The conformal $N$-point scalar correlator in coordinate space

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Abstract

We present a systematic derivation of the form of correlators of $N$ operators in a Conformal Field Theory in $d > 2$ dimensions and the exchange-symmetry constraints that the functions of the dimensionless cross-ratios obey for $N > 3$. 
1 Introduction

Quantum Field Theories may run into a fixed point (or a fixed line etc.) in their phase diagram. Then the space-time symmetry of the system may be enhanced from Poincare to Conformal symmetry. The most familiar case is that of a Gaussian fixed point where the theory is free in which case, if massless, it may be described by a free Conformal Field Theory (CFT) \[1, 2\]. All correlators in such theories are constrained by the requirement of their invariance under the action of the conformal group. The 2 and 3-point functions are completely fixed up to normalization, while the 4-point function is only partially constrained, with a 2-parameter freedom remaining after all conformal Ward identities have been imposed.

In this letter we present the computation of the correlator of \(N\) scalar operators in CFT in coordinate space\(^1\) and give two explicit examples. First we rederive the 4-point function and then we give the example of the 6-point function which has not appeared before.

2 The scalar correlator in \(x\)-space

The form of the correlator of four scalar operators \(\mathcal{O}(x_i)\) of the same scaling dimension \(\Delta\) in \(d(> 2)\) dimensions, located at space-time points \(x_i\) in a CFT is constrained by the conformal symmetry \(SO(2, d)\) to

\[
\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle = R_4 g(u, v), \quad R_4 = \frac{1}{x_{12}^2 x_{34}^2},
\]

where \(x_{ij} = |x_i - x_j|\). The conformally invariant cross-ratios \(u\) and \(v\) are defined as

\[
u = \frac{x_{12} x_{34}}{x_{13} x_{24}}, \quad v = \frac{x_{23} x_{14}}{x_{13} x_{24}}.
\]

The function \(g(u, v)\) remains unconstrained by the conformal symmetry itself, but satisfies additional relations, obtained by the requirement that the correlator, in Euclidean space, is symmetric under the interchange \(x_i \leftrightarrow x_j\), symbolized by the notation \((ij)\). The action \((ij)\) on a function of coordinates induces an action denoted as \(g_{ij}\). Invariance of the correlator under all possible such exchanges imposes the two exchange-symmetry constraints

\[
g(u, v) = g\left(\frac{u}{v}, \frac{1}{v}\right), \quad g(u, v) = \left(\frac{u}{v}\right)^{2\Delta} g(v, u).
\]

In the absence of additional input, like the Operator Product Expansion, these are (the only) independent constraints on \(g\). Recall finally that \(R_4\) and \(g(u, v)\) are both and separately conformally invariant, so that the correlator in Eq. (2.1) can be seen as being factorized in coordinate space, in the product of at least two invariant substructures.

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\(^1\)An earlier attempt is \[3\]. There is also recent interesting activity to express such correlators in momentum space \[4, 5, 6, 7\].

\(^2\)We will switching back and forth from Minkowski to Euclidean signature-in which case the space-time symmetry is \(SO(1, d + 1)\)-depending on the situation.

\(^3\)We will abuse this notation and sometimes use the same notation for the vector \(x_i^\mu - x_j^\mu\) itself. Which is the correct reading, should be clear from the context. When \(x_{ij}\) is raised to an even power the two are equivalent.
The simplest way to derive the conditions in Eq. (2.3)—consider for simplicity a CFT in four dimensions but the generalization to arbitrary dimensions is straightforward—is to embed the system in a flat space-time of two dimensions higher \([8]\), with metric of signature \((-\,+,\,+,\,+,\,+,-\)) parametrized by the coordinates \(y^A, A = \mu, 5, 6\) and project back to the original 4d space by the null-cone condition \(y^A y_A = 0\) and the identification for the 4d coordinates

\[
x^\mu = \frac{y^\mu}{y^+}, \quad y^+ = y_5^+ + y_6^+.
\]  

(2.4)

An advantage of this procedure is that the conformal transformations are just rotations and/or boosts in the 6-dimensional space and the only non-zero invariants constructed from the coordinates are the inner products

\[
y_i \cdot y_j = -\frac{1}{2} (y_i^+ y_j^+)(x_i - x_j)^2.
\]  

(2.5)

It can be shown that the fields \(O_q(x) = (y^+)^\Delta q \Phi(x, y^+)\) depend only on \(x\), have scaling dimension \(\Delta q\) and for a conformally invariant correlator the \(\Phi\)'s must contribute to it terms proportional to the product of all possible inner products \(y_i \cdot y_j\). The 4-point function for example must be of the form

\[
\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4) \rangle \sim \frac{\prod_{a=1,\ldots,4} (y_a^+)^{\Delta_a}}{\prod_{i,j} (y_i \cdot y_j)^{e_{ij}}}, \quad 1, 2, 3 = i < j = 2, \ldots, 4
\]  

(2.6)

Imposing the self consistency condition that the right hand side is \(y_a^+\)-independent and restricting to identical scalar operators, we arrive at Eq. (2.1). By acting on the result with \(g_{12}\) and \(g_{13}\) and requiring invariance of the correlator, we obtain Eq. (2.3).

This methodology can be straightforwardly generalized. For the correlator of \(N\) scalar operators

\[
\langle O_1(x_1) \cdots O_N(x_N) \rangle \sim \frac{\prod_{a=1,\ldots,N} (y_a^+)^{\Delta_a}}{\prod_{i,j} (y_i \cdot y_j)^{e_{ij}}}, \quad 1, \ldots, N - 1 = i < j
\]  

(2.7)

the conditions that restrict its form stem from the requirement of its independence from the \(y^+\)'s, as before. Clearly we have more unknowns than equations so we must decide which \(e_{ij}\) to solve for. Since we have \(N\) equations, we have to pick \(N\) exponents. Any loss of generality involved in this choice will be lifted by the exchange-symmetry constraints. A convenient choice is to solve for \(e_{1i}, i = 2, \ldots, N\) and \(e_{23}\). Defining the vectors \(E = (e_{12}, \ldots, e_{1N}, e_{23})\), \(D = (\Delta_1, \Delta_i - (\sigma_1^N + \rho_2^i))\) the equation to be solved for \(E\) is \((T\) stands for transpose

\[
ME^T = DT.
\]  

(2.8)

In the above we have defined the partial sums

\[
\sigma_1^N = e_{i,i+1} + e_{i,i+2} + \cdots + e_{i,N} \\
\rho_2^i = e_{2,i} + e_{3,i} + \cdots + e_{i-1,i}
\]  

(2.9)
where \( i, j = 2, \cdots, N \) and \( \sigma_i^j \) and \( \rho_i^j \) are non-zero only when \( i < j \). The matrix \( M \) is

\[
M = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 1 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
\end{bmatrix}
\]  

(2.10)

For \( N > d + 2 \) correlators degeneracies originating from the necessary linear dependence of some of the \( y_i \) may start to arise that must be dealt with [8]. They appear by making the above matrix have some linearly dependent rows and columns. In this case these rows/columns must be moved to the right hand side of Eq. (2.8), into \( D \). As a result, some of the exponents in \( E \) will not be independent. We will not complicate our analysis any further by such a possibility since apart from this technicality the logic is the same as for the non-degenerate case \( N \leq d + 2 \).

The easiest way to solve this system of equations is to discard the first row and last column, which leaves an \( N - 1 \) dimensional unit submatrix in \( M \), trivially invertible. The solution is given though in terms of \( e_{23} \) due to the missing row and column. Fortunately we can solve for \( e_{23} \) separately, by combining for example the sum of all \( N - 1 \) equations with the constraint that comes from the observation that Eq. (2.7) must be invariant under the trivial rescaling \( y_i \to \lambda y_i \). The result is

\[
2e_{23} = -\Delta_1 + \Delta_2 + \cdots + \Delta_N - 2(\rho_2^1 + \cdots + \rho_2^N)
\]  

(2.11)

and then the \( N - 1 \) dimensional system of equations collapses to

\[
2e_{1i} = 2\Delta_i - 2(\sigma_i^1 + \rho_2^i),
\]  

(2.12)

where the only thing to remember is to substitute for \( e_{23} \) from Eq. (2.11) when it appears in either \( \sigma_i^N \) or \( \rho_2^N \) which occurs twice, once in \( e_{12} \) and once in \( e_{13} \). It is illuminating to show the explicit form of the final solution:

\[
2e_{23} = (-\Delta_1 + \Delta_2 + \cdots + \Delta_N) - 2(\rho_2^1 + \cdots + \rho_2^N)
\]

\[
2e_{12} = 2\Delta_2 - (-\Delta_1 + \Delta_2 + \cdots + \Delta_N) - 2\sigma_2^N + 2(\rho_2^1 + \cdots + \rho_2^N)
\]

\[
2e_{13} = 2\Delta_3 - (-\Delta_1 + \Delta_2 + \cdots + \Delta_N) - 2\sigma_3^N + 2(\rho_2^1 + \cdots + \rho_2^N)
\]

\[
2e_{1i} = 2\Delta_i - 2(\sigma_i^1 + \rho_2^i), \quad i = 4, \cdots, N
\]  

(2.13)

where \( \sigma_2^{N \cdot} = \sigma_2^N - e_{23} \). Then

\[
\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_N(x_N) \rangle = \frac{1}{x_1^{2e_{12}} \cdots x_N^{2e_{1N}} x_{23}^{2e_{23}} \prod_{ij} x_{ij}^{2e_{ij}}}
\]  

(2.14)
where in the product: $i = 2, \cdots, N - 1$ and $j = i + 1, \cdots, N$ and $ij \neq 23$. To prepare this expression for an exchange-symmetry analysis we first define

\[
\begin{align*}
    \Delta_{23} &= -\Delta_1 + \Delta_2 + \cdots + \Delta_N \\
    \Delta_{12} &= 2\Delta_2 - (\Delta_1 + \Delta_2 + \cdots + \Delta_N) \\
    \Delta_{13} &= 2\Delta_3 - (\Delta_1 + \Delta_2 + \cdots + \Delta_N) \\
    \Delta_{1i} &= 2\Delta_i, \quad i = 4, \cdots, N
\end{align*}
\] (2.15)

and write the correlator as

\[
\langle O_1(x_1) \cdots O_N(x_N) \rangle = R_N \frac{x_{23}^{2(\rho_1^2 + \cdots + \rho_N^2)} \prod_{i=1}^{N} x_{1i}^{2(\sigma_i^N + \rho_i^2)}}{x_{12}^{-2\rho_2^2 + 2(\rho_2^2 + \cdots + \rho_N^2)} x_{13}^{-2\sigma_1^N + 2(\rho_3^2 + \cdots + \rho_N^2)} \prod_{ij} x_{ij}^{2e_{ij}}}
\] (2.16)

with the same restrictions on the $i, j$ indices as above and

\[
R_N = \frac{1}{x_{23}^{\Delta_{23}} \prod_{a=2}^{N} x_{1a}^{\Delta_{1a}}}. 
\] (2.17)

The geometric interpretation says that if we think of the correlator as a sort of a representation of a discrete metric on the points \{\(x_1, \cdots, x_N\)\} then \(R_N\) is its radial part, the rest is the angular part and rotations correspond to transformations that exchange \(x_i \leftrightarrow x_j\). When the \(N\) operators are distinct, the gauging of the exchange-symmetry group does not leave the triangle defined by any three points invariant (apart from the identity action) and the radial part \(R_N\) will contain a 123 sector, corresponding to the triangle defined by \(x_1, x_2, x_3\). For \(N = 3\) this is a conformally invariant structure. When the operators in the correlator are identical, it may happen that a non-trivial (not an identity) combination of the exchange-symmetry group elements leaves the 123 triangle invariant and then the corresponding sector has no reason to appear in \(R_N\). Instead, all information for structures built from triangles is contained in the angular part \(f\). Such is the case of the \(N = 4\) correlator of identical scalars.

The statement of exchange-symmetry (in Euclidean \(x\)-space) is that

\[
g_{1a} \langle O_1(x_1) \cdots O_N(x_N) \rangle = \langle O_1(x_1) \cdots O_N(x_N) \rangle, \quad a = 2, \cdots, N \] (2.18)

since the \((1a)\) generate the permutation group \(S_N\). We also define here the important quantity

\[
J_a = R_N^{-1}(g_{1a}R_N), \quad (2.19)
\]

a sort of discrete version of a Jacobian, originating from the transformation induced by the \(g_{1a}\). The only ingredient we are missing are the conformally invariant cross-ratios. These can be straightforwardly obtained from Eq. (2.16) by collecting all the \(x_{mn}\) under a fixed power \(2e_{kl}\). This is not a unique decomposition of the correlator but is easy to generalize. Then, we obtain the \(2(N - 3)\) conformally invariant, order two, cross-ratios

\[
u_{2k} = \frac{x_{23} x_{1k}}{x_{12} x_{2k}}, \quad \nu_{3k} = \frac{x_{23} x_{1k}}{x_{12} x_{3k}}, \quad k = 4, \cdots, N \] (2.20)
and the $\frac{1}{2}(N-3)(N-4)$, order three, cross-ratios $\left[\begin{array}{c}
\end{array}\right]$ (2.21)

These are ratios of 3-point functions but being dimensionless moduli, appear in the angular part of the correlator. The counting is right, since $1 + 2 + \cdots + (N-4) + 2(N-3) = \frac{1}{2}N(N-3)$. It seems that cross-ratios of higher order do not form and any higher order cross-ratio can be expressed in terms of the order two and the order three ratios, may it be of even or odd order. An interesting fact is that while for $N = 4$ we see only order two cross-ratios and for $N = 6$ the order two are twice as many as the order three ones, for large $N$ the order three cross ratios start to dominate. Note also the useful identities $g_{23}u_{2k} = u_{3k}$, $g_{jk}u_{ji} = u_{ki}$ ($k \neq i$), $g_{ik}u_{ji} = u_{jk}$ ($k \neq j$) (2.22) which tell us that we can start from $u_{24}$ and $u_{35}$ and generate all other cross-ratios by acting on them with the elements of $S_N$. The last step is to generalize in the expression for the correlator the part that depends on the unfixed exponents to a general function of its conformally invariant cross-ratios, which we will refer to also as the conformal coordinates:

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_N(x_N) \rangle = R_N f_{1 \cdots N}(u_{24}, \cdots , u_{2N}, u_{34}, \cdots , u_{4N}, \cdots , u_{N-1,N}) \rangle .$$

Now we are done, since one can use these expressions and obtain explicitly all $N$-correlators of scalar operators of scaling dimension $\Delta_i$ from Eq. (2.16) and their $N - 1$ cross symmetry constraints ($a = 2, \cdots , N$):

$$f_{q_1 \cdots q_N}(u_{24}, \cdots , u_{2N}, u_{34}, \cdots , u_{3N}, \cdots , u_{N-1,N}) =$$

$$J_a f_{q_1 \cdots q_N}(g_{1a}u_{24}, \cdots , g_{1a}u_{2N}, g_{1a}u_{34}, \cdots , g_{1a}u_{3N}, \cdots , g_{1a}u_{N-1,N}) .$$

To illustrate the general process we give two examples. We first rederive the $N = 4$ correlator and then present the $N = 6$ correlator for the simple case of identical operators, in which case $\Delta_4 = \Delta$ and $f_{q_1 \cdots q_N} = f$.

### 2.1 The $N = 4$ correlator

For $N = 4$ there are two coordinates of the type Eq. (2.20):

$$u_{24} = \frac{x_{23}x_{14}}{x_{13}x_{24}}, \quad u_{34} = \frac{x_{23}x_{14}}{x_{12}x_{34}} \quad (2.25)$$

and no coordinates of the type Eq. (2.21). Also, for identical operators $\Delta_{12} = \Delta_{13} = 0$ and $\Delta_{14} = \Delta_{23} = 2\Delta$. The correlator in this case is

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle = R_4 f(u_{24}, u_{34}), \quad R_4 = \frac{1}{x_{14}^{2\Delta} x_{23}^{2\Delta}} \quad (2.26)$$

$^4$The existence of these has been noticed in [9] for $N = 5$. 
The action of the generators of $S_4$ on the conformal coordinates are

\[
\begin{align*}
    g_{12} u_{24} &= \frac{1}{u_{24}}, &
    g_{12} u_{34} &= u_{34} \\
    g_{13} u_{24} &= \frac{u_{24}}{u_{34}}, &
    g_{13} u_{34} &= \frac{1}{u_{34}} \\
    g_{14} u_{24} &= u_{34}, &
    g_{14} u_{34} &= u_{24}
\end{align*}
\]

and the three Jacobians are

\[
J_2 = u_{24}^{2\Delta}, \quad J_3 = u_{34}^{2\Delta}, \quad J_4 = 1.
\]

These imply the three exchange-symmetry constraints

\[
\begin{align*}
    g_{12} : \quad f(u_{24}, u_{34}) &= u_{24}^{2\Delta} f(\frac{1}{u_{24}}, u_{34}) \\
    g_{13} : \quad f(u_{24}, u_{34}) &= u_{34}^{2\Delta} f(u_{24}, \frac{1}{u_{34}}) \\
    g_{14} : \quad f(u_{24}, u_{34}) &= f(u_{34}, u_{24})
\end{align*}
\]

We should make three comments here. One is related to the observation that in Eq. (2.3) we presented only two exchange-symmetry constraints and here we just found three. What happens is that out of the three covariant constraints in Eq. (2.29) only two are independent, as it is easy to check that $g_{12} g_{13} g_{12} \sim g_{14}$, where the $\sim$ sign indicates not a group theory relation between $S_N$ elements but an equivalence of their action on the correlator and the $u_{24}, u_{34}$. In other words, the transformation with the unit Jacobian in Eq. (2.29) for example is not independent. This seems to be a reflection of the fact that one can bring the four points $x_1, x_2, x_3, x_4$ on a plane by conformal transformations, thus the trivial Jacobian. Furthermore, one can place these points on the corners of a tilted rectangle that the gauging of the exchange-symmetry group turns into a square. As a result, the exchange symmetry effectively reduces to the dihedral group $D_4$ and if the freedom to choose which three points define the plane on which the fourth point is projected is taken into account, the symmetry reduces further to $D_3$, which is isomorphic to $S_3$. The latter is generated by $g_{12}$ and $g_{13}$ indeed. Thus, the $g_{14}$ operation can not be independent. The second comment is that according to our previous geometric arguments we expect to see no 3-point subsector in $R_4$ as the information about the invariance of the triangles inside the parallelogram under rotations about its two diagonals, are contained in the action of the $g_{14}$. Indeed, we saw that $\Delta_{12} = \Delta_{13} = 0$ and the radial part of the correlator $R_4$ contains only two disconnected $x_{ij}$’s ($x_{14} x_{23}$ in the $u_{24}, u_{34}$ angular coordinates and $x_{12} x_{34}$ in the $u, v$ coordinates). The third comment is that the two independent constraints in Eq. (2.29) are equivalent to the ones in Eq. (2.3) by a coordinate change, even though the trivial Jacobian transformation in the $(u_{24}, u_{34})$ coordinates maps to a non-trivial one in the $(u, v)$ coordinates and vice versa.

### 2.2 The $N=6$ correlator

Let us look at a slightly more complicated example, the one of the conformal correlator of six scalar operators. The algorithm we described then yields via Eq. (2.20) and Eq. (2.21) the nine...
with the radial prefactor

\[ R_6 = \frac{1}{x_{12}\Delta_{12} x_{13}\Delta_{13} x_{14}\Delta_{14} x_{15}\Delta_{15} x_{16}\Delta_{16} x_{23}\Delta_{23}}. \]  

In the case of scalar operators of the same scaling dimensions \( \Delta \) this reduces to

\[ \langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_6) \rangle = \left( \frac{x_{12} x_{13}}{x_{14} x_{15} x_{16} x_{23} x_{24}} \right)^{2\Delta} f \left( u_{24}, u_{25}, u_{26}, u_{34}, u_{35}, u_{36}, u_{45}, u_{46}, u_{56} \right). \]  

The five corresponding exchange-symmetry constraints are

\[ g_{12} : \quad f \left( u_{24}, u_{25}, u_{26}, u_{34}, u_{35}, u_{36}, u_{45}, u_{46}, u_{56} \right) = \frac{1}{u_{24}} \left( \frac{1}{u_{25}} \frac{1}{u_{26}} \frac{u_{24} u_{25} u_{26}}{u_{34} u_{35} u_{36}} \frac{u_{26} u_{24} u_{25} u_{26}}{u_{25} u_{24} u_{25} u_{26}} \right) \]  

\[ g_{13} : \quad f \left( u_{24}, u_{25}, u_{26}, u_{34}, u_{35}, u_{36}, u_{45}, u_{46}, u_{56} \right) = \frac{1}{u_{34}} \left( \frac{1}{u_{35}} \frac{1}{u_{36}} \frac{u_{34} u_{35} u_{36}}{u_{24} u_{25} u_{26}} \frac{u_{36} u_{34} u_{35} u_{36}}{u_{26} u_{24} u_{25} u_{26}} \right) \]  

\[ g_{14} : \quad f \left( u_{24}, u_{25}, u_{26}, u_{34}, u_{35}, u_{36}, u_{45}, u_{46}, u_{56} \right) = \frac{1}{u_{24} u_{25}} \left( \frac{1}{u_{34}} \frac{1}{u_{35}} \frac{1}{u_{36}} \frac{u_{24} u_{34} u_{35}}{u_{25} u_{34} u_{35}} \frac{u_{25} u_{24} u_{34} u_{35}}{u_{26} u_{24} u_{34} u_{35}} \right) \]  

\[ g_{15} : \quad f \left( u_{24}, u_{25}, u_{26}, u_{34}, u_{35}, u_{36}, u_{45}, u_{46}, u_{56} \right) = \frac{1}{u_{24} u_{25} u_{26}} \left( \frac{1}{u_{34}} \frac{1}{u_{35}} \frac{1}{u_{36}} \frac{u_{24} u_{34} u_{35} u_{36}}{u_{25} u_{34} u_{35} u_{36}} \frac{u_{25} u_{24} u_{34} u_{35} u_{36}}{u_{26} u_{24} u_{34} u_{35} u_{36}} \right) \]  

\[ g_{16} : \quad f \left( u_{24}, u_{25}, u_{26}, u_{34}, u_{35}, u_{36}, u_{45}, u_{46}, u_{56} \right) = \frac{1}{u_{24} u_{25} u_{26} u_{34}} \left( \frac{1}{u_{34}} \frac{1}{u_{35}} \frac{1}{u_{36}} \frac{u_{24} u_{34} u_{35} u_{36}}{u_{25} u_{34} u_{35} u_{36}} \frac{u_{25} u_{24} u_{34} u_{35} u_{36}}{u_{26} u_{24} u_{34} u_{35} u_{36}} \right) \]  

There is no trivial Jacobian, so we expect these constraints to be independent.
3 Conclusion

We computed the correlator of $N$ scalar operators in CFT, in coordinate space and gave two explicit examples, for $N = 4$ and $N = 6$. We found that in the $N = 6$ case there appear order three conformally invariant cross-ratios of six $x_{ij}$’s, in addition to the well known order two cross-ratios of the $N = 4$ case. We also gave the corresponding exchange-symmetry constraints associated with correlators of $N$ scalar operators with $N \geq 4$.

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