A CHARACTERIZATION OF HILBERT $C^*$-MODULES OVER FINITE DIMENSIONAL $C^*$-ALGEBRAS

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Abstract. We show that the unit ball of a full Hilbert $C^*$-module is sequentially compact in a certain weak topology if and only if the underlying $C^*$-algebra is finite dimensional. This provides an answer to the question posed in J. Chmieliński et al [Perturbation of the Wigner equation in inner product $C^*$-modules, J. Math. Phys. 49 (2008), no. 3, 033519].

1. Introduction and preliminaries

Let $\mathcal{A}$ be a $C^*$-algebra. A linear space $\mathcal{M}$ that is an algebraic left $\mathcal{A}$-module with $\lambda(ax) = a(\lambda x) = (\lambda a)x$ for $x \in \mathcal{M}$, $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$, is called a pre-Hilbert $\mathcal{A}$-module (or an inner product $\mathcal{A}$-module) if there exists an $\mathcal{A}$-valued inner product on $\mathcal{M}$, i.e., a mapping $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ satisfying

(i) $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$;
(ii) $\langle ax, y \rangle = a \langle x, y \rangle$;
(iii) $\langle x, y \rangle^* = \langle y, x \rangle$;
(iv) $\langle x, x \rangle \geq 0$;
(v) $\langle x, x \rangle = 0 \Leftrightarrow x = 0$,

for all $x, y, z \in \mathcal{M}$, $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$. Conditions (i) and (iii) yield the fact that the inner product is conjugate-linear with respect to the second variable. It follows from the definition that $\|x\|_{\mathcal{M}} := \sqrt{\|\langle x, x \rangle\|_{\mathcal{A}}}$ is a norm on $\mathcal{M}$, whence $\mathcal{M}$ becomes a normed left $\mathcal{A}$-module.

A pre-Hilbert $\mathcal{A}$-module $\mathcal{M}$ is called a Hilbert $C^*$-module if it is complete with respect to this norm. We say that a Hilbert $\mathcal{A}$-module $\mathcal{M}$ is full if the linear subspace $\langle \mathcal{M}, \mathcal{M} \rangle$ of $\mathcal{A}$ generated by $\{\langle x, y \rangle : x, y \in \mathcal{M}\}$ is dense in $\mathcal{A}$. The simplest examples are usual Hilbert spaces as Hilbert $\mathcal{C}$-modules, and $C^*$-algebras as Hilbert $C^*$-modules over themselves via $\langle a, b \rangle = ab^*$.

2000 Mathematics Subject Classification. Primary 46L08; Secondary 46L05, 46L10.

Key words and phrases. Hilbert $C^*$-module, finite dimensional $C^*$-algebra, $C^*$-algebra of compact operators.
The concept of a Hilbert $C^*$-module has been introduced by Kaplansky [6] and Paschke [11]. For more information we refer the reader e.g. to monographs [7, 9].

Despite a formal similarity of definitions, it is well known that Hilbert $C^*$-modules may lack many properties familiar from Hilbert space theory. In fact, it turns out that properties of a $C^*$-module reflect (or originate from) the properties of the underlying $C^*$-algebra.

A particularly well behaved class is the class of Hilbert $C^*$-modules over $C^*$-algebras of compact operators. There are several nice characterizations of such modules (see e.g. [1, 4, 5, 8, 13]). In our proofs we make use of orthonormal bases which exist only in Hilbert $C^*$-modules over $C^*$-algebras of compact operators (see [1, 2]). Recall that a system of vectors $\{\varepsilon_i : i \in I\}$ in a Hilbert $A$-module $M$ is said to be an orthonormal basis for $M$ if it satisfies the following conditions:

1. $p_i := \langle \varepsilon_i, \varepsilon_i \rangle \in A$ is a projection such that $p_i A p_i = C p_i$ for every $i \in I$;
2. $\langle \varepsilon_i, \varepsilon_j \rangle = 0$ for every $i, j \in I, i \neq j$;
3. $\{\varepsilon_i : i \in I\}$ generates a norm-dense submodule of $M$.

If $\{\varepsilon_i : i \in I\}$ is an orthonormal basis (with the above properties (1), (2), (3)) for $M$ then the reconstruction formula $x = \sum_{i \in I} \langle x, \varepsilon_i \rangle \varepsilon_i$ holds for every $x \in M$, with the norm convergence.

Various specific properties of Hilbert $C^*$-modules turn out to be particularly useful in applications. An interesting example of investigations of this type is a recent study of the stability of Wigner equation (see [3] and the references therein). In particular, the main result in [3] is obtained for Hilbert $C^*$-modules satisfying the following condition:

[H] For each norm-bounded sequence $(x_n)$ in $M$, there exist a subsequence $(x_{n_k})$ of $(x_n)$ and an element $x_0 \in M$ such that the sequence $(\langle x_{n_k}, y \rangle)$ converges to $\langle x_0, y \rangle$ in norm for any $y \in M$.

Notice that in case of a Hilbert space condition [H] is clearly satisfied: this is simply the fact that the unit (and hence each) ball in a Hilbert space is weakly sequentially compact.

It is proved in [3, Proposition 2.1] that a Hilbert $A$-module $M$ satisfies condition [H] whenever the underlying $C^*$-algebra is finite dimensional. In this note we prove the converse, i.e., we show that condition [H] is an exclusive property of the class of Hilbert $C^*$-modules over finite dimensional $C^*$-algebras. In this way we obtain a new characterization of such modules and answer a question posed in [3] concerning condition [H].
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2. The result

For a Hilbert space $H$ we denote by $\mathbb{B}(H)$ and $\mathbb{K}(H)$ the $C^*$-algebras of all bounded, resp. compact operators acting on $H$. We begin with a proposition that reduces the discussion to the class of $C^*$-algebras of compact operators.

**Proposition 2.1.** Suppose that $\mathcal{M}$ is a full Hilbert $C^*$-module over a $C^*$-algebra $\mathcal{A}$, which satisfies condition $[H]$. Then $\mathcal{A}$ is isomorphic to a $C^*$-algebra of (not necessarily all) compact operators acting on some Hilbert space.

**Proof.** Let us fix $y \in \mathcal{M}$. Consider the map $T_y : \mathcal{M} \to \mathcal{M}$ given by $T_y(x) = \langle y, x \rangle y$.

Obviously, $T_y$ is a bounded anti-linear operator.

Let $(x_n)$ be a norm-bounded sequence in $\mathcal{M}$ and let $(x_{n_k})$ be a subsequence of $(x_n)$ such that, for some $x_0 \in \mathcal{M}$, $\lim_{k \to \infty} \langle x_{n_k}, y \rangle = \langle x_0, y \rangle$ for all $y \in \mathcal{M}$. Then $\lim_{k \to \infty} \langle y, x_{n_k} \rangle y = \langle y, x_0 \rangle y$ for all $y \in \mathcal{M}$. This can be restated in the following way: for each norm-bounded sequence $(x_n)$ in $\mathcal{M}$, the sequence $(T_y(x_n))$ has a convergent subsequence. Hence, $T_y$ is a compact operator. Moreover, by the hypothesis, this is true for each $y \in \mathcal{M}$.

By [1, Proposition 1], (4) $\Rightarrow$ (1), there is a faithful representation $\pi : \mathcal{A} \to \mathbb{B}(H)$ of $\mathcal{A}$ on some Hilbert space $H$ such that $\pi(\langle y, y \rangle) \in \mathbb{K}(H)$. This holds for every $y \in \mathcal{M}$, so, by polarization, $\pi(\langle x, y \rangle) \in \mathbb{K}(H)$ for all $x, y \in \mathcal{M}$, and therefore $\pi(\mathcal{A}) \subseteq \mathbb{K}(H)$. □

By the preceding proposition, condition $[H]$ can only be satisfied in Hilbert $C^*$-modules over $C^*$-algebras of compact operators. (Here, and in the sequel, we identify $\mathcal{A}$ with $\pi(\mathcal{A})$, where $\pi$ is the representation from the preceding proof.) However, even if the underlying algebra is a $C^*$-algebra of compact operators, one still cannot conclude that condition $[H]$ is satisfied.

We demonstrate this fact in the following two examples.

**Example 2.2.** Consider a separable infinite dimensional Hilbert space $H$ with an orthonormal basis $(\varepsilon_n)$. We shall regard $\mathbb{K}(H)$ as a Hilbert $C^*$-module over itself via the inner product $\langle a, b \rangle = ab^*$. Let us show that $\mathbb{K}(H)$ does not satisfy $[H]$.

For $n \in \mathbb{N}$, denote by $p_n$ the orthogonal projection to $\text{span}\{\varepsilon_1, \ldots, \varepsilon_n\}$. Obviously, the sequence $(p_n)$ is norm-bounded.

Suppose that there exist a subsequence $(p_{n_k})$ and a compact operator $a \in \mathbb{K}(H)$ such that $\lim_{k \to \infty} \langle p_{n_k}, y \rangle = \langle a, y \rangle$ for all $y \in \mathbb{K}(H)$. This means $p_{n_k} y^* \to ay^*$ for all $y \in \mathbb{K}(H)$, which in turn gives us $p_{n_k} y \xi \to ay \xi$ for all $y \in \mathbb{K}(H)$ and for all $\xi \in H$. In particular, for every
\( n \in \mathbb{N}, \) we can take \( y = p_n - p_{n-1} \) (that is the orthogonal projection to the one-dimensional subspace spanned by \( \varepsilon_n \)) and \( \varepsilon = \varepsilon_n. \) Then the preceding relation yields \( a\varepsilon_n = \varepsilon_n \) for all \( n \in \mathbb{N}; \) i.e., \( a \) is the identity operator. Since \( \dim H = \infty, \) this is not a compact operator. Thus, the assumed property \([H]\) leads to a contradiction.

Recall that, by \([2, \text{Example 2}], \) \( \dim_{\mathbb{K}(H)} \mathbb{K}(H) = \dim H. \)

Our following example shows that even a Hilbert \( \mathbb{K}(H) \)-module \( \mathcal{M} \) such that \( \dim_{\mathbb{K}(H)} \mathcal{M} < \infty \) need not have property \([H].\)

**Example 2.3.** (cf. \([2, \text{Example 1}]) \) Let \( H \) be a Hilbert space. For \( \xi, \eta \in H \) define \( \langle \xi, \eta \rangle = e_{\xi, \eta} \in \mathbb{K}(H), \) where \( e_{\xi, \eta}(\nu) = \langle \nu | \eta \rangle \xi. \) Also, for \( a \in \mathbb{K}(H), \) define a left action on \( \xi \in H \) in a natural way as the action of the operator \( a \) on the vector \( \xi. \)

In this way \( H \) becomes a left Hilbert \( \mathbb{K}(H) \)-module. Notice that the resulting norm coincides with the original norm on \( H. \)

We also know that \( \dim_{\mathbb{K}(H)} H = 1. \) Indeed, if \( \varepsilon \) is an arbitrary unit vector then each \( \xi \in H \) admits a representation of the form \( \xi = \langle \xi, \varepsilon \rangle \varepsilon \) (because \( \langle \xi, \varepsilon \rangle \varepsilon = e_{\varepsilon, \varepsilon}(\varepsilon) = (\varepsilon | \varepsilon )\xi = \xi \)). This means that \( \{\varepsilon\} \) is an orthonormal basis for \( H, \) regarded as a \( \mathbb{K}(H) \)-module.

Notice that the entire preceding discussion was independent on the (usual) dimension of the underlying space \( H. \) Suppose now that \( H \) is a separable infinite dimensional Hilbert space. We claim that then \( H, \) as a Hilbert \( \mathbb{K}(H) \)-module, does not satisfy \([H].\)

To see this, let us fix an orthonormal basis \( (\varepsilon_n) \) for \( H. \) The sequence \( (\varepsilon_n) \) is obviously norm-bounded. Suppose that there exist a subsequence \( (\varepsilon_{n_k}) \) of \( (\varepsilon_n) \) and \( \varepsilon_0 \in H \) such that \( \lim_{k \to \infty} \langle \varepsilon_{n_k}, \varepsilon \rangle = \langle \varepsilon_0, \varepsilon \rangle \) for all \( \varepsilon \in H. \) In particular, this would imply \( \lim_{k \to \infty} \langle \varepsilon_{n_k}, \varepsilon_1 \rangle = \langle \varepsilon_0, \varepsilon_1 \rangle, \) i.e., \( \|e_{\varepsilon_{n_k}, \varepsilon_1} - e_{\varepsilon_0, \varepsilon_1}\| \to 0. \) But, \( \|e_{\varepsilon_{n_k}, \varepsilon_1} - e_{\varepsilon_0, \varepsilon_1}\| = \sup_{\|\eta\| = 1} \|e_{\varepsilon_{n_k}, \varepsilon_1}(\eta) - e_{\varepsilon_0, \varepsilon_1}(\eta)\| = \sup_{\|\eta\| = 1} \|e_{\varepsilon_0, \varepsilon_1}(\eta)|\varepsilon_0 - (e_{\varepsilon_{n_k}, \varepsilon_1} - \varepsilon_0)\| = \|e_{\varepsilon_{n_k}, \varepsilon_1} - \varepsilon_0\| \) and the last expression obviously does not converge to 0 as \( k \to \infty. \)

**Remark 2.4.** Suppose that \( \mathcal{M} \) is an arbitrary Hilbert \( C^* \)-module over a \( C^* \)-algebra \( \mathcal{A} \) of compact operators. It is well known that there is a family \( (H_j), j \in J, \) of Hilbert spaces such that \( \mathcal{A} = \bigoplus_{j \in J} \mathbb{K}(H_j). \) Furthermore, it then follows that \( \mathcal{M} = \bigoplus_{j \in J} \mathcal{M}_j, \) where \( \mathcal{M}_j = \overline{\mathbb{K}(H_j) \mathcal{M}} \) (i.e., \( \mathcal{M} \) is an outer direct sum of \( \mathcal{M}_j \)'s, where each \( \mathcal{M}_j \) is a full Hilbert \( \mathbb{K}(H_j) \)-module).

Now, by \([2, \text{Theorem 3}] \) and the preceding example, we conclude that if there exists \( j_0 \in J \) such that \( \dim H_{j_0} = \infty \) then \( \mathcal{M}_{j_0} \) cannot satisfy \([H]. \) Consequently, \( \mathcal{M} \) does not satisfy \([H]. \) Namely, if \( \dim \mathcal{M}_{j_0} = d \) (here \( d \) can be an arbitrary cardinal number), then, by Theorem 3
from \([2]\), \(M_{j_0}\) is an orthogonal sum of \(d\) copies of \(\mathbb{K}(H_{j_0})H_{j_0}\), and, by Example \([2,3]\) just one copy of \(\mathbb{K}(H_{j_0})H_{j_0}\) is enough to ruin property \([H]\).

From the preceding discussion we conclude that if \(\mathcal{M}\) is a full Hilbert \(C^*\)-module satisfying \([H]\), then \(\mathcal{M}\) is necessarily a Hilbert \(C^*\)-module over a \(C^*\)-algebra \(\mathcal{A}\) of compact operators. Moreover, \(\mathcal{A}\) has to be of the form \(\mathcal{A} = \bigoplus_{j \in J} \mathbb{K}(H_j)\) and each \(H_j\) must be finite dimensional. If, moreover, \(J\) is of finite cardinality, then \(\mathcal{A}\) is finite dimensional. Next we show that if card \(J = \infty\) with \(\dim H_j < \infty\) for all \(j \in J\), then again \(\mathcal{M}\) cannot satisfy \([H]\).

First, in this situation, since \(J\) as a set of an infinite cardinality contains a countable subset \(J'\), \(\mathcal{M} = \bigoplus_{j \in J} M_j\) can be written as the orthogonal sum of the form \(\mathcal{M} = \left(\bigoplus_{j \in J'} M_j\right) \bigoplus \left(\bigoplus_{j \in J \setminus J'} M_j\right)\). Thus, \(\mathcal{M}\) contains, as an orthogonal summand, a submodule of the form \(\mathcal{M}' = \bigoplus_{n \in \mathbb{N}} \mathcal{M}_n\), where each \(\mathcal{M}_n\) is a module over \(\mathbb{K}(H_n)\) and \(\dim H_n < \infty\). Moreover, each \(\mathcal{M}_n\) is, by \([2, \text{Theorem 3}]\), unitarily equivalent to the orthogonal sum of \(d_n = \dim \mathcal{M}_n\) copies of \(\mathbb{K}(H_n)H_n\); i.e., \(\mathcal{M}_n \simeq \bigoplus_{j=1}^{d_n} \mathbb{K}(H_n)H_n\).

If we take just one copy of each \(\mathbb{K}(H_n)H_n\), we conclude that \(\mathcal{M}'\) (and hence \(\mathcal{M}\)) contains, as an orthogonal summand, a submodule of the form \(\mathcal{M}'' \simeq \bigoplus_{n=1}^{\infty} \mathbb{K}(H_n)H_n\). It is now enough to prove that \(\mathcal{M}''\) does not satisfy \([H]\) and this can be argued essentially in the same way as in Example \([2,3]\).

Observe that \(\mathcal{M}''\) is also a Hilbert \(C^*\)-module over a direct sum \(\bigoplus_{n=1}^{\infty} \mathbb{K}(H_n) \subset \mathbb{K}(H)\), where \(H = \bigoplus_{n=1}^{\infty} H_n\) is an infinite dimensional Hilbert space. For each \(n \in \mathbb{N}\) take a unit vector \(\varepsilon_n \in H_n \subset H\). Let \(x_n = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, 0, 0, \ldots), n \in \mathbb{N}\). Notice that \(\langle x_n, x_n \rangle = \sum_{i=1}^{n} e_{\varepsilon_i, \varepsilon_i}\). Since this is an orthogonal projection onto an \(n\)-dimensional subspace of \(H\), we have \(\|x_n\| = 1\); thus, \(x_n\) is a norm-bounded sequence in \(\mathcal{M}''\). Suppose now that there exists a subsequence \((x_{n_k})\) and \(x_0 = (\xi_1, \xi_2, \ldots) \in \mathcal{M}''\) such that \(\lim_{k \to \infty} \langle x_{n_k}, y \rangle = \langle x_0, y \rangle\) for all \(y \in \mathcal{M}''\). Inserting \(y = (\varepsilon_1 - \xi_1, 0, 0, \ldots)\) we obtain \(\|\langle x_{n_k}, y \rangle - \langle x_0, y \rangle\| = \|\langle \varepsilon_1 - \xi_1, \varepsilon_1 - \xi_1 \rangle\| \to 0\), which implies \(\xi_1 = \varepsilon_1\). Similarly, for \(y = (0, \varepsilon_2 - \xi_2, 0, \ldots)\) we obtain \(\xi_2 = \varepsilon_2\) and, proceeding in the same way, \(\xi_n = \varepsilon_n\) for all \(n \in \mathbb{N}\). This gives us \(x_0 = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots)\), which is impossible since this sequence does not belong to \(\mathcal{M}''\).

After all, combining the preceding discussion with Proposition 2.1 from \([3]\), we get our main result.

**Theorem 2.5.** A full Hilbert \(C^*\)-module over a \(C^*\)-algebra \(\mathcal{A}\) satisfies condition \([H]\) if and only if \(\mathcal{A}\) is a finite dimensional \(C^*\)-algebra.

**Remark 2.6.** We may ask ourselves if one could replace condition \([H]\) with a weaker one:
[H′] For each norm-bounded sequence \((x_n)\) in \(\mathcal{M}\) and for every \(y \in \mathcal{M}\) there exists a subsequence \((x_{n_k})\) of \((x_n)\) such that the sequence \((\langle x_{n_k}, y \rangle)\) converges in norm.

Observe that [H′] is sufficient to prove Proposition 2.1, so full Hilbert \(C^*\)-modules with property [H′] have to be over \(C^*\)-algebras of compact operators. Also, it is obvious that [H′] is fulfilled in every Hilbert \(C^*\)-module over a finite dimensional \(C^*\)-algebra. However, our next example shows that [H′] does not characterize these Hilbert modules; in other words, [H′] is not sufficient for [H].

Consider a separable infinite dimensional Hilbert space \(H\) and the \(C^*\)-algebra \(\mathcal{A} \subset \mathbb{K}(H)\) of all diagonal (with respect to a fixed orthonormal basis) operators with diagonal entries converging to 0. Let \(\mathcal{M} = \mathcal{A}\). Then \(\mathcal{A}\) is a Hilbert \(C^*\)-module whose underlying \(C^*\)-algebra \(\mathcal{A}\) is infinite dimensional. By the preceding theorem, the Hilbert \(C^*\)-module \(\mathcal{A}\) cannot satisfy [H].

On the other hand, since \(\mathcal{A}\) is a Hilbert \(C^*\)-module over the (commutative) \(C^*\)-algebra \(\mathcal{A}\) of compact operators, by [1, Theorem 4] (see also its proof), all mappings \(T_y : \mathcal{A} \to \mathcal{A}\) given by \(T_y(x) = \langle y, x \rangle y\) are compact. But here we have \(T_y(x) = yx^*y = x^*y^2\) for all \(y \in \mathcal{A}\). In particular, taking self-adjoint \(y\) we get that \(x \mapsto x^*y\) is compact for every positive \(y \in \mathcal{A}\), and since positive elements of a \(C^*\)-algebra span the whole \(C^*\)-algebra, we get that the operator \(x \mapsto x^*y = \langle y, x \rangle\) is compact for every \(y \in \mathcal{A}\). This shows that our Hilbert \(C^*\)-module \(\mathcal{A}\) satisfies [H′].

Acknowledgement. The authors are thankful to the referee for several valuable suggestions for improving and clarifying the original manuscript.

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