A Classification of Dark Matter Candidates with Primarily Spin-Dependent Interactions with Matter

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ABSTRACT: We perform a model-independent classification of Weakly Interacting Massive Particle (WIMP) dark matter candidates that have the property that their scattering off nucleons is dominated by spin-dependent interactions. We study renormalizable theories where the scattering of dark matter is elastic and arises at tree-level. We show that if the WIMP-nucleon cross section is dominated by spin-dependent interactions the natural dark matter candidates are either Majorana fermions or real vector bosons, so that the dark matter particle is its own anti-particle. In such a scenario, scalar dark matter is disfavored. Dirac fermion and complex vector boson dark matter are also disfavored, except for very specific choices of quantum numbers. We further establish that any such theory must contain either new particles close to the weak scale with Standard Model quantum numbers, or alternatively, a $Z'$ gauge boson with mass at or below the TeV scale. In the region of parameter space that is of interest to current direct detection experiments, these particles naturally lie in a mass range that is kinematically accessible to the Large Hadron Collider (LHC).
1. Introduction

While it is now well established that about 80% of the matter in the universe is non-luminous and non-baryonic, the exact nature of the particles that constitute this dark matter remains a mystery. It is widely known, however, that weak scale stable particles that interact with visible matter with strength comparable to the weak force naturally tend to have the right relic abundance to explain observations. Then perhaps the simplest candidates for dark matter are such Weakly Interacting Massive Particles (WIMPs). Many well-motivated extensions of the Standard Model (SM) contain WIMP dark matter candidates that have been shown to yield the correct relic abundance, for example supersymmetry \[1, 2\], extra dimensional theories \[3, 4\], little Higgs models \[5, 6\] and the left-right twin Higgs model \[7\].

A variety of current and future experiments are involved in the search for WIMP dark matter. These include direct detection experiments, which aim to directly observe dark matter, indirect detection experiments, which search for the annihilation products of dark matter, and collider experiments, which hope to directly produce dark matter. The signal in current direct detection experiments is the kinetic energy transferred to a nucleus after it scatters off a dark matter particle. The energies involved are less than or of order 10 keV, which is well below the typical nuclear energy scales. Therefore, at these energies, the WIMP sees the entire nucleus as a single unit, with a net mass, charge and spin.

The magnitude of the WIMP-nucleus scattering cross section is extremely sensitive to the exact form of the interactions of the dark matter particle with the individual nucleons. In theories where the WIMP couples primarily to the spin of the nucleon, the corresponding interactions are labelled as spin-dependent. On the other hand, in theories where the WIMP-nucleon cross section is insensitive to the spin of the nucleon, the corresponding interactions are labelled as spin-independent. For a given WIMP-nucleon cross section, the corresponding WIMP-nucleus cross section is in general significantly larger for spin-independent interactions than for spin-dependent interactions. Why is this? Spin-independent scattering tends to be coherent, receiving contributions from all the nucleons in the nucleus. The corresponding WIMP-nucleus cross section is enhanced by a factor of $A^2$, where $A$ is the mass number of the nucleus. On the other hand, since the spins of nucleons in a nucleus tend to cancel in pairs, there is no such enhancement in the spin-dependent WIMP-nucleus cross section. For this reason, direct detection experiments are much more sensitive to spin-independent interactions than to spin-dependent interactions. In particular, the current bound on spin-independent WIMP-nucleon interactions is about five orders of magnitude stronger than the corresponding bound on spin-dependent interactions \[8, 9, 10, 11\].

Clearly, the prospects for direct detection of dark matter depend crucially on whether WIMP-nucleon interactions are primarily spin-dependent or spin-independent. Recently, several authors have studied the relative sizes of the spin-dependent and spin-independent contributions to the WIMP-nucleon cross section \[12, 13, 14, 15\]. In this paper we perform a model-independent classification of dark matter candidates whose interactions with nucleons are primarily spin-dependent. We study renormalizable theories where dark matter scattering
is elastic and arises at tree-level. For each theory we calculate at lowest order the form of the effective operator that governs WIMP-nucleon scattering, and determine whether it leads to spin-dependent or spin-independent interactions. Our results are in excellent agreement with what would be expected from a naive operator analysis. We find that the dark matter candidates where spin-dependent interactions can naturally dominate are Majorana fermions and real vector bosons, so that the dark matter particle is its own anti-particle. However, the converse is not true. Specifically, the fact that the dark matter particle is a Majorana fermion or real vector boson is not by itself enough to guarantee that spin-dependent interactions dominate.

If the WIMP-nucleon cross section is primarily spin-dependent, scalar dark matter is disfavored. Dirac fermion and complex vector boson dark matter are also disfavored, except for very specific choices of quantum numbers. If dark matter is composed of scalars, WIMP-nucleon scattering is purely spin-independent. The interactions of Dirac fermions and complex vector bosons with nucleons always tend to have a significant spin-independent component, unless the dark matter sector possesses a discrete symmetry similar in nature to charge conjugation, but under which the SM fields are invariant.

Theories where the tree-level WIMP-nucleon cross section is primarily spin-dependent share a feature which is potentially of great significance for colliders. We find that any such theory must contain either

- new particles at the weak scale charged under at least one of the SM gauge groups, or
- a $Z'$ gauge boson with mass at the TeV scale or below.

We show that in the region of parameter space that is of interest for current direct detection experiments, the masses of these new particles naturally lie in a range that is kinematically accessible to the Large Hadron Collider (LHC). The results of direct detection experiments that are looking for WIMPs with spin-dependent interactions with nuclei are therefore highly correlated with dark matter searches at the LHC.

The layout of the paper is as follows. In section 2, we distinguish using an effective field theory approach the dark matter candidates that have a primarily spin-dependent WIMP-nucleon cross section. We classify the various models in terms of the spin and quantum numbers of both the dark matter particle and of the intermediate particle mediating the interaction. We highlight the favored scenarios in section 3, and analyze in greater detail their consequences. In section 4, we investigate the link between direct detection experiments and collider experiments. We conclude in section 5.

2. Classification of models

2.1 Operator Analysis

In this section we identify and classify theories that have predominantly spin-dependent couplings to nucleons. WIMP dark matter is constrained to be neutral under both electromagnetism and color [16]. This implies that in renormalizable theories WIMPs do not
scatter off gluons at tree-level. Our approach will therefore be to first identify all the renormalizable theories which generate tree-level WIMP-quark scattering. For each such theory we find the form of the effective operators that contribute to the WIMP-nucleon cross section. Each operator leads to either spin-dependent or spin-independent scattering, enabling us to distinguish theories where spin-dependent interactions dominate. We limit our analysis to purely elastic scattering, leaving the more complicated cases of inelastic dark matter \cite{17,18,19} and form factor dark matter \cite{20,21,22} for future work. - However, before proceeding, we first perform an operator analysis of the various possibilities. Similar operator studies have been performed in \cite{13,23}. The matrix element for dark matter-nucleus scattering will in general involve the expectation values of quark bilinear operators inside the nucleus of the form $\langle N|\bar{q}\Gamma q|N\rangle$, where $\Gamma$ represents any of $\gamma^5$, $\gamma^\mu$, $\gamma^\mu\gamma^5$, $\sigma^{\mu\nu}$ or simply the identity $\mathbb{I}$. Of these, $\bar{q}\gamma^5 q$ always leads to cross sections that are velocity suppressed in the non-relativistic limit, and is therefore typically sub-dominant. Furthermore, the operators $\bar{q}q$, $\bar{q}\gamma^5 q$ and $\bar{q}\sigma^{\mu\nu} q$ violate the approximate chiral symmetry of QCD, leading to cross sections that are suppressed by the quark masses (unless there are additional sources of chiral symmetry breaking in the theory). For these reasons the operators $\bar{q}\gamma^\mu q$ and $\bar{q}\gamma^\mu\gamma^5 q$, when present, in general tend to dominate the cross section. Of these, the temporal component of $\bar{q}\gamma^\mu q$ leads to spin-independent interactions, and the spatial component of $\bar{q}\gamma^\mu\gamma^5 q$ leads to spin-dependent interactions. The spatial components of $\bar{q}\gamma^\mu q$ and the temporal component of $\bar{q}\gamma^\mu\gamma^5 q$ are velocity suppressed. The problem of identifying theories where the dominant interactions of dark matter with nucleons are sizeable and spin-dependent is therefore largely equivalent to finding theories where the couplings of dark matter to quarks are such that the operator $\bar{q}\gamma^\mu q$ is present in the low energy effective Lagrangian, whereas the operator $\bar{q}\gamma^\mu q$ is not.

However, in some cases neither $\bar{q}\gamma^\mu q$ nor $\bar{q}\gamma^\mu\gamma^5 q$ is generated at leading order. The dominant contribution to the cross section may then arise from one or more of the operators which are chirality suppressed. Of these $\bar{q}q$ always generates spin-independent interactions while $\bar{q}\sigma^{\mu\nu} q$ always generates spin-dependent interactions in the non-relativistic limit. The operator $\bar{q}\gamma^5 q$ generates spin-independent interactions, but these are additionally velocity suppressed and usually negligible.

Consider first the case of scalar dark matter. What are the possible operators that involve scalars coupling to $\bar{q}\gamma^\mu\gamma^5 q$ at leading order? Consider a complex scalar field, which we denote by $\phi$. One obvious candidate operator is

$$\phi^\dagger \partial_\mu \phi \bar{q}\gamma^\mu\gamma^5 q. \quad (2.1)$$

However, for both temporal and spatial $\mu$ the corresponding matrix element is velocity suppressed. It is straightforward to verify that this result applies in general to any operator that couples scalar dark matter to $\bar{q}\gamma^\mu\gamma^5 q$. No such constraint arises in the cases of the operator $\bar{q}\gamma^\mu q$. For this reason, the dominant interactions of complex scalar dark matter are in general spin-independent.
The case of real scalar dark matter is slightly different. After integration by parts the operator above can be rewritten as

\[ \phi \partial_\mu \phi \bar{q} \gamma^\mu q \rightarrow \phi^2 \partial_\mu (\bar{q} \gamma^\mu q). \] (2.2)

This operator does not contribute to the tree-level WIMP-nucleon cross section, as a consequence of the equation of motion (since \( \bar{q} \gamma^\mu q \) corresponds to a conserved current). In such a scenario the leading contribution to the cross section is expected to arise from the chirality suppressed operator

\[ \phi^2 \bar{q} q. \] (2.3)

Therefore, in the case of scalar dark matter, WIMP-nucleon scattering is in general spin-independent.

We move on to the case of fermionic dark matter. If we denote the dark matter candidate by a four component spinor \( \chi \), the obvious candidate operators at leading order are

\[ \bar{\chi} \gamma^\mu \chi \bar{q} \gamma^\mu \gamma^5 q \] and \[ \bar{\chi} \gamma^\mu \gamma^5 \chi \bar{q} \gamma^\mu \gamma^5 q. \] (2.4)

While the first of these operators is velocity suppressed, the second does indeed lead to spin-dependent interactions. However, for spin-dependent interactions to dominate, the operator

\[ \bar{\chi} \gamma^\mu \chi \bar{q} \gamma^\mu q \] (2.5)

must be absent. Now, if \( \chi \) is a Majorana fermion, then \( \bar{\chi} \gamma^\mu \chi \) identically vanishes and so this operator is automatically absent. However, in the case of Dirac fermions, in the absence of any symmetry to forbid this operator we expect that it will be present. One symmetry that can forbid this operator is a discrete symmetry under which the dark matter field \( \chi \) is interchanged with \( \chi^c \), its charge conjugate, while the SM fields are invariant. If the theory does not possess such a symmetry, we expect that the WIMP-nucleon scattering cross section will have a sizeable spin-independent component. As for the operator

\[ \bar{\chi} \gamma^\mu \gamma^5 \chi \bar{q} \gamma^\mu q \] (2.6)

which also gives rise to spin-independent interactions, it turns out that the corresponding matrix elements are velocity suppressed.

The final possibility is that the dark matter particle is a vector boson. We first consider the case where the corresponding vector field is real, so that the dark matter particle is its own anti-particle. If we denote the dark matter candidate by \( B_\mu \), in the non-relativistic limit the physical degrees of freedom correspond to spatial \( \mu \). The operator

\[ \epsilon_{\mu\nu\lambda\sigma} B^\nu \partial_\lambda B^\sigma \bar{q} \gamma^\mu \gamma^5 q \] (2.7)
gives rise to spin-dependent interactions. The corresponding operator but with \( \bar{q}\gamma^\mu\gamma^5q \) replaced by \( \bar{q}\gamma^\mu q \) is velocity suppressed. There also exist other operators that arise at the same order such as

\[
B^\nu \partial_\mu B_\nu \bar{q}\gamma^\mu\gamma^5q \quad \text{and} \quad B^\nu \partial_\mu B_\nu \bar{q}\gamma^\mu q.
\] (2.8)

However, these do not contribute significantly to the tree-level cross section. The effects of the first operator are velocity suppressed, while the second does not contribute significantly as a consequence of the equation of motion. Other operators that arise at the same order include

\[
B^\nu \partial_\mu B_\nu \bar{q}\gamma^\mu\gamma^5q \quad \text{and} \quad B^\nu \partial_\mu B_\nu \bar{q}\gamma^\mu q.
\] (2.9)

However, it is straightforward to verify that these lead to interactions that are always velocity suppressed. Therefore real vector boson dark matter naturally leads to spin-dependent interactions.

Finally we come to the case of complex vector dark matter. The operator

\[
\epsilon^{\mu\nu\lambda\sigma} B^\nu B^\sigma \bar{q}\gamma_\lambda \gamma^5 q
\] (2.10)

gives rise to spin-dependent interactions. However, the operator

\[
B^\nu \partial_\mu B^\nu \bar{q}\gamma^\mu q.
\] (2.11)

can now generate spin-independent interactions, and must be forbidden by a symmetry for spin-dependent interactions to dominate. One possibility is a discrete symmetry under which the vector field is interchanged with its charge conjugate \( B_\mu \leftrightarrow -B_\mu^\dagger \), while the SM fields are invariant. In the absence of such a symmetry we expect that the WIMP-nucleon cross section will in general have a sizeable spin-independent component.

Our conclusions from the operator analysis are that scalar dark matter always leads to spin-independent interactions, while Majorana fermion and real vector boson dark matter naturally lead to spin-dependent interactions in the chiral limit. In the cases where dark matter is composed of Dirac fermions or complex vector bosons, we expect that the WIMP-nucleon cross section will in general have a sizeable spin-independent component, except for very specific choices of quantum numbers. In what follows we firm up this conclusion by studying each of these cases in more detail. Our approach will be to consider all renormalizable theories which lead to tree-level WIMP-quark scattering. For each theory we calculate the quark bilinears that appear in the effective operators which contribute to the WIMP-nucleon cross section, enabling us to immediately identify those theories where scattering is primarily spin-dependent.

### 2.2 Scalar Dark Matter

We begin by considering the case of scalar dark matter. In general the scalar may be real, or it may be complex. Both cases are qualitatively similar, but with subtle differences. In general,
the tree-level scattering of a scalar $\phi$ with quarks can occur through any of the Feynman diagrams shown in Fig 1.

Our approach will be to start from the renormalizable Lagrangian that corresponds to each diagram separately, integrate out the intermediate particle, and find the form of the effective Lagrangian. We are particularly interested in interactions which generate a spin-dependent $(\bar{q}\gamma^\mu \gamma^5 q)$ coupling to quarks.

**Diagram 1a: t-channel vector exchange**

We note that this diagram only exists if the scalar particle is complex. Here the vector boson $Z$ being exchanged may correspond either to the $Z$ of the SM, or to a new $Z'$ which has been added to the SM. We do not consider the scenario where $Z$ corresponds to the SM photon since the photon always couples to the vector bilinear $\bar{q}\gamma^\mu q$, and we expect spin-independent scattering will dominate. At the renormalizable level, the form of the interaction between a complex scalar and a vector boson is fixed. In unitary gauge,

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m_Z^2 Z^\mu Z_\mu + a(\phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger) Z^\mu + \bar{q}\gamma^\mu(\alpha - \beta \gamma^5) q \ Z_\mu. \quad (2.12)$$

We now integrate out the vector particle using its equation of motion,

$$Z_\mu = \left[ (\partial^2 + m_Z^2)g^\beta\mu - \partial^\mu \partial^\beta \right]^{-1} \left[ -a(\phi^\dagger \partial_\beta \phi - \phi \partial_\beta \phi^\dagger) - \bar{q}\gamma^\beta(\alpha - \beta \gamma^5) q \right]. \quad (2.13)$$
The inverse of the differential operator above can be simplified using
\[
\left( \partial^2 + m_Z^2 g^{\beta \mu} - \partial^\mu \phi \partial^\beta \phi \right)^{-1} \equiv \frac{g_{\beta \mu} + \frac{\partial_\beta \partial_\mu}{\partial^2 + m_Z^2}}{\partial^2 + m_Z^2}.
\] (2.14)

In order to elucidate the operator structure, we expand the above in powers of $\partial/m_Z$, keeping only the zeroth and first order terms. For t-channel processes such as this one, this corresponds to assuming that the net momentum transfer in the process is much less than $m_Z$, which is certainly valid in the non-relativistic regime. Under this approximation,
\[
Z^\mu \simeq \frac{1}{m_Z^2} \left[ -a(\phi^\dagger \partial^\mu \phi - \phi \partial^\mu \phi^\dagger) - \bar{q} \gamma^\mu (\alpha - \beta \gamma^5) q \right].
\] (2.15)

This leads to the effective Lagrangian
\[
\mathcal{L}_{\text{eff}} \simeq \left[ a(\phi^\dagger \partial^\mu \phi - \phi \partial^\mu \phi^\dagger) + \bar{q} \gamma^\mu (\alpha - \beta \gamma^5) q \right] \frac{1}{2m_Z^2} \left[ -a(\phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger) - \bar{q} \gamma_\mu (\alpha - \beta \gamma^5) q \right].
\] (2.16)

Keeping only terms in the effective Lagrangian relevant to our process, this reduces to
\[
\mathcal{L}_{\text{eff}} \simeq -a \frac{\alpha}{m_Z^2} (\phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger) \bar{q} \gamma^\mu (\alpha - \beta \gamma^5) q.
\] (2.17)

In the low energy limit the spatial components of the derivative are suppressed. Therefore, the axial-vector contribution vanishes in the non-relativistic limit and we are left with
\[
\mathcal{L}_{\text{eff}} \simeq -a \frac{\alpha}{m_Z^2} (\phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger) \bar{q} \gamma^\mu q.
\] (2.18)

We see from this that, as expected, scattering is spin-independent.

**Diagram 1b: t-channel scalar exchange**

The dark matter field in this process may either be real or complex. This constitutes the primary detection channel in several scalar dark matter models [24, 25, 26, 27, 28]. If the dark matter field particle is complex, the general renormalizable Lagrangian has the form
\[
\mathcal{L} = \frac{1}{2} (\partial h)^2 - \frac{1}{2} m_h^2 h^2 - a \phi^\dagger \phi h - \bar{q}(\alpha - \beta \gamma^5)qh.
\] (2.19)

The case where the scalar field is real is very similar and may be recovered simply by setting $\phi^\dagger = \phi$ in the above Lagrangian. Here $h$ could represent the SM Higgs, or more generally any real scalar that couples to quarks. We follow the same procedure as earlier. Using the equation of motion for the intermediate scalar $h$, we integrate it out,
\[
h = (\partial^2 + m_h^2)^{-1} \left[ -a \phi^\dagger \phi - \bar{q}(\alpha - \beta \gamma^5)q \right].
\] (2.20)
As before, working in an expansion in powers of $\partial/m_h$ and neglecting terms of order $\partial^2/m_h^2$ or higher, we have
\begin{equation}
  h = -\frac{a\phi^\dagger\phi + \bar{q}(\alpha - \beta\gamma^5)q}{m_h^2}.
\end{equation}

The relevant part of the effective Lagrangian is then
\begin{equation}
  \mathcal{L}_{\text{eff}} \simeq \frac{\phi^\dagger\phi}{m_h^2} \left[a\alpha\bar{q}q - a\beta\bar{q}\gamma^5q\right].
\end{equation}

As noted earlier, the spinor $\bar{q}\gamma^5q$ vanishes in the non-relativistic limit. We are left with
\begin{equation}
  \mathcal{L}_{\text{eff}} \simeq \frac{a\alpha}{m_h^2} \phi^\dagger\phi \bar{q}q.
\end{equation}

which generates purely spin-independent interactions. The reason for the apparent disagreement with the naive operator analysis of the previous subsection is that the interaction Lagrangian we started from here explicitly breaks chiral symmetry. Therefore it is not surprising that the leading operator in the effective theory also breaks chiral symmetry.

**Diagram 1c: s- and u-channel fermion exchange**

In this case there are two distinct possibilities, depending on whether the scalar field is real or complex. For a complex scalar field there is only one diagram, which may either be s-channel or u-channel, depending on whether it is the particle or anti-particle being scattered. On the other hand, for a real scalar both s- and u-channel diagrams are present.

For the complex scalar, the calculation proceeds as follows. We start with the general form of the Lagrangian,
\begin{equation}
  \mathcal{L} = \bar{Q}(i\partial - m_Q)Q - \bar{q}(\alpha - \beta\gamma^5)Q\phi^\dagger - \bar{Q}(\alpha^* + \beta^*\gamma^5)q\phi.
\end{equation}

The equation of motion for $Q$ takes the form,
\begin{equation}
  Q = \frac{i\partial + m_Q}{\partial^2 + m_Q^2} \left[-(\alpha^* + \beta^*\gamma^5)q\phi\right].
\end{equation}

Derivatives acting on the quark fields can be neglected, since these effects are generally suppressed by powers of $\Lambda_{\text{QCD}}/m_Q$ (or $\Lambda_{\text{QCD}}/m_\phi$) relative to other contributions. We expand in $\partial/m_Q$ as before, keeping only zeroth order and first order terms. For s-channel and u-channel processes in the non-relativistic limit, this corresponds to the assumption that the WIMP mass squared is much less than the mass squared of the particle being exchanged, $m_\phi^2 \ll m_Q^2$. Although $m_\phi < m_Q$ is required for WIMP decays to be kinematically forbidden, it does not follow that $m_\phi^2 \ll m_Q^2$ is necessarily satisfied. However, this approximation suffices to determine the leading term in the low-energy effective interaction that emerges from
this class of theories, and determine whether it leads to spin-dependent or spin-independent interactions. We will relax this assumption in the next section. Then

$$Q = -m_Q (\alpha^* + \beta^* \gamma^5) q \frac{1}{m_Q^2} \phi - \gamma^\mu (\alpha^* + \beta^* \gamma^5) q \frac{i \partial^\mu}{m_Q} \phi. \quad (2.26)$$

The relevant terms in the effective Lagrangian are given by

$$\mathcal{L}_{\text{eff}} \simeq \frac{1}{m_Q^2} \bar{q} (\alpha - \beta \gamma^5) \phi^\dagger \left( m_Q (\alpha^* + \beta^* \gamma^5) q \phi + i \gamma^\mu (\alpha^* + \beta^* \gamma^5) q \partial_\mu \phi \right). \quad (2.27)$$

In the non-relativistic limit the derivative picks out the time direction, and therefore only the vector quark current survives in the second term above. We are left with

$$\mathcal{L}_{\text{eff}} \simeq \frac{1}{m_Q} (|\alpha|^2 - |\beta|^2) \bar{q} \phi^\dagger \phi + \frac{i}{m_Q} (|\alpha|^2 + |\beta|^2) \bar{q} \gamma^\mu q \phi^\dagger \partial_\mu \phi. \quad (2.28)$$

We see from this that the only interactions that are generated have the scalar and vector forms, both of which lead to spin-independent scattering. As expected, in the chiral limit (when \(\alpha = \pm \beta\)) the scalar contribution vanishes and we are left with just the vector interaction. For the real scalar, the effective Lagrangian is obtained by simply setting \(\phi^\dagger = \phi\) in the equation above. In this case the second term in the equation above does not contribute, as may be verified by integrating by parts. Then the leading contribution to the cross section arises from a chiral symmetry breaking effect, as expected from our operator analysis. In summary, we never generate sizeable spin-dependent interactions at low energies in the case of scalar dark matter.

### 2.3 Fermionic Dark Matter

We now move on to the case of fermionic dark matter. It is convenient to consider the cases of the Dirac fermion and Majorana fermion separately.

#### 2.3.1 Dirac Fermion Dark Matter

We begin with the case where dark matter consists of Dirac fermions. In what follows we require that the Lagrangian has a symmetry under which the Dirac field \(\chi\) can be arbitrarily rephased, \(\chi \rightarrow \chi \exp (i\theta)\). This is necessary to ensure that the Dirac nature of \(\chi\) is maintained at loop level, so that \(\chi\) does not split into two Majorana states. At tree-level the interaction of the dark matter particle with quarks can arise in any of four different ways as shown in the diagrams below. Diagrams 2a and 2b arise in some specific theories with Dirac fermion dark matter [29, 30].

**Diagram 2a: t-channel vector exchange**

The most general renormalizable Lagrangian corresponding to this process takes the form

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m_Z^2 Z^\mu Z_\mu + \bar{\chi} \gamma^\mu (\alpha - \beta \gamma^5) \chi Z_\mu + \bar{q} \gamma^\mu (\bar{\alpha} - \bar{\beta} \gamma^5) q Z_\mu. \quad (2.29)$$
Figure 2: Dirac fermion dark matter scattering diagrams

The analysis is similar to that of diagram 1a. In particular, since the external states are still non-relativistic, the analysis of integrating out the mediator particle is exactly same as before. Once again we neglect terms of order $\partial^2/m_Z^2$, leading to

$$Z_\mu \simeq g_{\mu\nu} \frac{1}{m_Z^2} \left[ -\bar{\chi}\gamma^\nu (\alpha - \beta \gamma^5) \chi - \bar{q}\gamma^\nu (\tilde{\alpha} - \tilde{\beta} \gamma^5) q \right]. \quad (2.30)$$

The effective Lagrangian can be immediately read off as

$$\mathcal{L}_{\text{eff}} = \left[ \bar{\chi}\gamma^\mu (\alpha - \beta \gamma^5) \chi + \bar{q}\gamma^\mu (\tilde{\alpha} - \tilde{\beta} \gamma^5) q \right] \frac{g_{\mu\nu}}{2m_Z^2} \left[ -\bar{\chi}\gamma^\nu (\alpha - \beta \gamma^5) \chi - \bar{q}\gamma^\nu (\tilde{\alpha} - \tilde{\beta} \gamma^5) q \right]. \quad (2.31)$$

We note that the mixed terms above do not survive in the non-relativistic limit. Keeping only terms relevant to our process, we are left with

$$\mathcal{L}_{\text{eff}} \simeq -\frac{1}{2m_Z^2} \left[ \alpha \bar{\chi}\gamma^\mu \chi \bar{q}\gamma^5 q + \beta \bar{\chi}\gamma^\mu \gamma^5 \chi \bar{q}\gamma^\mu q \right]. \quad (2.32)$$
We see that in general both spin-dependent and spin-independent interaction terms arise. However, if either $\alpha$ or $\tilde{\alpha}$ is zero, the interaction is purely spin-dependent. The case $\alpha = 0$ corresponds to the limit where the dark matter sector possesses a symmetry under which $\chi \leftrightarrow \chi^c$, while the SM fields are invariant. We will consider this possibility in greater detail in the next section.

**Diagram 2b: t-channel scalar exchange**

The relevant part of the Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}(\partial h)^2 - \frac{1}{2}m_h^2h^2 - \bar{\chi}(\alpha - \beta \gamma^5)\chi h - \bar{q}(\tilde{\alpha} - \tilde{\beta} \gamma^5)q h. \quad (2.33)$$

We follow the same procedure as earlier to integrate out $h$. We solve the equation of motion for $h$, and ignore terms of order $\partial^2 / m_h^2$ and higher,

$$h \simeq -\frac{1}{m_h^2} \left[ \bar{\chi}(\alpha - \beta \gamma^5)\chi + \bar{q}(\tilde{\alpha} - \tilde{\beta} \gamma^5)q \right]. \quad (2.34)$$

This leads to the following effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \left[ \bar{\chi}(\alpha - \beta \gamma^5)\chi + \bar{q}(\tilde{\alpha} - \tilde{\beta} \gamma^5)q \right] \frac{1}{2m_h^2} \left[ \bar{\chi}(\alpha - \beta \gamma^5)\chi + \bar{q}(\tilde{\alpha} - \tilde{\beta} \gamma^5)q \right]. \quad (2.35)$$

Keeping only the terms relevant to our process that survive in the non-relativistic limit,

$$\mathcal{L}_{\text{eff}} \simeq \frac{\alpha \tilde{\alpha}}{m_h^2} \bar{\chi} \chi \bar{q} q. \quad (2.36)$$

This leads to purely spin-independent interactions that are suppressed in the limit of exact chiral symmetry. As before, the reason for the apparent disagreement with the naive operator analysis of the previous subsection is that the interaction Lagrangian we started from explicitly breaks chiral symmetry.

**Diagram 2c: s- and u-channel vector exchange**

We start from the following Lagrangian,

$$\mathcal{L} = -\frac{1}{2} |\partial_{\mu}X_{\nu} - \partial_{\nu}X_{\mu}|^2 + m_X^2 X_{\mu}^\dagger X_{\mu} + \bar{\chi} \gamma^\mu(\alpha - \beta \gamma^5)q X_{\mu} + \bar{q} \gamma^\mu(\alpha^* - \beta^* \gamma^5)\chi X_{\mu}^\dagger. \quad (2.37)$$

We can integrate out the colored vector boson $X_{\mu}$ using its equation of motion. Dropping terms of order $\partial^2 / m_X^2$ and restricting to terms relevant to our process, we get

$$\mathcal{L}_{\text{eff}} \simeq -\frac{1}{m_X^2} \left[ \bar{\chi} \gamma^\nu(\alpha - \beta \gamma^5)q \bar{q} \gamma_{\nu}(\alpha^* - \beta^* \gamma^5)\chi \right]. \quad (2.38)$$

We can use the Fierz identities listed in Appendix A to rewrite the bilinears above. After dropping terms which vanish in the non-relativistic limit, the effective Lagrangian takes the form

$$\mathcal{L} \simeq \frac{1}{m_X^2} \left[ (|\alpha|^2 - |\beta|^2)\bar{q} q \bar{\chi} \chi - \frac{1}{2}(|\alpha|^2 + |\beta|^2)(\bar{q} \gamma^\mu \bar{q} \gamma_{\mu} \chi + \bar{q} \gamma^5 \gamma^\mu \gamma^5 q \chi_{\mu} \gamma^5 \chi) \right]. \quad (2.39)$$
We see that in general both spin-dependent and spin-independent interactions are generated, and with comparable magnitudes. The scalar term vanishes in the chiral limit ($\alpha = \pm \beta$). This conclusion is not unexpected because there are no values of the parameters $\alpha$ and $\beta$ for which the theory enjoys a $\chi \leftrightarrow \chi^c$ symmetry. While it is in fact possible to incorporate such a symmetry in a manner consistent with the rephasing symmetry of $\chi$, this requires adding additional fields to the theory. We will therefore not consider this possibility further.

**Diagram 2d: s- and u-channel scalar exchange**

The part of the Lagrangian relevant to this process has the general form

$$L = |\partial \Phi|^2 - m_\Phi^2 |\Phi|^2 - \bar{\chi}(\alpha - \beta \gamma^5)q\Phi - \bar{q}(\alpha^* + \beta^* \gamma^5)\chi\Phi^\dagger.$$  \hspace{1cm} (2.40)

Using the equation of motion for $\Phi$ (ignoring terms of order $\partial^2/m_\Phi^2$), we get

$$\Phi \simeq -\frac{\bar{q}(\alpha^* + \beta^* \gamma^5)\chi}{m_\Phi^2}.$$  \hspace{1cm} (2.41)

This leads to the effective Lagrangian

$$L_{\text{eff}} \simeq \frac{1}{m_\Phi^2} \left[ |\alpha|^2 (\bar{\chi}q)(\bar{q}\chi) - |\beta|^2 (\bar{\chi}\gamma^5 q)(\bar{q}\gamma^5 \chi) + \alpha \beta^* (\bar{\chi}q)(\bar{q}\gamma^5 \chi) - \alpha^* \beta (\bar{\chi}\gamma^5 q)(\bar{q}\chi) \right].$$  \hspace{1cm} (2.42)

After a Fierz rearrangement, neglecting terms which are velocity suppressed we are left with

$$L_{\text{eff}} \simeq -\frac{1}{4m_\Phi^2} \left[ (|\alpha|^2 - |\beta|^2)(\bar{q}q\bar{\chi} \chi) + \frac{1}{2} \bar{q} \sigma^{\mu \nu} q \bar{\chi} \sigma_{\mu \nu} \chi \right] + (|\alpha|^2 + |\beta|^2)(\bar{q}\gamma^\mu q \bar{\chi}\gamma^\mu \chi - \bar{\chi}\gamma^\mu \gamma^5 q \bar{\chi}\gamma^\mu \gamma^5 \chi).$$  \hspace{1cm} (2.43)

As in the previous case, we see that both spin-independent and spin-dependent interactions are necessarily generated. As expected, the scalar and tensor contributions vanish in the chiral limit. Once again, this conclusion could have been anticipated because there are no values of the parameters $\alpha$ and $\beta$ that are consistent with a $\chi \leftrightarrow \chi^c$ interchange symmetry. Incorporating such a symmetry without violating the rephasing symmetry of $\chi$ necessarily requires expanding upon this minimal field content, and we will therefore not expand on this possibility.

We see that in the case that the dark matter particle is a Dirac fermion, in the chiral limit most interactions necessarily generate both spin-dependent and spin-independent scattering. The solitary exception to this general rule arises in the case of vector boson exchange in the t-channel, and then only for very specific choices of charges. We will return to this possibility in the next section.

### 2.3.2 Majorana Fermion Dark Matter

We now move on to the case where the dark matter is Majorana. Due to the fact that the Majorana fermion is its own anti-particle, those bilinears which are odd under charge
conjugation vanish for Majorana spinors.

\[
\begin{array}{|c|}
\hline
\text{Bilinear} & C \\
\hline
\bar{\chi}\chi & + \\
i\bar{\chi}\gamma^5\chi & + \\
\bar{\chi}\gamma^\mu\chi & - \\
\bar{\chi}\gamma^\mu\gamma^5\chi & + \\
\bar{\chi}\sigma^{\mu\nu}\chi & - \\
\hline
\end{array}
\]

(2.44)

Therefore, for a Majorana fermion, the vector and the tensor bilinears vanish. With this in mind, we can now use our earlier results from the Dirac fermion case with only minor modifications. Majorana fermions arise naturally in supersymmetric theories of dark matter, in particular the processes corresponding to diagrams 3a, 3b and 3d [1].

**Diagram 3a: t-channel vector exchange**

Setting the vector coupling of $\chi$ to zero in the effective Lagrangian for the Dirac case in equation (2.31), we find

\[
\mathcal{L}_{\text{eff}} = \left[ \bar{\chi}\gamma^\mu(-\beta\gamma^5)\chi + \bar{q}\gamma^\mu(\bar{\alpha} - \bar{\beta}\gamma^5)q \right] \frac{g_{\mu\nu}}{2m_Z^2} \left[ -\bar{\chi}\gamma^\nu(-\beta\gamma^5)\chi - \bar{q}\gamma^\nu(\bar{\alpha} - \bar{\beta}\gamma^5)q \right].
\]  

(2.45)
Note that we have chosen a convention where assigning the canonically normalized Majorana field unit charge under $Z$ corresponds to setting $\beta = \frac{1}{2}$. Keeping only terms relevant to our process, and dropping interactions which are velocity suppressed we see that

$$L_{\text{eff}} \simeq -\frac{\beta^2}{m_Z^2} \bar{\chi} \gamma^\mu \gamma^5 \chi \bar{q} \gamma^\mu \gamma^5 q.$$ (2.46)

We see that only spin-dependent couplings are generated, in agreement with our operator analysis.

**Diagram 3b: t-channel scalar exchange**

The analysis here is very similar to that of the Dirac case considered earlier, and also leads to interactions that are purely spin-independent.

**Diagram 3c: s- and u-channel vector exchange**

If we again work to zeroth order in $\partial^2/m_X^2$, we can just copy the result from the Dirac case in equation (2.39), dropping the vector interaction term. Hence

$$L_{\text{eff}} \simeq \frac{1}{m_X^2} \left[ (|\alpha|^2 - |\beta|^2) \bar{q} q \bar{\chi} \chi - \frac{1}{2} (|\alpha|^2 + |\beta|^2) (\bar{q} \gamma^\mu \gamma^5 q \bar{\chi} \gamma^\mu \gamma^5 \chi) \right].$$ (2.47)

The dominant interaction is spin-dependent. There is also a spin-independent contribution which vanishes in the limit of exact chiral symmetry ($\alpha = \pm \beta$), and which can therefore naturally be small. Thus, this is another class of interactions which generates primarily spin-dependent interactions.

**Diagram 3d: s- and u-channel scalar exchange**

This case is again very similar to that of Dirac fermions except for the vector and tensor couplings, which vanish for a Majorana fermion. The effective Lagrangian is then

$$L_{\text{eff}} \simeq -\frac{1}{4m_\Phi^2} \left[ (|\alpha|^2 - |\beta|^2) \bar{q} q \bar{\chi} \chi - (|\alpha|^2 + |\beta|^2) \bar{q} \gamma^\mu \gamma^5 q \bar{\chi} \gamma^\mu \gamma^5 \chi \right].$$ (2.48)

As expected from our operator analysis, this interaction generates exclusively spin-dependent cross sections if chiral symmetry ($\alpha = \pm \beta$) is imposed.

To summarize, when the dark matter particle is a Majorana fermion, the non-relativistic WIMP-nucleon cross section is dominated by spin-dependent interactions in the chiral limit, in perfect agreement with the operator analysis.

**2.4 Vector Dark Matter**

We now consider the case of dark matter being a vector field, which may be real or complex. In the case that the vector field $B_\mu$ is complex, we require that the Lagrangian be invariant under the rephasing of $B_\mu$ by an arbitrary amount, $B_\mu \rightarrow B_\mu \exp (i\theta)$. This is to ensure that the degeneracy of the two real vector fields that constitute $B_\mu$ is radiatively stable.
There are three classes of diagrams that can contribute to the scattering of vector dark matter with SM quarks at tree-level. Diagrams 4b and 4c arise in the context of universal extra dimensions [31, 32] and also in the context of the littlest Higgs with T-parity [33].

Figure 4: Vector dark matter scattering diagrams

**Diagram 4a: t-channel vector exchange**

An interaction of this form arises only if the vector field is complex. The reason is that the triple gauge boson coupling is proportional to the completely anti-symmetric structure constant $f^{abc}$, which requires the presence of at least three distinct real vector dark matter fields. For simplicity we assume that the interaction is mediated by a single real vector boson $Z$. Our results can easily be generalized to more complicated cases. We let the gauge index take values 1, 2, 3, and denote the structure constant $f^{123}$ by $f$. We define the dark matter field $B_{\mu}$ in terms of the $a = 1, 2$ components as

$$B_{\mu} = \frac{A_{\mu}^1 + i A_{\mu}^2}{\sqrt{2}},$$

and associate the intermediate particle $Z_\mu$ with the gauge index $a = 3$. We start from the following Lagrangian, dropping terms which are irrelevant for this process.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \frac{1}{2} m_Z^2 Z_{\mu}Z^{\mu} + g\bar{q}\gamma^\mu (\alpha - \beta \gamma^5)qZ_{\mu}$$

$$\simeq -\frac{1}{4} (\partial_{\mu}Z_{\nu} - \partial_{\nu}Z_{\mu})^2 + \frac{1}{2} m_Z^2 Z_{\mu}Z^{\mu} + g\bar{q}\gamma^\mu (\alpha - \beta \gamma^5)qZ_{\mu}$$

$$+ \left[ igf \left( Z^\mu B^{\nu \dagger} (\partial_\mu B_\nu - \partial_\nu B_\mu) + B^\dagger_\mu B_\mu \partial^\mu Z^\nu \right) + \text{h.c.} \right].$$

(2.50)
From the equation of motion for \( Z_\mu \), keeping terms upto first order in \( \partial/m_Z \), we find
\[
Z_\mu \simeq \frac{1}{m_Z^2} \left[ g \bar{q} \gamma^\mu (\alpha - \beta \gamma^5) q + igf \left( B_\mu^\dagger \partial^\nu B^\nu - B_\nu^\dagger \partial^\nu B^\mu + B_\nu \partial^\nu B^\mu B^\mu \right) - igf \partial^\nu \left( B_\nu^\dagger B^\mu - B^\mu B^\nu B^\nu \right) \right].
\] (2.51)

Eliminating \( Z_\mu \) using its equation of motion and keeping only the terms relevant for this process that survive in the non-relativistic limit, we obtain the effective Lagrangian
\[
\mathcal{L}_{\text{eff}} \simeq \frac{i\alpha g^2 f}{m_Z^2} \bar{q} \gamma^\mu q.
\] (2.52)

This generates purely spin-independent interactions.

**Diagram 4b: t-channel scalar exchange**

We first start with the case of a real vector dark matter field. The relevant part of the Lagrangian has the general form
\[
\mathcal{L} = \frac{1}{2} (\partial h)^2 - \frac{1}{2} m_h^2 h^2 + a B_\mu B^\mu h - \bar{q}(\bar{\alpha} - \bar{\beta} \gamma^5) q h.
\] (2.53)

Integrating out \( h \), and ignoring terms of order \( \partial^2/m_h^2 \) and higher, we find
\[
h \simeq \frac{a B_\mu B^\mu - \bar{q}(\bar{\alpha} - \bar{\beta} \gamma^5) q}{m_h^2}.
\] (2.54)

The relevant terms in the effective Lagrangian are then
\[
\mathcal{L}_{\text{eff}} \simeq - \frac{a\bar{\alpha}}{m_h} B_\mu B^\mu \bar{q} q.
\] (2.55)

This diagram generates spin-independent interactions that are suppressed in the chiral limit. It is straightforward to verify that in the case of complex vector dark matter, the conclusion is exactly the same. The Lagrangian we began from explicitly breaks chiral symmetry, which is why this result differs from our naive operator analysis.

**Diagram 4c: s- and u-channel fermion exchange**

We again start with the case that the dark matter field is real. The most general renormalizable Lagrangian for such a process takes the form
\[
\mathcal{L} = \bar{Q} \left( i \gamma^\mu - m_Q \right) Q + \bar{q} \gamma^\mu (\alpha - \beta \gamma^5) q B_\mu + \bar{Q} \gamma^\mu (\alpha^* - \beta^* \gamma^5) q^* B_\mu.
\] (2.56)

Integrating out \( Q \) at tree-level we find
\[
Q = \frac{i \gamma^\mu}{\partial^2 + m_Q^2} \left[ \gamma^\mu (\alpha^* - \beta^* \gamma^5) q B_\mu \right].
\] (2.57)
This leads to the effective Lagrangian

\[ \mathcal{L}_{\text{eff}} = \bar{q} \gamma^\nu (\alpha - \beta \gamma^5) B_\nu \frac{i \partial}{\partial^2 + m_Q^2} \left[ \gamma^{\mu} (\alpha^* - \beta^* \gamma^5) q \right] B^\mu. \]  

(2.58)

Once again keeping only zeroth order and first order terms in an expansion in powers of \( \partial/m_Q \) we find

\[ \mathcal{L}_{\text{eff}} = \frac{1}{m_Q^2} \left[ m_Q \bar{q} \gamma^\nu (\alpha - \beta \gamma^5) \gamma^{\mu} (\alpha^* - \beta^* \gamma^5) q B_\nu B_\mu \right] 
+ \frac{i}{m_Q^2} \left[ \bar{q} \gamma^\nu (\alpha - \beta \gamma^5) \gamma^{\alpha} \gamma^{\mu} (\alpha^* - \beta^* \gamma^5) q B_\nu \partial_\alpha B_\mu \right]. \]  

(2.59)

where we have neglected terms where the derivatives act on the quark fields, and terms which vanish in the non-relativistic limit. After integrating by parts and using the identity \( \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\gamma \gamma^\rho = 2 \epsilon^{\mu \nu \rho \sigma} \gamma^\alpha \gamma^\rho \gamma^\gamma \gamma^\delta \gamma^\sigma \), this reduces to

\[ \mathcal{L}_{\text{eff}} = \frac{1}{m_Q^2} \left[ m_Q \left( |\alpha|^2 - |\beta|^2 \right) \bar{q} q B_\mu B^\mu \right] 
- \epsilon^{\mu \nu \rho \sigma} \frac{1}{m_Q^2} \left[ \left( |\alpha|^2 + |\beta|^2 \right) \bar{q} \gamma_\rho \gamma^\gamma q - (\alpha \beta^* + \beta \alpha^*) \bar{q} \gamma_\rho q \right] B_\nu \partial_\sigma B_\mu. \]  

(2.60)

In the low energy limit, the derivative picks out the \( \sigma = 0 \) component in the second term. The \( \epsilon \) tensor then requires the other indices to be spatial. Therefore the term proportional to the quark vector current is velocity suppressed. The effective Lagrangian is then

\[ \mathcal{L}_{\text{eff}} = \frac{1}{m_Q^2} \left[ m_Q \left( |\alpha|^2 - |\beta|^2 \right) \bar{q} q B_\mu B^\mu - \epsilon^{\mu \nu \rho \sigma} \left( |\alpha|^2 + |\beta|^2 \right) \bar{q} \gamma_\rho \gamma^\gamma q B_\nu \partial_\sigma B_\mu \right]. \]  

(2.61)

The spin-independent contribution vanishes in the limit of exact chiral symmetry, and can therefore naturally be small. Therefore, the spin-dependent cross section naturally dominates in this case.

We now move on to the case where the vector dark matter field is complex. Then only the u-channel diagram contributes. The Lagrangian we start from is modified to

\[ \mathcal{L} = \bar{Q} (i \partial - m_Q) Q + \bar{q} \gamma^\nu (\alpha - \beta \gamma^5) Q B_\nu + \bar{Q} \gamma^{\mu} (\alpha^* - \beta^* \gamma^5) q B^\dagger_\mu. \]  

(2.62)

The equation of motion for \( \bar{Q} \) leads to

\[ Q = \frac{i \partial}{\partial^2 + m_Q^2} \left[ \gamma^{\mu} (\alpha^* - \beta^* \gamma^5) q B^\dagger_\mu \right]. \]  

(2.63)

The effective Lagrangian is then

\[ \mathcal{L}_{\text{eff}} = \bar{q} \gamma^\nu (\alpha - \beta \gamma^5) B_\nu \frac{i \partial}{\partial^2 + m_Q^2} \left[ \gamma^{\mu} (\alpha^* - \beta^* \gamma^5) q B^\dagger_\mu \right]. \]  

(2.64)
Working to first order in $\partial/m_Q$, we are left with

$$L_{\text{eff}} = \frac{B_\mu B_\nu^\dagger}{m_Q} \bar{q} \left( (|\alpha|^2 - |\beta|^2) \gamma^\mu \gamma^\nu + (\alpha^* \beta - \alpha \beta^*) \gamma^\mu \gamma^\nu \gamma^5 \right) q + \frac{iB_\mu \partial_\alpha B_\nu^\dagger}{2m_Q^2} \bar{q} \left( (|\alpha|^2 + |\beta|^2) \gamma^\mu \gamma^\alpha \gamma^\nu - (\alpha^* \beta + \alpha \beta^*) \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^5 \right) q. \quad (2.65)$$

Here we have neglected the action of derivatives on quark fields. To distinguish spin-independent and spin-dependent contributions, we make use of the following identities.

$$\gamma^\mu \gamma^\nu = g^{\mu\nu} - i\sigma^{\mu\nu}$$
$$\gamma^\mu \gamma^\nu \gamma^5 = g^{\mu\nu} \gamma^5 + \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \sigma_{\alpha\beta}$$
$$\gamma^\mu \gamma^\alpha \gamma^5 = g^{\mu\alpha} \gamma^5 \gamma^\nu - g^{\mu\nu} \gamma^\alpha + g^{\nu\alpha} \gamma^\mu + i\epsilon^{\mu\alpha\nu\rho} \gamma^\rho \gamma^5$$
$$\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^5 = g^{\mu\alpha} \gamma^5 \gamma^\nu - g^{\mu\nu} \gamma^\alpha \gamma^5 + g^{\nu\alpha} \gamma^\mu \gamma^5 + i\epsilon^{\mu\alpha\nu\rho} \gamma^\rho \gamma^5. \quad (2.66)$$

Dropping terms that do not survive in the non-relativistic limit, we find the form of the effective Lagrangian to be

$$L_{\text{eff}} = \left| \alpha \right|^2 - \left| \beta \right|^2 \left( B_\mu^\dagger B^\mu q - iB_\mu B_\nu^\dagger \bar{q} \sigma^{\mu\nu} q \right) + \frac{i}{m_Q^2} \left[ -B_\mu^\dagger \left( \partial_\alpha B_\nu^\dagger \right) \bar{q} \gamma^\alpha q + i\epsilon^{\mu\alpha\nu\rho} B_\mu \left( \partial_\alpha B_\nu^\dagger \right) \bar{q} \gamma^\rho \gamma^5 q \right]. \quad (2.67)$$

We see that even in the limit of exact chiral symmetry, both spin-independent and spin-dependent interactions are present. This is exactly as expected, since there are no values of the parameters $\alpha$ and $\beta$ for which the theory admits a $B_\mu \leftrightarrow -B_\mu^\dagger$ interchange symmetry. It is not possible to incorporate such a symmetry in a manner consistent with the rephasing symmetry of $B_\mu$ unless additional fields are added to the theory.

In conclusion, we see that in the chiral limit, the results of scalar, real vector and Majorana fermion dark matter are in perfect agreement with the operator analysis we performed earlier. Scalar dark matter always leads to spin-independent interactions, while in the limit of exact chiral symmetry Majorana fermions and real vector bosons give rise to spin-dependent interactions. There is always a sizeable spin-independent component to the WIMP-nucleon cross-section in the cases of Dirac fermion and complex vector boson dark matter, except for very specific choices of quantum numbers. From this it follows that primarily spin-dependent WIMP-nucleon cross sections are closely associated with theories where the dark matter particle is its own anti-particle.

### 3. Models with spin-dependent couplings

We now study in more detail the cases which naturally lead to purely spin-dependent interactions in the chiral limit. We continue to work in the non-relativistic regime, and under the
assumption that the momentum transfer in the scattering process is much less than both the dark matter mass and the characteristic nuclear scales. We however no longer assume that the dark matter mass is much less than the mass of the particle mediating the interaction.

3.1 Dirac fermion

**t-channel vector exchange**

![Figure 5: Dirac dark matter scattering through t-channel vector exchange](image)

We return to the Lagrangian

\[ L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_Z^2 Z^\mu Z_\mu + \bar{\chi} \gamma^\mu (\alpha - \beta \gamma^5) \chi Z_\mu + \bar{q} \gamma^\mu (\tilde{\alpha} - \tilde{\beta} \gamma^5) q Z_\mu. \]  

(3.1)

Since the momentum in the propagator in the case of t-channel exchange is negligible compared to the dark matter mass, we can continue to use the effective Lagrangian defined in equation (2.31),

\[ L_{\text{eff}} \simeq -\frac{1}{m_Z^2} \left[ \alpha \bar{\chi} \gamma^\mu \chi \bar{q} \gamma_\mu q + \beta \bar{\chi} \gamma^\mu \gamma^5 \chi \bar{q} \gamma_\mu \gamma^5 q \right]. \]  

(3.2)

This leads to purely spin-dependent scattering only in the cases when either \( \alpha \) or \( \tilde{\alpha} \) is zero, while both \( \beta \) and \( \tilde{\beta} \) are non-zero.

Consider first the possibility that \( \tilde{\alpha} = 0 \). Clearly the charges of the quarks under the SM \( Z \) are such that this criterion is not satisfied. The crucial question is then whether a \( Z' \) can exist for which such a charge assignment is phenomenologically viable. The requirement that \( \tilde{\beta} \neq 0 \) implies that the left- and right-handed quarks have different charges under the \( Z' \). This in turn means that either the SM Higgs is charged under the \( Z' \), or alternatively that the quark masses arise from non-renormalizable interactions. We consider each of these possibilities in turn.

If the SM Higgs is charged under the \( Z' \), when electroweak symmetry is broken the SM \( Z \) will mix with the \( Z' \). The mixing angle is of order \( (m_Z/m'_Z)^2 \). After diagonalization the dark matter particle will acquire a charge under the \( Z \) of order \( (m_Z/m'_Z)^2 \). This will generate a sizeable spin-independent contribution to the WIMP-nucleon cross section. We conclude that this approach is not viable.

If the SM Higgs is not charged under the \( Z' \), then the quark masses must arise from non-renormalizable interactions involving both the SM Higgs and the Higgs field \( H' \) that breaks the gauge symmetry corresponding to the \( Z' \). While this is perhaps adequate for the light
quarks, it is somewhat unsatisfactory for the top quark, whose mass is large. This problem can be avoided by assigning the three generations different charges under the $Z'$, but at the expense of a new source of flavor violation close to the weak scale. In addition there is a chirality suppressed spin-independent contribution to the WIMP-nucleon cross section from $H'$ exchange, which may be sizeable. We conclude that this scenario is disfavored, although perhaps not excluded.

We now consider the alternate scenario where $\alpha = 0$. If the Dirac fermion $\chi$ has no interactions beyond those in the Lagrangian above, a simple change of variables allows the theory with $\alpha = 0$ to be rewritten as a theory of two degenerate Majorana fermions with identical charges under $Z$. This theory is therefore not distinct from the case of Majorana fermion dark matter to be considered in the next section. If $\chi$ does have additional interactions, however, this theory of Dirac fermion dark matter is in general not equivalent to a theory of Majorana fermion dark matter. We conclude that this class of theories of Dirac fermion dark matter can indeed give rise to primarily spin-dependent WIMP-nucleon scattering at tree-level, but only if the WIMP carries very specific charges under the gauge boson. We will not consider this possibility further.

### 3.2 Majorana fermion

#### t-channel vector exchange

![Figure 6: Majorana dark matter scattering through t-channel vector exchange](image)

We start from the Lagrangian

$$L = -\frac{1}{4} F_{\mu
u} F^{\mu\nu} + \frac{1}{2} m_Z^2 Z_{\mu} Z_{\mu} + \bar{\chi} \gamma_{\mu} (-\beta \gamma^5) \chi Z_{\mu} + \bar{q} \gamma_{\mu} (\tilde{\alpha} - \beta \gamma^5) q Z_{\mu}. \quad (3.3)$$

We can again neglect momentum-dependent terms in the propagator, leading to the matrix element

$$\mathcal{M} = \frac{2 \beta \tilde{\beta}}{m_Z^2} \bar{u}_\chi \gamma_{\mu} \gamma^5 u_\chi \langle \bar{q} \gamma_{\mu} \gamma^5 q \rangle. \quad (3.4)$$

The matrix elements $\langle \bar{q} \gamma_{\mu} \gamma^5 q \rangle$ etc. are defined in Appendix B, while $u_\chi, \bar{u}_\chi$ are the familiar spinors corresponding to the dark matter fermion. We see that this diagram does indeed lead to purely spin-dependent cross sections. The gauge boson exchanged in this process may be either the SM $Z$, or a new $Z'$. If it is the SM $Z$, then the dark matter particle must either constitute the neutral component of a single representation of SM SU(2)$_L$ or arise
as a linear combination of the neutral components of different SU(2)\textsubscript{L} representations. There is no such constraint if the gauge boson is a \(Z'\). The physics of these two cases is therefore very different, and so we consider them separately.

Exchange of the Standard Model Z

Consider first the case of the SM \(Z\). Since the \(Z\) is light and all its couplings are fixed, this scenario is already somewhat constrained by experiments. Let us understand the nature of these bounds. As explained above, the dark matter must either be the neutral component of an SU(2)\textsubscript{L} representation, or a linear combination of the neutral components of different SU(2)\textsubscript{L} representations and SM singlets. If the dark matter is not part of a linear combination, the direct detection constraints arising from spin-dependent WIMP-nucleon interactions are very strong. For example, a Majorana neutrino in a pure SU(2)\textsubscript{L} doublet representation has been excluded by XENON as a dark matter candidate \[10\]. For this reason we expect that in this class of theories the WIMP will be a linear combination of the neutral components of different SU(2)\textsubscript{L} representations, which allows the possibility of weaker couplings to the \(Z\), thereby loosening the direct detection constraints.

In addition to this spin-dependent contribution to the WIMP-nucleon cross section from \(Z\) exchange, there is also necessarily a spin-independent contribution from SM Higgs exchange. We now explain the origin of this effect. The coupling of any chiral field in an SU(2)\textsubscript{L} representation to the \(Z\) is proportional to \(I_3 + Q \sin^2 \theta_W\), where \(I_3\) is the weak-isospin, \(Q\) is the electric charge and \(\theta_W\) is the weak mixing angle. It follows that for the neutral component to have non-zero charge under the \(Z\), the representation must carry hypercharge. Therefore, until electroweak symmetry is broken, it cannot acquire a Majorana mass. This implies that the Majorana mass of any dark matter particle charged under the \(Z\) can arise only as an electroweak symmetry breaking effect, from couplings involving the Higgs. The conclusion is that in this class of theories there is a spin-independent contribution to the WIMP-nucleon cross section mediated by the Higgs, which is in general correlated with the spin-dependent cross section mediated by the \(Z\) \[15\].

In order to obtain a quantitative understanding of the constraints on this scenario, we choose a benchmark model. We consider a theory where the dark matter particle arises as a linear combination of the neutral components of two SM SU(2)\textsubscript{L} doublets which have hypercharges \(Y = \pm \frac{1}{2}\).

\[
\xi = \begin{pmatrix} \xi^+ \\ \xi_0 \end{pmatrix}, \quad \xi^c = \begin{pmatrix} \xi_0^c \\ -\xi^- \end{pmatrix}.
\]

(3.5)

In addition to a Dirac mass for \(\xi\) and \(\xi^c\), we include a non-renormalizable operator that gives rise to a Majorana mass term for the neutral component of \(\xi\).

\[
\mathcal{L} \supset -\frac{(H^\dagger \xi)^2}{\Lambda} + \text{h.c.}
\]

(3.6)
Here $H$ is the SM Higgs doublet. This non-renormalizable operator can be generated by integrating out a SM singlet. The Higgs acquires a vacuum expectation value,

$$\langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix},$$

where $v = 246$ GeV. We generate a Majorana mass term and a Yukawa coupling for the neutral component of the field $\xi_0$. We denote the physical Higgs field of the SM that emerges after electroweak symmetry breaking by $h$. The Lagrangian now contains,

$$\mathcal{L} \supset -\frac{g}{\cos \theta_W} \left( \bar{\xi}^c_0 \frac{1}{2} \sigma^\mu \xi_0 - \bar{\xi}^c_0 \frac{1}{2} \sigma^\mu \xi^c_0 \right) Z_\mu$$

$$- \left[ \frac{1}{2} \left( \frac{\xi^c_0}{\xi_0} \frac{\xi_0}{\xi^c_0} \right) \begin{pmatrix} 0 \\ M \\ M \end{pmatrix} \begin{pmatrix} \xi^c_0 \\ \xi_0 \\ h \end{pmatrix} + y_\xi \xi_0 \xi_0 h + \text{h.c.} \right],$$

where $m = v^2/\Lambda$ and $y_\xi = m/v$. The lighter eigenstate of the two mass eigenstates $\xi_D$ is the dark matter field,

$$\xi_D = \cos \phi \xi_0 + \sin \phi \xi^c_0.$$  

(3.9)

where $\phi$ is the mixing angle. The couplings of $\xi_D$ in the mass basis are given by

$$\mathcal{L} \supset -\frac{g \cos 2\phi}{2 \cos \theta_W} \bar{\xi}_D \gamma^\mu \gamma^5 \xi_0 Z_\mu - [y_\xi \cos^2 \phi \xi_D h + \text{h.c.}].$$

(3.10)

Translating this to couplings with a four-component Majorana fermion, $\chi$, to be consistent with the rest of our analysis, we find,

$$\mathcal{L} \supset \frac{g \cos 2\phi}{4 \cos \theta_W} \bar{\chi} \gamma^\mu \gamma^5 \chi Z_\mu - y_\xi \cos^2 \phi \bar{\chi} \chi h.$$  

(3.11)

The coupling to the $Z$ is suppressed by $\cos 2\phi$. In the limit $m \ll M$, this is simply $-m/2M$. The dark matter mass in this limit is approximately equal to $M$. Therefore, we see that at higher dark matter masses the XENON spin-dependent bounds can be avoided. However, since the coupling of the WIMP to the Higgs $y_\xi$ is proportional to the Majorana mass term $m$ and independent of $M$, the spin-independent cross section is insensitive to the dark matter mass in this limit. Therefore the spin-dependent and spin-independent bounds are somewhat complementary, with the latter more effective for larger values of the dark matter mass.

In figure 7 we display the interplay between spin-dependent and spin-independent direct detection bounds in the benchmark model. The upper (lower) curve with positive slope shows the current (future) bounds on spin-dependent scattering from the XENON experiment as a function of the dark matter mass. The upper (lower) curve with negative slope shows the maximum value of the spin-dependent scattering cross section that is consistent with the current (future) spin-independent direct detection constraints, arising from the Higgs
Figure 7: The maximum spin-dependent WIMP-nucleon cross section in the benchmark model consistent with the current (blue solid) and projected (green dashed) bounds on the spin-independent cross section from direct detection experiments. Also shown are current (blue dotted) and projected (green dotted) bounds on spin-dependent scattering from the XENON experiment. The black dot-dashed line shows where the WIMP-Higgs Yukawa coupling $y_\xi = 1$.

If the gauge boson being exchanged is not the SM $Z$ but a new $Z'$, there is considerably more flexibility with regard to charge assignments. However, the severe constraints from precision electroweak measurements, direct production and four-fermion point interactions [34, 35] on the mass and couplings of any new gauge boson limits the possible signal in direct detection experiments. In addition to the spin-dependent contribution to the WIMP-nucleon cross section, there may also be a spin-independent contribution if the SM Higgs mixes with the
Higgs that gives mass to the $Z'$. However, whether this effect arises or not is dependent on the $Z'$ charge assignments, and therefore does not give rise to a robust bound. This is another difference from the case of the SM $Z$.

**s- and u-channel vector exchange**

\[
\begin{align*}
\text{(a)} & \quad p_1 p_2 X \rightarrow q \\
\text{(b)} & \quad p_1 p_2 X \rightarrow q
\end{align*}
\]

Figure 8: Majorana dark matter scattering through s- and u-channel vector exchange

The Lagrangian corresponding to this process is

\[
\mathcal{L} = -\frac{1}{2} [\partial_\mu X_\nu - \partial_\nu X_\mu]^2 + m_X^2 X_\mu X^\mu + \bar{\chi} \gamma^\mu (\alpha - \beta \gamma^5) q X_\mu + \bar{q} \gamma^\mu (\alpha^* - \beta^* \gamma^5) \chi X^\mu. \tag{3.12}
\]

The matrix element for this process may be obtained from the Feynman diagrams. After a Fierz rearrangement, it reduces to

\[
\mathcal{M} \approx \frac{2}{(m_X^2 - m_\chi^2)} \left[ (|\alpha|^2 - |\beta|^2)(\bar{q}q) \bar{u}_\chi u_X - \frac{1}{2} (|\alpha|^2 + |\beta|^2)(\bar{q} \gamma^\mu \gamma^5 q) \bar{u}_\chi \gamma_\mu \gamma^5 u_X \right]. \tag{3.13}
\]

We see that in the chiral limit, scattering is purely spin-dependent, as expected. Note that the vector particle being exchanged is a SM color triplet.

**s- and u-channel scalar exchange**

\[
\begin{align*}
\text{(a)} & \quad p_1 p_2 \Phi \rightarrow q \\
\text{(b)} & \quad p_1 p_2 \Phi \rightarrow q
\end{align*}
\]

Figure 9: Majorana dark matter scattering through s- and u-channel scalar exchange

The Lagrangian corresponding to this process is

\[
\mathcal{L} = |\partial \Phi|^2 - m_\Phi^2 |\Phi|^2 - \bar{\chi} (\alpha - \beta \gamma^5) q \Phi - \bar{q} (\alpha^* + \beta^* \gamma^5) \chi \Phi^\dagger. \tag{3.14}
\]
After using Fierz identities and ignoring terms which are velocity suppressed, we can write the amplitude in terms of the expectation values of quark currents in the nucleus.

$$\mathcal{M} = -\frac{1}{2(m_Q^2 - m^2)} \left[ \left( |\alpha|^2 - |\beta|^2 \right) \bar{u}_X u_X \langle \bar{q} q \rangle - \left( |\alpha|^2 + |\beta|^2 \right) \bar{u}_X \gamma^\mu \gamma^5 u_X \langle \bar{q} \gamma_\mu \gamma^5 q \rangle \right].$$  \hspace{1cm} (3.15)

As expected the cross section is purely spin-dependent in the chiral limit. Again, note that the scalar being exchanged in this process transforms as a fundamental under SM color.

### 3.3 Real vector boson

#### s- and u-channel fermion exchange

The only diagram seen to give spin-dependent scattering was diagram 4c. The relevant Lagrangian takes the form

$$\mathcal{L} = \bar{Q}(i\partial - m_Q)Q + \bar{q} \gamma^\mu (\alpha - \beta \gamma^5)Q B^\mu + \bar{Q} \gamma^\mu (\alpha^* - \beta^* \gamma^5)q B^\mu.$$  \hspace{1cm} (3.16)

The corresponding matrix element is

$$\mathcal{M} = -\frac{2\epsilon_\mu(p_1)\epsilon^*_\nu(p_3)}{m_B^2 - m_Q^2} \left[ m_Q(|\alpha|^2 - |\beta|^2)g^{\mu\nu}\langle \bar{q} q \rangle - i(|\alpha|^2 + |\beta|^2)m_B \epsilon_{\mu\nu\rho\sigma}\langle \bar{q} \gamma_\rho \gamma^5 q \rangle \right].$$  \hspace{1cm} (3.17)

This is purely spin-dependent in the chiral limit. The fermion being exchanged in this process is again a SM color triplet.

### 3.4 Summary

Our results are summarised in Table 1. We see that the dark matter candidates which naturally lead to sizeable spin-dependent WIMP-nucleon cross sections without correspondingly large spin-independent cross sections are Majorana fermions and real vector bosons. These theories share the feature that the dark matter particle is its own anti-particle. We stress however that the fact that the dark matter particle is a Majorana fermion or real vector boson is not sufficient by itself to guarantee that WIMP-nucleon cross sections are primarily spin-dependent. This is only true if the chirality suppressed t-channel scalar exchange contribution is either absent or sub-dominant. While this can naturally be the case it is certainly not guaranteed.

The interactions of scalars with nucleons are spin-independent. The cross sections of Dirac fermions and complex vector bosons in general tend to have a sizeable spin-independent
| Dark Matter         | Mediator  | Process | Scattering |
|---------------------|-----------|---------|------------|
| Scalar              | $Z, Z'$   | ![Diagram](image1) | SI         |
|                     | $h$       | ![Diagram](image2) | SI         |
|                     | $Q$       | ![Diagram](image3) | SI         |
| Dirac Fermion       | $Z, Z'$   | ![Diagram](image4) | SI, SD$^\dagger$ |
|                     | $h$       | ![Diagram](image5) | SI         |
|                     | $X$       | ![Diagram](image6) | SI, SD     |
|                     | $\Phi$    | ![Diagram](image7) | SI, SD     |
| Majorana Fermion    | $Z, Z'$   | ![Diagram](image8) | SD         |
|                     | $h$       | ![Diagram](image9) | SI         |
|                     | $X$       | ![Diagram](image10) | SD in chiral limit |
|                     | $\Phi$    | ![Diagram](image11) | SD in chiral limit |
| Real Vector         | $h$       | ![Diagram](image12) | SI         |
|                     | $Q$       | ![Diagram](image13) | SD in chiral limit |
| Complex Vector      | $Z, Z'$   | ![Diagram](image14) | SI         |
|                     | $h$       | ![Diagram](image15) | SI         |
|                     | $Q$       | ![Diagram](image16) | SI, SD     |

Table 1: A summary of our results for WIMP-nucleon scattering, for each dark matter candidate and mediator. In the Feynman diagrams, scalars are represented by dashed lines, fermions by solid lines and vector bosons by wavy lines. Of the mediators, $h$, $Z'$ and the SM $Z$ are neutral under both electromagnetism and color, while $X$, $\Phi$ and $Q$ transform as triplets under color and carry electric charge.

$^\dagger$Can be primarily SD for specific choices of $Z'$ charges
component unless the dark sector possesses a discrete symmetry similar to charge conjugation, but under which the SM fields are invariant. In general incorporating such a symmetry requires complicating the theory by adding multiple mediators, except in the case of Dirac fermion dark matter interacting through t-channel vector exchange, when very specific choices of charge assignments allow such a symmetry to be realized.

What are the constraints from flavor on the class of theories we are considering? For WIMP-nucleon scattering mediated by the SM $Z$, flavor bounds are automatically satisfied. This also applies to t-channel $Z'$ exchange, provided the couplings of the $Z'$ are flavor-diagonal. However, couplings of the type quark-WIMP-mediator are very strongly constrained by precision flavor experiments, as these couplings will in general have a non-trivial flavor structure. In general, we need different mediators to couple to up-type and down-type quarks. Further, in order to satisfy these bounds, it may be necessary to introduce either multiple flavors of the mediating particle or multiple flavors of the dark matter particle, while incorporating some version of a GIM mechanism. These considerations, while important, do not impact our conclusions and we therefore leave this for future work.

In summary, we see that in each case where the spin-dependent contribution naturally dominates, there are either new particles charged under the SM gauge groups, or a new $Z'$ gauge boson. In the next section we investigate the implications of this result for the LHC.

4. Implications for Colliders

The analysis above reveals that in the chiral limit there is only one effective operator for each of Majorana and real vector boson dark matter which leads to spin-dependent interactions with nucleons. Therefore, a spin-dependent scattering signal in a direct detection experiment implies a model-independent lower bound on the coefficient of this operator. This in turn places limits on the masses of the particles mediating dark matter interactions, which can be searched for in collider experiments such as the LHC. For WIMPs whose primary interactions with nucleons are spin-dependent, direct detection experiments and collider searches are therefore highly correlated. In what follows we consider Majorana fermion dark matter and real vector boson dark matter in turn, and explore the region of parameter space which is accessible to direct detection experiments, and to the LHC.

4.1 Majorana Fermion Dark Matter

Starting from the effective Lagrangian

$$L_{\text{eff}} = d_q \bar{\chi} \gamma^\mu \gamma^5 \chi \bar{q} \gamma_\mu \gamma^5 q,$$  

we can derive the cross section (see Appendix B),

$$\sigma_0 = \frac{16m^2 \lambda^2}{\pi (m_\chi + m_N)^2} \left[ \sum_{q=u,d,s} d_q \lambda_q \right]^2 J_N(J_N + 1).$$
Table 2: Quark spin fractions in the proton and neutron [37, 38]

|       | proton       | neutron      |
|-------|--------------|--------------|
| $\Delta_u$ | $0.78 \pm 0.02$ | $-0.48 \pm 0.02$ |
| $\Delta_d$ | $-0.48 \pm 0.02$ | $0.78 \pm 0.02$ |
| $\Delta_s$ | $-0.15 \pm 0.02$ | $-0.15 \pm 0.02$ |

For a free nucleon, $\lambda_q$ is given simply by $\Delta^n_q$, the spin fraction of the nucleon carried by quark $q$. $J_N$ is the angular momentum of the nucleus, equal to $\frac{1}{2}$ for free nucleons, while $m_\chi$ and $m_N$ are the dark matter mass and the mass of the nucleus respectively. The quark spin fractions in the proton and the neutron are shown in Table 2.

This cross section is bounded by current experiments. The KIMS [11] experiment is currently the most sensitive to the WIMP-proton coupling while the XENON experiment [10] is the most sensitive to the WIMP-neutron coupling. Direct detection experiments translate their limits into bounds on the WIMP-nucleon cross section using a model-independent framework [36]. We see from figure 11a that if the model predicts identical couplings to neutrons and protons, then the XENON experiment provides the more stringent bound. These experiments will probe new parameter space in the near future, with the XENON experiment in particular expected to improve the current limit by two orders of magnitude.

The values of $d_u$, $d_d$ and $d_s$ which appear in equation (4.2) depend on the flavor structure of the theory. As explained earlier, dark matter scattering mediated by a $Z'$ is automatically consistent with constraints from flavor provided that the couplings of the $Z'$ are flavor diagonal. Therefore $d_d = d_s$ in this class of models. For theories where WIMP-nucleon scattering is mediated by $X_\mu$ or $\Phi$, the flavor structure is more complicated. In general, a GIM mechanism is required to ensure that flavor bounds are satisfied. We therefore allow the possibility of multiple mediators, with each mediator associated with a different quark flavor. In general the mediators that couple to left- and right-handed quarks are also distinct. Flavor constraints are satisfied provided that the mediators corresponding to different flavors of quarks are degenerate, and their couplings are flavor diagonal.

We now consider in turn the various theories of Majorana fermion dark matter which naturally lead to primarily spin-dependent WIMP-nucleon cross sections. For each theory we explore the range of dark matter and mediator masses which leads to a signal in direct detection experiments, and the implications for the LHC.

- Via the SM $Z$
  The axial couplings of the SM $Z$ to quarks are proportional to $I_3$, i.e. $d_u = -d_d = -d_s$. From figure 7 we see that the current spin-independent bounds imply that for the spin-dependent WIMP-nucleon cross section to be accessible to upcoming experiments the dark matter mass must lie below about 400 GeV. Therefore, in most of the parameter space that these experiments will probe the dark matter particle $\chi^0$ and its charged partner $\chi^+$ are kinematically accessible to the LHC. Although this result was obtained
in the context of a specific benchmark model, we do not expect the results in the general case to differ significantly.

- Via a $Z'$
  On comparison with equation (2.46), we see that the value of the coefficient is
  \[
  d_q = -\frac{\beta q}{m_{Z'}^2}.
  \] (4.3)
  As for the SM $Z$, we set $d_u = -d_d = -d_s$, since this choice is naturally consistent with flavor constraints on new physics, and yields a conservative estimate for the mediator mass. The values $\beta$ and $\tilde{\beta}$ equal to a $\frac{1}{2}$ correspond to chiral fermions having unit charge under the $Z'$. We use these as representative values. In figure 12 we have shown the range of values of the $Z'$ mass that would lead to a signal in direct detection experiments at the current bound, or within two orders of magnitude of the current bound. Unfortunately, the allowed range of masses is disfavored by precision electroweak measurements, direct production and four-fermion point interactions [34, 35], except perhaps for very specific charge assignments.

- Via colored vector bosons, $X_\mu$
  \[
  d_q = -\frac{|\alpha q|^2 + |\beta q|^2}{2(m_{X,q}^2 - m_{\chi}^2)}
  \] (4.4)
  In the chiral limit, the $X$ vector bosons which couple to left- and right-handed quarks are distinct particles. In the absence of tuning, it is therefore natural for either the left- or right-handed contribution to dominate. For the left-handed contribution, if the dark matter particle is a SM singlet we expect that $d_u = d_d$ as a consequence of the SM $SU(2)_L$ symmetry. Further, flavor constraints require $d_d = d_s$. Then the values of $\Delta_u$, $\Delta_q$ and $\Delta_s$ imply that there are large cancellations among the contributions of the different left-handed quarks to the WIMP-nucleon cross section, which is therefore somewhat suppressed. For the right-handed contribution, this cancellation can be avoided if $d_u \gg d_d(= d_s)$, or vice versa.

In figures 12 and 13 we have plotted the range of values of the $X$ masses that would lead to a signal at current direct detection experiments. In figure 12 we have set $d_d = d_s = 0$ and $d_u \neq 0$, with $\alpha_u = -\beta_u = \frac{1}{2}$, corresponding to one natural possibility for the contribution from right-handed quarks. In figure 13, on the other hand, we have set $d_u = d_d = d_s$, with $\alpha_q = \beta_q = \frac{1}{2}$ corresponding to the contribution from left-handed quarks. We see that away from the resonance region at $m_{\chi} = m_X$, the colored vector boson masses lie at a TeV or below in all of parameter space. They are therefore kinematically accessible to the LHC, and can be pair-produced through strong interactions. The signal is jets + missing energy. Recent model-independent studies
Figure 11: Current direct detection bounds on the spin-dependent (above) and spin-independent (below) dark matter-nucleon cross sections. The spin-independent bound assumes that dark matter has equal couplings to protons and neutrons.
Figure 12: Estimates for the mediator masses if direct detection experiments see a signal near the present bound (above) or two orders of magnitude below the present bound (below). The colored mediators $X$, $\Phi$ and $Q$ are assumed to couple only to up-type quarks, while the charges of the $Z'$ are assumed to be proportional to the charges of the SM $Z$. The masses of $X$, $\Phi$ and $Q$ must lie above the red dashed line, which corresponds to where the mediator mass is equal to the dark matter mass.
Figure 13: Estimates for the colored mediator masses if direct detection experiments see a signal near the present bound (above) or two orders of magnitude below the present bound (below), assuming equal couplings to up- and down-type quarks. The signal in direct detection experiments is suppressed due to cancellations arising from quark spin fractions, leading to lower values of the mediator masses.
of dark matter signals at the LHC involving jets + missing energy may be found, for example, in [39, 40], [41], [42].

- Via colored scalars, $\Phi$

\[ d_q = \frac{|\alpha_q|^2 + |\beta_q|^2}{4(m_{\Phi,q}^2 - m_{\chi}^2)} \]  

(4.5)

As explained earlier, in the chiral limit the mediators of left- and right-handed interactions are distinct. In figures 12 and 13 we have shown the range of values of the $\Phi$ mass that would lead to a signal at current direct detection experiments. In figure 12 we have set $d_u \neq 0$ and $d_d = d_s = 0$, with $\alpha_u = -\beta_u = \frac{1}{2}$, corresponding to the right-handed contribution. In figure 13 we have set $d_u = d_d = d_s$ with $\alpha_q = \beta_q = \frac{1}{2}$, corresponding to the left-handed contribution. We see that away from the resonance region the $\Phi$ masses lie below a TeV, which is promising for the LHC.

4.2 Real Vector Boson Dark Matter

The effective Lagrangian for WIMP-nucleon scattering in the case of real vector boson dark matter takes the form,

\[ \mathcal{L} = b_q (\partial \sigma B_\mu) B_\rho q \tilde{q} \gamma^5 q \sigma^{\rho \mu \alpha}. \]  

(4.6)

The corresponding cross section is

\[ \sigma_0 = \frac{8m_{\chi}^2 m_N^2}{3\pi (m_{\chi} + m_N)^2} \left[ \sum_{q=u,d,s} b_q \lambda_q \right]^2 J_N(J_N + 1). \]  

(4.7)

Here $b_q$ is related to the mass and couplings of the colored fermions $Q$ mediating the interaction,

\[ b_q = \frac{|\alpha_q|^2 + |\beta_q|^2}{(m_{Q,q}^2 - m_{\Phi,q}^2)}. \]  

(4.8)

We seek to explore the range of masses of $Q$ and $B_\mu$ that give rise to a signal at current direct detection experiments, and the resulting implications for the LHC. As in the cases of $X$ and $\Phi$, in the chiral limit the mediators $Q$ that couple to left-and right-handed quarks are in general different particles, and the mediators corresponding to different flavors are also distinct. Flavor constraints are satisfied provided the mediators associated with different flavors are degenerate, and their couplings are flavor diagonal. For concreteness we employ exactly the same conventions as earlier. Specifically, we first consider $b_u \neq 0$, $b_d = b_s = 0$ with $\alpha_u = -\beta_u = \frac{1}{2}$, corresponding to the contribution from right-handed quarks. The results are plotted in figure 12. We then consider $b_u = b_d = b_s$ with $\alpha_q = \beta_q = \frac{1}{2}$, corresponding to the contribution from left-handed quarks. The results are plotted in figure 13. From the figures, we see that a signal at current spin-dependent direct detection experiments implies that the masses of the colored fermions lie at a TeV or below, which is within the kinematic reach of the LHC.
5. Conclusions

We have classified dark matter candidates that have the property that WIMP-nucleon scattering is dominated by spin-dependent interactions. We have established that in this scenario the natural dark matter candidates are Majorana fermions or real vector bosons, while scalars are disfavored. Dirac fermion and complex vector boson dark matter are also disfavored except for very specific choices of quantum numbers. Therefore spin-dependent WIMP-nucleon cross sections are closely associated with theories where the dark matter particle is its own antiparticle. Furthermore, we have shown that such theory predicts either new particles close to the weak scale with SM quantum numbers, or a new $Z'$ gauge boson with mass at or below the TeV scale. In the region of parameter space that is of interest to current direct detection experiments, these particles naturally lie in a mass range that is kinematically accessible to the Large Hadron Collider (LHC).

These results also have implications for experiments involving neutrino telescopes searching for the products of dark matter annihilation in the sun. The rate of dark matter capture in the sun is controlled by the cross section for WIMP-nucleon scattering, which therefore impacts the signal. These experiments currently provide stronger limits on the spin-dependent WIMP-proton cross section than direct detection experiments \cite{43, 44, 45}. A dark matter signal at neutrino telescopes, if arising from spin-dependent WIMP-proton scattering, can be translated into limits on the mediator masses. This also has implications for the LHC. We plan to present the results of this analysis in a separate publication \cite{46}.

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Appendices

A. Relevant Fierz Identities

We list here the Fierz identities that are necessary to obtain our results. The results presented are for anti-commuting fields. Throughout these appendices, unless explicitly stated otherwise, we follow the conventions of Peskin and Schroeder [47].

\[
\bar{q}\chi\bar{g} = -\frac{1}{4} \left[ \bar{q}q\chi + \bar{q}\gamma^\mu q\gamma_\mu \chi + \bar{q}\gamma^5 q\chi\gamma^5 \chi - \bar{q}q\gamma^5 q\chi\gamma_\mu \gamma^5 \chi + \frac{1}{2} \bar{q}q \sigma^{\mu\nu} q\chi \sigma_{\mu\nu} \chi \right] \\
\tag{A.1}
\]

\[
\bar{q}\gamma^5 \chi\bar{q} = -\frac{1}{4} \left[ \bar{q}q\chi - \bar{q}\gamma^5 q\gamma_\mu \chi + \bar{q}\gamma^5 q\chi\gamma^5 \chi + \bar{q}q\gamma^5 q\chi\gamma_\mu \gamma^5 \chi + \frac{1}{2} \bar{q}q \sigma^{\mu\nu} q\chi \sigma_{\mu\nu} \chi \right] \\
\tag{A.2}
\]

\[
\bar{q}\gamma^5 \chi\bar{q} = -\frac{1}{4} \left[ \bar{q}q\gamma^5 \chi + \bar{q}\gamma^5 q\gamma_\mu \chi \gamma^5 \chi + \bar{q}\gamma^5 q\chi\gamma^5 \chi - \bar{q}\gamma^5 q\chi\gamma_\mu \chi - \frac{1}{2} \bar{q}q \sigma^{\mu\nu} q\chi \sigma^{\alpha\beta} \chi \right] \\
\tag{A.3}
\]

\[
\bar{q}\gamma^\mu \chi\bar{q} = -\left[ \bar{q}q\chi - \frac{1}{2} \bar{q}\gamma^\mu q\gamma_\mu \chi - \bar{q}\gamma^5 q\chi\gamma^5 \chi - \frac{1}{2} \bar{q}q \gamma^5 q\chi\gamma_\mu \gamma^5 \chi \right] \\
\tag{A.4}
\]

\[
\bar{q}\gamma^\mu \chi\gamma_\mu \gamma^5 q = -\left[ \bar{q}q\chi - \frac{1}{2} \bar{q}\gamma^\mu q\gamma_\mu \chi + \bar{q}\gamma^5 q\chi\gamma^5 \chi - \frac{1}{2} \bar{q}q \gamma^5 q\chi\gamma_\mu \gamma^5 \chi \right] \\
\tag{A.5}
\]

\[
\bar{q}\gamma^\mu \chi\gamma_\mu \gamma^5 q = -\left[ \bar{q}q\chi - \frac{1}{2} \bar{q}\gamma^\mu q\gamma_\mu \chi + \bar{q}\gamma^5 q\chi\gamma^5 \chi \right] - \frac{1}{2} \bar{q}q \gamma^5 q\chi\gamma_\mu \gamma^5 \chi \\
\tag{A.6}
\]

B. Fermionic dark matter scattering cross section

Let $\chi$ represent a fermionic dark matter particle, $q$ a quark, $n$ a nucleon and $N$ a nucleus. We consider elastic scattering of the WIMP off a nucleus

\[\chi(p_1) + N(p_2) \to \chi(p_3) + N(p_4).\]

We can write down the effective Lagrangian at the partonic level in the general case as follows,

\[\mathcal{L} = \bar{\chi}(\alpha + \beta\gamma^5)\chi \bar{q}(\alpha + \beta\gamma^5)q + \lambda_\eta \bar{\chi} \Gamma^\mu \chi \bar{q} \tilde{\Gamma}_\mu q + c_\eta \bar{\chi} \Lambda^{\mu\nu} \chi \bar{q} \tilde{\Lambda}_{\mu\nu} q. \tag{B.1}\]

where the $\mathbb{1}$, $\gamma^5$, $\Gamma$ and $\Lambda$ matrices span the Dirac matrix subspace.

The average speed of $\chi$ in the halo is 300 km/s. This means that the scattering with nuclei in direct detection experiments is highly non-relativistic and calls for an appropriate treatment. In particular, cross sections which are velocity suppressed are generally smaller by a factor of order $10^{-6}$, and can be neglected.

It is important to identify the terms which are relevant in the low-energy limit. This can be done schematically as follows. We can expand a fermion field $\psi \sim au + b^\dagger v$ where $u$ and $v$
are independent spinor solutions to Dirac equation. In the non-relativistic limit, \( u, v \) reduce to

\[
\begin{align*}
u & = \left( \frac{\sqrt{p.\sigma\xi}}{\sqrt{p.\sigma\xi}} \right)_{\text{NR limit}} \sqrt{m} \left( \frac{\eta}{\xi} \right) .
\end{align*}
\] (B.2)

From this it follows that,

\[
\begin{align*}ar{\psi}\psi & \approx 2m \left[ a^\dagger a + b^\dagger b \right] \quad \text{(B.3)}
\end{align*}
\]

\[
\begin{align*}ar{\psi}\gamma_5\psi & \approx 0 \quad \text{(B.4)}
\end{align*}
\]

\[
\begin{align*}ar{\psi}\gamma^\mu\psi & \approx 2m \left[ a^\dagger a + bb^\dagger \right] \delta^{\mu0} \quad \text{(B.5)}
\end{align*}
\]

\[
\begin{align*}ar{\psi}\gamma^\mu\gamma^5\psi & \approx 2m \left[ a^\dagger a \left( \xi^\dagger \sigma^i \xi \right) + bb^\dagger \left( \eta^\dagger \sigma^i \eta \right) \right] \delta^{\mu i} \quad \text{(B.6)}
\end{align*}
\]

\[
\begin{align*}ar{\psi}\sigma^{\mu\nu}\psi & \approx 2m \left[ a^\dagger a \left( \xi^\dagger \sigma^k \xi \right) + bb^\dagger \left( \eta^\dagger \sigma^k \eta \right) \right] \delta^{\mu i} \delta^{\nu j} \epsilon^{ijk} \quad \text{(B.7)}
\end{align*}
\]

We note the following features.

1. The scalar and vector bilinears just yield the number operator. Therefore, when these operators are evaluated in the nuclear state, they add coherently, generating spin-independent interactions. The axial-vector and the tensor bilinears on the other hand yield the spin-operator, and hence couple to the net spin of the nucleus.

2. The vector interaction picks out the temporal component, while the axial-vector interaction picks out the spatial component. Therefore, 4-fermion operators mixing these will be velocity suppressed.

This analysis is schematic and does not account for the complications arising from strong nuclear dynamics in the case of quark bilinears. However, as we shall see later, incorporating these effects does not alter these conclusions. From this it follows that only the following terms survive in the non-relativistic regime:

\[
\begin{align*}
\mathcal{L} = a_q \bar{\chi} \gamma_5 \chi q + b_q \bar{\chi} \gamma_5 \chi q + d_q \bar{\chi} \gamma_5 \chi q + c_q \bar{\chi} \sigma^{\mu\nu} \chi \sigma_{\mu\nu} q .
\end{align*}
\] (B.8)

The first two terms lead to spin-independent scattering, while the remaining two lead to spin-dependent scattering. The particle \( \chi \) could be either a Dirac or a Majorana fermion. We specify the distinction where applicable.

Since the momentum transfers in WIMP-nucleus scattering are generally small compared to the characteristic nuclear scales, dark matter cross sections are expressed in terms of the cross section at zero momentum transfer, \( \sigma_0 \). This is itself a good approximation to the total cross section \( \sigma \) if the nucleus is small. For scattering off larger nuclei, the momentum-transfer dependence of the cross section can be parametrized into a form factor \( F(|\vec{q}|) \). Define,

\[
\begin{align*}
\sigma_0 = \int_0^{4\mu^2v^2} d|\vec{q}|^2 \frac{d\sigma(q = 0)}{d|\vec{q}|^2} .
\end{align*}
\] (B.9)
where $\mu$ is the reduced mass, and $v$ is the velocity. The actual cross section can be written in terms of $\sigma_0$,

$$
\sigma = \int d|q|^2 \frac{d\sigma}{d|q|^2} 
= \int d|q|^2 F^2(|q|) \frac{d\sigma(q = 0)}{d|q|^2} 
= \frac{\sigma_0}{4\mu^2 v^2} \int d|q|^2 F^2(|q|).
$$

(B.10)

(B.11)

(B.12)

It is conventional to work with $\sigma_0$ to calculate event rates, since the form factor integral is independent of the details of the particle physics model (it only depends on the mass of the dark matter particle). The form factor obviously satisfies $F(0) = 1$. Numerical or analytical estimates for form factors are available [49, 48].

**B.1 Scalar Interaction**

We begin with the Lagrangian,

$$
\mathcal{L} = a_q \bar{\chi} \chi \bar{q} q.
$$

(B.13)

Dirac and Majorana particles can both have scalar interactions. We assume a Dirac particle to start with. We work out the amplitude in detail in this case to make the definitions of the various matrix elements clear. The $S$-matrix element arising from this interaction has the form

$$
\mathcal{M}_{if} \delta^{(4)}(p_1 + p_2 - p_3 - p_4) = \sum_q \int d^4 x \ a_q \langle \chi_f \chi_i | \bar{q}(x) q(x) | N_f N_i \rangle.
$$

(B.14)

We can separate the co-ordinate dependence using the translation operator. Consider the matrix element,

$$
\langle N_f(p_4) | \bar{q}(x) q(x) | N_i(p_2) \rangle = \langle N_f(p_4) | e^{iP \cdot x} \bar{q}(0) e^{-iP \cdot x} q(0) e^{-iP \cdot x} | N_i(p_2) \rangle = \langle N_f(p_4) | \bar{q}(0) q(0) | N_i(p_2) \rangle e^{-i(p_2 - p_4) \cdot x}.
$$

(B.15)

In the limit of small momentum transfer this becomes

$$
\langle N_f(p_4) | \bar{q}(x) q(x) | N_i(p_2) \rangle \approx \langle N_f | \bar{q} q \ | N_i \rangle e^{-i(p_2 - p_4) \cdot x}.
$$

(B.16)

where $|N_i\rangle$ represents a state corresponding to a nucleus at rest. Therefore, in the low energy limit, we can write the amplitude in terms of the quark matrix elements in nuclear states at rest. These matrix elements here on are position independent low-energy quantities. The amplitude is then

$$
\mathcal{M}_{if} \delta^{(4)}(p_1 + p_2 - p_3 - p_4) = \sum_q \int d^4 x \ a_q \langle N_f | \bar{q} q | N_i \rangle \langle \chi_f | \bar{\chi} \chi | \chi_i \rangle e^{-i(p_1 + p_2 - p_3 - p_4) \cdot x}
= \sum_q a_q \langle N_f | \bar{q} q | N_i \rangle \langle \chi_f | \bar{\chi} \chi | \chi_i \rangle \delta^{(4)}(p_1 + p_2 - p_3 - p_4).
$$

(B.17)
Therefore,
\[ \mathcal{M}_{if} = \sum_q a_q \langle N_f | \bar{q}q | N_i \rangle \langle \chi_f | \bar{\chi} \chi | \chi_i \rangle. \]  \tag{B.18} 

In this expression we are employing the conventional relativistic normalization for the one particle states,
\[ \langle N(p) | N(q) \rangle = 2E_p \delta^{(3)}(p - q). \]  \tag{B.19} 

On the other hand, the nuclear physics matrix elements we seek to determine are generally expressed in terms of states normalized according to the non-relativistic convention,
\[ \langle \tilde{N}(p) | \tilde{N}(q) \rangle = \delta^{(3)}(p - q). \]  \tag{B.20} 

Therefore, in the non-relativistic convention,
\[ \mathcal{M}_{if} = 4m_\chi m_N \sum_q a_q \langle \tilde{N}_f | \bar{q}q | \tilde{N}_i \rangle. \]  \tag{B.21} 

In equation (B.21), we use the low-energy expression (equation (B.3)) for \( \chi \), which gives us a factor of \( 2m_\chi \). The other factor of \( 2m_N \) comes from the relative factor in relativistic normalization. In the case where \( \chi \) is Majorana, both the \( a^\dagger a \) and \( b^\dagger b \) terms would contribute, giving an additional factor of two over equation (B.21). The nuclear states \( |\tilde{N}\rangle \) are now normalized non-relativistically. We must now evaluate the quark operator matrix element in the nuclear state.

The matrix element of the light quarks \( (q = u, d, s) \) in the neutron and proton can be computed in chiral perturbation theory from measurements of pion-nucleon sigma term \([50, 51, 52]\).

\[ \langle n | m_q \bar{q}q | n \rangle = m_n f_{Tq}^{(n)}. \]  \tag{B.22} 

where \( n \) represents either the proton or the neutron.

The heavy quarks contribute to the mass of the nucleon through the triangle diagram \([53]\). Using the heavy quark expansion, it can be shown that the matrix element for heavy quarks is,
\[ \langle n | m_q \bar{q}q | n \rangle = \frac{2}{27} m_n \left( 1 - \sum_{q=u,d,s} f_{Tq}^{(n)} \right). \]  \tag{B.23} 

for \( q = c, b, t \). Thus, we can define effective coupling of the dark matter with protons as
\[ \frac{f_p}{m_p} = \sum_{q=u,d,s} a_q \frac{f_{Tq}^{(p)}}{m_q} + \frac{2}{27} \left( 1 - \sum_{q=u,d,s} f_{Tq}^{(p)} \right) \sum_{q=c,b,t} \frac{a_q}{m_q}. \]  \tag{B.24}
An analogous expression holds for the coupling to neutrons. We use $f^{(p)}_{Tu} = 0.020 \pm 0.004$, $f^{(p)}_{Td} = 0.026 \pm 0.005$, $f^{(n)}_{Tu} = 0.014 \pm 0.003$, $f^{(n)}_{Td} = 0.036 \pm 0.008$, and $f^{(p,n)}_{Ts} = 0.118 \pm 0.062$ [37].

The scalar interaction couples left and right-handed quarks. Therefore, we expect the coupling $a_q$ to be proportional to the mass of the quarks (unless there are additional sources of chiral symmetry breaking in the theory). Therefore, the ratio $a_q/m_q$ is generally not large even for small quark masses.

Performing the sum over the entire nucleus gives us the following expression for the S-matrix element,

$$\mathcal{M}_{if} = 4m_\chi m_N [Zf_p + (A - Z)f_n].$$

(B.25)

This leads to the following expression for the cross section of a Dirac dark matter particle at zero-momentum transfer,

$$\sigma_0 = \frac{\mu^2}{\pi} [Zf_p + (A - Z)f_n]^2.$$ 

(B.26)

where $\mu$ is the reduced mass of the WIMP-nucleus system.

The only difference for a Majorana particle in the calculation leading up to here is the factor of two noted earlier. Therefore, the corresponding cross section for a Majorana particle is simply

$$\sigma_0 = \frac{4\mu^2}{\pi} [Zf_p + (A - Z)f_n]^2.$$ 

(B.27)

### B.2 Vector Interaction

The vector interaction also contributes to the spin-independent coupling. The Majorana fermion does not couple to the vector current, so the following discussion applies only to a Dirac fermion.

$$\mathcal{L} = b_q \bar{\chi} \gamma^\mu \chi \bar{q} \gamma_\mu q.$$ 

(B.28)

The calculation proceeds as in the previous case. Writing the matrix element in the low-energy limit,

$$\mathcal{M}_{if} = \sum_q b_q \langle \chi_f | \bar{\chi} \gamma^\mu \chi | \chi_i \rangle \langle N_f | \bar{q} \gamma_\mu q | N_i \rangle$$ 

(B.29)

$$= 4m_\chi m_N \sum_q b_q \langle \bar{N}_f | \bar{q} \gamma_\mu q | \bar{N}_i \rangle.$$ 

(B.30)

The sea-quarks and the gluons do not contribute to the vector current. The valence quark contributions all add up due to the conservation of the vector current. Therefore, the coupling to protons and neutrons is now simply given as,

$$b_p = 2b_u + b_d$$ 

(B.31)

$$b_n = b_u + 2b_d.$$ 

(B.32)
When the sum over the entire nucleus is performed, we get a form very similar to the scalar case considered earlier,

$$\mathcal{M}_{if} = 4m_\chi m_N [Z b_p + (A - Z) b_n].$$  \hspace{1cm} (B.33)

This leads to the cross section

$$\sigma_0 = \frac{\mu^2}{\pi} [Z b_p + (A - Z) b_n]^2.$$ \hspace{1cm} (B.34)

**B.3 Axial-Vector Interaction**

The axial-vector interaction will be seen to be spin-dependent. The Lagrangian is

$$\mathcal{L} = d_q \bar{\chi} \gamma^\mu \gamma^5 \chi q \gamma^5.$$ \hspace{1cm} (B.35)

This leads to the following matrix element in the limit of zero momentum transfer

$$\mathcal{M}_{if} = \sum_{q=u,d,s} d_q \langle \chi_f | \bar{\chi} \gamma^\mu \gamma^5 \chi | \chi_i \rangle \langle N_f | \bar{q} \gamma^5 q | N_i \rangle.$$ \hspace{1cm} (B.36)

The sum is only over the light quarks because heavy quarks do not contribute significantly to the spins of neutrons or protons. As before we go to the non-relativistic normalization,

$$\mathcal{M}_{if} = 4m_\chi m_N \sum_{q=u,d,s} d_q \langle \chi_f | \bar{\chi} \gamma^\mu \gamma^5 \chi | \chi_i \rangle \langle N_f | \bar{q} \gamma^5 q | N_i \rangle.$$ \hspace{1cm} (B.37)

Recalling the expressions for the spinors in the non-relativistic limit,

$$\langle \chi_f | \bar{\chi} \gamma^\mu \gamma^5 \chi | \chi_i \rangle = 2 \langle \chi_f | (S_\chi)_i | \chi_i \rangle \delta^\mu_i.$$ \hspace{1cm} (B.38)

We can write the quark spin operator in terms of the spin expectation values of the proton and the neutron in the nucleus,

$$\langle \bar{N}_f | \bar{q} \gamma^5 q | \bar{N}_i \rangle = 2 \delta^\mu_i \left( \langle \bar{N}_f | (S_p)_i | \bar{N}_i \rangle + \langle \bar{N}_f | (S_n)_i | \bar{N}_i \rangle \right).$$ \hspace{1cm} (B.39)

where $\Delta^p_n$ is the part of the spin of the nucleon $n$ carried by quark $q$. The nuclear state is specified by the angular momentum quantum numbers $(J_N, J_{Nz})$. Using the Wigner-Eckart theorem, we can write the combination of the spin-operators above in terms of the nuclear spin operator \([31]\). \[S_p\] + \[S_n\] = \lambda_q \langle \bar{N}_f | \bar{J}_N | \bar{N}_i \rangle. \hspace{1cm} (B.40)

Conventionally, the angular momentum matrix elements are reported in the $z$-projection in the highest $M_J$ state \([54]\).

$$\langle S \rangle \equiv \langle J, M_J = J | S_z | J, M_J = J \rangle.$$ \hspace{1cm} (B.41)
Then, the constant of proportionality is given by

\[ \lambda_q = \frac{\langle S_p \rangle}{J_N} \Delta_q^p + \frac{\langle S_n \rangle}{J_N} \Delta_q^n, \] (B.42)

where \( \langle S_{p,n} \rangle/J_N \) is the fraction of the nuclear spin carried by protons or neutrons. For example, we can estimate \( \lambda_q \) in single-particle shell model of nuclei [55, 56]. Here the spin of the nucleus is due to the unpaired nucleon \( n \). Then

\[ \lambda_q = \Delta_q^p \langle \tilde{N}_f | \vec{S}_n \cdot \vec{J}_N | \tilde{N}_i \rangle \cdot \langle \tilde{N}_f | \vec{S}_X \cdot \vec{J}_X | \tilde{N}_i \rangle. \] (B.43)

However, the value of \( \lambda_q \) is different in cases when the shell-model fails, and must then be estimated numerically.

We can now write the scattering amplitude in terms of nuclear spin,

\[ \mathcal{M}_{if} = 16 m_e m_N \sum_{q=u,d,s} d_q \lambda_q \langle \tilde{N}_f | \vec{S}_n \cdot \vec{J}_N | \tilde{N}_i \rangle \cdot \langle \tilde{N}_f | \vec{S}_X \cdot \vec{J}_X | \tilde{N}_i \rangle. \] (B.44)

Squaring and summing/averaging over final and initial states,

\[ \langle |\mathcal{M}|^2 \rangle = \frac{1}{2(2J_N+1)} \sum_{\chi_i,\chi_f,N_i,N_f} |\mathcal{M}_{if}|^2 \]
\[ = \frac{256 m_e^2 m_N^2}{2(2J_N+1)} \left[ \sum_{q=u,d,s} d_q \lambda_q \right] \left[ \sum_{q=u,d,s} \lambda_q \right] \frac{1}{2} \sum_{N_f} \langle \tilde{N}_f | J_N^2 | \tilde{N}_f \rangle \]
\[ = 64 m_e^2 m_N^2 \left[ \sum_{q=u,d,s} d_q \lambda_q \right] \left[ \sum_{q=u,d,s} \lambda_q \right] J_N(J_N+1). \] (B.45)

Hence the cross section in the NR limit is given as

\[ \sigma_0 = \frac{4 \mu^2}{\pi} \left[ \sum_{q=u,d,s} d_q \lambda_q \right] \left[ \sum_{q=u,d,s} \lambda_q \right] J_N(J_N+1). \] (B.46)

We can repeat this exercise for the Majorana fermion, with the only difference again being a factor of two in the low energy expression for the bilinear. Therefore, the expression for cross section for the Majorana particle is just 4 times the cross section for a Dirac particle.

\[ \sigma_0 = \frac{16 \mu^2}{\pi} \left[ \sum_{q=u,d,s} d_q \lambda_q \right] \left[ \sum_{q=u,d,s} \lambda_q \right] J_N(J_N+1). \] (B.47)

Values of \( d_q \) are obtained from theory while the value \( \lambda_q \) depends on the nucleus. For scattering off free protons (neutrons), \( \lambda_q \) reduces to \( \Delta_q^p (\Delta_q^n) \).
B.4 Tensor Interaction

We now consider the tensor interaction

\[ \mathcal{L} = \sum_q b_q \bar{\chi} \sigma^{\mu \nu} \chi \bar{q} \sigma_{\mu \nu} q. \]  

(B.48)

In the non-relativistic limit, this interaction will also yield spin-dependent interactions. Again, the current in the case of a Majorana fermion vanishes, so the following applies to a Dirac fermion only. We note that in the non-relativistic limit the bilinears that arise are very similar to the axial-vector case. Therefore, we can adapt that calculation to this after accounting for some extra factors. Specifically

\[ M_{\text{tensor}} = \sum_{q=u,d,s} b_q \langle \chi_f, N_f | \bar{\chi} \chi_{\mu \nu} \bar{q} \sigma_{\mu \nu} q | \chi_i, N_i \rangle \]  

(B.49)

\[ = 2 M_{\text{axial-vector}}. \]  

(B.50)

Since everything else including the kinematic factors are the same, this simply translates into a factor of 4 in the cross section. Thus,

\[ \sigma_0 = \frac{16 \mu^2}{\pi} \left( \sum_{q=u,d,s} b_q \lambda_q \right)^2 J_N (J_N + 1). \]  

(B.51)

C. Vector dark matter scattering cross section

We saw that in the case of real vector boson dark matter the effective interactions that survive in the non-relativistic limit have either the scalar or the axial-vector form. Using the analysis above, we can easily calculate the cross sections in these cases.

C.1 Scalar Interaction

The scalar term in the Lagrangian is written as

\[ \mathcal{L} = a_q m_B B_\mu B^\mu \bar{q} q. \]  

(C.1)

The additional factor of mass in the definition is simply to maintain the analogy with the four-fermion interaction. Then, in the limit of zero momentum transfer

\[ M_{ij} = 4m_N m_B [Z f_p + (A - Z) f_n] \epsilon^{\mu *}(p_3) \epsilon_\mu (p_1). \]  

(C.2)

where \( p_3 \approx p_1 \approx 0 \). The quantities \( f_p \) and \( f_n \) are defined exactly as in the scalar case.

\[ \frac{f_p}{m_p} = \sum_{q=u,d,s} \frac{f_T(p)}{m_q} + \frac{2}{27} \left( 1 - \sum_{q=u,d,s} f_T(p) \right) \sum_{q=c,b,t} \frac{a_q}{m_q}. \]  

(C.3)

This leads to the cross section,

\[ \sigma_0 = \frac{\mu^2}{\pi} [Z f_p + (A - Z)f_n]^2, \]  

(C.4)

where \( \mu \) is the reduced mass.
C.2 Axial-Vector Interaction

The form of the Lagrangian is

\[ \mathcal{L} = b_q (\partial_\sigma B^\mu) B^\nu \bar{q} \gamma_\alpha \gamma^5 q e^{\sigma\mu\nu}. \] (C.5)

This leads to the matrix element

\[ \mathcal{M}_{if} = 4 b_q m_N m_B \epsilon_{\mu\nu\alpha} \epsilon^*_{\mu}(p_3) \epsilon_{\nu}(p_1) \langle \bar{N}_f | \bar{q} \gamma_\alpha \gamma^5 q | N_i \rangle, \] (C.6)

where as before \( \vec{p}_3 \approx \vec{p}_1 \approx 0 \).

\[ \langle |\mathcal{M}|^2 \rangle = \frac{128}{3(2J_N + 1)} m_N^2 m_B^2 \left[ \sum_{q=u,d,s} b_q \lambda_q \right]^2 \sum_{N_f} \langle \bar{N}_f | J_N^2 | N_f \rangle. \] (C.7)

The corresponding cross section is then

\[ \sigma_0 = \frac{8 \mu^2}{3\pi} \left[ \sum_{q=u,d,s} b_q \lambda_q \right]^2 J_N(J_N + 1), \] (C.8)

where \( \mu \) is again the reduced mass of the WIMP-nucleus system.

References

[1] J. R. Ellis, J. S. Hagelin, D. V. Nanopoulos, K. A. Olive, and M. Srednicki, Supersymmetric relics from the big bang, Nucl. Phys. B238 (1984) 453–476.

[2] K. Griest, Cross sections, relic abundance, and detection rates for neutralino dark matter, Phys. Rev. D38 (1988) 2357.

[3] E. W. Kolb and R. Slansky, Dimensional Reduction in the Early Universe: Where Have the Massive Particles Gone?, Phys. Lett. B135 (1984) 378.

[4] G. Servant and T. M. P. Tait, Is the lightest Kaluza-Klein particle a viable dark matter candidate?, Nucl. Phys. B650 (2003) 391–419, [hep-ph/0206071].

[5] A. Birkedal-Hansen and J. G. Wacker, Scalar dark matter from theory space, Phys. Rev. D69 (2004) 065022, [hep-ph/0306161].

[6] J. Hubisz and P. Meade, Phenomenology of the littlest Higgs with T-parity, Phys. Rev. D71 (2005) 035016, [hep-ph/0411264].

[7] E. M. Dolle and S. Su, Dark Matter in the Left Right Twin Higgs Model, Phys. Rev. D77 (2008) 075013, [arXiv:0712.1234].

[8] The CDMS-II Collaboration, Z. Ahmed et. al., Results from the Final Exposure of the CDMS II Experiment, arXiv:0912.3592.

[9] E. Aprile, The XENON100 dark matter experiment, AIP Conf. Proc. 1115 (2009) 355–360.
[10] J. Angle et al., *Limits on spin-dependent WIMP-nucleon cross-sections from the XENON10 experiment*, Phys. Rev. Lett. 101 (2008) 091301, [arXiv:0805.2939].

[11] KIMS Collaboration, S. K. Kim, *New results from the KIMS experiment*, J. Phys. Conf. Ser. 120 (2008) 042021.

[12] G. Bertone, D. G. Cerdeno, J. I. Collar, and B. C. Odom, *WIMP identification through a combined measurement of axial and scalar couplings*, Phys. Rev. Lett. 99 (2007) 151301, [arXiv:0705.2502].

[13] V. Barger, W.-Y. Keung, and G. Shaughnessy, *Spin Dependence of Dark Matter Scattering*, Phys. Rev. D78 (2008) 056007, [arXiv:0806.1962].

[14] G. Belanger, E. Nezri, and A. Pukhov, *Discriminating dark matter candidates using direct detection*, Phys. Rev. D79 (2009) 015008, [arXiv:0810.1362].

[15] T. Cohen, D. J. Phalen, and A. Pierce, *On the Correlation Between the Spin-Independent and Spin-Dependent Direct Detection of Dark Matter*, arXiv:1001.3408.

[16] G. D. Starkman, A. Gould, R. Esmailzadeh, and S. Dimopoulos, *Opening the window on strongly interacting dark matter*, Phys. Rev. D41 (1990) 3594.

[17] D. Tucker-Smith and N. Weiner, *Inelastic dark matter*, Phys. Rev. D64 (2001) 043502, [hep-ph/0111138].

[18] D. Tucker-Smith and N. Weiner, *The status of inelastic dark matter*, Phys. Rev. D72 (2005) 063509, [hep-ph/0402065].

[19] Y. Cui, D. E. Morrissey, D. Poland, and L. Randall, *Candidates for Inelastic Dark Matter*, JHEP 05 (2009) 076, [arXiv:0901.0557].

[20] B. Feldstein, A. L. Fitzpatrick, and E. Katz, *Form Factor Dark Matter*, arXiv:0908.2991.

[21] S. Chang, A. Pierce, and N. Weiner, *Momentum Dependent Dark Matter Scattering*, arXiv:0908.3192.

[22] Y. Bai and P. J. Fox, *Resonant Dark Matter*, JHEP 11 (2009) 052, [arXiv:0909.2900].

[23] M. Beltran, D. Hooper, E. W. Kolb, and Z. C. Krusberg, *Deducing the nature of dark matter from direct and indirect detection experiments in the absence of collider signatures of new physics*, Phys. Rev. D80 (2009) 043509, [arXiv:0808.3384].

[24] J. McDonald, *Gauge Singlet Scalars as Cold Dark Matter*, Phys. Rev. D50 (1994) 3637–3649, [hep-ph/0702143].

[25] C. P. Burgess, M. Pospelov, and T. ter Veldhuis, *The minimal model of nonbaryonic dark matter: A singlet scalar*, Nucl. Phys. B619 (2001) 709–728, [hep-ph/0011335].

[26] R. Barbieri and L. J. Hall, *Improved naturalness and the two Higgs doublet model*, hep-ph/0510243.

[27] L. Lopez Honorez, E. Nezri, J. F. Oliver, and M. H. G. Tytgat, *The inert doublet model: An archetype for dark matter*, JCAP 0702 (2007) 028, [hep-ph/0612275].

[28] D. Majumdar and A. Ghosal, *Dark Matter candidate in a Heavy Higgs Model - Direct Detection Rates*, Mod. Phys. Lett. A23 (2008) 2011–2022, [hep-ph/0607067].
[29] K. Agashe and G. Servant, *Baryon number in warped GUTs: Model building and (dark matter related) phenomenology*, *JCAP* **0502** (2005) 002, [hep-ph/0411254].

[30] D. E. Brahm and L. J. Hall, *U(1)-prime Dark Matter*, *Phys. Rev.* **D41** (1990) 1067.

[31] H.-C. Cheng, J. L. Feng, and K. T. Matchev, *Kaluza-Klein dark matter*, *Phys. Rev. Lett.* **89** (2002) 211301, [hep-ph/0207125].

[32] G. Servant and T. M. P. Tait, *Elastic scattering and direct detection of Kaluza-Klein dark matter*, *New J. Phys.* **4** (2002) 99, [hep-ph/0209262].

[33] A. Birkedal, A. Noble, M. Perelstein, and A. Spray, *Little Higgs dark matter*, *Phys. Rev.* **D74** (2006) 035002, [hep-ph/0603077].

[34] T. Appelquist, B. A. Dobrescu, and A. R. Hopper, *Nonexotic neutral gauge bosons*, *Phys. Rev.* **D68** (2003) 035012, [hep-ph/0212073].

[35] Particle Data Group Collaboration, C. Amsler et. al., *Review of particle physics*, *Phys. Lett.* **B667** (2008) 1.

[36] D. R. Tovey, R. J. Gaitskell, P. Gondolo, Y. Ramachers, and L. Roszkowski, *A new model-independent method for extracting spin-dependent cross section limits from dark matter searches*, *Phys. Lett.* **B488** (2000) 17–26, [hep-ph/0005041].

[37] J. R. Ellis, A. Ferstl, and K. A. Olive, *Re-evaluation of the elastic scattering of supersymmetric dark matter*, *Phys. Lett.* **B481** (2000) 304–314, [hep-ph/0001005].

[38] G. K. Mallot, *The spin structure of the nucleon*, hep-ex/9912040.

[39] J. Alwall, M.-P. Le, M. Lisanti, and J. G. Wacker, *Model-Independent Jets plus Missing Energy Searches*, *Phys. Rev.* **D79** (2009) 015005, [arXiv:0809.3264].

[40] E. Izaguirre, M. Manhart, and J. G. Wacker, *Bigger, Better, Faster, More at the LHC*, arXiv:1003.3886.

[41] Q.-H. Cao, C.-R. Chen, C. S. Li, and H. Zhang, *Effective Dark Matter Model: Relic density, CDMS II, Fermi LAT and LHC*, arXiv:0912.4511.

[42] M. Beltran, D. Hooper, E. W. Kolb, Z. A. C. Krusberg, and T. M. P. Tait, *Maverick dark matter at colliders*, arXiv:1002.4137.

[43] IceCube Collaboration, R. Abbasi et. al., *Limits on a muon flux from neutralino annihilations in the Sun with the IceCube 22-string detector*, *Phys. Rev. Lett.* **102** (2009) 201302, [arXiv:0902.2460].

[44] IceCube Collaboration, R. Abbasi, *Limits on a muon flux from Kaluza-Klein dark matter annihilations in the Sun from the IceCube 22-string detector*, arXiv:0910.4480.

[45] G. Wikstrom and J. Edsjo, *Limits on the WIMP-nucleon scattering cross-section from neutrino telescopes*, *JCAP* **0904** (2009) 009, [arXiv:0903.2986].

[46] P. Agrawal, Z. Chacko, C. Kilic, and R. K. Mishra, *Direct Detection Constraints on Dark Matter Event Rates in Neutrino Telescopes, and Collider Implications*, arXiv:1003.5905.

[47] M. E. Peskin and D. V. Schroeder, *An Introduction to quantum field theory*. Reading, USA: Addison-Wesley, 1995.
[48] G. Jungman, M. Kamionkowski, and K. Griest, *Supersymmetric dark matter*, *Phys. Rept.* **267** (1996) 195–373, [hep-ph/9506380].

[49] A. Gould, *Cosmological density of WIMPs from solar and terrestrial annihilations*, *Astrophys. J.* **388** (1992) 338–344.

[50] T. P. Cheng, *Chiral symmetry and the Higgs nucleon coupling*, *Phys. Rev.* **D38** (1988) 2869.

[51] H.-Y. Cheng, *Low-energy interactions of scalar and pseudoscalar higgs bosons with baryons*, *Physics Letters B* **219** (1989), no. 2-3 347 – 353.

[52] J. Gasser, H. Leutwyler, and M. Sainio, *Sigma-term update*, *Physics Letters B* **253** (1991), no. 1-2 252 – 259.

[53] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *Remarks on Higgs Boson Interactions with Nucleons*, *Phys. Lett.* **B78** (1978) 443.

[54] M. T. Ressell *et. al.*, *Nuclear shell model calculations of neutralino - nucleus cross-sections for Si-29 and Ge-73*, *Phys. Rev.* **D48** (1993) 5519–5535.

[55] M. W. Goodman and E. Witten, *Detectability of certain dark-matter candidates*, *Phys. Rev.* **D31** (1985) 3059.

[56] J. Engel, *Nuclear form-factors for the scattering of weakly interacting massive particles*, *Phys. Lett.* **B264** (1991) 114–119.