Recent progress in the study of solitons and black holes in non-Abelian field theories coupled to gravity is reviewed. New topics include gravitational binding of monopoles, black holes with non-trivial topology, Lue-Weinberg bifurcation, asymptotically AdS lumps, solutions to the Freedman-Schwarz model with applications to holography, non-Abelian Born-Infeld solutions.

1 Introduction

During the past decade many surprising phenomena were discovered in the study of gravitating solitons and black holes involving Yang-Mills (YM) fields. A key role in this development has been played by gravitational sphalerons, known as Bartnik-McKinnon (BK) particle-like solutions to the Einstein-Yang-Mills (EYM) equations. The associated black holes, violating no-hair and uniqueness theorems, attracted much attention and helped to clarify a number of beliefs persistent in the Einstein theory. Other conceptually important results were obtained in the study of self-gravitating magnetic monopoles in gauge theories with spontaneous symmetry breaking. Meanwhile, for many years this research basically served only as a theoretical laboratory to study gravity in unusual conditions. Direct physical applications were unknown until the recent progress in superstring theory culminated in the discovery of D-branes and the related gauge theory/supergravity correspondence. It turned out that some gravitating lumps have non-trivial ten-dimensional interpretation and are extremely useful in the holographic treatment of the $\mathcal{N} = 1$ supersymmetric gauge theory. The goal of this talk is to give a brief overview of the subject with an emphasis on the results obtained during the last three years. A review of the work done before 1998 was given in where we address the reader for references prior to November 1998, citing only names and years.

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aThis is a written version of the talk given at the 16th International Conference on General Relativity and Gravitation, held on July 15-21, 2001, in Durban, South Africa
Neither pure gravity, nor flat space Yang-Mills (YM) theory admit static particle-like solutions. Gravity is purely attractive, and no static equilibrium is possible, unless some repulsive matter is introduced. On the contrary, the YM field is repulsive: due to scale invariance, the trace of the energy momentum tensor is zero, so the positivity of the energy density implies the positivity of the sum of principal pressures. Gravity breaks scale invariance, therefore the coupled Einstein-Yang-Mills (EYM) theory is expected to contain particle-like solutions. There is also a topological argument similar to Manton-Taub explanation of the existence of the sphaleron in the Weinberg-Salam theory. In that theory the Higgs broken phase manifold is $S^3$. Non-triviality of the third homotopy group $\pi_3(S^3)$ is an indication of the presence of non-contractible loops in the configuration space for asymptotically flat solutions, and, hence, saddle points on the energy surface. The corresponding unstable solution is the sphaleron; it sits on the top of the potential barrier separating topologically distinct Yang–Mills vacua. It was argued (Gal’tsov and Volkov (1991), Sudarsky and Wald (1992)) that the same argument can be applied to the gauge group itself, in fact $SU(2) \sim S^3$, once Higgs is replaced by another attractive field of non-topological nature, like gravity or dilaton.

Historically, particle-like $SU(2)$ EYM solutions were discovered numerically by Bartnik and McKinnon (1988) before this sphaleron interpretation was suggested. The metric is spherically symmetric and is parameterized by two functions $N, \sigma$ of a radial variable:

$$ds^2 = N \sigma^2 dt^2 - \frac{dr^2}{N} - r^2 d\Omega,$$

while the YM field is purely magnetic: $A = (1-w)(T_\theta \sin \theta d\varphi - T_\varphi d\theta)$, where $w$ is a real function of $r$ and $T_\theta, T_\varphi$ are spherical projections of the SU(2) generators. This is a configuration with the unit winding number. Higher winding numbers lead to non-spherical solutions. An embedded Abelian magnetic monopole with unit magnetic charge is $w \equiv 0$, while $w = \pm 1$ correspond to the neighboring topologically distinct YM vacua. Particle-like solutions are characterized by regularity at the origin

$$w = 1 - br^2 + O(r^4), \quad N = 1 - 4b^2 r^2 + O(r^4),$$

and asymptotic flatness: $N \to 1 - 2M/r$ with finite mass $M$, together with $\sigma \to 1$. This implies that the YM field asymptotically stays in one of the vacua

$$w = \pm(1 - a/r) + O(1/r^2).$$
Parameters $b$, $M$, $a$ can be determined via numerical matching. Solutions exist for a discrete increasing sequence $b_n$ on the semi-open interval $[b_1 = 0.4537, b_\infty = 0.706)$, with masses $M_n$ converging to unity as $n \to \infty$ in the units $m_{Pl}/g$, where $g$ is the gauge coupling constant. The function $w_n$ has $n$ zeroes and tends to $(-1)^n$, solutions exist for any finite $n$. The limiting solution, corresponding to $b_\infty$, is asymptotically non-flat.

For the lowest $n = 1$ BK solution, $w$ is a decreasing function varying between $w = 1$ and $w = -1$ in exactly the same way, as in the electroweak sphaleron. Odd-$n$ solutions all have sphaleron nature, while even-$n$ ones are topologically trivial. A vacuum to vacuum path was constructed explicitly (Gal’tsov and Volkov (1991)), showing that the Chern-Simons number is $1/2$ for odd-$n$ solutions and zero for even $n$. Sphaleron interpretation is supported by the fact that the $n$-th solution has exactly $n$ odd-parity negative modes, corresponding to excitations in the electric sector. In addition, there are also $n$ even-parity negative modes exhibiting gravitational instability of BK solutions. Spherically symmetric magnetic configurations of the SU(N) YM field are described by $N - 1$ real functions, the corresponding BK-type solutions have a nodal structure determined by the set of $N - 1$ integers (Kleihaus, Kunz and Sood (1995-1996)).

Magnetic configurations with the winding number $\nu$ could be obtained by replacing $\varphi \to \nu \varphi$ in the YM ansatz, but from the EYM equations it follows that there are no $\nu \neq 1$ spherically symmetric solutions other than embedded Abelian. Essentially non-Abelian configurations with $\nu \geq 2$ can be axially symmetric, in which case purely magnetic field is parameterized by four real functions subject to one gauge condition. Static axially symmetric regular solutions of the BK type were found numerically for lower $\nu = 2, 3, 4$ (Kleihaus and Kunz (1997)), they are likely to exist for all $\nu$. The SU(2) solutions are characterized by a node number $n$, their masses increase both with growing $\nu$ and $n$, approaching $M = \nu$ for $n \to \infty$. The metric is parameterized by three functions entering the Papapetrou ansatz; they exhibit toroidal or spheroidal shape of the equipotential surfaces. These solutions can be interpreted as superposition of BK particles aligned along $z$-axis, their Chern-Simons number is $\nu/2$. It remains unknown, whether static asymptotically flat EYM solutions exist without axial symmetry, like multimonopoles with $\nu \geq 3$.

3 Gravitating monopoles

3.1 Gravitational excitations

Magnetic monopole in the Georgy-Glashow model possesses a mass $M_{mon} \sim \nu/g$, where $\nu$ is the vacuum expectation value of Higgs, and a radius $R_{mon} \sim$
When the monopole gravitational radius \( R_G \sim GM_{\text{mon}} \) approaches \( R_{\text{mon}} \), one can expect new phenomena to occur. With this motivation, Lee, Nair and Weinberg (1992), Ortiz (1992), Breitenlohner, Forgacs and Maison (1992, 1995), see also, have investigated the coupled EYM-Higgs (EYMH) static spherically symmetric system and found that the existence of the maximal value of the dimensionless parameter \( \alpha = \sqrt{Gv} \), which is proportional to the ratio of the mass of \( W \)-boson in this theory to the Planck’s mass, such that self-gravitating monopoles cease to exist for \( \alpha > \alpha_{\text{cr}} \). This is not surprising, since for \( \alpha \sim 1 \) the monopole radius and its gravitational radius become comparable, so for large \( \alpha \) the monopole would be inside the event horizon.

An important new feature is the possibility of gravitational excitation of monopoles. Note that the YM function \( w \) in the monopole solution starts with \( w = 1 \) at the origin and monotonically tends to zero at infinity. There are other solutions in which \( w \) oscillates \( n \) times around zero before to enter the asymptotic regime, these correspond to higher values of the parameter \( b \) in (2), increasing with \( n \), for each \( \alpha \). On the phase diagram \((b, \alpha)\) one observes the lower branch extending up to \( \alpha_{\text{cr}} \) (ground state monopole) where it bifurcates giving rise to the upper branch (excited monopole with one \( w \)-node). This upper branch bifurcates in its turn giving rise to the next \( n = 2 \) branch and so on. All upper branches in the limit \( \alpha \to 0 \) tend to BK\( n \) EYM solutions after a suitable rescaling. Solutions belonging to the main branch are stable, like flat-space monopoles, while BK-excited monopoles are unstable, like BK particles. Excited monopoles can be regarded as bound states of monopoles with BK particles inside. Solutions near the main bifurcation point were recently investigated in detail by Lue and Weinberg, revealing new surprising features. Other recent work on gravitating monopoles includes the study of dyons, SU(3) monopoles, EYMH solutions with non-minimal gravitational coupling, YMH coupled to Brans-Dicke gravity.

### 3.2 Gravitational binding of monopoles

Like monopoles at rest in the flat space-time do not interact in the BPS limit where the Higgs field is massless, and a scalar attraction is precisely compensated by a magnetic repulsion. Away from the BPS bound, the Higgs field is massive, and the scalar force falls off exponentially, so magnetic repulsion dominates, and the energy of multimonopole solutions per winding number is always larger than the energy of the spherical \( \nu = 1 \) monopole. For \( \nu \geq 3 \) the BPS multimonopoles with no axial symmetry also exist.

It was found that gravity can bind monopoles provided the Higgs self-interaction constant \( \lambda \) is small enough. It was verified that the energy per
unit winding number for $\nu = 2, 3$ aligned gravitating monopoles is smaller than the energy of a single $\nu = 1$ monopole. In the case of monopole-antimonopole pairs, gravity enlarges the space of solutions: beside the ground state pair solution, there are BK-excitations which bifurcate at critical $\alpha$. The upper branch in the limit of vanishing $\alpha$ correspond to the (rescaled) BK$_1$ solution, which is spherically symmetric. An excited monopole-antimonopole pair linking to BK$_2$ was also found, this is conjectured to be valid for all $n$.

Other flat-space particle-like solutions include sphalerons in the Weinberg-Salam model and Skyrmions. Similarly to monopoles, gravitating sphalerons exist up to finite $\alpha$ and link to the BK-excitations (the bifurcation pattern is somewhat different from that in monopoles, for details see). When Higgs self-interaction constant increases, new sphaleron solutions, currently called bi-sphalerons, bifurcate from the main branch (Kunz and Brihaye (1989), Yaffe (1990)). Their response to gravity was studied in. For Skyrmions the self-gravitating problem was studied by Luckock (1987) well before the corresponding monopole problem was investigated. More detailed study was undertaken by Heusler, Droz and Straumann (1991) and Bizon and Chmaj (1992). The moduli space again exhibits bifurcations with BK-excited branches.

4 Hairy black holes

4.1 Violation of black hole rules

Soon after discovery of BK solutions, it was shown that there also exist black hole solutions with similar structure of fields outside the event horizon. For them, instead of regularity conditions at the origin, the regularity of the event horizon is required. The (non-degenerate) horizon is a linear zero of $N: N(r_h) = 0$, $N'(r_h) > 0$, on which $w$ starts with a finite value $w_h = w(r_h)$. This quantity serves as a parameter labeling solutions, another parameter is the horizon radius $r_h$. One finds black hole solutions for any $r_h$ and a discrete sequence of $w_h$ within the interval $0 < w_h < 1$ (Volkov and Gal’tsov (1989), Kunzle and Masood-ul-Alam (1990), Bizon (1990)).

The EYM black holes violate a naive no-hair conjecture, since the magnetic YM field outside the horizon is not associated with conserved charges. Also, a uniqueness property is violated: an infinite number of different solutions exists with equal ADM mass and zero Coulomb charges. Black hole counterparts to axially symmetric higher-$\nu$ solutions were found as well (Kleihaus and Kunz, 1997), these obviously violate the Israel theorem, valid for vacuum and electrovacuum, stating that static black holes should be spherically symmetric. Further, the Birkhoff theorem is no longer valid for the YM matter, since ‘spin from isospin’ phenomenon gives rise to field modes, which
are spherically symmetric, but time-dependent. Finally, the (electro)vacuum staticity theorem does not hold (though admits a non-Abelian generalization), and the Einstein-Maxwell circularity theorem is violated. All this stimulated a widespread interest to black holes with YM hair as a laboratory to test various long-standing beliefs of the black hole physics.

An internal structure of EYM black holes turned out to be particularly interesting (Donets, Gal’tsov and Zotov (1996). There are discrete sequences of black holes which have either Schwarzschild or Reissner-Nordström (RN) type internal structure, but a generic black hole exhibits a very unusual interior. The metric function $N$ oscillates with an exponentially growing amplitude remaining always negative, but approaching zero closer and closer in subsequent cycles. Each time when such an 'almost' Cauchy horizon is reached, the mass function starts to inflate, making $N$ to turn back, so true Cauchy horizons never form. These oscillations never end, and the approach to the singularity is non-analytic. All solutions, except for a discrete RN-type set (of zero measure in the moduli space) satisfy the strong cosmic censorship hypothesis (no internal Cauchy horizons).

4.2 Monopole-black hole transition

Gravitating Georgy-Glashow model solutions with $\alpha > \alpha_{cr}$ contain an event horizon inside the monopole core. This phenomenon is rather generic (Kastor and Trashen (1992)), it occurs each time when the lagrangian leads to the relation $\varepsilon + p_r \sim N$ at a 'would be' event horizon (the sum of the energy density and the radial pressure). It is so for the EYMH systems with triplet or complex doublet Higgs and the Skyrme model, all these theories admit hairy black holes. In fact, the first such black hole was discovered in the Skyrme model ( Luckock and Moss (1986)), it is particularly interesting as an example of a stable hairy black hole. Note, that purely EYM black holes are unstable, $n$-node solution inherits from BK $2n$ negative modes. Skyrme black holes violate even a weak form of the no-hair conjecture, ascribing the no-hair property only to stable black holes. For the recent work on gravitating Skyrme model see [13]. Dyonic EYMH black holes were discussed in [14].

A monopole-black hole transition near the bifurcation threshold was recently studied in detail by Lue and Weinberg [15], see also a review [16]. Globally regular and black hole space-times are qualitatively different, nevertheless, transition between them passes through a continuous family of configurations. Near the threshold $\alpha \sim \alpha_{cr}$ the metric function $N$ develops a minimum at some point, attaining a value very close to zero. This regular space-time has properties close to those of black holes. One can associate with near-critical solutions
an entropy, since information about the quasi-interior region is accessible with larger and larger time delay as one approaches the critical point. This entropy is likely to be compatible with the Bekenstein entropy. Lue-Weinberg bifurcation was also discussed in other models.

4.3 Static non-spherical black holes

Black hole counterparts to axially-symmetric multimonopole solutions were constructed possessing a deformed event horizon with a constant surface gravity. These disprove a recent modified uniqueness conjecture formulated in terms of the 'isolated horizons' language. Axial dyonic black holes were constructed in. An interesting case is the neutral black hole with a magnetic dipole hair. It is obtained by inserting a horizon between two centers in the monopole-antimonopole solution. Black holes in higher \( \nu \) magnetic monopoles may be non-axisymmetric, therefore violating Hawking’s 'strong rigidity' (Ridgway and Weinberg (1995)). Though no explicit solutions (even numerical) were obtained so far, the argument showing their existence is simple: there is an instability of the magnetically charged (embedded Abelian) Reissner-Nordström solution, which may have any magnetic charge, with respect to massive vector field present in the spontaneously broken gauge theory, in modes which are not axially symmetric. Moreover, again due to 'spin from isospin', there are no spherically symmetric states for \( \nu \neq 1 \). Going beyond the linear approximation, one is able to show that for \( \nu = 2 \) the instability results in an axially symmetric configuration, for \( \nu = 3 \) — tetrahedral, for \( \nu = 4 \) — cubic, etc. In fact, such crystal-type structures are known in the flat space BPS multimonopole systems. Note that the internal structure of hairy black holes with scalar fields is very different than described above for the EYM case: near the singularity the scalar field dominates and the mass-function exhibit a 'power-low inflation' (Gal’tsov, Donets and Zotov (1977), Breitenlohner, Lavrelashvili and Maison (1998)). For the Skyrme case see.

4.4 YM hair and quantum coherence

String infinite symmetries may be responsible for the maintenance of the quantum coherence in the black hole evaporation. These symmetries could encode all information about an enormous number of massive string excitations accumulated in the black hole state. Initially this idea was implemented using essentially two-dimensional black holes (quantum W-hair). An attempt to construct a four-dimensional model was made invoking hairy black holes with the \( SU(\infty) \) YM field. The \( su(N) \) algebra in the limit \( N \to \infty \) can be mapped to Poisson brackets defined on a unit 'internal' sphere. The static spherically
symmetric ansatz, elaborated by Kunzle (1991) for finite \( N \), then gives rise to the YM function depending on a radial variable and an internal angle \( \theta \). In presence of the large negative cosmological constant (with respect to the energy scale of the YM field) an approximate solution was obtained describing a black hole with hair, depending on infinite number of parameters.

5 Asymptotically AdS lumps

5.1 Nodeless solutions for EYM system

Recent interest to AdS/CFT correspondence stimulated an investigation of the EYM systems in presence of the negative cosmological constant \( \Lambda \). Hairy black holes in the SU(2) theory were first studied in \( \cite{24} \) and then (together with regular solutions) in \( \cite{25} \). Asymptotic conditions now follow from a finiteness of the mass \( M \), which is a limiting value of the mass function \( m(r) \) defined via

\[
N = 1 - 2m(r)/r - \Lambda r^2/3 \quad \text{also } \sigma \to 1.
\]

This no more implies \( w \to \pm 1 \), instead any constant value \( w_\infty \) is allowed. This means that solutions will have non-zero magnetic charge \( Q_m = (1 - w_\infty^2) \). Since the asymptotic condition on \( w \) is relaxed, one finds solutions for continuously varying parameter \( b \) in \( \cite{2} \) for each number of nodes \( n \) including zero. Topologically, the nodeless solutions have no sphaleronic nature, so it can be expected that they do not have odd parity negative modes. This was proved to be the case both in the spherically symmetric sector \( \cite{24,25} \) and for non-spherical perturbations \( \cite{26} \). The gravitational (even parity) perturbations were studied in the spherical sector in \( \cite{25} \) and for non-spherical excitations in \( \cite{26} \). Spherical modes are stable in the \( n = 0 \) sector for \( w_\infty < 1/\sqrt{3} \). Another news is that the ‘non-Abelian baldness’ (forbidding AF SU(2) dyons, Gal’tsov and Ershov (1989,1990)) is no longer valid: there are essentially non-Abelian solutions with an electric sector in the SU(2) theory. One finds dyon solutions with finite \( M \) for continuously varying \( b \) (and the corresponding parameter in the electric sector) and any number of nodes \( n \) including \( n = 0 \).

Although these AdS monopoles and dyons are very different from those in the AF EYMH theory, one finds again the bifurcation patterns similar to described above: branches with neighboring \( n \) get connected via bifurcation points in the parameter space. In particular, the mass of nodeless monopole solutions as a function of the magnetic charge exists till some critical charge value, then it turns back forming an upper unstable \( n = 1 \) branch. This feature, first observed in \( \cite{25} \), was further studied in \( \cite{27} \). Axially symmetric EYM solutions with higher winding numbers were investigated for \( \Lambda < 0 \) in a recent paper \( \cite{28} \). Transition from negative to positive \( \Lambda \) apparently reveals a fractal structure of the moduli space \( \cite{25} \).
5.2 Asymptotically AdS monopoles and sphalerons

Solutions of the EYMH system with triplet Higgs were studied in \cite{29,30}. Contrary to the EYM case, no essentially new features appear as compared with the $\Lambda = 0$ theory. Like in the AF case, they exist in a limited region of the parameter $Gv^2$, the critical value being somewhat smaller. The theory with the doublet Higgs in presence of the negative cosmological constant was investigated in \cite{31}. Again, the situation is rather similar to the $\Lambda = 0$ case: the electric component of the YM field is forbidden, no arbitrary $w_\infty$ is possible, a critical value of the Newton constant is observed where solutions cease to exist. However, non-zero $\Lambda$ implies a power law fall-off of the fields at infinity. An extensive study of perturbations of asymptotically AdS solitons and black holes was performed recently \cite{26,32} confirming earlier results on stability of nodeless solutions.

5.3 Black holes with non-spherical topology

Once asymptotic flatness is abandoned, there is no more 'topological censorship', forbidding black holes with non-spherical topology of the event horizon. Namely, for the negative cosmological constant, two-dimensional section of the horizon may be an arbitrary genus Riemann surface \cite{33,34,35}. Generalization of topological censorship for higher genus black holes was discussed in \cite{36}. Asymptotically AdS solutions provide non-trivial examples of black holes with the winding number one which are topologically different from the BK type solutions. One has merely to replace the spherical element $d\Omega$ in (1) by flat or hyperbolic $d\Omega_k = d\theta^2 + f_k(\theta)d\varphi^2$, with $k = 1, 0, -1$ and $f_1 = \sin \theta, f_0 = \theta, f_{-1} = \sinh \theta$. Purely magnetic nodeless $n = 0$ black holes with $k = -1, 0$ were constructed numerically in \cite{37}. For $k = 0$ the mass function remains positive definite elsewhere, so the solutions have positive mass. For the negative curvature $k = -1$, there are black holes not only with positive, but also with zero or negative mass. Solutions with $k = 0$ are stable both in odd and even parity sectors, while the hyperbolic solutions $k = -1$ are stable provided $w_\infty > 1$. Black hole solutions with non-spherical topology do not have globally regular counterparts.

6 Rotation

6.1 Failure of circularity

Mechanical equilibrium inside the flat space static compact field configurations generically will be destroyed once they are forced to rotate, so a priori
it is unlikely that one could endow solitons with an arbitrary angular momentum. On the other hand, in quantum theory, magnetic monopoles generically acquire discrete angular momentum (spin) due to interaction with fermions (fermion zero modes). This is closely related to supersymmetric embeddings, so the possibility of proper rotational zero modes (involving bosonic fields) is an important question. It is known that there are no such modes satisfying Bogomol’nyi equation. More recently the absence of non-Bogomol’nyi rotational zero modes for EYMH particle-like solutions was proved by Brodbeck and Heusler (1997) and Brodbeck, Heusler, Straumann and Volkov (1997). One serious complication to study rotation non-perturbatively is a failure of the circularity theorem for non-Abelian matter (Heusler and Straumann, 1993). The problem is that the Ricci-circularity condition for the stationary axially-symmetric space-time is not ensured by just imposing the same symmetries on the YM field. Therefore a Papapetrou ansatz does not necessarily hold. The most general parameterizations was derived in 38. It is a 2D dilaton gravity, with the matter sector including the 2D Yang-Mills field, two 2D Higgs fields, as well as two scalar moduli, the dilaton and two 2D Kaluza-Klein two-forms.

6.2 Rotational zero modes

Meanwhile, in the linearized theory, the number of odd-parity zero modes involved in rotation is considerably less. An analysis of perturbations of the EYMH system, extending previous work on this subject (see 1 for a review and references), was performed in 39, 40, showing, in particular, stability of the Schwarzschild and Reissner-Nordström black holes with respect to odd-parity EYM perturbations. It was confirmed also that only \( l = 1 \) zero modes, consisting of three real-valued functions, are responsible for (infinitesimal) rotation of both EYM and EYMH solitons and black holes. At this level circularity does hold. The set includes \( h_{\kappa\varphi} \) metric perturbation and two isotopic variations of \( A_\ell \). They satisfy a coupled system of linear ordinary differential equations of the second order, so the overall order of the system is six. A necessary condition for globally defined continuously varying zero modes to exist is a non-trivial intersection of the moduli spaces of local solutions near the origin (the event horizon black holes) and at spatial infinity. A local analysis reveals that the moduli space at infinity is four-dimensional for EYM (without Higgs) and three-dimensional for EYMH. In both cases the dimension of the moduli space near the origin (solitons) is three, while that at the horizon is four. Thus, for solitons involving scalar fields, the intersection of two three-dimensional moduli spaces inside the full six-dimensional space is trivial. For black holes inside magnetic monopoles, there is a one-parameter intersection
space, so event horizons may be rotating. For EYM (BK) particle-like solutions the intersection space is also one-dimensional, and more careful analysis shows that the electric charge and angular momentum are excited simultaneously.

6.3 Rotation and charge

An essential difference between EYMH solitons and BK particles is that the binding is non-gravitational in the first case and gravitational in the second. So it is perhaps not surprising that BK solutions possess rotational zero modes, while magnetic monopoles do not. For black holes gravitational attraction is dominating, so zero modes do exist in both cases. These may be regarded as another admissible hair for static solutions. Another interesting feature is that the charge and rotational degrees of freedom are connected in a different ways for different solutions. Recall that the magnetic YM configuration in BK particles is of the dipole type. An intuition based on the properties of the Kerr-Newmann solution suggest to view the magnetic moment of charged rotating solutions as due to Faraday effect. For rotating BK the situation is inverse: magnetic dipole configuration is primary, while rotation leads to charging the particle. For EYM black holes the angular momentum and the electric charge are independent parameters, but the relationship between the angular velocity of the event horizon and the angular momentum defined at infinity is non-trivial. This is again related to the 'spin from isospin' phenomenon: there is an extra contribution to the angular momentum from the black hole hair. In particular, there exist black holes with zero angular velocity of the horizon (static), but non-zero angular momentum (and non-zero charge). Alternatively, there are black holes with rotating horizons, but zero angular momentum, as seen from infinity. A non-perturbative numerical analysis of rotating EYM black holes with an assumption of circularity was reported recently\[^4\], confirming consistency of this assumption at non-linear level.

7 Links to superstrings

7.1 Dilaton

Since the YM field is scale-invariant, the only dimensional parameter of the EYM theory is the Planck mass. It is natural, therefore, to look for physical applications in the superstring theory. The first step to incorporate BK solutions into this framework consisted in including the dilaton. Particle-like solutions similar to BK were found in the flat YMD theory by Lavrelashvili and Maison (1992). When gravity is also taken into account, one finds a surprising equality of the ADM mass and the dilaton charge of BK-type solutions (Donets
and Gal’tsov, 1993). This was shown to be is a consequence of the scaling symmetry of the system of equation leading to the relation $g_{tt} = \exp(2\phi)$ (Donets, Gal’tsov and Volkov, 1993). Another enigmatic feature was observed for the $n = 1$ EYMD solution: it turned out that the parameter $b$ in (3) is rational, $b_1 = 1/6$, while $a_1 = 2M$ (Lavrelashvili and Maison, 1993). Efforts to explain these ‘magic’ numbers have led to a fascinating development: before going to this, let us mention some recent work on gravitating lumps with dilaton: axially symmetric EYMD configurations, Georgy-Glashow-dilaton monopoles and black holes, dilaton binding in multimonopoles.

7.2 Freedman–Schwarz model

In a search of hidden supersymmetry tentatively underlying these features, Chamseddine and Volkov (1997) have investigated the $\mathcal{N} = 4$ $SU(2) \times SU(2)$ gauged supergravity also known as Friedman-Schwarz (FS) model. Its bosonic sector includes two YM fields, dilaton and axion (which can be set zero for static purely magnetic configurations) with the dilaton potential, proportional to $-(g_1^2 + g_2^2) \exp(-2\phi)$. Formally one can derive the $SU(2)$ EYMD equations for $g_2 = ig_1$. This trick does not destroy supersymmetry: the supergravity Bogomol’nyi equations survive. Later it was shown by Volkov that such ‘complexified’ theory corresponds to the Euclidean FS model, and the associated Bogomol’nyi equations admit indeed one-node $w(r)$ as a solution (though still not found analytically). If one passes to the Lorentzian metric in this solution (this does not return us to the initial Lorentzian FS model), one obtains the EYMD analog of the $BK_1$, which is, of course, not supersymmetric (and unstable), but still bears an imprint of supersymmetry of its Euclidean counterpart.

Leaving only one $SU(2)$ field in $N=4$ supergravity (‘half-gauged’ FS model), one gets the lagrangian $L = -R + 2(\nabla \phi)^2 - e^{2\phi} F^2/2 + e^{-2\phi}$ which differs from the simple EYMD one by the dilaton potential. This theory admits an analytic solution

$$ds^2 = dt^2 - dr^2 - R^2 d\Omega, \quad R^2 = 2r \coth(r + c) - w^2 - 1, \quad w = r \sinh^{-1}(r + c), \quad e^{2\phi} = R^{-1} \sinh(r + c),$$

where the metric is presented in the string frame, and $c$ is a real parameter. The solution preserves $1/4$ of the initial supersymmetry of the FS model for all $c$. Asymptotically $w \to 0$, so this is a charge one monopole. However, the metric asymptotically is not flat and not AdS. The $c = 0$ solution is globally regular, an expansion of $w$ near the origin is of the type (3) with ‘magic’ $b = 1/6$. For $c \neq 0$ ($c$ is assumed to be non-negative), the radius $R$ shrinks at
some finite $r$, which is a curvature singularity. In the limit $c \to \infty$ one gets an Abelian dilatonic monopole.

### 7.3 Holographic interpretation

This solution has a ten-dimensional counterpart in Type I (pure) supergravity; in fact, as was shown by Volkov (1997), the $D = 4$ FS model corresponds to compactification of the latter on $S^3 \times S^3$. Somewhat unexpectedly, it was found\(^2\) that the same solution describes wrapping of a NS5 brane of the IIB theory on a shrinking $S^2$. This opens a way to construct a holographic description for the $\mathcal{N} = 1$ super Yang-Mills. A particularly interesting aspect of the latter theory, which can be investigated using this duality, is the chiral symmetry breaking transition. The non-Abelian component of the gauge field serves as the order parameter: unbroken chiral symmetry corresponds to having only Abelian YM component, $w = 0$. Thus, the BK-type solutions correspond to broken chiral symmetry, while a formation of an Abelian black hole signals the symmetry restoration. So one needs a detailed information about solutions including black holes; this was the subject of the recent paper by Gubser, Tseytlin and Volkov.\(^5\) Other solutions than (4) are non-BPS, they include globally regular solutions which start at the origin with $b \neq 1/6$. Like for asymptotically AdS BK solutions, the parameter $b$ here is continuous, for $b < 1/6$ $w$-solutions are still nodeless, while for $1/6 < b < 1/2$, solutions with more and more nodes appear, with the limiting solution $n = \infty$ starting with $b = 1/2$. These solutions may be regarded as BK-excited monopoles. This part of the solution space is somewhat similar to solutions of the pure EYM theory with a negative cosmological constant. However, now an asymptotic value of $w$ is zero, indicating the charge one monopole topology, while in the AdS case the magnetic (and electric) charges are continuously varying. An interesting new feature is that there is a discrete subspace of solutions with $b = b_n$ possessing a finite energy. Apart from these, there are non-trivial solutions with $w \equiv \pm 1, 0$, which have different ten-dimensional counterparts.

Black hole solutions are determined by the horizon value $w_h$. For $w_h = 0$ solutions are Abelian, while non-zero values $w_h$ give rise to non-Abelian black holes. These contain a discrete subclass of finite-energy solutions for which $w$ falls off exponentially at infinity. Black holes with toroidal event horizon in the $U(1) \times U(1)$ sector of the FS model were constructed in\(^3\).

### 7.4 Gauss-Bonnet

One earlier attempt to incorporate ‘stringy’ effects was to include higher-curvature terms in the lagrangian. First $\alpha'$ correction to the heterotic string ef-
fective action includes the Gauss-Bonnet term, $\beta \exp(2\phi)R_{GB}$, $R_{GB} = R_{\mu\nu\lambda\tau}R^{\mu\nu\lambda\tau} - 4R_{\mu\nu}R^{\mu\nu} + R^2$, which produces a non-trivial effect in four dimensions due to the dilaton factor. Note by passing that this term does not generate derivatives higher than the second order. Remarkably, in the static spherical space-time this term does not destroy the scale invariance of the equations of motion, so a non-trivial first integral still exists (Donets and Gal’tsov, 1993) implying the equality $M = D$. However, solutions exist only for restricted values of the coefficient $\beta < \beta_n$, e.g. $n = 1$ solution disappears for $\beta_1 = 0.37$, the critical value decreases for higher $n$. The question was raised whether quadratic curvature terms (Gauss-Bonnet) can stabilize negative modes. It is known that Gauss-Bonnet acts as a 'hair tonic' opening a channel of neutral black holes with a dilaton charge (Kanti, Mavromatos, Rizos, Tamvakis and Winstanley, 1996), and these are linearly stable in absence of the vector fields (the same authors, 1998). Actually, in the EYM(D) static spherically symmetric theory with the Gauss-Bonnet term there exists four types of black holes: i) neutral dilatonic (YM sector non-excited), ii) usual BK type, iii) holes with the unit magnetic charge (embedded Abelian, electric sector non-excited) and vi) holes with an electric charge, also embedded Abelian. It was found in that those solutions whose perturbations involve the YM magnetic function (ii, iii) remain unstable (in particular, sphaleronic instabilities of ii) are untouched), while neutral and electric solutions are stable, reflecting stability of Schwarzschild and Reissner-Nordström black holes.

8 Born-Infeld

Another aspect related to superstrings consist in investigation of solitons and black holes in the non-Abelian SU(N) Dirac-Born-Infeld theory, arising as an effective theory on N D3-branes. If branes are coincident, one deals with the purely massless theory, while for separated branes there is a Higgs potential with the vacuum expectation value equal to the brane separation. The simplest case is the pure YM theory with the Born-Infeld (BI) type action

$$S = \frac{\beta^2}{4\pi} \text{tr} \left\{ 1 - \sqrt{-\det(g_{\mu\nu} + \beta^{-1} F_{\mu\nu})} \right\} d^4x$$

(5)

In the non-Abelian case, $F_{\mu\nu}$ is matrix-valued, and one can a priori take the trace in the action in different ways (see for details). Two definitions are the most reliable: the symmetrized trace suggested by Tseytlin, which well covers the lower orders of the perturbative string effective action, and the 'ordinary trace-square root' form, which is a direct non-Abelian generalization of the four-dimensional U(1) BI action obtained by evaluating the determinant under
the square root. For the static spherically symmetric SU(2) magnetic ansatz these definitions lead to somewhat different one-dimensional lagrangians:

\[ L_{str} = \beta^2 r^2 (1 - R) \quad R = (1 + V^2 + 2K^2)^{1/2}, \]

where \( V^2 = (1 - w^2)^2 / (2\beta^2 r^4) \), \( K^2 = w^2 / (2\beta^2 r^2) \), and

\[ L_{str} = \beta^2 r^2 \left[ 1 - (1 + V^2)^{1/2} + K^2 A (1 + V^2)^{-1/2} \right] \]

with \( A = W^{-1} \arctan W \), \( W = (1 + V^2)^{1/2} (V^2 - K^2)^{-1/2} \).

8.1 NBI particle-like solutions (sphalerons)

The BI lagrangian breaks scale invariance introducing a new new scale parameter — BI critical field \( \beta \), which should be regarded as a manifestation of non-locality of the underlying string theory. Therefore, the no-go theorem for classical glueballs in the flat space YM theory is overruled, and one is led to look for such solutions in the non-Abelian Born-Infeld (NBI) theory. The proof of existence and numerical results for flat space NBI particle-like solutions were presented in [56] for the ordinary trace (6) and in [55] for the symmetrized trace (7) lagrangians, they have qualitatively similar properties. Solutions form a sequence of the BK type, but the difference is that now the parameter \( b_n \) in the expansion (2) is rather large and rapidly growing with the node number \( n \), for example, \( b_1 = 12.7 \), \( b_2 = 887 \) in the case (6) (these values are even larger for the model (7)). Solutions have sphaleron features and are expected to be unstable. When gravity is taken into account [57], one finds that \( b_n \) as a function of the dimensionless parameter \( G\beta \), continuously interpolates between these values for small \( G\beta \), and the BK values for large \( G\beta \).

8.2 Damping of gigantic oscillations inside YM black holes

Black hole solutions in the Einstein-NBI theory were also constructed [57], [58], outside the horizon they are qualitatively similar to those in the EYM case. However, an internal structure of the ENBI black holes is drastically different. In fact, one could expect that violent oscillations inside the EYM black holes should be modified in quantum theory: already in the first inflationary cycle the mass function attains values exceeding the Planck mass. Although the ENBI theory is still classical, it incorporates string \( \alpha' \) quantum corrections in a non-perturbative way, so one could expect that the problem of the over-Planckian masses in the EYM black holes will be resolved. This is indeed the case [58]: the internal behavior of the mass function now is perfectly smooth,
though the singularity still bears non-analytic features. Namely, the function $w$ attains a final value $w_0$, and there exist a two-parameter family of solutions for which the mass function is analytic in the Schwarzschild-type singularity. This family, however, is not generic, like in the EYM case. But now a generic three-parametric solution can be obtained in the vicinity of the singularity as the series expansion in terms of $r^{1/2}$. Mass function again attains a finite value, but one finds that the singularity is much weaker than in the Schwarzschild case. The scalar curvature diverges only as $R \sim r^{-3/2}$. Note that non-analytic behavior of the mass function $m(r)$ in the singularity is also observed in the Abelian Einstein-Born-Infeld theory coupled to dilaton.

8.3 NBI monopoles

Adding triplet Higgs to the NBI lagrangian, one finds flat space monopoles\cite{60,61}, which exhibit features, similar to those of gravitating monopoles. In view of the existence of the flat-space NBI sphalerons, this is not surprising. Monopole (and dyon) solutions exist for $\beta$ varying from infinity to a finite value $\beta_{cr}$, which was shown\cite{62} to be a bifurcation point giving rise to the branch of excited monopoles with one-node $w$ (higher node excitations exist as well). In a non-commutative $U(1)$ monopole (with Higgs) was also constructed following the D-brane in B-field / non-commutative gauge theory correspondence. Note that it is the symmetrized trace version of the theory which leads to the BPS equations coincident with the YMH ones. Effect of gravity on the NBI monopoles was discussed in \cite{63,64}. Recently the NBI model with doublet Higgs was considered\cite{65}, it was found that the sphalerons in this theory interpolate between the usual electroweak sphaleron and the NBI sphalerons of\cite{56}.

9 Conclusion

Basic properties of gravitating non-Abelian solitons and hairy black holes are fairly well understood now and exhibit certain universality: character of gravitational excitations, bifurcation patterns of moduli spaces, dependence of spectra on asymptotic conditions, phenomena near 'almost' horizons. Our list of references is almost complete for the period 1999 – November 2001 within the covered topics. However, many related directions remained outside the scope of the talk: global monopoles, boson stars, solutions to the gravity coupled Abelian Higgs model, cosmological solutions with non-Abelian fields, topological inflation, critical collapse of the YM field, solutions to higher-dimensional EYM systems. More complete update to the review\cite{1} will be given elsewhere.
Acknowledgments

The author is grateful to LOC for hospitality during the conference and to Gravity Research Foundation for support. The work was also supported in part by the Russian Foundation for Basic Research under grant 00-02-16306.

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