Thermodynamics of a Schwarzschild-like black hole with a minimum observable length and the radiation process of a thin accretion disc around it

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We study quantum gravity effects on the thermodynamic character and the radiation process of the thin accretion disks around Schwarzschild-like black hole. The quantum gravity correction is invoked through the framework of generalization of uncertainty which is equivalent to the renormalization group improved quantum gravity and maintain the limit of the asymptotically safe preposition of gravity. It admits a free parameter that encodes the quantum effects on the spacetime geometry. It allows us to study how the thermal properties of the black hole itself and the the accretion around it disk are modified in the quantum regime. We computed explicitly the entropy, temperature, free energy, and enthalpy of the modified black hole and show its variation with with the free parameter that encodes the quantum effects. We explicitly make estimations of quantum correction to the time averaged energy flux, the temperature of the disk, the differential luminosity, and the conversion efficiency of accreting mass into radiation. We observe a conspicuous shifting of the radius of the innermost stable circular orbit (ISCO) toward small values together with an enhancement of the maximum of the values of the average thermal radiation and greater conversion efficiency of accreting mass into radiation compared to the classical gravity scenario.

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I. INTRODUCTION

The black hole is a remarkable solution to the classical field equations thanks to Einstein within the context of general relativity theory. In an astrophysical sense, it is the regions of spacetime that gets deformed undergoing gravitational collapse. The solutions of the equation are inflected with singularities inside the event horizon. Therefore it loses its prophetic character and it is no longer possible to extend classical formulation of spacetime inside the event horizon. This forces to think about general relativity as an effective theory of gravity that remains valid solely up to certain energy scales. At high energy scales, such as the Planck scale, it is expected that a full theory of quantum gravity will resolve the unphysical singularities in the spacetime manifold. So the predictively be rehabilitated. However, attempts of describing gravity within the framework of quantum theory face the dander of the perturbative non-renormalizable character of general relativity. As a consequence, different approaches have surfaced, namely a loop quantum gravity spin foams [1–5], string theory [6–8], etc.. Another promising proposal to deal with this downside is that the Asymptotic Safety scenario which uses the useful techniques of the functional renormalization group. The existence of a non-Gaussian fixed point of the gravitational renormalization group flow that controls the behavior of the idea at trans-Planckian energies is that the main speculation within this construction. The physical degrees of freedom interact predominantly with anti-screening within the neighborhood of the non-Gaussian fixed point which renders physical quantities safe from unphysical divergences within the vicinity of the Plank scale which renders...
physical quantities safe from unphysical divergences. It is the fact that asymptotic safety defines a harmonious and prognostic scientific proposition among the frame of quantum field theory we couldn’t ignore its eventualty, still, it stands as a vaticination since a rigorous evidence for the existence of the non-Gaussian fixed point continues to be lacking. It is the fact that AS defines a consistent and prognostic scientific theory for gravity among the framework of quantum theory we could not ignore its potential, however, it stands as a prediction since a rigorous existence proof for the NGFP continues to be lacking. There is, however, substantial evidence supporting the existence of the non-trivial renormalization group fixed point at the center of this construction. The importance of black holes as testing ground for gravity theories within the strong field regime has motivated numerous studies on the implications of asymptotic safety for black hole physics, most of them geared toward determining quantum corrections to the classical metrics.

The generalization of uncertainty is a fascinating extension that has its origin in string theory and loop quantum gravity. In the article, the author analyzes quantum gravity corrections to the accretion onto black holes within the context of asymptotically safe gravity. The asymptotic safety was maintained there by invoking the running Newton’s constant proposition. The identical asymptotic safety could be contemplated within the Generalize Uncertainty Principle (GUP) framework which we will attempt to explore and study the physical quantities connected to the accretion process maintaining AS during this composition. The usage of generalized scientific theory based on GUP, which renders asymptotic safety, is used here to account for quantum gravity corrections to the Schwarzschild metrics. Quantum spacetime with running gravitational coupling that is included into the scenario gets manifested in effective mass.

Investigating the gravity-induced quantum interference pattern, followed by the Experiment with Gedanken-experiment to determine the weight of photon it has been established that the running Newton’s law applies to photons. Examining the quantum interference pattern caused by gravity, and then the thought Gedanken-experiment for determining the weight of a photon, it has been established that the running Newton gravitational constant can be stimulated by the principles of generalized uncertainty, which ends up in quantum gravity corrections to Schwarzschild region metrics. The enhanced quantum-corrected metric is used here cherish the Schwarzschild metric indeed and will be used to study the quantum-corrected thermodynamics of the black hole and the thermal radiation process of thin accretion disc around the black hole.

The paper is organized as follows. In Sec. II we amend quantum gravity correction to the the Schwarzschild metric by the use of GUP and study the impact of GUP on the nature of horizon of this GUP inspired quantum gravity corrected Schwarzschild metric. In Sec. III we describe the the thermodynamical characteristics of this modified black hole. In Sec. IV is devoted to the Description of the geodesic in the GUP inspired quantum corrected spacetime geometry. In Sec. V, we study the impact of GUP parameter in Mass accretion rate and differential luminosity for the thin accretion Sec. VI contains a brief summary and discussion.

II. INSERTION OF QUANTUM GRAVITY EFFECT INTO THE SCHWARZSCHILD METRIC

Before jumping into the formulation of having a modified black hole endowed with quantum gravity effect through GUP perspective it would be beneficial to give a brief description of GUP. Let us turn into that.

A. Description GUP with minimum measurable length

In recent times it has been noticed that various approaches to quantum gravity including string theory, noncommutative geometry, loop quantum gravity, predict the existence of a minimum measurable length of the order of Planck length. It leads to different generalizations of usual uncertainty relation in the context of quantum gravity, although Heisenberg uncertainty principle is the cornerstone of the formulation of quantum theory that puts a fundamental limit on the precision of measuring the position and momentum. In the Heisenberg uncertainty principle, $\Delta x \to 0$ leads to $\Delta p \to \infty$, therefore, the standard Heisenberg uncertainty relation $\Delta x \Delta p \geq \hbar$ becomes skimpy to explain the existence of a minimum measurable length. Thus, it necessitates the replacement of Heisenberg uncertainty principle by the Generalized Uncertainty Principle (GUP) to accommodate the possibility of minimum measurable length. In the article, Kempf showed that a generalized uncertainty relation could be defined by

$$\Delta x_\perp \Delta p_\perp \geq \hbar \delta_{kl} \left( 1 + \eta \left( (\Delta p)^2 + \langle p \rangle^2 \right) \right), \quad (1)$$

where $\eta$ is GUP parameter which has the definition $\eta = \frac{\eta_0}{M_{PL} c^2}$, where $M_{PL}$ is the Planck mass and $\eta_0$ is parameter of the order of unity. The expression has the potential to accommodate the minimum measurable length within
the revised principle. It is straightforward to see that for the above GUP, the minimum non-zero length is found out to be

$$(\Delta x)_{Min} = h\sqrt{\eta} \sqrt{1 + \eta \langle p \rangle^2},$$

(2)

where setting $\langle p \rangle = 0$ results to the absolute minimal measurable length:

$$(\Delta x)_{Min} = h\sqrt{\eta} = \sqrt{\eta} l_{Pl},$$

(3)

where $l_{Pl} = \left(\frac{G \hbar c^3}{2}\right)^{\frac{1}{2}} \approx 10^{-35} m$ is the Planck length. This generalized uncertainty corresponds to the following deformed commutation relation between position and momentum

$$[x_k, p_l] = i\hbar \delta_{kl} (1 + \eta p^2),$$

(4)

where $p^2 = \sum_k p_k^2$. What follows next is an attempt to have a modification of the relation (1) and (4) in one dimension by

$$\Delta x \Delta p \geq \hbar (1 + \eta (\Delta p)^2),$$

(5)

$$[x, p] = i\hbar (1 + \eta p^2) = i\hbar \zeta,$$

(6)

where $\zeta = 1 + \eta p^2$. Hence, Eqn. (5) can be rewritten as

$$\Delta x \Delta p \geq \hbar \zeta.$$  

(7)

B. Schwarzschild metric endowed with quantum gravity correction

Let us now formulate a Schwarzschild-like spacetime metric where quantum gravity correction gets induced through generalized uncertainty principle keeping in sight on the Gedanken-experiment initially proposed by Einstein. Einstein made a trial to exhibit the violation of the indeterminacy principle through a Gedanken-experiment which was purported to measure the load of photons [26–28]. He assumed a box containing photon gas with a totally reflective wall was suspended by a spring scale. There was a system inside the box that caused the shutter to open and shut at moment $\tau$ for the time interval $\Delta t$, which allowed to passing out just one photon. A clock capable of showing extremely high precision measurement of your time could be accustomed measure the interval $\Delta \tau$, and at the same time the mass difference of the box would determine the energy of the emitted photon as per Einstein’s assumption, the amount required for photon radiation is precisely $\Delta \tau \rightarrow 0$ which may be lead to the violation of the uncertainty relation for energy and time, i.e. $\Delta E \Delta \tau \rightarrow 0$. However Bohr argued that [27], Einstein’s deduction wasn’t flawless since he neglected the time-dilation effect which might play a significant role due to the difference of gravitational potential. Based on general relativity, when altitude changes, the speed of your time flow also changes thanks to the change in their gravitational potential. Thus, for the put down the box, the time uncertainty $\Delta \tau$, in terms of the vertical position uncertainty $\Delta x$, would be expressed as

$$\Delta \tau = g \Delta x c^2 \tau,$$

(8)

where $\tau$ represents the time period of weighing the photon. As it is known, according to the quantum theorem, the uncertainty relation in energy and time of the photon is express as $\Delta E \Delta \tau \geq \hbar$ which after substituting Eqn. (8) turns into

$$\Delta E \geq \frac{\hbar c^2}{gt \Delta x}.$$  

(9)

Let us now look at the relation between the weight of the photon in the Gedanken-experiment and the corresponding quantities in quantum mechanics in the GUP framework [11]. If we focus on the original position of the pointer on the box before opening the shutter, we will find that after releasing photon in the box the pointer moves up with reference to its original position. To get back the pointer in its original position in a period of time $\tau$, some weights equal to the weight of the photon must be added to the box. If we now use Eqn. (9), having accuracy in measuring the position $\Delta x$ as marked by the indicator of the clock, the minimum uncertainty in momentum $\Delta P_{Min}$ will be

$$\Delta P_{Min} = \frac{\zeta \hbar}{\Delta x},$$

(10)
since the quantum weight limit of a photon is \( g\Delta m \), so in a period of time \( \tau \) the smallest photon weight will be equal to \( \zeta h/\tau \Delta x = \Delta p_{Min}/\tau \leq g\Delta m \). Now, using Eqn. (10) we also find that

\[
\zeta h = \Delta x \Delta p_{Min} \leq g\tau \Delta x \Delta m.
\]

If the relation \( \Delta E = c^2\Delta m \), is used for \( \Delta m \) the Eqn. (11) can be written down as

\[
\Delta E \geq \frac{\zeta \hbar c^2}{\tau \Delta x}.
\]

where \( \zeta \) refers to the the GUP effects. Note that in the absence of GUP framework, i.e. when \( \eta \rightarrow 0 \), the standard energy-time uncertainty relation Eqn. (9) is reobtained.

A careful look on the standard uncertainty relation between energy and time (9) and the generalized uncertainty relation (17) gives rise to an interpretation that gravitational field strength \( g \) is modified to \( \bar{g} \). Then, replacing \( g \) by \( \bar{g} \) we have

\[
\bar{g} = \frac{g}{\zeta} = \frac{G_0 M}{\zeta R^2}.
\]

Hence, using (13), the modified Schwarzschild metric turns into

\[
d s^2 = - \left( 1 - \frac{2G_0 M}{\zeta c^2 r} \right) c^2 dt^2 + \left( 1 - \frac{2G_0 M}{\zeta c^2 r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \( G_0 \) stands for universal gravitational constant. On the other hand, as stated in some other literature \[26, 30\], when two virtual particles with energies \( \Delta E \) are at a distance \( \Delta S \) from each other, the tidal force between them is obtained by

\[
F = \frac{2G_0 M \Delta E}{r^3} \Delta x.
\]

So, the uncertainty in momentum is given by

\[
\Delta p = F \Delta t = \frac{2G_0 M \Delta E}{r^3} \Delta x \Delta t,
\]

where \( \Delta t \) represents the life time of the particle.

If virtual particles turns into real particles having the exposure of tidal force, the uncertainty relations \( \Delta p \Delta x \geq \hbar \) and \( \Delta E \Delta t \geq \hbar \) can be used with reasonably well justifiable manner. Therefore, using these uncertainty relations in (10), we find that

\[
(\Delta p)^2 \geq \frac{2\hbar^2 G_0 M}{c^2 r^3}.
\]

Accordingly, it can be written that \( p^2 \approx (\Delta p)^2 \approx \frac{2\hbar^2 G_0 M}{c^2 r^3} \). Hence applying this modified uncertainty relation, we obtain the Schwarzschild metric in the presence of minimal measurable length as

\[
d s^2 = - \left( 1 - \frac{2G_0 M r^2}{c^2 (r^3 + \eta 2\hbar^2 G_0 M)} \right) c^2 dt^2 + \left( 1 - \frac{2G_0 M r^2}{c^2 (r^3 + \eta 2\hbar^2 G_0 M)} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\]

This metric specifies a set of spacetime that depends on different scales of momentum via the modified mass of the black hole. For dimensionless case, the modified Schwarzschild metric (18) reads

\[
d s^2 = - \left( 1 - \frac{2M r^2}{r^3 + 2\eta M} \right) dt^2 + \left( 1 - \frac{2M r^2}{r^3 + 2\eta M} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\]

In the following sections, for the sake of simplicity, we use the modified Schwarzschild metric (18) in the form of

\[
d s^2 = - f(r\eta) c^2 dt^2 + \frac{1}{f(r\eta)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]
where
\[ f(r \eta) = 1 - \frac{2 G_0 M r^2}{c^2 \left( r^3 + \eta \frac{2 \hbar^2 G_0 M}{c^2} \right)} = 1 - \frac{2 M G_0}{r^3 + 2 \eta \hbar^2 (\frac{M G_0}{c^2})} = 1 - \frac{2 M}{r^3} G(r \eta), \]

where
\[ G(r \eta) \equiv \frac{G_0 r^3}{r^3 + 2 \eta \hbar^2 (\frac{M G_0}{c^2})}. \]

So for \( \eta \to 0 \), \( G(r \eta) \to G_0 \) and the quantum correction disappears and we get back to the Schwarzschild metric. If we consider the metric in equation in natural unit setting \( c = 1 \) and \( G_0 = 1 \) in \( \text{[19]} \) to find out the position of the Horizon we need to have the solution of the equation
\[ 1 - \frac{2 M r^2}{r^3 + 2 \eta M} = 0, \]

which has the solution
\[ r_H = \frac{2}{3} M + \frac{4}{3} M \cos \frac{1}{3} \cos^{-1} \left( 1 - \frac{27}{8} \frac{\eta}{M^2} \right), \]

provided the mass of the black hole satisfy the condition \( M > M_c \), where \( M_c = \frac{27}{16} \eta \). When \( M \leq M_c \) it fails to describe any horizon in the spacetime geometry since equation \( \text{[23]} \) can not provide any positive solution in that situation. Plots of the improved metric coefficient \( f(r) \) for different value of mass \( M \) with \( \eta = \frac{16}{27} \) and \( \eta = 0 \) (classical) are given below.

![Plots of the improved metric coefficient](image)

**FIG. 1:** Plots of the improved metric coefficient \( f(r \eta) \) for \( M = 0.4 < M_c \) (yellow), \( M = M_c \) (red), and \( M = 1.8 > M_c \) (blue), with \( M_c = 1 \) corresponding to \( \eta = \frac{16}{27} \). The black line shows the classical \( f_0(r) \) for \( M = 1.8 \) corresponding to \( \eta = 0 \) and green line shows the effect with \( \tilde{M} = 0 \).

To see the nature of the horizon closely and the thermodynamical characteristics of this gup-inspired black hole we rewrite the Lapse function in the following form.
\[ f(r) = 1 - \frac{2 G_0 M r^2}{r^3 + 2 G_0 M r^2 \hbar^2} = \frac{K(r, r_{sr}, \eta)}{r^3 + \tilde{\eta}^2 r_{sr}} \]

where
\[ K(r, r_{sr}, \eta) = r^3 - r_{sr} r^2 + \tilde{\eta}^2 r_{sr} \]

\[ r_{sr} = \frac{2 G_0 M}{c^2}, M \text{ is the gravitational energy-mass of the system and } \tilde{\eta} = \eta \hbar^2, \text{ The lapse function can be expand as follows} \]
\[ f(r) \simeq 1 - \frac{2 G_0 M}{c^2 r} + \frac{4 G_0^2 M^2 \tilde{\eta}^2}{r c^4} - ..... \]
which is asymptotically flat. But, when \( r << \eta \), it becomes similar to that of de Sitter i.e.,

\[
f(r) \simeq 1 - \frac{r^2}{\eta^2} + \frac{c^2 r^5}{2G_0 M \eta^4} - \ldots
\]  

(28)

So it is flat and regular at \( r = 0 \).

And, accordingly Eqn. (16) and (18) can be expressed in terms of \( \eta \). In natural unit \( \hbar = 1 \). With this approximation the possible horizons is obtain seating \( (K = \eta) \). Eqn. (26) is containing the polynomial expression of \( (K) \). To have a solution using Carnado’s method we define \( r = z + \frac{2G_0 M}{3c^2} = z + \frac{r_{sr}}{3} \). Eqn. (26) can be written in its canonical form as

\[
K(r; r_{sr}, \eta) \equiv z^3 - \frac{\xi_1}{4} z - \left( \frac{\xi_2}{4} \right)
\]  

(29)

where \( \xi_1 \) and \( \xi_2 \) are the Weierstrab invariants given by

\[
\xi_1 = \frac{4r_{sr}^2}{3} = \frac{4(2G_0 M)^2}{3c^4} > 0
\]  

(30)

\[
\xi_2 = \frac{8r_{sr}^3}{27} \left[ 1 - \left( \frac{\hbar^2 \eta^2}{\eta^2} \right)^2 \right] = \frac{8(2G_0 M)^3}{27c^6} \left[ 1 - \frac{\eta^2}{\tilde{\eta}_0^2} \right]
\]  

(31)

And \( \tilde{\eta}_0 \) was conveniently defined as

\[
\tilde{\eta}_0 \equiv \sqrt{\frac{2}{27}} r_{sr} = \sqrt{\frac{2}{27}} \frac{(2G_0 M)}{c^2}
\]  

(32)

Accordingly, the cubic discriminant associated with the polynomial (29) becomes

\[
\Delta_c = \frac{g_2^3}{16} \left[ 1 - \left( \frac{\xi_1}{4} \right)^2 \right]
\]  

(33)

which vanishes trivially at

\[
\tilde{\eta} = \tilde{\eta}_1 = 0
\]  

(34)

and at a nontrivial \( \tilde{\eta} \) at

\[
\tilde{\eta} = \tilde{\eta}_2 = \frac{2}{\sqrt{27}} \frac{(2G_0 M)}{c^2}
\]  

(35)

The sign of the discriminant \( \Delta_c \) determines the the nature of the roots. It is a cubic equation. It is known to have following possibilities of having its root. For \( \Delta_c > 0 \) when \( \tilde{\eta} \) lies within the range \( (\tilde{\eta}_1 < \tilde{\eta} < \tilde{\eta}_2) \), a multiple root for \( \Delta_c = 0(\tilde{\eta}_1 = \tilde{\eta}_2), \) and a pair of complex root along with a real root \( \Delta_c < 0(\tilde{\eta}_2 < \tilde{\eta} < \infty) \).

When \( \tilde{\eta} \) lies in the domain \( \tilde{\eta}_1 < \tilde{\eta} < \tilde{\eta}_0 \), the fundamental Eqn. (29) retains its standard form

\[
4z^3 - \xi_1 z - \xi_2 = 0,
\]  

(36)

Here we have the scope compared equation (26) with the trigonometric identity

\[
4\cos^3 \gamma - 3\cos \gamma - \cos 3\gamma = 0,
\]  

(37)

and obtained roots by applying some standard methods and recalling \( r = z + \frac{r_{sr}}{3} = z + \sqrt{\frac{\xi_1}{12}} \)

\[
\mathcal{R}_+ \equiv \frac{r_+}{r_{sr}} = \frac{1 + 2\cos \gamma}{3}
\]  

(38)

\[
\mathcal{R}_- \equiv \frac{r_-}{r_{sr}} = \frac{1 - \cos \gamma}{3} + \frac{\sqrt{3}}{3} \sin \gamma
\]  

(39)
\[ R_n = \frac{r_n}{r_s} = \frac{1 - \cos \gamma}{3} - \frac{\sqrt{3}}{3} \sin \gamma \quad (40) \]

for \( \tilde{\eta}_1 \leq \tilde{\eta} \leq \tilde{\eta}_0 \), where

\[ \gamma = \frac{1}{3} \cos^{-1} \left( 1 - \frac{\tilde{\eta}^2}{\tilde{\eta}_0^2} \right) = \frac{1}{3} \cos^{-1} \left( 1 - \frac{27 \tilde{\eta}c^4}{8 \tilde{\eta}_0^4 M^2} \right) \quad (41) \]

However, when \( \tilde{\eta} \) lies in the domain \( \tilde{\eta}_0 \leq l \leq \tilde{\eta}_2 \), Eqn. \( (29) \) can be written down as

\[ 4z^3 - x_{i1}z + |\xi_2| = 0. \quad (42) \]

in this case we exploit the known trigonometric identity to find the solution

\[ 4\sin^3 \gamma - 3\sin \gamma + \sin 3\gamma = 0 \quad (43) \]

Like the former one, we obtained three real roots which are given by

\[ R_+ \equiv \frac{r_+}{r_s} = \frac{1 - \sin \gamma}{3} + \frac{\sqrt{3}}{3} \cos \gamma \quad (44) \]

\[ R_- \equiv \frac{r_-}{r_s} = \frac{1 + 2\sin \gamma}{3} \quad (45) \]

\[ R_n \equiv \frac{r_n}{r_s} = \frac{1 - \sin \gamma}{3} - \frac{\sqrt{3}}{3} \cos \gamma \quad (46) \]

where

\[ \gamma = \frac{1}{3} \sin^{-1} \left( \frac{\tilde{\eta}^2}{\tilde{\eta}_0^2} \right) - 1 = \frac{1}{3} \sin^{-1} \left( \frac{27 \tilde{\eta}c^4}{8 \tilde{\eta}_0^4 M^2} - 1 \right) \quad (47) \]

Let us now see some general properties related to obtained roots. If \( R_+(\gamma) \) is the function given by Eqns. \( (38) \) or \( (44) \) with \( \gamma = \gamma(r_s, l) \) given by Eqns. \( (11) \) or \( (17) \), then in the domain \( l_1 \leq l \leq l_2 \) we obtain the following

\[ \left( \frac{dR_+}{d\gamma} \right) = A(\gamma) \quad (48) \]

\[ r_s \left( \frac{\partial \gamma}{\partial r_s} \right)_l = -l \left( \frac{\partial \gamma}{\partial l} \right) _{r_s} = B(\gamma) \quad (49) \]

\[ r_s \left( \frac{\partial R_+}{\partial r_s} \right)_l = -l \left( \frac{\partial R_+}{\partial l} \right) _{r_s} = C(\gamma) \quad (50) \]

where

\[ C(\gamma) = A(\gamma)B(\gamma) \quad (51) \]

In the domain \( \tilde{\eta}_1 \leq \tilde{\eta} \leq \tilde{\eta}_0 \), the functions \( A(\gamma), B(\gamma) \) and \( C(\gamma) \) can be expressed as

\[ A(\gamma) = -\frac{2}{3} \sin \gamma, \quad (52) \]

\[ B(\gamma) = -\frac{2}{3} (\csc 3\gamma - \cot 3\gamma), \quad (53) \]

\[ C(\gamma) = \frac{4}{9} \sin \gamma (\csc 3\gamma - \cot 3\gamma). \quad (54) \]
However, in the domain $\bar{\eta}_0 \leq \bar{\eta} \leq \bar{\eta}_2$ the functions $A(\gamma), B(\gamma)$ and $C(\gamma)$, are given by

$$A(\gamma) = -\frac{1}{3}(\cos\gamma + \sqrt{3}\sin\gamma)$$  \hspace{1cm} (55)

$$B(\gamma) = -\frac{2}{3}(\sec3\gamma + \tan3\gamma)$$  \hspace{1cm} (56)

$$C(\gamma) = \frac{2}{9}(\cos\gamma + \sqrt{3}\sin\gamma)(\sec3\gamma + \tan3\gamma)$$  \hspace{1cm} (57)

This all about the properties related to the nature of the roots of the lapse function of this GUP inspired black hole.

Now we are in a position to study the impact of GUP parameter on the thermodynamical characteristics of this black hole to which we now turn in the following section

### III. THERMODYNAMICS OF THE GUP INSPIRED QUANTUM GRAVITY CORRECTED BLACK HOLE

We consider the Bekenstein-Hawking approach to study the thermodynamics of this GUP inspired black hole [36–39, 41]. It is necessary to keep in mind that through this procedure, the construction of the thermodynamics is implicitly established in the Carath Aodory framework, in which, the existence of a function, termed as metrical entropy, is ensured along with the absolute temperature. The approach is based on the on the following postulate that the event horizon area of a black hole cannot decrease. It increases in most of the transformations of the black hole [42]. Therefore following using the Bekenstein approach we consider $A = 4\pi r_s^2$ to be the area of the event horizon. Therefore the entropy is then given by

$$S = \frac{k_B}{4} \frac{4\pi r_s^2}{l_p^2}$$  \hspace{1cm} (58)

where $k_B$ is the Boltzmann constant and $l_p$ is the Planck length. It is therefore useful to define the entropy function as

$$S(r_s, \bar{\eta}) = \frac{4l_p^2 S}{4\pi k_B} = r_s^2 \bar{R}_+^2 (r_s, \bar{\eta})$$  \hspace{1cm} (59)

$$\left(\frac{\partial S}{\partial r_s}\right) dr_s + \left(\frac{\partial S}{\partial \bar{\eta}}\right) d\bar{\eta}$$  \hspace{1cm} (60)

which allows us to define the temperature parameter $T$ as

$$\frac{1}{T} = \left(\frac{\partial S}{\partial r_s}\right) \bar{\eta}$$  \hspace{1cm} (61)

solving it using Eqns. (58), (60) we have

$$T = \frac{1/(2r_s)}{\bar{R}_+ (\bar{R}_+ + r_s, \partial \bar{R}_+ / \partial r_s)}$$  \hspace{1cm} (62)

$$T = \frac{T_s}{\bar{R}_+ (\bar{R}_+ + C\gamma)}$$  \hspace{1cm} (63)

where $T_s = \frac{1}{2r_s}$ is the temperature corresponding to the Schwarzschild black hole. For small values of $\bar{\eta}$, $(\bar{\eta} << r_s)$, the temperature remains near $T_s$ and under this case $T$ can be expanded as

$$T \approx T_s \left(1 - \frac{3\bar{\eta}^4}{r_s^4} - \frac{20\bar{\eta}^6}{r_s^6} - \ldotsight)$$  \hspace{1cm} (64)
\[ T \simeq T_s \left( 1 - \frac{3\eta^2\hbar^4c^8}{(2G_0M)^4} - \frac{20\eta^3\hbar^6c^{12}}{(2G_0M)^6} - \frac{117\eta^4\hbar^8c^{16}}{(2G_0M)^8} - \cdots \right) \]  \hfill (65)

FIG. 2: Variation of Temperature parameter $T$ as a function of $\eta$. It has maximum value $T_s = \frac{1}{2r_{sr}}$ for values of $\eta < 0.16$ and minimum value 0 at $\eta = 0.3328$.

If we then generalized mechanical force is now defined by

\[ F_{\tilde{\eta}} = -T \left( \frac{\partial S}{\partial \tilde{\eta}} \right)_{r_{sr}} = \frac{r_{sr}}{\eta} \frac{C(\gamma)}{(R_+(\gamma) + C(\gamma))} \]  \hfill (66)

Introduction of the conjugated canonical pair $(\eta, F_{\tilde{\eta}})$ allow us to define free energy $(F_e)$ through the following relation

\[ F_e \equiv F_{\eta \eta} = \frac{C(\gamma)}{(R_+(\gamma) + C(\gamma))} r_{sr} \]  \hfill (67)

Note that when $\tilde{\eta} \ll r_{sr}$, the free energy behaves as

\[ F_e \simeq \frac{2\tilde{\eta}^2}{r_{sr}} \left( 1 + \frac{3\tilde{\eta}^2}{r_{sr}^2} + \frac{12\tilde{\eta}^4}{r_{sr}^4} + \cdots \right) \]  \hfill (68)

\[ F_e \simeq \frac{2\eta\hbar^2c^2}{(2G_0M)} \left( 1 + \frac{3\eta^2\hbar^4c^4}{(2G_0M)^2} + \frac{12\eta^4\hbar^8c^8}{(2G_0M)^4} + \cdots \right) \]  \hfill (69)

FIG. 3: Variation of free energy parameter $F_e$ as a function of $\eta$. It has minimum value 0 at $\eta = 0$ and maximum value $r_{sr}$ at $\eta = 0.3328$. 
If we combine the first and second laws of thermodynamics, it enables us to write

$$TdS \geq dr_s - F\eta d\eta$$

From Eqn. (60), entropy is a homogeneous function of the second order in \(r_s\) and \(l\). Accordingly, Euler’s theorem requires

$$2TS = r_{sr} - F\eta = r_{sr} - F_c$$

This is the fundamental equation or the Gibbs-Duhem relation for this GUP-inspired black hole. Also, the equation of state (EoS) relating the thermal state variable, \(T\), to the mechanical variables of the system, can be obtained by combining Eqns. (63) and (66):

$$\frac{F\tilde{\eta}}{T} = \frac{2r_{sr}R_+(\gamma)C(\gamma)}{\tilde{\eta}}$$

If the mass-energy \(r_{sr}\) can be treated as free energy in the sense that the work can be stored in the form of potential energy and that can be recovered later. In fact, the total mass-energy differential can be rewritten from Eqn. (70) as

$$dr_s \leq TdS + F\eta d\tilde{\eta}$$

where the inequality holds for spontaneous changes. Hence, for reversible changes,

$$T = \left(\frac{\partial r_{sr}}{\partial S}\right)_{\tilde{\eta}}$$

$$F\tilde{\eta} = \left(\frac{\partial r_{sr}}{\partial l}\right)_S$$

From these, one of the Maxwell relations is obtained as

$$\left(\frac{\partial T}{\partial \tilde{\eta}}\right)_S = \left(\frac{\partial F\eta}{\partial S}\right)_{\tilde{\eta}}$$

If we consider the black hole as a closed and isolated system, we can write

$$\Delta r_s = \Delta Q - \delta W$$

for changes in the black hole’s mass-energy, where \(\Delta Q\) is the heat flux across the surface of the black hole, and \(\Delta W\) is the work. In a reversible process, with \(S\) and \(\tilde{\eta}\) as constants, we can now express the corresponding enthalpy as

$$H \equiv r_{sr} - F\eta = 2TS$$

which is obtained by adding an additional energy, that accounts for a mechanical coupling. Using Eqns. (48)-(50), we get

$$H = \frac{2G_0M}{c^2} \frac{R_+}{(R_+ + C\gamma)}$$

When \(\tilde{\eta} << r_{sr}\), the enthalpy behaves as

$$H \simeq r_{sr} \left(1 - \frac{2\tilde{\eta}}{r_{sr}^2} \frac{6\tilde{\eta}^2}{r_{sr}^3} - \frac{24\tilde{\eta}^3}{r_{sr}^4} - \ldots\right)$$

$$H \simeq \frac{(2G_0M)}{c^2} \left(1 - \frac{2\eta h^2 c^4}{(2G_0M)^2} \frac{6\eta^2 h^4 c^8}{(2G_0M)^4} - \ldots\right)$$
Finally, by adding thermal coupling term, we can express the Helmholtz free energy in terms of $r_{sr}$ in the following form

$$F_H = r_{sr} - TS = TS + \Xi$$  \hspace{1cm} (82)

Using equations (48) - (50), the above expression becomes

$$F_H = \frac{r_{sr}}{2} \left[ \frac{R_+(\gamma) + 2C(\gamma)}{R_+(\gamma) + C(\gamma)} \right]$$  \hspace{1cm} (83)

This all about the thermodynamics of this GUP inspired Black hole. In the following section we will study the impact of GUP parameter on the accretion process on to the black hole associated with the GUP inspired quantum corrected spacetime.

**IV. DESCRIPTION OF THE GEODESIC IN THE GUP INSPIRED QUANTUM CORRECTED SPACETIME GEOMETRY**

In this article we are intended to study the accretion phenomena onto a spherically symmetric Schwarzschild black hole employing a modified uncertainty relation that admits a quantum gravity correction that finds its place holding the hand of the concept of minimal measurable length. Although, other models which are associated with the GUP
correspond to the concept of minimum measurable length and maximum measurable momentum length simultaneously. This type of problem is amenable to have an answer for all kinds of available generalization of uncertainty relation \[31–35, 44–49\]. We will only consider the generalization related to the existence of a minimal length. During this context we consider steady, accretion onto a modified static and spherically symmetric Schwarzschild black hole. We obtain the critical point, critical fluid velocity, temperature, mass accretion rate, and observed total integrated flux within the proposed GUP framework.

The line element in Eqn. (19) can be written down within the form of a Schwarzschild-like metric by introducing an effective mass \( M_{\text{eff}}(r) \) which could be a function of the radial coordinate and depends on the free parameter \( \tilde{\xi} \), that is

\[
ds^2 = -\left(1 - 2 \frac{M_{\text{eff}}(r)}{r}\right) dt^2 + \left(1 - 2 \frac{M_{\text{eff}}(r)}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \tag{84}\]

with

\[
M_{\text{eff}}(r) = \frac{M}{(1 + \frac{2M\eta}{r^{3/2}})} \tag{85}\]

Putting \( \tilde{\eta} = \frac{M\eta}{M^{3/2}} = \frac{\eta}{M^{1/2}} \) and \( x = \frac{r}{M} \), Eqns. (85) can be rewritten as

\[
M_{\text{eff}}(r) = \frac{M}{(1 + \frac{2\tilde{\eta}}{x^{3/2}})} \tag{86}\]

Plots of the effective mass per unit mass as a function of \( x \) is shown below.

\[\text{FIG. 6: Plots of the effective mass as a function of } x \text{ for } \eta = \tilde{\eta}_c = 16/27 \text{ (red)} \text{ and } \eta = 0.3 \text{ (blue). The vertical dashed lines indicate the ISCO for the same values of } \tilde{\eta} \text{ respectively.}\]

Using this definition \[80\] we can write down the Lagrangian from the modified metric endowed with quantum correction replacing \( M \) by \( M \to M_{\text{eff}}(r) \).

\[
\mathcal{L} = \frac{1}{2} \left[-F(r)\dot{t}^2 + \frac{1}{F(r)} \dot{r}^2 + r^2 \dot{\theta}^2 \right] \tag{87}\]

where \( \left(1 - 2 \frac{M_{\text{eff}}(r)}{r}\right) = F(r) \) and the over dots referees to derivative with respect to the affine parameter. We have restricted ourselves to the equatorial plane setting \( (\theta = \pi/2, \dot{\theta} = 0) \). This indeed does not loose any generality. From Eqn. (87), the generalized momenta are are computed as follows

\[
p_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = - \left(1 - 2 \frac{M_{\text{eff}}(r)}{r}\right) \dot{t} = -K, \tag{88}\]

\[
p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = \left(1 - 2 \frac{M_{\text{eff}}(r)}{r}\right)^{-1} \dot{r} = B \tag{89}\]
\[ p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{r} \dot{\phi} = A \]  

(90)

The constants \( K \) and \( A \) are representing the energy and angular momentum per unit rest mass of the particle respectively. The canonical Hamiltonian is abstained by the Legendre transformation

\[ \mathcal{H} = p_t \dot{t} + p_r \dot{r} + p_\phi \dot{\phi} - \mathcal{L} \]  

(91)

and it reads

\[ \mathcal{H} = -K \dot{t} + B \dot{r} + A \dot{\phi} = -\frac{1}{2} \]  

(92)

Note that it is independencies of time so is a constant quantity. The particle is assumed at rest at infinity that allows to set the righthand side of the equation equal to \(-1\). We obtain the the equation corresponding to the energy plugging in Eqns. (88) and (90) in to the Eqn.(92)

\[ \frac{1}{2} \left( \frac{dr}{dt} \right)^2 + V_{eff}(r) = \frac{1}{2}(K^2 - 1), \]  

(93)

where \( V_{eff}(r) \) is given by

\[ V_{eff}(r) = \frac{M_{eff}(r)}{r} + \frac{\hbar^2}{2r^2} - \frac{M_{eff}(r)A^2}{r^3} \]  

(94)

\( V_{eff}(r) \) is nothing but the the effective potential per unit mass in this situation. Let us now proceed to find out the equation of the orbit of the massive particle of effective mass \( M_{eff}(r) \). To this end we define

\[ \dot{r} = \frac{dr}{d\tau} \]  

(95)

By using Eqn. (90) we can write

\[ \dot{r} = \frac{dr}{d\phi} = \frac{A}{r} \frac{dr}{d\phi} \]  

(96)

It is beneficial to make the change of variable \( u = 1/r \) at this stage. In terms of \( u \) Eqn. (93) can be written down as

\[ \left( \frac{du}{d\phi} \right)^2 + u = \left( \frac{K^2 - 1}{A^2} \right) + \frac{2uM_{eff}(u)}{A^2} + 2u^3M_{eff}(u) \]  

(97)

using Eqn. (90). Eqn. (97) leads us to the following equation of the orbit after undergoing the differentiation with respect to \( \phi \)

\[ \frac{d^2u}{d\phi^2} + u = \frac{M}{(1 + 2nu^3)} \left[ \frac{1}{A^2} - \frac{6M \eta u^3}{A^2(1 + 2nu^3)} + 3u^2 - \frac{6M \eta u^5}{(1 + 2nu^3)} \right] \]  

(98)

### A. Condition of circular orbits of the massive particles

To determine the fundamental equations describing the time-averaged radial disk structure, we first explicitly calculate the precise expression of angular momentum \( A \), the energy \( K \) per unit mass and the angular velocity \( \omega \) of particles taking possession of the circular trajectories.

For circular orbits within the equatorial plane \( \dot{r} = 0 \). Therefore that \( r \) is constant, and consequently \( u = 1/r \) may be a constant indeed. We now define two dimensionless quantities \( x = r/M \) and \( \tilde{\eta} = \frac{M \eta}{r^3} = \frac{\eta}{M r^2} \). From Eqn.(98) leads us we obtain an expression for the specific angular momentum (angular momentum per unit mass) \( A \) and if we rewrite \( A \) in terms of the dimensionless quantities \( x \) and \( \tilde{\eta} \) it reads

\[ A = \frac{x^2 M \sqrt{(x^3 - 4\tilde{\eta})}}{\sqrt{(x^6 - 3x^5 + 4\tilde{\eta}x^3 + 4\tilde{\eta}^2)}} \]  

(99)
Putting $\bar{A} = \frac{A}{M}$, variation of $\bar{A}$ with $x$ is shown below.

FIG. 7: The angular momentum $\bar{A} = \frac{A}{M}$ vs. $x$ for several values of $\tilde{\eta}$. From left to right: $\tilde{\eta} = \tilde{\eta}_C = \frac{16}{27}$ (red), $\tilde{\eta} = 0.3$ (blue). The black solid curve corresponds to the classical case $\tilde{\eta} = 0$.

Making use of the condition of having circular orbit $\dot{r} = 0$ in Eqn. (93) we can have the expression of the specific energy (energy per unit mass) $K$, and substituting the expression for $\bar{A}$ the specific energy $K$ can be expressed as

$$K = \sqrt{\frac{(x^3 - 2x^2 + 2\tilde{\eta}) (x^6 - 2x^5 + 4\tilde{\eta}x^3 - 4\tilde{\eta}x^2 + 4\tilde{\eta}^2)}{(x^3 + 2\tilde{\eta}) (x^6 - 3x^5 + 4\tilde{\eta}x^3 + 4\tilde{\eta}^2)}}.$$  

(100)

Upon substituting Eqn. (99) into Eqn. (94), the effective potential acquires the following form

$$V_{eff} = -\frac{x^2}{(x^3 + 2\tilde{\eta})} \left[ 1 - \frac{x^2(x^3 - 4\tilde{\eta})}{(x^6 - 3x^5 + 4\tilde{\eta}x^3 + 4\tilde{\eta}^2)} - \frac{x^2(x^3 - 4\tilde{\eta})}{2(x^6 - 3x^5 + 4\tilde{\eta}x^3 + 4\tilde{\eta}^2)} \right].$$  

(101)

Plot of $V_{eff}$ as function of $x$ for the same values of the free parameter $\tilde{\eta}$ in the range $0 \leq \tilde{\eta} \leq 16/27$ is shown in Figure 4.

FIG. 8: Plot of the effective potential for different values of $\tilde{\eta}$. From left to right: $\tilde{\eta} = \tilde{\eta}_C = \frac{16}{27}$ (red), $\tilde{\eta} = 0.3$ (blue). The black solid curve refers to the classical case $\tilde{\eta} = 0$.

Using Eqns. (99) and (100) the angular velocity can be computed from Eqns. (88) and (90) as

$$\omega = \frac{\dot{\phi}}{t} = \frac{\sqrt{(x^3 - 4\tilde{\eta})(x^3 - 2x^2 + 2\tilde{\eta})}}{M \sqrt{(x^3 + 2\tilde{\eta})(x^6 - 2x^5 + 4\tilde{\eta}x^3 - 4\tilde{\eta}x^2 + 4\tilde{\eta}^2)}}.$$  

(102)
Note that $V_{eff}$, $\tilde{A}$, $K$ and $\omega$ reduce to the corresponding classical expressions in the limit $\tilde{\eta} \to 0$.

At the local minima of the effective potential the possession of circular orbits materialize. Thus, the radius of the innermost stable circular geodesic orbit $x_{isco}$ is found out from the condition

$$\frac{d^2V_{eff}}{dx^2} = 0.$$  \hspace{1cm} (103)

Note that here it will appear in a dimension less manner. On the other hand ISCO can be calculated from the following condition [50].

$$\frac{dA}{dx} = \frac{dK}{dx} = 0.$$  \hspace{1cm} (104)

We are furnishing the ISCO of the thin disk for a set of selected values of $\tilde{\eta}$ in a tabular form

| $\tilde{\eta}$ | $x_{isco}$  |
|---------------|-------------|
| $\frac{16}{27}$ | 5.58396    |
| 0.3          | 5.80455    |
| 0            | 6          |

TABLE I: Values of $x_{isco}$ for different $\tilde{\eta}$.

![Figure 9](image)

FIG. 9: The sketch of effective potential for the values of $\tilde{A}$, evaluated at ISCO. From left to right: $\tilde{\eta} = \tilde{\eta}_C = \frac{16}{27}$ (red), $\tilde{\eta} = 0.3$ (blue). The classical case is described by the black solid curve where $\tilde{\eta} = 0$.

| $\tilde{\eta}$ | $K_{isco}$ | $\epsilon$ |
|---------------|------------|-------------|
| $\frac{16}{27}$ | 0.939752   | 6.0248      |
| 0.3          | 0.941393   | 5.8607      |
| 0            | 0.942809   | 5.7191      |

TABLE II: The energy per unit mass at the ISCO and the efficiency $\epsilon$ of the conversion of the accreted mass into radiation for several values of $\tilde{\eta}$.

where, due to the fact that $A$ is constant for circular orbits, the derivatives must be calculated for $V_{eff}$ as given by
Eqn. (94). This yields
\[
\frac{d^2V_{\text{eff}}}{dx^2} = \frac{-(x^6 - 28\hat{\eta}x^3 + 8\hat{\eta}^2)}{(x^3 + 2\hat{\eta})^3} + \frac{3(x^3 - 4\hat{\eta})}{(x^6 - 3x^5 + 4\hat{\eta}x^3 + 4\hat{\eta}^2)} + \frac{6x^5(x^3 - 4\hat{\eta})}{(x^3 + 2\hat{\eta})^2(x^6 - 3x^5 + 4\hat{\eta}x^3 + 4\hat{\eta}^2)} - \frac{18x^8(x^3 - 4\hat{\eta})}{(x^3 + 2\hat{\eta})^3(x^6 - 3x^5 + 4\hat{\eta}x^3 + 4\hat{\eta})}.
\]

(105)

V. MASS ACCRETION RATE FOR THIN ACCRETION DISK

A general relativistic treatment of an accretion disk around a black hole was reported in the pioneering articles [50, 51]. Let us consider a simple non-relativistic model of an accretion disk around a compact central object. Recent investigations on this issue in different perspectives are [52–55]. Here it is assumed that matter spirals inwards by losing angular momentum which, because of turbulent viscous density being transferred outward through the disk. As the gas moves inwards, it loses gravitational energy and heat over the surroundings by emitting thermal radiation [56]. This model assumes that disk is in a quasi-steady state lying in the equatorial plane of an stationary, axially-symmetric spacetime background. The disk material is assumed to be moving in nearly geodesic circular orbit. The disk is so thin that its maximum thickness $D$ satisfies $D/2R << 1$ where $R$ refers to the characteristic radius of the disk. The heat produced by stress and dynamical friction is efficiently emitted in the form of radiation substantially from the surface of the disk. The amounts describing the thermal characteristics of the disk are averaged over the azimuthal angle $\phi = \frac{2\pi}{\Delta \tau}$, over the thickness $D$, and over the time scale $\Delta \tau$. Here $\tau$ is the time that the gas takes to flow inward through a distance $2D$. With these propositions, the time-averaged radius of the disk is attained from the laws of conservation of rest mass energy and angular momentum. The integration of the equation of mass conservation reveals that the mass accretion rate remains constant for this process

\[
\dot{M} = \frac{dM}{d\tau} = -2\pi \Sigma(r)v^r = \text{constant},
\]

(106)

where $v^r$ and $\sigma$ are respectively the radial velocity and surface density of the accretion disk. Now the combined criteria of the energy and angular momentum conservation leads us to find out the expression of the differential of the luminosity $dL_{\infty}$ at infinity [54, 57].

\[
\frac{dL_{\infty}}{dlnr} = 4\pi r \sqrt{-g} F(r),
\]

(107)

where $F$ is the the flux of radiant energy emitted from the upper face of disk in the local frame of the accreting fluid. Let us call it as $B(r) \equiv BMx$ for later convenience. It has the expressed in terms of the specific angular momentum $\mathcal{A}$, the specific energy $\mathcal{K}$ and the angular velocity $\Omega$ which reads

\[
F(r) = -\frac{\dot{M}}{4\pi \sqrt{-g}} \frac{1}{(\mathcal{K} - \Omega \mathcal{A})^2} \frac{d\Omega}{dr} \int_{r_{\text{isco}}}^{r} (\mathcal{K} - \Omega \mathcal{A}) \frac{d\mathcal{A}}{dr} dr,
\]

(108)

where $\sqrt{-g} = r$ it remains the same for both for the GUP-improved quantum corrected metric and for the classical Schwarzschild metric. The numerical integration of Eqn. (108) gets simplified by using the relation $\frac{d\mathcal{K}}{dr} = \Omega \frac{d\mathcal{A}}{dr}$ and integrating by parts we have

\[
\int_{r_{\text{isco}}}^{r} (\mathcal{K} - \Omega \mathcal{A}) \frac{d\mathcal{A}}{dr} dr = \mathcal{K}h - \mathcal{K}_{\text{isco}} \mathcal{A}_{\text{isco}} - 2 \int_{r_{\text{isco}}}^{r} \mathcal{A} \frac{d\mathcal{A}}{dr} dr.
\]

(109)

Plot of energy flux per unit accretion rate $\frac{E(x)}{M}$ from a thin accretion disk around a GUP-improved Schwarzschild black hole as a function of $x$ is shown below.
Plots of the differential luminosity at infinity per unit accretion rate from a thin disk around a GUP-improved Schwarzschild black hole as a function of $x$ is shown below.

The radiation emitted can be considered as a black body radiation with the temperature given by

$$T(r) = \sigma^{-rac{1}{4}} F(r)^{rac{1}{4}},$$

as it is assumed that during the accretion process the disk in thermodynamic equilibrium. Here $\sigma$ stands for Stefan-Boltzmann constant.

The radial profile of the temperature of the accretion disk (more precisely, the radial profile $\frac{1}{4} F(r)^{rac{1}{4}}$), is shown for different values of the dimensionless free parameter $\tilde{\eta}$ namely, for the critical value $\tilde{\eta}_c = 16/27$, for $\tilde{\eta} = 0.3$, and for the classical solution $\tilde{\eta} = 0$. 
$\eta$ = 0.0
$\eta$ = 0.3
$\eta$ = 16/27

FIG. 11: Radial profiles of the temperature per unit accretion rate of a thin accretion disk around a GUP-improved Schwarzschild black hole for $\tilde{\eta} = \tilde{\eta}_c = \frac{16}{27}$ (red), $\tilde{\eta} = 0.3$ (blue). The black solid curve corresponds to the temperature of the disk around a Schwarzschild black hole in general relativity $\tilde{\eta} = 0$.

FIG. 12: Differential luminosity at infinity per unit accretion rate from a thin disk around a GUP-improved Schwarzschild black hole for $\tilde{\eta} = \tilde{\eta}_c = \frac{16}{27}$ (red), $\tilde{\eta} = 0.3$ (blue). The black solid curve is the energy flux from the disk around a classical Schwarzschild black hole ($\tilde{\eta} = 0$).

What follows next is a tabular presentation of furnishing maximum value and increase in maximum value of the time averaged energy flux per unit accretion rate $\frac{F(x)}{M}$, the differential luminosity per unit accretion rate $B(x)$ and the radial profile of the temperature of the accretion disk $T(x)$ at $\tilde{\eta} = 0, 0.3$, and $\frac{16}{27}$, respectively (table-III) to observe at a glance the change of the peak values of $\frac{F(x)}{M}$, $B(x)$ and $T(x)$ in percentage due to the presence of quantum correction endowed through the GUP framework.

| $\tilde{\eta}$ | $\frac{F(x)}{M}$ | Increase in max value in % | $B(x)$ | Increase in max value in % | $T(x)$ | Increase in max value in % |
|----------------|-------------------|--------------------------|--------|--------------------------|--------|--------------------------|
| 0              | 0.000001367       | -                        | 0.0233514 | 8.55                    | 0.0608144 | -                       |
| 0.3            | 0.00001484        | 8.55                     | 0.0238780 | 2.25                    | 0.0620756 | 2.07                     |
| 16/27          | 0.00001630        | 19.23                    | 0.0244795 | 4.83                    | 0.0635401 | 4.48                     |

TABLE III: Table containing maximum value and increase in maximum value of the time averaged energy flux per unit accretion rate $\frac{F(x)}{M}$, the differential luminosity per unit accretion rate $B(x)$ and the radial profile of the temperature of the accretion disk $T(x)$ at $\tilde{\eta} = 0, 0.3$, and $\frac{16}{27}$, respectively.

A comparative study our outcome with the outcome acquired in [55] shows that the percentage change of the peak values of physical quantities $\frac{F(x)}{M}$, $B(x)$ and $T(x)$ is somewhat smaller than that of the values acquired in [55]. We have additionally seen a conjoint impact of it in table-II where we find that the mathematical worth of $\epsilon$ is additionally observably substantially lower in this present circumstance.

VI. SUMMARY AND CONCLUSION

In this article, we have studied quantum gravity corrections to the thermal properties of a relativistic thin accretion disk around a GUP improve Schwarzschild black hole within the IR-limit of the asymptotic safety situation for quantum gravity. We have calculated, precisely, the corrections to the time-averaged energy flux, the differential luminosity at infinity, the disk temperature, and conjointly the conversion potency of accreting mass into radiation in comparison to the predictions of classical general relativity theory.
We have found that an increase in the parameter $\eta$ that encodes the quantum effects within the GUP framework, not solely results in a shifting of the radius of the inner fringe of the disk and also the ISCO, toward smaller values, but, as a consequence, we have found a tendency to rising the energy radiated far off from the disk together with an increase in temperature of the disk. We have noticed a bent to conjointly rise in the differential luminosity reaching the observer far away at infinity along with a higher conversion potency of accreting mass into radiation. Besides, a shifting of the height of the radial profiles the thermal properties toward smaller values of the radial coordinate have conjointly been observed.

In [52], it has been shown that experimental knowledge reveals that this model is associated with nursing correct one at low luminosities however it is not so well appropriate at high luminosities, a regime that a skinny accretion disk provides a much better description [58, 59]. Throughout this text it has been reinstated everywhere once more. Once again, our investigation shows that quantum gravity within the physics of black hole manifests itself at distances even larger than the radius of the ISCO and do not seem to be restricted inside of the horizon so greatly to its immediate neighborhood. Our result encourages the study of quantum gravity effects on several realistic black holes environments, like accretion onto quantum improved Kerr, Kerr-Sen black hole as a varied or complementary path to confront the predictions of asymptotic safety with astronomical observation.

Data availability Statement: It is a theoretical paper. Data used here are generated from numerical computation

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