Data-driven controller design for nonlinear systems: a two degrees of freedom architecture

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Abstract—In this paper, the $D^2$-IBC (Data-Driven Inversion Based Control) approach for nonlinear control is introduced and analyzed. The method does not require any a-priori knowledge of the system dynamics and relies on a two degrees of freedom scheme, with a nonlinear controller and a linear controller running in parallel. In particular, the former is devoted to stabilize the system around a trajectory of interest, whereas the latter is used to boost the closed-loop performance. The paper also presents a thorough stability and performance analysis of the closed-loop system.

I. INTRODUCTION

Consider a nonlinear discrete-time SISO system in regression form:

$$y_{t+1} = g(y_t, u_t, \xi_t)$$

where $u_t \in U \subset \mathbb{R}$ is the input, $y_t \in \mathbb{Y}$ is the output,$\xi_t \in \Xi \subset \mathbb{R}^{n_{\xi}}$ is a disturbance including both process and measurement noises, and $n$ is the system order. $U$ and $\Xi$ are compact sets. In particular, $U \equiv [u_\text{min}, u_\text{max}]$ accounts for input saturation.

Suppose that the system (1) is unknown, but a set of measurements is available:

$$D \equiv \{\tilde{u}_t, \tilde{y}_t\}_{t=1}^{L}$$

where $\tilde{u}_t$ and $\tilde{y}_t$ are bounded for all $t = 1, 2, \ldots, L$. The accent $\sim$ is used to indicate the input and output samples of the data set (2).

Let $\mathcal{Y}^{0} \subset \mathbb{R}^{n}$ be a set of initial conditions of interest for the system (1) and, for a given initial condition $y_0 \in \mathcal{Y}^{0}$, let $\mathcal{Y}(y_0) \subseteq \ell_{\infty}$ be a set of output sequences of interest.

To accomplish this task, we use the feedback control structure depicted in Figure 1, where $S$ is the system (1), $K^{nl}$ is a nonlinear controller, $K^{lin}$ is a linear controller, $r_t \in Y$ is the reference, and $Y \subset \mathbb{R}$ is a compact set where the output sequences of interest lie.

K$^{nl}$ is used to stabilize the system (1) around the trajectories of interest, while $K^{lin}$ allows us to further reduce the tracking error (especially in steady-state conditions). $K^{nl}$ is designed through the NIC (Nonlinear Inversion Control) approach presented in [1]. $K^{lin}$ is designed using a suitably modified version of the VRFT (Virtual Reference Feedback Tuning) method introduced in [2]. As shown in Sections III and IV, the design of both the controller is performed from data and is based on system inversion, hence the name $D^2$-IBC (Data-Driven Inversion Based Control).

Besides control design, other main contributions of the paper are a closed-loop stability analysis and a study on the performance enhancement given by the linear controller.

II. NOTATION

A column vector $x \in \mathbb{R}^{n_x \times 1}$ is denoted as $x = (x_1, \ldots, x_{n_x})$. A row vector $x \in \mathbb{R}^{1 \times n_x}$ is denoted as $x = [x_1, \ldots, x_{n_x}]^\top$, where $\top$ indicates the transpose.

A discrete-time signal (i.e. a sequence of vectors) is denoted with the bold style: $x = (x_1, x_2, \ldots)$, where $x_t \in \mathbb{R}^{n_x \times 1}$ and $t = 1, 2, \ldots$ indicates the discrete time; $x_{i,t}$ is the $i$th component of the signal $x$ at time $t$.

A regressor, i.e. a vector that, at time $t$, contains $n$ present and past values of a variable, is indicated with the bold style and the time index: $x_t = (x_{t-1}, \ldots, x_{t-n})$.

The $\ell_p$ norms of a vector $x = (x_1, \ldots, x_{n_x})$ are defined as

$$\|x\|_p = \begin{cases} \left( \sum_{i=1}^{n_x} |x_i|^p \right)^{1/p}, & p < \infty, \\ \max_i |x_i|, & p = \infty. \end{cases}$$

The $\ell_{\infty}$ norm is also used to denote the absolute value of a scalar: $\|x\|_{\infty} = |x|$ for $x \in \mathbb{R}$.
The $\ell_p$ norms of a signal $x = (x_1, x_2, \ldots)$ are defined as

$$
\|x\|_p \doteq \left\{ \begin{array}{ll}
\left(\sum_{i=1}^{\infty} \sum_{t=1}^{n_i} |x_{i,t}|^p \right)^{\frac{1}{p}}, & p < \infty, \\
\max_{i,t} |x_{i,t}|, & p = \infty,
\end{array} \right.
$$

where $x_{i,t}$ is the $i$th component of the signal $x$ at time $t$. These norms give rise to the well-known $\ell_p$ Banach spaces.

III. NONLINEAR CONTROLLER DESIGN

The nonlinear controller design is based on the method presented in [1]. The first step of this method is to identify the following optimization problem is solved to perform such an identification can be found in [3] or [1].

The nonlinear controller design is based on the method in the proposed architecture is to compensate for model-inversion errors and boost the control performance by assigning a desired dynamics to the resulting nonlinearly-compensated system.

The Virtual Reference Feedback Tuning (VRFT) method [2], [5] is here suitably adapted to be applicable in the $D^2$-IBC setting and employed to design the linear controller.

![Diagram](image)

Figure 2. The “virtual reference” rationale: the data is collected on the real system (solid) and applied to controller identification in the “virtual loop” (dashed).

Let the desired behavior for the closed-loop system be given by a linear asymptotically stable model $M$.

The “virtual reference” rationale to design $K^{lin}$ achieving $M$ without identifying any model of the system is based on the following observation, illustrated in Figure 2. In a “virtual” operating condition where the closed-loop system behaves exactly as $M$, the “virtual reference” signal $r^v_t$ would be given as the output of the inverse of $M$, say $M^{-1}$, when it is fed by $y_t$.

Obviously, since $M^{-1}$ is likely to be non-causal, $r^v_t$ could be computed only off-line using the available data set. However, in such a setting, both the trajectory of the fictitious signal $r^v_t$ and the subsequent “virtual error” $e^v_t = r^v_t - y_t$ could be calculated. This fact means that the optimal controller achieving $M$ in closed-loop is the dynamical system giving $u^{lin}_t = u_t - u^{nl}_t$ as an output when fed by $e^v_t$. The command input $u^{lin}_t$ is the output of the extended PID controller (6), which can then be designed based on the data set [2].

Let $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_0)$ be the input sequence of the data set and let $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_0)$ be obtained from the off-line filtering of $r^v_t$ and $y_t$, $t = 1, \ldots, 0$, with the controller $K^{nl}$ derived in Section III.

Therefore, the control design problem can be turned into an identification problem, where the optimal controller with the structure in (6) is the one with parameter vector

$$
\theta = \arg \min_{\theta \in \Theta} \left\| \delta u - u^{lin}(\theta) \right\|_2^2
$$

where $\delta u = \tilde{u} - \tilde{u}^{nl}$ and $u^{lin}(\theta) = (u^{lin}_{1-L}(\theta), \ldots, u^{lin}_0(\theta))$. In [2], it is shown how the problem [7] can be solved by means of convex optimization.

V. CLOSED-LOOP ANALYSIS

A. Stability analysis

The feedback system of Figure 1 is described by

$$
\begin{align*}
\dot{y}_t &= g(y_t, u_t, \xi_t) \\
u_t &= u_t^{lin} + u^{nl}_t \\
u^{nl}_t &= K^{nl}(r_{t+1}, y_{t+1}, u^{nl}_{t-1}) \\
u^{lin}_t &= K^{lin}(r_t - y_t, u^{nl}_{t-1})
\end{align*}
$$

where $\xi_t$ is the initial condition for the controller and $g$ is the plant function.
where $K^{nl}$ and $K^{lin}$ are two functions, chosen to be Lipschitz continuous, describing the nonlinear and linear controllers.

The assumptions required to guarantee the stability of this feedback system are now introduced and commented.

**Assumption 1:** The function $g$ in (1) is Lipschitz continuous on $Y^n \times U^n \times \Xi^n$. Without loss of generality, it is also assumed that $Y^n \times U^n \times \Xi^n$ contains the origin.

This assumption is mild, since most real-world dynamic systems are described by functions that are Lipschitz continuous on a compact set. Note anyway that all what presented in this paper can be easily extended to the case where $g$ is the sum of a Lipschitz continuous function plus a discontinuous but bounded function.

From Assumption 1 it follows that $g$ can be written as

$$ g(y_t, u_t, \xi_t) = g^o(y_t, u_t) + g^\xi(t) $$

where $g^o(y_t, u_t) = g(y_t, u_t, 0)$, $g^\xi_t \in \mathbb{R}^{1 \times n}$ : $\|g^\xi_t\|_\infty \leq \gamma \xi$ for some $\gamma \xi < \infty$. Assumption 1 implies that the residue function

$$ \Delta(y_t, u_t) = g^o(y_t, u_t) - f(y_t, u_t) $$

is Lipschitz continuous on $Y^n \times U^n \times \Xi^n$. In particular, a finite and non-negative constant $\gamma_y$ exists, such that

$$ \|\Delta(y, u) - \Delta(y', u')\| \leq \gamma_y \|y - y'\|_\infty $$

for all $y, y' \in Y^n$. $\gamma_y \leq 1$.

**Assumption 2:** $\gamma_y \leq 1$.

The meaning of this assumption is clear: it requires that $f$ describes accurately the variability of $g$ with respect to $y_t$.

In order to introduce the next assumption, the following stability notion is needed.

**Definition 1:** A nonlinear (possibly time-varying) system with input $u_t$, output $y_t$ and noise $\xi_t$ is finite-gain $\ell_\infty$ stable on $(Y^0, U, \Xi)$ if finite and non-negative constants $\Gamma_u, \Gamma_\xi$ and $\Lambda$ exist such that

$$ \|y\|_\infty \leq \Gamma_u \|u\|_\infty + \Gamma_\xi \|\xi\|_\infty + \Lambda $$

for any $(y_0, u, \xi) \in Y^0 \times U \times \Xi$, where $u = (u_1, u_2, \ldots), \xi = (\xi_1, \xi_2, \ldots)$ and $y = (y_1, y_2, \ldots)$.

Note that this finite-gain stability definition is more general than the standard one, which corresponds to the case $U = \ell_\infty$ and $\Xi = \ell_\infty$, see e.g. [6].

Now, consider that the difference equation (3), where $u_t$ is given in (9), defines a dynamical system with inputs $y_t$ and $r_{t+1}$, and output $\hat{y}_{t+1}$ ($u_t, u^{nl}_t$ and $u^{lin}_t$ are internal variables). This system is finite-gain $\ell_\infty$ stable on $(\ell_\infty, \ell_\infty, \ell_\infty)$:

$$ \|\hat{y}\|_\infty \leq \Gamma_y \|y\|_\infty + \Gamma_r \|r\|_\infty + \Lambda_f $$

with $\Gamma_y, \Gamma_r, \Lambda_f < \infty$. In fact, the system is formed by the cascade connection of the controller and the model (3). The controller provides a command input $u_t$ bounded in the compact set $U$. The model is a static Lipschitz continuous function of a regressor consisting in past values of $u_t$ and $y_t$.

**Assumption 3:** $\Gamma_y < 1 - \gamma_y$.

This assumption is not restrictive: It is certainly satisfied if $\mu = 0$ and the reference $r = (r_1, r_2, \ldots)$ is a system solution (i.e. $r_{t+1}$ is in the range of $f(y_{t}, \cdot)$ for all $t$). Indeed, in this case, $\hat{y}_{t+1} = r_{t+1}$, $\forall t$, since $K^{nl}$ performs an exact inversion of the model, see (9) ($K^{lin}$ gives a null input signal). This implies that $\Gamma_y = 0$, $\Gamma_r = 1$ and $\Lambda_f = 0$. Hence, if a “not too large” $\mu$ is chosen and the reference is “not too far” from a system solution, supposing that inequality (9) holds with a “small” $\gamma_y$ is completely reasonable. The meaning of Assumption 3 is that, in order to guarantee closed-loop stability, the controller must perform an effective right-inversion of the system and this inversion should depend as less as possible on the current working point $y_t$. Note that the bound (9) implies that, if the model (9) is exact, the designed controller stabilizes the closed loop system (a direct consequence of Theorem 1 below).

From Assumption 3 it follows that the system defined by the difference equation $(\hat{y}_{t+1} = r_t - f(y_{t-1}, u_{t-1}),$ where $u_t$ is given in (9), is finite-gain $\ell_\infty$ stable on $(0, Y, \gamma, Y)$:

$$ \|\hat{e}\|_\infty \leq \Gamma_y \|y\|_\infty + \Gamma_r \|r\|_\infty + \Lambda_e $$

(10)

with $\Gamma_y < 1 - \gamma_y$ and $\Gamma_r, \Lambda_e < \infty$. As discussed above, in “reasonable” working conditions, $\hat{y}_t \cong r_t$, implying that $\Gamma_y \cong 0$ and $\Gamma_r \cong 0$.

Closed-loop stability of the system (8) is stated by the following result, which also provides a bound on the tracking error.

**Theorem 1:** Let Assumptions 1-3 hold. Then:

(i) For any initial condition $y_0 \in Y^0$, the feedback system (8) is finite-gain $\ell_\infty$ stable on $(Y^0, Y^S, \Xi)$:

$$ \|y\|_\infty \leq \frac{1}{1 - \Gamma_y} \left( \Gamma_r \|r\|_\infty + \gamma \|y\|_\infty + \Lambda_y \right) $$

where $\Lambda_y = \Lambda_f + \max_{u \in U^n} \|\Delta(0, u)\|_\infty < \infty$ and $Y^S = \{ r \in Y(y_0) : y_t \in Y, \forall t, \forall \xi \in \Xi \}$.

(ii) The tracking error $\hat{e} = r - y$ is bounded as

$$ \|\hat{e}\|_\infty \leq \frac{1}{1 - \Gamma_y} \left( \Gamma_{err} \|r\|_\infty + \gamma \|y\|_\infty + \Lambda_e + \Delta \|\Delta\|_\infty \right) $$

where $\Gamma_{err} = \Gamma_y + \Gamma_r$ and $\|\Delta\|_\infty$ is the functional $L_\infty$ norm of $\Delta$, evaluated over $Y^n \times U^n$.

**Proof.** (i) The feedback system of Figure 1 is described by

$$ y_{t+1} = g(y_t, u_t, \xi_t) = \hat{y}_{t+1} + \delta y_t $$

(11)

where

$$ \hat{y}_{t+1} = f(y_t, u_t), \delta y_t = \Delta(y_t, u_t) + g^\xi(t) $$

and $u_t$ is given by (9).

From (11) and Assumption 3 the following inequalities hold:

$$ \|y_{t+1}\|_\infty \leq \|\hat{y}_{t+1}\|_\infty + \|\delta y_t\|_\infty $$

$$ \leq \Gamma_r \|r\|_\infty + \Gamma_y \|y\|_\infty + \Lambda_f + \|\delta y_t\|_\infty $$

(12)
In order to derive a bound on \(\|\delta y_t\|_\infty\) consider that,
\[
\|\Delta (y_t, u_t)\|_\infty - \|\Delta (0, u_t)\|_\infty \\
\leq \|\Delta (y_t, u_t) - \Delta (0, u_t)\|_\infty \leq \gamma_y \|y_t\|_\infty.
\]
This inequality is due to Assumption 2 and holds for any \(r \in \mathcal{Y}^S\). Indeed, \(\mathcal{Y}^S\) is the set of all reference sequences for which the system output remains in the domain where \(\Delta\) is Lipschitz continuous with respect to \(y_t\) with constant \(\gamma_y\).

It follows that
\[
\|\delta y_t\|_\infty \leq \|\Delta (y_t, u_t)\|_\infty + \gamma_y \|y_t\|_\infty + \gamma_y \|\xi_t\|_\infty + \Delta
\]
\[
\leq \gamma_y \|y_t\|_\infty + \gamma_y \|\xi_t\|_\infty + \Delta
\]
\[
\leq \gamma_y \|y_t\|_\infty + \gamma_y ||\xi|\|_\infty + \Delta
\]
where
\[
\hat{\Delta} = \max_{u \in U} \|\Delta (0, u)\|_\infty.
\]

Note that \(\hat{\Delta} < \infty\) since \(\Delta\) is Lipschitz continuous and \(U\) is a compact set, implying that \(\gamma_y < \infty\).

From (12) and (13), we obtain:
\[
\|y_{t+1}\|_\infty \leq \Gamma_r \|r\|_\infty + \Gamma_y \|y\|_\infty + \Lambda_f
\]
\[
+ \gamma_y \|y\|_\infty + \gamma_y \|\xi\|_\infty + \Delta.
\]
Since this inequality holds for all \(t\), we have that
\[
\|y\|_\infty \leq \Gamma_r \|r\|_\infty + \Gamma_y \|y\|_\infty + \Lambda_f
\]
\[
+ \gamma_y \|y\|_\infty + \gamma_y \|\xi\|_\infty + \Delta,
\]
which yields the following bound:
\[
\|y\|_\infty \leq \frac{1}{1 - \Gamma_r - \gamma_y} (\Gamma_r \|r\|_\infty + \gamma_y \|\xi\|_\infty + \Lambda_f + \Delta)
\]
where it has been considered that, by Assumptions 3 and 2,
\[
\Gamma_y + \gamma_y < 1.
\]
(ii) From (11), we have that
\[
e_t = r_t - y_t = r_t - \hat{y}_t - \delta y_t.
\]
Then,
\[
\|e_t\|_\infty \leq \|r_t - \hat{y}_t\|_\infty + \|\delta y_{t-1}\|_\infty.
\]
The term \(\|r_t - \hat{y}_t\|_\infty = \|\delta y_{t-1}\|_\infty\) is bounded according to (10). The following bound on the term \(\|\delta y_{t-1}\|_\infty\) is considered:
\[
\|\delta y_{t-1}\|_\infty \leq \|\Delta\|_\infty + \gamma_y \|\xi\|_\infty,
\]
which holds for any \(r \in \mathcal{Y}^S\). Thus,
\[
\|e\|_\infty \leq \Gamma_y \|y\|_\infty + \Gamma_y \|r\|_\infty
\]
\[
+ \Lambda_f + \|\Delta\|_\infty + \gamma_y \|\xi\|_\infty
\]
\[
\leq \Gamma_y \|e\|_\infty + \Gamma_y \|r\|_\infty + \Gamma_y \|r\|_\infty
\]
\[
+ \Lambda_f + \|\Delta\|_\infty + \gamma_y \|\xi\|_\infty.
\]
The claim follows.

Theorem 1 can be interpreted as follows. Two main conditions are sufficient to guarantee closed-loop stability. First, the model must describe accurately the model rate of variation with respect to \(y_t\) (i.e., the constant \(\gamma_y\) in Assumption 2 must be small). Second, the controller has to perform an effective inversion of the model (Assumption 3). These conditions allow for closed-loop stability and lead to the tracking error bound given in Theorem 1. It can be noted that this error is reduced if the model provides a “small” prediction error (\(\|\Delta\|_\infty\) is a measure of the prediction error on the whole model domain). In summary, the model should thus satisfy two requirements: it must be accurate in describing the dependence on \(y_t\) and, at the same time, in reproducing the system output. Note that, in the proposed control scheme, the model does not work in simulation but in prediction.

These results hold for any reference \(r \in \mathcal{Y}^S\), where \(\mathcal{Y}^S\) is the set of all sequences of interest for which the system output remains in the domain where \(\Delta\) is Lipschitz continuous with respect to \(y_t\), with fixed constant \(\gamma_y\). Clearly, this domain must be well explored by the data (2). In order to ensure the accuracy properties described above.

A reliable indication for generating suitable references can be the following: a reference signal should be a solution (or an approximate solution) of the system to control, i.e., a signal \(r = (r_1, r_2, \ldots)\) for which, at each time \(t\), a \(u_t\) exists giving \(y_{t+1} = g(y_t, u_t, u_{t-1}, \ldots, u_{t-n+1}, \xi_t) \approx r_{t+1}\). More in general, the reference trajectory must be compatible with the physical properties of the system to control. For instance, in a second order mechanical system, the two states are typically a position and a velocity. Thus, the position reference can be generated as a sequence of values ranging in the physical domain of this variable with “not too high” variations (no other particular indications are required here). The velocity reference can obviously be generated as the derivative of the position reference. Note anyway that reference design is a well-known open problem which arises for most nonlinear identification and control methods.

Remark 1: The stability analysis developed in the present paper is substantially different from the one in [7]. Indeed, no model is identified in [7]. The controller (directly designed from data) is seen as an approximation of some unknown ideal controller. The stability conditions depend on the quality of this approximation. In the present paper, no ideal controllers are assumed. The stability conditions are related to the quality of the identified model.

B. Properties of the linear controller

The stability analysis of Section V-A has been carried out considering the nonlinear and linear controllers together, as a unique block. The importance of the nonlinear controller is evident from the above results and discussions. An analysis is now carried out, showing that the linear controller is fundamental to further increase the tracking precision and robustness of the feedback system.

First of all, let us introduce the following assumption on the closed-loop system with the nonlinear controller.

Assumption 4: Let \(S'\) be the system formed by the feedback interconnection of \(S\) and \(K^{nl}\), having inputs \(u_t^{lin}\), \(r_t\) and \(\xi_t\) and output \(e_t = r_t - y_t\). The action \(u_t^{lin}\) of the linear controller \(K^{lin}\) and \(\xi_t\) are sufficiently small, so that \(S'\) is assumed to be characterized by an LTI (Linear Time Invariant) behavior.

This assumption is justified by the fact that the nonlinear controller \(K^{nl}\), if correctly designed, brings the system close
to a desired trajectory. It is thus reasonable supposing that the behavior of the system in a sufficiently small neighborhood of the trajectory is linear. Note also that the assumption is quite mild, as no specific dynamic description is required for $S'$. The variations of the signals from a given operating trajectory are simply required to be “small”.

Under Assumption 4, the overall control system of Figure [1] can be represented in an LFT (Linear Fractional Transformation) fashion as in Figure [3] where

$$ z_t = e_t, \quad w_t = [r_t, \xi_t]^T, $$

$$ \begin{bmatrix} z_t \\ e_t \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} w_t \\ u_{lin}^t \end{bmatrix} $$

and the $G_{ij}$’s are unknown transfer functions for all $i,j$. Notice that these transfer functions may even be unstable. Instead, with $K_{lin}$, not only the overall system is stable for Theorem 1 but the steady-state performance is definitely enhanced, as illustrated by the following result.

**Theorem 2:** Under Assumption 4 the system $S'$ with a linear controller $K_{lin}$ of type (6) designed according to the D$^2$IBC approach, is such that

(i) the steady state tracking error for a reference step excitation is zero;

(ii) any constant disturbance $\xi_t$ gives zero steady-state contribution to $e_t$.

**Proof.** From robust control theory [8], the transfer function between $w_t$ and $z_t$ in the scheme of Fig. 3 is

$$ T_{zw} = G_{11} + G_{12}K_{lin}(I - G_{22}K_{lin})^{-1}G_{21}. $$

Since, in our case, $z_t = e_t$, then $G_{11} = G_{21}, G_{12} = G_{22}$ and

$$ T_{zw} = G_{11} + G_{22}K_{lin}(I - G_{22}K_{lin})^{-1}G_{11}. $$

(i) Now consider only the contribution of $r_t$ on $z_t$ and let $G_{11} = [G_{11}^r G_{11}^f]$. The transfer function between $r_t$ and $z_t = e_t$ is the first element of $T_{zw} = [T_{xr}, T_{xz}]$, that is

$$ T_{xr} = G_{11}^r + \frac{G_{22}G_{11}^rK_{lin}}{1 - G_{22}K_{lin}} = \frac{1}{1 - G_{22}K_{lin}}. $$

Since the controller $K_{lin}$ of type (6) contains an integrator and the overall controller is such that the final system is stable (i.e. the numerator of $1 - G_{22}K_{lin}$ must have only stable roots), $T_{xr}$ turns out to have a derivative action which gives zero steady-state response to any reference step for all $G_{11}$ and $G_{22}$.

The variations of the signals from a given operating trajectory are simply required to be “small”.

The system $S'$ with the linear controller is such that

$$ T_{zt} = G_{11}^f + \frac{G_{22}G_{11}^fK_{lin}}{1 - G_{22}K_{lin}} = \frac{1}{1 - G_{22}K_{lin}}. $$

contains a derivator.

Notice that, without $K_{lin}$, the relationship between $w_t$ and the tracking error $e_t$ can in principle be anything. Intuitively, if the nonlinear operation inverted by $K_{lin}$ is accurate enough, the asymptotic effect of the external disturbances on the error will be small. However, Theorem 2 shows that only the linear controller can guarantee that such an effect is exactly zero.

**Remark 2:** Notice that the above results may hold also when $S'$ is linear but not time-invariant, provided that the linear relationships can be written - for any $i,j$ - as $G_{ij} = G_{ij} + \Delta G$, where $G_{ij}$ is a time-invariant nominal term and $\Delta G$ is another time-invariant term upper-bounding the time-varying dynamics, namely a system with a bound $\delta G < \infty$ such that $\|\Delta G\|_{\infty} \leq \delta G$.

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