On Idempotency of Linear Combinations of a Quadratic or a Cubic Matrix and an Arbitrary Matrix

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Abstract. Let $A$ be a quadratic or a cubic $n \times n$ nonzero matrix and $B$ be an arbitrary $n \times n$ nonzero matrix. In this study, we have established necessary and sufficient conditions for the idempotency of the linear combinations of the form $aA + bB$, under the some certain conditions imposed on $A$ and $B$, where $a, b$ are nonzero complex numbers.

1. Introduction and Preliminary Results

Let $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{C}^{mxn}$, and $\mathbb{C}^n$ denote the sets of complex numbers, nonzero complex numbers, all $m \times n$ complex matrices, and all $n \times n$ complex matrices, respectively. $0$, $0_n$, and $I_n$ stand for a zero matrix of appropriate size, a zero matrix of order $n$, and an identity matrix of order $n$, respectively. The symbol $\oplus$ will denote the direct sum of matrices. Moreover, a matrix $A \in \mathbb{C}^n$ is called an idempotent, an involutive, and an $\{\alpha, \beta\} -$ quadratic matrix if $A^2 = A$, $A^2 = I_n$, and $(A - \alpha I_n)(A - \beta I_n) = 0$ with $\alpha, \beta \in \mathbb{C}$, respectively [1]. It is noteworthy that an idempotent and an involutive matrix are a $\{1, 0\} -$ quadratic matrix and a $\{1, -1\} -$ quadratic matrix, respectively. As in above, we will call a matrix $A \in \mathbb{C}^n$ as an $\{\alpha, \beta, \gamma\} -$ cubic matrix if $(A - \alpha I_n)(A - \beta I_n)(A - \gamma I_n) = 0$ with $\alpha, \beta, \gamma \in \mathbb{C}$. Involutive, idempotent, tripotent, and quadratic matrices (that is, some special cases of cubic matrices) have been comprehensively studied in the literature (for example [1–6, 8–11]). Moreover, they have applications to digital image encryption [12].

Consider a linear combination of the form

$$K = aA + bB, \ A, B \in \mathbb{C}^n, \ a, b \in \mathbb{C}^*.$$ (1)

Recently, under some conditions, it has been studied some problems of characterizing all situations where a linear combination of the form (1) is a special type of matrix when $A$ and $B$ are special types of matrices (see, for example, [2–4, 9–11]). Liu et al. characterize the involutiveness of the form (1) when $A$ is a quadratic or a tripotent matrix and $B$ is an arbitrary matrix [8].

The aim of this paper is to obtain the necessary and sufficient conditions for $K = aA + bB$ to be an idempotent matrix, where $A$ is a quadratic or a cubic matrix and $B$ is an arbitrary matrix with some certain conditions. Moreover, some examples are given related to the obtained results.

It was established a useful expression for quadratic matrices in [9]. Now the following lemma, inspired by it, can be given for cubic matrices.
Lemma 1.1. Let $A \in \mathbb{C}^n$. The following statements are equivalent.

(a) There exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\alpha \neq \beta$, $\alpha \neq \gamma$, $\beta \neq \gamma$ and
\[(A - aI_n)(A - \beta I_n)(A - \gamma I_n) = 0.\] (2)

(b) $A$ is diagonalizable and its spectrum $\sigma(A)$ is a subset of $\{\alpha, \beta, \gamma\}$.

(c) There exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\alpha \neq \beta$, $\alpha \neq \gamma$, $\beta \neq \gamma$ and three idempotent matrices $X, Y, Z \in \mathbb{C}^n$ such that $A = \alpha X + \beta Y + \gamma Z$, $X + Y + Z = I_n$, and $XY = YX = XZ = ZX = ZY = 0$.

(d) There exist $a, b, c \in \mathbb{C}$ such that $a + b + c = 0$, and $A = aX + bY + cI_n$.

Proof.

(a) $\Rightarrow$ (b): It is clear from the fact that a matrix is diagonalizable if and only if every eigenvalue of it has multiplicity 1 as a zero of its minimal polynomial [7, Corollary 3.3.10].

(b) $\Rightarrow$ (c): Let $A$ be a diagonalizable matrix and $\sigma(A) \subset \{\alpha, \beta, \gamma\}$, then there exists a nonsingular matrix $S \in \mathbb{C}^n$ such that
\[A = S(\alpha I_p + \beta I_q + \gamma I_r)S^{-1}\]
with $p, q, r \in \{0, \ldots, n\}$ and $p + q + r = n$. Let $X = S(I_p \oplus 0 \oplus 0)S^{-1}$, $Y = S(0 \oplus I_q \oplus 0)S^{-1}$, and $Z = S(0 \oplus 0 \oplus I_r)S^{-1}$. Observe that $A = \alpha X + \beta Y + \gamma Z$, $X + Y + Z = I_n$, and $XY = YX = XZ = ZX = ZY = 0$ as desired.

(c) $\Rightarrow$ (d): Since $A = \alpha X + \beta Y + \gamma Z$ and $Z = I_n - X - Y$, we can write
\[A = (\alpha - \gamma)X + (\beta - \gamma)Y + \gamma I_n,
\]
and the desired result is obtained by taking $a = \alpha - \gamma$, $b = \beta - \gamma$, and $c = \gamma$.

(d) $\Rightarrow$ (a): Since $X$ commutes with $Y$ and they are idempotent, there exists a nonsingular matrix $S \in \mathbb{C}^n$ such that $X = S(I_p \oplus 0 \oplus 0)S^{-1}$ and $Y = S(0 \oplus I_q \oplus 0)S^{-1}$ with rank $(X) = p$ and rank $(Y) = q$ [7, Theorem 1.3.12]. So, it can be written
\[A = aS(I_p \oplus 0 \oplus 0)S^{-1} + bS(0 \oplus I_q \oplus 0)S^{-1} + cS(I_p \oplus I_q \oplus I_{n-p-q})S^{-1}
= S((a + c)I_p \oplus (b + c)I_q \oplus cI_{n-p-q})S^{-1}\]
by the hypothesis. Let $\alpha = a + c$, $\beta = b + c$, and $\gamma = c$. Hence, we have
\[A - aI_n = S(0 \oplus (\beta - a)I_q \oplus (\gamma - \alpha)I_{n-p-q})S^{-1},\]
\[A - \beta I_n = S((\alpha - \beta)I_p \oplus 0 \oplus (\gamma - \beta)I_{n-p-q})S^{-1},\]
and
\[A - \gamma I_n = S((\alpha - \gamma)I_p \oplus (\beta - \gamma)I_q \oplus 0)S^{-1}.\]

So, the proof is completed. \Box

Therefore some properties have been given for $[\alpha, \beta, \gamma]$ – cubic matrices. In view of the fact that a cubic matrix can be written as in (2), some results previously worked about special type of matrices can be generalized. Now we can give the main results.
2. Main Results

In this section, we will investigate the idempotency of the linear combination of the form (1), under some certain conditions. The following result, concerning with a cubic and an arbitrary matrix, is striking.

**Theorem 2.1.** Let \( \alpha, \beta, \gamma \in \mathbb{C} \) with \( \alpha \neq 0, \alpha \neq \beta, \alpha \neq \gamma, \beta \neq \gamma \). Moreover, let \( A \) and \( B \in \mathbb{C}^{n \times 1} \) be an \( [\alpha, \beta, \gamma] \) - cubic matrix and an arbitrary matrix, respectively. Furthermore, let \( A^2BA = A^2B \) and \( K = aA + bB \) with \( a, b \in \mathbb{C} \). Then \( K \) is an idempotent matrix if and only if there exists a nonsingular matrix \( V \in \mathbb{C}^{n} \) such that

\[
A = V \begin{pmatrix} \alpha I_p & 0 & 0 \\ 0 & \beta I_q & 0 \\ 0 & 0 & \gamma I_{n-p-q} \end{pmatrix} V^{-1}
\]

and \( B \) satisfies one of the following cases.

(a) \( \alpha = 1, \beta = 0, \) and \( a\gamma = 1. \)

\[
B = V \begin{pmatrix} I_r & 0 & 0 & 0 \\ -1 I_{r-r} & 0 & 0 & 0 \\ 0 & B_{42} & 0 & 0 \\ 0 & B_{43} & 0 & 0 \end{pmatrix} V^{-1},
\]

being \( B_{42} \in \mathbb{C}^{\times(r-r)}, B_{43} \in \mathbb{C}^{(r-r)\times s}, B_{62} \in \mathbb{C}^{(r-s)\times(n-p-q)}, \) and \( B_{72} \in \mathbb{C}^{(n-p-q)\times(r-r)} \) arbitrary.

(b) \( \alpha = 1, \gamma = 0, \) and \( a\beta = 1. \)

\[
B = V \begin{pmatrix} I_r & 0 & 0 & 0 \\ -1 I_{r-r} & 0 & 0 & 0 \\ 0 & B_{42} & 0 & 0 \\ 0 & B_{53} & 0 & 0 \end{pmatrix} V^{-1},
\]

being \( B_{42} \in \mathbb{C}^{\times(r-r)}, B_{53} \in \mathbb{C}^{(r-r)\times s}, B_{73} \in \mathbb{C}^{(n-p-q)\times(r-r)}, \) and \( B_{62} \in \mathbb{C}^{(n-p-q)\times(r-r)} \) arbitrary.

(c) \( \beta = 1, \gamma = 0, \) and \( a\alpha = 1. \)

\[
B = V \begin{pmatrix} 0 & 0 & B_{22} & 0 \\ 0 & -1 I_s & 0 & 0 \\ 0 & 0 & -1 I_{n-s} & 0 \\ 0 & 0 & B_{63} & 0 \end{pmatrix} V^{-1},
\]

being \( B_{22} \in \mathbb{C}^{\times(s-s)}, B_{63} \in \mathbb{C}^{(n-p-q)\times s}, B_{63} \in \mathbb{C}^{(r-r)\times s}, \) and \( B_{63} \in \mathbb{C}^{(n-p-q)\times s} \) arbitrary.

(d) \( \gamma = 1, \beta = 0, \) and \( a\alpha = 1. \)

\[
B = V \begin{pmatrix} 0 & 0 & 0 & B_{32} \\ 0 & -1 I_s & 0 & 0 \\ 0 & B_{42} & 0 & 0 \\ 0 & 0 & -1 I_{n-p-q} \end{pmatrix} V^{-1},
\]

being \( B_{32} \in \mathbb{C}^{\times(n-p-q)}, B_{42} \in \mathbb{C}^{(s-s)\times s}, B_{63} \in \mathbb{C}^{(n-p-q)\times s}, \) and \( B_{63} \in \mathbb{C}^{(s-s)\times s} \) arbitrary.
Proof. From Lemma 1.1, we can write a cubic matrix $A$ as
\[ A = U (aI_p + bI_q + cI_{n-p-q}) U^{-1}, \]
where $p, q \in [0, \ldots, n], p + q \leq n$ and $U \in \mathbb{C}^n$ is a nonsingular matrix. Let us write $B = U \begin{pmatrix} B_1 & B_2 & B_3 \\ B_4 & B_5 & B_6 \\ B_7 & B_8 & B_9 \end{pmatrix} U^{-1},$
where $B_1 \in \mathbb{C}^p, B_5 \in \mathbb{C}^q$. Observe that under the hypotheses $A^2BA = A^2B$ and $\alpha \neq 0$, one has
\begin{align*}
B_1 & = aB_1, & B_2 & = \beta B_2, & B_3 & = \gamma B_3, \\
\beta^2 B_4 & = \alpha \beta^2 B_4, & \beta^2 B_5 & = \beta^3 B_5, & \beta^2 B_6 & = \beta^2 \gamma B_6, \\
\gamma^2 B_7 & = \alpha \gamma^2 B_7, & \gamma^2 B_8 & = \beta \gamma^2 B_8, & \gamma^2 B_9 & = \gamma^2 B_9.
\end{align*}
Let us assume that $K$ is an idempotent matrix. Hence,
\begin{align*}
b^2(B_2B_4 + B_3B_7) + (a_1b + bB_1)^2 &= a_1b + bB_1, \\
ab(a + \beta)B_2 + b^2(B_2B_5 + B_3B_6 + B_1B_2) &= bB_2, \\
ab(a + \gamma)B_3 + \beta^2(B_3B_4 + B_3B_8 + B_1B_3) &= bB_3, \\
\beta^2(B_4B_6 + B_5B_8) + (a_1b + bB_1)^2 &= \beta^2 B_4, \\
ab(a + \beta)B_7 + b^2(B_7B_1 + B_8B_4 + B_3B_7) &= bB_7, \\
ab(a + \gamma)B_8 + \beta^2(B_6B_5 + B_6B_9 + B_3B_8) &= bB_8, \\
\beta^2(B_5B_6 + B_8B_9) + (a_1b + bB_1)^2 &= \beta^2 B_5.
\end{align*}
The proof can be divided into following cases depending on the scalars $\alpha, \beta, \gamma$.

(i) Let $\alpha = 1$.

From (3), it is seen that $B_2$ and $B_3$ are zero matrices. Depending on the $\beta$, let us split this case into two cases.

(i-1) Let $\beta = 0$.

From (3), it is seen that $B_5$ and $B_9$ are zero matrices. Reorganizing the equations of (4) it follows that
\begin{align*}
(a_1b + bB_1)^2 &= a_1b + bB_1, \\
ab(bB_5)^2 &= bB_5, \\
ab(bB_6)^2 &= bB_6, \\
ab+B_7 + b^2(B_2B_4 + B_3B_7) &= bB_7.
\end{align*}
From the first and second equations in (5), it is clear that $a_1b + bB_1$ and $bB_5$ are idempotent. Since an idempotent matrix is a $[1,0]-$ quadratic matrix, there exist $r \in [0, \ldots, p], s \in [0, \ldots, q]$ and nonsingular matrices $V_1 \in \mathbb{C}^p, V_2 \in \mathbb{C}^q$ such that
\[ a_1b + bB_1 = V_1 \begin{pmatrix} I_r & 0 \\ 0 & I_{p-r} \end{pmatrix} V_1^{-1}, \quad bB_5 = V_2 \begin{pmatrix} I_s & 0 \\ 0 & I_{q-s} \end{pmatrix} V_2^{-1}, \]
respectively. So, we obtain that
\begin{align*}
B_1 &= V_1 \begin{pmatrix} 1 & 0 \\ b & I_{p-r} \end{pmatrix} V_1^{-1}, \\
B_5 &= V_2 \begin{pmatrix} 1 & 0 \\ 0 & I_{q-s} \end{pmatrix} V_2^{-1}.
\end{align*}
Let $B_6$ and $B_7$ be written as
\begin{align*}
B_6 &= V_2 \begin{pmatrix} B_{6_1} \\ B_{6_2} \end{pmatrix}, \\
B_7 &= \begin{pmatrix} B_{7_1} \\ B_{7_2} \end{pmatrix} V_1^{-1},
\end{align*}
where $B_{6_1} \in \mathbb{C}^{x(n-p-q)}$ and $B_{7_1} \in \mathbb{C}^{(n-p-s)xr}$. Substituting (6), (7) into the forth and fifth equations in (5) yield
\[ (ab - b) V_2 \begin{pmatrix} B_{6_1} \\ B_{6_2} \end{pmatrix} + b^2 V_2 \begin{pmatrix} I_{r-s} & 0 \\ 0 & I_{p-r-s} \end{pmatrix} \begin{pmatrix} B_{6_1} \\ B_{6_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.\]
and

\[
(ab(1 + \gamma) - b) \left( \begin{bmatrix} B_{71} & B_{72} \end{bmatrix} \right) V_{1}^{-1} + b^2 \left( \begin{bmatrix} B_{71} & B_{72} \end{bmatrix} \right) \left( \begin{bmatrix} \frac{1}{b} I_p & 0 \\ 0 & \frac{1}{b} I_{p-r} \end{bmatrix} \right) V_{1}^{-1} = \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right).
\]

Therefore, it can be written

\[
\left( \begin{bmatrix} a\gamma B_{61} \\ (a\gamma - 1) B_{62} \end{bmatrix} \right) = \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \quad \text{and} \quad \left( \begin{bmatrix} a\gamma B_{71} \\ (a\gamma - 1) B_{72} \end{bmatrix} \right) = \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right).
\]

Moreover, from the third equation in (5), it is clear that \( a\gamma = 1 \). Hence, \( B_{6} \) and \( B_{7} \) reduce to

\[
B_{6} = V_{2} \left( \begin{bmatrix} 0 \\ B_{62} \end{bmatrix} \right) \quad \text{and} \quad B_{7} = \left( \begin{bmatrix} 0 \\ B_{72} \end{bmatrix} \right) V_{1}^{-1},
\]

where \( B_{6} \in \mathbb{C}^{(q-r) \times (n-p-q)} \) and \( B_{7} \in \mathbb{C}^{(n-p-q) \times (p-r)} \) are arbitrary matrices.

Lastly, let

\[
B_{4} = V_{2} \left( \begin{bmatrix} B_{41} & B_{42} \\ B_{43} & B_{44} \end{bmatrix} \right) V_{1}^{-1},
\]

where \( B_{4} \in \mathbb{C}^{\times r} \). Substituting (6), (8), and (9) into the sixth equation in (5) yields

\[
\left( \begin{bmatrix} B_{41} \\ bB_{62} B_{72} - B_{44} \end{bmatrix} \right) = \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right).
\]

Therefore, \( B_{4} \) turns into

\[
B_{4} = V_{2} \left( \begin{bmatrix} 0 \\ B_{43} \\ bB_{62} B_{72} \end{bmatrix} \right) V_{1}^{-1},
\]

where \( B_{43} \in \mathbb{C}^{(p-r)} \) and \( B_{44} \in \mathbb{C}^{(n-p-q)} \times \) are arbitrary matrices.

Let us define \( V := U \left( V_{1} \oplus V_{2} \oplus I_{n-p-q} \right) \). In view of (6), (8), and (10) we obtain that

\[
A = U \left( \begin{bmatrix} I_p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma I_{n-p-q} \end{bmatrix} \right) U^{-1}
\]

\[
= V \left( \begin{bmatrix} V_{1}^{-1} & 0 & 0 \\ 0 & V_{2}^{-1} & 0 \\ 0 & 0 & I_{n-p-q} \end{bmatrix} \right) \left( \begin{bmatrix} I_p & 0 & 0 \\ 0 & 0 & \gamma I_{n-p-q} \end{bmatrix} \right) \left( \begin{bmatrix} V_{1} & 0 & 0 \\ 0 & V_{2} & 0 \\ 0 & 0 & I_{n-p-q} \end{bmatrix} \right) V^{-1}
\]

\[
= V \left( \begin{bmatrix} I_p & 0 & 0 \\ 0 & 0 & \gamma I_{n-p-q} \end{bmatrix} \right) V^{-1}
\]

and

\[
B = U \left( \begin{bmatrix} \frac{1}{b} I_r & 0 \\ 0 & \frac{1}{b} I_{p-r} \end{bmatrix} \right) V_{1}^{-1} \left( \begin{bmatrix} 0 \\ 0 \\ 0 \\ V_{2} \left( \begin{bmatrix} B_{41} \\ bB_{62} B_{72} \end{bmatrix} \right) V_{1}^{-1} \end{bmatrix} \right) \left( \begin{bmatrix} \frac{1}{b} I_r & 0 \\ 0 & \gamma I_{n-p-q} \end{bmatrix} \right) \left( \begin{bmatrix} V_{1} \\ V_{2} \left( \begin{bmatrix} 0 \\ B_{62} \end{bmatrix} \right) V_{1}^{-1} \end{bmatrix} \right) U^{-1}
\]

\[
= V \left( \begin{bmatrix} \frac{1}{b} I_r & 0 & 0 & 0 \\ 0 & \frac{1}{b} I_{p-r} & 0 & 0 \\ 0 & 0 & \frac{1}{b} I_r & 0 \\ B_{41} & bB_{62} B_{72} & 0 & B_{62} \\ 0 & B_{72} & 0 & 0 \end{bmatrix} \right) V^{-1}
\]

\[
= V \left( \begin{bmatrix} \frac{1}{b} I_r & 0 & 0 & 0 \\ 0 & \frac{1}{b} I_{p-r} & 0 & 0 \\ 0 & 0 & \frac{1}{b} I_r & 0 \\ B_{41} & bB_{62} B_{72} & 0 & B_{62} \\ 0 & B_{72} & 0 & 0 \end{bmatrix} \right) V^{-1}
\]
which establishes part (a).

(i-2) Let \( \beta \neq 0 \).

From (3), it is seen that \( B_5 \) and \( B_6 \) are zero matrices. Reorganizing the equations of (4) it follows that

\[
(aI_p + bB_1)^2 = aI_p + bB_1, \quad (aI_q + bB_9)^2 = aI_{n-p-q} + bB_6,
\]

\[
ab (1 + \beta) B_4 + b^2 B_5 B_1 = bB_4, \quad ab (\beta + \gamma) B_8 + b^2 B_8 B_8 = B_8,
\]

\[
ab (1 + \gamma) B_7 + b^2 (B_7 B_4 + B_5 B_4 + B_3 B_7) = bB_7. \tag{11}
\]

From the first equation in (11), it is obvious that \( aI_p + bB_1 \) is an idempotent matrix. Thus, there exist \( r \in [0, \ldots, p] \) and a nonsingular matrix \( Y_1 \in C^p \) such that

\[
B_1 = Y_1 \begin{pmatrix} 0 & 1 - \frac{a}{I_p} \ I_r \ 0 & 0 \end{pmatrix} \ Y_1^{-1}. \tag{12}
\]

From the second and third equations in (11), it is clear that \( a\beta = 1 \) and \( aI_{n-p-q} + bB_6 \) is idempotent. However, from the last equation in (3), \( \gamma = 0 \) or \( \gamma = 1 \) or \( B_9 = 0 \). It is clear that \( \gamma \neq 0 \). Moreover, if \( B_9 = 0 \) then \( a\gamma = 0 \) or \( a\gamma = 1 \). But this latter equality contradicts the hypothesis \( \beta \neq \gamma \). Thus \( \gamma \) must be zero. So, there exist \( t \in [0, \ldots, n - p - q] \) and a nonsingular matrix \( Y_2 \in C^{(n-p-q)} \) such that

\[
B_9 = Y_2 \begin{pmatrix} 1 & I_t \ 0 & 0 \end{pmatrix} \ Y_2^{-1}. \tag{13}
\]

Let \( B_4 \) and \( B_8 \) be written as

\[
B_4 = \begin{pmatrix} B_{4_1} & B_{4_2} \end{pmatrix} Y_1^{-1} \quad \text{and} \quad B_8 = Y_2 \begin{pmatrix} B_{8_1} \ B_{8_2} \end{pmatrix}, \tag{14}
\]

where \( B_{4_1} \in C^{p \times r}, B_{8_1} \in C^{q \times q} \). Substituting (12), (14) and (13) into the forth and fifth equations in (11) yield

\[
(abB_{4_1} - (a\beta - 1) B_{4_2}) = \begin{pmatrix} 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a\beta B_{8_1} \ (a\beta - 1) B_{8_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

respectively. Moreover using \( a\beta = 1 \), \( B_4 \) and \( B_8 \) reduce to

\[
B_4 = \begin{pmatrix} 0 & B_{4_2} \end{pmatrix} Y_1^{-1} \quad \text{and} \quad B_8 = Y_2 \begin{pmatrix} 0 \ B_{8_2} \end{pmatrix}, \tag{15}
\]

where \( B_{4_2} \in C^{p \times (n-p-q)} \) and \( B_{8_2} \in C^{(n-p-q) \times q} \) are arbitrary matrices.

Lastly, let

\[
B_7 = Y_2 \begin{pmatrix} B_{7_1} & B_{7_2} \\ B_{7_3} & B_{7_4} \end{pmatrix} Y_1^{-1}, \tag{16}
\]

where \( B_{7_1} \in C^{r \times r} \). Substituting (12), (13), (15), and (16) into the sixth equation in (11) yields

\[
\begin{pmatrix} B_{7_1} & 0 \\ 0 & bB_{8_2} B_{4_2} - B_{7_1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

So, \( B_7 \) reduces to

\[
B_7 = Y_2 \begin{pmatrix} 0 & B_{7_2} \\ B_{7_3} & bB_{8_2} B_{4_2} \end{pmatrix} Y_1^{-1}, \tag{17}
\]

where \( B_{7_2} \in C^{r \times (n-p-q)} \) and \( B_{7_2} \in C^{(n-p-q) \times r} \) are arbitrary matrices.

Let us define \( V := U \left( Y_1 \oplus I_q \oplus Y_2 \right) \). In view of (12), (13), (15), and (17) we obtain that
\[ A = U \begin{pmatrix} I_p & 0 & 0 \\ 0 & \beta I_q & 0 \\ 0 & 0 & 0_{n-p-q} \end{pmatrix} U^{-1} \]

\[ = V \begin{pmatrix} Y_1^{-1} & 0 & 0 \\ 0 & I_{q} & 0 \\ 0 & 0 & Y_2^{-1} \end{pmatrix} \begin{pmatrix} I_p & 0 & 0 \\ 0 & \beta I_q & 0 \\ 0 & 0 & 0_{n-p-q} \end{pmatrix} \begin{pmatrix} Y_1 & 0 & 0 \\ 0 & I_{r} & 0 \\ 0 & 0 & Y_2 \end{pmatrix} V^{-1} \]

\[ = V \begin{pmatrix} I_p & 0 & 0 \\ 0 & \beta I_q & 0 \\ 0 & 0 & 0_{n-p-q} \end{pmatrix} V^{-1} \]

and

\[ B = U \begin{pmatrix} Y_1 \left( \frac{1}{\gamma} I_p \right) & 0 & 0 \\ 0 & \frac{1}{\gamma} I_{p-r} & 0 \\ 0 & 0 & Y_2 \end{pmatrix} Y_3^{-1} \begin{pmatrix} I_p \left( \frac{1}{\gamma} I_{p-r} \right) & 0 & 0 \\ 0 & I_{q} & 0 \\ 0 & 0 & Y_2 \end{pmatrix} Y_4^{-1} \begin{pmatrix} I_p & 0 & 0 \\ 0 & \beta I_q & 0 \\ 0 & 0 & 0_{n-p-q-t} \end{pmatrix} V^{-1} \]

which yields part (b).

(ii) Let \( \beta = 1 \).

From (3), it is seen that \( B_1, B_3, B_4, \) and \( B_6 \) are zero matrices. Reorganizing the equations of (4) it can be written

\[ \left( a^2 I_p \right)^2 = a a I_p, \quad \left( a I_q + b B_2 \right)^2 = a I_q + b B_5, \quad \left( a y I_{n-p-q} + b B_9 \right)^2 = a y I_{n-p-q} + b B_9 \]

\[ ab (a + 1) B_2 + b^2 B_9 B_5 = b B_{22}, \quad ab (a + y) B_7 + b^2 B_9 B_7 = b B_7, \]

\[ ab (1 + y) B_9 + b^2 (B_7 B_2 + B_5 B_7 + B_9 B_8) = b B_9. \]

(18)

It is clear that \( a a = 1 \) and \( a I_q + b B_9 \) is an idempotent matrix from the first and second equations in (18), respectively. There exist \( s \in \{0, \ldots, q\} \) and a nonsingular matrix \( T_1 \in \mathbb{C}^n \) such that

\[ B_9 = T_1 \begin{pmatrix} \frac{1}{\gamma} I_s & 0 \\ 0 & \frac{1}{\gamma} I_{n-s} \end{pmatrix} T_1^{-1}. \]

(19)

From the third equation in (18), it is clear that \( a y I_{n-p-q} + b B_9 \) is idempotent. However, from the last equation in (3), \( \gamma = 0 \) or \( \gamma = 1 \) or \( B_9 = 0 \). It is obvious that \( \gamma \neq 1 \). Moreover, if \( B_9 = 0 \) then \( a y = 0 \) or \( a y = 1 \). But this latter equality contradicts the hypothesis \( \alpha \neq \gamma \). Thus \( \gamma \) must be zero. So, there exist \( t \in \{0, \ldots, n-p-q\} \) and a nonsingular matrix \( T_2 \in \mathbb{C}^{(n-p-q)} \) such that

\[ B_9 = T_2 \begin{pmatrix} \frac{1}{\gamma} I_t & 0 \\ 0 & 0_{n-p-q-t} \end{pmatrix} T_2^{-1}. \]

(20)

Let \( B_2 \) and \( B_7 \) be written as

\[ B_2 = \begin{pmatrix} B_{21} & B_{22} \end{pmatrix} T_1^{-1} \quad \text{and} \quad B_7 = T_2 \begin{pmatrix} B_{71} \\ B_{72} \end{pmatrix}. \]

(21)
where $B_2 \in \mathbb{C}^{p\times s}$ and $B_7 \in \mathbb{C}^{s\times p}$. Substituting (19), (21) and (20), (21) into the forth and fifth equations in (18) yield \( (aaB_2, (aa - 1)B_2, ) = (0, 0) \) and \( (aaB_7, (aa - 1)B_7, ) = (0, 0) \), respectively. Moreover, since \( aa = 1 \), we obtain that

\[
B_2 = \begin{pmatrix} 0 & B_2 \end{pmatrix} T_1^{-1} \quad \text{and} \quad B_7 = T_2 \begin{pmatrix} 0 & B_7 \end{pmatrix},
\]

where $B_2 \in \mathbb{C}^{p\times(q-s)}$ and $B_7 \in \mathbb{C}^{(n-p-q-1)\times p}$ are arbitrary matrices.

Let

\[
B_8 = T_2 \begin{pmatrix} B_8_1 & B_8_2 \end{pmatrix} T_1^{-1},
\]

where $B_8_1 \in \mathbb{C}^{ts\times s}$. Substituting (19), (20), (22), and (23) into the sixth equation in (18) yields

\[
\begin{pmatrix} B_8_1 & 0 \\ 0 & bB_7, B_2, - B_8_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

So, $B_8$ turns into

\[
B_8 = T_2 \begin{pmatrix} 0 & B_8_2 \\ B_8_1 & bB_7, B_2, \end{pmatrix} T_1^{-1},
\]

where $B_8_1 \in \mathbb{C}^{t\times(q-s)}$ and $B_8_2 \in \mathbb{C}^{(n-p-q-1)\times t}$ are arbitrary matrices. Let us define $V := U(I_p \oplus T_1 \oplus T_2)$. In view of (19), (20), (22), and (24) we obtain that

\[
A = U \begin{pmatrix} \alpha I_p & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & 0_{n-p-q} \end{pmatrix} U^{-1}
\]

and

\[
B = U \begin{pmatrix} 0_p & \left( \begin{array}{c} 0 \\ B_{22} \\ \frac{1}{p} B_{22} \end{array} \right) T_1^{-1} & 0 \\ 0 & T_1 \left( \begin{array}{c} \frac{1}{p} I_q \\ 0 \\ \frac{1}{q} I_q \end{array} \right) T_1^{-1} & 0 \\ T_2 \left( \begin{array}{c} 0 \\ B_{22} \\ B_{80} \\ bB_7, B_2, \end{array} \right) T_1^{-1} & T_2 \left( \begin{array}{c} \frac{1}{q} I_q \\ 0 \\ 0_{n-p-q-1} \end{array} \right) T_2^{-1} \end{pmatrix} U
\]

which establishes part (c).
Let $\gamma = 1$.

From (3), it is easily seen that $B_1, B_2, B_7$, and $B_8$ are zero matrices. Reorganizing the equations of (4) it follows that

\[
\begin{align*}
(\alpha I_p)^2 &= a\alpha I_p, \\
(\alpha\beta I_1 + bB_5)^2 &= a\beta I_1 + bB_5, \\
(\alpha I_{n-p-q} + bB_9)^2 &= aI_{n-p-q} + bB_9,
\end{align*}
\]

\[
\begin{align*}
\alpha \beta I_1 B_3 + B^2_1B_3 &= bB_3, \\
\alpha \beta I_1 B_4 + \beta^2 B_1 B_4 &= bB_4.
\end{align*}
\]

(25)

It is clear that $\alpha \beta = 1$ and $a\beta I_1 + bB_5$ is an idempotent matrix from the first and second equations in (25), respectively. However from the fifth equation in (3), $\beta = 0$ or $\beta = 1$ or $B_5 = 0$. It is obvious that $\beta \neq 1$. Moreover, if $B_5 = 0$ then $a\beta = 0$ or $a\beta = 1$. But this latter equality contradicts the hypothesis $\alpha \neq \beta$. Thus $\beta$ must be zero. So, there exist $s \in \{0, \ldots, q\}$ and a nonsingular matrix $Z_1 \in \mathbb{C}^l$ such that

\[
B_5 = Z_1 \begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix} Z_1^{-1}.
\]

(26)

Moreover, from the third equation in (25), it is clear that $aI_{n-p-q} + bB_9$ is idempotent. Then there exist $t \in \{0, \ldots, n-p-q\}$ and a nonsingular matrix $Z_2 \in \mathbb{C}^{(n-p-q)}$ such that

\[
B_9 = Z_2 \begin{pmatrix} I_{n-p-q} & 0 \\ 0 & 0 \end{pmatrix} Z_2^{-1}.
\]

(27)

Let $B_1$ and $B_4$ be written as

\[
B_3 = \begin{pmatrix} B_3_1 & B_3_2 \end{pmatrix} Z_2^{-1} \quad \text{and} \quad B_4 = Z_1 \begin{pmatrix} B_4_1 \\ B_4_2 \end{pmatrix},
\]

(28)

where $B_3_1 \in \mathbb{C}^{r \times l}$ and $B_4_1 \in \mathbb{C}^{r \times p}$. Substituting (27), (28) and (26), (28) into the forth and fifth equations in (25) it is obtained that

\[
\begin{pmatrix} \alpha aB_3_1 \\ (\alpha a - 1)B_3_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a\alpha B_4_1 \\ (\alpha a - 1)B_4_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

respectively. Moreover using $\alpha \beta = 1$, $B_3$ and $B_4$ turn to

\[
B_3 = \begin{pmatrix} 0 & B_3_2 \end{pmatrix} Z_2^{-1} \quad \text{and} \quad B_4 = Z_1 \begin{pmatrix} 0 \\ B_4_2 \end{pmatrix},
\]

(29)

where $B_3_2 \in \mathbb{C}^{(n-p-q-\gamma) \times l}$ and $B_4_2 \in \mathbb{C}^{(r-q) \times p}$ are arbitrary matrices.

Lastly, let

\[
B_6 = Z_1 \begin{pmatrix} B_6_1 & B_6_2 \\ B_6_3 & B_6_4 \end{pmatrix} Z_2^{-1},
\]

(30)

where $B_6_1 \in \mathbb{C}^{s \times l}$. Substituting (26), (27), (29), and (30) into the sixth equation in (25) it is obtained that

\[
\begin{pmatrix} B_6_1 \\ 0 \\ bB_4_1 B_3_2 - B_6_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

So, $B_6$ reduces to

\[
B_6 = Z_1 \begin{pmatrix} 0 & B_6_2 \\ B_6_3 & B_6_4 \end{pmatrix} Z_2^{-1},
\]

(31)

where $B_6_2 \in \mathbb{C}^{s \times (n-p-q-\gamma)}$ and $B_6_4 \in \mathbb{C}^{(r-q) \times l}$ are arbitrary matrices.

Let us define $V := U \left( I_p \oplus Z_1 \oplus Z_2 \right)$. In view of (26), (27), (29), and (31) we obtain that
\[ A = \begin{array}{c}
V \begin{pmatrix}
I_p & 0 & 0 \\
0 & Z_1^{-1} & 0 \\
0 & 0 & Z_2^{-1}
\end{pmatrix} \begin{pmatrix}
I_p & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & I_{n-p-q}
\end{pmatrix} \begin{pmatrix}
I_p & 0 & 0 \\
0 & Z_1 & 0 \\
0 & 0 & Z_2
\end{pmatrix} \end{array} \]
and
\[ B = \begin{array}{c}
V \begin{pmatrix}
0_p & 0 & 0 \\
\frac{1}{\beta} I_{s} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
\frac{1}{\alpha} I_{t} & 0 & 0 \\
\frac{1}{\alpha} I_{t} & 0 & 0 \\
\frac{1}{\alpha} I_{t} & 0 & 0
\end{pmatrix} \end{array} \]
which establishes part (d).

(iv) Let \( \beta \neq 1, \alpha \neq 1, \) and \( \gamma \neq 1. \)
From (3), it is easily seen that \( B_1, B_2, \) and \( B_3 \) are zero matrices. Depending on the \( \beta, \) let us split this case into two cases.

(iv-1) Let \( \beta = 0. \)
From (3), it is seen that \( B_7, B_8, \) and \( B_9 \) are zero matrices. Reorganizing the equations of (4) it can be written
\[
\begin{align*}
(a\alpha I_p)^2 &= a \alpha I_p, \\
(bB_3)^2 &= bB_3, \\
(ab\gamma I_n)^2 &= a \gamma I_{n-p-q},
\end{align*}
\]
and
\[
\begin{align*}
ab\alpha B_4 + b^2 B_3 B_4 &= bB_4, \\
ab \gamma B_9 + b^2 B_3 B_9 &= bB_9.
\end{align*}
\]
(32)
It is clear that \( a\alpha = 1 \) and \( a \gamma = 1 \) from the first and third equations in (32), respectively. However these equalities contradict the hypothesis of \( a \neq \gamma. \) So, in this case, there is no matrix form of \( B \) such that the linear combination matrix \( K \) is idempotent.

(iv-2) Let \( \beta \neq 0. \)
From (3), it is seen that \( B_4, B_5, \) and \( B_6 \) are zero matrices. Reorganizing the equations of (4) it follows that
\[
\begin{align*}
(a\alpha I_p)^2 &= a \alpha I_p, \\
(ab\gamma I_n)^2 &= ab\gamma I_n, \\
(a \gamma I_{n-p-q} + bB_3)^2 &= a \gamma I_{n-p-q} + bB_3,
\end{align*}
\]
and
\[
\begin{align*}
ab (\alpha + \gamma) B_7 + b^2 B_3 B_7 &= bB_7, \\
ab (\beta + \gamma) B_8 + b^2 B_3 B_8 &= bB_8.
\end{align*}
\]
(33)
It is obvious that \( a\alpha = 1 \) and \( a \beta = 1 \) from the first and second equations in (33), respectively. However these equalities contradict the hypothesis of \( a \neq \beta. \) So, in this case, there is no matrix form of \( B \) such that the linear combination matrix \( K \) is idempotent.

Thus, the necessity has been proved. The sufficiency is obvious.

In the following example it is sought scalars such that the linear combination of a cubic matrix and an arbitrary matrix is an idempotent matrix.

Example 2.2. Let
\[
A = \begin{pmatrix}
-1 & 0 & -1 & 0 \\
-2 & 1 & -1 & 0 \\
2 & 0 & 2 & 0 \\
4 & -2 & 3 & -1
\end{pmatrix}
\]
and

\[ B_1 = \begin{pmatrix} -1 & 0 & 5 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & -1 \\ 2 & 0 & 0 & -2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -1 & 0 & -1 & 0 \\ -4 & 2 & -2 & 0 \\ 2 & 0 & 2 & 0 \\ 5 & -2 & 3 & 0 \end{pmatrix}. \]

Let us find all ordered pair \((a, b)\) such that \(K_i\) is idempotent, \(i = 1, 2\), where \(a, b \in C^*\) and \(K_i = aA + bB_i\). It is clear that \(A\) is a \([-1, 1, 0]\) – cubic matrix (note that it is not tripotent) and \(A^2B_iA = A_iB_i, i = 1, 2\). Moreover,

\[ K_1 = \begin{pmatrix} -a - b & 0 & 5b - a & b \\ -2a & a & -a - b & 0 \\ 2a + b & 0 & 2a + b & -b \\ 4a + 2b & -2a & 3a & -a - 2b \end{pmatrix}, \quad K_2 = \begin{pmatrix} -a - b & 0 & -a - b & 0 \\ -2a - 4b & a + 2b & -a - 2b & 0 \\ 2a + 2b & 0 & 2a + 2b & 0 \\ 4a + 5b & -2a - 2b & 3a + 3b & -a \end{pmatrix}. \]

\[ K_1^2 = \begin{pmatrix} -a^2 + 15ab + 8b^2 & -2ab & -a^2 + 8ab & -ab - 8b^2 \\ -2a^2 - ab - b^2 & a^2 & -a^2 - 14ab - b^2 & b^2 - ab \\ 2a^2 - 3ab - 2b^2 & 2ab & (2a + b)(a + 6b) - 3ab & ab + 2b^2 \\ 2a^2 - 13ab - 6b^2 & 4ab & a^2 + 17ab + 10b^2 & a^2 + 5ab + 6b^2 \end{pmatrix}, \]

\[ K_2^2 = \begin{pmatrix} (a + b)^2 & 0 & -a^2 - 2ab - b^2 & 0 \\ -2a^2 - 8ab - 8b^2 & (a + 2b)^2 & -(a + 2b)^2 & 0 \\ 2a^2 + 4ab + 2b^2 & 0 & 2(a + 2b)^2 & 0 \\ 2a^2 + 10ab + 9b^2 & -4ab - 4b^2 & a^2 + 6ab + 5b^2 & a^2 \end{pmatrix}. \]

The solution sets of nonlinear equations \(K_1^2 = K_1\) and \(K_2^2 = K_2\) are \((0, 0)\) and \((0, 0, -1, 1)\), respectively, \(i = 1, 2\). While the pair \((-1, 1)\) implies the idempotency of \(K_2\), there is no appropriate pair \((a, b)\) to imply that \(K_1\) is idempotent. Because the matrix \(B_1\) should have been in the form of the matrix \(B\), in the part (c) of Theorem 2.1, but \(B_1\) does not match the desired form. However, \(B_2\) satisfies aforementioned form of the matrix \(B\).

**Example 2.3.** Let us define \(A\) as in the previous example and let us find \(a \in C^*\) and all matrices \(B \in C^4\) such that \(A^2BA = A^2B\) and \(aA + B\) is idempotent. A diagonalized form of \(A\) and the matrix \(V\) that diagonalize it are

\[ A = V \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} V^{-1} \quad \text{and} \quad V = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & -2 & -2 & -1 \\ 1 & -1 & -3 & -1 \end{pmatrix}. \]

By using the notations of Theorem 2.1, let us assume \(\alpha = -1, \beta = 1, \gamma = 0\). Then only part (c) of Theorem 2.1 can be implied, so we get \(a = 1/\alpha = -1, p = 1, q = 2, s \in \{0, 1, 2\}\), and \(t \in \{0, 1\}\). Therefore, depending on the appearing and disappearing blocks of \(V^{-1}BV\), it can be written the following possible cases:

| POSSIBILITIES FOR V^{-1}BV |
|-----------------------------|
| t/s | s = 0 | s = 1 | s = 2 |
|-----|-------|-------|-------|
| t = 0 | \( \begin{pmatrix} 0 & c & \hat{d} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ e & e.c & e.d & 0 \end{pmatrix} \) | \( \begin{pmatrix} 0 & j & 0 \\ 0 & 2 & 0 \end{pmatrix} \) | \( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ w & x & y \end{pmatrix} \) |
| t = 1 | \( \begin{pmatrix} 0 & f & g & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & h & i & 1 \end{pmatrix} \) | \( \begin{pmatrix} 0 & 0 & u & 0 \\ 0 & 2 & 0 \end{pmatrix} \) | \( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) |
where \( c, d, e, f, g, h, i, j, k, l, u, v, w, x, \) and \( y \) are arbitrary complex numbers.

Involutiveness of the linear combination of the form (1) of a quadratic matrix and an arbitrary matrix under the condition \( ABA = BA \) was considered in [8]. It is noteworthy that the above theorem gives the solution to the problem of the idempotency of linear combination of the form (1) of a cubic matrix and an arbitrary matrix under the condition \( A^2BA = A^2B \). Then it may be interesting to reconsider of the same problem under the condition \( ABA = BA \).

**Theorem 2.4.** Let \( \alpha, \beta, \gamma \in \mathbb{C} \) with \( \alpha \neq 0, \alpha \neq \beta, \alpha \neq \gamma, \beta \neq \gamma \). Moreover, let \( A \) and \( B \) be \( C^n \setminus \{0\} \) be an \( \{\alpha, \beta, \gamma\}\) – cubic matrix and an arbitrary matrix, respectively. Furthermore, let \( ABA = BA \) and \( K = aA + bB \) with \( a, b \in \mathbb{C} \). Then \( K \) is an idempotent matrix if and only if there exists a nonsingular matrix \( V \in \mathbb{C}^n \) such that

\[
A = V \begin{pmatrix}
\alpha I_p & 0 & 0 \\
0 & \beta I_q & 0 \\
0 & 0 & \gamma I_{n-p-q}
\end{pmatrix} V^{-1}
\]

and \( B \) satisfies one of the following cases.

(a) \( \alpha = 1, \beta = 0, \) and \( a\gamma = 1. \)

\[
B = V \begin{pmatrix}
\gamma^{-1} I_p & 0 & 0 & B_{32} & 0 \\
0 & -\gamma I_{n-p-q} & B_{23} & bB_{22}B_{32} & B_{32} \\
0 & 0 & \frac{1}{b} I_s & 0 & 0 \\
0 & 0 & 0 & 0_{n-s} & 0 \\
0 & 0 & 0 & 0_{n-p-q} & 0
\end{pmatrix} V^{-1},
\]

being \( B_{32} \in C^{\gamma(n-s)}, B_{23} \in C^{(\rho-r)s}, B_{32} \in C^{(\rho-r)n-s}, \) and \( B_{32} \in C^{(n-p-s)(n-q-s)} \) arbitrary.

(b) \( \alpha = 1, \gamma = 0, \) and \( a\beta = 1. \)

\[
B = V \begin{pmatrix}
\beta^{-1} I_p & 0 & 0 & B_{32} & 0 \\
0 & -\beta I_{n-p-q} & B_{23} & bB_{22}B_{32} & B_{32} \\
0 & 0 & \frac{1}{b} I_s & 0 & 0 \\
0 & 0 & 0 & 0_{n-s} & 0 \\
0 & 0 & 0 & 0_{n-p-q} & 0
\end{pmatrix} V^{-1},
\]

being \( B_{32} \in C^{\beta(r-s)}, B_{23} \in C^{(\rho-r)s}, B_{32} \in C^{(\rho-r)n-s}, \) and \( B_{32} \in C^{(n-p-s)(n-q-s)} \) arbitrary.

(c) \( \beta = 1, \gamma = 0, \) and \( a\alpha = 1. \)

\[
B = V \begin{pmatrix}
0 & 0 & 0 & 0 & B_{32} \\
\frac{1}{a\alpha} I_p & 0 & 0 & 0 & B_{62} \\
0 & \frac{1}{a\alpha} I_q & 0 & 0 & bB_{32} \times B_{32} \\
0 & 0 & 0 & \frac{1}{b} I_s & 0 \\
0 & 0 & 0 & 0 & 0_{n-p-q-t}
\end{pmatrix} V^{-1},
\]

being \( B_{32} \in C^{\alpha(n-p-q-t)}, B_{42} \in C^{(\rho-r)s}, B_{62} \in C^{\alpha(n-p-q-t)}, \) and \( B_{62} \in C^{(n-p-q-t)} \) arbitrary.

(d) \( \gamma = 1, \beta = 0, \) and \( a\alpha = 1. \)

\[
B = V \begin{pmatrix}
0 & 0 & B_{32} & 0 & 0 \\
0 & \frac{1}{b} I_s & 0 & 0 & 0 \\
0 & 0 & 0 & 0_{n-s} & 0 \\
0 & 0 & 0_{n-p-q} & \frac{1}{a\alpha} I_t & 0 \\
B_{72} & B_{82} & bB_{72}B_{82} & 0 & -\frac{1}{a\alpha} I_{n-p-q-t}
\end{pmatrix} V^{-1},
\]

being \( B_{32} \in C^{\alpha(n-s)}, B_{72} \in C^{(n-p-q-t)s}, B_{82} \in C^{\alpha(n-s)}, \) and \( B_{82} \in C^{(n-p-q-t)s} \) arbitrary.
The proof of Theorem 2.4 is similar to the proof of Theorem 2.1, so it is omitted.

**Remark 2.5.** Note that α must be different from 1 in Theorems 2.1 and 2.4 since the hypotheses of α ≠ β, α ≠ γ, β ≠ γ. There is same situation in some parts of some results in the sequel.

A is considered as a cubic matrix in Theorem 2.1. It may be interesting to reconsider A as a quadratic matrix under the same condition $A^2BA = A^2B$.

**Theorem 2.6.** Let $\alpha, \beta \in \mathbb{C}, \alpha \neq 0, \alpha \neq \beta$. Moreover, let $A$ and $B \in \mathbb{C}^n \setminus \{0\}$ be an $[\alpha, \beta]$-quadratic matrix and an idempotent matrix, respectively. Furthermore, let $A^2BA = A^2B$ and $K = aA + bB$ with $a, b \in \mathbb{C}$. Then $K$ is an idempotent matrix if and only if there exists a nonsingular matrix $V \in \mathbb{C}^n$ such that

$$A = V \begin{pmatrix} \alpha I_p & 0 \\ 0 & \beta I_{n-p} \end{pmatrix} V^{-1}$$

and $B$ satisfies one of the following cases.

(a) $\beta \neq 1$.

$$B = V \begin{pmatrix} I_q & 0 & 0 & 0 \\ -\frac{\alpha}{b} I_{p-q} & 0 & 0 \\ 0 & Z_2 & 1-\frac{\alpha}{b} I_r & 0 \\ Z_3 & 0 & 0 & 1-\frac{\alpha}{b} I_{n-p-r} \end{pmatrix} V^{-1},$$

being $Z_2 \in \mathbb{C}^{px(n-p)}$ and $Z_3 \in \mathbb{C}^{(n-p-r)nxq}$ arbitrary.

(b) $\beta = 1$ and $a\alpha = 1$.

$$B = V \begin{pmatrix} 0 & 0 & Y_2 \\ 0 & a^{-1} I_r & 0 \\ 0 & 0 & \frac{1}{a} I_{n-p-r} \end{pmatrix} V^{-1},$$

being $Y_2 \in \mathbb{C}^{px(n-p-r)}$ arbitrary.

**Proof.** From Theorem 2.1 in [9], we can write a quadratic matrix $A$ as

$$A = U \begin{pmatrix} \alpha I_p & 0 \\ 0 & \beta I_{n-p} \end{pmatrix} U^{-1},$$

where $p \in \{0, \ldots, n\}$ and $U \in \mathbb{C}^n$ is a nonsingular matrix. We can represent $B$ as $B = U \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} U^{-1}$, where $X \in \mathbb{C}^r$. In view of the hypotheses $A^2BA = A^2B$ and $a \neq 0$ we can write

$$\alpha X = X, \quad \beta Y = Y, \quad \alpha \beta Z = \beta^2 Z, \quad \beta^2 T = \beta T.$$

Now let us assume that $K$ is an idempotent matrix then we can write

$$\left(\alpha a I_p + bX\right)^2 + b^2 YZ = \alpha a I_p + bX, \quad ab (a + \beta) Y + b^2 (XY + YT) = bY,$$

$$ab (a + \beta) Z + b^2 (ZX + TZ) = bZ, \quad (a \beta I_{n-p} + bT)^2 = a \beta I_{n-p} + bT.$$

Depending on the scalars $\alpha$ and $\beta$ we have the following cases.

(i) Let $\beta \neq 1$.

From (34), it is seen that $Y = 0$. Reorganizing the equations of (35), it can be written

$$\left(\alpha a I_p + bX\right)^2 = \alpha a I_p + bX, \quad (a \beta I_{n-p} + bT)^2 = a \beta I_{n-p} + bT,$$

$$ab (a + \beta) Z + b^2 (ZX + TZ) = bZ.$$

(36)
It is clear that \( a\alpha I_p + bX \) and \( a\beta I_{n-p} + bT \) are idempotent matrices from the first and second equations in (36), respectively. Since an idempotent matrix is an \([1,0]-\)quadratic matrix, there exist \( q \in \{0, \ldots, p\} \), \( r \in \{0, \ldots, n-p\} \) and nonsingular matrices \( S_1 \in \mathbb{C}^p, S_2 \in \mathbb{C}^{(n-p)} \) such that

\[
X = S_1 \left( \frac{1-a\alpha}{r} I_q \ 0 \right) S_1^{-1} \quad \text{and} \quad T = S_2 \left( \frac{1-a\beta}{p} I_{r} \right) S_2^{-1}.
\]

Let us write \( Z \) as

\[
Z = S_2 \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & 0 \end{pmatrix} S_1^{-1},
\]

where \( Z_1 \in \mathbb{C}^{r \times q} \). Substituting (37) and (38) into the third equation in (36) it is obtained that

\[
\begin{pmatrix} bZ_1 & 0 \\ 0 & -bZ_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Then \( Z \) reduces to

\[
Z = S_2 \begin{pmatrix} 0 & Z_2 \\ Z_3 & 0 \end{pmatrix} S_1^{-1},
\]

where \( Z_2 \in \mathbb{C}^{(r-p) \times q} \) and \( Z_3 \in \mathbb{C}^{(n-p-r) \times q} \) are arbitrary matrices.

Let us define \( V := U(S_1 \oplus S_2) \). In view of (37) and (39), we obtain that

\[
A = U \begin{pmatrix} aI_p & 0 \\ 0 & \beta I_{n-p} \end{pmatrix} U^{-1}
= V \left( S_1^{-1} 0 \\ 0 S_2^{-1} \right) \begin{pmatrix} aI_p & 0 \\ 0 & \beta I_{n-p} \end{pmatrix} \left( S_1 0 \\ 0 S_2 \right) V^{-1}
= V \begin{pmatrix} aI_p & 0 \\ 0 & \beta I_{n-p} \end{pmatrix} V^{-1}
\]

and

\[
B = U \begin{pmatrix} \begin{pmatrix} 1-a\alpha & 0 \\ 0 & \frac{1-a\alpha}{r} I_{p-q} \end{pmatrix} S_1^{-1} & 0 \\ S_2 & S_2 \begin{pmatrix} 1-a\beta & 0 \\ 0 & \frac{1-a\beta}{p} I_{n-p-r} \end{pmatrix} S_2^{-1} \end{pmatrix} U^{-1}
= V \begin{pmatrix} \begin{pmatrix} 1-a\alpha & 0 \\ 0 & \frac{1-a\alpha}{r} I_{p-q} \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1-a\beta & 0 \\ 0 & \frac{1-a\beta}{p} I_{n-p-r} \end{pmatrix} \end{pmatrix} V^{-1}
\]

which establishes part (a).

(ii) Let \( \beta = 1 \).

From the first and third equations in (34), we obtain \( X = 0 \) and \( Z = 0 \), respectively. Reorganizing the equations of (35), it is obtained that

\[
(aa)^2 I_p = a\alpha I_p, \quad (aI_{n-p} + bT)^2 = aI_{n-p} + bT, \quad ab (a + 1) Y + b^2 YT = bY.
\]
It is obvious that $a\alpha = 1$ and $aI_{n-p} + bI$ is an idempotent matrix from the first and second equations in (40), respectively. Hence, there exist $r \in \{0, \ldots, n-p\}$ and a nonsingular matrix $S \in \mathbb{C}^{(n-r)}$ such that

$$T = S \begin{pmatrix} \frac{1}{\alpha}I_r & 0 \\ 0 & \frac{1}{\alpha}I_{n-r} \end{pmatrix} S^{-1}. \tag{41}$$

Let us write $Y$ as

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} S^{-1}, \tag{42}$$

where $Y_1 \in \mathbb{C}^{p \times r}$. Substituting (41) and (42) into the third equation in (40) yields

$$\left( \begin{array}{cc} b(a\alpha)Y_1 \\ b(a\alpha - 1)Y_2 \end{array} \right) S^{-1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $a\alpha = 1$, $Y$ reduces to

$$Y = \begin{pmatrix} 0 \\ Y_2 \end{pmatrix} S^{-1},$$

where $Y_2 \in \mathbb{C}^{p \times (n-p-r)}$ is an arbitrary matrix.

Hence, we can easily write

$$A = U \left( aI_p \oplus I_{n-p} \right) U^{-1} = U \left( I_p \oplus S \right) \left( aI_p \oplus I_{n-p} \right) \left( I_p \oplus S^{-1} \right) U^{-1}$$

and

$$B = U \begin{pmatrix} 0_p & 0 \\ 0 & S \begin{pmatrix} \frac{1}{\alpha}I_r & 0 \\ 0 & \frac{1}{\alpha}I_{n-r} \end{pmatrix} S^{-1} \end{pmatrix} U^{-1}$$

$$= U \left( I_p \oplus S \right) \begin{pmatrix} 0_p & 0 \\ 0 & \frac{1}{\alpha}I_r \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\alpha}I_{n-r} \end{pmatrix} \left( I_p \oplus S^{-1} \right) U^{-1}.$$

The necessity part of the proof is completed by defining $V$ as $V := U \left( I_p \oplus S \right)$. The sufficiency is obvious. $\Box$

**Example 2.7.** Let

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & -2 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \end{pmatrix}.$$

Let us find all ordered pair $(a, b)$ such that $K$ is idempotent, where $a, b \in \mathbb{C}^*$ and $K = aA + bB$. It is clear that $A$ is a $\{1, 2\}$-quadratic matrix and $A^2BA = A^2B$. Moreover,

$$K = \begin{pmatrix} 2a & 2a-b \quad 3a \\ 0 & 3a+b \\ 0 & b-2a \end{pmatrix} \begin{pmatrix} -a-b \\ 2a+2b \\ -a \end{pmatrix} + \begin{pmatrix} 4b-2a \quad 4b-2a \end{pmatrix} \begin{pmatrix} 2a \quad 2a-b \\ 3a \quad -a-b \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$K^2 = \begin{pmatrix} 4a^2 & 6a^2 - 2ab - 5b^2 & 9a^2 - 6b^2 \\ 0 & 7a^2 + 2ab - b^2 & 6a^2 - 4ab - 2b^2 \\ 0 & -6a^2 + 2ab + 5b^2 & -5a^2 + 6b^2 \end{pmatrix} \begin{pmatrix} 3a^2 - 4ab - b^2 \\ -3a^2 - 2ab + b^2 \\ 3a^2 + 2ab - b^2 \end{pmatrix}.$$
From the idempotency of $K$, it is obtained that $(a, b) \in \{(0, 0)\}$. Therefore, it is seen that there is no appropriate pair $(a, b)$ to imply that $K$ is idempotent. Note that, $V = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 \end{pmatrix}$ and it diagonalize $A$. Moreover, $B$ is not compatible with the part (a) of Theorem 2.6.

**Example 2.8.** Let $A = \begin{pmatrix} 5 & -4 & 0 \\ 8 & -7 & 0 \\ -4 & 4 & 1 \end{pmatrix}$. Let us find $a \in \mathbb{C}^*$ and all matrices $B \in \mathbb{C}^{3}$ such that $A^2BA = A^2B$ and $aA + B$ is idempotent. A diagonalized form of $A$ and the matrix $V$ that diagonalize it are

$$A = V \begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} V^{-1} \text{ and } V = \begin{pmatrix} -1 & 0 \\ -2 & 1 \\ 0 & 1 \end{pmatrix}.$$

Using the notations of Theorem 2.6, let us assume $\alpha = -3$ and $\beta = 1$. Then only part (b) of Theorem 2.6 can be implied, so we get $a = -1/3$, $p = 1$, and $r \in \{0, 1, 2\}$. Therefore, depending on the appearing and disappearing blocks of $V^{-1}BV$, the following possible cases of $V^{-1}BV$ are obtained for the values of $r = 0, 1, 2$, respectively.

$$\begin{pmatrix} 0 & c & d \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & e \\ 0 & 4/3 & 0 \\ 0 & 0 & 4/3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $c$, $d$, and $e$ are arbitrary complex numbers.

It can be interesting to consider the above theorem under the condition in Theorem 2.4.

**Theorem 2.9.** Let $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$, $\alpha \neq \beta$. Moreover, let $A$ and $B \in \mathbb{C}^{n} \setminus \{0\}$ be an $[\alpha, \beta]$–quadratic matrix and an arbitrary matrix, respectively. Furthermore, let $ABA = BA$ and $K = aA + bB$ with $a, b \in \mathbb{C}$. Then $K$ is an idempotent matrix if and only if there exists a nonsingular matrix $V \in \mathbb{C}^{n}$ such that

$$A = V \begin{pmatrix} \alpha I_p & 0 \\ 0 & \beta I_{n-p} \end{pmatrix} V^{-1}$$

and $B$ satisfies one of the following cases.

(a) $\beta \neq 1$.

$$B = V \begin{pmatrix} \frac{1-\alpha}{\beta} I_q & 0 & 0 & Y_2 \\ 0 & \frac{1-\alpha}{\beta} I_{p-q} & Y_3 & 0 \\ 0 & 0 & \frac{1-\beta}{\alpha} I_r & 0 \\ 0 & 0 & 0 & \frac{1-\beta}{\alpha} I_{n-p-r} \end{pmatrix} V^{-1},$$

being $Y_2 \in \mathbb{C}^{p \times (n-p-r)}$ and $Y_3 \in \mathbb{C}^{(p-r) \times r}$ arbitrary.

(b) $\beta = 1$ and $a\alpha = 1$.

$$B = V \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{a-1}{ab} I_r & 0 \\ Z_2 & 0 & \frac{1}{ab} I_{n-p-r} \end{pmatrix} V^{-1},$$

being $Z_2 \in \mathbb{C}^{(n-p-r) \times p}$ arbitrary.

The proof of Theorem 2.9 is similar to the proof of Theorem 2.6, so it is omitted. Under the different conditions from the previous results, the problem of idempotency of the linear combination of the form (1) of a quadratic matrix and an arbitrary matrix is reconsidered in the following three results.
Theorem 2.10. Let $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$, $\alpha \neq \beta$. Moreover, let $A$ and $B \in \mathbb{C}^{n \setminus \{0\}}$ be an $\{\alpha, \beta\}$-quadratic matrix and an arbitrary matrix, respectively. Furthermore, let $BAB = AB^2$ and $K = \alpha A + bB$ with $a, b \in \mathbb{C}$. Then $K$ is an idempotent matrix if and only if there exists a nonsingular matrix $V \in \mathbb{C}^n$ such that

$$A = V \begin{pmatrix} \alpha I_p & 0 \\ 0 & \beta I_{n-p} \end{pmatrix} V^{-1}$$

and $B$ satisfies one of the following cases.

(a) $\beta = 0$ and $\alpha = 1$.

$$B = V \begin{pmatrix} 0 & 0 & 0 & Y_2 \\ 0 & 1/\beta I_{p-q} & 0 & 0 \\ 0 & 0 & 1/\beta I_r & 0 \\ Z_3 & 0 & 0 & 0_{n-p-r} \end{pmatrix} V^{-1},$$

being $Y_2 \in \mathbb{C}^{q \times (n-p-r)}$, $Z_3 \in \mathbb{C}^{(n-p-r) \times q}$ arbitrary and $Y_2 Z_3 = 0$, $Z_3 Y_2 = 0$.

(b) $\beta = 0$ and $\alpha \neq 1$.

$$B = V \begin{pmatrix} 1/\alpha I_q & 0 & 0 & Y_2 \\ 0 & 1/\alpha I_{p-q} & 0 & 0 \\ 0 & 0 & 1/\beta I_r & 0 \\ 0 & 0 & 0 & 0_{n-p-r} \end{pmatrix} V^{-1},$$

being $Y_2 \in \mathbb{C}^{q \times (n-p-r)}$ arbitrary.

(c) $\beta \neq 0$ and $\alpha \beta = 1$.

$$B = V \begin{pmatrix} 1/\alpha I_q & 0 & 0 & 0 \\ 0 & 1/\alpha I_{p-q} & Y_3 & 0 \\ 0 & 0 & 1/\beta I_r & 0 \\ 0 & 0 & 0 & -1/\beta I_{n-p-r} \end{pmatrix} V^{-1},$$

being $Y_3 \in \mathbb{C}^{(p-q) \times r}$ arbitrary.

(d) $\beta \neq 0$ and $aa = 1$.

$$B = V \begin{pmatrix} 0_q & 0 & 0 & 0 \\ 0 & 1/\alpha I_{p-q} & 0 & 0 \\ 0 & 0 & 1/\beta I_r & 0 \\ Z_3 & 0 & 0 & -1/\beta I_{n-p-r} \end{pmatrix} V^{-1},$$

being $Z_3 \in \mathbb{C}^{(n-p-r) \times q}$ arbitrary.

(e) $\beta \neq 0$, $\alpha \beta \neq 1$, and $aa \neq 1$.

$$B = V \begin{pmatrix} 1/\alpha I_q & 0 & 0 & 0 \\ 0 & 1/\alpha I_{p-q} & 0 & 0 \\ 0 & 0 & 1/\beta I_r & 0 \\ 0 & 0 & 0 & -1/\beta I_{n-p-r} \end{pmatrix} V^{-1}.$$

Proof. We can write a quadratic matrix $A$ as

$$A = U \left( \alpha I_p \otimes \beta I_{n-p} \right) U^{-1},$$
where $p \in \{0, \ldots, n\}$ and $U \in \mathbb{C}^n$ is a nonsingular matrix. We can represent $B$ as $B = U \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} U^{-1}$, where $X \in \mathbb{C}^r$. Observe that under the hypotheses $BAB = AB^2$, $a \neq 0$, and $a \neq \beta$, one has

$$
YZ = 0, \ YT = 0, \ ZX = 0, \ ZY = 0.
$$

(49)

Let us assume that $K$ is an idempotent matrix then one can deduce that

$$
(\alpha a I_p + bX)^2 + b^2 YZ = \alpha a I_p + bX, \ \ ab (\alpha + \beta) Y + b^2 (XY + YT) = bY,
$$

$$
ab (\alpha + \beta) Z + b^2 (ZX + TZ) = bZ, \ b^2 ZY + (ab I_{n-p} + bT)^2 = ab I_{n-p} + bT.
$$

(50)

Considering (49) and (50), we get the following equalities

$$
(\alpha a I_p + bX)^2 = \alpha a I_p + bX, \ (ab I_{n-p} + bT)^2 = ab I_{n-p} + bT,
$$

$$
ab (\alpha + \beta) Y + b^2 XY = bY, \ ab (\alpha + \beta) Z + b^2 TZ = bZ.
$$

(51)

It is clear that $\alpha a I_p + bX$ and $ab I_{n-p} + bT$ are idempotent matrices from the first and second equations in (51), respectively. So, there exist $q \in \{0, \ldots, p\}$, $r \in \{0, \ldots, n - p\}$ and nonsingular matrices $S_1 \in \mathbb{C}^r$ and $S_2 \in \mathbb{C}^{(n-p)}$ such that

$$
X = S_1 \begin{pmatrix} \frac{1-a}{n} I_q & 0 \\ -\frac{a}{n} I_{p-q} \end{pmatrix} S_1^{-1},
$$

(52)

$$
T = S_2 \begin{pmatrix} \frac{1-a}{n} I_r & 0 \\ -\frac{a}{n} I_{n-p-r} \end{pmatrix} S_2^{-1}.
$$

(53)

Let $Y$ and $Z$ be written as

$$
Y = S_1 \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} S_2^{-1} \text{ and } Z = S_2 \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} S_1^{-1},
$$

(54)

where $Y_1 \in \mathbb{C}^{n \times r}$ and $Z_1 \in \mathbb{C}^{r \times n}$. Besides, defining $V := U (S_1 \oplus S_2)$, it follows that

$$
A = U \begin{pmatrix} aI_p & 0 \\ 0 & \beta I_{n-p} \end{pmatrix} U^{-1}
$$

$$
= U (S_1 \oplus S_2) \begin{pmatrix} S_1^{-1} & 0 \\ 0 & S_2^{-1} \end{pmatrix} \begin{pmatrix} aI_p & 0 \\ 0 & \beta I_{n-p} \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} (S_1^{-1} \oplus S_2^{-1}) U^{-1}
$$

$$
= V \begin{pmatrix} aI_p & 0 \\ 0 & \beta I_{n-p} \end{pmatrix} V^{-1}.
$$

Substituting (52), (54), and (53), (54) into the third and forth equations in (51) it is obtained that

$$
\begin{pmatrix} \alpha b Y_1 & \alpha b Y_2 \\ (\alpha b - 1) Y_3 & (\alpha b - 1) Y_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ \begin{pmatrix} a a Z_1 & a a Z_2 \\ (aa - 1) Z_3 & (aa - 1) Z_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
$$

(55)

Depending on the $\alpha b$ and $aa$, we have the following cases for $Y$ and $Z$.

(i) Let $\alpha b = 0$ and $aa = 1$.

It is clear that $Y_3, Y_4$ and $Z_1, Z_2$ are zero matrices from the equations in (55). So, $Y$ and $Z$ reduce to

$$
Y = S_1 \begin{pmatrix} Y_1 & Y_2 \\ 0 & 0 \end{pmatrix} S_2^{-1} \text{ and } Z = S_2 \begin{pmatrix} 0 & 0 \\ Z_3 & Z_4 \end{pmatrix} S_1^{-1}.
$$
Substituting \( X, Y, Z, \) and \( T \) into (49), \( Y \) and \( Z \) are obtained as

\[
Y = \begin{pmatrix} 0 & Y_2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Z = S_2 \begin{pmatrix} 0 & 0 \\ Z_3 & 0 \end{pmatrix} S_1^{-1},
\]

where \( Y_2 \in \mathbb{C}^{q \times (p-r)} \), \( Z_3 \in \mathbb{C}^{(n-p-r) \times q} \) are arbitrary matrices and \( Y_2 Z_3 = 0, \) \( Z_3 Y_2 = 0 \). Therefore, \( B \) we get as

\[
B = U \begin{pmatrix} S_1 \left( \begin{array}{cc} 0 & 0 \\ 0 & \frac{-1}{p} I_{p-q} \end{array} \right) S_1^{-1} & S_1 \left( \begin{array}{cc} 0 & Y_2 \\ 0 & 0 \end{array} \right) S_2^{-1} \\ S_2 \left( \begin{array}{cc} Z_3 & 0 \\ 0 & 0 \end{array} \right) S_1^{-1} & S_2 \left( \begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array} \right) S_2^{-1} \end{pmatrix} U^{-1}
\]

\[
= U \left( S_1 \oplus S_2 \right) \begin{pmatrix} 0 & 0 & 0 & Y_2 \\ 0 & \frac{-1}{p} I_{p-q} & 0 & 0 \\ Z_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0_{n-p-r} \end{pmatrix} \left( S_1^{-1} \oplus S_2^{-1} \right) U^{-1}.
\]

which establishes part (a).

(ii) Let \( a \beta = 0 \) and \( a \alpha \neq 1 \).

From the equations in (55), it is clear that \( Y_3, Y_4, \) and \( Z \) are zero matrices. Thus, \( Y \) is as in (56) and then

\[
B = U \left( S_1 \oplus S_2 \right) \begin{pmatrix} \frac{1-a \alpha}{b} I_q & 0 & 0 & Y_2 \\ 0 & \frac{-a \alpha}{p} I_{p-q} & 0 & 0 \\ 0 & 0 & I_r & 0 \\ Z_3 & 0 & 0 & 0_{n-p-r} \end{pmatrix} \left( S_1^{-1} \oplus S_2^{-1} \right) U^{-1},
\]

where \( Y_2 \in \mathbb{C}^{q \times (p-r)} \) is an arbitrary matrix. So, it is completed part (b).

(iii) Let \( a \beta = 1 \) and \( a \alpha \neq 1 \).

It is obvious that \( Y_1, Y_2, \) and \( Z \) are zero matrices from the equations in (55). So, \( Y \) reduces to

\[
Y = S_1 \begin{pmatrix} 0 & 0 \\ Y_3 & Y_4 \end{pmatrix} S_2^{-1}.
\]

Using (53) and (57) in the second equation of (49), we get the equality \( \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{p} Y_4 \end{pmatrix} = 0_{px(n-p)} \). Therefore

\[
Y = S_1 \begin{pmatrix} 0 & 0 \\ Y_3 & 0 \end{pmatrix} S_2^{-1},
\]

where \( Y_3 \in \mathbb{C}^{(p-q) \times r} \) is an arbitrary matrix and
Let \( a \) and \( \beta \) be arbitrary matrices, respectively. Furthermore, let 
\[
B = U \left( S_{1} \begin{pmatrix} \frac{1}{a-b} I_{q} & 0 & 0 \\ 0 & -\frac{1}{b} I_{p-q} \end{pmatrix} S_{1}^{-1} \right) S_{2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} S_{2}^{-1} U^{-1} 
\]
\[
= U \left( S_{1} \oplus S_{2} \right) \begin{pmatrix} \frac{1}{a-b} I_{q} & 0 & 0 \\ 0 & -\frac{1}{b} I_{p-q} & Y_{3} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left( S_{1}^{-1} \oplus S_{2}^{-1} \right) U^{-1}. 
\]
which completes part (c).

(iv) Let \( a\beta \neq 0, a\beta \neq 1, \) and \( a\alpha = 1. \)
From the equations in (55), it is clear that \( Y = 0 \) and the form of \( Z \) is as in (56). Hence,
\[
B = U \left( S_{1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{b} I_{p-q} \end{pmatrix} S_{1}^{-1} \right) \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} S_{2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} S_{2}^{-1} U^{-1} 
\]
\[
= U \left( S_{1} \oplus S_{2} \right) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{b} I_{p-q} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \left( S_{1}^{-1} \oplus S_{2}^{-1} \right) U^{-1}, 
\]
where \( Z_{3} \in C^{(n-p)\times q} \) is an arbitrary matrix. So, the part (d) of the proof is completed.

(v) Let \( a\beta \neq 0, a\beta \neq 1, \) and \( a\alpha \neq 1. \)
From the equations in (55), it is easily seen that \( Y = 0 \) and \( Z = 0 \) thus,
\[
B = U \left( S_{1} \begin{pmatrix} \frac{1}{a-b} I_{q} & 0 & 0 \\ 0 & -\frac{1}{b} I_{p-q} \end{pmatrix} S_{1}^{-1} \right) \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} S_{2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} S_{2}^{-1} U^{-1} 
\]
\[
= U \left( S_{1} \oplus S_{2} \right) \begin{pmatrix} \frac{1}{a-b} I_{q} & 0 & 0 \\ 0 & -\frac{1}{b} I_{p-q} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \left( S_{1}^{-1} \oplus S_{2}^{-1} \right) U^{-1}. 
\]
Therefore, the part of the necessity of the proof is completed.

Now it is evident that if \( A \) is represented as in (43) and \( B \) is represented as in (44), (45), (46), (47) or (48) and the scalars \( a\alpha, a\beta \) satisfy corresponding conditions, then \( K^{2} = K. \)

Theorem 2.10 is given under the condition \( BAB = AB^{2} \). Premultiplying this condition by \( A \) leads to \( A^{2}B^{2} = (AB)^{2} \). A similar result can be given below under this new condition.

**Theorem 2.11.** Let \( a, \beta \in C^{*} \) with \( a \neq \beta. \) Moreover, let \( A \) and \( B \) be \( C^{n} \setminus \{0\} \) be an \( \{a, \beta\} \) – quadratic matrix and an arbitrary matrix, respectively. Furthermore, let \( A^{2}B^{2} = (AB)^{2} \) and \( K = aA + bB \) with \( a, b \in C^{*}. \) Then \( K \) is an idempotent matrix if and only if there exists a nonsingular matrix \( V \in C^{n} \) such that
\[
A = V \begin{pmatrix} aI_{p} & 0 \\ 0 & \beta I_{n-p} \end{pmatrix} V^{-1}
\]
and \( B \) satisfies one of the following cases.
(a) $a \beta = 1$.
\[
B = V \begin{pmatrix}
\frac{1-a^2}{b} I_q & 0 & 0 & 0 \\
0 & \frac{-a^2}{b} I_{p-q} & Y_3 & 0 \\
0 & 0 & I_r & 0 \\
0 & 0 & 0 & \frac{1}{b} I_{n-p-r}
\end{pmatrix} V^{-1},
\]
being $Y_3 \in \mathbb{C}^{(p-q) \times r}$ arbitrary.

(b) $a \alpha = 1$.
\[
B = V \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{-1}{b} I_{p-q} & 0 & 0 \\
0 & 0 & \frac{1-a^2}{b} I_r & 0 \\
Z_3 & 0 & 0 & \frac{-a}{b} I_{n-p-r}
\end{pmatrix} V^{-1},
\]
being $Z_3 \in \mathbb{C}^{(n-p-r) \times q}$ arbitrary.

(c) $a \alpha \neq 1$ and $a \beta \neq 1$,
\[
B = V \begin{pmatrix}
\frac{1-a^2}{b} I_q & 0 & 0 & 0 \\
0 & \frac{-a^2}{b} I_{p-q} & 0 & 0 \\
0 & 0 & \frac{1-a^2}{b} I_r & 0 \\
0 & 0 & 0 & \frac{-a}{b} I_{n-p-r}
\end{pmatrix} V^{-1}.
\]

The proof of this theorem is omitted since it is very similar to proof of Theorem 2.10.

**Remark 2.12.** Note that the matrices $B$ given in the last parts of Theorem 2.10 and 2.11 commute with the corresponding matrices $A$ while there is no such necessity in other results.

Lastly, let us give the following theorem.

**Theorem 2.13.** Let $\alpha, \beta, \in \mathbb{C}$, $\alpha \neq 0, \alpha \neq \beta$, and $(\alpha, \beta) \notin \{(-1, 1), (1, -1)\}$. Moreover, let $A$ and $B \in \mathbb{C}^{n} \setminus \{0\}$ be an $(\alpha, \beta)$-quadratic matrix and an arbitrary matrix, respectively. Furthermore, let $A^2 B A = B A$ and $K = a A + b B$ with $a, b \in \mathbb{C}$. Then $K$ is an idempotent matrix if and only if there exists a nonsingular matrix $V \in \mathbb{C}^n$ such that
\[
A = V \begin{pmatrix}
\alpha I_p & 0 \\
0 & \beta I_{n-p}
\end{pmatrix} V^{-1}
\]
and $B$ satisfies one of the following cases.

(a) $\beta = 0$.
\[
B = V \begin{pmatrix}
\frac{1-a^2}{b} I_q & 0 & 0 & Y_2 \\
0 & \frac{-a^2}{b} I_{p-q} & Y_3 & 0 \\
0 & 0 & I_r & 0 \\
0 & 0 & 0 & 0_{n-p-r}
\end{pmatrix} V^{-1},
\]
being $Y_2 \in \mathbb{C}^{p \times (n-p-r)}$ and $Y_3 \in \mathbb{C}^{(p-q) \times r}$ arbitrary.

(b) $\beta^2 \neq 1, \beta \neq 0$, and $a \beta = 1$.
\[
B = V \begin{pmatrix}
\frac{1-a^2}{b} I_q & 0 & 0 & W \\
0 & \frac{-a^2}{b} I_{p-q} & I_r & 0 \\
0 & 0 & I_{n-p} & 0
\end{pmatrix} V^{-1},
\]
being $W \in \mathbb{C}^{(p-q) \times (n-p)}$ arbitrary.
\( \beta^2 = 1 \) and \( \alpha \beta = 1 \).

\[
B = V \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 - q \beta & 0 \\
Z & 0 & - n p \beta \mathbf{I}_{n-p}
\end{pmatrix} V^{-1}, \tag{61}
\]

being \( Z_2 \in \mathbb{C}^{(n-p) \times p} \) arbitrary.

**Proof.** Let us write a quadratic matrix \( A \) as

\[
A = U \left( a \mathbf{I}_p \oplus \beta \mathbf{I}_{n-p} \right) U^{-1},
\]

where \( p \in \{0, \ldots, n\} \) and \( U \in \mathbb{C}^n \) is a nonsingular matrix. We can represent \( B \) as \( B = U \left( \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} \right) U^{-1} \) where \( X \in \mathbb{C}^p \). In view of the hypotheses \( A^2 \mathbf{B}A = \mathbf{B}A \) and \( \alpha \neq 0 \) we can write

\[
a^2 \mathbf{X} = \mathbf{X}, \quad \alpha^2 \beta \mathbf{Y} = \beta \mathbf{Y}, \quad \beta^2 \mathbf{Z} = \mathbf{Z}, \quad \beta^3 \mathbf{T} = \beta \mathbf{T}. \tag{62}
\]

Let us assume that \( K \) is an idempotent matrix then it follows that

\[
\begin{align*}
\left( a a \mathbf{I}_p + b X \right)^2 &= a a \mathbf{I}_p + b X, \\
abla \left( a \beta \mathbf{I}_{n-p} + b T \right)^2 &= a \beta \mathbf{I}_{n-p} + b T, \\
abla \left( a \alpha + b \right) \mathbf{Y} + b^2 \left( \mathbf{X} \mathbf{Y} + \mathbf{Y} \mathbf{T} \right) &= b \mathbf{Y}, \\
abla \left( a \alpha + b \right) \mathbf{Z} + b^2 \left( \mathbf{X} \mathbf{Z} + \mathbf{Z} \mathbf{T} \right) &= b \mathbf{Z}.
\end{align*} \tag{63}
\]

The proof can be split into following cases, depending on the scalar \( \beta \).

(i) Let \( \beta^2 \neq 1 \).

From (62), it is seen that \( Z = 0 \). Reorganizing the equations of (63), it can be written

\[
\begin{align*}
\left( a a \mathbf{I}_p + b X \right)^2 &= a a \mathbf{I}_p + b X, \\
abla \left( a \beta \mathbf{I}_{n-p} + b T \right)^2 &= a \beta \mathbf{I}_{n-p} + b T, \\
abla \left( a \alpha + b \right) \mathbf{Y} + b^2 \left( \mathbf{X} \mathbf{Y} + \mathbf{Y} \mathbf{T} \right) &= b \mathbf{Y}.
\end{align*} \tag{64}
\]

It is clear that \( a a \mathbf{I}_p + b X \) and \( a \beta \mathbf{I}_{n-p} + b T \) are idempotent matrices from the first and second equations in (64), respectively. Then, there exist \( q \in \{0, \ldots, p\}, r \in \{0, \ldots, n - p\} \) and nonsingular matrices \( S_1 \in \mathbb{C}^p, S_2 \in \mathbb{C}^{(n-p)} \) such that

\[
X = S_1 \left( \begin{pmatrix} 1 - q \beta & 0 \\ - q \beta \mathbf{I}_{p-q} \end{pmatrix} \right) S_1^{-1} \quad \text{and} \quad T = S_2 \left( \begin{pmatrix} 1 - q \beta & 0 \\ - q \beta \mathbf{I}_{n-p} \end{pmatrix} \right) S_2^{-1}. \tag{65}
\]

Let us write \( Y \) as

\[
Y = S_1 \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} S_2^{-1} \tag{66},
\]

where \( Y_1 \in \mathbb{C}^{p \times r} \). Substituting (65) and (66) into the third equation in (64) yields

\[
\begin{pmatrix} b Y_1 \\ 0 \\ - b Y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

Then \( Y \) reduces to

\[
Y = S_1 \begin{pmatrix} 0 \\ Y_3 \\ 0 \end{pmatrix} S_2^{-1} \tag{67},
\]

where \( Y_2 \in \mathbb{C}^{(p \times (n-p))} \) and \( Y_3 \in \mathbb{C}^{(p \times (p-n))} \) are arbitrary matrices.

Let us define \( V := U \left( S_1 \oplus S_2 \right) \). In view of (65) and (67) we obtain that
\[
A = U \begin{pmatrix} \alpha I_p & 0 \\ 0 & \beta I_{n-p} \end{pmatrix} U^{-1} \\
= V \begin{pmatrix} S_1^{-1} & 0 \\ 0 & S_2^{-1} \end{pmatrix} \begin{pmatrix} \alpha I_p & 0 \\ 0 & \beta I_{n-p} \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} V^{-1} \\
= V \begin{pmatrix} \alpha I_p & 0 \\ 0 & \beta I_{n-p} \end{pmatrix} V^{-1}
\]

and
\[
B = U \begin{pmatrix} S_1 \left( \frac{1-\alpha a}{b} I_q - \frac{-\alpha p}{b} I_{p-q} \right) S_3^{-1} & S_1 \left( 0 Y_2 \right) S_3^{-1} \\ 0 & S_2 \left( \frac{1-\alpha a}{b} I_r - \frac{-\alpha p}{b} I_{r-p} \right) S_2^{-1} \end{pmatrix} U^{-1} \\
= V \begin{pmatrix} \frac{1-\alpha a}{b} I_q & 0 \\ 0 & \frac{1-\alpha a}{b} I_{p-q} \\ 0 & Y_2 \\ 0 & 0 \end{pmatrix} \frac{1-\alpha a}{b} I_r S_2^{-1} + \frac{-\alpha p}{b} I_{r-p} \right) V^{-1}
\]

which yields part (a) for \( \beta = 0 \) and yields part (b) for \( \beta \neq 0 \) (then from (62), \( T = 0 \)).

(ii) Let \( \beta^2 = 1 \).

From the first and second equations in (62) and considering hypotheses \((\alpha, \beta) \notin \{(-1, 1), (1, -1)\}\) and \(\alpha \neq \beta\), it is obvious that \(X = 0\) and \(Y = 0\). Reorganizing the equations of (63), it can be written
\[
(aa)^2 I_p = aa I_p, \quad (a \beta I_{n-p} + b T)^2 = a \beta I_{n-p} + b T, \quad a b (\alpha + \beta) Z + b^2 T Z = b Z.
\]

It is explicit that \(aa = 1\) and \(a \beta I_{n-p} + b T\) is an idempotent matrix from the first and second equations in (68).

So, there exist \(r \in \{0, \ldots, n - p\}\) and a nonsingular matrix \(S \in C^{(n-p)}\) such that
\[
T = S \begin{pmatrix} \frac{1-aa}{b} I_q & 0 \\ 0 & \frac{1-aa}{b} I_{n-p} \end{pmatrix} S^{-1}.
\]

Let us write \(Z\) as
\[
Z = S \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix},
\]

where \(Z_1 \in C^{(n-p)}\). Substituting (69) and (70) into the third equation in (68), it is obtained that
\[
\begin{pmatrix} (aa) Z_1 \\ (aa - 1) Z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

Since \(aa = 1\), \(Z\) turns into \(Z = S \begin{pmatrix} 0 \\ Z_2 \end{pmatrix}\), where \(Z_2 \in C^{(n-p-r)\times p}\) is an arbitrary matrix.

Hence, we can easily write
\[
A = U \left( (\alpha I_p) \oplus (\beta I_{n-p}) \right) U^{-1} = U \left( (I_p) \oplus S \right) \left( (\alpha I_p) \oplus (\beta I_{n-p}) \right) \left( (I_p) \oplus S^{-1} \right) U^{-1}
\]

and
\[
B = U \begin{pmatrix} 0 & 0 \\ S \begin{pmatrix} 0 \\ Z_2 \end{pmatrix} & S \begin{pmatrix} \frac{1-aa}{b} I_q \\ 0 \end{pmatrix} \end{pmatrix} \frac{1-aa}{b} I_{n-p} S^{-1} U^{-1}
\]
\[ \begin{pmatrix} 0_p & 0 \\ 0 & \alpha \beta \delta \iota \kappa \lambda \\ Z_2 & 0 \end{pmatrix} (I_p \oplus S^{-1}) \begin{pmatrix} 0_p & 0 \\ 0 & \alpha \beta \delta \iota \kappa \lambda \\ Z_2 & 0 \end{pmatrix}^{-1}. \]

The necessity part of the proof is completed by defining \( V := U (I_p \oplus S) \).

Now, it is evident that if \( A \) is represented as in (58) and \( B \) is represented as in (59), (60) or (61) and the scalars \( \alpha, \beta \) satisfy corresponding conditions, then \( K^2 = K \). \( \square \)

**Example 2.14.** Let

\[ A = \begin{pmatrix} 5 & 0 & 6 & -3 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & -1 & 0 \\ 6 & 0 & 6 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & -4 \end{pmatrix} \]

and let us find all ordered pair \((a, b)\) such that \( K = aA + bB \) is idempotent, where \( a, b \in \mathbb{C}^* \). It is clear that \( A \) is a \([-1, 2]\) - quadratic matrix and \( A^2 B A = BA \). Moreover,

\[ K = \begin{pmatrix} 5a + 2b & 0 & 6a + 2b & -3a - 2b \\ 0 & 2a & -3a & 0 \\ 0 & 0 & -a & 0 \\ 6a + 4b & 0 & 6a + 4b & -4a - 4b \end{pmatrix} \]

and

\[ K^2 = \begin{pmatrix} 7a^2 - 4ab - 4b^2 & 0 & 6a^2 - 4ab - 4b^2 & -3a^2 + 4ab + 4b^2 \\ 0 & 4a^2 & -3a^2 & 0 \\ 0 & 0 & a^2 & 0 \\ 6a^2 - 8ab - 8b^2 & 0 & 6a^2 - 8ab - 8b^2 & -2a^2 + 8ab + 8b^2 \end{pmatrix}. \]

From the idempotency of \( K \), it is obtained that \((a, b) \in \{(0, 0), (0, -1/2)\} \). Although \( A \) is a \([-1, 2]\) - quadratic matrix, the form of the matrix \( B \) is not compatible with the part (a) of Theorem 2.13. Therefore, it is seen that there is no appropriate pair \((a, b)\) to imply that \( K \) is idempotent.

**Example 2.15.** Let \( A \) be as in the previous example and let us find \( a \in \mathbb{C}^* \) and all matrices \( B \in \mathbb{C}^4 \) such that \( A^2 B A = BA \) and \( aA - B \) is idempotent. A diagonalized form of \( A \) and the matrix \( V \) that diagonalize it are

\[ A = V \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} V^{-1} \quad \text{and} \quad V = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -1 \end{pmatrix}, \]

compatible with (58). Using the notations of Theorem 2.13, we can assume \( a = -1, b = 2 \). Then only part (b) of Theorem 2.13 can be implied, so we get \( a = 1/\beta = 1/2, n = 4, p = 2, \) and \( q \in \{0, 1, 2\} \). Therefore depending on the appearing and disappearing blocks of \( V^{-1} B V \), it can be written the following possible cases:

| POSSIBILITIES OF \( V^{-1} B V \) | \( q = 0 \) | \( q = 1 \) | \( q = 2 \) |
|-----------------------------|----------------|----------------|----------------|
| \( \begin{pmatrix} -1/2 & 0 & e & f \\ 0 & -1/2 & g & h \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \) | \( \begin{pmatrix} -3/2 & 0 & 0 & 0 \\ 0 & -1/2 & m & s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \) | \( \begin{pmatrix} -3/2 & 0 & 0 & 0 \\ 0 & -3/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \) |

where \( e, f, g, h, m, \) and \( s \) are arbitrary complex numbers.

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