Restriction of Laplace-de Rham operator on one-forms: from $\mathbb{R}^{n+2}$ and $\mathbb{R}^{n+1}$ ambient spaces to embedded (A)dS$_n$ submanifolds

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The Laplace-de Rham operator acting on a one-form $a$: $\Box a$, in $\mathbb{R}^{n+2}$ or $\mathbb{R}^{n+1}$ spaces is restricted on $n$-dimensional pseudo-spheres. This includes, in particular, the $n$-dimensional de Sitter and Anti-de Sitter space-times. The restriction is designed to extract the corresponding $n$-dimensional Laplace-de Rham operator acting on the corresponding $n$-dimensional one-form on pseudo-spheres. Explicit formulas relating these two operators are given in each situation. The converse problem, of extending an $n$-dimensional operator composed of the sum of the Laplace-de Rham operator and additional terms to ambient spaces Laplace-de Rham operator, is also studied. We show that for any additional term this operator on the embedded space is the restriction of Laplace-de Rham operator on the embedding space.

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I. INTRODUCTION

This paper deals with the restriction from different embedding spaces ($\mathbb{R}^{n+2}$ and $\mathbb{R}^{n+1}$) to the $n$-dimensional de Sitter (dS) and Anti-de Sitter (AdS) space-times of the Laplace-de Rham operator, hereafter generically denoted by $\Box$, acting on one-form or scalar fields.

Embedding curved space-times in flat higher dimensional space is an old and still pursued topic (see [1] for a wealth of references and [2–5] for current endeavors), often with an interest in the description of geodesics and determination of Hawking temperature. Here, on the contrary, the present work is focused on the restriction of differential operators from the embedding space to the space-time (the embedded submanifold). Such an approach has been used in the study of quantum field theories from an ambient space point of view [6–22]. It has also been already used in the case of quantum mechanics constrained to a surface in which the restricted (scalar) laplacian shows additional terms due to the embedding [23–26].

The lowering of dimensionality, between embedding and embedded spaces, leads to constraints or additional degrees of freedom for fields on which operators act, depending on the situation. For instance, the restriction of $\Box A$, $A$ being a vector field, to a lower dimensional embedded manifold, implies that the number of components of $A$ has to be reduced. For the restricted operator, these “additional” components become “additional” degrees of freedom, they may be used to impose conditions such as, for instance, transversality or homogeneity on the field. This kind of technique is of common use when the restricted operator is already known. See, in particular, the work initiated by Dirac [6, 7], extended by Fronsdal [8–13] and continued by others [14–22], where one builds in the embedding space a tensor calculus isomorphic to that of the embedded space.

Here, we take a different approach, no conditions are imposed on the one-form or scalar fields in the embedding space (neither transversality nor homogeneity), and the restricted operator retains its most general form. To summarize, an explicit relation between the Laplace-de Rham operator on the embedding space and that on the embedded space is proved for both $n+1$ and $n+2$ dimensional cases. Precisely, the restricted operator is decomposed into two parts, the former being the Laplace-de-Rham operator of the considered pseudo-sphere, the latter contains terms involving intrinsic geometrical structures (normal derivative to the manifold, Lie derivative along dilation vectors, . . . ) and the field over the

1 Such an ambient space formalism has been rediscovered recently in [27], seemingly unknowingly of a body of work spanning more than thirty years.
embedding space (that is unrestricted to the manifold). This last term, that we shall denote by AT (for “Additional Term”) in the sequel, is thus determined through

\[(\Box_d\alpha)_\Sigma = \Box_\Sigma\alpha_\Sigma + AT,\]

where \(\Box_d\) is the Laplace-de-Rham operator on the \(d\)-dimensional embedding space, \(\Sigma\) is the embedded space, \(\Box_\Sigma\) is the Laplace-de-Rham operator on \(\Sigma\), and \((\Box_d\alpha)_\Sigma,\ \alpha_\Sigma\), are restrictions, in a sense to be precised later, of respectively \(\Box_d\alpha\) and \(\alpha\).

The generic form of AT encompass many known situations as, in particular, the one where the restricted operator corresponds to a one-form (or scalar) field which belongs to an unitary irreducible representations of the (A)dS group \([28]\). Moreover, this form allows us to draw general conclusions on the freedom left in the interplay between operators in embedding and embedded spaces. Precisely, the geometry of the embedding being given, the additional term AT appearing in the restriction from the embedding space to the embedded pseudo-sphere is completely determined by the field. Conversely, if one considers on the embedded pseudo-sphere an operator \(\Box_\Sigma\beta + \chi(\beta)\) where \(\beta\) is a scalar field or a 1-form on \(\Sigma\), and \(\chi\) is any expression depending (or not) on \(\beta\) we will show that there always exists an extension \(\varpi(\beta)\) of the field \(\beta\) to the embedding space for which AT is \(\chi\), explicitly:

\[(\Box_d\varpi(\beta))_\Sigma = \Box_\Sigma\beta + \chi(\beta).\]

There is therefore no constraint on the additional term which can be any desired expression.

The considered embedding spaces \(\mathbb{R}^{n+2}\) (respectively \(\mathbb{R}^{n+1}\)) are the \(n + 2\) (respectively \(n + 1\)) dimensional real vector spaces endowed with the metric \(\eta\) left invariant under the conformal group SO(2, \(n\)) (respectively the isometrical groups SO(1, \(n\)) or SO(2, \(n-1\)) depending on whether we consider dS or AdS space-time). Note that, we consider the embedding space \(\mathbb{R}^{n+2}\) as our main goal to allow us further generalizations (see the conclusion Sec. VII). In this respect, the embedding space \(\mathbb{R}^{n+1}\) appears as an intermediate step.

The paper is organized as follows. In Sec. II we summarize the steps and the methods used to derive the expression of the restricted operators, Sec. III gives the expression of the Laplace-de Rahm operators on one-form and scalars in terms of general frames and anholonomy coefficients. The restriction from \(n + 1\)-dimensional space is considered in Sec. IV, it is generalized to the restriction from \(n + 2\)-dimensional space in Sec. V. The relation between restriction and extension is examined in Sec. VI. In Sec. VII we conclude and
consider the possibility of the extension of this work to other space-times. Some frequently used formulas are collected in appendix A1, the calculation of the co-differential of a co-frame $\delta(e^a)$, used in particular in Sec. III, is detailed in appendix A2. The calculations of the formulas stated in the paper are detailed in A3–A7.

Throughout the paper except otherwise specified $a, b, c$ denote abstract indexes, while $\mu, \nu, \ldots = 0, \ldots, n - 1$, and $A, B, \ldots = 0, \ldots, n$ and, finally, $\alpha, \beta, \ldots = 0, \ldots, n + 1$ are related, respectively, to $\Sigma_n$, $\mathbb{R}^{n+1}$, and $\mathbb{R}^{n+2}$. The canonical coordinates of a point $y$ of $\mathbb{R}^{n+2}$ (respectively $\mathbb{R}^{n+1}$) are denoted $\{y^a\}$ (respectively $\{y^4\}$) and their associated Cartesian orthonormal basis is denoted $\{\partial_a\}$ (respectively $\{\partial_A\}$).

II. GEOMETRIC CONTEXT AND METHOD

We proceed in three main steps. We first express the Laplace-de Rham operator on a $d$-dimensional pseudo-Riemannian oriented manifold $(M, g)$, in terms of any orthonormalized (local) field of frames and their associated anholonomy coefficients.

The $n + 1$-dimensional case is then tackled. The metric manifold $(M, g)$ is here the embedding space $\mathbb{R}^{n+1}$ endowed with a pseudo-metric $\eta$. We define specific frames of $\mathbb{R}^{n+1}$ adapted to any sub-manifold of $M$. They are built from orthonormal frames of $M$ completed by its outgoing normal, and extended to the embedding space. Anholonomic coefficients of these frames are then computed for the pseudo-sphere. The expressions of, both scalar and one-form, Laplace-de Rham operator obtained before are then particularized to these frames and coefficients and then restricted to the $n$-dimensional pseudo-sphere. The resulting expressions are rewritten using intrinsic, frame independent, quantities.

The restriction of the Laplace-de Rham operator to $n$-dimensional (A)dS space from the embedding $\mathbb{R}^{n+2}$ space is finally considered. The (A)dS space is obtained as the intersection of the null cone of $\mathbb{R}^{n+2}$ and an $n + 1$-dimensional hyper-plane. In that hyper-plane, the (A)dS space is a pseudo-sphere and we recover the geometry of the $\mathbb{R}^{n+1}$ case. The frames designed for the restriction in the $\mathbb{R}^{n+1}$ situation are extended to frames adapted to the $\mathbb{R}^{n+2}$ situation. The restriction of operators is performed in complete analogy to the $\mathbb{R}^{n+1}$ case. Here again, the restricted operators are re-expressed under an intrinsic form.
III. EXPRESSION OF $\square a$ AND $\square \phi$ IN ORTHO-NORMAL FRAMES

Let $(M, g)$ a $d$-dimensional metric pseudo-Riemannian manifold endowed (at least locally) by a (pseudo-) ortho-normalized frame $e_a$, $a = 1, \ldots, d$, with structure coefficients $c^a_{bc}$ which satisfy

$$[e_a, e_b] = c^c_{ab} e_c$$

and $de^a = -\frac{1}{2} c^a_{be} e^b \wedge e^c$.

Then, on a one-form or a scalar field, $a$ and $\phi$ respectively, the Laplace-de Rham operator reads

$$\square a = \eta^{bc} e_c(e_b(a_a)) + e^c e_c(a^b) + \eta^{bc} e_c(e_b(a_d)) - \eta^{bd} \eta_{mn} e^c_{mn} e_c(a^c)$$

$$+ \eta^{bd} c^p_{ac} e_c(a_a) - \eta^{bd} a_c e_b(c^p_{da}) - \eta^{bd} a_c e_b(c^p_{ca}) + a^b e_a(c^p_{ab})$$

$$- \frac{1}{2} a_c \eta^{mf} \eta^{bd} \eta_{mn} e^c_{mn} e_f e^c_{da} e^a,$$  \hspace{1cm} (1)

and

$$\square \phi = \eta^{ab} [e_a e_b(\phi) + e_a(\phi) e^p_b].$$  \hspace{1cm} (2)

The second of these two equalities is obtained by a straightforward calculation detailed in appendix A 3. Note that the derivation of Eq. (2) is useful for that of Eq. (1). The derivation of this last expression results from a direct, but more cumbersome, calculation detailed in appendix A 4.

IV. RESTRICTION FROM $\mathbb{R}^{n+1}$

A. Adapted frames

Let $M_n$ be any $n$-dimensional submanifold isometrically embedded in some oriented $d$-dimensional pseudo-euclidian space $E$. In this situation, for our purpose, it will be convenient to use a so-called local adapted frame, that is to say a field of frames of $E$ adapted to this embedding. This is defined as follows: let $Q$ be a point of $M_n$ and $U_Q$ a $E$-neighborhood of $Q$. A field of positively oriented ortho-normal frames of $E$: $e_A$, $A = 0, \ldots, d - 1$ on $U_Q$ is said to be adapted to $M_n$ iff at each point of $V_Q := M_n \cap U_Q$, $e_\mu, \mu = 0, \ldots, n - 1$ is a local field of ortho-normal frames of $M_n$. Note that, consequently, $e_n, \ldots, e_{d-1}$ are orthogonal to $M_n$. 
In this section, we restrict $M_n$ to be the $n$-dimensional pseudo-sphere denoted $\Sigma_n$ in $\mathbb{R}^{n+1}$ and defined through

$$y^2 := y^A y_A = -\epsilon H^{-2},$$

$H$ being a positive constant and $\epsilon = \pm 1$. We assume that $\mathbb{R}^{n+1}$ is oriented and that $\Sigma_n$ is oriented thanks to its outer normal. In the following, we focus on the de Sitter space obtained with $\epsilon = 1$, the signature of $E$ being $(1, n)$, and the Anti-de Sitter space obtained with $\epsilon = -1$, the signature of $E$ being $(2, n-1)$. Nevertheless the final results (5) and (6) of this section, concerning the $\mathbb{R}^{n+1}$ embedding, remain valid in other cases and, in particular, for the euclidean sphere $S_n$ obtained with $\epsilon = -1$, the signature of $E$ being $(n+1, 0)$. This is no more the case concerning the $\mathbb{R}^{n+2}$ embedding.

A set of adapted frames of $\Sigma_n$ can be constructed on $\mathbb{R}^{n+1}$ as follows. In a neighborhood (in the sense of $\Sigma_n$) $V_Q$ of some point $Q$ of $\Sigma_n$, let $e_\mu$ be a field of direct ortho-normal frames of $\Sigma_n$ defined at each point of $V_Q$. This field of frames is then extended to a field on $\Sigma_n$ of frames of $\mathbb{R}^{n+1}$ by adding the unit outer normal to $\Sigma_n$: $e_n$ which is, from the very definition of $M_n$, non isotropic. Then, this whole field of frames $e_A$ is extended to each point of the cone $U_Q = C(V_Q)$ of $\mathbb{R}^{n+1}$, which is the union of all the open half-lines coming from the origin of $\mathbb{R}^{n+1}$ and crossing $V_Q$. This is done by moving each frame identically to itself along the half-lines that intercept the origin of the frame. To this end, each component, in the canonical basis of $\mathbb{R}^{n+1}$, of a vector that belongs to the frame of $U_Q$ is required to be homogeneous of degree zero. Namely, the $e_B^A$, defined through $e_A = e_B^A \partial_B$, are homogeneous of degree zero and, more explicitly, since for $y \in U_Q$, we have $y/\sqrt{|y|^2} \in V_Q$ we define $e_B^A(y)$ through $e_B^A(y) = e_B^A(y/\sqrt{|y|^2})$. We note that the unit outer normal explicitly reads

$$e_n = \frac{1}{\sqrt{|y|^2}} D,$$

where $D$ is the dilation vector $D = y^A \partial_A$. We finally remark, that these choices lead to $\eta_{nn} = \eta^{nn} = \text{sgn}(y^2) = -\epsilon$ (in a neighborhood of $Q$).

**B. Coefficients of anholonomy**

We now compute the coefficients of anholonomy in the field of adapted frames defined Sec. IV A using the general relation $[e_b, e_c] = c_{bc}^a e_a$. 

We first note that since \( \{e_\mu\} \) is a frame of \( \Sigma_n \), a sub-manifold of \( \mathbb{R}^{n+1} \), one has \( [e_\mu, e_\nu] = c^{\lambda}_{\mu\nu} e_\lambda \). A first consequence is that the coefficients with Greek-index only do not change when restricted to \( \Sigma_n \); they will appear inside the Laplace-de Rham operator on \( \Sigma_n \) only and there is no need to compute them explicitly, a second consequence is that \( c^{\mu\nu}_{n} = 0 \).

To obtain \( c^{\nu}_{\mu n} \) we recall from Sec. IV A that the coefficients \( e_\nu^\mu \) are homogeneous functions of degree zero, then on an arbitrary homogeneous function \( \varphi \) of degree \( r \) one has:

\[
[e_n, e_\mu] \varphi = \frac{1}{\sqrt{|y^2|}} De_\mu \varphi - e_\mu \frac{1}{\sqrt{|y^2|}} D \varphi
= \frac{r - 1}{\sqrt{|y^2|}} e_\mu \varphi - e_\mu \left( \frac{1}{\sqrt{|y^2|}} \right) r \varphi - \frac{1}{\sqrt{|y^2|}} r e_\mu \varphi
= -\frac{e_\mu}{\sqrt{|y^2|}} \varphi,
\]

where, in the r.h.s of the second line, we used the fact that \( e_\mu(y^2) = 0 \) since \( e_\mu(y^2) = \langle dy^2, e_\mu \rangle \).

The anholonomy coefficients do not depend on the choice of \( \varphi \), then the last line of the above calculation shows that

\[
c^{\nu}_{\mu n} = \frac{1}{\sqrt{|y^2|}} \delta_\mu^\nu \quad \text{and} \quad c^n_{AB} = 0.
\]

We note that this expression leads to \( c^A_{A n} = n/\sqrt{|y^2|} \) which corresponds to the result of Eq. (A 2).

Finally, the derivatives of the coefficients of anholonomy are obtained thanks to their homogeneity: since \( e^B_A \) is homogeneous of degree zero, \( e_A = e^B_A \partial_B, \{\partial_B\} \) being the canonical basis of \( \mathbb{R}^{n+1} \), is homogeneous of degree \( -1 \), it follows from the general relation \( [e_b, e_c] = e^a_{bc} e_a \), that \( c^A_{BC} \) are homogeneous of degree \( -1 \), consequently

\[
e_n(c^A_{BC}) = -\frac{1}{\sqrt{|y^2|}} c^A_{BC} \quad \text{and} \quad e_\lambda(c^{\nu}_{\mu n}) = 0.
\]

C. Restriction to the pseudo-sphere \( \Sigma_n \)

1. Restriction of the one-form operator

We now restrict the one-form \( \Box_{n+1} \alpha \) to the sphere \( \Sigma_n \). This is obtained thanks to the pullback \( l^* \) of the canonical injection \( l : \Sigma_n \to \mathbb{R}^{n+1} \). If \( \alpha \) is a \( p \)-form on \( \mathbb{R}^{n+1} \) we introduce, for future convenience the notation \( \alpha_\Sigma := l^* \alpha \). From the definition of an adapted frame
Sec. IV A one has $l^*e^\mu = e^\mu$ and $l^*e^n = 0$, then

$$\alpha_\Sigma = l^*\alpha = (l^*\alpha_\Lambda)l^*(e^A) = (\alpha_\mu)_\Sigma e^\mu.$$  

As a consequence, in the restriction of $\Box_{n+1} a$ from Eq. (1) only the $\mu$-component remains, the contribution to $e^n$ being mapped to zero, that is: $(\Box_{n+1} a)_\Sigma = [(\Box_{n+1} a)_\mu]_\Sigma e^\mu =: [B_\mu e^n]_\Sigma$. The r.h.s. of this last expression divides into two parts: one that contains terms with Greek indexes only, another, that we shall denote $nB$, contains terms indexed by $n$. The former leads after restriction to the Laplace-de Rham operator on $\Sigma_n$ acting on $a_\Sigma$: $\Box_\Sigma (a_\Sigma)$, the latter is calculated in appendix A 5, one finds

$$nB = -\epsilon \frac{1}{|y^2|} [D^2(a_\mu)e^\mu + (n-1)D(a_\mu)e^\mu + 2e_\mu(i_\phi a)e^\mu + (n-2)a_\mu e^\mu].$$  

Then, the restricted Laplace-de Rham operator on the one form: $(\Box_{n+1} a)_\Sigma = \Box_\Sigma a_\Sigma + nB_\Sigma$, reads

$$(\Box_{n+1} a)_\Sigma = \Box_\Sigma a_\Sigma - \epsilon H^2 [D^2(a_\mu)e^\mu + (n-1)D(a_\mu)e^\mu + 2d_\Sigma(i_\phi a) + (n-2)a_\Sigma],$$

where $d_\Sigma = l^*d$. Note that for a transverse one-form ($y^A a_A = 0$) the term $d_\Sigma(i_\phi a)$ vanishes. Also, for $a$ homogeneous of degree $r$ (thus $a_\mu$ homogeneous of degree $r - 1$) the above expression reduces to

$$(\Box_{n+1} a)_\Sigma = \Box_\Sigma a_\Sigma - \epsilon H^2 [(r+1)(r+n-2)a_\Sigma + 2d_\Sigma(i_\phi a)].$$  

Returning to the general case of non-homogeneous, non-transverse one-form $a$, one observes that $L_\phi e^A = e^A$ implying $D(a_\Lambda)e^A = L_\phi(a) - a$, and $D^2(a_\Lambda)e^A = L_\phi^2(a) - 2L_\phi(a) + a$. Making these replacements in the term $nB_\Sigma$ leads to the intrinsic form

$$(\Box_{n+1} a)_\Sigma = \Box_\Sigma a_\Sigma - \epsilon H^2 [L_\phi^2(a) + (n-3)L_\phi(a) + 2d(i_\phi a)]_\Sigma,$$

where no reference frame is involved. Note that this formula, with $\epsilon = -1$ is still valid in the case of the euclidean sphere ($\Sigma_n = S_n$) embedded in the euclidean space $\mathbb{R}^{n+1}$. This can be proved through a straightforward adaptation of our proof.

2. Restriction of the scalar operator

For completeness, we note that the scalar operator can be restricted along the same lines as the one-form. The splitting between Greek-indexed and $n$-indexed term in Eq. (2) leads
again to a first term which restricts in $\Box_\Sigma \phi_\Sigma$ and a second term which, keeping the notation of Sec. IV C 1 for simplicity, reads

$$n^B := \eta^{nn} [c^2_n(\phi) + e_n(\phi)c^A_{an}].$$

Thanks to the expressions of $\eta^{nn}, e_n$ and $c^A_{an}$ Sec. IV A-IV B one obtains

$$n^B := -\frac{\epsilon}{|y|^2} (D^2 + (n-1)D) \phi.$$ 

The restriction of the Laplace-de Rham operator then reads

$$(\Box_{n+1}\phi)_\Sigma = \Box_\Sigma \phi_\Sigma - \epsilon H^2 (D^2(\phi) + (n-1)D(\phi))_\Sigma,$$

where again no particular frame (or coordinates) are involved. Once again this formula, with $\epsilon = -1$ is still valid in the case of the euclidean sphere ($\Sigma_n = S_n$) embedded in the euclidean space $\mathbb{R}^{n+1}$.

We note that if $\phi$ is homogeneous of degree $r$ the above expression reduces to

$$(\Box_{n+1}\phi)_\Sigma = \Box_\Sigma \phi_\Sigma - \epsilon H^2 r(n + 1) \phi_\Sigma.$$ 

**V. RESTRICTION FROM $\mathbb{R}^{n+2}$**

**A. Adapted frames of $\mathbb{R}^{n+2}$**

Hereafter, $\mathbb{R}^{n+2}$ stands for the oriented pseudo-euclidean space, the signature of the metrics being $(2, n)$.

The spaces of (Anti-)de Sitter, can be obtained as the intersection $\Sigma_n$ of the null cone $C$ of $\mathbb{R}^{n+2}$ and a $n + 1$-dimensional plane $P$ defined by an homogeneous polynomial $f(y)$ of degree one. We assume that the normal vector field of $P$ is nowhere isotropic and we note $F = \sharp df$ this normal vector, and also $\epsilon = \text{sgn}(F^2)$. The plane $P$ is endowed with the metric induced by that of $\mathbb{R}^{n+2}$ and is identified with the $\mathbb{R}^{n+1}$ of the previous section $P \simeq \mathbb{R}^{n+1}$.

Without loss of generality, we consider that the canonical Cartesian coordinates basis of $\mathbb{R}^{n+2}$ is such that

$$f(y) = Hy^{n+1}, \ H > 0. \quad (7)$$

This implies that

$$F = \sharp df = \epsilon H \partial_{n+1}. \quad (8)$$
The position of $F$, relative to the null cone of $\mathbb{R}^{n+2}$, “space-like” ($F^2 < 0$) or “time-like” ($F^2 > 0$), thus determines $\Sigma_n$ to be the de Sitter or Anti-de Sitter space. Writing the $\mathbb{R}^{n+2}$ metric as $\eta = \text{diag}(+, -, \ldots, -, -\epsilon, \epsilon)$, with $\epsilon = \pm 1$, allows us to set $\Sigma_n$ to be one of these spaces. In effect, the equations for the intersection of $C$ and $P$

$$
\begin{align*}
C(y) &= y^\mu y_\mu - \epsilon (y^n)^2 + \epsilon (y^{n+1})^2 = 0, \\
f(y) &= Hy^{n+1} = 1,
\end{align*}
$$

define $\Sigma_n$ as the de Sitter, or Anti-de Sitter space for, respectively, $\epsilon = +1$, and $\epsilon = -1$.

With this convention for the metric one has

$$
D = y^\alpha \partial_\alpha = D_P + D_{n+1},
$$

where $D_P := y^A \partial_A$ is the dilation operator of $P$, that appears in the $\mathbb{R}^{n+1}$ reduction Sec. IV, and $D_{n+1} := y^{n+1} \partial_{n+1}$.

Now, a field of frames adapted to $\Sigma_n$ in $\mathbb{R}^{n+2}$ is built as follows. We first build the adapted frames to $\Sigma_n$ in $U_Q \subset P$: $\{e_\mu, e_n\}$, as described in Sec. IV A. Then, one supplements each frame $\{e_\mu, e_n\}$ by the normal to $P$: $\partial_{n+1}$, located at the same point of $P$. Finally, we define the cylinder $W_Q$ which is the union of all the “vertical” lines directed by $\partial_{n+1}$ and crossing $U_Q$. We then extend the frames $\{e_\mu, e_n, \partial_{n+1}\}$ of $P$ to $W_Q \subset \mathbb{R}^{n+2}$ by translating, without change, along the vertical lines. Namely, for $y \in W_Q$ we note $y_P$ the orthogonal projection of $y$ on $P$ and define $e_\alpha^B(y)$ through the formula $e_\alpha^B(y) = e_\alpha^B(y_P) = e_\alpha^B(y^0, \ldots, y^n)$ and, writing for simplicity $e_{n+1} = \partial_{n+1}$, $e_{n+1}^A = e_{n+1}^A = 0$ and $e_{n+1}^{n+1} = 1$. We thus obtained an adapted frame $e_\alpha, \alpha = 0, \ldots, n+1$ fulfilling the important property $D(e_\alpha^B) = F(e_\alpha^B) = 0$ (the coefficients are homogeneous of degree 0 and do not depend on $y^{n+1}$). This implies that $[e_\mu, \partial_{n+1}] = [e_n, \partial_{n+1}] = 0$. That is, all anholonomy coefficients containing the index $n+1$ are equal to zero.

### B. Restriction to (A)dS spaces

#### 1. Restriction of the one-form operator

In the present paragraph we derive the restriction of the one form $\Box_{n+2}\alpha$ to the submanifold $\Sigma_n$, the (A)dS space. We consider Eq. (1) as the starting point, we apply on it the pullback of the canonical injection map $m : \Sigma_n \to \mathbb{R}^{n+2}$. If $\alpha$ is a $p$-form on $\mathbb{R}^{n+2}$ we
introduce, for future convenience the notation $\alpha_\Sigma := m^* \alpha$. From the definition of an adapted frame (see Sec. IV A) one has $m^* e^\mu = e^\mu$ and $m^* e^n = m^* e^{n+1} = 0$, then, if $\alpha$ is a 1-form,

$$\alpha_\Sigma = m^* \alpha = (m^* \alpha_\alpha) m^*(e^\alpha) = (\alpha_\mu) e^\mu.$$ 

The calculation parallels that of Sec. IV C.1. The r.h.s. of the pullback by $m$ of Eq. (1) divides now into three parts: the first one, containing only Greek indexes $\mu, \nu, \ldots$, leads after restriction to the Laplace-de Rham operator on $\Sigma$, the (A)dS space, the others are calculated in a straightforward way in appendix

$$m^*(\square_{n+2} a) = \square_\Sigma a_\Sigma + m^* \left[ \epsilon (\partial^2_{n+1} a_\mu) e^\mu - \epsilon H^2 \left( \mathcal{L}^2_D(a) + (n-3) \mathcal{L}_D(a) + 2 d(i_D a) \right) \right].$$

Finally, the second term of the r.h.s. of this equation can be recast under an intrinsic form, that is without reference to specific frames, the calculation, given in appendix A 6, leads to

$$m^*(\square_{n+2} a) = \square_\Sigma a_\Sigma + m^* \left[ -\epsilon H^2 \left( \mathcal{L}^2_D + (n-3) \mathcal{L}_D + 2 d_{aD} \right) + 2 \mathcal{L}_D \mathcal{L}_D \right. $$

$$+ (n-4) \mathcal{L}_D + 2 d_{aD} \left[ a \right]$$

$$= \square_\Sigma a_\Sigma + m^* \left[ -\epsilon H^2 \left( \mathcal{L}^2_D + (n-3) \mathcal{L}_D + 2 d_{aD} \right) + 2 \mathcal{L}_D \mathcal{L}_D \right.$$

$$+ n \mathcal{L}_D + 2 d_{aD} \left[ a \right],$$

where $[F,D] = 2F$ (due to the homogeneity of $f$) has been used.

2. Restriction of the scalar operator

Let us consider again the scalar case. The Eq. (2) for a, not necessarily homogeneous, scalar field $\phi$ splits, as in the one-form case, into three terms corresponding to those containing Greek indexes only, those involving the index $n$, and those indexed by $n+1$. The part relating to Greek indexes only, leads, upon reduction, to the scalar Laplace-de Rham operator on $\Sigma$, the (A)dS space, the others are calculated in a straightforward way in appendix
The result reads

$$m^*(\Box_{n+2}\phi) = \Box_{\Sigma}\phi_{\Sigma} - \epsilon H^2 \left[D^2\phi + (n - 1)D\phi\right]_{\Sigma} + [2FD\phi + (n - 2)F\phi]_{\Sigma}$$

$$= \Box_{\Sigma}\phi_{\Sigma} - \epsilon H^2 \left[D^2\phi + (n - 1)D\phi\right]_{\Sigma} + [2DF\phi + (n + 2)F\phi]_{\Sigma}. \quad (12)$$

For $\phi$ homogeneous of degree $r$, the above expression reduces to

$$\Box_{n+2}\phi = \Box_{\Sigma}\phi_{\Sigma} - \epsilon H^2 r(r + n - 1)\phi_{\Sigma} + (2r + n - 2)F(\phi)_{\Sigma}. \quad (13)$$

Note that for $n = 4$ we retrieve our previous result (Eq. 9 of [22]) as a particular case.

**VI. THE ADDITIONAL TERMS**

We observe that all restricted operators we obtained in previous sections Eqs. (5, 6, 11, 12), share the common structure:

$$(\Box_d\alpha)_{\Sigma} = \Box_{\Sigma}\alpha_{\Sigma} + \text{AT}(\alpha), \quad (13)$$

where $\alpha$ is a one-form or a scalar field and AT stands for “Additional Term”.

A natural question is whether the Additional Term has some characteristic depending on the embedding used in this paper. In order to address this question, we are interested in the converse situation in which one considers as a starting point the expression

$$\Box_{\Sigma}\beta + \chi(\beta), \quad (14)$$

where $\beta$ and $\chi$ live on $\Sigma$ and are respectively a one-form or scalar field, and an arbitrary operator. We will show that it is possible to choose an extension, to the embedding space, of $\beta$ such that the Additional Term corresponding to this extension is exactly equal to $\chi$. In order to be more explicit, we consider the restriction from $\mathbb{R}^{n+2}$, $m$ being the canonical injection from the embedded to the embedding space, and $m^*$, acting on $p$-forms, being the operation of restriction. We prove in the following that one can build a right inverse $\varpi_\chi$ (also called a section in that context) of $m^*$ (that is satisfying $m^* \circ \varpi_\chi = \text{Id}$) such that $\text{AT} \circ \varpi_\chi = \chi$ that is to say Eq. (13) becomes

$$(\Box_d\varpi_\chi(\beta))_{\Sigma} = \Box_{\Sigma} + \chi(\beta).$$

In other words, all additional terms $\chi$ on the embedded pseudo-sphere $\Sigma$ can be obtained by the choice of an appropriate extension of the one-form or scalar field. We will prove that this result is also valid in the $\mathbb{R}^{n+1}$ context which we first tackle.
A. Additional terms in the \( \mathbb{R}^{n+1} \) context

We recall that \( l \) is the canonical injection of \( \Sigma \) into \( \mathbb{R}^{n+1} \) and \( l^* \) the corresponding restriction for scalar and one-forms.

Let us first define the extension by homogeneity in the \( \mathbb{R}^{n+1} \) context. Let \( y \in \mathbb{R}^{n+1} \) a point such that \( \text{sgn}(y^\lambda y_\lambda) = -\epsilon \), this condition ensures that the half-line emanating from the center of \( \mathbb{R}^{n+1} \) and containing \( y \) intercepts \( \Sigma \). Let \( \rho \) be a map over \( \Sigma \), and for \( r \in \mathbb{R} \), let us set

\[
\rho^{(r)}(y) = \rho \left( \frac{y}{H \sqrt{|y_\lambda y^\lambda|}} \right) \left( H \sqrt{|y_\lambda y^\lambda|} \right)^r. \tag{15}
\]

Since \( y/(H \sqrt{|y_\lambda y^\lambda|}) \in \Sigma \), then \( \rho^{(r)} \) is homogeneous of degree \( r \) and \( (\rho^{(r)})_\Sigma = \rho \) by construction. In regards to the one-forms, one can define the extension by homogeneity of order \( s \) for any one-form \( h = h_\mu e^\mu \) on \( \Sigma \) through

\[
h^{(s)} = h^{(s-1)}_\mu e^\mu.
\]

Remark that, since \( D \) is along the (outer) normal to \( \Sigma \), any one-forms \( k = k_\mu e^\mu \) on \( E \) satisfy the transversality condition \( i_D k = 0 \). This is the case for \( h^{(s)} \) which is thus an extension of \( h \), transverse and homogeneous of degree \( s \). From Eq. (5) we see that \( AT \), in the present situation, involves the Lie derivative \( \mathcal{L}_D \), let us determine its action on \( h^{(s)} \), one has successively

\[
\mathcal{L}_D(h^{(s)}) = i_D dh^{(s-1)}_\mu e^\mu + d i_D h^{(s-1)}_\mu e^\mu
\]

\[
= D(h^{(s-1)}_\mu e^\mu) + h^{(s-1)}_\mu i_D de^\mu
\]

\[
= (s-1)h^{(s-1)}_\mu e^\mu - \frac{1}{2} h^{(s-1)}_\mu \sqrt{|y^2|} i_n c_{AB} e^A \wedge e^B
\]

\[
= (s-1)h^{(s)} - \frac{1}{2} h^{(s-1)}_\mu \sqrt{|y^2|} c_{AB} (\delta^A_n e^B - \delta^B_n e^A)
\]

\[
= (s-1)h^{(s)} - h^{(s-1)}_\mu \sqrt{|y^2|} c_{nA} e^A
\]

\[
= sh^{(s)},
\]

where transversality has been used to eliminate the second term on the r.h.s. in the first line. Now, in Eq. (14) let \( b := \beta \) be a one-form field on \( \Sigma \). Let us determine the section \( \varpi_\chi \) such that \( AT \circ \varpi_\chi(b) = \chi(b) \). It is sufficient to set

\[
\varpi_\chi(b) = b^{(0)} - \epsilon H^{-2}(\chi^{(s)} - \chi^{(0)}),
\]
with \( s \) solution of \( S(s) = 1 \) where \( S(X) = X^2 + (n - 3)X \). Note first, that applying \( l^* \) to both members of the above relation shows that \( \varpi \) is a section of \( l^* \), then one can prove the above statement. Starting from Eq. (5), we obtain:

\[
AT = -\epsilon H^2 l^* \circ [S(\mathcal{L}_D) + 2d(i_D)],
\]

and

\[
AT(\varpi_\chi(b)) = -\epsilon H^2 l^* S(\mathcal{L}_D) \varpi_\chi(b)
\]

\[
= -\epsilon H^2 l^* S(\mathcal{L}_D) [b^{(0)} - \epsilon H^{-2}(\chi^{(s)}(b) - \chi^{(0)}(b))]
\]

\[
= 0 + l^* S(\mathcal{L}_D)(\chi^{(s)}(b))
\]

\[
= S(s) l^* (\chi^{(s)}(b))
\]

\[
= \chi(b),
\]

where the transversality of \( \chi^{(s)}(b) \) has been used in the first and the third line.

The scalar case can be treated in a similar way. Let \( \psi := \beta \) be a scalar field on \( \Sigma \). One can verify readily that \( \varpi_\chi \) defined through

\[
\varpi_\chi(\psi) = \psi^{(0)} - \epsilon H^{-2}(\chi^{(s)}(\psi) - \chi^{(0)}(\psi)),
\]

where \( s \) is a solution of the equation \( Q(s) := s^2 + (n-1)s = 1 \), is a section of \( l^* \) (\( l^* \circ \varpi_\chi = \text{Id} \)). For this choice of section one has \( AT \circ \varpi_\chi = \chi \) and, once again, for all operator \( \chi \) on \( \Sigma \) there exists a section of \( l^* \) such that \( AT = \chi \).

### B. Additional terms in the \( \mathbb{R}^{n+2} \) context

As in the previous section, we have to find \( \varpi'_\chi \) a section of \( m^* \) such that \( AT' \circ \varpi'_\chi = \chi \) where \( AT' \) is the additional term in \( \mathbb{R}^{n+2} \) context (see Eq. (11))

\[
AT' = m^* \left[ -\epsilon H^2 (S(\mathcal{L}_D) + 2d(i_D)) + 2\mathcal{L}_D \mathcal{L}_F + n\mathcal{L}_F + 2di_F \right].
\]

Let \( p \) be the orthogonal projection of \( \mathbb{R}^{n+2} \) onto \( P \sim \mathbb{R}^{n+1} \) (remember that the normal vector of \( P \) is non-isotropic), one can verify that \( l = p \circ m \) and, as a consequence, \( p^* \varpi_\chi \) is a section of \( m^* \). We claim that \( \varpi'_\chi := p^* \varpi_\chi \) fulfills \( AT' \circ \varpi'_\chi = \chi \), which will allow us to conclude.
Before proving that, we need some remarks on \( p^* \). Let \( h = h_A e^A \) a one-form on \( P \sim \mathbb{R}^{n+1} \), then we have
\[
(p^* h)(y^0, \ldots, y^n, y^{n+1}) = h_A(y^0, \ldots, y^n) e^A.
\]
This proves several facts. First \( \mathcal{L}_F \circ p^* = i_F \circ p^* = 0 \), second \( p^* h \) is homogeneous of degree \( s \) as soon as \( h \) is, and finally \( i_D p^* h = 0 \) as soon as \( i_D h = 0 \). Then, we have
\[
AT' \circ \varphi^'_\chi = m^* \left[ -\epsilon H^2 (S(L_D) + 2di_D) + 2\mathcal{L}_D \mathcal{L}_F + n\mathcal{L}_F + 2di_F \right] \circ p^* \circ \varphi_\chi
= -\epsilon H^2 m^* S(L_D) \circ p^* \circ \varphi_\chi
= -\epsilon H^2 m^* S(s) p^* \circ \varphi_\chi
= -\epsilon H^2 l^* \circ \varphi_\chi
= \chi,
\]
and we again see that any operator \( \Box_{\Sigma} b + \chi(b) \) on \( \Sigma \) can be reached as a restriction of the operator \( \Box_{n+2} a \) where \( a \) is an appropriate extension of \( b \).

The scalar case can be treated in the same way: replacing \( \varphi_\chi \) with \( p^* \circ \varphi_\chi \) leads to the same result as before.

**VII. CONCLUSION**

The keystone of our construction is the existence of an adapted frame for which the commutation relations, leading to anholonomic coefficients, are easily calculated. This adapted frame is deeply related to the pseudo-sphere and the maximal symmetry of (A)dS. Other space-times are no longer maximally symmetric. Nevertheless, many of them share the same conformal group \( \text{SO}(2,n) \), including Friedmann-Lemaître-Robertson-Walker spaces. This should allow us, in a future work, to generalize to these spaces the results obtained here.
Appendix A: Details on calculations

1. Frequently used expression

Frequent use are made of the following usual formulas (see for instance [29]):

\[
\delta = (-1) * \hat{\eta}, \\
*^{-1} = \text{sgn}(g) * \eta^{d+1}, \\
* \omega = \text{sgn}(g) \text{ et } * 1 = \omega, \\
\mathcal{L}_V \omega = \text{div}(V) \omega = - \delta(\tilde{V}) \omega, \\
\mathcal{L}_V = i_V d + d i_V, \text{ (Cartan formula).}
\]

In an ortho-normal basis:

\[
\eta^{ab} = i^a j^b + j^b i^a, \\
*(e^a \wedge \cdots \wedge e^b) = i^b \cdots i^a \omega, \\
* i^a \cdots i^b \omega = \text{sgn}(g) \eta^{d+1} e^b \wedge \cdots \wedge e^a.
\]

2. Calculation of \( \delta(e^a) \)

\[
\delta(e^a) = - \text{sgn}(g) * d * e^a \\
= - \text{sgn}(g) * d i^a \omega \\
= - \text{sgn}(g) \eta^{aa} * d(-1)^{a-1} e^1 \wedge \cdots \wedge \tilde{\omega} e^a \wedge \cdots \wedge e^d \\
= \text{sgn}(g) (-1)^a \eta^{aa} * J,
\]

with \( J = d(e^1 \wedge \cdots \wedge \tilde{\omega} e^a \wedge \cdots \wedge e^d) \). Thus, using exceptionally the symbol \( \sum \) to indicate summations, one has successively
\[ J = \sum_{p<a} (-1)^{p-1} e^1 \wedge \cdots \wedge d e^p \wedge \cdots \wedge e^a \wedge \cdots \wedge e^d \]

\[ + \sum_{p>a} (-1)^{p} e^1 \wedge \cdots \wedge e^a \wedge \cdots \wedge d e^p \wedge \cdots \wedge e^d \]

\[ = \sum_{p<a} (-1)^{p} e^1 \wedge \cdots \wedge (\sum_{b<c} c^p_{bc} e^b \wedge e^c) \wedge \cdots \wedge e^a \wedge \cdots \wedge e^d \]

\[ + \sum_{p>a} (-1)^{p+1} e^1 \wedge \cdots \wedge e^a \wedge \cdots \wedge (\sum_{b<c} c^p_{bc} e^b \wedge e^c) \wedge \cdots \wedge e^d \]

\[ = \sum_{p<a} (-1)^{p} e^1 \wedge \cdots \wedge (c^p_{pa} e^p \wedge e^a) \wedge \cdots \wedge e^a \wedge \cdots \wedge e^d \]

\[ + \sum_{p>a} (-1)^{p+1} e^1 \wedge \cdots \wedge e^a \wedge \cdots \wedge (c^p_{ap} e^a \wedge e^p) \wedge \cdots \wedge e^d \]

\[ = \sum_{p<a} (-1)^{p} (-1)^{p-a+1} c^p_{pa} \omega + \sum_{p>a} (-1)^{p+1} (-1)^{a+1-p} c^p_{ap} \omega \]

\[ = (-1)^{a+1} \omega \sum_{p \neq a} c^p_{pa}. \]

Finally, returning to the Einstein summation convention, one gets

\[ \delta(e^a) = -\eta^{ab} c^p_{pb}. \]
3. Derivation of Eq. (2)

The expression Eq. (2) for the scalar field is obtained through the following steps:

\[
\Box \phi = -\delta d\phi \\
= -\delta e_a(\phi) e^a \\
= -*^{-1} d * \tilde{\eta} e_a(\phi) e^a \\
= -\text{sgn}(g) * \tilde{\eta}^{d+1} d * (-1)e_a(\phi) e^a \\
= \text{sgn}(g) * (-1)^{d(d+1)} de_a(\phi) i^a\omega \\
= \text{sgn}(g) * (d(e_a(\phi)) \wedge \tilde{i}^a\omega + e_a(\phi) di^a\omega) \\
= \text{sgn}(g) * (e_b(e_a(\phi)))e^b \wedge \tilde{i}^a\omega + e_a(\phi) \mathcal{L}_{\tilde{e}^a}(\omega)) \\
= \text{sgn}(g) * (e_b(e_a(\phi)))j^b \tilde{r}^a\omega + e_a(\phi) \mathcal{L}_{\tilde{e}^a}(\omega)) \\
= \text{sgn}(g) * (\eta^{ab} e_b(e_a(\phi)))\omega - e_a(\phi) \delta(e^a) \omega \\
= \eta^{ab} (e_b(e_a(\phi)) - e_a(\phi)\delta(e^a)) \\
= \eta^{ab}[e_a e_b(\phi) + e_a(\phi) c^a_{pb}],
\]

where the expression of the term \(\delta(e^a)\), derived in appendix A2, has been used in the last line.

We note for future reference that \(\delta\alpha\), \(\alpha\) being a one-form, is obtained by the replacement of \(e_a(\phi)\) by \(\alpha_a\) in the above calculation leading to

\[
\delta(\alpha) = -\eta^{ab} e_a(\alpha_b) + \alpha_a \delta(e^a). \tag{A1}
\]

4. Derivation of Eq. (1)

In order to expresses \(\Box a\) in an ortho-normal basis. We first compute the Maxwell operator:

\[
-\delta d(a_a e^a) = -\delta d(a_a) \wedge e^a + -\delta a_a de^c. 
\]
Calculation of $I$:

$$I = -\delta e_b(a_a)e^b \land e^a$$

$$= (-1)^d \text{sgn}(g) * d * e_b(a_a)e^b \land e^a$$

$$= (-1)^d \text{sgn}(g) * de_b(a_a)i^a_i^b \omega$$

$$= (-1)^d \text{sgn}(g) \left( \frac{*d(e_b(a_a))i^a_i^b \omega}{I_1} + \frac{*e_b(a_a)di^a_i^b \omega}{I_2} \right).$$

For the term $I_1$ one has successively:

$$I_1 = *d(e_b(a_a))i^a_i^b \omega$$

$$= *e_c((e_b(a_a)))^{j^c_{\eta^a - i^a} j^b_{\omega}}$$

$$= *e_c((e_b(a_a)))^{j^c_{\eta^a} j^b_{\omega}}$$

$$= e_a((e_b(a^a)))^{j^b_{\omega}} - e_c((e_b(a_a)))^{i^a \eta^b_{\omega}}$$

$$= \text{sgn}(g)(-1)^{d+1} \left( e_a((e_b(a^a)))e^b - \eta^b_{\omega} e_c((e_b(a_a))) e^a \right)$$

$$= \text{sgn}(g)(-1)^{d+1}(e_a e_b(a^a)e^b - \eta^b_{\omega} e_c e_b(a_a) e^a).$$

For the term $I_2$ one has successively:

$$I_2 = *e_b(a_a)di^a_i^b \omega$$

$$= *e_b(a_a)(\mathcal{L}_{\tilde{e}^a} - i^a d)i^b \omega$$

$$= *e_b(a_a) \left[ \mathcal{L}_{\tilde{e}^a}(e^b) \omega + e^b \mathcal{L}_{\tilde{e}^a}(\omega) - i^a \mathcal{L}_{\tilde{e}^a} \omega \right]$$

$$= *e_b(a_a) \left[ [e^a, e^b] \omega - \delta(e^a) i^b \omega + \delta(e^b) i^a \omega \right]$$

$$= *e_b(a_a) \left[ [\eta^{ac}\eta^{bm} c^l_{cm} \eta_{lf}f^l \omega - \delta(e^a) i^b \omega + \delta(e^b) i^a \omega \right]$$

$$= (-1)^{d+1} \text{sgn}(g)e_b(a_a) \left[ \delta(e^b) e^a + \eta^{ac}\eta^{bm} \eta_{lf}c^l_{cm} e^f - \delta(e^a)e^b \right]$$

$$= (-1)^{d+1} \text{sgn}(g)e_b(a_a) \left[ -\eta^{bd} p^c_{pd} e^a + \eta^{ac}\eta^{bm} \eta_{lf}c^l_{cm} e^f + \eta^{ad} p^c_{pd} e^b \right]$$

$$= (-1)^{d+1} \text{sgn}(g) \left[ -\eta^{bd} p^c_{pd} e_b(a_a) + \eta^{bm} \eta_{ma} c^a_{cm} e_b(a^c) + p^c_{pd} e_a(a^b) \right] e^a;$$

where Eq. (A.2) for $\delta(e^a)$ as been used.
We now compute $II$. We first remark that $II$ rewrites

$$II = -\delta a_e d^c$$
$$= -\delta a_e (\frac{1}{2} c^e_{ab} e^a \land e^b$$
$$= -\frac{1}{2} a_e c_{ab} e^b \land e^a,$$

which is nothing but $I$ with $e_b(a_a)$ in place of $\frac{1}{2} a_e c_{ab}$. Thus, one can recast $II$ under the form $II = (-1)^d \text{sgn}(g)(I_1 + I_2)$, where the terms $I_1$ and $I_2$ are evaluated as follow:

$$I_1 = \frac{1}{2} \text{sgn}(g) (-1)^{d+1} [\eta^{a b} e_a (a_e c^e_{ab}) e^b - \eta^{c b} e_c (a_d c^d_{ab}) e^a]$$
$$= \frac{1}{2} \text{sgn}(g) (-1)^{d+1} [\eta^{a b} e_a (a_e c^e_{ab}) e^b - \eta^{c b} e_c (a_d c^d_{ab}) e^a]$$

$$= \text{sgn}(g) (-1)^{d+1} [\eta^{a b} e_a (a_e c^e_{ab}) e^b]$$

and

$$I_2 = \frac{1}{2} \text{sgn}(g) (-1)^{d+1} a_e c^e_{ab} [\delta(e^b) e^a + \eta^{m n} \eta^b e^c_{ab} e^d - \delta(a_e) e^b]$$

$$= \text{sgn}(g) (-1)^{d+1} \left[ \eta^{m n} a_e c^e_{ba} e^p e^m + \frac{1}{2} a_e \eta^{mn} \eta^p e^c_{ab} e^b \right] e^a.$$ 

The Maxwell part of $\Box$ being determined one focus on the second part, namely: $-d\delta(a)$.

Thanks to Eq. (A1) one obtains

$$III = -d\delta(a)$$

$$= -d[-c_a(a^a) + a_a \delta(e^a)]$$

$$= e_b [e_a(a^a) + a^a c^p_{pa}] e^b$$

$$= e_b (e_a(a^a)) e^b + e_b (a^a) c^p_{pa} e^b + a^a e_b (c^p_{pa}) e^b.$$

Finally

$$\Box a = (-1)^d \text{sgn}(g)(I_1 + I_2 + II_1 + II_2) + III,$$

which is Eq. (1).
5. Expression of the term $nB$ of Eq. (3)

From Eq. (1) the terms containing the $n$ index denoted $nB$ reads

\[ nB = \sum_{m=1}^{9} nB^m, \]

where $nB^m$ is the $m$-th term of the r.h.s. of Eq. (1):

\[ nB^1 = n^m e_n(e_n(a_\mu)) e^\mu; \quad nB^2 = (e^n_{\mu\nu} e_n(a^\nu) + e^n_{\nu\mu} e_\nu(a^n) + e^n_{\mu n} e_n(a^n)) e^\mu \text{ etc.} \]

Thanks to the relation $a_n = i e_n a = i D a / \sqrt{|y^2|}$, and the fact that $e_\mu(y^2) = 0$, one obtains

\[ nB^1 = -\frac{1}{|y^2|} (D^2(a_\nu) - D(a_\mu)) e^\mu; \quad nB^2 = -\frac{1}{|y^2|} e_\mu(i D a) e^\mu; \quad nB^3 = -\frac{1}{|y^2|} i D (a_\mu) e^\mu; \]
\[ nB^4 = \epsilon \frac{n}{|y^2|} a_\mu e^\mu; \quad nB^5 = 0; \quad nB^6 = \frac{1}{|y^2|} a_\mu e^\mu. \]

Gathering these terms gives Eq. (3).

6. Intrinsic form for Eq. (10)

We now rewrite the second term of the r.h.s. of Eq. (10). We now show that

\[ m^* ((\partial^2_{n+1} a_\mu) e^\mu) = m^* (H^{-2} L^2_P(a)). \quad (A2) \]

To begin with, we have $\mathcal{L}_{\partial n+1} (e^\mu) = 0$, since

\[ \mathcal{L}_{\partial n+1} (e^\mu) = d i_{n+1} e^\mu + i_{n+1} d e^\mu \]
\[ = 0 - \frac{1}{2} i_{n+1} \epsilon^\mu_{\alpha\beta} e^\alpha \wedge \beta \]
\[ = 0, \]

in which we used $\epsilon^\mu_{\alpha n+1} = 0$. As a consequence, $\mathcal{L}^2_{n+1} (a_\mu) e^\mu = \mathcal{L}^2_{n+1} (a_\mu e^\mu)$ and the result follows using Eq. (8).

Thanks to Eqs. (9)–(8) and $y^{n+1} = H^{-1} f$ one has $D_\rho = D - \epsilon H^{-2} f F$, which allows us to eliminate $D_\rho$ from Eq. (10). This is achieved as follows.

One first rewrites $\mathcal{L}_{\partial n+1}$ thanks to the Cartan formula under the form:

\[ \mathcal{L}_\phi = \phi \mathcal{L}_V + d\phi \wedge i_V, \]
\( \phi \) being a scalar and \( V \) a vector. Then
\[
\mathcal{L}_{D_{n+1}} = y^{n+1} \mathcal{L}_{\partial_{n+1}} + dy^{n+1} \wedge i_{\partial_{n+1}}.
\]  
(A3)

using Eq. (8) and \( y^{n+1} = H^{-1} f \) this expression rewrites \( \mathcal{L}_{D_{n+1}} = \epsilon H^{-2}(f \mathcal{L}_F + df \wedge i_F) \), noting that \( m^* f = 1, m^* df = 0 \), one obtains
\[
m^* \mathcal{L}_{D_{n+1}} = m^*(\epsilon H^{-2} \mathcal{L}_F),
\]  
(A4)

where, here and after, we note, for any operator \( A \), \( m^* A \) instead of \( m^* \circ A \). This allows us to recast \( m^* \mathcal{L}_{D_P} \) under the form
\[
m^* \mathcal{L}_{D_P} = m^* (\mathcal{L}_D - \epsilon H^{-2} \mathcal{L}_F).
\]  
(A5)

We now consider \( m^* \mathcal{L}_{D_P}^2 = m^* \left( \mathcal{L}_D^2 + \mathcal{L}_{D_{n+1}}^2 - [\mathcal{L}_D, \mathcal{L}_{D_{n+1}}]_+ \right) \), and first compute \( m^* \mathcal{L}_{D_{n+1}}^2 \). From Eq. (A3), taking into account \( m^* dy^{n+1} = 0 \) and \( m^* y^{n+1} = H^{-1} \), one has
\[
m^* \mathcal{L}_{D_{n+1}}^2 = m^* \left( y^{n+1} \mathcal{L}_{\partial_{n+1}} + dy^{n+1} \wedge i_{\partial_{n+1}} \right) \left( y^{n+1} \mathcal{L}_{\partial_{n+1}} + dy^{n+1} \wedge i_{\partial_{n+1}} \right)
\]
\[
= m^* y^{n+1} \mathcal{L}_{\partial_{n+1}} y^{n+1} \mathcal{L}_{\partial_{n+1}} + m^* y^{n+1} \mathcal{L}_{\partial_{n+1}} dy^{n+1} \wedge i_{\partial_{n+1}}
\]
\[
= m^* y^{n+1} \left( \mathcal{L}_{\partial_{n+1}} (y^{n+1} \mathcal{L}_{\partial_{n+1}} + y^{n+1} \mathcal{L}_{\partial_{n+1}}^2) \right) + m^* y^{n+1} \mathcal{L}_{\partial_{n+1}} (dy^{n+1} \wedge i_{\partial_{n+1}})
\]
\[
= m^* \left( y^{n+1} \mathcal{L}_{\partial_{n+1}} + (y^{n+1})^2 \mathcal{L}_{\partial_{n+1}}^2 \right).
\]

Then we determine the term \( m^* [\mathcal{L}_D, \mathcal{L}_{D_{n+1}}]_+ \). Noting that \([D, D_{n+1}] = 0 \) and thus \([\mathcal{L}_D, \mathcal{L}_{D_{n+1}}] = 0 \) one has
\[
m^* [\mathcal{L}_D, \mathcal{L}_{D_{n+1}}]_+ = m^* (2 \mathcal{L}_{D_{n+1}} \mathcal{L}_D)
\]
\[
= m^* \left( 2 \left( y^{n+1} \mathcal{L}_{\partial_{n+1}} + dy^{n+1} \wedge i_{\partial_{n+1}} \right) \mathcal{L}_D \right)
\]
\[
= m^* 2 \left( y^{n+1} \mathcal{L}_{\partial_{n+1}} \mathcal{L}_D \right),
\]

where \( m^* dy^{n+1} = 0 \) has been used.

Gathering the above results and using Eq. (8) and \( y^{n+1} = H^{-1} f \) gives
\[
m^* \mathcal{L}_{D_P}^2 = m^* \left( \mathcal{L}_D^2 + H^{-4} \mathcal{L}_F^2 + \epsilon H^{-4} \mathcal{L}_F^2 (1 - 2 \mathcal{L}_D) \right).
\]  
(A6)

The last term of Eq. (10) containing \( D_P \), rewrites as
\[
m^* d_{D_P} = m^* (d_D - \epsilon H^{-2} d_F).
\]  
(A7)
Finally, using the above relations Eqs. (A2), (A5), (A6) and (A7), the second term of the r.h.s. of Eq. (10):

\[ X := m^* \left[ \epsilon (\partial^2_{n+1} a_\mu) e^\mu - \epsilon H^2 \left( \mathcal{L}^2_D (a) + (n - 3) \mathcal{L}_D (a) + 2 d(i_D a) \right) \right], \]

writes

\[ X = m^* \left[ -\epsilon H^2 \left( \mathcal{L}^2_D + (n - 3) \mathcal{L}_D + 2d_i \right) + 2 \mathcal{L}_F \mathcal{L}_D + (n - 4) \mathcal{L}_F + 2d_i \right] (a). \]

7. Derivation of Eq. (12)

The Eq. (2), as for the one-form, splits into three terms:

\[ r \mathcal{C} = \eta^\mu \nu \left[ e_\mu e_\nu (\phi) + e_\nu (\phi) c^\alpha_{\alpha n} \right], \]

\[ n \mathcal{C} = \eta^{\mu n} \left[ e_\nu e_n (\phi) + e_n (\phi) c^\alpha_{\alpha n} \right], \]

\[ n^+ l \mathcal{C} = \eta^{n+1} \left[ \partial_{n+1} \partial_{n+1} (\phi) + \partial_{n+1} (\phi) c^\alpha_{\alpha n+1} \right]. \]

As already said \( m^* r \mathcal{C} = \Box_\Sigma \phi_\Sigma \). Now, using the notations of Sec. VA, and \( y^2_p : = y^4 y_A \), the two remaining terms read

\[ n \mathcal{C} = \eta^{\mu n} \left[ e_\nu e_n (\phi) + e_n (\phi) c^\alpha_{\alpha n} \right] \]

\[ = -\epsilon \left[ \frac{1}{\sqrt{y^2_p}} (D_p) \frac{1}{\sqrt{|y^2_p|}} (D_p) \phi + \frac{n}{|y^2_p|} (D_p) \phi \right] \]

\[ = -\epsilon \left[ \frac{1}{y^2_p} \left( (D_p)^2 \phi - (D_p) \phi + n(D_P) \phi \right) \right] \]

\[ = -\epsilon \left[ \frac{1}{y^2_p} \left( D^2 \phi - 2D_{n+1} D \phi + (y^{n+1})^2 \partial^2_{n+1} \phi + D_{n+1} \phi + (n - 1) (D - D_{n+1}) \phi \right) \right], \]

and \( n^+ l \mathcal{C} = \epsilon \partial^2_{n+1} \phi. \)

Using \( y^{n+1} = H^{-1} f \) one remarks that the term \( n^+ l \mathcal{C} \) simplify with a term in \( n \mathcal{C}: \)

\[ m^* \left( -\epsilon \left[ \frac{y^{n+1}}{y^2_p} \right] \right. \left. \partial^2_{n+1} \phi + \epsilon \partial^2_{n+1} \phi \right) = -\epsilon \left[ \frac{y^{n+1}}{y^2_p} \right] - \epsilon H^{-2} \partial^2_{n+1} \phi + \epsilon \partial^2_{n+1} \phi = 0, \]

and one thus has successively

\[ m^* (\Box_{n+2}) = \Box_\Sigma \phi_\Sigma - \epsilon H^2 \left[ D^2 \phi - 2D_{n+1} D \phi + D_{n+1} \phi + (n - 1)(D \phi - D_{n+1} \phi) \right], \]

\[ = \Box_\Sigma \phi_\Sigma - \epsilon H^2 \left[ D^2 \phi - 2D_{n+1} D \phi + D_{n+1} \phi + (n - 1)(D \phi) \right], \]

\[ = \Box_\Sigma \phi_\Sigma - \epsilon H^2 \left[ D^2 \phi + (n - 1) D \phi - 2H^{-1} \partial_{n+1} D \phi + (2 - n)H^{-1} \partial_{n+1} \phi \right], \]

\[ = \Box_\Sigma \phi_\Sigma - \epsilon H^2 \left[ D^2 \phi + (n - 1) D \phi - 2\epsilon H^{-2} F D \phi + \epsilon (2 - n) H^{-2} F \phi \right], \]

\[ = \Box_\Sigma \phi_\Sigma - \epsilon H^2 \left[ D^2 \phi + (n - 1) D \phi \right] + \left[ 2F D \phi + (n - 2) F \phi \right], \]
which is Eq. (12).

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