Application of three body stability to globular clusters I: the tidal radius

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ABSTRACT

The tidal radius is commonly analytically determined by equating the tidal field of the galaxy to the gravitational potential of the cluster. This gives a radius at which stars can move from orbiting the cluster centre to independently orbiting the galaxy. In this paper the tidal radius of a globular cluster is estimated using a novel approach from the theoretical standpoint of the general three-body problem. This is achieved by an analytical formula for the transition radius between stable and unstable orbits in a globular cluster.

A stability analysis, outlined by Mardling (2008), is used here to predict the occurrence of unstable stellar orbits in the outermost region of a globular cluster in a distant orbit around a galaxy. It is found that the eccentricity of the cluster-galaxy orbit has a far more significant effect on the tidal radius of globular clusters than previous theoretical results have found. A simple analytical formula is given for determining the transition between stable and unstable orbits, which is analogous to the tidal radius for a globular cluster. This tidal radius estimate is interior to previous estimates and gives the innermost region from which stars can random walk to their eventual escape from the cluster.

Key words: gravitation, stellar dynamics, methods: analytical, stars: kinematics, globular clusters: general

1 INTRODUCTION

The tidal radius of a globular cluster (GC) is defined as the point at which stars will escape the cluster’s potential well and become part of the galactic halo. It is typically calculated by considering the equipotential point between the cluster and the tidal field of the galaxy (see King [1962]; Read et al. [2006]). The aim of this paper is to determine the boundary between tidally stable and unstable orbits in a cluster potential, where an unstable orbit refers to a star orbiting inside the globular cluster that will eventually escape the cluster. This stability boundary is analogous to the tidal radius of a globular cluster and is predicted using the stability analysis of Mardling (2008) for GCs on eccentric galactic orbits and with arbitrary mass ratios.

For the purposes of applying this stability analysis, the star-cluster-galaxy system is approximated as follows. A star of mass $m_i$ orbits a particle representing the total cluster mass $M_C$, which itself orbits the galaxy, taken as a particle of mass $M_G$. Each of these particles is treated as a point mass so that we can treat the system as the general three-body problem.

Simplifying the star-cluster-galaxy system to a single three-body problem ignores the effect of mutual interactions between stars in the cluster, which results in two-body relaxation being ignored. This means that in a real cluster stars will be able to diffuse over the predicted tidal radius and escape the cluster from orbits that were initially in stable regions. However the very long relaxation timescale in these regions means that this effect is negligible.

By approximating the cluster potential as a point mass in the stability analysis we are assuming that stars spend most of their time in the outermost regions of the cluster. Similarly by treating the galaxy as a point mass the discussion is limited to GCs, which spend most of their time at distances greater than approximately 5 kpc. The assumption of a point mass potential for the galaxy for distant cluster orbits is also justified since the mass inside 6.4 kpc is consistent with that of a point mass of roughly $10^{11} M_\odot$, based on the reasonably well known orbits of the globular clusters NGC 2419 and NGC 7006 (Bellazzini [2004]).

This paper is structured in the following way. A brief
overview of the Mardling stability criterion is presented in Section 2. This stability analysis, applied to the three-body system corresponding to a point mass approximation for the star, cluster and galaxy systems, is presented in Section 3. A model cluster is simulated in isolation to get the distribution of star-cluster orbital eccentricities in Section 4. Section 5 applies the stability boundary to calculate the tidal radius for a case study globular cluster of mass \( M_c = 10^5 M_{\odot} \), which is compared to previous theoretical work from the literature in Section 6. Conclusions and a discussion of the observational and numerical simulation implications of this work are summarised in Section 7.

2 MARDLING STABILITY CRITERION

We first summarise the Mardling stability criterion (MSC) and outline the algorithm used to determine the stability of any three-body system. The criterion for the MSC is given in Mardling (2008), and interested readers are referred to this study for details which have been omitted here. The criterion below covers the more general case where the inner and outer orbits are relatively inclined by an angle \( \gamma \) (Mardling 2011). The inner orbit refers to the orbit of the binary composed of point masses \( m_1 \) and \( m_2 \), while the outer orbit refers to the orbit of \( m_3 \) with the centre of mass associated with the combined mass \( m_{12} = m_1 + m_2 \).

We will use the common terminology of a system being in resonance if the ratio of the outer to inner periods (\( \sigma = T_o/T_i \)) is within a distance \( \Delta \sigma \) of a particular \( \{m:n\} \) resonance. When a system has initial conditions such that it resides in two resonances simultaneously then it is possible for the resonance angles to librate inside either resonance at any time. This jumping between resonances can occur at any time and means that two three-body systems with initially very close conditions will quickly diverge once one jumps to a different resonance, leaving the other behind. This sensitivity to changes in the initial conditions is characteristic of chaotic motion and is used by the MSC to predict unstable systems.

For a range of orbits, as exists for stars in globular clusters, the inclination between the inner and outer binary orbits is not restricted to the coplanar case. The method given in Mardling (2008) for calculating the resonance widths does not include the effect of the relative inclination \( I \). The effect of inclination on the resonance width is under development (Mardling 2011) and we present here a summary of the inclination factors relevant to this paper. In addition to the inclination, the phase of the orbit also changes over time due to the non-Keplerian motion in the centre (see equation 23 for the GC potential model). For simplicity the maximum possible resonance width is adopted which is equivalent to taking the resonance angle as zero.

From the formulation given in Mardling (2008) and including the inclination terms from Mardling (2011) the resonance width is given by

\[
\Delta \sigma_{mn} = 2 \sqrt{A_{mn}}
\]

where the leading quadrupole term in the spherical expansion of the disturbing function one can write

\[
A_{mn} = \max(A_{2n12}, A_{2n10}, A_{2n1-2}, A_{0n10})
\]

since \( m = -l, 0, l \) and

\[
A_{mn} = -6 \epsilon_{2m}^{(2m')} \gamma_{2m'}(e_i) F_{n}^{(2m)}(e_o) \gamma_{2mn'}(I)
\]

\[
\times \frac{m_1^{3/2} m_{12}^{3/2}}{m_0^{3/2} m_{123}^{3/2}} \frac{\left( m_{012} - m_{01} m_{123} \right)}{m_{123}^{3/2}}
\]

with \( \epsilon_{2m} = 3/8 \) and \( \epsilon_{2m'} = 1/4 \). The approximate dependence on the inner eccentricity is given by the functions

\[
\begin{align*}
\epsilon_{2}^{(20)}(e_i) & \approx \frac{e_i}{9216} (-9216 + 1152 e_i^2 - 48 e_i^4 + e_i^6) \\
\epsilon_{2}^{(22)}(e_i) & \approx -\frac{e_i^3}{15360} (4800 + 1880 e_i^2 + 1091 e_i^4) \\
\epsilon_{2}^{(22)}(e_i) & \approx -3 e_i + \frac{13}{8} e_i^3 + \frac{5}{192} e_i^5 - \frac{227}{3072} e_i^7
\end{align*}
\]

which is valid for \( 0 \leq e_i \leq 1 \). The errors between this approximate expression and the exact integral expression are less than 1% for \( e_i < 0.8 \) and < 0.1% for \( e_i < 0.63 \) (Mardling 2008). The dependence of equation 44 on the outer eccentricity is approximated by the asymptotic expression

\[
F_{n}^{(22)}(e_o) \approx \frac{8 \pi \sqrt{2} \pi}{3} \left( 1 - e_o^{3/4} e^{-n \xi(e_o)} \right)
\]

\[
F_{n}^{(20)}(e_o) \approx \frac{1}{\sqrt{2} \pi} \left( 1 - e_o^{3/4} e^{-n \xi(e_o)} \right)
\]

where

\[
\xi(e_o) = \left( \text{Cosh}^{-1} \left( \frac{1}{e_o} \right) - \sqrt{1 - e_o^2} \right)
\]

The relevant inclination factors are

\[
\begin{align*}
\gamma_{222}(I) & = \frac{1}{4} \left( 1 + \cos(I) \right)^2 \\
\gamma_{220}(I) & = \sqrt{\frac{3}{8}} \sin(I) \\
\gamma_{22-2}(I) & = \frac{1}{4} \left( 1 - \cos(I) \right)^2 \\
\gamma_{200}(I) & = \frac{1}{2} \left( 3 \cos^2(I) - 1 \right)
\end{align*}
\]

As a system evolves on a secular timescale the resonance widths vary and are at their maximum when \( e_i = \max \). Therefore the inner eccentricity in equation 44 needs to be modified for induced eccentricity in the inner orbit due to the outer orbit and the secular octopole term. The maximum eccentricity that can be dynamically induced in the inner eccentricity by the outer orbit following one passage is given by

\[
i_{i}^{\text{ind}} = \left[ e_i(0)^2 - 2 \beta_s e_i(0) \sin(\sigma_{201}) + \beta_s^2 \right]^{1/2}
\]

where \( e_i(0) \) denotes the initial inner eccentricity, and \( \beta_s = \frac{9 \pi}{2 n} \left( \frac{m_3}{m_{123}} \right) \left( 1 - e_o \right)^3 F_{n}^{(22)}(e_o) \)

where \( F_{n}^{(22)}(e_o) \) is given by equation 45. The eccentricity correction due to the secular octopole term, which is non-zero if \( m_1 \neq m_2 \), is

\[
i_{i}^{\text{oct}} = \left\{ \begin{array}{ll}
(1 + \alpha) e_i^q, & \alpha \leq 1 \\
e_i(0) + 2e_i^q, & \alpha > 1
\end{array} \right.
\]

where

\[
\alpha = \left| 1 - \frac{e_i(0)}{e_i^q} \right|
\]
and in the limit $e_i << 1$

$$e_i^{eq} = \frac{(5/4)e_m m_3 (m_1 - m_2)(a_i/a_0)^2 \sigma(1 - e_i^2)^{-1/2}}{m_1 m_2 - m_1 m_3 \sqrt{a_i/a_0} \sigma \sqrt{1 - e_i^2}}$$

(15)

which is sufficiently accurate to determine the stability boundary for all values of $e_i$ throughout this paper.

For inclined systems an additional secular effect is important, this is known as the Kozai effect (see Innanen et al. 1997) and involves a relationship between the eccentricity and inclination such that the maximum eccentricity induced by the Kozai mechanism is

$$e_K = \sqrt{\frac{1}{6} \left| Z + 1 - 4A^2 + \sqrt{D} \right|}$$

(16)

where

$$A = \cos I \sqrt{1 - e_i(0)^2}$$

(17)

$$Z = (1 - e_i(0)^2)(1 + \sin^2 I)$$

$$+ 5e_i(0)^2 (\sin \omega \sin I)^2$$

(18)

and

$$D = 16A^4 - 20A^2 - 8A^2 Z - 10Z + Z^2 + 25$$

(19)

which all depend on the initial eccentricity of the inner binary, $e_i(0)$, the initial relative inclination between the inner and outer orbits, $I$, and $\omega = \omega_o - \omega_i$.

The eccentricity determined by Equation (16) is the maximum possible eccentricity that comes out of this Kozai cycle that gives the maximum resonance width (Mardling 2011). This means that the maximum eccentricity induced by the Kozai mechanism ($e_K$) must also be included in the inner eccentricity functions given in equation (4). This is achieved by replacing $e_i$ in these equations with the theoretical maximum inner eccentricity given by

$$e_i = \max(e_i^{ind}, e_i^{oct}, e_K)$$

(20)

where $e_i^{ind}$ is the induced eccentricity due to the outer binary orbit (equation 11) and $e_i^{oct}$ is the eccentricity correction due to the octopole term (equation 10). As the Kozai mechanism relates the eccentricity and inclination then the inclination used in equations (11) - (10) must be replaced by the maximum possible inclination over a Kozai cycle ($I_K$). The maximum inclination is given by

$$\cos(I_K) = \frac{A}{\sqrt{1 - e_K}}$$

(21)

$$\sin(I_K) = \sqrt{1 - \cos^2 I_K}.$$  

(22)

Each resonance width is calculated using the maximum $e_i$ (equation 20) and inclination $I_K$ (equations 21 and 22) together with equation (13). For the application to GC orbits in a galactic tidal field a given set of system parameters ($m_1$, $m_2$, $m_3$, $e_o$, and $R_P$) are fixed and the stability for each particular set of $e_i(0)$, $\sigma$ and inclination $I$ is determined.

3 STABILITY IN GLOBULAR CLUSTERS

We next use the stability analysis outlined previously to examine how stability depends on the proximity of a star to the centre of the cluster. The MSC consists of a stability analysis to determine particular orbital configurations of the three masses which are unstable to the escape of one of the masses. In the case of a star-cluster-galaxy system the only energetically possible end state for such an unstable configuration is the escape of the star from the cluster. Therefore stars on orbits that are found to be unstable are predicted to eventually escape the cluster.

Once again mass potentials are assumed for the galaxy and cluster, which is valid for the star-cluster and cluster-galaxy distances of interest here. The three body system is then described by a star $m_1 = 1$ $M_\odot$, the globular cluster $M_C = 10^6 M_\odot$ and the galaxy $M_G = 10^{11} M_\odot$. In the notation of the stability analysis in Section 2 the orbit of the star-cluster is referred to as the inner orbit composed of $m_1 = M_C$ and $m_2 = m_1$, and the orbit of the cluster-galaxy as the outer orbit (where $m_3 = M_G$).

Using the MSC a boundary between predominately unstable and stable orbits is sought, separated by a period ratio $\sigma_o$. The transition between the unstable exterior of the cluster and stable interior is characterised by two additional period ratio values $\sigma_{min}$ and $\sigma_{max}$. The difference between $\sigma_{max}$ and $\sigma_{min}$ represents the width of the transition region and can be used to estimate a maximum and minimum tidal radii range for globular clusters. A conceptualisation of the different stability regions for stellar orbits within a cluster is shown in Figure 1.

To determine the stability boundary inside a cluster we need to average over the eccentricity and inclination for the star-cluster orbit. In the case of the relative inclination between the star-cluster orbit and the cluster-galaxy orbit this is trivial, assuming a $\cos I$ distribution. However the eccentricity distribution is not so straightforward. The approximation of such a distribution function is taken up in the next section.

Figure 1. Conceptualisation of the stability of stellar orbits in a globular cluster. The distance from the cluster centre associated with the transition from unstable (dark shading) to stable orbits (unshaded inner region) is indicated by the ratio of outer to inner periods $\sigma_o$. The region where orbits can be found in either unstable or stable configurations is shown as a light shading between $\sigma_{min}$ and $\sigma_{max}$. The cluster is truncated at the maximum theoretical tidal radius which is equivalent to $R_{King}$ given by equation (33).
potential is equivalent to the Plummer potential, and the velocities to be distributed such that the combined gravitational of mass $M_b$ reduces to the potential for a point mass then used to give an eccentricity. The distribution of eccentricities was sought, and the Beta distribution was found to provide the required fit. The Beta distribution is given by

$$f(e_i) = \frac{1}{B(\alpha, \beta)} e_i^{\alpha-1} (1 - e_i)^{\beta-1}$$

(24)

where $B(\alpha, \beta)$ is the Beta function and the mean of the distribution is given by $\alpha/(\alpha + \beta)$. For the fitting function plotted as a grey curve in Figure 2, $\alpha = 2.691732$, $\beta = 3$, and $B(\alpha, \beta) = 0.042898$. These values ensure the same mean eccentricity value of $e_i = 0.47$ for the fitting function as for the distribution. The cumulative probability distribution for equation (24) is

$$F(e_i) = \frac{1}{B(\alpha, \beta)} \left(0.371508e_i^{\alpha} - 0.541751e_i^{\alpha+1} + 0.213141e_i^{\alpha+2}\right),$$

(25)

which is used when averaging over the eccentricity distribution to derive the fraction of unstable orbits in the next section.

4 ECCENTRICITY DISTRIBUTION OF STARS IN AN ISOLATED PLUMMER SPHERE

The MSC requires an inner eccentricity to determine the stability for a given system. For the orbits inside a globular cluster this eccentricity distribution is not obvious. In this section a simple simulation of $N = 10^4$ particles is made to estimate a realistic eccentricity distribution. This number of particles was found to give sufficient resolution for the required distribution, while being computationally efficient.

A globular cluster is modelled using a Plummer sphere with gravitational potential given by Binney & Tremaine (1987).

$$\Phi = -\frac{GM_C}{r^{2+b}}$$

(23)

where $G$ is the gravitational constant, $M_C$ is the mass of the cluster, $r$ is the radial distance, and $b$ is a parameter chosen to describe the compactness of the cluster. For $b = 0$ the Plummer potential reduces to the potential for a point mass of mass $M_C$. For simplicity a value of $b = 1$ is assumed from herein.

Numerically modelling such a cluster requires $N$ particles to be distributed such that the combined gravitational potential is equivalent to the Plummer potential, and the velocities satisfy a Maxwellian distribution. The distribution of particles is achieved using the mass enclosed within a particular radius along with von Neumann’s rejection technique for random sampling from a distribution to determine the velocities. Here the same procedure as that of Aarseth et al. (1974) is followed, except for the alterations made to account for the cluster compactness parameter $b$ and for truncating the cluster at the King radius (equation (24)). Further details of this method can be found in Kennedy (2008).

To calculate the orbit of each particle a Bulirsch-Stoer integrator (Press et al. 1986) was used until a minimum and maximum distance could reliably be determined, which was then used to give an eccentricity. The distribution of eccentricities for particle orbits in a cluster is shown in Figure 2.

An integrable function that fits the eccentricity distribution was sought, and the Beta distribution was found to provide the required fit. The Beta distribution is given by

$$f(e_i) = \frac{1}{B(\alpha, \beta)} e_i^{\alpha-1} (1 - e_i)^{\beta-1}$$

(24)

where $B(\alpha, \beta)$ is the Beta function and the mean of the distribution is given by $\alpha/(\alpha + \beta)$. For the fitting function plotted as a grey curve in Figure 2, $\alpha = 2.691732$, $\beta = 3$, and $B(\alpha, \beta) = 0.042898$. These values ensure the same mean eccentricity value of $e_i = 0.47$ for the fitting function as for the distribution. The cumulative probability distribution for equation (24) is

$$F(e_i) = \frac{1}{B(\alpha, \beta)} \left(0.371508e_i^{\alpha} - 0.541751e_i^{\alpha+1} + 0.213141e_i^{\alpha+2}\right),$$

(25)

which is used when averaging over the eccentricity distribution to derive the fraction of unstable orbits in the next section.

5 APPLICATION TO GLOBULAR CLUSTERS AND APPROXIMATING THE TIDAL RADIUS

To demonstrate the stability analysis process the MSC is applied to a co-planar system with outer eccentricity $e_o = 0.5$ following the method given in Section 2. The resonance widths for $n:1$ type resonances are determined from equation (1) and are shown in Figure 3 (a) as a function of $\sigma = \nu_i/\nu_o = T_i/T_o$ for $m_2/m_1 = 10^{-6}$ and $m_3/m_1 = 10^{5}$.

Regions where the system resides in a single resonance are shaded green in Figure 3 (a) and the boundary of this region (the separatrix) is indicated by a black curve. The resonance width calculated by equation (1) is the distance of the separatrix from exact resonance ($n : 1$). Regions where two or more resonances overlap are shaded in red to indicate theoretically unstable orbits. In the context of globular clusters, stars on these orbits are expected to eventually escape from the cluster.

Unstable systems can still occur near the separatrix (see Figure 15 of Marck (2003)), which means that the predicted unstable regions are a conservative estimate. The criterion of an unstable system being any system that resides in two resonances simultaneously - this is adopted as a quick diagnostic and gives a good estimate as to where most unstable regions of $\sigma$-$e_i$ space occur.

By averaging over the inner eccentricity using equation (25) the fraction of unstable orbits as a function of $\sigma$ can be determined. This averaging is realised by numerically determining the minimum ($e_{min}$) and maximum ($e_{max}$) values for the unstable region shown in Figure 3 (a) for a particular $\sigma$. The fraction of unstable orbits predicted for a co-planar system with a fixed outer eccentricity is then

$$f_{unstable}(\sigma) = \frac{Pr(e_{min} < e_i < e_{max})}{F(e_{max}) - F(e_{min})}$$

(26)

where $F(e_i)$ is the cumulative distribution function for the eccentricities of particles in a Plummer sphere given by equation (25). The fraction of unstable orbits for the co-planar $e_o = 0.5$ system with $\phi_o = 0$ is shown in Figure 3 (b). The dashed red line in this figure indicates $\sigma_u$, which is the lowest
The tidal radius from three body stability

used to calculate the resonance width and hence the stability of orbits.

In this case one expects that \( \cos(I) \) is uniformly distributed between -1 and 1 and so the probability distribution function for the relative inclination can be written as

\[
g(I) = \frac{1}{\pi} (1 + \cos 2I)
\]

(27)

where \( 0 \leq I \leq \pi \). The probability of the inclination being in the range \( I_j - \Delta I/2 \leq I \leq I_j + \Delta I/2 \) is then given by

\[
P(I_j) = \left| \int_{I_j - \Delta I/2}^{I_j + \Delta I/2} g(I) dI \right| ,
\]

(28)

except at the end points where it is given by

\[
P(I_j = 0) = \left| \int_0^{\Delta I/2} g(I) dI \right|
\]

(29)

\[
P(I_j = \pi) = \left| \int_{\pi - \Delta I/2}^{\pi} g(I) dI \right|
\]

(30)

where a resolution of \( \Delta I = \pi/6 \) is used here. The resonance widths for determining the stability of the star-cluster-galaxy system are now calculated using equations (1) and (2), which includes additional terms to allow for a variable inclination.

Using the probability distribution given by equations (28) to (30), the fraction of unstable orbits becomes

\[
f_u(\sigma) = \sum_{j=1}^{N_{inc}} \left[ F(e_{max}) - F(e_{min}) \right] P(I_j)
\]

(31)

where \( N_{inc} \) is the total number of inclination values (set at 12 here) spaced in increments of \( \Delta I = 2\pi/N_{inc} \) and \( F(e) \), \( e_{max} \) and \( e_{min} \) have been defined in the previous section.

The fraction of unstable orbits after averaging over the relative inclination and eccentricities of stellar orbits within a cluster, as determined by equation (31), is shown in Figure 4 (a) to (c) for \( e_o = 0.2 \), 0.5 and 0.8 respectively. The dashed vertical line in this figure again indicates the lowest \( \sigma \) value where the fraction of unstable orbits drops beneath 10% \( (\sigma_u) \). The dotted lines indicate \( \sigma_{min} \), the lowest \( \sigma \) value for which \( f_{unstable} < 0.95 \), and \( \sigma_{max} \), the highest \( \sigma \) value for which \( f_{unstable} > 0.05 \).

From Figure 4 the transition from unstable to stable orbits \( (\sigma_u) \) increases as outer eccentricity \( (e_o) \) increases. This is due to the dependence of the resonance width on the combination \( n\xi(e_o) \) in equation 5, where \( n \) refers to the \( n : 1 \) resonance. This quantity is always positive and as it increases the resonance width rapidly falls to zero. Unstable systems therefore require \( n\xi(e_o) \) to be as close to zero as possible, which is achieved for high values of \( n \) and \( \sigma \) if \( e_o \) is also high. Physically, this reflects the fact that an exponentially small amount of energy is exchanged between the inner (star-cluster) and outer (cluster-galaxy) orbits when their orbits are very wide (Mardling 2008).

The width of the transition from unstable to stable orbits also increases with \( e_o \) as seen in the progression of panels (a) through (c) of Figure 4. Note that the basic structure of peak stability occurring at integer values of \( \sigma \) and stability increasing with increasing \( \sigma \) is consistent for all eccentricities. This phenomenon is expected since resonance widths
from the $n : 1$ and $n + 1 : 1$ resonances (where $n < \sigma < n + 1$) overlap at the midpoint between these resonances, as seen previously in Figure 5(a) for the coplanar case.

The results for all eccentricity values are shown in Figure 4 from which a near exponential dependence of $\sigma_u$ on $e_o$ is seen. This is also true for the width of the transition between unstable and stable orbits as represented by $\sigma_{\text{min/max}}$ and shown as dotted curves on each side of $\sigma_u$ in Figure 5.

Note that the coarseness in the curves for all $\sigma$ values associated with the width of this transition are shown as dotted lines. Note the increase in the range of $\sigma$ as $e_o$ increases.

The transition value of $\sigma_u$ will be used in the next section to calculate the tidal radius for a given perigalacticon and eccentricity of the cluster orbit about the galaxy. The width of the transition from unstable to stable orbits, given by $\sigma_{\text{min/max}}$, will be used to provide approximate error bars associated with the tidal radius.

This tidal radius calculation is intended to be generally applicable to the entire system of globular clusters. However, in determining $\sigma(e)$ the mass ratios have been taken as constant. This is implicitly reflected in the stellar cluster model used to determine the probability distribution for the eccentricity of stellar orbits within the cluster (equation 28). This cluster consisted of $1 M_\odot$ particles in a cluster of mass $M_C = 10^5 M_\odot$, which will not be true of all clusters.

Changing the cluster mass is expected to have a negligible effect on the resonance width calculation and therefore on the stability boundary as a function of eccentricity. This can be demonstrated by considering the mass dependence of equation (3) when $m_2 = m_1 \ll M_C = m_1$ and $m_1 = M_C \ll M_G = m_3$, which can be simplified to

$$\Delta \sigma \propto \sqrt{1 + \left(\frac{a_o}{a_i}\right)\left(\frac{M_\odot}{M_G}\right)}$$

where for distant globular clusters $a_o \sim 10$ kpc, $a_i \sim 10$ pc and $M_G = 10^{11} M_\odot$ means that this term is independent of mass for $10^4 \lesssim M_C/M_\odot \lesssim 10^6$. We can therefore apply the $\sigma(e)$ relationship derived in this section to all cluster masses for distant globular clusters.

6 COMPARISON WITH PREVIOUS THEORETICAL WORK

Two theoretical estimates for the tidal radius from the literature are compared to the tidal radius derived from the MSC method. The first estimate from the literature is the most commonly used tidal radius estimate in the field derived by King (1962) and will be referred to as the King radius. The second is an extended analytical determination that was also compared to N-body simulations for a single set of orbital parameters, this determination is given in Read et al. (2008) and will be referred to as the Read radius.

To aid comparison between these estimates the eccentricity dependence is separated out so that the tidal radius can be written as

$$r_t = R_p \left(\frac{M_C}{M_G}\right)^{1/3} f(e)$$

where $M_C$ and $M_G$ are the masses of the cluster and galaxy respectively, $e$ is the eccentricity of the clusters orbit around the galaxy, and $R_p$ is the distance of closest approach to the galaxy, referred to as the perigalacticon.

The simplest case to determine analytically is to consider a star located where the acceleration on the star in the rotating frame is zero and the velocity of the star relative to the cluster centre is also zero. Such a star will be on a radial
orbital with respect to the centre of the cluster. For a star on a radial orbit and using point mass potentials for the cluster and galaxy the eccentricity dependence of the tidal radius is given by (King 1962)

$$f(e) = k(3 + e)^{-1/3}$$  \hspace{1cm} (34)

where the constant $k \sim 0.7$ was introduced by Keenan (1981) to better fit observations of the galactic globular clusters.

Recently the analysis of King (1962) has been extended to stars on circular orbits on either prograde or retrograde orbits (Read et al. 2006). A star has zero acceleration in a coordinate frame rotating with the motion of a point mass cluster if the eccentricity dependence of the distance from the cluster is given by (Read et al. 2006)

$$f(e) = \left(\frac{1}{1 + e}\right)^{1/3} \left(\frac{\sqrt{a^2 + \frac{1}{1 + e}} - a}{1 + \frac{2}{1 + e}}\right)^{2/3}$$  \hspace{1cm} (35)

where $\alpha = 0$ denotes a star on a radial orbit inside the cluster and reduces to equation (34) in this case. Non-radial motion is restricted to the case of stars on circular orbits and is described in the tidal radius equation by setting $\alpha = 1$ or $-1$ for prograde and retrograde orbits respectively (note that $\alpha = -1$ gives $r_t$ greater than the King radius). For later comparison we define the Read radius as equation (35) with $\alpha = 1$, representing an easily tidally stripped cluster. The Read radius compared well with two N-body simulations of $10^7 M_\odot$ satellite clusters using $N = 10^7$ particles with orbital parameters of perigalacticon $R_P/R_{1/2} = 267$ and eccentricity $e_o = 0.0$ and perigalacticon $R_P/R_{1/2} = 77$ and eccentricity $e_o = 0.57$ (Read et al. 2006). A summary of alternate equations for the tidal radius designed to fit observations can be found in Bellazzini (2004).

For the stability boundary determined in Section 2 to be useful it must first be converted into an equivalent radius from the cluster centre. Assuming that the gravitational potential of the cluster is well approximated by a point mass plus orbital parameters of perigalacticon $R_P/R_{1/2}$ and eccentricity $e_o$.

By converting the period ratio shown in Figure 5 into an equivalent semi-major axis ($a_i$) via $T_o = \sigma T_i$ and using the resulting $a_i$ as $r_i$ then the eccentricity dependence itself as defined by $f(e)$ in equation (35) can be determined. The values for $f(e)$ as determined from the period ratio $\sigma$ dependence on eccentricity (Figure 5) are shown as the data points in Figure 6. These data can be fit by

$$f(e) = \exp \left[ \sum_{i=0}^{N-7} a_i e^i \right]$$  \hspace{1cm} (36)

where the coefficients are given in Table 1 for the minimum, maximum and indicative (chaos) radii.

From Figure 6 we see that the boundary between stable and unstable orbits occurs interior to both the Read and King tidal radius estimates. However the stability boundary is not perfectly equivalent to the tidal radius. This is because although chaotic orbits will eventually result in the escape of the star from the globular cluster, this occurs via a random walk process. Therefore there will be stars remaining outside the distance determined to be the stability boundary, sometimes for many crossing times. These stars will have a different velocity dispersion profile to what is expected for a relaxed and isolated globular cluster, a fact that forms the basis for the second paper in this series (Kennedy 2011).

The dangers of assuming that the tidal radius acts as an instant remover of stars has also been pointed out by Fukushige & Heggie (2000). They found for GCs on circular orbits that the escape timescales for stars beyond the tidal radii could be long enough to allow some stars to stay in this region indefinitely. This means that all predicted tidal radii will be lower limits, as stars outside this region can still be close to the GC while remaining formally unbound.

### Table 1. Coefficients for fit to $R_{\text{chaos}}$ and the minimum and maximum extents of the marginally chaotic zone. For radii $r < R_{\text{Min}}$ all orbits are expected to be stable, whereas for $r > R_{\text{Max}}$ they are expected to be chaotic.

| $e_o$  | $R_{\text{chaos}}$ | $R_{\text{Min}}$ | $R_{\text{Max}}$ |
|--------|---------------------|------------------|------------------|
| 0.0    | -0.897753           | -0.892899        | -0.888868        |
| 0.57   | 16.8616             | 10.3382          | 13.844           |
| 0.67   | -74.0133            | -45.1384         | -63.0546         |
| 0.75   | 177.136             | 108.206          | 163.39           |
| 0.8    | -236.158            | -144.186         | -239.155         |
| 0.85   | 163.999             | 99.5267          | 182.311          |
| 0.9    | -46.3734            | -27.6654         | -56.3393         |

### Figure 6. The eccentricity dependence of the tidal radii associated with the MSC prediction are shown as data points along with a solid line fit given by equation (36), the King radius as dashed lines and the Read radius as dotted lines. Coefficients to fitting functions given in Table 1 along with the minimum (red) and maximum (green) fits.

## 7 DISCUSSION AND CONCLUSIONS

The tidal radius determined here differs from previous estimates in that it emerges naturally from a stability analysis of the general three-body problem. It was found that the eccentricity of the cluster-galaxy orbit has a far more significant effect on the tidal radius of globular clusters than previous theoretical results have found.

The approach adopted here means that what is actually predicted is the boundary between stable and unstable orbits for stars inside a cluster, which is interior to previous values for all parameters of the cluster-galaxy orbit. This
was found to be the case in comparison to the commonly used tidal radius by King (1962) and to the more recent radius by Read et al. (2006). One key difference between the tidal radius estimation presented here and previously is that a much stronger dependence on the eccentricity of the cluster-galaxy orbit is predicted here. This dependence can be studied using N-body simulations, which would also give insight into how long stars on unstable orbits take to random walk out of the cluster.

From a practical point of view a major outcome of this work is the derivation of an easy to use tidal radius of the form

\[ r_t = R_p \left(\frac{M_C}{M_G}\right)^{1/3} f(e) \]  

(37)

where \( f(e) \) is given by a function (equation 36) fitted to stability results determined analytically from the stability of the general three-body problem. This has a wide range of applications, including the effect of unstable orbits on the velocity dispersion profile for Milky Way globular clusters, which is taken up in Kennedy (2011).

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