Fukaya category and Fourier transform

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Abstract. We construct a version of Fourier transform for families of real tori. This transform defines a functor from certain category associated with a symplectic family of tori to the category of holomorphic vector bundles on the dual family (the dual family has a natural complex structure). In the 1-dimensional case, the former category is closely related to the Fukaya category.

Introduction

This paper grew out from attempts to understand better the homological mirror symmetry for elliptic curves. The general homological mirror conjecture formulated by M. Kontsevich in [10] asserts that the derived category of coherent sheaves on a complex variety is equivalent to (the derived category of) the Fukaya category of the mirror dual symplectic manifold. This equivalence was proved in [15] for the case of elliptic curves and dual symplectic tori. However, the proof presented in [15] is rather computational and does not give a conceptual construction of a functor between two categories. In the present paper we fill up this gap by providing such a construction. We also get a glimpse of what is going on in the higher-dimensional case.

The idea is to use a version of the Fourier transform for families of real tori which generalizes the well-known correspondence between smooth functions on a circle and rapidly decreasing sequences of numbers (each function corresponds to its Fourier coefficients). On the other hand, this transform can be considered as a $C^\infty$-version of the Fourier-Mukai transform. Roughly speaking, given a symplectic manifold $M$ with a Lagrangian tori fibration, one introduces a natural complex structure on the dual fibration $M^\vee$. We say that $M^\vee$ is mirror dual to $M$. Then our transform produces a holomorphic vector bundle on $M^\vee$ starting from a Lagrangian submanifold $L$ of $M$ transversal to all fibers and a local system on $L$. We prove that the Dolbeault complex of this holomorphic vector bundle is isomorphic to some modification of the de Rham complex of the local system on $L$. In the case of an elliptic curve, we check that all holomorphic vector bundles on $M^\vee$ are obtained in this way. Also we construct a quasi-isomorphism of our modified de Rham complex with the complex that computes morphisms in the Fukaya category between $L$ and some fixed Lagrangian submanifold (which corresponds to the trivial...
line bundle on $M^\vee$). One can construct a similar quasi-isomorphism for arbitrary pair of Lagrangian submanifolds in $M$ (which are transversal to all fibers). The most natural way to do it would be to use tensor structures on our categories. The slight problem is that we are really dealing with dg-categories rather than with usual categories and the axiomatics of tensor dg-categories does not seem to be understood well enough. Hence, we restrict ourself to giving a brief sketch of how these structures look in our case in Sections 1.4, 2.4, and 3.4. It seems that to compare Fukaya complex with our modified de Rham complex in higher-dimensional case we need a generalization of Morse theory for closed 1-forms (cf. [12], [13]) together with a version of the result of Fukaya and Oh in [5] comparing Witten complex with Floer complex.

The study of mirror symmetry via Lagrangian fibrations originates from the conjecture of [16] that all mirror dual pairs of Calabi-Yau are equipped with dual special Lagrangian tori fibration. The geometry of such fibrations and their compactifications is studied in [7], [8], and [9]. In particular, the construction of a complex structure on the dual fibration can be found in these papers. On the other hand, K. Fukaya explains in [3] how to construct a complex structure (locally) on the moduli space of Lagrangian submanifolds (equipped with rank 1 local systems) of a symplectic manifold $M$, where Lagrangian submanifolds are considered up to Hamiltonian diffeomorphisms of $M$. Presumably these two constructions are compatible and one can hope that for some class of Lagrangian submanifolds the speciality condition picks a unique representative in each orbit of Hamiltonian diffeomorphisms group. Our point of view is closer to that of Fukaya: we do not equip our symplectic manifold with a complex structure, so we cannot consider special geometry. However, we do not consider the problem of compactifying the dual fibration and we do not know how to deal with Lagrangian submanifolds which intersect some fibers non-transversally. So it may well happen that special geometry will come up in relation with one of these problems.

The simplest higher-dimensional case in which our construction can be applied is that of a (homogeneous) symplectic torus equipped with a Lagrangian fibration by affine Lagrangian subtori. The corresponding construction of the mirror complex torus and of holomorphic bundles associated with affine Lagrangian subtori intersecting fibers transversally coincides with the one given by Fukaya in [3]. However, even in this case the homological mirror conjecture still seems to be far from reach (for dimensions greater than 2). Note that the construction of the mirror dual complex torus to a given (homogeneous) symplectic torus $T$ requires a choice of a linear Lagrangian subtorus in $T$. For different choices we obtain different complex tori. The homological mirror conjecture would imply that the derived categories on all these complex tori are equivalent (to be more precise, some of these categories should be twisted by a class in $H^2(T^\vee, O^*)$). This is indeed the case and follows from the main theorem of [14]. The corresponding equivalences are generalizations of the Fourier-Mukai transform.

While we were preparing this paper, N. C. Leung and E. Zaslow told us that they invented the same construction of a holomorphic bundle coming from a Lagrangian submanifold.

0.1. Organization. Section 1 contains the basic definitions and a sketch of the results of this paper.
In Section 2, we deal with a single real torus. We define the Poincaré bundle that lives on the product of our torus and the dual torus, and then use it to define a modified Fourier transform, which in this simple case is just the correspondence between bundles with unitary connections on a torus and sky-scraper sheaves on the dual torus.

Section 3 contains generalization of these results to families of tori. We describe the holomorphic sections of a vector bundle on the complex side in terms of rapidly decreasing sections of some bundle corresponding to its “Fourier transform” (notice that not every holomorphic vector bundle has this “Fourier transform”). Here we also analyze the case of elliptic curve.

Section 5 is devoted to interpreting the Floer cohomologies in our terms (i.e., using the spaces of rapidly decreasing sections of some bundles). This result is valid for elliptic curves only.

0.2. Notation. We work in the category of real $C^\infty$-manifolds. The words “a bundle on a manifold $X$” mean a (finite-dimensional) $C^\infty$-vector bundle over $\mathbb{C}$ on $X$. We usually identify a vector bundle with the corresponding sheaf of $C^\infty$-sections. For a manifold $X$, $T_X \to X$ (resp. $T_X^\vee \to X$) is the real tangent (resp. cotangent) bundle, $\Omega^{-1}(X)$ (resp. $\Omega^1(X)$) is the space of complex vector fields (resp. complex differential forms). If $X$ carries a complex structure, $T_X^{0,1} \subset T_X \otimes \mathbb{C}$ stands for the subbundle of anti-holomorphic vector fields. $\text{Diff}(X)$ is the algebra of differential operators on $X$ with $C^\infty(X) \otimes \mathbb{C}$-coefficients.

Let $F$ be a vector bundle on a manifold $X$, $\nabla_F : F \to F \otimes \Omega^1(X)$ a connection. We define the curvature $\text{curv} \nabla_F \in \Omega^2(X) \otimes \text{End}(F)$ of $\nabla_F$ by the usual formula: $\langle (\text{curv} \nabla_F), \tau_1 \wedge \tau_2 \rangle = \langle (\nabla_F)[\tau_1, \tau_2] - [(\nabla_F)\tau_1, (\nabla_F)\tau_2] \rangle$ for any $\tau_1, \tau_2 \in \Omega^{-1}(X)$. Here $\langle \cdot, \cdot \rangle$ stand for the natural pairing $\wedge^2 \Omega^1(X) \times \wedge^2 \Omega^{-1}(X) \to C^\infty(X)$ defined by $\langle \mu_1 \wedge \mu_2, \tau_1 \wedge \tau_2 \rangle = \langle \mu_1, \tau_2 \rangle \langle \mu_2, \tau_1 \rangle - \langle \mu_1, \tau_1 \rangle \langle \mu_2, \tau_2 \rangle$.

A local system $\mathcal{L}$ on a manifold $X$ is a vector bundle $F_{\mathcal{L}}$ together with a connection $\nabla_{\mathcal{L}}$ in $F_{\mathcal{L}}$ such that $\text{curv} \nabla_{\mathcal{L}} = 0$ (in other words, $\nabla_{\mathcal{L}}$ is flat). The fiber $\mathcal{L}_x$ of a local system $\mathcal{L}$ over $x \in X$ equals the fiber $(F_{\mathcal{L}})_x$. For any $x \in X$, a local system $\mathcal{L}$ defines the monodromy $\text{mon}(\mathcal{L}, x) : \pi_1(X, x) \to GL(\mathcal{L}_x)$. We say that a local system is unitary if for any $x \in X$, there is a Hermitian form on $\mathcal{L}_x$ such that $\text{mon}(\mathcal{L}, x)(\gamma)$ are unitary for all $\gamma \in \pi_1(X, x)$ (it is enough to check the condition for one point on each connected component of $X$).

For a manifold $X$ and $\tau \in \Omega^1(X)$, we denote by $O_X(\tau)$ the trivial line bundle together with the connection $\nabla = d + \tau$. In particular, $O_X := O_X(0)$ stands for the trivial local system on $X$.

1. Main results

1.1. Let $(M, \omega)$ be a symplectic manifold, $p : M \to B$ a surjective smooth map with Lagrangian fibers. Suppose that the fibers of $M \to B$ are isomorphic to a torus $(\mathbb{R}/\mathbb{Z})^n$. Fix a Lagrangian section $0_M : B \to M$. We call such collection $(p : M \to B, \omega, 0_M)$ (or, less formally, the map $p : M \to B$) a symplectic family of tori.

The symplectic form induces a natural flat connection on $T_B$ (using the canonical isomorphism $R^1p_*\mathbb{R} = T_B$) and an identification $M = T_B^\vee/\Gamma$, where $\Gamma$ is a horizontal lattice in $T_B^\vee$ ($\Gamma$ is dual to $\Gamma^\vee := R^1p_*\mathbb{Z} \subset R^1p_*\mathbb{R} = T_B$). This identification agrees with the symplectic structure, so $\Gamma \subset T_B^\vee$ is Lagrangian.
Hence the connection on $T_B$ is symmetric (in the sense of (11)). Recall that a connection $\nabla$ on $T_B$ is called symmetric if $\nabla_{\tau_1}(\tau_2) - \nabla_{\tau_2}(\tau_1) = [\tau_1, \tau_2]$ for any $\tau_1, \tau_2 \in \Omega^{-1}(B)$.

**Remark 1.** In particular, we see that $M \to B$ is locally (on $B$) isomorphic to $(V/\Gamma) \times U$ for some vector space $V$, a lattice $\Gamma \subset V$, and an open subset $U \subset V^\vee$. Besides, we see that the connection on $T_B$ induces a natural flat connection on $T_M$.

Consider the family of dual tori $M^\vee := T_B/\Gamma^\vee$. The connection on $T_B$ yields a natural isomorphism $T_{M^\vee} = (p^\vee)^*T_B \oplus (p^\vee)^*T_B$ such that the differential of $p^\vee : M^\vee \to B$ coincides with the first projection. So one can define a complex structure on $M^\vee$ using the operator $J : T_{M^\vee} \to T_{M^\vee} : (\xi_1, \xi_2) \mapsto (-\xi_2, \xi_1)$. The complex manifold $M^\vee$ is called the mirror dual of $M$.

For any torus $X = V/\Gamma$, the dual torus $X^\vee = V^\vee/\Gamma^\vee$ can be interpreted as a moduli space of one-dimensional unitary local systems on $X$. So there is a natural universal $X^\vee$-family of local systems on $X$. We can interpret this family as a bundle with a connection on $X \times X^\vee$ (see Section 2.1 for details). If we apply this constructions to fibers of $p$, we get a canonical bundle $\mathcal{P}$ on $M \times_B M^\vee$ together with a connection $\nabla_{\mathcal{P}}$ on $\mathcal{P}$ ($\nabla_{\mathcal{P}}$ is not flat).

Suppose we are given a Lagrangian submanifold $i : L \to M$ which is transversal to fibers of $p$, and a local system $\mathcal{L}$ on $L$. We also assume that $p|_L : L \to B$ is proper. Define the **Fourier transform** of $(L, \mathcal{L})$ by the formula

$$\text{Four}(L, \mathcal{L}) := (p_{M^\vee})_*(((i \times \text{id})^*\mathcal{P}) \otimes ((p_L)^*\mathcal{L}))$$

Here $p_{M^\vee} : L \times_B M^\vee \to M^\vee$, $(i \times \text{id}) : L \times_B M^\vee \to M \times_B M^\vee$, and $p_L : L \times_B M^\vee \to L$ are the natural maps. The map $p_{M^\vee}$ is a proper unramified covering, so $\text{Four}(L, \mathcal{L})$ is a bundle with connection on $M^\vee$.

**Theorem 1.1.** (i) The $\partial$-component of the connection on $\text{Four}(L, \mathcal{L})$ is flat (so $\text{Four}(L, \mathcal{L})$ can be considered as a holomorphic vector bundle on $M^\vee$); (ii) If $B \simeq (\mathbb{R}/\mathbb{Z})$, any holomorphic vector bundle on $M^\vee$ is isomorphic to $\text{Four}(L, \mathcal{L})$ for some $(L, \mathcal{L})$.

**Remark 2.** There is an analogue of the above theorem for the case when the fibration does not have a global Lagrangian section. In this case, the dual complex manifold $M^\vee$ carries a canonical cohomology class $e \in H^2(M^\vee, O_{M^\vee}^*)$, hence one has the corresponding twisted category of coherent sheaves (cf. (3)). The analogue of $\text{Four}(L, \mathcal{L})$ will be an object in this twisted category. We will consider this generalization in more details elsewhere. Also it would be interesting to find an analogue of our construction for Lagrangian foliations. In the case of a torus, this should lead to the functor considered by Fukaya in (2).

**1.2.** Let $(L, \mathcal{L})$ be as before.

Consider the natural map $u : T_B^\vee \to M$ (the “fiberwise universal cover”). Set $\tilde{L} := u^{-1}(L)$. Denote by $u_L^*\mathcal{L}$ the pull-back of $\mathcal{L}$ to $\tilde{L}$ and by $\tau$ the restriction of the canonical 1-form from $T_B^\vee$ to $\tilde{L}$. Since $\tilde{L} \subset T_B^\vee$ is Lagrangian, $\tau$ is closed, so by adding $-2\pi \tau$ to the connection on $u_L^*\mathcal{L}$ we get a new local system $\tilde{\mathcal{L}} := (u_L^*\mathcal{L}) \otimes O_{\tilde{L}}(-2\pi \tau)$.

Denote by $C^\infty(\tilde{L})$ the space of $C^\infty$-sections of $\tilde{L}$. Since $\tilde{L} \to B$ is an unramified covering, we have an embedding $\text{Diff}(B) \to \text{Diff}(\tilde{L})$. Set
(1.2) \( S(\mathcal{L}) := \{ s \in C^\infty(\mathcal{L}) | Ds \text{ is rapidly decreasing for any } D \in \text{Diff}(B) \} \)

Here a section \( s \) of \( \mathcal{L} \) is called rapidly decreasing if \( \lim_{||g|| \to \infty, g \in \Gamma_x} s((x, \tau + g)) ||g||^k = 0 \) for any \((x, \tau) \in \mathcal{L} \times M \mathcal{T}_B^+ = \mathcal{L} \) and \( k > 0 \). Here \( \Gamma_x \) stands for the fiber of \( \Gamma \) over \( x \in B \). Since \( s((x, \tau + g)) \in \mathcal{L}_{(x, \tau + g)} = \mathcal{L}_{(x, \tau)} \), the definition makes sense. Besides, it does not depend on the choice of a norm \( || \cdot || \) on \( \mathcal{T}^B \). Clearly, \( S(\mathcal{L}) \) is a \( \text{Diff}(B) \)-module.

**Theorem 1.2.** The de Rham complex \( \text{DR}(\mathcal{L}) \) of the \( \text{Diff}(B) \)-module \( S(\mathcal{L}) \) is isomorphic to the Dolbeault complex of \textbf{Four}(L, \mathcal{L}).

**1.3.** Suppose \( B \simeq \mathbb{R}/\mathbb{Z} \). Fix an orientation on \( B \).

Let \( L, \mathcal{L} \) be as before. Moreover, we suppose that \( \mathcal{L} \) is quasi-unitary, that is, for any \( x \in L \) all eigenvalues of \( \text{mon}(\mathcal{L}, x) \) are of absolute value 1 (it follows from Lemma 1.3 that this condition is not too restrictive). We also assume that \( L \) meets the zero section \( 0_M(B) \subset M \) transversally.

As before, \( \mathcal{L} = u^{-1}(L) \subset T^B \). Suppose \( \tilde{c} \in \mathcal{L} \) lies on the zero section \( 0_{\mathcal{T}^B}(B) \subset T^B \). Then in a neighborhood of \( \tilde{c}, \mathcal{L} \subset T^B \) is the graph of some \( \mu \in \Omega^1(B), d\mu = 0 \). Denote by \( b \in B \) the image of \( \tilde{c} \in \mathcal{L} \). In a neighborhood of \( b, \mu = df \) for some \( f \in C^\infty(B) \). We say that \( \tilde{c} \) is positive (resp. negative) if \( f \) has a local minimum (resp. maximum) at \( b \). Denote by \( \{ \tilde{c}^+_k \} \subset \mathcal{L} \) (resp. \( \{ \tilde{c}^-_l \} \subset \mathcal{L} \)) the set of all positive (resp. negative) points of intersection with the zero section.

Let \( \gamma \subset \mathcal{L} \) be an arc with endpoints \( \tilde{c}^+_k \) and \( \tilde{c}^-_l \). We say that \( \gamma \) is simple if it does not intersect the zero section. Denote by \( M(\gamma) : \mathcal{L}_{\tilde{c}^+_k} \to \mathcal{L}_{\tilde{c}^-_l} \) the monodromy of \( \mathcal{L} \) along \( \gamma \) (the monodromy is the product of the monodromy of \( u^*_A \mathcal{L} \) and \( \exp(2\pi A) \), where \( A \) is the oriented area of the domain restricted by \( \gamma \) and the zero section). Set \( d(\gamma) = M(\gamma) \) if the direction from \( \tilde{c}^+_k \) to \( \tilde{c}^-_l \) along \( \gamma \) agrees with the orientation of \( B \), and \( d(\gamma) = -M(\gamma) \) otherwise.

Set \( F^0 := \oplus_k \mathcal{L}_{\tilde{c}^+_k}, F^1 := \oplus_l \mathcal{L}_{\tilde{c}^-_l} \). Consider the operator \( d : F^0 \to F^1 \) whose "matrix elements" are \( d_{kl} : \mathcal{L}_{\tilde{c}^+_k} \to \mathcal{L}_{\tilde{c}^-_l} = \sum \gamma d(\gamma) \). Here the sum is taken over all simple arcs \( \gamma \) with endpoints \( \tilde{c}^+_k, \tilde{c}^-_l \) (there are at most two of them).

**Remark 3.** Since \( L \) meets the fibers of \( M \to B \) transversely, there is a canonical choice of lifting \( L \to M \) to \( L \to \text{Gr}L(T_M) \). Here \( \text{Gr}L(T_M) \to M \) is the fibration whose fiber over \( m \in M \) is the manifold of Lagrangian subspaces in \( T_M(m) \) (the Lagrangian Grassmanian of \( T_M(m) \)). \( \text{Gr}L(T_M) \to \text{Gr}L(T_M) \) is its fiberwise universal cover. This implies that the corresponding Floer cohomologies are equipped with a natural \( \mathbb{Z} \)-grading. Since \( L \) is also transversal to the zero section \( 0_M(B) \subset M \), we may compute the space (or, more precisely, the complex) of morphisms for the pair \( L, 0_M(B) \) in the Fukaya category. It is easy to see that the complex coincides with \( \mathcal{F}(\mathcal{L}) : F^0 \to F^1 \).

**Theorem 1.3.** The complex \( \mathcal{F}(\mathcal{L}) \) is quasi-isomorphic to \( \text{DR}(\mathcal{L}) \).

**Construction of a quasi-isomorphism \( \mathcal{F}(\mathcal{L}) \to \text{DR}(\mathcal{L}) \).** Consider distributions with values in \( \mathcal{L} \) that are rapidly decreasing smooth sections of \( \mathcal{L} \) outside some compact set. Let \( S(\mathcal{L})^D \) be the space of such distributions. Denote by \( \text{DR}(\mathcal{L})^D \)
the de Rham complex associated with the Diff(\(B\))-module \(S(\tilde{L})^D\). The inclusion \(S(\tilde{L}) \rightarrow S(\tilde{L})^D\) induces a quasi-isomorphism \(\text{DR}(\tilde{L}) \rightarrow \text{DR}(\tilde{L})^D\). Now let us define a morphism \(\mathcal{F}(\tilde{L}) \rightarrow \text{DR}(\tilde{L})^D\).

For \(\tilde{c}^+_k\), denote by \(C^+_k\) the maximal (open) subinterval \(I \subset \tilde{L}\) such that \(\tilde{c}^+_k \subset C^+_k\) and \(\tilde{c}^-_l \notin C^+_k\) for any \(l\) (\(I\) may be infinite). The morphism \(F^0 \rightarrow S(\tilde{L})^D\) sends \(v \in \tilde{L}_{\tilde{c}^+_k} \rightarrow f\) such that \(f\) vanishes outside \(C^+_k\), \(f\) is horizontal on \(C^+_k\), and \(f(\tilde{c}^+_k) = v\). The morphism \(F^1 \rightarrow S(\tilde{L})^D \otimes \Omega^1(B)\) sends \(v \in \tilde{L}_{\tilde{c}^-_l}\) to \(v \otimes \delta_{\tilde{c}^-_l}\). Here \(\delta_{\tilde{c}^-_l}\) is the delta-function at \(\tilde{c}^-_l\).

**Remark 4.** All this machinery works in a more general situation. Namely, we can consider a symplectic family of tori \(M \rightarrow B\) together with a closed purely imaginary horizontal form \(\omega^L\). Then we can work with the category of submanifolds \(L \subset M\) together with a bundle \(\mathcal{L}\) on \(L\) and a connection \(\nabla_{\mathcal{L}}\) such that \(L \rightarrow B\) is a finite unramified covering and \(\text{curv} \nabla_{\mathcal{L}} = 2\pi(\omega + \omega^f)|_L\).

1.4. The pairs \((L, \mathcal{L})\) of the kind considered above form a category. One can define the (fiberwise) convolution product in this category using the group structure on the fibers. However, the support of the convolution product does not need to be a smooth Lagrangian submanifold, so to have a tensor category, we have to consider a slightly different kind of objects (see Section 3.3).

After these precautions, we have a tensor category \(\text{Sky}(M/B)\). One easily sees that there is a canonical (i.e., functorial) choice of the dual object \(c^\vee\) for any \(c \in \text{Sky}(M/B)\). For any \(c \in \text{Sky}(M/B)\), we have the de Rham complex \(\text{DR}(c)\) (defined in a way similar to what we do for \((L, \mathcal{L})\)). Now we can use these data to define another “category” \(\text{Sky}(M/B)\): we set \(\text{Ob}(\text{Sky}(M/B)) := \text{Ob}(\text{Sky}(M/B))\), \(\text{Hom}_{\text{Sky}(M/B)}(c_1, c_2) := \text{DR}(c_2 \ast c_1^\vee)\), where \(\ast\) stands for the convolution product. It is not a “plain” category, but a “dg-category”. Similarly, the category of holomorphic vector bundles on \(M^\vee\) has a structure of a tensor dg-category (the morphism complex from \(L_1\) to \(L_2\) is the Dolbeault complex of \(L_2 \otimes L_1^\vee\)). Then the isomorphism of Theorem 1.2 induces a fully faithful tensor functor between tensor dg-categories.

2. Fourier transform on tori

2.1. Poincaré bundle. Let \(X\) be a torus (that is, a compact commutative real Lie group). Then \(X = V/\Gamma\) for \(V := H_1(X, \mathbb{R}), \Gamma := H_1(X, \mathbb{Z})\). The dual torus is \(X^\vee := V^\vee/\Gamma^\vee\) (\(V^\vee := \text{Hom}(V, \mathbb{R}) = H^1(X, \mathbb{R}), \Gamma^\vee := \text{Hom}(\Gamma, \mathbb{Z}) = H^1(X, \mathbb{Z})\)).

**Definition 2.1.** A Poincaré bundle for \(X\) is a line bundle \(\mathcal{P}\) on \(X \times X^\vee\) together with a connection \(\nabla_{\mathcal{P}}\) such that the following conditions are satisfied:

(i) \(\nabla_{\mathcal{P}}\) is flat on \(X \times \{x^\vee\}\), and the monodromy is \(\pi_1(X) = H_1(X, \mathbb{Z}) \rightarrow U(1) : \gamma \mapsto \exp(2\pi\sqrt{-1}(x^\vee, \gamma))\) (we denote by \((\cdot, \cdot)\) not only the natural pairing \(V^\vee \times V \rightarrow \mathbb{R}\), but also the induced pairings \(V^\vee / \Gamma^\vee \times \Gamma / \Gamma \rightarrow \mathbb{R} / \mathbb{Z}\); and \(V^\vee / \Gamma^\vee \times \Gamma \rightarrow \mathbb{R} / \mathbb{Z}\);

(ii) \(\nabla_{\mathcal{P}}\) is flat on \(\{x\} \times X^\vee\), and the monodromy is \(\pi_1(X^\vee) = H^1(X, \mathbb{Z}) \rightarrow U(1) : \gamma^\vee \mapsto \exp(-2\pi\sqrt{-1}(\gamma^\vee, x))\);

For any \((x, x^\vee) \in X \times X^\vee, \delta v \in V = T_xX, \delta v^\vee \in V^\vee = T_{x^\vee}X^\vee\), we have

\[\langle \text{curv}(\nabla_{\mathcal{P}}), \delta v \wedge \delta v^\vee \rangle = -2\pi\sqrt{-1}(\delta v^\vee, \delta v).\]

Clearly, \((\mathcal{P}, \nabla_{\mathcal{P}})\) is defined up to an isomorphism by (i), (i\(\vee\)), (ii). Furthermore, we always fix an identification \(\iota : \mathcal{P}_{(0,0)} \simeq \mathbb{C}\), so the collection \((\mathcal{P}, \nabla_{\mathcal{P}}, \iota)\) is defined up to a canonical isomorphism.
A Poincaré bundle allows us to identify $X^\vee$ with the moduli space of unitary local system on $X$ (and vice versa).

**Remark 5.** Suppose $V$ carries a complex structure $J : V \to V$. Define the complex structure on $V^\vee$ using $-J^\vee$. Then $X$, $X^\vee$, and $X \times X^\vee$ are complex manifolds. Let $\mathcal{P}$ be a Poincaré bundle for $X$. It is easy to see that $\nabla_{\mathcal{P}}$ is “flat in $\overline{\partial}$-direction” (i.e., the curvature $\nabla_{\mathcal{P}}$ vanishes on $\bigwedge^2 T^*_{X \times X^\vee}^\vee$). Hence $\mathcal{P}$ can be considered as a holomorphic line bundle on $X \times X^\vee$. Actually, $\mathcal{P}$ is in this case isomorphic to the “complex” Poincaré bundle (i.e., the universal bundle that comes from the interpretation of $X^\vee$ as a moduli space of holomorphic line bundles on $X$).

The following lemma is straightforward.

**Lemma 2.2.** Consider the local system $F := O_{V \times X^\vee}(2\pi \sqrt{-1}(dx^\vee, v))$. Here $dx^\vee \in \Omega^1(X^\vee) \otimes V^\vee$ is the natural form with values in $V^\vee$. Lift the natural action of $\Gamma = H_1(X, \mathbb{Z})$ on $V$ to $F$ by $(g)(f)(v, x^\vee) = \exp(-2\pi \sqrt{-1}(x^\vee, g))f(v - g, x^\vee)$. Then the corresponding line bundle with connection on $X \times X^\vee$ is a Poincaré bundle.

Consider the natural projection $u \times id : V \times X^\vee \to X \times X^\vee$. Then $(u \times id)^* \mathcal{P}$ is identified with $F$. We denote by $\text{Exp}(-2\pi \sqrt{-1}(x^\vee, v))$ the section of $(u \times id)^* \mathcal{P}$ that corresponds to $1 \in F$.

**Remark 6.** Let $\mathcal{P}$ be a Poincaré bundle for $X$, $\sigma' : X^\vee \times X \to X \times X^\vee : (x^\vee, x) \mapsto (-x, x^\vee)$. Then $(\sigma')^* \mathcal{P}$ is a Poincaré bundle for $X^\vee$.

### 2.2. Sky-scraper sheaves.

Given a finite set $S \subset X$ and (finite-dimensional) $\mathbb{C}$-vector spaces $F_s$ for all $s \in S$, we can define the corresponding (finite semisimple) sky-scraper sheaf $F$ on $X$ by $F(U) = \oplus_{s \in S \cap U} F_s$ for $U \subset X$. Denote by $\text{Sky}(X)$ the category of sky-scraper sheaves on $X$ ($\text{Sky}(X)$ is a full subcategory of the category of sheaves of vector spaces on $X$). Any sky-scraper sheaf is naturally a $C^\infty(X)$-module, and morphisms of sky-scraper sheaves agree with the action of $C^\infty(X)$.

For $F \in \text{Sky}(X)$, define the Fourier transform of $F$ by

\begin{equation}
\text{Four } F := (p_{X^\vee})_* \left((p_X^* F) \otimes \mathcal{P} \right).
\end{equation}

Here $p_X : X \times X^\vee \to X$ and $p_{X^\vee} : X \times X^\vee \to X^\vee$ are the natural projections.

**Four** $F$ is a locally free sheaf of rank $\dim H^0(X, F)$, so we interpret **Four** $F$ as a vector bundle on $X^\vee$. The connection $\nabla$ on $\mathcal{P}$ induces a flat unitary connection on **Four** $F$. So **Four** can be considered as a functor $\text{Sky}(X) \to \text{Loc}_u(X^\vee)$, where $\text{Loc}_u(X^\vee)$ is the category of unitary local systems on $X^\vee$. This functor is an equivalence of categories.

### 2.3. Rapidly decreasing sections.

For a sheaf $F \in \text{Sky}(X)$, set $\bar{F} := u^* F$, where $u : V \to X$ is the universal cover. The group $\Gamma := H_1(X, \mathbb{Z})$ acts on $V = H_1(X, \mathbb{R})$ and $\bar{F}$ is $\Gamma$-equivariant. We say that a section $s \in H^0(V, \bar{F})$ is rapidly decreasing if $\lim_{||g|| \to \infty, g \in \Gamma} s(x + g) ||g||^k = 0$ for any $x \in V$, $k > 0$ (the definition does not depend on the choice of a norm $|| \cdot ||$ on $V$). Denote by $S(\bar{F})$ the space of all rapidly decreasing sections of $\bar{F}$.

Take $F \in \text{Sky}(X)$, $f \in S(\bar{F})$. Set
(2.2) \[ \text{Four}_F f(x^\vee) = \sum_{v \in V} f(v) \exp(-2\pi i (x^\vee, v)) \]

The following lemma is clear:

**Lemma 2.3.** Let \( F \in \Sky(X) \). Then \( \text{Four}_F : \mathcal{S}(\bar{F}) \to C^\infty(\text{Four}(F)) \) is an isomorphism. Here \( C^\infty(\text{Four}(F)) \) is the space of \( C^\infty \)-sections of the local system \( \text{Four}(F) \).

2.4. Convolution. For \( F_1, F_2 \in \Sky(X) \), one can define their convolution product by \( F_1 \star F_2 := m_*((p_1^* F_1) \otimes (p_2^* F_2)) \), where \( m, p_1, p_2 : X \times X \to X \) are the group law, the first projection, and the second projection respectively. This gives a structure of a tensor category on \( \Sky(X) \) (the unit, dual element, and commutativity and associativity isomorphisms are easily defined). Then \( \text{Four} : \Sky(X) \to \Loc_n(X^\vee) \) is naturally a tensor functor (the tensor structure on \( \Loc_n(X^\vee) \) is the “usual” tensor product). Moreover, \( \text{Four}(F_1 \star F_2) = \text{Four}(F_1) \otimes \text{Four}(F_2) \) for any \( F_1, F_2 \in \Sky(X) \).

Besides, it is easy to define the natural convolution product \( S(\hat{\star}) : S(\bar{F}_1) \otimes S(\bar{F}_2) \to S((\bar{F}_1 \star \bar{F}_2)) \). This makes \( S(\hat{\bullet}) \) a tensor functor. One can check that \( \text{Four}_\bullet : S(\hat{\bullet}) \to C^\infty(\text{Four}(\hat{\bullet})) \) is actually an isomorphism of tensor functors (i.e., for any \( F_1, F_2 \in \Sky(X) \) the diagram

\[
\begin{array}{ccc}
S(\bar{F}_1) \otimes S(\bar{F}_2) & \xrightarrow{\phi} & C^\infty(\text{Four}(F_1)) \otimes C^\infty(\text{Four}(F_2)) \\
\downarrow & & \downarrow \\
S(\bar{F}_1 \star \bar{F}_2) & \xrightarrow{\phi} & C^\infty(\text{Four}(F_1) \otimes \text{Four}(F_2))
\end{array}
\]

commutes).

**Example 2.4.** Let \( F \) be the unit object in \( \Sky(X) \) (i.e., \( \text{supp} F = \{0\} \) and \( F_0 = \mathbb{C} \)). Then \( \bar{F} \) is a trivial sheaf on \( \Gamma = H^1(X, \mathbb{Z}) \). Clearly, \( \text{Four}(F) = O_X \) is the trivial local system on \( X \). In this case, the isomorphism \( (\text{Four}_F)^{-1} : C^\infty(X^\vee) \to \mathcal{S}(F) \) maps any \( C^\infty \)-function to its Fourier coefficients. Since \( F \star F = F \), the commutativity of (2.3) in this case is the well-known formula for the Fourier coefficients of the product.

3. Relative sky-scraper sheaves

Let \( p : M \to B \) be a symplectic family of tori. In this section, we construct “relative versions” of the objects from the previous section.

3.1. A transversally immersed Lagrangian manifold is a couple \((L, i)\), where \( i : L \to M \) is a morphism of \( C^\infty \)-manifolds such that \( p \circ i : L \to B \) is a proper finite unramified covering and \( i^*(\omega) = 0 \).

Consider the category \( \Sky(M/B) \), whose objects are triples \((L, i, \mathcal{L})\), where \((L, i)\) is a transversally immersed Lagrangian submanifold, and \( \mathcal{L} \) is a local system on \( L \).

**Remark 7.** Take any \((L_1, i_1, \mathcal{L}_1), (L_2, i_2, \mathcal{L}_2) \in \Sky(M/B)\). Consider \( L'_1 \to L'_2 := L_1 \times_M L_2 \). Denote by \( L_{1 \to 2} \subset L'_{1 \to 2} \) the maximal closed submanifold whose images in \( L_1, L_2 \) are open (if \( L_1 \) and \( L_2 \) are just “usual” Lagrangian submanifolds, \( L_{1 \to 2} \) is the union of common connected components of \( L_1 \) and \( L_2 \)). Let
\( p_1 : L_1 \to L_1, \, p_2 : L_1 \to L_2 \) be the natural projections. By definition, morphisms from \((L_1, i_1, L_1)\) to \((L_2, i_2, L_2)\) are horizontal morphisms \( p_1^i \mathcal{L}_1 \to p_2^i \mathcal{L}_2 \). The composition is defined in the natural way.

3.2. Let \( p^\vee : M^\vee \to B \) be the mirror dual of \( M \to B \). Take \((L, i, \mathcal{L}) \in \text{Sky}(M/B)\). One can easily define the (relative) Poincaré bundle \( \mathcal{P} \) on \( M \times B \). It carries a natural connection \( \nabla_{\mathcal{P}} \). Now define the (fiberwise) Fourier transform \( \text{Four}(L, i, \mathcal{L}) \) by the formula \( (1.1) \).

**Proof of Theorem** \( 3.1 \) (i). The natural map \( L \times B \to M^\vee \) is an unramified covering, so the complex structure on \( M^\vee \) induces a complex structure on \( L \times B \). Let \( (\mathcal{P}_B, \nabla_{\mathcal{P}_B}) \) be the Poincaré bundle on \( M \). It is enough to prove \( \text{curv}(\nabla_{\mathcal{P}_B}) \) vanishes on \( T_{L \times B}^{0,1} \).

The statement is local on \( B \), so we may assume \( M = X \times B \), \( M^\vee = X^\vee \times B \) for a torus \( X \). Denote by \( p_{X \times X^\vee} : X \times B \to X \) the natural projection, and by \( (\mathcal{P}_X, \nabla_{\mathcal{P}_X}) \) the Poincaré bundle of \( X \). \( p_{X \times X^\vee}^*(\mathcal{P}_X) = (\mathcal{P}_M, \nabla_{\mathcal{P}_M}) \), so \( \text{curv}(\nabla_{\mathcal{P}_M}) = p_{X \times X^\vee}^* \text{curv}(\nabla_{\mathcal{P}_X}) \). Since \( \text{curv}(\nabla_{\mathcal{P}_X}) \) is a scalar multiple of the natural symplectic form on \( X \times X^\vee \), it is enough to notice that \( p_{X \times X^\vee}^* \) maps \( T_{L \times B}^{0,1}(x) \) to a Lagrangian subspace of \( T_{X \times X^\vee}(x) \) for any \( x \in L \times B \).

3.3. **Proof of Theorem** \( 3.2 \). Consider the “fiberwise universal cover” \( u : T_B^\vee \to M \). For any \((L, i, \mathcal{L}) \in \text{Sky}(M/B)\), set \( \tilde{L} := L \times_{M} T_B^\vee \). Recall that \( \tilde{L} = u^*_L(L) \otimes O_{\mathcal{L}}(-2\pi i) \), where \( u_L : \tilde{L} \to L \) is the natural homomorphism and \( \tau \) is the pull-back of the natural 1-form on \( T_B^\vee \).

For any \( D \in \text{Diff}(B) \), we consider its pull-back \( \tilde{p}^*D \in \text{Diff}(\tilde{L}) \) (since \( \tilde{p} : \tilde{L} \to B \) is an unramified covering, the pull-back is well defined). Since \( \tilde{L} \) carries a canonical flat connection, we can apply \( \tilde{p}^*D \) to \( s \in \mathcal{C}^\infty(\tilde{L}) \). Denote by \( \mathcal{S}(\tilde{L}) \) the set of all sections \( s \in \mathcal{C}^\infty(\tilde{L}) \) such that \( (\tilde{p}^*D)s \) is (fiberwise) rapidly decreasing for any \( D \in \text{Diff}(B) \). \( \mathcal{S}(\tilde{L}) \) is a \( \text{Diff}(B) \)-module.

Just like in the “absolute” case (Lemma 2.3), the Fourier transform (formula \( 2.2 \)) yields a canonical isomorphism \( \mathcal{S}(\tilde{L}) \to \mathcal{C}^\infty(\text{Four}(L, i, \mathcal{L})) \) for any \( L, i, \mathcal{L} \).

The natural morphism \( (dp^\vee) \otimes \mathbb{C} : T_{M^\vee} \otimes \mathbb{C} \to (p^\vee)^*T_B \otimes \mathbb{C} \) induces an isomorphism \( T_{M^\vee}^{0,1} \to (p^\vee)^*T_B \otimes \mathbb{C} \). So we have an embedding of Lie algebras \( \Omega^{-1}(B) \to \mathcal{C}^\infty(T_{M^\vee}^{0,1}) \subset \Omega^{-1}(M^\vee) \). The Lie algebra \( \mathcal{C}^\infty(T_{M^\vee}^{0,1}) \) of anti-holomorphic vector fields acts on \( \mathcal{C}^\infty(\text{Four}(L, i, \mathcal{L})) \) (by Theorem \( 3.1 \) (i)), so \( \mathcal{C}^\infty(\text{Four}(L, i, \mathcal{L})) \) has a natural structure of a \( \text{Diff}(B) \)-module. One easily checks that the de Rham complex associated with this \( \text{Diff}(B) \)-module is identified with the Dolbeault complex of \( \text{Four}(L, i, \mathcal{L}) \).

The following lemma implies Theorem \( 3.1 \).

**Lemma 3.1.** The isomorphism \( \mathcal{S}(\tilde{L}) \to \mathcal{C}^\infty(\text{Four}(L, i, \mathcal{L})) \) agrees with the structures of \( \text{Diff}(B) \)-modules.

**Proof.** Again, we may assume \( M = B \times X \) for a torus \( X = V/\Gamma \). Consider the natural maps \( p_{V \times X^\vee} : \tilde{L} \times B \to M^\vee \to V \times X^\vee, \) \( p_M : \tilde{L} \times B \to M \), and \( p_{T_B^\vee} : L \times B \to T_B^\vee \times B \to T_B^\vee \).

\( \text{Exp}(-2\pi \sqrt{-1}(dx^\vee, v)) \) can be considered as a horizontal section of \( p_{T_B^\vee}^* (\mathcal{P}_M) \otimes p_{V \times X^\vee}^* (O_{V \times X^\vee}(-2\pi \sqrt{-1}(dx^\vee, v))) \). Now the statement follows from the fact that \( 1 \) is a holomorphic section of \( O_{\tilde{L} \times B \times V^\vee}(-2\pi p_{T_B^\vee}^* \tau - 2\pi \sqrt{-1}p_{V \times X^\vee}^* dx^\vee, v) \) (i.e., the \( \tilde{\mathcal{O}} \)).
component of the connection vanishes on $1$). Here $\tau$ stands for the natural 1-form on $T^*_B$, and the complex structure on $\tilde{L} \times_B M^\vee$ is that induced by $\tilde{L} \times_B M^\vee \to M^\vee$. \hfill \Box

### 3.4. Proof of Theorem 1.1(ii)
This result is actually proved in [15]. Our proof is slightly different in that it makes use of connections.

Let $F$ be a holomorphic bundle on the elliptic curve $M^\vee$. It is enough to consider the case of indecomposable $F$.

The following statement is a reformulation of [15, Proposition 1] (which in turn is a consequence of M. Atiyah’s results [1]):

**Proposition 3.2.** An indecomposable bundle $F$ on $M^\vee$ is isomorphic to $\pi_{r,!*}(L \otimes N)$, where $\pi_r : M^\vee \to M^\vee$ is the isogeny corresponding to an (unramified) cover $B_r \to B$, $L$ is a line bundle on $M^\vee$, and $N$ is a unipotent bundle on $M^\vee$ (i.e., $N$ admits a filtration with trivial factors).

**Four** agrees with passing to unramified covers $B_r \to B$, besides, **Four** transforms the convolution product in $\text{Sky}(M/B)$ to the tensor product of holomorphic vector bundles (see Section 3.3 for the definition of the convolution product). So it suffices to consider the following cases:

**Case 1.** Let $F = l$ be a line bundle on $M^\vee$. Our statement in this case follows from the following easy lemma:

**Lemma 3.3.** $l$ carries a $C^\infty$-connection $\nabla_l$ such that the following conditions are satisfied:

i) $\nabla_l$ agrees with the holomorphic structure on $l$ (i.e., the $\bar{\partial}$-component of $\nabla_l$ coincide with the canonical $\bar{\partial}$-differential);

ii) The curvature $\text{curv} \nabla_l$ is a horizontal (1,1)-form on $M^\vee$ (in terms of the canonical connection);

iii) The monodromies of $\nabla_l$ along the fibers of $M^\vee \to B$ are unitary.

**Case 2.** Let $F = N$ be a unipotent bundle on $M^\vee$. To complete the proof, it is enough to notice that $N$ carries a flat connection $\nabla_N$ such that $\nabla_N$ agrees with the holomorphic structure and $\nabla_N$ is trivial along the fibers of $M^\vee \to B$.

### 3.5. Remarks on tensor dg-categories
For any $(L_1, i_1, L_1), (L_2, i_2, L_2) \in \text{Sky}(M/B)$, set $L := L_1 \times_B L_2$, $L := p_1^*(L_1) \otimes p_2^*(L_2)$ (here $p_i : L \to L_i$ is the natural projection). Consider the composition $i := m \circ (i_1 \times i_2) : L_1 \times_B L_2 \to M \times_B M \to M$, where $m : M \times_B M \to M$ is the group law $(x_1, x_2) \mapsto x_1 + x_2$. Clearly, $(L, i, L) \in \text{Sky}(M/B)$. $(L_1, i_1, L_1) \star (L_2, i_2, L_2) := (L, i, L)$ is the convolution product of $(L_1, i_1, L_1)$ and $(L_2, i_2, L_2)$. The convolution product naturally extends to a structure of tensor category on $\text{Sky}(M/B)$ (the unit object, dual objects, and associativity/commutativity constraints are defined in a natural way). Notice that there is a functorial choice of dual object.

Just as in Section 3.4, the convolution product induces a functorial morphism of $\text{Diff}(B)$-modules $S(L_1) \otimes S(L_2) \to S(L_3)$ for any $(L_1, i_1, L_1), (L_2, i_2, L_2) \in \text{Sky}(M/B)$. $(L_1, i_1, L_1) \star (L_2, i_2, L_2) := (L_1, i_1, L_1) \star (L_2, i_2, L_2)$. So $S(\star)$ is a tensor functor from $\text{Sky}(M/B)$ to the category of $\text{Diff}(B)$-modules.

Just as we say in Section 4.4, we define a tensor dg-category $\text{Sky}(M/B)$ by setting $\text{Ob}(\text{Sky}(M/B)) := \text{Ob}(\text{Sky}(M/B))$, $\text{Hom}_{\text{Sky}(M/B)}(c_1, c_2) := \text{DR}(c_1 \star c_2^\vee)$. 


4. Connection with the Fukaya category

4.1. Hamiltonian diffeomorphisms. In this section, we prove some results about tensor dg-category $\text{Sky}(M/B)$. We do not use these facts anywhere, so the part may be skipped. However, the results give some clarification to the connection between $\text{Sky}(M/B)$ and the original category considered by Fukaya [4].

Fix $\mu \in \Omega^1(B)$ such that $d\mu = 0$. $\mu$ can be considered as a section of $T_B^\vee$. Denote by $i_\mu : B \to M$ the image of this section via the fiberwise universal cover $T_B^\vee \to M$. Set $c_\mu := (B, i_\mu, O_B(2\pi \mu)) \in \text{Sky}(M/B)$. The following statement follows from the definitions:

**Proposition 4.1.** $c_\mu \simeq 1_{\text{Sky}(M/B)}$ in $\text{Sky}(M/B)$.

Now let $A : M \to M$ be any symplectic diffeomorphism that preserves the fibration $M \to B$. It is easy to see that $A$ preserves the action of $T_B^\vee$ on $M$, so $A$ corresponds to some $\mu \in \Omega^1(B)$. Since $A$ preserves the symplectic structure, $d\mu = 0$. Now we can consider the “automorphism” $c \mapsto c_\mu \ast c$ of $\text{Sky}(M/B)$. Note that if $(L', i_{L'}, \mathcal{L}') = (L, i_L, \mathcal{L}) \ast c_\mu$, then $i_{L'}(L') = A(i_L(L))$.

In particular, if $A$ is Hamiltonian (that is, there is $f \in C^\infty(B)$ such that $\mu = df$), we get the following statement:

**Corollary 4.2.** The map $(L, i_L, \mathcal{L}) \mapsto (L, A \circ i_L, \mathcal{L})$ extends to an automorphism of $\text{Sky}(M/B)$.

**Proof.** It is enough to notice that $O_B(2\pi \mu)$ is a trivial local system if $\mu = df$, so $c_\mu \simeq (B, i_\mu, O_B)$ and $(L, i_L, \mathcal{L}) \ast (B, i_\mu, O_B) = (L, A \circ i_L, \mathcal{L})$ for any $(L, i_L, \mathcal{L}) \in \text{Sky}(M/B)$.

From now on, we suppose that $B$ is a torus.

Denote by $\text{Sky}(M/B)^{QU}$ the full subcategory of $\widetilde{\text{Sky}}(M/B)$ formed by triples $(L, i, \mathcal{L})$ with quasi-unitary $\mathcal{L}$ (that is, all the eigenvalues of all the monodromy operators are of absolute value 1).

**Lemma 4.3.** The natural embedding $\widetilde{\text{Sky}}(M/B)^{QU} \to \text{Sky}(M/B)$ is an equivalence of categories.

**Proof.** We should prove that for any $(L, i, \mathcal{L}) \in \widetilde{\text{Sky}}(M/B)$ there is $(L', i', \mathcal{L}') \in \text{Sky}^{QU}(M/B)$ such that $(L, i, \mathcal{L}) \simeq (L', i', \mathcal{L}')$ in $\text{Sky}(M/B)$. It is enough to prove this statement for indecomposable $\mathcal{L}$ and connected $L$.

Choose a point $x \in L$. For $\gamma \in \pi_1(L)$, we denote the monodromy along $\gamma$ by $\text{mon}(\gamma) \in GL(L_x)$. For any loop $\gamma \in \pi_1(L)$, all the eigenvalues of $\text{mon}(\gamma)$ are of the same absolute value (otherwise $L$ is decomposable).

Consider the homomorphism $\mu : \pi_1(L) \to \mathbb{R}_+ := \{a \in \mathbb{R} | a > 0\}$ : $\gamma \mapsto |\det(\text{mon}(\gamma))|^{1/d}$. Since $L \to B$ is a finite covering, $\pi_1(L) \subset H_1(B, \mathbb{Z}) \subset H_1(B, \mathbb{R})$ is a lattice. So $\mu$ induces log $\mu \in \text{Hom}(\pi_1(L), \mathbb{R}) = H^1(B, \mathbb{R})$.

Choose an invariant 1-form $\tilde{\mu}$ on $B$ that represents $-\frac{\log \mu}{2\pi} \in H^1(B, \mathbb{R})$. Clearly, $(L, i, \mathcal{L}) \ast c_\mu \in \text{Sky}(M/B)^{QU}$.

**Remark 8.** Suppose $M$ and $B$ are tori (in particular, they have a Lie group structure), and $p : M \to B$ is a group homomorphism. Assume also $\omega$ is translation invariant. Let $i : L \to M$ be a transversally immersed Lagrangian submanifold.
We say that \((L, i)\) is linear if for any connected component \(L_j \subset L\), one has \(i(L_j) = m + L'\) for some \(m \in M\) and some Lie subgroup \(L' \subset M\). Consider the full subcategory \(\widetilde{\text{Sky}}(M/B)^{L_N} \subset \widetilde{\text{Sky}}(M/B)\) that consists of \((L, i, \mathcal{L})\) such that \((L, i)\) is linear and \(\mathcal{L}\) is quasi-unitary. It can be proved (in a way similar to the proof of Lemma 4.3) that \(\widetilde{\text{Sky}}(M/B)^{L_N} \to \widetilde{\text{Sky}}(M/B)\) is an equivalence of categories.

4.2. Proof of Theorem 1.3. In this section we give a different construction of the quasi-isomorphism between \(F(\mathcal{L})\) and \(\text{DR}(\mathcal{L})\). Identification of this quasi-isomorphism with that constructed in Section 1.3 is left to reader.

Consider the de Rham complex \(\text{DR}(\mathcal{L}) := \mathcal{L}(\mathcal{L}) \to \mathcal{S}(\mathcal{L}) \otimes_{C^\infty(B)} \Omega^1(B)\). Recall that \(\{\tilde{c}_j^\perp\} \subset \tilde{L}\) is the set of all “negative” points whose images lie on the zero section \(0_{T^*_B}(B) \subset T^*_B\).

Denote by \(F(0)\) the set of all \(f \in \mathcal{S}(\tilde{L})\) such that \(f\) is horizontal in a neighborhood of \(\{\tilde{c}_j^\perp\}\) and by \(F(1)\) the set of all \(\mu \in \mathcal{S}(\tilde{L}) \otimes_{C^\infty(B)} \Omega^1(B)\) such that \(\mu\) vanishes in a neighborhood of \(\{\tilde{c}_j^\perp\}\). Since \(d(F(0)) \subset F(1)\), we have a complex \(\text{DR}'(\mathcal{L}) : F(0) \to F(1)\). Moreover, the natural map \(\text{DR}'(\mathcal{L}) \to \text{DR}(\mathcal{L})\) is a quasi-isomorphism.

Now let \(\tilde{L}_j \subset \tilde{L}\) be a connected component of \(\tilde{L} \setminus \{\tilde{c}_j^\perp\}\). Set \(\mathcal{S}(\tilde{L}_j) := \{s \in \mathcal{S}(\tilde{L}) : \text{supp } s \subset \tilde{L}_j\}\), \(F_j(1) := \mathcal{S}(\tilde{L}_j) \otimes_{C^\infty(B)} \Omega^1(B)\). Denote by \(F_j^\prime\) the set of all sections \(s \in C^\infty(\tilde{L}_j, B)\) such that \(\text{supp } s\) is contained in a compact set \(C \subset L\) and \(s\) is horizontal in some neighborhood of \(\{\tilde{x}_j^\perp\}\). Set \(F_j(0) := \mathcal{S}(\tilde{L}_j) + F_j^\prime\). Clearly \(d\) yields a morphism \(d_j : F_j(0) \to F_j(1)\), so we have a complex \(\text{DR}(\mathcal{L}) : F_j(0) \to F_j(1)\).

The restriction map induces a morphism of complexes \(\text{DR}'(\mathcal{L}) \to \text{DR}(\mathcal{L})\). So we have a map \(\text{DR}'(\mathcal{L}) \to \oplus_j \text{DR}(\mathcal{L})_j\). Moreover, one can see that this map is included into a short exact sequence

\[
0 \to \text{DR}'(\mathcal{L}) \to \oplus_j \text{DR}(\mathcal{L})_j \to F^1 \to 0
\]

(see Section 1.3 for the definition of \(F^1\)).

Proposition 4.4. i) The map \(d_j : F_j(0) \to F_j(1)\) is surjective for any \(j\); 
ii) If the image of \(\tilde{L}_j \subset \tilde{L}\) does not intersect \(0_{T^*_B}(B) \subset T^*_B\), then \(d_j\) is injective; 
iii) Suppose the image of \(\tilde{c}_k^\perp \in \tilde{L}_j\) lies on \(0_{T^*_B}(B) \subset T^*_B\). Set \(F''_j := \ker d_j\). Then the map \(F''_j \to \tilde{L}_j \to \tilde{L}_j^\perp : s \mapsto s(\tilde{c}_k^\perp)\) is injective.

Proof. Choose an isomorphism \(t : B \to B \oplus \mathbb{R}/\mathbb{Z}\). We may assume that \(t\) agrees with the natural connection on \(T_B\). There are three possibilities:

Case 1: \(M \to B \to B \oplus \mathbb{R}/\mathbb{Z}\) induces an isomorphism \(t : \tilde{L}_j \to B \oplus \mathbb{R}/\mathbb{Z}\). In this case, the image of \(\tilde{L}_j\) does not intersect \(0_{T^*_B}(B) \subset T^*_B\). It is easy to see that the monodromy of \(\tilde{L}_j|_{\tilde{L}_j}\) does not have 1 as its eigenvalue, hence the de Rham cohomology groups of \(\tilde{L}_j|_{\tilde{L}_j}\) vanish. So \(d_j\) is injective and i), ii) follow.

Case 2: \(M \to B \to B \oplus \mathbb{R}/\mathbb{Z}\) induces an isomorphism \(t : \tilde{L}_j \to (t_1, t_2) := \{t \in \mathbb{R} : t_1 < t < t_2\}\) for some \(t_1, t_2 \in \mathbb{R}\).
In this case, there is a unique $\tilde{C}_k^j \in \tilde{L}_j$ whose image lies on $0t_\beta^j \subset T_\beta^\vee$. Besides, $F^{(1)}_j = H^0_c(L_j, \mathcal{E}|_{L_j} \otimes_{C=\tilde{L}_j} \Omega^1(L_j))$ (here $H^0_c$ stands for the space of sections with compact support). Now i) and ii) are obvious.

**Case 3:** $M \to B \to \mathbb{R}/\mathbb{Z}$ induces an isomorphism $\tilde{L}_j \tilde{\cong} (t_1, t_2)$ where either $t_1 = -\infty$, or $t_2 = \infty$ (or both). Without loss of generality, we assume $t_1 = -\infty$.

Denote by $\tau \in \Omega^1(L_j)$ the pull-back of the natural 1-form on $T_B$. It is easy to see there are (unique) $a, b \in \mathbb{R}$ such that $\tau_0 := -2\pi \tau - (ta + b)dt$ is “bounded” in the following sense: there is $C \in \mathbb{R}$ such that for any connected closed subset $U \subset \tilde{L}_j$ we have

\[
\int_U \tau_0 < C
\]

Choose an isomorphism $\phi : \mathcal{E}|_{L_j} \tilde{\cong} (O_L, ((at + b)dt)^n)$ (where $n$ is the dimension of $\mathcal{E}|_{L_j}$). Set $\frac{df}{dt} := \frac{df}{dt} + at + b$. Denote by $\hat{S}(t_1, t_2)$ the space of $f \in C^\infty(t_1, t_2)$ such that $\lim_{t \to \infty} x^l \frac{df}{dt} = 0$ for any $k, l \geq 0$. If $t_2 < \infty$, we denote by $S^0(t_1, t_2) \subset \hat{S}(t_1, t_2)$ (resp. $S^1(t_1, t_2) \subset \hat{S}(t_1, t_2)$) the subspace of functions $f$ such that $\frac{df}{dt} = 0$ (resp. $f = 0$) in a neighborhood of $t_2$. If $t_2 = \infty$, we set $S^0(t_1, t_2) = S^1(t_1, t_2) = \hat{S}(t_1, t_2)$. \[4.2\] implies that $\phi$ induces isomorphisms $F^{(0)}_j \tilde{\cong} (S^0(t_1, t_2))^n$, $F^{(1)}_j \tilde{\cong} (S^1(t_1, t_2))^n$. The differential $d_j$ corresponds to $\frac{d}{dt} dt$.

There are two possibilities:

**Case 3a:** $a > 0$, the image of $\tilde{L}_j$ intersects $0t_\beta^j \subset T_\beta^\vee$ in exactly one point. Without loss of generality we may assume this point corresponds to $t = 0$. Now for any $g \in (S^1(t_1, t_2))^n$, a generic solution to $\frac{df}{dt} = g$ is given by

\[
f(x) = \exp(-(ax^2/2 + bx))(\int_0^x g(t) \exp(at^2/2 + bt)dt + C)
\]

where $C \in \mathbb{C}^n$. It is easy to see $f \in (S^0(t_1, t_2))^n$ for any $C$. i) and ii) follow.

**Case 3b:** $a < 0$, the image of $\tilde{L}_j$ does not meet the zero section, $t_2 < \infty$. For any $g \in (S^1(t_1, t_2))^n$ the formula

\[
f(x) = \exp(-(ax^2/2 + bx))(\int_{-\infty}^x g(t) \exp(at^2/2 + bt)dt + C)
\]

gives a generic solution to $\frac{df}{dt} = g$ ($C \in \mathbb{C}^n$). However, $f \in (S^0(t_1, t_2))^n$ if and only if $C = 0$. This implies i) and iiia).

Hence $H^1(\oplus_j \mathcal{D}R(L_j)) = \left\{ \begin{array}{ll} F^0_j, & j = 0 \\ 0, & \text{otherwise} \end{array} \right.$ . To complete the proof, it is enough to notice that the map $F^0 = H^0(\oplus_j \mathcal{D}R(L_j)) \to F^1$ induced by \[4.3\] coincides with that defined in Section 1.3.

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