Probabilistic entailment and iterated conditionals

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Abstract. In this paper we exploit the notions of conjoined and iterated conditionals, which are defined in the setting of coherence by means of suitable conditional random quantities with values in the interval $[0,1]$. We examine the iterated conditional $(B|K)|A|H$, by showing that $A|H$ p-entails $B|K$ if and only if $(B|K)|(A|H) = 1$. Then, we show that a p-consistent family $\mathcal{F} = \{E_1|H_1, E_2|H_2\}$ p-entails a conditional event $E_3|H_3$ if and only if $E_3|H_3 = 1$, or $(E_3|H_3)QC(S) = 1$ for some nonempty subset $S$ of $\mathcal{F}$, where $QC(S)$ is the quasi conjunction of the conditional events in $S$. Then, we examine the inference rules \textit{And}, \textit{Cut}, \textit{Cautious Monotonicity}, and \textit{Or} of System P and other well known inference rules (\textit{Modus Ponens}, \textit{Modus Tollens}, \textit{Bayes}). We also show that $QC(\mathcal{F})|C(\mathcal{F}) = 1$, where $C(\mathcal{F})$ is the conjunction of the conditional events in $\mathcal{F}$. We characterize p-entailment by showing that $\mathcal{F}$ p-entails $E_3|H_3$ if and only if $(E_3|H_3)|C(\mathcal{F}) = 1$. Finally, we examine Denial of the antecedent and Affirmation of the consequent, where the p-entailment of $(E_3|H_3)$ from $\mathcal{F}$ does not hold, by showing that $(E_3|H_3)|C(\mathcal{F}) \neq 1$.

1 Introduction

The new paradigm psychology of reasoning is characterized by using probability theory instead of classical bivalent logic as a normative background theory (see, e.g., Gilio & Over, 2012; Oaksford & Chater, 2007; Over, 2009; Elqayam & Over, 2012; Pfeifer & Douven, 2014; Pfeifer, 2013; Politzer & Baratgin, 2015). One of the key topics of the new paradigm psychology of reasoning is how people interpret and reason about conditionals (see, e.g., Douven, 2016; Edgington, 1997; Politzer, Over, & Baratgin, 2010; Evans & Over, 2004; Pfeifer & Kleiter, 2003, 2010; Pfeifer & Tulkki, 2017; Oaksford & Chater, 2003; Over & Cruz, 2018). How people interpret and reason about conditionals was also one of the key topics in the (old) logic-based

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paradigm psychology of reasoning, which dominated the 20th century experimental psychology of reasoning. While human interpretation of conditionals was labeled as “irrational” or “defective”, since the participants’ responses deviated from the semantics of the material conditional, rationality was revisited and rehabilitated within the new probabilistic paradigm: specifically, the majority of participants

- treat negated antecedents as irrelevant for evaluating whether a conditional holds, and
- evaluate their degrees of belief in conditionals by respective conditional probabilities (and not by the probability of the material conditional).

These findings speak for the conditional event interpretation, and against the material conditional interpretation, of conditionals.

Among various interpretations of probability, we advocate and use the coherence-based approach to probability (see, e.g., Berti, Miranda, & Rigo, 2017; Biazzo, Gilio, Lucasiewicz, & Sanfilippo, 2000; Capotorti, Lad, & Sanfilippo, 2005; Coletti & Scozzafava, 2002; Coletti, Petturiti, & Vantaggi, 2016; Gilio, Pfeifer, & Sanfilippo, 2016; Gilio & Sanfilippo, 2013, 2013d, 2014; Sanfilippo, 2012; Walley, Pelessoni, & Vicig, 2004), which traces back to Bruno de Finetti (1937/1980, 1970/1974). From a psychological point of view, it is evident that probability serves to measure degrees of belief and not some objective quantity in the world: this is in line with de Finetti provocative ontological motto “Probability does not exist” (1970/1974, Preface). The probabilistic approach based on coherence is thus characterized by subjective, and not by objective, probabilities. Methodologically, the approach based on coherence principle differs in many respects to standard approaches to probabilities. We mention two of them which highlight the psychological plausibility of our approach.

First, contrary to many approaches to probability, the coherence-based approach does not require a complete algebra. For drawing a probabilistic modus ponens inference, for example, an algebra could be constructed from the constituents derived from the involved events in the inference rule. This is psychologically plausible, as the reasoning person may focus on only what is considered to be relevant for drawing the inference.

Second, conditional probability is a primitive notion and it is not defined by the fraction of the joint and the marginal probabilities: the standard definition of $P(C|A)$ by $\frac{P(A \land C)}{P(A)}$ requires to assume that $P(A) > 0$, as a fraction over zero is undefined. Probabilistic approaches which define conditional probabilities in this way can therefore not properly manage zero antecedent probabilities. The subjective probabilistic approach allows for managing zero antecedent probabilities; moreover, zero probabilities are even exploited for reducing the complexity of the probabilistic inference. Another aspect of defining conditional probability directly is that the degree of belief in a conditional If $A$, then $C$ can be given in a direct way by the reasoner without presupposing knowledge about $P(A \land C)$ and $P(A)$: even as in everyday life it may be impracticable to evaluate the latter two
probabilities, people do assess conditionals. For example, if we want to assess our degree of belief in the conditional that *If I take the train at six, I am at home at seven*, we can do that directly, without thinking first about the unconditional probabilities of *I take the train at six and I am at home at seven* and of *I take the train at six*.

In some recent papers of Gilio and Sanfilippo the notions of conjoined and iterated conditionals have been introduced as suitable conditional random quantities (Gilio & Sanfilippo, 2013a, 2013b, 2014, 2017a). These new objects extend the usual notions of conjunction and conditioning from the case of unconditional events to the case of conditional events. For instance, we developed a semantics for examples like the following (which was presented by Douven, 2016, p. 45):

(I) *If the mother is angry if the son gets a B, then she will be furious if the son gets a C*,

which is an iterated (or nested) conditional. It consists of a conditional in its antecedent

(A) *if the son gets a B, then the mother is angry,*

and a conditional in its consequent

(C) *if the son gets a C, then the mother is furious.*

Of course, the degree of belief in (I) cannot be something like a conditional probability, as the famous triviality results by Lewis (1976) have shown. Rather, we conceive iterated conditionals like (I) as conditional random quantities (and not as conditional events) and measure the degree of belief in such objects by previsions \(P\) (not by probabilities \(P\); Gilio & Sanfilippo, 2014; Gilio, Over, Pfeifer, & Sanfilippo, 2017; Sanfilippo, Pfeifer, Over, & Gilio, 2018). We will explain the formal details below. Interestingly, when we considered the uncertainty propagation rule for the generalized probabilistic modus ponens (Sanfilippo, Pfeifer, & Gilio, 2017), where the degree of beliefs are propagated, for instance, from "*The cup broke if dropped*" \((A|H)\), and "*if the cup broke if dropped, then the cup was fragile* \((C|A|H)\)" to "*the cup was fragile* \((C)\)", we observed, that the uncertainty propagation rules coincide with those of the non-iterated probabilistic modus ponens (i.e., from \(P(A) = x\) and \(P(C|A) = y\) infer \(xy \leq P(C) \leq xy + 1 - x\)). Likewise, we have shown that the uncertainty propagation rules of the iterated version of Centering coincide with the respective (non-iterated) probability propagation rules (Sanfilippo et al., 2018). Thus, a remarkable aspect of the definitions of nested conditionals in terms of conditional random quantities preserve some well known classical results.

The main result of this paper may be also related to an analogue result derived from the *deduction theorem*. This theorem implies that if an argument is logically valid (or if the premises logically entail the conclusion), then the argument can be transformed into a *logically true conditional*, s.t., the premises are combined by conjunction and form the antecedent and the conclusion forms the
consequent of the resulting conditional, which is then a tautology. For example, the logically valid modus ponens (where \( A \rightarrow C \) denotes the material conditional \( \overline{A} \lor C \) and \( \models \) denotes logical entailment),

\[ \{ A, A \rightarrow C \} \models C, \]

can be transformed by the deduction theorem into the following conditional, which is a tautology (and *vice versa*), that is:

\[ (A \land (A \rightarrow C)) \rightarrow C = (A \land (\overline{A} \lor C)) \lor C = \Omega. \]

Instead of logical entailment, however, we consider in this paper the probabilistic entailment (p-entailment), as introduced by Adams [1972][1998]. Let \( C(\mathcal{F}) \) denote the conjunction of the conditional events in a p-consistent family \( \mathcal{F} \). We study, in analogy to the deduction theorem, whether the claim “a conditional event \( E|H \) is p-entailed by a p-consistent family \( \mathcal{F} \) of conditional events” is equivalent to the claim “the prevision in the iterated conditional \( (E|H)|C(\mathcal{F}) \) is equal to 1”. We examine many cases related to this aspect; in particular, we examine some inference rules of System P and other well known inference rules.

We remark that this basic relation, between p-entailment and iterated conditioning, appears in its most elementary form when we consider two non-impossible events \( A \) and \( B \) in the case where \( A \subseteq B \), that is where \( A \land \overline{B} = \emptyset \). In this case \( P(A) \leq P(B) \) and then \( A \) p-entails \( B \), that is \( P(A) = 1 \) implies \( P(B) = 1 \), and the unique coherent assessment on \( B|A \) is \( P(B|A) = 1 \). Therefore, by recalling that in the framework of the betting scheme, when we pay \( P(B|A) = x \), we receive \( B|A = AB + x\overline{A} \); when \( A \subseteq B \), it holds that \( A \) p-entails \( B \) and \( B|A = AB + 1 \cdot \overline{A} = A + \overline{A} = 1 \). Conversely, if \( B|A = 1 \), then \( P(B|A) = 1 \); moreover,

\[ P(B) = P(B|A)P(A) + P(B|\overline{A})P(\overline{A}) = P(A) + P(B|\overline{A})P(\overline{A}), \]

and when \( P(A) = 1 \) it follows that \( P(B) = 1 \), so that \( A \) p-entails \( B \).

The outline of the paper is as follows. In Section 2 we give some preliminaries on the notions of coherence and p-entailment for conditional random quantities, which assume values in \([0, 1]\). In Section 3 after recalling the notions of conjoined and iterated conditionals, we show that a conditional event \( A|H \) p-entails another conditional event \( B|K \) if and only if \( (B|K)|(A|H) = 1 \). Moreover, we show that a p-consistent family of two conditional events \( \{ E_1|H_1, E_2|H_2 \} \) p-entails a conditional event \( E_3|H_3 \) if and only if it holds that \( (E_3|H_3)|QC(E_1)|H_1, E_2|H_2) = 1 \), where \( QC(E_1)|H_1, E_2|H_2) \) denotes the quasi conjunction of \( E_1|H_1, E_2|H_2 \). We also characterize p-entailment of \( E_3|H_3 \) from the family \( \{ E_1|H_1, E_2|H_2 \} \) by the property that \( E_3|H_3 = 1 \), or \( (E_3|H_3)|QC(S) = 1 \) for some nonempty \( S \subseteq \{ E_1|H_1, E_2|H_2 \} \). In Section 4 we suitably generalize the notion of iterated conditioning; then, we examine some inference rules of System P and other well known inference rules. The generalization of the notion of iterated conditioning is necessary in order to examine the OR rule. In Section 5 we give two results which relate p-entailment and iterated conditioning. The first result
shows that the iterated conditional having as antecedent and consequent the conjunction and the quasi conjunction of two conditional events, respectively, is equal to 1. The second result characterizes the p-entailment of the conditional event \( E_3|H_3 \) from a p-consistent family \( \{ E_1|H_1, E_2|H_2 \} \) by the property that the iterated conditional \( (E_3|H_3)|(E_1|H_1) \land (E_2|H_2) \) is equal to 1. Finally, we examine two examples where the p-entailment of the conditional event \( E_3|H_3 \) from a p-consistent family \( \{ E_1|H_1, E_2|H_2 \} \) does not hold. We also show that in these cases \( (E_3|H_3)|(E_1|H_1) \land (E_2|H_2) \) does not coincide with 1.

2 Preliminaries

In our approach events represent uncertain facts described by (non ambiguous) logical propositions. An event \( A \) is a two-valued logical entity which is either true (\( T \)), or false (\( F \)). The indicator of an event \( A \) is a two-valued numerical quantity which is 1, or 0, according to whether \( A \) is true, or false, respectively. We use the same symbol to refer to an event and its indicator. We denote by \( \Omega \) the sure event and by \( \emptyset \) the impossible one (notice that, when necessary, the symbol \( \emptyset \) will denote the empty set). Given two events \( A \) and \( B \), we denote by \( A \land B \), or simply by \( AB \), the intersection, or conjunction, of \( A \) and \( B \), as defined in propositional logic; likewise, we denote by \( A \lor B \) the union, or disjunction, of \( A \) and \( B \). We denote by \( \overline{A} \) the negation of \( A \). Of course, the truth values for conjunctions, disjunctions and negations are defined as usual. Given any events \( A \) and \( B \), we simply write \( A \subseteq B \) to denote that \( A \) logically implies \( B \), that is \( AB = \emptyset \), which means that it is necessary that \( A \) and \( B \) cannot both be true. Given two events \( A, H \), with \( H \neq \emptyset \), the conditional event \( A|H \) is defined as a three-valued logical entity which is true (\( T \)), or false (\( F \)), or void (\( V \)), according to whether \( AH \) is true, or \( A\overline{H} \) is true, or \( \overline{H} \) is true, respectively. Given a conditional event \( A|H \) with \( P(A|H) = x \), then for (the indicator of) \( A|H \) we have \( A|H = xH \in \{ 1, 0, x \} \) (Sanfilippo et al., 2018, Appendix A.3). We recall below the notion of logical implication of Goodman and Nguyen (1988) for conditional events (see also Gilio & Sanfilippo, 2013d).

Definition 1. Given two conditional events \( A|H \) and \( B|K \) we define that \( A|H \) logically implies \( B|K \) (denoted by \( A|H \subseteq B|K \) if and only if \( AH \) is true implies \( BK \) is true and \( BK \) is true implies \( AH \) is true); i.e., \( AH \subseteq BK \) and \( \overline{B}K \subseteq \overline{A}H \).

A generalization of the Goodman and Nguyen logical implication to conditional random quantities has been given by Pelessoni & Vicig (2014). The notions of p-consistency and p-entailment of Adams (1975) were formulated for conditional events in the setting of coherence by Gilio and Sanfilippo (2010) (see also Gilio, 2012; Gilio & Sanfilippo, 2011, 2013d).

Definition 2. Let \( \mathcal{F}_n = \{ E_i|H_i, \ i = 1, \ldots, n \} \) be a family of \( n \) conditional events. Then, \( \mathcal{F}_n \) is p-consistent if and only if the probability assessment \( (p_1, p_2, \ldots, p_n) = (1, 1, \ldots, 1) \) on \( \mathcal{F}_n \) is coherent.
Definition 3. A p-consistent family $F_n = \{E_i|H_i, \ i = 1, \ldots , n\}$ p-entails a conditional event $E|H$ (denoted by $F_n \models_p E|H$) if and only if for any coherent probability assessment $(p_1, \ldots , p_n, z)$ on $F_n \cup \{E|H\}$ it holds that: if $p_1 = \cdots = p_n = 1$, then $z = 1$.

Of course, when $F_n$ p-entails $E|H$, there may be coherent assessments $(p_1, \ldots , p_n, z)$ with $z \neq 1$, but in such cases $p_i \neq 1$ for at least one index $i$. We say that the inference from a p-consistent family $F_n$ to $E|H$ is p-valid if and only if $F_n$ p-entails $E|H$. We recall the well known notion of quasi conjunction among conditional events:

Definition 4. Given a family $F_n = \{E_i|H_i, \ i = 1, \ldots , n\}$ of $n$ conditional events, the quasi conjunction of the conditional events in $F_n$ is defined as

$$QC(F_n) = \bigwedge_{i=1}^{n} (\overline{E_i} \lor E_i)(\bigvee_{i=1}^{n} H_i).$$

Moreover, we recall the following characterization of p-entailment (Gilio & Sanfilippo 2013d):

Theorem 1. Let a p-consistent family $F_n = \{E_i|H_i, \ i = 1, \ldots , n\}$ and a conditional event $E|H$ be given. The following assertions are equivalent:

1. $F_n$ p-entails $E|H$;
2. The assessment $P = (1, \ldots , 1, z)$ on $F = F_n \cup \{E|H\}$, where $P(E_i|H_i) = 1$, $i = 1, \ldots , n$, $P(E|H) = z$, is coherent if and only if $z = 1$;
3. The assessment $P = (1, \ldots , 1, 0)$ on $F = F_n \cup \{E|H\}$, where $P(E_i|H_i) = 1$, $i = 1, \ldots , n$, $P(E|H) = 0$, is not coherent;
4. Either there exists a nonempty $S \subseteq F_n$ such that $QC(S)$ implies $E|H$, or $H \subseteq E$;
5. There exists a nonempty $S \subseteq F_n$ such that $QC(S)$ p-entails $E|H$.

We also recall the characterization of the p-entailment for two conditional events (Gilio & Sanfilippo, 2013d, Theorem 7):

Theorem 2. Given two conditional events $A|H, B|K$, with $AH \neq \emptyset$. It holds that

$$A|H \Rightarrow_p B|K \iff A|H \subseteq B|K,$$

or $K \subseteq B \iff \Pi \subseteq \{(x, y) \in [0, 1]^2 : x \leq y\}$, where $\Pi$ is the set of coherent assessments $(x, y)$ on $\{A|H, B|K\}$.
(2014) the notions of conjoined, disjoined, and iterated conditionals have been studied in the framework of conditional random quantities. In particular, the next result establishes some conditions under which two conditional random quantities $X|H$ and $Y|K$ coincide (Gilio & Sanfilippo, 2014, Theorem 4):

**Theorem 3.** Given any events $H \neq \emptyset$ and $K \neq \emptyset$, and any random quantities $X$ and $Y$, let $\Pi$ be the set of the coherent prevision assessments $P(X|H) = \mu$ and $P(Y|K) = \nu$.

(i) Assume that, for every $(\mu, \nu) \in \Pi$, the values of $X|H$ and $Y|K$ always coincide when $H \lor K$ is true; then $\mu = \nu$ for every $(\mu, \nu) \in \Pi$.

(ii) For every $(\mu, \nu) \in \Pi$, the values of $X|H$ and $Y|K$ always coincide when $H \lor K$ is true if and only if $X|H = Y|K$.

### 3 Generalized System P and Compound Conditionals

In this section we recall the notions of conjunction and iterated conditioning. Then, we show that $A|H$ p-entails $B|K$ if and only if $(B|K)| (A|H) = 1$. Moreover, we show that $\{E_1|H_1, E_2|H_2\}$ p-entails $E_3|H_3$ if and only if $(E_3|H_3)|QC(E_1|H_1, E_2|H_2) = 1$.

### 3.1 Exploring conjunction and iterated conditioning

We recall below the definition of conjunction of two conditional events $A|H$ and $B|K$ (Gilio & Sanfilippo, 2013b, 2013a, 2014). Different approaches to compounded conditionals, not based on coherence, have been developed by other authors (see, e.g., Kaufmann, 2009; McGee, 1989).

**Definition 5.** Given any pair of conditional events $A|H$ and $B|K$, with $P(A|H) = x$ and $P(B|K) = y$, we define their conjunction as the conditional random quantity $(A|H) \land (B|K) = Z | (H \lor K)$, where $Z = \min \{A|H, B|K\}$.

In betting terms, $z = P[(A|H) \land (B|K)]$ represents the amount you agree to pay, with the proviso that you will receive the quantity:

$$ (A|H) \land (B|K) = \begin{cases} 
1, & \text{if } AHBK \text{ is true}, \\
0, & \text{if } AH \lor BK \text{ is true}, \\
x, & \text{if } BK \text{ is true}, \\
y, & \text{if } AHK \text{ is true}, \\
z, & \text{if } HK \text{ is true}.
\end{cases} \quad (1) $$

From (1), it follows that the conjunction $(A|H) \land (B|K)$ is the following random quantity

$$ (A|H) \land (B|K) = 1 \cdot AHBK + x \cdot BK + y \cdot AK + z \cdot HK. \quad (2) $$

We observe that if $H = K$, then $BK = AHK = \emptyset$, so that $(A|H) \land (B|K) = ABH + zHK$; moreover, $AB|H = ABH + pHK$, where $p = P(AB|H)$. We notice
that \((A|H) \land (B|H)\) and \(AB|H\) coincide when \(H\) is true; then, by Theorem 3
\(z = p;\) thus,
\[(A|H) \land (B|H) = AB|H.\] (3)

We recall that, given any coherent assessment \((x, y)\) on \(\{A|H, B|K\}\), with \(A, H, B, K\) logically independent, and with \(H \neq \emptyset, K \neq \emptyset\), the extension \(z = \mathbb{P}[(A|H) \land (B|K)]\) is coherent if and only if the following Fréchet-Hoeffding bounds are satisfied (Gilio & Sanfilippo, 2014, Theorem 7):
\[
\max\{x + y - 1, 0\} = z' \leq z \leq z'' = \min\{x, y\}.\] (4)

Note that the bounds in (4) coincide with the bounds for the conjunction of unconditional probabilities (i.e., if \(P(A) = x\) and \(P(B) = y\), then \(\max\{x + y - 1, 0\} \leq P(AB) \leq \min\{x, y\}\)).

We now turn to recalling and discussing the notion of iterated conditioning (see, e.g., (Gilio & Sanfilippo, 2013a, 2013b, 2014)).

**Definition 6 (Iterated conditioning).** Given any pair of conditional events \(A|H\) and \(B|K\), with \(AH \neq \emptyset\), the iterated conditional \((B|K)|(A|H)\) is defined as the conditional random quantity
\[
(B|K)|(A|H) = (B|K) \land (A|H) + \mu \overline{A}|H,\] (5)
where \(\mu = \mathbb{P}[(B|K)|(A|H)]\).

**Remark 1.** Notice that we assumed that \(AH \neq \emptyset\) to give a non-trivial meaning to the notion of the iterated conditional. Indeed, if \(AH\) were equal to \(\emptyset\), that is \(A|H = 0\), then it would be the case that \(\overline{A}|H = 1\) and \((B|K)|(A|H) = (B|K)\lor (A|H) = \mu \overline{A}|H = \mu\) would follow; that is, \((B|K)|(A|H)\) would coincide with the (indeterminate) value \(\mu\). Similarly in the case of \(B|\emptyset\) (which is of no interest): the trivial iterated conditional \((B|K)|0\) is not considered in our approach.

We observe that, by linearity of prevision, it holds that

\[
\mu = \mathbb{P}((B|K)|(A|H)) = \mathbb{P}((B|K) \land (A|H)) + \mu P(\overline{A}|H) = z + \mu (1 - x),
\]
from which it follows that \(z = \mu x\). Here, when \(x > 0\), we obtain \(\mu = \frac{z}{x} \in [0, 1]\).

Notice that \(z + \mu (1 - x), i.e., \(\mu\), is the value of \((B|K)|(A|H)\) when \(\overline{\overline{A}|K}\) is true. Then, by observing that
\[
\overline{A|H} \lor \overline{A|H}B|K \lor \overline{A|H}B|K \lor \overline{\overline{A}|H} = \overline{A|H} \lor \overline{\overline{A}|H} ,
\]
we obtain

\[
(B|K)|(A|H) = \begin{cases} 
1, & \text{if } AHBK \text{ is true,} \\
0, & \text{if } AHBK \text{ is true,} \\
y, & \text{if } AH\overline{BK} \text{ is true,} \\
x + \mu(1-x), & \text{if } \overline{ABK} \text{ is true,} \\
\mu(1-x), & \text{if } \overline{ABK} \text{ is true,} \\
\mu, & \text{if } \overline{AHBK} \text{ is true,} \\
\mu, & \text{if } \overline{AHBK} \text{ is true,} \\
\mu, & \text{if } \overline{AH} \lor \overline{BK} \text{ is true.}
\end{cases}
\]

In particular, when \( x = 0 \), it holds that

\[
(B|K)|(A|H) = \begin{cases} 
1, & \text{if } AHBK \text{ is true,} \\
0, & \text{if } AHBK \text{ is true,} \\
y, & \text{if } AH\overline{BK} \text{ is true,} \\
1, & \text{if } AHBK \text{ is true,} \\
y, & \text{if } AH\overline{BK} \text{ is true,} \\
\mu, & \text{if } \overline{AHBK} \text{ is true,} \\
\mu, & \text{if } \overline{AHBK} \text{ is true,} \\
\mu, & \text{if } \overline{AH} \lor \overline{BK} \text{ is true.}
\end{cases}
\]

As we can see, in order that the prevision assessment \( \mu \) on \( (B|K)|(A|H) \) be coherent, \( \mu \) must belong to the convex hull of the values 0, \( y \), \( 1 \); that is, (also when \( x = 0 \)) it must be that \( \mu \in [0,1] \).

**Proposition 1.** Given two conditional events \( A|H \) and \( B|K \), it holds that

\[
A|H \subseteq B|K \implies (A|H) \land (B|K) = A|H.
\]

**Proof.** We set \( P(A|H) = x \), \( P(B|K) = y \), and \( P[(A|H) \land (B|K)] = z \). As \( A|H \subseteq B|K \), it holds that \( AHBK = AH\overline{BK} = \overline{ABK} = \emptyset \) and \( AHBK = AH \) (Gilio & Sanfilippo [2013d], Remark 3). Then,

\[
(A|H) \land (B|K) = AHBK + x\overline{BK} + y\overline{AH} + z\overline{BK} = AHBK + x\overline{BK} + z\overline{BK}.
\]

Moreover,

\[
A|H = AH + x\overline{H} = AH + x\overline{BK} + x\overline{BK}.
\]

We notice that \( (A|H) \land (B|K) \) and \( A|H \) coincide when \( H \lor K \) is true. Then, \( z = x \) follows from Theorem 3. Therefore, \( (A|H) \land (B|K) = A|H \). \( \square \)

The following theorem shows that a conditional \( A|H \) \( \mu \)-entails another conditional \( B|K \) if and only if the unique coherent prevision assessment for the corresponding iterated conditional \( (B|K)|(A|H) \) is equal to one.

**Theorem 4.** Given two \( (\mu\text{-consistent}) \) conditional events \( A|H \) and \( B|K \), it holds that

\[
A|H \Rightarrow \mu B|K \iff (B|K)|(A|H) = 1.
\]
Proof. $(\Rightarrow)$. We distinguish two cases: (i) $A|H \subseteq B|K$; (ii) $K \subseteq B$. Case (i). We remark that if $A|H \subseteq B|K$, then $A|H \leq B|K$ and $P(A|H) \leq P(B|K)$; moreover, $(A|H) \land (B|K) = A|H$. Then, by defining $P((B|K)|(A|H)) = \mu$, $P(A|H) = x$, we obtain

$$(B|K)|(A|H) = (A|H) \land (B|K) + \mu \overline{A}|H = A|H + \mu \overline{A}|H = \begin{cases} 1, & \text{if } AH \text{ is true,} \\ \mu, & \text{if } \overline{A}H \text{ is true,} \\ x + \mu(1-x), & \text{if } \overline{H} \text{ is true.} \end{cases}$$

By linearity of prevision, we obtain

$$P((B|K)|(A|H)) = \mu = P(A|H) + \mu P(\overline{A}|H) = x + \mu(1-x);$$

which implies that

$$(B|K)|(A|H) = \begin{cases} 1, & \text{if } AH \text{ is true,} \\ \mu, & \text{if } \overline{A}H \lor \overline{H} \text{ is true.} \end{cases}$$

In order for $\mu$ to be coherent, $\mu$ must belong to the convex hull of the set $\{1\}$; i.e. $\mu = 1$. In other words, given two conditional events $A|H$ and $B|K$, with $A|H \subseteq B|K$, it holds that: $P((B|K)|(A|H)) = 1$. Thus $(B|K)|(A|H) = 1$.

Case (ii). If $K \subseteq B$ it holds that $P(B|K) = y = 1$ and $B|K = 1$. Then, $(A|H) \land (B|K) = (A|H)|(H \lor K) = A|H$ (see Gilio & Sanfilippo 2013a, Remark 4). Moreover, $(B|K)|(A|H) = A|H + \mu \overline{A}|H$ and by linearity of prevision it holds that $\mu = x + \mu(1-x)$. Then,

$$(B|K)|(A|H) = \begin{cases} 1, & \text{if } AH \text{ is true,} \\ \mu, & \text{if } \overline{A}H \text{ is true,} \\ x + \mu(1-x), & \text{if } \overline{H} \text{ is true.} \end{cases}$$

Then, by coherence, $\mu = 1$ and $(B|K)|(A|H) = 1$.

Thus, $p$-entailment of $B|K$ from $A|H$ implies $(B|K)|(A|H) = 1$.

$(\Leftarrow)$. Assume that $(B|K)|(A|H) = 1$, so that the unique coherent assessment for $P((B|K)|(A|H))$ is $\mu = 1$. Then, by observing that $P((A|H) \land (B|K)) \leq P(B|K) = y$, it follows that

$$P((A|H) \land (B|K)) = P((B|K)|(A|H))P(A|H) = P(A|H) = x \leq y.$$ 

Then, when $x = 1$, it holds that $y = 1$; that is, $A|H$ $p$-entails $B|K$. \qed

**Corollary 1.** Let three conditional events $E_1|H_1$, $E_2|H_2$, and $E_3|H_3$ be given, where $\{E_1|H_1, E_2|H_2\}$ is $p$-consistent. The quasi conjunction $QC(E_1|H_1, E_2|H_2)$ $p$-entails $E_3|H_3$ if and only if $(E_3|H_3)|QC(E_1|H_1, E_2|H_2) = 1$.

*Proof.* The assertion directly follows by applying Theorem 1 with $A|H = QC(E_1|H_1, E_2|H_2)$ and $B|K = E_3|H_3$. \qed

In the next result we characterize the $p$-entailment of $E_3|H_3$ from the family $\{E_1|H_1, E_2|H_2\}$ by the property that $E_3|H_3 = 1$, or $(E_3|H_3)|QC(S) = 1$ for some nonempty $S \subseteq \{E_1|H_1, E_2|H_2\}$.
Theorem 5. Let three conditional events \(E_1|H_1\), \(E_2|H_2\), and \(E_3|H_3\) be given, where \(\{E_1|H_1, E_2|H_2\}\) is \(p\)-consistent. Then, the family \(\{E_1|H_1, E_2|H_2\}\) \(p\)-entails \(E_3|H_3\) if and only if at least one of the following conditions is satisfied: (i) \(E_3|H_3 = 1\); (ii) \((E_3|H_3)|(E_1|H_1) = 1\); (iii) \((E_3|H_3)|(E_2|H_2) = 1\); (iv) \((E_3|H_3)|QC(E_1|H_1, E_2|H_2) = 1\).

Proof. \((\Rightarrow)\). By Theorem 1 as \(\{E_1|H_1, E_2|H_2\}\) \(p\)-entails \(E_3|H_3\), it follows that \(QC(S) \subseteq E_3|H_3\) for some \(\emptyset \neq S \subseteq \{E_1|H_1, E_2|H_2\}\), or \(H_3 \subseteq E_3\). If \(H_3 \subseteq E_3\), then \(P(E_3|H_3) = 1\) and \(E_3|H_3 = 1\). If \(S = \{E_i|H_i\}\), for \(i = 1, 2, 3\), by Theorem 3 it holds that \((E_3|H_3)|(E_i|H_i) = 1\). If \(S = \{E_1|H_1, E_2|H_2\}\), then by Corollary 3 it holds that \((E_3|H_3)|QC(E_1|H_1, E_2|H_2) = 1\).

\((\Leftarrow)\). If \(E_3|H_3 = 1\) then the unique coherent assessment on \(E_3|H_3\) is \(P(E_3|H_3) = 1\). This means that \(H_3 \subseteq E_3\) and then \(E_1|H_1, E_2|H_2\) \(p\)-entails \(E_3|H_3\).

Finally, if \((E_3|H_3)|QC(E_1|H_1, E_2|H_2) = 1\), then by Corollary 3 it holds that \(QC(E_1|H_1, E_2|H_2)\) \(p\)-entails \(E_3|H_3\) and hence, by Theorem 3 \(\{E_1|H_1, E_2|H_2\}\) \(p\)-entails \(E_3|H_3\).

4 Iterated conditionals and some inference rules

In this section we examine some inference rules with \(\{E_1|H_1, E_2|H_2\}\) as the premise set, and \(E_3|H_3\) as the conclusion, by showing that, if \(\{E_1|H_1, E_2|H_2\} \Rightarrow \ E_3|H_3\), then \((E_3|H_3)|(E_1|H_1) \land (E_2|H_2) = 1\). The notion of conjunction of three conditional events is given below (Gilio & Sanfilippo, 2017a).

Definition 7. Given a family of three conditional events \(\mathcal{F} = \{E_1|H_1, E_2|H_2, E_3|H_3\}\), we set \(P(E_i|H_i) = x_i, i = 1, 2, 3, P[(E_i|H_i) \land (E_j|H_j)] = x_{ij} = x_{ji}, i \neq j\). The conjunction \(\mathcal{C}(\mathcal{F}) = (E_1|H_1) \land (E_2|H_2) \land (E_3|H_3)\) is defined as the conditional random quantity

\[
\mathcal{C}(\mathcal{F}) = (E_1|H_1) \land (E_2|H_2) \land (E_3|H_3) = \begin{cases} 
1, & \text{if } E_1H_1E_2H_2E_3H_3 \text{ is true} \\
0, & \text{if } E_1H_1 \lor E_2H_2 \lor E_3H_3 \text{ is true}, \\
x_1, & \text{if } E_1H_2E_3H_3 \text{ is true}, \\
x_2, & \text{if } E_1H_1E_3H_3 \text{ is true}, \\
x_3, & \text{if } E_3H_1E_2H_3 \text{ is true}, \\
x_{12}, & \text{if } E_1H_2E_3H_3 \text{ is true}, \\
x_{13}, & \text{if } E_2H_1E_3H_3 \text{ is true}, \\
x_{23}, & \text{if } E_3H_1E_2H_3 \text{ is true}, \\
x_{123}, & \text{if } E_1H_2H_3 \text{ is true}
\end{cases}
\]

(9)

where \(x_{123} = P[\mathcal{C}(\mathcal{F})]\).

We recall below the definition of the object \((E_3|H_3)|(E_1|H_1) \land (E_2|H_2)\), which is under study in (Gilio & Sanfilippo, 2017b).
Definition 8. Let be given three conditional events \(E_1|H_1\), \(E_2|H_2\), and \(E_3|H_3\), with \((E_1|H_1) \land (E_2|H_2) \neq 0\). We denote by \((E_3|H_3)\left((E_1|H_1) \land (E_2|H_2)\right)\) the conditional random quantity

\[
(E_1|H_1) \land (E_2|H_2) \land (E_3|H_3) + \mu(1 - (E_1|H_1) \land (E_2|H_2)),
\]

where \(\mu = \mathbb{P}[(E_3|H_3)\left((E_1|H_1) \land (E_2|H_2)\right)]\).

Remark 2. We observe that, defining \(\mathbb{P}[(E_1|H_1) \land (E_2|H_2) \land (E_3|H_3)] = t\) and \(\mathbb{P}[(E_1|H_1) \land (E_2|H_2)] = z\), by the linearity of prevision it holds that \(\mu = t + \mu(1 - z)\); then, \(t = \mu z\), that is

\[
\mathbb{P}[(E_1|H_1) \land (E_2|H_2) \land (E_3|H_3)] = \mathbb{P}[(E_3|H_3)\left((E_1|H_1) \land (E_2|H_2)\right)]\mathbb{P}[(E_1|H_1) \land (E_2|H_2)].
\]

Modus Ponens: \(\{C|A, A\} \Rightarrow_p C\). It holds that \((C|A) \land A = AC = QC(AC)\); then, by Theorem 4 as \(AC \subseteq C\) it follows that

\[
C((C|A) \land A) = C(QC((C|A), A) = C|AC = 1.
\]

This can be seen as an analogy to the fact that the modus ponens is logically valid in logic and that the probabilistic modus ponens is p-valid.

Modus Tollens: \(\{C|A, \overline{C}\}\Rightarrow_p \overline{A}\). It holds that \((C|A) \land \overline{C} = x\overline{AC}\), where \(x = P(C|A)\), while \(QC(C|A, \overline{C}) = \overline{AC}\); then, assuming \(x > 0\), we obtain

\[
\overline{A}((C|A) \land \overline{C}) = \overline{A} \land (C|A) \land \overline{C} + \mu(1 - (C|A) \land \overline{C})) = \begin{cases} 
\mu, & \text{if } A \lor C \text{ is true,} \\
\mu x + (1 - \mu x), & \text{if } \overline{AC} \text{ is true.}
\end{cases}
\]

By coherence it must be the case that \(\mu = x + \mu(1 - x)\), i.e., \(x = \mu x\), which implies \(\mu = x + \mu(1 - x) = 1\); therefore,

\[
\overline{A}((C|A) \land \overline{C}) = 1.
\]

This can be seen as an analogy to the fact that the modus tollens is logically valid in logic and that the probabilistic modus tollens is p-valid. Notice that, if \(x = 0\), then \((C|A) \land \overline{C} = 0\) and the object \(\overline{A}((C|A) \land \overline{C}) = \overline{A}|0 = \mu\), which is indeterminate (see Remark 1).

Bages. We note that \((E|AH) \land (H|A) = EH|A = QC(E|AH, H|A)\); then, as \(EH|A \subseteq H|EA\), by Theorem 2 it holds that \(E|AH, H|A \Rightarrow_p H|EA\). Moreover, by Theorem 4 it follows that

\[
(H|EA)((E|HA) \land H|A) = (H|EA)((QC((E|HA), H|A) = (H|EA)(EH|A) = 1.
\]

In particular, if \(A = \Omega\), we obtain \((H|E)(EH) = 1\).
4.1 And, Cut, and Cautious Monotonicity of System P

In this section we consider the following inference rules of System P (Kraus, Lehmann, & Magidor [1990]): And, Cut, and Cautious Monotonicity (short: CM). System P is a basic nonmonotonic reasoning which allows for retracting conclusions in the light of new premises. The probabilistic versions of the rules of System P are p-valid (Adams [1975]; Biazzo, Gilio, Lukasiewicz, & Sanfilippo [2002]; Gilio [2002]). Experimental evidence supports the psychological plausibility of System P (see, e.g., Da Silva Neves, Bonnefon, & Raufasnte [2002]; Pfeifer & Kleiter [2003, 2005]; Schurz [2005]).

And rule: \( \{B\mid A, C\mid A\} \Rightarrow p \ BC\mid A \). By formula (3), it holds that \( (B\mid A) \land (C\mid A) = BC\mid A = QC(B\mid A, C\mid A) \); then, by Theorem 4 as \( BC\mid A \subseteq BC\mid A \) it follows that

\[
(BC\mid A)\cdot((C\mid A) \land (B\mid A)) = (BC\mid A)\cdot QC(B\mid A, C\mid A) = (BC\mid A)\cdot (BC\mid A) = 1.
\]

Cut rule: \( \{C\mid AB, B\mid A\} \Rightarrow p C\mid A \). We note that \( (C\mid AB) \land (B\mid A) = BC\mid A = QC(C\mid AB, B\mid A) \); then, by Theorem 4 as \( BC\mid A \subseteq C\mid A \) it follows that

\[
(C\mid A)\cdot((C\mid AB) \land (B\mid A)) = (C\mid A)\cdot QC(C\mid AB, B\mid A) = (C\mid A)\cdot (BC\mid A) = 1.
\]

CM rule: \( \{C\mid A, B\mid A\} \Rightarrow p C\mid AB \). By formula (3), it holds that \( (C\mid A) \land (B\mid A) = BC\mid A = QC(C\mid A, B\mid A) \); then, by Theorem 4 as \( BC\mid A \subseteq C\mid AB \) it follows that

\[
(C\mid AB)\cdot((C\mid A) \land (B\mid A)) = (C\mid AB)\cdot QC(C\mid A, B\mid A) = (C\mid AB)\cdot (BC\mid A) = 1.
\]

4.2 Or rule of System P

We recall that the Or rule is p-valid, that is \( \{C\mid A, C\mid B\} \Rightarrow p C\mid (A \lor B) \). The next result shows that the conclusion of the Or rule, \( C\mid (A \lor B) \), given the conjunction of the premises, \( (C\mid A) \land (C\mid B) \), coincides with 1.

Theorem 6. Given a p-consistent family \( \{C\mid A, C\mid B\} \) it holds that

\[
(C\mid (A \lor B))\cdot((C\mid A) \land (C\mid B)) = 1.
\]

Proof. By Definition 8 we obtain

\[
(C\mid (A \lor B))\cdot((C\mid A) \land (C\mid B)) = (C\mid (A \lor B))\cdot (C\mid A) \land (C\mid B) + \mu [1 - (C\mid A) \land (C\mid B)],
\]

where \( \mu = P([C\mid (A \lor B)]\cdot (C\mid A) \land (C\mid B)) \). We set \( P(C\mid A) = x, P(C\mid B) = y, \) and \( P((C\mid A) \land (C\mid B)) = z; \) then,

\[
(C\mid A) \land (C\mid B) = \begin{cases} 1, & \text{if } ABC \text{ is true}, \\ 0, & \text{if } (A \lor B) \neg C \text{ is true}, \\ x, & \text{if } \neg ABC \text{ is true}, \\ y, & \text{if } \neg A \neg BC \text{ is true}, \\ z, & \text{if } \neg AB \text{ is true}. \end{cases}
\]
Moreover, by defining \( P[(C|(A \lor B)) \land (C|A) \land (C|B)] = t \), we obtain
\[
(C|(A \lor B)) \land (C|A) \land (C|B) = \begin{cases} 
1, & \text{if } ABC \text{ is true,} \\
0, & \text{if } (A \lor B)\overline{C} \text{ is true,} \\
x, & \text{if } \overline{ABC} \text{ is true,} \\
y, & \text{if } \overline{AB}C \text{ is true,} \\
t, & \text{if } \overline{AB} \text{ is true.}
\end{cases}
\]

As we can see, \((C|(A \lor B)) \land (C|A) \land (C|B)\) and \((C|A) \land (C|B)\) coincide when \( A \lor B \) is true; then, by Theorem 3 it holds that \( t = z \), so that
\[
(C|(A \lor B)) \land (C|A) \land (C|B) = (C|A) \land (C|B).
\]

Then,
\[
(C|(A \lor B)) \land (C|A) \land (C|B) = (C|A) \land (C|B) + \mu[1 - (C|A) \land (C|B)],
\]
and by the linearity of prevision we obtain \( \mu = z + \mu(1 - z) \), so that \( z = \mu z \).

Moreover, by (10) we obtain
\[
(C|(A \lor B)) \land (C|A) \land (C|B) = \begin{cases} 
1, & \text{if } ABC \text{ is true,} \\
x + \mu(1 - x), & \text{if } \overline{ABC} \text{ is true,} \\
y + \mu(1 - y), & \text{if } AB\overline{C} \text{ is true,} \\
\mu, & \text{if } \overline{AB}C \text{ is true,} \\
\mu, & \text{if } \overline{AB} \text{ is true,}
\end{cases}
\]

which reduces to
\[
(C|(A \lor B)) \land (C|A) \land (C|B) = \begin{cases} 
1, & \text{if } ABC \text{ is true,} \\
x + \mu(1 - x), & \text{if } \overline{ABC} \text{ is true,} \\
y + \mu(1 - y), & \text{if } AB\overline{C} \text{ is true,} \\
\mu, & \text{if } \overline{AB} \lor \overline{C} \text{ is true.}
\end{cases}
\]

In order to prove that \((C|(A \lor B)) \land (C|A) \land (C|B) = 1\), we distinguish the following cases: (a) \( z > 0 \); (b) \( z = x = y = 0 \); (c) \( z = 0, x > 0, y > 0 \); (d) \( z = y = 0, x > 0 \); (e) \( z = x = 0, y > 0 \).

Case (a). By recalling that \( z = \mu z \), as \( z > 0 \) it follows that \( \mu = 1 \); then, \( y + \mu(1 - y) = x + \mu(1 - x) = 1 \), so that
\[
(C|(A \lor B)) \land (C|A) \land (C|B) = \begin{cases} 
1, & \text{if } ABC \lor AB\overline{C} \lor \overline{ABC} \text{ is true,} \\
\mu, & \text{if } \overline{ABC} \lor \overline{C} \text{ is true.}
\end{cases}
\]

Then, by coherence, \( \mu = 1 \) and \((C|(A \lor B)) \land (C|A) \land (C|B) = 1\).

Case (b). As \( x = y = 0 \), it holds that \( x + \mu(1 - x) = y + \mu(1 - y) = \mu \); then
\[
(C|(A \lor B)) \land (C|A) \land (C|B) = \begin{cases} 
1, & \text{if } ABC \text{ is true,} \\
\mu, & \text{if } \overline{ABC} \text{ is true.}
\end{cases}
\]
and, by coherence, $\mu = 1$; thus, $(C|(A \vee B))((C|A) \land (C|B)) = 1$

Case (c). By coherence, $\mu$ is a linear convex combination of the values $1, y + \mu(1 - y)$, and $x + \mu(1 - x)$, that is,

$$\mu = \lambda_1 + \lambda_2(y + \mu(1 - y)) + \lambda_3(x + \mu(1 - x)),$$

(11)

with $\lambda_h \geq 0, h = 1, 2, 3$, and $\lambda_1 + \lambda_2 + \lambda_3 = 1$. The equation (11) can be written as

$$\mu(\lambda_1 + \lambda_2 y + \lambda_3 x) = \lambda_1 + \lambda_2 y + \lambda_3 x,$$

where $\lambda_1 + \lambda_2 y + \lambda_3 x > 0$; then $\mu = y + \mu(1 - y) = x + \mu(1 - x) = 1$ and $(C|(A \vee B))((C|A) \land (C|B)) = 1$

Case (d). As $y = 0$ it holds that $y + \mu(1 - y) = \mu$; then,

$$(C|(A \vee B))((C|A) \land (C|B)) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ x + \mu(1 - x), & \text{if } A\overline{BC} \text{ is true,} \\ \mu, & \text{if } \overline{BC} \text{ is true.} \end{cases}$$

By coherence, $\mu$ is a linear convex combination of the values $1, x + \mu(1 - x)$, that is

$$\mu = \lambda_1 + \lambda_2(x + \mu(1 - x))$, \quad \lambda_1 \geq 0, \ \lambda_2 \geq 0, \ \lambda_1 + \lambda_2 = 1.$$  

(12)

The equation (12) can be written as $\mu(\lambda_1 + \lambda_2 x) = \lambda_1 + \lambda_2 x$, where $\lambda_1 + \lambda_2 x > 0$; then, $\mu = x + \mu(1 - x) = 1$ and $(C|(A \vee B))((C|A) \land (C|B)) = 1$

Case (e). Since $x = 0$, it holds that $x + \mu(1 - x) = \mu$; then,

$$(C|(A \vee B))((C|A) \land (C|B)) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ y + \mu(1 - y), & \text{if } A\overline{BC} \text{ is true,} \\ \mu, & \text{if } \overline{AC} \text{ is true.} \end{cases}$$

By coherence, $\mu$ is a linear convex combination of the values $1, y + \mu(1 - y)$, that is

$$\mu = \lambda_1 + \lambda_2(y + \mu(1 - y))$, \quad \lambda_1 \geq 0, \ \lambda_2 \geq 0, \ \lambda_1 + \lambda_2 = 1.$$  

(13)

The equation (13) can be written as $\mu(\lambda_1 + \lambda_2 y) = \lambda_1 + \lambda_2 y$, where $\lambda_1 + \lambda_2 y > 0$; then, $\mu = y + \mu(1 - y) = 1$ and $(C|(A \vee B))((C|A) \land (C|B)) = 1$. \hfill \Box

Remark 3. We observe that

$$(QC(C|A, C|B) = (\overline{A} \lor C) \land (\overline{C} \lor C))((A \lor B) = C|(A \lor B).$$

Then, the statement of Theorem 6 amounts to say that the iterated conditional $QC(C|A, C|B)((C|A) \land (C|B))$ is equal to 1. This aspect will be analyzed in general in the next section.

5 Iterated conditionals and p-entailment

In this section we give two results which relate p-entailment and iterated conditioning. In the next result, by defining $\mathcal{F} = \{E_1|H_1, E_2|H_2\}$, $QC(\mathcal{F}) = QC(E_1|H_1, E_2|H_2)$ and $C(\mathcal{F}) = (E_1|H_1) \land (E_2|H_2)$, we show that, under p-consistency of $\mathcal{F}$, the iterated conditional $QC(\mathcal{F})|(C(\mathcal{F}))$ is equal to 1.
We distinguish the following cases:

**Case (a).** Since the case that $\mu$ is the amount to be paid in order to receive 1, or additionally on $H$, we set $P(E_1|H_1) = x_1, P(E_2|H_2) = x_2, P(C) = x_{12}, P[C \land QC(F)] = \eta$. Moreover, we set $P[QC(F)|C(F)] = \mu$. Then,

$$QC(F)|C(F) = C(F) + \mu(1 - C(F)).$$

It can be verified that the possible values of the random vector $(C(F), C(F) \land QC(F))$ are

$$(1, 1), (0, 0), (x_1, x_1), (x_2, x_2), (x_{12}, \eta).$$

The value $(x_{12}, \eta)$ is associated to the constituent $\overline{H_1 \overline{H}_2}$. As we can see, conditionally on $H_1 \lor H_2$ being true, $C(F)$ and $C(F) \land QC(F)$ coincide; then, by Theorem 3, $x_{12} = \eta$, so that $QC(F) = C(F)$. Then,

$$QC(F)|C(F) = C(F) + \mu(1 - C(F)) = \begin{cases} 1, & \text{if } C(F) = 1, \\ \mu, & \text{if } C(F) = 0, \\ x_1 + \mu(1 - x_1), & \text{if } C(F) = x_1, \\ x_2 + \mu(1 - x_2), & \text{if } C(F) = x_2, \\ x_{12} + \mu(1 - x_{12}), & \text{if } C(F) = x_{12}. \end{cases}$$

By the linearity of prevision, we obtain $\mu = x_{12} + \mu(1 - x_{12})$, that is $x_{12} = \mu x_{12}$. Then,

$$QC(F)|C(F) = C(F) + \mu(1 - C(F)) = \begin{cases} 1, & \text{if } C(F) = 1, \\ x_1 + \mu(1 - x_1), & \text{if } C(F) = x_1, \\ x_2 + \mu(1 - x_2), & \text{if } C(F) = x_2, \\ \mu, & \text{if } C(F) = 0, \text{ or } C(F) = x_{12}. \end{cases}$$

We distinguish the following cases:

(a) $x_1 = x_2 = 0$; (b) $x_1 > 0, x_2 > 0$; (c) $x_1 > 0, x_2 = 0$; (d) $x_2 = 0, x_1 > 0$.

Case (a). Since $x_1 = x_2 = 0$, it holds that $x_1 + \mu(1 - x_1) = x_2 + \mu(1 - x_2) = \mu$, so that $QC(F)|C(F) \in \{1, \mu\}$. Based on the betting scheme, $\mu = P[QC(F)|C(F)]$ is the amount to be paid in order to receive 1, or $\mu$, according to whether the event $(C(F) = 1)$ is true, or false, respectively. Then, by coherence, it must be the case that $\mu = 1$. Therefore, $QC(F)|C(F) = 1$.

Case (b). By coherence, $\mu$ must be a linear convex combination of the values 1, $x_1 + \mu(1 - x_1)$, and $x_2 + \mu(1 - x_2)$, that is,

$$\mu = \lambda_1 + \lambda_2 (x_1 + \mu(1 - x_1)) + \lambda_3 (x_2 + \mu(1 - x_2)), \quad (14)$$

with $\lambda_h \geq 0, h = 1, 2, 3$, and $\lambda_1 + \lambda_2 + \lambda_3 = 1$. The equation (14) can be written as

$$\mu(\lambda_1 + \lambda_2 x_1 + \lambda_3 x_2) = \lambda_1 + \lambda_2 x_1 + \lambda_3 x_2,$$
where \( \lambda_1 + \lambda_2 x_1 + \lambda_3 x_2 > 0 \); then, \( \mu = x_1 + \mu(1 - x_1) = x_2 + \mu(1 - x_2) = 1 \) and \( QC(\mathcal{F})|\mathcal{C}(\mathcal{F}) = 1 \).

Case (c). As \( x_1 = 0 \), it holds that \( x_1 + \mu(1 - x_1) = \mu \), so that
\[
QC(\mathcal{F})|\mathcal{C}(\mathcal{F}) \in \{1, x_2 + \mu(1 - x_2), \mu\}.
\]

Then, by coherence, \( \mu \) must be a linear convex combination of the values \( 1, x_2 + \mu(1 - x_2) \), that is
\[
\mu = \lambda_1 + \lambda_2 [1_2 + \mu(1 - x_1)], \quad \lambda_1 + \lambda_2 = 1, \quad \lambda_1 \geq 0, \quad \lambda_2 \geq 0.
\]

It follows that \( \mu(\lambda_1 + \lambda_2 x_2) = \lambda_1 + \lambda_2 x_2 \), with \( \lambda_1 + \lambda_2 x_2 > 0 \). Then, \( \mu = 1 \) and \( QC(\mathcal{F})|\mathcal{C}(\mathcal{F}) = 1 \).

Case (d). As \( x_2 = 0 \), it holds that \( x_2 + \mu(1 - x_2) = \mu \), so that \( QC(\mathcal{F})|\mathcal{C}(\mathcal{F}) \in \{1, x_1 + \mu(1 - x_1), \mu\} \). Then, by coherence, \( \mu \) must be a linear convex combination of the values \( 1, x_1 + \mu(1 - x_1) \), that is
\[
\mu = \lambda_1 + \lambda_2 [1_2 + \mu(1 - x_1)], \quad \lambda_1 + \lambda_2 = 1, \quad \lambda_1 \geq 0, \quad \lambda_2 \geq 0.
\]

It follows that \( \mu(\lambda_1 + \lambda_2 x_1) = \lambda_1 + \lambda_2 x_1 \), with \( \lambda_1 + \lambda_2 x_1 > 0 \). Then, \( \mu = 1 \) and \( QC(\mathcal{F})|\mathcal{C}(\mathcal{F}) = 1 \).

Therefore, from the p-consistency of the family \( \mathcal{F} \) it follows that \( QC(\mathcal{F})|\mathcal{C}(\mathcal{F}) = 1 \).

The next theorem shows that the p-entailment of a conditional event \( E_3|H_3 \) from a p-consistent family \( \{E_1|H_1, E_2|H_2\} \) is equivalent to the iterated conditional \( (E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) \) being equal to 1.

**Theorem 8.** Let three conditional events \( E_1|H_1, E_2|H_2, \) and \( E_3|H_3 \) be given, where \( \{E_1|H_1, E_2|H_2\} \) is p-consistent. Then, \( \{E_1|H_1, E_2|H_2\} \) p-entails \( E_3|H_3 \) if and only if \( (E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = 1 \).

**Proof.** (\( \Rightarrow \)). We observe that by p-consistency \( E_1|H_1, E_2|H_2 \neq \emptyset \) and then \( (E_1|H_1) \wedge (E_2|H_2) \neq \emptyset \). By Theorem \( 8 \) \( \{E_1|H_1, E_2|H_2\} \) p-entails \( E_3|H_3 \) if and only if it holds that \( QC(S) \subseteq E_3|H_3 \) for some \( \emptyset \neq S \subseteq \{E_1|H_1, E_2|H_2\} \), or \( H_3 \subseteq E_3 \). We observe that, when \( H_3 \not\subseteq E_3 \), it holds that \( S = \{E_1|H_1\} \), or \( S = \{E_2|H_2\} \), or \( S = \{E_1|H_1, E_2|H_2\} \). We show that the iterated conditional may be represented as
\[
(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = (E_1|H_1) \wedge (E_2|H_2) + \mu(1 - (E_1|H_1) \wedge (E_2|H_2)).
\]

where \( \mu = P[(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))] \).

We distinguish the following four cases:

(i) \( H_3 \subseteq E_3 \);

(ii) \( H_3 \not\subseteq E_3 \) and \( E_1|H_1 \subseteq E_3|H_3 \);

(iii) \( H_3 \not\subseteq E_3 \) and \( E_2|H_2 \subseteq E_3|H_3 \);

(iv) \( H_3 \not\subseteq E_3 \) and \( QC(E_1|H_1, E_2|H_2) \subseteq E_3|H_3 \).
Case (i). If $H_3 \subseteq E_3$, then $E_3|H_3 = P(E_3|H_3) = 1$. We set $P(E_i|H_i) = x_i$, $P[E_i|H_i] \wedge (E_j|H_j)] = x_{ij}$ and we recall that

$$\max\{x_i + x_j - 1, 0\} \leq x_{ij} \leq \min\{x_i, x_j\}.$$  

Then, as $x_3 = 1$, we obtain $x_{13} = x_1, x_{23} = x_2$; it follows that for the random vector $((E_1|H_1) \wedge (E_2|H_2), (E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3))$ the possible values are

$$((1, 1), (0, 0), (x_1, x_1), (x_2, x_2), (x_{12}, x_{12}), (x_{12}, x_{123}),$$

where $x_{123} = P[(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)] = \mu$. As we can see, conditionally on $H_1 \lor H_2 \lor H_3$ being true, $(E_1|H_1) \wedge (E_2|H_2)$ and $(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)$ coincide; then, by coherence, $x_{12} = x_{123}$, so that $(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)$ and $(E_1|H_1) \wedge (E_2|H_2)$ coincide. Then, (15) is satisfied.

Case (ii). As $E_1|H_1 \subseteq E_3|H_3$, by Proposition 1 it holds that $E_1|H_1 \wedge E_3|H_3 = E_1|H_1$ and $(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3) = (E_1|H_1) \wedge (E_2|H_2)$. Then, (15) is satisfied.

Case (iii). As $E_2|H_2 \subseteq E_3|H_3$, by Proposition 1 it holds that $E_2|H_2 \wedge E_3|H_3 = E_2|H_2$ and $(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3) = (E_1|H_1) \wedge (E_2|H_2)$. Then, (15) is satisfied.

Case (iv). By taking into account that $QC[E_1|H_1, E_2|H_2] \subseteq E_3|H_3$, the set of possible values of the random vector

$$(E_1|H_1) \wedge (E_2|H_2), QC(E_1|H_1, E_2|H_2), (E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3),$$

as shown in Table 1 is

$$\{(1, 1, 1), (0, 0, 0), (x_1, 1, x_1), (x_2, 1, x_2), (x_{12}, \nu_{12}, x_{12}), (x_{12}, \nu_{12}, x_{123})\},$$

where $x_1 = P(E_1|H_1)$, $x_2 = P(E_2|H_2)$, $x_{12} = P[(E_1|H_1) \wedge (E_2|H_2)]$, $\nu_{12} = PQC(E_1|H_1, E_2|H_2)$, $x_{123} = P[(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)]$. As we can see, conditionally on $H_1 \lor H_2 \lor H_3$ being true (i.e., $\overline{H_1}\overline{H_2}\overline{H_3}$ being false), $(E_1|H_1) \wedge (E_2|H_2)$ and $(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)$ coincide; then, by Theorem 3 it holds that $x_{12} = x_{123}$, so that $(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3) = (E_1|H_1) \wedge (E_2|H_2)$. Then, (15) is satisfied.

Now, by using the representation (15), for the iterated conditional we obtain

$$(E_3|H_3)[((E_1|H_1) \wedge (E_2|H_2))] = \begin{cases} 1, & \text{if } E_1|H_1 E_2|H_2 \text{ is true,} \\ \mu, & \text{if } \overline{E_1|H_1} \lor \overline{E_2|H_2} \text{ is true,} \\ x_1 + \mu(1 - x_1), & \text{if } \overline{E_1|H_1} \overline{E_2|H_2} \text{ is true,} \\ x_2 + \mu(1 - x_2), & \text{if } E_1|H_1 \overline{E_2|H_2} \text{ is true,} \\ x_{12} + \mu(1 - x_{12}), & \text{if } E_1|H_1 \overline{E_2|H_2} \text{ is true.} \end{cases}$$

Moreover, by the linearity of prevision it holds that

$$\mu = P(E_3|H_3)[((E_1|H_1) \wedge (E_2|H_2))] = x_{12} + \mu(1 - x_{12});$$
Probabilistic entailment and iterated conditionals

\[
\begin{array}{|c|c|c|c|}
\hline
C_h & (E_1|H_1) \land (E_2|H_2) & QC(E_1|H_1, E_2|H_2) & (E_1|H_1) \land (E_2|H_2) \land (E_3|H_3) \\
\hline
E_1H_1E_2H_2E_3H_3 & 1 & 1 & 1 \\
E_1H_1E_2H_2E_3H_5 & 0 & 0 & 0 \\
E_1H_1E_2H_2\overline{E}_3H_5 & 0 & 0 & 0 \\
E_1H_1E_2\overline{H}_2E_3H_5 & 0 & 0 & 0 \\
E_1H_1\overline{E}_2H_2E_3H_5 & x_2 & 1 & x_2 \\
E_1H_1E_2H_2E_3H_3 & 0 & 0 & 0 \\
E_1H_1E_2H_2\overline{H}_3H_5 & 0 & 0 & 0 \\
E_1H_1H_2E_3H_5 & 0 & 0 & 0 \\
E_1H_1E_2H_2E_3H_5 & 0 & 0 & 0 \\
E_1H_1\overline{E}_2H_2E_3H_5 & 0 & 0 & 0 \\
E_1H_1\overline{H}_2E_3H_5 & 0 & 0 & 0 \\
E_1H_1\overline{H}_2\overline{E}_3H_5 & 0 & 0 & 0 \\
E_1H_1\overline{H}_2\overline{H}_3H_5 & 0 & 0 & 0 \\
E_1H_1E_2E_3H_5 & x_1 & 1 & x_1 \\
E_1H_1E_2\overline{E}_3H_5 & 0 & 0 & 0 \\
E_1H_1\overline{E}_2\overline{E}_3H_5 & 0 & 0 & 0 \\
E_1H_1\overline{E}_2\overline{H}_3H_5 & 0 & 0 & 0 \\
\hline
\end{array}
\]

Table 1. Possible values of the random vector \(((E_1|H_1) \land (E_2|H_2), QC(E_1|H_1, E_2|H_2), (E_1|H_1) \land (E_2|H_2) \land (E_3|H_3))\), under the assumption that \(QC(E_1|H_1, E_2|H_2) \subseteq E_3|H_5\).
from which it follows that \( x_{12} = \mu x_{12} \). Then, \( (18) \) becomes

\[
(E_3|H_3)((E_1|H_1) \land (E_2|H_2)) = \begin{cases} 
1, & \text{if } E_1H_1E_2H_2 \text{ is true}, \\
x_1 + \mu(1 - x_1), & \text{if } \overline{E}_1E_2H_2 \text{ is true}, \\
x_2 + \mu(1 - x_2), & \text{if } E_1H_1\overline{E}_2 \text{ is true}, \\
\mu, & \text{if } \overline{E}_1H_2 \lor \overline{E}_1 \lor \overline{E}_2H_2 \text{ is true}.
\end{cases}
\]

(17)

In order to prove that \( (E_3|H_3)((E_1|H_1) \land (E_2|H_2)) = 1 \), as already done in the proof of Theorem \( \text{[3]} \) we distinguish the following cases: (a) \( x_{12} > 0 \); (b) \( x_{12} = x_1 = x_2 = 0 \); (c) \( x_{12} = x_1 > 0, x_2 > 0 \); (d) \( x_{12} = x_2 = 0, x_1 > 0 \); (e) \( x_{12} = x_1 = 0, x_2 > 0 \).

Case (a). As \( x_{12} > 0 \) and \( x_{12} = \mu x_{12} \), it follows that \( \mu = 1 \) and then \( x_1 + \mu(1 - x_1) = x_2 + \mu(1 - x_2) = 1 \). Therefore, \( (E_3|H_3)((E_1|H_1) \land (E_2|H_2)) = 1 \).

Case (b). As \( x_1 = x_2 = 0 \), it holds that \( x_1 + \mu(1 - x_1) = x_2 + \mu(1 - x_2) = \mu \), so that \( (E_3|H_3)((E_1|H_1) \land (E_2|H_2)) \in \{1, \mu\} \). We observe that, based on the metaphor of the betting scheme, \( \mu = \mathbb{P}[(E_3|H_3)((E_1|H_1) \land (E_2|H_2))] \) is the amount to be paid in order to receive 1, or \( \mu \), according to whether \( E_1H_1E_2H_2 \) is true, or false, respectively. Then, by discarding the case where it is received back what has been paid, coherence requires that \( \mu = 1 \). Therefore \( (E_3|H_3)((E_1|H_1) \land (E_2|H_2)) = 1 \).

Case (c). By coherence, \( \mu \) must be a linear convex combination of the values \( 1, x_1 + \mu(1 - x_1), \) and \( x_2 + \mu(1 - x_2) \), that is,

\[
\mu = \lambda_1 + \lambda_2(x_1 + \mu(1 - x_1)) + \lambda_3(x_2 + \mu(1 - x_2)),
\]

with \( \lambda_h \geq 0, h = 1, 2, 3 \), and \( \lambda_1 + \lambda_2 + \lambda_3 = 1 \). The equation \( (18) \) can be written as

\[
\mu(\lambda_1 + \lambda_2x_1 + \lambda_3x_2) = \lambda_1 + \lambda_2x_1 + \lambda_3x_2,
\]

where \( \lambda_1 + \lambda_2x_1 + \lambda_3x_2 > 0 \); then, \( \mu = x_1 + \mu(1 - x_1) = x_2 + \mu(1 - x_2) = 1 \) and \( (E_3|H_3)((E_1|H_1) \land (E_2|H_2)) = 1 \).

Case (d). As \( x_2 = 0 \), it holds that \( x_2 + \mu(1 - x_2) = \mu \), so that

\[
(E_3|H_3)((E_1|H_1) \land (E_2|H_2)) \in \{1, x_1 + \mu(1 - x_1), \mu\}.
\]

Then, by coherence, \( \mu \) must be a linear convex combination of the values \( 1, x_1 + \mu(1 - x_1) \), that is

\[
\mu = \lambda_1 + \lambda_2[1 + \mu(1 - x_1)], \quad \lambda_1 + \lambda_2 = 1, \quad \lambda_1 \geq 0, \quad \lambda_2 \geq 0.
\]

It follows that \( \mu(\lambda_1 + \lambda_2x_1) = \lambda_1 + \lambda_2x_1 \), with \( \lambda_1 + \lambda_2x_1 > 0 \). Then, \( \mu = 1 \) and \( (E_3|H_3)((E_1|H_1) \land (E_2|H_2)) = 1 \).

Case (e). As \( x_1 = 0 \), it holds that \( x_1 + \mu(1 - x_1) = \mu \), so that

\[
(E_3|H_3)((E_1|H_1) \land (E_2|H_2)) \in \{1, x_2 + \mu(1 - x_2), \mu\}.
\]

Then, by coherence, \( \mu \) must be a linear convex combination of the values \( 1, x_2 + \mu(1 - x_2) \), that is

\[
\mu = \lambda_1 + \lambda_2[2 + \mu(1 - x_2)], \quad \lambda_1 + \lambda_2 = 1, \quad \lambda_1 \geq 0, \quad \lambda_2 \geq 0.
\]
It follows that \( \mu(\lambda_1 + \lambda_2x_2) = \lambda_1 + \lambda_2x_2 \), with \( \lambda_1 + \lambda_2x_2 > 0 \). Then, \( \mu = 1 \) and 
\[(E_3|H_3)|((E_1|H_1) \land (E_2|H_2)) = 1.\]

\((\Leftarrow)\). Assume that 
\[(E_3|H_3)|((E_1|H_1) \land (E_2|H_2)) = 1,\]
so that the unique coherent prevision assessment on 
\[(E_3|H_3)|((E_1|H_1) \land (E_2|H_2))\]
is \( \mu = 1 \). From Remark 2 it holds that 
\[x_{123} = \mu x_{12} = x_{12}.\]
Moreover, \( x_{123} \leq x_3 \) (Gilio & Sanfilippo, 2017a; Equation (8)) and 
\[x_{12} \geq \max\{x_1 + x_2 - 1, 0\}\] (see Equation (4)). Then, it holds that 
\[\max\{x_1 + x_2 - 1, 0\} \leq x_{12} = x_{123} \leq x_3,\]
and, when \( x_1 = x_2 = 1 \), it follows that \( x_{12} = x_{123} = x_3 = 1 \). Therefore, 
\[\{E_1|H_1, E_2|H_2\}\] p-entails \( E_3|H_3 \).

**Remark 4.** We recall that \( \{E_1|H_1, E_2|H_2\} \) p-entails \( QC(E_1|H_1, E_2|H_2) \) (QAND rule, see, e.g., Gilio & Sanfilippo, 2011, 2013c). Then, Theorem 7 follows by applying Theorem 8 with \( E_3|H_3 = QC(E_1|H_1, E_2|H_2) \). Similar comments can be made for the inference rules examined in Section 4.

In the examples below we show that if \( \{E_1|H_1, E_2|H_2\} \) does not p-entail \( E_3|H_3 \), the iterated conditional \( (E_3|H_3)|(E_1|H_1) \land (E_2|H_2)\) does not coincide with 1.

**Example 1 (Denial of the antecedent).** We consider the rule where the premise set is \( \{\overline{A}, C|A\} \) and the conclusion is \( C \). As is well known, that Denial of the antecedent is neither logically valid in logic nor p-valid in probability logic. Indeed, by defining \( P(\overline{A}) = x, P(C|A) = y, P(\overline{C}) = z \), it holds that
\[P(\overline{C}) = z = 1 - P(C) = 1 - [P(C|A)P(A) + P(C|\overline{A})P(\overline{A})] = 1 - y(1 - x) - P(C|\overline{A})x;\]

Then, when \( x = y = 1 \), we obtain \( z = 1 - P(C|\overline{A}) \in [0, 1] \); thus, \( \{\overline{A}, C|A\} \) does not p-entail \( \overline{C} \). Then, by Theorem 8 the iterated conditional \( \overline{C}|(\overline{A} \land (C|A)) \) does not coincide with 1. Indeed, by defining \( P[\overline{C}|(\overline{A} \land (C|A))] = \mu \), it holds that
\[\overline{C}|(\overline{A} \land (C|A)) = \overline{C} \land \overline{A} \land (C|A) + \mu(1 - \overline{A} \land (C|A)) = \begin{cases} 
\mu \text{, if } AC \text{ is true,} \\
\mu \text{, if } \overline{AC} \text{ is true,} \\
\mu(1 - y) \text{, if } \overline{AC} \text{ is true,} \\
y + \mu(1 - y) \text{, if } \overline{AC} \text{ is true.}
\end{cases}\]

If \( y = 1 \), we obtain
\[\overline{C}|(\overline{A} \land (C|A)) = \begin{cases} 
\mu \text{, if } AC \text{ is true,} \\
\mu \text{, if } \overline{AC} \text{ is true,} \\
1 \text{, if } \overline{AC} \text{ is true}, \\
0 \text{, if } \overline{AC} \text{ is true,}
\end{cases}\]

with \( \mu \) being coherent, for every \( \mu \in [0, 1] \). Therefore, \( \overline{C}|(\overline{A} \land (C|A)) \neq 1 \).

**Example 2 (Affirmation of the consequent).** We consider the rule where the premise set is \( \{C, C|A\} \) and the conclusion is \( A \). Affirmation of the consequent is
neither logically valid in logic nor p-valid in probability logic. Indeed, by defining $P(C) = x$, $P(C \mid A) = y$, $P(A) = z$, and $P(C \mid \neg A) = t$, it holds that

$$P(C) = x = P(C \mid A)P(A) + P(C \mid \neg A)P(\neg A) = yz + t(1 - z).$$

Then, when $x = y = 1$, we obtain $1 = z + t - zt$, that is $z(1 - t) = (1 - t)$. Therefore, when $t < 1$, it follows that $z = 1$. In other words, by adding the negated default in \cite{Gilio et al., 2016}, it holds that

$$P(C) = 1, P(C \mid A) = 1, P(C \mid \neg A) < 1 \Rightarrow P(A) = 1.$$ 

But in general (where no assumptions are made about $P(C \mid \neg A)$), $z \in [0, 1]$; thus p-entailment of $A$ from $\{C, C \mid A\}$ does not hold. Then, by Theorem 5, the iterated conditional $A[\{C \land (C \mid A)\}]$ does not coincide with 1. Indeed, by defining $P[A[\{C \land (C \mid A)\}] = \mu$, it holds that

$$A[\{C \land (C \mid A)\}] = A \land C \land (C \mid A) + \mu(1 - C \land (C \mid A)) = \begin{cases} 1, & \text{if } AC \text{ is true,} \\ \mu(1 - y), & \text{if } \neg AC \text{ is true,} \\ \mu, & \text{if } \neg C \text{ is true.} \end{cases}$$

If $y = 1$, we obtain

$$A[\{C \land (C \mid A)\}] = \begin{cases} 1, & \text{if } AC \text{ is true,} \\ 0, & \text{if } \neg AC \text{ is true,} \\ \mu, & \text{if } \neg C \text{ is true.} \end{cases}$$

with $\mu$ being coherent, for every $\mu \in [0, 1]$. Therefore, $A[\{C \land (C \mid A)\}] \neq 1$.

As another example, we could consider Transitivity, where $\{C \mid B, B \mid A\}$ is the premise set and $C \mid A$ is the conclusion. The p-entailment does not hold, indeed the assessment $(1, 1, z)$ on $\{C \mid B, B \mid A, C \mid A\}$ is coherent for any $z \in [0, 1]$. Then, by Theorem 5, the iterated conditional $A[\{C \mid A\}] | (C \mid B) \land (B \mid A)$ does not coincide with 1. But, by adding the negated default $P(\neg A \mid (A \lor B)) < 1$ it holds that (Gilio et al. 2016, Theorem 5)

$$P(C \mid B) = 1, P(B \mid A) = 1, P(\neg A \mid (A \lor B)) < 1 \Rightarrow P(C \mid A) = 1.$$ 

6 Concluding remarks

The results of this paper are based on the notions of conjoined conditionals and iterated conditionals. These objects, introduced in recent papers by Gilio and Sanfilippo, are defined in the setting of coherence by means of suitable conditional random quantities with values in the interval $[0, 1]$. By exploiting the logical implication of Goodman and Nguyen, we have shown that $A[H]$ p-entails $\neg B[K]$ if and only if $(\neg B[K]) | (A[H]) = 1$. Moreover, we have shown that a p-consistent family $\mathcal{F} = \{E_1[H_1], E_2[H_2]\}$ p-entails a conditional event $E_3[H_3]$ if
and only if $E_3|H_3 = 1$, or $(E_3|H_3)|QC(S) = 1$ for some nonempty subset $S$ of $F$.

We have also applied our result considered the inference rules And, Cut, Cautious Monotonicity, and Or of System P and the inference rules Modus Ponens, Modus Tollens, and Bayes. We have also shown that the iterated conditional $QC(F)|C(F)$ is equal to 1 for every p-consistent family $F = \{E_1|H_1, E_2|H_2\}$. Then, we have characterized the p-entailment of $E_3|H_3$ from a p-consistent family $F$ by showing that it amounts to the condition $(E_3|H_3)|C(F) = 1$.

Finally, we examined two examples (Denial of the Antecedent and Affirmation of the Consequent) when the p-entailment of the conditional event $E_3|H_3$ from a p-consistent family $\{E_1|H_1, E_2|H_2\}$ does not hold by also showing that $(E_3|H_3)((E_1|H_1) \wedge (E_2|H_2)) \neq 1$. Concerning the Affirmation of the Consequent, we also showed that (a kind of conditional) p-entailment holds if we add a suitable negated default in the set of premises. Psychologically, this could serve as a new explanation why some people interpret Affirmation of the Consequent as a valid argument form. Indeed, this argument form plays an important rôle in abductive reasoning in philosophy of science (e.g., where conclusions about possible causes/diseases are derived from effects/symptoms). Future work is needed to explore such applications of the presented theory and to explore further formal desiderata also related to the deduction theorem.

References

Adams, E. W. (1975). *The logic of conditionals*. Dordrecht: Reidel.
Adams, E. W. (1998). *A primer of probability logic*. Stanford: CSLI.
Berti, P., Miranda, E., & Rigo, P. (2017). Basic ideas underlying conglomerability and disintegrability. *International Journal of Approximate Reasoning*, 88(Supplement C), 387 - 400.
Biazzo, V., & Gilio, A. (2000). A generalization of the fundamental theorem of de Finetti for imprecise conditional probability assessments. *International Journal of Approximate Reasoning*, 24(2-3), 251-272.
Biazzo, V., Gilio, A., Lukasiewicz, T., & Sanfilippo, G. (2002). Probabilistic logic under coherence, model-theoretic probabilistic logic, and default reasoning in System P. *Journal of Applied Non-Classical Logics*, 12(2), 189-213.
Biazzo, V., Gilio, A., Lukasiewicz, T., & Sanfilippo, G. (2005). Probabilistic logic under coherence: Complexity and algorithms. *Annals of Mathematics and Artificial Intelligence*, 45(1-2), 35-81.
Capotorti, A., Lad, F., & Sanfilippo, G. (2007). Reassessing accuracy rates of median decisions. *The American Statistician*, 61(2), 132–138.
Coletti, G., Petturiti, D., & Vantaggi, B. (2016). Conditional belief functions as lower envelopes of conditional probabilities in a finite setting. *Information Sciences*, 64–84.
Coletti, G., & Scozzafava, R. (2002). *Probabilistic logic in a coherent setting*. Dordrecht: Kluwer.
Da Silva Neves, R., Bonnefon, J.-F., & Raufaste, E. (2002). An empirical test of patterns for nonmonotonic inference. *Annals of Mathematics and Artificial Intelligence, 34*, 107-130.

de Finetti, B. (1937/1980). Foresight: Its logical laws, its subjective sources. In *Studies in subjective probability* (p. 55-118). Huntington: Krieger.

de Finetti, B. (1970/1974). *Theory of probability* (Vols. 1, 2). Chichester: John Wiley & Sons.

Douven, I. (2016). *The epistemology of indicative conditionals: Formal and empirical approaches*. Cambridge: Cambridge University Press.

Edgington, D. (1995). On conditionals. *Mind, 104*, 235-329.

Elqayam, S., & Over, D. E. (2012). Probabilities, beliefs, and dual processing: The paradigm shift in the psychology of reasoning. *Mind and Society, 11*(1), 27-40.

Evans, J. S. B. T., & Over, D. E. (2004). *If*. Oxford: Oxford University Press.

Gilio, A. (2002). Probabilistic reasoning under coherence in System P. *Annals of Mathematics and Artificial Intelligence, 34*, 5-34.

Gilio, A. (2012). Generalizing inference rules in a coherence-based probabilistic default reasoning. *International Journal of Approximate Reasoning, 53*(3), 413–434.

Gilio, A., Over, D., Pfeifer, N., & Sanfilippo, G. (2017). Centering and compound conditionals under coherence. In M. B. Ferraro et al. (Eds.), *Soft methods for data science* (Vol. 456, pp. 253–260). Cham: Springer.

Gilio, A., & Over, D. E. (2012). The psychology of inferring conditionals from disjunctions: A probabilistic study. *Journal of Mathematical Psychology, 56*, 118–131.

Gilio, A., Pfeifer, N., & Sanfilippo, G. (2016). Transitivity in coherence-based probability logic. *Journal of Applied Logic, 14*, 46–64.

Gilio, A., & Sanfilippo, G. (2010). Quasi Conjunction and p-entailment in non-monotonic reasoning. In C. Borgelt et al. (Eds.), *Combining soft computing and statistical methods in data analysis* (Vol. 77, p. 321-328). Springer-Verlag.

Gilio, A., & Sanfilippo, G. (2011). Quasi conjunction and inclusion relation in probabilistic default reasoning. In W. Liu (Ed.), *Symbolic and quantitative approaches to reasoning with uncertainty* (Vol. 6717, p. 497-508). Springer Berlin / Heidelberg.

Gilio, A., & Sanfilippo, G. (2013a). Conditional random quantities and iterated conditioning in the setting of coherence. In L. C. van der Gaag (Ed.), *Ecsqaru 2013* (Vol. 7958, pp. 218–229). Berlin, Heidelberg: Springer.

Gilio, A., & Sanfilippo, G. (2013b). Conjunction, disjunction and iterated conditioning of conditional events. In *Synergies of soft computing and statistics for intelligent data analysis* (Vol. 190, pp. 399–407). Berlin: Springer.

Gilio, A., & Sanfilippo, G. (2013c). Probabilistic entailment in the setting of coherence: The role of quasi conjunction and inclusion relation. *International Journal of Approximate Reasoning, 54*(4), 513–525. doi: 10.1016/j.ijar.2012.11.001
Gilio, A., & Sanfilippo, G. (2013d). Quasi conjunction, quasi disjunction, t-norms and t-conorms: Probabilistic aspects. *Information Sciences*, 245, 146–167.

Gilio, A., & Sanfilippo, G. (2014). Conditional random quantities and compounds of conditionals. *Studia Logica*, 102(4), 709-729. doi: 10.1007/s11225-013-9511-6

Gilio, A., & Sanfilippo, G. (2017a). Conjunction and disjunction among conditional events. In S. Benferhat, K. Tabia, & M. Ali (Eds.), *lea/aie 2017*, part ii (Vol. 10351, pp. 85-96). Cham: Springer International Publishing.

Gilio, A., & Sanfilippo, G. (2017b). *Iterated conditioning, coherence, and penalty criterion*. (Working Paper)

Goodman, I. R., & Nguyen, H. T. (1988). Conditional Objects and the Modeling of Uncertainties. In M. M. Gupta & T. Yamakawa (Eds.), *Fuzzy computing* (pp. 119–138). North-Holland.

Jackson, F. (Ed.). (1991). *Conditionals*. Oxford: Oxford University Press.

Kaufmann, S. (2009). Conditionals right and left: Probabilities for the whole family. *Journal of Philosophical Logic*, 38, 1-53.

Kraus, S., Lehmann, D., & Magidor, M. (1990). Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence*, 44, 167-207.

Lewis, D. (1976). Probabilities of conditionals and conditional probabilities. *Philosophical Review*, 85, 297-315. (Reprint with postscript in *Jackson*, 1991, 76–101); the page references are to the reprint

McGee, V. (1989). Conditional probabilities and compounds of conditionals. *Philosophical Review*, 98, 485–541.

Oaksford, M., & Chater, N. (2003). Conditional probability and the cognitive science of conditional reasoning. *Mind & Language*, 18(4), 359-379.

Oaksford, M., & Chater, N. (2007). *Bayesian rationality: The probabilistic approach to human reasoning*. Oxford: Oxford University Press.

Over, D. E. (2009). New paradigm psychology of reasoning. *Thinking and Reasoning*, 15, 431–438.

Over, D. E., & Cruz, N. (2018). Probabilistic accounts of conditional reasoning. In L. Macchi, M. Bagassi, & R. Vialem (Eds.), *International handbook of thinking and reasoning*. Hove Sussex: Psychology Press.

Pfeifer, N., & Vicig, P. (2014). The goodman–nguyen relation within imprecise probability theory. *International Journal of Approximate Reasoning*, 55(8), 1694 - 1707.

Pfeifer, N. (2013). The new psychology of reasoning: A mental probability logical perspective. *Thinking & Reasoning*, 19(3–4), 329–345.

Pfeifer, N., & Douven, I. (2014). Formal epistemology and the new paradigm psychology of reasoning. *The Review of Philosophy and Psychology*, 5(2), 199–221.

Pfeifer, N., & Kleiter, G. D. (2003). Nonmonotonicity and human probabilistic reasoning. In *Proceedings of the 6th workshop on uncertainty processing* (p. 221-234). Hejnice: September 24–27, 2003.
Pfeifer, N., & Kleiter, G. D. (2005). Coherence and nonmonotonicity in human reasoning. *Synthese, 146*(1-2), 93-109.

Pfeifer, N., & Kleiter, G. D. (2010). The conditional in mental probability logic. In M. Oaksford & N. Chater (Eds.), *Cognition and conditionals: Probability and logic in human thought* (pp. 153–173). Oxford: Oxford University Press.

Pfeifer, N., & Tulkki, L. (2017). Conditionals, counterfactuals, and rational reasoning. An experimental study on basic principles. *Minds and Machines, 27*(1), 119–165.

Politzer, G., & Baratgin, J. (2015). Deductive schemas with uncertain premises using qualitative probability expressions. *Thinking & Reasoning, 22*(1), 78–98.

Politzer, G., Over, D. E., & Baratgin, J. (2010). Betting on conditionals. *Thinking & Reasoning, 16*(3), 172–197.

Sanfilippo, G. (2012). From imprecise probability assessments to conditional probabilities with quasi additive classes of conditioning events. In *Proceedings of the twenty-eighth conference on uncertainty in artificial intelligence, UAI-2012, Catalina Island, United States, August 15–17* (pp. 736–745). Corvallis: AUAI Press.

Sanfilippo, G., Pfeifer, N., & Gilio, A. (2017). Generalized probabilistic modus ponens. In A. Antonucci, L. Cholvy, & O. Papini (Eds.), *Symbolic and quantitative approaches to reasoning with uncertainty: 14th european conference, ECSQARU 2017 Lugano, Switzerland, July 10–14, 2017* (Vol. 10369, pp. 480–490). Springer International Publishing. doi: 10.1007/978-3-319-61581-3_43

Sanfilippo, G., Pfeifer, N., Over, D. E., & Gilio, A. (2018). Probabilistic inferences from conjoined to iterated conditionals. *International Journal of Approximate Reasoning, 93*(Supplement C), 103–118.

Schurz, G. (2005). Non-monotonic reasoning from an evolution-theoretic perspective: Ontic, logical and cognitive foundations. *Synthese, 1-2*, 37-51.

Walley, P., Pelessoni, R., & Vicig, P. (2004). Direct algorithms for checking consistency and making inferences from conditional probability assessments. *Journal of Statistical Planning and Inference, 126*(1), 119-151.