On a hierarchy of infinite-dimensional spaces and related Kolmogorov-Gelfand widths

O. Kounchev
Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences
and
IZKS, University of Bonn
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Abstract
Recently the theory of widths of Kolmogorov-Gelfand has received a great deal of interest due to its close relationship with the newly born area of Compressed Sensing. It has been realized that widths reflect properly the sparsity of the data in Signal Processing. However fundamental problems of the theory of widths in multidimensional Theory of Functions remain untouched, as well as analogous problems in the theory of multidimensional Signal Analysis. In the present paper we provide a multidimensional generalization of the original result of Kolmogorov by introducing a new hierarchy of infinite-dimensional spaces based on solutions of higher order elliptic equation.

1 Introduction
Recent interest to the theory of widths (especially to Gelfand widths) has been motivated by applications in Compressed Sensing (CS). In a certain sense the central idea of CS is rooted in the theory of widths, cf. e.g. [9], [7], [8], [30]. However, apparently this strategy works smoothly only in the case of representation of one-dimensional signals, while an adequate approach to multivariate signals is missing – one reason may be found by analogy in the fact that the theory of Kolmogorov-Gelfand widths fits properly only for one-dimensional function spaces (as pointed out below, e.g. in formula (23)). Recently, a new multivariate Wavelet Analysis was developed based on
solutions of elliptic partial differential equations ([15]), in particular "poly-
harmonic subdivision wavelets" were introduced (cf. [10], [21]); in order to
apply CS ideas to these wavelets it would require essential generalization
of the theory of widths for infinite-dimensional spaces. We start with this
motivation to study a generalization of the Kolmogorov-Gelfand theory of
widths by introducing a new hierarchy of infinite-dimensional spaces based
on solutions of higher order elliptic equations. However, there is a different
perspective on the present research: its main purpose is to introduce this new
hierarchy and to apply it to the theory of widths as a testing field. One may
expect also that this development would throw a new light on the nature of
sparsity in multidimensional Signal Analysis.

In his seminal paper [14] Kolmogorov has introduced the theory of widths
and has applied it ingeniously to the following set of functions defined in the
compact interval:

\[ K_p := \left\{ f \in AC^{p-1}([a,b]) : \int_0^1 |f^{(p)}(t)|^2 \, dt \leq 1 \right\}. \]  

(1)

In the present paper we study a natural multivariate generalization of the
set \( K_p \) which in a domain \( B \subset \mathbb{R}^n \) is given by

\[ K^*_p := \left\{ u \in H^{2p}(B) : \int_B |\Delta^p u(x)|^2 \, dx \leq 1 \right\} \],

(2)

where \( \Delta^p \) is the \( p \)-th iterate of the Laplace operator \( \Delta = \sum_{j=1}^n \partial^2 / \partial x_j^2 \); we
consider more general sets \( K^*_p \) given in (22) below. We generalize the notion
of width in the framework of the Polyharmonic Paradigm, and obtain analogs
to the one-dimensional results of Kolmogorov.

The Polyharmonic Paradigm has been announced in [15] as a new ap-
proach to Multidimensional Mathematical Analysis, which is based on solu-
tions of higher order elliptic partial differential equations as opposed to the
usual concept which is based on algebraic and trigonometric polynomials of
several variables. It has proved to be very successful in the Moment Problem
[20], Approximation and Interpolation [17], [18], and Spline Theory [19], [15].

The main objective of the present research is a new development of the
Polyharmonic Paradigm. It provides a new hierarchy of infinite-dimensional
spaces of functions which are used for a generalization of the Kolmogorov’s
theory of widths. This new hierarchy generalizes the usual hierarchy of finite-
dimensional subspaces \( X_N \) of the space \( C^\infty(I) \) for an interval \( I \subset \mathbb{R} \). The
crux of this notion of hierarchy is the following: Let the domain \( D \subset \mathbb{R}^n \), be
compact with sufficiently smooth boundary $\partial D$. Then the $N$–dimensional subspaces in $C^\infty (I)$ will be generalized by spaces of solutions of elliptic equations (and by more general spaces introduced in Definition 14 below):

$$X_N = \{ u : P_{2N}u (x) = 0, \text{ for } x \in D \} \subset L_2 (D);$$

(3)

here $P_{2N}$ is an elliptic operator of order $2N$ in the domain $D$. Respectively, the simplest version of our generalization of Kolmogorov’s theorem about widths finds the extremizer of the following problem

$$\inf_{X_N} \text{dist} (X_N, K_p^*),$$

where $K_p^*$ is the set defined in (2) and $X_N$ is defined in (3) for arbitrary elliptic operator $P_{2N}$ of order $2N$; for the complete formulation see Theorem 22 below.

What is the reason to take namely solutions of elliptic equations in the multidimensional case is explained in the following section.

1.1 The hierarchy of infinite-dimensional spaces - a justification via Chebyshev systems

Let us give a heuristic outline of the motivation and the main idea of this new hierarchy of spaces, by explaining how it appears as a natural generalization of the finite-dimensional subspaces of $C^{N-1} ([a, b])$ in a compact interval $[a, b]$ in $\mathbb{R}$.  

First of all, let us understand the structure of the finite-dimensional subspaces of $C^{N-1} ([a, b])$: It is important to note that for a ”general position” (or ”generic”) $N$–dimensional subspace $X_N \subset C^{N-1} (I)$ in the interval $I = (a, b)$, there exists a finite or infinite number of subintervals $I_{k,j} = (a_{k,j}, b_{k,j})$ with $\bigcup I_{k,j} = [a, b]$, and a basis $\{ u_k \}_{k=1}^N$,

$$X_N = \text{span} \{ u_k \}_{k=1}^N,$$

(4)

(here span denotes the linear closure) where the Wronski determinants satisfy

$$\varepsilon_{k,j} W (u_1 (t), u_2 (t), \ldots , u_k (t)) > 0 \quad \text{for } t \in I_{k,j}$$

(5)

with $\varepsilon_{k,j} = 1$ or $-1$.

A simplest example would be the space $X_2 := \text{span} \{ 1, t^2 \}$ considered on the interval $[-1, 1]$, where the Wronskian $W (1, t^2)$ changes sign at $0$.\footnote{Note that there are cases where the system of functions $\{ u_k \}_{k=1}^N$ has dimension}
Since A. Markov it is known that the positivity condition (5)-(6) is characteristic for Extended Complete Chebyshev systems (called ECT-systems in [13], chapter 11; cf. also [23], chapter 2, section 5, Theorem 5.1). For that reason, we may formulate our important observation by saying that a "general position" $N$-dimensional space $X_N \subset C^N(I)$ is a piecewise Extended Complete Chebyshev system of order $N$.

We will remind some basic properties related to Extended Complete Chebyshev Systems. The following fundamental result describes their structure (cf. [13], chapter 11, Theorem 1).

**Proposition 1** Let us assume that the space $X_N$ in (4) restricted to some subinterval $J \subset I$ has sign-definite Wronskians as in (5)-(6). Then the restriction of the space $X_N$ to the interval $J$ is a space of solutions for an ODE

$$L_N^J u(t) = 0, \quad \text{for } t \in J;$$

(7) here $L_N^J$ is an ordinary differential operator of order $N$ in $J$, given by

$$L_N^J \left( t; \frac{d}{dt} \right) = \prod_{k=1}^{N} \frac{1}{\rho_k(t)}$$

(8)

where the functions $\rho_k$ satisfy $\rho_k(t) > 0$ in $J$.

Let us remark that the weight functions $\rho_k$ may be chosen in different ways, cf. [26]. If we put for the Wronskians

$$W_k := W(u_1(t), u_2(t), \ldots, u_k(t))$$

then the functions $\rho_k(t)$ may be written as

$$\rho_1 = W_1 = u_1, \quad \rho_2 = W_2/W_1^2$$

$$\rho_k = W_k W_{k-2}/W_{k-1}^2 \quad \text{for } k \geq 3,$$

(cf. [31], [23], section 5, chapter 2, Theorem 5.1, or [13], chapter 11).

Obviously, the operator $L_N$ has a non-negative leading coefficient and is in this sense one-dimensional "elliptic".

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$N$ but its Wronskian is 0 on a whole interval, e.g. the system of two functions \(\{t^2, \chi (t) t^2 - \chi (-t) t^2\}\) in the interval \([-1, 1]\), where $\chi$ is the Heaviside function. However this is an exception, hence not "generic". One may try to make the last precise: By introducing a proper topology/metric in the set $S_N$ of all $N$-dimensional spaces in $C^N(I)$, e.g. by taking the distance between the unit spheres in two spaces, we may prove that in the set $S_N$ those spaces having Wronskian equal to 0 are a "small set" in the sense of second category of Baire.
Remark 2  The detailed proof of Proposition 1 is a part of the general theory of Chebyshev systems developed by A. Markov, S. Bernstein, M. Krein and others, in which the Extended Complete Chebyshev systems are a special case which are of interest for us. Their theory is presented in detail in the above mentioned monographs [23] and [13], whereby in the first reference the case of non-differentiable systems is emphasized.

Let us mention that the space $X_N$ generated by a Chebyshev system is often called Haar space, cf. [27]. Thus one may also say that a generic $N$–dimensional subspace of $C^{N-1}(I)$ is piecewise Haar space.

First, we will seek for a generalizable framework for the Extended Complete Chebyshev systems which we have obtained on every subinterval $I_j$. The interpolation framework seems to be suitable: Let us note that condition (5) has equivalent formulation as Hermite interpolation, in particular, for arbitrary $t_0 \in (a,b)$ and constants $\{c_k\}_{k=0}^{N-1}$, it is possible to solve the interpolation problem

$$u^{(k)}(t_0) = c_k \quad \text{for } k = 0, 1, ..., N - 1$$

where $u \in X_N$.

At this point it is important to emphasize that we will select judiciously, and generalize in the multivariate case, only those $2M$–dimensional subspaces $X_{2M} = \text{span} \{u_k\}_{k=1}^{2M} \subset C^{2M-1}(I)$ which satisfy a rather specific interpolation property:

**Definition 3** We say that the space $X_{2M} \subset C^{2M-1}(I)$ has the Dirichlet BVP property, if for every subinterval $I_1 = [a_1, b_1] \subset I$, and for arbitrary constants $\{c_k, d_k\}_{k=0}^{M-1}$, the (Dirichlet) boundary value problem

$$u^{(k)}(a_1) = c_k \quad \text{for } k = 0, 1, ..., M - 1$$

$$u^{(k)}(b_1) = d_k \quad \text{for } k = 0, 1, ..., M - 1$$

has a solution $u \in X_{2M}$.

**Remark 4** Let us assume that in the space $X_{2M}$ there exists an Extended Complete Chebyshev system in $I$ (i.e. a system satisfying positivity of the Wronskians (5)-(6) in $I$). Then $X_{2M}$ satisfies Definition 3 which follows from the very definition of Extended Complete Chebyshev systems, cf. [13], chapter 11. Thus the Extended Complete Chebyshev systems provide the main bulk of examples for Definition 3.
One may consider the solvability of problem (10)-(11) as a "parametrization" of the space $X_{2M}$ by the Dirichlet boundary values $\{c_k, d_k\}_{k=0}^{M-1}$, and this important property will be generalized to the multivariate case.

We are interested in the BVP interpretation which follows from Proposition 1: Since the space $X_{2M}$ may be represented as

$$X_{2M} = \{ u : L_{2M}u = 0 \quad \text{for} \quad t \in I \},$$

for an elliptic operator $L_{2M}$, then the solvability of problem (10)-(11) in the space $X_{2M}$ may be considered as a special case of the multidimensional theory for Elliptic Boundary Value Problems (BVP), and it is a classical BVP in the one-dimensional ODEs as well, cf. [28].

In view of the last observation, we seek for a multidimensional generalization of problem (10)-(11). Let $D$ be a bounded domain in $\mathbb{R}^n$ and consider the subspaces of $L_2(D)$. The space of solutions of an elliptic equation generalizing equation (7) may be considered as a natural generalization of the space $X_{2M}$. Indeed, if

$$X_{2M} = \{ u : P_{2M}u (x) = 0 \quad \text{for} \quad x \in D \},$$

(12)

where $P_{2M} (x; D_x)$ is an elliptic differential operator in the domain $D$, then the natural generalization to problem (10)-(11) is an Elliptic BVP, as for example the Dirichlet problem which may be considered for subdomains $D_1$ in $D$, namely

$$P_{2M}u (x) = 0 \quad \text{for} \quad x \in D_1$$

(13)

$$\left( \frac{\partial}{\partial n} \right)^k u (y) = c_k (y) \quad \text{for} \quad y \in \partial D_1, \quad \text{for} \quad k = 0, 1, ..., M - 1.$$  

(14)

Let us remind that the Dirichlet problem is well-known to be solvable for data $\{c_k (y)\}_{k=0}^{M-1}$ from a proper Sobolev or Hölder space on the boundary $\partial D_1$. An important point is that for a large class of operators $P_{2M}$ every solution of (13)-(14) may be approximated by solutions in the whole domain $D$, i.e. by elements of $X_{2M}$. This may be considered as a substitute of the interpolation property (10)-(11) in the one-dimensional case. Very important hint for identifying the operators $P_{2M}$ which represent Multidimensional Chebyshev systems is provided by formula (8). This is the main reason for the judicious choice of the special class of operators $P_{2M}$ in Definition 14 below, as they mimic the operators in (8) and serve our purposes.

Making analogy with the one-dimensional case (10)-(11), we may say that here the space $X_{2M}$ defined in (12) is "parametrized" by the Dirichlet boundary conditions (13), however the "parameter" $\{c_k (y)\}_{k=0}^{M-1}$ runs a

\[2\]This generalization has been discussed in detail in [16].
function space. Hence, the spaces $X_{2M}$ may be considered as a natural generalization of the one-dimensional Extended Complete Chebyshev systems and we call them Multidimensional Chebyshev systems.

After we have the Multidimensional Chebyshev systems in our disposal, the next step will be to introduce the multivariate generalization of the $N$-dimensional subspaces of $C^\infty(I)$. We will define them in Definition 14 below as subspaces $X_N$ of functions in $C^\infty(D)$ which are piecewise solutions of (regular) elliptic differential operators of order $2N$. We will say that $X_N$ has "Harmonic Dimension $N$" and we will write

$$\text{hdim}(X_N) = N,$$

see Definition 14 below. Kolmogorov's notion of $N$-width (and in a similar way Gelfand's width) is naturally generalized for symmetric sets by the notion of "Harmonic $N$-width" defined by putting

$$\text{hd}_N(S) := \inf_{\text{hdim}(X_N)=N} \text{dist}(X_N, S),$$

see Definition 21 below. The main result of the present paper is the computation of

$$\text{hd}_N(K_p^*) \quad \text{for} \; N \leq p,$$

where $K_p^*$ is defined in (2) and more generally in (22).

1.2 Plan of the paper

To facilitate the reader, in section 2 we provide a short summary of the original Kolmogorov's results. For the same reason, in section 3 we provide a short reminder on Elliptic BVP. In section 4 we prove the representation of the "cylindrical ellipsoid" set $K_p^*$ in principal axes which generalizes the one-dimensional representation of Kolmogorov, cf. Theorem 12 below. In section 5 we introduce the notion of Harmonic Dimension, and the First Kind spaces of Harmonic Dimension $N$. Based on it we define Harmonic Widths which generalize Kolmogorov's widths. In section 6, in Theorem 22 we prove a genuine analog to Kolmogorov's theorem about widths. It says that among all spaces $X_N$ having Harmonic Dimension $N$, some special space $\tilde{X}_N$ provides the best approximation to the set $K_p^*$ in problem

$$\inf_{\tilde{X}_N} \text{dist}(X_N, K_p^*),$$

and this space $\tilde{X}_N$ is identified by the principal axes representation provided by Theorem 12. In section 7 we introduce Second Kind spaces of Harmonic
Dimension $N$ and formulate a further generalization of Theorem 22. Apparently, the First and Second Kind spaces having Harmonic Dimension $N$ provide the maximal generalization in the present framework.

A special case of the present results is available in [22], and might be instructive for the reader to start with.

A final remark to our generalization is in order. In our consideration we will not strive to achieve a maximal generality. As it is clear, especially in the applications to the theory of widths even in the one-dimensional case we may consider not all $N$-dimensional subspaces but ”almost all” $N$-dimensional subspaces of $C^\infty(D)$ in some sense, or a class of $N$-dimensional subspaces which are dense (in a proper topology) in the set of all other $N$-dimensional subspaces. This ”genericity” point of view is essential in our multivariate generalization since it will allow us to avoid burdensome proofs necessary in the case of the bigger generality of the construction. For the same reason we will not consider elliptic pseudo-differential operators although almost all results have a generalization for such setting.

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2 Kolmogorov’s results - a reminder

In order to make our multivariate generalization transparent we will recall the original results of Kolmogorov provided in his seminal paper [14]. Kolmogorov has considered the set $K_p$ defined in (1). He proved that this is an ellipsoid by constructing explicitly its principal axes. Namely, he considered the eigenvalue problem

\begin{align}
(-1)^p u^{(2p)}(t) &= \lambda u(t) \quad \text{for} \ t \in (0,1) \\
u^{(p+j)}(0) = u^{(p+j)}(1) &= 0 \quad \text{for} \ j = 0,1,\ldots,p-1.
\end{align}

Kolmogorov used the following properties of problem (15)-(16) (cf. [25], Chapter 9.6, Theorem 9, p. 146, or [28], [29]):

Proposition 5 Problem (15)-(16) has a countable set of non-negative real eigenvalues with finite multiplicity. If we denote them by $\lambda_j$ in a monotone
order, they satisfy $\lambda_j \to \infty$ for $j \to \infty$. They satisfy the following asymptotic $\lambda_j = \pi^2 p_j^{2p} (1 + O(j^{-1}))$. The corresponding orthonormalized eigenfunctions $\{\psi_j\}_{j=1}^{\infty}$ form a complete orthonormal system in $L_2([0,1])$. The eigenvalue $\lambda = 0$ has multiplicity $p$ and the corresponding eigenfunctions $\{\psi_j\}_{j=1}^{p}$ are the basis for the solutions to equation $u^{(p)}(t) = 0$ in the interval $(0,1)$.

Further, Kolmogorov provided a description of the axes of the "cylindrical ellipsoid" $K_p$, from which an approximation theorem of Jackson type easily follows (cf. [25], chapter 4 and chapter 5).

**Proposition 6** Let $f \in L_2([a,b])$ have the $L_2$—expansion

$$f(t) = \sum_{j=1}^{\infty} f_j \psi_j(t).$$

Then $f \in K_p$ if and only if

$$\sum_{j=1}^{\infty} f_j^2 \lambda_j \leq 1.$$

For $N \geq p + 1$ and every $f \in K_p$ holds the following estimate (Jackson type approximation):

$$\left\| f - \sum_{j=1}^{N} f_j \psi_j(t) \right\|_{L_2} \leq \frac{1}{\sqrt{\lambda_{N+1}}} = O \left( \frac{1}{(N+1)^p} \right). \quad (17)$$

However, Kolmogorov didn’t stop at this point but asked further, whether the linear space $\bar{X}_N := \{\psi_j\}_{j=1}^{N}$ provides the "best possible approximation among the linear spaces of dimension $N$" in the following sense: If we put

$$d_N(K_p) := \inf_{\bar{X}_N} \text{dist} (\bar{X}_N, K_p) \quad (18)$$

the main result he proved in [14] says

$$d_N(K_p) = \text{dist} (\bar{X}_N, K_p). \quad (19)$$

Here we have used the notations, to be used also further,

$$\text{dist} (X, K_p) := \sup_{y \in K_p} \text{dist} (X, y) \quad (20)$$

$$\text{dist} (X, y) = \inf_{x \in X} \|x - y\|. \quad (21)$$
Hence, by inequality (17), equality (19) reads as
\[ d_N(K_p) = \frac{1}{\sqrt{\lambda_{N+1}}} \quad \text{for } N \geq p \]
\[ d_N(K_p) = \infty \quad \text{for } N = 0, 1, ..., p - 1. \]

**Definition 7** The left quantity in (18) is called Kolmogorov \( N \)-width, while the best approximation space \( \tilde{X}_N \) is called extremal (optimal) subspace (cf. this terminology in [33], [25], [29]).

Thus the main approach to the successful application of the theory of widths is based on a Jackson type theorem by which a special space \( \tilde{X}_N \) is identified. Then one has to find, among which subspaces \( X_N \) is \( \tilde{X}_N \) the extremal subspace. Put in a different perspective: one has to find as wide class of spaces \( X_N \) as possible, among which \( \tilde{X}_N \) is the extremal subspace.

Now let us consider the following set which is a natural multivariate generalization of the above set \( K_p \) defined in (1): For a bounded domain \( B \) in \( \mathbb{R}^n \) we put (more generally than (2))
\[ K_p^* := \left\{ u \in H^{2p}(B) : \int_B |L_{2p} u(x)|^2 \, dx \leq 1 \right\}, \tag{22} \]
where \( L_{2p} \) is a strongly elliptic operator in \( B \). Let us remark that the Sobolev space \( H^{2p}(B) \) is the multivariate version of the space of absolutely continuous functions on the interval with a highest derivative in \( L_2 \) (as in (1)). An important feature of the set \( K_p^* \) is that it contains an infinite-dimensional subspace
\[ \left\{ u \in H^{2p}(B) : L_{2p} u(x) = 0, \quad \text{for } x \in B \right\}. \]
Hence, all Kolmogorov widths are equal to infinity, i.e.
\[ d_N(K_p^*) = \infty \quad \text{for } N \geq 0 \tag{23} \]
and no way is seen to improve this if one remains within the finite-dimensional setting.

The main purpose of the present paper is to find a proper setting in the framework of the Polyharmonic Paradigm which generalizes the above results of Kolmogorov.

### 3 A reminder on Elliptic Boundary Value Problems

Let us specify the properties of the domains and the elliptic operators which we will consider. In what follows we assume that the domain \( D \), the differ-
ential operators and the boundary operators satisfy conditions for regular Elliptic BVP. Namely, we give the following:

Definition 8 We will say that the system of operators \( \{ A; B_j, \ j = 1, 2, \ldots, m \} \) forms a regular Elliptic BVP in the domain \( D \subset \mathbb{R}^n \) if the following conditions hold:

1. The operator
   \[
   A(x, D_x) = \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\alpha|} D^\alpha a_{\alpha\beta}(x) D^\beta
   \]
   is a differential operator with a principal part defined as
   \[
   A_0(x, D_x) = \sum_{|\alpha|+|\beta| = 2m} (-1)^{|\alpha|} a_{\alpha\beta}(x) D^{\alpha+\beta}.
   \]

It is uniformly strongly elliptic, i.e. for every \( x \in D \) holds
\[
c_0 |\xi|^{2m} \leq |A_0(x, \xi)| \leq c_1 |\xi|^{2m} \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}.
\]

2. The domain \( D \) is bounded and has a boundary \( \partial D \) of the class \( \mathcal{C}^{2m} \).

3. For every pair of linearly independent real vectors \( \xi, \eta \) and \( x \in \overline{D} \) the polynomial in \( z \), \( A_0(x, \xi + z\eta) \) has exactly \( m \) roots with positive imaginary parts.

4. The coefficients of \( A \) are in \( \mathcal{C}^\infty(D) \). The boundary operators \( B_j(x, D) = \sum_{|\alpha| \leq m_j} b_{j,\alpha}(x) D^\alpha \) form a normal system, i.e. their principal symbols are non-characteristic, i.e. satisfy \( B_{j,0}(x, \xi) = \sum_{|\alpha| = m_j} b_{j,\alpha}(x) \xi^\alpha \neq 0 \) for every \( x \in \partial D \) and \( \xi \neq 0 \), \( \xi \) is normal to \( \partial D \) at \( x \); they have pairwise different orders \( m_j \) which satisfy \( m_j < 2m \) for \( 1 \leq j \leq m \), and their coefficients \( b_{j,\alpha} \) belong to \( \mathcal{C}^\infty \) in \( \partial D \).

5. At any point \( x \in \partial D \) let \( \nu \) denote the outward normal to \( \partial D \) at \( x \) and let \( \xi \neq 0 \) be a real vector in the tangent hyperplane to \( \partial D \) at \( x \). The polynomials in \( z \) given by \( B_{j,0}(x, \xi + z\nu) \) are linearly independent modulo the polynomial \( \prod_{k=1}^{m_j} (z - z_k^+(\xi)) \) where \( z_k^+(\xi) \) denote the roots of \( A_0(x, \xi + z\eta) \) with positive imaginary parts.

Remark 9 With minor differences the above definition is available in [24] (conditions (i)-(iii) in chapter 2, section 5.1); in [32] (sections 5.11 and 5.12); in [12] (chapter 20); in [15] (section 23.2, p. 473).
Let us define a special system of boundary operators called Dirichlet. We put
\[ B_j = \left( \frac{\partial}{\partial n} \right)^{j-1} \quad \text{for} \quad j = 1, 2, \ldots, p - 1 \]
\[ S_j = \left( \frac{\partial}{\partial n} \right)^{p+j-1} \quad \text{for} \quad j = 1, 2, \ldots, p - 1. \]
Obviously,
\[ \text{ord}(B_j) = j - 1, \quad \text{ord}(S_j) = p + j - 1. \]
Let us denote by \( L^*_2 \) the operator formally adjoint to the elliptic operator \( L_2 \). There exist boundary operators \( C_j, T_j \), for \( j = 1, 2, \ldots, p - 1 \), such that
\[ \text{ord}(T_j) = 2p - j, \quad \text{ord}(C_j) = p - j \]
and the following Green's formula holds:
\[
\int_B \left( L_2 u \cdot v - u \cdot L^*_2 v \right) \, dx = \sum_{j=0}^{p-1} \int_{\partial B} \left( S_j u \cdot C_j v - B_j u \cdot T_j v \right) \, d\sigma_y; \quad (24)
\]
here \( \partial_n \) denotes the normal derivative to \( \partial B \), for functions \( u \) and \( v \) in the classes of Sobolev, \( u, v \in H^{2p}(B) \) (cf. [24], Theorem 2.1 in section 2.2, chapter 2, and Remark 2.2 in section 2.3).

For us the following eigenvalue problem will be important to consider for \( U \in H^{2p}(B) \), which is analogous to problem (15)-(16):
\[
L^*_2 L_2 U(x) = \lambda U(x) \quad \text{for} \quad x \in B \quad (25)
\]
\[ B_j L_2 U(y) = S_j L_2 U(y) = 0, \quad \text{for} \quad y \in \partial B, \quad j = 0, 1, \ldots, p - 1 \quad (26)\]
where \( \partial_n \) denotes the normal derivative at \( y \in \partial B \). It is obvious that the operator \( L^*_2 L_2 \) is formally self-adjoint, however the BVP (25)-(26) is not a nice one. Since a direct reference seems not to be available, we provide its consideration in the following theorem which is an analog to Proposition 5.

**Theorem 10** Let the operator \( L_2 \) be uniformly strongly elliptic in the domain \( B \). Then problem (25)-(26) has only real non-negative eigenvalues.

1. The eigenvalue \( \lambda = 0 \) has infinite multiplicity with corresponding eigenfunctions \( \{ \psi_j \}_{j=1}^{\infty} \) which represent an orthonormal basis of the space of all solutions to the equation \( L_2 U(x) = 0 \), for \( x \in B \).

2. The positive eigenvalues are countably many and each has finite multiplicity, and if we denote them by \( \lambda_j \) ordered increasingly, they satisfy \( \lambda_j \to \infty \) for \( j \to \infty \).
3. The orthonormalized eigenfunctions, corresponding to eigenvalues \( \lambda_j > 0 \), will be denoted by \( \{ \psi_j \}_{j=1}^{\infty} \). The set of functions \( \{ \psi_j \}_{j=1}^{\infty} \cup \{ \psi'_j \}_{j=1}^{\infty} \) form a complete orthonormal system in \( L_2(B) \).

**Remark 11** Problem (25)-(26) is well known to be a non-regular elliptic BVP, as well as non-coercive variational, cf. [1] (p. 150) and [24] (Remark 9.8 in chapter 2, section 9.6, and section 9.8).

The proof is provided in the Appendix below, section 8.

4. The principal axes of the ellipsoid \( K_p^* \) and a Jackson type theorem

Here we will find the principal axes of the ellipsoid \( K_p^* \) defined as

\[
K_p^* := \left\{ u \in H^{2p}(B) : \int_B |L_{2p}u(x)|^2 \, dx \leq 1 \right\},
\]

(27)

where \( L_{2p} \) is a uniformly strongly elliptic operator in \( B \).

We prove the following theorem which generalizes Kolmogorov’s one-dimensional result from Proposition 6, about the representation of the ellipsoid \( K_p \) in principal axes.

**Theorem 12** Let \( f \in K_p^* \). Then \( f \) is represented in a \( L_2 \)-series as

\[
f(x) = \sum_{j=1}^{\infty} f_j \psi_j(x) + \sum_{j=1}^{\infty} f_j' \psi'_j(x),
\]

where by Theorem 10 the eigenfunctions \( \psi'_j \) satisfy \( \Delta^p \psi'_j(x) = 0 \) while the eigenfunctions \( \psi_j \) correspond to the eigenvalues \( \lambda_j > 0 \), and also

\[
\sum_{j=1}^{\infty} \lambda_j f_j^2 \leq 1.
\]

(28)

Vice versa, every sequence \( \{ f'_j \}_{j=1}^{\infty} \cup \{ f_j \}_{j=1}^{\infty} \) with

\[
\sum_{j=1}^{\infty} |f'_j|^2 + \sum_{j=1}^{\infty} |f_j|^2 < \infty
\]

and \( \sum_{j=1}^{\infty} \lambda_j f_j^2 \leq 1 \) defines a function \( f \in L_2(B) \) which is in \( K_p^* \).
Proof. (1) According to Theorem 10, we know that arbitrary \( f \in L_2 (B) \) is represented as

\[
f (x) = \sum_{j=1}^{\infty} f'_j \psi_j (x) + \sum_{j=1}^{\infty} f_j \psi_j (x)
\]

\[
\| f \|^2_{L_2} = \sum_{j=1}^{\infty} |f'_j|^2 + \sum_{j=1}^{\infty} |f_j|^2 < \infty
\]

with convergence in the space \( L_2 (B) \).

(2) From the proof of Theorem 10, we know that if we put

\[
\psi_j \phi_j (x) = L_2^p \psi_j (x) \quad \text{for } j \geq 1,
\]

then the system of functions

\[
\frac{\psi_j \phi_j (x)}{\sqrt{\lambda_j}} \quad \text{for } j \geq 1
\]

is orthonormal sequence which is complete in \( L_2 (B) \).

(3) We will prove now that if \( f \in L_2 (B) \) then \( f \in K_\ast^p \) iff

\[
\sum_{j=1}^{\infty} f_j^2 \lambda_j \leq 1.
\]

Indeed, for every \( f \in H^{2p} (B) \) we have the expansion

\[
f (x) = \sum_{j=1}^{\infty} f'_j \psi_j (x) + \sum_{j=1}^{\infty} f_j \psi_j (x).
\]

We want to see that it is possible to differentiate termwise this expansion, i.e.

\[
L_{2p} f (x) = \sum_{j=1}^{\infty} f_j L_{2p} \psi_j (x) = \sum_{j=1}^{\infty} f_j \phi_j (x)
\]

Since \( \left\{ \frac{\phi_j}{\sqrt{\lambda_j}} \right\}_{j \geq 1} \) is a complete orthonormal basis of \( L_2 (B) \) it is sufficient to see that

\[
\int_B L_{2p} f (x) \phi_j dx = \int_B \left( \sum_{j=1}^{\infty} f_j L_{2p} \psi_j (x) \right) \phi_j dx.
\]
Due to the boundary properties of $\phi_j$ and since $\phi_j = L_{2p}\psi_j$, we obtain
\[
\int_B L_{2p}f(x)\phi_j dx = \int_B f(x)L_{2p}^*\phi_j dx = \lambda_j \int_B f\psi_j dx = \lambda_j f_j.
\]
On the other hand
\[
\int_B \left( \sum_{k=1}^{\infty} f_k \phi_k(x) \right) \phi_j dx = \lambda_j f_j.
\]
Hence
\[
L_{2p}f(x) = \sum_{j=1}^{\infty} f_j L_{2p}\psi_j(x) = \sum_{j=1}^{\infty} f_j \phi_j(x) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} f_j \frac{\phi_j(x)}{\sqrt{\lambda_j}}
\]
and since $\left\{\frac{\phi_j}{\sqrt{\lambda_j}}\right\}_{j \geq 1}$ is an orthonormal system, it follows
\[
\|L_{2p}f\|^2_{L_2} = \sum_{j=1}^{\infty} \lambda_j f_j^2.
\]
Thus if $f \in K_p$ it follows that $\sum_{j=1}^{\infty} \lambda_j f_j^2 \leq 1$.

Now, assume vice versa, that $\sum_{j=1}^{\infty} f_j^2 \lambda_j \leq 1$ holds together with $\sum_{j=1}^{\infty} |f_j|^2 + \sum_{j=1}^{\infty} |f_j'|^2 < \infty$. We have to see that the function
\[
f(x) = \sum_{j=1}^{\infty} f_j' \psi_j'(x) + \sum_{j=1}^{\infty} f_j \psi_j(x)
\]
belongs to the space $H^{2p}(B)$. Based on the completeness and orthonormality of the system $\left\{\frac{\phi_j(x)}{\sqrt{\lambda_j}}\right\}_{j=1}^{\infty}$ we may define the function $g \in L_2$ by putting
\[
g(x) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} f_j \frac{\phi_j(x)}{\sqrt{\lambda_j}} = \sum_{j=1}^{\infty} f_j \phi_j(x); 
\]
it obviously satisfies $\|g\|_{L_2} \leq 1$. 

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From the local solvability of elliptic equations ([24]) there exists a function \( F \in H^{2p}(B) \) which is a solution to equation \( L_{2p}F = g \). Let its representation be

\[
F(x) = \sum_{j=1}^{\infty} f_j \psi_j'(x) + \sum_{j=1}^{\infty} F_j \psi_j(x)
\]

with some coefficients \( F_j \) satisfying \( \sum_j |F_j|^2 < \infty \). As above we obtain

\[
\lambda_j \int_B F \psi_j dx = \int_B F L_{2p}^* L_{2p} \psi_j dx = \int_B L_{2p} F \cdot L_{2p} \psi_j dx = \int_B g \cdot \phi_j dx
\]

which implies \( F_j = f_j \). Hence, \( F = f \) and \( f \in H^{2p}(B) \). This ends the proof.

We are able to prove finally a Jackson type result as in Proposition 6.

**Theorem 13** Let \( N \geq 1 \). Then for every \( N \geq 1 \) and every \( f \in K^*_p \) holds the following estimate:

\[
\left\| f - \sum_{j=1}^{\infty} f_j \psi_j'(x) - \sum_{j=1}^{N} f_j \psi_j(x) \right\|_{L_2} \leq \frac{1}{\sqrt{\lambda_{N+1}}}
\]

**Proof.** The proof follows directly. Indeed, due to the monotonicity of \( \lambda_j \), and inequality (28), we obtain

\[
\left\| f - \sum_{j=1}^{\infty} f_j \psi_j'(x) - \sum_{j=1}^{N} f_j \psi_j(x) \right\|_{L_2}^2 = \sum_{j=N+1}^{\infty} f_j^2 \lambda_j \leq \frac{1}{\lambda_{N+1}} \sum_{j=N+1}^{\infty} f_j^2 \lambda_j \leq \frac{1}{\lambda_{N+1}}
\]

This ends the proof.

5 Introducing the Hierarchy and Harmonic Widths

In the present section we introduce the simplest representatives of the class of domains having Harmonic Dimension \( N \), which are called **First Kind** domains. They are piece-wise solutions to regular elliptic equations.
Definition 14 Let $D \subset \mathbb{R}^n$ be a bounded domain. For an integer $M \geq 1$ we say that the linear subspace $X_M \subset L_2(D)$ is of **First Kind** and has **Harmonic Dimension** $M$, and write

$$\text{hdim}(X_M) = M,$$

(29)

if the following conditions are fulfilled:

1. There exists a finite number of domains $D_j$ with **piece-wise smooth** boundaries $\partial D_j$ (which guarantees the validity of Green’s formula (24)), which are pairwise disjoint, i.e. $D_i \cap D_j = \emptyset$ for $i \neq j$, and such that we have the domain partition

$$D = \bigcup_j D_j.$$  

(30)

2. We assume that for $j = 1, 2, ..., k$ the **factorization operators** $Q_j(x, D_x)$ are **uniformly strongly elliptic** in the domain $D$ and, the functions $\rho_j$ defined in $D$ are infinitely smooth and satisfy

$$Z := \bigcup_{j=1}^k \{ x \in D : \rho_j(x) = 0 \} \subset \bigcup_{j=1}^k \partial D_j \setminus \partial D.$$  

We assume that $\text{ord}(Q_j) = 2N_j$ and $\sum_{j=1}^k 2N_j = 2M$. Define the operator

$$P_{2M}(x, D_x) u(x) = \left( \prod_{j=1}^k Q_j(x, D_x) \frac{1}{\rho_j(x)} \right) u(x)$$

(31)

for the points $x \in D$ where it is correctly defined (out of the set $Z$).

We specify the interface conditions: Let us denote by $u_i = u|_{D_i}$ the restriction of $u$ to $D_i$. If for some indexes $i \neq j$, the intersection $H := \partial D_i \cap \partial D_j$ has nonempty interior in the relative topology of $\partial D_i$ (hence also in $\partial D_j$) then the following **interface conditions** hold on $H$ in the sense of traces:

$$\left( \frac{\partial}{\partial n_x} \right)^k u_i(x) = \left( \frac{\partial}{\partial n_x} \right)^k u_j(x) \quad \text{for } k = 0, 1, ..., 2M - 1;$$

(32)

here the vector $n_x$ denotes one of the normals at $x$ to the surface $\partial D_i \cap \partial D_j$.

We define the space $X_M$ by putting

$$X_M = \left\{ u \in H^{2M}(D) : P_{2M} u(x) = 0, \quad \text{for } x \in \bigcup_{j=1}^k D_j, \right. \text{ and } u \text{ satisfies the interface conditions (32)} \right\}.$$  

(33)

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Remark 15 1. In [22] we considered the case of spaces $X_N$ of Harmonic Dimension $N$ defined by a single elliptic operator $P_{2N}$ (i.e. $P_{2N} = Q_1$) and a trivial partition of $D$, i.e. $D = D_1$.

2. Let us comment on the interface conditions (32) in Definition 14. Let us assume that we have an elliptic operator $P_{2N}$ with smooth coefficients defined on $D$ and that a non-trivial partition $\bigcup D_j$ is given. Due to the piecewise smoothness of the boundaries $\partial D_j$ we may apply the Green formula, and from the interface conditions (32) it follows that ”analytic continuation” is possible, hence every function in $X_N$ is a solution to $P_{2N}u = 0$ in the whole domain $D$ (see similar result in [15], Lemma 20.10, and the proof of Theorem 20.11).

3. One may choose a different set of interface conditions which are equivalent to (32), see [15] (Remark 20.12), and [24] (Lemma 2.1 in chapter 2).

4. The spaces $X_N$ defined in Definition 14 mimic in a natural way the one-dimensional case: the operator $P_{2M}$ (31) is similar to the operator (8) in Proposition 1.

5. The operator $\prod_j \rho_j \cdot P_{2M}$ does not have a singularity in the principal symbol but eventually only in the lower order coefficients.

Here is a simple non-trivial example to Definition 14:

$$D_1 = \{ x : |x| < 1 \}, \quad D_2 = \{ x : 1 < |x| < 2 \}$$

$$D = \{ x : |x| < 2 \}$$

$$P_1^1 (x; D_x) u(x) = \Delta \frac{1}{1 - |x|} \Delta u(x) \quad \text{for } x \in D_1$$

$$P_1^2 (x; D_x) u(x) = -\Delta \frac{1}{1 - |x|} \Delta u(x) \quad \text{for } x \in D_2,$$

where $\Delta$ is the Laplace operator. Typical elements of $X_2$ are the functions $u$ which are obtained as solutions to

$$\Delta u = (1 - |x|) w \quad \text{in } D,$$

where $\Delta w = 0$ in $D$.

The following result shows that we may construct a lot of solutions belonging to the set $X_M$ of Definition 14. We call these ”direct solutions”.

**Proposition 16** There is a set of boundary conditions $B_\ell$, $\ell = 1, 2, \ldots, M$ on $\partial D$ such that problem

$$P_{2M}u(x) = 0 \quad \text{for } x \in D$$

$$B_\ell u(y) = h_\ell(y) \quad \text{for } y \in \partial D, \text{ and } \ell = 1, 2, \ldots, M$$
is solvable for arbitrary data \( \{h_\ell\}_{\ell=1}^M \) from the corresponding Sobolev spaces, i.e. \( h_\ell \in H^{2M-\operatorname{ord}(B_\ell)-1/2}(\partial D) \), and the solution has the maximal regularity, i.e. \( u \in H^{2M}(D) \).

**Proof.** For every \( j \) with \( 1 \leq j \leq k \) we choose boundary operators \( B_{j,m} \) for \( m = 1, 2, \ldots, N_j \) for a regular elliptic BVP \( \{Q_j(x,D_x); B_{j,m}, \ m = 1, 2, \ldots, N_j\} \).

If we have the data function \( f \) on \( D \) and \( h^{(j)} = \{h_{j,m}\}_{m=1}^{N_j} \) on the boundary \( \partial D \) then the solution of the elliptic BVP

\[
Q_j w = f \quad \text{on } D \\
B_{j,m} w = h_{j,m}, \quad \text{on } \partial D, \text{ for } m = 1, 2, \ldots, N_j
\]

in case it exists will be denoted by \( I_j(f,h^{(j)}) \).\(^3\) We may write inductively

\[
u = \rho_k I_k (\cdots \rho_2 I_2 (\rho_1 I_1 (0; h^{(1)}); h^{(2)})) \cdots .
\]

For simplicity of notation let us assume that \( k = 2 \). Then the boundary conditions satisfied by \( u \) are obtained from

\[
Q_1 w = 0 \\
B_{1,m} w = h_{1,m}, \quad \text{for } m = 1, 2, \ldots, N_1
\]

and

\[
Q_2 \left( \frac{1}{\rho_2} u \right) = \rho_1 w \\
B_{2,m} \frac{1}{\rho_2} u = h_{2,m}, \quad \text{for } m = 1, 2, \ldots, N_2
\]

hence, we obtain

\[
B_{1,m} \left( \frac{1}{\rho_1} Q_2 \left( \frac{1}{\rho_2} u \right) \right) = h_{1,m}, \quad \text{for } m = 1, 2, \ldots, N_1.
\]

Thus we see that the system of boundary operators on \( \partial D \)

\[
B_{2,m} \frac{1}{\rho_2} u, \quad \text{for } m = 1, 2, \ldots, N_2 \\
B_{1,m} \left( \frac{1}{\rho_1} Q_2 \left( \frac{1}{\rho_2} u \right) \right) \quad \text{for } m = 1, 2, \ldots, N_1
\]

\(^3\)For the solvability recall that there is a finite number of conditions which have to be satisfied by the data \( \{f,h_{j,m}\} \) which guarantee the solvability, cf. [24] (Theorem 5.3, chapter 2, section 5.3).
is normal. Let us put

\[ B_j u = B_{2,j} \frac{1}{\rho_2} u \quad \text{for } j = 1, 2, \ldots, N_2 \]

\[ B_{N_2+j} u = B_{1,j} \left( \frac{1}{\rho_1} Q_2 \left( \frac{1}{\rho_2} u \right) \right) \quad \text{for } j = 1, 2, \ldots, N_1. \]

A simple direct check shows that the orders of the system of operators

\[ \{ B_j : j = 1, 2, \ldots, N_1 + N_2 \} \]

differ, and also satisfy the condition for being "non-characteristic" on the boundary, cf. Definition 8, item 4). We may proceed inductively to prove the statement for arbitrary \( k \geq 3 \).

\[ \mathbf{\square} \]

**Remark 17** Apparently, one may prove that the set of "direct solutions" obtained in Proposition 16 is dense in the whole space \( X_M \) defined in Definition 14.

The following fundamental theorem shows that, as in the one-dimensional case, on arbitrary small sub-domain \( G \) in \( D \) with \( G \cap (\bigcup \partial D_j) = \emptyset \), the space \( X_M \) with \( \hdim(X_M) = M \) has the same Harmonic Dimension \( M \). From a different point of view, it shows that a theorem of Runge-Lax-Malgrange type is true also for elliptic operators with singular coefficients of the type of operators \( P_{2M} \) considered in Definition 14.

**Theorem 18** Let the First Kind space \( X_M \) satisfy Definition 14 with

\[ \hdim(X_M) = M. \]

Assume that the elliptic operator \( P_{2M} \) which corresponds to the space \( X_M \) has factorization operators \( Q_j \) (from (31)) satisfying condition \( (U)_4 \) for uniqueness in the Cauchy problem in the small.\(^4\) Let \( G \) be a compact subdomain in some \( D_j \), i.e. \( G \cap (\bigcup \partial D_j) = \emptyset \). Then the set of "direct solutions" considered in Proposition 16 is dense in \( L_2(G) \) in the space

\[ \{ u \in H^{2M}(G) : P_{2M} u = 0 \quad \text{in } G \}. \]

\(^4\)The differential operator \( P \) satisfies condition \( (U)_4 \) for uniqueness in the Cauchy problem in the small in \( G \) provided that if \( G_1 \) is a connected open subset of \( G \) and \( u \in C^r(G_1) \) is a solution to \( P^* u = 0 \) and \( u \) is zero on a non-empty subset of \( G_1 \) then \( u \) is identically zero. Elliptic operators with analytic coefficients satisfy this property (cf. [4], part II, chapter 1.4; [5], p. 402).
Proof. For simplicity of notations we assume that for the elliptic operator $P_{2M}$ associated with $X_M$, by Definition 14, we have only two factorizing operators $Q_1$ and $Q_2$, i.e. $P_{2M}u = Q_1 \frac{1}{\rho_1} Q_2 \left( \frac{1}{\rho_2} u \right)$.

Let us take a solution $u \in H^{2M}(G)$ to $P_{2M}u = 0$ in $G$. We have

$$Q_1 \frac{1}{\rho_1} Q_2 \left( \frac{1}{\rho_2} u \right) = 0 \quad \text{in} \quad G$$

and we use the solutions $I_j$ for the Elliptic BVP (34)-(35) considered in the domain $G$, to express arbitrary solution as

$$u = \rho_2 I_2 \left( \rho_1 I_1 \left( 0; h^{(1)} \right) ; h^{(2)} \right),$$

where the boundary data $h^{(1)}$ and $h^{(2)}$ are arbitrary in proper Sobolev spaces. By the approximation theorem of Runge-Lax-Malgrange type (cf. [5], Theorem 4, and references there), which uses essentially property $(U)_s$ of operator $Q_1^*$, we obtain a function $w_\varepsilon$ which is a solution to $Q_1 w_\varepsilon = 0$ in $D$ and such that

$$\| I_1 \left( 0; h^{(1)} \right) - w_\varepsilon \|_{L^2(G)} < \varepsilon.$$ 

Next we apply the same approximation argument but with non-zero right-hand side $\rho_1 I_1 \left( 0; h^{(1)} \right)$ (cf. [6]) to prove the existence of a function $v_\varepsilon$ such that

$$\| I_2 \left( \rho_1 I_1 \left( 0; h^{(1)} \right) ; h^{(2)} \right) - v_\varepsilon \| < C \varepsilon$$

for some constant $C > 0$, where the constant $C$ depends on the functions $\rho_j$. Thus we obtain the function

$$u_\varepsilon = \rho_2 v_\varepsilon$$

which satisfies

$$\| u_\varepsilon - u \|_{L^2(G)} < C_1 \varepsilon,$$

and is a "direct solution" in the sense of Proposition 16.

The following theorem studies the orthogonal complement $X_N \ominus X_M$ of two First Kind spaces where $M < N$. While we will not need the whole generality of the result proved, the proof shows that $X_N \ominus X_M$ has at least $\text{hdim} = N - M$.

**Theorem 19** Let $M < N$ and the First Kind spaces $X_M$, $X_N$ satisfy Definition 14 with

$$\text{hdim} (X_M) = M, \text{ hdim} (X_N) = N.$$

Assume that the elliptic operator $P'_{2N}$, which is associated with the space $X_N$, has (by (31)) factorization operators $Q_j$ satisfying condition $(U)_s$ for
uniqueness in the Cauchy problem in the small (as in Theorem 18). Then 
the space $Y = X_N \setminus X_M$ is infinite-dimensional.

**Proof.** (1) Let, by Definition 14, the partition $\bigcup D_j$ and the operator $P_{2M}$
correspond to $X_M$, while the partition $\bigcup D'_j$ and the operator $P'_{2N}$
correspond to $X_N$. Assume that $D_1 \cap D'_1 \neq \emptyset$. Then we will choose a subdomain $G$
which is compactly supported in $D_1 \cap D'_1$.

Further we will fix our attention to the subdomain $G$ where both operators $P_{2M}$
and $P'_{2N}$ are uniformly strongly elliptic and will construct a subset of $X_N \cap X_M$ restricted to the domain $G$. Let us be more precise: If we denote by

$$X_N^G := \{ u : H^{2N}(G) : P_{2N}'u = 0 \text{ in } G \}$$  \hspace{1cm} (36)

then we will construct an infinite-dimensional subspace of $X_N^G \cap X_M^G$.

(2) For the uniformly strongly elliptic operator $P_{2M}$ on the domain $G$
we choose the Dirichlet system of boundary operators $B_j = \frac{\p}{\p n}$, for $j \geq 1$, which are iterates of the normal derivative $\p/\p n$ on the boundary $\p G$. As
already mentioned the system of operators $\{ P_{2M}; \frac{\p}{\p n} : j = 0, 1, ..., M - 1 \}$
on $G$ forms a regular Elliptic BVP (this is the Dirichlet Elliptic BVP for the
operator $P_{2M}$) (cf. [24], Remark 1.3 in section 1.4, chapter 2).

We complete the system $\{ B_j \}_{j=1}^M$ by the system of boundary operators
$S_j = \frac{\p^{M-1+j}}{\p n^{M-1+j}}$ for $j = 1, 2, ... M$. Hence, the system composed $\{ B_j \}_{j=1}^M \cup \{ S_j \}_{j=1}^M$
is a Dirichlet system of order $2M$ (cf. [24], Definition 2.1 and Theorem 2.1 in
section 2.2, chapter 2). Further, by [24] (Theorem 2.1), there exists a unique
Dirichlet system of order $2M$ of boundary operators $\{ C_j, T_j \}_{j=1}^M$ which is
uniquely determined as the adjoint to the system $\{ B_j, S_j \}_{j=1}^M$, and the Green
formula (24) holds on the domain $G$. We will use this below.

(3) In the domain $G$ we consider the elliptic operator $P_{2N}'P_{2M}^\ast$. As a
product of two strongly elliptic operators it is such again. By a standard
construction cited above (cf. [24], Theorem 2.1, section 2.2, chapter 2), we
may complete the Dirichlet system of operators $\{ B_j, S_j \}_{j=1}^M$ with $N - M$
boundary operators $R_j = \frac{\p^{2M-1+j}}{\p n^{2M-1+j}}$, $j = 1, 2, ..., N - M$. Again by the above
cited theorem, the Dirichlet system of boundary operators

$$\{ B_j, S_j \}_{j=1}^M \cup \{ R_j \}_{j=1}^{N-M}$$

covers the operator $P_{2N}'P_{2M}^\ast$. Finally, we consider the solutions $g \in H^{2N+2M}(G)$

to the following Elliptic BVP:

$$P_{2N}'P_{2M}^\ast g(x) = 0 \hspace{1cm} \text{for } x \in G \hspace{1cm} (37)$$

$$B_j g(y) = S_j g(y) = 0 \hspace{1cm} \text{for } j = 0, 1, ..., N - 1, \text{ for } y \in \p G \hspace{1cm} (38)$$

$$R_j g(y) = h_j(y) \hspace{1cm} \text{for } j = 1, 2, ..., N - M, \text{ for } y \in \p G. \hspace{1cm} (39)$$

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We may apply a classical result [24] (the existence Theorem 5.2 and Theorem 5.3 in chapter 2), to the solvability of problem (37)-(39) in the space $H^{2M+2N} (G)$.

(4) Let us check the properties of the function $P^*_{2M}g$ where $g$ satisfies (37)-(39). First of all, it is clear from (37) that $P^*_{2M}g \in X^G_N$ where we have used the notation (36).

By Green’s formula (24), applied for the operator $P^*_{2M}$ and for $u = g$ we obtain
\[
\int_G P^*_{2M}g \cdot v dx = 0 \quad \text{for all } v \text{ with } P_{2M}v = 0
\]
which implies that the function $P^*_{2M}g$ satisfies $P^*_{2M}g \perp X^G_{2M}$ ($X^G_{2M}$ defined as (36)).

By the general existence theorem for Elliptic BVP used already above (cf. [24], Theorem 5.3, the Fredholmness property), we know that a solution $g$ to problem (37)-(39) exists for those boundary data \( \{h_j\}_{j=1}^{N-M} \) which satisfy only a finite number of linear conditions (cf. [24], conditions (5.18)); these are determined by the solutions to the homogeneous adjoint Elliptic BVP. Hence, it follows that the space $Y^G_{N-M}$ of the functions $P^*_{2M}g$ where $g$ is a solution to (37)-(39) is infinite-dimensional.

(5) Let us construct a subspace of $X_N \setminus X_M$ which is infinite-dimensional. We use the obvious inclusion $X_{N|G} \subset X^G_N$, $X_{M|G} \subset X^G_M$, where for a space of functions $Y \subset L^2 (B)$ the space $Y|G$ consists of the restrictions of the elements of $Y$ to the domain $G$.

First of all, we find an orthonormal basis $\{v_j\}_{j \geq 1}$ in the infinite-dimensional space $Y^G_{N-M}$ (where the norm is $\|\cdot\|_{L^2(G)}$); by the Gram-Schmidt orthonormalization we obtain functions $g_j$ such that $v_j = P^*_{2M}g_j$ for $j \geq 1$.

Let us put $\varepsilon_j = \frac{1}{2^{j-1}}$ and use the density Theorem 18 to choose $u_j \in H^{2N} (D)$ with
\[
\|u_j - v_j\|_{L^2(G)} \leq \varepsilon_j \quad \text{for } j \geq 1.
\]

The orthogonality of $v_j$ to $X^G_M$ infers $\text{dist} (u_j|G, X^G_M) \geq 1 - \varepsilon_j$ in the $L^2 (G)$ norm. Hence, $\text{dist} (u_j, X_M) \geq 1 - \varepsilon_j$ in the $L^2 (D)$ norm, hence $u_j \notin X_M$.

Let us see that for every choice of the constants $\alpha_j$ holds
\[
\sum_{j=1}^{N-1} \alpha_j u_j \neq u_N.
\]
Indeed, by the triangle inequality for the norm $\|\cdot\|_{L^2(G)}$ it follows

$$1 + \sum_{j=1}^{N-1} |\alpha_j|^2 = \left\| v_N - \sum_{j=1}^{N-1} \alpha_j v_j \right\|$$

$$= \left\| v_N - u_N + u_N - \sum_{j=1}^{N-1} \alpha_j u_j + \sum_{j=1}^{N-1} \alpha_j u_j - \sum_{j=1}^{N-1} \alpha_j v_j \right\|$$

$$\leq \varepsilon_N + \left\| u_N - \sum_{j=1}^{N-1} \alpha_j u_j \right\| + \sum_{j=1}^{N-1} |\alpha_j| \varepsilon_j$$

or

$$1 - \varepsilon_N + \sum_{j=1}^{N-1} \left( |\alpha_j|^2 - |\alpha_j| \varepsilon_j \right) \leq \left\| u_N - \sum_{j=1}^{N-1} \alpha_j u_j \right\|.$$

Obviously

$$1 - \varepsilon_N + \sum_{j=1}^{N-1} \left( \frac{\varepsilon_j^2}{4} - \frac{\varepsilon_j}{2} \varepsilon_j \right) \leq 1 - \varepsilon_N + \sum_{j=1}^{N-1} \left( |\alpha_j|^2 - |\alpha_j| \varepsilon_j \right)$$

and since the left-hand side always exceed 1/4, this ends the proof that the system of functions $\{u_j|G\}_{j\geq1}$ is linearly independent. Hence, the system $\{u_j\}_{j\geq1}$ is linearly independent in the whole domain $D$.

As noted above $u_j \notin X_M$, hence span $\{u_j\}_{j\geq1}$ is the infinite-dimensional space we sought. The proof is finished.

We have the following prototype of Theorem 19, proved in [22].

**Corollary 20** Let $M \leq N$ and $X_M, X_N$ satisfy Definition 14 with

$$\text{hdim}(X_M) = M, \quad \text{hdim}(X_N) = N.$$

Assume that the differential operators $P_{2M}$ and $P_{2N}$, associated with $X_M$ and $X_N$, have trivial factorization operators by the definition (31), and trivial domain partitions $D = D_1$ and $\tilde{D} = D_1'$ by (30). Then the space of solutions of the Elliptic BVP (37)-(39) where $G = D$ is a subspace of the space

$$Y = X_N \ominus X_M.$$

The proof may be derived from the proof of Theorem 19 where we have put $G = D$. Note that we do not need the $(U)_s$ condition for the operator.
$P^2_{2N}$. Hence, strictly speaking, Corollary 20 is not a special case of Theorem 19.

Now we provide a generalization of Kolmogorov’s notion of width from formula (18); without restricting the generality we assume that we work only with symmetric subsets.

**Definition 21** Let $A$ be a centrally symmetric subset in $L^2(B)$. For fixed integers $M \geq 1$ and $N \geq 0$ we define the corresponding **Harmonic Width** by putting

$$h_{dM,N}(K) := \inf_{X_M,F_N} \text{dist} (X_M \oplus F_N, A),$$

where $\inf_{X_M,F_N}$ is taken over all spaces $X_M, F_N \subset C^\infty(B)$ with

$$\text{hdim} (X_M) = M \quad \text{dim} (F_N) = N.$$

### 6 Generalization of Kolmogorov’s result about widths

Next we prove results which are analogs to the original Kolmogorov’s results about widths in (19).

We denote by $F_N$ a **finite-dimensional** subspace of $L^2(B)$ of dimension $N$. We denote the special subspaces for an elliptic operator $P^2_p = L^2_p$ by

$$\tilde{X}_p := \{u \in H^{2p}(B) : L^2_p u(x) = 0, \quad \text{for } x \in B \},$$

and the special finite-dimensional subspaces

$$\tilde{F}_N := \{\psi_j : j \leq N\}_{\text{lin}}$$

where $\psi_j$ are the eigenfunctions from Theorem 10.

**Theorem 22** Let $K^*_p$ be the set defined in (27) as

$$K^*_p := \left\{u \in H^{2p}(B) : \int_B |L^2_p u(x)|^2 \, dx \leq 1 \right\},$$

with a constant coefficient operator $L^2_p$ which is uniformly strongly elliptic in the domain $B$. Let $X_M$ be a First Kind subspace of $L^2(B)$ of Harmonic Dimension $M$, according to Definition 14, i.e.

$$\text{hdim} (X_M) = M,$$
and let $N \geq 0$ be arbitrary.

1. If $M < p$ then

\[ \text{dist} \left( X_M \bigoplus F_N, K^*_p \right) = \infty. \]

Hence,

\[ \inf_{X_M, F_N} \text{dist} \left( X_M \bigoplus F_N, K^*_p \right) = \infty \]

or equivalently,

\[ \text{hd}_{M,N} (K^*_p) = \infty. \]

2. If $M = p$ then

\[ \inf_{X_p, F_N} \text{dist} \left( X_p \bigoplus F_N, K^*_p \right) = \text{dist} \left( \tilde{X}_p \bigoplus \tilde{F}_N, K^*_p \right), \]

i.e.

\[ \text{hd}_{p,N} (K^*_p) = \text{dist} \left( \tilde{X}_p \bigoplus \tilde{F}_N, K^*_p \right). \]

Remark 23 In both cases we see that the special spaces $\tilde{X}_M \bigoplus \tilde{F}_N$ are extremizers among the large class of spaces $X_M \bigoplus F_N$.

Proof. 1. If we assume that $X_M$ and $\tilde{X}_p$ are transversal the proof is clear since $\tilde{X}_p \subset K^*_p$ and there will be an infinite-dimensional subspace in $\tilde{X}_p \subset K^*_p$ containing at least one infinite axis with direction $f \in \tilde{X}_p \setminus X_M$, such that

\[ \text{dist} \left( X_M \bigoplus F_N, f \right) > 0 \]

which implies

\[ \text{dist} \left( X_M \bigoplus F_N, K^*_p \right) = \infty. \]

If they are not transversal we remind that operators with analytic coefficients satisfy the $(U)_s$ condition, and we may apply Lemma 24.

2. For proving the second item, let us first note that $\tilde{X}_p \subset X_p \bigoplus F_N$. Indeed, since $\tilde{X}_p \subset K^*_p$ the violation of $\tilde{X}_p \subset X_p \bigoplus F_N$ would imply that there exists an infinite axis $f$ in $K^*_p$ not contained in $X_p \bigoplus F_N$ which would immediately give

\[ \text{dist} \left( X_p \bigoplus F_N, K^*_p \right) = \infty. \]

Using the notations of Definition 14, there exists a finite cover $\bigcup D_j = B$, and by Lemma 27 (applied for $M = N = p$) it follows that on every subdomain $D_j$ holds $P_{2p}^{j} = C_j(x) L_{2p}$ for some function $C_j(x)$. Thus we see that every $u \in X_p$ is a piecewise solution of $L_{2p} u = 0$ on $B$, satisfying the interface conditions (32) in Definition 14. Here we use an uniqueness theorem for
“analytic continuation” across the boundary argument (proved directly by Green’s formula (24) as in [15], Lemma 20.10 and the proof of Theorem 20.11, p. 422) that \( u \in \tilde{X}_p \), hence \( X_p = \tilde{X}_p \).

Further we follow the usual way as in [25] to see that \( \tilde{F}_N \) is extremal among all finite-dimensional spaces \( F_N \), i.e.

\[
\inf_{F_N} \text{dist} \left( \tilde{X}_p \oplus F_N, K_p^* \right) = \text{dist} \left( \tilde{X}_p \oplus \tilde{F}_N, K_p^* \right).
\]

This ends the proof.

We prove the following fundamental result which shows the mutual position of two subspaces:

**Lemma 24** Assume the conditions of Theorem 19. Let the integer \( M_1 \geq 0 \). Then

\[
\text{dist} \left( X_M \oplus F_{M_1}, X_N \right) = \infty.
\]

The proof follows directly from Theorem 19 since a finite-dimensional subspace \( F_{M_1} \) would not disturb the arguments there.

We obtain immediately the following result.

**Corollary 25** Let us denote by \( U_{N+1} \) the unit ball in \( X_{N+1} \) in the \( L_2(B) \) norm. Then

\[
\text{dist} \left( X_N, U_{N+1} \right) = 1.
\]

**Remark 26** Lemma 24 and especially the above Corollary may be considered as a generalization in our setting of a theorem of Gohberg-Krein of 1957 (cf. [25], Theorem 2 on p. 137) in a Hilbert space.

We need the following intuitive result which is however not trivial.

**Lemma 27** Let for the strongly elliptic differential operators

\[
L_{2N} = P_{2N} (x; D_x)
\]

and \( P_{2M} = P_{2M} (x; D_x) \) of orders respectively \( 2N \leq 2M \) in the domain \( B \), the following inclusion hold

\[
X_N \cap H^{2M} (B) \subset X_M \setminus F,
\]

or

\[
\{ u \in H^{2M} (B) : L_{2N} u (x) = 0, \quad x \in B \} \subset \{ u \in H^{2M} (B) : P_{2M} u (x) = 0, \quad x \in B \} \setminus F,
\]

where \( F \subset L_2 (B) \) is a finite-dimensional subspace of \( L_2 (B) \). Then

\[
P_{2M} (x, D_x) = P'_{2M-2N} (x, D_x) L_{2N} (x, D_x)
\]

(42)

for some strongly elliptic differential operator \( P'_{2M-2N} \) of order \( 2M - 2N \).
Proof. It is clear that the arguments for proving equality (42) are purely local, and it suffices to consider only \( x_0 = 0 \), or we assume that the operator \( L_{2N} \) has constant coefficients.

First, we assume that the polynomial \( L_{2N} (\zeta) \) is irreducible. Then we consider the roots of the equation

\[
L_{2N} (\zeta) = 0 \quad \text{for} \quad \zeta \in \mathbb{C}^n.
\]

If \( \zeta \) is a solution to (43) then the function \( v(x) = \exp (\langle \zeta, x \rangle) \) is a solution to equation \( L_{2N} v = 0 \) in the whole space. Hence

\[
P_{2M} v = P_{2M} (x_0; D_x) v(\zeta) = 0,
\]

and by a well-known result on division of polynomials in algebra [34] (Theorem 9.7, p. 26), the statement of the theorem follows.

Now let us assume that \( L_{2N} \) is reducible and decomposed in two irreducible factors \( L_{2N} = Q_2 Q_1 \), which may be equal. Obviously, both polynomials \( Q_1 \) and \( Q_2 \) are uniformly strongly elliptic. Since the solutions to \( Q_1 u = 0 \) are also solutions to \( L_{2N} \) it follows by the above that

\[
P_{2M} (x, D_x) = P'_{2M-2N_1} (x, D_x) Q_1 (D_x)
\]

where \( 2N_1 \) is the order of the operator \( Q_1 \). Further, following the standard arguments in [24], by the uniform strong ellipticity of the operator \( Q_1 \), for every \( \zeta \in \mathbb{C}^n \), and for arbitrary \( s \geq 2N_1 \), there exists a solution \( u \in H^s (B) \) to equation

\[
Q_1 u_{\zeta} (x) = e^{\langle \zeta, x \rangle} \quad \text{for} \quad x \in B.
\]

Let \( \zeta \in \mathbb{C}^n \) be a solution to equation \( Q_2 (\zeta) = 0 \). Obviously,

\[
L_{2N} u_{\zeta} = 0
\]

hence, by the above it follows

\[
P_{2M} (x, D_x) u_{\zeta} = P'_{2M-2N_1} (x, D_x) Q_1 (D_x) u_{\zeta} = P'_{2M-2N_1} (x, \zeta) = 0.
\]

It follows that \( P'_{2M-2N_1} (x_0, \zeta) = 0 \). We proceed inductively if \( L_{2N} \) has more than two irreducible factors.

\[\blacksquare\]

7 Second Kind spaces of Harmonic Dimension \( N \) and widths

In order to make things more transparent, in Definition 14 we avoided the maximal generality of the notions and considered only First Kind spaces of
Harmonic Dimension $N$. Let us explain by analogy with the one-dimensional case how do the "Second Kind" spaces of Harmonic Dimension $N$ appear.

In the one-dimensional case, if we have a finite-dimensional subspace $X_N \subset C^N(I)$ then for a point $x_0 \in I$ the space

$$Y := \{ u \in X_N : u(x_0) = 0 \}$$

is an $(N-1)$-dimensional subspace. We would like that our notion of Harmonic Dimension $N$ behave in a similar way. For example, if $X_N$ is defined as a set of solutions of an elliptic operator $P_{2N}$ by

$$X_N := \{ u \in H^{2N}(B) : P_{2N}u = 0 \ \text{in} \ B \}$$

then it is natural to expect that the space

$$Y := \{ u \in X_N : u = 0 \ \text{on} \ \partial B \}$$

has Harmonic Dimension $N-1$. A simple example is the space

$$Y = \{ u \in H^4(B) : \Delta^2u = 0 \ \text{in} \ B, \ u = 0 \ \text{on} \ \partial B \}.$$

On the other hand, it is Theorem 19 and Corollary 20 above which show that such Second Kind spaces of Harmonic Dimension $N$ appear in a natural way when we consider the space $X_N \ominus X_M$ based on solutions of Elliptic BVP (37)-(39).

We give the following definition.

**Definition 28** For an integer $M \geq 1$ we say that the linear subspace $X_M \subset L^2(D)$ is of Second Kind and has **Harmonic Dimension** $M$, and write

$$\text{hdim } (X_M) = M,$$

if it satisfies all conditions of Definition 14 however with an elliptic operator $P_{2N}$, with $N \geq M$ and all elements $u \in X_M$ satisfy $N - M$ boundary conditions

$$B_j u = 0 \quad \text{on} \ \partial D, \ j = 1, 2, \ldots, N - M.$$

Here the boundary operators $\{B_j\}_{j=1}^{N-M}$ are a **normal system** of boundary operators defined on $\partial D$, by Definition 8, item 4).

By a technique similar to the already used we may prove the following results which generalize Theorem 22. We assume that $K_p^*$ is the set defined by (27) with a strongly elliptic constant coefficients operator $L_{2p}$. The space $\tilde{X}_p$ is defined by (40) and the space $\tilde{F}_L$ by (41).

The following theorem is a generalization of item 1) in Theorem 22.
Theorem 29 Let $M < p$ and $L \geq 0$ be arbitrary integer. Let $X_M$ be a Second Kind space with Harmonic Dimension $N$, i.e.

$$\text{hdim}(X_M) = M.$$ 

Let $F_L$ be an $L$–dimensional subset of $L_2(B)$. Then

$$\text{dist} \left( X_M \bigoplus F_L, K_p^* \right) = \infty.$$ 

The proof of Theorem 29 follows with minor modifications of Lemma 24 (Theorem 19).

It is more non-trivial to consider the case $N = p$. First we must prove the following result.

Lemma 30 Let $X_p$ be a Second Kind space of Harmonic Dimension $p$ and $L \geq 0$ be an arbitrary integer. Let $F_L$ be an $L$–dimensional subset of $L_2(B)$. Then

$$\text{dist} \left( X_p \bigoplus F_L, K_p^* \right) < \infty$$

implies

$\tilde{X}_p \subset X_p$. (44)

Let the elliptic operator $P_{2M}$ and the boundary operators $\{B_j\}_{j=1}^{M-p}$ be associated with $X_p$ by Definition 28. Then (44) implies the following factorizations:

$$P_{2M} = P'_{2M-2p} L_{2p}$$

$$B_j = B'_j L_{2p} \quad \text{for } j = 1, 2, ..., M - p.$$ 

The operator $P'_{2M-2p}$ is uniformly strongly elliptic in $D$, and the boundary operators $\{B'_j\}_{j=1}^{M-p}$ form a normal system which covers the operator $P'_{2M-2p}$.

Finally, the following generalization of item 2) in Theorem 22 may be proved. It shows that one needs to take into account the index of the Elliptic BVP involved.

Theorem 31 Let us consider those spaces $X_p$ of Second Kind with Harmonic Dimension $p$ for which

$$\text{dist} \left( X_p \bigoplus F_L, K_p^* \right) < \infty$$

with associated operators $P_{2M}$ and boundary operators $\{B_j\}_{j=1}^{M-p}$. Following the notations of Lemma 30, let us denote by $\mathcal{N}$ the following space of solutions $w \in H^{2M-2p}(D)$ of the Elliptic BVP on the domain $D$:

$$P'_{2M-2p} w = 0 \quad \text{on } D$$

$$B'_j w = 0, \quad \text{on } \partial D, \quad \text{for } j = 1, 2, ..., M - p$$

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Then the following equality holds

\[
\inf_{X_p, F_L} \{ \text{dist} (X_p \bigoplus F_L, K_p^*) : \dim (\mathcal{N}) + L = L_1 \} = \text{dist} \left( \tilde{X}_p \bigoplus \tilde{F}_{L_1}, K_p^* \right).
\]

From the theory of Elliptic BVP is known that \( \dim (\mathcal{N}) < \infty \) (cf. [24], Theorem 5.3, chapter 2, section 5.3). Let us denote by \( \{ w_s \}_{s=1}^{\dim(\mathcal{N})} \) a basis of the space \( \mathcal{N} \), and by \( u_s \) a fixed solution to \( L_2 p u_s = w_s \). The main point in the proof of Theorem 31 is that arbitrary solution \( u \) to equation \( P_{2M} u = 0 \) may be expressed as

\[
u = \sum_{s=1}^{\dim(\mathcal{N})} \lambda_s u_s + v
\]

where \( v \) is a solution to \( L_2 p v = 0 \).

8 Appendix, Proof of Theorem 10

**Proof.** (1) We consider the following auxiliary elliptic eigenvalue problem

\[
L_2 p L_2^* \phi (x) = \lambda \phi (x) \quad \text{on } B, \tag{45}
\]

\[
B_j \phi (y) = S_j \phi (y) = 0 \quad \text{for } j = 0, 1, ..., p - 1, \text{ for } y \in \partial B. \tag{46}
\]

Since this is the Dirichlet problem for the operator \( L_2 p L_2^* \), it is a classical fact that (45)-(46) is a regular Elliptic BVP considered in the Sobolev space \( H^{2p} (B) \), as defined in Definition 8. Also, it is a classical fact that the Dirichlet problem is a self-adjoint problem (cf. [24], Remark 2.4 in section 2.4 and Remark 2.6 in section 2.5, chapter 2).

Hence, we may apply the main results about the Spectral theory of regular self-adjoint Elliptic BVP. We refer to [11] (section 3 in chapter 2, p. 122, Theorem 2.52) and to references therein.

By the uniqueness Lemma 32 the eigenvalue problem (45)-(46) has only zero solution for \( \lambda = 0 \). It has eigenfunctions \( \phi_k \in H^{2p} (B) \) with eigenvalues \( \lambda_k > 0 \) for \( k = 1, 2, 3, ... \) for which \( \lambda_k \to \infty \) as \( k \to \infty \).

(2) Next, in the Sobolev space \( H^{2p} (B) \), we consider the problem:

\[
L_2 p L_2^* \varphi (x) = \phi_k (x) \quad \text{on } B, \tag{47}
\]

\[
B_j \varphi (y) = S_j \varphi (y) = 0 \quad \text{for } j = 0, 1, ..., p - 1, \text{ for } y \in \partial B. \tag{48}
\]

Obviously, the Elliptic BVP defined by problem (47)-(48) coincides with the Elliptic BVP defined by (45)-(46) up to the right-hand sides, and all
remains there hold as well. Hence, problem (47)-(48) has unique solution $\varphi_k \in H^{2p}(B)$. We put
\[ \psi_k = L_{2p}^* \varphi_k. \]
Hence, $L_{2p} \psi_k = \phi_k$. We infer that on the boundary $\partial B$ hold the equalities $B_j L_{2p} \psi_k = B_j \phi_k$ and $S_j L_{2p} \psi_k = S_j \phi_k$; since $\phi_k$ are solutions to (45)-(46) it follows
\[ B_j L_{2p} \psi_k (y) = S_j L_{2p} \psi_k (y) = 0 \quad \text{for} \quad j = 0, 1, \ldots, p-1, \quad \text{for} \quad y \in \partial B. \quad (49) \]

We will prove that $\psi_k$ are solutions to problem (25)-(26), they are mutually orthogonal, and they are also orthogonal to the space $\{ v \in H^{2p} : L_{2p} v = 0 \}$. [1]

(3) Let us see that
\[ L_{2p}^* L_{2p} \psi_k = \lambda_k \psi_k. \]
By the definition of $\psi_k$ this is equivalent to
\[ L_{2p}^* L_{2p} L_{2p}^* \varphi_k = \lambda_k L_{2p}^* \varphi_k; \]
from $L_{2p} L_{2p}^* \varphi_k = \phi_k$ this is equivalent to
\[ L_{2p}^* \phi_k = \lambda_k L_{2p}^* \varphi_k \]
On the other hand, by the basic properties of $\phi_k$ and $\varphi_k$, we have obviously $L_{2p} L_{2p}^* \phi_k = \lambda_k L_{2p} L_{2p}^* \varphi_k$, hence
\[ L_{2p} L_{2p}^* (\phi_k - \lambda_k \varphi_k) = 0. \]
Note that both $\phi_k$ and $\varphi_k$ satisfy the same zero Dirichlet boundary conditions, namely (46) and (48). Hence, by the uniqueness Lemma 32 it follows that $\phi_k - \lambda_k \varphi_k = 0$ which implies $L_{2p}^* L_{2p} \psi_k = \lambda_k \psi_k$. Thus we see that $\psi_k$ is a solution to problem (25)-(26) and does not satisfy $L_{2p} \psi = 0$.

(4) The orthogonality to the subspace $\{ v \in H^{2p} : L_{2p} v = 0 \}$ follows easily from the Green formula (24) applied to the operator $L_{2p}^* L_{2p}$,
\[
\int_D \left( L_{2p}^* L_{2p} \psi_k \cdot v - L_{2p} \psi_k \cdot L_{2p} v \right) dx \\
= \sum_{j=0}^{2p-1} \int_{\partial D} \left( S_j L_{2p} \psi_k \cdot C_j v - B_j L_{2p} \psi_k \cdot T_j v \right)
\]
in which substitute the zero boundary conditions (49) of $\psi_k$, and equality
\[
\int_D L_{2p}^* L_{2p} \psi_k \cdot v dx = \lambda_k \int_D \psi_k \cdot v dx.
\]

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The orthonormality of the system $\{\psi_k\}_{k=1}^\infty$ follows now easily by the equality
\[ \lambda_k \int \psi_k \psi_j dx = \int L^*_{2p} L_{2p} \psi_k \psi_j dx = \int L_{2p} \psi_k L_{2p} \psi_j dx = \int \phi_k \phi_j dx \]
and the orthogonality of the system $\{\phi_k\}_{k=1}^\infty$.

(5) For the completeness of the system $\{\psi_k\}_{k=1}^\infty$, let us assume that for some $f \in L_2(B)$ holds
\[ \int_B f \cdot \psi_k dx = \int_B f \cdot \psi'_k dx = 0 \quad \text{for all } k \geq 1. \] (50)
Then the Green formula (24) implies
\[ 0 = \lambda_k \int_B f \cdot \psi_k dx = \int_B f \cdot L^*_{2p} L_{2p} \psi_k dx = \int_B L_{2p} f \cdot L_{2p} \psi_k dx \]
\[ = \int_B L_{2p} f \cdot \phi_k dx \quad \text{for all } k \geq 1. \]
By the completeness of the system $\{\phi_k\}_{k=1}^\infty$, this implies that $L_{2p} f = 0$. From the second orthogonality in (50) follows that $f \equiv 0$, and this ends the proof of the completeness of the system $\{\psi'_j\}_{j=1}^\infty \cup \{\psi_j\}_{j=1}^\infty$.

■

We have used above the following simple result.

**Lemma 32** The solution to problem (45)-(46) for $\lambda = 0$ is unique.

**Proof.** From Green’s formula (24) we obtain
\[ \int_B [L_{2p} \phi]^2 dx - \int \phi \cdot L^*_{2p} L_{2p} \phi dx = \sum_{j=1}^P \int_{\partial B} (S_j \phi \cdot C_j L_{2p} \phi - B_j \phi \cdot T_j L_{2p} \phi) \, d\sigma_y, \]
hence $L_{2p} \phi = 0$.

Now for arbitrary $v \in H^{2p}(B)$ by the same Green’s formula we obtain
\[ \int_B (L_{2p} \phi \cdot v - \phi \cdot L^*_{2p} v) dx = \sum_{j=1}^P \int_{\partial B} (S_j \phi \cdot C_j v - B_j \phi \cdot T_j v) \, d\sigma_y = 0, \]
hence
\[ \int_B \phi \cdot L^*_{2p} v dx = 0. \]
From the local existence theorem for elliptic operators (cf. [24]) it follows that for arbitrary $f \in L_2(B)$ we may solve the elliptic equation $L^*_{2p} v = f$ with $v \in H^{2p}(B)$. From the density of $H^{2p}(B)$ in $L_2(B)$ we infer $\phi \equiv 0$.

This ends the proof.

■

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9 Conclusion and open problems

As in Approximation, Spline and Wavelet Theory ([18], [15]), in the present research solutions of higher order elliptic equation have shown flexibility which enabled a natural multidimensional generalization of Kolmogorov’s theory of widths with successful application to multidimensional sets $K_p^*$. Also, new features of Jackson type theorems have been disclosed in Theorem 22, which shows that one needs components of different dimensions: $X_p$ and $F_L$ are of different types.

It may come as a big surprise, but the present research shows this unambiguously, that in many issues one has to give up the convenient simplistic understanding of the multidimensional case, in particular by realizing that the finite-dimensional subspaces in $C^N(D)$, for domains $D \subset \mathbb{R}^n$ for $n \geq 2$, do not serve the same job as the finite-dimensional subspaces in $C^N(D)$ for intervals $D \subset \mathbb{R}^1$, and one has to replace them by a lot more sophisticated objects, namely by the spaces having Harmonic Dimension $N$. This is the main conclusion of the present research based on the successful application of the new Harmonic Widths to explaining the structure of the sets $K_p^*$.

Beyond the motivational problems mentioned in the Introduction, one may formulate several other open problems:

1. First of all, one has to study basic questions about the sets having Harmonic Dimension, by considering the sets $X_M \bigcap X_N$, $X_M \bigoplus X_N$, $X_M \bigotimes X_N$, and finding their Harmonic Dimension (if it exists!), etc.

2. Secondly, one has to prove a generalization of a theorem of S. Bernstein about differentiable Markov systems (or, in the case of differentiability, Extended Complete Chebyshev systems, in the terminology of [13]). As remarked in [23] (after the proof of Theorem 4.2 in chapter 2), S. Bernstein dealt with even stronger statement, namely, he was seeking Descartes systems (cf. [23]). This needs the factorization of elliptic PDOs into $N$ elliptic operators of second order. These operators will be obviously pseudo-differential, [12].

3. In this context, one has to check that the maximal generality of the theory in the present paper will be achieved by considering elliptic pseudo-differential operators.

4. New Jackson type theorems are suggested by the widths reasons: the simplest way to state them is to consider spaces defined by

$$\left\{ u : |L_{2p}u(x)| \leq 1 \quad \text{for } x \in D \right\}.$$
By arguments similar to the proof of Theorem 22 one is convinced that a reasonable Jackson type theorem may be proved only for operators $P_{2N}$ of the form $P_{2N} = P_{2N-2p}^* L_{2p}$, i.e. one has to approximate through functions $u_N$ in the spaces $\{ u : P_{2N} u = 0 \text{ in } D \}$. In the case of polyharmonic operator Jackson type results have been proved in [17].

5. One has to find a proper discrete version of the present research which will be essential for the applications to Compressed Sensing, compare the role of Gelfand’s widths in [9], [8].

6. Although we have mentioned the Chebyshev systems in passing, an important point of the present research is the generalization of the Extended Complete Chebyshev systems of order $N$ (discussed in more detail in [16]) which is the ground for the spaces having Harmonic Dimension $N$. One has to specify more precisely which are the elliptic differential/pseudodifferential operators acceptable for a Multidimensional Chebyshev system. This has to be considered in the context of S. Bernstein’s one-dimensional result, mentioned in Proposition 1.

7. In the same direction, let us recall that one-dimensional Chebyshev systems are important for the qualitative theory of ODEs, in particular for Sturmian type of theorems, cf. e.g. [2], [3]. There has been a long search for proper multidimensional generalizations of Chebyshev systems. The standard generalization by means of zero set property fails to produce a non-trivial multidimensional system and this is the content of the theorem of Mairhuber, cf. the thorough discussion in [23] (chapter 2, section 1.1). In general, zero set properties and intersections are not a reliable reference point for multidimensional Analysis. Indeed, let us recall that polyharmonic (and even harmonic) functions do not have simple zero sets, however they are solutions to nice Dirichlet problems (13) and for that reason are considered to be a genuine Multidimensional Chebyshev system as we have defined it in (12).

V.I. Arnold discusses the importance of the Chebyshev systems in his Toronto lectures, June 1997, Lecture 3: Topological Problems in Wave Propagation Theory and Topological Economy Principle in Algebraic Geometry. Fields Institute Communications, available online at http://www.pdmi.ras.ru/~arnsem/Arnold/arn-papers.html. On p. 8 he writes that "Even the Sturm theory is missing in higher dimensions. This is an interesting phenomenon. All attempts that I know to extend Sturm theory to higher dimensions failed. For instance, you can find such an attempt in the Courant-Hilbert’s book, in chapter 6,
but it is wrong. The topological theorems about zeros of linear combinations for higher dimensions, which are attributed there to Herman, are wrong even for the standard spherical Laplacian." The attempts to mimic multivariate Chebyshev systems are present in the works of V.I. Arnold in the context of multivariate Sturm type of theorems, see in particular problem 1996-5 in [3]. In view of these efforts of V.I. Arnold, one might try to apply the present framework for obtaining multidimensional Sturm type theorems.

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