A STUDY OF BIFURCATION PARAMETERS IN TRAVELLING WAVE SOLUTIONS OF A DAMPED FORCED KORTEWEG DE VRIES-KURAMOTO SIVASHINSKY TYPE EQUATION

Mudassar Imran
International Center for Applied Mathematics and Computational Bioengineering
Department of Mathematics and Natural Sciences
Gulf University for Science and Technology, Kuwait

Youssef Raffoul and Muhammad Usman*
300 College Park
Department of Mathematics, University of Dayton
Dayton, Ohio 45469-2316, USA

Chi Zhang
Department of Mechanical Engineering, 300 College Park
University of Dayton
Dayton, Ohio 45469, USA

Abstract. In this work, we consider an ordinary differential equation obtained from a damped externally excited Korteweg de Vries-Kuramoto Sivashinsky (KdV-KS) type equation using traveling coordinates. We also include controls and delays and use an asymptotic perturbation method to analyze the stability of the traveling wave solutions. The existence of bounded solutions is presented as well. We consider the primary resonance defined by the detuning parameter. External-excitation and frequency-response curves are shown to exhibit jump and hysteresis phenomena (discontinuous transitions between two stable solutions) for the KdV-KS type equation. We have obtained the existence of the bounded solutions of the system obtained from an ordinary differential equation associated with the KdV-KS equation and also show the global stability for a special case when there is no external force.

1. Introduction. The Korteweg-de Vries (KdV) equation has a very interesting history going back to 1834, when a naval engineer Scott Russell observed a solitary wave (he then called it a great wave of translation) [38]; the associated KdV equation for solitary waves was derived in 1895 by D. J. Korteweg and his Ph.D. student G. de Vries in their famous paper [21]. Although the equation describing the phenomena observed by Scott Russell bears the name KdV, it was first obtained by Boussinesq in 1877 [4, 5, 6, 7]. It was solved by inverse scattering method by Gardner et al. in 1967 [11]. Since then, the KdV equation has attracted attention of physical scientists and of mathematicians ([12, 14], for example.) Since the soliton phenomenon can

2010 Mathematics Subject Classification. Primary: 35B20, 35B32, 35B20, 35B32, 34Cxx, 37C75, 93Dxx, 34Exx.
Key words and phrases. Korteweg de Vries equation, Kuramoto-Sivashinsky equation, KdV-KS equation, KS-type equation, bifurcations, steady state solutions, asymptotic perturbation method.
Authors name are in alphabetical order.
* Corresponding author: Muhammad Usman.
be observed when the nonlinearity is of the same order as cubic dispersion, such a situation arises in hydrodynamics, physics, or acoustics; for example, the situation can arise in magnetosonic waves, sound waves and shallow water waves [12]. In particular, the equation is now commonly accepted as a mathematical model for the unidirectional propagation of small-amplitude long waves in nonlinear dispersive systems. In simplified form, the Korteweg-de Vries equation reads

$$u_t - 6uu_x + u_{xxx} = 0,$$  \hspace{1cm} (1)

where \(x\) and \(t\) denote position and time and \(u = u(x,t)\) the wave surface.

In 1977, Gregory I. Sivashinsky derived an equation for a “laminar flame front” [40, 41]. At the same time, Yoshiki Kuramoto, developed the same equation while modeling diffusion-induced chaos [22, 23, 24] in a study of the Belousov-Zabotinskii reaction in three dimensions. Their joint discovery is known as the Kuramoto-Sivashinsky Equation. The one-dimensional Kuramoto-Sivashinsky equation (henceforth KS) in normalized form is

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} = 0,$$

where the parameter \(\nu\) is positive and plays the role of the viscosity of the system [36]. This equation arises in many physical problems, such as plasma physics [8], in chemical physics for propagation of concentration waves ([20, 22, 23]), and models of flame propagation [20, 40, 3, 43, 9, 22, 23].

As described in [3], the Korteweg-de Vries equation with a Kuramoto-Sivashinsky dissipative term appended is given by

$$u_t + \frac{1}{2}(u_x)^2 + au_{xxx} + bu_{xx} + cu_{xxxx} = 0, \quad a, b, c \in \mathbb{R},$$

which is also known as the Benney-Lin Equation (See [2, [3] [26]).

In this paper, we shall study damped and externally forced KdV-Kuramoto-Sivashinsky (henceforth KdV-KS) type equation. In particular, we shall study the following forced KdV-KS-type equation

$$u_t + 2Buu_x + Au_{xxx} - \mu u_{xx} = f \cos k(x + ct),$$  \hspace{1cm} (2)

where \(c\) and \(k\) represent the velocity and wave constants respectively and \(\mu\) and \(f\) are nonnegative constants proportional to the strength of the dispersion and forcing respectively.

The study of bifurcations in the traveling waves of nonlinear partial differential equations has been studied for a long time. The periodic doubling bifurcations in traveling waves solutions of the Kuramoto-Sivashinsky equation was studied in [19, 18] with emphasis on periodic solutions. A computational study of bifurcation diagrams of steady states of a KS type equation by difference methods is by Changpin [25]. Qiong-Wei [37] studied bifurcations in modified KS equation with higher order nonlinearity. The effect of dissipation, damping, dispersion and external forcing play an important role in the study of bifurcations in traveling wave solutions of third and fourth order partial differential equations. The traveling wave solutions of the Burgers-KdV equation with a fourth order term is studied by Mansour in [32].

The KdV type equations with damping and forcing terms have been studied for bifurcations in steady state solutions, for example, [12, 14, 27]. Among many techniques used in this type of research include the methods of averaging, multiple time scale and asymptotic perturbation methods [34, 42, 15, 35, 33]. In [27], the
authors have used asymptotic perturbation method to study the bifurcation control in the Burgers-KdV equation. After using the traveling coordinates it becomes an ordinary differential equation that is integrated to get a lower order differential equation. In [10], authors included damping and dissipation terms that make the associated differential equation irreducible to lower order and extended the asymptotic perturbation methods to study the bifurcation in steady states and also studied the Benjamin-Bona-Mahony equation (BBM equation) [1]. In this work, the authors have studied a nonlinear analysis of KdV-KS equation, with controls and delays, and studied various bifurcation parameters and obtained some theorem on stability with respect to these parameters.

In section 2, we use traveling coordinates to get an associated differential equation. We employ an asymptotic perturbation method to obtain a normal form. Using polar coordinates, we have a system of two differential equations. We also obtain a travelling wave solution of the equation. In Section 3 we show the existence of bounded solutions for the system. Section 4 deals with the steady states of the unforced and forced system and also presents the stability analysis and bifurcation curves. Finally, section 5 is devoted to a brief conclusion of our findings.

2. Steady state solutions and asymptotic perturbation method. Using traveling coordinates
\[ u(z) = u(x, t), \quad z = x + ct, \]
in (2), where \( t = \frac{d}{dx} \), to obtain a third order ordinary differential equation, and in here we also include controls terms with delays to get
\[
c u'(z) + 2Bu(z)u'(z) + Au'''(z) - \mu u'''(z) - C_1 \epsilon^2 u'(z - z_0) - C_2 \epsilon^2 u''(z - z_0) = f \cos(kz),
\]
Consider a perturbed equation
\[
c u'(z) + 2B \epsilon u(z)u'(z) + Au'''(z) - \mu \epsilon^2 u'''(z) - C_1 \epsilon^2 u'(z - z_0) - C_2 \epsilon^2 u''(z - z_0) = f \epsilon^2 \cos(kz),
\]
where \( \epsilon \) represents a non-dimensional parameter. Integrate (3) once with respect to \( z \) to obtain
\[
c u(z) + B \epsilon u(z)^2 + Au''(z) - \mu \epsilon^2 u''(z) - C_1 \epsilon^2 u(z - z_0) - C_2 \epsilon^2 u'(z - z_0) = f \epsilon^2 \frac{\sin(kz)}{k}.
\]
To study the primary resonance \( (\omega \approx k) \) first define an external detuning parameter \( \sigma \) through the relation
\[
\omega = k + \epsilon^2 \sigma \quad \text{where} \quad \omega^2 = c/A,
\]
where \( k \) represents the forcing frequency and \( \omega \) represents the frequency of linear undamped and unforced system. Define a slow time scale
\[ \tau = \epsilon^2 z, \]
then express \( u \) analytically as a function of the parameter \( \epsilon \), this is known as asymptotic perturbation method or asymptotic Fourier series expansion [27, 28, 29, 30, 31, 10], in particular
\[
u(z) = \epsilon \psi_0(\tau; \epsilon) + (\psi_1(\tau; \epsilon) e^{-ikz} + \psi_1^*(\tau; \epsilon) e^{ikz}) + \epsilon (\psi_2(\tau; \epsilon) e^{-2ikz} + \psi_2^*(\tau; \epsilon) e^{2ikz}) + \ldots
\]
where \( \psi_m(\tau; \epsilon) \) is assumed to be analytic in \( \epsilon \). We shall employ the notation \( \psi_m^{(0)} = \psi_m \) for \( m \neq 1 \) and \( \psi_1 = \psi \). Consider only the lowest order \( (i = 0) \) to obtain

\[
u(z) = \epsilon \psi_0(\tau; \epsilon) + (\psi(\tau; \epsilon) e^{-ikz} + \psi^*(\tau; \epsilon) e^{ikz}) + \epsilon (\psi_2(\tau; \epsilon) e^{-2ikz} + \psi_2^*(\tau; \epsilon) e^{2ikz}) + \epsilon^2 (\psi_3(\tau; \epsilon) e^{-3ikz} + \psi_3^*(\tau; \epsilon) e^{3ikz}) + h.o.t.
\]

(7)

Now differentiate (7) with respect to \( z \) to obtain \( u', u'' \) and \( u''' \). Considering only terms up to the order of \( \epsilon^2 \) we have

\[
u'(z) = \epsilon^2 \left( \psi_\epsilon e^{-ikz} + \psi_\epsilon^* e^{ikz} \right) + \epsilon \left( -2ik\psi_\epsilon e^{-2ikz} + 2ik\psi_\epsilon^* e^{2ikz} \right)
- ik\psi_\epsilon e^{-ikz} + ik\psi_\epsilon^* e^{ikz} + \epsilon^2 \left(-3ik\psi_\epsilon^* e^{-3ikz} + 3ik\psi_\epsilon e^{3ikz} \right) + h.o.t.
\]

(8)

\[
u''(z) = \epsilon^2 \left(-2ik\psi_\epsilon e^{-ikz} + 2ik\psi_\epsilon^* e^{ikz} \right) + \epsilon \left(-4k^2\psi_\epsilon e^{-2ikz} - 4k^2\psi_\epsilon^* e^{2ikz} \right)
- k^2\psi_\epsilon e^{-ikz} - k^2\psi_\epsilon^* e^{ikz} - \epsilon^2 \left(9k^2\psi_\epsilon^* e^{-3ikz} + 9k^2\psi_\epsilon e^{3ikz} \right) + h.o.t.
\]

(9)

\[
u'''(z) = \epsilon^2 \left(-3k^2\psi_\epsilon e^{-ikz} - 3k^2\psi_\epsilon^* e^{ikz} \right) + \epsilon \left(8ik^3\psi_\epsilon e^{-2ikz} - 8ik^3\psi_\epsilon^* e^{2ikz} \right)
+ ik\psi_\epsilon e^{-ikz} - ik\psi_\epsilon^* e^{ikz} + \epsilon^2 \left(27ik^3\psi_\epsilon e^{-3ikz} - 27ik^3\psi_\epsilon^* e^{3ikz} \right) + h.o.t.
\]

(10)

\[
u(z - z_0) = \epsilon \psi_0 + \left( \psi e^{-ik(z - z_0)} + \psi^* e^{ik(z - z_0)} \right) + \epsilon \left( \psi_2 e^{-2ik(z - z_0)} + \psi_2^* e^{2ik(z - z_0)} \right)
+ \epsilon^2 \left( \psi_3 e^{-3ik(z - z_0)} + \psi_3^* e^{3ik(z - z_0)} \right) + h.o.t.
\]

(11)

\[
u'(z - z_0) = \epsilon^2 \left( \psi_\epsilon e^{-ik(z - z_0)} + \psi_\epsilon^* e^{ik(z - z_0)} \right) - ik\psi_\epsilon e^{-ik(z - z_0)} + ik\psi_\epsilon^* e^{ik(z - z_0)}
+ \epsilon \left( -2ik\psi_\epsilon e^{-2ik(z - z_0)} + 2ik\psi_\epsilon^* e^{2ik(z - z_0)} \right)
+ \epsilon^2 \left(-3ik\psi_\epsilon^* e^{-3ik(z - z_0)} + 3ik\psi_\epsilon e^{3ik(z - z_0)} \right) + h.o.t.
\]

(12)

Substituting (8)–(12) into (3), and replace \( k^2 = \omega^2 - 2k\sigma^2 \) and \( k^3 = k(\omega^2 - 2k\sigma^2) \), we obtain the following

\[
u(\epsilon \psi(\tau; \epsilon) + (\psi(\tau; \epsilon) e^{-ikz} + \psi^*(\tau; \epsilon) e^{ikz}) + \epsilon(\psi_2(\tau; \epsilon) e^{-2ikz}) + \psi_2^*(\tau; \epsilon) e^{2ikz})
+ \epsilon^2 (\psi_3(\tau; \epsilon) e^{-3ikz} + \psi_3^*(\tau; \epsilon) e^{3ikz}) + \epsilon \left( \psi_2^* e^{-ikz} + \psi^* e^{ikz} \right) + \epsilon^2 \left( -3\psi_\epsilon e^{-ikz} - 3\psi_\epsilon^* e^{ikz}\right)
+ \epsilon k^2 \omega^2 \left(8i\psi_2 e^{-2ikz} - 8i\psi_2^* e^{2ikz} \right) + k\omega^2 \left( i\psi e^{-ikz} - i\psi^* e^{ikz} \right)
- 2\mu \omega^2 \epsilon^2 \left( i\psi e^{-ikz} - i\psi^* e^{ikz} \right) + \epsilon^2 k^2 \omega^2 \left( 27i\psi_3 e^{-3ikz} - 27i\psi_3^* e^{3ikz}\right)
+ \mu \epsilon^2 \omega^2 \left( \psi e^{-ikz} + \psi^* e^{ikz} \right) - C_1 \epsilon^2 \left( \psi e^{-ik(z - z_0)} + \psi^* e^{ik(z - z_0)} \right)
- C_2 \epsilon^2 \left( ik\psi e^{-ik(z - z_0)} - ik\psi e^{-ik(z - z_0)} \right) = f e^{i\sin(kz)} / k.
\]

(13)

Next, we equate coefficients of \( e^{-ikz} \) of order \( \epsilon^2 \) to obtain

\[
2B(\psi_0 \psi + \psi^*_\psi_2) - 3A\psi_\tau \omega^2 - 2A\sigma \omega^2 i\psi + \mu \omega^2 \psi - C_1 \epsilon \psi e^{ikz} + C_2 ik\psi e^{ikz} = \frac{f}{2k}.
\]

(14)
Equate coefficients of order $\epsilon$ and coefficients of $e^{-2ikz}$ of order $\epsilon$ to obtain

$$\psi_0 = \frac{-2B}{c} |\psi|^2, \quad \psi_2 = \frac{-B\psi^2(1 - 8ki)}{c(1 + 64k^2)}. \quad (15)$$

Next, we substitute (15) into (14) to obtain the following normal form

$$\psi_\tau + \left[ \left( \frac{4B^2}{3c^2} + \frac{2B^2}{3c^2(1 + 64k^2)} - i \frac{16B^2k}{3c^2(1 + 64k^2)} \right) \right] |\psi|^2 \psi$$

$$+ \left( \frac{C_1 \cos(kz_0)}{3c} - \frac{\mu}{3A} + \frac{C_2 k \sin(kz_0)}{3c} \right) \psi$$

$$+ i \left( \frac{C_1 \sin(kz_0)}{3c} - \frac{C_2 k \cos(kz_0)}{3c} + \frac{2\sigma}{3} \right) \psi + i \frac{f}{6kc} = 0. \quad (16)$$

Let us denote:

$$\alpha_1 = \frac{4B^2}{3c^2} + \frac{2B^2}{3c^2(1 + 64k^2)}, \quad \alpha_2 = -\frac{16B^2k}{3c^2(1 + 64k^2)},$$

$$\beta_1 = \frac{C_1 \cos(kz_0)}{3c} - \frac{\mu}{3A} + \frac{C_2 k \sin(kz_0)}{3c} = C_3 \cos(kz_0 - \phi) - \frac{\mu}{3A},$$

$$\beta_2 = \frac{C_1 \sin(kz_0)}{3c} - \frac{C_2 k \cos(kz_0)}{3c} + \frac{2\sigma}{3} = C_3 \sin(kz_0 - \phi) + \frac{2\sigma}{3},$$

$$\delta = \frac{f}{6kc},$$

where $\cos \phi = \frac{C_1}{3c}$, $\sin \phi = \frac{C_2 k}{3c}$, and $C_3 = \sqrt{\left( \frac{C_1}{3c} \right)^2 + \left( \frac{C_2 k}{3c} \right)^2}$.

This will simplify (16) to obtain the normal form

$$\psi_\tau + (\alpha_1 + i\alpha_2) |\psi|^2 \psi + (\beta_1 + i\beta_2) \psi + i\delta = 0. \quad (18)$$

Next, we introduce polar form in (18) and write $\psi(\tau) = \rho(\tau) e^{i\theta(\tau)}$ to obtain the following system of equations:

$$\frac{d\rho}{d\tau} = -\alpha_1 \rho^3 - \beta_1 \rho - \delta \sin \theta,$$

$$\frac{d\theta}{d\tau} = -\alpha_2 \rho^3 - \beta_2 \rho - \delta \cos \theta. \quad (19)$$

Considering terms up to order $\epsilon$, in (6) the solution is

$$u(z) = \epsilon \psi_0 + \psi e^{-ikz} + \psi^* e^{ikz} + \epsilon(\psi_2 e^{-2ikz} + \psi_2^* e^{2ikz}),$$

$$= \epsilon \left( \frac{-2B}{c} \rho^2 \right) + \epsilon \left[ \frac{-B\rho^2 e^{2i\theta}(1 - 8ki)}{c(1 + 64k^2)} e^{-2ikz} + \frac{-B\rho^2 e^{-2i\theta}(1 + 8ki)}{c(1 + 64k^2)} e^{2ikz} \right]$$

$$+ (\rho e^{i\theta - iz} + \rho e^{-i\theta + iz}). \quad (20)$$

Then we set $\epsilon = 1$ to obtain the travelling wave solution of (2)

$$u(z) = \frac{-2B}{c} \rho^2 - \frac{B\rho^3}{c(1 + 64k^2)} [2\cos(2kz - \theta) - 16k \sin(2kz - \theta)] + 2\rho \cos(kz - \theta),$$

$$\quad (21)$$

where $\rho$ and $\theta$ are governed by (19).
3. Existence of bounded solutions for the system (19) for $\rho > 0$. Next, we consider the nonlinear system of differential equations

$$\frac{d\rho}{d\tau} = -\alpha_1 \rho^3 - \beta_1 \rho - \delta \sin(\theta),$$  \hspace{1cm} (22)

$$\frac{d\theta}{d\tau} = -\alpha_2 \rho^2 - \beta_2 - \frac{\delta}{\rho} \cos(\theta),$$  \hspace{1cm} (23)

and show that it has a solution that remains bounded for all $\tau \geq \tau_0 > 0$. Since our problem in nonlinear, we will invert it to an integral equation problem and then use the Schauder’s fixed point theorem over a bounded space of admissible functions and show the existence of a bounded solution. By the variations of parameters formula, one can easily show that $\rho(\tau)$ is a solution of (22) if and only if $\rho(\tau)$ satisfies

$$\rho(\tau) = \rho_0 e^{-\beta_1 (\tau - \tau_0)} + \int_{\tau_0}^{\tau} \left( -\alpha_1 \rho^3(s) - \delta \sin(\theta(s)) \right) e^{-\beta_1 (\tau - s)} ds, \hspace{0.5cm} \tau \geq \tau_0 \geq 0.$$  

Now equation (23) is totally nonlinear in $\theta$ and hence the variation of parameters is of limited use. To get around such difficulty, we create a linear term. Thus, for $\mu_1 > 0$, we put (23) in the form

$$\frac{d\theta}{d\tau} = -\mu_1 \theta - \alpha_2 \rho^2 - \beta_2 + \mu_1 \theta - \frac{\delta}{\rho} \cos(\theta(s)).$$  \hspace{1cm} (24)

By the variations of parameters formula, one can easily show that $\theta(\tau)$ is a solution of (23) if and only if $\theta(\tau)$ satisfies

$$\theta(\tau) = \theta_0 e^{-\mu_1 (\tau - \tau_0)} + \int_{\tau_0}^{\tau} \left( -\alpha_2 \rho^2(s) - \beta_2 + \mu_1 \theta(s) - \frac{\delta}{\rho} \cos(\theta(s)) \right) e^{-\mu_1 (\tau - s)} ds, \hspace{0.5cm} \tau \geq \tau_0 > 0.$$  

We begin by stating the following theorem.
**Theorem 3.1 (Schauder’s Fixed Point Theorem).** [13, Theorem 3.2, p. 119] Let $X$ be a Banach space. Assume that $K$ is a convex (not necessarily closed) subset of $X$. If $T: K \rightarrow K$ is compact, then $T$ has at least one fixed point in $K$.

**Theorem 3.2.** Suppose $\beta_1 > 0$, and let $M_1, M_2$ and $m_1$ be positive constants such that

$$M_1 \geq |\rho_0| + \frac{(|\alpha_1| M_1^3 + \delta)}{\beta_1},$$

and

$$M_2 \geq |\theta_0| + \frac{|\alpha_2| M_2^2 + |\beta_2| + \frac{\delta}{m_1} + (1 - \delta/4) (\frac{m_1}{2})^{1/3}}{\mu_1} M_1^{4/3},$$

then each of the (22) and (23) has a bounded solution.

**Proof.** Let $X$ be the Banach space of all bounded and continuous real valued functions $(x, y)$, endowed by the norm

$$|| (x, y) ||= \max \left\{ \sup_{t \in [\tau_0, \infty]} |x(t)|, \sup_{t \in [\tau_0, \infty]} |y(t)| \right\}.$$ 

We define a subset $\mathbb{M}$ of $X$ as follows:

$$\mathbb{M} = \{(\phi, \eta) : \phi, \eta \in X | \phi(\tau_0) = \rho_0, \eta(\tau_0) = \theta_0 \text{ and } ||\phi|| \leq M_1, ||\eta|| \leq M_2 < x^*\},$$

where $x^*$ is a root of $\cos(x) = 1 - \frac{x^4}{2}$ on $(0, 1)$.

Now, for $(\phi, \eta) \in \mathbb{M}$ we can define an operator $\tilde{E} : \mathbb{M} \rightarrow X$ by

$$\tilde{E} (\phi, \eta)(t) = ((\tilde{E}_1(\phi, \eta)(t), (\tilde{E}_2(\phi, \eta)(t))),$$

where

$$(\tilde{E}_1(\phi, \eta)(t) = \rho_0 e^{-\beta_1 (\tau - \tau_0)} + \int_{\tau_0}^{\tau} (\phi(s) - \delta \sin(\eta(s))) e^{-\beta_1 (\tau - s)} ds, \tau \geq \tau_0 \geq 0,$$

and

$$(\tilde{E}_2(\phi, \eta)(t) = \theta_0 e^{-\mu_1 (\tau - \tau_0)} + \int_{\tau_0}^{\tau} (\phi(s) - \beta_2 + \mu_1 \eta(s) - \frac{\delta}{\rho} \cos(\eta(s))) e^{-\mu_1 (\tau - s)} ds, \tau \geq \tau_0 \geq 0.$$ 

First, we note that $\tilde{E}$ maps $\mathbb{M}$ into itself. Indeed, if $(\phi, \eta) \in \mathbb{M}$, then from (27) we have that Now for $\tau \geq \tau_0 \geq 0$, and $\varphi \in \mathbb{M}$ we have that

$$|| (\tilde{E}_1(\phi, \eta)(t)) || \leq |\rho_0| + \int_{\tau_0}^{\tau} |(\phi(s) - \delta \sin(\eta(s))) e^{-\beta_1 (\tau - s)} ||ds,$$

and

$$\leq |\rho_0| + (|\alpha_1| M_1^3 + \delta) \int_{\tau_0}^{\tau} e^{-\beta_1 (\tau - s)} ds,$$

$$\leq |\rho_0| + \frac{(|\alpha_1| M_1^3 + \delta)}{\beta_1} (1 - e^{-\beta_1 (\tau - \tau_0)}),$$

$$\leq M_1, \text{ by } (25).$$
Since \( \rho > 0 \), there exists a constant \( m_1 \leq M_1 \) such that \( \rho \geq m_1 \). Now for \( \tau \geq \tau_0 > 0 \), and \( \varphi, \rho \in M \) we have that

\[
\| (\tilde{E}_2(\phi, \eta)(\tau)) \| \leq |\theta_0| + \left| \alpha_2 |M^2 + \mu_1 \eta + |\beta_2| + \left( 1 - \frac{\eta^4}{2} \right) \right| \int_{\tau_0}^{\tau} e^{-\mu_1 (\tau-s)} ds \leq \| (\tilde{E}_2(\phi, \eta)(\tau)) \| \leq \left| \theta_0 \right| + \left( \frac{\alpha_2 |M^2 + |\beta_2| + \delta (1 - \delta/4)}{\mu_1} \right),
\]

This proves that \( \tilde{E} \) maps \( M \) into itself.

Now, we have to show that \( \tilde{E} \) is continuous. Let \( \{(x^t, y^t)\} \) be a sequence in \( M \) such that,

\[
\lim_{t \to \infty} \| (x^t, y^t) - (x, y) \| = 0.
\]

Since \( M \) is closed, we have \( (x, y) \in M \). Then by the definition of \( \tilde{E} \) we have

\[
\| \tilde{E}(x^t, y^t) - \tilde{E}(x, y) \| = \max \left\{ \sup_{t \in [\tau_0, \infty)} \left| (\tilde{E}_1((x^t, y^t))(\tau)) - (\tilde{E}_1(x, y))(\tau) \right|, \right. \\
\left. \sup_{t \in [\tau_0, \infty)} \left| (\tilde{E}_2(x^t, y^t))(\tau)) - (\tilde{E}_2(x, y))(\tau) \right| \right\},
\]

in which for \( \varphi, \rho \in M \), we have that

\[
\left| (\tilde{E}_1(\varphi^t, \phi^t))(\tau)) - (\tilde{E}_1(\varphi, \phi))(\tau) \right| \leq \int_{\tau_0}^{\tau} \left| \left( - \alpha_1 |\varphi^3| - \delta \sin(\phi') + \alpha_1 \varphi^3(s) + \delta \sin(\phi) \right) e^{-\beta_1 (\tau-s)} \right| ds
\]

\[
\leq \int_{\tau_0}^{\tau} \left| |\alpha_1| |\varphi^3 - \varphi^3(s)| + \delta |\sin(\phi') - \sin(\phi)| \right| e^{-\beta_1 (\tau-s)} ds
\]

\[
\leq \int_{\tau_0}^{\tau} \left| |\alpha_1| |\varphi^3 - \varphi^3(s)| + \delta |\sin(\phi') - \sin(\phi)| \right| ds.
\]

The continuity of \( \varphi^3 \) and \( \sin(\phi) \) along with the Lebesgue dominated convergence theorem implies that

\[
\lim_{t \to \infty} \sup_{\tau \in [\tau_0, \infty)} \left| (\tilde{E}_1(\varphi^t, \phi^t))(\tau)) - (\tilde{E}_1(\varphi, \phi))(\tau) \right| = 0.
\]

In a similar fashion, for \( \varphi, \phi \in M \), we have that

\[
\left| (\tilde{E}_2(\varphi^t, \phi^t))(\tau)) - (\tilde{E}_2(\varphi, \phi))(\tau) \right| \leq \int_{\tau_0}^{\tau} \left| \left( - \alpha_2 |\varphi^2| - \mu_1 \varphi - \delta \cos(\phi') \right) e^{-\mu_1 (\tau-s)} \right| ds
\]

\[
- \int_{\tau_0}^{\tau} \left| \left( - \alpha_2 |\varphi^2| - \mu_1 |\varphi - \varphi| + \delta \left| \cos(\varphi') \right| - \cos(\phi) \right) \right| e^{-\mu_1 (\tau-s)} ds.\]
The continuity of $\phi^2$, $\varphi$, and $\frac{\cos(\varphi)}{\varphi}$ along with the Lebesgue dominated convergence theorem implies that
$$\lim_{l \to \infty} \sup_{\tau \in [\tau_0, \infty)} |\tilde{E}_2(\varphi^l, \phi^l)(\tau) - \tilde{E}_2(\varphi, \phi)(\tau)| = 0.$$ This shows that $\tilde{E}$ is continuous.

Finally, we have to show that $\tilde{E}_M$ is precompact. Let $\{ (x^l, y^l) \}$ be a sequence in $M$. Then for each $t \in [t_0, \infty)_T$, $\{ (x^l(t), y^l(t)) \}$ is a bounded sequence of pairs of real numbers. This shows that $\{ (x^l(t), y^l(t)) \}$ has a convergent subsequence. By the diagonal process, we can construct a convergent subsequence $\{ (x^l_k, y^l_k) \}$ of $\{ (x^l, y^l) \}$ in $M$. Since $\tilde{E}$ is continuous, we know that $\{ \tilde{E}(x^l_k, y^l_k) \}$ has a convergent subsequence in $\tilde{E}_M$. This means $\tilde{E}_M$ is precompact.

By using Schauder’s fixed point theorem, we can conclude that there exist $(x, y) \in M$ such that $(x, y) = \tilde{E}(x, y)$.

Next we deal with the case $\beta_1 < 0$.

**Theorem 3.3.** Let $\gamma > 0$ be constant such that
$$\gamma + \beta_1 > 0.$$ Assume there is a positive constant $M_1$, be such that
$$M_1 \geq |\rho_0| + \left( |\alpha_1| M_1^3 + \gamma M_1 + \delta \right) \frac{\gamma + \beta_1}{\gamma + \beta_1}. \tag{29}$$ In addition we assume (26), where $M_1$ satisfies (29) for the same $m_1$ as in Theorem 3.2. Then each of the (22) and (23) has a bounded solution.

**Proof.** Rewrite (22) as
$$\frac{d\rho}{d\tau} = - (\gamma + \beta_1) \rho + \gamma \rho - \alpha_1 \rho^3 - \delta \sin(\theta). \tag{30}$$ Then using the variation of parameters we may define the mapping $\tilde{E}_1 : M \to M$, by
$$\tilde{E}_1(\phi, \eta)(\tau) = \rho_0 e^{-(\gamma + \beta_1)(\tau - \tau_0)} + \int_{\tau_0}^{\tau} \left( - \alpha_1 \varphi^3(s) + \gamma \varphi - \delta \sin(\eta) \right) e^{-(\gamma + \beta_1)(\tau - s)} ds, \tag{31}$$ where $\tau \geq \tau_0 > 0$.

The rest of the proof follows along the lines of the proof of Theorem 3.2, using the maps $\tilde{E}_1$ and $\tilde{E}_2$, where the maps are given by (31) and (28), respectively. This completes the proof.  

4. **Stability analysis of steady states and bifurcation curves.** For the steady state solutions $(\rho_0, \theta_0)$, setting $\frac{d\rho}{d\tau} = 0$ and $\frac{d\theta}{d\tau} = 0$ in (19) and eliminating $\theta$ from the fixed points of (19), we get the following equation
$$\delta^2 = (\alpha_1 \rho_0^3 + \beta_1 \rho_0)^2 + (\alpha_2 \rho_0^3 + \beta_2 \rho_0)^2, \tag{32}$$ which is called the external excitation-response curve. Then we first consider a special case:
In the absence of external forcing \( \delta = 0 \),
\[
\alpha_1 \rho_0^2 + \beta_1 = 0, \quad \text{which implies } \rho_0^2 = -\frac{\beta_1}{\alpha_1} \quad \text{or} \quad \rho_0 = 0,
\]
\[
\alpha_2 \rho_0^2 + \beta_2 = 0, \quad \text{which implies } \rho_0^2 = -\frac{\beta_2}{\alpha_2} \quad \text{or} \quad \rho_0 = 0.
\] (33)

For nontrivial solution, \( \rho_0^2 > 0 \), so we have \( \frac{\alpha_1}{\beta_1} < 0 \) and \( \frac{\alpha_2}{\beta_2} < 0 \), which in turn implies \( \beta_1 < 0 \), and \( \beta_2 > 0 \). In the coming section, we study the stability of these steady state solutions. The above system has a trivial \((0,0)\) and a nontrivial steady state of the form \((\rho_0, \theta_0)\).

When \( \delta = 0 \) and \( \rho_0 \neq 0 \), we have
\[
\frac{d\rho}{d\tau} = -\alpha_1 \rho^3 - \beta_1 \rho, \\
\frac{d\theta}{d\tau} = -\alpha_2 \rho^2 - \beta_2,
\] (34)
hence the dynamics does not depend upon the \( \theta \) as
\[
\frac{d\rho}{d\tau} = -\alpha_1 \rho_0^3 - \beta_1 \rho_0 = -\rho_0(\alpha_1 \rho^2 + \beta_1).
\]
The state \((\rho_0, \theta_0)\) will be stable if \( \frac{d\rho}{d\tau} < 0 \), which implies \( 0 < \rho_0 < \sqrt{-\frac{\beta_1}{\alpha_1}} \) or \(-\sqrt{-\frac{\beta_1}{\alpha_1}} < \rho_0 < 0 \).

We consider the forced case \((\delta \neq 0 \text{ and } \rho_0 > 0)\)
\[
\frac{d\rho}{d\tau} = -\alpha_1 \rho^3 - \beta_1 \rho - \delta \sin \theta, \\
\frac{d\theta}{d\tau} = -\alpha_2 \rho^2 - \beta_2 - \frac{\delta}{\rho} \cos \theta.
\] (35)

Consider small perturbations \( \delta \rho \) and \( \delta \theta \) in \( \rho_0 \) and \( \theta_0 \) respectively. Namely, let
\[
\rho = \rho_0 + \delta \rho, \quad \theta = \theta_0 + \delta \theta.
\]

Linearization of (35) about \((\rho_0, \theta_0)\) yields the Jacobian matrix
\[
\mathbf{J} = \begin{pmatrix}
-3\alpha_1 \rho_0^2 - \beta_1 & -\delta \cos \theta_0 \\
-2\alpha_2 \rho_0 + \frac{\delta}{\rho_0} \cos \theta_0 & -\frac{\delta^2}{\rho_0^2} \sin \theta_0
\end{pmatrix}.
\]

We get the characteristic polynomial \( \lambda^2 + p \lambda + q = 0 \), with
\[
p = 2(\beta_1 + 2\alpha_1 \rho_0^2) \quad \text{and} \quad q = (\alpha_1 \rho_0^2 + \beta_1)(3\alpha_1 \rho_0^2 + \beta_1) + (\alpha_2 \rho_0^2 + \beta_2)(3\alpha_2 \rho_0^2 + \beta_2).
\]

By Routh-Hurwitz criterion, the steady state is asymptotically stable if and only if \( p > 0 \) and \( q > 0 \). Thus the two roots will have negative real parts, the necessary and sufficient condition is
\[
2(\beta_1 + 2\alpha_1 \rho_0^2) > 0,
\]
and
\[
(\alpha_1 \rho_0^2 + \beta_1)(3\alpha_1 \rho_0^2 + \beta_1) + (\alpha_2 \rho_0^2 + \beta_2)(3\alpha_2 \rho_0^2 + \beta_2) > 0.
\]

We now determine the stability of the steady state solutions and prove the slope-stability theorems using implicit differentiation. We know that we must have two
real negative roots to make the Routh stability which means both $p$ and $q$ must larger than zero. We recall here the equation (32) for steady state solutions
\[ \delta^2 = (\alpha_1 \rho_0^3 + \beta_1 \rho_0)^2 + (\alpha_2 \rho_0^3 + \beta_2 \rho_0)^2. \]
We then apply the Implicit Function Theorem to get the following derivative in order to analyze the stability of the KdV-KS equation
\[ \frac{d\rho_0}{df} = \frac{f}{18 \pi^2 \rho_0^2}. \tag{36} \]
Using $q = (\alpha_1 \rho_0^2 + \beta_1)(3\alpha_1 \rho_0^2 + \beta_1) + (\alpha_2 \rho_0^2 + \beta_2 \rho_0)(3\alpha_2 \rho_0^2 + \beta_2)$, we rewrite (36) as
\[ q \frac{d\rho_0}{df} = \frac{f}{36 A^2 k^6 \rho_0}. \tag{37} \]

**Theorem 4.1.** If $d\rho/df > 0$ then the steady state solution is stable. Otherwise, the steady state solution is unstable.

**Proof.** For the fixed points $(\rho_0, \theta_0)$ of (35), the Jacobian matrix associated with the linear system has a characteristics equation of the form
\[ \lambda^2 + p \lambda + q = 0. \]
Eigenvalues have negative real parts if $p > 0$ and $q > 0$.

Thus, $d\rho_0/df$ and $q$ have the same sign by (37); a solution is stable if and only if $d\rho/df > 0$, otherwise it is unstable. \qed

Next, figure 2 exhibits the curve $\rho(f)$, where $f$ is the amplitude of the external force. This is a well-known curve called the operating curve or external excitation response curve or a bifurcation diagram [39]. We set $c = k = 1$, and $\omega = 1$, hence study primary resonance. Another curve that is of importance is the curve between the parameters $\sigma$ and $\rho$ called as the frequency-response curve. The figure 3 shown the change in this curve with increasing value of nonlinearity. Figure 4 shows the softening of the frequency response curve with increasing values of $B$.  

![Figure 2. External Excitation Response Curve](image-url)
Figure 3. Frequency-Response Curve without delay

Figure 4. Frequency-Response Curve with varying B

Theorem 4.2. (Bendixson-Dulac criterion) Suppose there exists a continuously differentiable function $\beta(x,y)$ defined on a simply connected domain $G$. Suppose that the function: $\frac{\partial}{\partial x} (\beta f) + \frac{\partial}{\partial y} (\beta g)$ doesn’t change sign in $G$. Then there are no periodic solutions of $x' = f(x,y)$, $y' = g(x,y)$ in the region $G$. 
\[
\frac{d\rho}{d\tau} = -\alpha_1 \rho^3 - \beta_1 \rho - \delta \sin(\theta) = f(\rho, \theta),
\]
\[
\frac{d\theta}{d\tau} = -\alpha_2 \rho^2 - \beta_2 \rho - \frac{\delta}{\rho} \cos(\theta) = g(\rho, \theta),
\]
we set \( \beta = 1 \)
\[
\frac{d}{d\rho} \beta f + \frac{d}{d\theta} \beta g = -3\alpha_1 \rho^2 - \beta_1 + \frac{\delta}{\rho} \sin(\theta),
\]
\[
= -2(2\alpha_1 \rho^2 + \beta_1) = -2p
\]
By the theorem, we know that there is no periodic solutions.

**Theorem 4.3.** (Global Stability) The nontrivial steady state of system (2.17) when \( \delta = 0 \) is globally asymptotically stable when it exists.

This result follows from above theorems and Poincare-Bendixson theory. Any solution that starts with positive initial condition will not converge to a trivial solution. Moreover all the solutions are bounded and there is no periodic orbit. This implies that the nontrivial solution is globally asymptotically stable.

**Theorem 4.4.** (Global Stability) The nontrivial solution is globally asymptotically stable when it exists.

5. **Conclusion.** In this work, we have considered a forced perturbed KdV-KS equation with controls and delays. The stability of steady state solutions is discussed with respect to bifurcation parameters. The standard curves, which are also known as external excitation response curve and frequency response curve, show the bifurcation phenomena as the transition between the stable and unstable solution on hysteresis curve. The discontinuous transition between the stability corresponds to the vertical tangent as shown by the slope stability theorem between the \( \rho \) and \( f \). We have also obtained the existence of the bounded solutions of the system obtained from an ordinary differential equation associated with the KdV-KS equation. We also show the global stability for a special case when \( \delta = 0 \). Finally, we have considered the 1 : 1 resonance in our work using asymptotic perturbation method. Similar techniques can be generalized to more cases of \( p : q \), where \( p \) and \( q \) are relatively prime integers as discussed in [17, 33] by Henrard and Meyer for modified Duffings equation and forced van der Pol equation. In this case, higher order terms need to be included in the system (6).

**Acknowledgments.** Corresponding author would like to thank Dr. Ken Meyer at the University of Cincinnati for his valuable guidance and insight on the method of averaging.

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Received January 2017; revised June 2017.

E-mail address: mimran@asu.edu
E-mail address: yraffoul@udayton.edu
E-mail address: musmani@udayton.edu
E-mail address: sjdashuai1@126.com