On direct product of algebraic sets over groups II

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Abstract. We consider systems of group equations of the structure $S = S_1(X) \cup S_2(Y)$, where the set of variables $X, Y$ are disjoint. Suppose we know the radicals of the systems $S_i$. However we prove that the radical of the whole system $\text{Rad}(S)$ may contain equations which are not derived from equations from $\text{Rad}(S_i)$.

1. Introduction
Consider a group $G$. We say that a set $Y \subseteq G^n$ is algebraic over $G$ if it can be expressed as the solution set of a certain system of equations over $G$. One easily prove that the direct product of algebraic sets is algebraic again. In other words, the set $Y_1 \times Y_2$ is defined by

$$\{u_{1j}(x_1, x_2, \ldots, x_{n_1}) = 1 \mid j \in J_1\} \cup \{u_{2j}(y_1, y_2, \ldots, y_{n_2}) = 1 \mid j \in J_2\}$$

for $S_i = \{u_{ij}(x_1, x_2, \ldots, x_{n_i}) = 1 \mid j \in J_i\}$.

The direct product of algebraic sets $Y_1, Y_2$ inherits many properties of the sets $Y_i$. For example, if both $Y_i$ are irreducible so is $Y = Y_1 \times Y_2$ ([2]). On the other hand, there exist properties of $Y$ which do not follow from the corresponding properties of the sets $Y_i$.

Also we may consider the following problem: what properties $Y_1 \times Y_2$ are derived from the sets $Y_i$. In other words, can we describe the radical $\text{Rad}(Y_1 \times Y_2)$ using the sets of equations $\text{Rad}(Y_1), \text{Rad}(Y_2)$ (remind that $\text{Rad}(Y)$ is the maximal system with the solution $Y$)?

The current paper continues the studies of [1]. We remind the reader the formal statement of the central problem (Problems 1,2 below) and prove that it has a negative solution for free non-abelian groups (Theorem 3.3).

2. Main definitions
Following [2], we give the main definitions of algebraic geometry over groups.

We consider any group $G$ below as an algebraic structure of the language $L = \{\cdot, -1, 1\} \cup \{g \mid g \in G\}$. It is easy to see that every $L$-term with variables $X = \{x_1, x_2, \ldots, x_n\}$ actually belongs to the free product $G * F(X)$. An equation is said to be an equality $w(X) = 1$, for an $L$-term $w(X)$. The solution set of a system of equations (system, for shortness) $S$ in a group $G$ is denoted by $V(S) \subseteq G^n$. A pair of equations $w(X) = 1$, $u(X) = 1$ is called equivalent over $G$ if $V(w(X) = 1) = V(u(X) = 1)$. This equivalence relation is denoted by $w(X) \sim u(X)$.

The radical $\text{Rad}(S)$ over a group $G$ is the following set

$$\{w(X) = 1 \mid V(S) \subseteq V(w(X) = 1)\}.$$
Remark 2.1. We know that any group equation has 1 in the right part. Hence we write below $w(X) \in S$ or $w(X) \in \text{Rad}(S)$ for shortness.

Let $[S]$ be the normal closure of a system $S$ (see the definition in [2]). One can directly prove that, $[S] \subseteq \text{Rad}(S)$.

A system $S$ may depend on variables $X = \{x_1, x_2, \ldots, x_n\}$, and it does not depend on a variable set $Y$. We have $Z = X \cup Y$ and use the denotation

$$V^Z(S) = V(S) \times G^m,$$

$$\text{Rad}^Z(S) = \{ w(Z) = 1 \mid V^Z(S) \subseteq V^Z(w(Z) = 1) \}.$$

Roughly speaking, $V^Z(S)$ is all solutions of $S \cup \{ y_1 y_i^{-1} = 1 \mid 1 \leq i \leq m \}$.

The central problems of our paper is the following:

**Problem 1.** Fix a group $G$ and disjoint set of variables $X, Y$. Is the following equality

$$\text{Rad}^Z(S_1(X) \cup S_2(Y)) = \{ [\text{Rad}^Z(S_1(X)), \text{Rad}^Z(S_2(Y))] \}$$

true for arbitrary systems $S_1, S_2$ and $Z = X \cup Y$?

**Problem 2.** Under all denotation from Problem 1 is the following equality

$$nf(\text{Rad}^Z(S_1(X) \cup S_2(Y))) = nf([\text{Rad}^Z(S_1(X)), \text{Rad}^Z(S_2(Y))])$$

true for every systems $S_1(X), S_2(Y)$?

Obviously, the negative solution of Problem 2 implies the negative solution of Problem 1.

Let us take a free group $G$. According to the properties of free product, one can give the natural definition of a normal form in $G * F(X)$.

3. Free groups

Below in this section $G$ is a non-abelian free group and $a, b$ be its free generators.

**Lemma 3.1.** An equation

$$w(x) = 1, \ w(x) \in G * F(x)$$

is satisfied by all points $g \in G$ iff the word $w(x)$ equals 1 in the group $G * F(x)$.

**Proof.** Assume that $w$ is the product

$$u_1 x^{n_1} u_2 x^{n_2} \ldots u_k x^{n_k}, \ u_i \in G \setminus \{1\}, \ n_i \in \mathbb{Z} \setminus \{0\}$$

(the proof for other types of the word $w$ is similar).

One can choose a sufficiently large $n$ such that the reduced form of $a^n u_i a^{-n}$ starts with $a$ and finishes with $a^{-1}$ for any $u_i$ which is not a power of $a$. Let $v_i$ be the reduced form of $a^n u_i a^{-n}$. Obviously, each $v_i$ stats and finishes with $a^\pm 1$. Then the element

$$q = v_1 b^{n_1} v_2 b^{n_2} \ldots v_k b^{n_k}$$

is not 1 in $G$. We have

$$a^{-n} q a^n = a^{-n} (a^n u_1 a^{-n} b^{n_1} a^n u_2 a^{-n} b^{n_2} \ldots a^n u_k a^{-n} b^{n_k}) a^n = a^n u_k a^{-n} b^{n_k} a^n a^n = w(a^{-n} b a^n) \neq 1$$

that contradicts the condition.
Let $w(x, y) \in G * F(x, y)$. One can write $w(x, y)$ as a product of letters

$$w(x, y) = z_1 z_2 \ldots z_n, \; z_i \in V \cup G,$$

where either $z_i \in \{x^{\pm 1}, y^{\pm 1}\}$ or $z_i$ is a free generator of $G$.

**Lemma 3.2.** Suppose

$$w(x, y) \in \text{Rad}^{(x,y)}([x, a] = 1)$$

is of the form (3). Then there exists a set of pairs $M \subseteq \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ such that

(i) $(i, j) \in M$ if $(j, i) \in M$;

(ii) if $(i, j) \in M$ then $z_i = z_j^{-1}$ and $z_i, z_j \in \{x^{\pm 1}, y^{\pm 1}\}$;

(iii) for any $z_i \in \{x^{\pm 1}, y^{\pm 1}\}$ there exists $z_j = z_i^{-1}$ such that $(i, j) \in M$

(iv) if $(i, k) \in M$, $z_j \in \{x^{\pm 1}, y^{\pm 1}\}$ and $i < j < k$ then there exists $l$ with $(j, l) \in M$ and $i < l < k$.

**Proof.** The last condition of the set $M$ implies the absence of the following overlappings (an edge between indexes $\alpha, \beta$ means $(\alpha, \beta) \in M$):

![Forbidden overlappings in $M$.](image)

**Sketch of the proof.** By the condition, $w(x, y) = 1$ should satisfy any point $(a^m, g)$, $m \in \mathbb{Z}$, $g \in G$. Let $m = 2 \# a$, where $\# a$ is the total number of occurrences of the letters $a^{\pm 1}$ in $w(x, y)$. Then the letters $a$ from $w(x, y)$ cannot completely cancel any expression $x := a^m$. Thus, for any expression $x := a^m$ in $w(x, y)$ there exists a letter $x^{-1}$ which cancels almost all letters $a$ from $x$. If such $x$ and $x^{-1}$ are $i$-th and $j$-th letters in $w(x, y)$ we put $(i, j) \in M$.

Since the word $w(a^m, y)$ is reduced to 1 for any $y$, then for any letter $z_i = y^x$ in $w(x, y)$ there exists $y^{-x}$ which cancels $y$ in $w(a^m, y)$ (Lemma 3.1). Thus, it allows to define pairs $(i, j) \in M$ for the letters $y$.

Finally, $M$ contains the information about the cancellative pairs of the word $w(x, y)$.

**Formal proof.** Let $m = 2 \# a$, where $\# a$ is the total number of occurrences of the letters $a^{\pm 1}$ in $w(x, y)$. By the condition, the word $w(a^m, g)$ is reduced to 1 in $G$.

Let us introduce new letters $\alpha_m$ indexed by integers $m \in \mathbb{Z}$. We change $z_i$ to $\alpha_m$ by the following rule:

$$z_i = \begin{cases} 
\alpha_m & \text{if } z_i = x, \\
\alpha_m^{-1} & \text{if } z_i = x^{-1}
\end{cases}$$

Thus, the word $w(\alpha_m, \alpha_m^{-1}, y)$ is reduced to 1 if put $\alpha_m = a^m$, $\alpha_m^{-1} = a^{-m}$. Let us fix the pairs of cancellative letters and describe the cancellation process by the following rules.

(i) during the process, a letter $\alpha_m$ ($\alpha_m^{-1}$) may be changed to a new letter $\alpha_k$ ($k \in \mathbb{Z}$);

(ii) if a constant $a^\varepsilon$ ($\varepsilon \in \{-1, 1\}$) cancels with $a^{-\varepsilon}$ from $\alpha_k$ ($k \in \mathbb{Z}$) we denote the result of this cancellation by $\alpha_{k-\varepsilon}$;

(iii) if there cancel constant letters $a, a^{-1}$ from $\alpha_k$ and $\alpha_l$ ($k, l \in \mathbb{Z}$) the cancellation result equals to the following product of constants:

$$\alpha_k \alpha_l = \begin{cases} 
qq & \text{if } k - l \geq 0, \\
k - l \text{ times} & a^{-1}a^{-1} \ldots a^{-1} & \text{if } k - l < 0 \\
k - l \text{ times}
\end{cases}$$
Let $z_i$ ($z_j$) be the letter of $w(x, y)$ which was initially substituted to $\alpha_k$ (respectively, $\alpha_l$). Then we add a pair $(i, j)$ to $M$.

Since $m$ is large enough, the third rule above appears for each $\alpha_k$. Therefore, for any $i$ with $z_i = x^\varepsilon$ there exists a unique $j$ such that $z_j = y^{-\varepsilon}$ and $(i, j) \in M$.

Since for $x := a^m$ the word $w(x, y)$ is reduced to 1 for any $y \in G$, then for any letter $z_i = y^\varepsilon$ in $w(x, y)$ there exists $z_j = y^{-\varepsilon}$ which cancels with $y$ in $w(a^m, y)$ (Lemma 3.1). We add the pair $(i, j)$ to the set $M$.

The last property of the set $M$ obviously holds, since any set of cancellation pairs clearly has this property.

Applying to $M$ the symmetric closure, we obtain a set of pairs which satisfies the first property in the lemma statement.

**Theorem 3.3.** Let $G$ be a non-abelian free group and $a \in G$ be a free generator. For 

$$S_1 = \{[x, a] = 1\}, \quad S_2 = \{[y, a] = 1\}$$

the equality (2) fails.

**Proof.** Since the centralizer of any nontrivial element in $G$ is cyclic, the radical of the system $S = S_1 \cup S_2$ contains $[x, y]$.

Assume that 

$$nf([x, y]) = [x, y] \in nf([Rad(S_1), Rad(S_2)])$$

In other words, there exist 

$$w_i(x, y) \in Rad^Z(S_1) \cup Rad^Z(S_2)$$

such that the product 

$$w_1(x, y)w_2(x, y) \ldots w_k(x, y)$$

equals $[x, y]$ in $G \ast F(x, y)$.

Any word $w_i$ is a product of letters 

$$w_i = z_jz_{j+1} \ldots z_{j+|w_i|}$$

such that $j = \sum_{\alpha=1}^{i-1} |w_\alpha| + 1$. According to Lemma 3.2, there exist sets $M_1, M_2, \ldots, M_k$ of pairs. Denote 

$$M = \bigcup_{i=1}^{k} M_i.$$ 

Reducing the product (4) in $G \ast F(x, y)$ to $[x, y]$, we fix some order of cancellations in (4).

Suppose we have a cancellation of letters $z_i, z_j \in \{x^{\pm 1}, y^{\pm 1}\}$ from different words $w_i, w_j$. Hence, there exists pairs 

$$(i_0, i), (j, j_0) \in M.$$ 

We put 

$$M := (M \setminus \{(i_0, i), (j, j_0)\}) \cup \{i_0, j_0\}.$$ 

The transformation (5) is shown at the following figure.
One can directly prove that $M$ satisfies all properties from Lemma 3.2 after every transformation (5). Hence, the normal form of (4) should have the set $M$ which satisfies Lemma 3.2. However, the word $[x, y]$ does not admit any set $M$ with properties from Lemma 3.2. Hence, $[x, y]$ can not be obtained by multiplications and reductions of words from $[\text{Rad}^Z(S_1) \cup \text{Rad}^Z(S_2)]$.

Thus, $[x, y]$ is not a normal form of any equation from $[\text{Rad}^Z(S_1) \cup \text{Rad}^Z(S_2)]$.

References
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[2] Daniyarova E, Myasnikov A and Remeslennikov V 2017 Algebraic geometry over algebraic structures *Novosibirsk: SO RAN*