SCATTERING BY CURVATURES, RADIATIONLESS SOURCES, TRANSMISSION EIGENFUNCTIONS AND INVERSE SCATTERING PROBLEMS

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ABSTRACT. The paper is concerned with several separate but intriguingly connected topics arising in the theory of wave propagation. They include the geometrical characterizations of radiationless sources, non-radiating impinging waves and interior transmission eigenfunctions, and their applications to inverse scattering problems. A major novel discovery and technical ingredient in our study is a certain localization and geometrization property in the aforementioned wave scattering scenarios.

It is first shown that a scatterer, which might be an active source or an inhomogeneous index of refraction, must not be invisible if its support is sufficiently small, namely, the scattered wave pattern cannot be identically vanishing. In fact, we prove much more general results by deriving explicit relationships between the intensity of the scatterer and its diameter in these cases. These results significantly generalize the classical ones on radiationless/invisible bodies supported in a ball of uniform content. Then we localize and geometrize the “smallness” results to the case where there is a high-curvature point on the boundary of the support of the scatterer. We derive explicit relationships between the intensity of a radiating source, or a radiating scattered field for the inhomogeneous refraction case, locally at the aforementioned boundary point and the curvature of that point. The results can be used to derive geometric characterizations of radiationless sources or non-radiating waves near high-curvature points, as well as to classify non-invisible objects.

As significant applications of our study, we first derive a certain intrinsic geometric property of interior transmission eigenfunctions near a high-curvature points. This is of independent interest in spectral theory. We also apply the obtained results to the study of Schiffer’s problem in inverse scattering theory which concerns the determination of the support/shape of an unknown/inaccessible scatterer by a single scattering pattern measured away from the scatterer. We establish unique determination results for this problem in certain scenarios of practical interest. These are the first results for Schiffer’s problem with generic smooth scatterers.

The results obtained in this work are of fundamental importance in the theory of direct and inverse wave scattering, and there should be more profound physical implications underlying them.

Keywords radiationless sources, invisible, transmission eigenfunctions, inverse shape problems, geometrical properties, curvature, single far-field pattern

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1. INTRODUCTION

Visibility and invisibility are two main themes in wave scattering and both of them lie at the heart of scientific inquiry and technological development. To be more specific, we consider two types of “visible” and “invisible” scenarios. The first one is concerned with radiationless/non-radiating monochromatic sources, and the other one is concerned with non-radiating waves that impinge against a certain given scatterer consisting of
an inhomogeneous index of refraction. In this article, we establish certain geometrical characterizations of those invisible objects. The geometrical characterizations enable us to classify the radiationless sources and non-radiating impinging waves as well as the “invisible” scatterers, and moreover, they also help us to establish the visibility results in certain scenarios of practical importance.

The study of non-radiating sources has a long and colourful history, and there exists a vast amount of literature devoted to this intriguing topic. We refer to [27] for an excellent account of the historical development. The origin of non-radiating sources lay in the theory of the extended rigid electron, initiated by Sommerfeld [71, 72] and others [62]. Later, Ehrenfest [23], Schott [69, 70], Bohm and Weinstein [12] and Goedecke [29] theoretically predicted the existence of non-radiating sources. It was also postulated by those authors that non-radiating charge distributions might be used as models for elementary particles and Geodecke even suggested that such distributions might lead to a “theory of nature”. In more modern times, the mathematical properties of non-radiating sources have been more explicitly and systematically investigated [11, 21, 22, 25, 26, 37, 44, 60]. In this work, we discover more elegant characterisations of radiationless sources. We first establish an explicit relationship between the intensity of a source and the diameter of its support. The relationship immediately suggests that if the support of a generic source is sufficiently small, then it must not be radiationless, that is, its radiating pattern cannot be identically zero. This result generalises the classical result on radiationless sources which states that for a source supported in a ball with a constant intensity, if the radius of the ball is sufficiently small, then the source must be radiating.

Next, we localize and geometrize the result stated above for small scatterers. Instead, we consider a source with a generic intensity and supported in a bounded domain of arbitrary size. It is supposed that on the boundary of the support of the source there is a point with a relatively high curvature. We establish a quantitative relationship between the intensity of the source at the high-curvature point and the corresponding curvature at that point. This result readily implies that if the intensity of the source is not vanishing at a boundary point of its support and the curvature of that boundary point is sufficiently high, then the source must be radiating, no matter what the rest of the source is. That also means that a radiationless source must be nearly-vanishing near high-curvature points on the boundary of its support, and the higher the curvature the lower its intensity there. Our study significantly extends the relevant one in a recent article [4] by one of the authors, which proves the vanishing behaviour near singular corner points of a radiationless source.

The technical arguments developed for treating the geometric characterisations of radiationless sources pave the way for further studying the scattering from an inhomogeneous index of refraction due to an incident wave field. We are also interested in the case when there is no scattering, that is, invisibility occurs. Here, the question of interest for such invisibility phenomena is that for a given scatterer of inhomogeneous index of refraction, what conditions should an incident wave fulfil so that it propagates uninterrupted after impinging on the scatterer. That is, we would like to characterise all the non-radiating incident waves for a given scatterer. This perspective naturally leads to the so-called interior transmission eigenvalue problem. In fact, for a given inhomogeneous scatterer, a non-radiating wave must be an interior transmission eigenfunction associated to the scatterer. Hence, in order to characterize the non-radiating waves, one should first characterize the interior transmission eigenfunctions associated with the given inhomogeneous index of refraction.
The study of the interior transmission eigenvalue problem has a long history in inverse scattering theory. It was first introduced by Colton and Monk [18] and Kirsch [45]. The problem is a type of non-elliptic and non-self-adjoint eigenvalue problem, so its study is mathematically interesting and challenging. The existing studies in the literature mainly focus on the spectral properties of the transmission eigenvalues, including the existence, discreteness, infiniteness and Weyl’s laws. Generally, the results for transmission eigenvalues follow in a sense the results in spectral theory for the Laplacian in a bounded domain; see e.g. [9, 13, 17, 50, 61, 67] as well as a recent survey article [14] and the references cited therein. However, the transmission eigenfunctions reveal certain distinct and intriguing features. In general, the eigenfunctions do not form a complete set in \( L^2(\Omega) \), but certain generalized transmission eigenfunctions do [9, 65]. Here, \( \Omega \) signifies the domain in question, namely the support of the inhomogeneous index of refraction. In [10, 64], it is proved that the transmission eigenfunctions cannot be analytically extended across a corner with an interior angle less than \( \pi \). In [7, 8], geometric structures of transmission eigenfunctions were discovered for the first time in the literature. It is shown that the eigenfunctions vanish at a corner with an interior angle less than \( \pi \), while localize at a corner with an interior angle bigger than \( \pi \).

The spectral results above are of significant interest in pure mathematics. On the other hand, their implications to the invisibility phenomenon in wave scattering theory can be briefly described as follows. There exists a smallest positive transmission eigenvalue depending on the size of the scatterer as well as its refractive index. This implies that if the size of the scatterer is small enough (compared to the wavelength), then it cannot be invisible. That is, a small-size index of refraction scatters every incident wave field nontrivially. The vanishing of transmission eigenfunctions near a corner indicates that the corner scatters every incident wave nontrivially unless the wave vanishes at the corner. Physically speaking, a corner on the support of a scatterer makes the scatterer more visible or more detectable. In this article, we derive more geometric structures of transmission eigenfunctions that are of significant mathematical and practical interest. First, we establish a relationship among the value of the transmission eigenfunction, the diameter of the domain and the underlying refractive index, which indicates that if the domain is sufficiently small, then the transmission eigenfunction is nearly vanishing. Then we further localize and geometrize this “smallness” result. Briefly, it is proved that the interior transmission eigenfunctions must be nearly vanishing at a high-curvature point on the boundary. Moreover, the higher the curvature, the smaller the eigenfunction must be at the high-curvature point. The nearly vanishing behaviour readily implies that as long as the shape of a scatterer possesses a highly curved part, then it scatters every incident wave field nontrivially unless the wave is vanishingly small at the highly curved part. The practical implication of our result is significant since it indicates that even if a scatterer has a very smooth shape, significant scattering can be caused due to the curvature of the shape. This is in sharp contrast to the existing studies which establish the cause of scattering from the singularities of the shape, a mathematical fact which is to be expected from a physical point of view. On the other hand, the novel geometric structure of the transmission eigenfunctions is of significant mathematical interest for its own sake for the spectral theory of the transmission eigenvalue problem.

In addition, there is a different perspective on invisibility in wave scattering, which concerns the design of material structures such that for a given set of incident waves, no scattering would occur. This is also referred to as the cloaking technology and has received considerable attentions in the literature in recent years. It is actually shown [32, 52, 63]
that one can make material structures that are invisible with respect to the probing by any incident wave. However, those structures employing singular refractive indices that are unrealistic for fabrication. It is a fundamental question in cloaking theory, whether one can employ non-singular materials to achieve perfect invisibility. Our result, on the nearly vanishing of the transmission eigenfunctions, implies that in general, the use of singular materials for a perfect cloaking device is inevitable. Indeed, considering the incident plane waves of the form $e^{ix \cdot \xi}$, with $\xi \in \mathbb{R}^n$, which are usually used for probing and they are non-vanishing everywhere in the space. According to our discussion above, the high curvature of the shape of a regular inhomogeneous index of refraction scatters the plane waves nontrivially in general. This point has also been explored in our work [5] where it was proved that a corner of an inhomogeneous index of refraction scatters an incident wave not only nontrivially but also stably, as long as the incident wave is not vanishing at the corner point. The current study pushes this viewpoint to the much more general and practically interesting case of smooth shapes. This also motivates the following. To achieve invisibility for a given material structure, one should position the structure such that its corners or high-curvature points are located where the amplitude of the incident waves vanish. Finally, we would also like to mention in passing that there are some related research on the so-called approximate invisibility cloaking which employs regular mediums and tries to diminish the scattering effect; see e.g. [2,3,20,47,54,55] as well as the survey articles [30,31,58] and the references cited therein.

The visibility issue in the theory of wave scattering has a very strong practical background and is usually referred to as the inverse scattering problem. It is concerned with the extraction of knowledge of the underlying object, which is unknown or inaccessible, from the associated radiating wave patterns measured away from the object. If the underlying object is an active source which generates the radiating pattern, then one has the so-called inverse source problem. In the case that the underlying object is an inhomogeneous index of refraction, one sends an incident wave for probing and the inhomogeneity interrupts the wave propagation to generate the wave pattern to be measured. This is called the inverse medium problem. Inverse source problems arise in a variety of important applications including detection of hazardous chemicals, medical imaging, photoacoustic and thermoacoustic tomography, brain imaging, artificial intelligence in gesture computing and others. The inverse medium scattering problems are central to many industrial and engineering developments including radar and sonar, geophysical exploration, medical imaging and non-destructive testing. There is a rich theory on the inverse scattering problems and it is impossible for us to provide a comprehensive review on this topic. We refer to the research monographs [16, 41, 68] for discussions on these and other related developments on the inverse scattering problems.

In this paper, we are mainly concerned with the inverse problem of recovering the shape or support of an object, independent of its content, by a single measurement of the far-field pattern. This inverse problem is also referred to as Schiffer’s problem in the literature. The Schiffer problem was originally posed for impenetrable obstacles, i.e., the waves cannot penetrate inside the object and only exist in the exterior of the object. M. Schiffer was the first to show that a sound-soft obstacle can be uniquely determined by infinitely many far-field patterns. Schiffer’s proof was based on a spectral argument for the Dirichlet Laplacian and appeared as a private communication in the monograph by Lax and Phillips [51]. The requirement of infinitely many far-field patterns was relaxed to a finite number by Colton and Sleeman [19] depending on the a priori knowledge of the size of the obstacle. The uniqueness for the sound-hard obstacle case with infinitely many
far-field patterns was established by Kirsch and Kress [46]. Using infinitely many far-field patterns, Isakov established that the shape of a penetrable inhomogeneous medium can be uniquely determined [42]. However, it is widely conjectured that the uniqueness for Schiffer’s problem follows when using a single far-field pattern [16,41]. The breakthrough on this problem when having a single far-field pattern was made for polyhedral obstacles [1,15,24,56,57,59,66]. In [38], it was proved that a non-analytic Lipschitz obstacle can be uniquely recovered by at most a few far-field patterns. Recently, there is growing interest in establishing the uniqueness for Schiffer’s problem in determining the shape of an active source or a penetrable index of refraction using a single far-field pattern, but mainly restricted to the polyhedral support [4–6,39,40]. It is worth mentioning that in [48,49], for a given far-field pattern, it is shown that the convex hull of any source or scattering inhomogeneity that can produce the far-field pattern can be uniquely determined. In this paper, using the obtained results on the geometric structures of non-radiating sources and non-radiating waves, we establish uniqueness results for Schiffer’s problem in determining an active source or an inhomogeneous medium with a single far-field pattern in certain scenarios of practical interest. These are the first unique determination results for Schiffer’s problem concerning scatterers with smooth shapes.

The rest of the paper is organized as follows. In Section 2, we consider the radiationless source and its geometric characterizations. In Section 3, we consider the non-radiating waves and transmission eigenfunctions and their geometric characterizations. Section 4 is devoted to the study of Schiffer’s problem.

2. Geometrical characterizations of radiationless sources

2.1. Wave scattering from an active source. Let $f : \mathbb{R}^n \to \mathbb{C}$ be a function having compact support, $f = \chi_{\Omega} \varphi$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain$^*$ and $\varphi \in L^\infty(\mathbb{R}^n)$, $\varphi \neq 0$ in a neighbourhood of $\partial \Omega$. The set $\Omega$ is the external shape of $f$ while $\varphi$ describes the intensity of the source at various points in $\Omega$. We assume that $\varphi$ and $\Omega$ do not depend on the wavenumber $k$ that’s fixed. In other words we are considering monochromatic scattering. The source $f$ produces a scattered wave $u \in H^2_{\text{loc}}(\mathbb{R}^n)$ given by the unique solution to

$$(\Delta + k^2)u = f, \quad \lim_{r \to \infty} r^{\frac{n-1}{2}} (\partial_r - i k) u = 0$$

(2.1)

where $r = |x|$ for $x \in \mathbb{R}^n$. The limit in (2.1) is known as the Sommerfeld radiation condition which characterizes the outgoing nature of the radiating wave. By the limiting absorption principle (cf. [74]), the solution to (2.1) can be computed as follows,

$$u = (\Delta + k^2)^{-1} f = \lim_{\epsilon \to +0} (\Delta + (k - i \epsilon))^{-1} f$$

$$= - \lim_{\epsilon \to +0} \int_{\mathbb{R}^n} \frac{e^{ix \cdot \xi} \hat{f}(\xi)}{\xi^2 - (k - i \epsilon)^2} d\xi,$$

(2.2)

where $\hat{f}(\xi) := \mathcal{F} f(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} \, dx$ signifies the Fourier transform of $f$. Inverting the Fourier transform in (2.2), one has the following integral representation,

$$u = (\Delta + k^2)^{-1} f = -\frac{i}{4} \left( \frac{k}{2\pi} \right)^{\frac{n-2}{2}} \int_{\mathbb{R}^n} |x - y|^\frac{2-n}{2} H^{(1)}_{\frac{n-2}{2}} (k|x - y|) f(y) \, dy,$$

(2.3)

$^*$We shall consider only bounded domains $\Omega$ for which $H_0^2(\Omega) = \{ u|_{\Omega} \mid u \in H^2(\mathbb{R}^n), u = 0 \text{ in } \mathbb{R}^n \setminus \overline{\Omega} \}$. 
where $H_{(n-2)/2}^{(1)}$ is the first-kind Hankel function of order $(n-2)/2$. Stationary phase applied to (2.3) yields that

$$u(x) = \frac{e^{ik|x|}}{|x|^{(n-1)/2}} C_{n,k} \int_{\mathbb{R}^n} e^{-ik\hat{x} \cdot y} f(y) \, dy + O(|x|^{-\frac{n}{2}}), \quad |x| \to \infty,$$

(2.4)

where $\hat{x} := x/|x| \in S^{n-1}$, $x \in \mathbb{R}^n \setminus \{0\}$, and

$$C_{n,k} = \frac{-i}{\sqrt{8\pi}} \left( \frac{k}{2\pi} \right)^{\frac{n-2}{2}} e^{-\frac{(n-1)\pi}{4} i}.$$

The far-field pattern of $u$ is given by

$$u_\infty(\hat{x}) := C_{n,k} \int_{\mathbb{R}^n} e^{-ik\hat{x} \cdot y} f(y) \, dy = (2\pi)^n C_{n,k} \mathcal{F}f(k\hat{x}) \in L^2(S^{n-1}).$$

(2.5)

As discussed earlier, we are particularly interested in the case where the source $f$ does not radiate, and one has $u_\infty \equiv 0$. By the Rellich lemma (cf. [16]), which establishes the one-to-one correspondence between the wave field and its far-field pattern, one has $u = 0$ in the unbounded component of $\mathbb{R}^n \setminus \Omega$. Hence, in such a case, the source is also referred to as non-radiating or radiationless. According to (2.5), one clearly has $\hat{f}(k\hat{x}) \equiv 0$ for $\hat{x} \in S^{n-1}$ for a radiationless source. Hence, characterizing radiationless sources, one actually characterizes functions with compact supports whose Fourier transforms vanish on the sphere of radius $k$.

A classical example is given by a source of constant intensity supported on a ball, namely the source is of the form $f = c_0 \chi_{B(0,r_0)}$, where $c_0 \in \mathbb{C}$, $c_0 \neq 0$ and $B_{r_0} := B(0,r_0)$ is a central ball of radius $r_0 \in \mathbb{R}_+$. By the properties of the Bessel functions (cf. [33, B.3]), one has $\mathcal{F}\chi_{B_{r_0}}(k\hat{x}) = \gamma J_{n/2}(kr_0)$, where $\gamma > 0$ is a constant depending on the dimension $n$, wavenumber $k$ and radius $r_0$ of the ball. Here $J_{n/2}$ is a Bessel function of order $n/2$. Hence if the radius of the support of the source, measured in units of wavenumber, is a zero of the Bessel function, then the source is radiationless. In particular, by using the fact that there is a smallest positive zero of the Bessel function, a sufficiently small $kr_0$ implies that the source must be radiating. In what follows, we shall first generalize this classical example to a more general scenario. Indeed, we establish a quantitative relationship between the intensity of a generic radiating source and the diameter of its support. This relationship readily implies that if a source’s support is sufficiently small (compared to its intensity), then it must be radiating.

We further localize and geometrize the result concerning sources of small support. By localizing, we mean that instead of considering a source with a small support, we consider a source whose support is “locally small”. We know that for a domain with a small diameter, the curvature of its boundary surface must be large. Hence, we naturally characterize the “locally small” domain as the existence of a boundary point where the boundary surface curvature is very large. This is referred to as the “geometrization” of the “local smallness”. That is, we consider sources whose support contains a high-curvature point on its boundary. We basically extend the result on the source with a sufficiently small support to this “locally small” case. In fact, we establish a certain quantitative relationship between the intensity of the source at the high-curvature point and the corresponding curvature. This relationship readily implies that if the support of a source contains a boundary point with a sufficiently high curvature, then it must be radiating. It also implies that the intensity of a radiationless source must be nearly vanishing near a high-curvature point on the boundary of its support.
2.2. **Small-support sources must be radiating.** In this section, we establish that a source with a relatively small support compared to its intensity must be radiating. Consider the scattering problem (2.1); we have

**Theorem 2.1.** Let $n \geq 2$, $R_m, k \in \mathbb{R}_+$, $0 \leq \alpha \leq 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain of diameter at most $R_m$ and whose complement is connected. Then there exists a positive constant $C$, depending only on $k, R_m$ and $n$, such that if $\varphi \in C^\alpha(\overline{\Omega})$ satisfies

$$\sup_{\partial \Omega} |\varphi| \geq C \left( \frac{\text{diam}(\Omega)}{\|\varphi\|_{C^\alpha(\overline{\Omega})}} \right) \alpha,$$

then $\chi_\Omega \varphi$ radiates a non-zero far-field pattern at wavenumber $k$.

An immediate consequence of Theorem 2.1 is that for a source with a given strength, namely $\|\varphi\|_{C^\alpha(\overline{\Omega})}$ is fixed, if the size of its support is sufficiently small (depending on the wavenumber $k$) and $\sup_{\partial \Omega} |\varphi|$ has a positive lower bound, then it must be radiating; or in another word, if it is radiationless with a sufficiently small support, then its intensity must be nearly vanishing on the boundary of its support. Particularly, if a radiationless source has a constant intensity and its support is sufficiently small, then it must be nearly vanishing. One can also infer that the size of a radiationless source must have a positive lower bound that depends on its intensity and wavenumber. More precisely, we have

**Corollary 2.2.** Let $n \geq 2$, $R_m, k \in \mathbb{R}_+$, $0 \leq \alpha \leq 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain of diameter at most $R_m$ and whose complement is connected. Consider a source of the form $\chi_\Omega \varphi$ with $\varphi \in C^\alpha(\overline{\Omega})$. If the source is radiationless, then there exists a positive constant $C$, depending only on $k, R_m, n$, such that

$$\left( \frac{\text{diam}(\Omega)}{\|\varphi\|_{C^\alpha(\overline{\Omega})}} \right) \alpha \geq C \sup_{\partial \Omega} |\varphi| \|\varphi\|_{C^\alpha(\overline{\Omega})}.$$  

(2.7)

In order to prove Theorem 2.1, we first derive some auxiliary results.

**Lemma 2.3.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a domain of diameter at most $R_m \in \mathbb{R}_+$. Let $u \in H^2_0(\Omega)$ satisfy

$$(\Delta + k^2)u = \varphi$$

for some $\varphi \in L^\infty(\Omega)$ and $k \geq 0$. Then

$$\|u\|_{C^{1,\beta}(\overline{\Omega})} \leq C \|\varphi\|_{L^\infty(\Omega)}$$

(2.8)

for any $0 \leq \beta < 1$ and some finite constant $C = C(n, k, R_m, \beta)$.

**Proof.** We first extend $u$ by zero outside of $\Omega$ into a ball of radius $2R_m$ or slightly larger if $k^2$ is a Dirichlet-eigenvalue for $-\Delta$. By a bit of abuse of notation, we still denote the extended function as $u$. Clearly, $u \in H^2(B_{2R_m})$ satisfies

$$(\Delta + k^2)u = \chi_\Omega \varphi \quad \text{in} \ B_{2R_m}, \quad u = 0 \quad \text{on} \ \partial B_{2R_m}.$$  

(2.9)

By Corollary 8.7 in [28], we first have from (2.9) that

$$\|u\|_{H^1(B_{2R_m})} \leq C(k, R_m) \|\varphi\|_{L^2(\Omega)}.$$  

(2.10)

Then by further applying Corollary 8.35 in [28], we have

$$\|u\|_{C^{1,\beta}(B_{2R_m})} \leq C(n, k, R_m, \beta) \left( \|u\|_{H^1(B_{2R_m})} + \|\varphi\|_{L^\infty(\Omega)} \right)$$

for any $0 \leq \beta < 1$, which together with (2.10) readily yields (2.8). The proof is complete. \qed
Lemma 2.4. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $k \in \mathbb{R}_+$ be a fixed wavenumber. Let $u_0, u \in H^2(\Omega)$ and $\varphi \in L^\infty(\Omega)$ satisfy
\[(\Delta + k^2)u = \varphi, \quad \Delta u_0 = 0\] in $\Omega$. If $w = 0$ and $\partial_\nu w = 0$ on $\Gamma \subset \partial \Omega$ then
\[
\int_\Omega (\varphi - k^2 u)u_0 dx = \int_{\partial \Omega \setminus \Gamma} (u_0 \partial_\nu u - u \partial_\nu u_0) d\sigma,
\]
where $\nu$ is the exterior unit normal vector on $\partial \Omega$.

Proof. By (2.11), we first have $\varphi - k^2 u = \Delta u$. They using integration by parts, one can deduce
\[
\int_\Omega (\varphi - k^2 u)u_0 dx = \int_\Omega (\Delta uu_0 - u \Delta u_0) dx = \int_{\partial \Omega} (u_0 \partial_\nu u - u \partial_\nu u_0) d\sigma,
\]
which completes the proof. \hfill \Box

Lemma 2.5. Let $B \subset \mathbb{R}^n$ be a ball of radius $R$. Let $0 \leq \alpha \leq 1$ and $f \in C^\alpha(\overline{B})$ satisfy $\int_B f(x) dx = 0$. Then
\[
\sup_{\partial B} |f| \leq C R^\alpha \|f\|_{C^\alpha(\overline{B})}.
\]

Proof. For any $p \in \partial B$ and $x \in B$, since $f \in C^\alpha(\overline{B})$, we have
\[
|f(p) - f(x)| \leq \|f\|_{C^\alpha(\overline{B})}|x - p|^\alpha \leq \|f\|_{C^\alpha(\overline{B})}(2R)^\alpha, \quad x \in B.
\]
On the other hand, by using $\int_B f(x) dx = 0$, we have
\[
f(p) \int_B dx = \int_B (f(p) - f(x)) dx.
\]
Plugging the estimate in (2.15) to (2.16), one has by straightforward calculations that
\[
|f(p)| \leq 2^\alpha R^\alpha \|f\|_{C^\alpha(\overline{B})},
\]
which readily implies (2.14) by noting that $p$ can be any point on $\partial B$. The proof is complete. \hfill \Box

Proposition 2.6. Let $B \subset \mathbb{R}^n$, $n \geq 2$ be a ball of radius $R \leq R_m$. Let $u \in H^2(B)$ satisfy
\[(\Delta + k^2)u = \varphi\] for some $\varphi \in C^\alpha(\overline{B})$, $0 \leq \alpha \leq 1$ and $k \geq 0$. If $u \in H^2_0(B)$ then
\[
\sup_{\partial B} |\varphi| \leq CR^\alpha \|\varphi\|_{C^\alpha(\overline{B})}
\]
for some finite constant $C = C(n, R_m, k)$.

Proof. Using Lemma 2.3, one can first derive that
\[
\|u\|_{C^\alpha} \leq C(n, k, R_m)\|\varphi\|_{C^\alpha}.
\]
Set $f = \varphi - k^2 u$ and $u_0 = 1$. By (2.12) in Lemma 2.4 and the fact that $u \in H^2_0(B)$, one can show that
\[
\int_B f(x) dx = \int_B (\varphi - k^2 u)(x) dx = 0.
\]
By virtue of (2.19), we can apply Lemma 2.5, which in combination with (2.18) yields that
\[ \sup_{\partial B} |f| \leq CR^\alpha \| \varphi \|_{C^\alpha(B)}. \] (2.20)
Finally, noting that \( u = 0 \) on \( \partial B \), one has \( f = \varphi - k^2 u = \varphi \) on \( \partial B \), which together with (2.20) immediately proves (2.17). The proof is complete. □

Finally, we show that Proposition 2.6 holds for general domains as well.

**Proposition 2.7.** Let \( \Omega \subset \mathbb{R}^n, n \geq 2 \) be a bounded Lipschitz domain of diameter at most \( R_m \). Let \( k \geq 0 \) and \( 0 \leq \alpha \leq 1 \). Let \( \varphi \in C^\alpha(\overline{\Omega}) \) and \( u \in H^2_0(\Omega) \) satisfy
\[ (\Delta + k^2)u = \varphi \]
in \( \Omega \). Then
\[ \sup_{\partial \Omega} |\varphi| \leq C \left( \text{diam}(\Omega) \right) ^\alpha \| \varphi \|_{C^\alpha(\overline{\Omega})} \] (2.21)
for a finite positive constant \( C = C(n, R_m, k) \).

**Proof.** Let \( f = \varphi - k^2 u \). We see that \( f \in C^\alpha(\overline{\Omega}) \) with the estimate
\[ \| f \|_{C^\alpha(\overline{\Omega})} \leq C(n, R_m, k) \| \varphi \|_{C^\alpha(\overline{\Omega})} \] (2.22)
by Lemma 2.3, and also that \( f = \varphi \) on \( \partial \Omega \). Setting \( u_0 = 1 \), Lemma 2.4 implies that \( \int_{\Omega} f(x)dx = 0 \). Let \( p \) be any point belonging to \( \partial \Omega \). Then by writing \( f(x) = f(p) + (f(x) - f(p)) \), one can clearly show that
\[ \varphi(p)m(\Omega) = - \int_{\Omega} (f(x) - f(p))dx, \] (2.23)
where \( m(\Omega) \) signifies the measure of \( \Omega \). Using the Hölder continuity of \( f \), one has \( |f(x) - f(p)| \leq \| f \|_{C^\alpha(\overline{\Omega})} |x - p|^{\alpha} \), which together with (2.23) yields that
\[ |\varphi(p)|m(\Omega) \leq \| f \|_{C^\alpha(\overline{\Omega})} \int_{\Omega} |x - p|^{\alpha}dx \leq \| f \|_{C^\alpha(\overline{\Omega})} m(\Omega) \left( \text{diam}(\Omega) \right) ^\alpha. \] (2.24)
Finally, by combining (2.22), (2.24), one can show (2.21). The proof is complete. □

We are in a position to present the proof of Theorem 2.1.

**Proof of Theorem 2.1.** We prove the theorem by a reductio ad absurdum. Suppose that \( (\Delta + k^2)u = \chi_{\Omega} \varphi \) and \( u \in H^2_{\text{loc}}(\mathbb{R}^n) \) is radiating with a zero far-field pattern. By Rellich’s lemma, we know that \( u = 0 \) in \( \mathbb{R}^n \setminus \overline{\Omega} \). Clearly, the restriction of \( u \) to \( \Omega \) is in \( H^2_0(\Omega) \), and therefore Proposition 2.7 holds for \( u \). By taking the constant \( C \) in (2.6) slightly larger than that in (2.21), one readily obtains a contradiction, which immediately implies that one must have \( u^*_{\infty} \neq 0 \). The proof is complete. □

Next, we localize and geometrize the result in Theorem 2.1. To that end, we first introduce the admissible \( K \)-curvature point in the next subsection for our study.
2.3. Admissible $K$-curvature boundary points. In this section, we introduce the admissible $K$-curvature boundary points that shall be used throughout the rest of the paper. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $p \in \partial \Omega$ be a fixed point. We next detail the conditions for $p$ to be an admissible $K$-curvature point.

**Definition 2.8.** Let $K, L, M, \delta$ be positive constants and $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. A point $p \in \partial \Omega$ is said to be an admissible $K$-curvature point with parameters $L, M, \delta$ if the following conditions are fulfilled; see Figure 1 for a schematic illustration.

1. Up to a rigid motion, the point $p$ is the origin $x = 0$ and $e_n = (0, \ldots, 0, 1)$ is the interior unit normal vector to $\partial \Omega$ at 0.

2. Set $b = \sqrt{M/K}$ and $h = 1/K$. There is a $C^3$ function $\omega : B(0, b) \to \mathbb{R}^+ \cup \{0\}$ with $B(0, b) \subset \mathbb{R}^{n-1}$ such that if

$$\Omega_{b,h} = B(0, b) \times (-h, h) \cap \Omega,$$

(2.25)

then

$$\Omega_{b,h} = \{ x \in \mathbb{R}^n \mid |x'| < b, -h < x_n < h, \omega(x') < x_n < h \},$$

(2.26)

where $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$.

3. The function $\omega$ in Item 2 satisfies

$$\omega(x') = K|x'|^2 + O(|x'|^3), \quad x' \in B(0, b),$$

(2.27)

4. We have $M \geq 1$ and there are $0 < K_- \leq K \leq K_+ < \infty$ such that

$$K_- |x'|^2 \leq \omega(x') \leq K_+ |x'|^2, \quad |x'| < b,$$

$$M^{-1} \leq \frac{K_+}{K} \leq M, \quad K_+ - K_- \leq L K^{1-\delta}.$$

5. The intersection $V = \Omega_{b,h} \cap \mathbb{R}^{n-1} \times \{h\}$ is a Lipschitz domain.

It is worth of noting that the usual curvature at the point $p$ of the boundary surface is high if $K$ is large in Definition 2.8. A simple example for an admissible $K$-curvature boundary point is that locally near $p$, $\partial \Omega$ is the part of a paraboloid, namely, $\omega(x') = K|x'|^2$. In such a case, one can easily determine the values of parameters $L, M, \delta$ to

![Figure 1. The boundary neighbourhood $\Omega_{b,h}$ of a high-curvature point.](image)
fulfill the requirements in Definition 2.8. However, we allow the presence of more general geometries near \( p \), and this can be guaranteed by the following lemma.

**Lemma 2.9.** Assume that \( \omega(x') = K|x'|^2 + \mathcal{O}(|x'|^3) \) is a \( C^3 \)-function. Let \( L, \delta > 0 \), \( M \geq 1 \) and

\[
c_n = \frac{1}{c_n} \sup_{x' \in \mathbb{R}^{n-1}} \left| x' \right|^3 \sum_{|\beta|=3} \beta!.
\]

Let \( f(K) = \max_{|\alpha|,|\beta|=3} \sup_{|x'|<b} |\partial^\alpha \omega(x')|/\beta! \) where \( b = \sqrt{M}/K \). Assume that

\[
f(K) \leq \min \left( \frac{M - 1}{c_n M^{3/2}} K^2, \frac{L}{2c_n \sqrt{M}} K^{2-\delta} \right).
\]

Set

\[
K_- = K - c_n f(K) b \quad \text{and} \quad K_+ = K + c_n f(K) b.
\]

Then one has

\[
M^{-1} \leq \frac{K_-}{K} \leq K \leq \frac{K_+}{K} \leq M, \quad K_+ - K_- \leq L K^{1-\delta}
\]

and

\[
K_- |x'|^2 \leq \omega(x') \leq K_+ |x'|^2 \quad \text{when } |x'| < b.
\]

**Proof.** By Taylor’s theorem, we first have

\[
\omega(x') = K|x'|^2 + \sum_{|\beta|=3} R_\beta(x') x'^\beta
\]

where the functions \( R_\beta, |\beta|=3 \) satisfy

\[
|R_\beta(x')| \leq \max_{|\alpha|=3} \sup_{|x'|<b} |\partial^\alpha \omega(x')|/\beta! = f(K).
\]

Then one has

\[
|\omega(x') - K|x'|^2| \leq c_n f(K) |x'|^3,
\]

where \( c_n \) is positive and finite because \( |x_j x_k x_l| \leq |x'||x'||x'| = |x'|^3 \). Let \( K_-\), \( K_+ \) be defined in (2.29). Then if \( |x'| < b \), one can directly verify that

\[
K_- |x'|^2 \leq K |x'|^2 - c_n f(K) |x'|^3 \quad \text{and} \quad K |x'|^2 + c_n f(K) |x'|^3 \leq K_+ |x'|^2.
\]

Hence for \( |x'| < b \), there holds

\[
K_- |x'|^2 \leq K |x'|^2 - c_n f(K) |x'|^3 \leq \omega(x').
\]

The upper bound \( \omega(x') \leq K_+ |x'|^2 \) follows from a similar argument.

Next, we prove the bounds for \( K_+ - K_- \) and \( K_\pm/K \). Recall the assumed upper bound on \( f(K) \) and that \( b = \sqrt{M}/K \). One has by straightforward calculations that

\[
\frac{K_-}{K} = 1 - \frac{c_n \sqrt{M} f(K)}{K^2} \geq 1 - (1 - 1/M) = 1/M,
\]

\[
\frac{K_+}{K} = 1 + \frac{c_n \sqrt{B} f(K)}{K^2} \leq 1 + (M - 1) = M,
\]

and

\[
K_+ - K_- = 2c_n \sqrt{M} f(K)/K \leq L K^{1-\delta}.
\]

The proof is complete. \( \Box \)
In the subsequent study, the concept of Rellich’s lemma and unique continuation principle shall play an important role. They are the fundamental tools of bringing information from the far-field to the near-field, and a fortiori to the boundary of the scatterer. In essence if the far-field pattern is known, then the scattered wave is known in the component of the complement of the scatterer that is unbounded. Since the scatterer may be hollow and we are studying boundary behaviour, we define for which boundary points the information from the far-field pattern can be propagated from infinity.

**Definition 2.10.** Let \( U \subset \mathbb{R}^n \) be an open set and \( p \in \mathbb{R}^n \). We say that \( p \) is connected to infinity through \( U \) if there is a continuous path \( \gamma : \mathbb{R}_+ \to U \) such that \( \lim_{s \to 0} \gamma(s) = p \), and \( \lim_{s \to \infty} |\gamma(s)| = \infty \).

**Lemma 2.11.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain and \( u^s, u'^s \in H^2_{loc}(\mathbb{R}^n) \cap C^0(\mathbb{R}^n) \) satisfy the Sommerfeld radiation condition and \( (\Delta + k^2)u = (\Delta + k^2)u' = 0 \) in \( \mathbb{R}^n \setminus \overline{\Omega} \). If \( u^s = u'^s \) and \( p \in \mathbb{R}^n \) is connected to infinity through \( \mathbb{R}^n \setminus \overline{\Omega} \), then \( u^s(p) = u'^s(p) \).

**Proof.** Elliptic regularity and Rellich’s lemma (e.g. Lemma 2.11 in [16]) imply that \( u^s = u'^s \) outside a large ball. Let \( \gamma : \mathbb{R}_+ \to \mathbb{R}^n \setminus \overline{\Omega} \) be a path as in Definition 2.10. For each \( s \in \mathbb{R}_+ \) let \( r(s) = d(\gamma(s), \Omega) \) be the distance from \( \gamma(s) \) to \( \Omega \), and note that it is positive since \( \mathbb{R}^n \setminus \overline{\Omega} \) is open. Let \( U = \bigcup_{s \geq 0} B(\gamma(s), r(s)) \). Then \( U \) is a connected open set, \( p \in U \) and \( U \) reaches infinity. The latter implies that \( u^s = u'^s \) in an open ball in \( U \), and by analyticity we have \( u^s = u'^s \) in \( U \). Continuity implies the same conclusion at \( p \). The proof is complete. \( \square \)

### 2.4. Geometric structure of radiationless sources at admissible K-curvature points

In this subsection, we derive the geometric characterization of a radiationless source at admissible K-curvature points on the boundary of its support. We have

**Theorem 2.12.** Let \( \Omega \subset \mathbb{R}^n, n \geq 2 \) be a bounded domain with diameter at most \( D \). Consider an active source of the form \( \varphi \chi_\Omega \) with \( \varphi \in C^\alpha(\overline{\Omega}), 0 < \alpha < 1 \). Assume that \( p \in \partial \Omega \) is an admissible K-curvature point with parameters \( L, M, \delta \) and \( K \geq e \). Assume further that \( p \) is connected to infinity through \( \mathbb{R}^n \setminus \overline{\Omega} \). Then for any given wavenumber \( k \in \mathbb{R}_+ \) and Hölder-smoothness index \( \alpha \) there exists a positive constant \( \mathcal{E} = \mathcal{E}(\alpha, \delta, n, D, L, M, k) \in \mathbb{R}_+ \) such that if

\[
\frac{|\varphi(p)|}{\max(1, \|\varphi\|_{C^0(\overline{\Omega})})} \geq \mathcal{E}(\ln K)^{(n+3)/2}K^{-\min(\alpha, \delta)/2},
\]

then the source \( \chi_\Omega \varphi \) radiates a non-zero far-field pattern at wavenumber \( k \).

**Corollary 2.13.** Consider a source of the form \( \chi_\Omega \varphi \) and \( p \in \partial \Omega \) be an admissible K-curvature point as described in Theorem 2.12. Suppose the strength of the source is bounded, namely \( \|\varphi\|_{C^\alpha(\overline{\Omega})} \leq \mathcal{M} \) is bounded. If the source is radiationless, then there exists a constant \( \mathcal{C} = \mathcal{C}(\alpha, \delta, n, D, L, M, k, \mathcal{M}) \) such that

\[
|\varphi(p)| \leq \mathcal{C}(\ln K)^{(n+3)/2}K^{-\min(\alpha, \delta)/2}.
\]

That is, if \( K \) is sufficiently large at \( p \in \partial \Omega \), then the intensity of a radiationless source must be nearly vanishing at that high-curvature boundary point.

**Remark 2.14.** Comparing Theorems 2.1 and 2.12, one readily sees that the global geometrical parameter \( \text{diam}(\Omega) \) in (2.6) is replaced by the local geometrical parameter \( K \) in (2.31). Hence, Theorem 2.12 is a local and geometrized version of Theorem 2.1.
Remark 2.15. We would like to point out that according to our discussion made after (2.5), all the geometric properties established in Theorems 2.1 and 2.12 and Corollaries 2.2 and 2.13 can be equally formulated for functions whose Fourier transforms vanish on a given sphere.

To prove Theorem 2.12, we need to derive the following auxiliary technical results.

Lemma 2.16. Let $\Omega \subset \mathbb{R}^n$ be a domain and $0 \in \partial \Omega$ be an admissible $K$-curvature point with parameters $L, M, \delta$. Let $\Omega_{b,h}$ be given in (2.25) in Definition 2.8 associated with $0 \in \partial \Omega$. Let $u_0(x) = \exp(\rho \cdot x)$ where $\rho \in \mathbb{C}^n$, $\rho \cdot \rho = 0$ and assume that $w \in H^2(\Omega_{b,h}) \cap C^0(\Omega_{b,h})$, $\varphi \in L^\infty(\Omega_{b,h})$ satisfy $(\Delta + k^2)w = \varphi$ for some $k > 0$, and $w = \partial_{w}w = 0$ on $\Omega_{b,h} \cap \partial \Omega$. There holds,

$$
\varphi(0) \int_{x_n > K|x|^2} e^{\rho x} dx = \varphi(0) \int_{x_n > \max(h,K|x|^2)} e^{\rho x} dx + \varphi(0) \left( \int_{K|x|^2 < x_n < h} e^{\rho x} dx - \int_{\Omega_{b,h}} e^{\rho x} dx \right) - \int_{\Omega_{b,h}} e^{\rho x} (\varphi(x) - \varphi(0) - k^2(w(x) - w(0))) dx + \int_{\partial\Omega_{b,h} \cap \partial \Omega} (e^{\rho x} \partial_w w - w \partial_{\varphi} e^{\rho x}) d\sigma. \tag{2.33}
$$

Proof. By straightforward calculations, one can first simplify the right-hand side of (2.33) to be

$$
\varphi(0) \int_{x_n > K|x|^2} e^{\rho x} dx - \int_{\Omega_{b,h}} e^{\rho x} (\varphi(x) - k^2(w(x) - w(0))) dx + \int_{\partial\Omega_{b,h} \cap \partial \Omega} (e^{\rho x} \partial_w w - w \partial_{\varphi} e^{\rho x}) d\sigma. \tag{2.34}
$$

Noting that $w(0) = 0$, one has from (2.33) and (2.34) a similar identity to that in (2.12). Since $\rho \cdot \rho = 0$, it is clear that $\Delta u_0 = 0$, and hence the claim follows from Lemma 2.4 because $\Omega_{b,h}$ is a Lipschitz domain. The proof is complete. \hfill \Box

Lemma 2.17. Let $K \in \mathbb{R}_+$, $\rho \in \mathbb{C}^n$ with $\Re \rho_n < 0$, and $C_{\infty} = \{ x \in \mathbb{R}^n \mid x_n > K|x|^2 \}$. Then one has

$$
\int_{C_{\infty}} e^{\rho x} dx = \frac{1}{-\rho_n} \left( \frac{\pi}{-\rho_n K} \right)^{(n-1)/2} \exp \left( -\frac{\rho' \cdot \rho'}{4\rho_n K} \right), \tag{2.35}
$$

where $\rho' \cdot \rho' = \rho_1^2 + \cdots + \rho_{n-1}^2$.

Proof. We have

$$
\int_{C_{\infty}} e^{\rho x} dx = \int_{\mathbb{R}^{n-1}} e^{\rho' x'} \int_{K|x|^2} e^{\rho_n x_n} dx_n dx' = \frac{1}{\rho_n} \int_{\mathbb{R}^{n-1}} e^{\rho_n K|x|^2 + \rho' \cdot x'} dx'. \tag{2.36}
$$

It is easily seen that the right-hand side term of (2.36) is the product of integrals of the form $\int_{\mathbb{R}^{n-1}} \exp(\rho_n K x_j^2 + \rho_j x_j) dx_j$ with $j = 1, \ldots, n - 1$. The integration formula for a complex Gaussian gives

$$
\int_{-\infty}^{\infty} e^{At^2 + Bt} dt = \sqrt{-\frac{\pi}{A}} \exp \left( -\frac{B^2}{4A} \right)
$$
when \( \Re A < 0 \). We have \( \Re \rho_n K < 0 \) and thus

\[
\int_{-\infty}^{\infty} e^{\rho_n K x_j^2 + \rho_j x_j} dx_j = \sqrt{-\frac{\pi}{\rho_n K}} \exp \left( -\frac{\rho_j^2}{4\rho_n K} \right), \quad j = 1, \ldots, n - 1.
\]  

(2.37)

The claim follows by plugging (2.37) into (2.36), along with some straightforward calculations. The proof is complete. \( \square \)

Lemma 2.18. Let \( \tau, K, h \in \mathbb{R}_+ \), \( s \geq 0 \) and \( C_h = \{ x \in \mathbb{R}^n \mid h > x_n > K|x'|^2 \} \). Then there holds

\[
\int_{C_h} e^{-\tau x_n} |x|^s dx \leq C_{n,s} \left( h + K^{-1} \right)^{\frac{s}{2}} h^{\frac{n+1}{2}} K^{-\frac{n-1}{2}}, \quad s \leq 0.
\]

(2.38)

Proof. First, we note that a horizontal slice of the paraboloid \( C_h \) has a radius \( \sqrt{x_n/K} \) at the height \( x_n \). Hence

\[
\int_{C_h} e^{-\tau x_n} |x|^s dx = \int_0^h \int_{B(0, \sqrt{x_n/K})} \left( x_n^2 + |x'|^2 \right)^{s/2} dx' dx_n.
\]

(2.39)

By using the polar coordinates \( x' = r \theta, \theta \in S^{n-2}, r = |x| \), one has that

\[
\int_0^h \int_{B(0, \sqrt{x_n/K})} \left( x_n^2 + |x'|^2 \right)^{s/2} dx' dx_n = \int_0^h \int_{0}^{\sqrt{x_n/K}} (x_n^2 + r^2)^{s/2} r^{n-2} dr dx_n.
\]

(2.40)

Taking the upper bound of the integrands, once for \( r \), and then for \( x_n \), one can further show that

\[
\int_0^h \int_{0}^{\sqrt{x_n/K}} (x_n^2 + r^2)^{s/2} r^{n-2} dr dx_n \leq \sigma(S^{n-2}) \int_0^h \int_{0}^{\sqrt{x_n/K}} (x_n^2 + r^2)^{s/2} r^{n-2} dr dx_n \leq \sigma(S^{n-2}) \left( h + \frac{1}{K} \right)^{\frac{s}{2}} \left( \frac{h}{K} \right)^{\frac{n-1}{2}} \left( h^{1+\frac{s}{2}} K^{-\frac{n-1}{2}} \right).
\]

(2.41)

Finally, by combining (2.39), (2.40) and (2.41), one readily has (2.38). The proof is complete. \( \square \)

Lemma 2.19. Let \( \tau, K, h \in \mathbb{R}_+ \) and \( C_\ell = \{ x \in \mathbb{R}^n \mid \ell > x_n > K|x'|^2 \} \) for any \( \ell \in \mathbb{R}_+ \cup \{ \infty \} \). Then there holds

\[
\int_{C_\infty \setminus C_h} e^{-\tau x_n} dx \leq C_n \frac{1 + (\tau h)^{\frac{n-1}{2}} K^{\frac{n-1}{2}}}{\tau^{\frac{n+1}{2}} K^{\frac{n+1}{2}}} e^{-\tau h},
\]

(2.42)

where \( C_n \) depends only on \( n \).
Proof. The proof proceeds as that of Lemma 2.18 but with \( s = 0 \) and \( x_n \in (h, \infty) \) instead of \( x_n \in (0, \infty) \). Changing to polar coordinates \( x' = r\theta, \theta \in \mathbb{S}^{n-2}, r = |x'| \), and integrating \( \int_0^{r_{\max}} r^{n-2}dr = r_{\max}^{n-1}/(n - 1) \) with \( r_{\max} = \sqrt{x_n/K} \), one has that
\[
\int_{C_\infty \setminus C_h} e^{-\tau x_n} dx = \int_0^h e^{-\tau x_n} \int_{B(0,\sqrt{x_n/K})} dx' dx_n = \sigma(S^{n-2}) \int_0^h e^{-\tau x_n} \int_0^{\sqrt{x_n/K}} r^{n-2}dr dx_n.
\]
Next, using the change of variables \( t = \tau x_n \), one can further show that
\[
\frac{\sigma(S^{n-2})}{n - 1} \int_h^\infty e^{-\tau x_n} \left( \frac{x_n}{K} \right)^{\frac{n-1}{2}} dx_n = \frac{\sigma(S^{n-2})}{n - 1} \frac{1}{\tau K} \int_{\tau h}^\infty e^{-t} t^{\frac{n-1}{2}} - 1 dt.
\]
Switching to \( s = t - \tau h \) in the last integral in (2.44) allows us to estimate it as follows. Recall the definition of the \( \Gamma \)-function \( \Gamma(m) = \int_0^\infty e^{-s}s^{m-1}ds \), and also that \( (A+B)^{n-1} \leq \max(1, 2(1/2, A^{a-1} + B^{a-1}) \) for \( a > 1 \). We thus have
\[
\int_{\tau h}^\infty e^{-t} t^{\frac{n-1}{2}} dt = e^{-\tau h} \int_0^\infty e^{-s} (s + \tau h)^{\frac{n-1}{2}} - 1 ds \leq e^{-\tau h} \max(1, 2\frac{n-1}{2}) \left( \Gamma \left( \frac{n+1}{2} \right) + (\tau h)^{\frac{n-1}{2}} \right).
\]
Finally, by combining (2.43), (2.44) and (2.45), one can readily verify (2.42). The proof is complete. \( \square \)

Lemma 2.20. Let \( \tau, K_-, K_+, h \in \mathbb{R}_+ \) with \( K_+ > K_- \) and denote \( C_\pm = \{ x \in \mathbb{R}^n \mid K_\pm|x|^2 < x_n < h \} \). Then there hold
\[
\int_{C_\setminus C_+} e^{-\tau x_n} dx = \frac{\sigma(S^{n-2})}{n - 1} \left( \left( \frac{1}{K_-} \right)^{\frac{n-1}{2}} - \left( \frac{1}{K_+} \right)^{\frac{n-1}{2}} \right) \frac{1}{\tau} \gamma(\tau h, \frac{n+1}{2}),
\]
where
\[
\gamma(x, a) := \int_0^x \exp(-t)t^{a-1}dt, \quad a \in C,
\]
is the lower incomplete gamma function.

Proof. The proof can be proceeded as that of Lemma 2.19. We first note that the horizontal cut at the height \( x_h \) is an annulus of external radius \( \sqrt{x_n/K_-} \) and internal radius \( \sqrt{x_n/K_+} \). By using the polar coordinates in integrals and the fact that \( \int_a^b r^{n-2}dr = (b^{n-1} - a^{n-1})/(n - 1) \), one can deduce as follows
\[
\int_{C_\setminus C_+} e^{-\tau x_n} dx = \int_0^h e^{-\tau x_n} \int_{\sqrt{x_n/K_-} < |x'| < \sqrt{x_n/K_+}} dx' dx_n = \sigma(S^{n-2}) \int_0^h e^{-\tau x_n} \int_{\sqrt{x_n/K_-}}^{\sqrt{x_n/K_+}} r^{n-2}dr dx_n.
\]
Finally, by combining (2.43), (2.44) and (2.45), one can readily verify (2.42). The proof is complete. \( \square \)
Next, using again the change of variables $t = \tau x_n$ in the last integral in (2.48), along with the definition of the incomplete $\Gamma$-function, one can further show that
\[
\int_0^h e^{-\tau x_n} x_n^{\frac{n-1}{2}} dx_n = \frac{1}{\tau^{\frac{n-1}{2}}} \int_0^\tau e^{-t \frac{n+1}{2}} dt. \tag{2.49}
\]
Finally, by combining (2.48) and (2.49), one can readily show (2.46). The proof is complete. \qed

**Proposition 2.21.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a domain and $p \in \partial \Omega$ be an admissible $K$-curvature point with parameters $L, M, \delta$. Let $\Omega_{b,h}$ be introduced in (2.25) in Definition 2.8 associated with $p \in \partial \Omega$.

Let $x$ be the coordinates for which $p = 0$ in Definition 2.8. Let $u_0(x) = \exp(\rho \cdot x)$, where $\rho = i\tau e_1 - \tau e_n$ with $\tau \in \mathbb{R}_+$, and assume that $w \in H^2(\Omega_{b,h}) \cap C^{1,\beta}(\Omega_{b,h})$, $0 < \beta \leq 1$, and $\varphi \in L^\infty(\Omega_{b,h})$ satisfy $(\Delta + k^2)w = \varphi$ in $\Omega_{b,h}$ for some $k > 0$, and $w = \partial_x w = 0$ on $\Omega_{b,h} \cap \partial \Omega$. Then there holds
\[
C_{n,\alpha}\|\varphi(p)\|_\Omega \leq (1 + (\tau h)^{(n-1)/2})e^{\gamma(1/2 - h)} + \left(\frac{K}{K_-}\right)^{\frac{n-1}{2}} - \left(\frac{K}{K_+}\right)^{\frac{n-1}{2}} e^{\gamma(1/2 - h)} + \|\varphi\|_{C^n} + h^2\|w\|_{C^{1,\beta}}(h + K_-)^{n/2}h^{(n+1)/2}(K/K_-)^{(n-1)/2}e^{\gamma(1/2 - h)}(2.50)
\] 
\[
+ h^{\beta + (n-1)/2}(K/K_-)^{(n-1)/2}(1 + \tau h)^{(n+1)/2}e^{\gamma(1/2 - h)}\|w\|_{C^{1,\beta}},
\]
where $C_{n,\alpha}$ is a positive number.

**Proof.** In what follows, we make use of the coordinates $x$ in Definition 2.8, and hence $p$ is represented by $x = 0$. First, by Lemma 2.16, we have
\[
\varphi(0) \int_{x_n > K|x|^2} e^{\rho x} dx = \varphi(0) \int_{x_n > \max(h,K|x|^2)} e^{\rho x} dx \\
+ \varphi(0) \left( \int_{K|x|^2 < x_n < h} e^{\rho x} dx - \int_{\Omega_{b,h}} e^{\rho x} dx \right) \\
- \int_{\Omega_{b,h}} e^{\rho x} (\varphi(x) - \varphi(0) - k^2(w(x) - w(0))) dx \\
+ \int_{\partial \Omega_{b,h} \setminus \partial \Omega} (e^{\rho x} \partial_x w - w \partial_x e^{\rho x}) d\sigma = \varphi(0) \cdot I_1 + \varphi(0) \cdot I_2 + I_3 + I_4,
\] 
where
\[
I_1 := \int_{x_n > \max(h,K|x|^2)} e^{\rho x} dx, \tag{2.52}
\]
\[
I_2 := \int_{K|x|^2 < x_n < h} e^{\rho x} dx - \int_{\Omega_{b,h}} e^{\rho x} dx, \tag{2.53}
\]
\[
I_3 := - \int_{\Omega_{b,h}} e^{\rho x} (\varphi(x) - \varphi(0) - k^2(w(x) - w(0))) dx, \tag{2.54}
\]
\[
I_4 := \int_{\partial \Omega_{b,h} \setminus \partial \Omega} (e^{\rho x} \partial_x w - w \partial_x e^{\rho x}) d\sigma. \tag{2.55}
\]
With the help of Lemmas 2.17 to 2.20, we next estimate the terms $I_j$, $j = 1, \ldots, 4$. 
First of all Lemma 2.17 implies that

\[
\int_{x_n > K|x'|^2} e^\rho x \, dx = \left( \frac{\pi}{K} \right)^{(n-1)/2} \frac{1}{\tau^{(n+1)/2}} \exp \left( -\frac{\tau}{4K} \right),
\]

which together with Lemma 2.19 gives

\[
|I_1| = \left| \int_{x_n > \max(h,K|x'|^2)} e^\rho x \, dx \right| \leq \int_{x_n > \max(h,K|x'|^2)} e^{-\tau x^2} \, dx \leq C_n \frac{1 + (\tau h)^{n+1}}{\tau^{n+1} K^{n+1}} e^{-\tau h},
\]

where \( C_n \) depends only on the dimension \( n \).

We proceed with the estimates of the integral terms \( I_2 \) and \( I_3 \), respectively, in (2.53) and (2.54). Recall Definition 2.8 and let \( K_- \) and \( K_+ \) be as in Item 4 therein. In the definition, the distances \( b,h > 0 \) were chosen such that \( h \leq K_- b^2 \). Hence the paraboloids \( x_n = K_\pm |x'| \) do not touch the sides of the cylinder \( \left\{ x \mid |x'|_n < b, -h < x_n < h \right\} \). Set

\[
P_{b,h,\pm} = \left\{ x \in \mathbb{R}^n \mid K_\pm |x'|^2 < x_n < h \right\}.
\]

According to our discussion above and Item 4 of Definition 2.8, one can see that \( P_{b,h,-} \subset \Omega_{b,h} \subset P_{b,h,+} \). Hence, one can show that

\[
\left| \int_{K|x'|^2 < x_n < h} e^\rho x \, dx - \int_{\Omega_{b,h}} e^\rho x \, dx \right| \leq \int_{\{K|x'|^2 < x_n < h\} \Delta \Omega_{b,h}} e^{-\tau x_n} \, dx
\]

where and also in what follows, for two sets \( A \) and \( B \), \( A \Delta B := (A \cup B) \setminus (A \cap B) \) signifies the symmetric difference of the two sets. Next, by Lemma 2.20, we can further estimate that

\[
\int_{P_{b,h,-} \setminus P_{b,h,+}} e^{-\tau x_n} \, dx = C_n \left( K_-^{n+1} - K_+^{n+1} \right) \tau^{n+1} \gamma \left( \tau h, \frac{n+1}{2} \right),
\]

where by the definition in (2.47), one clearly has

\[
\gamma \left( \tau h, \frac{n+1}{2} \right) \leq \Gamma \left( \frac{n+1}{2} \right).
\]

Finally, by combining (2.59), (2.60) and (2.61), one has

\[
|I_2| \leq C_n \left( K_-^{n+1} - K_+^{n+1} \right) \tau^{n+1} \Gamma \left( \frac{n+1}{2} \right)
\]

For the third term \( I_3 \), we note that \( w \in C^{1,\beta} \) and so is also in \( C^\alpha \). Hence, there holds

\[
|\varphi(x) - \varphi(0) - k^2 (w(x) - w(0))| \leq (\|\varphi\|_{C^\alpha} + k^2 \|w\|_{C^\alpha}) |x|^\alpha.
\]
On the other hand, we recall that \( \Omega_{b,h} \subset P_{b,h,-} \). By applying Lemma 2.18 to estimate the integral on the second line below, one can deduce that

\[
|I_3| = \left| \int_{\Omega_{b,h}} e^{\rho x}(\varphi(x) - \varphi(0) - k^2(w(x) - w(0))) \, dx \right|
\leq \left( \|\varphi\|_{C^\alpha(\overline{P_{b,h}})} + k^2\|w\|_{C^\alpha(\overline{P_{b,h}})} \right) \int_{P_{b,h,-}} e^{-\tau x^2} |x|^\alpha \, dx
\leq C_{n,\alpha} \left( \|\varphi\|_{C^\alpha(\overline{P_{b,h}})} + k^2\|w\|_{C^\alpha(\overline{P_{b,h}})} \right) \left( h + K_\gamma^{-n/2} \right)^{(n+\alpha+1)/2} K_\gamma^{-(n-1)/2}. \tag{2.64}
\]

For the last boundary integral term \( I_4 \) in (2.55), we first note that \( V := \partial \Omega_{b,h} \setminus \partial \Omega \) is actually a horizontal slice because \( h \leq K_\gamma b^2 \) and \( P_{b,h,-} \supset \Omega \). Hence \( V = U \times \{h\} \) for some bounded domain \( U \subset \mathbb{R}^{n-1} \). Its measure is at most the measure of a slice of \( P_{b,h,-} \), so \( \sigma(U) \leq \sigma(S^{n-1})(h/K_\gamma)(n-1)/2 \). On the other hand we know that \( w = 0 \) and \( \partial_n w = 0 \) on \( \partial \Omega \), so \( \partial_n w = 0 \) too, and any point of \( V \) has a distance at most \( h \) from \( \partial \Omega \).

The definition of the Hölder-norm implies that

\[
|\partial_n w(x', h)| \leq \|w\|_{C^{1,\beta}}(h - \omega(x'))^\beta,
\tag{2.65}
\]
where we recall that the graph of the function \( \omega \) defines \( \Omega \) and also that \( x' \in U \) with \( \omega(x') \leq h \). On the other hand the graph stays above the zero line, \( \omega(x') \geq 0 \), and hence one obviously has from (2.65) that

\[
|\partial_n w(x', h)| \leq \|w\|_{C^{1,\beta}} h^\beta. \tag{2.66}
\]

Next, the fundamental theorem of calculus implies that

\[
w(x', h) = \int_{\omega(x')}^h \partial_n w(x', s) \, ds.
\]

Thus, combining with (2.66) and recalling that \( 0 < \omega(x') < h \), one has

\[
|w(x', h)| \leq \|w\|_{C^{1,\beta}} \int_{\omega(x')}^h s^\beta \, ds \leq \|w\|_{C^{1,\beta}} h^{1+\beta}/(1 + \beta),
\]

which readily implies with (2.66) and \( \sigma(U) \leq C_n(h/K_\gamma)^{(n-1)/2} \) that

\[
|I_4| = \left| \int_V \left( e^{\rho x} \partial_n w - w \partial_\nu e^{\rho x} \right) \, d\sigma \right|
\leq e^{-\tau h} \int_U \left( |\partial_n w(x', h)| + \tau |w(x', h)| \right) \, dx'
\leq C_{n,\beta} h^{\beta+(n-1)/2} K_\gamma^{-(n-1)/2} (1 + \tau h) e^{-\tau h} \|w\|_{C^{1,\beta}(\overline{\Omega_{b,h}})}. \tag{2.67}
\]

Finally, by adding up (2.56), (2.57), (2.60), (2.64), (2.67) and multiplying both sides by \( K^{(n-1)/2} \tau^{(n+1)/2} \exp(\tau/4K) \), one can obtain (2.58). The proof is complete. \( \square \)

Next, we proceed to derive a critical inequality with the help of Proposition 2.21 by properly choosing the parameters appearing therein.

**Proposition 2.22.** Let \( \Omega \subset \mathbb{R}^n, n \geq 2 \) be a bounded domain and \( w \in H_0^2(\Omega) \) satisfy

\[
(\Delta + k^2)w = \varphi
\]
for some \( \varphi \in L^\infty(\Omega) \) and \( k > 0 \). Let \( p \in \partial \Omega \) be an admissible \( K \)-curvature point with parameters \( L, M, \delta \).
If \( \varphi \) restricted to \( \Omega_{h,b} \) from Definition 2.8 is \( C^\alpha \)-smooth, \( 0 < \alpha < 1 \), and \( \Omega \) has a diameter at most \( D \) then

\[
| \varphi(p) | \leq \mathcal{E} \max \left( 1, \| \varphi \|_{C^\alpha} \right) \left( \ln K \right)^{(n+3)/2} K^{-\min(\alpha, \delta)/2} \tag{2.68}
\]

for some \( \mathcal{E} = \mathcal{E}(\alpha, \delta, n, D, L, M, k) \in \mathbb{R}_+ \) depending only on \( \alpha, \delta, n, D, L, M, k \).

**Proof.** First, we have by Lemma 2.3 that

\[
\| w \|_{C^{1,\beta}(\Omega)} \leq C_{D,n,\beta,k} \| \varphi \|_{L^\infty(\Omega)}
\]

for some finite constant \( C_{D,n,\beta,k} \). This gives the required function regularity for applying Proposition 2.21. Hence, for any \( \tau \in \mathbb{R}_+ \), by Proposition 2.21 and assuming without loss of generality that \( p = 0 \), we have

\[
C_{n,k,\alpha} | \varphi(0) | \sqrt{\pi} \leq (1 + (\tau h)^{(n+1)/2}) e^{\tau(\frac{1}{4\pi} - h)} + \left( \left( \frac{K}{K_-} \right)^{\frac{n-1}{2}} - \left( \frac{K}{K_+} \right)^{\frac{n-1}{2}} \right) e^{\frac{\tau}{4\pi}}
\]

\[
+ \| \varphi \|_{C^n} (h + K_{-1}^{-1})^{n/2} h^{(n+n+1)/2} (K/K_-)^{(n+1)/2} \tau^{-3/2} e^{\tau h} + h^{\beta}(n-1)/2 (K/K_-)^{(n-1)/2} (1 + \tau h) |\alpha + (\tau h)| e^{\tau h} \| w \|_{C^{1,\beta}}. \tag{2.69}
\]

Let us start by estimating the difference of powers of \( K/K_- \) and \( K/K_+ \). Recall that \( 1/M \leq K/K \leq M \) by Item 4 of Definition 2.8. Consider the function \( f(r) = r^{-\delta} \) with \( f'(r) = -sr^{-\delta-1} \). By the mean value theorem

\[
| f(r_+) - f(r_-) | \leq \sup_{r_- < \xi < r_+} | f'(\xi) | | r_+ - r_- | = C_{s,M} | r_+ - r_- |
\]

when \( 1/M \leq r_- < r_+ \leq M \). Recall also that \( |K_+ - K_-| \leq LK^{1-\delta} \) by Item 4. Hence

\[
\left| \left( \frac{K}{K_-} \right)^{\frac{n-1}{2}} - \left( \frac{K}{K_+} \right)^{\frac{n-1}{2}} \right| \leq C_{n,M} \left( K_+^{\frac{1}{K}} - K_-^{\frac{1}{K}} \right) = C_{n,L,M} K^{-\delta}, \tag{2.70}
\]

for some finite constant \( C_{n,L,M} \).

Next, we recall that \( h = 1/K \) and \( b = \sqrt{M} / K \) by Definition 2.8, and that \( K/K_- \leq M \). Applying (2.70) to (2.69), estimating the constants and then dividing them to the left-hand side give

\[
C_{n,k,\alpha,L,M} | \varphi(0) | \sqrt{\pi} \leq (1 + (\tau / K)^{(n-1)/2}) e^{-3\tau/K} + K^{-\delta} e^{3\tau/K} + \| \varphi \|_{C^n} K^{-(n+2\alpha+1)/2} \tau^{-3/2} e^{\tau/K} + K^{-\beta}(n-1)/2 (1 + \tau / K) \tau^{(n+1)/2} e^{-3\tau/K} \| w \|_{C^{1,\beta}}. \tag{2.71}
\]

Choose \( \tau = 4K \ln K^\gamma \) for some \( \gamma \in \mathbb{R}_+ \) to be specified in what follows. Since \( K \geq e \), after dividing by the constants and max(1, \| \varphi \|_{C^n}) , (2.71) can be further estimated above by

\[
(\ln K)^{(n-1)/2} K^{-3\gamma} + K^{-\delta} + (\ln K)^{3/2} K^{1-\gamma/2} \alpha + (\ln K)^{(n+3)/2} K^{1-\beta-3\gamma}. \tag{2.72}
\]

One can directly verify that the quantity in (2.72) tends to zero as \( K \to \infty \) if \( 0 < \gamma < \min(\alpha, \delta) \) and \( 3\gamma > 1 - \beta \). These two conditions are fulfilled if one chooses \( \beta = 1 - \min(\alpha, \delta) \) and \( \gamma = \min(\alpha, \delta)/2 \). For the final form of the upper bound, using the fact that \( (\ln r)^{a1} \leq a1 r^m / a2 e^r \) for any \( a1, a2 > 0, r \geq e \), each of the terms in (2.72) can then be estimated above by

\[
C_{n,L,M,\alpha}(\ln K)^{(n+3)/2} K^{-\min(\alpha, \delta)/2} \tag{2.73}
\]
for some positive constant $C_{n,L,M,\delta,\alpha}$. By combining our discussion above, one arrives at (2.68). The proof is complete. □

**Proof of Theorem 2.12.** The proof follows from Proposition 2.22. Let $\tilde{\Omega}$ be the interior of the complement of the unbounded component of $\mathbb{R}^n \setminus \Omega$, in other words $\tilde{\Omega}$ is $\Omega$ with holes filled up. If $(\Delta + k^2)u = \chi_{\Omega} \varphi$ and $u \in H^2_\text{loc}(\mathbb{R}^n)$ radiates a zero far-field pattern, then by the Rellich lemma $u|_{\tilde{\Omega}} \in H^2_0(\tilde{\Omega})$. Moreover $\chi_{\Omega} \varphi = \varphi \in C^\alpha$ in the set $\Omega_{b,h}$, where the latter is the notation introduced in Definition 2.8. Hence, one can readily show the claim in the theorem by Proposition 2.22. The proof is complete. □

3. **Geometrical characterizations of non-radiating waves and transmission eigenfunctions**

In Section 2, we consider the wave scattering due to an active source that generates the wave propagation. In this section, we consider a different scattering scenario where one uses an incident field to generate the wave propagation in a uniform and homogeneous space. There is an inhomogeneous medium scatterer located in the space. The medium scatterer is passive and is characterized by its index of refraction which is different from that of the ambient space. The presence of the inhomogeneity interrupts the wave propagation and produces the wave scattering. Since we shall be considering scattering at a fixed wavenumber, one can also formulate such a scattering problem in the context of quantum scattering, where the refractive index is replaced by a potential function and the wavenumber is the energy level. However, in order to be more definite in our description, we stick to the former case of medium scattering in our subsequent discussion.

We are mainly concerned with the scenario that no wave scattering is generated; that is, the incident wave passes through the medium without being interrupted. In such a case, the incident field is referred to as a non-scattering wave. We aim to geometrically characterize the non-scattering waves associated with a given medium scatterer. The critical observation is that the incident wave interacting with the medium scatterer generates an active source which connects to our previous study on radiationless sources in Section 2. Nevertheless, due to the interaction of the incident wave and the medium scatterer, some new physical phenomena manifest. Mathematically, we also need to introduce technically new ingredients to deal with the new situation. Furthermore, the study in Section 2 enables us to derive a certain elegant geometric structure of the so-called interior transmission eigenfunctions, which is of independent interest in spectral theory. In what follows, we first introduce medium scattering and non-scattering waves, invisibility cloaking and transmission eigenvalue problems. Then we study the geometrical characterization of non-scattering waves and its implication to invisibility cloaking. Finally, we derive the intrinsic geometric structure of the interior transmission eigenfunctions.

### 3.1. Wave scattering from an inhomogeneous medium.

Let $V \in L^\infty(\mathbb{R}^n)$ be a complex-valued function such that $\Im V \geq 0$ and $\text{supp}(V) \subset \Omega$. The function $V$ signifies the index of refraction of an inhomogeneous medium supported in $\Omega$. Let $u^i(x)$ be an incident field which is an entire solution to the Helmholtz equation

$$\Delta u^i + k^2 u^i = 0 \quad \text{in} \quad \mathbb{R}^n. \quad (3.1)$$

For specific examples, one can take the incident field to be a plane wave $u^i(x) = \exp(ik\theta \cdot x)$, where $\theta \in \mathbb{S}^{n-1}$ signifies an incident direction, or a Herglotz wave which is the
superposition of plane waves of the form
\[ u^i(x) = \int_{S^{n-1}} g(\theta) \exp(ik\theta \cdot x) \, d\sigma(\theta), \quad g \in L^2(S^{n-1}). \]

The presence of the inhomogeneity interrupts the propagation of the incident field. Let \( u^s \) signify the perturbation to the incident wave field. It is also called the scattered field. Set \( u = u^i + u^s \) to be the total wave field. Medium scattering is governed by the following Helmholtz system
\[
(\Delta + k^2(1 + V))u = 0, \quad u = u^i + u^s, \quad \text{in } \mathbb{R}^n,
\]
\[
\lim_{r \to \infty} r^{-\frac{n-1}{2}}(\partial_r - ik)u^s = 0.
\]
(3.2)

Here \( u \in H^2_{loc}(\mathbb{R}^n) \) and satisfies the following Lippmann-Schwinger equation,
\[
u = u^i - k^2(\Delta + k^2)^{-1}(Vu),
\]
where the integral operator \((\Delta + k^2)^{-1}\) is defined in (2.3). Similar to (2.5), one can have the far-field pattern of \( u^s \) from (3.3) by the stationary phase approximation,
\[
u^\infty(\hat{x}) = -k^2 C_{n,k} \mathcal{F}(Vu)(\hat{x}) \in L^2(S^{n-1}).
\]
(3.4)

It is noted that in (3.4), \( u \) in the right-hand side is the unknown total wave field, which is in sharp difference to (2.5) for the source scattering and is responsible for the major new technical difficulty of the study in the present section compared to that in Section 2.

Similarly to the scattering by an active source, we are also particularly interested in the case that there is no scattering associated with the configuration consisting of the incident wave \( u^i \) and the inhomogeneous medium \((\Omega, V)\). If this occurs, then \( u^i \) is referred to as a non-scattering incident field. Suppose that \( u^s_\infty \equiv 0 \) and by the Rellich lemma, one immediately has that \( u^s = 0 \) in the unbounded component of \( \mathbb{R}^n \setminus \Omega \). If \( \Omega \) is simply connected, by setting \( w = u^i|_{\Omega} \), one can readily verify that there holds
\[
(\Delta + k^2)w = 0 \quad \text{in } \Omega,
\]
\[
(\Delta + k^2(1 + V))u = 0 \quad \text{in } \Omega,
\]
\[
w, u \in L^2(\Omega), \quad u - w \in H^2_0(\Omega).
\]
(3.5) (3.6) (3.7)

It is pointed out that the last condition means that \( u = w \) and \( \partial_\nu u = \partial_\nu w \) on \( \partial \Omega \), which come from the standard transmission condition on the total wave field \( u \) in (3.2) across \( \partial \Omega \), along with the fact that \( u^s = 0 \) in \( \mathbb{R}^n \setminus \Omega \).

Equations (3.5) to (3.7) are referred to as the interior transmission problem in the literature as discussed in the introduction. If for some \( k \in \mathbb{R}_+ \), there exists a pair of nontrivial solutions to (3.5)–(3.7), then \( k \) is called a transmission eigenvalue and \( w, u \) are said to be the corresponding eigenfunctions. According to our discussion above, we know that if no scattering occurs for the Helmholtz system (3.2), i.e. invisibility, then the restrictions of the total wave \( u \) and the incident wave \( u^i \) form a pair of transmission eigenfunctions with the wavenumber being the transmission eigenvalue. On the other hand, for a pair of transmission eigenfunctions \( w \) and \( u \), it is easily seen that if \( w \) can be (analytically) extended from \( \Omega \) to the whole space \( \mathbb{R}^n \) as an entire solution to the Helmholtz equation (3.1), which is still denoted by \( w \), then \( w \) is a non-scattering incident wave field.

A well-known example is if \( \Omega \) is a central ball and \( V \) is radially symmetric. Then there exist non-scattering incident waves, which in turn, by our discussion above, implies
the existence of transmission eigenfunctions. Because of the radial symmetry they can be analytically extended as entire solutions to the Helmholtz equation. It is widely believed in the literature that in general transmission eigenfunctions cannot be analytically extended to the whole space as an entire solution to the Helmholtz equation associated with a generic \((\Omega, V)\). When none of them can be extended, this means that the inhomogeneous index of refraction scatters nontrivially every incident field.

Per our discussion in Section 1, transmission eigenfunctions cannot be analytically extended across a corner on \(\partial \Omega\). That means, corner singularities scatter every entire incident wave nontrivially \([10, 39]\). In what follows, we shall first establish a result applicable in many more situations by showing that if there is an admissible \(K\)-curvature \(m\) on \(\partial \Omega\), then it must scatter every generic entire incident field nontrivially. Our study follows a similar spirit to that in Section 2 on the source scattering in a localized point on \(K\) applicable in many more situations by showing that if there is an admissible incident wave nontrivially \([10, 39]\). In what follows, we shall first establish a result applicable in many more situations by showing that if there is an admissible \(K\)-curvature \(m\) on \(\partial \Omega\), then it must scatter every generic entire incident field nontrivially. Our study follows a similar spirit to that in Section 2 on the source scattering in a localized and geometerized manner. To that end, we present some preliminary results on the direct scattering problem \((3.2)\).

**Definition 3.1.** Let \(R_m, M, k\in \mathbb{R}_+\) and \(B_{R_m}\subset \mathbb{R}^n\), \(n\geq 2\) be a ball of radius \(R_m\). Let \(V\in L^\infty(B_{R_m})\) be extended by zero to \(\mathbb{R}^n\) and \(\|V\|_{L^\infty}\leq M\). Let \(u^i\) be an incident field satisfying \((3.1)\) and \(\|u^i(x)\|\leq 1\) for any \(x\in \mathbb{R}^n\). Consider the scattering described by \((3.2)\). It is said to be uniformly well-posed if there is such a unique \(u^s\in H^2_{loc}(\mathbb{R}^n)\) and it satisfies \(\|u^s\|_{H^2(B_{R_m})}\leq C\) for some constant \(C = C(R_m, M, k)\), depending only on \(R_m, M\) and \(k\).

Since the Helmholtz system \((3.2)\) is linear, by a scaling argument if necessary, it is convenient for us to assume \(|u^i|\leq 1\) in Definition 3.1 and also in what follows. The Fredholm theorem is applicable to the Lippman-Schwinger equation \((3.3)\) and hence one should always have for \((3.2)\) that

\[\|u^s\|_{H^2(B_{R_m})}\leq C(V, k),\]

where \(C(V, k)\) depend on \(V\) and \(k\). In Definition 3.1, it is required more that the upper bound depends only on \(R_m, \|V\|_{L^\infty}\) and \(k\). This can be regarded as requiring a certain compactness property of the scattering system \((3.2)\) with respect to the refractive index. In principle, this can be fulfilled by imposing a certain restriction on the refractive index, say e.g. it is from a finite-dimensional function space. However, in order to appeal for a more general study, we shall not explore this point in the current study and in what follows if needed, we shall simply assume the uniform well-posedness of the scattering system. Nevertheless, in the next lemma, we present a sufficient condition that can guarantee the uniform well-posedness of the scattering system associated with general refractive indices.

**Lemma 3.2.** Let \(R_m, M, k\in \mathbb{R}_+\), \(V\in L^\infty(B_{R_m})\) and \(u^i\) be as described in Definition 3.1. Consider the scattering system \((3.2)\) associated with this configuration. Then \(C_0 = \|(\Delta + k^2)^{-1}\|_{L^2(B_{R_m})\to L^2(B_{R_m})}\) is independent on \(k\). Moreover if \(k^2\|V\|_{L^\infty}\leq 1/(2C_0)\) then

\[\|u\|_{H^2(B_{R_m})}\leq C(1 + k^2) \quad \text{and} \quad \|u^s\|_{H^2(B_{R_m})}\leq Ck^2(1 + k^2)\|V\|_{L^\infty(B_{R_m})}\]

for some constant \(C = C(R_m, n)\).

**Proof.** By Theorem 8.2 in \([16]\), we know that the integral operator \((\Delta + k^2)^{-1}\) is bounded from \(L^2(B_{R_m})\to H^2(B_{R_m})\) with a norm at most \(C(R_m, n)(1 + k^2) < \infty\), where \(C(R_m, n)\) depends only on \(R_m\) and \(n\). By applying the stationary phase method when \(k\) is large, one
Lemma 2.3 implies that $u \parallel C$. Theorem 8.16 in [28] implies that $u$ and $(\Delta + k^2)u$ estimate for $f$, which is bounded by the a-priori constants according to the first part of the proof. This in turn implies that $f \in C^\alpha(\Omega)$ with a norm bounded by the a-priori constants, because $u = u^i + u^s$ and $u^i \in C^\alpha(\Omega)$ by the assumptions.

We have $(\Delta + k^2)u^s = f$ and $u^s \in H^2_0(\Omega), f \in C^\alpha(\Omega)$. By Proposition 2.7 and the estimate for $f$ we have

$$\sup_{\partial\Omega} |k^2 \varphi u| \leq C(\text{diam}(\Omega))^\alpha$$

3.2. Geometric characterizations of non-scattering incident fields and transmission eigenfunctions. We are now in a position to geometrically characterize non-scattering configurations associated with the medium scattering system (3.2) and the interior transmission eigenfunctions. First, we show that a generic inhomogeneous medium with a sufficiently small support scatters any generic incident field nontrivially. Indeed, we have

Theorem 3.3. Let $D, M_p, M_i, k \in \mathbb{R}_+, 0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded Lipschitz domain of diameter at most $D$ whose complement is connected, and let $V = \chi_\Omega \varphi$ with $\varphi \in C^\alpha(\mathbb{R}^n)$ having $\|\varphi\|_{C^\alpha(\mathbb{R}^n)} \leq M_p$. Let $u^i$ be an entire incident field satisfying $\|u^i\|_{L^\infty(\mathbb{R}^n)} \leq 1$ and $\|u^i\|_{C^{1,\alpha}(\mathbb{R}^n)} \leq M_i$. Assume that the Helmholtz system (3.2) is uniformly well-posed in the sense of Definition 3.1. Then there exists a positive constant $C$, depending only on $D, M_p, M_i$ and $k, n, \alpha$ such that if

$$\sup_{\partial\Omega} |\varphi u^i| > C(\text{diam}(\Omega))^\alpha$$

then $u^s_{\infty} \neq 0$. In other words if $\Omega$ is small compared with the material properties $\varphi$ and the modulus of the incident wave $|u^i|$, then there is scattering.

Proof. We proceed by a reductio ad absurdum. Assume contrarily that $u^s_{\infty} = 0$, and then by the Rellich lemma $u^s = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$. Let $B \supset \overline{\Omega}$ be a ball of diameter $2D$ containing $\Omega$. First, we note that $u$ is also the unique solution to the following PDE system: $v \in H^1(B)$ and $(\Delta + k^2(1 + V) - c)v = -cu$ with $v = u^i$ on $\partial B$ for a suitable large constant $c > 0$. Theorem 8.16 in [28] implies that $u = v \in L^\infty(B)$ with a bound

$$\|u\|_{L^\infty(B)} \leq \|u^i\|_{L^\infty(B)} + C\|u\|_{L^2(B)}$$

for some constant $C = C(n, k, D)$. On the other hand, the uniform well-posedness in Definition 3.1 implies $\|u\|_{L^2(B)} \leq C(D, M_p, k)$. Hence $u \in L^\infty(B)$ with a bound depending only on the a-priori constants.

Let $f = -k^2Vu$. Note that

$$(\Delta + k^2)u^s = f, \quad u^s \in H^2_0(\Omega) \quad \text{and} \quad \|f\|_{L^\infty(B)} \leq C(n, k, D, M_p).$$

Lemma 2.3 implies that $u^s \in C^\alpha(\Omega)$ with a norm bounded by the supremum of $|f|$, which is bounded by the a-priori constants according to the first part of the proof. This in turn implies that $f \in C^\alpha(\Omega)$ with a norm bounded by the a-priori constants, because $u = u^i + u^s$ and $u^i \in C^\alpha(\Omega)$ by the assumptions.

We have $(\Delta + k^2)u^s = f$ and $u^s \in H^2_0(\Omega), f \in C^\alpha(\Omega)$. By Proposition 2.7 and the estimate for $f$ we have

$$\sup_{\partial\Omega} |k^2 \varphi u| \leq C(\text{diam}(\Omega))^\alpha$$
for some finite constant \( C = C(\alpha, D, M_p, M_i, k, n) \). However, it is noted that \( u^* = 0 \) on \( \partial \Omega \). Hence \(|\varphi u^*| \leq C(\text{diam}(\Omega))^\alpha \) on \( \partial \Omega \) where \( C = C/k^2 \). Thus we have reached a contradiction with (3.8) and hence the assumption of \( u^*_\infty = 0 \) is false. The proof is complete. \( \square \)

**Remark 3.4.** Consider a specific and practical case where the incident field \( u^i \) is taken to be a plane wave \( \exp(ik \cdot x) \), \( \theta \in \mathbb{S}^{n-1} \). For such a case, one clearly has \(|u^i| = 1\). According to Theorem 3.3, if an inhomogeneous index of refraction has a positive lower bound on the boundary of its support and its support is sufficiently small, then it scatters every incident plane wave nontrivially; that is, it cannot be identically invisible under the plane wave probing.

Next, we localize and geoniterate the “smallness” result of Theorem 3.3.

**Theorem 3.5.** Let \( L, \delta, D, M_p, M_i, k \in \mathbb{R}_+ \), \( M > 1 \), \( 0 < \alpha < 1 \), \( n \geq 2 \) be the a-priori constants. Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain which has diameter at most \( D \) and let \( p \in \partial \Omega \) be an admissible \( K \)-curvature point with parameters \( L, M, \delta \) and \( K \geq e \), and \( p \) is connected to infinity through \( \mathbb{R}^n \setminus \overline{\Omega} \). Let \( V = \chi_{\Omega} \varphi \) with \( \varphi \in C^0(\mathbb{R}^n) \) and \( |\varphi|_{\text{C}^0(\overline{\Omega})} \leq M_p \). Let \( u^i \) be an entire incident plane wave nontrivially; that is, it cannot be identically invisible under the plane wave probing.

Then \( u^s \neq 0 \).

In other words an admissible high-curvature point on \( \partial \Omega \) scatters every generic incident field nontrivially independent of the other parts of the inhomogeneous index of refraction.

**Proof.** Assume contrarily that \( u^s_\infty = 0 \). Then by the Rellich lemma we know \( u^s = 0 \) in the unbounded component of \( \mathbb{R}^n \setminus \overline{\Omega} \). Let \( B \) be a ball of diameter \( 2D \) that contains \( \overline{\Omega} \). Then \( u^s = \partial_s u^s = 0 \) on its boundary. It is noted that \( u \) is a solution to the following PDE system:

\[
\begin{align*}
\varphi v + (\Delta + k^2(1 + V) - c)v = -cu & \quad \text{in } B \setminus \overline{\Omega}, \\
\partial_n v & = u^i \quad \text{on } \partial B.
\end{align*}
\]

By taking \( c > 0 \) large enough, Theorem 8.16 in [28] gives the unique solvability of the above PDE system, and so \( u = v \), and moreover it gives the integrability \( u \in L^\infty(B) \) with a bound \( \|u\|_{L^\infty(B)} \leq \|u^i\|_{L^\infty(B)} + C\|u\|_{L^2(B)} \) for some constant \( C = C(n, k, D) \). The uniform well-posedness of the scattering problem gives by Definition 3.1 implies \( \|u\|_{L^2(B)} \leq C(D, M_p, k) \). Hence \( u \) is bounded in \( B \) with a bound depending only on the a-priori constants.

Set \( f = -k^2Vu \). Then \( \|f\|_{L^\infty(B)} \leq C(n, k, D, M_p) \) and

\[
(\Delta + k^2)u^s = f, \quad u^s \in H^2_0(B).
\]

Lemma 2.3 implies that \( u^s \in C^0(\overline{B}) \) with a norm bounded by the supremum of \( |f| \), which is bounded by a-priori constants according to the first part of the proof. Since \( u = u^i + u^s \) and \( u^i \) is Hölder-continuous, this further implies that \( f \in C^0(\overline{\Omega}) \) with a norm bounded by the a-priori constants.

Consider the domain \( \Omega \) now, and let \( \tilde{\Omega} \supset \Omega \) be open and simply connected such that \( \partial \tilde{\Omega} \subset \partial \Omega \). We have \( (\Delta + k^2)u^s = f \) and \( u^s \in H^2_0(\overline{\Omega}) \) because \( u^s = 0 \) in the unbounded component of \( \mathbb{R}^n \setminus \overline{\Omega} \). The source \( f \) is Hölder-continuous in \( \Omega \), and hence it is obviously Hölder-continuous in \( \overline{\Omega}_{b,h} \) (cf. Definition 2.8 for the notation used here). By Proposition 2.22 and the estimate for \( f \) we have

\[
|f(p)| \leq C(\ln K)^{(n+3)/2} K^{-\min(\alpha, \delta)/2} (3.10)
\]
for some finite constant $C$ depending on the a-priori parameters. Set $C = C/k^2$ and recall that $f = -k^2V(u^i + u^s)$ with $u^s = 0$ at $x = p$. Thus we have reached a contradiction with (3.9) and so the assumption of $u^s_\infty = 0$ is false. The proof is complete. \hfill \Box

**Corollary 3.6.** Consider the scattering configuration described in Theorem 3.5 and assume that there is no scattering, namely $u^s_\infty \equiv 0$. Then

$$|\varphi(p)||u^i(p)| \leq \psi(K), \quad \psi(K) := C((\ln K)^{(n+3)/2}K^{-\min(\alpha,\delta)/2}, \quad (3.11)$$

where $C$ is a positive constant depending only on the a-priori constants. It can be straightforwardly verified that $\lim_{K \to +\infty} \psi(K) = 0$.

Hence, if the medium’s refractive index is not vanishing at a high-curvature point, then the incident field must be nearly vanishing at the high-curvature point.

The rest of the subsection is devoted to the study of the geometric structures of the interior transmission eigenfunctions in Equations (3.5) to (3.7). Before that, we would like to point out that if $(w,u)$ is a pair of transmission eigenfunctions associated with the eigenvalue $k$, then $(\alpha w,\alpha u)$, $\alpha \in \mathbb{C}\setminus\{0\}$, is obviously a pair of transmission eigenfunctions associated with $k$ as well. Hence, in what follows, we shall always normalize the transmission eigenfunctions in our study.

**Theorem 3.7.** Let $D, k \in \mathbb{R}_+, 0 < \alpha \leq 1, n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain of diameter at most $D$, and $V \in C^\alpha(\Omega)$, $\inf_{\Omega} |V| > 0$. Suppose that $k$ is an interior transmission eigenvalue and $u, w \in L^2(\Omega)$ is a pair of transmission eigenfunctions associated with $k$. If $u \in C^\alpha(\Omega)$ and $\|u\|_{C^\alpha(\Omega)} = 1$, then there holds

$$\sup_{\partial\Omega} |u| \leq C((\ln(\Omega))^{n/2}\|V\|_{C^\alpha(\Omega)}/\inf_{\partial\Omega} |V|, \quad (3.12)$$

where $C$ is a positive constant depending only on $\alpha, D, k$ and $n$.

**Proof.** Let $f = -k^2Vu$. Then $((\Delta + k^2)(u - w) = f$ in $\Omega$, $u - w \in H^2_0(\Omega)$ and $\|f\|_{C^\alpha} \leq k^2\|V\|_{C^\alpha}\|u\|_{C^\alpha}$. Proposition 2.7 implies that $\sup_{\partial\Omega} |f| \leq C(\ln(\Omega))^{\alpha}\|f\|_{C^\alpha}$ for some constant $C = C(\alpha, D, k, n)$. The claim follows after dividing by $k^2\inf_{\partial\Omega} |V|$. \hfill \Box

Equation (3.12) establishes the relationship among the value of the transmission eigenfunction, the diameter of the domain and the underlying refractive index. It indicates that if the domain is sufficiently small, then the transmission eigenfunction is nearly vanishing.

The following theorem localizes and geometrizes the “smallness” result of Theorem 3.7.

**Theorem 3.8.** Let $L, \delta, D, k \in \mathbb{R}_+, M > 1, 0 < \alpha < 1, n \geq 2$ be the a-priori constants. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain which has a diameter at most $D$. Assume that $p \in \partial\Omega$ is an admissible $K$-curvature point with parameters $L, M, \delta$ and $K \geq e$, and let $V \in C^\alpha(\Omega)$. Suppose that $k$ is an interior transmission eigenvalue and $u, w \in L^2(\Omega)$ is a pair of transmission eigenfunctions associated with $k$. If $u \in C^\alpha(\Omega)$ with $\|u\|_{C^\alpha(\Omega)} = 1$, then $\Omega_{b,h}$ is defined in Definition 2.8 associated with the point $p$, then there holds

$$|u(p)| \leq C((\ln K)^{(n+3)/2}k^{-\min(\alpha,\delta)/2}\|V\|_{C^\alpha}/|V(p)| \quad (3.13)$$

where $C$ is a positive constant depending only on the a-priori constants.

**Proof.** Set $f = -k^2Vu$ and note that $\|f\|_{C^\alpha(\Omega)} \leq k^2\|V\|_{C^\alpha(\Omega)}\|u\|_{C^\alpha(\Omega)}$. Then $((\Delta + k^2)(u - w) = f = u - w \in H^2_0(\Omega)$. Proposition 2.22 immediately yields that

$$|f(p)| \leq C\|V\|_{C^\alpha}(\ln K)^{(n+3)/2}k^{-\min(\alpha,\delta)/2},$$
for some constant $C$ depending on the a-priori parameters. The claim follows after dividing by $k^2 |V(p)|$. \hfill \Box

3.3. Implications to invisibility cloaking. Finally, we discuss briefly some interesting implications of our results established in the present section to invisibility cloaking. Per our discussion in the introduction, a cloaking device is certain stealth technology that make an object invisible with respect to certain wave measurements. To ease our discussion, let us consider the probing/incident fields to be plane waves which are nonvanishing everywhere in the space and have modulus 1. By the “local” result in Theorem 3.5, one concludes that the shape of a cloaking device cannot be curved severely since the high-curvature part can cause significant scattering, which in turn can make the device more “visible”. Moreover, in our earlier work [5] it is shown that corner singularities on the support of a scatterer can also cause significant scattering. These results suggest that a practical cloaking device should possess a smooth and round shape.

On the other hand, if an object possesses a corner part or a highly-curved part, does it mean that it is easier for detecting? The answer is yes. Indeed, the geometric structure of the transmission eigenfunctions derived in Theorem 3.8 can fulfil this detecting purpose. In fact, there is algorithmic development in [53] on the construction of the interior transmission eigenfunctions associated with an inhomogeneous medium through the corresponding far-field patterns. Hence, with the measurement of the far-field data, one can first derive the corresponding interior transmission eigenfunctions, then the highly-curved part of the a scatterer can be detected as the place where the transmission eigenfunction is nearly vanishing according to Theorem 3.8. Indeed, this is the core of the detecting algorithm proposed in [53] where it made use of the geometric structure of transmission eigenfunctions near corners derived in our work [7]. Clearly, with the novel geometric property derived in Theorem 3.8, the method in [53] can be equally extended to detecting the highly-curved part of an inhomogeneous scatterer.

4. Uniqueness results for inverse scattering problems

In this section, we consider the application of the results established so far in the current article to the inverse scattering problem. The inverse problem associated with the medium scattering system (3.2) can be described as identifying $(\Omega, V)$ by knowledge of the corresponding far-field pattern $u^s_\infty(\hat{x}; u^i)$. By introducing an abstract operator $\mathcal{S}$ defined via (3.2) that sends the scatterer $(\Omega, V)$ to the corresponding far-field pattern, the inverse problem can formulated as the following nonlinear equation,

$$\mathcal{S}(\Omega, V) = u^s_\infty(\hat{x}; u^i). \quad (4.1)$$

We are particularly interested in a single far-field measurement for the inverse problem (4.1) by recovering $\Omega$, independent of the medium content $V$. By a single far-field measurement, we mean that the far-field data are collected in all the observation direction $\hat{x} \in \mathbb{S}^{n-1}$ but corresponding to a single incident wave field $u^i$. It can be easily verified that the inverse problem is formally determined in such a case. We are mainly concerned with the uniqueness issue. That is, the sufficient condition to guarantee that for two scatterers $(\Omega, V)$ and $(\Omega', V')$, $\mathcal{S}(\Omega, V) = \mathcal{S}(\Omega', V')$ if and only if $(\Omega, V) \equiv (\Omega', V')$. This problem is underdetermined for a single measurement, but solvable for infinitely many incident-wave–far-field pairs [73]. But for the inverse shape problem described above, the uniqueness issue can be stated as proving $\Omega \equiv \Omega'$ if $\mathcal{S}(\Omega, V) = \mathcal{S}(\Omega', V')$ without
Lemma 3.2, one can deduce that there are constants and let \( C \) for some positive constant \( C \). Knowing Theorem 4.1.

On the other hand, by the Sobolev embeddings in two and three dimensions and Lemma 3.2, respectively, signify the far-field patterns associated with the scatterers \((\Omega, V)\) and \((\Omega', V')\), defined through the scattering system (3.2)-(3.4) with the incident field being \( u^i \). Then there exist two positive constants \( C_1 \) and \( C_2 \), depending only on the a-priori constants such that if \( u_{s\infty} = u'^{s\infty} \) then \( \Omega \) cannot have a component \( U \) with \( \overline{U} \cap \overline{\Omega} = \emptyset \) whose boundary points can be connected to infinity through \( \mathbb{R}^n \setminus \overline{\Omega} \cup \overline{\Omega'} \) and \( \text{diam}(U) < C_1 \) if \( k \leq C_2 \).

Proof. Let \( u \) and \( u' \), respectively, signify the total wave fields associated with \( u_{s\infty} \) and \( u'^{s\infty} \). Set \( w = u - u' \). Then \( u_{s\infty} = 0 \) and \((\Delta + k^2)w = 0\) in \( \mathbb{R}^n \setminus \overline{\Omega} \cup \overline{\Omega'} \), and so \( w = 0 \) in the unbounded component of that same set by the Rellich lemma. Assume contrarily that such an open set \( U \) would exist. Because it is a component of \( \Omega \), and does not touch \( \Omega' \), it must also be a component of \( \Omega \cup \Omega' \). Because the complements of \( \Omega \) and \( \Omega' \) are connected, \( U \) is simply connected. Moreover it can be connected to infinity through \( \mathbb{R}^n \setminus \overline{\Omega} \cup \overline{\Omega'} \). This implies that \( w \in H_0^2(U) \) and \((\Delta + k^2)w = -k^2 \varphi u \) in \( U \). Thus Proposition 2.7 implies

\[
\sup_{\partial U} |\varphi u| \leq C \left( \text{diam}(U) \right)^{\alpha} \| \varphi \|_{C^\alpha(\partial U)} \| u \|_{C^\alpha(\partial U)}
\]

for some positive constant \( C = C(\alpha, R_m, k) \).

On the other hand, by the Sobolev embeddings in two and three dimensions and Lemma 3.2, one can deduce that there are constants \( C(M_p) \) and \( C(k, R_m) \) such that \( \|u\|_{C^\alpha(\overline{B}_{R_m})} \leq C(k, R_m) \) when \( k^2 \leq C(M_p) \). This and the lemma further imply that there is a sufficiently small \( C_2 = C_2(R_m, M_p, m_i) \) such that if \( k \leq C_2 \) then \( |u^s| \leq m_i/2 \) in \( B_{R_m} \). Therefore, one has \( |u| \geq |u^i| - |u^s| \geq m_i/2 \) in \( B_{R_m} \). By applying the above estimates and the lower bound \( m_p \) on \( \varphi \) to (4.3), one readily has

\[
m_p m_i/2 \leq C \left( \text{diam}(U) \right)^{\alpha} M_p C(k, R_m)
\]

which is impossible when \( \text{diam}(U) \) is smaller than some constant depending only on the a-priori parameters \( k, R_m, \alpha, M_p, m_p, m_i \). This contradiction immediately yields that \( (\Omega, V) \) and \( (\Omega', V') \) must produce different far-field patterns. The proof is complete.

Corollary 4.2. Assume the situation of Theorem 4.1, but with \( \text{diam}(\Omega), \text{diam}(\Omega') < C_1 \) and \( k \leq C_2 \). If \( \Omega \cap \Omega' = \emptyset \) then \( u_{s\infty} \neq u'^{s\infty} \).

Corollary 4.3. Under the situation of Theorem 4.1, assume further that \( \Omega, \Omega' \) are well-separated collections of small scatterers, namely

\[
\Omega = \bigcup_{j=1}^M \Omega_j, \quad \Omega' = \bigcup_{l=1}^N \Omega'_l
\]

where \( \Omega_j, \Omega'_l \) each have a diameter at most \( C_1 \), and \( d(\Omega_{j_1}, \Omega_{j_2}), d(\Omega'_{l_1}, \Omega'_{l_2}) > 2C_1 \) for \( j_1 \neq j_2, l_1 \neq l_2 \). Then if \( u_{s\infty} = u'^{s\infty} \) we have \( M = N \), and under a re-indexing \( \Omega_j \cap \Omega'_j \neq \emptyset \) for \( j = 1, \ldots, M \).
Proof. Since the components are well-separated and small, a component of \( \Omega' \) can only intersect at most one component of \( \Omega \). If \( M > N \) then \( \Omega \) has more components than \( \Omega' \), and so one of them has a positive distance from \( \Omega' \) too. Then by using Theorem 4.1 one can arrive at the conclusion. \( \square \)

Remark 4.4. Corollary 4.2 basically indicates that if two scatterers are of sufficiently small sizes (might be with different medium contents) and produce the same far-field pattern, then they must be very close to each other in the sense that their supports must have a nonempty intersection.

Remark 4.5. On the other hand, Corollary 4.3 implies that the exact number and approximate locations of well-separated scatterers are uniquely determined by a single far-field measurement, a question studied in [36]. The above proof works also for the inverse source problem in which case there is no requirement on having a low wavenumber. This gives a formal proof for the numerical results in [34,35].

**Theorem 4.6.** Let \( L, \delta, R_{\alpha}, k, m_i, M_p, m_p \in \mathbb{R}_+ \), \( M > 1 \), \( 0 < \alpha \leq 1/2 \) be the a-priori constants. Let \( \Omega, \Omega' \subset B_{R_{\alpha}} \subset \mathbb{R}^n \), \( n \in \{2,3\} \) be bounded domains with connected complements, and let \( V = \chi_{\Omega} \varphi, V' = \chi_{\Omega'} \varphi' \) with \( \varphi, \varphi' \in C^0(\overline{\Omega}) \) such that
\[
|\varphi|, |\varphi'|, \|\varphi\|_{C^0(\overline{\Omega})}, \|\varphi'\|_{C^0(\overline{\Omega})} \leq M_p. \tag{4.5}
\]
Let \( u \) be an entire incident field satisfying (3.1) and \( m_i \leq |u| \leq 1 \). Let \( u^s_\infty \) and \( u^s \), respectively, signify the far-field patterns associated with the scatterers \( (\Omega, V) \) and \( (\Omega', V') \), defined through the scattering system (3.2)-(3.4) with the incident field being \( u \). Then there exist two positive constants \( C_1 \) and \( C_2 \), depending only on the a-priori constants such that if \( u^s_\infty = u^s \), \( k < C_2 \), then \( \Omega \setminus \Omega' \) cannot have an admissible \( K \)-curvature point \( p \) with parameters \( L, M, \delta \) and \( K > C_1 \), and satisfying \( d(p, \Omega') < \sqrt{1 + M/K} \) and connected to infinity through \( \mathbb{R}^n \setminus (\Omega \cup \Omega') \).

Proof. Let \( u \) and \( u' \), respectively, signify the total wave fields associated with \( u^s_\infty \) and \( u^s \), and \( w = u - u' \). One has that \( (\Delta + k^2)w = 0 \) in \( \mathbb{R}^n \setminus (\Omega \cup \Omega') \) and thus by the Rellich lemma \( w = 0 \) in \( \Sigma \), where \( \Sigma \) is the unbounded connected component of \( \mathbb{R}^n \setminus (\Omega \cup \Omega') \). Set \( U = \mathbb{R}^n \setminus \Sigma \), then clearly \( U \supset \Omega \cap \Omega' \). One easily sees that \( w \in H^2_0(U) \) and \( (\Delta + k^2)w = f \) in \( U \) where
\[
f = -k^2 \chi_\Omega \varphi u - k^2 \chi_{\Omega'} \varphi' u'. \tag{4.6}
\]
Let \( p \in \partial \Omega \setminus \Omega' \) be the admissible \( K \)-curvature point as stated in the theorem and consider the set \( \Omega_{b,h} \) associated with \( p \) as specified in Definition 2.8. Because \( \Omega' \) is of a distance \( \sqrt{1 + M/K} \) from \( p \), it does not intersect the rectangular neighbourhood \( B(0,b) \times (-h,h) \) that is used to define \( \Omega_{b,h} \). Since \( p \) can be joined to infinity without passing through \( \Omega \cup \Omega' \), we have \( p \in \partial U \) and actually \( \Omega_{b,h} = U_{b,h} \). Moreover, we have \( \int_{U_{b,h}} f = -k^2 \varphi u \).

Let us estimate \( \|f\|_{C^0(\overline{U_{b,h}})} \). As in the proof of Theorem 4.1, the Sobolev embeddings in two and three dimensions and Lemma 3.2 imply that there exist constants \( C(k, R_m) \) and \( C(M_p) \) such that \( \|u\|_{C^0(\overline{B(R_m)})} \leq C(k, R_m) \) when \( k^2 \leq C(M_p) \). Hence there holds
\[
\|f\|_{C^0(\overline{U_{b,h}})} \leq C(k, M_p, R_m). \tag{4.7}
\]
By (4.7), one can apply Proposition 2.22 to have
\[
|f(p)| \leq \mathcal{E}(\ln K)^{(n+3)/2} K^{-\min(\alpha, \delta)/2}. \tag{4.8}
\]
for some constant $E = E(\alpha, \delta, n, R_m, L, M, k, M_p) \in \mathbb{R}_+$. On the other hand, Lemma 3.2 implies that there is $C_2 = C_2(R_m, M_p, m_i)$ such that if $k \leq C_2$ then $\|u^s\|_{L^\infty(B_{R_m})} \leq m_i/2$. This shows that the total wave field $u$ does not vanish since one clearly has $|u| \geq |u^i| - |u^s| \geq m_i/2$. Moreover there is the lower bound $m_p$ on $\varphi$. By combining the above estimates into (4.6) and (4.7), one finally arrives at

$$k^2 m_i m_p / 2 \leq E(\ln K)^{(n+3)/2} K^{-\min(\alpha,\delta)/2}$$

which is impossible when $K$ is sufficiently small. This contradiction immediately yields that $\Omega \setminus \Omega'$ cannot have an admissible $K$-curvature point as stated in the theorem. The proof is complete. \hfill \Box

Theorem 4.6 states a local uniqueness result for the inverse shape problem which basically indicates that if two scatterers produce the same far-field pattern associated with a single incident wave, then the difference of the two scatterers cannot have a high-curvature point. On the other hand, if there is sufficient a-priori knowledge about the shape of the underlying scatterer, one can also obtain global uniqueness. As an illustrative example, one may consider an equilateral triangle in $\mathbb{R}^2$ with the three vertices being locally mollified to be admissible $K$-curvature points with sufficiently large $K$. Clearly, if two such kind of scatterers produce the same far-field pattern, then by Theorem 4.6 they are approximately the same in the sense that their corresponding mollified vertices must be around distance $K^{-1}$ from each other, respectively. Otherwise the difference of the two scatterers would possess a high-curvature point.

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