INVARIANCE PRINCIPLE
AND NON-COMPACT CENTER FOLIATIONS

SYLVAIN CROVISIER, MAURICIO POLETTI

ABSTRACT. We prove a generalization of a so called “invariance principle” for partially hyperbolic diffeomorphisms: if an invariant probability measure has all its center Lyapunov exponents equal to zero then the measure admits a center disintegration that is invariant by stable and unstable holonomies. This was known for systems admitting a foliation by compact center leaves, and we extend it to a larger class which contains discretized Anosov flows.

We use our result to classify measures of maximal entropy and study physical measures for perturbations of the time-one map of Anosov flows.

1. INTRODUCTION

Hyperbolic systems are usually studied through their invariant foliations and invariant measures: we can relate them by disintegrating the measures along the leaves of the foliations. This may be extended to more general settings.

Given a measure $\mu$, Oseledets theorem associates at $\mu$-almost every point $x$ an invariant splitting $T_x M = E^1(x) \oplus \cdots \oplus E^\ell(x)$ and Lyapunov exponents $\lambda^1(x) < \lambda^2(x) < \cdots < \lambda^\ell(x)$ which describe the exponential separation rate of orbits along the direction $E^i(x)$. When $f$ is $C^r$ ($r > 1$) and $\lambda^i(x)$ is positive, Pesin theory associates an unstable manifold $W^{u,i}(x)$ tangent to $E^1(x) \oplus \cdots \oplus E^\ell(x)$, which allows to disintegrate $\mu$ along the unstable leaves as measures $\mu^i_x$. Ledrappier and Young have introduced [24] an entropy $h^i(f, \mu)$ along the leaves $W^{u,i}$. When $h^i(f, \mu) = h^{i+1}(f, \mu)$, Ledrappier and Xie have proved [25] that the disintegration $\mu^i_x$ coincides $\mu$-almost everywhere with the disintegration $\mu^{i+1}_x$; in particular it is supported on the leaf $W^{u,i+1}(x)$.

In this work we consider a different setting. The diffeomorphism $f$ is $C^1$ and partially hyperbolic: there exists an invariant splitting

$$TM = E^s \oplus E^c \oplus E^u,$$

Date: October 28, 2022.
2010 Mathematics Subject Classification. 37C40, 37D25, 37D30.
Key words. Lyapunov exponents, Partial hyperbolicity, Invariant measures.
S.C. and M.P. were partially supported by the ERC project 692925 NUHGD. M.P. was also partially supported by Instituto Serrapilheira, grant “Jangada Dinâmica: Impulsionando Sistemas Dinâmicos na Região Nordeste” and Fondation Louis D.-Institut de France (project coordinated by Marcelo Viana).
a Riemannian metric $\| \cdot \|$, continuous functions $0 < \nu < \gamma < \hat{\gamma}^{-1} < \hat{\nu}^{-1}$ with $
u$, $\hat{\nu} < 1$ such that, for any unit vectors $v^s \in E^s(x)$, $v^c \in E^c(x)$, $v^u \in E^u(x)$, 
$$
\| Df(x)v^s \| < \nu(x) < \gamma(x) < \| Df(x)v^c \| < \hat{\gamma}(x)^{-1} < \hat{\nu}(x)^{-1} < \| Df(x)v^u \|. 
$$

All three sub-bundles $E^s$, $E^c$, $E^u$ are assumed to have positive dimension.

We are aimed to describe the measure $\mu$ along $E^c$, even when some of the Lyapunov exponents along this bundle vanish.

1.1. Invariant foliations and holonomies. The stable and unstable bundles $E^s$ and $E^u$ are uniquely integrable and their integral manifolds form two transverse continuous foliations $\mathcal{W}^s$ and $\mathcal{W}^u$, whose leaves are immersed sub-manifolds of the same class of differentiability as $f$. These foliations are referred to as the strong-stable and strong-unstable foliations. They are invariant under $f$, in the sense that

$$
 f(\mathcal{W}^s(x)) = \mathcal{W}^s(f(x)) \quad \text{and} \quad f(\mathcal{W}^u(x)) = \mathcal{W}^u(f(x)),
$$

where $\mathcal{W}^s(x)$ and $\mathcal{W}^u(x)$ denote the leaves passing through $x \in M$. One says that $f$ is accessible if every $x, y \in M$ can be connected by a path which is the union of finitely many $C^1$ paths tangent to $\mathcal{W}^s$ or $\mathcal{W}^u$ leaves.

We say that $f$ is dynamically coherent if there also exist invariant foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ tangent to the bundles $E^s \oplus E^c$ and $E^c \oplus E^u$ respectively: we call these foliations center-stable and center-unstable. The intersection of these foliations defines a center foliation $\mathcal{W}^c$: for any $x \in M$, $\mathcal{W}^c(x) = \mathcal{W}^{cs}(x) \cap \mathcal{W}^{cu}(x)$. Observe that the center and unstable foliations sub-foliate the center-unstable manifolds.

The Riemannian metric induces a distance $d^c$ along the leaves of $\mathcal{W}^c$.

**Definition 1.1.** The partially hyperbolic and dynamically coherent diffeomorphism $f$ is quasi-isometric in the center if there exist $K_0 \geq 1$ and $c_0 > 0$ such that for every $x, y \in M$ satisfying $\mathcal{W}^c(x) = \mathcal{W}^c(y)$ and every $n \in \mathbb{Z}$,

$$
 K_0^{-1}d^c(x, y) - c_0 \leq d^c(f^n(x), f^n(y)) \leq K_0d^c(x, y) + c_0.
$$

This holds for instance when the center leaves are compact and form a fiber bundle. There is an important class of examples, called discretized Anosov flows \cite{5 27} which are quasi-isometric in the center, but have non compact one-dimensional center leaves; for these systems, each center-leaf is individually fixed, i.e: $f(x) \in \mathcal{W}^c(x)$ for every $x \in M$. Perturbations of the time-one map of Anosov flows are of this kind.

Given an arc $\gamma$ that connects two points $x, y$ inside a leaf of $\mathcal{W}^u$, one defines the unstable holonomy between neighborhoods of $x$ and $y$ inside $\mathcal{W}^c(x)$ and $\mathcal{W}^c(y)$. The holonomy does not extend in general to the whole center leaves; indeed for $z \in M$, the manifolds $\mathcal{W}^c(z)$ and $\mathcal{W}^u(z)$ may have several intersections. However when $f$ is quasi-isometric in the center, this does not hold for a large set of points $z$ and there exist global holonomies.
**Proposition 1.2.** Let $f$ be a partially hyperbolic, dynamically coherent, quasi-isometric in the center, $C^1$ diffeomorphism and $\mu$ be an $f$-invariant measure. Then there exists a full measure set $X \subset M$ such that for any $x, y \in X$ with $W^{cu}(x) = W^{cu}(y)$ the following holds:

For any $z \in W^c(x)$, the leaves $W^u(z), W^c(y)$ intersect at a unique point, denoted $h^u_{x,y}(z)$. The map $h^u_{x,y}: W^c(x) \to W^c(y)$ is a homeomorphism.

The map $h^u_{x,y}$ is called unstable holonomy between $W^c(x)$ and $W^c(y)$.

**1.2. Invariant measures under holonomies.** If $\mu$ is a probability measure on $M$, its Rokhlin disintegration induces a Radon measure $\mu^c_x$ along the leaf $W^c(x)$ of $\mu$-almost every point $x$, which is well defined up to a factor: when $W^c(x) = W^c(y)$, there exists $K_{x,y} > 0$ such that $\mu^c_x = K_{x,y} \mu^c_y$.

We say that the center disintegration $\{\mu^c_x\}_{x \in M}$ of $\mu$ is invariant under unstable holonomies (or $u$-invariant) if for $\mu$-almost every $x, y$ satisfying $W^{cu}(x) = W^{cu}(y)$ there exists $K_{x,y} > 0$ such that $(h^u_{x,y})^*(\mu^c_x) = K_{x,y} \mu^c_y$, where the holonomy map $h^u_{x,y}$ is uniquely defined by Proposition 1.2.

The entropy of an $f$-invariant probability measure is denoted by $h(f, \mu)$. The Ledrappier-Young entropy along $E^u$ is called unstable entropy, denoted $h^u(f, \mu)$. One denotes analogously by $h^s(f, \mu)$ the stable entropy along $E^s$.

Now we can state our main theorem.

**Theorem A.** Let $f$ be a partially hyperbolic, dynamically coherent, quasi-isometric in the center, $C^1$ diffeomorphism and let $\mu$ be an ergodic measure. If $h(f, \mu) = h^u(f, \mu)$, then the center disintegration $\{\mu^c_x\}_{x \in M}$ is $u$-invariant.

By [24] and [10], when $f$ has more regularity, we have the following result.

**Corollary 1.3.** Let $f$ be a partially hyperbolic, dynamically coherent, quasi-isometric in the center, $C^r$, $r > 1$, diffeomorphism and let $\mu$ be an ergodic measure. If all the center Lyapunov exponents of $\mu$ are non-positive, then the center disintegration $\{\mu^c_x\}_{x \in M}$ is $u$-invariant.

This kind of result is known as an “Invariance Principle”. Ledrappier [22] proved a version for the projective action of linear cocycles and invariant measures whose Lyapunov exponents coincide. This has been generalized by Avila and Viana [3] for smooth cocycles: an application of their result shows that if $f$ is partially hyperbolic with compact center leaves that form a fiber bundle, then ergodic measures whose center Lyapunov exponents are all non-positive have $u$-invariant center disintegrations $\mu^c$. More recently Tahzibi and Yang [36] have proved a version of the invariance principle whose statement involves the entropy: for partially hyperbolic diffeomorphisms which are skew-products over an Anosov diffeomorphism, an ergodic measure is $u$-invariant if and only if its unstable entropy coincides with the

---

1 We believe that this also holds for $C^1$ diffeomorphisms, compare for instance with [17].

2 It is different from the invariance principle in probability.
entropy of its projection in the base. This implies\cite{3} Theorem A when the center leaves are compact and form a fiber bundle.

There are many applications of the invariance principle, for instance: genericity of positive Lyapunov exponents \cite{37, 2, 12, 30, 28}, existence of physical measures \cite{38}, properties of the measures of maximal entropy \cite{32}, rigidity of the perturbations of time-one maps of Anosov flows \cite{4}.

The main novelty of our result is that we do not need any kind of fiber bundle structure of the center manifold, we only need the quasi-isometric property on the center. This allows to extend many results for more general partially hyperbolic maps (see the following sections to see some of them).

Mainly the invariance principle is used to establish the su-invariance of the center disintegration of the measure when the center exponents are zero and then use some additional property of the measure and of the diffeomorphism to extend the (originally defined almost everywhere) disintegration to a continuous disintegration on the support of the measure: for instance when the dynamics is fibered over a hyperbolic base, one may require that the projected measure in the base satisfies a product structure; when the diffeomorphism is conservative, one may require the accessibility of the partially hyperbolic structure.

For measures whose center Lyapunov exponents vanish and having a local product structure, Avila-Viana’s invariance principle provides a continuous extension of the center disintegrations over the support of the measure. In our setting, we obtain a similar statement: however since the holonomies along the invariant foliations are in general defined locally, we have to localize the center measures.

We first define the local unstable holonomies: there exists $\delta_0, \varepsilon_0 > 0$ such that for any $x, y \in M$ with $d(x, y) < \varepsilon_0$, the plaques $B^u_{\delta_0}(x)$ and $B^c_{\delta_0}(y)$ intersect at a unique point denoted by $h^u_{y}(x)$, where $B^*_c(x)$ is the $\varepsilon$-ball centered at $x$ inside the leaf $W^s(x)$, for $* \in \{s, c, u, cs, cu\}$.

**Definition 1.4.** Given a set $X \subseteq M$ and $\delta > 0$, a family of local center measures, $\{\nu^c_x\}_{x \in X}$, is a family of Radon measures $\nu^c_x$ supported on $B^c_{\delta}(x)$ for each $x \in X$.

It extends the center disintegrations $\{\mu^c_x\}$ of a measure $\mu$ if $\mu(X) = 1$ and if for $\mu$-a.e. point $x$ there exists $K_x > 0$ such that $\mu^c_x|_{B^c_{\delta/2}(x)} = K_x.\nu^c_x|_{B^c_{\delta/2}(x)}$.

It is $f$-invariant if there is $\varepsilon > 0$ such that, for any $x \in X$, there exists $K_x > 0$ satisfying $f_* (\nu^c_x|_{B^c_{\delta}(x)}) = K_x.\nu^c_{f(x)}|_{f(B^c_{\delta}(x))}$.

It is $u$-invariant if there is $\varepsilon > 0$ such that, for any $x, y \in X$ with $y \in B^c_{\varepsilon}(x)$, there exists $K_{x,y} > 0$ satisfying

$$h^u_{y}(x)\ast (\nu^c_x|_{B^c_{\delta}(x)}) = K_{x,y}.\nu^c_{y}|_{h^u_{y}(B^c_{\delta}(x))}.$$
We define analogously the \( s\text-invariance. \)

Let \( \mathcal{N}^{cs\times u}_{\varepsilon}(z) \) be the product neighborhood of \( z \) which is the image of \( B^{u}_{\varepsilon}(z) \times B^{cs}_{\varepsilon}(z) \) under the homeomorphism \( (x, y) \mapsto h^{u}_{\varepsilon}(y) \).

**Definition 1.5.** A probability measure \( \mu \) has local \( cs\times u\)-product structure if there exists \( \varepsilon > 0 \) such that, for any \( x \in \text{supp}(\mu) \), the measure \( \nu := \mu|_{\mathcal{N}^{cs\times u}_{\varepsilon}(z)} \) is equivalent to a product measure \( \nu^{u} \times \nu^{cs} \) with respect to the product structure on \( \mathcal{N}^{cs\times u}_{\varepsilon}(z) \). See also Section 6.

As we will see below the local product structure is satisfied by natural classes of measures (equilibrium measures \[11, 13\] and some Gibbs u-states).

The following theorem can be applied to \( C^{r} \), \( r > 1 \), diffeomorphisms and measures with vanishing center exponents and a local \( cs\times u\)-product structure: indeed by \[24\] and \[10\], vanishing center exponents for a \( C^{r}\)-diffeomorphism imply that \( h^{s}(f, \mu) = h^{u}(f, \mu) = h(f, \mu) \).

**Theorem B.** Let \( f \) be a partially hyperbolic, dynamically coherent, quasi-isometric in the center, \( C^{1} \) diffeomorphism and let \( \mu \) be an ergodic measure.

If \( h^{s}(f, \mu) = h^{u}(f, \mu) = h(f, \mu) \) and if \( \mu \) has local \( cs\times u\)-product structure, then there exists a family of local center measures \( \{\nu^{c}_{x}\}_{x \in \text{supp}(\mu)} \) on the support of \( \mu \) which is continuous, \( f \)-invariant, \( s \)-invariant, \( u \)-invariant and extends the center disintegration of \( \mu \).

Let us mention that for conservative perturbations of the time-one map of Anosov flows, Avila, Viana and Wilkinson \[4\] constructed an artificial fiber bundle with compact leaves over \( f \); although center leaves of \( f \) are non-compact, this allows them to use \[3\] (or its version \[2\] for cocycles over conservative partially hyperbolic maps) in order to establish an su-invariance of the center disintegrations of the volume when its center Lyapunov exponents vanish. In this context our result allows us to work directly with the disintegration on the the center manifolds instead to consider an artificial fiber bundle as in \[4\]; it allows us to recover more properties of the \( \mu^{c} \) disintegration for measures with some product structure that are not volume.

**1.3. Consequence (1): Measures of maximal entropy (m.m.e.).** The variational principle asserts that the topological entropy of \( f \) is equal to the supremum of the entropies of its invariant probabilities. When \( f \) is partially hyperbolic with one-dimensional center, there exists a probability measure of maximal entropy, i.e. which realizes the supremum \[39\]. When the center foliation is a fibration with compact leaves, this has been studied in \[32\]. For transitive Anosov flows, there exists a unique measure of maximal entropy; Buzzi, Fisher and Tahzibi have studied the properties of the m.m.e. for diffeomorphisms close to their time-one map. The following improves \[11\] Theorem 1.1].

**Theorem C.** If \( (\phi_{t}) \) is a transitive Anosov \( C^{r} \) flow \((2 \leq r \leq \infty)\) on a compact manifold \( M \), then there is a \( C^{1} \) open set \( \mathcal{U} \subset \text{Diff}^{r}(M) \) whose
closure contains \( \phi_1 \), such that any \( f \in \mathcal{U} \) admits exactly two ergodic measures of maximal entropy: one with positive center exponent and one with negative central exponents; in particular both measures are Bernoulli.

A natural setting for this result are the already mentioned discretized Anosov flows, (also called with flow type in [11]), i.e. partially hyperbolic diffeomorphisms (1) which are dynamically coherent, (2) whose center foliation is one-dimensional, (3) which act like a flow in the center: there exists \( L > 0 \) such that \( f(x) \in \mathcal{W}^c(x) \) and \( d^c(f(x), x) < L \) for each \( x \in M \).

Margulis has constructed [25] measures of maximal entropy for the geodesic flow of manifolds with negative curvature: they are obtained from a family of measures carried by the unstable leaves, known as a Margulis system of measures. [11] extends this construction for \( C^2 \) discretized Anosov flows whose strong-stable and strong-unstable foliations are minimal: if \( \mu \) is a m.m.e. with non-positive center exponent, then its disintegrations \( \mu^u \) along the leaves of \( \mathcal{W}^u \) coincide with a system of Margulis measures, implying that \( \mu \) has a csxu-product structure.

We refine this result giving more information on the measure when the center Lyapunov exponent \( \lambda^c(\mu) \) vanishes; this answers some of the open questions left by this paper (see [11] Questions 2 and 3).

**Theorem D.** Let \( f \) be a \( C^2 \) discretized Anosov flow such that \( \mathcal{W}^s, \mathcal{W}^u \) are minimal, and \( \mu \) be an ergodic m.m.e. satisfying \( \lambda^c(\mu) = 0 \). Then the center disintegrations \( \mu^c_x \) do not contain atoms.

Moreover, if \( f \) is accessible, then (1) the m.m.e. is unique, (2) the disintegrations \( \mu^c_x \) are absolutely continuous with respect to Lebesgue and (3) \( f \) is the time-one map of a continuous flow (as regular as \( f \) along \( \mathcal{W}^c \)-leaves).

The next result shows the continuity of the m.m.e. It is analogous to [36, Theorem B] (when the center leaves are compact).

**Corollary 1.6.** Let \( f \) be \( C^2 \) discretized Anosov flow, such that \( \mathcal{W}^s, \mathcal{W}^u \) are minimal, and \( (f_n) \) be a sequence of \( C^2 \) diffeomorphisms converging to \( f \) in \( \text{Diff}^1(M) \), with ergodic measures \( \mu_n \) such that \( h(f_n, \mu_n) \to h_{\text{top}}(f) \) and \( \lambda^c(\mu_n) \geq 0 \).

Then \( (\mu_n) \) converges to an ergodic m.m.e. \( \mu \) satisfying \( \lambda^c(\mu) \geq 0 \). In particular if \( f \) has two hyperbolic ergodic m.m.e., there exists \( \varepsilon > 0 \) such that any ergodic measure with entropy larger than \( h_{\text{top}}(f) - \varepsilon \) is hyperbolic (i.e. its center exponent does not vanish).

1.4. **Consequence (2): Physical measures.** An invariant measure \( \mu \) is called physical if its basin

\[
B(\mu) := \left\{ x \in M; \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} = \mu \right\}
\]

has positive Lebesgue measure.
Sinai [35], Ruelle [34] and Bowen [9] have proved that $C^r$ uniformly hyperbolic diffeomorphisms (with $r > 1$) have finitely many physical measures that describes the statistical behavior of Lebesgue almost every point: in this case the physical measures coincide with the SRB measures, i.e. with measures whose disintegrations along their unstable manifolds are absolutely continuous.

We build physical measures for some perturbations of the time-one map of transitive Anosov flows.

**Theorem E.** If $(\phi_t)$ is a transitive Anosov $C^r$ flow (1 < $r$ ≤ ∞) on a compact manifold $M$, then there is a $C^1$ open set $U \subset \text{Diff}^r(M)$ whose closure contains $\phi_1$, such that any $f \in U$ admits a unique physical measure. Its basin has full volume in $M$ and its center exponent is negative.

In the particular case where $(\phi_t)$ is the geodesic flow on a $C^{\infty}$ surface with negative curvature, Dolgopyat showed [16] that generic 1-parameter deformations of the time-1 map of the flow exhibit a unique physical measure whose basin has full volume.

For partially hyperbolic diffeomorphisms, measures having absolutely continuous disintegrations along the leaves of the strong unstable foliation have been introduced and studied in [29] and are called *Gibbs u-states*; they are natural candidates to be physical measures [14, 20]. Assuming a weak contraction or expansion in the center, one can conclude that these systems admit physical measures [1, 8], but there also exist transitive examples with no physical measures [14].

This question is better understood under a regularity condition on the cs-foliation: we say that $\mathcal{W}^{cs}$ is absolutely continuous if zero Lebesgue measure sets are preserved by holonomies between strong-unstable transversals along cs-leaves. In this case, the Gibbs u-states have a cs×u-product structure.

Viana-Yang have proved [38] that for accessible partially hyperbolic diffeomorphisms whose center foliation form a fiber bundle with compact center leaves and whose cs-foliation is absolutely continuous, then every Gibbs u-state is a physical measures; moreover there exist at most finitely many ergodic ones. The next result extends some results of [38] to our setting and is used for proving Theorem E.

**Theorem F.** Let $f$ be a $C^r$ diffeomorphism (r > 1),

- which is partially hyperbolic and accessible,
- with quasi-isometric one-dimensional center,
- whose u-foliation is minimal,
- whose cs-foliations is absolutely continuous,

then any Gibbs u-state is a physical measure whose basin has full volume. Moreover it is unique and its center Lyapunov exponent is non-positive.
1.5. **More about systems which are quasi-isometric in the center.**

The quasi-isometric condition on the center is equivalent to require the existence of $0 < r_1 \leq r_2 \leq r_3$ such that for every $x \in M$ and $n \geq 1$

$$B^c_{r_1}(f^n(x)) \subset f^n(B^c_{r_2}(x)) \subset B^c_{r_3}(f^n(x)).$$

**Remark 1.7.** Only the first inclusion (called *non-shrinking center condition*) is used in the proof of the existence of global holonomies (Proposition 1.2), see Section 2.

As we mentioned, natural examples of such systems are (1) partially hyperbolic diffeomorphisms whose center foliation is a fibration with compact leaves, (2) $C^1$ perturbations of the time-one map of Anosov flows.

We remark that the quasi-isometric condition is also invariant under linear cocycle extension. For instance, if $f \in \text{Diff}^1(M)$ is partially hyperbolic, then any fiber-bunched $C^1$ map $A : M \to GL(d, \mathbb{R})$ induces a projective cocycle $F : M \times \mathbb{R}P^{d-1} \to M \times \mathbb{R}P^{d-1}$; this is a partially hyperbolic diffeomorphism with center manifold $W^c_F = W^c_f \times \mathbb{R}P^{d-1}$. Assuming that $f$ acts quasi-isometrically on the center, $F$ also acts quasi-isometrically on the center.

Many results based on the invariance principle discuss the generic positivity of Lyapunov exponents in the space of Hölder linear cocycles over some fixed dynamics with some hyperbolicity property, see for example [37], [2], [30]. Theorem A may extend these results to new classes of partially hyperbolic systems.

1.6. **Strategy of the proof and organization of the paper.** The idea of the proof is the following: Take some cu-square $R$ with $u \times c$-product structure, the $u$-invariance of $\mu^c_x$ is equivalent to have a product measure $\mu^u_x \times \mu^c_x$ with respect to the coordinates given by this product structure. We want to prove that for any partition $\mathcal{P}$ by vertical strips (a partition on the $u$-direction $c$-saturated inside a cu-square) the measure of the strip is equal to the measure of the $u$-disintegration on each $u$-leaf, in other words $\mu^{cu}_x(P)$ is equal to the conditional measure $\mu^u_x(P)$ for every $P \in \mathcal{P}$ and $x$ on the square $R$.

To do this we calculate the difference between the entropy of $\mathcal{P}$ and its entropy relative to the partition on unstable leaves, we call this difference the transverse entropy of $\mathcal{P}$ and denote it by $H^{tr}(\mathcal{P})$ (see Section 3). By some Jensen inequality the transverse entropy is non-negative, moreover is zero if and only $\mu^{cu}_x(P)$ is equal to the conditional measure $\mu^u_x(P)$ for every $P \in \mathcal{P}$ and $x$ in the cu-square $R$.

Let us forget the stable direction for now and let us suppose that there exists some cu-square $R$ and a partition $\mathcal{P}$ as before with positive transverse entropy. We want to construct a sequence of partitions by vertical strips $\mathcal{P}_n$ such that, for a large measure of points, $f^n(\mathcal{P}_n(x))$ contains the original cu-square $R$, and the partition $f^n\mathcal{P}_{n+1}$ refines $\mathcal{P}$.

This will imply $H^{tr}(\mathcal{P}_n) \geq nH^{tr}(\mathcal{P})$. We also check that the sequence $(\mathcal{P}_n)$ satisfies
\[\limsup_{n \to \infty} \frac{1}{n} \Pi^n(P_n) \leq h_\mu(f) - h_\nu^u(f).\] Consequently, if \(h_\mu(f) = h_\nu^u(f)\) we get a contradiction.

Let us discuss the construction of the partitions \(P_n\). As the center direction is not necessarily expanded it is generally difficult to find iterates such that \(f^n(P_n(x))\) contains the cu-square \(R\). For that reason, we actually define \(P_n\) in a set which is larger on the center and contains the original cu-square. Choosing sets which are sufficiently large along the center allows to apply the quasi-isometric property and ensure the covering property for many iterates. Here appears a technical difficulty: we a priori have a good product structure only for sets that are small in the center direction. For that reason, we have to extract measurable subset with full measure, which we have a good product structure and a large center size, as stated in Proposition 1.2: this is done in Section 2.

Another problem is that some atoms of the partitions \(P_n\) are very small in the u-direction, even after iterations: the unstable expansion under \(f^n\) may not be enough to ensure that the images of the elements of \(P_n\) cover the square. To overpass this difficulty, we need to require a property of small boundary for the unstable partition \(P\): for a large set of points \(x\), the \(f^{n-1}\) iterate of \(P_n(x)\) achieves some uniform size along the unstable direction: this is done in Section 4.

The precise construction of the partitions \(P_n\) is done in Section 5. The proof of Theorem B is in Section 6, those of Theorems C and D are contained in Section 7 and Theorem F is proved in Section 8.

1.7. Notations and general definitions.

1.7.1. Balls. As before we denote by \(d^*\) the distance along the leaves of \(W^*\), \(* \in \{u,c,s,cs,\text{cu}\}\), that are induced by the Riemannian metric.

Let us recall that for any \(x \in M\) and \(* \in \{u,c,s,cs,\text{cu}\}\), we denote by \(B^*_\varepsilon(x)\) the \(\varepsilon\)-ball centered at \(x\) inside \(W^*(x)\).

For any \(l, \delta > 0\), we define
\[
B^{cs}_{l,\delta}(x) = \bigcup \{B^c_l(y) : y \in B^s_\delta(x)\},
\]
\[
B^{cu}_{l,\delta}(x) = \bigcup \{B^c_u(y) : y \in B^s_\delta(x)\}.
\]

It will be convenient to fix a scale \(\varepsilon_0 > 0\) and to define the local plaques \(W^*_{loc}(x) := B^{cs}_{\varepsilon_0}(x)\). Choosing \(\varepsilon_0\) small ensures that for any \(x,y \in M\) the intersections \(W^s_{loc}(x) \cap W^u_{loc}(y)\) contain at most one point.

1.7.2. Disintegrations. Given a probability measure \(\mu\), its Rohklin disintegrations along the leaves of the foliation \(W^*\) are denoted by \(\{\mu^*_z\}\) (each measure is defined up to a factor and it is finite on subsets of \(W^*_z\) that are compact for the intrinsic topology).

When \(R \subset M\) is a measurable subset with positive measure and \(\{W^*_R\}\) is a measurable partition of \(R\) induced by the foliation \(W^*\), the measures \(\mu^*_R\) can be normalized: to almost every point \(z\), one associates a probability measure \(\mu^*_R(z)\) on \(W^*_R(z)\) such that:
– $\mu^*_R, z$ is constant on each set $W^*_R(z)$,
– for each measurable set $A$, the map $z \mapsto \mu^*_R, z(A)$ is measurable,
– $\int_R \mu^*_R, z(A) d\mu = \mu(A)/\mu(R)$.

1.7.3. Holonomies. Let us consider some foliations $W^*$ and two transverse sections $S_1, S_2$. A map $h^*: S_1 \to S_2$ is an holonomy map along the leaves of $W^*$ if there exists a continuous map $H: S_1 \times [0,1] \to M$ such that for every $(x,t) \in S_1 \times [0,1]$, one has $H(x,t) \in W^*(x)$ and $H(x,1) = h^*(x)$.

Holonomies along $W^*(x)$ are also called $*$-holonomies.

2. Global product structure

In this section we consider a dynamically coherent partially hyperbolic diffeomorphism that satisfies half of Definition 1.1, it does not shrink the center, i.e. there exist $K_0, c_0 > 0$ such that for every $x, y \in M$ in a same center leaf, and for every $n \geq 0$,

$$K_0^{-1} d^c(x,y) - c_0 \leq d^u(f^n(x), f^n(y)).$$

We prove Proposition 1.2 and build sets with a fibered $c \times u$-product structure (see Section 2.3).

2.1. Existence of global $c \times u$-product structure. Every $W^{cu}$ leaf is subfoliated by the transverse foliations $W^c$ and $W^u$, hence has a local product structure. In general this structure does not extend globally, see Figure 1.

We say that a set $X$ has a global $c \times u$-product structure if for any $x, y \in X$ in a same $W^{cu}$ leaf, the intersection $W^c(x) \cap W^u(y)$ contains exactly one point, which furthermore belongs to $X$. The main result of this section implies Proposition 1.2 and asserts that, for any invariant probability measure, there exists a measurable set with full measure which has a global $c \times u$-product structure.

![Figure 1. A leaf $W^{cu}$ without global product structure.](image)

**Theorem 2.1.** Let $f$ be a dynamically coherent partially hyperbolic diffeomorphism which does not shrink the center and $\mu$ be an $f$-invariant probability. Then there is a total measure set $X$, saturated by center leaves such that for any $x \in X$, and any $y \in W^{cu}(x)$, the unstable leaf of $y$ intersects the center leaf of $x$ in a unique point.
In particular, for each \( x \in X \) there exists a global continuous projection by \( u \)-holonomy \( h^u : \mathcal{W}^{cu}(x) \to \mathcal{W}^c(x) \). It induces for each \( x, y \in X \) in a same cu-leaf a homeomorphism \( h^u_{x,y} : \mathcal{W}^c(x) \to \mathcal{W}^c(y) \).

2.2. **su-sections with a product structure.** Between two unstable plaques that are close enough, there exists a well-defined center-stable holonomy (the holonomy that minimizes the distance inside center-stable leaves).

**Proposition 2.2.** For any \( x \in M \) and \( r > 0 \), there are \( \delta, \delta' > 0 \) such that for any \( y \) that is \( \delta \)-close to \( x \) there exists a unique holonomy \( h^{cs}_{x,y} : B^u_r(x) \to \mathcal{W}^u(y) \) defined by a continuous map \( H : B^u_r(x) \times [0,1] \to M \) satisfying:

- its image contains \( y \),
- for each \( z \in B^u_r(x) \), the distance \( d^{cs}(z, H(z,t)) \) is smaller than \( \delta' \).

Moreover, for any \( y, y' \) \( \delta \)-close to \( x \), the sets \( h^{cs}_{x,y}(B^u_r(x)) \) and \( h^{cs}_{x,y'}(B^u_r(x)) \) are disjoint or equal.

**Proof.** Take \( B^u_r(x) \) and for each \( \delta' > 0 \) take \( V_{\delta'}(x) = \bigcup_{y \in B^u_r(x)} B^{cs}_{\delta'}(y) \). By transversality of cs and \( u \) foliations, for \( \delta' \) sufficiently small \( V_{\delta}(x) \) is a tubular neighborhood of \( B^u_r(x) \). Hence \( B^{cs}_{\delta'}(z) \cap B^{cs}_{\delta'}(y) \neq \emptyset \) for \( z, y \in B^u_r(x) \) if and only if \( y = z \). By continuity of the \( u \) foliation and since \( \mathcal{W}^u(x) \) is homeomorphic to the space \( \mathbb{R}^d \), there exists \( \delta > 0 \) and \( r' > r \) such that if \( d(x,y) < \delta \) then \( B^{cs}_{\delta'}(y) \) intersects every \( B^{cs}_{\delta'}(z) \), with \( z \in B^u_r(x) \). Moreover (having chosen \( \delta' > 0 \) small) we can assume that \( B^{cs}_{2r'}(y) \cap B^{cs}_{\delta'}(z) \) contains only one point \( h^{cs}_{x,y}(z) \).

One then defines a continuous map \( H : B^{u}_{r'}(x) \times [0,1] \to M \) satisfying \( H(z,0) = z \) and \( H(z,1) = h^{cs}_{x,y}(z) \), by considering for each \( z \in B^{u}_{r'}(x) \) the geodesic inside \( B^{cs}_{\delta'}(z) \) that connects \( z \) and \( h^{cs}_{x,y}(z) \) (it is parametrized by \([0,1]\) with constant speed). Note that any cs-holonomy from \( B^{u}_{r'}(x) \) to \( \mathcal{W}^{u}(y) \), defined by a map \( H' : B^u_r(x) \times [0,1] \to M \) satisfying \( H'(z,t) \in B^{cs}_{\delta'}(z) \) for every \( t \in [0,1] \) coincides with \( h^{cs}_{x,y}(z) \), by uniqueness of the intersections \( B^{u}_{2r'}(y) \cap B^{u}_{\delta'}(z) \).

Let us now consider two points \( y, y' \) that are \( \delta \) close to \( x \) and satisfying \( B^{u}_{r'}(y) \cap B^{u}_{r'}(y') \neq \emptyset \). For each \( z \in B^{u}_{r'}(x) \), observe that \( h^{cs}_{x,y}(z), h^{cs}_{x,y'}(z) \in B^{u}_{2r'}(p) \cap B^{cs}_{\delta'}(z) \). This again implies that \( h^{cs}_{x,y}(z) = h^{cs}_{x,y'}(z) \). \( \square \)

The previous proposition justifies the following notation.

**Notation.** For any \( x \in M \), \( r > 0 \), let us consider \( \delta > 0 \) small and the holonomies \( h^{cs}_{x,y} \) as in the statement of Proposition 2.2. Then we denote

\[
\mathcal{W}^{cs\cdot u}_{r,\delta}(x) := \bigcup_{y \in B^u_r(x)} h^{cs}_{x,y}(B^u_r(x)),
\]

The set \( \mathcal{W}^{cs\cdot u}_{r,\delta}(x) \) has a \( cs\times u \)-product structure: it is homeomorphic to the product \( B^u_r(x) \times B^u_r(x) \) through the map \( \Psi : (z,y) \mapsto h^{cs}_{x,y}(z) \).
Definitions. To any points \( p = h^{cs}_{x,y}(z) \) and \( q = h^{cs}_{x,y'}(z') \) in \( W^{su}_{r}(x) \), one associates the local product \([p, q] := h^{cs}_{x,y}(z')\). It belongs to \( W^{su}(p) \cap W^{cs}(q)\). A subset \( Z \subset W^{su}_{r}(x) \) has a \( cs \times u \)-product structure if
\[
\forall p, q \in Z, \quad [p, q] \in Z.
\]
One sets \( W^{su}_{Z}(p) = \{ q \in Z : [p, q] = q \} \) and \( W^{cs}_{Z}(p) = \{ q \in Z : [q, p] = q \} \).

2.3. Sets with a fibered \( c \times u \)-product structure. The next key lemma does not essentially depend on the dynamics.

Lemma 2.3. Under the setting of Theorem 2.1, for any \( 0 < L^- < L^+ \), there exist \( x_0 \in \text{supp}(\mu) \), \( L \in (L^-, L^+) \), \( r > 0 \) small such that for any \( \delta > 0 \) small, the set \( Z := \{ z \in W^{su}_{r,\delta}(x_0) : \text{Card}(B^{c}_{10L}(z) \cap W^{su}_{r}(x_0)) = 1 \} \) satisfies:

(i) \( Z \) has a \( cs \times u \)-product structure.
(ii) \( B^{c}_{4L}(z) \) and \( B^{c}_{4L}(z') \) are disjoint for any \( z \neq z' \) in \( Z \).
(iii) \( \mu(\cup_{z \in Z \cap U} B^{c}_{\ell}(z)) > 0 \), for any \( \ell > 0 \) and any neighborhood \( U \) of \( x_0 \).
(iv) \( \text{Card}(B^{c}_{r}(y) \cap B^{c}_{2L}(z_2)) = 1 \), for any \( y \in W^{su}_{r}(z_1) \), \( z_1 \in Z \), \( y \in B^{c}_{4L}(z) \); this gives a homeomorphism \( h^{u}_{z_1,z_2} \) from \( B^{c}_{2L}(z_1) \) to a subset of \( B^{c}_{L+1}(z_2) \) containing \( B^{c}_{L-1}(z_2) \).
(v) \( \text{Card}(B^{c}_{r}(y) \cap B^{c}_{2L}(z_2)) = 1 \), for any \( y \in W^{su}_{r}(z_1) \), \( z_1 \in Z \), \( y \in B^{c}_{4L}(z_1) \); this gives a homeomorphism \( h^{u}_{z_1,z_2} \) from \( B^{c}_{2L}(z_1) \) to a subset of \( B^{c}_{L+1}(z_2) \) containing \( B^{c}_{L-1}(z_2) \).

![Figure 2. The set \( \bigcup_{z \in Z} B^{c}_{2L}(z) \).](attachment:image.png)

Proof. We start with preliminary constructions. We take \( x_0 \in \text{supp}(\mu) \), \( L \in (L^-, L^+) \) and \( r, \gamma > 0 \) small such that:

(a) \( W^{su}_{r}(x_0) \cap B^{c}_{10L}(x_0) = \{ x_0 \} \),
(b) \( d(W^{su}_{r}(x_0), \partial B^{c}_{10L}(z)) > \gamma \) for \( z \in W^{su}_{r}(x_0) \),
(c) \( 2 \text{diam}(W^{su}_{r}(x_0)) < \gamma \),
We have thus proved \( z \) contradicting Property (b). Consequently since it is the composition of a \( u \)-holonomy with a \( s \)-holonomy. Taking \( z \) for each \( z \in B^c_{L}(x) \), satisfies \( B^c_{L-1}(y) \subset \mathfrak{h}(B^c_{L}(x)) \subset B^c_{L+1}(y) \).

Let us define \( \mathfrak{T} := \bigcup_{y \in B_{11L}(x_0)} W^{su}_{r,r}(y) \). For each \( p \in \mathfrak{T} \), let \( W^{su}_{r,r}(p) := W^{su}_{r,r}(\pi(p)) \), \( W^{u}(p) := W^{u}_{loc}(p) \cap W^{su}_{r,r}(p) \) and \( W^{ws}(p) := W^{ws}_{loc}(p) \cap W^{su}_{r,r}(p) \).

Since \( \pi^{-1}(\pi(p)) = W^{su}_{r,r}(p) \) has a \( cs \times u \)-product structure, one can also write \( W^{u}(p) = W^{u}_{\pi^{-1}(\pi(p))}(p) \) and \( W^{ws}(p) = W^{ws}_{\pi^{-1}(\pi(p))}(p) \). Up to reducing \( r > 0 \), the set \( \mathfrak{T} \) is a tubular neighborhood of \( B^c_{11L}(x_0) \), i.e. for any \( p \in \mathfrak{T} \):

\( (e) \) there exists a unique point \( \pi(p) \in B^c_{11L}(x_0) \), such that \( p \in W^{su}_{r,r}(\pi(p)) \).

The map \( \pi : \mathfrak{T} \to B^c_{11L}(x_0) \) is continuous. There are \( \delta, \varepsilon \in (0, r) \) such that:

\( (f) \) \( B^c_{L}(z) \subset \mathfrak{T} \), for every \( z \in W^{su}_{r,r}(x_0) \),

\( (g) \) for \( * = ws \) or \( u \), and for every \( z, z' \in W^{su}_{r,r}(x_0) \) with \( z' \in W^{u}_{r,r}(z) \), one has \( W^{u}_{\delta,\delta}(y) \cap B^c_{10L+\gamma/2}(z') \neq \emptyset \), for every \( y \in B^c_{10L}(z) \),

\( (h) \) \( d^c(x, y) = \varepsilon \Rightarrow \frac{\varepsilon}{10} \leq d^c(\pi(x), \pi(y)) \leq \frac{11\varepsilon}{10} \), for every \( z \in W^{su}_{r,r}(x_0) \) and \( x, y \in B^c_{L}(z) \).

Note that for \( z \in W^{su}_{r,r}(x_0) \), the restriction of \( \pi \) to \( B^c_{L}(z) \) is locally invertible, since it is the composition of a \( u \)-holonomy with a \( s \)-holonomy. Taking \( \delta \) smaller, these properties imply for \( z \in W^{su}_{r,r}(x_0) \) and \( y \in B^c_{3L}(z) \):

\( (i) \) for any path \( \psi \subset B^c_{3L}(z) \) containing \( z \) and length \( < 3L \), \( \pi(\psi) \) is homotopic (endpoints fixed) in \( B^c_{4L}(x_0) \) to an arc of length \( < 4L \),

\( (j) \) any path \( \psi' \subset B^c_{11L}(x_0) \) containing \( \pi(y) \) and length \( < 4L \), \( \pi(\psi') \) has a continuous lift \( \psi \subset B^c_{10L}(z) \) for \( \pi \), which is homotopic (endpoints fixed) in \( B^c_{10L}(z) \) to an arc of length \( < 5L \).

We then define as in the statement of the lemma:

\[ Z := \{ z \in W^{su}_{r,r}(x_0) : Card\{B^c_{10L}(z) \cap W^{su}_{r,r}(x_0)\} = 1\}, \]

and check Items (i), (ii), (iv) and (v). In order to check Item (iii), one may need to change the point \( x_0 \) (and hence \( L, r, \gamma, \delta, \varepsilon \)), as we explain below.

**Item (i).** We first prove that \( Z \) has a \( cs \times u \)-product structure.

**Claim.** For any \( z \in Z \), \( z' \in W^{u}_{r,r}(z) \cap W^{su}_{\delta,\delta}(x_0) \) and \( * = ws \) or \( u \) we have \( W^{su}_{r,r}(x_0) \cap B^c_{10L}(z') \subset W^{su}_{r,r}(z) \).

**Proof.** Let \( z'' \in W^{su}_{r,r}(x_0) \cap B^c_{10L}(z') \). By Property (g) above, there exists a point \( z''' \in B^c_{10L+\gamma/2}(z) \cap W^{u}_{r,r}(z'') \). Since \( z \in Z \), we have either \( z''' = z \) or \( d^c(z'', z) \geq 10L \). Observe that the latter can not happen because this will imply (with (c)) that there exists \( y \in \partial B^c_{10L}(z) \) with \( d^c(z'', y) \leq \gamma/2 \), so

\[ d(W^{su}_{r,r}(x_0), \partial B^c_{10L}(z)) \leq d(W^{su}_{r,r}(x_0), y) \leq \gamma/2 + \text{diam} W^{su}_{r,r}(x_0) < \gamma, \]

contradicting Property (b). Consequently \( z''' = z \) and hence \( z \in W^{u}_{r,r}(z'') \).

We have thus proved \( z' \in W^{u}_{r,r}(z) \) as announced. □
Let us take $z, z' \in Z$ and consider $z'' = [z, z']$. For any point $y \in B_{10L}^c(z'') \cap W_{r,r}^{su}(x_0)$ the claim applied twice implies that $y \in W_T^u(z) \cap W_T^{us}(z')$, so $y = z''$. We have thus proved that $\text{Card}(B_{10L}^c(z'') \cap W_{r,r}^{su}(x_0)) = 1$ and then $z'' \in Z$, concluding the proof of Item (i).

**Item (ii).** It is a direct consequence of the definition of $Z$.

**Item (iv).** Its proof is based on the next property.

**Claim.** For every $z \in Z$, the projection $\pi : B_{3L}^c(z) \to B_{10L}^c(x_0)$ is injective.

**Proof.** Let us assume by contradiction that there exist $z \in Z$ and $y_1 \neq y_2 \in B_{3L}^c(z)$ such that $\pi(y_1) = \pi(y_2) =: y'$. There exists a geodesic path $\psi_1 \subset B_{3L}^c(z)$ from $z$ to $y_1$ whose length is smaller than $3L$. See Figure 3.

![Figure 3: Proof of Lemma 2.3 Item (iv)](image)

The path $\pi(\psi_1)$ connects $x_0$ to $y'$ and, by Property (i), is homotopic (endpoints fixed) to an arc $\psi_1'$ with length smaller than $4L$. By Property (j), it admits a lift $\tilde{\psi}_1$ containing $y_2$ which is homotopic (endpoints fixed) in $B_{10L}^c$ to an arc $\psi_2$ with length smaller than $5L$. Note that $\psi_2$ is contained in a $B_{5L}^c(z)$ and connects $y_2$ to some point in $(\pi|_{B_{10L}(z)})^{-1}(x_0)$. By definition of $Z$ this endpoint is necessarily $z$, hence $\psi_2$ actually contained in $B_{4L}^c(z)$.

We deduce that $\psi_1$ and $\tilde{\psi}_1$ have the same endpoint $z$. The homotopy between $\pi(\psi_1)$ and $\psi_1'$ can thus be lifted as an homotopy (endpoints fixed) between $\psi_1$ and $\tilde{\psi}_1$. Consequently the arcs $\psi_1, \tilde{\psi}_1$ are homotopic and have the same endpoints. This implies $y_1 = y_2$, a contradiction.

Let $z_1 \in Z$, $z_2 \in W^u_T(z_1)$ and $y \in B_{2L}^c(z_1)$. The points in $B_{2L}^u(y) \cap B_{2L}^c(z_2)$ belong to $\mathcal{F}$ (by Property (f)) and have the same projection by $\pi$ (since they are all contained in a ball $B_{2L}^u(y)$). The injectivity in the previous claim implies that $B_{2L}^u(y) \cap B_{2L}^c(z_2)$ contains at most one point and by Property (g) the intersection is non-empty. Hence $\text{Card}(B_{2L}^u(y) \cap B_{2L}^c(z_2)) = 1$. By Property (d), we have that $B_{L-1}^c(z_2) \subset h_{z_1,z_2}^u(B_L^c(z_1)) \subset B_{L+1}^c(z_2)$. Item (iv) is now proved.
By construction the points $T$ of $\mathcal{X}$ have obtained for $x_0$. Moreover one can require that:

and build a sequence of points $L$ of $\mathcal{X}$ such that one can modify slightly $\varepsilon > 0$ small enough, one can furthermore require the following additional property:

$k_1$ every $y \in \mathcal{W}^{su}_{r_1,r_1}(x_1)$ belongs to some $B^c_{L/3}(z)$ with $z \in \mathcal{W}^{su}_{r_1,r_1}(x_0)$; furthermore $B^c_{10L}(z)$ intersects $\mathcal{W}^{su}_{r_1,r_1}(x_0)$ twice.

While Item (iii) is not satisfied, one repeats inductively the constructions and build a sequence of points $x_n$ in the support of $\mu$ and numbers $\delta_n, r_n$. Moreover one can require that:

$k_n$ every $y \in \mathcal{W}^{su}_{r_n,r_n}(x_n)$ belongs to a $B^c_{L/2^n}(z)$, $z \in \mathcal{W}^{su}_{r_{n-1},r_{n-1}}(x_{n-1})$; furthermore $B^c_{10L}(z)$ intersects $\mathcal{W}^{su}_{r_{n-1},r_{n-1}}(x_{n-1})$ twice.

By construction the points $y \in \mathcal{W}^{su}_{r_n,r_n}(x_n)$ belong to some $B^c_{\frac{2^n-1}{2n-1}L}(z)$, where $z \in \mathcal{W}^{su}_{r_1,r_1}(x_0)$. Moreover by Property ($k_n$), the plaque $B^c_{10L+\frac{2^n-1}{2n-1}L}(z)$ intersects $\mathcal{W}^{su}_{r_1,r_1}(x_0)$ at least one more time than the plaque $B^c_{10L+\frac{2^n-1}{2n-1}L}(x_{n-1})$.

One deduces that $B^c_{10L+\frac{2^n-1}{2n-1}L}(x_n)$ intersects $\mathcal{W}^{su}_{r_1,r_1}(x_0)$ at least $n+1$ times.

The intersection points of a leaf $\mathcal{W}^c(z)$ with $\mathcal{W}^{su}_{r_1,r_1}(x_0)$ are separated from each other by a uniform distance $2\varepsilon > 0$ inside $\mathcal{W}^c(z)$. Since the volume of $B^c_{10L+\frac{2^n-1}{2n-1}L}(x_n)$ is uniformly bounded in $n$, the number of its intersection points with $\mathcal{W}^{su}_{r_1,r_1}(x_0)$ is bounded. This shows that the construction has to stop after some step $n$. We then replace $x_0, r, \delta, Z$ by $x_n, r_n, \delta_n$. During the construction the number $L$ has slightly changed also. Item (iii) is now satisfied, while Items (i), (ii), (iv), (v) remain unchanged. This ends the proof of Lemma 2.3.

2.4. Proof of Theorem 2.1. We first prove that for each $x \in M$ and any $y \in \mathcal{W}^{cu}(x)$, the intersection $\mathcal{W}^c(x) \cap \mathcal{W}^u(y)$ is non-empty.

Claim. For any $x \in M$, the union $\mathcal{W}^u\mathcal{W}^c(x)$ of the leaves $\mathcal{W}^u(z)$ for $z \in \mathcal{W}^c(x)$ coincides with $\mathcal{W}^{cu}(x)$.

Proof. Note that it is enough to prove that there exists $\delta > 0$ such that for any $x \in M$ and any $y \in \mathcal{W}^u\mathcal{W}^c(x)$, the $\delta$-neighborhood of $y$ in $\mathcal{W}^{cu}(x)$ is also contained in $\mathcal{W}^u\mathcal{W}^c(x)$. The uniform transversality of the foliations $\mathcal{W}^u$ and $\mathcal{W}^c$ ensures that for $\varepsilon > 0$ small enough and for any $x$, the $\varepsilon$-neighborhood of $\mathcal{W}^c(x)$ in $\mathcal{W}^{cu}(x)$ is contained in $\mathcal{W}^u\mathcal{W}^c(x)$.
Let us fix $x \in M$ and $y \in \mathcal{W}^u(x)$. Since $f$ does not shrink the center, for $n$ large enough, the preimage of the $\delta$-neighborhood of $y$ by $f^n$ in $\mathcal{W}^{cu}(x)$ is contained in the $\varepsilon$-neighborhood of $\mathcal{W}^c(f^{-n}(x))$, hence in $\mathcal{W}^u\mathcal{W}^c(f^{-n}(x))$. By invariance of the foliations, the $\delta$-neighborhood of $y$ in $\mathcal{W}^{cu}(x)$ is contained in $\mathcal{W}^u\mathcal{W}^c(x)$ as required. \hfill \Box

Let us consider the invariant set $X$ of points $x$ which satisfy the conclusion of Theorem 2.1. It is enough to prove that it has positive $\mu_0$-measure for any ergodic measure $\mu_0$. Indeed if $X$ does not have not full measure for some invariant measure $\mu$, one would find an ergodic component of $\mu$ which gives measure zero to $X$, a contradiction.

Let us fix an ergodic measure $\mu_0$ and let us assume by contradiction that there exists a full measure set $\tilde{X}$ of points $x$ such that there are two different points $y_1, y_2 \in \mathcal{W}^c(x)$ with $\mathcal{W}^u(y_1) = \mathcal{W}^u(y_2)$. Note that one can reduce the set $\tilde{X}$ and assume that some number $L^- > 0$ satisfies, for each such $x, y_1, y_2$, the inequalities $d^u(y_1, y_2) < L^-$ and $L^- > K_0(\max\{d^c(x, y_1), d^c(x, y_2)\} + c_0)$ where $K_0, c_0 > 0$ are the numbers which appears in the definition at the beginning of Section 2 (the dynamics does not shrink the center). We also set $L^+ = L^- + 1$. Lemma 2.3 gives us $r > 0$ and a set $Z$. Item (iii) of Lemma 2.3 and the ergodicity of $\mu_0$ ensure that there exists $x \in \tilde{X}$ which has arbitrarily large backward iterates $f^{-n}(x)$ which belong to plaques $B^c_L(z)$ for some $z \in Z$.

Let $y_1, y_2 \in \mathcal{W}^c(x)$ be the points associated to $x$. Since $n$ is large, one has $d^u(f^{-n}(y_1), f^{-n}(y_2)) < r$. Since $f$ does not shrink the center and $K_0(d^c(x, y_1) + c_0) < L^-$, one gets $d^c(f^{-n}(x), f^{-n}(y_1)) < L^-$, so that $f^{-n}(y_1)$ and $f^{-n}(y_2)$ belongs to a plaque $B^c_L(z)$ for some $z \in Z$. Setting $z_1 = z_2 = z$, one has found a point $y \in B^c_L(z)$ such that $\mathcal{W}^u(y)$ intersects $B^c_L(z)$. This contradicts Lemma 2.3 Item (iv). Hence $\mu_0(X) > 0$ and Theorem 2.1 is proved. \hfill \Box

2.5. An additional property. Having proved Theorem 2.1 one can improve Lemma 2.3 and obtain additional property on the $Z$:

**Lemma 2.4.** Let us consider $Z$ as in Lemma 2.3. Then, there exists an invariant full measure set $\Omega \subset M$ such that $Z$ contains any point $z' \in W^{su}(x_0)$ satisfying $B^c_{2L}(z') \cap \Omega \neq \emptyset$ and $W^u(z') \cap Z \neq \emptyset$.

**Proof.** Let $\Omega$ coincide with the full measure set $X$ given by Theorem 2.1 and consider $r, L > 0$ and a set $Z$ as in Lemma 2.3. Let us consider $z \in Z$ and $z' \in W^{su}(x_0) \cap W^u(z)$, such that $B^c_{2L}(z') \cap \Omega \neq \emptyset$. We have to prove that $z'$ belongs to $Z$. By Theorem 2.1 since $\mathcal{W}^c(z')$ meets $X$, the unstable leaf $\mathcal{W}^u(z)$ intersects $\mathcal{W}^c(z')$ in a unique point, which has to be $z'$. Let $z'' \in B^c_{10L}(z') \cap W^{su}(x_0)$. By the first claim in the proof of Lemma 2.3, $z'' \in W^{u}(z) \cap W^{c}(z')$, hence $z' = z''$. We have proved that Card($B^c_{10L}(z') \cap W^{su}(x_0)) = 1$, and $z'$ belongs to $Z$. \hfill \Box
This property will be used twice: first to check a covering property (Proposition 5.8), then to conclude the proof of Theorem A (Section 5.5).

3. TRANSVERSE ENTROPY ON A PRODUCT SPACE

In this section we temporarily abandon the dynamics and define the notion of transverse entropy. We then establish a criterion for u-invariance in section 3.3.

3.1. Transverse entropy of a partition. Let us consider two standard Borel spaces $X^c, X^u$ and a probability measure $\mu^{cu}$ on the product $X^{cu} := X^u \times X^c$. To any point $x = (x^u, x^c)$, one associates the horizontal $X^u(x) := X^c \times \{x^u\}$ and vertical $X^c(x) := \{x^c\} \times X^u$.

Rokhlin’s theorem [33] also associates horizontal and vertical disintegrations of $\mu^{cu}$, i.e. collections $\{\mu^u_x\}$ and $\{\mu^c_x\}$ of probabilities on the horizontal $X^u(x)$ and vertical $X^c(x)$ of almost every point $x$.

For any measurable set $R \subset X^{cu}$ with positive $\mu^{cu}$-measure, we define

$$\mu^{cu}_R := \frac{\mu^{cu}|_R}{\mu^{cu}(R)},$$

$$\mu^{cu}_{R,x} := \frac{\mu^u_x|_{R \cap X^u(x)}}{\mu^u_x(R \cap X^u(x))} \quad \text{for } \mu^{cu}_R \text{-almost every } x.$$

If $\mathcal{P}$ is a finite measurable partition of $X^{cu}$ we denote by $\mathcal{P}(x)$ the atom which contains $x$ and by $\mathcal{P}|_R$ the partition induced by $\mathcal{P}$ on $R$. We denote $\mathcal{P}' \prec \mathcal{P}$ when $\mathcal{P}$ is a partition finer than $\mathcal{P}'$. We then introduce the entropy of the partition $\mathcal{P}|_R$ and the entropy of $\mathcal{P}|_R$ along the horizontals:

$$H_{\mu^{cu}}(R, \mathcal{P}) = - \int_R \log \mu^{cu}_R(R \cap \mathcal{P}(x))d\mu^{cu}_R(x),$$

$$H^u_{\mu^{cu}}(R, \mathcal{P}) = - \int_R \log \mu^u_{R,x}(R \cap \mathcal{P}(x) \cap X^u(x))d\mu^{cu}_R(x).$$

Definition 3.1. The transverse entropy of the partition $\mathcal{P}|_R$ for the measure $\mu^{cu}_R$ with respect to the horizontals is:

$$H^{tr}_{\mu^{cu}}(R, \mathcal{P}) := H_{\mu^{cu}}(R, \mathcal{P}) - H^u_{\mu^{cu}}(R, \mathcal{P}).$$

When there is no ambiguity we omit the subindex $\mu^{cu}$.

3.2. Properties of the transverse entropy.

Proposition 3.2. The following properties hold:

(i) $H^{tr}(R, \mathcal{P}) \geq 0$.

(ii) $H^{tr}(R, \mathcal{P}) = 0$ if and only if for $\mu^{cu}$-almost every $x \in R$,

$$\mu^u_{R,x}(R \cap \mathcal{P}(x) \cap X^u(x)) = \mu^{cu}_R(R \cap \mathcal{P}(x)).$$

(iii) If $\mathcal{P}' \prec \mathcal{P}$ then $H^{tr}(R, \mathcal{P}') \leq H^{tr}(R, \mathcal{P})$. Moreover

$$H^{tr}(R, \mathcal{P}) = H^{tr}(R, \mathcal{P'}) + \sum_{P' \in \mathcal{P}'} \mu^{cu}_{R,P'}H^{tr}(R \cap P', \mathcal{P}).$$
(iv) If $R' \subset R$ with $X^u(x) \cap R = X^u(x) \cap R'$ for $\mu^{cu}$-almost every $x \in R'$, then

$$H^{tr}(R, \mathcal{P}) \geq \mu^{cu}_R(R')H^{tr}(R', \mathcal{P}).$$

**Proof.** Let $\alpha$ be the partition into local unstable sets $X^u(x) \cap R$ of $R$, and let $\tilde{\mu}$ be measure induced on the quotient $R/\alpha$. Then,

$$H^u(R, \mathcal{P}) = \int_{R/\alpha} \sum_{P \in \mathcal{P}} -\mu^u_x(P) \log \mu^u_x(P) d\tilde{\mu}(x).$$

Jensen inequality implies:

$$H^u(R, \mathcal{P}) \leq \sum_{P \in \mathcal{P}} \left( \int_{R/\alpha} \mu^u_x(P) d\tilde{\mu}(x) \right) \log \left( \int_{R/\alpha} \mu^u_x(P) d\tilde{\mu}(x) \right) = H(R, \mathcal{P})$$

and equality holds if and only if, for every $P \in \mathcal{P}$, $\mu^u_x(P \cap X^u(x) \cap R)$ is constant for $\mu^{cu}$-almost every $x$, proving (i) and (ii).

For (iii) observe that

$$H(R, \mathcal{P}) = H(R, \mathcal{P}') + \sum_{P' \subset \mathcal{P}'} \int_{P'} -\log \mu^{cu}_R(P') d\mu^{cu}_R.$$

As $\frac{1}{\mu^u_R(P')} \mu^{cu}_R = \mu^{cu}_{R \cap \mathcal{P}}$, we have

$$H(R, \mathcal{P}) = H(R, \mathcal{P}') + \sum_{P' \subset \mathcal{P}'} \mu^{cu}_R(P') H(R \cap P', \mathcal{P}).$$

An analogous formula is true for $H^u$, and taking the difference we get

$$H^{tr}(R, \mathcal{P}) = H^{tr}(R, \mathcal{P}') + \sum_{P' \subset \mathcal{P}'} \mu^{cu}_R(P') H^{tr}(R \cap P', \mathcal{P}).$$

For (iv) observe that $H^u(R, \mathcal{P}) = \int_{R^c} -\log \mu^u_{R,x}(\mathcal{P}(x)) d\mu^{cu}_R(x)$, and

$$H(R, \mathcal{P}) = \int_{R^c} -\log \mu^u_{R,x}(\mathcal{P}(x)) d\mu^{cu}_R(x) + \int_{R^c} -\log \mu^u_{R,x}(\mathcal{P}(x)) d\mu^{cu}_R(x).$$

If $R'$ is $u$-saturated inside $R$, we have $\mu^u_{R,x} = \mu^u_{R'^x}$. Then,

(1) $$H^u(R, \mathcal{P}) = \mu^{cu}_R(R')H^u(R', \mathcal{P}) + \mu^{cu}_R(R \backslash R')H^u(R \backslash R', \mathcal{P}).$$

Now observe that

$$H(R, \mathcal{P}) = \sum_{P \in \mathcal{P}} -\mu^{cu}_R(P) \log \mu^{cu}_R(P).$$

For any fixed $P \in \mathcal{P}$ we have

$$\mu^{cu}_R(P) = \mu^{cu}_R(R') \mu^{cu}_R(P \cap R') + \mu^{cu}_R(R \backslash R') \mu^{cu}_{R \backslash R'} (P \cap (R \backslash R')).$$
Then applying Jensen inequality to the function $\phi(x) = -x \log x$ we have

(2) $H(R, \mathcal{P}) \geq \sum_{P \in \mathcal{P}} \mu(R') \phi(\mu(R \cap R')) + \mu(R \setminus R') \phi(\mu(R \setminus R'))$

$= \mu(R') H(R', \mathcal{P}) + \mu(R \setminus R') H(R \setminus R', \mathcal{P}).$

Subtracting (1) from (2) we get

\begin{align*}
H^{tr}(R, \mathcal{P}) &\geq \mu(R') H^{tr}(R', \mathcal{P}) + \mu(R \setminus R') H^{tr}(R \setminus R', \mathcal{P}) \\
&\geq \mu(R') H^{tr}(R', \mathcal{P}).
\end{align*}

\[ \square \]

**Corollary 3.3.** Consider a sequence of measurable partitions $(\mathcal{P}_n)$ satisfying $\mathcal{P}_n \prec \mathcal{P}_{n+1}$. Then for every $n \in \mathbb{N},$

\begin{align*}
H^{tr}(R, \mathcal{P}_n) &= H^{tr}(R, \mathcal{P}_0) + \int \sum_{j=0}^{n-1} \sum_{P_j \in \mathcal{P}_j} \chi_{P_j}(x) H^{tr}(R \cap P_j, \mathcal{P}_{j+1}) d\mu(R, x).
\end{align*}

**Proof.** By Proposition 3.2 (iii)

\[ H^{tr}(R, \mathcal{P}_n) = H^{tr}(R, \mathcal{P}_{n-1}) + \sum_{\mathcal{P}_{n-1} \in \mathcal{P}_{n-1}} \mu(\mathcal{P}_{n-1}) H^{tr}(R \cap \mathcal{P}_{n-1}, \mathcal{P}_n). \]

Inductively we have

\begin{align*}
H^{tr}(R, \mathcal{P}_n) &= H^{tr}(R, \mathcal{P}_0) + \sum_{j=0}^{n-1} \sum_{P_j \in \mathcal{P}_j} \mu(\mathcal{P}_j) H^{tr}(R \cap P_j, \mathcal{P}_{j+1}),
\end{align*}

so we can write this as the announced formula. \[ \square \]

### 3.3. Criterion for $u$-invariance

In the present setting, the center disintegration $\{\mu^c\}$ is $u$-invariant if and only if $\mu^c$ is a product $\mu^u \times \mu^c$.

**Corollary 3.4.** Let us consider a sequence of measurable partitions $(\mathcal{P}_n)$ which generates $X^u$, and let us define the partitions $\mathcal{P}_n := \mathcal{P}_n \times X^c$ of $X$.

Then $\mu^c$ is a product $\mu^u \times \mu^c$ if and only if $H^{tr}(X^u, \mathcal{P}_n) = 0$ for each $n$.

**Proof.** By Rokhlin disintegration theorem we can write $\mu^c = \int_{X^u} \nu^c d\mu^c(x)$, we can identify $X^u \times \{x^c\}$ with $X^u$, now by Proposition 3.2 (ii) $\mu^c(\mathcal{P}_n(x)) = \mu^c(\mathcal{P}_n(x))$ if and only if $H^{tr}(X^u, \mathcal{P}_n) = 0$. As $\mathcal{P}_n$ generates the sigma algebra of $X^u$ then we also have $\nu^c(\mathcal{P}(A) = \mu^c(A \times X^c)$ for any measurable set $A \subset X^u$, so this implies $\nu^c$ is constant equal to $\mu^c(A) := \mu^c(A \times X^c).$ \[ \square \]

### 4. Partitions with small boundary

We here construct a special class of local partitions inside an unstable plaque of a partially hyperbolic diffeomorphism. The construction is standard and goes back to [23].
4.1. Definition of partitions with small boundary. Let us consider:
- a point \( x_0 \in M \),
- an unstable ball \( D^u := B^u_R(x_0) \) of radius \( \delta > 0 \) inside \( W^u(x_0) \),
- a finite measurable partition \( \mathcal{P}^u \) of \( D^u \) such that Lebesgue a.e. point \( x \in D^u \) belongs to the interior of \( \mathcal{P}^u(x) \) (relative to \( W^u(x_0) \)).

Let \( \partial \mathcal{P}^u \) be the boundary of \( \mathcal{P}^u \) inside \( W^u(x_0) \), and for \( L > 0 \) let
\[
\partial_L^cs \mathcal{P}^u = \bigcup_{x \in \partial \mathcal{P}^u} B^u_{L}(x).
\]
We also fix \( \lambda \in (0, 1) \) such that \( \lambda > \limsup_{n \to +\infty} \| Df^{-n} |_{E^u} \|^{1/n} \).

**Definition 4.1.** We say that \( \partial_L^cs \mathcal{P}^u \) has small measure (or that \( \mathcal{P}^u \) has small boundary when \( L \) is fixed) if there exists \( \lambda' \in (\lambda, 1) \) and \( C > 0 \) such that
\[
\mu \{ x \in M; d^u(\partial_L^cs \mathcal{P}^u, x) < \lambda^n \} \leq C \lambda'^n, \quad \forall n \geq 0.
\]
When \( \mathcal{P}^u = \{ D^u \} \) we say that \( D^u \) has small boundary.

4.2. Existence of partitions with small boundary. The partitions are obtained thanks to the next statement.

**Proposition 4.2.** For every \( L, \delta_0 > 0 \) there exists \( \delta \) arbitrarily close to \( \delta_0 \) such that \( D^u := B^u_R(x_0) \) has small boundary. Moreover, for any \( \varepsilon > 0 \), there exists a partition \( \mathcal{P}^u \) of \( D^u \) into sets with diameter smaller than \( \varepsilon \) which has small boundary.

The proof requires two lemmas. The first is proved as [11] Lemma A.1.

**Lemma 4.3.** Let \( \nu \) be a finite measure supported on an interval \([0, R]\). Then for any \( 0 < \lambda'' < \lambda' < 1 \), there is a full Lebesgue measure subset \( I \subset (0, R) \) with the following property. For every \( t \in I \), there is \( C_t > 0 \) such that:
\[
\nu([t - \lambda''^n, t + \lambda'^n]) \leq C_t \lambda'^n, \quad \forall n \geq 0.
\]

The second one asserts that cs-holonomies are Hölder continuous. The result is classical in the case of s-holonomies. In the case of cs-holonomies, we use the fact that \( f \) is quasi-isometric in the center.

**Lemma 4.4.** For any \( L > 0 \), there exist \( r, \alpha, C > 0 \) such that if \( D, D' \) are two subsets of unstable leaves with diameter smaller than \( r \) and if \( h : D \to D' \) is a cs-holonomy satisfying \( d^c(y, h(y)) < L \) for each \( y \in D \), then \( h \) is \( (\alpha, C) \)-Hölder continuous, i.e. for any \( y_1, y_2 \in D \) it satisfies:
\[
d^u(h(y_1), h(y_2)) \leq C d^u(y_1, y_2)^\alpha.
\]

**Proof.** Recall that we have fixed a small number \( \varepsilon_0 \) which measures the size of the local manifolds.

Let us consider two close points \( y_1, y_2 \) in a same leaf of \( W^u \) and their image by a cs-holonomy \( h \) satisfying \( d^c(h(y_1), h(y_2)) < L \). There exists two arcs \( \gamma_i : [0, 1] \to M \) connecting \( y_i \) to \( h(y_i) \), with length smaller than \( L \), contained in cs-leaves. These arcs are arbitrarily close if \( y_1, y_2 \) are close enough, i.e. if \( r \) is chosen small. One can thus require that \( \gamma_1(t), \gamma_2(t) \) are contained in a same local unstable leaf for each \( t \in [0, 1] \).
By forward iteration, the arcs separate in the unstable direction: there exists a first time \( N \) such that \( d^u(f^N(\gamma_1(t_0)), f^N(\gamma_2(t_0))) \geq \varepsilon_0 \) for some \( t_0 \in [0, 1] \). Note that there exists \( C_1 > 0 \) uniform such that

\[
d^u(h(y_1), h(y_2)) e^{C_1 N} \leq \varepsilon_0.
\]

Since \( f \) is quasi-isometric in the center, the diameter of the arcs \( f^N \circ \gamma_i \) is bounded by \( K_0L + \varepsilon_0 \), independently from \( N \). In particular there exists a constant \( \eta > 0 \) (which does not depend on \( y_1, y_2 \), nor \( N \)) such that \( d^u(f^N(\gamma_1(t)), f^N(\gamma_2(t))) \geq \eta \) for all \( t \in [0, 1] \). One deduces that there exists \( C_2 > 0 \) uniform such that

\[
d^u(y_1, y_2)e^{C_2 N} \geq \eta.
\]

Combining estimates (4) and (3), one gets the announced inequality with

\[\alpha = C_1/C_2 \text{ and } C = \varepsilon_0 \eta^{-\alpha}.\]

**Proof of Proposition 4.2.** Let \( \alpha > 0 \) be given by Lemma 3.4. We fix arbitrarily \( \lambda'' < \lambda' \) in \( (\lambda^\alpha, 1) \).

Let us fix \( \eta > 0 \) small and introduce the disc \( D_0^u = B_{0, +\eta}(x_0) \) which will contain our constructions. By Proposition 2.2, if \( \gamma > 0 \) is small, then the set \( \mathcal{N}_{\text{cs} \times u} := \bigcup_{x \in \mathcal{P}^u(x_0)} B_{\gamma, x}^c(x_0) \) has a cs\,-\,u\,-\,product structure: it is the image of \( D^u \times B_{\gamma}^c(x_0) \) by the homeomorphism \( \Psi : (x, y) \mapsto h_{x,y}(x) \).

**First case:** \( L > 0 \) is small so that \( \Lambda := \bigcup_{x \in D^u} B_{L, x}^c(x_0) \) is contained in \( \mathcal{N}_{\text{cs} \times u} \). Each partition \( \mathcal{P}^u \) of \( D_0^u \) can be extended to a partition of \( \mathcal{N}_{\text{cs} \times u} \) as \( \mathcal{P} = \Psi(\mathcal{P}^u \times B_{\gamma}^c(x_0)) \). We have \( \partial \mathcal{P} \supset \partial_{\text{cs}} \mathcal{P}^u \), so it is enough to prove that \( \partial \mathcal{P} \) has small measure.

Given any subset \( A \subset D_0^u \), we call *strip of base* \( A \) its cs-saturation inside \( \mathcal{N}_{\text{cs} \times u} \) of \( A \), i.e. the set \( \Psi(A \times B_{\gamma}^c(x_0)) \).

Let \( \{x_1, \ldots, x_m\} \) be a \( \frac{2}{\delta} \)-dense subset of \( D_0^u \) and \( S(x, b) \) be the strip of base \( B_b^c(x_i) \) for \( b > 0 \). For each \( x_i \) we define a finite measure \( \nu_i \) on \((0, \frac{\delta}{2})\) by:

\[\nu_i([a, b]) = \mu(S(x_i, b) \setminus S(x_i, a)).\]

By Lemma 3.3 for Lebesgue a.e. \( \varepsilon_i \in (\frac{1}{4}, \frac{\delta}{2}) \) there is \( C_i > 0 \) satisfying

\[\mu(S(x_i, \varepsilon_i + \lambda^m) \setminus S(x_i, \varepsilon_i - \lambda^m)) < C_i \lambda^n, \quad \forall n \geq 0.\]

Let \( \partial_{\text{cs}}^c B_{x}^u(\varepsilon_i)(x_i) := \bigcup_{y \in \partial B_{x}(\varepsilon_i)} B_{y}^c(y). \) By \( \alpha \)-Hölder continuity of the cs-holonomy (see Lemma 4.4) and since \( \lambda'' < \lambda' \), one gets for \( n \) large enough:

\[\{x \in M : d(\partial_{\text{cs}}^c B_{x}^u(\varepsilon_i)(x_i), x) < \lambda^n\} \subset S(x_i, \varepsilon_i - \lambda^m) \setminus S(x_i, \varepsilon_i + \lambda^m),\]

So \( \partial_{\text{cs}}^c B_{x}^u(\varepsilon_i)(x_i) \) has small measure. As there are finitely many \( i \) we can take the same \( \varepsilon' := \varepsilon_i \) for every \( i = 1, \ldots, m \), smaller than \( \frac{\delta}{2} \).

Analogously one can define a measure \( \nu_0 \) on \( (\delta_0 - \eta, \delta_0 + \eta) \) by setting \( \nu_0([a, b]) = \mu(S(x_0, b) \setminus S(x_0, a)) \). We can thus get \( \delta \) with \( |\delta - \delta_0| < \eta \) such that \( \partial_{\text{cs}} B_{x_0}^u(x_0) \) has small measure.

Let us take \( D' := B_{\delta}(x_0) \) and let \( \mathcal{P}' = \mathcal{P}'(x_1, \ldots, x_m, \varepsilon') \) be the partition of \( D' \) generated by intersecting the sets \( B_{x_i}^u(x_i), i = 1, \ldots, m \). The proof of the proposition is done in this case.
General case: L is arbitrary. We cover M by finitely many sets with a cs×u-product structure. We choose points $z_j$ and number $\delta_j > 0$ with $0 \leq j \leq \ell$ and introduce sets of the form $N_{j}^{cs\times u} = \Psi_{j}(D_{j}^{u} \times B_{y_{j}}^{cs}(z_{j}))$ which cover M, where $D_{j} = B_{\delta_{j}}^{u}(z_{j})$. We also take $D_{0}^{u} = B_{\delta_{0}+\eta}(x_{0})$ as before.

For each $x \in D_{j}^{u}$ and $0 < j \leq \ell$ we consider the geodesic arcs $\psi : [0, 1] \to B_{2L}^{cs}(x)$ with length less than $2L$, that connect x to some point $y \in D_{j}^{u}$. There are finitely many such arcs. Each of them defines some cs-holonomy $h_{j}$ from a neighborhood $\Sigma \subset D_{0}^{u}$ of $x$ to a neighborhood of $y$ in $D_{j}^{u}$. One has thus associated to the point $x$ a finite number of holonomy maps $h_{j}^{x}$, and one can assume that they are defined on the same domain $\Sigma \subset D_{0}^{u}$. Each holonomy $h_{j}^{x}$ takes its values in a disc $D_{j(k)}^{u}$ and by construction:

\[ B_{j}^{cs}(y) \subset \bigcup_{k} \Psi_{j(k)}(\{h_{j}^{x}(y)\} \times B_{y_{j}}^{cs}(z_{j(k)})), \quad \text{for each } y \in \Sigma. \]

We then consider some $\frac{\varepsilon}{4}$-dense subset $\{x_{1}, \ldots, x_{m}\}$ of $D_{0}^{u}$ whose associated domains $\Sigma_{1}, \ldots, \Sigma_{m}$ cover $D_{0}^{u}$. To each point $x_{i}$ are also associated finitely many cs-holonomies $h_{i}^{x}$: $\Sigma_{i} \to D_{j(k)}^{u}$ and denote $y_{i}^{k} := h_{i}^{x}(x_{i})$. To each of them, one considers for $b > 0$ the strip in $N_{j(k)}^{cs\times u}$ defined by:

\[ S(y_{i}^{k}, b) = \Psi_{j(k)}(h_{i}^{x}(B_{y_{i}^{k}}^{u}(x_{i}))) \times B_{y_{j}}^{cs}(z_{j(k)}), \]

and the measure $\nu_{i}^{k}$ on $(0, \frac{\varepsilon}{2})$ by

\[ \nu_{i}^{k}(a, b) = \mu(S(y_{i}^{k}, b) \setminus S(x_{i}^{k}, a)). \]

As before shows that there exists a total Lebesgue measure subset of $\varepsilon' \in (\frac{\varepsilon}{4}, \frac{\varepsilon}{2})$ such that each set $\Psi_{j(k)}(h_{i}^{x}(\partial B_{y_{i}^{k}}^{u}(x_{i}))) \times B_{y_{j}}^{cs}(z_{j(k)})$ has small measure. Analogously there exists $\delta$ with $|\delta - \delta_{0}| < \eta$ such that each set $\Psi_{j(k)}(h_{i}^{x}(\partial B_{y_{i}^{k}}^{u}(x_{0}))) \times B_{y_{j}}^{cs}(z_{j(k)})$ has small measure.

As before we take $D^{u} := B_{\delta_{0}}^{u}(x_{0})$ and let $P^{u} = P^{u}(x_{1}, \ldots, x_{m}, \varepsilon')$ be the partition of $D^{u}$ generated by intersecting the sets $B_{\varepsilon'}^{u}(x_{i})$, $i = 1, \ldots, m$. Its
atoms have diameter smaller than $\epsilon$. Moreover from \[5\], we get
\[
\partial_L^{cs} P^u(x_1, \ldots, x_k, \varepsilon') \subset \bigcup_{i,k} \Psi_{j(k)}(b_i^k(\partial B^u_{\varepsilon'}(x_i)) \times B_{\gamma^k}^{cs}(z_{j(k)}))
\]

\[
\partial_L^{cs} D^u \subset \bigcup_{i,k} \Psi_{j(k)}(b_i^k(\partial D^u) \times B_{\gamma^k}^{cs}(z_{j(k)}))
\]
hence $D^u$ and $\partial P^u$ have small boundary, concluding the proof of Proposition 4.2. $\square$

4.3. Size of local unstable manifolds. The small boundary property implies that for a large set of point $x$, many backward iterates of a local unstable manifold $B^u_\rho(x)$ do not intersect $\partial_L^{cs} P^u$.

**Proposition 4.5.** Let $L > 0$ and let $P^u$ be a partition of $D^u$ with small boundary. Then, for each $\beta > 0$ there exists $\rho = \rho(\beta) > 0$ such that the set
\[
M_\beta^n := \left\{ x \in M, \bigcup_{0 \leq j \leq n} f^j(\partial_L^{cs} P^u) \cap B^u_\rho(f^n(x)) = \emptyset \right\}
\]
satisfies $\mu(M \setminus M_\beta^n) < \beta$ for every $n \geq 0$.

**Proof.** The definition of small boundary fixes $C > 0$ and $\lambda' \in (\lambda, 1)$. Given $\beta > 0$, we choose $n_0$ such that $C \sum_{j=n_0}^{\infty} \lambda^j < \frac{\beta}{3}$ and $\rho > 0$ small such that
\[
\mu\{ y | f^{-j} B^u_\rho(y) \cap \partial_L^{cs} P^u \neq \emptyset \} < \frac{\beta}{3n_0}, \quad \forall 0 \leq j \leq n_0
\]
and
\[
f^{-j} B^u_\rho(y) \subset B^u_{\lambda^j}(f^{-j}(y)) \quad \forall 0 \leq j, \forall y \in M.
\]
For any $n \geq 0$, the complement of $M_\beta^n$ decomposes in two sets: the first
\[
\bigcup_{j=0}^{n_0} \{ x | f^{-j} B^u_\rho(f^n(x)) \cap \partial_L^{cs} P^u = \emptyset \}
\]
has measure smaller than $\beta/3$ by our choice of $\rho$; the second
\[
\bigcup_{j=n_0}^{n} \{ x | f^{-j} B^u_\rho(f^n(x)) \cap \partial_L^{cs} P^u \neq \emptyset \} \subset \bigcup_{j=n_0}^{n} \{ x | B^u_\lambda(f^{n-j}(x)) \cap \partial_L^{cs} P^u \neq \emptyset \}
\]
is contained in $\bigcup_{j=n_0}^{n} \{ x | d^n(\partial_L^{cs} P^u, f^{n-j}(x)) < \lambda^j \}$ which has $\mu$-measure less than $C \sum_{j=n_0}^{\infty} \lambda^j < \frac{\beta}{3}$ by definition 4.1. $\square$

**Remark 4.6.** The previous proposition remains true if we replace $f$ by some iterate $f^m$, since the set $M_\beta^n(f^m)$ for $f^m$ contains the set $M_\beta^{nm}$ for $f$. 
5. Transverse Entropy of a Diffeomorphism

We now prove Theorem A. Intermediate steps are stated in section 5.2.1. Note that the proof will use both inequalities in the definition of quasi-isometric center. We will iterate center-unstable plaques forwardly, and the lower bound is used in order to guarantee that the plaque does not collapse under iterations. But the upper bound is also used, so that the plaque keep bounded center geometry and can be compared to a reference center-unstable plaque, even after a large iterate.

5.1. Preliminary constructions.

5.1.1. Initial setting. In the whole section, we consider:

- a partially hyperbolic diffeomorphism $f$ quasi-isometric in the center: there is $K > 1$ such that for $l > 0$ large enough, $x \in M$, and $n \geq 0$,

$$B_{L/K}(f^n(x)) \subset f^n(B_L^c(x)) \subset B_{Kl}^c(f^n(x)),$$

- an ergodic probability $\mu$.

We consider three large center scales $\ell_1, \ell_2, L_0$ satisfying:

$$4K\ell_1 < \ell_2 < \frac{1}{4K}L_0.$$

Proposition 2.3 can be restated as follows:

**Proposition 5.1.** There exist $x_0 \in \text{supp}(\mu)$, $L > 0$ close to $L_0$, $\delta > 0$ small, and a measurable set $Z \subset W^\text{su}_{\delta,0}(x_0)$ with the following properties:

(a) The set $Z$ has a $cs\times u$-product structure.

(b) Two sets $B^c_{\delta L}(z), B^c_{\delta L}(z')$ are disjoint for any $z \neq z'$ in $Z$.

(c) For any $z \in Z$, $z' \in W^u_{\delta}(z)$, the $u$-holonomy defines a homeomorphism $h^u_{z,z'}$ from $B^c_{\delta L}(z)$ to a subset of $B^c_{\delta L+1}(z')$ containing $B^c_{\delta L-1}(z')$.

(d) For any $z \in Z$, $z' \in W^s_{\delta}(z)$, the $s$-holonomy defines a homeomorphism $h^s_{z,z'}$ from $B^c_{\delta L}(z)$ to a subset of $B^c_{\delta L+1}(z')$ containing $B^c_{\delta L-1}(z')$.

(e) $\mu(\cup_{z \in Z \cap U} B^c_{\delta L/2}) > 0$, for any neighborhood $U$ of $x_0$.

(f) There exists an invariant full measure set $\Omega$ such that for any $z \in Z$,

$$\left\{ z' \in W^u_{\delta}(z) \cap W^\text{su}_{\delta,0}(x_0) : B^c_{2\delta L}(z') \cap \Omega \neq \emptyset \right\} \subset Z.$$

**Proof.** The Proposition 2.3 provides us with a point $x_0$, some $L$ close to $L_0$, and a small number $r > 0$. Fixing $\delta > 0$ small, one also gets a measurable set $Z \subset W^\text{su}_{\delta,0}(x_0)$ which satisfies properties (a-e). The last property (f) is given by Lemma 2.3. \qed

We fix $r > 0$ small. Note that we can reduce $\delta > 0$ keeping the previous properties. One can thus furthermore require (from Proposition 1.1):

(g) For any $z \in Z$, $z' \in W^s_{\delta}(z)$, we have $B^c_{\delta L}(z') \subset B^c_{\delta L+1}(z)$.

(h) For any $z \in Z$, $z' \in W^s_{\delta}(z)$ (resp. $W^u_{\delta}(z)$) and $y \in B^c_{\delta L}(z)$, we have $d(h^s_{z,z'}(y), y) < r$ (resp. $d(h^u_{z,z'}(y), y) < r$).

(i) $D^u := B^c_{\delta}(x_0)$ has small boundary $\partial^u_{\delta L}D^u$ (as in definition 4.1).
5.1.2. The regions $R_1 \subset R_2 \subset \mathcal{N}$. We first define $\mathcal{N} \subset \bigcup_{z \in \mathbb{Z}} B_1^c(z)$, using
Proposition 5.1(a-c): a point $y \in B_1^c(z)$ belongs to $\mathcal{N}$ if for any $z' \in \mathcal{W}_Z^u(z)$
we have $h^u_{z,z'}(y) \in B_1^c(z')$. We then associate the sets:

$$
\mathcal{W}_N^c(y) := B_1^c(z) \cap \mathcal{N}, \quad \mathcal{W}_N^u(y) := \{h^u_{z,z'}(y), z' \in \mathcal{W}_Z^u(z)\},
$$

$$
\mathcal{W}_N^{cs}(y) := \bigcup_{z' \in \mathcal{W}_Z^2(z)} B_1^c(z'), \quad \mathcal{W}_N^{cu}(y) := \bigcup_{z' \in \mathcal{W}_Z^2(z)} B_1^c(z')
$$

This induces measurable partitions $\mathcal{W}_N^c, \mathcal{W}_N^u, \mathcal{W}_N^{cs}, \mathcal{W}_N^{cu}$ of $\mathcal{N}$. Note also
that $\mathcal{W}_N^u(y) = \bigcup_{y' \in \mathcal{W}_N^u(y)} \mathcal{W}_N^u(y')$ has a $c\times u$-product structure and that
$B_{L-1}(z) \subset \mathcal{W}_N^c(z) \subset B_L^c(z)$ for each $z \in \mathbb{Z}$. Since $\mathcal{W}^s$ and $\mathcal{W}^u$ are not
jointly integrable, $\mathcal{W}_N^{cs}(y)$ do not have in general a $c\times s$-product structure.

One defines $R_1$ by cutting $\mathcal{N}$ in the center at scale $\ell_1$: it is the set of points
$y \in \mathcal{N}$ such that $\mathcal{W}_N^u(y)$ is contained in $\bigcup_{z \in \mathbb{Z}} B_1^c(z)$. By Proposition 5.1(e),

$$(7) \quad \mu(R_1) > 0.$$

For any $y \in R_1$ and $* \in \{c, u, cu, cs\}$, we define the set $\mathcal{W}_{R_1}^*(y) := \mathcal{W}_N^* \cap R_1$.
The families $\mathcal{W}_{R_1}$ are measurable partitions of $R_1$.

One defines $R_2$ analogously, by cutting $\mathcal{N}$ at scale $\ell_2$ and measurable
partitions $\mathcal{W}_{R_2}^*$. The choices of $\ell_1, \ell_2, L$ imply the following.

**Lemma 5.2.** If $\delta$ is small enough, then for any $n \in \mathbb{Z}$ and any $x \in \mathcal{N}$:

- if $x \in R_1 \cap f^{-n}(R_1)$, we have $f^n(\mathcal{W}_{R_2}^c(x)) \subset \mathcal{W}_{R_1}^c(f^n(x))$,
- if $x \in R_2 \cap f^{-n}(R_2)$, we have $f^n(\mathcal{W}_{R_2}^c(x)) \subset \mathcal{W}_c(f^n(x))$,
- if $x \in R_2$, we have $f^n(\mathcal{W}_{R_2}^c(x)) \subset B_{L-1}^c(f^n(x))$.

**Proof.** Taking $\delta > 0$ small, for any $z \in \mathbb{Z}$, we have $\mathcal{W}_{R_2}^c(z) \supset B_{0.9\ell_2}(z)$.

Let us take $x \in R_1 \cap f^{-n}(R_1)$. For any $y \in \mathcal{W}_{R_2}^c(f^n(x))$, we have
$d^c(y, f^n(x)) < 2\ell_1$. The quasi-isometric property, the choice of $K$ and of
the scales $\ell_1, \ell_2$ give $d^c(f^{-n}(y), x) < 2\ell_1 K < 0.9\ell_2 - \ell_1$. Since $x \in B_1^c(z)$
for some $z \in \mathbb{Z}$, one deduces $f^{-n}(y) \in B_{0.9\ell_2}(z) \subset \mathcal{W}_{R_2}^c(x)$.

This gives the first property. The second and third ones are proved analogously. □

5.1.3. Separation of center plaques. Given $z, z' \in \mathbb{Z}$ and $n \geq 0$ we say that
$z$ does not separate from $z'$ until time $n$ if $f^n h_{z,z''}^u(y) \in \mathcal{W}_{R_2}^u(f^n(y))$ for any
$y \in \mathcal{W}_{R_2}^c(z)$ and $0 \leq j \leq n$ where $z'' = [z, z']$.

**Lemma 5.3.** If $r$ is small enough, then for each $n \geq 0$, the property

"$z$ does not separate from $z'$ until time $n$"

is an equivalence relation on $\{z \in \mathbb{Z}, f^n \mathcal{W}_{R_2}^c(z) \subset \mathcal{N}\}$. (By symmetry of
the relation, one can thus say that "$z, z'$ do not separate until time $n"$.)

**Proof.** We first claim that, by taking $r$ small enough, the separation property holds for any point $y \in \mathcal{W}_N^c(z)$ (rather than $\mathcal{W}_{R_2}^c(z)$) at return times to $\mathcal{N}$:
Claim. If $z$ does not separate from $z'$ until time $n$ and $f^n(W^s_{R_2}(z) \cup W^c_{R_2}(z')) \subset \mathcal{N}$, then $f^j h^u_{z,z'}(y) \in W^u_{loc}(f^j(y))$ for any $y \in W^s_{N}(z)$ and $0 \leq j \leq n$.

Proof. By (h), if $\bar{y} \in W^s_{R_2}(z)$, then $d(f^j(\bar{y}), f^j(h^u_{\bar{z},z'}(\bar{y}))) < r$ for $j = n$, hence for any $0 \leq j \leq n$ since $E^u$ is contracted by backward iterates. Assuming $r$ small enough, and using that $f^j(W^c_{N}(z))$ has diameter bounded by $2KL$, one deduces that $f^j(W^c_{N}(z))$ and $f^j(W^s_{N}(z))$ are close for $0 \leq j \leq n$. Hence $f^j h^u_{z,z'}(y)$ is close to $f^j(y)$ for $y \in W^s_{N}(z)$.

The relation is obviously reflexive. We prove that it is symmetric. Let us take $z, z' \in Z$ such that $f^n(W^c_{R_2}(z) \cup W^s_{R_2}(z')) \subset \mathcal{N}$, and $z$ does not separate from $z'$ until time $n$. We define $z'' := [z, z']$ and $z''' := [z', z]$. Given $y' \in W^s_{R_2}(z')$ let $y'' = h^u_{z',z''}(y')$. We have to prove that $f^j(y'') \in W^u_{loc}(f^j(y'))$ for every $0 \leq j \leq n$.

Let $y'' := h^u_{z',z''}(y')$ and $y = h^u_{z,z''}(y'')$. By (h), we have $d^u(y, y'') < r$ and $d^s(y', y'') < r$. Moreover as $f^m W^s_{R_2}(z) \cup f^m W^c_{R_2}(z') \subset \mathcal{N}$ and since $z$ does not separate from $z'$, the previous claim implies $f^j(y'') \in W^u_{loc}(f^j(y'))$ for every $0 \leq j \leq n$. We also have $d^u(f^m(y), f^m(y'')) < r$. Since $E^u$ (resp. $E^s$) is contracted by backward (resp. forward) iterates, we can conclude for any $0 \leq j \leq n$:

\begin{equation}
\text{(8)} \quad d^u(f^j(y), f^j(y'')) < r \quad \text{and} \quad d^s(f^j(y'), f^j(y'')) < r.
\end{equation}

Fix $0 \leq j \leq n$. We set $y_1 := f^j(y)$, $y_2 := f^j(y'')$, $y_3 := f^j(y')$, $y_4 := f^j(y''')$.

Claim. If $r > 0$ is small, and if the points $y_1, y_2, y_3, y_4$ satisfy $d^u(y_1, y_2) < r$, $d^s(y_2, y_3) < r$, $y_4 \in W^cs_{loc}(y_0) \cap W^u_{loc}(y_3)$ then $d^u(y_3, y_4) < 2r$.

Proof. For $r$ small, the points $y_i$ are close and belongs to a chart where the bundles $E^s, E^c, E^u$ are close to constant bundles which correspond to coordinate axes. The claim can be concluded by estimating successively the coordinates of the points $y_i$.

By the claim and \textbf{[1]}, $d^u(f^j(y'), f^j(y''')) < 2r$, hence $f^j(y''') \in W^u_{loc}(f^j(y'))$.

As this holds for all $0 \leq j \leq n$, $z'$ does not separate from $z$ in time $n$.

For the transitivity, we fix $z_1, z_2, z_3$ with $f^n(W^c_{R_2}(z_1) \cup W^s_{R_2}(z_2) \cup W^c_{R_2}(z_3))$ contained in $\mathcal{N}$, such that $z_1$ does not separate from $z_2$ until time $n$ and $z_2$ does not separate from $z_3$ until time $n$. Take $z_{i,j} = [z_i, z_j], i \neq j, 1 \leq i, j \leq 3$. Let $x_1 \in W^c_{R_2}(z_1)$ and $x_{1,i} = h^u_{z_{1,i},z_{1,i}}(x_1)$, for $i = 2, 3$.

For every $0 \leq j \leq n$ we have

\begin{equation}
\text{(9)} \quad d^u(f^j(x_1), f^j(x_{1,3})) \leq d^u(f^j(x_1), f^j(x_{1,2})) + d^u(f^j(x_{1,2}), f^j(x_{1,3})).
\end{equation}

Because $z_1$ does not separate from $z_2$ and the centers come back to $\mathcal{N}$, analogously to \textbf{[1]} we have $d^u(f^j(x_1), f^j(x_{1,2})) < r$.

Let $y_4 = f^j x_{1,2}$, $y_3 = f^j x_{1,3}$, $y_2 = f^j h^u_{z_{1,3},z_{2,3}}(x_{1,3})$, $y_1 = f^j h^u_{z_{2,3},z_{2,2}}(y_3)$ and $y_0 = f^j h^s_{z_{2,2},z_{1,2}}(y_1)$. As $z_2$ does not separate from $z_3$ until time $n$ and $f^n W^s_{R_2}(z_i) \subset \mathcal{N}$, $i = 2, 3$, the first claim implies again $d^u(y_1, y_2) < r$. 

The second claim and [1] give \( d^u(f^j(x_1), f^j(x_{1,3})) \leq 3r \), so that \( f^j(x_{1,3}) \in W^u_{loc}(f^j(x_1)) \). This concludes the transitivity. \( \square \)

5.1.4. **Pre-partitions** \( \mathcal{Q}_n \). We define for each \( n \geq 0 \) a partition \( \mathcal{Q}_n \) of \( R^2 \) saturated by \( W^c_{R^2} \) sets in the following way. Two sets \( W^c_{R^2}(z) \), \( W^c_{R^2}(z') \) are contained in the same atom if:

- either both \( f^n(W^c_{R^2}(z)) \) and \( f^n(W^c_{R^2}(z')) \) are not contained in \( N \),
- or \( f^n(W^c_{R^2}(z) \cup W^c_{R^2}(z')) \subset N \) and \( z, z' \) do not separate until time \( n \).

5.2. **A criterion for zero transverse entropy.**

5.2.1. **Integrated transverse entropy.** Consider a partition \( \mathcal{P}^u = \{ P^u \} \) of \( D^u \) with small boundary \( \partial^{cs}_{\mathcal{P}^u} \). It induces a partition \( \mathcal{P}_{R^1} \) of \( R^1 \) (also denoted by \( \mathcal{P} \)) into sets \( P \) of the form:

\[
P = \bigcup_{y \in \mathcal{P}^u} W^c_{R^1}(y).
\]

Each set \( W^c_{R^1}(z) \) has a product structure \( W^c_{R^1}(z) \times W^u_{R^1}(z) \). One introduces its transverse entropy for the measure \( \mu_{R^1,z}^{cu} \) on \( W^c_{R^1}(z) \), as in section 3:

\[
H^{tr}(R^1, \mathcal{P})(z) := H^{tr}(W^c_{R^1}(z), \mathcal{P} | W^c_{R^1}(z)).
\]

One then defines the **integrated transverse entropy** by

\[
\overline{H}^{tr}(R^1, \mathcal{P}) = \int_{R^1} H^{tr}(R^1, \mathcal{P})(z)d\mu(z).
\]

We are going to prove that the partition \( \mathcal{P} \) has zero transverse entropy.

**Proposition 5.4.** If \( h(f, \mu) = h^u(f, \mu) \), then every partition \( \mathcal{P}^u \) of \( D^u \) with small boundary induces a partition \( \mathcal{P} \) of \( R^1 \) satisfying \( \overline{H}^{tr}(R^1, \mathcal{P}) = 0 \).
5.2.2. The partitions $P^I_n$. On the set $\mathcal{N}$ we can define a partition from $P^u$ using the fibered structure in the same way as we defined $P$ on $R_1$:

$$P_N = \{ \cup_{y \in P^u} W^u_{\mathcal{N}}(y); P^u \in P^u \}.$$

Now we define a partition $\bar{Q}_n$ of $R_2$ which refines $Q_n$. For each $x \in R_2$:

- either $f^n(Q_n(x)) \not\subset \mathcal{N}$ and $\bar{Q}_n(x) = Q_n(x)$,
- or $f^n(Q_n(x)) \subset \mathcal{N}$ and $\bar{Q}_n(x) := f^{-n}(P_N f^n(x)) \cap Q_n(x)$.

We then define the partition $P^I_n := \vee_{k=0}^n \bar{Q}_k$ of $R_2$.

5.2.3. Intermediate steps. Proposition 5.4 is a consequence of the next ones.

**Proposition 5.5.** If $\overline{\text{H}^{tr}}(R_1, P) > 0$, then $\liminf_{n} \overline{\text{H}^{tr}}(R_2, P^I_n) > 0$.

**Proposition 5.6.** $\limsup_{n} \overline{\text{H}^{tr}}(R_2, P^I_n) \leq \mu(R_2)(h(f, \mu) - h^u(f, \mu))$.

They are proved in the two following sections.

5.3. Persistence of the transverse entropy. In this section we assume $\overline{\text{H}^{tr}}(R_1, P) > 0$ and prove Proposition 5.5

5.3.1. Choice of parameters. We first select some numbers.

- We fix $\eta > 0$ and a measurable subset $R'_1 \subset R_1$ which is a union of sets $W^u_{R'_1}(z)$ and which satisfies

  $$\mu(R'_1) > 0 \text{ and } \overline{\text{H}^{tr}}(R_1, P)(z) > \eta \text{ for every } z \in R'_1.$$

- We fix $0 < 3\beta < \mu(R'_1)^2$ and then $\rho > 0$ as given by Proposition 4.5.

- There is $0 < \rho' < \rho$ as follows. For any $x$ and $y \in B^u_{L+1}(x)$, there is a $cs$-holonomy map $h^{cs} : B^u_{\rho'}(x) \rightarrow W^u(y)$ (a priori not unique) satisfying $h^{cs}(x) = y$ and $h^{cs}(\zeta) \in B^u_{2L}(\zeta)$ for each $\zeta \in B^u_{\rho'}(x)$.

- We fix $m \geq 1$ and $g := f^m$ such that for every $z \in M$

  $$g(B^u_{\rho'}(z)) \supset B^u_{\rho}(g(z)).$$

- By remark 4.6 Proposition 4.5 is satisfied for $g = f^m$ and $\rho$: the sets

  $$M^u_{\beta} := \left\{ x \in M, \bigcup_{j=0}^n g^i(\partial_{2L}P^u) \cap B^u_{\rho}(g^m(x)) = \emptyset \right\}$$

  satisfy $\mu(M \setminus M^u_{\beta}) < \beta$ for every $n \geq 0$.

5.3.2. The partition $P^g_n$. For $n \geq 0$, we define the partition $P^g_n$ of $R_2$ by $P^g_n := \vee_{j=0}^n \bar{Q}_{nm}$. Note that the partition $P^g_{nm}$ defined for $f$ is thinner than $P^g_n$ for $g$. Hence for proving Proposition 5.5 it is enough to show that

$$\liminf_{n} \overline{\text{H}^{tr}}(R_2, P^g_n) > 0.$$

In the following we only work with the map $g$ and the partitions $P^g_n$ will be denoted by $P_n$ in order to keep the notations simpler.
5.3.3. Properties. We first see that the iterate $g^n(\mathcal{P}_n)$ somehow refines $\mathcal{P}_N$.

**Proposition 5.7.** If $x \in R_2 \cap g^{-n}(R_2)$, then $g^n(\mathcal{P}_n(x)) \subset \mathcal{P}_N(g^n(x))$.

**Proof.** By Lemma 5.2 since $g^n(x) \in R_2$ we have $g^n(\mathcal{W}_R^c(x)) \subset \mathcal{N}$, hence $g^n(\mathcal{Q}_{nm}(x)) \subset \mathcal{N}$. The definition of $\mathcal{Q}_{nm}$ and $\mathcal{P}_n$ conclude. $\Box$

Now we prove that when a point $x \in M_{\beta}^{n-1}$ comes back to $R_1$ by $g^n$, the image of $\mathcal{P}_{n-1}(x)$ covers $R_1$.

**Proposition 5.8.** For $\mu$-almost every point $x \in R_1 \cap M_{\beta}^{n-1} \cap g^{-n}(R_1)$, the set $\mathcal{W}_{R_1}^{\mu}(g^n(x))$ is essentially included inside the image $g^n(\mathcal{P}_{n-1}(x))$, i.e. $\mathcal{W}_{R_1}^{\mu}(g^n(x)) \setminus g^n(\mathcal{P}_{n-1}(x))$ has zero $\mu^{\mu}(x)$-measure.

**Proof.** Let us assume by contradiction that this is false and let us take $y$ such that $g^n(y) \in \mathcal{W}_{R_1}^{\mu}(g^n(x)) \setminus g^n(\mathcal{P}_{n-1}(x))$. We may assume that $x$ and $y$ belong to the full measure set $\Omega$ introduced in Proposition 5.1. (f). As the set $\mathcal{W}_{R_1}^{\mu}(g^n(x))$ has a c×u-product structure, there exists $y'$ such that $g^n(y') \in \mathcal{W}_{R_1}^{\mu}(g^n(x)) \cap \mathcal{W}_{R_1}^{\mu}(g^n(y))$. By Lemma 5.2 the set $\mathcal{W}_{R_1}^{\mu}(g^n(x)) \setminus g^n(\mathcal{P}_{n-1}(x))$ is saturated by plaques $\mathcal{W}_R^c(z)$, hence $g^n(y') \notin g^n(\mathcal{P}_{n-1}(x))$. By definition of $\mathcal{P}_{n-1}$, there exists $0 \leq k < n$ such that $\tilde{Q}_{mk}(y') \neq \tilde{Q}_{mk}(x)$. Two cases have to be considered.

**First case:** $g^k\mathcal{W}_{R_2}(x)$ and $g^k\mathcal{W}_{R_2}(y')$ are contained in $\mathcal{N}$, but in different element of $\mathcal{P}_N$. Let $z$ be the intersection point between $\mathcal{W}_{\mathcal{N}}^{cs}(g^k(x))$ and $\mathcal{W}_{\mathcal{N}}^{cs}(x_0)$. By (g) we have $z \in B_{L+1}^{cs}(g^k(x))$. We can thus consider a cs-holonomy map $\mathcal{H}^{cs}$ between $B_{L}^{cs}(g^k(x))$ and a subset of $\mathcal{W}_{\mathcal{N}}^{cs}(x_0)$ satisfying $\mathcal{H}^{cs}(\zeta) \in B_{2L}^{cs}(\zeta)$.

Since $g^k(y') \in \mathcal{W}_{R_1}^{\mu}(g^n(x))$, the choice of $g$ gives $g^{n-1}(y') \in B_{L+1}^{cs}(g(g^{n-1}(x)))$. Let us consider an arc $\gamma$ joining $g^k(x), g^k(y')$ in $B_{L+1}^{cs}(g(g^{n-1}(x)))$. It is contained in $B_{L}^{cs}(g^k(x))$, hence one can considers its image by $\mathcal{H}^{cs}$.

Since $g^k(x), g^k(y')$ are not contained in the same element of $\mathcal{W}_{L}^{\mu}(x_0)$, the image of $\gamma$ by $\mathcal{H}^{cs}$ meets the boundary of $\mathcal{P}_n$ inside $\mathcal{W}_{\mathcal{N}}^{cs}(x_0)$. We have thus proved that $g^{-j}B_{L}^{cs}(g^n(x))$ intersects $\partial^{cs}_{2L} \mathcal{P}_n$, with $j = n - 1 - k$. This is a contradiction since $x$ belongs to $M_{\beta}^{n-1}$.

**Second case:** $g^k\mathcal{W}_{R_2}(x)$ or $g^k\mathcal{W}_{R_2}(y')$ is contained in $\mathcal{N}$, but not both. Let us assume for instance that $g^k(\mathcal{W}_{R_2}(x)) \subset \mathcal{N}$ (the other situation is similar) and let $z \in Z$ be the point such that $g^k(x) \in B_{L}^{cs}(z_x)$.

The argument is similar to the first case. We consider a cs-holonomy map $\mathcal{H}^{cs}$, satisfying $\mathcal{H}^{cs}(\zeta) \in B_{2L}^{cs}(\zeta)$, between $B_{L}^{cs}(g^k(x))$ and a subset of $\mathcal{W}_{\mathcal{N}}^{cs}(x_0)$: it may be obtained by first projecting by center holonomy on $\mathcal{W}_{\mathcal{N}}^{cs}(z_x)$, and then by a local cs-holonomy on $\mathcal{W}_{\mathcal{N}}^{cs}(x_0)$. In particular any point $\zeta \in B_{L}^{cs}(g^k(x))$ whose projection $\mathcal{H}^{cs}(\zeta)$ belongs to $D^{cs} = \mathcal{W}_{\mathcal{N}}^{cs}(x_0)$ is contained in a center plaque $B_{L+1}(z)$ for some $z \in \mathcal{W}_{2L}(z_x) \cap \mathcal{W}_{\mathcal{N}}^{cs}(x_0)$. There
exists an arc $\gamma \subset B^u_{\rho}(g^k(x))$ joining $g^k(x)$ and $g^k(y')$. By construction, the image $h^\epsilon(g^k(x))$ belongs to $D^u$.

We claim that the image $h^\epsilon(g^k(y'))$ does not belong to $D^u$. If the claim does not hold, $g^k(y')$ would be contained in some set $B^u_{L+1}(z')$, with $z' \in \mathcal{W}^u_{\delta}(x_0) \cap \mathcal{W}^u_{28}(z_x)$. Note that $g^k(y)$ belongs to $B^u_{L-1}(g^k(y'))$, by Lemma 5.2. Consequently $B^u_{2L}(z')$ contains $g^k(y) \in \Omega$. By Proposition 5.1(f), this implies $z' \in Z$. The set $\mathcal{W}^c_{R_2}(y')$ is the image of $\mathcal{W}^c_{R_2}(x)$ by unstable holonomy (which is uniquely defined). Hence $g^k\mathcal{W}^c_{R_2}(y')$ is the image of $g^k\mathcal{W}^c_{R_2}(x)$ by unstable holonomy. Since $g^k\mathcal{W}^c_{R_2}(x) \subset \mathcal{N}$ and since $\mathcal{N}$ is invariant by the unstable holonomy $h^\epsilon_{\mathcal{N},z',z''}$, we also have $g^k\mathcal{W}^c_{R_2}(y') \subset \mathcal{N}$, a contradiction. The claim holds.

Since $h^\epsilon(g^k(y'))$ does not belong to $D^u$, the projection $h^\epsilon(\gamma)$ meets the boundary of $D^u$, and $g^{-j}B^u_{\rho}(g^{n-1}(x))$ intersects $\partial_{2L}D^u$, with $j = n - 1 - k$. This is a contradiction since $x$ belongs to $M^u_{\beta}$. □

The following controls the diameter of the iterates of the partitions $\mathcal{P}_n$.

**Proposition 5.9.** For any $\epsilon > 0$ there exists $r' > 0$ (large) and for every $n \geq 0$ there exists $Y_n \subset R_2$ satisfying $\mu(R_2 \setminus Y_n) < \epsilon$ such that if $x \in Y_n$ then $g^n(\mathcal{P}_n(x) \cap \mathcal{W}^u_{R_2}(x))$ has u-diameter smaller than $r'$.

The proof uses the following lemma.

**Lemma 5.10.** Let $g : M \to M$ be a measurable transformation preserving a probability measure $\mu$, and let $A$ be a positive measure subset. We define

$$A^0_N = \{ x \in A \text{ such that } g^j(x) \notin A \text{ for } n - N \leq j \leq n \}.$$  

Then for every $\epsilon > 0$ there is $N > 0$ such that $\mu(A^0_N) < \epsilon$ for every $n > N$.

**Proof.** Let us consider the ergodic decomposition of $\mu = \int \mu_e d\tilde{\mu}(e)$, where each $\mu_e$ is ergodic and gives total measure to disjoint sets $M_e$.

Let $E = \{ e \text{ such that } \mu_e(A) > 0 \}$ and $M' = \cup_{e \in E} M_e$. Since $\mu(A) > 0$, we have $\mu(M') > 0$ and $g^{-n}(A) \subset M'$ for every $n \in \mathbb{Z}$.

By ergodicity, for each $e \in E$ and for $\mu_e$ almost every $x \in M_e$, there exists a smallest $N_e(x) \geq 0$ such that $g^{N_e(x)}(x) \in A$. Then we can define $\Psi : M' \to N$ by $\Psi(x) = N_e(x)$ if $x \in M_e$.

Now take $M_N = \{ x, \Psi(x) \geq N + 1 \}$ and take $N$ sufficiently large such that $\mu(M_N) \leq \epsilon$, then $A^0_N = A \cap g^{-n+N}M_N$. □

**Proof of Proposition 5.9.** To prove the proposition, it is enough to bound uniformly the u-diameter of $g^n(\mathcal{P}_n(x) \cap \mathcal{W}^u_{R_2}(x))$ for $n$ large enough. Let $A = R_2$ and from Lemma 5.10 take $N$ large such that $\mu(A^0_N) \leq \epsilon_0$ for any $n \geq N$. Let $Y_n := A \setminus A^0_N$ so that $\mu(R_2 \setminus Y_n) < \epsilon_0$ as required.

Let us choose any $x \in Y_n$. By definition of $Y_n$ and $A^0_N$, there exists $n - N \leq j \leq n$ such that $g^j(x) \in A$. By Proposition 5.7, $g^{-j}(\mathcal{P}_j(x)) \subset \mathcal{P}_{N}(g^j(x)) \subset \mathcal{N}$; hence by (h) the u-diameter of $g^j(\mathcal{P}_j(x))$ is smaller than $r'$.
Proposition 3.2 item (iii) and the choice of $\eta$ included in $B^{(11)} H_{tr}$.

5.3.4. Proof of Proposition 5.5. The idea for proving (11) is to decompose $H^{tr}(R_2, \mathcal{P}_n)(z)$ by using corollary 3.3 and to estimate $H^{tr}(R_2 \cap \mathcal{P}_{n-1}(z), \mathcal{P}_n)(z)$ by comparing it with $H^{tr}(R_1, \mathcal{P})(g^n(z))$ when $g^n(z) \in R_1$. This is done by using inclusions (up to a zero measure set)

$$W^{cu}_{R_1}(g^n(z)) \subset g^n \mathcal{P}_{n-1}(z) \subset B^{cu}_{L,r^n}(g^n(z)).$$

We recall that $B^{cu}_{L,r^n}(g^n(z))$ has been defined in section 1.7.1.

We have fixed $\beta > 0$ in section 5.3.1. We apply Proposition 5.9 with $\varepsilon_0 := \beta$ to find $r' > 0$ and $Y_n$ for each $n \in \mathbb{N}$. By Lusin theorem, one can replace the set $R'_1$ (chosen at section 5.3.1) by a subset (with measure arbitrarily close to $\mu(R'_1)$) where $z \mapsto \mu^u_{R_1}(B_{L,r^n}(z))$ varies continuously. Note that $z \mapsto \mu^u_{R_1}(R'_1)$ is constant on each cu-plaque $W^{cu}_{R_1}(z)$. We can thus replace $R'_1$ by a subset with measure arbitrarily close to $\mu(R'_1)$ where $z \mapsto \mu^u_{R_1}(R'_1)$ varies continuously. Consequently we can find $\alpha > 0$ such that for every $z \in R'_1$,

$$\mu^u_{z}(R'_1) > \alpha \mu^u_{z}(B^{cu}_{L,r^n}(z)).$$

Since the new set $R'_1$ has been obtained by removing a small measure set of cu-plaques $W^{cu}_{R_1}(z)$, the condition $H^{tr}(R_1, \mathcal{P})(z) > \eta$ in section 5.3.1 is still satisfied.

Let $z$ in a full measure subset of $R_1 \cap M^{n-1}_\beta \cap Y_n$ such that $g^n(z) \in R'_1$. By Proposition 5.8, $W^{cu}_{R_1}(g^n(z))$ is essentially included in $g^n(\mathcal{P}_{n-1}(z))$. By Proposition 5.2 items (iv)

$$H^{tr}(g^n(\mathcal{P}_{n-1}(z) \cap W^{cu}_{R_2}(z)), g^n \mathcal{P}_n) \geq \frac{\mu^u(W^{cu}_{R_1}(g^n(z)))}{\mu^u(g^n(\mathcal{P}_{n-1}(z) \cap W^{cu}_{R_2}(z)))} H^{tr}(R_1, g^n \mathcal{P}_n)(g^n(z)).$$

Proposition 5.9 and Lemma 5.2 imply that $g^n(\mathcal{P}_{n-1}(z) \cap W^{cu}_{R_2}(z))$ is included in $B^{cu}_{L,r^n}(g^n(z))$. With the choice of $\alpha$ one thus gets

$$H^{tr}(g^n(\mathcal{P}_{n-1}(z) \cap W^{cu}_{R_2}(z)), g^n \mathcal{P}_n) > \alpha H^{tr}(R_1, g^n \mathcal{P}_n)(g^n(z)).$$

By Proposition 5.7 the restriction of $g^n \mathcal{P}_n$ to $R_1$ is finer than $\mathcal{P}$, so by Proposition 5.2 item (iii) and the choice of $\eta$ in section 5.3.1 we get

$$H^{tr}(g^n(\mathcal{P}_{n-1}(z) \cap W^{cu}_{R_2}(z)), g^n \mathcal{P}_n) > \alpha \eta.$$
Let $B_n = R_1 \cap M^{n-1}_\beta \cap Y_n$. By corollary 3.3 the $\mu$-invariance, (11), at any $x$,

$$\mathcal{H}^\nu(R_2, \mathcal{P}_n)(x) \geq \int \sum_{j=1}^{n} \sum_{P_j \in \mathcal{P}_j} \chi_{R_1}(g^j z) \chi_{B_j}(z) \mathcal{H}^\nu(P_{j-1} \mathcal{P}_j) d\mu^\nu_{R_2,z}(z) \geq \alpha \eta \int \sum_{j=1}^{n} \mu^\nu_{R_2,z}(g^{-j} R_1' \cap R_1').$$

Integrating on $R_2$ and using the measure estimates on $Y_j$ and $M^j_\beta$, we get

$$\mathcal{H}^\nu(R_2, \mathcal{P}_n) \geq \alpha \eta \left( \frac{1}{n} \sum_{j=0}^{n-1} \mu(g^{-j} A \cap A) - 2\beta \right).$$

Since $\mu$ is not necessarily ergodic for $g$ we will need the following lemma.

**Lemma 5.12.** Let $g : M \to M$ be a measurable map, $\mu$ be an invariant measure and $A \subset M$. Then $\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mu(g^{-j} A \cap A) \geq \mu(A)^2$.

**Proof.** By Von Neumann ergodic theorem

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mu(g^{-j} A \cap A) = \int \chi_A P \chi_A d\mu,$$

where $P$ is the orthogonal projection on invariant functions in $L^2(\mu)$. Then $\langle \chi_A, P \chi_A \rangle = \langle P \chi_A, P \chi_A \rangle = \langle P \chi_A, 1 \rangle \geq \langle P \chi_A, 1 \rangle^2 = \mu(A)^2$. \qed

Taking the lim inf on equation (12) and applying Lemma 5.12 we have

$$\liminf_{n \to +\infty} \frac{1}{n} \mathcal{H}^\nu(R_2, \mathcal{P}_n) \geq \alpha \eta (\mu(R_1')^2 - 2\beta) > 0.$$

This concludes the proof of Proposition 5.5. \qed

### 5.4. Upper bound on the transverse entropy

We prove Proposition 5.6.

**Lemma 5.13.** $\limsup_{n \to +\infty} \frac{1}{n} \int \mathcal{H}^\nu(R_2, \mathcal{P}_n^f) d\mu(z) \leq h(f, \mu) \mu(R_2)$.

**Proof.** We first extend the partition $\mathcal{P}_N$ to $M$:

$$\mathcal{P}_N' := \mathcal{P}_N \cup \{M \setminus N\}.$$

In order to have a partition which refines $R_2$ by sets with small diameters, we consider a finite measurable partition $\mathcal{A}$ of $M$ whose atoms have diameter much smaller than the diameter of the local unstable manifolds $W^u_{loc}(x)$ and define:

$$\tilde{\mathcal{P}} := \mathcal{P}_N' \vee \{R_2, M \setminus R_2\} \vee \mathcal{A}.$$

For each $n \geq 0$ we consider the dynamical partition

$$\tilde{\mathcal{P}}_n^f = \bigvee_{j=0}^{n-1} f^{-j} \tilde{\mathcal{P}}.$$
Lemma 5.14. \[ \liminf_{n \to +\infty} \frac{1}{n} \int_{R_2} H_{\mu_z}^c (R_2, \mathcal{P}_n^f) d\mu(z) \geq \mu(R_2) h^u(f, \mu). \]

Proof. We have that \( H_{\mu_z}^c (R_2, \mathcal{P}_n^f) = \int_{R_2} -\log \mu_z^u(\mathcal{P}_n^f(x)) d\mu_z^c(x) \).
By [24] Proposition 7.2.1 and Sections 9.2, 9.3] for \( \mu \) almost every \( x \in M \)

\[
\lim_{n \to 0} \liminf_{n \to +\infty} -\frac{1}{n} \log \mu_x^u(B^u_n(x,n)) = h^u(f,\mu),
\]

where \( B^u_n(x,n) = \cap_{j=0}^{n} f^{-j} B^u_n(f^j(x)) \). Since \( f \) expands uniformly in the unstable direction, given any \( \eta' > \eta > 0 \) there exists \( n_0 \) such that

\[
B^u_\eta(x,n) \subset B^u_{\eta'}(x,n) \subset B^u_{2\eta}(x,n-n_0).
\]

Hence

\[
\lim_{n \to +\infty} \liminf_{n \to +\infty} -\frac{1}{n} \log \mu_x^u(B^u_\eta(x,n)) = \lim_{n \to +\infty} \liminf_{n \to +\infty} -\frac{1}{n} \log \mu_x^u(B^u_{\eta'}(x,n)),
\]

\[
\lim_{n \to +\infty} \frac{1}{n} \log \mu_x^u(B^u_\eta(x,n)) = h^u(f,\mu).
\]

Fix some \( \varepsilon > 0 \). As noticed in remark 5.11 Proposition 5.9 also holds for the diffeomorphism \( f \): this gives some number \( r' > 0 \) and for each \( n \geq 0 \) a set \( Y_n \). There are \( n_0 \geq 1, R' \subset R_2 \) with \( \mu(R_2 \setminus R') < \varepsilon \) such that for every \( x \in R' \) and \( n \geq n_0 \),

\[
-\frac{1}{n} \log \mu_x^u(B^u_\eta(x,n)) \geq h^u(f,\mu) - \varepsilon.
\]

For \( x \in Y_n \) we have \( f^n(\mathcal{P}_n^f(x) \cap W^u_{R_2}(x)) \subset B^u_n(f^n(x)) \) and

\[
\frac{1}{n} \int_{R_2} -\log \mu_x^u(\mathcal{P}_n^f(x))d\mu \geq \frac{1}{n} \int_{R' \cap Y_n} -\log \mu_x^u(\mathcal{P}_n^f(x))d\mu \geq \frac{1}{n} \int_{R' \cap Y_n} -\log \mu_x^u(B^u_\eta(x,n))d\mu \geq \mu(R' \cap Y_n)(h^u(f,\mu) - \varepsilon) \geq (\mu(R_2) - 2\varepsilon)(h^u(f,\mu) - \varepsilon).
\]

Since \( \varepsilon > 0 \) is arbitrary, this implies the inequality. \( \square \)

The definition of \( \bar{H}^{tr}_{\mu^u}(R_2, \mathcal{P}_n^f) \) and Lemmas 5.13 and 5.14 together conclude the proof of Proposition 5.6. \( \square \)

5.5. Proof of Theorem A. By Theorem 2.1 there exists a set \( X \) with full \( \mu \)-measure which has a global \( c \times u \)-product structure inside each leaf \( W^u(x) \). We have to prove that the measure \( \mu^u_{\sigma^u} \) on \( W^u(x) \) is a product.

Let us fix \( \ell > 0 \) large and let us apply the construction of the beginning of section 5 with scales \( L, \ell_2, \ell_1 > K \ell \). This provides a set \( R_1 \) centered at some point \( x_0 \), with positive \( \mu \)-measure (by (7)) and where Proposition 5.4 holds. By Proposition 4.2 it can be applied to partitions with small boundary \( \mathcal{P}^u \) of \( D^u \) into subsets with arbitrarily small diameter, hence which generate the \( \sigma \)-algebra of \( D^u \). Since \( Z \) has a \( c \times u \)-product structure, they induce partitions of each set \( W^u_{\sigma}(y) \), which generate their \( \sigma \)-algebra. Let \( \mathcal{P} \) be the partitions of \( R_1 \) induced by the partitions \( \mathcal{P}^u \). Proposition 5.4 implies that for \( \mu \)-almost every point \( z \in R_1 \), the transverse entropies \( H^u(W^u_{\mathcal{P}^u}(z), \mathcal{P}|W^u_{\mathcal{P}^u}(z)) \) vanish. By corollary 5.4 each probability measure \( \mu^u_{\mathcal{P}^u}(z) \) is a product on the corresponding set \( W^u_{\mathcal{P}^u}(z) \) for almost every \( z \in R_1 \).
Almost every point \( x \in X \) has arbitrarily large backward iterates in \( R_1 \) close to \( x_0 \). We consider the large compact subset with product structure \( \mathcal{W}^{cu}_{\ell_1}(x) \), i.e. the set of intersection points \( \mathcal{W}^c(z) \cap \mathcal{W}^u(y) \) for \((y, z) \in B_\varepsilon^c(x) \times B_\varepsilon^c(x)\). By the quasi-isometric property, there exists \( n \geq 1 \) large such that \( f^{-n}(\mathcal{W}^c_{\ell_1}(x)) \subset \bigcup_{z \in \mathcal{W}^c(x)(x_0)} B_{K\ell}(z) \). Note that \( K\ell \) is smaller than the center size \( \ell_1 \) of \( R_1 \). Then Property (f) in Proposition \[5.1\] implies that there exists a full measure set \( \Omega \) such that \( f^{-n}(\mathcal{W}^{cu}_{\ell_1}(x)) \cap \Omega \subset \mathcal{W}^{cu}_{R_1}(\zeta) \) for some \( \zeta \in R_1 \).

In particular, \( \mu^{cu}_{f^{-n}}(x) = \Psi(x,y) \mu^{cu}_{\ell_1}(x) \) is a product. By invariance, this proves that \( \mu^{cu}_{x|\mathcal{W}^{cu}_{\ell_1}(x)} \) is a product. As \( \ell \) is arbitrary, this concludes that \( \mu^{cu}_{x} \) is a product.

**6. Measures with a Local Product Structure**

In this section we prove Theorem B: we show that if an invariant measure has some local product structure then its sub-invariant disintegrations on the center plaques \( \mathcal{W}^{c}_{\ell_1}(x) \) vary continuously with respect to \( x \).

Let us consider two transverse foliations \( \mathcal{F} \) and \( \mathcal{G} \), two points \( x, y \) such that \( \mathcal{G}(x) = \mathcal{G}(y) \), two discs \( D^\mathcal{F}(x) \subset \mathcal{F}(x) \), \( D^\mathcal{G}(y) \subset \mathcal{G}(y) \) and some holonomy map \( h_{x,y}^\mathcal{F} : D^\mathcal{F}(x) \to D^\mathcal{G}(y) \) along the leaves of \( \mathcal{G} \).

**Definition 6.1.** The disintegration \( \{\mu^\mathcal{F}_x\}_{x \in \mathcal{M}} \) of \( \mu \) along the leaves of \( \mathcal{F} \) is \( \mathcal{G} \)-quasi invariant if \( \mu^\mathcal{F}_y \) is absolutely continuous with respect to \( h_{x,y}^\mathcal{G} \mu^\mathcal{F}_x \) for \( \mu \)-almost every \( x, y \in \mathcal{M} \).

This is a weaker property than the invariance under \( \mathcal{F} \)-holonomies.

Take \( x_0 \in \mathcal{M} \) and \( \varepsilon > 0 \), let \( \mathcal{N}^{cs \times u}_{\varepsilon}(x_0) \) be the set with \( cs \times u \)-product structure as in definition \[1.5\] i.e. \( \Psi : B^u_\varepsilon(x_0) \times B^{cs}_\varepsilon(x_0) \to \mathcal{N}^{cs \times u}_{\varepsilon}(x_0) \), \( \Psi(x, y) = h^u_\varepsilon(y) \).

**Proposition 6.2.** \( \mu^u_\varepsilon \) is \( cs \)-quasi-invariant if and only if \( \mu \) has local \( cs \times u \)-product structure.

**Proof.** Fix the total measure set \( \mathcal{M}' \) where the \( cs \)-quasi invariance is satisfied, take \( x_0 \in \mathcal{M}' \) and such that \( \mu^{cs}_{x_0} \) almost every \( y \in \mathcal{W}^{cs}(x_0) \) belongs to \( \mathcal{M}' \), also take \( \varepsilon > 0 \) such that \( \mathcal{N}^{cs \times u}_{\varepsilon}(x_0) \) has \( cs \times u \)-product structure. We are going to work on \( B^u_\varepsilon(x_0) \times B^{cs}_\varepsilon(x_0) \sim \mathcal{N}^{cs \times u}_{\varepsilon}(x_0) \), let \( \tilde{\mu} = \Psi^* \mu \big|_{\mathcal{N}^{cs \times u}_{\varepsilon}(x_0)} \).

As \( x_0 \) and \( \varepsilon \) are fixed, to simplify the notation we write \( B^* := B^*_\varepsilon(x_0) \) for \( * = cs, u \).

By Rokhlin disintegration formula we can write

\[
\tilde{\mu} = \int_{\{x_0\} \times B^{cs}} \mu^u_y d\mu^{cs}(y),
\]

where \( \mu^{cs} \) is the projection of \( \tilde{\mu} \) by \( \pi^u : B^u \times B^{cs} \to \{x_0\} \times B^{cs}, (x^u, y^{cs}) \mapsto (x_0, y^{cs}) \).

Observe that using these coordinates with product structure the \( cs \)-holonomies are of the form \( h_{x_0}^{cs}(x^u, y^{cs}), (x^u, y^{cs}) \mapsto (x, y^{cs}) \). Then the \( cs \)-quasi invariance implies that for \( \mu^{cs}_{x_0} \) almost every \( y \in \{x_0\} \times B^{cs} \) we have that
\[ \mu^u_y = \rho(\cdot, y)\mu^u_{x_0}. \]
So we get that \( \tilde{\mu} = \rho\mu^u_{x_0} \times \mu^{cs}. \) The reciprocal affirmation is trivial. \( \square \)

Observe that the same proof shows that \( \mu^{cs}_x \) is \( u \) quasi invariant if the measure is absolutely continuous with measure \( \mu^u \times \mu^{cs} \), as a consequence, we get that \( cs \)-quasi invariance of \( \mu^u \) is equivalent to \( u \)-quasi invariance of \( \mu^{cs} \).

**Proposition 6.3.** If \( \mu^s \) is \( cu \) quasi invariant and \( \mu^c \) is \( u \)-invariant then \( \mu^{cs} \) is \( u \)-quasi invariant.

**Proof.** Using the local product structure of \( cs \) manifolds we can find \( \varepsilon > 0 \) sufficiently small, such that for every \( x \in M \), \( \Psi : B^c_\varepsilon(x) \times B^c_\varepsilon(x) \to W^{cs}(x) \), \( \Psi(z, y) = h^{cu}_{x, x}(y) \) is a homeomorphism over its image, let \( N^{sc}_\varepsilon(x) \) be this image.

Let \( M' \) be the set of points that satisfy the \( cu \) quasi invariance of \( \mu^s \) and the \( u \) invariance of \( \mu^c \). Take \( x_0 \in M \) and \( y_0 \in W^u(x_0) \) such that \( \mu^{cs}_{x_0}(M \setminus M') = \mu^c_{y_0}(M \setminus M') = 0. \) There exists \( 0 < \varepsilon' \leq \varepsilon \) such that the \( u \)-holonomy \( h^u : N^{cs}_{\varepsilon'}(x_0) \to W^{cs}(y_0) \) is a well defined homeomorphism over its image. Up to taking \( \varepsilon' \) smaller we can assume that \( h^u(N^{cs}_{\varepsilon'}(x_0) \subset N^{sc}_\varepsilon(y_0)). \) See Figure 6.

![Figure 6](image_url)

**Figure 6.** Definition the holonomy \( h^u : N^{cs}_{\varepsilon'}(x_0) \to N^{cs}_\varepsilon(y_0). \)

Take any set \( A \subset N^{cs}_{\varepsilon'}(x_0) \), to conclude we need to prove that \( \mu^{cs}_{x_0}(A) = 0 \) if and only if \( \mu^{cs}_{y_0}(h^u(A)) = 0. \) As the normalization of the measures does not matter to prove this, from now on we restrict \( \mu^{cs}_{x_0} \) to \( N^{cs}_{\varepsilon'}(x_0) \) and normalize such that \( \mu^{cs}_{x_0}(N^{cs}_{\varepsilon'}(x_0)) = 1. \)

We have
\[
\mu^{cs}_{x_0}(A) = \int_{B^c_{\varepsilon'}(x_0)} \mu^c_{z}(A) d\pi^c_{x_0} \mu^{cs}_{x_0}(z),
\]
where \( \pi^c : N^{cs}_{\varepsilon'}(x_0) \to B^c_{\varepsilon'}(x_0) \) is the natural projection using the center holonomy.

Observe that by the \( cu \) quasi invariance of \( \mu^s \), for \( \mu^s_{x_0} \) almost every \( x \in B^c_{\varepsilon'}(x_0) \) we have that \( \mu^s_{x} \) is absolutely continuous to \( h^{cu}_{x_0-x_0} \mu^s_{x_0}. \) Now observe
that for any $C \in \mathcal{B}^s_c(x_0)$

$$\pi^c_x \mu^c_{x_0}(C) = \int_{\mathcal{B}^s_c(x_0)} \mu^s_{x_0, x}(C) d\pi^c_{x_0}(y),$$

where $\pi^s : \mathcal{N}^s_c(x_0) \to \mathcal{B}^s_c(x_0)$ is the projection using the stable holonomy.

So we get that $\pi^c_x \mu^c_{x_0}$ is absolutely continuous with respect to $\mu^s_{x_0}$. Then we can write

$$\mu^c_{x_0}(A) = \int_{\mathcal{B}^s_c(x_0)} \mu^c_z(A) \rho(z) d\mu^s_{x_0}(z),$$

where $\rho$ is a positive function.

Analogously we have that

$$\mu^c_{y_0}(h^u(A)) = \int_{\mathcal{B}^s_c(y_0)} \mu^c_z(h^u(A)) \tilde{\rho}(z') d\mu^s_{y_0}(z'),$$

for some $\tilde{\rho}$ positive function. Here we normalize such that $\mu^c_{y_0}(h^u(\mathcal{N}^c_c(x_0))) = 1$, so $h^u_x \mu^c_z = \mu^c_{h^u(z)}$ for $\mu^c_{x_0}$ almost every $z \in \mathcal{N}^c_c(x_0)$.

Define the cs holonomy $h^{cu} : \mathcal{B}^s_c(x_0) \to \mathcal{B}^s_c(y_0)$ such that $h^{cu}(z) \in \mathcal{B}^c_c(h^u(z)) \cap \mathcal{B}^s_c(y_0)$. Now using the invariance of $\mu^c$ by $u$ holonomies we have

$$\mu^c_{y_0}(h^u(A)) = \int_{\mathcal{B}^s_c(y_0)} \mu^c_{h^{cu-1}(z')}(A) \tilde{\rho}(z') d\mu^s_{y_0}(z'),$$

see Figure 7.

![Figure 7](image_url)

**Figure 7.** $\mu^c_z(A) = \mu^c_z(h^u(A))$.

Then again by the cu quasi invariance of $\mu^s$ we have

$$\mu^c_{y_0}(h^u(A)) = \int_{\mathcal{B}^s_c(y_0)} \mu^c_z(A) \tilde{\rho}(h^{cu}(z)) \rho'(z) d\mu^s_{x_0}(z'),$$

For some $\rho'$ positive function. So from (13) and (14) we get $\mu^c_{x_0}(A) = 0$ if and only if $\mu^c_{y_0}(h^u(A)) = 0$. □

**Proposition 6.4.** If $\mu^u$ is cs-quasi invariant and $\mu^c$ is su-invariant then there exists a disintegration $x \mapsto [\mu^c_x]$ on the support of $\mu$ that is continuous and su-invariant.
Proof. Take $M'$ the set of total measure with su-invariance, by hypothesis

$$\mu(M') = 1.$$ 

There exists a dense set of points $x \in \text{supp}(\mu)$ such that $\mu_{x}^{cs}$ almost every point in $\mathcal{W}_{x}^{cs}(x)$ and $\mu_{x}^{s}$ almost every point in $\mathcal{W}_{x}^{s}(x)$ are contained on $M'$. Fix one of this points and denote it by $x_0$.

For $\gamma > 0$ sufficiently small there exists $\Psi : B_{\gamma}^{u}(x_0) \times B_{\gamma}^{cs}(x_0) \rightarrow N_{\epsilon}^{cs}(x_0)$ as in definition 4.3. Take $\varepsilon > 0$ sufficiently small such that $N_{\epsilon}^{cs}(x_0) \subset B_{\varepsilon}^{cs}(x_0)$, where $N_{\epsilon}^{cs}(x_0)$ is defined as in Proposition 6.8. Now take $V = \Psi(B_{\gamma}^{u}(x_0) \times N_{\epsilon}^{cs}(x_0))$, see Figure 8. For $x \in V$, we define $\mathcal{W}_{V}^{*}(x) := \mathcal{W}_{loc}^{*}(x) \cap V$, for $* = c, u, s, cu, cs$.

![Figure 8. Definition of $V$.](image-url)

By $c \times s$-product structure of $N_{\epsilon}^{cs}(x_0)$, for every $y, z \in N_{\epsilon}^{cs}(x_0)$ we have an $s$-holonomy $h^{s}_{t, y, z} : \mathcal{W}_{V}^{c}(y) \rightarrow \mathcal{W}_{V}^{c}(z)$ where $h^{s}_{t, y, z}(t)$ is the unique intersection of $\mathcal{W}_{V}^{c}(t)$ with $\mathcal{W}_{V}^{c}(z)$, moreover it is a homeomorphism. Take the center disintegration $V \ni x \mapsto \mu_{x}^{c}$ normalized by $\mu_{x}^{c}(\mathcal{W}_{V}^{c}(x)) = 1$.

Take some $y \in M' \cap \mathcal{W}_{V}^{s}(x_0)$ and its corresponding $\mu_{y}^{s}$, then for every $z \in \mathcal{W}_{V}^{c}(x_0)$ we can define $\tilde{\mu}_{z}^{c} = h_{y, z}^{s} \mu_{y}^{c}$, by the $s$-invariance inside $M'$ we have that $\tilde{\mu}_{z}^{c} = \mu_{z}^{c}$ for $\mu_{z}^{cs}$ almost every $z \in N_{\epsilon}^{cs}(x_0)$.

Now by construction of $V$ we have that for every $x \in \mathcal{W}_{V}^{c}(x_0)$ and any $y \in \mathcal{W}_{V}^{s}(x)$ there exist an $u$-holonomy $h^{u}_{x, y, z} : \mathcal{W}_{V}^{c}(x) \rightarrow \mathcal{W}_{V}^{c}(y)$ such that $h^{u}_{x, y, z}(t)$ is the only intersection point of $\mathcal{W}_{V}^{u}(t)$ and $\mathcal{W}_{V}^{c}(y)$, and it is a homeomorphism.

Now for every $y$ and $x$ as before we define $\tilde{\mu}_{y}^{c} = h^{u}_{x, y} \mu_{x}^{c}$. Take $y$ such that $\mu_{y}^{cs}(M') = 1$, by the u-quasi invariance of $\mu^{cs}$, the projection of $M' \cap \mathcal{W}_{V}^{cs}(x_0)$ to $\mathcal{W}_{V}^{c}(y)$ by the u-holonomy has total $\mu_{y}^{cs}$ measure, so the u-invariance of $\mu^{c}$ gives $\tilde{\mu}_{y}^{c} = \mu_{y}^{c}$ for every $y$ in the intersection of $M' \cap \mathcal{W}_{V}^{c}(y)$ and the u-projection of $M' \cap \mathcal{W}_{V}^{c}(x_0)$. As this intersection has total $\mu_{y}^{cs}$ measure for $\mu$ almost every $y \in V$ we get that $\tilde{\mu}_{y}^{c} = \mu_{y}^{c}$ $\mu$-almost every $y \in V$.

So $y \mapsto \tilde{\mu}_{y}^{c}$ is a disintegration of $\mu$ restricted to $V$, it is continuous and u-invariant.

By Proposition 6.8 we can exchange the role of $s$ and $u$ to get a continuous disintegration in a neighborhood $V'$ of $x_0$ that is $s$-invariant. As this two disintegration coincide $\mu$-almost everywhere (up to a multiplicative
factor) on $V \cap V'$ we conclude the proof covering the support of $\mu$ by these neighborhoods.

**Proposition 6.5.** Let $f \in C_r$, $r > 1$ be a partially hyperbolic accessible center bunched diffeomorphism with quasi isometric center. Then for every measure $\mu$ with $csu$-product structure, full support and zero center exponent, the center disintegrations $\{\mu^c_x\}$ are absolutely continuous with respect to the Lebesgue measure.

**Proof.** Fix a normalization of the disintegration, for example $\mu^c_x(\mathcal{W}^c_{loc}(x)) = 1$. Take any $x$ and $y \in M$, by accessibility there exists su-path from $x$ to $y$, moreover by compactness of $M$ the number of legs and the lengths of each leg can be taken uniformly bounded (independent of the $x$ and $y$) see [40, Lemma 4.5].

Fix $x, y \in M$, take $h : B^c_\gamma(x) \to \mathcal{W}^c(y)$ a composition stable and unstable holonomies given by the su-path defined on $B^c_\gamma(x)$, for some $\gamma > 0$ given by the su path connecting $x$ and $y$, i.e: $h(x) = y$. By the absolute continuity of the stable and unstable holonomies $h$ is absolutely continuous and as the number of legs and lengths is bounded the jacobian of $h$ is uniformly bounded by some constant $K > 1$.

Then for every $\epsilon > 0$ sufficiently small we have

$$K^{-1} \text{vol}^c(B^c(x)) \leq \text{vol}^c(h(B^c(x))) \leq K \text{vol}^c(B^c(x)).$$

By the invariance of $\mu^c$ we get

$$K'_{xy}^{-1} \frac{\mu^c_x(B^c(x))}{\text{vol}^c(B^c(x))} \leq \frac{\mu^c_y(h(B^c(x)))}{\text{vol}^c(h(B^c(x)))} \leq K'_{xy} \frac{\mu^c_y(B^c(x))}{\text{vol}^c(B^c(x))},$$

where $K'_{xy} \geq K$ is a constant that depends on the normalization, so it depends on $x, y$ but not on $\epsilon$.

Now the center bunching condition implies that the holonomies are Lipschitz inside center stable/unstable manifolds, uniformly in paths with bounded lengths, see [31, Theorem B]. Then it exists $C > 0$ such that $B^c_{C-1}(y) \subset h(B^c_{C}(x)) \subset B^c_{C'}(y)$. Taking $\epsilon \to 0$ we get that if the Radon-Nikodym derivatives on $x$ and $y$ exists,

$$K'_{xy}^{-1} \frac{d\mu^c_x}{d\text{vol}^c}(x) \leq \frac{d\mu^c_y}{d\text{vol}^c}(y) \leq K'_{xy} \frac{d\mu^c_x}{d\text{vol}^c}(x).$$

(15)

If $\mu^c$ has a part that is singular with respect to Lebesgue it will exists $y$ such that $\frac{d\mu^c_x}{d\text{vol}^c}(y) = \infty$, on the other hand there exists $x$ such that $\frac{d\mu^c_y}{d\text{vol}^c}(x)$ is finite, so this will contradict (15). Analogously if $\text{vol}^c$ has a singular part with respect to $\mu^c$ it will contradict (15).

**Proposition 6.6.** Let $f \in C^r$, $r > 1$ be a discretized Anosov flow and accessible diffeomorphism, if there exists an ergodic $f$ invariant measure $\mu$, with $csu$-product structure, fully supported and zero center exponent then $f$ is the time one map of a topological Anosov flow $C^1$ along the center.
Proof. By Propositions 6.4 and 6.5 we have that $\mu^c$ is absolutely continuous with respect to Lebesgue. Take the disintegration $x \mapsto \mu^c_x$ normalized such that $\mu^c_x([x,f(x)]) = 1$ on open leaves and extend by continuity on closed leaves, as the holonomy commutes with the dynamics we have that this normalization is preserved by the holonomies and $\mu^c_x$ is continuous.

Moreover from the proof of Proposition 6.5, once we have an su-invariant normalization of $x \mapsto \mu^c_x$, we get that $K'_xy = K$ for every $x,y \in \mathcal{M}$ in (15), so

\[
(16) \quad \frac{d\mu^c_y}{d\text{vol}^c} = \frac{1}{Jh(x)} \frac{d\mu^c_x}{d\text{vol}^c},
\]

where $h$ is a holonomy map (composition of stable and unstable holonomies) from $B^c_x(x)$ to $\mathcal{W}^c(y)$ and $Jh$ is his jacobian with respect to the Lebesgue measure on the corresponding center leaves.

Define $\Delta(x) = \frac{d\mu^c_x}{d\text{vol}^c}(x)$, as $\Delta(x)$ is the limit when $\epsilon \to 0$ of the continuous functions $\frac{\mu^c_x(B^c_\epsilon(x))}{\text{vol}^c(B^c_\epsilon(x))}$, it has some continuity point, then using (16) we get that $\Delta(x)$ is continuous everywhere.

Now define the vector field $Z_0 : \mathcal{M} \to T\mathcal{M}$ that $Z_0(x)$ is the positively oriented unit vector on $E^c$, then take $Z(x) = \frac{1}{\Delta(x)} Z_0(x)$.

Then if $\phi_t$ is the flow generated by $Z$ we have that $\mu^c_x(x, \phi_t(x)) = t$, by our normalization of $\mu^c$ we conclude that $\phi_1(x) = f(x)$.

\[\square\]

Remark 6.7. When the center foliations forms a fiber bundle the invariant measure $\mu$ can be projected to a hyperbolic homeomorphism, see for example [38]. In this cases the product structure necessary for the extension to a continuous su-invariant disintegration is that this projected measure is absolutely continuous with a product measure of the stable and unstable sets, see [3], [38].

In our setting as the center foliation in general do not form a fiber bundle we don’t have a natural projection to a quotient space. A natural definition for this projective product structure can be the following: We say that $\mu$ has projective product structure if for $\mu$-almost every $x \in \mathcal{M}$ there exists $r, \delta, \ell > 0$ such that $x \in V = \cup_{z \in \mathcal{W}^s_{r,\delta}(x)} B^s_\ell(z)$, with $\ell$ sufficiently small such that $B^s_\ell(z)$ are disjoint for different $z \in \mathcal{W}^s_{r,\delta}(x)$, with the property that the projection by center discs $\pi^c : V \to \mathcal{W}^s_{r,\delta}(x)$ satisfies that $\nu = \pi^c_* \mu |_V$ is absolutely continuous with a product measure $\nu^s \times \nu^u$ via the homeomorphism $B^s_x(x) \times B^u_x(x) \to \mathcal{W}^s_{r,\delta}(x)$.

Theorem 6.4 can probably be adapted to measures having projective product structure, but as in our applications is easier to check cs or cu quasi invariance we didn’t explore this definition.

7. MEASURES OF MAXIMAL ENTROPY

In this section we prove Theorem D, Theorem C and Corollary 1.6.
**Proof of Theorem D.** By [11, Theorem 3.2] we have that $\mu^x_u$ is cs-quasi-invariant and has full support, then by Propositions [6.4] we have a continuous (up to a multiplicative factor) $\mu^x_c$. As in proposition [6.5] we normalize the measure $\mu^x_c$ such that $\mu^x_c([x, f(x)]) = 1$ on open leaves and extend by continuity on closed leaves, as the holonomy commutes with the dynamics we have that this normalization is preserved by the holonomies and $\mu^x_c$ is continuous.

Assume that we have some $x \in M$ with $\mu^x_c(x) > 0$, using the su-invariance we can assume that the center manifold of $x$ is not compact, take the interval $[x, f(x)] \subset W^c(x)$, by the minimality of u-foliations we have an infinite sequence of different points $y_i \in [x, f(x)]$ such that $W^s(y_i) \cap W^u(x) \neq \emptyset$. So we can define a sequence of maps $h_i$ that are compositions of stable and unstable holonomies such that $h_i(x) = y_i$.

By the su-invariance we have that $\mu^x_c(y_i) = \mu^x_c(x)$, a contradiction because $\mu^x_c([x, f(x)]) = 1$.

Now if $f$ is accessible, by Proposition [6.5] the center disintegration is absolutely continuous with Lebesgue, moreover by Proposition [6.6] we conclude that $\phi_1(x) = f(x)$.

**Proof of Theorem C.** Take $V$ be the open set given by [11, Theorem 1.1]. We claim that there exists an open and dense set $U \subset V$ such that every $f \in U$ has a compact center leaf such that the restriction of $f$ to this leaf is Morse-Smale.

Assuming the claim we conclude that the only invariant measures supported on this center leaves are atomic, so the existence of m.m.e. with zero exponents will contradict Theorem D. Hence, we have that in $U$ all m.m.e. have non-zero Lyapunov exponents. By [11, Theorem 1.1] this implies that there are exactly two, one with positive and one with negative center Lyapunov exponent that are Bernoulli.

To prove the claim first observe that the condition of having a Morse-Smale compact center leaf is open. Now to prove that is dense, take any $f \in V$, as $f$ is a discretized Anosov flow it has a fixed compact center leaf $W^c(p)$, in other words $W^c(p)$ is a circle. Morse-Smale are dense in the circle so we can perturb $f$ such that $W^c(p)$ is still fixed and the restriction to $W^c(p)$ is Morse-Smale.

**Proof of Corollary 1.6.** By hypothesis we have that $\mu_n$ has non-negative center Lyapunov exponent, then $h^s(f_n, \mu_n) = h(f_n, \mu_n)$, let $\mu$ be any accumulation point of $\mu_n$, by [41] we have that $\limsup_{n \to \infty} h^s(f_n, \mu_n) \leq h^s(f, \mu)$, so by hypothesis we get that $h^s(f, \mu) = h_{\text{top}}(f)$, in particular $\mu$ is a m.m.e. of $f$. If $f$ has a unique m.m.e. with zero center exponent, as in Theorem D then we conclude.

So assume we are in the case that $f$ has two ergodic m.m.e. one with positive center exponent $\mu^+$ and one with negative $\mu^-$, then $\mu = a\mu^+ + b\mu^-$. Assume by contradiction that $b \neq 0$, by [19] the stable entropy is an affine function of the measure, so we conclude that $h^s(f, \mu^-) = h_{\text{top}}(f)$ and as $\mu^-$
has negative exponent we get $h^s(f,\mu^-) = h^u(f,\mu^-) = h(f,\mu)$, so we can apply again Theorem B and as in Theorem D conclude that $\mu^-$ has Lebesgue disintegration on the center and is a topological flow smooth on the center, a contradiction because $\mu^-$ has negative center Lyapunov exponent.  

8. Physical Measures

First we prove the following proposition.

**Proposition 8.1.** Let $f$ be a $C^r$ partially hyperbolic diffeomorphism whose $u$-foliation is minimal, if there exist a Gibbs $u$-state with all his center exponents negative then it is the unique Gibbs $u$-state and its basin have full Lebesgue measure.

**Proof.** Let $\mu$ an Gibbs $u$-state of $f$ with $\lambda^c(\mu) < 0$, then there exists some set $A \subset W^u(x)$ for some $x \in M$ with $\text{vol}^c(A) > 0$ and $\delta > 0$ such that, for every $x \in A$, $B^c_\delta(x)$ is contained on a Pesin stable manifold and $A \subset B(\mu)$. Assume that there exists another Gibbs $u$-state $\mu'$, take $W^u(y)$ such that $\text{vol}^u(W^u(y) \setminus B(\mu')) = 0$, as $W^u(y)$ is dense and the Pesin stable manifolds are absolutely continuous we get that $W^u(y)$ intersects $\cup_{x \in A} B^c_\delta(x)$ on a $\text{vol}^u$ positive measure subset, so $B(\mu') \cap B(\mu) \neq 0$ which implies that $\mu' = \mu$.

As Lebesgue for almost every point $x$, the accumulation points of the measures $\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}$ are Gibbs $u$-state (see [7, Theorem 11.16]), the uniqueness implies that the the basin has full Lebesgue measure. 

Now we need the following result, a version of this theorem for partially hyperbolic skew product can be found on [38].

**Proposition 8.2.** Let $f$ be a partially hyperbolic diffeomorphism, quasi isometric in the center, $\dim E^c = 1$ with minimal $u$-foliation and the center stable foliation is absolutely continuous, then every ergodic $u$-Gibbs $\nu$ has $\lambda^c(\nu) \leq 0$.

Observe that for this result we do not require $f$ to be accessible.

**Proposition 8.3.** Let $f$ be a partially hyperbolic diffeomorphism quasi isometric on the center, then given $c > 0$ and $l \in \mathbb{N}$, for every $r > 0$ there exists $n_r \in \mathbb{N}$ such that $\# B^c(x,r) \cap \Gamma_{c,l} \leq n_r$, where

$$\Gamma_{c,l} = \left\{ x \in M : \liminf \frac{1}{n} \sum_{i=1}^{n} \left\| Df^{-l} \big| _{E^c(f^l(x))} \right\|^{-1} > c \right\}.$$

**Proof.** First observe that [38, Lemma 3.8] remains valid in our setting, so there exists $\delta > 0$ such that for every neighborhood $U \subset W^c(x)$ of $x \in \Gamma_{c,l}$ has $\liminf \frac{1}{n} \sum_{j=0}^{n-1} \text{vol}^c(f^j(U)) \geq \delta$.

Now take $V = \sup_{y \in M} \text{vol}^c(B^c_{K+r+q}(y))$ then take $n_r > V/\delta$. Suppose that there exists different $y_j \in B^c_{r}(x) \cap \Gamma_{c,l}$ for $j = 1, \ldots, n_r$, take disjoint
neighborhoods $y_j \in U_j \subset B^c_\epsilon(x)$, now by the quasi-isometric center property,

$$V \geq \frac{1}{n} \sum_{i=0}^{n-1} \text{vol}^c(f^{i\Lambda}(B^c_\epsilon(x))) \geq \frac{1}{n} \sum_{j=1}^{n_r} \frac{1}{n} \sum_{i=0}^{n-1} \text{vol}^c(f^{i\Lambda}(U_j)),$$

taking the lim inf we get $V \geq n_r \delta$ a contradiction. □

Adapting directly \cite{38} Lemma 3.9 we can find $k_0$ and an ergodic u-Gibbs $\nu_\ast$ such that

$$\int \log \left\| Df^{-k_0} \right\|_{E^c(x)}^{-1} \nu_\ast(x) > 0.$$

Let $\lambda$ be the value of this integral, let $g = f^{k_0}$ and

$$\Gamma = \left\{ x \in M : \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \log \left\| Dg^{-1} \right\|_{E^c(g^j(x))}^{-1} = \lambda \right\},$$

by ergodicity $\nu_\ast(\Gamma) = 1$.

**Corollary 8.4.** Given $r > 0$, there exist $n_r \geq 1$ such that $\# B^c_\epsilon(x) \cap \Gamma < n_r$ for every $x \in N$

**Proof.** Use Proposition 8.3 with $c = \lambda/2$ and $l = k_0$, then use that $\Gamma \subset \Gamma_{c,l}$. □

From now on we assume that $\dim E^c = 1$ and by contradiction assume that $\lambda^c(\mu) > 0$, by Katok \cite{21} there exists hyperbolic periodic points. Take some periodic point $p$, now up to take a power of the period we have that $f^k : W^c(p) \to W^c(p)$ is a diffeomorphism that preserve the orientation and has a fixed point $p$, if $W^c(p)$ is a circle we know that any periodic point has the same period, this means that are fixed by $f^k$, if $W^c(p)$ is homeomorphic to $\mathbb{R}$ we know that every for every $x \in W^c(p)$, $f^{kn}(x)$ or goes to infinity or converges to a $f^k$ fixed point, as $f$ is quasi-isometric on the center it implies that it converges to a fixed point $p'$, up to taking $n$ converging to $-\infty$ we can assume that $p' \neq p$. This gives us a $f^k$ fixed interval $I \subset W^c(p)$ with $p$ and $p'$ as its boundary.

From now on take $\ell_0 = I$ as before when $W^u(p)$ is homeomorphic to $\mathbb{R}$ or $\ell_0 = W^c(p)$ when it is a compact circle. Observe that the support of $\mu$ is u-saturated then as the u foliation is minimal, $\mu$ have full support. The previous conclusion can also be applied for $\nu_\ast$ and $g$.

As $\nu_\ast$ is u-Gibbs for $\nu_\ast$-almost every $x \in M$ we have $\text{vol}^u(W^u(x) \setminus \Gamma) = 0$ so we can find a u disk $D^u$ that intersect $W^s(\ell_0) = \cup_{z \in \ell_0} W^s(z)$ and $\text{vol}^u(D^u \setminus \Gamma) = 0$.

**Lemma 8.5.** Every $x \in D^u \cap W^s(\ell_0)$ belongs to the strong stable manifold of some $f^k$ fixed point $y \in \ell_0$ of $f$.

This is an adaptation of \cite{38} Lemma 3.11] to our setting, for completeness we explain all the details.
Proof. Let $r = \text{diam}^c(\ell_0)$, take $n_{r+1}$ given by Corollary 8.4 and $k_0$ to be the period of $p$. Take $y_j = g^j(y)$, $x_j = g^j(x)$ and $D^u_j$ a disk of uniform size inside $g^i(D^u)$ centered at $x_j$.

First let us assume that $y$ is not periodic, and let $y^\ast$ be an accumulation point of $y_j$, up to taking a subsequence let us suppose that $y_j \to y^\ast$, take $n_1$ given by Corollary 8.4 with $r = 1$. First take $j_0$ sufficiently large such that there is a uniform neighborhood of $y^\ast$ in $W^u(y^\ast)$ such that the cs holonomy is well defined from $D^u_j$ to this neighborhood. Take $\Gamma_0 = \cap_{j \geq j_0} \pi^{cs}_j(D^u_j \cap \Gamma)$ where $\pi^{cs}_j$ is the local cs holonomy.

Take $w \in \Gamma_0$ close to $y^\ast$ and $w_j$ such that $w_j \in W^c(w) \cap \pi^{cs-1}_j(w)$, as $\pi^{cs-1}_j(w) \in \Gamma$ and $\Gamma$ is saturated by stable manifolds we get that $w_j \in \Gamma$, take $y_{j_0}, \ldots, y_{j_0+n_1}$, observe that the distance from $y_j$ to $w_j$ converges to zero when $w$ converges to $y^\ast$. Taking $j_0$ sufficiently large and $w$ close to $y^\ast$ we can ensure that $w_j \in B^c_j(w)$ for $j_0 \leq j \leq j_0 + n_1$, and also the $w_j$ all different. This gives a contradiction to Corollary 8.4 because $w_j \in \Gamma$.

Now as $f^k : \ell_0 \to \ell_0$ preserves orientation and has a fixed point, every point in $\ell_0$ is $f^k$ fixed. □

Now we adapt [38, Lemma 3.12].

Lemma 8.6. Every point $z \in \ell_0$ is periodic for $f$.

Proof. Take any $z$ in the interior of $\ell_0$ with respect to the topology of $W^c(p)$, as before take $x \in M$ such that $\text{vol}^u(W^u(x) \setminus \gamma) = 0$, now as $W^u(x)$ is dense it intersects $W^s(\ell_0)$ arbitrarily close to $z$, as $z$ is on the interior of $\ell_0$ we can take $z' \in \ell_0 \cap W^s(x')$ with $x' \in W^u(x)$, by Lemma 8.5 we get that $z'$ is periodic, so it is fixed for $f^k$ (every periodic point on $\ell_0$ have the same period) , as $z'$ is arbitrarily close to $z$ we get that $z$ is also periodic.

□

Now we can conclude

Proof of Proposition 8.2 Assume that $\nu$ has positive center Lyapunov exponents, then it is hyperbolic, this implies that the measure $\nu_*$ is also hyperbolic. Taking $p$ as before by Lemma 8.6 we get that $\ell_0$ is periodic in particular $p$ is not a hyperbolic periodic point of $f$, a contradiction. □

So from Proposition 8.1 and 8.2 we get the following result

Proposition 8.7. Let $f$ be a $C^r$ diffeomorphism ($r > 1$),

- which is partially hyperbolic,
- with quasi-isometric one-dimensional center,
- whose $u$-foliation is minimal,
- whose cs-foliations is absolutely continuous,

then either: there exists a unique Gibbs $u$-state with negative center Lyapunov exponent or every Gibbs $u$-state has zero center Lyapunov exponents.
Observe that until now we do not assume that $f$ is accessible, this will be necessary for conclude the uniqueness of $u$-Gibbs with zero center Lyapunov exponent.

**Proof of Theorem F.** Observe that as the $u$ foliation is minimal then every $u$-Gibbs is fully supported. We will need the following lemma

**Lemma 8.8.** If $\lambda^c = 0$ then every $u$-Gibbs is a $cu$-Gibbs and its basin has full Lebesgue measure. In particular, there exists at maximum one ergodic $u$-Gibbs with zero exponent.

**Proof.** By Propositions 6.4, 6.2 and 6.5 we have that $\mu^c$ is absolutely continuous with respect to Lebesgue. We claim that $\mu^{cu}$ is absolutely continuous with respect to Lebesgue. To prove this take a $cu$ leaf and inside take a small set that has a product structure $D^u \times D^c$. Restricted to this set we have

$$\mu^{cu} = \int_{D^c} \mu^u_y d\pi^u_x \mu^{cu},$$

where $\pi^u : D^u \times D^c \to D^c$ is the natural projection. As $\mu$ is an $u$-state we have that $\mu^u_x$ is the Lebesgue measure along $W^u(y)$, it is left to prove that $\pi^u_x \mu^{cu}$ is the Lebesgue measure on $D^c$.

To see this observe that for any measurable set $A \subset D^u$, and any $x \in D^u$, $\mu^c_x (W^c(x) \cap (\pi^u_x)^{-1}(A)) = 0$ if and only if $\text{vol}^c(W^c(x) \cap (\pi^u_x)^{-1}(A)) = 0$, then by the absolutely continuity of $W^u$ this occurs if and only if $\text{vol}^c(A) = 0$, proving the claim.

Now we are going to prove that there exists $\rho > 0$ such that for every $x \in \text{supp}(\mu)$, $\text{vol}(B_\rho(x) \setminus B(\mu)) = 0$, as $\mu$ has full support this will conclude the Lemma.

Let $B$ be the set such that the Birkhoff sum converges for every continuous function. Now take $x \in \text{supp}(\mu)$ take a cu-disk $D^{cu}$ centered at $x$ and $D = \cup_{y \in D^{cu}} W^s_{loc}(y)$, then there exist some $x'$ arbitrarily close to $x$ such that $\text{vol}^c_x (W^{cu}(x') \setminus B) = 0$ and $W^s_{loc}(y) \cap W^{cu}(x') \neq \emptyset$ for every $y \in D^{cu}$ then by the absolute continuity of the stable foliation Lebesgue almost every $y \in D^{cu}$ has $W^s_{loc}(y) \cap B \neq \emptyset$, as $B(\mu)$ is $s$-saturated $\text{vol}(D \setminus B(\mu)) = 0$. As $D$ contains a uniform size ball we conclude. \[\square\]

Then by Proposition 8.7 we have that every $u$-Gibbs measure has non-positive center Lyapunov exponent, moreover if there exists one with negative exponent then it is the unique one, it is a Physical measure and its basin have full Lebesgue measure. If every measure has zero center exponent by Lemma 8.8 we get that it is unique and is a physical measure with full Basin. \[\square\]

**8.1. Proof of Theorem E.** Let $(\phi_t)$ be a $C^r$ transitive Anosov flow, $1 < r \leq \infty$. We need the following lemma

**Lemma 8.9.** There exists $f$ arbitrarily close to $\phi_1$ in $\text{Diff}^r(M)$ such that:
• $f$ preserves the foliations $W^{cs}, W^{cu}$ of $\phi_1$,
• there exists a $f$-invariant compact center leaf $W^c(P)$ such that the restriction $f|_{W^c(P)}$ has a Morse-Smale dynamics,
• for any diffeomorphism $C^1$ close to $f$, the foliation $W^u$ is dynamically minimal (i.e. the orbit of any leaf of $W^u$ is dense in $M$),
• $f$ is accessible.

Proof. Let $Z$ be the vector field of the Anosov flow. We follow [6] to build a diffeomorphism $f$ arbitrarily close to $\phi_1$ with a hyperbolic periodic point $P$ such that for any perturbation $g$ of $f$, the homoclinic intersections of the continuation of $P$ are dense in $M$. See the proof of Theorem A, pages 395–395 and of Theorem 2.1 there. The perturbation is done along the orbits of the flow $Z$ and the (compact) center leaf of $P$ supports a Morse-Smale dynamics after the perturbation, so that the two first items are satisfied.

Note that in [6], the diffeomorphism $f$ is a perturbation of the time-$\tau$ map of the flow where $\tau$ is the period of a periodic point of $(\phi_t)$. The proof can be easily adapted in the following way: first one perturbs the parametrization of the flow so that:

– there is a center leaf $C$ and an integer $k \geq 1$ such that $\phi_k|_C = \text{id}$, 
– there are two points $P \neq Q$ in $C$ and an iterate $m$ such that the intersection $f^m(W^u(P)) \cap W^s(Q)$ is non empty.

Up to replace $Q$ by an iterate, one can also assume that there exists a center curve $\gamma \subset C$ which does not contain any other iterate of $P$ and $Q$. We then perturb $\phi_1$ in a neighborhood of $\gamma$ and of the orbit of a point in the intersection $f^m(W^u(P)) \cap W^s(Q)$. After perturbation, the periodic point $P$ is attracting along the center.

We then claim that the foliation $W^u$ for $f$ is dynamically minimal. Note that the foliation $W^{cs}$ for $f$ coincides with the center stable foliation of the transitive Anosov flow, hence is minimal. One deduces that any leaf of $W^u$ intersects the local center-stable leaf of $P$. Since the homoclinic class of $P$ coincides with $M$, the (periodic) orbit of $W^u(P)$ is dense. In particular, for any $\varepsilon > 0$, there exists $N \geq 1$ such that for any disc of radius 1 in a strong unstable leaf, the $N^{\text{th}}$ iterate is close to a large strong unstable disc in $W^u(f^k(P))$ for some $k$, hence is $\varepsilon$-dense. This concludes the dynamical minimality. Note that this applies to any diffeomorphism that is $C^1$-close to $f$.

In order to prove the accessibility, we perform a new perturbation along the orbits of $(\phi_t)$. Note that this will not affect the properties that we obtained previously.

We say that $f$ is jointly integrable if there exists a foliation whose leaves are tangent to the bundle $E^s \oplus E^u$. From [18, Lemma A.4.3], it is possible to perturb the dynamics near the periodic point $P$ so that after the perturbation this property does not hold. The perturbation is done by composition by the time-$\delta$ map of a flow transverse to $E^s \oplus E^u$ and in our case one can simply choose the flow generated by $Z$. 


We recall that the accessibility class $AC(x)$ of a point $x$ is the set of points $y$ that can be connected by a path which is piecewise tangent to $E^s$ or $E^u$. Once the joint integrability has been broken, there exists a point $x$ whose accessibility class $AC(x)$ is open, see [15, Lemma 3]. By the dynamical minimality of the unstable foliation we can joint any point to an iterate of $AC(x)$. This implies that the iterates of $AC(x)$ cover $M$. Since accessibility classes are either disjoint or equal, and that $M$ is connected (since it supports a transitive flow), we have $M = AC(x)$ and $f$ is accessible.

**Proof of Theorem E.** Let $f_0$ be the diffeomorphism given by Lemma 8.9 and let $\mu$ be a $u$-Gibbs of $f_0$, as $f_0$ is $C^1$ close to the time one map of a flow then it is a discretized Anosov flow, see [11]. By construction $W^{cs}$ coincides with the center stable foliation of the Anosov flow $(\phi_t)$, so in particular is absolutely continuous. By Theorem E there exists a unique $u$-Gibbs measure $\mu$; moreover it is a physical measure with non-positive center exponent.

By minimality of the strong unstable foliation $W^u$, the $u$-Gibbs measure has full support. As the $cs$ foliation coincide with the one of the flow then it is absolutely continuous with respect to Lebesgue and the $cs$ holonomy between unstable manifolds is also absolutely continuous, so as $\mu$ is a $u$-Gibbs this implies that is $\mu^u$ is $cs$ quasi invariant. If the center exponent is zero, by Proposition 6.3 we have that $f_0$ is the time one map of a topological flow, smooth along the center. In particular $f_0$ acts like a smooth flow restricted to the compact center curve $W^c(P)$: this is a contradiction because $f_0 |_{W^c(P)}$ has a Morse-Smale dynamics. So we conclude that the center exponent of $\mu$ is negative.

By [11] there exists a $C^1$ open neighborhood $U$ of $f_0$ in Diff$(M)$ such that for every diffeomorphism $f$ in $U$, every $u$-Gibbs measure has negative center exponent, hence is an SRB measure. The robustness in Lemma 8.9 shows that unstable foliation of $f$ is dynamically minimal. Hence if there exists two $u$-Gibbs measures, they are homoclinically related. Since $f$ is $C^r$, $r > 1$, this is enough to conclude that both measures coincide. Hence $f$ has a unique physical measure. By [14, Corollary 1.2], its basin has full volume.

**References**

[1] J. F. Alves, C. Bonatti, and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Invent. Math.* 140 (2000), 351–398.

[2] A. Avila, J. Santamaria, and M. Viana. Holonomy invariance: rough regularity and applications to Lyapunov exponents. *Astérisque* 358 (2013), 13–74.

[3] A. Avila and M. Viana. Extremal Lyapunov exponents: an invariance principle and applications. *Invent. Math.* 181 (2010), 115–189.

[4] A. Avila, M. Viana, and A. Wilkinson. Absolute continuity, Lyapunov exponents and rigidity I: geodesic flows. *J. Eur. Math. Soc.* 17 (2015), 1435–1462.

[5] T. Barthelmé, S. Fenley, and R. Potrie. Collapsed anosov flows and self orbit equivalences. Preprint arXiv:2008.06547.

[6] C. Bonatti and L. J. Díaz. Persistent nonhyperbolic transitive diffeomorphisms. *Annals of Math.* 143 (1996), 357–396.
[7] C. Bonatti, L. J. Díaz, and M. Viana. *Dynamics beyond uniform hyperbolicity*. Encyclopaedia of Mathematical Sciences **102**, Springer-Verlag, 2005.

[8] C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.* **115** (2000), 157–193.

[9] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*. Lect. Notes in Math. **470**, Springer Verlag, 1975.

[10] A. Brown. Smoothness of stable holonomies inside center-stable manifolds. *Ergod. Th & Dynam. Sys.* (2021), 1–26.

[11] J. Buzzi, T. Fisher, and A. Tahzibi. A dichotomy for measures of maximal entropy near time-one maps of transitive anosov flows. *Ann. Sci. Éc. Norm. Supér.* **55** (2022), 969–1002.

[12] L. Chao, K. Marin, and J. Yang. Lyapunov exponents of partially hyperbolic volume-preserving maps with 2-dimensional center bundle. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **35** (2018), 1687–1706.

[13] V. Climenhaga, Y. Pesin, and A. Zelerowicz. Equilibrium states in dynamical systems via geometric measure theory. *Bulletin Amer. Math. Soc.* **56** (2019), 569–610.

[14] S. Crovisier, D. Yang, and J. Zhang. Empirical measures of partially hyperbolic attractors. *Communications in Mathematical Physics* **375** (2020), 725–764.

[15] D. Dolgopyat. Stability of accessibility. *Ergod. Th. & Dynam. Sys.* **23** (2003), 1717–1731.

[16] F. Ledrappier. Positivity of the exponent for stationary sequences of matrices. *Lect. Notes in Math.* **1186** (1986), 56–73.

[17] G. A. Margulis. Certain measures associated with u-flow on compact manifolds. *Functional Analysis and Its Applications* **4** (1970), 55–67.

[18] S. Martinchich. Global stability of discretized anosov flows. Preprint arXiv:2204.03825.

[19] D. Obata and M. Poletti. Positive exponents for random products of conservative surface diffeomorphisms and some skew products. *J. Dynam. Differential Equations* **34** (2022), 2405–2428.

[20] Y. B. Pesin and Y. G. Sinai. Gibbs measures for partially hyperbolic attractors. *Ergod. Th & Dynam. Sys.* **2** (1982), 417–438.

[21] C. Pugh, M. Shub, and A. Wilkinson. Hölder foliations. *Duke Math. J.* **86** (1997), 517–546.
[32] F. Rodriguez Hertz, M. A. Rodriguez Hertz, A. Tahzibi, and R. Ures. Maximizing measures for partially hyperbolic systems with compact center leaves. *Ergod. Th & Dynam. Sys.* **32** (2012), 825–839.

[33] V. Rokhlin. On the fundamental ideas of measure theory. *Mat. Sbornik N.S.* (1949), 107–150, and *Amer. Math. Soc. Translation* **1952** (1952), 55 pp.

[34] D. Ruelle. A measure associated with Axiom A attractors. *Amer. J. Math.* **98** (1976), 619–654.

[35] Y. Sinai. Gibbs measures in ergodic theory. *Uspehi Mat. Nauk* **27** (1972), 21–69.

[36] A. Tahzibi and J. Yang. Invariance principle and rigidity of high entropy measures. *Trans. Amer. Math. Soc.* **371** (2019), 1231–1251.

[37] M. Viana. Almost all cocycles over any hyperbolic system have nonvanishing Lyapunov exponents. *Annals of Math.* **167** (2008), 643–680.

[38] M. Viana and J. Yang, Physical measures and absolute continuity for one-dimensional center direction. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **30** (2013), 845–877.

[39] M. Viana, G. Liao, and J. Yang. The entropy conjecture for diffeomorphisms away from tangencies. *J. Eur. Math. Soc.* **15** (2013), 2043–2060.

[40] A. Wilkinson. The cohomological equation for partially hyperbolic diffeomorphisms. *Astérisque* **358** (2013), 75–165.

[41] J. Yang. Entropy along expanding foliations. *Advances in Mathematics* **389** (2021), 107893.

**Sylvain Crovisier**: CNRS-Laboratoire de Mathématiques d’Orsay, UMR 8628, Université Paris-Saclay, Orsay Cedex 91405, France

E-mail: sylvain.crovisier@universite-paris-saclay.fr

**Mauricio Poletti**: Departamento de Matemática, Universidade Federal do Ceará, Campus do PICI, Bloco 914, CEP 60455-760, Fortaleza – CE, Brasil.

E-mail: mpoletti@mat.ufc.br