NEW INEQUALITIES FOR THE SPECTRAL GEOMETRIC MEAN

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ABSTRACT. The main goal of this article is to present new inequalities for the spectral geometric mean $A^t B$ of two positive definite operators $A, B$ on a Hilbert space. The obtained results complement many known inequalities for the geometric mean $A^\# B$. In particular, explicit comparisons between $A^t B$ and $A^\# B$ are given, Ando-type inequalities are presented for $A^t B$ and some other consequences.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the $C^*$-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. If $A \in \mathcal{B}(\mathcal{H})$ is such that $\langle Ax, x \rangle \geq 0$ for all $x \in \mathbb{C}^n$, $A$ is said to be positive semi-definite. If in addition $A$ is invertible, it is positive definite. The class of positive definite matrices in $\mathcal{M}_n$ will be denoted by $\mathcal{B}(\mathcal{H})^+$.

The weighted geometric mean of $A, B \in \mathcal{B}(\mathcal{H})^+$ is defined by the equation

\begin{equation}
A^t B = A^\frac{t}{2} \left( A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right)^t A^\frac{1}{2}, \quad 0 \leq t \leq 1.
\end{equation}

When $t = \frac{1}{2}$, we simply write $A^\# B$ instead of $A^\frac{1}{2} B$. An interesting geometric meaning of $A^\# B$ is that it is a midpoint of $A$ and $B$ for a natural Finsler metric (Thompson’s part metric) on the cone of positive definite operators [16, 17].

The geometric mean $\tilde{z}_t$ is a special case of operator means. Recall that an operator mean $\sigma$ on $\mathcal{B}(\mathcal{H})^+$ is a binary operation defined by

\begin{equation}
A \sigma B = A^\frac{t}{2} f \left( A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right) A^\frac{1}{2},
\end{equation}

where $f : (0, \infty) \to (0, \infty)$ is an operator monotone function, with $f(1) = 1$. Examples of other operator means are the weighted arithmetic and harmonic mean, defined respectively by

$A \nabla_t B = (1 - t) A + t B$ and $A!_t B = \left((1 - t)A^{-1} + t B^{-1}\right)^{-1}, 0 \leq t \leq 1.$

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It is well known that

(1.2) \quad \quad A!t B \leq A^t B \leq A\nabla t B, \quad 0 \leq t \leq 1.

In [5], the spectral geometric mean was defined by

\[ A^t_{\sharp} B = (A^{-1}_{\sharp} B)^{\frac{1}{2}} A (A^{-1}_{\sharp} B)^{\frac{1}{2}}. \]

Then the weighted spectral geometric mean was defined in [12] by

\[ A^t_{\sharp t} B = (A^{-1}_{\sharp} B)^t A (A^{-1}_{\sharp} B)^t, \quad 0 \leq t \leq 1. \]

In fact, simple manipulations of the above definition implies that \( A^t_{\sharp t} B \) is a unique positive definite solution \( X \) of the following equation

\[(A^{-1}_{\sharp} B)^t = A^{-1}_{\sharp} X.\]

We refer the reader to [10] as a recent reference for further algebraic and geometric meaning of \( A^t_{\sharp t} B \).

Our main target in this article is to present new inequalities for \( A^t_{\sharp t} B \), in a way that complements those inequalities known for \( A^t B \).

In particular,

- The relations between \( \langle A^t_{\sharp t} B x, x \rangle \) and \( \langle Ax, x \rangle^{1-t} \langle Bx, x \rangle^t \) are given.
- Upper and lower bounds of \( A^t_{\sharp t} B \) in terms of \( A^t B \) are given.
- Ando-type inequalities are given for \( \sharp t \), where the relation between \( \Phi(A^t B) \) and \( \Phi(A)\sharp t \Phi(B) \) are described in both ways.
- Kantorovich-type inequality is presented to find an upper bound of \( \Phi(A^{-1}_{\sharp} B \Phi(A)^{-1}. \)
- An Ando-Hiai inequality for the spectral geometric mean is given to describe the relation between \( (A^t_{\sharp} B)^r \) and \( A^r\sharp t B^r \).
- A detailed discussion of the Kantorovich constant is presented, as a by product of our results.

For this, we need to remind the reader of some well established results for \( A^t_{\sharp t} B \). We also refer the reader to the obtained inequalities for spectral geometric mean that simulates the corresponding result for the geometric mean. This will help the reader better follow the order of the results.

**Lemma 1.1.** [1] Let \( A, B \in B(\mathcal{H}) \in B(\mathcal{H})^+ \), and let \( \Phi \) be a unital positive linear map on \( B(\mathcal{H}) \). Then for any \( 0 \leq t \leq 1 \),

\[ \Phi(A^t_{\sharp t} B) \leq \Phi(A)\sharp t \Phi(B). \]
In particular, for any unit vector $x \in \mathcal{H}$,
\[
(1.3) \quad \langle A^* t B x, x \rangle \leq \langle Ax, x \rangle^{1-t} \langle B x, x \rangle^t.
\]

We refer the reader to Theorems 2.1 and 2.3 below for the spectral geometric mean versions of Lemma 1.1.

In this paper, we use the generalized Kantorovich constant for $0 < m < M$ and $t \in \mathbb{R}$:
\[
K(m, M, t) = \frac{(mM^t - Mm^t)}{(t-1)(M-m)} \left( \frac{t-1}{t} \frac{M^t - m^t}{mm^t - Mm^t} \right)^t.
\]
It is known that this recovers the (original) Kantorovich constant when $t = -1$ and $t = 2$, that is, $K(m, M, 2) = K(m, M, -1) = \frac{(M+m)^2}{4Mm}$, whose constant was appeared in [9, p.142] as the so-called Kantorovich inequality. In Section 3, we study the generalized Kantorovich constant is defined by
\[
(1.4) \quad K(x, t) := \frac{(x^t - x)}{(t-1)(x-x)} \left( \frac{t-1}{t} \frac{x^t - 1}{x^t - x} \right)^t, \quad x > 0, \quad t \in \mathbb{R}.
\]
We see that $K(x, 1, t) = K(x, t)$. We use same symbol $K$ as the generalized Kantorovich constant. However, this never makes confusion to the readers by the different numbers of their variables.

**Lemma 1.2.** [2] Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that $mA \leq B \leq MA$ for some scalars $0 < m \leq M$, and let $\Phi$ be a unital positive linear map. Then for any $0 \leq t \leq 1$,
\[
\Phi(A)^* t \Phi(B) \leq \frac{1}{K(m, M, t)} \Phi(A^* t B).
\]

The spectral geometric mean version of Lemma 1.2 is stated below in Theorem 2.4.

**Lemma 1.3.** [11] Let $A, B \in \mathcal{B}(\mathcal{H})^+$ such that $m_1 \leq A \leq M_1$, $m_2 \leq B \leq M_2$ and let $\Phi$ be a unital positive linear map on $\mathcal{B}(\mathcal{H})$. Then
\[
\Phi(A)^* \Phi(B) \leq \frac{\sqrt{M_1 M_2 + \sqrt{m_1 m_2}}}{2\sqrt{M_1 m_1 M_2 m_2}} \Phi(A^* t B).
\]
In particular, for any unit vector $x \in \mathcal{H}$,
\[
\sqrt{\langle Ax, x \rangle \langle Bx, x \rangle} \leq \frac{\sqrt{M_1 M_2 + \sqrt{m_1 m_2}}}{2\sqrt{M_1 m_1 M_2 m_2}} \langle A^* t B x, x \rangle.
\]

The Choi-Davis inequality is stated next [3, 4].

**Lemma 1.4.** Let $f : J \rightarrow \mathbb{R}$ be an operator convex function, and let $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ be a unital positive linear mapping. Then
\[
(1.5) \quad f(\Phi(A)) \leq \Phi(f(A)),
\]
for all self-adjoint operators \( A \in \mathcal{B}(\mathcal{H}) \) with spectra in the interval \( J \).

In particular, \( \Phi(A)^{-1} \leq \Phi(A^{-1}) \), for any invertible self-adjoint operator \( A \).

The inequality (1.5) is reversed when \( f \) is operator concave.

In particular, if \( 0 \leq t \leq 1 \), then
\[
\Phi(A^t) \leq \Phi(A)^t,
\]
for any positive definite operator \( A \).

**Lemma 1.5.** [13] Let \( A \in \mathcal{B}(\mathcal{H})^+ \) be such that \( m \leq A \leq M \), and let \( \Phi \) be a positive linear map on \( \mathcal{B}(\mathcal{H}) \). Then for any \( 0 \leq t \leq 1 \),
\[
\Phi(A)^t \leq \frac{1}{K(m, M, t)} \Phi(A^t)
\]
In particular, for any unit vector \( x \in \mathcal{H} \),
\[
\langle Ax, x \rangle^t \leq \frac{1}{K(m, M, t)} \langle A^t x, x \rangle
\]

**Lemma 1.6.** [14] Let \( A \in \mathcal{B}(\mathcal{H}) \) such that \( 0 < m \leq A \leq M \) and let \( \Phi \) be a positive unital linear mapping. Then
\[
\Phi(A^{-1}) \leq \frac{(M + m)^2}{4Mm} \Phi(A)^{-1}.
\]
In particular, if \( x \in \mathcal{H} \) is any unit vector, then
\[
\langle A^{-1} x, x \rangle \leq \frac{(M + m)^2}{4Mm} \langle Ax, x \rangle^{-1},
\]
for any positive definite \( A \).

We refer the reader to Corollary 2.4 below for the corresponding inequality that the spectral geometric mean fulfills similar to Lemma 1.6.

While the inequality \( A \leq B \) does not imply \( A^2 \leq B^2 \), for arbitrary self-adjoint operators in general, the following useful weaker version holds.

**Lemma 1.7.** [6] Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be such that \( 0 < A \leq B \) and \( m \leq A \leq M \). Then
\[
A^2 \leq \frac{(M + m)^2}{4Mm} B^2.
\]

**Lemma 1.8.** [18] Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be such that \( m \leq A, B \leq M \) for some scalars \( 0 \leq m \leq M \), and let \( 0 \leq r \leq 1 \). Then for any \( 0 \leq t \leq 1 \),
\[
(A \nabla_t B)^r \leq \frac{1}{K(m, M, t)} A^r \nabla_t B^r.
\]

**Lemma 1.9.** [15] Let \( A \in \mathcal{B}(\mathcal{H}) \) be a positive semi-definite operator. If \( x \in \mathcal{H} \) is a unit vector, then
(i) \( \langle Ax, x \rangle^r \leq \langle A^r x, x \rangle \) for all \( r \geq 1 \).
(ii) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r$ for all $0 \leq r \leq 1$.

(iii) If $A$ is invertible, then $\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle$ for all $r \leq 0$.

We notice that the function $f(x) = x^t, 0 \leq t \leq 1$ is operator monotone on $[0, \infty)$. This together with Lemma 1.1 implies that

\begin{equation}
\Phi(A^*_t B)^t \leq (\Phi(A^*_t \Phi(B)))^t, 0 \leq t \leq 1
\end{equation}

for any positive definite operators $A, B$.

2. Main results

In this section, we present our main results, where we show some comparisons between the geometric and spectral geometric means first, then we discuss some Ando-type inequalities for the spectral geometric mean.

2.1. Some comparisons between $A^*_t B$ and $A^*_{n}B$. The first result is the spectral geometric version of (1.3).

**Theorem 2.1.** Let $A, B \in \mathcal{B}(H)^+$ be such that $m \leq A, B \leq M$, and let $0 \leq t \leq 1$. Then for any unit vector $x \in H$,

\[
\langle A^*_t Bx, x \rangle \leq \frac{(M^1 + t + M^1 + t)^2}{4M^1 + tM^1 + t} \langle Ax, x \rangle^{1-t} \langle Bx, x \rangle^t.
\]

**Proof.** If $m \leq A, B \leq M$, then

\[
\frac{m^1 + t}{M^t} \leq X = (A^{-1}^*_t B)^t A(A^{-1}^*_t B)^t \leq \frac{M^1 + t}{m^t},
\]

and

\[
\frac{1}{M} \leq A^{-1} \leq \frac{1}{m}.
\]

On the other hand, we know that $A^*_t B$ is a unique positive definite solution $X$ of the following equation

\begin{equation}
(A^{-1}^*_t B)^t = A^{-1}^*_t X.
\end{equation}

Let $X = A^*_t B$. For any unit vector $x \in H$,

\[
\xi \sqrt{\langle A^{-1} x, x \rangle \langle X x, x \rangle} \leq \langle A^{-1}^*_t X x, x \rangle \quad \text{(by Lemma 1.3)}
\]

\[
= \langle (A^{-1}^*_t B)^t x, x \rangle
\]

\[
\leq \langle (A^{-1}^*_t B) x, x \rangle^t \quad \text{(by Lemma 1.9)}
\]

\[
\leq \langle A^{-1} x, x \rangle^{\frac{t}{2}} \langle B x, x \rangle^{\frac{t}{2}} \quad \text{(by Lemma 1.1)},
\]
where \( \xi = \frac{2\sqrt{M^{1+t}m^{1+t}}}{M^{1+t} + m^{1+t}} \). Thus,

\[
\langle Xx, x \rangle \leq \frac{1}{\xi^2} \langle A^{-1}x, x \rangle^{t-1} \langle Bx, x \rangle^t,
\]

which implies,

\[
\langle A_t^\ast Bx, x \rangle \leq \frac{1}{\xi^2} \langle A^{-1}x, x \rangle^{t-1} \langle Bx, x \rangle^t.
\]

Again by Lemma 1.9,

\[
\langle A_t^\ast Bx, x \rangle \leq \frac{1}{\xi^2} \langle Ax, x \rangle^{1-t} \langle Bx, x \rangle^t.
\]

□

The relation between weighted spectral geometric mean, arithmetic mean, and harmonic mean is stated in the following result, in a way that simulates (1.2) for the geometric mean.

This follows from Theorem 2.1, the weighted arithmetic-geometric mean, and the fact that \( (A^{-1}B)^{-1} = A_t^\ast B \).

**Corollary 2.1.** Let \( A, B \in \mathcal{B}(H)^+ \) be such that \( 0 < m \leq A, B \leq M \), and let \( 0 \leq t \leq 1 \). Then

\[
\frac{4M^{1+t}m^{1+t}}{(M^{1+t} + m^{1+t})^2} A_tB \leq A_t^\ast B \leq \frac{(M^{1+t} + m^{1+t})^2}{4M^{1+t}m^{1+t}} A_t^\ast B.
\]

**Remark 2.1.** The inequalities in Corollary 2.1 can be improved as follows. Recall the following refinement of the weighted arithmetic–geometric mean inequality [19]

\[
\left( \frac{(1 + h)^2}{4h} \right)^r a^{1-t}b^t \leq (1-t)a + tb
\]

where \( h = b/a \) and \( r = \min \{t, 1-t\} \). Let \( x \) be a unit vector. If we replace \( a \) and \( b \) by \( \langle Ax, x \rangle \) and \( \langle Bx, x \rangle \) with \( m \leq A \leq m' < M' \leq B \leq M \) and noting that the function

\[
f(h) = \frac{(1+h)^2}{4h}
\]

is increasing for \( h \geq 1 \), we get

\[
\left( \frac{(M' + m')^2}{4M'm'} \right)^r \langle Ax, x \rangle^{1-t} \langle Bx, x \rangle^t \leq \langle (A_t^\ast B)x, x \rangle.
\]

Thus,

\[
A_t^\ast B \leq \left( \frac{M^{1+t} + m^{1+t}}{4M^{1+t}m^{1+t}} \right) \left( \frac{4M'm'}{(M' + m')^2} \right)^r A_t^\ast B,
\]

whenever \( m \leq A \leq m' < M' \leq B \leq M \). The same inequality holds when \( m \leq B \leq m' < M' \leq A \leq M \).

The following result intends to give a relationship between the weighted spectral geometric mean and the weighted geometric mean.
Corollary 2.2. Let $A, B \in B(\mathcal{H})$ be such that $0 < m \leq A, B \leq M$, and let $0 \leq t \leq 1$. Then

$$A^{\#}tB \leq \frac{1}{K\left(\frac{M}{m}, m, t\right)} \frac{(M^1 + m^1 + t)^2}{4M^1 + m^1 + t^2} A^{\#}tB.$$ 

Proof. If $m \leq A, B \leq M$, then $\frac{m}{M} A \leq B \leq \frac{M}{m} A$. In this case, by Lemma 1.2, we get

$$\Phi(A^{\#}tB) \leq \frac{1}{K\left(\frac{M}{m}, m, t\right)} \Phi(A^{\#}tB),$$

for any positive unital linear mapping $\Phi$. In particular,

$$(Ax, x)^{1-t}(Bx, x)^t \leq \frac{1}{K\left(\frac{M}{m}, m, t\right)} (A^{\#}tBx, x).$$

Now, by Theorem 2.1, we have

$$A^{\#}tB \leq \frac{1}{K\left(\frac{M}{m}, m, t\right)} \frac{(M^1 + m^1 + t)^2}{4M^1 + m^1 + t^2} A^{\#}tB.$$ 

$\square$

A reverse of Theorem 2.1 can be stated as follows.

Theorem 2.2. Let $A, B \in B(\mathcal{H})$ be such that $0 < m \leq A, B \leq M$, and let $0 \leq t \leq 1$. Then for any unit vector $x \in \mathcal{H},$

$$(Ax, x)^{1-t}(Bx, x)^t \leq \frac{1}{K^2\left(\sqrt{\frac{m}{M}}, \sqrt{\frac{M}{m}}, t\right)} \left(\frac{(M^1 + m^1 + t)^2}{4M^1 + m^1 + t^2}\right)^{1-t} (A^{\#}tBx, x),$$

Proof. We have

$$\sqrt{\langle A^{-1} x, x \rangle \langle X x, x \rangle} \geq \langle A^{-1} x, x \rangle$$ (by Lemma 1.1)

$$= \langle (A^{-1} x, x) \rangle^{t}$$

$$\geq K\left(\sqrt{\frac{m}{M}}, \sqrt{\frac{M}{m}}, t\right) \langle A^{-1} x, x \rangle^{t}$$ (by Lemma 1.5)

$$\geq K\left(\sqrt{\frac{m}{M}}, \sqrt{\frac{M}{m}}, t\right) \left(\frac{2\sqrt{M}m}{M + m}\right)^{t} \langle A^{-1} x, x \rangle^{t}$$ (by Lemma 1.3).

Thus,

$$\langle A^{\#}tBx, x \rangle \geq K^2\left(\sqrt{\frac{m}{M}}, \sqrt{\frac{M}{m}}, t\right) \left(\frac{4M^1 + m^1 + t^2}{(M + m)^2}\right)^{t} \langle A^{-1} x, x \rangle^{t-1} \langle Bx, x \rangle.$$ 

Now, by Lemma 1.6, we have

$$(Ax, x)^{1-t}(Bx, x)^t \leq \frac{1}{K^2\left(\sqrt{\frac{m}{M}}, \sqrt{\frac{M}{m}}, t\right)} \left(\frac{(M^1 + m^1 + t)^2}{4M^1 + m^1 + t^2}\right)^{1-t} (A^{\#}tBx, x),$$

as desired. $\square$
Now we are ready to present a reversed version of Corollary (2.2).

**Corollary 2.3.** Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that $0 < m \leq A, B \leq M$, and let $0 \leq t \leq 1$. Then

$$A_{\sharp t}B \leq \frac{1}{K^2 \left( \sqrt{\frac{M}{m}}, \sqrt{\frac{m}{M}} \right) (M + m)^{\frac{1}{4} + t} A_{\sharp t}B.}$$

A detailed discussion of the relation between Corollaries 2.2 and 2.3 is given in Proposition 3.2. The fact that the proposition is stated later in this paper is due to the detailed computations of the generalized Kantorovich constant, which we prefer to state independently to avoid disturbance of the readers.

2.2. **Ando-type inequalities for $A_{\sharp t}B$.** We begin this section by presenting the following Ando-type inequality, similar to the celebrated result presented in Lemma 1.1.

**Theorem 2.3.** Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive definite such that $0 < m \leq A, B \leq M$ and let $\Phi$ be a positive unital linear mapping. Then

$$\Phi(A_{\sharp t}B) \leq \beta \Phi(A)_{\sharp t}\Phi(B), \quad 0 \leq t \leq 1,$$

where $\beta = \frac{(M + m)^{2t} (M^{1+t} + m^{1+t})^4}{4^{2t+1} M^{2+3t} m^{2+3t}}$.

**Proof.** Let $X = A_{\sharp t}B$ and $Y = \Phi(A)_{\sharp t}\Phi(B)$. Then

$$\Phi \left( A^{-1}_{\sharp t} X \right) = \Phi \left( (A^{-1}_{\sharp t} B)^\sharp \right) \quad \text{(by (2.1))}$$

$$\leq \Phi \left( A^{-1}_{\sharp t} B \right)^\sharp \quad \text{(by Lemma 1.4)}$$

$$\leq \left( \Phi \left( A^{-1}_{\sharp t} \Phi(B) \right) \right)^\sharp \quad \text{(by (1.6))}$$

$$\leq \left( C_1 \Phi(A)^{-1}_{\sharp t} \Phi(B) \right)^\sharp \quad \text{(by Lemma 1.6)}$$

$$= C_1^\sharp \left( \Phi(A)^{-1}_{\sharp t} \Phi(B) \right)^\sharp$$

$$= C_1^\sharp \Phi(A)^{-1}_{\sharp t} Y,$$

where $C_1 = \frac{(M + m)^2}{4 M m}$. On the other hand, using the reversed Ando’s inequality, we obtain

$$\Phi \left( A^{-1}_{\sharp t} X \right) \geq C_2 \Phi(A^{-1}_{\sharp t}) \Phi(X) \quad \text{(by Lemma 1.3)}$$

$$\geq C_2 \Phi(A)^{-1}_{\sharp t} \Phi(X) \quad \text{(by Lemma 1.4)}$$

where $C_2 = \frac{2 \sqrt{M^{1+t} m^{1+t}}}{M^{1+t} + m^{1+t}}$. Thus, we have shown that

$$C_2 \Phi(A)^{-1}_{\sharp t} \Phi(X) \leq C_1^\sharp \Phi(A)^{-1}_{\sharp t} Y.$$
This leads to
\[ C_2 \left( \Phi(A)^{\frac{1}{2}} \Phi(X) \Phi(A)^{\frac{1}{2}} \right)^{\frac{1}{2}} \leq C_1^{\frac{1}{2}} \left( \Phi(A)^{\frac{1}{2}} Y \Phi(A)^{\frac{1}{2}} \right)^{\frac{1}{2}}. \]

Now, by Lemma 1.7, we infer that
\[ C_2^2 \left( \Phi(A)^{\frac{1}{2}} \Phi(X) \Phi(A)^{\frac{1}{2}} \right) \leq C_1^t K_2 \left( \Phi(A)^{\frac{1}{2}} Y \Phi(A)^{\frac{1}{2}} \right), \]
where \( K_2 = \frac{(M^{1+t} + m^{1+t})^2}{4 M^{1+t} m^{1+t}} \), which in turns implies
\[ \Phi(A^\#_t B) \leq \frac{C_1^t K_2}{C_2^2} \Phi(A)^{\#_t} \Phi(B). \]

This completes the proof. \( \square \)

Having shown Theorem 2.3 as an Ando-type inequality for the spectral geometric mean, the following theorem intends to give a reversed version of Theorem 2.3.

**Theorem 2.4.** Let \( A, B \in B(\mathcal{H}) \) be positive definite such that \( 0 < m \leq A, B \leq M \) and let \( \Phi \) be a positive unital linear mapping. Then
\[ \Phi\left( A^{\#_t} \Phi(B)^{-1\#} \right) \leq \frac{1}{K^2 \left( \frac{m}{M}, \frac{M}{m}, t \right)} \left( \frac{M + m}{2 \sqrt{Mm}} \right)^{2t} \frac{(M^{1+t} + m^{1+t})^2}{4 M^{1+t} m^{1+t}} \Phi\left( A^\#_t B \right). \]

**Proof.** Let \( X = A^\#_t B \). Then
\[
\begin{align*}
\Phi\left( A^{-1\#} X \right) &= \Phi\left( \left( A^{-1\#} B \right)^t \right) \quad \text{(by (2.1))} \\
&\geq K \left( \frac{m}{M}, \frac{M}{m}, t \right) \Phi\left( A^{-1\#} B \right)^t \quad \text{(by Lemma 1.5)} \\
&\geq K \left( \frac{m}{M}, \frac{M}{m}, t \right) \left( \frac{2 \sqrt{Mm}}{M + m} \Phi\left( A^{-1} \right)^{\#} \Phi(B) \right)^t \quad \text{(by Lemma 1.2)} \\
&= K \left( \frac{m}{M}, \frac{M}{m}, t \right) \left( \frac{2 \sqrt{Mm}}{M + m} \Phi\left( A^{-1} \right)^{\#} \Phi(B) \right)^t \\
&\geq K \left( \frac{m}{M}, \frac{M}{m}, t \right) \left( \frac{2 \sqrt{Mm}}{M + m} \right)^t \Phi\left( A^{-1} \right)^{\#} \Phi(B)^t \quad \text{(by Lemma 1.4)} \\
&= K \left( \frac{m}{M}, \frac{M}{m}, t \right) \left( \frac{2 \sqrt{Mm}}{M + m} \right)^t \Phi\left( A^{-1} \right)^{\#} Y, \\
\end{align*}
\]
where \( Y = \Phi(A)^{\#_t} \Phi(B) \); see (2.1). This implies
\[
\Phi\left( A^{-1\#} Y \right) \leq \frac{1}{K \left( \frac{m}{M}, \frac{M}{m}, t \right)} \left( \frac{M + m}{2 \sqrt{Mm}} \right)^{2t} \Phi\left( A^{-1} \right)^{\#} \Phi(X). 
\]
Therefore,
\[
\left( \Phi(A)^{\frac{t}{2}} Y \Phi(A)^{\frac{t}{2}} \right)^{\frac{1}{t}} \leq \frac{1}{K \left( \frac{m}{M}, \frac{M}{m}, t \right)} \left( \frac{M + m}{2\sqrt{Mm}} \right)^{t} \left( \Phi(A)^{\frac{1}{2t}} \Phi(X) \Phi(A)^{\frac{1}{2t}} \right)^{\frac{1}{t}}.
\]

By Lemma 1.7,
\[
\Phi(A) \zeta_t \Phi(B) \leq \frac{1}{K^2 \left( \frac{m}{M}, \frac{M}{m}, t \right)} \left( \frac{M + m}{2\sqrt{Mm}} \right)^{2t} \frac{(M^{1+t} + m^{1+t})^2}{4M^{1+t}m^{1+t}} \Phi(A^\sharp t B),
\]
as desired. \( \square \)

An upper bound of \( \Phi(A)^{\zeta_t} \Phi(A)^{-1} \) in terms of the Kantorovich constant is usually stated as the Kantorovich inequality. In the following, we present this inequality for the spectral geometric mean.

**Corollary 2.4.** (Operator Kantorovich inequality for spectral geometric mean) Let \( A \in \mathcal{B}(\mathcal{H}) \) be positive definite such that \( 0 < m \leq A \leq M \) and let \( \Phi \) be a positive unital linear mapping. If \( 0 \leq t \leq 1 \), then

\[
\Phi(A) \zeta_t \Phi(A^{-1}) \leq \frac{1}{K^2 \left( \frac{m}{M}, \frac{M}{m}, t \right)} \left( \frac{M + m}{2\sqrt{Mm}} \right) \frac{(M^{1+t} + m^{1+t})^2}{4M^{1+t}m^{1+t}}.
\]

**Corollary 2.5.** Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be such that \( 0 < m \leq A, B \leq M \), and let \( 0 \leq t \leq 1 \). Then

\[
A\nabla_t B \leq \frac{m \nabla_\lambda M}{m^2 \lambda M} K^2 \left( \frac{M + m}{2\sqrt{Mm}} \right)^{1+t} \frac{(M + m)^2}{4Mm} A^\sharp t B,
\]

where \( \lambda = \min \{ t, 1 - t \} \).

**Proof.** It has been shown in [8] that
\[
(2.2) \quad A\nabla_t B \leq \frac{m \nabla_\lambda M}{m^2 \lambda M} A^\sharp t B.
\]

Now, the result follows by combining (2.2) and Corollary 2.3. \( \square \)

**Theorem 2.5.** (Ando-Hiai inequality for spectral geometric mean) Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be such that \( 0 < m \leq A, B \leq M \), and let \( 0 \leq r \leq 1 \). Then

\[
(A^\sharp t B)^r \leq \min \{ \kappa_1(m, M, r, t, \lambda), \kappa_2(m, M, r, t, \lambda) \} A^r t B^r,
\]

where

\[
\kappa_1(m, M, r, t, \lambda) := \frac{m^r \nabla_\lambda M^r}{m^2 \lambda M^r} K^2 \left( \frac{M^r + m^r}{4M^r m^r} \right)^{1+t} \frac{(M + m)^2}{4Mm} K^{2r} \left( \frac{M^{1+t} + m^{1+t}}{4M^{1+t}m^{1+t}} \right)^{r(1+t)}
\]

and

\[
\kappa_2(m, M, r, t, \lambda) := \frac{m^r \nabla_\lambda M^r}{m^2 \lambda M^r} K^2 \left( \frac{M^r + m^r}{4M^r m^r} \right)^{1+t} \frac{(M + m)^2}{4Mm} \left( \frac{M^{1+t} + m^{1+t}}{4M^{1+t}m^{1+t}} \right)^{r},
\]
for $0 \leq t \leq 1$ and $\lambda = \min \{t, 1 - t\}$.

**Proof.** In Corollary 2.3, we replace $A$ and $B$ by $A^{-1}$ and $B^{-1}$, respectively. Then we have

$$
A^\natural_t B \leq \frac{1}{K^2 \left( \sqrt[4]{M}, \sqrt{m}, t \right)} \left( \frac{(M + m)^2}{4Mm} \right)^{1+t} A^\natural_t B,
$$

since $(A^{-1}A^\natural B^{-1})^{-1} = A^\natural_t B$, $(A^{-1}A^\natural B^{-1})^{-1} = A^\natural_t B$ and

$$
K^{-2} \left( \sqrt[4]{1/M}, \sqrt{1/m}, t \right) \left( \frac{(1/m + 1/M)^2}{4MM} \right)^{1+t} = K^{-2} \left( \sqrt[4]{1/M}, \sqrt{m}, t \right) \left( \frac{(M + m)^2}{4Mm} \right)^{1+t}.
$$

Since the function $f(x) = x^r$ $(0 \leq r \leq 1)$ is operator monotone, we have

$$
K^{2r} \left( \sqrt[4]{m/M}, \sqrt{M/m}, t \right) \left( \frac{4Mm}{(M + m)^2} \right)^{r(1+t)} (A^\natural_t B)^r
\leq (A^\natural_t B)^r \quad \text{(by (2.3))}
\leq (A\nabla_t B)^r \quad \text{(by the weighted arithmetic–geometric operator mean inequality)}
\leq \frac{1}{K(m, M, t)} A^{\natural} \nabla_t B^r \quad \text{(by Lemma 1.8)}
\leq \frac{m^{r} \nabla M^{r}}{K(m, M, t) K^{2r} \left( \sqrt[4]{m/M}, \sqrt{M/m}, t \right)} \left( \frac{(M^{r} + m^{r})^2}{4Mr} \right)^{1+t} (A^\natural_t B^r) \quad \text{(by Corollary 2.5)}.
$$

Consequently,

$$(A^\natural_t B)^r \leq \frac{m^{r} \nabla M^{r}}{K(m, M, t) K^{2r} \left( \sqrt[4]{m/M}, \sqrt{M/m}, t \right)} \left( \frac{(M^{r} + m^{r})^2}{4Mr} \right)^{1+t} (A^\natural_t B^r).$$

Similarly we have,

$$(A^\natural_t B)^r \leq \left( \frac{(M^{1+t} + m^{1+t})^2}{4M^{1+t}m^{1+t}} \right)^{r} (A\nabla_t B)^r \quad \text{(by Corollary 2.1)}
\leq \left( \frac{(M^{1+t} + m^{1+t})^2}{4M^{1+t}m^{1+t}} \right)^{r} \frac{1}{K(m, M, t)} A^{\natural} \nabla_t B^r \quad \text{(by Lemma 1.8)}
\leq \frac{m^{r} \nabla M^{r}}{m^{r} \nabla M^{r}} \left( \frac{(M^{1+t} + m^{1+t})^2}{4M^{1+t}m^{1+t}} \right)^{r} \left( \frac{(M^{r} + m^{r})^2}{4Mr} \right)^{1+t} (A^\natural_t B^r) \quad \text{(by Corollary 2.5)}.$$
Remark 2.2. There is no ordering of two constants $\kappa_1(m, M, r, t, \lambda)$ and $\kappa_2(m, M, r, t, \lambda)$ appearing in Theorem 2.5. To compare them, it suffices to consider the function

$$\delta(x, t) := \left(\frac{(x + 1)^2}{4x}\right)^{t+1} - \frac{(x^{t+1} + 1)^2}{4x^{t+1}} K^2(x, t), \quad (x > 1, \ 0 \leq t \leq 1),$$

by putting $x := M/m > 1$. Then we have $\delta(10, 0.1) \simeq 0.10068$ and $\delta(10, 0.9) \simeq -10.011$.

Although we generally have the inequality

$$\left(\frac{(x + 1)^2}{4x}\right)^{t+1} \leq \frac{(x^{t+1} + 1)^2}{4x^{t+1}}$$

for $x > 1$ and $0 \leq t \leq 1$ by elementary calculations, the fact $K^2(x, t) \leq 1$ for $0 \leq t \leq 1$, affects this comparison.

3. On the generalized Kantorovich constant

In our discussion of the above inequalities, different constants have shown up. In this section, we attempt to give a detailed discussion of these constants. This helps better understand the stated results. The conclusion of this section is a hidden relation between Corollaries 2.2 and 2.3.

In the following proposition, we present a monotonicity behavior of the generalized Kantorovich constant $K(x, t)$.

Proposition 3.1. The generalized Kantorovich constant $K(x, t)$ satisfies the following monotonicity properties in $x$.

(i) If $t < 0$ or $t > 1$, then $K(x, t)$ is monotone decreasing when $0 < x < 1$, and monotone increasing when $x > 1$. $K(x, t)$ takes the minimum value 1 when $x = 1$.

(ii) If $0 < t < 1$, then $K(x, t)$ is monotone increasing when $0 < x < 1$, and monotone decreasing when $x > 1$. $K(x, t)$ takes the maximum value 1 when $x = 1$.

Proof. We calculate

$$\frac{dK(x, t)}{dx} = \frac{f_t(x)g_t(x)}{(1-t)(1-x^t)(x-1)^2} \left(\frac{t-1}{t}x^{t-1} - 1\right)^t,$$

where

$$f_t(x) := 1 - x^t + xt - t, \quad g_t(x) := tx^{t-1} - 1 + (1-t)x^t.$$

Then we have

$$\frac{df_t(x)}{dx} = t(1-x^{t-1}), \quad \frac{dg_t(x)}{dx} = t(1-t)x^{t-2}(x-1).$$

(i) For the case $t < 0$ or $t > 1$, we have $(1-t)(1-x^t) < 0$ for $0 < x < 1$, and $(1-t)(1-x^t) > 0$ for $x > 1$. We also find that $f_t(x) \leq f_t(1) = 0$ and $g_t(x) \leq g_t(1) = 0$ for all $x > 0$. Thus we have $\frac{dK(x, t)}{dx} \leq 0$ for $0 < x < 1$, and $\frac{dK(x, t)}{dx} \geq 0$ for $x > 1$. 
(ii) For the case $0 < t < 1$, we have $(1-t)(1-x^t) > 0$ for $0 < x < 1$, and $(1-t)(1-x^t) < 0$ for $x > 1$. We also find that $f_t(x) \geq f_t(1) = 0$ and $g_t(x) \geq g_t(1) = 0$ for all $x > 0$. Thus we have $\frac{dK(x,t)}{dx} \geq 0$ for $0 < x < 1$, and $\frac{dK(x,t)}{dx} \leq 0$ for $x > 1$.

Finally we note that $\lim_{x \to 1} K(x,t) = 1$, see \cite[Theorem 2.54 (iii)]{7} for example. \hfill \Box

**Remark 3.1.** The case $0 < x < 1$ in Proposition 3.1 can be proven by the fact $K(1/x,t) = K(x,t)$ for any $t > 0$ and $x > 0$, \cite[Theorem 2.54 (i)]{7}.

Proposition 3.1 can be stated equivalently in the following form, where

\begin{equation}
K(m,M,t) := \frac{(mM^t - Mm^t)}{(t-1)(M-m)} \left( \frac{t-1}{t} \frac{M^t - m^t}{mM^t - Mm^t} \right)^t, \quad (0 < m < M)
\end{equation}

**Corollary 3.1.** Let $0 < m_1 < M_1$ and $0 < m_2 < M_2$ such that $\frac{M_1}{m_1} \leq \frac{M_2}{m_2}$. If $0 < t < 1$, then $K(m_1, M_1, t) \geq K(m_2, M_2, t)$. If $t < 0$ or $t > 1$, then $K(m_1, M_1, t) \leq K(m_2, M_2, t)$.

To better understand these constants, and to reach our goal, we need the following lemma.

**Lemma 3.1.** (I) Let $0 \leq t \leq 1$. For $x > 1$, the following inequality holds

\begin{equation}
\frac{x+1}{x-1} \geq (1-t)^2 \left( \frac{x+x^t}{x-x^t} \right) + t^2 \left( \frac{x^t+1}{x^t-1} \right).
\end{equation}

For $0 < x < 1$, we have the inequality (3.4) below.

(II) Let $t \geq 1$ or $t \leq 0$. For $x > 1$, the following inequality holds

\begin{equation}
\frac{x+1}{x-1} \leq (1-t)^2 \left( \frac{x+x^t}{x-x^t} \right) + t^2 \left( \frac{x^t+1}{x^t-1} \right).
\end{equation}

For $0 < x < 1$, we have the inequality (3.3) above.

**Proof.** (I) We set the function

$$f(x,t) := \frac{x+1}{x-1} - (1-t)^2 \left( \frac{x+x^t}{x-x^t} \right) + t^2 \left( \frac{x^t+1}{x^t-1} \right), \quad (x > 1, \ 0 \leq t \leq 1).$$

Since $f(x,1-t) = f(x,t)$, we have only to prove $f(x,t) \geq 0$ for $x > 1$ and $0 \leq t \leq 1/2$. Then we calculate

$$\frac{df(x,t)}{dt} = -\frac{2g(x,t)}{(x-x^t)^2(x^t-1)^2},$$

where

$$g(x,t) := (x^t-1)(x^t-x)h(x,t) + x^t(logx)k(x,t)$$

in which

$$h(x,t) := (2t-1)x^t(1-x) + x^{2t} - x,$$
and \[ k(x, t) := (t - 1)^2 x(x^2 + 1) - t^2(x^2 + x^2) + 2(2t - 1)x^{t+1}. \]

(i) We have
\[
\frac{dh(x, t)}{dx} = x^t - 1 + 2t^2(x - 1)x^t - 1 + tx^t - 1(2x^t - x - 1), \quad \frac{d^2h(x, t)}{dx^2} = t(1 - 2t)x^{t-2}l(x, t),
\]
where
\[ l(x, t) := t(x - 1) + x + 1 - 2x^t. \]

Then we have
\[
\frac{dl(x, t)}{dx} = t + 1 - 2tx^{t-1}, \quad \frac{d^2l(x, t)}{dx^2} = -2t(t - 1)x^{t-2} \geq 0, \quad (x > 1, \ 0 \leq t \leq 1/2).
\]

Consequently, \( \frac{dl(x, t)}{dx} \geq \frac{d^2l(x, t)}{dx^2} \geq 1 - t \geq 0 \) which implies \( l(x, t) \geq l(1, t) = 0. \)

Since \( 1 - 2t \geq 0, \) this means \( \frac{d^2h(x, t)}{dx^2} \geq 0 \) so that we have \( \frac{dh(x, t)}{dx} \geq \frac{d^2h(x, t)}{dx^2} = 0 \) which implies \( h(x, t) \geq h(1, t) = 0 \) for \( x > 1 \) and \( 0 \leq t \leq 1/2. \)

(ii) Similarly, we calculate
\[
\frac{dk(x, t)}{dx} = 2t^3(x - 1)x^{2t-1} + 2t(x^t - 1) + (x^t - 1)^2 - t^2(3x^{2t} - 4x^t + 2x - 1),
\]
\[
\frac{d^2k(x, t)}{dx^2} = -2t(t + 1)(1 - 2t)x^t - 1 - t^2(1 - 2t)x^{2t-2} - (t - 1)^2(2t + 1)x^{2t+1},
\]
\[
\frac{d^2k(x, t)}{dx^2} = 2t(2t - 1)(t - 1)x^{t-3}u(x, t), \quad \text{where}
\]
\[
u(x, t) := (t + 1)x - 2t^2x^t + (t - 1)(2t + 1)x^{t+1}.
\]

Then we have
\[
\frac{du(x, t)}{dx} = t + 1 - 2t^3x^{t-1} + (t + 1)(t - 1)(2t + 1)x^t,
\]
\[
\frac{d^2u(x, t)}{dx^2} = t(t - 1)x^{t-2}(2t^2(x - 1) + 3tx + x) \leq 0, \quad (x > 1, \ 0 \leq t \leq 1/2).
\]

Thus \( \frac{du(x, t)}{dx} \leq \frac{du(1, t)}{dx} = t(t - 1) \leq 0 \) which implies \( u(x, t) \leq u(1, t) = 0. \) Since \( 2t - 1 \leq 0, \) this means \( \frac{d^3k(x, t)}{dx^3} \leq 0 \) so that \( \frac{d^2k(x, t)}{dx^2} \leq \frac{d^2k(1, t)}{dx^2} = 0 \) which implies \( \frac{dk(x, t)}{dx} \leq \frac{dk(1, t)}{dx} = 0. \) Therefore \( k(x, t) \leq k(1, t) = 0 \) for \( x > 1 \) and \( 0 \leq t \leq 1/2. \)

From (i) and (ii), we have \( g(x, t) \geq 0, \) namely we have \( \frac{df(x, t)}{dt} \geq 0 \) which implies \( f(x, t) \geq f(x, 0) = 0. \)

Finally, replacing \( x \) by \( 1/x \) in (3.3), we have the reversed inequality of (3.3) for \( 0 < x < 1 \) and \( 0 \leq t \leq 1. \)
(II) As we stated in the beginning of (I), we have the symmetric property for \( f(x, t) \) such that \( f(x, 1 - t) = f(x, t) \). Therefore it is sufficient to consider the case \( t \geq 1 \). Putting \( t := 1/s \leq 1 \) in (3.3), we have

\[
\frac{x + 1}{x - 1} \geq \left( 1 - \frac{1}{s} \right)^2 \frac{x + x^{1/s}}{x - x^{1/s}} + \frac{1}{s^2} \frac{x^{1/s} + 1}{x^{1/s} - 1}, \quad (x > 1, \ s \geq 1).
\]

Multiplying \( s^2 > 0 \) to both sides and putting \( y := x^{1/s} > 1 \), we have

\[
\frac{y + 1}{y - 1} \leq (1 - s)^2 \frac{y + y^s}{y - y^s} + s^2 \frac{y^s + 1}{y^s - 1}, \quad (y > 1, \ s \geq 1),
\]

which shows the inequality of (3.4).

Finally, replacing \( x \) by \( 1/x \) in (3.4), we have the reversed inequality of (3.4) for \( 0 < x < 1 \).

\[\square\]

Applying Lemma 3.1, we have the following bounds of Kantorovich constant \( K(x, t) \).

**Theorem 3.1.** Define 
\[
L(x, t) := \left( \frac{x^t + x}{x + 1} \right) \left( \frac{x^t + 1}{x^t + x} \right)^t
\]
for \( x > 0 \) and \( t \in \mathbb{R} \). Then

(I) For \( 0 \leq t \leq 1 \), we have

\[
(3.5) \quad L(x, t) \leq K(x, t), \quad (x > 0).
\]

(II) For \( t \geq 1 \) or \( t \leq 0 \) we have

\[
(3.6) \quad L(x, t) \geq K(x, t), \quad (x > 0).
\]

**Proof.** (I) The equality holds for the special case \( x = 1 \), since \( K(1, t) = \lim_{x \to 1} K(x, t) = 1 \) for \( t \in \mathbb{R} \), by l’Hospital’s theorem. We firstly assume \( x > 1 \). By elementary calculations, we have \( L(1/x, t) = L(x, t) \) and it is known [7, Theorem 2.54 (i)] that \( K(1/x) = K(x, t) \) for all \( t \in \mathbb{R} \) and \( x > 0 \). Therefore it is sufficient to prove (3.5) for \( t \geq 1 \) and \( x > 1 \).

The inequality (3.5) is equivalent to the following inequality

\[
(3.7) \quad \frac{(1 - t)(x - 1)(x + x^t)}{(x + 1)(x - x^t)} \leq \left( \frac{(1 - t)(x^t - 1)(x + x^t)}{t(x - x^t)(x^t + 1)} \right)^t.
\]

From the ordering of the weighted means, we generally have the inequality \( a^t \geq \frac{a}{(1 - t)a + t} \) for \( a > 0 \) and \( 0 \leq t \leq 1 \). In order to prove the inequality (3.7), it is sufficient to prove

\[
(3.8) \quad \frac{(1 - t)(x - 1)(x + x^t)}{(x + 1)(x - x^t)} \leq \frac{a_{x,t}}{(1 - t)a_{x,t} + t}; \quad a_{x,t} := \frac{(1 - t)(x^t - 1)(x + x^t)}{t(x - x^t)(x^t + 1)} > 0.
\]
By elementary calculations with \(1 \leq x^t \leq x\) for \(x \geq 1\) and \(0 \leq t \leq 1\), the inequality given in (3.8) is equivalent to
\[
\frac{(1-t)(x-1)(x + x^t)}{(x+1)(x - x^t)} \leq \frac{(1-t)(x^t - 1)(x^t + x)}{(1-t)^2(x^t - 1)(x^t + x) + t^2(x - x^t)(x^t + 1)}
\]
\[\Leftrightarrow (x+1)(x^t - 1)(x - x^t) \geq (x-1)\{(1-t)^2(x^t - 1)(x^t + x) + t^2(x - x^t)(x^t + 1)\}
\]
\[\Leftrightarrow \frac{x+1}{x-1} \geq \frac{(1-t)^2}{x^t - 1} + \frac{t^2}{x^t - 1},
\]
which is true for \(x > 1\) and \(0 \leq t \leq 1\) thanks to the inequality (3.3) in Lemma 3.1.

(II) We prove the inequality (3.6). By elementary calculations, we have \(L(x, 1-t) = L(x, t)\) for all \(t \in \mathbb{R}\) and \(x > 0\). It is also known [7, Theorem 2.54 (ii)] that \(K(x, 1-t) = K(x, t)\) for all \(t \in \mathbb{R}\) and \(x > 0\). Therefore it is sufficient to prove (3.6) for \(t \geq 1\) and \(x > 0\).

Putting \(t := \frac{1}{s} \leq 1\) in (3.5), we have
\[
\left(\frac{x^{1/s} + x}{x+1}\right) \left(\frac{x^{1/s} + 1}{x^{1/s} + x}\right)^{1/s} \leq \left(\frac{s^{1/s} - x}{1/s - 1}(x-1)\right)^{1/s} \left(\frac{1}{s}(x^{1/s} - 1)\right)^{1/s}, \quad (x > 0, \ s \geq 1).
\]

Taking \(s\)-th power of the both sides and then taking the inverse of the both sides, we have
\[
\left(\frac{x^{1/s} + x}{x^{1/s} + 1}\right)^{s} \geq \frac{1}{s} \left(\frac{x-1}{x^{1/s} - 1}\right)^{s}, \quad (x > 0, \ s \geq 1).
\]

Putting \(y := x^{1/s} > 0\), we have
\[
\left(\frac{y^{s} + y}{y+1}\right)^{s} \geq \frac{1}{s} \left(\frac{y^{s} - 1}{y^{s} + 1}\right)^{s}, \quad (y > 0, \ s \geq 1),
\]
which shows the inequality (3.6).

This completes the proof. \(\square\)

Note that we have
\[
L(x, 1/2) = \frac{\sqrt{x} (\sqrt{x} + 1)^2}{x+1} \leq \frac{2\sqrt{x}}{\sqrt{x} + 1} = K(x, 1/2)
\]
and
\[
L(x, 2) = L(x, -1) = \frac{(x^2 + 1)^2}{x(x+1)^2} \geq \frac{(x+1)^2}{4x} = K(x, 2) = K(x, -1)
\]
for special cases.

**Corollary 3.2.** (I) For \(0 \leq t \leq 1\), we have
\[
(3.9) \quad K(x^2, t) \leq K^2(x, t), \quad (x > 0).
\]

(II) For \(t \geq 1\) or \(t \leq 0\), we have
\[
(3.10) \quad K(x^2, t) \geq K^2(x, t), \quad (x > 0).
\]
Proof. (I) By elementary calculations, we see the inequality (3.9) is equivalent to the inequality:

\[
\frac{(x^{2t} - x^2)}{(t-1)(x^2 - 1)} \left( \frac{t-1}{t} x^{2t} - 1 \right)^t \leq \frac{(x^t - x)}{(t-1)^2(x-1)^2} \left( \frac{t-1}{t^2} (x^t - x) \right)^t
\]

which is equivalent to (3.5).

(II) Since \( K(x, 1-t) = K(x, t) \) for all \( t \in \mathbb{R} \), we have only to prove the inequality (3.10) for \( t \geq 1 \). From (I), we have the inequality (3.9) for \( x > 0 \) and \( 0 \leq t \leq 1 \). As stated in the proof of (I), the inequality (3.9) is equivalent to the inequality (3.5). In the inequality (3.5), we put \( t := 1/s \) for \( s \geq 1 \) and \( x^{1/s} := y > 0 \). Then the inequality (3.5) is written as

\[
\left( \frac{y^s + y}{y^s + 1} \right) \left( \frac{y + 1}{y^s + y} \right)^{1/s} \leq \frac{s(y^s - y)}{(s-1)(y^s - 1)} \left( \frac{(s-1)(y-1)}{y^s - y} \right)^{1/s}
\]

by elementary calculations. Taking \( s \)-th power of the both sides and then taking the inverse of the both sides, we have,

\[
\left( \frac{y^s + y}{y + 1} \right) \left( \frac{y + 1}{y^s + y} \right) \leq \frac{y^s - y}{(s-1)(y-1)} \left( \frac{(s-1)(y^s - 1)}{s(y^s - y)} \right)
\]

which is equivalent to the inequality \( K(y^2, s) \geq K^2(y, s) \) for \( s \geq 1 \) and \( y > 0 \), by elementary calculations. Thus the inequality (3.10) is true for \( x > 0 \), under the assumption \( t \geq 1 \) or \( t \leq 0 \).

□

Now we are ready to present the relation between Corollaries 2.2 and 2.3.

**Proposition 3.2.** Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be such that \( 0 < m \leq A, B \leq M \), and let \( 0 \leq t \leq 1 \). Then

(3.11) \( A_{\#t} B \leq \eta(m, M, t) A_{\#t} B \leq \Gamma(m, M, t) A_{\#t} B \),

where \( \eta(m, M, t) \) and \( \Gamma(m, M, t) \) are shown in Corollary 2.3 and Corollary 2.2, respectively in the following:

\[
\eta(m, M, t) := \frac{1}{K^2 \left( \sqrt{\frac{m}{M}}, \sqrt{\frac{M}{m}}, t \right)} \left( \frac{(M+m)^2}{4Mm} \right)^{1+t}, \quad \Gamma(m, M, t) := \frac{1}{K \left( \frac{m}{M}, \frac{M}{m}, t \right)} \left( \frac{(M^1+t + m^{1+t})}{4M^{1+t}m^{1+t}} \right).
\]

Proof. Corollary 2.2 states

\( A_{\#t} B \leq \Gamma(m, M, t) A_{\#t} B \).

Now replace \( A \) and \( B \) by \( A^{-1} \) and \( B^{-1} \), respectively. This will lead to replacing \( m \) and \( M \) by \( \frac{1}{m} \) and \( \frac{1}{M} \), respectively. Noting that \( \Gamma(m, M, t) = \Gamma \left( \frac{1}{M}, \frac{1}{m}, t \right) \), we find that

\( A^{-1}_{\#t} B^{-1} \leq \Gamma(m, M, t) A^{-1}_{\#t} B^{-1} \),
which implies
\[(A^{-1}B)^{-1} \geq \Gamma(m,M,t)^{-1} (A^{-1}B)^{-1}.\]
This is equivalent to
\[(3.12) \quad A^*_t B \leq \Gamma(m,M,t) A^*_t B.\]
Corollary 2.3 also states
\[(3.13) \quad A^*_t B \leq \eta(m,M,t) A^*_t B.\]
In the sequel, we show \(\eta(m,M,t) \leq \Gamma(m,M,t)\). We firstly show
\[(3.14) \quad \left(\frac{(M + m)^2}{4Mm}\right)^{1+t} \leq \left(\frac{(M^{1+t} + m^{1+t})^2}{4M^{1+t}m^{1+t}}\right).\]
Since the equality holds in (3.14) when \(t = 0\), we assume \(0 < t \leq 1\). The Hölder inequality for
\(a_1, a_2, b_1, b_2 > 0\) with \(\frac{1}{p} + \frac{1}{q} = 1\) and \(p, q > 1\) states that
\(a_1 b_1 + a_2 b_2 \leq (a_1^p + a_2^p)^{1/p} (b_1^q + b_2^q)^{1/q}\).
Putting \(a_1 := 1, \ a_2 := 1, \ b_1 := M, \ b_2 := m\) and \(p := \frac{t+1}{t} > 1, \ q := \frac{t+1}{t} > 1\), we have
\[1 \cdot M + 1 \cdot m \leq \left(1^{t+1} + 1^{t+1}\right)^{t+1} \left(M^{t+1} + m^{t+1}\right)^{t+1}.
Taking \((t + 1)\)-th power of the both sides, we have \((M + m)^{t+1} \leq 2^t (M^{t+1} + m^{t+1})\) which is equivalent to (3.14). Thus we have the inequality (3.14) for \(0 < m < M\) and \(0 \leq t \leq 1\). We secondly prove
\[(3.15) \quad K\left(\frac{m}{M}, \frac{M}{m}, t\right) \leq K^2\left(\sqrt{\frac{m}{M}}, \sqrt{\frac{M}{m}}, t\right).\]
Putting \(x := \sqrt{\frac{M}{m}} > 1\) for \(0 < m < M\), we have the relation with the different symbol given in (3.2):
\[K\left(\frac{m}{M}, \frac{M}{m}, t\right) = K\left(\frac{1}{x}, x, t\right) = K(x^2, t), \quad K\left(\sqrt{\frac{m}{M}}, \sqrt{\frac{M}{m}}, t\right) = K\left(\sqrt{\frac{1}{x}}, \sqrt{x}, t\right) = K(x, t).\]
With these and the obtained inequality (3.9) for \(0 \leq t \leq 1\), we have the inequality (3.15) for \(0 < m < M\) and \(0 \leq t \leq 1\). The inequality (3.15) together with (3.14) implies that
\(\eta(m,M,t) \leq \Gamma(m,M,t)\). \(\square\)

From Proposition 3.2, we can claim that Corollary 2.3 gives the tighter bound than Corollary 2.2.

The following is the final remark on our bound \(L(x,t)\) appeared in Theorem 3.1, for the generalized Kantorovich constant \(K(x,t)\).
Remark 3.2. It is known [7, Theorem 2.54 (iv)] that $K(x,t)$ is decreasing for $t < 1/2$ and increasing for $t > 1/2$. Therefore $K(x,t)$ takes the minimum value $\frac{2x^{1/4}}{\sqrt{x} + 1}$ when $t = 1/2$. It is natural to ask whether the following inequality holds or not:

$$L(x,t) \leq \frac{2x^{1/4}}{\sqrt{x} + 1}, \quad (x > 0, \ 0 \leq t \leq 1).$$

However, we have the following example.

$$\frac{2x^{1/4}}{\sqrt{x} + 1} - L(x,t) \simeq -0.0171811, \quad (x = 10, \ t = 0.1).$$

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