A Convergent Post-processed Discontinuous Galerkin Method for Incompressible Flow with Variable Density

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Abstract
We propose a linearized semi-implicit and decoupled finite element method for the incompressible Navier–Stokes equations with variable density. Our method is fully discrete and shown to be unconditionally stable. The velocity equation is solved by an $H^1$-conforming finite element method, and an upwind discontinuous Galerkin finite element method with post-processed velocity is adopted for the density equation. The proposed method is proved to be convergent in approximating reasonably smooth solutions in three-dimensional convex polyhedral domains.

Keywords Navier–Stokes equations · Variable density · Transport equation · Discontinuous Galerkin methods

Mathematics Subject Classification 65N30 · 65L12

1 Introduction
In this article we consider numerical approximation to incompressible flow with variable density, described by the following hyperbolic-parabolic system of partial differential equations (PDEs):

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho u) &= 0 &\text{in } \Omega \times (0, T], \\
\rho \partial_t u + \rho (u \cdot \nabla) u + \nabla p - \mu \Delta u &= 0 &\text{in } \Omega \times (0, T], \\
\nabla \cdot u &= 0 &\text{in } \Omega \times (0, T],
\end{align*}
\] (1.1)

\[\rho \text{ } \partial_{t} \text{ } \rho \text{ } + \text{ } \nabla \cdot \text{ } (\rho \text{ } u) \text{ } = \text{ } 0 \text{ } \quad \text{in } \Omega \times (0, T], \quad (1.1a)
\]
\[\rho \partial_{t} \text{ } u \text{ } + \rho \text{ } (u \cdot \nabla) \text{ } u \text{ } + \text{ } \nabla \text{ } p \text{ } - \mu \Delta \text{ } u \text{ } = \text{ } 0 \text{ } \quad \text{in } \Omega \times (0, T], \quad (1.1b)
\]
\[\nabla \cdot \text{ } u \text{ } = \text{ } 0 \text{ } \quad \text{in } \Omega \times (0, T], \quad (1.1c)
\]
in a convex polyhedral domain $\Omega \subset \mathbb{R}^d$, with $d \in \{2, 3\}$, up to a given time $T$, with the following boundary and initial conditions:

\begin{align}
  u &= 0 & \text{on } \partial \Omega \times [0, T], \\
  \rho &= \rho \,^0 & \text{in } \Omega \at t = 0.
\end{align}

(1.2a) (1.2b)

In this model, $\rho : \Omega \rightarrow \mathbb{R}$, $u : \Omega \rightarrow \mathbb{R}^d$ and $p : \Omega \rightarrow \mathbb{R}$ are the density, velocity and pressure of the fluid, respectively, and $\mu > 0$ is the viscosity constant of the fluid. The initial value of the density is assumed to satisfy the following physical condition:

\[ \rho_{\text{min}} := \min_{x \in \Omega} \rho \,^0(x) > 0. \] (1.3)

For smooth initial values satisfying the positivity condition (1.3), existence and uniqueness of smooth solutions of (1.1) in two dimensions were proved in [9, 16, 21]. Hence, this problem does not generate shock waves in finite time (at least in 2D). Existence and uniqueness of smooth solutions in three dimensions remains open similarly as the Navier–Stokes equations with constant density.

Numerical approximation to the coupled system (1.1) were studied with many different numerical methods, including projection methods [2, 5, 12, 19, 22], fractional-step methods [13, 14], backward differentiation formulae [18], and the discontinuous Galerkin (DG) method [20]. The stability of several numerical methods was proved in [12, 19, 22]. Convergence of a DG method and a staggered non-conforming finite element method were proved based on compactness arguments in [20] and [17], respectively, without explicit convergence rates.

Since the variable density introduces considerable difficulties to error analysis of the coupled nonlinear system, as mentioned in [22], error analysis has been done only in a few articles. The main difficulty is to prove boundedness of numerical solutions to both $\rho$ and $u$, as well as a positive lower bound of the numerical solution to $\rho$, uniformly with respect to the temporal stepsize and spatial mesh size. An error estimate for the single velocity equations (1.1b) was presented in [15] for the methods proposed in [13, 14], where the numerical solutions $\rho_n \,^h$, $n = 1, \ldots, N$, of the density equation were assumed to have positive upper and lower bound uniformly with respect to the temporal stepsize and spatial mesh size; see [15, Conjectures in Remark 4.2]. An error estimate for a fractional-step temporally semidiscrete method was presented in [3] under the assumption that the numerical solution of density has positive upper and lower bounds uniformly with respect to the temporal stepsize. The first complete error estimate of fully discrete FEM for the coupled system (1.1) was presented in [8] for the two-dimensional problem based on $H^3$ regularity assumption on the solution.

The analysis in [8] utilizes an error splitting approach, which involves analyzing the error of full discretization based on uniform regularity estimates for the temporally semidiscrete solutions. However, the analysis in [8] cannot be directly extended to three dimensions due to the presence of $H^1$-conforming finite element solution of $u$ in the density equation, which requires proving $W^{1, \infty}$-boundedness of the numerical solution to $u$ in order to obtain an error estimate for the density equation. This limits the analysis in [8] to two dimensions and solutions with $H^3$ regularity. Hence, error estimates for the three-dimensional problem based only on $H^{2+\alpha}$ spatial regularity of solutions (more realistic in general convex polyhedra) still remain open.

The objective of this article is to introduce a fully discrete, linearized semi-implicit, decoupled and unconditionally stable FEM for the coupled system (1.1)–(1.2) such that error analysis can be done in three dimensions under more realistic $H^{2+\alpha}$ regularity assumptions on the solution in a convex polyhedron. To this end, we propose an upwind DG method for
the density equation with post-processed velocity, and $H^1$-conforming FEM for the velocity equation. The key to error analysis in three dimensions is the post-processing of velocity, which projects the $H^1$-conforming finite element solution of $u$ to the divergence-free subspace of the Raviart–Thomas element space. This post-processing has a significant influence on the error analysis: it allows us to derive an error estimate without proving the $W^{1,\infty}$-boundedness of the numerical solution to $u$.

In Sect. 2, we present the main results of this paper, including the numerical method and error estimate. The proof of the main theorem is presented in Sect. 3.

## 2 Main Results

### 2.1 Notation

Let $\Omega$ be convex polygon/polyhedron in $\mathbb{R}^d$, and denote by $\nu$ the outward unit normal vector on the boundary $\partial\Omega$. We define the following function spaces on $\Omega$:

$$H^1(\Omega) := \{ v \in L^2(\Omega) : \nabla v \in L^2(\Omega)^d \}, \quad (2.1)$$

$$\dot{H}^1(\Omega) := \{ v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega \}, \quad (2.2)$$

$$\tilde{L}^2(\Omega) := \{ v \in L^2(\Omega) : \int_{\Omega} v \, dx = 0 \}, \quad (2.3)$$

$$H(\text{div}, \Omega) := \{ v \in L^2(\Omega)^d : \nabla \cdot v \in L^2(\Omega) \}. \quad (2.4)$$

For any nonnegative integer $r$, we denote by $P^r_{dG}(T_h)$ the scalar-valued discontinuous Galerkin finite element space of degree up to $r$, built on a quasi-uniform partition $T_h$ of $\Omega$ into tetrahedra (with $T_h$ denoting the set of tetrahedra, and $h$ denoting the mesh size). The outward unit normal vector on the boundary $\partial K$ of a tetrahedron $K \in T_h$ is denoted by $\nu_K$.

We define $\text{RT}^1(T_h)$ to be the $H(\text{div}, \Omega)$-conforming Raviart–Thomas finite element spaces of order 1, i.e.,

$$\text{RT}^1(T_h) := \{ w \in H(\text{div}, \Omega) : w|_K \in P_1(K)^d + x P_1(K), \forall K \in T_h \}. \quad (2.5)$$

We also define the following finite element spaces:

$$P^i(T_h) := P^i_{dG}(T_h) \cap H^1(\Omega), \quad (2.6)$$

$$P^{ib}(T_h) := P^{ib}_{dG}(T_h) \text{ enriched by a bubble function(cf. [4] and [6, Section 7.1]),} \quad (2.7)$$

$$\tilde{P}^i(T_h) := \{ v \in P^i(T_h) : \int_{\Omega} v \, dx = 0 \}, \quad (2.8)$$

$$\text{RT}^1_0(T_h) := \{ v_h \in \text{RT}^1(T_h) : \nabla \cdot v_h = 0 \text{ in } \Omega \text{ and } v_h \cdot \nu = 0 \text{ on } \partial\Omega \}. \quad (2.9)$$

We denote by $P^{RT}_h : L^2(\Omega)^d \rightarrow \text{RT}^1_0(T_h)$ the $L^2$-orthogonal projection, defined by

$$(v - P^{RT}_h v, w_h) = 0 \quad \forall w_h \in \text{RT}^1_0(T_h), \quad \forall v \in L^2(\Omega)^d. \quad (2.10)$$

Similarly, we denote by $P^{dG}_h : L^2(\Omega) \rightarrow P^2_{dG}(T_h)$ the $L^2$-orthogonal projection defined by

$$(v - P^{dG}_h v, w_h) = 0 \quad \forall w_h \in P^2_{dG}(T_h), \quad \forall v \in L^2(\Omega). \quad (2.11)$$
The finite element space $\mathbf{P}_1^b(T_h)^d \times \tilde{\mathbf{P}}_1(T_h)$ satisfies the inf-sup condition (cf. [4, 6])

$$
\|q_h\|_{L^2(\Omega)} \leq C \sup_{v_h \in \mathbf{P}_1^b(T_h), v_h \neq 0} \frac{|(\nabla \cdot v_h, q_h)|}{\|v_h\|_{H^1(\Omega)}}, \quad \forall \, q_h \in \tilde{\mathbf{P}}_1(T_h),
$$

and therefore is stable in approximating the Stokes and Navier–Stokes equations. This inf-sup condition is required in practical computation for the numerical method to be stable, but is not used in our error analysis.

We denote by

$$(u, v) = \sum_{K \in T_h} \int_K uv \, dx \quad \langle u, v \rangle_{\partial K_s} = \int_{\partial K_s} uv \, ds$$

the inner product of $L^2(\Omega)$ and $L^2(\partial K_s)$, respectively, where $\partial K_s$ is a subset of $\partial K$ for a tetrahedron $K \in T_h$. For a function $v$ uniformly continuous on each tetrahedron $K \in T_h$, we define

$$
\{v\} = \frac{1}{2} (v^+ + v^-) \quad \text{and} \quad [v] = (v^- - v^+)\nu_K
$$

(2.13)

to be the average and jump of the function $v$ defined on the boundary $\partial K$ for $K \in T_h$, with $v^+$ and $v^-$ denoting the exterior and interior traces of the function. If $F = K \cap K'$ is a common face of two tetrahedra $K$ and $K'$, then the jump $[v]$ on $F$ is independent of the definitions using $K$ and $K'$.

To guarantee the positivity of the numerical solution of density $\rho$, we denote by $\chi \in W^{1,\infty}(\mathbb{R})$ the cut-off function defined by

$$
\chi(s) = \begin{cases} 
\frac{1}{2} \rho_{\min} & \text{if } s < \frac{1}{2} \rho_{\min}, \\
\frac{1}{2} \rho_{\min} \leq s \leq \frac{3}{2} \rho_{\max}, \\
\frac{3}{2} \rho_{\max} & \text{if } s > \frac{3}{2} \rho_{\max},
\end{cases}
$$

where

$$
\rho_{\min} := \min_{x \in \Omega} \rho^0(x) \quad \text{and} \quad \rho_{\max} := \max_{x \in \Omega} \rho^0(x).
$$

(2.14)

The cut-off function defined above has the following conditions:

$$
\chi(s) = s \quad \forall s \in \left[\frac{1}{2} \rho_{\min}, \frac{3}{2} \rho_{\max}\right],
$$

(2.15a)

$$
\frac{1}{2} \rho_{\min} \leq \chi(s) \leq \frac{3}{2} \rho_{\max} \quad \forall s \in \mathbb{R}.
$$

(2.15b)

### 2.2 The Numerical Method and Its Convergence

Let $t_n = n\tau, n = 0, 1, \ldots, N$, be a uniform partition of the time interval $[0, T]$ with stepsize $\tau = T/N$. For a given function $u_h^{n-1}$ at time $t = t_{n-1}$, we denote by $\partial K_n^-(\partial K_n^+)$ the numerical inflow (outflow) boundary of the tetrahedron $K \in T_h$ at time $t = t_n$, defined by

$$
\partial K_n^- := \{x \in \partial K : (P^R_h \cdot u_h^{n-1} \cdot v_K)(x) < 0\}, \quad \partial K_n^+ := \{x \in \partial K : (P^R_h \cdot u_h^{n-1} \cdot v_K)(x) > 0\}.
$$
We consider the following fully discrete linearized FEM for (1.1)–(1.2) (based on a reformulation of the system as shown in [8,(1.5)–(1.7))): for given $(\rho_h^{n-1}, u_h^{n-1}) \in P^{2}_{dG}(T_h) \times \hat{P}^{1b}(T_h)^d$, find $(\rho_h^n, u_h^n, p_h^n) \in P^{2}_{dG}(T_h) \times \hat{P}^{1b}(T_h)^d \times \hat{P}^1(T_h)$ satisfying the equations

\[
(D_\tau \rho_h^n, \psi_h) + ((P^R_{h} u_h^{n-1} \cdot \nabla)\rho_h^n, \psi_h) - \sum_{K \in T_h} (P^R_{h} u_h^{n-1} \cdot [\rho_h^n], \psi_h)_{\partial K^n} = 0,
\]

\[
(\chi(\rho_h^n) D_\tau u_h^n, v_h) + \frac{1}{2} (D_\tau \chi(\rho_h^n) u_h^n, v_h) - \frac{1}{2} (\chi(\rho_h^n) u_h^{n-1}, \nabla (u_h^n \cdot v_h)) + (\chi(\rho_h^n) u_h^{n-1}, \nabla u_h^n, v_h) + (\mu \nabla u_h^n, \nabla v_h) - (p_h^n, \nabla \cdot v_h) = 0,
\]

\[
(\nabla \cdot u_h^n, q_h) = 0,
\]

for all test functions $(\psi_h, v_h, q_h) \in P^{2}_{dG}(T_h) \times \hat{P}^{1b}(T_h)^d \times \hat{P}^1(T_h)$, where

\[
D_\tau \rho_h^n = \frac{\rho_h^n - \rho_h^{n-1}}{\tau}, \quad D_\tau \chi(\rho_h^n) := \frac{\chi(\rho_h^n) - \chi(\rho_h^{n-1})}{\tau}
\]

are the backward Euler difference quotients of corresponding functions. The initial values of the numerical solutions are simply chosen to be

\[
\rho_h^0 = P^G_{d} \rho^0 \quad \text{and} \quad u_h^0 = I_h u^0,
\]

where $I_h : \hat{C} (\Omega)^d \to \hat{P}^{1b}(T_h)^d$ is the globally continuous nodal interpolation operator.

The proposed method (2.16) has unconditional energy stability, i.e., substituting $\psi_h = \rho_h^n$ and $v_h = u_h^n$ into (2.16), and using the relation

\[
\sum_{K \in T_h} (P^R_{h} u_h^{n-1} \cdot \nabla \frac{1}{2} |\rho_h^n|^2)_K = \sum_{K \in T_h} (P^R_{h} u_h^{n-1} \cdot \frac{1}{2} |\rho_h^n|^2)_{\partial K^n} - \sum_{K \in T_h} (P^R_{h} u_h^{n-1} \cdot \frac{1}{2} |\rho_h^n|^2)_{\partial K^n}
\]

\[
= \sum_{K \in T_h} (P^R_{h} u_h^{n-1} \cdot v_K, \frac{1}{2} |\rho_h^n|^2)_{\partial K^n} + \sum_{K \in T_h} (P^R_{h} u_h^{n-1} \cdot v_K, \frac{1}{2} |\rho_h^n|^2)_{\partial K^n}
\]

\[
- \sum_{K \in T_h} (P^R_{h} u_h^{n-1} \cdot v_K [(\rho_h^n)_+ - (\rho_h^n)_-])_{\partial K^n}
\]

\[
= - \sum_{K \in T_h} (P^R_{h} u_h^{n-1} \cdot v_K, \frac{1}{2} [(\rho_h^n)_+ - (\rho_h^n)_-]^2)_{\partial K^n}
\]

\[
greateror_equal_to 0,
\]

one can obtain the following energy inequality:

\[
\frac{1}{2} ||\rho_h^n||^2_{L^2(\Omega)} + \int_{\Omega} \frac{1}{2} \chi(\rho_h^n)|u_h^n|^2 \mathrm{d}x + \tau \mu ||\nabla u_h^n||^2_{L^2(\Omega)}
\]

\[
\leq \frac{1}{2} ||\rho_h^{n-1}||^2_{L^2(\Omega)} + \int_{\Omega} \frac{1}{2} \chi(\rho_h^{n-1})|u_h^{n-1}|^2 \mathrm{d}x.
\]

Since (2.16) is a linearly implicit method, the energy inequality above implies existence and uniqueness of numerical solutions without any condition on the time stepsize or spatial mesh size (setting $\rho_h^{n-1} = 0$ and $u_h^{n-1} = 0$ in (2.18) yields that the homogeneous linear system associated to (2.16) has only zero solution).
In this article, we prove convergence of the numerical method (2.16) under the following regularity assumption on the exact solution: for some $\alpha \in (0, \frac{1}{2})$

\[
\rho \in C([0, T]; H^{2+\alpha}(\Omega)), \quad \partial_t \rho \in C\left([0, T]; H^1(\Omega)\right), \quad \partial_t^2 \rho \in C\left([0, T]; L^2(\Omega)\right),
\]
\[
u \in C([0, T]; H^2(\Omega)), \quad \partial_t \nu \in C\left([0, T]; H^2(\Omega)\right), \quad \partial_t^2 \nu \in C\left([0, T]; L^2(\Omega)\right),
\]
\[
p \in C([0, T]; H^1(\Omega)), \quad \partial_t p \in C\left([0, T]; H^1(\Omega)\right),
\]

(2.19)

The spatial regularity in (2.19) is only slightly more than $H^2$, which is weaker and more reasonable than the regularity assumptions in [8] (which requires $H^3$ regularity of the solution) for this problem in a convex polygon or polyhedron. For the simplicity of notation, we denote by

\[
u^n = \nu(\cdot, t_n) \quad \text{and} \quad \rho^n = \rho(\cdot, t_n)
\]

the exact solutions $\nu$ and $\rho$ at time level $t = t_n$.

The main theoretical result of this article is the following theorem.

**Theorem 2.1** Under the regularity assumption (2.19) and stepsize restriction $\tau = o(h^{d/2})$, there exists a positive constant $h^*$ such that when $h \leq h^*$ the fully discrete solutions given by (2.16) satisfy the following error estimate:

\[
\max_{1 \leq n \leq N} \left( \|\nu^n - \nu_h^n\|_{L^2(\Omega)} + \|\rho^n - \rho_h^n\|_{L^2(\Omega)} \right) + \left( \sum_{n=1}^{N} \tau \|\nu^n - \nu_h^n\|_{H^1(\Omega)}^2 \right)^{1/2} \leq C \left( \tau + h^{\frac{3}{2} + \alpha} \right),
\]

where constant $C$ may depend on the exact solution $(\rho, \nu, p)$ and $T$.

The proof of Theorem 2.1 is presented in the next section. Throughout, we denote by $C$ a generic positive constant that may be different at different occurrences and may depend on the exact solution $(\rho, \nu, p)$ and $T$, but is independent of the mesh size $h$ and stepsize $\tau$.

**Remark 2.2** The convergence rates in Theorem 2.1 is limited by the regularity of solutions and the nature of hyperbolic equation of $\rho$. It is known that even for linear hyperbolic equations, the DG method generally loses half-order convergence; see [10, Corollary 2.32]. Once the error estimates for velocity and density are obtained, a weaker error estimate for the pressure (losing additional half order in time and one order in space) can be obtained by using the method in [8], which we omit in this paper. An error estimate for pressure without losing additional order of accuracy is still missing for this problem even in two dimensions.

### 3 Error Analysis

#### 3.1 Preliminary Results

We denote by $\mathcal{F}_h^I$ and $\mathcal{F}_h^B$ the set of all interior and boundary faces of $T_h$, respectively, and define $\mathcal{F}_h := \mathcal{F}_h^I \cup \mathcal{F}_h^B$ to be the collection of all faces. For an interior face $F = \partial K \cap \partial K'$ with $K, K' \in T_h$, the average and jump defined in (2.13), initially defined on $\partial K$ and $\partial K'$, respectively, coincide on the face $F$ and can be rewritten as

\[
\{\phi\} := \frac{1}{2}(\phi + \phi') \quad \text{and} \quad [\phi] := \phi_K + \phi_{K'} \quad \text{on} \quad F,
\]
where the last inequality is due to (3.3). This estimate of (see [7, §4.4, Corollary 4.4.7]):

\[ C \]

where the constant addition, the Lagrange interpolation \( I_h \) 

\[ v \]

This implies 

\[ \text{(3.2)} \]

It is known that the Stokes–Ritz projection has the following approximation property (cf. [24, Lemma 17.1]):

\[ \parallel u \parallel_{\Omega} \leq C \parallel u \parallel_{H^1(\Omega)} \]

Note that all finite element functions satisfy the following “inverse inequality” (see [7, §4.5]):

\[ \parallel w \parallel_{W^{s_1+1,q}(\Omega)} \leq C h^{s_1+1+2q} \parallel w \parallel_{W^{s_1,q}(\Omega)} \quad \text{for } 0 \leq s_1 \leq 1, \quad 1 \leq p \leq q \leq \infty. \]

where the constant \( C \) depending on the finite element space of \( w \) (but independent of \( h \)). In addition, the Lagrange interpolation \( I_h : \hat{C}(\Omega)^d \to \hat{P}^1(\Omega) \) has the following error bound (see [7, §4.4, Corollary 4.4.7]):

\[ \parallel u - I_h u \parallel_{L^2(\Omega)} \leq C h^2 \parallel u \parallel_{H^2(\Omega)} \quad \text{and} \quad \parallel u - I_h u \parallel_{L^\infty(\Omega)} \leq C h^{2-d} \parallel u \parallel_{H^2(\Omega)}. \]
By using the two estimates above, from (3.6) one can obtain
\[
\|u^n - \hat{u}_h^n\|_{L^\infty(\Omega)} \leq \|u^n - I_h u^n\|_{L^\infty(\Omega)} + \|I_h u^n - \hat{u}_h^n\|_{L^\infty(\Omega)} \\
\leq C h^{-\frac{d}{2}} \|u^n\|_{H^2(\Omega)} + C h^{-\frac{d}{2}} \|I_h u^n - \hat{u}_h^n\|_{L^2(\Omega)} \\
\leq C h^{-\frac{d}{2}} \|u^n\|_{H^2(\Omega)} + C h^{-\frac{d}{2}} \|I_h u^n - u^n\|_{L^2(\Omega)} + C h^{-\frac{d}{2}} \|u^n - \hat{u}_h^n\|_{L^2(\Omega)} \\
\leq C h^{-\frac{d}{2}} \leq C h^\frac{1}{2}, \quad \text{for } d = 2, 3. \quad (3.7)
\]

The Stokes–Ritz projection (\(\hat{u}_h^n, \hat{p}_h^n\)) will serve as an intermediate solution for comparison with the numerical solution (\(u_h^n, p_h^n\)). With the approximation property (3.6), it suffices to estimate the error \(e_u^n = u_h^n - \hat{u}_h^n\) and \(e_p^n = p_h^n - \hat{p}_h^n\) for the velocity equation.

To control the coupling term in the hyperbolic density equation, the following discrete Sobolev embedding inequality will be used in the error analysis.

**Lemma 3.1** In a convex polyhedron (or polygon) \(\Omega\), the following inequality holds:
\[
\|P^RT_h v\|_{L^6(\Omega)} \leq C \|v\|_{H^1(\Omega)} \quad \forall v \in H^1(\Omega)^d \quad \text{and} \quad v \cdot v = 0 \quad \text{on} \partial \Omega. \quad (3.8)
\]

**Proof** Let \(RT_h^1(T_h)\) be the subspace of \(RT^1(T_h)\) with the boundary condition \(\sigma_h \cdot v = 0\) on \(\partial \Omega\) for \(\sigma_h \in RT_h^1(T_h)\). We define \((\sigma_h, \phi_h) \in RT^1(T_h) \times P^1_{dG}(T_h)\) to be the solution of the following mixed finite element equations:
\[
(\sigma_h, \eta_h) + (\phi_h, \nabla \cdot \eta_h) = (v, \eta_h) \quad \forall \eta_h \in RT_h^1(T_h), \quad (3.9a) \\
(\nabla \cdot \sigma_h, \varphi_h) = 0 \quad \forall \varphi_h \in P^1_{dG}(T_h). \quad (3.9b)
\]
The second equation above implies \(\nabla \cdot \sigma_h = 0\). This together with the boundary condition \(\sigma_h \cdot v = 0\) on \(\partial \Omega\) implies that \(\sigma_h \in RT_h^1(T_h)\), which is defined in (2.9). By choosing \(\eta_h \in RT_h^0(T_h)\) in the first equation we obtain \(\sigma_h = P^RT_h v\), i.e., the \(L^2\) projection of \(v\) onto \(RT_h^1(T_h)\).

The partial differential equations to which the mixed method (3.9) approximates is
\[
\sigma - \nabla \phi = v, \\
\nabla \cdot \sigma = 0,
\]
with boundary condition \(\sigma \cdot v = 0\) on \(\partial \Omega\). Thus
\[
\begin{cases}
-\Delta \phi = \nabla \cdot v & \text{in } \Omega, \\
-\partial_n \phi = 0 & \text{on } \partial \Omega.
\end{cases}
\]

By the regularity of the Neumann problem in a convex polyhedron (cf. [11, Theorem 3.2.1.3 and Theorem 3.1.3.3]), we have
\[
\|\phi\|_{H^2(\Omega)} \leq C \|\nabla \cdot v\|_{L^2(\Omega)} \leq C \|v\|_{H^1(\Omega)}.
\]
By the standard error estimate of the mixed FEM (cf. [24, Theorem 17.1]), we have
\[
\|\sigma_h - \Pi^RT_h \sigma\|_{L^2(\Omega)} \leq C h \|\phi\|_{H^2(\Omega)} \leq C h \|v\|_{H^1(\Omega)}.
\]
This implies that, via the inverse inequality,
\[
\|\sigma_h - \Pi^RT_h \sigma\|_{L^6(\Omega)} \leq C h^{-\frac{d}{3}} \|\sigma_h - \Pi^RT_h \sigma\|_{L^2(\Omega)} \leq C h^{1-\frac{d}{3}} \|v\|_{H^1(\Omega)}. \quad (3.10)
\]
Then, using the $L^2$-orthogonal projection $P_{h}^{dG} : L^2(\Omega)^d \rightarrow P_{dG}^2(T_h)^d$. In the case $d \in \{2, 3\}$ we obtain, by using the triangle inequality,

$$\|\sigma_h\|_{L^6(\Omega)} \leq \|\sigma_h - \Pi_h^{RT} \sigma\|_{L^6(\Omega)} \leq \sum_{K \in T_h} \|\Pi_h^{RT} \sigma - P_{h}^{dG} \sigma\|_{L^6(\Omega)} + \sum_{F \in \mathcal{F}_h} h_F^{-d} \|P_{h}^{dG} \sigma\|_{L^6(\Omega)}$$

where the last inequality uses (3.10) and (3.4). This proves the desired result in Lemma 3.1.

Let $H^1(T_h)$ be the broken $H^1$ space, consisting of functions which are in $H^1(K)$ for all tetrahedra $K \in T_h$, equipped with the norm

$$\|\varphi\|_{H^1(T_h)} := \left( \sum_{K \in T_h} \|\nabla \varphi\|_{L^2(K)}^2 + \sum_{F \in \mathcal{F}_h} \frac{1}{h_F} \left\|\frac{\mu}{\omega}\right\|_{L^2(F)}^2 \right)^{\frac{1}{2}},$$

where $h_F$ denotes the diameter of face $F$, equivalent to the diameter of tetrahedron $K$ containing face $F$ according to the shape regularity of the partition. The $H^1(T_h)$-stability of the $L^2$-orthogonal projection $P_{h}^{dG}$ is presented in the following lemma.

**Lemma 3.2** The $L^2$ projection operator $P_{h}^{dG} : L^2(\Omega) \rightarrow P_{dG}^2(T_h)$ defined in (2.11) satisfies the following estimate:

$$\|P_{h}^{dG} \varphi\|_{H^1(T_h)} \leq C \|\varphi\|_{H^1(T_h)} \quad \forall \varphi \in H^1(T_h).$$

**Proof** For any $K \in T_h$ the following standard $L^2$ and $H^1$ approximation properties hold:

$$\|P_{h}^{dG} \varphi - \varphi\|_{L^2(K)} \leq C h_K \|\nabla \varphi\|_{L^2(K)}$$

and

$$\|\nabla (P_{h}^{dG} \varphi - \varphi)\|_{L^2(K)} \leq C \|\nabla \varphi\|_{L^2(K)}.$$

By the trace inequality on the tetrahedron $K$ and the above approximation properties, we have

$$h_F^{-1} \|P_{h}^{dG} \varphi - \varphi\|_{L^2(\partial K)} \leq C \left( h_K^{-1} \|P_{h}^{dG} \varphi - \varphi\|_{L^2(K)} + \|\nabla (P_{h}^{dG} \varphi - \varphi)\|_{L^2(K)} \right)$$

$$\leq C \|\nabla \varphi\|_{L^2(K)}.$$

Hence,

$$\|P_{h}^{dG} \varphi - \varphi\|_{H^1(T_h)}^2 = \sum_{K \in T_h} \left( \|\nabla (P_{h}^{dG} \varphi - \varphi)\|_{L^2(K)}^2 + h_F^{-1} \left\|\frac{\mu}{\omega}\right\|_{L^2(\partial K)}^2 \right)$$

$$\leq C \sum_{K \in T_h} \|\nabla \varphi\|_{L^2(K)}^2 \leq C \|\varphi\|_{H^1(T_h)}^2.$$

The desired result follows from the above inequality and the triangle inequality.

### 3.2 Mathematical Induction

We define the following error functions:

$$e_{\rho, h}^n = P_{h}^{dG} \rho^n - \rho_h^n,$$

$$e_{u, h}^n = \tilde{u}_h^n - u_h^n,$$

$$e_{\rho, h}^n = \tilde{P}_h^n - \rho_h^n.$$
For a given $1 \leq m \leq N$, we assume that the data $\rho_h^{n-1}$ and $u_h^{n-1}$, $n = 1, 2, \ldots, m$ are given and satisfying the following inequalities (errors on the previous time level are sufficiently small in some sense):

\begin{align}
\max_{1 \leq n \leq m} \| e_h^{n-1} \|_{L^\infty(\Omega)} & \leq \frac{1}{4} \rho_{\min}, \quad (3.12a) \\
\max_{1 \leq n \leq m} \| e_h^{n-1} \|_{L^2(\Omega)} & \leq h \frac{3}{2} + \frac{\tau}{2}, \quad (3.12b) \\
\max_{1 \leq n \leq m} \| e_h^{n-1} \|_{L^\infty(\Omega)} & \leq 1, \quad (3.12c) \\
\max_{1 \leq n \leq m} \| P_h^{RT} u_h^{n-1} - u^n \|_{L^\infty(\Omega)} & \leq 2, \quad (3.12d) \\
\sum_{n=0}^{m} \tau \| e_h^{n-1} \|_{H^1(\Omega)}^2 & \leq (\kappa + h^3) \tau^3, \quad (3.12e)
\end{align}

where $\kappa$ is a sufficiently small constant to be determined later in (3.28)–(3.29). Then we prove that the numerical solution $(\rho_h^n, u_h^n, p_h^n) \in P_2^0(T_h) \times \tilde{P}_{1b}(T_h) \times \tilde{P}^1(T_h)$ given by (2.16) satisfies the following inequalities:

\begin{align}
\max_{0 \leq n \leq m} \| e_h^n \|_{L^\infty(\Omega)} & \leq \frac{1}{4} \rho_{\min}, \quad (3.13a) \\
\max_{0 \leq n \leq m} \| e_h^n \|_{L^2(\Omega)} & \leq h \frac{3}{2} + \frac{\tau}{2}, \quad (3.13b) \\
\max_{0 \leq n \leq m} \| e_h^n \|_{L^\infty(\Omega)} & \leq 1, \quad (3.13c) \\
\max_{0 \leq n \leq m} \| P_h^{RT} u_h^n - u^n \|_{L^\infty(\Omega)} & \leq 2, \quad (3.13d) \\
\sum_{n=0}^{m} \tau \| e_h^n \|_{H^1(\Omega)}^2 & \leq (\kappa + h^3) \tau^3. \quad (3.13e)
\end{align}

If this can be proved then, by mathematical induction, (3.13) holds for all $1 \leq n \leq N$. To use mathematical induction, we emphasize that all the generic constants below will be independent of $m$ (but may depend on $T$).

The induction Assumption (3.12a) implies that

\begin{align*}
\| I_h^{DG} \rho^{n-1} - \rho_h^{n-1} \|_{L^\infty(\Omega)} & \leq \| I_h^{DG} \rho^{n-1} - P_h^{DG} \rho_h^{n-1} \|_{L^\infty(\Omega)} + \| e_h^{n-1} \|_{L^\infty(\Omega)} \\
& \leq C h^{-\frac{d}{2}} \| I_h^{DG} \rho^{n-1} - P_h^{DG} \rho_h^{n-1} \|_{L^2(\Omega)} + \| e_h^{n-1} \|_{L^\infty(\Omega)} \\
& \leq C h^{-\frac{d}{2}} \| \rho^{n-1} \|_{H^2(\Omega)} + \frac{1}{4} \rho_{\min} \leq \frac{3}{8} \rho_{\min}, \quad \text{when } h \text{ is sufficiently small.}
\end{align*}

Since the nodal interpolation $I_h^{DG} \rho^{n-1}$ satisfies

\begin{align*}
\| \rho^{n-1} - I_h^{DG} \rho^{n-1} \|_{L^\infty(\Omega)} & \leq C h^{-\frac{d}{2}} \| \rho^{n-1} \|_{C^\frac{1}{2}(\Omega)} \\
& \leq C h^{-\frac{d}{2}} \| \rho^{n-1} \|_{H^2+\sigma(\Omega)} \quad \text{(Sobolev embedding $H^{2+\sigma}(\Omega) \hookrightarrow C^{\frac{1}{2}}(\Omega)$)} \\
& \leq \frac{1}{8} \rho_{\min} \quad \text{when } h \text{ is sufficiently small},
\end{align*}
it follows that (by using the triangle inequality)
\[ \|\rho^{n-1} - \rho_h^{n-1}\|_{L^\infty(\Omega)} \leq \frac{1}{2} \rho_{\min}, \]
which implies
\[ \frac{1}{2} \rho_{\min} \leq \rho_h^{n-1}(x) \leq \frac{3}{2} \rho_{\max}, \quad n = 1, \ldots, m, \tag{3.14} \]
in view of the definition of \( \rho_{\min} \) and \( \rho_{\max} \) in (2.14).

Similarly, the error estimate (3.6) for the Stokes–Ritz projection and (3.12c) imply that
\[
\|u_h^{n-1}\|_{L^\infty(\Omega)} \leq \|e_{u,h}^{n-1}\|_{L^\infty(\Omega)} + \|I_h u^{n-1} - I_h u_h^{n-1}\|_{L^\infty(\Omega)} + \|I_h u_h^{n-1} - u_h^{n-1}\|_{L^\infty(\Omega)} + \|u_h^{n-1}\|_{L^\infty(\Omega)} \leq 1 + h^{-\frac{d}{2}} \|\hat{a}_h^{n-1}\|_{L^2(\Omega)} + Ch^\frac{1}{2} \|u^{n-1}\|_{C^\frac{1}{2}(\Omega)} + \|u_h^{n-1}\|_{L^\infty(\Omega)} \leq 1 + Ch^2 \|u_h^{n-1}\|_{H^2(\Omega)} + Ch^\frac{1}{2} \|u^{n-1}\|_{H^1(\Omega)} + \|u_h^{n-1}\|_{L^\infty(\Omega)} \leq 2 + \|u_h^{n-1}\|_{L^\infty(\Omega)} \quad \text{(when} h \text{ is sufficiently small)}. \tag{3.15} \]

Meanwhile, (3.12d) implies
\[
\max_{1 \leq n \leq m} \|P^RT_h u_h^{n-1}\|_{L^\infty(\Omega)} \leq 2 + \|u\|_{L^\infty(0,T;L^\infty(\Omega))}. \tag{3.16} \]

The boundedness of numerical solutions in (3.14)–(3.16) will be used in the following error analysis in estimating the nonlinear terms.

### 3.3 Estimates for \( e_{\rho,h} \)

From (1.1a) we know that the exact solution \( \rho^n \) satisfies the equation
\[
(D_\tau \rho^n, \varphi_h) + (u^{n-1} \cdot \nabla \rho^n, \varphi_h) = (R^n_{\rho}, \varphi_h) \quad \forall \varphi_h \in P^2_{dG}(T_h) \tag{3.17} \]
with
\[
R^n_{\rho} = D_\tau \rho^n - \partial_t \rho^n + (u^{n-1} - u^n) \cdot \nabla \rho^n. \]

Subtracting (2.16a) from (3.17) yields
\[
(D_\tau(\rho^n - P^\rho_{dG} \rho^n), \varphi_h) + (D_\tau e_{\rho,h}, \varphi_h)
+ ((P^RT_h u_h^{n-1}) \cdot \nabla (\rho^n - P^\rho_{dG} \rho^n), \varphi_h) + ((P^RT_h u_h^{n-1}) \cdot \nabla e_{\rho,h}^n, \varphi_h)
- \sum_{K \in T_h} (P^RT_h u_h^{n-1} \cdot \|\rho^n - P^\rho_{dG} \rho^n\|, \varphi_h)_{\partial K} - \sum_{K \in T_h} (P^RT_h u_h^{n-1} \cdot \|e_{\rho,h}^n\|, \varphi_h)_{\partial K}
+ ((u^{n-1} - P^RT_h u_h^{n-1}) \cdot \nabla \rho^n, \varphi_h) + (P^RT_h (u^{n-1} - u_h^{n-1}) \cdot \nabla \rho^n, \varphi_h)
= (R^n_{\rho}, \varphi_h) \quad \forall \varphi_h \in P^2_{dG}(T_h). \tag{3.18} \]

On a face \( F \in \mathcal{F}_h^1 \) we denote by \( P^\rho_{dG} \rho^n \) the value of \( P^\rho_{dG} \rho^n \) from the in-flow side. Then, by using integration by parts, we have
\[
(P^RT_h u_h^{n-1} \cdot \nabla (\rho^n - P^\rho_{dG} \rho^n), \varphi_h) - \sum_{K \in T_h} (P^RT_h u_h^{n-1} \cdot \|\rho^n - P^\rho_{dG} \rho^n\|, \varphi_h)_{\partial K}
\]
\[ -(P_{ht}^{RT} u_{ht}^{-1} (\rho^n - P_{ht}^{dG} \rho^n), \nabla \varphi_h) + \sum_{K \in T_h} \langle (P_{ht}^{RT} u_{ht}^{-1} - \nabla \varphi_h)(\rho^n - P_{ht}^{dG} \rho^n), \varphi_h \rangle_{\partial K} \times \forall \varphi_h \in P_{dG}^2(T_h). \]

Then, substituting this identity into (3.18), we obtain
\[
(D_{\tau} (\rho^n - P_{ht}^{dG} \rho^n), \varphi_h) + (D_{\tau} e_{\rho,h}^{n-1}, \varphi_h)
- (P_{ht}^{RT} u_{ht}^{-1} (\rho^n - P_{ht}^{dG} \rho^n), \nabla \varphi_h)
+ \sum_{K \in T_h} \langle (P_{ht}^{RT} u_{ht}^{-1} \cdot v_K)(\rho^n - P_{ht}^{dG} \rho^n), \varphi_h \rangle_{\partial K}
+ (P_{ht}^{RT} u_{ht}^{-1} \cdot \nabla e_{\rho,h}^{n}, \varphi_h) - \sum_{K \in T_h} \langle P_{ht}^{RT} u_{ht}^{-1} \cdot [e_{\rho,h}^{n}], \varphi_h \rangle_{\partial K^n}
+ ((u_{ht}^{n-1} - P_{ht}^{RT} u_{ht}^{-1}) \cdot \nabla \rho^n, \varphi_h) + (P_{ht}^{RT} (u_{ht}^{n-1} - u_{ht}^{-1}) \cdot \nabla \rho^n, \varphi_h)
= (R^n_{\rho}, \varphi_h) \quad \forall \varphi_h \in P_{dG}^2(T_h),
\]

which can be rewritten as
\[
(D_{\tau} e_{\rho,h}^{n-1}, \varphi_h) + (P_{ht}^{RT} u_{ht}^{-1} \cdot \nabla e_{\rho,h}^{n}, \varphi_h) - \sum_{K \in T_h} \langle P_{ht}^{RT} u_{ht}^{-1} \cdot [e_{\rho,h}^{n}], \varphi_h \rangle_{\partial K^n}
= -(D_{\tau} (\rho^n - P_{ht}^{dG} \rho^n), \varphi_h) + (P_{ht}^{RT} u_{ht}^{-1} (\rho^n - P_{ht}^{dG} \rho^n), \nabla \varphi_h)
- \sum_{K \in T_h} \langle (P_{ht}^{RT} u_{ht}^{-1} \cdot v_K)(\rho^n - P_{ht}^{dG} \rho^n), \varphi_h \rangle_{\partial K}
- ((u_{ht}^{n-1} - P_{ht}^{RT} u_{ht}^{-1}) \cdot \nabla \rho^n, \varphi_h) - (P_{ht}^{RT} (u_{ht}^{n-1} - u_{ht}^{-1}) \cdot \nabla \rho^n, \varphi_h)
+ (R^n_{\rho}, \varphi_h)
=: \sum_{j=1}^{6} E^n_{j}(\varphi_h). \tag{3.19}
\]

Since \( \nabla \cdot (P_{ht}^{RT} u_{ht}^{-1}) = 0 \) and \( (P_{ht}^{RT} u_{ht}^{-1}) \cdot v|_{\partial \Omega} = 0 \), it can be verified that
\[
(P_{ht}^{RT} u_{ht}^{-1} \cdot \nabla e_{\rho,h}^{n}, e_{\rho,h}^{n}) - \sum_{K \in T_h} \langle P_{ht}^{RT} u_{ht}^{-1} \cdot [e_{\rho,h}^{n}], e_{\rho,h}^{n} \rangle_{\partial K^n}
= \frac{1}{2} \sum_{F \in F_h^d} \| (P_{ht}^{RT} u_{ht}^{-1} \cdot \nabla F) + \frac{1}{2} \| e_{\rho,h}^{n} \|_{L^2(F)}^2, \tag{3.20}
\]

which is similar as (2.17). Since \( E^n_{j}(\varphi_h) = (D_{\tau} (\rho^n - P_{ht}^{dG} (D_{\tau} \rho^n), \varphi_h) = 0 \) for any \( \varphi_h \in P_{dG}^2(T_h) \), substituting \( \varphi_h = e_{\rho,h}^{n} \) into (3.19) yields
\[
\frac{1}{2} D_{\tau}(\| e_{\rho,h}^{n} \|_{L^2(\Omega)}^2) + \sum_{F \in F_h^d} \| (P_{ht}^{RT} u_{ht}^{-1} \cdot \nabla F) + \frac{1}{2} \| e_{\rho,h}^{n} \|_{L^2(F)}^2 \leq \sum_{j=1}^{6} |E^n_{j}(e_{\rho,h}^{n})|. \tag{3.21}
\]

In the following, we estimate \( |E^n_{j}(e_{\rho,h}^{n})| \) for \( j = 2, \ldots, 6 \).

We notice that \( \nabla \cdot (P_{ht}^{RT} u_{ht}^{-1}) = 0 \) in \( \Omega \), and \( P_{ht}^{RT} u_{ht}^{-1} \in RT^1(T_h) \). Thus we have \( P_{ht}^{RT} u_{ht}^{-1} \in P^1(T_h)^d \). Then by the definition of \( P_{ht}^{dG} \rho^n \), we have
\[
|E^n_{2}(e_{\rho,h}^{n})| = |(P_{ht}^{RT} u_{ht}^{-1} (\rho^n - P_{ht}^{dG} \rho^n), \nabla e_{\rho,h}^{n})| = 0. \tag{3.22}
\]
Since the value of $P_h^{RT} u_h^{n-1} \cdot v_F (\rho^n - \widetilde{P}_h^{dG} \rho^n)$ on a face $F \subseteq \partial K$ is independent of the tetrahedron containing the face $F$, and $P_h^{RT} u_h^{n-1} \cdot v_F (\rho^n - \widetilde{P}_h^{dG} \rho^n) = 0$ on the boundary faces, it follows that

$$|E_3^n(e_{\rho,h}^n)| = \left| \sum_{F \in \mathcal{F}_h^I} \left( P_h^{RT} u_h^{n-1} \cdot v_F (\rho^n - \widetilde{P}_h^{dG} \rho^n), [e_{\rho,h}^n] \cdot v_F \right)_F \right| \leq \sum_{F \in \mathcal{F}_h^I} \left\| P_h^{RT} u_h^{n-1} \cdot v_F \right\|_1^2 (\rho^n - \widetilde{P}_h^{dG} \rho^n) \left\| e_{\rho,h}^n \right\|_{L^2(F)}^2 + \frac{1}{4} \sum_{F \in \mathcal{F}_h^I} \left\| P_h^{RT} u_h^{n-1} \cdot v_F \right\|_1^2 \left\| e_{\rho,h}^n \right\|_{L^2(F)}^2 \leq \| P_h^{RT} u_h^{n-1} \|_{L^\infty(\Omega)} \sum_{F \in \mathcal{F}_h^I} \left\| \rho^n - \widetilde{P}_h^{dG} \rho^n \right\|_{L^2(F)}^2 + \frac{1}{4} \sum_{F \in \mathcal{F}_h^I} \left\| P_h^{RT} u_h^{n-1} \cdot v_F \right\|_1^2 \left\| e_{\rho,h}^n \right\|_{L^2(F)}^2 \leq C \sum_{K \in \mathcal{T}_h} \left( h^{-1} \| \rho^n - \widetilde{P}_h^{dG} \rho^n \|_{L^2(K)}^2 + h \| \rho^n - \widetilde{P}_h^{dG} \rho^n \|_{H^1(K)}^2 \right) + \frac{1}{4} \sum_{F \in \mathcal{F}_h^I} \left\| P_h^{RT} u_h^{n-1} \cdot v_F \right\|_1^2 \left\| e_{\rho,h}^n \right\|_{L^2(F)}^2 \leq Ch^{-1} h^{4+2\alpha} \| \rho^n \|_{H^{2+\alpha}(\Omega)}^2 + \frac{1}{4} \sum_{F \in \mathcal{F}_h^I} \left\| P_h^{RT} u_h^{n-1} \cdot v_F \right\|_1^2 \left\| e_{\rho,h}^n \right\|_{L^2(F)}^2 \right) (3.23)

where we have the inequality $ab = (\sqrt{2}a)(b/\sqrt{2}) \leq \frac{1}{2} (\sqrt{2}a)^2 + \frac{1}{4} (b/\sqrt{2})^2 = a^2 + b^2/4$.

For $\alpha \in (0, 1/2)$, by using the Sobolev embedding $H^{2+\alpha}(\Omega) \hookrightarrow W^{1, \frac{6}{1+\alpha}}(\Omega)$ and $H^2(\Omega) \hookrightarrow W^{3+\alpha, \frac{3}{1+\alpha}}(\Omega)$ (cf. [1, Theorem 7.43]), we have

$$|E_4^n(e_{\rho,h}^n)| = \left| \left( (u^{n-1} - P_h^{RT} u^{n-1}) \cdot \nabla \rho^n, e_{\rho,h}^n \right) \right| \leq \| u^{n-1} - P_h^{RT} u^{n-1} \|_{L^\frac{3}{1+\alpha}(\Omega)} \left\| \nabla \rho^n \right\|_{L^\frac{6}{1+\alpha}(\Omega)} \left\| e_{\rho,h}^n \right\|_{L^2(\Omega)} \right) \text{(Hölder’s inequality)} \leq C \left( \| u^{n-1} - I_h u^{n-1} \|_{L^\frac{3}{1+\alpha}(\Omega)} + \| I_h u^{n-1} - P_h^{RT} u^{n-1} \|_{L^\frac{2}{1+\alpha}(\Omega)} \right) \left\| \rho^n \right\|_{H^{2+\alpha}(\Omega)} \left\| e_{\rho,h}^n \right\|_{L^2(\Omega)} \leq C \left( \| u^{n-1} - I_h u^{n-1} \|_{L^\frac{3}{1+\alpha}(\Omega)} + h^{-\frac{1}{2}+\alpha} \| I_h u^{n-1} - P_h^{RT} u^{n-1} \|_{L^2(\Omega)} \right) \left\| e_{\rho,h}^n \right\|_{L^2(\Omega)} \leq C h^{\frac{3}{2}+\alpha} \left( \| u^{n-1} \|_{W^{\frac{3}{2}, \frac{3}{1+\alpha}}(\Omega)} + \| u^{n-1} \|_{H^2(\Omega)} \right) \left\| e_{\rho,h}^n \right\|_{L^2(\Omega)} \right) \text{(by using (3.4))} \leq C e^{-1} h^{3+2\alpha} + \epsilon \left\| e_{\rho,h}^n \right\|_{L^2(\Omega)} \right) \leq C e^{-1} h^{3} + \epsilon \left\| e_{\rho,h}^n \right\|_{L^2(\Omega)} \right) \leq C E_5^n(e_{\rho,h}^n) \left| \left( (u^{n-1} - P_h^{RT} u^{n-1}) \cdot \nabla \rho^n, e_{\rho,h}^n \right) \right| \leq \| P_h^{RT} (u^{n-1} - u^{n-1}) \|_{L^\frac{3}{1+\alpha}(\Omega)} \left\| \nabla \rho^n \right\|_{L^\frac{6}{1+\alpha}(\Omega)} \left\| e_{\rho,h}^n \right\|_{L^2(\Omega)} \leq C \left( \| P_h^{RT} (u^{n-1} - u^{n-1}) \|_{L^\frac{3}{1+\alpha}(\Omega)} + \| P_h^{RT} e_{\rho,h}^n \|_{L^\frac{3}{1+\alpha}(\Omega)} \right) \left\| \rho^n \right\|_{H^{2+\alpha}(\Omega)} \left\| e_{\rho,h}^n \right\|_{L^2(\Omega)} \right) \leq C (3.24)$
From now on we will remove the cut-off function $\chi$.

By the last inequality and the induction assumption (3.12e), we have

$$|E^2_{n}(e_{\rho,h})| \leq C \tau \left( \| \partial_t^2 \rho \|_{L^2(\Omega)} + \| \partial_t \rho \|_{L^2(\Omega)} \right) \| e_{n}^{\rho,h} \|_{L^2(\Omega)} \leq C \tau^2 + C \| e_{\rho,h}^{n} \|_{L^2(\Omega)}.$$

(3.26)

Substituting (3.22)–(3.24) into (3.21), we obtain for $1 \leq n \leq m$,

$$D_{\tau} \| e_{\rho,h}^{n} \|_{L^2(\Omega)}^2 + \frac{1}{4} \sum_{F \in F^h} \| (P_{h}^{RT} u_{h}^{n-1}) \cdot \nu_{F} \|_{F}^2 \| e_{\rho,h}^{n} \|_{L^2(F)}^2 \leq C \epsilon^{-1} \left( \tau^2 + h^{3+2\alpha} \right) + \epsilon \| e_{u,h}^{n-1} \|_{H^1(\Omega)}^2 + C \epsilon^{-1} \| e_{\rho,h}^{n} \|_{L^2(\Omega)}.$$

By choosing $\epsilon = 1$ and applying Grönwall’s inequality, we have

$$\max_{1 \leq n \leq m} \| e_{\rho,h}^{n} \|_{L^2(\Omega)}^2 + \frac{m}{4} \sum_{F \in F^h} \| (P_{h}^{RT} u_{h}^{n-1}) \cdot \nu_{F} \|_{F}^2 \| e_{\rho,h}^{n} \|_{L^2(F)}^2 \leq C \| e_{\rho,h}^{0} \|_{L^2(\Omega)}^2 + C \left( \tau^2 + h^{3+2\alpha} \right) + C \sum_{n=1}^{m} \| e_{u,h}^{n-1} \|_{H^1(\Omega)}^2 \leq C \left( \tau^2 + h^{3+2\alpha} \right) + C \sum_{n=1}^{m} \| e_{u,h}^{n-1} \|_{H^1(\Omega)}^2.$$

(3.27)

By the last inequality and the induction assumption (3.12e), we have

$$\max_{1 \leq n \leq m} \| e_{\rho,h}^{n} \|_{L^2(\Omega)} \leq C \left( \tau + h^{3+\alpha} + (\kappa \frac{1}{2} + h^{\frac{\alpha}{2}})h^{\frac{3}{2}} \right),$$

(3.28a)

$$\max_{1 \leq n \leq m} \| e_{\rho,h}^{n} \|_{L^\infty(\Omega)} \leq C h^{-\frac{d}{2}} \max_{1 \leq n \leq m} \| e_{\rho,h}^{n} \|_{L^2(\Omega)} \leq C \left( h^{-\frac{d}{2}} \tau + h^{\alpha} + \kappa \frac{1}{2} + h^{\frac{\alpha}{2}} \right).$$

(3.28b)

Since all the constants $C$ above are independent of $\kappa$, by choosing a sufficiently small $\kappa$ the inequality (3.28b) implies

$$\max_{1 \leq n \leq m} \| e_{\rho,h}^{n} \|_{L^\infty(\Omega)} \leq \frac{1}{4} \rho_{\min}$$

(3.29)

when

$$\tau \leq \kappa h^{\frac{d}{2}}$$

and $h$ is sufficiently small.

(3.30)

In this case,

$$\frac{1}{2} \rho_{\min} \leq \rho_{h}^{m}(x) \leq \frac{3}{2} \rho_{\max}.$$

(3.31)

As a result,

$$\chi(\rho_{h}^{n}) = \rho_{h}^{n} \text{ for } 0 \leq n \leq m.$$

(3.32)

From now on we will remove the cut-off function $\chi$ on $\rho_{h}^{n}$.
3.4 Estimates for $D_\tau e^n_{\rho,h}$

We estimate $\|D_\tau e^n_{\rho,h}\|_{L^2(\Omega)}$ by substituting $\varphi_h = D_\tau e^n_{\rho,h}$ in (3.19). Since $E^n(D_\tau e^n_{\rho,h}) = (D_\tau \rho^n - P_h^{DG}(D_\tau \rho^n), D_\tau e^n_{\rho,h}) = 0$, we have

$$\|D_\tau e^n_{\rho,h}\|^2_{L^2(\Omega)} \leq \sum_{j=2}^{6} |E^n(D_\tau e^n_{\rho,h})| + |((P_h^{RT}u_h^{n-1}) \cdot \nabla e^n_{\rho,h}, D_\tau e^n_{\rho,h})| + \left| \sum_{K \in T_h} \langle P_h^{RT}u_h^{n-1} \cdot [e^n_{\rho,h}], D_\tau e^n_{\rho,h} \rangle \right|_{\partial K^n}$$

$$=: \sum_{j=2}^{8} E^n_j(D_\tau e^n_{\rho,h}).$$

By using the inverse and trace inequalities, we have

$$|E^n_2(D_\tau e^n_{\rho,h})| \leq \|P_h^{RT}u_h^{n-1}\|_{L^\infty(\Omega)} \|\rho^n - P_h^{DG}\rho^n\|_{L^2(\Omega)} C h^{-1} \|D_\tau e^n_{\rho,h}\|_{L^2(\Omega)}$$

$$\leq C h^{-1} \|\rho^n - P_h^{DG}\rho^n\|_{L^2(\Omega)} \|D_\tau e^n_{\rho,h}\|_{L^2(\Omega)}$$

$$|E^n_3(D_\tau e^n_{\rho,h})| \leq \|P_h^{RT}u_h^{n-1}\|_{L^\infty(\Omega)} \left( C h^{-1/2} \|\rho^n - P_h^{DG}\rho^n\|_{L^2(\Omega)} + h^{1/2} \|\nabla (\rho^n - P_h^{DG}\rho^n)\|_{L^2(\Omega)} \right) \|D_\tau e^n_{\rho,h}\|_{L^2(\Omega)}$$

$$\leq C h^{-1} \|\rho^n - P_h^{DG}\rho^n\|_{L^2(\Omega)} \|D_\tau e^n_{\rho,h}\|_{L^2(\Omega)}$$

$$|E^n_4(D_\tau e^n_{\rho,h})| \leq \|P_h^{RT}u_h^{n-1}\|_{L^\infty(\Omega)} \|\nabla \rho^n\|_{L^2(\Omega)} \|\nabla (\rho^n - P_h^{DG}\rho^n)\|_{L^2(\Omega)} \|D_\tau e^n_{\rho,h}\|_{L^2(\Omega)}$$

$$\leq C \|\nabla \rho^n\|_{L^2(\Omega)} \|D_\tau e^n_{\rho,h}\|_{L^2(\Omega)}$$

and

$$|E^n_5(D_\tau e^n_{\rho,h})| \leq C \|\nabla e^n_{\rho,h}\|_{L^2(\Omega)} \left( \|u^{n-1} - P_h^{RT}u_h^{n-1}\|_{L^3(\Omega)} + \|P_h^{RT}(u^{n-1} - u_h^{n-1})\|_{L^3(\Omega)} \right) \|D_\tau e^n_{\rho,h}\|_{L^2(\Omega)}$$

$$\leq C \left( \|u^{n-1} - P_h^{RT}u_h^{n-1}\|_{L^3(\Omega)} + \|P_h^{RT}(u^{n-1} - u_h^{n-1})\|_{L^3(\Omega)} \right) \|D_\tau e^n_{\rho,h}\|_{L^2(\Omega)}$$

$$\leq C \left( h^{-2/6} + h^{-1/6} \|P_h^{RT}e^n_{u,h}\|_{L^2(\Omega)} \right) \|D_\tau e^n_{\rho,h}\|_{L^2(\Omega)}$$

$$\leq C \left( h^{-2/6} + h^{-1/6} \|P_h^{RT}e^n_{u,h}\|_{L^2(\Omega)} \right) \|D_\tau e^n_{\rho,h}\|_{L^2(\Omega)}$$
Substituting the estimates of $E^n_j(D\tau e^n_{\rho,h})$, $j = 2, \ldots, 8$, into (3.33), we obtain

$$\|D\tau e^n_{\rho,h}\|_{L^2(\Omega)} \leq C \tau^{-1} \left( \|e^n_{\rho,h}\|_{L^2(\Omega)} + \|e^n_{u,h}\|_{L^2(\Omega)} \right) + C(\tau + h).$$

By using (3.12b) and (2.8a), we have

$$\|D\tau e^n_{\rho,h}\|_{L^2(\Omega)} \leq C \tau^{-1} \left( \tau + h^{3+\alpha} + \kappa \frac{1}{2} h^{3} + h^{\frac{3}{2}+\frac{\alpha}{2}} + \tau^\frac{5}{6} \right).$$

The following estimate of $D\tau (\rho^n - \rho^n_h)$ is also needed in our error estimation for velocity.

**Lemma 3.3** The following inequality holds:

$$|(D\tau e^n_{\rho,h}, \varphi_h)| \leq C \varphi_h H^1(\Omega) \left( \|e^n_{\rho,h}\|_{L^2(\Omega)} + \|e^n_{u,h}\|_{L^2(\Omega)} + \tau + h^2 \right) \quad \forall \varphi_h \in P^2_dG(\Omega)h.$$

**Proof** According to (3.19), we have

$$\begin{aligned}
(D\tau (\rho^n - \rho^n_h), \varphi_h) &= \sum_{j=2}^{6} E^n_j(\varphi_h) - (P^RT_h u^n_{h-1} \cdot \nabla e^n_{\rho,h}, \varphi_h) \\
&\quad + \sum_{K \in \Omega} (P^RT_h u^n_{h-1} \cdot \|e^n_{\rho,h}\|, \varphi_h)\partial^n_{\alpha}

&= \sum_{j=2}^{6} E^n_j(\varphi_h) + (P^RT_h u^n_{h-1} \cdot \nabla \varphi_h, e^n_{\rho,h}) \\
&\quad - \sum_{K \in \Omega} (P^RT_h u^n_{h-1} \cdot \nu_K \varphi^n_{\rho,h}, \varphi_h)\partial K,
\end{aligned}$$

where we have used integration by parts, and $\varphi^n_{\rho,h}$ denotes the value of $e^n_{\rho,h}$ from the influx side on a face $F \subset \partial K$.

By the definition of the $L^2$-projection $P^dG_h$, we have

$$\begin{aligned}
|E^n_2(\varphi_h)| &= |(P^RT_h u^n_{h-1}(\rho^n - \rho^n_h), \nabla \varphi_h)| \\
&\leq C \left\|P^RT_h u^n_{h-1}\right\|_{L^\infty(\Omega)} \left\|\rho^n - P^dG_h \rho^n\right\|_{L^2(\Omega)} \left\|\varphi_h\right\|_{H^1(\Omega)} \\
&\leq C h^2 \left\|\rho^n\right\|_{H^2(\Omega)} \left\|\varphi_h\right\|_{H^1(\Omega)}.
\end{aligned}$$

Similarly as the estimates in (3.23), we have

$$\begin{aligned}
|E^n_3(\varphi_h)| &= \left| \sum_{F \in \mathcal{F}^I_h} \left\{ P^RT_h u^n_{h-1}(\rho^n - P^dG_h \rho^n), [\varphi_h]_F \right\} \right| \\
&\leq \left( \sum_{F \in \mathcal{F}^I_h} h^F \|P^RT_h u^n_{h-1}(\rho^n - P^dG_h \rho^n)\|_{L^2(F)}^2 \right)^{1/2} \cdot \left( \sum_{F \in \mathcal{F}^I_h} h^{-1}_F \left\|\varphi_h\right\|^2_{L^2(F)} \right)^{1/2} \\
&\leq C \left( \sum_{K \in \Omega} \|P^RT_h u^n_{h-1}\|_{L^\infty(K)}^2 \left\|\rho^n - P^dG_h \rho^n\right\|^2_{L^2(K)} + h^2 \left\|\rho^n - P^dG_h \rho^n\right\|^2_{H^1(K)} \right)^{1/2} \\
&\quad \cdot \left( \sum_{F \in \mathcal{F}^I_h} h^{-1}_F \left\|\varphi_h\right\|^2_{L^2(F)} \right)^{1/2}.
\end{aligned}$$
where the last inequality uses definition (3.11) of the norm \( \| \varphi \|_{H^1(T_h)} \).

By using integration by parts in \( E_4^n(e^n_{\rho,h}) \) and \( E_5^n(e^n_{\rho,h}) \), we have

\[
|E_4^n(e^n_{\rho,h})| = |(u^{n-1} - P_h^{RT} u^{n-1}) \cdot \nabla \rho^n, \varphi_h)| \\
\leq \| \rho^n \|_{L^\infty(\Omega)} \| u^{n-1} - P_h^{RT} u^{n-1} \|_{L^2(\Omega)} \| \nabla \varphi_h \|_{L^2(\Omega)} \\
+ \| \rho^n \|_{L^\infty(\Omega)} \sum_{F \in \mathcal{F}_h^I} h_F \| u^{n-1} - P_h^{RT} u^{n-1} \|_{L^2(F)}^2 \left( \sum_{F \in \mathcal{F}_h^I} h_F^{-1} \| \varphi_h \|_{L^2(F)}^2 \right)^{1/2} \\
\leq C(\| u^{n-1} - P_h^{RT} u^{n-1} \|_{L^2(\Omega)} + \| u^{n-1} - P_h^{RT} u^{n-1} \|_{H^1(\Omega)}) \| \varphi_h \|_{H^1(T_h)} \\
\leq C h^2 \| u^{n-1} \|_{H^2(\Omega)} \| \varphi_h \|_{H^1(T_h)}. 
\] (3.37)

The term \( E_6^n(e^n_{\rho,h}) \) can be estimated in the same way as (3.24), i.e.,

\[
|E_6^n(e^n_{\rho,h})| \leq C \tau \left( \| \delta_t^2 \rho \|_{L^2(\Omega)} + \| \delta_t u \|_{L^3(\Omega)} \| \nabla \rho \|_{L^6(\Omega)} \right) \| e^n_{\rho,h} \|_{L^2(\Omega)} \leq C \tau \| e^n_{\rho,h} \|_{L^2(\Omega)}. 
\] (3.38)

The last two terms in (3.35) can be estimated by

\[
\left| (P_h^{RT} u_h^{n-1} \cdot \nabla \varphi_h, e^n_{\rho,h}) \right| \leq C \| e^n_{\rho,h} \|_{L^2(\Omega)} \| \varphi_h \|_{H^1(T_h)} 
\] (3.40)

and

\[
\sum_{K \in T_h} (P_h^{RT} u_h^{n-1} \cdot \nu_K e^n_{\rho,h}, \varphi_h)_{\partial K} \\
= \sum_{F \in \mathcal{F}_h^I} \left( P_h^{RT} u_h^{n-1} e^n_{\rho,h}, \varphi_h \right)_F \\
\leq \| P_h^{RT} u_h^{n-1} \|_{L^\infty(\Omega)} \left( \sum_{F \in \mathcal{F}_h^I} h_F \| e^n_{\rho,h} \|_{L^2(F)}^2 \right)^{1/2} \left( \sum_{F \in \mathcal{F}_h^I} h_F^{-1} \| \varphi_h \|_{L^2(F)}^2 \right)^{1/2} 
\]
\[ \leq C \left( \sum_{K \in T_h} \| e_{\rho,h}^n \|^2_{L^2(K)} \right)^{\frac{1}{2}} \left( \sum_{F \in F^h} h_F^{-1} \| \| \varphi_h \| \|^2_{L^2(F)} \right)^{\frac{1}{2}} \text{(inverse trace inequality)} \]

\[ \leq C \| e_{\rho,h}^n \|_{L^2(\Omega)} \| \varphi_h \|_{H^1(T_h)}. \]  

(3.41)

Substituting (3.36)–(3.41) into (3.35) yields the desired result of Lemma 3.3. \( \square \)

### 3.5 Estimates for \( e_{u,h}^n \)

From (1.1b) one can see that the exact solution \( u^n \) satisfies the equation

\[ (\rho^{n-1} D_\tau u^n, v_h) + \frac{1}{2}((D_\tau \rho^n) u^n, v_h) + \frac{1}{2}((\rho^n u^{n-1} \cdot \nabla) u^n, v_h) \]

\[ - \frac{1}{2}((\rho^n u^{n-1} \cdot \nabla) v_h, u^n) + (\mu \nabla u^n, \nabla v_h) - (p^n, \nabla \cdot v_h) \]

\[ = (R_u^n, v_h) \quad \forall v_h \in \hat{P}^{1b}(T_h), \]  

(3.42)

with a defect \( R_u^n \), which has the following expression:

\[ R_u^n = (\rho^{n-1} - \rho^n) D_\tau u^n + \rho^n (D_\tau u^n - \partial_\tau u^n) + \frac{1}{2}(D_\tau \rho^n - \partial_\tau \rho^n) u^n \]

\[ + \rho^n (u^{n-1} - u^n) \cdot \nabla u^n. \]  

(3.43)

Under the regularity assumption (2.19), we have

\[ \| R_u^n \|_{L^2(\Omega)} \leq C \tau. \]  

(3.44)

We also note that Eq. (2.16b) can be rewritten as (removing the cut-off function \( \chi \) in view of (3.32))

\[ (\rho_h^{n-1} D_\tau e_{u,h}^n, v_h) + \frac{1}{2}(D_\tau \rho_h^n e_{u,h}^n, v_h) + \frac{1}{2}((\rho_h^n e_{u,h}^{n-1} \cdot \nabla) e_{u,h}^n, v_h) \]

\[ - \frac{1}{2}((\rho_h^n e_{u,h}^{n-1} \cdot \nabla) v_h, e_{u,h}^n) + (\mu \nabla e_{u,h}^n, \nabla v_h) - (p_h^n, \nabla \cdot v_h) = 0 \]

Subtracting the above equations from (3.42) yields

\[ \left[ (\rho_h^{n-1} D_\tau e_{u,h}^n, v_h) + (\rho_h^{n-1} D_\tau (u^n - \tilde{u}_h^n), v_h) \right] \]

\[ + (\rho^{n-1} - \rho_h^{n-1}) D_\tau u^n, v_h) \]

\[ + \frac{1}{2} \left[ (D_\tau \rho_h^n e_{u,h}^n, v_h) + (D_\tau (\rho^n - \rho_h^n) e_{u,h}^n, v_h) + (D_\tau \rho_h^n (u^n - \tilde{u}_h^n), v_h) \right] \]

\[ + \frac{1}{2} (D_\tau (\rho^n - \rho_h^n) u^n, v_h) \]

\[ + \frac{1}{2} (((\rho_h^n u_h^{n-1} - \tilde{u}_h^{n-1}) \cdot \nabla) u^n, v_h) + ((\rho_h^n e_{u,h}^{n-1} \cdot \nabla) u^n, v_h) \]

\[ + \frac{1}{2} (((\rho^n - \rho_h^n) u^{n-1} \cdot \nabla) u^n, v_h) \]

\[ - \frac{1}{2} \left[ ((\rho_h^n u_h^{n-1} - \tilde{u}_h^{n-1}) \cdot \nabla) e_{u,h}^n, e_{u,h}^n \right] + ((\rho_h^n u_h^{n-1} - \tilde{u}_h^{n-1}) \cdot \nabla) v_h, u^n - \tilde{u}_h^n \right] \]
Then, substituting \( v_h = e_{u,h}^n \) into the above equation and using the property \( (e_{p,h}^n \cdot \nabla e_{u,h}^n) = 0 \) (which is a consequence of \((2.16c)\)), we obtain the following error equation of \( e_{u,h}^n \):

\[
\begin{align*}
(\rho_{h}^n D_{T} e_{u,h}^n, e_{u,h}^n) + \frac{1}{2} (D_{T} \rho_{h}^n e_{u,h}^n, e_{u,h}^n) + (\mu \nabla e_{u,h}^n, \nabla e_{u,h}^n) \\
&= - (\rho_{h}^n D_{T} (u^{n} - \hat{u}_{h}^{n-1}), e_{u,h}^n) \\
&\quad - ((\rho_{h}^{n-1} - \rho_{h}^n) D_{T} u^{n}, e_{u,h}^n) \\
&\quad - \frac{1}{2} [(D_{T} (\rho_{h}^{n} - \rho_{h}^{n-1}) e_{u,h}^n, e_{u,h}^n) + (D_{T} \rho_{h}^n (u^{n} - \hat{u}_{h}^{n-1}), e_{u,h}^n)] \\
&\quad - \frac{1}{2} [(\rho_{h}^n u_{h}^{n-1} \cdot \nabla) e_{u,u}^n, e_{u,u}^n) + ((\rho_{h}^n u_{h}^{n-1} \cdot \nabla)(u^{n} - \hat{u}_{h}^{n-1}), e_{u,u}^n)] \\
&\quad - \frac{1}{2} [(\rho_{h}^n (u^{n-1} - \hat{u}_{h}^{n-1}) \cdot \nabla) u^{n}, e_{u,u}^n) + ((\rho_{h}^n e_{u,u}^{n-1} \cdot \nabla) u^{n}, e_{u,u}^n)] \\
&\quad + \frac{1}{2} [((\rho_{h}^n - \rho_{h}^{n-1}) u^{n-1} \cdot \nabla) u^{n}, e_{u,u}^n) \\
&= \sum_{j=1}^{11} F_{j}^{n}. 
\end{align*}
\]  

Since \( (\rho_{h}^n u_{h}^{n-1} \cdot \nabla) e_{u,u}^n, e_{u,u}^n) \) appears with opposite signs in both \( F_{5}^{n} \) and \( F_{8}^{n} \), it follows that

\[
\begin{align*}
(\rho_{h}^{n-1} D_{T} e_{u,u}^n, e_{u,u}^n) + \frac{1}{2} (D_{T} \rho_{h}^n e_{u,u}^n, e_{u,u}^n) + (\mu \nabla e_{u,u}^n, \nabla e_{u,u}^n) &= \sum_{j=1}^{11} \hat{F}_{j}^{n} + (R_{u}^{n}, e_{u,u}^n). 
\end{align*}
\]  

where \( \hat{F}_{j}^{n} = F_{j}^{n} \) for \( j \neq 5, 8 \), and

\[
\begin{align*}
\hat{F}_{5}^{n} &= -\frac{1}{2} ((\rho_{h}^n u_{h}^{n-1} \cdot \nabla)(u^{n} - \hat{u}_{h}^{n-1}), e_{u,u}^n), \\
\hat{F}_{8}^{n} &= \frac{1}{2} ((\rho_{h}^n u_{h}^{n-1} \cdot \nabla)e_{u,u}^n, u^{n} - \hat{u}_{h}^{n}).
\end{align*}
\]
In the following, we estimate $|\hat{F}_j^n|$ for $j = 1, \ldots, 11$.

First, we note that $(D_t \tilde{\rho}_h^n, D_t \tilde{p}_h^n)$ is actually the Stokes–Ritz projection of $(D_t u^n, D_t p^n)$. Therefore, (3.6) implies

$$
\|D_t u^n - D_t \tilde{\rho}_h^n\|_{L^2(\Omega)} \leq C h^2 \left(\|D_t u^n\|_{H^2(\Omega)} + \|D_t p^n\|_{H^1(\Omega)}\right) \leq C h^2.
$$

By using this result and the property $\|\tilde{\rho}_h^n\|_{L^\infty(\Omega)} \leq \frac{3}{2} \rho_{\max}$, we have

$$
|\hat{F}_1^n| = |(\rho_{\rho,h}^{-1} D_t (u^n - \tilde{\rho}_h^n), \epsilon_{u,h}^n)| \leq C h^2 \|\epsilon_{u,h}^n\|_{L^2(\Omega)} \leq C \epsilon h^4 + \epsilon \|\epsilon_{u,h}^n\|_{L^2(\Omega)}^2.
$$

Second, we have

$$
|\hat{F}_2^n| = 2 \left\| (\rho_{\rho,h}^{-1} - \rho_{\rho,h}^{-1}) D_t u^n, \epsilon_{u,h}^n \right\| \leq \left\| \rho_{\rho,h}^{-1} - \rho_{\rho,h}^{-1} \right\|_{L^2(\Omega)} \|D_t u^n\|_{L^1(\Omega)} \|\epsilon_{u,h}^n\|_{L^2(\Omega)} \leq C \left\| \rho_{\rho,h}^{-1} - \rho_{\rho,h}^{-1} \right\|_{L^2(\Omega)} \|D_t u^n\|_{H^1(\Omega)} \|\nabla\epsilon_{u,h}^n\|_{L^2(\Omega)} \quad \text{(Sobolev embedding)}
$$

$$
\leq C \left( \|\rho_{\rho,h}^{-1} - \rho_{\rho,h}^{-1} \right\|_{L^2(\Omega)} \|D_t u^n\|_{H^1(\Omega)} + \|\epsilon_{u,h}^n\|_{L^2(\Omega)} \right) \|D_t u^n\|_{H^1(\Omega)} \|\nabla\epsilon_{u,h}^n\|_{L^2(\Omega)}
$$

$$
\leq C \|\epsilon_{u,h}^n\|_{L^2(\Omega)} \left( h^2 + \|\epsilon_{u,h}^n\|_{L^2(\Omega)}^2 \right) + \epsilon \|\nabla\epsilon_{u,h}^n\|_{L^2(\Omega)}^2.
$$

|\hat{F}_3^n| = \frac{1}{2} \left\| (D_t (\rho_{\rho,h}^{-1} - \rho_{\rho,h}^{-1}) e_{u,h}^n, e_{u,h}^n) + (D_t \rho_{\rho,h}^{-1} (u^n - \tilde{\rho}_h^n), e_{u,h}^n) \right\|
$$

$$
\leq \left\| D_t (\rho_{\rho,h}^{-1} - \rho_{\rho,h}^{-1}) \right\|_{L^2(\Omega)} \|e_{u,h}^n\|_{L^2(\Omega)} + \|D_t \rho_{\rho,h}^{-1}\|_{L^2(\Omega)} \|u^n - \tilde{\rho}_h^n\|_{L^2(\Omega)} \|\nabla e_{u,h}^n\|_{L^2(\Omega)}
$$

$$
+ \left\| D_t \rho_{\rho,h}^{-1} \right\|_{H^1(\Omega)} \|u^n - \tilde{\rho}_h^n\|_{L^2(\Omega)} \|\nabla e_{u,h}^n\|_{L^2(\Omega)}
$$

$$
\leq C \left( \|\epsilon_{u,h}^n\|_{L^2(\Omega)} \left( h^2 + \|\epsilon_{u,h}^n\|_{L^2(\Omega)}^2 \right) + \|u^n - \tilde{\rho}_h^n\|_{L^2(\Omega)} \|\nabla e_{u,h}^n\|_{L^2(\Omega)} \right)
$$

$$
\leq C \left( \|\epsilon_{u,h}^n\|_{L^2(\Omega)} \left( h^2 + \|\epsilon_{u,h}^n\|_{L^2(\Omega)}^2 \right) + \|u^n - \tilde{\rho}_h^n\|_{L^2(\Omega)} \|\nabla e_{u,h}^n\|_{L^2(\Omega)} \right) + C \epsilon h^4.
$$

|\hat{F}_4^n| = \frac{1}{2} \left\| ((\rho_{\rho,h}^{-1} - \rho_{\rho,h}^{-1}) \cdot \nabla)(u^n - \tilde{\rho}_h^n), e_{u,h}^n \right\|
$$

$$
\leq \left\| ((\rho_{\rho,h}^{-1} - \rho_{\rho,h}^{-1}) \cdot \nabla)(u^n - \tilde{\rho}_h^n), e_{u,h}^n \right\|
$$

$$
+ \left\| ((\rho_{\rho,h}^{-1} - \rho_{\rho,h}^{-1}) u^n - \tilde{\rho}_h^n, e_{u,h}^n) \right\|
$$

$$
+ \left\| ((\rho_{\rho,h}^{-1} u^n - \tilde{\rho}_h^n, e_{u,h}^n) \right\|
$$

$$
|\hat{F}_5^n| = \frac{1}{2} \left\| ((\rho_{\rho,h}^{-1} - \rho_{\rho,h}^{-1}) \cdot \nabla)(u^n - \tilde{\rho}_h^n), e_{u,h}^n \right\|
$$

$$
= \left\| ((\rho_{\rho,h}^{-1} - \rho_{\rho,h}^{-1}) \cdot \nabla)(u^n - \tilde{\rho}_h^n), e_{u,h}^n \right\|
$$

$$
+ \left\| ((\rho_{\rho,h}^{-1} - \rho_{\rho,h}^{-1}) u^n - \tilde{\rho}_h^n, e_{u,h}^n) \right\|
$$

$$
+ \left\| ((\rho_{\rho,h}^{-1} u^n - \tilde{\rho}_h^n, e_{u,h}^n) \right\|
$$

$$
= 2 \left\| ((\rho_{\rho,h}^{-1} - \rho_{\rho,h}^{-1}) \cdot \nabla)(u^n - \tilde{\rho}_h^n), e_{u,h}^n \right\|
$$

$$
\leq C \|u^n - \tilde{\rho}_h^n\|_{L^2(\Omega)} \|\nabla e_{u,h}^n\|_{L^2(\Omega)} + C \epsilon h^4.
$$

\(\text{(use}\partial_t \rho_{\rho,h}^{-1} + \nabla \cdot (\rho_{\rho,h}^{-1} \rho_{\rho,h}^{-1}) = 0)\)

$$
\leq C \|u^n - u_{h}^{-1}\|_{L^2(\Omega)} \|\nabla e_{u,h}^n\|_{L^2(\Omega)} + C \epsilon h^4.
$$
\[ + C \| \rho^n - \rho^n_h \|_{L^2(\Omega)} \| u^{n-1} \|_{L^\infty(\Omega)} \| \nabla (u^n - \hat{u}_h^n) \|_{L^2(\Omega)} \| e^{a, h}_{u, u} \|_{L^\infty(\Omega)} \\
+ \| \rho^n \|_{L^\infty(\Omega)} \tau \| \partial_t u \|_{L^\infty(0, T; L^2(\Omega))} \| \nabla (u^n - \hat{u}_h^n) \|_{L^2(\Omega)} \| e^{a, h}_{u, h} \|_{L^6(\Omega)} \\
+ C (\| \partial_t \rho^n \|_{H^1(\Omega)} + \| \omega^n \|_{L^\infty(\Omega)} \| u^n \|_{L^\infty(\Omega)}) \| u^n - \hat{u}_h^n \|_{L^2(\Omega)} \| e^{a, h}_{u, h} \|_{H^1(\Omega)} \]

\[ \leq C h^{-\frac{1}{2}} \| u^{n-1} - u_h^{n-1} \|_{L^2(\Omega)} \| u^n - \hat{u}_h^n \|_{H^1(\Omega)} \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)} \] (inverse inequality)

\[ + C h^{-\frac{1}{2}} \| \rho^n - \rho^n_h \|_{L^2(\Omega)} \| u^n - \hat{u}_h^n \|_{H^1(\Omega)} \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)} \\
+ C \tau \| u^n - \hat{u}_h^n \|_{H^1(\Omega)} \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)} + C \| u^n - \hat{u}_h^n \|_{L^2(\Omega)} \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)} \leq C h^2 (\| u^{n-1} \|_{L^2(\Omega)} + \| \rho^n \|_{L^2(\Omega)} + \| \rho^n \|_{L^2(\Omega)}) \| u^n - \hat{u}_h^n \|_{L^2(\Omega)} + C (\| \rho^n \|_{L^2(\Omega)} + \| \rho^n \|_{L^2(\Omega)}) \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)} \leq (h + \epsilon) \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)}^2 + C (\| \rho^n \|_{L^2(\Omega)} + \| \rho^n \|_{L^2(\Omega)}) \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)} \leq C (h^2 + \| u^n \|_{H^2(\Omega)} \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)}) \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)} \leq C \epsilon^{-1} (h^4 + \| \rho^n \|_{L^2(\Omega)})^2 \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)}^2, \]

\[ |\tilde{F}_8^n| = \frac{1}{2} \|(\rho^n - \rho^n_h) u^{n-1} \cdot \nabla) u^n, e^{a, h}_{u, h} \| \leq C \left( \| \rho^n - P_{h}^{\text{DG}} \rho^n \|_{L^2(\Omega)} + \| e^{a, h}_{\rho, h} \|_{L^2(\Omega)} \right) \| u^{n-1} \|_{L^\infty(\Omega)} \| \nabla u^n \|_{L^2(\Omega)} \| e^{a, h}_{u, h} \|_{L^6(\Omega)} \leq C \left( h^2 \| \rho^n \|_{H^2(\Omega)} + \| e^{a, h}_{\rho, h} \|_{L^2(\Omega)} \right) \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)} \leq \epsilon \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)}^2 + C \epsilon^{-1} (h^4 + \| e^{a, h}_{\rho, h} \|_{L^2(\Omega)})^2, \]

\[ |\tilde{F}_9^n| = \frac{1}{2} \|(\rho^n u^{n-1} - \hat{u}_h^{n-1}) \cdot \nabla) e^{a, h}_{u, h}, u^n \| \leq \frac{3}{2} \rho_{\max} \| u^{n-1} - \hat{u}_h^{n-1} \|_{L^2(\Omega)} \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)} \| u^n - \hat{u}_h^n \|_{L^2(\Omega)} \leq C \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)} \| u^n - \hat{u}_h^n \|_{L^2(\Omega)} \] (by (3.15))

\[ \leq \epsilon \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)}^2 + C \epsilon^{-1} h^4, \]

\[ |\tilde{F}_{10}^n| = \frac{1}{2} \|(\rho^n (u^{n-1} - \hat{u}_h^{n-1}) \cdot \nabla) e^{a, h}_{u, h}, u^n \| \leq \frac{3}{2} \rho_{\max} \| u^{n-1} - \hat{u}_h^{n-1} \|_{L^2(\Omega)} \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)} \| u^n - \hat{u}_h^n \|_{L^2(\Omega)} \leq C \| u^{n-1} - \hat{u}_h^{n-1} \|_{L^2(\Omega)} \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)} \| u^n - \hat{u}_h^n \|_{L^2(\Omega)} \leq C (h^2 + \| e^{a, h}_{u, h} \|_{L^2(\Omega)}) \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)} \leq \epsilon \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)}^2 + C \epsilon^{-1} (h^4 + \| e^{a, h}_{u, h} \|_{L^2(\Omega)})^2, \]

\[ |\tilde{F}_{11}^n| = \frac{1}{2} \|(\rho^n - \rho^n_h) u^n \cdot \nabla) e^{a, h}_{u, h}, u^n \| \leq \left( \| \rho^n - P_{h}^{\text{DG}} \rho^n \|_{L^2(\Omega)} + \| e^{a, h}_{\rho, h} \|_{L^2(\Omega)} \right) \| u^{n-1} \|_{L^\infty(\Omega)} \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)} \| u^n \|_{L^\infty(\Omega)} \leq C \| \rho^n - P_{h}^{\text{DG}} \rho^n \|_{L^2(\Omega)} + \| e^{a, h}_{\rho, h} \|_{L^2(\Omega)} \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)} \leq (h^2 \| \rho^n \|_{H^2(\Omega)} + \| e^{a, h}_{\rho, h} \|_{L^2(\Omega)}) \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)} \leq \epsilon \| \nabla e^{a, h}_{u, h} \|_{L^2(\Omega)}^2 + C \epsilon^{-1} (h^4 + \| e^{a, h}_{\rho, h} \|_{L^2(\Omega)})^2. \]
Under regularity (2.19), the truncation error defined in (3.43) satisfies $\|R^n_u\|_{L^2(\Omega)} \leq C\tau$. Therefore,

$$|\hat{F}^n_{11}| \leq C\tau\|e^n_{u,h}\|_{L^2(\Omega)} \leq C\tau^2 + C\|e^n_{u,h}\|_{L^2(\Omega)}^2.$$ 

It remains to estimate $|\hat{F}^n_4|$. To this end, we use the following inequality and error estimate:

$$\|e^n_{u,h}\|_{L^\infty(\Omega)} \leq Ch^{-\frac{1}{2}}\|e^n_{u,h}\|_{H^1(\Omega)}$$

for $d = 2, 3$, (3.50)

$$\|u^n - \tilde{u}^n_h\|_{L^\infty(\Omega)} \leq C\tau^\frac{1}{2}(\|u^n\|_{H^2(\Omega)} + \|P^n\|_{H^1(\Omega)})$$

for $d = 2, 3$, (3.51)

$$\|D\tau\rho^n - P^n_d\rho^n\|_{H^{-1}(\Omega)} \leq C\tau^2\|D\tau\rho^n\|_{H^1(\Omega)}.$$ (3.52)

where the second inequality can be proved by using (3.6) combined with inverse inequality and triangle inequality. By using Lemmas 3.3 and 3.2, and (3.50)–(3.52), we have

$$\hat{F}^n_4 = \frac{1}{2}(D\tau(\rho^n - \rho^n_h)u^n_h, e^n_{u,h})$$

$$= \frac{1}{2}(D\tau e^n_{\rho,h}, P^n_h(u^n_h \cdot e^n_{u,h})) + \frac{1}{2}(D\tau(\rho^n - P^n_h\rho^n), u^n_h \cdot e^n_{u,h})$$

$$\leq C||P^n_h\{u^n_h \cdot e^n_{u,h}\}|_{H^1(\Omega)}(\|e^n_{\rho,h}\|_{L^2(\Omega)} + \|e^n_{\rho,h} - 1\|_{L^2(\Omega)} + \tau + h^2)$$

$$+ Ch^2\|D\tau\rho^n\|_{H^1(\Omega)}\|u^n_h e^n_{u,h}\|_{H^1(\Omega)}$$

$$\leq C||\nabla\{u^n_h \cdot e^n_{u,h}\}|_{L^2(\Omega)}(\|e^n_{\rho,h}\|_{L^2(\Omega)} + \|e^n_{\rho,h} - 1\|_{L^2(\Omega)} + \tau + h^2)$$

$$\leq C(||e^n_{\rho,h}\|_{L^\infty(\Omega)} + \|u^n - \tilde{u}^n_h\|_{L^\infty(\Omega)} + \|\nabla e^n_{\rho,h}\|_{L^3(\Omega)}$$

$$+ \|\nabla u^n - \tilde{u}^n_h\|_{L^{\frac{3}{2}}(\Omega)})(\|e^n_{\rho,h}\|_{L^2(\Omega)} + \|e^n_{\rho,h} - 1\|_{L^2(\Omega)} + \tau + h^2)$$

$$\leq C(h^{-\frac{1}{2}}\|\nabla e^n_{\rho,h}\|_{L^2(\Omega)} + h^\frac{1}{2})\|\nabla e^n_{\rho,h}\|_{L^2(\Omega)}(\|e^n_{\rho,h}\|_{L^2(\Omega)} + \|e^n_{\rho,h} - 1\|_{L^2(\Omega)} + \tau + h^2)$$

$$\leq Ch^{-\frac{1}{2}}\|e^n_{\rho,h}\|_{L^2(\Omega)}\|\nabla e^n_{\rho,h}\|_{L^2(\Omega)} + Ch^2\|\nabla e^n_{\rho,h}\|_{L^2(\Omega)}$$

$$+ C(\|e^n_{\rho,h} - 1\|_{L^2(\Omega)} + \tau + h^2)h^{-\frac{1}{2}}\|\nabla e^n_{\rho,h}\|_{L^2(\Omega)}$$

$$+ C(\|e^n_{\rho,h} - 1\|_{L^2(\Omega)} + \tau + h^2)h^\frac{1}{2}\|\nabla e^n_{\rho,h}\|_{L^2(\Omega)}$$

$$\leq \left[Ch^{-\frac{1}{2}}(\tau + h^3 + \frac{h^5}{2} + \kappa^\frac{1}{2}) + Ch^2(\tau + h^3 + \frac{h^5}{2} + \kappa^\frac{1}{2} + \tau + h^2)\right]\|\nabla e^n_{\rho,h}\|_{L^2(\Omega)}$$

$$+ Ch^\frac{1}{2}\|\nabla e^n_{\rho,h}\|_{L^2(\Omega)} + Ch^\frac{1}{2}\|e^n_{\rho,h}\|_{L^2(\Omega)} + Ch^\frac{1}{2}(\|e^n_{\rho,h} - 1\|_{L^2(\Omega)}^2 + \tau^2 + h^4),$$

where we have used (3.28) and (3.12b) in deriving the last inequality. With the stepsize restriction $\tau \leq \kappa h^\frac{1}{4}$, the inequality above furthermore implies

$$\hat{F}^n_4 \leq Ch^\frac{1}{2}(\|\nabla e^n_{\rho,h}\|_{L^2(\Omega)} + \|e^n_{\rho,h}\|_{L^2(\Omega)}^2 + \|e^n_{\rho,h} - 1\|_{L^2(\Omega)}^2) + C(\tau^2 + h^4).$$
By substituting the consistency error (3.44) and the estimates of \(F_j, j = 1, \ldots, 11\), into (3.47), we obtain
\[
\frac{1}{2}D_\tau\|\sqrt{\rho h^k e^k_{u,h}}\|^2_{L^2(\Omega)} + \mu\|e^\mu_{u,h}\|_{H^1(\Omega)}^2 \\
\leq (\epsilon + Ch^\frac{1}{2})\|\nabla e^\mu_{u,h}\|^2_{L^2(\Omega)} + Ce^{-1}\|e^\mu_{\rho,h}\|^2_{L^2(\Omega)} + \|e^{\mu-1}\|_{L^2(\Omega)}^2 + \|e^{\mu-1}_{h,h}\|_{L^2(\Omega)}^2 + C\epsilon^{-1}(\tau^2 + h^4).
\]

By choosing sufficiently small \(\epsilon\) and \(h\), the first term on the right-hand side can be absorbed by the left-hand side. Then, summing up the inequality above for \(n = 1, \ldots, k\), we obtain for \(k = 1, \ldots, m\),
\[
\frac{1}{2}\|\sqrt{\rho h^k e^k_{u,h}}\|^2_{L^2(\Omega)} + \mu\sum_{n=0}^k \tau\|e^\mu_{u,h}\|_{H^1(\Omega)}^2 \\
\leq C\sum_{n=0}^k \tau\|e^\mu_{\rho,h}\|^2_{L^2(\Omega)} + \|e^{\mu-1}_{\rho,h}\|^2_{L^2(\Omega)} + \|e^{\mu-1}_{u,h}\|^2_{L^2(\Omega)} + C(\tau^2 + h^4).
\]

Summing up \(\lambda \times (3.27)\) and (3.53), we obtain
\[
\lambda\|e^\mu_{\rho,h}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\sqrt{\rho h^k e^k_{u,h}}\|^2_{L^2(\Omega)} + \mu\sum_{n=0}^k \tau\|e^\mu_{u,h}\|_{H^1(\Omega)}^2 \\
\leq C(\tau^2 + h^{3+2\alpha}) + C\lambda\sum_{n=0}^k \tau\|e^\mu_{u,h}\|_{H^1(\Omega)}^2 \\
+ C\tau\sum_{n=1}^k \left(\|e^\mu_{\rho,h}\|^2_{L^2(\Omega)} + \|e^{\mu-1}_{\rho,h}\|^2_{L^2(\Omega)} + \|e^{\mu-1}_{u,h}\|^2_{L^2(\Omega)}\right).
\]

By choosing \(\lambda\) small enough, the term \(C\lambda\sum_{n=0}^k \tau\|e^\mu_{u,h}\|_{H^1(\Omega)}^2\) can be absorbed by the left-hand side and we obtain for \(1 \leq k \leq m\)
\[
\|e^\mu_{\rho,h}\|^2_{L^2(\Omega)} + \|\sqrt{\rho h^k e^k_{u,h}}\|^2_{L^2(\Omega)} + \sum_{n=0}^k \tau\|e^\mu_{u,h}\|_{H^1(\Omega)}^2 \\
\leq C(\tau^2 + h^{3+2\alpha}) + C\tau\sum_{n=1}^k \left(\|e^{\mu-1}_{\rho,h}\|^2_{L^2(\Omega)} + \|e^\mu_{\rho,h}\|^2_{L^2(\Omega)} + \|e^{\mu-1}_{u,h}\|^2_{L^2(\Omega)}\right).
\]

Applying Grönwall’s inequality to (3.54) and using (3.14) and (3.31), we have
\[
\max_{1 \leq n \leq m} \left(\|e^\mu_{\rho,h}\|^2_{L^2(\Omega)} + \|e^\mu_{u,h}\|^2_{L^2(\Omega)}\right) + \sum_{n=1}^m \tau\|e^\mu_{u,h}\|_{H^1(\Omega)}^2 \leq C(\tau^2 + h^{3+2\alpha}).
\]

For \(\tau \leq \kappa h^2\) and sufficiently small \(\kappa\) and \(h\), the inequality above implies
\[
\|e^\mu_{u,h}\|_{L^2(\Omega)} \leq h^{\frac{3+2\alpha}{2}} + \tau^\delta, \\
\|e^\mu_{u,h}\|_{L^\infty(\Omega)} \leq C\tau^{-\delta}\|e^\mu_{u,h}\|_{L^2(\Omega)} \leq h^{-\delta}(\tau + h^{\frac{3+2\alpha}{2}}) \leq 1, \\
\|P^RT_h u^n - u^n\|_{L^\infty(\Omega)} \leq \|P^RT_h (u^n_h - u^n)\|_{L^\infty(\Omega)} + \|P^RT_h u^n - u^n\|_{L^\infty(\Omega)} \\
\leq C\tau^{-\delta}\|P^RT (u^n_h - u^n)\|_{L^2(\Omega)} + \|P^RT_h u^n - u^n\|_{L^\infty(\Omega)}.
\]
\[ \leq C h^{-\frac{q}{2}} (\tau + h^{\frac{1}{2} + \alpha}) \]
\[ \leq C \kappa + C h^\alpha \]
\[ \leq 2, \]
\[ \sum_{n=0}^{m} \tau \| \nabla e_u^n \|_{L^2(\Omega)}^2 \leq C (\tau^2 + h^{3+2\alpha}) \leq C (\kappa^2 h^3 + h^{3+2\alpha}) \]
\[ \leq (\kappa + h^\alpha) h^3 \text{ (when } \kappa \text{ and } h \text{ are sufficiently small).} \]

This proves (3.13b, 3.13c, 3.13d, 3.13e). Since (3.13a) has been proved in (3.29), the mathematical induction is closed. Consequently, the estimates (3.13) hold for \( m = N \) (with the same constants), which imply the desired estimate in Theorem 2.1.

### 4 Numerical Experiments

In this section, we present numerical examples to illustrate the convergence of the numerical method shown in Theorem 2.1. All the computations are performed by Firedrake [23].

In order to test the order of convergence, we consider the following equations with source terms \( f \) and \( g \):

\[ \partial_t \rho + \nabla \cdot (\rho u) = f \quad \text{in } \Omega \times (0, T], \]
\[ \rho \partial_t u + \rho (u \cdot \nabla) u + \nabla p - \mu \Delta u = g \quad \text{in } \Omega \times (0, T], \]
\[ \nabla \cdot u = 0 \quad \text{in } \Omega \times (0, T], \]

with \( T = 0.25 \) and \( \mu = 0.001 \). The source terms \( f \) and \( g \) are constructed by substituting an exact solution to the equations. The errors of the numerical solutions with stepsize \( \tau \) and mesh size \( h \) are denoted by

\[ \| E^{\tau, h}_{\rho} \|_{\ell^\infty(L^2)} = \max_{1 \leq n \leq N_T} \| \rho^n_h - \rho^n \|_{L^2}, \quad \| E^{\tau, h}_{u} \|_{\ell^\infty(L^2)} = \max_{1 \leq n \leq N_T} \| u^n_h - u^n \|_{L^2}, \]

The convergence order in space is computed by using the following formula,

\[ \text{convergence order} = \frac{\log(\frac{E^{\tau_1, h}_\rho}{E^{\tau_2, h}_\rho})}{\log(h_1/h_2)} \]

with a sufficiently small stepsize \( \tau \). The convergence order in time is computed by using the following formula,

\[ \text{convergence order} = \frac{\log(\frac{E^{\tau_1, h}_\rho}{E^{\tau_1, h}_\rho})}{\log(\tau_1/\tau_2)} \]

with a sufficiently small mesh size \( h \).

In two dimensions, we consider the problem on the unit square \( \Omega = (0, 1) \times (0, 1) \) with the following exact solution:

\[ \left\{ \begin{array}{l}
\rho = 2 + x(x - 1) \cos(\sin(t)) + y(y - 1) \sin(\sin(t)), \\
u = (\sin^2(\pi x) \sin(2\pi y), -\sin(2\pi x) \sin^2(\pi y)), \\
p = tx + y - \frac{t + 1}{2}.
\end{array} \right. \]

We test the convergence order in space by choosing a sufficiently small stepsize \( \tau = 1/2^{048} \) so that the error from temporal discretization is negligible in comparison with the error from...
spatial discretization. The errors of the numerical solutions are presented in Table 1, where second-order convergence is observed for both \( \rho \) and \( u \). This is consistent with the theoretical result in Theorem 2.1.

The convergence order in time is computed by choosing \( h = \tau^{1/2} \) and presented in Table 2, where first-order convergence in time is observed. This is also consistent with the theoretical result in Theorem 2.1.

In three dimensions, we consider the problem in a unit cube \( \Omega = (0, 1) \times (0, 1) \times (0, 1) \) with the follow exact solution:

\[
\begin{align*}
\rho(x, y, z, t) &= 2 + \frac{1}{3} \left( \sin(\pi x) + \sin(\pi y) + \sin(\pi z) \right) \sin(\pi t + \frac{\pi}{2}), \\
u(x, y, z, t) &= \left( \sin^2(\pi x) \sin(2\pi y) \sin(2\pi z), \sin(2\pi x) \sin^2(\pi y) \sin(2\pi z), \\
&\quad -2 \sin(2\pi x) \sin(2\pi y) \sin^2(\pi z) \right), \\
p(x, y, z, t) &= t(x + y) + z - t + \frac{1}{2}.
\end{align*}
\]

The errors of the numerical solutions and the convergence orders in space and time are presented in Tables 3 and 4, respectively. Second-order convergence in space and first-order convergence in time are observed, which are consistent with the theoretical result in Theorem 2.1.
Table 4 Temporal convergence with $h = \tau^{1/2}$

| $\tau$  | $\| E_{\rho}^{\tau,h} \|_{L^\infty(L^2)}$ | Convergence order | $\| E_u^{\tau,h} \|_{L^\infty(L^2)}$ | Convergence order |
|--------|----------------------------------|-----------------|----------------------------------|-----------------|
| 1/256  | 3.45e−03                         |                 | 3.74e−02                         |                 |
| 1/324  | 2.72e−03                         | 1.01            | 2.83e−02                         | 1.19            |
| 1/400  | 2.20e−03                         | 1.01            | 2.19e−02                         | 1.21            |
| 1/484  | 1.81e−03                         | 1.01            | 1.74e−02                         | 1.23            |
| 1/576  | 1.52e−03                         | 1.02            | 1.40e−02                         | 1.23            |

Table 5 Spatial convergence with $\tau = 1/2048$

| $h$     | $\| E_{\rho}^{\tau,h} \|_{L^\infty(L^2)}$ | Convergence order | $\| E_u^{\tau,h} \|_{L^\infty(L^2)}$ | Convergence order |
|---------|----------------------------------|-----------------|----------------------------------|-----------------|
| 1/10    | 3.10e−03                         |                 | 8.99e−02                         |                 |
| 1/12    | 2.17e−03                         | 1.94            | 6.19e−02                         | 2.04            |
| 1/14    | 1.59e−03                         | 2.04            | 4.40e−02                         | 2.21            |
| 1/16    | 1.20e−03                         | 2.09            | 3.23e−02                         | 2.32            |

Table 6 Temporal convergence with $h = \tau^{1/2}$

| $\tau$  | $\| E_{\rho}^{\tau,h} \|_{L^\infty(L^2)}$ | Convergence order | $\| E_u^{\tau,h} \|_{L^\infty(L^2)}$ | Convergence order |
|--------|----------------------------------|-----------------|----------------------------------|-----------------|
| 1/576  | 4.95e−04                         |                 | 1.25e−02                         |                 |
| 1/676  | 4.18e−04                         | 1.06            | 1.03e−02                         | 1.19            |
| 1/784  | 3.57e−04                         | 1.06            | 8.66e−03                         | 1.18            |
| 1/900  | 3.09e−04                         | 1.05            | 7.37e−03                         | 1.17            |
| 1/1024 | 2.70e−04                         | 1.05            | 6.36e−03                         | 1.15            |

In the two examples above, the exact solutions are sufficiently smooth. Finally, we also consider an exact solution which is not sufficiently smooth,

$$\begin{cases}
\rho = 2 + g(x, c) \cos(t) + \left( g(y, c) + g(z, c) \right) \sin(t), \\
u = \left( \sin^2(\pi x) \sin(2\pi y) \sin(2\pi z), \sin(2\pi x) \sin^2(\pi y) \sin(2\pi z) \right), \\
\quad -2 \sin(2\pi x) \sin(2\pi y) \sin^2(\pi z)), \\
p = t(x + y) + z - \frac{t + 1}{2},
\end{cases}$$

(4.7)

where

$$g(x, c) = \left| x - \frac{1}{2} \right|^c$$

(4.8)

with $c = 1.51$ on unit cube $\Omega = (0, 1) \times (0, 1) \times (0, 1)$. This exact solution satisfies $\rho \in H^{2+\alpha}(\Omega)$ for some $\alpha \in (0, 0.01)$. The errors of the numerical solutions and the convergence orders in space and time are presented in Table 5 and Table 6, respectively. Again, second-order convergence in space and first-order convergence in time are observed, which are consistent with the theoretical result in Theorem 2.1.
5 Conclusions

We have present error analysis for a fully discrete, linearized semi-implicit and decoupled FEM for the coupled system (1.1) describing incompressible flow with variable density. Compared to the previous work in [8], the error analysis in this paper is obtained in three dimensions under more realistic $H^{2+\alpha}$ regularity assumptions on the solution in a convex polyhedron. In the numerical method for the velocity equation (2.16b), we have added a stabilization term

$$ \frac{1}{2} \left( D \chi (\rho_h^n u_h^n, v_h) - \frac{1}{2} \chi (\rho_h^n u_h^{n-1}, \nabla (u_h^n \cdot v_h)) \right), $$

which helps to stabilize the velocity equation and therefore yields the energy inequality (2.18) unconditionally, which holds also for small viscosity $\mu$. Since our error analysis strongly relies on the viscosity in the momentum equations, we have not considered the convection dominate case in this paper. But the energy inequality (2.18) implies that the method at least maintains the energy stability of the numerical solution in the convection dominant case. The error analysis for system (1.1) in the convection dominant case is more challenging and remains open.

Author Contributions BL, WQ and ZY have participated sufficiently in the work to take public responsibility for the content, including participation in the concept, method, analysis and writing. All authors certify that this material or similar material has not been and will not be submitted to or published in any other publication.

Funding The work of B. Li and Z. Yang was supported in part by National Natural Science Foundation of China (NSFC Grant 12071020) and an internal grant of The Hong Kong Polytechnic University (Project 4-ZZKQ). Weifeng Qiu is supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CityU 11302718).

Availability of Data and Materials Not applicable.

Code Availability Not applicable.

Declarations

Conflict of interest No conflict of interest exists.

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