$T = 0$ Partition Functions for Potts Antiferromagnets on Lattice Strips with Fully Periodic Boundary Conditions

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Abstract

We present exact calculations of the zero-temperature partition function for the $q$-state Potts antiferromagnet (equivalently, the chromatic polynomial) for families of arbitrarily long strip graphs of the square and triangular lattices with width $L_y = 4$ and boundary conditions that are doubly periodic or doubly periodic with reversed orientation (i.e. of torus or Klein bottle type). These boundary conditions have the advantage of removing edge effects. In the limit of infinite length, we calculate the exponent of the entropy, $W(q)$ and determine the continuous locus $B$ where it is singular. We also give results for toroidal strips involving "crossing subgraphs"; these make possible a unified treatment of torus and Klein bottle boundary conditions and enable us to prove that for a given strip, the locus $B$ is the same for these boundary conditions.

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1 Introduction

The \(q\)-state Potts antiferromagnet (AF) \([1, 2]\) exhibits nonzero ground state entropy, \(S_0 > 0\) (without frustration) for sufficiently large \(q\) on a given lattice \(\Lambda\) or, more generally, on a graph \(G = (V, E)\) defined by its set of vertices \(V\) and edges joining these vertices \(E\). This is equivalent to a ground state degeneracy per site \(W > 1\), since \(S_0 = k_B \ln W\). Such nonzero ground state entropy is important as an exception to the third law of thermodynamics \([4]\).

There is a close connection with graph theory here, since the zero-temperature partition function of the above-mentioned \(q\)-state Potts antiferromagnet on a graph \(G\) satisfies

\[
Z(G, q, T = 0)_{PAF} = P(G, q)
\]

where \(P(G, q)\) is the chromatic polynomial expressing the number of ways of coloring the vertices of the graph \(G\) with \(q\) colors such that no two adjacent vertices have the same color (for reviews, see \([5] - [7]\)). The minimum number of colors necessary for such a coloring of \(G\) is called the chromatic number, \(\chi(G)\). Thus

\[
W(\{G\}, q) = \lim_{n \to \infty} P(G, q)^{1/n}
\]

where \(n = |V|\) is the number of vertices of \(G\) and \(\{G\} = \lim_{n \to \infty} G\). At certain special points \(q_s\) (typically \(q_s = 0, 1, ..., \chi(G)\)), one has the noncommutativity of limits

\[
\lim_{q \to q_s} \lim_{n \to \infty} P(G, q)^{1/n} \neq \lim_{n \to \infty} \lim_{q \to q_s} P(G, q)^{1/n}
\]

and hence it is necessary to specify the order of the limits in the definition of \(W(\{G\}, q_s)\) \([8]\). Denoting \(W_{qn}\) and \(W_{nq}\) as the functions defined by the different order of limits on the left and right-hand sides of \((1.3)\), we take \(W \equiv W_{qn}\) here; this has the advantage of removing certain isolated discontinuities that are present in \(W_{nq}\).

Using the expression for \(P(G, q)\), one can generalize \(q\) from \(\mathbb{Z}_+\) to \(\mathbb{C}\). The zeros of \(P(G, q)\) in the complex \(q\) plane are called chromatic zeros; a subset of these may form an accumulation set in the \(n \to \infty\) limit, denoted \(\mathcal{B} [9]\), which is the continuous locus of points where \(W(\{G\}, q)\) is nonanalytic. \([9]\) The maximal region in the complex \(q\) plane to which one can analytically continue the function \(W(\{G\}, q)\) from physical values where there is nonzero ground state entropy is denoted \(R_1\). The maximal value of \(q\) where \(\mathcal{B}\) intersects the (positive) real axis is labelled \(q_c(\{G\})\). This point is important since it separates the interval \(q > q_c(\{G\})\) on the positive real \(q\) axis where the Potts model (with \(q\) extended from \(\mathbb{Z}_+\) to \(\mathbb{R}\)) exhibits nonzero ground state entropy (which increases with \(q\), asymptotically approaching

\[1\]For some families of graphs \(\mathcal{B}\) may be null, and \(W\) may also be nonanalytic at certain discrete points.
$S_0 = k_B \ln q$ for large $q$, and which for a regular lattice $\Lambda$ can be calculated approximately via large-$q$ series expansions) from the interval $0 \leq q \leq q_c(\{G\})$ in which $S_0$ has a different analytic form. Early calculations of chromatic polynomials for $L_y = 2$ strips of the square lattice with periodic longitudinal boundary conditions were performed in \cite{10} (see also the related works \cite{11}-\cite{15}).

Here we present exact calculations of the chromatic polynomials for strips of the square and triangular lattice with transverse width $L_y = 4$ (i.e. transverse cross sections forming squares) and arbitrarily great length $L_x$ with the following boundary conditions: (i) $(PBC_y, PBC_x) = \text{toroidal}$, and (ii) $(PBC_y, TPBC_x) = \text{Klein bottle}$, where $PBC_i$ denotes periodic boundary conditions in the $i$'th direction and $TPBC_x$ denotes periodic longitudinal boundary conditions with an orientation-reversal (twist)\footnote{The boundary conditions $(PBC_y, PBC_x)$ and $(PBC_y, TPBC_x)$ can be implemented in a manner that is uniform in the length $L_x$; as noted before \cite{10}, the boundary conditions $(TPBC_y, PBC_x)$ (different type of Klein bottle) and $(TPBC_y, TPBC_x)$ (projective plane) require different identifications as $L_x$ varies and will not be considered here.} These extend our previous calculations of chromatic polynomials for width $L_y = 3$ on the square \cite{16} and triangular \cite{17} lattices with torus and Klein bottle boundary conditions.

A major motivation for using boundary conditions that are fully periodic or fully periodic with reversed orientation (here, toroidal and Klein bottle) is the well-known fact that if one imposes periodic boundary conditions in a certain direction, this removes edge effects in that direction. Clearly the most complete removal of such edge effects is achieved if one imposes fully periodic boundary conditions (including the possibility of orientation reversal). This also has an important related consequence pertaining to the uniformity of the lattice. To discuss this, we first recall two definitions from mathematical graph theory. The degree $\Delta$ of a vertex of a graph is the number of edges connected to it. A $\Delta$-regular graph is a graph in which all vertices have the same degree, $\Delta$. An infinite regular lattice has the property that each vertex (site) on the lattice has the same degree, i.e., coordination number. For the two types of lattices considered here, namely square and triangular, the coordination number is 4 and 6, respectively. It is advantageous to deal with finite sections of regular lattices having boundary conditions that preserve the $\Delta$-regular property of the infinite lattice. Fully periodic periodic boundary conditions, and the reversed-orientation periodic boundary conditions considered here, have the merit of preserving this property of $\Delta$-regularity; in contrast, this is not the case if one uses boundary conditions that are free in one or more directions. In previous studies with families of lattice strip graphs of arbitrarily great length with periodic or reversed-orientation periodic longitudinal boundary conditions and free transverse boundary conditions (i.e., cyclic or Möbius strips), it was shown that, in the
$L_x \to \infty$ limit, the resultant locus $\mathcal{B}$ exhibits, for finite width $L_y$, a number of properties expected to hold for the locus $\mathcal{B}$ on the infinite 2D lattice, including (i) passing through $q = 0$, (ii) passing through $q = 2$, (iii) passing through a maximal real point, $q_c$, and (iv) enclosing one or more regions including the interval $0 < q < q_c$. In contrast, if one uses free longitudinal boundary conditions, it was found that properties (i) and (iv) do not hold, and properties (ii) and (iii) do not, in general, hold; rather, one anticipates that these would be approached in the limit $L_y \to \infty$. It was thus inferred that the key condition to guarantee that these properties hold is the presence of periodic (or reversed-orientation periodic) longitudinal boundary conditions. This thus provides a third motivation for calculations with doubly periodic boundary conditions, since one expects that the resultant loci $\mathcal{B}$ will exhibit the features (i)-(iv) already for finite $L_y$, and this was confirmed by the study of $L_y = 3$ strips of the square and triangular lattices. As will be seen, our exact results for $L_y = 4$ again support this inference. A fourth motivation for this study is that, as was shown in the earlier calculations of chromatic polynomials for strips of the square and triangular lattices with width $L_y = 3$, the use of Klein bottle, as opposed to torus, boundary conditions has the effect of simplifying the structure of the resultant chromatic polynomial. This thus elucidates the effect of the topology of the surface on which the family of strip graphs is embedded with the structure of the chromatic polynomial. In addition to those listed, some previous related calculations of chromatic polynomials for families of graphs with periodic longitudinal boundary conditions are in Refs. [10]-[26].

In general, the $L_y \times L_x$ strips of the square and triangular lattice have $n = |V| = L_y L_x$ vertices and, for the number of edges $|E| = (\Delta/2)n$ the values $|E| = 2n$ and $|E| = 3n$ respectively. (For $L_x = 2$, some of these strip graphs involve multiple edges joining pairs of vertices and hence are multigraphs rather than proper graphs; we shall be interested primarily in the cases $L_x \geq 3$ where there are no multiple edges.)

We label a particular type of strip graph as $G_s$ or just $G$ and the specific graph of width $L_y$ and length $L_x$ vertices as $(G_s, L_y \times L_x, BC_y, BC_x)$. A generic form for chromatic polynomials for recursively defined families of graphs, of which strip graphs $G_s$ are special cases, is

$$P(G_s, L_y \times L_x, BC_y, BC_x, q) = \sum_{j=1}^{N_{G_s,\lambda}} c_{G_s,j}(q)(\lambda_{G_s,j}(q))^m$$  \hspace{1cm} (1.4)

where $c_{G_s,j}(q)$ and the $N_{G_s,\lambda}$ terms $\lambda_{G_s,j}(q)$ depend on the type of strip graph $G_s$ but are independent of $m$. The $\lambda_{G_s,j}$ are the (nonzero) eigenvalues of the coloring matrix. We shall denote the total number of different eigenvalues of the coloring matrix for a recursive family of graphs $G_s$ as $N_{G_s,\lambda,tot}$. Clearly $N_{G_s,\lambda,tot} = N_{G_s,\lambda}$ if there is no zero eigenvalue, and
For a given type of strip graph $G_s$, we denote the sum of the coefficients $c_{G_s,j}$ as

$$C_{G_s} \equiv C(G_s) = \sum_{j=1}^{N_{G_s,\lambda}} c_{G_s,j}.$$  \hspace{1cm} (1.5)

According to a general theorem, for a strip $G_s$ of the square or triangular lattice with torus boundary conditions $[21, 26]$, \hspace{1cm} (1.6)

$$C(G_s, L_y \times L_x, PBC_y, PBC_x) = P(C_{Ly}, q), \hspace{0.5cm} G_s = sq, tri$$

where $C_n$ denotes the circuit graph with $n$ vertices and $P(C_n, q) = (q - 1)^n + (q - 1)(-1)^n$. Further, for a strip of the square or triangular lattice with Klein bottle boundary conditions $[26]$

$$C(G_s, L_y \times L_x, PBC_y, TPBC_x) = 0, \hspace{0.5cm} G_s = sq, tri.$$  \hspace{1cm} (1.7)

2 \hspace{0.5cm} \textbf{L_y = 4 Strip of the Square Lattice with (PBC}_y, PBC_x)$$\textbf{)}$

In general, for a strip of the square lattice of size $L_y \times L_x$ with (PBC$_y$, PBC$_x$), i.e., toroidal boundary conditions, for $L_y \geq 2$ and $L_x \geq 2$, the chromatic number is given by

$$\chi(sq, L_y \times L_x, PBC_y, PBC_x) = \begin{cases} 2 & \text{if } L_y \text{ is even and } L_x \text{ is even} \\ 3 & \text{otherwise} \end{cases}$$  \hspace{1cm} (2.1)

Thus, in the present case with $L_y = 4$, it follows that $\chi = 2$ for even $L_x$ and $\chi = 3$ for odd $L_x$. We calculate the chromatic polynomial $P$ by a systematic, iterative use of the deletion-contraction theorems as in our earlier work $[21, 24]$ and a coloring matrix method $[12]$. For the $L_y = 4$ strip graphs of the square lattice with torus boundary conditions (labelled st4), we find $N_{st4,\lambda} = 33$ and

$$P(sq, 4 \times L_x, PBC_y, PBC_x, q) = \sum_{j=1}^{33} c_{st4,j} (\lambda_{st4,j})^{L_x}$$  \hspace{1cm} (2.2)

where

$$\lambda_{st4,1} = 1$$ \hspace{1cm} (2.3)

$$\lambda_{st4,2} = 1 - q$$ \hspace{1cm} (2.4)

$$\lambda_{st4,3} = 2 - q$$ \hspace{1cm} (2.5)

$$\lambda_{st4,4} = 3 - q$$ \hspace{1cm} (2.6)
The remaining twelve $\lambda_{st4,j}$’s for $22 \leq j \leq 33$ are roots of four cubic equations,

$$\xi^3 + (q^3 - 6q^2 + 16q - 14)\xi^2 - (q - 1)(q^4 - 9q^3 + 31q^2 - 55q + 43)\xi$$

$$-(q - 3)(q - 1)^2(q^3 - 6q^2 + 12q - 10) = 0$$

with roots $\lambda_{st4,j}$ for $j = 22, 23, 24,$

$$\xi^3 + (q - 4)(q^2 - 6q + 12)\xi^2 - (q - 3)(q^4 - 11q^3 + 45q^2 - 81q + 59)\xi$$

$$-(q^6 - 15q^5 + 91q^4 - 285q^3 + 488q^2 - 442q + 170) = 0$$
with roots $\lambda_{st4,j}$ for $j = 25, 26, 27$,

$$
\xi^3 - 2(q^2 - 6q + 12)\xi^2 + (q^4 - 13q^3 + 59q^2 - 113q + 83)\xi
$$

$$
+ (q^5 - 13q^4 + 62q^3 - 135q^2 + 141q - 60) = 0
$$

(2.20)

with roots $\lambda_{st4,j}$ for $j = 28, 29, 30$, and

$$
\xi^3 - 2(q^2 - 6q + 10)\xi^2 + (q^4 - 13q^3 + 59q^2 - 113q + 75)\xi
$$

$$
+ (q^5 - 13q^4 + 64q^3 - 149q^2 + 167q - 72) = 0
$$

(2.21)

with roots $\lambda_{st4,j}$ for $j = 31, 32, 33$.

The corresponding coefficients are

$$
c_{st4,1} = q^4 - 8q^3 + 20q^2 - 15q + 1
$$

(2.22)

$$
c_{st4,2} = \frac{1}{2}c_{st4,4} = c_{st4,6} = \frac{1}{3}(q - 1)(q^2 - 5q + 3)
$$

(2.23)

$$
c_{st4,3} = \frac{1}{6}(q - 2)(4q^2 - 13q - 3)
$$

(2.24)

$$
c_{st4,j} = \frac{2}{3}q(q - 2)(q - 4) \text{ for } j = 5, 20, 21
$$

(2.25)

$$
c_{st4,j} = \frac{1}{2}(q - 1)(q - 2) \text{ for } j = 7, 18, 19
$$

(2.26)

$$
c_{st4,j} = (q - 1)(q - 2) \text{ for } j = 31, 32, 33
$$

(2.27)

$$
c_{st4,j} = \frac{1}{2}q(q - 3) \text{ for } j = 8, 14, 15, 28, 29, 30
$$

(2.28)

$$
c_{st4,j} = q(q - 3) \text{ for } j = 16, 17
$$

(2.29)

$$
c_{st4,j} = 1 \text{ for } j = 9, 10, 11
$$

(2.30)

$$
c_{st4,j} = 2(q - 1) \text{ for } j = 12, 13
$$

(2.31)

$$
c_{st4,j} = q - 1 \text{ for } 22 \leq j \leq 27.
$$

(2.32)

The sum of these coefficients is equal to $P(C_4, q) = q(q - 1)(q^2 - 3q + 3)$, as dictated by the $L_y = 4$ special case of our general result (1.6).

The singular locus $B$ for the $L_x \to \infty$ limit of the strip of the square lattice with $L_y = 4$ and toroidal boundary conditions is shown in Fig. [1]. For comparison, chromatic zeros are
Figure 1: Singular locus $\mathcal{B}$ for the $L_x \to \infty$ limit of the strip of the square lattice with $L_y = 4$ and toroidal boundary conditions. For comparison, chromatic zeros are shown for $L_x = 30$ (i.e., $n = 120$).
calculated and shown for length $L_x = 30$ (i.e., $n = 120$ vertices). The locus $B$ crosses the real axis at the points $q = 0$, $q = 2$, and at the maximal point $q = q_c$, where

$$q_c = 2.7827657... \quad \text{for} \quad \{G\} = (sq, 4 \times \infty, PBC_y, PBC_x). \quad (2.33)$$

As is evident from Fig. 1, the locus $B$ separates the $q$ plane into different regions including the following: (i) $R_1$, containing the semi-infinite intervals $q > q_c$ and $q < 0$ on the real axis and extending outward to infinite $|q|$; (ii) $R_2$ containing the interval $2 < q < q_c$; (iii) $R_3$ containing the real interval $0 < q < 2$; and (iv) the complex-conjugate pair $R_4, R_4^*$ centered approximately at $q = 2.9 \pm 1.3i$. The (nonzero) density of chromatic zeros has the smallest values on the curve separating regions $R_1$ and $R_3$ in the vicinity of the point $q = 0$ and on the curve separating regions $R_2$ and $R_3$ in the vicinity of the point $q = 2$.

In region $R_1$, $\lambda_{st,10}$ is the dominant $\lambda_{G,j}$, so

$$W = (\lambda_{st,10})^{1/4}, \quad q \in R_1. \quad (2.34)$$

This is the same as $W$ for the corresponding $L_x \to \infty$ limit of the strip of the square lattice with the same width $L_y = 4$ and cylindrical ($PBC_y, FBC_x$) boundary conditions, calculated in [28]. This equality of the $W$ functions for the $L_x \to \infty$ limit of two strips of a given lattice with the same transverse boundary conditions and different longitudinal boundary conditions in the more restrictive region $R_1$ defined by the two boundary conditions is a general result [20, 22].

In region $R_2$, the largest root of the cubic equation (2.20) is dominant; we label this as $\lambda_{st,28}$ so that

$$|W| = |\lambda_{st,28}|^{1/4}, \quad q \in R_2 \quad (2.35)$$

(in regions other than $R_1$, only $|W|$ can be determined unambiguously [8]). Thus, $q_c$ is the relevant solution of the equation of degeneracy in magnitude $|\lambda_{st,10}| = |\lambda_{st,28}|$. In region $R_3$,

$$|W| = |\lambda_{st,26}|^{1/4}, \quad q \in R_3. \quad (2.36)$$

In regions $R_4, R_4^*$,

$$|W| = |\lambda_{st,22}|^{1/4}, \quad q \in R_4, \quad R_4^*. \quad (2.37)$$

In [25] we have listed values of $W$ for a range of values of $q$ for the $L_x \to \infty$ limit of various strips of the square lattice, including $(sq, 4 \times \infty, PBC_y, FBC_x)$. Since $W$ is independent of $BC_x$ for $q$ in the more restrictive region $R_1$ defined by $FBC_x$ and $(T)PBC_x$ (which is the $R_1$ defined by $PBC_x$ here), it follows, in particular, that

$$W(4 \times \infty, PBC_y, (T)PBC_x, q) = W(4 \times \infty, PBC_y, FBC_x, q) \quad \text{for} \quad q \geq q_c \quad (2.38)$$
Table 1: Values of $W(sq, L_y \times \infty, PBC_y, (T)PBC_x, q)$ for low integral $q$ and for respective $q_c$.

| $L_y$ | $BC_y$ | $BC_x$ | $|W_{q=0}|$ | $|W_{q=1}|$ | $|W_{q=2}|$ | $q_c$ | $W_{q=q_c}$ |
|-------|--------|--------|----------|----------|----------|------|------------|
| 3     | P      | (T)P   | 2.35     | 1.91     | 1.44     | 3    | 1.26       |
| 4     | P      | (T)P   | 2.58     | 2.11     | 1.64     | 2.78 | 1.44       |

where $q_c$ was given above in (2.33). For low integral values of $q$ we list the values of $|W(q)|$ for this strip in Table 1, together with corresponding values given in [25] for $W$ in the $L_x \to \infty$ limit of the $L_y = 3$ strip with $(PBC_y, (T)PBC_x)$.

For various lengths $L_x$, some of the chromatic zeros (those near to the origin) have support for $\text{Re}(q) < 0$, but the locus $B$ itself only has support for $\text{Re}(q) \geq 0$. We have encountered this type of situation in earlier work [18, 22]. The property that $B$ only has support for $\text{Re}(q) \geq 0$ can be demonstrated by carrying out a Taylor series expansion of the degeneracy equation

$$|\lambda_{st4,10}| = |\lambda_{st4,25}|$$

near the origin, which is, numerically,

$$|44.1 - 52.0q + 27.6q^2 + O(q^3)| = |44.1 - 32.2q + 8.8q^2 + O(q^3)|.$$  \hspace{1cm} (2.39)

More generally, consider a degeneracy equation determining a curve on $B$ which, in the vicinity of the origin $q = 0$, has the form

$$|a_0 + a_1q + a_2q^2 + O(q^3)| = |a_0 + b_1q + b_2q^2 + O(q^3)|$$  \hspace{1cm} (2.40)

where the coefficients $a_i$ and $b_i$ are real and nonzero, $a_1 \neq b_1$, and, without loss of generality, we can take $a_0 > 0$. Writing $q$ in polar coordinates as $q = re^{i\theta}$ and expanding for small $r$, eq. (2.40) reduces, to order $r$, to the equation $a_0(a_1 - b_1)r \cos \theta = 0$, which has as its solution $\theta = \pm \pi/2$. Thus the curve $B$ defined by a degeneracy equation of the form (2.40) passes through the origin vertically. In order to determine in which direction (right or left) the curve bends away from the vertical as one moves away from the origin, let us write $q = q_r + iq_i$, where $q_r$ and $q_i$ are real, with $q_r^2 + q_i^2 = r^2 \ll 1$. Substituting, expanding, and using the fact that the curve $B$ passes vertically through the origin so that near this point $|q_r|$ is small compared with $|q_i|$, we find, to this order,

$$q_r = \frac{2a_0(a_2 - b_2) + b_2^2 - a_1^2 |q_i^2|}{2a_0(a_1 - b_1)}.$$  \hspace{1cm} (2.41)

Thus if the right-hand side of this equation is positive (negative), the curve $B$ bends to the right (left) into the half-plane with $\text{Re}(q) > 0$ ($\text{Re}(q) < 0$) as one moves away from the origin. For the degeneracy equation (2.39), the right-hand side of eq. (2.41) is positive, so $B$
Table 2: Properties of $P$, $W$, and $B$ for strip graphs $G_s$ of the square (sq) and triangular (tri) lattices with periodic longitudinal boundary conditions ($FBC_y$, $(T)PBC_x$) (cyclic and Möbius) and $(PBC_y$, $(T)PBC_x$) (torus and Klein bottle). The properties apply for a given strip of type $G_s$ of size $L_y \times L_x$; some apply for arbitrary $L_x$, such as $N_{G_s,\lambda}$, while others apply for the infinite-length limit, such as the properties of the locus $B$. The entry 37(38) for $N_{G_s,\lambda}$ means that $P$ has 37 different $\lambda_j$’s, but the coloring matrix also has a zero eigenvalue that does not contribute to $P$. The column denoted eqs. describes the numbers and degrees of the algebraic equations giving the $\lambda_{G_s,j}$; for example, $\{9(1),6(2),4(3)\}$ indicates that there are nine linear equations, six quadratic equations and four cubic equation. The column denoted BCR lists the points at which $B$ crosses the real $q$ axis; here the largest of these is $q_c$ for the given family $G_s$. Column labelled “SN” refers to whether $B$ has support for negative $Re(q)$, indicated as yes (y) or no (n).

| $G_s$ | $L_y$ | $BC_y$ | $BC_x$ | $N_{G_s,\lambda}$ | eqs. | BCR | $q_c$ | SN | ref. |
|-------|-------|--------|--------|-------------------|------|-----|-------|-----|------|
| sq    | 3     | P      | P      | 8                 | $\{8(1)\}$ | 0, 2, 3 | 3     | n   | 16   |
| sq    | 3     | P      | TP     | 5                 | $\{5(1)\}$ | 0, 2, 3 | 3     | n   | 16   |
| sq    | 4     | P      | P      | 33                | $\{9(1),6(2),4(3)\}$ | 0, 2, 2.78 | 2.78 | n   | here |
| sq    | 4     | P      | TP     | 22                | $\{7(1),3(2),3(3)\}$ | 0, 2, 2.78 | 2.78 | n   | here |
| sq    | 1     | F      | P      | 2                 | $\{2(1)\}$ | 0, 2 | 2     | n   | 8    |
| sq    | 2     | F      | (T)P   | 4                 | $\{4(1)\}$ | 0, 2 | 2     | n   | 8    |
| sq    | 3     | F      | (T)P   | 10                | $\{5(1),1(2),1(3)\}$ | 0, 2, 2.34 | 2.34 | y   | 19-21 |
| sq    | 4     | F      | (T)P   | 26                | $\{4(1),1(2),2(3),1(4),2(5)\}$ | 0, 2, 2.49 | 2.49 | y   | 25   |
| tri   | 3     | P      | P      | 11                | $\{5(1),3(2)\}$ | 0, 2, 3.72 | 3.72 | n   | 17   |
| tri   | 3     | P      | TP     | 5                 | $\{5(1)\}$ | 0, 2, 3.72 | 3.72 | n   | 17   |
| tri   | 4     | P      | P      | 37(38)            | $\{5(1),4(2),2(3),3(4),1(6)\}$ | 0, 2, 4 | 4     | n   | here |
| tri   | 4     | P      | TP     | 12(13)            | $\{4(1),1(2),2(3)\}$ | 0, 2, 4 | 4     | n   | here |
| tri   | 2     | F      | (T)P   | 4                 | $\{2(1),1(2)\}$ | 0, 2, 3 | 3     | n   | 20   |
| tri   | 3     | F      | (T)P   | 10                | $\{3(1),2(2),1(3)\}$ | 0, 2, 3 | 3     | n   | 17   |
| tri   | 4     | F      | P      | 26                | $\{1(1),2(4),1(8),1(9)\}$ | 0, 2, 3, 3.23 | 3.23 | y   | 17   |

bends to the right near the origin. As is evident from Fig. 1, as one moves farther away from the origin, the curve $B$ bends farther to the right, so that $B$ has no support for $Re(q) < 0$. This is to be contrasted with the situation for (the $L_x \rightarrow \infty$ limit of) sufficiently wide strips with cyclic or Möbius boundary conditions (the $L_x \rightarrow \infty$ limits of a given strip with cyclic boundary conditions is the same as the limit with Möbius boundary conditions), where it was found that for widths $L_y = 3, 4$ for the square lattice [20, 25] and for width $L_y = 4$ for the triangular lattice [17], $B$ did have some support for $Re(q) < 0$. A comparison of some properties of $B$ in the present case and for other strips with periodic or orientation-reversing periodic longitudinal boundary conditions is given in Table 2.
3  $L_y = 4$ Strip of the Square Lattice with $(PBC_y, TPBC_x)$

In general, for the strip graph of the square lattice with even width $L_y$ and $(PBC_y, TPBC_x)$, i.e., Klein bottle boundary conditions, we find that $\chi = 4$ if $L_x = 2$ and, for $L_x \geq 3$,

$$
\chi(sq, L_y \times L_x, PBC_y, TPBC_x) = \begin{cases} 
2 & \text{if } L_x \text{ is odd} \\
3 & \text{if } L_x \text{ is even}
\end{cases} \tag{3.1}
$$

For this strip (labelled $sk4$) we calculate that $N_{sk4,\lambda} = 22$ and

$$
P(sq, 4 \times L_x, PBC_y, TPBC_x, q) = \sum_{j=1}^{22} c_{sk4,j} \cdot (\lambda_{sk4,j})^{L_x} \tag{3.2}
$$

The nonzero terms $\lambda_{sk4,j}$ are identical to a subset of the terms $\lambda_{st4,j}$’s for the same strip with torus boundary conditions. The 11 terms that occur in the chromatic polynomial (2.2) for toroidal boundary conditions but are absent in the chromatic polynomial (3.2) for the Klein bottle case are

$$
\lambda_{st4,j} , j = 4, 5, 12, 13, 16, 17, 20, 21, 31, 32, 33 . \tag{3.3}
$$

We have

$$
\lambda_{sk4,j} = \lambda_{st4,j} \text{ for } 1 \leq j \leq 3 \tag{3.4}
$$

$$
\lambda_{sk4,j} = \lambda_{st4,j+2} \text{ for } 4 \leq j \leq 9 \tag{3.5}
$$

$$
\lambda_{sk4,j} = \lambda_{st4,j+4} \text{ for } j = 10, 11 \tag{3.6}
$$

$$
\lambda_{sk4,j} = \lambda_{st4,j+6} \text{ for } j = 12, 13 \tag{3.7}
$$

$$
\lambda_{sk4,j} = \lambda_{st4,j+8} \text{ for } 14 \leq j \leq 22 . \tag{3.8}
$$

The corresponding coefficients are

$$
c_{sk4,1} = 1 \tag{3.9}
$$

$$
c_{sk4,2} = q - 1 \tag{3.10}
$$

$$
c_{sk4,3} = \frac{1}{2}(q - 1)(q - 2) \tag{3.11}
$$

$$
c_{sk4,4} = -(q - 1) \tag{3.12}
$$

$$
c_{sk4,5} = -c_{st4,7} = -\frac{1}{2}(q - 1)(q - 2) \tag{3.13}
$$

$$
c_{sk4,6} = c_{st4,8} = \frac{1}{2}q(q - 3) \tag{3.14}
$$

$$
c_{sk4,7} = -c_{st4,9} = -1 \tag{3.15}
$$
\[ c_{sk4,j} = c_{st4,j+2} = 1 \quad \text{for} \quad j = 8, 9 \]  
(3.16)
\[ c_{sk4,j} = -c_{st4,j+4} = -\frac{1}{2}q(q - 3) \quad \text{for} \quad j = 10, 11 \]  
(3.17)
\[ c_{sk4,j} = -c_{st4,j+6} = -\frac{1}{2}(q - 1)(q - 2) \quad \text{for} \quad j = 12, 13 \]  
(3.18)
\[ c_{sk4,j} = -c_{st4,j+8} = -(q - 1) \quad \text{for} \quad 14 \leq j \leq 16 \]  
(3.19)
\[ c_{sk4,j} = c_{st4,j+8} = q - 1 \quad \text{for} \quad 17 \leq j \leq 19 \]  
(3.20)
\[ c_{sk4,j} = c_{st4,j+8} = \frac{1}{2}q(q - 3) \quad \text{for} \quad 20 \leq j \leq 22 \]  
(3.21)

The sum of these coefficients is zero, as dictated by the \( L_y = 4 \) special case of the general result (1.7) above.

Because none of the terms \( \lambda_{st4,j} \) in (3.3) that is present in (2.2) and absent in (3.2) is dominant, it follows that in the limit \( L_x \to \infty \), the \( W \) functions are the same for both of these boundary conditions, and hence, so is the singular locus \( B \). Below we shall prove in general that this must be the case; that is, in the limit \( L_x \to \infty \), a strip of the square (or triangular) lattice of width \( L_y \) with \((PBC_y, PBC_x)\) (torus) boundary conditions yields the same \( W \) function and singular locus \( B \) as the corresponding strip with \((PBC_y, TPBC_x)\) (Klein bottle) boundary conditions.

4 \hspace{1cm} \( L_y = 4 \) Strip of the Triangular Lattice with \((PBC_y, PBC_x)\)

By similar methods, we have calculated the chromatic polynomials for strips of the triangular lattice with width \( L_y = 4 \), arbitrarily great length \( L_x \), and torus boundary conditions (labelled \( tt4 \)). In general, for a strip of the triangular lattice of size \( L_y \times L_x \) with toroidal boundary conditions, for \( L_y \geq 3 \) and \( L_x \geq 3 \), the chromatic number is given by

\[
\chi(tri, L_y \times L_x, PBC_y, PBC_x) = \begin{cases} 
3 & \text{if } L_y = 0 \mod 3 \text{ and } L_x = 0 \mod 3 \\
4 & \text{otherwise}
\end{cases} \quad (4.1)
\]

Thus, in the present case, \( \chi = 4 \), independent of \( L_x \). In the notation of eq. (1.4) we find \( N_{tt4,\chi} = 37 \) and

\[
P(tri, 4 \times L_x, PBC_y, PBC_x, q) = \sum_{j=1}^{37} c_{tt4,j}(\lambda_{tt4,j})^{L_x} \quad (4.2)
\]

where

\[
\lambda_{tt4,1} = 2 \quad (4.3)
\]
\[
\lambda_{tt4,(2,3)} = \sqrt{2}e^{\pm i\pi/4} \quad (4.4)
\]
\[ \lambda_{tt4,4} = 2(3 - q) \] (4.5)
\[ \lambda_{tt4,5} = 3 - q \] (4.6)
\[ \lambda_{tt4,6} = -2(2q - 9) \] (4.7)
\[ \lambda_{tt4,7} = 2(q - 3)^2 \] (4.8)

\[ \lambda_{tt4,(8,9)} = \frac{(q - 3)}{2} \left[ q^3 - 9q^2 + 33q - 48 \right] \pm(q - 4)(q^4 - 10q^3 + 43q^2 - 106q + 129)^{1/2} \] (4.9)

\[ \lambda_{tt4,(10,11)} = \pm i \sqrt{3} (q - 3) \] (4.10)
\[ \lambda_{tt4,(12,13)} = \pm i (q - 2) \sqrt{2(q - 3)(q - 4)} . \] (4.11)

The \( \lambda_{tt4,j} \)’s for \( 14 \leq j \leq 19 \) are roots of two cubic equations,
\[ \xi^3 + 2(q^3 - 12q^2 + 51q - 75)\xi^2 \]
\[ -4(q - 3)^3(q^2 - 7q + 13)\xi - 8(q - 3)^4(q^2 - 5q + 5) = 0 \] (4.12)
with roots \( \lambda_{tt4,j}, j = 14, 15, 16 \), and
\[ \xi^3 - 2(2q^2 - 17q + 39)\xi^2 + 2(q^4 - 17q^3 + 100q^2 - 244q + 214)\xi \]
\[ +4(q - 3)(q^4 - 11q^3 + 44q^2 - 76q + 46) = 0 \] (4.13)
with roots \( \lambda_{tt4,j}, j = 17, 18, 19 \). The \( \lambda_{tt4,j} \)’s for \( 20 \leq j \leq 31 \) are roots of three quartic equations,
\[ \xi^4 + 2(q^3 - 9q^2 + 29q - 34)\xi^3 + 2(q^3 - 9q^2 + 29q - 34)^2\xi^2 \]
\[ +4(q - 3)^2(q^2 - 5q + 5)(q^3 - 9q^2 + 29q - 34)\xi + 4(q - 3)^4(q^2 - 5q + 5)^2 = 0 \] (4.14)
with roots \( \lambda_{tt4,j}, 20 \leq j \leq 23 \),
\[ \xi^4 - 2(q^2 - 7q + 14)\xi^3 + 2(q^2 - 7q + 14)^2\xi^2 \]
\[ +4(q - 3)(q^2 - 7q + 14)(q^2 - 6q + 7)\xi + 4(q - 3)^2(q^2 - 6q + 7)^2 = 0 \]
with roots $\lambda_{tt, j}$, $24 \leq j \leq 27$, and
\[
\xi^4 + 2(3q - 11)\xi^3 + 2(3q - 11)^2\xi^2 \\
+2(3q - 11)(3q^2 - 18q + 23)\xi + (3q^2 - 18q + 23)^2 = 0
\]
(4.16)

with roots $\lambda_{tt, j}$, $28 \leq j \leq 31$. Finally, the $\lambda_{tt, j}$'s for $32 \leq j \leq 37$ are roots of an equation of degree six:
\[
\xi^6 - 2(q - 5)(2q - 7)\xi^5 + 2(q - 5)^2(2q - 7)^2\xi^4 \\
+8(q - 4)^2(3q^3 - 29q^2 + 89q - 85)\xi^3 + 4(3q^3 - 28q^2 + 84q - 79)^2\xi^2 \\
+8(q - 3)^2(q^2 - 5q + 5)(3q^3 - 28q^2 + 84q - 79)\xi + 8(q - 3)^4(q^2 - 5q + 5)^2 = 0
\]
(4.17)

Each of the three quartic equations above has roots of the form $a_\ell e^{\pm i\pi/4}$, $b_\ell e^{\pm i\pi/4}$, where $\ell = 1, 2, 3$ indexes the quartic equation, so
\[
\lambda_{tt, j} = a_1 e^{\pm i\pi/4} \quad \text{for} \; j = 20, 21 \\
\lambda_{tt, j} = b_1 e^{\pm i\pi/4} \quad \text{for} \; j = 22, 23 \\
\lambda_{tt, j} = a_2 e^{\pm i\pi/4} \quad \text{for} \; j = 24, 25 \\
\lambda_{tt, j} = b_2 e^{\pm i\pi/4} \quad \text{for} \; j = 26, 27 \\
\lambda_{tt, j} = a_3 e^{\pm i\pi/4} \quad \text{for} \; j = 28, 29 \\
\lambda_{tt, j} = b_3 e^{\pm i\pi/4} \quad \text{for} \; j = 30, 31
\]
(4.18) (4.19) (4.20) (4.21) (4.22) (4.23)

where the values of $a_\ell$ and $b_\ell$, $\ell = 1, 2, 3$ are determined by these quartic equations. Similarly, the roots of the sixth-order equation are of the form $c_\ell e^{\pm i\pi/4}$, $\ell = 1, 2, 3$, i.e.,
\[
\lambda_{tt, j} = c_1 e^{\pm i\pi/4} \quad \text{for} \; j = 32, 33 \\
\lambda_{tt, j} = c_2 e^{\pm i\pi/4} \quad \text{for} \; j = 34, 35
\]
(4.24) (4.25)
\[ \lambda_{tt4,j} = c_3 e^{\pm i\pi/4} \quad \text{for} \quad j = 36, 37 \] (4.26)

where the values of \( c_\ell, \ell = 1, 2, 3 \) follow from eq. (4.17). Below we shall comment further on these phase factors.

The corresponding coefficients are

\[ c_{tt4,1} = \frac{1}{4} q(q-2)(q-3)^2 \] (4.27)

\[ c_{tt4,2} = c_{tt4,3} = \frac{1}{12} (q-1)(q-2)(3q^2 - 11q - 6) \] (4.28)

\[ c_{tt4,j} = \frac{1}{2} (q-1)(q-2) \quad \text{for} \quad j = 4, 7 \quad \text{and} \quad 32 \leq j \leq 37 \] (4.29)

\[ c_{tt4,5} = \frac{2}{3} q(q-2)(q-4) \] (4.30)

\[ c_{tt4,j} = \frac{1}{3} q(q-2)(q-4) \quad \text{for} \quad j = 10, 11 \quad \text{and} \quad 28 \leq j \leq 31 \] (4.31)

\[ c_{tt4,6} = \frac{1}{3} (q-1)(q^2 - 5q + 3) \] (4.32)

\[ c_{tt4,8} = c_{tt4,9} = 1 \] (4.33)

\[ c_{tt4,j} = \frac{1}{2} q(q-3) \quad \text{for} \quad j = 12, 13, 17, 18, 19 \quad \text{and} \quad 24 \leq j \leq 27 \] (4.34)

\[ c_{tt4,j} = q-1 \quad \text{for} \quad j = 14, 15, 16 \quad \text{and} \quad 20 \leq j \leq 23 \] (4.35)

Formally, we have also found a zero eigenvalue,

\[ \lambda_{tt4,38} = 0 \] (4.36)

with coefficient (multiplicity)

\[ c_{tt4,38} = \frac{1}{12} q(q-1)(3q^2 - 17q + 40) \] (4.37)

Although this term does not contribute to the chromatic polynomial (1.4), the corresponding coefficient does contribute to the sum of multiplicities, i.e. to the total dimension of the space of coloring configurations, given by (1.6). The sum of all of the coefficients, including that corresponding to the zero eigenvalue, is equal to \( P(C_4, q) = q(q-1)(q^2 - 3q + 3) \), which is an \( L_y = 4 \) special case of (1.6).

The singular locus \( \mathcal{B} \) for the \( L_x \to \infty \) limit of the strip of the triangular lattice with \( L_y = 4 \) and toroidal boundary conditions is shown in Fig. 2. For comparison, chromatic
Figure 2: Singular locus $B$ for the $L_x \to \infty$ limit of the strip of the triangular lattice with $L_y = 4$ and toroidal boundary conditions. For comparison, chromatic zeros are shown for $L_x = 30$ (i.e., $n = 120$).
zeros are calculated and shown for length $L_x = 30$ (i.e., $n = 120$ vertices). The locus $B$ crosses the real axis at the points $q = 0, q = 2$, and at the maximal point $q = q_c$, where

$$q_c = 4 \text{ for } \{G\} = (tri, 4 \times L_x, PBC_y, PBC_x).$$

At this point there are several degeneracies of magnitudes of eigenvalues; these occur for $\lambda_j$ with $j = 1, 4, 6, 7, 8, 9$ and $14 \leq j \leq 19$.

As is evident from Fig. 2, the locus $B$ separates the $q$ plane into different regions including the following (we use the same symbols as for the $L_y = 4$ toroidal strip of the square lattice, but it is understood that the regions are specific to this section): (i) $R_1$, containing the semi-infinite intervals $q > 4$ and $q < 0$ on the real axis and extending outward to infinite $|q|$, (ii) $R_2$ containing the interval $2 < q < 4$, and (iii) $R_3$ containing the real interval $0 < q < 2$ Again, the (nonzero) density of chromatic zeros has the smallest values on the curve separating regions $R_1$ and $R_3$ in the vicinity of the point $q = 0$ and on the curve separating regions $R_2$ and $R_3$ in the vicinity of the point $q = 2$.

In region $R_1$, $\lambda_{tt,8}$ is the dominant $\lambda_{G,j}$, so

$$W = (\lambda_{tt,8})^{1/4}, \quad q \in R_1.$$  \hfill (4.39)

This is the same as $W$ for the corresponding $L_x \to \infty$ limit of the strip of the triangular lattice with the same width $L_y = 4$ and cylindrical $(PBC_y, FBC_x)$ boundary conditions, calculated in [28].

In region $R_2$,

$$|W| = |\lambda_{tt,17}|^{1/4}, \quad q \in R_2$$

where $\lambda_{tt,17}$ is the root of the cubic equation (4.13) that has the maximal magnitude for $2 < q < 4$. In region $R_3$,

$$|W| = |\lambda_{tt,14}|^{1/4}, \quad q \in R_3.$$  \hfill (4.41)

There are no other regions containing nonzero intervals of the real axis besides $R_j, j = 1, 2, 3$. However, our previous calculations for various families of graphs [20, 17] have shown that $B$ can include pairs of extremely small complex-conjugate sliver regions. We have not made an exhaustive search for these in the present case.

Corresponding to eq. (2.38) for the toroidal or Klein bottle and cylindrical strips of the square lattice, we have

$$W(tri, 4 \times \infty, PBC_y, (T)PBC_x, q) = W(tri, 4 \times \infty, PBC_y, FBC_x, q) \text{ for } q \geq 4.$$  \hfill (4.42)

Hence the values of $W(tri, 4 \times \infty, PBC_y, FBC_x, q)$ for various values of $q \geq 4$ given in [25] (see also [17]) are also applicable here. For low integral values of $q$ we list the values of
Table 3: Values of $W(tri, L_y \times \infty, PBC_y, (T)P BC_x, q)$ for low integral $q$ and for respective $q_c$.

| $L_y$ | $BC_y$ | $BC_x$ | $|W_{q=0}|$ | $|W_{q=1}|$ | $|W_{q=2}|$ | $|W_{q=3}|$ | $q_c$ | $W_{q=q_c}$ |
|-------|--------|--------|-----------|-----------|-----------|-----------|-------|-------------|
| 3     | P      | (T)P   | 3.17      | 2.62      | 2         | 1.71      | 3.72  | 1.41        |
| 4     | P      | (T)P   | 3.44      | 2.86      | 2.25      | 1.83      | 4     | 1.19        |

$|W(q)|$ for this strip in Table 3, together with corresponding values given in [23, 17] for $W$ in the $L_x \to \infty$ limit of the $L_y = 3$ strip with $(PBC_y, (T)PBC_x)$.

The locus $B$ only has support for $Re(q) \geq 0$. This can be demonstrated by carrying out a Taylor series expansion of the degeneracy equation $|\lambda_{tt4,8}| = |\lambda_{tt4,14}|$ near the origin, which is, numerically,

$$|140.15 - 141.25q + 57.6q^2 + O(q^3)| = |140.15 - 93.4q + 21.9q^2 + O(q^3)|.$$ (4.43)

This equation is of the form (2.40), and, using eq. (2.41), we verify that $B$ bends to the right as one moves away from the origin. Farther away from the origin, one can see from Fig. 2 that $B$ continues to move into the half-plane with $Re(q) > 0$, so that the conclusion stated above follows, that this locus has no support for $Re(q) < 0$.

5  $L_y = 4$ Strip of the Triangular Lattice with $(PBC_y, TPBC_x)$

The strip of the triangular lattice with width $L_y = 4$, arbitrarily great length $L_x$, and $(PBC_y, TPBC_x) =$ Klein bottle boundary conditions, labelled $tk4$, has (for $L_x \geq 2$) chromatic number

$$\chi(tri, 4 \times L_x, PBC_y, TPBC_x) = \begin{cases} 4 & \text{if } L_x \text{ is even} \\ 5 & \text{if } L_x \text{ is odd} \end{cases}$$ (5.1)

In the notation of eq. (1.4) we find $N_{tk4,\lambda} = 12$ and

$$P(tri, 4 \times L_x, PBC_y, TPBC_x, q) = \sum_{j=1}^{12} c_{tk4,j}(\lambda_{tk4,j})^{L_x}$$ (5.2)

where

$$\lambda_{tk4,1} = \lambda_{tt4,1} = 2$$ (5.3)
$$\lambda_{tk4,2} = \lambda_{tt4,4} = 2(3 - q)$$ (5.4)
$$\lambda_{tk4,3} = \lambda_{tt4,6} = -2(2q - 9)$$ (5.5)
$$\lambda_{tk4,4} = \lambda_{tt4,7} = 2(q - 3)^2$$ (5.6)
\( \lambda_{tk4,j} = \lambda_{tt4,j+3} \) for \( j = 5, 6 \)  \hspace{1cm} (5.7)

\( \lambda_{tk4,j} = \lambda_{tt4,j+7} \) for \( 7 \leq j \leq 12 \).  \hspace{1cm} (5.8)

The corresponding coefficients are

\[
c_{tk4,1} = \frac{1}{2}q(q - 2)(q - 3) \hspace{1cm} (5.9)
\]

\[
c_{tk4,2} = c_{tt4,4} = \frac{1}{2}(q - 1)(q - 2) \hspace{1cm} (5.10)
\]

\[
c_{tk4,3} = -(q - 1) \hspace{1cm} (5.11)
\]

\[
c_{tk4,4} = -c_{tt4,7} = -\frac{1}{2}(q - 1)(q - 2) \hspace{1cm} (5.12)
\]

\[
c_{tk4,5} = c_{tk4,6} = c_{tt4,8} = c_{tt4,9} = 1 \hspace{1cm} (5.13)
\]

\[
c_{tk4,j} = c_{tt4,j+7} = q - 1 \hspace{1cm} \text{for} \hspace{1cm} j = 7, 8, 9 \hspace{1cm} (5.14)
\]

\[
c_{tk4,j} = c_{tt4,j+7} = \frac{1}{2}q(q - 3) \hspace{1cm} \text{for} \hspace{1cm} j = 10, 11, 12 \hspace{1cm} (5.15)
\]

The coloring matrix also has another eigenvalue, namely,

\[
\lambda_{tk4,13} = 0 \hspace{1cm} (5.16)
\]

with multiplicity

\[
c_{tk4,13} = -\frac{1}{2}q(q - 1)^2 \hspace{1cm} (5.17)
\]

Hence, the total number of distinct eigenvalues of the coloring matrix for this strip is \( N_{tk4,\lambda,\text{tot}} = N_{tk4,\lambda} + 1 = 13 \). The sum of all of the coefficients, including that for the zero eigenvalue, is zero; this is an \( L_y = 4 \) special case of (1.7).

6 Cyclic and Toroidal Crossing-Subgraph Strips of the Square Lattice

6.1 General

It is worthwhile to include here some results on certain related families of strip graphs since these give insight into the structure of the chromatic polynomials for the various strips with longitudinal boundary conditions which are periodic or periodic with reversed orientation. Let us consider first a strip of the square lattice of fixed width \( L_y \) and arbitrarily great length \( L_x \) constructed as follows. As before, the longitudinal (horizontal) direction on the strip to be \( x \) and the transverse (vertical) direction to be \( y \). Label the vertices of two successive
transverse slices of the strip, starting at the top as \((1, 2, \ldots, L_y)\) and \((1', 2', \ldots, L'_y)\). First, consider the case of free transverse boundary conditions, for which these transverse slices of the strip are line (path) graphs with \(L_y\) vertices. Connect these with edges linking vertices 1 to \(L'_y\), 2 to \((L_y - 1)'\), 3 to \((L_y - 2)'\), and so forth. For example, for \(L_y = 2\), we connect 1 to 2' and 2 to 1'; for \(L_y = 3\), we connect 1 to 3', 2 to 2', and 3 to 1', etc. for other values of \(L_y\). An example of this crossing-subgraph strip of the square lattice of width \(L_y = 3\) is given in Fig. 3(a).

We impose periodic longitudinal boundary conditions. We shall denote this crossing-subgraph strip (labelled \(cg\)) of the square (\(sq\)) lattice as \(cg(sq, L_y \times L_x, FBC_y, PBC_x)\). We observe that

\[
    cg(sq, L_y \times L_x, FBC_y, PBC_x) = \begin{cases} 
    (sq, L_y \times L_x, FBC_y, PBC_x) & \text{if } L_x \text{ is even} \\
    (sq, L_y \times L_x, FBC_y, TPBC_x) & \text{if } L_x \text{ is odd}
\end{cases} \qquad (6.1.1)
\]

Figure 3: Illustrative crossing-subgraph strip graphs of the square lattice with (a) \((FBC_y, PBC_x) = \text{cyclic}\) and (b) \((PBC_y, PBC_x) = \text{toroidal type}\). For these, \(L_y = 3\) and \(L_x = 4\). Vertices are indicated with \(\bullet\) (points where edges cross without a symbol \(\bullet\) are not vertices.)
That is, for even (odd) $L_x$, this crossing-subgraph strip reduces to the cyclic (Möbius) strip of the square lattice. Secondly, consider the case where we impose periodic transverse boundary conditions; we denote this toroidal crossing-subgraph strip of the square lattice as $cg(sq, L_y \times L_x, PBC_y, PBC_x)$. In this case the transverse slices are circuit graphs with $L_y$ vertices. An example of this toroidal crossing-subgraph strip of the square lattice with width $L_y = 3$ is shown in Fig. 3(b). We have

$$cg(sq, L_y \times L_x, PBC_y, PBC_x) = \begin{cases} (sq, L_y \times L_x, PBC_y, PBC_x) & \text{if } L_x \text{ is even} \\ (sq, L_y \times L_x, PBC_y, TPBC_x) & \text{if } L_x \text{ is odd} \end{cases} \quad (6.1.2)$$

That is, for even (odd) $L_x$, this crossing-subgraph strip reduces to the strip of the square lattice with torus (Klein bottle) boundary conditions. Given the relations (6.1.1) and (6.1.2), it follows that a knowledge of the chromatic polynomial for the cyclic crossing-subgraph strip of width $L_y$ of the square lattice is equivalent to a knowledge of the chromatic polynomials for the strip of this lattice with width $L_y$ with both cyclic and Möbius boundary conditions and, similarly, a knowledge of the chromatic polynomial for the toroidal crossing-subgraph strip of the square lattice is equivalent to a knowledge of the chromatic polynomials for the strip of this lattice with both torus and Klein bottle boundary conditions.

The sum of the coefficients for the $L_y \times L_x$ cyclic crossing-subgraph strip of the square lattice is

$$C_{cg(sq, L_y \times L_x, FBC_y, PBC_x)} = P(T_{L_y}, q) \quad (6.1.3)$$

where $T_n$ is the tree graph on $n$ vertices, and $P(T_n, q) = q(q - 1)^{n-1}$. The sum of the coefficients for the $L_y \times L_x$ toroidal crossing-subgraph strip of the square or triangular lattice is

$$C_{cg(G_x, L_y \times L_x, PBC_y, PBC_x)} = P(C_{L_y}, q) \quad \text{for } G_x = sq, tri \quad (6.1.4)$$

as in (1.6).

We have carried out explicit calculations of chromatic polynomials for a number of crossing-subgraph strips (labelled $cg$) and have related the results to those for strips with cyclic/Möbius and torus/Klein bottle boundary conditions. We concentrate here on strips of the square lattice and discuss those of the triangular lattice below. We find that a certain subset of the terms in (1.4) for the crossing-subgraph strips occur in opposite-sign pairs of the form $\pm \lambda_{cg,j}$. Consider the coefficients for the $\pm \lambda_{cg,j}$’s in each pair: in some cases, these are different, while in others they are the same. Let us denote the number of $\lambda_{cg,j}$’s comprising opposite-sign pairs such that the members of each pair have different (the same) coefficients as $N_{cg,opd,\lambda}$ ($N_{cg,ops,\lambda}$), respectively. The number of remaining $\lambda_{cg,j}$’s that are not members of an opposite-sign pair is denoted $N_{cg,up,\lambda}$, where $up$ means “unpaired”. Clearly

$$N_{cg,\lambda} = N_{cg,up,\lambda} + N_{cg,opd,\lambda} + N_{cg,ops,\lambda} \quad (6.1.5)$$
For the cyclic and Möbius strips that we have studied, we find $N_{c_{\text{g}},\text{ops},\lambda} = 0$, while for torus and Klein bottle strips, $N_{c_{\text{g}},\text{ops},\lambda}$ is, in general, nonzero. For even $L_x$, where, according to the identities (6.1.1) and (6.1.2), the cyclic (toroidal) crossing-subgraph strip reduces to the cyclic (toroidal) strip of the square lattice, two such opposite-sign terms reduce to a single term as follows (the subscripts $jp, jm$ denote $j, \pm$)

$$c_{c_{\text{g}},\text{cyc},jp}(\lambda_{c_{\text{g}},\text{cyc},j})^{L_x} + c_{c_{\text{g}},\text{cyc},jm}(-\lambda_{c_{\text{g}},\text{cyc},j})^{L_x} = (c_{c_{\text{g}},\text{cyc},jp} + c_{c_{\text{g}},\text{cyc},jm})(\lambda_{c_{\text{g}},\text{cyc},j})^{L_x} = c_{\text{sq},\text{cyc},j}(\lambda_{c_{\text{g}},\text{cyc},j})^{L_x} \quad (6.1.6)$$

For odd $L_x = m$ the cyclic (toroidal) crossing-subgraph strip reduces to the Möbius (Klein bottle) strip, and the pair of opposite-sign terms reduces to a single term as follows:

$$c_{c_{\text{g}},\text{cyc},jp}(\lambda_{c_{\text{g}},\text{cyc},j})^{L_x} + c_{c_{\text{g}},\text{cyc},jm}(-\lambda_{c_{\text{g}},\text{cyc},j})^{L_x} = (c_{c_{\text{g}},\text{cyc},jp} - c_{c_{\text{g}},\text{cyc},jm})(\lambda_{c_{\text{g}},\text{cyc},j})^{L_x} = c_{\text{sq},\text{Mb},j}(\lambda_{c_{\text{g}},\text{cyc},j})^{L_x} \quad (6.1.7)$$

where the subscript $Mb$ denotes Möbius. In particular, if $\lambda_{c_{\text{g}},j}$ is one of the $N_{c_{\text{g}},\text{ops},\lambda}$ terms with $c_{c_{\text{g}},jp} = c_{c_{\text{g}},jm}$, then the terms in (6.1.7) cancel each other, leaving no contribution. As noted above, in our studies, we have found that this can happen for Klein bottle strips, since $N_{c_{\text{g}},\text{ops},\lambda} \neq 0$ for these, but not for Möbius strips, since $N_{c_{\text{g}},\text{ops},\lambda} = 0$ for these. The inverse relations connecting the coefficients for the terms $\pm\lambda_{c_{\text{g}},\text{cyc},j}$ in the chromatic polynomial for the crossing-subgraph cyclic strip to the coefficients $c_{\text{sq},\text{cyc},j}$ and $c_{\text{sq},\text{Mb},j}$ in the cyclic and Möbius strips are thus

$$c_{c_{\text{g}},\text{cyc},jp} = \frac{1}{2}(c_{\text{sq},\text{cyc},j} + c_{\text{sq},\text{Mb},j}) \quad (6.1.8)$$

$$c_{c_{\text{g}},\text{cyc},jm} = \frac{1}{2}(c_{\text{sq},\text{cyc},j} - c_{\text{sq},\text{Mb},j}) \quad (6.1.9)$$

and similarly, for the coefficients for the terms $\pm\lambda_{c_{\text{g}},\text{torus},j}$ in the chromatic polynomial for the toroidal crossing-subgraph strip in terms of the coefficients $c_{\text{sq},\text{torus},j}$ and $c_{\text{sq},\text{Kb},j}$ in the torus and Klein bottle ($Kb$) strips,

$$c_{c_{\text{g}},\text{torus},jp} = \frac{1}{2}(c_{\text{sq},\text{torus},j} + c_{\text{sq},\text{Kb},j}) \quad (6.1.10)$$

$$c_{c_{\text{g}},\text{torus},jm} = \frac{1}{2}(c_{\text{sq},\text{torus},j} - c_{\text{sq},\text{Kb},j}) \quad (6.1.11)$$

From these considerations, we derive the following general formula:

$$N_{\text{sq},L_y,\text{cyc},\lambda} = N_{c_{\text{g}},\text{sq},L_y,\text{cyc},\lambda} - \frac{1}{2}N_{c_{\text{g}},\text{sq},L_y,\text{cyc},\text{opd},\lambda} \quad (6.1.12)$$
and, since $N_{cg, ops, \lambda} = 0$ for the cyclic crossing-graph strips of the square lattice that we have studied, the same formula applies to the corresponding Möbius strips with $N_{sq, L_y, cyc, \lambda}$ replaced by $N_{sq, L_y, Mb, \lambda}$. Further,

$$N_{sq, L_y, torus, \lambda} = N_{cg, sq, L_y, torus, \lambda} - \frac{1}{2}N_{cg, sq, L_y, torus, opd, \lambda} - \frac{1}{2}N_{cg, sq, L_y, torus, ops, \lambda}$$ \tag{6.1.13}

$$N_{sq, L_y, Kb, \lambda} = N_{cg, sq, L_y, torus, \lambda} - \frac{1}{2}N_{cg, sq, L_y, torus, opd, \lambda} - N_{cg, sq, L_y, torus, ops, \lambda}$$ \tag{6.1.14}

Thus,

$$N_{sq, L_y, torus, \lambda} - N_{sq, L_y, Kb, \lambda} = \frac{1}{2}N_{cg, sq, L_y, torus, ops, \lambda}$$ \tag{6.1.15}

In the limit $L_x \to \infty$, the cyclic or toroidal crossing-subgraph strip of a given width $L_y$ yields a $W$ function via (1.2) and hence a singular locus $B$. An important theorem can be proved from this by observing that we can take this limit using even or odd values of $L_x$; in the even-$L_x$ case, we obtain the function $W$ and locus $B$ for the strip with torus boundary conditions, while in the odd-$L_x$ case, we obtain the $W$ function and $B$ for the strip with Klein bottle boundary conditions. Since the original limit exists, all three of these limits must be the same. This proves the following theorem:

**Theorem 1:** The $W$ function and singular locus $B$ are the same for the $L_x \to \infty$ limit of strip of the square lattice with width $L_y$ and length $L_x$ whether one imposes $(PBC_y, PBC_x)$ or $(PBC_y, TPBC_x)$, i.e. torus or Klein bottle boundary conditions.

Since the chromatic polynomials for these two sets of boundary conditions involve different numbers of terms, this was not, *a priori* obvious. This feature was first noticed in [16] and was shown there to be a consequence of the fact that none of the terms $\lambda_{st3,j}$ for the torus case that were absent in the Klein bottle case was dominant; here we have succeeded in explaining why this had to be true; if it were not, then the respective loci $B$ would be different, but this is impossible, as a consequence of our present theorem. Thus, a corollary to the theorem is

**Corollary 1:** Consider an $L_y \times L_x$ strip of the square lattice with torus or Klein bottle boundary conditions, and denote the set of nonzero eigenvalues that contribute to (1.4) for these two respective strips as $\lambda_{stL_y,j}, j = 1, ..., N_{stL_y,\lambda}$ and $\lambda_{skL_y,j}, j = 1, ..., N_{skL_y,\lambda}$. Focus on the set of eigenvalues $\lambda_{stL_y,j}$ that do not occur among the set $\lambda_{skL_y,j}$ (the number of these is given by eq. (6.1.15)); none of these can be dominant eigenvalues.

In a similar manner, one can prove that the locus $B$ for the $L_x \to \infty$ limits of the strips of the square lattice with cyclic and Möbius boundary conditions are the same without using...
as input the identity of terms $\lambda_{sq,ly,cy,j} = \lambda_{sq,ly,Mb,j}$ that we have observed in our studies [21, 25]. Below we shall also prove a similar theorem for strips of the triangular lattice with torus and Klein bottle boundary conditions.

We recall that in [26] we observed from our work that the coefficients $c_{G,j}$ in (1.4) for cyclic and M"{o}bius strips of the square lattice are Chebyshev polynomials; in particular, for a given degree $d$ polynomial, there is a unique coefficient with this degree, and it is given by

$$c^{(d)} = U_{2d}\left(\frac{\sqrt{q}}{2}\right)$$

where $U_n(x)$ is the Chebyshev polynomial of the second kind, defined by

$$U_n(x) = \sum_{j=0}^{[\frac{n}{2}]} (-1)^j \binom{n-j}{j} (2x)^{n-2j}$$

where $[\frac{n}{2}]$ means the integral part of $\frac{n}{2}$. The first few of these coefficients are $c^{(0)} = 1$, $c^{(1)} = q - 1$, $c^{(2)} = q^2 - 3q + 1$, $c^{(3)} = q^3 - 5q^2 + 6q - 1$, etc. We also found that the eigenvalues $\lambda_{G,j}$ were the same for cyclic and M"{o}bius strips of the square (and triangular) lattices of a given width that we considered. We established the transformation rules specifying how a coefficient of a given degree changes when one switches from the cyclic to M"{o}bius strip of the square lattice [26]:

$$c^{(0)} \rightarrow \pm c^{(0)}$$

$$c^{(2k)} \rightarrow \pm c^{(k-1)}, 1 \leq k \leq \left[\frac{L_y}{2}\right]$$

$$c^{(2k+1)} \rightarrow \pm c^{(k+1)}, 0 \leq k \leq \left[\frac{L_y - 1}{2}\right].$$

Following the notation of [26], denote the number of terms (eigenvalues) $\lambda_{G,j}$ with coefficients $c^{(d)}$ as $n_P(L_y, d)$. We concentrate on the case of the square strip here and suppress the $sq$ in the notation. This satisfies

$$N_{L_y,\lambda} = \sum_{d=0}^{L_y} n_P(L_y, d)$$

where in the notation used here, $N_{L_y,\lambda}$ refers to the quantity denoted $N_{P,L_y,\lambda}$ in [26]. We gave general formulas for $n_P(L_y, d)$ and $N_{L_y,\lambda}$; in particular, here we shall need the following ones:

$$n_P(L_y, 0) = M_{L_y-1}$$

and

$$n_P(L_y, 1) = M_{L_y}$$
where the Motzkin number $M_n$ is given by
\begin{equation}
M_n = \sum_{j=0}^{n} (-1)^j C_{n+1-j} \binom{n}{j}
\end{equation}
(6.1.24)
and
\begin{equation}
C_n = \frac{1}{n+1} \binom{2n}{n}
\end{equation}
(6.1.25)
is the Catalan number. For the total number of terms, we obtained the result
\begin{equation}
N_{Ly,\lambda} = 2(L_y - 1)! \sum_{j=0}^{|L_y|} \frac{(L_y - j)}{(j!)^2(L_y - 2j)!}.
\end{equation}
(6.1.26)
Now our transformation formulas (6.1.18)-(6.1.20) imply that the only cases where the coefficients in (1.4) for the cyclic and Möbius strips can be the same, up to sign, are for $c^{(0)} = 1$ and $c^{(1)}$. If these coefficients are the same, then, by eq (6.1.9), $c_{cg,jm} = 0$, while if they are opposite in sign, then, by eq (6.1.8), $c_{cg,jp} = 0$; hence, in either case, there is only one term from the possible pair $\pm \lambda_{cg,sq,cyc,j}$ contributing to (1.4) for the cyclic crossed-subgraph strip. For all of the other coefficients $c_{sq,cyc,j}$ in (1.4) for the cyclic strips, our transformation formulas (6.1.18)- (6.1.20) imply that $c_{sq,Mb,j} \neq \pm c_{sq,cyc,j}$, so that both $c_{cg,jp}$ and $c_{cg,jm}$ are nonzero and both of the corresponding pair $\pm \lambda_{cg,sq,cyc,j}$ contribute to (1.4) for the cyclic crossed-subgraph strip. It follows that the total number of terms for the cyclic crossed-subgraph strip of the square lattice is given by
\begin{equation}
N_{cg,sq,Ly,\lambda} = 2N_{sq,cyc,Ly,\lambda} - n_P(L_y, 0) - n_P(L_y, 1).
\end{equation}
(6.1.27)
Substituting our results from [26] for each of the quantities on the right-hand side, we obtain, for the number of unpaired terms for the crossed-subgraph strip, the relation
\begin{equation}
N_{cg,sq,Ly,up,\lambda} = M_{Ly-1} + M_{Ly}.
\end{equation}
(6.1.28)
for the number of terms comprising members of opposite-sign pairs, the relation
\begin{equation}
N_{cg,sq,Ly,opd,\lambda} = 2(N_{cg,sq,Ly,\lambda} - N_{sq,cyc,Ly,\lambda})
= 2\left[2(L_y - 1)! \sum_{j=0}^{|L_y|} \frac{(L_y - j)}{(j!)^2(L_y - 2j)!} - M_{Ly-1} - M_{Ly}\right]
\end{equation}
(6.1.29)
and for the total number of (nonzero) terms,
\begin{equation}
N_{cg,sq,Ly,\lambda} = 4(L_y - 1)! \sum_{j=0}^{|L_y|} \frac{(L_y - j)}{(j!)^2(L_y - 2j)!} - M_{Ly-1} - M_{Ly}.
\end{equation}
(6.1.30)
6.2 $L_y = 2$ Cyclic Crossing-Subgraph Strip of the Square Lattice

For example, for $L_y = 2$, using the Motzkin numbers $M_1 = 1$, $M_2 = 2$, the total number of terms is $N_{cg,sq,2,\lambda} = 5$, with $N_{cg,sq,2,up,\lambda} = 3$ and $N_{cg,sq,2,opd,\lambda} = 2$. We have explicitly calculated the chromatic polynomial for this case using iterated deletion-contraction and coloring matrix methods and find (with the shorthand $cgs2$ for $cg(sq, 2 \times L_x, cyc)$)

$$P(cg(sq, 2 \times L_x, cyc)) = \sum_{j=1}^{5} c_{cgs2,j}(\lambda_{cgs2,j})^{L_x}$$

(6.2.1)

where

$$\lambda_{cgs2,j} = \pm 1, j = 1, 2$$

(6.2.2)

$$\lambda_{cgs2,3} = 3 - q$$

(6.2.3)

$$\lambda_{cgs2,4} = q - 1$$

(6.2.4)

$$\lambda_{cgs2,5} = q^2 - 3q + 3$$

(6.2.5)

with coefficients

$$c_{cgs2,1} = \frac{1}{2}(c^{(2)} - c^{(0)}) = \frac{1}{2}q(q - 3)$$

(6.2.6)

$$c_{cgs2,2} = \frac{1}{2}(c^{(2)} + c^{(0)}) = \frac{1}{2}(q - 1)(q - 2)$$

(6.2.7)

$$c_{cgs2,j} = c^{(1)} = q - 1 \text{ for } j = 3, 4$$

(6.2.8)

$$c_{cgs2,5} = c^{(0)} = 1.$$  

(6.2.9)

For even $L_x$, this chromatic polynomial for the width $L_y = 2$ crossing graph strip of the square lattice reduces to the result for the regular cyclic strip of the square lattice with $N_{sq,2,cyc,\lambda} = 4$, namely, with $L_x = m$,

$$P(sq, 2 \times L_x, cyc.) = (q^2 - 3q + 1) + (q - 1)\left[(3 - q)^m + (1 - q)^m\right] + (q^2 - 3q + 3)^m$$

(6.2.10)

while for odd $L_x$, the chromatic polynomial (6.2.1) reduces to the result for the $L_y = 2$ Möbius strip with $N_{sq,2,Mb,\lambda} = 4$, namely

$$P(sq, 2 \times L_x, Mb) = -1 + (q - 1)\left[(3 - q)^m - (1 - q)^m\right] + (q^2 - 3q + 3)^m.$$  

(6.2.11)
6.3 \( L_y = 3 \) Cyclic Crossing-Subgraph Strip of the Square Lattice

The chromatic polynomial for the \( L_y = 3 \) cyclic crossing-subgraph strip of the square lattice (labelled \( cgs3 \)) can be calculated directly or from the known results for the chromatic polynomials of the \( L_y = 3 \) cyclic [20] and Möbius [21] strips of the square lattice. It is worthwhile to display the results here because of the unified understanding that they give concerning the structures of the chromatic polynomials for the cyclic and Möbius strips. From our general formulas above we have \( N_{cg, sq, 3, \lambda} = 14 \) with \( N_{cg, sq, 3, up, \lambda} = 6 \) and \( N_{cg, sq, 3, opd, \lambda} = 8 \). For the respective even and odd values of \( L_x \) where the chromatic polynomial reduces to that for the \( L_y = 3 \) cyclic and Möbius strips of the square lattice, we have \( N_{sq, 3, cyc, \lambda} = N_{sq, 3, Mb, \lambda} = 14 - \frac{1}{2} \times 8 = 10 \), in agreement with the previous calculations in [20, 21]. We find

\[
P(\text{cg}(sq, 3 \times L_x, cyc)) = \sum_{j=1}^{14} c_{cgs3,j}(\lambda_{cgs3,j})^{L_x}
\]  

(6.3.1)

where

\[
\lambda_{cgs3,j} = \pm 1 \text{ , } j = 1, 2
\]  

(6.3.2)

\[
\lambda_{cgs3,j} = \pm (q - 1) \text{ , } j = 3, 4
\]  

(6.3.3)

\[
\lambda_{cgs3,j} = \pm (q - 2) \text{ , } j = 5, 6
\]  

(6.3.4)

\[
\lambda_{cgs3,j} = \pm (q - 4) \text{ , } j = 7, 8
\]  

(6.3.5)

\[
\lambda_{cgs3,9} = (q - 2)^2
\]  

(6.3.6)

\[
\lambda_{cgs3,10} = \lambda_{sq3,6}
\]  

(6.3.7)

\[
\lambda_{cgs3,11} = \lambda_{sq3,7}
\]  

(6.3.8)

where \( \lambda_{sq3,j} \) for \( j = 6, 7 \) were given in eq. (3.10) of [20], and

\[
\lambda_{cgs3,j} = \lambda_{sq3,j-4} \text{ , } 12 \leq j \leq 14
\]  

(6.3.9)

where \( \lambda_{sq3,j} \) for \( j = 8, 9, 10 \) were defined by eq. (3.11) of [20].

The corresponding coefficients are

\[
c_{cgs3,1} = \frac{1}{2}(c^{(3)} - c^{(2)}) = \frac{1}{2}(q - 2)(q^2 - 4q + 1)
\]  

(6.3.10)

\[
c_{cgs3,2} = \frac{1}{2}(c^{(3)} + c^{(2)}) = \frac{1}{2}q(q - 1)(q - 3)
\]  

(6.3.11)

\[
c_{cgs3,j} = \frac{1}{2}(c^{(2)} - c^{(0)}) = \frac{1}{2}q(q - 3) \text{ for } j = 3, 6, 7
\]  

(6.3.12)

27
\[ c_{cg3,j} = \frac{1}{2}(c^{(2)} + c^{(0)}) = \frac{1}{2}(q - 1)(q - 2) \quad \text{for } j = 4, 5, 8 \quad (6.3.13) \]
\[ c_{cg3,j} = c^{(1)} = q - 1 \quad \text{for } j = 9, 12, 13, 14 \quad (6.3.14) \]
\[ c_{cg3,j} = 1, \ j = 10, 11. \quad (6.3.15) \]

6.4 \( L_y = 4 \) Cyclic Crossing-Subgraph Strip of the Square Lattice

From our calculations of the chromatic polynomials for the width \( L_y = 4 \) cyclic and Möbius strips of the square lattice [25], we have obtained the chromatic polynomial for the \( L_y = 4 \) cyclic crossing-subgraph strip (labelled \( cg4 \)). In accord with our general formulas, we get \( N_{cg,sq,4,\lambda} = 39 \) with \( N_{cg,sq,4,up,\lambda} = 13 \) and \( N_{cg,sq,4,opd,\lambda} = 26 \). For respective even and odd \( L_x \), the reduction to the \( L_y = 4 \) cyclic and Möbius strips has \( N_{sq,4,cyc,\lambda} = N_{sq,4,Mb,\lambda} = 39 - (1/2)\times 26 = 26. \) We omit listing the terms and their coefficients here since they are lengthy and our previous examples are sufficient to illustrate our general formulas.

In passing, we note that the cyclic crossing-subgraph strip of the triangular lattice does not yield either the cyclic or Möbius strip of this lattice for even or odd \( L_x \). This provides a further understanding of our earlier findings that the coefficients for the \( L_y = 2 \) [20] and \( L_y = 3 \) [17] Möbius strips of the triangular lattice are not polynomials in \( q \).

6.5 \( L_y = 3 \) Toroidal Crossing-Subgraph Strip of the Square Lattice

For the toroidal crossing-subgraph strip of the square lattice with width \( L_y = 3 \) (labelled \( cgst3 \)) we find \( N_{cgst3,\lambda} = 12 \) with \( N_{cgst3,up,\lambda} = 4 \), \( N_{cgst3,opd,\lambda} = 2 \), and \( N_{cgst3,ops,\lambda} = 6 \) (see Table [4]). Our result is

\[ P(cg(sq, 3 \times L_x, PBC_y, PBC_z)) = \sum_{j=1}^{12} c_{cgst3,j}(\lambda_{cgst3,j})^{L_x} \quad (6.5.1) \]

with

\[ \lambda_{cgst3,j} = \pm 1, j = 1, 2 \quad (6.5.2) \]
\[ \lambda_{cgst3,3} = 1 - q \quad (6.5.3) \]
\[ \lambda_{cgst3,j} = \pm(q - 2) \quad \text{for } j = 4, 5 \quad (6.5.4) \]
\[ \lambda_{cgst3,j} = \pm(q - 4) \quad \text{for } j = 6, 7 \quad (6.5.5) \]
\[ \lambda_{cgst3,8} = q - 5 \quad (6.5.6) \]
\[ \lambda_{cgst3,j} = \pm(q - 2)^2 \quad \text{for } j = 9, 10 \quad (6.5.7) \]
\[ \lambda_{cgst3,11} = -(q^2 - 7q + 13) \quad (6.5.8) \]

28
\[ \lambda_{gst3,12} = q^3 - 6q^2 + 14q - 13 \quad (6.5.9) \]

with corresponding coefficients

\[ c_{gst3,1} = \frac{1}{2} (q - 2)(q^2 - 4q + 1) \quad (6.5.10) \]
\[ c_{gst3,2} = \frac{1}{2} q(q^2 - 6q + 7) \quad (6.5.11) \]
\[ c_{gst3,j} = \frac{1}{2} (q - 1)(q - 2) \quad \text{for } j = 3, 6, 7 \quad (6.5.12) \]
\[ c_{gst3,j} = \frac{1}{2} q(q - 3) \quad \text{for } j = 4, 5, 8 \quad (6.5.13) \]
\[ c_{gst3,j} = q - 1 \quad \text{for } 9 \leq j \leq 11 \quad (6.5.14) \]
\[ c_{gst3,12} = 1. \quad (6.5.15) \]

Hence, in the even-\( L_x \) case, the reduction to the chromatic polynomial for the \( L_y = 3 \) toroidal strip of the square lattice (labelled \( st3 \)) has, by eq. (6.1.13), \( N_{st3,\lambda} = 12 - 1 - 3 = 8 \) terms, while in the odd-\( L_x \) case, the reduction to the chromatic polynomial for the \( L_y = 3 \) Klein bottle strip of the square lattice (labelled \( sk3 \)) has \( N_{sk3,\lambda} = 12 - 1 - 6 = 5 \). These are in agreement with the results that were obtained in [16]. The resultant coefficients for the torus and Klein bottle strips can be computed from the analogues of formulas (6.1.6) and (6.1.7) and agree with those given in [16].

These correspondences shed further light on the coefficients that enter into the chromatic polynomials for strip graphs with torus and Klein bottle boundary conditions. We recall that these are not of the simple form expressible in terms of Chebyshev polynomials of the second kind that we found for strip graphs with cyclic and Möbius boundary conditions in [26]. Among other differences, there is not a unique coefficient with a given degree \( d \) in \( q \); for example, for degree \( d = 2 \), both \((q - 1)(q - 2)/2 \) and \( q(q - 3)/2 \) as coefficients. However, the reduction of the chromatic polynomials for the toroidal crossing-subgraph strips to the respective chromatic polynomials for the torus and Klein bottle strips yields relations linking the coefficients in the latter to Chebyshev polynomials, such as (6.2.6) and (6.2.7). In the Appendix we list our results for the \( L_y = 4 \) crossing-subgraph toroidal strip of the square lattice (labelled \( cgst4 \))

### 7 Toroidal Crossing-Subgraph Strips of the Triangular Lattice

We consider here a strip of the triangular lattice of fixed width \( L_y \) and arbitrarily great length \( L_x \) constructed as follows. As before, label the vertices of two successive transverse slices
Table 4: Numbers of nonzero terms $\lambda_{G_s,j}$, $N_{G_s,up,\lambda}$, $N_{G_s,opd,\lambda}$, and $N_{G_s,ops,\lambda}$ for crossing-subgraph strips of the square and triangular lattices. See text for notation.

| $G_s$ | $L_y$ | $BC_y$ | $BC_x$ | $N_{G_s,\lambda}$ | $N_{G_s,up,\lambda}$ | $N_{G_s,opd,\lambda}$ | $N_{G_s,ops,\lambda}$ |
|-------|-------|--------|--------|-------------------|----------------------|------------------------|---------------------|
| cg,sq | 3     | P      | P      | 12                | 4                    | 2                      | 6                   |
| cg,sq | 4     | P      | P      | 48                | 18                   | 8                      | 22                  |
| cg,sq | 2     | F      | P      | 5                 | 3                    | 2                      | 0                   |
| cg,sq | 3     | F      | P      | 14                | 6                    | 8                      | 0                   |
| cg,sq | 4     | F      | P      | 39                | 13                   | 26                     | 0                   |
| cg,tri| 3     | P      | P      | 12                | 4                    | 2                      | 6                   |
| cg,tri| 4     | P      | P      | 40                | 10                   | 4                      | 26                  |

of the strip, starting at the top, as $(1, 2, \ldots, L_y)$ and $(1', 2', \ldots, L'_y)$. With periodic transverse boundary conditions, these transverse slices form circuit graphs, $C_{L_y}$. Connect these with edges linking vertices 1 to $L'_y$, 2 to $(L_y - 1)'$, and so forth. This forms the toroidal crossing-subgraph strip of the square lattice. Next, add diagonal edges as illustrated for the $L_y = 3$ case in Fig. 4. Finally, impose periodic longitudinal boundary conditions. This yields the crossing-subgraph (cg) strip of the triangular (tri) lattice with toroidal boundary conditions. We shall label this strip as $cg(tri, L_y \times L_x, PBC_y, PBC_x)$.

We first observe that

$$cg(tri, L_y \times L_x, PBC_y, PBC_x) = tri(L_y \times L_x, PBC_y, PBC_x) \quad \text{if} \quad L_x = 0 \mod 2L_y \quad (7.1)$$
i.e., for $L_x = 0 \mod 2L_y$, the toroidal crossing-subgraph strip of the triangular lattice reduces to the toroidal strip of this lattice. Associated with this result, there are terms in the
chromatic polynomial for the resultant toroidal strip of the triangular lattice corresponding to opposite-sign pairs of $\lambda_{cgtL_y,j}$'s in the chromatic polynomial for the toroidal crossing-subgraph strip that appear with phase factors of the form $e^{\pm \pi i/L_y}$ and certain products thereof. For $L_y = 3$ these are $e^{\pm 2\pi i/3}$, while for $L_y = 4$ they are $e^{\pm \pi i/4}$, $\ell = 1, 2$. It is thus necessary to distinguish between terms $\lambda_{cgtL_y,j}$ in the chromatic polynomial for the toroidal crossing-subgraph strip of the triangular lattice that correspond to terms in the toroidal strip of the triangular lattice that do, or do not, have the form of complex-conjugate pairs of terms $\lambda_{ttL_y,j}$ with complex prefactors such as $e^{\pm 2\pi i/3}$. (where the subscript $ttL_y$ refers to the toroidal strip of the triangular lattice with width $L_y$). We thus define $N_{cg,tri,L_y,torus,ops,r,\lambda}$ and $N_{cg,tri,L_y,torus,ops,i,\lambda}$ respectively as the number of $\lambda_{cgtL_y,j}$'s that comprise opposite-sign pairs such that the members of each pair have the same coefficient and correspond to a $\lambda_{ttL_y,j}$ that is real (is a member of a pair with complex prefactor) for real $q$. The resultant general formula relating these numbers of terms is

$$N_{tri,L_y,torus,\lambda} = N_{cg,tri,L_y,torus,\lambda} - \frac{1}{2} N_{cg,tri,L_y,torus,ops,\lambda} - \frac{1}{2} N_{cg,tri,L_y,torus,ops,r,\lambda} \quad (7.2)$$

If (i) $L_y$ is odd and $L_x$ is odd, or (ii) if $L_y$ is even and $L_x = 1 \mod 4$, then the toroidal crossing-subgraph strip of the triangular lattice reduces to the strip of this lattice with Klein bottle boundary conditions ($PBC_y, TPBC_x$). In this case, the reduction of the number of terms is determined by eq. (6.1.14) with the obvious replacement of square by triangular lattice strip, i.e.,

$$N_{tri,L_y,Kb,\lambda} = N_{cg,tri,L_y,torus,\lambda} - \frac{1}{2} N_{cg,tri,L_y,torus,ops,\lambda} - N_{cg,tri,L_y,torus,ops,\lambda} . \quad (7.3)$$

Thus,

$$N_{tri,L_y,torus,\lambda} - N_{tri,L_y,Kb,\lambda} = N_{cg,tri,L_y,torus,ops,\lambda} - \frac{1}{2} N_{cg,tri,L_y,torus,ops,r,\lambda} . \quad (7.4)$$

In the limit $L_x \to \infty$, the width $L_y$ toroidal crossing-subgraph strip of the triangular lattice yields a $W$ function via (1.2) and hence a singular locus $B$. As before, we can take this limit using values of $L_x$ such that the crossing-subgraph strip reduces to the strip of the triangular lattice with either torus or Klein bottle boundary conditions. Given that the original limit exists, all three of these limits must be the same. This proves

Theorem 2: The $W$ function and singular locus $B$ are the same for the $L_x \to \infty$ limit of strip of the triangular lattice with width $L_y$ and length $L_x$ whether one imposes ($PBC_y, PBC_x$) or ($PBC_y, TPBC_x$), i.e. torus or Klein bottle boundary conditions.

Hence also
Corollary 2: Consider an $L_y \times L_x$ strip of the triangular lattice with torus or Klein bottle boundary conditions, and denote the set of nonzero eigenvalues that contribute to (1.4) for these two respective strips as $\lambda_{ttL_y,j}$, $j = 1, \ldots, N_{ttL_y,\lambda}$ and $\lambda_{tkL_y,j}$, $j = 1, \ldots, N_{tkL_y,\lambda}$. Focus on the set of eigenvalues $\lambda_{ttL_y,j}$ that do not occur among the set $\lambda_{tkL_y,j}$ (the number of these is given by (7.4)); none of these can be dominant eigenvalues.

For the $L_y = 3$ crossing-subgraph toroidal strip of the triangular lattice (labelled $cgt3$) we find that $N_{cgt3,\lambda} = 12$ and calculate

$$
P(cg(tri, 3 \times L_x, torus)) = \sum_{j=1}^{12} c_{cgt3,j} (\lambda_{cgt3,j})^{L_x} \tag{7.5}
$$

where

$$
\lambda_{cgt3,j} = \pm 1 \quad \text{for } j = 1, 2 \tag{7.6}
$$

$$
\lambda_{cgt3,j} = \pm 2 \quad \text{for } j = 3, 4 \tag{7.7}
$$

$$
\lambda_{cgt3,j} = \pm (2q - 7) \quad \text{for } j = 5, 6 \tag{7.8}
$$

$$
\lambda_{cgt3,7} = 2 - q \tag{7.9}
$$

$$
\lambda_{cgt3,8} = 3q - 14 \tag{7.10}
$$

$$
\lambda_{cgt3,9} = -2(q - 4)^2 \tag{7.11}
$$

$$
\lambda_{cgt3,j} = \pm (q^2 - 5q + 7) \quad \text{for } j = 10, 11 \tag{7.12}
$$

$$
\lambda_{cgt3,12} = q^3 - 9q^2 + 29q - 32 \tag{7.13}
$$

The corresponding coefficients are

$$
c_{cgt3,j} = \frac{1}{2} q(q - 1)(q - 2) \quad \text{for } j = 1, 2 \tag{7.14}
$$

$$
c_{cgt3,3} = \frac{1}{6} (q - 1)(q - 2)(q - 3) \tag{7.15}
$$

$$
c_{cgt3,4} = \frac{1}{6} q(q - 1)(q - 5) \tag{7.16}
$$

$$
c_{cgt3,j} = \frac{1}{2} (q - 1)(q - 2) \quad \text{for } 5 \leq j \leq 7 \tag{7.17}
$$

$$
c_{cgt3,8} = \frac{1}{2} q(q - 3) \tag{7.18}
$$

$$
c_{cgt3,j} = q - 1 \quad \text{for } 9 \leq j \leq 11 \tag{7.19}
$$

32
\[ c_{cg3,12} = 1. \]  
(7.20)

Thus, \( N_{cg3,up,\lambda} = 4, N_{cg3,opd,\lambda} = 2, \) and \( N_{cg3,ops,\lambda} = 6. \) Hence, for \( L_x = 0 \text{ mod } 6, \) where the chromatic polynomial reduces to that for the \( L_y = 3 \) toroidal strip of the triangular lattice, the number of \( \lambda_j \)'s is reduced, according to the general formula (7.2), to \( N_{tt3,\lambda} = 12 - 1 = 11. \) These numbers agree with our previous result \[17\], 

\[ P(tri, 3 \times L_x, torus) = \sum_{j=1}^{11} c_{tt3,j}(\lambda_{tt3,j})^{L_x} \]  
(7.21)

where

\[ \lambda_{tt3,1} = -2 \]  
(7.22)

\[ \lambda_{tt3,2} = q - 2 \]  
(7.23)

\[ \lambda_{tt3,3} = 3q - 14 \]  
(7.24)

\[ \lambda_{tt3,4} = -2(q - 4)^2 \]  
(7.25)

\[ \lambda_{tt3,5} = q^3 - 9q^2 + 29q - 32 \]  
(7.26)

\[ \lambda_{tt3,j} = e^{\pm 2\pi i/3} \text{ for } j = 6, 7 \]  
(7.27)

\[ \lambda_{tt3,j} = (q^2 - 5q + 7)e^{\pm 2\pi i/3} \text{ for } j = 8, 9 \]  
(7.28)

\[ \lambda_{tt3,j} = -(2q - 7)e^{\pm 2\pi i/3} \text{ for } j = 10, 11 \]  
(7.29)

with coefficients

\[ c_{tt3,1} = \frac{1}{3}(q - 1)(q^2 - 5q + 3) \]  
(7.30)

\[ c_{tt3,j} = \frac{1}{2}(q - 1)(q - 2) \text{ for } j = 2, 10, 11 \]  
(7.31)

\[ c_{tt3,3} = \frac{1}{2}q(q - 3) \]  
(7.32)

\[ c_{tt3,j} = q - 1 \text{ for } j = 4, 8, 9 \]  
(7.33)

\[ c_{tt3,5} = 1 \]  
(7.34)

\[ c_{tt3,6} = c_{tt3,7} = \frac{1}{6}q(q - 1)(2q - 7). \]  
(7.35)

Similarly, for odd \( L_x \) where the \( L_y = 3 \) crossing-subgraph strip reduces to the \( L_y = 3 \) Klein bottle strip of the triangular lattice, the number of nonzero terms is reduced, according to the general formula (7.3), to \( N_{tk3,\lambda} = 12 - 1 - 6 = 5, \) in agreement with our previous calculation \[17\],

\[ P(tri, 3 \times L_x, PBC_y, TPBC_x) = \sum_{j=1}^{5} c_{tk3,j}(\lambda_{tk3,j})^{L_x} \]  
(7.36)
where

\[
\begin{align*}
\lambda_{tk,1} &= \mu_{t3,1} = -2 \\
\lambda_{tk,2} &= \mu_{t3,2} = q - 2 \\
\lambda_{tk,3} &= \mu_{t3,3} = 3q - 14 \\
\lambda_{tk,4} &= \mu_{t3,4} = -2(q - 4)^2 \\
\lambda_{tk,5} &= \mu_{t3,5} = q^3 - 9q^2 + 29q - 32
\end{align*}
\]  

(7.37)

(7.38)

(7.39)

(7.40)

(7.41)

with coefficients

\[
\begin{align*}
c_{tk,1} &= -(q - 1) \\
c_{tk,2} &= -\frac{1}{2}(q - 1)(q - 2) \\
c_{tk,3} &= \frac{1}{2}q(q - 3) \\
c_{tk,4} &= q - 1 \\
c_{tk,5} &= 1
\end{align*}
\]  

(7.42)

(7.43)

(7.44)

(7.45)

(7.46)

In the Appendix we give our results for the \(L_y = 4\) crossing-subgraph toroidal strip of the triangular lattice.

8 Concluding Discussion

We comment further here on some features of our results.

1. Our exact calculations of the singular loci \(B\) for \(L_y = 4\) strips of the square and triangular lattice with toroidal or Klein bottle conditions exhibit the following features, as did the earlier calculations for the \(L_y = 3\) strips of these lattices in \([16, 17]\): \(B\) (i) passes through \(q = 0\), (ii) passes through \(q = 2\), (iii) passes through a maximal real point, thereby defining a \(q_c\), and (iv) encloses one or more regions including the interval \(0 < q < q_c\) \([8]\). As noted above, we also found that these four features hold for the \((L_x \to \infty\) limit of) strips with cyclic and Möbius boundary conditions, which leads to the inference that the key condition is the existence of periodic (or reversed-orientation periodic) longitudinal boundary conditions.

2. Previous exact calculations of \(B\) for cyclic and Möbius strips of the square and Möbius lattice of various widths \([8, 20, 21, 17, 25]\) are consistent with the inference that as \(L_y\) increases, the outer envelope of \(B\) moves outward, i.e. if \(L_y > L_y'\), then the
outer envelope of $\mathcal{B}$ for $L_y$ encloses that for $L'_y$ [22]. In particular, $q_c$ is a nondecreasing function of $L_y$. However, we have also shown that neither of these properties holds for strips with $(PBC_y, FBC_x)$, i.e., cylindrical, boundary conditions [28, 17, 25]. Our present results show that for strips of the square and triangular lattices with toroidal or Klein bottle boundary conditions, $(PBC_y, (T)PBC_x)$, the outer envelope of $\mathcal{B}$ does not, in general, move monotonically outward as one increases the width. This is illustrated in Figs. 5 and 6, which show the boundaries $\mathcal{B}$ for the $L_y = 3$ and $L_y = 4$ strips of, respectively, the square and triangular lattices with torus or Klein bottle boundary conditions. See also Table 2. This monotonic (non monotonic) behavior of the outer envelope is reminiscent of the monotonic (non monotonic) behavior of the $W$ function for free (periodic) transverse boundary conditions discussed in [30] (see also [17, 25]).

3. As a special aspect of this outer envelope, $q_c$ decreases from 3 to approximately 2.78 for the (infinite-length limit of the) square-lattice strip with toroidal or Klein bottle

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{Comparison of the singular loci $\mathcal{B}$ for the $L_x \to \infty$ limit of the strips of the square lattice with $L_y = 3$ (darker curve) and $L_y = 4$ (lighter curve) with toroidal boundary conditions (or equivalently, Klein bottle boundary conditions, which yield the same $\mathcal{B}$ for a given $L_y$).}
\end{figure}
Figure 6: Comparison of the singular loci $\mathcal{B}$ for the $L_x \to \infty$ limit of the strips of the triangular lattice with $L_y = 3$ (darker curve) and $L_y = 4$ (lighter curve) with toroidal boundary conditions (or equivalently, Klein bottle boundary conditions, which yield the same $\mathcal{B}$ for a given $L_y$).
boundary conditions when one increase the width from \( L_y = 3 \) to \( L_y = 4 \). In contrast, \( q_c \) increases from about 3.72 to 4 for the (infinite-length limit of the) triangular-lattice strip with toroidal or Klein bottle boundary conditions when one increases \( L_y \) from 3 to 4. Related to this, the calculation of \( P \) and \( W \) and \( B \) for the \( L_y = 3 \) strip of the square lattice with toroidal or Klein bottle boundary conditions in [14] showed that \( q_c = 3 \) for (the \( L_x \rightarrow \infty \) limit of) that strip, and hence showed that \( q_c \) for a finite-width, infinite-length strip of a given lattice can be the same as for the limit of infinite width, i.e. the full 2D infinite lattice, since \( q_c = 3 \) for the square lattice [31] (for general upper bounds, see [32]). So far, this was an isolated example. Our calculation in the present paper provides a second example of this phenomenon: \( q_c \) for the infinite-length limit of the \( L_y = 4 \) strip of the triangular lattice with toroidal or Klein bottle boundary conditions has the value \( q_c = 4 \), which is equal to the value [33] for the full 2D triangular lattice, i.e. the \( L_y \rightarrow \infty \) limit of the strip. Parenthetically, we note that rigorous bounds on \( q_c \) have been given in [32].

4. In all of the cases of strips of the square and triangular lattice with periodic or reversed-orientation periodic longitudinal boundary conditions for which we have performed exact calculations of the chromatic polynomials and have determined the respective singular loci \( B \), we have found the following results for the coefficients corresponding to the dominant terms in various regions: (i) in region \( R_1 \), this coefficient has been proved to be unity [22]; (ii) in the region containing the interval \( 0 < q < 2 \), the coefficient is \( c^{(1)} = q - 1 \) for the strips of the square and triangular lattice with cyclic and torus b.c., the square strips with Möbius and Klein bottle b.c. and the triangular strips with Klein bottle b.c. (the coefficients are not, in general, polynomials for Möbius strips of the triangular lattice); and (iii) for the observed complex-conjugate pairs of regions, the coefficients are also \( q - 1 \) for cyclic and torus b.c. and \( \pm(q - 1) \) for Möbius (sq case) and Klein bottle b.c. [8, 19, 21, 17, 25]. A fourth finding is that (iv) for the torus/Klein bottle boundary conditions, in the cases that we have studied, we have found that the coefficient corresponding to the dominant \( \lambda_{G,j} \) in the region containing the interval \( 2 < q < q_c \), for each respective \( q_c \), is \( q(q - 3)/2 \).

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9 Appendix: Further Results on Crossing-Subgraph Strips

9.1 $L_y = 4$ Toroidal Crossing-Subgraph Strip of the Square Lattice

For the $L_y = 4$ crossing-subgraph toroidal strip of the square lattice (labelled \(cgst4\)) we find that $N_{cgst4,\lambda} = 48$ and calculate

\[
P(cg(sq, 4 \times L_x, torus)) = \sum_{j=1}^{48} c_{cgst4,j}(\lambda_{cgst4,j})^{L_x} \tag{9.1.1}
\]

where

\[
\lambda_{cgst4,j} = \pm 1 \text{ for } j = 1, 2 \tag{9.1.2}
\]

\[
\lambda_{cgst4,j} = \pm (q - 1) \text{ for } j = 3, 4 \tag{9.1.3}
\]

\[
\lambda_{cgst4,j} = \pm (q - 2) \text{ for } j = 5, 6 \tag{9.1.4}
\]

\[
\lambda_{cgst4,j} = \pm (q - 3) \text{ for } j = 7, 8 \tag{9.1.5}
\]

\[
\lambda_{cgst4,j} = \pm (q - 4) \text{ for } j = 9, 10 \tag{9.1.6}
\]

\[
\lambda_{cgst4,j} = \pm (q - 5) \text{ for } j = 11, 12 \tag{9.1.7}
\]

\[
\lambda_{cgst4,13} = -(q^2 - 5q + 5) \tag{9.1.8}
\]

\[
\lambda_{cgst4,14} = q^2 - 5q + 7 \tag{9.1.9}
\]

\[
\lambda_{cgst4,15} = -(q - 1)(q - 3) \tag{9.1.10}
\]

\[
\lambda_{cgst4,j} = \lambda_{st4,j-6} \text{ for } j = 16, 17 \tag{9.1.11}
\]

(where $\lambda_{st4,j}$ for $j = 10, 11$ were given in the text in eq. (2.12)),

\[
\lambda_{cgst4,j} = \pm \lambda_{st4,12} \text{ for } j = 18, 19 \tag{9.1.12}
\]

\[
\lambda_{cgst4,j} = \pm \lambda_{st4,13} \text{ for } j = 20, 21 \tag{9.1.13}
\]

\[
\lambda_{cgst4,j} = -\lambda_{st4,j-8} \text{ for } j = 22, 23 \tag{9.1.14}
\]

\[
\lambda_{cgst4,j} = \pm \lambda_{st4,16} \text{ for } j = 24, 25 \tag{9.1.15}
\]

\[
\lambda_{cgst4,j} = \pm \lambda_{st4,17} \text{ for } j = 26, 27 \tag{9.1.16}
\]

\[
\lambda_{cgst4,j} = -\lambda_{st4,j-10} \text{ for } j = 28, 29 \tag{9.1.17}
\]

\[
\lambda_{cgst4,j} = \pm \lambda_{st4,20} \text{ for } j = 30, 31 \tag{9.1.18}
\]
\[ \lambda_{\text{cgst4}, j} = \pm \lambda_{\text{st4}, 21} \text{ for } j = 32, 33 \]  \hfill (9.1.19)

\[ \lambda_{\text{cgst4}, j} = -\lambda_{\text{st4}, j-12} \text{ for } 34 \leq j \leq 36 \]  \hfill (9.1.20)

\[ \lambda_{\text{cgst4}, j} = \lambda_{\text{st4}, j-12} \text{ for } 37 \leq j \leq 42 \]  \hfill (9.1.21)

\[ \lambda_{\text{cgst4}, j} = \pm \lambda_{\text{st4}, 31} \text{ for } j = 43, 44 \]  \hfill (9.1.22)

\[ \lambda_{\text{cgst4}, j} = \pm \lambda_{\text{st4}, 32} \text{ for } j = 45, 46 \]  \hfill (9.1.23)

\[ \lambda_{\text{cgst4}, j} = \pm \lambda_{\text{st4}, 33} \text{ for } j = 47, 48 \]  \hfill (9.1.24)

The corresponding coefficients are

\[ c_{\text{cgst4,1}} = \frac{1}{2} (q - 1)(q^3 - 7q^2 + 13q - 2) \]  \hfill (9.1.25)

\[ c_{\text{cgst4,2}} = \frac{1}{2} q(q - 3)(q^2 - 5q + 5) \]  \hfill (9.1.26)

\[ c_{\text{cgst4,j}} = \frac{1}{6} q(q - 1)(q - 5) \text{ for } j = 3, 12 \]  \hfill (9.1.27)

\[ c_{\text{cgst4,4}} = \frac{1}{6} (q - 1)(q - 2)(q - 3) \]  \hfill (9.1.28)

\[ c_{\text{cgst4,j}} = \frac{1}{3} q(q - 2)(q - 4) \text{ for } j = 5, 9, 10 \text{ and } 30 \leq j \leq 33 \]  \hfill (9.1.29)

\[ c_{\text{cgst4,6}} = \frac{1}{6} (q - 2)(q - 3)(2q + 1) \]  \hfill (9.1.30)

\[ c_{\text{cgst4,j}} = \frac{1}{6} (q - 1)(q^2 - 5q + 3) \text{ for } j = 7, 8 \]  \hfill (9.1.31)

\[ c_{\text{cgst4,11}} = \frac{1}{6} (q - 1)(q - 2)(q - 3) \]  \hfill (9.1.32)

\[ c_{\text{cgst4,j}} = \frac{1}{2} (q - 1)(q - 2) \text{ for } j = 13, 28, 29 \text{ and } 43 \leq j \leq 48 \]  \hfill (9.1.33)

\[ c_{\text{cgst4,j}} = \frac{1}{2} q(q - 3) \text{ for } j = 14 \text{ and } 22 \leq j \leq 27, \text{ 40 } \leq j \leq 42 \]  \hfill (9.1.34)

\[ c_{\text{cgst4,j}} = 1 \text{ for } j = 15, 16, 17 \]  \hfill (9.1.35)

\[ c_{\text{cgst4,j}} = q - 1 \text{ for } 18 \leq j \leq 21 \text{ and } 34 \leq j \leq 39 \]  \hfill (9.1.36)

Thus, \( N_{\text{cgst4,opd,} \lambda} = 8 \) and \( N_{\text{cgst4,ops,} \lambda} = 22 \) and hence for the even and odd \( L_x \) values where the chromatic polynomial for this crossing-subgraph strip reduces to the respective chromatic polynomial for the \( L_y = 4 \) strip with torus and Klein bottle boundary conditions, we have, from eqs. (6.1.13) and (6.1.14), \( N_{\text{st4,} \lambda} = 48 - (1/2) \ast (8 + 22) = 33 \) and \( N_{\text{sk4,} \lambda} = 48 - (1/2) \ast 8 - 22 = 22 \), in agreement with our calculations in the text.
9.2 $L_y = 4$ Toroidal Crossing-Subgraph Strip of the Triangular Lattice

For the $L_y = 4$ crossing-subgraph toroidal strip of the triangular lattice (labelled $cgt4$) we find that there are $N_{cgt4,\lambda} = 40$ different nonzero $\lambda_{cgt4,j}$ terms that enter into the chromatic polynomial and that the coloring matrix also has a zero eigenvalue, so that the total number of eigenvalues of the coloring matrix for this strip is $N_{cgt4,\lambda,tot} = 41$. We calculate

$$P(cg(tri, 4 \times L_x, torus)) = \sum_{j=1}^{40} c_{cgt4,j}(\lambda_{cgt4,j})^{L_x}$$

(9.2.1)

where

$$\lambda_{cgt4,j} = \pm 2 \quad \text{for} \quad j = 1, 2$$

(9.2.2)

$$\lambda_{cgt4,j} = \pm \sqrt{2} \quad \text{for} \quad j = 3, 4$$

(9.2.3)

$$\lambda_{cgt4,5} = 2(3 - q)$$

(9.2.4)

$$\lambda_{cgt4,j} = \pm (3 - q) \quad \text{for} \quad j = 6, 7$$

(9.2.5)

$$\lambda_{cgt4,j} = \pm 2(2q - 9) \quad \text{for} \quad j = 8, 9$$

(9.2.6)

$$\lambda_{cgt4,10} = -2(q - 3)^2$$

(9.2.7)

$$\lambda_{cgt4,j} = \lambda_{tt4,j-3} \quad \text{for} \quad j = 11, 12$$

(9.2.8)

where $\lambda_{tt4,j}$ for $j = 8, 9$ were given above in eqs. (4.9),

$$\lambda_{cgt4,j} = \pm \sqrt{3}(q - 3) \quad \text{for} \quad j = 13, 14$$

(9.2.9)

$$\lambda_{cgt4,j} = \pm (q - 2)\sqrt{2(q - 3)(q - 4)} \quad \text{for} \quad j = 15, 16$$

(9.2.10)

$$\lambda_{cgt4,j} = \lambda_{tt4,j-3} \quad \text{for} \quad 17 \leq j \leq 22$$

(9.2.11)

The twelve terms $\lambda_{cgt4,j}$ for $23 \leq j \leq 34$ are related to the $\lambda_{tt4,j}$’s that are the roots of the quartic equations (4.14)-(4.16) as follows, where the $a_\ell$ and $b_\ell$ were defined in eqs. (4.18)-(4.23):

$$\lambda_{cgt4,j} = \pm \sqrt{2}a_1 \quad \text{for} \quad j = 23, 24$$

(9.2.12)

$$\lambda_{cgt4,j} = \pm \sqrt{2}b_1 \quad \text{for} \quad j = 25, 26$$

(9.2.13)

$$\lambda_{cgt4,j} = \pm \sqrt{2}a_2 \quad \text{for} \quad j = 27, 28$$

(9.2.14)

$$\lambda_{cgt4,j} = \pm \sqrt{2}b_2 \quad \text{for} \quad j = 29, 30$$

(9.2.15)

$$\lambda_{cgt4,j} = \pm \sqrt{2}a_3 \quad \text{for} \quad j = 31, 32$$

(9.2.16)
\[ \lambda_{cgt4,j} = \pm \sqrt{2}b_3 \quad \text{for} \quad j = 33, 34 . \] (9.2.17)

The six terms \( \lambda_{cgt4,j} \) for \( 35 \leq j \leq 40 \) are related to the \( \lambda_{tt4,j} \)'s that are the roots of the sixth-degree equation (4.17) as follows, where the \( c_\ell \) were defined in eqs. (4.24)-(4.26):

\[ \lambda_{cgt4,j} = \pm \sqrt{2}c_1 \quad \text{for} \quad j = 35, 36 \] (9.2.18)
\[ \lambda_{cgt4,j} = \pm \sqrt{2}c_2 \quad \text{for} \quad j = 37, 38 \] (9.2.19)
\[ \lambda_{cgt4,j} = \pm \sqrt{2}c_3 \quad \text{for} \quad j = 39, 40 . \] (9.2.20)

The corresponding coefficients are

\[ c_{cgt4,1} = \frac{1}{8}q(q - 1)(q - 2)(q - 3) \] (9.2.21)
\[ c_{cgt4,2} = \frac{1}{8}q(q - 2)(q - 3)(q - 5) \] (9.2.22)
\[ c_{cgt4,j} = \frac{1}{12}(q - 1)(q - 2)(3q^2 - 11q - 6) \quad \text{for} \quad j = 3, 4 \] (9.2.23)
\[ c_{cgt4,j} = \frac{1}{2}(q - 1)(q - 2) \quad \text{for} \quad j = 5, 10 \quad \text{and} \quad 35 \leq j \leq 40 \] (9.2.24)
\[ c_{cgt4,j} = \frac{1}{3}q(q - 2)(q - 4) \quad \text{for} \quad j = 6, 7, 13, 14 \quad \text{and} \quad 31 \leq j \leq 34 \] (9.2.25)
\[ c_{cgt4,8} = \frac{1}{6}(q - 1)(q - 2)(q - 3) \] (9.2.26)
\[ c_{cgt4,9} = \frac{1}{6}q(q - 1)(q - 5) \] (9.2.27)
\[ c_{cgt4,j} = 1 \quad \text{for} \quad j = 11, 12 \] (9.2.28)
\[ c_{cgt4,j} = \frac{1}{2}q(q - 3) \quad \text{for} \quad j = 15, 16 \quad \text{and} \quad 20 \leq j \leq 22, \quad 27 \leq j \leq 30 \] (9.2.29)
\[ c_{cgt4,j} = q - 1 \quad \text{for} \quad 17 \leq j \leq 19 \quad \text{and} \quad 23 \leq j \leq 26 . \] (9.2.30)

Finally, the coloring matrix has a zero eigenvalue,

\[ \lambda_{cgt4,41} = 0 \] (9.2.31)

with multiplicity

\[ c_{cgt4,41} = \frac{1}{12}q(q - 1)(3q^2 - 17q + 40) . \] (9.2.32)

It follows from the general relation (7.1) that for \( L_x = 0 \mod 8 \), this \( L_y = 4 \) crossing-subgraph strip of the triangular lattice reduces to the regular \( L_y = 4 \) toroidal strip of the triangular lattice. This gives insight into the occurrence of the phase factors in several of
the $\lambda_{tt4,j}$ terms. For this strip, we have $N_{cg4,up,\lambda} = 2$, $N_{cg4,opd,\lambda} = 4$, and $N_{cg4,ops,\lambda} = 26$. Further, $N_{cg4,ops,r,\lambda} = 2$ and $N_{cg4,ops,i,\lambda} = 24$. Hence, by the general relation (7.2), for $L_x = 0 \mod 8$, where the $L_y = 4$ crossing-subgraph strip reduces to the toroidal strip of the triangular lattice, the number of nonzero terms is reduced to $N_{tt4,\lambda} = 40 - 2 - 1 = 37$. For $L_x = 1 \mod 4$ where the $L_y = 4$ crossing-subgraph strip reduces to the Klein bottle strip of the triangular lattice, the number of nonzero terms is reduced, according to the general formula (7.3), to $N_{tk4,\lambda} = 40 - 2 - 26 = 12$. These numbers agree with our exact calculations presented in the text.

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