EQUIVARIANCE, BRST AND SUPERSPACE

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The structure of equivariant cohomology in non-abelian localization formulas and topological field theories is discussed. Equivariance is formulated in terms of a nilpotent BRST symmetry, and another nilpotent operator which restricts the BRST cohomology onto the equivariant, or basic sector. A superfield formulation is presented and connections to reducible (BFV) quantization of topological Yang-Mills theory are discussed.

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1. Introduction

In the present paper we study equivariant (i.e. basic) cohomology, a concept which is acquiring increasing attention in modern theoretical physics. This interest is mainly a consequence of the relevance that equivariant cohomology has to various localization formulas, quantum integrability and topological field theories.

Here we shall be mostly interested in equivariant cohomology relevant to localiza-

The original localization formula by Duistermaat and Heckman (DH) \[1\] concerns exponential integrals over symplectic manifolds, i.e. classical partition functions. The theorem states that if a Hamiltonian \(H\) on a \(2n\)-dimensional compact symplectic manifold with symplectic two-form \(\omega\) generates the Poisson action of a torus, the stationary phase approximation is exact

\[
\int \omega^n \exp\{-i\phi H\} = 1
\]

In \[2, 3\] it was noted that the underlying structure in the DH formula was that of equivariant cohomology with respect to the torus acting on the manifold: The integrand depends only on the equivariant extension \(\omega + \phi H\) of the symplectic two-form, and the exactness of stationary phase approximation is a consequence of an equivariant version of Stokes theorem.

Infinite dimensional generalizations of DH were introduced in \[4, 5\] and loop space extensions were considered in \[6\]. In particular, in these papers localization proofs of index theorems for Dirac operators and their equivariant extensions with respect to Lie groups were related to equivariant cohomology. The formalism was also shown to be relevant in a geometric formulation of Poincare supersymmetric field theories \[7\].

A generalization to include a non-abelian group action was presented by Witten in \[8\], who applied it to two dimensional Yang-Mills theory. In \[9\] this approach was combined with \[6\], to localize path integrals with Hamiltonians that are \textit{a priori} arbitrary functions of generators of circle actions. In particular, the equivariant exterior derivative was written as a sum of a nilpotent part related to the group action, and
a kinetic part related to a model independent loop space circle action.

For a long time, equivariant cohomology has also played a central rôle in topological field theories [10]. Indeed, topological field theories share many aspects with the generalized Duistermaat-Heckman systems; Most notably their path integrals can be evaluated exactly using a localization method. Topological field theories also possess a nilpotent BRST-like symmetry, and they can be viewed as BRST gauge fixings of underlying trivial theories.

In [11] it was first realized that the structure determined by the (nilpotent) BRST-operator in four dimensional topological Yang-Mills theory [12] is that of equivariant cohomology. The argument was strengthened in [13], where the BRST structure was identified with basic cohomology, and in [14] where the Weil algebra structure of topological Yang-Mills theory was clarified.

The approach of [13] resolves elegantly the problem of nontriviality of observables in topological Yang-Mills theory. In particular, the structure of basic cohomology introduced there shows that in addition of computing the BRST cohomology it is necessary to restrict onto the basic forms. In [15] the approach of [13] was subsequently related to the mathematical theory of BRST [16].

In the following we develop the BRST description of equivariant cohomology from the point of view of localization formulas. We are particularly interested in exposing the intimate relationship between the equivariant structure underlying localization formulas and topological Yang-Mills theory, as formulated in [17, 18]. For this, we first briefly review symplectic action of a Lie group in Section 2. In section 3, we present the different models of equivariant cohomology and in section 4, we consider the construction of equivariant cohomology operators and in particular equivariant extensions of the symplectic two-form in the framework of localization formulas. In Section 5, we introduce a superfield formalism and in section 6, we discuss the connection of the construction with the Batalin-Fradkin-Vilkovisky approach to reducible constrained systems.
2. Symplectic actions and Lie groups

In the present paper we are interested in the equivariant cohomology which is related to the action of a connected Lie group $G$ as local diffeomorphisms on a symplectic manifold $M$. The dimension of $M$ is $2n$, and local coordinates on $M$ are denoted by $z^k, k = 1 \ldots 2n$.

The $G$-action on $M$ is generated by vector fields $\mathcal{X}_a, a = 1 \ldots m$ that realize the commutation relations of $G$,

\[
[\mathcal{X}_a, \mathcal{X}_b] = f^{abc} \mathcal{X}_c
\]

with $f^{abc}$ the structure constants of the Lie algebra $\mathfrak{g}$ of $G$.

With $\mathcal{X}$ a generic vector field on $M$, we denote contraction along $\mathcal{X}$ by $i_\mathcal{X}$. In particular, the basis of contractions corresponding to the Lie algebra generators $\{\mathcal{X}_a\}$ is denoted by $i_{\mathcal{X}_a} \equiv i_a$. The pertinent Lie-derivatives

\[
\mathcal{L}_a = di_a + i_a d
\]

with $d$ the exterior derivative on the exterior algebra $\Omega(M)$ of $M$, then generate the $G$-action on $\Omega(M)$,

\[
[\mathcal{L}_a, \mathcal{L}_b] = f^{abc} \mathcal{L}_c .
\]

In addition we have the Lie-derivative action on the contraction:

\[
[i_a, \mathcal{L}_b] = f^{abc} i_c .
\]

The symplectic two-form

\[
\omega = \frac{1}{2} \omega_{kl} dz^k \wedge dz^l
\]

on $M$ is closed and nondegenerate. Locally,

\[
\omega = d\vartheta .
\]

where the one-form $\vartheta$ is the symplectic potential. We shall assume that the action of $G$ is symplectic so that it preserves the symplectic structure,

\[
\mathcal{L}_a \omega = di_a \omega = 0 .
\]
If the one-forms $i_a \omega$ are exact (for this the triviality of $H^1(M, \mathbb{R})$ is sufficient), we can then introduce the momentum map $H : M \rightarrow g^*$ where $g^*$ is the dual Lie algebra. When evaluated on a vector field $\mathcal{X}$, the momentum map $H$ yields the corresponding Hamiltonian $H_\mathcal{X}(z)$ by

$$i_{\mathcal{X}} \omega = -dH_\mathcal{X}$$

or in local coordinates,

$$\mathcal{X} = \omega^{kl} \partial_k H_\mathcal{X} \partial_l .$$

For the Lie algebra $g$ this yields a one-to-one correspondence between the vector fields $\mathcal{X}_a$ (and corresponding Lie-derivatives $\mathcal{L}_a$) and certain functions $H_a$ on $M$, the components of the momentum map

$$H = \phi^a H_a ,$$

where $\{\phi^a\}$ is a (symmetric) basis of the dual Lie algebra $g^*$.

The Poisson bracket of the Hamiltonians $H_a$ is defined by

$$\{H_a, H_b\} = \omega(\mathcal{X}_a, \mathcal{X}_b) = \mathcal{L}_a H_b .$$

From the Jacobi identity for $g$ we then get the homomorphism

$$\mathcal{X}_{\{H_a, H_b\}} = [\mathcal{X}_a, \mathcal{X}_b] .$$

However, the inverse is not necessarily true: The Hamiltonian function which corresponds to the commutator of two group generators may differ from the Poisson bracket of the pertinent Hamiltonian functions,

$$\{H_a, H_b\} = f^{abc} H_c + \kappa_{ab} .$$

Here $\kappa_{ab}$ is the 2-cocycle in the Lie-algebra cohomology of $g$, and the appearance of a cocycle can be related to the possible noninvariance of the symplectic potential under $G$: From (2) it follows that

$$i_a \vartheta = H_a + h_a$$

with some functions $h_a$ on $M$, and the definition of the Poisson bracket implies the cocycle in (5) is given by

$$\kappa_{ab} = f^{abc} h_c - \mathcal{L}_a h_b + \mathcal{L}_b h_a .$$
Thus, only when $\kappa_{ab} = 0$ for all $a, b$ does the $G$-action of the vector fields $X_a$ lift isomorphically to the Poisson action of the corresponding Hamiltonians $H_a$ on $M$.

3. Models for equivariant cohomology

We are interested in the equivariant cohomology $H^*_G(M)$ associated with the symplectic action of the Lie group $G$ on the manifold $M$. For this we first note that $H^*_G(M)$ is essentially the deRham cohomology of $M$ mod($G$). If the action of $G$ is free i.e. the only element of $G$ which acts trivially is the unit element, the quotient space $M/G$ is well defined and the $G$-equivariant cohomology of $M$ coincides with the ordinary cohomology of $M$ mod($G$), that is $H^*_G(M) = H^*(M/G)$.

For the non-free action of a compact group $G$ there exists three different approaches to model $H^*_G(M)$ using differential forms on $M$ and polynomial functions and forms on the Lie algebra $g$ of $G$. The two classical models are the Cartan and Weil ones, described e.g. in [3, 20]. These two are interpolated by the BRST model, which is relevant to the BRST structure of topological field theories. The BRST model is discussed e.g. in [13] and [15] and the interrelations between the different models are clarified in [13].

**The Cartan Model:** The simplest example of a group action on the symplectic manifold $M$ is that of the action of the circle $G = S^1 = U(1)$, determined by a vector field $X$ as the generator of the Lie-algebra $u(1)$ of $U(1)$. In order to describe the corresponding equivariant cohomology of $M$ we introduce the following equivariant exterior derivative operator on $M$ [3, 21]

$$s = d - \phi \ i_X .$$

(8)

where the sign is chosen for later convenience. The factor $\phi$ is a real parameter, and the operator $s$ acts on the whole deRham complex $\Omega(M)$ of differential forms on $M$.

The square of $s$ is the Lie-derivative with respect to $X$,

$$s^2 = -\phi (di_X + i_X d) = -\phi L_X .$$

(9)

Thus on the subcomplex $\Omega_{U(1)}$ of $U(1)$-invariant exterior forms, $s$ is nilpotent and
defines an exterior differential operator. The cohomology of $s$ on this subcomplex defines the equivariant cohomology $H^*_U(1)$ of the manifold $M$.

We are especially interested in the equivariant extension of the symplectic two-form $\omega$. With $H$ the momentum map of the circle action on $M$, i.e. the Hamiltonian corresponding to $X$ we get from the definition (3),

\[(d - \phi i_X)(\omega - \phi H) = 0 , \quad (10)\]

which identifies $\omega - \phi H$ as the equivariant extension of the symplectic two-form. Of particular interest are related integrals over $M$ that can be evaluated by localization methods based on (11), such as the DH integration formula

\[\int \omega^n \exp\{-i\phi H\} = (-i)^n n! \int \exp\{i(\omega - \phi H)\} = \frac{1}{\phi^n} \sum_{dH=0} \exp(-i\phi H) \sqrt{\text{det}||\partial_{ij}H||} \quad (11)\]

From (11) we see that the integrand is an equivariantly closed form, and localization to the critical points of $H$ follows from changing the representative of (11) in the equivariant cohomology class.

In order to generalize for a non-abelian group $G$, the parameter $\phi$ must first be properly interpreted. For this we identify it as a generator of the algebra of polynomials on $u(1)$, i.e. as a basis element of the symmetric algebra $S(u(1)^*)$ over the dual of the Lie-algebra of $U(1)$. The operator (8) then acts on the complex $S(u(1)^*) \otimes \Omega(M)$, and from (8) we conclude that on the $U(1)$-invariant subcomplex $(S(u(1)^*) \otimes \Omega(M))^{U(1)}$ the action of $s$ is nilpotent, and the equivariant cohomology is the $s$-cohomology of $(S(u(1)^*) \otimes \Omega(M))^{U(1)}$. As shown in (8), the operations of evaluating $\phi$ and formation of cohomology commute for abelian group actions, so that the results coincide independently of the interpretation of $\phi$. This model for equivariant cohomology is called the abelian Cartan model.

For a free $U(1)$-action we have

\[(S(u(1)^*) \otimes \Omega(M))^{U(1)} = S(u(1)^*) \otimes \Omega(M mod U(1)). \quad (11)\]

From this we see that the multipliers $\phi$ play in this case no cohomological role, and the equivariant cohomology really restricts to the cohomology of the quotient space $M mod U(1)$. 

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In order to generalize the Cartan model to non-abelian compact Lie groups $G$, we consider the algebra $S(g^*)$ of polynomials on $g$ as the symmetric algebra on the dual $g^*$ of $g$. With $m$ the dimension of $g$, the basis of $S(g^*)$ which is dual to some basis $\{X_a\}$ of $g$ is $\phi^a \ (a = 1, \ldots m)$.

In analogy with (8) we introduce the nonabelian equivariant exterior derivative

$$s = d - \phi^a i_a \quad (12)$$

which squares to the Lie-derivative

$$s^2 = -\phi^a \mathcal{L}_a \ . \quad (13)$$

As a consequence $s$ determines a nilpotent exterior derivative operator on the $G$-invariant subcomplex

$$\Omega_G(M) = (S(g^*) \otimes \Omega(M))^G \quad (14)$$

of the exterior algebra on $M$. Elements of $\Omega_G(M)$ are equivariant differential forms, that is mappings $\mu$ from $g$ to $\Omega(M)$ which fulfill the equivariance condition $\mu(\gamma \cdot X) = \gamma \cdot \mu(X)$ for $\gamma \in G$, $X \in g$ and $G$ operating by the adjoint action on $g$.

The cohomology of $s$ which we denote $H^*_s(\Omega_G(M))$, then gives the Cartan model for the $G$-equivariant cohomology on $M$.

*The Weil Algebra*: In order to formulate the equivariant cohomology using nilpotent operators, we also need to introduce anticommuting ghosts corresponding to the factors $\phi^a$. The corresponding algebra is the Weil algebra of $g$, which is defined as the tensor product of the exterior and symmetric algebras on $g^*$,

$$W(g) = S(g^*) \otimes \Omega(g^*)$$

where elements of $\Omega(g^*)$ are multilinear antisymmetric forms on $g$, generated by anticommuting basis of one-forms $\eta^a \ (a = 1 \ldots m)$. We introduce the grading one to $\eta^a$, and grading two to the commuting basis elements $\phi^a$ of $S(g^*)$.

The Weil model of equivariant cohomology is based on the Weil algebra. Due to the similarities of the Weil and the BRST models, we will handle them together, later on. Here we only discuss the Weil algebra.
In the following we shall construct the differential calculus in a Hamiltonian manner. For this we realize derivatives and internal multiplications using Poisson brackets. For example contraction of the one-forms $\eta^a$ is realized by anticommuting $P_a$ and derivation with respect to $\psi^a$ is realized by commuting $\pi_a$ using Poisson brackets
\[
\{\pi_a, \phi^b\} = \{P_a, \eta^b\} = \delta^b_a .
\]
In terms of these variables, the coadjoint action of $G$ on $W(\mathfrak{g})$ is generated by the derivations
\[
L_a = -f^{abc}(\phi_b \pi_c + \eta_b \mathcal{P}_c) ,
\]
the action being
\[
\{L_a, \phi^b\} = f^{abc} \phi^c , \quad \{L_a, \eta^b\} = f^{abc} \eta^c
\]
and
\[
\{L_a, L_b\} = f^{abc} L_c .
\]

Next we introduce a couple of differential operators on $W(\mathfrak{g})$. We first define the "abelian" differential
\[
d_o = \phi^a \mathcal{P}_a
\]
which identifies $\phi^a$ as the differential of $\eta^a$. The nonvanishing actions are
\[
d_o \eta^a = \phi^a , \quad d_o \pi_a = -\mathcal{P}_a .
\]
The second operator we define is
\[
d_g = -f^{abc}(\eta^a \phi_b \pi_c + \frac{1}{2} \eta^a \eta^b \mathcal{P}_c) ,
\]
which computes the $W(\mathfrak{g})$-valued Lie algebra cohomology of $\mathfrak{g}$. This is readily seen by re-writing (18) in terms of (13):
\[
d_g = \eta^a L_a + \frac{1}{2} f^{abc} \eta^a \eta^b \mathcal{P}_c ,
\]
which is of the familiar form of a Lie algebra coboundary operator, or a BRST operator related to the constraints $\{L_a\}$ acting on $W(\mathfrak{g})$.

The sum of the derivations (16) and (18) is the Weil differential
\[
d_w = d_o + d_g ,
\]
with the actions
\[ d_w \eta^a = \phi^a - \frac{1}{2} f^{abc} \eta^b \eta^c, \]
\[ d_w \phi^a = -f^{abc} \eta^b \phi^c. \]  
(21)

These three operators are all nilpotent derivations of degree one
\[ d^2_w = d^2_o = d^2_g = 0 \]
and act as exterior derivations on \( W(g) \).

From (21) we conclude that the cohomology of \( d_w \) on \( W(g) \) is trivial. Indeed, \( d_w \) can be obtained from \( d_o \) by a canonical transformation which is of the functional form
\[ Q \rightarrow e^{-\Phi} Q e^{\Phi} = Q + \{Q, \Phi\} + \frac{1}{2} \{\{Q, \Phi\}, \Phi\} + \ldots \]  
(22)
with generating function
\[ \Phi = \Phi_1 = f^{abc} \eta^a \eta^b \pi_c. \]  
(23)
That is,
\[ d_w = e^{-\Phi_1} d_o e^{\Phi_1}. \]
In particular, we conclude that the cohomology of \( d_w \) must also be trivial.

In (21) we immediately recognize the action of an exterior derivative on a connection one-form \( A \sim \eta^a \) and a curvature two-form \( F \sim \phi^a \) of a principal \( G \)-bundle, i.e. the definitions of the curvature and the Bianchi identity,
\[ dA = F - \frac{1}{2} [A, A], \]
\[ dF = -[A, F]. \]

These relations explain the relevance of the Weil algebra as a universal model of connections on \( G \)-bundles. The connection and curvature define a unique homomorphism from \( W(g) \) to the exterior algebra \( \Omega(P) \) over the bundle, known as the Weil homomorphism which carries the algebraic connection and curvature \( (\eta^a, \phi^a) \) to the geometric ones \( (A, F) \) [20, 14]. This is also why \( W(g) \) appears in equivariant cohomology theory: it models the universal bundle isomorphically on the level of cohomology. The universal bundle, being a contractible space with free \( G \)-action, can be used to
lift a non-free $G$-action on $M$ to a free $G$-action on a related space with equivalent homotopy. This leads to the topological definition of equivariant cohomology 

Finally, we note that the action of $d_w$ on the contraction $\mathcal{P}_a$ yields the corresponding generator (13) of the coadjoint action:

$$\{d_w, \mathcal{P}_a\} \equiv d_w\mathcal{P}_a + \mathcal{P}_a d_w \equiv \{d_g, \mathcal{P}_a\} = L_a .$$

In particular, the derivation $L_a$ has the natural structure of a Lie-derivative on $W(g)$ that commutes with our differentials,

$$\{d_w, L_a\} = \{d_g, L_a\} = \{d_o, L_a\} = 0 .$$

The BRST Model: We are now in a position to define the BRST model of equivariant cohomology. For this we consider the tensor product $W(g) \otimes \Omega(M)$ of the Weil algebra with the exterior algebra over $M$. In analogy with the canonical realization of the Weil algebra, we realize derivation with respect to the coordinates $z^k$ on $M$ canonically by $p_k$, represent the one-forms $dz^k$ by anticommuting variables $c^k$ and the contraction operating on $c^k$ by $\bar{c}_k$. The pertinent Poisson brackets are

$$\{p_k, z^l\} = \{\bar{c}_k, c^l\} = \delta_k^l \quad (24)$$

We introduce a grading of the variables by defining $gr(\eta, \phi, c, z) = (1, 2, 1, 0)$. In terms of (24) the exterior derivative on $\Omega(M)$ is

$$d = c^k p_k$$

and the contraction and Lie derivative with respect to the vector fields $\mathcal{X}_a$ are

$$\iota_a = \mathcal{X}^k_a \bar{c}_k$$

$$\mathcal{L}_a = \mathcal{X}^k_a p_k + c^k \partial_k \mathcal{X}^l_a \bar{c}_l$$

We recall the exterior derivative (16) and define the following exterior derivative on $W(g) \otimes \Omega(M)$,

$$s_o = d + d_o \quad (25)$$
Since the cohomology of $d_o$ is trivial, we conclude that the cohomology of $s_o$ on $W(g) \otimes \Omega(M)$ equals the deRham cohomology of $d$ on $\Omega(M)$.

By introducing the canonical conjugation (22), we obtain from (25)

$$s_w = e^{-\Phi_1}s_oe^{\Phi_1} = d + d_w,$$

(26)

where $d_w$ is the Weil differential (21). This is the differential of the Weil model of equivariant cohomology [3].

If we introduce a further conjugation with

$$\Phi_2 = -\eta^a i_{\chi_a},$$

(27)

we then find the following nilpotent graded derivation of degree one on $W(g) \otimes \Omega(M)$ [13, 14],

$$s = e^{-\Phi_2}(d + d_w)e^{\Phi_2} = d + d_w - \phi^a i_a + \eta^a L_a,$$

(28)

which gives the BRST model for equivariant cohomology. It is the natural nilpotent extension of (12), with the ghost version of the non-nilpotency of the Cartan model (13), and the Weil differential $d_W$ taking care of nilpotency. By construction the cohomology of $s$ on $W(g) \otimes \Omega(M)$ equals the deRham cohomology of $d$ on $\Omega(M)$.

As shown in [13], by appropriately restricting $s$ to a subcomplex of $W(g) \otimes \Omega(M)$ we obtain the $G$-equivariant cohomology of $M$. Indeed, if we consider the $\eta^a$-independent (which restricts onto $S(g^*) \otimes \Omega(M)$) and $(L_a + L_a)$ -invariant (which picks up the $G$-invariant part) subcomplex, we recover the algebra $\Omega_G(M)$ of the Cartan model (14). Moreover, after this restriction the BRST operator (28) reduces to the Cartan model differential (12).

Correspondingly, in the Weil model, after restricting to the basic subcomplex, defined to be horizontal (annihilated by $\mathcal{P}_a + i_a$) and $G$-invariant, the operator (26) describes equivariant cohomology.

In order to properly restrict the domain of $s$, following [13] we introduce another nilpotent operator $\tilde{W}$ such that its kernel coincides with the desired $G$-invariant, $\eta^a$-independent subcomplex. For this we introduce another copy of the Weil algebra, $\tilde{W}(g)$. We denote the generators of $\tilde{W}(g)$ by $\bar{\phi}^a$ and $\bar{\eta}^a$, they are the $g^*$-valued
coefficients corresponding to $\eta^a$ independence (generated by $P_a$) and $G$-invariance (generated by $L_a + L_a$), respectively. Consequently the desired nilpotent operator $\mathcal{W}$ must include the terms

$$\mathcal{W} = \bar{\eta}^a (L_a + L_a) - \bar{\phi}^a P_a + ...$$

In order to complete the construction of $\mathcal{W}$, we specify its action on $\tilde{W}(g)$. Indeed, if we define this action to coincide with the action of the Lie algebra coboundary operator $d_\tilde{g}$ (18) on $\tilde{W}(g)$, we find that the following operator

$$\mathcal{W} = \bar{\eta}^a (L_a + L_a) - \bar{\phi}^a P_a \quad (29)$$

is nilpotent. If we also extend the action of $s$ to $\tilde{W}(g)$ by

$$s = d + d_w + d_{\bar{\phi}} - \phi^a i_a + \eta^a L_a \quad (30)$$

we then find that $s$ and $\mathcal{W}$ satisfy the nilpotent algebra

$$\{s, s\} = \{\mathcal{W}, s\} = \{\mathcal{W}, \mathcal{W}\} = 0 \quad (31)$$

and the $G$-equivariant cohomology of $M$ is isomorphic to the cohomology of $s$, restricted to the kernel of $\mathcal{W}$. This determines the BRST model for the $G$-equivariant cohomology of $M$ which is relevant for the construction of nonabelian generalizations [8] of the Duistermaat-Heckman integration formula (11). Loop space generalizations of the constructions above can be found in [22].

The restriction onto the basic subcomplex in the Weil model can be formulated using a nilpotent operator as well. The natural choice is

$$\mathcal{W}_w = d_\tilde{g} + \bar{\eta}^a (L_a + L_a) - \bar{\phi}^a (P_a + i_a) \quad (32)$$

with the corresponding extension of (26)

$$s_w = d + d_w + d_{\bar{\phi}} \quad (33)$$

Operators (32) and (33) are canonical transformations of (29) and (30), respectively, with the generating function $-\Phi_2$. Thus they obey an algebra similar to (31).
4. Non-abelian equivariant symplectic two-forms

In (10) we presented the abelian equivariant extension of the symplectic two-form $\omega$. We shall now construct the most general non-abelian equivariant extension of $\omega$ on the complex $W(\mathfrak{g}) \otimes \Omega(M)$, corresponding to the symplectic action of the non-abelian Lie-group $G$ on the symplectic manifold $M$. In analogy with (11), this non-abelian equivariant extension can then be used as the starting point for constructing non-abelian generalizations of the DH integral that can be evaluated using (non-abelian) localization methods.

We consider the Poisson bracket realization of $G$,

$$\{ H_a, H_b \} = f^{abc} H_c \, , \tag{34}$$

with $H_a$ functions defined on the manifold $M$. In order to construct the most general non-abelian generalization of (10), we then introduce the following Ansatz,

$$H_0 = \omega + \alpha \eta^a dH_a + \beta \phi^a H_a + \gamma f^{abc} \eta^a H_b dH_c \, , \tag{35}$$

which is the most general form of degree two that can be constructed on $W(\mathfrak{g}) \otimes \Omega(M)$ in terms of the variables that appear in the BRST model of the $G$-equivariant cohomology. We shall now determine the parameters in (35) by requiring $G$-equivariance, i.e. that $H_0$ is annihilated both by $s$ and $W$ of the BRST model.

By demanding

$$s_0 \mathcal{H} = 0 \tag{36}$$

we first get the conditions

$$\alpha = -\beta \, , \quad \gamma = 0 \, . \tag{37}$$

We then introduce the canonical transformation (22) generated by

$$\Phi_T = \Phi_1 + \Phi_2 \, , \tag{38}$$

with $\Phi_1$ and $\Phi_2$ defined in (23, 27). This yields for $H_0$,

$$H_0 \rightarrow \exp\{ -\Phi_T \} \mathcal{H}_0 \exp\{ \Phi_T \} = \mathcal{H}$$
Explicitly,
\[ \mathcal{H} = \omega - \alpha \phi^a H_a + (\alpha - 1) \eta^a dH_a + \frac{1}{2} (1 - \alpha) f^{abc} \eta^a \eta^b H_c \]

As a consequence of (36), \( \mathcal{H} \) satisfies the condition \( s \mathcal{H} = 0 \). In order to restrict it to the subcomplex \( \Omega_G(M) \) we then require that
\[ \mathcal{W} \mathcal{H} = 0 \]
which sets
\[ \alpha = 1 \]
and yields
\[ \mathcal{H} = \omega - \phi^a H_a \]
(39)
as the most general \( G \)-equivariant extension of the symplectic two-form \( \omega \). We note that the final result (39) coincides with the nonabelian equivariant extension of \( \omega \) introduced in [8].

In a number of applications to two-dimensional integrable models (most notably the KdV model) we obtain the following generalization: instead of (34), the Hamiltonians \( H_a \) obey the centrally extended Lie algebra
\[ \{ H_a, H_b \} = f^{abc} H_c + \kappa_{ab} , \]
with \( \kappa_{ab} \) the Lie-algebra two-cocycle. Now the most general Ansatz of degree two for the equivariant extension of \( \omega \) is
\[ \mathcal{H}_o = \omega + \alpha \eta^a dH_a + \beta \phi^a H_a + \gamma f^{abc} \eta^a H_b dH_c + \mu \kappa_{ab} \eta^a \eta^b \]

Requiring
\[ s_o \mathcal{H}_o = 0 \]
we then find, in addition to (37), the condition
\[ \mu = 0 . \]

Performing the canonical transformation (22, 38) we get
\[ \mathcal{H} = \omega - \alpha \phi^a H_a + (\alpha - 1) \eta^a dH_a + \frac{1}{2} (1 - \alpha) f^{abc} \eta^a \eta^b H_c + (\alpha - \frac{1}{2}) \kappa_{ab} \eta^a \eta^b . \]
If we set $\alpha = 1$ we get the following two-form,

$$H = \omega - \phi^a H_a + \frac{1}{2} \kappa_{ab} \eta^a \eta^b.$$  \hspace{1cm} (40)

However, if we operate on $H$ by $W$, we find

$$W(\kappa_{ab} \eta^a \eta^b) = (\bar{\phi}^a \eta^b - \phi^a \bar{\eta}^b - f^{cda} \bar{\eta}^c \eta^d \eta^b) \kappa_{ab},$$

so that the restriction to $\Omega_G(M)$ cannot be implemented. Indeed, we have recovered the fact [3] that equivariant extensions of the symplectic two-form are in one-to-one correspondence to the Poisson liftings of the symplectic action of the group. However, using (6, 7), we can write (40) in the trivially $s$-closed form

$$H = s(\vartheta + \eta^a h_a),$$

and we conclude that it is still possible to derive localization formulas for Hamiltonians constructed from a central extension of a non-abelian Lie algebra.

5. Superspace Formulation

In the previous sections we have developed the BRST picture of equivariant cohomology in terms of the bosonic coordinates $z^k$ on $M$ and fermionic variables $c^k \sim dz^k$, and two Weil algebras over the Lie algebra $g$ acting on $M$ generated by $\eta^a, \phi^a$ and $\bar{\eta}^a, \bar{\phi}^a$ respectively. Obviously the coordinates $z^k, \phi^a$ and $\bar{\phi}^a$ can be interpreted as bosonic coordinates in a superspace, with corresponding superpartners $c^k, \eta^a$ and $\bar{\eta}^a$. In order to represent the exterior algebra on this superspace in a Hamiltonian framework, we introduce the pertinent conjugate variables with the nonvanishing Poisson brackets

$$\{p_k, z^l\} = \{\bar{c}_k, c^l\} = \{\pi_a, \phi^b\} = \{\mathcal{P}_a, \eta^b\} = \{\bar{\pi}_a, \bar{\phi}^b\} = \{\bar{\mathcal{P}}_a, \bar{\eta}^b\} = \delta^b_a. \hspace{1cm} (41)$$

The Hamiltonian realization of the abelian derivation on $\Omega(M) \otimes W(g) \otimes \bar{W}(g)$ is then an extension of (25) that acts on $\bar{W}(g)$ as well:

$$s_o = d + d_o + d_\theta = c^k p_k + \phi^a \mathcal{P}_a + \bar{\phi}^a \bar{\mathcal{P}}_a. \hspace{1cm} (42)$$
The supercanonical transformation generated by (38) relates (42) to the full BRST operator (30). By performing the inverse canonical transformation generated by $-\Phi_T$ on the restriction operator (29), we get a description of equivariant cohomology on the level of the abelian exterior derivative (42). In particular, by construction this transformed restriction operator

$$W_0 = W + \phi^a(f^{abc}\eta^b\pi_c - i_a) = d_g + \bar{\eta}^a(\mathcal{L}_a + L_a) - \bar{\phi}^a(\mathcal{P}_a - f^{abc}\eta^b\pi_c + i_a)$$

(43)

obeys

$$\{W_0, s_0\} = \{W_0, W_0\} = 0.$$ 

Now we want to combine the coordinates (41) into superfields on some underlying superspace. For this, we introduce a N=2 superspace with two grassmannian directions $\theta$ and $\bar{\theta}$, and define the superfields

$$A^k = z^k + \theta^c$$
$$E_k = \bar{c}_k\bar{\theta} + \theta\bar{p}_k$$
$$A^{a}_\theta = \eta^a + \phi^a\theta$$
$$E_{\theta,a} = \bar{\eta} + \bar{\phi}_a\theta$$
$$A^{a}_{\bar{\theta}} = -\bar{\eta} - \bar{\phi}_a\theta$$
$$E_{\bar{\theta},a} = \bar{\theta}\pi_a + \theta\bar{p}_a$$

(44)

The fields $\{A^k\}$ generate the exterior algebra $\Omega(M)$, the $\theta$-component $A_{\theta}$ generate the Weil algebra $W(g)$ and the $\bar{\theta}$-component the extra Weil algebra $\bar{W}(g)$.

Notice that these superfields are truncated: In order to get the full, untruncated superfields it is necessary to double the number of component fields. In the case of BRST quantization of a constrained system these extra fields would be related to a BRST gauge fixing of the theory, but in the following such fields are not relevant. We refer to [17] and [18], where this aspect has been discussed in the context of topological Yang-Mills theory.

From (44) we conclude, that the superfields (44) satisfy the (properly truncated) superspace Poisson brackets

$$\{E_{\alpha,a}(\zeta), A_{\beta}^b(\zeta')\} = g_{\alpha\beta} \delta^b_a \delta(\zeta - \zeta'),$$

16
where \( \zeta \) denotes \( \theta \) and \( \bar{\theta} \) collectively and \( \alpha \) labels the components of \( E \) and \( A \) in (44), e.g. \( E_\alpha \) has components \((E_{\alpha a}, E_{\alpha \bar{\theta}}, E_{\bar{\alpha} \bar{\theta}})\), and the metric in \( \theta \)-space is antisymmetric: 
\[
g_{\theta \bar{\theta}} = -g_{\bar{\theta} \theta} = -g_{\theta \bar{\theta}} = g_{\bar{\theta} \theta} = 1.
\]
The \( \delta \)-function in the anticommuting variables is the appropriate truncation of
\[
\delta(\zeta - \zeta') = \theta \bar{\theta} - \theta' \bar{\theta} - \theta \bar{\theta}' + \theta' \bar{\theta}'.
\]
corresponding to our truncation of the superfields (44).

We define the (truncated) supergenerators of infinitesimal \( G \)-transformations
\[
D^k_a = -X^k_a + (dX^k_a) \theta = -X^k_a + \epsilon^l(\partial_l X^k_a) \theta
\]
and the corresponding generators
\[
G_a = D^k_a E_k = \bar{\theta} X^k_a \bar{c}_k - dX^k_a \bar{c}_k \theta \bar{\theta} = \bar{\theta} i_a - \theta \bar{\theta} L_a
\]
that satisfy the Lie-algebra (34) in the superspace,
\[
\{G_a(\zeta), G_b(\zeta')\} = -f^{abc} G_c(\zeta) \delta(\zeta - \zeta').
\]
On the superfields, (43) generate the superspace gauge transformations,
\[
\{G_a(\zeta), A^k(\zeta')\} = D^k_a(\zeta') \delta(\zeta - \zeta')
\]
\[
\{G_a(\zeta), E_k(\zeta')\} = -(\partial_k G_a(\zeta')) \delta(\zeta - \zeta'),
\]
where the truncated \( \delta \)-functions give e.g. \( D^k_a(\zeta') \) after integrating over the supercoordinates \( \zeta \).

We also introduce covariant derivation and components of the gauge generators in the \( \theta \)-direction:
\[
D_{\theta,a}^b = \delta_a^b \partial_\theta + f^{abc} A_\theta^c
\]
\[
G_{\theta,a} = g_{\theta \bar{\theta}} D_{\theta,a}^b E_{\theta,b},
\]
and similarly for \( \bar{\theta} \).

We are now in a position to introduce the superfield representations of the various quantities we have introduced previously. Indeed, we find that the generator of
coadjoint action on the Weil algebra \((15)\) and the Lie algebra cohomology differential \((18)\) have the following representations in terms of the superfields:

\[
L_a = -\int d\bar{\theta} d\theta f^{abc} A_\theta^b \varepsilon_{\bar{c},c} \\
d_g = -\frac{1}{2} \int d\bar{\theta} d\theta f^{abc} A_\theta^a A_\theta^b \varepsilon_{\bar{c},c} \tag{47}
\]

In addition we note that

\[
A_\theta^a G_a = \phi^a \eta^a - \eta^a L_a, \tag{48}
\]

and that the abelian differential \((42)\) can be expressed as the generator of \(\theta\)-translations:

\[
s_\theta = g^\alpha\beta(\partial_\theta A_\alpha^a) \varepsilon_{\beta,a}. \tag{49}
\]

(Here we use the convention that a summation over \(a\) is understood only if the corresponding fields carry a representation of \(G\) - generically in the \(\theta\) and \(\bar{\theta}\) components. An integration over \(d\bar{\theta} d\theta\) is also understood here and in the following whenever it is plausible.)

Combining these, we finally get the following superfield representation of the BRST operator \((30)\) for the equivariant cohomology:

\[
s = g^\alpha\beta(\partial_\theta A_\alpha^a) \varepsilon_{\beta,a} + A_\theta^a G_a - \frac{1}{2} f^{abc} A_\theta^a A_\theta^b \varepsilon_{\bar{c},c}. \tag{50}
\]

In particular, in the last two terms we recognize the functional form \((19)\) of a BRST operator related to the constraint algebra \(\{G_a, G_b\} = f^{abc} G_c\), with \(A_\theta\) viewed as the ghost field.

In order to obtain a superspace representation of \((29)\) it is easiest to work in terms of the canonically transformed \((43)\). Using \((47, 48)\) we then get

\[
W_o = A_\theta^a (G + G_\theta)_a - \frac{1}{2} f^{abc} A_\theta^a A_\theta^b \varepsilon_{\bar{c},c}. \tag{51}
\]

Hence we conclude that superspace functions invariant under \(\theta\)-translations describe \(G\)-equivariant cohomology on \(M\), provided that we restrict them to the kernel of \(W_o\).

Notice that the superspace representation \((51)\) has the functional form of a conventional BRST operator related to the constraints \((G + G_\theta)_a\) generating the action of
on $\Omega(M) \otimes W(g)$, i.e. on the $A^k$ and $A_\theta$ sectors of superspace. The corresponding ghosts $A_\theta$ generate the extra Weil algebra $\bar{W}(g)$. In particular, the reducibility of the BRST operator (30) is in some sense lifted when the theory is formulated in the superspace.

6. Relation to first stage reducible constraints

We observe that the superfield formalism we have developed here is identical to the superfield formulation of four dimensional topological Yang-Mills theory developed in [17, 18]; the only difference is that the dependence on space coordinates $\vec{x}$ has been truncated. In the present section we shall discuss this connection, and in particular how the BRST model for equivariant cohomology is related to the Hamiltonian approach to constrained quantization developed by Batalin, Fradkin and Vilkovisky (BFV) [23].

In the BFV approach to Hamiltonian BRST quantization of constrained systems, the BRST operator appears as a nilpotent operator that encompasses all information about the algebra of constraints. For example, nilpotency of (51) in the framework of a constrained system would immediately tell us that the constraints $(G + G_\theta)_a$ satisfy the algebra (46).

In the case of a first class, first stage reducible constrained system we are dealing with constraints $F_a = 0, a = 1...m$ with Poisson brackets that close with some structure functions $C^{abc}$,

$$\{F_a, F_b\} = C^{abc}F_c$$

and reducibility implies that there exists $k \leq m$ linear relations between the constraints of the form

$$B^a_i F_a = 0$$

with some multipliers $B^a_i$. In the BFV approach we attach to the constraints ghost fields $c^a$ together with their canonical conjugates $\bar{c}_a$, and interpret the reducibility condition as an extra constraint acting on the conjugate ghosts,

$$B^a_i \bar{c}_a = 0.$$
which gives rise to ghost for ghost fields $\phi^i$ that are bosonic fields with ghost number (i.e. grading) 2. This reducibility is then incorporated into the nilpotency of a BRST operator in a systematic manner, as explained in [23].

Here it is sufficient to consider - in analogy with four dimensional topological Yang-Mills theory [12, 10] - a constraint algebra that consists of two intrinsically irreducible sets, a set $\{E_k\}$ of abelian constraint functionals and a (not bigger) set of constraint functionals $\{G_a\}$ which generate a non-abelian Lie algebra with structure constants $f^{abc}$. These two sets of constraint functionals then form a reducible constraint algebra with multipliers $\delta_a^b$ and $-D_a^k$ respectively, obtained by setting

$$G_a - D_a^k E_k = 0 \quad \forall a .$$

so that the structure functions of the constraint algebra are

$$C^{abc} = f^{abc}$$
$$C^{abk} = 0$$
$$C^{akl} = \{D_a^l, E_k\}$$

This is exactly the case which leads to equivariant cohomology: The abelian constraint functionals $E_k$ can be identified as conjugate momenta of local coordinates $z_k$ on some manifold, and the constraints $E_k = 0$ imply independence of the coordinates, i.e. that we are interested in cohomological properties of the corresponding manifold. The ghosts of the topological constraints constitute a basis for one-forms on the manifold. The nonabelian constraints generate the group $G$ acting on the manifold, and the cohomology of the abelian BRST operator (exterior derivative)

$$s = c^a E_a$$

reduces to the $G$-equivariant cohomology. The ghosts of the nonabelian constraints are the $\eta^a$ fields, which generate the Weil algebra together with the ghosts for ghosts $\phi^a$. Finally, equation (52) is just the generating vector field (1) written in component form.

In particular, the formalism presented here is identical to that found in the Hamiltonian quantization of the four dimensional topological Yang-Mills theory [17, 18], where the reducibility equation (52) is the Gauss law constraint.
In the case of group actions on finite dimensional spaces we already have a BRST operator \((28)\) which encorporates all information about the constraint algebra \(\{E_k, G_a\}\). In particular, the correspondence between the constraint algebra and exterior calculus is

\[
\begin{align*}
    d &= c^a E_a \\
i_a &= D^k_a \bar{c}_k \\
L_a &= G_a + c^k C^{alk} \bar{c}_l
\end{align*}
\]

Writing \((28)\) in terms of these, we get

\[
s = c^a E_a + \eta^a G_a + \phi^a (\mathcal{P}_a - D^k_a \bar{c}_k) - \frac{1}{2} f^{abc} \eta^a \eta^b \mathcal{P}_c - C^{akl} \eta^a c^k \bar{c}_l - f^{abc} \eta^a \phi^b \pi_c.
\]

(53)

where we recognize terms corresponding both to the two sets of original constraints and to the ghost constraint, terms related to the structure functions \(C^{abc}\), and an additional term related to the Lie-algebra cohomology operator \((18)\).

As shown by \((26, 28)\) the cohomology captured by \((53)\) is just the deRham cohomology of the abelian BRST operator \(d = c^a E_a\). To get a nontrivial answer, a restriction onto the basic subcomplex should be made, using \(\mathcal{W}\). This is an easy way to establish the nontriviality of observables in topological Yang-Mills.

As discussed in \([18]\), the minimal BRST operator \((53)\) can be extended by adding more fields. For gauge fixing purposes (and to maintain manifest Lorentz invariance in field theory applications), the Lagrange multipliers corresponding to the constraints should be made dynamical. Doing this, extra abelian constraints arise, which express that the multiplier momenta must vanish. The corresponding ghost fields are the so called anti-ghosts. In this way the fields related to the constrained system \(\{E_k = 0, G_a = 0\}\) fall naturally in three sets \([18]\):

- The first set includes the fields related to the topological constraint \(E_k = 0\): the coordinate \(z\), the ghost \(c\), an antighost and a multiplier, as well as the corresponding momenta.

- The second set includes the Weil algebra generated by the ghost for ghost \(\phi\) and the ghost \(\eta\) of the constraint \(G_a = 0\), and an antighost and a multiplier for the same.

- The third set includes an antighost and a multiplier for the ghost constraint. Again, to maintain manifest Lorentz invariance in field theory applications, some
extra fields have to be defined. These are the extra ghosts of \[23\], a commuting and an anticommuting canonical pair for each reducibility equation, \textit{i.e.} for each \(G_a\). We recognize in the extraghost the generators \(\bar{\phi}, \bar{\eta}\) of our extra Weil algebra \(\bar{W}(g)\).

The fields in each of these three sets define a component of the superconnection \(\{A_k, A_{\theta}, A_{\bar{\theta}}\}\) in \([14]\) in the absence of all multiplier and antighost fields, in analogy \([17, 18]\) with four dimensional topological Yang-Mills theory. In particular, \((50)\) reproduces the BRST operator of topological Yang-Mills.

Analogues to conjugations \((23, 27)\) can also be found in \([17, 18]\). In \([18]\) other conjugations are introduced as well. The most interesting one is generated by

\[
\Phi_3 = f^{abc}\eta^a\bar{\eta}^b\bar{\pi}^c,
\]

which lifts the abelian action of \(d_0\) on \(\bar{W}(g)\) to the coadjoint action:

\[
s \to s' = s + \eta^a\bar{L}_a + f^{abc}\phi^a\bar{\eta}^b\bar{\pi}^c \equiv s + A_{\theta}^a D_{\theta}^{ab} E_{\bar{\theta}}^b.
\]

On the superspace level this BRST operator is related to the \textit{full} supespace Gauss law, extended to act on \(A_{\theta}, E_{\theta}\) as well:

\[
\{E_{\theta,a}, s'\} = g^{a\beta} D_{a,a}^{\beta} E_{\beta,b} \equiv (G + G_{\theta} + G_{\bar{\theta}})_a.
\]

Notice however, that in \(s'\) the roles of \(A_{\theta}, E_{\theta}\) as superspace ghosts as in \((51)\) has been lost.

### 7. Conclusions

Following \([13]\) we have formulated equivariant cohomology in the context of localization formulas in terms of two nilpotent operators, the BRST operator \(s\) and the restriction operator \(W\). In addition, we have developed a superfield formalism for equivariant BRST using a N=2 superspace with fermionic coordinates \(\theta, \bar{\theta}\). We have found, that in this superspace formalism all variables relevant to localization can be combined into a single superconnection \(A\).

Furthermore, we have shown that the BRST operator can be conjugated to the translation operator in the \(\theta\)-direction in superspace, and the restriction operator acquires the form \((51)\) of a conventional BRST operator related to the \textit{superspace} action.
of the $A_k$ and $A_\theta$ parts of the *superspace* Gauss law, with the remaining superfields $A_\bar{\theta}$ acting as superghosts of this superconstraint. The connection of equivariant cohomology and BFV quantization of four dimensional topological Yang-Mills theory becomes then transparent.

Depending of the interpretation of the fields $A_k$, the superfield formalism presented here describes both equivariant cohomology in the symplectic setting relevant to localization, and the BRST structure of (cohomological) topological field theories. From this we conclude that there should be a unified description of localization in the symplectic loop space [1], the supersymmetric loop space [7] and in the case of topological field theory [12]. Indeed, this is consistent with the mathematical conjecture [24] that all lower dimensional integrable models could be obtained as dimensional reductions of 4-dimensional self-dual Yang-Mills theory, which is intimately connected with topological Yang-Mills.

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