Generic triangular solutions of the reflection equation: \( U_q(\widehat{sl}_2) \) case

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Abstract
We consider intertwining relations of the triangular \( q \)-Onsager algebra, and obtain generic triangular boundary \( K \)-operators in terms of the Borel subalgebras of \( U_q(\widehat{sl}_2) \). These \( K \)-operators solve the reflection equation.

Keywords: triangular \( q \)-Onsager algebra, \( K \)-operator, reflection equation, \( L \)-operator

1. Introduction

The reflection equation [1] is a fundamental object in quantum integrable systems with open boundary conditions [2]. It has the following expression:

\[
R_{12} \left( \begin{array}{c} x \\ y \end{array} \right) K_1(x) \bar{R}_{12} (xy) K_2(y) \bar{R}_{12} \left( \begin{array}{c} 1 \\ xy \end{array} \right) K_1(x) \bar{R}_{12} \left( \begin{array}{c} \overline{x} \\ \overline{y} \end{array} \right), \quad x, y \in \mathbb{C}^x,
\]

(1.1)

where \( R(x) \) and \( \bar{R}(x) \) are solutions (R-matrices) of the Yang–Baxter equation and \( K(x) \) is a \( K \)-matrix. The indices 1 and 2 denote the space on which the operators act non-trivially. In particular, there is a \( 2 \times 2 \) matrix solution [3–5] of the reflection equation associated with the \( 4 \times 4 \) \( R \)-matrices of the six-vertex model (for the two-dimensional fundamental representation of \( U_q(\widehat{sl}_2) \): see (2.24) and (2.25)):

\[
K(x) = \begin{pmatrix}
 x^0 \epsilon_+ + x^{-s} \epsilon_- & \frac{k_+(x^0 - x^{-s})}{q - q^{-1}} \\
\frac{k_-(x^0 - x^{-s})}{q - q^{-1}} & x^{-s} \epsilon_+ + x^0 \epsilon_-
\end{pmatrix},
\]

(1.2)

where \( k_\pm \) and \( \epsilon_\pm \) are scalar parameters. These \( R \)-matrices are evaluations of more general operators called \( L \)-operators: \( L_{12}(x), \bar{L}_{12}(x) \in U_q(\widehat{sl}_2) \otimes \text{End}(\mathbb{C}^2) \) (see (2.22) and (2.23)). Namely,
they are given by $R_{12}(x) = (\pi \otimes 1)L_{12}(x), \bar{R}_{12}(x) = (\pi \otimes 1)\bar{L}_{12}(x)$, where $\pi$ is the fundamental representation of $U_q(sl_2)$. In this context, a natural problem is to explore the solutions of the reflection equation associated with the $L$-operators:

$$L_{12} \left( \begin{array}{c} x \\ y \end{array} \right) K_i(x)\bar{L}_{12} \left( \begin{array}{c} y \\ x \end{array} \right) K_2(y)L_{12} = K_2(y)\bar{L}_{12} \left( \begin{array}{c} y \\ x \end{array} \right) K_i(x)L_{12} \left( \begin{array}{c} x \\ y \end{array} \right) ,$$

(1.3)

where $K(x)$ is a generic $1$-operator in $U_q(sl_2)$. In this paper, we propose generic triangular solutions $K(x)$ of (1.3) associated with the triangular $K$-matrices ((1.2) for $k_+ = 0$ or $k_- = 0$) in terms of the elements of the Borel subalgebras of $U_q(sl_2)$ (see (3.14), (3.27), (3.29) and (3.30)). Evaluation of the generic $K$-operator in the fundamental representation of $U_q(sl_2)$ reproduces the $2 \times 2$ triangular $K$-matrices (1.2): $K(x) = \pi(K(x))$. In the context of Baxter $Q$-operators for integrable systems with open boundaries, generic diagonal $K$-operators$^2$ ($k_+ = k_- = 0$ case) for $U_q(sl_2)$ were previously proposed in [6]. This paper extends these to the triangular case in part. Although it is beyond the scope of the present paper, we expect that our results will be useful to construct Baxter $Q$-operators for integrable systems with triangular boundaries (by taking limits of $K$-operators as discussed in [6]). We also remark that there are solutions of the reflection equation for the symmetric tensor representations of $U_q(A_{n-1}^{(1)})$ [21] and $U_q(\widehat{sl}_2)$ [22]. The directions of their results [21, 22] are different from ours in that their solutions are not expressed in terms of generators of symmetry algebras and thus depend on representations. As for the rational case ($q = 1$ case; the XXX-model associated with the Yangian $Y(sl_2)$), diagonal $K$-operators for Baxter $Q$-operators appeared first in [8], and a generic non-diagonal $K$-operator was proposed recently$^3$ in [9].

In section 2, we review the quantum algebras $U_q(\widehat{sl}_2)$ and $U_q(sl_2)$ and associated $R$-and $L$-operators. In section 3, we recall the triangular $q$-Onsager algebra $O_q(\widehat{sl}_2)$ [10], which is the underlying symmetry for the XXZ-spin chain with triangular boundary conditions. It is a co-ideal subalgebra (see [11]) of $U_q(\widehat{sl}_2)$. We solve the intertwining relations of the form

$$\ev_{x} \cdot (a)K(x) = K(x)\ev_{x}(a)$$

for $a \in O_q(\widehat{sl}_2)$.

(1.4)

where $\ev_{x}$ is an evaluation map $\ev_{x} : O_q(\widehat{sl}_2) \to U_q(sl_2)$ with the spectral parameter $x \in \mathbb{C}^\times$. We show$^4$ that solutions of these intertwining relations also solve the reflection equation (1.3).

In section 4, we discuss the connection to the $q$-Onsager algebra [13, 14], which is the symmetry algebra for the case $k_+ k_- \neq 0$. In the appendix, we review miscellaneous formulas which follow from the $q$-deformed Hadamard formula [15]. Throughout this paper, we assume that the deformation parameter is of the form $q$ is not a root of unity. We also identify the multiplicative unit element $I$ of the quantum algebra.
multiplied by a complex number $b \in \mathbb{C}$ with $b$ by an algebra embedding from the field of scalars into the associative algebra: $b \mathbf{1} = b$. We use the following notation:

**Notation.**

- For any elements $X, Y$ of the quantum algebras, we define the $q$-commutator by $[X, Y]_q = XY - qYX$. In particular, we set $[X, Y]_1 = [X, Y]$.
- We introduce an expression $(x; q)_k = \prod_{j=0}^{k-1} (1 - xq^j)$. In particular, we define $(x; q)_\infty = \lim_{k \to \infty} (x; q)_k = \prod_{j=0}^{\infty} (1 - xq^j)$ for $|q| < 1$. For more detail, see for example, page 47 in [17].
- We define a $q$-analogue of the exponential function by $\exp_q(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k}{(k)_q!} = ((1 - q)x; q)_\infty^{-1}$ for $|q| < 1$, $|(1 - q)x| < 1$, where $(k)_q! = (1)_q (2)_q \cdots (k)_q$, $(0)_q! = 1$, $(k)_q = (1 - q^k)/(1 - q)$. For more detail, see for example, page 47 in [17].
- We will use free parameters $s_0, s_1 \in \mathbb{Z}$. In particular, we set $s = s_0 + s_1$.

2. Quantum algebras and $L$-operators

In this section, we review quantum algebras and associated $R$- and $L$-operators. We basically follow the convention in [6]. We also refer to [16, 17] for review on this subject.

2.1. The quantum affine algebra $U_q(\hat{sl}_2)$

Let us start from the definition of the algebra.

**Definition 2.1.** The quantum affine algebra $U_q(\hat{sl}_2)$ (at level 0, i.e. the quantum loop algebra) is a Hopf algebra generated by the generators $e_i, f_i, q^{\xi h_i}$ for $i \in \{0, 1\}$ and $\xi \in \mathbb{C}$ obeying the following relations:

\[ q^{0 h_0} = q^0 = 1, \quad q^{\xi h_i} q^{\eta h_0} = q^{(\xi + \eta) h_0}, \quad q^{0 h_0} q^{0 h_1} = 1, \quad (2.1) \]

\[ [e_i, f_j] = h_{ij} q^{h_i - h_j} - q^{-h_j} q^{h_i h_j}, \quad q^{\xi h_i} e_j q^{-\xi h_i} = q^\xi [e_j, q^{\xi h_i}], \quad q^{\xi h_i} f_j q^{-\xi h_i} = q^{-\xi h_i} [f_j, q^{\xi h_i}], \quad (2.2) \]

\[ [e_i, [e_i, [e_i, e_j]_{q^2}]]_{q^2} = [f_i, [f_i, [f_i, f_j]_{q^2}]]_{q^2} = 0 \quad i \neq j, \quad \xi, \eta \in \mathbb{C}, \quad (2.3) \]

where $(a_{ij})_{0 \leq i, j \leq 1}$ is the Cartan matrix

\[ (a_{ij})_{0 \leq i, j \leq 1} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \]

The algebra has automorphisms $\sigma$ and $\tau$ defined by

\[ \sigma(e_0) = e_1, \quad \sigma(f_0) = f_1, \quad \sigma(q^{h_0}) = q^{h_1}, \]

\[ \sigma(e_1) = e_0, \quad \sigma(f_1) = f_0, \quad \sigma(q^{h_1}) = q^{h_0}, \quad \sigma(q) = q \quad (2.4) \]

and

\[ \tau(e_i) = f_i, \quad \tau(f_i) = e_i, \quad \tau(q^{h_i}) = q^{-h_i}, \quad \tau(q) = q, \quad i = 0, 1. \quad (2.5) \]
The algebra also has an anti-automorphism $\iota$ defined by\(^5\)

$$
\iota(e_i) = q^{-1 - h_i} f_i, \quad \iota(f_i) = e_i q^{1 + h_i}, \quad \iota(q_i) = q_i, \quad i = 0, 1.
$$

(2.6)

Recall that this means $\sigma(ab) = \sigma(a)\sigma(b)$, $\tau(ab) = \tau(a)\tau(b)$ and $\iota(ab) = \iota(b)\iota(a)$ for $a, b \in U_q(\mathfrak{sl}_2)$. We use the following co-product $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$:

$$
\Delta(e_i) = e_i \otimes 1 + q^{-h_i} \otimes e_i,
\Delta(f_i) = f_i \otimes q^{h_i} + 1 \otimes f_i,
\Delta(q_i) = q_i \otimes q_i.
$$

(2.7)

We will also use the opposite co-product defined by

$$
\Delta' = p \circ \Delta, \quad p \circ (X \otimes Y) = Y \otimes X, X, Y \in U_q(\mathfrak{sl}_2).
$$

(2.8)

Anti-pode, co-unit and grading element $d$ are not used in this paper.

The Borel subalgebra $\mathcal{B}_+$ and $\mathcal{B}_-$ are generated by $e_i, q_i^{\pm h}$ and $f_i, q_i^{\pm h}$, respectively, where $i \in \{0, 1\}, \xi \in \mathbb{C}$. There exists a unique element $[18, 19] \mathcal{R}$ in a completion of $\mathcal{B}_+ \otimes \mathcal{B}_-$ called the universal $R$-matrix which satisfies the following relations

$$
\Delta'(a) = \mathcal{R} \Delta(a) \quad \text{for all} \quad a \in U_q(\mathfrak{sl}_2),
(\Delta \otimes 1) \mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{23},
(1 \otimes \Delta) \mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{12},
$$

(2.9)

where\(^6\) $\mathcal{R}_{12} = \mathcal{R} \otimes 1$, $\mathcal{R}_{23} = 1 \otimes \mathcal{R}$, $\mathcal{R}_{13} = (p \otimes 1) \mathcal{R}_{23}$. The Yang–Baxter equation

$$
\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}
$$

(2.10)

follows from these relations (2.9).

### 2.2. The quantum algebra $U_q(\mathfrak{sl}_2)$

**Definition 2.2.** The quantum algebra $U_q(\mathfrak{sl}_2)$ is generated by the elements $E, F, q^{\pm H}$ for $\xi \in \mathbb{C}$ obeying the following relations:

$$
q^{H^0} = q^0 = 1, \quad q^{H^0} q^{H^H} = q^{(\xi + \eta)H}, \quad q^{H^0} E q^{-H^H} = q^{2\xi} E,
$$

$$
q^{H^0} F q^{-H^H} = q^{-2\xi} F, \quad [E, F] = \frac{q^H - q^{-H}}{q - q^{-1}}, \quad \xi, \eta \in \mathbb{C}.
$$

(2.11)

The upper (resp. lower) Borel subalgebra is generated by $E, q^{\pm H}$ (resp. $F, q^{\pm H}$). The Casimir element

$$
C = FE + \frac{q^{H+1} + q^{-H-1}}{(q - q^{-1})^2} = EF + \frac{q^{H-1} + q^{-H+1}}{(q - q^{-1})^2}
$$

(2.12)

\(^5\) For any $a \in \mathbb{C}$ and a Cartan element $\mathcal{H}$, we denote $q^a q^\mathcal{H}$ as $q^{a+\mathcal{H}}$.

\(^6\) We will use similar notation for the $L$-operators to indicate the space on which they non-trivially act.
is central in \( U_q(\mathfrak{sl}_2) \). There are an automorphism

\[
\sigma(E) = F, \quad \sigma(F) = E, \quad \sigma(q^{iH}) = q^{-iH}, \quad \sigma(q) = q, 
\]

and an anti-automorphism

\[
\iota(E) = q^{-H-1}F, \quad \iota(F) = Eq^{H+1}, \quad \iota(q^{iH}) = q^{-iH}, \quad \iota(q) = q 
\]

of the algebra. They are \( U_q(\mathfrak{sl}_2) \) analogues of (2.4) and (2.6), respectively. There is an algebra homomorphism called evaluation map\(^7\) \( \text{ev}_q: U_q(\mathfrak{sl}_2) \mapsto U_q(\mathfrak{sl}_2) \),

\[
e_0 \mapsto x^0F, \quad f_0 \mapsto x^{-30}E, \quad q^0 \mapsto q^{-5H}, \\
e_1 \mapsto x^1E, \quad f_1 \mapsto x^{-31}F, \quad q^1 \mapsto q^{-3H},
\]

where \( x \in \mathbb{C}^* \) is the spectral parameter. We set\(^8\)

\[
\sigma(s_0) = s_1, \quad \sigma(s_1) = s_0, \\
\iota(s_0) = s_0, \quad \iota(s_1) = s_1.
\]

One can check consistency of these: \( \sigma \circ \text{ev}_q = \text{ev}_q \circ \sigma \) and \( \iota \circ \text{ev}_q = \text{ev}_q \circ \iota \). The fundamental representation \( \pi \) of \( U_q(\mathfrak{sl}_2) \) is given by \( \pi(E) = E_{12}, \pi(F) = E_{21} \) and \( \pi(q^{iH}) = q^iE_{11} + q^{-i}E_{22} \), where \( E_{ij} \) is the \( 2 \times 2 \) matrix unit whose \((k,l)\)-element is \( \delta_{kl}\delta_{ij} \). The composition \( \pi_x = \pi \circ \text{ev}_q \) gives an evaluation representation of \( U_q(\mathfrak{sl}_2) \). In case we consider the fundamental representation, we define an algebra automorphism \( \sigma \) and an algebra anti-automorphism \( \iota \) of the algebra of \( 2 \times 2 \) matrices over \( \mathbb{C} \) by

\[
\sigma(E_{ij}) = E_{3-i,3-j}, \\
\iota(E_{ij}) = E_{ji}, \quad i, j = 1, 2.
\]

We have an identity of algebra homomorphisms \( \pi \circ \sigma = \sigma \circ \pi \) and an identity of algebra anti-homomorphisms \( \pi \circ \iota = \iota \circ \pi \), which justifies our use of the same symbol for different maps.

2.3. \( L \)-operators

The so-called \( L \)-operators are images of the universal \( R \)-matrix, which are given by \( L(xy^{-1}) = \phi(xy^{-1})(\text{ev}_q \otimes \pi_y)R, \quad \overline{L}(xy^{-1}) = \phi(x^{-1}y)(\text{ev}_q \otimes \pi_y)\overline{R}_{21} \), where \( x, y \in \mathbb{C}^* \), and \( \phi(xy^{-1}) \) is an overall factor whose explicit expression will not be used in this paper. They are solutions of the intertwining relations, which follow from (2.9):

\[
(\text{ev}_q \otimes \pi_y)\Delta(a)L(xy^{-1}) = L(xy^{-1})(\text{ev}_q \otimes \pi_y)\Delta(a), \\
(\text{ev}_q \otimes \pi_y)\Delta(a)\overline{L}(xy^{-1}) = \overline{L}(xy^{-1})(\text{ev}_q \otimes \pi_y)\Delta(a) \quad \forall a \in U_q(\mathfrak{sl}_2).
\]

\(^7\) We emulate [20] and consider the general gradation of the algebra.

\(^8\) The operations to permute the free integer parameters \((m_0, m_1)\) in the evaluation map are originally independent of the automorphisms or anti-automorphisms of the algebras. But we synchronize these and purposely use the same notation.

A similar remark holds true for (3.4) and (3.5).
In this section, we consider intertwining relations of the triangular 3. The reflection equation and its solutions (2.27) and (2.28) and (2.29) swap under the map  

\[ R_{12} \left( \frac{y}{x} \right) K_1(x) R_{12} \left( \frac{x}{y} \right) K_2(y) = K_2(y) R_{12} \left( \frac{1}{xy} \right) K_1(x) R_{12} \left( \frac{x}{y} \right), \]  

(3.1)

Explicitly, they read

\[
\begin{align*}
\mathbf{L}(x) &= \left( \begin{array}{cccc}
q^{-x} - q^{-1}x^s q^{-x} & (q - q^{-1})x^s F q^{-x} & 0 & 0 \\
0 & 1 - x^s & (q - q^{-1})x^s & 0 \\
0 & 0 & 1 - x^s & 0 \\
0 & 0 & 0 & q - q^{-1}x^s
\end{array} \right), \tag{2.22}
\end{align*}
\]

\[
\begin{align*}
\mathbf{\overline{L}}(x) &= \left( \begin{array}{cccc}
q^{-x} - q^{-1}x^{-s} q^{-x} & (q - q^{-1})x^{-s} F q^{-x} & 0 & 0 \\
0 & 1 - x^{-s} & (q - q^{-1})x^{-s} & 0 \\
0 & 0 & 1 - x^{-s} & 0 \\
0 & 0 & 0 & q - q^{-1}x^{-s}
\end{array} \right). \tag{2.23}
\end{align*}
\]

One can check that these \( \mathbf{L} \)-operators satisfy the unitarity and the crossing unitarity conditions (cf equations (3.5) and (3.6) in [6]). Evaluating the first space of these \( \mathbf{L} \)-operators in the fundamental representation, we obtain \( \mathbf{R} \)-matrices of the 6-vertex model.

\[
\begin{align*}
\mathbf{R}(x) &= q^{\frac{1}{2}}(\pi \otimes 1)\mathbf{L}(x) = \left( \begin{array}{cccc}
q^{-x} & 0 & 0 & 0 \\
0 & 1 - x^s & (q - q^{-1})x^s & 0 \\
0 & 0 & 1 - x^s & 0 \\
0 & 0 & 0 & q - q^{-1}x^s
\end{array} \right), \tag{2.24}
\end{align*}
\]

\[
\begin{align*}
\mathbf{\overline{R}}(x) &= q^{\frac{1}{2}}(\pi \otimes 1)\mathbf{\overline{L}}(x) = \left( \begin{array}{cccc}
q^{-x^{-s}} & 0 & 0 & 0 \\
0 & 1 - x^{-s} & (q - q^{-1})x^{-s} & 0 \\
0 & 0 & 1 - x^{-s} & 0 \\
0 & 0 & 0 & q - q^{-1}x^{-s}
\end{array} \right). \tag{2.25}
\end{align*}
\]

These satisfy Yang–Baxter relations, which follow from (2.10):

\[
\begin{align*}
R_{12}(x/y)\overline{R}_{13}(x/z)R_{23}(y/z) &= R_{23}(y/z)\overline{R}_{13}(x/z)R_{12}(x/y), \quad x, y, z \in \mathbb{C}^*, \tag{2.26}
\end{align*}
\]

\[
\begin{align*}
\overline{R}_{12}(x/y)\overline{R}_{13}(x/z)\overline{R}_{23}(y/z) &= \overline{R}_{23}(y/z)\overline{R}_{13}(x/z)\overline{R}_{12}(x/y), \tag{2.27}
\end{align*}
\]

\[
\begin{align*}
L_{12}(x/y)L_{13}(x/z)R_{23}(y/z) &= R_{23}(y/z)L_{13}(x/z)L_{12}(x/y), \tag{2.28}
\end{align*}
\]

\[
\begin{align*}
\overline{L}_{12}(x/y)\overline{L}_{13}(x/z)\overline{R}_{23}(y/z) &= \overline{R}_{23}(y/z)\overline{L}_{13}(x/z)\overline{L}_{12}(x/y). \tag{2.29}
\end{align*}
\]

One can check \( (\sigma \otimes 1)\mathbf{L}(x) = \mathbf{L}(x) \), \( (\sigma \otimes 1)\overline{\mathbf{L}}(x) = \overline{\mathbf{L}}(x) \), \( (\sigma \otimes \sigma)\mathbf{R}(x) = \mathbf{R}(x) \), \( (\sigma \otimes 1)\mathbf{R}(x) = \mathbf{R}(x) \), \( (\sigma \otimes \sigma)\overline{\mathbf{R}}(x) = \overline{\mathbf{R}}(x) \), \( (\sigma \otimes 1)\overline{\mathbf{L}}(x) = \overline{\mathbf{L}}(x^{-1}) \), \( (\sigma \otimes 1)\overline{\mathbf{L}}(x) = \overline{\mathbf{L}}(x^{-1}) \), \( (\sigma \otimes \sigma)\overline{\mathbf{R}}(x) = \overline{\mathbf{R}}(x^{-1}) \). Thus (2.26)–(2.29) are invariant under the map \( \sigma \otimes 1 \otimes \sigma \); (2.26) and (2.27) swap and (2.28) and (2.29) swap under the map \( \iota \otimes \sigma \otimes \iota \).

3. The reflection equation and its solutions

In this section, we consider intertwining relations of the triangular \( q \)-Onsager algebra and obtain generic \( \mathbf{K} \)-operators in terms of the Borel subalgebras of \( U_q(sl_2) \). These \( \mathbf{K} \)-operators give solutions of the reflection equation associated with the \( \mathbf{L} \)-operators.

3.1. Reflection equation

We start from the following form of the reflection equation [1] for the \( R \)-matrices (2.24) and (2.25):

\[
R_{12} \left( \frac{y}{x} \right) K_1(x) R_{12} \left( \frac{x}{y} \right) K_2(y) = K_2(y) R_{12} \left( \frac{1}{xy} \right) K_1(x) R_{12} \left( \frac{x}{y} \right), \tag{3.1}
\]
where $x, y \in \mathbb{C}^\times$, $K_1(x) = K(x) \otimes 1$, $K_2(y) = 1 \otimes K(y)$. The most general solution of the reflection equation (3.1) is given by (see [3–5])

$$K(x) = \begin{pmatrix} x^0 \epsilon_+ + x^{-i_1} \epsilon_- & k_+ (x^t - x^{-t}) \frac{k_- (x^t - x^{-t})}{q - q^{-1}} \\ k_+ (x^t - x^{-t}) \frac{k_- (x^t - x^{-t})}{q - q^{-1}} & x^{-i_0} \epsilon_+ + x^t \epsilon_- \end{pmatrix}, \quad (3.2)$$

where $k_\pm$ and $\epsilon_\pm$ are scalar parameters. We assume $\epsilon_+ \epsilon_- \neq 0$ since we will deal with solutions which contain $\epsilon_+^{-1}$ or $\epsilon_-^{-1}$. We would like to consider the reflection equation for the $L$-operators (2.22) and (2.23):

$$L_{12} \left( \frac{x}{x} \right) K_1(x) L_{12} (xy) K_2(y) = K_2(y) L_{12} \left( \frac{1}{xy} \right) K_1(x) L_{12} \left( \frac{y}{y} \right), \quad (3.3)$$

and solve this with respect to the $K$-operator $K(x)$. The reflection equation (3.1) is the image of (3.3) for $\pi \otimes 1$. We set

$$\sigma(\epsilon_+) = \epsilon_-, \quad \sigma(\epsilon_-) = \epsilon_+, \quad \sigma(k_+) = k_- \quad \text{for} \quad k_+ \neq 0, \quad \sigma(k_-) = k_+ \quad \text{for} \quad k_- \neq 0; \quad (4.3)$$

$$\iota(\epsilon_+) = \epsilon_-, \quad \iota(\epsilon_-) = \epsilon_+, \quad \iota(k_+) = k_- \quad \text{for} \quad k_+ \neq 0, \quad \iota(k_-) = k_+ \quad \text{for} \quad k_- \neq 0. \quad (5.3)$$

The reflection equations (3.1) and (3.3) are invariant under the action of $\sigma \otimes \sigma$ and $\iota \otimes \iota$ if $k_+ k_- \neq 0$; while those for the upper triangular $K$-matrix ((3.2) with $k_- = 0$) and the lower triangular $K$-matrix ((3.2) with $k_+ = 0$) swap one another. Thus we can derive the generic lower triangular $K$-operator from the upper triangular one by $\sigma$ or $\iota$.

### 3.2. The triangular $q$-Onsager algebra

The triangular $q$-Onsager algebra is an underlying symmetry algebra of triangular solutions of the reflection equation.

**Definition 3.1.** The triangular $q$-Onsager algebra $O_q^t(\mathfrak{sl}_2)$ [10] is generated by the generators $^0T_0, T_1, \hat{P}_1$ obeying the following relations

$$[T_1, [T_1, \hat{P}_1]_{q^2}] = k_+ q(q + q^{-1})^2[T_0, T_1], \quad [T_0, [T_0, \hat{P}_1]_{q^2}] = k_+ q^{-1}(q + q^{-1})^2[T_0, T_1], \quad [T_1, T_0]_{q^{-2}} = \epsilon_+ \epsilon_-(1 - q^{-2}), \quad (3.6)$$

where $k_+, \epsilon_+, \epsilon_- \in \mathbb{C}$.

The algebra $O_q^t(\mathfrak{sl}_2)$ can be realized in terms of the generators of $U_q(\mathfrak{sl}_2)$ as follows:

$$T_0 = k_+ q_1 q_1^{h_1} + \epsilon_+ q_{h_1}, \quad T_1 = k_+ f_0 + \epsilon_- q_{h_0}, \quad (3.7)$$

$^9$We assume the central element $\Gamma = 1$.

$^{10}$The convention used in equation (2.12) in the original paper [10] is related to (3.7) by the automorphism (2.5) of $U_q(\mathfrak{sl}_2)$ and the replacement $k_i \to k_i$ and $q \to q^{-1}$ (under the condition $h_0 + h_1 = c = 0, \Gamma = 1$).
\[ \hat{\mathcal{P}}_1 = -(q^2 - q^{-2})(\epsilon_-, qf_1q^h_0 + \epsilon_+ e_0) + k_+q^{-1}([f_1, f_0]_{q^2} + [e_1, e_0]_{q^2}) + \hat{p}, \]

where \( \hat{p} \in \mathbb{C} \). From now on we identify \( O'_q(\hat{sl}_2) \) with the corresponding subalgebra of \( U_q(\hat{sl}_2) \) and note that it is a right coideal, i.e. \( \Delta(O'_q(\hat{sl}_2)) \subseteq O'_q(\hat{sl}_2) \otimes U_q(\hat{sl}_2) \) which follows from the following formulas for the generators:

\[ \Delta(T_0) = T_0 \otimes q^{h_0} + 1 \otimes (k_+q^{-1}q^{h_0}), \quad \Delta(T_1) = T_1 \otimes q^{h_0} + 1 \otimes (k_+f_0), \]
\[ \Delta(\hat{\mathcal{P}}_1) = \hat{\mathcal{P}}_1 \otimes 1 - (q^2 - q^{-2})(T_1 \otimes f_1q^{-1} + T_0 \otimes e_0) + 1 \otimes k_+q^{-1}([f_1, f_0]_{q^2} + [e_1, e_0]_{q^2}). \]

(3.8)

Equivalently, the subalgebra \( O'_q(\hat{sl}_2) \) is a left coideal with respect to the opposite co-product:

\[ \Delta'(O'_q(\hat{sl}_2)) \subseteq U_q(\hat{sl}_2) \otimes O'_q(\hat{sl}_2). \]

One can apply the evaluation map to (3.7) and obtain a homomorphism \( O'_q(\hat{sl}_2) \hookrightarrow U_q(\hat{sl}_2); \)

\[ \text{ev}_\epsilon(T_0) = k_+qx^{s_1}Eq^H + \epsilon_+q^H, \]
\[ \text{ev}_\epsilon(T_1) = k_+x^{-s_0}E + \epsilon_-q^{-H}, \]
\[ \text{ev}_\epsilon(\hat{\mathcal{P}}_1) = -(q^2 - q^{-2})F(\epsilon_-qx^{-s_1}q^{-H} + \epsilon_+x^{s_0}) + k_+ \left( -(q - q^{-1})(x^s + x^{-s})C + \frac{q + q^{-1}}{q - q^{-1}}(x^s_qq^{-H} + x^{-s}q^{-H}) \right) + \hat{p}. \]

We remark that the algebras generated by \( \{T_0' = \sigma(T_0), T_1' = \sigma(T_1), \hat{\mathcal{P}}_1' = \sigma(\hat{\mathcal{P}}_1)\} \) and \( \{T_1' = \sigma(T_0), T_0' = \sigma(T_1), \hat{\mathcal{P}}_1' = \sigma(\hat{\mathcal{P}}_1)\} \) (under the realization (3.7)) satisfy (3.6) with replacement \( k_+ \to k_- \) and are right co-ideals of \( U_q(\hat{sl}_2) \) with respect to the co-product (2.7).

### 3.3. Solutions of the intertwining relations

The intertwining relations (equivalent to (1.4)) associated with the triangular \( q \)-Onsager algebra are given by:

\[ \text{ev}_\epsilon(T_0) = K(x)\text{ev}_\epsilon(T_0), \quad \text{ev}_\epsilon(T_1) = K(x)\text{ev}_\epsilon(T_1), \quad \text{ev}_\epsilon(\hat{\mathcal{P}}_1) = K(x)\text{ev}_\epsilon(\hat{\mathcal{P}}_1), \]

for \( a \in \{T_0, T_1, \hat{\mathcal{P}}_1\} \). Explicitly, we have

\[ (k_+qx^{s_1}Eq^H + \epsilon_+q^H)K(x) = K(x)(k_+qx^{s_1}Eq^H + \epsilon_+q^H), \]
\[ (k_+x^{-s_0}E + \epsilon_-q^{-H})K(x) = K(x)(k_+x^{-s_0}E + \epsilon_-q^{-H}), \]

(3.11)
\( (3.12) \)

and

\[ \left( -(q^2 - q^{-2})F(\epsilon_-qx^{-s_1}q^{-H} + \epsilon_+x^{s_0}) + k_+ \frac{q + q^{-1}}{q - q^{-1}}(x^s_qq^{-H} + x^{-s}q^{-H}) \right)K(x) \]
\[ = K(x) \left( -(q^2 - q^{-2})F(\epsilon_-qx^{-s_1}q^{-H} + \epsilon_+x^{s_0}) + k_+ \frac{q + q^{-1}}{q - q^{-1}}(x^s_qq^{-H} + x^{-s}q^{-H}) \right). \]

(3.13)
In the third relation (3.13), we omit the trivial contribution from the terms on the central element and the scalar. We will prove the following theorem.

**Theorem 3.2.** The following $K$-operator solves the intertwining relations (3.10).\(^{11}\)

\[
K(x) = x^{n_H} \exp_{q^{-1}} \left( \alpha_+ E q^H \right) \begin{pmatrix}
\frac{t^{-1} x^q q^{-H-1}; q^{-2}}{t^{-1} x^q q^{-H-1}; q^{-2}} & \infty \\
\frac{t^{-1} x^q q^{-H-1}; q^{-2}}{t^{-1} x^q q^{-H-1}; q^{-2}} & \infty
\end{pmatrix} \exp_{q^{-1}} \left( \alpha_+ E q^H \right)
\]

\[
= x^{n_H} \begin{pmatrix}
\frac{q^{-1} x (k x^{-\infty} E + \epsilon_+ q^{-H})}{q^{-1} x (k x^{-\infty} E + \epsilon_+ q^{-H})} & \infty \\
\frac{q^{-1} x (k x^{-\infty} E + \epsilon_+ q^{-H})}{q^{-1} x (k x^{-\infty} E + \epsilon_+ q^{-H})} & \infty
\end{pmatrix}
\]

for $k = 0$, $|q| > 1$,

where we set $\alpha_+ = -\frac{q k x^{-\infty}}{q - q^{-1}}$, $x = q^u$, $x^{n_H} = q^{u n_H}$, $u \in \mathbb{C}$.

**Proof.** First we rewrite (3.11) and (3.12) in terms of $K(x) = x^{n_H} \tilde{K}(x)$ to get

\[
(k_+ q x^{-\infty} E + \epsilon_+ q^H) \tilde{K}(x) = \tilde{K}(x) (k_+ q x^H E + \epsilon_+ q^H),
\]

(3.15)

\[
(k_+ x^{-\infty} E + \epsilon_- q^{-H}) \tilde{K}(x) = \tilde{K}(x) (k_+ x^{-\infty} E + \epsilon_- q^{-H}).
\]

(3.16)

We find that $\theta_{\nu}(T_1)$ is transformed to a Cartan element of $U_q(sl_2)$ by the following similarity transformation (cf (A2) and (A3))\(^{12}\):

\[
\exp_{q^{-1}} \left( \alpha_+ E q^H \right) (k_+ q x^{-\infty} E + \epsilon_- q^{-H}) \exp_{q^{-1}} \left( \alpha_+ E q^H \right) = \epsilon_- q^{-H},
\]

(3.17)

where $\alpha_+ = -\frac{q k x^{-\infty}}{q - q^{-1}}$. Thus $\tilde{K}(x) = \exp_{q^{-1}} \left( \alpha_+ E q^H \right) K_0(x) \exp_{q^{-1}} \left( \alpha_+ E q^H \right)$ solves (3.16) if $K_0(x)$ commutes with any Cartan element of $U_q(sl_2)$. In this case, (3.15) boils down to

\[
E (\epsilon_- q^{1+H} + \epsilon_+ x^{-H}) K_0(x) = K_0(x) E (\epsilon_+ q^{1+H} + \epsilon_- x^H),
\]

(3.18)

which is equivalent to an intertwining relation for generic diagonal solutions of the reflection equation (equation (4.8) in [6]). Making use of a generic diagonal solution\(^{13}\)

\[
K_0(x) = \begin{pmatrix}
\frac{t^{-1} x^q q^{-H-1}; q^{-2}}{t^{-1} x^q q^{-H-1}; q^{-2}} & \infty \\
\frac{t^{-1} x^q q^{-H-1}; q^{-2}}{t^{-1} x^q q^{-H-1}; q^{-2}} & \infty
\end{pmatrix} \exp_{q^{-1}} \left( -\frac{\epsilon_-}{\epsilon_+ (q - q^{-1})} x^q q^{-H} \right) \exp_{q^{-1}} \left( -\frac{\epsilon_-}{\epsilon_+ (q - q^{-1})} x^q q^{-H} \right)
\]

for $|q| > 1$,

(3.19)

\(^{11}\)The formula for $|q| < 1$ can be obtained by replacement: $\exp_{q^{-1}}(x) = ((1 - q^{-2})x; q^{-2})_\infty \exp_{q^{-1}}(-x) = ((q^2 - 1)x; q^2)_\infty$. We also remark that (3.14) can be rewritten as $K(x) = R(x^{-1})^{-1} x^{n_H} R(x)$, where $R(x) = \exp_{q^{-1}} \left( -\frac{\epsilon_-}{\epsilon_+ (q - q^{-1})} x^q q^{-H} \right) \exp_{q^{-1}} \left( -\frac{q k x^{-\infty}}{q - q^{-1}} x^{n_H} E q^H \right)$.

\(^{12}\)A similar procedure was used in [9] for the rational case.

\(^{13}\)There is a freedom on an overall factor $f(q^2)$ for a fixed $\xi \in \mathbb{C}$ which commutes with the generator $E$. Here $f(y)$ is a series on $y \in \mathbb{C}$ whose coefficients are central elements of $U_q(sl_2)$ or complex numbers and satisfies $f(qy^2) = f(y)$. 

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which satisfies \((3.18)\) \((x^0 H K_0(x)\) corresponds to equation \((4.9)\) in \([6]\), we find that \((3.14)\) satisfies \((3.11)\) and \((3.12)\).

Next, we show that \((3.14)\) satisfies \((3.13)\). Making use of \((A2)\) and \((A4)\), and taking note on the fact that \(K_0(x)\) commutes with the Cartan element and the central element, we find that \((3.13)\) reduces to a combination of \((3.18)\)

\[
F(\epsilon_+ + \epsilon_- x^1 q^{1-H}) K_0(x) = K_0(x) F(\epsilon_+ + \epsilon_- x^{-1} q^{1-H}),
\]

which is equivalent to another intertwining relation for generic diagonal solutions of the reflection equation (equation \((4.7)\) in \([6]\); thus \((3.19)\) satisfies \((3.20)\).) The second equality of \((3.14)\) follows from the inverse of \((3.17)\). \(\square\)

The solution \((3.14)\) can be well defined at least for finite dimensional representations of \(U_q(sl_2)\) since there is a finite \(n\) such that \((E q^H)^n = 0\) and \(q^{-H}\) and \(x^0 H\) are finite size diagonal matrices (on appropriate base vectors). In particular for the fundamental representation, the solution \((3.14)\) reproduces the \(2 \times 2\) triangular matrix solution \((3.2)\) for \(k_0 = 0\) of the reflection equation \((3.1)\):

\[
\pi(K(x)) = \kappa(x) \begin{pmatrix} x^0 \epsilon_+ + x^{-1} \epsilon_- & \frac{k_+(x^1 - x^{-1})}{q - q^{-1}} \\ 0 & x^{-1} \epsilon_+ + x^1 \epsilon_- \end{pmatrix},
\]

where \(\kappa(x)\) is an overall factor defined by

\[
\kappa(x) = \left( \frac{\epsilon_+ x^1 q^{1-H}; q^{-2}}{\epsilon_+ x^{-1}; q^{-2}} \right)_\infty.
\]

### 3.4. Solutions of the reflection equation

We remark that the intertwining relations \(r_i((\text{ev}_i \otimes \text{ev}_j) \Delta'(a)) = ((\text{ev}_{i-1} \otimes \text{ev}_{j+1}) \Delta'(a)) r_i, i = 1, 2, \) for any \(a \in O_q'(s l_2)\) follow from \((2.20)\), \((2.21)\) and \((3.10)\), where the right hand side and the left hand side of \((3.3)\) are denoted \(^{14}\) as \(r_1\) and \(r_2\), respectively. This suggests the following theorem.

**Theorem 3.3.** The \(K\)-operator \((3.14)\) is a solution of the reflection equation \((3.3)\).

We have proven this by lengthy direct computation.

**Proof.** Expanding \((3.3)\) with respect to \(y\) and multiplying Cartan elements on both sides, we find four different types of relations on \(K(x)\), two of which are identical to \((3.11)\) and \((3.12)\), and the other two have the form:

\[
- (q - q^{-1})^2 x^{-s_1} (k_+ q^{-1} x^0 E + \epsilon_- q^{-1-H} + \epsilon_+ x') K(x) F q^{-H} + (q - q^{-1})^2 x^1 F q^{-H} K(x) \\
\times (k_+ q x^{-s_1} x^1 E + \epsilon_- q^{-1-H} + \epsilon_+ x') + k_+(x^1 - x^{-1} q^{-2}) (K(x) - q^{-H} K(x) q^{-H}) = 0
\]

and

\[
E q^H K(x)(\epsilon_+ x^0 q^{H+1} + \epsilon_- x^{-s_1}) = (\epsilon_+ x^{-s_1} q^{H+1} + \epsilon_- x^0) K(x) E q^H.
\]

\(^{14}\) Here \(K(x)\) in \(r_i\) is assumed to be the one in \((3.10)\).
We show that the assumption (3.14) satisfies (3.23) and (3.24) leads to trivial identities. First, we apply (3.12) for (3.23) and replace $FE$ and $EF$ with Cartan elements and the Casimir element based on (2.12). Then we take the term containing $F$ to the right side of $K(x)$ by (3.13), and derive an equation on $K_0(x)$ by (A2) and (A4). Taking note on the relation

$$EK_0(x) = K_0(x) \left( \frac{1 + \frac{\epsilon_+}{\epsilon_-} x^q q^{1-H}}{1 + \frac{\epsilon_+}{\epsilon_-} x'^q q^{1-H}} \right) E,$$

which follows from (3.19), we arrive at the following relation

\[
K_0(x) \left[ -\left( q^H - \lambda \alpha_+ + \frac{\epsilon_- + \epsilon_+ x^q q^{1-H}}{\epsilon_+ + \epsilon_- x'^q q^{1-H}} E q^{2H+1} \right) \left( \lambda \alpha_- q^{-1} x^{-r} - \lambda^2 \epsilon_+ x^0 \alpha_+ q^{H-1} \right) C \\
+ \lambda F \left( \epsilon_+ x^0 + \epsilon_- x^{-r} q^{1-H} \right) - \alpha_+ E \left( \lambda \epsilon_+ x^0 q^{1-H+1} - k_+ x^{-r} q^{2H-1} \right) \\
+ \alpha_+ E \left( \epsilon_+ x^0 q^{2H-2} + \epsilon_- x^{-r} q^{H-1} - k_+ \lambda^{-1} x^{-s} q^{H-2} \right) \\
+ \lambda q^{-1} F \left( \epsilon_+ x^0 + \epsilon_- x^{-r} q^{1-H} \right) C + \lambda q^{-2} F \left( \epsilon_+ x^0 + \epsilon_- x^{-r} q^{1-H} \right) \\
- \alpha_+ E \left( \lambda \epsilon_+ x^0 q^{1-H} - k_+ x'^q q^{2H-1} \right) + \epsilon_- x^{-r} q^{H-2} \right) = 0,
\]

\[
\lambda = q - q^{-1}.
\]

Expansion of the part $[\ldots]$ in (3.26) is lengthy and involved; but one can check that it is indeed 0 if one replaces $FE$ and $EF$ with Cartan elements and the Casimir element based on (2.12). By using (A2), one can show that (3.24) reduces to (3.18) (or one may use (3.25)).

The solution of the reflection equation (3.3) associated with another triangular solution (3.2) with $k_+ = 0$ follows from (3.14)

$$K(x) = \iota((3.14)) = x^{nH} \exp_{q^{-1}} (\alpha_- F) \left( \frac{x^{-1q^{H-1}}}{\epsilon_+} q^{-2} \right) \exp_{q^{-1}} (\alpha_- F)$$

\[
= x^{nH} \left( \frac{x^{-1q^{H-1}} (k_- q x^0 F q^{-H} + \epsilon_- q^{-H}) q^{-2}}{\epsilon_+} \right)_{\infty},
\]

for $k_+ = 0$, $|q| > 1,$

\[
(3.27)
\]

where $\alpha_- = q^{-1} x^{0q_0} (\alpha_+) = -\frac{k_- q^{0q_0}}{q^{-1} q^{H-1}}$. In particular for the fundamental representation, the solution (3.27) reproduces the $2 \times 2$ triangular matrix solution ((3.2) for $k_+ = 0$) of the reflection equation (3.1):

$$\pi(K(x)) \equiv \kappa(x) \left( \begin{array}{cc}
\frac{x^0 \epsilon_- + x^{-s} \epsilon_-}{k_- (x^r - s^{-r})} & 0 \\
0 & q^{-1} x^{0q_0} + x^{0q_0} \epsilon_-
\end{array} \right),$$

\[
(3.28)
\]

where $\kappa(x)$ is the overall factor defined by (3.22).
We obtain two more solutions of the reflection equation (3.3) with a different prefactor. The first one is

\[
 K(x) = \sigma((3.14)) = x^{-s_i H} \exp_{q^2}^{-1} \left( \beta_- F q^{-H} \right) \left( \frac{-\epsilon_i x^2 q^{H-1}; q^{-2}}{-\epsilon_i x^{-1} q^{H-1}; q^{-2}} \right)_{\infty} \exp_{q^2}^{-1} \left( \beta_- F q^{-H} \right) \\
 = x^{-s_i H} \left( \frac{-q^{-1} \epsilon_i (k_- x^{-s_i} F + \epsilon_+ q^H); q^{-2}}{-q^{-1} \epsilon_i (k_- x^{-s_i} F + \epsilon_+ q^H); q^{-2}} \right)_{\infty} 
\]

for \( k_+ = 0, \ |q| > 1 \),

(3.29)

where \( \beta_- = \sigma(\alpha_-) = -\frac{q k_+ - \epsilon_+}{q - q^{-1} \epsilon_+} \). The second one is

\[
 K(x) = \sigma(\sigma((3.14))) = \sigma((3.27)) \\
 = x^{-s_i H} \exp_{q^2}^{-1} \left( \beta_+ E \right) \left( \frac{-\epsilon_i x^2 q^{H-1}; q^{-2}}{-\epsilon_i x^{-1} q^{H-1}; q^{-2}} \right)_{\infty} \exp_{q^2}^{-1} \left( \beta_+ E \right) \\
 = x^{-s_i H} \left( \frac{-q^{-1} \epsilon_i (k_+ x^s F q^H + \epsilon_+ q^H); q^{-2}}{-q^{-1} \epsilon_i (k_+ x^s F q^H + \epsilon_+ q^H); q^{-2}} \right)_{\infty} 
\]

for \( k_- = 0, \ |q| > 1 \),

(3.30)

where \( \beta_+ = \sigma(\alpha_-) = -\frac{k_+ \epsilon_+}{q - q^{-1} \epsilon_+} \).

We remark that our generic triangular solutions (3.14), (3.27), (3.29) and (3.30) reduce to the generic diagonal solutions [6] at \( k_+ = k_- = 0 \).

4. Discussion

In this paper, we solved the intertwining relations of the triangular \( q \)-Onsager algebra under the evaluation map, and obtained generic triangular solutions of the reflection equation. A natural problem in this direction will be to generalize our solutions to the most general case \( k_+ k_- \neq 0 \), which is related to the \( q \)-Onsager algebra [13, 14]. The \( q \)-Onsager algebra \( O_q(sU(2)) \) (with the central element \( \Gamma = 1 \)) is generated by two elements \( W_0, W_1 \) and unit obeying the following relations.

\[
 \left[ W_0, \left[ W_0, W_1 \right]_{q^{-2}} \right]_{q^2} = (q + q^{-1})^2 k_+ k_- \left[ W_0, W_1 \right],
\]

\[
 \left[ W_1, \left[ W_1, W_0 \right]_{q^{-2}} \right]_{q^2} = (q + q^{-1})^2 k_+ k_- \left[ W_1, W_0 \right].
\]

(4.1)
This algebra is realized\(^{15}\) in terms of the generators of \(U_q(sl_2)\):

\[
W_0 = k_+ q e_1 q^{h_1} + k_- f_1 + \epsilon_+ q^{h_1},
\]

\[
W_1 = k_- q e_1 q^{h_0} + k_+ f_0 + \epsilon_- q^{h_0}.
\]

(4.2)

Note that two of the generators (3.7) of the triangular \(q\)-Onsager algebra are specializations of these: \(T_0 = W_0|_{k_0 = 0}\), \(T_1 = W_1|_{k_1 = 0}\). In order to obtain generic non-diagonal solutions of the reflection equation (3.3) for \(k_+ k_- \neq 0\), we must solve the intertwining relations

\[
ev_{\pm 1}(W_0|_{k_0 = 0})K(x) = K(x)ev_{\pm 1}(W_0),
\]

(4.3)

\[
ev_{\pm 1}(W_1|_{k_1 = 0})K(x) = K(x)ev_{\pm 1}(W_1).
\]

(4.4)

Our generic triangular solutions are deformation of the generic diagonal solution \((k_+ = k_- = 0\) case)\(^{16}\) [6]. In view of this, it would be tempting to see if the following substitution works.

\[
K(x) = \lambda^{i_{ij} H} \left( \frac{2^{-1} q x^{E + k_- q x^{\gamma E} - h} + k_+ q x^{\gamma E} - h}{-2^{-1} q x^{E + k_- q x^{\gamma E} - h} + k_+ q x^{\gamma E} - h}; q^{-2} \right)^\infty
\]

\[
= \lambda^{i_{ij} H} \left( \frac{2^{-1} q x^{E + k_- q x^{\gamma E} - h} + k_+ q x^{\gamma E} - h}{-2^{-1} q x^{E + k_- q x^{\gamma E} - h} + k_+ q x^{\gamma E} - h}; q^{-2} \right)^\infty
\]

(4.5)

In fact, this reduces to (3.14) at \(k_- = 0\) and (3.27) at \(k_+ = 0\). However, (4.5) seems not satisfy (4.3), although it solves (4.4). How this could be resolved remains to be clarified. We expect that recent solutions of the reflection equation for the symmetric tensor representations of \(U_q(A_{n-1}^{(1)})\) [21]\(^{17}\) (see also, [22]) and the rational case \(Y(sl_2)\) [9] are great clues for this.

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Appendix. \(q\)-analogue of Hadamard formula

In this section, we review useful formulas derived from the \(q\)-deformed Hadamard formula [15] for elements \(A\) and \(B\) in \(U_q(sl_2)\):

\[
\exp_q(B)\exp_q^{-1}(A) = \sum_{k=0}^\infty \frac{B_k}{(k)_q!}.
\]

(A1)

\(^{15}\)The convention used in equation (2.7) in the paper [10] is related to (4.2) by the automorphism (2.5) of \(U_q(sl_2)\) and the replacement \(k_0 \to k_1\) and \(q \to q^{-1}\) (under the condition \(k_0 + k_1 = c = 0, \Gamma = 1\)).

\(^{16}\)The generic diagonal solution corresponds to \(x^{i_{ij} H}K_0(x)\), where \(K_0(x)\) is defined in (3.19).

\(^{17}\)See also [23], in which all invertible solutions of the reflection equation are classified for the vector representation, including all triangular ones.
where $B_0 = B, B_{k+1} = [A, B_k]_q$. For $a, b, c \in \mathbb{C}$, one can prove the following relations (as formal series) based on induction and the property of the $q$-exponential function $\exp_q^{-1}(x) = \exp_q^{-1}(-x)$.

\[
\exp_q^{-2} (aE_q^b) \exp^{-1} (aE_q^b) = \sum_{j=0}^{\infty} \frac{(q^2; q^2)_{j}}{(q^2; q^2)} (a(1-q^{-2})E_q^{b+1})/q^j, \\
(A2)
\]

\[
\exp_q^{-2} (aE_q^b) q^H \exp^{-1} (aE_q^b) = \sum_{j=0}^{\infty} \frac{(q^2; q^2)_{j}}{(q^2; q^2)} (a(1-q^{-2})E_q^{b+1})/q^j, \\
(A3)
\]

\[
\exp_q^{-2} (aE_q^b) F_q^H \exp^{-1} (aE_q^b) = F_q^H + a(1-q^{-2})q^{-2b} \sum_{j=1}^{\infty} \frac{(a(1-q^{-2})E_q^{b+1})^{-1}}{(q^2; q^2)} \left\{ (q^2, q^2)_{j} C_{q}^{b+c+1} H^{-1} \right\} \\
(A4)
\]

\[
\exp_q^{-1} (aE_q^b) q^H \exp_q^{-2} (aE_q^b) = \sum_{j=0}^{\infty} \frac{(q^2; q^2)_{j}}{(q^2; q^2)} (-a(1-q^{-2})E_q^{b+1})/q^j, \\
(A5)
\]

\[
\exp_q^{-1} (aE_q^b) E_q^H \exp_q^{-2} (aE_q^b) = \sum_{j=0}^{\infty} \frac{(q^2; q^2)_{j}}{(q^2; q^2)} (-a(1-q^{-2})E_q^{b+1})/q^j, \\
(A6)
\]

\[
\exp_q^{-1} (aE_q^b) F_q^H \exp_q^{-2} (aE_q^b) = F_q^H - a(1-q^{-2})q^{-2b} \sum_{j=1}^{\infty} \frac{(-a(1-q^{-2})E_q^{b+1})^{-1}}{(q^2; q^2)} \left\{ (q^2, q^2)_{j} C_{q}^{b+c+1} H^{-1} \right\} \\
(A7)
\]

\[
\exp_q^{-2} (aF_q^b) q^H \exp_q^{-1} (aF_q^b) = \sum_{j=0}^{\infty} \frac{(q^2; q^2)_{j}}{(q^2; q^2)} (a(1-q^{-2})F_q^{b+1})/q^j, \\
(A8)
\]
\[
\exp_{q^{-2}} (aFq^{bH}) Fq^{dH} \exp_{q^{-1}}^{-1} (aFq^{bH}) = \sum_{j=0}^{\infty} \left( \frac{q^{2(b+j)}; q^{-2}}{(q^{-2}; q^{-2})} \right) (a(1 - q^{-2})Fq^{bH})/Fq^{dH},
\]

\[
\exp_{q^{-2}} (aFq^{bH}) Eq^{dH} Fq^{dH} = Eq^{dH} + a(1 - q^{-2})q^{bH} \sum_{j=1}^{\infty} \left( \frac{a(1 - q^{-2})Fq^{bH}}{(q^{-2}; q^{-2})} \right) \left\{ \left( q^{-2(b+c+1)}; q^{-2} \right) Cq^{(b+c+1)H} \right. \\
- \left. \left( q^{-2(b+c+1)}; q^{-2} \right) (q^{-2}Cq^{(b+c+1)H-1} + (q^{-2}Cq^{(b+c+1)H+1}) \right\} \right. \\
\exp_{q^{-2}}^{-1} (aFq^{bH}) q^{dH} \exp_{q^{-2}} (aFq^{bH}) = \sum_{j=0}^{\infty} \left( \frac{q^{2c}; q^{2j}}{(q^{2}; q^{2})} \right) (-a(1 - q^{2})Fq^{bH})/q^{dH},
\]

\[
\exp_{q^{-2}}^{-1} (aFq^{bH}) Fq^{dH} \exp_{q^{-2}} (aFq^{bH}) = \sum_{j=0}^{\infty} \left( \frac{q^{2b-2c}; q^{2j}}{(q^{2}; q^{2})} \right) (-a(1 - q^{2})Fq^{bH})/Fq^{dH},
\]

\[
\exp_{q^{-2}}^{-1} (aFq^{bH}) Eq^{dH} \exp_{q^{-2}} (aFq^{bH}) = Eq^{dH} - a(1 - q^{2})q^{bH} \sum_{j=1}^{\infty} \left( \frac{-a(1 - q^{2})Fq^{bH}}{(q^{2}; q^{2})} \right) \left\{ \left( q^{-2(b+c+1)}; q^{2} \right) Cq^{(b+c)H} \right. \\
- \left. \left( q^{-2(b+c+1)}; q^{2} \right) (q^{2}Cq^{(b+c+1)H-1} + (q^{2}Cq^{(b+c+1)H+1}) \right\} \right. \\
\]

The relation \(q^{k}; q_{k} = (-1)^{k}q^{k}+\frac{ik-1}{2}(q^{-n}; q^{-1})_{k}\) is useful to modify the above relations.

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