Measuring an entropy in heavy ion collisions

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We propose to use the coincidence method of Ma to measure an entropy of the system created in heavy ion collisions. Moreover we estimate, in a simple model, the values of parameters for which the thermodynamical behaviour sets in.

Existence of thermodynamical equilibrium in heavy ion collisions is an important question. Many phenomenological models critically depend on this assumption, and consequently there exists an ongoing theoretical debate of this problem. Alternatively one should develop methods which would allow to answer this question directly by the experiments. For example how to check the saddle point relation

\[ \frac{\partial S(E, n)}{\partial E} \bigg|_{n} = \frac{1}{T}, \]

which is central in a thermodynamical description. Out of four observables entering above equation (energy \(E\), multiplicity \(n\) and temperature \(T\)), the entropy \(S\) is most difficult to directly measure in experiment. In this talk the recent proposal to determine entropy experimentally is discussed.

The Boltzmann relation \(S(E, n) = \log \Gamma(E, n)\) reduces the problem to measuring the density of states \(\Gamma(E, n)\) of a system with given energy and multiplicity. To this end we have proposed to employ the coincidence method advocated by Ma. Suppose that the phase-space is divided into multidimensional cells. Suppose furthermore that our system occupies \(\Gamma\) cells (with a uniform probability). Each cell represents a different state of the system. Our problem is to calculate \(\Gamma\). Let us select randomly \(\mathcal{N}\) configurations of the system. The main advantage of this approach is that \(\mathcal{N} \sim \sqrt{\Gamma} \ll \Gamma\) is sufficient. These configurations occupy some cells. The average occupation number of a cell is \(\mathcal{N}/\Gamma \ll 1\). Under this condition, the average number of pairs in the same cell is \((\mathcal{N}/\Gamma)^2\) The total number of coincidences \(\mathcal{N}_c\) is the sum over all cells \(\mathcal{N}_c = \Gamma(\mathcal{N}/\Gamma)^2\), hence

\[ \Gamma = \frac{\mathcal{N}_c}{\mathcal{N}_t}, \]

where \(\mathcal{N}_t \approx \mathcal{N}^2\) is the total number of pairs. Generalization for nonequivalent configurations is also available.

To define meaningfully a coincidence of two states, we discretize particle momenta \(p_i = an_i\) with some discretization scale \(a\). As usual \(a\) should be chosen such that computations are practically feasible and at the same time reproduce continuum physics.

We have tested the method in the case of the classical gas of noninteracing, nonrelativistic particles with mass \(m\) in \(d\) dimensions. The discretized expression for the number of momentum states of \(N\) degrees of freedom with the total energy \(E = Ma^2/2m\) reads

\[ \Gamma(M, N) = \sum_{n_1, \ldots, n_N, n_1^2 + \ldots + n_N^2 = M} 1, \]

where the momenta \(p_i = an_i\) as above.

Numbers \(\Gamma(M, N)\) satisfy simple recursion relation which will be used to calculate them exactly. These will serve to benchmark the performance of our Monte Carlo.

Two questions were addressed: a) how large numbers \(\Gamma\) can be reproduced by the coincidence method with present computers and b) how well is the thermodynamic limit, \(M, N \to \infty, M/N = \)
ε-fixed,
\[
\frac{1}{N} \log \Gamma(M, N) \cong \frac{1}{2}[\log(\epsilon) + \log(2\pi) + 1].
\]  
(4)
approximated within available window of \(M\) and \(N\).

A sample of runs is summarized in Table 1. Instead of generating \(N\) configurations of \(N\) integer-valued momenta \(\{n_1, n_2, \ldots, n_N\}_k, k = 1, \ldots, N\), we have uniquely labelled each, \(k\)-th say, configuration by an integer index \(I_k, k = 1, \ldots, \Gamma(M, N)\). Consequently each Monte Carlo run consisted of a generation of a sample of \(N\) integer indices, \((I_1, I_2, \ldots, I_N), 1 \leq I_k \leq \Gamma(M, N), k = 1, \ldots, N\), uniformly distributed in the whole space of available states. Then we counted the number of coincidences \(\hat{N}_c\), i.e., the number of pairs \((I_j, I_k)\) such that \(I_j = I_k\). The estimate for the number of all states (column 5 of Table 1) is then
\[
\hat{\Gamma} = \mathcal{N}(N - 1)/\hat{N}_c.
\]  
(5)
Multinomial nature of the above process allows for simple calculation of the distribution of \(N_c\). In particular, the dispersion of \(N_c\) reads (\(\mathcal{N} \gg 1\))
\[
\sigma^2[N_c] = 2 < N_c > (1 + 2\mathcal{N}/\Gamma + \mathcal{N}^2/2\Gamma^2),
\]  
(6)
with \(< N_c > = \mathcal{N}^2/\Gamma\), cf.\(\mathcal{R}\). This gives for the relative error of the determination of \(\Gamma\) after \(\mathcal{N}\) trials \(\sqrt{\sigma^2}\Gamma/\Gamma = \sqrt{2(\mathcal{N}^2/\Gamma + \mathcal{N}^2/2\Gamma^2)}\) and for the estimate of the error \(\sigma\Gamma/\Gamma = \sqrt{2/\hat{N}_c}\). The last estimate is quoted in column 6-th while the actual relative deviation is in the last column. It is evident from these formulas that the coincidence method works for much smaller number of trials (\(\sim \sqrt{\Gamma}\)) than the standard approach which measures average occupation of a single state. The estimated error is steadily decreasing like \(1/\mathcal{N}\) and actual deviation follows the suit albeit with some fluctuations. It is interesting to note that the errors decrease as a number of trials and not as \(1/\sqrt{\mathcal{N}}\), see \(\mathcal{R}\) for more details. Substantial improvement in the performance can be achieved if naive counting of pairs is replaced by ”binwise” counting, i.e. a set of generated indices \(\{I_1, \ldots, I_N\}\) is first sorted. This trick reduces the computing effort from \(O(\mathcal{N}^2)\) to \(O(\mathcal{N} \log \mathcal{N})\). We find that the Monte Carlo results are well under control and show that the method is quite reliable. With the sorting trick it is practical for \(\Gamma\) of the order of \(10^{10}\). We will discuss now if this is sufficient to see the onset of thermodynamic properties.

Figure 1 shows the entropy density as a function of the scaling variable \(\epsilon = M/N\). Statistical errors of MC results (and the deviation from the exact discrete values given by \(\Gamma(M, N)\)) are much smaller that the size of symbols. Exact values are very close to the continuum expressions for the volume of the nonrelativistic phase space (solid lines). Considered as a function of \(\epsilon\) and \(N\) they obviously show a substantial \(N\)-dependence. The \(N\) varies from 8 (lowest curve) to 24 in this plot. On the other hand, the deviation from the ultimate scaling limit, (Eq.(4), the uppermost curve), varies from 8 (lowest curve) to 24 in this plot. With \(N\) starting from 12, deviations from the infinite system are smaller than 20%. Note that \(N\) denotes the number of degrees of freedom, which in \(d\) space dimensions corresponds to \(\mathcal{N}/d\) particles.

Second, we test the saddle point relation \(\mathcal{R}\) with the temperature eliminated with the aid of
the equipartition relation $E/N = T/2$. Upon discretization it reads

$$\log \left( \frac{\Gamma(M+1,N)}{\Gamma(M,N)} \right) = \frac{N}{2M+1}. \quad (7)$$

This equation is tested in Fig.2, where a half of the inverse of the left hand side, as obtained from simulations, is plotted as a function of $\epsilon$. Solid line represents the right hand side \[3\]. Similarly to the previous case agreement is very good for $N \geq 12$. It was necessary to reduce MC errors to the level of 1%-3% in order to achieve this agreement. Of course this test is much more sensitive than the previous one since it requires precise measurement of the derivatives.

To conclude, the coincidence method is satisfactory in practice for the number of degrees of freedom (or number of cells) up to $\sim 30$. This is sufficient to see the signatures of the thermal equilibrium. For more than 12 degrees of freedom the scaling of the entropy density is confirmed with the accuracy better than 20\%. The saddle point relation, coupled with the equipartition principle, $\partial S/\partial E = 1/2\epsilon$ is also very well reproduced.

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3Of course $\epsilon = (M+1/2)/N$ in this case.

| $N$     | $M$     | $N'$     | $N_c$     | $\Gamma$   | $\sigma/\Gamma$ | $\delta/\Gamma$ |
|---------|---------|----------|-----------|------------|-----------------|-----------------|
| 800 000 | 10      | 750 000  | 12        | 4.68749$_{10}^+10$ | 0.40        | 0.13            |
| 3 000 000| 188     | 1 500 000| 74        | 3.04054$_{10}^+10$ | 0.16        | 0.27            |
| 13      | 24      | 3 000 000| 256       | 3.51562$_{10}^+10$ | 0.09        | 0.15            |

Table 1

Monte Carlo results for $\hat{\Gamma}(M,N)$ (col.5). The third and fourth column give the number of generated configurations $N'$, and the number of observed coincidences $N_c$. In the last two columns we quote the Monte Carlo estimate of the relative error, see the text, and the actual relative deviation $\delta/\Gamma = |\hat{\Gamma} - \Gamma|/\Gamma$ from the exact value $\Gamma$ also quoted in the Table.

![Figure 2. Testing the relation (7). Symbols as in Fig.1, solid line corresponds to the thermodynamical limit.](image)

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