Edge-Corner Correspondence: Boundary-Obstructed Topological Phases with Chiral Symmetry

Motohiko Ezawa
Department of Applied Physics, University of Tokyo, Hongo 7-3-1, 113-8656, Japan

The bulk-edge correspondence characterizes topological insulators and superconductors. We generalize this concept to the bulk-corner correspondence and the edge-corner correspondence in two dimensions. In the bulk-corner (edge-corner) correspondence, the topological number is defined for the bulk (edge), while the topological phase is evidenced by the emergence of zero-energy corner states. It is shown that the boundary-obstructed topological phases recently proposed are the edge-corner-correspondence type, while the higher-order topological phases are classified into the bulk-corner-correspondence type and the edge-corner-correspondence type. We construct a simple model exhibiting the edge-corner correspondence based on two Chern insulators having the $s$-wave, $d$-wave and $s_{\pm}$-wave pairings. It is possible to define topological numbers for the edge Hamiltonians, and we have zero-energy corner states in the topological phase. The emergence of zero-energy corner states is observable by measuring the impedance resonance in topological electric circuits.

Topological insulators and superconductors are one of the most studied fields in this decade\cite{28}. A topological number is defined for the bulk. The bulk gap must close at a topological phase transition point because the topological number cannot change its quantized value without gap closing. Consequently, the topological phase is evidenced by the emergence of gapless edge states although the bulk is gapped. This phenomenon is well known as the bulk-edge correspondence.

In this paper, we construct a simple model realizing the edge-corner correspondence and the boundary-obstructed topological phase in order to make a clear understanding of these phenomena. The model consists of two Chern insulators with the opposite Chern numbers. We introduce pairing interactions including the $s$-wave, $d$-wave and $s_{\pm}$-wave pairings. The system has chiral symmetry. The topological class is BDI, which allows a topological phase in one dimension but none in two dimensions\cite{23}. Accordingly, a topological phase transition may occur, where the edge gap is closed while the band gap is open, and zero-energy corner states emerge in the topological phase. It is possible to define the topological numbers $\Gamma_x$ and $\Gamma_y$ for the edge Hamiltonians along the $x$ and $y$ axes, and the topological number $\Gamma_x\Gamma_y$ for the bulk. We point out that the edge-corner correspondence is observable by the impedance measurement in electric circuits.

**Model Hamiltonian:** We start with the Chern insulator on square lattice, whose Hamiltonian is given by

\[
H_{\text{Chern}} = M \sigma_z + \lambda_x \sigma_x \sin k_x + \lambda_y \sigma_y \sin k_y,
\]

with Pauli matrices $\sigma_z$, spin-orbit interactions $\lambda_x$ and $\lambda_y$, and

\[
M = t_x \cos k_x + t_y \cos k_y - \mu,
\]

where $t_x$ and $t_y$ are hopping parameters, and $\mu$ is the chemical potential.
The system belongs to the class BDI, where there is no topological complexity, $\Xi$ complex conjugation. The system has particle-hole symmetry, $\Delta$. Here, $\Delta$ is the energy of $\Delta_0 = 0.5t$ and $\Delta_{s^\pm} = 0$ for simplicity. The bulk gap is independent of the magnitude of $\Delta$.

**Edge theory:** We show the band structure of a nanoribbon for various $\Delta_d$ in Fig 1(b1)~(b3). The edge states become gapless at a certain phase-transition point as in Fig 1(b2). In order to determine this point and to construct the topological phase diagram, we derive the edge Hamiltonians along the $x$ and $y$ axes.

We first construct the edge Hamiltonian along the $y$ axis. We make a Taylor expansion at the $\Gamma$ point and decompose the Hamiltonian (6) into the unperturbed Hamiltonian $H_0$ and the perturbed Hamiltonian $H_1$, $H = H_0 + H_1$, with

$$H_0 = \tau_2 [\sigma_z (2^{-1} t_x \partial_x^2 + t_x + t_y - \mu) - i \lambda_x \sigma_x \partial_x],$$

$$H_1 = \lambda_y k_y \tau_z \sigma_y + (\Delta_0 + \Delta_x + \Delta_y) \tau_x + 2^{-1} \Delta_y \tau_z \partial_y^2,$$

where we have omitted insignificant $k_y^2$ terms. Setting $H_0 = \tau_2 H_0^\dagger$, we solve the eigen equation of $H_0$ as $\psi = (\psi_1, i \psi_1)$ with $\psi_1 = e^{-i \kappa x \sigma_y k_y^2}$, where the penetration depth is

$$\kappa = \left( \eta \lambda_x \pm \sqrt{\lambda_x^2 + 2 t_x m} \right) / t_x,$$

with $m = \mu - t_x - t_y$. By taking an expectation value of $H_1$ by $\psi$, we obtain the edge Hamiltonian along the $y$ axis as

$$H_y = \eta_y \lambda_y k_y \tau_z + \Delta_y \tau_x,$$

with

$$\tilde{\Delta}_0 = \Delta_0 + \Delta_y m / t_x,$$

where $\alpha = x, y$; $\eta_x = 1$ for the lower and $\eta_x = -1$ for upper edges. The edge gap closes at $\Delta_x = 0$. Similarly we obtain the edge Hamiltonian for the $x$ axis as

$$H_x = \eta_x \lambda_x k_x \tau_z + \tilde{\Delta}_0 \tau_x,$$

where $\eta_y = 1$ for the right and $\eta_y = -1$ for left edges. The edge gap closes at $\Delta_y = 0$.

**Edge symmetries:** The edge Hamiltonian $H_\alpha$ has chiral symmetry, $\{ H, \tau_y \} = 0$. The system has time-reversal symmetry, $T \mathcal{H} (k) T^\dagger = \mathcal{H} (-k)$ with $T = \sigma_x \tau_z K$, where $K$ takes complex conjugation. The system has particle-hole symmetry, $\Xi \mathcal{H} (k) \Xi^\dagger = - \mathcal{H} (-k)$ with $\Xi = \sigma_x \tau_z K$. Hence, the system belongs to the class BDI, where there is no topological phase in two dimensions. In addition, the system has parity symmetry $\mathcal{P} \mathcal{H} (k) \mathcal{P}^\dagger = \mathcal{H} (-k)$ with $\mathcal{P} = \sigma_z$.

**Bulk:** The energy spectrum of $H$ is given by

$$E = \pm \sqrt{E_{\text{Chern}}^2 + E_\Delta^2},$$

where

$$E_{\text{Chern}} = \sqrt{M^2 + \lambda_x^2 \sigma_x^2 + \lambda_y^2 \sigma_y^2},$$

is the energy of $H_{\text{Chern}}$, and

$$E_\Delta = \Delta_0 + \Delta_x \sigma_x k_x + \Delta_y \sigma_y k_y$$

is the energy of $H_\Delta$. The bulk-band gap is

$$2E (0, 0) = 2 \sqrt{(t_x + t_y - \mu)^2 + \Delta_0^2}.$$  

The bulk gap $E$ never closes when the $E_{\text{Chern}}$ is gapped.

We show the band structure for various $\Delta_d$ in Fig 1(a1)~(a3) by setting $t_x = t_y = \lambda_x = \lambda_y = \mu = t$,

FIG. 1: Band structures of (a) the bulk and (b) a nanoribbon. The nanoribbon bands consist of the bulk bands (in cyan) and the edge bands (in blue). The edge becomes gapless only at the topological phase transition point as in (b2), while the bulk gap is always open.

We have used parameters $\Delta_d = 0$ for (a1) and (b1), $\Delta_d = 0.5t$ for (a2) and (b2), and $\Delta_d = t$ for (a3) and (b3). We have set $t_x = t_y = \lambda_x = \lambda_y = t$, $\Delta_0 = 0.5t$ and $\Delta_{s^\pm} = 0$. The nanoribbon width is $W = 20$.
Similarly, the zero-energy solution along the \( y \)-axis is derived by solving (18), or

\[
\psi_y(y) = \exp \left[ \mp \frac{\eta_y}{\lambda_y} \bar{\Delta}_y y \right].
\]  

The wave functions are well defined when they converge, leading to the condition,

\[
\bar{\Delta}_x \bar{\Delta}_y < 0.
\]  

The zero-energy corner states emerge only in this case.

**Topological phase diagram:** We construct the topological phase diagram in the \((\bar{\Delta}_x, \bar{\Delta}_y)\) plane. As we have just derived, the condition for the emergence of the zero-energy corner states is given by (25), or \(\bar{\Delta}_x \bar{\Delta}_y < 0\), where the system is topological. On the other hand, the system is trivial for \(\bar{\Delta}_x \bar{\Delta}_y > 0\), because there are no zero-energy states. The topological phase diagram is determined by these conditions as in Fig.2(d), with the phase boundary being given by \(\bar{\Delta}_x \bar{\Delta}_y = 0\).

We may define the topological number for the bulk, which reproduces the phase diagram determined in terms of \(\bar{\Delta}_x \bar{\Delta}_y\). With the use of \(\Gamma_\alpha\) defined by (21), it is given by

\[
\Gamma_{xy} = \Gamma_x \Gamma_y,
\]  

where \(\Gamma_{xy} = 1\) for the trivial phase and \(\Gamma_{xy} = -1\) for the topological phase as in Fig.2(d).

**Electric-circuit realization:** Electric circuits provide us with an ideal playground to realize various topological phases.\(^{32,33}\) Especially, higher-order topological phases\(^{32,33}\) are simulated based on electric circuits. Topological corner states are observed by impedance resonance.\(^{32,33}\)

Electric-circuit realization is based on the correspondence\(^{22,23}\) between the Hamiltonian \(H\) and the circuit Laplacian \(J\) such that \(J = \omega H\). The Hamiltonian \(H_{\text{chiral}}\) is already discussed in electric circuits.\(^{20}\) The circuit corresponding to Hamiltonian \(H\) can be constructed in a similar manner. The capacitance contribute to \(\omega C\), while the inductance contributes to \(1/\omega L\) in the circuit Laplacian. It corresponds to the positive and negative hoppings in the Hamiltonian. The imaginary hoppings are represented by the operational amplifiers\(^{20}\). We tune the frequency \(\omega\) to be the critical frequency \(\omega_0 = 1/\sqrt{LC}\) so that the circuit Laplacian is identical to the Hamiltonian.

Admittance is obtained by the eigenvalue of the circuit Laplacian\(^{32}\), which corresponds to the eigen energy of the Hamiltonian. We show the admittance spectrum as a function of \(\omega\) in Fig.3(a). The zero-admittance state appears in the topological phase, while it is absent in the trivial phase. The circuit Laplacian is explicitly given by

\[
J = \begin{pmatrix}
  f_1 & g_1 & h & 0 \\
  g_2 & f_2 & 0 & h \\
  h & 0 & f_2 & g_2 \\
  0 & h & g_1 & f_1
\end{pmatrix}
\]  

FIG. 2: (a) Energy spectrum in square geometry. The bulk and edge bands are shown in cyan and blue, respectively. The bulk gap is open at the topological phase transition point. The horizontal axis is \(\Delta_i/t\), while the vertical axis is the energy in unit of \(t\). We have used a square with size \(N = 32\). (b) Bulk gap (in cyan) and edge gap (in blue) as a function of \(\Delta_i/t\), which are obtained analytically. The horizontal lines (in magenta) represent zero-energy corner states. (c) LDOS for zero-energy corner states. All other parameters are the same as in Fig.1. (d) Topological phase diagram is determined by the sign of \(\bar{\Delta}_x \bar{\Delta}_y\). We can assign the topological number \(\Gamma_\alpha\) to this phase diagram, where \(\Gamma_{xy} = 1\) (\(\Gamma_{xy} = -1\)) stands for the trivial (topological) phase.

it is calculated as

\[
\Gamma_\alpha = -\eta_\alpha \text{sgn} \left[ \lambda_\alpha \bar{\Delta}_\alpha \right].
\]  

We can differentiate the trivial and topological phases by the conditions \(\bar{\Delta}_x > 0\) and \(\bar{\Delta}_x < 0\), respectively.

**Corner theory:** We calculate numerically the energy spectrum and show it as a function of \(\Delta_i\) for square geometry in Fig.2(a). It consists of the bulk and edge states shown in blue and cyan, respectively. On the other hand, we can derive analytically the bulk-band gap as in Eq. (12) and the edge-band gap as in Eq. (17), which we show in Fig.2(b). The numerical and analytical results agree very well. Additionally, we have shown zero-energy corner states in magenta in Figs.2(a) and (b), which we derive analytically in what follows. It is prominent that the edge gap closes only at the phase transition point \((\Delta_x = 0)\). We show the local density of states (LDOS) for the zero-energy corner states in Fig.2(c).

We study the corner states analytically. They are described by the Jackiw-Rebbi solutions\(^{34}\) of the edge Hamiltonians. The zero-energy solution along the \(x\)-axis is derived by solving (18), or

\[
H_x = -i\eta_x \lambda_x \tau_z \partial_x + \bar{\Delta}_x \tau_x.
\]  

The solution is obtained as \(\psi_x(x) = (\psi_{x1}, i\eta_x \psi_{x1})\) with

\[
\psi_x(x) = \exp \left[ \pm \frac{\eta_x}{\lambda_x} \bar{\Delta}_x x \right].
\]  

Similarly, the zero-energy solution along the \(y\)-axis reads

\[
\psi_y(y) = \exp \left[ \mp \frac{\eta_y}{\lambda_y} \bar{\Delta}_y y \right].
\]
It is straightforward to design electric circuits corresponding to this circuit Laplacian.

We analyze the impedance\(^{(1)}\), which is defined by \(Z_{ab} = V_a/I_b = G_{ab}\) where \(G = J^{-1}\) is the Green function. It diverges at the frequency where the admittance is zero (\(J = 0\)). Note that the impedance and the admittance have inverse relation. Taking the nodes \(a\) and \(b\) at two corners, we show the impedance in the trivial and topological phases in Figs. 3(b1) and (b2), respectively. A strong impedance peak is observed at the critical frequency \(\omega_0\) in the topological phase, while it is absent in the trivial phase. It signals the emergence of zero-energy corner states. Hence, the edge-corner correspondence is observable in electric circuits.

**Discussions:** We have presented a classification of various higher-order topological phases based on the bulk-corner and edge-corner correspondences. We have constructed a simple model exhibiting the edge-corner correspondence, where topological numbers are defined for edges. It provides us with a clear picture of the boundary-obstructed topological phase transition. Furthermore, we have argued that the emergence of topological corner states is detectable by observing impedance peaks in topological electric circuit. The merit of electric circuits is that we can tune the values of elements relatively easily, which enables to realize a topological phase transition.

The generalization to three dimensions is straightforward. We may consider bulk-hinge, surface-hinge correspondences for the second-order topological phases\(^{(2)}\). We may also consider bulk-corner, surface-corner and hinge-corner correspondences for the third-order topological phases\(^{(3)}\).

The author is very much grateful to N. Nagaosa for helpful discussions on the subject. This work is supported by the Grants-in-Aid for Scientific Research from MEXT KAKENHI (Grants No. JP17K05490 and No. JP18H03676). This work is also supported by CREST, JST (JPMJCR16F1).

---

1. M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
2. X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
3. M. Leijnse and K. Flensberg, Semicond. Sci. Technol. 27, 124003 (2012).
4. C. W. J. Beenakker, Annu. Rev. Con. Mat. Phys. 4, 113 (2013).
5. See Supplementary Material for the generalization of the bulk-boundary correspondence to include the second-order topological phases and the third-order topological phases in three dimensions.
6. W. A. Benalcazar, B. A. Bernevig, and T. L. Hughes, 357, 61 (2017).
7. F. Zhang, C.L. Kane and E.J. Mele, Phys. Rev. Lett. 110, 046404 (2013).
8. F. Schindler, A. Cook, M. G. Vergniory, and T. Neupert, in APS March Meeting (2017).
9. Y. Peng, Y. Bao, and F. von Oppen, Phys. Rev. B 95, 235143 (2017).
10. J. Langbehn, Y. Peng, L. Trifunovic, F. von Oppen, and P. W. Brouwer, Phys. Rev. Lett. 119, 246401 (2017).
11. Z. Song, Z. Fang, and C. Fang, Phys. Rev. Lett. 119, 246402 (2017).
12. W. A. Benalcazar, B. A. Bernevig, and T. L. Hughes, Phys. Rev. B 96, 245115 (2017).
13. F. Schindler, A. M. Cook, M. G. Vergniory, Z. Wang, S. S. P. Parkin, B. A. Bernevig, and T. Neupert, Science Advances 01 Jun 2018: Vol. 4, no. 6, eaat0346.
14. C. Fang, L. Fu, Science Advances 5, eaat2374 (2019).
15. M. Ezawa, Phys. Rev. Lett. 120, 026801 (2018).
16. M. Ezawa, Phys. Rev. B 98, 045125 (2018).
17. M. Geier, L. Trifunovic, M. Hoskam, and P. W. Brouwer, Phys. Rev. B 97, 205135 (2018).
18. M. Ezawa, Phys. Rev. B 97, 155305 (2018).
19. E. Khalaf, Phys. Rev. B 97, 205136 (2018).
20. Z. Yan, F. Song and Z. Wang, Phys. Rev. Lett. 121, 096803 (2018)
21. Q. Wang, C.-C. Liu, Y.-M. Lu and F. Zhang, Phys. Rev. Lett. 121, 186801 (2018).
22. Y. Wang, M. Lin, T. L. Hughes, Phys. Rev. B 98, 165144 (2018).
23. E. Khalaf, W. A. Benalcazar, T. L. Hughes and R. Queiroz, arXiv:1908.00011.
24. K. Asaga and T. Fukui, arXiv:2002.04209
25. J. Claes and T. L. Hughes, arXiv:2005.09659

---

FIG. 3: (a) Admittance spectrum in square geometry (a1) for the trivial phase and (a2) for the topological phase. The horizontal axis is \(\omega/\omega_0\). There are four-fold degenerate zero-admittance states at \(\omega_0 = \omega\) in the topological phase. They are marked by a black circle in (a2). (b) Impedance as a function of \(\omega\) (b1) in the trivial phase and (b2) in the topological phase. There is a strong peak at \(\omega = \omega_0\) in the topological phase, as marked by a black circle in (a2). The horizontal axis is \(\omega/\omega_0\). We have used a square with size \(N = 16\).

\[
\begin{align*}
\left(1\right) & \quad f_1 = i\omega \left(C_x \cos k_x + C_y \cos k_y\right) - i\omega C_x - i\omega C_y + i\omega C_m,
\left(2\right) & \quad f_2 = \left(i\omega L_x\right)^{-1} \cos k_x + \left(i\omega L_y\right)^{-1} \cos k_y - \left(i\omega L_x\right)^{-1} - \left(i\omega L_y\right)^{-1} - \left(i\omega L_m\right)^{-1},
\left(3\right) & \quad g_1 = i\omega C_x e^{ik_x} + \left(i\omega L_x\right)^{-1} e^{-ik_x} + R^{-1} \left(e^{ik_x} - e^{-ik_x}\right),
\left(4\right) & \quad g_2 = \left(i\omega L_x\right)^{-1} e^{ik_x} + i\omega C_x e^{-ik_x} + R^{-1} \left(e^{ik_x} - e^{-ik_x}\right),
\left(5\right) & \quad h = i\omega C_{\Delta 0} + i\omega C_{\Delta x} \cos k_x + i\omega C_{\Delta y} \cos k_y.
\end{align*}
\]
26. A. Tiwari, A. Jahin and Y. Wang, arXiv:2005.12291
27. X. Wu, W. A. Benalcazar, Y. Li, R. Thomale, C.-X. Liu, and J. Hu, arXiv:2003.12204
28. A. P. Schnyder, Shinsei Ryu, A. Furusaki and A. W. W. Ludwig, Phys. Rev. B 78, 195125 (2008).
29. G. R. Stewart, Reviews of Modern Physics 83, 1589 (2011)
30. P. J. Hirschfeld, M. M. Korshunov and I. I. Mazin, Reports on Progress in Physics 74, 124508 (2011)
31. R. Jackiw and C. Rebbi, Phys. Rev. D 13 3398 (1976)
32. S. Imhof, C. Berger, F. Bayer, J. Brehm, L. Molenkamp, T. Kiessling, F. Schindler, C. H. Lee, M. Greiter, T. Neupert, R. Thomale, Nat. Phys. 14, 925 (2018).
33. C. H. Lee , S. Imhof, C. Berger, F. Bayer, J. Brehm, L. W. Molenkamp, T. Kiessling and R. Thomale, Communications Physics, 1, 39 (2018).
34. T. Helbig, T. Hofmann, C. H. Lee, R. Thomale, S. Imhof, L. W. Molenkamp and T. Kiessling, Phys. Rev. B 99, 161114 (2019).
35. Y. Lu, N. Jia, L. Su, C. Owens, G. Juzeliunas, D. I. Schuster and J. Simon, Phys. Rev. B 99, 020302 (2019).
36. K. Luo, R. Yu and H. Weng, Research (2018), ID 6793752.
37. E. Zhao, Ann. Phys. 399, 289 (2018).
38. M. Ezawa, Phys. Rev. B 98, 201402(R) (2018).
39. M. Serra-Garcia, R. Susstrunk and S. D. Huber, Phys. Rev. B 99, 020304 (2019).
40. T. Hofmann, T. Helbig, C. H. Lee, M. Greiter, R. Thomale, Phys. Rev. Lett. 122, 247702 (2019).
41. M. Ezawa, Phys. Rev. B 100, 045407 (2019)
42. M. Ezawa, cond-mat/arXiv:1907.06911
43. M. Ezawa, Phys. Rev. B 99, 201411(R) (2019).
44. M. Ezawa, Phys. Rev. B 99, 121411(R) (2019).
Supplemental Material

Edge-Corner Correspondence: Boundary-Obstructed Topological Phases with Chiral Symmetry

Motohiko Ezawa
Department of Applied Physics, University of Tokyo, Hongo 7-3-1, 113-8656, Japan

S1. Second-order topological phases in three dimensions

The second-order topological phases in three dimensions are characterized by the phenomena that gapless hinge states emerge although the bulk and the surface are gapped. They are classified into the bulk-hinge-correspondence type and the surface-hinge-correspondence type according to the geometrical object upon which the topological numbers are defined. Hereafter, let us abbreviate the A-B-correspondence type as the A-B type. (i) In the bulk-hinge type, the bulk gap closes at the phase transition point. The topological number is defined for the bulk. (ii) In the surface-hinge type, the surface gap closes but the bulk gap remains to open at the phase transition point. The topological number is defined for the surface.

These phenomena are summarized as

\[
\text{(i) bulk-hinge correspondence:} \quad \begin{array}{c|c|c|c|c}
\text{trivial} & \text{PTP} & \text{topological} \\
\hline
\text{bulk} & o & \times & o \\
\text{surface} & o & \times & o \\
\text{hinge} & o & \times & o \\
\end{array}
\] (S1)

\[
\text{(ii) surface-hinge correspondence:} \quad \begin{array}{c|c|c|c|c}
\text{trivial} & \text{PTP} & \text{topological} \\
\hline
\text{bulk} & o & o & o \\
\text{surface} & o & \times & o \\
\text{hinge} & o & \times & o \\
\end{array}
\] (S2)

S2. Third-order topological phases in three dimensions

The third-order topological phases in three dimensions are characterized by the phenomena that zero-energy corner states emerge although the bulk, the surface and the hinge are gapped. They are classified into three types, the bulk-corner type, surface-corner type and the hinge-corner type, according to the geometrical object upon which the topological numbers are defined. (i) In the bulk-corner type, the bulk gap closes at the phase transition point. The topological number is defined for the bulk. (ii) In the surface-corner type, the surface gap closes but the bulk gap remains to open at the phase transition point. The topological number is defined for the surface. (iii) In the hinge-corner type, the hinge gap closes but the bulk and surface gaps remain to open at the phase transition point. The topological number is defined for the hinge.

These phenomena are summarized as

\[
\text{(i) bulk-corner correspondence:} \quad \begin{array}{c|c|c|c|c|c|c}
\text{trivial} & \text{PTP} & \text{topological} \\
\hline
\text{bulk} & o & \times & o \\
\text{surface} & o & \times & o \\
\text{hinge} & o & \times & o \\
\text{corner} & o & \times & \times \\
\end{array}
\] (S3)

\[
\text{(ii) surface-corner correspondence:} \quad \begin{array}{c|c|c|c|c|c|c}
\text{trivial} & \text{PTP} & \text{topological} \\
\hline
\text{bulk} & o & o & o \\
\text{surface} & o & \times & o \\
\text{hinge} & o & \times & o \\
\text{corner} & o & \times & \times \\
\end{array}
\] (S4)

\[
\text{(iii) hinge-corner correspondence:} \quad \begin{array}{c|c|c|c|c|c|c}
\text{trivial} & \text{PTP} & \text{topological} \\
\hline
\text{bulk} & o & o & o \\
\text{surface} & o & o & o \\
\text{hinge} & o & \times & o \\
\text{corner} & o & \times & \times \\
\end{array}
\] (S5)