A note on Lawson homology for smooth varieties with small Chow groups

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Abstract

Let X be a smooth projective variety of dimension n on which rational and homological equivalence coincide for algebraic p-cycles in the range $0 \leq p \leq s$. We show that the homologically trivial sector of rational Lawson homology $L_p H_k(X, \mathbb{Q})_{hom}$ vanishes for $0 \leq n - p \leq s + 2$. This is an analogue of a theorem of C. Peters in “dual dimensions”. Together with Peters’ theorem we get that the natural transformation $L_p H_k(X, \mathbb{Q}) \to H_k(X, \mathbb{Q})$ is injective for all $p$ and $k$ when $X$ is a smooth projective variety of dimension 4 and $\text{Ch}_0(X) = \mathbb{Z}$.

1 Introduction

In this paper, all projective varieties are defined over $\mathbb{C}$. Let $X$ be a projective variety with dimension $n$. Let $Z_p(X)$ be the space of algebraic $p$-cycles on $X$.

The Lawson homology $L_p H_k(X)$ of $p$-cycles is defined by

$$L_p H_k(X) = \begin{cases} \pi_{k-2p}(Z_p(X)), & k \geq 2p; \\ 0, & k < 2p \end{cases}$$

where $Z_p(X)$ is given a natural topology (cf. [F], [L2]). For a general discussion of Lawson homology, see the survey paper [L2].

In [FM], Friedlander and Mazur showed that there are natural maps, called cycle class maps

$$\Phi_{p,k} : L_p H_k(X) \to H_k(X).$$
Definition 1.1 \( L_pH_k(X)_{\text{hom}} := \ker\{\Phi_{p,k} : L_pH_k(X) \to H_k(X)\}; \) \( L_pH_k(X,\mathbb{Q})_{\text{hom}} := L_pH_*(X)_{\text{hom}} \otimes \mathbb{Q}. \)

C. Peters proved the following result by using the decomposition of the diagram for the smooth varieties with small Chow groups first shown by Bloch and Srinivas [BS] and generalized by Paranjape [Pa], Laterveer [Lat] and others:

Theorem 1.1 (Peters [Pe]) Let \( X \) be a smooth projective variety for which rational and homological equivalence coincide for \( p \)-cycles in the range \( 0 \leq p \leq s \) (that is, in the terminology of [Lat], \( X \) has small chow groups up to rank \( s \)). Then \( L_pH_*(X)_{\text{hom}} \otimes \mathbb{Q} = 0 \) in the range \( 0 \leq p \leq s + 1 \).

By carefully checking the proof of Peters, we discover the symmetry of the decomposition of the diagonal \( \Delta_X \subset X \times X \) and note that the proof works for \( p \)-cycles with \( 0 \leq n - p \leq s + 2 \).

In this note, we will use the tools of Lawson homology and the methods and notations given in [Pe] (and the references therein) to show the following main result:

Theorem 1.2 Let \( X \) be a smooth projective variety of dimension \( n \) for which rational and homological equivalence coincide for \( p \)-cycles in the range \( 0 \leq p \leq s \). Then \( L_pH_*(X)_{\text{hom}} \otimes \mathbb{Q} = 0 \) in the range \( 0 \leq n - p \leq s + 2 \).

For convenience, we introduce the following definition:

Definition 1.2 A smooth projective variety \( X \) over \( \mathbb{C} \) is called rationally connected if there is a rational curve through any 2 points of \( X \). A necessary condition for \( Z \) to be rationally connected is that \( \text{Ch}_0(X) \cong \mathbb{Z} \).

For equivalent descriptions of this definition, see the paper of Kollár, Miyaoka and Mori [KMM].

Corollary 1.1 Let \( X \) be a smooth projective variety with \( \text{dim}(X) = 4 \) and \( \text{Ch}_0(X) \cong \mathbb{Z} \). Then \( L_pH_k(X)_{\text{hom}} \otimes \mathbb{Q} = 0 \) for all \( p \) and \( k \). In particular, all the smooth hypersurfaces in \( \mathbb{P}^5 \) with degree less or equal than 5 have this property (cf. [Ro]).

Remark 1.1 It is shown by the author in [H] that for any smooth projective rational variety \( X \) of \( \text{dim}(X) = 4 \), \( L_pH_k(X)_{\text{hom}} = 0 \) for any \( p \) and \( k \). Hence the nontriviality of \( L_pH_k(X)_{\text{hom}} \) for some \( p,k \) for a rationally connected fourfold \( X \) would imply irrationality of \( X \).

Corollary 1.2 Let \( X \) be a general cubic hypersurface of dimension less than or equal to 6, then \( L_*(H_*(X)_{\text{hom}} \otimes \mathbb{Q} = 0 \).
Remark 1.2 Laterveer [La1] showed that Griffiths groups are torsion for a general cubic hypersurface of dimension less than or equal to 6. For the general cubic sevenfold in $\mathbb{P}^8$, Albano and Collino showed that $\text{Griff}_3(X)$ (which is $\cong L_3H_6(X)_{\text{hom}}$ by Friedlander in [F]) is nontrivial even after tensoring with $\mathbb{Q}$.

Remark 1.3 This work was done in Spring of 2005 as part of my Ph. D. thesis. It was included in my research statement and put on my web page http://www.math.sunysb.edu/~wenchuan/job/rs.pdf in November 2005. I recently learned that M. Voineagu has independently obtained this result (cf. [Vn]).

2 The Proof of the main result

The proof of the main theorem is based on: the Lemma 12 in [Pe], the decomposition of the diagonal given in [Pa], and the computation of Lawson homology of codimension 1 cycles for a smooth projective variety given by Friedlander [F].

For convenience, we write the results we need as follows:

Theorem 2.1 (Friedlander [F]) Let $X$ be any smooth projective variety of dimension $n$. Then we have the following isomorphisms

$$
\begin{align*}
L_{n-1}H_{2n}(X) & \cong \mathbb{Z}, \\
L_{n-1}H_{2n-1}(X) & \cong H_{2n-1}(X, \mathbb{Z}), \\
L_{n-1}H_{2n-2}(X) & \cong H_{n-1,n-1}(X, \mathbb{Z}) = \text{NS}(X) \\
L_{n-1}H_k(X) & = 0 \text{ for } k > 2n.
\end{align*}
$$

Remark 2.1 From this theorem we have $L_{n-1}H_*(X)_{\text{hom}} = 0$ for any smooth projective variety $X$ with $\text{dim}(X) = n$.

Now we need to review some definitions about the action of correspondences. Let $X$ and $Y$ be smooth projective varieties with $\text{dim}(X) = n$. For $\alpha \in \mathcal{Z}_{n+d}(X \times Y)$, one puts

$$
\alpha_*(u) = (p_2)_*[p_1^*(u) \cdot \alpha], \quad u \in \mathcal{Z}_p(X)
$$

where $(p_2)_*$ is the proper push-forward, $p_1^*$ is the flat pull back and the "\cdot" denotes the intersection product of cycles [Pe, definition 10]. In this way, $\alpha_*$ gives a correspondence homomorphism

$$
\alpha_* : \mathcal{Z}_p(X) \to \mathcal{Z}_{p+d}(Y).
$$

This $\alpha_*$ induces a map (also denoted by $\alpha_*$) on Lawson homology groups

$$
\alpha_* : L_pH_k(X) \to L_{p+d}H_{k+2d}(Y)
$$
which depends only on the class of $\alpha$ in the Chow group of $X \times Y$ modulo algebraic equivalence. For the details of the argument here, see [Pe, section 1 C.]

The key Lemma we need was given by Peters as follows:

**Proposition 2.1** ([Pe], Lemma 12) Assume that $X$ and $Y$ are smooth projective varieties and let $\alpha \subset X \times Y$ be an irreducible cycle of dimension $\dim(X) = n$, supported on $V \times W$, where, $V \subset X$ is a subvariety of dimension $v$ and $W \subset Y$ a subvariety of dimension $w$. Let $\tilde{V}$, resp. $\tilde{W}$ be a resolution of singularities of $V$, resp. $W$ and let $i : \tilde{V} \to X$ and $j : \tilde{W} \to Y$ be the corresponding morphisms. With $\tilde{\alpha} \subset \tilde{V} \times \tilde{W}$ the proper transform of $\alpha$ and $p_1$, resp. $p_2$ the projections from $X \times Y$ to the first, resp. the second factor, there is a commutative diagram

$$
\begin{array}{ccc}
L_{p-n+v+w}H_{k+2(v+w-n)}(\tilde{V} \times \tilde{W}) & \xrightarrow{\tilde{\alpha}_*} & L_pH_k(\tilde{V} \times \tilde{W}) \\
\uparrow p_1^* & & \downarrow (p_2)_* \\
L_{p-n+v}H_{k+2(v-n)}(\tilde{V}) & \xrightarrow{i_*} & L_pH_k(W) \\
\uparrow i^* & & \downarrow j_* \\
L_pH_k(X) & \xrightarrow{\alpha_*} & L_pH_k(Y).
\end{array}
$$

Here $i^*$ is induced by the Gysin homomorphism, $p_1^*$ is the flat pull-back, and $(p_2)_*$ and $j_*$ come from proper push forward. In particular, $\alpha_* = 0$ if $p < n - v$ or if $p > w$. Moreover, $\alpha_{n-v}$ acts trivially on $L_{n-v}H_*(X)_{\text{hom}}$, while $\alpha_w$ acts trivially on $L_wH_*(X)_{\text{hom}}$.

There is a corollary of this proposition given by Peters:

**Corollary 2.1** ([Pe], Corollary 13) An irreducible cycle $\alpha \subset X \times X$ supported on a product variety $V \times W$ with $\dim V + \dim W = n = \dim(X)$ acts trivially on $L_0H_*(X)_{\text{hom}}$.

Combining Friedlander’s result (Theorem 2.1) and Peters’ Lemma (Proposition 2.1), we have the following:

**Corollary 2.2** Under the assumptions of Proposition 2.1, we have that $\alpha_{w-1}$ acts trivially on $L_{w-1}H_*(X)_{\text{hom}}$.

Now we want to recall some results about the decomposition of the diagonal given in [BS] and generalized by Paranjape [Pa] and Laterveer [Lat] with more general triviality hypotheses on the Chow group as stated in Theorem 1.1. Since the decomposition of diagonal is symmetric, we have the following version of the diagonal (cf. [VS], Theorem 10.29):

**Theorem 2.2** Let $X$ be a smooth projective variety. Assume that for $p \leq s$, the maps

$$
cl : CH_p(X) \otimes \mathbb{Q} \to H^{2n-2p}(X, \mathbb{Q})
$$

are injective. Then there exists a decomposition

$$
\Delta_X = \alpha^{(0)} + \cdots + \alpha^{(s)} + \beta \in CH^n(X \times X) \otimes \mathbb{Q},
$$

where $\Delta_X$ is the diagonal of $X$ and $\alpha^{(i)}$ are cycles of dimension $n-i$ in $X$.
where $\alpha^{(p)}$ is supported in $V_p \times W_{n-p}$, $p = 0, \ldots, s$ with $\dim V_p = p$ and $\dim W_{n-p} = n-p$, and $\beta$ is supported in $X \times W_{n-s-1}$.

Using the above theorem and Corollary 2.1, we deduce that the identity acts as $\beta$ on the homologically zero part of the Lawson homology $L_*H_*(X)_{\text{hom}}$. Applying Proposition 2.1 and Corollary 2.2, we have the following main result:

**Theorem 2.3** Let $X$ be a smooth projective variety such that the maps

$$cl : CH_p(X) \otimes \mathbb{Q} \to H^{2n-2p}(X, \mathbb{Q})$$

are injective for $p \leq s$. Then $L_{n-p}H_*(X)_{\text{hom}} \otimes \mathbb{Q} = 0$ for $p = 0, \ldots, s+1, s+2$.

As the application, we get Corollary 1.1 immediately.

Recall a result in [Pa] and [S], i.e., the general cubic hypersurface $X$ of dimension greater than or equal to 5 has $\text{Ch}_1(X) \cong \mathbb{Z}$ (Certainly $\text{Ch}_0(X) \cong \mathbb{Z}$ by Roitman [Ro]). Hence we have the following

**Corollary 2.3** Let $X$ be a general cubic hypersurface of dimension less than or equal 6, then $L_*H_*(X)_{\text{hom}} = 0$. \hfill $\blacksquare$

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