ALGEBRAIC PROPERTIES OF QUASI-FINITE COMPLEXES

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Abstract. A countable CW complex $K$ is quasi-finite (as defined by A. Karasev [21]) if for every finite subcomplex $M$ of $K$ there is a finite subcomplex $e(M)$ such that any map $f : A \to M$, where $A$ is closed in a separable metric space $X$ satisfying $X \tau K$, has an extension $g : X \to e(M)$. Levin’s [26] results imply that none of the Eilenberg-MacLane spaces $K(G, 2)$ is quasi-finite if $G \neq 0$. In this paper we discuss quasi-finiteness of all Eilenberg-MacLane spaces. More generally, we deal with CW complexes with finitely many nonzero Postnikov invariants.

Here are the main results of the paper:

Theorem 0.1. Suppose $K$ is a countable CW complex with finitely many nonzero Postnikov invariants. If $\pi_1(K)$ is a locally finite group and $K$ is quasi-finite, then $K$ is acyclic.

Theorem 0.2. Suppose $K$ is a countable non-contractible CW complex with finitely many nonzero Postnikov invariants. If $\pi_1(K)$ is nilpotent and $K$ is quasi-finite, then $K$ is extensionally equivalent to $S^1$.

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1. Introduction

The notation $K \in AE(X)$ or $X \tau K$ means that any map $f : A \to K$, $A$ closed in $X$, extends over $X$.

**Theorem 1.1** (Chigogidze). For each countable simplicial complex $P$ the following conditions are equivalent:

1. $P \in AE(X)$ implies $P \in AE(\beta(X))$ for any normal space $X$.
2. There exists a $P$-invertible map $p : X \to I^\omega$ of a metrizable compactum $X$ with $P \in AE(X)$ onto the Hilbert cube.

Karasev [21] gave an intrinsic characterization of countable complexes $P$ satisfying 1.1 and called them **quasi-finite complexes**.

**Definition 1.2.** A CW complex $K$ is called **quasi-finite** if there is a function $e$ from the family of all finite subcomplexes of $K$ to itself satisfying the following property: For every separable metric space $X$ such that $K$ is an absolute extensor of $X$ and for every map $f : A \to M$, $A$ closed in $X$, $f$ extends to $g : X \to e(M)$.

For subsequent generalizations of quasi-finiteness see [22] and [2]. In particular, it is shown in [2] that a countable CW complex $K$ is quasi-finite if and only if $X \tau K$ implies $\beta(X) \tau K$ for all separable metric spaces $X$. That is an improvement of 1.1.

The first example of a non-quasi-finite CW complex was given by Dranishnikov [9] who showed that $K(Z,4)$ admits a separable metric space $X$ satisfying $X \tau K(Z,4)$ but not $\beta(X) \tau K(Z,4)$ (see [10] for other examples of such $X$). In [11] it was shown that all $K(G,n)$, $n \geq 3$ and $G \neq 0$, admit a separable metric space $X$ so that $\dim_G(X) = n$ but $\dim_G(\beta(X)) > n$ (see also [23] for related results). Finally, Levin [26] established a result implying the same fact for all $K(G,n)$ so that $G \neq 0$ and $n \geq 2$. The only remaining case among Eilenberg-Maclane spaces are complexes $K(G,1)$.

**Problem 1.3.** Characterize groups $G$ such that $K(G,1)$ is quasi-finite. What are the properties of the class of groups $G$ such that $K(G,1)$ is quasi-finite?

Problem 1.3 was the main motivation of this paper. More generally, we discuss quasi-finiteness of complexes with finitely many non-trivial Postnikov invariants.
2. Truncated cohomology

One of the main tools of this paper is truncated cohomology used for the first time by Dydak and Walsh [17] in their construction of an infinite-dimensional compactum $X$ of integral dimension 2.

Given a pointed CW complex $L$ and a pointed space $X$ we define $h_k^L(X)$ as the $(-k)$-th homotopy group of the function space $\text{Map}_*(X, L)$, the space of base-point preserving maps whose base-point is the constant map. Since we are interested in Abelian groups, $k$ ranges from minus infinity to $-2$. Also, spaces $X$ of interest in this paper are countable CW complexes.

CW complexes $L$ for which truncated cohomology $h^*_* L$ is of most use are those with finite homotopy groups. In that case $h^*_* L$ is continuous in the sense that any map $f : K \to \Omega^k L$ that is phantom (that means all restrictions $f|M$ are homotopically trivial for finite subcomplexes $M$ of $K$) must be homotopically trivial if $K$ is a countable CW complex. In case of $L$ having finite homotopy groups, Levin [25] (see Proposition 2.1) proved that $h^*_* L$ is strongly continuous: any map $f : N \to \Omega^k L$, $N$ being a subcomplex of $K$, that cannot be extended over a countable CW complex $K$, admits a finite subcomplex $M$ of $K$ such that $f|_{M \cap N}$ cannot be extended over $M$.

Since we are interested in vanishing of truncated cohomology $h^*_* L$, the remainder of this section is devoted to weak contractibility of mapping spaces.

We first recall a result that in the literature is known as the Zabrodsky Lemma (see Miller [29], Proposition 9.5, and Bousfield [1], Theorem 4.6 as well as Corollary 4.8).

**Lemma 2.1.** Let $F \to E \to B$ be a fibration where $B$ has the homotopy type of a connected CW complex. Let $X$ be a space. If $\text{Map}_*(F, X)$ is weakly contractible, the induced map $\text{Map}_*(B, X) \to \text{Map}_*(E, X)$ is a weak homotopy equivalence. ■

**Definition 2.2.** Let $\mathcal{P}$ be a set of primes. By a $\mathcal{P}$-complex we mean a finite CW complex $K$ that is simply connected and all its homotopy groups are $\mathcal{P}$-groups. That is, homotopy groups of $K$ are finite and the order of each element is a product of primes belonging to $\mathcal{P}$.

A CW complex $K$ is a co-$\mathcal{P}$-complex if for some $k$ the mapping space $\text{Map}_* (\Sigma^k K, L)$ is weakly contractible for all $\mathcal{P}$-complexes $L$.

**Lemma 2.3.** If $K$ is one of the following

1. The classifying space $BG$ of a Lie group $G$ with a finite number of path components,
2. A connected infinite loop space whose fundamental group is a torsion group,
3. A simply connected CW complex with finitely many homotopy groups,
then $\text{Map}_*(K, L)$ is weakly contractible for all nilpotent finite complexes $L$ with finite homotopy groups.

**Proof.** Let $L$ be a finite nilpotent complex with finite homotopy groups. The hypotheses render $L$ complete with respect to Sullivan’s finite completion (see [31]). Thus case (1) follows from Friedlander and Mislin [15], Theorem 3.1, while case (2) follows from McGibbon [28], Theorem 3. Case (3) follows from [24] and (2) by induction over the number of non-trivial homotopy groups of $K$. See more details in the proof of $\mathcal{P}$- [25]. ■
Proposition 2.4. A finite product (or a finite wedge) of co-$\mathcal{P}$-complexes is a co-$\mathcal{P}$-complex.

Proof. In case of a finite wedge the proof is quite simple as $\text{Map}_s(K \vee P, L)$ is the product of $\text{Map}_s(K, L)$ and $\text{Map}_s(P, L)$. For the finite product one can use induction plus an observation that 2.1 can be applied to a fibration $F \to K \to P$ and yield that $K$ is a co-$\mathcal{P}$-complex if both $F$ and $P$ are co-$\mathcal{P}$-complexes.

Proposition 2.5. Let $\mathcal{P}$ be a set of primes. Suppose $K_s$, $s \in S$, is a a family of CW complexes. If there is a natural number $k$ so that all function spaces $\text{Map}_s(\Sigma^k K_s, L)$ are weakly contractible for all $\mathcal{P}$-complexes $L$, then the wedge $K = \bigvee_{s \in S} K_s$ is a co-$\mathcal{P}$-complex. Moreover, if $S$ is countable and each $K_s$ is countable, then the weak product $\prod_{s \in S} K_s$ is a co-$\mathcal{P}$-complex.

Proof. The case of the wedge is left to the reader. If $S$ is countable, then each finite product $K_T = \prod_{s \in T} K_s$ has the property that $\text{Map}_s(K_T, \Omega^k L)$ is weakly contractible for any $\mathcal{P}$-complex $L$ as in the proof of 2.4. Using the fact that truncated cohomology with respect to $\Omega^k L$ is continuous, one gets that $K' = \prod_{s \in S} K_s$, being the direct limit of $K_T$, also has the property that $\text{Map}_s(K', \Omega^k L)$ is weakly contractible.

Definition 2.6. Let $\mathcal{P}$ be a set of primes and let $G$ be a group. $G$ is called a co-$\mathcal{P}$-group if $K(G, 1)$ is a co-$\mathcal{P}$-complex.

By Miller’s theorem, all locally finite groups are co-$\mathcal{P}$-groups, where $\mathcal{P}$ is the set of all primes. Another example would consist of all acyclic groups. Divisible groups would serve as well. Note that by the Zabrodsky Lemma 2.1, a group extension $N \to G \to Q$ implies that under the assumption that $N$ is a co-$\mathcal{P}$-group, $G$ is a co-$\mathcal{P}$-group if and only if $Q$ is.

Definition 2.7. Let $K$ be a connected CW complex. We say that $K$ has finitely many unstable Postnikov invariants if for some $k \geq 0$, the $k$-connected cover $K<k>$ of $K$ is an infinite loop space. As usual, $K<k>$ is the (homotopy) fibre of the $k$-th Postnikov approximation $K \to P_k(K)$.

Note that infinite loop spaces (in particular infinite symmetric products) and Postnikov pieces are special cases.

Lemma 2.8. Suppose $\mathcal{P}$ is a set of primes. Let $K$ be a connected CW complex with finitely many unstable Postnikov invariants. $K$ is a co-$\mathcal{P}$-complex if and only if $G = \pi_1(K)$ is a co-$\mathcal{P}$-group.

Proof. Let $L$ be a $\mathcal{P}$-complex. Let $\tilde{K}$ be the universal cover of $K$. If $K$ is itself an infinite loop space, so is $\tilde{K}$, and therefore the space $\text{Map}_s(\tilde{K}, L)$ is weakly contractible by Theorem 3 of McGibbon 28. Otherwise for some $i \geq 1$ the $i$-connected cover $\tilde{K}<i>$ of $\tilde{K}$ is an infinite loop space. Consider the fibration sequence $\tilde{K}<i> \to \tilde{K} \to P_i\tilde{K}$ where $P_i\tilde{K}$ is the $i$-th Postnikov approximation of $\tilde{K}$. The space $\text{Map}_s(\tilde{K}<i>, L)$ is weakly contractible by Theorem 3 of McGibbon 28. It follows essentially from Zabrodsky 32, Theorem D, and the fact that $L$ is Sullivan-complete, that the mapping space $\text{Map}_s(P_i\tilde{K}, L)$ is weakly contractible (see also McGibbon 28, Theorem 2). Thus by Lemma 241 also the space $\text{Map}_s(\tilde{K}, L)$ is
weakly contractible. The space $\bar{K}$ sits in the fibration sequence $\bar{K} \to K \to K(G, 1)$ and another application of Lemma 2.4 renders the spaces $\text{Map}_*(K(G, 1), L)$ and $\text{Map}_*(K, L)$ weakly equivalent.

**Lemma 2.9.** Let $\mathcal{P}$ be a nonempty set of primes. If $G$ is a nilpotent group that is local away from $\mathcal{P}$, then it is a co-$\mathcal{P}$-group.

**Proof.** Let $\mathcal{P}'$ denote the set of primes not in $\mathcal{P}$. The hypotheses on $G$ render $K(G, 1)$ a $\mathcal{P}'$-local space. By the fundamental theorem of localization of nilpotent spaces it follows that the homology of $K(G, 1)$ is also $\mathcal{P}'$-local. Let $\cdots \to L_3 \to L_2 \to L_1 \to L_0$ denote the refined Postnikov tower for $L$. That is, $L_0$ is a point and for each $i$, the fibration $L_i \to L_{i-1}$ is principal with fibre $K(G_i, k_i)$ where $G_i$ is $p$-torsion abelian. Note that $L$ is weakly equivalent to the inverse limit $\lim_{\leftarrow} L_i$, and since $K(G, 1)$ is a CW complex it suffices to show that $\text{Map}_* (K(G, 1), \lim_{\leftarrow} L_i)$ is weakly contractible. This latter space is homeomorphic with the inverse limit $\lim_{\leftarrow} \text{Map}_*(K(G, 1), L_i)$. Since the fibrations are principal, the Puppe sequence shows that we only need to consider reduced cohomology $\tilde{H}^*(K(G, 1); G_i)$ with coefficients in $G_i$. Since $H_*(G)$ is local away from $\mathcal{P}$ it follows by the universal coefficient theorem that $\tilde{H}^*(K(G, 1); G_i)$ is trivial.

**Corollary 2.10.** Suppose $\mathcal{P}$ is a set of primes and $G$ is a nilpotent group with Abelianization $\text{Ab}(G)$. If $\text{Ab}(G)/\text{Tor}(\text{Ab}(G))$ is $\mathcal{P}$-divisible, then $G$ is a co-$\mathcal{P}$-group.

**Proof.** By Lemma 2.4 $\text{Ab}(G)/\text{Tor}(\text{Ab}(G))$ is $\mathcal{P}$-divisible if and only if $G$ is local away from $\mathcal{P}$.

3. Homology and cohomology of quasi-finite CW complexes

In this section we deal with (co)homological properties of quasi-finite complexes. First, we need a generalization of Theorem II of [13].

**Theorem 3.1.** Suppose $K$ is a countable CW complex and $h_\ast$ is a generalized reduced homology theory such that $h_\ast(K) = 0$. For any CW complex $P$ and any $\alpha \in h_\ast(P) \setminus \{0\}$ there is a compactum $X$ and a map $f : A \to P$ from a closed subset $A$ of $X$ such that $X \tau K, \alpha = f_\ast(\gamma)$ for some $\gamma \in h_\ast(A)$ and $\gamma = 0$ in $h_\ast(X)$.

**Proof.** Replacing $P$ by the carrier of $\alpha$ we may assume $P$ is finite. Compactum $X$ is built as in Theorem II of [13]. We start with $X_1 = \text{Cone}(P)$, $A_1 = P$ and build an inverse sequence $(X_n, A_n)$ of compact polyhedra so that for every extension problem $g : B \to K$, $B$ closed in $X_n$, there is $m > n$ and a map $G : X_m \to K$ extending $g \circ p_m^n : B' \to K$, where $p_m^n : X_m \to X_n$ is the bonding map and $B' = (p_m^n)^{-1}(B)$. For each $n$ we have $\gamma_n \in h_\ast(A_n)$ which vanishes in $h_\ast(X_n)$. In the inductive step we pick an extension problem $g : B \to K$, $B$ closed in $X_n$, create an extension $G : X_n \to \text{Cone}(K)$, and consider the pull-back $E$ of the projection $K \times I \to \text{Cone}(K)$ under $G$. The projection $p : E \to X_n$ has fibers being either homeomorphic to $K$ or single points. Therefore $h_\ast(p)$ is an isomorphism and one can pick a finite subpolyhedron $A_{n+1}$ of $E$ carrying $\gamma_{n+1} \in h_\ast(A_{n+1})$ which gets mapped to $\gamma_n$ under $h_\ast(p)$. Since $\gamma_{n+1}$ vanishes in $h_\ast(E)$, it vanishes in a finite subpolyhedron $X_{n+1}$ of $E$ containing $A_{n+1}$. Since there are only countably many extension problems to be solved (see [5] or [10]) that process produces an inverse sequence whose inverse limit $(X, A)$ satisfies $X \tau K$ and one has $\gamma \in h_\ast(A)$ so that $\gamma$ vanishes in $h_\ast(X)$ and $f_\ast(\gamma) = \alpha$, where $f : A \to P = A_1$ is the projection. ■
Theorem 3.2. Suppose $K$ is a countable CW complex and $h^\ast$ is a strongly continuous truncated cohomology theory such that $h^\ast(K) = 0$. For any countable CW complex $P$ and any $\alpha \in h^\ast(P) \setminus \{0\}$ there is a compactum $X$ and a map $f : A \to P$ from a closed subset $A$ of $X$ such that $X\tau K$ and there is no $\gamma \in h^\ast(X)$ satisfying $\gamma|_A = f^\ast(\alpha)$.

Proof. We can reduce the proof to the case of $P$ being a finite polyhedron as there is a finite subcomplex $M$ of $P$ so that $\alpha|_M \neq 0$ and that $M$ can be used instead of $P$. Compactum $X$ is built as in [51]. We start with $X_1 = Cone(P)$, $A_1 = P$ and built an inverse sequence $(X_n, A_n)$ of compact polyhedra so that for every extension problem $g : B \to K$, $B$ closed in $X_n$, there is $m > n$ and a map $G : X_m \to K$ extending $g \circ p^n_m : B' \to K$, where $p^n_m : X_m \to X_n$ is the bonding map and $B' = (p^n_m)^{-1}(B)$. Also, for each $n$ the pullback $\alpha_n$ of $\alpha$ under $A_n \to A_1$ does not extend over $X_n$. In the inductive step we pick an extension problem $g : B \to K$, $B$ closed in $X_n$, create an extension $G : X_n \to Cone(K)$, and consider the pull-back of the projection $K \times I \to Cone(K)$ under $G$. The projection $p : E \to X_n$ has fibers being either homeomorphic to $K$ or single points. Therefore $p^\ast = h^\ast(p)$ is an isomorphism. Since $p^\ast(\alpha_n)$ does not extend over $E$, there is a finite subpolyhedron $X_{n+1}$ of $E$ such that $p^\ast(\alpha_n)$ restricted to $A_{n+1} = X_{n+1} \cap p^{-1}(A_n)$ does not extend over $X_{n+1}$. Since there are only countably many extension problems to be solved (see [8] or [10]) that process produces an inverse sequence whose inverse limit $(X, A)$ satisfies $X\tau K$ and the projection $f : A \to P = A_1$ has the property that there is no $\gamma \in h^\ast(X)$ satisfying $\gamma|_A = f^\ast(\alpha)$. \hfill \Box

Recall that, given a map $i : M \to N$, $X\tau i$ means that for any map $f : A \to M$, $A$ closed in $X$, there is a map $g : X \to N$ extending $i \circ f$.

Theorem 3.3. Suppose $K$ is a countable CW complex and $i : M \to N$ is a map of CW complexes such that $X\tau K$ implies $X\tau i$ for all compacta $X$.

1. If $h_\ast$ is a generalized reduced homology theory such that the inclusion induced homomorphism $h_\ast(M) \to h_\ast(N)$ is not trivial, then $h_\ast(K) \neq 0$.
2. If $h^\ast$ is a truncated strongly continuous cohomology theory such that the inclusion induced homomorphism $h^\ast(N) \to h^\ast(M)$ is not trivial and $M$ is countable, then $h^\ast(K) \neq 0$.

Proof. We may assume $i$ is an inclusion.

1. Suppose $\alpha \in h_\ast(M)$ does not become $0$ in $h_\ast(N)$. As in [51] pick a map $f : A \to M$ of a closed subset of a compactum $X$ so that $X\tau K$ and $\gamma$ equals $0$ in $h_\ast(c(M))$ for some $\gamma \in h_\ast(A)$ satisfying $f_\ast(\gamma) = \alpha$. If $f$ extends to $g : X \to N$, then $\alpha = f_\ast(\gamma)$ becomes $0$ in $h_\ast(N)$, a contradiction.

2. Suppose $\alpha \in h^\ast(N)$ and $\alpha|_M \neq 0$. We may reduce this case to $M$ finite by switching to a finite subcomplex $L$ of $M$ with the property $\alpha|_L \neq 0$. As in [52] pick a map $f : A \to M$ of a closed subset of a compactum $X$ so that $X\tau K$ and $f^\ast(\alpha|_M)$ does not extend over $X$. If $f : A \to M$ extends to $g : X \to N$, then $g^\ast(\alpha) \in h^\ast(X))$ extends $f^\ast(\alpha|_M)$, a contradiction. \hfill \Box

Theorem 3.4. Suppose $\mathcal{P}$ is a set of primes. Let $K$ be a connected countable co-$\mathcal{P}$-complex. If $K$ is quasi-finite, then it is $Z_{(\mathcal{P})}$-acyclic.

Proof. Assume $K$ is quasi-finite and not $Z_{(\mathcal{P})}$-acyclic. Replace $K$ with $\Sigma K$ (using [2]) if necessary to ensure $H_k(K; Z_{(\mathcal{P})}) \neq 0$ for some $k \geq 2$. Let $\alpha_K \in H_k(K; Z_{(\mathcal{P})})$ be nonzero. Since $K$ is the colimit of its finite subcomplexes, $\alpha_K$ is
the image of $\alpha_M \in H_k(M; \mathbb{Z}(p))$ for some finite subcomplex $M$ of $K$. Certainly the image of $\alpha_M$ under $H_k(M; \mathbb{Z}(p)) \to H_k(e(M); \mathbb{Z}(p))$ is nontrivial. Thus Lemma 4.2 yields a $\mathcal{P}$-complex $L$ with the restriction morphism $[e(M), \Omega^2 L] \to [M, \Omega^2 L]$ nontrivial. This is to say that $h^*(e(M)) \to h^*(M)$ is nontrivial where $h^*$ is the truncated cohomology theory defined by virtue of $\Omega^2 L$. The hypotheses on $L$ ensure strong continuity of $h^*$. Thus the nontriviality of $h^*(e(M)) \to h^*(M)$ contradicts (2) of Theorem 5.3.

Corollary 3.5. Let $K$ be a countable CW complex with finitely many nonzero homotopy groups and $G = \pi_1(K)$ nilpotent. Suppose that $G$ is not torsion. If $K$ is quasi-finite, the group $FG = G/\text{Tor}(G)$ (and thus also $\text{Ab}(G)/\text{Tor}(\text{Ab}(G))$) is not divisible by any prime $p$.

Proof. Suppose that, on the contrary, $FG$ is divisible by a prime $p$, hence local away from $p$. Since $G$ is not torsion and is nilpotent, also $\text{Ab}(G)$ is not torsion, hence certainly $H_1(K) \otimes \mathbb{Z}(p)$ is nontrivial. Thus Theorem 5.3 yields a contradiction.

Corollary 3.6. Let $K$ be a simply connected countable CW complex with at least one and at most finitely many nontrivial homotopy groups. Then $K$ is not quasi-finite.

Corollary 3.7. Suppose $G$ is a locally finite countable group. If $K(G, 1)$ is quasi-finite, then $G$ is acyclic.

However, there are some countable acyclic groups $G$ for which $K(G, 1)$ are also not quasi-finite. Cencelj and Repovš [13], using results of Dranishnikov and Repovš [13] showed in §5 that the minimal grope $M^*$ which is $K(\pi_1(M^*), 1)$ is not quasi-finite. This holds also for the fundamental group of any grope: For a grope $M$ let $\gamma(m)$ denote the maximal number of handles on the discs with handles used in the construction of the $m$-stage of $M$. Modify the inverse limit construction of the example of [8] replacing every simplex in the triangulation of the $k$-th element of the inverse system by the $n$-th stage of the grope which has every generator replaced by a disc with $\gamma((kn))$ handles.

4. Ljubljana complexes

Definition 4.1. A connected CW complex $L$ is called a Ljubljana complex if there is a co-AP-complex $K$, AP being the set of all primes, such that, for any compactum $X$, the conditions $X\tau L$ and $X\tau K(H_1(K), 1)$ imply $X\tau K$.

Lemma 4.2. Suppose $F \to E \to B$ is a fibration of connected CW complexes. If $F$ is a co-AP-complex, AP being the set of all primes, and $B$ is a Ljubljana complex, then $E$ is a Ljubljana complex.

Proof. Notice $\pi_1(E) \to \pi_1(B)$ is an epimorphism (use the long exact sequence of a fibration) which implies $H_1(E) \to H_1(B)$ is an epimorphism.

Pick a co-AP-complex $K$ such that $X\tau K$ and $X\tau K(H_1(B), 1)$ imply $X\tau B$ for all compacta $X$. Let $M$ be the wedge of $F$, $K$, $K(Q, 1)$, and $X\tau K(Z/p^\infty, 1)$ for all primes $p$. By [8] and the Miller Theorem, $M$ is a co-AP-complex. Suppose $X$ is a compactum such that $X\tau M$ and $X\tau K(H_1(E), 1)$. By [8] one gets $X\tau K(H_1(B), 1)$ which, together with $X\tau K$, implies $X\tau B$. Since $X\tau F$ and $X\tau B$, we infer $X\tau E$. ■
Corollary 4.3. Let $L$ be a connected CW complex with nilpotent fundamental group. If $L$ has finitely many unstable Postnikov invariants, then $L$ is a Ljubljana complex.

Proof. Notice that the universal cover $\tilde{L}$ of $L$ is a co-$AP$-complex by \ref{AP}. We get $K(\pi_1(L), 1)$ is a Ljubljana complex by \ref{Ljubljana}. The fibration $\tilde{L} \to L \to K(\pi_1(L), 1)$ implies $L$ is a Ljubljana complex.

Definition 4.4. A connected CW complex $L$ is called extensionally Abelian if $X\tau K(H_n(L), n)$ for all $n \geq 1$ imply $X\tau K$ for all compacta $X$.

Proposition 4.5. Each extensionally Abelian complex $L$ is a Ljubljana complex.

Proof. Let $K$ be the weak product of $K(H_n(L), n)$, $n \geq 2$. By (2) of \ref{AP}, $K$ is a co-$AP$-complex. Clearly, $X\tau K$ and $X\tau K(H_1(L), 1)$ imply $X\tau K(H_n(L), n)$ for all $n \geq 1$. Thus $X\tau L$.

Proposition 4.6. A finite wedge (or finite product) of Ljubljana complexes is a Ljubljana complex.

Proof. Let $L$ be the wedge (or the product) of Ljubljana complexes $L_s$, $s \in S$, where $S$ is finite. For each $s \in S$ choose a co-$AP$-complex $K_s$ such that for any compactum $X$ the conditions $X\tau K_s$ and $X\tau K(H_1(L_s), 1)$ imply $X\tau L_s$. Let $K$ be the wedge of all $K_s$. By \ref{AP}, it is a co-$AP$-complex. Notice that $H_1(L_s)$ is a retract of $H_1(L)$ for each $s \in S$. Therefore any compactum $X$ satisfying

- a. $X\tau K(H_1(L), 1)$,
- b. $X\tau K$,

also satisfies $X\tau K(H_1(L_s), 1)$ for each $s \in S$. Hence $X\tau L_s$ for each $s \in S$ which implies $X\tau L$.

There is a connection between Ljubljana complexes and co-$\mathcal{P}$-complexes.

Proposition 4.7. Suppose $K$ is a countable Ljubljana complex. If $\mathcal{P}$ is a set of primes such that $H_1(K)/\text{Tor}(H_1(K))$ is $\mathcal{P}$-divisible, then $K$ is a co-$\mathcal{P}$-complex.

Proof. Choose a co-$AP$-complex $L$ such that, for any compactum $X$, the conditions $X\tau L$ and $X\tau K(H_1(K), 1)$ imply $X\tau K$. Let $\mathcal{P}'$ be the complement of $\mathcal{P}$ in the set of all primes. Consider $K'$, the wedge of $L$, $K(Z(\mathcal{P}', 1))$, $K(Q, 1)$, and all $K(\mathbb{Z}/p, 1)$ ($p$ ranging through all primes). By \ref{AP} and \ref{AP}, $K'$ is a co-$\mathcal{P}$-complex. Since $X\tau K'$ implies $X\tau K$ for all compacta, \ref{AP} implies that there is $k \geq 0$ such that the truncated cohomology of $K$ with respect to $\Omega^k L$, $L$ any $\mathcal{P}$-complex, is trivial. Thus $K$ is a co-$\mathcal{P}$-complex.

Theorem 4.8. Suppose $K$ is a countable Ljubljana complex such that $\Sigma^m K$ is equivalent (over the class of compacta) to a quasi-finite countable complex $L$ for some $m \geq 0$. If $K$ is not acyclic, then it is equivalent to $S^1$.

Proof. We may assume $L$ is simply connected as $\Sigma^{m+1} K$ is equivalent to $\Sigma L$ (see \ref{AP}) and $\Sigma L$ is quasi-finite by \ref{AP}.

Suppose $K$ is not equivalent to $S^1$. Choose a co-$AP$-complex $P$ such that conditions $X\tau P$ and $X\tau K(H_1(K), 1)$ imply $X\tau K$. Let $k \geq 2$ be a number such that all maps $\Sigma^k P \to R$ are null-homotopic if $R$ is an $AP$-complex and $n \geq k$.

Step 1. $L$ is not contractible as otherwise $\Sigma^m K$ would have to be contractible implying $K$ being acyclic.
Step 2. $L$ is not acyclic as it is not contractible by Step 1.

Step 3. Since $X\tau K$ implies $X\tau K(H_1(K), 1)$, the group $H_1(K)$ has the property of $H_1(K)/\text{Tor}(H_1(K))$ being divisible by some prime $p$. Indeed, if $H_1(K)/\text{Tor}(H_1(K))$ is not divisible by any prime, then the Bockstein basis of $H_1(K)$ consists of all Bockstein groups and $X\tau K(H_1(K), 1)$ implies $X\tau S^1$ by Bockstein First Theorem. Since $X\tau K$ implies $X\tau K(H_1(K), 1)$ and $X\tau S^1$ implies $X\tau K$ for any compactum $X$, $K$ is equivalent to $S^1$ over compacta.

Let $e$ be the function of $L$.

Case 1: $H_*(K)$ is a torsion group. There is $M$ such that $H_*(M) \to H_*(e(M))$ is not trivial. By 7.1 there is a map $f : \Sigma^k(e(M)) \to J$ such that $f|_{\Sigma^k(M)}$ is not trivial, $J$ is simply connected, and all homotopy groups of $J$ are finite. Consider the wedge $N$ of $P$ and $K(\bigoplus \mathbb{Z}/q, 1)$. Notice $X\tau N$ implies $X\tau K(H_1(K), 1)$. Therefore $X\tau N$ implies $X\tau K$ which, in turn, implies $X\tau L$ and $X\tau i_M$, where $i_M : M \to e(M)$. Since $\text{Map}_*(N, \Omega^k(J))$ is weakly contractible, $\Sigma^k$ implies homotopy triviality of $f|_{\Sigma^k(M)}$, a contradiction.

Case 2: $H_*(K)$ is not a torsion group. Notice $H_*(L)$ is not torsion as well. Indeed if $H_*(L)$ is torsion, we could find a finite dimensional compactum $Y$ of high rational dimension but all torsion dimensions equal 1. Such compactum satisfies $Y\tau L$ but $Y\tau \Sigma^n K$ fails as it implies rational dimension of $Y$ to be at most $m + n$, where $H_*(K)$ is not torsion. There is $M$ such that the image of $H_*(M) \to H_*(e(M))$ is not torsion. Therefore there is $n > 0$ such that $H_n(M; \mathbb{Z}/(p)) = H_n(e(M); \mathbb{Z}/(p))$ is not trivial. By 7.2 there is a map $f : \Sigma^k(e(M)) \to J$ such that $f|_{\Sigma^k(M)}$ is not trivial, $J$ is simply connected, and all homotopy groups of $J$ are finite $p$-groups. Consider the wedge $N$ of $P$ and $K(\mathbb{Z}[1/p] \oplus \mathbb{Z}/p, 1)$. Using 7.3 and 7.4 one gets $X\tau N$ implies $X\tau K$ which, in turn, implies $X\tau L$ and $X\tau i_M$, where $i_M : M \to e(M)$. Since $\text{Map}_*(N, \Omega^k(J))$ is weakly contractible, $\Sigma^k$ implies homotopy triviality of $f|_{\Sigma^k(M)}$, a contradiction.

**Corollary 4.9.** Suppose $G$ is a nontrivial nilpotent group. If $K(G, 1)$ is quasi-finite, then it is equivalent, over the class of paracompact spaces, to $S^1$.

5. Application to cohomological dimension theory

**Theorem 5.1.** Suppose $G \neq 1$ is a countable group such that $\text{dim}_G(\beta(X)) = 1$ for every separable metric space $X$ satisfying $\text{dim}_G(X) = 1$. If $G$ is nilpotent, then $\text{dim}_G(X) \leq 1$ implies $\text{dim}(X) \leq 1$ for all paracompact spaces $X$.

**Proof.** By an improvement of Chigogidze’s Theorem 1.1 contained in [2], $K(G, 1)$ is quasi-finite. Therefore 1.1 says $K(G, 1)$ is equivalent to $S^1$ over compacta. A result in [2] says that $K(G, 1)$ is equivalent to $S^1$ over paracompact spaces which completes the proof.

6. Appendix A

**Lemma 6.1.** Let $p$ be a natural number and let $D_p$ be the class of groups $G$ such that $\text{Ab}(G)/\text{Tor}(\text{Ab}(G))$ is $p$-divisible, where $\text{Ab}(G)$ is the Abelianization of $G$. If $f : G \to H$ is an epimorphism and $G \in D_p$, then $H \in D_p$. 

Lemma 6.2. Let $p$ be a natural number and let $\mathcal{D}_p$ be the class of groups $G$ such that $H/Tor(H)$ is $p$-divisible, where $H$ is the Abelianization of $G$. If $G, H$ are Abelian and $G \in \mathcal{D}_p$, then $G \otimes H \in \mathcal{D}_p$.

Proof. It suffices to show that for each element $a$ of $G \otimes H$ there is $b \in G \otimes H$ and an integer $k \neq 0$ such that $k \cdot a + kp \cdot b = 0$. That in turn can be reduced to generators of $G \otimes H$ of the form $g \otimes h$. Pick $u \in G$ and an integer $k \neq 0$ such that $k \cdot g + kp \cdot u = 0$. Now $k \cdot (g \otimes h) + kp \cdot (u \otimes h) = 0$.

We recall a result of Robinson (see [20], 5.2.6) on the relation between a nilpotent group and its abelianization.

Proposition 6.3 (Robinson). Let $\mathcal{N}$ denote the category of nilpotent groups. Let $\mathcal{P}$ be a class of groups in $\mathcal{N}$ with the following properties.

1. For $A$ and $B$ abelian, $B \in \mathcal{P}$, any quotient of $A \otimes B$ belongs to $\mathcal{P}$.
2. For $K, Q \in \mathcal{P}$, an extension $1 \to K \to G \to Q \to 1$ in $\mathcal{N}$ implies $G \in \mathcal{P}$.

Suppose that $G \in \mathcal{N}$. If $Ab(G)$ belongs to $\mathcal{P}$, so does $G$.

We note the following corollary.

Corollary 6.4. Let $G$ be a nilpotent group and set $H = Ab(G)$. If $H/Tor(H)$ is $p$-divisible, so is $G/Tor(G)$.

Proof. Define the class $\mathcal{D}_p$ by letting a nilpotent group $G$ belong to $\mathcal{D}_p$ if and only if $F_{p}(G) = G/Tor_{p}(G)$ is $p$-divisible where $Tor_{p}(G)$ denotes the $p$-torsion subgroup of $G$. Note that $F_{p}(G)$ is $p$-divisible if and only if $G/Tor(G)$ is, hence it suffices to check properties (1) and (2) of Proposition 6.3.

As for (1) it follows from 6.1 and 6.2.

For (2), note that $F_{p}$ is a functor $\mathcal{N} \to \mathcal{N}$. Let $1 \to K \to G \to Q \to 1$ be an extension in $\mathcal{N}$. We apply $F_{p}$. Since $Tor_{p}(K) = K \cap Tor_{p}(G)$, the morphism $F_{p}(K) \to F_{p}(G)$ is injective. Evidently, $q: F_{p}(G) \to F_{p}(Q)$ is surjective. Moreover, $F_{p}(K)$ is a subset of the kernel of $q$. Assume that $K$ belongs to $\mathcal{D}_p$. If $q(\xi) = 1$ for some $\xi \in F_{p}(G)$, then $\xi^{p^i} \in F_{p}(K)$ for large enough $i$. By assumption on $K$, the group $F_{p}(K)$ is $p$-divisible, hence $\xi^{p^i} = \eta^{p^i}$ for an element $\eta \in F_{p}(K)$. But $F_{p}(G)$ is free of $p$-torsion (and nilpotent), so the equality $\xi^{p^i} = \eta^{p^i}$ in $F_{p}(G)$ implies $\xi = \eta$ (see for example Hilton, Mislin, Roitberg [20], Corollary 2.3). Therefore in fact $\xi \in F_{p}(K)$, i.e. $ker q = F_{p}(K)$. This is to say that $1 \to F_{p}(K) \to F_{p}(G) \to F_{p}(Q) \to 1$ is an extension in $\mathcal{N}$. If, in addition, $Q$ belongs to $\mathcal{D}_p$ then $F_{p}(K)$ and $F_{p}(Q)$ are $p$-divisible and free of $p$-torsion, and as such local away from $p$. Therefore so is $F_{p}(G)$, by Corollary 2.5 of 20.

Lemma 6.5. Let $f: G \to H$ be an epimorphism of Abelian groups and let $X$ be a compactum. If

a. $X \tau K(G, 1)$

b. $X \tau K(Q, 1)$

c. $X \tau K(Z/p^\infty, 1)$ for all primes $p$,

then $X \tau K(H, 1)$. 
Proof. Suppose $X\tau K(H,1)$ fails. This can only happen if there is a Bockstein group $F$ in the Bockstein basis $\sigma(H)$ such that $\text{dim}_F(X) > 1$. That group must be either $Z(p)$ or $Z/p$ for some $p$. $Z(p)$ belongs to $\sigma(H)$ if and only if $H/\text{Tor}(H)$ is not divisible by $p$ in which case $Z(p)$ belongs to $\sigma(G)$ by Corollary 6.6 and $\text{dim}_F(X) \leq 1$ by Bockstein First Theorem. Therefore $F = Z/p$ which means that $\text{Tor}(H)$ is not divisible by $p$. Now, either $G$ is not divisible by $p$ or its torsion group is not divisible by $p$ implying $\text{dim}_F(X) \leq 1$, a contradiction. $\blacksquare$

**Corollary 6.6.** Let $G$ be a nilpotent group with Abelianization $\text{Ab}(G)$ and let $X$ be a compactum. If

- $X\tau K(\text{Ab}(G),1)$,
- $X\tau K(Q,1)$,
- $X\tau K(Z/p^\infty,1)$ for all primes $p$,

then $X\tau K(G,1)$.

Proof. Consider the lower central series of $G$: $G = \Gamma^1 G \supset \Gamma^2 G \supset \ldots \supset \Gamma^n G \supset \ldots$. Let $F_i = \Gamma^n G/\Gamma^{i+1} G$. Since there is an epimorphism from $F_i \otimes \text{Ab}(G)$ to $F_{i+1}$, where $\text{Ab}(G)$ is the Abelianization of $G$, $X\tau K(F_i,1)$ for all $i$ by 6.5. We proceed by induction on $c - i$ ($c$ being the nilpotency class of $G$) showing that $X\tau K(\Gamma^i G,1)$.

If $c - i = 0$, then $\Gamma^i G = F_i$ and we are done. Since the sequence $1 \to \Gamma^{i+1} G \to \Gamma^i G \to F_i \to 1$ is exact, one uses a fibration $\chi(\Gamma^{i+1} G,1) \to \chi(\Gamma^i G,1) \to K(F_i,1)$ to conclude $X\tau K(\Gamma^i G,1)$ given $X\tau K(\Gamma^{i+1} G,1)$. That constitutes the inductive step and, as $\Gamma^1 G = G$, we get $X\tau K(G,1)$. $\blacksquare$

**Corollary 6.7.** Let $p$ be a natural number and let $D_p$ be the class of groups $G$ such that $H/\text{Tor}(H)$ is $p$-divisible, where $H$ is the Abelianization of $G$. If $G \in D_p$ is nilpotent and $X\tau K\left(Z[\frac{1}{p}] \oplus Z/p,1\right)$, then $X\tau K(G,1)$ for any compactum $X$.

**Corollary 6.8.** Let $G$ be a nilpotent group. If the Abelianization $\text{Ab}(G)$ of $G$ is a torsion group and $X\tau K\left(\bigoplus_p Z/p,1\right)$, then $X\tau K(G,1)$ for any compactum $X$.

### 7. Appendix B

In this Appendix we prove results allowing us to detect homology via maps to finite complexes with finite homotopy groups.

**Lemma 7.1.** Let $A$ be a finite CW complex and $\alpha \in H_k(A)$ a nontrivial element where $k \geq 2$. There exists a finite $(k-1)$-connected CW complex $B$ with finite homotopy groups and a map $f: A \to B$ with $\beta = f_*(\alpha)$ nontrivial. Furthermore, if $\alpha$ is of infinite order in $H_k(A)$, then $\beta$ may be assumed to be of order $r$ for any given natural $r \geq 2$.

Proof. With the exception of the statements about the connectedness and the order, this is precisely Lemma 2.1 of Levin. In the course of proving the cited lemma, Levin constructs a $(k-1)$-connected complex $L$, and he makes $\beta$ of order 2 if $\alpha$ has infinite order. The generalization to arbitrary $r$ is trivial. $\blacksquare$

**Lemma 7.2.** Let $M$ be a finite CW complex, and let $\mathcal{P}$ be a nonempty set of primes. Let $\alpha \in H_k(M; Z(p))$ be a nontrivial element for some $k \geq 2$. Then there exists a finite $(k-1)$-connected CW complex $N$ with $\mathcal{P}$-torsion homotopy groups and a map $f: M \to N$ with $f_*(\alpha)$ nontrivial.
Proof. The assumption is that there exists an element $\alpha \in H_k(M)$ which is either $P$-torsion or it has infinite order. We can apply Lemma 7.1 to obtain a $(k-1)$-connected finite complex with finite homotopy groups $N'$ and a map $f': M \to N'$ with $\beta' = f'_*(\alpha)$ nontrivial of order all of whose prime divisors belong to $P$. Let $N' \to N$ be localization at the set $P$. Then $\beta'$ will map to nontrivial $\beta$ under localization $\tilde{H}_*(N') \to \tilde{H}_*(N) = \tilde{H}_*(N') \otimes \mathbb{Z}(P)$ and $N$ is (homotopy equivalent to) the finite complex as in the statement of the lemma. ■

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