ON CLOSED SUBALGEBRAS OF $C_B(X)$

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Abstract. For a completely regular space $X$ and a non-vanishing self-adjoint closed subalgebra $H$ of $C_B(X)$ which separates points from closed sets in $X$ we construct the Gelfand spectrum $\text{sp}(H)$ of $H$ as an open subspace of the compactification of $X$ generated by $H$. The simple construction of $\text{sp}(H)$ enables easier examination of its properties. We illustrate this by an example showing that the space $\text{sp}(H)$ is separable metrizable if and only if $H$ is countably generated.

1. Introduction

By a space we mean a topological space. We follow the definitions of [6], in particular, completely regular spaces and compact spaces are Hausdorff (consequently, locally compact spaces are completely regular), and Lindelöf spaces are regular. The field of scalars is the complex field $\mathbb{C}$, however, all results remain true (with exactly the same proof) in the real setting.

Let $X$ be a space. We denote by $C_B(X)$ the set of all continuous bounded scalar-valued mappings on $X$. The set $C_B(X)$ is a Banach algebra with pointwise addition and multiplication and the supremum norm. We denote by $C_0(X)$ the set of all $f$ in $C_B(X)$ which vanish at infinity (i.e., $|f|^{-1}(\epsilon, \infty)$ is compact for all $\epsilon > 0$). For any $f$ in $C_B(X)$, the cozero-set of $f$ is defined to be $X \setminus f^{-1}(0)$ and is denoted by $\text{Coz}(f)$.

In [7], assuming that $X$ is a completely regular space, we have represented a non-vanishing self-adjoint closed subalgebra $H$ of $C_B(X)$ which has local units as $C_0(Y)$ for some locally compact space $Y$. (Here, by $H$ having local units we mean that for every closed subspace $C$ of $X$ and every neighborhood $U$ of $C$ in $X$ contained in the support of some element of $H$ there is some $h$ in $H$ which equals 1 on $C$ and vanishes outside $U$.) Our purpose here is to improve our previous result by replacing the rather “ad hoc” assumption that $H$ has local units by the more standard assumption that $H$ separates points from closed sets in $X$. This will be accomplished by considering the compactification $\alpha_X H$ of $X$ generated by $H$ instead of the Stone–Čech compactification of $X$. Our approach here, apart from standardizing our assumption on $H$, enables us to derive certain properties of the space $Y$ (which coincide with the Gelfand spectrum of $H$). We illustrate this by an example in which we show that the space $Y$ is separable metrizable if and only if $H$ is countably generated. We conclude with the description of examples of completely regular spaces $X$ and non-vanishing self-adjoint closed subalgebras $H$.

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of \(C_B(X)\) which separate points from closed sets in \(X\), so that our conclusion may apply.

For other results of the same nature dealing with non-vanishing closed ideals of \(C_B(X)\) (where \(X\) is a completely regular space) we refer to [8–11].

2. The representation theorem

We begin with the following definition.

**Definition 2.1.** Let \(X\) be a space. A subset \(H\) of \(C_B(X)\)
- is self-adjoint if \(H\) contains the complex conjugate \(\overline{h}\) of any element \(h\) in \(H\) (where \(\overline{h}(x) = \overline{h(x)}\) for any \(x\) in \(X\)).
- is non-vanishing if for any \(x\) in \(X\) there is some \(h\) in \(H\) such that \(h(x) \neq 0\).
- separates points in \(X\) if for any distinct elements \(x\) and \(y\) in \(X\) there is some \(h\) in \(H\) such that \(h(x) \neq h(y)\).
- separate points from closed sets in \(X\) if for any closed subspace \(C\) of \(X\) and any \(x\) in \(X\setminus C\) there is some \(h\) in \(H\) such that \(h(x)\) is not in \(\text{cl}\ C\).

We need the following well known theorem. We include the proof here for notational convenience and completeness.

**Notation and Lemma 2.2.** Let \(X\) be a completely regular space and let \(H\) be a subset of \(C_B(X)\) which separates points from closed sets in \(X\). For any \(h\) in \(H\), let \(I_h = \text{cl}_{\overline{C}_h} C_h(X)\). Let

\[
e : X \longrightarrow \prod_{h \in H} I_h
\]

be the evaluation map where

\[
x \longmapsto \{h(x)\}_{h \in H}
\]

for any \(x\) in \(X\). Then, \(e\) is a homeomorphic embedding. Let

\[
\alpha_H X = \text{cl}_{\prod_{h \in H} I_h} e(X)
\]

and identify \(X\) with its image \(e(X)\). Then \(\alpha_H X\) is a compactification of \(X\) on which every element of \(H\) extends continuously. For any \(h\) in \(H\), we denote by \(h_\alpha\) the (unique) continuous extension of \(h\) on \(\alpha_H X\).

**Proof.** It is known that \(e\) is injective if \(H\) separates points in \(X\) and \(e\) is a homeomorphic embedding if \(H\) further separates points from closed sets in \(X\). (See Theorem 2.3.20 of [3].) By our assumption \(H\) separates points from closed sets in \(X\) which also implies that \(H\) separates points in \(X\) (as all one point sets are closed in \(X\)). Note that \(\prod_{h \in H} I_h\) is compact (as is a product of compact spaces) and therefore, its closed subspace \(\alpha_H X\) is also compact. It is clear that \(\alpha_H X\) contains \(e(X)\) (= \(X\)) as a dense subspace. Therefore \(\alpha_H X\) is a compactification of \(X\). Note that \(\pi_h(e(x)) = h(x)\) for every \(x\) in \(X\), where \(h\) is in \(H\) and \(\pi_h : \alpha_H X \rightarrow \mathbb{C}\) is the projection onto the \(h\)-th coordinate. Since \(X\) is identified with \(e(X)\), \(\pi_h\) continuously extends \(h\). Note that continuous extensions of \(h\) coincide, as are identical on the dense subspace \(e(X)\) (= \(X\)) of \(\alpha_H X\). ■

In Theorem 2.9 of [7], assuming that \(X\) is a completely regular space, we have represented a non-vanishing self-adjoint closed subalgebra \(H\) of \(C_B(X)\) which has local units as \(C_0(Y)\) for some locally compact space \(Y\) (where as pointed previously,
by $H$ having local units it is meant that for every closed subspace $C$ of $X$ and every neighborhood $U$ of $C$ in $X$ contained in the support of some element of $H$ there is some $h$ in $H$ which equals 1 on $C$ and vanishes outside $U$.) The space $Y$ has been constructed as an open subspace of the Stone–Čech compactification of $X$. The requirement on $H$ to have local units is rather “ad hoc.” In the next theorem, we improve Theorem 2.9 of [7] by replacing the assumption that $H$ has local units by the more standard assumption that $H$ separates points from closed sets in $X$. This will be done by considering the compactification $\alpha_H X$ of $X$ generated by $H$ instead of the Stone–Čech compactification of $X$.

In the proof of the next theorem we use the following corollary of the Stone–Weierstrass theorem. (See Chapter V of [5], Theorem 8.1 and Corollary 8.3.)

**Lemma 2.3.** Let $Y$ be a locally compact space and let $G$ be a non-vanishing self-adjoint closed subalgebra of $C_0(Y)$ which separates points in $Y$. Then $G = C_0(Y)$.

Also, we need to use the following version of the Banach–Stone theorem stating that the topology of a locally compact space $Y$ determines and is determined by the normed algebraic properties of $C_0(Y)$. (See Theorem 7.1 of [2]. It turns out the even algebraic properties of $C_0(Y)$ suffice to determine the topology of $Y$; see [1].)

**Lemma 2.4.** For locally compact spaces $Y$ and $Z$, the normed algebras $C_0(Y)$ and $C_0(Z)$ are isometrically isomorphic if and only if the spaces $Y$ and $Z$ are homeomorphic.

We are now at a place to prove our representation theorem.

**Theorem 2.5.** Let $X$ be a completely regular space and let $H$ be a non-vanishing self-adjoint closed subalgebra of $C_B(X)$ which separates points from closed sets in $X$. Then $H$ is isometrically isomorphic to $C_0(Y)$ for some locally compact space

$$Y = \bigcup_{\alpha \in H} \text{Coz}(h_\alpha).$$

The space $Y$ is unique (up to homeomorphism) with this property. Moreover, the following are equivalent:

1. $H$ is unital.
2. $H$ has 1.
3. $Y$ is compact.
4. $Y = \alpha_H X$.

**Proof.** Let

$$\lambda_H X = \bigcup_{h \in H} \text{Coz}(h_\alpha).$$

Note that $\lambda_H X$ contains $X$, as for any $x$ in $X$, since $H$ is non-vanishing, there is some $h$ in $H$ such that $h(x) \neq 0$, but $h_\alpha(x) = h(x)$. For any $h$ in $H$ we let

$$h_\lambda = h_\alpha|_{\lambda_H X} : \lambda_H X \to \mathbb{C}.$$ 

Note that $h_\lambda$ extends $h$, as $h_\alpha$ does and $\lambda_H X$ contains $X$. We define the mapping

$$\phi : H \to C_0(\lambda_H X)$$

by $\phi(h) = h_\lambda$ for any $h$ in $H$, and we check that $\phi$ is an isometric isomorphism.
Let $h$ be in $H$. For any $\epsilon > 0$ we have
\[
|h_\lambda|^{-1}(\epsilon, \infty) = \lambda X \cap |h_\alpha|^{-1}(\epsilon, \infty).
\]
But the latter is $|h_\alpha|^{-1}(\epsilon, \infty)$, and is therefore, being closed in $\alpha X$, is compact. That is, $\phi(h)$ is in $C_0(\lambda X)$. That $\phi$ is a homomorphism and is injective follows from the observation that any two continuous mappings on $\lambda X$ which agree on the (dense) subspace $X$ of $\lambda X$ agree on the whole $\lambda X$. (As an example, $(f + g)_\lambda$ and $f_\lambda + g_\lambda$ coincide, as they are both identical to $f + g$ on $X$.) Note that for an $h$ in $H$, by continuity of $h$, we have
\[
|h_\lambda|(\lambda X) = |h_\lambda|(\text{cl}_\lambda X) = \text{cl}_C|h_\lambda|(X) = \text{cl}_C|h|(X) \subseteq [0, ||h||].
\]
Thus $||h_\lambda|| \leq ||h||$. That $||h|| \leq ||h_\lambda||$ is clear, as $h_\lambda$ extends $h$. Therefore $||h_\lambda|| = ||h||$. This shows that $\phi$ preserves norm. To conclude the proof we need to show that $\phi$ is surjective. By Lemma 2.3 we suffice to check that $\phi(H)$ is a non-vanishing self-adjoint closed subalgebra of $C_0(\lambda X)$ which separates points in $\lambda X$.

As it is observed in the above lines $\phi(H)$ is a subalgebra of $C_0(\lambda X)$, which is also closed in $C_0(\lambda X)$, as is an isometrically isomorphic image of a complete normed space. The fact that $\phi(H)$ is non-vanishing on $\lambda X$ follows from the definition of $\lambda X$. (Indeed, for any $z$ in $\lambda X$ we have $h_\alpha(z) \neq 0$, and in particular $h_\lambda(z) \neq 0$, for some $h$ in $H$.) It is also clear that $\phi(H)$ is self-adjoint, as $(h_\lambda) = (\overline{h})_\lambda$ for any $h$ in $H$ (since the two mapping coincide with $\overline{h}$ on the dense subspace $X$ of $\lambda X$). We check that $\phi(H)$ separates points in $\lambda X$. We suffice to check that $\{h_\alpha : h \in H\}$ separates points in $\alpha X$. But this follows, as if $s = \{s_h\}_{h \in H}$ and $t = \{t_h\}_{h \in H}$ are distinct elements of $\alpha X$, then $s_h \neq t_h$ for some $h$ in $H$, and then by Lemma 2.2 (and its proof), we have
\[
h_\alpha(s) = \pi_h(s) = s_h \neq t_h = \pi_h(t) = h_\alpha(t).
\]
Note that $\lambda X$ is locally compact, as (by its definition) is open in the compact space $\alpha X$. Lemma 2.3 now implies that
\[
\phi(H) = C_0(\lambda X).
\]
This shows that $H$ is isometrically isomorphic to $C_0(\lambda X)$. The uniqueness assertion of the theorem follows from Lemma 2.4.

Finally, we check that (1)–(4) are equivalent. To see that (1) implies (2), let $u$ be a unit in $H$. For any $x$ in $X$ let $h_x$ be an element of $H$ which does not vanish at $x$. Then $u(x)h_x(x) = h_x(x)$ which implies that $u(x) = 1$. That (2) implies (4) is clear and follows from the definition of $\lambda X$. That (2) implies (1), and (4) implies (3) are also clear. Finally, (3) implies (1), as if $Y$ is compact then $C_0(Y)$ coincides with $C_B(Y)$ and is therefore unital (as $C_B(Y)$ is). But then $H$ is unital, as is isometrically isomorphic to $C_0(Y)$ by the first part.

**Remark 2.6.** In Theorem 2.5 for a completely regular space $X$ and a non-vanishing self-adjoint closed subalgebra $H$ of $C_B(X)$ which separates points from closed sets in $X$ we have proved that $H$ and $C_0(\lambda X)$ are isometrically isomorphic. On the other hand (in the case when the field of scalars is $\mathbb{C}$) by the commutative Gelfand–Naimark theorem, $H$, being a Banach algebra, is isometrically isomorphic to $C_0(Y)$ with $Y$ being the spectrum (or the maximal ideal space) of $H$ with the Gelfand (or Zariski) topology. The uniqueness part of Theorem 2.5 implies that $\lambda X$ and the spectrum of $H$ coincide. Indeed, our result here has the advantage
that it explicitly constructs the spectrum of $H$ as a subspace of a compactification of $X$ with a known structure. As we will see next, this may provide some information which is not generally expected to be derivable from the standard Gelfand theory. For this purpose, Theorem 2.9 may be considered as a particular example, in which we prove that the spectrum of $H$ is separable metrizable if $H$ is countably generated.

The above remark motivates to introduce the following notation.

**Definition 2.7.** Let $X$ be a completely regular space and let $H$ be a non-vanishing self-adjoint closed subalgebra of $C_B(X)$ which separates points from closed sets in $X$. We denote by $\text{sp}(H)$ the unique locally compact space $Y$ such that $H$ and $C_0(Y)$ are isometrically isomorphic.

Our next theorem should be known to various authors; we deduce it here, however, as an easy corollary of our representation theorem. We need to use the following lemma.

**Lemma 2.8.** Let $X$ be a completely regular space and let $H$ be a subset of $C_B(X)$ which separates points from closed sets in $X$. Then

$$\alpha_{\overline{H}} X = \alpha_H X.$$ 

Here the bar denotes the closure in $C_B(X)$.

**Proof.** Note that $\overline{H}$ separates points from closed sets in $X$, as $H$ does. By the construction in Lemma 2.2 we have

$$X \subseteq \alpha_H X \subseteq \alpha_{\overline{H}} X,$$

with $X$ being dense in the latter. This implies that $\alpha_{\overline{H}} X \subseteq \alpha_H X$ which together with the above relation proves the lemma.

In the following theorem we use the known fact that a locally compact metrizable space $Y$ is $\sigma$-compact if and only if $C_0(Y)$ is separable. (See Theorem 3.5.17 of [4].) Observe that in locally compact metrizable spaces $\sigma$-compactness and separability coincide. (See Corollary 4.1.16 and Exercise 3.8.C(b) of [6].)

For a space $X$ and a subset $G$ of $C_B(X)$ we define the algebra (closed algebra, respectively) generated by $G$ as the smallest subalgebra (closed subalgebra, respectively) of $C_B(X)$ which contains $G$. We denote by $\langle G \rangle$ the subalgebra of $C_B(X)$ generated by $G$. Note that the closed subalgebra of $C_B(X)$ generated by $G$ is then the closure $\overline{\langle G \rangle}$ of $G$ in $C_B(X)$.

**Theorem 2.9.** Let $X$ be a completely regular space and let $H$ be a non-vanishing self-adjoint closed subalgebra of $C_B(X)$ which separates points from closed sets in $X$. Then $H$ is countably generated if and only if $\text{sp}(H)$ is separable and metrizable.

**Proof.** Suppose that $\text{sp}(H)$ is a separable metrizable space. Since $\text{sp}(H)$ is locally compact, $\text{sp}(H)$ is then $\sigma$-compact, and therefore $C_0(\text{sp}(H))$ is separable. But then $H$ is also separable (as is isometrically isomorphic to $C_0(\text{sp}(H))$) and therefore countably generated.

Suppose that $H$ is countably generated. Let

$$H = \langle h_1, h_2, \ldots \rangle,$$
where \( h_1, h_2, \ldots \) are in \( H \) and the bar denotes the closure in \( C_B(X) \). Note that 
\[
\langle h_1, h_2, \ldots \rangle = \left\{ \sum_{i \in I} c_i h_{k_i}^{p_i} \cdots h_{k_n}^{p_n} : I \text{ is finite}, c_i \in \mathbb{C}, k_j, p_j \in \mathbb{N} \text{ for } j = 1, \ldots, n_i \right\}.
\]
Let 
\[
Q = \left\{ \sum_{i \in I} q_i h_{k_i}^{p_i} \cdots h_{k_n}^{p_n} : I \text{ is finite}, q_i \in \mathbb{Q} \times \mathbb{Q}, k_j, p_j \in \mathbb{N} \text{ for } j = 1, \ldots, n_i \right\}.
\]
One can check that \( Q \) is dense in \( \langle h_1, h_2, \ldots \rangle \), and is therefore dense in \( H \), i.e., \( H = \overline{Q} \). Also, \( Q \) separates points from closed sets in \( X \). (To see this, let \( C \) be a closed subspace of \( X \) and let \( x \) be an element of \( X \setminus C \). Then there is an element \( h \) in \( H \) such that \( h(x) \) is not in \( \text{cl}_C(h)(C) \). Let \( \epsilon > 0 \) such that \( |h(x) - h(c)| > \epsilon \) for every \( c \) in \( C \). Let \( f \) be an element of \( Q \) such that \( \|f - h\| < \epsilon/3 \). Then 
\[
|f(x) - f(c)| \geq |h(x) - h(c)| - |f(c) - h(c)| - |f(x) - h(x)| > \epsilon - \epsilon/3 - \epsilon/3 = \epsilon/3
\]
for every \( c \) in \( C \), and therefore \( f(x) \) is not in \( \text{cl}_C(f)(C) \).) By Lemma 2.5 we have 
\[
\alpha_H X = \alpha_Q X.
\]
Note that \( Q \) is countable. Therefore \( \alpha_Q X \) is a separable metrizable space, as \( \alpha_Q X \) (by the construction in Lemma 2.2) will be a subspace of a countable product of spaces \( \mathbb{C} \). But then \( \lambda_H X \) is also a separable metrizable space, as \( \lambda_H X \) is a subspace of \( \alpha_H X \) (= \( \alpha_Q X \)). Finally, observe that \( \lambda_H X \) is the spectrum of \( H \) by Theorem 2.5 (and its proof).

In the following we give examples of completely regular spaces \( X \) and non-vanishing self-adjoint closed subalgebras \( H \) of \( C_B(X) \) which separate points from closed sets in \( X \), thus, satisfying the assumption in Theorem 2.5.

**Example 2.10.** Let \( \mathcal{P} \) be a topological property which is closed hereditary (in the sense that every closed subspace of a space with \( \mathcal{P} \) has \( \mathcal{P} \)) and preserved under countable closed unions of subspaces (in the sense that every space which is a countable union of closed subspaces with \( \mathcal{P} \) has \( \mathcal{P} \)). (Examples of such topological properties are the Lindelöf property, subparacompactness, submetacompactness, the submeta-Lindelöf property, weakly \( \theta \)-refinability and weakly \( \delta \theta \)-refinability; see Theorems 7.1 and 7.3 of [3].) Let \( X \) be a completely regular locally-\( \mathcal{P} \) space (in the sense that every element of \( X \) has a neighborhood in \( X \) with \( \mathcal{P} \)). Let
\[
H = \left\{ f \in C_B(X) : |f|^{-1}([\epsilon, \infty)) \text{ has } \mathcal{P} \text{ for all } \epsilon > 0 \right\}.
\]
Then \( H \) is a non-vanishing self-adjoint closed subalgebras of \( C_B(X) \) which separates points from closed sets in \( X \), as we now check. (In Theorem 3.3.9 of [11] it has been shown that \( H \) is a non-vanishing closed ideal of \( C_B(X) \).) That \( H \) is closed under addition and multiplication follows from the fact that for any two elements \( f \) and \( g \) in \( H \) and \( \epsilon > 0 \) we have 
\[
|f + g|^{-1}([\epsilon, \infty)) \subseteq |f|^{-1}([\epsilon/2, \infty)) \cup |g|^{-1}([\epsilon/2, \infty)),
\]
where the latter (being the union of two closed subspaces with \( \mathcal{P} \)) has \( \mathcal{P} \), and 
\[
|fg|^{-1}([\epsilon, \infty)) \subseteq |f|^{-1}([\epsilon/M, \infty))
\]
where \( M > \|g\| \). Therefore, \( |fg|^{-1}([\epsilon, \infty)) \) and \( |f + g|^{-1}([\epsilon, \infty)) \), being closed subspaces of spaces with \( \mathcal{P} \), have \( \mathcal{P} \). Similarly, one can check that \( H \) is closed under scalar multiplication. Therefore, \( H \) is a subalgebra of \( C_B(X) \). To check that
$H$ is a closed subalgebra of $C_B(X)$, let $h_n \to f$, where $h_1, h_2, \ldots$ is a sequence in $H$ and $f$ in $C_B(X)$. Then, for every $\epsilon > 0$ we have

$$|f|^{-1}([\epsilon, \infty)) \subseteq \bigcup_{n=1}^{\infty} |h_n|^{-1}([\epsilon/2, \infty)),$$

where the latter (being a countable union of closed subspaces with $\mathcal{P}$) has $\mathcal{P}$, and therefore, so does its closed subspace $|f|^{-1}([\epsilon, \infty))$. Therefore, $f$ is in $H$, and thus $H$ is closed in $C_B(X)$. It is clear that $H$ is self-adjoint. To check that $H$ is non-vanishing and separates points from closed sets in $X$, let $C$ be a closed subspace of $X$ and let $x$ be in $X \setminus C$. Since $X$ is locally-$\mathcal{P}$, there is a neighborhood $U$ of $x$ in $X$ which has $\mathcal{P}$. Let $W = U \cap (X \setminus C)$. Then $W$ is a neighborhood of $x$ in $X$. By complete regularity of $X$ there is some $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(y) = 0$ for all $y$ outside $W$, in particular, $f(C) = 0$. Note that $f$ is in $H$, as $|f|^{-1}([\epsilon, \infty)) \subseteq U$ for every $\epsilon > 0$, $U$ has $\mathcal{P}$, and $|f|^{-1}([\epsilon, \infty))$ is closed in $U$.

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