Effects of quantum statistics near the critical point of nuclear matter

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Effects of quantum statistics for nuclear matter equation of state are analyzed in terms of the recently proposed quantum van der Waals model. The system pressure is expanded over a small parameter $\delta = n(mT)^{-1/2}g(1-bn)^{-1}$, where $n$ and $T$ are, respectively, the particle number density and temperature, $m$ and $g$ the particle mass and degeneracy factor. The parameter $b$ corresponds to the van der Waals excluded volume. The corrections due to quantum statistics for the critical point values of $T_c$, $n_c$, and the critical pressure $P_c$ are found within the linear and quadratic orders over $\delta$. These approximate analytical results appear to be in a good agreement with exact numerical calculations in the quantum van der Waals model for interacting Fermi particles: the symmetric nuclear matter ($g = 4$) and the pure neutron matter ($g = 2$). They can be also applied to the system of interacting Bose particles like the matter composed of $\alpha$ nuclei.

I. INTRODUCTION

A study of nuclear matter – an interacting system of protons and neutrons – has a long history; see, e.g., Refs. [1–11]. Realistic versions of the nuclear matter equation of state includes both the attractive and repulsive forces between protons and neutrons. A theoretical description of the thermodynamical behavior of this matter leads to the liquid-gas first-order phase transition which ends at the critical point. Such a behavior is rather similar to that in the atomic gases and liquids.

In the present paper the recently proposed van der Waals model of quantum statistics. The system pressure is expanded over a small temperature, $T$ and not too large particle densities. Within these restrictions, the number of nucleons becomes a conserved number, and the chemical potential of this system regulates the number density of nucleons. We do not include the Coulomb forces and make no differences between protons and neutrons (both these particles are named as nucleons). In addition, under these restrictions the non-relativistic treatment becomes very accurate and is adopted in our studies.

The paper is organized as follows. In Sec. II we remind some results of the ideal Bose and Fermi gases. In Sec. III the quantum statistics effects near the critical point are studied for the symmetric nuclear matter. These results are then extended to the pure neutron matter and pure $\alpha$ matter. Section IV summarizes the paper.

II. IDEAL QUANTUM GASES

The pressure $P(T, \mu)$ plays the role of the thermodynamical potential in the grand canonical ensemble (GCE) where temperature $T$ and chemical potential $\mu$ are independent variables. The particle number density $n(T, \mu)$, entropy density $s(T, \mu)$, and energy density $\varepsilon(T, \mu)$ are given as

$$n = \left( \frac{\partial P}{\partial \mu} \right)_T, \quad s = \left( \frac{\partial P}{\partial T} \right)_\mu, \quad \varepsilon = Ts + \mu n - P \quad . \quad (1)$$

We start with the GCE expressions for the pressure $P_{id}(T, \mu)$ and particle number density $n_{id}(T, \mu)$ for the ideal non-relativistic quantum gas [13, 20],

$$P_{id} = \frac{g}{3} \int \frac{dp}{(2\pi\hbar)^3} \frac{p^2}{m} \left[ \exp\left( \frac{p^2}{2mT} - \frac{\mu}{T} \right) - 1 \right]^{-1}, \quad (2)$$

$$n_{id} = g \int \frac{dp}{(2\pi\hbar)^3} \left[ \exp\left( \frac{p^2}{2mT} - \frac{\mu}{T} \right) - 1 \right]^{-1}, \quad (3)$$

where $m$ and $g$ are, respectively, the particle mass and degeneracy factor. The value of $\theta = -1$ corresponds to the Fermi gas, $\theta = 1$ to the Bose gas, and $\theta = 0$ is the Boltzmann (classical) approximation when effects of the quantum statistics are neglected.

Equations (2) and (3) can be expressed in terms of the power series over fugacity $z \equiv \exp(\mu/T)$ for $z \leq 1$:

$$P(T, z) \equiv \frac{gT}{\theta \Lambda^3} L_{1/2}(\theta z) = \frac{gT}{\theta \Lambda^3} \sum_{k=1}^{\infty} \frac{(\theta z)^k}{k^{3/2}} \quad , \quad (4)$$

$$n(T, z) \equiv \frac{g}{\theta \Lambda^3} L_{3/2}(\theta z) = \frac{g}{\theta \Lambda^3} \sum_{k=1}^{\infty} \frac{(\theta z)^k}{k^{3/2}} \quad , \quad (5)$$

where

$$\Lambda \equiv \hbar \sqrt{\frac{2\pi}{mT}} \quad . \quad (6)$$

1 The units with Boltzmann constant $\kappa_B = 1$ are used. We keep the Plank constant in the formulae to illustrate the effects of quantum statistics, but put $\hbar = \hbar/2\pi = 1$ in all numerical calculations. For simplicity, we omitted here and below the subscript $id$ for the “ideal gas” everywhere where it will not lead to a misunderstanding.
is the de Broglie heat wavelength [27], and \( \text{Li}_n \) is the polylogarithmic function [24, 25]. The values of \( \mu > 0 \), i.e., \( z > 1 \), are forbidden in the ideal Bose gas. The point \( \mu = 0 \) corresponds to an onset of the Bose-Einstein condensation in the system of bosons. For fermions, any values of \( \mu \) are possible, i.e., integrals (2) and (3) exist for \( \theta = -1 \) at all real values of \( \mu \). However, the power series (4) and (5) are convergent at \( z \leq 1 \) only. For the Fermi statistics at \( z > 1 \), the integral representation of the corresponding polilogarithmic function can be used. Particularly, at \( z \to \infty \) one can use the asymptotic Sommerfeld expansion of the Li\(_n\)(\( -z \)) functions over \( 1/\ln^2|z| \) [24].

Figure 1 shows lines of the constant values of fugacity \( z \) in the \( n-T \) plane for the ideal Fermi gases (a) and (b), and Bose gas (c), see Eq. (14). Figures 1 (a) corresponds to the isospin symmetric ideal nucleon gas (i.e., an equal number of protons and neutrons). We take \( m \approx 938 \text{ MeV} \) neglecting a small difference between proton and neutron masses. The degeneracy factor is then \( g = 4 \) which takes into account two spin and two isospin states of nucleon. The ideal neutron gas with \( g = 2 \) is presented in Fig. 1 (b), and the ideal Bose gas of \( \alpha \)-particles (\( g = 1 \) and \( m \approx 3727 \text{ MeV} \)) is shown in Fig. 1 (c).

At \( z \ll 1 \), only one term \( k = 1 \) is enough in Eqs. (4) and (5) which leads to the classical ideal gas relation

\[
P = nT. \tag{7}
\]

Note that the result (7) follows automatically from Eqs. (2) and (3) at \( \theta = 0 \). As seen from Fig. 1, the classical Boltzmann approximation at \( z \ll 1 \) is valid for large \( T \) and small \( n \) region of the \( n-T \) plane. However, the classical ideal gas equation (7) becomes wrong at small \( T \) and/or large \( n \). The entropy density of the classical ideal gas reads

\[
s = \left( \frac{\partial P}{\partial T} \right)_\mu = n \left( \frac{5}{2} - \frac{\mu}{T} \right), \tag{8}
\]

and it becomes negative at \( \mu/T > 5/2 \). The particle number density in the classical ideal gas at \( \mu/T = 5/2 \) equals to

\[
n = \frac{g}{\hbar^3} \left( \frac{mT}{2\pi} \right)^{3/2} \exp \left( \frac{5}{2} \right). \tag{9}
\]

The relation (9) is shown by a dashed red line in Fig. 1. Under this line, the entropy density of the classical ideal gas becomes negative. This happens at small \( T \) and/or large \( n \) and means a contradiction with the third law of thermodynamics. Quantum statistics solves this problem: Eqs. (2) and (3) guarantee \( s \geq 0 \) at all \( T \) and \( n \), and \( s = 0 \) at \( T = 0 \).

Inverting the \( z^k \) power series in Eq. (3), one finds the power expansion of \( z \) over the parameter \( \epsilon \) (see, e.g., Ref. [21]),

\[
\epsilon = -\frac{\theta n \Lambda^3}{4\sqrt{2}} g = -\theta \frac{\hbar^3 \pi^{3/2} n}{2 g (mT)^{3/2}}. \tag{10}
\]

This expansion is inserted then to Eq. (11). At small \( |\epsilon| < 1 \) the expansion of the pressure over the powers of \( \epsilon \) is rapidly convergent, and a few first terms give already a good approximation of the quantum statistics effects. Taking the three terms, \( k = 1, 2, \) and 3, in Eqs. (14) and (9), one obtains a classical gas result (7) plus the corrections due to the effects of quantum statistics:

\[
P(T,n) = nT \left[ 1 + \epsilon - c_2 \epsilon^2 + O(\epsilon^3) \right], \tag{11}
\]

where \( c_2 = 4[16/(9\sqrt{3}) - 1] \approx 0.106 \). For brevity, we call the linear and quadratic \( \epsilon \)-terms in Eq. (11) as the first and second (order) quantum corrections.

Equation (11) demonstrates explicitly a deviation of the quantum ideal gas pressure from the classical ideal gas value (7): the Fermi statistics leads to an increasing of the classical pressure, while the Bose statistics to its
decreasing. This is often interpreted \cite{20} as the effective Fermi ‘repulsion’ and Bose ‘attraction’ between quantum particles.

III. QUANTUM VAN DER WAALS MODEL

Recently, the van der Waals (vdW) equation of state was extended by taking into account the effects of quantum statistics in Ref. \cite{18}. The pressure function of the quantum vdW (QvdW) model can be presented as \cite{18}

\[ P(T, n) = \frac{nT}{1 - nb} - an^2, \tag{12} \]

\[ n_{id}(T, \mu^*) = \frac{n}{1 - bn}, \tag{13} \]

where \( P_{id} \) and \( n_{id} \) are respectively given by Eqs. (2) and (3). The constants \( a > 0 \) and \( b > 0 \) are responsible for respectively attractive and repulsive interactions between particles. The QvdW model given by Eqs. (12) and (13) satisfies the following conditions:

1. In the Boltzmann approximation, i.e., at \( \theta = 0 \) in Eqs. (2) and (3), the QvdW model \cite{12} and \cite{13} is reduced to the classical vdW model \cite{20}.

\[ P = \frac{nT}{1 - nb} - an^2. \tag{14} \]

Note that the classical vdW model \cite{14} is further reduced to the ideal classical gas \cite{17} at \( a = 0 \) and \( b = 0 \).

2. At \( a = 0 \) and \( b = 0 \) the QvdW model \cite{12} and \cite{13} is transformed to the quantum ideal gas Eqs. (2) and (3).

3. The QvdW model \cite{12} and \cite{13}, in contrast to its classical version \cite{14}, satisfies the third law of thermodynamics by having a non-negative entropy with \( s = 0 \) at \( T = 0 \).

We fix the model parameters \( a \) and \( b \) using the ground state properties of the symmetric nuclear matter (see, e.g., Ref. \cite{25}): at \( T = 0 \) and \( n = n_0 = 0.16 \text{ fm}^{-3} \), one requires \( P = 0 \) and the binding energy per nucleon \( \epsilon(T = 0, n = n_0)/n_0 = -16 \text{ MeV} \). With the step-like Fermi momentum distribution for nucleons the analytical expressions for the thermodynamical quantities at \( T = 0 \) are obtained. One then finds from the above requirements

\[ a = 392.8 \text{ MeV} \cdot \text{fm}^3, \quad b = 3.35 \text{ fm}^3. \tag{15} \]

These values are very close to those found in Ref. \cite{13}. Small differences appear because of the non-relativistic formulation used in the present studies.

In what follows, the first and second quantum correction of the QvdW model will be considered. This is analogous to that in Eq. (11) for the ideal quantum gas. Expanding \( P_{id}[T, n_{id}(T, \mu^*)] \) in Eq. (12) over the small parameter \( \delta \) (with \( \theta = -1 \) for fermions and \( \theta = 1 \) for bosons),

\[ \delta = \frac{\epsilon}{1 - bn} = -\theta \frac{\bar{b}^3 \pi^{3/2} n}{2 g (1 - bn)(mT)^{3/2}}, \tag{16} \]

one obtains

\[ P(T, n) = \frac{nT}{1 - bn} \left[ 1 + \delta - c_2 \delta^2 + O(\delta^3) \right] - an^2, \tag{17} \]

where \( c_2 \) is the same small number coefficient as in Eq. (11). Similarly to the ideal gases, the quantum correction in Eq. (17) increases with the particle number density \( n \) and decreases with the system temperature \( T \), particle mass \( m \), and degeneracy factor \( g \). A new feature of the quantum effects in the system of particle with the vdW interactions is the additional factor \( (1 - bn)^{-1} \) in the quantum correction \( \delta \), i.e., the quantum statistics becomes stronger due to the repulsive interactions between particles.

The vdW model, both in its classical form \cite{14} and in its QvdW extension \cite{12} and \cite{13}, describes the first order liquid-gas phase transition. The critical point (CP) of this transition satisfies the following equations:

\[ \left( \frac{\partial P}{\partial n} \right)_T = 0, \quad \left( \frac{\partial^2 P}{\partial n^2} \right)_T = 0. \tag{18} \]

Using Eq. (17) in the first approximation in \( \delta \), one derives from Eq. (18) the system of two equations for the CP parameters \( n_c \) and \( T_c \) at the same first order:

\[ \frac{T}{2an (1 - nb)^2} \left[ 1 + 2\delta \right] = 1, \tag{19} \]

\[ \frac{bT}{a (1 - nb)^3} \left[ 1 + \left( 1 + 2nb \right) \frac{\delta}{bn} \right] = 1. \tag{20} \]

Solving the system \cite{19} and \cite{20}, one finds in the same first order approximation over \( \delta \):

\[ T_c^{(1)} \approx T_c^{(0)} (1 - 2\delta_0) \approx 19.0 \text{ MeV}, \]

\[ n_c^{(1)} \approx n_c^{(0)} (1 - 2\delta_0) \approx 0.065 \text{ fm}^{-3}. \tag{21} \]

In Eq. (21),

\[ T_c^{(0)} = \frac{8a}{27b} \approx 29.2 \text{ MeV}, \quad n_c^{(0)} = \frac{1}{3b} \approx 0.100 \text{ fm}^{-3}, \]

\[ P_c^{(0)} = \frac{a}{27b^2} \approx 1.09 \text{ MeV} \cdot \text{fm}^{-3} \tag{22} \]

are the CP parameters of the classical vdW model, i.e., they are found from Eq. (18) for the equation (14). The parameter \( \delta_0 \) in Eq. (21) is given by Eq. (19) calculated at the CP \cite{22}, i.e., at \( n = n_c^{(0)} \) and \( T = T_c^{(0)} \). Substituting Eq. (21) for the critical temperature \( T_c^{(1)} \) and density \( n_c^{(1)} \) into Eq. (17) at the same first order, for the CP pressure \( P_c^{(1)} \) one obtains

\[ P_c^{(1)} \approx 0.48 \text{ MeV} \cdot \text{fm}^{-3}. \tag{23} \]

The numerical calculations within the full QvdW model \cite{12} and \cite{13} give

\[ T_c \approx 19.7 \text{ MeV}, \quad n_c \approx 0.072 \text{ fm}^{-3}, \]

\[ P_c \approx 0.52 \text{ MeV} \cdot \text{fm}^{-3}. \tag{24} \]

These our results \cite{23} appear to be essentially the same as those obtained in Ref. \cite{18}.\footnote{The multi-component QvdW model with different \( a \) and \( b \) parameters for protons and neutrons was discussed in Refs. \cite{26,27}.}
Table I. Results for the CP parameters of the symmetric nuclear matter \((g = 4)\).

| Critical points | Eq. (22) | First order correction | QvdW |
|----------------|---------|------------------------|------|
| \(T_c \) [MeV] | 29.2 | 19.0 | 19.7 |
| \(n_c \) [fm\(^{-3}\)] | 0.100 | 0.065 | 0.072 |
| \(P_c \) [MeV · fm\(^{-3}\)] | 1.09 | 0.48 | 0.52 |

Table II. Results for the CP parameters for the neutron matter \((g = 2)\).

| Critical points | Eq. (22) | First order correction | QvdW |
|----------------|---------|------------------------|------|
| \(T_c \) [MeV] | 29.2 | 8.7 | 10.8 |
| \(n_c \) [fm\(^{-3}\)] | 0.100 | 0.030 | 0.051 |
| \(P_c \) [MeV · fm\(^{-3}\)] | 1.09 | 0.13 | 0.20 |

A summary of the results for the CP parameters is presented in Table I. A difference of the results for the classical vdW model \((22)\) and QvdW model \((24)\) demonstrates a role of the effects of Fermi statistics at the CP of the symmetric nuclear matter. The size of these effects appears to be rather significant. On the other hand, it is remarkable that the first order correction \((21)\) and \((23)\) reproduce these effects of quantum statistics with a high accuracy. The second order correction in the expansion \((17)\) leads to an improvement of the CP parameters, \(T_c^{(2)} = 19.7\) MeV, \(n_c^{(2)} = 0.072\) fm\(^{-3}\) and \(P_c^{(2)} = 0.53\) MeV·fm\(^{-3}\), which become closer to the numerical results \((24)\) of the full QvdW model.

The CP parameters of the classical vdW model \((22)\) depend on the interaction parameters \(a\) and \(b\), and they are not sensitive to the values of particle mass \(m\) and degeneracy factor \(g\). The effects of quantum statistics change this conclusion. The parameter \(\delta\) given by Eq. \((16)\) is proportional to \(m^{-3/2} \cdot g^{-1}\), i.e., the effects of quantum statistics become smaller for heavier particles (e.g., for the light nuclei admixture in the nuclear matter) or for large values of degeneracy factor \(g\). Particularly, for \(g \gg 1\) the quantum statistics effects become negligible, and the QvdW model \((12,13)\) is reduced at \(T > 0\) to the classical vdW model \((14)\). These results are in agreement with the numerical calculations in the recent paper \((27)\).

The effects of quantum statistics become larger in the neutron matter within the QvdW model with the same \(a\) and \(b\) parameters, but \(g = 2\) instead of \(g = 4\) for the symmetric nuclear matter. This leads to essentially larger effects of the quantum statistics and stronger changes of the CP parameters. These results are summarized in Table II.

\(^3\) The properties of the pure neutron matter appear to be very sensitive to the values of the vdW parameters \(a\) and \(b\) for neutrons. This is discussed in Ref. \((23)\).

Figure 2. Lines of the constant values of the parameter \(\delta\) \([Eq. (16)\]) in the \(T-n\) plane for the symmetric nuclear matter (a), neutron matter (b), and \(|\alpha|\) for \(\alpha\)-matter (c). The boundaries of the liquid-gas mixed phase regions are shown in (a) and (b) by solid blue lines. The red close dots, blue squares, and asterisks show the CP parameters in the respectively QvdW model within the first order correction, the full QvdW model, and classical vdW model.

Table II

In Fig. 2 the expansion parameter \(\delta\) entered to Eq. \((17)\) is shown in the \(n-T\) plane in the QvdW model for the symmetric nuclear matter (a) and pure neutron matter (b), and the parameter \(-\delta\) for the pure \(\alpha\) matter (c). For the symmetric nuclear matter in Fig. 2(a) the values of \(\delta\) in a vicinity of the CP is rather small, \(\delta \approx 0.2\). This explains a good accuracy of the first order corrections of the quantum statistics effects.

For the neutron matter shown in Fig. 2(b) the parameter of the quantum correction is \(\delta \approx 0.5\) near the CP. The correction due to the quantum statistics becomes indeed larger than that for the symmetric nuclear matter.
This also explains why an agreement of the first-order quantum results for the CP parameters with their exact QvdW numerical values is worse than in the case of symmetric nuclear matter. The second order correction in Eq. (17) leads to the critical values $T_c^{(2)} = 10.6$ MeV, $n_c^{(2)} = 0.052$ fm$^{-3}$, and $P_c^{(2)} = 0.20$ MeV fm$^{-3}$. This improves significantly an agreement with results of the full QvdW model calculations presented in Table II.

The calculations for interacting $\alpha$ particles within the QvdW model are shown in Fig. 2 (c). For illustrative purposes we take the vdW parameters $a$ and $b$ of the pure $\alpha$-matter to reproduce the estimates $T_c = 11.2$ MeV and $n_c = 0.013$ fm$^{-3}$ from Ref. [28] within the classical vdW model. One finds a rather small value $|\delta| \approx 0.05$ for the $\alpha$ matter in a vicinity of the CP, $\delta < 0$ [see Fig. 2 (c)]. Note that the Bose and Fermi statistics lead to the same absolute values of the first order quantum correction (17), but with the different signs. Thus, the quantum Bose effects change the CP parameters into the opposite direction, i.e., $T_c$ and $n_c$ are moved to larger values due to the Bose effects in comparison with their classical vdW values. As seen from Fig. 2 (c), there is about a 10% increase of the $T_c^{(0)}$ and $n_c^{(0)}$ values due to the Bose statistics correction.

At small $T$ and/or large $n$, the parameter $\delta \sim n(1 - bm)^{-1}$. $T^{-3/2}$ becomes large. In this region of the phase diagram, the QvdW model should be treated within the full quantum statistical formulation (12) and (18). In the $n$-$T$ region with $\delta > 1$ shown in Fig. 2 (a) and (b) by the dashed lines the first order quantum approximation looses their meaning.

The mixed gas-liquid phases of the symmetric nuclear matter and pure neutron matter correspond to the $n$-$T$ regions under the blue solid lines presented in Fig. 2 (a) and (b), respectively. The physical states inside the mixed phase correspond to the equilibrium of the gas and liquid components with equal values of $T$, $\mu$, and $P$. However, these components have different particle number densities, $n_{\text{gas}} < n_{\text{liq}}$. They become very different, $n_{\text{gas}} \ll n_{\text{liq}}$, at small temperatures. In this case, the effects of quantum statistics are small for the gas component ($\delta < 1$) but are rather large ($\delta > 1$) for the liquid one. For the Bose $\alpha$ particle system large quantum effects mean a possibility of the Bose-Einstein condensation. Thus, both effects – the first-order phase transition and a formation of the Bose condensate – should be treated simultaneously (see, e.g., Ref. [28]). These questions are however outside of the scope of the present paper.

Rough estimates give $|\delta_0| \ll 1$ in most cases of the CP for different atomic gases [29] and, thus, small effects of quantum statistics. This happens, despite of much smaller values of the atomic critical temperature $T_A \ll T_c$ in comparison with the nuclear matter $T_c$ values. The small effects of quantum statistics in the atomic systems in their gas and liquid states is a consequence of very small atomic densities as compared to the nuclear ones, $n_A \ll n_c$. One exception from these arguments is the atomic system of He-4. The experimental value for its CP parameters [29, 30], $T_c^{(0)}(\text{He}) = 9.2$ MeV and $n_c^{(0)}(\text{He}) = 0.013$ fm$^{-3}$ lead to the estimate $\delta_0^{(0)}(\text{He}) \approx -0.1$. Therefore, our consideration within the first order quantum correction over $\delta$ looks rather suitable to the analysis of the quantum statistics effects near the CP of this atomic system.

IV. SUMMARY

The QvdW equation of state has been used to study the quantum statistics effects in a vicinity of the critical point of nuclear matter. To obtain the analytical expressions, the first order quantum statistics correction over the small parameter $\delta$ is considered. An explicit dependence on the system parameters is demonstrated. Particularly, the CP position depends on the particle mass $m$ and degeneracy factor $g$. Such a dependence is absent within the classical vdW model. The quantum corrections to the CP parameters of the symmetric nuclear matter appear to be quite significant. For example, the value of $T_c^{(0)} = 29.2$ MeV in the classical vdW model decreases to the value $T_c^{(1)} = 19.0$ MeV. On the other hand, this approximate analytical result within the first order quantum correction is already close to the numerical value of $T_c = 19.7$ MeV obtained by the numerical calculations within the full QvdW model. The quantum correction of the CP parameters becomes even larger for the pure neutron matter. In this case, the classical value of $T_c^{(0)} = 29.2$ MeV decreases to $T_c^{(1)} = 8.7$ MeV which is still close to the numerical value of the full QvdW model value $T_c = 10.8$ MeV, and the second order corrections improves this agreement.

Our consideration is straightly extended to the system of interacting bosons. The quantum corrections have different signs for fermions and bosons. An example of the pure $\alpha$ matter has been considered. The CP temperature $T_c^{(0)} = 11.2$ MeV for the the classical vdW model of $\alpha$ particles increases by about of 10% to $T_c^{(1)} = 12.3$ MeV for $\alpha$-matter within the first order approximation in the QvdW model.

ACKNOWLEDGMENTS

We thank D.V. Anchishkin, B.I. Lev, A. Motornenko, R.V Poberezhnyuk, A.I. Sanzhur, and V. Vovchenko, for many fruitful discussions. The work of S.N.F. and R.V Poberezhnyuk, A.I. Sanzhur, and V. Vovchenko, was supported by the Program “Fundamental researches in high energy physics and nuclear physics (international collaboration)” at the Department of Nuclear Physics and Energy of the National Academy of Sciences of Ukraine. The work of M.I.G. was supported by the Program of Fundamental Research of the Department of Physics and Astronomy of the National Academy of Sciences of Ukraine.
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