Invariance Principle for the Random Wind-Tree Process

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Abstract. Consider a point particle moving through a Poisson distributed array of cubes all oriented along the axes—the random wind-tree model introduced in Ehrenfest–Ehrenfest (1912) as reported by Ehrenfest, Ehrenfest (Begriffliche Grundlagen der statistischen Auffassung in der Mechanik Encykl. d. Math. Wissensch. IV 2 II, Heft 6, 90 S (1912) (Translated:) The conceptual foundations of the statistical approach in mechanics. Dover Books on Physics, 1912). We show that in the joint Boltzmann–Grad and diffusive limit this process satisfies an invariance principle. That is, the process converges in distribution to Brownian motion in a particular scaling limit. In a previous paper (2020) (Lutsko, Tóth in Commun. Math. Phys. 379:589–632, 2020) the authors used a novel coupling method to prove the same statement for the random Lorentz gas with spherical scatterers. In this paper we show that, despite the change in dynamics, a similar strategy with some modification can be used to prove the invariance principle for the random wind-tree model. The key differences from our previous work are that the individual path segments of the underlying Markov process are no longer fully independent and the geometry of recollisions is simpler.

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1. Introduction

In this paper we consider the motion of a point particle through an array of randomly placed, oriented in parallel cubes in $\mathbb{R}^3$—the so-called random wind-tree process [6]. In a recent paper [13] the authors showed that the random Lorentz gas (i.e., a similar process with spherical rather than cubic scatterers [12]) satisfies an invariance principle in a particular scaling limit which is intermediate between the kinetic and purely diffusive time scales. In this paper we prove an invariance principle for the wind-tree process in a similar
intermediate regime. The proof will follow similar lines. However, there are two key differences: in the Lorentz gas, after collision with a randomly placed scatterer (in 3 dimensions) the velocity is redistributed independently of the initial velocity; while for the wind-tree process the velocities form a genuine Markov chain. On the other hand as the collisions are simpler in the wind-tree setting, the necessary geometric estimates follow with significantly less effort.

More formally let $P$ be a Poisson point process of intensity $\varrho > 0$ in $\mathbb{R}^3$ (our results hold for general dimension $d \geq 3$; however to reduce notation we restrict to $d = 3$). Let $Q_r$ be a cube of side length $r$ oriented parallel with the axes, and let $P + Q_r$ be an array of obstacles/scatterers. We consider the trajectory of a point particle $X^{r,\varrho}(t)$ starting at the origin ($X^{r,\varrho}(0) = 0$) with a fixed initial velocity of unit length. The particle then flies in straight lines, reflecting elastically off of the obstacles. In this setting the origin is in $(P + Q_r)^c$ with probability tending to 1; hence such a trajectory is well defined (see [13, (2)] for more details).

A fundamental open problem for both the random wind-tree model and the random Lorentz gas is to prove an invariance principle in the diffusive limit. That is, in the limit

$$\frac{X^{r,\varrho}(Tt)}{\sqrt{T}}, \quad T \to \infty,$$

does the scaled process converge weakly to a Wiener process? In our previous paper we showed that the Lorentz gas satisfies an invariance principle in the limit (1) if we simultaneously take the low-density limit under certain constraints. The aim for this paper is to replicate that result for the wind-tree model (Fig. 1).
1.1. Scaling and Main Result

Fix a probability vector \( p = (p_1, p_2, p_3) \) with \( p_i > 0 \) for all \( i \), and let \( |p| = \sqrt{p_1^2 + p_2^2 + p_3^2} \). The state-space of velocities is then

\[
\Omega := \left\{ v \in \mathbb{S}_1^2 : |v_i| = \frac{p_i}{|p|}, \ i = 1, 2, 3 \right\}.
\]

(2)

Fix the initial velocity \( \dot{X}^{r,\varrho}(0^+) \in \Omega \). We study the process \( t \mapsto X^{r,\varrho}(t) \) in the joint Boltzmann–Grad and diffusive scaling limit. That is, for \( t \in [0, 1] \) we consider the limit:

\[
r \to 0, \quad r^2 \varrho \to |p|^{-1}, \quad T(r) \to \infty,
\]

\[
t \mapsto \frac{X(tT)}{\sqrt{T}}.
\]

(3)

We have dropped the dependence on \( r \) and \( \varrho \) in the notation (thus \( X^{r,\varrho}(t) = X(t) \)). Moreover, we choose \( r^2 \varrho \to |p|^{-1} \) since \( |p|^{-1} \) is the cross-sectional area of the cube as viewed by the particle; thus, the mean free flight will be of length 1. With that, the main result of this paper is the following invariance principle:

**Theorem 1.** Consider the intermediate scaling limit (3) such that \( \lim_{r \to 0} T(r)r^2 = 0 \) then

\[
\left\{ t \mapsto T^{-1/2}X(tT) \right\} \implies \left\{ t \mapsto B(t) \right\}
\]

(4)

as \( r \to 0 \) in the averaged-quenched sense (see below). Where \( B(t) \) is a Wiener process with covariance matrix \( M = \text{diag}(v_1^2, v_2^2, v_3^2) \) in \( \mathbb{R}^3 \).

The proof follows from a joint construction of \( t \mapsto X(t) \) and a second Markovian process which we introduce in Sect. 2.2. In Sect. 2.4 we state and outline the proof of the main technical theorem of the paper (Theorem 2). Theorem 1 is then a straightforward corollary of Theorem 2.

**Remark 1.** For the Lorentz gas we proved the same theorem with the asymptotic constraint \( \lim_{r \to 0} T(r)r^2 |\log r|^2 = 0 \). The reason for this logarithmic correction are those collisions for which the angle between incoming and outgoing velocities is small. In the wind-tree model the velocity of the point particle is restricted to a fixed discrete set; hence, the logarithmic factor can be removed.

In this context there are two relevant limits one could take:

(Q) **Quenched limit:** For almost all (i.e., typical) realizations of the underlying Poisson point process, with averaging over the random initial velocity and the initial position of the particle (for the Lorentz gas in the quenched setting one could get away with just averaging over the initial velocity [3]).

(AQ) **Averaged-quenched (a.k.a. annealed) limit:** Averaging over the random placement of the scatterers.

This paper (and our previous paper [13]) are in the averaged-quenched setting.
1.2. Related Work

While we cannot hope to give an exhaustive account we present here some of the related work. The wind-tree model was introduced in the famous monograph by Paul and Tatiana Ehrenfest [6, Appendix to Section 5] as a simplified model to explain the return to equilibrium of the velocity distribution of a gas. It is noteworthy that, in defining the model P. and T. Ehrenfest considered randomly placed scatterers oriented along the axes (as we have done), however the periodic wind-tree model (often referred to as the Ehrenfest model)—where rectangular scatterers are centered at the points of a hypercubic lattice—is the better studied model. This owes to the fact that the periodic setting can be studied using methods from parabolic dynamical systems. While the random wind-tree model is less well understood, it is of significant interest as a stochastic process and a model for diffusion in particle systems.

2D Periodic Wind-Tree: The periodic wind-tree model in 2 dimensions has been the focus of a lot of recent research. In this setting the billiard flow is parabolic (i.e., close orbits diverge polynomially). Thus (unlike for the periodic Lorentz gas—see, for example, the survey [15]) the tools of hyperbolic dynamics cannot be used. Instead the standard approach is to use the so-called Katok–Zemliakov construction (see [18]), which allows one to replace the billiard flow by linear flow on translation surfaces.

There have not yet been any theorems concerning the diffusive limit or an invariance principle for the periodic wind-tree process. However there have been a number of interesting and contrasting results concerned with the speed of diffusion and exceptional trajectories. Hardy and Weber [10] showed that some specific directions diffuse at a rate of \( \log T \log \log T \), while Delecroix–Hubert–Lelièvre [5] showed that typical (with respect to angle) trajectories satisfy the superdiffusive polynomial diffusion rate \( T^{2/3} \). Additionally Delecroix [4] showed that for any rectangular scatterer, there is a set of diverging trajectories with positive Hausdorff measure, while Hubert–Lelièvre–Troubetzkoy [11] and then Avila and Hubert [1] showed that the billiard flow is recurrent for almost every direction. Finally Fraczek and Ulcigrai [7] proved that generically the billiard flow is not ergodic.

Random Wind-Tree and Lorentz gas: At the moment there are fewer rigorous results about the random wind-tree model and the random Lorentz gas than their periodic counterparts. Gallavotti [8], [9] used classical (probabilistic) methods to show that in the (annealed) Boltzmann–Grad limit (i.e., (3) with \( T \) constant) both models obey a linear Boltzmann equation with different collision terms. For a wide class of Lorentz gas models with spherically symmetric scattering potentials, Spohn [17] and Boldrighini–Bunimovich–Sinai (for the hard-core random Lorentz gas) [3] showed that in the Boltzmann–Grad limit the Lorentz gas converges in the annealed, respectively, quenched sense to a Markovian flight process. To our knowledge all the previous work on these random models has been in the Boltzmann–Grad limit and for finite time intervals. The holy grail—the invariance principle in the diffusive
limit—remains open for both models. Our previous paper [13] represents partial progress in this direction for the Lorentz gas—in that we showed that the Lorentz gas does satisfy an invariance principle, provided we simultaneously take the Boltzmann–Grad limit—and Theorem 1 likewise represents a step in this direction for the wind-tree model.

Recently Marklof and Strömbergsson [16] prove convergence to a limiting transport process for a wide class of spherically symmetric potentials and scatterer configurations. In particular this approach subsumes these previous results on the random Lorentz gas [8], [9], [17], [3] and covers many other cases (periodic or quasi-crystals) all with spherically symmetric scattering potentials.

While the random wind-tree process with Poisson distributed scatterers was only previously treated by Gallavotti, there have been other efforts to understand the random setting. In a recent paper [14], Málaga Sabogal and Troubetzkoy consider a set of wind-tree configurations endowed with the Hausdorff topology. They show that in this topologically random setting, the wind-tree flow has infinite ergodic index in almost every direction. In particular, in that setting they are able to prove rigorously the Ansatz which motivated Ehrenfest and Ehrenfest to propose this model. Namely by applying ergodic theorems they showed that the velocities of a cloud of initially parallel particles will decorrelate. That said, there has not been any results concerning the random wind-tree model with Poisson distributed scatterers.

2. Coupling Construction

2.1. State-Space and Notation

Returning now to the random wind-tree model, for the rest of the paper we assume the initial velocity is fixed to be $v_0 \in \Omega$. This will aid in the exposition but can be assumed without loss of generality, since the time taken to reach this velocity is exponentially bounded.

At each collision one component of the velocity changes sign. For $i = 1, 2, 3$, let $\vartheta_i : \mathbb{R}^3 \to \mathbb{R}^3$ be such that $\vartheta_i(v)_j = (-1)^{\delta_{i,j}}v_j$ for $j = 1, 2, 3$. During a collision, for each direction $i = 1, 2, 3$, the probability $P(v \mapsto \vartheta_i(v)) = p_i$. For any $v \in \Omega$ and $w \in \Omega_v$ denote the set of accessible velocities after one collision starting from $v$, namely

$$\Omega_v = \{ w \in \Omega : w = \vartheta_i(v) \text{ for some } 1 \leq i \leq 3 \}. \quad (5)$$

Let $m_v$ denote the measure on $\Omega_v$ which selects $\vartheta_i(v)$ with probability $p_i$. Moreover, for $v \in \Omega$ and $w \in \Omega_v$ let $B(v, w) \subset \partial Q_r$ be the face of the cube $Q_r$ such that a particle traveling with velocity $v$ colliding with that face would adopt the velocity $w$. Formally, for $v \in \Omega$ and $w = \vartheta_k(v)$

$$B(v, w) = \left\{ b \in \partial Q_r : b_k = -\frac{v_k}{|v_k|}r \right\}. \quad (6)$$
2.2. Markovian Flight Process

Let \( \{u_n\}_{n=0}^{\infty} \) be a realization of the following Markov chain on \( \Omega: u_1 = v_0 \) and then for all \( i \geq 1, \) \( u_{i+1} \) are independently selected from \( \Omega_{u_i} \) according to the measure \( m_{u_i}. \) For later use let \( u_0 \in \Omega_{u_1}. \) Let

\[
\{\xi_n\}_{n=1}^{\infty} \sim EXP(1) \tag{7}
\]

be i.i.d exponentially distributed flight times and let

\[
Y_n := \sum_{i=1}^{n} y_i \quad , \quad y_n := \xi_n u_n \tag{8}
\]

denote the discrete Markovian Flight Process. To define the continuous process, for \( t \in \mathbb{R} \) let

\[
\tau_n := \sum_{i=1}^{n} \xi_i \quad , \quad \nu_t := \max\{n : \tau_n \leq t\} \quad , \quad \{t\} := t - \tau_{\nu_t}, \tag{9}
\]

that is \( \tau_n \) are the scattering times, \( \nu_t \) is the label of the most recent scattering, and \( \{t\} \) is the time since the previous scattering, at time \( t. \) Now define

\[
Y(t) := Y_{\nu_t} + u_{\nu_t+1}\{t\} \tag{10}
\]

to be the (continuous) Markovian Flight Process. Note that the processes \( t \mapsto Y(t) \) and \( \{Y_n\}_{n=1}^{\infty} \) do not depend on \( r. \)

For later use we introduce the following virtual scatterers:

\[
Y'_k := Y_k + \beta_k \quad , \quad \beta_k \sim UNI(-B(u_k, u_{k+1})) \quad , \quad k \geq 0 \tag{11}
\]

\[
S^Y_n := \{Y'_k \in \mathbb{R}^3, \quad 0 \leq k \leq n\} \quad , \quad n \geq 0.
\]

In words \( Y'_k \) is the position of a scatterer if it had caused the \( k^{th} \) collision (of course \( Y \) is independent of any scatterers, thus the term virtual). Note also that we assume there is a virtual collision at time 0; this has no effect on the definition of the model. However, it will ease the notation. One difference with the random Lorentz gas is that the position of a scatterer associated to a velocity jump is not uniquely determined. Therefore we select from among the possible virtual scatterers uniformly.

For later use we introduce the sequence of indicators \( \epsilon_j = \mathbb{1}\{\xi_j < 1\} \) and the corresponding distributions \( EXP(1|1) := \text{distrib}(\xi_j|\epsilon_j = 1) \) and similarly \( EXP(1|0) = \text{distrib}(\xi_j|\epsilon_j = 0). \) We refer to \( \xi := (\epsilon_j)_{j \geq 0} \) as the signature of the sequence \( (\xi_j)_{j \geq 0}. \)

2.3. Joint Construction

Our goal for this section is to construct the physical wind-tree and Markovian processes on the same probability space. We construct the wind-tree process as an exploration process: in that the process explores its environment as time moves forward. For convenience for what follows we will also construct a third auxiliary process, \( \{t \mapsto Z(t)\}, \) coupled to the \( X \) and \( Y \) processes. The auxiliary process, which we call either the forgetful or myopic process, is only used in Sects. 4–6. Hence some readers may wish to ignore it until later. Indeed if we
only wanted to prove Theorem 1 for times of order $o(r^{-1})$ (we do this in Sect. 3), then this myopic process does not play a role and can be ignored.

The construction will proceed inductively on certain (as yet unspecified) time intervals. To simplify the explanation, first we will explain how the processes $X$ and $Z$ are constructed on a given time interval, given certain random data. Then, we will explain how the random data are generated to enable the coupling to $\{t \mapsto Y(t)\}$ and we will explain on which time intervals these processes are defined.

Throughout the construction we label the velocity of $\dot{X}(t) =: V(t)$, $\dot{Y}(t) =: U(t)$ and $\dot{Z}(t) = W(t)$.

2.3.1. Building $X$ on $[\hat{\tau}_{n-1}, \hat{\tau}_n)$. We label the intervals of construction of $X$ by $[\hat{\tau}_{n-1}, \hat{\tau}_n)$. In Sect. 2.3.4 we will make precise what these $\hat{\tau}$ are.

To construct $X$ on an interval $[\hat{\tau}_{n-1}, \hat{\tau}_n)$, given a position $X(\hat{\tau}_{n-1}) = X_{n-1} \in \mathbb{R}^3$, a velocity $V(\hat{\tau}_{n-1}) \in \Omega$ and $S_{n-1}^X \subset \mathbb{R}^{n-1} \cup \{\star\}$ a finite set of points (where $\star$ is a fictitious point at infinity with $\inf_{x \in \mathbb{R}^3} |x - \star| = \infty$ which will aid in the exposition) perform the following algorithm:

**Step 1 Mechanical flight on** $S^X_{n-1}$ in $[\hat{\tau}_{n-1}, \hat{\tau}_n)$: The trajectory $t \mapsto X(t)$ on $t \in [\hat{\tau}_{n-1}, \hat{\tau}_n)$ is defined to be free motion, with initial position $X_{n-1}$ and velocity $V(\hat{\tau}_{n-1})$, and with reflective collisions on $Q_r + S^X_{n-1}$.

**Step 2 Attempt Fresh Collision**: Suppose, we are given a velocity $\tilde{w}_{n+1} \in \Omega_{V(\hat{\tau}_n)}$ and an impact parameter $\hat{\beta}_n \in -B(V(\hat{\tau}_n), \tilde{w}_{n+1})$. Set

$$X'' := X(\hat{\tau}_n) + \hat{\beta}_n \quad (12)$$

Now
- If $0 < s \leq \hat{\tau}_{n-1} : X(s) \in X'' + Q_r$, then let $X'_n := \star$, and $V(\hat{\tau}_n^+) = V(\hat{\tau}_n^-)$.
- If not, then $X'_n := X''$, and $V(\hat{\tau}_n^+) = \tilde{w}_{n+1}$.

Now set $S^X_n = S^X_{n-1} \cup \{X'_n\}$.

We say: on the interval $[\hat{\tau}_{n-1}, \hat{\tau}_n)$ the process $\{t \mapsto X(t)\}$ attempts a fresh collision at $\hat{\tau}_n$ with data $(\tilde{w}_{n+1}, \hat{\beta}_n)$.

We will make precise the distributions of $\tilde{w}_{n+1}$ and $\hat{\beta}_n$ in the construction below. Note that if, given a $\tilde{w}_{n+1}$ and a $\hat{\beta}_n$, we build $X$ on the interval $[\hat{\tau}_{n-1}, \hat{\tau}_n)$, then after the construction we have sufficient information to build $X$ on the interval $[\hat{\tau}_n, \hat{\tau}_{n+1})$ (provided we are given another pair $\tilde{w}_{n+2}, \hat{\beta}_{n+1}$).

The construction of $t \mapsto X(t)$ can differ from $t \mapsto Y(t)$ in two ways. First, in [Step 1] if $X(t)$ encounters a previously placed scatterer (i.e., an element in $Q_r + S^X_{n-1}$), then it will obey this recollision. Second, in [Step 2], a collision may be rejected (c.f the first bullet point) if the attempted collision is shadowed.

The proof of Theorem 1 follows by controlling these possible mismatches.

2.3.2. Building $Z$ on $[\hat{\tau}_{n-1}, \hat{\tau}_n)$. We call the process $\{t \mapsto Z(t)\}$ forgetful in that the process only respects direct mismatches (see Fig. 2 for a diagram).

That is, recollisions with the immediately preceding scatterer, or shadowed events where the scattering is shadowed by the immediately preceding path segment.
Suppose that we are given a time interval \([\tilde{\tau}_n - 1, \tilde{\tau}_n)\). Assume further, we are given a position \(Z(\tilde{\tau}_n - 1) = Z_{n-1}\), velocity \(W(\tilde{\tau}_n - 1) \in \Omega\), and a pair \(S^Z_{n-1} = \{Z'_{n-1}, Z''_{n-2}\} \subset \mathbb{R}^3 \cup \{\star\}\).

**Step 1 Mechanical flight on** \(S^Z_{n-1}\) **in** \([\tilde{\tau}_n - 1, \tilde{\tau}_n)\): The trajectory \(t \mapsto Z(t)\) on \(t \in [\tilde{\tau}_n - 1, \tilde{\tau}_n)\) is defined to be free motion starting at position \(Z(\tilde{\tau}_n - 1)\) and with velocity \(W(\tilde{\tau}_n - 1)\) with reflective collisions on \(Q_r + S^Z_{n-1}\).

**Step 2 Attempt Fresh Collision:** Suppose that we are given a velocity \(\tilde{w}_{n+1} \in \Omega_{W(\tilde{\tau}_n)}\) and an impact parameter \(\tilde{\beta}_n \in -B(W(\tilde{\tau}_n - 1), \tilde{w}_{n+1})\). Set

\[
Z'' := Z(\tilde{\tau}_n) + \tilde{\beta}_n
\]

Now
- If there exists an \(s \in (\tilde{\tau}_n - 2, \tilde{\tau}_n - 1)\) such that \(Z(s) \in Z'' + Q_r\) then let \(Z'_n := \star\), and \(W(\tilde{\tau}_n^+) = W(\tilde{\tau}_n^-)\).
- If not, then \(Z'_n := Z''\), and \(Z(\tilde{\tau}_n^+) = \tilde{w}_{n+1}\).

Now set \(S^Z_n = \{Z'_n, Z''_{n-1}\}\).

Similarly we say that on the interval \([\tilde{\tau}_n - 1, \tilde{\tau}_n)\) the process \(\{t \mapsto Z(t)\}\) attempts a fresh collision at \(\tilde{\tau}_n\) with data \((\tilde{w}_{n+1}, \tilde{\beta}_n)\).

### 2.3.3. Parity.
Consider just the processes \(\{t \mapsto Y(t)\}\) and \(\{t \mapsto X(t)\}\), the idea behind the coupling is the following:

- \(X(0) = Y(0)\) and the velocities are initially parallel.
- \(X\) and \(Y\) then run parallel until one of two possible mismatches occurs:
  - A **recollision**, which corresponds to a collision with a previously placed scatterer during 2.3.1 of Subsect. 2.3.1.
  - A **shadowed collision**, which corresponds to \(X'_n = \star\) in 2.3.1 of Subsect. 2.3.1.
- After a mismatch the two velocity processes proceed independently.
- When the two velocities happen to coincide, we recouple the two processes and they run parallel until the next mismatch.
However there is a problem with this setup as we have described it. Note that there are two parity classes: \((-v, (\vartheta_i(v)))_{i=1,2,3}\) and \((v, (\vartheta_i(\vartheta_j(v)))_{i\neq j}\) (note that \(\vartheta_i(\vartheta_j(v)) = \vartheta_j(\vartheta_i(v))\)). The Markov process \((u_n)_{n\in\mathbb{N}}\) alternates between these two classes. The problem is that if there is a parity mismatch between \(V(t)\) and \(U(t)\) at a given time, then as long as the two processes experience fresh collisions at the same times, only another mismatch can restore the parity. This is too long to wait. Therefore we need to alter the sequence of collision times to restore parity. For this we will make use of Lemma 1. For future use, we define the equivalence relation \(u \sim v\) if \(u\) and \(v\) are in the same parity class.

**Lemma 1.** Let \((\xi_j)_{j \geq 1}\) and \(\xi''\) be i.i.d. \(\text{EXP}(1)\)-distributed random variables. Let \(\tau_n := \sum_{j=1}^{n} \xi_j\), \(n \geq 0\) be the PPP of intensity 1 on the half-line \(\mathbb{R}_+\), with the gap-sequence \((\xi_j)_{j \geq 1}\). Define a new sequence \((\tau'_n)_{n \geq 0}\) as follows:

- If \(\xi'' < \xi_1\), then \(\tau'_1 = \xi''\) and \(\tau'_{n} = \tau_{n-1}\), for \(n \geq 2\). (That is: insert \(\xi''\) as the first point and leave the rest as they are.)
- If \(\xi_1 < \xi''\), then \(\tau'_n = \tau_{n+1}\), for \(n \geq 1\). (That is: delete the first point \(\tau_1\) and leave the rest as they are.)

The newly formed point process \((\tau'_n)_{n \geq 0}\) will itself be a PPP of intensity 1 on the half-line \(\mathbb{R}_+\). In other words, the new gap sequence \(\xi'_j := \tau'_j - \tau'_{j-1}\), \(j \geq 1\), will be i.i.d. \(\text{EXP}(1)\)-distributed.

**Proof of Lemma 1.** Let \(n \geq 2\) and \(s_j \in [0, \infty)\), \(1 \leq j \leq n\) and write

\[
P (s_j < \xi'_j, 1 \leq j \leq n) = P (\{s_j < \xi'_j, 1 \leq j \leq n\} \cap \{\xi'' < \xi_1\}) + P (\{s_j < \xi'_j, 1 \leq j \leq n\} \cap \{\xi_1 < \xi''\})
\]

\[
= P (\{s_1 < \xi'' < \xi_1 - s_2\} \cap \{s_{j+1} < \xi_j, 2 \leq j \leq n-1\}) + P (\{\xi_1 < \xi''\} \cap \{s_1 < \xi_1 + \xi_2\} \cap \{s_{j-1} < \xi_j, 3 \leq j \leq n+1\})
\]

\[
= P (s_1 < \xi'' < \xi_1 - s_2) P (s_{j+1} < \xi_j, 2 \leq j \leq n-1) + P (\{\xi_1 < \xi''\} \cap \{s_1 < \xi_1 + \xi_2\}) P (s_{j-1} < \xi_j, 3 \leq j \leq n+1)
\]

\[
= (P (s_1 < \xi'' < \xi_1 - s_2) + P (\{\xi_1 < \xi''\} \cap \{s_1 < \xi_1 + \xi_2\}) e^{-s_2}) \prod_{j=3}^{n} e^{-s_j}. \tag{14}
\]

For \(\xi_1, \xi_2, \xi''\) i.i.d. \(\text{EXP}(1)\)-distributed random variables it is straightforward to compute

\[
P (s_1 < \xi'' < \xi_1 - s_2) = \frac{1}{2} e^{-2s_1 - s_2} \tag{15}
\]

\[
P (\{\xi_1 < \xi''\} \cap \{s_1 < \xi_1 + \xi_2\}) = e^{-s_1} - \frac{1}{2} e^{-2s_1}. \tag{16}
\]

Finally, form (14), (15) and (16) we readily get

\[
P (s_j < \xi'_j, 1 \leq j \leq n) = \prod_{j=1}^{n} e^{-s_j}.
\]
2.3.4. Joint Coupling. Assume \( \{ t \mapsto Y(t) \} \) is constructed as in Sect. 2.2. We will construct the \( X \) and \( Z \) processes inductively on the intervals \( [\tau_{2n}, \tau_{2n+2}) \) as follows: First set \[
X(0) = X_0 = 0, \quad V(0^+) = u_1, \quad X'_0 = \hat{\beta}_0 = \beta_0, \quad S_0^X = \{X_0'\} \]
\[
Z(0) = Z_0 = 0, \quad W(0^+) = u_1, \quad W'_0 = \tilde{\beta}_0 = \beta_0, \quad S_0^Z = \{Z'_0, Z'_{-1}\} \]
where \( Z'_{-1} = \star \).

Now, our goal is to construct all three processes inductively on the intervals \( [\tau_{2n}, \tau_{2n+2}) \). However, if there is a parity mismatch, we must use Lemma 1 to restore parity. Hence let \( n \in \mathbb{N} \), sample an exponential time \( \zeta_n \sim \text{EXP}(1) \) independent of the entire history. There are 7 possible parity situations arranged and labeled in the following table:

| Parity at time \( \tau_{2n}^+ \) | \( \zeta_n \leq \xi_{2n+1} \) | \( \zeta_n > \xi_{2n+1} \) |
|-------------------------------|----------------|----------------|
| \( U \not\equiv V \equiv W \) | A | C |
| \( U \not\equiv V \not\equiv W \) | D | E |
| \( U \equiv W \not\equiv V \) | F | G |

For completeness of the construction we define all of these cases; however on our time scales we will (w.h.p) only see situations A, B, and C.

On the interval \( [\tau_{2n}, \tau_{2n+2}) \) the \( X \) and \( Z \) processes attempt fresh collisions at the following times:

| Situation | \( X \) | \( Z \) |
|-----------|--------|--------|
| A | \( \tau_{2n+1}, \tau_{2n+2} \) | \( \tau_{2n+1}, \tau_{2n+2} \) |
| B | \( \tau_{2n+1}, \tau_{2n+2} \) | \( \tau_{2n+1}, \tau_{2n+2} \) |
| C | \( \tau_{2n+1}, \tau_{2n+2} \) | \( \tau_{2n+1}, \tau_{2n+2} \) |
| D | \( \tau_{2n+1}, \tau_{2n+2} \) | \( \tau_{2n+1}, \tau_{2n+2} \) |
| E | \( \tau_{2n+1}, \tau_{2n+2} \) | \( \tau_{2n+1}, \tau_{2n+2} \) |
| F | \( \tau_{2n+1}, \tau_{2n+2} \) | \( \tau_{2n+1}, \tau_{2n+2} \) |
| G | \( \tau_{2n+1}, \tau_{2n+2} \) | \( \tau_{2n+1}, \tau_{2n+2} \) |

Again, the idea here is: if the \( X \) processes matches the parity of the \( Y \)-process, then it attempts fresh collisions at the same times as the \( Y \)-process. If not, then it uses \( \zeta_n \) to attempt either 1 or 3 collisions during the time it takes \( Y \) to perform two collisions. As a consequence, if there are no mismatches, then parity will be restored at the end of the time interval. Likewise of the \( Z \)-process.

In what follows the following coupling rule will dictate the random variables \( \hat{\beta}_n, \hat{w}_n, \tilde{\beta}_n, \tilde{w}_n \) used in the attempted fresh collisions. **For the \( Z \)-process:** If the \( Z \)-process attempts a fresh collision at time \( t_a \), sample \( \tilde{w} \) from \( \Omega_{W(t_a)} \) according to the measure \( m_{W(t_a)} \) and sample \( \tilde{\beta} \) from...
−B(W(t_a^-), \tilde{w}) both independent of the past. We now attempt to couple W
with \( U \) at \( t_a \):

- **Couple W to U:** If \( W(t_a^-) = U(t_a^-) \) and \( t_a = \tau_a \) for some \( n \), attempt a
  fresh collision at \( Z(t_a) \) using data \((\beta_n, u_{n+1})\).
- **W is independent of U:** Otherwise attempt a fresh collision at \( Z(t_a) \) using
data \((\beta, \tilde{w})\).

**For the X-process:** If the X-process attempts a fresh collision at time \( t_a \),
sample \( \hat{w} \) from \( \Omega_{\tau_a} \) according to the measure \( m_{\tau_a} \) and sample \( \hat{\beta} \) from

\[ -B(V(t_a^-), \hat{w}) \]

both independent of the past. We now couple \( V \) to either \( U \) and/or \( W \) if possible:

- **Couple V to U:** If \( V(t_a^-) = U(t_a^-) \) and \( t_a = \tau_a \) for some \( n \) attempt a fresh
  collision at \( X(t_a) \) using \((\beta_n, u_{n+1})\).
- **Couple V to W:** If \( V(t_a^-) = W(t_a^-) \) and the \( Z \) process also attempts a
  fresh collision independent of \( U \) at time \( t_a \), attempt a fresh collision at
  \( X(t_a) \) using \((\hat{\beta}, \hat{w})\).
- **V is independent of U and W:** Otherwise attempt a fresh collision at
  \( X(t_a) \) using \((\hat{\beta}, \hat{w})\).

After this construction we have constructed three processes. For the wind-
tree exploration process \{\( t \mapsto X(t) \}\), the attempted fresh collision times are
\{\( \tau_n \)\} \( n \in \mathbb{N} \), by Lemma 1 these form a (temporal) Poisson point process on \( \mathbb{R}_+ \); the
scatterers are placed at positions \( \{X_n\} \subset \mathbb{R}^3 \cup \{\star\} \); and the impact parameters
are \{\( \beta_n \)\} \( n \in \mathbb{N} \). Moreover, the attempted velocities after collisions are \{\( \hat{w}_n \)\} \( n \in \mathbb{N} \),
these velocities are attempted since, in 2.3.1 the attempted collision may be
rejected (i.e., \( X_n = \star \)). Because of the Poisson distribution of the scatterers in
\( \mathbb{R}^3 \), and using Lemma 1, this process is distributed like the original wind-tree
model in the annealed sense as described in the introduction.

For the process \{\( t \mapsto Z(t) \}\), the attempted fresh collision times are
\{\( \tau_n \)\} \( n \in \mathbb{N} \), which by Lemma 1 form a (temporal) Poisson point process on \( \mathbb{R}_+ \);
the scatterers are placed at positions \( \{Z_n\} \subset \mathbb{R}^3 \cup \{\star\} \); and the impact parameters
are \{\( \beta_n \)\} \( n \in \mathbb{N} \). The attempted velocities for the Z-process are \{\( \hat{w}_n \)\} \( n \in \mathbb{N} \).

**2.4. Main Technical Result and Method Proof**

The main result we prove is the following

**Theorem 2.** Let \( T = T(r) \) be such that \( \lim_{r \to 0} T(r) = \infty \) and \( \lim_{r \to 0} r^2 T(r) = 0 \). Then for any \( \delta > 0 \)

\[
\lim_{r \to 0} \mathbf{P} \left( \sup_{0 \leq t \leq T} |X(t) - Y(t)| > \delta \sqrt{T} \right) = 0. \tag{18}
\]

From here Theorem 1 follows as a consequence of the classical Donsker’s
invariance principle [2]: that is, the process \( t \mapsto Y(t) \) is a true Markov process;
hence Donsker’s original invariance principle does not apply directly; however
in what follows we will show how to separate \( Y \) into i.i.d mean 0 pieces with
finite second moment. Thus Donsker’s principle will imply that \( t \mapsto \frac{Y(t\sqrt{T})}{\sqrt{T}} \)}


converges to a Wiener process in the diffusive scaling. Therefore the process \( t \mapsto X(t) \) does as well. We omit the details of this final step, and the rest of the paper is devoted to proving Theorem 2.

The strategy of proof is the same as in [13]. We begin with the joint realization of the Markovian flight process and the wind-tree exploration process described above. During the two mismatch events (recollisions and shadowed scatterings) the two velocity processes diverge. In either case the two processes are decoupled until recoupling is possible, at which point the two processes are recoupled and proceed parallel to each other until the next mismatch.

The proof then follows two steps. In Sect. 3 we show that such mismatches occur only on time scales of order \( r^{-1} \). Hence until such times both process are (w.h.p) in the same position and Theorem 2 follows immediately for \( T = o(r^{-1}) \). Note that this intermediate result is a statement about the Markovian flight process and thus is a purely probabilistic statement. During the rest of the paper we show that on time scales of order \( o(r^{-2}) \) only (geometrically) simple mismatches occur. During such mismatches the separation between \( X \) and \( Y \) is of order \( \mathcal{O}(1) \). Hence on the time scales of Theorem 2 there are \( o(Tr) \) mismatches. During each mismatch the two processes separate by a distance of order \( \mathcal{O}(1) \), hence up to \( T = o(r^{-2}) \), \( \frac{|X(T(r)) - Y(T(r))|}{\sqrt{T}} \rightarrow 0 \), thus proving (18). Sections 4–6 are devoted to formalizing this argument.

The reason for introducing the forgetful process \( \{ t \mapsto Z(t) \} \) is that the forgetful process will satisfy additional independence properties exploited in the proof. Thus during the second stage of the proof, we will in fact show that the forgetful and Markovian processes do not diverge too much. Then we show that with high probability the wind-tree and forgetful processes are in fact the same on these time scales (i.e., we show that with probability tending to 1 as \( r \rightarrow 0 \), the direct mismatches defining the \( Z \)-process are the only ones seen by the \( X \)-process).

Remark on dimension: As with the Lorentz gas, because of the recurrence of the random walk the same proof does not yield the result in 2 dimensions. For the Lorentz gas the geometry of mismatches imposed another reason that the proof cannot be extended to 2 dimensions. However for the wind-tree model the mismatches have a far simpler geometry and thus this obstruction is not present in 2 dimensions and the only obstruction is the recurrence/transience issue.

2.5. \( r \)-Consistency and \( r \)-Compatibility

The proof will hinge on two definitions which we present now for a general process (i.e., this could be a segment of any of the above mentioned processes). Let

\[
\begin{align*}
&n \in \mathbb{N}, \quad \tau_0 \in \mathbb{R}, \quad Z_0 \in \mathbb{R}^3, \quad U_0, \ldots, U_{n+1} \in \Omega \quad t_1, \ldots, t_n \in \mathbb{R}_+,
\end{align*}
\]

be given, such that either \( U_{i+1} \in \Omega_{U_i} \) or \( U_{i+1} = U_i \) for all \( 0 \leq i \leq n \). Moreover fix a set of vectors \( \beta_j \in B(U_j, U_{j+1}) \) (if \( U_j = U_{j+1} \) we set \( \beta_j = \star \) ) and define
for \( j = 0, \ldots, n, \)
\[
\tau_j := \tau_0 + \sum_{k=1}^{j} t_k, \quad \mathcal{Z}_j := \mathcal{Z}_0 + \sum_{k=1}^{j} t_k U_k, \quad \mathcal{Z}'_j := \mathcal{Z}_j + \beta_j
\]
and for \( t \in [\tau_j, \tau_{j+1}], j = 0, \ldots, n, \)
\[
\mathcal{Z}(t) := \mathcal{Z}_j + (t - \tau_j) U_{j+1}.
\]

We call the piece-wise linear trajectory \((\mathcal{Z}(t) : \tau_0^- < t < \tau_n^+) \text{ mechanically } r\text{-consistent}\) if
\[
\mathcal{Z}(t) = \mathcal{Z}(t') \quad \text{for} \quad t < t' < \tau_j
\]
and \((\mathcal{Z}(t) : \tau_0^- < t < \tau_n^+) \text{ mechanically } r\text{-inconsistent}\) if \((19)\) fails.

Given two finite pieces of mechanically \(r\text{-consistent trajectories } (\mathcal{Z}_a(t) : \tau_{a,0}^- < t < \tau_{a,n_a}^+) \text{ and } (\mathcal{Z}_b(t) : \tau_{b,0}^- < t < \tau_{b,n_b}^+), \text{ defined over non-overlapping time intervals: } [\tau_{a,0}, \tau_{a,n_a}] \cap [\tau_{b,0}, \tau_{b,n_b}] = \emptyset \text{ with } \tau_{a,n_a} \leq \tau_{b,0}, \text{ we will call them mechanically } r\text{-compatible}\) if
\[
\mathcal{Z}_a(t) = \mathcal{Z}_b(t) \quad \text{for} \quad t < t' < \tau_j
\]
and \((\mathcal{Z}_a(t) : \tau_0^- < t < \tau_n^+) \text{ mechanically } r\text{-incompatible}\) if \((20)\) fails.

### 3. No Mismatches Till \(T = o(r^{-1})\)

#### 3.1. Excursions

Unlike for the 3-dimensional Lorentz gas, the directions of path segments of the Markovian flight process are not independent. To decompose the process \(t \mapsto Y(t)\) into i.i.d segments we introduce *excursions*. Let
\[
\gamma := \min\{i > 1 : u_{i+1} = v_0\}
\]
and define a *pack* to be a collection
\[
\omega := (\gamma; \{u_i\}_{i=1}^{\gamma}, \{\beta_i\}_{i=1}^{\gamma}, \{\xi_i\}_{i=1}^{\gamma}),
\]
u_{\gamma} \in \Omega_{v_0}, and for all \(i > 1, u_i \neq v_0 \text{ and } u_{i-1} \in \Omega_{u_i}. \text{ Given a pack we consider the process } t \mapsto Y(t) \text{ associated to it via the rules set forth in Sect. 2.2---call the process built from such a pack, an excursion. That is, an excursion is a small segment of a Markovian flight process, which starts with velocity } v_0 \text{ such that the next velocity after the excursion is } v_0.

#### 3.2. Concatenation

For \(n = 1, 2, 3, \ldots\) consider infinitely many independent packs:
\[
\omega_n = (\gamma_n; \{u_{n,i}\}_{i=1}^{\gamma_n}, \{\beta_{n,i}\}_{i=1}^{\gamma_n}, \{\xi_{n,i}\}_{i=1}^{\gamma_n}).
\]
For each pack define the associated flight process \(t \mapsto Y_n(t)\) together with the discrete process \(\{Y_{n,i}\}_{i=0}^{\gamma_n}. \text{ Denote}
\[
\theta_n := \sum_{i=1}^{\gamma_n} \xi_{n,i}, \quad \overline{Y}_n := Y_{n,\gamma_n}.
\]
Define the following variables
\[ \Gamma_0 = 0, \quad \Gamma_n = \Gamma_{n-1} + \gamma_n, \quad \text{for } n \geq 1 \]
\[ \nu_n := \max\{m : \Gamma_n \leq n\}, \quad \{n\} := n - \Gamma_{\nu_n}. \]
Likewise
\[ \Theta_0 = 0, \quad \Theta_n = \Theta_{n-1} + \theta_n, \quad \text{for } n \geq 1 \]
\[ \nu_t := \max\{m : \Theta_n \leq t\}, \quad \{t\} := t - \Theta_{\nu_t}. \]

Now define the following three processes: the end-point process with \( \Xi_0 = 0 \)
\[ \Xi_n := \sum_{k=1}^{n} Y_k, \]
the concatenated discrete Markovian flight process with \( Y_0 = 0 \)
\[ Y_n := \Xi_{\nu_n} + Y_{\nu+n+1}\{n\}, \]
and the continuous concatenated Markovian flight process with \( Y(0) = 0 \)
\[ Y(t) := \Xi_{\nu_t} + Y_{\nu+1}\{t\}. \]
The advantage of this decomposition is that the different excursions making up the process \( Y \) are i.i.d steps with exponentially decaying tails. That is, the end-point \( (\Xi_n)_{n \geq 0} \) is a random walk with i.i.d steps with exponentially decaying tails.

### 3.3. Occupation Measures

Define the following occupation measures for a set \( A \subset \mathbb{R}^3 \)
\[ G(A) := \mathbb{E} (|\{1 \leq k < \infty : Y_k \in A\}|), \quad H(A) := \mathbb{E} (|\{0 < t < \infty : Y(t) \in A\}|), \]
\[ g(A) := \mathbb{E} (|\{1 \leq k \leq \gamma_1 : Y_k \in A\}|), \quad h(A) := \mathbb{E} (|\{0 < t < \Theta_1 : Y(t) \in A\}|), \]
\[ R(A) := \mathbb{E} (|\{1 \leq k < \infty : \Xi_k \in A\}|). \]

**Lemma 2.** The following upper bounds hold for any measurable set \( A \subset \mathbb{R}^3 \)
\[ R(A) \leq K(A) + L_{v_0}(A), \quad (22) \]
\[ g(A) \leq M(A) + L_{v_0}(A), \quad h(A) \leq M(A) + L_{v_0}(A), \quad (23) \]
\[ G(A) \leq K(A) + L_{v_0}(A), \quad H(A) \leq K(A) + L_{v_0}(A), \quad (24) \]
where
\[ K(dx) := C \min\{1, |x|^{-1}\} dx, \quad M(dx) := Ce^{-c|x|} dx \]
\[ L_{v_0}(A) := C \int_{0}^{\infty} \mathbb{1}\{tv_0 \in A\} e^{-ct} dt \]

This lemma is slightly different from the Lorentz gas case as \( L_{v_0} \) takes into account the discrete state-space of velocities. However the end result (Proposition 1) remains the same.
Proof. To bound $g(A)$ let

$$g_1(A) := \mathbb{P}(Y_1 \in A) = C \int_0^\infty \mathbb{I}\{tv_0 \in A\} e^{-t} dt.$$ 

We have fixed the initial velocity to be $u_1 = v_0$; therefore the points $\{Y_k - Y_1\}_{k=1}^{\gamma_1}$ are independent of the initial step $Y_1$. Therefore write

$$g_2(A) := \mathbb{E}(\{1 \leq k \leq \gamma_1 : Y_k - Y_1 \in A\}),$$

and note that

$$g(A) = \int_{\mathbb{R}^3} g_2(A-x)g_1(dx). \quad (25)$$

Similarly we can write

$$h_1(A) := \mathbb{E}(\{t \leq \tau_1 : Y(t) \in A\}) = C \int_0^\infty \mathbb{I}\{tv_0 \in A\} e^{-\max(1,t)} dt,$$

$$h_2(A) := \mathbb{E}(\{\tau_1 \leq t \leq \Theta_1 : Y(t) - Y_1 \in A\})$$

$$h(A) = \int_{\mathbb{R}^3} h_2(A-x)g_1(dx) + h_1(A). \quad (26)$$

Now the bounds (23) follow by inserting the bounds:

$$g_2(\{x : |x| > s\}) \leq Ce^{-cs}, \quad h_2(\{x : |x| > s\}) \leq Ce^{-cs}$$

$$g_2(\mathbb{R}^3) = \mathbb{E}(\gamma_1) < \infty, \quad h_2(\mathbb{R}^3) = \mathbb{E}(\Theta_1 - \tau_1) < \infty \quad (27)$$

into (25) and (26). That is,

$$g(A) \leq \int_{A^c} g_2(\{y : |y| > |x|\})dx + C \int_A g_1(dx) \leq M(A) + L_{v_0}(A) \quad (28)$$

and likewise for $h(A)$.

Now, to achieve (22) note that since $\gamma_1 > 1$

$$\mathbb{P}(\Xi_1 \in A) \leq \mathbb{E}(\{2 \leq k \leq \gamma_1 : Y_k \in A\}) \leq g(A) \quad (29)$$

Hence the density of distribution of $\Xi_1$ is bounded by the density of $g$. Moreover, because $\mathbb{P}(\theta_1 > s) \leq Ce^{-cs}$ for some $C < \infty$ and $c > 0$, we know that the density of distribution of $\Xi_1$ has exponentially decaying tails. Therefore $\Xi$ is a random walk, with i.i.d steps, and step distribution bounded by $g$ with exponentially decaying tails. Hence a standard random walk argument implies (22).

(24) then follows by writing (using the fact that the different excursions are i.i.d)

$$G(A) = g(A) + \int_{\mathbb{R}^3} g(A-x)R(dx), \quad H(A) = h(A) + \int_{\mathbb{R}^3} h(A-x)R(dx)$$

and inserting (22) and (23).
3.4. Inter-Excursion Mismatches

Let \( t \rightarrow Y^*(t) \) denote a Markovian flight process with associated virtual scatterers \( Y^{*'} \in S^{*'} \) and initial velocity \( u^*_1 \in -\Omega v_0 \). Let \( t \rightarrow Y(t) \) be a second Markovian flight process with associated virtual scatterers \( S^{*} \), and initial velocity \( v_0 \).

We think of \( Y^* \) as the process run backward in time. Define the events

\[
\tilde{W}_j := \{ (Y(t) - Y'_{k'}) : 0 < t < \Theta_{j-1}, \quad \Gamma_{j-1} < k \leq \Gamma_j \} \cap Q_r \neq \emptyset \},
\]

\[
\tilde{W}_j := \{ (Y_k - Y(t)) : 0 \leq k < \Gamma_{j-1}, \quad \Theta_{j-1} < t < \Theta_j \} \cap Q_r \neq \emptyset \},
\]

\[
\tilde{W}_j^* := \{ (Y^*_k - Y(t)) : 0 < t < \Theta_{j-1}, \quad 0 < k \leq \gamma \} \cap Q_r \neq \emptyset \},
\]

\[
\tilde{W}_j^* := \{ (Y^{*'}_k - Y(t)) : 0 < k \leq \Gamma_{j-1}, \quad 0 < t < \Theta_j \} \cap Q_r \neq \emptyset \},
\]

\[
\tilde{W}_\infty := \{ (Y^*_k - Y(t)) : 0 < t < \infty, \quad 0 < k \leq \gamma \} \cap Q_r \neq \emptyset \},
\]

\[
\tilde{W}_\infty := \{ (Y^{*'}_k - Y(t)) : 0 < k < \infty, \quad 0 < t < \Theta_j \} \cap Q_r \neq \emptyset \}.
\]

In words \( \tilde{W}_j \) is the event that during the \((j-1)^{th}\) excursion, a collision of \( Y \) is (virtually) shadowed by a previous excursion. And \( \tilde{W}_j \) is the event that during the \((j-1)^{th}\) excursion the process (virtually) recollides with a scatterer from an earlier excursion.

It readily follows that

\[
P(\tilde{W}_j) = P(\tilde{W}_j^*) \leq P(\tilde{W}_{j+1}^*) \leq P(\tilde{W}_\infty^*), \quad P(\tilde{W}_j) = P(\tilde{W}_j^*) \leq P(\tilde{W}_{j+1}^*) \leq P(\tilde{W}_\infty^*). \tag{30}
\]

By the union bound

\[
P(\tilde{W}_\infty^*) \leq \sum_{z \in Z^3} P(\{1 < k < \infty : Y^*_k \in B_{zr,2r}\} \neq \emptyset) P(\{0 < t \leq \theta : Y(t) \in B_{zr,2r}\} \neq \emptyset)
\]

\[
\leq \sum_{z \in Z^3} (2r)^{-1} E(\{1 < k < \infty : Y^*_k \in B_{zr,2r}\}) \cdot E(\{0 < t \leq \theta : Y(t) \in B_{zr,3r}\}),
\]

\[
P(\tilde{W}_\infty^*) \leq \sum_{z \in Z^3} P(\{0 < t < \infty : Y^*(t) \in B_{zr,2r}\} \neq \emptyset) P(\{1 \leq j \leq \gamma : Y_j \in B_{zr,2r}\} \neq \emptyset)
\]

\[
\leq \sum_{z \in Z^3} (2r)^{-1} E(\{0 < t < \infty : Y^*(t) \in B_{zr,3r}\}) \cdot E(\{1 \leq j \leq \gamma : Y_j \in B_{zr,2r}\}). \tag{31}
\]
3.5. Computations

(31) implies that

\[
P \left( \tilde{W}_\infty^* \right) \leq (2r)^{-1} \sum_{z \in \mathbb{Z}^3} H^*(B_{zr,3r})g(B_{zr,2r})
\]

\[
P \left( \hat{W}_\infty^* \right) \leq (2r)^{-1} \sum_{z \in \mathbb{Z}^3} G^*(B_{zr,3r})h(B_{zr,2r})
\]

(32)

where \( G^* \) is defined like \( G \), except that in this instance the initial velocity is chosen from \(-\Omega_{v_0}\) rather than fixed to be \( v_0 \).

**Lemma 3.** The following bounds hold for some \( C < \infty \) and any \( v \in \Omega \)

\[
\sum_{z \in \mathbb{Z}^3} K(B_{zr,3r})M(B_{zr,2r}) \leq Cr^3, \quad \sum_{z \in \mathbb{Z}^3} L_v(B_{zr,3r})M(B_{zr,2r}) \leq Cr^3
\]

\[
\sum_{z \in \mathbb{Z}^3} K(B_{zr,3r})L_v(B_{zr,2r}) \leq Cr^3, \quad \sum_{z \in \mathbb{Z}^3} L_v(B_{zr,3r})L_w(B_{zr,2r}) \leq Cr^2.
\]

for \( v \neq w \in \Omega \)

**Proof.** The following bounds follow immediately from the definitions of \( K, M, \) and \( L_v \)

\[
K(B_{zr,3r}) \leq Cr^3, \quad M(B_{zr,3r}) \leq Cr^3 e^{-cr|z|}, \quad L_v(B_{zr,3r}) \leq Cr^3 \delta_{0,z} + Cr \mathbb{1} \{ \exists t > 0 : vt \cap B_{zr,3r} \} (1 - \delta_{0,z})e^{-cr|z|}.
\]

From here

\[
\sum_{z \in \mathbb{Z}^3} K(B_{zr,3r})M(B_{zr,2r}) \leq Cr^6 \sum_{z \in \mathbb{Z}^3} e^{-cr|z|} \leq Cr^3 \int_{\mathbb{R}^3} e^{-c|z|} dz \leq Cr^3
\]

where we use a Riemann integral to go from the first line to the second. Likewise

\[
\sum_{z \in \mathbb{Z}^3} K(B_{zr,3r})L_v(B_{zr,2r}) \leq Cr^6 + Cr^4 \sum_{z \in (\mathbb{Z}^3)^*} \mathbb{1} \{ \exists t > 0 : vt \cap B_{zr,3r} \} e^{-cr|z|} \leq Cr^6 + C'r^4 \sum_{z=1}^{\infty} e^{-c|rvz|} \leq Cr^6 + Cr^3 \int_0^{\infty} e^{-c|vt|} dt \leq Cr^3
\]

(34)

where from the first line to the second we approximate the points \( zr \in r\mathbb{Z}^3 \) close to the line \( vt \) by the points \( rvz \) for \( z \in \mathbb{Z} \).

Similarly

\[
\sum_{z \in \mathbb{Z}^3} L_v(B_{zr,3r})M(B_{zr,2r}) \leq Cr^6 + Cr^4 \sum_{z \in (\mathbb{Z}^3)^*} \mathbb{1} \{ \exists t > 0 : vt \cap B_{zr,3r} \} e^{-2cr|z|}
\]
the bound then follows as it did in (34).

Finally,
\[
\sum_{z \in \mathbb{Z}^3} L_v(B_{zr, 3r}) L_w(B_{zr, 2r}) \\
\leq C r^2 \sum_{z \in (\mathbb{Z}^3)^*} \mathbb{1}\{\exists t > 0 : vt \cap B_{zr, 3r}\} \mathbb{1}\{\exists t > 0 : wt \cap B_{zr, 3r}\} e^{-2cr|z|} \\
\leq C r^2 e^{-c r} \leq C r^2,
\]

since \(v \neq w\) only finitely many \(z \in \mathbb{Z}\) contribute to the sum, from which the second line follows.

Note that Lemma 2 is stated for \(G\) and \(H\) and not \(G^*\) and \(H^*\). However similar bounds hold for the backward excursions. Thus (omitting these details), we use Lemma 2 to insert Lemma 3 into (32) to get:

**Proposition 1.** There exists a constant \(C > 0\) such that for all \(j \geq 1\)
\[
\mathbb{P}(\hat{W}_j) \leq Cr, \quad \mathbb{P}(\tilde{W}_j) \leq Cr.
\]

### 3.6. Mismatches within one Excursion

Define the following indicator functions
\[
\begin{align*}
\hat{\eta}_j &= \hat{\eta}(y_{j-2}, y_{j-1}, y_j) := \mathbb{1}\left\{ \min_{0 \leq t \leq \xi_j-2} (tu_{j-2} + y_{j-1} + \beta_{j-1}) \in Q_r \right\} \\
\tilde{\eta}_j &= \tilde{\eta}(y_{j-2}, y_{j-1}, y_j) := \mathbb{1}\left\{ \min_{0 \leq t \leq \xi_j} (y_{j-1} + tu_j - \beta_{j-2}) \in Q_r \right\} \\
\eta_j &= \max\{\hat{\eta}_j, \tilde{\eta}_j\}
\end{align*}
\]

In words, \(\hat{\eta}_j\) is the event that the \((j-1)\)-labeled collision is shadowed by the immediately preceding path (i.e., a direct shadowing event). And \(\tilde{\eta}_j\) is the event that during the \(j\)th path segment there is a recollision with the immediately preceding obstacle (i.e., a direct recollision)—see the left hand side of Fig. 2.

**Lemma 4.** For any \(i, j < \gamma\) with \(i \neq j\) there exists a \(C < \infty\) such that
\[
\begin{align*}
\mathbb{E}(\eta_j) &\leq C r \quad (37) \\
\mathbb{E}(\eta_j \eta_i) &\leq C r^2 \quad (38)
\end{align*}
\]

(38) is not needed to prove the result for \(T = o(r^{-1})\), however, will be used to prove Theorem 2.

**Proof of Lemma 4.** Suppose \(u_{j-2} = U\). Then throughout the two subsequent collisions we know for some \(i = 1, 2, 3\) - \((u_{j-1})_i = (u_j)_i = U_i\) (i.e., one coordinate of the velocity remains unchanged). Thus to (directly) recollide with \(Y_{j-2} + Q_r\) we require \(\xi_{j-1} < C r\) which implies (37). The same is true for shadowing events, that is \(\hat{\eta}_j = 1\) implies \(\xi_{j-1} > C r\) for some constant.

(38) follows for the same reason. Suppose \(i \neq j\), then for \(\eta_j \eta_i = 1\), requires \(\max\{\xi_{j-1}, \xi_{i-1}\} < C r\) for some constant. As these are independent exponentials, (38) is immediate.
Lemma 4 controls the probability of a direct mismatch. However we also need to control indirect mismatches. To that end define

\[ \tilde{\eta}_j^o := \mathbb{I} \left\{ \min_{0 \leq t \leq \tau_j - 3} (Y(t) - Y'_{j-1}) \in Q_r \right\} \]

\[ \bar{\eta}_j^o := \mathbb{I} \left\{ \min_{\tau_{j-1} \leq t \leq \tau_j} \left( \min_{0 \leq k \leq j-3} (Y(t) - Y'_k) \right) \in Q_r \right\} \]

\[ \eta_j^o := \max\{\tilde{\eta}_j^o, \bar{\eta}_j^o\} \]  \hspace{1cm} (39)

In words \( \tilde{\eta}_j^o \) is the indicator that an indirect (virtual) shadowing event occurs and \( \bar{\eta}_j^o \) is the event an indirect (virtual) recollision occurs. That is a mismatch which involves more than the immediately preceding obstacle or path.

**Lemma 5.** For any \( 3 < j \leq \gamma \) there exists a constant \( C > 0 \) such that

\[ \mathbb{E} (\eta_j^o) \leq C \gamma^2 r^2 \]  \hspace{1cm} (40)

**Proof of Lemma 5.** Under time reversal Markovian flight processes remain Markovian flight process, while recollisions become shadowed events. Hence recollisions and shadowing events happen with the same probability and thus we may restrict to proving the statement for recollisions.

By the union bound

\[ \mathbb{E} (\bar{\eta}_j^o) \leq \sum_{k \leq j-3} \mathbb{P} \left( \left\{ \min_{\tau_{j-1} \leq t \leq \tau_j} (Y(t) - Y'_k) \in Q_r \right\} \right) . \]  \hspace{1cm} (41)

Write \( \mathcal{A}_k = \{\min_{\tau_{j-1} \leq t \leq \tau_j} (Y(t) - Y'_k) \in Q_r\} \) - the event there is an indirect recollision after \( k-1 \) fresh collisions. To have an indirect recollision requires at least three distinct velocities along the path, thus

\[ \mathbb{P} (\mathcal{A}_k) = \mathbb{P} (\mathcal{A}_k \cap \exists i \in [k+1, j-2] : u_i \neq u_j, u_{j-1}) . \]

Moreover at each collision exactly one of the velocity coordinates changes sign. Hence we know \( u_j \) and \( u_{j-1} \) differ by a sign change in one coordinate; therefore the event in the right hand side of (3.6) implies there is a third velocity which is linearly independent of \( u_j \) and \( u_{j-1} \). Therefore

\[ (3.6) = \mathbb{P} (\mathcal{A}_k \cap \exists i \in [k+1, j-2] : u_i, u_j, u_{j-1} \text{ lin. ind.}) \]

Moreover note that if we fix \( i \)

\[ \mathcal{A}_k = \{ \min_{0 \leq t \leq \xi_j} (\xi_i u_i + \xi_{j-1} u_{j-1} + tu_j - s_i) \in Q_r \} \]

where

\[ s_i = \sum_{l=k+1}^{j-2} u_l \xi_l. \]

Let \( B_i \) denote the event \( u_i, u_{j-1}, u_j \) are linearly independent. In this case

\[ \mathbb{P} (\mathcal{A}_k) \leq \sum_{i=k+1}^{j-2} \mathbb{P} (B_i \cap \mathcal{A}_k) \]
\[ \leq \sum_{i=k+1}^{j-2} \mathbb{E} \left( \mathbb{P} \left( B_i \cap \left\{ \min_{0 \leq t \leq \xi_j} (\xi_i u_i + \xi_{j-1} u_{j-1} + tu_j - s_i) \in Q_r \right\} \mid s_i \right) \right). \]

Lemma 6 (below) implies that the probability inside the expectation is bounded by \( Cr^2 \). As \( j - 2 - k \leq \gamma \) this implies
\[ \mathbb{P}(A_k) \leq C \gamma r^2. \]
Inserting this into (41) then implies (40)

**Lemma 6.** Suppose \( U_1, U_2, U_3 \in \Omega \) are linearly independent and \( \xi_1, \xi_2, \xi_3 \sim \text{EXP}(1) \) are i.i.d exponentials. Then there exists a constant \( C < \infty \) such that for any \( s \in \mathbb{R}^3 \)
\[ \mathbb{P} \left( \min_{0 \leq t \leq \xi_3} (U_1 \xi_1 + U_2 \xi_2 + U_3 t - s) \in Q_r \right) \leq Cr^2. \] (42)

**Proof.** We can assume
\[ U_1 = (\nu_1, \nu_2, \nu_3), \quad U_2 = (-\nu_1, \nu_2, \nu_3), \quad U_3 = (-\nu_1, -\nu_2, \nu_3) \]
in which case for any \( t \leq \xi_3 \)
\[ U_1 \xi_1 + U_2 \xi_2 + U_3 t = ((\xi_1 - \xi_2 - t)\nu_1, (\xi_1 + \xi_2 - t)\nu_2, (\xi_1 + \xi_2 + t)\nu_3). \] (43)
Therefore the event on the left hand side of (42) is the event that there exists a \( t \leq \xi_3 \) satisfying the system of inequalities
\[
\begin{align*}
s_1 - \frac{r}{2} & \leq (\xi_1 - \xi_2 - t)\nu_1 \leq s_1 - \frac{r}{2} \\
s_2 - \frac{r}{2} & \leq (\xi_1 + \xi_2 - t)\nu_2 \leq s_2 - \frac{r}{2} \\
s_3 - \frac{r}{2} & \leq (\xi_1 + \xi_2 + t)\nu_3 \leq s_3 - \frac{r}{2}
\end{align*}
\]
solving these equations, we find that regardless of \( t \) there exist \( c_1, c_2, C_1, C_2 \) such that
\[ \xi_1 \in [c_1 - C_1 r, c_1 + C_1 r], \quad \xi_2 \in [c_2 - C_2 r, c_2 + C_2 r] \]
since \( \xi_1 \) and \( \xi_2 \) are i.i.d exponentials (42) follows immediately.

4. Beyond the Naïve Coupling

In the following sections we extend the results of Sect. 3 to times on the order \( o(r^{-2}) \). In order to reduce the amount of notation we will use the same notation for the analogous objects and will give the redefinitions explicitly. Recall the definition of the process \( \{ t \mapsto Z(t) \} \) given in Sect. 2.3. We will split the process \( \{ t \mapsto Z(t) \} \) into legs (similar to the excursions of the previous section).
4.1. Legs

Similar to Sect. 3.1 we split \( t \mapsto Z(t) \) into legs. However to ensure that the different legs are independent we impose the restriction that each leg begins and ends with two path segments of length greater than 1. Let \( \tilde{\xi}_n = \tilde{\tau}_n - \tilde{\tau}_{n-1} \) for all \( n \geq 1 \). Let

\[
\gamma := \min\{i > 1 : \tilde{\xi}_{i-1} ; \tilde{\xi}_i ; \tilde{\xi}_{i+1} > 1, \quad \tilde{w}_{i+1} = \tilde{w}_1 = v_0\}. \tag{44}
\]

Note that the condition on \( \tilde{\xi}_i \) implies that \( \gamma \in \{2\} \cup \{5, \ldots \} \). If we define \( \theta := \sum_{i=1}^\gamma \tilde{\xi}_i \), then

\[
P (\gamma > s) \leq Ce^{-cs}, \quad P (\theta > s) \leq Ce^{-cs}. \tag{45}
\]

The definition of a pack is then similar to Subsection 3.1: a pack is a collection

\[
\varpi := \left( \gamma ; \{\tilde{\xi}_i\}_{i=1}^\gamma , \{\tilde{\beta}_i\}_{i=1}^\gamma , \{\tilde{w}_i\}_{i=1}^\gamma \right).
\]

Given a pack we consider the process \( t \mapsto Z(t) \) associated to it via the rules set forth in Subsect. 2.3 and call such a segment a leg. Note that, in order to have a direct mismatch at step \( n \) requires that \( \tilde{\xi}_{n-1} < Cr \) for some constant \( C < \infty \). Hence the beginning and end of a leg are Markovian steps.

Furthermore given a pack \( \varpi \) a backward leg is defined to be

\[
(\theta ; Z^*(t); 0 \leq t \leq \theta)
\]

where

\[
Z^*(t) = Z(\theta - t, \varpi^*) - Z(\varpi^*)
\]

(we use the notation \( Z(t, \varpi) \) to denote the forward forgetful process built from the pack \( \varpi \)) where

\[
\varpi^* := (\gamma ; \{\xi_{\gamma-j}\}_{j=0}^{\gamma-1} , \{\beta_{\gamma-j}\}_{j=0}^{\gamma-1} , \{w_{\gamma-j}\}_{j=0}^{\gamma-1})
\]

As before denote

\[
Z_j^* := Z^*(\tilde{\tau}_j), \quad 0 \leq j \leq \gamma, \quad Z^* = Z_\gamma^*.
\]

Note the processes \( t \mapsto Z(t) \) and \( t \mapsto Z^*(t) \) do not have the same distribution.

4.2. Concatenation

Let \( \varpi_n = (\gamma_n ; \{\tilde{\xi}_{n,j}\}_{j=1}^{\gamma_n} , \{\tilde{\beta}_{n,j}\}_{j=1}^{\gamma_n} , \{\tilde{w}_{n,j}\}_{j=1}^{\gamma_n}) \), \( n \geq 1 \), be a sequence of i.i.d packs and consider the associated forward legs \( (Z_n(t) : 0 \leq t \leq \theta_n) \), \( (Z_{n,j} : 1 \leq j \leq \gamma_n) \) and backward legs \( (Z_{n,j}^*(t) : 0 \leq t \leq \theta_n) \), \( (Z_{n,j}^* : 1 \leq j \leq \gamma_n) \).

To construct the concatenated forward and backward processes \( t \mapsto Z(t) \), \( t \mapsto Z^*(t) \), \( 0 \leq t < \infty \), define for \( n \in \mathbb{Z}_+ \) and \( t \in \mathbb{R}_+ \)

\[
\Gamma_n := \sum_{k=1}^n \gamma_k, \quad \nu_n := \max\{m : \Gamma_m \leq n\}, \quad \{n\} := n - \Gamma_{\nu_n}, \tag{46}
\]

\[
\Theta_n := \sum_{k=1}^n \theta_k, \quad \nu_t := \max\{m : \Theta_m < t\}, \quad \{t\} := t - \Theta_{\nu_t}.
\]
The concatenated (multi-leg) forward and backward \( Z \)-processes are
\[
\Xi_n := \sum_{k=1}^{n} Z_k, \quad Z_n := \Xi_{\nu_n} + Z_{\nu_n+1, \{n\}}, \quad Z(t) := \Xi_{\nu_t} + Z_{\nu_t+1, \{t\}},
\]
\[
\Xi^*_n := \sum_{k=1}^{n} Z^*_k, \quad Z^*_n := \Xi^*_{\nu_n} + Z^*_{\nu_n+1, \{n\}}, \quad Z^*(t) := \Xi^*_{\nu_t} + Z^*_{\nu_t+1, \{t\}}.
\]

(47)

4.3. Mismatches in a Leg

Let \( \varpi = (\gamma; \{\xi_j\}_{j=1}^{\gamma}, \{\beta_j\}_{j=1}^{\gamma}, \{\tilde{\omega}_j\}_{j=1}^{\gamma}) \) be a pack. Let \( u \in \Omega_{v_0} \) a velocity and \( \beta_0 \in B(u, v_0) \) an impact parameter.

Let \( t \mapsto X(t) \) be the wind-tree process coupled to the pack \( \varpi \). This, given the processes \( t \mapsto Y(t) \) and \( t \mapsto Z(t) \) follow the rules in Subsect. 2.3 until time \( \tau_\gamma \).

Consider the jointly realized triple \((Y(t), X(t), Z(t)) : 0^- < t < \theta^+\) — a Markovian flight process, a wind-tree exploration process and a forgetful process all coupled to \( \varpi \). The time interval \( 0^- < t < \theta^+ \) indicates that the velocity immediately prior to the position at 0 is \( u \); there is a collision with a scatterer at \( \beta_0 \), and at \( \theta^+ \) the velocity of \( Y \) and \( Z \) is \( w \).

Proposition 2. There exists a \( C < \infty \) such that for all \( w \in \Omega \) and \( u \in \Omega_w \) and \( \beta_0 \in B(u, w) \)
\[
P \left( X(t) \neq Z(t) : 0^- < t < \theta^+ \right) \leq r^2.
\]
(48)

This proposition will be proved in Sect. 6.

4.4. Inter-Leg Mismatches

Consider a forgetful process \( t \mapsto Z(t) \) built from legs. Define the following events
\[
\hat{W}_j := \{\{Z(t) - Z'_k : 0 < t < \Theta_{j-1}, \Gamma_{j-1} < k \leq \Gamma_j \} \cap Q_r \neq \emptyset\},
\]
\[
\tilde{W}_j := \{\{Z'_k - Z(t) : 0 \leq k < \Gamma_{j-1}, \Theta_{j-1} < t < \Theta_j \} \cap Q_r \neq \emptyset\},
\]
(49)
i.e., \( \hat{W}_j \) is the event a collision during the \( j^{th} \) leg is (virtually) shadowed by a path segment in a previous leg. \( \tilde{W}_j \) is the event that during the \( j^{th} \) leg the process (virtually) collides with an obstacle placed during a previous leg.

Proposition 3. There exists a \( C < \infty \) such that for all \( j \geq 1 \),
\[
P \left( \hat{W}_j \right) \leq C r^2, \quad P \left( \tilde{W}_j \right) \leq C r^2.
\]
(50)
The proof of this proposition is the content of Sect. 5.

5. Proof of Proposition 3

The proof of Proposition 3 follows the similar lines to that of Proposition 1. However as we have redefined legs, we shall go through the full proof. In this section we redefine the Green’s functions \( g, h, G, \) and \( H \).
5.1. Occupation Measures

Let $t \mapsto Z(t)$ be a forward forgetful process with initial velocity $v_0$ and $t \mapsto Z^*(t)$ a backward process with initial velocity in $\Omega_{-\tilde{w}_1}$ (distributed according to $m_{-v_0}$). Define the events

\[ \tilde{W}_j^* := \{ |\{ Z^*(t) - Z_k^* : 0 < t < \Theta_j, 0 < k \leq \gamma \} \cap Q_r \neq \emptyset \}, \]

\[ \tilde{W}_j := \{ |\{ Z_k^* - Z(t) : 0 < k \leq \Gamma_j, 0 < t < \theta \} \cap Q_r \neq \emptyset \}, \]

\[ \tilde{W}_\infty^* := \{ |\{ Z^*(t) - Z_k^* : 0 < t < \infty, 0 < k \leq \gamma \} \cap Q_r \neq \emptyset \}, \]

\[ \tilde{W}_\infty := \{ |\{ Z_k^* - Z(t) : 0 < k < \infty, 0 < t < \theta \} \cap Q_r \neq \emptyset \}. \]

The same calculation as (30), (31), and (32) implies

\[ P(\tilde{W}_j) \leq P(\tilde{W}_\infty^*) \leq (2r)^{-1} \sum_{z \in \mathbb{Z}^3} H^*(B_{2r,3r})g(B_{2r,2r}), \]

\[ P(\tilde{W}_j) \leq P(\tilde{W}_\infty) \leq (2r)^{-1} \sum_{z \in \mathbb{Z}^3} G^*(B_{2r,3r})h(B_{2r,2r}), \]

where the right hand side is in terms of the following Green’s functions: for $A \subset \mathbb{R}^3$

\[ g(A) := E(|\{ 1 \leq k \leq \gamma : Z_k \in A \}|), \quad g^*(A) := E(|\{ 1 \leq k \leq \gamma : Z_k^* \in A \}|), \]

\[ h(A) := E(|\{ 0 < t \leq \theta : Z(t) \in A \}|), \quad h^*(A) := E(|\{ 0 < t \leq \theta : Z^*(t) \in A \}|), \]

\[ R^*(A) := E(|\{ 1 \leq n < \infty : Z_n \in A \}|), \quad K^*(A) := E(|\{ 0 < t < \infty : Z^*(t) \in A \}|). \]

Note that

\[ G^*(A) = g^*(A) + \int_{\mathbb{R}^3} g^*(A - x)R^*(dx), \]

\[ H^*(A) = h^*(A) + \int_{\mathbb{R}^3} h^*(A - x)R^*(dx). \] (52)

5.2. Bounds

**Lemma 7.** The following bounds hold for any Borel set $A \subset \mathbb{R}^3$

\[ g(A) \leq M(A) + \tilde{L}_{v_0}(A), \quad g^*(A) \leq M(A) + \tilde{L}^\perp_{v_0}(A), \] (53)

\[ h(A) \leq M(A) + L_{v_0}(A), \quad h^*(A) \leq M(A) + L^\perp_{v_0}(A), \] (54)

\[ R^*(A) \leq K(A) + \tilde{L}^\perp_{v_0}(A), \quad K^*(A) \leq K(A) + L^\perp_{v_0}(A), \] (55)

\[ G^*(A) \leq K(A) + \tilde{L}^\perp_{v_0}(A), \quad H^*(A) \leq K(A) + L^\perp_{v_0}(A), \] (56)

where $K$, $L_{v_0}$, and $M$ are as defined in Lemma 2 and

\[ L^\perp_{v_0}(A) := C \sum_{w \in \Omega_{-v_0}} \int_0^\infty 1_{\{tw \cap A\}} e^{-ct} dt, \]

\[ \tilde{L}_{v_0}(A) := C \int_1^\infty 1_{\{tv_0 \cap A\}} e^{-ct} dt, \]
\[
\tilde{L}_{v_0}^\perp(A) := C \sum_{w \in \Omega_{-v_0}} \int_1^\infty \mathbb{1}\{tw \cap A\} e^{-ct} dt.
\]

**Proof.** The proof of this lemma follows the same lines as the proof of Lemma 2; however the legs in this section are conditioned to have the first step longer than 1. (55) follows from the fact that the steps of \( \Xi_\ast^\ast \) are i.i.d with exponentially decaying tails and the density of each step is bounded by \( g^\ast(dx) \).

To bound \( g(A) \) write:

\[
g(A) = \int_{\mathbb{R}^3} g_2(A - x) g_1(dx),
\]

\[
g_1(A) := P(\{Z_1 \in A\}) = C \int_1^\infty \mathbb{1}\{tv_0 \in A\} e^{-t} dt,
\]

\[
g_2(A) := E(\{|1 \leq k \leq \gamma_1 : Z_k - Z_1 \in A\}).
\]

This follows since \( Z_k - Z_1 \) is independent of \( Z_1 \) for every \( k \geq 2 \). (53) then follows in the same way as did (23) in Lemma 2 from the bounds

\[
g_2(\{x : |x| > s\}) \leq Ce^{-cs}, \quad g_2(\mathbb{R}^3) = E(\gamma) < \infty.
\]

For \( g^\ast(A) \) write

\[
g^\ast(A) = E(\{|1 \leq k \leq \gamma_1 : Z_k \in A\})
\]

\[
\leq \sum_{w \in \Omega_{-v_0}} E(\{|1 \leq k \leq \gamma_1 : Z_k \in A\} | \tilde{w}_1 = w) =: \sum_{w \in \Omega_{-v_0}} g_{w}^\ast(A),
\]

where \( \tilde{w}_1 := \tilde{Z}^\ast(0^+) \). As for \( g(A) \) we now split

\[
ge_{w}^\ast(A) = \int_{\mathbb{R}^3} g_{2,w}(A - x) g_{1,w}(dx)
\]

\[
g_{1,w}^\ast(A) := P(\{Z_1 \in A \mid \tilde{w}_1 = w\})
\]

\[
g_{2,w}^\ast(A) := E(\{|1 \leq k \leq \gamma_1 : Z_k - Z_1 \in A\} \mid \tilde{w}_1 = w).
\]

Our bound for \( g^\ast(A) \) now follows the same lines as for \( g(A) \). \( h^\ast(A) \) is very similar.

The bounds on \( G^\ast \) and \( H^\ast \) follow by inserting the bounds for \( g^\ast, h^\ast, R^\ast \) into (52).

### 5.3. Computations

**Lemma 8.** The following bounds hold for some \( C < \infty \) and \( r \) small enough

\[
\sum_{z \in \mathbb{Z}^3} \tilde{L}_{v_0}^\perp(B_{2rz,3r}) L_{v_0}(B_{2rz,2r}) = 0, \quad \sum_{z \in \mathbb{Z}^3} L_{v_0}^\perp(B_{2rz,3r}) \tilde{L}_{v_0}(B_{2rz,2r}) = 0,
\]

\[
\sum_{z \in \mathbb{Z}^3} K(B_{2rz,3r}) \tilde{L}_{v_0}(B_{2rz,2r}) \leq Cr^3, \quad \sum_{z \in \mathbb{Z}^3} \tilde{L}_{v_0}^\perp(B_{2rz,3r}) M(B_{2rz,2r}) \leq Cr^3,
\]

\[
\sum_{z \in \mathbb{Z}^3} L_{v_0}^\perp(B_{2rz,3r}) M(B_{2rz,2r}) \leq Cr^3.
\]
Proof. These bounds follow by observing

\[
\tilde{L}_{v_0}(B_{zr,3r}) \leq C \mathbb{I}\{\exists t \geq 1 : B_{zr,3r} \cap v_0t \} e^{-cr|z|},
\]

\[
\tilde{L}^\perp_{v_0}(B_{zr,3r}) \leq C \sum_{\omega \in \Omega - v_0} \mathbb{I}\{\exists t \geq 1 : B_{zr,3r} \cap wt \} e^{-cr|z|},
\]

(57)

and (33). With that the first two bounds are trivial. The third bound follows from:

\[
\sum_{z \in \mathbb{Z}^3} K(B_{zr,3r}) \tilde{L}_{v_0}(B_{zr,2r}) \leq C r^6 + C r^4 \sum_{\omega \in \Omega - v_0} \sum_{z \in (\mathbb{Z}^3)^*} \mathbb{I}\{\exists t \geq 3r : B_{zr,3r} \cap wt \} e^{-cr|z|} \leq C r^3,
\]

where in the last line we approximate the sum by an integral in the same way as we did in (34).

Note that by (57)

\[
\sum_{z \in \mathbb{Z}^3} \tilde{L}^\perp_{v_0}(B_{zr,3r})M(B_{zr,2r}) \leq \sum_{z \in \mathbb{Z}^3} L^\perp_{v_0}(B_{zr,3r})M(B_{zr,2r}).
\]

Moreover by (33) and (57)

\[
\sum_{z \in \mathbb{Z}^3} L^\perp_{v_0}(B_{zr,3r})M(B_{zr,2r}) \leq Cr^6 + Cr^4 \sum_{\omega \in \Omega - v_0} \mathbb{I}\{\exists t \geq 3r : B_{zr,3r} \cap wt \} e^{-2cr|z|} \leq Cr^3.
\]

Proposition 3. The proof of Proposition 3 follows by inserting the bounds in Lemma 7 into (51) and then applying Lemma 8.

6. Proof of Proposition 2

In the setting of Sect. 4.3 the proof of Proposition 2 will follow from considering the following indicator functions

\[
\tilde{\eta}_j := \mathbb{I}\left\{\min_{\tilde{t}_{j-1} < t < \tilde{t}_j} (Z(t) - Z_{j-2}) \in Q_r \right\}
\]

\[
\hat{\eta}_j := \mathbb{I}\left\{\min_{\tilde{t}_{j-3} < t < \tilde{t}_{j-2}} (Z(t) - Z(\tilde{t}_{j-1}) - \tilde{\beta}_{j-1}) \in Q_r \right\}
\]

\[
\eta_j := \max\{\tilde{\eta}_j, \hat{\eta}_j\}
\]

(58)

In particular, \(\eta_j\) is the probability of a mismatch for the \(Z\)-process in immediately before the \(j^{th}\) leg. It is important to note, the simple geometric fact (which follows simply from the fact that the collision angles are bounded) that
\( \eta_j^* = 1 \) implies \( \tilde{\xi}_{j-1} < Cr \) for some constant \( C < \infty \). This fact will make the geometric estimates vastly easier than for the Lorentz gas, where the equivalent statement is false.

The following statements will provide the proof of Proposition 2

\[
P \left( \{ \mathcal{X}(t) \neq Z(t) : 0^- \leq t \leq \theta^+ \} \cap \{ \sum_{j=1}^\gamma \eta_j > 1 \} \right) \leq Cr^2, \quad (59)
\]

\[
P \left( \{ \mathcal{X}(t) \neq Z(t) : 0^- \leq t \leq \theta^+ \} \cap \{ \sum_{j=1}^\gamma = 0 \} \right) \leq Cr^2, \quad (60)
\]

\[
P \left( \{ \mathcal{X}(t) \neq Z(t) : 0^- \leq t \leq \theta^+ \} \cap \{ \sum_{j=1}^\gamma = 1 \} \right) \leq Cr^2. \quad (61)
\]

6.1. Proof of (59)

The simple geometric fact stated in the previous section implies

\[
P \left( \sum_{j=1}^\gamma \eta_j > 1 \right) \leq \frac{\gamma^2}{2} \max_{1 \leq j < k \leq \gamma} P(\eta_j \eta_k = 1) \leq C\gamma^2 r^2.
\]

(59) now follows from the exponential tail bounds (45).

6.2. Proof of (60)

On \( \{ \sum_{j=1}^\gamma \eta_j = 0 \} \), the process \( \{ t \mapsto Z(t) \} \) is distributed like a Markovian flight process. Hence the event in (60) can be written

\[
\{ \mathcal{X}(t) \neq Z(t) : 0^- \leq t \leq \theta^+ \} \cap \left\{ \sum_{j=1}^\gamma = 0 \right\} = \{ \exists 3 \leq j \leq \gamma : \eta_j^o = 1 \} \cap \left\{ \sum_{j=1}^\gamma = 0 \right\}
\]

where \( \eta_j^o \) is the indicator of an indirect mismatch, as defined in (39). Therefore using Lemma 5

\[
P \left( \{ \mathcal{X}(t) \neq Z(t) : 0^- \leq t \leq \theta^+ \} \cap \{ \sum_{j=1}^\gamma = 1 \} \right)
\]

\[
\leq \gamma \max_{3 \leq j \leq \gamma} P(\eta_j^o = 1)
\]

\[
\leq C\gamma^3 r^2.
\]

Thus (60) again follows from the exponential tail bounds (45).
### 6.3. Proof of (61)

Given a $\gamma \in \{2\} \cup \{5, \ldots \}$, a signature $\xi$ (recall the definition of a signature given at the end of Sect. 2.2) compatible with the definition of a pack, and a fixed label $3 < k < \gamma$. Let $V_1, V_2 \in \Omega$ and let $\omega$ be a pack with signature $\xi$ and $\tilde{w}_{k-2} = V_1$ and $\tilde{w}_{k+1} = V_2$ (we assume $V_1$ and $V_2$ are compatible with this definition).

- On $0^- < t \leq \tilde{\tau}_{k-1} - Z^{(k)}(t) = Y(t)$, conditioned such that $\tilde{w}_{k-2} = V_1$.
- On $\tilde{\tau}_{k-1} < t \leq \tilde{\tau}_k - Z^{(k)}(t)$ is constructed like the $Z$-process, conditioned such that the final velocity is $\tilde{w}_k \in \Omega_{V_2}$.
- On $\tilde{\tau}_k < t < \tilde{\tau}_\gamma - Z^{(k)}(t) = Y(t)$ a Markovian flight process starting at $Z^{(k)}(\tilde{\tau}_k)$, conditioned such that $\tilde{w}_{k+1} = V_2$.

On $\{\eta_j = \delta_{j,k} : 1 \leq j \leq \gamma\} - Z^{(k)}$ is distributed like $Z$. The reason for conditioning on $V_1$ and $V_2$ is to ensure the following three parts are independent:

\[
\begin{align*}
(Z^{(k)}(t) : 0^- < t \leq \tilde{\tau}_{k-3}) &\sim (Y(t) : 0^- < t \leq \tilde{\tau}_{k-3}), \\
(Z^{(k)}(\tilde{\tau}_{k-3} + t) - Z^{(k)}(\tilde{\tau}_{k-3}) : 0 \leq t \leq \tilde{\tau}_k - \tilde{\tau}_{k-3}), \\
(Z^{(k)}(\tilde{\tau}_k + t) - Z(\tilde{\tau}_k) : 0 \leq t < \theta^+ - \tilde{\tau}_k).
\end{align*}
\]

(62)

Let $A^{(k)}_{a,a}$, $1 \leq a \leq 3$ be the event that the $a$-th part of the trajectory is $r$-inconsistent. For $1 \leq a < b \leq 3$ we denote $A^{(k)}_{a,b}$ the event that the $a$ and $b$-th parts are $r$-incompatible. Therefore to prove (61) we will bound

\[
\max_{\xi, k, V_1, V_2} \mathbb{P} \left( \{\tilde{\eta}_k = 1\} \cap A^{(k)}_{a,b} \mid \xi, V_1, V_2 \right),
\]

(63)

\[
\max_{\xi, k, V_1, V_2} \mathbb{P} \left( \{\tilde{\eta}_k = 1\} \cap \{\tilde{\eta}_k = 0\} \cap A^{(k)}_{a,b} \mid \xi, V_1, V_2 \right),
\]

6.4. Bounds

First notice that $A^{(k)}_{1,1}, A^{(k)}_{3,3}$ and $A^{(k)}_{1,3}$ involve only Markovian segments; hence the following estimates follow readily from Lemmas 2, 3, 4, and 5:

\[
\max_{\xi, k, V_1, V_2} \mathbb{P} \left( \{\tilde{\eta}_k = 1\} \cap A^{(k)}_{a,b} \mid \xi, V_1, V_2 \right) \leq C\gamma^3 r^2,
\]

(64)

\[
\max_{\xi, k, V_1, V_2} \mathbb{P} \left( \{\tilde{\eta}_k = 1\} \cap \{\tilde{\eta}_k = 0\} \cap A^{(k)}_{a,b} \mid \xi, V_1, V_2 \right) \leq C\gamma^3 r^2,
\]

Therefore there remain 6 bounds.

Note that during middle segment in (62) the velocity of $Z^{(k)}(t)$ is restricted to only three possible velocities. Thus one component of the velocity remains unchanged throughout this segment. Therefore the middle segment can only be $r$-inconsistent if two of the path segments are shorter than $Cr$ for
some constant $C < \infty$. Thus
\[
P \left( \{ \tilde{\eta}_k = 1 \} \cap A_{2,2}^{(k)} \mid \xi, V_1, V_2 \right) \leq C r^2, \\
P \left( \{ \tilde{\eta}_k = 1 \} \cap \{ \tilde{\eta}_k = 0 \} \cap A_{2,2}^{(k)} \mid \xi, V_1, V_2 \right) \leq C r^2.
\]

(65)

It remains to prove
\[
P \left( \{ \tilde{\eta}_k = 1 \} \cap A_{b,2}^{(k)} \mid \xi, V_1, V_2 \right) \leq C \gamma r^2,
\]
\[
P \left( \{ \tilde{\eta}_k = 1 \} \cap \{ \tilde{\eta}_k = 0 \} \cap A_{b,2}^{(k)} \mid \xi, V_1, V_2 \right) \leq C \gamma r^2,
\]

(66)

Thus we will only prove (66) for $b = 3$ as the proof for $b = 1$ is the same. Given a set $A \subset \mathbb{R}^3$ define the following occupation measures for the third part of (62)
\[
G_{x}^{(k)}(A) := E \left( \# \{ 1 \leq j \leq \gamma - k : Z^{(k)}(\tau_{j+k}) - Z^{(k)}(\tau_k) \in A \} \mid \epsilon_{k+j} : 1 \leq j \leq \gamma - k, V_2 \right),
\]
\[
H_{x}^{(k)}(A) := E \left( \# \{ 1 \leq j \leq \gamma - k : \tilde{Y}(\tau_{j}) \in A \} \mid \epsilon_{k+j} : 1 \leq j \leq \gamma - k, V_2 \right),
\]

(65)

where $t \mapsto \tilde{Y}(t)$ is a Markovian flight process with initial velocity in $\Omega_{V_2}$. Similarly
\[
\tilde{G}_{x}^{(k)}(A) := E \left( \# \{ 1 \leq j \leq 3 : Z^{(k)}(\tau_{k-j}) - Z^{(k)}(\tau_k) \in A \} \cdot \tilde{\eta}_k \mid \xi, V_1, V_2 \right),
\]
\[
\tilde{H}_{x}^{(k)}(A) := E \left( \{ \tilde{\tau}_{k-3} \leq t \leq \tilde{\tau}_k : Z^{(k)}(t) - Z^{(k)}(\tilde{\tau}_k) \in A \} \mid \tilde{\eta}_k \cdot (1 - \tilde{\eta}_k) \mid \xi, V_1, V_2 \right),
\]
\[
\tilde{G}_{x}^{(k)}(A) := E \left( \# \{ 1 \leq j \leq 3 : Z^{(k)}(\tilde{\tau}_{k-j}) - Z^{(k)}(\tilde{\tau}_k) \in A \} \cdot \tilde{\eta}_k \cdot (1 - \tilde{\eta}_k) \mid \xi, V_1, V_2 \right),
\]
\[
\tilde{H}_{x}^{(k)}(A) := E \left( \{ \tilde{\tau}_{k-3} \leq t \leq \tilde{\tau}_k : Z^{(k)}(t) - Z^{(k)}(\tilde{\tau}_k) \in A \} \mid \tilde{\eta}_k \cdot (1 - \tilde{\eta}_k) \mid \xi, V_1, V_2 \right).
\]

As the middle and last parts in (62) are independent the following bounds apply
\[
P \left( \{ \tilde{\eta}_k = 1 \} \cap A_{3,2}^{(k)} \mid \xi, V_1, V_2 \right)
\leq C r^{-1} \left( \int_{\mathbb{R}^3} G_{x}^{(k)}(B_{x,2r}) \tilde{H}_{x}^{(k)}(dx) + \int_{\mathbb{R}^3} H_{x}^{(k)}(B_{x,3r}) \tilde{G}_{x}^{(k)}(dx) \right),
\]

(67)

By (24) there exists a constant $C < \infty$ such that
\[
G_{x}^{(k)}(B_{x,2r}) \leq CF(x), \quad H_{x}^{(k)}(B_{x,2r}) \leq CF(x)
\]

(68)
where \( F : \mathbb{R}^3 \to \mathbb{R}_+ \)
\[
F(x) = r \{ |x| \leq r \} + \frac{r^3}{|x|^2} \{ r < |x| \leq 1 \} + \frac{r^3}{|x|} \{|x| > 1 \}
+ re^{-c|x|} \mathbb{1} \{ \exists t > 0 : B_{x,2r} \cap tV \} \{|x| > r \}.
\]

For simplicity we will only treat the first term on the right hand side in the second line of (67) (this is the most difficult); the other terms can be dealt with similarly.

Since during the middle section of (62) one component of the velocity does not change sign, we can conclude
\[
\hat{G}^{(k)}(B_{0,s}), \tilde{G}^{(k)}(B_{0,s}) \leq Crs, \quad \hat{H}^{(k)}(B_{0,s}), \tilde{H}^{(k)}(B_{0,s}) \leq Crs,
\]
and
\[
\hat{G}^{(k)}(\mathbb{R}^3), \tilde{G}^{(k)}(\mathbb{R}^3) \leq Cr, \quad \hat{H}^{(k)}(\mathbb{R}^3), \tilde{H}^{(k)}(\mathbb{R}^3) \leq Cr.
\]

First note that by (69)
\[
\int_{|x| > r} re^{-c|x|} \mathbb{1} \{ \exists t > 0 : B_{x,2r} \cap tV \} \hat{H}^{(k)}(dx)
\leq Cr^2 \int_{|x| > r} e^{-c|x|} \mathbb{1} \{ \exists t > 0 : B_{x,2r} \cap tV \} dx
\leq Cr^4 \int_{t > r} e^{-c|tV|} dt \leq Cr^4
\]
and
\[
\int_{|x| > 1} \frac{r^3}{|x|} \hat{H}^{(k)}(B_{x,2r}) \leq Cr^4.
\]
Finally let \( \tilde{F}(u) = r \{ u \leq r \} + \frac{r^3}{u^2} \{ r < u \leq 1 \} \), then by applying integration by parts
\[
\int_{|x| < 1} \tilde{F}(|x|) \tilde{H}^{(k)}(dx) \leq C \int_0^1 \tilde{F}(u) d\tilde{H}^{(k)}(B_{0,u})
= Cr^3 \tilde{H}^{(k)}(B_{0,1}) - C \int_0^1 \tilde{H}^{(k)}(B_{0,u}) \tilde{F}'(u) du
\leq Cr^4 + Cr^4 \int_r^1 u^{-2} du
\leq Cr^4 + Cr^3.
\]

(66) follows by inserting these bounds into (67).

6.5. Proof of Theorem 2 - concluded

The proof of Theorem 2 now follows the same lines as [13, Sect. 7] repeated here for completeness.

Let \( \{ t \mapsto Y(t) \} \) be a Markovian flight process. Let \( \{ t \mapsto Z(t) \} \) be a coupled forgetful process. We split \( \{ t \mapsto Z(t) \} \) into i.i.d legs \( (Z_n(t) : 0 \leq t \leq \theta_n) \),
each associated to an i.i.d pack \( \varpi_n = (\gamma_n; \{ \xi_{n,j} \}_{j=1}^{\gamma_n}, \{ \beta_{n,j} \}_{j=1}^{\gamma_n}, \{ \tilde{w}_{n,j} \}_{j=1}^{\gamma_n}) \).

In addition, to each leg \((Z_n(t) : 0 \leq t \leq \theta_n)\) we associate a wind-tree process coupled to that leg \((X_n(t) : 0 \leq t \leq \theta_n)\). From these components we construct the concatenated auxiliary process

\[
\mathcal{X}(t) = \sum_{k=1}^{\nu_t} \mathcal{X}(\theta_n) + \mathcal{X}_{\nu_t+1}(\{t\}).
\]  

Note that \( t \mapsto \mathcal{X}(t) \) is not a physical process. Each leg is independent of the others. Finally let \( t \mapsto X(t) \) be the true wind-tree process, coupled to \( t \mapsto Y(t) \) and \( t \mapsto Z(t) \) as in Sect. 2.3.

We will use Propositions 2 and 3 to prove that until time \( T = T(r) = o(r^{-2}) \) the processes \( t \mapsto X(t) \), \( t \mapsto \mathcal{X}(t) \), and \( t \mapsto Z(t) \) coincide with high probability.

For this define the (discrete) stopping times

\[
\rho := \min \{ n : X_n(t) \neq Z_n(t), 0 \leq t \leq \theta_n \}
\]

\[
\sigma := \min \{ n : \max \{ \mathbb{1}_{\tilde{w}_n}, \mathbb{1}_{\tilde{w}_n} > 0 \} = 1 \},
\]

and note that by construction

\[
\inf \{ t : Z(t) \neq X(t) \} \geq \Theta_{\min_{\rho, \sigma}} - 1.
\]

**Lemma 9.** Let \( T = T(r) \) such that \( \lim_{r \to \infty} T(r) = \infty \) and \( \lim_{r \to \infty} r^2 T(r) = 0 \). Then

\[
\lim_{r \to 0} \mathbb{P}(\Theta_{\min_{\rho, \sigma}} - 1 < T) = 0.
\]  

**Lemma 10.** Let \( T = T(r) \) such that \( \lim_{r \to \infty} T(r) = \infty \) and \( \lim_{r \to \infty} r^2 T(r) = 0 \). Then for any \( \delta > 0 \)

\[
\lim_{r \to 0} \mathbb{P}\left( \max_{0 \leq t \leq T} |Y(t) - Z(t)| > \delta \sqrt{T} \right) = 0.
\]  

**Proof of Lemma 9.**

\[
\mathbb{P}(\Theta_{\min_{\rho, \sigma}} - 1 < T) \leq \mathbb{P}\left( \rho \leq 2 \mathbb{E}(\theta)^{-1} T \right) + \mathbb{P}\left( \sigma \leq 2 \mathbb{E}(\theta)^{-1} T \right) + \mathbb{P}\left( \sum_{j=1}^{2 \mathbb{E}(\theta)^{-1} T} \theta_j < T \right) + \mathbb{P}\left( \sum_{j=1}^{2 \mathbb{E}(\theta)^{-1} T} \theta_j < T \right) 
\]

\[
\leq Cr^2 T + Cr^2 T + Ce^{-cT},
\]

where \( C < \infty \) and \( c > 0 \). The first term on the right hand side of (74) is bounded by union bound and (48) from Proposition 2. Likewise the second term is bounded by union bound Proposition 3. In bounding the third term we use a large deviation upper bound for the sum of independent \( \theta_j \)-s.

Finally (72) readily follows from (74).
Proof of Lemma 10. Note first that
\[
\max_{0 \leq t \leq T} |Y(t) - Z(t)| \leq \sum_{j=1}^{\nu_T+1} \eta_j \left( \sum_{i=j}^{\nu_j'} \xi_i \right),
\]
with \(\nu_T\) and \(\eta_j\) defined in (9), respectively, and \(\nu_j'\) is \(\nu_j\) from (46) (the label of the leg containing \(j\)). Hence,
\[
P \left( \max_{0 \leq t \leq T} |Y(t) - Z(t)| > \delta \sqrt{T} \right)
\leq P \left( \sum_{j=1}^{2T} \eta_j \left( \sum_{i=j}^{\nu_j'} \xi_i \right) > \delta \sqrt{T} \right) + P \left( \nu_T > 2T \right)
\leq C \delta^{-1} \sqrt{T} r + e^{-cT},
\] (75)
with \(C < \infty\) and \(c > 0\). The first term on the right hand side of (75) is bounded by Markov’s inequality and the bound
\[
E \left( \eta_j \left( \sum_{i=j}^{\nu_j'} \xi_i \right) \right) \leq Cr.
\]
To see this recall the exponential tail bound for \(\gamma\) (45). The bound on the second term follows from a straightforward large deviation estimate on \(\nu_T \sim POI(T)\).

Finally (73) readily follows from (75).

(18) is a direct consequence of Lemmas 9 and 10 and this concludes the proof of Theorem 2.

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