Asymptotics of the heat equation with ‘exotic’ boundary conditions or with time dependent coefficients

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Abstract

The heat trace asymptotics are discussed for operators of Laplace type with Dirichlet, Robin, spectral, D/N, and transmittal boundary conditions. The heat content asymptotics are discussed for operators with time dependent coefficients and Dirichlet or Robin boundary conditions.

1 Introduction

Standard Dirichlet and Neumann boundary conditions appear in numerous physical applications, some of which are nicely described at this Conference. In certain cases physics requires consideration of more ‘exotic’ boundary value
problems. For example, divergences of the Casimir energy in non-static, but reasonably slow varying, external fields are related to the asymptotics of the Schrödinger equation with a time dependent Hamiltonian. After the Wick rotation the latter are defined by the heat trace asymptotics for operators with time dependent coefficients. It is easy to imagine a physical experiment when temperature of a part of the surface of a body is kept constant while the heat flow from the other part to the outside is negligible. Such physical experiments are described by the D/N boundary value problem. Transmittal boundary conditions appear in the case of semi-transparent surfaces or when the geometry of the manifold is not smooth. The most fashionable example (and the closest to the topic of the present Conference) of the non-smooth geometries is given by the brane world scenario \[33\]. Spectral boundary conditions are of relevance in one-loop quantum cosmology and supergravity \[15, 16\]. Furthermore, given their nice transformation properties under chiral rotations and supertranslations there is little doubt that study of spectral boundary conditions is also useful.

Let \( D \) be an operator of Laplace type acting on the space of smooth sections to a vector bundle \( V \) over a compact Riemannian manifold \( M \) of dimension \( m \) with smooth boundary \( \partial M \). Let \( D_B \) be the realization of \( D \) with Dirichlet or Robin boundary conditions; we will consider more ‘exotic’ boundary conditions presently. Let \( e^{-tD_B} \) be the fundamental solution of the heat equation; \( u := e^{-tD_B}\phi \) is determined by the equations:

\[
\begin{align*}
    u(x; 0) &= \phi, \quad B u = 0, \quad \text{and} \quad (\partial_t + D)u = 0.
\end{align*}
\]

Let \( f \) be a smooth localizing or smearing function. We define the smeared heat trace:

\[
a(f; D, B)(t) := \text{Tr}_{L^2}(fe^{-tD_B}).
\]

As \( t \downarrow 0 \) there is an asymptotic series \[25, 26, 34\]

\[
a(f; D, B) \sim \sum_{n \geq 0} t^{(n-m)/2}a_n(f, D, B).
\]

(1)

The asymptotic heat trace coefficients may be decomposed as the sum of an interior and a boundary contribution:

\[
a_n(f, D, B) = a_n^M(f, D) + a_n^{\partial M}(f, D, B).
\]

The invariants \( a_n^M \) and \( a_n^{\partial M} \) are computable as integrals of local geometric invariants.

Let \( \psi \) be an auxiliary section to \( V \) defined over the boundary \( \partial M \) and let the potential \( p \) measure internal heat sources and sinks. Let \( u \) be the temperature distribution defined by the inhomogeneous equations:

\[
\begin{align*}
    u(x; 0) &= \phi, \quad B u = \psi, \quad \text{and} \quad (\partial_t + D)u = p.
\end{align*}
\]

With Dirichlet boundary conditions, we keep the boundary at constant temperature \( \psi \); with Neumann boundary conditions, we pump heat into \( M \) at a
rate defined by $\psi$ to control the heat flow in the normal direction. Let $\rho$ be the specific heat of the manifold. The total heat content

$$\beta(p, \phi, \psi, \rho; D, \mathcal{B})(t) := \int_M u \rho$$

has an asymptotic expansion as $t \downarrow 0$

$$\beta \sim \sum_{n \geq 0} \beta_n(p, \phi, \psi, \rho; D, \mathcal{B}) t^{n/2}.$$

The heat content asymptotics $\beta_n$ can be decomposed as the sum of an interior and a boundary contribution given by locally computable invariants.

The coefficients $a_n$ and $\beta_n$ encode spectral information about the global geometry of the manifold. In Section 2 we discuss the interior invariants $a^M_n$. These invariants vanish if $n$ is odd. In Theorem 2.1, we give formulas \[18\] for the invariants $a_n$ for $n = 0, 2, 4$; formulas for the invariants $a_6 \[18\]$ and $a_8 \[1, 3\]$ are known.

In Section 3, we define the Dirichlet and Robin boundary operator - see equation (2). In Theorem 3.1, we give formulas \[9, 27, 30\] for the associated boundary correction terms $a^n_M$ if $n \leq 4$; formulas for $a_5$ are known \[1\].

In Section 4, we define transmittal boundary conditions - see equation (3). In Theorem 4.1, we give formulas \[8, 23, 29\] for the boundary correction terms $a_n^\Sigma$ if $n \leq 3$; formulas for $a_4$ are known \[23\].

In Section 5, we discuss spectral boundary conditions. In contrast to Dirichlet, Robin and transmittal boundary conditions, spectral boundary conditions are non-local. In Theorem 5.1, we give formulas for the boundary correction terms if $n \leq 3 \[13, 21\]$. Apart from normalizing factors involving powers of $4\pi$, the formulas of Theorems 2.1, 3.1, and 4.1 involve coefficients which are independent of the dimension $m$ of the underlying manifold. In contrast, the formulae of Theorem 5.1 are very dimension dependent. This is one of the notable features of spectral boundary conditions.

In Section 6, we consider a time dependent family $\mathcal{D} = \mathcal{D}_t$ of operators of Laplace type. The heat temperature distribution is defined by:

$$u(x; 0) = \phi, \quad B u = 0, \quad (\partial_t + \mathcal{D}_t) u = 0.$$ 

The map $\phi \rightarrow u$ is described by a smooth kernel function $K$ with the property that:

$$u(x; t) = \int_M K(t, x, y; \mathcal{D}, \mathcal{B}) \phi(y) dy.$$ 

The heat trace asymptotics are then defined not by the heat trace but directly in terms of the kernel function:

$$a(f, \mathcal{D}, \mathcal{B})(t) := \int_M f \text{Tr} K(t, x, x; \mathcal{D}, \mathcal{B})$$

$$\sim \sum_n t^{(n-m)/2} a_n(f, \mathcal{D}, \mathcal{B}).$$
In Theorem 6.1 we give formulas for the interior invariants. We define boundary conditions in equation (4) which are time dependent. In Theorem 6.2, we give formulas for the boundary correction in the heat equation asymptotics.

In Section 7, we give a brief discussion of the D/N problem [4, 12, 14]. Here, in contrast to other boundary conditions, there is not a classical asymptotic expansion at the $a_3$ level.

In Section 8, we discuss the heat content asymptotics. In Theorem 8.1, we give formulae [5, 7, 20, 24] for the invariants $\beta_n$ for $n \leq 4$ for Dirichlet or Robin boundary conditions. The coefficients which appear do not depend on the dimension $m$. In the static setting, partial results are available for $\beta_5$ [5, 6].

## 2 Interior Heat Trace Coefficients

We introduce the following notational conventions to describe the interior heat trace coefficients $a^M_n$. Let Greek indices $\mu, \nu$ range from 1 to $m$ and index a local coordinate frame. Let Latin indices $i, j, k, l$ range from 1 to $m$ and index an orthonormal frame. We adopt the Einstein convention and sum over repeated indices. The operator $D$ determines a connection $\nabla$ and an endomorphism $E$ so that

$$D = -(\text{Tr } \nabla^2 + E).$$

If we express

$$D = -(g^{\mu\nu}\partial_\mu \partial_\nu + a^\mu \partial_\mu + b)$$

relative to a local system of coordinates, then the connection 1 form $\omega$ and the endomorphism $E$ are given by:

$$\omega_3 = \tfrac{1}{2}g_{\mu\delta}(a^\nu + g^{\mu\sigma}\Gamma^{\nu}_{\mu\sigma}I), \quad \text{and}$$

$$E = b - g^{\nu\mu}(\partial_\nu \omega_\mu + \omega_\nu \omega_\mu - \omega_\sigma \Gamma^{\nu}_{\nu\mu} \omega_\sigma).$$

If $D = \delta d$ is the scalar Laplacian, then the connection $\nabla$ is trivial and the endomorphism $E$ vanishes. More generally, if $D = (d\delta + \delta d)_p$ is the Laplacian on $p$ forms, then $\nabla$ is the Levi-Civita connection and $E$ is given by the Weitzenböck formulas [19]. If $D$ is the spin Laplacian, then $\nabla$ is the spin connection and with our sign convention $E = -\frac{1}{4}\tau$ where $\tau$ is the scalar curvature.

Let ‘;' denote multiple covariant differentiation with respect to the connection on $V$ and the Levi-Civita connection of $M$. Let

$$\rho_{ij} := R_{ikkj} \quad \text{and} \quad \tau := \rho_{ii}$$

be the Ricci tensor and the scalar curvature. Let $\Omega$ be the curvature of the connection $\nabla$. If $A$ is a scalar invariant, we let $\text{Tr } (A) := \text{Tr } (AI)$.

The invariants $a^M_n$ vanish if $n$ is odd. If $n$ is even and if $n \leq 4$, then we have

### Theorem 2.1
1. \( \alpha_0^M(f, D) = (4\pi)^{-m/2} \int_M \text{Tr} (f) \).
2. \( \alpha_2^M(f, D) = (4\pi)^{-m/2} \frac{1}{4} \int_M f \text{Tr} (\tau + 6E) \).
3. \( \alpha_4^M(f, D) = (4\pi)^{-m/2} \frac{1}{384} \int_M f \text{Tr} \{ 60E_{kk} + 60\tau E + 180E^2 + 30\Omega_{ij}\Omega_{ij} + 12\tau_{kk} + 5\tau^2 - 2|\rho|^2 + 2|R|^2 \} \).

3 Heat Trace Asymptotics for Robin and Dirichlet Boundary Conditions

Near the boundary, let Roman indices \( a, b \) range from 1 to \( m - 1 \) and index a local orthonormal frame \( \{ e_a \} \) for the tangent bundle of \( \partial M \). We let \( e_m \) be the inward unit normal. Let 
\[
L_{ab} := (\nabla e_a e_b, e_m)
\]
be the second fundamental form. Decompose the boundary of \( M \) as the disjoint union of two closed (possibly empty) sets:
\[
\partial M = C_N \cup C_D.
\]
Let \( u_{;m} \) be the covariant derivative of \( u \) with respect to the inward unit normal using the natural connection defined by \( D \). Let the boundary operator
\[
B u := u|_{C_D} \oplus (u_{;m} + Su)|_{C_N}
\]
define Dirichlet boundary conditions on \( C_D \) and Robin boundary conditions on \( C_N \). Let ‘;’ denote multiple covariant differentiation of tensors defined on \( \partial M \) with respect to the connection on \( V \) and the Levi-Civita connection of the boundary. Note that ‘;’ and ‘:’ differ by the second fundamental form. We have \[9, 27, 30\]:

Theorem 3.1

1. \( \alpha_0^{\partial M}(f, D, B) = 0 \).
2. \( \alpha_2^{\partial M}(f, D, B) = -(4\pi)^{(1-m)/2} \frac{1}{6} \int_{C_D} \text{Tr} (f) + (4\pi)^{(1-m)/2} \frac{1}{4} \int_{C_N} \text{Tr} (f) \).
3. \( \alpha_4^{\partial M}(f, D, B) = (4\pi)^{-m/2} \frac{1}{384} \int_{C_D} \text{Tr} \{ 2fL_{aa} - 3f_{;m} \} \\
+ (4\pi)^{-m/2} \frac{1}{6} \int_{C_N} \text{Tr} \{ 2fL_{aa} + 12fS + 3f_{;m} \} \).
4. \( \alpha_6^{\partial M}(f, D, B) = -(4\pi)^{(1-m)/2} \frac{1}{384} \int_{C_D} \text{Tr} \{ f(96E + 16\tau - 8\rho_{mm} + 7L_{aa}L_{bb} - 10L_{ab}L_{ab} - 30f_{;m}L_{aa} + 24f_{;mm}) + (4\pi)^{(1-m)/2} \frac{1}{384} \int_{C_N} \text{Tr} \{ f(96E + 16\tau - 8\rho_{mm} + 13L_{aa}L_{bb} + 2L_{ab}L_{ab} + 96SL_{aa} + 192S^2) + f_{;m}(6L_{aa} + 96S) \\
+ 24f_{;mm} \} \).
work in [8, 29].

\[ \omega \] is a chiral tensor - if we interchange the roles of + and − with these boundary conditions. Let

\[ a \]

The invariants

\[ D \]

\[ a \]

normal derivatives of \( \phi \) satisfy

\[ \nu \]

Thus \( \phi \) satisfies transmittal boundary conditions if \( \phi \) is continuous and if the normal derivatives of \( \phi^+ \) match to the normal derivatives of \( \phi^- \) modulo the impedance transmission term \( \Xi \). We let \( D_B \) be the realization of \( D = (D^+, D^-) \) with these boundary conditions. Let \( f = (f^+, f^-) \) be smooth on \( M^\pm \) and continuous on \( \Sigma \); we impose no matching condition on the normal derivatives of \( f \). Let

\[ a(f, D, B) = Tr L^2(f e^{-tD_B}) \sim \sum_{n \geq 0} t^{(n-m)/2} a_n(f, D, B). \]

We can decompose the invariants \( a_n \) in the form:

\[ a_n(f, D, B) = a_n^M(f, D) + a_n^M(-f, D) + a_n^\Sigma(f, D, B). \]

The invariants \( a_n^M, a_n^\Sigma \) can be computed using the formulas of Theorem 2.1. Let \( \nu^\pm \) be the inward normals of \( \Sigma \subset M^\pm \); \( \nu = \nu^+ = -\nu^- \). We let

\[ \omega_a := \nabla_a^+ - \nabla_a^- \] and \( L^\pm_a := \pm(\nabla_{c_a} c_b, \nu) \).

The tensor \( \omega_a \) measures the failure of the connections \( \nabla^\pm \) to agree on \( \Sigma \); it is a chiral tensor - if we interchange the roles of + and −, then this tensor changes sign. The tensors \( L^\pm_a \) are the second fundamental forms of the inclusions \( \Sigma \subset M^\pm \). We refer to [23] for the proof of the following theorem; see also related work in [8, 29].

4 Transmittal boundary conditions

Let \( \partial M \) be empty. We suppose given a hypersurface \( \Sigma \) which divides \( M \) into two smooth components \( M^\pm \). We also suppose given operators of Laplace type \( D^\pm \) on \( M^\pm \). Let \( \nu \) be the inward normal of \( \Sigma \subset M^+ \). For \( \phi = (\phi^+, \phi^-) \), we define:

\[ B\phi := \{ \phi^+|_{\Sigma} - \phi^-|_{\Sigma} \} \]

\[ \oplus \{ (\nabla^\nu \phi^+)|_{\Sigma} - (\nabla^\nu \phi^-)|_{\Sigma} - \Xi \phi^+|_{\Sigma} \}. \]

Thus \( \phi \) satisfies transmittal boundary conditions if \( \phi \) is continuous and if the normal derivatives of \( \phi^+ \) match to the normal derivatives of \( \phi^- \) modulo the impedance transmission term \( \Xi \). We let \( D_B \) be the realization of \( D = (D^+, D^-) \) with these boundary conditions. Let \( f = (f^+, f^-) \) be smooth on \( M^\pm \) and continuous on \( \Sigma \); we impose no matching condition on the normal derivatives of \( f \). Let

\[ a(f, D, B) = Tr L^2(f e^{-tD_B}) \sim \sum_{n \geq 0} t^{(n-m)/2} a_n(f, D, B). \]

We can decompose the invariants \( a_n \) in the form:

\[ a_n(f, D, B) = a_n^M(f, D) + a_n^M(-f, D) + a_n^\Sigma(f, D, B). \]

The invariants \( a_n^M, a_n^\Sigma \) can be computed using the formulas of Theorem 2.1. Let \( \nu^\pm \) be the inward normals of \( \Sigma \subset M^\pm \); \( \nu = \nu^+ = -\nu^- \). We let

\[ \omega_a := \nabla_a^+ - \nabla_a^- \] and \( L^\pm_a := \pm(\nabla_{c_a} c_b, \nu) \).

The tensor \( \omega_a \) measures the failure of the connections \( \nabla^\pm \) to agree on \( \Sigma \); it is a chiral tensor - if we interchange the roles of + and −, then this tensor changes sign. The tensors \( L^\pm_a \) are the second fundamental forms of the inclusions \( \Sigma \subset M^\pm \). We refer to [23] for the proof of the following theorem; see also related work in [8, 29].

6
Theorem 4.1

1. \( a_0^\Sigma(f, D, \Xi) = 0 \).
2. \( a_1^\Sigma(f, D, \Xi) = 0 \).
3. \( a_2^\Sigma(f, D, \Xi) = (4\pi)^{-m/2} \frac{1}{6} \int_{\Sigma} \text{Tr} \left\{ 2f(L_{aa}^+ + L_{aa}^-) - 6f\Xi \right\} \).
4. \( a_3^\Sigma(f, D, \Xi) = (4\pi)^{1-m/2} \frac{1}{384} \int_{\Sigma} \text{Tr} \left\{ \frac{3}{2} f(L_{aa}^+ L_{ab}^- + L_{aa}^- L_{ab}^+) + 3f(L_{ab} L_{ab}^+ + L_{ab}^- L_{ab}^-) + 9(L_{aa}^+ + L_{aa}^-)(f_{\mu^+} + f_{\mu^-}) + 48f\Xi^2 - 24f\omega_a\omega_a - 24f(L_{aa}^+ + L_{aa}^-)\Xi - 24(f_{\mu^+} + f_{\mu^-})\Xi \right\} \).

5 Spectral boundary conditions

Let \( P : C^\infty(E_1) \to C^\infty(E_2) \) be an elliptic complex of Dirac type; this means that the associated second order operators \( P^*P \) and \( PP^* \) are of Laplace type. Since such an elliptic complex does not in general admit local boundary conditions \([2]\), we impose spectral boundary conditions. Let \( \gamma \) be the leading symbol of the operator \( P \). Then \( \gamma + \gamma^* \) defines a unitary Clifford module structure on \( E_1 \oplus E_2 \).

Let \( \nabla = \nabla_1 \oplus \nabla_2 \) be a compatible unitary connection \([10]\). This means that:

\[
\nabla(\gamma + \gamma^*) = 0, \quad \text{and} \quad (\nabla s, \tilde{s}) + (s, \nabla \tilde{s}) = d(s, \tilde{s}).
\]

In general this auxiliary connection will not coincide with the connections associated to the Laplacians \( \Delta_1 = P^*P \) and \( \Delta_2 = PP^* \). We expand

\[
P = \gamma^i \nabla_i + \psi
\]

where \( \psi \) is a smooth linear map from \( E_1 \) to \( E_2 \). We parallel translate frames for \( E \) along the normal geodesics rays defined by the inward unit normal. Relative to such a gauge, we have \( \nabla_m = \partial_m \). We set \( x^m = 0 \) to define the tangential operator \( B \) on \( C^\infty(E_1|_{\partial M}) \):

\[
B := (\gamma^m)^{-1}\{\gamma^a \nabla_a + \psi\}.
\]

Let \( B^* \) be the adjoint of \( B \) relative to the structures on the boundary and let \( \Theta \) be an auxiliary self-adjoint operator. We define

\[
A := \frac{B + B^*}{2} + \Theta
\]

Let \( \mathcal{B} \) denote projection on the span of the eigenspaces corresponding to the non-negative spectrum of \( A \). Let \( P_{\mathcal{B}} \) be the associated realization of \( P \) and
let $D_B := (P_B)^*P_B$. Results of Grubb and Seeley [14] show that there is an
asymptotic series as $t \downarrow 0$ of the form:

$$
\text{Tr} L^2 \{ f e^{-tD_B} \}
\sim \sum_{0 \leq k \leq m-1} a_k(F, D, B) t^{(k-m)/2}
$$

modulo terms which are $O(t^{-\frac{m}{2}})$. (There is in fact a complete asymptotic series
with log terms, but we shall only be interested in the first few terms in the
series). We shall assume $m \geq 4$ so the terms $a_n$ for $n \leq 3$ are well defined. Let

$$
\hat{\psi} := \gamma_m^{-1} \psi, \quad \text{and}
C(m) := \Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{1}{2}\right)^{-1}\Gamma\left(\frac{m+1}{2}\right)^{-1}.
$$

We refer to [13, 21] for the proof of the following theorem:

**Theorem 5.1** We have

1. $a_0(f, D, B) = (4\pi)^{-m/2} \int_M \text{Tr} \{ f \}$.

2. $a_1(f, D, B) = (4\pi)^{-(m-1)/2} \left( \frac{2}{m} (C(m) - 1) \int_M \text{Tr} \{ f \} \right)$.

3. $a_2(f, D, B) = a_2^B(f, D) + (4\pi)^{-m/2} \int_M f \text{Tr}\{ \frac{1}{2} [\hat{\psi} + \hat{\psi}^*] + \frac{1}{4}(1 - \frac{4\pi}{m} C(m)) L_{aa} \} - \frac{m-1}{2} (1 - \frac{4\pi}{m} C(m)) \text{Tr} \{ f \} m$.

4. $a_3(f, D, B) = (4\pi)^{-(m-1)/2} \int_M f \text{Tr}\{ \left( \frac{1}{m}(1 - \frac{C(m)}{m}) \right) [\hat{\psi} + \hat{\psi}^*] \} + \frac{1}{16}(5 - 2m + \frac{7m-4m^2}{m-2} C(m)) \hat{\psi}^* \hat{\psi}^* + \frac{1}{32(m-1)} (2m - 3 - \frac{2m^2 - 6m + 4}{m-2} C(m) - 1) \gamma_a^T \gamma_a^T \hat{\psi}^* \hat{\psi}^* + \frac{1}{16(m-1)} (1 + \frac{4m - 2m^2}{m-2} C(m)) \gamma_a^T \gamma_a^T \hat{\psi}^* \hat{\psi}^* - \frac{1}{48} \left( \frac{m-1}{m} C(m) - 1 \right) \tau + \frac{1}{48} \left( 1 - \frac{4m - 10}{m-2} C(m) \right) \rho_{mm} + \frac{1}{48(m+1)} \left( \frac{17m+5m}{4} - \frac{3m^2 - 8m}{m-2} C(m) \right) + \frac{1}{8(m-1)} (- \frac{17m^2}{8} + 4m^2 - 11m^2 + 5m - 1 C(m)) + \frac{1}{8(m-1)} C(m) \Theta \Theta + \frac{1}{m-1} \gamma_a^T \Theta^T \Theta \} + \frac{4m^2 - 11m^2 + 5m - 1 C(m)}{8(m-1)} \text{Tr} \{ I \} + \frac{m-1}{16(m-1)} (2C(m) - 1) f_{mm} \text{Tr} \{ I \}$.

### 6 Time dependent coefficients

Previously, we have considered static operators. Let $\mathcal{D}$ be an operator of Laplace
type where the coefficients are time dependent. We expand

$$
\mathcal{D}u := Du + \sum_{r>0} t^r \{ G_{r,i} u_{ij} + F_{r,i} u_{ij} + E_r u \}
$$

and consider time dependent Dirichlet and Robin boundary conditions:

$$
Bu := u|_{C_D} \oplus (u_m + Su + t(T_a u_a + S_1 u))|_{C_N}.
$$
We consider the heat equation:

\[ u(x; 0) = \phi, \quad Bu = 0, \quad (\partial_t + D)u = 0. \]

There is a smooth kernel function \( K(t, x, y, D, B) \) so that we may express:

\[ u(x; t) = \int_{M} K(t, x, y, D, B)\phi(y). \]

We take the fiber trace to define

\[ a(f, D, B) = \int_{M} f \text{Tr} K(t, x, D, B) \sim \sum_{n=0} \epsilon^{(n-m)/2} a_n(f, D, B). \]

This agrees with the previous definition if \( D \) is static. We refer to [22] for the proof of the following two results which give the additional terms in the asymptotic expansion arising from the time dependent nature of the coefficients:

**Theorem 6.1**

1. \( a_0^M(f, D) = a_0^M(f, D) \).

2. \( a_2^M(f, D) = a_2^M(f, D) + (4\pi)^{-m/2} \frac{1}{\sqrt{3\pi}} \int_M f \text{Tr} (\frac{1}{2}G_{1,ii}) \).

3. \( a_4^M(f, D) = a_4^M(f, D) + (4\pi)^{-m/2} \frac{1}{3\sqrt{3\pi}} \int_M f \text{Tr} (45G_{1,ii}G_{1,ijk} + \frac{45}{2}G_{1,ij}G_{1,ij} + 60G_{2,ii} - 180G_{1,ii}R_{jkkj} - 30G_{1,ij}R_{ikkj} + 90G_{1,ii}E + 60F_{1,ii} + 15G_{1,ii;jj} - 30G_{1,ij;ij}). \)

Let \( B_0 \) denote the associated static boundary conditions. We have:

**Theorem 6.2**

1. \( a_n^{M}(f, D, B_0) = a_n^{M}(f, D, B_0) \) for \( n \leq 2 \).

2. \( a_3^{M}(f, D, B_0) = a_3^{M}(f, D, B_0) + (4\pi)^{(1-m)/2} \frac{1}{3\sqrt{3\pi}} \int_{C_D} f \text{Tr} (-24G_{1,aa}) + (4\pi)^{(1-m)/2} \frac{1}{3\sqrt{3\pi}} \int_{C_N} f \text{Tr} (24G_{1,aa}). \)

3. \( a_4^{M}(f, D, B_0) = a_4^{M}(f, D, B_0) + (4\pi)^{-m/2} \frac{1}{3\sqrt{3\pi}} \int_{C_D} f \text{Tr} \{30G_{1,aa}L_{bb} - 60G_{1,mm}L_{ab} + 30G_{1,ab}L_{ab} + 30G_{1,mm;m} - 30G_{1,aa;m} + 60G_{1,aa;a} - 30F_{1,m} \} + f_{in} \text{Tr} \{-45G_{1,aa} + 45G_{1,mm} \} + (4\pi)^{-m/2} \frac{1}{3\sqrt{3\pi}} \int_{C_N} f \text{Tr} \{30G_{1,aa}L_{bb} + 120G_{1,mm}L_{ab} - 150G_{1,ab}L_{ab} - 60G_{1,mm;m} + 60G_{1,aa;m} + 150F_{1,m} + 180S_{1,aa} + 180S_{1,mm} + 360S_{1} \} + f_{in} \text{Tr} \{45G_{1,aa} - 45G_{1,mm} \}. \)
7 The D/N Problem

In Section 3, we assumed that $C_N \cap C_D$ was empty to define the boundary operator $B$ of equation (2). This meant that the Neumann and Dirichlet components did not interact. In this section, we suppose that $\Sigma := C_D \cap C_N$ is a non-empty smooth submanifold of $\partial M$ of dimension $m - 2$.

We can motivate this more generalized setting with a physical example. Let $M$ be a solid ball which floats in ice water. The part of the boundary of the ball which is in air satisfies Neumann conditions and the part which is in the water satisfies Dirichlet conditions. Thus $B$ is defined by complementary spherical caps about the north and south poles of the ball which intersect in a circle of latitude.

The setting where $\Sigma$ is not empty is known in the literature as the N/D problem. It has been investigated extensively from the functional analytic point of view [28, 31, 32, 35]. It is natural to conjecture the asymptotic expansion described in (1) could be generalized to this setting by adding an extra integral over $\Sigma$ of some suitably chosen local invariant. Preliminary computations [4, 12] suggest the additional correction term for $n = 2$ is given by:

$$a_2^\Sigma = -\pi (4\pi)^{-m/2} \int_\Sigma \text{Tr}(f).$$

However, it has been shown [14] that the asymptotic expansion does not exist with locally computable coefficients at the $a_3$ level. Thus probably either log terms arise or non-local terms arise; it is also possible, of course, that no asymptotic expansion exists.

8 Heat Content Asymptotics

Let $D$ be a time dependent family of operators of Laplace type. Let $\psi(y; t)$ be a smooth section to $V$ defined over $\partial M$. On the Neumann boundary component $C_N$, we use a Neumann heat pump to pump heat into $M$ at a rate defined by $\psi$; in this setting, the parameter $S$ controls the coupling between the heat transfer and the temperature difference on the Neumann component. On the Dirichlet component we use a Dirichlet heat pump to keep the temperature at $\psi$. Let $p$ be a heat source. The temperature distribution $u = u_{p, \phi, \psi}(x; t)$ which is defined by these data is the solution to the equations:

$$(\partial_t + D)u = p, \quad u(x; 0) = \phi, \quad B u = \psi.$$ 

Let $\rho$ be the specific heat; we regard $\rho$ as a section to the dual bundle $V^*$ and let $\langle \cdot, \cdot \rangle$ denote the dual pairing between $V$ and $V^*$. The total heat energy content $\beta$ is defined by $\beta(t) := \int_M u p$. We expand $\beta$ in an asymptotic series as $t \downarrow 0$ to define the associated heat content asymptotics:

$$\beta \sim \sum_n t^{n/2} \beta_n(p, \phi, \psi, \rho; D, B).$$
Let $\tilde{D}$ and $\tilde{B}$ be the dual operator and dual boundary condition on the dual bundle $V^*$. We summarize results of \cite{3, 3, 21, 22}:

**Theorem 8.1**

1. $\beta_0(p, \phi, \psi, \rho; D, B) = \int_M \langle \phi, \rho \rangle$.
2. $\beta_1(p, \phi, \psi, \rho; D, B) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_{C_D} \{ \langle \phi - \psi, \rho \rangle \}$.
3. $\beta_2(p, \phi, \psi, \rho; D, B) = -\int_M \{ \langle D\phi, \rho \rangle - \langle p_0, \rho \rangle \}$
   \[ + \int_{C_D} \{ \langle \frac{4}{3}L_{aa}(\phi - \psi), \rho \rangle - \langle \phi - \psi_, \rho_m \rangle \} \]
   \[ + \int_{C_N} \{ \langle (B\phi - \psi), \rho \rangle \} \].
4. $\beta_3(p, \phi, \psi, \rho; D, B) = -\frac{\sqrt{2}}{\sqrt{\pi}} \int_{C_D} \{ \langle p_0, \rho \rangle - \langle \frac{2}{3}D\phi, \rho \rangle \}
   \[ - \frac{2}{3}\langle \phi - \psi_0, \tilde{D}\rho \rangle + \frac{1}{3}\langle \phi - \psi_0, \rho_a \rangle - \frac{2}{3}\langle \psi_1, \rho \rangle + \langle \frac{1}{3}E + \frac{1}{12}L_{aa}L_{bb}
   \[ - \frac{L_{ab}L_{ac} - \frac{1}{12}R_{abam}L_{ab} + \frac{1}{32}R_{abkm} + \frac{1}{32}F_{ac} \rangle
   \[ + \int_{C_N} \{ \langle (B\phi - \psi), \rho \rangle \}. \]

5. $\beta_4(p, \phi, \psi, \rho; D, B) = \frac{1}{2} \int_M \{ \langle p_1, \rho \rangle - \langle Dp_0, \rho \rangle + \langle D\phi, \tilde{D}\rho \rangle
   \[ - \frac{1}{3} \langle L_{aa}(\phi - \psi), \rho \rangle - \frac{1}{3} \langle \phi - \psi, \rho_m \rangle \}
   \[ + \int_{C_D} \{ \langle \frac{4}{3}L_{aa}(p_0, \rho) - \frac{1}{3}(p_0, \rho_m \rangle
   \[ \frac{1}{3} \langle L_{ab}(\phi - \psi), \rho \rangle + \langle \frac{1}{3}E_m + \frac{1}{12}L_{ab}L_{cc}
   \[ + \frac{1}{3} \langle L_{ab}(\phi - \psi), \rho \rangle - \frac{1}{3} \langle L_{ab}(\phi - \psi), \rho_m \rangle
   \[ + \int_{C_N} \{ \langle (B\phi - \psi), \rho \rangle \}. \]

**9 Remarks**

We have presented explicit combinatorial formulas for both the heat content and the heat trace asymptotics. One of our motivations in computing these invariants was to see if there was a direct combinatorial link between the invariants; there does not seem to be one immediately evident although techniques used in the computation of both the heat content and the heat trace asymptotics share certain common features and in principle there are methods which might permit both to be computed simultaneously. Another example of an asymptotic formula involving geometric data arises from expanding the volume of a tube of radius $r$ around a submanifold $N$ embedded in an ambient manifold, see for example \cite{24}. Again, there does not seem to be any direct combinatorial link between these asymptotic formulae and those we have presented here.
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