Asymptotic expansions with exponential, power, and logarithmic functions for non-autonomous nonlinear differential equations

DAT CAO AND LUAN HOANG

Abstract. This paper develops further and systematically the asymptotic expansion theory that was initiated by Foias and Saut in (Ann Inst H Poincaré Anal Non Linéaire, 4(1):1–47 1987). We study the long-time dynamics of a large class of dissipative systems of nonlinear ordinary differential equations with time-decaying forcing functions. The nonlinear term can be, but not restricted to, any smooth vector field which, together with its first derivative, vanishes at the origin. The forcing function can be approximated, as time tends to infinity, by a series of functions which are coherent combinations of exponential, power and iterated logarithmic functions. We prove that any decaying solution admits an asymptotic expansion, as time tends to infinity, corresponding to the asymptotic structure of the forcing function. Moreover, these expansions can be generated by more than two base functions and go beyond the polynomial formulation imposed in previous work.

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1. Introduction

This work is motivated by a deep result by Foias and Saut [12], which is on the long-time behavior of solutions of the Navier–Stokes equations, and its later developments.
in [3, 4, 13, 16–19, 21]. In the original work [12], Foias and Saut studied the Navier–Stokes equations written in the functional form (on an appropriate infinite-dimensional space) as

$$u_t + Au + B(u, u) = 0,$$

where $A$ is a linear operator, and $B$ is a bilinear form. They established the following asymptotic expansion, as $t \to \infty$,

$$u(t) \sim \sum_{k=1}^{\infty} q_k(t)e^{-\mu_k t},$$

(1.2)

where $q_k(t)$’s are polynomials in $t$, and $\mu_k$ increases to infinity. Roughly speaking, expansion (1.2) means that for each $N$, the solution $u(t)$ can be approximated by the finite sum

$$s_N(t) := \sum_{k=1}^{N} q_k(t)e^{-\mu_k t},$$

in the sense that the remainder $u(t) - s_N(t)$ decays exponentially faster than the fastest decaying mode $e^{-\mu_N t}$ in $s_N(t)$, see Definition 1.4 for the precise meaning.

Expansion (1.2) is studied in more detail in [6–10, 13, 16] regarding its convergence, approximation in Gevrey spaces, associated invariant nonlinear manifolds and normal form, and connection to the theory of Poincaré–Dulac normal form, applications to statistical solutions and turbulence theory, etc. A similar expansion to (1.2) is also established in [18] for the Navier–Stokes equations of rotating fluids. Besides the Navier–Stokes equations, expansion (1.2) was obtained and studied for other ordinary differential equations (ODEs) [19], and dissipative wave equations [21]. The last two mentioned papers deal with equations with more general nonlinearity than the quadratic term $B(u, u)$ in (1.1). However, they are still autonomous systems.

Regarding non-autonomous systems, recent papers [3, 4, 17] extend the Foias–Saut result to the Navier–Stokes equations with time-dependent forces, that is,

$$u_t + Au + B(u, u) = f(t),$$

(1.3)

where the force $f(t)$ decays to zero as $t \to \infty$. In [17], asymptotic expansion (1.2) for a solution $u(t)$ of (1.3) is obtained under the condition that

$$f(t) \sim \sum_{k=1}^{\infty} p_k(t)e^{-\mu_k t},$$

(1.4)

where $p_k$’s are appropriate polynomials. The papers [3, 4] consider the forces that decay not as fast as exponential functions. It is obtained, among other things, that if

$$f(t) \sim \sum_{k=1}^{\infty} \eta_k t^{-\mu_k}, \text{ respectively, } f(t) \sim \sum_{k=1}^{\infty} \eta_k (\ln t)^{-\mu_k},$$

(1.5)
where \( \eta_k \)'s are constant vectors (in functional spaces), then there exist constant vectors \( \xi_k \)'s such that

\[
    u(t) \sim \sum_{k=1}^{\infty} \xi_k t^{-\mu_k}, \quad \text{respectively, } \quad u(t) \sim \sum_{k=1}^{\infty} \xi_k (\ln t)^{-\mu_k}. \tag{1.6}
\]

However, the fact that \( \eta_k \) and \( \xi_k \) are independent of \( t \) makes the expansions in (1.5) and (1.6) less than full counterparts of the original (1.2).

The current paper aims to combine two approaches: one in [19,21] for general equations and the other in [3,4,17] for general forcing functions. To make the ideas clear, we avoid, in this paper, complicated issues about global existence, uniqueness, and regularity that often arise in nonlinear partial differential equations (PDEs). Thus, we choose to work with systems of ODEs (in finite-dimensional spaces) with general nonlinearity, and explore various types of forcing functions. We describe the systems of differential equations of our interest and explain the main ideas now.

**Notation.** The following notation will be used throughout the paper.

- \( \mathbb{N} = \{1, 2, 3, \ldots\} \) denotes the set of natural numbers, and \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \).
- For a vector \( y \in \mathbb{R}^n \), its Euclidean norm is denoted by \( |y| \).
- Let \( f \) be a nonnegative function defined in a neighborhood of the origin in \( \mathbb{R}^n \). For a number \( \alpha > 0 \), we write \( f(y) = \mathcal{O}(|y|^\alpha) \) as \( y \to 0 \), if there are positive numbers \( r \) and \( C \) such that \( f(y) \leq C|y|^\alpha \) for all \( x \in \mathbb{R}^n \) with \( |x| < r \).
- Let \( f, h : [T_0, \infty) \to [0, \infty) \) for some \( T_0 \in \mathbb{R} \). We write \( f(t) = \mathcal{O}(h(t)) \) (implicitly means as \( t \to \infty \)) if there exist numbers \( T \geq T_0 \) and \( C > 0 \) such that \( f(t) \leq Ch(t) \) for all \( t \geq T \).
- Let \( T_0 \in \mathbb{R} \), functions \( f, g : [T_0, \infty) \to \mathbb{R}^n \), and \( h : [T_0, \infty) \to [0, \infty) \). We will conveniently write \( f(t) = g(t) + \mathcal{O}(h(t)) \) to indicate \( |f(t) - g(t)| = \mathcal{O}(h(t)) \).
- In particular, when \( g = 0 \), the expression \( f(t) = \mathcal{O}(h(t)) \) means \( |f(t)| = \mathcal{O}(h(t)) \).
- For long-time estimates, we will algebraically manipulate the above big-O notation in our calculations. For example, suppose \( u(t) \) and \( v(t) \) are \( \mathbb{R}^n \)-valued functions with \( u(t) = \mathcal{O}(e^{-t}) \) and \( v(t) = \mathcal{O}(e^{-2t}) \). We can manipulate (and read from left to right)

\[
    u(t) + v(t) = \mathcal{O}(e^{-t}) + \mathcal{O}(e^{-2t}) = \mathcal{O}(e^{-t}), \quad |u(t)|v(t) = \mathcal{O}(e^{-t})\mathcal{O}(e^{-2t}) = \mathcal{O}(e^{-3t}).
\]

Let \( n \in \mathbb{N} \) be fixed throughout the paper. Consider the following system of nonlinear ODEs in \( \mathbb{R}^n \):

\[
    y' = -Ay + G(y) + f(t), \tag{1.7}
\]

where \( A \) is an \( n \times n \) constant (real) matrix, \( G \) is a vector field on \( \mathbb{R}^n \), and \( f \) is a function from \((0, \infty)\) to \( \mathbb{R}^n \).

**Assumption 1.1.** Matrix \( A \) is a diagonalizable matrix with positive eigenvalues.
This assumption is common in studying the dissipative dynamical systems, although it is not as general as [21]. It helps us simplify the calculations and displays the key features of the dissipative dynamics.

**Assumption 1.2.** Function $G : \mathbb{R}^n \to \mathbb{R}^n$ has the following properties.

(i) $G$ is locally Lipschitz.

(ii) There exist functions $G_m : \mathbb{R}^n \to \mathbb{R}^n$, for $m \geq 2$, each is a homogeneous polynomial of degree $m$, such that for any $N \geq 2$, there exists $\delta > 0$ such that

$$
\left| G(y) - \sum_{m=2}^N G_m(y) \right| = O(|y|^{N+\delta}) \text{ as } y \to 0.
$$

We formally write Assumption 1.2(ii) as an expansion

$$
G(y) \sim \sum_{m=2}^\infty G_m(y) \text{ as } y \to 0.
$$

It is clear that if $G$ is a $C^\infty$-function with $G(0) = 0$ and $G'(0) = 0$, then $G$ satisfies Assumption 1.2. Note that we do not require the convergence of the formal series on the right-hand side of (1.9). Even when the convergence occurs, the limit is not necessarily the function $G(y)$. For instance, if $h : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $|y|^{-\alpha} h(y) \to 0$ as $y \to 0$ for all $\alpha > 0$, then $G$ and $G + h$ have the same expansion (1.9).

Next, we investigate the class of functions for $f(t)$ and the forms of expansions that can be obtained. Since this paper involves different vector-valued polynomials of several variables, we clarify their definition here.

**Definition 1.3.** Let $X$ be a real linear space. For $m \in \mathbb{N}$, a function $p : \mathbb{R}^m \to X$ is a polynomial if

$$
p(z) = \sum a_\alpha z^\alpha \text{ for } z \in \mathbb{R}^m,
$$

where the sum is taken over finitely many multi-index $\alpha \in \mathbb{Z}^m_+$, and $a_\alpha$’s are vectors in $X$.

In particular, when $m = 1$, a function $p : \mathbb{R} \to X$ is a polynomial if

$$
p(t) = \sum_{k=0}^N a_k t^k \text{ for } t \in \mathbb{R},
$$

where $N \geq 0$, and $a_k$’s are vectors in $X$.

Examining the expansions in (1.2), (1.4), (1.5) and (1.6), we aim to establish some results for the forcing function of the general form

$$
f(t) \sim \sum_{k=1}^\infty p_k(\phi(t)) \psi(t)^{-\gamma_k},
$$

where $p_k$’s are $\mathbb{R}^n$-valued polynomials of one variable, and $\gamma_k$’s are positive numbers.
Clearly, when \( \psi(t) = e^t \) and \( \phi(t) = t \), (1.10) gives (1.4). When \( \psi(t) = t \), resp., \( \psi(t) = \ln t \), (1.10) resembles (1.5). Because of the presence of the time-dependent function \( \phi(t) \) in (1.10), it will remove the restriction of having only constant vectors \( \eta_k \)'s in (1.5). However, it is not clear what \( \phi(t) \) might be. Nonetheless, we provide the rigorous definition for (1.10) here and will specify some natural choices for \( \phi(t) \) later.

Definition 1.4. Let \((\psi, \phi)\) be a pair of real-valued functions defined on \((T, \infty)\) for some \(T \in \mathbb{R}\) such that

\[
\lim_{t \to \infty} \psi(t) = \infty = \lim_{t \to \infty} \phi(t), \tag{1.11}
\]

and

\[
\lim_{t \to \infty} \frac{\phi^\lambda(t)}{\psi(t)} = 0 \quad \text{for all } \lambda > 0. \tag{1.12}
\]

Let \((X, \|\cdot\|_X)\) be a normed space, and \(g\) be a function from \((T', \infty)\) to \(X\) for some \(T' \in \mathbb{R}\).

(i) Let \((\gamma_k)_{k=1}^\infty\) be a divergent, strictly increasing sequence of positive numbers. We say

\[
g(t) \sim \sum_{k=1}^\infty p_k(\phi(t))\psi(t)^{-\gamma_k}, \tag{1.13}
\]

where each \(p_k : \mathbb{R} \to X\) is a polynomial, if for any \(N \geq 1\), there exists \(\mu > \gamma_N\) such that

\[
\left\| g(t) - \sum_{k=1}^N p_k(\phi(t))\psi(t)^{-\gamma_k} \right\|_X = \mathcal{O}(\psi(t)^{-\mu}). \tag{1.14}
\]

(ii) Let \(N \in \mathbb{N}\), and \((\gamma_k)_{k=1}^N\) be positive and strictly increasing. We say

\[
g(t) \sim \sum_{k=1}^N p_k(\phi(t))\psi(t)^{-\gamma_k}, \tag{1.15}
\]

where each \(p_k : \mathbb{R} \to X\) is a polynomial, for \(1 \leq k \leq N\), if

\[
\left\| g(t) - \sum_{k=1}^N p_k(\phi(t))\psi(t)^{-\gamma_k} \right\|_X = \mathcal{O}(\psi(t)^{-\lambda}) \quad \text{for any } \lambda > 0. \tag{1.16}
\]

We call \(\psi\) and \(\phi\) the base functions for expansions (1.13) and (1.15), with \(\psi\) being primary, and \(\phi\) being secondary.

In case \(X\) is a finite-dimensional normed space, all norms on \(X\) are equivalent. Hence, the above definitions of (1.13) and (1.15) are independent of the particular norm \(\|\cdot\|_X\).

One can see, again, that expansions (1.2) and (1.4) correspond to (1.13) with \((\psi, \phi) = (e^t, t)\). The next two examples for (1.13) are \((\psi, \phi) = (t, \ln t)\) and \((\psi, \phi) = (\ln t, \ln \ln t)\), with notation \(\ln \ln t = \ln(\ln t)\), which yield the expansions that not only
cover but also are more general than those in (1.5) and (1.6). It turns out that corresponding to these three cases of \((\psi, \phi)\) and asymptotic expansion (1.10) for \(f(t)\), we can prove that any decaying solution \(y(t)\) of (1.7) admits a similar asymptotic expansion

\[
y(t) \sim \sum_{k=1}^{\infty} q_k(\phi(t)) \psi(t)^{-\mu_k},
\]

(1.17)

where \(\mu_k\)'s are positive numbers appropriately generated based on the powers \(\gamma_k\)'s in (1.10).

This is the starting point of the current paper that explains the main ideas. More sophisticated expansions and all technicalities will be presented in detail below.

The paper is organized as follows: Section 2 sets up the background for Eq. (1.7) and develops essential tools for the paper. It contains the approximation lemmas for solutions of linear ODEs, see Lemmas 2.2, 2.9 and Corollary 2.10. They are simple but important building blocks for the complicated nonlinear theory developed in later sections. Especially, Lemma 2.9 will enable us to deal with a much larger class of forcing functions. In Sect. 3, we prove the basic existence result in Theorem 3.1 for solutions of the studied ODEs. A specific asymptotic estimate, as time tends to infinity, which corresponds to the decay of the forcing function, is established in Theorem 3.2. These will be used to obtain the first term in the asymptotic expansion of the solutions. Section 4 contains our first main result, Theorem 4.10, corresponding to the expansions in Definition 1.4, with specific types of \((\psi, \phi)\) indicated in Definition 4.2. It fully justifies (1.17) and removes the limitation of the previous work [3,4] mentioned in the remark after (1.6). Moreover, we emphasize that the calculations in the proof of Theorem 4.10 will crucially serve the further developments in the next section. In Sect. 5, we investigate expansions that are generated by more than two base functions, see Definition 5.1. They can consist of many secondary base functions and allow the functions \(p_k\)'s in (1.13) to be more than just polynomials, i.e., the powers can be real numbers, not just nonnegative integers. Therefore, compared to those in Sect. 4 with the same primary base function, these expansions are more precise approximations of the forcing function and solutions. Despite not yet covering the case of exponential primary base function, they are rather significant deviations from the polynomial-based formulation for \(q_k\)'s and \(p_k\)'s in the asymptotic expansions (1.2) and (1.13), respectively. The case of purely iterated logarithmic functions is treated in Sect. 5.1, see Theorem 5.6. The case of mixed power and iterated logarithmic functions is treated in Sect. 5.2, see Theorem 5.8. Typical cases of expansions with the three functions such as power, logarithmic \((\ln t)\) and iterated logarithmic \((\ln \ln t)\), see (5.32) and (5.36), are explored in Corollaries 5.10 and 5.11. In Appendix 5.2, we briefly discuss a specific aspect of our results, namely their application to the Galerkin approximations to nonlinear PDEs.

We comment that the approach presented in the current paper is based on the result and ideas of Foias and Saut in [12]. For our specific problem of obtaining large-time
asymptotic expansions for decaying solutions, it is direct, simple and does not resort to the normal form theory for ODEs, see, e.g., [1, 2, 20].

Finally, it is worth mentioning that our results can be extended to the Navier–Stokes equations or other PDEs with appropriate settings, by combining this paper’s techniques with the methods in [3, 4] or [21].

2. Preliminaries

For the estimation of linear and multi-linear mappings in this paper, we recall the definition of their norms here. If \( m \in \mathbb{N} \) and \( \mathcal{L} \) is an \( m \)-linear mapping from \( \mathbb{R}^{m \times n} \) to \( \mathbb{R}^n \), the norm of \( \mathcal{L} \) is defined by

\[
\| \mathcal{L} \| = \max \{ |\mathcal{L}(y_1, y_2, \ldots, y_m) : y_j \in \mathbb{R}^n, |y_j| = 1, \text{ for } 1 \leq j \leq m \}.
\]

Then, \( \| \mathcal{L} \| \) is a number in \([0, \infty)\), and, for any \( y_1, y_2, \ldots, y_m \in \mathbb{R}^n \), one has

\[
|\mathcal{L}(y_1, y_2, \ldots, y_m)| \leq \| \mathcal{L} \| \cdot |y_1| \cdot |y_2| \cdots |y_m|.
\] (2.1)

Now, we examine Eq. (1.7) further. Regarding its linear part, thanks to Assumption 1.1, we can denote by \( \Lambda_j \), for \( 1 \leq j \leq n \), the eigenvalues of \( A \) which are positive and increasing in \( j \). Then, there exists an invertible matrix \( S \) such that

\[
A = S^{-1}A_0S, \quad \text{where } A_0 = \text{diag}[\Lambda_1, \Lambda_2, \ldots, \Lambda_n].
\] (2.2)

Denote the distinct eigenvalues of \( A \) by \( \lambda_j \)'s which are strictly increasing in \( j \), i.e.,

\[
0 < \lambda_1 = \Lambda_1 < \lambda_2 < \ldots < \lambda_d = \Lambda_n, \quad \text{with } 1 \leq d \leq n.
\]

For \( 1 \leq k, \ell \leq n \), let \( E_{k\ell} \) be the elementary \( n \times n \) matrix \((\delta_{ki}\delta_{\ell j})_{1 \leq i, j \leq n}\), where \( \delta_{ki} \) and \( \delta_{\ell j} \) are the Kronecker delta symbols.

For an eigenvalue \( \lambda \) of \( A \), define

\[
\hat{R}_\lambda = \sum_{1 \leq j \leq n, \lambda_j = \lambda} E_{jj} \quad \text{and} \quad R_\lambda = S^{-1}\hat{R}_\lambda S.
\]

The following are immediate facts.

(a) If \( \lambda \) is an eigenvalue of \( A \), then

\[
R^2_\lambda = R_\lambda, \quad AR_\lambda = R_\lambda A = \lambda R_\lambda.
\] (2.3) (2.4)

(b) If \( \lambda \) and \( \mu \) are distinct eigenvalues of \( A \), then

\[
R_\lambda R_\mu = 0.
\] (2.5)
(c) One has

\[ I_n = \sum_{j=1}^{d} R_{\lambda_j}, \tag{2.6} \]

and, for any \( y \in \mathbb{R}^n \),

\[ |y| \leq \sum_{j=1}^{d} |R_{\lambda_j}y| \leq (\sum_{j=1}^{d} \|R_{\lambda_j}\||y|. \tag{2.7} \]

For the nonlinear part of Eq. (1.7), we consider condition (1.9). For each \( m \geq 2 \), there exists an \( m \)-linear mapping \( G_m \) from \( \mathbb{R}^{m \times n} \) to \( \mathbb{R}^n \) such that

\[ G_m(y) = \mathcal{G}_m(y, y, \ldots, y) \text{ for } y \in \mathbb{R}^n. \tag{2.8} \]

By (2.1), one has, for any \( y_1, y_2, \ldots, y_m \in \mathbb{R}^n \), that

\[ |\mathcal{G}_m(y_1, y_2, \ldots, y_m)| \leq \|\mathcal{G}_m\| \cdot |y_1| \cdot |y_2| \cdots |y_m|. \tag{2.9} \]

In particular,

\[ |G_m(y)| \leq \|\mathcal{G}_m\| \cdot |y|^m \quad \forall y \in \mathbb{R}^n. \tag{2.10} \]

It follows (1.8), when \( N = 2 \), and (2.10), for \( m = 2 \), that \( |G(y)| = \mathcal{O}(|y|^2) \) as \( y \to 0 \). Thus, there exists numbers \( c_*, r_* > 0 \) such that

\[ |G(y)| \leq c_*|y|^2 \quad \forall y \in \mathbb{R}^n \text{ with } |y| < r_. \tag{2.11} \]

Next, we obtain elementary results on long-time asymptotic estimates for integrals and approximations for solutions of linear ODEs. They will play important roles in later developments of the paper.

We will often use the following simple fact

\[ \int_{0}^{t} e^{-\alpha(t-\tau)} e^{-\beta \tau} \, d\tau = \begin{cases} \mathcal{O}(e^{-\min\{\alpha, \beta\}t}), & \text{if } \alpha \neq \beta, \\ \mathcal{O}(te^{-\alpha t}), & \text{if } \alpha = \beta. \end{cases} \tag{2.12} \]

We recall here and make concise [17, Lemma 4.2], which originates from the work of Foias and Saut [12].

**Lemma 2.1.** Let \((X, \| \cdot \|_X)\) be a Banach space. Let \( p : \mathbb{R} \to X \) be a polynomial, and \( g \in C([0, \infty), X) \) satisfy

\[ \|g(t)\|_X \leq Me^{-\delta t} \quad \forall t \geq 0, \text{ for some } M, \delta > 0. \]

Suppose that \( y \in C([0, \infty), X) \) solves the equation

\[ y'(t) + \beta y(t) = p(t) + g(t), \quad \text{for } t > 0, \]

where \( \beta \) is a constant in \( \mathbb{R} \).
In case $\beta < 0$, assume further that
\[
\lim_{t \to \infty} (e^{\beta t} \|y(t)\|_X) = 0.
\] (2.13)

Then, there exists a unique $X$-valued polynomial $q(t)$ such that
\[
q'(t) + \beta q(t) = p(t), \quad \text{for } t \in \mathbb{R},
\] (2.14)
and
\[
\|y(t) - q(t)\|_X = O(e^{-\varepsilon t}) \quad \text{for any } \varepsilon \in (0, \varepsilon_{\delta, \beta}),
\] (2.15)
where
\[
\varepsilon_{\delta, \beta} = \begin{cases} 
\min\{\delta, \beta\} & \text{if } \beta > 0, \\
\delta & \text{otherwise}.
\end{cases}
\] (2.16)

Proof: The uniqueness of $q(t)$ is due to Lemma 4.1 with $N = 1$, $y_1 = 0$, $(\psi, \phi) = (e^t, t)$ and estimate (2.15). In fact, following [17, Lemma 4.2], the polynomial $q(t)$ is explicitly defined by
\[
q(t) = \begin{cases} 
e^{-\delta t} \int_0^t e^{\beta \tau} p(\tau) \, d\tau & \text{if } \beta > 0, \\
y(0) + \int_0^{\infty} g(\tau) \, d\tau + \int_0^t p(\tau) \, d\tau & \text{if } \beta = 0, \\
-e^{-\beta t} \int_t^{\infty} e^{\beta \tau} p(\tau) \, d\tau & \text{if } \beta < 0,
\end{cases}
\] (2.17)

and satisfies the following estimates.

If $\beta > 0$ then
\[
\|y(t) - q(t)\|_X \leq \left(\|y(0) - q(0)\|_X + \frac{M}{|\beta - \delta|}\right) e^{-\min\{\delta, \beta\}|t|}, \quad t \geq 0, \text{ for } \beta \neq \delta,
\] (2.18)
and
\[
\|y(t) - q(t)\|_X \leq (\|y(0) - q(0)\|_X + Mt)e^{-\varepsilon \delta t}, \quad t \geq 0, \text{ for } \beta = \delta.
\] (2.19)

If either ($\beta = 0$) or ($\beta < 0$ with (2.13)), then
\[
\|y(t) - q(t)\|_X \leq \frac{M}{|\beta| + \delta} e^{-\delta |t|}, \quad t \geq 0.
\] (2.20)

Combining (2.18), (2.19) and (2.20), we deduce the concise estimate (2.15).

Using the basic Lemma 2.1, we derive, below, an efficient approximation lemma for linear ODEs. It will be utilized in the proof of the main result of Sect. 4, Theorem 4.10.

Lemma 2.2. Let $p(t)$ be an $\mathbb{R}^n$-valued polynomial and $g : [T, \infty) \to \mathbb{R}^n$, for some $T \in \mathbb{R}$, be a continuous function satisfying $|g(t)| = O(e^{-\alpha t})$ for some $\alpha > 0$. Suppose $\lambda > 0$ and $y \in C([T, \infty), \mathbb{R}^n)$ is a solution of
\[
y'(t) = -(A - \lambda I_n)y(t) + p(t) + g(t), \quad \text{for } t \in (T, \infty).
\] (2.21)
If \( \lambda > \lambda_1 \), assume further that
\[
\lim_{t \to \infty} (e^{(\lambda_\ast - \lambda)t} |y(t)|) = 0, \text{ where } \lambda_\ast = \max\{\lambda_j : 1 \leq j \leq d, \lambda_j < \lambda\}. \tag{2.22}
\]

Then, there exists a unique \( \mathbb{R}^n \)-valued polynomial \( q(t) \) such that
\[
q'(t) = -(A - \lambda I_n) q(t) + p(t), \text{ for } t \in \mathbb{R}, \tag{2.23}
\]
and
\[
|y(t) - q(t)| = \mathcal{O}(e^{-\varepsilon t}) \text{ for all } \varepsilon \in (0, \delta), \tag{2.24}
\]
where \( \delta = \min\{\varepsilon_{\alpha, \lambda_j - \lambda} : 1 \leq j \leq d\} > 0 \), with \( \varepsilon_{\alpha, \lambda_j - \lambda} \) being defined as in (2.16).

**Proof.** Similar to Lemma 2.1, the uniqueness of \( q(t) \) comes from Lemma 4.1 with \( X = \mathbb{R}^n \), \( N = 1 \), \( \gamma_1 = 0 \), \( (\psi, \phi) = (e^t, t) \) and the exponential decay in (2.24).

Let \( 1 \leq j \leq d \). Applying \( R_{\lambda_j} \) to Eq. (2.25) with \( \lambda = \lambda_j \), \( y(t) = R_{\lambda_j} y(t) + R_{\lambda_j} p(t) + R_{\lambda_j} g(t) \), for \( t > T. \tag{2.25} \)

We apply Lemma 2.1 to Eq. (2.25) with \( X = R_{\lambda_j}(\mathbb{R}^n) \), \( \| \cdot \|_X = | \cdot | \),
\[
\beta := \lambda_j - \lambda, \quad y(t) := R_{\lambda_j} y(T + t), \quad p(t) := R_{\lambda_j} p(T + t), \quad g(t) := R_{\lambda_j} g(T + t).
\]

In case \( \beta < 0 \), we have \( e^{\beta t} |R_{\lambda_j} y(T + t)| \leq e^{(\lambda_\ast - \lambda)t} |y(T + t)| \), which goes to zero as \( t \to \infty \), thanks to (2.22). Thus, condition (2.13) is satisfied. Therefore, it follows (2.15) and (2.14) that
\[
|R_{\lambda_j} y(T + t) - \tilde{q}_j(t)| = \mathcal{O}(e^{-\varepsilon t}) \quad \forall \varepsilon \in (0, \varepsilon_{\alpha, \lambda_j - \lambda}), \tag{2.26}
\]
where \( \tilde{q}_j(t) \) is an \( R_{\lambda_j}(\mathbb{R}^n) \)-valued polynomial that satisfies
\[
\tilde{q}_j'(t) = - (\lambda_j - \lambda) \tilde{q}_j(t) + R_{\lambda_j} p(T + t) \text{ for } t \in \mathbb{R}. \tag{2.27}
\]

Define \( q(t) = \sum_{j=1}^d \tilde{q}_j(t - T) \) for \( t \in \mathbb{R} \). Then, \( q(t) \) is an \( \mathbb{R}^n \)-valued polynomial and, by properties (2.3) and (2.5), \( R_{\lambda_j} q(T + t) = \tilde{q}_j(t) \). By using the last relation and replacing \( T + t \) with \( t \), we have from (2.26) and (2.27) that
\[
|R_{\lambda_j} (y(t) - q(t))| = \mathcal{O}(e^{-\varepsilon t}) \quad \forall \varepsilon \in (0, \delta), \tag{2.28}
\]
\[
R_{\lambda_j} q'(t) = -(A - \lambda I_n) R_{\lambda_j} q(t) + R_{\lambda_j} p(t) \text{ for } t \in \mathbb{R}. \tag{2.29}
\]

Summing up (2.28) in \( j \) from 1 to \( d \), and using the first inequality in (2.7), we obtain (2.24).

Finally, summing up (2.29) in \( j \) from 1 to \( d \), and using identity (2.6), we obtain (2.23). \( \Box \)

**Remark 2.3.** We recall, below, some well-known observations from [12,13].
(a) Regarding Lemma 2.1, if \( \beta \neq 0 \), then the polynomial \( q(t) \) is independent of solution \( y(t) \), see formula (2.17). In fact, it is the unique polynomial solution of (2.14). In case \( \beta = 0 \), \( q(t) \) depends on \( y(0) \).

(b) Consequently, if \( \lambda \) is not an eigenvalue of \( A \), then the polynomial \( q(t) \) in Lemma 2.2 is the unique polynomial solution of (2.23), and independent of \( y(t) \). In case \( \lambda \) is an eigenvalue of \( A \), then \( q(t) \) depends on \( R_\lambda y(T) \).

The following types of functions will be used in our asymptotic expansions.

**Definition 2.4.** Define the iterated exponential and logarithmic functions as follows:

\[
E_0(t) = t \quad \text{for} \quad t \in \mathbb{R}, \quad \text{and} \quad E_{m+1}(t) = e^{E_m(t)} \quad \text{for} \quad m \in \mathbb{Z}_+, \quad t \in \mathbb{R},
\]

\[
L_0(t) = t \quad \text{for} \quad t \in \mathbb{R}, \quad \text{and} \quad L_{m+1}(t) = \ln(L_m(t)) \quad \text{for} \quad m \in \mathbb{Z}_+, \quad t > E_m(0).
\]

Let \( m \in \mathbb{Z}_+ \) and \( k \in \mathbb{N} \), define \( \mathbb{R}^k \)-valued function \( L_{m,k} \) by

\[
L_{m,k}(t) = (L_{m+1}(t), L_{m+2}(t), \ldots, L_{m+k}(t)) \quad \text{for} \quad t > E_{m+k-1}(0).
\]

In particular, denote \( L_k = L_{0,k} \) for \( k \in \mathbb{N} \).

Throughout the paper, we will use the convention

\[
\sum_{k=1}^{0} a_k = 0 \quad \text{and} \quad \prod_{k=1}^{0} a_k = 1.
\]

For \( m \in \mathbb{Z}_+ \), note that \( L_m(t) \) is positive and increasing for \( t > E_m(0) \), and

\[
\lim_{t \to \infty} L_m(t) = \infty, \quad \lim_{t \to \infty} \frac{L_{m+k}(t)^\lambda}{L_m(t)} = 0 \quad \text{for all} \quad k \in \mathbb{N}, \quad \lambda > 0.
\]

For \( m \in \mathbb{N} \), the first and second derivatives of \( L_m(t) \) are

\[
L'_m(t) = \frac{1}{t \prod_{k=1}^{m-1} L_k(t)},
\]

\[
L''_m(t) = -\frac{1}{t^2} \cdot \frac{1}{\prod_{k=1}^{m-1} L_k(t)} \left\{ 1 + \sum_{j=1}^{m-1} \frac{1}{\prod_{p=1}^{j} L_p(t)} \right\}.
\]

Clearly, \( L''_m(t) < 0 \) on \((E_m(0), \infty)\), hence,

\[
L_m \quad \text{is concave on} \quad (E_m(0), \infty) \quad \text{for all} \quad m \in \mathbb{N}.
\]

(2.30)

In the next two lemmas, we obtain asymptotic estimates for some integrals which will appear in later proofs.

**Lemma 2.5.** Let \( m \in \mathbb{Z}_+ \) and \( \lambda > 0, \sigma > 0 \) be given. For any number \( T_* > E_m(0) \), there exists a number \( C > 0 \) such that

\[
\int_{0}^{t} e^{-\sigma(t-\tau)} L_m(T_* + \tau)^{-\lambda} d\tau \leq C L_m(T_* + t)^{-\lambda} \quad \text{for all} \quad t \geq 0.
\]

(2.31)
Proof. We recall [4, Lemma 3.1], which states as follows.

Let \( F \) be a continuous, decreasing function from \([0, \infty)\) to \([0, \infty)\). For any \( \sigma > 0 \) and \( \theta \in (0, 1) \), one has

\[
\int_0^t e^{-\sigma(t-\tau)} F(\tau) \, d\tau \leq \frac{1}{\sigma} \left( F(0) e^{-(1-\theta)\sigma t} + F(\theta t) \right) \quad \forall t \geq 0. \tag{2.32}
\]

Case \( m = 0 \). Then, \( T_* > 0 \). Applying inequality (2.32) to \( F(t) = (T_* + t)^{-\lambda} \) and \( \theta = 1/2 \) gives

\[
\int_0^t e^{-\sigma(t-\tau)} (T_* + \tau)^{-\lambda} \, d\tau \leq \frac{1}{\sigma} \left( T_*^{-\lambda} e^{-\sigma t/2} + (T_* + t/2)^{-\lambda} \right) \quad \forall t \geq 0. \tag{2.33}
\]

Clearly, there exists a number \( C_1 > 0 \) such that

\[
e^{-\sigma t/2} \leq C_1 (T_* + t)^{-\lambda} \quad \text{for all } t \geq 0. \tag{2.34}
\]

Also, for \( t \geq 0 \),

\[
(T_* + t/2)^{-\lambda} = 2^\lambda (2T_* + t)^{-\lambda} \leq 2^\lambda (T_* + t)^{-\lambda}. \tag{2.35}
\]

From (2.33), (2.34) and (2.35), we obtain (2.31).

Case \( m \geq 1 \). Let \( F(t) = L_m(T_* + t)^{-\lambda} \) and fix \( \theta \in (0, 1) \). By (2.32),

\[
\int_0^t e^{-\sigma(t-\tau)} L_m(T_* + \tau)^{-\lambda} \, d\tau \leq \frac{1}{\sigma} \left( L_m(T_*)^{-\lambda} e^{-(1-\theta)\sigma t} + L_m(T_* + \theta t)^{-\lambda} \right) \quad \forall t \geq 0. \tag{2.36}
\]

Similar to (2.34), there is a number \( C_2 > 0 \) such that

\[
e^{-(1-\theta)\sigma t} \leq C_2 L_m(T_* + t)^{-\lambda} \quad \text{for all } t \geq 0. \tag{2.37}
\]

By the concavity of \( L_m \), see (2.30), we have

\[
L_m(T_* + \theta t) = L_m((1 - \theta)T_* + \theta (T_* + t)) \geq (1 - \theta) L_m(T_*) + \theta L_m(T_* + t).
\]

Since \( L_m(T_*) > 0 \), it follows that \( L_m(T_* + \theta t) \geq \theta \rho_m(T_* + t) \), and we have

\[
L_m(T_* + \theta t)^{-\lambda} \leq \theta^{-\lambda} L_m(T_* + t)^{-\lambda}. \tag{2.38}
\]

We obtain (2.31) from (2.36), (2.37) and (2.38).

□

Lemma 2.6. Let \( \sigma > 0 \), \( N \in \mathbb{N} \), and \( k_j \in \mathbb{N} \), \( a_j \in \mathbb{R} \) for \( j = 1, 2, \ldots, N \). Denote

\[k_* = \max\{k_j : 1 \leq j \leq N\} \quad \text{and let } T_* > E_{k_*}(0)\).

Then, it holds, for all \( \lambda \in (0, 1) \), that

\[
\int_0^t e^{-\sigma(t-\tau)} \prod_{j=1}^N (L_{k_j}(T_* + \tau))^{a_j} \, d\tau = \frac{1}{\sigma} \prod_{j=1}^N (L_{k_j}(T_* + t))^{a_j} + \mathcal{O}(t^{-\lambda}). \tag{2.39}
\]
In particular, for $k \in \mathbb{N}$, $T_* > E_k(0)$, and any $\alpha \in \mathbb{R}$, $\lambda \in (0, 1)$, one has
\[
\int_0^t e^{-\sigma(t-\tau)}(L_k(T_* + \tau))^{\alpha} \, d\tau = \frac{1}{\sigma}(L_k(t + T_*))^{\alpha} + O(t^{-\lambda}). \tag{2.40}
\]

**Proof.** Denote $F(t) = \prod_{j=1}^N (L_{kj}(t))^{\alpha_j}$ and $I = \int_0^t e^{-\sigma(t-\tau)} F(T_* + \tau) \, d\tau$. Integrating by parts gives
\[
I = \frac{1}{\sigma} \left( F(T_* + t) - F(T_*) e^{-\sigma t} \right) - \sum_{j=1}^N \frac{1}{\sigma} \int_0^t e^{-\sigma(t-\tau)} \frac{\alpha_j F(T_* + \tau)}{(T_* + \tau) \prod_{p=1}^{kj} L_p(T_* + \tau)} \, d\tau. \tag{2.41}
\]

Let $\lambda$ be any number in the interval $(0, 1)$. Note that
\[
F(T_*) e^{-\sigma t} = O(t^{-\lambda}). \tag{2.42}
\]

Also, there exists, for each $j = 1, 2, \ldots, N$, a positive number $C_j > 0$ such that
\[
\left| \frac{F(T_* + t)}{(T_* + t) \prod_{p=1}^{kj} L_p(T_* + t)} \right| \leq C_j (T_* + t)^{-\lambda}, \quad \forall t \geq 0.
\]

By this estimate and inequality (2.31) with $m = 0$,
\[
\int_0^t e^{-\sigma(t-\tau)} \frac{F(T_* + \tau)}{(T_* + \tau) \prod_{p=1}^{kj} L_p(T_* + \tau)} \, d\tau = O(t^{-\lambda}). \tag{2.43}
\]

Then, combining (2.41), (2.42) and (2.43), we obtain (2.39). Inequality (2.40) is a special case of (2.39) when $N = 1, k_1 = k$, and $\alpha_1 = \alpha$. \hfill \Box

Our results will cover more complicated expansions than (1.13). They involve the following type of functions which are more general than the polynomials in Definition 1.3.

**Definition 2.7.** For $z = (z_1, z_2, \ldots, z_k) \in (0, \infty)^k$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{R}^k$, with $k \in \mathbb{N}$, denote
\[
z^\alpha = \prod_{j=1}^k z_j^{\alpha_j}.
\]

Let $X$ be a real linear space and $k \in \mathbb{N}$. Define $\mathcal{P}(k, X)$ to be the set of functions $p : (0, \infty)^k \to X$ of the form
\[
p(z) = \sum_{\alpha \in I} c_\alpha z^\alpha \text{ for } z \in (0, \infty)^k, \tag{2.44}
\]
where $I$ is a non-empty, finite subset of $\mathbb{R}^k$, and $c_\alpha$’s are vectors in $X$. Below are immediate observations about Definition 2.7.
Lemma 2.8. The following statements hold true.

(i) If $p_j \in \mathcal{P}(k, X_j)$ for $1 \leq j \leq m$, where $m \geq 1$, $X_j$’s are linear spaces, and $L$ is an $m$-linear mapping from $\prod_{j=1}^{m} X_j$ to $X$, then $L(p_1, p_2, \ldots, p_m) \in \mathcal{P}(k, X)$.

(ii) If $p \in \mathcal{P}(k, X)$ and $L : X \to Y$ is a linear mapping, then $Lp \in \mathcal{P}(k, Y)$.

(iii) If $p \in \mathcal{P}(k, \mathbb{R}^n)$ and $1 \leq j \leq n$, then the canonical projection $\pi_j p$, that maps $p$ to its $j$th component, belongs to $\mathcal{P}(k, \mathbb{R})$.

(iv) If $p \in \mathcal{P}(k, \mathbb{R})$ and $q \in \mathcal{P}(k, X)$, then the product $pq \in \mathcal{P}(k, X)$.

Consequently, if $p_j \in \mathcal{P}(k, \mathbb{R})$ for $1 \leq j \leq m$, then $p_1 p_2 \ldots p_m \in \mathcal{P}(k, \mathbb{R})$.

(v) If $p \in \mathcal{P}(k, \mathbb{R}^n)$ and $q$ is a polynomial from $\mathbb{R}^n$ to $X$, then the composition $q \circ p$ belongs to $\mathcal{P}(k, X)$.

(vi) In case $X$ is a normed space, if $p \in \mathcal{P}(k, X)$, then so is each partial derivative $\partial p(z)/\partial z_j$, for $z = (z_1, z_2, \ldots, z_k) \in (0, \infty)^k$ and $1 \leq j \leq k$.

Proof. Within this proof, all summations $\sum$ are meant to have finitely many terms.

We prove (i) first. For $z \in (0, \infty)^k$, we have

$$L(p_1(z), p_2(z), \ldots, p_m(z)) = L\left(\sum_{\alpha_1 \in \mathbb{R}^k} c_{\alpha_1}^{(1)} z^{\alpha_1}, \sum_{\alpha_2 \in \mathbb{R}^k} c_{\alpha_2}^{(2)} z^{\alpha_2}, \ldots, \sum_{\alpha_m \in \mathbb{R}^k} c_{\alpha_m}^{(m)} z^{\alpha_m}\right)$$

$$= \sum_{\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}^k} L(c_{\alpha_1}^{(1)}, c_{\alpha_2}^{(2)}, \ldots, c_{\alpha_m}^{(m)}) z^{\alpha_1} z^{\alpha_2} \ldots z^{\alpha_m}$$

$$= \sum_{\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}^k} L(c_{\alpha_1}^{(1)}, c_{\alpha_2}^{(2)}, \ldots, c_{\alpha_m}^{(m)}) \sum_{j=1}^{m} \alpha_j.$$
Therefore, \( L(p_1, p_2, \ldots, p_m) \in \mathcal{P}(k, X) \).

Then, (ii) is a consequence of (i) when \( m = 1 \), and (iii) is a consequence of (ii) with \( L = \pi_j \).

For the first part of (iv), we apply (i) to \( m = 2 \), \( X_1 = \mathbb{R} \), \( X_2 = X \), and \( L : \mathbb{R} \times X \to X \) defined by \( L(p, q) = pq \). For the second part, we apply the first part many times with \( X = \mathbb{R} \).

We now prove (v). Suppose \( p = (p_1, p_2, \ldots, p_n) \) and, for \( z \in \mathbb{R}^k \), \( q(z) = \sum_{\alpha \in \mathbb{Z}^k_+} c_\alpha z^\alpha \), where \( c_\alpha \in X \). For \( z \in (0, \infty)^k \), we have

\[
q(p(z)) = \sum_{\alpha \in \mathbb{Z}^k_+} c_\alpha p(z)^\alpha = \sum_{\alpha_1, \ldots, \alpha_k \in \mathbb{Z}^k_+} c_\alpha \prod_{j=1}^k p_j(z)^{\alpha_j}.
\]

By (iii), each \( p_j \in \mathcal{P}(k, \mathbb{R}) \), and by (iv), noticing that \( \alpha_j \in \mathbb{Z}^k_+ \), \( p_j(z)^{\alpha_j} \in \mathcal{P}(k, \mathbb{R}) \). By (iv), again, \( \prod_{j=1}^k p_j(z)^{\alpha_j} \) and, then, \( c_\alpha \prod_{j=1}^k p_j(z)^{\alpha_j} \in \mathcal{P}(k, X) \). Summing finitely many such terms yields \( p(q(z)) \in \mathcal{P}(k, X) \).

Finally, (vi) is obviously true. \( \square \)

The next result is similar to Lemma 2.2, but allows the function \( p \) in Eq. (2.21) to take a more general form.

**Lemma 2.9.** Given \( m \in \mathbb{Z}_+, k \in \mathbb{N} \), and a number \( t_0 > E_{m+k}(0) \). Let \( p \in \mathcal{P}(k, \mathbb{R}^n) \), and let function \( g \in C([t_0, \infty), \mathbb{R}^n) \) satisfy

\[
|g(t)| = \mathcal{O}(L_m^{-\alpha}(t)) \text{ for some } \alpha > 0.
\] (2.47)

Suppose \( y \in C([t_0, \infty), \mathbb{R}^n) \) is a solution of

\[
y' = -Ay + p(L_{m,k}(t)) + g(t) \text{ on } (t_0, \infty).
\] (2.48)

Then, there exists \( \delta > 0 \) such that

\[
|y(t) - A^{-1} p(L_{m,k}(t))| = \mathcal{O}(L_m(t)^{-\delta}).
\] (2.49)

**Proof.** In this proof, \( \sum_{\alpha} \) denotes a sum over finitely many \( \alpha \in \mathbb{R}^k \). Let

\[
p(z) = \sum_{\alpha} c_\alpha z^\alpha \text{ for } z \in (0, \infty)^k,
\] (2.50)

where each \( c_\alpha \) is constant vector in \( \mathbb{R}^n \).

Let \( 1 \leq j \leq d \). Applying \( R_{\lambda_j} \) to Eq. (2.48) gives

\[
(R_{\lambda_j} y)' = -\lambda_j (R_{\lambda_j} y) + R_{\lambda_j} p(L_{m,k}(t)) + R_{\lambda_j} g(t) \text{ on } (t_0, \infty).
\]

By the variation of constants formula, we obtain, for \( t \geq 0 \),

\[
R_{\lambda_j} y(t_0 + t) = R_{\lambda_j} y(0) e^{-\lambda_j t} + \int_0^t e^{-\lambda_j (t-\tau)} R_{\lambda_j} p(L_{m,k}(t_0 + \tau)) d\tau
\]
\[ + \int_0^t e^{-\lambda_j (t-\tau)} R_{\lambda_j} g(t_0 + \tau) d\tau.
\] (2.51)
Obviously, the first term on the right-hand side of (2.51) is of $O(L_m(t_0 + \tau)^{-\delta})$ for any $\lambda > 0$. For the last term on the right-hand side of (2.51), using property (2.47) and then applying Lemma 2.5, we have

$$\left| \int_0^t e^{-\lambda_j(t-\tau)} R_{\lambda_j} g(t_0 + \tau) d\tau \right| \leq C \int_0^t e^{-\lambda_j(t-\tau)} L_m(t_0 + \tau)^{-\gamma} d\tau = O(L_m(t_0 + \tau)^{-\gamma}).$$

Rewriting the second term on the right-hand side of (2.51) with the use of the explicit form (2.50) of $p(z)$, and applying (2.39) of Lemma 2.6 give, for any $\gamma \in (0, 1)$,

$$\int_0^t e^{-\lambda_j(t-\tau)} R_{\lambda_j} p(L_{m,k}(t_0 + \tau)) d\tau = \sum_a \left( \int_0^t e^{-\lambda_j(t-\tau)} L_{m,k}(t_0 + \tau)^a d\tau \right) R_{\lambda_j} c_a$$

$$= \sum_a \frac{1}{\lambda_j} L_{m,k}(t_0 + \tau)^a R_{\lambda_j} c_a + O(t^{-\gamma})$$

$$= A^{-1} R_{\lambda_j} \left( \sum_a c_a L_{m,k}(t_0 + \tau)^a \right) + O(L_m(t_0 + \tau)^{-\gamma}).$$

Combining the above, we obtain

$$R_{\lambda_j} y(t_0 + t) = A^{-1} R_{\lambda_j} p(L_{m,k}(t_0 + t)) + O(L_m(t_0 + t)^{-\delta})$$

for some $\delta > 0$, which implies

$$R_{\lambda_j} y(t) = A^{-1} R_{\lambda_j} p(L_{m,k}(t)) + O(L_m(t)^{-\delta}).$$

Summing up this equation in $j$ from 1 to $d$, and using (2.6) give

$$y(t) = A^{-1} p(L_{m,k}(t)) + O(L_m(t)^{-\delta}),$$

which proves (2.49). □

**Corollary 2.10.** Let $m \in \mathbb{Z}_+$ and $j \in \mathbb{N}$ such that $j > m$, and let $T_0 > E_j(0)$. Assume $g$ is a continuous function from $[T_0, \infty)$ to $\mathbb{R}^n$ satisfying $|g(t)| = O(L_m(t)^{-\alpha})$ for some $\alpha > 0$. Let $p : \mathbb{R} \to \mathbb{R}^n$ be a polynomial, and $y \in C([T_0, \infty), \mathbb{R}^n)$ be a solution of

$$y' = -Ay + p(L_j(t)) + g(t)$$

on $(T_0, \infty)$.

Then, there exists $\delta > 0$ such that

$$|y(t) - A^{-1} p(L_j(t))| = O(L_m(t)^{-\delta}). \quad (2.52)$$

**Proof.** Let $k = j - m$, define function $\tilde{p}(z_1, z_2, \ldots, z_k) = p(z_k)$. Note that $\tilde{p} \in \mathcal{P}(k, \mathbb{R}^n)$ and $\tilde{p}(L_{m,k}(t)) = p(L_j(t))$. Applying Lemma 2.9 with $\tilde{p}$ replacing $p$, we obtain (2.52) from (2.49). □
3. Basic existence result and large-time estimates

We establish basic facts for Eq. (1.7). First, we have the global existence result for small initial data and forcing function.

**Theorem 3.1.** There are positive numbers $\varepsilon_0$ and $\varepsilon_1$ such that if $y_0 \in \mathbb{R}^n$ with $|y_0| < \varepsilon_0$, and $f \in C([0, \infty), \mathbb{R}^n)$ with

$$\|f\|_\infty := \sup \{|f(t)| : t \in [0, \infty)\} < \varepsilon_1,$$

then there exists a unique solution $y \in C^1([0, \infty), \mathbb{R}^n)$ of (1.7) on $[0, \infty)$ with $y(0) = y_0$. If, in addition,

$$\lim_{t \to \infty} f(t) = 0,$$

then

$$\lim_{t \to \infty} y(t) = 0.$$  

**Proof.** (a) First, we consider the special case when $A = A_0$, where $A_0$ is in (2.2). We have

$$(Ay) \cdot y \geq \lambda_1 |y|^2 \forall y \in \mathbb{R}^n.$$  

Let $r_*$ and $c_*$ be as in (2.11). Set

$$C_0 = \min \left\{ r_*, \frac{\lambda_1}{4c_*} \right\}, \quad \varepsilon_0 = \min \left\{ \frac{C_0}{2}, c_* \right\}, \quad \text{and} \quad \varepsilon_1 = \frac{\lambda_1 C_0}{2\sqrt{2}}.$$  

Suppose $|y_0| < \varepsilon_0$ and $\|f\|_\infty < \varepsilon_1$. Note that $|y_0| < C_0$. By the local existence and uniqueness theory, see, e.g., [5,14], there exists a maximal $T \in (0, \infty]$ such that there is a unique solution $y(t)$ on $[0, T)$ that satisfies

$$|y(T)| < C_0 \text{ on } [0, T).$$

We claim $T = \infty$. Suppose this is not true, then, by (3.6), the local existence result and the maximality of $T$, the solution $y(t)$ exists on $[0, T')$ for some $T' > T$ and we have

$$|y(T)| = C_0.$$  

Let $\varepsilon > 0$ be arbitrary. Taking the dot product of (1.7) with $y(t)$, then using property (3.4), Cauchy–Schwarz’s inequality, and estimate (2.11), we obtain, for $t \in (0, T)$,

$$\frac{1}{2} \frac{d}{dt} |y|^2 = -Ay \cdot y + G(y) \cdot y + f \cdot y \leq -\lambda_1 |y|^2 + c_* |y|^2 |y| + |f||y|.$$  

Applying Cauchy’s inequality to the last product yields

$$\frac{1}{2} \frac{d}{dt} |y|^2 \leq -(\lambda_1 - c_* |y| - \varepsilon)|y|^2 + \frac{|f|^2}{4\varepsilon} \text{ on } (0, T).$$
Taking \( \varepsilon = \lambda_1/2 \) in (3.8), and utilizing estimate (3.6) for the first \(|y|\) on its right-hand side, for \( t \in (0, T) \), we have that

\[
\frac{d}{dt}|y|^2 \leq -(\lambda_1 - 2c_0C_0)|y|^2 + \frac{|f|^2}{\lambda_1} \quad \text{on } (0, T).
\]

Set \( \alpha_0 = \lambda_1/2 > 0 \). With \( C_0 \) defined in (3.5), we have \( \lambda_1 - 2c_0C_0 \geq \alpha_0 \), and, hence,

\[
\frac{d}{dt}|y|^2 \leq -\alpha_0|y|^2 + \frac{|f|^2}{\lambda_1} \quad \text{on } (0, T).
\]  

By Gronwall’s inequality and assumption (3.1), we have for \( t \in [0, T) \) that

\[
|y(t)|^2 \leq |y_0|^2 e^{-\alpha_0 t} + \frac{\varepsilon_0^2}{\lambda_1 \alpha_0} \leq \varepsilon_0^2 + \frac{\varepsilon_1^2}{\lambda_1 \alpha_0} \leq \frac{C_0^2}{2}.
\]

Letting \( t \to T^- \), we obtain \( y(T) \leq C_0/\sqrt{2} \), which contradicts (3.7). Therefore \( T = \infty \).

Now, assume (3.2). We have (3.9) holds for all \( t > 0 \). Applying [15, Lemma 3.9] to (3.9) gives

\[
0 \leq \limsup_{t \to \infty} |y(t)|^2 \leq \limsup_{t \to \infty} \frac{|f(t)|^2}{\alpha_0 \lambda_1} = 0.
\]

This proves (3.3).

(b) Consider the general case of \( A \) as in (2.2). We make use of the transformations

\[
z = Sy, \quad \tilde{G}(z) = SG(S^{-1}z), \quad \tilde{f}(t) = Sf(t).
\]  

Then, Eq. (1.7) is equivalent to

\[
z' = -A_0z + \tilde{G}(z) + \tilde{f}(t).
\]  

There exists a constant \( c \geq 1 \) such that

\[
ce^{-1}|y| \leq |z| \leq c|y| \quad \text{and} \quad c^{-1}|f| \leq |\tilde{f}| \leq c|f|.
\]  

Thanks to the equivalent norms in (3.12), \( \tilde{G} \) and \( \tilde{f} \) have similar properties to \( G \) and \( f \), respectively. By part (a) applied to Eq. (3.11), we obtain the results for \( z(t) \) and \( \tilde{f}(t) \). Then, thanks to relation (3.12) again, the results for \( y(t) \) and \( f(t) \) follow. We omit the details.

There are examples of \( G(y) \) such that the global solution \( y(t) \) exists even when \( |y_0| \) is not small. Theorem 3.1 only guarantees that the set of global solutions of our interest is not empty. Certainly, not all solutions decay to zero. For example, even when \( f = 0 \), the system (1.7) may have a nonzero steady state.

When \( f(t) \) has more specific decay than (3.2), then we can obtain corresponding large-time estimates for \( y(t) \).
Theorem 3.2. Let $\psi(t) = e^t$ or $\psi(t) = L_k(t)$ for some integer $k \geq 0$. Assume there is $T \geq 0$ such that $f \in C((T, \infty))$ and

$$|f(t)| = O(\psi(t)^{-\alpha}) \text{ for some } \alpha > 0. \quad (3.13)$$

Let $y(t)$ be a solution of (1.7) on $(T, \infty)$ and satisfy

$$\liminf_{t \to \infty} |y(t)| < \varepsilon_0, \quad (3.14)$$

where $\varepsilon_0$ is as in Theorem 3.1. Then,

$$y(t) = \begin{cases} O(\psi(t)^{-\min[\lambda_1, \alpha]}), & \text{if } \psi(t) = e^t, \; \alpha \neq \lambda_1, \\ O(t \psi(t)^{-\alpha}), & \text{if } \psi(t) = e^t, \; \alpha = \lambda_1, \\ O(\psi(t)^{-\alpha}), & \text{if } \psi(t) = L_k(t). \end{cases} \quad (3.15)$$

Proof. Similar to the proof of Theorem 3.1, by using the transformations in (3.10) and property (3.12), we can assume $A$ takes the form (2.2). Note in this case that

$$\|e^{-tA}\| = e^{-\lambda_1 t} \text{ for all } t \geq 0. \quad (3.16)$$

Let $\varepsilon_1$ be as in Theorem 3.1. By (3.14) and (3.13), there exists $t_\ast > T$ such that

$$|y(t_\ast)| < \varepsilon_0 \text{ and } \sup\{|f(t) : t \geq t_\ast\} < \varepsilon_1.$$ 

Note also that $f$ satisfies (3.2). Applying Theorem 3.1 yields (3.3). Denote $\mu = \min[\lambda_1, \alpha]$. Let $c_\ast$ be as in (2.11), and $\delta$ be an arbitrary positive number with $\delta < \min(2\varepsilon_0c_\ast, \mu)$. By assumption (3.13) and the proved fact (3.3), there exist $t_0 > t_\ast$ and $C > 0$ such that $y \in C^1([t_0, \infty), \mathbb{R}^n)$ and

$$|y(t)| \leq \frac{\delta}{2c_\ast} < \varepsilon_0, \quad |f(t)| \leq C\psi(t)^{-\alpha} < \varepsilon_1 \text{ for all } t \geq t_0. \quad (3.17)$$

Then, (3.8), with the choice $\varepsilon = \delta/2$, and the estimates in (3.17) yield

$$\frac{d}{dt}|y(t_0 + t)|^2 \leq -2(\lambda_1 - \delta/2 - \delta/2)|y(t_0 + t)|^2 + C^2\delta^{-1}\psi(t_0 + t)^{-2\alpha}$$

$$= -2(\mu - \delta)|y(t_0 + t)|^2 + C^2\delta^{-1}\psi(t_0 + t)^{-2\alpha}$$

for all $t > 0$. By Gronwall’s inequality, we have, for $t > 0$,

$$|y(t_0 + t)|^2 \leq |y(t_0)|^2 e^{-2(\mu-\delta)t} + C^2\delta^{-1} \int_0^t e^{-2(\mu-\delta)(t-\tau)}\psi(t_0 + \tau)^{-2\alpha}d\tau.$$ 

Estimating the last integral, either by using (2.12) in case $\psi(t) = e^t$, noticing that $\alpha > \mu - \delta$, or by applying Lemma 2.5 in case $\psi(t) = L_k(t)$, we obtain

$$|y(t_0 + t)|^2 = \begin{cases} O(\psi(t_0 + t)^{-2(\mu-\delta)}) \quad & \text{if } \psi(t) = e^t, \\ O(\psi(t_0 + t)^{-2\alpha}) \quad & \text{if } \psi(t) = L_k(t). \end{cases} \quad (3.18)$$
Then, the last estimate in (3.15) follows from the second estimate in (3.18). Consider $\psi(t) = e^{t}$ now. Choose, additionally, $\delta < \mu/4$ such that $2(\mu - \delta) \neq \lambda_1$. For any $T_0 \geq t_0$, and $t \geq 0$, by variation of constant formula,

$$y(T_0 + t) = e^{-tA}y(T_0) + \int_{0}^{t} e^{-(t-\tau)A}\{G(y(T_0 + \tau)) + f(T_0 + \tau)\}d\tau.$$ 

Selecting $T_0$ sufficiently large, using (3.16), (2.11), the first estimate in (3.18), and (3.13), we obtain

$$|y(T_0 + t)| \leq e^{-\lambda_1 t}|y(T_0)| + C'\int_{0}^{t} e^{-(\lambda_1 - \delta)(t-\tau)} + e^{-\alpha \tau}d\tau$$

for some $C' > 0$. Note, by using (2.12), that

$$\int_{0}^{t} e^{-\lambda_1(t-\tau)} e^{-2(\mu - \delta)\tau}d\tau = O(e^{-\min\{\lambda_1, 3\mu/2\}t}) = O(e^{-\mu t}).$$

Then, estimating $\int_{0}^{t} e^{-\lambda_1(t-\tau)} e^{-\alpha \tau}d\tau$ by using (2.12) again, we obtain

$$y(T_0 + t) = \begin{cases} O(e^{-\mu t}), & \text{if } \alpha \neq \lambda_1, \\ O(te^{-\mu t}), & \text{if } \alpha = \lambda_1. \end{cases}$$

By shifting time $T_0 + t$ back to $t$, we obtain the first two estimates in (3.15).

□

4. Expansions with one secondary base function

In this section, we study the asymptotic expansions of the form (1.13) in Definition 1.4. They are expressed in terms of one primary base function $\psi$ and one secondary base function $\phi$.

We explain Definition 1.4 further now.

(a) The first limit in (1.11) obviously follows the second one and relation (1.12). Nonetheless, we stated both limits in (1.11) to make a clear presentation.

(b) It follows (1.11) and (1.12) that

$$\lim_{t \to \infty} \frac{\phi(t)^{\lambda}}{\psi(t)^{\alpha}} = 0 \text{ for all } \lambda \in \mathbb{R}, \alpha > 0. \quad (4.1)$$

(c) If $g(t) = \sum_{k=1}^{N} p_k(\phi(t))\psi(t)^{-\gamma_k}$, then clearly the expansion (1.15) holds.

(d) Consider (1.15). Suppose $(\gamma_k)_{k=1}^{N}$ can be extended as to a sequence $(\gamma_k)_{k=1}^{\infty}$ as in (i). Set $p_k = 0$ for $k > N$. Then, thanks to (1.16), we obtain (1.13). This means the finite expansion (1.15) is a special case of (1.13). Such an extension is, of course, not unique, and some may be more appropriate than the others, see, e.g., Scenarios 4.7 and 4.8.
Another type of asymptotic expansions in [3, Definition 4.1] is
\[ g(t) \sim \sum_{k=1}^{\infty} \xi_k \psi_k(t), \tag{4.2} \]
where \( \xi_k \)'s are constant vectors, and \( (\psi_k)_{k=1}^{\infty} \) is a general system of decaying functions.

On the one hand, \( p_k(\phi(t)) \) in (1.13) is more general than \( \xi_k \) in (4.2). On the other hand, the \( \psi_k \) in (4.2) is more general than \( \psi^{-\gamma_k} \) in (1.13).

It turns out that the asymptotic expansion (1.13), for any given function \( g(t) \), is unique. This is a direct consequence of the following lemma.

**Lemma 4.1.** Let the normed space \( (X, \| \cdot \|_X) \) and functions \( \psi, \phi \) be as in Definition 1.4. Given a function \( g : (T', \infty) \to X \) for some \( T' \in \mathbb{R} \). Let \( 0 \leq \gamma_1 < \ldots < \gamma_N \) for some \( N \in \mathbb{N} \). Suppose \( p_1, \ldots, p_N : \mathbb{R} \to X \) are polynomials such that (1.14) holds for some \( \mu > \gamma_N \). Then, such polynomials \( p_1, \ldots, p_N \) are unique.

**Proof.** Suppose \( \hat{p}_k \)'s are \( X \)-valued polynomials, for \( 1 \leq k \leq N \), such that
\[
\left\| g(t) - \sum_{k=1}^{N} \hat{p}_k(\phi(t)) \psi(t)^{-\gamma_k} \right\|_X = \mathcal{O}(\psi(t)^{-\hat{\mu}}) \quad \text{for some } \hat{\mu} > \gamma_N.
\]

For each \( k \), let \( h_k = p_k - \hat{p}_k \), which is an \( X \)-valued polynomial. By the triangle inequality, we have
\[
\left\| \sum_{k=1}^{N} h_k(\phi(t)) \psi(t)^{-\gamma_k} \right\|_X \leq \left\| \sum_{k=1}^{N} p_k(\phi(t)) \psi(t)^{-\gamma_k} - g(t) \right\|_X + \left\| g(t) - \sum_{k=1}^{N} \hat{p}_k(\phi(t)) \psi(t)^{-\gamma_k} \right\|_X
\]
\[
(4.3)
\]
hence
\[
\left\| \sum_{k=1}^{N} h_k(\phi(t)) \psi(t)^{-\gamma_k} \right\|_X = \mathcal{O}(\psi(t)^{-\mu_*}), \quad \text{where } \mu_* = \min\{\mu, \hat{\mu}\} > \gamma_N.
\]

Multiplying this equation by \( \psi(t)^{\gamma_1} \) and making use of the property (4.1), we have
\[
\lim_{t \to \infty} h_1(\phi(t)) = 0. \tag{4.4}
\]

Because \( h_1 \) is a polynomial, together with (4.4) and the fact \( \phi(t) \to \infty \) as \( t \to \infty \), we deduce \( h_1 = 0 \). Repeating this argument for the remaining \( h_k \)'s, we obtain \( h_k = 0 \) for all \( k = 1, \ldots, N \). \( \square \)

As a side note, the power \( \gamma_1 \) in Lemma 4.1 is in \( [0, \infty) \), which is more general than the positive \( \gamma_1 \) in expansion (1.13). This small alteration aims to provide a short argument at the beginning of the proofs of Lemmas 2.1 and 2.2.

Of course, there are many choices of \( (\psi, \phi) \). We will develop our theory for the following three typical cases.
**Definition 4.2.** We define three types of pair of functions \((\psi, \phi)\).

Type 1: \((\psi, \phi) = (e^t, t)\),

Type 2: \((\psi, \phi) = (t, \ln t)\), and

Type 3: \((\psi, \phi) = (L_{n_*(t)}, L_{n_*(t)})\), with \(n_* > m_* \geq 1\).

Clearly, the functions \((\psi, \phi)\) in Definition 4.2 satisfy conditions (1.11) and (1.12) in Definition 1.4. For the rest of this section, \((\psi, \phi)\) is one of the three types in Definition 4.2.

We note that

\[
(\psi'(t), \phi'(t)) = \begin{cases} 
(\psi(t), 1), & \text{for Type 1,} \\
(1, \psi(t)^{-1}), & \text{for Type 2,} \\
(\mathcal{O}(\psi(t)^{-\alpha}), \mathcal{O}(\psi(t)^{-\beta})) & \forall \alpha, \beta > 0, \text{ for Type 3.}
\end{cases}
\]  

We now focus on the differential equation of our interest—Eq. (1.7). We need a basic requirement on its forcing function \(f(t)\).

**Assumption 4.3.** There exists a number \(T_f \geq 0\) such that \(f\) is continuous on \([T_f, \infty)\).

More specific conditions on \(f\) will be specified later for each result.

**Definition 4.4.** Let \(S\) be a subset of \(\mathbb{R}\).

We say \(S\) preserves the addition if \(x + y \in S\) for all \(x, y \in S\).

We say \(S\) preserves the unit increment if \(x + 1 \in S\) for all \(x \in S\).

The main assumption on \(f\) for this section is the following.

**Assumption 4.5.** The function \(f(t)\) admits the asymptotic expansion, in the sense of Definition 1.4 with \(X = \mathbb{R}^n\) and \(\|\cdot\|_X = |\cdot|\),

\[
f(t) \sim \sum_{k=1}^\infty p_k(\phi(t))\psi(t)^{-\mu_k}, \tag{4.6}
\]

where \(p_k\)'s are \(\mathbb{R}^n\)-valued polynomials, \((\mu_k)_{k=1}^\infty\) is a divergent, strictly increasing sequence of positive numbers. Moreover, the set \(S := \{\mu_k : k \in \mathbb{N}\}\) satisfies

(a) \(S\) preserves the addition.

(b) In case of Type 1, \(S\) contains all eigenvalues of \(A\).

(c) In case of Type 2, \(S\) preserves the unit increment.

Note from (b) of Assumption 4.5 that in case of Type 1, one has

\[
\mu_1 \leq \lambda_1. \tag{4.7}
\]

Below, we discuss typical scenarios when Assumption 4.5 holds.
**Scenario 4.6.** The forcing function $f(t)$ in (1.7) has the following expansion, in the sense of Definition 1.4,

$$f(t) \sim \sum_{k=1}^{\infty} \hat{p}_k(\phi(t))\psi(t)^{-\alpha_k} \text{ in } \mathbb{R}^n,$$

(4.8)

where $\hat{p}_k$’s are polynomials from $\mathbb{R}$ to $\mathbb{R}^n$, and $(\alpha_k)_{k=1}^{\infty}$ is a divergent, strictly increasing sequence of positive numbers.

For Type 1, let $S$ be the joint additive semigroup generated by both $\lambda_j$’s and $\alpha_j$’s, i.e.,

$$S = \left\{ \sum_{j=1}^{k} \lambda_{s_j} + \sum_{j=1}^{m} \alpha_{\ell_j} : k, m \in \mathbb{Z}_+, k^2 + m^2 > 0, 1 \leq s_j \leq d, \ell_j \in \mathbb{N} \right\}.$$

(4.9)

For Type 2, let $S$ be defined by

$$S = \left\{ k + \sum_{j=1}^{m} \alpha_{\ell_j} : k \in \mathbb{Z}_+, m \in \mathbb{N}, \ell_j \in \mathbb{N} \right\}.$$

(4.10)

For Type 3, let $S$ be the additive semigroup generated by $\alpha_k$’s, i.e.,

$$S = \left\{ \sum_{j=1}^{m} \alpha_{\ell_j} : m \in \mathbb{N}, \ell_j \in \mathbb{N} \right\}.$$

(4.11)

Re-arrange the set $S$ as a sequence

$$S = (\mu_k)_{k=1}^{\infty}$$

of strictly increasing positive numbers.

(4.12)

Then, $(\mu_k)_{k=1}^{\infty}$ and $S$ satisfy the properties in Assumption 4.5. Note that the set $S$ in (4.9), (4.10), (4.11) contains the sequence $(\alpha_k)_{k=1}^{\infty}$. Therefore, we can rewrite expansion (4.8), after re-indexing $\hat{p}_k$’s, as the expansion (4.6). For example, the first term in expansion (4.6) is identified by

$$\mu_1 = \begin{cases} 
\min\{\lambda_1, \alpha_1\}, & \text{for Type 1}, \\
\alpha_1, & \text{for Types 2 and 3},
\end{cases}$$

$$p_1 = \begin{cases} 
0, & \text{for Type 1 with } \lambda_1 < \alpha_1, \\
\hat{p}_1, & \text{for Type 1 with } \lambda_1 \geq \alpha_1, \text{ and for Types 2 and 3}.
\end{cases}$$

**Scenario 4.7.** In case of finite approximation, as in Definition 1.4(ii),

$$f(t) \sim \sum_{k=1}^{N} \hat{p}_k(\phi(t))\psi(t)^{-\alpha_k} \text{ in } \mathbb{R}^n,$$

(4.13)
then, corresponding to Type 1, 2, 3, the set $S$ is defined by formulas (4.9), (4.10), (4.11) with restriction $1 \leq \ell_j \leq N$. This set $S$ is still infinite, can be arranged as sequence $(\mu_k)_{k=1}^\infty$ as in Scenario 4.6, and contains $(\alpha_k)_{k=1}^N$. Hence, again, we can rewrite (4.13) as (4.6).

Scenario 4.8. Function $f$ decays faster than any exponential functions, i.e., $e^{\alpha t} f(t) \to 0$ as $t \to \infty$ for any $\alpha > 0$. (This includes the case $f = 0$.) Then, we only consider Type 1, and let $S$ be the semigroup generated by the spectrum of $A$, i.e.,

$$
S = \left\{ \sum_{j=1}^{k} \lambda_{s_j} : k \in \mathbb{N}, \ 1 \leq s_j \leq d \right\}.
$$

Same as in Scenarios 4.6 and 4.7, we can write (4.6) with $p_k = 0$ for all $k \in \mathbb{N}$.

Regarding expansion (4.6), denote $\tilde{p}_k(t) = p_k(\phi(t))$, and

$$
f_k(t) = \tilde{p}_k(t)\psi(t)^{-\mu_k} \quad \text{and} \quad \tilde{f}_N(t) = \sum_{k=1}^{N} f_k(t). \quad (4.14)
$$

Then, according to Definition 1.4, for any $N \in \mathbb{N}$, one has

$$
|f(t) - \tilde{f}_N(t)| = \left| f(t) - \sum_{k=1}^{N} f_k(t) \right| = O(\psi(t)^{-\mu_N - \varepsilon_N}) \quad \text{for some } \varepsilon_N > 0. \quad (4.15)
$$

The type of solutions of Eq. (1.7) that will be the subject of our analysis is the following.

Assumption 4.9. There exists a number $T_0 \geq 0$ such that $y \in C^1((T_0, \infty))$ is a solution of (1.7) on $(T_0, \infty)$, and $y(t) \to 0$ as $t \to \infty$.

Our first main result on the asymptotic expansion for solutions of (1.7) is the next theorem. In this theorem, we denote

$$
\varepsilon = \begin{cases} 
1, & \text{for Type 1,} \\
0, & \text{for Types 2 and 3.} 
\end{cases}
$$

Theorem 4.10. Let Assumptions 4.3 and 4.5 hold. Let $y(t)$ be a solution of (1.7) as in Assumption 4.9. Then, there exist unique polynomials $q_k : \mathbb{R} \to \mathbb{R}^n$, for $k \in \mathbb{N}$, such that $y(t)$ admits the expansion, in the sense of Definition 1.4,

$$
y(t) \sim \sum_{k=1}^{\infty} q_k(\phi(t))\psi(t)^{-\mu_k}. \quad (4.16)
$$

Moreover, the polynomials $q_k$’s solve, on $\mathbb{R}$, the following equations:

$$
Aq_1 + \varepsilon(q'_1 - \mu_1 q_1) = p_1, \quad (4.17)
$$
and, for \( k \geq 2 \),

\[
Aq_k + \epsilon (q_k' - \mu_k q_k) = \sum_{m \geq 2} \sum_{\mu_j + \mu_j + \ldots + \mu_j = \mu_k} G_m(q_{j_1}, q_{j_2}, \ldots, q_{j_m}) + p_k + \chi_k,
\]

where \( \chi_k : \mathbb{R} \to \mathbb{R}^n \) is a polynomial defined, in cases of Types 1 and 3, by \( \chi_k = 0 \), and, in case of Type 2, by

\[
\chi_k = \begin{cases} 
\mu \lambda q_\lambda - q_\lambda', & \text{if there exists } \mu \in [1, k-1] \text{ such that } \mu + 1 = \mu_k, \\
0, & \text{otherwise.} 
\end{cases}
\]

We provide some explanations to the theorem above.

(a) In (4.18), it is understood that \( j_1, j_2, \ldots, j_m \geq 1 \). Also, if the set of indices is empty, then the sum is understood to be zero.

(b) Consider the double summation in (4.18). Since \( m \geq 2 \) and \( \mu_j > 0 \), we have each \( \mu_j \leq k - 1 \), hence \( j_i \leq k - 1 \). Therefore, those terms \( q_{j_i}'s \) in (4.18) come from the previous step.

(c) For any numbers \( M \geq \mu_k / \mu_1 \) and \( Z \geq k - 1 \), one has

\[
\sum_{m \leq M} \sum_{1 \leq j_1, j_2, \ldots, j_m \leq Z} = \sum_{m \geq 2} \sum_{2 \leq m \leq M} \sum_{\mu_j + \mu_j + \ldots + \mu_j = \mu_k}.
\]

We quickly verify (4.20). Since, obviously, the sum on the left-hand side is part of sum on the right-hand side, it suffices to show the reverse. Consider the right-hand side of (4.20). Firstly, due to (b), we have \( j_i \leq k - 1 \leq Z \). Secondly, we have \( m \mu_1 \leq \sum_{i=1}^m \mu_j = \mu_k \), which yields \( m \leq \mu_k / \mu_1 \leq M \).

Therefore, thanks to (4.20), the double summation in (4.18), in fact, is a finite sum.

(d) For our convenience, define

\[
\chi_1 = 0 \text{ as a function from } \mathbb{R} \text{ to } \mathbb{R}^n.
\]

Consider formula (4.18) even for \( k = 1 \). Then, there are no indices to satisfy the conditions in the double sum on the right-hand side of (4.18). Hence, by convention in (a), it is 0. Therefore, (4.18) for \( k = 1 \), in fact, reads as (4.17).

(e) With the observation in (d), we can combine (4.17) and (4.18) into

\[
Aq_k + \epsilon (q_k' - \mu_k q_k) = \sum_{m \geq 2} \sum_{\mu_j + \mu_j + \ldots + \mu_j = \mu_k} G_m(q_{j_1}, q_{j_2}, \ldots, q_{j_m}) + p_k + \chi_k,
\]

for all \( k \in \mathbb{N} \).

(f) The index \( \lambda \) in (4.19), if exists, is obviously unique. Thus, \( \chi_k \) is well-defined.
Proof of Theorem 4.10. The uniqueness of polynomials \( q_k \)'s comes from Lemma 4.1. It remains to establish (4.16)–(4.18).

For \( N \in \mathbb{N} \), denote by \( (T_N) \) the statement: Equation (4.22) holds true on \( \mathbb{R} \), for \( k = 1, 2, \ldots, N \), and

\[
\left| y(t) - \sum_{k=1}^{N} q_k(\phi(t))\psi(t)^{-\mu_k} \right| = O(\psi(t)^{-\mu_N-\delta_N}) \text{ for some } \delta_N > 0.
\]

We will find, by induction, polynomials \( q_k : \mathbb{R} \to \mathbb{R}^n \), for \( k \in \mathbb{N} \), such that \( (T_N) \) holds true for all \( N \in \mathbb{N} \). In the calculations below, \( t \) will be sufficiently large.

First step. Let \( N = 1 \). Note, by the triangle inequality, (4.15), (4.14) and (4.1), that

\[
|f(t)| \leq |f(t) - f_1(t)| + |f_1(t)| = O(\psi(t)^{-\mu_1-\delta_1}) + O(\psi(t)^{-\mu_1+\delta})
\]

for all \( \delta > 0 \). This yields

\[
f(t) = O(\psi(t)^{-\mu_1+\delta}) \quad \forall \delta > 0. \tag{4.23}
\]

Applying Theorem 3.2, with the use of property (4.23), gives

\[
y(t) = O(\psi(t)^{-\mu_1+\delta}) \quad \forall \delta > 0. \tag{4.24}
\]

Let \( w_0(t) = \psi(t)^{\mu_1}y(t) \). Then,

\[
w_0' = \psi^{\mu_1}y' + \mu_1\psi^{\mu_1-1}y' = \psi^{\mu_1}(-Ay + G(y) + f) + \mu_1\psi^{\mu_1-1}y'.
\]

Thus,

\[
w_0' = -Aw_0 + \psi^{\mu_1}G(y) + \psi^{\mu_1}(f - f_1) + \psi^{\mu_1}f_1 + \mu_1\psi^{\mu_1-1}y'. \tag{4.25}
\]

Now fix \( \delta > 0 \) such that \( \delta < \min\{1, \mu_1/2\} \). By (2.11) and (4.24) we have

\[
\psi(t)^{\mu_1}|G(y(t))| = \psi(t)^{\mu_1}O(\psi(t)^{-2\mu_1+2\delta}) = O(\psi(t)^{-(\mu_1-2\delta)}). \tag{4.26}
\]

By (4.15),

\[
\psi(t)^{\mu_1}|f(t) - f_1(t)| = \psi(t)^{\mu_1}O(\psi(t)^{-\mu_1-\delta_1}) = O(\psi(t)^{-\delta_1}). \tag{4.27}
\]

By formula of \( \psi'(t) \) in (4.5) and estimate (4.24),

\[
\mu_1\psi(t)^{\mu_1-1}\psi'(t)y(t) = \begin{cases} 
\mu_1 w_0(t) & \text{for Type 1}, \\
\psi(t)^{\mu_1-1}O(\psi(t)^{-\mu_1+\delta}) & \text{for Type 2}, \\
\psi(t)^{\mu_1-1}O(\psi(t)^{-\lambda})O(\psi(t)^{-\mu_1+\delta}) & \forall \lambda > 0, \text{ for Type 3}.
\end{cases}
\]

Thus,

\[
\mu_1\psi(t)^{\mu_1-1}\psi'(t)y(t) = \begin{cases} 
\mu_1 w_0(t) & \text{for Type 1}, \\
O(\psi(t)^{-(1-\delta)}) & \text{for Types 2 and 3}.
\end{cases} \tag{4.28}
\]
Combining (4.25), (4.26), (4.27), (4.28) with the fact $\psi(t)^{\mu_1}f_1(t) = p_1(\phi(t))$, we arrive at
\[ w'_0(t) = -(A - \epsilon_1 I_n)w_0(t) + p_1(\phi(t)) + O(\psi(t)^{-\delta_1}), \quad (4.29) \]
where $\delta_1 = \min\{\mu_1 - 2\delta, \epsilon_1, 1 - \delta\} > 0$.

In case of Type 1, we apply Lemma 2.2 to equation (4.29) taking $\lambda = \mu_1$. Thanks to (4.7), we have $\lambda \leq \lambda_1$, hence there is no need to check condition (2.22).

In case of Types 2 and 3, we apply Corollary 2.10 to Eq. (4.29) noticing that $\epsilon = 0$. Then, there exist a polynomial $q_1 : \mathbb{R} \to \mathbb{R}^n$ and a number $\delta_1 > 0$ such that (4.17), which is (4.22) for $k = 1$, holds on $\mathbb{R}$, and
\[ |w_0(t) - q_1(\phi(t))| = O(\psi(t)^{-\delta_1}). \]

Multiplying the last equation by $\psi(t)^{-\mu_1}$ yields
\[ |y(t) - q_1(\phi(t))\psi(t)^{-\mu_1}| = O(\psi(t)^{-\mu_1-\delta_1}). \]

Therefore, statement $(T_1)$ holds true.

**Induction step.** Let $N \geq 1$. Suppose there are polynomials $q_k$’s, for $1 \leq k \leq N$, such that the statement $(T_N)$ is true.

For $k = 1, 2, \ldots, N$, let $\tilde{q}_k(t) = q_k(\phi(t))$, and
\[ y_k(t) = \tilde{q}_k(t)\psi(t)^{-\mu_k}, \quad \tilde{y}_N(t) = \sum_{k=1}^{N} y_k(t), \quad v_N(t) = y(t) - \tilde{y}_N(t). \quad (4.30) \]

By the induction hypothesis, we have
\[ v_N(t) = O(\psi(t)^{-\mu_N-\delta_N}) \text{ for some } \delta_N > 0. \quad (4.31) \]

Note from (4.1) that
\[ |y_k(t)| = O(\psi(t)^{-\mu_k+\delta}), \quad \forall \delta > 0. \]

Then,
\[ |\tilde{y}_N(t)| \leq \sum_{k=1}^{N} |y_k(t)| = O(\psi(t)^{-\mu_1+\delta}), \quad \forall \delta > 0. \quad (4.32) \]

(a) Let $w_N(t) = \psi(t)^{\mu_N+1}v_N(t)$. We write an appropriate differential equation for $w_N(t)$. We have
\[ w'_N = \mu_{N+1}\psi^{\mu_N+1}y'v_N + \psi^{\mu_N+1}(y' - \sum_{k=1}^{N} y'_k) \]
\[ = \mu_{N+1}\psi^{\mu_N+1}y'v_N + \psi^{\mu_N+1}(-Ay + G(y) + f) - \psi^{\mu_N+1}\sum_{k=1}^{N} y'_k. \]
Using $\psi'$ in (4.5) and estimate (4.31), we have
\[
\mu_{N+1} \psi(t) \mu_{N+1} \psi'(t) v_N(t)
= \begin{cases} 
\mu_{N+1} w_N(t) & \text{for Type 1}, \\
\mu_{N+1} \psi(t) \mu_{N+1} \psi'(t) \mathcal{O}(\psi(t)^{-\mu - \delta_N}) & \text{for Type 2}, \\
\mu_{N+1} \psi(t) \mu_{N+1} \psi'(t) \mathcal{O}(\psi(t)^{-\lambda}) \mathcal{O}(\psi(t)^{-\mu - \delta_N}), \forall \lambda > 0, & \text{for Type 3}.
\end{cases}
\]

Note that $\mu_{N+1} \leq \mu_N + 1$ for Type 2, thanks to Assumption 4.5(c). For Type 3, take $\lambda > \mu_N + 1 - \mu_{N+1}$. Then,
\[
\mu_{N+1} \psi(t) \mu_{N+1} \psi'(t) v_N(t) = \epsilon \mu_{N+1} w_N(t) + \mathcal{O}(\psi(t)^{-\delta_N}).
\]

Also,
\[
\psi^{\mu_{N+1}} A y = \psi^{\mu_{N+1}} A (\tilde{y}_N + v_N) = \psi^{\mu_{N+1}} A \tilde{y}_N + A w_N,
\]
and
\[
\psi(t)^{\mu_{N+1}} f(t) = \psi(t)^{\mu_{N+1}} \left[ f_N(t) + f_{N+1}(t) + \mathcal{O}(\psi(t)^{-\mu_{N+1} - \varepsilon_{N+1}}) \right]
= \psi(t)^{\mu_{N+1}} f_N(t) + \tilde{p}_{N+1}(t) + \mathcal{O}(\psi(t)^{-\varepsilon_{N+1}}).
\]
Thus,
\[
u_N' = -(A - \epsilon \mu_{N+1} I_n) w_N - \psi(t)^{\mu_{N+1}} A \tilde{y}_N + \psi(t)^{\mu_{N+1}} G(y) + \tilde{p}_{N+1}(t)
+ \psi(t)^{\mu_{N+1}} \tilde{f}_N(t) - \psi(t)^{\mu_{N+1}} \sum_{k=1}^{N} y_k' + \mathcal{O}(\psi(t)^{-\min[\delta_N, \varepsilon_{N+1}]}). \tag{4.33}
\]

Below, two terms $\psi(t)^{\mu_{N+1}} G(y(t))$ and $\psi(t)^{\mu_{N+1}} \sum_{k=1}^{N} y_k'(t)$ in (4.33) will be further calculated.

(b) We calculate $\psi(t)^{\mu_{N+1}} G(y(t))$. Letting $\delta = \mu_1/2$ in (4.24) yields
\[
y(t) = \mathcal{O}(\psi(t)^{-\mu_1/2}). \tag{4.34}
\]
Let $M_{N+1}$ be the smallest integer such that
\[
M_{N+1} \geq \frac{2 \mu_{N+1}}{\mu_1}. \tag{4.35}
\]
Note that $M_{N+1} \geq 2$. By (1.8), there exists $\theta_N > 0$ such that
\[
|G(y) - \sum_{m=2}^{M_{N+1}} G_m(y)| = \mathcal{O}(|y|^{M_{N+1} + \theta_N}) \text{ as } y \to 0. \tag{4.36}
\]
We calculate and estimate, using (4.36) and (4.34),
\[
\psi(t)^{\mu_{N+1}} G(y(t)) = \psi(t)^{\mu_{N+1}} \left\{ \sum_{m=2}^{M_{N+1}} G_m(y(t)) + \mathcal{O}(|y(t)|^{M_{N+1} + \theta_N}) \right\}
= \psi(t)^{\mu_{N+1}} \sum_{m=2}^{M_{N+1}} G_m(y(t)) + \psi(t)^{\mu_{N+1}} \mathcal{O}(\psi(t)^{-3(M_{N+1} + \theta_N)\mu_1/2}).
\]
Thus,
\[
\psi(t)^{\mu_{N+1}} G(y(t)) = \psi(t)^{\mu_{N+1}} \sum_{m=2}^{M_{N+1}} G_m(y(t)) + O\left(\psi(t)^{-\delta'_{N+1}}\right),
\] (4.37)

where \(\delta'_{N+1} = (M_{N+1} + \theta_N)\mu_1/2 - \mu_{N+1}\), which is a positive number thanks to (4.35).

For each \(G_m(y(t))\) in (4.37), we rewrite and estimate it, using (2.8) and (2.9), as
\[
G_m(y(t)) = G_m(\tilde{y}_N + v_N) = G_m(\tilde{y}_N + v_N, \ldots, \tilde{y}_N + v_N)
\]
\[
= G_m(\tilde{y}_N, \ldots, \tilde{y}_N) + \sum_{m=1}^{M_{N+1}} \mathcal{O}(|\tilde{y}_N(t)|^k|v_N(t)|^{m-k})
\]
\[
= G_m(\tilde{y}_N(t)) + \mathcal{O}(|v_N(t)|^2) + \mathcal{O}(|\tilde{y}_N(t)||v_N(t)|).
\]
The last two terms are estimated, by using (4.31) and (4.32) with \(\delta = \delta_N/2\), by
\[
\mathcal{O}(\psi(t)^{-2(\mu_{N}+\delta_N)}) + \mathcal{O}(\psi(t)^{-\mu_1+\delta_N/2}\psi(t)^{-\mu_{N+1}-\delta_N}).
\]

Since \(\mu_{N+1} \leq \mu_N + \mu_1 \leq 2\mu_N\), we obtain
\[
G_m(y(t)) = G_m(\tilde{y}_N(t)) + \mathcal{O}(\psi(t)^{-\mu_{N+1}-\delta_N/2}).
\] (4.38)

Summing up (4.38) in \(m\) and combining with (4.37), we obtain
\[
\psi(t)^{\mu_{N+1}} G(y(t)) = \psi(t)^{\mu_{N+1}} \sum_{m=2}^{M_{N+1}} G_m(\tilde{y}_N(t)) + \mathcal{O}(\psi(t)^{-\delta_N/2}) + \mathcal{O}\left(\psi(t)^{-\delta_{N+1}}\right).
\] (4.39)

We continue to manipulate
\[
\sum_{m=2}^{M_{N+1}} G_m(\tilde{y}_N(t)) = \sum_{m=2}^{M_{N+1}} \sum_{1 \leq j_1, j_2, \ldots, j_m \leq N} \frac{G_m(\tilde{q}_{j_1}(t), \tilde{q}_{j_2}(t), \ldots, \tilde{q}_{j_m}(t))}{\psi(t)^{\mu_{j_1}+\mu_{j_2}+\cdots+\mu_{j_m}}}.
\] (4.40)

Note from (2.9), the fact that each \(\tilde{q}_{j_i}\) is a polynomial, and relation (4.1), that
\[
|G_m(\tilde{q}_{j_1}(t), \tilde{q}_{j_2}(t), \ldots, \tilde{q}_{j_m}(t))| \leq \|G_m\| \cdot \prod_{i=1}^{m} |q_{j_i}(\phi(t))| = \mathcal{O}(\psi(t)^{\delta}) \quad \forall \delta > 0.
\] (4.41)

Thanks to Assumption 4.5(a), the set \(S\) preserves the addition. Hence, the sum \(\mu_{j_1} + \mu_{j_2} + \cdots + \mu_{j_m}\) belongs to \(S\), and, thus, it must be \(\mu_k\) for some \(k \geq 1\). Therefore, we can split the sum in (4.40) into three parts:
\[
\mu_{j_1} + \mu_{j_2} + \cdots + \mu_{j_m} = \mu_k \text{ for } k \leq N, \ k = N + 1 \text{ and } k \geq N + 2.
\]
Corresponding to the last part, i.e., $\mu_k \geq \mu_{N+2}$, taking into account (4.41), the summand in (4.40) is

$$\frac{G_m(\tilde{q}_{j_1}(t), \tilde{q}_{j_2}(t), \ldots, \tilde{q}_{j_m}(t))}{\psi(t)^{\mu_{j_1} + \mu_{j_2} + \cdots + \mu_{j_m}}} = O(\psi(t)^{-\mu_{N+2} - \delta}) \quad \forall \delta > 0.$$ 

Thus, we rewrite (4.40) as

$$\sum_{m=2}^{M_{N+1}} G_m(\tilde{y}_N(t)) = \sum_{k=1}^{N+1} \tilde{Q}_k(t) \psi(t)^{\mu_k} + O(\psi(t)^{-\mu_{N+2} - \delta}) \quad \forall \delta > 0,$$

where, for $1 \leq k \leq N + 1$,

$$\tilde{Q}_k(t) = \sum_{m=2}^{M_{N+1}} \sum_{\substack{1 \leq j_1, j_2, \ldots, j_m \leq N \\mu_{j_1} + \mu_{j_2} + \cdots + \mu_{j_m} = \mu_k}} G_m(\tilde{q}_{j_1}(t), \tilde{q}_{j_2}(t), \ldots, \tilde{q}_{j_m}(t)) = Q_k(\phi(t)),$$

with $Q_k : \mathbb{R} \to \mathbb{R}^n$ being defined by

$$Q_k(z) = \sum_{m=2}^{M_{N+1}} \sum_{\substack{1 \leq j_1, j_2, \ldots, j_m \leq N \\mu_{j_1} + \mu_{j_2} + \cdots + \mu_{j_m} = \mu_k}} G_m(q_{j_1}(z), q_{j_2}(z), \ldots, q_{j_m}(z)) \quad \text{for} \ z \in \mathbb{R}.$$  

(4.44)

For $1 \leq k \leq N + 1$, we note that $N \geq k - 1$, and, thanks to (4.35), $M_{N+1} > \mu_{N+1}/\mu_1 > \mu_k/\mu_1$. By relation (4.20),

$$\sum_{m=2}^{M_{N+1}} \sum_{\substack{1 \leq j_1, j_2, \ldots, j_m \leq N \\mu_{j_1} + \mu_{j_2} + \cdots + \mu_{j_m} = \mu_k}} = \sum_{m=2}^{M_{N+1}} \sum_{\substack{1 \leq j_1, j_2, \ldots, j_m \leq N \\mu_{j_1} + \mu_{j_2} + \cdots + \mu_{j_m} = \mu_k}}.$$

(4.45)

Thus, we have

$$Q_k = \sum_{m=2}^{M_{N+1}} \sum_{\substack{1 \leq j_1, j_2, \ldots, j_m \leq N \\mu_{j_1} + \mu_{j_2} + \cdots + \mu_{j_m} = \mu_k}} G_m(q_{j_1}, q_{j_2}, \ldots, q_{j_m}).$$  

(4.46)

Combining (4.39) with (4.42) gives, for any $\delta > 0$,

$$\psi(t)^{\mu_{N+1}} G(y(t)) = \tilde{Q}_{N+1}(t) + \psi(t)^{\mu_{N+1}} \sum_{k=1}^{N} \tilde{Q}_k(t) \psi(t)^{\mu_k}$$

$$+ O(\psi(t)^{-\mu_{N+2} + \mu_{N+1} + \delta}) + O(\psi(t)^{-\delta_{N+1}}) \quad \text{for} \ \delta > 0.$$

(4.47)
Combining (4.33) with (4.47) gives
\[ w_N' = -(A - \epsilon \mu_{N+1} I_n) w_N + \tilde{Q}_{N+1} + \tilde{p}_{N+1} \]
\[ + \psi(t)^{\mu_{N+1}} \sum_{k=1}^{N} \psi(t)^{-\mu_k} \left\{ -A \tilde{q}_k + \tilde{Q}_k + \tilde{p}_k \right\} \]
\[ - \psi(t)^{\mu_{N+1}} \sum_{k=1}^{N} y_k' + \mathcal{O}(\psi(t)^{1-\rho_{N+1}}) + \mathcal{O}(\psi(t)^{-\mu_{N+2} + \mu_{N+1} + \delta}), \quad (4.48) \]

for any \( \delta > 0 \), where \( \epsilon'_{N+1} = \min\{\delta N/2, \epsilon_{N+1}, \delta'_{N+1}\} > 0 \).

(c) We calculate \( \psi(t)^{\mu_{N+1}} \sum_{k=1}^{N} y_k' \). Define \( \tilde{\chi}_k(t) = \chi_k(\phi(t)) \) for \( 1 \leq k \leq N + 1 \), and let
\[ \theta = \begin{cases} 1, & \text{for Type 2}, \\ 0, & \text{for Types 1 and 3}. \end{cases} \]

Let \( \lambda > 0 \) be arbitrary. Using (4.5), we have
\[ y_k' = \epsilon_q^k(\phi(t)) \phi'(t) \psi(t)^{-\mu_k} - \mu_k q_k(\phi(t)) \psi(t)^{-\mu_k - 1} \psi'(t) \]
\[ = \epsilon[q_k^0(\phi(t)) - \mu_k q_k(\phi(t)) \psi(t)^{-\mu_k} + \theta[q_k(\phi(t)) - \mu_k q_k(\phi(t))]] \psi(t)^{-\mu_k - 1} + (1 - \epsilon)(1 - \theta)\mathcal{O}(\psi(t)^{-\lambda}). \]

Summing up in \( k \) from 1 to \( N \) gives
\[ \sum_{k=1}^{N} y_k' = \sum_{k=1}^{N} \epsilon[q_k^0(\phi(t)) - \mu_k q_k(\phi(t)) \psi(t)^{-\mu_k} - \theta J(t) + \mathcal{O}(\psi(t)^{-\lambda}), \quad (4.49) \]

where \( J(t) = \sum_{k=1}^{N} [\mu_k q_k(\phi(t)) - q_k^0(\phi(t))] \psi(t)^{-\mu_k - 1} \).

Consider \( J(t) \) when \( \theta = 1 \), i.e., in the case of Type 2. Note, by Assumption 4.5(c), that \( \mu_k + 1 = \mu_s \) for a unique \( s \in \mathbb{N} \), and, in case \( 1 \leq s \leq N + 1 \),
\[ [\mu_k q_k(\phi(t)) - q_k^0(\phi(t))] \psi(t)^{-\mu_k - 1} = \chi_s(\phi(t)) \psi(t)^{-\mu_s}. \]

Splitting the sum in \( J(t) \) into \( s \leq N, s = N + 1 \) and \( s \geq N + 2 \), we obtain
\[ \theta J(t) = \sum_{1 \leq k \leq N} \tilde{\chi}_k(t) \psi(t)^{-\mu_k} + \tilde{\chi}_{N+1}(t) \psi(t)^{-\mu_{N+1}} + \theta \sum_{1 \leq k \leq N} \frac{\mu_k q_k(\phi(t)) - q_k^0(\phi(t))}{\psi(t)^{\mu_k + 1}}. \]

Note, for \( \mu_k + 1 \geq \mu_{N+2} \), we have
\[ \frac{\mu_k q_k(\phi(t)) - q_k^0(\phi(t))}{\psi(t)^{\mu_k + 1}} = \mathcal{O}(\psi(t)^{-\mu_{N+2} + \delta}) \quad \forall \delta > 0. \quad (4.51) \]

For any \( \delta > 0 \), selecting \( \lambda > \mu_{N+2} \) in (4.49), and using (4.50), (4.51), we obtain
\[ \sum_{k=1}^{N} y_k' = \sum_{k=1}^{N} \left\{ \epsilon[q_k^0(\phi(t)) - \mu_k q_k(\phi(t))] - \tilde{\chi}_k(t) \right\} \psi(t)^{-\mu_k} \]
\[ - \tilde{\chi}_{N+1}(t) \psi(t)^{-\mu_{N+1}} + \mathcal{O}(\psi(t)^{-\mu_{N+2} + \delta}). \]
\[
\psi(t)^{\mu N+1} \sum_{k=1}^{N} y_k^j(t) = \psi(t)^{\mu N+1} \sum_{k=1}^{N} \left[ \epsilon[q_k^j(\phi(t)) - \mu_k q_k(\phi(t))] - \tilde{\chi}_k(t) \right] \psi(t)^{-\mu_k} - \tilde{\chi}_{N+1}(t) + \mathcal{O}(\psi(t)^{-\mu N+1+\delta}).
\]

(4.52)

Combing (4.48) with (4.52), and selecting \( \delta = (\mu_{N+2} - \mu_{N+1})/2 \) give

\[
w_N' = -(A - \epsilon \mu_{N+1} I_n) w_N + \hat{\Phi}_{N+1}(\phi(t)) + \psi(t)^{\mu N+1} \sum_{k=1}^{N} \psi(t)^{-\mu_k} \hat{\Phi}_k(t) + \mathcal{O}(\psi(t)^{-\hat{\delta}_{N+1}}),
\]

(4.53)

where \( \hat{\delta}_{N+1} = \min(\epsilon N_{+1}^{'}, (\mu_{N+2} - \mu_{N+1})/2) > 0 \), \( \hat{\Phi}_{N+1} : \mathbb{R} \to \mathbb{R}^n \) is the polynomial defined by

\[
\hat{\Phi}_{N+1} = Q_{N+1} + p_{N+1} + \chi_{N+1},
\]

and, for \( 1 \leq k \leq N \),

\[
\hat{\Phi}_k(t) = \Phi_k(\phi(t)), \text{ with } \Phi_k = -A q_k + Q_k + p_k - \epsilon(q_k^{' - \mu_k q_k}) + \chi_k.
\]

By the induction hypothesis and (4.46), we have \( \Phi_k = 0 \) for \( k = 1, \ldots, N \). Thus,

\[
w_N' = -(A - \epsilon \mu_{N+1} I_n) w_N + \hat{\Phi}_{N+1}(\phi(t)) + \mathcal{O}(\psi(t)^{-\hat{\delta}_{N+1}}).
\]

(4.54)

(d) Using Eq. (4.54), we apply Lemma 2.2 in case of Type 1, or Corollary 2.10 in case of Types 2 and 3, to polynomial \( p = \hat{\Phi}_{N+1} \).

We check condition (2.22) for Type 1 with \( \lambda = \mu_{N+1} \). Let \( \lambda_* \) be as in (2.22). Then, \( \lambda_* < \mu_{N+1} \). Also, since \( \lambda_* \) is an eigenvalue of \( A \), we have, thanks to Assumption 4.5(b), \( \lambda_* \in \mathcal{S} \). Hence \( \lambda_* \leq \mu_N \). Thus,

\[
e^{(\lambda_* - \mu_{N+1})t} |w_N(t)| = e^{\lambda_* t} |v_N(t)| \leq e^{\mu N t} |v_N(t)| \to 0 \text{ as } t \to \infty.
\]

Hence (2.22) is satisfied.

Then, by Lemma 2.2 and Corollary 2.10, there exist a polynomial \( q_{N+1} : \mathbb{R} \to \mathbb{R}^n \) and a number \( \delta_{N+1} > 0 \) such that

\[
|w_N(t) - q_{N+1}(\phi(t))| = \mathcal{O}(\psi(t)^{-\delta_{N+1}}),
\]

(4.55)

and \( q_{N+1}(z) \) solves

\[
A q_{N+1}(z) + \epsilon[q_{N+1}'(z) - \mu_{N+1} q_{N+1}(z)] = \hat{\Phi}_{N+1}(z), \quad z \in \mathbb{R}.
\]

Multiplying (4.55) by \( \psi(t)^{-\mu_{N+1}} \) gives

\[
|y(t) - \sum_{k=1}^{N+1} q_k(\phi(t)) \psi(t)^{-\mu_k}| = \mathcal{O}(\psi(t)^{-\mu_{N+1} - \delta_{N+1}}).
\]
Thus, the statement \((T_{N+1})\) holds true.

**Conclusion.** By the induction principle, the statement \((T_N)\) is true for all \(N \in \mathbb{N}\). Note that the polynomial \(q_{N+1}\) is constructed without changing the previous \(q_k\) for \(1 \leq k \leq N\). Therefore, the polynomials \(q_k\) exist for all \(k \in \mathbb{N}\), for which \((T_N)\) is true for all \(N \in \mathbb{N}\). Consequently, we obtain the asymptotic expansion (4.16) with the polynomials \(q_k\)'s satisfying (4.17) and (4.18). \(\square\)

For convenience in comparisons, we write formulas in Theorem 4.10 for three cases of \((\psi, \phi)\) explicitly.

**Type 1.** The expansions (4.6) and (4.16) are

\[
 f(t) \sim \sum_{k=1}^{\infty} p_k(t)e^{-\mu_k t} \quad \text{and} \quad y(t) \sim \sum_{k=1}^{\infty} q_k(t)e^{-\mu_k t} = \sum_{k=1}^{\infty} y_k(t),
\]

where, following the concise form (4.22),

\[
 q_k' + (A - \mu_k I_n)q_k = \sum_{m \geq 2} \sum_{\mu_{j_1} + \mu_{j_2} + \ldots + \mu_{j_m} = \mu_k} G_m(q_{j_1}, q_{j_2}, \ldots, q_{j_m}) + p_k \quad \text{for} \quad k \in \mathbb{N},
\]

or, equivalently,

\[
 y_k' + Ay_k = \sum_{m \geq 2} \sum_{\mu_{j_1} + \mu_{j_2} + \ldots + \mu_{j_m} = \mu_k} G_m(y_{j_1}, y_{j_2}, \ldots, y_{j_m}) + f_k \quad \text{for} \quad k \in \mathbb{N}.
\]

**Type 2.** The expansions (4.6) and (4.16) are

\[
 f(t) \sim \sum_{k=1}^{\infty} p_k(\ln t)t^{-\mu_k} \quad \text{and} \quad y(t) \sim \sum_{k=1}^{\infty} q_k(\ln t)t^{-\mu_k},
\]

where

\[
 q_k = A^{-1}\left\{ \sum_{m \geq 2} \sum_{\mu_{j_1} + \mu_{j_2} + \ldots + \mu_{j_m} = \mu_k} G_m(q_{j_1}, q_{j_2}, \ldots, q_{j_m}) + p_k + \chi_k \right\} \quad \text{for} \quad k \in \mathbb{N},
\]

with \(\chi_k\) defined by (4.21) and (4.19).

If \(\mu_k = k\) for all \(k \in \mathbb{N}\), then \(\chi_k = (k - 1)q_{k-1} - q'_k - 1\) for all \(k \geq 2\).

In particular, if \(p_k = \eta_k = \text{const.}\) for all \(k\), then \(q_k = \tilde{\xi}_k = \text{const.}\) for all \(k\). This type of expansions is studied in [4] for the Navier–Stokes equations.

**Type 3.** The expansions (4.6) and (4.16) are

\[
 f(t) \sim \sum_{k=1}^{\infty} p_k(L_{n_1}(t)L_{n_2}(t))^{-\mu_k} \quad \text{and} \quad y(t) \sim \sum_{k=1}^{\infty} q_k(L_{n_1}(t)L_{n_2}(t))^{-\mu_k},
\]

where

\[
 q_k = A^{-1}\left\{ \sum_{m \geq 2} \sum_{\mu_{j_1} + \mu_{j_2} + \ldots + \mu_{j_m} = \mu_k} G_m(q_{j_1}, q_{j_2}, \ldots, q_{j_m}) + p_k \right\} \quad \text{for} \quad k \in \mathbb{N}.
\]
Remark 4.11. We have the following comparisons.

(a) In case of Type 1, the asymptotic expansion, in general, depends on the individual solution \(y(t)\), and \(q_k\) is determined by solving an ODE. In cases of Types 2 and 3, all solutions have the same expansion, and \(q_k\) is determined by some algebraic operations.

(b) For Types 1 and 2, the time derivative in (1.7) is reflected on the construction of \(q_k\)'s, see (4.56) and (4.19), respectively. This is not the case for Type 3, see (4.58).

(c) In case of Type 1 and when \(\mu_k\) is not an eigenvalue of \(A\), thanks to Remark 2.3(b), the function \(q_k(t)\) is the unique polynomial solution of (4.56), which depends only on the forcing function \(f(t)\) and the previous \(q_j(t)\)'s, for \(1 \leq j \leq k\). Consequently, if two solutions of (1.7) have the same \(q_k\)'s for \(1 \leq k \leq N\), for some \(N \in \mathbb{N}\), then they have the same \(q_k\)'s for all \(k > N\), and, hence, for all \(k \in \mathbb{N}\). This property is not available in infinite-dimensional spaces.

5. Expansions with multiple secondary base functions

This section aims to generalize the results in Sect. 4. Section 5.1 will generalize expansions for Type 3, while Sect. 5.2 for Type 2. With the notation in Definitions 2.4 and 2.7, we can consider a general form of expansions in the following.

Definition 5.1. Let \((X, \| \cdot \|_X)\) be a normed space, and \(g\) be a function from \((T, \infty)\) to \(X\) for some \(T \in \mathbb{R}\). Let \(m_* \in \mathbb{Z}_+\).

(i) Let \((\gamma_k)_{k=1}^{\infty}\) be a divergent, strictly increasing sequence of positive numbers, and \(n_k \in \mathbb{N}, p_k \in \mathcal{P}(n_k, X)\) for each \(k \in \mathbb{N}\). We say

\[
g(t) \sim \sum_{k=1}^{\infty} p_k(L_{m_*,n_k}(t))L_{m_*}(t)^{-\gamma_k},
\]

if, for each \(N \in \mathbb{N}\), there is some \(\mu > \gamma_N\) such that

\[
\left\| g(t) - \sum_{k=1}^{N} p_k(L_{m_*,n_k}(t))L_{m_*}(t)^{-\gamma_k} \right\|_X = O(L_{m_*}(t)^{-\mu}).
\]

(ii) Let \(N \in \mathbb{N}, (\gamma_k)_{k=1}^{N}\) be positive and strictly increasing, and \(n_k \in \mathbb{N}, p_k \in \mathcal{P}(n_k, X)\), for \(k = 1, 2, \ldots, N\). We say

\[
g(t) \sim \sum_{k=1}^{N} p_k(L_{m_*,n_k}(t))L_{m_*}(t)^{-\gamma_k},
\]

if it holds for all \(\lambda > 0\) that

\[
\left\| g(t) - \sum_{k=1}^{N} p_k(L_{m_*,n_k}(t))L_{m_*}(t)^{-\gamma_k} \right\|_X = O(L_{m_*}(t)^{-\lambda}).
\]
In particular, when \( m_\ast = 0 \), expansion (5.1) reads as

\[
g(t) \sim \sum_{k=1}^{\infty} p_k(L_{n_k}(t)) t^{-\gamma_k}.
\]

We have the following remarks on Definition 5.1.

(a) Similar to Definition 1.4, we call \( L_{m_\ast}(t) \) the primary base function of expansion (5.1), and \( L_{m_\ast+j}(t) \) with \( 1 \leq j \leq n_k \) and \( k \in \mathbb{N} \), the secondary base functions.

(b) Comparing two expansions (1.13) and (5.1) when they have the same primary base function, the latter is more general than the former, even in the case (5.1) has only one secondary base function. It is due to the fact that the functions \( p_k \)’s belong to a larger class, see remark (a) after Definition 2.7.

(c) Because function \( p_k(L_{m_\ast,n_k}(t)) \) is not restricted to only nonnegative integer powers of \( L_{m_\ast+j}(t) \)’s, the asymptotic expansion (5.1), in fact, is a more precise approximation of \( g(t) \) compared to (1.13) for Type 3 in Definition 4.2.

Note that

\[
\lim_{t \to \infty} \frac{L_{m_\ast,k}(t)'^{\alpha}}{L_{m_\ast}(t)'^{\delta}} = 0 \text{ for any } \alpha \in \mathbb{R}, \text{ and } \delta > 0.
\]

Similar to Lemma 4.1, we have the following uniqueness of the approximation (5.2).

**Lemma 5.2.** Let \( (X, \| \cdot \|_X) \) be a normed space. Given a function \( g : (T, \infty) \to X \) for some \( T \in \mathbb{R} \). Let \( N \in \mathbb{N} \), numbers \( n_k \in \mathbb{N} \) and \( \gamma_k \in \mathbb{R} \), for \( 1 \leq k \leq N \), such that \( 0 \leq \gamma_1 < \gamma_2 < \ldots < \gamma_N \). Suppose there exist \( p_k \in \mathcal{P}(n_k, X) \), for \( 1 \leq k \leq N \), such that (5.2) holds for some \( \mu > \gamma_N \). Then, such functions \( p_1, p_2, \ldots, p_N \) are unique.

**Proof.** Suppose \( \hat{p}_k \in \mathcal{P}(n_k, X) \), for \( 1 \leq k \leq N \), satisfy

\[
\left\| g(t) - \sum_{k=1}^{N} \hat{p}_k(L_{m_\ast,n_k}(t)) L_{m_\ast}(t)^{-\gamma_k} \right\|_X = \mathcal{O}(L_{m_\ast}(t)^{-\hat{\mu}}) \text{ for some } \hat{\mu} > \gamma_N.
\]

Let \( h_k = p_k - \hat{p}_k \) for \( 1 \leq k \leq N \). By triangle inequality, see (4.3), we have

\[
\left\| \sum_{k=1}^{N} h_k(L_{m_\ast,n_k}(t)) L_{m_\ast}(t)^{-\gamma_k} \right\|_X = \mathcal{O}(L_{m_\ast}(t)^{-\hat{\mu}}) \text{ for } \hat{\mu} = \min\{\mu, \hat{\mu}\} > \gamma_N.
\]

Multiplying this equation by \( L_{m_\ast}(t)^{\gamma_1} \) yields

\[
\|h_1(L_{m_\ast,n_k}(t))\|_X = \mathcal{O}(L_{m_\ast}(t)^{-\delta}) \text{ for } \delta = \hat{\mu} - \gamma_1 > 0.
\]

Suppose \( h_1 \neq 0 \). Write \( h_1(z) \) as a finite sum \( \sum c_{\beta} z^\beta \) for \( z \in (0, \infty)^{n_k} \), where \( c_{\beta} \)'s are nonzero vectors in \( X \), and \( \beta \)'s are distinct powers in \( \mathbb{R}^{n_k} \). We use the lexicography order for the powers \( \beta \)'s in \( \mathbb{R}^{n_k} \). If \( \alpha, \beta \) are the powers in \( \mathbb{R}^{n_k} \), and \( \alpha > \beta \), then

\[
\lim_{t \to \infty} \frac{L_{m_\ast,n_k}(t)^{\beta}}{L_{m_\ast,n_k}(t)^{\alpha}} = 0.
\]
Let $\beta_*$ be the maximum power among those $\beta$’s in the formula of $h_1(z)$. Then, multiplying (5.4) by $(\mathcal{L}_{m_*,n_k}(t))^{-\beta_*}$ and passing $t \to \infty$, making use of (5.3) and (5.5), we obtain $c_{\beta_*} = 0$, which is a contradiction. Thus, we have $h_1 = 0$. Repeating this argument gives $h_k = 0$, hence, $p_k = \hat{p}_k$, for all $k = 1, 2, \ldots, N$. \hfill \Box

Throughout this section, $f(t)$ is a forcing function as in Assumption 4.3, and $y(t)$ is a solution of (1.7) as in Assumption 4.9.

5.1. Iterated logarithmic expansions

This subsection studies the expansions that contain only iterated logarithmic functions.

**Assumption 5.3.** The function $f(t)$ admits the asymptotic expansion, in the sense of Definition 5.1 with $X = \mathbb{R}^n$ and $\| \cdot \|_X = | \cdot |$,

\[
f(t) \sim \sum_{k=1}^{\infty} p_k(L_{m_*,n_k}(t))L_{m_*}(t)^{-\mu_k},
\]

where $m_* \in \mathbb{N}$, $(\mu_k)_{k=1}^{\infty}$ is a divergent, strictly increasing sequence of positive numbers, the set $S := \{\mu_k : k \in \mathbb{N}\}$ preserves the addition, $n_k$ is increasing in $k$, but not necessarily strictly increasing, and $p_k \in \mathcal{P}(n_k, \mathbb{R}^n)$.

Below are two typical cases for Assumption 5.3 to hold.

**Scenario 5.4.** Let $m_* \in \mathbb{N}$. Assume, in the sense of Definition 5.1, that

\[
f(t) \sim \sum_{k=1}^{\infty} \hat{p}_k(L_{m_*,\hat{n}_k}(t))L_{m_*}(t)^{-\alpha_k},
\]

where $\hat{n}_k \in \mathbb{N}$, $\hat{p}_k \in \mathcal{P}(\hat{n}_k, \mathbb{R}^n)$, and $(\alpha_k)_{k=1}^{\infty}$ is a divergent, strictly increasing sequence of positive numbers.

Let the set $S$ be defined by (4.11) and be re-ordered as the sequence $(\mu_k)_{k=1}^{\infty}$ as in (4.12). Then, $(\alpha_k)_{k=1}^{\infty}$ becomes a subsequence of $(\mu_k)_{k=1}^{\infty}$, and, by re-indexing $\hat{p}_k$ and $\hat{n}_k$, we rewrite (5.7) as

\[
f(t) \sim \sum_{k=1}^{\infty} \hat{p}_k(L_{m_*,\hat{n}_k}(t))L_{m_*}(t)^{-\mu_k}.
\]

Let $n_k = \max\{\hat{n}_j : 1 \leq j \leq k\}$. By embedding (2.46), we have $\mathcal{P}(\hat{n}_k, \mathbb{R}^n) \subset \mathcal{P}(n_k, \mathbb{R}^n)$ and define $p_k = \mathcal{I}_{\hat{n}_k,n_k}\hat{p}_k$. Then, $p_k \in \mathcal{P}(n_k, \mathbb{R}^n)$, and, thanks to (2.45), $\hat{p}_k(L_{m_*,\hat{n}_k}(t)) = p_k(L_{m_*,n_k}(t))$. Thus, we can rewrite (5.8) as (5.6).

**Scenario 5.5.** Assume, similar to (5.7), we have the finite expansion, in the sense of Definition 5.1,

\[
f(t) \sim \sum_{k=1}^{N} \hat{p}_k(L_{m_*,\hat{n}_k}(t))L_{m_*}(t)^{-\alpha_k},
\]

for some $N \in \mathbb{N}$. Let $S$ be defined by (4.11) for $1 \leq \ell_j \leq N$. Then, similar to Scenario 5.4, we can rewrite (5.9) as (5.6).
Our second main result on the asymptotic expansion for solutions of (1.7) is the following.

**Theorem 5.6.** Under Assumption 5.3, the solution \( y(t) \) admits the asymptotic expansion, in the sense of Definition 5.1,

\[
y(t) \sim \sum_{k=1}^{\infty} q_k(L_{m_*,n_k}(t))L_{m_*}(t)^{-\mu_k},
\]

where

\[
q_k \in \mathcal{P}(n_k, \mathbb{R}^n) \text{ for all } k \in \mathbb{N},
\]

and are defined recursively by

\[
q_k = \begin{cases} A^{-1}p_1, & \text{for } k = 1, \\ A^{-1}\left( \sum_{m \geq 2} \sum_{j_1, j_2, \ldots, j_m \geq 1} G_m(q_{j_1}, q_{j_2}, \ldots, q_{j_m}) + p_k \right), & \text{for } k \geq 2. \end{cases}
\]

We quickly verify, by induction, that the definition of \( q_k \) is valid and (5.11) holds true.

First, because \( p_1 \in \mathcal{P}(n_1, \mathbb{R}^n) \) and \( q_1 = A^{-1}p_1 \), we have \( q_1 \in \mathcal{P}(n_1, \mathbb{R}^n) \), thanks to Lemma 2.8(ii).

Let \( k \geq 2 \). Suppose \( q_j \in \mathcal{P}(n_j, \mathbb{R}^n) \) for all \( 1 \leq j \leq k - 1 \). By Remark (c) after the statement of Theorem 4.10, the sums on the right-hand side of (5.12) is over finitely many \( m \)’s and \( 1 \leq j_1, j_2, \ldots, j_k \leq k - 1 \). Thus, \( n_{j_i} < n_k \), for \( 1 \leq i \leq k \), and, thanks to embedding (2.46), we can consider each \( q_{j_i} \) belonging to \( \mathcal{P}(n_k, \mathbb{R}^n) \). Together with \( p_k \in \mathcal{P}(n_k, \mathbb{R}^n) \) and, again, Lemma 2.8(ii), we obtain \( q_k \in \mathcal{P}(n_k, \mathbb{R}^n) \).

Then, by the induction principle, \( q_k \in \mathcal{P}(n_k, \mathbb{R}^n) \) for all \( k \in \mathbb{N} \).

**Proof of Theorem 5.6.** The proof is similar to that of Theorem 4.10 for Type 3. We sketch it here.

Replace the statement \((T_N)\) in the proof of Theorem 4.10 by the following:

\[
\left| y(t) - \sum_{k=1}^{N} q_k(L_{m_*,n_k}(t))L_{m_*}(t)^{-\mu_k} \right| = \mathcal{O}(L_{m_*}(t)^{-\mu_N-\delta_N}) \text{ for some } \delta_N > 0.
\]

Let \( \psi(t) = L_{m_*}(t), \tilde{p}_k(t) = p_k(L_{m_*,n_k}(t)) \), and define \( f_k(t), \tilde{f}_N(t) \) as in (4.14).

First, we notice that

\[
|\psi'(t)| = \mathcal{O}(\psi(t)^{-\lambda}) \quad \forall \lambda > 0.
\]

One can verify (4.23) and, hence, (4.24).

We prove that \((T_N)\) is true for all \( N \in \mathbb{N} \). We proceed by induction. Consider \( t \) sufficiently large in all calculations below.
First step: $N = 1$. Set $w_0(t) = \psi(t)^{\mu_1}y(t)$. We follow the calculations in the proof of Theorem 4.10 for Type 3 with $\epsilon = \theta = 0$. We obtain the same Eq. (4.29) for $w_0$. This gives

$$w_0' = -Aw_0 + p_1(\mathcal{L}_{m, n_1}(t)) + \mathcal{O}(\psi(t)^{-\delta_1})$$

for some $\delta_1 > 0$.

Applying Lemma 2.9 to this equation for $w_0$ yields the existence of a number $\delta_1 > 0$ such that

$$|w_0(t) - A^{-1}p_1(\mathcal{L}_{m, n_1}(t))| = \mathcal{O}(\psi(t)^{-\delta_1}).$$

Noting that $A^{-1}p_1 = q_1 \in \mathcal{P}(n_1, \mathbb{R}^n)$, and multiplying the preceding equation by $\psi(t)^{-\mu_1}$ gives

$$|y(t) - q_1(\mathcal{L}_{m, n_1}(t))\psi(t)^{-\mu_1}| = \mathcal{O}(\psi(t)^{-\mu_1-\delta_1}).$$

Then, $(T_1)$ holds true.

Induction step: $N \geq 1$. Assume $(T_N)$. Let $\tilde{q}_k(t) = q_k(\mathcal{L}_{m, n_k}(t))$, define $y_k(t)$, $\tilde{y}_N(t)$ and $v_N(t)$ as in (4.30).

Let $w_N(t) = \psi(t)^{\mu_{N+1}}v_N(t)$. Same calculations as in parts (a) and (b) of the proof of Theorem 4.10, we obtain Eq. (4.48) again, which yields

$$w_N' = -Aw_N + \tilde{Q}_{N+1} + \tilde{p}_{N+1} + \psi(t)^{\mu_{N+1}}\sum_{k=1}^{N} \psi(t)^{-\mu_k}\left\{-A\tilde{q}_k + \tilde{Q}_k + \tilde{p}_k\right\}$$

$$- \psi(t)^{\mu_{N+1}}\sum_{k=1}^{N} y_k' + \mathcal{O}(\psi(t)^{-\epsilon_{N+1}'}) , \quad (5.13)$$

for some $\epsilon_{N+1}' > 0$. Here, same as (4.43), (4.44), (4.45), and (4.46), for $1 \leq k \leq N + 1$,

$$\tilde{Q}_k(t) = Q_k(\mathcal{L}_{m, n_k}(t)) \quad \text{with } Q_k \text{ being defined by (4.46).} \quad (5.14)$$

Note that $\psi'(t), L'_k(t) = \mathcal{O}(t^{-\lambda})$ for all $\lambda \in (0, 1)$, and, by (5.11) and Lemma 2.8(vi), $\partial q_k/\partial z_j \in \mathcal{P}(n_k, \mathbb{R}^n)$ for $1 \leq j \leq n_k$. Hence,

$$\frac{d}{dt} q_k(\mathcal{L}_{m, n_k}(t)) = \sum_{j=1}^{n_k} \frac{\partial q_k}{\partial z_j}(\mathcal{L}_{m, n_k}(t))L'_{m, j}(t) = \mathcal{O}(t^{-\lambda}) \quad \forall \lambda \in (0, 1).$$

Thus,

$$y_k'(t) = \frac{d}{dt} q_k(\mathcal{L}_{m, n_k}(t))\psi(t)^{-\mu_k} - \mu_k q_k(\mathcal{L}_{m, n_k}(t))\psi(t)^{-\mu_k-1}\psi'(t)$$

$$= \mathcal{O}(t^{-\lambda}) \quad \forall \lambda \in (0, 1).$$

Consequently, $y_k(t) = \mathcal{O}(\psi(t)^{-\mu_{N+1}-\epsilon_{N+1}'})$. From this and (5.13), we obtain (4.53) again:

$$w_N' = -Aw_N + \tilde{\Phi}_{N+1}(\mathcal{L}_{m, n_{N+1}}(t)) + \psi(t)^{\mu_{N+1}}\sum_{k=1}^{N} \psi(t)^{-\mu_k}\Phi_k(\mathcal{L}_{m, n_k}(t)) + \mathcal{O}(\psi(t)^{-\delta_{N+1}}),$$
where \( \hat{\delta}_{N+1} = \varepsilon''_{N+1} > 0 \), and \( \hat{\Phi}_{N+1} = Q_{N+1} + P_{N+1} \), with \( \Phi_k = -A q_k + Q_k + p_k \) for \( 1 \leq k \leq N \).

By (5.12), \( \Phi_k = 0 \) for \( 1 \leq k \leq N \). We therefore obtain

\[
\dot{w}_N = -A w_N + \hat{\Phi}_{N+1} (L_{m_+, n_{N+1}} (t)) + O (\psi (t) - \hat{\delta}_{N+1}). \tag{5.15}
\]

Using relation in (2.46), we can view \( \hat{\Phi}_{N+1} \) as an element in the class \( P (n_{N+1}, \mathbb{R}^n) \).

By applying Lemma 2.9 to Eq. (5.15) with \( y = w_N \) and \( p = \hat{\Phi}_{N+1} \), there exists \( \delta_{N+1} > 0 \) such that

\[
|w_N (t) - A^{-1} \hat{\Phi}_{N+1} (L_{m_+, n_{N+1}} (t))| = O (\psi (t) - \delta_{N+1}). \tag{5.16}
\]

Note that \( A^{-1} \hat{\Phi}_{N+1} = q_{N+1} \) which belongs to \( P (n_{N+1}, \mathbb{R}^n) \), thanks to Lemma 2.8(ii). Multiplying Eq. (5.16) by \( \psi (t) - \mu_{N+1} \) gives

\[
|v_N (t) - q_{N+1} (L_{m_+, n_{N+1}} (t)) \psi (t)^{-\mu_{N+1}}| = O (\psi (t)^{-\mu_{N+1} - \delta_{N+1}}). \tag{5.17}
\]

This proves \((T_{N+1})\). Therefore, by the induction principle, \((T_N)\) holds for all \( N \in \mathbb{N} \), which implies the asymptotic expansion (5.10).

\[\square\]

**Remark 5.7.** If all functions \( p_k \)'s in (5.6) are polynomials, then, by induction, all \( q_k \)'s in (5.10) are also polynomials.

**5.2. Mixed power and iterated logarithmic expansions**

For motivation, we consider a simple case when \( f (t) = t^{-1} \ln \ln t \) for large \( t \). Then, the expansion of solution \( y(t) \) should contain at least a term \( t^{-1} \ln \ln t \). It yields that \( y'(t) \) should contain

\[
\frac{1}{t^2 \ln t} \quad \text{and} \quad \frac{\ln \ln t}{t^2}.
\]

By Eq. (1.7), these terms should be in the expansion of \( y(t) \) as well. Therefore, we need to have some form of expansions with combinations of functions \( t, \ln t, \) and \( \ln \ln t \). In fact, even more general result holds true as showed in the following theorem.

**Theorem 5.8.** Let \( (\mu_k)_{k=1}^{\infty} \) be a divergent, strictly increasing sequence of positive numbers that preserves the addition and the unit increment. Let \( (n_k)_{k=1}^{\infty} \) be an increasing sequence in \( \mathbb{N} \). Suppose

\[
f (t) \sim \sum_{k=1}^{\infty} p_k (L_{n_k} (t)) t^{-\mu_k}, \quad \text{where} \ p_k \in P (n_k, \mathbb{R}^n). \tag{5.17}
\]

Then, the solution \( y(t) \) admits the asymptotic expansion

\[
y(t) \sim \sum_{k=1}^{\infty} q_k (L_{n_k} (t)) t^{-\mu_k}, \tag{5.18}
\]
where \( q_k \in \mathcal{P}(n_k, \mathbb{R}^n) \) is defined by

\[
q_k = \begin{cases} 
A^{-1} p_1, & \text{for } k = 1, \\
A^{-1} \left( \sum_{m \geq 2} \sum_{j_1 + j_2 + \ldots + j_m = k} g_m(q_{j_1}, q_{j_2}, \ldots, q_{j_m}) + p_k + \chi_k \right), & \text{for } k \geq 2,
\end{cases}
\]

with \( \chi_k \in \mathcal{P}(n_k, \mathbb{R}^n) \) being

\[
\chi_k(z) = \begin{cases}
\mu \lambda q_k(z) - \sum_{j=1}^{n_k} \frac{1}{z_1 z_2 \ldots z_{j-1}} \frac{\partial q_k(z)}{\partial z_j}, & \text{if there is } \lambda \geq 1 \\
0, & \text{such that } \mu \lambda + 1 = \mu_k,
\end{cases}
\]

for \( z = (z_1, z_2, \ldots, z_{n_k}) \in (0, \infty)^{n_k} \).

**Proof.** Similar to the verification after Theorem 5.6, and thanks to Lemma 2.8(iv),(vi), we can validate that \( q_k \) and \( \chi_k \) belong to \( \mathcal{P}(n_k, \mathbb{R}^n) \) for all \( k \in \mathbb{N} \).

Set \( \psi(t) = t \). For \( k \in \mathbb{N} \), we denote \( \tilde{p}_k(t) = p_k(L_{n_k}(t)) \) and define \( f_k(t) \) and \( \tilde{f}_N(t) \) as in (4.14). Again, one can verify (4.23) and, hence, (4.24).

It suffices to prove, for all \( N \in \mathbb{N} \), that

\[
\left| y(t) - \sum_{k=1}^{N} q_k(L_{n_k}(t)) t^{-\mu_k} \right| = O(t^{-\mu_N - \delta_N}) \quad \text{for some } \delta_N > 0.
\]

We prove it by induction. Let \( (T_N) \) denote the statement (5.21). Again, we consider \( t \) sufficiently large for the rest of the proof.

**First step:** \( N = 1 \). Let \( w_0(t) = t^{\mu_1} y(t) \). Same as (4.29) of Theorem 4.10 for Type 2, we have

\[
w_0' = -Aw_0 + p_1(L_{n_1}(t)) + O(t^{-\delta_1}) \quad \text{for some } \delta_1 > 0.
\]

Applying Lemma 2.9 to Eq. (5.22), there exists \( \delta_1 > 0 \) such that

\[
\left| w_0(t) - A^{-1} p_1(L_{n_1}(t)) \right| = O(t^{-\delta_1}).
\]

Dividing (5.23) by \( t^{\mu_1} \) gives

\[
\left| y(t) - t^{-\mu_1} q_1(L_{n_1}(t)) \right| = O(t^{-\mu_1 - \delta_1}).
\]

This estimate proves \( (T_1) \).

**Induction step:** \( N \geq 1 \). Assume \( (T_N) \). Denote \( \tilde{q}_k(t) = q_k(L_{n_k}(t)) \) and define \( y_k(t) \), \( \tilde{y}_N \), \( v_N \) as in (4.30).

Let \( w_N(t) = t^{\mu_N+1} v_N(t) \). Same as (4.48) for Type 2,

\[
w_N' = -Aw_N + \tilde{Q}_{N+1} + \tilde{p}_{N+1} + t^{\mu_N+1} \sum_{k=1}^{N} t^{-\mu_k} \left\{ -A\tilde{q}_k + \tilde{Q}_k + \tilde{p}_k \right\} - t^{\mu_N+1} \sum_{k=1}^{N} y'_k + O(t^{-\epsilon_{N+1}}),
\]

(5.24)
for some $\varepsilon''_{N+1} > 0$. Similar to (5.14), we have, for $1 \leq k \leq N + 1$, the function $\tilde{Q}_k(t)$ is $Q_k(\mathcal{L}_{n_k}(t))$ with $Q_k$ defined by (4.46).

We calculate

$$y'_k(t) = -\mu_k q_k(\mathcal{L}_{n_k}(t)) + \sum_{j=1}^{n_k} \frac{1}{L_1(t)L_2(t) \ldots L_{j-1}(t)} \frac{\partial q_k}{\partial z_j}(\mathcal{L}_{n_k}(t)) \left. t^{-\mu_k-1}. \right]$$

(5.25)

Let $\tilde{\chi}_k(t) = \chi_k(\mathcal{L}_{k}(t))$ for $1 \leq k \leq N + 1$. Note in (5.25) that $\mu_k + 1 = \mu_s$ for some $s \in \mathbb{N}$. Summing up (5.25) in $k$ from 1 to $N$ and split the sum into three parts corresponding to $s \leq N, s = N + 1$ and $s \geq N + 2$, we obtain

$$\sum_{k=1}^{N} y'_k(t) = -\sum_{k=1}^{N} t^{-\mu_k} \tilde{\chi}_k(t) - t^{-\mu_{N+1}} \tilde{\chi}_{N+1}(t) + O(t^{-\mu}),$$

for some $\mu > \mu_{N+1}$. Thus,

$$t^{\mu_{N+1}} \sum_{k=1}^{N} y'_k(t) = -t^{\mu_{N+1}} \sum_{k=1}^{N} t^{-\mu_k} \tilde{\chi}_k(t) - \tilde{\chi}_{N+1}(t) + O(t^{-\mu + \mu_{N+1}}).$$

(5.26)

From (5.26) and (5.24), it follows

$$w'_N = -A w_N + \hat{\Phi}_{N+1}(\mathcal{L}_{n_{N+1}}(t)) + t^{\mu_{N+1}} \sum_{k=1}^{N} t^{-\mu_k} \Phi_k(\mathcal{L}_{n_k}(t)) + O(t^{-\delta_{N+1}}),$$

where $\hat{\delta}_{N+1} = \min\{\varepsilon''_{N+1}, \mu - \mu_{N+1}\} > 0,$

$$\hat{\Phi}_{N+1} = Q_{N+1} + p_{N+1} + \chi_{N+1},$$

$$\Phi_k = -A q_k + p_k + Q_k + \chi_k, \quad 1 \leq k \leq N.$$

By (5.19), we have $\Phi_k = 0$ for $k = 1, \ldots, N$. Thus,

$$w'_N = -A w_N + \hat{\Phi}_{N+1}(\mathcal{L}_{n_{N+1}}(t)) + O(t^{-\delta_{N+1}}).$$

Applying Lemma 2.9 to this equation for $w_N$, there exists $\delta_{N+1} > 0$ such that

$$\left| w_N(t) - A^{-1} \hat{\Phi}_{N+1}(\mathcal{L}_{n_{N+1}}(t)) \right| = O(t^{-\delta_{N+1}}).$$

(5.27)

Noting that $A^{-1} \hat{\Phi}_{N+1} = q_{N+1}$, and dividing Eq. (5.27) by $t^{\mu_{N+1}}$ give

$$\left| w_N(t) - t^{-\mu_{N+1}} q_{N+1}(\mathcal{L}_{n_{N+1}}(t)) \right| = O(t^{-\mu_{N+1}-\delta_{N+1}}).$$

This implies ($T_{N+1}$) and completes the induction Proof of (5.21) for all $N \in \mathbb{N}$. □
Example 5.9. Case $n_k = 2$ for all $k \in \mathbb{N}$. Assumption (5.17) becomes

$$f(t) \sim \sum_{k=1}^{\infty} p_k(\ln t, \ln \ln t) t^{-\mu_k} \quad \text{with } p_k \in \mathcal{P}(2, \mathbb{R}^n).$$  \hfill (5.28)

Then, the conclusion (5.18) becomes

$$y(t) \sim \sum_{k=1}^{\infty} q_k(\ln t, \ln \ln t) t^{-\mu_k},$$  \hfill (5.29)

where each $q_k \in \mathcal{P}(2, \mathbb{R}^n)$ is defined by (5.19), with $\chi_k$ in (5.20) becoming

$$\chi_k(z) = \begin{cases} \mu \lambda q_\lambda(z) - \frac{\partial q_\lambda(z)}{\partial z_1} - \frac{1}{z_1} \cdot \frac{\partial q_\lambda(z)}{\partial z_2}, & \text{if there exists } \lambda \geq 1 \text{ such that } \\ 0, & \text{otherwise,} \end{cases}$$  \hfill (5.30)

for $z = (z_1, z_2) \in (0, \infty)^2$.

Corollary 5.10. Let $(\mu_k)_{k=1}^{\infty}$ be as in Theorem 5.8. Assume

$$f(t) \sim \sum_{k=1}^{\infty} \frac{1}{t^{\mu_k}} \left( \sum_{j=1}^{n_k} p_{k,j}(\ln t) \ln^{\beta_{k,j}(t)} \right),$$  \hfill (5.31)

where $n_k \geq 1$, $\beta_{k,j} \geq 0$, and $p_{k,j}$'s are polynomials from $\mathbb{R}$ to $\mathbb{R}^{n}$ for $1 \leq j \leq n_k$.

Then, there exist natural numbers $J_k$'s, for $k \in \mathbb{N}$, $\mathbb{R}^n$-valued polynomials of one variable $q_{k,j}$'s and nonnegative numbers $\gamma_{k,j}$'s, for $1 \leq j \leq J_k$, such that

$$y(t) \sim \sum_{k=1}^{\infty} \frac{1}{t^{\mu_k}} \left( \sum_{j=1}^{J_k} q_{k,j}(\ln t) \ln^{\gamma_{k,j}(t)} \right).$$  \hfill (5.32)

Proof. We follow Example 5.9. Expansion (5.31) is of the form (5.28), where

$$p_k(z_1, z_2) = \sum_{j=1}^{n_k} p_{k,j}(z_2) z_1^{-\beta_{k,j}} z_2^1,$$

with $p_{k,j}$ being $\mathbb{R}^n$-valued polynomials of one variable. Then, we have expansion (5.29). We prove that (5.29), in fact, is of the form (5.32). Define

$$F = \left\{ h : (0, \infty)^2 \rightarrow \mathbb{R}^n \text{ such that } h(z_1, z_2) \text{ is a finite sum of the products } q(z_2) z_1^{-\gamma} \right\}.$$

for some polynomial $q : \mathbb{R} \rightarrow \mathbb{R}^n$ and number $\gamma \geq 0$. \hfill (5.33)

To establish (5.32), it suffices to have that $q_k \in F$ for all $k \in \mathbb{N}$. We will prove this by induction in $k$. First, we have $p_k \in F$ for all $k \in \mathbb{N}$.
Because \( q_1 = A^{-1}p_1 \) and \( p_1 \in F \), we have \( q_1 \in F \). Let \( k \geq 2 \), and assume \( q_s \in F \) for all \( 1 \leq s \leq k - 1 \). Then, for \( 1 \leq s \leq k - 1 \), we can write

\[
q_s(z_1, z_2) = \sum_{j=1}^{J_s} q_{s,j}(z_2)z_1^{-\gamma_{s,j}},
\]

where \( q_{s,j}'s \) are \( \mathbb{R}^n \)-valued polynomials of one variable, and \( \gamma_{s,j} \geq 0 \). Note that

\[
\frac{\partial q_s(z_1, z_2)}{\partial z_1} = \sum_{j=1}^{J_s} -\gamma_{s,j}q_{s,j}(z_2)z_1^{-\gamma_{s,j}-1},
\]

\[
\frac{1}{z_1} \frac{\partial q_s(z_1, z_2)}{\partial z_2} = \sum_{j=1}^{J_s} q'_{s,j}(z_2)z_1^{-\gamma_{s,j}-1}.
\]

(5.34)

Consider \( q_k \) defined by (5.19) and (5.30). It follows formula (5.30) and calculations in (5.34) that \( \chi_k \in F \). Then, it is obvious that each term on the right-hand side of (5.19) belongs to \( F \), therefore, so does \( q_k \). By the induction principle, we conclude \( q_k \in F \) for all \( k \in \mathbb{N} \) and complete the proof.

Below is a particular case when the polynomial \( q_{k,j} \) in (5.32) can be determined more explicitly.

**Corollary 5.11.** Let \( (\mu_k)_{k=1}^{\infty} \) and \( (\beta_j)_{j=1}^{\infty} \) be divergent, strictly increasing sequences that preserve the addition and the unit increment, with \( \mu_1 > 0 \) and \( \beta_1 \geq 0 \). Assume

\[
f(t) \sim \sum_{k=1}^{\infty} \frac{1}{t^{\mu_k}} \left( \sum_{j=1}^{\infty} \frac{p_{k,j}(\ln \ln t)}{\ln^{\beta_j}(t)} \right),
\]

(5.35)

where \( p_{k,j}'s \) are the polynomials which, for each \( k \in \mathbb{N} \), differ from zero for only finitely many \( j \)'s. Then,

\[
y(t) \sim \sum_{k=1}^{\infty} \frac{1}{t^{\mu_k}} \left( \sum_{j=1}^{\infty} \frac{q_{k,j}(\ln \ln t)}{\ln^{\beta_j}(t)} \right),
\]

(5.36)

where \( q_{k,j}'s \) are the polynomials defined by

\[
q_{1,j} = A^{-1}p_{1,j} \quad \text{for} \ j \in \mathbb{N},
\]

(5.37)

and, for \( k \geq 2, j \in \mathbb{N} \),

\[
q_{k,j} = A^{-1} \left\{ \sum_{m \geq 2} \sum_{\mu_j_1+\mu_j_2+\ldots+\mu_j_m=\mu_k} \mathcal{G}_m(q_{j_1,l_1}, q_{j_2,l_2}, \ldots, q_{j_m,l_m}) + p_{k,j} + \chi_{k,j} \right\},
\]

(5.38)
with

\[
\chi_{k,j} = \begin{cases} 
\mu_\lambda q_{\lambda,j} - q'_{\lambda,\ell} + \beta \ell q_{\lambda,\ell}, & \text{if there exist } \lambda, \ell \geq 1 \text{ such that } \\
0, & \mu_\lambda + 1 = \mu_k, \beta \ell + 1 = \beta j,
\end{cases}
\]

(5.39)

**Proof.** First of all, one can verify, by induction in \(k\), that the function \(q_{k,j}\), for each \(k \in \mathbb{N}\), differs from the zero function only for finitely many \(j\)'s.

We apply Corollary 5.10 to \(\beta_{k,j} = \beta j\) for all \(k, j\). Under our assumptions, expansion (5.35) is the same as (5.28) with

\[
p_k(z_1, z_2) = \sum_{j=1}^{\infty} p_{k,j}(z_2) z_1^{-\beta_j}.
\]

(5.40)

Then, we have expansion (5.29) with \(q_k\) defined by (5.19) and (5.30). It suffices to establish, for all \(k \in \mathbb{N}\), that

\[
q_k(z_1, z_2) = \sum_{j=1}^{\infty} q_{k,j}(z_2) z_1^{-\beta_j},
\]

(5.41)

where \(q_{k,j}\) are defined by (5.37) and (5.38). We prove (5.41) by induction.

When \(k = 1\), thanks to (5.19) and (5.40),

\[
q_1(z_1, z_2) = A^{-1} p_1(z_1, z_2) = \sum_{j=1}^{\infty} A^{-1} p_{1,j}(z_2) z_1^{-\beta_j},
\]

thus (5.41) holds true for \(k = 1\).

Let \(k \geq 2\), assume formula (5.41) is true for \(q_1, q_2, \ldots, q_{k-1}\).

Set \(B = \{\beta_j : j \in \mathbb{N}\}\). Let \(F\) be the set of functions defined by (5.33) with the restriction \(\gamma \in B\). Because \(B\) preserves the addition, we have \(G_m(h_1, \ldots, h_m) \in F\) whenever \(h_1, \ldots, h_m \in F\). Therefore, the double sum in (5.19) belongs to \(F\) and can be rewritten as

\[
\sum_{j=1}^{\infty} \frac{1}{z_1^{\beta_j}} \left\{ \sum_{m \geq 2} \sum_{\mu_{j_1} + \mu_{j_2} + \ldots + \mu_{j_m} = \mu_k} G_m(q_{j_1,j_1}, q_{j_2,j_2}, \ldots, q_{j_m,j_m}) \right\}.
\]

(5.42)

We have from (5.30) that

\[
\chi_k(z_1, z_2) = \mu_\lambda \sum_{j=1}^{\infty} q_{\lambda,j}(z_2) z_1^{-\beta_j} + \sum_{j=1}^{\infty} \beta_j q_{\lambda,j}(z_2) z_1^{-\beta_j+1} - \frac{1}{z_1} \sum_{j=1}^{\infty} \frac{q'_{\lambda,j}(z_2)}{z_1^{\beta_j}},
\]

(5.43)

if there is \(\lambda\) such that \(\mu_\lambda + 1 = \mu_k\), or \(\chi_k(z_1, z_2) = 0\), otherwise.
Since $\beta_j + 1 \in B$ for all $j$, we can rewrite (5.43) as

$$\chi_k(z_1, z_2) = \sum_{j=1}^{\infty} \chi_{k, j}(z_2),$$

(5.44)

where $\chi_{k, j}$, for $j \in \mathbb{N}$, are polynomials defined as in formula (5.39).

Combining (5.19) with (5.42), (5.40) and (5.44), we obtain formula (5.41) for $q_k$.

By the induction principle, (5.41) holds true for all $k \in \mathbb{N}$. The proof is complete.

□

Example 5.12. Suppose $\mu_k = k$ and $\beta_j = j$ for all $k, j \in \mathbb{N}$. Then, $q_{1, j} = A^{-1} p_{1, j}$ for all $j \in \mathbb{N}$, and, for $k \geq 2$, $j \in \mathbb{N}$,

$$q_{k, j} = A^{-1} \left\{ \sum_{m \geq 2} \sum_{j_1 + j_2 + \ldots + j_m = k} G_m(q_{j_1, l_1}, q_{j_2, l_2}, \ldots, q_{j_m, l_m}) + p_{k, j} + \chi_{k, j} \right\},$$

where $\chi_{k, j} = (k - 1)q_{k-1, j} - q'_{k-1, j-1} + (j - 1)q_{k-1, j-1}$.

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A Appendix

We discuss a particular application of our results to numerical approximations of a nonlinear PDE problem using ODE systems. The presentation below is focused on the ideas without showing technical details.

Consider the Navier–Stokes Eq. (1.3) with a given initial data $u(0) = u_0$. For $m \in \mathbb{N}$, let $P_m$ denote the orthogonal projection to the first $m$ eigenspaces (corresponding to the first $m$ distinct eigenvalues) of the Stokes operator $A$.

The Galerkin approximation problem is

$$\frac{d u_m}{dt} + A u_m + B_m(u_m, u_m) = P_m f, \quad u_m(0) = P_m u_0,$$

(A.1)

where $B_m(u, u) = P_m B(u, u)$. For each $m \in \mathbb{N}$, the approximate system (A.1) is an ODE system in a finite-dimensional space, and $B_m(\cdot, \cdot)$ is a bilinear form. Thus, the results obtained in previous sections apply.

Consider Type 1, 2, 3 problems as in Sect. 4, that is,

$$f(t) \sim \sum_{k=1}^{\infty} p_k(\phi(t)) \psi(t)^{-k},$$

(A.2)

where the base functions $\phi(t)$ and $\psi(t)$ are given in Definition 4.2.
Then, the solutions $u(t)$ and $u_m(t)$ have the asymptotic expansions

$$u(t) \sim \sum_{k=1}^{\infty} q_k(\phi(t))\psi(t)^{-k} \quad \text{and} \quad u_m(t) \sim \sum_{k=1}^{\infty} q_k^{(m)}(\phi(t))\psi(t)^{-k},$$

respectively. \hfill (A.3)

The question is whether $q_k^{(m)}$ converges to $q_k$ as $m \to \infty$ in a certain sense. First, we roughly have

$$B_m(u, u) \to B(u, u), \quad P_m u_0 \to u_0 \quad \text{and} \quad P_m f \to f \quad \text{as} \quad m \to \infty. \hfill \text{(A.4)}$$

(The normed spaces in which the convergences hold depend on the regularity of $u, u_0$ and $f$.)

For Types 2 and 3, the polynomials $q_k$'s are independent of the solution $u(t)$, depend only on $p_k$ and $B(\cdot, \cdot)$. Similarly, for each $m \in \mathbb{N}$, the polynomials $q_k^{(m)}$'s are independent of the individual solution $u_m(t)$, depend only on $P_m p_k$ and $P_m B(\cdot, \cdot)$. With the convergences in (A.4) and explicit formulas (4.57) and (4.58), it is likely that the coefficients of $q_k^{(m)}$ converge to its corresponding coefficients of $q_k(t)$, as $m \to \infty$.

For Type 1, we consider the case $u(t)$ is a unique, regular solution on $[0, \infty)$. The construction of polynomial $q_k$, respectively $q_k^{(m)}$, depends on the long-time values of $u(t)$, respectively $u_m(t)$. Therefore, determining the convergence of $q_k^{(m)}$ to $q_k$, as $m \to \infty$, is more subtle than in the case of Types 2 and 3. However, we only consider the convergence for each fixed $k$, and, in light of many related estimates in previous work such as [8,9,11,13], it may still be possible to prove such a convergence.

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Dat Cao  
Department of Mathematics and Statistics  
Minnesota State University, Mankato  
Mankato MN 56001  
USA  
E-mail: dat.cao@mnsu.edu

Luan Hoang  
Department of Mathematics and Statistics  
Texas Tech University  
1108 Memorial Circle  
Lubbock TX 79409–1042  
USA  
E-mail: luan.hoang@ttu.edu

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