On Linear Difference Equations over Rings and Modules *

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Abstract

In this note we develop a coalgebraic approach to the study of solutions of linear difference equations over modules and rings. Some known results about linearly recursive sequences over base fields are generalized to linearly (bi)recursive (bi)sequences of modules over arbitrary commutative ground rings.

Introduction

Although the theory of linear difference equations over base fields is well understood, the theory over arbitrary ground rings and modules is still under development. It is becoming more interesting and is gaining increasingly special importance mainly because of recent applications in coding theory and cryptography (e.g. [HN99], [KKMMN99]).

In a series of papers E. Taft et al. (e.g. [PT80], [LT90], [Taf95]) developed a coalgebraic aspect to the study of linearly recursive sequences over fields. Moreover L. Grünenfelder et al. studied in ([GO93], [GK97]) the linearly recursive sequences over finite dimensional vector spaces. Linearly recursive (bi)sequences over arbitrary rings and modules were studied intensively by A. Nechaev et al. (e.g. [Nec96], [KKMN95], [Nec93]), however the coalgebraic approach in their work was limited to the field case. Generalization to the case of arbitrary commutative ground rings was studied by several authors including V. Kurakin ([Kur94], [Kur00] & [Kur02]) and eventually Abuhlail, Gomez-Torrecillas and Wisbauer [AG-TW00].

In this note we develop a coalgebraic aspect to the study of solutions of linear difference equations over arbitrary rings and modules. For some of our results we assume that the

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ground ring is artinian. Our results generalize also previous results of us in [AG-TW00] and [Abu01 Kapitel 4]. A standard reference for the theory of linearly recursive sequences over rings and modules is the comprehensive work of A. Mikhalev et al. [KKMN95]. For the theory of Hopf algebras the reader may refer to any of the classical references (e.g. Swe69, Abe80 and Mon93).

With \( R \) we denote a commutative ring with \( 1_R \neq 0_R \) and with \( U(R) = \{ r \in R \mid r \text{ is invertible} \} \) the group of units of \( R \). The category of \( R \)-(bi)modules will be denoted by \( \mathcal{M}_R \). For an \( R \)-module \( M \), we call an \( R \)-submodule \( K \subset M \) pure (in the sense of Cohn), if for every \( R \)-module \( N \) the induced map \( \iota_K \otimes \text{id}_N : K \otimes_R N \to M \otimes_R N \) is injective.

For an \( R \)-algebra \( A \) and an \( A \)-module \( M \), we call an \( A \)-submodule \( K \subset M \) \( R \)-cofinite, if \( M/K \) is f.g. in \( \mathcal{M}_R \). For an \( R \)-algebra \( A \) we denote by \( \mathcal{K}_A \) the class of \( R \)-cofinite ideals. If \( A \) is an \( R \)-algebra with \( \mathcal{K}_A \) a filter, then we define for every left \( A \)-module \( M \) the finite dual right \( A \)-module

\[
M^\circ := \{ f \in M^* \mid \text{Ke}(f) \supset IM \text{ for some } A \text{-ideal } I \text{ with } A/I \text{ f.g.} \}.
\]

With \( \mathbb{N} \) resp. \( \mathbb{Z} \) we denote the set of natural numbers resp. the ring of integers. Moreover we set \( \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\} \). For an \( n \times n \) matrix \( M \) over \( R \) we denote the characteristic polynomial with \( \chi(M) \). The identity matrix of order \( n \) over \( R \) is denoted by \( E_n \). For an \( m \times n \) matrix \( A \) and a \( k \times l \) matrix \( B \), the Kronecker product (tensor product) of \( A \) and \( B \) is the \( mk \times nl \) matrix

\[
A \otimes B := \begin{bmatrix}
a_{11} \cdot B & a_{12} \cdot B & \ldots & a_{1n} \cdot B \\
a_{21} \cdot B & a_{22} \cdot B & \ldots & a_{2n} \cdot B \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} \cdot B & a_{m2} \cdot B & \ldots & a_{mn} \cdot B
\end{bmatrix}
\]

1 Preliminaries

Let \( M \) be an \( R \)-module and

\[
M[x] := M[x_1, \ldots, x_k], \quad M[x, x^{-1}] := M[x_1, x_1^{-1}, \ldots, x_k, x_k^{-1}].
\]

We consider the polynomial ring \( R[x] \) and the ring of Laurent polynomials \( R[x, x^{-1}] \) as commutative \( R \)-algebras with the usual multiplication and the usual unity. For every \( R \)-module \( M \), \( M[x] \) (resp. \( M[x, x^{-1}] \)) is an \( R[x] \)-module (resp. an \( R[x, x^{-1}] \)-module) with action induced from the \( R \)-module structure on \( M \) and we have moreover canonical \( R \)-module isomorphisms

\[
M[x] \simeq M \otimes_R R[x] \simeq M^{(\mathbb{N}_0^k)} \quad \text{and} \quad M[x, x^{-1}] \simeq M \otimes_R R[x, x^{-1}] \simeq M^{(\mathbb{Z}^k)}.
\]

For \( n = (n_1, \ldots, n_k) \in \mathbb{N}_0^k \) resp. \( z = (z_1, \ldots, z_k) \in \mathbb{Z}^k \) we set \( x^n := x_1^{n_1} \cdot \cdots \cdot x_k^{n_k} \) resp. \( x^z := x_1^{z_1} \cdot \cdots \cdot x_k^{z_k} \).
1.1. Let $M$ be an $R$-module, $l = (l_1, ..., l_k) \in \mathbb{N}_0^k$ and consider the system of linear difference equations (ab. SLDE)

$$
\begin{align*}
    x_{n+(l_1,0,...,0)} &+ \sum_{i=1}^{l_1} p(1,l_1-i)(n)x_{n+(l_1-i,0,...,0)} &= g_1(n), \\
    x_{n+(0,l_2,0,...,0)} &+ \sum_{i=1}^{l_2} p(2,l_2-i)(n)x_{n+(0,l_2-i,0,...,0)} &= g_2(n), \\
    & \vdots & \vdots & \vdots & \vdots \\
    x_{n+(0,...,0,l_k)} &+ \sum_{i=1}^{l_k} p(k,l_k-i)(n)x_{n+(0,...,0,l_k-i)} &= g_k(n),
\end{align*}
$$

(3)

where the $p_{ji}$'s are $R$-valued functions and the $g_j$'s are $M$-valued functions defined for all $n \in \mathbb{N}_0^k$. If the $g_j$'s are identically zero, then (3) is said to be a homogenous SLDE. If the $p_{ji}$'s are constants, then (3) is said to be a SLDE with constant coefficients.

1.2. For an $R$-module $M$ and $k \geq 1$ let

$$
S_M^{<k>} := \{ u : \mathbb{N}_0^k \to M \} \cong M^{\mathbb{N}_0^k}
$$

be the $R$-module of $k$-sequences over $M$. If $M$ (resp. $k$) is not mentioned, then we mean $M = R$ (resp. $k = 1$). For $f(x) = \sum_i a_i x^i \in R[x]$ and $w \in S_M^{<k>}$ define

$$
f(x) \rightarrow w = u \in S_M^{<k>}, \text{ where } u(n) := \sum_i a_i w(n+i) \text{ for all } n \in \mathbb{N}_0^k.
$$

(4)

With this action $S_M^{<k>}$ is an $R[x]$-module. For subsets $I \subset R[x]$ and $L \subset S_M^{<k>}$ consider the annihilator submodules

$$
\text{An}_{S_M^{<k>}}(I) = \{ w \in S_M^{<k>} | f \rightarrow w = 0 \text{ for every } f \in I \},
$$

$$
\text{An}_{R[x]}(L) = \{ h \in R[x] | h \rightarrow u = 0 \text{ for every } u \in L \}.
$$

Note that $\text{An}_{S_M^{<k>}}(I) \subset S_M^{<k>}$ is an $R[x]$-submodule and $\text{An}_{R[x]}(L) \subset R[x]$ is an ideal.

1.3. A polynomial $f(x) \in R[x]$ is called monic, if its leading coefficient is $1_R$. For every monic polynomial $f(x) = x^l + a_{l-1}x^{l-1} + ... + a_1x + a_0 \in R[x]$, the companion matrix of $f$ is defined to be the $l \times l$ matrix

$$
S_f := \begin{bmatrix}
    0_R & 0_R & \cdots & 0_R & -a_0 \\
    1_R & 0_R & \cdots & 0_R & -a_1 \\
    0_R & 1_R & \cdots & 0_R & -a_2 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0_R & 0_R & \cdots & 1_R & -a_{l-1}
\end{bmatrix}
$$

(5)

$S_f$ is a matrix that has $f(x)$ as its characteristic polynomial as well as its minimum polynomial (Jon73 Theorem 4.18).
Definition 1.4. An ideal \( I \triangleleft R[x] \) will be called monic, if it contains a non-empty subset of monic polynomials
\[
\{ f_j(x_j) = x_j^{l_j} + a_{j-1}^{(j)} x_j^{l_j-1} + \ldots + a_1^{(j)} x_j + a_0^{(j)} \mid j = 1, \ldots, k \}.
\] (6)

In this case the polynomials (6) are called elementary polynomials and \( (f_1(x_1), \ldots, f_k(x_k)) \triangleleft R[x] \) an elementary ideal. A monic polynomial \( q(x) \in R[x] \) is called reversible, if \( q(0) \in U(R) \). An ideal \( I \triangleleft R[x, x^{-1}] \) will be called reversible, if it contains a subset of reversible polynomials \( \{ q_1(x_1), \ldots, q_k(x_k) \} \).

1.5. Let \( M \) be an \( R \)-module. We call \( u \in S_M^{<k>} \) a linearly recursive \( k \)-sequence (resp. a linearly birecursive \( k \)-sequence), if \( An_{R[x]}(u) \) is a monic ideal (resp. a reversible ideal).

Note that \( R[x] \) is a polynomial ring and its monomial ideals are called elementary ideals. Thus, if \( An_{R[x]}(u) \) contains a non-empty subset of reversible \( k \)-sequences, then \( u \) will be called a reversible \( k \)-sequence and \( \{ q_1(x_1), \ldots, q_k(x_k) \} \) will be called a monic \( R \)-sequences. The subsets \( \mathcal{L}_M^{<k>} \subseteq S_M^{<k>} \) of linearly recursive \( k \)-sequences and \( \mathcal{B}_M^{<k>} \subseteq S_M^{<k>\text{rev}} \) of linearly birecursive \( k \)-sequences are obviously \( R[x] \)-submodules.

1.6. ([MN96] Page 170) The lexicographical linear order \( (\preceq) \) on \( \mathbb{N}_0^k \) is defined as follows: for \( i = (i_1, \ldots, i_k) \) and \( n = (n_1, \ldots, n_k) \in \mathbb{N}_0^k \), we say \( i \preceq n \), if the first number in the sequence of integers
\[
(n_1 + \ldots + n_k) - (i_1 + \ldots + i_k), \ n_1 - i_1, \ldots, n_k - i_k
\]
that is different from zero is positive.

Let \( M \) be an \( R \)-module, \( F := \{ f_1(x_1), \ldots, f_k(x_k) \} \subset R[x] \) a subset of monic polynomials with \( \deg(f_j(x_j)) = l_j \) for \( j = 1, \ldots, k, 1 := (l_1, \ldots, l_k) \) and \( I_F := (f_1, \ldots, f_k) \triangleleft R[x] \). Note that the natural order \( (\preceq) \) on \( \mathbb{N}_0^k \) induces on \( \mathbb{N}_0^k \) a partial order and we define the polyhedron \( \Pi_F = \Pi(1) := \{ i \in \mathbb{N}_0^k \mid i \leq 1 \} \). The initial polyhedron of values of \( \omega \in \mathcal{L}_M^{<k>} \) is defined as \( \omega(\Pi_F) := \{ \omega(i) \mid i \in \Pi_F \} \). For \( l = l_1 \cdot \ldots \cdot l_k \) the points of the polyhedron \( \Pi_F \) build a chain \( 0 = i_0 \preceq i_1 \preceq \ldots \preceq i_{l-1} \) and we can write \( \omega(\Pi_F) \) as an initial vector of values \( (\omega(0), \omega(i_1), \ldots, \omega(i_{l-1})) \) \( \in M^l \).

Let \( \omega \in An_{S_M^{<k>}}(f_1(x_1), \ldots, f_k(x_k)) \), where \( f_j(x_j) \) is monic for \( j = 1, \ldots, n \) and write for every \( n = (n_1, \ldots, n_k) \in \mathbb{N}_0^k : \)
\[
x_j^{n_j} = h_j(x_j)f_j(x_j) + r_j(x_j), \ \text{where } \deg(r_j(x_j)) < l_j.
\]
If we set
\[
g^{(n)}(x) := \prod_{j=1}^{k} r_j(x_j) = \sum_{i \in \Pi_F} a_i^{(n)} x_i \text{ and } v := x^n \rightarrow \omega = g^{(n)}(x) \rightarrow \omega,
\]
then
\[
\omega(n) = v(0) = \sum_{i \in \Pi_F} a_i^{(n)} \omega(i) \text{ for every } n \in \mathbb{N}_0^k.
\]
Consequently \( \omega \) is completely determined by the initial polyhedron of values \( \omega(\Pi_F) \). For \( t \in \Pi_F \) define the sequence \( e^F_t \in \text{An}_{S^<_k}(I_F) \) with initial polyhedron of values \( e^F_t(i) = \delta_{i,t} \) for all \( i \in \Pi_F \). The sequence \( e^F_{i-1} \) is called the impulse sequence of \( \text{An}_{S^<_k}(I_F) \).

**Examples**

We give now some examples of linearly recursive sequences. For more examples the reader may refer to [KKMN95].

**Example 1.7.** (Geometric progression). Let \( M \) be an \( R \)-module, \( m \in M \), \( r \in R \) and consider \( w \in S_M \) given by

\[
w(n) := r^nm \quad \text{for every} \quad n \in \mathbb{N}_0.
\]

Then \( w \in L_M \) with initial condition \( w(0) = m \) and elementary characteristic polynomial \( f(x) = x - r \). Moreover \( \text{An}_{R[x]}(w) = R[x](x - r) + R[x]\text{An}_R(r) \).

**Example 1.8.** (Arithmetic progression). Let \( M \) be an \( R \)-module, \( \{p, q\} \subset M \) and consider \( w \in S_M \) given by

\[
w(n) := p + nq \quad \text{for every} \quad n \in \mathbb{N}_0.
\]

Then \( w \in L_M \) with initial vector \( (p, p + q) \) and elementary characteristic polynomial \( f(x) = (x - 1)^2 \). If \( \text{An}_R(q) = 0 \), then \( f(x) \) is a unique minimal polynomial of \( w \). If \( r \in \text{An}_R(q) \), then \( f_r(x) = (x - 1)^2 + r(x - 1) \) is another minimal polynomial of \( w \).

**Remark 1.9.** An example of a non linearly recursive sequences over \( \mathbb{Z} \) is the sequence of prime positive numbers \( \{2, 3, 5, 7, ...\} \).

**Example 1.10.** Let \( E = \{f_1(x), ..., f_k(x)\} \subset R[x] \) be a subset of monic polynomials.

1. Let \( M \) be an \( R \)-module, \( u_i \in \text{An}_{S_M}(f_i) \) for \( i = 1, ..., k \) and consider \( u := u_1 + ... + u_k \in S_M^<_k \) defined by \( u(n) = u_1(n_1) + ... + u_k(n_k) \). Then \( u \in \text{An}_{S_M^<_k}(g_1(x_1), ..., g_k(x_k)) \), where for \( i = 1, ..., k \):

\[
g_i(x_i) = \begin{cases} f_i(x_i), & f_i(1_R) = 0_R \\ f_i(x_i)(x_i - 1_R), & \text{otherwise.} \end{cases}
\] (7)

2. Let \( M_1, ..., M_k \) be \( R \)-modules, \( u_i \in \text{An}_{S_{M_i}}(f_i) \) for \( i = 1, ..., k \), \( M := M_1 \oplus ... \oplus M_k \) and consider \( u \in S_{M}^<_k \) defined by \( u(n) := (u_1(n_1), ..., u_k(n_k)) \). Then \( u \in \text{An}_{S_M^<_k}(g_1(x_1), ..., g_k(x_k)) \), where the \( g_i \)'s are defined as in (7).

3. Let \( u_i \in \text{An}_{S_R}(f_i) \) for \( i = 1, ..., k \) and consider \( u \in S_R^<_k \) defined by \( u(n) := u_1(n_1) \cdot ... \cdot u_k(n_k) \). Then \( u \in \text{An}_{S_R^<_k}(f_1(x_1), ..., f_k(x_k)) \) and

\[
\text{An}_{S_R^<_k}(f_1(x_1), ..., f_k(x_k)) \simeq \text{An}_{S_R}(f_1) \otimes_R ... \otimes_R \text{An}_{S_R}(f_k).
\]
4. Let $M_1, \ldots, M_k$ be $R$-modules, $u_i \in \text{An}_{S_M}(f_i)$ for $i = 1, \ldots, k$, $M := M_1 \otimes_R \cdots \otimes_R M_k$ and consider $u \in S_M^{<k>}$ defined by $u(n) := u_1(n_1) \otimes \cdots \otimes u_k(n_k)$. Then $u \in \text{An}_{S_M^{<k>}}(f_1(x_1), \ldots, f_k(x_k))$ and

$$\text{An}_{S_M^{<k>}}(f_1(x_1), \ldots, f_k(x_k)) \cong \text{An}_{S_M}(f_1) \otimes_R \cdots \otimes_R \text{An}_{S_M}(f_k).$$

**Admissible $R$-bialgebras and Hopf $R$-algebras**

For every $R$-coalgebra $(C, \Delta_C, \varepsilon_C)$ there is a dual $R$-algebra $C^* := \text{Hom}_R(C, R)$ with multiplication the so called convolution product

$$(f \star g)(c) := \sum f(c_1)g(c_2) \text{ for all } f, g \in C^*, c \in C$$

and unity $\varepsilon_C$. Although every algebra $A$ has a dual coalgebra, if the ground ring is hereditary noetherian (e.g. a field), the existence of dual coalgebras of algebras over an arbitrary commutative ground rings is not guaranteed!! One way to handle this problem is to restrict the class of $R$-algebras, for which the dual $R$-coalgebras are defined.

**Definition 1.11.** Let $A$ be an $R$-algebra (resp. an $R$-bialgebra, a Hopf $R$-algebra). Then we call $A$:

1. an $\alpha$-algebra (resp. an $\alpha$-bialgebra, a Hopf $\alpha$-algebra), if $K_A$ is a filter and $A^o \subset R^A$ is pure.

2. cofinitary, if $K_A$ is a filter and for every $I \in K_A$ there exists an $A$-ideal $T \supseteq I$ with $A/T$ f.g. and projective.

**1.12.** Let $H$ be an $R$-bialgebra and consider the class of $R$-cofinite $H$-ideals $K_H$. We call $H$ an admissible $R$-bialgebra, if $H$ is cofinitary and $K_H$ satisfies the following axioms:

$$(\text{A1}) \quad \forall I, J \in K_H \text{ there exists } L \in K_H, \text{ s.t. } \Delta_H(L) \subseteq I \otimes_R H + H \otimes_R J$$

(8)

and

$$(\text{A2}) \quad \exists I \in K_H, \text{ s.t. } \text{Ke}(\varepsilon_H) \supseteq I.$$ (9)

We call a Hopf $R$-algebra $H$ an admissible Hopf $R$-algebra, if $H$ is cofinitary, $K_H$ satisfies (A1), (A2) and

$$\text{(A3) \quad for every } I \in K_H \text{ there exists } J \in K_H, \text{ s.t. } S_H(J) \subseteq I.$$ (10)

**Remark 1.13.** It follows from the proof of [AG-TL01 Proposition 4.2.], that every cofinitary $R$-algebra (resp. $R$-bialgebra, Hopf $R$-algebra) is an $\alpha$-algebra (resp. an $\alpha$-bialgebra, a Hopf $\alpha$-algebra). By ([Abu01 Lemma 2.5.6.]) every cofinitary bialgebra (Hopf algebra) over a noetherian ground ring is admissible.
Proposition 1.14. ([Abu01, Proposition 2.4.13, Proposition 2.5.7])

1. If \( A \) is a cofinitary \( R \)-algebra, then \( A^\circ \) is an \( R \)-coalgebra. If \( H \) is an admissible \( R \)-bialgebra (resp. an admissible Hopf \( R \)-algebra), then \( H^\circ \) is an \( R \)-bialgebra (resp. a Hopf \( R \)-algebra).

2. Let \( R \) be noetherian. If \( A \) is an \( \alpha \)-algebra (resp. an \( \alpha \)-bialgebra, a Hopf \( \alpha \)-algebra), then \( A^\circ \) is an \( R \)-coalgebra (resp. an \( R \)-bialgebra, a Hopf \( R \)-algebra).

Proposition 1.15. Let \( A \) be an \( \alpha \)-algebra (resp. an \( \alpha \)-bialgebra, a Hopf \( \alpha \)-algebra), \( B \) a cofinitary \( R \)-algebra (resp. \( R \)-bialgebra, Hopf \( R \)-algebra) and consider the canonical map \( \sigma : A^\circ \otimes_R B^\circ \to (A \otimes_R B)^\circ \). Then:

1. \( \sigma \) is injective.

2. If \( R \) is noetherian, then \( \sigma \) is an isomorphism of \( R \)-coalgebras (resp. \( R \)-bialgebras, Hopf \( R \)-algebras).

Proof. 1. The proof is along the lines of the proof of [Kur02, Proposition 5].

2. The proof is along the lines of the proof of [AG-TL01, Theorem 4.10].

The proof of [AG-TL01, Lemma 4.12] can be generalized to get

Lemma 1.16. For any set of reversible polynomials \( \{q_1(x_1), ..., q_k(x_k)\} \subseteq R[x] \) we have an isomorphism of \( R \)-algebras

\[
R[x]/(q_1(x_1), ..., q_k(x_k)) \simeq R[x, x^{-1}]/(q_1(x_1), ..., q_k(x_k)).
\]

Lemma 1.17. ([Kur02, Proposition 1]) Let \( R \) be an arbitrary commutative ring.

1. An ideal \( I \prec R[x] \) is \( R \)-cofinite, iff it’s monic. Consequently every \( R \)-cofinite \( R[x] \)-ideal contains an ideal \( \overline{I} \prec R[x] \), such that \( R[x]/\overline{I} \) is free of finite rank. In particular \( R[x] \) is cofinitary.

2. An ideal \( I \prec R[x, x^{-1}] \) is \( R \)-cofinite, iff it’s reversible. Consequently every \( R \)-cofinite \( R[x, x^{-1}] \)-ideal contains an ideal \( \overline{I} \prec R[x, x^{-1}] \), such that \( R[x, x^{-1}]/\overline{I} \) is free of finite rank. In particular \( R[x, x^{-1}] \) is cofinitary.

2 Linearly (bi)recursive sequences

In this section we study the linearly (bi)recursive \( k \)-sequences over \( R \)-modules, where \( R \) is an arbitrary commutative ground ring.
2.1. Let \((G, \mu_G, \epsilon_G)\) be a (commutative) monoid. Considering the elements of the basis \(G\) as group-like elements, the monoid algebra \(RG\) becomes a (commutative) cocommutative \(R\)-bialgebra \((RG, \mu, \eta, \Delta_g, \epsilon_g)\), where
\[
\Delta_g(x) = x \otimes x \text{ and } \epsilon_g(x) = 1_R \text{ for every } x \in G.
\]

If \(G\) is a group, then \(RG\) is a Hopf \(R\)-algebra with antipode
\[
S_g : RG \to RG, \quad x \mapsto x^{-1} \text{ for every } x \in G.
\]

2.2. Bialgebra structures on \(R[x]\). Consider the commutative monoid \(G\) generated by \(\{x_j \mid j = 1, ..., k\}\). Then \(R[x] = RG\) has the structure of a commutative cocommutative \(R\)-bialgebra \((R[x], \mu, \eta, \Delta_g, \epsilon_g)\), where \(\mu\) is the usual multiplication, \(\eta\) is the usual unity and
\[
\Delta_g : R[x] \to R[x] \otimes_R R[x], \quad x_j^n \mapsto x_j^n \otimes x_j^n, \quad \forall \ n \geq 0, j = 1, ..., k,
\]
\[
\epsilon_g : R[x] \to R, \quad x_j^n \mapsto 1_R, \quad \forall \ n \geq 0, j = 1, ..., k.
\]

On the other hand \(R[x; p] = (R[x], \mu, \eta, \Delta_g, \epsilon_g)\) is a commutative cocommutative Hopf \(R\)-algebra, where \(\mu\) is the usual multiplication, \(\eta\) is the usual unity and
\[
\Delta_p : R[x] \to R[x] \otimes_R R[x], \quad x_j^n \mapsto \sum_{t=0}^{n} \binom{n}{t} x_j^t \otimes x_j^{n-t}, \quad \forall \ n \geq 0, j = 1, ..., k,
\]
\[
\epsilon_p : R[x] \to R, \quad x_j^n \mapsto \delta_{n,0}, \quad \forall \ n \geq 0, j = 1, ..., k,
\]
\[
S_p : R[x] \to R[x], \quad x_j^n \mapsto (-1)^n x_j^n, \quad \forall \ n \geq 0, j = 1, ..., k.
\]

Remarks 2.3. 1. Let \(R\) be an integral domain, then it follows by [Grbi09, Theorem 1.3.6] that for every set \(G\), the class of group-like elements of the \(R\)-coalgebra \(RG\) is \(G\) itself. Then one can show as in the field case [CG93], that \(R[x; g]\) and \(R[x; p]\) are the only possible \(R\)-bialgebra structures on \(R[x]\) with the usual multiplication and the usual unity.

2. The \(R\)-bialgebra \(R[x; g]\) has no antipode, because the group-like elements in a Hopf \(R\)-algebra should be invertible.

The proof of the following result depends mainly on arguments of [Kur02, Theorem 2]:

Proposition 2.4. Let \(R\) be an arbitrary commutative ring. Then \(R[x; g]\) is an admissible \(R\)-bialgebra and \(R[x; p]\) is an admissible Hopf \(R\)-algebra. Hence \(R[x; g]^\circ\) is an \(R\)-bialgebra and \(R[x; p]^\circ\) is a Hopf \(R\)-algebra.

Proof. Denote with \((R[x], \Delta, \epsilon)\) either of the cofinitary \(R\)-bialgebras \(R[x; g]\) and \(R[x; p]\). Let \(I, J \triangleleft R[x]\) be \(R\)-cofinite ideals and assume w.l.o.g. that \(R[x]/I\) and \(R[x]/J\) are free of finite rank (see Lemma 1.17). Let \(\beta\) be a basis of the free \(R\)-module \(B := R[x]/I \otimes_R R[x]/J\) and consider the \(R\)-algebra morphism \(\overline{\Delta} := (\pi_I \otimes \pi_J) \circ \Delta : R[x] \to R[x]/I \otimes_R R[x]/J\). For \(j = 1, ..., k\) let \(M_j\) be the matrix of the \(R\)-linear map
\[
T_j : B \to B, \quad b \mapsto \overline{\Delta}(x_j)b
\]
w.r.t. $\beta$ and $\chi_j(\lambda)$ its characteristic polynomial. Then $\chi_j(\overline{\Delta}(x_j)) = 0$ for $j = 1, ..., k$. Since $\overline{\Delta}$ is an $R$-algebra morphism, it follows that $\chi_j(x_j) \in \ker(\overline{\Delta}) = \Delta^{-1}(I \otimes_R R[x] + R[x] \otimes_R J)$ for $j = 1, ..., k$. If we set $L := (\chi_1(x_1), ..., \chi_k(x_k)) \subset R[x]$, then $\Delta(L) \subset I \otimes_R R[x] + R[x] \otimes_R J$, i.e. $K_{R[x]}$ satisfies axiom (9). Note that $R[x]/\ker(\varepsilon) \simeq R$, hence $K_{R[x]}$ satisfies axiom (9). Consequently $R[x; g]$ and $R[x; p]$ are admissible $R$-bialgebras. Consider now the Hopf $R$-algebra $R[x; p]$ with the bijective antipode $S_p$. For every ideal $I \lhd R[x]$, $S^{-1}_p(I) \lhd R[x; p]$ is an ideal and we have an isomorphism of $R$-modules $R[x]/S^{-1}_p(I) \simeq R[x]/I$, hence $K_{R[x; p]}$ satisfies axiom (10). Consequently $R[x; p]$ is an admissible Hopf $R$-algebra. The last statement follows now by Proposition 1.14.

If $M$ is an arbitrary $R$-module, then we have obviously an isomorphism of $R[x]$-modules

$$\Phi_M : M[x]^* \rightarrow S_{M^*}^{<k>}, \; \alpha \mapsto [n \mapsto m \mapsto \alpha(mx^n)]$$

with inverse $u \mapsto [mx^n \mapsto u(n)(m)]$.

**Proposition 2.5.** Let $M$ be an $R$-module. Then (11) induces an isomorphism of $R[x]$-modules

$$M[x]^* \simeq L^{<k>}_M.$$  \hspace{1cm} (12)

**Proof.** Consider the $R[x]$-module isomorphism $M[x]^* \cong S_{M^*}^{<k>}$ (11). Let $\alpha \in M[x]^*$. Then there exists an $R$-cofinite $R[x]$-ideal $I$, such that $I \vDash \alpha = 0$. So $I \hookrightarrow \Phi(\alpha) = \Phi(I \hookrightarrow \alpha) = 0$, i.e. $I \subset \text{An}_{R[x]}(\Phi(\alpha))$. By Lemma 1.17 (1) $I$ is monic, i.e. $\Phi(\alpha) \in L^{<k>}_M$.

On the other hand, let $u \in L^{<k>}_M$. By definition $J := \text{An}_{R[x]}(u)$ is a monic ideal and it follows by Lemma 1.17 (1) that $J \lhd R[x]$ is $R$-cofinite. For $\alpha := \Phi^{-1}(u)$ we have $J \vDash \alpha = J \rightarrow \Phi^{-1}(u) = \Phi^{-1}(J \rightarrow u) = 0$, i.e. $\alpha \in M[x]^*$.

**2.6. The coalgebra structure on $L^{<k>}$.**

By Lemma 1.17 (1) $(R[x], \mu, \eta)$ is a cofinitary $R$-algebra, where $\mu$ is the usual multiplication and $\eta$ is the usual unity. Hence $(R[x]^*, \mu^*, \eta^*)$ is (by Proposition 1.14) an $R$-coalgebra, where

$$\mu^* : R[x]^* \rightarrow R[x]^* \otimes_R R[x]^*, \; f \mapsto [x_i^s \otimes x_j^t \mapsto f(x_i^s x_j^t), s, t \geq 0, i, j = 1, ..., k],$$

$$\eta^* : R[x]^* \rightarrow R, \; f \mapsto f(1_R).$$

So $L^{<k>}_M \simeq R[x]^*$ has the structure of an $R$-coalgebra with counity

$$\varepsilon_{L^{<k>} : L^{<k>} \rightarrow R, \; u \mapsto u(0).}$$

and comultiplication described as follows (see [KKMN93, Proposition 14.16]):

Let $u \in L^{<k>}, \{f(x_{k_1}), ..., f(x_{k_l})\} \subset \text{An}_{R[x]}(u)$ a subset of elementary characteristic polynomials with $\deg(f_j(x_j)) = l_j$ and $1 := (l_1, ..., l_k)$. So we have for all $n, i \in \mathbb{N}_0^k$:

$$u(n + i) = (x^i \rightarrow u)(n) = (\sum_{t \leq i - 1} (x^i \rightarrow u)(t) \cdot e^p_t)(n) = \sum_{t \leq i - 1} (x^t \rightarrow u)(i) \cdot e^p_t(n).$$
The comultiplication of $\mathcal{L}^{<k>}$ is given then by
\[
\Delta_{\mathcal{L}^{<k>}} : \mathcal{L}^{<k>} \rightarrow \mathcal{L}^{<k>} \otimes_R \mathcal{L}^{<k>}, \quad u \mapsto \sum_{t \leq 1} (x^t \mapsto u) \otimes e_t^F. \tag{14}
\]

**Example 2.7.** Consider the Fibonacci sequence $F = (0, 1, 1, 2, 3, 5, ...).$ Clearly $F$ is given by
\[
F(0) = 0, \quad F(1) = 1, \quad F(n + 2) = F(n + 1) + F(n) \quad \text{for all } n \geq 0,
\]
i.e. $F \in \mathcal{L}_Z$ with initial vector $(0, 1)$ and elementary characteristic polynomial $f(x) = x^2 - x - 1 \in \mathbb{Z}[x].$ By (14) one can easily calculate
\[
\Delta_{\mathcal{L}_Z}(F) = F \otimes_Z (x \mapsto F) + (x \mapsto F) \otimes_Z F - F \otimes_Z F.
\]

**2.8. The $R$-bialgebra $(\mathcal{L}_R^{<k>; g}).$** Consider the $R$-bialgebra $R[x; g].$ Then $\mathcal{S}^{<k>} \simeq R^{\mathbb{N}_0} \simeq R[x; g]^* \simeq$ is an $R$-algebra with multiplication given by the Hadamard product
\[
\ast_g : \mathcal{S}^{<k>} \otimes_R \mathcal{S}^{<k>} \rightarrow \mathcal{S}^{<k>}, \quad u \otimes v \mapsto [n \mapsto u(n)v(n)] \tag{15}
\]
and the unity
\[
\eta_g : R \rightarrow \mathcal{S}^{<k>}, \quad 1_R \mapsto [n \mapsto 1] \quad \text{for every } n \in \mathbb{N}_0^k. \tag{16}
\]

By Propositions 2.4 and 2.5 $(\mathcal{L}_R^{<k>; g}) \simeq R[x; g]^*$ has the structure of an $R$-bialgebra with the coalgebra structure described in 2.6 the Hadamard product (15) and the unity (16).

**2.9. The Hopf $R$-algebra $(\mathcal{L}_R^{<k>; p}).$** Consider the Hopf $R$-algebra $R[x; p].$ Then $\mathcal{S}^{<k>} \simeq R^{\mathbb{N}_0^k} \simeq R[x; p]^* \simeq$ is an $R$-algebra with multiplication given by the Hurwitz product
\[
\ast_p : \mathcal{S}^{<k>} \otimes_R \mathcal{S}^{<k>} \rightarrow \mathcal{S}^{<k>}, \quad u \otimes v \mapsto [n \mapsto \sum_{t \leq n} \binom{n}{t} u(t)v(n - t)] \tag{17}
\]
and the unity
\[
\eta_p : R \rightarrow \mathcal{S}^{<k>}, \quad 1_R \mapsto [n \mapsto \delta_{n0}] \quad \text{for every } n \in \mathbb{N}_0^k. \tag{18}
\]

By Propositions 2.4 and 2.5 $(\mathcal{L}_R^{<k>; p}) \simeq R[x; p]^*$ has the structure of a Hopf $R$-algebra with the coalgebra structure described in 2.6 the Hurwitz product (17), the unity (18) and the antipode
\[
S_{\mathcal{L}^{<k>}} : \mathcal{L}^{<k>} \rightarrow \mathcal{L}^{<k>}, \quad u \mapsto [i \mapsto (-1)^i u(i)].
\]

**Proposition 2.10.** ([Kur02 Theorem 3]) Let $u$ and $v$ be linearly recursive sequences over $R$ of orders $m, n$ and with characteristic polynomials $f(x), g(x)$ respectively. Then

1. $u \ast_g v$ is a linearly recursive sequence over $R$ of order $m \cdot n$ and characteristic polynomial $\chi(S_f \otimes S_g);$

2. $u \ast_p v$ is a linearly recursive sequence over $R$ of order $m \cdot n$ and characteristic polynomial $\chi(S_f \otimes E_n + E_m \otimes S_g).$
Example 2.11. Let $R$ be any ring and $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty \in S_R$ be solutions of the difference equations

\[
x_{n+3} - x_{n+2} + x_{n-1} - x_n = 0; \quad x_0 = 0, \quad x_1 = 1, \quad x_2 = 2;
\]
\[
y_{n+2} - y_{n+1} + y_n = 0; \quad y_0 = 1, \quad y_1 = 0.
\]

Then $\{x_n\}_{n=0}^\infty$ is a linearly recursive sequence over $R$ with characteristic polynomial $f(x) = x^3 - x^2 + x - 1$ and $\{y_n\}_{n=0}^\infty$ is a linearly recursive sequence over $R$ with characteristic polynomial $g(x) = x^2 - x + 1$.

Notice that

\[
S_f \otimes S_g = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & -1 \\
0 & 1 & 1
\end{bmatrix} \otimes \begin{bmatrix}
0 & -1 \\
1 & 1
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & -1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}.
\]

Hence $\{z_n\}_{n=0}^\infty := \{x_n\}_{n=0}^\infty \ast g \{y_n\}_{n=0}^\infty$ is by Proposition 2.10 a linearly recursive sequence over $R$ with characteristic polynomial

\[
\chi(S_f \otimes S_g) = x^6 - x^5 + x^3 - x + 1,
\]

i.e. $\{z_n\}_{n=0}^\infty$ is a solution of the difference equation

\[
z_{n+6} - z_{n+5} + z_{n+3} - z_{n+1} + z_n = 0 \text{ with initial vector } (0,0,-2,-1,0,1).
\]

The following table gives the first 11 terms of the sequences $\{z_n\}_{n=0}^\infty$:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|---|----|
| $x_n$ | 0 | 1 | 2 | 1 | 0 | 1 | 2 | 1 | 0 | 1 | 2 |
| $y_n$ | 1 | 0 | -1 | -1 | 0 | 1 | 1 | 0 | -1 | -1 | 0 |
| $z_n$ | 0 | 0 | -2 | -1 | 0 | 1 | 2 | 0 | 0 | -1 | 0 |

Example 2.12. Consider the sequences $\{x_n\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty$ of the previous example. Then

\[
S_f \otimes E_2 + E_3 \otimes S_g = \begin{bmatrix}
0 & -1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & -1 & -1 & 0 \\
0 & 1 & 1 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 & 1 & 2
\end{bmatrix}.
\]
By Proposition 2.10 \( \{z_n\}_{n=0}^{\infty} = \{x_n\}_{n=0}^{\infty} \ast_p \{y_n\}_{n=0}^{\infty} := \{ \sum_{j=0}^{n} \binom{n}{j} x_j \cdot y_{n-j} \}_{n=0}^{\infty} \) is a linearly recursive sequence over \( R \) with characteristic polynomial

\[
\chi(S_\ell \otimes E_2 + E_3 \otimes S_\ell) = x^6 - 5x^5 + 14x^4 - 25x^3 + 28x^2 - 15x + 3.
\]

Hence \( \{z_n\}_{n=0}^{\infty} \) is a solution of the difference equation

\[
z_{n+6} - 5z_{n+5} + 14z_{n+4} - 25z_{n+3} + 28z_{n+2} - 15z_{n+1} + 3z_n = 0
\]

with initial vector \((0, 1, 2, -2, -16, -29)\).

The following table gives the first 9 terms of the sequences \( \{z_n\}_{n=0}^{\infty} \):

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-------|---|---|---|---|---|---|---|---|---|
| \( x_n \) | 0 | 1 | 2 | 1 | 0 | 1 | 2 | 1 | 0 |
| \( y_n \) | 1 | 0 | -1 | -1 | 0 | 1 | 1 | 0 | -1 |
| \( z_n \) | 0 | 1 | 2 | -2 | -16 | -29 | -12 | 29 | 0 |

### 2.13. Cofree comodules

Let \( C \) be an \( R \)-coalgebra. A right \( C \)-comodule \( (M, \varrho_M) \) is called cofree, if there exists an \( R \)-module \( K \), such that \( (\varrho_M) \simeq (K \otimes_R C, id_K \otimes \Delta_C) \) as right \( C \)-comodules. Note that if \( K \simeq R^{(\Lambda)} \), a free \( R \)-module, then \( M \simeq R^{(\Lambda)} \otimes_R C \simeq C^{(\Lambda)} \) as right \( C \)-comodules (this is one reason of the terminology cofree).

As a direct consequence of Lemma 1.17 we get

**Corollary 2.14.** Let \( M \) be an \( R[x] \)-module. Then we have an isomorphism of \( R[x]^{\circ} \)-comodules

\[
L_{M^{\circ}}^{<k>} \simeq M[x]^{\circ} \otimes_R R[x]^{\circ} \simeq M^* \otimes_R L_R^{<k>}.
\]

In particular \( M[x]^{\circ} (L_{M^{\circ}}^{<k>}) \) is a cofree \( R[x]^{\circ} \)-comodule \( (L_R^{<k>}-comodule) \).

### 3. Linearly (bi)recursive bisequences

In this section we consider the linearly (bi)recursive \( k \)-bisequences and the reversible \( k \)-sequences over \( R \)-modules, where \( R \) is an arbitrary commutative ground ring. We generalize results of [LT90] and [KKMN95] concerning the bialgebra structure of the linearly recursive sequences over a base field to the case of arbitrary artinian ground rings.

**3.1.** Let \( M \) be an \( R \)-module, \( l=(l_1, \ldots, l_k) \in \mathbb{N}_0^k \) and consider the system of linear bidifference equations (ab. SLBE)

\[
\begin{align*}
x_{z+(l_1,0,\ldots,0)} & \leftarrow \sum_{i=1}^{l_1} p(1,l_1-i) \cdot \langle z \rangle \cdot x_{z+(l_1-i,0,\ldots,0)} = g_1(z), \\
x_{z+(0,l_2,0,\ldots,0)} & \leftarrow \sum_{i=1}^{l_2} p(2,l_2-i) \cdot \langle z \rangle \cdot x_{z+(0,l_2-i,0,\ldots,0)} = g_2(z), \\
& \cdots \\
& \cdots \\
x_{z+(0,\ldots,0,l_k)} & \leftarrow \sum_{i=1}^{l_k} p(k,l_k-i) \cdot \langle z \rangle \cdot x_{z+(0,\ldots,0,l_k-i)} = g_k(z),
\end{align*}
\]

(19)
where the \( p_{ji} \)'s are \( R \)-valued functions and the \( g_j \)'s are \( M \)-valued functions defined for all \( z \in \mathbb{Z}^{<k>} \). If the \( g_j \)'s are identically zero, then (19) is said to be a homogenous SLBE. If the \( p_{ji} \)'s are constants, then (19) is said to be a SLBE with constant coefficients.

### 3.2. Bisequences

For an \( R \)-module \( M \) and \( k \geq 0 \) let

\[
\tilde{\mathcal{S}}^{<k>_M} := \{ \tilde{\nu} : \mathbb{Z}^k \to M \} \cong M^{\mathbb{Z}^k}
\]

be the \( R \)-module of \( k \)-bisequences over \( M \). If \( M \) (resp. \( k \)) is not mentioned, then we mean \( M = R \) (resp. \( k = 1 \)). For \( \tilde{\omega} \in \tilde{\mathcal{S}}^{<k>_M} \) and \( f(x) = \sum a_i x^i \in R[x, x^{-1}] \) define

\[
f(x) \to \tilde{\omega} = \tilde{\nu} \in \tilde{\mathcal{S}}^{<k>_M}, \text{ where } \tilde{\nu}(z) := \sum_i a_i \tilde{\omega}(z + i) \text{ for all } z \in \mathbb{Z}^k.
\]

With this action \( \tilde{\mathcal{S}}^{<k>_M} \) becomes an \( R[x, x^{-1}] \)-module. For subsets \( I \subset R[x, x^{-1}] \) and \( Y \subset \tilde{\mathcal{S}}^{<k>_M} \) consider

\[
\begin{align*}
\text{An}_{\tilde{\mathcal{S}}^{<k>_M}}(I) &= \{ \tilde{\omega} \in \tilde{\mathcal{S}}^{<k>_M} | g \to \tilde{\omega} = 0 \text{ for every } g \in I \}, \\
\text{An}_{R[x, x^{-1}]}(Y) &= \{ h \in R[x, x^{-1}] | h \to \tilde{\nu} = 0 \text{ for every } \tilde{\nu} \in Y \}.
\end{align*}
\]

Obviously \( \text{An}_{\tilde{\mathcal{S}}^{<k>_M}}(I) \subset \mathcal{S}^{<k>_M} \) is an \( R[x, x^{-1}] \)-submodule and \( \text{An}_{R[x, x^{-1}]}(Y) \subset R[x, x^{-1}] \) is an ideal.

**Definition 3.3.** Let \( M \) be an \( R \)-module. We call \( \tilde{\omega} \in \tilde{\mathcal{S}}^{<k>_M} \) a linearly recursive \( k \)-bisequence (resp. a linearly birecursive \( k \)-bisequence), if \( \text{An}_{R[x]}(\tilde{\omega}) \) is a monic ideal (resp. a reversible ideal). Note that a \( k \)-bisequence \( \tilde{w} \in \tilde{\mathcal{S}}^{<k>_M} \) is linearly recursive, if it’s a solution of a homogenous SLBE with constants coefficients of the form (19). The subsets \( \tilde{\mathcal{L}}^{<k>_M} \subset \tilde{\mathcal{S}}^{<k>_M} \) of linearly recursive \( k \)-bisequences and \( \tilde{\mathcal{B}}^{<k>_M} \subset \tilde{\mathcal{S}}^{<k>_M} \) of linearly birecursive \( k \)-bisequences over \( M \) are obviously \( R[x, x^{-1}] \)-submodules.

### Reversible sequences over modules

**3.4.** Let \( M \) be an \( R \)-module. A \( k \)-bisequence \( \tilde{u} \) is said to be a reverse of \( u \in \mathcal{S}^{<k>_M} \), if \( \tilde{u}|_{\mathbb{N}^k_0} = u \) and \( \text{An}_{R[x]}(\tilde{u}) = \text{An}_{R[x]}(u) \). A linearly recursive \( k \)-sequence \( u \) will be called reversible, if \( u \) has a reverse \( \tilde{u} \in \tilde{\mathcal{L}}^{<k>_M} \). With \( \mathcal{R}^{<k>_M} \subset \mathcal{L}^{<k>_M} \) we denote the \( R[x] \)-submodule of reversible \( k \)-sequences over \( M \).

**Lemma 3.5.** (Compare [KKMN95, Proposition 14.11]) Let \( R \) be artinian.

1. Every monic ideal \( I \subset R[x] \) contains a subset of monic polynomials

\[
\{ x_j^{d_j} q_j(x_j) | q_j(x_j) \text{ is reversible for } j = 1, \ldots, k \}. \tag{20}
\]
2. Let $M$ be an $R$-module. Then every linearly recursive $k$-bisequence over $M$ is linearly birecursive (i.e. $\mathcal{B}_M^{<k>} = \mathcal{L}_M^{<k>}$.)

**Proof.** 1. By [AM69, 8.7] every commutative artinian ring is (up to isomorphism) a direct sum of local artinian rings. W.l.o.g. let $R$ be a local artinian ring. The Jacobson radical of $R$

\[ J(R) = \{ r \in R | r \text{ is not invertible in } R \} \]

is nilpotent, hence there exists a positive integer $n$, such that $J(R)^n = 0$. Let $I$ be a monic ideal with a subset of monic polynomials \( \{ g_1(x_1), ..., g_k(x_k) \} \subset I \). If $g_j(x_j) \equiv f_j(x_j) \pmod{J(R)[x_j]}$ for $j = 1, ..., k$, then $g_j(x_j)|f_j(x_j)^n$, where $n$ is the index of nilpotency of the ideal $J(R)$. Hence $f_j(x_j)^n \in I$. If we write $f_j(x_j)^n = x_j^{d_j} q_j(x_j)$ with $(x_j, q_j(x_j)) = 1$, then $q_j(0) \in U(R)$, i.e. $q_j(x_j)$ is a reversible polynomial for $j = 1, ..., k$.

2. Let $\tilde{u}$ be a linearly recursive $k$-bisequence over $M$. If $R$ is artinian, then $\mathrm{An}_{R[x]}(\tilde{u})$ contains by (1) a subset of monic polynomials \( \{ x_j^{d_j} q_j(x_j) | q_j(x_j) \text{ is reversible for } j = 1, ..., k \} \). Then for every $z \in \mathbb{Z}^k$ we have $(q_j(x_j) \rightarrow \tilde{u})(z_1, ..., z_j, ..., z_k) = (x_j^{d_j} q_j(x_j) \rightarrow \tilde{u})(z_1, ..., z_j - d_j, ..., z_k) = 0$. Hence \( q_j(x_j) | i = 1, ..., k \} \subset \mathrm{An}_{R[x]}(\tilde{u}) \), i.e. $\mathrm{An}_{R[x]}(\tilde{u})$ is a reversible ideal.

**3.6. Backsolving.** Let $M$ be an $R$-module. Let $u$ be a linearly recursive sequence over $M$ and assume that $\mathrm{An}_{R[x]}(u)$ contains some monic polynomial of the form $x^d q(x) = x^d(a_0 + a_1 x + ... + a_{l-1} x^{l-1} + x^l)$, $a_0 \in U(R)$. Then

\[ a_0 u(j + d) + a_1 u(j + d + 1) + ... + a_{l-1} u(j + d + l - 1) + u(j + d + l) = 0 \text{ for all } j \geq 0 \]

and we get by Backsolving a unique linearly birecursive bisequence $\tilde{u} \in \mathrm{An}_{\mathcal{S}_d}(q(x))$ with $\tilde{u}(n) = u(n)$ for all $n \geq d$. The bisequence $\tilde{u} \equiv 0$ in case $l = 0$ and is given for $l \neq 0$ by

\[ \tilde{u}(z) := \begin{cases} u(z), & z \geq d \\ -a_0^{-1} (a_1 \tilde{u}(z + 1) + ... + a_{l-1} \tilde{u}(z + l - 1) + \tilde{u}(z + l)), & z < d. \end{cases} \]

If there are two bisequences $\tilde{v}, \tilde{w} \in \mathrm{An}_{\mathcal{S}_d}(q(x))$ with $\tilde{v}(n) = u(n) = \tilde{w}(n)$ for all $n \geq d$, then one can easily show by backsolving using $q(x)$ that $\tilde{v} = \tilde{w}$. Moreover we claim that $\mathrm{An}_{R[x]}(\tilde{u}) = \mathrm{An}_{R[x]}(u)$. It’s obvious that $\mathrm{An}_{R[x]}(\tilde{u}) \subseteq \mathrm{An}_{R[x]}(u)$. On the other hand assume $g(x) = \sum_{j=0}^{m} b_j x^j \in \mathrm{An}_{R[x]}(u)$. We prove by induction that $(g \rightarrow \tilde{u})(z) = 0$ for all $z \in \mathbb{Z}$. First of all, note that for all $z \geq d$ we have $(g \rightarrow \tilde{u})(z) = (g \rightarrow u)(z) = 0$. Now let $z_0 < d$ and assume that $(g \rightarrow \tilde{u})(z) = 0$ for $z \in \{ z_0, z_0 + 1, ..., z_0 + l - 1 \} \subseteq \mathbb{Z}$. Then we have for
z = z_0 - 1:

\[(g \rightarrow \tilde{u})(z_0 - 1) = \sum_{j=0}^{m} b_j \tilde{u}(j + z_0 - 1)\]

\[= \sum_{j=0}^{m} b_j (\sum_{i=1}^{l} -a_0^{-1} a_i \tilde{u}(j + z_0 - 1 + i))\]

\[= -\sum_{i=1}^{l} a_0^{-1} a_i \sum_{j=0}^{m} b_j \tilde{u}(j + z_0 - 1 + i)\]

\[= -\sum_{i=1}^{l} a_0^{-1} a_i (g \rightarrow \tilde{u})(z_0 - 1 + i)\]

\[= 0.\]

If \(u\) is a linearly recursive \(k\)-sequence over \(M\) with \(k > 1\) and \(A_{R[x]}(u)\) contains a set of monic polynomials \(\{x_j^d : q_j(x)\mid q_j\text{ is reversible for } j = 1, \ldots, k\}\), then we get by backsolving through \(q_j(x_j)\) along the \(j\)-th row for \(j = 1, \ldots, k\) a unique linearly birecursive \(k\)-bisequence \(\tilde{u} \in \mathbb{A}_{\mathcal{S}_M^{<k>}}(\{q_1(x_1), \ldots, q_k(x_k)\})\) with \(\tilde{u}(n) = u(n)\) for all \(n \geq d\) and it follows moreover that \(A_{R[x]}(\tilde{u}) = A_{R[x]}(u)\).

**Lemma 3.7.** Let \(M\) be an \(R\)-module.

1. Every birecursive \(k\)-sequence over \(M\) is reversible with unique reverse (which we denote by Rev(u)). Moreover \(\mathcal{B}_{M}^{<k>}\) becomes a structure of an \(R[x, x^{-1}]\)-module through \(f \rightarrow u := (f \rightarrow \text{Rev}(u))|_{\mathcal{B}_{M}^{<k>}}\).

2. If \(R\) is artinian, then every reversible \(k\)-sequence over \(M\) is birecursive as well (i.e. \(\mathcal{B}_{M}^{<k>} = \mathcal{R}_{M}^{<k>}\)).

**Proof.** Let \(M\) be an \(R\)-module.

1. If \(u \in \mathcal{B}_{M}^{<k>}\), then \(A_{R[x]}(u)\) contains a set of reversible polynomials \(\{q_j(x_j)\mid j = 1, \ldots, k\}\) and we get by backsolving (see 3.6) a unique linearly birecursive \(k\)-bisequence \(\tilde{u} \in \mathbb{A}_{\mathcal{S}_M^{<k>}}(\{q_1(x_1), \ldots, q_k(x_k)\})\) with \(\tilde{u}(n) = u(n)\) for all \(n \in \mathbb{N}_0^k\). For the bisequence \(\tilde{u}\) we have as shown above \(A_{R[x]}(\tilde{u}) = A_{R[x]}(u)\), i.e. \(\tilde{u}\) is a reverse of \(u\). The last statement is obvious.

2. By (1) \(\mathcal{B}_{M}^{<k>} \subseteq \mathcal{R}_{M}^{<k>}\). If \(R\) is artinian and \(u \in \mathcal{R}_{M}^{<k>}\) with reverse \(\tilde{u}\), then \(A_{R[x]}(u) = A_{R[x]}(\tilde{u})\) is by Lemma 3.6 (2) reversible, i.e. \(u \in \mathcal{B}_{M}^{<k>}\).

**Example 3.8.** The Fibonacci sequence \(F = (0, 1, 1, 2, 3, 5, \ldots)\) has elementary characteristic polynomial \(f(x) = x^2 - x - 1\). Since \(f(0) = -1\) is invertible in \(\mathbb{Z}\), we conclude that \(F\) is reversible with reverse

\[\text{Rev}(F)(z) = \begin{cases} f(z) & z \geq 0 \\ \text{Rev}(F)(z + 2) - \text{Rev}(F)(z + 1) & z < 0. \end{cases}\]
The following tables lists some of the terms of the bisequence \( \text{Rev}(F) \in \text{An}_{S_z}(x^2 - x - 1) : \)

| \( z \) | ... | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|---|
| \( \text{Rev}(F)(z) \) | ... | -3 | 2 | -1 | 1 | 0 | 1 | 1 | 2 | 3 |

**Lemma 3.9.** We have an isomorphism of \( R[x, x^{-1}] \)-modules

\[
\widehat{B}_M^{<k>} \simeq B_M^{<k>}. \tag{21}
\]

**Proof.** By Lemma 3.4 we have the well defined \( R[x, x^{-1}] \)-linear map

\[
\text{Rev}(\cdot): B_M^{<k>} \rightarrow \widehat{B}_M^{<k>}, \quad u \mapsto \text{Rev}(u).
\]

It’s easy to see that \( \text{Rev}(\cdot) \) is bijective with inverse \( \widetilde{u} \mapsto \widetilde{u}_{|b_0^k} \).

**3.10.** Let \( M \) be an \( R \)-module. We call a \( k \)-sequence \( u \in S_M^{<k>} \) periodic (resp. degenerating), if \( x^d(x^t \rightarrow u) = 0 \) for some \( d \in \mathbb{N}_0 \) and \( t \in \mathbb{N}^k \) (resp. \( x^d \rightarrow u = 0 \) for some \( d \in \mathbb{N}_0 \)). It’s clear that the subsets \( P_M^{<k>} \subseteq L_M^{<k>} \) of periodic \( k \)-sequences and \( D_M^{<k>} \subseteq L_M^{<k>} \) of degenerating \( k \)-sequences are \( R|x|\)-submodules.

**Remark 3.11.** ([KKMN95], Proposition 5.2]) If \( M \) is a finite \( R \)-module, then every linearly recursive sequence over \( M \) is periodic (i.e. \( P_M^{<1>} = L_M^{<1>} \)).

**Proposition 3.12.** ([KKMN95], Proposition 5.27]) Let \( R \) be an arbitrary commutative ring, \( M \) an \( R \)-module and denote with \( L_M^{<k>} \) the set of reversible periodic \( k \)-sequences over \( M \). Then we have an isomorphism of \( R[x] \)-modules

\[
P_M^{<k>} \simeq D_M^{<k>} \oplus L_M^{<k>} \tag{22}
\]

The following result generalizes Proposition 3.12 and describes the \( R[x] \)-module structure of arbitrary linearly recursive \( k \)-sequences of \( R \)-modules, where \( R \) is an artinian commutative ground ring:

**Proposition 3.13.** Let \( M \) be an \( R \)-module. If \( R \) is artinian, then we have isomorphisms of \( R[x] \)-modules

\[
L_M^{<k>} \simeq D_M^{<k>} \oplus \widehat{L}_M^{<k>} = D_M^{<k>} \oplus \tilde{B}_M^{<k>} \simeq D_M^{<k>} \oplus B_M^{<k>} = D_M^{<k>} \oplus \mathcal{R}_M^{<k>}. \tag{23}
\]

**Proof.** Let \( R \) be artinian and \( M \) an \( R \)-module. If \( u \) is a linearly recursive sequence over \( M \), then \( \text{An}_{R[x]}(u) \) contains by Lemma 3.6 (1) a set of monic polynomials \( \{ x^dq_j(x_j) \mid q_j(x_j) \text{ is reversible for } j = 1, \ldots, k \} \). By backsolving (see 3.6) we have a well defined morphism of \( R[x] \)-modules

\[
\gamma: L_M^{<k>} \rightarrow \tilde{L}_M^{<k>}, \quad \gamma \rightarrow \tilde{u}, \tag{24}
\]

where \( \tilde{u} \) is the unique linearly birecursive bisequence \( \tilde{u} \in \text{An}_{S_M}(q_1, \ldots, q_k) \) with \( \tilde{u}(n) = u(n) \) for all \( n \geq d \). It’s clear that \( \text{Ker}(\gamma) = D_M^{<k>} \). On the other hand, there is a morphism of \( R[x] \)-modules

\[
\beta: \tilde{L}_M^{<k>} \rightarrow L_M^{<k>}, \quad \tilde{w} \rightarrow \tilde{w}_{|b_0^k}. \tag{25}
\]
It’s obvious that $\gamma \circ \beta = \text{id}_{\tilde{L}^{<k>}_M}$, hence the following exact sequence of $R[x]$-modules splits

$$0 \rightarrow \mathcal{D}^{<k>}_M \rightarrow \mathcal{L}^{<k>}_M \xrightarrow{\gamma} \tilde{L}^{<k>}_M \rightarrow 0,$$

i.e. $\mathcal{L}^{<k>}_M \simeq \mathcal{D}^{<k>}_M \oplus \tilde{L}^{<k>}_M$. Since $R$ is artinian, we have by Lemmata 3.5 (2) and 3.7 (2) $\tilde{L}^{<k>}_M = \tilde{B}^{<k>}_M$ and $B^{<k>}_M = R^{<k>}_M$. We are done now by the isomorphism of $R[x]$-modules $B^{<k>}_M \simeq \tilde{B}^{<k>}_M$ (Lemma 3.9).

### 3.14. The Hopf $R$-algebra $R[x, x^{-1}]$.

Consider the commutative group $G$ generated by $\{x_j \mid j = 1, \ldots, k\}$. Then the ring of Laurent polynomials $R[x, x^{-1}] = RG$ has the structure of a commutative cocommutative Hopf $R$-algebra $(R[x, x^{-1}], \mu, \eta, \Delta, \varepsilon, S)$, where $\mu$ resp. $\eta$ are the usual multiplication resp. the usual unity and

$$
\begin{align*}
\Delta : \quad & R[x, x^{-1}] \rightarrow R[x, x^{-1}] \otimes_R R[x, x^{-1}], & x^z_j & \mapsto x^z_j \otimes x^z_j, & \forall z \in \mathbb{Z}, j = 1, \ldots, k, \\
\varepsilon : \quad & R[x, x^{-1}] \rightarrow R, & x^z_j & \mapsto 1_R, & \forall z \in \mathbb{Z}, j = 1, \ldots, k, \\
S : \quad & R[x, x^{-1}] \rightarrow R[x, x^{-1}], & x^z_j & \mapsto x^{-z}_j, & \forall z \in \mathbb{Z}, j = 1, \ldots, k.
\end{align*}
$$

(26)

**Proposition 3.15.** Let $R$ be an arbitrary commutative ring. Then $R[x, x^{-1}]$ is an admissible Hopf $R$-algebra and $R[x, x^{-1}]^\circ$ is a Hopf $R$-algebra.

**Proof.** Notice that $R[x, x^{-1}]$ is a cofinitary Hopf $R$-algebra by Lemma 1.17 (2). Consider the proof of Proposition 2.4 and replace $R[x]$ with $R[x, x^{-1}]$. Then the map

$$T_j : B \rightarrow B, \ b \mapsto \overline{\Delta}(x_j)b$$

is invertible with inverse

$$\overline{T}_j : B \rightarrow B, \ b \mapsto \overline{\Delta}(x_j^{-1})b.$$ 

Then the matrix $M_j$ of $T_j$ is invertible and $\chi_j(0) \in U(R)$ for $j = 1, \ldots, k$. Consequently $K_{R[x, x^{-1}]}$ satisfies axiom (8). Since $R[x, x^{-1}]/\text{Ke}(\varepsilon) \simeq R$, $K_{R[x, x^{-1}]}$ satisfies axiom (7). Consider the bijective antipode $S$ of $R[x; p]$. For every ideal $I \lhd R[x, x^{-1}]$, $S^{-1}(I) \lhd R[x, x^{-1}]$ is an ideal and we have an isomorphism of $R$-modules $R[x, x^{-1}]/S^{-1}(I) \simeq R[x, x^{-1}]/I$. Hence $K_{R[x, x^{-1}]}$ satisfies axiom (10). Consequently $R[x, x^{-1}]$ is an admissible Hopf $R$-algebra. The last statement follows now by Proposition 1.14.

For every $R$-module $M$ we have an isomorphism of $R[x, x^{-1}]$-modules

$$\Psi_M : M[x, x^{-1}]^* \rightarrow \tilde{S}^{<k>}_M, \ \varphi \mapsto [z \mapsto [m \mapsto \varphi(mx^z)]]$$

(27)

with inverse $\tilde{\varphi} \mapsto [m \mapsto \tilde{\varphi}(m) \in M[x, x^{-1}]].$

As in the proof of Proposition 2.5 we get

**Proposition 3.16.** Let $R$ be an arbitrary ring. Then $\tilde{S}^{<k>}_M$ induces an isomorphism of $R[x, x^{-1}]$-modules

$$M[x, x^{-1}]^\circ \simeq \tilde{B}^{<k>}_M.$$ 

(28)
Proof. Consider the isomorphism of \( R[x, x^{-1}] \)-modules \( M[x, x^{-1}]^* \xrightarrow{\Psi^*} \tilde{S}^{<k>}_{M^*} \) (27). Let \( \mathfrak{r} \in M[x, x^{-1}]^\circ \). Then \( I \rightarrow \mathfrak{r} = 0 \) for some \( R \)-cofinite \( R[x, x^{-1}] \)-ideal \( I \triangleleft R[x, x^{-1}] \) and so \( \tilde{S}(\mathfrak{r}) = \tilde{S}(I \triangleleft \mathfrak{r}) = 0 \). By Lemma 3.15 (2) \( I \) is a reversible ideal and so \( \operatorname{An}_{R[x]}(u) \supseteq I \cap R[x] \) is a reversible ideal, i.e. \( \Psi(\mathfrak{r}) \) is linearly birecursive.

On the other hand, let \( \tilde{u} \in \tilde{B}^{<k>}_{M^*} \). Then \( \operatorname{An}_{R[x]}(\tilde{u}) \) is by definition a reversible ideal, i.e. it contains a subset of reversible polynomials \( \{g \} \). Note that for arbitrary \( g \in R[x, x^{-1}] \) we have \( \tilde{u} \rightarrow \Psi^{-1}(\tilde{u}) = \Psi^{-1}(g \rightarrow \tilde{u}) = \Psi^{-1}(g \rightarrow (q_j \rightarrow \tilde{u})) = 0 \) for \( j = 1, \ldots, k \). By Lemma 3.17 (2) the reversible ideal \( \tilde{S}(\mathfrak{r}) \) of \( R[x, x^{-1}] \) is \( R \)-cofinite, i.e. \( \Psi^{-1}(\tilde{u}) \in M[x, x^{-1}]^\circ \).

3.17. The Hopf \( R \)-algebra structures on \( \tilde{B}^{<k>} \) and \( B^{<k>} \). Let \( R \) be an arbitrary ring and consider the Hopf \( R \)-algebra \( R[x, x^{-1}] \). Then \( \tilde{S}^{<k>} \simeq R^{<k>} \simeq R[x, x^{-1}]^* \) is an \( R \)-algebra with the Hadamard product
\[
\star : \tilde{S}^{<k>} \otimes_R \tilde{S}^{<k>} \rightarrow \tilde{S}^{<k>}, \quad \tilde{u} \otimes \tilde{v} \mapsto [z \mapsto \tilde{u}(z)\tilde{v}(z)]
\]
and the unity
\[
\eta : R \rightarrow \tilde{S}^{<k>}, \quad 1_{R} \mapsto [z \mapsto 1_{R}] \text{ for every } z \in \mathbb{Z}^k.
\]
By Proposition 3.13 \( R[x, x^{-1}]^\circ \) is a Hopf \( R \)-algebra. So \( B^{<k>} \simeq R[x, x^{-1}]^\circ \) inherits the structure of a Hopf \( R \)-algebra \( (B^{<k>}, \star, \eta, \Delta_B^{<k>}, \varepsilon_B^{<k>}, S_B^{<k>}) \), where \( \star \) is the Hadamard product (13), \( \eta \) is the unity (16) and
\[
\begin{align*}
\Delta_B^{<k>} : B^{<k>} &\rightarrow B^{<k>} \otimes_R B^{<k>}, \quad u \mapsto \sum_{t \leq 1} (x^t \mapsto u) \otimes e^{F}_t, \\
\varepsilon_B^{<k>} : B^{<k>} &\rightarrow R, \quad u \mapsto u(0), \\
S_B^{<k>} : B^{<k>} &\rightarrow B^{<k>}, \quad u \mapsto [n \mapsto \operatorname{Rev}(u)(-n)].
\end{align*}
\]
Moreover \( \tilde{B}^{<k>} \simeq R[x, x^{-1}]^\circ \) becomes a Hopf \( R \)-algebra \( (\tilde{B}^{<k>}, \star, \eta, \Delta_{\tilde{B}}^{<k>}, \varepsilon_{\tilde{B}}^{<k>}, S_{\tilde{B}}^{<k>}) \), where \( \star \) is the Hadamard product (29), \( \eta \) is the unity (30) and
\[
\begin{align*}
\Delta_{\tilde{B}}^{<k>} : \tilde{B}^{<k>} &\rightarrow \tilde{B}^{<k>} \otimes_R \tilde{B}^{<k>}, \quad \tilde{u} \mapsto \sum_{t \leq 1} \operatorname{Rev}(x^t \mapsto \tilde{u}_{hb}(t)) \otimes e^{F}_t, \\
\varepsilon_{\tilde{B}}^{<k>} : \tilde{B}^{<k>} &\rightarrow R, \quad \tilde{u} \mapsto \tilde{u}(0), \\
S_{\tilde{B}}^{<k>} : \tilde{B}^{<k>} &\rightarrow \tilde{B}^{<k>}, \quad \tilde{u} \mapsto [z \mapsto \tilde{u}(-z)].
\end{align*}
\]
Note that with these structures the isomorphism \( B^{<k>} \simeq \tilde{B}^{<k>} \) of Lemma 3.9 turns to be an isomorphism of Hopf \( R \)-algebras.

The following theorem extends the corresponding result from the case of a base field \[LT90\] Page 124] (see also \[KKMN95\] 14.15]) to the case of arbitrary artinian ground rings:

Theorem 3.18. If \( R \) is artinian, then there are isomorphisms of \( R \)-bialgebras
\[
\mathcal{L}^{<k>} \simeq \mathcal{D}^{<k>} \oplus \tilde{\mathcal{L}}^{<k>} = \mathcal{D}^{<k>} \oplus \tilde{\mathcal{B}}^{<k>} \simeq \mathcal{D}^{<k>} \oplus B^{<k>} = \mathcal{D}^{<k>} \oplus R^{<k>}.
\]
Proof. Consider the isomorphism $\mathcal{L}^{<k>} \simeq \mathcal{D}^{<k>} \oplus \tilde{\mathcal{L}}^{<k>}$ (23). With the help of Lemma 3.16 and 3.5 one can show as in [L190, Seite 123], that $\phi : \mathcal{L}^{<k>} \rightarrow \tilde{\mathcal{L}}^{<k>}$ (24) and $\beta : \tilde{\mathcal{L}}^{<k>} \rightarrow \mathcal{L}^{<k>}$ (25) are in fact bialgebra morphisms. Obviously $\text{Ke}(\phi) = \mathcal{D}^{<k>} \subset \mathcal{L}^{<k>}$ is an $\mathcal{L}^{<k>}$-subbialgebra and we are done.

As an analog to Corollary (2.14) we get

Corollary 3.19. Let $M$ be an $R[x, x^{-1}]$-module. Then we have an isomorphism of $R[x, x^{-1}]^\circ$-comodules

$\tilde{\mathcal{L}}^{<k>}_M \simeq M[x, x^{-1}] \circ \simeq M \bigotimes_R R[x, x^{-1}] \circ \simeq M \bigotimes_R \tilde{\mathcal{L}}^{<k>}_R$.

In particular $M[x, x^{-1}] \circ (\tilde{\mathcal{L}}^{<k>}_M)$ is a cofree $R[x, x^{-1}] \circ$-comodule ($\tilde{\mathcal{L}}^{<k>}_R$-comodule).

As a consequence of [Abu01, Satz 2.4.7] and [Abu01, Folgerung 2.5.10] we get

Corollary 3.20. Let $R$ be noetherian and consider the $R$-bialgebra $R[x; g] \circ$ (resp. the Hopf $R$-algebra $R[x; p] \circ$, the Hopf $R$-algebra $R[x, x^{-1}] \circ$). If $A$ is an $\alpha$-algebra (resp. an $\alpha$-bialgebra, a Hopf $\alpha$-algebra), then we have isomorphism of $R$-coalgebras (resp. $R$-bialgebras, Hopf $R$-algebras)

$A[x; g] \circ \simeq A \bigotimes_R R[x; g] \circ$, $A[x; p] \circ \simeq A \bigotimes_R R[x; p] \circ$ and $A[x, x^{-1}] \circ \simeq A \bigotimes_R R[x, x^{-1}] \circ$.

(32)

3.21. Representative functions. Let $G$ be a monoid (a group) and consider the $R$-algebra $B = R^G$ with pointwise multiplication. Then $B$ is an $RG$-bimodule under the left and right actions

$$(xf)(x) = f(xy) \text{ and } (fy)(x) = f(yx) \text{ for all } x, y \in G.$$ We call $f \in R^G$ an $R$-valued representative function on the monoid $G$, if $(RG)f(RG)$ is finitely generated as an $R$-module. If $R$ is noetherian, then the subset $\mathcal{R}(G) \subset R^G$ of all representative functions on $G$ is an $RG$-subbimodule. Moreover we deduce from [AG-TW00, Theorem 2.13, Corollary 2.15] that in case $(RG) \circ \subset R^G$ is pure, we have an isomorphism of $R$-bialgebras (Hopf $R$-algebras) $\mathcal{R}(G) \simeq (RG) \circ$.

Corollary 3.22. Let $R$ be noetherian.

1. Considering the monoid $(N_0^k, +)$ we have isomorphisms of $R$-bialgebras

$\mathcal{R}(N_0^k) \simeq R[x; p] \circ \simeq \mathcal{L}^{<k>}_R$.

2. Considering the monoid $(Z^k, +)$ we have isomorphisms of Hopf $R$-algebras

$\mathcal{R}(Z^k) \simeq R[x, x^{-1}] \circ \simeq \tilde{\mathcal{B}}^{<k>}_R \simeq \mathcal{B}^{<k>}_R$.

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