A canonical thickening of $\mathbb{Q}$ and the dynamics of continued fraction transformations

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Abstract

We construct a countable family of open intervals contained in $(0,1]$ whose endpoints are quadratic surds and such that their union is a full measure set. We then show that these intervals are precisely the monotonicity intervals of the entropy of $\alpha$-continued fractions, thus proving a conjecture of Nakada and Natsui.

1 Introduction

In many areas of mathematics, the space of parameters of a family of mathematical objects is itself an object of the same type. A well-known example of this phenomenon in dynamics is the Mandelbrot set, whose local geometry reflects the geometry of the Julia set of the quadratic polynomial corresponding to a given point.

The goal of this paper is to study a family of dynamical systems known as $\alpha$-continued fraction transformations, showing that the intervals in parameter space where a stability condition holds can themselves be described by means of regular continued fraction expansions. The family $\{T_\alpha\}_{\alpha \in (0,1]}$ of $\alpha$-continued fraction transformations has been defined in [6]; the most striking feature of this family may be that the entropy $h(T_\alpha)$ is not a monotone function of the parameter $\alpha$ (see [5]), and it is not even smooth everywhere.

Rather, Nakada and Natsui (7) showed that the entropy is locally monotone on intervals $I$ of parameters which satisfy the following matching condition

$$\exists N, M \in \mathbb{N}_+ : T^N_\alpha(\alpha) = T^M_\alpha(\alpha - 1) \quad \forall \alpha \in I$$

as well as some other technical conditions. Such intervals will be called matching intervals, and their union will be referred to as the matching set.

In [7], Nakada and Natsui exhibited three infinite families of matching intervals, where the entropy is, respectively, increasing, decreasing, and constant. Moreover, they conjectured:

Conjecture 1.1. The matching set has full measure in $(0,1]$ (hence it is dense).
A numerical study of the conjecture has been carried out in [1]: the goal of this paper is to prove the existence of the structures numerically observed there, thus proving conjecture 1.1.

The main tool to analyze the matching set will be regular continued fraction expansions; in fact, this matching set can be perfectly described without even mentioning the dynamics of \( \alpha \)-transformations. Let us briefly explain why.

It is well known that any rational value \( r \in \mathbb{Q} \) can be expressed as a finite continued fraction expansion of either even or odd length. This fact, usually perceived as a nuisance, will give us the chance to perform the following “natural” construction:

1. For any rational number \( r \in \mathbb{Q} \cap (0,1] \) we consider its two regular continued fraction expansions, namely:

\[
[0; a_1, \ldots, a_n] = [0; a_1, \ldots, a_n - 1, 1]
\]

\( a_n \geq 2 \)

We will associate to any such \( r \) the interval \( I_r \) whose endpoints are the quadratic surds

\[
[0; a_1, \ldots, a_n] \quad \text{and} \quad [0; a_1, \ldots, a_n - 1, 1]
\]

Such an \( I_r \) will be called the \textit{quadratic interval} generated by \( r \).

2. We will consider the union of all quadratic intervals

\[
\mathcal{M} := \bigcup_{r \in \mathbb{Q} \cap (0,1]} I_r
\]

The object of section 2 will be to understand the structure of the open dense set \( \mathcal{M} \), which can be summarized in the

\textbf{Theorem 1.2.} \textit{The set} \( \mathcal{M} \text{ has full Lebesgue measure in} \ (0,1], \text{ but its complement has Hausdorff dimension} \ 1.} \]

Although the family of quadratic intervals \( \{I_r\}_{r \in \mathbb{Q}} \) will have substantial overlapping, there is a subfamily that covers \( \mathcal{M} \) exactly. More precisely, a quadratic interval \( I_r \) will be called \textit{maximal} if it is not properly contained in any other quadratic interval. It turns out that every quadratic interval is contained in some maximal one, and distinct maximal quadratic intervals do not intersect (lemma 2.6): thus \( \mathcal{M} \) is the disjoint union of this collection of maximal intervals. This suggests that \( (0,1] \setminus \mathcal{M} \) should have a Cantor-like structure; this is only partially true because \( (0,1] \setminus \mathcal{M} \) is not perfect. Indeed, the presence of isolated points is a consequence of the \textit{period-doubling} phenomenon (see subsection 3.3): if \( r := [0; a_1, \ldots, a_n] \in \mathbb{Q} \) with \( n \) odd and \( I_r \) is a maximal quadratic interval, then \( r' := [0; a_1, \ldots, a_n, a_1, \ldots, a_n] < r \) generates \( I_{r'} \) which is maximal as well, and the quadratic surd \( \alpha := [0; a_1, \ldots, a_n] \) is a common endpoint, which is obviously not contained in any quadratic interval.

In the second part of the paper (section 3) we prove that that this set \( \mathcal{M} \) is closely connected to the matching intervals. More precisely we prove
Theorem 1.3. Let $a \in \mathbb{Q} \cap (0, 1]$ such that $I_a$ is maximal. Then there exist positive integers $N, M$ such that

$$T^N_\alpha(\alpha) = T^M_\alpha(\alpha - 1) \quad \forall \alpha \in I_a$$

Moreover, the entropy function $\alpha \mapsto h(T_\alpha)$ is monotone on $I_a$.

The proof of the theorem relies on the fact that an algebraic matching condition stronger than (1) holds everywhere on $M$; by theorem 1.2 this condition holds for almost every parameter.

Moreover, the set defined by the algebraic matching condition contains the matching set defined by Nakada and Natsui and the difference between them is countable (see appendix), hence they have the same measure and conjecture 1.1 follows.

Our method also gives us an explicit control over the combinatorics of matchings: given any rational number, we are able to determine which maximal interval it belongs to and the matching exponents $(N, M)$, hence the local behaviour of entropy (constant, increasing or decreasing). Conversely, one can use such knowledge to produce families of matching intervals with prescribed properties.

Finally, section 4 contains a few technical tools we use throughout the paper, including a criterion to compare purely periodic quadratic surds (String Lemma 2.12) and an explicit characterization of either of the finite continued fraction expansions which generate a maximal quadratic interval (lemma 2.13).

It is worth noting that the phenomenon we describe is strongly reminiscent of the theory of circle maps (see e.g. [8], chap. 7.2.): in that case, around each rational rotation number, in the parameter space there is a region ('Arnold tongue') where the dynamics is still periodic ('mode-locking'), in such a way that on the critical line the complement of the union of all Arnold tongues has measure zero (even though its Hausdorff dimension is strictly smaller than 1, differently from our case [2]).

Recently S. Katok and I. Ugarcovici have studied another family of transformations, called $(a, b)$-continued fractions, which seem to share various features with the transformation $T_\alpha$ (see [4]): it would be worth investigating more closely the connection between these systems in order to see whether the two different approaches can lead to a deeper understanding of both.

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2 Thickening $\mathbb{Q}$

Let $S = (s_1,\ldots, s_n)$ be a finite string of positive integers: we will use the notation

$$[0; S] := [0; s_1, \ldots, s_n] = \frac{1}{s_1 + \frac{1}{s_2 + \cdots + \frac{1}{s_n}}}$$

Moreover, $\bar{S}$ will be the periodic infinite string $SSS\ldots$ and $[0; \bar{S}]$ the quadratic surd with purely periodic continued fraction $[0; \frac{s_1}{s_1}, \ldots, \frac{s_n}{s_n}]$. The symbol $|S|$ will denote the length of the string $S$. We will denote the denominator of the rational number $r$ as $\text{den}(r)$.

2.1 Pseudocenters

Let us start out by defining a useful tool in our analysis of intervals defined by continued fractions.

**Lemma 2.1.** Let $J = ]\alpha, \beta[ \in \mathbb{R}, |\alpha - \beta| < 1$. Then there exists a unique rational $p/q \in J$ such that $q = \min \{q' \geq 1 : p'/q' \in J \}$.

**Proof.** Let $d := \min \{q \geq 1 : p/q \in J \}$. If $d = 1$ we are done. Let $d > 1$ and assume by contradiction that $\frac{c}{d} < \frac{k}{d} < \frac{c+1}{d}$, both belong to $J$. Then there exists $k \in \mathbb{Z}$ such that $\frac{k}{d-1} < \frac{c}{d} < \frac{k+1}{d-1}$, hence $cd - c < kd < cd - c$, which is a contradiction since $kd$ is an integer. $\square$

**Definition 2.1.** The number $\frac{p}{q}$ which satisfies the properties of the previous lemma will be called the pseudocenter of $J$.

**Lemma 2.2.** Let $\alpha, \beta \in ]0, 1[ \in \mathbb{Q}$ be two irrational numbers with c.f. expansions $\beta = [0; s_0, b_1, b_2, \ldots]$ and $\alpha = [0; s_0, a_1, a_2, \ldots]$, where $S$ stands for a finite string of positive integers. Assume $b_0 > a_0$. Then the pseudocenter of the interval $J$ with endpoints $\alpha$ and $\beta$ is $r = [0; S, a_0 + 1](= [0; S, a_0, 1])$

**Proof.** Suppose there exists $s \in \mathbb{Q} \cap J$ with $\text{den}(s) < \text{den}(r)$. Since $s \in J$, then $s = [0; S, s_0, s_1, \ldots, s_k]$ with $a_0 \leq s_0 \leq b_0$ and $k \geq 0$. The choice $s_0 \geq a_0 + 1$ gives rise to $\text{den}(s) \geq \text{den}(r)$, so $s_0 = a_0$. On the other hand, $[0; S, a_0]$ does not belong to the interval, so $k \geq 1$ and $s_1 \geq 1$, still implying $\text{den}(s) \geq \text{den}(r)$. $\square$

2.2 Quadratic intervals

**Definition 2.2.** Let $0 < a < 1$ be a rational number with c.f. expansion $a = [0; a_1, \ldots, a_N] = [0; a_1, \ldots, a_N - 1, 1]$, $a_N \geq 2$
We define the quadratic interval $I_a$ associated to $a$ to be the open interval with endpoints

$$[0; a_1, \ldots, a_N, a_N] \quad \text{and} \quad [0; a_1, \ldots, a_N, a_N - 1, 1]$$

Moreover, we define $I_1 := (\sqrt{\frac{5}{2}} - 1, 1)$ (recall that $\sqrt{\frac{5}{2}} - 1 = [0; 1]$).

Note that the ordering of the endpoints in (2) depends on the parity of $N$: given $a \in \mathbb{Q}$, we will denote by $A^+$ and $A^-$ the two strings of positive integers which represent $a$ as a continued fraction, with the convention that $A^+$ is the string of even length and $A^-$ the string of odd length, so that $I_a = ([0; A^-], [0; A^+])$, $a = [0; A^+] = [0; A^-]$.

**Example**

If $a = \frac{1}{3} = [0; 3] = [0; 2, 1]$, $[0; A^+] = [0; 2, 1]$, $[0; A^-] = [0; 3]$, $I_a = (\sqrt{\frac{5}{2}} - 3, \sqrt{\frac{5}{2}} - 1)$

Note that $a$ is the pseudocenter of $I_a$, hence $I_a = I_{a'} \iff a = a'$.

**Lemma 2.3.**

1. If $\xi \in I_a$, then $a$ is a convergent to $\xi$.
2. If $I_a \cap I_b \neq \emptyset$, then either $a$ is a convergent to $b$ or $b$ is a convergent to $a$.
3. If $I_a \subseteq I_b$ then $b$ is convergent to $a$, hence $\text{den}(a) < \text{den}(b)$.

**Proof.** 1. Since $\xi \in I_a$, either $\xi = [0; a_1, \ldots, a_n, \ldots]$ or $\xi = [0; a_1, \ldots, a_N - 1, 1, \ldots]$. In the first case the claim holds; in the second case one has to notice that neither $[0; a_1, \ldots, a_N - 1]$ nor all elements of the form $[0; a_1, \ldots, a_N - 1, k, \ldots]$ with $k \geq 2$ belong to $I_a$, so $k = 1$ and $a$ is a convergent of $\xi$.
2. Fix $\xi \in I_a \cap I_b$. By the previous point, both $a$ and $b$ are convergents of $\xi$, hence the rational with the shortest expansion is a convergent of the other.
3. From 1. since $a \in I_a \subseteq I_b$. \hfill \qed

**Definition 2.3.** A quadratic interval $I_a$ is maximal if it is not properly contained in any $I_b$ with $b \in \mathbb{Q} \cap [0, 1]$.

The interest in maximal quadratic intervals lies in the

**Proposition 2.4.** Every quadratic interval $I_a$ is contained in a unique maximal quadratic interval.

A good way to visualize the family of quadratic intervals is to plot, for any rational $a$, the geodesic $\gamma_a$ on the hyperbolic upper half plane with the same endpoints as $I_a$, as in the following picture: one can see the maximal intervals corresponding to the “heighest” geodesics, in such a way that every $\gamma_a$ has some maximal geodesic (possibly itself) above it and no two maximal $\gamma_a$ intersect.
The proof of proposition 2.4 will be given in two lemmas:

**Lemma 2.5.** Every quadratic interval \( I_a \) is contained in some maximal quadratic interval.

**Proof.** If \( I_a \) were not contained in any maximal interval, then there would exist an infinite chain \( I_a \subsetneq I_{a_1} \subsetneq I_{a_2} \subsetneq \ldots \) of proper inclusions, hence by the lemma every \( a_i \) is a convergent of \( a \), but rational numbers can only have a finite number of convergents. \( \square \)

**Lemma 2.6.** If \( I_a \) is maximal then for all \( a' \in \mathbb{Q} \cap (0,1) \)

\[ I_a \cap I_{a'} \neq \emptyset \implies I_{a'} \subset I_a, \]

and equality holds iff \( a = a' \). In particular, distinct maximal intervals do not intersect.

**Proof.** We need the following lemma, which we will prove in section 4.

**Lemma 2.7.** If \( I_a \cap I_b \neq \emptyset , I_a \setminus I_b \neq \emptyset \) and \( I_b \setminus I_a \neq \emptyset \), then either \( I_a \) or \( I_b \) is not maximal.

Let now \( I_{a_0} \) be the maximal interval which contains \( I_{a'} \). Since \( I_a \cap I_{a_0} \neq \emptyset \), by lemma 2.7 either \( I_a \subseteq I_{a_0} \) or \( I_{a_0} \subseteq I_a \), hence by maximality \( I_a = I_{a_0} \) and \( I_{a'} \subseteq I_a \). Since \( a \) is the pseudocenter of \( I_a \), \( I_a = I_{a'} \implies a = a' \). \( \square \)

### 2.3 Hausdorff dimension

In this section we prove theorem 1.2, which states that the exceptional set \( E := [0,1] \setminus \mathcal{M} \) has zero Lebesgue measure but Hausdorff dimension equal to 1. The key tool of the proof is the following lemma, which establishes a connection between \( E \) and numbers of bounded type.

**Lemma 2.8.**

(i) Let \( \xi \in E = [0,1] \setminus \mathcal{M} \). Then \( \xi \) is irrational and \( \xi = [0; a_1, \ldots, a_n, \ldots] \) with \( a_j \leq a_1 \) for all \( j \in \mathbb{N}^+ \)

(ii) Let \( \xi = [0; a_1, \ldots, a_n, \ldots] \) be an irrational number such that \( a_k \leq a_1 - 1 \) for all \( k \geq 2 \). Then \( \xi \) does not belong to any \( I_a \) for any \( a \in \mathbb{Q} \cap (0,1) \).
Proof. Since $\xi \notin \mathcal{M}$ then $\xi \notin \mathbb{Q}$. If $\xi$ has the infinite c.f. expansion $\xi = [0; a_1, \ldots, a_n, \ldots]$ with $a_k > a_1$ for some $k \in \mathbb{N}^+$ then $x$ lies between $r := [0; a_1, \ldots, a_{k-1}]$ and $\alpha := [0; a_1, \ldots, a_{k-1}]$; therefore $x \in I_r \subset \mathcal{M}$. Let $a = [0; A^+] = [0; A^-]$, so that $I_a = ([0; A^-], [0; A^+])$. If $\xi \in I_a$, by lemma 2.3 $a$ is a convergent of $\xi$, so either $\xi = [0; A^+, \ldots]$ or $\xi = [0; A^-, \ldots]$.

In the first case $\xi = [0; A^+, s, \ldots]$ with $s < a_1$, so $\xi > [0; A^-] = [0; A^+, a_1, \ldots]$; in the second one, $\xi = [0; A^-, s, \ldots]$ with $s < a_1$ and therefore $\xi < [0; A^-] = [0; A^-, a_1, \ldots]$.

Proof. (of Theorem 1.2) Lemma 2.8 implies that $\mathcal{E}$ is contained in the set of numbers of bounded type, hence it has Lebesgue measure zero.

On the other hand, let $N \geq 1$, and define

$$C_N := \{ x = [0; a_1, \ldots] \mid a_k \leq N \ \forall k \geq 1 \}$$

$$E_N := \left[ \frac{1}{N+1}, \frac{1}{N} \right] \cap \mathcal{E}$$

By lemma 2.10 and lemma 2.8 for $N \geq 2, E_N \supseteq \phi(C_{N-1})$ where $\phi(x) := x \mapsto \frac{1}{N+x}$. Since $\phi$ is a bi-Lipschitz map, it preserves Hausdorff dimension, so

$$\dim_H C_{N-1} = \dim_H \phi(C_{N-1}) \leq \dim_H E_N \leq \dim_H C_N$$

Since it is well-known (3) that $\sup_{N \to \infty} \dim_H C_N = 1$ and $\mathcal{E} = \cup_N E_N$, the claim follows.

Remark. A similar way of stating the same result would be to say that for every $\xi \in \mathbb{Q} \cap \left( \frac{1}{N+1}, \frac{1}{N} \right)$

$$B \left( \frac{p}{q}, \frac{1}{(N+2)q^2} \right) \subseteq I_{p/q} \subseteq B \left( \frac{p}{q}, \frac{1}{(N-1)q^2} \right)$$

This means that in any fixed subinterval $\left( \frac{1}{N+1}, \frac{1}{N} \right)$ the size of the geodesic over $I_{p/q}$ is comparable to the diameter of the horocycles $\partial B \left( \frac{p}{q}, \frac{1}{Nq^2} \right)$ (which, for any fixed $N$, all lie in the same $SL_2(\mathbb{Z})$-orbit). The picture shows this comparison for $N = 10$. 

\[ \text{Diagram showing the comparison} \]
2.4 The bisection algorithm

We will now describe an algorithmic way to produce all maximal intervals, as announced in [1], sect. 4.1. This will also provide an alternative proof of the fact the $\mathcal{M}$ has full measure.

Let $F$ be a family of disjoint open intervals which accumulate only at 0, i.e. such that for every $\epsilon > 0$ the set $\{J \in F : J \cap [\epsilon, 1] \neq \emptyset\}$ is finite, and denote $F = \bigcup_{J \in F} J$. The complement $[0, 1] \setminus F$ will then be a countable union of closed disjoint intervals $C_j$, which we refer to as gaps. Note that some $C_j$ may well be a single point. To any gap which is not a single point we can associate its pseudocenter $c \in \mathbb{Q}$ as defined in the previous sections, and moreover consider the interval $I_c$ associated to this rational value. The following proposition applies.

**Proposition 2.9.** Let $I_a$ and $I_b$ be two maximal intervals such that the gap between them is not a single point, and let $c$ be the pseudocenter of the gap. Then $I_c$ is a maximal interval and it is disjoint from both $I_a$ and $I_b$.

**Proof.** Pick $I_{c_0}$ maximal such that $I_c \subseteq I_{c_0}$, so by lemma 2.3 $\text{den}(c_0) \leq \text{den}(c)$. On the other hand, since maximal intervals do not intersect, then $I_{c_0}$ is contained in the gap and since $c$ is pseudocenter, then $\text{den}(c) \leq \text{den}(c_0)$ and equality holds only if $c = c_0$.

The proposition implies that if we add to the family of maximal intervals $\mathcal{F}$ all intervals which arise as gaps between adjacent intervals then we will get another family of maximal (hence disjoint) intervals, and we can iterate the procedure.

For instance, let us start with the collection $\mathcal{F}_1 := \{I_{1/n}, n \geq 1\}$. All these intervals are maximal, since the continued fraction of their pseudocenters has only one digit (apply lemma 2.3).

Let us construct the families of intervals $\mathcal{F}_n$ recursively as follows:

$$G_n := \{C \text{ connected component of } [0, 1] \setminus F_n\}$$

$$\mathcal{F}_{n+1} := \mathcal{F}_n \cup \{I_r : r \text{ pseudocenter of } C, C \in G_n, C \text{ not a single point}\}$$

(where $F_n$ denotes the union of all intervals belonging to $\mathcal{F}_n$).

It is thus clear that the union $\mathcal{F}_\infty := \bigcup \mathcal{F}_n$ will be a countable family of maximal intervals. The union of all elements of $\mathcal{F}_\infty$ will be denoted by $F_\infty$; its complement (the set of numbers which do not belong to any of the intervals produced by the algorithm) has the following property:

**Lemma 2.10.** $[0, 1] \setminus F_\infty$ consists of irrational numbers of bounded type; more precisely, the elements of $(\frac{1}{n+1}, \frac{1}{n}] \setminus F_\infty$ have partial quotients bounded by $n$.

**Proof.** Let $\gamma = [0;c_1, c_2, ..., c_n, ...] \notin F_\infty$; we claim that $c_k \leq c_1$ for all $k \in \mathbb{N}$. Since $\gamma \notin F_\infty$, $\forall n \geq 1$ we can choose $J_n \in G_n$ such that $\gamma \in J_n$. Clearly, $J_{n+1} \subseteq J_n$. Furthermore, $\gamma$ cannot be contained in either $I_{\frac{1}{c_1}}$ nor $I_{\frac{1}{c_1+1}}$, so all $J_n$ are
produced by successive bisection of the gap \([0; c_1, 1], [0; c_1])\), hence by lemma 2.2 for every \(n\), the endpoints of \(J_n\) are quadratic surds with c.f. expansion bounded by \(c_1\). It may happen that there exists \(n_0\) such that \(J_n = \{\gamma\} \forall n \geq n_0\), so \(\gamma\) is an endpoint of \(J_{n_0}\), hence it is irrational and \(c_1\)-bounded. Otherwise, let \(p_n/q_n\) be the pseudocenter of \(J_n\); by uniqueness of the pseudocenter, \(\text{diam} J_n \leq 2/q_n\), and \(q_{n+1} > q_n\) since \(J_{n+1} \subseteq J_n\). This implies \(\gamma\) cannot be rational, since the minimum denominator of a rational sitting in \(J_n\) is \(q_n \to +\infty\). Moreover, \(\text{diam} J_n \to 0\), so \(\gamma\) is limit point of endpoints of the \(J_n\), which are \(c_1\)-bounded, hence \(\gamma\) is also \(c_1\)-bounded.

\[\text{Proposition 2.11.} \quad \text{The family } \mathcal{F}_\infty \text{ is precisely the family of all maximal intervals; hence } \mathcal{F}_\infty = \mathcal{M}.\]

\[\text{Proof.} \quad \text{If } I_c \text{ a maximal interval does not belong to } \mathcal{F}_\infty, \text{ then its pseudocenter belongs to the complement of } \mathcal{F}_\infty, \text{ but the previous lemma asserts that this set does not contain any rational.} \]

Note that proposition 2.11 and lemma 2.10 provide another way of seeing that the complement of \(\mathcal{M}\) consists of numbers of bounded type, hence it has full measure.

### 2.5 Maximal intervals and strings

In order to get a finer control on the maximality properties of quadratic intervals, we introduce a systematic description of the continued fraction expansions in terms of strings and develop a few tools in order to characterize the expansions of those rational numbers which give rise to maximal intervals.

Let us start out with some notation. If \(S = (s_1, \ldots, s_n)\) is a finite string of positive integers and \(x\) a real number, we will denote

\[0; S] := \frac{1}{s_1 + \frac{1}{\ddots + \frac{1}{s_n}}} \quad \quad [0; S + x] := \frac{1}{s_1 + \frac{1}{\ddots + \frac{1}{s_n + x}}}\]

We will also introduce a total ordering on the space of finite strings of given length: given two distinct finite strings \(S\) and \(T\) of equal length, let \(l := \min\{i : S_i \neq T_i\}\). We will set

\[S < T := \begin{cases} S_l < T_l \text{ if } l \equiv 0 \mod 2 \\ S_l > T_l \text{ if } l \equiv 1 \mod 2 \end{cases} \]

The exact same definition also gives a total ordering on the space of infinite strings. Note that if \(S\) and \(T\) have equal length \(L \in \mathbb{N} \cup \{\infty\}\),

\[S < T \iff [0; S] < [0; T]\]

i.e. this ordering can be obtained by pulling back the order structure on \(\mathbb{R}\), via identification of a string with the value of the corresponding c.f.

The following lemma is the essential tool used to compare two purely periodic infinite strings:
Lemma 2.12. Let $S$, $T$ be two nonempty, finite strings. Then the pair of infinite strings $S$, $T$ is ordered in the same way as the pair $ST$, $TS$; namely

$$ST \lesssim TS \iff S \lesssim T.$$ 

Finally, we can give an explicit characterization of the c.f. expansion of those rationals which are pseudocenters of maximal intervals:

Proposition 2.13. Let $a = [0; A] \in \mathbb{Q} \cap (0, 1]$. The following are equivalent:

(i) $I_a$ is maximal.

(ii) If $A = ST$ with $S$, $T$ finite nonempty strings, then either $ST < TS$ or $ST = TS$ with $T = S$, $|S|$ odd.

Moreover, if $[0; ST]$ is maximal, then $[0; T] > [0; ST]$.

For the sake of readability, we postpone the proofs of these results to section 4.

3 Application to $\alpha$-continued fractions

After having investigated the properties of the maximal set itself, this section will be devoted to studying its relation with the parameter space of $\alpha$-continued fractions.

3.1 Matching intervals

Let $\alpha \in (0, 1]$. Recall that the $\alpha$-continued fraction expansion is given by the map $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ defined by $T_\alpha(0) = 0$ and

$$T_\alpha(x) = \frac{\epsilon_\alpha(x)}{x} - c_\alpha(x) \quad \text{for } x \neq 0$$

with

$$\epsilon_\alpha(x) := \text{Sign}(x) \quad c_\alpha(x) := \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor$$

Moreover, one can represent the encoding with the matrices in $GL(2, \mathbb{Z})$

$$M_{\alpha,x,n} = \begin{pmatrix} 0 & \epsilon_\alpha(x) \\ 1 & c_\alpha(x) \end{pmatrix} \cdots \begin{pmatrix} 0 & \epsilon_\alpha(T_\alpha^{n-1}(x)) \\ 1 & c_\alpha(T_\alpha^{n-1}(x)) \end{pmatrix} = \begin{pmatrix} p_{n-1,\alpha}(x) & p_{n,\alpha}(x) \\ q_{n-1,\alpha}(x) & q_{n,\alpha}(x) \end{pmatrix}$$

so that

$$x = \frac{p_{n-1,\alpha}(x)x_n + p_{n,\alpha}(x)}{q_{n-1,\alpha}(x)x_n + q_{n,\alpha}(x)} \quad \text{with } x_n = T_\alpha^n(x). \quad (3)$$

We will be interested in the metric entropy $h(T_\alpha)$ of these transformations as a function of $\alpha$; in [7], a series of matching conditions were introduced in order to define intervals in the parameter space where the entropy function $\alpha \mapsto h(T_\alpha)$ is monotone. In the same spirit, we will define
Definition 3.1. The value \( \alpha \in [0,1] \) is said to satisfy an algebraic matching condition of order \((N, M)\) when the following matrix identity holds:

\[
M_{\alpha, \alpha, N} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} M_{\alpha, \alpha-1, M} \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \quad (N, M)_{\text{alg}}
\]

We will be interested in the set

\[M_{\text{alg}} = \{ \alpha \in [0,1] \text{ s.t. } \exists N, M \in \mathbb{N} : \alpha \text{ satisfies } (N, M)_{\text{alg}} \}\]

To get some intuition of what this condition means from a dynamic point of view, one should note that \((N, M)_{\text{alg}}\) implies

\[T_\alpha^{N+1}(\alpha) = T_\alpha^{M+1}(\alpha - 1)\]

The formal proof of this result is given in the appendix, together with a thorough discussion of the relationship between our algebraic matching condition and the conditions originally considered by Nakada and Natsui.

The main result will be:

**Theorem 3.1.** Let \( a \in \mathbb{Q} \cap (0,1] \) such that \( I_a \) is maximal, and let \( a = [0; a_1, \ldots, a_n], \) \( n \) even. If we define

\[N := \sum_{j \text{ even}} a_j \quad M := \sum_{j \text{ odd}} a_j\]

then for every \( x \in I_a \), the matching condition \((N, M)_{\text{alg}}\) holds.

**Corollary 3.2.** \( M_{\text{alg}} \) has full Lebesgue measure in \((0,1]\).

**Proof.** By theorem 3.1, \( M_{\text{alg}} \) contains \( M \), which has full measure by theorem 1.2. \(\Box\)

Since it can be proved (see appendix) that the difference between \( M_{\text{alg}} \) and the matching set defined by Nakada and Natsui is countable, this also establishes conjecture 1.1.

### 3.2 Anatomy of maximal orbits

The first step in the proof of theorem 3.1 will be to describe explicitly the first few steps of the orbit of any point inside a maximal interval \( I_a \): we will start by establishing the

**Lemma 3.3.** Let \( a \in \mathbb{Q} \cap (0,1] \) be the pseudocenter of a maximal \( I_a = (\alpha^-, \alpha^+) \).

1. Let \( a \leq x < \alpha^+ \), so that we can write \( x = [0; a_1, \ldots, a_n + y] \) with \( 0 \leq y < \alpha^+ \), \( a = [0; a_1, \ldots, a_n] \) with \( n \equiv 0 \mod 2 \). Then

\[-1; b, a_{k+1}, \ldots, a_n + y] > \alpha^+ - 1 \quad \forall 1 \leq b \leq a_k, \ 1 < k \leq n\]
2. Let $\alpha^- < x \leq a$, so that $x = [0; a_1, \ldots, a_n + y]$ with $0 \leq y < \alpha^-$, $a = [0; a_1, \ldots, a_n]$ with $n \equiv 1 \pmod 2$ (note this is the representation of $a$ in c.f. other than the one given in the previous point). Then

\[
[-1; b, a_k, \ldots, a_n + y] > \alpha^+ - 1 \quad \forall 1 \leq b \leq a_k, \ 1 < k \leq n
\]

Proof. 1. Let $S := (a_1, \ldots, a_{k-1})$, $T := (a_k, \ldots, a_n)$ and $c := [0; T]$. By lemma 2.13 and 2.12

\[
TS \geq ST \Rightarrow TS \geq ST \Rightarrow [0; STS] \geq [0; ST]
\]

Moreover,

\[
TS \geq ST \Rightarrow TST \geq STT \Rightarrow T \geq ST
\]

Now, $I_c \cap I_a = \emptyset$ since $I_a$ is maximal and the denominator of $c$ is smaller than the denominator of $a$, hence $[0; T] > [0; ST]$. Since $b \leq a_k$ and $0 \leq y < \alpha^+$, for $k$ even we have

\[
[-1; b, a_{k+1}, \ldots, a_n + y] \geq [-1; T, y] \geq [-1; T, \alpha^+] = [-1; TS] \geq [-1; ST] = \alpha^+ - 1
\]

and for $k$ odd,

\[
[-1; b, a_{k+1}, \ldots, a_n + y] \geq [-1; T, y] \geq [-1; T] > [-1; ST] = \alpha^+ - 1
\]

2. Let $S := (a_1, \ldots, a_{k-1})$, $T := (a_k, \ldots, a_n)$ and $c := [0; T]$. If $k$ is odd

\[
TS \geq ST \Rightarrow TTS \leq TST \Rightarrow T \leq TS
\]

Moreover, $T \geq ST$ as in the previous point, and since $I_a \cap I_c = \emptyset$, then $[0; T] \geq \alpha^+$, so $[0; TS] \geq [0; T] \geq \alpha^+$; hence,

\[
[-1; b, a_{k+1}, \ldots, a_n + y] \geq [-1; T, y] \geq [-1; T, \alpha^+] = [-1; TS] \geq \alpha^+ - 1.
\]

For $k$ even, by the last point of proposition 2.13 $[0; T] > [0; ST]$, and since $I_a \cap I_c = \emptyset$, $[0; T] > \alpha^+$; thus,

\[
[-1; b, a_{k+1}, \ldots, a_n + y] \geq [-1; T, y] \geq [-1; T] > \alpha^+ - 1
\]

An immediate corollary is the explicit description of the orbit of the pseudocenter which explains an empirical rule given in 11.

**Corollary 3.4.** Let $a := [0; a_1, a_2, \ldots, a_n]$, $(n \geq 1)$ and let $I_a$ be maximal; then the orbits of $a$ and $a - 1$ are as follows:

\[
a = [0; a_1, a_2, \ldots, a_n] \quad a - 1 = [-1; a_1, a_2, \ldots, a_n]
\]

\[
T_a(a) = [-1; a_2, \ldots, a_n] \quad T_a(a - 1) = [-1; a_1 - 1, a_2, \ldots, a_n]
\]

\[
T_a(a_2)(a) = [-1; 1, a_3, \ldots, a_n] \quad T_a(a_2)(a - 1) = [-1; 1, a_2, \ldots, a_n]
\]

\[
T_a(a_2+1)(a) = [-1; a_4, \ldots, a_n] \quad T_a(a_2+1)(a - 1) = [-1; a_3, \ldots, a_n]
\]

\[
T_a^N(a) = 0 \quad T_a^M(a - 1) = 0
\]
where (see also [1], pg. 23)

\[ N = \sum_{j \text{ even}} a_j, \quad M = \sum_{j \text{ odd}} a_j, \quad \text{if } n \text{ is even} \]
\[ N = 1 + \sum_{j \text{ even}} a_j, \quad M = -1 + \sum_{j \text{ odd}} a_j, \quad \text{if } n \text{ is odd} \]

We will now prove that an algebraic matching condition holds for any pseudo-center of a maximal interval.

**Proposition 3.5.** Let \( a \in \mathbb{Q} \cap (0, 1] \) so that \( I_a \) is maximal, and let \( N \) and \( M \) be given by the previous corollary. Then \( a \) satisfies the matching condition \((N, M)_{\text{alg}}\).

**Proof.** We will make use of the following lemma:

**Lemma 3.6.** For \( \alpha < \frac{\sqrt{5} - 1}{2} \), one has \( q_{n+1,\alpha}(x) > q_{n,\alpha}(x) \geq 1 \) for every \( n \geq 0 \) and every \( x \in [\alpha - 1, \alpha] \).

**Proof.** By definition, \( q_{0,\alpha}(x) = 1 \) and \( q_{1,\alpha}(x) = c_{1,\alpha}(x) \geq 2 \) (the latter only for \( \alpha < \frac{\sqrt{5} - 1}{2} \)). By induction, \( q_{n+1,\alpha}(x) = c_{n+1,\alpha}(x)q_{n,\alpha}(x) + c_{n+1,\alpha}(x)q_{n-1,\alpha}(x) \geq 2q_{n,\alpha}(x) - q_{n-1,\alpha}(x) > q_{n,\alpha}(x) \).

Since it is easy to see that all values of \( \alpha > \frac{\sqrt{5} - 1}{2} \) satisfy a matching condition of order \((1, 2)\), we can restrict our attention to the case in which we can apply lemma 3.6. We will denote \( p_k := p_{k,\alpha}(\alpha) \) and \( p'_k := p_{k,\alpha}(\alpha - 1) \). Let \((N, M)\) be given by corollary 3.4 such that

\[ T^N_a(a) = 0 \quad \text{and} \quad T^M_a(a - 1) = 0 \]

By equation (3),

\[ a = p_N/q_N \quad a - 1 = p'_M/q'_M \]

and since \( \gcd(p_N, q_N) = \gcd(p'_M, q'_M) = 1 \) (because \( \det M_{a,x,k} = \pm 1 \)),

\[ q_N = q'_M \quad p_N = p'_M + q'_M \quad (4) \]

Now, corollary 3.4 implies \( \epsilon_a(T^N_a(a)) = \epsilon_a(T^N_a(a - 1)) = -1 \) for \( 1 \leq i \leq N - 1, 1 \leq j \leq M - 1 \), hence

\[ \det M_{a,a,N} = -1 \quad \det M_{a,a-1,M} = 1 \]

by writing out the two determinants and summing up

\[ p_{N-1}q_N - p_Nq_{N-1} + p'_{M-1}q'_M - p'_Mq'_{M-1} = 0 \]

and by using (4)

\[ q'_M(p_{N-1} + p'_{M-1} - q_{N-1}) = p'_M(q'_M + q_{N-1}) \]

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Now, $q'_M$ and $p'_M$ are coprime, hence $q'_M(q'_{M-1} + q_{N-1})$, and by lemma 3.6

$$0 < q'_M(q'_{M-1} + q_{N-1}) < 2q'_M,$$

so

$$q'_M = q'_{M-1} + q'_{N-1} \quad p'_M = p_{N-1} + p'_{M-1} - q_{N-1}$$

which yields precisely the algebraic matching condition

$$M_{a,a,N} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} M_{a,a-1,M} \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

The final step will be to prove that all points in $I_a$ have the same convergents as the pseudocenter.

**Lemma 3.7.** Let $I_a$ be maximal, and $x \in I_a$, $N$, $M$ as in corollary 3.4. Then

$$M_{x,x,k} = M_{a,a,k} \quad \forall 1 \leq k \leq N$$

$$M_{x,x-1,h} = M_{a,a-1,h} \quad \forall 1 \leq h \leq M$$

**Proof.** If $x \geq a$, we can write $x = [0; A+y]$ with $|A| \equiv 0 \mod 2$ and $0 \leq y < \alpha^+$; from corollary 3.4

$$x = [0; a_1, a_2, \ldots a_n + y] \quad x - 1 = [-1; a_1, a_2, \ldots a_n + y]$$

$$M_{a,a-1,i}(x) = [-1; a_2, \ldots a_n + y] \quad M_{a,a-1,i-1}(x - 1) = [-1; a_1 - 1, a_2, \ldots a_n + y]$$

$$M_{a,a-1,a_2}(x) = [-1; a_3, a_2, \ldots a_n + y] \quad M_{a,a-1,a_2-1}(x - 1) = [-1; a_2, a_3, \ldots a_n + y]$$

$$M_{a,a-1,a_2+1}(x) = [-1; a_4, \ldots a_n + y] \quad M_{a,a-1,a_2,a_1}(x - 1) = [-1; a_3, a_4, \ldots a_n + y]$$

$$\ldots$$

$$M_{a,a-1,a_2}(x) = [-1; a_3, \ldots a_n + y] \quad M_{a,a-1,a_2+1}(x - 1) = [-1; a_4, \ldots a_n + y]$$

$$M_{a,a,N}(x) = [-1; 1 + y] \quad M_{a,a-1,M}(x - 1) = y$$

and again from the lemma,

$$M_{a,a,k}^{-1}(x) \in (\alpha^+ - 1, 0) \subseteq (x - 1, 0) \quad 1 \leq k \leq N$$

$$M_{a,a-1,h}^{-1}(x) \in (\alpha^+ - 1, 0) \subseteq (x - 1, 0) \quad 1 \leq h \leq M - 1$$

hence

$$M_{x,x,k} = M_{a,a,k} \quad 1 \leq k \leq N$$

$$M_{x,x-1,h} = M_{a,a-1,h} \quad 1 \leq h \leq M - 1.$$
hence, $y < [0; A^{k+1}] \leq [0; A + y] = x$ and $M^{-1}_{a,a-1,M}(x - 1) \in (0, x)$, so $M_{a,a-1,M} = M_{x,x-1,M}$.

The case $x \leq a$ is similar: the only non-negative element of the orbit this time is

$$M^{-1}_{a,a,N}(x) = y \quad \text{with} \quad 0 \leq y < \alpha$$

which, since $\alpha - x$, still implies $M_{a,a,N} = M_{x,x,N}$.

**Proof of theorem 3.1** Let $x \in I_a$, a maximal. By proposition 3.5 $M_{a,a,N}$ and $M_{a,a-1,M}$ are related by the identity $(N, M)_{\text{alg}}$. Since by lemma 3.7 $M_{x,x,N} = M_{a,a,N}$ and $M_{x,x-1,M} = M_{a,a-1,M}$, the algebraic matching condition $(N, M)_{\text{alg}}$ holds also for $x$.

In order to complete the proof of theorem 1.3 we are left with proving that the entropy is monotone on every maximal $I_a$:

**Proposition 3.8.** Let $I_a$ be a maximal quadratic interval, and let $N$ and $M$ be as in theorem 3.1 then the function $\alpha \mapsto h(T_\alpha)$ is:

(i) strictly increasing if $N < M$

(ii) constant if $N = M$

(iii) strictly decreasing if $N > M$

on the whole interval $I_a$.

The proof is just an adaptation of the one given in [7] (see appendix): let us just remark that we are able to establish explicit bounds for the domain of validity of their entropy formula, which was previously just claimed to work locally. Moreover, $N$ and $M$ are now given in terms of the c.f. expansion of $a$, so it becomes immediate to establish which of the cases (i)-(ii)-(iii) holds in a neighbourhood of any given rational number.

### 3.3 Period doubling

Another feature observed in [1] (sect. 4.2.) was the production of infinite chains of adjacent matching intervals via period doubling; more formally,

**Proposition 3.9.** Let $a$ be the pseudocenter of a maximal interval $I_a$, and write $a = [0; A^-] = [0; A^+]$ with $|A^-| \equiv 1 \mod 2$. Then $a' := [0; A^-A^-]$ is the pseudocenter of a maximal interval.

The proposition follows immediately from lemma 4.4 which will be proved in next section. By applying the proposition repeatedly, one gets the

**Corollary 3.10.** Let $I_a$ be a maximal (hence matching) interval. Then there is a countable chain of matching intervals

$$\cdots < I_{a_{n+1}} < I_a < \cdots < I_{a_1} = I_a$$

such that $I_{a_n}$ and $I_{a_{n+1}}$ are adjacent, and $\lim_{n \to \infty} a_n := a_\infty > 0$. 

Note that the proposition also gives a recursive algorithm to generate the c.f. expansion of the limit point \( a_\infty \): an explicit computation for the chain generated by \( I_{1/2} \) is contained in [1], sect. 4.2.

4 String techniques

This section contains the proofs of a few technical lemmata about the string ordering mentioned in the rest of the paper.

4.1 String formalism

To prove our results we shall need to fix some notation to manipulate the strings of partial quotients.

If \( A, B \) are two finite strings composed with the alphabet \( \mathbb{N}_+ \) we denote

- \( A' \) the twin string of the finite string \( A \) i.e. the string such that the finite c.f.’s \([0, A]\) and \([0, A']\) represent the same rational number;
- \( AB \) the concatenation of \( A \) and \( B \); \( Ab \) will denote the concatenation of the finite string \( A \) with the one-letter string \((b)\);
- \( A^n \) the concatenation of \( n \) copies of \( A \) (\( A^0 \) is the empty string);
- \( \overline{A} \) means the endless concatenation of \( A \);
- \(|A|\) the length of \( A \);
- \((A)_i^j\) the substring of \( A \) going from the \( i \)-th figure to the \( j \)-th figure of \( A \);
- \( A \subseteq B \) means that \( A \) is a prefix of \( B \), i.e. there exists \( B_1 \) such that \( B = AB_1 \).

We will be interested in the alternating lexicographic order structure on the space of finite or infinite strings as defined in section 2.5. Note that, the set of finite strings \( S \) is a semigroup for the operation of concatenation. Associating a finite string \( S \) to the fractional map \((x \mapsto [0; S + x])\) yields a natural action of the semigroup \( S \) on \( \mathbb{R}_+ \). Let us also recall that the map \((x \mapsto [0; S + x])\) is increasing if \( |S| \) is even and decreasing if \( |S| \) is odd, in particular odd convergents of any \( x \) are greater than \( x \) while even convergents are smaller. Moreover, if \( x := [0; S, a + x'] \) and \( y := [0; S, b + y'] \) with \( a > b \in \mathbb{N}_+ \), \( x', y' \in [0, 1] \), then \( x > y \) if \( |S| \) is even and \( x < y \) if \( |S| \) is odd.

In the following we shall need some effective criterion to compare infinite periodic strings \( S, T \): as soon as \( |S| \neq |T| \) this becomes a nontrivial task. The next section will deal this issue.
4.2 String Lemma

**Lemma 4.1.** Let $S, T$ be two nonempty strings. Then the pair of infinite strings $S, T$ is ordered in the same way as the pair $ST, TS$; namely

$$ST \leq TS \iff S \leq T.$$ 

**Proof.** If $ST = TS$ we can prove that there exists another string $P$ and integers $k, h \in \mathbb{N}$ such that $S = P^k$, $T = P^h$, hence $S = T$. In fact, we proceed by induction on $n := \max\{S, T\}$. For $n = 1$ the claim is obviously true. Assume now we have proved this claim for all pairs of strings of length strictly less than $n$, and let $S, T$ be a pair of strings of maximal length $n$. We may assume that $0 < |T| < |S| \leq n$, the cases $|T| = 0$ and $|T| = |S|$ being trivial. The hypothesis $TS = ST$ implies that $T$ is a prefix of $S$, namely $S = TS_1$ therefore $TS = ST$ translates into $TS_1 = S_1T$. Since $\max\{|T|, |S_1|\} < |S|$ we use the inductive hypothesis to conclude that $T = P^k$, $S_1 = P^h$, and therefore $S = P^{k+h}$.

If $ST \neq TS$, then $d := \min\{j \in \mathbb{N} : (ST)_j \neq (TS)_j\} \leq s + t$. By Lemma 4.2 with $n = d - 1$ one has

$$(ST)_1^d = (S)_1^d \quad (TS)_1^d = (T)_1^d$$

hence the pair $(S, T)$ is ordered in the same way as $(ST, TS)$. □

**Lemma 4.2.** Let $S, T$ be two nonempty strings, $s := |S|, t := |T|, n \in \mathbb{N}$, $0 \leq n < s + t$. If $(ST)_1^n = (TS)_1^n$ then

$$\begin{cases} 
(S)_1^{n+1} = (ST)_1^{n+1} \quad (\ast) \\
(T)_1^{n+1} = (TS)_1^{n+1} \quad (\ast\ast)
\end{cases}$$

**Proof of lemma 4.2.** We can assume $|T| \leq |S|$. We can split the proof in three cases, depending on the relation between $n$ and the lengths data $t$ and $s$.

**Case 1:** $0 \leq n < t$. In this case both $(\ast)$ and $(\ast\ast)$ trivially hold.

**Case 2:** $n < s$, $kt \leq n < (k + 1)t$ for some $k \geq 1$. Hypothesis (i) implies that $T^k$ is a prefix of $S$, i.e. $S = T^kS_1$. On the other hand

- $\overline{S}$ coincides with $ST$ on the first $s$ figures $\xrightarrow{n \leq s} (\ast)$ holds;
- $\overline{T}$ coincides with $TS$ on the first $(k + 1)t$ figures $\xrightarrow{n < (k + 1)t} (\ast\ast)$ holds;

**Case 3:** $s \leq n < s + t$. Hypothesis (i) implies that $S$ is a prefix of $T^k$ (with $k = \lceil \frac{n}{t} \rceil$), i.e. $S = T^{k-1}T_0$, $T = T_0T_1$. Thus

$$(S)_1^{s+t} = T^{k-1}T_0T_0T_1 = ST \quad (T)_1^{s+t} = T^kT_0 = TS$$

So $(\ast)$ and $(\ast\ast)$ are again both verified.

The following remark will be useful further on

**Remark.** Let $T, S$ be two nonempty strings and set $a := [0; ST], b := [0; S], I_a := [\alpha^-, \alpha^+]$ and $I_b := [\beta^-, \beta^+]$. Then
(i) If \(|S|\) is even then \(b < a\) and \(\beta^- < \alpha^-\);

(ii) If \(|S|\) is odd and \(T \neq (1)\), then \(b > a\) and \(\beta^+ > \alpha^+\)

**Lemma 4.3.** If \(I_a \cap I_b \neq \emptyset\), \(I_a \setminus I_b \neq \emptyset\) and \(I_b \setminus I_a \neq \emptyset\), then either \(I_a\) or \(I_b\) is not maximal.

**Proof.** By Lemma 2.3, without loss of generality, we may assume that \(a\) is a convergent of \(b\); hence we can write \(a = [0; A], b = [0; A^\ell A_0]\), where \(A_0 \neq \emptyset\) is a proper prefix of \(A\). Let \(a_0 := [0; A_0]\); we claim that the interval \(I_{a_0}\) contains either \(I_a\) or \(I_b\). There are several cases to be examined; in all cases the proof that the two intervals are nested, one inside the other, amounts to checking two inequalities: one of the two inequalities will be a trivial consequence of the previous remark while the other is harder, but it will follow from the String Lemma 2.12. We treat just one case in detail, and provide a table explains how to get the “hard” inequality for all the other cases. Let \(|A| = 0, |A_0| = 0, \alpha^+ = [0; A], \beta^+ = [0; A^\ell A_0]\).

\[
\alpha^+ < \beta^+ \iff \overline{A} < A^\ell A_0 \iff AA_0 < A \iff \overline{A} < \overline{A_0}
\]

so \(\alpha^+ < \alpha_0^+\) and, by the remark 4.2 \(\alpha_0^- < \alpha^-\) so that \(I_a \subseteq I_{a_0}\) where \(a_0 := [0; A_0]\).

| Cases | hypotheses used | hard inequality | aim |
|-------|-----------------|-----------------|-----|
| \(|A|\) even \(a < b\) \(\alpha^+ := [0; A]\) | \(|A_0|\) even \(a_0 < a < b < \alpha^+\) | \(A^\ell A_0 > A\) \(\beta^+ > \alpha^+\) | \(\overline{A} < \overline{A_0}\) \(\alpha^+ < \alpha_0^+\) | \(I_a \subseteq I_{a_0}\) |
| \(|A|\) even \(a < b\) \(\alpha^+ := [0; A]\) | \(|A_0|\) odd \(a^+ < b < a_0\) | \(A^\ell A_0 < A\) \(\beta^- < \alpha^+\) | \(\overline{A_0} < \overline{A}\) \(\alpha^- < \beta^-\) | \(I_b \subseteq I_{a_0}\) |
| \(|A|\) odd \(|A^\ell A_0|\) even | \(b < a\) \(b < \alpha^-\) \(\alpha^- := [0; \overline{A}]\) | \(A^\ell A_0 > A\) \(\beta^+ > \alpha^-\) | \(\overline{A_0} > A^\ell A_0\) \(\alpha_0^+ > \beta^+\) if \(\begin{cases} |A_0| & \text{even} \\ \ell & \text{even} \end{cases}\) \(\alpha^- < \alpha_0^-\) \(\alpha_0^- < \alpha^-\) if \(\begin{cases} |A_0| & \text{odd} \\ \ell & \text{odd} \end{cases}\) | \(I_b \subseteq I_{a_0}\) |
| \(|A|\) odd \(|A^\ell A_0|\) odd | \(b < a\) \(\alpha^- < b\) \(\alpha^- := [0; \overline{A}]\) | \(A^\ell A_0 > A\) \(\beta^+ > \alpha^-\) | \(\overline{A_0} > A^\ell A_0\) \(\alpha_0^+ > \beta^+\) if \(\begin{cases} |A_0| & \text{even} \\ \ell & \text{even} \end{cases}\) \(\alpha^- < \alpha_0^-\) \(\alpha_0^- < \alpha^-\) if \(\begin{cases} |A_0| & \text{odd} \\ \ell & \text{odd} \end{cases}\) | \(I_a \subseteq I_{a_0}\) |
Lemma 4.4. Let \( a_1 = [0, P], a_\ell = [0, P^\ell] \). The following are equivalent

(i) \( I_{a_\ell} \) is maximal;

(ii) \( I_{a_1} \) is maximal and

\[
\ell = 1 \quad \text{if } |P| \text{ is even,}
\ell \leq 2 \quad \text{if } |P| \text{ is odd.}
\]

Proof. [(i) \( \Rightarrow \) (ii)]. If \( |P| \) even, \( \ell > 1 \) and \( a_{\ell - 1} = [0; P^{\ell - 1}] \), then \( I_{a_{\ell - 1}} \supseteq I_{a_\ell} \) so that \( I_{a_\ell} \) can’t be maximal. If \( |P| \) is odd and \( \ell > 2 \), setting \( a_{\ell - 2} = [0; P^{\ell - 2}] \) then \( I_{a_{\ell - 2}} \supseteq I_{a_\ell} \); so, again, \( I_{a_\ell} \) can’t be maximal. To conclude the proof, we just need to prove that \( I_{a_1} \) is maximal. Let \( I_{a_1} \) be the maximal interval containing \( I_{a_1} \), so that \( a := [0; P] \) is a convergent of \( a_1 \). The function \( \phi(x) := [0; P + x] \) is injective, \( \phi : I_{a_1} \rightarrow \phi(I_{a_1}) = I_{\phi(a_1)} \), with \( \phi(a_1) := [0; PP] \); moreover \( \phi(I_{a_1}) = I_{\phi(a_1)} = I_{a_2} \). So

\[
I_{a_1} \cap I_{a_2} \neq \emptyset, \quad \phi(I_{a_1}) \subseteq \phi(I_{a_2}) = I_{\phi(a_2)}, \quad I_{a_2} = \phi(I_{a_1}) \subseteq \phi(I_{a_2}) = I_{\phi(a_2)}.
\]

Since \( I_{a_2} \) is maximal, \( I_{a_2} = I_{\phi(a_2)} \) and thence \( I_{a_1} = I_{a_2} \) is maximal.

[(ii) \( \Rightarrow \) (i)]. Let \( |P| \) be odd and \( \ell = 2 \) (otherwise there’s nothing to prove!); we have to show that \( I_{a_2} \) is maximal (if \( I_{a_1} \) is). Let \( I_{a_2} := [0; PP] \neq [0; P] = \alpha^+ = \alpha^- < a_1. \)

If \( I_{a_2} \) is the maximal interval containing \( I_{a_2} \), \( a := [0; A], |A| \) even, we have that \( I_\alpha \cap I_{a_1} = \emptyset \) and so \( \alpha = A \). Therefore \( A \) and \( P \) have a common period \( Q \); \( A = Q^m, P = Q^\ell \); on the other hand, by virtue of the implication [(i) \( \Rightarrow \) (ii)] (the one we have already proved) we get \( \ell = 1 \) (\( \ell = 1 \) is impossible, since \( |P| \) is odd) and therefore \( m = 2 \), so \( I_{a_2} = I_a \) is maximal. \( \square \)

Let \( S, T \) be two nonempty strings and

\[
a := [0; ST], \quad b := [0; S], \quad c := [0; T]; \quad I_a := [\alpha^-; \alpha^+], \quad I_b := [\beta^-; \beta^+], \quad I_c := [\gamma^-; \gamma^+]. \quad (5)
\]

Proposition 4.5. Let \( a = [0; A] \in \mathbb{Q} \cap (0, 1] \).

1. The following are equivalent:

(i) \( I_a \) is maximal.

(ii) If \( A = ST \) with \( S, T \) finite nonempty strings, then either \( ST < TS \) or \( ST = TS \) with \( T = S \), \( |S| \) odd.

Moreover, if \( a = [0; ST] \) and \( I_a \) is maximal, \( [0; T] > [0; ST] \) (i.e. \( a < c \)).

2. Moreover, if \( a = [0; ST] \) and \( I_a \) is maximal, \( [0; T] > [0; ST] \) (i.e. \( a < c \)).
Proof. (1)–[(i) ⇒ (ii)]. Let us use the notation introduced above in (5); by maximality of $I_a$ we immediately get that $b \notin I_a$ ($a, b \in \mathbb{Q}$ and $\text{den}(b) < \text{den}(a)$ - see also definition (2.2)); since $b \in I_b \setminus I_a \neq \emptyset$, maximality of $I_a$ and lemma (2.6) also imply that $I_a \cap I_b = \emptyset$.

Case 0. If $b$ is an even convergent of $a$ (i.e. if $|S|$ is even and $b < a$) then $I_b$ lies to the left of $I_a$ and hence $\beta^+ \leq \alpha^-$; since $[0; \overline{S}] = \beta^+$ and $[0; \overline{ST}] \in \{\alpha^\pm\}$, by String Lemma (2.12) we get $STS \leq STS$ and, since $|S|$ is even $ST \leq TS$.

Lemma (4.4) tells us that, since $|S|$ is even and $I_a$ is maximal, equality cannot hold.

Case 1. If $b$ is an odd convergent (i.e. if $|S|$ is odd and $b > a$) by the previous argument $\alpha^+ \leq \beta^-$. If $[0; \overline{ST}] = \alpha^+ = \beta^- = [0; \overline{S}]$ then, by lemma (4.4), $T = S$. If not, then $[0; \overline{ST}] < [0; \overline{S}]$; by String Lemma (2.12) $STS < SST$ and, since $|S|$ is odd, this implies that $TS > ST$ (which is the same conclusion as the previous case).

The first implication is thus proved.

(1)–[(ii) ⇒ (i)]. Assume $I_a$ is not maximal; then there exist two non-empty strings such that $a := [0; \overline{ST}]$, $b := [0; \overline{S}]$, $I_b$ is maximal and $I_b \supseteq I_a$ (which, in particular, implies that if $|S|$ is odd then $S \neq T$). Then $\alpha^+ \leq \beta^+$ and $\alpha^- \geq \beta^-$. Let us give a quick glance at the cases that can occur:

| $|S|$ | $|T|$ | $[0; \overline{ST}]$ | $[0; \overline{S}]$ | consequence of String Lemma | Conclusion |
|------|------|----------------|----------------|--------------------------|-----------|
| even | even | $\alpha^+$ | $\beta^+$ | $STS \leq SST$ | $ST \geq TS$ |
| even | odd | $\alpha^-$ | $\beta^+$ | $STS < SST$ | $ST > TS$ |
| odd  | even | $\alpha^-$ | $\beta^-$ | $STS \leq SST$ | $ST \geq TS$ |
| odd  | odd  | $\alpha^+$ | $\beta^-$ | $STS < SST$ | $ST > TS$ |

It is thus easy to realize that condition (ii) never holds.

(2) Let us now prove the second statement of the previous proposition. Since our claim concerns rational values, we may assume that $|S|$ is even (so that $\alpha^+ = [0; \overline{ST}]$). Let us rule out the “period doubling case” (i.e. $|S|$ odd and $S = T$): in this case $a < c$ because $c$ is an odd convergent of $a$. In all other cases the strict inequality $ST < TS$ holds and hence $STT < TST$.

Moreover we know that

- $\gamma := [0; \overline{T}]$ is an endpoint of $I_c$;
- $\gamma > \alpha^+$ (because $STT \leq TST$);
- $I_a \cap I_c = \emptyset$ because $I_c$ must contain points which are not in $I_a$, and $I_a$ is maximal (recall lemma (2.6)).

Therefore $c > a$ (and in fact $\alpha^+ \leq \gamma^-$ since $I_a \cap I_c = \emptyset$).
Let us point out that proposition 4.5 provides an effective algorithm to decide whether or not a string defines the pseudocenter of a maximal interval: it is sufficient to check that all its cyclical permutation produce strings which are strictly bigger (except if the exceptional case of period doubling occurs).

Appendix

(A) Comparison between matching conditions

Let us recall the matching conditions given in [7]:

\( (c-1) \quad \{ T^n_\alpha(\alpha) : 0 \leq n < k_1 \} \cap \{ T^m_\alpha(\alpha-1) : 0 \leq m < k_2 \} = \emptyset \)

\( (c-2) \quad M_{\alpha,\alpha,k_1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} M_{\alpha,\alpha-1,k_2} \Rightarrow T^{k_1}_\alpha(\alpha) = T^{k_2}_\alpha(\alpha-1) \)

\( (c-3) \quad T^{k_1}_\alpha(\alpha) = T^{k_2}_\alpha(\alpha-1) \notin \{ \alpha, \alpha-1 \} \)

The matching set is therefore

\[ M_{NN} := \{ \alpha \in (0,1) : (c-1), (c-2), (c-3) \text{ hold for some } (k_1, k_2) \} \]

Proposition 4.6. If \( \alpha \) satisfies the algebraic matching condition \( (N, M)_{\text{alg}} \), then \( T^{N+1}_\alpha(\alpha) = T^{M+1}_\alpha(\alpha-1) \).

Proof. By writing the identity \( (N, M)_{\text{alg}} \) in terms of Möbius transformations and evaluating it at \( \alpha \),

\[ T^N_\alpha(\alpha) + T^M_\alpha(\alpha-1) = -T^N_\alpha(\alpha)T^M_\alpha(\alpha-1) \]

which implies \( T^N_\alpha(\alpha) = 0 \Leftrightarrow T^M_\alpha(\alpha-1) = 0 \). If both are zero, the claim follows trivially since \( T_\alpha(0) = 0 \); if they are nonzero, one can write

\[ \frac{1}{T^N_\alpha(\alpha)} + \frac{1}{T^M_\alpha(\alpha-1)} = -1 \quad (6) \]

Now suppose \( \epsilon_\alpha(T^N_\alpha(\alpha)) = \epsilon \), and \( c_\alpha(T^N_\alpha(\alpha)) = c \) so that

\[ \frac{\epsilon}{T^N_\alpha(\alpha)} - c \in [\alpha-1, \alpha) \]

The fact that \( |T^N_\alpha(\alpha)| < 1 \) and \( \epsilon \) imply \( \epsilon(T^M_\alpha(\alpha-1)) = -\epsilon \), hence

\[ \frac{\epsilon}{T^M_\alpha(\alpha-1)} - c - \epsilon \in [\alpha-1, \alpha) \]

so \( c_\alpha(T^M_\alpha(\alpha-1)) = c + \epsilon \) and \( T^{N+1}_\alpha(\alpha) = T^{M+1}_\alpha(\alpha-1) \). \( \square \)
Proposition 4.7. Let \( I_a \) be a maximal quadratic interval, and let the two c.f. expansions of \( a \) be \( a = [0; A^+, A^-] \). Let \( N \) and \( M \) as is theorem 3.1 and let

\[
\tilde{I}_a := \{ \alpha \in I_a \text{ s.t. (c-1), (c-2), (c-3) hold with } k_1 = N + 1, k_2 = M + 1 \}
\]

Then

\[
I_a \setminus \tilde{I}_a \subseteq \{ \alpha \in [0; A^+, k], k \in \mathbb{N} \} \cup \{ \alpha = [0; A^-, k], k \in \mathbb{N} \}
\]

Proof. By the proof of the previous proposition, (c-2) holds for \( \alpha \in I_a \setminus \{ a \} \). By using the explicit description of the orbits as in corollary 3.4 and lemma 3.7, one can check (c-1) holds for every \( \alpha \in I_a \setminus \{ a \} \). Exceptions to (c-3) precisely correspond to \( \alpha = [0; A^-, k] \) or \( \alpha = [0; A^+, k] \).

Corollary 4.8. \( M \setminus \mathcal{M}_{NN} \) is a countable set.

(B) The entropy is monotone on maximal intervals

Let us now prove proposition 3.8:

Lemma 4.9. Let \( a \in \mathbb{Q} \cap (0, 1) \) such that \( I_a = (\alpha^-, \alpha^+) \) is maximal, let \( a = [0; A] \) be its c.f. expansion with \( |A| \equiv 0 \mod 2 \), and choose \( \alpha, \alpha' \) such that \( \alpha^- < \alpha < \alpha' < \alpha^+ \) and \( \alpha' \leq [0; A, a] \). Then

\[
\frac{h(T_{\alpha'})}{h(T_{\alpha})} = 1 + (M - N)\mu_{\alpha'}([\alpha, \alpha'])
\]

\[
\frac{h(T_{\alpha})}{h(T_{\alpha'})} = 1 + (N - M)\mu_{\alpha}([\alpha - 1, \alpha' - 1])
\]

where \( \mu_{\alpha} \) and \( \mu_{\alpha'} \) are the invariant densities of \( T_{\alpha} \) and \( T_{\alpha'} \), respectively.

Proof. Choose \( x \in (\alpha, \alpha') \). The proof proceeds exactly as in [7], thm. 2, once we show that

\[
M_{\alpha', x, k}^{-1}(x) \notin (\alpha, \alpha') \quad 1 \leq k \leq N
\]

\[
M_{\alpha, x-1, h}^{-1}(x - 1) \notin (\alpha - 1, \alpha' - 1) \quad 1 \leq h \leq M
\]

This follows directly from lemma 3.3 except for two cases: one in which \( h = M \) and \( x \geq a \), and the other in which and \( k = N \) and \( x \leq a \). In the first case one can write \( x = [0; A + y] \) with \( a = [0; A], \) \( |A| \equiv 0 \mod 2, 0 \leq y < \alpha^+ \). Then \( M_{\alpha, x-1, M}^{-1}(x - 1) = y < \alpha \) because

\[
[0; A + y] = x < \alpha' \leq [0; A + \alpha] \Rightarrow y < \alpha
\]

The second case is handled similarly.

Proof of proposition 3.8 Given \( \alpha, \alpha' \in I_a, \alpha < \alpha' \), let \( a_k := [0; A_k^+, \alpha] \) and \( k_0 := \max \{ k > 0 \text{ s.t. } a_k < \alpha' \} \). One can apply the lemma to each consecutive pair of the chain \( \alpha < \alpha_1 < \cdots < \alpha_{k_0} < \alpha \).
References

[1] C. Carminati, S. Marmi, A. Profeti, G. Tiozzo, *The entropy of α-continued fractions: numerical results*, arXiv:0912.2329v1 [math.DS]

[2] J. Graczyk, G. Świa̧tek, *Critical circle maps near bifurcation*, Comm. Math. Phys. 176 (1996), 2, 227–260

[3] V. Jarník, *Zur metrischen Theorie der diophantische Approximation*, Prace Mat.-Fiz. (1928-9), 91-106

[4] S. Katok, I. Ugarcovici, *Structure of attractors for (a,b)-continued fraction transformations*, arXiv:1004.4200v1 [math.DS]

[5] L. Luzzi, S. Marmi, *On the entropy of Japanese continued fractions*, Discrete and continuous dynamical systems, 20 (2008), 673-711

[6] H. Nakada, *Metrical theory for a class of continued fraction transformations and their natural extensions*, Tokyo J. Math. 4 (1981), 399-426

[7] H. Nakada, R. Natsui, *The non-monotonicity of the entropy of α-continued fraction transformations*, Nonlinearity 21 (2008), 1207-1225

[8] H. G. Schuster, W. Jost, *Deterministic chaos: an introduction*, fourth ed. (2005), Wiley