ON THE RESTRICTED SIZE RAMSEY NUMBER
FOR A PAIR OF CYCLES

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Abstract. For graphs $H_1, H_2$ by $r^*(H_1, H_2)$ we denote the minimum number of edges in a graph $G$ on $r(H_1, H_2)$ vertices such that $G \rightarrow (H_1, H_2)$. We show that for each pair of natural numbers $k, n$, $k \leq n$, where $k$ is odd and $n$ is large enough, we have

$$r^*(C_n, C_k) = \lceil (n + 1)(2n - 1)/2 \rceil.$$ 

1. Introduction

In the paper we use the standard arrow notation and write $G \rightarrow (H_1, H_2, \ldots, H_r)$ if any coloring of the edges of a graph $G$ with $r$ colors leads to a copy of $H_i$ in the $i$th color for some $i = 1, 2, \ldots, r$. Let us recall that the Ramsey number $r(H_1, H_2, \ldots, H_r)$ is the minimum number of vertices in graph $G$ such that $G \rightarrow (H_1, H_2, \ldots, H_r)$, while the size Ramsey number $\hat{r}(H_1, H_2, \ldots, H_r)$ is the minimum number of edges in a graph $G$ with such a property. In general Ramsey numbers are hard to find, yet it has been computed, or estimated, for some simple families of graphs. In particular, $r(H_1, H_2)$ is known for $H_1, H_2$ which are either paths or cycles (see Gerencsér, Gyárfás [11], Rosta [22], Faudree, Schelp [7], or Radziszowski [21] for a survey of results on Ramsey Theory). The size Ramsey number for paths and cycles are not so well understood. In the fundamental paper on this subject Beck [4] proved that $\hat{r}(P_n, P_n) = O(n)$. The exact value of the hidden constant is not known and finding it seems to be hard. The best estimates we have got so far are those by Bal, DeBiasio [3], and Dudek, Praškal [6]

$$(3.75 - o(1))n \leq \hat{r}(P_n, P_n) \leq 74n.$$ 

The size Ramsey number for cycles was studied by Haxell, Kohayakawa, and Łuczak [12] who proved that $\hat{r}(C_n, C_n) \leq An$ for some huge constant $A$. The value of the constant $A$ has been substantially improved by Javadi, Khoeini, Omidi, and Pokrovskiy [13], and then by Javadi and Miralaei [15]; both of the above papers also considered the case of more than one color. However, finding the exact value of the limit $\hat{r}(C_n, C_k)/n$ (or even prove that it exists) seems to be out of reach right now.

In this paper we study yet another version of Ramsey number. By the restricted size Ramsey number $r^*(H_1, H_2, \ldots, H_r)$ we denote the minimum number of edges in a graph $G$ on $r(H_1, H_2, \ldots, H_r)$ vertices such that $G \rightarrow (H_1, H_2, \ldots, H_r)$. Although this notion has been around for quite some time (see Faudree and Schelp [7]) there are rather few results on the restricted Ramsey number in the literature. The goal of this note is to find the exact value of $r^*(C_n, C_k)$ where $k$ is odd and $n$ is large. Note that for such choice of $k$ and $n$, $r(C_n, C_k) = 2n - 1$ and one of the colorings used to show the lower bound is when we divide $K_{2n-2}$ into two equal parts and color all edges between them in the second color.

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Theorem 1.1. There exists a constant $n_0$ such that for every $n \geq n_0$ and any odd $k$, with $3 \leq k \leq n$, we have

$$r^*(C_n, C_k) = \lceil (n + 1)(2n - 1)/2 \rceil.$$ 

The structure of the paper is the following. In the next section we derive Theorem 1.1 from two Lemmata 2.1 and 2.2 on colorings of a pseudorandom graph $\tilde{G}_n$. A precise definition of $\tilde{G}_n$, and the way one can construct it, is given in Section 3, while the proofs of Lemmata 2.1 and 2.2 can be found in Sections 4 and 5 respectively.

We conclude this part of the paper with some remarks and comments. Note that calculating $r^*(C_n, C_n)$, in a way, supplements results related to Shelp’s problem who asked about the minimum $a$ so that every graph $G$ on $N = r(C_n, C_n)$ vertices and the minimum degree $aN$ has the property $G \rightarrow (C_n, C_n)$ (see Benevides et al. [2] for the solution of Shelp’s problem, and Łuczak and Rahimi [19,20] for the case of three colors).

Finding $r^*(H_\ell, H_\ell)$ can be also viewed as a special case of the following question: given a graph $H_\ell$ on $n$ vertices, and a function $k = k(\ell)$, what is the minimum number of edges in a graph $G$ on $r(H_\ell, H_\ell) + k$ vertices such that $G \rightarrow (H_\ell, H_\ell)$? Although in this formulation this problem looks slightly artificial, a moment of reflection reveals that it could be (and usually is) related to the following question on random graphs: what is the largest $\ell$ so that with probability $1 - o(1)$ each coloring of the edges of the random graph $G(n, p)$ with two colors leads to a monochromatic copy of $H_\ell$? As far as we know this problem was first addressed and solved for paths by Letzter [17] when $p = p(n) = \Theta(1/n)$, whereas for cycles and all range of $p = p(n)$ it was treated in the recent paper of Araújo et al. [1].

Finally, one can ask about the value of $r^*(C_n, C_k)$ when $k \leq n$ is an even number. This question seems to be particularly interesting when $k$ is much smaller than $n$. For instance, in the simplest case when $k = 4$, we have $r(C_n, C_4) = n + 1$ and so one can expect that $r^*(C_n, C_4) = n^{1/2+o(1)}$, but at this moment the lower and upper bounds for $r^*(C_n, C_4)$ we can prove are both far from the conjectured $n^{1/2+o(1)}$.

2. Proof of Theorem 1.1

Here we describe how to show Theorem 1.1 using two Lemmata 2.1 and 2.2 proofs of which we postpone until the following sections of the article. As we shortly see the lower bound for $r^*(C_n, C_k)$ is quite simple to obtain, so the main part of the proof is to find a graph $\tilde{G}_n$ with $r(C_n, C_k)$ vertices and $r^*(C_n, C_k)$ edges, and verify that any coloring of edges of $\tilde{G}_n$ leads to either $C_n$ in the first color, or $C_k$ in the other color.

For $\tilde{G}_n$ we take a pseudorandom $(n+1)$-regular (or almost $(n+1)$-regular, if $n$ is even) graph on $2n−1$ vertices which we call $n$-fit; in the following Section 3 we give a precise definition of $n$-fit graphs and describe how to construct them. Our argument consists of two parts. Firstly, we use the Regularity Lemma to verify that any coloring of $\tilde{G}_n$ without monochromatic $C_n$ and $C_k$ in the appropriate colors resembles the extremal coloring of $K_{2n−2}$ which avoids such $C_n$ and $C_k$, i.e. it contains a large induced bipartite subgraph in one of the colors. Our result, proved in Section 4, can be stated as follows.

Lemma 2.1. There exists $n_1$ such that for every $n \geq n_1$ the following holds. If $k \leq n$ is odd and the edges of an $n$-fit graph $\tilde{G}_n$ are colored with two colors so that there are no copies of $C_n$ in the first color and no copies of $C_k$ in the second color, then there exists two disjoint subsets of vertices of $\tilde{G}_n$, $V'$ and $V''$, $|V'|, |V''| \geq 0.99n$, such that all edges between $V'$ and $V''$ are of the same color.

Then we supplement the above with the following result proved in Section 5.
Lemma 2.2. There exists \(n_2\) such that for every \(n \geq n_2\) the following holds. Let us suppose that the edges of an \(n\)-fit graph \(\tilde{G}_n\) are colored with two colors so that there exists two disjoint subsets of vertices \(\tilde{G}_n, V'\) and \(V''\), \(|V'|, |V''| \geq 0.99n\), such that all edges between \(V'\) and \(V''\) are of the same color. Then the subgraph induced by one of the colors contains a copy of cycle \(C_\ell\) for each \(3 \leq \ell \leq n\).

Now we are ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Clearly, for \(n \geq \max\{n_1, n_2\}\) we must have \(\tilde{G}_n \to (C_n, C_k)\) since otherwise Lemmata 2.1 and 2.2 lead to a contradiction. Hence \(r^*(C_n, C_k) \leq \lceil (n + 1)(2n - 1)/2 \rceil\).

Moreover, each graph \(G = (V, E)\) such that \(|V| = r(C_n, C_k) = 2n - 1\) and \(|E| < \lceil (n + 1)(2n - 1)/2 \rceil\), contains a vertex \(v\) of degree at most \(n\). Let \(V' \subseteq V\) be a set of \(n - 1\) vertices which contains all, except at most one, neighbors of \(v\), and \(V'' = V \setminus V'\).

Now color all edges between \(V'\) and \(V''\) with the second color and all other edges by the first color. Then, there are no \(C_n\) with edges colored in the first color and no odd cycles with edges colored in the second color. □

3. \(n\)-FIT GRAPHS

Our proofs of Lemmata 2.1 and 2.2 rely on the fact that in the graph \(\tilde{G}_n\) we are to color the edges are ‘uniformly’ distributed. The following definition makes it precise. Here, for two sets of vertices \(S\) and \(T\) of a graph \(G = (V, E)\), we put
\[e(S, T) = |\{(v, w) : v \in S, w \in T, \{v, w\} \in E\}|,\]
i.e. \(e(S, T)\) denote the number of edges with one end in \(S\), the second in \(T\), where all edges contained in \(S \cap T\) are counted twice. Moreover, by \(N_G(v) = N(v)\) we denote all neighbors of the vertex \(v\) in \(G\).

Definition 3.1. A graph \(G = (V, E)\) is called \(n\)-fit if the following holds:

(A) \(|V| = 2n - 1\) and \(|E| = \lceil (n + 1)(2n - 1)/2 \rceil\);

(B) the minimum degree of \(G\) is \(n + 1\) (i.e. \(G\) is \((n + 1)\)-regular if \(n\) is odd and it has \(2n - 2\) vertices of degree \(n + 1\) and one of degree \(n + 2\) if \(n\) is even);

(C) for all \(v, w \in V, v \neq w\), we have
\[|N(v) \cap N(w)| - n/2 \leq n^{0.7};\]

(D) for all \(S, T \subseteq V\) we have
\[|e(S, T) - |S||T|/2| \leq n^{1.7}.\]

The main result of this section states that for every \(n\) large enough one can find an \(n\)-fit graph.

Lemma 3.2. There exists \(n_0\) such that \(n\)-fit graphs exist for each \(n \geq n_0\).

Proof. Although it is expected that a random graph with a degree sequence as described in (A) and (B) fulfills also conditions (C) and (D), instead of employing somewhat complicated formula for the number of graphs with a given degree sequence found by Liebenau and Wierman [16], we use the standard binomial model of random graph, which, due to the independence of edges, is much easier to deal with.

Consider the random graph \(G_{2n-1}\) which is chosen at random from all graphs with \(2n - 1\) vertices, or, equivalently, a graph with vertex set \([2n - 1] = \{1, 2, \ldots, 2n - 1\}\) in which each edge appears independently with probability \(1/2\). It is easy to verify using Chernoff bounds that for \(n\) large enough with positive probability (in fact with probability \(1 - o(1)\)) the graph \(G_{2n-1}\) has the following properties:
(a) for every vertex $v$ we have
$$|\deg(v) - n| \leq n^{0.6};$$

(b) for every pair of vertices $v, w$
$$| |N(v) \cap N(w)| - n/2| \leq n^{0.6};$$

(c) for every two subsets of vertices $S, T$
$$|e(S, T) - |S||T|/2| \leq n^{1.6}.$$ 

Thus, let $\hat{G} = ([2n - 1], \hat{E})$ be a graph for which (a), (b), and (c) hold. Clearly, it has also properties (C) and (D) with a large margin. In order to adjust its degree sequence we modify slightly the edge set of $\hat{G}$ and show that there exists sets $E_1, E_2 \subseteq [2n - 1]^{(2)}$ such that the graph $\hat{G}_n = ([2n - 1], (\hat{E} \setminus E_1) \cup E_2)$ has all vertices of degree $n + 1$, except, perhaps one of degree $n + 2$, and furthermore the maximum degree of the graph $G_1 = ([2n - 1], E_1 \cup E_2)$ is smaller than $136n^{0.6}$, so (C) and (D) hold and the graph $G_n$ is $n$-fit.

Let us first delete from $\hat{G} = ([2n - 1], \hat{E})$ some edges so that the resulting graph has maximum degree $n + 1$. We do it recursively examining all vertices of $\hat{G}$ one by one. For each vertex $v$ we need to delete not more than $\deg(v) - (n + 1)$ surplus edges incident to $v$ which connect $v$ to those of its neighbors whose degrees changed the least during the procedure. Note that from (a) it follows that in the whole process we remove $|E'_1| \leq 2n^{1.6}$ edges. We want to argue, by a direct recursive argument, that a graph $([2n - 1], E'_1)$ has maximum degree at most $7n^{0.6}$. To this end note that because of $|E'_1| \leq 2n^{1.6}$, in each stage at most $2n/3$ vertices has more than $6n^{0.6}$ incident edges deleted. Thus, since by (a) and the recursive assumption in each step of the process the vertex $v$ has at least $n - n^{0.6} - 7n^{0.6} > 2n/3 + n^{0.6}$ neighbors in $\hat{G}$, we never delete any edge which join $v$ to its neighbor who already lost at least $6n^{0.6}$ edges incident to it. Because we remove at most $n^{0.6}$ edges incident to $v$, its degree in $([2n - 1], E'_1)$ is at most $7n^{0.6}$.

Now we need to add edges to $\hat{G} = ([2n - 1], \hat{E} \setminus E'_1)$ so that the resulting graph has the correct degree sequence. Note that by (a) and the estimate for $E'_1$ we need to add to it $|E''| \leq 4n^{1.6}$ edges. In fact we add to it $E_2, E_2 = 2|E''|$, edges and delete another $E'_1$, $|E'_1| = |E''|$, edges recursively in the following way. Let us suppose that two vertices, $v'$ and $v''$ have degree smaller than required. We find two other vertices, $w'$ and $w''$, such that $w'$ is not incident to $v'$, $w''$ is not incident to $v''$, but $w'$ and $w''$ are connected by an edge. Then we delete from the graph an edge $w'w''$ (and put it in the set $E'_1$) and add to the graph (and to the set $E_2$) edges $v'w'$ and $v''w''$. In order to assure that the graph $\hat{G} = ([2n - 1], E'_1 \cup E_2)$ has small maximum degree for each pair of vertices $v'$ and $v''$ we select $w'$ and $w''$ in the following way. We first choose a set $W'$ of $n/10$ non-neighbors of $v'$ which are neighbors of $v''$ and which have smallest degree in currently generated part of $\hat{G} = ([2n - 1], E'_1 \cup E_2)$. In a similar way $W'' \subseteq V \setminus W'$ is the set of $n/10$ vertices of smallest degree, counted in currently generated part of $\hat{G} = ([2n - 1], E'_1 \cup E_2)$, in the set $N(v') \setminus N(v'')$. Note that because of $|E'_1| \leq 12n^{1.6}$, the graph $\hat{G}$ contains at most $n/5$ vertices of degree larger than $120n^{0.6}$ and since both sets $N(v') \setminus N(v'')$
and \( N(v') \setminus N(v) \) have at least \( n/2 - O(n^{0.6}) \geq n/3 \) vertices in the above procedure none of the sets \( W' \) and \( W'' \) contains a vertex whose degree has been changed more than \( 120n^{0.6} \) times. Moreover, by (c), between sets \( W' \) and \( W'' \) there exist at least \( |W'| |W''| / 2 - O(n^{1.6}) = \Theta(n^2) \) edges, so we can always choose two adjacent vertices \( w' \) and \( w'' \) from these two sets. Hence, since by (a) and the first part of the proof all vertices of \( \hat{G} \) have degree at least \( n - 8n^{0.6} \), the maximum degree of \( \hat{G} = ([2n - 1], E'_n \cup E_2) \) is bounded from above by \( 128n^{0.6} + 1 \).

Thus, we have shown that the maximum degree of \( ([2n - 1], E'_n \cup E_2) \) is bounded from above by \( 136n^{0.6} \). Hence from (b) it follows that for any two vertices \( v \) and \( w \) in \( \hat{G}_n = ([2n - 1], (\hat{E} \setminus (E'_n \cup E'_2)) \cup E_2) \) we have
\[
|N(v) \cap N(w)| - n/2 \leq n^{0.6} + 272n^{0.6} \leq n^{0.7},
\]
and, by (c), for any two sets \( S \) and \( T \) of vertices of \( \hat{G}_n \)
\[
|e(S, T) - |S||T|/2| \leq n^{1.6} + 136n^{0.6} \max\{|S|, |T|\} \leq n^{1.7}. \tag*{\square}
\]

4. Proof of Lemma 2.1

The main tool in the proof of Lemma 2.1 is the Regularity Lemma so we start with recalling some definitions and basic facts (for a survey of results concerning the Regularity Lemma see Komlós and Simonovits [14]).

Let \( G = (V, E) \) be a graph and let \( V_1, V_2 \subseteq V \) be a pair of disjoint subsets of vertices of \( G \). The density of \( (V_1, V_2) \) is defined as
\[
d(V_1, V_2) = \frac{e(V_1, V_2)}{|V_1||V_2|}.
\]

We say that a pair \( (V_1, V_2) \) is \( \varepsilon \)-regular for some \( \varepsilon > 0 \) if for every pair of subsets \( W_1 \subseteq V_1 \) and \( W_2 \subseteq V_2 \) such that \( |W_1| \geq \varepsilon |V_1| \) and \( |W_2| \geq \varepsilon |V_2| \) we have
\[
d(V_1, V_2) - d(W_1, W_2) | \leq \varepsilon.
\]

We call an \( \varepsilon \)-regular pair \( (V_1, V_2) \) strongly \( \varepsilon \)-regular if every vertex \( v_1 \in V_1 \) has at least \( d(V_1, V_2)|V_2|/10 \) neighbors in \( V_2 \), and every vertex \( v_2 \in V_2 \) has at least \( d(V_1, V_2)|V_1|/10 \) neighbors in \( V_1 \).

The next result states some elementary results on regular pairs.

**Fact 4.1.** For every \( 0 < \varepsilon < d/100 \) there exists \( n' \) such that for every subsets \( V_1, V_2 \) of vertices of \( G = (V, E) \) such that \( |V_1|, |V_2| \geq n' \) the following holds.

(i) For each \( \varepsilon \)-regular pair \( (V_1, V_2) \) of density \( d \) there are sets \( W_1 \subseteq V_1 \), \( |W_1| \geq (1 - 2\varepsilon)|V_1| \), \( W_2 \subseteq V_2 \), \( |W_2| \geq (1 - 2\varepsilon)|V_2| \), such that the pair \( (W_1, W_2) \) is strongly \( \varepsilon \)-regular.

(ii) If \( (V_1, V_2) \) is a strongly \( \varepsilon \)-regular pair and \( v_1 \in V_1 \), \( v_2 \in V_2 \) then for every odd \( k \) such that
\[
3 \leq k \leq 2(1 - 2\varepsilon/d) \min\{|V_1|, |V_2|\}
\]
there exists a path of length \( k \) which starts at \( v_1 \) and ends in \( v_2 \).

**Proof.** Both parts of the above results are well known and direct consequence of the definition. Thus, part (i) is obtained by removing vertices of small degree from both sets of \( \varepsilon \)-regular pairs. A path required in (ii) can be built greedily, when in each step we choose a vertex which has a lot of neighbors among vertices which do not belong to the part of the path we generated so far. \( \quare \)
Finally, let us define an \( \varepsilon \)-regular partition of a graph \( G = (V, E) \) to be a partition of vertices of \( G \) into \( \ell \) parts \( V = W_1 \cup W_2 \cup \cdots \cup W_\ell \) so that

\[
(i) \ |W_i| - |W_j| \leq 1, \quad \text{for all } 1 \leq i < j \leq \ell;
(ii) \text{ among } \binom{\ell}{2} \text{ pairs } (W_i, W_j) \text{ all but at most } \varepsilon \ell^2/2 \text{ are } \varepsilon \text{-regular.}
\]

Our argument is based on the following version of the well known Szemerédi’s Regularity Lemma.

**Lemma 4.2.** For every \( \varepsilon > 0 \) there exists \( S \) with the following property. For every graph \( G = (V, E) \) whose edges were colored with two colors, i.e. \( E = R \cup B \), there is a partition \( V = W_1 \cup W_2 \cup \cdots \cup W_S \) of vertices of \( G \) into \( \ell \) parts, where \( 1/\varepsilon \leq \ell \leq S \), which is \( \varepsilon \)-regular for both graphs \( G_R = (V, R) \) and \( G_B = (V, B) \).

The main motivation of using the Regularity Lemma for studying the Ramsey numbers of sparse structures such as cycles is based on the following simple observation due to Luczak [18]. Let us suppose that we apply Lemma 4.2 to a graph with some \( \varepsilon > 0 \). We construct a (2-colored) \( \varepsilon \)-reduced graph \( G_\varepsilon \) by replacing each subset \( W_i \) by a single vertex \( w_i \), and if a pair \((W_i, W_j)\) is \( \varepsilon \)-regular in both colors we color the edge \( w_iw_j \) with the color which appear more frequently among edges joining these sets (in case of a draw we can choose any of the colors, say, the first one). In such a way we obtain a 2-coloring of edges of a graph \( K_\varepsilon \) obtained from the complete graph from which we removed not more than \( \varepsilon \ell^2 \) edges. Note that if \( G \) is \( n \)-fit and the edge \( w_iw_j \) of \( G_\varepsilon \) is colored with some color, the density of the pair \((W_i, W_j)\) in \( \tilde{G}_n \) in this color is at least \( 1/5 \). It turns out that, instead of looking for a long monochromatic cycle in \( G \), it is enough to find large matching contained in one monochromatic component in the auxiliary graph \( G_\varepsilon \).

In order to make this statement precise, we say that a graph has property \( M_\ell \) if there exists a matching saturating \([\ell]\) vertices which is contained in one non-bipartite component of this graph. It is well known (see Luczak [18] and Figaj and Luczak [8, 9]) that if a graph induced by one of the colors of 2-colored \( \varepsilon \)-reduced graph \( G_\varepsilon \) has property \( M_\ell \), then \( G \) contains cycles in this color for every length from \( 3\ell \) to \((t/s - 12\varepsilon)2n\). Here we need a slightly stronger statement.

**Lemma 4.3.** Let us suppose that an \( n \)-fit graph \( \tilde{G}_n = (V, E) \) is colored with two colors, red and blue, and \( G_\varepsilon \) is an \( \varepsilon \)-reduced graph for this coloring for some \( \varepsilon \), where \( 0 < \varepsilon < 10^{-5} \). Then, for any constant \( a \), \( 0 < a < 1 \), the following holds. If the subgraph induced in \( G_\varepsilon \) by one of the colors, say blue, has property \( M_\ell \), then there exists a set \( W \subseteq V \) such that \( |W| \geq (a - 2\varepsilon)2n \) and for every two vertices \( w, w' \in W \) and each \( \ell \), \( 3\ell \leq \ell \leq (a - 12\varepsilon)2n \), there exists a monochromatic path of length \( \ell \) joining \( w, w' \).

**Sketch of the proof of Lemma 4.3.** The proof follows closely the argument presented in [8, 9, 18] so we just outline it omitting technical details. Consider a blue component of \( G_\varepsilon \) which contains a matching \( M \) saturating \([\ell]\) vertices. Let \( U \) be a unicyclic blue subgraph which contains an odd cycle and all edges of \( M \). It is easy to see that then there exists a closed walk \( L \) which goes through every edge of \( U \) twice. Thus, for every two vertices \( w', w" \) of \( U \) there exists two walks \( L_\varepsilon(w', w") \) and \( L_\varepsilon(w', w") \subseteq L \), of even and odd lengths respectively, which start at \( w' \), end in \( w" \) and contain each edge of \( M \). Note that none of these walks is longer than \( 2s \).

Now take all pairs \((W_t, W_a)\) which correspond to the edges in the matching \( M \) and delete from each of them some vertices in order to make them strongly \( \varepsilon \)-regular. The union of all sets which belong to these pairs is our set \( W \). Note that, by Fact 4.1(i), \( |W| \geq (a - 2\varepsilon)2n \).
Next, let us choose \( \ell, 3s \leq \ell \leq (a - 12\varepsilon)2n \), and take any two vertices \( v_i, v_j \in W \) such that \( v_i \in W_i \) and \( v_j \in W_j \). Then there exists in \( G^\varepsilon_n \) a blue walk \( L(w_i, w_j) \) of length at most \( 2s \) and of the same parity as \( \ell \) from \( w_i \) to \( w_j \) which traverses all the edges from the matching \( M \). Now ‘lift’ this walk to a blue path \( P \) in \( G_n \) of length at most \( 2s + 4 \leq 3s \) which joins \( v_i \) and \( v_j \) and uses at least one edge from each pair \((W_t, W_u)\) which corresponds to an edge of the matching \( M \). A slight increase in the length is caused by the fact that the ends of the path \( P \), vertices \( v_i \) and \( v_j \), could have few or no neighbors outside the strongly \( \varepsilon \)-regular pairs they belong to; for instance \( v_i \in W_1 \) could have no neighbors outside the strongly regular pair \((W_t, W_k)\). In this case we need to start the path \( P \) with vertices \( v_i u_1 u_2 \), where \( u_1 \in W_k, u_2 \in W_i \), where \( u_2 \) has a lot of neighbors in the next set which correspond to the second vertex of the walk \( L(w_i, w_j) \). Once we have such a path \( P \) we can increase its length by replacing each edge contained in a strongly \( \varepsilon \)-regular pairs by an odd path of required length (see Fact 4.1(ii)). \( \Box \)

Note that the above results allow us to study properties of colorings of \( K^\varepsilon_s \) instead of colorings of an \( n \)-fit graph \( \tilde{G}_n \). Thus, we show that if a coloring of \( \tilde{G}_n \) avoids monochromatic cycles of required length (which means that in \( G^\varepsilon_n \) monochromatic graphs either have no property \( \tilde{M}_t \) or contain no triangles), then in \( G^\varepsilon_n \) there exists a large monochromatic bipartite component (see Lemmata 4.6 and 4.7 below). In the final step of the proof, we ‘lift’ a part of this component to a large monochromatic bipartite graph in \( \tilde{G}_n \).

We start with a few observations on a graph \( K^\varepsilon_s \) which, let us recall, is obtain from the complete graph \( K_s \) by removing at most \( \varepsilon s^2 \) edges.

**Fact 4.4.** All but at most \( \sqrt{\varepsilon}s \) vertices of \( K^\varepsilon_s \) have at least \((1 - 2\sqrt{\varepsilon})s\) neighbors.

**Proof.** Indeed, otherwise the number of ‘missing’ edges would be larger than

\[
(\sqrt{\varepsilon}s) \cdot (2\sqrt{\varepsilon}s)/2 = \varepsilon s^2,
\]

contradicting the definition of \( K^\varepsilon_s \). \( \Box \)

**Fact 4.5.** Let \( \varepsilon > 0, s \geq 1/\sqrt{\varepsilon} \), and let \( W \subseteq V, |W| \geq 11\sqrt{\varepsilon}s \), be any set of vertices of \( K^\varepsilon_s \). Then there exists \( W \subseteq V, |\tilde{W}| \geq |W| - 2\sqrt{\varepsilon}s \), such that the graph \( K^\varepsilon_s[\tilde{W}] \) spanned by \( \tilde{W} \) in \( K^\varepsilon_s \) is hamiltonian and contains a triangle (and so it is a non-bipartite connected graph with a perfect matching).

**Proof.** We let \( \tilde{W} \) to be a set of vertices of \( V(K^\varepsilon_s) \) obtained from \( W \) by removing all vertices with degree in \( K^\varepsilon_s \) less than \((1 - 2\sqrt{\varepsilon})s\) plus, perhaps, one vertex more so that \( |\tilde{W}| \) is even. Then, from Fact 4.4, we get

\[
|\tilde{W}| \geq |W| - \sqrt{\varepsilon}s - 1 \geq 9\sqrt{\varepsilon}s,
\]

and

\[
\delta(K^\varepsilon_s[\tilde{W}]) \geq |\tilde{W}| - 4\sqrt{\varepsilon}s > |\tilde{W}|/2.
\]

Hence, from Dirac’s Theorem, \( K^\varepsilon_s[\tilde{W}] \) is hamiltonian and the above condition also clearly implies that it contains a triangle. \( \Box \)

Now we can state and prove two main results concerning the coloring of \( K^\varepsilon_s \).

**Lemma 4.6.** For every positive real \( \varepsilon \leq 10^{-5} \) and integer \( s \geq 1/\sqrt{\varepsilon} \) the following holds. If the edges of \( K^\varepsilon_s \) are partitioned into two graphs \( B \) and \( R \) in such a way that \( R \) does not have property \( M_{(1/2+13\varepsilon)s} \) and \( K_3 \not\subseteq B \), then \( B \) contains an induced bipartite subgraph with a bipartition \((W_1, W_2)\), such that \( |W_1|, |W_2| \geq (1/2 - 13\varepsilon)s \).
Proof. Since $K_3 \not\subseteq B$, from Fact 4.5 it follows that $R$ contains no independent sets on $11\sqrt{\varepsilon}s$ vertices. Consequently, none of components of $R$ with more than $22\sqrt{\varepsilon}s$ vertices is bipartite. Hence, our assumption that the red graph $R$ does not have property $M_{(1/2+13\varepsilon)s}$ implies that $R$ contains no components with more than 

$$(1/2 + 13\varepsilon)s + 11\sqrt{\varepsilon}s \leq (1/2 + 12\sqrt{\varepsilon})s$$

vertices.

Furthermore, from Fact 4.4 we infer that at most $\sqrt{\varepsilon}s$ vertices of $K^*_s$ have less than $(1 - 2\sqrt{\varepsilon})s > 0.9s$ neighbors. Consequently, because of $K_3 \not\subseteq B$, the set of vertices of degrees larger than 0.9s spans in $R$ only two components on vertex sets $W_1$, $W_2$,

$$(1/2 + 12\sqrt{\varepsilon})s \geq |W_1| \geq |W_2| \geq s - |W_1| - \sqrt{\varepsilon}s \geq (1/2 - 13\sqrt{\varepsilon})s.$$ 

Now the assertion follows from the fact that each pair of vertices from $W_i$ has a blue common neighbor in $W_j$, $\{i, j\} = \{1, 2\}$ and thereby, since $K_3 \not\subseteq B$, all the edges contained in either $W_1$ or $W_2$, must be red. \hfill \Box

Lemma 4.7. For every positive real $\varepsilon \leq 10^{-5}$ and a large enough integer $s$ the following holds. If the edges of $K^*_s$ are partitioned into two graphs $B$ and $R$ in such a way that none of them has property $M_{(1/2+13\varepsilon)s}$, then one of the colors contains an induced bipartite subgraph with a bipartition $(W_1, W_2)$, such that $|W_1|, |W_2| \geq (1/2 - 4\sqrt{\varepsilon})s$.

Proof. Benevides et al. [2] proved that if $t$ is large enough, then for every graph $G$ on $t$ vertices with the minimum degree at least 0.74$t$ we have $G \rightarrow (P_{0.66t}, P_{0.66t})$. Thus, because of $1 - 3\sqrt{\varepsilon} > 0.74$, from Fact 4.4 it follows that in one of the colors, say red, there exists a monochromatic path $P$ whose length is at least

$$0.66(1 - \sqrt{\varepsilon})s \geq 0.65s > (1/2 + 14\varepsilon)s.$$ 

Since the red graph $R$ does not have property $M_{(1/2+13\varepsilon)s}$, the red component containing $P$ must be bipartite with bipartition $(W_1, W_2)$, where $|W_1| \geq |W_2|$. All the edges inside $W_1$ and $W_2$ are blue, so our assertion and Fact 4.5 imply that

$$0.51s \geq (1/2 + 13\varepsilon + 2\sqrt{\varepsilon})s \geq |W_1| \geq |W_2| \geq 0.65s - |W_1| \geq 0.14s \geq 11\sqrt{\varepsilon}s.$$ 

Consequently, using Fact 4.5 again we infer that the blue subgraph induced by $W_1 \cup W_2$ contains two large cycles $C_1$ and $C_2$ with at least

$$|W_1 \cup W_2| - 4\sqrt{\varepsilon}s \geq 0.65s - 4\sqrt{\varepsilon}s > (1/2 + 13\varepsilon)s$$

vertices combined, both contained in non-bipartite blue components. Therefore, because $B$ does not have property $M_{(1/2+13\varepsilon)s}$, there is no vertex in $T = V \setminus (W_1 \cup W_2)$ with blue neighbors in both, $W_1$ and $W_2$. This, in turn, in view of Fact 4.4 and (4.1), implies that $|T| \leq \sqrt{\varepsilon}s$, and thereby

$$|W_1| \geq |W_2| = s - |T| - |W_1| \geq (1/2 - 4\sqrt{\varepsilon})s.$$ 

This completes the proof of Lemma 4.7. \hfill \Box

Proof of Lemma 2.1. Let us assume that, for some large enough $n_1$, integers $n \geq n_1$, and odd $\ell \leq n$ are given. Suppose also that we color the edges of an $n$-fit graph $G_n$ with two colors, red and blue, so that there are no red copies of $C_n$ and no blue copies of $C_\ell$. We argue that there exists two disjoint subsets of vertices $G_n$, $V_1$ and $V_2$, $|V_1|, |V_2| \geq 0.99n$, such that all the edges between $V_1$ and $V_2$ are of the same color.

To this end we apply the Regularity Lemma to a coloring of $n$-fit graph $G_n$ with some small enough $\varepsilon > 0$, so that all arguments hold. Consider an $\varepsilon$-reduced graph $G^\varepsilon_n$ and denote by $R$ and $B$ its subgraphs consisting of red and blue edges respectively.
We first observe that from our assertion and Lemma 4.3 it follows that the red graph $R$ does not have property $M_{(1/2+13\varepsilon)s}$. For the same reason, if $\ell \geq 3s$, the blue graph $B$ does not have property $M_{(1/2+13\varepsilon)s}$. Moreover, if $\ell \leq 7s$, $B$ does not contain a blue triangle $K_3$. Indeed, note that for even $\ell$, $4 \leq \ell \leq 7s$, the existence of a cycle $C_{\ell}$ in an $\varepsilon$-regular pair $(V_1, V_2)$ is guaranteed by Fact 4.1. If in addition also pairs $(V_1, V_3)$ and $(V_2, V_3)$ are $\varepsilon$-regular, again by Fact 4.1(i), we can find in $V_3$ a vertex with many neighbors in both $V_1$ and $V_2$, and thereby, using the $\varepsilon$-regularity of the pair $(V_1, V_2)$, a cycle $C_{\ell}$ of all odd length $3 \leq \ell \leq 7s$.

Thus we can apply Lemma 4.6, if $\ell \leq 7s$, and Lemma 4.7 otherwise, to $R$ and $B$ concluding that one of these graphs contains an induced bipartite subgraph with a bipartition $(W_1, W_2)$, such that $|W_1|, |W_2| \geq (1/2 - 13\sqrt{\varepsilon})s$. If, in addition, $\ell \leq 7s$, this bipartite subgraph is blue. So assume first that $G_s[W_1], G_s[W_2] \subseteq R$.

Now, Fact 4.5 tells us that for both $i = 1, 2$, there exists $\widehat{W}_i \subseteq W_i$, such that the graph $G_s[\widehat{W}_i]$ is connected, non-bipartite, and contains a perfect matching. In other words, $G_s[\widehat{W}_i]$ has the property $M_{0.498s}$ and thereby, in view of Lemma 4.3, there exists a set $V'_i$ of vertices of $G_n$, which is contained in the union of the sets corresponding to vertices $w \in \widehat{W}_i$, such that $|V'_i| \geq (0.498 - 2\varepsilon)n \geq 0.95n$, and every two vertices of $V'_i$ are connected in $G_n$ by a red path $P_k$ of length $k$ for all $k$,

$$3s \leq k \leq 0.9n \leq (0.495 - 12\varepsilon)(2n).$$

Consequently, if there exist two disjoint red edges in $\hat{G}_n[V'_1, V'_2]$, one can connect their ends by red paths of lengths $[\ell/2]$ and $[\ell/2] - 2$, thus obtaining a red cycle $C_{\ell}$, of any length $\ell$, $7s \leq \ell \leq n$. But $C_{\ell} \not\subseteq R$ so if we remove from $\hat{G}_n[V'_1, V'_2]$ ends of one red edge (if there exists any) we remove them all. Thus, we arrive at $V_1, V_2 \subseteq V$ with $|V_i| \geq |V'_i| - 1$ for $i = 1, 2$ and all edges joining $V_1$ and $V_2$ in red color.

In case $G_s[\widehat{W}_i] \subseteq B$ we have $\ell \geq 7s$ and one can repeat the above reasoning with swapped colors. \hfill \qed

5. PROOF OF LEMMA 2.2

Our goal in this section is to establish Lemma 2.2. To this end, in what follows, we study an $n$-fit graph $\tilde{G}_n = (V, E)$ whose edges are colored with two colors, red and blue, so that there exist two disjoint sets $V_1, V_2 \subseteq V$, $|V_1| = |V_2| = 0.99n$, such that all edges between these two sets are colored with the same color, say blue. Denote by $R$ and $B$ subgraphs of $\tilde{G}_n$ induced by red and blue edges respectively. Set $W = V \setminus (V_1 \cup V_2)$. Moreover, we assume that $n$ is large enough so that all inequalities below hold.

We start with a few simple observations. The first one says that most vertices have a lot of neighbors in both sets, $V_1$ and $V_2$.

**Claim 5.1.** All vertices, except at most one, have at least $0.23n$ neighbors in each of the sets $V_1$ and $V_2$.

**Proof.** Let us suppose that there exists a vertex $w$ of $\tilde{G}_n$, which has fewer than $0.23n$ neighbors in one of the sets $V_1$, $V_2$, say $V_1$. Each vertex $v$ shares with $w$ at least $0.5n - n^{0.7} > 0.49n$ neighbors, and so

$$|N(v) \cap V_2| \geq |N(v) \cap N(w)| - |N(w) \cap V_1| - |W| > 0.49n - 0.23n - 0.02n > 0.23n.$$ 

Moreover,

$$|N(v) \cup N(w)| = |N(v)| + |N(w)| - |N(v) \cap N(w)| \geq 1.5n - n^{0.7} > 1.49n,$$
so

\[ |N(v) \cap V_i| \geq |N(v) \cup N(w)| - |V \setminus V_i| - |N(w) \cap V_i| \geq 1.49n - 1.01n - 0.23n > 0.23n. \]

If \( \tilde{G}_n \) contains a vertex which has fewer than 0.23\( n \) neighbors in one of the sets we call it *special* and denote by \( s \). In order to simplify the notation we shall always assume that if such special vertex exists, it does not belong to \( V_1 \cup V_2 \). Otherwise we just remove \( s \) from \( V_1 \cup V_2 \) and the tiny change of the size of one of the sets \( V_i, V_j \) would not affect our estimates, which are always quite crude.

Now we show that if there are vertices with a lot of blue neighbors in both \( V_1 \) and \( V_2 \), we are done.

**Claim 5.2.** If there exists a vertex \( w \in V \) with at least \( n^{0.9} \) blue neighbors in both sets \( V_1 \) and \( V_2 \), then the blue graph \( B \) contains a cycle \( C_\ell \) for each \( \ell \) with \( 3 \leq \ell \leq n \).

**Proof.** Note first that the pair \((V_1, V_2)\) is strongly \( \varepsilon \)-regular in the blue graph for every constant \( \varepsilon > 0 \) (in fact one can take even \( \varepsilon = n^{-0.01} \)). Thus, by Fact 4.1(ii), each blue edge joining vertices \( v_1 \in V_1, v_2 \in V_2 \), is contained in a cycle of each even length \( \ell_e \), where \( 4 \leq \ell_e \leq n \). The very same Fact 4.1(ii) implies that \( w \) belongs to a blue cycle of odd length \( \ell_o \), provided \( 5 \leq \ell_o \leq n \). Finally, by the definition of \( n \)-fit graphs, the sets \( N_B(w) \cap V_1 \) and \( N_B(w) \cap V_2 \) are joined by more than \( n^{1.8} / 5 \) edges and all of them are blue, hence \( B \) contains a lot of triangles. \( \square \)

Therefore, from now on we assume that all vertices have a lot of blue neighbors only in one of the sets \( V_1, V_2 \). In particular, in view of Claim 5.1, sets \( W_1 \) and \( W_2 \), defined for \( i = 1, 2 \),

\[ W_i = \{ w \in V : |N_R(w) \cap V_i| \geq 0.22n \text{ and } |N_B(w) \cap V_i| \leq n^{0.9} \}, \]

(5.1) cover the whole set \( V \) except, perhaps, the special vertex \( s \). Note also that \( V_i \subseteq W_i \) for \( i = 1, 2 \). Our next claim states that the subgraph spanned in the red graph by each of the sets \( W_i, i = 1, 2 \), is pancyclic. In fact we prove a slightly stronger statements.

**Claim 5.3.** For every \( i = 1, 2 \) and each \( \ell \), where \( 3 \leq \ell \leq |W_i| \), the subgraph spanned by \( W_i \) in the red graph contains a cycle of length \( \ell \). Moreover, if for some \( i = 1, 2 \) there exists a vertex \( v \notin W_i \) which has at least two red neighbors in \( W_i \), then the red graph contains a cycle of length \( |W_i| + 1 \).

**Proof.** Let us first find red cycles of length \( \ell \leq 0.97n \). To this end, take any vertex \( w \in V_i \), and partition \( V_i \setminus \{ w \} \) into two roughly equal sets \( V_i = W'_i \cup W''_i \) such that each of them contains at least \( 0.1n \) red neighbors of \( w \). Note that for each small constant \( \varepsilon > 0 \) the pair \((W'_i, W''_i)\) is \( \varepsilon \)-regular in \( \tilde{G}_n \) and, more importantly, by Claim 5.2, also in \( R \). Thus, by Fact 4.1(i) it contains a strongly \( \varepsilon \)-regular pair \((\tilde{W}'_i, \tilde{W}''_i)\) in \( R \) of almost the same size. From that we infer, in the same way as we did in Claim 5.2, that \( V_i \) contains cycles of all length \( \ell \) up to \( |V_i| - 30\varepsilon n \geq 0.97n \).

In search of longer cycles we use the well-known argument of Chvátal and Erdős [5]. Let us observe first that if \( v \) has at least two red neighbors in \( W_i \) then it is contained in a red cycle of length at least \( 0.96n \). Indeed, then one can split \( V_i \) into two equal sets such that one contains at least \( 0.1n \) red neighbors of one red neighbor of \( v \), the other contains at least \( 0.1n \) red neighbors of the other red neighbor of \( v \), and repeat verbatim our previous argument.

Now let us assume that we have already constructed a red cycle \( C_\ell = v_1v_2 \cdots v_\ell v_1 \) for some \( \ell \geq 0.96n \), and let \( w \) denote a vertex of \( W_i \) which does not belong to it.
Since \( w \) has at least 0.22\( n \) red neighbors in \( V_i \), at least 0.17\( n \) of them lay on \( C_\ell \). We say that a red neighbor of \( w \) in \( V(C_\ell) \) is good if its predecessor in \( C_\ell \) belongs to \( V_i \). Denote the set of predecessors in \( C_\ell \) of all good vertices by \( U \). Observe that because of \( |W_1 - V_i| \leq 0.02 n \), \(|U| \geq 0.1n\) and thereby \( G_n[U] \) contains \( O(n^2) \) edges. This however, in view of (5.1), implies that there are two red good neighbors of \( w \), say \( v_i \) and \( v_j \), \( i < j \), whose predecessors are adjacent in \( \hat{G}_n \) by a red edge. Then, the cycle
\[
v_1v_2 \cdots v_{i-1}v_{j-2} \cdots v_iwv_jv_{j+1} \cdots v_tv_1
\]
is a red cycle of length \( \ell + 1 \) (see Figure 5.1). In such a way we can include to the cycle all remaining vertices of \( W_i \) one by one.

\[\Box\]

Proof of Lemma 2.2. From Claim 5.3 it follows that we are done unless \(|W_1| = |W_2| = n - 1\), \( W_1 \cap W_2 = \emptyset \), and the special vertex \( s \) exists and has at most one red neighbor in each of the sets \( W_1 \) and \( W_2 \). Note that since \( \delta(\hat{G}_n) = n + 1 \), \( s \) must have in each of the sets \( W_1 \), \( W_2 \), at least one blue neighbor. Moreover, by Claim 5.2, in one of these sets, say \( W_2 \), \( s \) has fewer than \( n^{0.9} \) blue neighbors. Since the degree of \( s \) is at least \( n + 1 \) so it has at least \( n - 1 - n^{0.9} \) blue neighbors in \( W_1 \) and so
\[
|N_B(s) \cap V_1| > n - 1 - n^{0.9} - |W| > 0.97n. \tag{5.2}
\]

Now let \( w \) denote a blue neighbor of \( s \) in \( W_2 \). Note that if \( w \) has at least two red neighbors in \( V_1 \) we are done by Claim 5.3. Thus, in view of Claim 5.1, \( w \) must have at least \( 0.23n - 2 \geq 0.2n \) blue neighbors in \( V_1 \). We show that in this case the graph \( B \) contains a blue cycle \( C_\ell \) of any length \( \ell \), \( 3 \leq \ell \leq n \).

Let \( v_1 \) denote and blue neighbor of \( w \) in \( N_B(s) \cap V_1 \), \( v_2 \) stand for a blue neighbor of \( v_1 \) in \( V_2 \) such that \( v_2 \neq w \), and let \( v_3 \) be a blue neighbor of \( v_2 \) in \( \{N_B(s) \cap V_1 \} - \{v_1\} \). Note that from (5.2) and Claim 5.1 it follows that such vertices \( v_1 \), \( v_2 \), \( v_3 \) always exist. Then \( swv_1s \) and \( swv_1v_2v_3s \) are blue cycles of length 3 and 5, respectively. Moreover, the pair \((V_1 - \{v_1\}, V_2 - \{w\})\) is strongly \( \varepsilon \)-regular for each constant \( \varepsilon > 0 \), so Fact 4.1(ii) implies that for every odd length \( \ell_o \), \( 7 \leq \ell_o \leq n + 4 \) the path \( v_3svw_1v_2 \) is contained in an odd blue cycle of length \( \ell_o \) and the blue edge \( v_2v_3 \) belongs to an even blue cycle of length \( \ell_e \) for each \( 4 \leq \ell_e \leq n \). Thus, the blue graph \( B \) contains cycles of every length between 3 and \( n \). This completes the proof of Lemma 2.2. 

\[\Box\]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.2.png}
\caption{Constructing blue cycles. Short odd cycles \( swv_1s \) and \( swv_1v_2v_3s \) are clearly visible, while all odd cycles of length at least 7, as well as all even cycles, are constructed using wavy paths of appropriate lengths.}
\end{figure}


**References**

[1] Pedro Araújo, Matías Pavez-Signé, and Nicolás Sanhueza-Matamala, *Ramsey numbers of cycles in random graphs*. Submitted.

[2] F. S. Benevides, T. Łuczak, A. Scott, J. Skokan, and M. White, *Monochromatic cycles in 2-coloured graphs*, Combin. Probab. Comput. **21** (2012), no. 1-2, 57–87. MR2900048

[3] Deepak Bal and Louis DeBiasio, *New lower bounds on the size-Ramsey number of a path*, The Electronic Journal of Combinatorics **29** (2022), no. 1, #P1.18. MR4396465

[4] József Beck, *On size Ramsey number of paths, trees, and circuits. I*, Journal of Graph Theory **7** (1983), no. 1, 115–129. MR0693028

[5] V. Chvátal and P. Erdös, *A note on Hamiltonian circuits*, Discrete Math. **2** (1972), 111–113, DOI 10.1016/0012-365X(72)90079-9. MR297600

[6] Andrzej Dudek and Paweł Prałat, *On some multicolor Ramsey properties of random graphs*, SIAM Journal on Discrete Mathematics **31** (2017), no. 3, 2079–2092. MR3697158

[7] Ralph J. Faudree and Richard H. Schelp, *All Ramsey numbers for cycles in graphs*, Discrete Mathematics **8** (1974), no. 4, 313–329. MR0345866

[8] Agnieszka Figaj and Tomasz Łuczak, *The Ramsey number for a triple of long even cycles*, Journal of Combinatorial Theory, Series B **97** (2007), no. 4, 584–596.

[9] Agnieszka Figaj and Tomasz Łuczak, *The Ramsey numbers for a triple of long cycles*, Combinatorica **38** (2018), no. 4, 827–845.

[10] R. J. Faudree and R. H. Schelp, *A survey of results on the size Ramsey number*, Paul Erdös and his mathematics, II (Budapest, 1999), Bolyai Soc. Math. Stud., vol. 11, János Bolyai Math. Soc., Budapest, 2002, pp. 291–309. MR1954730

[11] László Gerencsér and András Gyárfás, *On Ramsey-type problems*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math **10** (1967), 167–170. MR0239997

[12] Penny Haxell, Yoshiharu Kohayakawa, and Tomasz Łuczak, *The induced size-Ramsey number of cycles*, Combinatorics, Probability and Computing **4** (1995), no. 3, 217–239. MR1356576

[13] Ramin Javadi, Farideh Khoeini, Gholam Reza Omidi, and Alexey Pokrovskiy, *On the size-Ramsey number of cycles*, Combinatorics, Probability and Computing **28** (2019), no. 6, 871–880. MR4045660

[14] J. Komlós and M. Simonovits, *Szemerédi’s regularity lemma and its applications in graph theory*, Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993), Bolyai Soc. Math. Stud., vol. 2, János Bolyai Math. Soc., Budapest, 1996, pp. 295–352. MR1395865

[15] Ramin Javadi and Meysam Miralaei, *Multicolor Size-Ramsey Number of Cycles*, arXiv preprint arXiv:2106.16023 (2021).

[16] Anita Liebenau and Nick Wormald, *Asymptotic enumeration of graphs by degree sequence, and the degree sequence of a random graph*, arXiv preprint arXiv:1702.08373 (2017).

[17] Shoham Letzter, *Path Ramsey number for random graphs*, Combin. Probab. Comput. **25** (2016), no. 4, 612–622, DOI 10.1017/S0963548315000279. MR3506430

[18] Tomasz Łuczak, *R(C_n, C_n, C_n) ≤ (4 + o(1))n*, Journal of Combinatorial Theory, Series B **75** (1999), no. 2, 174–187.

[19] Tomasz Łuczak and Zahra Rahimi, *On Schelp’s problem for three odd long cycles*, J. Combin. Theory Ser. B **143** (2020), 1–15, DOI 10.1016/j.jctb.2019.11.002. MR4089572

[20] , *Long monochromatic even cycles in 3-edge-coloured graphs of large minimum degree*, J. Graph Theory **99** (2022), no. 4, 691–714, DOI 10.1002/jgt.22760. MR4429175

[21] Stanisław Radziszowski, *Small ramsey numbers*, The Electronic Journal of Combinatorics **1000** (2021), DS1–Jan. MR1670625

[22] Vera Rosta, *On a Ramsey-type problem of JA Bondy and P. Erdős. I*, Journal of Combinatorial Theory, Series B **15** (1973), no. 1, 94–104. MR0332567
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