Duality and quantum-algebra symmetry of the $A_{N-1}^{(1)}$ open spin chain with diagonal boundary fields

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Abstract

We show that the transfer matrix of the $A_{N-1}^{(1)}$ open spin chain with diagonal boundary fields has the symmetry $U_q(SU(l)) \times U_q(SU(N-l)) \times U(1)$, as well as a “duality” symmetry which maps $l \leftrightarrow N-l$. We exploit these symmetries to compute exact boundary $S$ matrices in the regime with $q$ real.
1 Introduction

The richness of boundary phenomena is a dominant theme of contemporary theoretical physics. Well-known examples include catalysis of baryon decay, D-branes, particle production near event horizons, the Kondo problem, and edge excitations in the fractional quantum Hall effect. As much of the boundary phenomena of interest is nonperturbative, integrable models with boundaries provide a particularly valuable laboratory for its study [1], [2]. Magnetic chains associated with affine Lie algebras [3], [4] constitute large classes of such integrable models. We focus here on the simplest such class, namely, the $A_{N-1}^{(1)}$ spin chain. In addition to its value as a simple toy model, this model is of interest in its own right: it is related to loop models describing certain self-avoiding walks (see, e.g., [5] and references therein), and possibly also to the $A_{N-1}^{(1)}$ Toda field theory with imaginary coupling [6], [7].

Pasquier and Saleur observed [8] that a certain integrable open XXZ spin chain Hamiltonian has the quantum-algebra (or “quantum group”) [9] symmetry $U_q(SU(2))$. Kulish and Sklyanin later showed [10] that this symmetry extends to the full transfer matrix [11]. This result was then generalized [12] - [14] to higher rank: namely, the transfer matrix of the $A_{N-1}^{(1)}$ open spin chain without boundary fields has the symmetry $U_q(SU(N))$. This model was further investigated in Refs. [15], [16]. Moreover, the most general diagonal boundary interactions which preserve integrability were found [17], [18]. However, the question of what symmetry – if any – remains in the presence of such boundary interactions was not explored.

We show here that by turning on diagonal boundary fields, the $U_q(SU(N))$ symmetry is broken to $U_q(SU(l)) \times U_q(SU(N - l)) \times U(1)$. Moreover, we find a “duality” symmetry which relates $l \leftrightarrow N - l$. We exploit these symmetries to compute exact boundary $S$ matrices [2], which describe the scattering of the model’s excitations from the ends of the chain, in the regime with $q$ real. For the case $N = 2$, we recover the results of Refs. [19] and [20]; and for $q = 1$ we recover the recent results [21].

In Section 2 we review the construction of the open chain transfer matrix, and we exhibit its symmetries. In Section 3 we summarize the computation of boundary $S$ matrices. We conclude in Section 4 with a brief discussion of some possible generalizations of this work. Some technical details have been relegated to the Appendices.

2 The transfer matrix and its symmetries

There are two basic building blocks for constructing integrable open spin chains:
1. The $R$ matrix, which is a solution of the Yang-Baxter equation

\[ R_{12}(\lambda) R_{13}(\lambda + \lambda') R_{23}(\lambda') = R_{23}(\lambda') R_{13}(\lambda + \lambda') R_{12}(\lambda), \]  

(see, e.g., [22]). We assume that the $R$ matrix has the unitarity property

\[ R_{12}(\lambda) R_{21}(-\lambda) = 1, \]

where $R_{21}(\lambda) = \mathcal{P}_{12} R_{12}(\lambda) \mathcal{P}_{12} = R_{12}(\lambda)^{t_{12}}$, $t$ denotes transpose, and $\mathcal{P}_{12}$ is the permutation matrix\(^1\), and also the property (see also [23])

\[ R_{12}(\lambda)^{t_{11}} M_1 R_{12}(-\lambda - 2\rho)^{t_{2}} M_1^{-1} \propto 1, \]

with $M^t = M$ and

\[ [M_1 M_2, R_{12}(\lambda)] = 0. \]

2. The matrices $K^\pm$, which are solutions of the boundary Yang-Baxter equation [24]\(^2\)

\[ R_{12}(\lambda_1 - \lambda_2) K_1^+(\lambda_1) R_{21}(\lambda_1 + \lambda_2) K_2^+(\lambda_2) = K_2^+(\lambda_2) R_{12}(\lambda_1 + \lambda_2) K_1^+(\lambda_1) R_{21}(\lambda_1 - \lambda_2). \]

The corresponding transfer matrix $t(\lambda)$ for an open chain of $N$ spins is given by [11], [12]

\[ t(\lambda) = \text{tr} M_0 K_0^+(-\lambda - \rho)^t T_0(\lambda) K_0^- (\lambda) \hat{T}_0(\lambda), \]

where $\text{tr}_0$ denotes trace over the “auxiliary space” 0, $T_0(\lambda)$ is the monodromy matrix

\[ T_0(\lambda) = R_{0N}(\lambda) \cdots R_{01}(\lambda), \]

and $\hat{T}_0(\lambda)$ is given by

\[ \hat{T}_0(\lambda) = R_{10}(\lambda) \cdots R_{N0}(\lambda). \]

\(^1\)The permutation matrix is defined by $\mathcal{P} x \otimes y = y \otimes x$ for all vectors $x$ and $y$ in an $N$-dimensional complex vector space $C_N$.

\(^2\)We use here a $K^+$ which is related to the $K^\text{old}_+$ used in earlier work [12] - [18] by

\[ K^\text{old}_+(\lambda) = M K^+(\lambda - \rho)^t. \]

The matrix $K^\text{old}_+$ satisfies an equation more complicated than (2.4), but the expression for the transfer matrix in terms of $K^\text{old}_+$ is simple. The matrix $K^+$ which we use here obeys the same boundary Yang-Baxter equation as $K^-$, but the expression for the transfer matrix in terms of this $K^+$ (see Eq. (2.7)) is more complicated. The two approaches are of course equivalent.
(As is customary, we usually suppress the “quantum-space” subscripts 1, . . . , N.) Indeed, it can be shown that this transfer matrix has the commutativity property

$$[t(\lambda), t(\lambda')] = 0.$$  \hspace{1cm} (2.10)

In this paper, we consider the case of the $A_{N-1}^{(1)}$ $R$ matrix \[23\]

\begin{align*}
R_{12}(\lambda)_{jj,jj} &= 1, \\
R_{12}(\lambda)_{jk,jk} &= \frac{\sinh(-i\eta\lambda)}{\sinh(\eta(-i\lambda + 1))}, \quad j \neq k, \\
R_{12}(\lambda)_{jk,kj} &= \frac{\sinh(\eta)}{\sinh(\eta(-i\lambda + 1))} \exp(i\eta\lambda \text{sign}(j-k)), \quad j \neq k, \\
&\quad 1 \leq j, k \leq N,
\end{align*}  \hspace{1cm} (2.11)

which depends on the anisotropy parameter $\eta \geq 0$, and which becomes $SU(N)$ invariant for $\eta \to 0$. This $R$ matrix has the properties (2.2) and (2.3), with \[17\]

$$M_{jk} = \delta_{jk} e^{\eta(N-2j+1)}, \quad \rho = iN/2.$$  \hspace{1cm} (2.12)

Moreover, we consider the $N \times N$ diagonal $K$ matrices given by \[18\]

\begin{align*}
K^-(\lambda) &= K_{(l)}(\lambda, \xi_-), \\
K^+(\lambda) &= K_{(l)}(\lambda, \xi_+ - \frac{N}{2}),
\end{align*}  \hspace{1cm} (2.13)

where

$$K_{(l)}(\lambda, \xi) = \text{diag}(a, \ldots, a, b, \ldots, b),$$

$$a = \sinh(\eta(\xi + i\lambda)) e^{-i\eta\lambda}, \quad b = \sinh(\eta(\xi - i\lambda)) e^{i\eta\lambda},$$  \hspace{1cm} (2.14)

for arbitrary $\xi$, and any $l \in \{1, \ldots, N-1\}$. Eq. (2.14) is the most general diagonal solution of the boundary Yang-Baxter equation with the $A_{N-1}^{(1)}$ $R$ matrix, up to an irrelevant overall factor.

We shall usually denote by $t_{(l)}(\lambda, \xi_-, \xi_+)$ the corresponding open spin chain transfer matrix \[3\]

$$t_{(l)}(\lambda, \xi_-, \xi_+) = \text{tr}_0 M_0 K_{(l)} \exp(-\lambda - \rho, \xi_+ - \frac{N}{2}) T_0(\lambda) \ K_{(l)} \ o(\lambda, \xi_-) \ \hat{T}_0(\lambda).$$  \hspace{1cm} (2.15)

\[3\]The more general transfer matrix $t_{(l_+, l_-)}(\lambda, \xi_-, \xi_+)$ constructed with $K_{(l_+)}(\lambda, \xi)$ also forms a one-parameter commutative family. For simplicity, we consider here (as in \[21\]) the special case $l_+ = l_- = l$. 

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The corresponding open spin chain Hamiltonian \( H_{\text{open}} \) is related to the derivative of the transfer matrix at \( \lambda = 0 \):

\[
H_{\text{open}} = \frac{i}{4 \sinh(\eta \xi^-)} \left. \left( -\rho, \xi_+^\prime - \frac{N}{2} \right) \right|_{\lambda = 0} d \lambda \left( \lambda, \xi_+, \xi^-_+ \right) + \frac{i}{4} \left. \frac{d}{d \lambda} \log \left( \left. \left( -\rho, \xi_+^\prime - \frac{N}{2} \right) \right|_{\lambda = 0} \right) K(l) \left( \lambda, \xi_-, \xi_+^\prime \right) \right|_{\lambda = 0}
\]

\[
= \sum_{n=1}^{N-1} H_{nn+1} + \frac{i}{4 \sinh(\eta \xi^-)} \left. \left( -\rho, \xi_+^\prime - \frac{N}{2} \right) \right|_{\lambda = 0} d \lambda K(l) \left( \lambda, \xi_-, \xi_+^\prime \right)
\]

\[
+ \frac{\text{tr}_0 M_0 K(l) \left( -\rho, \xi_+^\prime - \frac{N}{2} \right) \mathcal{H}_{N0}}{\text{tr} MK(l) \left( -\rho, \xi_+^\prime - \frac{N}{2} \right)},
\]

where the two-site Hamiltonian \( H_{jk} \) is given by

\[
H_{jk} = \frac{i}{2} \left. P_{jk} d \lambda R_{jk}(\lambda) \right|_{\lambda = 0}.
\]

One can verify that the Hamiltonian is Hermitian.

The parameters \( \xi_\pm^{-1} \) may be regarded as certain boundary fields. For \( \xi \to \infty \), the \( K \) matrix (2.14) evidently becomes proportional to the identity matrix

\[
K(l)(\lambda, \xi) \sim \frac{1}{2} e^{\eta \xi} 1,
\]

and so the transfer matrix is \( U_q(SU(N)) \) invariant [13], [14]. We now exhibit exact symmetries of this transfer matrix for finite values of \( \xi_\pm \).

### 2.1 Duality

The duality symmetry of the transfer matrix is a remnant of the cyclic \( (Z_N) \) symmetry [25], [26] of the \( A_{N-1}^{(1)} \) \( R \) matrix. Indeed, the \( R \) matrix (2.11) satisfies

\[
U_1 R_{12}(\lambda) U_1^{-1} = V_2(-\lambda)^{-1} R_{12}(\lambda) V_2(-\lambda)
\]

\[
U_2 R_{12}(\lambda) U_2^{-1} = V_1(\lambda)^{-1} R_{12}(\lambda) V_1(\lambda),
\]

where \( U \) is the \( N \times N \) matrix

\[
U_{jk} = \delta_{j,k-1} + \delta_{N,j} \delta_{1,k}
\]

which has the property \( U^N = 1 \), and \( V(\lambda) \) is the matrix

\[
V(\lambda)_{jk} = e^{-i \eta \lambda} \delta_{j,k-1} + e^{i \eta \lambda} \delta_{N,j} \delta_{1,k}.
\]
The corresponding quantum-space operator $\mathcal{U}$ defined by

$$\mathcal{U} = U_1 U_2 \cdots U_N$$

therefore has the following action on the monodromy matrices

$$\mathcal{U} T_0(\lambda) \mathcal{U}^{-1} = V_0(\lambda)^{-1} T_0(\lambda) V_0(\lambda),$$

$$\mathcal{U} \hat{T}_0(\lambda) \mathcal{U}^{-1} = V_0(-\lambda)^{-1} \hat{T}_0(\lambda) V_0(-\lambda),$$

and the transfer matrix (2.15) transforms as follows:

$$\mathcal{U}^l t_{(l)}(\lambda, \xi_-, \xi_+) \mathcal{U}^{-l} = \text{tr}_0 \left\{ \left( V(-\lambda)^l MK_{(l)}(-\lambda - \rho, \xi_+ - \frac{\mathcal{N}}{2}) V(\lambda)^{-l} \right)_0 ight\}$$

$$\times T_0(\lambda) \left( V(\lambda)^l K_{(l)}(\lambda, \xi_-) V(-\lambda)^{-l} \right)_0 \hat{T}_0(\lambda).$$

(2.24)

We now observe that the $K$ matrix (2.14) satisfies

$$V(\lambda)^l K_{(l)}(\lambda, \xi_-) V(-\lambda)^{-l} = -e^{-2i(l-1)\eta \lambda} K_{(\mathcal{N}-l)}(\lambda, -\xi_-),$$

$$V(-\lambda)^l MK_{(l)}(-\lambda - \rho, \xi_+ - \frac{\mathcal{N}}{2}) V(\lambda)^{-l} = -e^{-2i(l-1)\eta \lambda} e^{\eta(N-2l)} MK_{(\mathcal{N}-l)}(-\lambda - \rho, -\xi_+ + \frac{\mathcal{N}}{2}),$$

$$l = 1, \ldots, \mathcal{N} - 1.$$  

(2.25)

We conclude that the transfer matrix has the “duality” transformation property

$$\mathcal{U}^l t_{(l)}(\lambda, \xi_-, \xi_+) \mathcal{U}^{-l} = e^{\eta(N-2l)} t_{(l')}(\lambda, \xi_-, \xi_+'),$$

(2.26)

where

$$\xi_-' = -\xi_-,$$

$$\xi_+ ' = -\xi_+ + \mathcal{N},$$

$$l' = \mathcal{N} - l.$$  

(2.27)

Notice that $\mathcal{U}^l \mathcal{U}^l = 1$. The transfer matrix is “self-dual” for $\xi_- = \xi_- ' = 0$, $\xi_+ = \xi_+ ' = \mathcal{N}/2$, and $l = l' = \mathcal{N}/2$.

### 2.2 Quantum algebra symmetry

In the defining representation of $SU(\mathcal{N})$, we identify (following the notations of our previous paper [21]) the raising and lowering operators

$$j^{+(k)} = e_{k,k+1}, \quad j^{-(k)} = e_{k+1,k}, \quad k = 1, \ldots, \mathcal{N} - 1.$$  

(2.28)

The prefactor $e^{\eta(N-2l)}$ can be absorbed in the definition of $K^+$, and is not significant.
which correspond to the simple roots, and the Cartan generators

\[ s^{(k)} = e_{k,k} - e_{k+1,k+1}, \quad k = 1, \ldots, N - 1, \]

(2.29)

where \( e_{k,l} \) are elementary \( N \times N \) matrices with matrix elements \( (e_{k,l})_{ab} = \delta_{k,a} \delta_{l,b} \). We denote by \( j_n^{(k)} \), \( s_n^{(k)} \) the generators at site \( n \), e.g.,

\[ s_n^{(k)} = 1 \otimes \ldots \otimes 1 \otimes s^{(k)} \otimes 1 \otimes \ldots \otimes 1, \quad n = 1, \ldots, N. \]

(2.30)

The corresponding generators of the quantum algebra \( U_q(SU(N)) \) which act on the full space of states are given by (see, e.g., [9], [16])

\[ J^{\pm}(k) = \sum_{n=1}^{N} q^{-s_{N}/2} \ldots q^{-s_{n+1}/2} j_n^{(k)} q^{s_{n-1}/2} \ldots q^{s_{1}/2}, \]

\[ S^{(k)} = \sum_{n=1}^{N} s_n^{(k)}, \quad k = 1, \ldots, N - 1. \]

(2.31)

These generators obey the commutation relations

\[ \left[ J^{+}(k), J^{-}(j) \right] = \delta_{k,j} \left[ S^{(k)} \right]_q, \quad \left[ S^{(k)}, J^{+}(j) \right] = (2\delta_{k,j} - \delta_{k-1,j} - \delta_{k+1,j}) J^{+}(j), \]

(2.32)

where \( [x]_q \equiv (q^x - q^{-x})/(q - q^{-1}) \).

We claim that the transfer matrix \( t_{(l)}(\lambda, \xi_-, \xi_+) \) has the invariance \( U_q(SU(l)) \times U_q(SU(N - l)) \times U(1) \). In particular, we shall now show that

\[ \left[ t_{(l)}(\lambda, \xi_-, \xi_+), S^{(k)} \right] = 0, \quad k = 1, \ldots, N - 1, \]

(2.33)

\[ \left[ t_{(l)}(\lambda, \xi_-, \xi_+), J^{\pm}(k) \right] = 0, \quad k \neq l, \]

(2.34)

where \( q = e^{-\eta} \). Readers who are interested primarily in the calculation of boundary \( S \) matrices may now wish to skip directly to Section 3, which may be read independently of the proof of Eqs. (2.33) and (2.34) that is presented below.

It is convenient to introduce, following Sklyanin [11], the quantity \( T_{(l)} 0(\lambda, \xi) \) defined by

\[ T_{(l)} 0(\lambda, \xi) = T_0(\lambda) K_{(l)} 0(\lambda, \xi) \tilde{T}_0(\lambda), \]

(2.35)

in terms of which the expression (2.13) for the transfer matrix acquires the more compact form

\[ t_{(l)}(\lambda, \xi_-, \xi_+) = \text{tr}_0 M_0 K_{(l)} 0(-\lambda - \rho, \xi_+ - \frac{N}{2}) T_{(l)} 0(\lambda, \xi_-). \]

(2.36)
The proof of the first relation (2.33) is straightforward. From the identity
\[
\left[ s_2^{(k)}, R_{12}(\lambda) \right] = - \left[ s_1^{(k)}, R_{12}(\lambda) \right],
\]
(2.37)

it follows that
\[
\left[ S^{(k)}, T_0(\lambda) \right] = - \left[ s_0^{(k)}, T_0(\lambda) \right],
\]
(2.38)

and similarly for \( \hat{T}_0(\lambda) \). Hence,
\[
\left[ S^{(k)}, T_0(\lambda, \xi) \right] = - \left[ s_0^{(k)}, T_0(\lambda, \xi) \right],
\]
(2.39)

where we have also used the fact that \( \left[ s_0^{(k)}, K_0(\lambda, \xi) \right] = 0 \). We conclude that
\[
\left[ S^{(k)}, T_0(\lambda, \xi_-), T_0(\lambda, \xi_+) \right] = - \text{tr}_0 \left[ S^{(k)}, M_0 K_0(\lambda, \xi_+ - \frac{N}{2}) T_0(\lambda, \xi_-) \right] = 0,
\]
(2.40)

where the final equality follows from the cyclic property of the trace.

Our proof of the second relation (2.34) relies on the well-known fact that in the limits \( \lambda \to \pm i\infty \), the monodromy matrix becomes an upper/lower triangular matrix whose elements can be expressed in terms of the \( U_q(SU(N)) \) generators. (See, e.g., \cite{9}, \cite{13}, \cite{16}.) It is therefore convenient to introduce notations for the corresponding limits of the \( R \) and \( T \) matrices:
\[
R_{12}(\lambda) \rightarrow e^{\mp \eta} R_{12}^\pm,
\]
\[
T_0(\lambda) \rightarrow e^{\mp \eta N} T_0^\pm,
\]
\[
\hat{T}_0(\lambda) \rightarrow e^{\mp \eta N} \hat{T}_0^\pm,
\]
(2.41)

for \( \lambda \to \pm i\infty \), with
\[
T_0^\pm = R_{0N}^\pm \cdots R_{01}^\pm,
\]
\[
\hat{T}_0^\pm = R_{10}^\pm \cdots R_{N0}^\pm.
\]
(2.42)

Schematically,
\[
T_0^+ \sim \begin{pmatrix}
* & J^{-(1)} & * \\
* & \ddots & * \\
0 & * & J^{-(N-1)}
\end{pmatrix} \sim \hat{T}_0^-.
\]

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where the diagonal matrix elements involve only the Cartan generators. We shall not need more explicit expressions for $T_0^\pm$ and $T_0^\pm$, which can be found e.g. in [16].

It is also important to realize that in these limits the $K$ matrices (2.14) become projectors:

$$K_{(l)}(\lambda, \xi) \sim -\frac{1}{2} e^{\pm 2i(\lambda - \eta)} \Pi_{(l)}^\pm \quad (2.44)$$

for $\lambda \to \pm i\infty$, where $\Pi_{(l)}^\pm$ are the orthogonal projection operators

$$\Pi_{(l)}^+ = \text{diag}(1, \ldots, 1, 0, \ldots, 0),$$
$$\Pi_{(l)}^- = \text{diag}(0, \ldots, 0, 1, \ldots, 1), \quad (2.45)$$

and therefore $\Pi_{(l)}^+ + \Pi_{(l)}^- = 1$.

It follows that for $\lambda \to \pm i\infty$, the quantity $\mathcal{T}_{(l)} 0(\lambda, \xi)$ defined in Eq. (2.35) tends (up to an irrelevant factor) to $\mathcal{T}_{(l)} 0^\pm$, where

$$\mathcal{T}_{(l)}^\pm 0 = T_0^\pm \Pi_{(l)}^\pm \hat{T}_0^\pm. \quad (2.46)$$

It is easy to show (see Eq. (A.4) in Appendix A) that $\mathcal{T}_{(l)}^\pm 0$ and $\hat{T}_{(l)}^\pm$ obey

$$\Pi_{(l)}^+ \mathcal{T}_{(l)}^\pm 0 \Pi_{(l)}^+ 0 = \mathcal{T}_{(l)}^\pm 0; \quad (2.47)$$

With the help of Eq. (2.43), one can also see that $\mathcal{T}_{(l)}^\pm 0$ depends on the generators $J_{(l)}^\pm(k)$ with $k < l$, while $\mathcal{T}_{(l)}^\pm 0$ depends on the generators $J_{(l)}^\pm(k)$ with $k > l$.

To prove the relation (2.34), it suffices to show that the transfer matrix commutes with $\tau_{(l)}^\pm$

$$[t_{(l)}(\lambda, \xi_-; \xi_+), \tau_{(l)}^\pm] = 0, \quad (2.48)$$

where $\tau_{(l)}^\pm$ are defined by

$$\tau_{(l)}^\pm = \text{tr}_0 P_0^\pm \mathcal{T}_{(l)}^\pm 0, \quad (2.49)$$

and $P^\pm$ are $\mathcal{N} \times \mathcal{N}$ matrices which obey

$$\Pi_{(l)}^\pm P^\pm \Pi_{(l)}^\pm = P^\pm, \quad (2.50)$$
but which are otherwise arbitrary. Indeed, with suitable choices of $P^+$ and $P^-$, one can project from $T^+_0$ and $T^-_0$ the generators $J^{\pm(k)}$ with $k \neq l$. For example, with $P^+ = \tilde{j}^{\pm(k)}$ one projects out $J^{\pm(k)}$ with $k < l$.

The quantities $\tau^{\pm}_l$ evidently have a structure similar to that of the transfer matrix. The proof of Eq. (2.48) is similar to the proof of the fundamental commutativity property (2.10) of the transfer matrix. Indeed, following Sklyanin, one can show using the properties (2.3), (2.4) that

$$
\tau^{\pm}_l t_l(\lambda, \xi_-, \xi_+),
\tau^{\pm}_l = \text{tr}_1 \left( P^+_1 T^+_l \right) \left[ M_2 K_l \right] \left( -\lambda - \rho, \xi_+ - \frac{N}{2} \right) t_2(\lambda, \xi_-) \right),
\vdots
= \text{tr}_1 \left( P^+_1 T^+_l \right) \left[ M_2 K_l \right] \left( -\lambda - \rho, \xi_+ - \frac{N}{2} \right) t_2(\lambda, \xi_-) \right),
\vdots
= \text{tr}_1 \left( P^+_1 T^+_l \right) \left[ M_2 K_l \right] \left( -\lambda - \rho, \xi_+ - \frac{N}{2} \right) t_2(\lambda, \xi_-) \right),
\vdots
= \text{tr}_1 \left( P^+_1 T^+_l \right) \left[ M_2 K_l \right] \left( -\lambda - \rho, \xi_+ - \frac{N}{2} \right) t_2(\lambda, \xi_-) \right),
\vdots
$$

Similarly,

$$
t_l(\lambda, \xi_-, \xi_+) = \text{tr}_2 \left( M_2 K_l \right) \left( -\lambda - \rho, \xi_+ - \frac{N}{2} \right) t_1 \left( P^+_1 T^+_l \right) \left[ M_2 K_l \right] \left( -\lambda - \rho, \xi_+ - \frac{N}{2} \right) t_2(\lambda, \xi_-) \right),
\vdots
= \text{tr}_2 \left( M_2 K_l \right) \left( -\lambda - \rho, \xi_+ - \frac{N}{2} \right) t_1 \left( P^+_1 T^+_l \right) \left[ M_2 K_l \right] \left( -\lambda - \rho, \xi_+ - \frac{N}{2} \right) t_2(\lambda, \xi_-) \right),
\vdots
= \text{tr}_2 \left( M_2 K_l \right) \left( -\lambda - \rho, \xi_+ - \frac{N}{2} \right) t_1 \left( P^+_1 T^+_l \right) \left[ M_2 K_l \right] \left( -\lambda - \rho, \xi_+ - \frac{N}{2} \right) t_2(\lambda, \xi_-) \right),
\vdots
= \text{tr}_2 \left( M_2 K_l \right) \left( -\lambda - \rho, \xi_+ - \frac{N}{2} \right) t_1 \left( P^+_1 T^+_l \right) \left[ M_2 K_l \right] \left( -\lambda - \rho, \xi_+ - \frac{N}{2} \right) t_2(\lambda, \xi_-) \right),
\vdots
$$

It is easy to see that the quantities in square brackets in Eqs. (2.51) and (2.52) are equal. Indeed, since $T_l(\lambda, \xi)$ obeys the boundary Yang-Baxter equation

$$
R_{12}(\lambda_1 - \lambda_2) T_l(1)(\lambda_1, \xi) R_{21}(\lambda_1 + \lambda_2) T_l(2)(\lambda_2, \xi) = T_l(2)(\lambda_2, \xi) R_{12}(\lambda_1 + \lambda_2) T_l(1)(\lambda_1, \xi) R_{21}(\lambda_1 - \lambda_2),
$$

by taking $\lambda_1 \to \pm i \infty$ we obtain the relations

$$
R_{12}^\pm T_l^+(R_{21}^\pm T_l^+(2)(\lambda, \xi) = T_l^+(2)(\lambda, \xi) R_{12}^\pm T_l^+(1) R_{21}^\pm.
$$

In order to prove the commutativity (2.48), it therefore suffices to show that

$$
\text{tr}(AC) = \text{tr}(BC),
$$

where

$$
A = R_{12}^\pm P^+_1 M_1^{-1} R_{21}^\pm K_l \left( -\lambda - \rho, \xi_+ - \frac{N}{2} \right),
$$

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\[ B = K(l) 2(-\lambda - \rho, \xi_+ - \frac{N}{2})R_{12}^+ P_1^\pm t_1 M_1^{-1} R_{21}^+, \]
\[ C = \left[ R_{12}^+ T_{(l)}^\pm 1 R_{21}^+ T_{(l)} 2(\lambda, \xi_-)M_1 M_2 \right]^{t_{12}}. \] (2.56)

We establish this result by showing that \( A - B \) and \( C \) are trace orthogonal matrices. That is, we show that for some projection operator \( \Pi \),
\[ \Pi C \Pi = C \]
\[ \Pi (A - B) \Pi = 0, \] (2.57)
from which it immediately follows that
\[ \text{tr} [(A - B) C] = \text{tr} [(A - B) \Pi C] = \text{tr} [\Pi (A - B) \Pi C] = 0. \] (2.58)

Indeed, in Appendix A we show that
\[ \Pi_{(l)}^\pm 1 \left( R_{12}^+ T_{(l)}^\pm 1 R_{21}^+ T_{(l)} 2(\lambda, \xi_-)M_1 M_2 \right) \Pi_{(l)}^\pm 1 = R_{12}^+ T_{(l)}^\pm 1 R_{21}^+ T_{(l)} 2(\lambda, \xi_-)M_1 M_2, \] (2.59)
\[ \Pi_{(l)}^\pm 1 \left( R_{12}^+ P_1^\pm t_1 M_1^{-1} R_{21} K(l) 2(\lambda, \xi_-) \right) \Pi_{(l)}^\pm 1 = \Pi_{(l)}^\pm 1 \left( K(l) 2(\lambda, \xi_-) R_{12}^+ P_1^\pm t_1 M_1^{-1} R_{21}^+ \right) \Pi_{(l)}^\pm 1, \] (2.60)
thereby completing the proof.

We have considered here symmetry transformations which leave the anisotropy parameter \( \eta \) unchanged. In Appendix C we briefly discuss related transformations, which however transform \( \eta \to -\eta \).

### 3 Boundary S Matrix

The eigenstates of the transfer matrix \( t(l)(\lambda, \xi_-, \xi_+) \) can be constructed by the so-called nested Bethe Ansatz using the pseudovacuum \( \omega(k) \) defined by
\[ \omega(k) = v(k) \otimes \cdots \otimes v(k), \quad k = 1, \ldots, N, \] (3.1)
where \( v(k) \) are the standard \( N \)-dimensional Cartesian basis vectors with components \( v(k) a = \delta_{k,a} \). The corresponding Bethe Ansatz equations (BAE) can be cast in the form
\[ h_{(l)}^{(j)}(\lambda^{(j)}) = J_{\alpha}^{(j)}, \quad \alpha = 1, \ldots, M^{(j)}, \quad j = 1, \ldots, N - 1, \] (3.2)

\(^5\text{We restrict to real } \lambda_{\alpha}^{(j)}, \text{ which suffices for computing boundary } S \text{ matrices.}\)
where \( h^{(j)}_{(l)}(\lambda) \) is the so-called counting function (explicit expressions are given below), and \( \{ J^{(j)}_\alpha \} \) are integers lying in certain ranges which serve as “quantum numbers” of the Bethe Ansatz (BA) states. We label the holes in the ranges by \( \{ \tilde{J}^{(j)}_\alpha \} \), \( \alpha = 1, \ldots, \nu^{(j)} \). The corresponding hole rapidities \( \{ \tilde{\lambda}^{(j)}_\alpha \} \) are defined by
\[
h^{(j)}_{(l)}(\tilde{\lambda}^{(j)}_\alpha) = \tilde{J}^{(j)}_\alpha, \quad \alpha = 1, \ldots, \nu^{(j)}. \tag{3.3}
\]

The nature of the ground state and excitations is similar to that of the corresponding isotropic model \cite{27,28,21}, except that Lie algebra symmetries are now replaced by their \( q \)-deformations. Indeed, the ground state is the BA state with no holes, i.e., with \( N - 1 \) filled Fermi seas. Moreover, the BA state with one hole in the \( j^{th} \) sea (i.e., \( \nu^{(k)} = \delta_{k,j} \) for \( k = 1, \ldots, N - 1 \)) is a particle-like excited state which belongs to the fundamental representation \( [j] \) of \( U_q(SU(N)) \), corresponding to a Young tableau with a single column of \( j \) boxes, as shown in Figure 1. This excitation carries energy \( s^{(j)}(\tilde{\lambda}^{(j)}) \) and “momentum”

\[
p^{(j)}(\tilde{\lambda}^{(j)}), \text{ where} \]
\[
\frac{d}{d\lambda} p^{(j)}(\lambda) = 2\pi s^{(j)}(\lambda), \tag{3.4}
\]
and the Fourier transform of \( s^{(j)}(\lambda) \) is given by
\[
\hat{s}^{(j)}(k) = \frac{\sinh \left( (N - j)\eta |k| \right)}{\sinh (N \eta |k|)}. \tag{3.6}
\]

\[\text{Figure 1: Young tableau with a single column of } j \text{ boxes, corresponding to the fundamental representation } [j] \text{ of } U_q(SU(N))\]

\[\text{Since the boundary interactions break the bulk } U_q(SU(N)) \text{ symmetry, states should – strictly speaking – be classified according to the unbroken symmetry } U_q(SU(l)) \times U_q(SU(N - l)) \times U(1). \text{ However, one expects that the effects of the boundary should be “small” at points of the chain that are far from the boundary. In particular, bulk multiparticle states should “approximately” form irreducible representations of } U_q(SU(N)), \text{ and therefore we refer to such states by } U_q(SU(N)) \text{ quantum numbers.}\]

\[\text{Our conventions for Fourier transforms are as follows:}\]
\[
f(\lambda) = \frac{\eta}{\pi} \sum_{k=-\infty}^{\infty} e^{-2\eta k \lambda} \hat{f}(k), \quad \hat{f}(k) = \int_{-\pi/2\eta}^{\pi/2\eta} d\lambda \, e^{2\eta k \lambda} f(\lambda). \tag{3.5}
\]
Introducing the lattice spacing $a$, and taking the continuum ($a \to 0$) and isotropic ($\eta \to 0$) limits with $\frac{4}{a} e^{-\frac{x^2}{a^2}} = \mu$ finite, one obtains a relativistic dispersion relation, with mass $m^{(j)} = \mu \sin(\pi j/N)$.

We define the boundary $S$ matrices $S_{(l)}^{\pm} [j]$ for a particle of type $[j]$ by the quantization condition

\[
\left( e^{i2p^{(j)}(\lambda)N} S_{(l)}^{+} [j] S_{(l)}^{-} [j] - 1 \right) |\tilde{\lambda}^{(j)}\rangle = 0 .
\]

We compute these $S$ matrices from the $1/N$ terms in the distribution of roots of the BAE for the corresponding BA states $|\tilde{\lambda}^{(j)}\rangle$. Indeed, we rely on the key identity

\[
\frac{1}{\pi} \frac{d}{d\lambda} p^{(j)}(\lambda) + \sigma^{(j)}(\lambda) - 2s^{(j)}(\lambda) = \frac{1}{N} \frac{d}{d\lambda} h^{(j)}(\lambda) ,
\]

where the “density” $\sigma^{(j)}(\lambda)$ defined by

\[
\sigma^{(j)}(\lambda) = \frac{1}{N} \frac{d}{d\lambda} h^{(j)}(\lambda)
\]

describes the distribution of roots of the BAE. This method is a generalization of the approach pioneered by Korepin and Andrei-Destri for calculating bulk two-particle $S$ matrices. (See also [33].)

In order to compute each of the boundary $S$ matrix elements, we shall need the densities $\sigma^{(j)}(\lambda)$ corresponding to each of the pseudovacua $\omega_{(1)}, \ldots, \omega_{(N)}$. It can be shown that the Bethe Ansatz equations (and hence, the counting functions $h^{(j)}(\lambda)$ and densities $\sigma^{(j)}(\lambda)$) corresponding to the first $l$ pseudovacua $\omega_{(1)}, \ldots, \omega_{(l)}$ are all the same; and the BAE corresponding to the last $N-l$ pseudovacua $\omega_{(l+1)}, \ldots, \omega_{(N)}$ are all the same. This result is quite plausible, in view of the $U_q(SU(l)) \times U_q(SU(N-l))$ symmetry of the transfer matrix. For completeness, we provide in Appendix B a proof of this result.

Corresponding to the first $l$ pseudovacua $\omega_{(1)}, \ldots, \omega_{(l)}$, the BAE for the transfer matrix $t_{(l)}(\lambda, \xi -, \xi +)$ are given by [18]

\[
1 = \left[ e_{2\xi_+ - l}^{(l)}(\lambda^{(l)}) e_{-(2\xi_+ - 2N+l)}^{(l)}(\lambda^{(l)}) \delta_{l,j} + (1 - \delta_{l,j}) \right]
\times \prod_{\beta=1}^{M^{(l-1)}} e_{-1}(\lambda^{(j)} - \lambda^{(j-1)}_{\beta}) e_{-1}(\lambda^{(j)} + \lambda^{(j-1)}_{\beta}) \prod_{\beta=1}^{M^{(l)}} e_{2}(\lambda^{(j)} - \lambda^{(j)}_{\beta}) e_{2}(\lambda^{(j)} + \lambda^{(j)}_{\beta})
\times \prod_{\beta=1}^{M^{(l+1)}} e_{-1}(\lambda^{(j)} - \lambda^{(j+1)}_{\beta}) e_{-1}(\lambda^{(j)} + \lambda^{(j+1)}_{\beta}) \left\{ \alpha = 1, \ldots, M^{(j)}, \quad j = 1, \ldots, N - 1, \right\}
\]

(3.10)
where

\[ e_n(\lambda) = \frac{\sin \eta \left( \lambda + \frac{in}{2} \right)}{\sin \eta \left( \lambda - \frac{in}{2} \right)}, \]

(3.11)

and \( M^{(0)} = N, \quad M^{(N)} = 0, \quad \lambda_\alpha^{(0)} = \lambda_\alpha^{(N)} = 0 \). The requirement that solutions of the BAE correspond to independent BA states leads to the restriction \( 0 < \lambda_\alpha^{(j)} < \frac{\pi}{2\eta} \). For simplicity, we restrict to “weak” boundary fields \( \xi_- > \frac{\pi}{2}(N - 1), \quad \xi_+ < N - \frac{1}{2} \). The energy as a function of \( \xi_- \) and \( \xi_+ \) is discussed for the case \( N = 2 \) in [19], [24].

Defining the counting function so that the Bethe Ansatz equations take the form (3.2), we find

\[
\begin{align*}
&h_{(l)}^{(j)}(\lambda) = \frac{1}{2\pi} \left\{ q_1(\lambda) + r_1(\lambda) + \left[ -q_{2\xi_- + l}(\lambda) + q_{2\xi_- - 2N + l}(\lambda) \right] \delta_{j,l} \\
&+ \sum_{\beta=1}^{M^{(j-1)}} \left[ q_1(\lambda - \lambda_\beta^{(j-1)}) + q_1(\lambda + \lambda_\beta^{(j-1)}) \right] - \sum_{\beta=1}^{M^{(j)}} \left[ q_2(\lambda - \lambda_\beta^{(j)}) + q_2(\lambda + \lambda_\beta^{(j)}) \right] \\
&+ \sum_{\beta=1}^{M^{(j+1)}} \left[ q_1(\lambda - \lambda_\beta^{(j+1)}) + q_1(\lambda + \lambda_\beta^{(j+1)}) \right] \right\},
\end{align*}
\]

(3.12)

where \( q_n(\lambda) \) and \( r_n(\lambda) \) are odd monotonic-increasing functions defined (for \( n > 0 \)) by

\[
\begin{align*}
&g_n(\lambda) = \pi + i \log e_n(\lambda), \\
&-\pi < g_n(\lambda) \leq \pi \quad \text{for} \quad -\frac{\pi}{2\eta} < \lambda \leq \frac{\pi}{2\eta}, \quad \text{and} \quad g_n(\lambda + \frac{\pi}{\eta}) = g_n(\lambda) + 2\pi, \\
&r_n(\lambda) = i \log g_n(\lambda), \quad g_n(\lambda) = \frac{\cos \eta \left( \lambda + \frac{in}{2} \right)}{\cos \eta \left( \lambda - \frac{in}{2} \right)}, \\
&-\pi < r_n(\lambda) \leq \pi \quad \text{for} \quad -\frac{\pi}{2\eta} < \lambda \leq \frac{\pi}{2\eta}, \quad \text{and} \quad r_n(\lambda + \frac{\pi}{\eta}) = r_n(\lambda) + 2\pi.
\end{align*}
\]

(3.13)

(3.14)

We consider now a multi-hole state, with the number of holes in each of the seas given by \( \nu^{(1)}, \ldots, \nu^{(N-1)} \). In the thermodynamic limit, we obtain a system of linear integral equations for the densities (3.4) by approximating the sums in \( h_{(l)}^{(j)}(\lambda) \) by integrals using [20]

\[ \frac{1}{N} \sum_{\alpha=1}^{M^{(j)}} g(\lambda_\alpha^{(j)}) \approx \int_0^{\pi/2\eta} g(\lambda') \sigma_{(l)}^{(j)}(\lambda') d\lambda' - \frac{1}{N} \sum_{\alpha=1}^{\nu^{(j)}} g(\tilde{\lambda}_\alpha^{(j)}) - \frac{1}{2N} \left[ g(0) + g(\frac{\pi}{2\eta}) \right]. \]

(3.15)

In this way we obtain

\[ \sum_{m=1}^{N-1} \left( (\delta + K)_{jm} \ast \sigma_{(l)}^{(m)} \right)(\lambda) = 2a_1(\lambda)\delta_{j,1} + \frac{1}{N} \left\{ a_2(\lambda) + b_2(\lambda) \right\}. \]
Using Fourier transforms, we obtain the solution

$$
\sigma_{(l)}(\lambda) = R_{(l)}(\lambda) + \delta\sigma_{(l)}^{(j)}(\lambda)
$$

(3.18)

where $$\sigma_{(l)}^{(j)}(\lambda)$$ is the symmetric density defined by

$$
\sigma_{(l)}^{(j)}(\lambda) = \begin{cases} 
\sigma_{(l)}^{(j)}(\lambda) & \lambda > 0 \\
\sigma_{(l)}^{(j)}(-\lambda) & \lambda < 0
\end{cases},
$$

(3.17)

* denotes the convolution

$$
(f \ast g)(\lambda) = \int_{-\pi/2\eta}^{\pi/2\eta} f(\lambda - \lambda') g(\lambda') d\lambda',
$$

(3.19)

Using Fourier transforms, we obtain the solution

$$
\sigma_{(l)}^{(j)}(\lambda) = 2\sigma_{(l)}^{(j)}(\lambda) + \delta\sigma_{(l)}^{(j)}(\lambda)
$$

$$
+ \frac{1}{N} \sum_{m=1}^{N-1} \left( R_{jm} \ast \left( a_2 + b_2 + (a_1 + b_1) \left( -1 + \delta_{m,1} + \delta_{m,N-1} \right) \right) \right)(\lambda)
$$

$$
+ \sum_{m=1}^{N-1} \sum_{a=1}^{\nu^{(m)}} \left[ \delta(\lambda - \tilde{\lambda}_a^{(m)}) \delta_{j,m} - R_{jm}(\lambda - \tilde{\lambda}_a^{(m)}) + (\tilde{\lambda}_a^{(m)} \rightarrow -\tilde{\lambda}_a^{(m)}) \right],
$$

(3.20)

where $$R_{jm}(\lambda)$$ has the Fourier transform (see, e.g., [25])

$$
\hat{R}_{mm'}(k) = \frac{e^{\eta|k|} \sinh (m_\eta|k|) \sinh ((N - m_\eta|k|)}{\sinh (N\eta|k|) \sinh (\eta|k|)}.
$$

(3.21)

with $$m_\eta = \max(m, m')$$ and $$m_\eta = \min(m, m')$$, and

$$
\delta\sigma_{(l)}^{(j)}(\lambda) = \frac{1}{N} \left( R_{jt} \ast \left( -a_{2\xi_- + l} + a_{2\xi_+ - 2N_+ l} \right) \right)(\lambda)
$$

(3.22)

has the dependence on the boundary parameters $$\xi_. \xi_\parallel$$. We note also

$$
\hat{\sigma}_n(k) = e^{-\eta|k|}, \quad \hat{b}_n(k) = (-)^k \hat{a}_n(k), \quad n > 0.
$$

(3.23)
The above density $\sigma^{(j)}_{(l)}(\lambda)$ corresponds to the first $l$ pseudovacua $\omega(1), \ldots, \omega(l)$. We also need the density $\sigma^{(j)}_{(l)}(\lambda)$ corresponding to the last $\mathcal{N} - l$ pseudovacua $\omega(l+1), \ldots, \omega(\mathcal{N})$, as already remarked. We recall that under duality, the transfer matrix transforms as (2.26)

$$\mathcal{U}^l \ t_{(l)}(\lambda, \xi_-, \xi_+) \mathcal{U}^{-l} \propto t_{(l')}(\lambda, \xi'_-, \xi'_+),$$

(3.24)

while the pseudovacuum $\omega(k)$ transforms as

$$\mathcal{U}^l \ \omega(k) = \omega(k-l),$$

(3.25)

where we identify $\omega(k) \equiv \omega(k+\mathcal{N})$. It follows [21] that the BAE for $t_{(l)}(\lambda, \xi_-, \xi_+)$ with pseudovacua $\omega(l+1), \ldots, \omega(\mathcal{N})$ are the same as the BAE for $t_{(l')}(\lambda, \xi'_-, \xi'_+)$ with $\omega(1), \ldots, \omega(l')$; i.e., the BAE are given by Eq. (3.10), except with $\xi_+ \rightarrow \xi'_+$ and $l \rightarrow l'$. It follows that the corresponding densities $\sigma^{(j)}_{(l)}(\lambda)$ are given by Eq. (3.20), except with

$$\delta \sigma^{(j)}_{(l)}(\lambda) = \frac{1}{N} \left( \mathcal{R}_{j,\mathcal{N}-l} * \left(a_{2\xi_-N+l} - a_{2\xi_+N+l}\right) \right)(\lambda).$$

(3.26)

For simplicity, we compute boundary $S$ matrices only for the cases [1] and $[\mathcal{N} - 1] = [1]$, for which the $S$ matrices act in the $\mathcal{N}$-dimensional complex vector spaces $\mathcal{C}_\mathcal{N}$ and $\overline{\mathcal{C}_\mathcal{N}}$, respectively.

We begin with the case [1], for which the boundary $S$ matrices $S^+_{(l)} [1]$ are diagonal $\mathcal{N} \times \mathcal{N}$ matrices of the form

$$S^+_{(l)} [1] = \text{diag}(\alpha^+_{(l)}, \ldots, \alpha^+_{(l)}, \beta^+_{(l)}, \ldots, \beta^+_{(l)}).$$

(3.27)

Choosing the state $|\tilde{\lambda}^{(1)}\rangle$ in Eq. (3.7) to be the BA state constructed with any of the first $l$ pseudovacua $\omega(1), \ldots, \omega(l)$ having one hole in sea 1 (i.e., $\nu^{(j)} = \delta_{j,1}$), it follows from the identity (3.8) that (up to a rapidity-independent phase factor)

$$\alpha^+_{(l)} \ \alpha^-_{(l)} \sim \exp \left\{ i2\pi N \int_0^{\tilde{\lambda}^{(1)}} \left( \sigma^{(1)}_{(l)}(\lambda) - 2s^{(1)}(\lambda) \right) d\lambda \right\},$$

(3.28)

where $\sigma^{(1)}_{(l)}(\lambda)$ is given by Eq. (3.20) with $j = 1$. We now observe that

$$\int_0^{\tilde{\lambda}^{(1)}} \left[ \mathcal{R} \left( \lambda - \tilde{\lambda}^{(1)} \right) + \mathcal{R} \left( \lambda + \tilde{\lambda}^{(1)} \right) \right] d\lambda = \int_0^{\tilde{\lambda}^{(1)}} 2\mathcal{R} \left( 2\lambda \right) d\lambda.$$

(3.29)

Moreover, we note the identity

$$\sum_{k=1}^{\infty} \frac{(1 - e^{-2\eta k\beta})(1 - e^{-2\eta k\gamma}) e^{-2\eta k\mu}}{1 - e^{-2\eta k}} \frac{1}{k} = \log \frac{\Gamma_{q^2}(\mu)\Gamma_{q^2}(\mu + \beta + \gamma)}{\Gamma_{q^2}(\mu + \beta)\Gamma_{q^2}(\mu + \gamma)} , \quad q = e^{-\eta},$$

(3.30)
as well as the $q$-analogue of the duplication formula \[ 34 \]

\[
(1 + q)^{2x-1} \Gamma_q(x) \Gamma_q(x + \frac{1}{2}) = \Gamma_q(2x) \Gamma_q(\frac{1}{2}) ,
\]

where $\Gamma_q(x)$ is the $q$-analogue of the Euler gamma function.

We find \[ 35 \]

\[
\alpha_{(l)}^- = q^{-i\tilde{\lambda}^{(1)}(1+l)}/S_0(\tilde{\lambda}^{(1)})
\]

\[
\times \frac{\Gamma_q^{2N} \left( \frac{1}{N} \left( \xi_- + l - \frac{1}{2} + i\tilde{\lambda}^{(1)} \right) \right) \Gamma_q^{2N} \left( \frac{1}{N} \left( \xi_- + N - \frac{1}{2} - i\tilde{\lambda}^{(1)} \right) \right)}{\Gamma_q^{2N} \left( \frac{1}{N} \left( \xi_- + l - \frac{1}{2} - i\tilde{\lambda}^{(1)} \right) \right) \Gamma_q^{2N} \left( \frac{1}{N} \left( \xi_- + N - \frac{1}{2} + i\tilde{\lambda}^{(1)} \right) \right)} ,
\]

\[
\alpha_{(l)}^+ = q^{-i\tilde{\lambda}^{(1)}(1-l+2N)}/S_0(\tilde{\lambda}^{(1)})
\]

\[
\times \frac{\Gamma_q^{2N} \left( \frac{1}{N} \left( \xi_+ - N + l - \frac{1}{2} - i\tilde{\lambda}^{(1)} \right) \right) \Gamma_q^{2N} \left( \frac{1}{N} \left( \xi_+ - \frac{1}{2} + i\tilde{\lambda}^{(1)} \right) \right)}{\Gamma_q^{2N} \left( \frac{1}{N} \left( \xi_+ - N + l - \frac{1}{2} + i\tilde{\lambda}^{(1)} \right) \right) \Gamma_q^{2N} \left( \frac{1}{N} \left( \xi_+ - \frac{1}{2} - i\tilde{\lambda}^{(1)} \right) \right)} ,
\]

where the prefactor $S_0(\tilde{\lambda})$ is given by

\[
S_0(\tilde{\lambda}) = \frac{\Gamma_q^{4N} \left( \frac{1}{N} \left( \frac{1}{2}(N - 1) - i\tilde{\lambda} \right) \right) \Gamma_q^{4N} \left( \frac{1}{N} \left( N + i\tilde{\lambda} \right) \right)}{\Gamma_q^{4N} \left( \frac{1}{N} \left( \frac{1}{2}(N - 1) + i\tilde{\lambda} \right) \right) \Gamma_q^{4N} \left( \frac{1}{N} \left( N - i\tilde{\lambda} \right) \right)} .
\]

Furthermore, choosing the state $|\tilde{\lambda}^{(1)}\rangle$ in Eq. (3.7) to be the BA state constructed with any of the last $N - l$ pseudovacua $\omega_{(l+1)}, \ldots, \omega_{(N)}$ having one hole in sea 1, we obtain the relation

\[
\frac{\beta_{(l)}^+}{\alpha_{(l)}^+} = \exp \left\{ i2\pi N \int_0^{\tilde{\lambda}^{(1)}} \left( \sigma_{(l)}^{(1)}(\lambda) - \sigma_{(l)}^{(1)}(\lambda) \right) d\lambda \right\} .
\]

We note that

\[
\sigma_{(l)}^{(1)}(\lambda) - \sigma_{(l)}^{(1)}(\lambda) = \delta \sigma_{(l)}^{(1)}(\lambda) - \delta \sigma_{(l)}^{(1)}(\lambda)
\]

\[
= \frac{1}{N} \left( a_{2\xi_- - 1}(\lambda) - a_{2\xi_+ - 2N + 2l - 1}(\lambda) \right) ,
\]

where $\delta \sigma_{(l)}^{(1)}$ and $\delta \sigma_{(l)}^{(1)}$ are given by Eqs. (3.22) and (3.20), respectively. We conclude

\[
\frac{\alpha_{(l)}^-}{\beta_{(l)}^-} = -e_{2\xi_-}(\tilde{\lambda}^{(1)}) ,
\]

\[
\frac{\beta_{(l)}^+}{\alpha_{(l)}^+} = -e_{2\xi_+ - 2N + 2l - 1}(\tilde{\lambda}^{(1)}) ,
\]

\[8\]Evidently, Eq. (3.28) determines the product $\alpha_{(l)}^-\alpha_{(l)}^+$. Further conditions are necessary to determine $\alpha_{(l)}^-$ and $\alpha_{(l)}^+$ separately. Following [31], we assume that the parts of $\alpha_{(l)}^-$ and $\alpha_{(l)}^+$ which do not originate from $\delta \sigma_{(l)}^{(1)}$ are equal.
where we have resolved the sign ambiguity by demanding that the $S$ matrix be proportional to the unit matrix for $\tilde{\lambda}(1) = 0$.

We remark that there are relations among the boundary $S$ matrices. Indeed, the above results for $S_{(l)}^{+}$ [1] are valid in the regime $\xi_{-} > \frac{1}{2}(N - 1)$, $\xi_{+} > N - \frac{1}{2}$, to which we now refer as regime I. Corresponding results for regime II defined as $\xi_{-} < -\frac{1}{2}(N - 1)$, $\xi_{+} < -N + \frac{1}{2}$ can be obtained by a similar analysis, starting with the BAE (3.10) and replacing $\xi_{-} \to -\xi_{-}$, $\xi_{+} \to -\xi_{+} + N$. We observe that the $S$ matrices in these two regimes are related by duality:

\[
\begin{align*}
\alpha^{(l)}_{(I)} (\xi_{\pm} \to \xi^{\prime}_{\pm}, l \to l') &= \beta^{(l)}_{(II)} (\xi_{\pm} \\
\beta^{(l)}_{(I)} (\xi_{\pm} \to \xi^{\prime}_{\pm}, l \to l') &= \alpha^{(l)}_{(II)},
\end{align*}
\]

where $\xi_{\pm}'$ and $l'$ are given by Eq. (2.27). Moreover, there is a relation between $S_{(l)}^{-}$ [1] and $S_{(l)}^{+}$ [1]:

\[
S_{(l)}^{-} (l) \mid_{\xi_{-} \to -\xi_{+} + N - l} = S_{(l)}^{+} (l) .
\]

Finally, we consider the case $[N - 1]$. The boundary $S$ matrices $S_{(l)}^{\pm}[N - 1]$ are diagonal $N \times N$ matrices of the form

\[
S_{(l)}^{\pm}[N - 1] = \text{diag} \left( \frac{\alpha^{\pm}}{l}, \ldots, \frac{\alpha^{\pm}}{l}, \frac{\beta^{\pm}}{N - l}, \ldots, \frac{\beta^{\pm}}{N - l} \right).
\]

For this case we must consider one hole in sea $N - 1$. Noting that

\[
\beta_{(l)}^{+} \beta_{(l)}^{-} \sim \exp \left\{ i2\pi N \int_{0}^{\tilde{\lambda}(N - 1)} \left( \sigma_{(l)}^{(N - 1)}(\lambda) - 2s^{(N - 1)}(\lambda) \right) d\lambda \right\},
\]

we obtain

\[
\begin{align*}
\beta_{(l)}^{-} &= q^{-i2\tilde{\lambda}(N - 1)(1 - l + N)/N} S_{0}(\tilde{\lambda}(N - 1)) \\
&\times \frac{\Gamma_{qN} \left( \frac{1}{N} \left( \xi_{-} + l + \frac{1}{2}(N - 1) - i\tilde{\lambda}(N - 1) \right) \right)}{\Gamma_{qN} \left( \frac{1}{N} \left( \xi_{-} + l + \frac{1}{2}(N - 1) + i\tilde{\lambda}(N - 1) \right) \right)} \\
\beta_{(l)}^{+} &= q^{-i2\tilde{\lambda}(N - 1)(1 + l + N)/N} S_{0}(\tilde{\lambda}(N - 1)) \\
&\times \frac{\Gamma_{qN} \left( \frac{1}{N} \left( \xi_{+} + l - \frac{1}{2}(N + 1) + i\tilde{\lambda}(N - 1) \right) \right)}{\Gamma_{qN} \left( \frac{1}{N} \left( \xi_{+} + l - \frac{1}{2}(N + 1) - i\tilde{\lambda}(N - 1) \right) \right)}.
\end{align*}
\]
where $S_0(\tilde{\lambda})$ is given in Eq. (3.33). Moreover,

$$\frac{\beta_-(l)}{\alpha_-(l)} = -e_{2\xi_- + 2l - N - 1(\tilde{\lambda}(N - 1))},$$

$$\frac{\alpha_+(l)}{\beta_+(l)} = -e_{2\xi_+ - N - 1(\tilde{\lambda}(N - 1))}.$$

(3.42)

For the case $N = 2$, we recover the results of Refs. [19] and [20]. In the isotropic limit $\eta \to 0$, we see that $q \to 1$ and $\Gamma_q(x) \to \Gamma(x)$; hence, we recover the recent results [21].

4 Discussion

The transfer matrix $t_0(\lambda, \xi_-, \xi_+)$ has the symmetry $U_q(SU(l)) \times U_q(SU(N - l)) \times U(1)$ as well as a duality symmetry $l \leftrightarrow N - l$, as we have shown in Eqs. (2.33), (2.34) and (2.26), respectively. It should be possible to show, generalizing [13], [16], [36], that the corresponding Bethe Ansatz states $| \rangle$ are highest weights of $U_q(SU(l)) \times U_q(SU(N - l))$. That is,

$$J^{+(k)}| \rangle = 0, \quad k \neq l.$$

(4.1)

This symmetry was used here for computing boundary $S$ matrices. We expect that this symmetry will also be valuable for computing other quantities of physical interest (e.g., form factors, correlation functions, etc.) and for elucidating the phase structure of the corresponding vertex model. Although we have considered here the noncritical regime, we expect that this approach can be extended to the critical regime $|q| = 1$.

The “mixed” boundary condition case $l_+ \neq l_-$ merits further investigation. One expects that the boundary $S$ matrix for one end of the chain should be independent of the boundary conditions at the other end. However, for mixed boundary conditions, the unbroken symmetry group of the transfer matrix is smaller, and therefore, the arguments presented here require further refinement.

We have seen that by turning on diagonal boundary interactions, the symmetry algebra of the transfer matrix is broken to a nontrivial subalgebra. We expect that this is a general phenomenon. More explicitly, consider the $R$ matrix associated with the affine Lie algebra $\hat{g}$ [3, 4]. It is known [13] that the corresponding open spin chain transfer matrix with $K = 1$ (i.e., without additional boundary interactions) has the symmetry $U_q(g_0)$, where $g_0$ is the maximal finite-dimensional subalgebra of $\hat{g}$. We expect that the transfer matrix with diagonal $K \neq 1$ will have as its symmetry a nontrivial subalgebra of $U_q(g_0)$.
We emphasize that we have considered here only diagonal boundary interactions. It would be interesting to look for symmetries of the transfer matrix with nondiagonal $K$ matrices. Unfortunately, even for the simplest case $A_1^{(1)}$, the Bethe Ansatz solution is not yet known, which precludes computation of the boundary $S$ matrix in the manner described here.

We also emphasize that here we have restricted our attention to exact symmetries of the finite chain. For the (semi) infinite chain, one expects additional affine symmetries [19],[37]. In field theory, such symmetries are generated by fractional-spin integrals of motion [38],[39],[7], which can exist even for nondiagonal boundary interactions [40].

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A Proof of Eqs. (2.59) and (2.60)

We prove here the relations (2.59), (2.60) which enter our proof of quantum-algebra symmetry of the transfer matrix.

We begin by noting the identities

$$
\Pi^{\pm}_{(l)} R^{\pm}_{12} \Pi^{\pm}_{(l)} = R^{\pm}_{12} \Pi^{\pm}_{(l)},
$$

which imply

$$
\Pi^{\pm}_{(l)} R^{\pm}_{21} \Pi^{\pm}_{(l)} = \Pi^{\pm}_{(l)} R^{\pm}_{21},
$$

and therefore

$$
\Pi^{\pm}_{(l)} T^{\pm}_{0} \Pi^{\pm}_{(l)} = T^{\pm}_{0} \Pi^{\pm}_{(l)},
$$

$$
\Pi^{\pm}_{(l)} \hat{T}^{\pm}_{0} \Pi^{\pm}_{(l)} = \Pi^{\pm}_{(l)} \hat{T}^{\pm}_{0}.
$$

It is now easy to show that

$$
\Pi^{\pm}_{(l)} T^{\pm}_{(l)} = T^{\pm}_{(l)}.
$$

Indeed, recalling Eq. (2.46),

$$
\Pi^{\pm}_{(l)} T^{\pm}_{(l)} = \Pi^{\pm}_{(l)} T^{\pm}_{0} \Pi^{\pm}_{(l)} \hat{T}^{\pm}_{0} \Pi^{\pm}_{(l)} = T^{\pm}_{0} \Pi^{\pm}_{(l)} \hat{T}^{\pm}_{0} = T^{\pm}_{(l)}.
$$
In a similar manner one can prove that

\[ \Pi_{(l)}^{\pm 1} \left( R_{12}^{\pm} \ T_{(l)}^{\pm 1} \ R_{21}^{\pm} \ T_{(l)}^{\pm 2} (\lambda, \xi) \right) \Pi_{(l)}^{\pm 1} = R_{12}^{\pm} \ T_{(l)}^{\pm 1} \ R_{21}^{\pm} \ T_{(l)}^{\pm 2} (\lambda, \xi) . \]  

(A.6)

This relation, together with the fact

\[ \Pi_{(l)}^{\pm} \ M \ \Pi_{(l)}^{\pm} = \Pi_{(l)}^{\pm} \ M = M \ \Pi_{(l)}^{\pm} , \]  

(A.7)

imply the first desired result (2.59).

We next note the identities

\[ \Pi_{(l)}^{\pm 1} \ R_{12}^{\pm} \ \Pi_{(l)}^{\pm 1} = \Pi_{(l)}^{\pm} \ R_{12}^{\pm} , \]  

(A.8)

which imply

\[ \Pi_{(l)}^{\pm 1} \ R_{21}^{\pm} \ \Pi_{(l)}^{\pm 1} = R_{21}^{\pm} \ \Pi_{(l)}^{\pm 1} . \]  

(A.9)

Let us denote by \( R^{\pm} \) the “projected” \( R \) matrix, i.e.,

\[ R_{12}^{\pm} = \Pi_{(l)}^{\pm 1} \ R_{12}^{\pm} \ \Pi_{(l)}^{\pm 1} . \]  

(A.10)

It is easy to see that this projected \( R \) matrix commutes with \( K_{(l)}^{2} (\lambda, \xi) \),

\[ [R_{12}^{\pm}, K_{(l)}^{2} (\lambda, \xi)] = 0 . \]  

(A.11)

Indeed, we see from Eq. (2.14) that

\[ K_{(l)} (\lambda, \xi) = a\Pi_{(l)}^{+} + b\Pi_{(l)}^{-} . \]  

(A.12)

Since \([R_{12}^{+}, \Pi_{(l)}^{+}]_2 = 0\), then \([R_{12}^{+}, \Pi_{(l)}^{-}]_2 = 0\), and the result (A.11) for \( R_{12}^{+} \) immediately follows. The proof for \( R_{12}^{-} \) is obtained in similar fashion.

All the pieces are now in place for proving the relation

\[ \Pi_{(l)}^{\pm 1} \left( R_{12}^{\pm} P_{1}^{\pm t_1} M_{1}^{-1} R_{21}^{\pm} K_{(l)}^{2} (\lambda, \xi) \right) \Pi_{(l)}^{\pm 1} = \Pi_{(l)}^{\pm 1} \left( K_{(l)}^{2} (\lambda, \xi) R_{12}^{\pm} P_{1}^{\pm t_1} M_{1}^{-1} R_{21}^{\pm} \right) \Pi_{(l)}^{\pm 1} , \]  

(A.13)

which is the second desired result (2.60). Indeed,

\[ \text{LHS} = R_{12}^{\pm} P_{1}^{\pm t_1} M_{1}^{-1} R_{21}^{\pm} K_{(l)}^{2} (\lambda, \xi) = K_{(l)}^{2} (\lambda, \xi) R_{12}^{\pm} P_{1}^{\pm t_1} M_{1}^{-1} R_{21}^{\pm} = \text{RHS} . \]  

(A.14)
B  Bethe Ansatz Equations for general pseudovacua

A priori, one expects that the Bethe Ansatz equations (BAE) depend on the choice of pseudovacuum. We show here that the BAE for the transfer matrix \( t(l)(\lambda, \xi_-, \xi_+) \) corresponding to the pseudovacua \( \omega(1), \ldots, \omega(l) \) are all the same; and that the BAE corresponding to the pseudovacua \( \omega(l+1), \ldots, \omega(N) \) are all the same.

We first observe that the operators \( (J^{\pm(k)})^N \), where \( N \) is the number of sites, transform one pseudovacuum into another:
\[
\begin{align*}
(J^{-}(k))^{N} \omega_{(j)} &= \omega_{(j+1)} \delta_{k,j}, \\
(J^{+}(k))^{N} \omega_{(j)} &= \omega_{(j-1)} \delta_{k,j-1}.
\end{align*}
\] (B.1)

The \( U_q(SU(l)) \times U_q(SU(N - l)) \) symmetry of the transfer matrix (2.34) implies the (weaker) result
\[
\left[ t(l)(\lambda, \xi_-, \xi_+), (J^{\pm(k)})^{N} \right] = 0, \quad k \neq l.
\] (B.2)

Let \( \Lambda^{(0)}_{(l)}(\lambda, \xi_-, \xi_+) \) denote the eigenvalue of the transfer matrix corresponding to the first pseudovacuum \( \omega_{(1)} \),
\[
t(l)(\lambda, \xi_-, \xi_+) \omega_{(1)} = \Lambda^{(0)}_{(l)}(\lambda, \xi_-, \xi_+) \omega_{(1)}.
\] (B.3)

Acting repeatedly on both sides of this equation with the lowering operator \( (J^{-}(k))^{N} \), we conclude that the transfer matrix has the same eigenvalue with each of the first \( l \) pseudovacua,
\[
t(l)(\lambda, \xi_-, \xi_+) \omega_{(k)} = \Lambda^{(0)}_{(l)}(\lambda, \xi_-, \xi_+) \omega_{(k)}, \quad k = 1, \ldots, l.
\] (B.4)

Similarly, let \( \Lambda^{(0)}_{(l)}(\lambda, \xi_-, \xi_+) \) be the eigenvalue of the transfer matrix corresponding to the last pseudovacuum \( \omega_{(N)} \),
\[
t(l)(\lambda, \xi_-, \xi_+) \omega_{(N)} = \Lambda^{(0)}_{(l)}(\lambda, \xi_-, \xi_+) \omega_{(N)}.
\] (B.5)

Acting on both sides with the raising operator \( (J^{+}(k))^{N} \), we see that the transfer matrix has the same eigenvalue with each of the last \( N - l \) pseudovacua,
\[
t(l)(\lambda, \xi_-, \xi_+) \omega_{(k)} = \Lambda^{(0)}_{(l)}(\lambda, \xi_-, \xi_+) \omega_{(k)}, \quad k = l + 1, \ldots, N.
\] (B.6)

We have shown so far that the vacuum eigenvalues of \( t(l)(\lambda, \xi_-, \xi_+) \) with \( \omega(1), \ldots, \omega(l) \) are the same, and those with \( \omega(l+1), \ldots, \omega(N) \) are the same. The eigenvalues of \( t(l)(\lambda, \xi_-, \xi_+) \) corresponding to general BA states are “dressed” pseudovacuum eigenvalues, with dressing
functions that can be deduced from properties of the transfer matrix such as analyticity, fusion, etc., independently of the choice of pseudovacuum. (See, e.g., [41], [42].) It follows that the expressions for the eigenvalues of the transfer matrix corresponding to general BA states constructed with pseudovacua $\omega(1), \ldots, \omega(l)$ are all the same, and analogously for the pseudovacua $\omega(l+1), \ldots, \omega(N)$. Since the BAE are the conditions that the eigenvalues have vanishing residues, we conclude that the BAE corresponding to the pseudovacua $\omega(1), \ldots, \omega(l)$ are all the same; and that the BAE corresponding to the pseudovacua $\omega(l+1), \ldots, \omega(N)$ are all the same.

We remark that the above proof could be streamlined (in particular, avoiding the argument of dressing the pseudovacuum eigenvalues) if one could construct invertible transformations $U(j)$ which keep the transfer matrix invariant and change the pseudovacuum,

$$U(j) \ t(l)(\lambda, \xi_- , \xi_+) \ U(j)^{-1} = t(l)(\lambda, \xi_- , \xi_+),$$

$$U(j) \ \omega(j) = \omega(j+1),$$

(B.7)

for $j \neq l$. For the isotropic case, such transformations are finite elements of the groups $SU(l)$ and $SU(N-l)$, which can easily be constructed [21]. Unfortunately, for the anisotropic case, we have not yet succeeded in constructing such transformations for general values of the number of sites, $N$. Such transformations are presumably related to the so-called $q$-Weyl group [43], and may have other interesting applications.

C Symmetry transformations which change the anisotropy parameter

The symmetries of the transfer matrix discussed in Section 2 leave the anisotropy parameter $\eta$ unchanged. We briefly discuss here related transformations, which however transform $\eta \to -\eta$. Let us now make the dependence on $\eta$ explicit, and denote the $R$ matrix (2.11) by $R_{12}(\lambda, \eta)$, and the $K$ matrix (2.14) by $K(l)(\lambda, \xi, \eta)$. Evidently,

$$R_{21}(\lambda, \eta) = R_{12}(\lambda, -\eta).$$

(C.1)

Moreover, let $U$ now be the antidiagonal $N \times N$ matrix

$$U = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ 1 & & & 1 \end{pmatrix}.$$  

(C.2)
One can show that

$$U_1 U_2 R_{12}(\lambda, \eta) U_1^{-1} U_2^{-1} = R_{12}(\lambda, -\eta),$$

$$U K_{(l)}(\lambda, \xi, \eta) U^{-1} = K_{(l')}(\lambda, \xi', -\eta),$$

(C.3)

where $l'$ and $\xi'_\pm$ are given by Eq. (2.27). The transfer matrix transforms under the corresponding quantum-space transformation $U$ as follows:

$$U t_{(l)}(\lambda, \xi_-, \xi_+, \eta) U^{-1} \propto t_{(l')}(\lambda, \xi'_-, \xi'_+, -\eta).$$

(C.4)

Like the duality transformation (2.26), this transformation maps $l \to l'$ and $\xi_\pm \to \xi'_\pm$, but it also maps $\eta \to -\eta$.

Furthermore, define the two $N \times N$ matrices

$$U^+_{(l)} = \begin{pmatrix}
1 & & & \\
& 1 & & \\
& & \ddots & \\
& & & 1
\end{pmatrix}, \quad U^-_{(l)} = \begin{pmatrix}
1 & & & \\
& 1 & & \\
& & \ddots & \\
& & & 1
\end{pmatrix}.$$

(C.5)

with an antidiagonal upper $l \times l$ block, and an antidiagonal lower $(N-l) \times (N-l)$ block, respectively. The corresponding quantum-space transformations $U^\pm_{(l)}$ have the following action on the transfer matrix

$$U^\pm_{(l)} t_{(l)}(\lambda, \xi_-, \xi_+, \eta) U^\pm_{(l)}^{-1} \propto t_{(l')}(\lambda, \xi'_-, \xi'_+, -\eta).$$

(C.6)

However, for $N > 3$, there are not enough of these transformations to determine all the elements of the boundary $S$ matrices.

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