Lie point symmetries of integrable evolution equations and invariant solutions

Andrei K. Svinin
Institute of System Dynamics and Control Theory
Siberian Branch of Russian Academy of Sciences
P.O. Box 1233, 664033 Irkutsk, Russia

Abstract

An integrable hierarchies connected with linear stationary Schrödinger equation with energy dependent potentials (in general case) are considered. Galilei-like and scaling invariance transformations are constructed. A symmetry method is applied to construct invariant solutions.

\[1\] e-mail: svinin@icc.ru
I. INTRODUCTION

Symmetries of partial differential equations are used for description of the general set of solutions, for producing families of solutions from known exact solutions, for construction of invariant solutions etc\textsuperscript{1–3}.

Integrable evolution equations (systems of equation), with the group point of view, has a remarkable property: they come in hierarchies of commuting flows which often it is possible to describe with the help of suitable recursion operator as

\[ u_{tm} = \Lambda^{m-1} K_1, \]

where $\Lambda$ is recursion operator and $K_1$ is some vector field. In many cases $K_1[u] = u_x$, which presents a shift of spatial variable $x \in \mathbb{R}^1$. The fact of compatibility of these flows make possible to say about simultaneous solution $u(x, t_2, t_3, ...)$ and investigate their group properties.

In this article we investigate integrable evolution equations (systems of equations) associated with stationary Schrödinger equation with energy-dependent potential. In this case, recursion operator $\Lambda$ is known in its explicit form. That allows us to establish transformation property of $\Lambda$. We construct one-parameter linear point transformation generated by the shift of spectral parameter $\lambda \rightarrow \lambda = \lambda - \epsilon$. Particular case of these transformations is well known Galilei transformation for Korteweg—de Vries equation. Also we construct scaling invariance group. We next apply standard symmetry method to construct simultaneous solutions, invariant w.r.t. Galilei-like group $G_{N,n}$. Corresponding ansatz involve the vector-function $U = U(X, T_2, ..., T_{N-2}, T_N)$, where $\{X = T_1, T_2, ..., T_{N-2}, T_N\}$ is the set of differential invariants. We show that this ansatz reduce first $N$ members of hierarchy

\[ u_{tm} = \Lambda^{m-1}[u]u_x = K_m[u], \quad m = 1, ..., N \]

to $(N - 2)$ members of that, i.e.,

\[ U_{T_m} = K_m[U], \quad m = 1, ..., N - 2, \]

with additional Galilean and scaling self-similarity constraint.

The plan of this article is as follows. In Sec. II, we construct symmetry transformation groups. Sec. III is devoted to construction $G_{N,n}$-invariant solutions.

II. INTEGRABLE HIERARCHIES AND SYMMETRY TRANSFORMATIONS

Let us recall some relevant notions which are useful throughout this paper. Let $M$ be a manifold of smooth vector-functions $u : \mathbb{R}^1 \rightarrow \mathbb{C}^n$. We denote by $A_u$, the algebra of polynomials in finite collection of variables $u^{(k)}_{ix}$.

**Definition 1:** Let $K[u] = (K_1[u], ..., K_n[u])^T \in T_u M$ is a vector field and $\Lambda : T_u M \rightarrow T_u M$ is a linear operator. The Gateaux derivatives of $K$ and $\Lambda$ with respect to $u$ in the
direction \( X \in T_u M \) are defined through the relations

\[
K'[u](X) = \frac{\partial}{\partial \tau} \bigg|_{\tau=0} K[u + \tau X], \quad \Lambda'[u](X) = \frac{\partial}{\partial \tau} \bigg|_{\tau=0} \Lambda[u + \tau X].
\]

**Definition 2:** The Lie derivatives of \( K \) and \( \Lambda \) in the direction \( X \) are defined, respectively, as

\[
L_X K = [X, K] = K'_{(X)} - X'_{(K)}, \quad L_X \Lambda = \Lambda'_{(X)} - [K', \Lambda].
\]

The linear space \( T_u M \) endowed with the commutator \([X, Y] = L_X Y\) bears the structure of the infinite Lie algebra \( \text{Vect} \).

**Definition 3:** The operator \( \Lambda : T_u M \to T_u M \) is called hereditary if its Nijenhuis torsion vanishes, i.e.

\[
T_\Lambda(X, Y) = [[\Lambda X, \Lambda Y] + [X, \Lambda Y]] + \Lambda^2 [X, Y] = 0
\]

for any \( X, Y \in T_u M \).

Vector fields \( \Lambda^{k-1} K_1 \) span abelian subalgebra in \( \text{Vect} \) if an operator \( \Lambda \) is hereditary and \( L_{K_1} \Lambda = 0 \).

It is known that, with each auxiliary linear equation

\[
\psi_{xx}(x, \lambda) + \sum_{i=1}^n u_i(x)(-\lambda)^{i-1} + (-\lambda)^n \psi(x, \lambda) = 0,
\]

one can associate the hierarchies of commuting flows

\[
u_{t_m} = K_m[u] = \Lambda^{m-1}[u] u_x, \quad u = (u_1, \ldots, u_n)^T
\]

with the hereditary recursion operator of the form

\[
\Lambda[u] = \begin{pmatrix}
0 & 0 & \cdots & 0 & \frac{1}{4} u_x^2 + u_1 + \frac{1}{2} u_{1x} \partial_x^{-1} \\
-1 & 0 & \cdots & 0 & u_2 + \frac{1}{2} u_{2x} \partial_x^{-1} \\
0 & -1 & \ddots & u_3 + \frac{1}{2} u_{3x} \partial_x^{-1} \\
\vdots & \ddots & \ddots & 0 & \vdots \\
0 & \cdots & \cdots & -1 & u_n + \frac{1}{2} u_{nx} \partial_x^{-1}
\end{pmatrix}
\]

It is obvious that \( L_{u_x} \Lambda = 0 \).

In what follows, we define the operator \( \partial_x^{-1} \) occurring in (3) so that \( \partial_x^{-1}(f) \in A_0^u \) for any \( f = f[u] \in \text{Im} \ \partial_x \subset A_0^u \), where \( A_0^u \subset A_u \) is a ring of differential polynomials in the fields \( u_i \) with zero constant terms.
A. Galilei-like symmetry transformations

In the previous paper\(^7\) we have proposed a construction of Galilei-like symmetry transformations for equations (2). To make this paper self-contained, we gives all proofs below.

Let us define the one-parameter linear transformation of the field \(u_i\) by the shift of the spectral parameter \(\lambda\) through the relation

\[
\sum_{i=1}^{n} u_i(x)(-\lambda)^{i-1} + (-\lambda)^n = \sum_{i=1}^{n} \overline{u}_i(x)(-\lambda + \epsilon)^{i-1} + (-\lambda + \epsilon)^n.
\]

Comparing the coefficients of the same powers of \(\lambda\), one obtains

\[
u_1 = F_1(u, \epsilon) = \sum_{i=1}^{n} u_i \epsilon^{i-1} + \epsilon^n, \quad \epsilon = 0, 1, \ldots, n-1.
\]

\[u_{i+1} = F_{i+1}(u, \epsilon) = \frac{1}{i!} \frac{\partial^i F_1(u, \epsilon)}{\partial \epsilon^i}, \quad i = 1, \ldots, n-1.\]

It is obvious that inverse transformation to (4) is given by

\[
u_1 = F_1(u, -\epsilon), \quad \nu_{i+1} = F_{i+1}(u, -\epsilon), \quad i = 1, \ldots, n-1.
\]

Eq. (4) can be rewritten in vector form

\[
u = F(\nu, \epsilon) = A(\epsilon) \nu + d(\epsilon),
\]

where \(A(\epsilon)\) is an upper diagonal matrix with units in main diagonal and \(d(\epsilon)\) is a vector.

From (4) we easy obtain

\[
A_{ik}(\epsilon) = C_{k-i}^{i-1} \epsilon^{k-i}, \quad k > i, \quad d_i(\epsilon) = C_{n-i}^{i-1} \epsilon^{n-i+1}.
\]

Here and in what follows the symbol \(C^q_p\) denotes binomial coefficient \((q\choose p)\).

**Lemma 1:** The recursion operator satisfy the following identity:

\[
\Lambda[u]|_{u = F(\nu, \epsilon)} = A(\epsilon) (\Lambda[\nu] + \epsilon) A^{-1}(\epsilon)
\]

**Proof:** Let us denote

\[
p(\epsilon) = p_0 + \ldots + p_{n-1} \epsilon^{n-1} + \epsilon^n = \left(\frac{1}{4} \partial_x^2 + u_1 + \frac{1}{2} u_1 x \partial_x^{-1}\right)|_{u_1 = F_1(\nu, \epsilon)}.
\]

By virtue of Eq. (4) we can write

\[
\Lambda(\epsilon) \equiv \Lambda[u]|_{u = F(\nu, \epsilon)} = \begin{pmatrix}
0 & 0 & \ldots & 0 & p(\epsilon) \\
-1 & 0 & \ldots & 0 & p'(\epsilon) \\
0 & -1 & \ldots & \frac{1}{2} p''(\epsilon) \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & -1 & \frac{1}{(n-1)!} p^{(n-1)}(\epsilon)
\end{pmatrix}
\]
It is obvious that \( \Lambda(0) = \Lambda[\mathfrak{u}] \). Since the matrix \( A(\epsilon) \) is nondegenerate, we can rewrite Eq. (8) as

\[
A(\epsilon) (\Lambda(0) + \epsilon) = \Lambda(\epsilon) A(\epsilon).
\]

The coefficients \( p_i \) are operators but it is obvious that when proving fulfillment of relation (9) we can operate with \( p_i \) like with a numbers.

We have

\[
A(\epsilon) \Lambda(0) = \begin{pmatrix}
-\epsilon & -\epsilon^2 & \ldots & -\epsilon^{n-1} & p(\epsilon) - \epsilon^n \\
-1 & -C_2^1 \epsilon & \ldots & -C_{n-1}^1 \epsilon^{n-2} & p'(\epsilon) - C_n^1 \epsilon^{n-1} \\
0 & -1 & \ldots & -C_{n-2}^2 \epsilon^{n-3} & \frac{1}{2!} p''(\epsilon) - C_n^2 \epsilon^{n-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & -1 & \frac{1}{(n-1)!} p^{(n-1)}(\epsilon) - C_n^{n-1} \epsilon
\end{pmatrix},
\]

\[
\epsilon A(\epsilon) = \begin{pmatrix}
\epsilon & \epsilon^2 & \epsilon^3 & \ldots & \epsilon^{n-1} & \epsilon^n \\
0 & \epsilon & C_2^1 \epsilon^2 & \ldots & C_{n-2}^1 \epsilon^{n-2} & C_{n-1}^1 \epsilon^{n-1} \\
0 & 0 & \epsilon & \ldots & C_{n-2}^2 \epsilon^{n-3} & C_{n-1}^2 \epsilon^{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & \epsilon
\end{pmatrix},
\]

\[
\Lambda(\epsilon) A(\epsilon) = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & p(\epsilon) \\
-1 & -\epsilon & -\epsilon^2 & \ldots & -\epsilon^{n-2} & p'(\epsilon) - \epsilon^{n-1} \\
0 & -1 & -C_2^1 \epsilon & \ldots & -C_{n-2}^1 \epsilon^{n-3} & \frac{1}{2!} p''(\epsilon) - C_n^1 \epsilon^{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & -1 & \frac{1}{(n-1)!} p^{(n-1)}(\epsilon) - C_n^{n-2} \epsilon
\end{pmatrix}.
\]

Comparing diagonal-wise matrix elements and using the relation

\[
C_{k-r}^k - C_{k-1}^{k-r-1} = C_{k-1}^{k-r},
\]

one can verify that (9) is fulfilled. □

The relation (8) defines transformation law of recursion operator \( \Lambda[\mathfrak{u}] \) w.r.t. (4), but it should be noted that operator \( \partial_x^{-1} \), occurring in \( \Lambda[\mathfrak{u}] \), now acts on a ring of differential polynomials in the transformed fields \( \mathfrak{u} \) and its action must be consistent with this transformation. Let \( \mathbf{F}^{-1}_*: A_\mathfrak{u} \to A[\mathfrak{u}] \) and \( \mathbf{F}^{-1}_*: A[\mathfrak{u}] \to A_\mathfrak{u} \) are maps of rings of differential polynomials generating by (4). Then we must to require that

\[
\mathbf{F}^{-1}_* \circ \partial_x^{-1}(f) \in A_\mathfrak{u}^0,
\]

(10)
for any $f[\mathbf{u}] \in \text{Im} \, \partial_x \subset A_0^\mathbf{u}$.

So, when computing vector fields $\Lambda^r[\mathbf{u}]\mathbf{u}_x$ we must to take into account the condition (I0). For example,

$$\partial_x^{-1}(\mathbf{u}_n) = \mathbf{u}_n + n\epsilon$$

(11)

because $F^s_1(\pi_n + n\epsilon) = u_n \in A_0^\mathbf{u}$. Taking into account (I1) we have

$$\Lambda[\mathbf{u}]\mathbf{u}_x = K_2[\mathbf{u}] + \frac{n}{2}\epsilon\mathbf{u}_x,$$

(12)

To compute $\Lambda^2[\mathbf{u}]\mathbf{u}_x$ there is a need to calculate the $n$th component of the vector field $K_2[\mathbf{u}]$. We have $K_2[\mathbf{u}] = \frac{1}{4}\mathbf{u}_{xxx} + \frac{3}{2}\mathbf{u}_x \in \text{Im} \, \partial_x$ in the case $n = 1$ and $K_{2,n}[\mathbf{u}] = -\mathbf{u}_{n-1,x} + \frac{3}{2}\mathbf{u}_n \mathbf{u}_{nx} \in \text{Im} \, \partial_x$ for $k \geq 2$. Taking into account (10), we obtain

$$\partial_x^{-1}(K_2[\mathbf{u}]) = \frac{1}{4}\mathbf{u}_{xx} + \frac{3}{4}\mathbf{u}_x^2 - \frac{3}{4}\epsilon^2,$$

(13)

$$\partial_x^{-1}(K_{2,n}[\mathbf{u}]) = -\mathbf{u}_{n-1} + \frac{3}{4}\mathbf{u}_n^2 + \left(\frac{1}{2}n(n-1) - \frac{3}{4}n^2\right)\epsilon^2$$

(14)

Using (12), (13) and (14) we obtain:

$$\Lambda^2[\mathbf{u}]\mathbf{u}_x = \Lambda[\mathbf{u}]K_2[\mathbf{u}] + \frac{n}{2}\epsilon\Lambda[\mathbf{u}]\mathbf{u}_x$$

$$= K_3[\mathbf{u}] + \left(\frac{1}{2}n(n-1) - \frac{3}{4}n^2\right)\epsilon^2\mathbf{u}_x + \frac{n}{2}\epsilon \left(K_2[\mathbf{u}] + \frac{n}{2}\epsilon\mathbf{u}_x\right)$$

$$= K_3[\mathbf{u}] + \frac{n}{2}\epsilon K_2[\mathbf{u}] + \left(\frac{1}{2}n(n-1) - \frac{1}{8}n^2\right)\epsilon^2\mathbf{u}_x.$$  

(15)

Using lemma 1 and Eqs. (12), (13) we obtain evolution equations on transformed functions $\mathbf{u}_i(x, t_2, t_3)$

$$\mathbf{u}_{t_2} = (\Lambda[\mathbf{u}] + \epsilon)\mathbf{u}_x$$

$$= K_2[\mathbf{u}] + \left(\frac{n}{2} + 1\right)\epsilon\mathbf{u}_x,$$

(16)

$$\mathbf{u}_{t_3} = (\Lambda[\mathbf{u}] + \epsilon)^2\mathbf{u}_x$$

$$= \Lambda^2[\mathbf{u}]\mathbf{u}_x + 2\epsilon\Lambda[\mathbf{u}]\mathbf{u}_x + \epsilon^2\mathbf{u}_x$$

$$= K_3[\mathbf{u}] + \frac{n}{2}\epsilon K_2[\mathbf{u}] + \left(\frac{1}{2}n(n-1) - \frac{1}{8}n^2\right)\epsilon^2\mathbf{u}_x + 2\epsilon \left(K_2[\mathbf{u}] + \frac{n}{2}\epsilon\mathbf{u}_x\right) + \epsilon^2\mathbf{u}_x$$

$$= K_3[\mathbf{u}] + \left(\frac{n}{2} + 2\right)\epsilon K_2[\mathbf{u}] + \frac{1}{2} \left(\frac{n}{2} + 1\right) \left(\frac{n}{2} + 2\right)\epsilon^2\mathbf{u}_x.$$  

(17)
Let us define two one-parameter linear transformations of independent variables

\[
\begin{pmatrix} x \\ t_2 \end{pmatrix} = B_2(\epsilon) \begin{pmatrix} x \\ t_2 \end{pmatrix} = \begin{pmatrix} 1 & \left(\frac{n}{2} + 1\right) \epsilon \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t_2 \end{pmatrix}
\]

and

\[
\begin{pmatrix} x \\ t_2 \\ t_3 \end{pmatrix} = B_3(\epsilon) \begin{pmatrix} x \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} 1 & \left(\frac{n}{2} + 1\right) \epsilon & \frac{1}{2} \left(\frac{n}{2} + 1\right) \left(\frac{n}{2} + 2\right) \epsilon^2 \\ 0 & 1 & \left(\frac{n}{2} + 2\right) \epsilon \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t_2 \\ t_3 \end{pmatrix}
\]

(18)

It is easy to check that matrices \(B_2(\epsilon)\) and \(B_3(\epsilon)\) satisfy group relation \(B(\epsilon)B(\epsilon_1) = B(\epsilon + \epsilon_1)\). By straightforward calculations, it can be verified that:

(i) system of evolution equation \(u_{t_2} = K_2[u] = \Lambda[u]u_x\) is invariant under transformations (4) and (18),

(ii) a pair of systems \(u_{t_2} = K_2[u] = \Lambda[u]u_x\) and \(u_{t_3} = K_3[u] = \Lambda^2[u]u_x\) is invariant under transformations (4) and (19).

We can undertake to generalize transformations (18) and (19). Let us denote \(t = (x, t_2, t_3, ..., t_N)^T\). We suppose that matrix \(B_N(\epsilon)\) defining transformation of independent variables

\[
\mathbf{t} = B_N(\epsilon)\mathbf{t}
\]

is upper diagonal matrix with units in main diagonal and is written in the form

\[
B_N(\epsilon) = \begin{pmatrix} 1 & b_{2,1}\epsilon & \ldots & b_{N,1}\epsilon^{N-1} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{N,N-1}\epsilon \\ 0 & \ldots & 0 & 1 \end{pmatrix}
\]

Here \(b_{k,l}\) are some coefficients to be determined. Necessary condition in order that (20) complete transformation (4) to define one-parameter group is the relation

\[
B_N(\epsilon)B_N(\epsilon_1) = B_N(\epsilon + \epsilon_1),
\]

which element-wise can be written as

\[
b_{m+r,m}(\epsilon + \epsilon_1)^r = b_{m+r,m}\left(\epsilon^r + \sum_{i=1}^{r-1} C_r^i \epsilon^i \epsilon_{1}^{r-i} + \epsilon_1^r\right)
\]

\[
= b_{m+r,m} \epsilon^r + \sum_{i=1}^{r-1} b_{m+r,m+i} b_{m+i,m} \epsilon^i \epsilon_{1}^{r-i} + b_{m+r,m} \epsilon_1^r.
\]

Thus the relation

\[
C_r^i b_{r+k,k} = b_{r+k,k} b_{k+i,k}
\]
must be fulfilled for each \( i = 1, \ldots, r - 1 \). The solution of Eq. (22) is uniquely given by

\[
b_{m+r,m} = \frac{1}{r!} \prod_{i=1}^{r} b_{k+i,k+i-1}.
\]

Thus to derive coefficients \( b_{ml} \) there is a need to calculate \( b_{m,m-1} \). In order to do that we observe that

\[
\bar{u}_{tm} = K_{m} \bar{u} + b_{m,m-1} \epsilon K_{m-1} \bar{u} + O(\epsilon^2).
\]  
(23)

Taking into account (12) we have

\[
\bar{u}_{tm} = \Lambda_{m} - 1 \bar{u} + \left( m - 1 \right) \epsilon \Lambda_{m-2} \bar{u} + O(\epsilon^2)
\]  
(24)

In what follows, we will denote the groups of one-parameter transformations (5), (20) by \( \mathcal{G}_{N,n} \). The infinitesimal generators (vector fields) of \( \mathcal{G}_{N,n} \) can be calculated immediately to yield

\[
v_{N,n} = \sum_{i=1}^{N-1} \left( \frac{n}{2} + i \right) t_{i+1} \frac{\partial}{\partial t_i} - \sum_{j=1}^{n} j u_{j+1} \frac{\partial}{\partial u_j},
\]

where \( u_{n+1} = 1 \). For convenience, here we denote \( t_1 = x \). As can be checked, respective characteristics are given by

\[
Q_{N,n}(u) = -\sum_{i=1}^{N-1} \left( \frac{n}{2} + i \right) t_{i+1} u_i - A'(0) u - d'(0),
\]

where \( A'(0) \) and \( d'(0) \) denote a valuation of derivatives with respect to group parameter calculated at \( \epsilon = 0 \), being \( A'(0) = \sum_{i=1}^{n-1} i E_{i+1,i} \) and

\[
d'(0) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ n \end{pmatrix}.
\]

Here \( E_{ij} \) denotes matrix with unit in \((i, j)\) place and zeros elsewhere.
Repeatedly applying recursion operator \( \Lambda[u] \) to \( Q_{N,n} \), we obtain infinite sequence of characteristics \( Q_{N,n}^{(i)} \). In particular, once applying \( \Lambda[u] \) to \( Q_{N,n} \) yields the characteristic

\[
Q_{N,n}^{(1)}(u) = -\sum_{i=1}^{N} \left( \frac{n}{2} + i - 1 \right) t_i u_{t_i} - J u,
\]

where \( J = \text{diag}(n, n-1, \ldots, 1) \). This characteristic corresponds to scaling invariance group. Below we construct scaling invariance transformations making use a transformation property of recursion operator \( \Lambda[u] \).

**B. Scaling symmetry transformations**

Let us investigate invariance properties of the systems (2) w.r.t. scaling transformations. We denote

\[
p_1 = \frac{1}{4} \partial_x^2 + u_1 + \frac{1}{2} u_{1x} \partial_x^{-1}, \quad p_i = u_i + \frac{1}{2} u_{ix} \partial_x^{-1}, \quad i = 2, \ldots, n.
\]

It is obvious that relation

\[
\begin{pmatrix}
\beta^{n+1} & 0 & \ldots & 0 \\
0 & \beta^n & \ldots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & \beta^2
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \ldots & 0 & p_1 \\
-1 & 0 & \ldots & 0 & p_2 \\
0 & -1 & \ldots & \vdots & p_3 \\
\vdots & \ddots & \ddots & 0 & \vdots \\
0 & \ldots & 0 & -1 & p_n
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & \ldots & 0 & \beta^n p_1 \\
-1 & 0 & \ldots & 0 & \beta^{n-1} p_2 \\
0 & -1 & \ldots & \vdots & \beta^{n-2} p_3 \\
\vdots & \ddots & \ddots & 0 & \vdots \\
0 & \ldots & 0 & -1 & \beta p_n
\end{pmatrix}
\begin{pmatrix}
\beta^n & 0 & \ldots & 0 \\
0 & \beta^{n-1} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & \beta
\end{pmatrix},
\]

where \( \beta \) is arbitrary number, is valid. In what follows, \( \beta \) will be group parameter.

Let us introduce scaling transformation of variables \( x \) and \( u_i \):

\[
x = \beta^{n/2} x, \quad u_i = \beta^{i-n-1} u_i.
\]

Consequently Eq. (23) we comes to the relation

\[
\Lambda[u]|_{u=A(\beta)} = A(\beta) (\beta \Lambda[u]) A^{-1}(\beta),
\]

where \( A(\beta) = \text{diag}(\beta^n, \ldots, \beta) \). Using the Eq. (24), we obtain equation

\[
\mathbf{m}_m = \beta^{(n/2)+m-1} \Lambda^{m-1}[\mathbf{m}] \mathbf{m}_m
\]
from which it follows that by scaling evolution parameter \( t_m \) as \( \tilde{t}_m = \beta^{(n/2)+m-1}t_m \) we arrive at

\[
\varpi_{t_m} = \Lambda^{m-1} \varpi \varpi_x
\]

Thus we have the following statement:

**Corollary 1:** The system of equations

\[
u_{t_m} = \Lambda^{m-1} [\varpi] \varpi_x
\]

with the recursion operator (27) is invariant under scaling transformation

\[
\varpi = \beta^{n/2} x, \quad \tilde{t}_m = \beta^{(n/2)+m-1} t_m, \quad \varpi_i = \beta^{i-n-1} u_i.
\]

One can assign, to each variable, scaling dimension \([\cdot]\), so that \([x] = n/2, [t_m] = (n/2) + m - 1\) and \([u_i] = i - n - 1\).

**III. INVARIANT SOLUTIONS**

Given Galilei-like symmetry group \( G_{N,n} \), corresponding one-parameter family of solutions of (2) is given by

\[
u(\epsilon, x, t_2, ..., t_N) = A(\epsilon)u(\varpi, \tilde{t}_2, ..., \tilde{t}_N) + d(\epsilon) \quad (28)
\]

Next we are going, using standard symmetry method, to construct \( G_{N,n} \)-invariant solutions of equations (2). To do this, we need to find the set of generators of \( G_{N,n} \)-invariant functions algebra.

**Proposition 1:** For the group \( G_{N,n} \), there exist functionally independent differential invariants in the form:

\[
T_j = t_j + \sum_{k=1}^{N-j-1} (-1)^k \alpha_{j,k} \frac{t_{j+k}^k t_{N-1}^k}{t_N^k}, \quad j = 1, ..., N - 2,
\]

\[
T_N = t_N,
\]

\[
v_j = u_j + \sum_{k=1}^{n-j} (-1)^k \beta_{j,k} u_{j+k}^k, \quad j = 1, ..., n - 1,
\]

\[
v_n = u_n + \frac{2n t_{N-1}}{(2N + n - 2)t_N},
\]

where the coefficients \( \alpha_{j,k} \) and \( \beta_{j,k} \) are positive rational numbers to be determined.

**Proof.** It is evident that \( T_N \) is an invariant. Consider \( T_j \), for \( j = 1, ..., N - 3 \). We have

\[
v_{N,n}(T_j) = \left( \frac{n}{2} + j \right) t_{j+1}
\]

\[- \left( \frac{n}{2} + j + 1 \right) \alpha_{j+1} \frac{t_{j+2} t_{N-1}}{t_N^k} + ... + (-1)^{N-j-1} \left( \frac{n}{2} + N - 1 \right) \alpha_{j,N-j-1} \frac{t_{N} t_{j}^{N-j-1}}{t_N^{N-j-1}}
\]

10
Collecting similar terms we obtain recurrence relations:

\[
+t_N \left\{ -\alpha_{j,1} \frac{t_{j+1}}{t_N} + \alpha_{j,2} \frac{2t_{j+2}t_{N-1}}{t_N} + \ldots + (-1)^{N-j-1} \alpha_{j,N-j-1} \frac{(N-j-1)t_{N-j-1}^N}{t_{N-j}^N} \right\} .
\]

Require that \( v_{N,n}(T_j) = 0 \). Collecting similar terms and equating it to zero, we obtain recurrence relations:

\[
t_{j+1} : \left( \frac{n}{2} + j \right) = \left( \frac{n}{2} + N - 1 \right) \alpha_{j,1},
\]

\[
\frac{t_{j+k}t_{N-1}^{N-k}}{t_{N-1}^{N-k}} : \left( \frac{n}{2} + j + k - 1 \right) \alpha_{j,k-1} = k \left( \frac{n}{2} + N - 1 \right) \alpha_{j,k}, \quad k = 2, \ldots, N-j-2,
\]

\[
\frac{t_{N-j-1}^{N-j-1}}{t_{N-j-2}^{N-j-2}} : \left( \frac{n}{2} + N - 2 \right) \alpha_{j,N-j-2} = (N-j) \left( \frac{n}{2} + N - 1 \right) \alpha_{j,N-j-1}.
\]

Separately, consider the case \( j = N-2 \). Assuming that \( v_{N,n}(T_{N-2}) = 0 \), we obtain relation

\[
v_{N,n}(T_{N-2}) = \left( \frac{n}{2} + N - 2 \right) t_{N-1} - 2 \alpha_{N-2,1} \left( \frac{n}{2} + N - 1 \right) t_{N-1} = 0
\]

from which it follows

\[
2 \left( \frac{n}{2} + N - 1 \right) \alpha_{N-2,1} = \left( \frac{n}{2} + N - 2 \right).
\]

So, all coefficients \( \alpha_{j,k} \) are uniquely defined by recurrence relations above. From these relations, we easy obtain

\[
\alpha_{j,k} = \frac{1}{k!} \left( \frac{n}{2} + N - 1 \right)^k \prod_{i=1}^{k} \left( \frac{n}{2} + j - 1 \right), \quad j = 1, \ldots, N-j-2,
\]

\[
\alpha_{j,N-j-1} = \frac{(N-j-1)}{(N-j)(N-j-1)!} \left( \frac{n}{2} + N - 1 \right)^{N-j-1} \prod_{i=1}^{N-j-1} \left( \frac{n}{2} + i + j - 1 \right).
\]

By simple calculations, it is easy to prove that \( v_n \) is an invariant of \( G_{N,n} \). Consider \( v_j \), for \( j = 1, \ldots, n-1 \). Requiring that \( v_{N,n}(v_j) = 0 \) we obtain

\[
v_{N,n}(v_j) = -n \left\{ -\beta_{j,1}u_{j+1} + 2\beta_{j,2}u_{j+2}u_n + \ldots + (-1)^{n-j-1}(n-j-1)\beta_{j,n-j-1}u_{n-1}u_n^{n-j-2} + (-1)^{n-j}(n-j)\beta_{j,n-j}u_n^{n-j} \right\} - ju_{j+1} + (j+1)\beta_{j,1}u_{j+2}u_n - (j+2)\beta_{j,2}u_{j+3}u_n^2 + \ldots
\]

\[
-(-1)^{n-j-1}(n-1)\beta_{j,n-j-1}u_n^{n-j} - (-1)^{n-j}n\beta_{j,n-j}u_n^{n-j} = 0.
\]

Collecting similar terms we obtain recurrence relations:

\[
u_{j+1} : n\beta_{j,1} = j,
\]
\[ u_{j+k}u_k^n : \quad kn\beta_{j,k} = (j + k - 1)\beta_{j,k-1}, \quad k = 2, \ldots, n - j - 1, \]

\[ u_n^{n-j} : \quad (n-j)\beta_{j,n-j} = (n-1)\beta_{j,n-j-1} - n\beta_{j,n-j}, \]

from which it follows that

\[ \beta_{j,k} = \frac{C_{j+k-1}^k}{n^k}, \quad k = 1, \ldots, n - j - 1 \]

\[ \beta_{j,n-j} = \frac{n-j}{n-j+1} \frac{C_{n-j}^{n-j}}{n^{n-j}}. \]

It is obvious that all invariants determined above are functionally independent and generate the algebra of \( G_{N,n} \)-invariant functions. □

It should be noted that \( [T_j] = [t_j] \) and \( [v_j] = [u_j] \). Denote \( T = (T_1 = X, T_2, \ldots, T_{N-2})^T \) and \( t = (x, t_2, \ldots, t_{N-2})^T \). In what follows, it will be useful following two lemmas.

**Lemma 2:** The relation

\[ T = B_{N-2}(\tau)t + c(\tau), \quad (30) \]

where

\[ \tau = -\frac{t_{N-1}}{\left(\frac{n}{2} + N - 1\right) t_N} \]

and \( c(\tau) = (c_1(\tau), \ldots, c_{N-2}(\tau))^T \), where

\[ c_i(\tau) = \frac{t_{N-1}\tau^{N-j-1}}{(N-j)(N-j-2)!} \prod_{k=1}^{N-j-1} \left(\frac{n}{2} + j + k - 1\right) \]

is fulfilled.

**Proof:** It is easy to prove the validity of the relation (30) by simple straightforward calculations. □

**Lemma 3:** The matrix \( A(\tau) \) and vector \( d(\tau) \) as functions of \( \tau \) satisfy differential equations:

\[ A^{-1}(\tau)A'(\tau) = A'(0), \quad A^{-1}(\tau)d'(\tau) = d'(\tau). \quad (31) \]

**Proof:** Taking into account group identities \( A(\tau + \tau_1) = A(\tau)A(\tau_1) \) and \( d(\tau + \tau_1) = A(\tau)d(\tau_1) + d(\tau) \), we easy obtain

\[ A'(\tau) = \lim_{\tau_1 \to 0} \frac{A(\tau + \tau_1) - A(\tau)}{\tau_1} = A(\tau) \lim_{\tau_1 \to 0} \frac{A(\tau_1) - I}{\tau_1} = A(\tau)A'(0), \]

\[ d'(\tau) = \lim_{\tau_1 \to 0} \frac{d(\tau + \tau_1) - d(\tau)}{\tau_1} = A(\tau) \lim_{\tau_1 \to 0} \frac{d(\tau_1)}{\tau_1} = A(\tau)d'(0) \]

From these relations, it follows Eq. (31). □
Now, using differential invariants (29), we can construct invariant solutions. By turns, we define more convenient variables $U_j$ by

$$U_n = v_n, \quad U_j = v_j - \sum_{k=1}^{n-j} (-1)^k \beta_{j,k} U_{j+k} u_n^k, \quad j = n-1, ..., 1.$$ 

Observe that

$$u_n = U_n + n\tau = U_n + C^{n-1}\tau.$$ 

Next

$$u_{n-1} = v_{n-1} + \beta_{n-1,n} u_n^2 = v_{n-1} + \beta_{n-1,n} \left( U_n^2 + 2n\tau U_n + n^2\tau^2 \right) = U_{n-1} + 2n\beta_{n-1,n}\tau U_n + n^2\beta_{n-1,n}\tau^2.$$ 

Using the relations

$$2n\beta_{n-1,n} = C^{n-2}_n, \quad n^2\beta_{n-1,n} = C^{n-2}_n$$

we obtain

$$u_{n-1} = U_{n-1} + C^{n-2}_{n-1}\tau U_n + C^{n-2}_{n}.$$ 

Thus, it is naturally to conjecture that the relation

$$u_{j} = U_{j} + \sum_{k=1}^{n-j} C^{n-1}_{j+k-1}\tau^k U_{j+k} + C^{n-1}_{n-1}\tau^{n-i-1}$$

(32)

holds for $i = 1, ..., n$. We observe that, consequently (3) and (7), the relation (32) can be rewritten in vector form as

$$\mathbf{u} = A(\tau) \mathbf{U} + \mathbf{d}(\tau).$$

Thus, we comes to the following ansatz for $G_{N,n}$-invariant solutions

$$\mathbf{u}(x, t_2, ..., t_n) = A(\tau) \mathbf{U}(X, T_2, ..., T_{N-2}, T_N) + \mathbf{d}(\tau).$$

(33)

In what follows, we will consider the case $N \geq 3$. Taking into account Eq. (28) and lemma 2, we obtain

**Proposition 2:** Substitution of the ansatz (33) into the equations (2), for $m = 1, ..., N-2$, yields

$$U_{T_m} = K_m[U], \quad m = 1, ..., N-2.$$ 

(34)

Here $K_m[U]$ are vector fields $K_m[u]$, where variables $u^{(k)}_{ix}$ are replaced by $U^{(k)}_{ix}$.

**Proposition 3:** Substitution of the ansatz (33) into the system

$$u_{t_{N-1}} = K_{N-1}[\mathbf{u}]$$

by virtue (34), yields following constraint:

$$\left( \frac{n}{2} + N - 1 \right) T_N K_{N-1}[\mathbf{U}] + \sum_{i=1}^{N-3} \left( \frac{n}{2} + i \right) T_{i+1} K_i[\mathbf{U}] + A'(0) \mathbf{U} + d'(0) = 0$$

(35)
Proof: We have

\[
\mathbf{u}_{N-1} = -\frac{1}{\left(\frac{n}{2} + N - 1\right) t_N} A'(\tau) \mathbf{U} - \frac{1}{\left(\frac{n}{2} + N - 1\right) t_N} \mathbf{d}'(\tau)
\]

\[+ A(\tau) \left\{ \mathbf{U}_X \left( -\alpha_{1,1} \frac{t_2}{t_N} + 2\alpha_{1,2} \frac{t_3 t_{N-1}}{t_N^2} - \ldots + (-1)^{N-3} (N-3) \alpha_{1,N-3} \frac{t_{N-2} t_{N-1}^{N-4}}{t_N^{N-3}} \right) + \mathbf{U}_T_2 \left( -\alpha_{2,1} \frac{t_3}{t_N} + 2\alpha_{2,2} \frac{t_4 t_{N-1}}{t_N^2} - \ldots \right) + \mathbf{U}_{T_{N-2}} \left( -2\alpha_{N-2,1} \frac{t_{N-1} t_{N-2}}{t_N^2} \right) \right\} = A(\tau) \left\{ \mathbf{K}_{N-1} [\mathbf{U}] + b_{N-1,1} \mathbf{K}_{N-2} \mathbf{K}_{N-2} [\mathbf{U}] + \ldots + b_{N-1,1} \tau^{N-2} \mathbf{U}_X \right\}.
\]

Taking into account Eq. (34), lemma 3 and the relations

\[
\left(\frac{n}{2} + N - 1\right) \alpha_{i,1} = \left(\frac{n}{2} + i\right),
\]

\[
k \left(\frac{n}{2} + N - 1\right) \alpha_{i,k} = \left(\frac{n}{2} + i\right) \alpha_{i+1,k-1},
\]

\[
(N - i) \left(\frac{n}{2} + N - 1\right) \alpha_{i,N-i-1} = \left(\frac{n}{2} + i\right) \alpha_{i+1,N-i-2} + \frac{b_{N-1,i}}{\left(\frac{n}{2} + N - 1\right)^{N-i-2}}
\]

for \(k = 2, \ldots, N - i - 2, \ i = 1, \ldots, N - 2\), we come to (33). \(\square\)

Proposition 4: Substitution of the ansatz (33) into the system

\[
\mathbf{u}_{t_N} = \mathbf{K}_N [\mathbf{u}]
\]

by virtue Eqs. (34) and (32), yields

\[
\mathbf{U}_{T_N} = \mathbf{K}_N [\mathbf{U}]
\]

(37)

Proof: We have

\[
\mathbf{u}_{t_N} = \frac{t_{N-1}}{\left(\frac{n}{2} + N - 1\right) t_N^2} A'(\tau) \mathbf{U} + \frac{t_{N-1}}{\left(\frac{n}{2} + N - 1\right) t_N^2} \mathbf{d}'(\tau)
\]

\[+ A(\tau) \left\{ \mathbf{U}_X \left( \alpha_{1,1} \frac{t_2 t_{N-1}}{t_N^2} - 2\alpha_{1,2} \frac{t_3 t_{N-1}^2}{t_N^3} \ldots + (-1)^{N-3} (N-3) \alpha_{1,N-3} \frac{t_{N-2} t_{N-1}^{N-4}}{t_N^{N-3}} \right) + \mathbf{U}_T_2 \left( \alpha_{2,1} \frac{t_3 t_{N-1}}{t_N^2} - 2\alpha_{2,2} \frac{t_4 t_{N-1}^2}{t_N^3} \ldots \right) \right\}
\]

\[\left(\frac{n}{2} + N - 2\right) \alpha_{1,N-2} \frac{t_{N-1} t_{N-1}^{N-1}}{t_N^2} \right) + \mathbf{U}_{T_{N-2}} \left( \alpha_{2,1} \frac{t_3 t_{N-1}}{t_N^2} - 2\alpha_{2,2} \frac{t_4 t_{N-1}^2}{t_N^3} \ldots \right). \]
\[-(N-1)N^{-4}(N-4)\alpha_{2,N-4} - \frac{t_{N-2}N^{-4}}{t_{N-3}}N^{-3}(N-3)\alpha_{2,N-3} - \frac{t_{N-2}N^{-2}}{t_{N-2}} + \ldots\]

\[+ U_{T_{N-2}} \left( \frac{\alpha_{N-2}}{t_N^2} \right) + U_{T_N} \right) = A(\tau) \left\{ K_N[U] + b_{N,N-1}T_{N-1}[U] + \ldots + b_{N,1}N^{-1}U_X \right\}.\]

Now taking into account Eq. (34), lemma 3, the relations (36) and the relation

\[(N-i-1) \left( \frac{n}{2} + N-1 \right) \alpha_{i,N-i-1} = \left( \frac{n}{2} + i \right) \alpha_{i+1,N-i-2} + \frac{b_{N,i}}{(\frac{n}{2} + N-1)^{N-i-1}}\]

we obtain (37). □

We observe that constraint (35) is Galilean self-similarity condition

\[-Q_{N,n}(U) = \sum_{i=1}^{N-1} \left( \frac{n}{2} + i - 1 \right) T_{i+1}K_i[U] + A'(0)U + d'(0) = 0,\]

where \(T_{N-1} = 0\). Consequence of (38) is scaling self-similarity condition

\[-Q_{N,n}^{(1)}(U) = \sum_{i=1}^{N} \left( \frac{n}{2} + i - 1 \right) T_iK_i[U] + JU = 0,\]

where \(T_{N-1} = 0\). By virtue (34) and (37), we can replace in Eq. (39)

\[K_i[U] \rightarrow U_{T_i},\]

for \(i = 1,\ldots,N, i \neq N-1\), to obtain the constraint

\[\left( \frac{n}{2} + N-1 \right) T_N U_{T_N} + \sum_{i=1}^{N-2} \left( \frac{n}{2} + i - 1 \right) T_iU_{T_i} + JU = 0.\]

Example. Separately consider the case \(N = 3\). The ansatz

\[u(x,t_2,t_3) = A(\tau)U(X,T_3) + d(\tau), \quad \tau = \frac{-t_2}{\left( \frac{n}{2} + 2 \right) t_3}, \quad X = x - \frac{(n+2)t_2^2}{2(n+4)t_3}, \quad T_3 = t_3\]

gives

\[\left( \frac{n}{2} + 2 \right) T_3K_2[U] + A'(0)U + d'(0) = 0,\]

\[\left( \frac{n}{2} + 2 \right) T_3U_{T_3} + \frac{n}{2}XU_X + JU = 0.\]

More explicitly (38) is given by

\[\frac{5}{2} T_3 \left( \frac{1}{4}U_{XXX} + \frac{3}{2}UU_X \right) + 1 = 0\]
for \( n = 1 \) and
\[
\left( \frac{n}{2} + 2 \right) T_3 \left( \frac{1}{4} U_{nXX} + U_1 U_{nX} + \frac{1}{2} U_n U_{1X} \right) + U_2 = 0,
\]
\[
\left( \frac{n}{2} + 2 \right) T_3 \left( -U_{i-1,X} + U_1 U_{nX} + \frac{1}{2} U_n U_{1X} \right) + i U_{i+1} = 0, \quad i = 2, \ldots, n-1 \tag{43}
\]
\[
\left( \frac{n}{2} + 2 \right) T_3 \left( -U_{n-1,X} + \frac{3}{2} U_n U_{nX} \right) + n = 0
\]
for \( n \geq 2 \). Eq. (41), which presents scaling self-similar constraint can be immediately solved to yield
\[
U_i(X, T_3) = \frac{1}{T_3^p} f_i(Z), \quad Z = \frac{X}{T_3^q} \tag{44}
\]
where
\[
p = \frac{2(n - i + 1)}{n + 4}, \quad q = \frac{n}{n + 4}.
\]
Introducing (44) into (42) and (43) we gives, respectively,
\[
\frac{5}{2} \left( \frac{1}{4} f_{ZZZ} + \frac{3}{2} f f_Z \right) + 1 = 0 \tag{45}
\]
for \( n = 1 \) and
\[
\left( \frac{n}{2} + 2 \right) \left( \frac{1}{4} f_{nZZZ} + f_1 f_{nZ} + \frac{1}{2} f_n f_{1Z} \right) + f_2 = 0,
\]
\[
\left( \frac{n}{2} + 2 \right) \left( -f_{i-1,Z} + f_i f_{nZ} + \frac{1}{2} f_n f_{iZ} \right) + i f_{i+1} = 0, \quad i = 2, \ldots, n-1 \tag{46}
\]
for \( n \geq 2 \).

According to Ablowitz—Ramani—Segur conjecture\(^8\), equations (43) and (46) should have the Painlevé property. Single integration of Eq. (45) gives first Painlevé transcendent\(^9\):
\[
f'' + \frac{6}{5} f^2 + \frac{8}{5} Z = 0
\]
whose canonical form is \( f'' = 6 f^2 + Z \). The system (46) in the case \( n = 2 \) is reduced to second Painlevé transcendent. Putting
\[
f_1 = \frac{3}{4} f^2 + \frac{2}{3} Z, \quad f_2 = f
\]
and once integrating, we obtain second Painlevé equation:
\[
f'' + 2 f^3 + \frac{8}{3} Z f + \alpha = 0
\]
whose canonical form is \( f'' = 2 f^3 + Z f + \alpha \).

The problem to be addressed: to investigate Painlevé property for the system (46), for every \( n \).
1. L.V. Ovsiannikov, *Group Analysis of Differential Equations*, (Academic, New York, 1982).

2. N.H. Ibragimov, *Transformation Groups Applied to Mathematical Physics*, (Reidel, Dordrecht, 1985)

3. P.J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate texts in Mathematics (Springer-Verlag, New York, 1986).

4. Y. Kosmann-Schwarzbach, Lett. Math. Phys. **38**, 421 (1996).

5. B. Fuchssteiner, Nonlinear Anal. Theor. Meth. Appl. **3**, 849 (1979).

6. L.M. Alonso, J. Math. Phys. **21**, 2342 (1980).

7. A.K. Svinin, J. Phys. A: Math. Gen. **32**, 5499 (1999).

8. M.J. Ablowitz, A. Ramani, H.J. Segur, J. Math. Phys. **21**, 715 (1980).

9. E.L Ince, *Ordinary Differential Equations*, (Dower, New York, 1956).