is a non-degenerate and hence an isolated point in \( M_{\pi'}^w \).

Next, invoking Proposition 2.17 we can find a perturbation (also denoted as \( \pi' \)) so that \( M^1_{\pi',II} \) is non-degenerate within \( C^l_{E,II} \), i.e. the tangential cohomologies \( H^1_{\pi',I} \) are trivial. To get the normal cohomologies \( H^1_{\pi',I} \) to vanish as well repeat the argument as for the trivial strata – the usage of perturbations vanishing on \( C^l_{E} \) ensure that \( M^1_{\pi',II} \) remains unchanged. At this stage \( M^w_{\pi',0,1,II} \) is non-degenerate and isolated within \( M^w_{\pi'} \). Non-degeneracy for \( M^w_{\pi'} \) is achieved by another perturbation of the sort in Proposition 2.16.
In the $SU(3)$-flat case, we proceed just as in the preceding except that we may skip
the initial step of perturbing near the trivial connection. This is because $H^1_{\partial} \cong H^1(Y)$
is always trivial. qed.

3 Definition of the Invariant

3.1 The Floer-Taubes operator

Let $x$ be a point in the configuration spaces $C_E$ or $A_F$. As in the preceding subsection
let $\delta^0_x$ denote the zeroth differential in the elliptic complex and $V$ the vector bundle
whose $L^2$-sections models the tangent space to the configuration space. Let $W$ denote
\text{ad}E$ in the SW-case and \text{ad}F in the $SU(3)$-case.

The Floer-Taubes operator (at $x$) is the ‘roll-up’ of of the full version of the fundamental elliptic complex of §2.6. It is the bounded operator from the $L^2$-sections to the $L^2$-sections of $W \oplus V$ given in block diagonal form:

\[
L^x = \begin{pmatrix}
0 & \delta^0_x \\
\delta^0_x & \delta^1_x
\end{pmatrix}.
\]  

(3.1)

As before $\delta^0_x^*$ is the formal $L^2$-adjoint of $\delta^0_x$. By construction the kernel of $L^x$ is just
$H^0_x \oplus H^1_x$ and the cokernel which we identify with the $L^2$-orthogonal of the image,
$H^0_x \oplus H^2_x$.

The proofs in the next two subsections are below.

3.1.1 Orientability

In order to orient the moduli space we consider the determinant line $\text{detind} L^x$ of the
family $L^x$ parameterized by $x$. This determinant line is equivariant with respect to
the gauge action and descends to a line bundle denoted as $\text{detind} \hat{L}^x$ on the quotient
space. The ‘orientability’ of the quotient space is a consequence of the following:

Lemma 3.1 The line bundle $\text{detind} \hat{L}^x$ is orientable, i.e. the pull-back over any
closed loop is a trivial line bundle over $S^1$.
With this lemma it is possible to define, as in the manner of Taubes, a relative sign between non-degenerate zeros of $\hat{\mathcal{A}}_x$ as the basis for a Poincare-Hopf index.

Let us now consider the Floer-Taubes operator along a reducible stratum $\mathcal{R}$. Let $x \in \mathcal{R}$. Then according to §2.7 we have a splitting of the bundles $W$ and $V$ into tangential and normal components. This induces a splitting of the Floer-Taubes operator

$$L^\pi_x = K^\pi_x \oplus N^\pi_x$$

into tangential and normal components acting on sections of $W^\tau \oplus V^\tau$ and $W^\nu \oplus V^\nu$ respectively (details below). Note that if we denote by $K_x$ and $N_x$ the operators in the unperturbed situation then $K^\pi_x = K_x + (L^\tau\pi)_x$ and $N^\pi_x = N_x + (L^\nu\pi)_x$ where we split $(L\pi)_x$ according to Proposition 2.11. Furthermore $N^\pi_x$ commutes with the action of $\text{stab}(x)$.

**Lemma 3.2** The family of operators $\{N^\nu_x\}$ always extends continuously over the trivial strata except in the SW-Type II case. In the SW-Type II case this becomes true after restrict the family to the subset of reductions $(A, \Phi) = (A_0 \oplus A_1, (\phi, 0))$ in a fixed splitting $E = L_0 \oplus L_1$.

This follows from the explicit descriptions of the normal operators, below.

Henceforth we shall assume a fixed splitting $E = L_0 \oplus L_1$ for all SW-Type II reductions, i.e. restrict to $\mathcal{C}^{II}(L_0, L_1)$. This is without any loss of generality as any other such splitting can be moved to the reference one by a gauge transformation.

The orientability of the quotient space of $\mathcal{R}$ is claimed by the next lemma.

**Lemma 3.3** The determinant line $\text{det}\text{ind} K^\pi$ descends to an orientable line bundle $\text{det}\text{ind} \hat{K}^\pi$ over the quotient space of $\mathcal{R}$.

### 3.1.2 Spectral Flow

The normal operator is a formally self-adjoint Fredholm operator when $\pi = 0$. In particular it is a $(L^2)$-symmetric operator with domain the $L^2_2$-sections of $W^\nu \oplus V^\nu$. In the presence of a non-trivial perturbation term Lemma 2.11 asserts that the normal operator continues to be symmetric on $L^2_2$-sections.

The next proposition is an observation in [15].
Proposition 3.4 The normal operator $N^\pi_x$ regarded as an unbounded operator on $L^2(W^\nu \oplus V^\nu)$ is essentially self-adjoint. It has only a real discrete spectrum with no accumulation points. Each eigenvalue is of finite multiplicity and the eigenvalues are unbounded in both directions in $\mathbb{R}$. The normal operator depend differentiably on the parameters $x$ and $\pi$.

An immediate consequence of this proposition is that the concept of spectral flow is well-defined for $N^\nu_x$. The spectral flow for a path $\gamma$ in the quotient space is taken to be the spectral-flow along any lift of $\gamma$; the independence of lift is clear because the operators $N^\pi_x$ and $N^\pi_{g(x)}$ are conjugate to each other.

Let $SF^{sw,I}_\nu$, $SF^{sw,II}_\nu$, $SF^{su,I}_\nu$ denote the complex spectral-flow in the various cases indicated.

Convention When working with spectral flow the initial and final operators in the family may have non-trivial kernel. Our convention will be the spectral flow across $-\varepsilon^2$ where $0 < \varepsilon \ll 1$.

Lemma 3.5 (a) Let $\gamma(t)$, $t \in [0,1]$ be a piecewise differentiable loop in $C^0_E/G_E = A^0_F/G_F$. Then

$$SF^{su,I}_\nu(\gamma) = 2\text{cs}(\hat{\gamma}(0)) - 2\text{cs}(\hat{\gamma}(1)) = -2SF^{sw,I}_\nu(\gamma).$$

Here $\hat{\gamma}$ is a lift of $\gamma$ and $\text{cs}$ is the Chern-Simons function on $C^0_E/G_E = A^0_F/G_F$ defined with respect to some fixed trivial basepoint connection.

(b) If $\gamma$ is a loop in $C^0_{II}/G_E$ then $SF^{sw,II}_\nu(\gamma) = 0$.

Remark 3.6 It is known that $\pi_1(A^0_F/G_F) \cong \mathbb{Z}$ and if $\gamma$ is a generator then $\text{cs}(\hat{\gamma}(0)) - \text{cs}(\hat{\gamma}(1)) = \pm 1$. This gives us our spectral flow calculation $SF^{su,I}_\nu(\gamma) = \pm 2$ and $SF^{sw,I}_\nu(\gamma) = \pm 1$ for a generator $\gamma$.

3.1.3 Details of the splittings and proofs

Let us first summarize the splittings stated in §2.7 in a more convenient form.

|       | Reducible | $W^\tau \oplus V^\tau$ | $W^\nu \oplus V^\nu$ |
|-------|-----------|-------------------------|-----------------------|
| SW    | Trivial   | $\{0\}$                 | $(A^{0+1} \otimes u(2)) \oplus (S \otimes \mathbb{C}^2)$ |
|       | Type I    | $A^{0+1} \otimes \text{ad}E$ | $S \otimes E$ |
|       | Type II   | $(A^{0+1} \otimes iR_0) \oplus (S \otimes L_0) \oplus (A^{0+1} \otimes iR_1)$ | $(A^{0+1} \otimes (L_0 \otimes \overline{L}_1)) \oplus (S \otimes L_1)$ |
| $SU(3)$ | Trivial   | $\{0\}$                 | $A^{0+1} \otimes \text{su}(3)$ |
|       | Type I    | $A^{0+1} \otimes \text{ad}E$ | $A^{0+1} \otimes (E \otimes \mathbb{L})$ |
The corresponding tangential operators in the unperturbed situation is as below:

| Reducible | Tangential Operator |
|-----------|---------------------|
| SW        |                     |
| Type I    | $K_{sw}^{sw, I}(\gamma, a) = (d^*_a, *d_A a + d_A \gamma)$ |
| Type II   | $K_{\alpha, \phi}^{sw, II}(\gamma, a, \phi, \xi, b) = (d^*_a a - B(\phi, \psi), *d_\alpha a - \{\phi \cdot \psi\}_0 a + d_\alpha \gamma, D_\alpha \phi + a \cdot \psi - \gamma \psi, d^* b, *db + d\xi)$ |
| SU(3)     |                     |
| Type I    | $K_{\alpha}^{su, I}(\gamma, a) = (d^*_a a, *d_\alpha a + d_\alpha \gamma)$ |

The corresponding normal operators in the unperturbed situation is as below:

| Reducible | Normal Operator |
|-----------|----------------|
| SW        |                 |
| Trivial   | $N_{\Theta}^{sw, 0}(\gamma, a, \phi) = (d^*_a, *da + d\gamma, (D \oplus D) \phi)$ |
| Type I    | $N_{\Theta}^{sw, I}(\phi) = D_A \phi$ |
| Type II   | $N_{\alpha, \psi}^{sw, II}(\gamma, a, \phi) = (d^*_a a - B(\phi, \psi), *d_\alpha a - \{\phi \cdot \psi\}_0 a + d_\alpha \gamma, D_\alpha \phi + a \cdot \psi - \gamma \psi)$ |
| SU(3)     |                 |
| Trivial   | $N_{\Theta}^{su, 0}(\gamma, a) = (d^*_a, *da + d\gamma)$ |
| Type I    | $N_{\alpha}^{su, I}(\gamma, a) = (d^*_a a, *d_\alpha a + d_\alpha \gamma)$ |

**Proof of Lemmas 3.1, 3.3** Assume Proposition 3.4. The lemmas follow from the unperturbed situation $\pi = 0$ by application of the deformation $t\pi, 0 \leq t \leq 1$. The assertions of Lemmas 3.1 and 3.3 are then equivalent to the condition that the respective operators (which are now formally self-adjoint elliptic) have even real spectral flow around any path that is a loop at the level of the quotient space. Let us now deal with the individual operators in turn.

$K_{sw}^{sw, I}$ and $K_{su}^{su, I}$ are the same operator, the (negative of the) boundary of the Anti-Self-Dual (ASD) operator in 4-dimensions. By the spectral-flow around closed loops is equal to the (negative of the) index of a twisted ASD operator on $Y \times S^1$. This index is well-known to be congruent to 0 mod 8. After a deformation we can decompose $K_{\alpha, \psi}^{sw, II}$ into a sum of three operators: $K \oplus D_\alpha \oplus K$ where $K(\gamma, a) = (d^*_a, *da + d\gamma)$. The Dirac operator $D_\alpha$ is clearly complex and thus even spectral flow. Being topological, $K$ has no spectral-flow. Thus $K_{\alpha, \psi}^{sw, II}$ has even real spectral flow around loops in the quotient space. A deformation of $L_{sw}^{sw, I}$ brings it into a direct sum $K_{\alpha, \psi}^{sw, II} \oplus N_{\alpha, \psi}^{sw, II}$. The complex linear nature of $N_{\alpha, \psi}^{sw, II}$ brings it into a direct sum $K_{\alpha}^{su, I} \oplus N_{\alpha}^{su, I}$. The complex linear nature of $N_{\alpha}^{su, I}$ means it always has even real spectral flow whereas the spectral-flow of $K_{\alpha, \psi}^{sw, II}$ we have already treated. The case of $L_{su}^{su, I}$ follows from the same line of reasoning as above, being the negative of the boundary of the ASD operator. qed.

**Proof of Lemma 3.5** Assuming Proposition 3.4 and after a deformation we may again assume $\pi = 0$. Item(a): The path $\hat{\gamma}$ defines a $U(2)$-connection $\hat{A}$ over $Y \times [0,1]$. Since $N_{\hat{A}}^{su, I}$ is the negative of the boundary of the $\hat{A}$-twisted ASD operator...
on $Y \times [0,1]$ (with the orientation $dydt$), by \[3\] the spectral flow of $N_{A}^{\text{su},I}$ along $\hat{\gamma}$ is equal to the negative of the index of the ASD-operator on $Y \times [0,1]$ with APS spectral boundary conditions. This index is computed from \[1], \[3\] to be

$$-2 \int_{Y \times [0,1]} c_2(\hat{A}) = -2\left(\text{cs}(\hat{\gamma}(0)) - \text{cs}(\hat{\gamma}(1))\right).$$

(3.3)

(Note: with the orientation $dydt$ the boundary of $Y \times [0,1]$ is oriented according to $Y \times \{0\} - Y \times \{1\}$ for Stokes’ Theorem to hold without any signs.) On the other hand $N_{A}^{\text{sw},I}$ is the boundary of the 4-dimensional Dirac operator coupled to $\hat{A}$ on $Y \times [0,1]$. Thus the spectral flow along $\hat{\gamma}$ is equal to the index of this 4-dimensional Dirac operator. This has index given by $-\int_{Y \times [0,1]} c_2(\hat{A})$. Item(b) can be seen by either a similar computation or the observation that $C^{0,II}/G_E$ is simply connected. qed.

Proof of Proposition 3.4 (\[15\]) we need to show that $N_{x}^{\pi}$ is essentially self-adjoint and has compact resolvent; then the spectrum is real and discrete. The absence of accumulation points and unboundedness as $\rightarrow \pm\infty$ follows by the Hilbert-Schmidt theorem applied to any resolvent of $N_{x}^{\pi}$. In the unperturbed case, $N_{x}$ is formally self-adjoint elliptic on a compact manifold and standard elliptic theory gives $N_{x}$ as essentially self-adjoint and with compact resolvent. In general we can regard $N_{x}^{\pi}$ as a perturbation of $N_{x}$ since $N_{x}^{\pi} = N_{x} + (L''\pi)_{x}$. By Lemma 2.11 \[2,11\] $(L''\pi)_{x}$ is symmetric with dense domain the $L^2_{2}$-sections, and thus is closable. Furthermore since $(L''\pi)_{x}$ is bounded as an operator on $L^2_{2}$-sections it follows that $(L''\pi)_{x}$ is relatively compact with respect to $N_{x}$. This in turn implies that $(L''\pi)_{x}$ has arbitrarily small relative bound with respect to $N_{x}$. Now standard stability theory \[10\] tells us $N_{x}^{\pi}$ is also essentially self-adjoint and has compact resolvent. qed.

3.2 The Main Theorem

Recall $E \rightarrow Y$ is our $U(2)$-bundle in SW-theory. Without loss, let us choose now in $SU(3)$-Casson $F = E \oplus \overline{\det E}$. Then given a connection $A$ on $E$, this induces the $SU(3)$ connection $A \oplus \det A$ on $F$. In this way we obtain an identification

$$C^{0,II}_{E}/G_E = A^{0,II}_{F}/G_F.$$

(3.4)

Henceforth we shall assume this identification.

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Lemma 3.7 There exists admissible non-degenerate perturbations $\pi, \pi'$ in $U(2)$-SW, $SU(3)$-Casson-Taubes respectively such that $\mathcal{M}_\pi^{sw,I} \cong \mathcal{M}_{\pi'}^{su,I}$.

Proof Follow proof of Proposition 2.13. qed.

Such a pair of perturbations we call non-degenerate compatible. In such a situation we shall simply write $\mathcal{M}_\pi^{I}$ for both $\mathcal{M}_\pi^{sw,I}$ and $\mathcal{M}_{\pi'}^{su,I}$, regarding them as being identified.

At a non-degenerate point $x \in \mathcal{M}_\pi^{sw,*}$ the kernel and cokernel of $L_{\pi}^{sw,\pi}(L_{\pi}^{su,\pi'})$ are trivial, $\tau$ any representative of $x$. This gives a canonical trivialization (with $\mathbb{R}$) of the determinant line $\text{detind} \hat{L}_{\pi}^{sw,\pi}(\text{detind} \hat{L}_{\pi}^{su,\pi'})$ at $x$. Thus in order to orient $\mathcal{M}_\pi^{sw,*}$ we fix the overall orientation of $\text{detind} \hat{L}_{\pi}^{sw,\pi}(\text{detind} \hat{L}_{\pi}^{su,\pi'})$ by fixing the orientation at one point, which we take to be $[\Theta]$ and then propagating the orientation from this point. This is well-defined by Lemma 3.1. At $\tau = \Theta$ the kernel and cokernel of $L_{\pi}^{sw,\pi}$ or $L_{\pi}^{su,\pi'}$ are identical, call them $H$ and specify the orientation of the determinant line at $[\Theta]$ by the rule $o(H) \wedge o(H)^*$ where $o(H)$ is any orientation of $H$ and $o(H)^*$ the dual orientation. We denote the orientation at $x$ by

$$\varepsilon(x) \in \{\pm 1\}.$$ 

In an identical manner non-degenerate points in the reducible strata $\mathcal{M}_\pi^{sw,I}, \mathcal{M}_\pi^{sw,II}, \mathcal{M}_{\pi'}^{su,I}$ are oriented but this time using the tangential operators $K_{A}^{sw,I,\pi}, K_{A_0}^{sw,II,\pi}, K_{A}^{su,I}$ respectively and invoking Lemma 3.3. Again we denote the orientation at $x$ by $\varepsilon(x)$, the usage will be clear from the context.

In the situation of compatible perturbations, the operators $K_{A}^{sw,I,\pi}$ and $K_{A}^{su,I,\pi'}$ coincide. Thus we identify them and simply write $K_{A}^{I,\pi}$. It is clear that the orientation at a non-degenerate point $x \in \mathcal{M}_\pi^{I}$ is the same in the $U(2)$-SW and $SU(3)$-Casson theories.

Assume henceforth that $\pi, \pi'$ are non-degenerate compatible perturbations. For the top (non-singular) strata contribution, let

$$\Lambda_{sw}^{*}(g, \pi) = \sum_{x \in \mathcal{M}_{g,\pi}^{sw,*}} \varepsilon(x), \quad \Lambda_{su}^{*}(g, \pi') = \sum_{x \in \mathcal{M}_{\pi'}^{su,*}} \varepsilon(x). \quad (3.5)$$

Counter-terms (0) We remind the reader that $g$ denotes the metric on our ZHS $Y$ and $D$ the canonical Dirac operator on the spinor bundle $S \to Y$. Denote by
$B$ the operator on $Y$ which is half the boundary of the signature operator in 4-dimensions. To these operators we may associate the Atiyah-Patodi-Singer (APS) spectral invariants

$$\eta(B), \quad \xi = \frac{1}{2} \left( \eta(D) + \dim C \ker D \right).$$

(3.6)

Let $\pi$ be a perturbation in $U(2)$-SW-theory and let $D^\pi_{\Theta}$ the normal operator at a trivial connection $\Theta$. Recall that this operator acts on $S \otimes E$. Now set

$$c(g, \pi) = \eta(B) + \frac{1}{8} \xi + \frac{1}{2} \left( \text{complex spectral flow of} \right)$$

$$\left\{ (1-t)D_{\Theta} + tD^\pi_{\Theta} \right\}_{t=0}$$

(3.7)

**Lemma 3.8** $c(g, \pi) \equiv \mu(Y) \mod 2$ where $\mu(Y) \in \{0,1\}$ is the Rokhlin invariant.

**Proof** This is discussed in [12] but we repeat it here for convenience. If $X$ is compact oriented spin 4-manifold with oriented boundary $Y$ then an application of the APS index theorems to $X$ shows that

$$\xi + \frac{1}{8} \eta(B) = -\text{Index } D^4 - \frac{1}{8} \text{sign } X.$$  

(3.8)

Here $D^4$ is the Dirac operator on $X$ and $\text{sign } X$ the signature. Thus we see that the left-side of (3.8) is always an integer. The mod2 reduction of the right-side only involves the signature term (since in four dimensions the Dirac operator is quaternionic linear and so its index is even) and therefore is just the Rokhlin invariant $\mu(Y)$. The complex spectral flow of the family $\left\{ (1-t)D_{\Theta} + tD^\pi_{\Theta} \right\}_{t=0}^{1}$ is always divisible by 4 since both $D_{\Theta}$ and $D^\pi_{\Theta}$ are each a double of a quaternionic linear operator. qed.

Now set, as the contribution from the SW-strata of the trivial connection:

$$\Lambda^0(g, \pi) = \frac{1}{2} c(g, \pi) \left( c(g, \pi) - 1 \right).$$

(3.9)

**Counter-terms (I)** For the Type I reducible strata contribution set

$$\Lambda^I(g, \pi, \pi') = \sum_{x \in M^I_{\pi}} \varepsilon(x) \left\{ (\text{SF}^\text{sw,I}_{\nu} + 2\text{SF}^\text{sw,I}_{\nu})[\Theta], x \right\} + 4c(g, \pi').$$

(3.10)

The spectral flow term is taken along any path in $C^0_{E,F} \cap G_E = A^0_{F} / G_F$ from $[\Theta]$ to $x$. This is well-defined independent of path by Lemma 3.5.
Counter-terms (II) Lastly, for the SW-Type II reducible strata set

$$\Lambda^I(g, \pi) = \sum_{x \in M^I_{\pi}} \varepsilon(x) \{ SF^{sw,I}_\nu ([\Theta], x) + c(g, \pi) \}. \quad (3.11)$$

This is again well-defined, by Lemma 3.5.

As a remark, the presence of the metric dependent terms $c(g, \pi)$ in $\Lambda^I$ and $\Lambda^{II}$ are inserted to cancel out spectral flow phenomena at $[\Theta]$ in expressions $SF^{sw,I}_\nu ([\Theta], x)$, $SF^{sw,II}_\nu ([\Theta], x)$ if we were to vary the metric.

The main result of this paper is:

**Theorem 3.9** Let $Y$ be an oriented closed integral homology 3-sphere with Riemannian metric $g$. Let $\pi$, $\pi'$ be non-degenerate compatible admissible perturbations in $U(2)$-SW, $SU(3)$-Casson respectively. Then the sum

$$\tau(Y) = \Lambda^*_{su}(g, \pi') + 2\Lambda^*_{sw}(g, \pi) + \Lambda^I(g, \pi, \pi') + 2\Lambda^{II}(g, \pi) + 2\Lambda^0(g, \pi)$$

is an integer and independent of $g$ and $\pi$, $\pi'$ and thus defines an oriented diffeomorphism invariant of $Y$.

Let

$$\lambda^{su(2)}_{g,\pi}(Y) = \sum_{x \in M^I_{\pi}} \varepsilon(x), \quad \lambda^{sw}_{g,\pi}(Y) = \sum_{x \in M^{sw,II}_{\pi}} \varepsilon(x) + c(g, \pi).$$

According to Taubes [15], $\lambda^{su(2)}_{g,\pi}(Y)$ is (up to a universal sign) twice Casson’s invariant. $\lambda^{sw}_{g,\pi}(Y)$ on the other hand is precisely the definition of the abelian SW-invariant for $Y$ (see for instance [12]). By [13], this is again (up to a universal sign) equal to Casson’s invariant.

**Corollary 3.10** Let $-Y$ denote $Y$ but with the reverse orientation. Then

$$\tau(-Y) = \tau(Y) + 2\lambda^{su(2)}_{g,\pi}(Y) + 2\lambda^{sw}_{g,\pi}(Y).$$

Thus an orientation independent invariant for $Y$ is given by

$$\overline{\tau}(Y) = \tau(Y) + \lambda^{su(2)}_{g,\pi}(Y) + \lambda^{sw}_{g,\pi}(Y).$$
Proof Under reversal of orientation of $Y$ the $\pi'$-perturbed flat equation transforms to the $(-\pi')$-perturbed equations. Thus we can identify the $SU(3)$-moduli spaces in either orientation. The Floer-Taubes operator switches to its negative. In the SW-case, reversal simply changes the Clifford action on the spinor bundle to its negative. Thus the SW-equation in the reversed orientation is equation to the original except that the Dirac component of the equation switches to its negative. If $\pi = (\ast k, l)$ is the non-degenerate perturbation originally used then $\pi = (\ast k, -l)$ is non-degenerate and admissible in the reversed context. If $(A, \Phi)$ is a (perturbed) solution in the original then $(A, -\Phi)$ is a (perturbed) solution in the reversed. Thus we can identify $\mathcal{M}_\pi^{sw}(-Y)$ with $\mathcal{M}_\pi^{sw}(Y)$. The Floer-Taubes operator also changes to its negative in the SW-context.

Under a reversal of orientation $Y \mapsto -Y$ we obtain the transformations

$$
\begin{align*}
\varepsilon(x) &\mapsto \varepsilon(x), x \in \mathcal{M}_\pi^{sw}, \mathcal{M}_\pi^{su}^* \\
\varepsilon(x) &\mapsto -\varepsilon(x), x \in \mathcal{M}_\pi^{sw,r}, \mathcal{M}_\pi^{su,r} \\
SF_{\nu}^{sw,I}([\Theta], x) &\mapsto -SF_{\nu}^{sw,I}([\Theta], x) - 2 \\
SF_{\nu}^{sw,I}([\Theta], x) &\mapsto -SF_{\nu}^{sw,I}([\Theta], x) \\
SF_{\nu}^{sw,II}([\Theta], x) &\mapsto -SF_{\nu}^{sw,II}([\Theta], x) - 1 \\
c(g, \pi) &\mapsto -c(g, \pi).
\end{align*}
$$

The non-trivial constants in the spectral flow terms are the corrections terms which equal the dimension of the kernel of the normal operators at $\Theta$. The orientation reversal formula easily follows. $\mathcal{F}(Y)$ is the average of $\tau(Y)$ and $\tau(-Y)$ and thus independent of orientation. qed.

Remark 3.11 If the Casson invariant of $Y$ is non-zero then by Corollary 3.10 at least one of $\tau(Y)$ or $\tau(-Y)$ is also non-zero. Since there are infinitely many $Y$ for which Casson’s invariant is non-zero we deduce that there are infinitely many integral homology spheres for which $\tau \neq 0$ for at least one orientation.

4 Proof of Theorem 3.9

The basic strategy of the proof is an extension of those in [12], [4] and [14]. Some (standard) portions of the argument are omitted and can be found in the cited references.
Standing Convention To simplify notation we often confuse a point say \( x \), in the quotient space \( \mathcal{Z}/\mathcal{G} \) with a representative of \( x \) in the context of the Floer-Taubes/normal/tangential operators as well as cohomology spaces. This is permissible as different representatives of the same gauge orbit give rise to conjugate operators. In particular we may regard the normal operators as being parameterized by \( \mathcal{Z}^r/\mathcal{G} \).

4.1 Parameterized Moduli Spaces

Let \((g_0, \pi_0, \pi_0')\) and \((g_1, \pi_1, \pi_1')\) be triples consisting of metric and non-degenerate admissible compatible perturbations. We wish to compare the moduli spaces for these two triples.

To this end we may form the corresponding parameterized moduli spaces in the SW and SU(3) cases:

\[
W_{\text{sw}} = \bigcup_{t \in [0, 1]} \mathcal{M}_{g_t, \pi_t}^{\text{sw}} \times \{t\} \subset \mathcal{C}_E/\mathcal{G}_E \times [0, 1],
\]

\[
W_{\text{su}} = \bigcup_{t \in [0, 1]} \mathcal{M}_{g_t, \pi_t}^{\text{su}} \times \{t\} \subset \mathcal{A}_F/\mathcal{G}_F \times [0, 1].
\]

(Note: in the SW-case, fix a model for the spinor bundle. Then regard the Clifford action etc., as varying with the metric.) We retain the usage of the (superscript) notations \(*, r, I, II\) etc. pertaining to the unparameterized moduli spaces in the parameterized setting. The parameterized moduli spaces can also be regarded as being formed from the \( t \)-dependent SW and SU(3)-flat equations over \( \mathcal{Z} \times [0, 1]: \)

\[
\tilde{X}_{\text{sw}}(A, \Phi, t) = 0, \quad \tilde{X}_{\text{su}}(A) = 0.
\]

It will be necessary to introduce perturbations in the parameterized context. An admissible perturbation \( \sigma \) in the parameterized context is defined in an analogous manner to the unparameterized case as a function \( \sigma: \mathcal{Z} \times [0, 1] \to L^2_2(V) \) with the additional condition that the support is contained in \( \mathcal{Z} \times (0, 1) \). The restriction of \( \sigma \) to a slice \( \mathcal{Z} \times \{t\} \) is clearly an admissible perturbation on \( Y \) itself.

Denote the \( \sigma, \sigma' \) perturbed versions of the parameterized moduli spaces by

\[
W_{\sigma}^{\text{sw}}, \quad W_{\sigma'}^{\text{su}}
\]

respectively. To save notation we shall use \( W_{\sigma} \) to denote either space.
**Proposition 4.1** $W_\sigma$ is always compact.

**Proof** Follows from Proposition 2.6. qed.

In compact notation we may write the $\sigma$-perturbed (4.2) as

$$\tilde{\mathcal{X}}_\sigma: \mathcal{Z} \times [0,1] \to L^2_2(V). \quad (4.4)$$

If we let the linearization be $\tilde{\delta}^1_{x,t}$ then this fits into an elliptic complex

$$L^2_3(W) \overset{\delta^0}{\longrightarrow} L^2_2(V) \oplus \mathbb{R} \overset{\tilde{\delta}^1_{x,t}}{\longrightarrow} L^2_1(V) \overset{\delta^0_*}{\longrightarrow} L^2(W) \quad (4.5)$$

with (harmonic) cohomologies $\tilde{H}^i_{x,t}$. The condition that $W^*_\sigma$ is cut out transversely is $\tilde{H}^1_{x,t} = \{0\}$, i.e. non-degenerate/regular.

Along a reducible strata again we have an orthogonal decomposition paralleling Proposition 2.12, with tangential and normal cohomologies $\tilde{H}^{1,\tau}_{x,t}$ and $\tilde{H}^{1,\nu}_{x,t}$ respectively. Call $W^\alpha_{\sigma}$, $\alpha \in \{\ast,0,I,II\}$ regular if $\tilde{H}^{1,\tau}_{x,t} = \{0\}$, $(x,t) \in W^\alpha_{\sigma}$. This is the condition of being cut out transversely in $\mathcal{Z}^\alpha_{\mathcal{G}} \times [0,1]$.

In the next proposition let $\mathcal{M}$ denote either $\mathcal{M}^{sw}$ or $\mathcal{M}^{su}$.

**Proposition 4.2** There are admissible perturbations $\sigma$ such that $W^{sw}_{\sigma}$ is a regular stratified compact singular cobordisms between $\mathcal{M}_{\pi_0}$ and $\mathcal{M}_{\pi_1}$, i.e. the following hold:

(a) Each individual strata $W^\alpha_{\sigma}$, $\alpha \in \{\ast,0,I,II\}$ is regular: it is a 1-manifold with boundary $\mathcal{M}_{\pi_0} \cup \mathcal{M}_{\pi_1}$ and with possibly a number of non-compact ends.

(b) Each end limits to a singular point, which lies in a reducible strata. There are only finitely many singular points.

(c) A neighbourhood of each singular point is diffeomorphic to $T = \{(r,t) \mid rt = 0, t \geq 0\} \subset \mathbb{R}^2$ where the edge $\{0\} \times (0,\infty)$ corresponds to the limiting end.

(d) Each reducible strata parameterizes the associated family of normal operators; the singular points are exactly where the family experiences spectral flow. The spectral flow at these points are always transverse

(e) Limiting ends only occur in the irreducible and in the SW-Type II strata. When a singular point lies in the SW-Trivial strata the corresponding limiting end lies in the SW-Type II reducible strata, and every limiting end of the SW-Type II strata lies in the SW-Trivial strata. There are no singular points on the SU(3)-Trivial strata.
This will be proven in §4.4.

**Remark 4.3** (a) The normal operator at \((x,t) \in W^r_\sigma\) is the normal operator at \(x\) in \(\mathcal{M}^r_{g_r, \pi_1}\). (b) Transverse spectral flow means all eigenvalues crosses zero transversely, modulo multiplicity if the operator has a complex or quaternionic structure, i.e. we have only simple eigenvalues over \(\mathbb{R}, \mathbb{C}\) or \(\mathbb{H}\).

**Proposition 4.4** Let \(\sigma, \sigma'\) be perturbations as in Proposition 4.2. \(W^{sw}_\sigma\) and \(W^{sw}_{\sigma'}\) admit a consistent orientation convention such that

\[
\partial(W^{sw}_\sigma \setminus \{\text{singular points}\}) = \mathcal{M}^{sw}_{\pi_1} \cup -\mathcal{M}^{sw}_{\pi_0}
\]

\[
\partial(W^{su}_{\sigma'} \setminus \{\text{singular points}\}) = \mathcal{M}^{su}_{\pi_1} \cup -\mathcal{M}^{su}_{\pi_0}.
\]

Note that the assertion of the proposition is inclusive of the reducible strata where we assign the orientation value +1 to the trivial connection \([\Theta]\). The existence of the claimed orientations is established by considering determinant line bundles and will be established in the section on orientation, §5.

The existence of the orientations on the reducible strata of the parameterized moduli spaces also allows us to assign to each singular point a value +1 or −1 according to the spectral flow of the normal operator at that point, moving in the direction of the given orientation.

**Proposition 4.5** Let \(\Xi^{-1}(0), \Xi: \mathbb{R} \times [0, \infty) \to \mathbb{R}, \Xi(x, y) = xy\) be a orientation preserving local model for a singular point, where the orientation on \(\mathbb{R} \times \{0\}\) is the usual orientation on \(\mathbb{R}\). Let \(\varepsilon \in \{\pm 1\}\) be the sign of the spectral flow of the normal operator at \(x = 0\) in the local model. Then the orientation on \(\{0\} \times [0, \infty)\) is given by \(-\varepsilon\) multiplied with the standard orientation on \([0, \infty)\).

The proof is also in §5.

### 4.2 The Main Argument

To show that the sum \(\tau(Y)\) in Theorem 3.9 is a diffeomorphism invariant we need to show that the defect

\[
2\Lambda^*_{sw}(g_1, \pi_1, \pi'_1) + \Lambda^*_{su}(g_1, \pi_1, \pi'_1) - 2\Lambda^*_{sw}(g_0, \pi_0, \pi'_0) - \Lambda^*_{su}(g_0, \pi_0, \pi'_0) \tag{4.6}
\]
exactly cancels the defects
\[
\begin{align*}
\Lambda^I(g_1, \pi_1, \pi'_1) - \Lambda^I(g_0, \pi_0, \pi'_0), \\
2\Lambda^{II}(g_1, \pi_1) - 2\Lambda^{II}(g_0, \pi_0), \\
2\Lambda^0(g_1, \pi_1) - 2\Lambda^0(g_0, \pi_0).
\end{align*}
\]
(4.7)

This is established by through the singular cobordisms \(W_{sw}^\sigma\) and \(W_{su}^\sigma\) of Proposition 4.2. To this end, without loss we may assume that \(W_{sw}^\sigma\) and \(W_{su}^\sigma\) are elementary singular cobordisms, by which we mean the occurrence of exactly one singular point (or none). According to Proposition 4.2 we have the following different types of elementary singular cobordisms for \(W_{sw}^\sigma\) and \(W_{su}^\sigma\).

- no singular points
- singular point is Type I reducible
- singular point is Type II reducible
- singular point is Trivial (connection)

In the case of all these types of elementary singular cobordisms except for the last (let us call it \(W_{sw}^{0}\)), the invariance by analysing the defects, is covered in [4] (see also [14], [12]) without any new idea. These cases correspond to the birth or death of new points at bifurcation into the highest (i.e. irreducible) strata in the parameterized moduli space. The defect (4.6) is cancelled by the first two defects in (4.7) with the last defect in (4.7) identically zero.

The case \(W_{sw}^{0}\) presents the new phenomena of birth/death of new points into the Type II strata at bifurcation. As such it represents a ‘second order’ defect, being the defect of the counter-term \(\Lambda^I(g, \pi, \pi')\). To simply matters more, we may assume that \(W_{sw}^{0}\) consists of components all of which are topologically closed intervals \([0, 1]\) except a single one which is topologically \([-1, 1] \times \{0\} \cup \{0\} \times [0, 1]\). (There are actually two subcases corresponding to where the boundary of the normal edge lies.) Clearly in this case the defect (4.6) is zero as well as the term \(\Lambda^I(g_1, \pi_1, \pi'_1) - \Lambda^I(g_0, \pi_0, \pi'_0)\) since the irreducible and Type I strata components are assumed to be closed intervals \([0, 1]\). Thus we only have to deal with the changes in the defect terms \(\Lambda^{II}\) and \(\Lambda^0\).

**Claim 4.6** \(\Lambda^{II}(g_1, \pi_1, \pi'_1) - \Lambda^{II}(g_0, \pi_0, \pi'_0) + \Lambda^0(g_1, \pi_1) - \Lambda^0(g_0, \pi_0) = 0\).

Given the claim, Theorem 3.9 is proven.
4.3 Proof of Claim 4.6

For convenience we change notation; assume the parameterization varies over $[-1, 1]$ instead of $[0, 1]$ so the initial metric, perturbation, etc. are now $g_{-1}$, $\pi_{-1}$ etc.

Any component of $W^{sw,0}$ which is a product makes no contribution to the defects $\Lambda^{II}(g_{1}, \pi_{1}, \pi'_{1}) - \Lambda^{II}(g_{-1}, \pi_{-1}, \pi'_{-1})$ and $\Lambda^{0}(g_{1}, \pi_{1}) - \Lambda^{0}(g_{-1}, \pi_{-1})$ so we focus our attention on the component of $W^{sw,0}$ which topologically is $[-1, 1] \times \{0\} \cup \{0\} \times [0, 1]$. Let $\epsilon = \pm 1$ be the sign of the $H$-spectral flow (or half the $C$-spectral flow) of the normal operator $N^{sw,0}$ at the singular point $(0, 0)$. By Lemma 4.5 the arc $\{0\} \times [0, 1]$ is oriented as $-\epsilon$ times the standard orientation on $[0, 1]$. We have two situations for the boundary point $p = (0, 1)$. Denote by Case A when this is in $M^{sw}_{\pi_{-1}}$ and Case B when in $M^{sw}_{\pi_{1}}$. Recall that $\epsilon(p) \in \{\pm 1\}$ denotes the orientation of $p$ as a point in $M^{sw}_{\pi_{-1}}$ or $M^{sw}_{\pi_{1}}$. Observe in Case A, $\epsilon(p) = \epsilon$ and in Case B, $\epsilon(p) = -\epsilon$. Then the defect

$$\Lambda^{II}(g_{1}, \pi_{1}, \pi'_{1}) - \Lambda^{II}(g_{-1}, \pi_{-1}, \pi'_{-1})$$

$$= \begin{cases} 
-\epsilon \left( \text{SF}^{sw,II}_{\nu}(\Theta, p) + c(g_{-1}, \pi_{-1}) \right) & \text{in Case A} \\
-\epsilon \left( \text{SF}^{sw,II}_{\nu}(\Theta, p) + c(g_{1}, \pi_{1}) \right) & \text{in Case B}.
\end{cases}$$

In the notation we implicitly assume that $\text{SF}^{sw,II}_{\nu}(\Theta, p)$ takes place in either $C_{E}^{r} \times \{-1\}$ or $C_{E}^{r} \times \{+1\}$ depending on where $p$ is located. The key observation is the following:

**Lemma 4.7**

$$\text{SF}^{sw,II}_{\nu}(\Theta, p) = \begin{cases} 
\frac{1}{2}(\epsilon - 1) & \text{in Case A} \\
-\frac{1}{2}(\epsilon + 1) & \text{in Case B}
\end{cases}$$

This shall be proven below. It follows then that

$$\Lambda^{II}(g_{1}, \pi_{1}, \pi'_{1}) - \Lambda^{II}(g_{-1}, \pi_{-1}, \pi'_{-1})$$

$$= \begin{cases} 
-\epsilon \left( \frac{1}{2}(\epsilon - 1) + c(g_{-1}, \pi_{-1}) \right) & \text{in Case A} \\
-\epsilon \left( -\frac{1}{2}(\epsilon + 1) + c(g_{1}, \pi_{1}) \right) & \text{in Case B}
\end{cases}$$

$$= \frac{1}{2}(\epsilon - 1) - \epsilon c(g_{-1}, \pi_{-1}),$$

where in the last line we use the relation $c(g_{1}, \pi_{1}) = c(g_{-1}, \pi_{-1}) + \epsilon$. To prove this recall from the definition that $c(g_{t}, \pi_{t})$ changes by the spectral flow of $D^{\pi_{t}}$ (with...
respect to metric $g_t$) acting on $S$, as $t$ varies. We claim this is exactly half the
$\mathbf{C}$-spectral flow of $N_{\Theta,t}^{\text{sw},0}$ as $t$ varies. After trivializing $E$ as $\mathbf{C}^2 \times Y$ using $\Theta$, it is
seen that $N_{\Theta,t}^{\text{sw},0} = K \oplus D^{\pi_t} \oplus D^{\pi_t}$ where $K$ is the deRham operator on $A^{0+1} \otimes \mathbf{C}$ (see §3.1.3). Since $K$ is topological it has no spectral flow so half the $\mathbf{C}$-spectral flow of
$N_{\Theta,t}^{\text{sw},0}$ is equal to the $\mathbf{C}$-spectral flow of $D^{\pi_t}$, as claimed.

To continue: on the other hand we easily see the defect

\[ A^0(g_1, \pi_1) - A^0(g_{-1}, \pi_{-1}) \]

\[ = \frac{1}{2} \left( c(g_{-1}, \pi_{-1}) + \varepsilon \right) \left( c(g_{-1}, \pi_{-1}) + \varepsilon - 1 \right) \]

\[ - \frac{1}{2} c(g_{-1}, \pi_{-1}) \left( c(g_{-1}, \pi_{-1}) - 1 \right) \]

\[ = \frac{1}{2} \left( 1 - \varepsilon \right) + \varepsilon c(g_{-1}, \pi_{-1}). \]

Hence the sum of the defects in zero and Claim 4.6 is established. qed.

Proof of Lemma 4.7 Let $A$ reduce as $\theta \oplus \theta$ in the splitting $E = L_0 \oplus L_1$ and
let $\Phi = (\phi, 0) \in L_2^2(S \otimes L_0) \oplus L_2^2(S \otimes L_1)$. Furthermore trivialize $L_i$ as $\mathbf{C} \times Y$ via the trivial connection $\theta$. Then the normal operator $N_{\phi,0}^{\text{sw},II,\pi}$ acts on sections of
$(A^{0+1} \otimes \mathbf{C}) \oplus S$ (§3.1.3). If $\phi = 0$, then $N_{\phi,0}^{\text{sw},II,\pi}$ decouples as $K \oplus D^{\pi_t}$ where $K$ is the
deRham operator on $A^{0+1} \otimes \mathbf{C}$ and $D^{\pi_t}$ is the $\pi_t$-perturbed Dirac operator on $S$ with
respect to metric $g_t$. Thus the spectral flow of $N_{\phi,0}^{\text{sw},II,\pi}$ along the arc $[-1, 0] \times \{0\}$ in
the local model for the singular point is $\frac{1}{2}(\varepsilon + 1)$ and along the arc $[0, 1] \times \{0\}$ it is
$\frac{1}{2}(\varepsilon - 1)$ (in our convention, ff. Prop. 3.4). Note that the kernel and cokernel of $K$ is
$\cong \mathbf{C}$, the constant functions, and therefore makes no contribution to spectral flow.

By assumption of transverse spectral flow, $D^{\pi,t}$ has kernel $\cong \mathbf{C}$ at the singular point
$(0, 0)$. Let $\phi$ be an element, say of unit length in the kernel. Then to first order, the family $N_s := N_{\phi,0}^{\text{sw},II,\pi}$, $s \in [0, 1]$ models the family $N_{\phi,0}^{\text{sw},II,\pi}$ along $\{0\} \times [0, 1]$ at
$(0, 0)$.

Sublemma 4.8 Let $N_0'$ be the derivative of $N_s$ at $s = 0$. Identify $\ker N_0$ with $\mathbf{C}^2$
via the basis $\{1, \phi\}$. Then the restriction of $N_0'$ to $\ker N_0$ followed by $L^2$-projection
onto the same is given by the matrix

\[
\begin{pmatrix}
0 & -1 \\
-1 & 0
\end{pmatrix}
\]
It easily follows from the sublemma that to 1st order the complex eigenvalues of the family $N_s$ at $s = 0$ and therefore $N_{sw,II}^{sw}$ along $\{0\} \times [0, 1]$ at $(0,0)$ is given by
\[
\begin{vmatrix}
-\lambda & -s \\
-s & -\lambda
\end{vmatrix} = 0,
\]
i.e. $\lambda = +s$ and $\lambda = -s$. Since along $\{0\} \times [0, 1]$ there is no other spectral flow (by assumption), the spectral flow must actually be $-1$. Thus in Case A,
\[
SF_{\nu}^{sw,II}([\Theta], p) = \frac{1}{2}(\varepsilon + 1) - 1 = \frac{1}{2}(\varepsilon - 1)
\]
and in Case B,
\[
SF_{\nu}^{sw,II}([\Theta], p) = -\frac{1}{2}(\varepsilon - 1) - 1 = -\frac{1}{2}(\varepsilon + 1).
\]
This proves the Lemma 4.7 modulo Sublemma 4.8. qed.

Proof of Sublemma 4.8 Our goal is to get an explicit expression for $N_s$. Let $\delta^{0,\nu}_{\theta,s\phi}$ denote the differentials in the normal component of orthogonal decomposition of the fundamental elliptic complex along $\mathcal{C}^{II}(L_0, L_1)$ (Proposition 2.12) extended over the trivial strata. Recall $L_i$ is trivialized as $\mathbb{C} \times Y$ by $\theta$. Then we have
\[
\delta^{0,\nu}_{\theta,s\phi} : L^2_2(A^0 \otimes \mathbb{C}) \to L^2_2(A^1 \otimes \mathbb{C}) \oplus L^2_2(S) \quad (4.9)
\]
\[
\xi \mapsto (d\xi, -\xi(s\phi)),
\]
\[
\delta^{1,\nu}_{\theta,s\phi} : L^2_2(A^1 \otimes \mathbb{C}) \oplus L^2_2(S) \to L^2_2(A^1 \otimes \mathbb{C}) \oplus L^2_1(S) \quad (4.10)
\]
\[
(a, \psi) \mapsto (*da - *\{\psi \cdot s\phi\}_0, D^{\nu,0}\psi + a \cdot (s\phi)).
\]
The $L^2$-adjoint of (4.9) is then
\[
\delta^{0,\nu}_{\theta,s\phi} : L^2_2(A^1 \otimes \mathbb{C}) \oplus L^2_2(S) \to L^2_2(A^0 \otimes \mathbb{C}) \quad (4.11)
\]
\[
(a, \psi) \mapsto d^*a - \langle \psi, s\phi \rangle_{\mathbb{C}}.
\]
$N_s$ is given by the block matrix
\[
\begin{pmatrix}
0 & \delta^{0,\nu}_{\theta,s\phi} \\
\delta^{0,\nu}_{\theta,s\phi} & \delta^{1,\nu}_{\theta,s\phi}
\end{pmatrix}
\]
Identify ker $N_0$ with $\mathbb{C}^2$ via the basis $\{1, \phi\}$. Let $N'_s$ denote the derivative of $N_s$ with respect to $s$. Denote by $\widehat{N}'_0$ the restriction of $N'_0$ to ker $N_0$ followed by $L^2$-projection onto the same. Then from (4.9), (4.10), (4.11) we have
\[
\widehat{N}'_0(z, w) = (-w, -z).
\]
This has matrix exactly as claimed in the sublemma. This completes the proof. qed.
4.4 Proof of Proposition 4.2

Let $W^\alpha_x$, $\alpha \in \{0, I, II\}$ denote a reducible strata. Assume it is regular; then it is a 1-manifold which parameterizes the family of normal operators $N^\alpha_x$. (Strictly speaking we ought to work with a lift of $W^\alpha_x$ to the configuration space $\mathcal{Z} \times [0, 1]$ but in any case the family will be unique up to conjugation.)

Call $W^\alpha_x$ normally transverse if given any $x \in W^\alpha_x$ there exists a 1-1 parameterization $J: (-\varepsilon, \varepsilon) \to W^\alpha_x$ of a neighbourhood of $x$ such that the pull-back family $N^\alpha_x \circ J$ is a transverse family with respect to spectral flow.

We need a standard result on the local structure of a bifurcation point which for instance is proven in [12]. However we include the proof since we shall need an ingredient from it in for §5.

Lemma 4.9 Assume $W^\alpha_x$ is regular and normally transverse. Let $x \in W^\alpha_x$ be a singular point. Then a local model for a neighbourhood $U$ of $(x,t)$ in $W^\sigma_x$ is

$$\Upsilon^{-1}(0),$$

where $\Upsilon: \mathbb{R} \times [0, \infty) \to \mathbb{R}$, $(x,y) \mapsto xy$ with $(0,0) \leftrightarrow x$, $\mathbb{R} \times \{0\} \cong U \cap W^\alpha_x$.

Proof $\Upsilon$ is the quadratic approximation for the Kuranishi obstruction map $\Xi$, below. To recall: let $\tau$ represent a tangent vector to $W^\sigma_x$ at $(x,t)$. ($(x,t)$ is fixed throughout this proof.) Then $\ker \tilde{L}_{x,t} = \mathbb{R}\{\tau\} \oplus H^{1,\nu}_x$ and $\text{coker} \tilde{L}_{x,t} = H^{1,\nu}_x$. Let $\Pi$ denote $L^2$-projection onto $H^{1,\nu}_x$. By the implicit function theorem there exists a map $f: \mathbb{R}\{\tau\} \oplus H^{1,\nu}_x \to (H^{1,\nu}_x)^\perp$ such that for all $r$ and $h$

$$\begin{align*}
(I - \Pi)(\tilde{\mathcal{K}}((t,x) + r\tau + h + f(r\tau, h)) &= 0, \\
f(0,0) &= 0, \quad df_{0,0} = 0.
\end{align*}
\tag{4.12}$$

Here $\tilde{\mathcal{K}}$ denotes the maps in (4.2). (Note: for convenience we henceforth reverse the order of the variables for $\tilde{\mathcal{K}}$.) The obstruction map $\Xi: \mathbb{R}\{\tau\} \oplus H^{1,\nu}_x \to H^{1,\nu}_x$ is given as

$$\Xi(r\tau, h) = \Pi \circ \tilde{\mathcal{K}}((t,x) + r\tau + h + f(r\tau, h)). \tag{4.13}$$

Assume, for convenience that $J'(0) = \tau$. Let $a + b \in X_x$ with $a \in X^\tau_x$ and $b \in X^\nu_x$. Then it is seen that

$$\tilde{\mathcal{K}}(t', x + a + b) = K^\alpha_{x,t'}(a) + N^\alpha_{x,t'}(b) + B(a + b, a + b) \tag{4.14}$$
where $B$ is a bilinear term. The expressions $B(a,a), B(b,b) \in X^\nu_x$ and $B(a,b), B(b,a) \in X^\nu_x$. The normal component of the second derivative of $\tilde{X}$ at $(t,x)$ in the pair of directions $(r\tau,h)$ is given by

$$r \left. \frac{d}{du}(N^\alpha \circ J(u)) \right|_{u=0} (h)$$

(4.15)

This in turn is exactly $c_0 r h$ where $c_0 \neq 0$ has the same sign as the spectral flow for $N^\alpha \circ J(u)$ at $u = 0$ (see [12]). In figuring the quadratic approximation for $\Xi$ in (4.13) we can drop the $f$ term since $df = 0$ at the origin. Thus the quadratic approximation for $\Xi(r\tau,h)$ is $c_0 r h$. After dividing out by the action of $\text{stab}(x)$ on $H^{1,\nu}_x$ we obtain $\Upsilon$. qed.

Proposition 4.2 largely follows directly from this lemma as soon as we can find an admissible perturbation $\sigma$ such that $W^\sigma_{\alpha}$ is regular and normally transverse for all $\alpha$, and $W^\sigma_{\alpha}^*$ is also regular. The remaining points to show are the nature of the limiting ends associated to each singular point, and the non occurrence of singular points on the $SU(3)$-trivial strata.

4.4.1 Local Models

Assume $W^\sigma_{\alpha}$ is regular but not yet normally transverse. Let $(x,t)$ be a point where spectral flow occurs for $N^\alpha$ and $J:(-\varepsilon,\varepsilon) \to W^\sigma_{\alpha}$ a 1-1 parameterization of a small neighbourhood of $(x,t)$, giving us a 1-parameter family of operators $N_s = N^\alpha \circ J(s)$.

Let $H^\nu = \ker N^\alpha_{x,t}$. This consists of $L^2$-sections and is independent of the completion $N = N^\alpha_{x,t}: L^2_{2+k}(W^\nu \oplus V^\nu) \to L^2_{1+k}(W^\nu \oplus V^\nu)$, $k \geq 0$ due to the standard elliptic estimates. Furthermore we have $L^2$-Hodge-decompositions into closed subspaces (and using the symmetry property of $N$):

$$L^2_{1}(W^\nu \oplus V^\nu) = \text{Ran}(N|_{L^2_3}) \oplus H^\nu$$

$$L^2_{2}(W^\nu \oplus V^\nu) = \text{Ran}(N|_{L^2_3}) \oplus H^\nu.$$ 

Thus $N$ by restriction, defines a Banach space isomorphism $\text{Ran}(N|_{L^2_3}) \to \text{Ran}(N|_{L^2_3})$.

Assume for $J$ that the parameterization $N_s = N^\alpha \circ J(s)$ is at least $C^2$; this means the second derivative $\frac{d^2 N_s}{ds^2}$ is a bounded map $L^2_{2}(W^\nu \oplus V^\nu) \to L^2_{1}(W^\nu \oplus V^\nu)$ and continuous in $s$ where the target space has the operator topology. Let $\Pi$ denote $L^2$-projection of $\text{Ran}(N|_{L^2_3}) \oplus H^\nu$ onto the first factor. Define

$$N^0_s = \Pi \circ N_s: \text{Ran}(N|_{L^2_3}) \to \text{Ran}(N|_{L^2_3}).$$

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By the implicit function theorem the exists a subinterval, which we can take to be $(-\varepsilon, \varepsilon)$ again, and a $C^2$-map $f : H^\nu \times (-\varepsilon, \varepsilon) \to \text{Ran}(N|_{L^2})$ such that the following hold:

\[
N^0_s(\phi + f(\phi, s)) = 0 \\
f(\phi, 0) = 0 \\
\frac{\partial f}{\partial s}(\phi, 0) = 0 \\
f(\cdot, s) \in B(H^\nu, \text{Ran}(N|_{L^2})).
\]

Thus solving $N_s(\phi) = 0$, $|s| \leq \varepsilon$ is equivalent to solving the equation

\[T(s)\phi := (I - \Pi) \circ N_s(\phi + f(\phi, s)) = 0, \quad \phi \in H^\nu,
\]

where $T : (-\varepsilon, \varepsilon) \to \text{Hom}(H^\nu)$. It is clear then that $\dim(\ker N_s) = \dim(\ker T(s))$. $T(s)$ is the local model for the family $N_s$.

**Lemma 4.10** (a) $T(s)$ is symmetric, i.e. $\langle T(s)\psi, \phi \rangle_{L^2} = \langle \psi, T(s)\phi \rangle_{L^2}$ for $\psi, \phi \in H^\nu$  
(b) $T(s)$ commutes with $\text{stab}(x)$.

Item (a) is proven by the computation

\[
\langle T(s)\psi, \phi \rangle_{L^2} = \langle N_s(\psi + f(s, \psi)), \phi \rangle_{L^2} \\
= \langle N_s(\psi + f(s, \psi)), \phi + f(s, \phi) \rangle_{L^2} \\
= \langle \psi + f(s, \psi), N_s(\phi + f(s, \phi)) \rangle_{L^2} \\
= \langle \psi, N_s(\phi + f(s, \phi)) \rangle_{L^2}.
\]

Item (b) follows easily from Proposition 2.11. Thus our local model is really a map into the linear space of symmetric operators on $H^\nu$ which commute with $\Gamma = \text{stab}(x)$:

\[T : (-\varepsilon, \varepsilon) \to \text{Sym}_\Gamma(H^\nu).
\]  

Let $B^0$ be a sufficiently small open neighbourhood of the origin in any finite dimensional vector subspace of the perturbations $P_0$ which vanish on the reducibles (we do not want the perturbation to move $W^\alpha$ itself). Then we have a parameterized local model

\[\hat{T} : (-\varepsilon/2, \varepsilon/2) \times B^0 \to \text{Sym}_\Gamma(H^\nu)
\]  

such that $\hat{T}(s, 0) = T(s), \ s \in (-\varepsilon/2, \varepsilon/2)$.  

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4.4.2 Completion of the argument

The SW-case Step1: start at the lowest strata \( W^0 = \{ [\theta] \} \times [0, 1] \) which is always regular (in the trivial strata). We need to find an admissible perturbation in \( P_0 \) which makes \( W^0 \) normally transverse. Let \((x_0, t_0) \in W^0\) and let \( N_s = N^0 \circ J(s) \) where \( J: (-\varepsilon, \varepsilon) \to W^0 \) parameterizes a sufficiently small neighbourhood of \((x_0, t_0)\) such that a local model \( s \mapsto T(s) \in \text{Sym}_H(H^\nu) \) for \( N_s \) is valid.

Let \( \sigma \in P_0 \). Then \( W^0 = W^0_\sigma \) (actually for the trivial strata this is valid for all \( \sigma \)) and \( N_s \) changes to a new family \( N^s_\sigma \) which is still parameterized by \((-\varepsilon, \varepsilon)\). If \( \sigma \) is sufficiently small then we continue to have a local model \( s \mapsto \hat{T}(s, \sigma), s \in (-\varepsilon/2, \varepsilon/2) \).

In particular \( N^s_\sigma \) has transverse spectral flow if and only if the family \( s \mapsto \hat{T}(s, \sigma) \) has transverse spectral flow. The last statement is in turn equivalent to the following: the transformations in \( \text{Sym}_H(H^\nu) \) which have non-trivial rank form a codimension one real subvariety, being the zeros of the determinant map to \( R \). Denote by \( V^{(k)} \) those which have real rank \( \geq 4k \geq 0 \). Then transverse spectral flow is the condition that \( s \mapsto \hat{T}(s, \sigma) \) is disjoint from \( V^{(k)}, k \geq 2 \) and meets \( V^{(1)} \) transversely.

If \( \pi \) is an admissible perturbation on \( \mathcal{Z} \) and \( h \) is a function with support in \((0, 1)\) then \( \sigma = (h\pi, 0) \) is an admissible perturbation on \( \mathcal{Z} \times [0, 1] \). Let \( s_0 \in (-\varepsilon/2, \varepsilon/2) \) and \( h \) have support near \( s_0 \). Then according to Proposition 2.13 there exists \( h_i \pi_i, i = 1, \ldots, n \) and \( r_i, i = 1, \ldots, n \) sufficiently small such that the linearization of \((r_1, \ldots, r_n) \mapsto \hat{T}(s_0, r_1 h_1 \pi_1 + \ldots + r_n h_n \pi_n)\) is surjective at \( r_1 = \ldots = r_n = 0 \).

Since surjectivity is an open condition this continues to hold for all \( s \) close to \( s_0 \).

By an open cover argument we may enlarge our set of \( h_i \pi_i \) (but keeping it finite) and a sufficiently small open neighbourhood \( B^0 \) of the origin in their span in \( P_0 \) such that \( \hat{T}: (-\varepsilon/3, \varepsilon/3) \times B^0 \to \text{Sym}_H(H^\nu) \) is submersion along \((-\varepsilon/3, \varepsilon/3) \times \{0\} \).

Thus by Sard-Smale there exists an admissible perturbation \( \sigma \) which makes \( W^0_\sigma \) normally transverse along the portion \( J(-\varepsilon/3, \varepsilon/3) \). Another open cover argument gives normal transversality at all points in \( W^0 \).

Step2: At this stage \( W^*_\sigma \) and \( W^{1, \text{II}}_\sigma \) are now regular in a neighbourhood of \( W^0 = W^0_\sigma \). In particular the local models for the bifurcation/singular points (Lemma 4.9) are valid for those in \( W^0 \). We claim that any bifurcation arc limiting into \( W^0 \) must come from the Type II strata. This follows from the observation that \( H^{1, \text{II}, \tau}_{c,t} = \text{stab}(x) \cdot H^{1, \text{II}, \tau}_{c,t} \) at such a bifurcation point. Here \( H^{1, \text{II}, \tau}_{c,t} \) is the first cohomology of the normal component of the fundamental elliptic complex along the Type II
strata (see Proposition 2.12), extended over the trivial connection \( \theta \). In fact \( H^{1,\nu}_{\theta,t} = H^{0,II,\nu}_{\theta,t} \oplus J(H^{1,II,\tau}_{\theta,t}) \) where \( J \) is the constant gauge transformation which switches the factors in the splitting \( E = L_0 \oplus L_1 = (C \oplus C) \times Y \). \( H^{1,II,\tau}_{\theta,t} \cong \mathbb{R} \) is the limiting tangent space to \( W^{II}_0 \) at the end which limits to \( ([\Theta], t) \).

The slice space at \((x_0, t_0)\) for the action of \( G \) on \( Z \times [0, 1] \) is \( X_{x_0} \times \mathbb{R} \). Let \((v, \tau) \in X_{x_0} \times \mathbb{R} \) and \( h \) a function with support in \((0, 1)\). Then by Proposition 2.16 there exists a \( \pi \) such that the perturbation \((h\pi, h\tau)\) is admissible on \( Z \times [0, 1] \) and \((h(t_0)\pi_{x_0}, h(t_0)\tau) = (v, \tau)\). Furthermore this perturbation can be assumed to be supported away from \( W^0 \). Thus by a standard transversality argument there exists an admissible \( \sigma' \) which makes \( W^{II}_{\sigma+\sigma'} \) regular (in \( C^{II}/G \times [0, 1] \)). As such it is a 1-manifold with boundary \( M^{SW,II}_{g_0,\pi_0} \cup M^{SW,II}_{g_1,\pi_1} \) and with ends (if any) which limit into \( W^0 \).

**Step3:** Repeat **Step1** but applied to \( W^{II}_{\sigma+\sigma'} \). That is we can find a further perturbation \( \sigma'' \in \mathcal{P}_0 \) such that \( W^{II}_{\sigma+\sigma'+\sigma''} \) is normally transverse. To simplify notation continue to denote by \( \sigma \) the perturbation \( \sigma + \sigma' + \sigma'' \). All bifurcations on \( W^{II}_{\sigma} \) are into \( W^* \).

**Step4:** Repeat **Step2** and **Step3** but applied to \( W^I_\sigma \). In this **Step2** since \( \Gamma \cong U(1) \), \( V^{(k)} \) are the symmetric maps which have real rank \( \geq 2k \geq 0 \). The rest of the argument proceeds as before.

**Step5:** Repeat **Step2** but applied to \( W^*_\sigma \) so as to make it regular. This completes the proof in the SW-case.

**The SU(3)-case** We follow the same argument as above but we may start at **Step2** as \( W^0 \) is always isolated. \( \text{qed.} \)

5 Orientation

We continue to enforce the notational conventions stated at the beginning of §4.

### 5.1 Convention for determinant lines

Let \( \{L_x : \mathcal{V}_0 \rightarrow \mathcal{V}_1\} \) be a family of Fredholm operators parameterized by \( x \in X \). The determinant line \( \text{det} \text{ind} L \) is the (real) line bundle over \( X \) whose fiber at \( x \) is formally \( A^{\text{max}}(\ker L_x) \otimes A^{\text{max}}(\coker L_x)^* \). In the context we have been working in \( L_x \).
is a compact perturbation of a first order elliptic operator over $Y$ with $\mathcal{V}_0 = L_{k+1}^2(V_0)$, $\mathcal{V}_1 = L_k^2(V_1)$ and $X$ either $C_E$ or $A_F$.

Given a differentiable path $\gamma: [a,b] \to X$ we can consider the family $L_{\gamma(t)}$ along the path $\gamma$. An orientation of $(\text{detind} L)_{\gamma(a)}$ can then be propagated to an orientation of $(\text{detind} L)_{\gamma(b)}$ along $\gamma$. Since the kernel of $L_{\gamma(t)}$ may jump as $t$ varies, to carry out this procedure we need to stabilize the pulled back family $(\gamma^* L)$ over $[a,b]$. This consists of the following data: a finite dimensional vector space $W$ and a map $\Psi: V_0 \oplus W \to V_1$ such that $\tilde{L}_t = L_{\gamma(t)} + \Psi: V_0 \oplus W \to V_1$ is surjective for all $t$. Then $(\gamma^* L)$ is realized as the line bundle with fiber at $t$ being $\Lambda^{\text{max}}(\ker \tilde{L}_t) \otimes \Lambda^{\text{max}} W^*$.

The issue we wish to address is how an orientation of a fiber of $\Lambda^{\text{max}}(\ker L_{\gamma(t)}) \otimes \Lambda^{\text{max}}(\coker L_{\gamma(t)})^*$ is to be carried over to the corresponding fiber of the stabilization, $\Lambda^{\text{max}}(\ker \tilde{L}_t) \otimes \Lambda^{\text{max}} W^*$. Since this is a fiberwise convention we henceforth simplify our discussion by considering only a single operator $L = L_x: \mathcal{V}_0 \to \mathcal{V}_1$ in the family.

Suppose $\bar{L} = L + \Psi: V_0 \oplus W \to \mathcal{V}_1 \subset \mathcal{V}_1 \oplus W$ is a stabilization. Regarding $L$ and $\bar{L}$ as two step chain complexes, we have the following exact sequence of complexes:

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{V}_0 & \xrightarrow{L} & \mathcal{V}_1 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{V}_0 \oplus W & \xrightarrow{\bar{L}} & \mathcal{V}_1 \oplus W & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & W & \xrightarrow{0} & W & \to & 0 \\
\end{array}
\] (5.1)

This gives a long exact sequence:

\[
0 \to \ker L \to \ker \bar{L} \to W \to \coker L \to \coker \bar{L} \to W \to 0.
\] (5.2)

Assuming inner products on all the spaces we can ‘roll-up’ the exact sequence into a single isomorphism

\[
\xi: \ker \bar{L} \oplus \coker L \oplus W \to \ker L \oplus W \oplus \coker \bar{L}.
\] (5.3)

Now an orientation of $\text{detind} L$ or $\text{detind} \bar{L}$ is equivalent to saying that an orientation for the kernel is determined by an orientation of the cokernel, and vice versa. Regarding an orientation $o(V)$ as a non-zero element in the highest exterior power of $V$, orient $\text{detind} \bar{L}$ from $\text{detind} L$ by the rule

\[
\xi^*(o(\ker L) \wedge o(W) \wedge o(\coker \bar{L})) = o(\ker \bar{L}) \wedge o(\coker L) \wedge o(W).
\] (5.4)
The rule is independent of choice of orientation of \( W \) used as well as the inner product since changes in this will only change \( \xi^* \) by a positive constant.

We collect some remarks:

(i) Allow \( \Psi \) to vary continuously with respect to say a real parameter and keeping the surjectivity condition; the spaces and maps in the sequence (5.2) will vary continuously. Thus if \( \Lambda^{\max}(\ker L) \otimes \Lambda^{\max}(\coker L)^* \) is oriented and one stabilization is homotopic to another, then the induced orientations are carried continuously onto each other.

(ii) The first statement in (i) remains true if in addition we allow \( L_s \) to vary continuously with the parameter \( 0 \leq s \leq 1 \) with the assumption the kernel does not jump and \( \tilde{L}_s = L_s + \Psi_s \) remains surjective. Therefore if \( \Lambda^{\max}(\ker L_{\gamma(s)}) \otimes \Lambda^{\max}(\coker L_{\gamma(s)})^* \) is a locally trivial family then the propagated orientation from \( s = 0 \) to \( s = 1 \) is consistent with the propagated orientation after stabilization.

(iii) It is not necessary to insist on stabilizations which are ‘surjective’ in (i) and (ii). For instance everything we have said so far is equally true applied to say the stabilization where \( \Psi \) is the zero map. The homotopy assertions continue to be true with the assumption the kernel does not jump in a continuous variation.

5.2 The Floer-Taubes operator in the parameterized context

This subsection contains the proof of Proposition 4.4. Retain the notation of §4.

‘Rolling up’ the complex (4.5) gives a (perturbed) elliptic operator

\[
\tilde{L}_{x,t} : L^2_2(W \oplus V) \oplus \mathbb{R} \to L^1_2(W \oplus V).
\] (5.5)

This operator has the property that the restriction to \( L^2_2(W \oplus V) \) for a fixed value of \( t \) gives the Floer-Taubes operator on \( Y \) with respect to parameter value \( t \). As in §2.7 along a reducible strata \( \mathcal{Z}^\alpha \times [0, 1] \) we can take the tangential component of \( \tilde{L}_{x,t} \):

\[
\tilde{K}^\alpha_{x,t} : L^2_2(W^\tau \oplus V^\tau) \oplus \mathbb{R} \to L^1_2(W^\tau \oplus V^\tau).
\] (5.6)
The determinant index \( \text{det ind} \tilde{L}_{x,t} \) determines an orientation of \( W^*_\sigma \) and \( \text{det ind} \tilde{K}^\alpha \) and orientation of \( W^*_{\sigma} \). An orientation of \( \text{det ind} \tilde{L}_{x,t} \) and \( \text{det ind} \tilde{K}^\alpha \) will immediately give Proposition 4.4 up to an overall sign in the various stratas. The issue is how to fix the overall orientation by so that the sign is correct. This in turn reduces to fixing the orientation along the trivial strata.

For convenience, denote either \( \tilde{L}_{x,t} \) or \( \tilde{K}^\alpha_{x,t} \) by \( \tilde{\Lambda}_{x,t} \), and \( \tilde{\Lambda}^0_{x,t} \) the restriction to the summand different from \( R \). At a point on the trivial strata, say represented by \( (\Theta, t) \), the kernel and cokernel (= \( L^2 \)-orthogonal of range) of \( \tilde{\Lambda}^0 \) coincide by self-adjointness. Thus we have

\[
\ker \tilde{\Lambda}_{\Theta,t} = \ker \tilde{\Lambda}^0_{\Theta,t} \oplus R, \quad \text{coker} \tilde{\Lambda}_{\Theta,t} = \text{coker} \tilde{\Lambda}^0_{\Theta,t} = \ker \tilde{\Lambda}^0_{\Theta,t}.
\]

The \( R \) summand in the kernel corresponds exactly to the tangent space to the trivial strata \( \{(\Theta)\} \times [0, 1] \).

**Proposition 5.1** Orient \( \tilde{\Lambda} \) by specifying the orientation of \( \text{det ind} \tilde{\Lambda} \) at \( (\Theta, t) \) by \( \overline{\sigma}(R) \wedge o(\ker \tilde{\Lambda}^0_{\Theta,t}) \wedge o(\ker \tilde{\Lambda}^0_{\Theta,t})^* \) where \( \overline{\sigma}(R) \) is the standard orientation. Then Proposition 4.4 holds with this choice.

**Proof** It suffices to consider irreducible strata of the constant family \( W^*_\sigma = \mathcal{M}_\pi \times [0, 1] \); the reducible stratas are treated in an identical manner using the tangential operators for that strata.

Let \( [x] \in \mathcal{M}_\pi^* \) and \( \gamma: [0, 1] \to Z \times \{t\} \) a path from \( (\Theta, t) \) to \( x \). The assumption that \( W^*_\sigma \) is a constant family gives

\[
\ker \tilde{\Lambda}_{\gamma(t)} = \ker \tilde{\Lambda}^0_{\gamma(t)} \oplus R, \quad \text{coker} \tilde{\Lambda}_{\gamma(t)} = \ker \tilde{\Lambda}^0_{\gamma(t)}.
\]

Note that \( \tilde{\Lambda}^0 \) is the family which orients \( \mathcal{M}_\pi^* \). Equation (5.8) gives us an isomorphism from \( \text{det ind} (\gamma^*\tilde{\Lambda}^0) \) to \( \text{det ind} (\gamma^*\tilde{\Lambda}) \) by the rule \( \omega \mapsto \overline{\sigma}(R) \wedge \omega \). This isomorphism carries the standard orientation for \( \text{det ind} \tilde{\Lambda}^0 \) at \( \Theta \) to the orientation of \( \text{det ind} \tilde{\Lambda}^0 \) at \( (\Theta, t) \) given in the statement of the proposition. Thus if \( \varepsilon(x) \) is the orientation of \( x \) in \( Z^*/G \) then the arc \( \{x\} \times [0, 1] = [0, 1] \) in \( W^*_\sigma \) is assigned the orientation \( \varepsilon(x)\overline{\sigma}(R) \).

Now it is straightforward to see that \( \partial W^*_\sigma = \mathcal{M}_\pi^* \times \{1\} - \mathcal{M}_\pi^* \times \{0\} \). qed.

### 5.3 Bifurcation Points

This subsection establishes Proposition 4.5.
5.3.1 Type I, II Strata

Let us assume that the various strata of $W_\sigma$ are oriented according to the convention in the preceding section. Let $(x, t_0)$ denote a bifurcation point on a Type I or II strata; parameterize a small neighbourhood by $J: (-\varepsilon, \varepsilon) \rightarrow W_\sigma$ with $J(0) = (x, t_0)$ and consistent with the orientation of the strata at $x$. Let us assume that the various strata of $x$ is consistent with the orientation of the strata at $x$. Let $\{v_s\}$ represents a non-zero tangent vector to the strata at $x$. We assume that $\tau$ is consistent with the orientation of the strata at $x$.

Let $v \in H^1_x$ be unit length. To 1st order, the bifurcation arc at $x$ is modelled by $\{sv\}$, $0 \leq s < \varepsilon$. Now define a 1-parameter family $\tilde{L}_s = \tilde{L}_{\xi(s)}$ where $\xi$ parametrizes (up to gauge equivalence) the bifurcation arc with $\xi(0) = x$ and $\xi'(0) = v$. We have

$$\ker \tilde{L}_0 = R\{\tau\} \oplus H^0_x \oplus H^1_x, \quad \text{coker} \tilde{L}_0 = H^0_x \oplus H^1_x$$

(5.9)

where we identify the cokernel with the $L^2$-orthogonal of the range. On the other hand for $s \neq 0$, $\ker \tilde{L}_s \cong R$ is the tangent space to the bifurcation arc, whilst $\text{coker} \tilde{L}_s = \{0\}$. Notice that $\lim_{s \to 0} \ker \tilde{L}_s = R\{v\}$.

Let $\gamma \in H^0_x$ be unit length. This is a tangent vector to $\text{stab}(x)$ at the identity. The derivative of the $\text{stab}(x)$ action on $H^1_x$ gives an action of $\gamma$ on $H^1_x$ which is a complex structure. We fix the complex structure by our choice of $\gamma$. Using $\tau$, $\gamma$ and $v$ as a basis we may express

$$\ker \tilde{L}_0 = R\{\tau\} \oplus R\{\gamma\} \oplus C\{v\}, \quad \text{coker} \tilde{L}_0 = R\{\gamma\} \oplus C\{v\}.$$ 

(5.10)

Our next goal will be to reduce $\tilde{L}_s$ to a more managable finite dimensional family $T_s$. The following discussion parallels subsection 4.4.1 so we shall be brief. $\tilde{L}_s$ is a bounded operator $L^2_{2+k}(W \oplus V) \oplus R \rightarrow L^2_{1+k}(W \oplus V)$, $k \geq 0$. The $L^2$-adjoint is a bounded operator $\tilde{L}_s^*: L^2_{2+k}(W \oplus V) \rightarrow L^2_{1+k}(W \oplus V) \oplus R$. We have the following $L^2$-decompositions:

$$L^2_1(W \oplus V) = \text{Ran}(\tilde{L}_0|_{L^2_1}) \oplus \ker (\tilde{L}_0^*|_{L^2_1})$$

$$L^2_2(W \oplus V) = \text{Ran}(\tilde{L}_0^*|_{L^2_2}) \oplus \ker (\tilde{L}_0|_{L^2_2}).$$

Let $\mathcal{V} = \ker (\tilde{L}_0|_{L^2_1})$, $\mathcal{W} = \ker (\tilde{L}_0^*|_{L^2_2})$. Denote by $\Pi_\mathcal{V}$, $\Pi_\mathcal{W}$ $L^2$-projection maps onto $\mathcal{V}$, $\mathcal{W}$ respectively. By the implicit function theorem the following holds:

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(i) for $|s| < \varepsilon$ there exists $f: \mathcal{V} \times (-\varepsilon, \varepsilon) \to \mathcal{V}^\perp$ such that

$$(I - \Pi_W) \circ \tilde{L}_s(v + f(v, s)) = 0$$

$$f(v, 0) = 0$$

$$\frac{\partial f}{\partial s}(v, 0) = 0$$

$$f(\cdot, s) \in B(\mathcal{V}, \text{Ran}(\tilde{L}_s^0|_{L_2^2}))$$

(ii) for $|s| < \varepsilon$ there exists $g: \mathcal{W} \times (-\varepsilon, \varepsilon) \to \mathcal{W}^\perp$ such that

$$(I - \Pi_V) \circ \tilde{L}_s^*(w + g(w, s)) = 0$$

$$g(w, 0) = 0$$

$$\frac{\partial g}{\partial s}(w, 0) = 0$$

$$g(\cdot, s) \in B(\mathcal{W}, \text{Ran}(\tilde{L}_0^0|_{L_2^2}))$$

These lead to the following isomorphisms:

$$\gamma_{1,s}: L_2^2(W \oplus V) \rightarrow L_2^2(W \oplus V)$$

$$(v, v') \mapsto (v, v' + f(v', s)), \quad v \in \mathcal{V}, v' \in \mathcal{V}^\perp$$

$$\gamma_{2,s}: L_1^2(W \oplus V) \rightarrow L_1^2(W \oplus V)$$

$$(w, w') \mapsto (w, w' + f(w', s)), \quad w \in \mathcal{W}, w' \in \mathcal{W}^\perp$$

The preceding proves:

**Lemma 5.2** The family $\tilde{L}_s^0 = \gamma_{1,s}^{-1} \circ \tilde{L}_s \circ \gamma_{1,s}$ is isomorphic to $\tilde{L}_s$. Define $T: (-\varepsilon, \varepsilon) \to \text{Hom}(\mathcal{V}, \mathcal{W})$ by the rule

$$T_s(v) = \Pi_W \circ \tilde{L}_s(v + f(v, s)).$$

Then $\tilde{L}_s^0$ splits orthogonally as $T_s \oplus \tilde{L}_s^1: \mathcal{V} \oplus \mathcal{V}^\perp \rightarrow \mathcal{W} \oplus \mathcal{W}^\perp$ where $\tilde{L}_s^1$ is an isomorphism. Lastly detind $\tilde{L} \equiv \text{detind} \tilde{L}_s^0 \equiv \text{detind} T$.

The last statement of the lemma is a consequence of the observation $\ker \tilde{L}_s = \ker \tilde{L}_s^0 = \ker T_s$ and $\text{coker} \tilde{L}_s = \text{coker} \tilde{L}_s^0 = \text{coker} T_s$. $T_s$ is the local model for $\tilde{L}_s$. Clearly detind $T$ inherits the orientation of detind $\tilde{L}$. 

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With the aim of determining the orientation of the bifurcation arc we may now exclusively work with \( T_s \) rather than \( \tilde{L}_s \). For \( s \neq 0 \), \( \ker T_s \cong \mathbb{R} \) and \( \operatorname{coker} T_s = \{0\} \) with \( \lim_{s \to 0} \ker T_s = \mathbb{R}\{v\} \). An orientation of \( \lim_{s \to 0} \ker T_s \) mutually determines an orientation of \( \ker T_s \) for \( s \neq 0 \) by continuity. Recall that \( \lim_{s \to 0} \ker T_s = \mathbb{R}\{v\} \) models (to first order) the bifurcation arc at \((x,t_0)\). We wish to get the induced orientation on \( \lim_{s \to 0} \ker T_s = \mathbb{R}\{v\} \) from \( \det \text{ind} \hat{T}_s \). Without loss of generality we can replace \( T_s \) with the linearized family

\[
\hat{T}_s = T_0 + s T'_0 = s T'_0
\]

(5.14)
since \( T_0 = 0 \). \( \det \text{ind} \hat{T}_s \) agrees with \( \det \text{ind} T \) at \( s = 0 \) so it is oriented according to our conventions (§5.2).

**Lemma 5.3** Identify \( \ker T_0 = \ker \tilde{L}_0 \) with \( \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C} \) and \( \operatorname{coker} T_0 = \operatorname{coker} \tilde{L}_0 \) with \( \mathbb{R} \oplus \mathbb{C} \) using the basis in (5.10). Let \( T'_0 \) denote the derivative of \( T_s \) at \( s = 0 \). Then

\[
T'_0(t, r, z) = (\Re(iz), c_0 t + ir)
\]

(5.15)

where \( c_0 \neq 0 \) has the same sign as the spectral flow for \( N_u \) at \( u = 0 \).

This will be proven below.

The proof of Proposition 4.5 in the Type I,II cases is now a direct consequence of the next lemma. For the linearized family, \( \ker \hat{T}_s = \mathbb{R}\{v\} \) and \( \operatorname{coker} \hat{T}_s = \{0\} \), \( s \neq 0 \).

**Lemma 5.4** Give \( \mathbb{R}\{v\} = \mathbb{R} \) the standard orientation \( \sigma(\mathbb{R}) \) of \( \mathbb{R} \). (With this orientation the bifurcation arc is pointing away from the reducible strata.) Let \( o'(\mathbb{R}) \) be the induced orientation on \( \mathbb{R}\{v\} = \mathbb{R} \) given by the orientation of \( \det \text{ind} \hat{T} \). Then \( o'(\mathbb{R}) = -\text{sign}(c_0) \sigma(\mathbb{R}) \).

**Proof** We shall stabilize the family \( \hat{T} \) explicitly and evaluate the propagated orientation on \( \det \text{ind} \hat{T} \), \( s \neq 0 \). Let \( \mathcal{V} = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C} \) and \( \mathcal{W} = \mathbb{R} \oplus \mathbb{C} \). Stabilize \( \hat{T}_s \) by

\[
\hat{T}_s + I : \mathcal{V} \oplus \mathcal{W} \to \mathcal{W} \oplus \mathcal{W}.
\]

(5.16)

Clearly this is surjective for all \( s \). Let \( \mathcal{V}_s = \ker(\hat{T}_s + I) \). Then \( \mathcal{V}_0 = \mathcal{V} \) and \( \{\mathcal{V}_s\} \) is a locally trivial family. Denote by \( p_1 \) the projection onto the 1st factor of \( \mathcal{V} \oplus \mathcal{W} \). Then \( p_1 : \mathcal{V}_s \to \mathcal{V} \) is an isomorphism. Choose an orientation \( o(\mathcal{W}) \) for \( \mathcal{W} \) (the result
we obtain will be seen to be independent of the choice). The reference orientation of detind $\tilde{T}$ at $s = 0$ dictates that the induced orientation $o(\mathcal{V})$ is $\tau \wedge o(\mathcal{W})$. The orientation propagated to $\mathcal{V}_s$ is obtained by pulling back via $p_1$; denote this as $o(\mathcal{V}_s) = p_1^* o(\mathcal{V})$. Thus detind $(\tilde{T} + I)$ at $s$ has the propagated orientation $p_1^* o(\mathcal{V}) \wedge o(\mathcal{W}^*)$.

On the other hand, since $\tilde{T} + I$ is a stabilization there is, according to our convention of §5.1, a canonical way of relating the orientation on $\mathcal{R}\{v\} = (\text{detind } \tilde{T})_s$ with $\Lambda_{\text{max}}^{\mathcal{V}_s} \otimes \Lambda_{\text{max}}^{\mathcal{W}^*} = (\text{detind } (\tilde{T} + I))_s$, $s \neq 0$. This is given by the exact sequence (5.2) and the rule (5.3). In our situation since $\text{coker } \tilde{T}_s = \{0\}$, $s \neq 0$, we have for $s \neq 0$, the exact sequence

$$0 \longrightarrow \mathcal{R}\{v\} \xrightarrow{\iota} \mathcal{V}_{s} \xrightarrow{p_2} \mathcal{W} \longrightarrow 0 \longrightarrow \mathcal{W} \xrightarrow{\text{Id}} \mathcal{W} \longrightarrow 0. \quad (5.17)$$

Here $\iota$ is the inclusion and $p_2$ is projection onto the 2nd factor of $\mathcal{V} \oplus \mathcal{W}$. Once we have chosen $o(\mathcal{W})$ as above, our stabilization convention requires that $\mathcal{V}_{s}$ has the induced orientation

$$o'(\mathcal{V}_s) = v \wedge p_2^* o(\mathcal{W}) = \varepsilon o(\mathcal{V}_s), \quad (5.18)$$

$\varepsilon \neq 0$. The lemma is proven once we can show $\text{sign}(\varepsilon) = -\text{sign}(c_0)$.

The composition $p_2 \circ (p_1)^{-1} : \mathcal{V} \to \mathcal{W}$ is given by $u \mapsto (u, -s T_0'(u)) \mapsto -s T_0'(u)$. If we identify $\mathcal{V}_s$ with $\mathcal{V}$ via $p_1$ then (5.18) is equivalent to $o'(\mathcal{V}) = -v \wedge (s T_0')^* o(\mathcal{W}) = \varepsilon o(\mathcal{V})$. Choose say $o(\mathcal{W}) = \gamma \wedge v \wedge i v$. Then according to Lemma 5.3 $(T_0')^* (\gamma \wedge v \wedge i v) = \frac{1}{s^3 c_0} i v \wedge \tau \wedge \gamma$. Thus

$$o'(\mathcal{V}) = -\frac{1}{s^3 c_0} v \wedge i v \wedge \tau \wedge \gamma = -\frac{1}{s^3 c_0} \tau \wedge o(\mathcal{W}) = -\frac{1}{s^3 c_0} o(\mathcal{V}). \quad (5.19)$$

Hence $\varepsilon = -\text{sign}(c_0)$. qed.

**Proof of Lemma 5.3** It suffices to replace $\tilde{L}_s$ with the ‘1st order bifurcation family’ $\tilde{L}_s = \tilde{L}_{x+sv}$ since the computation takes place at $s = 0$. In the rule (5.3) with $\tilde{L}$ replaced by $\tilde{L}$, we may drop the $f$ term when computing the derivative of the family at $s = 0$ since $\frac{\partial f}{\partial s}|_{s=0} = 0$. Then restricted to $\mathcal{R}\{\tau\} \oplus \mathcal{R}\{\gamma\} \oplus \mathcal{C}\{v\}$ we have

$$\tilde{L}_s(t \tau, r \gamma, z v) = (\delta^{0x}_{x+sv}(z v), \delta^{1x}_{x+sv}(t \tau, z v) + \delta^{0x}_{x+sv}(r \gamma)). \quad (5.20)$$

The derivative at $s = 0$ of $\delta^{0x}_{x+sv}(r \gamma)$ is $r \gamma \cdot v = ir v$ which by our convention $\gamma \cdot v$ defines the complex structure $i$ on $\mathcal{C}\{v\}$. The derivative at $s = 0$ for $\delta^{0x}_{x+sv}(z v)$ is then the adjoint of this complex multiplication and hence the real part of $\langle i, z \rangle \gamma = iz \gamma$.

The last term in (5.15) unaccounted for is $c_0 tv$ (recall $v$ is taken as one of the
basis vectors) which is the derivative at \( s = 0 \) for \( \tilde{\delta}^1_{x+zv}(t\tau, zv) \). This computation essentially reduces to that of the quadratic approximation for the obstruction map in the proof of Lemma 4.9. Following the notation introduced there and equation (4.14) we obtain

\[
\tilde{\delta}^1_{t', x+a+b}(t\tau, a' + b') = t \frac{d}{du} (K^\alpha \circ J(u)) \bigg|_{u=0} (a) + K_{t', x}(a') \\
+ t \frac{d}{du} (N^\alpha \circ J(u)) \bigg|_{u=0} (b) + N_{t', x}(b') \\
+ B(a + b, a' + b') + B(a' + b', a + b).
\]

Now replacing \( b \) with \( sv \) and taking the derivative with respect to \( s \) at \( s = 0 \) gives

\[
\frac{d}{ds} (\Pi_W \circ \tilde{\delta}^1_{t', x+a+b})(t\tau, b') = t \frac{d}{du} (N^\alpha \circ J(u)) \bigg|_{u=0} (v) = c_0 tv.
\]

This completes the computation of \( T'_0 \) noting that in (5.13) we can drop the \( f \) term since \( df = 0 \) at \((0,0)\). qed.

### 5.4 Trivial strata

Recall that in the SW-case, any bifurcation along the trivial strata ends up in the Type II strata. Fix a splitting \( E = L_0 \oplus L_1 \) and let \( \mathcal{C}(L_0) \subset \mathcal{C}_E \) denote the subset consisting of pairs \((A, \Phi)\) where \( A = A_0 \oplus \theta \) (\( \theta \) is a fixed trivial connection on \( L_1 \)) and \( \Phi = (\phi_0, 0) \) with respect to the splitting. It is clear that any Type II reducible is gauge equivalent to one in \( \mathcal{C}(L_0) \).

Let \( \mathcal{G}^0 \) be the subgroup of gauge transformations which take block diagonal form

\[
\begin{pmatrix}
  g & 0 \\
  0 & 1
\end{pmatrix}
\]

\( \mathcal{G}^0 \) preserves \( \mathcal{C}(L_0) \) setwise (but is not the largest subgroup to do so). \( \mathcal{C}(L_0) \) with the action of \( \mathcal{G}^0 \) is basically the context of \( U(1) \)-SW theory and we may regard the bifurcation we are concerned with as having along the trivial strata \( \{[\theta]\} \times [0,1] \) inside \( \mathcal{C}(L_0)/\mathcal{G}^0 \times [0,1] \).

At any \((A_0, \phi_0, \theta) \in \mathcal{C}(L_0)\) the tangential operator \( K^{sw,II} \) splits as \( K^0_{A_0, \phi_0} \oplus K \) where \( K \) is our (untwisted) deRham operator on \( A^{0+1} \). Thus we see that \( \text{detind} K^{sw,II} = \text{detind} K^0 \). We may regard \( K^0 \) as being the SW-Floer-Taubes operator for \( \mathcal{C}(L_0) \) and therefore the deformation theory, orientations, spectral flow, etc. are completely
determined in reference to $K^0$ which in turn is determined by $K^{sw,II}$. In our $U(1)$ gauge theory perspective the trivial strata has stabilizer $U(1) \subset \mathcal{G}^0$ and therefore we are back in the situation of the Type I,II bifurcations. In particular the proof of the orientation of the bifurcation arc now extends to this situation.

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