Competition between the superconductivity and nematic order in the high-\(T_c\) superconductor

Jing Wang\(^1\) and Guo-Zhu Liu\(^{1,2,3}\)

\(^1\) Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, People’s Republic of China
\(^2\) Max Planck Institut für Physik komplexer Systeme, D-01187 Dresden, Germany
E-mail: gzliu@ustc.edu.cn

New Journal of Physics 15 (2013) 073039 (20pp)
Received 17 April 2013
Published 23 July 2013
Online at http://www.njp.org/
doi:10.1088/1367-2630/15/7/073039

Abstract. We investigate the competition between the d-wave superconductivity and the nematic order in the high-\(T_c\) superconductor and examine the role played by the gapless fermionic degrees of freedom. Apart from the competitive interaction with the superconducting order parameter, the nematic order parameter couples strongly to gapless nodal quasiparticles. The interplay of these two kinds of interactions is analyzed by means of the renormalization group method. In case the fermionic degrees of freedom are entirely neglected, the competitive interaction between the two bosonic order parameters is strongly relevant and can lead to runaway behavior. However, these properties are fundamentally changed once the dynamics of the fermions are taken into account. At the nematic quantum critical point where an extreme fermion velocity anisotropy occurs, the superconducting and nematic order parameters are decoupled from each other. Consequently, the phase transitions are continuous and the d-wave superconductivity can coexist with the nematic order homogeneously. These results indicate that the gapless fermions can play an important role and should be carefully included in the theoretical description of competing orders.

\(^3\) Author to whom any correspondence should be addressed.

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1. Introduction

Unconventional superconductors usually refer to those superconductors that cannot be understood within the conventional Bardeen–Cooper–Schrieffer (BCS) theory. Notable examples of unconventional superconductors are the high-$T_c$ cuprate superconductor, the heavy fermion superconductor and the iron-based superconductor. Unlike BCS superconductors, unconventional superconductivity is generally driven by electron–electron interactions and often has a magnetic origin. Another interesting property of the unconventional superconductor is that its ground state is not unique. In addition to the defining superconducting state, unconventional superconductors also exhibit a variety of other symmetry-broken ground states, including antiferromagnetic, nematic and stripe states, upon tuning such parameters as doping and pressure [1–5]. A widely recognized notion is that the long-range superconducting order competes and, under certain circumstances, coexists with other long-range orders. The competition and possible coexistence between different orders can give rise to rich properties and hence have attracted intense theoretical and experimental interest in recent years.

The successful microscopic theory of competing orders has not yet been established to date, primarily because the pairing mechanism in most unconventional superconductors is still undetermined. A realistic and commonly used strategy is to build a low-energy effective field theory on phenomenological grounds. One can first write down the Ginzburg–Landau (GL) actions for two bosonic order parameters and then introduce certain coupling terms between these two scalar fields. Such a generalized GL model has recently been applied to describe the competing orders in a number of unconventional superconductors [6–15]. An early success of such a theoretical investigation is the prediction of a field-induced antiferromagnetic core in the superconducting vortices of high-$T_c$ superconductors [6]. This prediction was subsequently confirmed in experiments [16, 17]. Interestingly, experiments further found that the antiferromagnetic order not only exists in the vortex cores, but also extends into the superconducting region [16] and exhibits non-trivial spatial modulation [17, 18]. A phenomenological field theory that contains a simple quadratic–quadratic coupling term between the superconducting and antiferromagnetic order parameters was put forward to understand these new findings [7, 8].
Recently, the issue of competing orders has attracted revived interest. It is found that the competitive interaction between distinct orders can drive an instability, which gives rise to a general tendency of a first order transition \[11, 14\]. This phenomenon may account for the first order transition observed in some unconventional superconductors \[4\]. In addition, non-uniform glassy electronic phases and Brazovskii type transitions are predicted to emerge due to the competition between two long-range orders \[10\]. Another interesting observation is that the competition between superconducting and antiferromagnetic orders can help to judge the gap symmetry of iron-based superconductors \[9, 12\]. Furthermore, the competition between superconducting and nematic orders might be responsible for \[15\] the electronic anisotropy observed in the vortex state of the FeSe superconductor \[19\].

The effective field theory adopted in previous analyses of the competing orders normally contains only two bosonic order parameters. The fermionic degrees of freedom are usually completely integrated out in the spirit of the Hertz–Millis–Moriya (HMM) theory \[20–22\]. This integration procedure is expected to be applicable in systems that do not contain gapless fermionic excitations. For instance, the iron-based superconductors seem to have an s-wave energy gap, so the electronic excitations are fully gapped and can be safely integrated out \[9, 12\]. However, such integration manipulation is not always valid. Indeed, its validity has recently been questioned in several itinerant electron systems \[23–26\]. In the systems that exhibit gapless fermionic excitations, integrating out fermions may lead to singularities, especially in the vicinity of the quantum critical point (QCP). Actually, infrared singularities have been found on the border of several quantum phase transitions \[23–26\]. In order to properly describe the quantum critical behavior in these systems, it is more appropriate to maintain both bosonic order parameter and gapless fermions in the effective theory. When a long-range order competing with superconductivity also couples to gapless fermions, it would be interesting to go beyond the HMM theory and examine the role of gapless fermions. The recent analysis presented in \[13, 27\] did suggest non-trivial roles played by gapless fermions.

In various types of superconductors, the superconductivity may compete with several possible long-range orders. To examine the role of fermions, we wish to study a prototypical model which describes the competition between two distinct long-range orders, contains gapless fermions and in the meantime is technically controllable. In this paper, we choose to consider the competition between the superconductivity and the nematic order in the context of high-$T_c$ superconductors. In recent years, there has been increasing experimental evidence pointing toward the existence of an electronic nematic phase in some high-$T_c$ superconductors \[1–3, 28–31\], especially YBa$_2$Cu$_3$O$_{6+\delta}$ and Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$. According to these experiments, a nematic order is predicted to compete and coexist with the superconductivity, which is schematically plotted in figure 1. The nematic transition and the coupling of the nematic order to gapless fermions have stimulated intense research efforts \[1–3, 27, 32–42\]. From a field-theoretical viewpoint, the nematic order parameter is a real scalar field and does not carry a finite wave vector, which substantially simplifies theoretical calculations.

It is known that the high-$T_c$ superconductor has a $d_{x^2−y^2}$ energy gap, which vanishes at four nodes, $(\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$. Therefore, gapless nodal quasiparticles (qps) are present even at the lowest energy in the superconducting phase. These nodal qps are believed to be responsible for many of the anomalous low-temperature properties of the superconducting dome. When a nematic QCP exists somewhere in the superconducting dome, as shown in figure 1, the fluctuation of the nematic order parameter couples to gapless nodal qps. This coupling can generate non-Fermi
Figure 1. Schematic phase diagram on \((x, T)\)-plane of high-\(T_c\) superconductors. \(x\) represents doping concentration. \(x_1\) and \(x_2\) are the QCPs of the superconducting and nematic phase transitions, respectively.

liquid behaviors and other anomalous phenomena in the vicinity of the nematic QCP \([27, 36, 37, 39–42]\). In particular, the ratio \(\kappa\) between the gap velocity \(v_\Delta\) and the Fermi velocity \(v_F\) of nodal qps is driven to vanish, i.e. \(\kappa = v_\Delta/v_F \to 0\), by the critical nematic fluctuation \([37]\), leading to extreme velocity anisotropy. These unusual behaviors may have significant effects on the interplay between the superconductivity and the nematic order, which is the topic of this paper.

In this paper, we first write down an effective field theory that describes both the competitive interaction between two bosonic (superconducting and nematic) order parameters and the Yukawa-type coupling between the nematic order parameter and the nodal qps. We then carry out a detailed renormalization group (RG) analysis \([43]\) within this effective theory. Specifically, we will derive and solve the RG flow equations of all physical parameters so as to determine the possible stable fixed points. We will demonstrate that gapless fermionic degrees of freedom can fundamentally change the basic properties of the interplay between the superconductivity and the nematic order. In case all the fermions are entirely neglected, the ordering competition may be strong enough to produce runaway behavior and turn the continuous phase transitions to first order \([14]\). However, once the dynamics of gapless nodal qps are properly incorporated, a stable fixed point exists with the two originally competing bosonic order parameters decoupled from each other in the vicinity of the nematic QCP. As a result, both the superconducting and nematic transitions remain continuous. In addition, the d-wave superconductivity can coexist with the nematic order homogeneously. Our results indicate that it is important to include the dynamics of gapless fermions in the theoretic description of competing orders.

In section 2, we write down the effective action which contains two bosonic order parameters and gapless nodal qps. In section 3, we make RG calculations and derive the flow equations for all parameters in the effective action. In section 4, we present numerical solutions of the flow equations and discuss the physical implications. The paper ends in section 5 with a summary and conclusion.
2. Effective field theory of competing orders

We first need to write down an effective field theory to describe the competition between superconducting and nematic orders. This will be done largely on phenomenological grounds. In the phase diagram presented in figure 1, the horizontal axis is doping concentration $x$. The QCP of the superconducting transition is $x_1$, which is roughly $x_1 \approx 0.05$ in many high-$T_c$ superconductors. The anticipated QCP for a nematic transition is represented by $x_2$. So far, the precise value and even the very existence of $x_2$ have not yet been unambiguously determined. Here, we assume that $x_2$ is larger than $x_1$, which implies a bulk coexistence of superconducting and nematic orders.

In this system, there are three types of degrees of freedom: superconducting order parameter $\psi$, nematic order parameter $\phi$ and gapless nodal qps $\Psi$. The competition between superconducting and nematic orders can be described by a repulsive quadratic–quadratic coupling term, $\propto \psi^2 \phi^2$, which is widely adopted in the description of competing orders [12–14, 27]. In addition to this competitive interaction, the nematic order parameter $\phi$ also interacts with gapless nodal qps $\Psi$, which is usually described by a Yukawa-type coupling term. There is, however, no direct coupling between the superconducting order parameter and the nodal qps. Firstly, the nodal qps are excited from the $d_{x^2-y^2}$ gap nodes where the superconducting order parameter vanishes. Secondly, these qps are known to have a sharp peak and a very long lifetime in the superconducting dome in the absence of competing orders [44], so their coupling to $\psi$ must be quite weak. Furthermore, in this paper we are mainly interested in the physical properties in the close vicinity of the nematic QCP $x_2$, where the fluctuation of the nematic order parameter is critical. Unless the superconducting QCP $x_1$ coincides with, or is very close to, $x_2$, the fluctuation of the superconducting order parameter is not critical at $x_2$. Therefore, the coupling between the superconducting order parameter and the nodal qps is not as important as that between the critical nematic fluctuation and the nodal qps at $x_2$ and can simply be neglected.

On the basis of the above qualitative analysis, we can write down the following partition function:

$$ Z = \int D\psi D\phi D\bar{\Psi} D\Psi e^{\mathcal{S}}, \quad (1) $$

where the effective action is

$$ S = S_{\psi} + S_{\phi} + S_{\Psi} + S_{\psi\phi} + S_{\phi\phi}, \quad (2) $$

$$ S_{\psi} = \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \left(-2\alpha + q^2\right)\psi^2 + \frac{\beta}{2} \int d^2 r d\tau \psi^4, \quad (3) $$

$$ S_{\phi} = \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \left(-2r + q^2\right)\phi^2 + \frac{u}{2} \int d^2 r d\tau \phi^4, \quad (4) $$

$$ S_{\Psi} = \int \frac{d^3 k}{(2\pi)^3} \left[\Psi_{1i}^\dagger (-i\omega + v_{1k_x} \tau^z + v_{\Delta k_x} \tau^x) \Psi_{1i} + \Psi_{2i}^\dagger (-i\omega + v_{1k_y} \tau^z + v_{\Delta k_y} \tau^y) \Psi_{2i}\right], \quad (5) $$

$$ S_{\psi\phi} = \gamma \int d^2 x d\tau \psi^2 \phi^2, \quad (6) $$

$$ S_{\phi\phi} = \int d^2 x d\tau \lambda \left(\Psi_{1i}^\dagger \tau^z \Psi_{1i} + \Psi_{2i}^\dagger \tau^x \Psi_{2i}\right), \quad (7) $$

New Journal of Physics 15 (2013) 073039 (http://www.njp.org/)
Figure 2. (a) Free propagator of superconducting field $\psi$; (b) free propagator of nematic field $\phi$; and (c) free propagator of nodal QPs $\Psi$. 

![Figure 2](image)

Figure 3. (a) Polarization function for nematic field $\phi$; (b) Fermion self-energy correction due to nematic fluctuation.

![Figure 3](image)

where $\tau^{x,y,z}$ are Pauli matrices and the flavor index $i$ sums the integers from 1 to $N$. $\Psi_i^\dagger$ represents the nodal QPs excited from $(\frac{\pi}{2}, \frac{\pi}{2})$ and $(-\frac{\pi}{2}, -\frac{\pi}{2})$ points, while $\Psi_2^\dagger$ represents the other two. The physical flavor of the nodal qps, $N = 2$. Here, $r$ is the tuning parameter for a nematic transition with $r = 0$ at $x_2$. $v_{F,\Delta}$ are the Fermi velocity and gap velocity of the nodal qps, respectively. The competitive interaction term $\psi^2\phi^2$ has a positive coefficient, $\gamma > 0$. $S_{\psi\phi}$ represents the Yukawa coupling between the nematic order parameter and gapless nodal qps, with $\lambda$ being its coupling constant. The free propagators of all the fields are shown in figure 2.

In order to simplify calculations, it proves convenient to make two transformations [37]: $\phi \rightarrow \phi/\lambda$, and $r \rightarrow \lambda^2 r$. It is now easy to rewrite equation (7) as

$$S_{\psi\phi} = \int d^2x \, d\tau \phi (\Psi_i^\dagger \tau^x \Psi_i + \Psi_2^\dagger \tau^y \Psi_2).$$

(8)

The effective action represented by equation (2) was studied recently in [27]. It was demonstrated that both the superfluid density and the critical temperature $T_c$ are significantly suppressed at the nematic QCP $x_2$. However, the superconducting order parameter $\psi$ was assumed in [27] to be classical, which is valid only when $x_2$ is not close to $x_1$. In this paper, we go beyond such an approximation and consider the quantum fluctuations of both the superconducting and nematic order parameters.

As emphasized in [27], the gapless nodal qps can have an important impact on the competition between the superconductivity and the nematic order. The simplest way to include the fermionic degrees of freedom is to introduce the polarization function $\Pi(q)$, due to the nodal qps, into the effective action of the nematic order, $S_{\phi}$. To the leading order of $1/N$-expansion, the polarization function $\Pi(q)$ is represented by the one-loop Feynman diagram shown in figure 3(a) and formally given by [37, 42]

$$\Pi(q, \epsilon) = N \int \frac{d^3k d\omega}{(2\pi)^3} \text{Tr}[\tau^x G_0(k, \omega) \tau^y G_0(k+q, \omega+\epsilon)],$$

where

$$G_0(k, \omega) = \frac{1}{-\omega + v_F k_x \tau^x + v_{\Delta} k_y \tau^y}$$
is the free propagator for the nodal qps \( \Psi_1 \) (the free propagator for the nodal qps \( \Psi_2 \) can be similarly written down). The polarization function \( \Pi(\epsilon, \mathbf{q}) \) has already been calculated previously [27, 37] and is known to have the form

\[
\Pi(\mathbf{q}, \epsilon) = \frac{N}{16v_Fv_\Delta} \left[ \frac{\epsilon^2+v_F^2q_x^2}{\sqrt{\epsilon^2+v_F^2q_x^2+v_\Delta^2q_y^2}} + (q_x \leftrightarrow q_y) \right].
\] (9)

Including this term, the quadratic part of \( S_\phi \) becomes

\[
(-2r + q^2) \phi^2 \rightarrow [-2r + q^2 + \Pi(q)] \phi^2.
\] (10)

From the expression of polarization \( \Pi(q) \), it is easy to see that the inclusion of \( \Pi(q) \) does not change the dynamical exponent \( z = 1 \) of \( \phi \). However, \( \Pi(q) \propto q \), so it dominates over the kinetic term \( q^2 \) in the low energy regime. More importantly, the polarization \( \Pi(q) \) introduces two important quantities, the nodal qps’ Fermi velocity \( v_F \) and the gap velocity \( v_\Delta \), into the effective action of \( \phi \).

Although the polarization \( \Pi(q) \) represents the influence of the nodal qps, we cannot completely integrate out the nodal qps and drop them from the effective theory. These gapless nodal qps should be maintained for several reasons. Firstly, according to the general spirit of RG, one can safely integrate out high-energy modes at low energies. However, the nodal qps are gapless and hence exist even at the lowest energy. Integrating out the gapless fermions completely may lead to unphysical singularities [24–26]. Secondly, the coupling between the gapless fermions and the order parameter fluctuation often causes non-Fermi liquid behavior in observable quantities. In the case of nematic transition, the critical nematic fluctuation gives rise to fermion velocity renormalization and extreme anisotropy, which would be overlooked if the fermions are fully integrated out. As will be shown below, the velocity ratio \( \kappa = v_\Delta/v_F \) plays a non-trivial role.

The effective field theory contains seven parameters: \( \alpha, r, \beta, u, \gamma, v_F, v_\Delta \). They are all subjected to renormalizations due to the mutual interactions among three field operators: \( \psi, \phi \) and \( \Psi \). We will study the flow of these seven parameters under scaling transformations and eventually obtain seven RG equations. Since we study the \( \psi-\phi \) interaction and the \( \Psi-\phi \) interaction on equal footings, these seven RG equations are self-consistently coupled to each other. The low-energy behaviors of these parameters and the possible fixed points can be determined by solving these coupled RG equations.

### 3. Renormalization group calculations

In this section, we make a RG analysis and obtain the flow equations of all the aforementioned parameters. In order to examine the impacts of the gapless nodal qps, we go beyond the HMM theory and maintain the nodal qps throughout our calculations. We first analyze the coupling between the nematic order and the nodal qps and derive the RG equations for the fermion velocities, \( v_{F,\Delta} \). We then consider the competitive interaction between the superconducting and nematic order parameters and obtain the RG equations for the rest of the five parameters, \( (\alpha, r, \beta, u, \gamma) \), which depend on the fermion velocities, \( v_{F,\Delta} \). These seven equations are self-consistently coupled to each other since the fermion velocities \( v_{F,\Delta} \) appearing in the equations of \( (\alpha, r, \beta, u, \gamma) \) flow according to their own equations.
Our RG analysis will be performed within the framework presented in [43]. According to the RG theory [43], we first employ the following scaling transformations:

\[ k_i = k_i' e^{-l}, \]
\[ \omega = \omega' e^{-l}, \]
\[ q_i = q_i' e^{-l}, \]
\[ \epsilon = \epsilon' e^{-l}, \]

where \( i = x, y \) and \( l \) represent the scaling parameter. Next we need to determine how all of the field operators flow under these transformations. It is easy to know that the superconducting order parameter \( \psi \) should be re-scaled as

\[ \psi(k, \omega) = \psi'(k', \omega') e^{5l/2} \]

on the basis of its action equation (3). Similarly, the rescaling behavior of the nodal qps \( \Psi_{1,2} \) is found to be [37]

\[ \Psi_{1,2}(k, \omega) = \Psi_{1,2}'(k', \omega') e^{(4+C_1)l/2}, \]

where the explicit expression of constant \( C_1 \) will be defined below in equation (20).

However, determining the rescaling behavior of the nematic field \( \phi \) is much more complicated. In the standard RG theory [43], one should regard the kinetic term of \( \phi \) as the fixed point and obtain the rescaling behavior of \( \phi \) by requiring such a term invariant under the scaling transformations. Nevertheless, the kinetic term of nematic order \( S_\phi \) is irrelevant in the low-energy region; therefore, it cannot be regarded as the fixed point. One has to choose another term to serve as the fixed point.

As already demonstrated in section 2, the polarization function \( \Pi(q) \) generated by the strong interaction with gapless nodal qps dominates over the kinetic term at low energies. Naturally, one might expect to choose the polarization term, \( \Pi(q) \phi^2(q) \), to serve as the fixed point. However, we emphasize here that it is not appropriate to obtain the scaling of \( \phi \) by requiring \( \Pi(q) \) invariant under RG transformations. Equation (9) tells us that \( \Pi(q) \) contains two fermion velocities, \( \nu_{F,\Delta} \), which are apparently scale-dependent and flow strongly as \( l \) varies. If we insist on requiring that the polarization term is invariant under the scaling transformations, we have to assume that \( \nu_{F,\Delta} \) are always constants and do not flow with a varying \( l \). Under this assumption, the important property of singular fermion velocity renormalization cannot be properly taken into account in our theoretical description of competing orders. For these reasons, we believe it is not appropriate to adopt the polarization term as the fixed point of this model.

Since both the kinetic and polarization terms are not good choices for the fixed point, we have to choose another term from the effective action. It appears that the Yukawa coupling term, \( \lambda \phi \Psi^\dagger \tau^x \Psi \), is the only available candidate. However, there is a very important problem: how to treat the coupling constant \( \lambda \)? In the conventional perturbation theory, usually one can make a perturbative expansion in powers of \( \lambda \). Unfortunately, this scheme does not apply to this model because explicit RG calculations [2] have revealed that \( \lambda \) tends to diverge at the lowest energy. It was later realized that a reasonable route to access such a model is to fix \( \lambda \) at a certain finite value [36–38] and perform a perturbative expansion in powers of \( 1/N \), where \( N \) is apparently the flavor. In this formalism, \( \lambda \) is a constant and does not flow with running \( l \). One can first
sorb $\lambda$ into $\phi$ [37] and then require that the Yukawa coupling term is invariant under the scaling transformations. It is now easy to know that the nematic field transforms as [37, 38]
\[
\phi(\mathbf{q}, \epsilon) = \phi'(\mathbf{q}', \epsilon') e^{(2c_3-C_1)\epsilon},
\]
where $C_3$ will be defined below in equation (20). This expression is to be used in the following calculations.

3.1. Flow equations of $v_F$ and $v_\Delta$

The Yukawa-type interaction between the nematic order and the gapless nodal qps has recently been investigated in several papers [36, 37, 39–42]. It is well known that the Fermi velocity of nodal qps is indeed not equal to the gap velocity, i.e. $v_F \neq v_\Delta$. Experiments [44, 45] have determined that the velocity ratio $\kappa = v_\Delta/v_F \approx 0.1$. This ratio is a very important parameter since it enters into a number of observable quantities of high-$T_c$ superconductors, including electric conductivity [46, 47], thermal conductivity [47], superfluid density [47, 48] and $T_c$ [48]. An interesting property revealed and discussed in [36, 37, 39–42] is that the velocity anisotropy is significantly enhanced by the nematic fluctuation.

The calculation of nodal qps’ self-energy function and the derivation of the flow equations have already been presented in previous publications [37, 42] and therefore are not shown here. It is only necessary to summarize the basic calculations as well as the relevant results. To the leading order, the fermion self-energy is represented by the diagram in figure 3(b) and has the form

\[
\Sigma(\mathbf{k}, \omega) = \int \frac{d^2q}{(2\pi)^3} G_0(\mathbf{k} + \mathbf{q}, \omega + \epsilon) \frac{1}{\Pi(q)}. \tag{18}
\]

As shown in [37], it can be written as

\[
\frac{d\Sigma(\mathbf{k}, \omega)}{d\ln \Lambda} = C_1(-i \omega) + C_2v_Fk_x \tau^z + C_3v_\Delta k_y \tau^x, \tag{19}
\]

where

\[
C_1 = \frac{2(v_\Delta/v_F)}{N \pi^3} \int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \frac{x^2 - \cos^2 \theta - (v_\Delta/v_F)^2 \sin^2 \theta}{[x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta]^2} G(x, \theta),
\]
\[
C_2 = \frac{2(v_\Delta/v_F)}{N \pi^3} \int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \frac{\cos^2 \theta - x^2 - (v_\Delta/v_F)^2 \sin^2 \theta}{[x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta]^2} G(x, \theta),
\]
\[
C_3 = \frac{2(v_\Delta/v_F)}{N \pi^3} \int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \frac{x^2 + \cos^2 \theta - (v_\Delta/v_F)^2 \sin^2 \theta}{[x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta]^2} G(x, \theta),
\]

\[
G^{-1} = \frac{x^2 + \cos^2 \theta}{\sqrt{x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta}} + \frac{x^2 + \sin^2 \theta}{\sqrt{x^2 + \sin^2 \theta + (v_\Delta/v_F)^2 \cos^2 \theta}}.
\]

Using the Dyson equation, it is easy to get a renormalized fermion propagator

\[
G^{-1}_\psi(\mathbf{k}, \omega) = -i \omega + v_Fk_x \tau^z + v_\Delta k_y \tau^x - \Sigma(\mathbf{k}, \omega), \tag{20}
\]
which leads to the following RG equations:

\[
\frac{d v_F}{dl} = (C_1 - C_2)v_F, \quad (21)
\]

\[
\frac{d v_{\Delta}}{dl} = (C_1 - C_3)v_{\Delta}, \quad (22)
\]

\[
\frac{d(v_{\Delta}/v_F)}{dl} = (C_2 - C_3)(v_{\Delta}/v_F), \quad (23)
\]

where \( l > 0 \) is the running scale. A straightforward analysis showed that the ratio \( v_{\Delta}/v_F \) flows to zero at the lowest energy, giving rise to a novel fixed point of extreme velocity anisotropy \([37]\). Such a fixed point in turn leads to a number of non-trivial consequences, such as an unusual broadening of the spectral function \([36]\), non-Fermi liquid behavior \([39]\), an enhancement of the dc thermal conductivity \([40]\) and a suppression of the superconductivity \([27]\). To analyze the influence of velocity renormalization and especially the extreme anisotropy manifested at nematic QCP on the nature of a superconducting transition, we require that the constant fermion velocities appearing in the polarization \( \Pi(q) \) flow with the running scale \( l \) according to equations (21) and (22).

The extreme velocity anisotropy is a special feature of the nematic QCP, where \( r = 0 \) and the nematic fluctuation is critical. Away from nematic QCP, \( r \neq 0 \), so the nematic fluctuation leads only to a relatively unimportant renormalization of the fermion velocities. For \( r \neq 0 \), \( v_{F,\Delta} \) and therefore their ratios \( v_{\Delta}/v_F \) remain finite.

### 3.2. Flow equations of \( \alpha, r, \beta, u \) and \( \gamma \)

We next consider the competitive interaction between the nematic and superconducting order parameters. Our analysis follows closely the scheme presented in a recent work of She et al \([14]\). It was assumed in \([14]\) that all the fermionic degrees of freedom can be integrated out and their effects can be represented by the dynamical exponent \( z \). Compared with \([14]\), the main difference here is the inclusion of polarization \( \Pi(q) \) in the effective action of nematic order \( \phi \), which is supposed to reflect the influence of nodal qps. The corresponding (sub) action that describes the ordering competition is

\[
S_{\text{com}} = S_{\psi} + S_{\phi} + S_{\psi\phi}, \quad (24)
\]

where

\[
S_{\psi} = \frac{1}{2} \int \frac{d^2k}{(2\pi)^3} d\omega \left( -2\alpha + k^2 + \omega^2 \right) \psi^2 + \frac{\beta}{2} \int \prod_{m=1}^{4} \frac{d^2 k_m}{(2\pi)^3} d\omega_m \delta^2 \left( \sum k_m \right) \delta \left( \sum \omega_m \right) \psi^4,
\]

\[
S_{\phi} = \frac{1}{2} \int \frac{d^2q}{(2\pi)^3} d\epsilon \left[ -2r + q^2 + \epsilon^2 + \Pi(q) \right] \phi^2 + \frac{u}{2} \int \prod_{m=1}^{4} \frac{d^2 q_m}{(2\pi)^3} d\epsilon_m \delta^2 \left( \sum q_m \right) \delta \left( \sum \epsilon_m \right) \phi^4,
\]

\[
S_{\psi\phi} = \gamma \int \prod_{i=1,2} \frac{d^2 k_i}{(2\pi)^6} d\omega_i d^2 q_i d\epsilon_i \psi(k_i, \omega_i) \phi(q_i, \epsilon_i) \delta^2 (k_1 + k_2 + q_1 + q_2) \delta (\omega_1 + \omega_2 + \epsilon_1 + \epsilon_2).
\]

Before performing a standard RG analysis within this action, it is convenient to rescale the momenta and energy by \( \Lambda \), i.e. \( k \rightarrow k/\Lambda, \omega \rightarrow \omega/\Lambda \).
Each field operator can be separated into a slow mode and a fast mode, i.e.

\[
\psi = \psi_s + \psi_f, \quad (25)
\]

\[
\phi = \phi_s + \phi_f. \quad (26)
\]

After introducing a UV cutoff \( \Lambda \), we can define the slow mode of the superconducting order parameter as \( \psi_s = \psi(k) \) with \( 0 < k < e^{-1} \Lambda \) and the fast mode as \( \psi_f = \psi(k) \) with \( e^{-1} \Lambda < k < \Lambda \), using the formalism of [43]. Based on such modes of separation, the effective action (24) is decomposed into three parts: \( S^s \) that contains only the slow modes, \( S^f \) that contains only the fast modes and \( S^{sf} \) that contains both the slow and fast modes. More concretely, we have

\[
S_{\text{com}} = S^s + S^f + S^{sf}
\]

where

\[
S^s = \frac{1}{2} \int \frac{d^2q}{(2\pi)^3} \left[ -2r + \mathbf{q}^2 + \epsilon^2 + \sum_{q \neq 0} \right] \psi_s^2 + \frac{\beta}{2} \int \frac{d^2k_m d\omega_m}{(2\pi)^3} \delta^2 \left( \sum k_m \right) \delta \left( \sum \omega_m \right) \psi_s^4,
\]

\[
S^f = \frac{1}{2} \int \frac{d^2q}{(2\pi)^3} \left[ -2r + \mathbf{q}^2 + \epsilon^2 + \sum_{q \neq 0} \right] \psi_f^2 + \frac{\beta}{2} \int \frac{d^2k_m d\omega_m}{(2\pi)^3} \delta^2 \left( \sum k_m \right) \delta \left( \sum \omega_m \right) \psi_f^4,
\]

\[
S^{sf} = \int \frac{d^2k}{(2\pi)^3} \left[ (3\beta) \psi_1 \psi_1 \psi_s \psi_s + (3\mu) \phi_1 \phi_1 \phi_1 \phi_1 \right] + \frac{\mu}{2} \int \frac{d^2q_m d\epsilon_m}{(2\pi)^3} \delta^2 \left( \sum q_m \right) \delta \left( \sum \epsilon_m \right) \phi_1^4,
\]

\[
\times \delta \left( \omega_1 + \omega_2 + \epsilon_1 + \epsilon_2 \right) \left( \gamma \psi_1 \phi_1 \phi_1 + \gamma \psi_1 \psi_1 \phi_1 + 4 \gamma \psi_1 \psi_1 \phi_1 \phi_1 \right).
\]

After this decomposition, the partition function can be rearranged in the following way:

\[
Z = \int \mathcal{D} \psi_s \mathcal{D} \phi_s \mathcal{D} \Psi \mathcal{D} \Psi \int \mathcal{D} \psi_f \mathcal{D} \phi_f \times \exp \left[ \left( S^s + S^f_s + S_{\psi}^s \right) + \left( S^f + S_{\psi f}^f \right) + S^{sf} \right]
\]

\[
= \int \mathcal{D} \psi_s \mathcal{D} \phi_s \mathcal{D} \Psi \mathcal{D} \Psi \exp \left( S_{\psi}^s + S_{\psi}^s + S_{\psi}^s \right) \int \mathcal{D} \psi_f \mathcal{D} \phi_f \exp \left[ \left( S^f + S_{\psi f}^f \right) + S^{sf} \right]
\]

\[
= \int \mathcal{D} \psi_s \mathcal{D} \phi_s \mathcal{D} \Psi \mathcal{D} \Psi \exp \left( S_{\psi}^s + S_{\psi}^s + S_{\psi}^s \right)
\]

\[
= \int \mathcal{D} \psi_s \mathcal{D} \phi_s \mathcal{D} \Psi \mathcal{D} \Psi \exp \left( S_{\psi}^s \right). \quad (28)
\]
The next step is to integrate over all the fast modes and obtain an effective action of the slow modes. The functional integration can be performed using the standard diagrammatic techniques. The propagators for the superconducting order \( \psi \) and the nematic order \( \phi \), shown in figure 2, are

\[
G_\psi(k, \omega) = \frac{1}{-2\alpha + k^2 + \omega^2}, \tag{29}
\]

\[
G_\phi(q, \epsilon) = \frac{1}{-2r + q^2 + \epsilon^2 + \Pi(q)}.
\tag{30}
\]

The polarization appearing in \( G_\phi(q, \epsilon) \) reflects the presence of gapless nodal qps. As already pointed out, \( \Pi(q) \) dominates over the kinetic term \( q^2 \) in the low-energy regime. In order to further simplify the nematic propagator, we consider the close vicinity of the nematic QCP where \( r \) is very small. In this case, we are allowed to approximate the nematic propagator by

\[
G_\phi(q, \epsilon) \approx \frac{1}{-2r + \Pi(q)} \quad \text{(at low energy)}
\]

\[
\approx \frac{1}{\Pi(q) \left[ 1 - \frac{2r}{\Pi(q)} \right]}.
\]

\[
\approx \frac{1}{\Pi(q) + \frac{2r}{\Pi^2(q)}.} \tag{31}
\]

Now let us work in the spherical coordinates. We first define \( \epsilon \equiv \nu q_0 = \nu q \cos \theta \), \( q_1 = q \sin \theta \cos \varphi \) and \( q_2 = q \sin \theta \sin \varphi \), with \( \nu \) being the velocity of the nematic order parameter \( \phi \). For convenience, we could extract constant \( \nu \) from the energy \( \omega \) of the superconducting order parameter \( \psi \). Accordingly, the velocities of nodal qps, \( v_F \) and \( v_\Delta \), can be divided by \( \nu \). After this manipulation, \( v_F \) and \( v_\Delta \) become dimensionless. Now the polarization function can be written in the form

\[
\Pi(q, \theta, \varphi) = q D(\theta, \varphi), \tag{32}
\]

where the function

\[
D(\theta, \varphi) = \frac{N}{16 v_F v_\Delta} \left( \frac{\cos^2 \theta + v_F^2 \sin^2 \theta \cos^2 \varphi}{\sqrt{\cos^2 \theta + v_F^2 \sin^2 \theta \cos^2 \varphi + v_\Delta^2 \sin^2 \theta \sin^2 \varphi}} + \frac{\cos^2 \theta + v_F^2 \sin^2 \theta \sin^2 \varphi}{\sqrt{\cos^2 \theta + v_F^2 \sin^2 \theta \sin^2 \varphi + v_\Delta^2 \sin^2 \theta \cos^2 \varphi}} \right)
\]

is dimensionless. Before proceeding with the next calculations, it is helpful to define

\[
F_1 \equiv F_1(v_F, v_\Delta) = \int_0^\pi d\theta \int_0^{2\pi} d\varphi \frac{\sin \theta}{(2\pi)^3 D(\theta, \varphi)}, \tag{33}
\]

\[
F_2 \equiv F_2(v_F, v_\Delta) = \int_0^\pi d\theta \int_0^{2\pi} d\varphi \frac{\sin \theta}{(2\pi)^3 D^2(\theta, \varphi)}. \tag{34}
\]
With these arrangements, we can now turn to calculate the one-loop contribution to the RG equations of all the parameters.

3.2.1. $\alpha, r$. The diagrams contributing to $\alpha$ to the leading order are shown in figure 4(a). We perform the following calculations:

\[
S[\alpha] = \int_{v_0}^{v} \frac{d^3 q}{(2\pi)^3} \psi_s(q) \psi_s(-q) \left[ -2\alpha + 3\beta \int_{b}^{1} \frac{d^3 q'}{(2\pi)^3} G' \psi + \gamma \int_{b}^{1} \frac{d^3 q'}{(2\pi)^3} G' \phi \right]
\]

\[
\approx \int_{b}^{1} \frac{d^3 q}{(2\pi)^3} \psi_s(q) \psi_s(-q) (-2\alpha) \exp \left\{ -\frac{l}{2\alpha} \left[ \frac{3\beta}{2\pi^2} (1 + 2\alpha) + \gamma (F_1 + 2r F_2) \right] \right\}
\]

\[
= \int_{b}^{1} \frac{d^3 q}{(2\pi)^3} e^{-3l} \psi_s(q) \psi_s(-q) e^{l/2} e^{l/2} (-2\alpha')
\times \exp \left\{ -\frac{l}{2\alpha} \left[ \frac{3\beta}{2\pi^2} (1 + 2\alpha) + \gamma (F_1 + 2r F_2) \right] \right\}
\]

\[
= \int_{b}^{1} \frac{d^3 q}{(2\pi)^3} \psi_s(q) \psi_s(-q) (-2\alpha'),
\]

where

\[
\alpha' = \alpha \exp \left\{ 2l - \frac{l}{2\alpha} \left[ \frac{3\beta}{2\pi^2} (1 + 2\alpha) + \gamma (F_1 + 2r F_2) \right] \right\}.
\]

Its derivative with respect to running scale $l$ is

\[
\frac{d\alpha}{dl} = 2\alpha - \left[ \frac{3\beta}{4\pi^2} (1 + 2\alpha) + \frac{\gamma}{2} (F_1 + 2r F_2) \right].
\]
By calculating the diagrams shown in figure 4(b), the flow equation for parameter $r$ can be obtained similarly

$$\frac{d r}{d l} = \left[ 1 + 2(C_3 - C_1) \right] r - \left[ \frac{3u}{2} (F_1 + 2r F_2) + \frac{\gamma}{4\pi^2} (1 + 2\alpha) \right]. \quad (38)$$

In the absence of fermionic degrees of freedom, $F_1 = F_2 = \frac{1}{2\pi^2}$ and $[1 + 2(C_3 - C_1)] r$ replaced with $2r$, then the flow equation of $\alpha$ is identical to that of $r$ [14]. Such an ‘exchange symmetry’ is certainly broken by the gapless nodal qps via the polarization $\Pi(q)$ appearing in the effective action of nematic order $\phi$.

3.2.2. $\beta, u$. The one-loop corrections to $\beta$ are depicted in figure 5(a). By paralleling the steps performed in equation (36), we can similarly obtain

$$S[\beta] \equiv \int \prod_{m=1}^{4} \frac{d^3 k_m}{(2\pi)^3} \frac{d \omega_m}{\delta^2} \left( \sum k_m \right) \delta \left( \sum \omega_m \right) \frac{\beta'}{2} |\psi_s|^4,$$

where

$$\beta' = \beta \exp \left\{ 1 - \frac{2}{\beta} \left[ \frac{9\beta^2}{2\pi^2} + \gamma^2 (F_2 + 4r F_3) \right] \right\} l.$$  

It leads to

$$\frac{d\beta}{dl} = \beta - \left[ \frac{9\beta^2}{\pi^2} + 2\gamma^2 (F_2 + 4r F_3) \right]. \quad (39)$$

By calculating the diagrams shown in figure 5(b), the RG equation for $u$ is found to be

$$\frac{du}{dl} = [-1 + 4(C_3 - C_1)] u - \left[ \frac{\gamma^2}{\pi^2} + 18u^2 (F_2 + 4r F_3) \right]. \quad (40)$$

Once again, the influence of gapless nodal qps is reflected in the functions $F_{2,3}$ and $C_{1,3}$. 

---

**Figure 5.** One-loop corrections to quartic coefficients $\beta$ in (a) and $u$ in (b), respectively.
3.2.3. $\gamma$. Now we consider the flow of $\gamma$, which characterizes the strength of the competitive interaction. The one-loop corrections to the term $\gamma \psi^2 \phi^2$ have three diagrams, presented in figure 6. Following a similar procedure, we eventually obtain

$$S[\gamma] = \int \prod_{m=1}^{14} \frac{d^2 k_m}{(2\pi)^2} \frac{d \omega_m}{2\pi} \delta^2 \left( \sum k_m \right) \delta \left( \sum \omega_m \right) \gamma' \psi^2 \phi^2,$$

where

$$\gamma' = \gamma \exp \left\{ l - \left[ \frac{3\beta}{\pi^2} + 6u(F_2 + 4r F_3) + 8\gamma (F_1 + 2r F_2 + 2\alpha F_1) \right]l \right\}.$$

The flow equation of $\gamma$ is therefore given by

$$\frac{d\gamma}{dl} = \gamma \left\{ 2(C_3 - C_1) - \left[ \frac{3\beta}{\pi^2} + 6u(F_2 + 4r F_3) + 8\gamma (F_1 + 2r F_2 + 2\alpha F_1) \right] \right\}. \tag{41}$$

4. Results and analysis

In section 3, we obtained the RG equations for a number of relevant parameters. In order to specify the possible fixed point of the system under consideration, we need to solve these RG equations.

For later reference, it is useful to list all the RG equations obtained in the last section:

$$\frac{d\alpha}{dl} = 2\alpha - \left[ \frac{3\beta}{4\pi^2} (1 + 2\alpha) + \frac{\gamma}{2} (F_1 + 2r F_2) \right], \tag{42}$$

$$\frac{dr}{dl} = [1 + 2(C_3 - C_1)]r - \left[ \frac{3u}{2} (F_1 + 2r F_2) + \frac{\gamma}{4\pi^2} (1 + 2\alpha) \right], \tag{43}$$

$$\frac{d\beta}{dl} = \beta - \left[ \frac{9\beta^2}{\pi^2} + 2\gamma^2 (F_2 + 4r F_3) \right], \tag{44}$$

$$\frac{du}{dl} = [-1 + 4(C_3 - C_1)]u - \left[ \frac{\gamma^2}{\pi^2} + 18u^2 (F_2 + 4r F_3) \right], \tag{45}$$

$$\frac{d\gamma}{dl} = \gamma \left\{ 2(C_3 - C_1) - \left[ \frac{3\beta}{\pi^2} + 6u(F_2 + 4r F_3) + 8\gamma (F_1 + 2r F_2 + 2\alpha F_1) \right] \right\}, \tag{46}$$

$$\frac{dv_f}{dl} = (C_1 - C_2)v_f, \tag{47}$$

$$\frac{dv_\Delta}{dl} = (C_1 - C_3)v_\Delta. \tag{48}$$
Compared with the case in which the action contains only two bosonic order parameters, gapless nodal qps show their existence by entering into the three functions $F_{1,2,3}$, which are all functions of the fermion velocities, $v_{F,\Delta}$. We now address the influence of these nodal qps on the fixed-point properties of the interacting system.

For finite $r$, the quantum fluctuation of the nematic order and its coupling to gapless nodal qps are relatively weak and hence only lead to unimportant renormalizations of fermion velocities. Actually, the nematic fluctuation is singular only in the close vicinity of the nematic QCP where $r = 0$. Here, we focus on the nematic QCP and solve the above RG equations self-consistently, after taking into account the singular velocity renormalization of the nodal qps driven by the critical nematic fluctuation.

4.1. Theoretical analysis

Before solving the RG equations, it is useful to first make a simple theoretical analysis of the possible scaling behavior. At the nematic QCP, $r = 0$, so the RG equations can be simplified to

$$\frac{d\alpha}{dl} = 2\alpha - \left[\frac{3\beta}{4\pi^2} (1 + 2\alpha) + \frac{\gamma}{2} F_1\right], \quad (49)$$

$$\frac{d\beta}{dl} = \beta - \left[\frac{9\beta^2 + 2\gamma^2}{\pi^2} F_2\right], \quad (50)$$

$$\frac{du}{dl} = [-1 + 4(C_3 - C_1)]u - \left[\gamma^2 + 18u^2 F_2\right], \quad (51)$$

$$\frac{d\gamma}{dl} = \gamma \left[1 - \frac{1}{4\pi^2} \left(3\beta + 3u + 4\gamma\right)\right]. \quad (52)$$

Here, we are particularly interested in the behavior of the competitive coupling constant $\gamma$. The initial values of two quartic coefficients, $\beta$ and $u$, are taken to be positive to ensure that the system is stable. Numerical computations show that $F_{1,2,3}$ are all positive and driven to vanish as $l \to \infty$ due to the extreme velocity anisotropy, namely $v_{\Delta}/v_F \to 0$, at the nematic QCP. Moreover, since $C_3 - C_1 \to 0$ as $l \to \infty$, the right-hand side of equation (52) is actually negative in the low-energy region. As a result, the parameter $\gamma$ is expected to be strongly irrelevant at low energy. It thus turns out that the superconducting and nematic orders might be decoupled, which is directly owing to the presence of gapless nodal qps.

To examine the effects of gapless nodal qps, here we present a set of RG equations

$$\frac{d\alpha}{dl} = 2\alpha - \frac{1}{8\pi^2} \left[3\beta (1 + 2\alpha) + \gamma\right], \quad (53)$$

$$\frac{d\beta}{dl} = \beta - \frac{1}{4\pi^2} \left(9\beta^2 + \gamma^2\right), \quad (54)$$

$$\frac{du}{dl} = u - \frac{1}{4\pi^2} \left(9u^2 + \gamma^2\right), \quad (55)$$

$$\frac{d\gamma}{dl} = \gamma \left[1 - \frac{1}{4\pi^2} (3\beta + 3u + 4\gamma)\right]. \quad (56)$$
which are obtained without considering the fermionic degrees of freedom [14]. It is easy to see that the right-hand side of equation (56) is not always negative. These equations have been analyzed in [14]. It was found that the fixed point structure depends on the initial values of $\beta$ and $u$. For certain values of $\beta$ and $u$, the system has a stable fixed point where the coupling parameter $\gamma$ approaches a finite constant $\gamma^*$. In such a case, the superconducting and nematic orders experience a strong competitive interaction and tend to suppress each other. For other values of $\beta$ and $u$, the system has no stable fixed point, which implies the instability of the system and the appearance of the first order transition. In any case, the properties of the system, without including the nodal qps, are significantly different from those obtained after including the nodal qps.

The above analysis strongly suggests that gapless nodal qps can have a significant influence on the fixed points of this interacting system. In particular, including the dynamics of nodal qps may fundamentally change the nature of the competitive interaction between the superconducting and nematic orders.

4.2. Numerical results and physical implications

In order to confirm our qualitative analysis, we now solve the RG equations numerically. As already pointed out, the fermion velocities $v_{F,\Delta}$ are no longer constants at the nematic QCP: they become scale-dependent and their ratio $v_{\Delta}/v_F$ vanishes at the lowest energy. In this case, the fixed point should be obtained by self-consistently solving all the RG equations presented in equations (42)–(48). However, since the equations of $v_{F,\Delta}$ are relatively independent of all the others, they can be solved first. It is easy to get $v^*_F = 0$, $v^*_\Delta = 0$, in the limit $l \to \infty$. This immediately implies that $F_1 = F_2 = F_3 = 0$. Now the rest of the set of equations become much simpler, given by

$$\frac{d\alpha}{dl} = 2\alpha - \frac{3\beta}{4\pi^2} (1 + 2\alpha) = 0, \quad (57)$$

$$\frac{d\beta}{dl} = \beta - \frac{9\beta^2}{\pi^2} = 0, \quad (58)$$

$$\frac{du}{dl} = -u - \frac{\gamma^2}{\pi^2} = 0, \quad (59)$$

$$\frac{d\gamma}{dl} = -\gamma \frac{3\beta}{\pi^2} = 0. \quad (60)$$

These coupled equations have two solutions:

(1) $r^* = u^* = \gamma^* = 0$, $\beta^* = \alpha^* = 0$;

(2) $r^* = u^* = \gamma^* = 0$, $\beta^* = \frac{\pi^2}{9}$, $\alpha^* = \frac{1}{22}$. \quad (61)

One can check that the second solution corresponds to a stable fixed point and that the first one is unstable. It is important to notice that $\gamma^* = 0$ is at the stable fixed point, which means the competitive interaction between the superconducting and nematic order parameters is actually irrelevant. These results indicate that these two long-range orders are decoupled from each other at the nematic QCP. Hence, we can draw two conclusions. Firstly, due to this decoupling, the
superconducting and nematic long-range orders can coexist homogeneously. Secondly, both the superconducting and nematic phase transitions remain continuous, which is apparently different from the results of the first order transitions obtained without considering the fermionic degrees of freedom.

It is now interesting to further discuss the role played by gapless nodal qps. Currently, the quantum critical phenomena are usually investigated within HMM theory, which supposes that all the fermionic degrees of freedom can be entirely integrated out. If we use this scheme in our case, we would obtain an effective action that consists solely of two bosonic order parameters, $\psi$ and $\phi$. Gapless nodal qps only show their existence in the polarization $\Pi(q)$, which contributes a $\propto \Pi(q)\phi^2$ term to the effective Lagrangian of $\phi$. The fermion velocities, $v_{F,F}$, have to take bare values and cannot be renormalized, because the coupling between the nematic order and nodal qps cannot be properly accounted for once nodal qps are completely integrated out. It would not be possible to incorporate the extreme velocity anisotropy driven by a critical nematic fluctuation into the theoretical analysis. However, as demonstrated in the above calculations, such an extreme velocity anisotropy does have a significant effect on the RG trajectories. It is therefore necessary to include the dynamics of nodal qps.

In the absence of nodal qps, the propagator of the nematic order parameter behaves as $G_\phi \propto 1/q^2$. This propagator is strongly singular in the small momenta limit $q \to 0$. After including gapless nodal qps, the nematic propagator becomes $G_\phi \propto 1/q$, which is less singular compared with $G_\phi \propto 1/q^2$ as $q \to 0$. It is clear that the coupling with nodal qps weakens the critical fluctuation of the nematic order parameter, which in turn leads to the decoupling between the nematic and superconducting orders. Apparently, gapless nodal qps do play an important role and thus should be seriously considered in the effective theory of competing orders.

5. Summary and discussion

In summary, we have carried out a RG analysis within an effective low-energy field theory that describes the interplay between the superconductivity and the nematic order in the context of d-wave high-$T_c$ superconductors. Different from some previous theoretical treatments, we go beyond the HMM framework and incorporate the gapless nodal qps explicitly in our calculations. After analyzing the RG equations of a number of physical parameters, we have demonstrated that gapless nodal qps have a significant impact on the interplay between d-wave superconductivity and the nematic order. If the nodal qps are entirely neglected, the competition between the superconducting and nematic orders can result in runaway behavior, which in turn drives the first order transition [14]. However, including the dynamics of nodal qps can change this picture fundamentally and give rise to a stable fixed point with these two bosonic order parameters decoupled from each other in the vicinity of the nematic QCP. Therefore, both the superconducting and nematic phase transitions remain continuous. Moreover, the d-wave superconductivity can coexist with the nematic order homogeneously. These results indicate that it should be important to include the dynamics of gapless fermions in a theoretical description of the competing orders.

The competition between the superconductivity and the nematic order is just a simple example of the rich phenomena of ordering competition in unconventional superconductors. It would be more interesting to investigate the competition between superconductivity and antiferromagnetism, which is believed to be a fundamental issue not only in high-$T_c$ cuprate superconductors but also in heavy fermion and iron-based superconductors. Compared with
the case of the competing nematic order, the interplay between superconductivity and antiferromagnetism is more complicated and is expected to exhibit more interesting behaviors. For instance, the antiferromagnetic order parameter is a complex scalar field and carries a finite wave vector $Q$, which is often incommensurate. Furthermore, the antiferromagnetic order parameter may acquire a non-trivial dynamical exponent, $z \neq 1$, due to its coupling to gapless fermions, which would make RG calculations more involved [14]. Nevertheless, despite the technical difficulties, the general formalism presented in this paper can be applied to analyze the effects of the fermionic degrees of freedom on the interplay between superconductivity and antiferromagnetism.

Acknowledgments

We are grateful to Jian-Huang She for their very helpful communications and Jing-Rong Wang for their recent collaboration on relevant projects. GZL acknowledges support by the National Natural Science Foundation of China under grants numbers 11074234 and 11274286 and the Visitors Program of MPIPKS at Dresden.

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New Journal of Physics 15 (2013) 073039 (http://www.njp.org/)
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