A modification of the spiked harmonic oscillator is studied in the case for which the perturbation potential contains both an inverse power and a linear term. It is then possible to evaluate trial functions by solving an integral equation due to the occurrence of the linear term. The general form of such integral equation is obtained by using a Green-function method, and adding a modified Bessel function of second kind which solves an homogeneous problem with Dirichlet boundary condition at the origin.

Key words: quantum mechanics, perturbation theory.
1. INTRODUCTION

In the literature on quantum-mechanical problems, an important role is played by the “spiked” harmonic oscillator in three spatial dimensions. This is a system where the radial part $\psi(r)$ of the wave function is ruled by the Hamiltonian operator \[ \tilde{H}(\alpha, \lambda) \equiv -\frac{d^2}{dr^2} + r^2 + \frac{l(l+1)}{r^2} + \frac{\lambda}{r^{\alpha}}, \] in the sense that $\tilde{H}(\alpha, \lambda)$ acts on $\varphi(r) \equiv r\psi(r)$. In the $s$-wave case (i.e. when the angular momentum quantum number $l$ vanishes) $\tilde{H}(\alpha, \lambda)$ reduces to \[ H(\alpha, \lambda) \equiv -\frac{d^2}{dr^2} + r^2 + \frac{\lambda}{r^{\alpha}} \equiv H(\alpha, 0) + \lambda V, \] where $H(\alpha, 0)$ is formally equal to the simple harmonic oscillator Hamiltonian, $r$ belongs to the interval $[0, \infty]$ and $V \equiv r^{-\alpha}$. For any fixed value of $\lambda$, the potential term in (1.2) diverges as $r \to 0$ in such a way that the operator $H(\alpha, \lambda)$ acts on wave functions which vanish at the origin: \[ \varphi(0) = 0. \] More precisely, the imposition of Eq. (1.3) is necessary since not all functions in the domain of $H(\alpha, 0)$ are in the domain of $V$ [2]. Thus, when $\lambda \to 0$ and $\alpha$ is fixed, the operator $H(\alpha, \lambda)$ converges to an operator formally equal to the unperturbed operator $H_0 \equiv -\frac{d^2}{dr^2} + r^2$, but supplemented by the Dirichlet boundary condition (1.3) for all functions in its domain. This means that the unperturbed operator $H_0$, for which the boundary condition (1.3) is not necessary, differs from the limiting operator $H(\alpha, 0)$, for which Eq. (1.3) is instead necessary to characterize the domain.

The full potential in (1.1) or (1.2) inherits the name “spiked” from a pronounced peak near the origin for $\lambda > 0$, and its consideration is suggested by many concrete problems in chemical, nuclear and particle physics. Some important results within this framework are as follows.

(i) Development of singular perturbation theory, with application to the small-$\lambda$ expansion of the ground-state energy [1].

(ii) A variational method has been successfully applied to a large-coupling perturbative calculation of the ground-state energy [2].
(iii) Weak-coupling perturbative analysis [3], and its relation with the strong coupling regime for $\alpha$ in the neighbourhood of $\alpha = 2$.

(iv) A non-perturbative but absolutely convergent algorithm for the evaluation of eigenfunctions [4].

Note now that for $\alpha > 2$ the Hamiltonian operators (1.1) or (1.2) lead to a non-Fuchsian singularity [5] at $r = 0$ of the stationary Schrödinger equation, because their potential term has a pole of order $> 2$ therein. Section 2 outlines the method developed by Harrell [1] for dealing with such singularities in a perturbative analysis. A non-trivial extension is studied in Sec. 3, i.e. a model where the perturbation potential contains both an inverse power and a linear term. Concluding remarks are presented in Sec. 4.

2. THE HARRELL METHOD

The method developed by Harrell relies on the choice of suitable trial functions for self-adjoint operators (i.e. normalized vectors in their domain) and on the following lemma [1]:

**Lemma.** If $\psi_\lambda$ is a trial function for the self-adjoint operator $T + \lambda T'$, where both $T$ and $T'$ are self-adjoint and $E(0)$ is an isolated, non-degenerate stable eigenvalue of $T$, and $E(\lambda)$ is a continuous function such that the scalar product $\left( \psi_\lambda, [T + \lambda T' - E(\lambda)] \psi_\lambda \right)$ tends to 0 as $\lambda$ tends to 0, and

$$ \|[T + \lambda T' - E(\lambda)] \psi_\lambda\| = o \left( \sqrt{(\psi_\lambda, [T + \lambda T' - E(\lambda)] \psi_\lambda)} \right), $$

then the eigenvalue of $T + \lambda T'$ which converges to $E(0)$ is

$$ E(\lambda) = \left( \psi_\lambda, [T + \lambda T'] \psi_\lambda \right) + O \left( \|[T + \lambda T' - E(\lambda)] \psi_\lambda\|^{2} \right). $$

In a one-dimensional example, hereafter chosen for simplicity, let the full Hamiltonian be $H_0 + \lambda V$, where

$$ H_0 \equiv -\frac{d^2}{dx^2} + x^2, $$

and

$$ V(x) \equiv x^{-\alpha}. $$
If $\alpha = 4$, on denoting by $u_i$ the unperturbed eigenstates, a trial function can be chosen in the form [1]

$$\psi(x) = W(x; \lambda)u_i(x), \quad (2.5)$$

where (inspired by the JWKB approximation)

$$W(x; \lambda) = N(\lambda)\exp\left(-\int_x^\infty \sqrt{\lambda V(\xi)} d\xi\right) = N(\lambda)\exp\left(-\frac{\sqrt{\lambda}}{x}\right), \quad (2.6)$$

with $N(\lambda)$ a normalization factor that approaches 1 as $\lambda$ tends to 0. One then finds the formula [1]

$$[H_0 - E_i + \lambda V]Wu_i = 2W\frac{\sqrt{\lambda}}{x^2} \left(\frac{1}{x} - \frac{d}{dx}\right) u_i. \quad (2.7)$$

By virtue of Eq. (2.7) and of the Lemma at the beginning of this section one obtains the following formula for the energy eigenvalues of the spiked harmonic oscillator in one dimension with $\alpha = 4$:

$$E_i(\lambda) = E_i(0) + 2 \left( u_i, x^{-2} \left(\frac{1}{x} - \frac{d}{dx}\right) u_i \right) \sqrt{\lambda} + O(\lambda). \quad (2.8)$$

When $\alpha \neq 4$, one can use the identity (hereafter $W_\alpha$ replaces $W$)

$$\frac{d^2}{dx^2}(W_\alpha u_i) = W_\alpha u''_i + 2W'_\alpha u'_i + W''_\alpha u_i, \quad (2.9)$$

which implies [1]

$$[H_0 - E_i + \lambda V]W_\alpha u_i = \left[ -\frac{d^2 W_\alpha}{dx^2} - 2\frac{dW_\alpha}{dx} \frac{d}{dx} + \lambda VW_\alpha \right] u_i. \quad (2.10)$$

The idea is now to choose $W_\alpha$ in such a way that the action of $[H_0 - E_i + \lambda V]$ on $W_\alpha u_i$ involves again the action of the operator $(\frac{1}{x} - \frac{d}{dx})$ on $u_i$ (see Eq. (2.7)). For this purpose, Harrell imposed the differential equation

$$\left[ \frac{d^2}{dx^2} + 2\frac{d}{dx} - \frac{\lambda}{x^\alpha} \right] W_\alpha = 0, \quad (2.11)$$

so that Eq. (2.10) reduces indeed to (cf. Eq. (2.7))

$$[H_0 - E_i + \lambda V]W_\alpha u_i = 2\frac{dW_\alpha}{dx} \left(\frac{1}{x} - \frac{d}{dx}\right) u_i. \quad (2.12)$$
On defining
\[ \nu \equiv \frac{1}{(\alpha - 2)}, \]  
the solution of Eq. (2.11) can be expressed in the form
\[ W_\alpha(x; \lambda) = \frac{2\nu\lambda^{\nu}}{\Gamma(\nu)} x^{-\frac{1}{2}\nu} K_\nu(2\nu\sqrt{\lambda x} - \frac{\pi}{2}). \text{ (2.14)} \]

The energy eigenvalues of the spiked oscillator in one dimension with \( \alpha \geq 4 \) are then found to be [1]
\[ E_i(\lambda) = E_i(0) + 2 \frac{\Gamma(1 - \nu)}{\Gamma(1 + \nu)} \nu^{2\nu} \left( u_i, x^{-2} \left( \frac{1}{x} - \frac{d}{dx} \right) u_i \right) \lambda^\nu + O(\lambda^{2\nu}). \text{ (2.15)} \]

3. EXTENSION TO OTHER SINGULAR POTENTIALS

When the perturbation potential is not an inverse power, Eq. (2.11) is replaced by an equation which cannot generally be solved. Harrell has however shown that, if \( V \) is bounded away from \( x = 0 \) and lies in between \( x^{-\alpha} \) and \( x^{-\beta} \), with \( 0 < \alpha < \beta \), then the effect of \( V \) on the eigenvalues is not essentially different [1].

It therefore appears of interest to study cases in which \( V \) is not (a pure) inverse power, but does not obey the restrictions considered in Sec. 5 of Ref. [1]. For this purpose, we assume that (cf. (2.4))
\[ V(x) \equiv x^{-\alpha} + \kappa x. \text{ (3.1)} \]

By doing so we study a model that reduces to the spiked oscillator in the neighbourhood of the origin, whereas at large \( x \) it approaches an oscillator perturbed by a “Stark-like” term. The two terms in (3.1) are separately well studied in the literature, so that their joint effect provides a well motivated departure from the scheme studied in Sec. 5 of Ref. [1].

The exact solution of the counterpart of Eq. (2.11) when (3.1) holds is not available in the literature to our knowledge. For this purpose we study an integral equation, whose construction is as follows. On assuming the validity of Eq. (3.1) for the perturbation potential, Eq. (2.11) is replaced by the inhomogeneous equation
\[ LW_\alpha(x; \lambda) = f_\alpha(x; \lambda), \text{ (3.2)} \]
where

\[ L \equiv \frac{d^2}{dx^2} + \frac{2}{x} \frac{d}{dx} - \frac{\lambda}{x^\alpha}, \quad (3.3) \]

\[ f_\alpha(x; \lambda) \equiv \lambda \kappa x W_\alpha(x; \lambda). \quad (3.4) \]

Since \( V(x) \) approaches \( x^{-\alpha} \) as \( x \to 0 \), we require again the boundary condition studied in Ref. [1], i.e.

\[ W_\alpha(0) = 0. \quad (3.5) \]

We are here following an approach similar to the one leading to an integral equation for scattering problems, where the right-hand side of the stationary Schrödinger equation involves again the unknown function:

\[ (\Delta + k^2)\psi_k(\vec{x}) = \frac{2m}{\hbar^2} V(\vec{x})\psi_k(\vec{x}). \]

In our problem, following Ref. [1], we study one-dimensional Schrödinger operators with Dirichlet boundary conditions at 0 only on the space \( L^2([0, \infty)) \), to remove the degeneracy resulting from the decoupling of the two half-lines \((-\infty, 0)\) and \((0, \infty)\) (only odd eigenfunctions of the ordinary harmonic oscillator obey the Dirichlet condition at 0). Moreover, to be able to use all known results on one-dimensional boundary-value problems on closed intervals of the real line, we first work on the interval \((0, b)\) and then take the limit as \( b \to \infty \). More precisely, we start from a problem whose Green function \( G(x, \xi; \lambda) \) satisfies the equation

\[ LG = 0 \quad \text{for} \quad x \in (0, \infty) \quad \text{and} \quad \xi \in (0, \infty), \quad (3.6) \]

the boundary condition

\[ G(0, \xi; \lambda) = 0, \quad (3.7) \]

the summability condition (in that the integral of \( G \) over a closed interval of values of \( \lambda \) yields a square-integrable function of \( x \))

\[ G(x, \xi; \lambda) \in L^2_{(c)}(0, \infty), \quad (3.8) \]

and the continuity conditions

\[ \lim_{x \to \xi^+} G(x, \xi; \lambda) = \lim_{x \to \xi^-} G(x, \xi; \lambda), \quad (3.9) \]

\[ \lim_{x \to \xi^+} \frac{\partial G}{\partial x} - \lim_{x \to \xi^-} \frac{\partial G}{\partial x} = 1. \quad (3.10) \]
As is shown on page 442 of Ref. [5], $G(x, \xi; \lambda)$ is recovered on studying first the Green function $G_b(x, \xi; \lambda)$ for a regular problem on the interval $(0, b)$, and then taking the limit as $b \to \infty$, i.e.

$$\lim_{b \to \infty} G_b(x, \xi; \lambda) = G(x, \xi; \lambda).$$

(3.11)

Such a relation holds independently of the boundary condition imposed on $G_b(x, \xi; \lambda)$ at $x = b$ [5]. With this understanding, the full solution of the inhomogeneous equation (3.2) with boundary condition (3.5) is given by ($\gamma$ being a constant)

$$W_\alpha(x; \lambda) = \gamma \frac{2\nu^\nu \lambda^{\nu - \frac{1}{2}}}{\Gamma(\nu)} x^{-\frac{1}{2}} K_\nu(2\nu \sqrt{\lambda} x^{-\frac{1}{2}}) + \lim_{b \to \infty} \frac{\lambda \kappa}{b} \int_0^b G_b(x, \xi; \lambda) \xi W_\alpha(\xi; \lambda) d\xi,$$

(3.12)

where the first term on the right-hand side of (3.12) is the regular solution (2.14) of the homogeneous equation $L W_\alpha = 0$. The Green function $G_b(x, \xi; \lambda)$ has to obey the differential equation [5]

$$L G_b = 0 \text{ for } x \in (0, \xi) \text{ and } x \in (\xi, b),$$

(3.13)

the homogeneous boundary conditions [5]

$$G_b(0, \xi; \lambda) = 0,$$

(3.14)

$$G_b(b, \xi; \lambda) = 0,$$

(3.15)

and the continuity conditions [5]

$$\lim_{x \to \xi^+} G_b(x, \xi; \lambda) = \lim_{x \to \xi^-} G_b(x, \xi; \lambda),$$

(3.16)

$$\lim_{x \to \xi^+} \frac{\partial G_b}{\partial x} - \lim_{x \to \xi^-} \frac{\partial G_b}{\partial x} = 1.$$

(3.17)

To obtain the explicit form of $G_b(x, \xi; \lambda)$ one has to consider a non-trivial solution $u_0(x; \lambda)$ of the homogeneous equation $Lu = 0$ satisfying $u(0) = 0$, and a non-trivial solution $u_b(x; \lambda)$ of $Lu = 0$ satisfying $u(b) = 0$. By virtue of (3.13)–(3.15) one then finds [5]

$$G_b(x, \xi; \lambda) = A(\xi; \lambda) u_0(x; \lambda) \text{ if } x \in (0, \xi),$$

(3.18)

$$G_b(x, \xi; \lambda) = B(\xi; \lambda) u_b(x; \lambda) \text{ if } x \in (\xi, b),$$

(3.19)
where \( u_0 \) and \( u_b \) are independent. The continuity conditions (3.16) and (3.17) imply that \( A(\xi; \lambda) \) and \( B(\xi; \lambda) \) are obtained by solving the inhomogeneous system

\[
A(\xi; \lambda)u_0(\xi; \lambda) - B(\xi; \lambda)u_b(\xi; \lambda) = 0, \quad (3.20)
\]

\[
B(\xi; \lambda)u_b'(\xi; \lambda) - A(\xi; \lambda)u_0'(\xi; \lambda) = 1, \quad (3.21)
\]

which yields

\[
A(\xi; \lambda) = \frac{u_b(\xi; \lambda)}{\Omega(u_0, u_b; \xi; \lambda)}, \quad (3.22)
\]

\[
B(\xi; \lambda) = \frac{u_0(\xi; \lambda)}{\Omega(u_0, u_b; \xi; \lambda)}, \quad (3.23)
\]

where \( \Omega \) is the Wronskian of \( u_0 \) and \( u_b \). We now recall the Abel formula for \( \Omega \), according to which [5]

\[
\Omega(u_0, u_b; x; \lambda) = C(\lambda)e^{-v(x)}, \quad (3.24)
\]

where \( C(\lambda) \) is a constant and, for the operator \( L \) in (3.3), \( v \) is a particular solution of the equation [5]

\[
\frac{dv}{dx} = \frac{2}{x}, \quad (3.25)
\]

i.e.

\[
v(x) = \log x^2, \quad (3.26)
\]

which implies that

\[
\Omega = \frac{C}{x^2}. \quad (3.27)
\]

Thus, on defining as usual \( x_< \equiv \min(x, \xi), x_> \equiv \max(x, \xi) \), Eqs. (3.18), (3.19), (3.22), (3.23) and (3.27) imply that the Green function is expressed by

\[
G_b(x, \xi; \lambda) = \frac{\xi^2}{C(\lambda)}u_0(x_<; \lambda)u_b(x_>; \lambda). \quad (3.28)
\]

The integral equation (3.12) for \( W_\alpha \) becomes therefore

\[
W_\alpha(x; \lambda) = \gamma \frac{2^{\nu} \lambda^{\frac{3}{2}}}{\Gamma(\nu)} x^{-\frac{1}{2}} K_{\nu}(2\nu \sqrt{\lambda x} - \frac{1}{2})
\]

\[
+ \lim_{b \to \infty} \frac{\lambda \kappa}{C} \left[ u_b(x; \lambda) \int_0^x u_0(\xi; \lambda)\xi^3 W_\alpha(\xi; \lambda) d\xi 
\right. 
\]

\[
+ u_0(x; \lambda) \int_x^b u_b(\xi; \lambda)\xi^3 W_\alpha(\xi; \lambda) d\xi \right]. \quad (3.29)
\]
In Eq. (3.29), \( u_0(x; \lambda) \) and \( u_b(x; \lambda) \) can be chosen to be of the form
\[
\begin{align*}
  u_0(x; \lambda) &= C_0(\nu)x^{-\frac{1}{2}}K_\nu(2\nu\sqrt{\lambda}x^{\nu}), \\
  u_b(x; \lambda) &= C_b(\nu)x^{-\frac{1}{2}}I_\nu(2\nu\sqrt{\lambda}x^{\nu}),
\end{align*}
\] (3.30) (3.31)
in agreement with the homogeneous Dirichlet conditions at 0 and at \( b \), respectively. Before taking the limit as \( b \to \infty \) in Eq. (3.29) we can now regard the perturbation parameter \( \lambda \) as an eigenvalue. We are therefore studying, for finite \( b \), a Fredholm integral equation of second kind, whose general form is (the parameter \( a \) vanishes in our problem)
\[
\varphi(s) = f(s) + \lambda \int_a^b K(s, t)\varphi(t)dt.
\] (3.32)
If \( \lambda \) is an eigenvalue, a necessary and sufficient condition for the existence of solutions of Eq. (3.32) is that, for any solution \( \chi \) of the equation
\[
\chi(s) = \lambda \int_a^b K(t, s)\chi(t)dt,
\] (3.33)
the known term \( f(s) \) should satisfy the condition [6]
\[
\int_a^b f(s)\chi(s)ds = 0.
\] (3.34)
In our problem, \( f \) is the first term on the right-hand side of Eq. (3.29), the kernel \( K \) is given by \( G_b \) in (3.28), and Eq. (3.34) provides a powerful operational criterion. Indeed, one might try to use directly the theory of integral equations [6] on the interval \((0, \infty)\) instead of the limiting procedure in Eq. (3.29), but the necessary standard of rigour goes beyond our present capabilities.

4. CONCLUDING REMARKS

In the present letter we have exploited the fact that if a spiked harmonic oscillator is modified by the addition of a linear term, the full perturbation potential may be seen as consisting of an inverse power plus a term linear in the independent variable. It is then possible to evaluate the function \( W(x; \lambda) \) occurring in the trial function (2.5) by solving the inhomogeneous equation (3.2), which leads to the integral equation (3.29). This is involved, but leads in principle to a complete calculational scheme. Note that, if one tries to combine the term \( x^2 \) in the unperturbed Hamiltonian with the linear term in the
perturbation potential (3.1), one eventually moves the singular point away from the origin, whereas the spiked oscillator is (normally) studied by looking at the singular point at the origin.

It would be rather interesting, as a subject for further research, to consider suitable changes of independent variable in the investigation of non-Fuchsian singularities. For example, given the Hamiltonian operator

$$H \equiv -\frac{d^2}{dr^2} + \frac{b}{r^2} + \frac{a}{r^p}, \quad (4.1)$$

if one defines the new independent variable (cf. page 971 of Ref. [7])

$$\rho \equiv r^\gamma, \quad (4.2)$$

for a suitable parameter $\gamma$, the stationary Schrödinger equation becomes

$$\left[ \frac{d^2}{d\rho^2} + \left( 1 - \frac{1}{\gamma} \right) \frac{1}{\rho} \frac{d}{d\rho} + \frac{1}{\gamma^2 \rho^2} \left( E \rho^{\frac{2}{\gamma}} - a \rho^{\frac{(2-p)}{\gamma}} - b \right) \right] \varphi(\rho) = 0. \quad (4.3)$$

By construction, the larger is $\gamma$, the more Eq. (4.3) tends to its Fuchsian limit

$$\left[ \frac{d^2}{d\rho^2} + \left( 1 - \frac{1}{\gamma} \right) \frac{1}{\rho} \frac{d}{d\rho} + \frac{(E - a - b)}{\gamma^2 \rho^2} \right] \varphi(\rho) = 0, \quad (4.4)$$

for all values of $\rho$. This remark can be made precise by defining

$$\varepsilon \equiv \frac{1}{\gamma}, \quad (4.5)$$

$$F(\varepsilon) \equiv \varepsilon^2 \left( E \rho^{2\varepsilon} - a \rho^{(2-p)\varepsilon} - b \right), \quad (4.6)$$

and considering the asymptotic expansion at small $\varepsilon$ (and hence large $\gamma$)

$$F(\varepsilon) \sim \varepsilon^2 \left[ E - a - b + \varepsilon(2E - (2-p)a) \log \rho + O(\varepsilon^2) \right]. \quad (4.7)$$

The first “non-Fuchsian correction” of the limiting equation (4.4) is therefore

$$\left\{ \frac{d^2}{d\rho^2} + \left( 1 - \varepsilon \right) \frac{1}{\rho} \frac{d}{d\rho} + \frac{1}{\rho^2} \left[ \varepsilon^2((E - a - b) + \varepsilon(2E - (2-p)a) \log \rho) \right] \right\} \varphi(\rho) = 0. \quad (4.8)$$
Interestingly, logarithmic terms in the potential can be therefore seen to result from a sequence of approximations relating Eq. (4.3) to its Fuchsian limit (4.4). Moreover, all equations with Fuchsian singularities like Eq. (4.4) might be seen as non-trivial limits of stationary Schrödinger equations with non-Fuchsian singular points. It remains to be seen whether such properties can be useful in the investigation of the topics discussed in the previous sections.

Another topic for further research is the Schrödinger equation for perturbed stationary states of an isotropic oscillator in three dimensions written in the form

$$\left[ \frac{d^2}{dr^2} + k^2 - \mu^2 r^2 - \frac{l(l+1)}{r^2} - S(r) \right] \varphi(r) = 0, \quad (4.9)$$

where, having set (here $\mu \equiv \frac{m\omega}{\hbar}$)

$$V(r) \equiv \frac{2m}{\hbar^2} U(r) = \mu^2 r^2 + S(r), \quad (4.10)$$

the function $S$ represents the “singular” part of the potential according to our terminology. We look for exact solutions of Eq. (4.9) which can be written as

$$\varphi(r) = A(r) e^{B(r)} e^{-\frac{\mu r^2}{2}}. \quad (4.11)$$

The second exponential in (4.11) takes into account that, at large $r$, the term $\lambda^2 r^2$ dominates over all other terms in the potential (including, of course, $\frac{l(l+1)}{r^2}$), and has not been absorbed into $B(r)$ for later convenience. It is worth stressing that Eq. (4.11) is not a JWKB ansatz but rather a convenient factorization of the exact solution of Eq. (4.9).

We determine $B(r)$ from a non-linear equation by straightforward integration (see below), while the corresponding second-order equation for $A$ is rather involved.

Indeed, insertion of (4.11) into Eq. (4.9) leads to

$$\left\{ \frac{d^2}{dr^2} + 2(B' - \mu r) \frac{d}{dr} + \left[ k^2 - \mu - \frac{l(l+1)}{r^2} - 2\mu r B' + B'' + B'^2 - S(r) \right] \right\} A(r) = 0. \quad (4.12)$$

To avoid having coefficients of this equation which depend in a non-linear way on $B$ we choose the function $B$ so that

$$B'^2 - S(r) = 0, \quad (4.13)$$
which implies (up to a sign, here implicitly absorbed into the square root)

\[ B(r) = \int \sqrt{S(r)} \, dr. \] (4.14)

Hence one finds the following second-order equation for the function \( A \):

\[
\left\{ \frac{d^2}{dr^2} + 2(\sqrt{S} - \mu r) \frac{d}{dr} + \left[ k^2 - \mu - \frac{l(l+1)}{r^2} - 2\mu r \sqrt{S} + \frac{S'}{2\sqrt{S}} \right] \right\} A(r) = 0. \] (4.15)

It should be stressed that the step leading to Eq. (4.13) is legitimate but not compelling. For each choice of \( B(r) \) there will be a different equation for \( A(r) \), but in such a way that \( \varphi(r) \) remains the same (see (4.11)). Unfortunately, Eq. (4.15) remains too difficult, as far as we can see.

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