Functional CLT for non-Hermitian random matrices

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Abstract. For large dimensional non-Hermitian random matrices $X$ with real or complex independent, identically distributed, centered entries, we consider the fluctuations of $f(X)$ as a matrix where $f$ is an analytic function around the spectrum of $X$. We prove that for a generic bounded square matrix $A$, the quantity $\text{Tr} f(X)A$ exhibits Gaussian fluctuations as the matrix size grows to infinity, which consists of two independent modes corresponding to the tracial and traceless parts of $A$. We find a new formula for the variance of the traceless part that involves the Frobenius norm of $A$ and the $L^2$-norm of $f$ on the boundary of the limiting spectrum.

Résumé. On étudie les fluctuations de $f(X)$, où $X$ est une matrice aléatoire non-hermitienne de grande taille à coefficients i.i.d. (réels ou complexes), et $f$ une fonction analytique sur un domaine qui contient le spectre de $X$. On prouve que, pour une matrice carrée générique et bornée $A$, les fluctuations de la quantité $\text{Tr} f(X)A$ sont asymptotiquement gaussiennes et comportent deux modes indépendants, correspondant aux composantes traciale et de trace nulle de $A$. Une nouvelle formule est établie pour la variance de la composante de trace nulle, qui implique la norme de Frobenius de $A$ et la norme $L^2$ de $f$ sur la frontière du spectre limite.

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1. Introduction

A distinctive feature of eigenvalues of large dimensional random matrices as point processes is the strong correlation between them. In particular for a Hermitian random matrix, this feature reflects the original idea of Wigner that random matrices can serve as universal models for strongly correlated system. One empirical evidence, among many others, of the strong correlation is the size of the variances of their linear statistics. Consider the eigenvalues $\{\sigma_i\}_{i \leq N}$ of an $(N \times N)$ random matrix $X$ with independent, identically distributed (i.i.d.) entries with zero mean and variance $1/N$. Then the celebrated circular law \cite{EJP99, EJP01, EJP01b, EJP01c} states that the distribution of a randomly chosen eigenvalue $\sigma_i$, or equivalently the empirical spectral distribution (e.s.d.) of $X$, weakly converges to the uniform distribution on the unit disk $\mathbb{D}$. In this setting, the linear statistics $\text{Tr} f(X) = \sum_{i=1}^{N} f(\sigma_i)$ for a regular test function $f$ has fluctuations of constant order, which happens to be Gaussian \cite{EJP03, EJP04}. Compared to the classical central limit theorem where $\sigma_i$’s are independent random variables, the linear statistics of $X$ does not need to be scaled by $N^{-1/2}$ showing that its eigenvalues have strong correlation.

The question we ask in this paper is whether a \textit{functional} version of the central limit theorem holds for $X$. More precisely, for an i.i.d. random matrix $X$ above and a deterministic norm-bounded $(N \times N)$ matrix $A$, we are interested in the fluctuations of the functional statistics

\begin{equation}
\text{Tr} f(X)A := \frac{1}{2\pi i} \oint_{\gamma} f(z) \text{Tr}((z - X)^{-1}A)dz,
\end{equation}

where $f$ is analytic in a neighborhood $\mathcal{D}$ of the closed unit disk $\mathbb{D}$ and $\gamma$ is a positively oriented curve in $\mathcal{D}$ encircling the spectrum $\sigma(X)$ of $X$. Note that this quantity is well-defined only when $\sigma(X) \subset \mathcal{D}$, but we will see below that such an event has very high probability. Our main result, Theorem 2.3, proves that $\text{Tr} f(X)A$ has Gaussian fluctuations of constant order.

The main motivation is to identify further Gaussian modes beyond the traditionally studied linear eigenvalue statistics. Eventually $f(X)$ is a large matrix that contains many other modes beyond its trace. The simplest observable that detects the genuine matrix character of $f(X)$ is when we test it against a deterministic matrix $A$. In particular, by understanding
the limiting distribution of $\text{Tr}(X)A$ we can extend the elegant Gaussian free field (GFF) interpretation of the tracial CLT result, given in [21], to a conceptually novel matrix valued GFF.

We note that the analogous question with similar motivation for Wigner matrices has been answered in [8, 17]. For a Wigner matrix $W$, a Hermitian random matrix with independent, centered entries of variance $1/N$, it is well known that its e.s.d. converges to the semi-circle distribution in $[-2, 2]$ from [24] and that the linear statistics $\text{Tr}(W)$ has Gaussian fluctuations (see [18] for example). Furthermore, it was proved in [8] that the functional statistics $\text{Tr}(W)A$ has Gaussian fluctuations in the limit $N \to \infty$, where the test function can be chosen in both macro- and mesoscopic regimes, that is, $f(X) := g(N^\alpha(x - E))$ for some $\alpha \in (0, 1)$ and $E \in (-2, 2)$. Functional statistics in the Hermitian setting directly relates to overlaps of eigenstates with $A$ via spectral decomposition

$$\text{Tr}(f(X)A) = \sum_{i=1}^{N} f(\lambda_i) \langle u_i, Au_i \rangle, \quad Wu_i = \lambda u_i, \quad \|u_i\| = 1.\tag{1.2}$$

In our setting, since $X$ is not normal, the matrix-function $f(X)$ itself is defined only when $f$ is analytic in a domain $\mathcal{D}$ containing the spectrum of $X$. Hence all eigenvalues of $X$ must contribute to $f(X)$, and in particular we cannot localize in the spectrum when considering $f(X)$. This differentiates $\text{Tr}(f(X)A)$ from the tracial statistics $\text{Tr}(f(X))$, for which we can use the fact that

$$\text{Tr}(f(X)) = \sum_{i=1}^{N} f(\sigma_i)$$

to reduce the problem to analyzing the e.s.d. of $X$. The identity (1.2) enables us to define $\text{Tr}(f(X))$ for more general test functions $f$, and it was proved in [9] that $\text{Tr}(f(X))$ has Gaussian fluctuations when $f$ is $H^2$ or on a mesoscopic scale starting from this. Interestingly, the variance of $\text{Tr}(f(X)A)$ we found in Theorem 2.3 involves the norm of $L^2$-Hardy space, which does not seem to have canonical generalization to non-analytic functions, especially to those vanishing on the boundary $\partial \mathcal{D}$.

Another obstacle in studying the functional statistics $\text{Tr}(f(X)A)$ is the absence of Girko’s formula, which is a commonly used technique to study the eigenvalues of non-Hermitian random matrices. For example, Girko’s formula played a vital role in [1, 2, 9], and even in the original proofs of the circular law in [4, 14]. In terms of $X$, a version of Girko’s formula in [23] is as follows; for a smooth and compactly supported complex function $X$ along the imaginary axis, not that of $X$, so that we can apply robust approaches developed for Hermitian random matrices such as local laws. The most crucial drawback of Girko’s formula is that it applies only to tracial quantities involving the eigenvalues, not the matrix itself. In the same vein as the previous paragraph, the functional statistics $\text{Tr}(f(X)A)$ concerns both eigenvalues and eigenvectors, thus we need a different approach.

Nevertheless, we find that one of the four $(N \times N)$ blocks of $(W_z - \eta I)^{-1}$ can serve as an approximation to $(X - z)^{-1}$ when $\eta$ is small enough. Applying this observation to (1.1) we may work with the contour integral of the block of $(W_z - \eta I)^{-1}$, which effectively replaces Girko’s formula. We remark that a similar argument was used in [11] to express $f(X)g(X)^*$ as the contour integral of certain block of a Hermitian resolvent. Our argument requires $\eta$ to be very small, well below the optimal scale $\eta \gg 1/N$ typically appearing in local laws for $W_z$ (see [1] for instance). While it is hard to keep track of $(W_z - \eta I)^{-1}$ for $\eta \ll 1/N$ when $z$ is inside the limiting spectrum $\mathbb{D}$, since we are assuming $\overline{\mathbb{D}} \subset \mathcal{D}$, we can take the contour $\gamma$, hence $z$, in $\mathbb{D} \setminus \overline{\mathbb{D}}$. This allows us to apply the optimal local law for $W_z$ outside the spectrum with all ranges of $\eta$ established e.g. in [3].

After approximating $\text{Tr}(f(X)A)$ with certain functional statistics of $(W_z - \eta I)^{-1}$, we prove that its moments asymptotically satisfy a recursion which leads to convergence of moments via Wick’s theorem. Along the calculation, we find that quantities corresponding to the variances are normalized traces of product of three resolvents with deterministic matrices in between. We adapt the strategy of [8, 9] to derive their deterministic approximates, using local laws for product of two resolvents proved in [9]. Actual calculations in this part involve cumulant expansions in the following form as appeared in [15]; for a real or complex random variable $Z$ and a smooth function $f$ with bounded derivatives, we have

$$\mathbb{E}Z f(Z) = \sum_{k,l=0}^{\infty} \kappa(k + 1,l) \frac{k!}{k!l!} \mathbb{E}f^{(k)}(Z),$$

where $\kappa(k + 1,l)$ is the cumulant of order $k + l$. The cumulant expansions are valid under the assumption that $f$ is $C^\infty$ and has bounded derivatives.
where \( \partial^{(k,l)} := \frac{\partial^{k+l}}{\partial Z^k \partial \bar{Z}^l} \) and \( \kappa(k,l) \) are the mixed cumulants, that is,

\[
\begin{aligned}
\kappa(k,l) &\equiv \kappa_C(k,l) := (-i)^{k+l} \left[ \frac{\partial^{k+l}}{m^{k+l}} \log \mathbb{E} e^{i s Z + i \bar{Z}} \right]_{s,t=0} \quad \text{if } Z \text{ is complex}, \\
\kappa(k,l) &\equiv \kappa_R(k,l) := \kappa_C(k,0) \mathbb{I}(l = 0) \quad \text{if } Z \text{ is real}.
\end{aligned}
\]

We conclude this section with a brief history of Gaussian fluctuations in linear statistics of random matrices. For Hermitian random matrices, tracial CLT has been a classical topic and there have been many results with increasing generality in terms of regularity of \( f \); see \([5, 15, 18, 22]\) for instance. As mentioned above, a functional CLT for Wigner matrices was proved in \([8, 17]\), the former covering also mesoscopic regimes. For non-Hermitian matrices, \([10, 19, 20]\) presented tracial CLT on macroscopic scale for analytic test functions, and later in \([9]\) it was generalized to \( H^2 \) test functions both in macroscopic and partially mesoscopic scales.

The rest of this paper is organized as follows: In Section 2, we rigorously define our model and state our main result, Theorem 2.3. In Section 3, we explain the key components of our proof and prove our main result based on them. The CLT for the resolvents of \( W \) for complex-valued \( X \) is presented in Section 4, which is the main technical achievement of our paper. Section 5 collects computational and technical parts of the proof of the resolvent CLT. Finally, in Section 6 we prove the main result for real-valued \( X \).

1.1. Notational conventions

We introduce some conventions used throughout the paper. For each \( r > 0 \), \( \mathbb{D}_r \) denotes the open, centered disk in \( \mathbb{C} \) with radius \( r \) and \( \mathbb{D}_r^c \) stands for its complement in \( \mathbb{C} \). For integers \( m \) and \( n \), we write \([m, n] := [m, n] \cap \mathbb{Z} \). When \( m = 1 \), we further abbreviate \([n] \equiv [1, n]\). We denote the identity matrix of any dimension by \( I \). For an \((N \times N)\) matrix \( A \), we write \( (A) := N^{-1} \text{Tr} A \) for its normalized trace. For a \( 2N \times 2N \) matrix \( P \), we write \( \{P[k]\}_{k \in [2]} \) to denote its \((N \times N)\) blocks, and conversely for an \((N \times N)\) matrix \( A \), we denote by \( A^{(kl)} \) the \((2N \times 2N)\) matrix satisfying \( (A^{(kl)})^{(k'l')} = A_{kk'} \delta_{ll'} \); for example,

\[
P = \begin{pmatrix} P^{[11]} & P^{[12]} \\ P^{[21]} & P^{[22]} \end{pmatrix}, \quad A^{[12]} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}.
\]

We will often use that \( \text{Tr} P^{[ij]} A = \text{Tr} P A^{[ji]} \). We use the following abbreviations for sums over \([N]\):

\[
\sum_{\substack{a_1, \ldots, a_n \ni 1 \atop a_1, \ldots, a_n}}^{m} = \prod_{i=1}^{n} \sum_{a_i = 1}^{N}, \quad \sum_{\substack{a_1, \ldots, a_n \ni 1 \atop a_1, \ldots, a_n \neq b_1, \ldots, b_m}}^{(b_1, \ldots, b_m)} = \prod_{i=1}^{n} \sum_{a_i = 1}^{N} \sum_{a_i \neq b_1, \ldots, b_m}^{b_1, \ldots, b_m}, \quad b_1, \ldots, b_m \in [N].
\]

For a random variable, vector, or matrix \( X \), we write \([X] = X - \mathbb{E}[X] \). Given two \( N \)-dependent random vectors \( Y \) and \( Z \) of the same dimension \( n \) with all moments finite and a function \( f : \mathbb{N} \to [0, \infty) \), we write

\[
Y \equiv Z + O_m(f(N))
\]

if the following holds; for each fixed monic \(*\)-monomial \( p \) of degree \( k \), there is a constant \( C_{n,k} > 0 \) such that

\[
\|\mathbb{E}[p(Y)] - \mathbb{E}[p(Z)]\| \leq C_{n,k} f(N).
\]

In other words, \( Y \) and \( Z \) are \( f \)-close to each other in the sense of all moments. Note that this concept is really meaningful when \( Y \) and \( Z \) are of order 1 and \( f(N) \ll 1 \), and we apply it only in such occasions.

For two \( N \)-dependent random variables \( X \) and \( Y \) with \( Y \geq 0 \), we write \( X \prec Y \) or \( X = O_{\prec}(Y) \) if the following holds; for all fixed \( \epsilon, D > 0 \), there exists \( N_0(\epsilon, D) \in \mathbb{N} \) such that

\[
P \left( |X| > N^\epsilon Y \right) \leq N^{-D}, \quad \forall N \geq N_0(\epsilon, D).
\]

In this case we say that \( X \) is \textit{stochastically dominated} by \( Y \).

Finally, we use the standard big-O and \( \lesssim \) notations; for two functions \( f \) and \( g \) of \( N \) with \( g \geq 0 \), we write \( f \lesssim g \) or \( f = O(g) \) when there is a constant \( C > 0 \) such that \( f(N) \leq C g(N) \) for all \( N \).
2. Model and results

Definition 2.1. Let $\chi$ be either real or genuinely complex random variable that satisfies the following conditions:

- $\mathbb{E}[|\chi|] = 0$ and $\mathbb{E}[|\chi|^2] = 1$;
- for each $p \in \mathbb{N}$, there exists $C_p > 0$ such that $\mathbb{E}[|\chi|^p] \leq C_p$;
- $\mathbb{E}[\chi^2] = 0$ if $\chi$ is complex-valued$^1$.

We define $X$ to be the $(N \times N)$ random matrix whose entries are independent and identically distributed (i.i.d.) with the same law as $N^{-1/2} \chi$.

As mentioned in the introduction, a proper definition of $f(X)$ must be restricted to an event of the form $\sigma(X) \subset D$.

We introduce our choice of such an event in the next lemma, which also shows that this event has high probability.

Lemma 2.2. [3, Lemma 6.1] For each $\delta, \kappa > 0$, define the $N$-dependent event $\Omega_\delta(\kappa)$ as follows:

\[
\Omega_\delta(\kappa) := \left\{ \sup_{z \in \mathbb{D}_{1+\delta/2}} \|((X - z)(X - z)^*)^{-1}\| \leq \kappa^{-2} \right\}.
\]

For any fixed $\delta > 0$ and $\kappa \in (0, \delta/2)$, we have $\mathbb{P}[\Omega_\delta(\kappa)] > 1 - N^{-D}$ for all $D > 0$ when $N$ is sufficiently large.

Note that the definition (2.1) of $\Omega_\delta(\kappa)$ already implies $\sigma(X) \subset \mathbb{D}_{1+\delta/2}$ on the event. Next, we introduce our main result, the central limit theorem for $\mathbb{E}[f(X)A]$ restricted on $\Omega_\delta(\kappa)$:

Theorem 2.3. Let $n \in \mathbb{N}$, $\delta > 0$ and $\kappa \in (0, \delta/2)$ be fixed, $A_1, \cdots, A_n$ be $(N \times N)$ deterministic matrices with $\|A_i\| \leq 1$, and $f_1, \cdots, f_n$ be fixed analytic functions on $\mathbb{D}_{1+\delta}$ with $\|f_i\|_{L^\infty(D_{1+\delta/2})} \leq 1$. Define

\[
L_N(f_i, A_i) := \mathbb{E}_{\Omega_\delta(\kappa)} \left[ \mathbb{E}[f(X)A_i] \right], \quad i \in \llbracket n \rrbracket.
\]

Then the centered random vector $(\mathbb{E}[L_N(f_i, A_i)])_{i=1, \ldots, n}$ is approximately Gaussian, that is, for all $\epsilon > 0$ we have

\[
(\mathbb{E}[L_N(f_i, A_i)])_{i=1, \ldots, n} \approx (\xi(f_i, A_i))_{i=1, \ldots, n} + \mathcal{O}_n(N^{-1/2+\epsilon}),
\]

where $(\xi(f_i, A_i))_{i=1, \ldots, n}$ is a centered complex Gaussian vector whose covariance is given by

\[
\mathbb{E}[\xi(f_i, A_i)\xi(f_j, A_j)] = \frac{1}{\pi} \langle A_i^* A_j \rangle \langle f_i^* f_j \rangle_{L^2(\mathbb{D})} + \langle \hat{A}_i A_j^* \rangle \langle f_i f_j \rangle_{L^2(\mathbb{D})},
\]

\[
\mathbb{E}[\xi(f_i, A_i)\xi(f_j, A_j)] = \begin{cases} 0 & \text{if } \chi \text{ is complex}, \\ \frac{1}{\pi} \langle A_i A_j \rangle \langle f_i^* f_j \rangle_{L^2(\mathbb{D})} + \langle \hat{A}_i A_j^* \rangle \langle f_i f_j \rangle_{L^2(\mathbb{D})} & \text{if } \chi \text{ is real}. \end{cases}
\]

Here we denoted $f^*(z) = \overline{f(z)}$, $\hat{A} := A - \langle A \rangle$, and

\[
\langle f, g \rangle_{L^2(\mathbb{D})} := \frac{1}{2\pi} \int_{\partial \mathbb{D}} (f(z) - f(0))(g(z) - g(0))|dz|.
\]

Furthermore, the mean of $L_N(f, A)$ has the following asymptotics for all $\epsilon > 0$:

\[
\mathbb{E}[L_N(f, A)] - N\langle A \rangle f(0) = \begin{cases} \mathcal{O}(N^{-1/2+\epsilon}) & \text{if } \chi \text{ is complex}, \\ \langle A \rangle \left( \frac{f(1) + f(-1)}{2} - f(0) \right) + \mathcal{O}(N^{-1/2+\epsilon}) & \text{if } \chi \text{ is real}. \end{cases}
\]

Remark 2.4. In particular, for $n = 1$ and $A$ and $f$ as above, $L_N(f, A)$ has asymptotically the same distribution as the sum of three independent Gaussian random variables $\xi(f, \langle A \rangle I), \xi(f, \hat{A}_d)$, and $\xi(f, A_{od})$ where $A_d := \text{diag}(A_{ii})_{1 \leq i \leq N} - \langle A \rangle I$ and $A_{od} := A - \text{diag}(A_{ii})_{1 \leq i \leq N}$.

$^{1}$We made this commonly used assumption only for the sake of simplicity; our method can easily handle the exact dependence on arbitrary $\mathbb{E}(|\chi|^2) \in \mathbb{C}$ as well.
Remark 2.5. In fact, the matrix-function $f(X)$ can be defined for more general test functions $f$ under an additional assumption that all eigenvalues of $X$ are simple. In this case, $X$ admits the spectral decomposition

$$X = \sum_i \sigma_i r_i l_i^*$$

where $\{(l_i, r_i) : i \in [N]\}$ is a biorthogonal system of right and left eigenvectors subject to the normalization $l_i^* r_j = \delta_{ij}$. We may then define

$$f(X) := \sum_i f(\sigma_i) r_i l_i^*, \quad L_N(f, A) := \sum_i f(\sigma_i) l_i^* A r_i$$

for a general test function $f$, and it is easy to check that (2.6) is indeed consistent with (1.1) when $f$ is as in Theorem 2.3.

Even though using (2.6) in (2.2) serves as a proper definition of $L_N(f, A)$ under a mild assumption, working with (2.6) is prohibitively harder than (1.1). The most fundamental reason is that the spectral projectors $r_i l_i^*$ are very unstable and thus difficult to control, in contrast to Hermitian matrices which decompose into orthogonal spectral projections. For example, in [6] it is proved that $||l_i||^2 ||r_i||^2$ is typically of order $N$ and has a heavy-tailed distribution even after proper normalization. Moreover, all known results on $l_i$ and $r_i$’s including [6] are restricted to Ginibre ensemble, and their proofs exploit exact identities that are available only for Ginibre ensemble.

Finally, as mentioned in the introduction, the limiting variance as given in (2.4) only makes sense for analytic functions. Finding the correct generalization of (2.4) to generic test functions, for example by studying the case of Ginibre ensemble, would be an interesting, albeit highly nontrivial problem. We believe that it would require a very different approach from the present paper and thus left for future work.

2.1. Gaussian functional representation

The goal of this section is to find a representation for $\xi$ as a Gaussian Hilbert space on $H \otimes M_N(\mathbb{C})$, where $H$ is a Hilbert space of test functions and $M_N(\mathbb{C})$ is equipped with the inner product $\langle AB^* \rangle$.

In [21], it was proved that when $A = I$ the limiting Gaussian process $\xi(f, I)$ can be identified with the Gaussian free field conditioned to be harmonic outside the disk. The precise formulation as follows; we consider the Gaussian Hilbert space $\{h(f) : f \in H_0^1(\mathbb{C})\}$ on the Sobolev space $H_0^1(\mathbb{C})$ defined by

$$\mathbb{E}[h(f)] = 0, \quad \mathbb{E}[h(f) \overline{h(g)}] = \langle f, g \rangle_{H_0^1(\mathbb{C})}, \quad \mathbb{E}[h(f) h(g)] = \begin{cases} 0 & \text{if } \chi \text{ is complex}, \\ \langle f, g^* \rangle_{H_0^1(\mathbb{C})} & \text{if } \chi \text{ is real}, \end{cases}$$

where

$$H_0^1(\mathbb{C}) := \mathcal{C}_0^\infty(\mathbb{C})^\perp \|H_0^1(\mathbb{C})\|,$$

$$(f, g)_{H_0^1(\mathbb{C})} = \langle \nabla f, \nabla g \rangle_{L^2(\mathbb{C})}.$$

We decompose the Hilbert space $H_0^1(\mathbb{C})$ as the direct sum of three subspaces, $H_0^1(\mathbb{D})$, $H_0^1(\mathbb{D}^c)$, and their complement $H_0^1(\partial \mathbb{D}^c)^\perp$. From [21, Theorem 1.1 and Corollary 1.2] (see also Section 2.1 of [9]), we have that

$$\mathbb{E}[||\xi(f, I)||^2] = \frac{1}{4\pi} ||f'||_{L^2(\mathbb{D})} = \frac{1}{4\pi} ||P_0 f||_{H_0^1(\mathbb{C})}^2,$$

where $P_0$ denotes the orthogonal projection onto $H_0^1(\mathbb{D}) \oplus H_0^1(\partial \mathbb{D})^\perp$, the subspace consisting of functions harmonic outside the closed unit disk. Therefore $\xi(\cdot, I)$ can be identified with $(4\pi)^{-1/2} P_0 h$ where $P_0 h(f) := h(P_0 f)$.

For the functional part in $\xi(f, A)$, we consider the Gaussian field $u$ on the Hardy space $\mathcal{H}^2$ defined by

$$(2.7) \quad u(f) = \sum_{k=1}^\infty \xi_k \hat{f}(k), \quad f \in \mathcal{H}^2,$$

where $\{\xi_k : k \in \mathbb{N}\}$ is a collection of i.i.d. standard Gaussian variables in $\mathbb{C}$ if $\chi$ is complex-valued and in $\mathbb{R}$ if $\chi$ is real-valued, and

$$\mathcal{H}^2 := \{f \in L^2(\partial \mathbb{D}) : \hat{f}(k) = 0, \forall k < 0\}, \quad \hat{f}(k) := \frac{1}{2\pi} \int_{|z|=1} f(z) z^{-k} |dz|, \quad k \in \mathbb{Z}.$$
is considered as a Hilbert subspace of $L^2(\partial \mathbb{D})$. For a fixed $A \in M_N(\mathbb{C})$ with $\langle A \rangle = 0$, we have

$$E[|\xi(f, A)|^2] = E[|u(f)|^2]\langle AA^*\rangle, \quad E[\xi(f, A)^2] = E[u(f)^2]\langle AA^T\rangle.$$ 

Note that $E[|u(f)|^2] = 0$ when $\chi$ is complex.

Now we turn to finding an expression that accounts for the matrix part of $\xi$. First, for the tracial part, we consider $P_0 h \otimes I$ as a random functional on $H^1_0(\mathbb{C}) \otimes M_N(\mathbb{C})$ given by

$$P_0 h \otimes I)(f \otimes A) := h(P_0 f)\langle A \rangle.$$ 

It immediately follows that the covariances of (2.8) for different $(f, A)$’s match the first term of (2.4). Second, for the functional part, we consider $N^2$ i.i.d. copies $\{u_{ij} : i, j \in [N]\}$ of $u$ and define the functional

$$U(f, A) := \text{Tr} A U(f)^*, \quad U(f) \in M_N(\mathbb{C}), \quad U(f)_{ij} := \frac{1}{\sqrt{N}} u_{ij}(f).$$

In other words, $U(f)$ is an $(N \times N)$ random matrix whose entries are i.i.d. with the same law as $N^{-1/2} u(f)$ for each $f$. Then we have

$$E[U(f, A) U(g, B)] = (f, g)_{L^2(\partial \mathbb{D})} \langle AB^*\rangle, \quad E[U(f, A) U(g, B)] = \begin{cases} 0 & \text{if } \chi \text{ is complex}, \\ (f, g^*)_{L^2(\partial \mathbb{D})} \langle AB^\top\rangle & \text{if } \chi \text{ is real}. \end{cases}$$

Therefore, as in (2.8), we find that

$$P_1 U(f \otimes A) := U(P_1(f \otimes A))$$

has the same covariance as the functional part of (2.4), where $P_1$ denotes the orthogonal projection onto $H^2 \otimes \{ A \in M_N(\mathbb{C}) : \langle A \rangle = 0 \}$. Combining (2.8) and (2.9), we can express $\xi$ as the sum of two Gaussian functionals on $(H^1_0(\mathbb{C}) \cap H^2) \otimes M_N(\mathbb{C})$;

$$\xi = \frac{1}{\sqrt{4\pi}}(P_0 h \otimes I) + P_1 U.$$

3. Outline of the proof

In the rest of the paper, we often omit $N$ to write, for example, $L = L_N$, but every quantity should be considered $N$-dependent unless otherwise specified. Also we will focus on the case when $\chi$ is complex, and present how to modify the proof for the real case in Section 6.

One of the major obstacles in studying non-Hermitian matrices is that they have complex spectra so that the resolvent no longer has a regularizing effect. A widely used technique to circumvent this problem is Hermitization, whose precise definition is as follows.

Definition 3.1. For $z \in \mathbb{C}$, we define $W_z \in M_{2N}(\mathbb{C})$ to be the Hermitization of $X - z$, that is,

$$W_z := \begin{pmatrix} 0 & X - z \\ (X - z)^* & 0 \end{pmatrix},$$

and we abbreviate $W \equiv W_0$. We further denote $G_z(w) := (W_z - w)^{-1}$ for $w \in \mathbb{C}_+$ to be the resolvent of $W_z$.

It follows from the Schur complement formula that the resolvent $G_z(w)$ can be written in the following block form;

$$G_z(w) = \begin{pmatrix} w((X - z)(X - z)^* - w^2)^{-1} & ((X - z)(X - z)^* - w^2)^{-1}(X - z) \\ (X - z)^*((X - z)(X - z)^* - w^2)^{-1} & w((X - z)^*(X - z) - w^2)^{-1} \end{pmatrix}.$$ 

An immediate consequence of (3.1) is that $(G_z(i\eta))^{[12]} = G_z(i\eta)^{[21]}$.

Note that $W_z$ is a Hermitian random matrix with independent entries for which we have more options to approach, and its spectral properties have been studied extensively. In particular, our proof relies on the local law for $W_z$ established in [3]. We remark that the local law for a slightly different Hermitization was proved earlier in [7]. In order to rigorously
state the local law, we define the deterministic approximation to \( G_z(\eta) \). For each \( \eta > 0 \), we define \( M_z(\eta) \in M_{2N}(\mathbb{C}) \) to be the unique solution \( M \) of

\[
(3.2) \quad -MS[M] = \eta M + M(zI^{(12)} + z\Omega^{(21)}) + I, \quad S[P] := \begin{pmatrix} \langle P^{[22]} \rangle & 0 \\ 0 & \langle P^{[11]} \rangle \end{pmatrix}, \quad \forall P \in M_{2N}(\mathbb{C}),
\]

with positive definite imaginary part \( \text{Im} M = \frac{1}{\eta}(M - M^*) \). The equation (3.2) is an example of matrix Dyson equation, so that the existence and uniqueness of its solution follow from [16]. It easy to check from (3.2) that \( M_z(\eta) \) is the block constant matrix

\[
M_z(\eta) = \begin{pmatrix} m_z(\eta) & -zu_z(\eta) \\ -z\Omega u_z(\eta) & m_z(\eta) \end{pmatrix},
\]

where \( m_z(\eta) \) and \( u_z(\eta) \) satisfy

\[
(3.3) \quad -\frac{1}{m_z(\eta)} = \eta + m_z(\eta) - \frac{|z|^2}{\eta + m_z(\eta)}, \quad \text{Im } m_z(\eta) > 0, \quad u_z(\eta) = \frac{m_z(\eta)}{\eta + m_z(\eta)}.
\]

Finally, we recall from [2, Lemma 3.3] that \( \|M_z(\eta)\| \) is uniformly bounded in \( z \) and \( \eta \), that is,

\[
(3.4) \quad \|M_z(\eta)\| + |m_z(\eta)| + |u_z(\eta)| \lesssim 1.
\]

With these notations, the local law for \( W_z \) is stated as follows. For simplicity, we here present it only in the relevant regime, that is, outside the spectrum:

**Lemma 3.2.** [3, Lemma B.1] Let \( \delta > 0 \) be fixed and let \( A \) be a deterministic \((N \times N)\) matrix with \( \|A\| \leq 1 \); then the following hold uniformly over \( |z|^2 \geq 1 + \delta \) and \( \eta \in (0,1] \):

\[
\max_{i,j \in [2N]} |(G_z(\eta) - M_z(\eta))_{ij}| \lesssim \frac{1}{\sqrt{N}},
\]

\[
\max_{k, l \in [2]} \left| \langle A(G_z(\eta) - M_z(\eta))^{[kl]} \rangle \right| \lesssim \frac{1}{N}.
\]

**Remark 3.3.** In principle, the canonical definition of the operator \( S \) in the MDE, given by \( P \mapsto \mathbb{E}[W^* P W] \), slightly differs for the real and the complex symmetry class. When \( W \) is the Hermitization of a complex i.i.d. matrix, then \( \mathbb{E}[W^* P W] \) is given by (3.2). However, when \( W \) is the Hermitization of a real i.i.d. matrix \( X \), we find that the operator \( \tilde{S}[] := \mathbb{E}[W^* W] \) is given by

\[
(3.5) \quad \tilde{S}[P] = \begin{pmatrix} \langle P^{[22]} \rangle & N^{-1} \langle P^{[21]} \rangle \\ N^{-1} \langle P^{[12]} \rangle & \langle P^{[11]} \rangle \end{pmatrix},
\]

i.e. it has a tiny off-diagonal part. This would lead to the unique solution of the MDE with \( \tilde{S} \) which is not in a block diagonal form and its entries can mildly depend on \( N \) unlike \( M_z \). To avoid such complication we keep the same matrix Dyson equation with \( S \) defined in (3.2), and its block diagonal solution \( M_z \) for both the real and complex cases. This fact will play an important role in Section 6 when we prove the main result for real \( X \). Further remarks on local laws for non-Hermitian matrices with real entries can be found in [2, Remark 5.4].

 Typically one writes the logarithmic potential of the averaged eigenvalue distribution of \( X \) in terms of \( G_z(\eta) \), via Girko’s formula in [14]. Alternatively, being outside of the spectrum, we can take a rather direct approach to extract the eigenvalues of \( X \) from \( W_z \). We will use that whenever \( (X - z)(X - z)^* \) is invertible, it follows that

\[
(3.6) \quad \lim_{\eta \to 0} G_z^{[21]}(\eta) = (X - z)^* \lim_{\eta \to 0} \frac{1}{(X - z)(X - z)^* + \eta^2} = \frac{1}{(X - z)}.
\]

In fact, on the event \( \Omega_\delta(\kappa) \) we can quantify (3.6) with a concrete rate of convergence in terms of \( \eta \). Recall from Lemma 2.2 that the smallest singular value of \( (X - z) \) is bounded from below by \( \kappa \) on \( \Omega_\delta(\kappa) \), which has very high probability. Since the eigenvalues of \( W_z \) have the same modulus as the singular values of \( (X - z) \), we have

\[
(3.7) \quad \|G_z(\eta)\| \leq \|((X - z)(X - z)^*)^{-1} + \eta^2\|^{-1/2} \leq \kappa^{-1} \quad \text{on } \Omega_\delta(\kappa),
\]
uniformly over $|z| \geq 1 + \delta/2$ and $\eta > 0$. Using (3.1), this further implies
\[
\left\| G^{[21]}_z((i\eta) - (X-z))^{-1} \right\| = \left\| \frac{\eta^2}{((X-z)(X-z)^* + \eta^2)(X-z)} \right\| \leq \kappa^{-3}\eta^2
\]
uniformly over $|z| \geq 1 + \delta/2$, so that on $\Omega_\delta(\kappa)$ we have, for any $A \in M_N(\mathbb{C})$ with $\|A\| \leq 1$, that
(3.8) \[
|\text{Tr}((X-z)^{-1}A) - \text{Tr} G^{[21]}_z((i\eta)A)| \leq \kappa^{-3}N\eta^2
\]
As a consequence of Lemma 2.2 and (3.8), we can prove that the difference between $\|L(f,A)\|$ and the contour integral of $\|\text{Tr} G^{[21]}_z((i\eta)A)\|$ is negligible when $\eta$ is small. We here emphasize that the latter is not restricted on the event $\Omega_\delta(\kappa)$ unlike the former.

**Proposition 3.4.** Let $\eta = N^{-2}$ and suppose that the assumptions of Theorem 2.3 hold true. Then for all $\epsilon > 0$ we have
\[
(\|L(f,A)\|)_{i \in [n]} = -\left(\frac{1}{2\pi i} \oint_{\gamma} f(z) \text{Tr}(z^{-1}A)dz\right) + O_m(N^{-3+\epsilon}),
\]
where $\gamma$ is the positively oriented circle $\{z : |z| = 1 + \delta/2\}$.

**Proof.** For simplicity, we prove (3.9) when $n = 1$ since the proof for multi-dimensional vectors is identical except for a few minor algebraic modification. Also, we fix the parameters $\delta, \kappa > 0$ and will not keep track of their effect on convergence rates.

Taking the Cauchy integral of (3.8) over the path $\gamma = \{z : |z| = 1 + \delta/2\}$ leads to
(3.10) \[
L(f,A) = -\mathbb{I}_{\Omega_\delta(\kappa)} \frac{1}{2\pi i} \oint_{\gamma} f(z) \text{Tr}(z^{-1}A)dz = -\mathbb{I}_{\Omega_\delta(\kappa)} \frac{1}{2\pi i} \oint_{\gamma} f(z) \text{Tr}(G_z(i\eta))A dz + O(N\eta^2),
\]
so that the same equality holds with both sides centered. We also note from Lemma 3.2 and $\|M_z(i\eta)\| \lesssim 1$ that $\|\mathbb{I}_{\Omega_\delta(\kappa)} \text{Tr} G_z(i\eta)\| \lesssim 1$, which, together with (3.7), gives for any $p, \epsilon, D > 0$ that
(3.11) \[
\mathbb{E}[\|\mathbb{I}_{\Omega_\delta(\kappa)} \text{Tr} G_z(i\eta)\|] \leq N^\epsilon + N^{p-D}.
\]
By our choice of $\eta = N^{-2}$ we have $N\eta^2 = N^{-3}$, so that from $|x|^p - |y|^p \leq \sum_{k=1}^{p} (\binom{p}{k}) |x - y|^k |x|^{p-k}$ we get
\[
\left| \mathbb{E}\left[ |L(f,A)|^p - \left(\frac{1}{2\pi i} \oint_{\gamma} f(z)\mathbb{I}_{\Omega_\delta(\kappa)} \text{Tr} G_z(i\eta)\|^{[21]}A|dz\right)^p \right| \right| \lesssim N\eta^2 \int_{\gamma} \mathbb{E}[\|\mathbb{I}_{\Omega_\delta(\kappa)} \text{Tr} G_z(i\eta)\|] dz = O(N^{-3+\epsilon}),
\]
where we applied Hölder’s inequality to interchange the contour integral with the $L^p$ norm. Similar estimate holds for generic $*$-monomials of the form $L^p L^q$. In other words, we have shown that
(3.12) \[
\|L(f,A)\| = \left(\frac{1}{2\pi i} \oint_{\gamma} f(z)\mathbb{I}_{\Omega_\delta(\kappa)} \text{Tr} G_z(i\eta)\|^{[21]}A|dz\right) + O_m(N^{-3+\epsilon})
\]
for all $\epsilon > 0$.

On the other hand, by Lemma 2.2 we have for all $p > 0$ that
(3.13) \[
\mathbb{E}[\mathbb{I}_{\Omega_\delta(\kappa)^c} \text{Tr} G_z(i\eta)\|^{[21]}A|p = O(N^{p-D} \eta^{-p}) = O(N^{3p-D}),
\]
where we used the trivial bound $\|G\| \leq \eta^{-1}$. Enlarging $D$ depending on $p$ leads to
(3.14) \[
\mathbb{I}_{\Omega_\delta(\kappa)^c} \frac{1}{2\pi i} \oint_{\gamma} f(z) \text{Tr}(G_z(i\eta))A dz = O_m(N^{-D})
\]
for any fixed $D$. Then (3.14) together with (3.11) gives, after a binomial expansion, that
(3.15) \[
\frac{1}{2\pi i} \oint_{\gamma} \mathbb{I}_{\Omega_\delta(\kappa)} \text{Tr} G_z(i\eta)\|^{[21]}A|dz = \frac{1}{2\pi i} \oint_{\gamma} \mathbb{I}_{\text{Tr} G_z(i\eta)\|^{[21]}A|dz + O_m(N^{-D})
\]
for all fixed $D > 0$. Combining (3.12) and (3.15), we conclude (3.9) for any fixed $\epsilon > 0$.

**Remark 3.5.** As easily seen from the proof, our choice of $\eta = N^{-2}$ in Proposition 3.4 is purely cosmetic. In fact, the same proof applies whenever $\log \eta/\log N \in (-\infty, -1/2)$, in which case the upper bound in (3.9) becomes $O_m(N^{1+\epsilon}\eta^2)$.

In the following sections, we will prove that for $\eta = N^{-2}$, the random vector

$$- \left( \frac{1}{2\pi i} \oint_{\gamma} f_i(z) \mathrm{Tr}(G_z(in)^{[2]} A_i) \mathrm{d}z \right)_{i \in [n]}$$

has asymptotically the same moment as the Gaussian vector $\xi(f_i, A_i)$, which would imply Theorem 2.3 via Proposition 3.4. In this regard, for the rest of the paper we fix $\eta = N^{-2}$ and we suppress the spectral parameter to write $G_z \equiv G_z(in)$, $\mathcal{M}_z \equiv M_z(in)$ et cetera. We recall from [9, Eq. (3.7)] that the deterministic matrix $\mathcal{M}_z$ has the following asymptotic expansions in $\eta$ for $|z| > 1$:

$$\mathcal{M}_z = \left( \begin{array}{cc} \frac{1}{\eta(|z|^2 - 1)^{-1}} & -\frac{\eta^{-1}}{\eta(|z|^2 - 1)^{-1}} \\ -\frac{\eta^{-1}}{\eta(|z|^2 - 1)^{-1}} & \frac{1}{\eta(|z|^2 - 1)^{-1}} \end{array} \right) = O(\eta^2),$$

or equivalently,

$$m_z(\eta) = \frac{i\eta}{|z|^2 - 1} + O(\eta^2), \quad u_z(\eta) = \frac{1}{|z|^2} + O(\eta^2).$$

For $\eta$ fixed to be $N^{-2}$, we state our main technical result, the central limit theorem for the resolvents $(\mathrm{Tr} G_z^{[2]} A_i)_{i \in [n]}$ as follows. Its proof for complex and real $\chi$ is postponed to Sections 4 and 6, respectively.

**Proposition 3.6.** Let $\epsilon, \delta > 0$ be fixed and take a finite collection of deterministic matrices $A_1, \cdots, A_n \in M_N(\mathbb{C})$ with $\|A_i\| \leq 1$ and $z_1, \cdots, z_n \in \mathbb{D}_{1+\delta/2}$. Then, for $\eta = N^{-2}$, we have

$$\mathbb{E}[\mathrm{Tr} G_z^{[2]} A_i] + N\langle A_i \rangle \frac{1}{z_i} = \begin{cases} O(N^{-1/2+\epsilon}) & \text{if } \chi \text{ is complex,} \\ \frac{\langle A_i \rangle}{z_i(z_i^2 - 1)} + O(N^{-1/2+\epsilon}) & \text{if } \chi \text{ is real,} \end{cases}$$

where \{$(\zeta, A) : |z| > 1, A \in M_N(\mathbb{C})$\} is a centered Gaussian process with covariances

$$\mathbb{E}[\zeta(z, A)\zeta(w, B)] = \langle A \rangle \langle B \rangle \frac{1}{(1 - zw)^2} + \langle A \bar{B}^* \rangle \frac{1}{zw(zw - 1)},$$

$$\mathbb{E}[\zeta(z, A)\zeta(w, B)] = \begin{cases} 0 & \text{if } \chi \text{ is complex,} \\ \langle A \rangle \langle B \rangle \frac{1}{(1 - zw)^2} + \langle A \bar{B}^* \rangle \frac{1}{zw(zw - 1)} & \text{if } \chi \text{ is real.} \end{cases}$$

**Proof of Theorem 2.3.** First of all, by (3.9) we have

$$\mathbb{E}[\mathrm{Tr} G_z^{[2]} A_i]_{1 \leq i \leq n} = \left( \frac{1}{2\pi i} \oint_{\gamma} f_i(z) \mathrm{Tr} G_z^{[2]} A_i \mathrm{d}z \right)_{1 \leq i \leq n} + O_m(N^{-3+\epsilon}).$$

Then, for a fixed $*$-monomial $\prod_{i=1}^k a_{j_i}^{m_i}$ with a multiset $\{j_1, \cdots, j_k\}$ of indices in $[1, n]$ and $m_1 \in \{1, *\}$, we have

$$\mathbb{E}[\prod_{i=1}^k (f_i(z_i)\mathrm{Tr} G_z^{[2]} A_{j_i})]^{m_1} = \mathbb{E}[\prod_{i=1}^k (f_i(z_i))^{m_1}] + O(N^{-3+\epsilon}).$$
where we used that the estimate in (3.18) is uniform over \( z_i \)'s in \( \{ z : |z| \geq 1 + \delta/2 \} \). To calculate the right-hand side of (3.20), we use Wick’s theorem and the fact that (see Eq. (1.7) in [20] for a proof of the last identity)

\[
- \frac{1}{4\pi^2} \oint \oint f(z)g(w) \frac{dw}{w(zw - 1)} \frac{dz}{z(zw - 1)} = \frac{1}{2\pi} \int_{\partial \mathbb{D}} f(z)g(z)dz - f(0)g(0) = \langle f, g \rangle \mathcal{L}_2(\partial \mathbb{D}),
\]

Thus the covariance of contour integrals of \( \zeta(\cdot, A) \) coincide with that of \( \xi(f, A) \), hence combining with (3.20) leads to

\[
(2\pi)^{-k} \oint \cdots \oint \mathbb{E} \left[ \prod_{i=1}^{k} (f(z_i)\mathbb{I}[\text{Tr} G_{z_i}^{[21]} A_j])^{m_i} d(z_i) \right] = \mathbb{E} \left[ \prod_{i=1}^{k} \xi(f_j, A_j)^{m_i} \right] + O(N^{-1/2+\epsilon}),
\]

which implies (2.3).

In order to prove (2.5), we use (3.10) and (3.13) with \( p = 1 \) to get

\[
\mathbb{E}[L(f, A)] = \frac{1}{2\pi} \oint f(z)|\mathbb{E}[\text{Tr} G_{z}^{[21]} A]|dz + O(N^{-3}).
\]

Therefore (3.19) implies (2.5) as desired. This concludes the proof of Theorem 2.3.

We conclude this section with a remark on Proposition 3.6.

**Remark 3.7.** As in Proposition 3.4, the choice of \( \eta = N^{-2} \) does not play a crucial role in the sense that the same proof applies to smaller \( \eta \) as long as \( \log \eta / \log N \) remains bounded; for example, we may take \( \eta = N^{-100} \). However, taking smaller \( \eta \) does not improve the rate of convergence in (3.18) or (3.19), in contrast to Proposition 3.4.

### 4. Proof of Proposition 3.6

In this section, we prove that the joint moments of \( \mathbb{E}[\text{Tr} G_{z}^{[21]} A] \) converge to those of corresponding Gaussian variables, and the asymptotics (3.19) of the mean will be proved in Section 5.3. We first prove Proposition 3.6 when \( \chi \) is complex, and the proof for real-valued \( \chi \) will be given in Section 6.

For any \( p, q \in \mathbb{N} \), we consider the moment

\[
\mathbb{E} \left[ \prod_{i \in [p]} \chi_i \prod_{j \in [q]} \chi_j \right],
\]

where we defined

\[
\chi_i = \mathbb{I}[\text{Tr} G_{z}^{[21]} A_i], \quad i \in [p], \quad \chi_j = \mathbb{I}[\text{Tr}(G_{w}^{[21]} B_j^*)], \quad j \in [q],
\]

and abbreviated \( G_z \equiv G_z(i\eta) \). We further assume that there exist sets of indices \( S \subset [p] \) and \( T \subset [q] \) such that

\[
\text{Tr} A_i = 0 = \text{Tr} B_j, \quad i \in S, \quad j \in T, \quad A_i = I = B_j, \quad i \in S^c, \quad j \in T^c,
\]

where \( S^c = [p] \setminus S \) and \( T^c = [q] \setminus T \). We remark that calculating all joint moments as in (4.1) is equivalent to that of Proposition 3.6, since for any \( (N \times N) \) matrix \( A \) we may decompose \( \text{Tr} G_{z}^{[21]} A \) as

\[
\text{Tr} G_{z}^{[21]} A = \text{Tr} G_{z}^{[21]} [A + \langle A \rangle \text{Tr} G_{z}^{[21]}],
\]

so that the first and second terms correspond respectively to the cases \( i \in S \) and \( i \in S^c \). We further remark that it is necessary to separate the complex conjugates \( \chi^* \) from \( \chi \) in contrast to [9], for \( \text{Tr} G_{z}^{[21]} A \) no longer has the same form as \( \text{Tr}(G^*)^{[21]} A \) since \( (G^{[21]})^* = (G^*)^{[12]} = G^{[12]} \). To simplify the presentation, we abbreviate

\[
\chi' := \prod_{i \in [p]} \chi_i, \quad \chi'^{(i_1, \ldots, i_n)} := \prod_{i \in [p], i \neq i_1, \ldots, i_n} \chi_i,
\]

and define \( \chi' \) similarly. With these notations, we have the following asymptotic Wick formula:
Proposition 4.1. Let $\delta, \epsilon > 0$ be fixed and suppose that $z_i, w_j \in \mathbb{D}_{i+j}^c$ for all $i, j$. Then we have

$$
\mathbb{E}[\mathcal{X} \mathcal{Y}] = \sum_{P \in \text{Pair}(S; T)} \prod_{(i,j) \in P} V^0(z_i, w_j) (A_i B_i^*) \prod_{(i,j) \in Q} V(z_i, w_j) + O(N^{-1/2+\epsilon}),
$$

where $\text{Pair}(I, J)$ for $I \subset [p]$ and $J \subset [q]$ denotes the set of perfect matchings from $I$ to $J$ and

$$
V^0(z, w) = \frac{1}{z w (z w - 1)}, \quad V(z, w) = \frac{1}{(1 - z w)^2}.
$$

In particular, $\mathbb{E}[\mathcal{X} \mathcal{Y}]$ is nonzero only if $|S| = |T|$ and $|S^c| = |T^c|$, hence $p = q$.

Before proceeding to the proof of Proposition 4.1, we remark that the averaged local law gives a priori bounds on $\text{Tr} G_z^{[kl]} A$ for $k, l \in \{1, 2\}$. To see this, note that Lemma 3.2 and (3.16) imply

$$
|\text{Tr} G_z^{[kl]} A + \langle A \rangle \text{Tr} M_z^{[kl]}| = |\text{Tr} (G_z - M_z) A^{[kl]}| < 1,
$$

and by triangle inequality this leads to $||\text{Tr} G_z^{[kl]} A|| < 1$. In particular, we have

$$
|\text{Tr} G_z^{[kl]} A| < 1 \quad \text{if } \langle A \rangle = 0 \text{ or } k = l,
$$

where we used (3.16) and $\eta = N^{-2}$ for the case $k = l$. The same argument applies to $\mathcal{X}$ and $\mathcal{Y}$, so that

$$
\mathcal{X}_i = I[\text{Tr} G_z^{[21]} A_1] < 1, \quad \mathcal{Y}_j = ||\text{Tr} G_z^{[21]} B_j|| < 1.
$$

We will often use (4.6) to simply replace factors of $\mathcal{X}$ and $\mathcal{Y}$ by $N^\epsilon$ in various estimates.

Proof of Proposition 4.1. We aim to derive asymptotic Wick formulas for the joint moment in (4.3) with respect to the first index $1 \in [p]$. To be precise, we prove that

$$
\mathbb{E}[\mathcal{X} \mathcal{Y}] = \sum_{j \in T} \frac{\langle A_1 B_j^* \rangle}{z_j w_j (z_j w_j - 1)} \mathbb{E}[\mathcal{X}^{(1)} \mathcal{Y}^{(j)}] + O(N^{-1/2+\epsilon}) \quad \text{if } 1 \in S,
$$

$$
\mathbb{E}[\mathcal{X} \mathcal{Y}] = \sum_{j \in T^c} \frac{1}{(z_j w_j - 1)^2} \mathbb{E}[\mathcal{X}^{(1)} \mathcal{Y}^{(j)}] + O(N^{-1/2+\epsilon}) \quad \text{if } 1 \in S^c,
$$

hold uniformly over $z_i, w_j \in \mathbb{D}_{i+j}^c$.

Using the resolvent identity $(W_z - i\eta) G = I$, we write

$$
\eta \text{Tr} G_z^{[11]} A_1 = \text{Tr} X G_z^{[21]} A_1 - z \text{Tr} G_z^{[21]} A_1 - \text{Tr} A_1.
$$

Since $\text{Tr} A_1$ is deterministic, the identity $\mathbb{E}[X[I[Y]]] = \mathbb{E}[X[Y]]$ gives

$$
\mathbb{E}[\mathcal{X} \mathcal{Y}] = \mathbb{E}[\langle (X G_z^{[21]} A_1) I[Y] \mathcal{X}^{(1)} \mathcal{Y} \rangle] = \frac{\tau_1}{z_1} \mathbb{E}[\text{Tr} \left( (X G_z^{[21]} - i\eta G_z^{[11]}) A_1 \right) I[I[Y] \mathcal{X}^{(1)} \mathcal{Y}]]
$$

$$
= \frac{1}{z_1} \mathbb{E}[\text{Tr} X G_z^{[21]} I[I[Y] \mathcal{X}^{(1)} \mathcal{Y}]] + O(N^\epsilon),
$$

where we used (4.5) to the second term.

Now we calculate the first term on the right-hand side of (4.9) using cumulant expansions in (1.3);

$$
\mathbb{E}[\text{Tr} (X G_z^{[21]} A_1) I[I[Y] \mathcal{X}^{(1)} \mathcal{Y}]] = \sum_{a, b} \mathbb{E}[X_{ab} \text{Tr} \Delta_{ab} G_z^{[21]} A_1 I[I[Y] \mathcal{X}^{(1)} \mathcal{Y}]] = \frac{1}{N} \sum_{a, b} \mathbb{E} \left[ \text{Tr} \Delta_{ab} \partial_{ab}^{(0, 1)} [G_z^{[21]}] A_1 I[I[Y] \mathcal{X}^{(1)} \mathcal{Y}]]
$$

$$
+ \frac{1}{N} \sum_{i \in [2, p]} \sum_{a, b} \mathbb{E} \left[ \partial_{ab}^{(0, 1)} \mathcal{X}_i \text{Tr} \Delta_{ab} G_z^{[21]} A_1 \mathcal{X}^{(1, i)} \mathcal{Y} \right] + \frac{1}{N} \sum_{j \in [q]} \sum_{a, b} \mathbb{E} \left[ \partial_{ab}^{(0, 1)} \mathcal{Y}_j \text{Tr} \Delta_{ab} G_z^{[21]} A_1 \mathcal{X}^{(1)} \mathcal{Y}^{(j)} \right]
$$

$$
+ \sum_{k, l \in [2]} \frac{\kappa(k + 1, l)}{k l!} \sum_{a, b} \mathbb{E} \left[ \partial_{ab}^{(k, l)} \left( \text{Tr} \Delta_{ab} G_z^{[21]} A_1 I[I[Y] \mathcal{X}^{(1)} \mathcal{Y}]] \right) \right],
$$

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where we defined $\Delta_{ab} \in M_N(\mathbb{C})$ with $(\Delta_{ab})_{ij} = \delta_{ai}\delta_{bj}$, the cumulants

$$\kappa(k,l) := (-i)^{k+l} \left[ \frac{\partial^{k+l}}{\partial s^k \partial t^l} \log E[e^{isX_{11} + itX_{11}}] \right]_{s,t=0},$$

and $\partial^{(k,l)} := \frac{\partial^{k+l}}{\partial X_{ab} \partial \bar{X}_{ab}}$.

Note that the term in (4.10) corresponding to $\partial_{ba}^{(1,0)}$ vanishes since $\kappa(2,0) = E[X_{11}^2] = 0$. Applying $\partial_{ab}^{(0,1)}[G_{z_1}^{[kl]}] = -G_{z_1}^{[kl]} G_{z_1}^{[kl]}$ to the first term of (4.10), we find that

$$\frac{1}{N} \sum_{a,b} \text{Tr} \Delta_{ab} \partial_{ab}^{(1)}[G_{z_1}^{[21]}] A_1 = - \frac{1}{N} \sum_{a,b} \text{Tr} \Delta_{ab} G_{z_1}^{[22]} \Delta_{ba} G_{z_1}^{[11]} A_1 = - \langle G_{z_1}^{[22]} \rangle \text{Tr} G_{z_1}^{[11]} A_1.$$

Thus the contribution of the first term on the right-hand side of (4.10) reads

$$|E[(G_{z_1}^{[22]} \text{Tr} G_{z_1}^{[11]} A_1][\mathcal{X}(1)\mathcal{Y}])] = \frac{1}{N} |E[(\text{Tr} G_{z_1}^{[22]})(\text{Tr} G_{z_1}^{[11]} A_1)][\mathcal{X}(1)\mathcal{Y}])]| \leq N^{-1+\epsilon},$$

where we applied (4.5) to the first two traces and (4.6) to get $\mathcal{X}, \mathcal{Y} \prec 1$.

The same calculations for the second and third terms of (4.10) give

$$\frac{1}{N} \sum_{a,b} \partial_{ab}^{(0,1)}[\mathcal{X}_i] \text{Tr} \Delta_{ab} G_{z_1}^{[21]} A_1 = - \langle G_{z_1}^{[11]} A_i G_{z_1}^{[22]} G_{z_1}^{[21]} A_1 \rangle = - \mathcal{X}_i,$$

(4.12)

$$\frac{1}{N} \sum_{a,b} \partial_{ab}^{(0,1)}[\mathcal{Y}_j] \text{Tr} \Delta_{ab} G_{z_1}^{[21]} A_1 = - \langle G_{w_j}^{[12]} B_j^* G_{w_j}^{[12]} G_{z_1}^{[21]} A_1 \rangle = - \mathcal{Y}_j.$$

Substituting (4.11) and (4.12) into (4.10) and then plugging the result back to (4.9), we conclude

$$z_1 E[\mathcal{X}\mathcal{Y}] = - \sum_{i \in [2,p]} E[\mathcal{X}_i \mathcal{X}^{(1)} \mathcal{Y}] - \sum_{j \in [q]} E[\mathcal{Y}_j \mathcal{X}^{(1)} \mathcal{Y}^{(j)}]$$

$$+ \sum_{k,l \in \mathbb{Z}_+} \frac{\kappa(k+1,l)}{k! l!} \sum_{a,b} \left[ \partial_{ab}^{(k,l)} \left( \text{Tr} \Delta_{ab} G_{z_1}^{[21]} A_1 [\mathcal{X}(1) \mathcal{Y}] \right) \right] + O(N^{-1+\epsilon}).$$

(4.13)

We estimate each term of (4.13) using the following proposition to derive (4.7) and (4.8). Its proof is postponed to Section 5.

**Proposition 4.2.** Under the assumptions of Proposition 4.1, the following estimates hold uniformly over $z_i, w_j \in \mathbb{D}_{1+\delta}$ and for any fixed $k, l \in \mathbb{N}$ with $k + l \geq 2$:

$$E[(\mathcal{X}_i^2)] = O(N^{-2+\epsilon}), \quad E\left[\mathcal{Y}_j + \frac{(A_1 B_j^*)}{(z_1 w_j - 1)^2} + \frac{z_1 (A_1) (B_j^*)^2}{(z_1 w_j - 1)^2} \right] = O(N^{-2+\epsilon}),$$

(4.14)

$$\frac{\kappa(k+1,l)}{k! l!} \sum_{a,b} \partial_{ab}^{(k,l)} \left( \text{Tr} \Delta_{ab} G_{z_1}^{[21]} A_1 [\mathcal{X}^{(1)} \mathcal{Y}] \right) \lessgtr N^{-1/2-(k+l+4)/2}. $$

(4.15)

Armed with Proposition 4.2, we proceed with the proof of Proposition 4.1. We use Cauchy–Schwarz inequality to estimate

$$|E[\mathcal{X}_i \mathcal{X}^{(1)} \mathcal{Y}]| \leq E[(\mathcal{X}_i^2)]^{1/2} E[|\mathcal{X}^{(1)} \mathcal{Y}|^2]^{1/2} \leq N^{-1+\epsilon}$$

where we used (4.6) in the second inequality, and we can estimate the subleading term of $\mathcal{Y}_j$ similarly. Combining (4.13), (4.14), and (4.15) proves (4.7) and (4.8).

Since the specific choice of index 1 in (4.7) and (4.8) played no role in the argument, we can apply (4.7) and (4.8) iteratively for any index, yielding (4.3). This completes the proof of Proposition 4.1 for complex-valued $\chi$, modulo Proposition 4.2. □
5. Proof of Proposition 4.2 and (3.19)

The quantities $X_i$ and $\mathcal{Y}_j$ are traces of products of three resolvent, whose variants were studied in [9]. We will shortly see that similar calculations apply to $X_i$ and $\mathcal{Y}_j$, and in fact the local laws for product of two resolvent, Theorem 5.2 therein, can be used verbatim. To state the local laws for products, we introduce the relevant notations. For given $z, w \in \mathbb{C}$, we define the operator $\mathcal{B}(z, w)$ as

$$\mathcal{B}(z, w) := 1 - M_z(\eta) S_z \, M_w(\eta),$$

where $S$ was defined in (3.2). For a given deterministic $(2N \times 2N)$ matrix $B$, we further define

$$M_B(z, w) = \mathcal{B}(z, w)^{-1} [M_z(\eta) B M_w(\eta)].$$

Recall from [9, Eq. (6.4)] that the operator $\mathcal{B}(z, w)$ has four eigenvalues, $1, 1, \beta$, and $\beta^*$, given by

$$\beta, \beta^* := 1 - u_z u_w \Re (z \bar{w}) \pm \sqrt{m_z^2 m_w^2 - u_z^2 u_w^2 \Im (z \bar{w})^2},$$

where $m_z$ and $u_z$ are defined as solutions of (3.3). Applying the asymptotics in (3.17), for $|z|^2, |w|^2 > 1 + \delta$ we find that

$$\beta, \beta^* = |u_z u_w z \bar{w} - 1|^2 - m_z^2 m_w^2 \geq |(1 + \delta)^{-1} - 1| + O(\delta^2) \gtrsim \delta.$$

Since $|\beta|, |\beta^| \lesssim 1$ the inverse of $\mathcal{B}$ has $O(1)$ operator norm, in contrast to Lemma 6.1 of [9] where the parameters $z, w$ were allowed inside the unit disk. In particular, this allows us to strengthen the bound of [9, Theorem 5.2] in the following form.

**Lemma 5.1.** Let $\delta > 0$, $P, Q \in M_{2N}(\mathbb{C})$ and $x, y \in \mathbb{C}^{2N}$ be deterministic with $\|P\|, \|Q\|, \|x\|, \|y\| \leq 1$. Then the following hold uniformly over $z, w \in \{z \in \mathbb{C} : |z|^2 > 1 + \delta\}$:

$$|\langle x, (G_z P G_w - M_P(z, w)) y \rangle| \prec \frac{1}{\sqrt{N}},$$

$$|\langle Q (G_z P G_w - M_P(z, w)) \rangle| \prec \frac{1}{N},$$

where we abbreviated $G_z \equiv G_z(\in)$ and $G_w \equiv G_w(\in)$ with $\eta = N^{-2}$.

**Proof.** The proof is a straightforward modification of Theorem 5.2 in [9]. The only difference is that we use (5.2) and

$$|\langle A(G_z B G_w) \rangle| \prec 1,$$

which is due to (3.7). Using (3.7) and (5.3), we may replace all $\eta_z$ appearing in the proof of Theorem 5.2 in [9] by 1, for example in Eq. (5.16) therein. Also, we may skip the iteration of Eq. (5.14) in [9] since we can use (5.3) from the beginning. Finally, feeding (5.2) into Eq. (5.1) in [9] proves the result.

**Lemma 5.1** shows that $M_B$ can be used as a deterministic approximation to products of two resolvent with a deterministic matrix $B$ in between that arises along our calculations. First we record the small $\eta$ asymptotics of $M_B$. For $z, w \in \mathbb{D}_{1+\delta}$, we have

$$\mathcal{B}(z, w)_{|M_B^*} = \begin{pmatrix} M_B^{[1]} - m_z m_w M_B^{[22]} - z \bar{w} u_z u_w M_B^{[11]} & m_z m_w + w m_z u_w M_B^{[22]} + z u_z M_B^{[11]} \\ m_B^{[1]} + \bar{\bar{x}} u_z m_w M_B^{[22]} + w m_z u_w M_B^{[11]} & M_B^{[2]} - \bar{w} u_z u_w M_B^{[22]} - m_z M_B^{[11]} \end{pmatrix},$$

$$\mathcal{B}(z, w)_{|M_B} = M_B B M_w,$$

due to the definitions of $\mathcal{B}$ and $M_B$, where we abbreviated $M_B \equiv M_B(z, w)$. Taking the trace of each block, we find the following system of equations:

$$\begin{cases}
(1 - z \bar{w} u_z u_w) M_B^{[1]} - m_z m_w M_B^{[22]} = \langle (M_B B M_w)_{[1]} \rangle, \\
- m_z M_B^{[11]} + (1 - \bar{w} u_z u_w) M_B^{[22]} = \langle (M_B B M_w)_{[22]} \rangle.
\end{cases}$$
Solving this equation gives

\[
\begin{pmatrix}
\langle M_B^{[11]} \rangle \\
\langle M_B^{[22]} \rangle
\end{pmatrix} = 
\left( 1 - z_u u_w z_w \right)^{-1} - m_z^2 m_w^2 \left( 1 - u_z u_w z_w \right) \left( 1 - u_z u_w z_w \right)^{-1} \left( 1 - z_u u_w z_w \right)^{-1}
\]

\[
\begin{pmatrix}
\langle M_B^{[11]} \rangle \\
\langle M_B^{[22]} \rangle
\end{pmatrix} = 
\left( 1 - z_u u_w z_w \right)^{-1} - m_z^2 m_w^2 \left( 1 - u_z u_w z_w \right) \left( 1 - z_u u_w z_w \right)^{-1} \left( 1 - u_z u_w z_w \right)^{-1} + O(\eta^2),
\]

where we used (3.16) in the last equality. Now we may apply (3.16) to calculate the right-hand side of (5.4) for a given \( B \). For example when \( B = A^{[11]} \), we get

\[
M_z A^{[11]} M_w = \left( \begin{array}{cc}
0 & -z^{-1} \\
-z^{-1} & 0
\end{array} \right) \left( \begin{array}{c}
A_0 \\
0
\end{array} \right) = O(\eta) = \frac{1}{z w} A^{[22]} + O(\eta),
\]

and \( M_z A^{(kl)} M_w \) for other choices of \( k, l \) can be calculated in the exact same way. Substituting these into (5.4) gives

\[
\begin{pmatrix}
\langle M_A^{(11)} (z, w) \rangle \\
\langle M_A^{(11)} (z, w) \rangle
\end{pmatrix} = \frac{1}{z w} \left( \begin{array}{c}
A_0 \\
0
\end{array} \right) + O(\eta), \quad \begin{pmatrix}
\langle M_A^{(12)} (z, w) \rangle \\
\langle M_A^{(12)} (z, w) \rangle
\end{pmatrix} = O(\eta)
\]

\[
\begin{pmatrix}
\langle M_A^{(21)} (z, w) \rangle \\
\langle M_A^{(21)} (z, w) \rangle
\end{pmatrix} = O(\eta), \quad \begin{pmatrix}
\langle M_A^{(22)} (z, w) \rangle \\
\langle M_A^{(22)} (z, w) \rangle
\end{pmatrix} = \frac{1}{z w - 1} \left( \begin{array}{c}
A_0 \\
0
\end{array} \right) + O(\eta).
\]

Since \( M_B (z, w) = M_z (B + S[M_B (z, w)]) M_w \), asymptotics in (5.5) directly imply those for \( M_A (z, w) \) as follows.

\[
\begin{align*}
M_A^{(11)} (z, w) &= \frac{1}{z w} \left( A + \frac{\langle A \rangle}{z w - 1} \right)^{[22]} + O_{\|\|}(\eta), \quad M_A^{(12)} (z, w) = \frac{1}{z w} A^{[21]} + O_{\|\|}(\eta), \\
M_A^{(21)} (z, w) &= \frac{1}{z w} A^{[12]} + O_{\|\|}(\eta), \quad M_A^{(22)} (z, w) = \frac{1}{z w - 1} \left( A + \frac{\langle A \rangle}{z w - 1} \right)^{[11]} + O_{\|\|}(\eta),
\end{align*}
\]

where the remainders have operator norms of size \( O(\eta) \) due to (3.16).

Another notation we recall from [9] is the definition of self-renormalization:

\[
W f(W) := W f(W) - \mathbb{E}_W \tilde{W} (\partial_W f)(W),
\]

where the function \( f : M_{2N} (\mathbb{C}) \to M_{2N} (\mathbb{C}) \) has bounded derivative, \( \tilde{W} \) is the Hermitization of a complex Ginibre ensemble as in Definition 3.1, and \( (\partial_W f)(W) \) denotes the directional derivative along \( \tilde{W} \) evaluated at \( W \). Taking \( f(W) \) to be \( G_z \) gives

\[
W G_z \equiv W G_z + S[G_z] G_z,
\]

and from (3.2) we further have

\[
G_z - M_z = -M_z W G_z + M_z S[G_z - M_z] G_z.
\]

As we will see in the following sections, Lemma 3.2 proves that the second term of (5.8) is small since \( S \) acts as a partial trace, allowing us to consider \( S[G - M] \) as a constant of size \( O_{\sim}(N^{-1}) \) in effect.

5.1. Proof of (4.14)

Finally we are ready to prove (4.14). Since this section concerns only three spectral parameters \( z_1, z_2, \) and \( w_j \), we abbreviate \( G_1 \equiv G_{z_1}, G_1 \equiv G_{z_2}, G_j \equiv G_{w_j} \), and the same for \( M_z \)'s. We first apply (5.8) to the factor of \( G_i^{[22]} \) in \( \xi_i \):

\[
\begin{align*}
G_i^{[22]} A_i G_i^{[11]} A_i &= M_i^{[22]} G_i^{[21]} A_i G_i^{[11]} A_i - (M_i W G_i^{[22]} G_i^{[21]} A_i G_i^{[11]} A_i \\
&\quad + (M_i S[G_i - M_i] G_i^{[22]} G_i^{[21]} A_i G_i^{[11]} A_i) \\
&= (M_i^{[22]} G_i^{[21]} A_i G_i^{[11]} A_i + (M_i S[G_i I^{[22]} G_i] G_i^{[21]} A_i G_i^{[11]} A_i) \\
&\quad + (M_i S[G_i I^{[22]} G_i] G_i^{[21]} A_i G_i^{[11]} A_i) - (M_i W G_i I^{[22]} G_i A_i^{[11]} G_i^{[21]} A_i \\
&\quad + (M_i S[G_i - M_i] G_i]^{[22]} G_i^{[21]} A_i G_i^{[11]} A_i).
\end{align*}
\]

(5.9)
Note that in the second equality we applied (5.7) twice, namely
\[ W G_i I^{(22)} G_i = W G_i I^{(22)} G_i + S[G_i I^{(22)} G_i] G_i, \]
\[ W G_i I^{(22)} G_i A_i^{(11)} G_i = W G_i I^{(22)} G_i A_i^{(11)} G_i + S[G_i I^{(22)} G_i A_i^{(11)} G_i] G_i. \]

By Lemma 5.1, (5.5), and (5.6), the contribution of the first two terms on the right-hand side of (5.9) is given by
\[ \langle (M_i I^{(22)} + S[G_i I^{(22)} G_i]) G_i \rangle^{[21]} A_i G_i^{[11]} A_i \]
\[ = \langle (M_i I^{(22)} + S[M_i I^{(22)} (z_i, z_i)]) M_i A_i^{(1)} (z_i, z_i) \rangle^{[21]} A_i \rangle + O_{\prec} (N^{-1}) = O(\eta) + O_{\prec} (N^{-1}) = O_{\prec} (N^{-1}), \]
where the last equality is due to our choice of \( \eta = N^{-2}. \)

Before we proceed to the third term, we consider the last two terms of (5.9). The fourth term of (5.9) can be estimated using the exact same argument as in [9, Eq. (6.21)], except we use the norm bound (3.7) to replace all factors of \( \eta^{-1} \) by 1. As a result, for generic deterministic matrices \( A, A', A'' \in M_{2N}(\mathbb{C}) \) with bounded operator norms we find that
\[ \mathbb{E} \left[ \frac{1}{N} \text{Tr} W G_i A_i A' G_i A'' \right] = O(N^{-2}). \]

The normalized trace of the fourth term of (5.9) has exactly the same form with \( A = I^{(22)}, A' = A_i^{(11)}, \) and \( A'' = A_i^{(12)} M_i, \) so that its second moment is bounded by \( N^{-2}. \) For the last term in (5.9), we use Lemma 3.2 to each block of \( S[G - M] \) and apply the norm bound (3.7) to the rest, which leads to
\[ |\langle (M_i S[G_i - M_i] G_i^{[22]} G_i^{[21]} A_i G_i^{[11]} A_i) \rangle| \leq \frac{1}{N}. \]

In order to deal with the third term of (5.9), we repeat the same expansion as (5.9) without the last factor of \( A_i. \) To be precise, we write
\[ G_i I^{(22)} G_i A_i^{(11)} G_i = M_i I^{(22)} G_i A_i^{(11)} G_i + M_i S[G_i I^{(22)} G_i] G_i A_i^{(11)} G_i + M_i S[G_i I^{(22)} G_i A_i^{(11)} G_i] G_i \]
\[ - M_i W G_i I^{(22)} G_i A_i^{(11)} G_i + M_i S[G_i - M_i] G_i I^{(22)} G_i A_i^{(11)} G_i. \]

Then we take the partial trace over the (1, 1)-th block of (5.13) to get
\[ \langle (G_i I^{(22)} G_i A_i^{(11)} G_i) \rangle^{[11]} = \langle (M_i I^{(22)} + S[M_i I^{(22)} (z_i, z_i)]) M_i A_i^{(1)} (z_i, z_i) \rangle^{[11]} \]
\[ + \langle (M_i S[G_i I^{(22)} G_i] M_i) A_i^{(11)} G_i \rangle^{[11]} \]
\[ - \langle (M_i W G_i I^{(22)} G_i A_i^{(11)} G_i) \rangle^{[11]} \]
\[ = |z_i|^{-2} \langle G_i I^{(22)} G_i A_i^{(11)} G_i \rangle^{[22]} - e_{i1} + O_{\prec} (N^{-1}) \]
\[ = |z_i|^{-2} \langle G_i I^{(22)} G_i A_i^{(11)} G_i \rangle^{[22]} - e_{i1} + O_{\prec} (N^{-1}), \]
where we used respectively Lemma 5.1 and (5.6) in the first and second equalities, and abbreviated
\[ e_{i1} := \langle (M_i W G_i I^{(22)} G_i A_i^{(11)} G_i) \rangle^{[11]}. \]

By the same reasoning, taking the partial trace of the (2, 2)-th block of (5.13) gives
\[ \langle (G_i I^{(22)} G_i A_i^{(11)} G_i) \rangle^{[22]} = |z_i|^{-2} \langle G_i I^{(22)} G_i A_i^{(11)} G_i \rangle^{[11]} - e_{i2} + O_{\prec} (N^{-1}), \]
where \( e_{i2} \) is the (2, 2)-th partial trace of the same \( (2N \times 2N) \) matrix in the definition of \( e_{i1}. \) Solving (5.14) and (5.15) with respect to the partial traces leads to
\[ \| S[G_i I^{(22)} G_i A_i^{(11)} G_i] \| \leq \| \langle (G_i I^{(22)} G_i A_i^{(11)} G_i) \rangle^{[11]} \rangle + \| \langle (G_i I^{(22)} G_i A_i^{(11)} G_i) \rangle^{[22]} \rangle \leq |e_{i1}| + |e_{i2}| + O_{\prec} (N^{-1}). \]

Note that \( e_{i1} \) and \( e_{i2} \) both have the same form as (5.11), so that \( \mathbb{E}[|e_{i1}|^2], \mathbb{E}[|e_{i2}|^2] \) are \( O(N^{-2}). \) Therefore the normalized trace of the third term on the right-hand side of (5.9) has second moment of \( O(N^{-2+\epsilon}). \)
Plugging (5.10), (5.11), (5.12), and (5.16) into (5.9), we obtain that

\[(5.17) \quad \mathcal{X}_i = R_i + O(\eta(N^{-1}) \quad \text{where } R_i \text{ is a random variable with } \mathbb{E}[|R_i|^2] = O(N^{-2}) \text{ uniformly over } i. \]

This completes the proof of the first estimate in (4.14).

The proof of the second estimate in (4.14) follows similar lines, and the only difference is that the analogue of (5.10) no longer has negligible contribution. To be precise, the exact same reasoning as above proves

\[(5.18) \quad Q_j = \langle G^{[12]}_j J+j^1 A \rangle \mathcal{G}_j^{[12]} B_j^+ = \langle (M_j f^{(22)} + S(G_j f^{(22)} G_1)) G_1 \rangle + O(N^{-1}), \]

where \(R_j\) satisfies \(\mathbb{E}[|R_j|^2] = O(N^{-2}). \) To avoid repetition, we omit the detailed proof of (5.18) and just mention the key changes. The first term on the right-hand side of (5.18), in contrast to (5.10), concentrates around a deterministic value of constant order. This value can be calculated using \(\eta = N^{-2}, \) Lemma 5.1, and (5.6) as

\[(5.19) \quad \langle (M_j f^{(22)} + S(M_j (w_j, z_1))) M_A^{(11)} (z_1, w_j) \rangle + O(N^{-1}) \]

\[= z_1 \frac{\langle A_1 \rangle \langle B_j^+ \rangle}{1 - z_1 w_j} + \frac{\langle A_1 B_j^+ \rangle}{(1 - z_1 w_j)^2} + O(N^{-1}). \]

Combining (5.18) and (5.19) proves the second estimate in (4.14).

5.2. Proof of (4.15)

Prior to the proof of (4.15), we remark that Lemma 3.2, \(\eta = N^{-2}, \) and (3.16) imply the following;

\[(5.20) \quad G^{[k]}_{ab} \prec \delta_{ab} \mathbb{I}_{k \neq l} + N^{-1/2}. \]

In other words, \(G^{[k]}_{ab} \) is \(O(N^{-1/2}) \) unless it is a diagonal entry of either \(G^{[1]} \) or \(G^{[2]} \). This a priori bound will be repeatedly used throughout the remaining sections, since the calculations largely involve case-by-case estimates of resolvent entries.

We now turn to the proof of (4.15), whose left-hand side can be expanded as

\[(5.21) \quad \frac{\kappa(k, l + 1)}{k!!} \sum_{a, b} \partial_{ab}^{(k, l)} \langle (G_k^{[1]} A_1) ba [\mathcal{X}(1)] Y \rangle \]

\[= \sum_{a, b} \kappa(k, l + 1) \sum_{k_1 \leq k, l_1 \leq l} \left( \frac{\partial_{ab}^{(k_1, l_1)} \langle (G_k^{[1]} A_1) ba \rangle}{k_1 ! l_1 !} \right) \left( \frac{\partial_{ab}^{(k-k_1, l-l_1)} \|[\mathcal{X}(1)] Y \|}{(k-k_1)!(l-l_1)!} \right). \]

When \(k + l \geq 4, \) by (3.7) we have for all \(k_1 \leq k \) and \(l_1 \leq l \) that

\[\partial_{ab}^{(k_1, l_1)} \langle (G_k^{[1]} A_1) ba \rangle < 1, \quad \partial_{ab}^{(k-k_1, l-l_1)} \|[\mathcal{X}(1)] Y \| < 1. \]

Thus \(\kappa(k, l + 1) = O(N^{-(k+l+1)/2}) \) gives

\[\left| \sum_{a, b} \frac{\kappa(k, l + 1)}{k!!} \partial_{ab}^{(k, l)} \langle (G_k^{[1]} A_1) ba [\mathcal{X}(1)] Y \rangle \right| \ll N^{2-(k+l+1)/2}, \]

which exactly matches (4.15). Therefore we assume \(k + l \in \{2, 3\} \) in what follows.

We first consider the case when \(k_1 + l_1 = k + l, \) that is, when all the differentials operators in (5.21) act on the first factor \((G_k^{[1]} A_1) ba. \) Then the derivative consists of terms of the form

\[(5.22) \quad \text{Tr } G_1 A_1 \Delta G_1 \Delta_2 \cdots \Delta_{k+l} G_1 (A_1 \Delta_{ba})^{(12)}, \]
where $\Delta_i$'s are either $\Delta_{ab}^{(12)}$ or $\Delta_{aa}^{(21)}$, which reduces to products of $(k+l)$ entries of $G$ and an entry of $GA^{(12)}$. In particular, for any choice of $\Delta_i$, each entry of $G$ appearing in (5.22) must be one of the following:

$$G_{aa}^{[11]}, \quad G_{ab}^{[12]}, \quad G_{ba}^{[21]}, \quad G_{bb}^{[22]}.$$ 

Hence (5.20) proves that

$$| \text{Tr} \ G_1 \Delta_1 \cdots \Delta_{k+l} (A_1 \Delta_{ab})^{(12)} | \prec N^{-(k+l)/2} + \delta_{ab},$$

so that

$$\kappa(k, l + 1) \sum_{a,b} \mathbb{E} \left[ |[\mathcal{X}(1)Y][\partial_{ab}^{(k,l)} [(G_1^{[21]} A_1)_{ba}]] \right] = O(N^{-1/2 + \epsilon}).$$

Therefore we have that

$$\kappa(k, l + 1) \sum_{a,b} \mathbb{E} \left[ |[\mathcal{X}(1)Y][\partial_{ab}^{(k,l)} [(G_1^{[21]} A_1)_{ba}]] \right] = O(N^{-1/2 + \epsilon}).$$

Thus it suffices to consider the case where $k = l = 2$ and $k_1 + l_1 < k + l$, that is, we need to estimate the following quantity:

$$\sum_{a,b} \sum_{k,l} \kappa(|k|, |l| + 1) \frac{\partial_{ab}^{(k,l)}}{k_1! l_1!} \prod_{i \in [2, p]} \frac{\partial_{ab}^{(k,l)}}{k_i! l_i!} \prod_{j \in [q]} \frac{\partial_{ab}^{(k,l)}}{k_{p+j}! l_{p+j}!},$$

where $k$ and $l$ run over $(p + q)$-tuples of nonnegative integers with $|k| + |l| = 2$ and $k_1 + l_1 < |k| + |l|$, and we denoted $|k| = \sum_{i \in [p+q]} k_i$ and $|l| = \sum_{i \in [p+q]} l_i$. Here we used the fact that at least one differential operator acts on $[\mathcal{X}(1)Y]$, so that its derivative is equal to that of the uncentered quantity $\mathcal{X}(1)Y$.

In what follows, we present an upper bound of the contribution of the sum in (5.24) for $|k| + |l| = 2$ case by case. The case $|k| + |l| = 3$ will be handled in a similar fashion afterwards.

**Case 1:** $k_1 = 2$ or $l_1 = 2$ for some $i > 1$.

Due to similarity, we consider only the case $k_1 = 2$ for some $i \in [2, p]$. In this case, the whole contribution is

$$\kappa(2, 1) \frac{\mathcal{X}(1)Y}{2} \sum_{a,b} (G_1^{[21]} A_1)_{ba} g_{ab}^{(2,0)} [\text{Tr} G_i^{[21]} A_i] = \kappa(2, 1) \mathcal{X}(1)Y \sum_{a,b} (G_1^{[21]} A_1)_{ba} (\text{Tr} G_i^{[21]} \Delta_{ab} G_i^{[21]} \Delta_{ab} G_i^{[21]} A_i).$$

Note that the first factor on the right-hand side of (5.25) is given by

$$\left( G_1^{[21]} A_1 \right)_{ba} = (M_1^{[21]} A_1)_{ba} + O_\prec(N^{-1/2}) = - \frac{1}{z_1} (A_1)_{ba} + O_\prec(N^{-1/2}),$$

using Lemma 3.2, and second factor admits the following bound;

$$| \text{Tr} G_i^{[21]} \Delta_{ab} G_i^{[21]} \Delta_{ab} G_i^{[21]} A_i | = |(G_i^{[21]} A_i G_i^{[21]} A_i)_{ba} | < N^{-1/2} |(A_i)_{ba}| + N^{-1},$$

where we applied Lemma 5.1 and (5.6) to get

$$\left( G_i^{[21]} A_i G_i^{[21]} \right)_{ba} = (M_i A_i^{(12)} M_i^{[21]} A_i)_{ba} + O_\prec(N^{-1/2}) = |z_i|^{-2} (A_i)_{ba} + O_\prec(N^{-1/2}).$$

Combining with $\kappa(2, 1) = O(N^{-3/2})$ and (4.6), the left-hand side of (5.25) is stochastically dominated by

$$N^{-3/2} \sum_{a,b} |(A_1)_{ba} | |(A_i)_{ba}| | + N^{-1/2} \leq N^{-3/2}(\text{Tr} |A_1|^2 + \text{Tr} |A_2|^2) + N^{-1/2} < N^{-1/2} (1 + \|A_1\| + \|A_i\|),$$

which establishes (4.15) for this case. The same argument applies for $i > p$ and for $l_i = 2$. 
Case 2: $k_i = 1 = l_i$ for some $i > 1$.

As above, we focus on the case $i \in [2, p]$. In this case the contribution becomes

$$\kappa(1, 2) \chi^{(1-i)} \sum_{a,b} (G_1^{[2]} A_1)_{ab} \text{Tr}(G_1^{[2]} \Delta_{ab} G_1^{[1]} + (G_2^{[2]} \Delta_{ab} G_1^{[1]}) A_i)$$

$$= \kappa(1, 2) \chi^{(1-i)} \sum_{a,b} (G_1^{[2]} A_1)_{ab} \left( (G_i^{[1]})_{aa} (G_i^{[2]} A_i)_{ab} + (G_i^{[2]})_{bb} (G_i^{[1]} A_i)_{ab} \right).$$

Then we use Lemma 5.1 and (5.6) to get

$$(G_1^{[2]} A_1)_{bb} = (M_{A_1} (z_i, z_i))_{bb} + O_{\prec} (N^{-1/2}) = O_{\prec} (N^{-1/2}),$$

$$= (G_i^{[1]} A_i)_{bb} = (M_{A_1} (z_i, z_i))_{bb} + O_{\prec} (N^{-1/2}) = O_{\prec} (N^{-1/2}).$$

By (5.20), (5.26), and $\kappa(1, 2) = O(N^{-3/2})$, the contribution of this case is stochastically dominated by

$$(5.27) \quad N^{-5/2} \sum_{a,b} |(A_1)_{ba}| \leq N^{-3/2} (\text{Tr} |A_1|^2)^{1/2} \leq \|A_1\| N^{-1}.$$ 

Case 3: $k_1 + l_1 = 1$.

In this case, one derivative is acting on the first factor $\text{Tr}(G_1^{[2]} A_1 \Delta_{ab})$ in (5.24), resulting in a factor of

$$(5.28) \quad \partial^{(k_1, l_1)} \text{Tr} G_1^{[2]} A_1 \Delta_{ab} = \begin{cases} (G_1^{[2]} A_1)_{ba} \text{Tr}(G_1^{[2]} A_1)_{ba} & \text{if } k_1 = 1, \\ (G_1^{[2]})_{bb} (G_1^{[1]} A_1)_{aa} & \text{if } l_1 = 1. \end{cases}$$

By (5.20), this factor is $O_{\prec} (N^{-1/2})$. Then the remaining one differential should act on either $\chi_i$ or $Y_j$. If it acts on $\chi_i$, we get a factor of

$$(5.29) \quad \partial^{(k_1, l_1)} \chi_i = \begin{cases} (G_1^{[2]} A_1 G_1^{[21]})_{ab} \prec |(A_1)_{ba}| + N^{-1/2} & \text{if } k_1 = 1, \\ (G_1^{[21]})_{ab} \prec N^{-1/2} & \text{if } l_1 = 1. \end{cases}$$

Applying Cauchy–Schwarz inequality to $|(A_1)_{ba}|$ as in (5.27) proves that the total contribution is of order $O_{\prec} (N^{-1/2})$, and the same applies when the remaining differential acts on $Y_j$'s.

Case 4: $k_j + l_j = 1 = k_i + l_i$ for some $i \neq j > 1$.

By (5.29) and Cauchy–Schwarz inequality, the contribution of this case is dominated by

$$N^{-3/2} (\text{Tr} |A_1|^2 + \text{Tr} |B_j|^2) + N^{-2} \sum_{a,b} \left( |(A_1)_{ab}| + |(B_j)_{ab}| \right) + N^{-1/2} = O(N^{-1/2})$$

if $i \in [2, p]$ and $j \in [p + 1, p + q]$. The same reasoning applies to other choices of $i$ and $j$, proving (4.15).

The four cases above exhaust all possible choices of $(k, l)$ with $|k| + |l| = 2$.

Since the proof of (4.15) for $|k| + |l| = 3$ is almost the same, we only present its sketch. Since $\kappa(|k|, |l| + 1) = O(N^{-2})$, the sum $\kappa(|k|, |l| + 1) \sum_{a,b}$ becomes the average over $a$ and $b$, and we only need this average to be $O_{\prec} (N^{-1/2})$. For example, using (5.26) and (5.27), the contribution when $k_1 + l_1 = 0$ is bounded by

$$N^{-2+\epsilon} \sum_{a,b} |(A_1)_{ba}| + N^{-1/2+\epsilon} \leq N^{-1/2+\epsilon} (1 + \|A_1\|),$$

where we simply applied (4.6) and the norm bounds $\|G_z\|, \|A_i\|, \|B_j\| = O(1)$ to factors other than $\text{Tr} G_1^{[21]} A_1 \Delta_{ab}$. By the same reasoning, (5.28) and (5.29) exhaust the case when $k_i + l_i = 1$ for some $i \in [p + q]$. Since the only remaining case is $k_1 + l_1 = 3 = |k| + |l|$, which was already dealt with in (5.23), this completes the proof of (4.15).
5.3. Proof of (3.19)

In this section, we prove the asymptotics (3.19) of \( E[\text{Tr} G^{[21]} f] \). First, we repeat the same expansion using \( i\eta G + I = W_z G \) as in (4.10):

\[
E[i\eta \text{Tr} G^{[1]}_1 A_1] = E[\text{Tr} X G^{[1]}_1 A_1] - z_1 E[\text{Tr} G^{[21]}_1 A_1] - \text{Tr} A_1
\]

(5.30)

\[- z_1 E[\text{Tr} G^{[21]}_1 A_1] - \text{Tr} A_1 + \sum_{k,l \in Z^+} \frac{\kappa(k + 1, l)}{k!} \sum_{a,b} E[\partial_{ab}^{(k,l)} \{\text{Tr} \Delta_{ab} G^{[21]}_1 A_1\}] .
\]

Therefore, in order to prove (3.19), it suffices to prove that the left-hand side and the last term on the right-hand side of (5.30) are \( O(N^{-1/2+\epsilon}) \). Since the left-hand side is \( O_x(\eta) \) by (4.5), we may focus on the latter.

Note that \( \kappa(2, 0) = 0 \), so that the term corresponding to \((k, l) = (1, 0)\) vanishes. When \( k = 0 \) and \( l = 1 \), we have

\[\kappa(1, 1) \sum_{a,b} \partial_{ab}^{(0,1)} \{\text{Tr} \Delta_{ab} G^{[21]}_1 A_1\} = -(G^{[22]}_1 \text{Tr} G^{[1]}_1 A_1 < N(N^{-1} + \eta)^2 < N^{-1},\]

where we used Lemma 3.2 and \( \eta = N^{-2} \). Next, we trivially handle the case \( k + l \geq 4 \) using \( \kappa(k + 1, l) = O(N^{-(k+l+1)/2}) \) as in the previous section, that is,

\[\kappa(k + 1, l) \sum_{a,b} \partial_{ab}^{(k,l)} \{\text{Tr} \Delta_{ab} G^{[21]}_1 A_1\} < N^{-(k+l-3)/2} \quad \text{if} \ k + l \geq 4 .\]

Thus we have reduced the case to \( k + l \in \{2, 3\} \) and what we need to estimate is identical to the left-hand side of (5.25) except the centering. Therefore it suffices to recall that (5.25) followed from the bound without centering and that \( \kappa(k, l + 1) \) and \( \kappa(k + 1, l) \) are both \( O(N^{-(k+l+1)/2}) \). This completes the proof of (3.19), hence that of Proposition 3.6 for complex \( \chi \).

6. Extension to real random matrices

This section is devoted to the proof of Proposition 3.6 when \( \chi \) is real-valued. Since there is no, for the first time, the necessary modifications. We will repeatedly use that \((G^{[1]}_z)^\dagger = G^{[21]}_z \) and \((G^{[1]}_z)^\dagger = G^{[1]}_z \) for real-valued \( X \).

First, let us recall Proposition 4.1. For \( I \subset [p], J \subset [q] \), where \( \text{PairPart}(I, J) \) is the set of partitions of index set \( I \cup J \subset [p+q] \) into pairs and \( V_{k,\ell}^0 \) and \( V_{k',\ell'}^0 \) are defined by

\[
V_{k,\ell}^0 := V^0(z_i, \bar{z}_{i'}) \langle A_i A_j \rangle, \quad V_{i,\ell'}^0 := V^0(z_i, \bar{z}_{i'}),
\]

\[
V_{i,j+p}^0 := V^0(z_i, w_j) \langle A_i B_j \rangle, \quad V_{i,j+p}^0 := V^0(z_i, w_j),
\]

\[
V_{j+p,j'+p}^0 := V^0(w_j, w_{j'}) \langle B_j B_{j'} \rangle, \quad V_{j+p,j'+p}^0 := V^0(w_j, w_{j'}),
\]

for \( i, i' \in [p] \) and \( j, j' \in [q] \) with \( V^0 \) and \( V \) defined in (4.4). The second necessary modification is in (4.10); when \( \chi \) is a real random variable, we use the real cumulant expansion in (1.3):

\[
E[X_{ab} f(X_{ab})] = \sum_{k \in \mathbb{N}} \frac{\kappa(k + 1)}{k!} E[\partial_{ab}^{(k)} f(X_{ab})],
\]

where \( \kappa(k) \) and \( \partial_{ab}^{(k)} \) are defined by the exact same formulas as \( \kappa(k, 0) \) and \( \partial_{ab}^{(k,0)} \), respectively. The derivative of the resolvent is given by \( \partial_{ab} [G^{[k]}_z] = -G^{[k]}_z \Delta_{ab} G^{[2]}_z - G^{[k]}_z \Delta_{ba} G^{[1]}_z \). After applying these changes, the first term of (4.10)
becomes the expectation of

\[(6.2) \quad -\left(\left(G_{2z}^{(22)}\right)\Tr G_{2z}^{[1]} A_1 + \frac{1}{N} \sum_{a,b} (G_{2z}^{(21)})_{ba} (G_{2z}^{(21)} A)_{ab}\right) \parallel[X^{(1)} Y]\],

where the second term in the bracket was absent in the complex case. This term is written as

\[
\frac{1}{N} \sum_{a,b} (G_{2z}^{(21)})_{ba} (G_{2z}^{(21)} A)_{ba} = \langle (G_{2z}^{(21)} A G_{2z}^{[12]} - (G_{2z}^{(21)} A G_{2z}^{[12]}))_{ba} \rangle.
\]

We will see below that Lemma 5.1 and (5.5) still hold for real \(X\), so that the bound in (4.11) remains true for the analogue of the first term of (4.10) after an application of \(E[\parallel X Y] = E[X] I[Y]\). Similarly, (4.13) holds if we replace \(\tilde{x}_i\) and \(\tilde{y}_j\) respectively by

\[
\tilde{x}_i := -\frac{1}{N} \sum_{a,b} (G_{2z}^{(21)} A_1)_{ba} \partial_{a,b} \Tr G_{2z}^{[21]} A_i = \frac{1}{N} \sum_{a,b} (G_{2z}^{(21)} A_1)_{ba} \Tr G_{2z} (\Delta_{ab}^{(12)} + \Delta_{ba}^{(21)}) G_{2z}^{(21)} A_i
\]

\[
= \frac{1}{N} \sum_{a,b} (G_{2z}^{(21)} A_1)_{ba} \left( (G_{2z}^{(21)} A_1 G_{2z}^{[12]})_{ba} + (G_{2z}^{[11]} A_1 G_{2z}^{[22]})_{ab} \right) = \tilde{x}_i + \langle (G_{2z}^{(21)} A_1 G_{2z}^{[12]} A_i^{[22]} G_{2z}^{[12]}), \rangle
\]

\[
\tilde{y}_j := -\frac{1}{N} \sum_{a,b} (G_{2z}^{(21)} A_1)_{ba} \partial_{a,b} \Tr G_{2z}^{(12)} B_j^2 = \tilde{y}_j + \langle (G_{2z}^{(21)} A_1 G_{2z}^{[11]} Y_j G_{2z}^{[22]}), \rangle
\]

and we change remainders in cumulant expansions accordingly.

Therefore, it suffices to show the following analogues of (4.14) and (4.15) in order to prove (6.1):

\[
E \left[ \tilde{x}_i \left[ \frac{\langle A_1 A_1' \rangle \langle A_1' A_1 \rangle}{z_i (z_i^* - 1)} + \frac{z_i \langle A_1 \rangle \langle A_1' \rangle}{(z_i^* - 1)^2} \right] \right]^2 = O(N^{-2 - \epsilon}),
\]

\[
E \left[ \tilde{y}_j \left[ \frac{\langle A_1 B_j^* \rangle \langle A_1' B_j^* \rangle}{w_j (w_j^* - 1)} + \frac{z_i \langle A_1 \rangle \langle B_j^* \rangle}{(z_i^* - 1)^2} \right] \right]^2 = O(N^{-2 - \epsilon}),
\]

\[
\kappa (k + 1) \sum_{a,b} \partial_{a,b} \left[ \Tr (G_{2z}^{(21)} A_1) \parallel[X^{(1)} Y] \right] = N^{-1/2} (k - 4 + \epsilon/2).
\]

Note that the estimates in (6.4) are the same as (4.14) and (4.15) except for that of \(\tilde{x}_i\). Next, we explain how we modify the proof of (4.14) to prove the first two estimates of (6.4). The self-renormalization defined in (5.7) should be modified, where we took \(\widetilde{W} \) to be the Hermitization of a complex Ginibre ensemble. When \(X\) is real we take \(\widetilde{W} \) to be that of a real Ginibre ensemble and define \(W f(W)\) via the same formula. As a result, we have

\[
WG_{2z} = WG_z - \mathbb{E}_{\widetilde{W}} \mathbb{W}(\partial_{w} G_z) = WG_z + \tilde{S}[G_z] G_z,
\]

where \(\tilde{S}\) is defined in (3.5). In this way, the analogue of (5.8) becomes

\[
G_z - M_z = -M_z W G_z + M_z S[G_z - M_z] G_z + M_z D_z G_z, \quad D_z := \frac{1}{N} \left( \begin{array}{c} 0 \\ G_{2z}^{[12]} \\ 0 \end{array} \right),
\]

where we used \((G_{2z}^{[21]})^T = G_{2z}^{[21]}\). Nonetheless, Lemma 5.1 remains intact; following lines of [9, Eq. (5.5)], we have

\[
\langle G_z P G_w \rangle = \langle M_z P M_w \rangle + \langle M_z P (G_w - M_w) \rangle + \langle M_z W G_z P G_w \rangle + \langle M_z S[G_z P G_w] M_w \rangle
\]

\[
+ \langle M_z S[G_z P G_w] (G_w - M_w) \rangle + \langle M_z \tilde{S}[G_z] - S[M_z] G_z P G_w \rangle
\]

\[
= \langle M_z P M_w \rangle + \langle M_z S[G_z P G_w] M_w \rangle + \langle M_z W G_z P G_w \rangle + O_N(N^{-1}),
\]

where we used the fact that

\[
\| (S - \tilde{S})[G_z] \| \leq \frac{1}{N} \| G_z \| < \frac{1}{N}, \quad \| (S - \tilde{S})[G_z P G_w] \| \leq \frac{1}{N} \| G_z P G_w \| < \frac{1}{N}.
\]
Inspecting the proof of Proposition 5.3 in [9] which proves the bound \( \langle W G_z P G_w \rangle \ll N^{-1} \), one can see that the same applies to the real case since \( W \) still stands for the remainders of the (real) cumulant expansion. This leads to the same asymptotics for \( \langle G_z P G_w \rangle \) as in Lemma 5.1, that is, \( \tilde{B}^{-1}(M_z PM_w) \), and similarly isotropic local law is valid. Therefore Lemma 5.1 is true when \( X \) is real as well.

In the same spirit, using \( \| \tilde{S} - S \| \| P \| \leq N^{-1} \| P \| \) and \( \| G \| \ll 1 \), all arguments along the proof of (4.14) stay the same up to additional errors of \( O(\epsilon^{-1} N^{-1}) \). Thus the first two estimates of (6.4) reduce to

\[
\begin{align*}
\mathbb{E} \left[ \left| \langle G_z^{[12]} G_z^{[21]} A_1 G_z^{[11]} A_1 \rangle - \frac{(A_1 A_1^\top)}{z(z_1 z_1 - 1)} + \frac{z_1 (A_1 \langle A_1^\top \rangle)^2}{(z_1 z_1 - 1)^2} \right|^2 \right] &= O(N^{-2+\epsilon}), \\
\mathbb{E}[|\langle G_z^{[22]} G_z^{[21]} A_1 G_z^{[11]} B_j \rangle|^2] &= O(N^{-2+\epsilon}).
\end{align*}
\]

Since the left-hand sides of (6.6) have exactly the same form as \( \mathbb{E}_j \) and \( X_i \), respectively, the result immediately follows.

The proof of the last estimate in (6.4) is completely analogous to Section 5.2: we first exhaust the case \( k \geq 4 \) and use the exact same division of cases with \( k_i + l_i \) replaced by \( k_i \), where \( k_i \) denotes the number of differential operators \( \partial_{ab} \) hitting \( (G_1^{[21]} A_1)_{ba} \), \( X_i \), or \( Y_{i-p} \). Then the proof immediately follows once we observe from

\[
(\partial_{ab}^{(1,0)} + \partial_{ab}^{(0,1)}) G_z = G_z (\Delta_{12}^{12} + \Delta_{ab}^{21}) G_z
\]

that the real partial derivative has the same form as the sum of complex partial derivatives in the complex case;

\[
(\partial_{ab}^{(1,0)} + \partial_{ab}^{(0,1)}) G_z = G_z (\Delta_{ab}^{12} + \Delta_{ba}^{12}) G_z.
\]

Therefore, after expanding all real partial derivatives using (6.7), each individual term has already been covered in Section 5.2. This completes the proof of (6.4).

Finally, to prove (3.19) for real-valued \( X \), we first rewrite (5.30);

\[
\mathbb{E}[\eta \text{Tr} G_1^{[11]} A_1] = -z_1 \mathbb{E}[\text{Tr} G_1^{[21]} A_1] - \text{Tr} A_1 + \sum_{k \in \mathbb{N}} \frac{k(k+1)}{k!} \sum_{a,b} \mathbb{E}[\partial_{ab}^{(k)} \text{Tr} \Delta_{ab} G_1^{[21]} A_1].
\]

The only difference here is in the first term of the cumulant expansion. Taking \( k = 1 \), we get the same quantity as in (6.2),

\[
-\frac{1}{N} \sum_{a,b} \mathbb{E}[\text{Tr} G_1 (\Delta_{ab}^{12} + \Delta_{ba}^{21}) G_1 (A_1 \Delta_{ab})^{[12]}] = -\mathbb{E}[\langle (G_{z_1} A_1^{(11)} G_{z_1}^{[22]}) \rangle - \mathbb{E}[\langle G_{z_1} \text{Tr} G_{z_1}^{[11]} A_1 \rangle]],
\]

so that the first term is newly appeared in the real case. By Lemma 5.1 and (5.6), we have

\[
\mathbb{E}[\langle (G_{z_1} A_1^{(11)} G_{z_1}^{[22]}) \rangle] = \langle M_A^{[11]} (z_1, z_1)^{[22]} \rangle + O(N^{-1+\epsilon}) = \frac{\langle A_1 \rangle}{z_1^2 - 1}.
\]

Plugging this into (6.8) proves (3.19) for the real case. This concludes the proof of Proposition 3.6 for real-valued \( X \).

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