Geodesic regression

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May 3, 2020

Abstract

The theory of geodesic regression aims to find a geodesic curve which is an optimal fit to a given set of data. In this article we restrict ourselves to the Riemannian manifold of positive definite operators (matrices) on a Hilbert space of finite dimension. There is a unique geodesic curve connecting two positive definite operators, and it is given by the weighted geometric mean. The function that measures the squared Riemannian metric distance between an operator and a geodesic curve is not convex nor geodesically convex in the operators generating the curve. This is a marked difference to the situation in linear regression. The literature mainly tries to find numerical solutions that approximate the optimal curve in a single point.

We suggest to apply a distance measure slightly coarser than the Riemannian metric. The ensuing control function faithfully identifies geodesic curves, and it coincides with the standard control function based on the Riemannian metric for commuting operators. The control function constructed in this way is geodesically convex. We are therefore able to find a global and uniquely defined optimal fit to any given set of data. The generators of the geodesic curve may also be determined as the unique solution to two operator equations.

MSC2010: 53C22; 47A64; 47N10

Key words and phrases: Geodesic regression; geodesically convex function, derivative pricing, image registration, computer vision, machine learning.
1 Introduction

Linear regression is a classical method to solve the following problem: To a set of data $D = \{(x_i, y_i) \mid i = 1, \ldots, n\}$ consisting of pairs of real numbers, we seek an affine function $f(t) = at + b$ that gives the best fit to the data in the sense that the control function

$$F(a, b) = \sum_{i=1}^{n} (ax_i + b - y_i)^2,$$

which measures the sum of squared differences is minimized. Since the control function is convex, it is easy to derive that this problem has a unique solution, and to express it by a simple formula. Note that the affine functions are geodesic curves in $\mathbb{R}^2$.

We investigate problems, where the data set $D = \{(p_i, X_i) \mid i = 1, \ldots, n\}$ consists of pairs of numbers and elements in a Riemannian manifold. The aim is to find a geodesic curve in the manifold that gives the best fit to the given data. The problem stated in this generality is very difficult, and there is in the literature only partial results that give a first or second order approximation to the best fit in a single point. The problem of obtaining a suitable geodesic regression is important in derivative pricing, biomedical image registration, computer vision and machine learning. There is a large literature, cf. [6, 12, 5, 2, 14, 4] and the references therein, aiming at developing algorithms for the determination of geodesic curves, with applications in these fields of research.

We limit ourselves to investigate the problem where the Riemannian manifold $\mathcal{M}$ is the set of positive definite operators (matrices) acting on a Hilbert space of finite dimension. The geodesic curves in $\mathcal{M}$ are then given on the form

$$\gamma(p) = A\#_p B \quad p \in \mathbb{R},$$

generated by a pair $(A, B)$ of positive definite matrices, where $\#_p$ denotes the power mean. We may without loss of generality assume $0 \leq p_i \leq 1$ for $i = 1, \ldots, n$ in which case the power mean reduces to the weighted geometric mean. Even in this case the problem is rather difficult. One may write down a control function that sums squared differences as measured by the Riemannian metric, but this control function is neither convex nor geodesically convex in the generators of the geodesic curve, and this is exactly the origin of all the difficulties.
As a novel tool we introduce a distance measure that is slightly coarser than the Riemannian metric. For commuting operators, it coincides with the Riemannian metric. The associated control function is geodesically convex, and it faithfully identifies a geodesic curve by having global minimum equal to zero, if and only if the given data already are connected by a geodesic. We are then able to prove the existence of a global and uniquely defined best fit to any given set of data with respect to the control function. It may be numerically obtained by very fast algorithms searching for the global minimum of a geodesically convex function. It may also be obtained as the unique solution to two operator equations derived from the partial Riemannian gradients.

2 Geodesically convex functions

The powermean $A^p B$ of two elements $A, B \in \mathcal{M}$ is given by
\[
A^p B = B^{1/2} (B^{-1/2} AB^{-1/2})^p B^{1/2} \quad p \in \mathbb{R}
\]
and determines a unique geodetic curve going through $A$ and $B$. We note that $A^p B = A$ for $p = 1$, and $A^p B = B$ for $p = 0$. We refer to [10] for a general introduction to differential manifolds, and to [1, 13] for the geometry of $\mathcal{M}$.

**Definition 2.1.** A function $F : \mathcal{M}^k \rightarrow \mathbb{R}$ of $k$ variables is said to be geodesically convex if
\[
F(A_1^p B_1, \ldots, A_k^p B_k) \leq pF(A_1, \ldots, A_k) + (1 - p)F(B_1, \ldots, B_k)
\]
for $p \in [0, 1]$ and elements $A_1, \ldots, A_k; B_1, \ldots, B_k \in \mathcal{M}$. It is said to be strictly geodesically convex if there is equality only for $p = 0$ and $p = 1$.

The theory of geodesically convex functions is quite similar to the theory of convex functions. In particular, if a strictly geodesically convex function has a stationary point (the partial Riemannian gradients are zero), then it is a global minimum point. We proved in [9, Theorem 3.6] the following result.

**Theorem 2.2.** A trace function of the form,
\[
G(X) = \text{Tr} \, g(X) \quad g : (0, \infty) \rightarrow \mathbb{R},
\]
is geodesically convex in $\mathcal{M}$, if and only if
\[
g(t) = f(\log t) \quad t > 0
\]
for a convex function $f : \mathbb{R} \rightarrow \mathbb{R}$. 

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We call functions $g$ written on the form (1) for convex-log functions. Let $\mathcal{H}$ and $\mathcal{K}$ denote Hilbert spaces of finite dimension.

**Lemma 2.3.** Let $W: \mathcal{H} \to \mathcal{K}$ be an isometry, and let $Q = WW^*$ be the range projection in $B(\mathcal{K})$. The restriction of the trace function

$$G(X) = \text{Tr}_{\mathcal{H}} f(W^* \log(X)W)$$

to positive definite operators in the commutant of $Q$ is geodesically convex.

Note that the commutant of $Q$ in $B(\mathcal{K})$ also is a Riemannian manifold.

**Proof.** We may identify $\mathcal{H}$ with the subspace $\mathcal{L} = Q\mathcal{K}$ of $\mathcal{K}$, and $W$ with a unitary on $\mathcal{L}$. If $X \in B(\mathcal{K})$ commutes with $Q$, then

$$\text{Tr}_{\mathcal{H}} f(W^* X W) = \text{Tr}_{\mathcal{L}} f(X),$$

where $X$ on the right-hand side is identified with its restriction to the invariant subspace $\mathcal{L}$. If $X$ and $Y$ commute with $Q$, so do $\log X$, $\log Y$, and $\log(X\#_p Y)$. The statement now follows from Theorem 2.2.

**Theorem 2.4.** Let $f: \mathbb{R} \to \mathbb{R}$ be a convex function, and let $\omega_1, \omega_2$ be non-negative weights with sum one. The trace function

$$G(X_1, X_2) = \text{Tr} f(\omega_1 \log X_1 + \omega_2 \log X_2)$$

is geodesically convex in two positive definite matrices.

**Proof.** To block matrices

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

with entries in bounded operators $B(\mathcal{H})$ on $\mathcal{H}$ we set

$$\rho(X) = X \otimes I_2 = \begin{pmatrix} x_{11} & x_{12} & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 \\ 0 & 0 & x_{11} & x_{12} \\ 0 & 0 & x_{21} & x_{22} \end{pmatrix}$$

and note that $\rho$ is a $*$-homomorphism. To non-negative weights $\omega_1$ and $\omega_2$ with sum one we set

$$W = I_{\mathcal{H}} \otimes \begin{pmatrix} \omega_1^{1/2} \\ 0 \\ 0 \\ \omega_2^{1/2} \end{pmatrix},$$

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so $W$ is a block matrix with entries in $B(\mathcal{H})$. We set
\[ \Phi(X) = W^* \rho(X) W = \omega_1 x_{11} + \omega_2 x_{22} \]
and note that $\Phi$ is a unital completely positive linear map. In particular, $\Phi(I) = W^* W = 1_\mathcal{H}$. Therefore, $W$ is an isometry and the range projection is the block matrix
\[
Q = WW^* = \begin{pmatrix}
\omega_1 & 0 & 0 & \omega_1^{1/2} \omega_2^{1/2} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\omega_2^{1/2} \omega_1^{1/2} & 0 & 0 & \omega_2 \\
\end{pmatrix}
\]
with entries in $B(\mathcal{H})$. Furthermore,
\[
\rho(X)Q = \begin{pmatrix}
\omega_1 x_{11} & 0 & 0 & \omega_1^{1/2} \omega_2^{1/2} x_{11} \\
\omega_1 x_{21} & 0 & 0 & \omega_1^{1/2} \omega_2^{1/2} x_{21} \\
\omega_2^{1/2} \omega_1^{1/2} x_{12} & 0 & 0 & \omega_2 x_{12} \\
\omega_2^{1/2} \omega_1^{1/2} x_{22} & 0 & 0 & \omega_2 x_{22} \\
\end{pmatrix}
\]
and
\[
Q \rho(X)Q = (\omega_1 x_{11} + \omega_2 x_{22}) \otimes Q = \Phi(X) \otimes Q.
\]
If $X$ is a block diagonal matrix then
\[
\rho(X)Q = \begin{pmatrix}
\omega_{11} x_{111} & 0 & 0 & \omega_1^{1/2} \omega_2^{1/2} x_{111} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\omega_2^{1/2} \omega_1^{1/2} x_{22} & 0 & 0 & \omega_2 x_{22} \\
\end{pmatrix}
= Q \rho(X),
\]
so $\rho(X)$ leaves the subspace $QK$ invariant. We now consider diagonal block matrices
\[
X = \begin{pmatrix}
X_1 & 0 \\
0 & X_2
\end{pmatrix}
\quad \text{and} \quad
Y = \begin{pmatrix}
Y_1 & 0 \\
0 & Y_2
\end{pmatrix}
\]
and obtain
\[
f(\omega_1 \log(X_1 \#_p Y_1) + \omega_2 \log(X_2 \#_p Y_2)) = \\
= f(\Phi(\log(X \#_p Y))) = f(W^* \rho(\log(X \# Y)) W) \\
= f(W^* \log(\rho(X) \# \rho(Y)) W),
\]
where we used that $\rho$ is a $*$-homomorphism. We may now apply the Lemma 2.3 and obtain

$$\text{Tr}_{\mathcal{H}} f (\omega_1 \log(X_1 \#_p Y_1) + \omega_2 \log(X_2 \#_p Y_2))$$

$$= \text{Tr}_{\mathcal{H}} f (W^* \log(\rho(X) \# \rho(Y))W)$$

$$\leq p \text{Tr}_{\mathcal{H}} f (W^* \rho(\log(X))W) + (1 - p) \text{Tr}_{\mathcal{H}} f (W^* \rho(\log(Y))W)$$

$$= p \text{Tr}_{\mathcal{H}} f (W^* \rho(\log(X))W) + (1 - p) \text{Tr}_{\mathcal{H}} f (W^* \rho(\log(Y))W)$$

$$= p \text{Tr}_{\mathcal{H}} f (\Phi(X)) + (1 - p) \text{Tr}_{\mathcal{H}} f (\Phi(Y))$$

$$= p \text{Tr}_{\mathcal{H}} f (\omega_1 \log X_1 + \omega_2 \log X_2) + (1 - p) \text{Tr}_{\mathcal{H}} f (\omega_1 \log Y_1 + \omega_2 \log Y_2)$$

as desired. \qed

One might also consider the trace function of three variables,

$$G(X_1, X_2, X_3) = \text{Tr} f (\omega_1 \log X_1 + \omega_2 \log X_2 + \omega_3 \log X_3),$$

for a convex function $f$ and non-negative weights $\omega_1, \omega_2, \omega_3$ with sum one. This trace function, however, is not generally geodesically convex. Since the square function is convex and positively homogeneous of degree two we obtain:

**Corollary 2.5.** Let $\omega_1, \omega_k$ be non-negative numbers. The trace function

$$G(X_1, X_2) = \text{Tr} (\omega_1 \log X_1 + \omega_2 \log X_2)^2$$

is geodesically convex in two positive definite matrices.

### 3 The distance function

The space $B(\mathcal{H})_+$ of positive definite operators (matrices) on a Hilbert space $\mathcal{H}$ of finite dimension is a Riemannian manifold. The Riemannian metric $\delta_2$ is given by

$$\delta_2(A, B) = \| \log(A^{-1/2}BA^{-1/2}) \|_2 = (\text{Tr} \log^2(A^{-1/2}BA^{-1/2}))^{1/2}.$$ 

In addition to the metric properties it satisfies

$$\delta_2(A^{-1}, B^{-1}) = \delta_2(A, B).$$
The squared Riemannian distance between a positive definite operator \(X\) and the weighted geometric mean \(A\#_p B\) is given by
\[
\delta_2^2(X, A\#_p B) = \text{Tr} \log^2 \left( (X^{-1/2} A X^{-1/2})\#_p (X^{-1/2} B X^{-1/2}) \right)
\]
in terms of the function
\[
F_p(A, B) = \text{Tr} \log^2 (A\#_p B)
\]
defined for \(p \in [0, 1]\) and positive definite \(A\) and \(B\). Note that
\[
F_p(A, B) = \delta_2^2(I, A\#_p B).
\]
Numeric calculations reveal that \(F_p\) is neither convex nor geodesically convex.

**Theorem 3.1.** Let \(p \in [0, 1]\). The inequality
\[
F_p(A, B) \leq G_p(A, B) = \text{Tr} \left( p \log A + (1 - p) \log B \right)^2
\]
holds for positive definite \(A\) and \(B\). In addition,
\[
F_p(A, B) = G_p(A, B)
\]
for commuting operators, and \(G_p\) is geodesically convex.

**Proof.** We recall [9, Theorem 3.4] that
\[
\log(A\#_p B) \prec p \log A + (1 - p) \log B
\]
in terms of matrix majorization. There exists therefore, see for example [11, Theorem 3.6], a unital quantum channel \(T\) such that
\[
\log(A\#_p B) = T \left( p \log A + (1 - p) \log B \right).
\]
Since the square is operator convex and \(T\) is completely positive and unital, we thus have the inequality
\[
\log^2(A\#_p B) = \left( T \left( p \log A + (1 - p) \log B \right) \right)^2
\]
\[
\leq T \left( (p \log A + (1 - p) \log B)^2 \right),
\]
cf. [3, Main Corollary] and for generalizations [8, Corollary 2.3]. By taking the trace we obtain

$$F_p(A, B) = \text{Tr } \log^2(A\#_p B) \leq \text{Tr } (p \log A + (1 - p) \log B)^2 = G_p(A, B),$$

where we used that $T$ is trace preserving. The identity (2) for commuting operators is straightforward. The last statement follows from Corollary 2.5.

Note that both

$$F_p(tA, B) = F_p(A, t^{p/(1-p)} B) \quad \text{and} \quad G_p(tA, B) = G_p(A, t^{p/(1-p)} B)$$

for $0 < p < 1$ and $t > 0$. Therefore, $G_p$ is not strongly geodesically convex. However, this ambiguity disappears when we later define the control function and sum over contributions with different values of $p$.

### 3.1 The divergence $G_p(X; A, B)$

To $p \in [0, 1]$ and a positive definite operator $X$ we set

$$G_p(X; A, B) = G_p(X^{-1/2}AX^{-1/2}, X^{-1/2}BX^{1/2})$$

and note that the squared Riemannian distance

$$\delta_2(X, A\#_p B)^2 \leq G_p(X; A, B).$$

We call $G_p(X; A, B)$ a divergence since it gives a measure of the distance between $X$ and $A\#_p B$, see Proposition 3.3.

**Proposition 3.2.** Let $X$ be a positive definite operator. The divergence $G_p(X; A, B)$ is geodesically convex in positive definite $A, B$.

**Proof.** To $p \in [0, 1]$ and positive definite $A_1, A_2$ and $B_1, B_2$ we obtain

$$G_p(X; A_1\#_p A_2, B_1\#_p B_2)$$

$$= G_p(X^{-1/2}(A_1\#_p A_2)X^{-1/2}, X^{-1/2}(B_1\#_p B_2)X^{1/2})$$

$$= G_p((X^{-1/2}A_1X^{-1/2})\#_p(X^{-1/2}A_2X^{-1/2}),$$

$$\quad (X^{-1/2}B_1X^{-1/2})\#_p(X^{-1/2}B_2X^{1/2}))$$

$$\leq pG_p(X^{-1/2}A_1X^{-1/2}, X^{-1/2}B_1X^{-1/2}) +$$

$$\quad (1 - p)G_p(X^{-1/2}A_2X^{-1/2}, X^{-1/2}B_2X^{-1/2})$$

$$= pG_p(X; A_1, B_1) + (1 - p)G_p(X; A_2, B_2),$$
where we used congruence invariance of the weighted geometric mean.

**Proposition 3.3.** Take \( p \in [0, 1] \) and positive definite operators \( X, A \) and \( B \). Then
\[
G_p(X; A, B) = 0,
\]
if and only if \( X = A \#_p B \).

**Proof.** We realize that \( G_p(X; A, B) = 0 \) if and only if
\[
p \log(X^{-1/2}AX^{-1/2}) + (1 - p) \log(X^{-1/2}BX^{-1/2}) = 0.
\]
Since the logarithm is real analytic we may write the operator equation on the form
\[
pX^{-1/2} \log(AX^{-1})X^{1/2} + (1 - p)X^{-1/2} \log(BX^{-1})X^{1/2} = 0,
\]
and by multiplying with \( X^{1/2} \) from the left and \( X^{-1/2} \) from the right this is equivalent to
\[
p \log(AX^{-1}) + (1 - p) \log(BX^{-1}) = 0.
\]
This expression may be written on the form
\[
\log(AX^{-1})^p = \log(BX^{-1})^{-(1-p)} = \log(XB^{-1})^{1-p},
\]
where the exponents of the not necessarily self-adjoint matrices are defined by Cauchy’s integral formula. By taking the exponential function on both sides of the equation this is equivalent to
\[
(AX^{-1})^p = (XB^{-1})^{1-p}
\]
and thus
\[
AX^{-1} = (XB^{-1})^{(1-p)/p} = (XB^{-1})^{1/p}BX^{-1}.
\]
It follows that
\[
A = (XB^{-1})^{1/p}B \quad \text{and thus} \quad AB^{-1} = (XB^{-1})^{1/p}
\]
which is equivalent to
\[
XB^{-1} = (AB^{-1})^p.
\]
We can now solve for \( X \) and obtain
\[
X = (AB^{-1})^p B = B^{1/2}(B^{-1/2}AB^{-1/2})^p B^{1/2} = A \#_p B
\]
as desired. \(\square\)
4 The geodesic regression

Let a data set
\[ \mathcal{D} = \{(p_i, X_i) \mid i = 1, \ldots, n\} \]
be given that consists of numbers \(0 \leq p_i \leq 1\) for \(i = 1, \ldots, n\) and positive definite operators \(X_1, \ldots, X_n\). We define the control function
\[ G_D(A, B) = \sum_{i=1}^{n} G_{p_i}(X_i; A, B) \]
and note that it is geodesically convex in positive definite operators. The previous analysis now yields.

**Theorem 4.1.** The control function \(G_D(A, B) = 0\), if and only if the operators \(X_1, \ldots, X_n\) are joined by a geodesic curve \(\gamma(t)\) such that \(\gamma(p_i) = X_i\) for \(i = 1, \ldots, n\). If so then
\[ \gamma(p_i) = X_i = A\#_{p_i} B \quad i = 1, \ldots, n, \]
where \((A, B)\) is the uniquely defined pair in which \(G_D(A, B)\) vanishes.

The control function in (4) is strictly geodesically convex. Furthermore, if \(A\) and \(B\) tend to zero or to infinity then the control function tends to infinity. It is therefore not monotone, and it thus have a stationary point which is a unique global minimum.

**Theorem 4.2.** Let \((A, B)\) be the unique minimizer of the control function \(G_D(A, B)\). The geodesic curve
\[ \gamma(t) = A\#_p B \quad 0 \leq p \leq 1 \]
is the optimal fit to the given data set \(\mathcal{D}\) in terms of the control function (4).

5 The Riemannian gradients

Let \(p \in [0, 1]\) and let \(X\) be a positive definite operator. We consider trace functions of the form
\[ F(A, B) = \text{Tr}\, f\left(p \log(X^{-1/2}AX^{-1/2}) + (1-p) \log(X^{-1/2}BX^{-1/2})\right), \]
where $f$ is a real function defined in the real line. If we to an operator $Z$ put $Z_X = X^{-1/2}ZX^{-1/2}$ the function may be written as

$$F(A, B) = \text{Tr} f \left( p \log A_X + (1 - p) \log B_X \right).$$

We recall that the Fréchet differential of the logarithmic function is given by

$$d \log(X)V = \int_0^\infty (X + t)^{-1}V(X + t)^{-1}dt$$

implying that

$$(5) \quad \text{Tr} Sd \log(X)V = \text{Tr} S \int_0^\infty (X + t)^{-1}V(X + t)^{-1}dt = \text{Tr} Vd \log(X)S.$$

**Proposition 5.1.** The Riemannian gradients of $F$ are given by

$$\nabla_A F(A, B) = pAX^{-1/2}d \log(A_X)(f'(p \log A_X + (1 - p) \log B_X))X^{-1/2}A$$

$$\nabla_B F(A, B) = (1 - p)BX^{-1/2}d \log(B_X)(f'(p \log A_X + (1 - p) \log B_X))X^{-1/2}B.$$

**Proof.** We calculate the partial Fréchet differential

$$d_A F(A, B)V = p\text{Tr} df \left( p \log(A_X) + (1 - p) \log(B_X) \right)d \log(A_X)d_A(A_X)V.$$

Since $d_A(A_X)V = V_X$ and by using [7, Theorem 2.2] this becomes

$$d_A F(A, B)V = p\text{Tr} f'(p \log A_X + (1 - p) \log B_X)d \log(A_X)V_X.$$

The partial Riemannian gradient $\nabla_A F(A, B)$ is defined by the relation

$$< \nabla_A F(A, B) | V >_A = d_A F(A, B)V.$$

That is

$$\text{Tr} A^{-1} \nabla_A F(A, B)A^{-1}V = d_A F(A, B)V$$

$$= p\text{Tr} f'(p \log A_X + (1 - p) \log B_X)d \log(A_X)V_X$$

$$= p\text{Tr} d \log(A_X)(f'(p \log A_X + (1 - p) \log B_X)V_X$$

$$= p\text{Tr} X^{-1/2}d \log(A_X)(f'(p \log A_X + (1 - p) \log B_X))X^{-1/2}V$$

for all $V$, where we used (5). It follows that

$$A^{-1} \nabla_A F(A, B)A^{-1} = pX^{-1/2}d \log(A_X)(f'(p \log A_X + (1 - p) \log B_X))X^{-1/2}$$

from which the first statement follows. The second is obtained by symmetry. \(\square\)
Applying Proposition (5.1) for the function \( f(t) = t^2 \) we obtain

\[
\nabla_A G_p(X; A, B) = 2pAX^{-1/2}d\log(A_X) \left( p \log A_X + (1 - p) \log B_X \right) X^{-1/2} A
\]

with a similar expression for \( \nabla_A G_p(X; A, B) \). We note by Proposition 3.3 that the partial Riemannian gradient \( \nabla_A G_p(X; A, B) = 0 \), if and only if \( X = A\#_p B \) as expected.

The partial Riemannian gradients of the strictly geodesically convex control function \( G_D(A, B) \) defined in (4) vanish, if and only \((A, B)\) is the unique point minimizing \( G_D(A, B) \). Therefore,

**Theorem 5.2.** The unique minimizer \((A, B)\) of the geodesically convex control function \( G_D(A, B) \) is also the unique solution to the operator equations:

\[
\sum_{i=1}^{n} p_i X_i^{-1/2} d\log(A_{X_i}) \left( p_i \log A_{X_i} + (1 - p_i) \log B_{X_i} \right) X_i^{-1/2} = 0
\]

and

\[
\sum_{i=1}^{n} (1 - p_i) X_i^{-1/2} d\log(B_{X_i}) \left( p_i \log A_{X_i} + (1 - p_i) \log B_{X_i} \right) X_i^{-1/2} = 0.
\]

**References**

[1] Rajendra Bhatia. *Positive Definite Matrices*, volume 24 of *Princeton Series in Applied Mathematics*. Princeton University Press, 2007.

[2] M.M. Bronstein, J. Bruna, Y. LeCun, A. Szlam, and P. Vandergheynst. Geometric deep learning. Going beyond Euclidean data. *IEEE Signal Processing Magazine*, pages 18–42, July 2017.

[3] C. Davis. A Schwarz inequality for convex operator functions. *Proc. Amer. Math. Soc.*, 8:42–44, 1957.

[4] Z. Ding and et al. Fast predictive simple geodesic regression. *Medical Image Analysis*, 56:193–209, 2019.

[5] P. Thomas Fletcher. Geodesic regression and the theory of least squares on Riemannian manifolds. *Int J Comput Vis*, 105:171–185, 2013.
[6] C. Han, F.C. Park, and J. Kang. A geometric treatment of time-varying volatilities. Rev Quant Finan Acc, 49:1121–1141, 2017.

[7] F. Hansen and G.K. Pedersen. Perturbation formulas for traces on $C^*$-algebras. Publ. RIMS, Kyoto Univ., 31:169–178, 1995.

[8] Frank Hansen. Perspectives and completely positive maps. Annals of Functional Analysis, 8(2):168–176, 2017.

[9] Frank Hansen. Convex multivariate operator means. Linear Algebra and Its Applications, 564:209–224, 2019.

[10] Serge Lang. Differential and Riemannian Manifolds, volume 160 of Graduate Texts in Mathematics. Springer, 3. edition, 1995.

[11] Chi-Kwong Li and Yiu-Tung Poon. Interpolation by completely positive maps. Linear and Multilinear Algebra, 59(10):1159–1170, 2011.

[12] M. Moakher and Zérai. The Riemannian geometry of the space of positive-definite matrices and its application to the regularization of positive-definite matrix-valued data. J Math Imaging Vis, 40:171–187, 2011.

[13] Constantin P. Niculescu and Lars-Erik Persson. Convex Functions and Their Applications. CMS Books in Mathematics. Springer, 2. edition, 2018.

[14] X. Pennec, S. Joshi, and M. Nielsen. Mathematical methods for medical imaging. Int J Comput Vis, 105:109–110, 2013.