Phase estimation via quantum interferometry for noisy detectors

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The sensitivity in optical interferometry is strongly affected by losses during the signal propagation or at the detection stage. The optimal quantum states of the probing signals in the presence of loss were recently found. However, in many cases of practical interest, their associated accuracy is worse than the one obtainable without employing quantum resources (e.g. entanglement and squeezing) but neglecting the detector’s loss. Here we detail an experiment that can reach the latter even in the presence of imperfect detectors: it employs a phase-sensitive amplification of the signals after the phase sensing, before the detection. We experimentally demonstrated the feasibility of a phase estimation experiment able to reach its optimal working regime. Since our method uses coherent states as input signals, it is a practical technique that can be used for high-sensitivity interferometry and, in contrast to the optimal strategies, does not require one to have an exact characterization of the loss beforehand.

From the investigation of fragile biological samples, such as tissues [1] or blood proteins in aqueous buffer solution [2], to gravitational wave measurements [3, 4], the estimation of an optical phase \( \phi \) through interferometric experiments is an ubiquitous technique. For each input state of the probe, the maximum accuracy of the process, optimized over all possible measurement strategies, is provided by the quantum Fisher information \( I_q \) through the Quantum Cramér-Rao (QCR) bound [5, 6]. The QCR sets an asymptotically achievable lower bound on the mean square error of the estimation \( \delta \phi \geq (M I_q)^{-1/2} \), where \( M \) is the number of repeated experiments.

In the absence of noise and when no quantum effects (like entanglement or squeezing) are exploited in the probe preparation, the QCR bound scales as the inverse of the mean photon number, the Standard Quantum Limit (SQL). Better performances are known to be achievable when using entangled input signals [7, 12]. However, all experiments up to now have been performed using post-selection and cannot claim a sub-SQL sensitivity [13]. An alternative approach, exploited in gravitational wave interferometry, relies on combining an intense coherent beam with squeezed light on a beam-splitter, obtaining an enhancement in the sensitivity of a constant factor proportional to the squeezing factor [3, 4, 14]. Additionally, in the presence of the loss, the SQL can be asymptotically beaten only by a constant factor [3, 15, 16], so that sophisticated sub-SQL strategies [11, 17] (implemented up to now only for few photons) may not be worth the effort. This implies that, for practical high-sensitivity interferometry, the best resource exploitation (or, equivalently, the minimally invasive scenarios) currently entail strategies based on the use of a coherent state \( |\alpha\rangle \), i.e. a classical signal. Its QCR bound takes the form \( \delta \phi \geq (2M \eta |\alpha|^2)^{-1/2} \), where we consider separately the loss \( \mathcal{L}_\xi = 1 - \xi \) in the sensing stage and the loss \( \mathcal{L}_\eta = 1 - \eta \) in the overall detection process. Here we present the experimental realization of a robust phase estimation protocol that improves the above accuracy up to \( (2M \xi |\alpha|^2)^{-1} \), while still using coherent signals as input. It achieves the SQL of a system only affected by the propagation loss \( \mathcal{L}_\xi \), and not by the detection stage \( \mathcal{L}_\eta \).

Our scheme employs a conventional interferometric phase sensing stage that uses coherent-state probes. These are amplified with an optical parametric amplifier (OPA) carrying the phase after the interaction with the sample, but before the lossy detectors. No post-selection is employed to filter [12, 13] the output signal. The OPA (an optimal phase-covariant quantum cloning machine [18]) transfers the properties of the injected state into a field with a larger number of particles, robust under losses and decoherence [19]. Previous works addressed quantum signal amplification, namely quadrature signal, in a lossy environment adopting non-linear methods [20] and feedforward techniques [21]. At variance with these approaches our manuscript analyzes how the amplification of coherent states can be adopted for phase estimation purposes in a lossy environment. Specifically by studying the quantum Fisher information problem, we show that, by adopting the amplification-based strategy, the extracted information can achieve the quantum Cramér-Rao bound associated to the coherent probe state measured with a perfect detection apparatus. Since the amplification acts after the interaction of the probe state with the sample, our scheme is suitable for the analysis of fragile samples, e.g. optical microscopy of biological cells [22], or for single-photon interferometry [23] (where the small intensity of the probes achieved only limited accuracy).

**Theory** - The probe is a horizontally-polarized \((H)\) coherent state \( |\alpha\rangle_H |0\rangle_V \) with \( \alpha = |\alpha| e^{i\theta} \). The state is rotated in the \( \bar{\pi}_\pm = 2^{-1/2}(\pi_H \pm \pi_V) \) polarization basis, and the interaction with the sample induces a phase shift \( \phi \) between the \( \bar{\pi}_\pm \) polarization components: \( U_\phi \). The sample loss \( \mathcal{L}_\xi \) reduces the state amplitude to \( \beta = \sqrt{\xi} \alpha \). The maximum amount of information which can be extracted on the coherent probe state is encoded in the corresponding QCR bound \( \delta \phi \geq (M I_{SQL}^\xi)^{-1/2} \), where \( I_{SQL}^\xi = 2|\beta|^2 \). In the absence of amplification, the detection losses \( \mathcal{L}_\eta \) would increase the QCR to \( \delta \phi \geq (M \eta I_{SQL}^\xi)^{-1/2} \). To prevent this and to attain the previous bound, we implemented the operations shown in Fig. 1: a \( \lambda/4 \) wave-plate...
with optical axis at 45° and the OPA, described by the unitary
\( U_{\text{OPA}} = \exp[\alpha(\hat{a}_H^2 - \hat{a}_V^2) / 2 + \mathrm{h.c.}] \), where \( g = |g|e^{i\lambda} \) is
the amplifier gain, and \( \alpha_H \) and \( \alpha_V \) are the annihilation operators of
the two polarization modes. After the action of detection
losses \( 1 - \eta \), the state evolves into \( \rho_{\eta}^{g,\eta} \). The quantum Fisher
information \( I_{\text{ampl}} \) of the amplification strategy, evaluated on
the state \( \rho_{\eta}^{g,\eta} \) and quantifying the optimal performances
of the scheme, reads

\[
I_{\text{ampl}}^g(\beta, g, \eta) = 2|\beta|^2 \eta \frac{g^2 - g_{\text{eff}}}{\sqrt{1 + 4\eta(1 - \eta)^2}}, \tag{1}
\]

where \( g_{\text{eff}} = 1 / 4 \log((\eta e^{2g} + 1 - \eta) / (\eta e^{-2g} + 1 - \eta)) \), \( \eta = \sinh^2 g \), and we maximized the \( \phi \)-dependent quantum Fisher
information by choosing \( \phi = \pi / 2 - \lambda / 2 + \theta \). For \( \eta \gg (8\eta)^{-1} \) and \( |\beta|^2 \gg 1 / 2 \), we observe that \( I_{\text{ampl}}^g \)
approaches the SQL limit \( I_{\text{SQL}}^g \) (dash-dotted line in Fig. 1[b]). In other
words, increasing the amplifier gain, the effects of the detector loss
might be asymptotically removed [23].

Achieving the accuracy associated with quantum Fisher
bound \( I_{\text{ampl}}^g \) would need to use an optimal estimation
strategy which is difficult to characterize [6] and most likely
challenging to implement. To experimentally test our
proposal we decided hence to recover \( \phi \) by measuring (via lossy
detectors) the photon number difference \( D = n_H - n_V \)
between the two modes on the output state \( \rho_{\eta}^{g,\eta} \) after losses,
with \( n_x = a_x^2 a_x^\dagger \). Even though in general this scheme fails to
reach the accuracy bound of \( I_{\text{ampl}}^g \), in the limit of high gain \( g \)
and high amplitude \( \beta \) it allows us to reach the value of \( I_{\text{SQL}}^g \)
(hence of \( I_{\text{ampl}}^g \)). Indeed the resulting uncertainty can be
evaluated [5] as \( \delta \phi = \sigma(\langle D \rangle) \langle \delta D \rangle \), where \( \langle D \rangle \)
is the expectation value of \( D \) on the output state. A calculation of
the estimation error \( \delta \phi \) of the whole procedure shows that it
depends on the value of the phase \( \phi \) to be estimated. The maximum
sensitivity, that is, the minimum uncertainty \( \delta \phi_{\text{ampl}} \), is
obtained for \( \phi = \pi / 2 \) by setting \( \lambda = 2\theta \):

\[
\delta \phi_{\text{ampl}} = \frac{a^{1/2}(\eta, \eta)}{|\beta|^2 \sqrt{\eta(1 + 2\eta) + 2 \sqrt{\eta(1 + \eta)}}}, \tag{2}
\]

where \( a(\eta, \eta) = 2\pi(1 + \eta + 2\eta) + |\beta|^2[1 + 2\pi + \eta(6 + 8\eta)] \).
It is then clear that for \( \eta \gg (2\eta)^{-1} \) and \( |\beta|^2 \gg 1 / 2 \) we
have \( \delta \phi_{\text{ampl}} \approx (2|\beta|^2)^{-1/2} \), that is, the QCR bound of the state \( |\Psi_{\eta}^\phi \rangle \) (before the amplification and the detector loss) can
be attained by our detection strategy. We also notice that
the adopted data processing is optimal for a wide range of parameters.
This can be shown by evaluating the classical Fisher
information \( I_{\text{ampl}}^g \), which represents the maximum amount of information
that can be extracted from the probe state using our choice of measurement, optimizing over all possible
data-processing. In the present strategy, the sensitivity \( (\delta \phi_{\text{ampl}})^{-1} \)
closely tracks the \( I_{\text{ampl}}^g \) both for small and intermediate values
of \( \pi \). Furthermore, the trend of the two curves suggest a
close resemblance also in the high photon number regime (see
FIG. 1. (1) THEORETICAL ANALYSIS. (a) Scheme of the amplifier based protocol. (b) Comparison between the classical Fisher
information \( I_{\text{ampl}} \) (points) and the sensitivity \( (\delta \phi_{\text{ampl}})^{-1} \) (lines) for \( |\beta|^2 = 9 \). (2) COMPARISON WITH OTHER SCHEMES. (c) Conventional
(unamplified) coherent-state interferometry with sample and detection loss \( L_a \) and \( L_q \) respectively: it can achieve the SQL bound connected
to the quantum Fisher information (QFI) \( I_{\text{SQL}}^g \). (d) Interferometry based on the states that optimize the QFI in the presence of loss, proposed
in [8], with the corresponding QFI, \( I_{\text{opt}}^g \). (e) Comparison between the QFI for the three strategies (a),(c) and (d), for \( |\beta|^2 = 20 \) and \( g = 3.5 \),
normalized with respect to \( I_{\text{SQL}}^g \). Blue dash-dotted line: QFI of our method \( I_{\text{ampl}}^g \). Red solid line: QFI of the coherent state phase estimation
with loss of Fig. 1. Green dashed line: QFI of the optimal strategy of Fig. 1. \( I_{\text{opt}}^g \). [8].
Because of the dependence of \( \delta \phi \) on \( \phi \), to achieve the minimum error \( \delta \phi_{\text{amp}} \), an adaptive strategy \[26\] is necessary. In the Supplementary Material we show that it is sufficient to use a simple two-stage strategy in which we first find a rough estimate of the phase \( \phi_{\text{est}} \) employing conventional phase estimation methods, and then we use it to tune the zero-reference so that our scheme operates at its optimal working point detailed above. We also show that the resources employed in the first stage of this adaptive strategy are asymptotically negligible with respect to the resources employed in the second high-resolution stage.

**Efficiency of the phase estimation** - We now compare our method to other strategies, using as a benchmark the SQL \( \delta \phi \geq (MI_{\text{SQL}})^{-1/2} \), which would be achieved by a probe coherent state with \( |\beta|^2 \) average photons using lossless detectors. Consider now the case with no amplification, where a coherent state is subject to both the sample and detector loss (Fig.1b). This is the strategy conventionally used in interferometry \[27\]. Our method clearly always outperforms it, see the continuous line in Fig.1b. Furthermore, in a lossy scenario the present amplifier-based method achieves better performances than any quantum strategy. Recently, the optimal strategy in the presence of loss was derived \[8\] (Fig.1b). It employs the state that maximizes the quantum Fisher information in lossy conditions. Of course, this strategy cannot be beaten if one could access the optimal measurement that attains the QCR bound. Even though elegant proof-of-principle experiments exist \[10\], both this measurement and the creation of these states without using post-selection are beyond the reach of practical implementations for the foreseeable future, especially for states with large average photon-numbers. In addition, the form of these states strongly depends on the value of the loss \( L_\xi \): it may be unknown and its experimental evaluation typically requires irradiating the sample, which removes the advantage of using the optimal minimally-invasive states. In contrast, the present amplifier-based protocol uses readily available input states and detection strategies, and does not require a priori knowledge since the choice of the coherent state is independent of the value of the loss. Since our method is devised especially to counter the detector loss \( L_\eta \), we compare the performance of our states with the optimal state calculated for the total amount of loss \( L_\xi \), showing that our method can achieve better performance for the practically-relevant case of low values of \( \eta \) (see dashed line in Fig. 1b), where the detection strategy is clearly not optimized to achieve the QCR bound of the optimal states.

**Experimental Setup** - We now describe the experimental implementation in highly lossy conditions, showing that we can achieve a significative phase-sensitivity enhancement with respect to the coherent probe based strategy. The optical setup is reported in Fig. 2. To acquire the phase shift to be measured, the probe coherent state is injected into the sample, which is simulated by a Babinet-Soleil compensator that introduces a tunable phase shift \( \phi \) between the \( H \) and \( V \) polarizations. Subsequently, the probe state is superimposed spatially and temporally with a pump and injected into the OPA. In this experimental realization the phases of the pump and of the coherent state are not stabilized: this will reduce the achievable enhancement by a fixed numerical factor of 4. Note that such condition corresponds to the absence of an external phase reference. In contrast to previous realizations of parametric amplification of coherent states \[28\] which focused on the single-photon excitation regime, we could achieve a large value for the nonlinear gain, up to \( g = 3.3 \), corresponding to a number of generated photons per mode \( \bar{\eta} \sim 180 \) in spontaneous emission. In addition, our scheme is also able to exploit the polarization degree of freedom. After the amplification, the two output orthogonal polarizations were spatially divided and detected by two avalanche photodiodes. Their count rates are then subtracted to obtain the value of \( \langle D \rangle \), and recorded as a function of the phase \( \phi \), introduced by the Babinet.

**Experimental phase estimation** - The results of the experiment are reported in Fig. 3. An enhancement of \( \sim 200 \) in the counts rate for the former case is observed without significantly affecting the visibility of the fringe pattern (Fig. 3b), leading to an increased phase resolution. We measured the enhancement \( (\delta \phi_{\text{coh}}/\delta \phi_{\text{exp}})^2 \) achievable with our protocol \( \delta \phi_{\text{exp}} \) with respect to the conventional unamplified interferometry \( \delta \phi_{\text{coh}} \), in the \( \phi = \pi/2 \) working point (see Fig. 3b). The quantity \( (\delta \phi_{\text{coh}}/\delta \phi_{\text{exp}})^2 \) represents the fraction of additional runs \( \bar{M} \) of a coherent state phase estimation experiment in order to achieve the same performances of the amplifier-based strategy, with the two protocols compared for the same values of \( |\beta|^2 \) and \( \eta \). Our measurement shows a good agreement with the theoretical predictions. A significant enhancement up to a value of \( (\delta \phi_{\text{coh}}/\delta \phi_{\text{exp}})^2 = 186.3 \pm 9.3 \) has been achieved.
Then, we adopted a Bayesian approach in order to obtain an experimental enhancement (\(\delta \phi_{\text{coh}}/\delta \phi_{\exp}\))^2 evaluated at \(\phi = \pi/2\) as a function of the nonlinear gain \(g\) for \(|\beta|^2 \sim 5.8, \eta \sim 1.46 \times 10^{-4}\) (experiment: black diamond points; theory: black solid line) and \(|\beta|^2 \sim 22.8, \eta \sim 3.48 \times 10^{-5}\) (experiment: green star points; theory: green dashed line). (c)-(d) Experimental results for the phase estimation experiment performed with the amplifier based strategy \((g = 3.3, |\beta|^2 \sim 22.8, \eta \sim 3.48 \times 10^{-5})\) for different values of the phase. Estimated values of the phase \(\phi_{\exp}\) (c) and corresponding error \(\delta \phi_{\exp}\) (d). Points: experimental results. Blue solid lines: theoretical prediction given respectively by the true value of the phase \(\phi\) (c) and by the classical Fisher information (d). Red dashed line corresponds to the classical Fisher information for the adopted coherent state without amplification.

We then performed a phase estimation experiment with the amplifier based strategy for different values of the phase shift \(\phi\). To this end, for each chosen value of the phase we recorded the photon-counts in the two output detectors for \(M_{\text{exp}} = 7.5 \times 10^5\) subsequent pulses of the coherent state. Then, we adopted a Bayesian approach in order to obtain an estimate \(\phi_{\exp}\) for the phase and to evaluate the associated error \(\delta \phi_{\exp}\). The results are reported in Figs. 3c-d. We observe that the estimated values of the phase \(\phi_{\exp}\) are in good agreement with the corresponding true values \(\phi\), and that the estimation process reaches the Cramér-Rao bound. Furthermore, the obtained results clearly outperforms the coherent state strategy when no amplification is performed (red dashed line in Fig. 3b).

Conclusions and perspectives - We discuss a strategy for phase estimation in the presence of noisy detectors that can reach the performance of a lossless probe. This approach involves coherent states as input signals, thus not requiring any a priori characterization of the amount of losses, and phase sensitive amplification after the interaction with the sample and before detection losses. As a further perspective, our method could be exploited with different classes of probe states, including quantum resources such as squeezing, leading to sub- SQL phase estimation experiments in lossy conditions.

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Supplementary Material: phase estimation via quantum interferometry for noisy detectors

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In this supplementary material we elaborate on the material presented in the main text, giving more details on the experimental procedure and carefully deriving the formulas presented there. In Sec. I we describe the experiment and the evolution of the quantum state of the probe as it evolves through the apparatus. In Sec. II we calculate the explicit form of the output state of our apparatus. In Sec. III we calculate the quantum Fisher information of the output state, and in Sec. IV the classical Fisher information that results from fixing the deformation parameter. Finally, in Sec. VII we give the details of the theoretical model we employed to analyze the experimental data.

I. EXPERIMENTAL SETUP

The probe is a horizontally (H) polarized electromagnetic field prepared in the coherent state $|\alpha\rangle_H|0\rangle_V$ with $\alpha = |\alpha|e^{i\theta}$. It is sent through an interferometric setup to interact with the sample. The sample induces a phase shift $\phi$ on the system and is characterized by a loss $1 - \xi$. The aim of our apparatus is to determine $\phi$, while employing a low intensity signal. The phase shift is induced through a unitary transformation of the type

$$U_\phi = e^{-i(a_-)\alpha - \phi},$$

where $a_- = (a_H - a_V)/\sqrt{2}$ is the annihilation operator connected to the $-\text{ polarization}$. The loss is induced through a completely positive map $\mathcal{L}_\xi$ of the form

$$\mathcal{L}_\xi[\rho] = \sum_n A_n \rho(A_n)^\dagger,$$

where $\rho$ is an arbitrary state. Since the action of the phase unitary $U_\phi$ and of the loss $\mathcal{L}_\xi$ commute, we can consider these two as independent processes that occur during the interaction with the sample. The action of the loss map on a coherent state simply shifts its amplitude $\mathcal{L}_\xi[|\alpha\rangle\langle\alpha|] = |\sqrt{\xi}\alpha\rangle\langle\sqrt{\xi}\alpha|$, without changing the form of the state. Thus, our choice of coherent state probes will not depend on the noise characteristics of the sample. Consider first the unitary part of the interaction $U_\phi$: the state evolves as

$$|\Psi^\alpha_\phi\rangle = U_\phi|\alpha\rangle_H|0\rangle_V = e^{-i\phi/2}\alpha\cos(\phi/2)|H\rangle|e^{-i\phi/2}\alpha\sin(\phi/2)|V\rangle.$$  

Then, the action of the loss $\mathcal{L}_\xi$ reduces the amplitude of the coherent states so that, after the interaction of the sample, the probe has evolved to

$$|\Psi^\beta_{\phi}\rangle = |e^{-i\phi/2}\beta\cos(\phi/2)|H\rangle|e^{-i\phi/2}\beta\sin(\phi/2)|V\rangle, \quad (4)$$

with $\beta = \sqrt{\xi}\alpha$. When this state is measured by a homodyne detection apparatus, the error $\delta\phi$ on the phase $\phi$ reads $\delta\phi = (2|\beta|^2\eta)^{-1/2}$, where $\eta$ is the overall detection efficiency which takes into account losses and mode matching between the field and the local oscillator (spectral and spatial). To overcome the limitation induced by $\eta$, we consider the following strategy. Before the amplification, a relative phase-shift of $\pi/2$ is inserted between the $H$ and the $V$ polarization components by means of a $\lambda/4$ birefringent waveplate, leading to:

$$|e^{-i\phi/2}\beta\cos(\phi/2)|H\rangle - e^{-i\phi/2}\beta\sin(\phi/2)|V\rangle. \quad (5)$$

The resulting state is then injected in an optical parametric amplifier (OPA). The interaction Hamiltonian of the OPA is

$$\mathcal{H}_{OPA} = i\hbar\chi \left(a^\dagger_+ a^\dagger_-\right) + \text{h.c.} = i\hbar\chi \left(a_H^2 - a_V^2\right)/2 + \text{h.c.} \quad (6)$$

where $a_\pm = (a_H \pm a_V)/\sqrt{2}$, and $\chi$ is the parameter that quantifies the strength of the interaction. It corresponds to a unitary operation

$$U_{OPA} = \exp[g(a_H^2 - a_V^2)/2 + \text{h.c.}] \quad (7)$$

where $g = |g|e^{i\lambda} = \chi t$ is the amplifier gain ($t$ being the interaction time). Form the form of the unitary in (7), it is clear that the OPA is equivalent to two single-mode squeezers acting independently on the modes $H$ and $V$ with opposite phases, namely $U_{OPA} = S_H(-g) \otimes S_V(g)$, where $S_l(g) \equiv \exp[-g a_l^2/2 + \text{h.c.}], l = H,V$. 

After the amplification, the state has evolved to $|\Psi^\beta_{\phi,g}\rangle = U_{OPA}|\Psi^\beta_{\phi}\rangle$. Finally, it is detected by lossy detectors,
parametrized by a quantum efficiency $\eta$. These are equivalent to perfect detectors that measure the number of photons, preceded by a loss map $\mathcal{L}_\eta$ \cite{1}. The action of this map on the state $|\Psi_{\phi}^{\beta,g}\rangle$ produces the mixed state

$$\rho_{\phi}^{\beta,g,\eta} = \mathcal{L}_\eta[|\Psi_{\phi}^{\beta,g}\rangle\langle\Psi_{\phi}^{\beta,g}|]. \quad (8)$$

The explicit form of this state will be calculated in Sec. II.

The corresponding experimental setup for the present protocol is shown in Fig. 2 of the paper. The excitation source is a Ti:Sa laser system, consisting in a Ti:Sa mode-locked Mira900, whose output beam is injected into the Ti:Sa RegA9000 amplifier. The overall laser system can output a 1.5W beam at wavelength $\lambda_0 = 795$ nm. In a first nonlinear crystal, the output field is doubled in frequency through a second harmonic generation (SHG) process to generate the experiment pump beam at wavelength $\lambda_p = 397.5$ nm of power $P = 650$ W. The remainder of the 795 nm beam is then separated from the pump beam through a dichroic mirror, and is prepared in the coherent state $|\alpha\rangle_+$ by controlled attenuation, spectral filtering (IF) and polarizing optics. The coherent state probe then acquires the phase shift by interacting with the sample (in our case, a Babinet-Soleil compensator), and is then injected into the OPA after the acquisition of the phase.

II. STATE EVOLUTION

In this section we calculate the explicit form of the output state $\rho_{\phi}^{\beta,g,\eta}$ of our scheme, by exploiting some operatorial relations for Gaussian states. This will be useful to evaluate the quantum and classical Fisher informations in the following sections. The state impinging at the measurement stage after detection losses can be written in the form:

$$\rho_{\phi}^{\beta,g,\eta} = \mathcal{L}_\eta \left\{ S_H(g_H)S_V(g_V)\xi \left[ D_H(\alpha_H)D_V(\alpha_V)|0\rangle|0\rangle \right. \right.$$  
$$\left. + \left. D_H^\dagger(\beta_H)D_V^\dagger(\beta_V)S_H^\dagger(g_H)S_V^\dagger(g_V) \right]\right\} \quad (9)$$

where $D_l(\alpha) = \exp(\alpha a_l^{\dagger} - \alpha^{*} a_l)$ is the displacement operator such that $D(\alpha)|0\rangle = |\alpha\rangle$. The action of the lossy channel $\xi$ and of the displacement operators can be interchanged as

$$\mathcal{L}_\xi \left[ D_H(\alpha_H)D_V(\alpha_V)|0\rangle|0\rangle \right. \right.$$  
$$\left. + \left. D_H^\dagger(\beta_H)D_V^\dagger(\beta_V)|0\rangle|0\rangle \right]\right\} =$$  

$$= D_H(\beta_H)D_V(\beta_V)|0\rangle|0\rangle + D_H^\dagger(\beta_H)D_V^\dagger(\beta_V)\quad (10)$$

where $\beta_l = \sqrt{\xi} \alpha_l$. The output state then reads:

$$\rho_{\phi}^{\beta,g,\eta} = \mathcal{L}_\eta \left\{ S_H(g_H)S_V(g_V)D_H(\beta_H)D_V(\beta_V)|0\rangle|0\rangle \right.$$  
$$\left. + \left. D_H^\dagger(\beta_H)D_V^\dagger(\beta_V)S_H^\dagger(g_H)S_V^\dagger(g_V) \right]\right\} \quad (11)$$

The action of the squeezing operators and of the displacement operators can be now inverted according to

$$D(\alpha)S(g) = S(g)D(\alpha_+) \quad (12)$$
$$S(g)D(\alpha) = D(\alpha_-)S(g) \quad (13)$$

where $\alpha_\pm \equiv \alpha \cosh g \pm \alpha^{*} e^{i\lambda} \sinh g$. Using Eqs. (12, 13) we can write

$$S_l(g_l)D_l(\beta_l)|0\rangle = D_l(\gamma_l)S_l(g_l)|0\rangle \quad (14)$$

with $\gamma_l \equiv \beta_l \cosh g_l - \beta^{*}_l e^{i\lambda_l} \sinh g_l$. The output state can then be written as

$$\rho_{\phi}^{\beta,g,\eta} = \mathcal{L}_\eta \left\{ D_H(\gamma_H)D_V(\gamma_V)S_H(g_H)S_V(g_V)|0\rangle|0\rangle \right.$$  
$$\left. + \left. S_H^\dagger(g_H)S_V^\dagger(g_V)D_H^\dagger(\gamma_H)D_V^\dagger(\gamma_V) \right]\right\} \quad (15)$$

By interchanging the action of the loss $\mathcal{L}_\eta$ and of the displacement operators $D_l(\gamma_l)$, we obtain

$$\rho_{\phi}^{\beta,g,\eta} = D_H(\gamma_H)D_V(\gamma_V)\mathcal{L}_\eta \left\{ S_H(g_H)S_V(g_V)|0\rangle|0\rangle \right.$$  
$$\left. + \left. S_H^\dagger(g_H)S_V^\dagger(g_V)D_H^\dagger(\gamma_H)D_V^\dagger(\gamma_V) \right]\right\} \quad (16)$$

where $\gamma_l = \sqrt{\xi}\gamma_l$. Finally, by exploiting the identity \cite{B3} involving the action of $\mathcal{L}_\eta$ on squeezed vacuum states, we can express the output state after detection losses in the Gaussian form

$$\rho_{\phi}^{\beta,g,\eta} = D_H(\gamma_H)D_V(\gamma_V)S_H(g_{l\text{eff}})S_V(g_{l\text{eff}})\left[ \rho_{H}^{th}(N_{\text{eff}}) \otimes \right.$$  
$$\left. \rho_{V}^{th}(N_{\text{eff}}) \right] S_H^\dagger(g_{l\text{eff}}^\text{eff})S_V^\dagger(g_{l\text{eff}})D_H^\dagger(\gamma_H)D_V^\dagger(\gamma_V) \quad (17)$$

The expressions for $g_{l\text{eff}}^\text{eff}$ and $N_{l\text{eff}}^\text{eff}$ are reported in Eqs. \cite{B3, B6}.

A. Eigenvalues and Eigenvectors

From Eq. (17) one can calculate the spectrum of eigenvalues and eigenvectors of $\rho_{\phi}^{\beta,g,\eta}$. As a first step, we observe that the density matrix of the state takes the form of a separable state $\rho_{\phi}^{H} \otimes \rho_{\phi}^{V}$, where

$$\rho_{\phi}^{(l)} = D_l(\gamma_l)S_l(g_{l\text{eff}})\rho_{H}^{th}(N_{\text{eff}}^{(l)})S_l^\dagger(g_{l\text{eff}})D_l^\dagger(\gamma_l), \quad (18)$$

with $l = H, V$. Since the state for the two modes has the same Gaussian form, the joint spectrum can be obtained by analyzing directly the $\rho_{\phi}^{(l)}$ single-mode state. By expanding the density matrix in the Fock basis we obtain:

$$\rho_{\phi}^{(l)} = \sum_{n=0}^{\infty} \frac{(N_{\text{eff}}^{(l)})^n}{(1 + N_{\text{eff}}^{(l)})^{n+1}} D_l(\gamma_l)S_l(g_{l\text{eff}}^n)D_l^\dagger(\gamma_l) \quad (19)$$

The eigenvalues and the eigenvectors of the state $\rho_{\phi}^{(l)} = \sum_n \epsilon_n^{(l)} |\psi_n^{(l)}\rangle_l \langle \psi_n^{(l)}|$ are then respectively

$$\epsilon_n^{(l)} = \frac{(N_{\text{eff}}^{(l)})^n}{(1 + N_{\text{eff}}^{(l)})^{n+1}} \quad (20)$$

$$|\psi_n^{(l)}\rangle_l = D_l(\gamma_l)S_l(g_{l\text{eff}}^n)|n\rangle_l \quad (21)$$
Finally, the eigenvalues and the eigenvectors of the joint two-modes density matrix can be written as

$$\rho_{m,n}^{\beta,g,n} = \sum_{m,n=0}^{\infty} \theta_{m,n} |\Psi_{m,n}\rangle_{HV} \langle \Psi_{m,n}|$$  \hspace{1cm} (22)

Here $\theta_{m,n}$ and $|\zeta_m\rangle$ are respectively the eigenvalues and the eigenvectors of the density matrix, and $\epsilon_{m,n} = (\sigma_n - \sigma_m)^2/(\sigma_n + \sigma_m)$. In the case of the output density matrix $\rho_{m,n}^{\beta,g,n}$ of the amplifier-based protocol the eigenvalues and the eigenvectors are parametrized by the indices $(n, m)$, and the QFI is

$$I^g(\alpha, \xi, |g_i\rangle, \{\lambda_i\}, \eta) = \sum_{p,q=0}^{\infty} \frac{(\partial_\phi \sigma_{p,q})^2}{\sigma_{p,q}} + 2 \sum_{m,n} \epsilon_{i,j,m,n} |\langle \Psi_{i,j}| \partial_\phi \Psi_{m,n}\rangle|^2$$  \hspace{1cm} (26)

where

$$\epsilon_{i,j,m,n} = \frac{(\theta_{i,j} - \theta_{m,n})^2}{\theta_{i,j} + \theta_{m,n}}.$$  \hspace{1cm} (27)

We observe that, for the density matrix $\rho_{m,n}^{\beta,g,n}$, the eigenvalues $\theta_{m,n}$, \hspace{1cm} (22)\hspace{1cm} and the first term in Eq. \hspace{1cm} (26)\hspace{1cm} vanish. In order to calculate the second term, it is necessary to evaluate the following quantity: $|\langle \Psi_{i,j}| \partial_\phi \Psi_{m,n}\rangle|^2$. Such term can be written as

$$\langle \Psi_{i,j}| \partial_\phi \Psi_{m,n}\rangle = \langle \Psi_{i,j}| \partial_\phi (|\psi_{m,n}^{(1)}\rangle \otimes |\psi_{m,n}^{(2)}\rangle) \rangle =$$

$$= \langle \Psi_{i,j}| (\partial_\phi |\psi_{m,n}^{(1)}\rangle \otimes |\psi_{m,n}^{(2)}\rangle + |\psi_{m,n}^{(1)}\rangle \otimes (\partial_\phi |\psi_{m,n}^{(2)}\rangle) \rangle =$$

$$= 1 \langle \psi_{i,j}^{(1)}| (\partial_\phi |\psi_{m,n}^{(1)}\rangle \otimes |\psi_n^{(2)}\rangle) \rangle + \delta_{i,m} 2 \langle \psi_{i,j}^{(1)}| (\partial_\phi |\psi_{m,n}^{(2)}\rangle \otimes |\psi_n^{(1)}\rangle) \rangle.$$  \hspace{1cm} (28)

The latter can be evaluated by differentiating the displacement operator written in normally-ordered form:

$$\partial_\phi [D_l(\tilde{\gamma}_l)] = \partial_\phi [e^{-\frac{1}{2} F_l(\tilde{\gamma}_l)} e^{-\frac{1}{2} F_l(\tilde{\gamma}_l^*)}]$$  \hspace{1cm} (30)

By differentiating the three exponential with respect to $\phi$, and by exploiting the following commutation relation:

$$[a_l, e^{\tilde{\gamma}_l a_l^*}] = \tilde{\gamma}_l e^{\tilde{\gamma}_l a_l^*}$$  \hspace{1cm} (31)

the derivative of $D_l(\tilde{\gamma}_l)$ reads:

$$\partial_\phi [D_l(\tilde{\gamma}_l)] = [c^{(l)}_{\alpha,\xi,\eta,1,0} + F^{(l)}_{\alpha,\xi,\eta,1,0} (a_l, a_l^*)] D_l(\tilde{\gamma}_l).$$  \hspace{1cm} (32)

The scalar $c^{(l)}_{\alpha,\xi,\eta,1,0}$ and the operator $F^{(l)}_{\alpha,\xi,\eta,1,0}$ are respectively:

$$c^{(l)}_{\alpha,\xi,\eta,1,0} = \frac{1}{2} [\tilde{\gamma}_l (\partial_\phi \tilde{\gamma}_l^*) - (\partial_\phi \tilde{\gamma}_l) \tilde{\gamma}_l^*]$$  \hspace{1cm} (33)

$$F^{(l)}_{\alpha,\xi,\eta,1,0} (a_l, a_l^*) = (\partial_\phi \tilde{\gamma}_l) a_l^* - (\partial_\phi \tilde{\gamma}_l^*) a_l$$  \hspace{1cm} (34)

We obtain

$$\partial_\phi [D_l(\tilde{\gamma}_l)] = [c^{(l)}_{\alpha,\xi,\eta,1,0} + F^{(l)}_{\alpha,\xi,\eta,1,0} (a_l, a_l^*)] D_l(\tilde{\gamma}_l).$$  \hspace{1cm} (32)

By replacing the latter expressions in Eq. \hspace{1cm} (29), the scalar product $i \langle \psi_i | \partial_\phi \psi_m | \rangle = 1$ can be evaluated as

$$i \langle \psi_i | \partial_\phi \psi_m | \rangle = [c^{(l)}_{\alpha,\xi,\eta,1,0} + F^{(l)}_{\alpha,\xi,\eta,1,0} (a_l, a_l^*)] D_l(\tilde{\gamma}_l) S_l(g^{(l)} |m\rangle)\langle g^{(l)} | S_l(g^{(l)} |m\rangle).$$  \hspace{1cm} (35)

Such average value can be evaluated by exploiting the operatorial identities

$$S_l^{(g)} a S_l^{(g)} = a \cosh g - a^* \sinh g$$  \hspace{1cm} (36)

$$S_l^{(g)} a^* S_l^{(g)} = a^* \cosh g - ae^{-i \lambda} \sinh g$$  \hspace{1cm} (37)

$$D_l^{(g)} a D_l^{(g)} = a + \alpha$$  \hspace{1cm} (38)

$$D_l^{(g)} a^* D_l^{(g)} = a^* + \alpha^*.$$  \hspace{1cm} (39)

Note that the $\epsilon_{i,j,m,n}$ coefficients present the following symmetries,

$$\epsilon_{m,n,m,n} = 0$$  \hspace{1cm} (43)

$$\epsilon_{i,j,m,n} = \epsilon_{m,j,i,n}$$  \hspace{1cm} (44)

$$\epsilon_{i,j,m,n} = \epsilon_{i,n,m,j}$$  \hspace{1cm} (45)
By inserting Eqs. (28)-(40) in Eq. (26) and by exploiting the symmetries of the $\epsilon_{i,j,m,n}$ coefficients we obtain

$$I^q(\alpha, \xi, \{g_l\}, \{\lambda_l\}, \eta) = 4 \sum_{m,n=0}^{\infty} \left[|B_{\alpha,\xi,g_l,\lambda_l,\eta}^{(1)}|^2 (m+1) \times \epsilon_{m+1,m,n,n} + |B_{\alpha,\xi,g_l,\lambda_l,\eta}^{(2)}|^2 (n+1) \epsilon_{m,m,n,n+1}\right]$$

(46)

The QFI $I^q_{\text{amp}}(\alpha, \theta, \phi, \xi, g, \lambda, \eta)$ of the scheme is obtained by replacing $g_H \rightarrow -g$ and $g_V \rightarrow g$. This choice of the parameters is equivalent to the case described in the main paper (with $g_H \rightarrow -g$, $g_V \rightarrow g$ and the additional $\pi/2$ phase shift in the probe state) leading to the same expression for the QFI. We finally obtain

$$I^q_{\text{amp}}(\alpha, \xi, \eta) = \frac{2|\alpha|^2 \xi \eta}{\sqrt{1 + 4 \eta(1 - \eta) \sinh^2 g}} \times \{ \cosh[2(\gamma - g_{\text{eff}})] - \cos(\lambda + 2\phi - 2\theta) \sinh[2(\gamma - g_{\text{eff}})] \}$$

(47)

The optimal condition corresponds to the case $\cos(\lambda + 2\phi - 2\theta) = -1$, where the QFI is

$$I^q_{\text{amp}}(\alpha, \xi, g, \eta) = 2|\alpha|^2 \xi \eta \frac{e^{2(\gamma - g_{\text{eff}})}}{\sqrt{1 + 4 \eta(1 - \eta) \sinh^2 g}}$$

(48)

In Fig. 1 we report the trend of $I^q_{\text{amp}}$ normalized with respect to the SQL $I^q_{\text{SQL}}$, and we observe that for $n \gg (8 \eta)^{-1}$ and $|\beta|^2 \gg 1/2$ we have $I^q_{\text{amp}} \rightarrow I^q_{\text{SQL}}$. Again, the dependence of the QFI $I^q$ of (47) on the parameter $\phi$ to be estimated implies that to achieve its maximum $I^q_{\text{amp}}$, an adaptive strategy (see Sec. VI) is necessary.

IV. CLASSICAL FISHER INFORMATION FOR THE PHOTON-COUNTING MEASUREMENT

In this section we describe the calculation for the classical Fisher information associated with our scheme when photon-counting measurements are performed [Fig 2]. The output state of the protocol is described by the density matrix $\rho_{\beta, \gamma, \eta}$, while the measurement operators that describe photon-counting detectors are the projectors over Fock states

$$\Pi_{n(H), n(V)} = \Pi_{n(H)}^{(l)} \otimes \Pi_{n(V)}^{(l)}$$

(49)

where $\Pi_{n(l)}^{(l)} = |n(l)\rangle \langle n(l)|$, with $l = H, V$ labeling the optical mode. The probability distribution of the measurement outcomes can be evaluated as

$$p(n(H), n(V)|\phi) = \text{Tr}[\rho_{\beta, \gamma, \eta}^{(l)} \Pi_{n(H), n(V)}^{(l)}]$$

(50)

The classical Fisher information associated to the probability distributions of the measurement outcomes is given by the following expression (3):

$$I_{\phi} = \sum_{n,m=0}^{\infty} \frac{[\partial_{\phi} p(n(H), n(V)|\phi)]^2}{p(n(H), n(V)|\phi)}$$

(51)

For the amplifier-based protocol, the probability distribution $p(n(H), n(V)|\phi)$ can be separated in two independent single-mode contributions as

$$p(n(H), n(V)|\phi) = \prod_{l=H,V} p(n(l)|\phi)$$

(52)

Here, $\rho_{l}$ are the single-mode density matrices for modes $l = H, V$ and

$$p(n(l)|\phi) = \text{Tr}[\rho_{l} \Pi_{n(l)}^{(l)}]$$

(53)

In this case, the classical Fisher information can be separated in two single-mode contributions

$$I_{\phi} = \sum_{l=H,V} I_{\phi}^{(l)}$$

(54)

where

$$I_{\phi}^{(l)} = \sum_{n=0}^{\infty} \frac{[\partial_{\phi} p(n(l)|\phi)]^2}{p(n(l)|\phi)}$$

(55)
A. Photon-number distribution of the amplified coherent states

We begin by calculating the photon-number distribution of the amplified coherent states. The density matrix of the output state before the measurement stage is given by

$$
\rho_\phi^{\beta,g,\eta} = D_H(\gamma_H)D_V(\gamma_V)S_H(\hat{g}_\theta^{\text{eff}})S_V(\hat{g}_\delta^{\text{eff}}) \left[ \rho_H^{\theta,\text{eff}} \otimes ho_V^{\delta,\text{eff}} \right] S_H(\hat{g}_\theta^{\text{eff}})S_V(\hat{g}_\delta^{\text{eff}})D_H(\gamma_H)D_V(\gamma_V)
$$

(56)

to evaluate the photon-number distribution, we exploit the following identity between the elements of the density matrix expressed in the Fock basis $\rho = \sum_{n,m=0}^{\infty} \rho_{n,m} |n\rangle \langle m|$ and the Wigner function of a general single-mode state $\rho$,

$$
\rho_{n,m} = \pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dp W_\rho(x,p) W_{n,m}(x,p)
$$

(57)

where $W_{n,m}(x,p)$ is the Wigner function associated to the operator $|n\rangle \langle m|$. Here, the $|x, p\rangle$ operators are defined according to $\Delta^2 x \Delta^2 p \geq \frac{1}{16}$. The corresponding photon-number distribution can be recovered from the diagonal elements $\rho_{n,n}$ by exploiting the expression of the Wigner function of a Fock state:

$$
W_{n,n}(x, p) = \frac{2}{\pi} (-1)^n L_n [4(x^2 + p^2)] e^{-2(x^2 + p^2)}
$$

(58)

Since the density matrix of the state $\rho_\phi^{\beta,g,\eta} = \rho_\phi^{(H)} \otimes \rho_\phi^{(V)}$ is separable between the two modes, we can evaluate the distributions for the two components $\rho_\phi^{(l)}$ separately. The first step is the evaluation of the Wigner function for the single-mode density matrix:

$$
\rho_\phi^{(l)} = D_l(\gamma_l)S_l(\hat{g}_l^{\text{eff}})\rho_l^{(l,\text{eff})}S_l^{\dagger}(\hat{g}_l^{\text{eff}})D_l^{\dagger}(\gamma_l)
$$

(59)

The Wigner function for this state takes the following Gaussian form

$$
W_{\rho^{(l)}}(x_l, p_l) = \frac{2}{\pi} \frac{1}{1 + 2N_l^{\text{eff}}} e^{-2(x_l^2 + p_l^2)\sigma_l^{xx}} \times e^{-2(x_l^2 + p_l^2)\sigma_l^{pp}}
$$

(60)

where the first order and the second order moments are, respectively

$$
x_l^0 = \text{Re} [\gamma_l]
$$

(61)

$$
p_l^0 = \text{Im} [\gamma_l]
$$

(62)

and

$$
\sigma_l^{xx} = \cosh(2g_l^{\text{eff}}) + \cos \lambda_l \sinh(2g_l^{\text{eff}})
$$

(63)

$$
\sigma_l^{pp} = \cosh(2g_l^{\text{eff}}) - \cos \lambda_l \sinh(2g_l^{\text{eff}})
$$

(64)

$$
\sigma_l^{xp} = \sin \lambda_l \sinh(2g_l^{\text{eff}})
$$

(65)

Here, $g_l^{\text{eff}}$ and $\lambda_l$ are respectively the absolute values and the phase of the squeezing parameters $g_l^{\text{eff}}$. We can now proceed with the calculation of the single-mode photon-number distribution $p(n^{(l)}|\phi)$, which can be evaluated from the integral

$$
p(n^{(l)}|\phi) = \pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_l dp_l W_{\rho^{(l)}}(x_l, p_l) W_{n,m}(x_l, p_l)
$$

(66)

We first begin by performing the following rotation on the quadrature variables $(x_l, p_l) \rightarrow (x'_l, p'_l)$ of the $W_{\rho^{(l)}}(x, p)$ function:

$$
x'_l = x_l \cos \psi_l + p_l \sin \psi_l
$$

(67)

$$
p'_l = -x_l \sin \psi_l + p_l \cos \psi_l
$$

(68)

$$
x'_l^0 = x_l^0 \cos \psi_l + p_l^0 \sin \psi_l
$$

(69)

$$
p'_l^0 = -x_l^0 \sin \psi_l + p_l^0 \cos \psi_l
$$

(70)

where $\psi_l = \lambda_l / 2$. The Wigner function in this rotated quadrature set is

$$
W_{\rho^{(l)}}(x_l', p_l') = \frac{2}{\pi} \frac{1}{1 + 2N_l^{\text{eff}}} e^{-2[(x_l')^2 + (p_l')^2] \sigma_l^{xx}} \times e^{-2[(x_l')^2 + (p_l')^2] \sigma_l^{pp}}
$$

(71)

The same rotation is performed on the $W_{n,n}(x_l, p_l)$, which presents radial symmetry and hence its form is not affected by the rotation according to

$$
W_{n,n}(x_l', p_l') = \frac{2}{\pi} (-1)^n L_n [4((x_l')^2 + (p_l')^2)] e^{-2[(x_l')^2 + (p_l')^2]}
$$

(72)

We can then proceed with the evaluation of the integral (66). By performing the basis rotation $(x_l, p_l) \rightarrow (x'_l, p'_l)$ in the integration variable we obtain

$$
p(n^{(l)}|\phi) = \pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_l dp_l W_{\rho^{(l)}}(x_l', p_l') W_{n,m}(x_l', p_l')
$$

(73)

By expanding the Laguerre polynomials of the $W_{n,n}(x_l', p_l')$ function we obtain

$$
p(n^{(l)}|\phi) = \frac{4}{\pi} \frac{(-1)^n}{(1 + 2N_l^{\text{eff}})^n} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \sum_{j=0}^{k} \frac{(-4)^k}{k!} \left( \begin{array}{c} k \\ j \end{array} \right) \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_l dp_l (x_l')^j (p_l')^{k-j} e^{-2[(x_l')^2 + (p_l')^2]} \times e^{-2[(x_l')^2 + (p_l')^2] \sigma_l^{xx}}
$$

(74)
The integrals in $d\xi_i'$ and $dp_j'$ can be evaluated separately. We now define the following auxiliary functions

\[ \tilde{A}_{x_i} = 1 + \frac{e^{-2\eta_{\text{eff}}^x}}{1 + 2N_{\text{eff}}^x} \]  

(75)

\[ \tilde{B}_{x_i} = \frac{x_i^{\eta^x}}{1 + 2N_{\text{eff}}^x + e^{-2\eta_{\text{eff}}^x}} \]  

(76)

\[ \tilde{C}_{x_i} = \frac{(x_i^0)^2 e^{-2\eta_{\text{eff}}^x}}{1 + 2N_{\text{eff}}^x + e^{-2\eta_{\text{eff}}^x}} \]  

(77)

\[ \tilde{A}_{p_i} = 1 + \frac{e^{2\eta_{\text{eff}}^p}}{1 + 2N_{\text{eff}}^p} \]  

(78)

\[ \tilde{B}_{p_i} = \frac{x_i^0 e^{2\eta_{\text{eff}}^p}}{1 + 2N_{\text{eff}}^p + e^{2\eta_{\text{eff}}^p}} \]  

(79)

\[ \tilde{C}_{p_i} = \frac{(x_i^0)^2 e^{2\eta_{\text{eff}}^p}}{1 + 2N_{\text{eff}}^p + e^{2\eta_{\text{eff}}^p}} \]  

(80)

where the $\tilde{B}$ and the $\tilde{C}$ terms depend on the phase $\phi$. Finally, by exploiting the definition of the confluent hypergeometric functions $U(a, b, z)$, the single-mode photon number distribution can be written as:

\[ p(n^{(l)}|\phi) = \frac{2(-1)^n}{1 + 2N_{\text{eff}}^x} e^{-2(\tilde{C}_{x_i} + \tilde{C}_{p_i})} \sum_{k=0}^{n} \frac{k!}{k!} \left( \frac{n}{k} \right) \frac{k!}{k!} \]  

(81)

\[ \times U[-j, 1/2, -2\tilde{A}_{x_i}(\tilde{B}_{x_i})^2] U[-k+j, 1/2, -2\tilde{A}_{p_i}(\tilde{B}_{p_i})^2] \]  

\[ (\tilde{A}_{x_i})^{j+1/2}(\tilde{A}_{p_i})^{k-j+1/2} \]

\[ \times (85) \]

\[ (86) \]

\[ (87) \]

\[ (88) \]

\[ (89) \]

\[ \]
\( \phi \) is to measure the output photon-number difference \( D = c_H^\dagger c_H - c_V^\dagger c_V \) and to extrapolate the value of \( \phi \) from the dependence of \( \langle D \rangle \) on it. By exploiting the expressions for the field operators, the average of \( D \) on the state \( \rho_{\phi, \eta}^{\text{sql}} \) is

\[
\langle D \rangle = \eta |\alpha|^2 \xi \left[ \cos \phi (1 + 2\eta) + \cos (\phi + \lambda - 2\theta) 2\sqrt{\eta(1 + \eta)} \right] \tag{92}
\]

To evaluate the resolution \( \delta \phi \) on the estimated phase according to standard estimation theory, we need to calculate the fluctuations \( \sigma(\langle D \rangle) \) on the detected signal. Such quantity can be evaluated according to \( \sigma^2(\langle D \rangle) = \langle D^2 \rangle - \langle D \rangle^2 \). By evaluating the average values \( \langle (c_H^\dagger c_H)^2 \rangle \) and \( \langle (c_V^\dagger c_V)^2 \rangle \), we obtain:

\[
\sigma^2(\langle D \rangle) = \eta \left[ a(\eta, \eta) + \cos \phi \cos(\phi + \lambda - 2\theta)b(\eta, \eta) \right] \tag{93}
\]

where:

\[
a(\eta, \eta) = 2\pi(1 + \eta + 2\eta^\pi) + |\alpha|^2 \xi \left[ 1 + 2\pi + \eta^\pi(6 + 8\pi) \right] \tag{94}
\]

\[
b(\eta, \eta) = 2\sqrt{\eta(1 + \eta)}|\alpha|^2 \xi (1 + \eta + 4\eta^\pi) \tag{95}
\]

We note that both the signal and the fluctuations depend on the phase difference between the coherent beam \( \theta \) and the pump beam \( \lambda \). Finally, the resolution of this detection strategy can be evaluated according to standard estimation theory as

\[
\delta \phi = \frac{\sqrt{\sigma^2(\langle D \rangle)}}{|\partial \langle D \rangle / \partial \phi|} = \frac{\sqrt{a(\eta, \eta) + \cos \phi \cos(\phi + \lambda - 2\theta)b(\eta, \eta)}}{|\alpha|^2 \sqrt{\eta \xi} \left[ \cos(\phi + 2\pi) + \cos(\phi + \lambda - 2\theta) 2\sqrt{\eta(1 + \eta)} \right]} \tag{96}
\]

Its optimal operating point is achieved for \( \lambda - 2\theta = 0 \) and for a value of the actual phase of \( \phi = \pi/2 \), corresponding to the steepest point of the signal \( \langle D \rangle \). The error associated to the phase estimation process in this optimal working point reads:

\[
\delta \phi_{\text{ampl}} = \frac{a^{1/2}(\eta, \eta)}{|\alpha|^2 \xi \sqrt{\eta \xi} (1 + 2\pi + 2\sqrt{\eta \xi})} \tag{97}
\]

In Fig. 3 we report the value of \( (\delta \phi_{\text{ampl}}^{-1})^2 \) normalized with respect to the SQL \( I_{\text{SQL}}^\phi \). We note that for \( \eta \gg (2\eta)^{-1} \) and \( |\beta|^2 \gg 1/2 \) the QCR bound \( \delta \phi \geq (M2|\beta|^2)^{-1/2} \) of the state \( |\Psi^\phi_\eta \rangle \) (before the amplification and the detector loss) can be attained by our detection strategy.

The fact that \( \delta \phi \) depends on the parameter \( \phi \) we want to estimate implies that the optimal regime \( \delta \phi_{\text{ampl}} \) can be achieved only by employing an adaptive strategy, where some initial measurements are performed to get an estimate of \( \phi \) so that the apparatus can be employed in its optimal working point around \( \phi = \pi/2 \). This is addressed in the next section.

\section{Adaptive Protocol}

In this section we detail a simple two-stage adaptive scheme, where first a rough estimate of the parameter \( \phi \) is found, and then this estimate is employed in a second high-resolution stage of the protocol.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3.png}
\caption{Plot of \( (\delta \phi_{\text{ampl}}^{-1})^2 \) as a function of the nonlinear gain \( g \) of the amplifier and of the detection efficiency \( \eta \), with \( |\beta|^2 = 20 \), normalized with respect to \( I_{\text{SQL}}^\phi \).}
\end{figure}

\subsection{Bounds for a two-step adaptive protocol}

Let \( \phi \) be the parameter we want to estimate (the phase) and assume that it is encoded in two different families of states, i.e. the family \( \{\rho_{\phi}\}_\phi \) and the family \( \{\sigma_{\phi}\}_\phi \). For example, the first family can be identified with the states of the system at the output of the interferometer with no amplification is used. The second family instead is identified as the the state at the output of the interferometer when the amplifier is active and where we have set the phase reference in such a way that the apparatus gives optimal performances for \( \phi = 0 \). In what follows we will consider a two stage estimation strategy in which \( i) \) first we perform \( M_1 \) measurements on the state \( \rho_{\phi} \) of the first family to get a preliminary estimation of \( \phi \), and then \( ii) \) we perform \( M_2 \) measurement on the state \( \sigma_{\phi} \) of the second family to improve our estimation (of course in the second stage we are facilitated by the fact that we have already acquired some info on \( \phi \)).

Let then \( \vec{x} = (x_1, x_2, \cdots) \) the data extracted from the first set of measurement and \( \phi_{\text{ext}}^{(M_1)}(\vec{x}) \) the estimation function we use to get the preliminary estimation of \( \phi \). Using the quantum Cramer-Rao (QCR) bound we have

\[
\delta^2 \phi_{1} = \sum_{\vec{x}} P_{1}(\vec{x}) [\phi - \phi_{\text{ext}}^{(M_1)}(\vec{x})]^2 \geq \frac{1}{M_1 I_{1}^\phi(\phi)} \tag{99}
\]

where \( P_{1}(\vec{x}) \) are the probability of getting the outcomes \( \vec{x} \) when measuring \( \rho_{\phi}^{M_1} \) and \( I_{1}^\phi(\phi) \) is the quantum Fisher info associated with the family \( \{\rho_{\phi}\}_\phi \). For the sake of simplicity we assume that \( \phi_{\text{ext}}^{(M_1)}(\vec{x}) \) is unbiased, i.e.

\[
\sum_{\vec{x}} P_{1}(\vec{x}) [\phi - \phi_{\text{ext}}^{(M_1)}(\vec{x})] = 0 \tag{100}
\]

(generalization to the general case are possible).

In the second stage of the estimation we use the family \( \{\sigma_{\phi}\}_\phi \) where we modify the way the phase is mapped by rescaling it by \( \phi_{\text{ext}}^{(M_1)}(\vec{x}) \). This is possible for instance by changing the initial phase reference which effectively shifts the unknown phase \( \phi \) to \( \chi = \phi - \phi_{\text{ext}}^{(M_1)}(\vec{x}) \) : this is the new parameter we wish to recover. In the second stage, we perform
measurements on $\sigma_{\chi_{M_2}}$ obtaining the data $\vec{y} = (y_1, y_2, \cdots)$. We determine $\chi$ via the estimator $\chi_{est}(\vec{y})$ which again we assume to be unbiased, i.e.

$$\sum_{\vec{y}} P_2(\vec{y}) \left[ \chi - \chi_{est}(\vec{y}) \right] = 0 \; , \quad (101)$$

(here $P_2(\vec{y})$ is the probability of getting the outcomes $\vec{y}$ when measuring $\sigma_{\chi_{M_2}}$). The whole process can be described hence by introducing a joint estimator function

$$\phi_{est}(\vec{x}, \vec{y}) = \phi_{est}(\vec{x}) + \chi_{est}(\vec{y}) \; . \quad (102)$$

characterized by a probability distribution $P_1(\vec{x})P_2(\vec{y})$ and which (by construction) is unbiased, i.e.

$$\sum_{\vec{x}, \vec{y}} P_1(\vec{x})P_2(\vec{y}) \phi_{est}(\vec{x}, \vec{y}) = \phi \; . \quad (103)$$

Let us now compute the variance of the error associated with such estimator. Formally this is given by

$$\delta^2 \phi = \sum_{\vec{x}, \vec{y}} P_1(\vec{x}) P_2(\vec{y}) \left[ \phi - \phi_{est}(\vec{x}, \vec{y}) \right]^2$$

$$= \sum_{\vec{x}} P_1(\vec{x}) \left[ \sum_{\vec{y}} P_2(\vec{y}) \left[ \phi - \phi_{est}(\vec{x}, \vec{y}) \right]^2 \right]$$

$$= \sum_{\vec{x}} P_1(\vec{x}) \left[ \sum_{\vec{y}} P_2(\vec{y}) \left[ \phi - \phi_{est}(\vec{x}) - \chi_{est}(\vec{y}) \right]^2 \right]$$

$$= \sum_{\vec{x}} P_1(\vec{x}) \left[ \sum_{\vec{y}} P_2(\vec{y}) \left[ \chi - \chi_{est}(\vec{y}) \right]^2 \right]$$

$$\geq \sum_{\vec{x}} P_1(\vec{x}) \frac{1}{M_2 I_2^q(\chi)}$$

$$= \sum_{\vec{x}} P_1(\vec{x}) \frac{1}{M_2 I_2^q(\phi - \phi_{est}(\vec{x}))} \; , \quad (104)$$

where we used the QCR bound on the estimation of $\chi$ and where $I_2^q(\chi)$ is the quantum Fisher info of the state $\sigma(\chi)$. The above expression can now approximated by using the fact that for sufficiently large $M_1$, $\phi_{est}(\vec{x}) \simeq \phi$, i.e. $\chi \simeq 0$. This allows us to expand $I_2^q(\chi)$ around $\chi = 0$, i.e.

$$I_2^q(\phi - \phi_{est}(\vec{x})) \simeq I_2^q(0) + (\phi - \phi_{est}(\vec{x})) I_2^q(0)$$

$$+ (\phi - \phi_{est}(\vec{x}))^2 I_2^q(0)/2 \; , \quad (105)$$

which yields

$$\delta^2 \phi \simeq \frac{1}{M_2} \sum_{\vec{x}} P_1(\vec{x}) \left[ I_2^q(0) + (\phi - \phi_{est}(\vec{x})) I_2^q(0) \right]$$

$$+ (\phi - \phi_{est}(\vec{x}))^2 I_2^q(0)/2$$

$$\simeq \frac{1}{M_2} \sum_{\vec{x}} P_1(\vec{x}) \left[ 1 - (\phi - \phi_{est}(\vec{x})) \right] I_2^q(0)$$

$$- (\phi - \phi_{est}(\vec{x}))^2 I_2^q(0)/2$$

$$+ (\phi - \phi_{est}(\vec{x}))^2 \left[ I_2^q(0) - \frac{I_2^q(0)^2}{I_2^q(0)} \right]$$

$$= \frac{1}{M_2 I_2^q(0)} \left[ 1 - \delta^2 \phi \left( \frac{I_2^q(0)}{I_2^q(0)} - \frac{I_2^q(0)^2}{I_2^q(0)} \right) \right]$$

where we used Eq. (100) and the definition of $\delta^2 \phi$. Suppose now that $I_2^q(\chi)$ achieves its maximum for $\chi = 0$ (this is what happens thanks to our new choice of reference). This implies that $I_2^q(0) = 0$ and $I_2^q(\chi) \leq 0$. Therefore we get

$$\delta^2 \phi \geq \frac{1}{M_2 I_2^q(0)} \left[ 1 + \frac{|I_2^q(0)|}{2 I_2^q(0)} \right]$$

$$\geq \frac{1}{M_2 I_2^q(0)} \left[ 1 + \frac{|I_2^q(0)|}{2M_1 I_2^q(0)} \right] \; , \quad (106)$$

where in the last inequality we used the QCR bound (99).

Defining $\tilde{M} = M_1 + M_2$ the total number of measurements, we can write

$$\delta^2 \phi \geq \frac{1}{(1-p)M_2 I_2^q(0)} \left[ 1 + \frac{|I_2^q(0)|}{2pM_1 I_2^q(0) I_2^q(0)} \right] \; , \quad (107)$$

with $p = M_1/M$ begin the fraction of measurement we employ in the first step of the protocol. This equation provides the corrections to the accuracy we get when we adopt the adaptive strategy.

**Observation I:** It is worth comparing the above bound with the accuracy one could get if instead of performing the preliminary step one could have used all $M$ copies to perform only the estimation on the states $\sigma_{\phi}$. In this case the resulting accuracy would be $1/(M I_2^q(\phi))$. Do we gain something by going true the adaptive result? A positive answer would require

$$\frac{1}{(1-p)M_2 I_2^q(0)} \left[ 1 + \frac{|I_2^q(0)|}{2pM_1 I_2^q(0) I_2^q(0)} \right] \leq \frac{1}{M I_2^q(\phi)} \; , \quad (108)$$

which can be cast as

$$\frac{p + A}{p(1-p)} \leq B \; , \quad (109)$$

with $B = I_2^q(0)/I_2^q(\phi)$ and $A = \frac{|I_2^q(0)|}{2M_1 I_2^q(0) I_2^q(0)}$. Since by assumption $B \geq 1$ and $A \geq 0$, one can easily verify that there are value of $p$ which allows one to obtain Eq. (108) if $B$ is sufficiently large.
Observation II: For fixed $M$ we can optimize the right-hand-side of Eq. (107) with respect to $p$. This yields

$$p_{\text{opt}} = \sqrt{A^2 + A - A},$$

(110)

(notice that this is an increasing function of $A$ which is always positive and smaller than $1/2$ – the latter being the asymptotic value reached for $A \gg 1$). Consequently we can write

$$\delta^2 \phi \geq \frac{1}{(1-p)M^2_1(0)} \left[ 1 + \frac{1}{(1-p)M^2_1(0)} \left[ 1 + \frac{1}{2pM^2_1(\phi)P^2_2(0)} \right] \right],$$

(111)

Now, for $M \gg 1$ we have that $A \to 0$. Therefore we can write

$$\delta^2 \phi \geq \frac{1}{M^2_1(0)} \left[ 1 + 2\sqrt{A} \right],$$

This implies that the resources $M_1$ employed in the first stage of the protocol can be neglected, and the precision asymptotically approaches the QCR of the second stage: the term with the square root in (111) is asymptotically negligible.

B. Numerical simulation of a two-step adaptive protocol

Here we provide a numerical simulation of a two-step protocol tailored to reach the optimal performances, given by the maximum of the classical Fisher information $I_{\text{ampl}}$ in $\phi = \pi/2$ and $\lambda - 2\theta = 0$, for all the value of $\phi$. The two steps of the protocols are here described:

(I) In a first step, a coherent probe state without the amplification-stage (that is, by setting $g_H = g_V = 0$) is adopted to obtain a rough estimate $\phi_1$ of the phase.

(II) In a second step, the scheme is adjusted to the optimal working point by means of an additional phase shift $\psi$, which is tuned in order to set the overall phase of the interferometer to $\phi_{\text{tot}} = \phi + \psi \simeq \pi/2$. Furthermore, the difference between the pump beam phase $\lambda$ and the coherent state phase $\theta$ is set to $\lambda - 2\theta = 0$.

The data analysis on each step can be performed for instance by means of a Bayesian approach [2]. In Fig. 4 we report the results of a numerical simulation for $M = 10^5$ repeated measurements. We observe that, for all values of the phase $\phi \in [0, \pi]$ the error $\delta \phi$ reaches the maximum of the classical Fisher information, that is, $I_{\text{ampl}}$ evaluated at $\phi = \pi/2$ and $\lambda - 2\theta = 0$.

\begin{center}
\begin{figure}[h]
\begin{tabular}{cc}
(a) & (b) \\
\end{tabular}
\end{figure}
\end{center}

FIG. 4. Numerical simulation of a two step protocol for a phase estimation experiment with the amplifier-based strategy, for $q = 2$, $|\beta|^2 = 4$, and $\eta = 10^{-2}$, with $M = 10^5$ repeated measurements. (a) Estimated value $\hat{\phi}$ and (b) corresponding error $\delta \phi$ associated to the estimation process. Points: numerical simulation. Blue solid line: classical Fisher information of the amplifier-based protocol, which sets the bound for $\delta \phi$ without an adaptive strategy. Red dashed line: classical Fisher information for a coherent state protocol with the same parameters without the amplification strategy.

VII. MODELING THE EXPERIMENT

Here we discuss the theoretical model for the analysis of the experimental data of the protocol. In the implementation described in the main paper, no phase stabilization is performed on the optical path of the pump beam, hence the phase varies randomly at each experimental run. To model such effect, an average on the phase $\lambda$ with a uniform distribution $\mathcal{P}(\lambda) = \frac{1}{\pi}$ must be performed on both the signal and the fluctuations. In this case, the average signal in the two polarizations $H$ and $V$ is given by

$$\langle n_H \rangle = \eta \left[ \pi + |\alpha|^2 \xi (1 + 2\eta \sin^2(\phi/2)) \right],$$

(112)

$$\langle n_V \rangle = \eta \left[ \pi + |\alpha|^2 \xi (1 + 2\eta \sin^2(\phi/2)) \right].$$

(113)

The average number of the count rates $\langle D \rangle$ is then given by

$$\langle D \rangle = |\alpha|^2 \eta \xi \cos \phi (1 + 2\eta)$$

(114)

In the high losses regime investigated throughout the paper, the number of photons effectively impinging on the detector is smaller than one, since $\eta \langle n_H \rangle < 1$. In this regime, the single-photon counting process is described by a Poissonian statistics. Hence, the fluctuation on the difference signal can be evaluated as

$$\sigma^2 (\langle D \rangle) = \sigma^2 (\langle n_H \rangle) + \sigma^2 (\langle n_V \rangle) = \langle n_H \rangle + \langle n_V \rangle$$

(115)

By explicitly substituting the expressions for $\langle n_H \rangle$ and $\langle n_V \rangle$ we obtain the following expression for the phase estimation error

$$\delta \phi = \sqrt{\frac{2\pi + |\alpha|^2 (1 + 2\eta)}{|\alpha|^2 \xi \sqrt{\eta} (1 + 2\eta) \sin \phi}}$$

(116)

The optimal point is achieved for $\phi = \pi/2$, where the error $\delta \phi$ is

$$\delta \phi_{\text{exp}} = \sqrt{\frac{2\pi + |\alpha|^2 (1 + 2\eta)}{|\alpha|^2 \xi \sqrt{\eta} (1 + 2\eta)}}$$

(117)
Appendix A: Quantum Fisher Information

Here we briefly review the properties of the quantum Fisher information for mixed states. Let us consider a family of states $\sigma_\phi$ depending on a parameter $\phi$. Such family of states can be exploited to estimate the value of the parameter $\phi$. In local estimation theory, the maximum amount of information that can be extracted on the parameter $\phi$ with $M$ repeated measurements is given by the QFI $I_\phi^q$. More specifically, the variance of any estimator of the parameter $\phi$ satisfies the quantum Cramer-Rao inequality:

$$\delta^2 \phi \geq \frac{1}{M I_\phi^q}$$  \hspace{1cm} (A1)

Here, $I_\phi^q$ represents the optimization of the classical Fisher information over all possible choice of the quantum measurement. In general, the quantum Fisher information of the family of states $\sigma_\phi$ is given by the following definition:

$$I_\phi^q = \text{Tr} [\sigma_\phi L_\phi^2]$$  \hspace{1cm} (A2)

where $L_\phi$ is the symmetric logarithmic derivative of $\sigma_\phi$:

$$\partial_\phi \sigma_\phi = \frac{L_\phi \sigma_\phi + \sigma_\phi L_\phi}{2}$$  \hspace{1cm} (A3)

By expressing the density matrix in terms of its spectral decomposition $\sigma_\phi = \sum_m \sigma_m |\zeta_m\rangle \langle \zeta_m|$, the quantum Fisher information can be evaluated as [3]:

$$I_\phi^q = \sum_p \frac{(\partial_\phi \sigma_p)^2}{\sigma_p} + 2 \sum_{n,m} \epsilon_{n,m} |\langle \zeta_m | \partial_\phi \zeta_n \rangle|^2$$  \hspace{1cm} (A4)

Here, $\partial_\phi \sigma_p$ is the derivative of the eigenvalues with respect to $\phi$, and $|\partial_\phi \zeta_n\rangle$ is the derivative of the eigenvectors written in a $\phi$-independent basis $\{|k\rangle\}$:

$$|\partial_\phi \zeta_m\rangle = \sum_k (\partial_\phi \zeta_{mk}) |k\rangle$$  \hspace{1cm} (A5)

Finally, the coefficient $\epsilon_{n,m}$ is given by the following expression:

$$\epsilon_{n,m} = \frac{(\sigma_n - \sigma_m)^2}{\sigma_n + \sigma_m}$$  \hspace{1cm} (A6)

Appendix B: Mathematical relations

In this Appendix we report some mathematical relations exploited in the calculation of the Fisher information.

**Thermal state.** – The thermal single-mode state is defined as:

$$\rho_\text{th}(N) = \frac{1}{1 + N} \sum_{n=0}^{\infty} \chi^n |n\rangle \langle n|$$  \hspace{1cm} (B1)

with $\chi = N/(1 + N)$, where $N$ is the average number of photons of the state.

**Lossy squeezed vacuum.** – The state generated by the action of a lossy channel on the squeezed vacuum state can be written as according to [4]:

$$\mathcal{L}_\eta [S(g)|0\rangle\langle 0|S^\dagger(g)] = S^\dagger(g_{\text{eff}}) \rho_\text{th}(N_{\text{eff}}) S(g_{\text{eff}})$$  \hspace{1cm} (B2)

The effective modulus of the squeezing parameter $g_{\text{eff}}$ and the effective thermal noise $N_{\text{eff}}$ take the form:

$$g_{\text{eff}} = \frac{1}{4} \log \left( \frac{P}{M} \right)$$  \hspace{1cm} (B3)

$$N_{\text{eff}} = \frac{-1 + \sqrt{PM}}{2}$$  \hspace{1cm} (B4)

where:

$$P = \eta e^{2g} + 1 - \eta$$  \hspace{1cm} (B5)

$$M = \eta e^{-2g} + 1 - \eta$$  \hspace{1cm} (B6)

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[4] Aspachs M., Calsamiglia J., Muñoz-Tapia R., & Bagan E., Phase estimation for thermal Gaussian states, Phys. Rev. A 79, 033834 (2009).