Moment free deviation inequalities for linear combinations of independent random variables with power-type tails

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Abstract

We present order of magnitude estimates for the quantiles of non-negative linear combinations of non-negative random variables, as well as deviation inequalities for general linear combinations of independent random variables, under the assumption that all random variables satisfy the same power-type tail bound on $\mathbb{P}\{|X_i| > t\}$ of the form $t^{-q}$, $t^{-q/2}$ or $t^{-q/2}(\ln t)^{q/2}$, for $q > 2$. The third type is applicable in the nonlinear setting. In the situations we consider, these results improve on classical estimates of Nagaev.

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1 Introduction

In Latala’s 1997 paper [14], the problem of estimating $L_p$ norms of sums of independent random variables, i.e. $(\mathbb{E}|\sum_{i=1}^n X_i|^p)^{1/p}$, was reduced to the problem of evaluating a type

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of Orlicz norm. In the case where the distribution of each $X_i$ decays quicker than any power function, these moment estimates can often be used in conjunction with Markov’s inequality to obtain correct order of magnitude bounds on the quantiles of $\sum X_i$. In the setting where the tail probabilities decay like power functions, e.g. $\mathbb{P}\{|X_i| > t\} = (1+t)^{-q}$ for $q \in (2, \infty)$, the corresponding $L_p$ norms are finite only in a bounded range of $p$ and order of magnitude estimates on these norms do not contain enough information to recover correct order of magnitude estimates for the quantiles and tail probabilities of $\sum X_i$. In the example just given, one misses sub-Gaussian estimates in the central region of the distribution and is off by a poly-logarithmic factor in the tails.

Deviation inequalities for sums of heavy tailed random variables have been studied extensively, and results are given at varying degrees of precision, generality and usability, under various assumptions on the tails, in the asymptotic sense with $n \to \infty$ and in the non-asymptotic sense (i.e. quantitative bounds that hold for all $n$ or for $n > n_0$). We refer the reader to [11, 12, 14, 18] and the references therein for more details. Of particular relevance is a result of Nagaev (following an earlier result of Linnik), see for example [18, Theorem 1.9], which we simplify for convenience: If $q > 2$ and $(X_i)_{i=1}^n$ are symmetric and i.i.d. with $\mathbb{P}\{|X_i| > t\} = (1+t)^{-q}$ for $t > 0$, then

$$\mathbb{P}\left\{\frac{1}{\sqrt{n\mathbb{E}X_1^2}} \sum_{i=1}^{n} X_i > t\right\} = (1 + o(1)) (1 - \Phi(t)) + (1 + o(1)) n \mathbb{P}\{X_i > \sqrt{n}t\}$$

for $n \to \infty$ and $t \geq \sqrt{n}$. In the full statement the symmetry is not needed and the CDF may involve a slowly varying function. See the given reference for details. In the case of linear combinations of these same variables, one has

$$\mathbb{P}\left\{\sum_{i=1}^{n} a_i X_i > s\right\} \leq \left(1 + \frac{2}{\mathbb{E}X_1^2}\right)^p (\mathbb{E}\max\{0, X_1\}^p) s^{-p} \sum_{i=1}^{n} |a_i|^p + \mathbb{E}\left[\frac{2(p+2)^{-2}e^{-p}s^2}{\sum_{i=1}^{n} |a_i|^2 \mathbb{E}X_1^2}\right]$$

Here we are using [18, Corollary 1.8] where our $a_i X_i$ represents his $X_i$, and $p \in [2, q)$. This goes back to a 1971 result of Fuk and Nagaev [11] and implies

$$\mathbb{P}\left\{\sum_{i=1}^{n} a_i X_i > \frac{p+2}{2} e^{p/2} (\mathbb{E}X_1^2)^{1/2} t |a|_2 + \mathbb{E}\left(\max\{0, X_1\}^p\right)^{1/p} e^{t^2/(2p)} |a|_p\right\} \leq C e^{-t^2/2}$$

where $|a|_q = (\sum_{i=1}^{n} |a_i|^q)^{1/p}$ is the $l_q^n$ norm. For large values of $t$ the second term dominates, and to minimize the deviation requires $p$ close to $q$, but not too close. For $p > (q - 1)/2$,

$$\mathbb{E}\left(\max\{0, X_1\}^p\right)^{1/p} \leq \left(\frac{q}{2} \int_0^1 u^p du + \frac{q}{2} \int_1^\infty u^{-q-1+p} du\right)^{1/p} \leq \left(\frac{q}{q-p}\right)^{1/p}$$

with a lower bound if we include an extra factor of $2^{-(2+q)/p}$. The minimum of

$$e^{h(p)} := \exp\left(\frac{t^2}{2p} + \frac{1}{p} \ln \frac{q}{q-p}\right)$$
for $p \in [2, q)$ is achieved when
\[
\frac{t^2}{2} = \ln \left( 1 - \frac{1}{s} \right) + \frac{1}{s - 1} \quad \text{where} \quad s = \frac{q}{p}
\]
The solution satisfies
\[
1 + \frac{1}{t^2 / 2 + \ln t^2} \leq s \leq 1 + \frac{1}{t^2 / 2 + \ln(t^2 / 2)}
\]
so for large values of $t$ one has
\[
\Pr \left\{ \sum_{i=1}^{n} a_i X_i > C t^{2/q} e^{t^2/(2q)} |a|_q \right\} \leq C e^{-t^2/2} \quad (1)
\]
The contributions of this paper are twofold:

\(i\). An improved deviation inequality for linear combinations of independent random variables with power type decay, presented in Section 2, that removes the factor $t^{2/q}$ from

\(ii\). Order of magnitude estimates for non-negative linear combinations of non-negative random variables, and various other tools, that are used to prove the deviation inequality mentioned in \((i)\) above. These are also used in \([9]\) to prove a deviation inequality in the nonlinear setting. We must stress that for us, at least, and we hope for others, these tools are of significant independent interest and utility. They are presented in Section 3.

## 2 Main result

**Theorem 1** There exists a universal constant $C > 0$ such that the following is true. Let $n \in \mathbb{N}$, $2 < q < \infty$, let $a \in \mathbb{R}^n$ with $a \neq 0$, and let $(X_i)_{i=1}^{n}$ be a sequence of independent random variables such that for all $t > 0$,
\[
\Pr \{ |X| > t \} \leq 2(1 + t)^{-q} \quad (2)
\]
For all $t > 0$,
\[
\Pr \left\{ \left| \sum_{i=1}^{n} a_i X_i - \mathbb{E} \sum_{i=1}^{n} a_i X_i \right| > C_q \left( t |a|_2 + e^{t^2/(2q)} |a|_q \right) \right\} \leq C e^{-t^2/2} \quad (3)
\]
where $C_q > 1$ is a function of $q$.

**Notation and conventions:** $M$ denotes median, $C$, $c$ etc. denote positive universal constants that may take on different values at each appearance, whose values we do not necessarily control. $C_q$, $c_q$ etc. denote ‘constants’ that depend on $q$ (i.e. functions of $q$). $|a|_2$ is often denoted $|a|$. Our usage of the term ‘random variable’ is limited to the real valued case.
3 Non-negative linear combinations of non-negative i.i.d. RV

3.1 Sums of order statistics (case of equal coefficients)

Concentration inequalities for the binomial distribution and the study of order statistics of uniform \((0, 1)\) random variables are of course quite standard. See for example [3][25]. In this section we present several results, based on classical techniques like the exponential moment method and the Rényi representation of order statistics, tailored to our purposes.

Define \(\xi_1 : [0, 1] \to [0, 1]\) and \(\xi_2 : [0, \infty) \to (0, 1]\) by

\[
\xi_1(t) = e^t (1 - t) \quad \xi_2(t) = e^{-t} (1 + t)
\]

Lemma 2

\[
\xi_1^{-1}(t) \leq \min \left\{ \sqrt{2(1-t)}, 1 - e^{-1}t \right\} : 0 \leq t \leq 1
\]

\[
\xi_2^{-1}(t) \leq \begin{cases} 
\log t^{-1} + \log (1 + 4 \log t^{-1}) & : 0 < t \leq 2e^{-1} \\
\sqrt{2 \log t^{-1} + 10 (\log t^{-1})^{3/2}} & : 2e^{-1} \leq t \leq 1
\end{cases}
\]

Proof. The estimates for \(\xi_1^{-1}\) follow since \(\xi_1(t) \leq \min \{1 - t^2/2, e(1 - t)\}\). To estimate \(\xi_2^{-1}\) we re-write \(y = e^{-t} (1 + t)\) as \(z = t - \log (1 + t)\), where \(z = \log y^{-1}\). If \(z < 1 - \log 2\) then \(t < 1\), since \(t \mapsto t - \log (1 + t)\) is strictly increasing. Since \(\log (1 + t) = \sum_{i=1}^{\infty} (-1)^{i+1} j^{-1} t^j\) is alternating, with terms that decrease in absolute value, \(z = t - \log (1 + t) \geq t - (t - t^2/2 + t^3/3) \geq t^2/6\). But then \(z = t - \log (1 + t) \geq t - (t - t^2/2 + t^3/3) \) and so \(t^2/2 \leq z + t^3/3 \leq z + 2 \sqrt{2} z^{3/2}\). If \(z \geq 1 - \log 2\) then \(t \geq 1\) and \(\log (1 + t) \leq t \log (2)\) so \(z = t - \log (1 + t) \geq (1 - \log (2)) t\) and \(t \leq (1 - \log (2))^{-1} z\). But then \(t = z + \log (1 + t) \leq z + \log (1 + (1 - \log (2))^{-1} z)\). \(\blacksquare\)

Lemma 3 Let \((\gamma_i)_{i=1}^n\) be an i.i.d. sample from \((0, 1)\) with corresponding order statistics \((\gamma_{(i)})_{i=1}^n\) and let \(t > 0\). With probability at least \(1 - 3^{-1} n^2 \exp (-t^2/2)\), the following event occurs: for all \(1 \leq k \leq n\), \(\gamma_{(k)}\) is bounded above by both of the following quantities

\[
\frac{k}{n+1} \left( 1 + \xi_2^{-1} \left( \exp \left( \frac{-t^2 - 4 \log k}{2k} \right) \right) \right)
\]

\[
1 - \frac{n-k+1}{n+1} \left( 1 - \xi_1^{-1} \left( \exp \left( \frac{-t^2 - 4 \log (n-k+1)}{2(n-k+1)} \right) \right) \right)
\]

and with probability at least \(1 - C \exp (-t^2/2)\) the following event occurs: for all \(1 \leq k \leq n\),

\[
\gamma_{(k)} \leq 1 - \frac{n-k}{n} \exp \left( -c \max \left\{ \frac{(t + \sqrt{\log k}) \sqrt{k}}{\sqrt{n(n-k+1)}}, \frac{t^2 + \log k}{n-k+1} \right\} \right)
\]

\[
\leq \frac{k}{n} + c \frac{n-k}{n} \max \left\{ \frac{(t + \sqrt{\log k}) \sqrt{k}}{\sqrt{n(n-k+1)}}, \frac{t^2 + \log k}{n-k+1} \right\}
\]
Remark 4  For \( k \leq n/2 \), (5) gives a typical deviation about the mean at most \( C \sqrt{k \log k/n} \) but breaks down as \( t \to \infty \) and \( n, k \) are fixed. For \( k \geq n/2 \) (6) gives a typical deviation at most \( C \sqrt{(n-k+1) \log (n-k+1)/n} \), and remains non-trivial (i.e. \( < 1 \)) for all \( 1 \leq k \leq n \) as \( t \to \infty \). For \( k \leq n/2 \) (7) also gives a typical deviation of \( C \sqrt{k \log k/n} \); it is not quite as precise as (5) (which includes the exact function \( \xi_2 \)) for \( 0 < t < t_{n,k} \) but eventually improves upon (5) and remains non-trivial as \( t \to \infty \).

Proof of Lemma 3. If \( B \) has a binomial distribution with parameters \((n, p)\), and \( np \leq s < n \), then using the exponential moment method,

\[
P \{ B \geq s \} = P \{ e^{\lambda B} \geq e^{\lambda s} \} \leq e^{-\lambda s} (1 - p + pe^{\lambda})^n = \left( \frac{np}{s} \right)^s \left( \frac{n - np}{n - s} \right)^{n-s}
\]

See e.g. [3, Ex. 2.11 p48]. Let \(#(E)\) denote the number of \( 1 \leq i \leq n \) such that \( \gamma_i \in E \). Then (recycling the variable \( s \)),

\[
P \left\{ \gamma(k) \geq \frac{k + s \sqrt{k}}{n+1} \right\} = P \left\{ \# \left( \frac{k + s \sqrt{k}}{n+1}, 1 \right) \geq n-k+1 \right\} \leq \left( 1 - \frac{s \sqrt{k}}{n-k+1} \right)^{n-k+1} \left( 1 + \frac{s}{\sqrt{k}} \right)^{k-1} \left( \frac{k}{k-1} \right)^{k-1} \div \left( \frac{n+1}{n} \right)^n \leq \left( \xi_2 \left( \frac{s}{\sqrt{k}} \right) \right)^k \leq \frac{\exp \left( -\frac{t^2}{2} \right)}{k^2}
\]

provided

\[
s \geq \sqrt{k} \xi_2^{-1} \left( k^{-2/k} \exp \left( -\frac{t^2}{2k} \right) \right)
\]

We then apply the union bound over all \( 1 \leq k \leq n \). (6) follows the same lines:

\[
P \left\{ \gamma(k) \geq \frac{k + s \sqrt{n-k+1}}{n+1} \right\} = P \left\{ \# \left( \frac{k + s \sqrt{n-k+1}}{n+1}, 1 \right) \geq n-k+1 \right\}
\]

To prove (7), we make use of the Rényi representation of order statistics from the exponential distribution (which we heard of from [4, Theorem 2.5]): there exist i.i.d. standard exponential random variables \((Z_j)_n^1\) such that

\[
-l \log (1 - \gamma(k)) = \sum_{j=1}^k \frac{Z_j}{n-j+1}
\]

(this is an easy consequence of the fact that for all \( 1 \leq k \leq n \), the order statistics \((\gamma(j))_{k+1}^n\) are (after being re-scaled to fill \((0, 1)\)) independent of \((\gamma(j))_1^k\) and distributed as the order statistics from a sample of size \( n-k \). Thus we may write

\[
1 - \gamma(k) = (1 - \gamma(1)) \prod_{j=2}^k \left( 1 - \gamma(j) \right) \left( 1 - \gamma(j-1) \right)^{-1}
\]
which is the product of \( k \) independent variables. Concentration of \( \log (1 - \gamma_{(k)}) \) about its mean (with probability \( 1 - Ck^{-2} \exp(-t^2/2) \)) can now be studied using the basic estimate
\[
P \left\{ \left| \sum_{j=1}^{k} a_j (Z_j - 1) \right| > r \right\} \leq 2 \exp \left( -c \min \left\{ \frac{r}{|a|}, \frac{r}{|a|_\infty} \right\} \right)
\]
valid for all \( r > 0 \) and all \( a \in \mathbb{R}^k \). \( \Box \) is proved using the exponential moment method, see for example [3, Ex. 2.27 p50], or use [7][Theorem 3]. The result can be transferred back to \( \gamma_{(k)} \) using the transformation \( t \mapsto 1 - \exp(-t) \).

Recall the definition of the quantile function as a generalized inverse given above the statement of Theorem [1].

**Corollary 5** Let \( n \in \mathbb{N} \), \( \lambda \in [2, \infty) \), and let \( (Y_i)_{1}^{n} \) be an i.i.d. sequence of non-negative random variables, each with cumulative distribution \( \mathbb{F} \), quantile function \( \mathbb{F}^{-1} \), and corresponding order statistics \( (Y_{(i)})_{1}^{n} \). With probability at least \( 1 - 3^{-1}\pi^2 \exp(-\lambda^2/2) \), the following event holds: for all \( j, k \in \mathbb{Z} \) with \( 0 \leq j \leq k < n \),
\[
\sum_{i=n-k}^{n-j} Y_{(i)} \leq \mathbb{F}^{-1} \left( 1 - \frac{j + 1}{n + 1} \left( 1 - \xi_{j+1}^{-1} \left( \exp \left( -\frac{\lambda^2}{2} \frac{\log(j+1)}{j+1} \right) \right) \right) \right) + (n + 1) \int_{(j+1)/(n+1)}^{(k+1)/(n+1)} \mathbb{F}^{-1} \left( 1 - t \left( 1 - \xi_k^{-1} \left( \exp \left( -\frac{\lambda^2}{2} \frac{\log((n+1)t)}{(n+1)t} \right) \right) \right) \right) dt
\]

**Proof.** Let \( (\gamma_i)_{1}^{n} \) be an i.i.d. sample from the uniform distribution on \( (0, 1) \). Since \( (Y_{(i)})_{1}^{n} \) has the same distribution as \( (\mathbb{F}^{-1}(\gamma_i))_{1}^{n} \) we may assume without loss of generality that \( Y_{(i)} = \mathbb{F}^{-1}(\gamma_{(i)}) \). We now apply Lemma [3] to the random vector \( (\gamma_{(i)})_{1}^{n} \). If \( j = k \) we simply have one term. If \( j < k \) write
\[
\sum_{i=n-k}^{n-j} Y_{(i)} = Y_{(n-j)} + \sum_{i=n-k}^{n-j-1} Y_{(i)}
\]
and compare the sum to an integral using right hand endpoints the fact that the integrand is decreasing. Here we also use the fact that \( x \mapsto (\lambda^2 + 4 \log x)/x \) is decreasing provided \( \log x \geq 1 - \lambda^2/4 \), and we have assumed that \( \lambda \geq 2 \).

**Lemma 6** In Corollary [3] we can replace the upper bound for \( \sum_{i=n-k}^{n-j} Y_{(i)} \) with
\[
\mathbb{F}^{-1} \left( 1 - \frac{j + 1}{n + 1} e^{-\lambda^2 \frac{\log(j+1)}{2(j+1)}} \right) + \lambda^2 \int_{\frac{(k+1)}{\lambda^2} \exp\left( -\frac{\lambda^2}{2j(j+1)} \right)}^{\frac{(k+1)}{\lambda^2} \exp\left( -\frac{\lambda^2}{2j(j+1)} \right)} \mathbb{F}^{-1} \left( 1 - e^{-1-2/e} \frac{\lambda^2}{2(n+1)z} \right) \left\{ 1 + \frac{1}{z} \left( \log \left( e + \frac{1}{z} \right) \right)^{-2} \right\} dz
\]

**Proof.** By Corollary [5] and Lemma [2] \( \sum_{i=n-k}^{n-j} Y_{(i)} \) is bounded above by
\[
\mathbb{F}^{-1} \left( 1 - \frac{j + 1}{n + 1} e^{-\lambda^2 \frac{\log(j+1)}{2(j+1)}} \right) + (n + 1) \int_{(j+1)/(n+1)}^{(k+1)/(n+1)} \mathbb{F}^{-1} \left( 1 - e^{-1-2/e} t \exp \left( -\frac{\lambda^2}{2(n+1)t} \right) \right) dt
\]
Then set 
\[ s = \frac{\lambda^2}{2(n+1)} t^{-1} \]
and the integral becomes
\[ \frac{\lambda^2}{2} \int_{\frac{s}{2(n+1)}}^{\frac{s}{2(n+1)}+\lambda^2} F^{-1}\left(1 - e^{-1/2/e} \frac{\lambda^2}{2(n+1)} s^{-1} e^{-s}\right) s^{-2} ds \quad (9) \]
Setting \( z = q(s) = s^{-1} e^{-s} \) and using \( q'(s) = -q(s)(1 + 1/s) \),
\[ ds = \frac{dz}{q'(q^{-1}(z))} = \frac{dz}{q'(s)} = \frac{dz}{-q(s)(1 + s^{-1})} = \frac{-e^s dz}{s^{-1}(1 + s^{-1})} \]
The expression in (9) can then be written as
\[ \frac{\lambda^2}{2} \int_{\frac{2q(s)}{\lambda^2}}^{\frac{2q(s)}{\lambda^2}+\lambda^2} e^s \frac{s^{-2}}{1 + s} \quad (10) \]
We’d like to write \( e^s/(1 + s) \) as a function of \( x \), or at least bound it above by such a function, and we start by estimating it in terms of \( z = s^{-1} e^{-s} \in (0, \infty) \). When \( z \) is small \( s \) is large and \( s \geq (1/2) \log \frac{1}{z} - 1 \), and
\[ e^s \frac{s^{-1}}{1 + s} \leq C \frac{1}{z} \left(\log \frac{1}{z}\right)^{-2} \]
When \( z \) is large \( s \) is small and \( e^s/(1 + s) \leq C \). By continuity, for all \( z \in (0, \infty) \),
\[ e^s \frac{s^{-1}}{1 + s} \leq C \left\{ 1 + \frac{1}{z} \left(\log \left(e + \frac{1}{z}\right)\right)^{-2} \right\} \]
and we define
\[ C_0 = \sup \left\{ \frac{e^s}{1 + s} \left[ 1 + \frac{se^s}{s + e^s} \right]^{-1} : s \in (0, \infty) \right\} \]
to be the smallest possible value of \( C \). A numerical computation shows that \( 1 < C_0 < 2 \).

**Lemma 7** For all \( a, b \in (0, \infty) \) with \( a \leq b \) and all \( r \in \mathbb{R} \),
\[ \int_a^b x^{-r} dx \leq C \min \left\{ |1 - r|^{-1}, \log \frac{b}{a}\right\} \left( a^{1-r} + b^{1-r} \right) \quad (11) \]
where we define \( 0^{-1} = \infty \). If \( 0 < a < b < e^{-1} \) and \( r > 1 \) then
\[ \int_a^b x^{-r} \left(\log \frac{1}{x}\right)^{-2} dx \leq C \min \left\{ 1, \log \frac{\log \frac{a}{b}}{\log \frac{b}{a}} \right\} \left( \log \frac{1}{b}\right)^{-1} \]
\[ + \quad C \min \left\{ (r-1)^{-1}, \log \frac{b}{a}\right\} \left[ (r-1)^{-1} + \log \frac{1}{a}\right]^{-2} a^{1-r} \quad (12) \]
\[ \leq \quad C_r \min \left\{ 1, \log \frac{b}{a}\right\} \left( \log \frac{1}{a}\right)^{-2} a^{1-r} \quad (13) \]
The inequalities in (11) and (12) can be reversed by replacing $C$ with $c$, and (13) can be reversed by replacing $C_r$ with $c_r$.

**Proof.** Assume without loss of generality that $a < b$. (11) is Lemma 3 in [10] without the restriction that $a = 1$, and it follows from that lemma by a change of variables. For (12), set $t = (r - 1) \log(1/x)$, so the integral becomes

$$\int_{(r-1) \log(1/b)}^{(r-1) \log(1/a)} (r-1) e^t dt$$

(14)

Now $t^{-2} e^t$ is the same order of magnitude as $t^{-2} + (1 + t)^{-2} e^{1+t}$, which can be checked separately for $t \leq 1$ and $t > 1$. To integrate the first term of this integrand use (11) with 2 in place of $r$. To integrate the second term set $u = t + 1$ and note that the resulting integrand is the same order of magnitude as a function that has an instantaneous exponential growth rate that is bounded above and below by universal constants, i.e.

$$C(r-1) \int_{(r-1) \log(1/b)+1}^{(r-1) \log(1/a)+1} u^{-2} e^u du \leq C(r-1) \int_{(r-1) \log(1/b)+1}^{(r-1) \log(1/a)+1} (u+2)^{-2} e^u du$$

and for all $u \in [1, \infty)$,

$$\frac{2}{3} \leq \frac{d}{du} \ln((u+2)^{-2} e^u) \leq 1$$

so the integral is the same order of magnitude as

$$C(r-1) \min \left\{1, (r-1) \log \frac{b}{a} \right\} \left[(r-1) \log \frac{1}{a} + 3\right]^{-2} \exp((r-1) \log(1/a) + 1)$$

**Proposition 8** Consider the setting and assumptions of Corollary 5 and assume, in addition, that $p > 0$, $T \geq 1$ and that for all $\delta, x \in (0, 1)$,

$$H^*(\delta x) \geq T^{-1} \delta^{-1/p} H^*(x)$$

(15)

where $H^*(x) = F^{-1}(1-x)$. Then the upper bound for $\sum_{i=n-k}^{n-j} Y(i)$ can be replaced with

$$\left[1 + T \lambda^2 A \right] H^* \left( e^{-1-2/e} \frac{j+1}{n+1} \exp \left( \frac{-\lambda^2}{2(j+1)} \right) \right) + Cn \int_{\frac{k+1}{n+1} \exp \left( \frac{-\lambda^2}{2(k+1)} - 2/e \right)}^{1} H^*(x) dx$$

where $A = 0$ if $\lambda^2/2 \leq j+1$ and $A$ equals

$$C^{1+1/p} \min \left\{p, \lambda^2 \left( \frac{1}{j+1} - \min \left\{ \frac{\lambda^2/2}{k+1} \right\} \right) \right\} \left\{ p + 1 + \frac{\lambda^2}{j+1} \right\}^{-2}$$

$$+ C \min \left\{1, \log \left( \min \left\{ k+1, \frac{\lambda^2/2}{j+1} \right\} \right) \right\} \left\{ \frac{\lambda^2}{2(j+1)} \exp \left( \frac{\lambda^2}{2(j+1)} \right) \right\}^{-1/p} \left[ 1 + \frac{\lambda^2}{k+1} \right]^{-1}$$

if $\lambda^2/2 > j+1$. 

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Proof. By Lemma 6 and assumption (15), \( \sum_{i=n-k}^{n-j} Y(i) \) is (with the required probability) at most \( I + II + III \), where

\[
I = H^* \left( \frac{j + 1}{n + 1} e^{-1} \exp \left( -\frac{-\lambda^2}{2(j + 1)} \right) \right)
\]

\[
II = C \lambda^2 \int_{\frac{2(j+1)^2}{\lambda^2} \exp \left( -\frac{-\lambda^2}{2(j+1)} \right)}^{\lambda^2} H^* \left( -\frac{1 - 2/e \lambda^2}{2(n+1)} z \right) dz 
\]

\[
\leq Cn \int_{\frac{2(j+1)^2}{\lambda^2} \exp \left( -\frac{-\lambda^2}{2(j+1)} \right) - \frac{1 - 2/e}{2}}^{\frac{2(j+1)^2}{\lambda^2} - 1 - 2/e} H^* (x) dx 
\]

and \( III \) is the product of

\[
C \lambda^2 T H^* \left( -\frac{1 - 2/e \lambda^2}{2(j+1)} \right) \tag{16}
\]

and

\[
\int_{\min \left\{ e^{-1}, \frac{2(j+1)^2}{\lambda^2} \exp \left( -\frac{-\lambda^2}{2(j+1)} \right) \right\}}^{\max \left\{ e^{-1}, \frac{2(j+1)^2}{\lambda^2} \exp \left( -\frac{-\lambda^2}{2(j+1)} \right) \right\}} \left( \frac{2(j + 1)}{\lambda^2} \exp \left( -\frac{-\lambda^2}{2(j + 1)} \right) z^{-1} \right)^{1/p} \left\{ \frac{1}{z} \left( \log \left( e + \frac{1}{z} \right) \right)^{-2} \right\} dz 
\]

Here what we are doing is taking the term \( 1 + \frac{1}{z} \left( \log \left( e + \frac{1}{z} \right) \right)^{-2} \) which appears in Lemma 6 and expressing the corresponding integral as a sum of terms, one with coefficient 1 and another with coefficient \( \frac{1}{z} \left( \log \left( e + \frac{1}{z} \right) \right)^{-2} \). The second term only comes into play when the coefficient is at least \( c \), so we may restrict the integral in \( III \) to values of \( z \) in \((0, e^{-1})\). By Lemma 7 we can bound the integral in \( III \) above by the sum of

\[
C^{1+1/p} \min \left\{ p, \log \max \left\{ e, \frac{\lambda^2}{2(j+1)} \exp \left( \frac{\lambda^2}{2(j+1)} \right) \right\} \right\} \left( p + 1 + \frac{\lambda^2}{2(j+1)} \right)^{-2} 
\]

\[
\times \left[ 1 + \frac{2(j+1)}{\lambda^2} \exp \left( -\frac{-\lambda^2}{2(j+1)} \right) \right]^{1/p} 
\]

and

\[
C \left[ \frac{\lambda^2}{2(j+1)} \exp \left( -\frac{\lambda^2}{2(j+1)} \right) \right]^{-1/p} \left[ \max \left\{ 1, \log \left[ \frac{\lambda^2}{2(k+1)} \exp \left( \frac{\lambda^2}{2(k+1)} \right) \right] \right\} \right]^{-1} 
\]

\[
\times \min \left\{ 1, \log \max \left\{ 1, \log \left[ \frac{\lambda^2}{2(k+1)} \exp \left( \frac{\lambda^2}{2(k+1)} \right) \right] \right\} \right\} 
\]

Unless \( \lambda^2/(2(j+1)) \geq 1 \), the integral in \( III \) is zero because the interval of integration has length zero. So while bounding \( III \) we assume this is the case, and this allows for
simplification. By considering the cases $\lambda^2 \leq 2(k+1)$ and $\lambda^2 > 2(k+1)$ separately,

$$\min \left\{ \begin{array}{l} p, \log \frac{\max \left\{ e, \frac{\lambda^2}{2(j+1)} \exp \left( \frac{\lambda^2}{2(j+1)} \right) \right\}}{\max \left\{ e, \frac{\lambda^2}{2(k+1)} \exp \left( \frac{\lambda^2}{2(k+1)} \right) \right\}} \\ \lambda^2 \leq 2(k+1) \end{array} \right\}$$

$$\leq C \min \left\{ \begin{array}{l} p, \lambda^2 \left( \frac{1}{j+1} - \frac{1}{j+1} \min \left\{ \lambda^2/2, k+1 \right\} \right) + \log \frac{\min \left\{ \lambda^2/2, k+1 \right\}}{j+1} \\ \lambda^2 > 2(k+1) \end{array} \right\}$$

$$\leq C \min \left\{ \begin{array}{l} p, \lambda^2 \left( \frac{1}{j+1} - \frac{1}{\min \left\{ \lambda^2/2, k+1 \right\}} \right) \end{array} \right\}$$

Here we have used the fact that because the logarithm is 1-Lipschitz on $[1, \infty)$, for all $a, b \in [1, \infty)$ with $a \leq b$, $\log(1/a) - \log(1/b) \leq 1/a - 1/b$. We now prove two claims which help simplify another term.

**Claim:** for all $a, b \in [1, \infty)$ such that $a < b$ and $a \leq e + 1$ (say),

$$c \min \left\{ \frac{b-a}{a}, \log b \right\} \leq \log b - \log a \leq \min \left\{ \frac{b-a}{a}, \log b \right\}$$

**Proof of Claim:** The upper bound holds because the derivative of log is decreasing and because $\log a \geq 0$. For the lower bound, note that $a^2 \leq (e + 1)a$, so either $b \leq (e + 1)a$ or $b > a^2$. If $b \leq (e + 1)a$ then (by considering the derivative)

$$\log b - \log a \geq \frac{b-a}{b} \geq C \frac{b-a}{a}$$

and if $b > a^2$ then

$$\log b - \log a = (1/2) \log b$$

**Claim:** for all $s, t \in [1, \infty)$ with $s < t$,

$$\log \frac{t + \log t}{s + \log s} \leq C \log \frac{t}{s}$$

**Proof of Claim:** This is certainly true when $s \geq e$, because $x \mapsto (\log x)/x$ is decreasing on $(e, \infty)$, which implies the desired inequality with $C = 1$. For $s < e$, apply the first claim twice to get

$$\log \frac{t + \log t}{s + \log s} \leq C \min \left\{ \frac{(t-s) + \log t - \log s}{s + \log s}, \log(t + \log t) \right\} \leq C \min \left\{ \frac{t-s}{s}, \log(t) \right\}$$

which proves the second claim.

We now find a simplified upper bound for

$$\min \left\{ \begin{array}{l} 1, \log \frac{\max \left\{ 1, \log \left[ \frac{\lambda^2}{2(j+1)} \exp \left( \frac{\lambda^2}{2(j+1)} \right) \right] \right\}}{\max \left\{ 1, \log \left[ \frac{\lambda^2}{2(k+1)} \exp \left( \frac{\lambda^2}{2(k+1)} \right) \right] \right\}} \\ \lambda^2 \leq 2(k+1) \end{array} \right\}$$
When $\lambda^2 \geq 2(k+1)$ and when $\lambda^2 < 2(k+1)$ we get (respectively) as upper bounds using the second claim and using $\log \log(t e^t) \leq C \log t$ for $t \geq 1$,

$$\min \left\{ 1, \log \frac{k+1}{j+1} \right\} \quad \quad C \min \left\{ 1, \log \frac{\lambda^2}{2(j+1)} \right\}$$

In either case, we have the following upper bound

$$C \min \left\{ 1, \log \frac{\min \{k+1, \lambda^2/2\}}{j+1} \right\} = C \log \underbrace{\min \{e(j+1), k+1, \lambda^2/2\}}_{j+1}$$

The result of these simplifications is that the upper bound for the integral in $\text{III}$ reduces to the quantity $A$ as defined in the statement of the result. ■

**Remark 9** If in Proposition 8 we set $j = 0$, $k = n-1$ and assume that $p > 1$ and $T \leq C$ for any desired constant $C \geq 1$, the bound on $\sum_{i=1}^{n} Y_{(i)} = \sum_{i=1}^{n} Y_i$ can be replaced with

$$C \left( 1 + \lambda^{-2/p} e^{-\lambda^2/(2p)} \right) H^* \left( e^{-1-2/e} \frac{1}{n+1} \exp \left( -\frac{\lambda^2}{2} \right) \right) + C n \mathbb{E} Y_1$$

One also has

$$1 + \lambda^{-2/p} e^{-\lambda^2/(2p)} \min \{ \lambda^2, n \} \leq C p$$

**Proof.** When bounding $A$ we may assume without loss of generality that $A \neq 0$. Distribute $\lambda^2$ into $A$, bound the two minima in $A$ by $p$ and 1 respectively and use

$$\frac{\lambda^2 p}{(p+1+\lambda^2)^2} \leq 1 \quad \text{and} \quad \lambda^{2-2/p} \exp \left( -\frac{\lambda^2}{2p} \right) \leq C p$$

To bound the coefficient as in the statement, use $\min \{ \lambda^2, n \} \leq \lambda^2$ and optimize. ■

**Corollary 10** Let $p > 1$, $n \in \mathbb{N}$, $\lambda > 0$, and let $(Y_i)^n$ be an i.i.d. sequence of non-negative random variables, each with cumulative distribution $F(x) = \min \{ 1, x^{-p} \}$, quantile function $F^{-1}(x) = x^{-1/p}$, and corresponding order statistics $(Y_{(i)})^n$. With probability at least $1 - C \exp(-\lambda^2/2)$, the following event occurs: for all $j \in \mathbb{Z}$ with $0 \leq j \leq n/2$, $\sum_{i=1}^{n-j} Y_{(i)}$ is bounded above by

$$\frac{C p n}{p-1} + C \left( 1 + (j+1) \min \left\{ \left( \frac{\lambda^2}{p(j+1)} \right)^2, \left( \frac{\lambda^2}{p(j+1)} \right)^{-1} \right\} \right) \left( \frac{n}{j+1} \right)^{1/p} \exp \left( -\frac{\lambda^2}{2p(j+1)} \right)$$

**Note:** The condition $j \leq n/2$ is not necessary, but rather highlights the setting where the bound is most effective.

**Proof.** Proposition 8 with $k = n-1$ gives the estimate $\sum_{i=1}^{n-j} Y_{(i)} \leq I + II + III + IV$ where

$$I = C n \int_0^1 H^*(x) dx = \frac{C p n}{p-1}$$

$$II = H^* \left( e^{-1-2/e} \frac{j+1}{n+1} \exp \left( -\frac{\lambda^2}{2(j+1)} \right) \right) \leq C \left( \frac{n}{j+1} \right)^{1/p} \exp \left( -\frac{\lambda^2}{2p(j+1)} \right)$$
and $\text{III, IV} = 0$ unless $\lambda^2 \geq 2(j+1)$, in which case

$$III = C \lambda^2 \min\left\{ p, \frac{\lambda^2}{j+1} \right\} \max\left\{ p, \frac{\lambda^2}{j+1} \right\}^{-2} \left( e^{-1-2/e} \frac{j+1}{n+1} \exp\left( \frac{\lambda^2}{2(j+1)} \right) \right)^{1/p} \left( \frac{n}{j+1} \right)^{1/p} \exp\left( \frac{\lambda^2}{2p(j+1)} \right)$$

and $\text{IV}$ equals

$$C \lambda^2 \left[ \frac{\lambda^2}{2(j+1)} \exp\left( \frac{\lambda^2}{2(j+1)} \right) \right]^{-1/p} \left( 1 + \frac{\lambda^2}{n} \right)^{-1} H^* \left( e^{-1-2/e} \frac{j+1}{n+1} \exp\left( \frac{\lambda^2}{2(j+1)} \right) \right)$$

$$\leq C \lambda^{2-2/p} n^{1/p} \left( 1 + \frac{\lambda^2}{n} \right)^{-1}$$

for $\lambda \leq \sqrt{n}$, $\text{IV} \leq C \lambda^{2-2/p} n^{1/p} \leq Cn$ and for $\lambda > \sqrt{n}$, $\text{IV} \leq C \lambda^{-2/p} n^{-1/p} \leq Cn$. Either way, $\text{IV} \leq I$.

Further remarks under the tail condition $F(x) = \min\{1, x^{-p}\}$.

Setting $j = 0$ in Corollary [10] or applying Remark [9]

$$\mathbb{P}\left\{ \sum_{i=1}^{n} Y_i \geq \frac{Cpn}{p-1} + Cn^{1/p} \exp\left( \frac{\lambda^2}{2p} \right) \right\} < Ce^{-\lambda^2/2}$$

This gives the correct order of magnitude for $\sum_{i=1}^{n} Y_i$ in the i.i.d. case up to the value of $C$, since the same bound describes the order of magnitude of $n\mathbb{E}Y_i + \max_{1 \leq i \leq n} Y_i$.

Returning to the case of a general value of $j$, and setting $k = j+1$ and $s^k = \exp(\lambda^2/2)$, the bound in Corollary [10] can be written as

$$\mathbb{P}\left\{ \sum_{i=1}^{n-k+1} Y_{(i)} > \frac{Cpn}{p-1} + C \left( 1 + k \min\left\{ \left( \frac{\log s}{p} \right)^2, \left( \frac{\log s}{p} \right)^{-1} \right\} \right) \left( \frac{n}{k} \right)^{1/p} s^{1/p} \right\} \leq Cs^{-k}$$

for $s > 1$. Compare this to the following bound of Guédon, Litvak, Pajor, and Tomczak-Jaegermann [13, Lemma 4.4]: for all $s \in (1, \infty)$,

$$\mathbb{P}\left\{ \sum_{i=1}^{n-k+1} Y_{(i)} > \frac{12p (es)^{1/p}}{p-1} n \right\} \leq s^{-k}$$

### 3.2 Partial reduction to the case of equal coefficients (geometric approach)

#### 3.2.1 A norm for quantiles of linear functionals

Let $\mu$ be any probability measure on $\mathbb{R}^n$ not supported on any half space not containing the origin, and such that

$$\int_{\mathbb{R}^n} |\langle x, a \rangle| \, d\mu(x) < \infty \quad (17)$$
for all $a \in \mathbb{R}^n$, let $X = (X_i)_1^n$ be a random vector with distribution $\mu$, and let $F_a(t) = \mathbb{P}\{\sum_1^n a_i X_i \leq t\}$. Set $X^{(0)} = 0 \in \mathbb{R}^n$ and let $(X^{(j)})_{j=1}^\infty$ be an i.i.d. sample from $\mu$, let $\delta \in (0, 1/2)$, and let $N \sim \text{Pois}(\delta^{-1})$. A basic result in the theory of Poisson point processes is that the random measure
\[
\sum_{j=1}^N \delta (X^{(j)})
\]
is a Poisson point process with intensity $\delta^{-1} \mu$, where $\delta(x)$ denotes the Dirac point mass at $x$, not to be confused with $\delta \in (0, 1/2)$. The set
\[
\mathfrak{Z} = \mathbb{E} \text{conv} \{X_i\}_{i=1}^N := \left\{ x \in \mathbb{R}^n : \forall \theta \in S^{n-1}, \langle \theta, X \rangle \leq \mathbb{E} \max_{0 \leq j \leq N} \langle \theta, X^{(j)} \rangle \right\}
\]
is seen to be a compact convex set with nonempty interior (i.e. a convex body), in fact $0 \in \text{int}(\mathfrak{Z})$. Its dual Minkowski functional, given by $|a|_{\mathfrak{Z}^\circ} = \sup \{ \langle x, a \rangle : x \in \mathfrak{Z} \}$, can be expressed as
\[
|a|_{\mathfrak{Z}^\circ} = \mathbb{E} \max_{0 \leq j \leq N} \langle a, X^{(j)} \rangle = \delta^{-1} \int_0^{1-F_a(0)} F_a^{-1}(1-s) \exp(-\delta^{-1} s) \, ds \tag{18}
\]
This is because for $t > 0$, by definition of a Poisson point process,
\[
G(t) := \mathbb{P}\{\max_{0 \leq j \leq N} \langle a, X^{(j)} \rangle \leq t\} = \exp(-\delta^{-1} (1 - F_a(t)))
\]
so
\[
\mathbb{E} \max_{0 \leq j \leq N} \langle a, X^{(j)} \rangle = \int_0^1 G^{-1}(t) \, dt = \int_0^1 \frac{F_a^{-1}(1 - \delta \log t^{-1})}{F_a^{-1}(1)} \, dt
\]
This convex body is a modification of the expected convex hull of a fixed sample size used in [6] (see references therein) and is related to the dual (polar) of the convex floating body defined by deleting all half spaces with $\mu$ measure less than $\delta$, see [2, 5, 24]. Its advantage over the convex floating body is that there is an explicit formula for its Minkowski functional (by definition), and its advantage over the expected convex hull with a fixed sample size is the representation of its dual Minkowski functional in (18).

**Lemma 11** For all $a \in \mathbb{R}^n$ and all $0 < \delta < 1 - F_a(0)$,
\[
\mathbb{P}\left\{ \sum_{i=1}^n a_i X_i > 2 |a|_{\mathfrak{Z}^\circ} \right\} \leq \delta \log 2 \quad \mathbb{P}\left\{ \sum_{i=1}^n a_i X_i \geq (1 + R)^{-1} |a|_{\mathfrak{Z}^\circ} \right\} \geq \delta
\]
where
\[
R = \frac{\delta^{-1} \int_{1-\delta}^1 F_a^{-1}(t) \, dt}{F_a^{-1}(1 - \delta)}
\]
Proof. Comparing the mean $E$ and any median $M$,

$$\mathbb{P}\left\{ \sum_{i=1}^{n} a_i X_i > 2|a|_{3^p} \right\} \leq \mathbb{P}\left\{ \sum_{i=1}^{n} a_i X_i > M \max_{0 \leq j \leq N} \langle a, X^{(j)} \rangle \right\}$$

$$= -\delta \log G\left( M \max_{0 \leq j \leq N} \langle a, X^{(j)} \rangle \right) \leq \delta \log 2$$

On the other hand, from (18),

$$|a|_{3^p} \leq \delta^{-1} \int_{0}^{\delta} F_{a}^{-1} (1-s) \, ds + \delta^{-1} \int_{0}^{\infty} F_{a}^{-1} (1-\delta) \exp (-\delta^{-1} s) \, ds$$

so for all $\varepsilon \in (0, 1/2)$,

$$\mathbb{P}\left\{ \sum_{i=1}^{n} a_i X_i > (1 + \varepsilon)^{-1} (1 + R)^{-1} |a|_{3^p} \right\} \geq \mathbb{P}\left\{ \sum_{i=1}^{n} a_i X_i > (1 + \varepsilon)^{-1} F_{a}^{-1} (1 - \delta) \right\} > \delta$$

The role of $\varepsilon$ is a technicality related to the definition of the generalized inverse $F_{a}^{-1}$. ■

If, on the other hand, $\mu$ is supported on $[0, \infty)^n$ and for all $a \in \mathbb{R}^n$

$$\mathbb{P}\left\{ \sum_{i=1}^{n} |a_i| X_i > 0 \right\} > 0$$

and (17) holds, then

$$[a]_{\delta} = \mathbb{E} \max_{0 \leq j \leq N} \sum_{i=1}^{n} |a_i| X^{(j)}$$

(19)
as a function of $a$, is a norm, and Lemma 11 holds with $|\cdot|_{3^p}$ replaced with $|\cdot|_{\delta}$ and $F_a$ replaced with $F_{(|a_i|)_{1}^{q}}$. Assuming for simplicity that each $a_i \geq 0$, the version of (18) for $|\cdot|_{\delta}$ is

$$[a]_{\delta} = \delta^{-1} \int_{0}^{1} F_{a}^{-1} (1-s) \exp (-\delta^{-1} s) \, ds$$

(20)

3.2.2 A norm characterized by its values on $\{0,1\}^n$

For any $r \in [1, \infty)$ and $q \in (1, \infty)$ define

$$V_{r,q} = \left\{ \max \left\{ |u|_{1,r} , |u|_{q} \right\}^{-1} u : u \in \{0, \pm 1\}^n , u \neq 0 \right\} \quad E_{r,q} = \text{conv} (V_{r,q})$$

where $\text{conv}$ denotes convex hull. The Minkowski functional of $E_{r,q}$ is the norm $|x|_{r,q} = \inf \{ \lambda > 0 : x \in \lambda E_{r,q} \}$.

Lemma 12 For all $x \in \{0, \pm 1\}^n$, $|x|_{r,q} = \max \left\{ |x|_{1,r} , |x|_{1,q} \right\}$.

Proof. Since $V_{r,q} \subset \partial \left( B_1^{n} \cap r^{-1} B_q^{n} \right)$ and $B_1^{n} \cap r^{-1} B_q^{n}$ is convex, it follows that $V_{r,q} \subset \partial E_{r,q}$. ■
Lemma 13  If $\|\cdot\|$ is any norm on $\mathbb{R}^n$ and $\|x\| \leq |x|_{r,q}$ for all $x \in \{0, \pm 1\}^n$, then $\|x\| \leq |x|_{r,q}$ for all $x \in \mathbb{R}^n$.

Proof. This follows since

$$\text{conv} \left\{ |u|_{r,q}^{-1} u : u \in \{0, \pm 1\}^n, u \neq 0 \right\} \subseteq \text{conv} \left\{ \|u\|^{-1} u : u \in \{0, \pm 1\}^n, u \neq 0 \right\}$$

By Lemma 12, LHS is $E_{r,q}$, and RHS is a subset of the unit ball corresponding to $|\cdot|_{r,q}$.

The dual Minkowski functional of $E_{r,q}$ is defined by

$$|y|_{r,q}^o = \sup \left\{ \sum_{i=1}^n x_i y_i : x \in E_{r,q} \right\}$$

Recall that $(y[i])_1^n$ denote the non-increasing rearrangement of the absolute values of $(y_i)_1^n$.

Proposition 14  For all $x, y \in \mathbb{R}^n$,

$$|y|_{r,q}^o \leq 2 \sup \left\{ r^{-1} k^{-1/q} \sum_{i=1}^k y[i] : 1 \leq k \leq \min \left\{ r^{q/(q-1)}, n \right\} \right\} \leq 2 |y|_{r,q}^o \quad (21)$$

and

$$|x|_{r,q} \leq 4q^{-1} \left( |x|_1 + r \sum_{i=1}^n i^{-1+1/q} x[i] \right) \leq 16 |x|_{r,q} \quad (22)$$

Proof. The right hand inequality in (21) follows from the definition of $|y|_{r,q}^o$, since the supremum is an upper bound. For the left hand inequality, note that

$$|y|_{r,q}^o = \sup \left\{ \sum_{i=1}^n x_i y_i : x \in V_{r,q} \right\}$$

Since $V_{r,q}$ is invariant under coordinate permutations and coordinate sign changes, so is $E_{r,q}$, and

$$|y|_{r,q}^o = \left| (y[i])_1^n \right|_{r,q}^o = \sup \left\{ \max \left\{ k, rk^{1/q} \right\}^{-1} \sum_{i=1}^k y[i] : 1 \leq k \leq n \right\}$$

For $k \geq r^{q/(q-1)}$, $\max \left\{ k, rk^{1/q} \right\} = k$ and $k^{-1} \sum_{i=1}^k y[i]$ is non-increasing in $k$, so we may restrict our attention to values of $k$ such that $k \leq \left\lceil r^{q/(q-1)} \right\rceil$. The factor of 2 is the price we pay for neglecting $k = \left\lceil r^{q/(q-1)} \right\rceil$. For (22), assume without loss of generality that the coordinates of $x$ are strictly positive and strictly decreasing. Since the canonical embedding of a normed space into its bidual is an isometry,

$$|x|_{r,q} = \sup \left\{ \sum_{i=1}^n x_i y_i : |y|_{r,q}^o \leq 1 \right\} \quad (23)$$
Now evaluate this supremum by finding the appropriate \( y \), and replacing \( |y|_{r,q}^c \) with the equivalent quantity

\[
|y|^r = \sup \left\{ r^{-1} k^{1/q} \sum_{i=1}^k y[i] : 1 \leq k \leq \min \{ r^{q/(q-1)} / n \} \right\}
\]

Bounds on the coordinates of \( y \) are achieved by exploiting the fact that \( |y|^r \leq 1 \) and that \( y \) is a maximizer of \( \sum x_i y_i \). Including non-explicit constants of the form \( C_q, c_q \) may help to simplify the calculations. An alternative method is to notice that within the collection of points with positive decreasing coordinates, \( \partial E_{r,q} \) is contained in a hyperplane determined by \( n \) given points. ■

3.3 Partial reduction to the case of equal coefficients (combinatorial approach)

For \( n \in \mathbb{N} \) and \( 0 \leq k \leq n \), the symbol \( S(n, k) \) represents the number of ways to partition a set of cardinality \( n \) into a total of \( k \) nonempty subsets, taking \( S(n, 0) = 0 \). This is known as a Stirling number of the second kind. It follows that the number of functions \( f : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\} \) with \( \text{Range}(f) = k \) is equal to

\[
E(n, k) = \frac{n^!}{(n-k)^!} S(n, k)
\]

For \( k \geq 1 \), \( S(n+1, k) = kS(n, k) + S(n, k-1) \). This can easily be seen by taking a set of \( n+1 \) elements, setting one aside, and considering partitions where the distinguished element stands alone as a singleton and those where it does not.

**Lemma 15** For all \( n \in \mathbb{N} \) and \( n/2 \leq k \leq n \), \( k! S(n, k) \geq (n-k)! S(n, n-k) \), which can be written as \( E(n, k) \geq E(n, n-k) \).

**Proof.** Consider the lemma as a sequence of statements \((S_n)_{n=1}^\infty \). \( S_1 \) is seen to be true since \( E(1, 0) = 0 \) while \( E(1, 1) = 1 \). Suppose that \( S_n \) is true for some \( n \geq 1 \) and consider \( S_{n+1} \). If \( k = (n+1)/2 \) then the required inequality for \( S_{n+1} \) holds with equality. If \( k > (n+1)/2 \) then \( k \geq (n+2)/2 \) and by \( S_n \), \( k! S(n+1, k) = k! [kS(n, k) + S(n, k-1)] \) can be bounded below by

\[
k(n-k)! S(n, n-k) + k(n-k+1)! S(n, n-k+1) \\
\geq (n-k+1)! S(n, n-k) + (n-k+1)(n-k+1)! S(n, n-k+1) \\
\geq (n-k+1)! S(n+1, n-k+1)
\]

So \( S_{n+1} \) is true. ■

**Theorem 16** Let \( I = (I(i))_{i=1}^n \) be an i.i.d. sequence of random integers uniformly distributed in \( \{1, 2, \cdots, n\} \) and \( V = (V_i)_{i=1}^n \) an i.i.d. sequence of non-negative random variables independent of \( I \). Then for all \( b \in [0, \infty)^n \) and all \( t > 0 \),

\[
P \left\{ \sum_{i=1}^n b_i V_i \geq t \right\} \leq 2 P \left\{ \sum_{i=1}^n b_I(i) V_i \geq \frac{t}{2} \right\}
\]
Proof. Because the distributions in question do not depend on the underlying probability space, we may assume without loss of generality that this underlying probability space is rich enough to support the independent random variables that we introduce throughout the proof, and that it is non-atomic. Consider any $v \in [0, \infty)^n$ and let $\sigma$ be a random permutation uniformly distributed in the symmetric group $S_n$ and independent of $(I, V)$. Let

$$E = \{k \in \{1, 2, \cdots, n\} : \forall i \leq k, I(i) \neq I(k)\}$$

and

$$F = \{1, 2, \cdots, n\} \setminus E$$

Consider an i.i.d. collection of random bijections $(q_{G,H})_P$ indexed by

$$P = \{(G, H) \in \mathcal{P}([1, 2, \cdots, n]) \times \mathcal{P}([1, 2, \cdots, n]) : |G| = |H|\}$$

where $\mathcal{P}(\cdot)$ denotes power set and each $q_{G,H}$ is uniformly distributed among the collection of all bijections from $G$ to $H$. We do not exclude the element $(\emptyset, \emptyset)$ from $P$. We take $(q_{G,H})_P$ to be independent of $(I, V, \sigma)$. Let $\theta \in S_n$ be the random permutation defined as

$$\theta(i) = \begin{cases} I(i) & i \in E \\ q_{F;\{1,2,\cdots,n\}\setminus I(E)}(i) & i \in F \end{cases}$$

Note that $\theta$ is uniformly distributed in $S_n$, and independent of $(V, \sigma)$ because it is defined in terms of $I$ and $(q_{G,H})_P$.

Claim 1: $E$ is independent of $(\theta, \sigma)$.

Proof of Claim 1: Consider any $\theta^{(0)} \in S_n$ and $E_0 \subseteq \{1, 2, \cdots, n\}$ with $1 \in E_0$, and let $F_0 = \{1, 2, \cdots, n\} \setminus E_0$. Now $\{\theta = \theta^{(0)}\} \cap \{E = E_0\}$ is equal to

$$\left[\bigcap_{i \in E_0} \left\{I(i) = \theta^{(0)}_i\right\}\right] \cap \left[\bigcap_{i \in F_0} \left\{I(i) \in \left\{\theta^{(0)}_j : j < i, j \in E_0\right\}\right\}\right]$$

$$\cap \left\{q_{F_0;\{1,2,\cdots,n\}\setminus \theta^{(0)}(E_0)} = \theta^{(0)}|F_0\right\}$$

where $\theta^{(0)}|F_0$ denotes the restriction of $\theta^{(0)}$ to $F_0$. This can be seen by showing that set inclusion holds in both directions and noting that $I(i) = I(j)$ for some $j < i$ if and only if $I(i) = I(j)$ for some $j < i$ with $j \in E$. Since $(q_{G,H})_P$ is independent of $I$,

$$\mathbb{P}\left(\{\theta = \theta^{(0)}\} \cap \{E = E_0\}\right)$$

$$= \mathbb{P}\left(\left[\bigcap_{i \in E_0} \left\{I(i) = \theta^{(0)}_i\right\}\right] \cap \left[\bigcap_{i \in F_0} \left\{I(i) \in \left\{\theta^{(0)}_j : j < i, j \in E_0\right\}\right\}\right]\right)$$

$$\times \mathbb{P}\left(\left\{q_{F_0;\{1,2,\cdots,n\}\setminus \theta^{(0)}(E_0)} = \theta^{(0)}|F_0\right\}\right)$$

Since the coordinates of $I$ are independent of each other this reduces to

$$n^{-|E_0|} \prod_{i \in F_0} \left|\left\{\theta^{(0)}_j : j < i, j \in E_0\right\}\right| \frac{|F_0|!}{n!}$$

$$= n^{-n} \prod_{i \in F_0} \left|\left\{j : j < i, j \in E_0\right\}\right| \frac{|F_0|!}{n!}$$
Since this probability does not depend on \(\theta^{(0)}\) and
\[
\sum_{\theta^* \in S_n} P(\{\theta = \theta^*\} \cap \{E = E_0\}) = P(\{E = E_0\})
\]
we conclude that
\[
P(\{\theta = \theta^{(0)}\} \cap \{E = E_0\}) = \frac{1}{n!} P(\{E = E_0\}) = P(\{\theta = \theta^{(0)}\}) P(\{E = E_0\})
\]
which is enough to show that \(\theta\) and \(E\) are independent. Yet \(\sigma\) is independent of \((I, \theta)\) and therefore of \((E, \theta, \sigma)\), so the distribution of \((E, \theta, \sigma)\) is a product measure.

Claim 2: For any (deterministic) \(G_0, G^*_0 \subseteq \{1, 2, \cdots, n\}\) such that \(|G_0| = |G^*_0|\), the random variables
\[
\sum_{i \in G_0} b_{\theta(i)} v_{\sigma(i)} \quad \text{and} \quad \sum_{i \in G^*_0} b_{\theta(i)} v_{\sigma(i)}
\]
have the same distribution. Consequently, if \(G_1 \subseteq \{1, 2, \cdots, n\}\) and \(|G_0| \leq |G_1|\), then for all \(t > 0\),
\[
P\left(\sum_{i \in G_1} b_{\theta(i)} v_{\sigma(i)} \geq t\right) \geq P\left(\sum_{i \in G_0} b_{\theta(i)} v_{\sigma(i)} \geq t\right)
\]
Here we take \(\sum_{i \in \emptyset} = 0\).

Proof of Claim 2: We may assume that \(G_0\) and \(G^*_0\) are non-empty. Consider any fixed \(\omega \in S_n\) that maps \(G_0\) to \(G^*_0\). Then
\[
\sum_{i \in G_0} b_{\theta(i)} v_{\sigma(i)} = \sum_{i \in G^*_0} b_{\theta_{\omega^{-1}}(i)} v_{\sigma_{\omega^{-1}}(i)}
\]
As observed before, \(\theta\) and \(\sigma\) are independent and both uniformly distributed on \(S_n\), so the joint distribution of \((\theta, \sigma)\) in \(S_n \times S_n\) is the uniform distribution. Since \(\omega\) is fixed, the same can be said of \(\theta_{\omega^{-1}}\) and \(\sigma_{\omega^{-1}}\). Yet the distributions of
\[
\sum_{i \in G^*_0} b_{\theta_{\omega^{-1}}(i)} v_{\sigma_{\omega^{-1}}(i)} \quad \text{and} \quad \sum_{i \in G^*_0} b_{\theta(i)} v_{\sigma(i)}
\]
are the push-forward measures of the distributions of \((\theta_{\omega^{-1}}, \sigma_{\omega^{-1}})\) and \((\theta, \sigma)\) under the action of
\[
(\alpha, \beta) \mapsto \sum_{i \in G^*_0} b_{\alpha(i)} v_{\beta(i)}
\]
so these two sums have the same distribution. The last part of the claim follows by taking \(G' \subseteq G_1\) with \(|G'| = |G_0|\), using the fact that the terms are non-negative, and applying the first part of the claim to conclude that
\[
P\left(\sum_{i \in G_1} b_{\theta(i)} v_{\sigma(i)} \geq t\right) \geq P\left(\sum_{i \in G'} b_{\theta(i)} v_{\sigma(i)} \geq t\right) = P\left(\sum_{i \in G_0} b_{\theta(i)} v_{\sigma(i)} \geq t\right)
\]
Claim 3: Let $G_0$ and $G_1$ be random subsets of $\{1, 2, \ldots, n\}$, not necessarily uniformly distributed in the power set, and assume that for all $k \in \{0, 1, 2, \ldots, n\}$, $\mathbb{P}\{|G_1| \geq k\} \geq \mathbb{P}\{|G_0| \geq k\}$. Assume also that $(G_0, G_1)$ is independent of the ordered pair $(\theta, \sigma)$. Then for all $t > 0$,

$$
\mathbb{P}\left\{\sum_{i \in G_1} b_{\theta(i)} v_{\sigma(i)} \geq t\right\} \geq \mathbb{P}\left\{\sum_{i \in G_0} b_{\theta(i)} v_{\sigma(i)} \geq t\right\}
$$

Proof of Claim 3: Fix any sequence of sets $(G^{(k)})_0^n$ with $|G^{(k)}| = k$. By independence, for any $t > 0$,

$$
\mathbb{P}\left\{\sum_{i \in G_0} b_{\theta(i)} v_{\sigma(i)} \geq t\right\} = \sum_{k=0}^{n} \sum_{|G^*| = k} \mathbb{P}\left\{\sum_{i \in G_0} b_{\theta(i)} v_{\sigma(i)} \geq t \text{ and } G_0 = G^*\right\} = \sum_{k=0}^{n} \mathbb{P}\left\{\sum_{i \in G^*} b_{\theta(i)} v_{\sigma(i)} \geq t \right\} \mathbb{P}\{|G_0| = k\}
$$

By Claim 2 this can be written as

$$
\sum_{k=0}^{n} \sum_{|G^*| = k} \mathbb{P}\left\{\sum_{i \in G^{(k)}} b_{\theta(i)} v_{\sigma(i)} \geq t\right\} \mathbb{P}\{|G_0| = k\} = \sum_{k=0}^{n} \mathbb{P}\left\{\sum_{i \in G^{(k)}} b_{\theta(i)} v_{\sigma(i)} \geq t\right\} \mathbb{P}\{|G_0| = k\}
$$

Similarly,

$$
\mathbb{P}\left\{\sum_{i \in G_1} b_{\theta(i)} v_{\sigma(i)} \geq t\right\} = \sum_{k=0}^{n} \mathbb{P}\left\{\sum_{i \in G^{(k)}} b_{\theta(i)} v_{\sigma(i)} \geq t\right\} \mathbb{P}\{|G_1| = k\}
$$

By Claim 2 again

$$
\mathbb{P}\left\{\sum_{i \in G^{(k)}} b_{\theta(i)} v_{\sigma(i)} \geq t\right\}
$$

is a non-decreasing function of $k$. Because the distribution of $|G_0|$ is dominated by the distribution of $|G_1|$, this implies that Claim 3 is true.

Claim 4: For all $t > 0$,

$$
\mathbb{P}\left\{\sum_{i \in E} b_{\theta(i)} v_{\sigma(i)} \geq t\right\} \geq \mathbb{P}\left\{\sum_{i \in F} b_{\theta(i)} v_{\sigma(i)} \geq t\right\}
$$

Proof of Claim 4: By Lemma 15, the distribution of $|E|$ dominates the distribution of $|F|$. Because $E$ is independent of $(\theta, \sigma)$ and $F$ is a function of $E$, the ordered pair $(E, F)$ is independent of $(\theta, \sigma)$. Claim 4 now follows from Claim 3.
Claim 5: For all $t > 0$,

$$\mathbb{P}\left\{ \sum_{i=1}^{n} b_{\theta(i)} v_{\sigma(i)} \geq t \right\} \leq 2\mathbb{P}\left\{ \sum_{i=1}^{n} b_{I(i)} v_{\sigma(i)} \geq \frac{t}{2} \right\}$$

Proof of Claim 5: The LHS is bounded above by

$$\mathbb{P}\left\{ \sum_{i \in E} b_{\theta(i)} v_{\sigma(i)} \geq \frac{t}{2} \right\} + \mathbb{P}\left\{ \sum_{i \in E} b_{\theta(i)} v_{\sigma(i)} \geq \frac{t}{2} \right\} \leq 2\mathbb{P}\left\{ \sum_{i \in E} b_{\theta(i)} v_{\sigma(i)} \geq \frac{t}{2} \right\} = 2\mathbb{P}\left\{ \sum_{i \in E} b_{I(i)} v_{\sigma(i)} \geq \frac{t}{2} \right\}$$

which is bounded above by the RHS.

Claim 6: The Theorem is true.

Proof of Claim 6: Since $V$ has not entered the proof until now we can take it to be independent of everything else (assuming as we are that the underlying probability space is rich enough). So

$$\mathbb{P}\left\{ \sum_{i=1}^{n} b_{\theta(i)} V_{\sigma(i)} \geq t \right\} = \int_{[0, \infty)^n} \mathbb{P}\left\{ \sum_{i=1}^{n} b_{\theta(i)} v_{\sigma(i)} \geq t \right\} \, d\mathbb{P}_V(v)$$

where $\mathbb{P}_V$ is the distribution of $V$. By Claim 5 this is bounded above by

$$\int_{[0, \infty)^n} 2\mathbb{P}\left\{ \sum_{i=1}^{n} b_{I(i)} v_{\sigma(i)} \geq \frac{t}{2} \right\} \, d\mathbb{P}_V(v) = 2\mathbb{P}\left\{ \sum_{i=1}^{n} b_{I(i)} V_{\sigma(i)} \geq \frac{t}{2} \right\}$$

Now $\sum_{i=1}^{n} b_{I(i)} V_{\sigma(i)} = \sum_{i=1}^{n} b_{I(i) \sigma^{-1}(i)} V_i$. Since the coordinates of $I$ are independent and uniformly distributed in $\{1, 2, \cdots, n\}$, and $\sigma$ is independent of $I$, the distribution of $(I \sigma^{-1}(i))^n_{\sigma}$ is the same as that of $(I(i))^n_{\sigma}$. Since $V$ is independent of $(I, \sigma)$, this then implies that the distribution of $(b_{I(i) \sigma^{-1}(i)} V_i^n)_{\sigma}$ is the same as that of $(b_{I(i)} V_i^n)_{\sigma}$ and the theorem is proved.

### 3.4 Combining the geometric and combinatorial approaches

Throughout this section we fix $n \in \mathbb{N}$ and $q \in (2, \infty)$ and consider two sequences of i.i.d. non-negative random variables $(W_i^n)_{\sigma}$ and $(Y_i^n)_{\sigma}$ such that for all $t > 0$,

$$\mathbb{P}\{W_i > t\} = e^{q/2} (e + t)^{-q/2} (\ln(e + t))^{q/2} \quad \mathbb{P}\{Y_i > t\} = (1 + t)^{-q/2}$$

Let $(b_i^n)_{\sigma} \in (0, \infty)^n$ and let $(I(i)_{\sigma})_{\sigma}$ be an i.i.d. sequence of random integers uniformly distributed in $\{1, 2, \cdots, n\}$ as in Theorem 16. For $\delta \in (0, 1)$, let $[\cdot]_{\delta, W}$ be the norm as studied in Section 3.2.1 associated to the distribution of $(W_i^n)_{\sigma}$ (see in particular 19 and 20), and let $[\cdot]_{\delta, Y}$ be the corresponding norm associated to the distribution of $(Y_i^n)_{\sigma}$.
Proposition 17 For all $b \in [0, \infty)^n$ and all $t > 0$, with probability at least $1 - Ce^{-t^2/2}$,

$$
\sum_{i=1}^{n} b_i W_i \leq C_q |b|_1 + C_q \left( t^2 + \ln |b|_0 \right) e^{t^2/q} |b|_{q/2}
$$

(24)

Proof. Assume momentarily that each $b_i \neq 0$. For $t > 0$ let $G(t) = \mathbb{P} \left\{ b_{I(i)} W_i \geq t \right\}$. By independence, and Fubini’s theorem applied to $\{1, 2, \ldots , n\} \times [0, \infty)$,

$$
G(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{P} \left\{ W_i \geq \frac{t}{b_i} \right\} = \frac{1}{n} \sum_{i=1}^{n} e^{q/2} (e + tb_i^{-1})^{-q/2} \left( \ln(e + tb_i^{-1}) \right)^{q/2}
$$

and

$$
-tG'(t) = \frac{1}{n^2} \sum_{i=1}^{n} e^{q/2} t b_i^{-1} (e + tb_i^{-1})^{-q/2-1} \left( \ln(e + tb_i^{-1}) \right)^{q/2-1} \left( \ln(e + tb_i^{-1}) - 1 \right) \leq \frac{q}{2} G(t)
$$

Now $H = G^{-1}$ is the reflected quantile function of $b_{I(i)} W_i$ and by the inverse function theorem the inequality $-tG'(t) \leq \frac{q}{2} G(t)$ can be written as $-H'(t) \geq (2/q) H(t)/t$, and then as

$$
\frac{-d}{dt} \ln H(t) \geq \frac{2t^{-1}}{q}
$$

By FTC this implies that for all $\delta, x \in (0, 1)$, $H(\delta x) \geq \delta^{-2/q} H(x)$ and the assumption of Proposition 8 is satisfied with $p = q/2$ and $T = 1$. By the conclusion of that result (see Remark 9 for a simplification), with probability at least $1 - Ce^{-t^2/2}$,

$$
\sum_{i=1}^{n} b_{I(i)} W_i \leq C_q H \left( e^{-1-2/e} \frac{1}{n+1} e^{-t^2/2} \right) + Cn \int_{0}^{1} H(x)dx
$$

The second term represents $Cn \mathbb{E}(b_{I(i)} W_i) = Cn (\mathbb{E} b_{I(i)})(\mathbb{E} W_i) = C_q |b|_1$, and we now focus on the first term. Using $\ln(e + x) \leq C_q \ln(e + x^{q/2})$ valid for $x \geq 0$,

$$
G(s) \leq \frac{C_q}{n} \left( \sum_{j=1}^{n} (e + sb_j^{-1})^{-q/2} \right) \sum_{i=1}^{n} \frac{(e + sb_i^{-1})^{-q/2}}{\sum_{j=1}^{n} (e + sb_j^{-1})^{-q/2}} \left( \ln(e + (sb_i^{-1})^{q/2}) \right)^{q/2}
$$

Since $x \mapsto (\ln(e + x))^{q/2}$ for $x \in [0, \infty)$ is the same order of magnitude as a concave function (up to a factor of $C_q$), we may apply Jensen’s inequality to bound this above by,

$$
\frac{C_q}{n} \left( \sum_{j=1}^{n} (e + sb_j^{-1})^{-q/2} \right) \left[ \ln \left( e + \sum_{i=1}^{n} \frac{(e + sb_i^{-1})^{-q/2}}{\sum_{j=1}^{n} (e + sb_j^{-1})^{-q/2}} (sb_i^{-1})^{q/2} \right) \right]^{q/2} \leq \frac{C_q}{n} \left( \sum_{j=1}^{n} (e + sb_j^{-1})^{-q/2} \right) \left[ \ln \left( e + \frac{n}{\sum_{j=1}^{n} (e + sb_j^{-1})^{-q/2}} \right) \right]^{q/2}
$$

If $s \geq |b|_{\infty}$ this is at most

$$
C_q n^{-1} s^{-q/2} \left( \sum_{i=1}^{n} b_i^{q/2} \right) \left[ \ln \left( e + n s^{q/2} \left( \sum_{i=1}^{n} b_i^{q/2} \right)^{-1} \right) \right]^{q/2}
$$
Setting
\[ s = \max \left\{ |b|_\infty, C_q e^{t^2/q} |b|_{q/2} (t^2 + \ln n) \right\} \]
we see that indeed \( s \geq |b|_\infty\), and

\[ G(s) \leq e^{-1-2/e} \frac{1}{n+1} e^{-t^2/2} \]

so \( H \left( e^{-1-2/e} \frac{1}{n+1} e^{-t^2/2} \right) \leq s \). All of this implies that with probability at least \( 1 - C e^{-t^2/2} \),

\[ \sum_{i=1}^n b_{I(i)} W_i \leq C_q |b|_1 + C_q e^{t^2/q} |b|_{q/2} (t^2 + \ln n) \]

By Theorem 16 this implies that with the same probability,

\[ \sum_{i=1}^n b_i W_i \leq C_q |b|_1 + C_q e^{t^2/q} |b|_{q/2} (t^2 + \ln n) \]

If \( b \in [0, \infty)^n \) has exactly \( j \) non-zero coordinates then we may apply this result to the truncated vector \( b^* \in [0, \infty)^j \) to improve the \( \ln n \) to \( \ln j \), arriving at 24. ■

**Proposition 18** For all \( b \in [0, \infty)^n \) and all \( t > 0 \), with probability at least \( 1 - C e^{-t^2/2} \),

\[ \sum_{i=1}^n b_i W_i \leq C_q |b|_1 + C_q e^{t^2/q} |b|_{q/2} (t^2 + \ln n) \]

(25)

**Proof.** For any \( 1 \leq k \leq n \), taking \( b \) to be the vector with 1 for its first \( k \) coordinates and 0 for the remaining \( n - k \) coordinates, (24) implies that with probability at least \( 1 - C e^{-t^2/2} \),

\[ \sum_{i=1}^n b_i W_i \leq C_q k + C_q e^{t^2/q} k^{2/q} (t^2 + \ln k) \]

For \( t^2 > (q/2 - 1) \ln k \) the term \( \ln k \) can be dropped by increasing the value of \( C_q \). For \( t^2 \leq (q/2 - 1) \ln k \),

\[ C_q e^{t^2/q} k^{2/q} (t^2 + \ln k) \leq C_q k^{1/2+1/q} \ln k < C_q k \]

and still the term \( \ln k \) can be dropped. So the bound can be written as

\[ \sum_{i=1}^n b_i W_i \leq C_q \left( k + t^2 e^{t^2/q} k^{2/q} \right) \]

This can be written as

\[ F_b^{-1} \left( 1 - C e^{-t^2/2} \right) \leq C_q k + C_q k^{2/q} t^2 e^{t^2/q} \]
Set $\delta = C e^{-t^2/2}$. From the integral representation of $[\cdot]_{\delta W}$ in (20),

$$[b]_{\delta W} \leq C_q \delta^{-1} \int_0^1 \left[ k + k^{2/q} s^{-2/q} \left( \frac{C}{s} \right) \right] e^{-\delta^{-1} s} ds \leq C_q \left( k + k^{2/q} \delta^{-2/q} \log \delta^{-1} \right)$$

$$= C_q \left( |b|_1 + \delta^{-2/q} \log \delta^{-1} |b|_{q/2} \right) \leq C_q |b|_{r,q/2}$$

where $r = \delta^{-2/q} \log \delta^{-1}$ and the last inequality follows from Lemma 12 (i.e. the simplified formula for $|b|_{r,q/2}$ as a $\{0,1\}$-vector). Since this holds for any such $k$ and $b$, by Lemma 13 $[b]_r \leq C_q |b|_{r,q/2}$ for all $b \in \mathbb{R}^n$. (25) now follows by recalling Lemma 11 (that $[b]_{\delta W}$ bounds the quantiles of $\sum b_i W_i$), and using the general estimate for $|b|_{r,q/2}$ in (22). ■

**Lemma 19** For all $x \in \mathbb{R}^n$ with $x_1 \geq x_2 \geq \cdots \geq x_n$ and $x_1 \neq x_n$,

$$|x|_{q/2} \leq \left( \frac{|x|_1 - nx_n}{x_1-x_n} x_1^{q/2} + \frac{nx_1 - |x|_1}{x_1-x_n} x_n^{q/2} \right)^{2/q}$$

**Proof.** We maximize $f(z) = \sum z_i^{q/2}$ over the compact set $E$ of all $z \in [0, \infty)^n$ such that $|z|_1 = |x|_1$ and $x_1 = z_1 \geq z_2 \geq \cdots \geq z_n = x_n$. If $z \in E$ has the property that $z_i \notin \{x_1, x_n\}$ for more than one value of $i$, say $j$ and $k$ with $j < k$, we can assume (per definition) that $j$ is the least value for which this holds and $k$ is the greatest. But then there exists $\varepsilon > 0$ such that $y \in E$ where

$$y_i = \begin{cases} 
z_i & i \notin \{j,k\} \\
z_j + \varepsilon & i = j \\
z_k - \varepsilon & i = k 
\end{cases}$$

and $f(y) = (z_j + \varepsilon)^{q/2} + (z_k - \varepsilon)^{q/2} + \sum_{i \notin \{j,k\}} z_i^{q/2} > f(z)$. This follows because by convexity and comparing the slope of secant lines

$$(z_j + \varepsilon)^{q/2} + (z_k - \varepsilon)^{q/2} > z_j^{q/2} + (z_k - \varepsilon)^{q/2}$$

By excluding such points, the maximum occurs at a point $z$ such that

$$z_i = \begin{cases} 
x_1 & i < k \\
x_n & i > k 
\end{cases}$$

for some $1 \leq k \leq n - 1$. The value of $k$ is determined by the equation $|z|_1 = |x|_1$. To re-distribute the $l_1^n$ mass of $x$ to form $z$, we start each coordinate at 0, add $x_n$ to each coordinate, and then distribute the remaining total of $|x|_1 - nx_n$ in doses of $x_1 - x_n$ until we no longer have enough for a full dose. This implies that $k - 1 = [\alpha]$, where

$$\alpha = \frac{|x|_1 - nx_n}{x_1-x_n}$$

and it follows again by convexity that

$$f(z) = \left[ \alpha \right] x_1^{q/2} + \left( (\alpha - [\alpha]) x_1 + (1 - (\alpha - [\alpha])) x_n \right)^{q/2} + (n - 1 - [\alpha]) x_n^{q/2}$$

$$\leq \alpha x_1^{q/2} + (n - \alpha) x_n^{q/2}$$

Since $x \in E$, $|x|_{q/2} \leq f(z)^{2/q}$. ■
Proposition 20  For all $b \in [0, \infty)^n$ and all $t > 0$, with probability at least $1 - Ce^{-t^2/2}$,

$$\sum_{i=1}^{n} b_i W_i \leq C_q \left( |b|_1 + t^2 e^{t^2/q} |b|_{q/2} \right)$$  \hspace{1cm} (26)

Proof. Note that by Proposition 17 (see the explanation about removing the ln $k$ in the proof of Proposition 18), the result already holds as long as

$$|b|_{q/2} |b|_{10}^{c_q^\#} \leq C_q |b|_1$$  \hspace{1cm} (27)

where $c_q^\# > 0$ can be taken to be arbitrarily small and $C_q$ can be arbitrarily large. We fix the value of $c_q^\#$ to be the same at different appearances. Consider any $b \in [0, \infty)^n$ such that (27) is violated, and assume without loss of generality that $1 = b_1 \geq b_2 \geq b_3 \cdots \geq b_n$. Set $t = b_n$. We must have $t < 1/2$ otherwise (27) would hold. If $b_n > 0$ then by the assumption that (27) is violated and Lemma 19

$$|b|_1 \leq c_q n^{c_q^\#} \left( \left| \frac{|b|_1 - nt}{1 - t} \right| + \frac{n - |b|_1}{1 - t} t^{q/2} \right)^{2/q} \leq c_q n^{c_q^\#} \left( (|b|_1 - nt) + (n - |b|_1) t^{q/2} \right)^{2/q}$$

We now consider two cases. In Case I, $|b|_1 - nt \leq (n - |b|_1) t^{q/2}$ which leads to the contradiction

$$\frac{1}{n} |b|_1 \leq c_q n^{-1+c_q^\#+2/q} \left( 1 - \frac{1}{n} |b|_1 \right) t$$

This is a contradiction because LHS is the average size of a coordinate while RHS is less than the smallest coordinate. In Case II, $|b|_1 - nt > (n - |b|_1) t^{q/2}$ which leads to

$$t \leq \frac{1}{n} \left( |b|_1 - C_q n^{-c_q^\# q/2} |b|_1^{q/2} \right)$$

Using $s - As^{q/2} \leq C_q A^{-2/(q-2)}$ valid for $s \geq 0$, this is bounded above by

$$C_q n^{-1+c_q^\# q/(q-2)}$$  \hspace{1cm} (28)

This obviously holds also when $b = 0$. As we argued before, the same estimate can be applied to a truncated vector of $k$ coordinates, with $n$ replaced with $k$ in this estimate, as long as the truncated vector violates (27). Let $k$ be the largest integer such that the truncated vector $(b_i)_1^k$ satisfies (27). Such a value of $k$ exists because every element of $\mathbb{R}^1$ satisfies (27), and by our assumption that $b$ violates (27), $1 \leq k \leq n - 1$. For all $j > k$, it follows from the definition of $k$ that $(b_i)_1^j$ violates (27), so by applying (28) to this vector in dimension $j$, $b_j \leq C_q j^{-1+c_q^\# q/(q-2)}$. By Propositions 17 and 18 applied to $(b_i)_1^k$ and $(b_i)_{k+1}^n$ respectively, with probability at least $1 - 2Ce^{-t^2/2}$,

$$\sum_{i=1}^{n} b_i W_i \leq \sum_{i=1}^{k} b_i W_i + \sum_{i=k+1}^{n} b_i W_i$$

$$\leq C_q \left( \sum_{i=1}^{k} b_i + t^2 e^{t^2/q} \left( \sum_{i=1}^{k} b_i^{q/2} \right)^{2/q} \right) + \sum_{i=k+1}^{n} b_i + t^2 e^{t^2/q} \sum_{i=k+1}^{n} (i-k)^{-1+2/q} b_i$$

24
The reason for \((i - k)\) is that for \(i \geq k + 1\), \(b_i\) is the \((i - k)\)th coordinate of \((b_i)_{k+1}^n\). By our estimate on \(b_j\) for \(j > k\),

\[
\sum_{i=k+1}^{n} (i - k)^{-1+2/q} b_i = \sum_{i=1}^{n-k} i^{-1+2/q} b_{i+k} \leq C q \sum_{i=1}^{n-k} i^{-1+2/q} (i + k)^{-1+c^q q/(q-2)}
\]

Consider the case where \(n \geq 2k\) (the case \(n < 2k\) is similar, just with one less term). This is bounded above by

\[
C q k^{-1+c^q q/(q-2)} \sum_{i=1}^{k} i^{-1+2/q} + C q \sum_{i=k+1}^{\infty} i^{-2+2/q+c^q q/(q-2)} \leq C q k^{-1+2/q+c^q q/(q-2)}
\]

Since \(q > 2\) we can choose \(c^q q > 0\) so that \(-1 + 2/q + c^q q/(q - 2) < 0\). The bound on \(\sum_i^q b_i W_i\) then becomes

\[
C q \left( |b|_1 + t^2 e^{t^2/q} (1 + |b|_{q/2}) \right)
\]

and the 1 can be deleted by our assumption that \(b_1 = 1\).

**Proposition 21** For all \(b \in [0, \infty)^n\) and all \(t > 0\), with probability at least \(1 - Ce^{-t^2/2}\),

\[
\sum_{i=1}^{n} b_i Y_i \leq C q \left( |b|_1 + e^{t^2/q} |b|_{q/2} \right)
\]

**Proof.** The proof is almost identical to that of Proposition 17 but simpler because it does not involve the logarithmic term. Setting

\[
G(t) = \mathbb{P} \left\{ b_{\ell(i)} Y_i \geq t \right\} = \frac{1}{n} \sum_{i=1}^{n} \mathbb{P} \left\{ Y_i \geq \frac{t}{b_i} \right\} = \frac{1}{n} \sum_{i=1}^{n} (1 + t b_i^{-1})^{-q/2}
\]

This function satisfies the conditions of Proposition 8 with \(p = q/2\), \(j = 0\) and \(k = n - 1\), the reasoning for this is the same as in the proof of Proposition 17. So applying the simplified bound as in Remark 9 with probability at least \(1 - Ce^{-t^2/2}\),

\[
\sum_{i=1}^{n} b_{\ell(i)} Y_i \leq C \left( 1 + t^{-4/q} e^{-t^2/q} \min\{t^2, n\} \right) H \left( e^{-1-2/e} \frac{1}{n+1} e^{-t^2/2} \right) + C |b|_1 \mathbb{E} Y_1
\]

\[
\leq C \left( 1 + t^{-4/q} e^{-t^2/q} \min\{t^2, n\} \right) e^{t^2/q} |b|_{q/2} + C |b|_1 \mathbb{E} Y_1
\]

where \(H = G^{-1}\). By Theorem 16 the result can be transferred to \(\sum_{i=1}^{n} b_i Y_i\).

**4 Proof of Theorem 1**

First we assume that each \(X_i\) has a distribution that is symmetric about 0, and then we may assume without loss of generality that each \(a_i \geq 0\), and that \(|a| = 1\). We will use a standard trick in analysis of introducing random signs. Let \((\varepsilon_i)_{i=1}^{n}\) be an i.i.d. sequence of
Rademacher random variables, i.e. each $\varepsilon_i$ takes the values $\pm 1$ each with probability $1/2$, independent of $(X_i)_i^n$. By the assumed independence and symmetry, the vector $(\varepsilon_i |X_i|)_i^n$ has the same distribution as $(X_i)_1^n$, and

$$
P \left\{ \left| \sum_{i=1}^n a_i X_i \right| > t \right\} = \mathbb{P} \left\{ \left| \sum_{i=1}^n a_i \varepsilon_i |X_i| \right| > t \right\} = \int_{[0,\infty]^n} \mathbb{P} \left\{ \left| \sum_{i=1}^n x_i \varepsilon_i \right| > t \right\} d\mathbb{P}_{(a_i|X_i|)_1^n}(x)
$$

where $\mathbb{P}_{(a_i|X_i|)_1^n}$ is the distribution of $(a_i |X_i|)_1^n$. Since $(\varepsilon_i)_1^n$ are independent and sub-Gaussian, with universal constants,

$$\mathbb{P} \left\{ \left| \sum_{i=1}^n x_i \varepsilon_i \right| > t \right\} \leq C \exp \left( -\frac{ct^2}{|x|^2} \right)$$

So, using Proposition 21, the probability a few lines above is at most

$$C \mathbb{E} \exp \left( -\frac{ct^2}{\sum_1^n a_i^2 X_i^2} \right) \leq C \int_0^\infty u e^{-u^2/2} \exp \left( -\frac{ct^2}{C_q \sum a_i^2 + C_q e^{u^2/q} (\sum a_i^2)^{2/q}} \right) du$$

Here we are using the fact that the tail probabilities of $X_i^2$ decay as $t^{-q/2}$. This integral splits into two, the first one being

$$C \int_0^{\sqrt{2q \ln(|a|)/|a|_q}} u e^{-u^2/2} \exp \left( -\frac{ct^2}{C_q e^{u^2/q} (\sum a_i^2)^{2/q}} \right) du \leq C e^{-c_q t^2}$$

and the second one being

$$C \int_{\sqrt{2q \ln(|a|)/|a|_q}}^\infty u e^{-u^2/2} \exp \left( -\frac{ct^2}{C_q e^{u^2/q} (\sum a_i^2)^{2/q}} \right) du = C_q t^{-q} |a|_q \int_0^{C_q t^2 |a|^{-2}} e^{-\omega \ln(1+q/2)} d\omega$$

where we have set $\omega = c_q t^2 e^{-u^2/q} |a|_q^{-2}$. This implies

$$\mathbb{P} \left\{ \left| \sum_{i=1}^n a_i X_i \right| > t \right\} \leq C e^{-c_q t^2} + C_q t^{-q} |a|_q^q$$

which can be written as

$$\mathbb{P} \left\{ \left| \sum_{i=1}^n a_i X_i \right| > C_q \left( t |a| + e^{t^2/(2q)} |a|_q \right) \right\} \leq C e^{-t^2/2}$$

and therefore

$$\mathbb{P} \left\{ \left| \sum_{i=1}^n a_i X_i - M \sum_{i=1}^n a_i X_i \right| > C_q \left( t |a| + e^{t^2/(2q)} |a|_q \right) \right\} \leq C e^{-t^2/2}$$
When \( X_i \) are no longer assumed to be symmetric, apply what has been proved to \( X_i - X'_i \), where \( (X'_i)^n \) is an independent copy of \( (X_i)^n \), so

\[
\mathbb{P} \left\{ \left| \sum_{i=1}^{n} a_i X_i - \mathbb{M} \sum_{i=1}^{n} a_i X_i \right| - \left| \sum_{i=1}^{n} a_i X'_i - \mathbb{M} \sum_{i=1}^{n} a_i X'_i \right| > C_q \left( t |a| + e^{t^2/(2q)} |a|_q \right) \right\}
\]

is at most \( Ce^{-t^2/2} \). However, by independence, this probability can be expressed as the integral over \( \mathbb{R}^n \) of the function that maps \( x \) to

\[
\mathbb{P} \left\{ \left| \sum_{i=1}^{n} a_i X_i - \mathbb{M} \sum_{i=1}^{n} a_i X_i \right| - \left| \sum_{i=1}^{n} a_i X_i - \mathbb{M} \sum_{i=1}^{n} a_i X'_i \right| > C_q \left( t |a| + e^{t^2/(2q)} |a|_q \right) \right\}
\]

Integration is performed with respect to \( P_X \), the distribution of \( (X_i)_i^n \). However, setting

\[
E = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i \leq \sum_{i=1}^{n} a_i X'_i \right\}
\]

we see that \( P_X(E) \geq 1/2 \) and for all \( x \in E \),

\[
\mathbb{P} \left\{ \left( \sum_{i=1}^{n} a_i X_i - \mathbb{M} \sum_{i=1}^{n} a_i X_i \right) - \left( \sum_{i=1}^{n} a_i x_i - \mathbb{M} \sum_{i=1}^{n} a_i X'_i \right) > C_q \left( t |a| + e^{t^2/(2q)} |a|_q \right) \right\}
\]

\[
\geq \mathbb{P} \left\{ \sum_{i=1}^{n} a_i X_i - \mathbb{M} \sum_{i=1}^{n} a_i X_i > C_q \left( t |a| + e^{t^2/(2q)} |a|_q \right) \right\}
\]

So the probability in (30) is at most \( Ce^{-t^2/2} \) and at least

\[
\frac{1}{2} \mathbb{P} \left\{ \sum_{i=1}^{n} a_i X_i - \mathbb{M} \sum_{i=1}^{n} a_i X_i > C_q \left( t |a| + e^{t^2/(2q)} |a|_q \right) \right\}
\]

so

\[
\mathbb{P} \left\{ \sum_{i=1}^{n} a_i X_i - \mathbb{M} \sum_{i=1}^{n} a_i X_i > C_q \left( t |a| + e^{t^2/(2q)} |a|_q \right) \right\} \leq Ce^{-t^2/2}
\]

A similar argument implies

\[
\mathbb{P} \left\{ \sum_{i=1}^{n} a_i X_i - \mathbb{M} \sum_{i=1}^{n} a_i X_i < -C_q \left( t |a| + e^{t^2/(2q)} |a|_q \right) \right\} \leq Ce^{-t^2/2}
\]

and these last two estimates imply (3) with \( \mathbb{M} \) instead of \( \mathbb{E} \), but that deviation inequality gives a bound on the distance between \( \mathbb{M} \) and \( \mathbb{E} \), and using the triangle inequality we can then replace \( \mathbb{M} \) with \( \mathbb{E} \) (this is very standard).
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