Modeling and control of magnetorheological 6-DOF stewart platform based on multibody systems transfer matrix method

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Abstract
A 6-DOF vibration isolation system with magnetorheological damper (MRD) based on Stewart mechanism was designed to reduce the mechanical vibration more effectively. To make the isolation system perform well, an effective control method was also proposed. Using the linear multibody systems transfer matrix method (MSTMM), the dynamic model of the MRD 6-DOF vibration isolation system was established to obtain the dynamic characteristics. Firstly, based on the test mechanical property of magnetorheological damper, the electromagnetic coupling model of magnetorheological damper was established. Secondly, the transfer equation of 6-DOF vibration isolation system was established by linear MSTMM. And the state space equation was derived using the modal superposition method, which was based on the augmented eigenvector and body dynamics equation. Thirdly, the linear quadratic Gaussian control method of MRD 6-DOF vibration isolation system was designed. Finally, the numerical simulation was carried out. The results showed that MRD 6-DOF vibration isolation system with optimal control improved the dynamic performance of mechanical equipment effectively. Compared with passive vibration isolation system, the translation of the vibration isolation system in the X, Y, and Z directions, as well as the rotation around the X axis, Y axis and Z axis were reduced by 51.85%, 58.71%, 64.43%, 75.43%, 66.45% and 52.87%, respectively. Similarly, the percentage drops were 50.00%, 56.16%, 34.94%, 53.09%, 38.14% and 46.76% compared to the PID control based vibration isolation system, respectively. The established dynamic model and control strategy provided theoretical basis for relevant vibration isolation tests.

Keywords: magnetorheological damper (MRD), transfer matrix method, vibration isolation, LQG

(Some figures may appear in colour only in the online journal)

1. Introduction

Stewart mechanism was a parallel kinematic mechanism proposed by American engineer Stewart when designing flight simulator [1]. Figure 1 shows the structure of Stewart,
machine tool [2], vehicles [3], sensors [4], spacecraft [5], telescope [6] and so on.

The motion of the upper platform of Stewart mechanism was tracked by controlling the telescopic quantity of the six legs namely actuator. The currently used actuators mainly include hydraulic type [7], linear motor type [8], piezoelectric type [9] and magneto-rheological type [10]. Magnetorheological damper, namely MRD is widely used in mechanical vibration isolation system for the low energy consumption, simple structure and fast response. The dynamic model of MRD is the basis of establishing the system dynamic model and designing the control law of the system. At present, the main dynamic models of MRD were Bingham model, Bouc–Wen model, Dahl model, phenomenological model and non-parametric model [11]. Bouc–Wen model was widely used in describing the mechanical behavior of MRD because it can flexibly describe the mechanical behavior of various hysteretic dynamic systems. To accurately describe the mechanical behavior of MRD using Bouc–Wen model, it is necessary to identify the parameters according to the mechanical properties test of MRD. At present, the parameter identification methods of MRD dynamic model included genetic algorithm [12], particle swarm optimization [13], artificial fish swarm algorithm [14] and so on. Artificial fish swarm algorithm has the advantages of overcoming local extremum, insensitive initial value and parameter selection, strong robustness, easy realization, fast convergence and flexibility [15]. Therefore, artificial fish swarm algorithm was adopted to identify the parameters of MRD dynamic model in this study.

The control strategy of Stewart parallel mechanism was based on the dynamic modeling, and various dynamic methods has been studied. Shao et al [16], Wang et al [17], Dasgupta et al [18] and Mauricio et al [19] has used Newton–Euler (N–E) approach in their studies. Shao et al deduced the rigid-body dynamic model of the rigid Stewart manipulator using the N–E approach in five-hundred-meter aperture spherical radio telescope. Wang et al derived the simulation models of a 5 R mechanism and a 6-UPS platform. And the calculation results indicated that the closed-form algebraic loop solver is more efficient than the numerical ones. Dasgupta et al presented an inverse dynamic formulation of the Stewart platform manipulator with the gravity effects as well as the viscous friction at the joints using the N–E approach. Mauricio et al used N–E approach to model six degree of freedom flight simulator motion system. And a control approach for the motion control of a flight simulator motion base was presented. Staicu [20], Sun et al [21] and Hamidreza et al [22] used Lagrange equation in their researches. In Staicu’s studies, the inverse dynamics problem was solved using an approach based on the principle of virtual work, and the results in the framework of the Lagrange equations with their multipliers has been verified. Sun et al established the dynamic model of the small range six-axis accelerometer using the Lagrange equation, and the experiments showed that the accuracy of the theoretical analysis were 90.4% along the x-axis, 74.9% along the y-axis and 78.9% along the z-axis. Hamidreza et al made a comparison between the inverse dynamic solution based on Lagrange formulation and the direct dynamic solution of the Stewart platform by simulation with ADAMS commercial engineering package. Asadi et al [23], Huang et al [24] and Amit et al [25] used Kane’s method to model Stewart parallel mechanism. Base on the Kane’s method, Asadi et al presented a complete model of inverse dynamics of the most general Stewart platform manipulator, without any simplification on dynamic properties of its components. Huang et al established dynamic model of a new type of hydraulically driven 6-DOF parallel platform with higher frequency band using the Kane’s method. And the platform has been proved to have validity in high frequency applications. Amit et al presented modeling simulation and control of stuart type 6-DOF parallel manipulator base on the Kane’s method. A novel control architecture has been proposed, which not only has industry standard PID controller but also includes compensator for drastically improving tracking performance. Han et al [26] and Yang et al [27] used finite element method to model Stewart parallel mechanism. In Han’s research, the rigidity and flexibility coupling dynamic analysis of the 6–6 Stewart parallel mechanism and mechanism random errors effect on its motion were analyzed by the ADAMS/APDL and VC++ software. Yang et al investigated the structural performance of the 6-MVS using the finite element method. The analysis and simulation results showed that the 6-MVS can exactly produce the required micro-vibration spectrum. Wang et al [10] studied the parallel vibration isolation system based on magnetorheological technology. The numerical simulation results showed that the semi-active parallel vibration isolation system had a good vibration isolation effect on six freedoms. The control system was greatly limited for the lack of the specific dynamic equation and state equation in the established dynamic model by SimMechanics software. Chen et al [28] used linear MSTMM to model the Stewart parallel mechanism for the flexible multi-body systems. The presented method was validated by comparing the computational results of natural frequencies and mode shapes as well as dynamic responses with FEM. Rui et al [29, 30] proposed MSTMM to calculate dynamic response of complex mechanical system. It has been applied in 52 scientific research and key practical engineering in over 100 kinds of products. He concluded that MSTMM showed a much higher computation speed than other methods. The computational time ratio between MSTMM and Lagrange equation has been studied, showing a
much larger value of computational time ratio. Also, higher value corresponded to the higher number of degrees of freedom, with the largest value of $7 \times 10^9$ at $3 \times 10^6$ degrees of freedom. Thus, MSTMM showed a much larger advantages in the calculation of vibration characteristics and dynamic response of complex mechanical systems. Besides, global dynamic equation was not necessary during the calculation of dynamic response.

In this paper, a new 6-DOF vibration isolation system based on MRD was proposed and modeled by the linear MSTMM. By solving homogeneous linear algebraic equations, the vibration characteristics were calculated. And then its state space representation was derived by the modal superposition method based on the augmented eigenvector and body dynamics equation. Following were the main contents. Firstly, the electromagnetic coupling model and inverse model (current output model) of the MRD were established. Relative parameters were obtained by identifying the dynamic model of MRD using the artificial fish swarm algorithm. Secondly, a detailed description of the structure of MRD 6-DOF vibration isolation system was conducted. Thirdly, the dynamic model of the MRD 6-DOF vibration isolation system was conducted. Finally, numerical analysis was conducted on the dynamic response of the MRD 6-DOF vibration isolation system. It presented a better vibration isolation performance than that of the passive case and PID case.

2. Performance and dynamic modeling of MRD

According to MRD’s structure, there exist three types of MRD, i.e. single rod mono-tube, double rod mono-tube and single rod twin-tube. In the 6-DOF vibration isolation platform, the single rod mono-tube MRD (RD-8041-1, produced by Lord Company) was adopted. It is mainly composed of piston, piston rod, airbag, magnetorheological fluid (MRF), excitation coil and cylinder, as shown in figure 2. By changing the working current of excitation coil, the magnetic field intensity of damping channel was changed and then was the output damping force.

2.1. Mechanical property of MRD

To build the dynamic model of MRD 6-DOF vibration isolation system, the mechanical properties of MRD were tested first by an electro-hydraulic servo fatigue machine (type: LFV150 kN, the W + B GmbH, Switzerland) shown in figure 3. Sinusoidal signals excitation (amplitudes: 10 mm, frequencies: 1 Hz) was applied to the MRD with the driving currents of 0.0 A, 0.2 A, 0.4 A, 0.6 A, 0.8 A and 1.0 A, respectively. The output force–displacement curve and force–velocity curve are shown in figure 4. It can be seen that the damping force increases with the increase of current from 0 to 1 A and no longer increases significantly at 1 A. It means that when the current increases to 1 A, the MRD reaches the magnetic saturation. Besides, there exist two distinct regions, i.e. the pre-yield and post-yield regions in the force–velocity curve. An obvious hysteresis is observed in the pre-yield area, and larger hysteresis loop corresponds to larger current.

2.2. Dynamic modeling of MRD

2.2.1. Bouc–Wen model. Bouc–Wen model can flexibly describe the mechanical behavior of various hysteretic dynamic systems. And was widely used in describing the mechanical behavior of MRD with hysteretic characteristics. The structure of Bouc–Wen model is shown in figure 5,
which is composed of hysteretic system, spring and viscous damping in parallel [12].

The mathematical expressions of Bouc–Wen model are shown in equations (1) and (2):

\[ f_d = c_0 \dot{x} + k_0 x + \alpha z + f_0, \quad (1) \]

\[ z = -\gamma |\dot{z}| |\dot{z}|^{n-1} - \beta |\dot{z}|^{n} + A \dot{x}, \quad (2) \]

where, \( f_d \) denotes the damping force; \( z \) denotes the hysteresis operator; \( x \) and \( \dot{x} \) are the relative displacement and velocity at the two ends of the damper; \( k_0, c_0 \) and \( \alpha \) are the viscous coefficient, stiffness coefficient and hysteretic coefficient, respectively. They are the main parameters describing the mechanical properties of MRDs and have clear physical meanings; \( f_0 \) is the balance force, which is caused by the existence of compensators; \( \gamma, \beta \) and \( A \) are the shape control coefficients of hysteretic loops; \( n \) is the yield slope coefficient with a value of 2 usually. The shape of hysteretic loops was determined by the shape control coefficients and yield slope coefficients. Though they performed well in the shape control under various frequency excitations, they have no physical meaning.

According to the data of MRD performance test, the parameters of Bouc–Wen model are identified by artificial fish swarm algorithm [31], which is an efficient intelligent optimization algorithm. The basic steps include fish swarm initialization, foraging behavior, clustering behavior, tail-chasing behavior and random behavior. The flow chart is shown in figure 6.

The artificial fish swarm algorithm’s fitness function is shown in equation (3):

\[ J = \frac{1}{N_{\text{exp}}} \sqrt{\sum_{i=1}^{N_{\text{exp}}} (f_{\text{sim}} - f_{\text{exp}})^2}, \quad (3) \]

where, \( N_{\text{exp}} \) is the number of experimental data points; \( f_{\text{sim}} \) is the damping force of simulation; \( f_{\text{exp}} \) is the damping force of experiment. Table 1 lists relative parameters of the artificial fish swarm algorithm, such as quantities of artificial fish (\( N \)), maximum iteration number (MAXGEN), maximum number of foraging attempts (try_number), visual distance (Visual), crowding factor (delta) and step.

The identified parameters of the Bouc–Wen model at different currents are shown in table 2.

Table 2 shows that there has no certain regularity between the value of \( f_0 \) and current. \( f_0 \) is dependent on the

| Parameter | Value | Parameter | Value |
|-----------|-------|-----------|-------|
| \( N \)   | 100   | Visual    | 0.1   |
| MAXGEN    | 1000  | Delta     | 0.618 |
| Try_number| 200   | Step      | 0.01  |

![Figure 4](image_url) The experimental testing results (black dotted line) and Bouc–Wen model simulation results (red solid line) of MRD: (a) the force–displacement curve; (b) the force–velocity curve.

![Figure 5](image_url) Structural sketch of Bouc–Wen model.

Table 1. Parameters of artificial fish swarm algorithm.
initial position of each test. According to table 2, the relations between \( k, c, \alpha, \gamma, \beta, A \) and current are fitted by least square method. The fitting results are shown in figure 7.

The mathematical expressions are shown in equations (4)–(9):

\[
\begin{align*}
  k_0 &= 175.4I^3 - 321.1I^2 + 285.9I + 741.8, \\
  c_0 &= -1407.0I^3 + 121.1I^2 + 3890.0I + 905.5, \\
  \alpha &= 5886.0I^3 + 12360.0I^2 - 3122.0I + 257.8, \\
  \gamma &= 100 \cdot 200.0I^3 - 241 \cdot 000.0I^2 + 124 \cdot 200.0I + 43 \cdot 000.0, \\
  \beta &= -545.6I^3 - 107.0I^2 + 962.8I + 102.9, \\
  A &= -8908.0I^3 + 30 \cdot 660.0I^2 - 36 \cdot 600.0I + 14 \cdot 820.0. 
\end{align*}
\]

Substituting the fitted result of equations (4)–(9) into equations (1)–(2) obtain the theoretic relations of force–displacement and force–velocity. From simulation calculation results shown in figure 4, it can be seen that the force–displacement curve and force–velocity curve are in good agreement with the experimental data. In addition, the established dynamic model shows the same saturation phenomenon at 1 A as that of the experimental results.

2.2.2. The current output model of MRD (the inverse model of MRD). The dynamic model of MRD is based on the input variables of velocity, displacement and current. Since the output of the controller is the control force, it is necessary to establish the current output model of MRD (inverse model).

The dynamic model of MRD was described by the Bingham model, which well-expressed the force–displacement relationship [32]. It is a parallel mechanism and composed of a coulomb friction member and a viscous damper, as shown in figure 8. The damping force is expressed as equation (10):

\[
f_d = c_0 \dot{x} + f_c \text{sign}(\dot{x}),
\]

where, \( c_0 \) represents the viscous damping coefficient; \( f_c \) represents the coulomb damping force relative to the yield stress of the fluid.

Based on the experimental data of MRD, artificial fish swarm algorithm was adopted to identify parameters of the Bingham model. The artificial fish swarm algorithm’s fitness function is shown in equation (3). The parameters of the artificial fish swarm algorithm as shown in table 1. The identified parameters of the Bingham model at different currents are shown in table 3.

The relationship between identified parameters and current of Bingham model was fitted by the least-squares method, shown in figure 9. It can be seen that there are good linear fitness for the relationships of \( c_0-I \) and \( f_c-I \).

The fitting results are shown in equations (11) and (12):

\[
\begin{align*}
  c_0 &= 2945I + 932.9, \\
  f_c &= 916.5I + 52.97. 
\end{align*}
\]

Substituting the fitted result of \( c_0 \) and \( f_c \) into equation (10) obtain the theoretic relation of force–displacement and
force–velocity. From simulation calculation results shown in figure 10, it can be seen that the force–displacement curve and force–velocity curve (in the plastic zone) are in good agreement with the experimental data. In addition, the established dynamic model shows the same saturation phenomenon at 1 A as that of the experimental results. Therefore, the established dynamic model can well reflect the actual performance of MRD and the current output model.
3. MRD 6-DOF vibration isolation system

Figure 11 shows the structure of the MRD 6-DOF vibration isolation system and leg. The legs of 6-DOF vibration isolation system are composed of six MRDs and six springs. The MRD is parallel placed in the center circle of the spring. The legs are connected with the upper and lower platforms by spherical hinges at both ends. Six springs are used to balance the platform, and six MRDs provide semi-active control force for the vibration isolation platform to achieve the vibration isolation in six directions.

4. Dynamic modeling

Dynamic modeling is the foundation of the implementation of semi-active control. Traditional methods of dynamic modeling include N–E approach and Lagrange method, etc. The MSTMM is more superior for the fewer equations, higher computational efficiency and without the system overall dynamics equation. It has been widely used in dynamic modeling of mechanical systems by many scholars. In this paper, linear MSTMM is used to model the MRD 6-DOF vibration isolation system.

4.1. Dynamics model

Based on the characteristics of linear MSTMM, some assumptions on the MRD 6-DOF vibration isolation system are made:

- All frictional forces are ignored in the dynamic modeling.
- The spherical hinge is simplified to a spatial elastic hinge with large linear stiffness.
- Spring and MRD are simplified as massless hinge elements. Their mass and geometric relationships are simplified as sprung mass and unsprung mass. The sprung mass and the unsprung mass are connected by spring hinge and damper hinge.
- The spring is a spatial spring with great stiffness in other directions except the axis direction.
- The 6-DOF vibration isolation system is assumed to be a linear multi-body system.

The simplified MRD 6-DOF vibration isolation system diagrams are shown in figures 12 and 13. Circle $P_u$ and $P_d$ represent upper platform and lower platform, respectively. Points $A_u$, $B_u$ and $C_u$ respectively represent the geometric centers of two adjacent spatial elastic hinges on the upper platform. And the three points are equidistantly distributed on the circle $P_u$. Points $A_d$, $B_d$ and $C_d$ respectively represent the geometric centers of two adjacent spatial elastic hinges on the upper platform. And the three points are equidistantly distributed on the circle $P_d$. $D_{uj}$ and $D_{dj}$ ($j = 1, 2, \ldots, 6$) represent sprung mass and unsprung mass respectively. $K_j$ and $f_{dj}$ represent spatial spring and damping element respectively. The upper platform is considered as a spatial rigid body with six input and single output, and numbered as 1. Six sprung masses and six unsprung masses are considered as spatial rigid bodies with single input and single output, and numbered as 8–13 and 20–25, respectively. The six spatial springs, six upper spatial elastic hinges and six lower spatial elastic hinges are numbered as 14–19, 2–7 and 26–31, respectively. The damping force $f_{dj}$ generated by the MRD is applied to the connecting rigid body along the central axis of the leg, respectively. The origin of the inertial coordinate system $\alpha\beta\gamma$ describing the motion of the system is located at the center of the upper surface of the lower platform. The $z$-axis is parallel to the line $oA_0$, and the $x\gamma c$ plane coincides with the surface $A_dB_dC_d$. The dynamics model of MRD 6-DOF vibration isolation system consists of 13 rigid bodies and 18 hinges.

Figure 14 shows the topology diagram of the system, which describes the connections between multi-body system elements. The body and hinge elements are represented by the circle and arrow, respectively. And the transfer direction of state vectors is along the direction of arrows [29]. $I_r$ ($r = 1 \sim 6$) represent the input points of element 1 and output points of elements 2–7, respectively. $B_0$ located at the center of the upper surface of the upper platform is the output point of the system, which is kept free station. $B_j$ ($j = 1 \sim 6$) are the input points of the system, which are the connection point between element 0 (system boundary) and spatial elastic hinge 26–31. The topological diagram is used to describe the transfer relationship among state vectors. And then is applied to the deduce of the overall transfer equation of the system.
4.2. State vector

The state vector represents the mechanical state of any point in a mechanical system, including displacement and internal forces. In linear MSTMM [29], the state vectors in physical coordinates are represented as follows equation (14):

\[ z_{ij} = [x, y, z, \theta_x, \theta_y, \theta_z, m_x, m_y, m_z, q_x, q_y, q_z]^T, \]  

where, \( z_{ij} \) represents the state vector of point \( j \) on element \( i; \) \( x, y, z \) represent line displacement; \( \theta_x, \theta_y, \theta_z \) represent angular displacement; \( m_x, m_y, m_z \) represent internal torques; \( q_x, q_y, q_z \) represent internal forces.

Based on MSTMM, the vibration characteristics is calculated in modal coordinates [29]. The state vector in modal coordinates is defined as equation (15):

\[ Z_{ij} = [X, Y, Z, \Theta_x, \Theta_y, \Theta_z, M_x, M_y, M_z, Q_x, Q_y, Q_z]^T. \]  

The transformation relation of the state vector in the physical coordinates and modal coordinates is as follows equation (16):

\[ z_{ij} = Z_{ij}e^{jwt}. \]
where, \( e \), \( i \), \( \omega \) and \( t \) are the mathematical constant, imaginary unit, natural frequency and time, respectively.

4.3. Transfer and geometrical equations of element

4.3.1. Transfer and geometric equation of the upper platform

The upper platform is a spatial rigid body with six input ends and signal output end. The center of the upper platform’s upper surface is the output end. And the six input ends are the geometric centers of the six spatial elastic hinges. The schematic diagram is shown in figure 12. Using the input point \( I_1 \) as the coordinate origin, the body fixed frame \( I_1^b x^b y^b z^b \) is established. The body fixed frame is parallel to the inertial coordinate system \( oxyz \). And it is used to describe the relative positions of other points on the rigid body. In \( oxyz \) coordinate system, the transfer equation of rigid body 1 is defined as equation (17):

\[
Z_{I_1,o} = \sum_{n=1}^{6} U_{1,n} Z_{1,d,n},
\]

where, \( U_{1,n} \) is the transfer equation, shown as equation (18):

\[
U_{1,h} = \begin{bmatrix}
I_3 & -I_1^o & O_{3 \times 3} & O_{3 \times 3} \\
O_{3 \times 3} & I_3 & O_{3 \times 3} & O_{3 \times 3} \\
m_1\omega^2 I_{CO} - \omega^2 (m_1 I_1^o I_{1,C} + J_k) & I_3 & -I_1^o & I_3 \\
m_1\omega^2 I_3 & -m_1\omega^2 I_{1,C} & O_{3 \times 3} & I_3
\end{bmatrix},
\]

(18)

\[
U_{1,l} = \begin{bmatrix}
O_{3 \times 3} & O_{3 \times 3} & O_{3 \times 3} & O_{3 \times 3} \\
O_{3 \times 3} & O_{3 \times 3} & O_{3 \times 3} & O_{3 \times 3} \\
O_{3 \times 3} & O_{3 \times 3} & I_3 & I_1^o \\
O_{3 \times 3} & O_{3 \times 3} & O_{3 \times 3} & I_3
\end{bmatrix},
\]

(19)

where, \( m_1 \) is the mass of rigid body 1; \( J_k \) is the moment of inertia of the rigid body relative to point \( I_1 \) in the concatenated coordinate system; \( I_{CO} \) represents the cross product matrix of the position vectors of point E and point F; \( O_{3 \times 3} \) and \( I_3 \) represent the \( 3 \times 3 \) zero matrix and the \( 3 \times 3 \) identity matrix, respectively; \( O \) and \( C \) represent the output point and the center of mass, respectively. Based on the theory of rigid body kinematics, the geometric relationships among the points on the rigid body was obtained. And displacement of other input points on rigid body 1 can be described by input point \( I_1 \). The geometric equation is defined as equation (20):

\[
H_{1,l} Z_{1,l} = H_{1,h} Z_{1,h},
\]

(20)
The spatial elastic hinge transfer equation is as equation (30):

\[
\mathbf{U}_{i,j}^b = \begin{bmatrix}
I_3 & O_{3 \times 3} & O_{3 \times 3} & \mathbf{Ke} \\
O_{3 \times 3} & I_3 & O_{3 \times 3} & O_{3 \times 3} + \mathbf{K}' \\
O_{3 \times 3} & O_{3 \times 3} & I_3 & O_{3 \times 3} \\
O_{3 \times 3} & O_{3 \times 3} & O_{3 \times 3} & I_3
\end{bmatrix},
\]

(30)

where, \( \mathbf{Ke} \) and \( \mathbf{K}' \) represent the stiffness coefficients of spring; \( k_x', k_y', k_z' \) represent the torsional stiffness of torsion spring.

In the process of establishing the system overall transfer equation, the transfer equation of the spatial elastic hinge in the body fixed frame \( I_{6,6}^{b,b} \) should be transformed to the inertial coordinate system \( \alpha x\gamma \zeta \). It is realized by introducing the direction cosine matrix considering the similarity of the spatial elastic hinge and the rigid body of the leg.

4.3.4. Spatial spring transfer equation. The spatial spring is composed of three springs, which are perpendicular to each other and whose endpoints coincide with each other. The body fixed frame \( I_{6,6}^{b,b} \) is established along the axis of the three springs as shown in figure 13. The spatial spring transfer equation is as equation (33):

\[
\mathbf{Z}_{i,o}^b = \mathbf{U}_{i,j}^b \mathbf{Z}_{j,i}^b \quad (i = 14, \ldots, 19),
\]

(33)

where, \( \mathbf{U}_{i,j}^b \) is the transfer matrix of spatial spring in the body fixed frame \( I_{6,6}^{b,b} \), shown in equation (34). \( \mathbf{K} \) is the stiffness coefficient matrix, shown in equation (35):

\[
\mathbf{U}_{i,j}^b = \begin{bmatrix}
I_3 & O_{3 \times 3} & O_{3 \times 3} & \mathbf{K} \\
O_{3 \times 3} & I_3 & O_{3 \times 3} & O_{3 \times 3} \\
O_{3 \times 3} & O_{3 \times 3} & I_3 & O_{3 \times 3} \\
O_{3 \times 3} & O_{3 \times 3} & O_{3 \times 3} & I_3
\end{bmatrix},
\]

(34)

\[
\mathbf{K} = \begin{bmatrix}
-1/k_x & 0 & 0 \\
0 & -1/k_y & 0 \\
0 & 0 & -1/k_z
\end{bmatrix},
\]

(35)

where, \( k_x, k_y, k_z \) are the stiffness coefficients of the spatial spring in the body fixed frame \( I_{6,6}^{b,b} \).

The spatial spring transfer equation is similar to the rigid bodies of the legs. It is necessary to introduce the direction cosine matrix between the body fixed frame \( I_{6,6}^{b,b} \) and the inertial coordinate system \( \alpha x\gamma \zeta \) for transformation.

4.4. System overall transfer equation

Based on the automatic derivation theorem of the overall transfer equation in MSTMM [29] and the topological diagram of the MRD 6-DOF vibration isolation system, the system overall transfer equation is obtained. The overall transfer equation is composed of the master transfer equation.

\[
\mathbf{H}_{i,k} = \begin{bmatrix}
I_3 & O_{3 \times 3} & O_{3 \times 3} \\
O_{3 \times 3} & I_3 & O_{3 \times 3} \\
O_{3 \times 3} & O_{3 \times 3} & I_3
\end{bmatrix},
\]

(21)

4.3.2. Rigid body transfer equation of leg. The legs are simplified to 12 spatial rigid bodies with single input and single output, numbered as 8–13, 20–25, as shown in figure 14. The body fixed frame \( I_{1,1}^{b,b} \) is established with its input point as the coordinate origin. The \( x^b \) axis is along the axis of the leg, as shown in figure 13. In the body fixed frame, the transfer equation and as equation (22):

\[
\mathbf{Z}_{i,o}^b = \mathbf{U}_{i,j}^b \mathbf{Z}_{j,i}^b \quad (i = 8, \ldots, 13, 20, \ldots, 25),
\]

(22)

where

\[
\mathbf{U}_{i,j}^b = \begin{bmatrix}
I_3 & -\mathbf{l}_i & O_{3 \times 3} \\
O_{3 \times 3} & I_3 & O_{3 \times 3} \\
m_i\omega^3 & -\omega^2(m_i\omega^3l_i + J_i) & I_3 & \mathbf{l}_i
\end{bmatrix}
\]

(23)

where \( m_i \) is the mass of the rigid bodies 8–13, 20–25; and \( J_i \) is the moment of inertia of the rigid body relative to the input point.

Since the overall transfer equation of the system is established in the inertial coordinate system, it is necessary to introduce the direction cosine matrix \( \mathbf{A}_i \) between \( I_{6,6}^{b,b} \) and \( \alpha x\gamma \zeta \) to transform the transfer equation of rigid body in the body fixed frame into the inertial coordinate system. The relative relations and parameters are shown as equations (24)–(28):

\[
\mathbf{Z}_{i,o}^b = \mathbf{R}_i \mathbf{Z}_{a,i}^b = \mathbf{R}_i \mathbf{Z}_{i,o},
\]

(24)

\[
\mathbf{Z}_{i,o}^b = \mathbf{U}_{i,j}^b \mathbf{R}_i \mathbf{Z}_{i,j}^b = \mathbf{U}_{i,j}^b \mathbf{U}_{i,j}^b \mathbf{Z}_{i,j}^b,
\]

(25)

\[
\mathbf{Z}_{i,o}^b = \mathbf{R}_i^\top \mathbf{U}_{i,j}^b \mathbf{R}_i \mathbf{Z}_{i,j}^b = \mathbf{U}_{i,j}^b \mathbf{Z}_{i,j}^b,
\]

(26)

where

\[
\mathbf{R}_i = \text{diag}(\mathbf{A}_i, \mathbf{A}_i, \mathbf{A}_i, \mathbf{A}_i),
\]

(27)

\[
\mathbf{A}_i = \begin{bmatrix}
\mathbf{A}_i^x & \mathbf{A}_i^y & \mathbf{A}_i^z \\
\mathbf{A}_i^y & \mathbf{A}_i^x & \mathbf{A}_i^z \\
\mathbf{A}_i^z & \mathbf{A}_i^y & \mathbf{A}_i^x
\end{bmatrix}
\]

(28)

4.3.3. Spatial elastic hinge transfer equation. The spatial elastic hinge is composed of three directional springs and torsion springs. The body fixed frame \( I_{6,6}^{b,b} \) is established along the axis direction of the springs. And it is shown in figure 13. The transfer equation of spatial elastic hinge in the coordinate system \( I_{6,6}^{b,b} \) is as equation (29):

\[
\mathbf{Z}_{i,o}^b = \mathbf{U}_{i,j}^b \mathbf{Z}_{j,i}^b \quad (i = 2, \ldots, 7, 26, \ldots, 31),
\]

(29)

where, \( \mathbf{U}_{i,j}^b \) is the transfer matrix of spatial elastic hinge in the body fixed frame \( I_{6,6}^{b,b} \), and the expression is shown in equation (30):
and the geometric equation. According to the topological diagram of the system, the master transfer equation is equation (36):

\[
Z_{B_0} = T_{26} \cdot Z_{B_1} + T_{27} \cdot Z_{B_2} + T_{28} \cdot Z_{B_3} + T_{29} \cdot Z_{B_4} + T_{30} \cdot Z_{B_5} + T_{31} \cdot Z_{B_6},
\]

where,

\[
\begin{align*}
T_{26} &= U_{1,b} U_{2,b} U_{3,b} U_{4,b} U_{20,b} U_{26,b}, \\
T_{27} &= U_{1,b} U_{2,b} U_{3,b} U_{15,b} U_{21,b}, \\
T_{28} &= U_{1,b} U_{4,b} U_{10,b} U_{16,b} U_{22,b} U_{28,b}, \\
T_{29} &= U_{1,b} U_{5,b} U_{11,b} U_{17,b} U_{23,b} U_{29,b}, \\
T_{30} &= U_{1,b} U_{6,b} U_{12,b} U_{18,b} U_{24,b} U_{30,b}, \\
T_{31} &= U_{1,b} U_{7,b} U_{13,b} U_{19,b} U_{25,b} U_{31,b},
\end{align*}
\]

Combining the geometric equation of the input point on element 1 with the transfer equation of element 2–31, the system geometric equation can be obtained, shown in equation (38):

\[
\begin{align*}
G_{26} \cdot Z_{B_1} + G_{27} \cdot Z_{B_2} + G_{28} \cdot Z_{B_3} + G_{29} \cdot Z_{B_4} + G_{30} \cdot Z_{B_5} + G_{31} \cdot Z_{B_6} &= 0,
\end{align*}
\]

where, \( \theta \) is the \( 0 \) array.

\[
\begin{align*}
G_{26} &= -H_{1,b} U_{2,b} U_{8,b} U_{14,b} U_{15,b} U_{20,b} U_{26,b}, \\
G_{27} &= H_{1,b} U_{3,b} U_{9,b} U_{15,b} U_{21,b}, \\
G_{28} &= H_{1,b} U_{4,b} U_{10,b} U_{16,b} U_{22,b} U_{28,b}, \\
G_{29} &= H_{1,b} U_{5,b} U_{11,b} U_{17,b} U_{23,b} U_{29,b}, \\
G_{30} &= H_{1,b} U_{6,b} U_{12,b} U_{18,b} U_{24,b} U_{30,b}, \\
G_{31} &= H_{1,b} U_{7,b} U_{13,b} U_{19,b} U_{25,b} U_{31,b},
\end{align*}
\]

From equations (36) and (38), the system overall transfer equation is obtained, shown as equation (40):

\[
U_{all} Z_{all} = 0,
\]

where

\[
Z_{all} = [Z_{B_1}^T, Z_{B_2}^T, Z_{B_3}^T, Z_{B_4}^T, Z_{B_5}^T, Z_{B_6}^T, Z_{B_7}^T],
\]

\[
U_{all} = \begin{bmatrix}
-T & T_{26} & T_{27} & T_{28} & T_{29} & T_{30} & T_{31} & 1 \\
O & G_{26} & G_{27} & G_{28} & G_{29} & G_{30} & G_{31} & 0 \\
O & G_{26} & G_{27} & G_{28} & G_{29} & G_{30} & G_{31} & 0 \\
O & G_{26} & G_{27} & G_{28} & G_{29} & G_{30} & G_{31} & 0 \\
O & G_{26} & G_{27} & G_{28} & G_{29} & G_{30} & G_{31} & 0
\end{bmatrix}
\]

4.5. Vibration characteristics

Boundary conditions of the system is shown as equations (43)–(44):

\[
Z_{B_0} = [X, Y, Z, \Theta_x, \Theta_y, \Theta_z, 0, 0, 0, 0, 0, 0]_b^T,
\]

\[
Z_{B_j} = [0, 0, 0, 0, 0, M_i, M_j, M_k, Q_x, Q_y, Q_z]_b^T \times (j = 1, \cdots, 6),
\]

Substituting equations (43) and (44) into equation (40) obtains a linear homogeneous equations of unknown boundary state vectors, shown as equation (45):

\[
U_{all} Z_{all} = 0,
\]

where, \( Z_{all} \) is the column matrix composed of unknown boundary state vectors; \( U_{all} \) is the square matrix composed of corresponding columns of unknown boundary state variables in the overall transfer matrix, which is called the characteristic matrix.

For the undamped-free vibration system, if equation (45) has a nonzero solution, the determinant of the characteristic matrix should be zero, shown as equation (46):

\[
f(\omega) = \det(U_{all}) = 0.
\]

Equation (46) is an algebraic equation about the system natural frequency \( \omega_2 \), namely, the system frequency equation.

By substituting the calculated natural frequencies of each mode into equation (46), the unknown variables are solved by the singular value decomposition. Via the transfer equation of the element, the vibration mode is obtained by calculating the state vector of the connection point from the input boundary to the output boundary end.

4.6. State space representation

4.6.1. Body dynamics equation. The body dynamics equations of each element in its body fixed frame are shown as equation (47):

\[
M_i^b v_i^b + K_i^b v_i^b = f_i^b \quad (i = 1, 8, \cdots, 13, 20, \cdots, 25),
\]

where, \( M_i^b \) and \( K_i^b \) represent mass and stiffness parameter matrix, respectively; \( v_i^b \) and \( v_i^c \) are the column matrices of displacements and accelerations, respectively; \( f_i^b \) represent the column matrix of external force (including external torque) on the body.

(1) The body dynamics equation of rigid body 1 is shown in equation (48):

\[
M_1^b = \begin{bmatrix}
 m_1 I_1 & -m_1 I_1 \\
 m_1 I_1 & I_1
\end{bmatrix},
\]

\[
v_1^b = [x^b, y^b, z^b, \theta_x^b, \theta_y^b, \theta_z^b]_b^T, f_1^b = 0
\]

\[
K_1^b = \begin{bmatrix}
 -\sum_{r=1}^{6} I_1 D_i^1 |_{l_1} + I_1 D_i^3 |_{l_1} & \cdots & O_{3 \times 3} \\
 -\sum_{r=2}^{6} I_1 D_i^1 |_{l_1} + I_1 D_i^3 |_{l_1} & \cdots & \sum_{r=1}^{6} I_1 D_i^1 |_{l_1} - I_1 D_i^1 |_{l_1}
\end{bmatrix}
\]

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where \( d^1 \) and \( d^3 \) are linear operator, the expressions are shown in equations (49)–(50):

\[
\begin{align*}
D^1 \theta_{t.d,i} & = m_x \dot{v}_{x,i} \\
D^1 \theta_{t.d,i} & = m_y \dot{v}_{y,i} \\
D^1 \theta_{t.d,i} & = m_z \dot{v}_{z,i} \\
D^3 \theta_{t.d,i} & = m_x \dot{v}_{x,i} \\
D^3 \theta_{t.d,i} & = m_y \dot{v}_{y,i} \\
D^3 \theta_{t.d,i} & = m_z \dot{v}_{z,i} \\
D^3 \theta_{t.d,i} & = q_{x,i} \\
D^3 \theta_{t.d,i} & = q_{y,i} \\
D^3 \theta_{t.d,i} & = q_{z,i} \\
\end{align*}
\]

when \( d^1 \) acts on the angular displacement of a joining point, it represents all the internal moments except the damping moments. When \( d^3 \) acts on the displacement of a joining point, it represents all internal forces except damping forces.

(2) The body dynamics equations of rigid bodies 8–13, 20–25 are shown in equation (51):

\[
M^b = \begin{bmatrix} m_1 I_1 & -m_1 \hat{I}_{1,b} \\ m_1 \hat{I}_{1,b} \end{bmatrix}, \quad K^{bb} = \begin{bmatrix} -I_1 D^1 |_{\theta_1} + I_1 D^1 |_{\theta_1} & O_{3 \times 3} \\ I_1 D^1 |_{\theta_1} - I_1 D^1 |_{\theta_1} \end{bmatrix}
\]

The external force parameter matrix of rigid bodies 8–13 is shown in equation (52):

\[
f^b_i = -[f_{d,i}, 0, 0, 0, 0]^T \quad (i = 8, \cdots, 13),
\]

where, \( f_{d,i} \) represents the damping force provided by the MRD.

The external force parameter matrix of rigid bodies 20–25 is shown in equation (53):

\[
f^b_i = [f_{d,i} + F_{x,i}, F_{y,i}, F_{z,i}, M_{x,i}, M_{y,i}, M_{z,i}]^T \quad (i = 20, \cdots, 25),
\]

where, \( F_{x,i}, F_{y,i}, F_{z,i}, M_{x,i}, M_{y,i}, M_{z,i} \) are external forces and external torques generated by the displacement excitation of hinges 26–31 in the body fixed frame of each rigid body.

The body dynamics equation of rigid body 1 in inertial coordinate system is the same as that of equation (47). The body dynamics equation of rigid bodies 8–13, 20–25 needs to be transformed into the body dynamics equation in the inertial coordinate system through the direction cosine matrix, shown in equation (54): \( M_i v_{i,0} + K_i v_i = f_i \) \((i = 8, \cdots, 13, 20, \cdots, 25)\),

where, \( M_i = R_i^b M_i^b (R_i^b)^T \); \( K_i = R_i^b K_i^b (R_i^b)^T \); \( v_i = R_i^b v_i^b \); \( R_i^b \) is the direction cosine matrix of the body fixed frame and the inertial coordinate system.

(3) System body dynamics equation.

The body dynamics equation of the system is established from the body dynamics equation of the element, shown in equation (55):

\[
M v_t + K v = \sum_{j=1}^{12} (\Omega_j f_{d,j} + \Phi_j w_j),
\]

where, \( M = \text{diag}(M_1, M_8, \cdots, M_{13}, M_{20}, \cdots, M_{25}) \) and \( K = \text{diag}(K_1, K_8, \cdots, K_{13}, K_{20}, \cdots, K_{25}) \) are system mass parameter matrix and stiffness parameter matrix, respectively; \( v = [v_1^T, v_8^T, \cdots, v_{13}^T, v_{20}^T, \cdots, v_{25}^T]^T \) represents the displacement state vector; \( f_{d,j} \) and \( w_j = [F_{x,j}, F_{y,j}, F_{z,j}, M_{x,j}, M_{y,j}, M_{z,j}]^T \) respectively represent the damping force of the MRD and external force; \( \Omega_j \) and \( \Phi_j \) respectively represent the position and direction of damping force and external force.

4.6.2. State equation. According to linear MSTMM [29], the augmented eigenvector of the system is selected as equation (56):

\[
V^k = [V_1^{kT}, V_8^{kT}, \cdots, V_{13}^{kT}, V_{20}^{kT}, \cdots, V_{25}^{kT}]^T \quad (k = 1, 2, \cdots),
\]

where, \( k \) is the modal order.

The orthogonality of the augmented eigenvector [22] can be obtained, shown in equation (57):

\[
\begin{align*}
MV^k, V^* & = \delta_{kk} M^k \\
KV^k, V^* & = \delta_{kk} K^k
\end{align*}
\]

where, \( M^k \) represents the \( k \)th modal mass; and \( K^k = \omega_k^2 M^k \) represents the modal stiffness; when \( k = s, \delta_{ss} = 1 \); when \( k = s, \delta_{ss} = 0 \).

The system displacement vector of the body in the physical coordinates is expressed by augmented eigenvector, shown in equation (58):

\[
v \approx \sum_{k=1}^{m} V^k q^k(t),
\]

where, \( q^k(t) \) represents the generalized coordinates of the system; \( t \) is time.

Substituting equations (58) into (55) obtains equation (59):

\[
\sum_{k=1}^{m} MV^k \ddot{q}^k(t) + \sum_{k=1}^{m} KV^k q^k(t) = \sum_{j=1}^{12} (\Omega_j f_{d,j} + \Phi_j w_j).
\]

Making the inner product of equation (59) and \( V^s (s = 1, 2, \cdots m) \) obtained equation (60):

\[
M^s \dot{q}^k(t) + K^s q^k(t) = \sum_{j=1}^{12} (\langle \Omega_j, V^s \rangle f_{d,j} + \langle \Phi_j, V^s \rangle w_j).
\]
From equations (59) and (60), the differential equation of motion in generalized coordinates obtained, shown in equation (61):

\[ q^k(t) + \omega^2_k q^k(t) = b^k u + \tau^k w \quad (k = 1, \ldots, m), \]  

(61)

where, \( b^k = [b_{x}^{k}, \ldots, b_{1}^{k}] \), \( b_{i}^{k} = \langle \Omega_{i}, V^k \rangle / M^k (j = 1, \ldots, 12) \) are the position and direction of the damping force; \( u = [u_{01}, \ldots, u_{12}]^T \) is the damping force of MRD; \( \tau^k = [\tau_{x}^k, \ldots, \tau_{12}^k] \), \( \tau_{i}^k = \langle \Phi, V^k \rangle / M^k \) are the position and direction of the external force; \( w = [w_{1}^1, \ldots, w_{12}^1]^T \) is the external force caused by excitation.

The external force matrix of rigid body 8–13 is shown as equation (62):

\[
w = \begin{bmatrix} q_1^1, q_2^1, q_{1\times6}, q_{1\times6}, q_{1\times6}, q_{1\times6}, \ldots, q_{1\times6}, q_{1\times6}, \ldots, q_{1\times6}, q_{1\times6}, q_{1\times6}, q_{1\times6} \end{bmatrix}^T.
\]

(62)

The initial conditions of system generalized coordinates are shown in equation (63):

\[
\begin{align*}
q^k(0) &= \begin{bmatrix} (v(0), M V^k) \\ M^k \end{bmatrix} \\
\dot{q}^k(0) &= \begin{bmatrix} (\dot{v}(0), M V^k) \\ M^k \end{bmatrix} \\
(k &= 1, \ldots, m).
\end{align*}
\]

(63)

Taking the modal state variable is shown as equation (64):

\[
x^k = [q^k, \dot{q}^k]^T.
\]

(64)

State equation is shown as equation (65):

\[
x = Ax + Bu + \Gamma w,
\]

(65)

where

\[
x = \begin{bmatrix} x^1 \\ \vdots \\ x^m \end{bmatrix}, A = \text{diag}(A^1, A^2, \ldots, A^m), B = \begin{bmatrix} B^1 \\ \vdots \\ B^m \end{bmatrix},
\]

\[
A^k = \begin{bmatrix} 0 & 1 \\ -\omega^2_k & 0 \end{bmatrix}, B^k = \begin{bmatrix} \theta_{1\times12}^k \\ b^k \end{bmatrix},
\]

\[
\Gamma^k = \begin{bmatrix} \theta_{1\times72}^k \\ \Gamma^k \end{bmatrix} \quad (k = 1, \ldots, m).
\]

(66)

4.6.3. Output equation. The motion state of the upper platform of the MRD 6-DOF vibration isolation system is the main evaluation index of vibration isolation performance. Therefore, the output point displacement of rigid body 1 is taken as the output variable of the system, and the output equation is shown as equation (67):

\[
y = [x, y, \theta_{\alpha}, \theta_{\beta}, \theta_{\gamma}, \theta_{\delta}, \theta_{\epsilon}, \theta_{\zeta}]^T = \begin{bmatrix} I_3 & -I_6 \theta \phi \psi \theta \end{bmatrix} y = : (x, v_1)
\]

(67)

\[
\chi = [x_1^T, \ldots, x_6^T]^T, x_6(l = 1, \ldots, 6) \text{ represents the position and direction of the } \ell \text{th row output.}
\]

Making \( X_i = [x_i, \theta_{1,72}]^T \) obtains equation (68):

\[
y_i = \sum_{k=1}^{m} c^k q^k(t).
\]

(68)

Equation (69) can be obtained from equations (58) and (68):

\[ y_i = \sum_{k=1}^{m} c^k q^k(t), \]

(69)

where

\[ c^k = \langle X_i, V^k \rangle. \]

(70)

The output equation is shown (71):

\[ y = C x, \]

(71)

where

\[
C = [C_1^T, \ldots, C_6^T]^T, \quad C_l = [c_1^l, \ldots, c_6^l],
\]

\[
c_l^l = [c_l^k, \ldots, 0] \quad (l = 1, \ldots, 6).
\]

(72)

According to the above dynamic modeling, the transfer matrix method of multi-body system shows the following characteristics:

1. The overall transfer equation of the system is composed of the transfer equation of each element according to the corresponding transfer relationship, which makes the whole dynamic modeling process modular and simplified.

2. By introducing the volume dynamics equation, the complex process of formulating the system’s overall dynamics equation is avoided.

5. Semi-active control of 6-DOF vibration isolation system

5.1. LQG controller

The semi-active vibration isolation of the MRD 6-DOF isolation system was realized by adjustment of the damping force, which is achieved by the current control. Thus, designing the control strategy of system is of great necessity.

LQG control strategy is a mature state space design method developed in modern control theory. Its largest advantage is that the optimal control law with linear state feedback can be obtained [33, 34]. The block diagram of LQG controller is shown as figure 15.
The working performance of the MRD 6-DOF vibration isolation system is mainly reflected by the output end state of the upper platform. So the objective performance function of LQG control strategy is defined as equation (73):

$$J = \lim_{T \to \infty} \int_0^T (y^T Q_y y + u^T R u)dt.$$  \hspace{1cm} (73)

Substituting equations (71) into (73) obtains equation (74):

$$J = \lim_{T \to \infty} \int_0^\infty (x^T Q x + u^T R u)dt,$$  \hspace{1cm} (74)

where, $Q_y$ is the weighted matrix of the output variables and a positive semidefinite matrix; $R$ is the weighted matrix of the input variables and a positive definite matrix; $Q = C^T Q_y C$.

LQG controller is composed of a state estimator and the state feedback control law, as shown in figure 15. Kalman filter is used to estimate the state variables, shown in equation (75):

$$\dot{x} = Ax + Bu + K_e(y - C\hat{x}),$$  \hspace{1cm} (75)

where $K_e$ is the estimator gain matrix.

The state space expression of the controller is shown in equation (76):

$$\dot{x} = (A - BK_c - K_c C)\hat{x} + K_c y,$$

$$u = -K_c \hat{x},$$  \hspace{1cm} (76)

where, $K_c$ is the control gain.

$L_e$ and $K_c$ are unknown quantities, which can be obtained by solving the following two algebraic Riccati equations, shown in equations (77) and (78):

$$AS_e + S_e A^T - \frac{1}{R} S_e C^T C S_e + \bar{Q} = 0,$$  \hspace{1cm} (77)

$$S_e A + A^T S_e - \frac{1}{R} S_e B B^T S_e + \bar{Q} = 0,$$  \hspace{1cm} (78)

where, $\bar{Q}$ and $\bar{R}$ are weight matrix of the observer, the definition of which is similar to that of $Q$ and $R$.

The observer gain and the control gain are shown in equations (79) and (80):

$$K_c = \frac{1}{R} S_e C^T,$$  \hspace{1cm} (79)

$$K_c = \frac{1}{R} B^T S_e,$$  \hspace{1cm} (80)

5.2. LQG semi-active control

The damping force of MRD can be tracked by current adjustment only when the active control force is within the

### Table 4. The main parameters of the MRD 6-DOF vibration isolation system.

| Parameter | Value | Parameter | Value |
|-----------|-------|-----------|-------|
| $m_1$ (kg) | 235.62 | $\delta_{20}$ (m) | 0.05 |
| $m_2$ (kg) | 0.312 | $k_v$ (N m$^{-1}$) | 12 000 |
| $m_3$ (kg) | 0.585 | $k_e$ (N m$^{-1}$) | 300 000 |
| $r_1$ (m) | 0.3 | $k_e$ (N m$^{-1}$) | 200 000 |
| $r_2$ (m) | 0.25 | $k_v$ (N m$^{-1}$) | 400 000 |
| $h_1$ (m) | 0.15 | $k_v$ (N m rad$^{-1}$) | 6000 |
| $h_2$ (m) | 0.34 | $k_v$ (N m rad$^{-1}$) | 4000 |
| $a_{8}$ (m) | 0.04 | $k_v$ (N m rad$^{-1}$) | 5000 |

### Table 5. The first 15 natural frequencies.

| Mode | 1 | 2 | 3 | 4 | 5 |
|------|---|---|---|---|---|
| $\omega$ | 23.57 | 36.11 | 53.50 | 77.22 | 359.47 |
| Mode | 6 | 7 | 8 | 9 | 10 |
| $\omega$ | 601.59 | 668.50 | 729.17 | 729.64 | 791.95 |
| Mode | 11 | 12 | 13 | 14 | 15 |
| $\omega$ | 943.63 | 944.36 | 944.93 | 1001.48 | 1001.59 |

The working performance of the MRD 6-DOF vibration isolation system is mainly reflected by the output end state of the upper platform. So the objective performance function of LQG control strategy is defined as equation (73):

$$J = \lim_{T \to \infty} \int_0^\infty (y^T Q_y y + u^T R u)dt.$$  \hspace{1cm} (73)

The block diagram of LQG controller is shown in figure 15. The structure of the LQG semi-active control system is shown in figure 16.
controllable range (the damping force of MRD is saturated at its maximum working current). According to the relationship between the active control force and the leg velocity of the 6-DOF vibration isolation system, a force limiter is designed.

\[\begin{align*}
  x &= 0.01 \sin(10\pi t) \\
  y &= 0.01 \sin(16\pi t) \\
  z &= 0.01 \sin(10\pi t)
\end{align*}\]

\[\begin{align*}
  \theta_i &= 0.001 \sin(12\pi t) \\
  \theta_r &= 0.001 \sin(14\pi t) \quad \text{(rad)}, \\
  \theta_c &= 0.001 \sin(12\pi t)
\end{align*}\]

It is described in equation (81):

\[f_d = \begin{cases}
  F(l_{\text{max}}) \quad &\text{if } F(l_{\text{max}}) > 0 \text{ and } F(l_{\text{max}}) > |F(l_{\text{max}})| \\
  F(l_{\text{max}}) \quad &\text{if } F(l_{\text{max}}) < 0 \text{ and } F(l_{\text{max}}) < |F(l_{\text{max}})| \\
  0 \quad &\text{otherwise}
\end{cases}\]

(81)

where, \(f_d\) is the damping force of the MRD; \(F(l_{\text{max}})\) is the damping force of the MRD at the maximum input current; \(F(l_{\text{max}})\) is the damping force of the MRD at the maximum input current; \(F_r\) is the optimal control force of the controller; \(\dot{x}_b, \dot{x}_u\) represent the sprung mass and unsprung mass velocity, respectively.

The structure of the LQG semi-active control system of the MRD 6-DOF vibration isolation system is shown in figure 16. The semi-active control system is composed of a system controller and a damper controller. The system controller calculates the expected damping force according to the response of the MRD 6-DOF vibration isolation system. The damper controller enables the MRD to track the expected damping force by adjusting the control current. The execution flow is as follows: firstly, the controller calculates the active control force according to the measurement output, and inputs it into the force limiter to generate the expected damping force. Then, according to the expected damping force and the response of the 6-DOF vibration isolation system, the expected input current is calculated by the reverse model of the MRD and input into the MRD. Finally, the expected damping force is approximately realized by the MRD, and the purpose of 6-DOF vibration isolation is achieved. The inverse Bingham model was used to calculate the expected control current, as shown in equation (13).

6. Numerical results and discussion

The main parameters of the MRD 6-DOF vibration isolation system are calculated. The first 15 natural frequencies of the system are shown in table 5. The displacement excitation is shown in equation (82).

\[\text{The excitation is applied to the geometric center of the upper surface of the lower platform}\]

Because the system response is solved by the modal superposition method in MSTMM. It is an essential step to determine the order of the mode so that the system has the stationary solution. The first 9, 11 and 15 modes are used to solve the response of the passive system, and the solution results are shown in figure 17. Figures 17(a)–(f) are the linear and angular displacements of the output points of the upper platform in the directions of \(x, y, z\) respectively. The results show that the system response of the first 9, 11 and 15 order modes tend to be stable. So the response of the MRD 6-DOF vibration isolation system is accurately calculated.

Figure 18 is the simulation results of the MRD 6-DOF vibration isolation system based on the LQG semi-active control, the MRD 6-DOF vibration isolation system based on the PID semi-active control and the passive 6-DOF vibration isolation system. Figures 18(a)–(f) represent the simulation results of linear and angular displacement along \(x, y, z\) directions of the upper platform in the MRD 6-DOF vibration isolation system, respectively. And table 6 shows the RMS values of the results.

From figure 18 and table 6, it can be seen that the RMS values of the linear and angular displacements of the output point in \(x, y, z\) directions are respectively reduced by 51.85%, 58.71%, 64.43%, 75.43%, 66.45% and 52.87% compared with the passive 6-DOF vibration isolation system. When compared with the PID semi-active control, the percentage drops are 50.00%, 56.16%, 34.94%, 53.09%, 38.14% and 46.76%, respectively. It indicates that the MRD 6-DOF vibration isolation system with the LQG semi-active control is superior to that of other two cases. Furthermore, the MRD makes the 6-DOF vibration isolation system performs better in the actual application.

Figures 19(a)–(f) shows the relationship between the working current of six legs (MRDs) and time under LQG semi-active control. It can be seen that the working current of MRDs varies in the range of 0–1 A, showing obvious limitations of maximum and minimum. It indicates that the calculated current by LQG semi-active control is consistent with the working current of MRD design. It further proves the effectiveness of LQG semi-active control algorithm. Figure 20 shows the relationship between the elongation of six legs (MRDs) and time based on the optimal control algorithm. It proves the correctness of the numerical calculation, the elongation for the of six legs (MRDs) varies from \(-0.020\) to 0.020 m, which was in the working stroke range.
of MRDs. The relationship between the damping forces of the LQG semi-active control and time is shown in figure 21. It can be seen that the curves are not smooth, indicating that the dynamic model of the MRD is highly nonlinear and can realize semi-active control. Therefore, the six legs of the MRD 6-DOF vibration isolation

Figure 17. The response curve of first 9, 11 and 15 mode.
Figure 18. The response curve.
Figure 19. The Working Current of MRDs.

Table 6. The RMS values of results.

| Index   | $x$ (m) | $y$ (m) | $z$ (m) | $\Theta_x$ (rad) | $\Theta_y$ (rad) | $\Theta_z$ (rad) |
|---------|---------|---------|---------|------------------|------------------|------------------|
| PASSIVE | 0.0027  | 0.0155  | 0.0280  | 0.0464           | $6.871 \times 10^{-5}$ | $9.792 \times 10^{-5}$ |
| PID     | 0.0026  | 0.0146  | 0.0166  | 0.0243           | $3.726 \times 10^{-5}$ | $8.668 \times 10^{-5}$ |
| LQG     | 0.0013  | 0.0064  | 0.0108  | 0.0114           | $2.305 \times 10^{-5}$ | $4.615 \times 10^{-5}$ |
The system can produce the corresponding damping force according to the real-time attitude of the upper platform.

7. Conclusions

A new MRD 6-DOF vibration isolation system based on Stewart mechanism was designed to improve the performance of the 6-DOF vibration isolation system. The linear MSTMM was first used to establish the dynamic model of the MRD 6-DOF vibration isolation system. It has few equations, high computational efficiency and no need for overall dynamic equations. To make it perform well in the practical application, a real-time and effective LQG semi-active control method was proposed according to the dynamic characteristics of MRD. The research in this paper can provide a theoretical basis for the relevant vibration isolation test, and the related verification test will be described in the next article. The main works and conclusions were as follows:

1. The mechanical tests showed that the MRD presented good mechanical properties, the adjustable coefficient of damping force can reach 10 when applying current. Based on the experimental result, parameters of the dynamic model were identified using the artificial fish swarm algorithm, and the electromagnetic coupling model and inverse model (current output model) of the MRD were established, respectively.

2. Linear MSTMM, which has fewer equations, higher computational efficiency and without overall dynamic equations, was adopted to establishing the dynamic model of the MRD 6-DOF vibration isolation system. The vibration characteristics of the system were calculated. The state equation was derived by the augmented eigenvectors and the body dynamics equation combined with the modal superposition method.

3. Numerical results showed that the response of the 6-DOF vibration isolation system for the first 9, 11 and 15 modes solutions tended to be stable and can accurately display the real-time attitude of the output point. The MRD based on LQG semi-active control can produce the corresponding damping force according to the real-time attitude of the output point in the 6-DOF vibration isolation system. Compared with the passive system and the MRD 6-DOF vibration isolation system with PID semi-active control, the MRD 6-DOF vibration isolation system with LQG semi-active control showed better vibration isolation performance.

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