Limit cycles in the spectra of mass imbalanced many-boson system

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Abstract
The independence between few-body scales beyond the van der Waals universality is demonstrated for the extreme mass-imbalanced case of a specific many-boson system. This finding generalizes the scaling properties of universal tetramers to a broader class of heterogeneous few-boson systems. We assume two heavy atoms interacting with \((N - 2)\)-lighter ones at the unitary limit, using a particular case where no interactions are active between identical particles, by investigating the interwoven spectra of this many-body system for an arbitrary number of light bosons. A large mass-ratio between the particles allows us to treat this \(N\)-body system analytically, by solving an effective inverse-squared long-range interaction which is stablished for the two heavy bosons. For a cluster with \(N - 2\) light bosons \((N ≥ 4)\), we discuss the implications of the corresponding long-range potentials associated with different subsystem thresholds, implying in independent interwoven limit cycles for the correlation between the energies of excited \(N\)-body system. Our study with extreme mass-imbalanced few-boson bound states provides a fundamental understanding of the scaling behavior of their interwoven spectra. The novel insights enlarge the well-known Efimov physics paradigm and show the existence of different limit cycles, which could be probed by new experiments.

Keywords: ultracold gases, trapped gases, few-body systems, bound states, Efimov physics

(Some figures may appear in colour only in the online journal)

1. Introduction
The existence of a universal correlation between the binding energies of successive four-boson bound states (tetramers), for large two-body scattering lengths, was verified in reference [1] by solving the corresponding four-body Faddeev–Yakubovsky (FY) formalism⁶. The results obtained are related to the existence of additional scales [4, 5] not constrained by the well-known three-body Efimov physics [6], which actually refers to universal aspects of three-body systems close to the unitary limit (infinite two-body scattering length). The correlations, which were evidenced in reference [1], were further explored in [7], within a theoretical application to tetramer-dissociation data in ultra-cold gas of caesium atoms close to broad Feshbach resonances [8, 9]. For that, the shifts in the four-body recombination peaks, due to an effective range correction to the zero-range model close to the unitary limit, were considered. However, by considering a bound system with more than three particles in the unitary limit, a challenge was in establishing a more simplified description (possibly relying in some analytical procedure) to confirm the emergence of an independent new scale when a new particle is being added to the system. In a more general context of a many-body system,
the results for the independent spectra could be verified by the interwoven between the corresponding limit cycles.

In the context of cold atoms such independence between few-body scales is beyond the van der Waals universality [10–17]. The van der Waals universality is specific to the case of atoms interacting by the corresponding inverse power law potential [17], and it is verified close to a broad Feshbach resonance dominated by the open channel. It is due to the separation between the ranges of about one nanometer of the true short-range few-atom chemical potentials and the van der Waals length of several nanometers. At such distances, the atoms have an effective repulsive force which prevent them to explore shorter length scales in order to be sensitive to the details of the interatomic potentials. Therefore, the van der Waals length is the single short range scale that controls and determines the few-atom physics. This is supported by several experiments that have investigated the position of three and four-atom recombination peaks (see e.g. [18]). While this would suggest a general universality for all systems with arbitrary number of bosons (see e.g. [19–22]), it was shown in [23] that the ground-state energy and the structural properties for larger clusters of identical bosons interacting via a two-body zero-range force (regulated at finite range) are not universally determined by the three-body parameter. It was found theoretically that the results will depend on the specific form of the three-body regulator. As already found in the case of Brunnian (Borromean with arbitrary number of particles) systems with identical bosons [24], the binding energies in the unitary limit show a large variability when obtained with short-range interactions. In the recent review [25], the reader can find an updated discussion on the issues raised by the existence of scales beyond the three-body one in few-boson systems. Also, recent studies reported in [26] on the universality of scaling sizes and energy ratios, using quantum Monte Carlo simulations, have extended to tetramers some universal behaviors obtained for two- and three-body systems [27–29].

Experimental investigations on the predicted universality of Efimov states across broad and narrow Feshbach resonances have been performed in the last five years [30, 31] with the Lithium–Caesium ($^6\text{Li}$–$^{133}\text{Cs}$) mixtures, which are opening a new window of opportunities to test the independence between the few-body scales, from which the so called van der Waals universality may be broken. A single channel prescription is not enough to describe the position of the three-body recombination peaks, as the results so far obtained are evidencing a dependence of the position of the Efimov resonance on the Feshbach resonance strength, with a clear departure from the universal prediction for the narrow Feshbach resonance. Near a narrow Feshbach resonance, where a single channel description is poor, it is natural to presuppose that, beyond the expected large variations in the two-body scale (namely, the atom–atom scattering length, $\ell$), the three-body parameter can also change as suggested by the observations of resonant recombinations in the $^6\text{Li}$–$^{133}\text{Cs}$ experiments [31]. On theory grounds, such possibility can occur as three- and even four-body potentials are induced when the atomic trap set-up is tuned to be close to a narrow Feshbach resonance, such that the single channel description has to be supplemented by these induced potentials in the open channel, as shown in reference [5]. When this happens, one can disentangle the effects of the independent three, four and more body scales, which indeed are called to justify a thoroughly discussion of the independence of these scales for trapped atoms.

Relevant to mention is that, in the context of experimental realizations of atomic Bose–Einstein condensates, the application of Feshbach resonances to alter the atom–atom interaction was first demonstrated in cold-atom experiments with $^{23}\text{Na}$ [32] and with $^{85}\text{Rb}$ [33], with the theoretical aspects of the technique being reviewed in reference [34]. In brief, the origin of the induced few-particle forces can be understood by starting with the Feshbach decomposition of the Hilbert space in open channels ($P$-space) and closed channels ($Q$-space) with $P + Q = 1$. In the atom–atom Feshbach resonance, the $Q$-space represents the pair states in the potential well where it is bound, while the $P$-space contains the pair states in the lower potential well, where the open channel wave function meets asymptotically the continuum. To be concrete, the projectors $P$ and $Q$ will represent different spin states of the atom–atom pair, which are close to a Feshbach resonance (see e.g. [17]). For few-body systems, such as the ones constituted by three and four particles, the three- and four-body effective potentials appear in the single channel description, respectively, when the pair of particles virtually propagates in the $Q$-space, interacting with the spectators. On this regard, in [5] one can find a detailed discussion with illustrations.

The above mentioned example considers one Feshbach resonant pair interaction in the three atom system. By reducing the three-body coupled channel problem to the open single channel one, the effective Hamiltonian acting on this channel contains an effective potential, which has a non-connected three-body part corresponding to the resonant pair interaction and a connected three-body part that corresponds to an effective three-body interaction with intensity depending on the properties of the Feshbach resonance. In particular, the strength of the effective potential is enhanced for a narrow resonance, as the coupling between open and closed channels is larger in this case [17]. Note that, in the region where the effective three-atom potential is attractive, the length scale is larger than the van der Waals length.

The attraction dislocates the effective repulsive barrier to distances larger than the van der Waals radius ($\ell_{\text{vdw}}$), with the characteristic length scale increasing with respect to $\ell_{\text{vdw}}$. This implies in a dislocation of the position of the resonance in the three-body recombination toward larger absolute values of the scattering length. Indeed, this behavior was verified in recent experiments, as reported in [31]. By extending such example to systems with more particles, one should expect that, through Feshbach resonance mechanisms, not only the two-body scattering length is disposable to be tuned, but also the short-range scales related to three, four and more particles. In the presence of spectator particles, if the excitation of the Feshbach resonance is turned off, this example will reduce to the one discussed in [35]. In short, near a narrow Feshbach resonance, the induced few-body forces in the open channel can drive independently the corresponding physical scales. Then, by restricting our example to a four-particle system, observables such
as the position of the resonance in the recombination rates, or the scattering lengths for atom–dimer, atom–trimer and dimer–dimer, are not constrained only by the van der Waals length, as other larger few-atom length scales can play a role.

In view of the above considerations, we are motivated to search for a generalization of previous findings obtained in the case of a four-boson system with zero-range two-body interactions [1], in which a four-body scale was found necessary when the two- and three-boson scales are fixed, leading to the prediction of a new limit cycle for the four-boson system. Such limit cycle has a different geometrical ratio between the four-body energies in the unitary limit, as compared to the usual Efimov ratio, when the trimer energy is smaller compared to the tetramer energy. This suggested that the long-range effective potential for the tetrathers has a strength different from the trimer one. In addition, the energy of a four-body bound state can be changed by a four-body short range interaction, while keeping the universal correlation between two successive tetramer states. Narrow Feshbach resonance may allow to disentangle the trimer and tetramer binding energies. For more particles, it was suggested that a new experimental information is required for each new boson added to the system [4]. Narrow Feshbach resonances, which may include effects from many-body forces when the dynamics is reduced to a single channel, can be effectively described by few-body scales that change independently. In the particular case, when the interaction is dominated by the open channel, with the single channel description well established and only two-body forces being manifested, all short range scales should be determined by the van der Waals length. Considering such quite exciting possibilities for narrow Feshbach resonances, it is timely and demanding a study on the interwoven and independent cycles, which can emerge in the spectrum of a many particle system. In addition, the interest in few-body physics and the corresponding scaling behavior of the observables, following studies presented in [36, 37], is further strengthened by recent experimental observations of three-photon bound states in a quantum nonlinear medium, where three-photon bound states are viewed as photonic solitons in the quantum regime [38]. It was also pointed out in this reference that strong effective $N$-body forces in larger photonic molecules and clusters can allow studies which are not possible to be realized with conventional systems.

Furthermore, as considered by Petrov [39, 40] for three-bosons in the vicinity of a narrow Feshbach resonance, the so-called ‘energy-dependent scattering length’ $a(E)$ (as derived from the effective-range expansion) has an effective-range correction $R^*$ inversely proportional to the width of the resonance. So, for a narrow Feshbach resonance, this range $R^*$ can be a relevant parameter to be taken into account. It is also noticeable that, within the Born–Oppenheimer (BO) approximation for a heavy–heavy–light system, the three-body potential acquires a Coulomb character departing from the Efimov inverse-square behavior at distances $R \ll R^*$ [40]. This Coulomb-like character at short distances was also verified in [41] for the scattering of the heavy particle by the dimer formed the heavy–light system, near the unitary limit. However, besides being quite relevant to consider a more realistic model which encapsulates the range $R^*$ in the physics of deeply bound and more compact systems, in the present work we are mainly concerned with the tail of the effective potential which is not affected by the effective range in a significant way for the situation that $R^* \ll R \ll |a|$. Our aim is to study weakly-bound $N$-body systems constituted by two-heavy bosons and $N − 2$ light ones, close to the $N − 1$ threshold ($N \geq 4$) at the unitary limit, and the states that are characterized by sizes much larger than $R^*$. It is worthwhile to point out that [40] in the case of the heavy–heavy–light systems at distances $R \sim R^*$ the heavy–heavy wave function can be matched with the Efimov-like wave function, namely the one living in the long range potential inverse square potential, with the three-body parameter determined by $R^*$. In this situation, the three-body observables depend only on $a$ and $R^*$. For the deeply bound systems a departure from the Efimov-type scaling is expected when their sizes $\sim R^*$, this interesting case is beyond the present investigation. By using the BO approximation, where the light–heavy system is providing the interaction for the heavy–heavy system, we look for a simplified approach considering only the tail of the BO effective potential from which the main physics aspects will emerge associated with the few-body scales in correspondence with the number of bosons $N \geq 4$ near the unitary limit. The present analysis will also support previous studies on the existence of a four-body scale, which should be independent from the three-body one when considering a bosonic system with no short-range interaction constraints. In this direction, for identical four-boson system, a more involved numerical approach using the FY formalism can be found in [1, 5]. More recently, in [42], by resorting to an approximate form of the four-boson FY equations for the zero-range interaction, there is a demonstration on the requirement of such independent four-body scale, when studying the breaking of the scaling invariance (in the ultraviolet region) to a discrete one. This was performed by extending a previous study due to Danilov [43] for the three-boson system.

For our task in evidencing the existence of independent scales in a few-boson system, the well-known adiabatic BO approach, applied to a low-energy system with two-heavy and one light particles [44, 45], is extended to $N$-body system with an arbitrary number of identical ($N − 2$)-light particles, in which we assume that the identical particles are not interacting between each other. Actually, this assumption is within possible experimental realizations in cold-atom laboratories, considering mixtures of two kind of atoms (as, for example, in the already cited experiments [30, 31]), by tuning to zero the two-body scattering lengths of the identical (heavy–heavy and light–light) particles, via Feshbach resonance techniques [34]. So, for $N \geq 4$, in order to be strictly valid the adiabatic approach, we consider the two identical heavy particles ($\alpha = 1$ and $2$, with masses $m_\alpha$) are interacting with the light particles ($\beta = 3$ to $N$, with masses $m_\beta \ll m_\alpha$) near the unitary limit. Within this procedure, an effective two-body interaction emerges for the two-heavy particles. The limits of validity of the adiabatic approach is being verified in the case of a three-body system, by comparing with exact numerical approaches for different two-body interactions and mass ratios.

In the next section 2 we present details of the BO formalism applied to two-heavy and ($N − 2$)-light boson systems.
addition, the corresponding spectra of two-heavy and two-light particles are discussed in this section. In section 3, interwoven cycles are evidenced in the spectra, followed by the corresponding scaling plots, with particular focus in the tetramer case. Furthermore, a general discussion applied to systems beyond four-bodies is introduced in this section. Finally, concluding remarks and perspectives are summarized in section 4.

2. BO approximation

2.1. Two-heavy and \((N - 2)\)-light boson system

Here we briefly describe a generalization of the approach presented by Fonseca et al. [44] (see also a pedagogical presentation in [45]) for the case of a many-body mixture with two-species of particles, two-heavy and \((N - 2)\)-light ones, we define the corresponding coordinates as \(x_1, x_2\) for the two heavy particles, being \(x_j (j = 3, 4, \ldots, N)\) for the \((N - 2)\)-light particles. Next, we consider the minimal condition for the interactions, such that the identical particles are not interacting between each other, remaining only the heavy–light interactions. Within this condition, we define the relative coordinates as \(R = (x_1 - x_2)\) and \(r_{j(1,2, \ldots, N-2)} = (x_{j+2} - x_2/2)\). The corresponding Schrödinger equation is given by

\[
H \Psi = \left[ \frac{\hbar^2}{m_\alpha} \nabla^2_R + V_0(R) + \sum_{j=1}^{N-2} H_j \right] \Psi, \tag{1}
\]

where \(\Psi \equiv \Psi(r_1, r_2, \ldots, r_{N-2}, R)\) is the total wave function, \(V_0\) is the potential between the two-heavy particles, and \(H_j\) is a three-body Hamiltonian for the interaction between the two heavy particles with each light particle \(j\), given by

\[
H_j = -\frac{\hbar^2}{2\mu_{(2\alpha\beta)}} \nabla^2_r_j + \sum_{i=1}^{2} V_i \left( \frac{r_j + (-1)^j R}{2} \right), \tag{2}
\]

where \(\mu_{(2\alpha\beta)} \equiv 2m_\alpha m_\beta/(2m_\alpha + m_\beta)\) is the reduced mass for the \(\alpha\alpha\beta\) system and \(V_i\) is the interaction for the heavy–light system. As the masses of the particles are assumed such that \(m_\alpha \gg m_\beta\), the kinetic energy corresponding to the two-heavy particles is much smaller than the kinetic energy related to the heavy–light mass system in the center-of-mass of the three-body system. Therefore, in this limit for the three-body \(\alpha\alpha\beta\) system one can apply the well-known BO approximation [46], by separating the motion of the light–heavy particles from the two heavy ones. This approach can be straightforward extended to the other light particles being added to the system. However, as another obvious assumption when applying the BO approximation to such system with more that one light particle, we should also consider that the new light bosons being added are not so far apart from the formed subsystem, to avoid having a new slow degree of freedom competing with the slow motion of the heavy bosons.

Within the above assumption, the total wave function can be decomposed as

\[
\Psi = \Psi(r_1, r_2, \ldots, r_N, R) = \phi(R) \prod_{j=1}^{N-2} \psi_j(r_j),
\]

where \(R\) is a parameter in \(\psi_j(R)\).

With that all the \(N - 2\) light particles interacting in the same way with the heavy particles, the corresponding energy eigenvalues \(E_{N-2}(R) \equiv \langle N - 2 \rangle \hat{E}(R)\) will be the effective potential for the two heavy particles:

\[
\left[ -\frac{\hbar^2}{2\mu_{(2\alpha\beta)}} \nabla^2_R + \sum_{i=1}^{2} V_i \left( \frac{r_j + (-1)^j R}{2} \right) - E(R) \right] \psi_j(R_j) = 0
\]

and \(E_3\) being the energy solution for the system with two-heavy and one light boson. As the asymptotic behavior of \(\hat{E}(R)\) is not affected by \(V_0(R)\), we can assume \(V_0(R) = 0\) within our purpose. For the light–heavy particles one can take short-range separable interactions, with \(V_1\) and \(V_2\) having the operator form \(\lambda |j\rangle \langle j|\). In this way, the light–heavy particle system can easily be solved in momentum space by considering simple separable interactions with Yamaguchi form-factors such as \(g(p) \equiv 1/(p^2 + \beta^2)\). Another choice of form-factor parametrization can be found in [47], in which two parameters are used, allowing to reproduce low-energy phase shifts together with the corresponding dimer energy. Further, it is assumed a shallow bound state, \(-\hbar^2/(2\mu_{\alpha\beta}a^2)\), where \(\mu_{\alpha\beta}\) is the reduced mass and \(a \equiv a_{\alpha\beta}\) the light–heavy scattering length. Within these assumptions, the effective potential in the equation for \(\phi(R)\) is

\[
E_{N-2}(R) = -(N-2)\frac{\hbar^2 \kappa^2}{2\mu_{(2\alpha\beta)}} = -(N-2)\frac{\kappa^2}{\nu} \left( \frac{\hbar^2}{2m_\beta} \right), \tag{3}
\]

with

\[
\nu \equiv \mu_{(2\alpha\beta)}/m_\beta \quad \text{and} \quad \kappa \equiv \kappa(R),
\]

which will satisfy the relation

\[
\left[ \kappa - \left( \frac{1}{a} \right) R \right] = e^{-\kappa R}. \tag{5}
\]

The solution in the limit \(a \to \infty\) leads to

\[
E_{N-2}(R) = -(N-2)\frac{\gamma^2}{\nu R^2} \left( \frac{\hbar^2}{2m_\beta} \right), \tag{6}
\]

where \(\gamma = e^{-\gamma} = 0.5671433\). By relaxing the unitary limit, considering any other value for \(a\), the expression (5) for \(\kappa(R)\) can be fitted within a function

\[
\kappa(R) \approx \frac{1}{a} + \left( \frac{\gamma a}{R} + \frac{\nu}{a} \right) e^{-R/a}, \tag{7}
\]

where the constant \(\varepsilon\) is adjusted numerically. With good accuracy we obtain \(\varepsilon \equiv 0.185\). With the above expression for \(\kappa(R)\), the effective potential \(E_{N-2}(R)\) for the two-heavy particle system,

\[
E_{N-2}(R) = -\frac{(N-2)}{\nu a^2} \left[ 1 + \left( \frac{\gamma a}{R} + \varepsilon \right) e^{-R/a} \right] \frac{\hbar^2}{2m_\beta}, \tag{8}
\]

will satisfy both limits \(R \ll a\) and \(R \gg a\). Near the unitary limit, where \(R \ll a\), by keeping in the potential the next Coulomb-like term, the bound-state equation for \((N - 2)\)-light
and two-heavy particles is

\[ \left[ \frac{d^2}{dR^2} + \frac{m_\alpha}{2m_\beta} \left( \frac{N - 2}{\nu} \right) \left( \frac{\gamma^2}{R^2} + \frac{0.7008}{Ra} \right) + B_N \right] u = 0, \quad (9) \]

where \( B_N \equiv -m_\alpha E_N/h^2 \) are the redefined energies for the \( N \)-body bound-state spectra and \( u \equiv u(R) \equiv R\phi(R) \). In the present adiabatic approximation, with \( m_\alpha \gg m_\beta \), we have \( \nu \sim 1 \), such that \( m_{2\alpha\beta}/m_\alpha \) gives approximately the light to heavy mass ratio. Next, the light–heavy mass ratio is defined as \( A \equiv m_\beta/m_\alpha \).

The effective potential given by transcendental equation (5) is valid close to a shallow bound state for the light–heavy system, near the unitary limit, being more adequate for broad Feshbach resonance where the effective range \( (R_e) \) can be disregarded. For a narrow resonance, as shown in [40], one has to take into account in equation (5) the effective-range correction brought by \( R^* \), such that we should have

\[ \left[ \kappa - \frac{1}{a} + R^* \kappa^2 \right] R = e^{-R^*}, \quad (10) \]

instead of equation (5). In this case, in the region \( R_e \ll R \ll R^* \ll |a| \), the solution will give us a Coulomb-like potential for the heavy–light system [40], as

\[ \kappa^2 \sim (R R^*)^{-1}. \quad (11) \]

However, it was also pointed out that, in region \( R_e \ll R \ll R^* \ll |a| \), the adiabatic potential obtained from the solution of equation (10) recovers the \( 1/R^2 \) tail, which is the situation that we consider in the following.

For a radial potential \( \Lambda/R^2 \), where \( \Lambda \) is dimensionless, the system has no bound-state for \( \Lambda > -1/4 \), and is anomalous for \( \Lambda < -1/4 \) due to the singularity at \( R \to 0 \). There is no lower limit in the energy spectrum, which requires a regularization, such that \( R > r_1 \), where \( r_1 \) is a radial short-range cut-off. Therefore, for a boundary condition we fix the wave function to zero at \( R = r_1 \). It is important to note that the geometric scaling property is independent on the value of \( r_1 \). In the unitary limit (\( a \to \infty \)), for both broad (in the region \( R_e \ll R \ll |a| \)) and narrow (in the region \( R_e \ll R^* \ll R \ll |a| \)) Feshbach resonance, we have

\[ \left[ \frac{d^2}{dR^2} + \frac{s_N}{R^2} + \frac{1}{R^2} - B_N \right] u = 0, \quad (N \geq 3), \quad (12) \]

where \( s_N \equiv s_N(A) \equiv \sqrt{\left( \frac{2N + 3}{4N} \right) (N - 2)^2 - \frac{1}{4}} \) (function of the mass ratio) is defining the adiabatic scaling factor. For the corresponding three-body system (\( N = 3 \)), this scaling factor should correspond to the non-adiabatic one, which is usually defined as \( s_3 \) [48]. (In the following, we take \( s_3 \) as defining our adiabatic value for \( s_N \)).

In our simplified scheme, we are generalizing the BO approach to the case of two-heavy and \( (N - 2) \)-light bosons, in a way that we can obtain a general relation between the corresponding scaling factors with the case that we have just one-light boson:

\[ s_N^2 = (N - 2)s_3^2 + (N - 3)/4 \]

\[ \simeq (N - 2)s_3^2 + (N - 3)/4 \quad (N \geq 3), \quad (13) \]

which implies that \( s_{N+1} > s_{N-1} \), and therefore the geometrical ratio between the energies of two successive states of the \( N \)-particle system is smaller than the corresponding ratio for the \( (N - 1) \)-particle system. This pattern seems to persist even in the case where the BO approximation is not applicable like in what was found theoretically for the four and three-boson systems with a zero-range potential when \( B_4^{(1)}/B_4^{(0)} \sim 1/127 \), with \( B_3^{(0)} \ll B_3^{(1)} \) in the strict unitary limit (for zero two-body bound-state, \( B_{A\beta} = 0 \) [1]).

Therefore, the bound-state spectrum for the two-heavy and \( (N - 2) \)-light boson, with identical particles not interacting, is obtained by the solution of equation (12), which follows in exact analogy with the BO approach for the three-body case, where we have two-heavy and one-light bosons. As detailed in [44], the three-body spectrum is obtained from the zeros of a modified Bessel function of the second kind with pure imaginary order \( s_3 \) (as defined in [49]): \( u(R) = \sqrt{s_3^2 R} K_{s_3}(s_3 R) \), where \( s_3 \equiv \sqrt{B_3^{(0)}} \). From the condition that the wave-function must be zero at some short distance, with a cut-off regularizing the potential at \( R = r_1 \), for shallow bound-state levels, we have \( \sqrt{B_3^{(0)}} r_1 = e^{-n_3/r_1} \times f(s_3) \), where \( f(s_3) \) is a constant factor which does not depend on specific levels. From this solution, emerges the well-known geometric scaling of the three-body spectrum, with \( B_3^{(0)} = e^{-2n_3/r_1} B_3^{(0)} \quad (n = 0, 1, \ldots) \), as well as the fact that the bound-state energies are scaling with the inverse square of the cut-off at short distances, \( 1/r_1^2 \).

We should also note that, the boundary condition of the wave-function at long distances is given by the absolute value of the two-body scattering length, with the number of the levels in the spectrum being

\[ N_3 \simeq \frac{s_3}{\pi} \ln(|a|/r_1), \quad (14) \]

which is infinite in the unitary limit [44]. As we move away from the unitary limit, the number of trimers decrease with the ratio between adjacent binding energies following a scaling relation, as shown in figure 2 of [50] for the case of three-identical particles.

Before going to the next section where our aim is to analyze the inter-relation between the spectrum of a \( N \)-boson system with the spectrum of subsystems, it is of interest to check the extension of the validity of the adiabatic BO approach, close to unitary limit. For that, we are verifying numerically the \( s_3 \) values, obtained for the case with \( N = 3 \) (one light and two boson system) for different values of the mass-ratio \( A \equiv m_\beta/m_\alpha \ll 1 \), in comparison with the values of \( s_3 \) reported in [48]. The results presented in table 1 are illustrative on the accuracy of the BO approach, which improves as the mass ratio \( A \) decreases.

### 2.2. Two-heavy and two-light bosons

As discussed in the previous subsection, the solutions for the spectrum of two-heavy and \( (N - 2) \)-light bosons are obtained
by following in close analogy the same analytical expression as in the case of \( N = 3 \). Therefore, the bound-state wave functions presented in equation (12) are given by modified Bessel functions of the third kind with pure imaginary order \( is_3 \), such that \( u(R) = \sqrt{k^2 - 1} K_{is_3}(r_3 R) \), with \( s_3^2 = B_3 \). However, the cases with \( N \geq 4 \) will differ from the case of \( N = 3 \) by the boundary conditions. For example, in the case that \( N = 4 \) (two-heavy and two-light bosons), with the wave function vanishing at \( R = r_2 \), the shallow energy states in the spectrum are given by \( B_4^{(0)} = (r_2^{-1} e^{-i\pi/4}) \times f(s_4) \). Let us emphasize that the condition of having the wave-function vanishing at \( R = r_2 \) represents the information associated to a short-range four-body scale.

At the short range, the cut-off of the long-range Efimov-like potential is associated to a four-body short-range parameter; and, at the long range, associated to the size of the three-body system. In analogy with the three-body case, in which the spectrum is restricted by the size of the two-body bound-state, the four-body spectrum is restricted by the size of the three-body level that we are considering. Following this physically motivated picture, in the case of four-body system, one of the heavy particles can only probe the long-range potential if it is in a region within the tail of the remaining three-body bound state. If it is more distant it cannot interact with the other heavy particle through the Efimov-like long-range potential. Therefore, similar as in the three-body case, where the number of levels is given by equation (14), for the expected number of four-body levels attached to the three-body ground-state level we obtain

\[
A_4^{(0)} \approx \frac{s_4}{\pi} \ln \left( \sqrt{B_4^{(0)}}/r_2 \right),
\]

which shows that, by increasing a given three-body binding energy, the corresponding number can change according to this relation. Similar as in the three-body case, the ratio between adjacent tetramer energies will follow a scaling relation. This scaling relation was verified in [1], by solving the full FY four-body system, considering identical particles.

Therefore, the realization of a maximum (infinite) number of tetramers is possible only if the trimmer spectrum is collapsed in the ground state with zero bound-state energy. Otherwise, the maximum number is restricted by the size of the ground-state trimer.

In order to proceed, we see here the convenience to replace the previously mentioned labels \( n \) of the three-body Efimov spectrum, by \( n_3 \), with two-labels defining the possible four-body spectra (\( n_4 \) and \( n_3 \)), due to the fact that for each three-body level we can possible have a four-body spectrum. Therefore, the geometric scaling of the three-body spectrum is given by

\[
B_3^{(n_3)} = e^{-2n_3 \pi / s_3} B_3^{(0)} \quad (n_3 = 0, 1, \ldots), \quad B_3^{(0)} \sim r_1^{-1/2}. \tag{16}
\]

The factor \( s_3 \) was replaced by the exactly known values \( s_0 \), as given in table 1, in order to improve the approximation of our results obtained for the cases we have four or more particles. Therefore, from equation (13), the relation between trimmer (\( N = 3 \)) and tetramer (\( N = 4 \)) scaling factors, is given by \( s_4^2 = 2s_3^2 + 1/4 \). So, for a given level \( n_3 \), we have the following relation for the four-body levels:

\[
B_4^{(n_4)} = e^{-2n_4 \pi / s_4} B_4^{(0)} \quad (n_4 = 0, 1, \ldots). \tag{17}
\]

This is realizable for the four-body states below the ground-state trimer \((n_3 = 0)\), being limited in the other cases, as will be discussed.

The BO relation given by equation (13), for the specific case of \( N = 4 \) with two heavy and two light bosons, was also presented recently in [51], under the same simplified conditions where only the non-identical particles (heavy–light) have non-zero interactions. By considering \( A \ll 1 \) the BO approximation was shown to be fully consistent with non-adiabatic FY calculations.

3. Interwoven cycles

3.1. Three- and four-body spectra

Notice that, by considering a tetramer, with two-heavy and two-light particles, where only the light–heavy particles interact weakly (such that \( B_{1\beta} \) is close to zero), we should have a four-body spectrum \((\alpha \alpha \beta \beta)\) interconnected with two identical three-body spectrum \((\alpha \alpha \beta)\). We have the energy ratios for the trimer and the tetramer spectrum from the tail of the long-range potential. But we should point out the relation between the four-body and three-body levels. Concerning that, we have the schematic figure 1 to illustrate the dependences of the tetramers on the trimer energies. The effective BO potential for the two heavy particles, at large distances has the three-plus-one \((3 + 1)\) channel threshold, for each possible trimer state, as shown in the figure. The effective potential holds the tetramer bound states below the ground state trimer, otherwise tetramer resonances are placed in the effective potential below each excited trimer.

We should note that, in the standard BO approximation applied for two heavy plus one light boson, the cut-off of the long-range potential generated by the light boson attached to the heavy particles is given by the scattering length magnitude, which represents the dimer size and appears explicitly

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**Table 1.** Values of the scaling factor \( s_3 \) and \( e^{\pi/s_3} \), obtained by solving the adiabatic equation (12) in comparison with the respective exact values as reported in [48].

| \( A \)  | 0.1 | 0.05 | 0.04 | 0.03 | 0.02 | 0.01 | 0.001 |
|--------|-----|------|------|------|------|------|-------|
| \( s_3 \) | 1.1995 | 1.7456 | 1.9624 | 2.2784 | 2.8057 | 3.9891 | 12.675 |
| \( s_0 \) | 1.4682 | 1.9194 | 2.1142 | 2.4067 | 2.9084 | 4.0612 | 12.698 |
| \( e^{\pi/s_3} \) | 13.725 | 6.0483 | 4.9574 | 3.9703 | 3.0641 | 2.1980 | 1.2813 |
| \( e^{\pi/s_0} \) | 8.4977 | 5.1383 | 4.4192 | 3.6890 | 2.9452 | 2.1675 | 1.2807 |
in the formalism. The same approach is analogously extended to the tetramer case when we add one more light boson to the trimer, with an Efimov-like potential emerging between the heavy bosons. This analogy of the tetramer with the trimer case could be under question, as no size parameters appear explicitly in the formalism related to the trimers. However, in view of that, there is no need to appear explicitly any size parameters in the tetramer BO formalism, if the corresponding energy thresholds. The potential and all the energies are dimensionless, scaled by a factor 100 times the three-body ground-state energy, with a maximum of four particles, in the unitary limit, where the four-body system is damped at the size of the three-body ground-state. The horizontal dashed lines indicate the 3 + 1 dissociation threshold at the three-body energies $E_3^{(n)} = -(\hbar^2 / m_\alpha) B_3^{(n)} (n = 0$ is the ground state), with the ratio between them fixed by the Efimov factor $e^{2n/\pi}$. The arrows indicate schematically where the long-range potential in the 4-body system is damped, at the size of the nth 3-body state.

The correlation between the energies of successive tetramer levels is verified to be universal and again dominated by a long-range potential, with strength larger than the corresponding one for the trimer.

Table 2. In this table, we are applying the expression (13) to obtain the corresponding energy ratios in the spectrum of two-heavy and $(N - 2)$-light particles, when the interaction is restricted to inter-species atoms. For that, we are using the non-adiabatic scaling factor shown in table 1 for the three-body system.

| $A$ | 0.04 | 0.03 | 0.02 | 0.01 | 0.001 |
|-----|------|------|------|------|------|
| $s_0$ | 2.1142 | 2.4067 | 2.9084 | 4.0611 | 12.698 |
| $e^{-2\pi/s_0}$ | 19.529 | 13.609 | 8.6742 | 4.6979 | 1.6402 |
| $e^{-2\pi/s_0}$ | 7.9460 | 6.2118 | 4.5560 | 2.9739 | 1.4187 |
| $e^{-2\pi/s_0}$ | 5.3909 | 4.4197 | 3.4390 | 2.4321 | 1.3305 |
| $e^{-2\pi/s_0}$ | 4.2876 | 3.6137 | 2.9107 | 2.1580 | 1.2805 |

Figure 1. The effective long-range potential (in the unitary limit) between the two heavy particles, in the BO approximation of the two-light and two-heavy particle system, considering the different 3 + 1 thresholds. The potential and all the energies are dimensionless, scaled by a factor 100 times the three-body ground-state energy, with a maximum of four particles, in the unitary limit, where the four-body system is damped at the size of the three-body ground-state. The horizontal dashed lines indicate the 3 + 1 dissociation threshold at the three-body energies $E_3^{(n)} = -(\hbar^2 / m_\alpha) B_3^{(n)} (n = 0$ is the ground state), with the ratio between them fixed by the Efimov factor $e^{2n/\pi}$. The arrows indicate schematically where the long-range potential in the 4-body system is damped, at the size of the nth 3-body state.
To illustrate this last point, let us consider, for example, the tetramers between the ground and 1st excited trimer levels for $A = 0.01$. Now, suppose that one tetramer resonance within these two trimer levels has an energy close to the ground state trimer, then according to table 2, the next tetramer resonance following the geometrical ratio will have an energy for a recent effective-field theory approach related to the binding energies, beyond the van der Waals universality. See and effective short-range few-body forces in the open channel Feshbach resonances are present in the atom–atom system, this can be potentially found in cold atomic gases when narrow what was found for the four-bosons system in [1]. Such as situation can be realized for tetramers below the ground state trimer, as we have showed that for such large mass asymmetries no more than two tetramers are possible between two Efimov trimers. As shown, by varying the long-range cut-off, we observe the convergence of the results to the expected scaling limit.

Within the perspective of experimental realizations with large mass-imbalanced systems, one should expect that our idealized adiabatic predictions close to the unitary limit will be affected due to possible range corrections, which can only be more precisely assumed by solving the equation (10), in which we have considered the range $R^*$ as explicitly given for the narrow resonances. For broad resonances, instead of that, one should use the corresponding effective range. At the unitary limit, the long-range BO potential is cut at the trimer size, expected to be much larger than $R^*$ or the effective range. The consideration of the range correction to the BO potential is a straightforward extension of the present work, which can be performed by following the quite informative Petrov’s lectures [40]. Even found relevant at smaller distances when applied to specific mass-imbalanced systems, the range correction will not affect our main conclusions about the tail of the BO potential.

3.2. More than four bodies

Let us consider a generalization of the previous discussion addressing the three and four-body systems to more than four particles. We start with the five-body problem, with two-heavy and three-light bosons, where the interactions are only between heavy and light particles close to the unitary limit (as before). In this case, we have the 4 + 1 and the 3 + 2 thresholds: the five-body states that are bound will have the long range potential damped at the size of the four body state; as well as affected by the existing three-body bound state levels. Therefore, among the sub-system states, the question is to find out which one has the smallest size and cuts the long-range effective five-body potential. Nevertheless, the physical picture has to be extended for the five-body states when three and four-body excited states exist. Here, we still remain with the problem of finding the relevant threshold that determines the asymptotic value of the potential. To be simple, assuming that the 4 + 1 threshold is the relevant one and therefore
becoming apparently more difficult to identify which are the ones more effective in providing the range. However, one situation that can be considered more likely of being generalized is when we add one light boson to a $N$-body system which is in a Borromean state, considering that only one two-body scattering threshold exists, namely, the $(N - 1) + 1$ threshold. In this case, the interwoven spectrum is analogous to the one described for the trimer–tetramer case.

We illustrate the evolution of the states when the long-range cut-off is moved by considering the scaling function for the $N$-body system [two-heavy and $(N - 2)$-light bosons], analogous to the one written for the tetramer case (19). In the unitary limit ($B_{N3} = 0$), the scaling between the $N$-body energy levels, as one varies the corresponding long-range cut-off $R_{(N-1)}^N$ [where $i$ refer to the levels of the $(N - 1)$-system], for $N \geq 4$ is defined by

$$
\frac{B_{N1}^{(i+1)}(R_{(N-1)}, \tilde{E}_{(N-1)}^{(i+1)})}{B_{N1}^{(i)}(R_{(N-1)}, \tilde{E}_{(N-1)}^{(i)})} = \mathcal{G}_N \left( \frac{B_{N1}^{(0)}(R_{(N-1)}, \tilde{E}_{(N-1)}^{(0)})}{B_{N1}^{(0)}(R_{(N-1)}, \tilde{E}_{(N-1)}^{(0)})} \right),
$$

with $n$ labeling the energy level of the $N$-body system. This relation expresses the universal correlation between the binding energy ratios of successive states for two-heavy-boson systems, with $(N - 2)$-light bosons and with $(N - 3)$ light bosons. Note that the dependences on the other scales are wiped out as we assume that in this example the $N - 2$, $N - 3$, ..., are much larger than the $N - 1$ system. This can be regarded as a situation close to a Brunnian system [24]. Corresponding to the above described conditions, in the panel (b) of figure 2, we present our full results for the correlations obtained for $N = 3$ to 7, considering the mass ratio $A = 0.001$, where we demonstrate that the scaling is converging to the analytical expression given by $\mathcal{G}_N \left( e^{2\pi/s_N} \right) = e^{2\pi/s_N}$. A look on these results closer to the analytic limit is presented in the panel (c) of this figure. The patterns are the same as in the tetramer case, with the only difference being the Efimov factor which determines the strength of the long-range potential.

4. Summary and conclusion

Firstly, let us summarize the relevant physical aspects of the three and four-body interwoven energy spectra, which is presented in the first subsection related to interwoven cycles:

(a) The three-body thresholds for the ground and excited states give the asymptotic value of the four-body long range effective interaction.

(b) The separation between the asymptotic values of the long range effective potential follows the geometrical three-body energy ratio in the unitary limit, and provides the thresholds for the four-body states attached to the light–heavy–heavy trimers.

(c) The effective four-body long-range potential itself carries a proper geometrical scale different from the three-body one, $s_4 > s_3$ (where $s_3 = s_0$), exhibiting a proper limit cycle independent of the three-body one, which is damped at the size of the trimer in the unitary limit.

(d) The scaling function correlating the ratios between two close tetramer levels follows the same trend as in the
three-body Efimov case, but with a different scaling factor $\xi$.

e) For very-large mass asymmetry as considered in the present BO analysis, the ratio between the energies of the trimer levels is not much larger than the ratio between tetramer levels, such that we have no room for more than one tetramer level between two trimer levels.

Secondly, as discussed in the second subsection related to interwoven cycles, the case with more than four-body systems, with two-heavy and $(N - 2)$-light particles, becomes more complex due to the different scattering thresholds. However, some general properties can also be summarized as:

(a) The effective interaction between the two-heavy bosons has an Efimov-type potential at unitary limit, with strength increasing linearly with the number of light particles and a correspondingly decreasing geometrical separation between the bound states.

(b) The effective $N$-body long-range potential asymptotically goes to the lowest scattering threshold (see figure 1). For the simplest situation where only the $(N - 1)$-state is bound, it corresponds to its binding energy.

(c) The heavy–heavy effective interaction for the $N$-body system is damped at the size of bound-state levels of the $(N - 1)$-system. This determines the maximal number of possible weakly bound states composed by $(N - 2)$-light boson and two heavy ones.

The presented predicted interwoven spectrum for a mixture with two kind of strongly mass-asymmetric bosons near the unitary limit, as well as the existence of independent few-body scales, are expected to be confirmed in future experimental cold-atom investigations, which can provide more defined conclusions to previous related studies, such as the ones reported in references [52, 53, 55]. In order to verify the predicted interwoven spectrum in an experimental realization we suggest to consider ultracold quantum mixtures of two atomic species with strong mass asymmetry, such as the systems which are being investigated with ytterbium and lithium [56], as well as mixtures with other alkaline-earth atoms [57, 58]. In experiments with a given set of atoms, different narrow Feshbach resonances have to be exploited for the inter-species in order to control the two-body scattering length, such that the weakly-bound $N$-body systems close to the $N - 1$ threshold ($N \geq 4$) near the unitary limit $(\alpha a \gg R^*)$ will have sizes much larger than $R^*$. In this way, as an example, it should be possible to explore the correlations between the positions of the recombination resonances coming from two successive tetramer bound states crossing the continuum threshold, which will move along the correlation curve and break the van der Waals universality. All these exciting possibilities also demand further theoretical and experimental efforts in order to determine the characteristics of these induced effective few-atom forces in the open channels close to narrow Feshbach resonances.

Finally, within the perspective of further theoretical studies, we should mention on the exciting possibilities to obtain other analytical evidences on the few-body scales which could be verified experimentally. In this direction, one could consider the direct extension of the present analysis, by performing calculations with the FY formalism applied to mass-imbalanced systems, which would extend a previous full-numerical investigation shown in reference [1], given further support to recent studies on few-boson scale symmetry breaking to discrete one reported in reference [42]. Also within the context of few-body systems composed by ultracold fermions, with imbalanced mass and spin, one could envisage possible extensions of the approach pursued in the present work. The properties of such systems can be studied with large enough number of magnetic states to allow the identical fermions to occupy relative $s$-wave states. On this regard, new possibilities can be found by extending to three dimensions recent one-dimensional few-fermion investigations, as the ones reported in references [59, 60], when studying unconventional pairings and critical behaviors.

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