LOCAL HOLOMORPHIC MAPPINGS RESPECTING HOMOGENEOUS SUBSPACES ON RATIONAL HOMOGENEOUS SPACES

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ABSTRACT. Let $G/P$ be a rational homogeneous space (not necessarily irreducible) and $x_0 \in G/P$ be the point at which the isotropy group is $P$. The $G$-translates of the orbit $Qx_0$ of a parabolic subgroup $Q \subseteq G$ such that $P \cap Q$ is parabolic are called $Q$-cycles. We established an extension theorem for local biholomorphisms on $G/P$ that map local pieces of $Q$-cycles into $Q$-cycles. We showed that such maps extend to global biholomorphisms of $G/P$ if $G/P$ is $Q$-cycle-connected, or equivalently, if there does not exist a non-trivial parabolic subgroup containing $P$ and $Q$. Then we applied this to the study of local biholomorphisms preserving the real group orbits on $G/P$ and showed that such a map extend to a global biholomorphism if the real group orbit admits a non-trivial holomorphic cover by the $Q$-cycles. The non-closed boundary orbits of a bounded symmetric domain embedded in its compact dual are examples of such real group orbits. Finally, using the results of Mok-Zhang on Schubert rigidity, we also established a Cartan-Fubini type extension theorem pertaining to $Q$-cycles, saying that if a local biholomorphism preserves the variety of tangent spaces of $Q$-cycles, then it extends to a global biholomorphism when the $Q$-cycles are positive dimensional and $G/P$ is of Picard number 1. This generalizes a well-known theorem of Hwang-Mok on minimal rational curves.

1. Introduction

1.1. Extension theorem for $Q$-cycles-respecting maps.

Starting from the late 90s, Hwang and Mok has developed a theory to study Fano manifolds of Picard number 1 using their rational curves. A number of difficult problems in Algebraic Geometry have been solved using the theory, for example, those related to deformation rigidity of rational homogeneous spaces [10, 13], Lazarsfeld’s problem [11] and target rigidity [6]. The theory is based on a special kind of rational curves, called minimal rational curves, in which the word “minimal” means the degree of the rational curve with respect to some choice of ample line bundle is minimal among free rational curves. One of the basic themes of the theory is the rigidity phenomena of the holomorphic
mappings that interact nicely with minimal rational curves. To make things more concrete, it suffices for us to mention the so-called Cartan-Fubini type extension, which is a key ingredient in Hwang-Mok’s theory and its application to geometric problems. In the case of rational homogeneous spaces of Picard number 1 other than $\mathbb{P}^n$, an equidimensional version of the extension says that if a local biholomorphism preserves the set of tangent directions of minimal rational curves, then it extends to a global biholomorphism. There are generalizations to general uniruled projective manifolds [12] and also non-equidimensional situations [8, 14].

Roughly speaking, there are two components in the proofs for Cartan-Fubini type extensions mentioned above. Firstly, one proves that if a local holomorphic map preserves the set of tangent directions of minimal rational curves, then it actually maps pieces of minimal rational curves into minimal rational curves. We will say that such maps respect minimal rational curves. Then one proves an extension statement for the maps that respect minimal rational curves. In this article, we are going to study such extension phenomena for local biholomorphisms that respect homogeneous submanifolds (or cycles) of arbitrary dimension, in the case of rational homogeneous spaces of arbitrary Picard number. For maps that are only known to respect tangencies, our results together with those from the study of Schubert rigidity will also lead to rigidity statements for spaces of Picard number 1, which are analogous to the usual Cartan-Fubini type extensions. This will be addressed at the end of this section. Before stating our main results, we would like to point out that this kind of cycle-respecting properties appear naturally and ubiquitously in some areas in Several Complex Variables, e.g. in the study of proper holomorphic mappings among symmetric domains and holomorphic mappings that preserves real group orbits on rational homogeneous spaces. For example, since the work of Mok-Tsai [21] it is now well known that any proper holomorphic map between bounded symmetric domains respects symmetric subspaces of a particular kind; and in [23] it was discovered that in many cases a holomorphic mappings preserving the closed orbits of $SU(p, q)$ on Grassmannians respect certain subgrassmannians.

From now on, we let $G/P$ be a rational homogeneous space, where $G$ is a semisimple complex Lie group and $P \subset G$ is a parabolic subgroup. Let $x \in G/P$ be the base point at which the isotropy group is $P$ and let $Q \subset G$ be another parabolic subgroup such that $P \cap Q$ is parabolic. Then the orbit $Qx$ of $Q$ (under canonical left action) is a rational homogeneous subspace of $G/P$. This orbit and any of its translations by elements of $G$ will be called a $Q$-cycle. A $Q$-cycle is non-trivial (not a point nor equal to $G/P$) if $Q \not\subset P$ and $Q \not= G$. We remark that when $G/P$ is of Picard number 1 and is associated to a long root, then the minimal rational curves on $G/P$ are the $Q$-cycles for some $Q \subset G$. In the Cartan-Fubini type extensions established in the literature for minimal rational curves, the condition on Picard number (equal to 1) is always assumed for the source manifold, because otherwise two general points in the source manifold
cannot be connected by a chain of minimal rational curves, which is certainly an
obstruction for extension. In our situation regarding the $Q$-cycles in $G/P$, we
say that $G/P$ is $Q$-cycle-connected if any two points can be connected by a chain
of $Q$-cycles and we have the following simple criterion for $Q$-cycle-connectivity
which is very easy to check on the Dynkin diagram:

**$Q$-cycle-connectivity.** (see Proposition 2.6) $G/P$ is $Q$-cycle-connected if and
only if there does not exist any parabolic subgroup $R \subseteq G$ containing $P$ and $Q$,
or equivalently, when the set of marked nodes on the Dynkin diagram associated
to $P$ and $Q$ are disjoint.

A local biholomorphism defined on $G/P$ will be said to be $Q$-cycle-respecting
if it maps pieces of $Q$-cycles into $Q$-cycles (for the precise definition, see Sec-
tion 2.4). Now we are able to state our main result regarding the extension of
the germs of such maps:

**Theorem 1.1.** Let $G/P$ be a rational homogeneous space and $Q \subseteq G$ be a
parabolic subgroup such that $P \cap Q$ is parabolic. If $G/P$ is $Q$-cycle-connected,
or equivalently, if there does not exist any parabolic subgroup $R \subseteq G contain-
ing P$ and $Q$, then every germ of $Q$-cycle-respecting map on $G/P$ extends to a
biholomorphism of $G/P$.

Comparing with the previous works on Cartan-Fubini type ext ension, we first
of all do not need to impose the Picard number 1 condition and in particu-
lar, our theorem applies also to reducible rational homogeneous spaces. As
mentioned, the usual Picard number 1 condition was assumed in order to get
cycle-connectivity by minimal rational curves. Secondly, in these studies, af-
ter establishing the algebraic extension of the concerned local map, the Picard
number 1 condition and the fact that the cycles are one-dimensional had also
been used in an essential way to prove the univalence and holomorphicity of
the extension. Here in our theorem, the cycle-connectivity is taken as the only
hypothesis, which has an equivalent group-theoretic condition and can be easily
verified.

The novelty of our method is that instead of establishing the extension of
a $Q$-cycle-respecting germ $f$ directly on $G/P$ (or on the graph of $f$), in which
one usually only obtains meromorphic maps, we use the germ $f$ to construct
a sequence of holomorphic maps $\{F_k\}$ from a sequence of projective manifolds
$\{T^k_{x_0}\}$, called the $Q$-towers, to the target space. We want to emphasize that the
maps $\{F_k\}$ are holomorphic by construction and thus we do not need to deal with
indeterminacies nor essential singularities of meromorphic maps. Geometrically,
the $Q$-towers can be interpreted as some sort of universal families of the chains
of $Q$-cycles emanating from $x_0$ and there are evaluation maps $p_k : T^k_{x_0} \to G/P$
sending the towers to their images in $G/P$ (see [22] for the case of rational
curves). The $Q$-cycle-connectivity hypothesis then implies that for some positive
integer $N$, the evaluation map $p^N_{x_0} : T^N_{x_0} \to G/P$ is surjective. The second part
of our proof is then to show that \( F_N \) actually descends to a holomorphic map \( F : G/P \rightarrow G/P \), by exploiting the theory of \( P \)-action on \( G/P \).

1.2. Local biholomorphisms preserving real group orbits on \( G/P \).

We then apply Theorem 1.1 to the study of holomorphic maps pertaining to the real group orbits on \( G/P \). We recall first of all that a real Lie subgroup \( G_0 \subset G \) is called a real form of \( G \) if the complexification of its Lie algebra \( \mathfrak{g}_0 \) is the Lie algebra \( \mathfrak{g} \) of \( G \). Wolf has laid down the foundation of the action of \( G_0 \) on \( G/P \) in [25]. It is now well known that \( G_0 \) has only a finite number of orbits on \( G/P \), which will be called real group orbits, including in particular open orbit(s) and a unique closed orbit. Any open real group orbit is also called a flag domain in the literature. The bounded symmetric domains embedded in their compact duals are the most well known examples of flag domains. In Several Complex Variables, the study of proper holomorphic maps on domains is a classical topic and one is naturally led to consider local holomorphic maps that respect the boundary structures of the domains. In the case of open real group orbits (i.e. flag domains), their boundary is again a union of real group orbits which are also homogeneous CR submanifolds in \( G/P \). Thus, local holomorphic maps preserving real group orbits (or CR maps between real group orbits) arise very naturally in Several Complex Variables.

Our key observation here is that, in many cases a real group orbit \( O \subset G/P \) can be covered by a family of \( Q \)-cycles in a very special way that \( O \) and these \( Q \)-cycles are “tangled” under the holomorphic mappings from \( O \) or into \( O \). For the precise definition, see Definition 4.3. In such cases, we will say that \( O \) has a holomorphic cover of \( Q \)-type and the cover is said to be non-trivial if the covering \( Q \)-cycles are neither points nor the entire \( G/P \). Here we simply remark that the non-closed boundary orbits of a bounded symmetric domain embedded in its compact dual are examples of real group orbits having such kind of non-trivial covers. For more examples, see Example 4.6. Our main result for the real group orbits on a rational homogeneous space \( G/P \) can now be stated as follows.

**Theorem 1.2.** Let \( O \) be a real group orbit on \( G/P \) and \( U \subset G/P \) be a connected open set such that \( U \cap O \neq \emptyset \). Suppose \( O \) has a non-trivial holomorphic cover of \( Q \)-type for some parabolic subgroup \( Q \subset G \) and \( G/P \) is \( Q \)-cycle-connected. If \( f : U \rightarrow f(U) \subset G/P \) is a biholomorphism such that \( f(U \cap O) \subset O \), then \( f \) extends to a biholomorphism of \( G/P \).

As an illustration of Theorem 1.2, we specialize it for the case of bounded symmetric domains (which are allowed to be reducible):

**Corollary 1.3.** Let \( M \) be a compact Hermitian symmetric space and \( \Omega \subset M \) be the Borel embedding of the bounded symmetric domain \( \Omega \) dual to \( M \). Let \( G_0 := \text{Aut}(\Omega) \subset \text{Aut}(M) \) and \( O \subset \partial \Omega \) be \( G_0 \)-orbit which is neither open nor closed, and \( U \subset M \) be a connected open set such that \( U \cap O \neq \emptyset \). If \( f : U \rightarrow \)
\( f(U) \subset M \) is a biholomorphism such that \( f(U \cap O) \subset O \), then \( f \) extends to a biholomorphism of \( M \).

The closed \( G_0 \)-orbit in \( \partial \Omega \) is precisely the Shilov boundary of \( \Omega \) and it is a classical result of Alexander \cite{1} and Khenkin-Tumanov \cite{17} that the same extension holds in this case for irreducible bounded symmetric domains of at least dimension 2. For certain \( G_0 \)-orbits in a compact Hermitian symmetric space, similar extensions have also been established in the works including \cite{15}, \cite{16}, \cite{18}, \cite{20}.

In Theorem \ref{thm:cycle_connectivity}, if \( G/P \) is of Picard number 1, then cycle connectivity is automatic whenever the \( Q \)-cycles are non-trivial, so we have

**Corollary 1.4.** Let \( O \) be a real group orbit on \( G/P \) of Picard number 1 and \( U \subset G/P \) be a connected open set such that \( U \cap O \neq \emptyset \). Suppose \( O \) has a holomorphic cover of \( Q \)-type for some parabolic subgroup \( Q \subset G \) and \( Q \not\subset P \). If \( f : U \to f(U) \subset G/P \) is a biholomorphism such that \( f(U \cap O) \subset O \), then \( f \) extends to a biholomorphism of \( G/P \).

Corollary 1.4 covers in particular the closed \( SU(p,q) \)-orbits on the Grassmannian \( Gr(d, \mathbb{C}^{p+q}) \), where \( d < \min(p,q) \). These orbits include the boundaries of the so-called generalized balls, which have been studied in \cite{2}, \cite{23}, etc.

### 1.3. Cartan-Fubini type extension for \( Q \)-cycles.

As mentioned at the beginning of the introduction, the original Cartan-Fubini type extensions for minimal rational curves are proven for local holomorphic maps which are only assumed to preserve the tangent directions of minimal rational curves. Proving such maps actually respect minimal rational curves is a non-trivial part of the extension. Now if minimal rational curves are replaced by more general \( Q \)-cycles, we apply the results obtained by Mok-Zhang \cite{22} related to the so-called Schubert rigidity on rational homogeneous spaces (see \cite{22} for more background), which allow us to show that the preservation of tangencies imply the \( Q \)-cycle-respecting property. The detail will be given in Section 5 and the following extension is what we can obtain for \( Q \)-cycles, which parallels the usual Cartan-Fubini type extension for minimal rational curves.

**Theorem 1.5.** Let \( G/P \) be a rational homogeneous space of Picard number 1 and \( Q \subset G \) be a parabolic subgroup such that \( Q \not\subset P \) and \( P \cap Q \) is parabolic. If \( U \subset G/P \) is a connected open set and \( f : U \to f(U) \subset G/P \) is a biholomorphism such that it sends the tangent space of any \( Q \)-cycle to the tangent space of some \( Q \)-cycle, then \( f \) extends to a biholomorphism of \( G/P \).

## 2. \( Q \)-cycles, \( Q \)-towers and sheaf of \( Q \)-cycle-respecting maps

### 2.1. \( Q \)-cycles on \( G/P \).
Let $P, Q \subset G$ be parabolic subgroups of a complex simple Lie group $G$ such that $P \cap Q$ is parabolic. Then we have a double fibration

$$
\begin{align*}
\xymatrix{ & G/(P \cap Q) \ar[dl]_q \ar[dr]^p & \\
G/Q & & G/P
}\end{align*}
$$

Through the double fibration, any point $s \in G/Q$ defines a rational homogeneous subspace $\mathcal{P}_s := p(\mathcal{q}^{-1}(s))$ in $G/P$, which we will call a $Q$-cycle on $G/P$. If we consider the canonical left action of $Q$ on $G/P$, then any $Q$-cycle is just a $G$-translate of the $Q$-orbit $Qx_0 \cong Q/(P \cap Q)$, where $x_0 \in G/P$ is the point where $P$ is the isotropy group. Similarly, any point $x \in G/P$ gives a rational homogeneous subspace $\mathcal{Q}_x := q(\mathcal{p}^{-1}(x))$ in $G/Q$, called a $P$-cycle on $G/Q$.

**Remark.** We have allowed the cycles to be zero-dimensional but only the positive dimensional ones are relevant in the current article.

**Definition 2.1.** Let $k \in \mathbb{N}$. We call a $k$-tuple $(\mathcal{P}_{s_1}, \ldots, \mathcal{P}_{s_k})$ a chain of $Q$-cycles of length $k$, or simply a $k$-chain of $Q$-cycles, if $\mathcal{P}_{s_i} \cap \mathcal{P}_{s_{i+1}} \neq \emptyset$ for $1 \leq i \leq k - 1$.

Since $P \cap Q$ is parabolic, there exists a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a choice of simple roots $\Delta$ for $\mathfrak{h}$ such that $P, Q$ correspond to two different markings on the Dynkin diagram $\mathcal{D}(G)$ of $G$. Let $\psi_P$ and $\psi_Q$ be the set of marked nodes associated to $P, Q$ respectively. For a subset $\chi$ of $\Delta$ we say that $\chi$ separates $\psi_P$ and $\psi_Q$ if any connected subdiagram of the Dynkin diagram $\mathcal{D}(G)$ of $G$ containing both a node in $\psi_P$ and a node in $\psi_Q$ also contains a node in $\chi$. The smallest subset $\psi'_Q$ of $\psi_Q$ which separates $\psi_P$ and $\psi_Q$ is called the reduction of $\psi_Q$ mod $\psi_P$. The parabolic subgroup $Q'$ corresponding to $\psi'_Q$ is called the reduction of $Q$ mod $P$.

More generally, let $G$ be a complex semisimple Lie group having a decomposition $G = G_1 \times \cdots \times G_m$ into simple factors and $P, Q \subset G$ be parabolic subgroups such that $P \cap Q$ is parabolic. Then we can write $P = P_1 \times \cdots \times P_m$ and $Q = Q_1 \times \cdots \times Q_m$ such that $P_k, Q_k$ and $P_k \cap Q_k$ are parabolic subgroups of $G_k$ for every $k$. In this case, if $Q'_k$ is the reduction of $Q_k$ mod $P_k$ for all $k$, then $Q' := Q'_1 \times \cdots \times Q'_m$ is called the reduction of $Q$ mod $P$. We say that $Q$ is reduced mod $P$ if $Q = Q'$.

**Proposition 2.2 (26, Corollary of Theorem 3).** Let $G$ be a complex semisimple Lie group which is a direct product of simple factors. Let $P, Q \subset G$ be parabolic subgroups such that $P \cap Q$ is parabolic and let $Q'$ be the reduction of $Q$ mod $P$. The moduli space of $Q$-cycles on $G/P$ is $G/Q'$. More precisely, if we let $x_0 \in G/P$ be the point whose isotropy group is $P$, then we have $Q' = \{g \in G : gQx_0 = Qx_0\}$. 
If $Q$ is reduced mod $P$ and vice versa, we have the following interpretation of $G/(P \cap Q)$ which will be important for establishing our extension theorems. In what follows, we will denote by $Gr(d,T(M))$ the Grassmannian bundle of $d$-dimensional holomorphic tangent subspaces of a complex manifold $M$. In addition, for the sake of making a simpler statement, we take $Gr(0,T(M)) := M$.

**Proposition 2.3.** Let $G$ be a complex semisimple Lie group which is a direct product of simple factors. Let $P, Q \subset G$ be parabolic subgroups such that $P \cap Q$ is parabolic. Suppose $Q$ is reduced mod $P$ (resp. $P$ is reduced mod $Q$). Then $G/(P \cap Q)$ can be holomorphically embedded as a closed complex submanifold in $Gr(k,T(G/P))$, where $k = \dim\mathbb{C}(\mathcal{P}_s)$ for any $s \in G/Q$ (resp. $Gr(\ell,T(G/Q))$, where $\ell = \dim\mathbb{C}(Q_x)$ for any $x \in G/P$).

**Proof.** Suppose $Q$ is reduced mod $P$. We just need to consider the non-trivial cases where $\dim\mathbb{C}(\mathcal{P}_s) = k > 0$. We can regard $G/(P \cap Q)$ as the universal family of $Q$-cycles on $G/P$. Let $\zeta \in G/(P \cap Q)$ and $x := p(\zeta)$, $s := q(\zeta)$. Then by sending $\zeta$ to the holomorphic tangent space of $\mathcal{P}_s$ at $x$, we get a holomorphic map $h : G/(P \cap Q) \to Gr(k,T(G/P))$. Note that the $Q$-cycles are compact and the moduli of $Q$-cycles passing through a given point in $G/P$, which is just a $P$-cycle in $G/Q$ is also compact. It then follows that the holomorphic tangent space of a $Q$-cycle at any of its point already determines the cycle itself. (See for example [9], Lemma 3.4 for a proof of this.) Thus, $h$ is a holomorphic embedding. Similarly, $G/(P \cap Q)$ can be embedded in $Gr(\ell,T(G/Q))$ if $P$ is reduced mod $Q$. \qed

### 2.2. $Q$-towers on $G/P$.

We are now going to construct some projective manifolds, called the $Q$-towers, associated to $G/P$ which are essentially some kind of universal families of the $k$-chains of $Q$-cycles. In order to simplify the notations, we write $\mathcal{U} := G/(P \cap Q)$ and denote the canonical projections by $p : \mathcal{U} \to G/P$ and $q : \mathcal{U} \to G/Q$. We will define the $Q$-towers recursively. First of all, let the restriction of $q$ to $p^{-1}(x_0)$ be $q_1$. Consider the pullback of the bundle $q : \mathcal{U} \to G/Q$ by $q_1$ and denote it by $\tilde{q}_1 : q_1^*\mathcal{U} \to p^{-1}(x_0)$. Associated to the pullback, there is a map $\tilde{q}_1 : q_1^*\mathcal{U} \to \mathcal{U}$ such that $q \circ \tilde{q}_1 = q_1 \circ q_1$. To summarize, we have now

$$
\begin{array}{ccc}
q_1^*\mathcal{U} & \xrightarrow{\tilde{q}_1} & \mathcal{U} \\
\downarrow{q_1} & & \downarrow{q} \\
p^{-1}(x_0) & \xrightarrow{q_1} & G/Q \\
\end{array}
$$

We call $T^1_{x_0} := q_1^*\mathcal{U}$ the first $Q$-tower at $x_0$. Let $p_1 := p \circ \tilde{q}_1$. Geometrically, the image $p_1(T^1_{x_0}) \subset G/P$ is the union of all the $Q$-cycles passing through $x_0$ and $p_1$ is just the *evaluation map* when we regard $\tilde{q}_1 : T^1_{x_0} \to p^{-1}(x_0)$ as the universal family of $Q$-cycles passing through $x_0$. 


To construct the second $Q$-tower, consider the pullback of the bundle $p : U \to G/P$ by $p_1 : T^1_{x_0} \to G/P$ and denote it by $\tilde{p}_1 : \tilde{p}_1^* U \to T^1_{x_0}$. Thus, we have the commutative diagram

\[
\begin{array}{ccc}
\tilde{p}_1^* U & \xrightarrow{\tilde{p}_1} & U \\
\downarrow \tilde{p}_1 & & \downarrow p \\
T^1_{x_0} & \xrightarrow{p_1} & G/P
\end{array}
\]

for some $\tilde{p}_1 : \tilde{p}_1^* U \to U$. Now write $q_2 : \tilde{q}_2^* U \to G/Q$, where $q_2 := q \circ \tilde{p}_1$ and let $\tilde{q}_2 : \tilde{q}_2^* U \to \tilde{p}_1^* U$ be the pullback of $q : U \to G/Q$ by $q_2$. We now have the commutative diagram

\[
\begin{array}{ccc}
\tilde{q}_2^* U & \xrightarrow{\tilde{q}_2} & U \\
\downarrow \tilde{q}_2 & & \downarrow q \\
\tilde{p}_1^* U & \xrightarrow{q_2} & G/Q
\end{array}
\]

for some $\tilde{q}_2 : \tilde{q}_2^* U \to U$. Then the second $Q$-tower at $x_0$ is $T^2_{x_0} := \tilde{q}_2^* U$. It is equipped with the evaluation map $p_2 : \tilde{q}_2^* U \to G/P$, where $p_2 = p \circ \tilde{q}_2$.

Assume now for some $k \geq 2$, the $k$-th $Q$-tower $T^k_{x_0}$ together with the evaluation map $p_k : T^k_{x_0} \to G/P$ have been constructed. Consider the pullback $\tilde{p}_k : \tilde{p}_k^* U \to T^k_{x_0}$ of the bundle $p : U \to G/P$ by $p_k$ and let $\tilde{q}_k : \tilde{q}_k^* U \to U$ be the map such that $p \circ \tilde{p}_k = p_k \circ \tilde{q}_k$.

Write $q_{k+1} : p_{k+1}^* U \to G/Q$, where $q_{k+1} := q \circ \tilde{p}_k$ and let $\tilde{q}_{k+1} : \tilde{q}_{k+1}^* U \to p_{k+1}^* U$ be the pullback of $q : U \to G/Q$ by $q_{k+1}$, together with the map $\tilde{q}_{k+1} : \tilde{q}_{k+1}^* U \to U$. The $(k+1)$-th $Q$-tower at $x_0$ is $T^{k+1}_{x_0} := \tilde{q}_{k+1}^* U$, equipped with the evaluation map $p_{k+1} : \tilde{q}_{k+1}^* U \to G/P$, where $p_{k+1} = p \circ \tilde{q}_{k+1}$.

As for the first $Q$-tower, the image of $p_k(T^k_{x_0})$ is just the union of the images of all the $k$-chains $(\mathcal{P}_{s_1}, \ldots, \mathcal{P}_{s_k})$ of $Q$-cycles such that $x_0 \in \mathcal{P}_{s_1}$.

There is an alternative description for the $Q$-towers, as follows. By keeping track of each pullback procedure during the construction, it is not difficult to see that $T^k_{x_0}$ can be realized as the closed complex submanifold of $(G/P)^k \times (G/Q)^k$ consisting of the $2k$-tuples $(x_1, \ldots, x_k, s_1, \ldots, s_k)$ such that for every $j \in \{1, \ldots, k\}$, the $Q$-cycle $\mathcal{P}_{s_j}$ contains both $x_{j-1}$ and $x_j$. In this way, we see that $T^k_{x_0}$ is a projective manifold. The evaluation map $p_k$ is just the projection $(x_1, \ldots, x_k, s_1, \ldots, s_k) \mapsto x_k$. Furthermore, for $1 \leq j < k$, the projection $\pi_{k,j}$ defined by

$$
\pi_{k,j}(x_1, \ldots, x_k, s_1, \ldots, s_k) := (x_1, \ldots, x_j, s_1, \ldots, s_j)
$$

is clearly a holomorphic surjection from $T^k_{x_0}$ to $T^j_{x_0}$. 
It follows immediately from the definition that \( \pi_{k,j}^{-1}(x_1, \ldots, x_j, s_1, \ldots, s_j) \) can be canonically identified with \( T_{x_j}^{k-j} \). Since \( G/P \) is homogeneous and its automorphisms respect the \( Q \)-cycles, we have the biholomorphism \( T_{x_0}^k \cong T_{x_0}^j \) for every \( \ell \in \mathbb{N}^+ \) and \( x \in G/P \). Thus, whenever \( 1 \leq j < k \), we have the holomorphic fiber bundle

\[
T_{x_0}^{k-j} \rightarrow T_{x_0}^k \xrightarrow{\pi_{k,j}} T_{x_0}^j.
\]

**Definition 2.4.** Let \( D_{x_0}^k = \{(x_1, \ldots, x_k, s_1, \ldots, s_k) \in T_{x_0}^k : x_1 = \cdots = x_k = x_0 \} \).

We call \( \mathcal{D}_{x_0}^k \) the diagonal preimage of \( x_0 \) in \( T_{x_0}^k \). It is obvious that \( \mathcal{D}_{x_0}^1 = \mathcal{P}_1^{-1}(x_0) \), \( \mathcal{D}_{x_0}^k \subset \mathcal{P}_k^{-1}(x_0) \) and \( \pi_{k,j}(\mathcal{D}_{x_0}^k) = \mathcal{D}_{x_0}^j \) whenever \( 1 \leq j < k \).

### 2.3. \( Q \)-cycle-connectivity.

Using \( Q \)-cycles, we can define an equivalence relation \( \sim \) on \( G/P \) as follows. For any pair of two points \( x, y \in G/P \), we say that \( x \sim y \) if there is a chain of \( Q \)-cycles \( (\mathcal{P}_{s_1}, \ldots, \mathcal{P}_{s_k}) \) such that \( x \in \mathcal{P}_{s_1} \) and \( y \in \mathcal{P}_{s_k} \). Then the equivalence relation \( \sim \) is \( G \)-equivariant, i.e. \( x \sim y \) if and only if \( gx \sim gy \) for any \( g \in G \).

Thus there is a holomorphic surjective map from \( G/P \) to the space of equivalence classes of \( \sim \) (which is also a rational homogeneous space of \( G \)).

**Definition 2.5.** We say that \( G/P \) is \( Q \)-cycle-connected if for every \( x, y \in G/P \), there is a chain of \( Q \)-cycles connecting \( x \) and \( y \), i.e. \( x \sim y \).

**Proposition 2.6.** Let \( P, Q \subset G \) be parabolic subgroups such that \( P \cap Q \) is parabolic. Then the following are equivalent:

1. \( G/P \) is \( Q \)-cycle-connected;
2. There exists \( N \in \mathbb{N}^+ \) such that for every \( x, y \in G/P \), there is an \( N \)-chain of \( Q \)-cycles connecting \( x \) and \( y \);
3. There exists \( N \in \mathbb{N}^+ \) such that for every \( k \geq N \), the evaluation map \( \mathcal{P}_k : T_x^k \rightarrow G/P \) is surjective for any \( x \);
4. There does not exist any parabolic subgroup \( R \subset G \) containing \( P \) and \( Q \).

**Proof.** The implication (2) \( \Rightarrow \) (1) is trivial. To see that (1) implies (2), fix a base point \( x_0 \in G/P \) and consider the evaluation map of \( \mathcal{P}_k : T_{x_0}^k \rightarrow G/P \) for every \( k \). Each \( \mathcal{P}_k \) is a proper holomorphic map between irreducible projective varieties and thus \( \mathcal{P}_k(T_{x_0}^k) \) is an irreducible algebraic subvariety of \( G/P \). Recall that \( \mathcal{P}_k(T_{x_0}^k) \) is the union of the images of all the \( k \)-chains of \( Q \)-cycles emanating from \( x_0 \) and in particular, \( \mathcal{P}_k(T_{x_0}^k) \subset \mathcal{P}_{k+1}(T_{x_0}^{k+1}) \). Therefore, by considering the dimension, there is an \( N \in \mathbb{N}^+ \) such that \( \mathcal{P}_k(T_{x_0}^k) = \mathcal{P}_N(T_{x_0}^N) \) for every \( k \geq N \). But (1) implies that \( \mathcal{P}_N(T_{x_0}^N) \) cannot be a proper subset of \( G/P \) and hence we have \( \mathcal{P}_k(T_{x_0}^k) = \mathcal{P}_N(T_{x_0}^N) = G/P \) for every \( k \geq N \). The previous argument also demonstrates that (1) \( \Rightarrow \) (3) and the converse is trivial.
Suppose now $G/P$ is not $Q$-cycle-connected. Then as explained previously, there is a $G$-equivariant holomorphic fibration $\psi: G/P \to G/R$ for some parabolic subgroup $R \subseteq G$ containing $P$. Now let $x_0 \in G/P$ be the point whose isotropy group is just $P$. Then the $Q$-cycle $Qx_0$ is contained in $\psi^{-1}(\psi(x_0)) = Rx_0$ by our construction of $G/R$. Thus, for every $q \in Q$, there exists $r \in R$ such that $qx_0 = rx_0$ and hence $q^{-1}r \in P \subseteq R$. Therefore, $q \in R$ and it follows that $Q \subseteq R$. We have thus established $(4) \Rightarrow (1)$. Finally, if there exists a parabolic subgroup $R \subseteq G$ containing $P$ and $Q$, then any $Q$-cycle is contracted to a point by the $G$-equivariant projection $G/P \to G/R$ and it is immediate that $G/P$ is not $Q$-cycle-connected. Hence, $(1) \Rightarrow (4)$. □

Remark. Regarding the cycle-connectivity on complex homogeneous manifolds, we noted that a related but different statement is given by Kollár ([19], Theorem 2).

**Corollary 2.7.** If $G/P$ is of Picard number 1, then $(1)$ to $(4)$ in Proposition 2.6 hold for whenever $Q \not\subseteq P$, or equivalently, whenever the $Q$-cycles are of positive dimension.

Proof. The condition $(4)$ in Proposition 2.6 obviously holds since $P$ is a maximal parabolic subgroup. □

**Lemma 2.8.** The canonical left action of $P$ on $G/P$ has a unique open orbit. Furthermore, the open orbit is simply connected.

Proof. As $P$ is parabolic, it only has finitely many orbits in $G/P$ (e.g. see [3]). The irreducibility (as an algebraic variety) of $G/P$ implies that the union of all the open orbit(s) is connected since it is the complement of the union of the lower dimensional orbits and the latter is a proper complex algebraic subvariety $Z \subset G/P$. Therefore, there is just one open orbit and we denote it by $O$.

To show that $O$ is simply connected, we first consider the case when $G$ is simple. Fix a Borel subgroup $B$ contained in $P$. Then $Z$ is a union of Schubert varieties of $B$. Now consider the involution $\iota$ of the Dynkin diagram of $G$ defined by mapping a simple root $\alpha$ to $-\omega_0(\alpha)$, where $\omega_0$ is the longest element of the Weyl group of $G$. The involution $\iota$ is nontrivial if $G$ is of type $A_\ell$, $D_\ell$, and $E_6$, and is the identity otherwise. We will divide into two cases: (1) when $\phi_P$ is invariant under $\iota$; (2) when $\phi_P \neq \iota(\phi_P)$.

When $\phi_P = \iota(\phi_P)$, there is a point $x_\infty$ in $G/P$ whose isotropy group is the opposite parabolic subgroup $P^-$ (just take $x_\infty := \omega_0 x_0$, where $x_0$ is the point in $G/P$ at which the isotropy group is $P$). Let $U$ be the unipotent part of $P$. Then the open orbit $O$ is the $P$-orbit $P.x_\infty = U.x_\infty$ and is biholomorphic to $\mathbb{C}^n$, where $n = \dim\, G/P$.

When $\phi_P \neq \iota(\phi_P)$, then $Z$ has codimension $\geq 2$ if $\phi_P \cap \iota(\phi_P) = \emptyset$, and has codimension one if $\phi_P \cap \iota(\phi_P) \neq \emptyset$ (Proposition 6 of [24]). In the first case,
\(\mathcal{O}\) is simply connected because it is the complement of a complex subvariety of codimension at least 2 in \(G/P\) which is simply connected. In the second case, let \(\phi'\) be a subset of \(\phi_P\) such that \(\phi' \cap \iota(\phi_P) = \emptyset\) (just take \(\phi' := \phi_P - \iota(\phi_P)\)). Since \(\phi_P \neq \iota(\phi_P)\), \(\phi'\) is nonempty. Let \(P'\) be the corresponding parabolic subgroup of \(G\) and \(\tau : G/P \to G/P'\) be the projection map. Then the complement of the open \(P\)-orbit \(\mathcal{O}'\) in \(G/P'\) has codimension at least 2 in \(G/P'\) (Proposition 6 of [21]) and thus \(\mathcal{O}'\) is simply connected. Since \(\tau|_O : O \to \mathcal{O}'\) is a surjective map whose fibers are connected, \(O\) is simply connected.

Finally, when \(G\) is a direct product simple complex Lie groups, then \(O\) decomposes as a product correspondingly. Each factor of the product is simply connected by the argument above and thus their product \(O\) is simply connected. \(\square\)

**Proposition 2.9.** If \(N\) is a positive integer such that any \(x \in G/P\) can be connected to \(x_0\) by a chain of \(Q\)-cycles of length at most \(N\), then for every \(n \geq N\), the fibers of the evaluation map \(p_n : T^n_{x_0} \to G/P\) are connected.

**Proof.** Fix a positive integer \(n \geq N\) and let \(T := T^n_{x_0}, \rho := p_n\). Then \(\rho : T \to G/P\) is a holomorphic surjection between two projective manifolds and in particular, is proper. By Stein factorization, there are a projective variety \(Y\) and a holomorphic map \(\rho' : T \to Y\) with connected fibers and a finite holomorphic map \(\phi : Y \to G/P\) such that \(\rho = \phi \circ \rho'\). It suffices to show that \(\phi\) is actually biholomorphic.

By the homogeneity of \(G/P\) we can assume that the action of \(P\) on \(G/P\) fixes \(x_0\). Since the double fibration \(G/Q \leftarrow U \to G/P\) is \(G\)-equivariant, we see that \(P\) acts on \(T\) and \(\rho\) is \(P\)-equivariant. Recall that in the construction of Stein factorization, the variety \(Y\) is just the space of connected components of the fibers of \(\rho\). Thus, \(Y\) is acted by \(P\) accordingly and in the factorization both \(\phi\) and \(\rho'\) are \(P\)-equivariant.

As mentioned in the proof of Lemma 2.8 there are only finitely many \(P\)-orbits in \(G/P\). Consequently, \(Y\) also has only finitely many \(P\)-orbits since \(\phi : Y \to G/P\) is a finite map. Thus, \(Y\) has an open \(P\)-orbit. As \(T\) is irreducible and hence so is \(Y\), the union of the open orbit(s) of \(Y\) is connected since the union of the lower dimensional orbits is a proper complex algebraic subvariety. Therefore, there is just one open \(P\)-orbit on \(Y\) and we denote it by \(O_Y\). By Lemma 2.8 the unique open \(P\)-orbit \(O \subset G/P\) is simply connected. Now the \(P\)-equivariant map \(\phi\) maps \(O_Y\) onto \(O\). Since \(O_Y\) and \(O\) are \(P\)-orbits, the finite map \(\phi|_{O_Y}\) is necessarily unramified since the ramification locus must be \(P\)-invariant. Thus, \(\phi|_{O_Y} : O_Y \to O\) is a biholomorphism and hence \(\phi : Y \to G/P\) is a finite birational holomorphic map onto \(G/P\). Note that \(G/P\) is smooth and in particular, is a normal variety. By Zariski Main Theorem ([7] Corollary III.11.4), the inverse image of \(\phi\) of every point of \(G/P\) is connected and consequently, \(\phi\) is a biholomorphism. \(\square\)

2.4. Sheaf of \(Q\)-cycle-respecting maps \(\mathcal{B}\).
Let $f : U \subset G/P \to G/P$ be a local biholomorphism defined on an open set $U$. We say that $f$ is $Q$-cycle-respecting if for every $Q$-cycle $\mathcal{P}_s$ intersecting $U$, the image of each connected component of $\mathcal{P}_s \cap U$ is contained in some $Q$-cycle $\mathcal{P}_{s'}$. If we assign to every open set $U \subset G/P$ the set of such local biholomorphisms, it is easily seen that we get a presheaf $\mathcal{B}$ on $G/P$ and we will denote the sheaf associated to $\mathcal{B}$ by $\mathcal{B}$.

We call $\mathcal{B}$ the sheaf of $Q$-cycle-respecting maps. The set of sections of $\mathcal{B}$ on $U$ will be denoted by $\mathcal{B}(U)$ and for $\mathcal{S} \in \mathcal{B}(U)$, its germ at $x \in U$ will be denoted by $\mathcal{S}_x$. We will denote by $\mathcal{B}_x$ the stalk of $\mathcal{B}$ at $x$. By our definition, for an open set $U \subset G/P$, the sections in $\mathcal{B}(U)$ correspond to the holomorphic maps from $U$ to $G/P$ that are locally $Q$-cycle-respecting biholomorphism. If $f \in \mathcal{B}_x$, it is clear that the point $f(x)$ only depends on $f$ and we will denote it by $f(x)$.

The following proposition says essentially that for a given germ in $\mathcal{B}$, we can choose a local representative of it having a “stronger” $Q$-cycle-respecting behavior.

**Proposition 2.10.** Let $f : U \subset G/P \to G/P$ be a $Q$-cycle-respecting local biholomorphism defined on an open set $U$. There exists an open set $V_1 \subset U$ such that for any $Q$-cycle $\mathcal{P}_s$ intersecting $V_1$, we have $f(\mathcal{P}_s \cap V_1) \subset \mathcal{P}_{s'}$ for some $s' \in G/Q$ and the analogous condition also holds for $f^{-1}|_{V_2}$, where $V_2 = f(V_1)$.

**Proof.** Let $L$ be an ample line bundle on $G/P$. Then $L$ is very ample (Section 2.8 of [4]). Consider the embedding of $G/P$ into $\mathbb{P}(V)$ by $L$. Let $\mathcal{P}_0$ be the $Q$-orbit of $x_0$. Then $\mathcal{P}_0$ is the linear section $G/P \cap \mathbb{P}(V_0)$ of $G/P$ by $\mathbb{P}(V_0)$, where $\mathbb{P}(V_0)$ be the linear span of $\mathcal{P}_0$ in $\mathbb{P}(V)$ (Section 2.10 and 2.11 of [4]). Here we use the fact that $\mathcal{P}_0$ is a Schubert variety of $G/P$. Furthermore, since $G$ acts on $\mathbb{P}(V)$ linearly, $g\mathbb{P}(V)$ are linear spaces in $\mathbb{P}(V)$ and $g\mathcal{P}_0$ are linear sections $G/P \cap g\mathbb{P}(V_0)$. Take a neighbourhood $U'$ of $x_0$ in $U$ which is convex in the sense that $g\mathbb{P}(V_0) \cap U'$ are connected for any $g \in G$. Then $g\mathcal{P} \cap U' = g\mathbb{P}(V_0) \cap U'$ is connected for any $g \in G$. By the $Q$-cycle-respecting property of $f$, for any $Q$-cycle $\mathcal{P}_s$ intersecting $U'$, we have $f(\mathcal{P}_s \cap U') \subset \mathcal{P}_{s'}$ for some $s' \in G/Q$.

To get the analogous condition for $f^{-1}$, take a convex neighborhood $V''$ of $f(x_0)$ in $V' := f(U')$ in the same way as above and put $V_1 := f^{-1}(V'')$. Then we still have the property that for any $Q$-cycle $\mathcal{P}_s$ intersecting $V_1$, $\mathcal{P}_s \cap V_1 \subset \mathcal{P}_{s'}$ for some $s' \in G/Q$ (note. now the intersection $\mathcal{P}_s \cap V_1$ may be disconnected). By the convexity of $V''$, the analogous condition holds for $f^{-1}$ on $V_2 = f(V_1) = V''$. $\square$

For our purposes, given a sheaf $\mathcal{F}$ on a complex manifold $X$, we will equivalently regard a section $\mathcal{S} \in \mathcal{F}(U)$ on an open set $U \subset X$ as a continuous map from $U$ to the espace étalé $\tilde{\mathcal{F}}$ of $\mathcal{F}$, defined by $\mathcal{S}(x) = \mathcal{S}_x$, $x \in U$. Here we recall that the espace étalé of $\mathcal{F}$ is the set $\bigcup_{x \in X} \mathcal{F}_x$, equipped with the topology
generated by the base \( \{ \mathcal{G}(U) : \mathcal{G} \in \mathcal{I}(U), U \subset X \} \), where \( \mathcal{G}(U) \) is the image of \( U \) under the map \( \mathcal{G} \) just described.

The espace \( \text{étalé} \) of \( \mathcal{B} \) is a Hausdorff topological space by the identity theorem for holomorphic functions. In particular, for a continuous curve \( \Gamma : [0, 1] \to G/P \) and a germ \( \tilde{f} \in \mathcal{B}_{\Gamma(0)} \), there is at most one lifting \( \tilde{\Gamma} \) to the espace \( \text{étalé} \) of \( \mathcal{B} \) such that \( \tilde{\Gamma}(0) = \tilde{f} \). If such lifting exists, we call \( \tilde{\Gamma}(1) \in \mathcal{B}_{\Gamma(1)} \) the analytic continuation of \( \tilde{f} \) along \( \Gamma \).

Now let \( M \) be a complex manifold and \( g : M \to G/P \) be a holomorphic map. The inverse image \( g^{-1}\mathcal{B} \) of \( \mathcal{B} \) on \( M \) is by definition \( \{ (m, \epsilon) \in M \times \mathcal{E} : g(m) = \pi(\epsilon) \} \cong \bigcup_{m \in M} \mathcal{B}_{g(m)} \) is equipped with the subspace topology from \( M \times \mathcal{E} \), and \( \pi_M, \tilde{g} \) are the projections to \( M \) and \( \mathcal{E} \) respectively. We can define the notion of analytic continuation for \( g^{-1}\mathcal{B} \) analogously and it is related to the analytic continuation of \( \mathcal{B} \) as follows.

**Proposition 2.11.** Let \( \Gamma : [0, 1] \to M \) be a continuous curve and \( f_0 \in (g^{-1}\mathcal{B})_{\Gamma(0)} \). Suppose that \( f_1 \in (g^{-1}\mathcal{B})_{\Gamma(1)} \) is an analytic continuation of \( f_0 \) along \( \Gamma \), then through the canonical identifications \( \tilde{g} : (g^{-1}\mathcal{B})_{\Gamma(t)} \cong \mathcal{B}_{g(\Gamma(t))}, \) \( t \in [0, 1] \), the germ \( f_1 \in \mathcal{B}_{g(\Gamma(1))} \) is also the analytic continuation of \( f_0 \in \mathcal{B}_{g(\Gamma(0))} \) along \( g \circ \Gamma \).

**Proof.** It follows immediately from the commutative diagram above. \( \square \)

**Corollary 2.12.** Let \( i : U \hookrightarrow M \) be the inclusion of an open set \( U \) and regard a section \( \mathcal{G} \in (g^{-1}\mathcal{B})(U) \) as a continuous map \( \mathcal{G} : U \to g^{-1}\mathcal{E} \) such that \( \pi_M \circ \mathcal{G}|_U = i \). If \( Z \subset U \) is a connected set such that \( g|_Z \) is a constant, then \( \tilde{g} \circ \mathcal{G}|_Z \) is a constant.

**Proof.** Let \( g|_Z \equiv x \in G/P \). Then, by the commutative diagram above, \( \tilde{g} \circ \mathcal{G}(Z) \) is a connected set in \( \pi^{-1}(x) \). But by definition \( \pi^{-1}(x) \) is a discrete set in \( \mathcal{E} \) and hence \( \tilde{g} \circ \mathcal{G}|_Z \) is a constant. \( \square \)
3. Extension for germs of \(Q\)-cycle-respecting maps

3.1. Local extension.

Fix an arbitrary base point \(x_0 \in G/P\) and \(f \in B_{x_0}\).

We will first establish a local version of the extension. In what follows, we will adopt the notations in Section 2.2 and write \(U := G/(P \cap Q)\), and \(p : U \to G/P\) and \(q : U \to G/Q\) for the canonical projections. Recall that for \(x_0 \in G/P\), the set \(Q_{x_0} = q(p^{-1}(x_0))\) contains precisely the points \(s \in G/Q\) such that \(x_0 \in P_s\).

**Proposition 3.1.** Suppose \(P\) is reduced mod \(Q\) and vice versa. Let \(x_0 \in G/P\) and \(f \in B_{x_0}\) be a germ of \(Q\)-cycle-respecting map. There exist connected open sets \(V_1, V_2 \subseteq U := G/(P \cap Q)\) containing \(q^{-1}(Q_{x_0})\), \(q^{-1}(Q_{f(x_0)})\) respectively, and a biholomorphic map \(F : V_1 \to V_2\), a section \(\xi \in (\mathcal{P}^{-1})_q(V_1)\) such that, under the canonical identification \((\mathcal{P}^{-1})_q(V) \cong B_{p(\xi)}\) for \(\zeta \in V_1\), we have (i) \(p(F(\zeta)) = \xi(p(\zeta))\) for every \(\zeta \in V_1\); and (ii) \(\xi_{f^j} = f\) for every \(\xi \in \mathcal{P}^{-1}(x_0)\).

**Proof.** Since \(Q\) is reduced mod \(P\), by Proposition 2.3, \(U\) can be identified with a closed complex submanifold of the Grassmannian bundle \(Gr(k, T(G/P))\) of \(k\)-dimensional holomorphic tangent subspaces of \(G/P\), where \(k = \dim \mathcal{P}_s\) for any \(s \in G/Q\). Here, a point \(\zeta \in U\) is then identified with the holomorphic tangent space of \(\mathcal{P}_q(\zeta)\) at \(p(\zeta)\).

By Proposition 2.10, we can choose a local biholomorphism \(f : V_1 \to V_2\), representing the germ \(f \in B_{x_0}\), where \(V_1\) is a neighborhood containing \(x_0\) and \(V_2 = f(V_1)\), satisfying the following condition:

\((\dagger)\) For any \(Q\)-cycle \(P_s\) intersecting \(V_1\), we have \(f(P_s \cap V_1) \subseteq P_{s'}\) for some \(s' \in G/Q\) and the analogous condition also holds for \(f^{-1}\) on \(V_2\).

Under the identifications made on \(U\) as described above, the differential of \(f\) induces a local biholomorphism on \(U\)

\[
[df] : p^{-1}(V_1) \to p^{-1}(V_2),
\]

where \(p^{-1}(V_j)\) is regarded as a closed complex submanifold of \(Gr(k, T(V_j))\) for \(j = 1, 2\). Now take a point \(s \in G/Q\) such that \(P_s \cap V_1 \neq \emptyset\). By our hypotheses, \(f(P_s \cap V_1) \subseteq P_{s'}\) for some \(s' \in G/Q\). This implies that \([df]\) is fiber-preserving with respect to \(q\). Hence, if we let \(V_1^{\sharp} := q(p^{-1}(V_1))\) and \(V_2^{\sharp} := q(p^{-1}(V_2))\), which are connected open sets on \(G/Q\), then \([df]\) induces a holomorphic map \(f^{\sharp} : V_1^{\sharp} \to V_2^{\sharp}\), which is also a biholomorphism since by the condition \((\dagger)\), it is easily seen that \([d(f^{-1})] = [df]^{-1}\).

Let \(\ell = \dim \mathcal{Q}_x\) for any \(x \in G/P\). The differential of \(f^{\sharp}\) gives a biholomorphism

\[
[df^{\sharp}] : Gr(\ell, T(V_1^{\sharp})) \to Gr(\ell, T(V_2^{\sharp})),
\]
where $Gr(\ell, T(V^*_1))$ and $Gr(\ell, T(V^*_2))$ are the restrictions of the Grassmannian bundle of $\ell$-dimensional holomorphic tangent subspaces of $G/Q$ to $V^*_1$ to $V^*_2$ respectively. Now observe that as $P$ is also reduced mod $Q$, we can thus, by the same token, identify $U$ with a complex submanifold of $Gr(\ell, T(G/Q))$. Under such identification, we can regard $q^{-1}(V^*_1) \subset Gr(\ell, T(V^*_1))$ and $q^{-1}(V^*_2) \subset Gr(\ell, T(V^*_2))$ as closed complex submanifolds. We are going to show that $[df]_{\ell,T}$ sends $q^{-1}(V^*_1)$ onto $q^{-1}(V^*_2)$ and moreover, $[df]_{\ell,T}|_{p^{-1}(V_1)} = [df]$. Note that $p^{-1}(V_1)$ is open in $U$ and hence is open in $q^{-1}(V^*_2) = q^{-1}(q(p^{-1}(V_1)))$.

Take a point $\zeta \in p^{-1}(V_1)$ and let $x := p(\zeta)$ and $s := q(\zeta)$. When $\zeta$ is identified with a point in $Gr(\ell, T(G/Q))$, then $\zeta$ is the holomorphic tangent space of $Q_x$ at $s$. Through the double fibration, $Q_x$ consists of precisely the points $t$ such that $x \in P_t$. Since $f(P_t \cap V_1) \subset P_{f(t)}$, for every $t \in Q_x \cap V^*_1$, we have $f(x) \in P_{f(t)}$ and hence $f^*(Q_x \cap V^*_1) \subset Q_{f(x)}$ (it is an open inclusion since $f^*$ is holomorphic). By taking the differential of $f^*$, we get that $[df^*](\zeta)$ is the holomorphic tangent space of $Q_{f(x)}$ at $f^*(s)$. Therefore, when $[df^*](\zeta)$ is regarded as a point in $U$, we have
\[ p([df^*](\zeta)) = f(x) \quad \text{and} \quad q([df^*](\zeta)) = f^*(s). \]

Now recall that $f^*$ was defined by making use of the fact that $[df]$ is fiber-preserving with respect to $q$ and thus $f^*(s) = q([df](\zeta))$. In addition, we obviously have $p([df](\zeta)) = f(x)$ by our identifications and the definition of $[df]$. Since $(p, q) : G/(P \cap Q) \to G/P \times G/Q$ is injective, we deduce that $[df](\zeta) = [df^*](\zeta)$. Since $\zeta \in p^{-1}(V_1)$ is arbitrary, it follows that $[df^*] : q^{-1}(V^*_1) \to Gr(\ell, T(V^*_2))$ is a holomorphic extension of $[df] : p^{-1}(V_1) \to p^{-1}(V_2)$. Furthermore, as $p^{-1}(V_2) \subset q^{-1}(V^*_2) \subset Gr(\ell, T(V^*_2))$, and the latter inclusion is an embedding of a closed complex submanifold, we conclude that we have $[df^*]|_{q^{-1}(V^*_1)} \subset q^{-1}(V^*_2)$. Replacing $f$ by $f^{-1}$ and $V_1$ by $V_2$, we get a biholomorphism
\[ [df^*] : q^{-1}(V^*_1) \to q^{-1}(V^*_2). \]

Next, we are going to construct a section in $(p^{-1}\mathcal{U})(q^{-1}(V^*_1))$. Let $\zeta \in q^{-1}(V^*_1)$. Choose a connected open set $W \subset U$ such that $\zeta \in W \subset q^{-1}(V^*_1)$. Recall that we have previously shown that $[df^*] : q^{-1}(V^*_1) \to q^{-1}(V^*_2)$ is a holomorphic extension of $[df] : p^{-1}(V_1) \to p^{-1}(V_2)$. In particular, for any $x \in G/P$, $p \circ [df^*]$ is constant along each connected component of $p^{-1}(x) \cap q^{-1}(V^*_1)$ whenever the intersection is non-empty. This follows from the fact that $[df]$, as the differential of $f$, satisfies the same property on $p^{-1}(V_1)$, which is an open subset of $q^{-1}(V^*_1)$ and such analytic property is preserved in any holomorphic extension by the identity theorem for holomorphic functions. The same is true for $[df^*]^{-1}$. Thus, by choosing the open set $W$ sufficiently small, we see that $[df^*]|_W : W \to [df^*](W)$ descends to a local biholomorphism on $G/P$, which
we denote by \( f_W \). To see that \( f_W \) is \( Q \)-cycle-respecting, it suffices to note that \( f_W \) descends from \([df^2]\), which respects the fibers of \( q \) and this translates precisely to the condition that \( f_W \) is \( Q \)-cycle-respecting. Therefore, we deduce that \( f_W \in \mathcal{B}(p(W)) \). By taking the direct limit over \( W \), we then get a germ \( f_\xi \in \mathcal{B}_{p(\xi)} \).

Now we have obtained a map to the espace étalé of \( p^{-1}B \) over \( q^{-1}(V^2_1) \), denoted by

\[
\mathcal{F} : q^{-1}(V^2_1) \to \bigcup_{\zeta \in q^{-1}(V^2_1)} \mathcal{B}_{p(\zeta)},
\]

so that \( \mathcal{F}(\zeta) = f_\zeta \). To see that \( \mathcal{F} \) is indeed a section in \((p^{-1}B)(q^{-1}(V^2_1))\), we take a point \( \zeta \in q^{-1}(Q_{x_0}) \) and an open neighborhood \( W \subset q^{-1}(V^2_1) \) containing \( \zeta \), such that \([df^2]\) descends to \( f_W \in \mathcal{B}(p(W)) \), as before. Then for any \( \eta \in W \), it is by our construction that the germ of \( f_W \) at \( p(\eta) \) is just \( \mathcal{F}(\eta) = f_\eta \). That is, on the open neighborhood \( W \) of \( \zeta \), the map \( \mathcal{F}|_W \) is given by taking the germs of a section in \( \mathcal{B}(p(W)) \). Hence, \( \mathcal{F} \in (p^{-1}B)(q^{-1}(V^2_1)) \).

Now let \( V_j := q^{-1}(V^2_j) \) for \( j = 1, 2 \) and \( \mathcal{F} : V_1 \xrightarrow{\sim} V_2 \), where \( \mathcal{F} := [df^2]|_{V_1} \).

It remains to check that under the canonical identification \((p^{-1}B)_\zeta \cong \mathcal{B}_{p(\zeta)} \) for \( \zeta \in V_1 \), we have \( p(\mathcal{F}(\zeta)) = \mathcal{F}(p(\zeta)) = \mathcal{F}(\zeta)(p(\zeta)) \) for every \( \zeta \in V_1 \); and \( f = \mathcal{F}\xi = \mathcal{F}(\xi) \) for every \( \xi \in p^{-1}(x_0) \).

The first half is clear, it follows directly from how we construct \( \mathcal{F}(\zeta) \) by descending \([df^2]\) locally and the fact that \( \mathcal{F} \) is just \([df^2]\). For the second half, since \( p^{-1}(V_1) \) is contained in the domain of definition of \([df]\), if \( \xi \in p^{-1}(x_0) \) and \( W \) is an open neighborhood of \( \xi \) in \( U \) such that \( W \subset p^{-1}(V_1) \), then for every \( \eta \in W \), we have \( \mathcal{F}(\xi)(p(\eta)) = p([df^2](\eta)) = \mathcal{F}(p(\eta)) \). Thus, \( \mathcal{F}\xi = f \).  

\[\square\]

### 3.2. Global extension.

Recall that for every \( k \in \mathbb{N}^+ \), we have the \( k \)-th \( Q \)-tower \( p_k : T^k_{x_0} \to G/P \) at \( x_0 \). We will use the same symbol \( p_k \) for the evaluation map of \( T^k_{x_0} \). Using Proposition 3.1, we will now prove that a germ of \( Q \)-cycle-respecting map at \( x_0 \) induces a biholomorphic map on \( T^k_{x_0} \) and a global section of the inverse image of \( B \) related in a similar way as in Proposition 3.1.

**Proposition 3.2.** Suppose \( P \) is reduced mod \( Q \) and vice versa. Let \( x_0 \in G/P \) and \( f \in B_{x_0} \) be a germ of \( Q \)-cycle-respecting map. For every \( k \in \mathbb{N}^+ \), there exist a biholomorphic map \( F_k : T^k_{x_0} \to T^k_{x_0} \) and a section \( \mathcal{F}_k \in (p^{-1}B)(T^k_{x_0}) \) such that under the canonical identification \((p^{-1}B)_\mu \cong \mathcal{B}_{p_k(\mu)} \) for \( \mu \in T^k_{x_0} \), we have \((i) p_k(F_k(\mu)) = (\mathcal{F}_k)_\mu(p_k(\mu)) \) for every \( \mu \in T^k_{x_0} \); and \((ii) (\mathcal{F}_k)_\mu = f \) for every \( \nu \in D^k_{x_0} \subset T^k_{x_0} \), where \( D^k_{x_0} \) is the diagonal preimage of \( x_0 \) in \( T^k_{x_0} \). (Definition 2.7).
Proof. By definition, \( T^1_{x_0} \) is the pullback bundle \( q^*\mathcal{U} \) of \( q : \mathcal{U} \to G/Q \) by \( q_1 : p^{-1}(x_0) \to G/Q \), where \( q_1 \) is the restriction of \( q \) to \( p^{-1}(x_0) \). Since \( q_1 : p^{-1}(x_0) \to q_1(p^{-1}(x_0)) = Q_{x_0} \) is a biholomorphism, we see that \( T^1_{x_0} \) is biholomorphic to \( q^{-1}(Q_{x_0}) \). Similarly, we also have \( T^1_{f(x_0)} \cong q^{-1}(Q_{f(x_0)}) \). On the other hand, following the notations and the context in Proposition 3.1 if \( \zeta \in q^{-1}(Q_{x_0}) \), then
\[
q(\mathcal{F}(\zeta)) = q([df^2](\zeta)) = f^2(q(\zeta)) \in f^2(Q_{x_0}) \subset Q_{f(x_0)}
\]
and thus \( \mathcal{F}(\zeta) \in q^{-1}(Q_{f(x_0)}) \). Therefore, the restriction of \( \mathcal{F} \) gives a biholomorphism \( F^1_\alpha : q^{-1}(Q_{x_0}) \to q^{-1}(Q_{f(x_0)}) \), where \( F^1_\alpha := \mathcal{F}|_{q^{-1}(Q_{x_0})} \) and this canonically induces a biholomorphism \( F_\alpha : T^1_{x_0} \to T^1_{f(x_0)} \) through the identification \( T^1_{x_0} \cong q^{-1}(Q_{x_0}) \). Furthermore, under the same identification, the evaluation map \( p_1 : T^1_{x_0} \to G/P \) corresponds to the canonical projection \( p : q^{-1}(Q_{x_0}) \to G/P \). Therefore, by again restricting to \( q^{-1}(Q_{x_0}) \cong T^1_{x_0} \), the section \( \tilde{\alpha} \) obtained in Proposition 3.1 gives a section \( \tilde{\alpha}_1 \in (p_1^{-1}\mathcal{B})(T^1_{x_0}) \). The case for \( k = 1 \) is thus settled.

Suppose now the biholomorphism \( F_k \) and global section \( \tilde{\alpha}_k \) have been constructed for some \( k \geq 1 \). Consider the holomorphic fiber bundle \( \pi : T^k_{x_0} \to T^1_{x_0} \), where \( \pi := \pi_{k+1,k} \), which is defined just before Definition 2.4. Similarly, write \( \Pi : T^k_{f(x_0)} \to T^1_{f(x_0)} \) for the corresponding bundle at \( f(x_0) \).

Let \( \alpha \in T^k_{x_0} \). As we have the canonical biholomorphisms \( \pi^{-1}(\alpha) \cong T^1_{p_k(\alpha)} \) and \( \Pi^{-1}(F_k(\alpha)) \cong T^1_{p_k(F_k(\alpha))} \) and also \( (\tilde{\alpha}_k)_\alpha (p_k(\alpha)) = p_k(F_k(\alpha)) \), by using the germ \( (\tilde{\alpha}_k)_\alpha (p_k(\alpha)) \), we obtain, from the case \( k = 1 \), a biholomorphism \( F^{\alpha} : \pi^{-1}(\alpha) \to \Pi^{-1}(F_k(\alpha)) \), and a section \( \tilde{\alpha}^{\alpha} \in (p_k^{-1}\mathcal{B})(\pi^{-1}(\alpha)) \). Since \( \alpha \in T^k_{x_0} \) is arbitrary, we have thus set-theoretically constructed a bijective map \( F^{\alpha} : T^k_{x_0} \to T^k_{f(x_0)} \) and a map \( \tilde{\alpha}_{k+1} \) from \( T^k_{x_0} \) to the espace étalé of \( p_k^{-1}\mathcal{B} \). We will now argue that \( F_k \) is holomorphic and \( \tilde{\alpha}_{k+1} \) is a section in \( (p_k^{-1}\mathcal{B})(T^k_{x_0}) \).

Let \( \mu \in T^k_{x_0} \), \( \alpha := \pi(\mu) \in T^k_{x_0} \) and \( x := p_k(\alpha) \in G/P \). Since \( \tilde{\alpha}_k \in (p_k^{-1}\mathcal{B})(T^k_{x_0}) \), there are an open neighborhood \( V' \subset T^k_{x_0} \) of \( \alpha \), an open neighborhood \( V \subset G/P \) of \( x \), and a section \( \tilde{\alpha}_V \in \mathcal{B}(V) \) such that \( p_k(V') \subset V \) and \( (\tilde{\alpha}_k)_\eta = (\tilde{\alpha}_V)_{p_k(\eta)} \) for every \( \eta \in V' \).

By Proposition 3.1 we can, by choosing \( V \) satisfying the condition (†) therein, obtain a biholomorphic map \( \mathcal{F} : V \to \mathcal{F}(V) \subset \mathcal{U} \), where \( \mathcal{V} := q^{-1}(q(p^{-1}(V))) \), and a section \( \tilde{\alpha} \in (p_1^{-1}\mathcal{B})(\mathcal{V}) \) satisfying the conditions stated in Proposition 3.1. Since \( \pi : T^k_{x_0} \to T^1_{x_0} \) is continuous, there is a neighborhood \( V \subset T^{k+1}_{x_0} \) of \( \mu \) such that \( \pi(V) \subset V' \). Now for \( \eta \in V \), if we let \( \beta := \pi(\eta) \in V' \), since \( \pi^{-1}(\beta) \cong T^1_{p_k(\beta)} \cong q^{-1}(Q_{p_k(\beta)}) \) canonically, \( \eta \) corresponds to a point \( \eta' \in q^{-1}(Q_{p_k(\beta)}) \subset V' \). We let \( \varphi : \mathcal{V} \to \mathcal{V} \) be the map defined by \( \varphi(\eta) = \eta' \). From how we defined the \( Q \)-towers in Section 2.2, \( \varphi \) is actually \( \tilde{\alpha}_{k+1}|_V \), and in particular, it is continuous.
By tracing back how we constructed \((\mathfrak{g}_{k+1})_\eta\), we see that \((\mathfrak{g}_{k+1})_\eta = \mathfrak{f}_\varphi(\eta)\).

Since \(\mathfrak{F} \in (p^{-1}\mathcal{B})(\mathcal{V})\), there are open neighborhoods \(\mathcal{W} \subset \mathcal{V}\) of \(\varphi(\mu)\), an open neighborhood \(\mathcal{W} \subset G/P\) of \(p(\varphi(\mu))\) and a section \(\mathfrak{F}_W \in \mathcal{B}(\mathcal{W})\) such that \(p(\mathcal{W}) \subset \mathcal{W}\) and \(\mathfrak{F}_\lambda = (\mathfrak{F}_W)p(\lambda)\) for every \(\lambda \in \mathcal{W}\). Now note that \(p \circ \varphi = p_{k+1}: \mathcal{V} \subset \mathcal{T}_{x_0}^{k+1} \to G/P\). So if we define the open set \(\mathcal{W} := \varphi^{-1}(\mathcal{W}) \subset \mathcal{V} \subset \mathcal{T}_{x_0}^{k+1}\), then \(p_{k+1}(\mathcal{W}) \subset \mathcal{W}\) and for every \(\eta \in \mathcal{W}\), we have

\[
(\mathfrak{F}_{k+1})_\eta = \mathfrak{f}_\varphi(\eta) = (\mathfrak{F}_W)p(\varphi(\eta)) = (\mathfrak{F}_W)p_{k+1}(\eta).
\]

Since \(\mu \in \mathcal{T}_{x_0}^{k+1}\) is arbitrary, we now see that \(\mathfrak{F}_{k+1}\) is a section in \((p_{k+1}\mathcal{B})(\mathcal{T}_{x_0}^{k+1})\). Similarly, from \(F_{k+1}(\eta) = F(\varphi(\eta))\), it follows that \(F_{k+1}\) is holomorphic.

Finally, for \(\mu \in \mathcal{T}_{x_0}^{k+1}\), let \(\alpha = \pi(\mu)\), then

\[
p_{k+1}(F_{k+1}(\mu)) = p_{k+1}(F^\alpha(\mu)) = (\mathfrak{F}^\alpha)_\mu(p_{k+1}(\mu)) = (\mathfrak{F}_{k+1})_\mu(p_{k+1}(\mu)).
\]

Moreover, if \(\nu \in D_{x_0}^{k+1}\), then \(\beta := \pi(\nu) \in D_{x_0}^k\) and

\[
(\mathfrak{F}_{k+1})_\nu = (\mathfrak{F}^\beta)_\nu = (\mathfrak{F}_k)_\beta = f.
\]

Using Proposition \[2.11\], we can interpret the germs \((\mathfrak{f}_k)_\mu\) as analytic continuations of \(f\) along continuous curves in \(p_k(\mathcal{T}_{x_0}^k) \subset G/P\), as follows.

**Corollary 3.3.** Let \(k \in \mathbb{N}^+\), \(\nu \in D_{x_0}^k \subset \mathcal{T}_{x_0}^k\), \(\mu \in \mathcal{T}_{x_0}^k\) and \(\Gamma : [0,1] \to \mathcal{T}_{x_0}^k\) be a continuous curve such that \(\Gamma(0) = \nu\) and \(\Gamma(1) = \mu\). The germ \((\mathfrak{F}_k)_\mu \in (p_k^{-1}\mathcal{B})_\mu \cong \mathcal{B}_{p_k(\mu)}\) is the analytic continuation of \(f\) along the curve \(p_k \circ \Gamma\).

**Proof.** Regard the global section \(\mathfrak{F}_k\) as a continuous map from \(\mathcal{T}_{x_0}^k\) to the espace étale of \(p_k^{-1}\mathcal{B}\), then \(\mathfrak{F}_k \circ \Gamma\) is a lifting of \(\Gamma\) such that \(\mathfrak{F}_k \circ \Gamma(0) = \mathfrak{F}_k(\nu) = f\). Thus, \(\mathfrak{F}_k(\Gamma(1)) = \mathfrak{F}_k(\mu) = (\mathfrak{F}_k)_\mu\) is the analytic continuation of \(f\) along the curve \(\Gamma\). By Proposition \[2.11\], \((\mathfrak{F}_k)_\mu\) is the analytic continuation of \(f\) along \(p_k \circ \Gamma\).

**Remark.** For the case \(k = 1\) in Corollary \[3.3\], if we let \(\mu = (x, s) \in \mathcal{T}_{x_0}^1 \subset G/P \times G/Q\) (see the end of Section \[2.2\]) and choose \(\nu = (x_0, s) \in D_{x_0}^1 = p_1^{-1}(x_0)\), then we can find a continuous curve \(\gamma\) in \(P_s\) connecting \(x_0\) and \(x\). Such \(\gamma\) can always be lifted to a curve \(\Gamma\) in \(q^{-1}(s) \subset q^{-1}(Q_{x_0}) \cong \mathcal{T}_{x_0}^1\) connecting \(\nu\) and \(\mu\). Thus, for every \(\mu \in \mathcal{T}_{x_0}^1\), the germ \((\mathfrak{F}_1)_\mu\) is the analytic continuation of \(f\) along a curve contained in a \(Q\)-cycle passing through \(x_0\).

Now we can give the proof for Theorem \[1.1\].

**Proof of Theorem \[1.1\].** We can always take \(G\) to be a semisimple complex Lie group which is a direct product of simple factors. By Proposition \[2.2\], there
exists a unique parabolic subgroup $Q'$ properly containing $Q$ such that $Q'$ is reduced mod $P$ and we have the following diagram for the associated fibrations:

$$
\begin{array}{ccc}
G/(P \cap Q) & \rightarrow & G/P \\
\downarrow & & \downarrow \\
G/Q & \rightarrow & G/(P \cap Q') \\
\downarrow & & \downarrow \\
G/Q' & \rightarrow & G/P
\end{array}
$$

For such $Q'$, every $Q'$-cycle in $G/P$ is a $Q$-cycle and vice versa. In addition, $G/Q'$ effectively parameterizes the $Q'$-cycles (or $Q$-cycles). More explicitly, for $s, t \in G/Q$, we have $p_s = p_t$ if and only if $\omega(s) = \omega(t)$. Since the $Q$-cycles and $Q'$-cycles on $G/P$ coincide, it suffices to prove the proposition for $P$ and $Q'$. Thus, without loss of generality, we can assume that $Q$ is reduced mod $P$.

Let $x_0 \in G/P$ and $\mathfrak{f} \in \mathcal{B}_{x_0}$ be a germ of $Q$-cycle-respecting map. We first prove the theorem for the cases where $P$ is reduced mod $Q$.

By Proposition 2.9, there exists $N \in \mathbb{N}^+$ such that the evaluation map $p_N : \mathcal{T}_{x_0}^N \rightarrow G/P$ from the $N$-th $Q$-tower is surjective. Let $F_N : \mathcal{T}_{x_0}^N \rightarrow T^N_{f(x_0)}$ be the biholomorphism and $\mathfrak{f}_N \in (p_N^{-1}\mathcal{B})(\mathcal{T}_{x_0}^N)$ be the global section obtained from $\mathfrak{f}$ in Proposition 3.2.

By Proposition 2.9, for any $x \in G/P$, the preimage $p_N^{-1}(x)$ is connected. Thus, it follows from Corollary 2.12 that $(\mathfrak{f}_N)_\mu = (\mathfrak{f}_N)_\nu$ for every $\mu, \nu \in p_N^{-1}(x)$. Using (i) of Proposition 3.2, we now see that

$$p_N(F_N(\mu)) = (\mathfrak{f}_N)_\mu(p_N(\mu)) = (\mathfrak{f}_N)_\nu(p_N(\nu)) = p_N(F_N(\nu))$$

for every $\mu, \nu \in p_N^{-1}(x)$. That is, the map $p_N \circ F_N : \mathcal{T}_{x_0}^N \rightarrow G/P$ is constant on each preimage $p_N^{-1}(x)$. Hence, $F_N$ descends to a holomorphic map $F : G/P \rightarrow G/P$, which is actually biholomorphic because the same arguments apply to $F_N^{-1}$, which clearly descends to the inverse of $F$.

We are now going to prove that $F$ indeed extends the germ $\mathfrak{f}$. First of all, note that by the definition of the inverse image sheaf $p_N^{-1}\mathcal{B}$, the statement (ii) in Proposition 3.2 implies for each $\nu \in \mathcal{D}_{x_0}^N$, there are an open neighborhood $\mathcal{V} \subset \mathcal{T}_{x_0}^N$ of $\nu$, an open neighborhood $\mathcal{V} \subset G/P$ of $p_N(\mathcal{V}) \ni x_0$ and a $Q$-cycle-respecting map $\mathfrak{f} : \mathcal{V} \rightarrow G/P$ representing $\mathfrak{f}$, such that $(\mathfrak{f}_N)_\mu(p_N(\mu)) = f(p_N(\mu))$ for every $\mu \in \mathcal{V}$. Together with statement (i) in Proposition 3.2, we conclude that for every $x$ in the open set $p_N(\mathcal{V}) \subset G/P$ and any $\mu \in p_N^{-1}(x)$,

$$F(x) = p_N(F_N(\mu)) = (\mathfrak{f}_N)_\mu(p_N(\mu)) = f(p_N(\mu)) = f(x).$$

Thus, $F$ extends $\mathfrak{f}$ and the proof is complete for the cases where $P$ is reduced mod $Q$. 

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It remains to consider the general case where $P$ is not reduced mod $Q$. Define an equivalent relation $\approx$ on $G/P$ by $x \approx y$ for $x, y \in G/P$ if and only if $Qx = Qy$, or equivalently, $\cap_{x \in P_s} P_s = \cap_{y \in P_s} P_s$. (One direction is clear: if $Qx = Qy$, then $\cap_{x \in P_s} P_s = \cap_{y \in P_s} P_s$. To prove the converse, suppose that $\cap_{x \in P_s} P_s = \cap_{y \in P_s} P_s$. Since $y \in \cap_{y \in P_s} P_s$, $y$ is contained in $\cap_{x \in P_s} P_s$, i.e., $y$ is contained in $P_s$ for all $s \in \mathcal{Q}_x$. Hence $Q_x \subset Q_y$ and thus $Q_x = Q_y$.) Then the relation $\approx$ is equivariant under the action of $G$, that is, for any $g \in G$, $x \approx y$ if and only if $gx \approx gy$.

Hence there is a $G$-equivariant quotient map $\varpi$ from $G/P$ to the space $\tilde{X}$ of equivalent classes and, if we let $\tilde{\approx}$ be an equivalent relation on $\tilde{X}$ of $P$-cycles on $\tilde{G}/\tilde{P}$, then $\tilde{\approx}$ is contained in $\varpi$ and we have $\tilde{X} = G/\tilde{P}$. (In fact, $\tilde{P}$ is nothing but the reduction of $P$ mod $Q$.) For $s \in G/Q$ let $\tilde{P}_s := \varpi(P_s)$. Then $\tilde{P}_s$ are the $Q$-cycles on $G/\tilde{P}$. Now $\tilde{x} \in G/\tilde{P}$ is determined by $\tilde{Q}_x$.

Let $f : V_1 \to V_2$ be a local biholomorphism satisfying the condition $(\dagger)$ in the proof of Proposition 3.1. Then $f$ induces a biholomorphism $\tilde{f} : \tilde{V}_1 := \varpi(V_1) \to \tilde{V}_2 := \varpi(V_2)$, and $\tilde{f}$ satisfies the condition $(\dagger)$ with respect to the family $\{\tilde{P}_s : s \in G/Q\}$. Note that we still have $Q$-cycle-connectivity for the $Q$-cycles on $G/\tilde{P}$ since such property is inherited by $G$-equivariant projection. Now $P$ is also reduced mod $Q$, so we conclude that $\tilde{f}$ extends to a biholomorphism $\tilde{F} : G/\tilde{P} \to G/P$ by the case we settled before.

The automorphism $\tilde{F}$ is given by some element of $G$ except in the following cases:

1. $G/\tilde{P} = \mathbb{C}P^{2n-1}$ and $G = Sp(n, \mathbb{C})$;
2. $G/\tilde{P}$ is a spinor variety and $G = SO(2n + 1, \mathbb{C})$;
3. $G/\tilde{P} = \mathbb{Q}^2$ and $G = G_2$.

If $G/\tilde{P}$ is not one of (1) - (3), then $\tilde{F}$ comes from an element $g \in G$ and thus $f$ is also a restriction of the automorphism of $G/P$ given by the same $g$.

If $G/\tilde{P}$ is one of (1) - (3), then the marking $\phi_{\tilde{P}}$ associated to $\tilde{P}$ on the Dynkin diagram of $G$ consists of only one element which is an end of the diagram. Let $\phi_Q$ be the corresponding marking for $Q$. Since $\phi_{\tilde{P}}$ is the reduction of $\phi_P$ mod $\phi_Q$, it is contained in $\phi_P$. If $\phi_{\tilde{P}} \subsetneq \phi_P$, then there is a connected subdiagram of $G$ containing both a node in $\phi_P - \phi_{\tilde{P}}$ and a node in $\phi_Q$ which does not contain any element in $\phi_{\tilde{P}}$, contradicting to the fact that $\phi_{\tilde{P}}$ is the reduction of $\phi_P$ mod $\phi_Q$. Therefore, $\phi_{\tilde{P}} = \phi_P$. Hence, $G/\tilde{P} = G/P$ and $\tilde{F}$ is an extension of $f$.  \hfill $\Box$

4. Local biholomorphisms preserving real group orbits

4.1. Real group orbits of flag type.
Let $G/P$ be a rational homogeneous space and let $G_0$ be a real form of $G$, i.e. $G_0$ is the real analytic subgroup of $G$ corresponding to a real Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ such that $\mathfrak{g}_0 \oplus J\mathfrak{g}_0 = \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of $G$ and $J$ is the complex structure operator. Let $\tau$ denote the complex conjugation of $G$ over $G_0$.

We now collect some foundational results about the canonical action of $G_0$ on $G/P$ established by Wolf [25]. First of all, there are only a finite number of orbits, which will be called real group orbits. In particular, there are open orbit(s). Furthermore, it is also known that there is a unique closed orbit.

Let $\mathcal{O}$ be a $G_0$-orbit in $G/P$. A holomorphic arc in $\mathcal{O}$ is a holomorphic map $f : \Delta \to G/P$ such that $f(\Delta) \subset \mathcal{O}$, where $\Delta$ is the unit disk in $\mathbb{C}$. By a chain of holomorphic arcs in $\mathcal{O}$ connecting two points $x, y$ in $\mathcal{O}$, we mean a finite sequence of holomorphic arcs $f_1, \ldots, f_k$ in $\mathcal{O}$ such that $x \in f_1(\Delta)$ and $y \in f_k(\Delta)$ and $f_i(\Delta) \cap f_{i+1}(\Delta) \neq \emptyset$ for all $i = 1, \ldots, k - 1$. Define an equivalence relation on $\mathcal{O}$ as follows. Two elements $x, y \in \mathcal{O}$ are equivalent if and only if there is a chain of holomorphic arcs in $\mathcal{O}$ connecting $x$ and $y$. An equivalence class of this equivalence relation on $\mathcal{O}$ is called a holomorphic arc component of $\mathcal{O}$. We remark that a holomorphic arc component is not necessarily a complex submanifold of $\mathcal{O}$.

Fix $x \in \mathcal{O}$ and let $C$ be the holomorphic arc component of $\mathcal{O}$ containing $x$. Denote by $N_0$ the identity component of the normalizer $N_{G_0}(C) := \{g \in G_0 : gC = C\}$ of $C$ in $G_0$. Then $C$ is an $N_0$-orbit (Lemma 8.2 of [25]). The complexification $N$ of $N_0$ is a parabolic subgroup of $G$ such that $\tau N = N$. Moreover, $N_0$ is the identity component of the parabolic subgroup $N \cap G_0$ of $G_0$ (Theorem 8.5 of [25]). Let $s \in G/N$ be the point whose isotropy group is $N$ and let $\Sigma$ be the $G_0$-orbit containing $s$ in $G/N$. Since $\tau N = N$, $\Sigma$ is closed in $G/N$ and is totally real in the sense that $\Sigma$ is the set of real points of the complex projective variety $G/N$ defined over $\mathbb{R}$ (Theorem 3.6 of [25]).

**Definition 4.1** ([25], Definition 9.1). A real group orbit $\mathcal{O}$ is said to be partially complex if its holomorphic arc components are locally closed complex submanifolds of $G/P$; and of flag type if for $x \in \mathcal{O}$, the orbit $Nx$ is a rational homogeneous space, i.e. if $N \cap P_x$ is parabolic, where $P_x \subset G$ is the isotropy group of $x$.

**Remark.** The criterion for $\mathcal{O}$ being of flag type given in [25] is slightly different from the one given here but they are equivalent.

As mentioned, if $C$ is the holomorphic arc component containing $x$, then $C = N_0 x$. Therefore, $C$ is a real group orbit of the rational homogeneous space $Nx$ if $\mathcal{O}$ is of flag type.

**Example 4.2** (see Remark 9.23 in [25]). The following are examples of $G_0$-orbits which are partially complex and of flag type.
(1) The $G_0$-orbits in a compact Hermitian symmetric space $G/P$, where $G_0$ is the automorphism group of the bounded symmetric domain dual to $G/P$. It is well known that every holomorphic arc component is isomorphic to some bounded symmetric domain except for the closed orbit.

(2) The orbits of any $SU(p,q)$ acting $Gr(n,\mathbb{C}^{2n})$, where $p + q = 2n$ and $p < q$. For example, for the action of $SU(1,3)$ on $Gr(2,\mathbb{C}^4)$, the closed orbit has holomorphic arc components isomorphic to $\mathbb{P}^1$.

4.2. Holomorphic cover of subdiagram type.

Now we are going to apply our extension theorem for $Q$-cycle-respecting maps to study local holomorphic maps preserving a real group orbit on a rational homogeneous space $G/P$. Let $x_0 \in G/P$ be the point whose isotropy group is $P$. Let $G_0 \subset G$ be a real form and $O$ be the $G_0$-orbit containing $x_0$. Suppose $Q \subset G$ is a parabolic subgroup such that $P \cap Q$ is parabolic. As before we have the double fibration $G/Q \leftarrow G/(P \cap Q) \rightarrow G/P$. Now let $S$ be a $G_0$-orbit on $G/Q$ such that $x_0 \in p(q^{-1}(S))$. That is, we have

\[ G/(P \cap Q) \leftarrow \mathcal{S} \subset G/Q \rightarrow G/P \supset O \ni x_0. \]

We begin with the following definition, which roughly describes the situation in which a real group orbit can be written as a union of complex submanifolds lying on a family $Q$-cycles which are rigid under holomorphic mappings.

**Definition 4.3.** We say that $O$ has a *holomorphic cover of $Q$-type* if there exists a $G_0$-orbit $\mathcal{S} \subset G/Q$ such that, for every $s \in \mathcal{S}$, there is a real group orbit $O_s$ on $P_s := p(q^{-1}(s))$ satisfying

(i) $O = \bigcup_{s \in \mathcal{S}} O_s$; and

(ii) for each $s \in \mathcal{S}$ and any holomorphic map $h : W \rightarrow G/P$ defined on an open set $W \subset P_s$ with $h(W \cap O_s) \subset O$, we have that each connected component of $h(W \cap O_s)$ is contained in $O_t$ for some $t \in \mathcal{S}$.

**Definition 4.4.** We say that $O$ has a *holomorphic cover of subdiagram type* if it has a holomorphic cover of $Q$-type for some parabolic subgroup $Q \subset G$; and such a cover is said to be *non-trivial* if in addition $Q \not\subset P$ and $Q \neq G$.

The following proposition is the major motivation for the definitions above.

**Proposition 4.5.** If a real group orbit $O \subset G/P$ is partially complex and of flag type, then it has a holomorphic cover of subdiagram type.
Proof. Recall the notations in Section 4.1. Let $C \subset O$ be a holomorphic arc component containing a point $x \in O$ and $Q$ be the complexification of the normalizer of $C$ in $G_0$. We have known that $C \subset Qx$ is a real group orbit of $Qx$ which is a rational homogeneous space because $O$ is of flag type. Furthermore, since $O$ is partially complex, $C$ is a complex submanifold of $G/P$. Let $s \in G/Q$ be a point such that the corresponding $Q$-cycle is $Qx$. Define $S := G_0s \subset G/Q$ and for every $s' \in \mathcal{S}$, let $\mathcal{P}_{s'} := p(q^{-1}(s'))$. Then if we write $s' = gs$, where $g \in G_0$, then $C_{s'} := gC$ is independent of the choice of $g$ and also we have $C_{s'} \subset \mathcal{P}_{s'} \cap O$, which is the holomorphic arc component containing $gx$.

It remains to verify that the family $\{\mathcal{P}_{s'} : s' \in \mathcal{S}\}$ of $Q$-cycles together with the family $\{O_{s'} := C_{s'} \subset \mathcal{P}_{s'} : s' \in \mathcal{S}\}$ of real group orbits (which are complex submanifolds), is a holomorphic cover of $Q$-type for $O$. Let $s' \in \mathcal{S}$ and $W \subset \mathcal{P}_{s'}$ be a connected open set, and $h : W \to G/P$ be a holomorphic map such that $h(W \cap O_{s'}) \subset O$. Since $O_{s'}$ is a complex submanifold, it follows that each connected component of $h(W \cap O_{s'})$ is contained in some holomorphic arc component of $O$ and thus in $O_t$ for some $t \in \mathcal{S}$.

Example 4.6. The following real group orbits have a holomorphic cover of subdiagram type.

(1) By the previous proposition, any real group orbit which is partially complex and of flag type has a holomorphic cover of $Q$-type, where $Q$ is the complexification of the normalizer of a holomorphic arc component of it. Among the $G_0$-orbits in Example 4.2 (1), an orbit has a non-trivial holomorphic cover of subdiagram type if it is neither open nor closed. For the $G_0$-orbits in Example 4.2 (2), the closed orbit has a non-trivial holomorphic cover of subdiagram type.

(2) The closed orbit of $G_0 = SU(p,q)$ on $Gr(d, \mathbb{C}^{p+q})$, where $d < \min(p,q)$, has a non-trivial holomorphic cover of subdiagram type (23, Proposition 3.2 therein). In this case, the holomorphic cover is of $Q$-type, where $Q$ is the maximal parabolic subgroup of $G = SL(p+q, \mathbb{C})$ such that $G/Q = Gr(\min(p,q), \mathbb{C}^{p+q})$. Moreover, in this case we have $O_s = \mathcal{P}_{s} \cong Gr(d, \min(p,q))$.

4.3. Proof of Theorem 1.2.

We begin with the following simple observation that a real group orbit is always a set of uniqueness for holomorphic functions.

Lemma 4.7. Let $O$ be a real group orbit on $G/P$ for some real form $G_0 \subset G$. Let $U \subset G/P$ be a connected open subset such that $U \cap O \neq \emptyset$. Then there does not exist any proper complex analytic subvariety of $U$ containing $U \cap O$.

Proof. Let $x \in O$, $T_x(O)$ be the real tangent space of $O$ at $x$ and $J$ be the complex structure operator. Note that we have the canonical isomorphisms
$T_x(O) \cong g_0/(g_0 \cap p_x) \cong (g_0 + p_x)/p_x$, where $p_x \subset g$ is the Lie algebra of the isotropy group $P_x$ at $x$, then

$$T_x(O) + J(T_x(O)) \cong (g_0 + p_x)/p_x + J(g_0 + p_x)/p_x$$

$$= (g_0 + Jg_0 + p_x)/p_x$$

$$= g/p_x \cong T_x(G/P)$$

Hence, we see that $T_x(O)$ cannot be contained in a $J$-invariant proper tangent subspace at $x$ and the desired statement follows. □

**Proposition 4.8.** Let $O$ be a real group orbit on $G/P$. Suppose $U \subset G/P$ is a connected open subset such that $U \cap O \neq \emptyset$ and $u : U \to \mathbb{C}$ is a holomorphic function such that $u_{|U \cap O} \equiv 0$, then $u$ is identically zero on $U$.

**Proof.** It follows directly from Lemma 4.7 since the zero set of any non-trivial holomorphic function on $U$ is a proper complex analytic subvariety in $U$. □

**Proof of Theorem 1.2.** Let $O$ be a $G_0$-orbit in $G/P$ having a non-trivial holomorphic cover of subdiagram type. Then there is a parabolic subgroup $Q \triangleleft G$ such that $P \cap Q$ is parabolic and we have the double fibration

$$\begin{array}{ccc}
G/(P \cap Q) & \leftarrow & G/P \supset O \\
\downarrow q & & \downarrow p \\
S \subset G/Q & & \end{array}$$

where $S \subset G/Q$ is a real group orbit, and a family $\{O_s\}_{s \in S}$ of real group orbits of $P_s := p(q^{-1}(s))$ such that

1. $O = \bigcup_{s \in S} O_s$;
2. for each $s \in S$ and any holomorphic map $h : W \to G/P$ defined on an open set $W \subset P_s$, satisfying $h(W \cap O_s) \subset O$, we have that each connected component of $h(W \cap O_s)$ is contained in $O_t$ for some $t \in S$.

Replacing $U$ by a smaller open subset if necessary, we may assume that $P_s \cap U$ is connected for any $P_s$ intersecting $U$ (see the proof of Proposition 2.10).

Since $f$ is a biholomorphism such that $f(U \cap O) \subset O$, if we let $s \in S$ and fix a connected component of $O_s \cap U$, its image under $f$ is contained in some $O_t \subset P_t$ by condition (2). Now, since $O_s$ is a real group orbit of $P_s$, following directly from Proposition 4.8 we deduce that the image of a connected component of $P_s \cap U$ under $f$ is also contained in some $P_t$. We are going to show that this is true for every $Q$-cycle intersecting $U$, i.e., $f : U \subset G/P \to G/P$ is in fact $Q$-cycle-respecting.

As in the proof of Theorem 1.1 identify the universal family $U = G/(P \cap Q)$ with a closed complex submanifold of the Grassmannian bundle $Gr(k; T(G/P))$,
where \( k = \dim_C(\mathcal{P}_s) \). Let \( V := f(U) \). Then the differential \( df \) induces a map 
\[ [df] : Gr(k, T(U)) \to Gr(k, T(V)) \]. We claim that 

(a) \([df] \) maps \( p^{-1}(U) \) to \( p^{-1}(V) \);
(b) \([df] : p^{-1}(U) \to p^{-1}(V) \) is fiber-preserving with respect to \( q \), i.e., it
sends a fiber of \( q \) to a fiber of \( q \).

Since \( p^{-1}(V) \subset Gr(k, T(V)) \) is a closed complex submanifold, to say that 
\([df](p^{-1}(U)) \subset p^{-1}(V) \) is the same as saying that the pullbacks of the local 
defining holomorphic functions for \( p^{-1}(V) \) vanish on \( p^{-1}(U) \).

Let \((s_0, x_0) \in p^{-1}(U)\), where \( x_0 \in G/P \) (respectively, \( s_0 \in G/Q \)) is the base point of \( G/P \) with the isotropy group \( P \) (respectively, \( G/Q \) with the isotropy group \( Q \)). Take connected open neighborhoods \( \mathcal{S} \subset G/Q \) of \( s_0 \) and \( \chi \subset O_{s_0} \)
of \( x_0 \) so that \( \mathcal{S} \times \chi \) can be embedded into \( p^{-1}(U) \) as an open neighborhood of \((s_0, x_0)\) with \( q(\mathcal{S} \times \chi) = \mathcal{S} \). For each \( x \in \chi \) consider the intersection 
\( q^{-1}(\mathcal{S}) \cap (\mathcal{S} \times \{x\}) = (\mathcal{S} \cap \mathcal{S}) \times \{x\} \). We observe that each point \((s, x) \in \mathcal{S} \cap \mathcal{S} \) as a point in the universal family, corresponds to a holomorphic

tangent space of \( \mathcal{P}_s \) at \( x \) (since \( \mathcal{S} \) parametrizes such family). Furthermore, as
proven above, the image \( f(\mathcal{P}_s \cap U) \) is contained in some \( \mathcal{P}_t \). Hence we have
\[ [df](\mathcal{S} \cap \mathcal{S}) \times \{x\}) \subset p^{-1}(V) \).

Now the pullbacks of the local defining functions of \( p^{-1}(V) \) vanish on \( (\mathcal{S} \cap \mathcal{S}) \times \{x\}) \) for every \( x \in \chi \). Proposition \ref{prop:holomorphic_functions} implies that these holomorphic functions vanish identically on \( \mathcal{S} \times \{x\}) \) for every \( x \in \chi \) and hence on an open neighborhood in \( p^{-1}(U) \). Therefore, \([df](p^{-1}(U)) \subset p^{-1}(V) \).

To see (b) we first consider the case where \( s \in \mathcal{S} \cap U^2 \), where \( U^2 := q(p^{-1}(U)) \).
By the arguments at the beginning of the proof, we know that the fibers of \( q \) over \( \mathcal{S} \cap U^2 \) are preserved by \([df] \). On the other hand, the fiber-preserving property can
clearly be translated to the vanishing of a set of holomorphic functions defined
on \( U^2 \). We have just seen that these relevant holomorphic functions vanish on \( \mathcal{S} \cap U^2 \) and Proposition \ref{prop:holomorphic_functions} once again implies that they vanish identically on \( U^2 \) and therefore \([df] \) is everywhere fiber-preserving with respect to \( q \), which simply
translates to the statement that \( f \) is \( Q \)-cycle-respecting.

We have thus shown that \( f \) is a \( Q \)-cycle-respecting local biholomorphism and
by Theorem \ref{thm:local_biholomorphism} \( f \) extends to a biholomorphism of \( G/P \). \( \square \)

5. Local biholomorphisms respecting tangent spaces of \( Q \)-cycles

Recall that \( G/(P \cap Q) \) can be regarded as a closed complex submanifold \( \mathcal{C} \) of \( Gr(k, T(G/P)) \), namely, as the variety of tangent spaces of \( Q \)-cycles on \( G/P \).
We may ask whether we can have an extension theorem parallel to the original
Cartan-Fubini extension Theorem for minimal rational curves. That is, does a
local biholomorphism that is only known to respect tangent spaces of the $Q$-cycles extend to a global biholomorphism? In this section we will show that the answer is affirmative when $G/P$ is of Picard number one and give a proof of Theorem 1.5.

The question is whether the preservation of the variety of tangent spaces of $Q$-cycles implies the $Q$-cycle-respecting property. It can be rephrased in the following way.

**Question.** Let $S \subset G/P$ be a locally closed, connected complex submanifold which is an integral variety of $C \subset \text{Gr}(k, T(G/P))$, i.e., for every $p \in S$, there is a $Q$-cycle, depending on $p$, sharing the same tangent subspace with $S$ at $p$. Is $S$ itself an open subset of a $Q$-cycle?

An affirmative answer to the above question is sufficient to prove Theorem 1.5. However, in the case of minimal rational curves, i.e. when the $Q$-cycles are lines, then there exists some holomorphic map $h : \mathbb{P}^1 \to G/P$ such that $h(\mathbb{P}^1)$ is an integral variety of $C \subset \text{Gr}(1, T(G/P)) = \mathbb{P}(T(G/P))$ and $h(\mathbb{P}^1)$ is not a line (Section 6 of [5]). Nevertheless, the Cartan-Fubini extension for minimal rational curves still holds.

If $Q$-cycles are either non-linear or a maximal linear subspace, the answer to above question is affirmative when $G/P$ is of Picard number 1 with some exceptions, described as follows.

**Proposition 5.1.** Let $G/P$ be a rational homogeneous space of Picard number 1 and let $Q \subset G$ be a parabolic subgroup such that $P \cap Q$ is parabolic. Assume that the $Q$-cycles are either maximal linear subspaces of $G/P$ or non-linear homogeneous subspaces. If $S \subset G/P$ is a locally closed, connected complex submanifold which is an integral variety of $G/(P \cap Q) \subset \text{Gr}(k, T(G/P))$, then $S$ is an open subset of a $Q$-cycle unless

1. $G/P$ is of type $(B_\ell, \alpha_\ell)$ and $Q$ is associated with $\{\alpha_{\ell-1}, \alpha_\ell\}$;
2. $G/P$ is of type $(C_\ell, \alpha_\ell)$ and $Q$ is associated with $\{\alpha_{\ell-1}\}$;
3. $G/P$ is of type $(F_4, \alpha_1)$ and $Q$ is associated with $\{\alpha_3\}$;
4. $G/P$ is of type $(G_2, \alpha_2)$ and $Q$ is associated with $\{\alpha_1\}$.

**Proof.** The desired statement is included in Corollary 1.2 and Lemma 1.2 of [22].

**Proof of Theorem 1.5** If the $Q$-cycles are either non-linear homogeneous subspaces or maximal linear subspaces of $G/P$ other than the exceptions listed in Proposition 5.1, any local biholomorphism $f : U \to f(U) \subset G/P$ preserving the variety of tangent spaces of $Q$-cycles is $Q$-cycle-respecting by Proposition 5.1. By Theorem 1.1, $f$ can be extended to a biholomorphism of $G/P$. 

**Proof.** The desired statement is included in Corollary 1.2 and Lemma 1.2 of [22].
If the $Q$-cycles are non-maximal linear subspaces of $G/P$ or maximal linear spaces in the list of Proposition 5.1, then the preservation of the variety of tangent spaces of $Q$-cycles implies the preservation of the variety of tangent spaces of invariant lines (see the remark below), and thus, by Corollary 5.4 of [27], $f$ can be extended to a biholomorphism of $G/P$. □

**Remark.** If $P$ is associated to a long root, then lines in $G/P$ are $Q$-cycles for some $Q$. However, if $P$ is associated to a short root, then generic lines are not $Q$-cycles. By an invariant line we mean a line that is a $Q$-cycle. The Cartan-Fubini extension Theorem of Hwang-Mok [12] can be applied to a local biholomorphism preserving the set of tangent directions of lines in $G/P$, while Corollary 5.4 of [27] can be applied to a local biholomorphism preserving the set of tangent directions of invariant lines in $G/P$.

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