Remarks on Agemi-type structural condition for systems of semilinear wave equations

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April 22, 2019

Abstract: We consider a two-component system of cubic semilinear wave equations in two space dimensions satisfying the Agemi-type structural condition (Ag) but violating (Ag0) and (Ag+). For this system, we show that small amplitude solutions are asymptotically free as $t \to +\infty$.

Key Words: Semilinear wave equation; asymptotic behavior; Agemi-type condition.

2010 Mathematics Subject Classification: 35L71, 35B40.

1 Introduction

This paper is devoted to the study on large-time asymptotic behavior of solutions $u = (u_1, u_2)$ to

\[
\begin{align*}
\Box u_1 &= - (\partial_t u_2)^2 \partial_t u_1, \\
\Box u_2 &= - (\partial_t u_1)^2 \partial_t u_2,
\end{align*}
\tag{1.1}
\]

with the initial condition

\[
\begin{align*}
u_j(0, x) &= \varepsilon f_j(x), \\
\partial_t u_j(0, x) &= \varepsilon g_j(x),
\end{align*}
\tag{1.2}
\]

where $\varepsilon > 0$ is a small parameter, $\Box = \partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2$, and $f_j, g_j \in C^\infty_0(\mathbb{R}^2)$.

Before getting into the details, let us recall the backgrounds briefly to make clear why this system is of our interest. To put (1.1) in perspective, let us first consider more general systems in the form

\[
\Box u = F(\partial u), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d,
\tag{1.3}
\]

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with $C_0^\infty$-data of size $\varepsilon$, where $u = (u_j(t,x))_{1 \leq j \leq N}$, $\partial_0 = \partial / \partial t$, $\partial_k = \partial / \partial x_k$ ($1 \leq k \leq d$), $\Delta = \partial_1^2 + \cdots + \partial_d^2$, $\Box = \partial_t^2 - \Delta$ and $\partial u = (\partial_t u_j)_{0 \leq \alpha \leq d, 1 \leq j \leq N}$. $F = (F_j)_{1 \leq j \leq N}$ is an $\mathbb{R}^N$-valued $C^\infty$-function vanishing of order $p \geq 2$ in a neighborhood of 0 $\in \mathbb{R}^{N \times (1 + d)}$. If $p > 1 + 2/(d-1)$ and $\varepsilon$ is small enough, it is well-known that $(1.3)$ admits a unique global $C^\infty$-solution and it behaves like a solution to the free wave equation as $t \to \infty$, while if $p \leq 1 + 2/(d - 1)$, global existence fails to hold in general even when $\varepsilon > 0$ is arbitrarily small ($[10], [5]$ etc). In this sense, the power $p_c(d) := 1 + 2/(d - 1)$ is a critical exponent for nonlinearity perturbation. Note that $p_c(2) = 3$ and $p_c(3) = 2$. On the other hand, the small data global existence can hold for some class of nonlinearity of the critical power. One of the most successful examples is the so called null condition, which has been originally introduced by Christodoulou [4] and Klainerman [12] in three dimensional case and developed later by several authors (see [5], [8], [12], [1] etc., for the two-dimensional counterparts). We remark that the global solution $u$ under the null condition is asymptotically free in the sense that there exists a solution $u^+$ to the free wave equation $\Box u^+ = 0$ such that
\[
\lim_{t \to \infty} \|u(t) - u^+(t)\|_E = 0,
\]
where the energy norm $\| \cdot \|_E$ is defined by
\[
\|\phi(t)\|^2_E = \frac{1}{2} \int_{\mathbb{R}^d} \sum_{a=0}^d |\partial_a \phi(t,x)|^2 \, dx.
\]
When we restrict to the case where $d = 2$ and the nonlinearity is given by
\[
F_j(\partial u) = \sum_{a,b,c=0}^N \sum_{k,l,m=1}^2 C_{jklm}^{abc}(\partial_a u_k)(\partial_b u_l)(\partial_c u_m)
\]
with real constants $C_{jklm}^{abc}$, the null condition is satisfied if and only if $F_j^{\text{red}}(\omega, Y)$ vanishes identically on $S^1 \times \mathbb{R}^N$, where
\[
F_j^{\text{red}}(\omega, Y) = \sum_{k,l,m=1}^N \sum_{a,b,c=0}^2 C_{jklm}^{abc} \omega_a \omega_b \omega_c Y_k Y_l Y_m
\]
for $Y = (Y_j)_{1 \leq j \leq N} \in \mathbb{R}^N$ and $\omega = (\omega_1, \omega_2) \in S^1$, with the convention $\omega_0 = -1$.

Recently, a lot of efforts have been made for the study on weaker structural conditions than the null condition mentioned above which ensure the small data global existence (see e.g., [25], [27], [28], [2], [3], [9], [23], [16], [19], [17], [18], [14], [15], [6], etc). It should be emphasized that the situation becomes much more complicated because long-range nonlinear effects must be taken into account. In [18], the following condition has been introduced:

(Ag) There exists an $N \times N$-matrix valued continuous function $A = A(\omega)$ on $S^1$, which is a positive-definite symmetric matrix for each $\omega \in S^1$, such that
\[
Y \cdot A(\omega) F^{\text{red}}(\omega, Y) \geq 0, \quad (\omega, Y) \in S^1 \times \mathbb{R}^N,
\]
where the symbol $\cdot$ denotes the standard inner product in $\mathbb{R}^N$. 

\[
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\]
After the partial results \[23, 9, 19\], it has been shown in \[18\] that \((\text{Ag})\) implies the small data global existence for \((1.3)\)–\((1.4)\) in two space dimensions (see also \[21, 20, 24\] etc., for closely related works). We note that this condition is motivated by works of Rentaro Agemi in the late 1990’s. He tried to find a structural condition which covers not only the standard null condition but also the wave equations with cubic nonlinear damping such as \(\Box v = - (\partial_t v)^3\). Therefore it would be fair to call this the Agemi-type condition. As for the asymptotic behavior of the global solutions under \((\text{Ag})\), many interesting problems seem left unsolved. To the authors’ knowledge, only the following two cases \((\text{Ag}_+)\) and \((\text{Ag}_0)\) are well-understood:

\((\text{Ag}_+)\) There exist an \(\mathcal{A}(\omega)\) as in \((\text{Ag})\) and a positive constant \(C\) such that

\[
Y \cdot \mathcal{A}(\omega) F_{\text{red}}^\text{red}(\omega, Y) \geq C|Y|^4, \quad (\omega, Y) \in S^1 \times \mathbb{R}^N. \tag{1.5}
\]

Note that \((\text{1.5})\) is equivalent to

\[
Y \cdot \mathcal{A}(\omega) F^c_{\text{red}}(\omega, Y) \neq 0, \quad (\omega, Y) \in S^1 \times (\mathbb{R}^N \setminus \{0\}),
\]

if \((\text{Ag})\) is satisfied and \(F\) is cubic. Under \((\text{Ag}_+)\), the total energy \(\|u(t)\|_E\) decays like \(O((\log t)^{-1/4+\delta})\) as \(t \to +\infty\), where \(\delta > 0\) can be arbitrarily small. See \[18\] for the detail.

\((\text{Ag}_0)\) There exists an \(\mathcal{A}(\omega)\) as in \((\text{Ag})\) such that

\[
Y \cdot \mathcal{A}(\omega) F_{\text{red}}(\omega, Y) = 0, \quad (\omega, Y) \in S^1 \times \mathbb{R}^N.
\]

Note that \((\text{Ag}_0)\) is stronger than \((\text{Ag})\) if \(F\) is cubic (while it is equivalent to \((\text{Ag})\) in the quadratic case). Roughly speaking, it holds under \((\text{Ag}_0)\) that

\[
\partial u(t, x) \sim |x|^{-1/2} \hat{\omega}(x) V(t; |x| - t, x/|x|)
\]

as \(t \to \infty\), where \(\hat{\omega}(x) = (-1, x_1/|x|, x_2/|x|)\), and \(V(t; \sigma, \omega)\) solves

\[
\partial_t V = \frac{1}{t} Q(\omega, V)V
\]

with a suitable skew-symmetric matrix \(Q\) depending on \((\omega, V)\). In particular, decay of the total energy never occurs under \((\text{Ag}_0)\) except for the trivial solution. Typical example satisfying \((\text{Ag}_0)\) is

\[
\left\{ \begin{array}{l}
    \Box u_1 = -(\partial_t u_1)^2 \partial_t u_2, \\
    \Box u_2 = (\partial_t u_1)^3.
\end{array} \right.
\]

For more details on \((\text{Ag}_0)\), see \[17, 15\] and Chapter 10 in \[14\].

Now, let us turn back to our system \((1.1)\), that is the case where \(F_1(\partial u) = -(\partial_t u_2)^2 \partial_t u_1, F_2(\partial u) = -(\partial_t u_1)^2 \partial_t u_2\) and \(N = 2\) in \[(13)\]. We can easily check that \((\text{Ag})\) is satisfied by \[(11)\] with \(\mathcal{A}(\omega)\) being the \(2 \times 2\) identity matrix. Indeed we have \(Y \cdot F_{\text{red}}(\omega, Y) = 2Y_1^2Y_2^2\). Note
also that both (Ag+) and (Ag0) are violated. We observe that the system (1.1) possesses two conservation laws
\[
\frac{d}{dt} \left( \|u_1(t)\|_E^2 + \|u_2(t)\|_E^2 \right) = -2 \int_{\mathbb{R}^2} \left( \partial_t u_1(t, x) \right)^2 \left( \partial_t u_2(t, x) \right)^2 \, dx \tag{1.6}
\]
and
\[
\frac{d}{dt} \left( \|u_1(t)\|_E^2 - \|u_2(t)\|_E^2 \right) = 0. \tag{1.7}
\]
However, these are not enough to say something about the large-time asymptotics for \(u(t)\), and this is not trivial at all. To the authors’ knowledge, there are no previous results which cover the asymptotic behavior of solutions to (1.1)–(1.2). The aim of the present paper is to address this point. Several related issues will be discussed elsewhere.

The main result is as follows.

**Theorem 1.1.** Suppose that \(f, g \in C_0^\infty(\mathbb{R}^2)\) and \(\varepsilon\) is suitably small. Then the global solution \(u(t)\) to (1.1)–(1.2) is asymptotically free in the following sense: there exists \((f^+, g^+) \in \dot{H}^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)\) such that
\[
\lim_{t \to +\infty} \|u(t) - u^+(t)\|_E = 0,
\]
where \(u^+\) solves the free wave equation \(\Box u^+ = 0\) with \((u, \partial_t u)|_{t=0} = (f^+, g^+)\).

**Remark 1.1.** If \((f_1, g_1) = (f_2, g_2)\), then the system is reduced to the single equation \(\Box v = -\left(\partial_t v\right)^3\). Therefore we can adapt the result of [10], [18] to see that the total energy \(\|u(t)\|_E\) decays like \(O((\log t)^{-1/4+\delta})\) as \(t \to \infty\). On the other hand, if \(\|u_1(0)\|_E \neq \|u_2(0)\|_E\), then at least one component \(u_1\) or \(u_2\) tends to a non-trivial free solution because of the conservation law (1.7).

**Remark 1.2.** Our proof of Theorem 1.1 does not rely on the conservation laws (1.6) and (1.7) at all. For example, the same proof is valid for the system
\[
\begin{cases}
\Box u_1 = -|\nabla_x u_2|^2 \partial_t u_1, \\
\Box u_2 = -|\nabla_x u_1|^2 \partial_t u_2,
\end{cases}
\]
or more generally, any cubic terms satisfying the standard null condition can be added to the right-hand side of it.

**Remark 1.3.** The above theorem concerns only the forward Cauchy problem (i.e., for \(t > 0\)). For the backward Cauchy problem, it is not difficult to construct a blowing-up solution (with a suitable choice of \(f, g\)) based on the idea of [5]. This should be contrasted with the behavior of solutions under (Ag0).

## 2 Preliminaries

In this section, we collect several notations which will be used in the subsequent sections. For \(z \in \mathbb{R}^d\), we write \(\langle z \rangle = \sqrt{1 + |z|^2}\). We define
\[
S := t\partial_t + x_1 \partial_1 + x_2 \partial_2, \quad L_1 := t\partial_1 + x_1 \partial_t, \quad L_2 := t\partial_2 + x_2 \partial_t, \quad \Omega := x_1 \partial_2 - x_2 \partial_1,
\]
and we set \( \Gamma = (\Gamma_j)_{0 \leq j \leq 6} = (S, L_1, L_2, \Omega, \partial_0, \partial_1, \partial_2) \). For a multi-index \( \alpha = (\alpha_0, \alpha_1, \cdots, \alpha_6) \in \mathbb{Z}_+^7 \), we write \(|\alpha| = \alpha_0 + \alpha_1 + \cdots + \alpha_6 \) and \( \Gamma^\alpha = \Gamma_0^{\alpha_0} \Gamma_1^{\alpha_1} \cdots \Gamma_6^{\alpha_6} \), where \( \mathbb{Z}_+ = \{ n \in \mathbb{Z}; n \geq 0 \} \). We define \(| \cdot |_s\) by

\[
|\phi(t, x)|_s = \sum_{|\alpha| \leq s} |\Gamma^\alpha \phi(t, x)|.
\]

For \( x \in \mathbb{R}^2 \setminus \{0\} \), we write \( r := |x| \), \( \omega = (\omega_1, \omega_2) := x/|x| \), \( \omega^\perp = (\omega^\perp_1, \omega^\perp_2) := (-\omega_2, \omega_1) \), \( \partial_x := \omega_1 \partial_1 + \omega_2 \partial_2 \), and \( \partial_\pm := \partial_x \pm \partial_r \). Important relations are

\[
\partial_+ \partial_- (r^{1/2} \phi) = r^{1/2} \Box \phi + \frac{1}{4r^{3/2}} (4\Omega^2 + 1) \phi, \tag{2.1}
\]

\[
(t + r)(\partial_j - \omega_j \partial_r) = \omega^\perp_j (\Omega + \omega_1 L_2 - \omega_2 L_1), \quad j = 1, 2, \tag{2.2}
\]

\[
(t + r) \partial_+ = S + \omega_1 L_1 + \omega_2 L_2, \tag{2.3}
\]

and \( \partial_+ + \partial_- = 2\partial_x \), \( \partial_+ - \partial_- = 2\partial_r \). Next we set \( \Lambda_\infty = \{ (t, x) \in [0, \infty) \times \mathbb{R}^2; |x| \geq t/2 \geq 1 \} \) and \( \mathcal{D} = -2^{-1} \partial_\cdot \). Then we have the following.

**Lemma 2.1.** There exists a positive constant \( C \) such that

\[
||x|^{1/2} \partial \phi(t, x) - \hat{\phi}(x) \mathcal{D} (|x|^{1/2} \phi(t, x))| \leq C (t + |x|)^{-1/2} |\phi(t, x)|_1
\]

for \( (t, x) \in \Lambda_\infty \), where \( \hat{\phi}(x) = (-1, x_1/|x|, x_2/|x|) \).

This is a consequence of (2.2) and (2.3). See Corollary 3.3 in [19] for more detail of the proof.

### 3 The John–Hörmander reduction

In this section, we will make reductions of the problem along the approach exploited in [19], [17], [18], [15]. The essential idea goes back to John [11] and Hörmander [7] concerning detailed lifespan estimates for quadratic quasilinear wave equations in three space dimensions.

Let \( u = (u_1, u_2) \) be a smooth solution to (1.1)–(1.2) on \([0, \infty) \times \mathbb{R}^2\). Since \( f \) and \( g \) are compactly-supported, we can take \( R > 0 \) such that \( \text{supp } f \cup \text{supp } g \subset \{ x \in \mathbb{R}^2; |x| \leq R \} \). Then, by the finite propagation property, we have

\[
\text{supp } u(t, \cdot) \subset \{ x \in \mathbb{R}^2; |x| \leq t + R \} \tag{3.1}
\]

for \( t \geq 0 \). We define \( U = (U_1, U_2) \) by \( U_j(t, x) = \mathcal{D} (|x|^{1/2} u_j(t, x)) \), \( j = 1, 2 \). We also introduce \( H = (H_1, H_2) \) by

\[
H_1 = \frac{1}{2} \left( r^{1/2} (\partial_t u_2)^2 (\partial_t u_1) + \frac{1}{t} U_2^2 U_1 \right) - \frac{1}{8t^{3/2}} (4\Omega^2 + 1) u_1,
\]

\[
H_2 = \frac{1}{2} \left( r^{1/2} (\partial_t u_1)^2 (\partial_t u_2) + \frac{1}{t} U_1^2 U_2 \right) - \frac{1}{8t^{3/2}} (4\Omega^2 + 1) u_2.
\]
By (2.1), we have

\[
\begin{align*}
\partial_t U_1(t, x) &= \frac{-1}{2t} U_1(t, x) U_2(t, x)^2 + H_1(t, x), \\
\partial_t U_2(t, x) &= \frac{-1}{2t} U_1(t, x)^2 U_2(t, x) + H_2(t, x).
\end{align*}
\] (3.2)

The following lemma tells us that \( H \) can be regarded as a remainder if we have a good control of \( u \) near the light cone.

**Lemma 3.1.** There exists a positive constant \( C \) which may depend on \( R \) such that

\[
|H(t, x)| \leq C t^{-1/2} (|\partial u| + (t + |x|)^{-1} |u_1^1|)^2 |u_1| + C t^{-3/2} |u_2|
\] for \((t, x) \in \Lambda_{\infty, R} := \{(t, x) \in \Lambda_{\infty} ; |x| \leq t + R\}.

For the proof, see Lemma 2.8 in [18].

Next we recall the basic decay estimates satisfied by the global small amplitude solution \( u \) to (1.1)–(1.2). From the argument of Section 3 in [18], we already know the following.

**Lemma 3.2.** Let \( k \geq 4 \), \( 0 < \mu < 1/10 \) and \( 0 < (8k + 7) \nu < \mu \). Suppose that \( \varepsilon \) is suitably small. Then the solution \( u \) to (1.1)–(1.2) satisfies

\[
|u(t, x)|_{k+1} \leq C \varepsilon (t + |x|)^{-1/2 + \mu},
\] (3.4)

\[
|\partial u(t, x)|_k \leq C \varepsilon (t + |x|)^{-1/2 + \nu} (t - |x|)^{-\mu-1}
\]

and

\[
|\partial u(t, x)| \leq C \varepsilon (t + |x|)^{-1/2} (t - |x|)^{-\mu-1}
\] (3.5)

for \((t, x) \in [0, \infty) \times \mathbb{R}^2\), where \( C \) is a positive constant independent of \( \varepsilon \).

In what follows, we denote various positive constants by the same letter \( C \) which may vary from one line to another. From (3.4), (3.5), (3.3) and Lemma 2.1 we have

\[
|U(t, x)| \leq |x|^{1/2} |\partial u(t, x)| + |x|^{1/2} |\partial u(t, x) - \hat{\omega} U(t, x)| \leq C \varepsilon (t - |x|)^{-\mu-1}
\] (3.6)

and

\[
|H(t, x)| \leq C \varepsilon^2 t^{-1/2} (t + |x|)^{-\mu-1} (t - |x|)^{\mu-1} + C t^{-3/2} (t + |x|)^{-\mu-1/2} \\
\leq C \varepsilon t^{2\mu-3/2} (t - |x|)^{-\mu-1/2}
\] (3.7)

for \((t, x) \in \Lambda_{\infty, R}\). Remember that the weights \(|x|^{-1}, t^{-1}, (1 + t)^{-1}, \langle t + |x| \rangle^{-1}\) are equivalent to each other on \( \Lambda_{\infty, R}\). Indeed we have

\[
\langle t + |x| \rangle^{-1} \leq |x|^{-1} \leq 2t^{-1} \leq 3(1 + t)^{-1} \leq 3(R + 2) \langle t + |x| \rangle^{-1}.
\]
Now we make the final reduction. We set
\[ \Sigma = \{(t, x) \in [0, \infty) \times \mathbb{R}^2; |x| \geq t/2 = 1 \text{ or } |x| = t/2 \geq 1\} \]
and \( t_{0,\sigma} = \max\{2, -2\sigma\} \). Then, since the half line \( \{(t, (t + \sigma)\omega); t \geq 0\} \) meets \( \Sigma \) at the point \((t_{0,\sigma}, (t_{0,\sigma} + \sigma)\omega)\) for each \((\sigma, \omega) \in \mathbb{R} \times S^1\), we can see that
\[ \Lambda_{\infty,R} = \bigcup_{(\sigma,\omega) \in (-\infty,R] \times S^1} \{(t, (t + \sigma)\omega); t \geq t_{0,\sigma}\}. \]

We also note that there exists a positive constant \( c_0 \) depending only on \( R \) such that
\[ c_0^{-1}\langle \sigma \rangle \leq t_{0,\sigma} \leq c_0 \langle \sigma \rangle \tag{3.8} \]
for \( \sigma \in (-\infty, R] \). We set \( V_j(t; \sigma, \omega) = U_j(t, (t + \sigma)\omega) \) and \( K_j(t; \sigma, \omega) = H_j(t, (t + \sigma)\omega) \) for \((t; \sigma, \omega) \in [t_{0,\sigma}, \infty) \times \mathbb{R} \times S^1, j = 1, 2\). Then we can rewrite (3.2) as
\[ \begin{cases} 
\partial_t V_1(t) = \frac{-1}{2t} V_1(t)V_2(t)^2 + K_1(t), \\
\partial_t V_2(t) = \frac{-1}{2t} V_1(t)^2 V_2(t) + K_2(t),
\end{cases} \tag{3.9} \]
which we call the profile equation. It follows from (3.6) and (3.7) that
\[ |V(t; \sigma, \omega)| \leq C\varepsilon \langle \sigma \rangle^{-\mu - 1} \tag{3.10} \]
and
\[ |K(t; \sigma, \omega)| \leq C\varepsilon \langle \sigma \rangle^{-\mu - 1/2} t^{2\mu - 3/2} \tag{3.11} \]
for \((t, \sigma, \omega) \in [t_{0,\sigma}, \infty) \times (-\infty, R] \times S^1\).

At the end of this section, let us summarize what has been done so far. By Lemma 2.1 and (3.4), the leading part for \( \partial u_j(t, x) \) as \( t \to \infty \) could be given by \( |x|^{-1/2} \hat{\omega}(x)V_j(t; |x| - t, x/|x|) \), and, in view of (3.10)–(3.11), the evolution of \( V = (V_1, V_2) \) could be characterized by the system
\[ \partial_t V_1 = \frac{-1}{2t} V_1 V_2^2, \quad \partial_t V_2 = \frac{-1}{2t} V_1^2 V_2, \]
up to harmless remainder terms. Our strategy of the proof of Theorem 1.1 consists of two steps: the first is to investigate the asymptotic behavior of \( V(t; \sigma, \omega) \) as \( t \to +\infty \), and the second is to convert it into that of \( \partial u(t, x) \). They will be carried out in Sections 4 and 5, respectively.
4 Asymptotics of solutions to the profile equation

In this section, we focus on large-time behavior of \( V(t; \sigma, \omega) \) introduced in the previous section. The goal here is to show the following.

\[ \text{Proposition 4.1.} \] Let \( V = (V_j(t; \sigma, \omega))_{j=1,2} \) be as above. There exists \( V^+ = (V^+_j(\sigma, \omega))_{j=1,2} \in L^2(\mathbb{R} \times S^1) \) such that

\[
\lim_{t \to \infty} \int_{\mathbb{R}} \int_{S^1} \left| \chi_t(\sigma)V(t; \sigma, \omega) - V^+(\sigma, \omega) \right|^2 dS_\omega d\sigma = 0, \tag{4.1}
\]

where \( \chi_t: \mathbb{R} \to \mathbb{R} \) is a bump function satisfying \( \chi_t(\sigma) = 1 \) for \( \sigma > -t \) and \( \chi_t(\sigma) = 0 \) for \( \sigma \leq -t \).

Before going into the proof, let us introduce two simple lemmas.

\[ \text{Lemma 4.1.} \] Let \( C_0 > 0, C_1 \geq 0, p > 1, q > 1 \) and \( t_0 \geq 2 \). Suppose that \( \Phi(t) \) satisfies

\[
\frac{d\Phi(t)}{dt} \leq -\frac{C_0}{t} |\Phi(t)|^p + \frac{C_1}{t^q}
\]

for \( t \geq t_0 \). Then we have

\[
\Phi(t) \leq \frac{C_2}{(\log t)^{p^* - 1}}
\]

for \( t \geq t_0 \), where \( p^* \) is the Hölder conjugate of \( p \) (i.e., \( 1/p + 1/p^* = 1 \)), and

\[
C_2 = \frac{1}{\log 2} \left( (\log t_0)^{p^*} \Phi(t_0) + C_1 \int_{2}^{\infty} \frac{\log \tau}{\tau^q} d\tau \right) + \left( \frac{p^*}{C_0 p} \right)^{p^* - 1}.
\]

For the proof, see Lemma 4.1 of [18].

\[ \text{Lemma 4.2.} \] Let \( t_0 \geq 0 \) be given. For \( \lambda, Q \in C \cap L^1([t_0, \infty)) \), assume that \( y(t) \) satisfies

\[
\frac{dy(t)}{dt} = \lambda(t)y(t) + Q(t)
\]

for \( t \geq t_0 \). Then we have

\[
|y(t) - y^+| \leq C_3 \int_{t}^{\infty} \left( |y^+| |\lambda(\tau)| + |Q(\tau)| \right) d\tau
\]

for \( t \geq t_0 \), where

\[
C_3 = \exp \left( \int_{t_0}^{\infty} |\lambda(\tau)| d\tau \right)
\]

and

\[
y^+ = y(t_0) e^{ \int_{t_0}^{\infty} \lambda(\tau) d\tau } + \int_{t_0}^{\infty} Q(s) e^{ \int_{s}^{\infty} \lambda(\tau) d\tau } ds.
\]
Proof. Put

\[ \Phi(t; s) = \exp \left( \int_s^t \lambda(\tau) d\tau \right) \]

for \( s, t \in [t_0, \infty] \). Then we see that

\[ y(t) = \Phi(t; t_0)y(t_0) + \int_{t_0}^t \Phi(t; s)Q(s) ds = \Phi(t; \infty) y^+ - \int_t^\infty \Phi(t; s)Q(s) ds. \]

We also note that \( |\Phi(s; t)| \leq C_3 \) and that

\[ |\Phi(t; \infty) - 1| \leq C_3 \int_t^\infty |\lambda(\tau)| d\tau. \]

Therefore we obtain

\[ |y(t) - y^+| \leq |\Phi(t; \infty) - 1||y^+| + \int_t^\infty |\Phi(t; s)||Q(s)| ds \leq C_3|y^+| \int_t^\infty |\lambda(\tau)| d\tau + C_3 \int_t^\infty |Q(\tau)| d\tau, \]

as desired. \( \square \)

Proof of Proposition 4.1. We first show the pointwise convergence of \( V(t; \sigma, \omega) \) as \( t \to +\infty \). We note that (3.1) implies \( V(t; \sigma, \omega) = 0 \) if \( \sigma \geq R \). In what follows, we fix \( (\sigma, \omega) \in \times (-\infty, R] \times \mathbb{S}^1 \) and introduce

\[ \rho(t) = \rho(t; \sigma, \omega) := V_1(t; \sigma, \omega) K_1(t; \sigma, \omega) - V_2(t; \sigma, \omega) K_2(t; \sigma, \omega) \]

so that

\[ \frac{1}{2} \partial_t \left( (V_1(t))^2 - (V_2(t))^2 \right) = V_1(t) \partial_t V_1(t) - V_2(t) \partial_t V_2(t) = \rho(t). \]

It follows from (3.8), (3.10) and (3.11) that

\[ \int_{t_0, \sigma}^\infty |\rho(\tau; \sigma, \omega)| d\tau \leq \int_{t_0, \sigma}^\infty \sum_{j=1}^2 |V_j(\tau; \sigma, \omega) K_j(\tau; \sigma, \omega)| d\tau \leq \int_{t_0, \sigma}^\infty C \varepsilon^2 (\sigma)^{-3/2} r^{2\mu - 3/2} d\tau \leq C \varepsilon^2 (\sigma)^{-3/2} (t_0, \sigma)^{2\mu - 1/2} \leq C \varepsilon^2 (\sigma)^{2\mu - 2}. \]

Therefore we obtain

\[ (V_1(t; \sigma, \omega))^2 - (V_2(t; \sigma, \omega))^2 = (V_1(t_0, \sigma, \omega))^2 - (V_2(t_0, \sigma, \omega))^2 + 2 \int_{t_0, \sigma}^t \rho(\tau; \sigma, \omega) d\tau = m(\sigma, \omega) - r(t; \sigma, \omega) \]

(4.2)
for $t \geq t_{0,\sigma}$, where

$$m = m(\sigma, \omega) := (V_1(t_{0,\sigma}; \sigma, \omega))^2 - (V_2(t_{0,\sigma}; \sigma, \omega))^2 + 2 \int_{t_{0,\sigma}}^{\infty} \rho(\tau; \sigma, \omega) d\tau$$

and

$$r(t) = r(t; \sigma, \omega) := 2 \int_{t}^{\infty} \rho(\tau; \sigma, \omega) d\tau.$$  

Note that

$$|m| \leq |V(t_{0,\sigma})|^2 + C \int_{t_{0,\sigma}}^{\infty} |\rho(\tau)| d\tau \leq C \varepsilon^2 \langle \sigma \rangle^{2\mu - 2}$$

and

$$|r(t)| \leq C \int_{t}^{\infty} |\rho(\tau)| d\tau \leq C \varepsilon^2 \langle \sigma \rangle^{-3/2} t^{2\mu - 1/2}.$$

(4.3)

Now we divide the argument into three cases according to the sign of $m(\sigma, \omega)$ as follows.

- **Case 1:** $m(\sigma, \omega) > 0$. First we focus on the asymptotics for $V_2(t)$. By (3.9), (3.10), (3.11), (4.2) and (4.3), we have

$$\partial_t V_2(t) = -\frac{1}{2t} V_2(t)^3 - \frac{m}{2t} V_2(t) + \frac{r(t)}{2t} V_2(t) + K_2(t) \leq -\frac{1}{2t} V_2(t)^3 - \frac{m}{2t} V_2(t) + C \varepsilon \langle \sigma \rangle^{-\mu - 1/2} t^{2\mu - 3/2},$$

whence

$$\frac{1}{2} \partial_t \left( t^m V_2(t)^2 \right) = t^m V_2(t) \left( \partial_t V_2(t) + \frac{m}{2t} V_2(t) \right) \leq t^m \left( -\frac{1}{2t} V_2(t)^4 + C \varepsilon^2 \langle \sigma \rangle^{-3/2} t^{2\mu - 3/2} \right) \leq C \varepsilon^2 \langle \sigma \rangle^{-3/2} t^{2\mu + m - 3/2}.$$  

Integration in $t$ leads to

$$t^m V_2(t)^2 - (t_{0,\sigma})^m V_2(t_{0,\sigma})^2 \leq C \varepsilon^2 \langle \sigma \rangle^{-3/2} \int_{t_{0,\sigma}}^{t} \tau^{2\mu + m - 3/2} d\tau \leq C \varepsilon^2 \langle \sigma \rangle^{-3/2} (t_{0,\sigma})^{2\mu + m - 1/2} \leq C \varepsilon^2 \langle \sigma \rangle^{2\mu + m - 2}$$

for $t \geq t_{0,\sigma}$. Therefore we deduce that

$$|V_2(t)| \leq C \varepsilon \langle \sigma \rangle^{\mu + m/2 - 1/2} t^{-m/2}.$$  

(4.4)
In particular, $V_2(t) \to 0$ as $t \to +\infty$. Next we turn our attentions to the asymptotics for $V_1(t)$. Since $V_1(t)$ solves $V_1'(t) = \lambda(t)V_1(t) + Q(t)$ with $\lambda(t) = -V_2(t)^2/t$ and $Q(t) = K_1(t : \sigma, \omega)$, we can apply Lemma 4.2 to $V_1(t)$. Then we have

$$|V_1(t) - W_1^+| \leq C \int_t^\infty \left( \frac{|W_1^+||V_2(\tau)|^2}{\tau} + |K_1(\tau)| \right) d\tau,$$

where

$$W_1^+ = W_1^+(\sigma, \omega) = V_1(t_0, \sigma, \omega) e^{-\int_{t_0}^\infty V_1(\tau; \sigma, \omega)^2 \frac{d\tau}{\tau}} + \int_{t_0}^\infty K_1(\tau; \sigma, \omega) e^{-\int_{t_0}^\infty V_1(\tau; \sigma, \omega)^2 \frac{d\tau}{\tau}} d\tau.$$

By (3.10), (3.11) and (4.4), we have

$$|W_1^+| \leq |V_1(t_0, \sigma)| + \int_{t_0}^\infty |K_1(s)| ds \leq C \varepsilon(\sigma)^{\mu-1}$$

and

$$\int_t^\infty \left( \frac{|W_1^+||V_2(\tau)|^2}{\tau} + |K_1(\tau)| \right) d\tau \leq C \int_t^\infty \left( \frac{\varepsilon(\sigma)^{3\mu+m-3}}{\tau^{m+\mu}} + \frac{\varepsilon(\sigma)^{-\mu-1/2}}{\tau^{3/2-2\mu}} \right) d\tau \leq C \varepsilon(\sigma)^{3\mu+m-3} + C \varepsilon(\sigma)^{-\mu-1/2}.$$

Therefore we conclude that $V_1(t) \to W_1^+$ as $t \to +\infty$.

**Case 2:** $m(\sigma, \omega) < 0$. Similarly to the previous case, we have

$$\lim_{t \to \infty} |V_1(t; \sigma, \omega)| = 0, \quad \lim_{t \to \infty} |V_2(t; \sigma, \omega) - W_2^+(\sigma, \omega)| = 0,$$

where

$$W_2^+(\sigma, \omega) := V_2(t_0, \sigma, \omega) e^{-\int_{t_0}^\infty V_1(\tau; \sigma, \omega)^2 \frac{d\tau}{\tau}} + \int_{t_0}^\infty K_2(\tau; \sigma, \omega) e^{-\int_{t_0}^\infty V_1(\tau; \sigma, \omega)^2 \frac{d\tau}{\tau}} d\tau.$$

Remark that $|W_2^+| \leq C \varepsilon(\sigma)^{\mu-1}$.

**Case 3:** $m(\sigma, \omega) = 0$. By (3.3), (3.10), (3.11), (4.2) and (4.3), we have

$$\partial_t (V_1(t)^2) = -\frac{1}{t} V_1(t)^4 - \frac{r(t)}{t} V_1(t)^2 + 2V_1(t)K_1(t) \leq -\frac{1}{t} (V_1(t))^4 + C \varepsilon^2(\sigma)^{-3/2} t^{2\mu-3/2}$$

for $t \geq t_{0, \sigma}$. Thus we can apply Lemma 4.1 with $\Phi(t) = V_1(t)^2$ to obtain

$$|V_1(t)| \leq \frac{C}{\sqrt{\log t}} \to 0 \quad (t \to +\infty).$$

Also (4.2) gives us $V_2(t) = \sqrt{V_1(t)^2 + r(t)} \to 0$ as $t \to \infty$. 

Therefore we conclude that $V_1(t) \to W_1^+$ as $t \to +\infty$. 

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Summing up the three cases above, we deduce that $V(t; \sigma, \omega)$ converges as $t \to +\infty$ for each fixed $(\sigma, \omega) \in \mathbb{R} \times S^1$. In order to show (4.1), we set

$$V^+_1(\sigma, \omega) := \begin{cases} W^+_1(\sigma, \omega) & (m(\sigma, \omega) > 0), \\ 0 & (m(\sigma, \omega) \leq 0), \end{cases}$$

$$V^+_2(\sigma, \omega) := \begin{cases} 0 & (m(\sigma, \omega) \geq 0), \\ W^+_2(\sigma, \omega) & (m(\sigma, \omega) < 0), \end{cases}$$

and $V^+(\sigma, \omega) = (V^+_j(\sigma, \omega))_{j=1,2}$ for $(\sigma, \omega) \in \mathbb{R} \times S^1$. Then, by virtue of (4.5), we have $V^+ \in L^2(\mathbb{R} \times S^1)$ and

$$\left| \chi_t(\sigma)V(t; \sigma, \omega) - V^+(\sigma, \omega) \right|^2 \leq C\varepsilon^2(\sigma)^{2\mu-2} \in L^1(\mathbb{R} \times S^1)$$

for all $t \geq t_{0,\sigma}$. Moreover, it holds that

$$\lim_{t \to \infty} \left| \chi_t(\sigma)V(t; \sigma, \omega) - V^+(\sigma, \omega) \right|^2 = 0$$

for each fixed $(\sigma, \omega) \in \mathbb{R} \times S^1$. Consequently, Lebesgue’s dominated convergence theorem yields (4.1).

5 Proof of Theorem 1.1

We are going to prove Theorem 1.1. First we recall the following useful lemma.

Lemma 5.1 ([13] Theorem 2.1). For $\phi \in C \left( [0, \infty); \dot{H}^1(\mathbb{R}^2) \right) \cap C^1 \left( [0, \infty); L^2(\mathbb{R}^2) \right)$, the following two assertions (i) and (ii) are equivalent:

(i) There exists $(\phi^+_0, \phi^+_1) \in \dot{H}^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ such that

$$\lim_{t \to \infty} \| \phi(t) - \phi^+(t) \|_E = 0,$$

where $\phi^+ \in C \left( [0, \infty); \dot{H}^1(\mathbb{R}^2) \right) \cap C^1 \left( [0, \infty); L^2(\mathbb{R}^2) \right)$ is a unique solution to $\Box \phi^+ = 0$, $\phi^+(0) = \phi^+_0$, $\partial \phi^+(0) = \phi^+_1$.

(ii) There exists $\Phi = \Phi(\sigma, \omega) \in L^2(\mathbb{R} \times S^1)$ such that

$$\lim_{t \to \infty} \| \partial \phi(t, \cdot) - \hat{\omega}(\cdot) \Phi^+(t, \cdot) \|_{L^2(\mathbb{R}^2)} = 0,$$

where $\hat{\omega}(x) = (-1, x_1/|x|, x_2/|x|)$ and $\Phi^+(t, x) = |x|^{-1/2}\Phi(|x| - t, x/|x|)$.

By virtue of this lemma, to prove that $u_1$ is asymptotically free, it is sufficient to show

$$\lim_{t \to \infty} \| \partial u_1(t, \cdot) - \hat{\omega}(\cdot)V^+_1(\cdot) \|_{L^2(\mathbb{R}^2)} = 0 \quad (5.1)$$
for $V_1^+(\sigma, \omega)$ obtained in Section 4. To prove (5.1), we split
\[
\|\partial u_1(t, \cdot) - \hat{\omega}(\cdot)V_1^{+2}(t, \cdot)\|_{L^2(\mathbb{R}^2)}^2
= \int_{\mathbb{R}^2} |\partial u_1(t, x) - \hat{\omega}(x)| |x|^{-1/2} V_1^+(|x| - t, x/|x|) |dx
\leq 2 \int_{\mathbb{R}^2 \setminus \Lambda_\infty} |\partial u_1(t, x) - \hat{\omega}(x)| |x|^{-1/2} V_1(t; |x| - t, x/|x|) |^2 dx
+ 2 \int_{\Lambda_\infty} |\partial u_1(t, x) - \hat{\omega}(x)| |x|^{-1/2} V_1(t; |x| - t, x/|x|) |^2 dx
+ 2 \int_0^\infty \int_{\mathbb{S}^1} |\hat{\omega}(r\omega)V_1(t; r - t, \omega) - \hat{\omega}(r\omega) V_1^+(r - t, \omega)|^2 dS_\omega dr
=: J_1(t) + J_2(t) + J_3(t).
\]
To show the decay for $J_1(t)$, we note that $\langle t + |x| \rangle \leq C\langle t - |x| \rangle$ on $\mathbb{R}^2 \setminus \Lambda_\infty$. Then (3.4) and (5.10) imply
\[
J_1(t) \leq C\varepsilon^2 \int_{\mathbb{R}^2 \setminus \Lambda_\infty} (\langle t - |x| \rangle)^{-1} \langle t + |x| \rangle^{2\mu - 2} + |x|^{-1} \langle t - |x| \rangle^{2\mu - 2} dx
\leq C\varepsilon^2 \int_{\mathbb{R}^2 \setminus \Lambda_\infty} |x|^{-1} \langle t + |x| \rangle^{2\mu - 2} dx
\leq C\varepsilon^2 \int_0^\infty \int_{\mathbb{S}^1} (1 + t + r)^{2\mu - 2} dS_\omega dr
\leq C\varepsilon^2 (1 + t)^{2\mu - 1}.
\]
As for $J_2(t)$, we see from Lemma 2.1 and (3.4) that
\[
J_2(t) = 2 \int_{\Lambda_\infty} |x|^{-1} |x|^{1/2} \partial u_1(t, x) - \hat{\omega}(x) D (|x|^{1/2} u_1(t, x)) |^2 dx
\leq C \int_{\Lambda_\infty} |x|^{-1} \langle t + |x| \rangle^{-1} |u(t, x)|^2 dx
\leq C\varepsilon^2 \int_{\mathbb{R}^2} |x|^{-1} \langle t + |x| \rangle^{2\mu - 2} dx
\leq C\varepsilon^2 (1 + t)^{2\mu - 1}.
\]
Finally, it follows from (4.1) that
\[
J_3(t) \leq C \int_0^\infty \int_{\mathbb{S}^1} |V_1(t; r - t, \omega) - V_1^+(r - t, \omega)|^2 dS_\omega dr
\leq C \int_{-t}^\infty \int_{\mathbb{S}^1} |V_1(t; \sigma, \omega) - V_1^+(\sigma, \omega)|^2 dS_\omega d\sigma
\leq C \int_{\mathbb{R}} \int_{\mathbb{S}^1} |\chi(t) V_1(t; \sigma, \omega) - V_1^+(\sigma, \omega)|^2 dS_\omega d\sigma
\rightarrow 0
\]
as $t \to \infty$. Piecing them together, we arrive at (5.1). Similarly we have
\[
\lim_{t \to \infty} \|\partial u_2(t, \cdot) - \hat{\omega}(\cdot)V_2^+(t, \cdot)\|_{L^2(\mathbb{R}^2)} = 0,
\]
where $V_2^+$ is from Proposition 4.1. With the aid of Lemma 5.1 we conclude that $u_2$ is also
asymptotically free. \qed

Acknowledgments

The authors would like to thank Professor Soichiro Katayama, Dr. Yuji Sagawa and Daisuke
Sakoda for their useful conversations on this subject. The work of H. S. is supported by
Grant-in-Aid for Scientific Research (C) (No. 17K05322), JSPS.

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