Upper bound on the radii of regular ultra-compact star photonspheres

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Abstract

We investigate the photonsphere in the background of regular asymptotically flat compact stars. The analysis includes the general hairy compact star considering the matter fields’ backreaction on the metric in various gravity theories. We prove that the photonsphere of the compact star has an upper bound expressed in terms of the ADM mass of the spacetime. In the case of negative isotropic trace, a stronger upper bound can be obtained.

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I. INTRODUCTION

General relativity predicts that null bound geodesics may exist outside compact objects, such as black holes and horizonless compact stars [1, 2]. The null geodesics can provide valuable information about the structure and geometry of the curved spacetime. In particular, the circular null geodesics on which photon can orbit the central compact object are known as photonspheres. The photonspheres are of great importance from the astrophysical and theoretical aspects [3–13].

On the side astrophysics, it was found that photonspheres play an important role in the optical appearance of a compact star to external observers in the asymptotic region. For example, the strong gravitational lensing phenomenon by black holes is mainly due to the existence of null circular geodesics [14]. On the other side of theoretical studies, the photonsphere is useful in determining the effective length of the hair above the hairy black hole horizon [15, 20]. And it was also shown that the photonsphere provide the fast way to circle a black hole [21, 23]. In addition, it was found that stable photonspheres of compact star can trigger nonlinear instabilities to massless field perturbations [24, 30]. And the characteristic resonances of black holes are related to unstable circular null geodesics [3, 31–38].

The horizonless star with photonspheres is usually called regular ultra-compact star. Bounds on the compactness and photonsphere radii were studied. In the case of positive energy-momentum trace, the lower bound on the compactness parameter of horizonless ultra-compact star was studied in [39]. And the discussion was also extended to the regular ultra-compact star with negative energy-momentum trace [40]. In the black hole background, it was shown that the photonsphere radius has an upper bound expressed in terms of the total ADM mass of the spacetime [41]. Along this line, we try to examine whether there are similar bounds on photonspheres of horizonless ultra-compact stars.

This paper is planned as follows. In section II, we introduce the regular compact star with photonspheres in the asymptotically flat gravity. In section III, we analytically obtain upper bounds on the radius of regular compact star photonsphere. And the last section is devoted to our main results.
II. THE GRAVITY MODEL OF REGULAR COMPACT STARS

We consider static spherically symmetric horizonless ultra-compact star which possesses null circular geodesics. In Schwarzschild coordinates, the compact star geometry are described by the line element

$$ds^2 = -g(r)e^{-2\chi(r)}dt^2 + g^{-1}dr^2 + r^2(d\varphi^2 + \sin^2\varphi d\phi^2).$$

The solutions $\chi(r)$ and $g(r)$ are functions of the radial coordinate $r$. Regularity of the gravity at the center requires

$$g(r \to 0) = 1 + O(r^2) \quad \text{and} \quad \chi(0) < \infty.$$  

The spacetime at the infinity is asymptotically flat, which is characterized by

$$g(r \to \infty) = 1 \quad \text{and} \quad \chi(r \to \infty) = 0.$$  

We state that these spherically symmetric stars could be solutions of a perfect fluid coupled to the gravity background. According to Einstein equations $G^\mu_\nu = 8\pi T^\mu_\nu$, the anisotropic energy momentum tensor is

$$T^\mu_\nu = \text{diag}\{-\rho, -\rho, -\rho, p_r + p_r', 2p_r', -2p_r'' - \frac{2\rho'r''}{8\pi r^2} - \frac{2\rho''}{8\pi r^2}, -\rho - \frac{2\rho'}{8\pi r^2} + \frac{2\rho''}{8\pi r^2}, -\rho + \frac{2\rho'}{8\pi r^2} - \frac{2\rho''}{8\pi r^2}\}.$$  

Here we define $T^t_t = -\rho$, $T^r_r = p_r$ and $T^{\varphi\varphi} = T^\phi_\phi = p_T$ as the energy density, the radial pressure and the tangential pressure respectively. And the equations of metric solutions can be expressed as

$$g' = -8\pi r\rho + \frac{1-\rho}{r},$$

$$\chi' = \frac{-4\pi r(\rho + p)}{g},$$

The gravitational mass $m(r)$ within a sphere of radius $r$ is given by the integration

$$m(r) = \int_0^r 4\pi r'^2 \rho(r')dr'.$$  

And the metric solution can be put in the form

$$g = 1 - \frac{2m(r)}{r},$$

According to (6), a finite mass configuration is characterized by

$$r^3 \rho(r) \to 0 \quad \text{as} \quad r \to \infty.$$  

III. UPPER BOUNDS ON RADII OF COMPACT STAR PHOTONSpheres

In this part, we prove a generic upper bound on the photonsphere of compact stars. We firstly follow the analysis in \[2, 17, 25\] to obtain the characteristic equation of the photonsphere in the spherically symmetric compact star background. The conservation equation $T_{\nu\mu}^\mu$ has only one nontrivial component

$$T_{r,\mu}^\mu = 0. \quad (9)$$

Substituting equations (4) and (5) into (9), we arrive at

$$p'(r) = \frac{1}{2rg}[(3g - 1 - 8\pi r^2)p(\rho + p) + 2gT - 8gp] \quad (10)$$

with $T = -\rho + p + 2p_T$ as the trace of the energy momentum tensor.

With the pressure function $P(r) = r^2p$, the relation (10) can be transformed into

$$P'(r) = \frac{r}{2g}[N(\rho + p) + 2gT - 4gp], \quad (11)$$

where $N = 3g - 1 - 8\pi r^2p$.

We assume that the matter fields satisfy the dominant energy condition

$$\rho \geq |p|, \quad |p_T| \geq 0. \quad (12)$$

According to (8) and (12), the pressure function $P(r)$ has the asymptotical behavior

$$P(r \rightarrow 0) \rightarrow 0 \quad \text{and} \quad rP(r \rightarrow \infty) \rightarrow 0. \quad (13)$$

With relations (2), (3) and (13), the radial function $N(r)$ satisfies

$$N(r = 0) = 2 \quad \text{and} \quad N(r \rightarrow \infty) \rightarrow 2. \quad (14)$$

In the spherically symmetric spacetime, the photonsphere is characterized by

$$V_r = E^2 \quad \text{and} \quad V'_r = 0, \quad (15)$$

where $V_r$ is the effective radial potential that governs the null trajectories in the form

$$V_r = (1 - e^{2\chi})E^2 + g\frac{L^2}{r^2}. \quad (16)$$

Here $E$ is the conserved energy and $L$ is the conserved angular momentum in accordance with the independence of the metric (1) on both $t$ and $\phi$. 
Substituting Einstein equations (4) and (5) into (15) and (16), the photonsphere is determined by the characteristic relation \[39\]

\[N(r_\gamma) = 3g(r_\gamma) - 1 - 8\pi(r_\gamma)^2p = 0. \tag{17}\]

The roots of (17) correspond to the discrete radii of the null circular geodesics. For the case of a Schwarzschild black hole, there is \(g = 1 - \frac{2M}{r}, \chi = 0\) and \(p = \frac{1+g+r_\gamma^2-2rg_\gamma'}{8\pi r^2} = 0\), which yields the familiar \(r_\gamma = 3M\).

We define \(r_\gamma^{out}\) as the outermost photonsphere of the regular ultra-compact objects. From Eqs. (14) and (17), one deduces that the outermost photonsphere of the spherically symmetric horizonless ultra-compact objects satisfies the relation \[28, 39, 40\]

\[N'(r = r_\gamma^{out}) \geq 0. \tag{18}\]

We point out that spatially regular horizonless spacetimes usually possess an even number of photonspheres and the degenerate case of \(N'(r = r_\gamma) = 0\) may be characterized by odd number of photonspheres \[27, 28\].

From relations (4), (11) and (17), we get the function \[28, 39, 40\]

\[N'(r = r_\gamma) = \frac{2}{r_\gamma}[1 - 8\pi r_\gamma^2(\rho + p)]. \tag{19}\]

Putting (19) into (18), we obtain the inequality

\[8\pi(r_\gamma^{out})^2(\rho + p) \leq 1 \tag{20}\]

at the outermost photonsphere of the ultra-compact star.

With (12), (17) and (20), we obtains the relations \[40, 41\]

\[g(r_\gamma^{out}) = \frac{1 + 8\pi(r_\gamma^{out})^2p}{3} \geq \frac{1}{3} + \frac{8\pi(r_\gamma^{out})^2p}{3} \leq \frac{1}{3} + \frac{4\pi(r_\gamma^{out})^2(\rho + p)}{3} \leq \frac{1}{2} + \frac{1}{6} = \frac{1}{2}. \tag{21}\]

Considering the relations (7) and (21), we have

\[\frac{2m(r_\gamma^{out})}{r_\gamma^{out}} \geq \frac{1}{2}. \tag{22}\]

So we obtain an upper bound on the radii of photonspheres

\[r_\gamma \leq r_\gamma^{out} \leq 4m(r_\gamma^{out}) = 4 \int_0^{r_\gamma^{out}} 4\pi r^2\rho(r')dr' \leq 4 \int_0^{\infty} 4\pi r^2\rho(r')dr' = 4M. \tag{23}\]

We also consider the matter field configurations with the negative isotropic trace

\[T = -\rho + 3p < 0 \quad \text{or} \quad \rho > 3p. \tag{24}\]
According to (17), (20) and (24), we obtain the series of relations

\[
g(r_{\text{out}}) = \frac{1 + 8\pi (r_{\text{out}})^2 p}{3} \leq \frac{1 + 2\pi (r_{\text{out}})^2 (p + p)}{3} \leq \frac{1 + \frac{1}{4}}{4} = \frac{5}{12}.
\]  

(25)

Taking cognizance of relations (7) and (25), we arrive at the inequality

\[
\frac{2m(r_{\text{out}})}{r_{\text{out}}} \geq \frac{7}{12}.
\]  

(26)

So a stronger upper bound can be obtained for \( T < 0 \) as

\[
r_{\gamma} \leq r_{\text{out}} \leq \frac{24}{7} m(r_{\text{out}}) \leq \frac{24M}{7}.
\]  

(27)

We mention that the spherically symmetric asymptotically flat black hole photonsphere has an upper bound \( r_{\gamma} \leq 3M \) [41]. In the black hole, there is a condition \( N(r_H) \leq 0 \) at the horizon \( r_H \), which play an important role in the analysis. In this horizonless compact star, we have no such relation and instead there is \( N(0) = 2 > 0 \) at the center. For this reason, we cannot simply follow the analysis of black hole photonsphere in [41] to obtain the bound on the regular star photonsphere. We believe it is interesting to further search for stronger upper bounds on the compact star photonsphere and examine whether there is regular star photonsphere, which can saturates the bound (23) and (27) [43–51]. It is known that one way to construct hairy compact objects is enclosing the compact objects in a box [52–59]. So it is also very interesting to examine the photonsphere radius bound in the confined gravity.

**IV. CONCLUSIONS**

We studied photonspheres in the background of horizonless asymptotically flat ultra-compact stars. We showed that the radius of the compact star photonsphere is bounded from above by \( r_{\gamma} \leq 4M \), where \( r_{\gamma} \) is the radius of the photonsphere and M is the total ADM mass of the spacetime. In the case of negative isotropic trace, we obtained a stronger upper bound in the form \( r_{\gamma} \leq \frac{24M}{7} \). The analysis in this work can be applied to the general gravity model considering the matter fields’ backreaction on the compact star in various asymptotically flat gravity theories.

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[1] J. M. Bardeen, W. H. Press and S. A. Teukolsky, Rotating black holes: Locally nonrotating frames, energy extraction, and scalar synchrotron radiation, Astrophys. J. 178,347(1972).
[2] S. Chandrasekhar, The Mathematical Theory of Black Holes, (Oxford University Press, New York, 1983).
[3] Goebel, C. J., Comments on the “vibrations” of a Black Hole, Astrophysical Journal, vol. 172, p.1 L 95.
[4] Teo, E., Spherical Photon Orbits Around a Kerr Black Hole, General Relativity and Gravitation (2003) 35: 1909.
[5] Pedro V.P. Cunha, Carlos A.R. Herdeiro, Eugen Radu, Fundamental photon orbits: black hole shadows and spacetime instabilities, Phys. Rev. D 96(2017)no.2,024039.
[6] Jai Grover, Alexander Wittig, Black Hole Shadows and Invariant Phase Space Structures, Phys. Rev. D 96(2017)no.2,024045.
[7] Pedro V.P. Cunha, Carlos A.R. Herdeiro, Maria J. Rodriguez, Does the black hole shadow probe the event horizon geometry?, Phys. Rev. D 97(2018)no.8,084020.
[8] S. L. Shapiro and S. A. Teukolsky, Black holes, white dwarfs, and neutron stars: The physics of compact objects, New York, USA: Wiley(1983)645p.
[9] V. Cardoso, A. S. Miranda, E. Berti, H. Witek and V. T. Zanchin, Geodesic stability, Lyapunov exponents and quasinormal modes, Phys. Rev. D 79, 064016(2009).
[10] S. Hod, Spherical null geodesics of rotating Kerr black holes, Phys. Lett. B 718,1552(2013).
[11] Emanuel Gallo, J. R. Villanueva, Photon spheres in Einstein and Einstein-Gauss-Bonnet theories and circular null geodesics in axially-symmetric spacetimes, Phys. Rev. D 92(2015)no.6,064048.
[12] Zdenek Stuchlik, Jan Schec, Bobir Toshmatov, Jan Hladik, Jan Novotny, Gravitational instability of polytropic spheres containing region of trapped null geodesics: a possible explanation of central supermassive black holes in galactic halos, JCAP 1706(2017)no.06, 056.
[13] Zdenek Stuchlik, Stanislav Hledik, Jan Novotny, General relativistic polytropes with a repulsive cosmological constant, Phys. Rev. D 94(2016)no.10,103513.
[14] Ivan Zh. Stefanov, Stoytcho S. Yazadjiev, Galin G. Gyulchev, Connection between Black-Hole Quasinormal Modes and Lensing in the Strong Deflection Limit, Phys. Rev. Lett. 104(2010)251103.
[15] D. Núñez, H. Quevedo, and D. Sudarsky, Black Holes Have No Short Hair, Phys. Rev. Lett. 76, 571(1996).
[16] Shahar Hod, A no-short scalar hair theorem for rotating Kerr black holes, Class,Quant.Grav. 33(2016)114001.
[17] S. Hod, Hairy Black Holes and Null Circular Geodesics, Phys. Rev. D 84, 124030 (2011).
[18] Yun Soo Myung, Taeyou Moon, Hair mass bound in the Einstein-Born-Infeld black hole, Phys. Rev. D 86, 084047.
[19] Yan Peng, Hair mass bound in the black hole with nonzero cosmological constants, Phys. Rev. D 98(2018)104041.
[20] Yan Peng, Hair distributions in noncommutative Einstein-Born-Infeld black holes [arXiv:1808.07988]
[21] S. Hod, The fastest way to circle a black hole, Physical Review D 84, 104024 (2011).
[22] S. Hod, Fermat’s principle in black-hole spacetimes, Int. J. Mod. Phys. D 27 (2018) no.14, 1847025.
[23] Yan Peng, The shortest orbital period in scalar hairy kerr black holes [arXiv:1901.02601] [gr-qc].
[24] J. Keir, Slowly decaying waves on spherically symmetric spacetimes and ultracompact neutron stars, Classical Quantum Gravity 33, 135009 (2016).
[25] V. Cardoso, A.S. Miranda, E. Berti, H. Witek, and V.T. Zanchin, Geodesic stability, Lyapunov exponents and quasinormal modes, Phys. Rev. D 79, 064016 (2009).
[26] S. Hod, Upper bound on the radii of black-hole photon spheres, Phys. Lett. B 727, 345 (2013).
[27] P. V. P. Cunha, E. Berti, and C. A. R. Herdeiro, Light-Ring Stability for Ultracompact Objects, Phys. Rev. Lett. 119 (2017)251102.
[28] S. Hod, On the number of light rings in curved spacetimes of ultra-compact objects, Phys. Lett. B 776, 1 (2018).
[29] S. Hod, Upper bound on the gravitational masses of stable spatially regular charged compact objects, Phys. Rev. D 98, 064014 (2018).
[30] Yan Peng, On instabilities of scalar hairy regular compact reflecting stars, JHEP 1810(2018)185.
[31] Bahram Mashhoon, Stability of charged rotating black holes in the eikonal approximation, Phys. Rev. D 31(1985)no.2,290-293.
[32] S. Hod, Universal Bound on Dynamical Relaxation Times and Black-Hole Quasinormal Ringing, Phys. Rev. D 75(2007)064013.
[33] S. Hod, Black-hole quasinormal resonances: Wave analysis versus a geometric-optics approximation, Phys. Rev. D 80(2009)064004.
[34] Sam R. Dolan, The Quasinormal Mode Spectrum of a Kerr Black Hole in the Eikonal Limit, Phys. Rev. D 82(2010)104003.
[35] Yves Decanini, Antoine Folacci, Bernard Raffaelli, Unstable circular null geodesics of static spherically symmetric black holes, Regge poles and quasinormal frequencies, Phys. Rev. D 81(2010)104039.
[36] Yves Decanini, Antoine Folacci, Bernard Raffaelli, Resonance and absorption spectra of the Schwarzschild black hole for massive scalar perturbations: a complex angular momentum analysis, Phys. Rev. D 84(2011)084035.
[37] S. Hod, Resonance spectrum of near-extremal Kerr black holes in the eikonal limit, Phys. Lett. B 715(2012)348-351.
[38] Huan Yang, David A. Nichols, Fan Zhang, Aaron Zimmerman, Zhongyang Zhang, Yanbei Chen, Quasinormal-
mode spectrum of Kerr black holes and its geometric interpretation, Phys. Rev. D 86(2012)104006.

[39] S. Hod, Self-gravitating field configurations: The role of the energy-momentum trace, Physics Letters B 739(2014)383.

[40] S. Hod, Lower bound on the compactness of isotropic ultra-compact objects, Physical Review D 97, 084018(2018).

[41] S. Hod, Upper bound on the radii of black-hole photonspheres, Physics Letters B 727(2013)345.

[42] H. Bondi, Anisotropic spheres in general relativity, Mon. Not. Roy. Astr. Soc. 259,365 (1992).

[43] S. Hod, Charged reflecting stars supporting charged massive scalar field configurations, Eur. Phys. J. C 78 (2018) no.3, 173.

[44] S. Hod, Marginally bound resonances of charged massive scalar fields in the background of a charged reflecting shell, Physics Letters B 768(2017)97-102.

[45] Srijit Bhattacharjee, Sudipta Sarkar, No-hair theorems for a static and stationary reflecting star, Physical Review D 95, 084027 (2017).

[46] S. Hod, Charged massive scalar field configurations supported by a spherically symmetric charged reflecting shell, Physics Letters B 763, 275 (2016).

[47] Yan Peng, Studies of scalar field configurations supported by reflecting shells in the AdS spacetime, Eur. Phys. J. C 78(2018)680.

[48] Shahar Hod, Ultra-spinning exotic compact objects supporting static massless scalar field configurations, Physics Letters B 774(2017)582.

[49] Yan Peng, Scalar field configurations supported by charged compact reflecting stars in a curved spacetime, Physics Letters B 780(2018)144-148.

[50] Yan Peng, Scalar condensation behaviors around regular Neumann reflecting stars, Nucl. Phys. B 934 (2018) 459-465.

[51] Yan Peng, Static scalar field condensation in regular asymptotically AdS reflecting star backgrounds, Physics Letters B 782(2018)717-722.

[52] Pallab Basu, Chethan Krishnan, P. N. Bala Subramanian, Hairy Black Holes in a Box, JHEP 1611(2016)041.

[53] Nicolas Sanchis-Gual, Juan Carlos Degollado, Pedro J. Montero, Jos A. Font, Carlos Herdeiro, Explosion and final state of an unstable Reissner-Nordström black hole, Phys. Rev. Lett. 116(2016)141101.

[54] Sam R Dolan, Supakchai Ponglertsakul, Elizabeth Winstanley, Stability of black holes in Einstein-charged scalar field theory in a cavity, Phys. Rev. D 92(2015)124047.

[55] Yan Peng, Bin Wang, Yunqi Liu, On the thermodynamics of the black hole and hairy black hole transitions in the asymptotically flat spacetime with a box, Eur. Phys. J. C 78(2018)176.

[56] Yan Peng, Studies of a general flat space/boson star transition model in a box through a language similar to holographic superconductors, JHEP 07(2017)042.

[57] Elisa Maggio, Paolo Pani, Valeria Ferrari, Exotic Compact Objects and How to Quench their Ergoregion Instability, Phys. Rev. D 96(2017)104047.

[58] Shahar Hod, Ultra-spinning exotic compact objects supporting static massless scalar field configurations, Phys. Lett. B 774(2017)582.

[59] Carlos A. R. Herdeiro, Eugen Radu, Asymptotically flat black holes with scalar hair: A review, Int. J. Mod. Phys. D 24(2015)09,1542014.