ON COHESIVE ALMOST ZERO-DIMENSIONAL SPACES

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Abstract. We investigate C-sets in almost zero-dimensional spaces, showing that closed σC-sets are C-sets. As corollaries, we prove that every rim-σ-compact almost zero-dimensional space is zero-dimensional and that each cohesive almost zero-dimensional space is nowhere rational. To show these results are sharp, we construct a rim-discrete connected set with an explosion point. We also show every cohesive almost zero-dimensional subspace of (Cantor set)×R is nowhere dense.

1. Introduction

All spaces under consideration are separable and metrizable.

A subset A of a topological space X is called a C-set in X if A can be written as an intersection of clopen subsets of X. A σC-set is a countable union of C-sets. A space X is said to be almost zero-dimensional provided every point x∈X has a neighborhood basis consisting of C-sets in X.

A space X is cohesive if every point x∈X has a neighborhood which contains no non-empty clopen subset of X. Clearly every cohesive space is nowhere zero-dimensional. The converse is false, even for almost zero-dimensional spaces [10]. Spaces which are both almost zero-dimensional and cohesive include:

- Erdős space  \( \mathcal{E} = \{ x \in \ell^2 : x_i \in \mathbb{Q} \text{ for each } i < \omega \} \) and
- complete Erdős space  \( \mathcal{E}_c = \{ x \in \ell^2 : x_i \in \{0\} \cup \{1/n : n = 1, 2, 3, \ldots\} \text{ for each } i < \omega \} \),

where \( \ell^2 \) stands for the Hilbert space of square summable sequences of real numbers. Other examples include the homeomorphism groups of the Sierpiński carpet and Menger universal curve [33, 8], and various endpoint sets in complex dynamics [2, 31].

Almost zero-dimensionality of \( \mathcal{E} \) and \( \mathcal{E}_c \) follows from the fact that each closed \( \varepsilon \)-ball in either space is closed in the zero-dimensional topology inherited from \( \mathbb{Q}^\omega \), which is weaker than the \( \ell^2 \)-norm topology. The spaces are cohesive because all non-empty clopen subsets of \( \mathcal{E} \) and \( \mathcal{E}_c \) are unbounded in the \( \ell^2 \)-norm as proved by Erdős [18]. Thus, if we add a point \( \infty \) to \( \ell^2 \) whose neighborhoods are the complements of bounded sets then we have that \( \mathcal{E} \cup \{\infty\} \) and \( \mathcal{E}_c \cup \{\infty\} \) are connected. The following result is Proposition 5.4 in Dijkstra and van Mill [12].

**Proposition 1.1.** Every almost zero-dimensional cohesive space has a one-point connection. If a space has a one-point connection then it is cohesive.

Actually, open subsets of non-singleton connected spaces are cohesive, because cohesion is open hereditary [12, Remark 5.2]. More information on cohesion and one-point connectifications can be found in [1].

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In Section 3 of this paper we will show that every cohesive almost zero-dimensional space $E$ is homeomorphic to a dense subset $E' \subset \mathcal{E}_c$ such that $E' \cup \{\infty\}$ is connected. The result is largely a consequence of earlier work by Dijkstra and van Mill [12, Chapters 4 and 5]. We apply the embedding to show that every cohesive almost zero-dimensional subspace of $(\text{Cantor set}) \times \mathbb{R}$ is nowhere dense, and there is a continuous one-to-one image of complete Erdős space that is totally disconnected but not almost zero-dimensional.

In Section 4 of the paper we examine C-sets and the rim-type of almost zero-dimensional spaces. We say $X$ is rational at $x \in X$ if $x$ has a neighborhood basis of open sets with countable boundaries. In [32, §6 Example p.596], Nishiura and Tymchatyn implicitly proved that $D_e$, the set of endpoints of Lelek’s fan [27, §9], is not rational at any of its points. Results in [5, 6, 23] later established $D_e \simeq \mathcal{E}_c$, so $\mathcal{E}_c$ is nowhere rational. Working in $\ell^2$, Banakh [3] recently demonstrated that each bounded open subset of $\mathcal{E}$ has an uncountable boundary. We generalize these results by proving that each cohesive almost zero-dimensional space is nowhere rational. Moreover, every rim-$\sigma$-compact almost zero-dimensional space is zero-dimensional. We also find that in almost zero-dimensional spaces cohesion is preserved if we delete $\sigma$-compacta. These results follow from Theorem 4.3, which states that closed $\sigma$C-sets in almost zero-dimensional spaces are C-sets.

In Section 5 we will construct a rim-discrete connected space $\tau$ with an explosion point. The example is partially based on [30, Example 1], which was constructed by the second author to answer a question from the Houston Problem Book [7]. The pulverized complement of the explosion point will be a rim-discrete totally disconnected set which is not zero-dimensional, in contrast with Section 4 results. Additionally, the rim-discrete property guarantees the entire connected set has a rational compactification [35, 20, 19]. We therefore solve [7, Problem 79] in the context of explosion point spaces. Results from Section 4 indicate that this new solution is optimal.

In general, $ZD \implies AZD \implies TD \implies HD$, where we used abbreviations for zero-dimensional, almost zero-dimensional, totally disconnected, and hereditarily disconnected. In certain contexts, these implications can be reversed. For example,

$$HD \overset{(1)}{\implies} TD \overset{(2)}{\implies} AZD \overset{(3)}{\implies} ZD$$

for subsets of hereditarily locally connected continua [24, §50 IV Theorem 9]. As mentioned above, the implication (3) is valid in the larger class of subsets of rational continua. But [30, Example 1] and the example $\tau$ in Section 5 show that (1) and (2) are generally false in that context.

2. Preliminaries

A space $X$ is hereditarily disconnected if every connected subset of $X$ contains at most one point. A space $X$ is totally disconnected if every singleton in $X$ is a C-set. A point $x$ in a connected space $X$ is:

- a dispersion point if $X \setminus \{x\}$ is hereditarily disconnected;
- an explosion point if $X \setminus \{x\}$ is totally disconnected.

If $P$ is a topological property then a space $X$ is rim-$P$ provided $X$ has a basis of open sets whose boundaries have the property $P$. Rational $\equiv$ rim-countable. Zero-dimensional $\equiv$ rim-empty.

Throughout the paper, $\mathcal{E}$ will denote the middle-third Cantor set in $[0, 1]$. The coordinate projections in $\mathbb{R}^2$ are denoted $\pi_0$ and $\pi_1$; $\pi_0((x, y)) = x$ and $\pi_1((x, y)) = y$. We define $\nabla : [0, 1]^2 \to [0, 1]^2$ by $(x, y) \mapsto (xy + \frac{1}{2}(1 - y), y)$. The image of $\nabla$ is the region
enclosed by the triangle with vertices $(0, 1)$, $(\frac{1}{2}, 0)$, and $(1, 1)$. Note that $\triangledown | [0, 1] \times (0, 1]$ is a homeomorphism and $\triangledown^{-1}(\frac{1}{2}, 0) = [0, 1] \times \{0\}$. For each $X \subset \mathcal{E} \times (0, 1]$ we put

$$\triangledown X = \triangledown(X) \cup \{(\frac{1}{2}, 0)\}.$$  

The Cantor fan is the set $\triangledown(\mathcal{E} \times [0, 1]) = \triangledown(\mathcal{E} \times (0, 1])$, see Figure 1.

Given $X \subset \mathcal{E}$, a function $\varphi : X \to [0, 1]$ is upper semi-continuous (abbreviated USC) if $\varphi^{-1}[0, t)$ is open in $X$ for every $t \in [0, 1]$. Define

$$G_{\varphi}^0 = \{(x, \varphi(x)) : \varphi(x) > 0\};$$ and
$$L_{\varphi}^0 = \{(x, t) : 0 \leq t \leq \varphi(x)\}.$$  

We say $\varphi$ is a Lelek function if $\varphi$ is USC and $G_{\varphi}^0$ is dense in $L_{\varphi}^0$. Lelek functions with domain $\mathcal{E}$ exist, and if $\varphi$ is a Lelek function with domain $\mathcal{E}$ then $\triangledown L_{\varphi}^0$ is a Lelek fan, see Figure 2. For example, let $\| \|$ be the $\ell^2$-norm and identify $\mathcal{E}$ with the Cantor set $(\{0\} \cup \{1/n : n = 1, 2, 3, \ldots\})^\omega$. Define $\eta(x) = 1/(1 + \|x\|)$, where $1/\infty = 0$. Then $\mathcal{E}_c$ is homeomorphic to $G_{\eta}^0$, $\eta : \mathcal{E} \to [0, 1]$ is a Lelek function, and $\triangledown L_{\eta}^0$ is a Lelek fan; see [34] and the proof of [9, Theorem 3].

Figure 1. Cantor fan  
Figure 2. Lelek fan

3. Embedding into fans and complete Erdős space

Let $E$ be any non-empty cohesive almost zero-dimensional space. Dijkstra and van Mill proved: There is a Lelek function $\chi : X \to [0, 1]$ such that $E$ is homeomorphic to $G_{\chi}^0$ and hence $E$ admits a dense embedding in $\mathcal{E}_c$ [12, Proposition 5.10]. We observe:

**Theorem 3.1.** For the Lelek function $\chi$ constructed in [12], $\triangledown G_{\chi}^0$ is connected. Thus, there is a dense homeomorphic embedding $\alpha : E \hookrightarrow \mathcal{E}_c$ such that $\alpha(E) \cup \{\infty\}$ is connected.

**Proof.** In [12], $\chi$ is constructed via two USC functions $\varphi$ and $\psi$ which have the same zero-dimensional domain $X$. First, $\varphi$ is given by [12, Lemma 4.11] such that $E$ is homeomorphic to $G_{\varphi}^0$. And then, in the proof of [12, Lemma 5.8], $\psi$ is defined by

$$\psi(x) = \lim_{\varepsilon \to 0^+} \inf J_\varepsilon(x),$$

where $U_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$; and

$$J_\varepsilon(x) = \{t \in [0, 1] : U_\varepsilon(x) \times (t, 1) \cap G_{\varphi}^0 \text{ contains no non-empty clopen subset of } G_{\varphi}^0\}.$$
Notice that $J_ε(x)$ becomes larger as $ε$ decreases, so its infimum decreases. Thus $ψ(x)$ is well-defined. Finally, $χ$ is defined so that $(x, ϕ(x)) \mapsto (x, χ(x))$ is a homeomorphism and $χ ≤ ϕ − ψ$ [12, Lemma 4.9].

To prove that $∇G_0^0$ is connected, we let $A$ be any non-empty clopen subset of $G_0^0$ and show $0 ∈ π_1(A)$. Define $y = \inf{\{ϕ(x) : x ∈ π_0(A)\}}$ and let $ε > 0$. Pick an $x ∈ π_0(A)$ with $ϕ(x) < y − ε$. Since $\{⟨x, ϕ(x)⟩ : x ∈ π_0(A)\}$ is a clopen subset of $G_0^0$ and $X$ is zero-dimensional, $ψ(x) ≥ y$. We have $⟨x, χ(x)⟩ ∈ A$ and $π_1(⟨x, χ(x)⟩) = χ(x) ≤ ϕ(x) − ψ(x) < (y + ε) − y = ε$. Since $ε$ was an arbitrary positive number, this shows that $0 ∈ π_1(A)$.

We will now construct $α$. Since $χ$ is Lelek, $X$ is perfect, so we may assume $X$ is dense in $C$. Now $χ$ extends to a Lelek function $Y : C → [0, 1]$ such that $G_0^0$ is dense in $G_0^0$ [12, Lemma 4.8]. In particular, $∇L_0^0$ is a Lelek fan. By [5, 6] the Lelek fan is unique, so there is a homeomorphism $Ξ : ∇L_0^0 → ∇L_0^0$ (recall $η$ from Section 2). We observe that $Ξ(∇G_0^0) = ∇G_0^0 ≃ Ξ_c \cup \{∞\}$. So there is a homeomorphism $γ : ∇G_0^0 → Ξ_c \cup \{∞\}$. We know there is also a homeomorphism $β : E → ∇G_0^0$. Let $α = γ ◦ β$. Notice that $α(E) \cup \{∞\} = γ(∇G_0^0)$ is connected.

**Corollary 3.2.** If $Y$ is a complete space containing $E$, then there is a complete cohesive almost zero-dimensional space $E'$ such that $E ⊂ E' ⊂ Y$.

**Proof.** Let $α : E → Ξ_c$ be given by Theorem 3.1. Since $Y$ and $Ξ_c$ are both complete, Lavrentiev’s Theorem [17, Theorem 4.3.21] says $α$ extends to a homeomorphism between $G_0^0$-sets $E'$ and $A$ such that $E ⊂ E' ⊂ Y$ and $α(E) ⊂ A ⊂ Ξ_c$. Since $α(E)$ is dense in $Ξ_c$ and $α(E) \cup \{∞\}$ is connected, $A \cup \{∞\}$ is connected. So $E'$ is cohesive. □

**Theorem 3.3.** Every cohesive almost zero-dimensional subset of $C × R$ is nowhere dense.

**Proof.** Cohesion is open-hereditary [12, Remark 5.2]. By self-similarity of $C × R$, it therefore suffices to show there is no dense cohesive almost zero-dimensional subspace of $C × R$. Suppose on the contrary that $E$ is such a space. By Corollary 3.2 there is a complete cohesive almost zero-dimensional $X ⊂ C × R$ such that $E ⊂ X$. Then $X$ is a dense $C_0^0$-subset of $C × R$, so by [4, 25] there exists $c ∈ C$ such that $X \cap \{(c) × R\} = \{(c) × R\}$. Let $x = (c, r) ∈ X$. We obtain a contradiction by showing $X$ is zero-dimensional at $x$.

Let $V × (a, b)$ be any regular open subset of $C × R$ which contains $x$. There exist an $r_1 ∈ (a, r)$ and an $r_2 ∈ (r, b)$ such that $x_1 = (c, r_1)$ and $x_2 = (c, r_2)$ are in $X$. Since $X$ is totally disconnected, there are $C_0^0$-sets $W_1$ and $W_2$ such that $x_1 ∈ W_1$, $x_2 ∈ W_2$, and $x ∉ W_1 ∪ W_2$. Let $U_1, U_2 ⊂ V$ be $C_0^0$-clopen sets such that $x_i ∈ (U_i × (r_i)) \cap X ⊂ W_i$ for each $i ∈ \{1, 2\}$. Then $[(U_1 ∩ U_2) × [r_1, r_2] \setminus (W_1 ∪ W_2)] ∩ X$ is an $X$-clopen subset of $V × (a, b)$ which contains $x$. This shows that $X$ is zero-dimensional at $x$. □

Corollary 3.3 shows that a certain continuous one-to-one mapping of $Ξ_c$ is totally disconnected but not almost zero-dimensional. Define $f : Ξ_c → \{(0) ∪ \{1/n : n = 1, 2, 3, ...\}\}^ω × [0, 1]$ by $f(x) = (x, \frac{1 + \sin \|x\|}{2})$. Let $Y = f(Ξ_c)$. Clearly, $f$ is one-to-one and continuous, and $Y$ is totally disconnected. The example $Y$ is essentially the same as [29, Example X2] and therefore by [29, Propositions 3 and 5] $Y$ is dense in $C × [0, 1]$ and $∇Y$ is connected. Thus $Y$ is cohesive. By Theorem 3.3 $Y$ is not almost zero-dimensional. Both this example and the space $τ$ constructed in Section 5 show that Theorem 3.3 does not extend to totally disconnected spaces.
4. σC-sets and rim-type

**Remark 1.** If \( x \in A_0 \subset X \) with \( \partial A \) a C-set in \( X \) then there is a clopen set \( C \) with \( x \in C \) and \( C \cap \partial A = \emptyset \) and hence \( C \cap A_0 = C \cap \overline{A} \) is also clopen. Consequently, rim-C is equivalent to zero-dimensional.

**Lemma 4.1.** For every two disjoint C-sets in a space, there is a clopen set containing one and missing the other.

**Proof.** Identical to the proof of [16, Lemma 1.2.6]. □

**Theorem 4.2.** Let \( A \) be a subset of an almost zero-dimensional space \( X \). If there is a \( \sigma \)C-set \( B \) with \( \partial A \subset B \subset A \), then \( A \) is a C-set.

**Proof.** Suppose \( B = \bigcup\{B_i : i < \omega\} \) where each \( B_i \) is a C-set, and \( \partial A \subset B \subset \overline{A} \). To prove \( A \) is a C-set, it suffices to show that for every \( x \in X \setminus \overline{A} \) there is an \( X \)-clopen set \( C \) such that \( x \in C \subset X \setminus \overline{A} \).

Let \( x \in X \setminus \overline{A} \). By the Lindelöf property and almost zero-dimensionality, it is possible to write the open set \( X \setminus \overline{A} \) as the union of countably many C-sets in \( X \) whose interiors cover \( X \setminus \overline{A} \). The property of being a C-set is closed under finite unions, so there is an increasing sequence of C-sets \( D_0 \subset D_1 \subset \cdots \) with \( x \in D_0 \) and

\[
\bigcup\{D_i : i < \omega\} = \bigcup\{D_0^i : i < \omega\} = X \setminus \overline{A}.
\]

By Lemma 4.1, for each \( i < \omega \) there is an \( X \)-clopen set \( C_i \) such that \( D_i \subset C_i X \setminus B_i \).

Let \( C = \bigcap\{C_i : i < \omega\} \setminus A^0 \). Clearly, \( C \) is closed, \( x \in C \), and

\[
C \subset X \setminus (A^0 \cup B) = X \setminus \overline{A}.
\]

Further, if \( y \in C \) then there exists \( j < \omega \) such that \( y \in D_j^0 \). The open set \( D_j^0 \cap \bigcap\{C_i : i < j\} \) witnesses that \( y \in C^0 \). This shows \( C \) is open and thus clopen. □

**Theorem 4.3.** In an almost zero-dimensional space, every closed \( \sigma \)C-set is a C-set.

**Proof.** Given a closed \( \sigma \)C-set \( A \), apply Theorem 4.2 with \( B = A \). □

With Remark 1 we get:

**Corollary 4.4.** Every rim-\( \sigma \)C almost zero-dimensional space is zero-dimensional.

Since compacta are C-sets in totally disconnected spaces we also have:

**Corollary 4.5.** Every almost zero-dimensional space that is rim-\( \sigma \)-compact or rational is zero-dimensional.

A space is called nowhere rim-\( \sigma \)C (nowhere rim-\( \sigma \)-compact respectively nowhere rational) if no point has a neighborhood basis consisting of sets that have boundaries that are \( \sigma \)C-sets (\( \sigma \)-compact respectively countable). With Theorem 4.3 and Remark 1 we also find:

**Corollary 4.6.** Cohesive almost zero-dimensional spaces are nowhere rim-\( \sigma \)C and hence nowhere rim-\( \sigma \)-compact and nowhere rational.

Thus there are no rim-\( \sigma \)-compact or rational connected spaces \( Y \) with a point \( p \) such that \( Y \setminus \{p\} \) is almost zero-dimensional, using Proposition 1.1.

**Theorem 4.7.** If \( X \) almost zero-dimensional, \( Y = X \cup \{p\} \) is connected, and \( K \subset X \) is \( \sigma \)-compact, then \( Y \setminus K \) is connected.


**Proof.** Suppose $X$ almost zero-dimensional, $Y$ is connected, and $K \subset X$ is $\sigma$-compact. Striving for a contradiction suppose $Y \setminus K$ is not connected. Then $Y \setminus K$ is the union of two non-empty relatively closed subsets $A$ and $B$ such that $A \cap B = \emptyset$. We may assume that $p \in B$. The closures of $A$ and $B$ in the open set $Y \setminus (A \cap B)$ are disjoint, so they are contained in disjoint $Y$-open sets $U$ and $V$. Note that $\partial U$ in $Y$ is contained in $K$ and is therefore $\sigma$-compact and hence a $\sigma$C-set in the totally disconnected space $X$. By Theorem 4.3 $\partial A$ is a $C$-set in $X$. So by Remark 1 $U$ contains a nonempty clopen subset $C$ of $X$. Note that $X$ is open in $Y$ and $U$ is contained in the $Y$-closed set $Y \setminus B$ so $C$ is also clopen in $Y$. This violates the assumption that $Y$ is connected. \hfill $\square$

Since $\mathcal{E} \cup \{\infty\}$ and $\mathcal{E}_c \cup \{\infty\}$ are connected we have:

**Corollary 4.8.** Bounded neighborhoods in $\mathcal{E}$ and $\mathcal{E}_c$ do not have $\sigma$-compact boundaries.

Combining Theorem 4.7 with Proposition 1.1 we find:

**Theorem 4.9.** If $X$ is cohesive and almost zero-dimensional and $K \subset X$ is $\sigma$-compact, then $X \setminus K$ is cohesive.

For the spaces $\mathcal{E}$, $\mathcal{E}_c$, and $\mathcal{E}_c^\sigma$ there is a stronger result: in these spaces $\sigma$-compacta are negligible, see [13], [23], and [11].

A connected space $X$ is $\sigma$-connected if $X$ cannot be written as the union of $\omega$-many pairwise disjoint non-empty closed subsets. Note that the Sierpiński Theorem [17, Theorem 6.1.27] states that every continuum is $\sigma$-connected. Lelek [26, P4] asked whether every connected space with a dispersion point is $\sigma$-connected. Duda [15, Example 5] answered this question in the negative.

**Theorem 4.10.** If a space $X$ contains an open almost zero-dimensional subspace $O$ with $O \neq \emptyset$ and $X \setminus O \neq \emptyset$ then $X$ is not $\sigma$-connected.

**Proof.** We may assume that $X$ is connected. Since $O$ is almost zero-dimensional and open we can find for every $x \in O$ a $C$-set $A_x$ in $O$ that is closed in $X$ and with $x \in A_{x_0}$. Select a countable subcovering $\{B_i : i < \omega\}$ of $\{A_x : x \in O\}$. Since the union of two $C$-sets is a $C$-set we can arrange that $B_i \subset B_{i+1}$ for each $i < \omega$. Also we may assume that $B_0 = \emptyset$. Since $B_i$ is a $C$-set in $O$ we can find an $O$-clopen covering $C_i$ of $O \setminus B_i$. We may assume that $C_i = \{C_{ij} : j < \omega\}$ is countable. Moreover, by clopenness we can arrange that $C_i$ is a disjoint collection. Consider the countable closed disjoint covering

$$\mathcal{F} = \{\{X \setminus O\} \cup \{C_{ij} \cap B_{i+1} : i, j < \omega\}\} \setminus \{\emptyset\}$$

of $X$. If $\mathcal{F}$ is finite then $O$ is closed and hence clopen, violating the connectedness of $X$. Thus $X$ is not $\sigma$-connected. \hfill $\square$

Since every cohesive almost zero-dimensional space has a one-point connectification by Proposition 1.1 it produces an example in answer to Lelek’s question. These examples are explosion point spaces rather than just dispersion point spaces. In particular, we have that $\mathcal{E} \cup \{\infty\}$ and $\mathcal{E}_c \cup \{\infty\}$ are counterexamples. Note that $\mathcal{E}_c \cup \{\infty\}$ is complete which is optimal because $\sigma$-compact dispersion point spaces cannot exist.

5. A rim-discrete space with an explosion point

Let $\mathcal{E}$, $\nabla$ and $\nabla$ be as defined in Section 2. We will construct a function $\tau : P \to (0, 1)$ with domain $P \subset \mathcal{E}$ such that:
(1) \( \tau \) is a dense subset of \( \mathcal{C} \times (0, 1) \);
(2) \( \nabla \tau \) is connected; and
(3) \( \nabla \tau \) is rim-discrete.

Here we identify a function like \( \tau \) with its graph (in the product topology). Clearly \( \tau \) will be totally disconnected. Note that \( \tau \) cannot be almost zero-dimensional by (2), (3), and Corollary 4.5 or (1), (2), and Theorem 3.3.

5.1. Construction of \( Z \). We first describe a rim-discrete connectible set \( Z \subseteq \mathcal{C} \times \mathbb{R} \) similar to \( Y \) in [30, Example 1].

Let \( E \) be the set of endpoints of connected components of \( \mathbb{R} \setminus \mathcal{C} \). For each \( \sigma \in 2^{<\omega} \), let \( n = \text{dom}(\sigma) \) and define

\[
B(\sigma) = \left[ \sum_{k=0}^{n-1} \frac{2\sigma(k)}{3^{k+1}}, \sum_{k=0}^{n-1} \frac{2\sigma(k)}{3^{k+1}} + \frac{1}{3^n} \right] \cap \mathcal{C}.
\]

Here, \( B(\varnothing) = [0, 1] \cap \mathcal{C} = \mathcal{C} \). The set of all \( B(\sigma) \)'s is the canonical clopen basis for \( \mathcal{C} \).

Suppose \( \sigma \in 2^{<\omega} \), \( Q \) is a countable dense subset of \( B(\sigma) \setminus E \), and \( a \) and \( b \) are real numbers with \( a < b \). Fix an enumeration \( \{q_m : m < \omega\} \) for \( Q \), and define a function

\[
f = f_{(Q, \sigma, a, b)} : B(\sigma) \to [a, b]
\]

by the formula

\[
f(c) = a + (b - a) \cdot \sum \{2^{-m} : m < \omega \text{ and } q_m < c\}.
\]

Note that:

- \( f \) is well-defined and non-decreasing;
- \( f \upharpoonright B(\sigma) \setminus E \) is one-to-one;
- \( f \) has the same value at consecutive elements of \( E \);
- \( Q \) is the set of discontinuities of \( f \); and
- the discontinuity at \( q_m \) is caused by a jump of height \( (b - a) \cdot 2^{-m-1} \).

Let

\[
D = D_{(Q, \sigma, a, b)} = f \cup \bigcup \{ \{q_m\} \times [f(q_m), f(q_m) + (b - a) \cdot 2^{-m-1}] : m < \omega \}.
\]

Thus \( D \) is equal to (the graph of) \( f \) together with vertical arcs corresponding to the jumps in \( f \). Note that \( \pi_1(D) = [a, b] \) and \( D \) is compact.

Let \( \{Q^n_i : n, i < \omega\} \) be a collection of pairwise disjoint countable dense subsets of \( \mathcal{C} \setminus E \). As in [30, Example 1], it is possible to recursively define a sequence \( \mathcal{R}_0, \mathcal{R}_1, \ldots \) of finite partial tilings of \( \mathcal{C} \times \mathbb{R} \) so that for each \( n < \omega \):

i. \( \mathcal{R}_n \) consists of rectangles \( R^n_i = B(\sigma^n_i) \times [a^n_i, b^n_i], \) where \( i < |\mathcal{R}_n| < \omega, \sigma^n_i \in 2^n, \) and \( 0 < b^n_i - a^n_i \leq \frac{1}{n+1} \) for all \( i < |\mathcal{R}_n| \);

ii. the sets

\[
D^n_i = D_{(Q^n_i \cap B(\sigma^n_i), \sigma^n_i, a^n_i, b^n_i)}
\]

are such that \( D^n_i \cap D^n_j = \varnothing \) whenever \( k < n \) or \( i \neq j \);

iii. for every arc \( I \subseteq \mathcal{C} \times [-n, n+1] \setminus \bigcup \{D_k^i : k \leq n \text{ and } i < |\mathcal{R}_k| \} \) there are integers \( i < |\mathcal{R}_n|, k \leq n, \) and \( j < |\mathcal{R}_k| \) such that \( I \subseteq R^n_i \cup R^n_j \) and \( d(I, D^i_k) \leq \frac{1}{3^n} \), where \( d \) is the standard metric on \( \mathbb{R}^2 \).

Let \( M^n_i \) be the (discrete) set of midpoints of the vertical arcs in \( D^n_i \). The key difference between the sets \( M^n_i \) and the \( T^n_i(M) \) defined in [30, Example 1] is that here we have guaranteed \( \pi_0(M^n_i) \cap \pi_0(M^n_j) \subset Q^n_i \cap Q^n_j = \varnothing \) whenever \( n \neq k \) or \( i \neq j \), whereas a vertical line could intersect multiple \( T^n_i(M) \)'s.
Let \( \{D_n : n < \omega \} \) and \( \{M_n : n < \omega \} \) be the sets of all \( D^n \)'s and \( M^n \)'s, respectively. Properties (i) through (iii) guarantee the set \( Z = \mathcal{C} \times \mathbb{R} \setminus \bigcup \{D_n \setminus M_n : n < \omega \} \) is rim-discrete; see [30, Claims 1 and 3]. Essentially, \( \tau \) will be a subset of \( Z \) containing all \( M_n \)'s, but will be vertically compressed from \( \mathcal{C} \times \mathbb{R} \) into \( \mathcal{C} \times (0, 1) \).

5.2. Construction of \( \mathcal{G} \). We now construct a connected function \( \mathcal{G} \) on which \( \tau \) will be based. Let \( \xi : \mathbb{R} \to (0, 1) \) be a homeomorphism, e.g. \( \xi = \frac{1}{2} + \frac{1}{2} \arctan \). Let \( \phi : [0, 1] \to [0, 1] \) be the Cantor function [14], and put \( \Phi = \phi \times \xi \). Then each \( \Phi(D_n) \) is an arc which resembles the graph of \( \phi \) reflected across the diagonal \( x = y \). See Figure 3.

![Graph of \( \phi \) (black) and its “inverse” (green)](image)

Note that \( \phi(E) \) is the set of dyadic rationals in \([0, 1] \). Let

\[
g = (\phi(E) \times \{0\}) \cup \bigcup \{\Phi(M_n) : n < \omega\}.
\]

Since \( \pi_0 \upharpoonright M_n \) is one-to-one and the \( \pi_0(M_n) \)'s are pairwise disjoint, \( g \) is a function. Also,

\[
\text{dom}(g) = \phi(E) \cup \bigcup \{\pi_0(\Phi(M_n)) : n < \omega\}
\]

is countable and \( \text{ran}(g) \subset [0, 1] \). Our goal is to extend \( g \) to a connected function \( \mathcal{G} : [0, 1] \to (-1, 1) \). This will be accomplished with the help of two claims. By a \textit{continuum} we shall mean a compact connected metrizable space with more than one point.

**Claim 5.1.** Fix \( n < \omega \) and put \( D = D_n \) and \( M = M_n \). Let \( A \subset [0, 1] \) have a dense complement and let \( K \subset \Phi(D) \cup (A \times (-1, 1)) \) be a continuum. If \( |\pi_0(K)| > 1 \) then \( K \cap \Phi(M) \neq \emptyset \).

**Proof.** Let \( a \) and \( b \) be two points in \( K \) such that \( \pi_0(a) < \pi_0(b) \). Since \( \pi_0(K) \) is an interval contained in the union of the zero-dimensional set \( A \) and the interval \( \pi_0(D) \) we have \( \pi_0(K) \subset \pi_0(D) \). Noting that \( \pi_0(M) \) is dense in \( \pi_0(D) \) we find a \( p \in \Phi(M) \) such that \( \pi_0(p) \in (\pi_0(a), \pi_0(b)) \). If \( p \notin K \), then we can find \( c, d \in [0, 1] \setminus A \) such that \( U = [c, d] \times \{\pi_1(p)\} \) is disjoint from \( K \) and \( \pi_0(a) < c < \pi_0(p) < d < \pi_0(b) \). Then \( U \cup ([c] \times (\pi_1(p), 1)) \cup ([d] \times (-1, \pi_1(p)) \) separates \( K \) with \( a \) and \( b \) on opposite sides. This contradicts our assumption that \( K \) is connected. Therefore \( p \in K \). \( \square \)
Claim 5.2. Let \( A \subset [0,1] \) be any countable set, and let \( K \subset \bigcup \{ \Phi(D_n) : n < \omega \} \cup (A \times (-1,1)) \) be a continuum. If \( |\pi_0(K)| > 1 \) then \( K \cap \Phi(M_n) \neq \emptyset \) for some \( n < \omega \).

Proof. For each \( x \in [0,1] \), let \( K_x = K \cap (\{x\} \times (-1,1)) \). Let \( \mathcal{K} \) be the decomposition of \( K \) consisting of every connected component of every non-empty \( K_x \). Applying [17, Lemma 6.2.21] to the perfect map \( \pi_0 \restriction K \), we see that \( \mathcal{K} \) is upper semi-continuous. If \( q : K \to K' \) is the associated (closed) quotient mapping then \( K' \) is also a continuum. Consider the countable covering \( \mathcal{V} \) of \( K' \) consisting of the compacta \( q(K_x) \) for \( x \in A \) and \( q(\Phi(D_n) \cap K) \) for all \( n < \omega \). By the Baire Category Theorem there is an element of \( \mathcal{V} \) that has nonempty interior in \( K' \) and hence contains a (non-degenerate) continuum \( C' \) by [17, Theorem 6.1.25]. Each \( q(K_x) \) is zero-dimensional by [17, Theorem 6.2.24], so \( C' \subset q(\Phi(D_n) \cap K) \) for some \( n < \omega \). Since \( q \) is a closed monotone map, the pre-image \( C = q^{-1}(C') \) is a continuum by [17, Theorem 6.1.29]. Note that \( |\pi_0(C)| > 1 \) because otherwise \( C' \) would be a subset of some zero-dimensional \( q(K_x) \). If \( x \notin A \) then each connected component of \( K_x \) is contained in a single \( \Phi(D_k) \) by the Sierpiński Theorem [17, Theorem 6.1.27], because the \( \Phi(D_n) \)'s are disjoint. Thus \( q(\Phi(D_n) \cap K_x) \) is disjoint from \( q(\Phi(D_i) \cap K_x) \) for each \( i \neq n \). So \( C \subset (A \times (-1,1)) \cup \Phi(D_n) \). By Claim 5.1 we have that \( C \cap \Phi(M_n) \neq \emptyset \). \( \square \)

Now let \( \{x_\alpha : \alpha < c\} \) enumerate the set \( [0,1] \setminus \text{dom}(g) \). Let \( \{ K_\gamma : \gamma < c \} \) be the set of continua in \( [0,1] \times (-1,1) \) such that:

- \( K_\gamma \) is not contained in any vertical line; and
- \( K_\gamma \cap \Phi(M_n) = \emptyset \) for all \( n < \omega \).

For each \( \alpha < c \) let \( l_\alpha = (\{x_\alpha\} \times (-1,1)) \setminus \bigcup \{ \Phi(D_n) : n < \omega \} \). By transfinite induction we define for each \( \alpha < c \) an ordinal

\[
\gamma(\alpha) = \min \{ \gamma < c : l_\alpha \cap K_\gamma \neq \emptyset \text{ and } \gamma \neq \gamma(\beta) \text{ for any } \beta < \alpha \}.
\]

We verify that the one-to-one function \( \gamma : c \to c \) is well-defined. Let \( \alpha < c \) so \( x_\alpha \notin \text{dom}(g) \) and \( x_\alpha \notin \pi_0(\Phi(M_n)) \) for each \( n \). Since \( M_n \) contains the midpoints of all vertical intervals in \( D_n \) we have that \( \{x_\alpha\} \times (-1,1) \) contains at most one point of \( \Phi(D_n) \). Let \( A \) be the countable set

\[
\bigcup_{n<\omega} \pi_1(\{\{x_\alpha\} \times (-1,1) \cap \Phi(D_n)) \cup \Phi(M_n))
\]

If \( a \in (-1,1) \setminus A \) then \( K = [0,1] \times \{a\} \) misses every \( \Phi(M_n) \) so \( K = K_\beta \) for some \( \beta < c \). Also we have \( l_\alpha \cap K_\beta \neq \emptyset \). Since \( |(-1,1) \setminus A| = c \) we have that \( \gamma \) is well-defined.

For every \( \alpha < c \) choose a \( y_\alpha \in \pi_1(l_\alpha \cap K_{\gamma(\alpha)}) \). Define

\[
\overline{g} = g \cup \{ (x_\alpha, y_\alpha) : \alpha < c \}
\]

and note that \( \overline{g} : [0,1] \to (-1,1) \) is a function. To prove that the graph of \( \overline{g} \) is connected let \( K \) be a continuum in \( [0,1] \times (-1,1) \) such that \( |\pi_0(K)| > 1 \). We show that \( K \cap \overline{g} \neq \emptyset \). The set \( K \) intersects some \( \Phi(M_n) \), which is a subset of \( \overline{g} \), or \( K = K_\alpha \) for some \( \alpha < c \). By the contraposition of Claim 5.2, the projection \( A = \pi_0(K_\alpha \setminus \bigcup \{ \Phi(D_n) : n < \omega \}) \) is uncountable. \( A \) is a continuous image of a Polish space, so in fact it has cardinality \( c \) by [21, Corollary 11.20]. Since \( [0,1] \setminus \{x_\beta : \beta < c\} = \text{dom}(g) \) is countable, this means \( B = \{ \beta < c : l_\beta \cap K_\alpha \neq \emptyset \} \) has cardinality \( c \). Assuming that \( K_\alpha \cap \overline{g} = \emptyset \) we find that \( \alpha \) cannot be in the range of \( \overline{g} \). If \( \beta \in B \) then \( l_\beta \cap K_\alpha \neq \emptyset \) so by the definition of \( \gamma \) we have \( \gamma(\beta) < \alpha \). Thus \( \gamma \restriction B \) is a one-to-one function from \( B \) into \( \{ \delta : \delta < \alpha \} \) and we have the
desired contradiction. So (the graph of) $g$ intersects each continuum in $[0, 1] \times (-1, 1)$ not lying wholly in a vertical line. By [22, Theorem 2], $g$ is connected.

5.3. **Definition and properties of $\tau$.** Observe that $g \circ \phi \subset (C \times (-1, 1)) \cup ([0, 1] \times \{0\})$. Let $$\tau = (g \circ \phi) \cap (0, 1)^2.$$ The domain of $\tau$ is the set $P = \pi_0(\tau) \subset C$.

Let $X = \nabla(\tau \cap ((0, 1) \times [0, 1]))$. If $A$ is any clopen subset of $X$ with $\langle \frac{1}{2}, 0 \rangle \in A$, then $A = X$. Otherwise, $\nabla^{-1}(X \setminus A)$ would be a non-empty proper clopen subset of $\tau$, contrary to the fact that $\tau$ is connected. Therefore $X$ is connected. Note that $\nabla \tau \simeq X$, so $\nabla \tau$ is also connected. Finally, let $\Xi = \text{id}_C \times \xi$. By [30, Claims 3 and 4] and the construction of $Z$, $\nabla \Xi(Z)$ is rim-discrete. We have $\nabla \tau \subset \nabla \Xi(Z)$, so $\nabla \tau$ is rim-discrete.

5.4. **Two questions.** A continuum is **Suslinian** if it contains no uncountable collection of pairwise disjoint (non-degenerate) subcontinua [28]. The class of Suslinian continua is slightly larger than the class of rational continua.

**Question 1.** Can $\mathcal{E}_c$ be embedded into a Suslinian continuum?

**Question 2.** Can $\mathcal{E}_c$ be densely embedded into the plane $\mathbb{R}^2$?

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