Manipulability in a Group Activity Selection Problem

Andreas Darmann

Abstract We consider the aspect of strategic manipulation in the group activity selection problem GASP. In that problem, the goal is to assign agents to activities when the agents have preferences over the activities themselves and over the number of participants in the activities, that is preferences over pairs “(activity, group size)”, including the possibility “do nothing”. In such a setting the main, natural solution concept studied in Darmann et al. (2012) to aggregate the agents’ preferences into a group solution is maximum individual rationality: the aim is to assign the maximum number of agents to activities in a way such that no agent prefers doing nothing to the actual choice. In this paper, we consider the possibility of strategic manipulation involved in providing such solutions. We adapt and analyze well-known preference extensions such as the Kelly extension and the Gärdenfors extension in our setting. In addition, for some special cases inherent in the model of GASP we show that strategyproofness is provided by such an aggregation, while for others this is not the case.

1 Introduction

We consider the aspect of strategic manipulability in the following group activity selection problem GASP (introduced in Darmann et al. (2012)). In this setting, there is a set of agents and a set of activities to which the agents should be assigned, where each agent can take part in at most one activity. The agents’ preferences depend on the activity itself and the number of participants in that activity. As a particular example consider the organizer of a workshop who plans to set up social activities for the free afternoon, or a
company that wants to provide free sports classes for its employees in order to raise the overall satisfaction of the employees. Since these take place simultaneously, each agent can take part in at most one activity; of course, the choice of abstaining from any activity, i.e., doing nothing, should be a valid option as well. It is plausible to assume that the preferences of the agents do not depend on the activity alone but also on the number of agents taking part in the respective activity, since, e.g., a table tennis tournament with 40 agents and only one table will not be desired even by a passionate table tennis player.

A natural goal of the organizer now would be to find an assignment of agents to activities that maximizes the total number of agents taking part in some activity without forcing an agent to participate when she is not willing to.

More formally, in GASp the agents’ preferences are weak orders over pairs “(activity, group size)” including the possibility “do nothing” to which we refer as the void activity. The goal, of course, would be to assign agents to activities in a reasonable manner. A main requirement is that the assignment should be individually rational, meaning that no agent should be forced to take part in an alternative she deems unacceptable, i.e., would rather prefer doing nothing to. In Darmann et al. (2012), the problem of finding a maximum individually rational assignment – that is an assignment maximizing the number of agents assigned to a non-void activity in an individually rational assignment – is introduced and studied from a computational viewpoint, with the focus on special cases including increasing and decreasing preferences. Loosely speaking, increasing preferences mean that an agent would like as many other agents as possible to join the same activity. In the decreasing preferences case, an agent would like to share the same activity with as few other agents as possible. In this paper, we analyze the aspect of manipulability involved in providing maximum individually rational assignments with particular focus on the special cases of increasing and decreasing preferences.

Whether the aggregation of individual preferences into a group solution is susceptible to strategic manipulation is one of the central questions in social choice theory. Such an aggregation function (which outputs a single outcome) or aggregation correspondence (which outputs a set of outcomes) is strategyproof, and hence not manipulable, if no agent can be better off by misrepresenting her true preferences. In its classical framework, both strategyproofness of aggregation functions (see, for instance, Barberà (2010), and the seminal papers by Gibbard (1973) and Satterthwaite (1975)), and of aggregation correspondences (see, e.g., Barberà et al. (2001), Brandt & Brill (2011), Brandt & Geist (2014)) has been well-studied. In our context, we focus on the multi-valued aggregation correspondence whose output consists of the set of all maximum individually rational assignments. Clearly, comparing different assignments, an agent will prefer one which yields the best alternative for her. Comparing sets of assignments, however, is less obvious. Instead of asking the agents to give a ranking over all possible sets of outcomes (which requires to rank an exponential number of possibilities), the typical assumption is that the preferences over the single alternatives can be extended to preferences over sets of alternatives. Of course, such a preference extension can
be performed in various ways (see, e.g., Barberà et al. (2004) and Barberà (2010)). We study different preference extensions and the aspect of manipulability involved in each of them. With the Kelly-, Gärdenfors- and Fishburn extension (Kelly (1977), Gärdenfors (1976) and Fishburn (1972) respectively), we adapt the most prominent representatives of preference extensions to our setting. We also include the natural maxi-max extension and maxi-min extension (see Moretti & Tsoukiás (2012)) in our analysis: in an optimistic mindset, one might hope for the most-preferred among the possible alternatives; having a pessimistic view, one might be worried by the least-preferred among the alternatives.

Finally, both for the case of the considered aggregation correspondence and single-valued aggregation functions that output a specific maximum individually rational assignment, we provide manipulability and strategyproofness results respectively for some special cases of GASP.

Related work.

As mentioned above, Darmann et al. (2012) introduce the framework of GASP and its variant a-GASP, which corresponds to the setting of GASP under an approval voting scenario: for finding a desired assignment, the information taken into account is each agents’ induced partition of the set of alternatives into approved and disapproved ones (those ranked above and below “do nothing” respectively). Darmann et al. (2012) provide a computational complexity study of finding “good” assignments, where a main focus is laid on maximum individually rational assignments. In addition, for a-GASP they also introduce and briefly analyse stability concepts such as Nash, individual, or core stability from a computational viewpoint. Besides, Darmann et al. (2012) discuss the connection between GASP and hedonic coalition formation games, and point out that the setting of GASP can be embedded in the framework of hedonic games. It should be noted, however, that the model of GASP allows for a more succinct representation and inheres in natural special cases to which we draw our attention (see also Darmann et al. (2012)).

Darmann (2015) considers the variant o-GASP, in which the agents’ preferences are strict orders over the alternatives; in particular, Darmann (2015) extends the computational complexity study of finding maximum individually rational assignments and considers the problem of finding stable solutions in o-GASP from a computational viewpoint.

Strategyproofness of coalition formation rules has been studied in several papers, often axiomatically motivated. Rodríguez-Álvarez (2009) characterize single-lapping rules over the domain of additively representable or separable preferences by four axioms including strategyproofness. Papai (2004) uses strategyproofness as one of the characterizing axioms of these rules in general domains. Other works on strategyproofness of coalition formation games, in connection with core stability, include the works of Alcalde & Revilla (2004), Cechlarova & Romero-Medina (2001) and Sönmez (1999).
In a related work, Lee & Shoham (2015) introduce the stable invitation problem, both in an anonymous and non-anonymous variant. In the former variant, called ASIP, each agent has preferences over the number of attendees of an event, and the goal is a “good” invitation, i.e., a subset \( S \) of the agents such that each member of \( S \) agrees with the number of agents in \( S \). Note that finding an individually rational invitation in ASIP corresponds to finding an individually rational assignment in GASP with a single activity (see Lee & Shoham (2015)). Lee & Shoham (2015) discuss strategyproofness of single-valued mechanisms for ASIP in connection with stable and individually rational invitations. In particular, Lee & Shoham (2015) provide the impossibility result that a strategyproof mechanism for ASIP that always outputs an individually rational invitation cannot exist. We add to that result by showing that, in the terminology used in our paper, a strategyproof aggregation function outputting a maximum individually rational assignment cannot exist in the setting of GASP with only one activity (and hence in ASIP) when all agents’ preferences are decreasing, while for the increasing case each such aggregation function is strategyproof.

The paper is organized as follows. In Section 2 we present the model of GASP and the corresponding basic definitions (cf. Darmann et al. (2012)). The concepts of strategyproofness and the preference extensions involved are presented in Section 3, where also the relation between these concepts is discussed. In Section 4 we discuss manipulability for some special cases of GASP, for the considered aggregation functions and correspondence in general. Section 5 contains the aspect of strategic manipulation with respect to the dedicated preference extensions considered. Finally, in Section 6 we briefly discuss manipulability in the variants a-GASP and o-GASP.

2 Formal Model

In Section 2 we re-cite the model and basic definitions from Darmann et al. (2012).

**Definition 1** (see Darmann et al. (2012))

An instance of the Group Activity Selection Problem (GASP) is given by a set of agents \( N = \{1, \ldots, n\} \), a set of activities \( A = A^* \cup \{a_\emptyset\} \), where \( A^* = \{a_1, \ldots, a_m\} \), and a profile \( P \), which consists of \( n \) votes (one for each agent): \( P = (V_1, \ldots, V_n) \). The vote of agent \( i \) describes his preferences over the set of alternatives \( X = X^* \cup \{a_\emptyset\} \), where \( X^* = A^* \times \{1, \ldots, n\} \); alternative \((a, k)\), \( a \in A^* \), is interpreted as “activity \( a \) with \( k \) participants”, and \( a_\emptyset \) is the void activity.

The vote \( V_i \) of an agent \( i \in N \) (also denoted by \( \succcurlyeq_i \)) is a weak order over \( X^* \); its induced strict preference and indifference relations are denoted by \( \succ_i \) and \( \sim_i \), respectively. We set \( S_i = \{(a, k) \in X^* \mid (a, k) \succcurlyeq_i a_\emptyset\} \); we say that agent \( i \) approves of all alternatives in \( S_i \), and refer to the set \( S_i \) as the induced approval vote of agent \( i \).
An assignment for an instance \((N, A, P)\) of GASP is a mapping \(\pi : N \rightarrow A\); \(\pi(i) = a_0\) means that agent \(i\) does not participate in any activity. We set \(\pi^a = \{i \in N \mid \pi(i) = a\}\) for \(a \in A\); for \(i \in N\), let \(\pi_i := \{i' \in N \mid \pi(i') = \pi(i)\}\).

Abusing notation, we say that assignment \(\pi\) assigns agent \(i\) to alternative \((a, k)\) if \(\pi(i) = a\) with \(|\pi^a| = k\). As the main requirement considered for an assignment, no agent should be assigned to an alternative she deems unacceptable.

**Definition 2** (see Darmann et al. (2012)) Given an instance \((N, A, P)\) of GASP, an assignment \(\pi : N \rightarrow A\) is said to be individually rational if for every \(a \in A^*\) and every agent \(i \in \pi^a\) it holds that \((a, |\pi^a|) \in S_i\).

Clearly, in any instance \((N, A, P)\) of GASP the assignment \(\pi(i) = a_0\) for all \(i \in N\) is individually rational. As a consequence, an individually rational assignment exists in each instance of GASP. A main goal of a benevolent central authority, however, is to maximize the number of agents assigned to a non-void activity. Let \(\#(\pi) = |\{i \in N \mid \pi(i) \neq a_0\}|\) denote the total number of agents assigned to non-void activities.

**Definition 3** (see Darmann et al. (2012)) We say that \(\pi\) is maximum individually rational if \(\pi\) is individually rational and \(\#(\pi) \geq \#(\pi')\) for every individually rational assignment \(\pi'\).

Note that for determining maximum individually rational assignments only the sets \(S_i, i \in N\), are relevant. Thus, in what follows we will omit the alternatives ranked below \(a_0\) when considering such assignments.

### 2.1 Special Cases

We will consider some special cases of GASP in our analysis. For instance, we may distinguish between unique activities (such as playing table tennis when there is only one table) and activities that exist in multiple copies (such as going for a hike).

**Definition 4** (see Darmann et al. (2012)) Given an instance \((N, A, P)\) of GASP, we say that \(a, b \in A^*\) are equivalent if for each \(i \in N\) and every \(j \in \{1, \ldots, n\}\) it holds that \((a, j) \sim_i (b, j)\). An activity \(a \in A^*\) is copyable if \(A^*\) contains \(k \geq n\) activities that are equivalent to \(a\) (including \(a\) itself). An activity \(a \in A^*\) is simple if there is no \(b \in A^* \setminus \{a\}\) such that \(b\) is equivalent to \(a\).

In the case of copyable activities, in order to maintain a succinct representation each equivalence class in \(A^*\) is represented by one activity. Unless stated otherwise, the activities in an instance of GASP are simple.
We now turn to the two natural special cases of increasing and decreasing preferences. Informally speaking, an agent has increasing preferences with respect to an activity \(a\) if she prefers to participate in \(a\) together with as many other agents as possible; analogously, an agent has decreasing preferences with respect to an activity \(a\) if she wishes to share \(a\) with as few other agents as possible.

**Definition 5** Let \((N, A, P)\) be an instance of GASP. Agent \(i\)'s preferences are *increasing* with respect to activity \(a \in A^*\) if for each \(j < n\) with \((a, j) \succeq_i a_0\) we have \((a, j + 1) \succeq_i (a, j)\). Similarly, agent \(i\)'s preferences are *decreasing* with respect to activity \(a \in A^*\) if for each \(j > 1\) with \((a, j) \succeq_i a_0\) we have \((a, j - 1) \succeq_i (a, j)\).

An instance \((N, A, P)\) of GASP is *increasing* (respectively, *decreasing*) if the preferences of each agent \(i \in N\) are increasing (respectively, decreasing) with respect to each activity \(a \in A^*\).

### 3 Strategyproofness: Concepts

We study strategyproofness in connection with maximum individually rational assignments. In this respect, we focus on the aggregation correspondence \(C\) (called mir-aggregation correspondence), which outputs the set of all maximum individually rational assignments of a given instance of GASP. In contrast, (deterministic) aggregation functions (which we call mir-aggregation functions) output exactly one maximum individually rational assignment for each instance.

Given the set \(N\) of agents and \(A\) of activities, let \(S(N, A)\) denote the set of all instances of GASP with agent-set \(N\) and activity-set \(A\), and let \(\alpha(N, A) := \{\pi \mid \pi : N \rightarrow A\}\).

**Definition 6** Given an instance \(I = (N, A, P)\) of GASP, let \(\Pi(I)\) (or simply \(\Pi\)) be the set of maximum individually rational assignments in \(I\). The mapping \(C : S(N, A) \rightarrow 2^{\alpha(N, A)}\) with \(C(I) = \Pi(I)\) is called mir-aggregation correspondence.

We call a function \(f : S(N, A) \rightarrow \alpha(N, A)\) with \(f(I) \in \Pi(I)\) mir-aggregation function.

Let \(I = (N, A, P)\) be an instance of GASP. When it is obvious from the context to which instance we refer, we will write \(C\) instead of \(C(I)\) and \(f\) instead of \(f(I)\) respectively.

**Definition 7** A mir-aggregation function \(f\) is called *manipulable*, if there exist an instance \(I = (N, A, P)\) of GASP, an agent \(i \in N\) and a profile \(P'\) with \(P'|_{N \setminus \{i\}} = P'|_{N \setminus \{i\}}\) such that, with \(f(I) = \pi, I' = (N, A, P')\) and \(f(I') = \pi'\),

\[(\pi'(i), |\pi'_i|) \succ_i (\pi(i), |\pi_i|)\]

---

1 In Darmann et al. (2012) these two concepts are defined for a-GASP only and hence in a (slightly) different manner. E.g., agent \(i\)'s preferences are increasing with respect to activity \(a \in A^*\) if \(\{k \mid (a, k) \in S_i\} = [\ell_i^a, n]\) for some \(\ell_i^a \in N\).
holds, where \( \succ_i \) relates to the asymmetric part of \( \succeq_i \) in \( P \). In such a case, to refer to the respective instance and agent, we say that \( f \) is manipulable at instance \( I \) by agent \( i \).

\( f \) is called strategyproof, if \( f \) is not manipulable.

In order to consider manipulability of the mir-aggregation correspondence \( C \), we adapt the natural extension axiom (see Barberà et al. (2004)) which, formulated for sets of objects, states that rankings over singleton sets containing one object each should be consistent with the rankings over the objects. In our setting, the agents’ rankings over sets containing one assignment each should be consistent with the rankings over the respective alternatives assigned. More formally, any function \( \varepsilon \) that, given an instance \( I = (N, A, P) \) for all \( i \in N \) yields a preference relation \( \succeq_i^\varepsilon \) over sets of assignments (i.e., over \( 2^{\alpha(N,A)} \)), is called a preference extension, if for all \( \pi, \tilde{\pi} \in \alpha(N, A) \) the following holds:

\[
\{ \pi \} \succeq_i^\varepsilon \{ \tilde{\pi} \} \iff (\pi(i), |\pi_i|) \succeq_i (\tilde{\pi}(i), |\tilde{\pi}_i|)
\]

For a preference extension \( \varepsilon \), \( \succ_i^\varepsilon \) denotes the asymmetric part of \( \succeq_i^\varepsilon \). Abusing notation, we will also write \( \pi \succ_i \tilde{\pi} \) instead of \( (\pi(i), |\pi_i|) \succeq_i (\tilde{\pi}(i), |\tilde{\pi}_i|) \); in such a case, we say that \( i \) considers \( \pi \) at least as good as \( \tilde{\pi} \) (respectively, we say that \( i \) prefers \( \pi \) to \( \tilde{\pi} \) if \( \pi \succ_i \tilde{\pi} \), i.e., \( (\pi(i), |\pi_i|) \succ_i (\tilde{\pi}(i), |\tilde{\pi}_i|) \) holds).

**Definition 8** Let \( \varepsilon \) be a preference extension. \( C \) is \( \varepsilon \)-manipulable if there exist an instance \( I = (N, A, P) \), an agent \( i \in N \) and a profile \( P' \) with \( P'|_{N\setminus\{i\}} = P'|_{N\setminus\{i\}} \) such that, with \( I' = (N, A, P') \),

\[
\Pi(I') \succ_i^\varepsilon \Pi(I)
\]

holds, where \( \succ_i \) relates to asymmetric part of \( \succeq_i \) in \( P \). In such a case, to refer to the respective instance and agent, we say that \( C \) is \( \varepsilon \)-manipulable at instance \( I \) by agent \( i \).

\( C \) is \( \varepsilon \)-strategyproof, if \( C \) is not \( \varepsilon \)-manipulable.

In what follows, we apply particular representatives of preference extensions to our setting. These are the intuitive maxi-max and maxi-min extension (see Moretti & Tsoukiàs (2012)), as well as the well-known Kelly extension (Kelly (1977)), Gärdenfors extension (Gärdenfors (1976)), and Fishburn extension (Fishburn (1972)).

Let \( (N, A, P) \) be an instance of GASP. For \( X \in 2^{\alpha(N,A)} \), an assignment \( \pi \in X \) is \( i \)'s max-assignment in \( X \) (denoted by \( \text{max}_i X \)), if \( (\pi(i), |\pi_i|) \succeq_i (\tilde{\pi}(i), |\tilde{\pi}_i|) \) holds for all \( \tilde{\pi} \in X \); we denote the corresponding number \( |\pi_i| \) by \( m_X(i) \).

Analogously, \( \pi \in X \) is \( i \)'s min-assignment in \( X \) (denoted by \( \text{min}_i X \)), if \( (\pi(i), |\pi_i|) \succeq_i (\tilde{\pi}(i), |\tilde{\pi}_i|) \) holds for all \( \tilde{\pi} \in X \); in this case, the respective number \( |\pi_i| \) is denoted by \( m_X(i) \).

In the maxi-max extension, an agent prefers a set \( X \) of assignments to a set \( Y \) of assignments, if she prefers her max-assignment in \( X \) to her max-assignment in \( Y \), i.e., if she prefers the best alternative assigned to her by an assignment in \( X \) to the best alternative assigned to her by an assignment in \( Y \).
Similarly, in the maxi-min extension, an agent prefers a set $X$ of assignments to a set $Y$ of assignments, if she prefers her min-assignment in $X$ to her min-assignment in $Y$.

**Definition 9** The *maxi-max extension* is defined by: for $i \in N$ and $X, Y \in 2^a$, $X \supseteq X Y$ iff $(\max_i X, m_X^\supseteq X Y(i)) \supseteq_i (\max_i Y, m_Y^\supseteq X Y(i))$.

Analogously, the *maxi-min extension* is defined by: for $i \in N$ and $X, Y \in 2^a$, $X \supseteq X Y$ iff $(\min_i X, m_X^\supseteq X Y(i)) \supseteq_i (\min_i Y, m_Y^\supseteq X Y(i))$.

Adapting the Kelly extension (Kelly (1977)), an agent considers assignment set $X$ at least as good as assignment set $Y$ if she considers each alternative assigned to her by an assignment in $X$ at least as good as each alternative assigned to her by an assignment in $Y$.

**Definition 10** The *Kelly extension* is defined as follows. For $i \in N$ and $X, Y \in 2^a$, $X \supseteq X Y$ iff for every $x \in X, y \in Y$ we have $(x(i), |x_i|) \supseteq_i (y, |y_i|)$.

In our setting, in the Gärdenfors extension (Gärdenfors (1976)) an agent considers $X$ at least as good as $Y$ if the following holds: (i) $X$ can be “created” from $Y$ by adding (removing) assignments, and the agent considers each added (removed) assignment at least as good as each of the original assignments (not better than any of the remaining assignments); (ii) otherwise, the agent considers each assignment in $X \setminus Y$ at least as good as each assignment in $Y \setminus X$.

**Definition 11** The *Gärdenfors extension* is defined as follows. For $i \in N$ and $X, Y \in 2^a$, $X \supseteq X Y$ if one of the three following conditions is satisfied:

1. $X \subseteq Y$ and for all $x \in X, y \in Y \setminus X$ we have $(x(i), |x_i|) \supseteq_i (y, |y_i|)$.
2. $Y \subseteq X$ and for all $x \in X \setminus Y, y \in Y$ we have $(x(i), |x_i|) \supseteq_i (y, |y_i|)$.
3. neither $X \subseteq Y$ nor $Y \subseteq X$ and $(x(i), |x_i|) \supseteq_i (y(i), |y_i|)$ for all $x \in X \setminus Y, y \in Y \setminus X$.

Finally, the Fishburn extension requires for an agent to consider $X$ at least as good as $Y$ not only that (i) the agent considers each assignment in $X \setminus Y$ at least as good as each assignment in $Y \setminus X$, but also that (ii) the agent considers each alternative in $X \setminus Y$ at least as good as each assignment which is contained in both $X$ and $Y$, and each of the latter assignments at least as good as each assignment in $Y \setminus X$.

**Definition 12** The *Fishburn extension* (Fishburn (1972)) is defined as follows. For $i \in N$ and $X, Y \in 2^a$, $X \supseteq X Y$ if all of the following three conditions are satisfied:

1. $(x(i), |x_i|) \supseteq_i (y, |y_i|)$ for all $x \in X \setminus Y, y \in X \cap Y$.
2. $(y(i), |y_i|) \supseteq_i (z(i), |z_i|)$ for all $y \in X \cap Y, z \in Y \setminus X$.
3. $(x(i), |x_i|) \supseteq_i (z(i), |z_i|)$ for all $x \in X \setminus Y, z \in Y \setminus X$.

In the following section, we discuss the relation between the concepts of $\varepsilon$-strategyproofness for the preference extensions $\varepsilon$ considered.
3.1 Relation between the concepts

Considering manipulability of the mir-aggregation correspondence $C$ w.r.t. to the preference extensions introduced, assume that, given instance $I = (N, A, P)$ of GASP, agent $i \in N$ wants to manipulate at $I$ by misreporting her preferences. Let $I' = (N, A, P')$ with $P|_{N \setminus \{i\}} = P'|_{N \setminus \{i\}}$ be the respective instance; $S_i$ is the approval set of agent $i$’s true preferences (i.e., $S_i$ relates to $\succeq_i$ in $P$), and $S'_i$ is the approval set of agent $i$’s misreported preferences (i.e., $S'_i$ relates to $\succeq'_i$ in $P'$). Let $I^I := I(I)$ and $I'^I := I(I')$.

For the preference extensions considered (maxi-max-, maxi-min-, Kelly-, Gärdenfors-, and Fishburn extension) ranking disapproved alternatives, i.e., some $a \in A \setminus S_i$, above $a_0$ in $\succeq'_i$ is useless for agent $i$, because then $I'$ might contain a maximum individually rational assignment in which $i$ is assigned to an alternative she disapproves of. Putting things the other way, if $I' \succ_i I$ holds, then removing $a \in A \setminus S_i$ from $S'_i$ (ceteris paribus) results in an instance $I''$ with $I'' = I(I'')$ such that $I'' \succ_i I$ holds, for each of the dedicated extensions considered. Thus, for these extensions we can assume that

$$S'_i \subset S_i$$

(1)

holds. Obviously, (1) implies $\#(\pi') \leq \#(\pi)$ for $\pi' \in I'$, $\pi \in I$. In addition, we have either $I' \subseteq I$ (if $\#(\pi') = \#(\pi)$) or $I' \cap I = \emptyset$ (if $\#(\pi') < \#(\pi)$). As consequences, we can note that (with $X = I'$, $Y = I$) the second condition of Gärdenfors manipulability and the first condition of Fishburn manipulability are redundant.

As it turns out, the concepts of Kelly strategyproofness, Gärdenfors strategyproofness, and Fishburn strategyproofness coincide.

**Theorem 1** Given instance $I$ of GASP, for the mir-aggregation correspondence $C$ the following holds:

$C$ is Kelly manipulable at $I \iff C$ is Gärdenfors manipulable at $I \iff C$ is Fishburn manipulable at $I$.

**Proof** We argue that Fishburn manipulability of $C$ implies that $C$ is both Kelly and Gärdenfors manipulable.

Assume that $C$ is Fishburn manipulable. Using the above notation, let $I = (N, A, P)$ and $I' = (N, A, P')$ be the respective instances; in particular, for some $i \in N$ we have $I' \succ_{i}^{F} I$. Clearly, $I' \neq I$ holds.

Case I: $I' \cap I = \emptyset$. Then, $I' \setminus I = I'$ and $I \setminus I' = I$. In this case, we only need to consider condition 3 of the Fishburn extension; $I' \succ_{i}^{F} I$ hence means that for all $x \in I'$ and all $y \in I$ we have $x \succeq_{i} y$ while for one such pair we have $x \succ_{i} y$. Obviously, this corresponds to $C$ being Kelly manipulable.

Regarding the Gärdenfors extension, note that by $I' \cap I = \emptyset$ condition 3 of the Gärdenfors extension is crucial; clearly, $I' \succ_{i}^{G} I$ follows.

Case II: $I' \subset I$. Now, $I' \cap I = I'$ follows. Since $I' \setminus I = \emptyset$, we need to consider condition 2 of the Fishburn extension only: $I' \succ_{i}^{F} I$ thus means
that for all \( x \in I' \) and all \( y \in I \setminus I' \) we have \( x \succ_i y \) while for one such pair we have \( x \succeq_i y \). By condition 1 of Gärdenfors extension, this means \( I' \succ^V I \), i.e., \( C \) is Gärdenfors manipulable. Observe that \( I' \) contains only agent \( i \)'s most preferred assignments of \( I \), because otherwise there exist \( x \in I' \) and \( y \in I \) with \( x \prec_i y \). Obviously, for all \( x \in I' \) and all \( y \in I \) we have \( x \succeq_j y \), while for one such pair \( x \succ_i y \) holds due to Fishburn manipulability. Thus, \( C \) is Kelly manipulable.

It is not difficult to verify that by analogous argumentation it follows that (i) Kelly manipulability implies Fishburn manipulability and (ii) Gärdenfors manipulability implies Fishburn manipulability.

However, note that maxi-max strategyproofness does not imply maxi-min strategyproofness or vice versa; in addition, none of these concepts imply or are implied by Kelly strategyproofness (and thus by Gärdenfors- or Fishburn strategyproofness).

**Example 1** Consider instance \( I = (N, A, P) \) with \( N = \{1, 2, 3\} \), \( A^* = \{a\} \), \( V_1 = (a, 2) \succ_1 (a, 1) \succ_1 a_\emptyset \), and \( V_2 = V_3 = (a, 2) \succ a_\emptyset \). The set of maximum individually rational assignments in \( I \) is \( I = \{\pi, \mu, \lambda\} \) with \( \pi(1) = a, \pi(2) = a \) and \( \pi(3) = a_\emptyset, \mu(1) = \mu(3) = a \) and \( \mu(2) = a_\emptyset \), and \( \lambda(2) = \lambda(3) = a \) and \( \lambda(1) = a_\emptyset \). It is easy to see that agents 2 and 3 have no chance to manipulate. Assume agent 1 reports \( V_1' = (a, 1) \succ_1 a_\emptyset \) instead of \( V_1 \) (ceteris paribus). In the respective instance \( I' \), there is exactly one maximum individually rational assignment, namely \( \pi' \) with \( \pi'(1) = a \) and \( \pi'(2) = \pi'(3) = a_\emptyset \). Thus, \( C \) is maxi-min manipulable at instance \( I \) by agent 1 because \( \min_1 I' = (a, 1) \succ_1 a_\emptyset = \min_1 I \). On the other hand, since dropping \( (a, 1) \) from \( V_1 \) has no effect on the set of maximum individually rational assignments, it is not hard to verify that \( C \) is neither maxi-max nor Kelly manipulable.

**Example 2** Let instance \( I = (N, A, P) \) with \( N = \{1, 2\} \), \( A^* = \{a\} \), and \( V_1 = V_2 = (a, 1) \succ (a, 2) \sim a_\emptyset \) be given. The only maximum individually rational assignment in \( I \) is \( \pi \) with \( \pi(1) = \pi(2) = a \). Either agent can Kelly manipulate at instance \( I \). For instance, assume that agent 1 reports \( V_1' = (a, 1) \succ_1 a_\emptyset \) instead of \( V_1 \) (ceteris paribus). In the respective instance \( I' \), there are two maximum individually rational assignments, namely \( \lambda \) with \( \lambda(1) = a \) and \( \lambda(2) = a_\emptyset \) and \( \mu \) with \( \mu(1) = a_\emptyset \) and \( \mu(2) = a \). Note that both \( (\lambda(1), 1) \succ (\pi(1), 2) \) and \( a_\emptyset \sim (\pi(1), 2) \) hold. Thus, \( C \) is Kelly manipulable at instance \( I \) by agent 1. Also, it follows that \( C \) is maxi-max manipulable. On the other hand, it is easy to verify that \( C \) is not maxi-min manipulable at instance \( I \).

**Example 3** Consider instance \( I = (N, A, P) \) with \( N = \{1, 2\} \), \( A^* = \{a\} \), and \( V_1 = V_2 = (a, 1) \succ (a, 2) \succ a_\emptyset \). The only maximum individually rational assignment in \( I \) is \( \pi \) with \( \pi(1) = \pi(2) = a \). Either agent can maxi-max manipulate at instance \( I \). E.g., assume that agent 1 reports \( V_1' = (a, 1) \succ_1 a_\emptyset \) instead of \( V_1 \) (ceteris paribus). In the respective instance \( I' \), there are two maximum individually rational assignments, namely \( \lambda \) with \( \lambda(1) = a \) and \( \lambda(2) = a_\emptyset \)
and $\mu$ with $\mu(1) = a_\emptyset$ and $\mu(2) = a$. Now we have $(\lambda(1), 1) \succ_1 (\pi(1), 2)$ but $a_\emptyset \prec_1 (\pi(1), 2)$. Thus, $C$ is maxi-max manipulable but neither maxi-min- nor Kelly manipulable at instance $I$.

Example 4 In instance $I = (N, A, P)$ with $N = \{1, 2\}$, $A^* = \{a, b\}$, $V_1 = V_2 = (a, 1) \succ (b, 1) \succ a_\emptyset$, there are two maximum individually rational assignments: $\pi$ with $\pi(1) = a$ and $\pi(2) = b$, and $\sigma$ with $\sigma(1) = b$ and $\sigma(2) = a$. Again, assume that agent 1 reports $V'_1 = (a, 1) \succ_1 a_\emptyset$ instead of $V_1$ (ceteris paribus). In the respective instance $I'$, the only maximum individually rational assignment is $\lambda$ with $\lambda(1) = a$ and $\lambda(2) = b$. Now, due to $(\lambda(1), 1) = (\pi(1), 1)$ and $(\lambda(1), 1) \succ_1 (\sigma(1), 1)$ it follows that $C$ is Kelly manipulable and maxi-min manipulable. However, it is easy to see that $C$ is not maxi-max manipulable at instance $I$.

We proceed with a discussion of manipulability in special cases of GASP — both for the mir-aggregation correspondence and mir-aggregation functions — in Section 4. In Section 5, we analyse manipulability of correspondence $C$ with respect to the preference extensions considered.

4 Manipulability in Special Cases of GASP

Considering the aspect of manipulability in special cases of GASP, on the positive side, in a particular special case both the mir-aggregation correspondence $C$ and each mir-aggregation function are strategyproof (in the former case, for any preference extension $\varepsilon$). On the negative side, each mir-aggregation function and the correspondence $C$ are manipulable in the case of a single activity already, irrespective of whether the activity is copyable or simple.

In what follows, given instances $I = (N, A, P)$ and $I' = (N, A, P')$, let $\Pi := H(I)$ and $\Pi' := H(I')$.

Theorem 2 In GASP, (i) every mir-aggregation function is manipulable and (ii) $C$ is $\varepsilon$-manipulable for every preference extension $\varepsilon$, even when $A^*$ consists of one simple (or copyable) activity.

Proof Consider the following instance $(N, A, P)$ with $N = \{1, 2\}$ and $A^* = \{a\}$, where $V_1 = (a, 1) \succ_1 (a, 2) \succ_1 a_\emptyset$, i.e., $S_1 = \{(a, 1), (a, 2)\}$. Let $V_2 = (a, 2) \succ_2 a_\emptyset$, which means $S_2 = \{(a, 2)\}$. Clearly, irrespective of $a$ being simple or copyable, the only maximum individually rational assignment is $\pi(1) = \pi(2) = a$. Let $P'$ such that $V'_2 = V_2$ and $V'_1 = (a, 1) \succ_1 a_\emptyset$. Thus, $S'_1 = \{(a, 1)\}$ and $S'_2 = \{(a, 2)\}$. Then, the only maximum individually rational assignment is $\pi'(1) = a$, $\pi'(2) = a_\emptyset$ (again, irrespective of $a$ being copyable or simple). Since according to $V_1$ agent 1’s true preference is $(a, 1) \succ_1 (a, 2)$, agent 1 is better off with $(N, A, P')$ than with $(N, A, P)$. Due to $\Pi = \{\pi\}$ and $\Pi' = \{\pi'\}$, this means that every mir-aggregation function is manipulable. For each extension $\varepsilon$, $\{\pi'\} \succ_\varepsilon \{\pi\}$ holds by definition, i.e., $C$ is $\varepsilon$-manipulable. ■
Theorem 3 In GASP, when all agents have increasing preferences and \( A^* \) consists of one simple activity, then (i) every mir-aggregation function is strategyproof and (ii) \( C \) is \( \varepsilon \)-strategyproof for every preference extension \( \varepsilon \).

Proof Given an instance \( I = (N, A, P) \) with \( A^* = \{a\} \), where \( a \) is simple, and the agents have increasing preferences. First, we show that for \( I \), a maximum individually rational assignment is unique. Assume the opposite, and let \( \#(\pi) = k \) for \( \pi \in \Pi \). If \( |\Pi| \geq 2 \), then there are at least \( k+1 \) agents that approve of \( (a,k) \). By increasing preferences, each of these agents approve of \( (a,k+1) \) as well. Thus, there is an assignment \( \tilde{\pi} \) with \( \#(\tilde{\pi}) = k+1 \), which contradicts with the fact that \( \pi \) is a maximum individually rational assignment. Thus, \( \Pi = \{\pi\} \) for some assignment \( \pi \).

Now, let \( i \) misreport her preferences; i.e., let \( I' = (N, A, P') \) such that for some \( i \in N \), \( P'|_{N\setminus\{i\}} = P|_{N\setminus\{i\}} \) and \( V'_i \neq V_i \) holds. Due to increasing preferences, \( S_i = \{a\} \times [n_i, n] \), and \( S'_i = \{a\} \times [n'_i, n] \) hold for some \( n_i, n'_i \in \mathbb{N} \). Note that \( I' \) is also an instance of GASP with one simple activity and increasing preferences, thus \( \Pi' = \{\pi'\} \) for some assignment \( \pi' \). We show that \( (\pi(i), |\pi_i|) \lessdot (\pi'(i), |\pi'_i|) \) and thus \( \pi \succeq \pi' \) holds for any preference extension \( \varepsilon \).

- If \( n'_i = n_i \), then \( \pi' = \pi \) follows. Hence, \( i \) is clearly not better off in this case.
- If \( n'_i > n_i \), then any individually rational (w.r.t. \( P' \)) assignment in \( I' \) is individually rational (w.r.t. \( P \)) in \( I \). In particular, \( \#(\pi') \leq k \) holds. Now, we distinguish whether \( k \geq n'_i \) or \( k < n'_i \) holds.
  - In the former case, \( \pi \) is individually rational also in \( I' \), and thus \( \pi' = \pi \) follows. Again, \( i \) is not better off.
  - For the latter case \( (k < n'_i) \), \( \#(\pi') \leq k \) implies that \( \pi'(i) = a_q \) holds.
    Thus, \( i \) cannot be better off in \( I' \), because due to the individual rationality of \( \pi \), either \( \pi(i) = a_q \) or \( \pi(i) = a \) and \( (a,k) \in S_i \) holds.
- If \( n'_i < n_i \), we distinguish whether (i) \( k \geq n_i \) or (ii) \( k < n_i \) holds. In case (i), because \( A^* \) consists of only one simple activity, it is easy to see that \( \pi' = \pi \) holds. Therewith, \( i \) is not better off in \( I' \) than in \( I \).
  In case (ii), we get \( \pi(i) = a_q \). However, for each \( \ell \geq n_i \), the set of agents who approve of \( (a,\ell) \) in \( I \) coincides with the set of agents who approve of \( (a,\ell) \) in \( I' \). Thus, \( \#(\pi') < n_i \) follows. As a consequence, either \( \pi'(i) = a_q \) or \( \pi'(i) = a \) and \( (a,\#(\pi')) \notin S_i \) follows. Neither way agent \( i \) is better off in instance \( I' \).

Theorem 4 In GASP, when all agents have increasing preferences and \( A^* \) consists of two simple (or copyable) activities, then (i) every mir-aggregation function is manipulable and (ii) \( C \) is \( \varepsilon \)-manipulable for every preference extension \( \varepsilon \).

Proof Given instance \( I = (N, A, P) \) with \( N = \{1, 2\} \), \( A^* = \{a_1, a_2\} \), \( V_1 = (a_1, 2) \succ_1 (a_1, 1) \succ_1 (a_2, 2) \succ_1 a_0 \), and \( V_2 = (a_2, 2) \succ_2 a_0 \). The only maximum individually rational assignment is \( \pi(1) = \pi(2) = a_2 \). However, in instance
$I' = (N, A, P')$ with $V'_1 = (a_1, 2) \succ_1 (a_1, 1) \succ_1 a_0$ and $V'_2 = V_2$, the only maximum individually rational assignment is $\pi'$ with $\pi'(1) = a_1$ and $\pi'(2) = a_0$. By $(a_1, 1) \succ_1 (a_2, 2)$, we get that (i) every mir-aggregation function is manipulable, and (ii) $C$ is $\varepsilon$-manipulable for every preference extension $\varepsilon$ because $\{\pi'\} \succ_1^C \{\pi\}$ holds. ■

**Theorem 5** In GASP, when all agents have decreasing preferences and $A^*$ consists of only one simple activity, then every mir-aggregation function $f$ is manipulable.

**Proof** Assume there is a strategyproof mir-aggregation function $f$. Throughout this proof, let $N = \{1, 2\}$ and $A^* = \{a\}$, where $a$ is simple. Instance $I = (N, A, P)$ of GASP is given by

$$V_1 = V_2 = (a, 1) \succ (a, 2) \succ a_0$$

Since $a$ is simple, the assignment $\pi$, given by $\pi(1) = \pi(2) = a$ is the unique maximum individually rational assignment in $I$. Thus, $f(I) = \pi$ holds. Now, consider the following two instances, and the individually rational assignments $\lambda, \mu$ given by $\lambda(1) = a$, $\lambda(2) = a_0$ and $\mu(1) = a_0$, $\mu(2) = a$.

1. Let instance $I^{(1)} = (N, A, P^{(1)})$ with $V^{(1)}_1 = V_1$ and $V^{(1)}_2 = (a, 1) \succ a_0$. In $I^{(1)}$, there are two maximum individually rational assignments: $\lambda$ and $\mu$. Assume that in $I^{(1)}$ we have $f(I^{(1)}) = \mu$. Then, starting with instance $I$, agent 2 can manipulate by ranking $(a, 2)$ below $a_0$, i.e., removing $(a, 2)$ from $S_2$, since she prefers $(a, 1)$ to $(a, 2)$. Therefore, we must have $f(I^{(1)}) = \lambda$.

2. Let instance $I^{(2)} = (N, A, P^{(2)})$ with $V^{(2)}_1 = (a, 1) \succ a_0$ and $V^{(2)}_2 = V_2$. Analogously to 1., $f(I^{(2)}) = \lambda$ implies that, starting with instance $I$, agent 1 can manipulate by ranking $(a, 2)$ below $a_0$. Therefore, we must have $f(1) = a_0$ and $f(2) = a$ in instance $I^{(2)}$.

Finally, consider instance $I^{(3)} = (N, A, P^{(3)})$ with $V^{(3)}_1 = V^{(3)}_2 = (a, 1) \succ a_0$.

**Case I:** $f(I^{(3)}) = \lambda$. Then, starting with $I^{(3)}$, agent 2 can manipulate by reporting $V_2$ instead of $V^{(3)}_2$, i.e., “creating” $I^{(2)}$: this guarantees her to be the only agent assigned to $a$ (see 2.), which she prefers to $a_0$ in $V^{(3)}_2$.

**Case II:** $f(I^{(3)}) = \mu$. Starting with $I^{(3)}$, now agent 1 is able to manipulate by reporting $V_1$ instead of $V^{(3)}_1$, i.e., “creating” $I^{(1)}$. In this way, agent 1 is the only agent assigned to $a$ (see 1.), which she prefers to $a_0$ in $V^{(3)}_1$.

Thus, either choice of $f$ in $I^{(3)}$ offers a possibility to manipulate. Therefore, there is no strategyproof mir-aggregation function in the case of one simple activity and decreasing preferences. ■

Note that in GASP, when all agents have decreasing preferences and all activities are copyable, then there is an individually rational assignment that assigns each agent to one of her top-ranked alternatives. It is easy to verify that in this case an aggregation function is strategyproof if and only if it outputs such an assignment for each such instance.
Theorem 6 In GASP, when all agents have decreasing preferences and all activities are copyable, then a mir-aggregation function $f$ is strategyproof if and only if $f$ always outputs an assignment that assigns each agent to one of her top-ranked alternatives.

Finally, let us consider the case of increasing preferences and copyable activities with respect to mir-aggregation functions $f$. By Theorem 4 we know that each such function $f$ is manipulable in the case of two copyable activities. However, in the case of only one copyable activity, again there is an individually rational assignment that assigns each agent to one of her top-ranked alternative\(^2\). In fact, it is not hard to see that $f$ is strategyproof if and only if it outputs such an assignment for each such instance.

Theorem 7 In GASP, when all agents have increasing preferences and the set of activities consists of one copyable activity, a mir-aggregation function $f$ is strategyproof if and only if $f$ always outputs an assignment that assigns each agent to one of her top-ranked alternatives.

5 Manipulability of Correspondence $C$: Dedicated Extensions

In this section, we analyse the aspect of strategic manipulation of correspondence $C$ with respect to the preference extensions introduced in Section 3. An overview over the results is given in Table 1 (in the table, “sp” means strategyproof and “man” means manipulable).

| preferences, activities | Kelly/Gärdenfors/Fishburn | Maxi-max | Maxi-min |
|-------------------------|---------------------------|----------|----------|
| decreasing, 1 simple    | man                       | man      | sp       |
| decreasing, 1 copyable  | man                       | sp       | man      |
| decreasing, 2 simple    | man                       | man      | man      |
| decreasing, 2 copyable  | man                       | sp       | man      |
| increasing, 1 simple    | sp                        | sp       | sp       |
| increasing, 1 copyable  | man                       | sp       | man      |
| increasing, 2 simple/copyable | man               | man      | man      |

Table 1 Overview over the results regarding manipulability of correspondence $C$ w.r.t. different preference extensions.

5.1 Maxi-max & Maxi-min Manipulability

Theorem 8 In GASP, when all agents have decreasing preferences and $A^\ast$ consists of one simple activity, then the mir-aggregation correspondence $C$ is maxi-min strategyproof, but maxi-max manipulable.

\(^2\) assuming that the profile of the considered instance contains only alternatives $(a, k)$ approved by at least $k$ agents.
Proof Maxi-max manipulability follows from Example 2.

Assume that $C$ is maxi-min manipulable, i.e., there are instances $(N, A, P)$, $(N, A, P')$ and $i \in N$ such that $P'|_{N \setminus \{i\}} = P|_{N \setminus \{i\}}$, agent $i$'s true preferences are $\Pi$, but she is better off with misreporting her true preferences in terms of $V_i$. Since the preferences must be decreasing and $A^* = \{a\}$, that means that in $P'$, agent $i$ can only misreport by ranking approved alternatives below $a_{\emptyset}$, i.e., removing alternatives from her approval set. I.e., if agent $i$'s true preferences yield $S_i = \{a\} \times [1, n_i]$, then $S_i' = \{a\} \times [1, n_i']$ with $n_i' < n_i$. Let $\pi$ be a maximum individually rational assignment in $(N, A, P)$. We distinguish two cases.

Case I: $\#(\pi) \leq n_i'$. Then, $\Pi = \Pi'$ must hold, and hence by definition $C$ is strategyproof.

Case II: $\#(\pi) > n_i'$. If, for a maximum individually rational assignment $\pi'$ in $(N, A, P')$, $\#(\pi') > n_i'$ holds, then $\pi'(i) = a_{\emptyset}$ follows, and $i$ is not better off in $(N, A, P')$. Thus, $\#(\pi') \leq n_i'$ holds. Since we have $\#(\pi) > n_i'$ and $P'|_{N \setminus \{i\}} = P|_{N \setminus \{i\}}$ where all agents have decreasing preferences, this means that there are $\#(\pi) - 1 \geq n_i'$ agents $j \in N \setminus \{i\}$ such that $(a, \#(\pi) - 1) \in S_j$; i.e., there is a maximum individually rational assignment $\pi^*$ in $(N, A, P')$ with $\pi^*(i) = a_{\emptyset}$. As a consequence, $\min_i \Pi' = a_{\emptyset}$. Therewith, $(\min_i \Pi', m_{\Pi'}) \succ_i (\min_i \Pi, m_{\Pi})$ cannot hold because all assignments $\pi \in \Pi$ are individually rational.

Theorem 9 In GASP, when all agents have decreasing preferences and $A^*$ consists of copyable activities, then the mir-aggregation correspondence $C$ is maxi-max strategyproof, but maxi-min manipulable even if $A^*$ consists of only one copyable activity.

Proof Maxi-max Strategyproofness: Let $I = (N, A, P)$ and $I' = (N, A, P')$ such that for some $i \in N$, $P'|_{N \setminus \{i\}} = P|_{N \setminus \{i\}}$ and $V_i' \neq V_i$ holds. Let $k := \#(\pi)$ for $\pi \in \Pi$. By (1), $\#(\pi') \leq k$ holds for $\pi' \in \Pi'$. If $\pi'(i) = a_{\emptyset}$ for all $\pi' \in \Pi'$, we cannot have $\Pi' \succ_i^{\max} \Pi$ since each $\pi \in \Pi$ is individually rational. Assume $\pi'(i) = a$ with $a \neq a_{\emptyset}$ for some $\pi' \in \Pi'$. Take an arbitrary $\pi \in \Pi$. Let $J := \{g \in N \setminus \{i\} | \pi(g) \neq a_{\emptyset}\}$. Because the agents have decreasing preferences (and the activities are copyable), the assignment $\pi^*$ with $\pi^*(j) = \pi(j)$ for $j \in J$ and $\pi^*(i) = b$ for a copy $b$ of $a$ is individually rational. Observe that $\#(\pi^*) \geq k$ and hence $\#(\pi^*) = k$ follows. By (1), however, every individually rational assignment (w.r.t. $P'$) in $I'$ is individually rational (w.r.t. $P$) in $I$. Hence, from $\#(\pi^*) = k$ we can conclude that $\Pi' \subseteq \Pi$ follows. Hence, $\Pi' \succ_i^{\max} \Pi$ cannot be satisfied in this case either. Therewith, $C$ is maxi-max strategyproof.

Maxi-min Manipulability: Consider instance $I = (N, A, P)$ of GASP with $N = \{1, 2\}$ and $A^* = \{a\}$ where $a$ is copyable. Let $V_1 = V_2 = (a, 1) \succ (a, 2)$ for all $a_{\emptyset}$. There are two maximum individually rational assignments in $I$, $\pi$ with $\pi(1) = \pi(2) = a$ and $\pi'$ with $\pi'(1) = a$ and $\pi'(2) = b$ for a copy $b$ of $a$. If agent 1 reports $V_1' = (a, 1) \succ a_{\emptyset}$ (ceteris paribus), then in the new instance $I'$ assignment $\pi'$ is the only maximum individually rational assignment. By $(\min_i \Pi', m_{\Pi'}) = (a, 1) \succ (a, 2) = (\min_i \Pi, m_{\Pi})$ it follows that $C$ is maxi-min manipulable.
Theorem 10 In GASP, the mir-aggregation correspondence $C$ is maxi-max and maxi-min manipulable, even when all agents have decreasing preferences and $A^*$ consists of two simple activities.

Proof The theorem follows from Theorem 8 and Example 4 respectively.

In the case of increasing preferences, for every extension $\varepsilon$, $C$ is $\varepsilon$-strategyproof in the case of one simple activity (by Theorem 3), while $C$ is $\varepsilon$-manipulable in the case of two simple or copyable activities (by Theorem 4). For the case of increasing preferences and one copyable activity we get the following result.

Theorem 11 In GASP, when all agents have increasing preferences and $A^*$ consists of one copyable activity, then the mir-aggregation correspondence $C$ is maxi-max strategyproof, but maxi-min manipulable.

Proof Maxi-max Strategyproofness: Assume that $C$ is maxi-max manipulable. Then, there are instances $I := (N, A, P), I' = (N, A, P')$, where for some $i \in N$ $P' \setminus [N \setminus (i)] = P \setminus [N \setminus (i)]$ holds, such that $(\max, P', m^*_P) >_i (\max, P, m^*_P)$. Let $S_i = \{a\} \times [n_i, n]$ and $S'_i = \{a\} \times [n'_i, n]$. By (1), $n'_i > n_i$ holds. Thus, every individually rational assignment (w.r.t. $P'$) in $I'$ is individually rational (w.r.t. $P$) in $I$. This implies $\#(\pi') \leq k$ for $\pi' \in I'$, where $k := \#(\pi)$ for $\pi \in I$.

If $\#(\pi') = k$, then $I' \subseteq I$ follows. I.e., $C$ is not maxi-max manipulable.

If $\#(\pi') < k$, we distinguish whether (i) $k \geq n'_i$ or (ii) $k < n'_i$ holds. If (i) holds, then
\[
\{j \in N|(a, k) \in S_j\} = \{j \in N|(a, k) \in S'_j\}
\]
follows. Because we have increasing preferences, it holds that $(a, k) \in S_j$ for each $j \in N$ with $\pi(j) = a$ (for arbitrary $\pi \in I$). Hence, with $\#(\pi) = k$, (2) implies that in $I'$ there are $k$ agents who approve of $(a, k)$. I.e., $\#(\pi') \geq k$ follows, in contradiction with our assumption.

Thus, (ii) $k < n'_i$ holds. But then from $\#(\pi') \leq k$, $\pi'(i) = a_0$ immediately follows for all $\pi' \in I'$. However, $a_0 >_i (\pi(i), |\pi(i)|)$ cannot hold by the individual rationality of $\pi$, in contradiction with our assumption that $C$ is maxi-max manipulable.

Maxi-min Manipulability: Consider instance $I := (N, A, P)$ of GASP with $N = \{1, 2\}$ and $A^* = \{a\}$, with a copyable activity $a$. Let $V_1 = V_2 = (a, 2) >_1 (a, 1) >_1 a_0$, i.e., $S_1 = S_2 = \{(a, 1), (a, 2)\}$. The two maximum individually rational assignments for $(N, A, P)$ are $\lambda, \mu$ with $\lambda(1) = a, \lambda(2) = b$ for a copy $b$ of $a$, and $\mu(1) = \mu(2) = a$.

In $I' = (N, A, P')$, for agent 1 we have $V'_1 = (a, 2) >_1 a_0$, while $V'_2 = V_2$. Then, $\Pi' = \{\mu\}$. Hence, $(a, 2) >_1 (a, 1)$ implies that $(\min, \Pi', m^*_P) = (a, 2) >_1 (a, 1) = (\min, \Pi, m^*_P)$; i.e., $C$ is maxi-min manipulable.

5.2 Kelly/Gärdenfors/Fishburn Manipulability

In this section, by means of the Gärdenfors extension we consider the issue of manipulability with respect to the Kelly-, Gärdenfors-, and Fishburn extension
Finally, we consider strategic manipulation in connection with two natural GASP variants of GASP: the mir-aggregation correspondence $C$ is Kelly manipulable (and hence Gärdenfors and Fishburn manipulable) even when all agents have decreasing preferences and $A^*$ consists of one simple activity. As it turns out, this is also the case when $A^*$ consists of one copyable instead of one simple activity.

**Theorem 12** In GASP, when all agents have decreasing preferences and $A^*$ consists of one copyable activity, then the mir-aggregation correspondence $C$ is Gärdenfors manipulable.

Proof Consider instance $\mathcal{I} = (N, A, P)$ of GASP with $N = \{1, 2\}$ and $A^* = \{a\}$, where $a$ is copyable. Let $V_1 = V_2 = (a, 1) \succ_1 (a, 2) \succ_1 a\emptyset$. In $\mathcal{I}$, there are two maximum individually rational assignments $\lambda, \mu$: $\lambda(1) = \lambda(2) = a$, and $\mu(1) = a, \mu(2) = b$ for a copy $b$ of $a$. I.e., $\Pi = \{\lambda, \mu\}$ with $(\lambda(1), |\lambda_1|) = (a, 2)$, whereas $(\mu(1), |\mu_1|) = (a, 1)$. Now, in $\mathcal{I}' = (N, A', P')$, agent $1$ “drops” $(a, 2)$ and reports $V_1' = (a, 1) \succ_1 a\emptyset$ only, while $V_2' = V_2$. As a consequence $\Pi' = \{\mu\}$, i.e., $\mu$ is the only maximum individually rational assignment in $\mathcal{I}'$. Since $\Pi' \subset \Pi$ it suffices to check the first condition of Gärdenfors manipulability, which is satisfied since $(a, 1) \succ_1 (a, 2)$ holds, where $\Pi' = \{\mu\}$ and $\Pi \setminus \Pi' = \{\lambda\}$.

In addition, for the case of increasing preferences and one copyable activity we can state the following theorem.

**Theorem 13** In GASP, when all agents have increasing preferences and $A^*$ consists of one copyable activity, then the mir-aggregation correspondence $C$ is Gärdenfors manipulable.

Proof Consider instance $\mathcal{I} = (N, A, P)$ of GASP with $N = \{1, 2\}$ and $A^* = \{a\}$, for a copyable activity $a$. Let $V_1 = V_2 = (a, 2) \succ_1 (a, 1) \succ_1 a\emptyset$. Then, $\Pi = \{\lambda, \mu\}$ with $\lambda(1) = a, \lambda(2) = b$ for a copy $b$ of $a$, and $\mu(1) = \mu(2) = a$. In $\mathcal{I}' = (N, A', P')$, we have $V_1' = (a, 2) \succ_1 a\emptyset$, while $V_2' = V_2$. Then, $\Pi' = \{\mu\}$. With $\Pi \setminus \Pi' = \{\lambda\}$ and $(a, 2) \succ_1 (a, 1)$ it follows that $C$ is Gärdenfors manipulable.

6 Manipulability in variants of GASP

Finally, we consider strategic manipulation in connection with two natural variants of GASP: the approval voting scenario of a-GASP (see Darmann et al. (2012)) and the strict rankings setting of o-GASP (see Darmann (2015)). In the variant o-GASP, the agents’ preferences are strict orders over $X$. In a-GASP, each agent $i$’s preferences are trichotomous: for each $a \in A^*$, $a \succ_i a\emptyset$ or $a \preceq_i a\emptyset$ holds; in addition, $a \sim_i b$ holds for any two alternatives $a, b \in S_i$, and for any two alternatives $a, b \in A^* \setminus S_i$. In a-GASP, a manipulation of $C$ w.r.t. any of the considered preference extensions is impossible. To verify this, consider the following three cases. Firstly, assume that in instance $\mathcal{I}$ all maximum individually rational assignments assign agent $i$ to a non-void activity; since each alternative in $S_i$ is considered
equally good by agent \(i\), she cannot be better off by misreporting her preferences. Secondly, assume that in instance \(I\) all maximum individually rational assignments assign agent \(i\) to \(a_0\). Clearly, there is no chance for \(i\) to manipulate in this case as well, because a maximum individually rational assignment assigning \(i\) to a non-void activity in the “manipulated” instance \(I'\) must assign \(i\) to an alternative disapproved by \(i\) in her true preferences. Thirdly, assume that in instance \(I\) some maximum individually rational assignments assign \(i\) to \(a_0\) while others assign \(i\) to a non-void activity. For any maximum individually rational assignment \(\pi\) that assigns agent \(i\) to \(a_0\), \(\pi\) must be a maximum individually rational assignment in the “manipulated” instance \(I'\) as well; in particular, \(I' \subseteq \Pi\) holds. In addition, there is an assignment \(\lambda \in \Pi\) such that \((\lambda(i), |\lambda_i|) \succ_i a_0\) holds. It is not hard to verify that the two latter facts imply \(I' \not\subseteq \Pi\), for any of the considered extensions \(\varepsilon\).

Turning to \(o\)-GASP, obviously the strategyproofness results for GASP presented in the previous sections also hold for \(o\)-GASP. In addition, all but one of the special cases of GASP in which \(C\) was shown to be manipulable actually used instances of \(o\)-GASP, and hence these results hold for the latter variant as well. In Example 2, however, which shows that \(C\) is Kelly-/Gärdenfors-/Fishburn manipulable in GASP with decreasing preferences and one simple activity the agent’s preferences are not strict.

As it turns out, in contrast to the above result for the general setting of GASP, if the agents’ preferences are strict, then in the case of one simple activity and decreasing preferences \(C\) is Kelly-/Gärdenfors-/Fishburn strategyproof. However, when a second simple activity is added, \(C\) is manipulable.

**Theorem 14** In \(o\)-GASP, when all agents have decreasing preferences and \(A^*\) consists of one simple activity, then the mir-aggregation correspondence \(C\) is Gärdenfors strategyproof.

**Proof** Assume the opposite. Let \(I = (N, A, \Pi)\) and \(I' = (N, A, \Pi')\), with \(P_a|_{N \setminus \{i\}} = P'_a|_{N \setminus \{i\}}\) for some \(i \in N\) such that \(I' \succ_i \Pi\) holds. Let \(A^* = \{a\}\).

Let \(k := \#(\pi)\) for \(\pi \in \Pi\). If \((a, k) \notin S_i\), then \(\pi(i) = a_0\) holds for each \(\pi \in \Pi\) and thus \(i\) cannot manipulate. Thus, \((a, k) \in S_i\) holds, which implies that there is a \(\pi \in \Pi\) with \(\pi(i) = a\). Recall that \(S'_i \subset S_i\) holds (stated in (1)). Hence, if \((a, k) \in S'_i\), then \(II = I'I\) follows, in contradiction with \(I' \succ_i \Pi\).

Therefore,

\[
(a, k) \notin S'_i
\]  

(3)

holds.

Case I: \(\pi(i) \neq a_0\) for all \(\pi \in \Pi\). Then, \(|\Pi| = 1\) follows (otherwise, there are at least \(k + 1\) agents who approve of \((a, k)\), and hence an assignment of size \(k\) which assigns \(i\) to the void activity exists). Thus, the assignment \(\pi'(j) = \pi(j)\) for \(j \neq i\) and \(\pi^*(i) = a_0\) must be a maximum individually rational assignment in \(I'\), with \(#(\pi^*) = k - 1\). As a consequence, \(I' \cap \Pi = \emptyset\) holds. Thus, the last condition of Gärdenfors manipulability needs to be checked in this case, which entails that \(a_0 \succ_i (\pi(i), |\pi_i|)\) holds for \(\pi \in \Pi\). This, however, is clearly not satisfied because \(\pi\) is individually rational.
Case II: There is a \( \pi \in \Pi \) such that \( \pi(i) = a_\emptyset \). Then, \( \#(\pi') = k \) for \( \pi' \in \Pi' \) follows. Due to \( (a, k) \notin S' \) (stated in (3)), we can conclude that for all \( \pi' \in \Pi' \) we have \( \pi'(i) = a_\emptyset \). In particular, \( \Pi' \subseteq \Pi \) follows. Obviously, we can assume \( \Pi' \subset \Pi \), since \( \Pi' \subseteq \Pi \) contradicts with Gärdenfors manipulability. Thus, we need to check the first condition of Gärdenfors manipulability, which here requires \( a_\emptyset \preceq_i (\pi(i), |\pi|) \) for all \( \pi \in \Pi \setminus \Pi' \). Since there is an assignment \( \pi \in \Pi \setminus \Pi' \) with \( \pi(i) = a \) and \( \#(\pi) = k \), due to the fact that \( i \)'s preferences are strict orders over \( X \) this would require \( a_\emptyset \succ_i (a, k) \) which contradicts with the individual rationality of \( \pi \in \Pi \). ■

**Theorem 15** In o-GASP, when all agents have decreasing preferences and \( A^* \) consists of two simple activities, then the mir-aggregation correspondence \( C \) is Gärdenfors manipulable.

**Proof** Consider the instance \( \mathcal{I} = (N, A, P) \) of GASP with \( N = \{1, 2\} \), \( A^* = \{a_1, a_2\} \), with \( V_1 = (a_2, 1) \succ_1 (a_2, 2) \succ_1 (a_1, 1) \succ_1 a_\emptyset \) and \( V_2 = (a_2, 1) \succ_2 (a_2, 2) \succ_2 a_\emptyset \). Then, \( \Pi = \{\lambda, \mu\} \) with \( \lambda(1) = a_1 \) and \( \lambda(2) = a_2 \), and \( \mu(1) = a_2 \) and \( \mu(2) = a_\emptyset \). In \( \mathcal{I}' = (N, A, P') \), let \( V'_2 = V_2 \) but agent 1 “drops” \((a_1, 1)\): \( V'_1 = (a_2, 1) \succ_1 (a_2, 2) \succ_1 a_\emptyset \). Then, \( \Pi' = \{\mu\} \). Again, due to \( \Pi' \subset \Pi \) it suffices to check the first condition of Gärdenfors manipulability, which is satisfied since \( (a_2, 2) \succ_1 (a_1, 1) \) holds, where \( \Pi' = \{\mu\} \) and \( \Pi \setminus \Pi' = \{\lambda\} \). ■

7 Conclusion

We have analyzed the aspect of strategic manipulation in the group activity selection problem when the goal is to find maximum individually rational assignments. For the case of the mir-aggregation correspondence \( C \) (all maximum individually rational assignments of an instance are determined) we have given a detailed analysis with respect to prominent representatives of preference extensions. We have shown that in some special cases of GASP \( C \) is strategyproof for each preference extension, while in others \( C \) is manipulable irrespective of the preference extension considered. In addition, in some cases a strategyproof mir-aggregation function (that outputs a specific maximum individually rational assignment for each instance) does not exist, while in others each mir-aggregation function turns out to be strategyproof. An interesting direction for future research could be to characterize preference extensions and corresponding domains for which \( C \) is strategyproof.

Acknowledgements.

The author is grateful to Jérôme Lang and Christian Klamler for fruitful discussions.
References

Alcalde, J., & Revilla, P. 2004. Researching with whom? Stability and manipulation. *Journal of Mathematical Economics*, 40(8), 869–887.

Barberà, S. 2010. Strategy-proof social choice. Chap. 25, pages 731–832 of: Arrow, K., Sen, A., & Suzumura, K. (eds), *Handbook of Social Choice and Welfare*, vol. 2. Elsevier.

Barberà, S., Dutta, B., & Sen, A. 2001. Strategy-proof Social Choice Correspondences. *Journal of Economic Theory*, 101(2), 374 – 394.

Barberà, S., Bossert, W., & Pattanaik, P.K. 2004. Ranking Sets of Objects. Pages 893–977 of: Barberà, S., Hammond, P.J., & Seidl, C. (eds), *Handbook of Utility Theory*. Springer.

Brandt, F., & Brill, M. 2011. Necessary and Sufficient Conditions for the Strategyproofness of Irresolute Social Choice Functions. Pages 136–142 of: *Proceedings of the 13th Conference on Theoretical Aspects of Rationality and Knowledge*. TARK XIII.

Brandt, F., & Geist, C. 2014. Finding Strategyproof Social Choice Functions via SAT Solving. Pages 1193–1200 of: *Proceedings of the 2014 International Conference on Autonomous Agents and Multi-agent Systems*. AAMAS '14.

Cechlarova, K., & Romero-Medina, A. 2001. Stability in coalition formation games. *International Journal of Game Theory*, 29, 487–494.

Darmann, A. 2015. *Algorithmic Decision Theory: 4th International Conference, ADT 2015, Lexington, KY, USA, September 27-30, 2015, Proceedings*. Springer International Publishing. Chap. Group Activity Selection from Ordinal Preferences, pages 35–51.

Darmann, A., Elkind, E., Kurz, S., Lang, J., Schauer, J., & Woeginger, G. 2012. Group Activity Selection Problem. Pages 156–169 of: *Proceedings of the 8th International Conference on Internet and Network Economics*. WINE’12. Berlin, Heidelberg: Springer-Verlag.

Fishburn, P.C. 1972. Even-chance lotteries in social choice theory. *Theory and Decision*, 3(1), 18–40.

Gärdenfors, P. 1976. Manipulation of social choice functions. *Journal of Economic Theory*, 13, 217–228.

Gibbard, A. 1973. Manipulation of Voting Schemes: A General Result. *Econometrica*, 41(4), 587–601.

Kelly, J.S. 1977. Strategy-Proofness and Social Choice Functions without Single-Valuedness. *Econometrica*, 45(2), 439–446.

Lee, H., & Shoham, Y. 2015. Stable Invitations. Pages 965–971 of: *Twenty-Ninth AAAI Conference on Artificial Intelligence*. Association for the Advancement of Artificial Intelligence.

Moretti, S., & Tsoukiàs, A. 2012. Ranking Sets of Possibly Interacting Objects Using Shapley Extensions. In: Brewka, G., Eiter, T., & McIlraith, S. (eds), *Principles of Knowledge Representation and Reasoning: Proceedings of the Thirteenth International Conference, KR 2012, Rome, Italy, June 10-14, 2012*. AAAI Press.
Papai, S. 2004. Unique stability in simple coalition formation games. *Games and Economic Behavior, 48*(2), 337–354.

Rodríguez-Álvarez, C. 2009. Strategy-proof coalition formation. *International Journal of Game Theory, 38*(3), 431–452.

Satterthwaite, M. 1975. Strategy-proofness and Arrow’s conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory, 10*(2), 187–217.

Sönmez, Tayfun. 1999. Strategy-proofness and Essentially Single-valued Cores. *Econometrica, 67*(3), 677–689.