FOUR DIMENSIONAL CONFORMAL GRAVITY, CONFINEMENT, AND GALACTIC ROTATION CURVES

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Abstract We review the current status and prospects for the conformal invariant fourth order theory of gravity which has recently been advanced by Mannheim and Kazanas as a candidate alternative to the standard second order Einstein theory. We examine how it is possible in principle to replace the Einstein theory at all while retaining its tested features, and we appeal to the wisdom gleaned from particle physics to suggest a covariant alternative. Specifically, motivated by the underlying scale invariance of the fundamental strong, electromagnetic and weak interactions, we propose that, just like their inertial counterparts in the other fundamental theories, gravitational mass and length scales should also be induced dynamically by spontaneous breakdown in a theory of gravity which itself possesses no fundamental scales at all. Indeed, in a sense this viewpoint is even mandated by the equivalence principle since once inertial masses are dynamical, it is natural to expect gravitational masses to be dynamical too. Given the connection of conformal gravity to particle theory, we explore the theory as a microscopic fundamental theory of elementary particles, to thus give curvature a more prominent role in elementary particle dynamics, and in particular in the generation both of elementary particle masses and the extended bag-like models which are thought to describe them. We discuss the degree to which conformal gravity can compete with string theory as a consistent candidate microscopic theory of gravity. Additionally, we explore the theory as a macroscopic theory of gravity where the exact, non-perturbative, classical potential is found to be of the confining $V(r) = -\beta/r + \gamma r/2$ form, a form which enables us to provide an explanation for the general systematics of galactic rotational velocity curves without the need to assume the existence of copious amounts of dark matter; with this explanation of the curves apparently not being in conflict with the current round of microlensing observations.

(1) The Case for Reconsidering the Standard Einstein Theory

At the present time there is little doubt in the general community as to the correctness of the standard second order Einstein theory of gravity which is based on the action

$$I_{EH} = -\int d^4x (-g)^{1/2} R^\alpha_\alpha /16\pi G$$

and gravitational equations of motion of the form

$$R_{\mu\nu} - g_{\mu\nu} R^\alpha_\alpha /2 = -8\pi G T_{\mu\nu}$$

However given the fact that this theory when applied to currently available astrophysical and cosmological data then requires that the universe be composed of overwhelming amounts of non-luminous or dark matter, it is a well established scientific tradition to pause and question so startling an implication, and to at least consider the possibility that this need for dark matter might instead actually be signaling a possible breakdown of the
standard Newton-Einstein theory on the largest distance scales. Given this possibility, there is then some value in going over the standard theory step by step to see whether there is anything in the entire package which still has a chance to go wrong. Thus we seek to identify which aspects of the theory are well observationally established and which are less so, so that we can then determine what it is that available observational data actually mandate, with this information then to be used as a guide to identify potential candidate gravitational theories. We thus argue not that the standard theory might eventually fail, but rather, we find\textsuperscript{8,9} it to still have a few weak links and loose ends which leave it in an exposed position. Moreover, identifying such loose ends is of interest in and of itself, independent of the merits or otherwise of the conformal theory we consider here, since their very elucidation sharpens our understanding and appreciation of the standard theory.

While the equivalence principle is one of the key underpinnings of General Relativity, it is important to note that the equivalence principle itself does not address the question of what specific form the gravitational equations of motion should take. Rather its success implies only that gravity in fact be a metric theory, with the gravitational field to be described uniquely by a covariantly coupled metric. Since geodesic motion follows from the covariant conservation of the standard test particle energy-momentum tensor, and since this local conservation itself follows purely from covariance in any general coordinate invariant gravitational theory (something which was pointed out by Eddington as long ago as the very early days of Relativity), one can conclude that geodesic motion itself only entails that the gravitational force be a general coordinate scalar. The dependence of the geodesics on the detailed dynamics of the theory actually only comes through the fact that it is the gravitational field equations which then determine the explicit form of the Christoffel symbols which enter into the geodesic equations. Moreover, Eddington also noted that since the three classic tests of General Relativity only probe the Schwarzschild solution associated with the Ricci flat vacuum outside of a source, the three tests can also be satisfied by having some higher derivative of the Ricci tensor vanish instead (this of course being the case in higher order gravity), since the Schwarzschild solution is still an exact solution in such cases. Hence by studying the region exterior to a source no information is provided on what $T_{\mu\nu}$ might be equal to in the interior region where it is non-zero. Or, in other words, Einstein gravity is sufficient to explain the classic tests but not necessary.

As regards the actual structure of the equations of motion in the $T_{\mu\nu}$ non-zero region, it is curious to note that Einstein did not in fact derive the familiar second order equations of motion of Eq. (2) by appealing to some fundamental principle (which would incidentally be more in keeping with the philosophical route he followed to reach the equivalence principle); rather he simply postulated Eq. (2) by noting, first, that these equations reduce for weak gravity to the standard second order gravitational Poisson equation

$$\nabla^2 \phi(r) = g(r)$$  \hspace{1cm} (3)

with its familiar exterior Newtonian potential solution, viz.

$$\phi(r > R) = -\frac{1}{r} \int_0^R dr' g(r') r'^2$$  \hspace{1cm} (4)

for a spherically symmetric, static source with radius $R$; and that, second, Eq. (2) yields relativistic corrections to this non-relativistic theory. The observational confirmation of these corrections on terrestrial to solar system distance scales not only established the
validity of the Einstein theory on those scales but seems to have established it on all others too, even though many other theories could potentially have the same leading perturbative structure on a given distance scale while differing radically elsewhere. We note the one way nature of the argument. Second order Einstein implies second order Poisson which in turn implies Newton. The reverse however is not true, and at the present time the Einstein theory is only sufficient to yield Newton’s Law of Gravity, but not yet necessary, to thus open the door to candidate alternative theories of gravity even at this late date.

To explore this point further, consider instead a fourth order Poisson equation

\[ \nabla^4 B(r) = f(r). \] (5)

For a spherical source Eq. (5) can be completely integrated in a closed form\(^8\) to yield

\[ B(r > R) = -\frac{r}{2} \int_0^R dr' f(r') r'^2 - \frac{1}{6r} \int_0^R dr' f(r') r'^4 \] (6)

as its exact exterior solution. The fourth order Poisson equation (which incidentally emerges as an exact equation of the relativistic fourth order conformal gravity theory below) thus also contains the Newtonian potential in its solution, thus divorcing Newton’s Law of Gravity (in principle at least) from both the second order Poisson equation and the second order Einstein theory, to thus make it manifest that the Einstein theory is in fact only sufficient to yield Newton but not necessary. Since the fourth order Poisson equation also yields the linear potential term in Eq. (6) which would then dominate over Newton at large distances, our very ability to use pure Newtonian gravity on the largest distance scales comes from the assertion that the Newtonian term is the leading term at infinity, something which, while widely believed, is not definitively supported by any known data at the present time. Indeed, there is not yet any evidence for the validity of Newton’s Law of Gravity at all on galactic or larger distance scales as the whole dark matter issue makes apparent; in fact, if galactic rotation curve data were the only data we had (i.e. if we had no solar system information at all), we would not be able to extract out a Newtonian Law at all. Thus the validity of the Einstein Equations (as opposed to the validity of covariance itself) rests not so much on the relativistic corrections, but rather on the validity of the second order Poisson equation and its Newtonian solution on distance scales much larger than those on which the Newton-Poisson picture was first established; and their validity in turn requires the universe to be predominantly non-luminous.

Searches to actually ascertain the baryonic dark matter content of our galaxy are currently under way via the MACHO, EROS and OGLE microlensing observations (see the contribution of D. Bennett, PASCOS 94 proceedings). However, the current microlensing counting rates seem to be indicating that while there are in fact more faint low mass stars and brown dwarfs in the Milky Way than we had anticipated, there does not appear to be the amount needed for rotation curve systematics (comment by D. N. Spergel in his talk at the PASCOS 94 conference), and perhaps not even for galactic stability, with a mass density in lenses perhaps of the order of that in luminous matter in the galaxy, rather than the sought after factor of ten or so more. Moreover, if a brown dwarf spherical halo is to explain rotation curves, it must additionally be found to be distributed with a core radius which is a factor or so larger than the scale length \(R_0\) of the observed surface brightness (\(\sim \exp(-R/R_0)\)) of the luminous disk, something which is simply not known at the present time. In the fitting to rotation curves, a pure Newtonian exponential disk gives a peak in the rotation curve at around \(2R_0\) (to thus require something additional at larger radii,
either a halo with a bigger core radius or some new gravity which acts on these larger distances). Moreover, the Newtonian disk $M/L$ mass to light ratio (taken to be a constant for a given galaxy) is usually normalized in the standard dark matter fits (and also in the conformal gravity fits shown in Fig. (1)) to the inner region maximum in the rotation curve. In these so called maximum disk fits the needed $M/L$ ratio is usually found to be a factor larger than that actually detected optically in the solar neighborhood. Since most of the microlensing events that have so far been detected are due to lensing off the bulge of the galaxy, they are essentially confirming that there are many non-luminous sources in the plane of galaxy, to thus actually lend support to the maximum disk fitting commonly used. For the standard dark matter fits, less than maximum disk is also permitted, with the halo (whose distribution is totally unrelated to that of the disk) then being parametrized so it can contribute in the inner region as well as the outer. However, for the conformal gravity fits to be presented below, the new linear potential term is integrated over the same disk distribution as the Newtonian one and thus must be maximum disk or else the fit would fall below the data at all radii. Lensing off the bulge thus appears to be supporting the maximum disk fitting required of the conformal theory. At regards lensing in the direction of the Magellanic clouds which explores the spherical halo, it currently appears that there are just not enough events, and not even as many as off the bulge. This might actually prove to be a severe problem for the standard dark matter theory, since as the known amount of matter in the plane of the disk is found to increase, the spherical halo would have to contain all that much more matter again if it is indeed going to stabilize the now much heavier disk. (This is not a problem for the conformal theory though - D. M. Christodoulou (Ap. J. 372, 471 (1991)) has shown that disks are actually stable under conformal gravity potentials without the need for any spherical halo at all). While the microlensing counting rates will be better known in the near future, an ultimate shortfall in the number of halo lenses would mean either that the bulk of galactic dark matter is non-baryonic, or that a new gravity theory is required. Moreover, even if the requisite dark matter is all there galactically (which incidentally would not actually exclude the conformal theory but rather simply set a strict upper bound on the strength of the linear potential term in Eq. (6)), and even if the conformal theory should eventually fail observationally, that would still only leave the Einstein theory as a sufficient theory of gravity, with the challenge then being to find a new fundamental principle which would make it necessary also. (Indeed, it is the very absence of any such underlying fundamental principle which has engendered problems such as the notorious cosmological constant problem, a problem for which the standard theory has no current answer and simply ignores by fiat in Eq. (2)). Since no fundamental principle is currently forthcoming, it is thus valid to explore alternate covariant gravitational theories which do possess some such principle.

(2) Conformal Gravity as a Macroscopic Theory

In recent times it has become fashionable in particle physics to consider theories with local invariances and with no fundamental length scales at the level of the Lagrangian at all; and indeed today all of the other fundamental interactions (the strong, electromagnetic, and the weak interactions) are all thought to be local scaleless gauge theories which can only acquire mass or length scales through dynamics. Consequently, it is both attractive and a possible road to a unification of all the fundamental forces to entertain the idea that gravity should also be a theory with no fundamental length scale either, and that it should also obey some analogous local principle; and indeed the principle of local conformal invariance of the spacetime geometry (i.e. invariance under local stretchings of the form $g_{\mu\nu}(x) \rightarrow \exp(2\alpha(x))g_{\mu\nu}(x)$) will precisely serve this purpose since it forces upon us a
theory of gravity with no fundamental length scale at all. This theory of gravity is known as conformal or Weyl gravity, and it is the unique four dimensional theory of gravity which meets all of our above stated needs. Thus we propose that gravity be based not on the Einstein-Hilbert action but rather on the conformal invariant fourth order action

$$I_W = -\alpha \int d^4x (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa}$$  \hspace{1cm} (7)$$

where $C_{\lambda\mu\nu\kappa}$ is the conformal Weyl tensor and $\alpha$ is a purely dimensionless coefficient. Over the years many previous authors have considered gravity based on Eq. (7) (Refs. (1-9) and the reviews of S. L. Adler (Revs. Mod. Phys. 54, 729 (1982)) and of A. Zee (Ann. Phys. 151, 431 (1983)) give some of the bibliography), and the theory has had an up and down history, being pronounced dead many times but never actually being completely buried. Conformal invariance was initially introduced by Weyl who wanted to use the same $\alpha(x)$ for both the conformal transformations and the electromagnetic gauge transformations as a way to unify the two long range forces. Standing in the way of such a proposal was the immediate realization that conformal invariance implied that all particles had to be massless, and so the theory could not progress. However, with the advent of modern spontaneously broken gauge theories, it is now apparent that mass can still be generated in the vacuum in otherwise dimensionless theories, thereby permitting us to revisit the issue. Motivated by the way the Fermi constant is generated in spontaneously broken weak interactions, it is immediately suggested to try to dynamically induce the Einstein action with its Newtonian constant as the macroscopic low energy limit of a microscopic conformal theory, a program considered recently by Adler and by Zee. Such a program if successful would then be sufficient to yield the standard gravitational phenomenology (and then of course require dark matter). However, as we noted above, such a program is not necessary. Observation does not require the recovering of the Einstein Equations, only the recovering of their solutions in the kinematic region where the solutions have been tested. Thus Mannheim and Kazanas suggested to eschew looking for an effective low energy Einstein limit altogether, and instead to consider the fourth order theory in and of itself as a macroscopic gravitational theory in its own right, to try to find exact solutions to it, and to confront it with observation directly. (In passing we note that string theory, the current most popular microscopic gravitational theory, involves not only the second order Einstein term but all other orders too. Moreover, fourth order terms are even induced as radiative corrections in the non-renormalizable second order Einstein theory. Thus there is no dispute that fourth order terms play some role in physics - and our interest here is simply in using data directly to determine the relative strengths of specific terms).

Since conformal gravity possesses no intrinsic fundamental length scale (thereby immediately excluding any cosmological term and thus naturally addressing this longstanding open problem\(^1\)), its gravitational equations must thus be fourth order rather than second order ones with Eq. (2) then being replaced by

$$4\alpha W_{\mu\nu} = 4\alpha(W_{\mu\nu}^{(2)} - W_{\mu\nu}^{(1)}/3) = T_{\mu\nu}$$ \hspace{1cm} (8)$$

where $W_{\mu\nu}$ is given by

$$W_{\mu\nu}^{(1)} = 2g_{\mu\nu}(R_{\alpha\beta})^{\gamma\delta} - 2(R_{\alpha\beta})_{;\mu;\nu} - 2R_{\mu\nu} R_{\alpha\beta} + g_{\mu\nu} (R_{\alpha\beta})^2/2$$

$$W_{\mu\nu}^{(2)} = g_{\mu\nu}(R_{\alpha\beta})^{;\gamma\delta} + R_{\mu\nu ;\gamma\delta} - R_{\mu;\beta ;\gamma\delta} - R_{\nu;\beta ;\gamma\delta} - 2R_{\mu\beta} R_{\nu\gamma\delta} + g_{\mu\nu} R_{\alpha\beta} R_{\alpha\beta}/2 \hspace{1cm} (9)$$
so that the theory is a lot less tractable than the Einstein one. Despite the severe computational difficulties that the theory possesses, Mannheim and Kazanas\textsuperscript{2} have actually been able to find its complete and exact solution exterior to a static, spherically symmetric, gravitating source, viz.

\[-g_{00} = 1/g_{rr} = 1 - \beta(2 - 3\beta\gamma)/r - 3\beta\gamma + \gamma r - kr^2 \tag{10}\]

where the parameters $\beta$, $\gamma$ and $k$ are three dimensionful integration constants which appear in the solution but not in the equations of motion and thus serve to spontaneously break the scale symmetry. As we see, the solution turns out to be none other than the analog and extension of the exterior Schwarzschild vacuum solution of Einstein gravity. We find that in the fourth order theory the Schwarzschild solution still obtains except that it is augmented by a new confining-type gravitational potential term which grows linearly with distance, which can have a strength such that this new term is unimportant on solar distance scales (so that the successes of Einstein gravity on those distance scales remain intact), and which can then first become important only galactically. Since the gravitational potential of the theory grows with distance it then follows that the rotational velocities of stars in galaxies should not fall as a function of distance; and we have recently\textsuperscript{7} obtained some good first fitting to the rotation curves of four typical galaxies (each one being a representative of four characteristic categories of rotation curve which correlate velocity with luminosity for the relevant galaxies) in the conformal theory which we present here as Fig. (1). As we can see, the rotation data are able to tolerate the presence of a linearly rising potential and admit of reasonable fitting. (The fitting so far has only been done by integrating the linear and Newtonian terms over the observed luminous matter distributions assuming constant mass to light ratios. As the microlensing data become more precise it will of course become necessary to include such sources in the fitting as well). Intriguingly, we find from the fits that for a galaxy the coefficient of its linear term is typically of order the inverse of the Hubble radius, to thus suggest an intriguing cosmological (Machian?) connection. Thus the viewpoint of conformal gravity is that the theory of gravity needs to be revised on galactic and larger distance scales, and that since the new potential term does grow with distance, the deviations from Newton-Einstein should be even more pronounced on even larger distance scales, this also being in accord with the gross observational trend actually found on large scales. (Actually treating the theory on large distance scales is quite subtle because at some point the linear potentials of all of the rest of the galaxies in the universe become competitive with that of a given galaxy of interest, with the given galaxy seeing not the total effect of all the other galaxies - that only contributes to the overall Hubble flow - but rather the deviation from homogeneity, a deviation which necessitates developing a theory of galaxy formation in the conformal theory).

As regards the cosmological implications of the fourth order theory, so far one exact Robertson-Walker type solution has been found in a simple scalar field model,\textsuperscript{5} which yields a topologically open $k < 0$ universe which expands but then nonetheless recollapses (because of the infrared slavery associated with the linear confining potential), and then rebounds from a finite minimum radius to oscillate indefinitely. The model thus provides a closed form exact solution to a fully relativistic cosmology which possesses no spacetime singularity at all. Additionally, the cosmology possesses no flatness problem (the flatness problem is not a generic property of Robertson-Walker cosmologies, it is a specific feature of the Einstein Equations, and thus avoided in the conformal theory which simply has a different set of equations of motion). Since there is no flatness problem in the model, there is thus no need to appeal to the popular inflationary universe, so that its need for large amounts of cosmological dark matter is thereby avoided.
In the interior region inside of a spherically symmetric, static gravitational source we find\(^8\) that in the conformal theory the fourth order Poisson equation of Eq. (5) emerges as an exact all order classical equation where the parameter \(B(r)\) is now given as the metric component \(-g_{00}(r)\) and where the spherically symmetric source function is given as

\[
f(r) = 3(T^0_0 - T^r_r)/4\alpha B(r)
\]  

(11)

Recognizing that the radial piece of \(\nabla^4 B(r)\) can be written as \((rB)^{''''}/r\), we thus immediately see that our exterior metric of Eq. (10) emerges as the most general solution to the fourth order Laplace equation \(\nabla^4 B(r) = 0\). Moreover, we can also match the solution of Eq. (6) to that of Eq. (10) to enable us to express the parameters of the exterior solution in terms of moments of the interior matter distribution according to

\[
\beta(2 - 3\beta\gamma) = \frac{1}{6} \int_0^R dr' f(r')r'^4, \quad \gamma = -\frac{1}{2} \int_0^R dr' f(r')r'^2
\]  

(12)

We thus see the somewhat unanticipated outcome that even though the Green’s function \(-|r - r'|/8\pi\) of the fourth order \(\nabla^4\) operator is linear in the distance \(r\), after integrating Eq. (5) with it to actually obtain Eq. (6), we find that we obtain not merely the linear potential term, but also the \(1/r\) Newtonian term of Eq. (6) as well, even though no second order Laplacian operator \(\nabla^2\) is present anywhere in the fourth order theory. Thus we see that while a second order Poisson equation is sufficient to generate a \(1/r\) potential, it is not in fact necessary, with Newton’s Law obtaining in the fourth order theory also.

As a potential macroscopic gravitational theory, conformal gravity admits of a wealth of eventual astrophysical and cosmological testing; and, indeed, it is paramount to study the dynamics of clusters of galaxies in the theory (where the effects of the linear potential will be even more prominent than in the individual galaxies themselves) as well as the detailed structure of galactic gravitational lensing (the most sensitive consequence of dark matter); while on stellar distance scales it is important to study the decay of the orbit of a binary pulsar in the theory (a test of the field nature of gravity and of the existence of gravitational radiation reaction which is present in any relativistically covariant theory where gravitational information is communicated retardedly with a finite velocity). On cosmological distance scales it is necessary to study the implications of the theory as they impact on the isotropy of the cosmic microwave background, on the cosmological mass content of the universe, on the development of large scale structure, on the growth of inhomogeneities and galaxy formation, and on primordial nucleosynthesis. (The initial analyses (L. Knox and A. Kosowsky, Fermilab preprint Pub-93/322-A, and also D. Elizondo and G. Yepes, Ap. J. 428, 17 (1994)) of nucleosynthesis in the single scalar field model cosmology presented in Ref. (5) is that the theory produces sufficient primordial helium but apparently not enough of the other light elements. It is thus crucial to develop a full theory of galaxy formation to see whether the associated presence of inhomogeneities can improve the situation). All of these galactic and cosmological astrophysical phenomena can ultimately provide for definitive observational testing of both conformal gravity and the standard second order Einstein theory.

(3) Conformal Gravity as a Microscopic Theory

While the solution of Eq. (12) shows that the Newtonian term will be obtainable for any spherically symmetric matter distribution, we note that its strength is related to the fourth moment of \(f(r)\) rather than to the second one, the case which occurs in the familiar
second order Einstein theory. Since this fourth moment would vanish for a delta function source, we see that in order to obtain a Newtonian potential in the fourth order theory the source must be extended rather than pointlike. While this violates our standard second order intuition (but not any observational information incidentally - the $1/r^2$ potential is the exact exterior solution to the second order Poisson equation in Eq. (4) no matter how the source $g(r)$ behaves in the interior region, with a delta function source being sufficient but not at all necessary to yield the Newtonian solution in the exterior), it is not all that surprising since our experience with dynamical mass generation in the other fundamental interactions (which we recall motivated our choice of locally conformal invariant gravity in the first place) indicates that we should anyway expect elementary particles to be extended soliton or bag-like objects rather than pointlike ones, with the only new feature here being that curvature must now play a role in producing such structures.

To emphasize this point, consider scalar and spinor fields $\psi(x)$ and $S(x)$ coupled conformally to gravity with matter action

$$I_M = -\int d^4x(-g)^{1/2}[S^\mu S_\mu/2 + \lambda S^4 - S^2 R^\mu_\mu/12 + i\bar{\psi}\gamma^\mu(x)(\partial_\mu + \Gamma_\mu(x))\psi - hS\bar{\psi}\psi] \quad (13)$$

where $\Gamma_\mu(x)$ is the fermion spin connection and $h$ and $\lambda$ are dimensionless coupling constants. As we can see, when the Ricci scalar is non-zero, it can induce a tachyonic Higgs mass term into a theory with no fundamental scales and no fundamental $-\mu^2 S^2$ term at all. Thus we propose that even though the curvature outside of elementary particles may be small, nonetheless curvature effects within them may well be substantial (perhaps even induced by gluon exchange, which itself may even be another way of describing curvature effects simply because of the equivalence principle). With this curvature effect then giving the fermion a Higgs mass in Eq. (13), we may then find that such elementary particles only have small curvature effects on each other; i.e. that just like in the nuclear shell model, most of the interaction is used up in producing the self-consistent states in the first place, with these states then only interacting with each other through weak residual forces. In this way microscopic curvature could be very strong, with it then almost all being used up just in order to produce massive fermions in the first place, with these fermions then only having weak residual gravity, so that we have to go to large macroscopic systems before the effects can build up enough to become observable. To establish such a picture we must thus look for extended bag like fermionic solutions based on Eq. (13) and its analogs.

Since the solutions in our theory are extended according to the above discussion, there is some concern as to whether such extended objects are compatible with positivity of the source distribution $f(r)$. To this end we have constructed an explicit though only illustrative candidate source which is positive definite everywhere, and which may be thought to describe extended elementary particles. Its specific form is

$$f(r) = -2p\delta(r)/r^2 - (3q/2)[\nabla^2 - (r^2/12)\nabla^4][\delta(r)/r^2] \quad (14)$$

where $p$ and $q$ are new intrinsic parameters which characterize the fundamental source, and its positivity (for an appropriate choice of the signs of $p$ and $q$) may be made manifest by writing the $q$-dependent part of the source as the limit $\epsilon \to 0$ of

$$f(r) = 6q\epsilon(9r^4 - 3\epsilon^4 - 10\epsilon^2 r^2)/\pi(r^2 + \epsilon^2)^5 \quad (15)$$

with $f(r)$ then being positive in this limit while trapping a singularity at the origin. For the source function of Eq. (14) we find that Eq. (12) yields $\beta(2 - 3\beta\gamma) = q$ and $\gamma = p$, which
shows that, because of the specific nature of the trapped singularity, the strengths of the linear and Newtonian potential terms of a fundamental source are in principle independent, with there thus being no relation of the form $\gamma \sim \beta / R_0^2$ where $R_0$ is the radius of the source. Thus, as might perhaps have been expected in a higher derivative theory, an elementary particle now comes with two rather than one intrinsic mass scales, an intriguing result which requires further study. Since the weak gravity potential outside of a fundamental source is just given by $V(r) = -q/2r + pr/2$, we see that for macroscopic weak gravity bulk matter of mass $M$, the coefficient of its ensuing macroscopic $1/r$ potential term is given as $Nq/2$, i.e. directly proportional to the total number, $N$, of microscopic singularities just as in the Einstein case where the same coefficient is identified as $MG$. Thus the very deep singularity in Eq. (14) takes care of the macroscopic fourth moment integral and leads us to a macroscopic inverse square gravitational force when $r \ll 1/\gamma$ which is universal (because $\alpha$ is universally coupled in Eq. (7)), which is an extensive function of the number of particles (as long as binding effects are unimportant), and which for spherical bulk matter only depends on the distance from the center of the source, just as desired for Newton’s Law. Thus having a fundamental Newton constant is only sufficient to give universal gravitation, but not apparently necessary. ($G$ is never defined independently in Newtonian gravity, only the product $MG$ is ever measured gravitationally - Newton’s constant thus has a status not unlike that of Boltzmann’s constant which only ever appears in the product $kT$). In the above way we can thus build up the macroscopic potentials of stars from their fundamental constituents and then use the stars as sources to calculate the weak gravity galactic geodesics which are used to fit the rotation curves shown in Fig. (1). (Of course, simply because of their spherical symmetry, stars will have potentials of the form $V(r) = -\beta/r + \gamma r/2$ (numerically the $\beta\gamma$ product in the $1/r$ term in Eq. (10) is found to be negligible) because that is the exact potential in Eq. (10) no matter how these stellar coefficients depend on the microscopic substructure. Thus the phenomenological fitting of Fig. (1) is completely insensitive to specific details of the fundamental microscopic $f(r)$). While the source function given in Eq. (14) is not of course mandated as the only possible source appropriate to the theory, its very existence serves as a counter-example to the claim often made in the literature that in the fourth order theory Newton’s law is incompatible with positivity of the matter distribution, and thus enables us to dispose of an objection to the conformal theory which had previously hindered its development.

As a microscopic theory the conformal theory suffers from two well known difficulties, namely the theory is found to possess ghosts and anomalies in lowest order perturbation theory when quantized around flat spacetime. As regards the trace anomaly, we note, that unlike its triangle anomaly counterpart, the trace anomaly is renormalizable. Thus as well as possibly being canceled by a group theoretical interplay between the fundamental fields of the theory (as part of a unification?) or by a non-trivial interplay between graviton and matter field loops (or even by quantizing around some other geometric background), it is also in principle possible to cancel it non-perturbatively by a Gell-Mann Low coupling constant renormalization eigenvalue (see e.g. S. L. Adler, J. C. Collins, and A. Duncan, Phys. Rev. D15, 1712 (1977)) which would then restore the underlying scale invariance of the theory only with anomalous dimensions. (With the advent of asymptotic freedom, renormalization group fixed points away from the origin fell somewhat into disfavor. Nonetheless, they still represent a viable non-perturbative option for field theory which has never been formally excluded; and it was noted quite some time ago (P. D. Mannheim, Phys. Rev. D12, 1772 (1975)) that when the dimension of the fermion $\bar{\psi}\psi$ bilinear is reduced from 3 to 2 by a fixed point in QED, the vacuum then undergoes spontaneous breakdown and generates dynamical masses. It would thus be of some inter-
est to see if an analogous situation obtains in the conformal case). As regards linearizing around flat spacetime in general, we note that unlike Einstein gravity where the matter free background is Riemann flat Minkowski space, for the conformal theory the natural background is a vanishing Weyl tensor, with the background metric then not being flat but only conformal to flat. (From the point of view of maximally symmetric 4 spaces, de Sitter, anti de Sitter and Minkowski spaces all have the maximal 10 Killing vectors, and all are flat or conformal to flat. Thus the background for gravity should perhaps be maximally symmetric rather than flat, with explicit energy-momentum tensor sources to which gravity couples then lowering the symmetry in specific physically interesting cases). Further, given the non-asymptotically flat metric of Eq. (10), the flat spacetime limit may not be all that relevant to the theory, with the restoring linear potential possibly even confining the ghosts and removing them from the physical spectrum in the true vacuum altogether (perhaps with a Gell-Mann Low eigenvalue actually setting the ghost residue to zero just like the analogous Landau ghost cancellation in quantum electrodynamics - moreover, if the dimension of $g_{\mu\nu}$ could be changed by one whole unit at such an eigenvalue, the $1/q^4$ propagator could then be non-perturbatively modified into the ghost free $1/q^2$). As regards the ghost question, we note additionally that in a first order perturbation expansion around flat spacetime the fourth order theory yields a fourth order box operator, to thus yield corrections to the metric which grow with distance (just like the linear potential) and which hence become indefinitely large at large distances. Since the ghost states would appear on shell at low momenta, we see that their presence in the theory is inferred in precisely the kinematic region where first order perturbation theory becomes untrustworthy. (Nonetheless, the ghosts are still free to contribute at very short distances far off the mass shell, to thus maintain the power counting renormalizability that the fourth order theory is known to have - basically the $1/q^4$ propagator is equivalent to two $1/q^2$ propagators, one being a regular graviton, and the other a ghost graviton which cancels the familiar non-renormalizable infinities associated with a single Einstein propagator). Thus, if anything, the ghosts are signaling only that flat spacetime is not a good limit to the theory, something which would anyway be expected given the structure of the exact, non-perturbative solution of Eq. (10). It is thus both worthwhile and crucial to explore the ghost and anomaly issues further, and in particular to quantize the theory around the metric of Eq. (10) rather than around flat spacetime.

As a microscopic theory the conformal theory also has many positive features. It is known to be renormalizable (K. S. Stelle, Phys. Rev. D16, 953 (1977)), asymptotically free (E. Tomboulis, Phys. Letts. 97B, 77 (1980)), and as we have now seen, explicitly confining. In that sense then it begins to compete with QCD. Moreover, given the analog between local gauge invariance and local scale invariance (the gauge transformation is a complex phase on the fields while the scale transformation is real one), and given the intriguing analog between the conformal $C_\lambda^{\mu\nu\kappa}C_\lambda^{\mu\nu\kappa}$ and gauge $F_{\mu\nu}F^{\mu\nu}$ actions, local scale invariance might provide the road to unification of gravity with the other interactions by making gravity look a lot more like the others. While work still needs to be done on the ghost and anomaly questions, we recall that string theory, another candidate quantum gravitational theory, languished for many years until M. B. Green and J. H. Schwarz (Phys. Letts. 149B, 117 (1984)) found a very clever higher dimensional anomaly cancellation mechanism. Certainly the enormous effort that has been made on string theory by a whole generation of researchers has yet to be made on the conformal theory, and it too could possess surprises; indeed much of our current understanding of strings and of gauge theories and of particle physics in general grew out of a need to cancel ghosts, so a ghost often proves to be a highly informative diagnostic. Thus at the very least conformal gravity merits
further study. Since conformal gravity sets out to be a completely consistent quantum theory of gravity (perhaps via a Gell-Mann Low eigenvalue), it sets out to provide an alternative to string theory, and, moreover, to do so directly in four spacetime dimensions from the beginning. In fact the coupling constant $\alpha$ in the conformal action of Eq. (7) is only dimensionless in four spacetime dimensions. Moreover, since conformal gravity comes with no fundamental scale (it demotes Newton’s constant from fundamental status), it stands in sharp contrast to string theory which gives extra special status to Newton’s constant over all other dimensionful fundamental physical constants. It may, however, be possible to establish some connection between the conformal and string theories in the zero string tension limit where string theory then possesses no intrinsic scale. In fact if a connection could be made, then as long as the known consistency of string theory survives in this limit, we would then be able to infer the consistency of scaleless gravity too and thus provide for a string based mechanism for solving the conformal theory ghost problem non-perturbatively. As regards the general issue of the relative merits of string gravity and conformal gravity, we note that one of the nice features of having a conventional renormalizable gravitational theory such as conformal gravity is that, just like QED, there should then be a direct correspondence between the microscopic and macroscopic theories, with the matrix elements of the quantum field in coherent states of its quanta then having a chance to have equations of motion of the same generic form as those of the underlying quantum field itself, simply because the primary role of renormalization is to then convert bare propagators and vertices into dressed ones. In this way the action of Eq. (7) could then be used for both microscopic and macroscopic physics. To contrast, we note that in string theory, however, the connection between microscopic and macroscopic fields is somewhat remote. Further, we note that string theory is not so much an attempt to construct a consistent quantum gravitational theory, but rather to construct one which reduces to Einstein gravity at low energies. (It does not quite do that incidentally since while it nicely recovers the Einstein term it also leads to a huge Planck density cosmological term as well. String theory thus makes the cosmological constant problem more not less severe, and this might even be an indicator of its lack of viability). As we have now seen in our discussion of macroscopic gravity, there may not in fact be any need to actually recover the Einstein Equations anyway, with the central question for microscopic physics in the end actually being what exactly the non-relativistic gravitational potential is on the largest distance scales.

(4) Conformal Gravity and the Structure of the Energy-Momentum Tensor

As we noted earlier, it was only with the emergence of spontaneously broken mass generation in strong, electromagnetic and weak interactions, that it became possible to return to the scaleless conformal gravitational theory. However, it is quite remarkable that the general community has shown no interest in incorporating any of this new Higgs based physics into the standard gravitational Einstein theory. Indeed, one of the most curious and disquieting aspects of the entire contemporary approach to the role of mass in fundamental theory is that mass is treated as being of dynamical origin for the purposes of the strong, electromagnetic and weak interactions, and yet is treated as being purely mechanical and kinematical for gravitational purposes; with the source of Einstein gravity being taken to be not of the Weinberg-Salam or QCD form, but rather to simply be a collection of mechanical particles with their familiar kinematic energy-momentum tensors, i.e. to be of the form thought to prevail when General Relativity itself was first written down. Thus, at the present time, essentially all macroscopic tests of General Relativity simply assume that gravity is produced by perfect fluid sources which are purely kinematic
in structure possessing purely mechanical energy densities and pressures. This purely Newtonian description of motion (after correction for relativistic kinematics) forms the cornerstone of gravitational exploration, and is regarded as being so sacrosanct and obvious as to not even require any further justification, even though such sources have never been shown to emerge in gauge theories, and even though string theory gives little guidance as to the explicit structure of the right hand side of Eq. (2).

That this Newtonian model for $T_{\mu\nu}$ was actually considered at all is because of too much of a reliance on the flat space limit of gravity, a limit in which one only measures the mechanical motions of particles as they move about, i.e. in which one only measures energy and momentum changes. In flat spacetime the zero of energy is not observable, only the energy of the particle excitations with respect to the vacuum, and yet the wisdom obtained from these excitations is then written in a general covariant form and posited as the source of gravity without further question. Now gravity responds not merely to energy and momentum changes, but also to the zero of energy and momentum, something which is of course recognized as a part of the cosmological constant problem and then ignored. Perhaps even more serious than not knowing how to set the zero of energy is the fact that we now know that particles get their masses (which then characterize the one particle excitations via the energy-momentum relation $E^2 = k^2 + m^2$) from Higgs fields which also carry energy and momentum, with this energy and momentum also being ignored in the standard treatment, again without justification. Thus in order to correctly include all of the Higgs effects and determine the zero of energy consistently, we not only need a theory of mass generation in a theory with no intrinsic scales, we also need to incorporate its implications into the structure of the gravitational source.

Contemporary particle physics leads us quite naturally to matter actions such as the scaleless Higgs one exhibited in Eq. (13), and even if one does not want to make the jump to conformal gravity on the gravitational side of the gravitational equations of motion, one should thus at least use the energy-momentum tensor associated with Eq. (13), viz. 

$$ T_{\mu\nu} = i\bar{\psi}\gamma_\mu(x)[\partial_\nu + \Gamma_\nu(x)]\psi + 2S_\mu S_\nu/3 - g_{\mu\nu}S^\alpha S_\alpha/6 - SS_{\mu;\nu}/3 + g_{\mu\nu}SS^\alpha/3 $$

$$ -S^2(R_{\mu\nu} - g_{\mu\nu}R/2)/6 - g_{\mu\nu}\lambda S^4 $$

as the source on the right hand side of the Einstein Equations in Eq. (2) since Eq. (16) and its analogs embody much of the wisdom of particle physics. Use of the scalar field equation of motion

$$ S^\mu_{\mu} + SR^\mu_{\mu}/6 - 4\lambda S^3 + h\bar{\psi}\psi = 0 $$

and of the covariant Dirac equation

$$ i\gamma^\mu(x)[\partial_\mu + \Gamma_\mu(x)]\psi - hS\psi = 0 $$

enable us to show that the conformal energy-momentum tensor is both kinematically traceless and covariantly conserved just as it should be. From Eq. (16) it is possible to derive some crucial results for gravitational theory. First, as can be seen directly from Eq. (16), in the constant $S(x) = S_0$ gauge not only is the full $T_{\mu\nu}$ covariantly conserved, but two separate pieces of it, namely the kinematic fermion piece $T_{\mu\nu}^{\text{kin}} = i\bar{\psi}\gamma_\mu(x)[\partial_\nu + \Gamma_\nu(x)]\psi$ and the remaining Higgs dependent piece $T_{\mu\nu}(S_0)$, are independently covariantly conserved also. Thus, while the fermion could share energy and momentum with the Higgs field which gives it its mass, it turns out that in fact it does not. Hence the one particle excitations
of the fermion are the same as they would be if the fermion simply had a mechanical bare mass, with matrix elements of the fermionic piece of the energy-momentum tensor being the same as the $T^{\text{kin}}_{\mu\nu}$ conventionally used in the standard theory, so that its associated covariant conservation then gives geodesic motion for such particles in an external field. Second, an incoherent averaging over a bath of the fermions of Eq. (18) then gives the fermion kinetic term the form of a kinematic perfect fluid, viz. $T^{\text{kin}}_{\mu\nu} = (\rho + p) U_\mu U_\nu + pg_{\mu\nu}$, with its covariant conservation then giving the familiar Euler hydrostatic equation when the fermions are the source of the gravitational field. Hence we see that both geodesic motion and perfect fluid hydrodynamics still occur in theories with mass generation, and that in and of themselves, they are simply totally misleading guides as to the structure of the full energy-momentum tensor of gravitational theory, since even while the particle motions may decouple from the Higgs fields, the gravitational field itself is still very much sensitive to them. Since this latter Higgs energy and the attendant back reaction on the geometry contained in $T_{\mu\nu}(S_0)$ are simply ignored in the standard Newtonian based picture of gravitational sources (even though $T^{\alpha\beta}(S_0)$ is equal to $-T^{\alpha\beta}(\text{kin})$ and thus as large), at the present time the entire standard treatment of Eq. (2) must be regarded as suspect. Thus for instance, if the source of gravity is the traceless Eq. (16), it would then lead in the standard theory to $R^{\alpha\beta} = 0$ and thus to cosmologies in which the scale factor $R(t)$ grows as $t^{1/2}$ even in the matter dominated era ($\rho = nm$), thus making it quite unclear as to what the standard theory would then produce by way of nucleosynthesis. Thus, at the present time, the nucleosynthesis successes of the standard theory must be regarded purely as a coincidence until some explanation is found for why we can model the entire $T^{\mu\nu}$ in Eq. (2) simply as $T^{\text{kin}}_{\mu\nu}$. And, moreover, the ignoring of the back reaction of the Higgs field on the geometry is in fact the cosmological constant problem.

If the reader is now prepared to make the full conformal jump on the gravitational side as well we are thus led to replace the Einstein Equations altogether by Eq. (8) coupled to Eq. (16) and its analogs, with a symmetry, conformal invariance, then dictating the structure of both sides of the gravitational equations, something which has up till now not been achieved in the standard theory. Since the entire theory is now prescribed, the cosmological term has no freedom, with there being no kinematic one in the equations of motion, and with the back reaction on the geometry sharply constraining its value if one is induced dynamically, with the tracelessness constraint actually forcing its magnitude to be the same as that of the contribution of the positive energy matter fields which propagate cosmologically and not 120 orders of magnitude larger (not that the conformal theory even possesses a Planck scale anyway). However, to solve the cosmological constant problem completely we still need to consider the contribution of the negative energy modes which fill the vacuum. These, of course, are the modes which give masses to the fundamental particles in the first place, with their back reaction on the geometry causing particles to become extended in the first place, so again there are constraints. In fact, based on Eqs. (8), (16-18), the following model of elementary particles is suggested. These equations are found to admit of an exact solution in which a kinematic perfect fluid of fermions couples to a constant scalar field in a static Robertson-Walker geometry with spatial curvature $k$. In the solution the energy density is found to be given by $\rho = -\lambda S_0^4 - S_0^2 k/2$, so that the negative energy modes can thus support a closed $k > 0$ universe, with elementary particles then emerging as trapped three surfaces (black holes are only trapped two surfaces). (In passing we note that $\rho > 0$ essentially gives the $k < 0$ cosmology of Ref. (5)). Since classically no energy or momentum, or even the vacuum energy, can be transported across
trapped three surfaces, such classical systems may not suffer from the classical radiation reaction problem. Quantum mechanically they might also be closed and thus confining, though they may still be able to communicate with other particles by Hawking-like radiation. In this way then the negative vacuum energy of a filled Dirac sea could be nicely trapped inside of elementary particles and never be manifest on cosmological distance scales. (Within such elementary particles the curvature based string picture may still be able to emerge, though possibly with the old dual resonance model hadronic string tension instead). As we can thus see, particle physics has some interesting implications for the structure of gravitational sources, with the cosmological constant problem perhaps being not so much a problem for fundamental theory in general, but possibly only being a problem for theories based on the use of the Einstein-Hilbert action; with the problem then being not so much how to get rid of the cosmological constant at all, but rather how to get rid of it in an Newton-Einstein based theory where it has no reason not to be there.

Given that the Newton-Einstein theory was first established on solar distance scales in the weak gravity limit, the theoretically so far unjustified insistence on the use of the Einstein Equations for both larger distances and stronger coupling thus represents a quite enormous extrapolation to kinematic regimes where there is currently little observational guidance. Indeed, the Einstein theory insists that gravity be attractive on all scales, whilst in the conformal theory the back reaction \(-S^2R_{\mu\nu}/12\) term in Eq. (13) acts as an induced repulsive gravitational term (it has the opposite sign to the Einstein-Hilbert term) which prevents the cosmology of Ref. (5) from actually having any singularity. (That singularities may not be generic to gravity has also been emphasized by N. J. Cornish and J. W. Moffat in their recent UTPT-94-08 study of the non-symmetric gravitational alternative). The full perturbative gravitational expansion for the metric could thus look very different from that expected in the Einstein theory, and only observations on large distances and in strong fields can ultimately enable us to identify the rest of the series. This work has been supported in part by the Department of Energy under grant No. DE-FG02-92ER40716.00.

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Figure Caption
Figure (1). The calculated rotational velocity curves of Ref. (7) associated with stellar potentials \(V(r) = -\beta/r + \gamma r/2\) for four typical galaxies, the intermediate sized NGC3198, the compact luminous NGC2903, the large sized NGC5907, and the dwarf irregular DDO154 (which is calculated at both of two possible observational distances from the Milky Way). In each graph the bars show the observed data points with their quoted errors as a function of radial distance (in arc minutes) from the center of each galaxy. The full curve shows the overall theoretical galactic rotational velocity prediction (in km/s), while the two indicated dotted curves show the rotation curves that the separate stellar Newtonian and linear potentials would produce when integrated over the luminous matter distribution of each galaxy. No dark matter is assumed.
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