Invariant Stein domains in Stein symmetric spaces
and a non-linear complex convexity theorem

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Abstract

We prove a complex version of Kostant’s non-linear convexity theorem. Applications to the construction of $G$-invariant Grauert tubes of Riemannian symmetric $G/K$ spaces are given.

Introduction

Let $X = G/K$ be a semisimple non-compact Riemannian symmetric space. We may assume that $G$ is semisimple with finite center and we write $G = NAK$ for an Iwasawa decomposition of $G$. By our assumption, $G$ sits in its universal complexification $G_C$ and so $X \subseteq X_C := G_C/K_C$. Note that $X_C$ is a Stein symmetric space. Observe that the group $G$ does not act properly (i.e. with compact isotropy subgroups) on $G_C/K_C$ since $K_C$ is not compact. Write $U$ for the compact real form of $G_C$. The maximal connected subdomain of $GU_K/K_C \subseteq X_C$ on which $G$ acts properly and which contains $X$ was introduced in [AkGi90]. It is given by

$$\Xi := G \exp(i\Omega) K_C/K_C$$

where $\Omega$ is a polyhedral convex domain in $\mathfrak{a} = \text{Lie}(A)$ defined by

$$\Omega := \{ X \in \mathfrak{a} : (\forall \alpha \in \Sigma) \ |\alpha(X)| < \frac{\pi}{2} \}.$$ 

Here $\Sigma$ denotes the restricted root system with respect to $\mathfrak{a}$.

One of our principal aims is to construct a broad class of $G$-invariant Stein subdomains in $\Xi$. Let us remind the reader that it is a notoriously difficult problem to verify that certain unions of orbits of non-compact groups are Stein. For example the problem posed in [AkGi90] whether $\Xi$ is Stein was unsolved until the last year.

The Iwasawa decomposition on $G$ cannot be holomorphically extended to the whole group $G_C$; we have that $N_C A_C K_C \not\subseteq G_C$ is an open Zariski dense subset. The Iwasawa domain $\Xi_I$

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is defined as the maximal connected $G$-invariant subdomain in $X_{\mathbb{C}}$ which contains $X$ and is contained in $N_{\mathbb{C}}A_{\mathbb{C}}K_{\mathbb{C}}/K_{\mathbb{C}}$:

$$\Xi_I := (\bigcap_{g \in G} g(N_{\mathbb{C}}A_{\mathbb{C}}K_{\mathbb{C}}/K_{\mathbb{C}}))_0,$$

where $(\cdot)_0$ refers to the connected component containing $X$. Since $\Xi_I$ is the connected component of an open intersection of Stein domains, it is easy to see that $\Xi_I$ is Stein (cf. proof of Theorem 3.4).

The domains $\Xi$ and $\Xi_I$ again became of recent interest. One has $\Xi = \Xi_I$. Here are the results in chronological order:

- $\Xi \subseteq \Xi_I$ for all classical groups $G$ (cf. [KrSt01a]).
- $\Xi = \Xi_I$ for all classical groups $G$ (cf. [GiMa01]).
- $\Xi_I \subseteq \Xi$ for all $G$ (cf. [Ba01]).

In [KOS01] it will be shown that $\Xi \subseteq \Xi_I$ for the exceptional cases. Also it is announced in [Hu01] that $\Xi \subseteq \Xi_I$. This will then give us $\Xi = \Xi_I$ in the general case. In particular, $\Xi$ is Stein. Also, in the preprint [BHH01] a complex-geometric proof of the Steinness of $\Xi$ is given.

Geometrically, the domain $\Xi$ is a rather complicated object. If $G$ is a group of Hermitian type, then $\Xi = X \times X$ (cf. [BHH01], [GiMa01] or [KrSt01b]). If $X = G/K$ is classical, then there exists a group of Hermitian type $S \supseteq G$ with maximal compact subgroup $U \supseteq K$ such that $G/K$ is a real form of the Hermitian symmetric space $S/U$. In [BHH01] and [KrSt01b] a $G$-equivariant subdomain $\Xi_0 \subseteq \Xi$ was exhibited which is biholomorphic to $S/U$. Further one has $\Xi = \Xi_0 = S/U$ if and only if $\Sigma$ is of type $C_n$ or $BC_n$. In particular, if $\Sigma \neq C_n, BC_n$, then $S/U \subsetneq \Xi$ and the explicit geometric structure of $\Xi$ is very intricate. For many exceptional spaces $G/K$ the geometric structure is even more complicated.

The domain $\Xi$ (in [Gi98] it is called complex crown of $X$) is universal in the sense that many analytical and geometrical constructions on the Riemannian symmetric space $X$ extend to $\Xi$. In [KrSt01a] it was shown that the inclusion $\Xi \subseteq \Xi_I$ implies that all eigenfunctions on $X$ for the algebra of $G$-invariant differential operators $\mathbb{D}(X)$ extend holomorphically to $\Xi$. In [GiMa01] the problem $\Xi \subseteq \Xi_I$ is included in a broad class of geometrical problems connected with Matsuki duality. These geometrical problems include in particular the problem of the parametrization of compact complex cycles in flag domains (cf. [Wo92]).

Let us now come to the contents of this paper. Our first main result is a complex version of Kostant’s non-linear convexity theorem (cf. [Kos73]). Write $G \to A$, $g \mapsto a(g)$ for the middle projection in $G = NAK$.

**Theorem.** (Kostant) Let $X \in \mathfrak{a}$. Then

$$a(K \exp(X)) = \exp(\text{conv}(W X)),$$

where $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ denotes the Weyl group and $\text{conv}(\cdot)$ refers to the convex hull of $\cdot$.

One shows that the middle projection $a: G \to A$ holomorphically extends to

$$a: G \exp(i\Omega)K_{\mathbb{C}} \to A_{\mathbb{C}}.$$  

Then our main result is:

**Theorem A.** (Complex Convexity Theorem) For all $X \in \Omega$ we have that

$$a(G \exp(iX)) \subseteq A \exp(i \text{conv}(W X)).$$
Our convexity theorem features interesting applications to the geometry of the domain $\Xi$ and its generic subdomains which are defined as follows: Let $\omega \subseteq \Omega$ be a non-empty convex $W$-invariant open subset. Then we can form the domains

$$\Xi(\omega) = G \exp(i\omega)K_C/K_C.$$  

Note that $\Xi = \Xi(\Omega)$ and that the Iwasawa projection (1) naturally factors to a holomorphic mapping $a: \Xi \to A_C$. Finally we define for every $g \in G$ the horospherical tube $T(g, \omega) \subseteq X_C$ by

$$T(g, \omega) = g(N_CA\exp(i\omega)K_C/K_C).$$

As an application of Theorem A we now obtain:

**Theorem B.** Let $\omega$ be an open convex Weyl group invariant subset of $\Omega$. Then the following assertions hold:

(i) $a(\Xi(\omega)) = A\exp(i\omega)$.

(ii) $\Xi(\omega) = (\bigcap_{g \in G} T(g, \omega))_0$.

(iii) The domain $\Xi(\omega)$ is Stein.

Let us emphasize that in the case of $\Xi(\omega) = \Xi$, the inclusion $\Xi \subseteq \Xi_I$ means only the existence of an Iwasawa projection $a: \Xi \to A_C$; (i) in Theorem B gives a more precise information on the image of this projection. Further (ii) in the above theorem is a much stronger statement than (iii); in particular, (ii) implies (iii) since all horospherical tubes $T(g, \omega)$ are Stein.

Theorem B can be considered as an analogue of Lassalle’s results for compact symmetric spaces (cf. [La78]). We can interpret Theorem A and Theorem B as statements for $G$-orbits in $X_C$ intersecting $A\exp(i\Omega)K_C/K_C$ which in the case of compact symmetric spaces $G/K$ are true for arbitrary $G$-orbits (cf. [La78]).

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### 1. Notation

Let $G$ be a connected semisimple Lie group sitting inside a complexification $G_C$. We denote by $\mathfrak{g}$ and $\mathfrak{g}_C$ the Lie algebras of $G$ and $G_C$, respectively. Let $K < G$ be a maximal compact subgroup and $\mathfrak{k}$ its Lie algebra. Denote by $\theta: G \to G$ a Cartan involution which has $K$ as a fixed point set.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition attached to $\mathfrak{k}$. Take $\mathfrak{a} \subseteq \mathfrak{p}$ a maximal Abelian subspace and let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}) \subseteq \mathfrak{a}^*$ be the corresponding root system. Related to this root system is the root space decomposition according to the simultaneous eigenvalues of $\text{ad}(H), H \in \mathfrak{a}$:

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^\alpha,$$

here $\mathfrak{m} = \mathfrak{g}_\mathfrak{k}(\mathfrak{a})$ and $\mathfrak{g}^\alpha = \{X \in \mathfrak{g} : (\forall H \in \mathfrak{a}) [H, X] = \alpha(H)X\}$. For the choice of a positive system $\Sigma^+ \subseteq \Sigma$ one obtains the nilpotent Lie algebra $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}^\alpha$. Then one has the Iwasawa decomposition on the Lie algebra level

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}.$$
We write $A, N$ for the analytic subgroups of $G$ corresponding to $a$ and $n$. For these choices one has for $G$ the Iwasawa decomposition, namely, the multiplication map

$$N \times A \times K \to G, \quad (n, a, k) \mapsto n a k$$

In particular, every element $g \in G$ can be written uniquely as $g = n(g)a(g)\kappa(g)$ with each of the maps $\kappa(g) \in K$, $a(g) \in A$, $n(g) \in N$ depending analytically on $g \in G$. The last piece of structure theory we shall recall is the little Weyl group. We denote by $W = N_K(a)/Z_K(a)$ the Weyl group of $\Sigma(a, g)$.

Finally we define the domain

$$\Omega = \{X \in a: (\forall \alpha \in \Sigma) |\alpha(X)| < \frac{\pi}{2}\}.$$

Clearly $\Omega$ is convex and $W$-invariant.

## 2. The complex convexity theorem

Let us first give the relevant notation. Denote by $K_C, A_C, N_C$ the complexifications of $K, A$ and $N$. Then $N_C A_C K_C$ is a proper Zariski-open, hence dense subset of $G_C$. Throughout this paper we will assume that $\Xi \subseteq \Xi_I$ holds, i.e.,

$$G \exp(i\Omega) \subseteq N_C A_C K_C$$

(see the introduction for proofs of this statement). We now set $T_\Omega := A \exp(i\Omega) \subseteq A_C$. Then $G T_\Omega \subseteq N_C A_C K_C$.

One can show that one has a well defined holomorphic middle projection $N_C A_C K_C \to A_C/(A_C \cap K_C)$. Due to the simple connectedness of $T_\Omega$, this projection restricted to $G T_\Omega$ lifts to $A_C$ (cf. [KrSt01a, proof of Th. 1.8(iii)]) and we obtain an analytic mapping

$$G \times T_\Omega \to A_C, \quad (g, a) \mapsto a(ga)$$

holomorphic in the second variable such that

$$ga \in N_C a(ga) K_C$$

holds.

If $V$ is a vector space and $E \subseteq V$ is a subset, then we denote by $\text{conv}(E)$ the convex hull of $E$ in $V$.

**Theorem 2.1.** (Complex Convexity Theorem) Assume that $G$ is classical. Then we have for all $X \in \Omega$ that

$$a(G \exp(iX)) \subseteq A \exp(i \text{conv}(WX)).$$

The proof of Theorem 2.1 will be given in several steps.

Fix $a \in T_\Omega$ and consider the function

$$f_a: K \to a_C, \quad k \mapsto \log a(ka).$$

Further we write $p_a: g_C \to a_C$ for the projection along $\xi_C + n_C$. For $x \in G T_\Omega K_C \subseteq N_C A_C K_C$ we write $b(x) = n(x) a(x)$ for the triangular part of $x.$
Lemma 2.2. For any $a \in T_\Omega$, $k \in K$ and $X \in \mathfrak{k}$ we have

$$\frac{d}{dt} \bigg|_{t=0} f_a(\exp(tX)k) = p_{a_{\mathbb{C}}}(\Ad(b(k)a))^{-1}X).$$

Proof. This result can be easily deduced from the known case for $a \in A$ by analytic continuation. However, for convenience for the reader, we briefly recall the proof. We have

$$\frac{d}{dt} \bigg|_{t=0} f_a(\exp(tX)k) = \frac{d}{dt} \bigg|_{t=0} \log a(\exp(tX)ka) = \frac{d}{dt} \bigg|_{t=0} \log a(\exp(tX)b(ka)) = \frac{d}{dt} \bigg|_{t=0} \log a(\exp(tX)b(ka)) + \log a(ka) = \frac{d}{dt} \bigg|_{t=0} \log a(\exp(t\Ad(b(ka)^{-1})X)) = p_{a_{\mathbb{C}}}(\Ad(b(ka))^{-1}X),$$

proving the lemma. \(\blacksquare\)

Write $g_\mathbb{C}$ for $g_\mathbb{C}$ considered as a real Lie algebra and $\kappa_\mathbb{R}$ for the Cartan-Killing form on $g_\mathbb{C}$. Let $\kappa$ be the Cartan-Killing form on $g$ and recall the following relation between $\kappa$ and $\kappa_\mathbb{R}$:

$$(\forall X, X', Y, Y' \in g) \quad \kappa_\mathbb{R}(X + iyX', iyY') = 2(\kappa(X, X') - \kappa(Y, Y')).$$

For every $\lambda \in (g_\mathbb{C})^*$ we define $H_\lambda \in g_\mathbb{C}$ by $\lambda(X) = \kappa_\mathbb{R}(X, H_\lambda)$ for all $X \in g_\mathbb{C}$.

For every $\lambda \in (a_\mathbb{C})^*$ and $a \in T_\Omega$ we now define the function

$$f_{a, \lambda}: K \to \mathbb{R}, \quad k \mapsto \lambda(f_a(k)).$$

Lemma 2.3. Let $a \in T_\Omega$ and $k \in K$. Then the following assertions hold:

(i) For all $X \in \mathfrak{k}$ one has $\frac{d}{dt} \bigg|_{t=0} f_{a, \lambda}(\exp(tX)k) = \kappa_\mathbb{R}(X, \Ad(n(ka))H_\lambda)$.

(ii) We have $df_{a, \lambda}(k) = 0$ if and only if

$$\mathfrak{k} \perp_{\kappa_\mathbb{R}} \Ad(n(ka))H_\lambda.$$

Proof. (i) is immediate from Lemma 2.2 and the notations introduced from above. (ii) follows from (i). \(\blacksquare\)

Write $i\mathfrak{a}$ for the subspace of $(\mathfrak{a}_\mathbb{C})^*$ which vanishes on $\mathfrak{a}$. Our next goal is to determine the critical set of $f_{a, \lambda}$ for $\lambda \in i\mathfrak{a}_\mathbb{C}$.

We also denote by $\theta$ the holomorphic extension of the Cartan involution to $G_\mathbb{C}$. Further we write $G_\mathbb{C} \to G_\mathbb{C}$, $x \mapsto \overline{x}$ for the complex conjugation with respect to the real form $G$.

Lemma 2.4. Suppose that $X \in \Omega$ is regular. Set $a = \exp(iX)$ and let $k \in K$. Then the following implication holds

$$ka \in NA_\mathbb{C}K_\mathbb{C} \Rightarrow k \in N_K(a).$$

Proof. We write $ka = nbk'$ for $n \in N$, $b \in A_\mathbb{C}$ and $k' \in K_\mathbb{C}$. Then

$$ka^2k^{-1} = ka(\theta(ka)^{-1}) = nbk'(\theta(nbk'))^{-1} = nb^2\theta(n)^{-1}.$$

Set $x := ka^2k^{-1}$. Then

$$\overline{x} = ka^{-2}k^{-1} = x^{-1}.$$
Thus we obtain that
\[ x = \theta(n)b^{-2}n^{-1}, \]

Therefore \( x = \theta(n)b^{-2}n^{-1} \) and so
\[ x = \theta(n)b^{-2}n^{-1} \in A_C \theta(N_C). \]

Now let \( 0 < t \leq 1 \) and write \( a_t := \exp(itX) \) and accordingly we define \( x_t \). Then we obtain that
\[ \log x_t^2 = \log(k \exp(i4tX)k^{-1}) = Ad(k)(i4tX) \in a_C + \theta(N_C). \]

Thus the fact that \( H_\lambda \in i\mathfrak{a} \) is regular implies that \( Ad(n(ka))H_\lambda \in i\mathfrak{a} + (\mathfrak{n}_C \setminus \mathfrak{n}) \). But this contradicts \( Ad(n(ka))H_\lambda \in i\mathfrak{g} + \mathfrak{p} \), proving our proposition.

**Proposition 2.5.** Suppose that \( X \in \Omega \) and set \( a = \exp(iX) \). Let \( \lambda \in i\mathfrak{a}_R^* \) be such that \( H_\lambda \in i\mathfrak{a} \) is regular. Then we have
\[ df_{a,\lambda}(k) = 0 \iff k \in N_K(a). \]

**Proof.** The implication \( \iff \) follows immediately from Lemma 2.3 (ii).

\( \Rightarrow \) Suppose that \( df_{a,\lambda}(k) = 0 \). Then Lemma 2.3 (ii) implies that \( t\mathfrak{n}_\mathfrak{a} Ad(n(ka))H_\lambda \), or equivalently \( Ad(n(ka))H_\lambda \in i\mathfrak{g} + \mathfrak{p} \). Assume that \( k \notin N_K(a) \). By Lemma 2.4 we have \( n(ka) = \exp(Y) \) for some \( Y \in \mathfrak{n}_C \setminus \mathfrak{n} \). Hence
\[ Ad(n(ka))H_\lambda = H_\lambda + \sum_{\mathfrak{g} \in \mathfrak{g}} \left[ Y, H_\lambda \right] + \cdots. \]

Thus the fact that \( H_\lambda \in i\mathfrak{a} \) is regular implies that \( Ad(n(ka))H_\lambda \in i\mathfrak{a} + (\mathfrak{n}_C \setminus \mathfrak{n}) \). But this contradicts \( Ad(n(ka))H_\lambda \in i\mathfrak{g} + \mathfrak{p} \), proving our proposition.

**Proof of Theorem 2.1.** First we observe that it is sufficient to prove
\[ a(K \exp(iX)) \subseteq \exp(i \text{conv}(WX)) \]

for all \( X \in \Omega \). By a simple density/continuity argument we may further assume that \( X \) is regular. Suppose that there exists a \( k \in K \) such that \( a(k \exp(iX)) \notin \exp(i \text{conv}(WX)) \), or equivalently
\[ \text{Im} \log a(k \exp(iX)) \notin \text{conv}(WX). \]

Then we find a regular element \( \lambda \in i\mathfrak{a}_R^* \) such that
\[ f_{\exp(ix),\lambda}(k) > \max_{Y \in i \text{conv}(WX)} \lambda(Y). \]

But Proposition 2.5 implies that \( f_{\exp(ix),\lambda} \) takes its maximum at an element \( k \in N_K(a) \). Hence \( f_{\exp(ix),\lambda}(k) = \lambda(Ad(k)iX) \); a contradiction to our inequality from above.
3. Applications

Let now \( \omega \subseteq \Omega \) be an open Weyl group invariant convex set. Then we define the domain
\[
\Xi(\omega) = G \exp(i\omega)K_C/K_C.
\]
If \( \omega = \Omega \), then we set \( \Xi = \Xi(\Omega) \). Write \( \partial \Xi(\omega) \) for the topological boundary of \( \Xi(\omega) \) in \( G_C/K_C \).

Note the following properties of \( \Xi(\omega) \):

- \( \Xi(\omega) \) is open in \( G_C/K_C \) (cf. [AkGi90]).
- \( \Xi(\omega) \) is connected and \( G\)-invariant.
- \( G \) acts properly on \( \Xi \) (cf. [AkGi90]).
- One has \( G \exp(i\partial \omega)K_C/K_C \subseteq \partial \Xi(\omega) \). Moreover if \( \omega \subseteq \Omega \), then we have \( \partial \Xi(\omega) = G \exp(i\partial \omega)K_C/K_C \) (cf. [KrSt01b, Prop. 4.1]).

From our discussions in the previous section it is clear that we have a holomorphic projection
\[
a: \Xi_\omega \to A_C, \quad x \mapsto a(x)
\]
with \( x \in N_Ca(x)K_C/K_C \) for all \( x \in \Xi \). Finally we define the Abelian tube domain
\[
T_\omega := A \exp(i\omega) \subseteq A_C.
\]

An immediate consequence of our complex convexity theorem then is:

**Lemma 3.1.** We have
\[
a(\Xi(\omega)) = T_\omega.
\]

**Remark 3.2.** From Lemma 3.1 we obtain in particular that \( a(\Xi) \subseteq T_\Omega \). It is interesting to observe that this inclusion for \( G = \text{Sp}(n, \mathbb{R}) \) extends a result of Siegel.

Consider the vector space \( V := \text{Symm}(n, \mathbb{R}) \) of real symmetric matrices with its subcone of positive definite matrices \( V^+ \). Then we have the symmetric Siegel domain
\[
S^+ := V + iV^+ \subseteq V_C.
\]

Recall that \( S^+ \cong \text{Sp}(n, \mathbb{R})/U(n) \).

If we write \( \Delta_j \) for the \( j \)-th principal minor on \( V_C \), then Siegel’s Lemma says
\[
\Delta_j(z) \neq 0 \quad \text{for } z \in S^+
\]
and all \( 1 \leq j \leq n \). Now consider the rational functions
\[
\chi_j(z) = \frac{\Delta_j(z)}{\Delta_{j-1}(z)}
\]
on \( V_C \). Then the inclusion \( a(\Xi) \subseteq T_\Omega \) for \( G = \text{Sp}(n, \mathbb{R}) \) implies
\[
\text{Im } \chi_j(z) > 0 \quad \text{for } z \in S^+
\]
and for all \( j \). Note that (3.2) implies (3.1). To see this one identifies \( S^+ \) with the symmetric space \( \text{Sp}(n, \mathbb{R})/U(n) \). Then with \( S^- := \overline{S^+} \) one has a biholomorphism \( \Xi \cong S^+ \times S^- \) (cf. [GiMa01] or [KrSt01b]). Realizing \( A \) as diagonal matrices in \( G \), one then easily shows that \( a(\Xi) \subseteq T_\Omega \) implies (3.2).

The example discussed above admits a natural generalization to all tube domains \( V + iV^+ \) associated to an Euclidean Jordan algebra \( V \) and cone \( V^+ \).
For every \( g \in G \) we define the horospherical tube associated to \( \omega \) by

\[
T(g, \omega) := g(N_C T_\omega K_C/K_C) \subseteq G_C/K_C.
\]

Note that \( T(1, \omega) \) is biholomorphic to \( N_C \times T_\omega \). In particular, all horospherical tubes \( T(g, \omega) \) are Stein.

**Theorem 3.3.** For any non-empty open convex \( W \)-invariant set \( \omega \subseteq \Omega \) the domain \( \Xi(\omega) \) is the connected component of the intersection of horospherical tubes:

\[
\Xi(\omega) = \left( \bigcap_{g \in G} T(g, \omega) \right)_0.
\]

**Proof.** From Lemma 3.1 we obtain that

\[
\Xi(\omega) \subseteq N_C T_\omega K_C/K_C = a^{-1}(T(1, \omega)).
\]

Thus the fact that \( \Xi(\omega) \) is \( G \)-invariant and connected implies that \( \Xi(\omega) \subseteq \left( \bigcap_{g \in G} T(g, \omega) \right)_0 \). On the other hand we obtain from \( \Xi = \Xi_I \) (cf. the discussion in the introduction) and Lemma 3.1 that \( \Xi = \left( \bigcap_{g \in G} T(g, \Omega) \right)_0 \). From this and Lemma 3.1 we hence get

\[
\left( \bigcap_{g \in G} T(g, \omega) \right)_0 \subseteq \Xi \cap T(1, \omega) = \Xi(\omega),
\]

completing the proof of the theorem.

An interesting application of Theorem 3.3 is the following:

**Proposition 3.4.** Let \( \omega \subseteq \Omega \) be an open convex \( W \)-invariant set. Then the intersection

\[
I(\omega) := \bigcap_{g \in G} T(g, \omega)
\]

is open. In particular, \( I(\omega) \) and every connected component of \( I(\omega) \) is Stein. In particular, \( \Xi(\omega) \) is Stein.

**Proof.** (following a suggestion of Dmitri Akhiezer) Since \( G = KAN \) and since \( AN \) leaves \( T(1, \omega) \) invariant, we obtain

\[
I(\omega) = \bigcap_{k \in K} T(k, \omega).
\]

Hence \( I(\omega) \) is an intersection of open sets over the “compact parameter space” \( K \). In particular \( I \) is open. Since all horospherical tubes \( T(k, \omega) \) are Stein, we obtain that \( I \) is Stein. With \( I(\omega) \) all its connected components are Stein, concluding the proof of the theorem.

As a final application of the complex convexity theorem we prove a result on the characterization of the boundary of \( \Xi \).

**Proposition 3.5.** Let \( \omega \subseteq \Omega \) be an open convex \( W \)-invariant set. Let \( (z_n)_{n \in \mathbb{N}} \) be a sequence with \( z_n \to z_0 \in \partial \Xi(\omega) \). Then

\[
(\text{Im log } a(z_n))_{n \in \mathbb{N}}
\]

is a sequence in \( \omega \) and every accumulation point of this sequence lies in \( \partial \omega \).

**Proof.** Set \( X_n := \text{Im log } a(z_n) \). By Lemma 3.1 we have \( X_n \in \omega \). Since \( \overline{\omega} \) is compact in \( a \), we may assume that \( X_n \to X_0 \) with \( X_0 \in \mathfrak{Y} \).

If \( X_0 \not\in \partial \omega \), then we find a convex Weyl group invariant open set \( \omega_1 \) such that \( \overline{\omega_1} \subseteq \omega \) and \( X_n \in \omega_1 \). Then Theorem 3.3 implies that \( z_n \in \Xi(\omega_1) \) and so \( z_0 \in \Xi(\omega_1) \). Now \( \overline{\omega_1} \subseteq \Omega \) and so \( \partial \Xi(\omega_1) = G \exp(i\partial \omega_1)K_C/K_C \). Thus \( z_0 \in \Xi(\omega_1) \), contradicting the assumption \( z_0 \in \partial \Xi(\omega) \). This concludes the proof of the proposition.
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