MILNOR K-THEORY OF COMPLETE DISCRETE VALUATION RINGS WITH FINITE RESIDUE FIELDS

CHRISTIAN DAHLHAUSEN

ABSTRACT. Consider a complete discrete valuation ring \( \mathcal{O} \) with quotient field \( F \) and finite residue field. Then the inclusion map \( \mathcal{O} \rightarrow F \) induces a map \( \hat{K}_n^M \mathcal{O} \rightarrow \hat{K}_n^M F \) on improved Milnor K-theory. We show that this map is an isomorphism in degrees bigger or equal to 3. This implies the Gersten conjecture for improved Milnor K-theory for \( \mathcal{O} \). This result is new in the \( p \)-adic case.

1. INTRODUCTION

Let \( \hat{K}_n^M \) denote the improved Milnor K-theory as introduced by GABBER [Ga98] and mainly developed by KERZ [Ke10]. For fields, it coincides with the usual Milnor K-theory. For \( n \geq 1 \) and a discrete valuation ring \( \mathcal{O} \) with quotient field \( F \) and residue field \( \kappa \), there is the so-called Gersten complex

\[
0 \rightarrow \hat{K}_n^M \mathcal{O} \rightarrow \hat{K}_n^M F \rightarrow \hat{K}_{n-1}^M \kappa \rightarrow 0.
\]

This complex is known to be right-exact. Is it also exact on the left side? In the equicharacteristic case this was shown by KERZ [Ke10, Prop. 10 (8)]. As a slight extension to a special case of the mixed characteristic case, we show the exactness on the left side by proving the following result (Theorem 4.2).

Theorem. Let \( \mathcal{O} \) be a complete discrete valuation ring with quotient field \( F \) and finite residue field. Then for \( n \geq 3 \) the inclusion map \( i: \mathcal{O} \rightarrow F \) induces an isomorphism

\[
i_*: \hat{K}_n^M \mathcal{O} \xrightarrow{\cong} \hat{K}_n^M F
\]

on improved Milnor K-theory.

In the \( p \)-adic case this is new. We prove the theorem as follows: We show that \( \hat{K}_n^M \mathcal{O} \) is a divisible abelian group for \( n \geq 3 \) (Theorem 3.7) using results from the appendix of MILNOR’s book [Mi71]. Combining this with the unique divisibility of \( \hat{K}_n^M F \) (proved by SVITSKII [Si86]) and a comparison between algebraic and Milnor K-theory (proved by NESTERENKO and SUSLIN [NS90]) yields the theorem.

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2. Milnor K-theory

**Definition 2.1.** Let $A$ be a commutative ring with unit, $T, A^\times$ be the (non-commutative) tensor algebra of $A^\times$ over $\mathbb{Z}$, and $\text{StR}_*, A^\times$ the homogeneous ideal of $T, A^\times$ which is generated by the set $\{x \otimes y \in T_2 A^\times | x + y = 1\}$ (the so-called Steinberg relations). Define the **Milnor K-theory** of $A$ to be the graded ring

$$K^M_*A := T_* A^\times / \text{StR}_* A^\times .$$

For $x_1, \ldots, x_n \in A^\times$ let $\{x_1, \ldots, x_n\}$ denote the image of $x_1 \otimes \ldots \otimes x_n$ under the natural homomorphism $T_n A^\times \to K^M_n A$. Evidently, this yields a functor from commutative rings to abelian groups.

This notion behaves well if $A$ is a field or a local ring with infinite residue field [Ke09]. But some nice properties do not hold for arbitrary commutative rings (e.g. that the natural map to algebraic K-theory is an isomorphism in degree 2). For local rings, this lack is repaired by a generalisation due to Gabber [Ga98], the improved Milnor K-theory, which was mainly developed by Kerz [Ke10].

**Definition 2.2.** Let $A$ be a local ring and $n \in \mathbb{N} := \mathbb{Z}_{\geq 0}$. The subset

$$S := \{ \sum_{i \in \mathbb{N}^n} a_i \cdot t^\alpha \in A[t_1, \ldots, t_n] | (a_i | i \in \mathbb{N}^n) = A \}$$

of $A[t_1, \ldots, t_n]$ is multiplicatively closed, where $t^\alpha = t_1^{i_1} \cdots t_n^{i_n}$. Define the **ring of rational functions** (in $n$ variables) to be $A(t_1, \ldots, t_n) := S^{-1} A[t_1, \ldots, t_n]$. We obtain maps $\nu_A : A \to A(t)$ and $\nu_{t_1, t_2} : A(t) \to A(t_1, t_2)$ by mapping $t$ respectively to $t_1$ or $t_2$.

For $n \geq 0$ we define the $n$-th **improved Milnor K-theory** of $A$ to be

$$\hat{K}^M_n A := \ker [K^M_n A(t) \xrightarrow{\delta^M_n} K^M_n A(t_1, t_2)],$$

where $\delta^M_n := K^M_n(t_1) - K^M_n(t_2)$. By definition, we have an exact sequence

$$0 \longrightarrow \hat{K}^M_n A \xrightarrow{\iota_n} K^M_n A(t) \xrightarrow{\delta^M_n} K^M_n A(t_1, t_2).$$

In particular, for $n = 0$ we have $\hat{K}^M_0 = \ker (\mathbb{Z} \xrightarrow{0} \mathbb{Z}) = \mathbb{Z}$. As a direct consequence of the construction we obtain a natural homomorphism

$$K^M_* A \longrightarrow \hat{K}^M_* A.$$

We state some facts about Milnor K-theory of local rings.

**Proposition 2.3.** Let $A$ be a local ring. Then:

(i) $\hat{K}^M_1 A \cong A^\times$.

(ii) For $\alpha \in \hat{K}^M_n A$ and $\beta \in \hat{K}^M_n A$ we have $\alpha \beta = (-1)^n m \beta \alpha$, i.e. $\hat{K}^M_* A$ is graded-commutative.

(iii) For $x \in A^\times$ the equations $\{x, -x\} = 0$ and $\{x, x\} = \{x, -1\}$ hold in $\hat{K}^M_* A$.

(iv) There exists a natural homomorphism

$$\Phi_{MQ}(A) : \hat{K}^M_* A \longrightarrow K_* A.$$
where $K_*$ denotes algebraic $K$-theory due to Quillen. In degree 2, this is an isomorphism.

(v) If $A$ is a field, the natural homomorphism

$$K^*_M A \longrightarrow \hat{K}^*_M A$$

is an isomorphism.

(vi) If $A$ is a finite field, then $\hat{K}_n^* A \cong 0$ for $n \geq 2$.

Proof. Everything is due to Kerz and references within this proof refer to [Ke10] (unless stated otherwise). (i) and (ii) is [Prop. 10 (1), (2)]. The first statement of (iv) follows from [Thm. 7] and the existence of a natural map $K^*_M A \rightarrow K^*_A$. The second statement of (iv) and (iii) is [Prop. 10 (3)] combined with [Prop. 2]. (v) is [Prop. 10 (4)]. (vi) follows from (v) and [Mi71, Ex. 1.5]. Precisely, (ii) and (iii) rely on the corresponding facts for $K^*_M$; for proofs of them see e.g. [GS06, Prop. 7.1.1, Lem. 7.1.2]. □

Theorem 2.4 ([Ke10 Thm. A, Thm. B]). Let $A$ be a local ring. Then:

(i) The natural homomorphism

$$K^*_M A \longrightarrow \hat{K}^*_M A$$

is surjective.

(ii) If $A$ has infinite residue field, the homomorphism

$$K^*_M A \longrightarrow \hat{K}^*_M A$$

is an isomorphism.

Lemma 2.5. Let $O$ be a discrete valuation ring with quotient field $F$ and let $\pi \in O$ be a uniformising element. For $n \geq 1$ we have

$$K^*_n F = \{ \{ \pi, u_2, \ldots, u_n \}, \{ u_1, \ldots, u_n \} | u_1, \ldots, u_n \in O^\times \}.$$

Proof. This follows straightforwardly from Proposition 2.3 (iii) and the fact that every element $x \in F^\times$ has a description $x = u \pi^k$ for suitable $u \in O^\times$ and $k \in \mathbb{Z}$. □

Theorem 2.6 ([Mi70, Lem. 2.1], cf. [GS06, Prop. 7.1.4]). Let $O$ be a discrete valuation ring with quotient field $F$, discrete valuation $v : F^\times \rightarrow \mathbb{Z}$, and residue field $\kappa$. Then for every $n \geq 1$ there exists a unique homomorphism (the TAME SYMBOL)

$$\partial : K^*_M F \longrightarrow K^*_n \kappa$$

such that for every uniformising element $\pi \in O$ and all units $u_2, \ldots, u_n \in O^\times$ we have

$$\partial(\{ \pi, u_2, \ldots, u_n \}) = \{ u_2, \ldots, u_n \}.$$ We also write $\partial_\pi$ or $\partial_v$ to indicate the considered valuation.

Proposition 2.7. Let $O$ be a discrete valuation ring with quotient field $F$ and residue field $\kappa$. For $n \geq 1$ we have an exact sequence of groups

$$0 \longrightarrow V_n \longrightarrow K^*_n F \xrightarrow{\partial} K^*_n \kappa \longrightarrow 0,$$
where $V_n$ is the subgroup of $K^M_n F$ generated by
\[ \{ \{ u_1, \ldots, u_n \} \mid u_1, \ldots, u_n \in \mathcal{O}^\times \}. \]

As a consequence, we have an exact sequence
\[ \hat{K}^M_n \mathcal{O} \to \hat{K}^M_n F \to \hat{K}^M_{n-1} \kappa \to 0 \]
where $\hat{\delta}$ is induced by $\delta$ and the natural isomorphism of Proposition 2.3 (v).

**Proof.** For the exactness of the first sequence look at [GS06, Prop. 1.7.1]. The
exactness of the second sequence is shown by a diagram chase in the commu-
tative diagram
\[ \begin{array}{ccc}
\hat{K}^M_n \mathcal{O} & \xrightarrow{\iota} & \hat{K}^M_n F \\
\downarrow & & \downarrow \\
\hat{K}^M_n \mathcal{O} & \xrightarrow{\iota} & \hat{K}^M_n F \\
\end{array} \]

Proposition 2.8 ([Ge73, Thm. 1.3 a], cf. [We10, IV. Cor. 1.13; V. Cor. 6.9.2]).
Let $\mathcal{O}$ be a discrete valuation ring with quotient field $F$ and finite
residue field $\kappa$. Then for all $n \geq 0$ there is an exact sequence
\[ 0 \to K_n \mathcal{O} \to K_n F \to K_{n-1} \kappa \to 0. \]
Furthermore, $K_{2n} \kappa = 0$ for $n \geq 1$ [Qu72].

In particular, by using Proposition 2.3 (iv), there is a short exact sequence
\[ 0 \to K^M_2 \mathcal{O} \to K^M_2 F \to K^M_1 \kappa \to 0. \] (1)
The goal is to generalise the sequence (1) to arbitrary $n \geq 2$. We want to
know if for any discrete valuation ring $\mathcal{O}$ and all $n \geq 2$ the sequences
\[ 0 \to K^M_n \mathcal{O} \to K^M_n F \to K^M_{n-1} \kappa \to 0 \] (2)
are exact? For algebraic K-theory, GERSTEN conjectured that the analogous
sequences to (2) are exact for all $n \geq 2$ [Ge73]. Thus we refer to this question
as the Gersten conjecture for improved Milnor K-theory. In the case that
$\mathcal{O}$ is complete and $\kappa$ is finite this will be an immediate consequence of our main
result which relies on an analogous statement by NESTERENKO and SUSLIN
for local rings with infinite residue field and classical Milnor K-theory [NS90,
Thm. 4.1]. A proof based on their result will be presented in the appendix.

**Proposition 2.9** ([Ke10, Prop. 10 (6)]). Let $A$ be a local ring and $n \in \mathbb{N}$. Then
there exists a map
\[ \Phi_{MQ}(A) : \hat{K}^M_n A \to K_n A \]
as well as a map
\[ \Phi_{QM}(A) : K_n A \to \hat{K}^M_n A \]
such that the composition
\[ \hat{K}^M_n A \xrightarrow{\Phi_{MQ}(A)} K_n A \xrightarrow{\Phi_{QM}(A)} \hat{K}^M_n A \]
is multiplication with $\chi_n := (-1)^{n-1}.(n-1)!$. 

3. Divisibility of $\hat{K}_n^M \mathcal{O}$ for $n \geq 3$

In this section we prove that $\hat{K}_n^M \mathcal{O}$ is divisible for a complete discrete valuation ring $\mathcal{O}$ with finite residue field and $n \geq 3$. This result will be the key ingredient for the proof of our main result. First, we examine the divisibility prime to $p$.

**Definition 3.1.** For an abelian group $A$ and $m \in \mathbb{Z}$ we set $A/m := A/mA$, where $mA = \{ma | a \in A\}$. Thus $A/m \cong A \otimes_{\mathbb{Z}} \mathbb{Z}/m$.

**Lemma 3.2.** Let $\mathcal{O}$ be a complete discrete valuation ring with quotient field $F$ and finite residue field $\kappa$ of characteristic $p$. Furthermore, let $m \in \mathbb{Z}$ be such that $(p, m) = 1$. Then the canonical projection $\mathcal{O} \rightarrow \kappa$ induces an isomorphism

$$\frac{(K^M \mathcal{O})/m}{(K^M \kappa)/m} \cong (\mathcal{O}^\times/m \cong (U_1 \times \kappa^\times)/m \cong (\kappa^\times/m) \cong \kappa^\times/m).$$

of graded rings.

**Proof.** The claim is trivial for $K^M_0$. From Hensel's Lemma we obtain

$$\mathcal{O}^\times/m \cong (U_1 \times \kappa^\times)/m \cong (\kappa^\times/m) \cong \kappa^\times/m.$$\n
This implies the claim for $K^M_1$. Now let $n \geq 2$. Then we have the following commutative diagram with exact rows

$$\begin{array}{c}
0 \rightarrow \text{StR}_n \mathcal{O}^\times \rightarrow T_n \mathcal{O}^\times \rightarrow K^M \mathcal{O} \rightarrow 0 \\
\downarrow \downarrow \downarrow \\
0 \rightarrow \text{StR}_n \kappa^\times \rightarrow T_n \kappa^\times \rightarrow K^M \kappa \rightarrow 0,
\end{array}$$

where $T_n \mathcal{O}^\times$ and $T_n \kappa^\times$ denote the tensor algebras of $\mathcal{O}^\times$ and $\kappa^\times$ and

$$\text{StR}_n \mathcal{O}^\times = \{x_1 \otimes \ldots \otimes x_n \in T_n \mathcal{O}^\times | \exists i \neq j : x_i + x_j = 1\},$$

$$\text{StR}_n \kappa^\times = \{\bar{x}_1 \otimes \ldots \otimes \bar{x}_n \in T_n \kappa^\times | \exists i \neq j : \bar{x}_i + \bar{x}_j = 1\}.$$\n
Note that $\bar{\mathcal{T}} = \bar{T}$ in $\kappa$ implies $y \in U_1$. Tensoring the diagram (3) with $\mathbb{Z}/m$ we obtain the following commutative diagram with exact lines

$$\begin{array}{c}
\text{StR}_n \mathcal{O}^\times/m \rightarrow T_n \mathcal{O}^\times/m \rightarrow K^M \mathcal{O}/m \rightarrow 0 \\
\downarrow \downarrow \downarrow \\
\text{StR}_n \kappa^\times/m \rightarrow T_n \kappa^\times/m \rightarrow K^M \kappa/m \rightarrow 0.
\end{array}$$

It is easy to check that $(T_+ A)/m \cong T_+(A/m)$ for an arbitrary abelian group $A$; along with $\mathcal{O}^\times/m \cong \kappa^\times/m$ we see that $\beta$ is an isomorphism.

Now let’s show that $\alpha$ is surjective. Therefore let $n \in \mathbb{N}$ and $\bar{x}_1 \otimes \ldots \otimes \bar{x}_n \in \text{StR}_n \kappa^\times$. Without loss of generality we can assume that $\bar{x}_1 + \bar{x}_2 = \bar{1}$, i.e. $x_1 + x_2 =: u \in U_1$, hence $u^{-1} x_1 + u^{-1} x_2 = 1$ in $\mathcal{O}$. Thus $\zeta := u^{-1} x_1 \otimes u^{-1} x_2 \otimes x_3 \otimes \ldots \otimes x_n \in \text{StR}_n \mathcal{O}^\times$ satisfies $a(\zeta) = \bar{x}_1 \otimes \ldots \otimes \bar{x}_n$. So $\alpha$ is surjective.

Applying the five lemma to (4) demonstrates that $\gamma$ is an isomorphism. This concludes the proof. ☐

**Corollary 3.3.** Let $\mathcal{O}$ be a complete discrete valuation ring with quotient field $F$ and finite residue field $\kappa$ of characteristic $p$. Then for $n \geq 2$ the groups $K^M_n \mathcal{O}$ and $\hat{K}_n^M \mathcal{O}$ are $m$-divisible if $(p, m) = 1$. 
For \( n \geq 2 \) and \( m \in \mathbb{Z} \) such that \( (p,m) = 1 \) we have \((K^M_n \mathcal{O})/m \cong (K^M_n \kappa)/m = 0\), whence \( K^M_n \kappa = 0 \) since \( \kappa \) is a finite field (see Proposition 2.3 (iv)). Thus \( K^M_n \mathcal{O} \) is \( m \)-divisible. The surjective homomorphism \( K^M_n \mathcal{O} \rightarrow \hat{K}^M_n \mathcal{O} \) shows the claim for \( \hat{K}^M_n \mathcal{O} \).

Next we want to understand multiplication by \( p \) on Milnor K-theory. For that we need the following results.

**Proposition 3.4.** Let \( \mathcal{O} \) be a complete discrete valuation ring with quotient field \( F \) and finite residue field of characteristic \( p \).

(i) Let \( F \) contain all \( p \)-th roots of unity. Then \((\hat{K}^M_2 F)/p \) is cyclic of order \( p \).

(ii) If \( F \) contains only the trivial \( p \)-th root of unity, \((\hat{K}^M_2 F)/p \) is zero.

Here we use tacitly the fact that \( \hat{K}^M_* F \cong K^M_* F \) from Proposition 2.3 (v).

**Proof.** This is done in [Mi71], (i) is [Cor. A.12] and (ii) follows from [Lem. A.6] together with [Lem. A.4]. □

To treat a special case in our proof below, we need a norm map for a non-étale extension which is not covered literally by KERZ’ work. However, everything needed is basically already contained in [Ke09].

**Proposition 3.5.** Let \( A \) be a local and factorial domain and \( \iota: A \hookrightarrow B \) an extension of local rings such that \( B \cong A[X]/(\pi) \) for a monic irreducible polynomial \( \pi \in A[X] \). Then there exists a transfer morphism, the **Norm Map**,

\[ N_{B/A}: K^M_* B \rightarrow K^M_* A \]

satisfying the projection formula

\[ N_{B/A}(\langle \iota_* (x), y \rangle) = \{ x, N_{B/A}(y) \} \] (5)

for all \( x \in K^M_* A \) and \( y \in K^M_* B \).

**Proof.** This will be done in the appendix, see C.1 □

**Proposition 3.6** (Bass, Tate, cf. [Mi71 Cor. A.15]). Let \( \mathcal{O} \subset \mathcal{O}' \) be complete discrete valuation rings with quotient fields \( F \subset F' \) and finite residue fields. Then the norm map

\[ N_{F'/F}: \hat{K}^M_2 F' \rightarrow \hat{K}^M_2 F \]

is surjective.

With this we can prove the missing \( p \)-divisibility.

**Theorem 3.7.** Let \( \mathcal{O} \) be a complete discrete valuation ring with quotient field \( F \), finite residue field \( \kappa \) of characteristic \( p \), and \( n \geq 3 \). Then \( \hat{K}^M_n \mathcal{O} \) is divisible.

**Proof.** The homomorphism

\[ K^M_n \mathcal{O} \otimes K^M_{n-3} \mathcal{O} \rightarrow K^M_n \mathcal{O} \]

\[ \{ x_1, x_2, x_3 \} \otimes \{ x_4, \ldots, x_n \} \mapsto \{ x_1, \ldots, x_n \} \]
is surjective and for every \( n \geq 0 \) the homomorphism \( K^M_n \mathcal{O} \to \hat{K}_n^M \) is also surjective by Proposition 2.3 (8). By the commutative diagram

\[
\begin{array}{ccc}
K_3^M \mathcal{O} \otimes_{\mathbb{Z}} K_{n-3}^M & \longrightarrow & K_n^M \\
\downarrow & & \downarrow \\
\hat{K}_3^M \mathcal{O} \otimes_{\mathbb{Z}} \hat{K}_{n-3}^M & \longrightarrow & \hat{K}_n^M,
\end{array}
\]

we see that the lower map is also surjective and hence its target is divisible if \( \hat{K}_3^M \mathcal{O} \) is divisible. Thus it suffices to show that multiplication with \( \ell \) on \( \hat{K}_3^M \mathcal{O} \) is surjective for every prime number \( \ell \). Equivalently, we can show that \( (\hat{K}_3^M \mathcal{O})/\ell = 0 \) for every prime \( \ell \). The case \( \ell \neq p \) is the assertion of Corollary 3.3. All in all, it remains to show that \( (\hat{K}_3^M \mathcal{O})/p = 0 \).

According to Proposition 2.8 we have an exact sequence

\[
0 \longrightarrow K_2^M \mathcal{O} \overset{\iota}{\longrightarrow} K_2^M \mathcal{F} \overset{\hat{\delta}}{\longrightarrow} \hat{K}_1^M K \longrightarrow 0.
\]

Tensoring with \( \mathbb{Z}/p \) leads to an exact sequence

\[
\text{Tor}_1^Z(\mathbb{Z}/p, \hat{K}_1^M K) \longrightarrow (\hat{K}_2^M \mathcal{O})/p \longrightarrow (\hat{K}_2^M \mathcal{F})/p \longrightarrow \hat{K}_1^M K/p,
\]

as \( \hat{K}_1^M K \cong \kappa^\times \) is a finite cyclic group with order prime to \( p \). By Proposition 3.4, we get accordingly

\[
(K_2^M \mathcal{O})/p \cong (K_2^M \mathcal{F})/p \cong \begin{cases} 
\mathbb{Z}/p & \text{if } \mathcal{F} \text{ contains the } p\text{-th roots of unity} \\
0 & \text{else}
\end{cases} \tag{6}
\]

If this is zero, we are done. Thus we consider only the other case.

Now we suppose for a while that \(-1\) has a \( p\)-th root \( \sqrt[p]{-1} \) in \( \mathcal{F} \). In fact, this is always the case for \( p = 2 \) or if \( \mathcal{F} \) is a finite extension of \( \mathbb{F}_2((t)) \). Let \( \{u, v, w\} \in \hat{K}_3^M \mathcal{O} \) and let squared brackets denote residue classes modulo \( p \). If \([u, x] = 0\) in \((\hat{K}_3^M \mathcal{O})/p \cong \mathbb{Z}/p\) for all \( x \in \mathcal{O}^* \), then

\[
[u, v, w] = [u, v] \cdot [w] = 0 \in (\hat{K}_3^M \mathcal{O})/p.
\]

Otherwise, there is an \( x \in \mathcal{O}^* \) such that \([u, x]\) is a generator of \((\hat{K}_3^M \mathcal{O})/p\). Let \([v, w] = k \cdot [u, x]\) for a suitable \( k \in \mathbb{Z} \). Then

\[
[u, v, w] = k \cdot [u, u, x] = k \cdot [-1, u, x] = k p \cdot \sqrt[p]{-1} \cdot u, x = 0 \in (\hat{K}_3^M \mathcal{O})/p.
\]

If, in contrast, \(-1\) does not have a \( p\)-th root in \( \mathcal{F} \), we have \( p = 2 \) and \( \mathcal{F} \) is a finite extension of \( \mathbb{Q}_2 \). Thus \( \mathcal{O} \) is a local and factorial domain. We consider the finite field extension \( F' := \mathcal{F}[\sqrt{-1}] \). By [Se79, ch. 1, §6 (ii)], the valuation ring of \( F' \) is \( \mathcal{O}' := \mathcal{O}[\sqrt{-1}] \). This allows us to apply Proposition 3.5.

Due to finiteness, \( F' \) is also complete. The norm map \( N_{F'/F} : \hat{K}_2^M F' \to \hat{K}_2^M F \) is surjective due to Proposition 3.6 and induces a surjective map \( N_{F'/F} : (\hat{K}_2^M F')/p \to (\hat{K}_2^M F)/p \).
This induces a surjective map
\[ \tilde{\mathcal{N}}_{\mathcal{O}/\mathcal{O}'} : (\hat{K}_2^M \mathcal{O}')/p \to (\hat{K}_2^M \mathcal{O})/p \]
on the Milnor K-theory of their integers via the identification (10). Let \([x, y, z] \in (\hat{K}_2^M \mathcal{O})/p\). Decompose \([x, y, z] = [x] \cdot [y, z]\) such that \([x] \in (\hat{K}_1^M \mathcal{O})/p\) and \([y, z] \in (\hat{K}_2^M \mathcal{O})/p\). We find a preimage \(a \in \hat{K}_2^M \mathcal{O}'\) of \([y, z]\) under \(\tilde{\mathcal{N}}_{\mathcal{O}/\mathcal{O}'}\). The projection formula (5) yields
\[ \tilde{\mathcal{N}}_{\mathcal{O}/\mathcal{O}'}(i_*([x]) \cdot a) = [x] \cdot \tilde{\mathcal{N}}_{\mathcal{O}/\mathcal{O}'}(a) = [x] \cdot [y, z] = [x, y, z], \]
where \(i: \mathcal{O} \to \mathcal{O}'\) is the inclusion. Thus \((\hat{K}_3^M \mathcal{O}')/p \to (\hat{K}_3^M \mathcal{O})/p\) is onto. By the previous argument, the domain of this map is zero, hence its target, too. This concludes the proof. \(\square\)

4. THE ISOMORPHISM \(\hat{K}_n^M \mathcal{O} \xrightarrow{\cong} \hat{K}_n^M \mathcal{F}\) FOR \(n \geq 3\)

Our last ingredient is the following result by SIVITSKII:

\[ \text{Theorem 4.1 (} \text{Si86, p. 562)}\]. Let \(\mathcal{O}\) be a complete discrete valuation ring with quotient field \(\mathcal{F}\) and finite residue field, and let \(n \geq 3\). Then \(\hat{K}_n^M \mathcal{F}\) is an uncountable, uniquely divisible group.

The uncountability was found by BASS and TATE [BT73] and divisibility was shown by MILNOR [Mi70, Ex. 1.7]. The torsion-freeness was originally proved by SIVITSKII [Si86] with explicit calculations; there is also a more structural proof using the Norm Residue Theorem which was formerly known as the Bloch-Kato conjecture and which was proved by ROST and VOEVODSKY, for a proof consider [We10, VI. Prop. 7.1]. Now we are able to prove our main result.

\[ \text{Theorem 4.2}. \text{ Let } \mathcal{O} \text{ be a complete discrete valuation ring with quotient field } \mathcal{F} \text{ and finite residue field, and let } n \geq 3. \text{ Then the inclusion } i: \mathcal{O} \hookrightarrow \mathcal{F} \text{ induces an isomorphism } \]
\[ i_* : \hat{K}_n^M \mathcal{O} \xrightarrow{\cong} \hat{K}_n^M \mathcal{F} \]
on improved Milnor K-theory.

\[ \text{Proof}. \text{ Surjectivity. Let } \kappa \text{ be the residue field of } \mathcal{O}. \text{ According to Proposition 2.7 we have an exact sequence } \]
\[ \hat{K}_n^M \mathcal{O} \xrightarrow{i_*} \hat{K}_n^M \mathcal{F} \xrightarrow{\delta} \hat{K}_{n-1}^M \kappa \to 0. \]

Because \(\kappa\) is a finite field and \(n - 1 \geq 2\), we see that \(\hat{K}_{n-1}^M \kappa \cong K_{n-1}^M \kappa = 0\). Thus \(i_*\) is surjective.
**Injectivity.** Consider the commutative diagram

\[
\begin{array}{ccc}
\hat{K}_n^M \Omega & \xrightarrow{\psi} & \hat{K}_n^M F \\
\downarrow \chi_n & & \downarrow \chi_n \\
K_n^M \Omega & \xrightarrow{\psi} & K_n^M F
\end{array}
\]

(7)

where \(K_n(i)\) is injective due to Proposition 2.8. Let \(\alpha \in \ker(\psi)\). As \(\hat{K}_n^M \Omega\) is divisible we find \(\beta \in \hat{K}_n^M \Omega\) such that \(\alpha = \chi_n \beta\) where \(\chi_n = (-1)^{n-1}(n-1)\). Hence \(0 = \psi(\alpha) = \psi(\chi_n \beta) = \chi_n \psi(\beta)\).

We have \(\beta \in \ker(\psi)\) because \(\hat{K}_n^M F \cong K_n^M F\) is uniquely divisible by Theorem 4.1. Since \(K_n(i)\) is injective, we also have \(\ker(\psi) \subseteq \ker(\Phi_{MQ}(\Omega))\) by diagram chasing and the latter one is clearly a subset of \(\ker(\chi_n)\). We conclude that \(\alpha = \chi_n \beta = 0\). So \(\psi\) is injective, hence an isomorphism.

**Remark.** In fact, one could simplify this proof using only 2-divisibility for \(\hat{K}_n^M \Omega\). With that one could proof the theorem first for \(n = 3\) and gets entire divisibility of \(\hat{K}_n^M \Omega\) for free by Theorem 4.1. Then, one could do the proof for higher \(n\). In characteristic \(p \neq 2\), this would spare the proof of Theorem 3.7.

Immediately from Theorem 4.1 (respectively its proof) we obtain the following.

**Corollary 4.3.** Let \(\Omega\) be a complete discrete valuation ring with finite residue field. Then:

(i) The canonical map \(\hat{K}_n^M \Omega \to K_n \Omega\) is injective.

(ii) \(\hat{K}_n^M \Omega\) is uniquely divisible for \(n \geq 3\).

Furthermore, we are now able to generalise the exact sequence (1).

**Corollary 4.4.** Let \(\Omega\) be a complete discrete valuation ring with quotient field \(F\) and finite residue field \(\kappa\). Then we have for each \(n \geq 1\) an exact sequence

\[
0 \longrightarrow \hat{K}_n^M \Omega \xrightarrow{\psi} \hat{K}_n^M F \longrightarrow \hat{K}_{n-1}^M \kappa \longrightarrow 0.
\]

**APPENDIX A. THE RING OF RATIONAL FUNCTIONS**

The ring of rational functions plays a crucial role for the definition of improved Milnor K-theory. This section’s content relies on KERZ’S paper [Ke10] and elaborates some of the proofs. We start by recalling Definition 2.2 from above.

**Definition A.1.** Let \(A\) be a commutative ring and \(n \in \mathbb{N}\). The subset

\[
S := \left\{ \sum_{i \in \mathbb{N}^n} a_i \cdot t^i \in A[t_1, \ldots, t_n] \mid (a_i | i \in \mathbb{N}^n) = A \right\}
\]
is multiplicatively closed, where $t^t = t_1^{t_1} \cdot \ldots \cdot t_n^{t_n}$. Define the ring of rational functions (in $n$ variables) to be

$$A(t_1, \ldots, t_n) := S^{-1}A[t_1, \ldots, t_n].$$

We obtain maps $v: A \to A(t)$ and $t_1, t_2: A(t) \to A(t_1, t_2)$ by mapping $t$ respectively to $t_1$ or $t_2$.

**Remark.** For a field $F$ and $n \geq 1$ we have $F(t_1, \ldots, t_n) = \text{Frac}(F[t_1, \ldots, t_n])$.

**Lemma A.2** (cf. [Ke10, p. 6]). Let $(A, m, \kappa)$ be a local ring and $n \geq 1$. Then the ring $A(t_1, \ldots, t_n)$ is a local ring with maximal ideal $mA(t_1, \ldots, t_n)$ and infinite residue field $\kappa(t_1, \ldots, t_n)$.

**Proof.** Set $B := A[t_1, \ldots, t_n]$ and $n := mA[t_1, \ldots, t_n]$. Obviously, $n \triangleleft B$ is a prime ideal and

$$S = \left\{ \sum_{i \in \mathbb{N}_0} a_i \cdot t^i \in A[t_1, \ldots, t_n] \mid (a_i)_{t^i} \in I^n \right\} = A,$$

$$= \left\{ \sum_{i \in \mathbb{N}_0} a_i \cdot t^i \in A[t_1, \ldots, t_n] \mid \exists i \in \mathbb{N} : a_i \notin m \right\}$$

$$= B \setminus n,$$

hence $A(t_1, \ldots, t_n) = B_n$ is local with maximal ideal $nB_n$. Its residue field is

$$B_n/nB_n \cong (B/n)_n = \text{Frac}(\kappa[t_1, \ldots, t_n]) = \kappa(t_1, \ldots, t_n).$$

$\square$

**Lemma A.3.** Let $(A, m)$ be a local ring and $A \to B$ an integral ring extension such that $mB \triangleleft B$ is a prime ideal. Then $B$ is also local with maximal ideal $mB$.

**Proof.** According to the Going-Up theorem (see [Ma80, pt. I, ch. 2, Thm. 5]) the maximal ideals of $B$ are precisely the prime ideals of $B$ lying over $m$. But every prime ideal $p \triangleleft B$ with $A \cap p \ni m$ must contain $mB$. Thus it is the only maximal ideal of $B$. $\square$

**Lemma A.4.** Let $A$ be a ring, $S \subseteq A$ a multiplicatively closed subset and let $v: A \to S^{-1}A$ the canonical homomorphism. If $S$ does not contain any zero-divisors, then $v$ is injective. In this case a ring homomorphism $f_S: S^{-1}A \to B$ is injective if and only if the composition $f: A \to B$ is injective.

**Proof.** If $f$ is injective and $0 = f_S(a/b) = f(a)f(s)^{-1}$, then $f(a) = 0$ since $f(s)^{-1}$ is a unit. The rest is omitted. $\square$

**Proposition A.5.** Let $A \to B$ be a flat, local (i.e. the maximal ideal of $A$ generates the maximal ideal of $B$), and integral extension of local rings, and $n \geq 1$. Then the canonical homomorphism

$$\varphi: B \otimes_A A(t_1, \ldots, t_n) \xrightarrow{\cong} B(t_1, \ldots, t_n)$$

is an isomorphism. In particular, the induced homomorphism $A(t_1, \ldots, t_n) \to B(t_1, \ldots, t_n)$ is an extension of local rings with infinite residue fields.
Proof. For convenience, we write “t” for “t₁,...,tₙ”. Let m ⊲ A and n ⊲ B the maximal ideals, hence n = mB. We set mᵢ := mA[t] and nᵢ := nB[t] = mB[t]. We use some facts for ring maps corresponding to the permanence s of properties of schemes as stated in [GW10, Appendix C].

**Injectivity.** The flat base change A[t] → B[t] is injective. According to Lemma A.4, the localisation B[t] → B(t) is injective. Applying again Lemma A.4, this time to A[t] → A(t), and using the commutative diagram

\[
\begin{array}{ccc}
A[t] & \longrightarrow & B[t] \\
\downarrow & & \downarrow \\
A(t) & \longrightarrow & B(t),
\end{array}
\]

we see that A(t) → B(t) is injective, hence also B ⊗ₐ A(t) → B ⊗ₐ B(t) since B is a flat A-module. Thus the commutative diagram

\[
\begin{array}{ccc}
B ⊗ₐ A(t) & \longrightarrow & B(t) \\
\downarrow \psi & & \downarrow \\
B ⊗ₐ B(t)
\end{array}
\]

tells us that ψ is injective.

**Surjectivity.** The base change A(t) → B ⊗ₐ A(t) is an integral extension. By Lemma A.3, B ⊗ₐ A(t) is a local ring with maximal ideal mᵢ(B ⊗ₐ A(t)). Furthermore, we have a commutative diagram

\[
\begin{array}{ccc}
B ⊗ₐ A(t) & \longrightarrow & (A[t] ⊗ₐ A[t])A(t) \\
\downarrow \varphi & & \downarrow \varphi' \\
B(t) & \longrightarrow & B[t] ⊗ₐ A[t]A(t)
\end{array}
\]

Hence surjectivity for ϕ is equivalent to surjectivity for ϕ’. Consider the homomorphism

\[
\psi : B[t] \longrightarrow B[t] ⊗ₐ A[t]A(t)
\]

\[f \mapsto f \otimes 1\]

If \( f = \sum_{i=0}^{d} b_i t^i \in B[t]\backslash n_i \), then there is a j such that \( b_j \) is a unit in B. Then its image \( \psi(b_j) \) is a unit as well and hence \( \psi(f) \in (B[t] ⊗ₐ A[t]A(t))^\times \). Thus the universal property of localisation yields a homomorphism

\[
\psi' : B(t) \longrightarrow B[t] ⊗ₐ A[t]A(t)
\]
such that $\psi = \psi' \circ \iota$ where $\iota : B[t] \to B(t)$. The commutative diagram

\[
\begin{array}{ccc}
B[t] & \xrightarrow{\psi} & B(t) \\
\downarrow{\text{id}_{B[t]}} & & \downarrow{\text{id}_{B(t)}} \\
B[t] \otimes_A A(t) & \xrightarrow{\psi'} & B(t)
\end{array}
\]

and the uniqueness for lifts to the localisation show $\psi' \circ \psi' = \text{id}_{B(t)}$. So $\psi'$ is surjective.

\section*{Appendix B. Milnor K-theory and Algebraic K-theory}

This section is dedicated to Kerz’s Theorem \ref{thm:kerz} which we restate at this place again.

\begin{theorem}[\cite{Ke10} Prop. 10 (6)] Let $A$ be a local ring and $n \in \mathbb{N}$. Then there exists a natural map

$\Phi_{MQ}(A) : K^M_n A \to K_n A$

as well as a natural map

$\Phi_{QM}(A) : K_n A \to K^M_n A$

such that the composition

$K^M_n A \xrightarrow{\Phi_{MQ}(A)} K_n A \xrightarrow{\Phi_{QM}(A)} K^M_n A$

is multiplication with $\chi_n := (-1)^{n-1} \cdot (n-1)!$.

As there are few details given in \cite{Ke10}, we present more of them in this section. The statement relies on an analogous statement by Nesterenko and Suslin for local rings with infinite residue field and classical Milnor K-theory.

For any commutative ring $A$ there exists a graded-commutative product map $K_1(A) \otimes \ldots \otimes K_1(A) \to K_n(A)$ such that the Steinberg relations are satisfied \cite{We10} IV.1.10.1]. Thus there exists a graded map

$\Phi_{MQ}'(A) : K^M_n A \to K_n A$.

In the case of a local ring with an infinite residue field, there is also a map in the other direction.

\begin{theorem}[\cite{NS90} Thm. 4.1] Let $A$ be a local ring with infinite residue field and $n \in \mathbb{N}$. Then there exists a natural map

$\Phi_{QM}'(A) : K_n A \to K^M_n (A)$

such that the composition

$K^M_n A \xrightarrow{\Phi_{MQ}'(A)} K_n A \xrightarrow{\Phi_{QM}'(A)} K^M_n A$

is multiplication with $\chi_n = (-1)^{n-1} \cdot (n-1)!$. 
We say a few words on the construction of the map $\Phi_{QM}^\prime(A)$. By [NS90, Thm. 3.25] there are natural isomorphisms

$$H_n(GL_n(A)) \xrightarrow{\cong} H_n(GL(A))$$ and

$$H_n(GL_n(A))/H_n(GL_{n-1}(A)) \xrightarrow{\cong} K_n^M A.$$ for a local ring $A$ with infinite residue field. So a homomorphism $\Phi_{QM}^\prime(A)$ can be defined as the composition of homomorphisms such that the following diagram commutes.

$$\begin{array}{ccc}
K_n A & \xrightarrow{\pi_n(BGL(A)^+)} & H_n(BGL(A)^+) \\
\downarrow \Phi_{QM}^\prime(A) & & \downarrow \cong \\
K_n^M A & \xrightarrow{\cong} & H_n(GL_n(A))/H_n(GL_{n-1}(A)) \\
\downarrow & & \downarrow \\
\Phi_{QM}^\prime(A) & \xrightarrow{\delta^Q} & K_n A
\end{array}$$

For passing from the case of local rings with infinite residue fields to the case of arbitrary local rings, we consider the construction of improved Milnor K-theory in a more general setting. Given a functor $E$ from rings to abelian groups, we can associate to it an improved functor $\tilde{E}$ given by

$$\tilde{E}(A) := \ker\left[E(A(t)) \xrightarrow{\delta} E(A(t_1, t_2))\right]$$

where $A(t)$ and $A(t_1, t_2)$ are rings of rational functions and $\delta$ is the map $E(t_1) - E(t_2)$ (cf. Definition [A.1]). This construction is essentially due to GABBER [Ga98] and was investigated by KERZ [Ke10]. The latter one proved [Ke10, Prop. 9] that the canonical map $E(A) \rightarrow \tilde{E}(A)$ is an isomorphism if one of the following two conditions holds

(i) The ring $A$ is a local with infinite residue field.
(ii) The ring $A$ is local and $E$ admits compatible norm maps for all finite \'{e}tale extensions of local rings (cf. [Ke10, p. 5]).

Algebraic K-theory admits those norm maps (cf. [We10, IV.6.3.2]) where they are called “transfer maps”). Thus for any local ring $A$ we have a natural isomorphism

$$K_n A \xrightarrow{\cong} \hat{K}_n A. \quad (8)$$

**Proof of Theorem [B.1]** Let $A$ be a local ring. Then the rings $A(t)$ and $A(t_1, t_2)$ are local rings with infinite residue field, see Lemma [A.2] By naturality of the maps $\Phi_{MQ}$ and $\Phi_{QM}$ together with the isomorphism (8), we obtain the desired property of the kernel of the map as indicated in the diagram

$$\begin{array}{ccc}
K_n^M A & \xrightarrow{\ker} & K_n^M A(t) \\
\downarrow \Phi_{MQ}(A(t)) & & \downarrow \Phi_{MQ}(A(t_1, t_2)) \\
K_n A & \xrightarrow{\ker} & K_n A(t) \\
\downarrow \Phi_{QM}(A(t)) & & \downarrow \Phi_{QM}(A(t_1, t_2)) \\
K_n^M A & \xrightarrow{\ker} & K_n^M A(t) \\
\downarrow \Phi_{QM}(A(t)) & & \downarrow \Phi_{QM}(A(t_1, t_2)) \\
K_n^M A & \xrightarrow{\ker} & K_n^M A(t) \\
\downarrow \Phi_{QM}(A(t)) & & \downarrow \Phi_{QM}(A(t_1, t_2)) \\
K_n^M A & \xrightarrow{\ker} & K_n^M A(t)
\end{array}$$

By Theorem [B.2] the middle vertical composition is multiplication with the natural number $\chi_n$, hence this also holds for the left vertical composition. $\square$
Appendix C. The norm map

There is a norm map for Milnor K-theory of fields. More precisely, for every field extension $F \hookrightarrow F'$ there is a homomorphism $N_{F'/F} : K^M_{*} F' \rightarrow K^M_{*} F$ of graded $K^M_{*} F$-modules such that the following two conditions hold.

(i) **Functoriality.** For every tower $F \hookrightarrow F' \hookrightarrow F''$ of fields holds $N_{F'/F} = N_{F''/F} \circ N_{F'/F}$.

(ii) **Reciprocity.** For $\alpha \in K^M_{*} F(t)$ holds

$$\sum_{v} N_{\kappa(v)/F} \circ \partial_{v}(\alpha) = 0$$

where $v$ runs over all discrete valuations of $F(t)$ over $F$ and $\kappa(v)$ is the residue field of $v$.

We give a brief sketch of the construction of the norm map. For a detailed exposition we refer the reader to \cite{GS06} § 7.3.

For a finite field extension $F'/F$ such that $F' \cong F[X]/(\pi)$ where $\pi$ is monic and irreducible, the norm map is defined via the split exact Bass-Tate sequence

$$0 \longrightarrow K^M_{*} F \longrightarrow K^M_{*} F(X) \oplus_{P} \bigoplus_{P} K^M_{*}(F[X]/(P)) \longrightarrow 0,$$

where the sum is over all monic irreducible $P \in F[X]$ and $\oplus P$ is the well-defined sum of the tame symbols $\partial_{P}$ (with respect to the discrete valuation which is associated to the prime element $P$, see Theorem \ref{thm:Bass-Tate}). Taking coproducts over all values of $n$, we obtain an exact sequence of $K^M_{*} F$-modules. Let $\partial_{\infty}$ be the tame symbol corresponding to the negative degree valuation on $F(X)$. Its residue field is isomorphic to $F$. The composition $K^M_{*} F \rightarrow K^M_{*} F(\pi) \rightarrow K^M_{*} F$ vanishes since all elements in $F$ have degree zero.

By the universal property of the cokernel, we obtain a map $N$ as indicated in the diagram (where all the maps have degree zero).

$$0 \longrightarrow K^M_{*} F \longrightarrow K^M_{*} F(X) \oplus_{P} \bigoplus_{P} K^M_{*}(F[X]/(P)) \longrightarrow 0$$

By precomposing $N$ with the inclusion of the factor $K^M_{*}(F[X]/(\pi))$, we get the norm map

$$N_{F'/F} : K^M_{*} F' \rightarrow K^M_{*} F$$

of the extension $F'/F$.

All the maps in diagram \ref{eq:norm} are (graded) $K^M_{*} F$-linear. This is clear for the map $\iota_*$, which is induced by the inclusion $\iota : F \hookrightarrow F(X)$. With the explicit description of the tame symbol (see Theorem \ref{thm:tame_symbol}) we deduce linearity for the tame symbols $\partial : K^M_{*} F(X) \rightarrow K^M_{*} F$ where $\kappa = F$ (if $\partial = \partial_{\infty}$ or $\kappa = F[X]/(P)$ (if $\partial = \partial_{P}$ for a monic irreducible $P \in F[X]$). Let $\pi$ be a uniformiser and $O$ the valuation ring with respect to the valuation in question. Then $K^M_{*} F(X)$ is
additively generated by the set
\[ \{ (\pi, u_2, \ldots, u_n), \{u_1, \ldots, u_n\} \mid n \geq 1, u_1, \ldots, u_n \in \mathcal{O}^\times \} . \]
We note that \( \partial(\{u_1, \ldots, u_n\}) = 0 \) for all \( u_1, \ldots, u_n \in \mathcal{O}^\times \), hence linearity for those elements is clear. Now let \( x_1, \ldots, x_k \in F^\times \). Observe that \( F^\times \subseteq \mathcal{O}^\times \) and that the map \( F \to \kappa \) is injective. Thus we see \( K^M_\kappa F \)-linearity as follows.
\[
\partial\left( \{x_1, \ldots, x_k\} \cdot (\pi, u_2, \ldots, u_n) \right) = \partial\left( \{x_1, \ldots, x_k, \pi, u_2, \ldots, u_n\} \right) \\
= \partial\left( (-1)^k \{\pi, x_1, \ldots, x_k, u_2, \ldots, u_n\} \right) \\
= (-1)^k \{x_1, \ldots, x_k, \pi, u_2, \ldots, u_n\} \\
= (-1)^k \{x_1, \ldots, x_k\} \cdot \partial(\{\pi, u_2, \ldots, u_n\})
\]
The sign appears for \( \partial \) having degree \(-1\). Hence \( \partial_\infty \) and also the direct sum \( \bigoplus \partial_{\rho} \) in (10) are \( K^M_\kappa F \)-linear. Thus this also holds for the induced map \( N \) and (since the inclusion of a factor is also linear) as well for the norm map \( N_{F^p/F} \).

For an arbitrary finite field extension \( F(\alpha_1, \ldots, \alpha_n)/F \), we define the norm map via a decomposition
\[
F \subset F(\alpha_1) \subset F(\alpha_1, \alpha_2) \subset \ldots \subset F(\alpha_1, \ldots, \alpha_n).
\]
This construction is due to Bass and Tate \cite[§5]{BT73}. Its independence of the generating family \((\alpha_1, \ldots, \alpha_n)\) was proven by Kato \cite{Ka80}.

Kerz extended this to the realm of finite étale extensions of semi-local rings with infinite residue fields \cite{Ke09}. For improved Milnor K-theory, he also showed the existence of norm maps for finite étale extensions of arbitrary local rings by reducing to the case of infinite residue fields and classical Milnor K-theory \cite{Ke10}.

As a matter of fact, finite étale extensions \( A \to B \) of local rings are precisely those of the form \( B \cong A[t]/(\pi) \) with \( \pi \) monic irreducible and \( \text{Disc}(\pi) \in A^\times \). Our aim is to drop the condition with the discriminant. This is possible by restricting to factorial domains so that we can use a Bass-Tate-like sequence by Kerz. This is basically Kerz’ work though not stated literally by himself.

**Proposition C.1.** Let \( A \) be a local and factorial domain and \( \iota : A \to B \) an extension of local rings such that \( B \cong A[X]/(\pi) \) for a monic irreducible polynomial \( \pi \in A[X] \). Then there exists a norm map
\[
N_{B/A} : \hat{K}^M_\kappa B \to \hat{K}^M_\kappa A
\]
satisfying the projection formula
\[
N_{B/A}(\{\iota_*(x), y\}) = \{x, N_{B/A}(y)\}
\]
for all \( x \in \hat{K}^M_\kappa A \) and \( y \in \hat{K}^M_\kappa B \).

**Proof.** References within this proof refer to \cite[§4]{Ke09} unless said otherwise. For semi-local domains with infinite residue field, there is a split exact sequence
\[
0 \to \hat{K}^M_n A \to \hat{K}^n A \xrightarrow{\bigoplus \partial} \bigoplus_{p} \hat{K}^M_{n-1}(A[X]/(P)) \to 0,
\]
where the sum is over all monic irreducible \( P \in A[X] \), see [Thm. 4.4]. Here \( K_n^i A \) is an appropriate group generated by so-called feasible tuples of elements in \( \text{Frac}(A)(X) \). A tuple

\[
\begin{pmatrix}
\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}
\end{pmatrix}
\]

with \( p_i, q_i \in A[X] \) and all \( p_i/q_i \) in reduced form is feasible iff the highest nonvanishing coefficients of \( p_i, q_i \) are invertible in \( A \) and for irreducible factors \( u \) of \( p_i \) or \( q_i \) and \( v \) of \( p_j \) or \( q_j \) \((i \neq j)\), we have \( u = av \) with \( a \in A^\times \) or \((u, v) = 1\), see [Def. 4.1]. Now \( K_n^i A \) is defined by modding out relations for \( A^\times \)-linearity which yield a \( K_n^M A \)-module structure on \( K_n^i A \), and the relations

\[
(p_1, \ldots, p, 1 - p, \ldots, p_n) = 0 \quad \text{and} \quad (p_1, \ldots, p, -p, \ldots, p_n) = 0
\]

for \( p \in \text{Frac}(A)(X) \), see [Def. 4.2].

The map \( \oplus \partial p \) is similar to the tame symbol appearing in (9). First, for all monic irreducible \( P \in A[t] \) there are maps \( \partial_P: K_n^i A \to K_{n-1}^M A[t]/(P) \) satisfying

\[
\partial_P(P, p_2, \ldots, p_n) = \{p_2, \ldots, p_n\}
\]

for \( p_i \in A[X] \) such that \((P, p_i) = 1\) [Proof of Theorem. 4.4, Step 2]. Then \( \oplus \partial \) is the well-defined sum of those \( \partial_P \). Analogously, we define a map \( \partial_\infty: K_n^i A \to K_{n-1}^M A \) corresponding to the negative degree valuation by using \( X^{-1} \) as a uniformiser.

With this we can define norm maps in the case of an infinite residue field exactly as in the case of fields via a map \( N \) as indicated in the following diagram.

\[
\begin{array}{ccc}
0 & \longrightarrow & K_n^M A \\
& \downarrow \partial_\infty & \\
& \oplus \partial K_n^M (A[X]/(P)) & \longrightarrow & 0.
\end{array}
\]

Again, all the maps are \( K_n^M A \)-linear. For the map \( i \) this follows directly from the construction above. For the tame symbols this holds as in (11) since they satisfy (14). For the map \( N \) this is a formal consequence. Again, we obtain \( K_n^i A \)-linear norm map \( N_{B/A}: K_n^M B \to K_n^M A \) for \( B = A[X]/(\pi) \) by precomposing with the linear inclusion of the factor. The projection formula (12) is nothing else than the statement that the norm maps are \( K_n^M A \)-linear.

Now we treat the case of arbitrary residue fields. Let \( A \) be an arbitrary factorial and local domain and \( A \to B \) an extension of local rings such that \( B \cong A[X]/(\pi) \) for a monic irreducible polynomial \( \pi \in A[X] \).

Claim. \((A[X]/(\pi))(t_1, \ldots, t_k) \cong A(t_1, \ldots, t_k)[X]/(\pi)\).

To see this, we first observe that the right-hand side is isomorphic to the tensor product \( A(t_1, \ldots, t_k) \otimes A[X]/(\pi) \). By Proposition A.5 above, the left-hand side fulfills this as well.

If \( A \) has residue field \( \kappa \), then \( A(t_1, \ldots, t_k) \) has residue field \( \kappa(t_1, \ldots, t_k) \) by Lemma A.2. Hence we can reduce to the situation of infinite residue fields.
The diagram

\[
M_n^A(t) \xrightarrow{\delta} M_n^A(t) \xrightarrow{\delta} \oplus_P M_{n-1}(A(t)[X]/(P)) \\
\downarrow \delta \hspace{1cm} \downarrow \delta \\
M_n^B(t_1,t_2) \xrightarrow{\delta} M_n^A(t_1,t_2) \xrightarrow{\delta} \oplus_Q M_{n-1}(A(t_1,t_2)[X]/(Q))
\]

commutes where $\delta = (t_1)_* - (t_2)_*$, and $P$ and $Q$ run over all monic irreducible polynomials over $A(t)$ respectively over $A(t_1,t_2)$ (note that a monic irreducible polynomial over $A(t)$ is also monic irreducible over $A(t_1,t_2)$ both via $t_1$ and via $t_2$). Thus the right-hand square in the diagram

\[
K_n^M B \xrightarrow{\delta} K_n^M B(t_1,t_2) \\
\downarrow N_{B/A} \\
K_n^M A \xrightarrow{\delta} K_n^M A(t_1,t_2)
\]

commutes which enables us to define the desired norm map $N_{B/A}$ via restriction and the projection formula inherits from the case of infinite residue fields to the case of arbitrary residue fields. \qed

**References**

[BT73] Bass, Hyman; Tate, John: *The Milnor ring of a global field*. Springer Lect. Notes in Math. 342 (1973), pp. 349-446.

[Ge73] Gersten, Stephen: *Some Exact Sequences in the Higher K-theory of Rings*. Springer Lect. Notes in Math. 341 (1973), pp. 211-243.

[GS06] Gille, Philippe; Szamuely, Tamás: *Central Simple Algebras and Galois Cohomology*. Cambridge University Press (Cambridge), 2006.

[GW10] Görtz, Ulrich; Wedhorn, Torsten: *Algebraic Geometry I. Schemes. With Examples and Exercises*. Vieweg+Teubner, 2010.

[Ka80] Kato, Kazuya: *A generalisation of local class field theory by using K-groups, II*. J. Fac. Sci. Univ Tokyo, Vol. 27 (1980), pp. 603â–ÁŠ683.

[Ke09] Kerz, Moritz: *The Gersten conjecture for Milnor K-theory*. Invent. math. 175 (2009), pp. 1-33.

[Ke10] Kerz, Moritz: *Milnor K-theory of local rings with finite residue fields*. J. Algebric Geom. 19 (2010), pp. 173-191.

[Ma80] Matsumura, Hideyuki: *Commutative Algebra. Second Edition*. The Benjamin/Cummings Publishing Company, 1980.

[Mi70] Milnor, John: *Algebraic K-Theory and Quadratic forms*. Invent. Math. 9 (1970), pp. 318â–ÁŠ344.

[Mi71] Milnor, John W.: *Introduction to algebraic K-theory*. Annals of Math. Study 72, Princeton Univ. Press, 1971.

[NS90] Nesterenko, Yuri; Suslin, Andrei: *Homology of the general linear group over a local ring, and Milnorâ–ÁŽs K-theory*. Math. USSR-Izv. 34 (1990), no. 1, pp. 121-145.

[Qu72] Quillen, Daniel: *On the cohomology and K-theory of the general linear groups over a finite field*. Ann. of Math. 96 (1972), pp. 552-586.

[Si86] Sivitski, Igor: *On torsion in Milnor’s K-groups for a local field*. Math. USSR Sb. Vol. 54 (1986), No. 2, pp. 561-569.

[Se79] Serre, Jean Pierre: *Local fields*. Springer, 1979.

[We10] Weibel, Charles: *The K-book: an introduction to algebraic K-theory*. AMS (Heidelberg), 2013.
Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany

E-mail address: christian.dahlhausen(at)ur.de