SINGULAR CONTINUOUS SPECTRUM AND GENERIC FULL SPECTRAL/PACKING DIMENSION FOR UNBOUNDED QUASIPERIODIC SCHRÖDINGER OPERATORS

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Abstract. We proved that Schrödinger operators with unbounded potentials \((H_{\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + g(\theta + n\alpha)u_n f(\theta + n\alpha)\) have purely singular continuous spectrum on the set \(\{E : 0 < L(E) < \delta(\alpha, \theta; f, g)\}\), where \(\delta\) is an explicit function and \(L\) is the Lyapunov exponent. We only require \(f, g\) are Hölder continuous functions and \(f\) has finitely many zeros with weak non-degenerate assumptions. Moreover, we show that for generic \(\alpha\) and a.e. \(\theta\), the spectral measure of \(H_{\alpha,\theta}\) has full spectral/packing dimension.

1. Introduction and main results

Unbounded Schrödinger operators attract a lot of interests in both physics and math literatures, see e.g. [15, 5, 3, 13, 29, 28, 26, 17, 19, 14, 21, 23]. In general, high barriers of the operators lead to the absence of absolutely continuous spectrum or even pure point spectrum with localized eigenstates, see e.g. [28, 26, 17, 19]. One significant example of unbounded quasiperiodic Schrödinger operator is the Maryland model, proposed by Grempel, Fishman, and Prange [15] as a linear version of the quantum kicked rotor. Mathematically, it is interesting due to its richness of spectral theory, see Simon [7, 29]. A complete description of spectral transitions with respect to all parameters for the Maryland model was given recently by Jitomirskaya and Liu in [21]. There are also a number of multidimensional generalizations of Maryland model, see e.g. [5, 13]. On the other hand, there is a simple way of excluding the existence of point spectrum that relies on Gordon condition (a single/double almost repetition) of the potential going back to Gordon [16]; see also in [7, 8]. Most applications of the Gordon-type argument quantify the competition between the quality of repetitions and the Lyapunov growth, and are concentrated on bounded potentials, see e.g. [3, 4, 5]. Simon in [29] applied Gordon-type of arguments to Maryland model and obtained purely singular continuous spectrum for generic frequencies. Jitomirskaya and Liu in [21] obtained the sharp parameters region (both in phase and frequency) for singular continuous spectrum by the refined Gordon-type argument (see Theorem 3.4). This sharp result about the singular continuous spectrum was generalized by Jitomirskaya and Yang in [23] to unbounded quasiperiodic Schrödinger operators with meromorphic potentials.

Recently, there has been an increased interest in developing methods that don’t involve analyticity, see e.g. [27, 1, 31, 20]. The methods to exclude point spectrum, used in [29, 21, 23], strongly rely on the meromorphic potentials, where the singularities are analytic. It is still not clear how to obtain singular continuous spectrum in the best possible arithmetic regime for rough unbounded potential. This leads to our first motivation. In this paper, we study unbounded quasiperiodic Schrödinger operators with lower regularity assumptions. We obtain the absence of point spectrum in the sharp parameters regime. This generalizes the results of [23] to more general unbounded potentials.

Moreover, we want to study finer decompositions of the singular continuous measure of quasiperiodic Schrödinger operators with rough unbounded potentials. In the recent work of Jitomirskaya and
Zhang in [24], a quantitative version of Gordon-type results was obtained: a quantitative strengthening (multiple almost repetitions) implies quantitative continuity of the spectral measure. While the main application to the quasiperiodic case also requires the potential to be bounded. For unbounded operators, the difficulty lies in obtaining uniform upper semicontinuity of the Lyapunov exponent. This is resolved in Lemma 4.2 which leads to quantitative spectral continuity after incorporating the abstract scheme developed in [24]. The main consequence is the full spectral/packing dimension of the spectral measures for generic frequencies, which generalizes the results of [24] to unbounded potentials.

More precisely, we study lattice Schrödinger operators on \( \ell^2(\mathbb{Z}) \) of the form:

\[
(H_{\alpha, \theta} u)_n = u_{n+1} + u_{n-1} + \frac{g(\theta + n\alpha)}{f(\theta + n\alpha)} u_n,
\]

where \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) is the frequency, \( \theta \in \mathbb{T} := \mathbb{R} / \mathbb{Z} \) is the phase. Let \( C^{\tau_0} (\mathbb{T}, \mathbb{R}) \) be the space of \( \tau_0 \)-Hölder continuous functions with norm:

\[
\| f \|_{\tau_0} := \sup_{x \in \mathbb{T}} |f(x)| + \sup_{x, y \in \mathbb{T}} \frac{|f(x) - f(y)|}{|x - y|^{\tau_0}} < \infty
\]

for some \( 0 < \tau_0 \leq 1 \). We assume \( f, g \in C^{\tau_0} (\mathbb{T}, \mathbb{R}) \). We also require \( f \) to be in the following space of functions introduced in [18]:

\[
\mathcal{F}(\mathbb{T}, \mathbb{R}) := \left\{ f \in C^0(\mathbb{T}, \mathbb{R}) : \exists m \in \mathbb{N}^+, \theta_\ell \in \mathbb{T}, 0 < \tau_\ell \leq 1, \ell = 1, \cdots, m \\
\text{such that } \bar{f}(\theta) := \frac{f(\theta)}{\prod_{\ell=1}^m |\sin \pi (\theta - \theta_\ell)|^{\tau_\ell}} \in C^0(\mathbb{T}, \mathbb{R}) \text{ and } \inf_{\theta} |\bar{f}(\theta)| > 0 \right\}
\]

Let \( \theta_\alpha \) be the continued fraction approximants of \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \). We define index \( \delta \) as follows:

\[
\delta(\alpha, \theta) = \delta(\alpha, \theta; f, g) = \limsup_{n \to \infty} \frac{\sum_{\ell=1}^m \tau_\ell \ln \|q_n(\theta - \theta_\ell)\|_{\mathbb{R}/\mathbb{Z}} + \tau_{\min} \ln q_{n+1}}{q_n},
\]

where \( \tau_{\min} = \min_{0 \leq \ell \leq m} \tau_\ell \) and \( \|x\|_{\mathbb{R}/\mathbb{Z}} = \min_{z \in \mathbb{Z}} |x - z| \). Let \( L(E) \) be the Lyapunov exponent, see [24]. \( L(E) \) depends also on \( \alpha \) but we suppress it from the notation as we keep \( \alpha \) fixed.

Our main result is:

**Theorem 1.1.** Let \( f, g \) be \( \tau_0 \)-Hölder continuous, \( f \) be given as in (1.3) and let \( \delta(\alpha, \theta; f, g) \) be as in (1.4). Then \( H_{\alpha, \theta} \) has no eigenvalues on \( \{ E : L(E) < \delta(\alpha, \theta) \} \).

**Remark 1.** Absence of absolutely continuous spectrum follows from a.e. positivity of the Lyapunov exponents and holds for all unbounded potentials [28]. Then \( H_{\alpha, \theta} \) has purely singular continuous spectrum on \( \{ E : 0 < L(E) < \delta(\alpha, \theta) \} \).

It is easy to check that the class \( \mathcal{F}(\mathbb{T}, \mathbb{R}) \) in (1.3) contains all \( C^k, k \geq 1 \) continuous potentials with finitely many non-degenerate zeros \(^1\). In particular, it covers the potential studied in [23], where \( g \) is assumed to be Lipschitz continuous and \( f \) is analytic, where all \( \tau_\ell = 1, \ell = 0, \cdots, m \). The following examples where the potential is only Hölder continuous show that such generalization is indeed nontrivial.

**Example 1.1.** Let \( (H_{\alpha, \theta} u)_n = u_{n+1} + u_{n-1} + \frac{1}{f(\theta + n\alpha \mod 1)} u_n \), where

\[
f(\theta) = \begin{cases} 
0 & \text{, } \theta = 0; \\
\theta \cos(\frac{1}{\theta}) + 2\sqrt{\theta} & \text{, } 0 < \theta \leq \frac{1}{2}; \\
f(1 - \theta) & \text{, } \frac{1}{2} < \theta \leq 1.
\end{cases}
\]

\(^1\)We say \( \theta_0 \in \mathbb{T} \) is a non-degenerate zero of \( f \in C^k(\mathbb{T}, \mathbb{C}) \) if \( f(\theta_0) = 0 \) and \( f^{(k)}(\theta_0) \neq 0 \).
It is an easy exercise by definition that \( \theta \cos(\frac{1}{\theta}) \) is sharp \( \frac{1}{2} \)-Hölder continuous near 0. Moreover, 
\[
\frac{f(\theta)}{\sqrt{|\sin \pi \theta|}} \geq 1
\]
and is continuous on \( \mathbb{T} \). Therefore, \( f \in \mathcal{F} \) with \( \tau_1 = \frac{1}{2} \). Let 
\[
\delta(\alpha, \theta) = \limsup_{n \to \infty} \frac{\ln \|g_n \theta\|_{\mathbb{R}/\mathbb{Z}} + \ln q_{n+1}}{q_n}.
\]

Theorem 1.1 implies that \( H_{\alpha, \theta} \) has purely singular continuous spectrum on \( \{ E : 0 < L(E) < \frac{1}{2} \delta(\alpha, \theta) \} \).

**Example 1.2.** Let 
\[
(H_{\alpha, \theta} u)_n = u_{n+1} + u_{n-1} + h(\theta + n \alpha \mod 1) u_n,
\]
where
\[
h(\theta) = \begin{cases} 
\log \theta & , \ 0 < \theta \leq \frac{1}{2}; \\
\log(1 - \theta) & , \ \frac{1}{2} < \theta < 1.
\end{cases}
\]

Let \( \tilde{\delta}(\alpha, \theta) \) be as in (1.6). Let 
\[
g(\theta) = \begin{cases} 
0 & , \ \theta = 0; \\
\theta \log \theta & , \ 0 < \theta \leq \frac{1}{2}; \\
g(1 - \theta) & , \ \frac{1}{2} < \theta \leq 1.
\end{cases}
\]

Clearly, \( h(\theta) = \frac{g(\theta)}{\theta} \) on \( (0, 1) \) and \( g(\theta) \) is \( (1 - \epsilon) \)-Hölder continuous on \( [0, 1] \) for any \( 0 < \epsilon < 1 \). By Theorem 1.1 \( H_{\alpha, \theta} \) has purely singular continuous spectrum on \( \{ E : 0 < L(E) < (1 - \epsilon)\tilde{\delta}(\alpha, \theta) \} \) for any \( 0 < \epsilon < 1 \). It is also easy to check that exact the same thing holds true for \( \tilde{h}(\theta) = \frac{\chi_{(0,1/2)}(\theta)}{\theta} \log \theta - \chi_{(1/2,1)}(\theta) \log(1 - \theta) \).

Once we obtain singular continuous spectral measure, we want to move further to study the fractal dimension properties of the measure. Let \( \mu = \mu_{\alpha, \theta} \) be the spectral measure of the operator \( H_{\alpha, \theta} \). The fractal properties of \( \mu \) are closely related to the boundary behavior of its Borel transforms \( M(E + i\varepsilon) = \int \frac{\delta(\mathbb{R}/\mathbb{Z})}{E - \tau_0} \), see e.g. [10]. We are interested in the following local fractal dimension of \( \mu \). For any compact set \( I \subset \mathbb{R} \), let \( \mu_I \) be the restriction of \( \mu \) on \( I \). We say \( \mu_I \) is \( \gamma \)-spectral continuous if for some \( \gamma \in (0, 1) \) and \( \mu \) a.e. \( E \in I \), we have 
\[
\liminf_{\varepsilon \downarrow 0} \varepsilon^{1-\gamma} |M(E + i\varepsilon)| < \infty.
\]

Define the (upper) spectral dimension of \( \mu_I \) to be 
\[
\text{dim}_{\text{spe}}(\mu_I) = \sup \{ \gamma \in (0, 1) : \ \mu_I \text{ is } \gamma \text{-spectral continuous} \}.
\]

Consider another arithmetic index of \( \alpha \) defined by \( \beta(\alpha) = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n} \). It is well known that for a.e. \( \theta \in \mathbb{T} \), \( \delta(\alpha, \theta) = \tau_{\min} \cdot \beta(\alpha) \). Combine the Lyapunov growth estimates (see Lemma 4.2) and the recent work of Jitomirskaya-Zhang [24], we have the following quantitative spectral continuity for \( \mu_I \).

**Theorem 1.2.** Under the same assumption of Theorem 1.1 For any compact set \( I \subset \mathbb{R} \), there is a constant \( C = C(\|f\|_{\infty}, \|g\|_{\infty}, I) \) such that for a.e. \( \theta \in \mathbb{T} \) if \( \tau_{\min} \cdot \beta(\alpha) > C \), then \( \mu_I \) has positive spectral dimension. In particular, for a.e. \( \theta \), if \( \beta(\alpha) = \infty \), then \( \mu_I \) has full spectral dimension.

**Remark 1.2.** The Hausdorff/packing dimension of a (Borel) measure \( \mu \), namely, \( \text{dim}_{H}(\mu)/\text{dim}_{P}(\mu) \) is defined through the \( \limsup \) / \( \liminf \) (\( \mu \) almost everywhere) of its \( \gamma \)-derivative \( \limsup_{\varepsilon \downarrow 0} \frac{\ln \mu(E - \varepsilon, E + \varepsilon)}{\ln \varepsilon} \). It is well known (see e.g. [11, 24]) that the (local) packing dimension is bounded from below by
the spectral dimension in (1.11). Therefore, we have that for $\beta(\alpha) = \infty$ (which is a generic subset of $[0,1]$), the restriction of $\mu$ on any compact set has full packing dimension. On the other hand, it was showed in [30] that $\mu$ is Hausdorff singular ($\dim_H(\mu) = 0$) on $\{E : L(E) > 0\}$ for any $\alpha, \theta$. Therefore, we have a generic subset of frequencies $\alpha$, such that for a.e. $\theta$, $\mu_I$ is not exact dimensional (the Hausdorff dimension and the packing dimension do not equal).

2. Preliminaries: cocycle, Lyapunov exponent

From now on, we assume $f \in C^\tau_0$ and has the expression as in (1.3):

$$f(\theta) = \tilde{f}(\theta) \prod_{\ell=1}^{m} |\sin \pi(\theta - \theta_{\ell})|^{\tau_{\ell}}, \quad \theta_{\ell} \in \mathbb{T}, \quad \tau_{\ell} \in (0,1], \quad \ell = 1 \cdots, m, \quad \tilde{f} \in C^0 \text{ and } \inf \frac{1}{\tau} |\tilde{f}(\theta)| > 0.$$ 

Let $S = \bigcup_{k=1}^{q} \theta_{\ell} + \hat{\alpha} \mathbb{Z} + \mathbb{Z}$. We fix $E$ in the spectrum and $\theta \in S^c$.

A formal solution of the equation $H_{\alpha,\omega} u = Eu$ can be reconstructed via the following equation

$$(2.1) \quad \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A(\theta + n\alpha) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix},$$

where $A(\theta) = \begin{pmatrix} E - g(\theta) & -1 \\ 1 & 0 \end{pmatrix}$ is the so-called transfer matrix.

The pair $(\alpha, A)$ is the cocycle corresponding to the operator $[14]$. It can be viewed as a linear skew-product $(x, \omega) \mapsto (x + \alpha, A(x) \cdot \omega)$. Generally, one can define $M_n$ for an invertible cocycle $(\alpha, \mathbb{M})$ by $(\alpha, \mathbb{M})^n = (\alpha, M_n)$, $n \in \mathbb{Z}$ so that for $n \geq 0$:

$$(2.2) \quad M_n(x) = M(x + (n-1)\alpha)M(x + (n-2)\alpha) \cdots M(x),$$

and $M_{-n}(x) = M_n^{-1}(x - n\alpha)$.

The Lyapunov exponent of a cocycle $(\alpha, M)$ is defined by

$$L(\alpha, M) = \lim_{n \to \infty} \frac{1}{n} \int_{\mathbb{T}} \ln \|M_n(x)||dx.$$

Let $A(x) = \frac{1}{f(x)} D(x)$, where

$$(2.3) \quad D(x) = \begin{pmatrix} Ef(x) - g(x) & -f(x) \\ f(x) & 0 \end{pmatrix}$$

is the regular part of $A(x)$. It is convenient to assume $\int_{\mathbb{T}} \ln |f(x)| dx = 0$, otherwise it is enough to consider the constant scaling by $b = e^{\int_{\mathbb{T}} \ln |f(x)| dx}$ and $f_b := bf(\theta), g_b := bg(\theta)$ in the potential. We have

$$(2.4) \quad L(E) := L(\alpha, A) = L(\alpha, D).$$

Lemma 2.1. [1] Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}, \theta \in \mathbb{R}$ and $0 \leq j_0 \leq q_n - 1$ be such that

$$| \sin \pi(\theta + j_0 \alpha) | = \inf_{0 \leq j \leq q_n - 1} | \sin \pi(\theta + j \alpha) |,$$

then for some absolute constant $C > 0$,

$$-C \ln q_n \leq \sum_{j=0, j \neq j_0}^{q_n-1} \ln | \sin \pi(\theta + j \alpha) | + (q_n - 1) \ln 2 \leq C \ln q_n.$$
We will also use that the denominators of continued fraction approximants of \( \alpha \) satisfy
\[
\| k\alpha \|_{\mathbb{R}\setminus\mathbb{Z}} \geq \| q_n \alpha \|_{\mathbb{R}\setminus\mathbb{Z}}, \quad 1 \leq k < q_{n+1},
\]
and
\[
\frac{1}{2q_{n+1}} \leq \| q_n \alpha \|_{\mathbb{R}\setminus\mathbb{Z}} \leq \frac{1}{q_{n+1}}.
\]

(2.5)

A quick corollary of subadditivity and unique ergodicity is the following uniform upper semicontinuity result:

**Lemma 2.2** ([12], see also in [2, 22]). Suppose \((\alpha, A)\) is a continuous cocycle. Then for any \( \varepsilon > 0 \), there exists \( C(\varepsilon) > 0 \), such that for any \( x \in \mathbb{T} \) we have
\[
\| A_n(x) \| \leq C e^{n(L(A)+\varepsilon)}.
\]

**Remark 2.1.** The above result was first obtained by Furman in [12] in a relatively more general setting. Jitomirskaya and Mavi generalized the result to the case where \( A(x) \) is bounded in norm (from above) and has points of discontinuity with Lebesgue measure zero. In general, such uniform \( \limsup \) may not hold if \( A(x) \) is unbounded. In section 4, we obtain a weakened version (see Lemma 4.2) of such upper semicontinuity for quasiperiodic unbounded cocycles. The estimate plays important role in the proof of Theorem 1.2. It is also of independent interest in the study of the multiplicative ergodic theorem for uniquely ergodic systems.

**Remark 2.2.** Applying this to 1-dimensional continuous cocycles, we get that if \( g \) is a continuous function such that \( \ln |g| \in L^1(\mathbb{T}) \), then
\[
\left| \prod_{j=a}^{b} g(x + j\alpha) \right| \leq e^{(b-a+1)(\int \ln |g|d\theta + \varepsilon)}.
\]

3. Absence of Point Spectrum

By the definition of \( \delta(\alpha, \theta) \), for any \( \varepsilon > 0 \), there exists a subsequence \( q_{n_i} \) of \( q_n \) such that
\[
\sum_{\ell=1}^{m} \tau_{\ell} \ln \| q_{n_i} (\theta - \theta_{\ell}) \| + \tau_{\min} \ln q_{n_{i+1}} > \delta(\alpha, \theta) - \frac{\varepsilon}{4}.
\]

(3.1)

In this section, we omit the subindex \( n_i \) and still denote the subsequence satisfying (3.1) by \( q_n \) whenever it is clear.

3.1. **key lemmas.** The following lemmas are in spirit close to Lemma 4.2 in [23]. For all \( 1 \leq \ell \leq m \), let \( j_{\ell} \in [0, q_n) \) be such that the following holds: \( | \sin \pi (\theta - \theta_{\ell} + j_{\ell}\alpha) | = \inf_{0 \leq j < q_n} | \sin \pi (\theta - \theta_{\ell} + j\alpha) | \).

**Lemma 3.1.** If \( \delta(\alpha, \theta) > 0 \), then
\[
\prod_{\ell=1}^{m} | \sin \pi (\theta - \theta_{\ell} + j_{\ell}\alpha) |^{\tau_{\ell}} \geq \frac{e^{q_n(\delta - \frac{\varepsilon}{8})}}{q_n^{\tau_{\min}}},
\]
where \( \tau_{\min} = \min_{0 \leq \ell \leq m} \tau_{\ell} \).

**Proof.** By (3.1),
\[
\prod_{\ell=1}^{m} \| q_n (\theta - \theta_{\ell}) \|^{\tau_{\ell}} > \frac{e^{q_n(\delta - \frac{\varepsilon}{8})}}{q_n^{\tau_{\min}}},
\]

(3.3)
In particular,

\[(3.4) \quad \|q_n(\theta - \theta_\ell)\|^{\tau_\ell} > \frac{e^{q_n(\delta - \frac{4}{\tau_\ell})}}{q_{n+1}^{\tau_\ell}} > \frac{e^{q_n(\delta - \frac{4}{\tau_\ell})}}{q_{n+1}^{\tau_\ell}}\]

for any \(1 \leq \ell \leq m\). Therefore, \(\|q_n(\theta - \theta_\ell)\| \geq q_{n+1}^{(\delta - \frac{4}{\tau_\ell})} q_{n+1}^{-1} \geq 2q_n\) and

\[q_n | \sin \pi(\theta - \theta_\ell + j_\ell \alpha) | \geq 2q_n \|\theta - \theta_\ell + j_\ell \alpha\| \geq 2\|q_n(\theta - \theta_\ell)\| - 2j_\ell \|q_n \alpha\| \geq \|q_n(\theta - \theta_\ell)\|,\]

provided \(q_n\) large. Denote

\[(3.5) \quad \tau_{\text{sum}} = \sum_{\ell=1}^{m} \tau_\ell.\]

Then we have

\[
\prod_{\ell=1}^{m} | \sin \pi(\theta - \theta_\ell + j_\ell \alpha) | \geq \prod_{\ell=1}^{m} \frac{\|q_n(\theta - \theta_\ell)\|^{\tau_\ell}}{q_{n+1}^{\tau_\ell}} = \prod_{\ell=1}^{m} \frac{\|q_n(\theta - \theta_\ell)\|^{\tau_\ell}}{q_{n+1}^{\tau_\ell}} \prod_{\ell=1}^{m} \frac{1}{q_{n+1}^{\tau_\ell}} \geq \frac{e^{q_n(\delta - \frac{4}{\tau_\ell})}}{q_{n+1}^{\tau_\ell}} \cdot \frac{1}{q_{n+1}^{\tau_\ell}} \geq \frac{e^{q_n(\delta - \frac{4}{\tau_\ell})}}{q_{n+1}^{\tau_\ell}},
\]

provided \(q_n\) large.

\[\square\]

\textbf{Lemma 3.2.} Let \(f \in \mathcal{F}\) as defined in \((1.3)\) where

\[f(\theta) = \tilde{f}(\theta) \prod_{\ell=1}^{m} | \sin \pi(\theta - \theta_\ell) |^{\tau_\ell}, \quad \tilde{f} \in C^0, \quad \inf_{\tilde{f}} > 0.\]

Assume further that \(\int_T \ln |f(\theta)| d\theta = 0\), then

\[(3.6) \quad \prod_{j=0}^{q_n-1} |f(\theta + j\alpha)| \geq \frac{e^{q_n(\delta - \epsilon)}}{q_{n+1}^{\tau_{\text{min}}}},\]

where \(\tau_{\text{min}} = \min_{0 \leq \ell \leq m} \tau_\ell.\)

\textbf{Proof.} By Lemma 2.1,

\[(3.7) \quad \prod_{j=0, j \neq j_\ell}^{q_n-1} | \sin \pi(\theta - \theta_\ell + j\alpha) | \geq e^{-(\epsilon q_n - 1) \ln 2 - C \ln q_n} \geq e^{-q_n(\ln 2 - \epsilon)} q_{n+1}^{\tau_{\text{sum}}},\]

provided \(C \ln q_n < \frac{\epsilon}{\tau_{\text{sum}}}{q_n},\) in which \(C\) is the absolute constant in Lemma 2.1 and \(\tau_{\text{sum}}\) is given as in \((3.5)\).

Combine \((3.2)\) with \((3.7)\), we have
\[ \prod_{j=0}^{q_n-1} \prod_{\ell=1}^m |\sin(\theta - \theta_\ell + j\alpha)|^{\tau_\ell} = \prod_{\ell=1}^m \left( \prod_{j=0, j \neq \ell}^{q_n-1} |\sin(\theta - \theta_\ell + j\alpha)|^{\tau_\ell} \right) \cdot \left( \prod_{\ell=1}^m |\sin(\theta - \theta_\ell + j\alpha)|^{\tau_\ell} \right) \]

\[ \geq \left( \prod_{\ell=1}^m \left( e^{-q_n (\ln 2 + \frac{1}{2} \tau_{\text{sum}}^{-1})^{\tau_\ell}} \right) \cdot \frac{e^{q_n (\delta - \frac{3}{4})}}{q_{n+1}^{\tau_{\text{min}}}} \right) \cdot e^{-q_n (\tau_{\text{sum}} \ln 2 + \frac{1}{2})} \cdot \frac{e^{q_n (\delta - \frac{3}{4})}}{q_{n+1}^{\tau_{\text{min}}}} = e^{q_n (\delta - \frac{3}{4} - \tau_{\text{sum}} \ln 2)} q_{n+1}^{-\tau_{\text{min}}} \cdot \frac{e^{q_n (\delta - \frac{3}{4})}}{q_{n+1}^{\tau_{\text{min}}}}. \]

Notice \(|f(\theta)|^{-1}\) is continuous, by Remark 2.2 we have

\[ \prod_{j=0}^{q_n-1} |f(\theta_\ell + j\alpha)| \geq e^{q_n \left( \int_{\tau} |f(\theta)| \, d\theta - \frac{3}{4} \right)} \cdot \frac{e^{q_n (\delta - \frac{3}{4} - \tau_{\text{sum}} \ln 2)} q_{n+1}^{-\tau_{\text{min}}} \cdot \frac{e^{q_n (\delta - \frac{3}{4})}}{q_{n+1}^{\tau_{\text{min}}}}}. \]

By the well known integral \(\int_{\tau} \ln |\sin \theta| \, d\theta = -\ln 2\), it is easy to check that

\[ (3.8) \int_{\tau} \ln |f(\theta)| \, d\theta = \int_{\tau} \ln |\tilde{f}(\theta)| \, d\theta - \tau_{\text{sum}} \ln 2. \]

Therefore,

\[ \prod_{j=0}^{q_n-1} |f(\theta_\ell + j\alpha)| \geq e^{q_n \left( \int_{\tau} |\tilde{f}(\theta)| \, d\theta - \frac{3}{4} \right)} e^{q_n (\delta - \frac{3}{4} - \tau_{\text{sum}} \ln 2)} q_{n+1}^{-\tau_{\text{min}}} \geq e^{q_n \left( \int_{\tau} |f(\theta)| \, d\theta - \frac{3}{4} \right)} e^{q_n (\delta - \frac{3}{4} - \tau_{\text{sum}} \ln 2)} q_{n+1}^{-\tau_{\text{min}}} \]

\[ \boxdot \]

### 3.2. Proof of Theorem 1.1

Let \(A(x)\) be given as in (2.1). Direct computation shows that

\[ A^{-1}(x) = \frac{1}{f(x)} \left( \begin{array}{cc} f(x) & 0 \\ -f(x) & Ef(x) - g(x) \end{array} \right) \triangleq \frac{F(x)}{f(x)}. \]

Observe that \(F(\theta)\) and \(f(\theta)\) are both \(C^\gamma\) Hölder continuous functions. There exists constant \(\tilde{C}\) only depending the \(C^\gamma\)-norms of \(f\) and \(g\) given in (1.2) such that:

\[ (3.9) \sup_\theta \|F(\theta + q_n \alpha) - F(\theta)\| < \frac{\tilde{C} q_n^{-1}}{q_{n+1}^{-1}}, \sup_\theta \|f(\theta + q_n \alpha) - f(\theta)\| < \frac{\tilde{C}}{q_{n+1}^{-1}} \frac{q_n^{-\tau_{\text{min}}}}{q_{n+1}^{-\tau_{\text{min}}}}. \]

The following lemma follows immediately from (3.6) and (3.9). The proof is exactly the same as used for Lemma 3.2 in Section 5 of [23]. We omit the details here.

**Lemma 3.3.** Let \(\varphi\) be a solution to \(H_{\alpha, \theta} \varphi = E \varphi\) satisfying \(\left( \begin{array}{c} \varphi_0 \\ \varphi_{-1} \end{array} \right) = 1\). Let \(q_n\) be the subsequence given as in (3.6), we have the following estimates:

\[ (3.10) \| (A_{q_n}^{-1}(\theta) - A_{q_n}^{-1}(\theta - q_n \alpha)) \left( \begin{array}{c} \varphi_0 \\ \varphi_{-1} \end{array} \right) \| \leq e^{q_n (L(E) - \delta(\alpha, \theta) + 4\varepsilon)}, \]

and

\[ (3.11) \| (A_{q_n}^2(\theta) - A_{2q_n}(\theta)) \left( \begin{array}{c} \varphi_0 \\ \varphi_{-1} \end{array} \right) \| \leq e^{q_n (L(E) - \delta(\alpha, \theta) + 4\varepsilon)}. \]
Lemma 4.1. For any \( q_n \) large enough, 
\[
\| (A_{q_n}^2(\theta) - A_{2q_n}(\theta)) \left( \varphi_0 \right) \| \leq e^{-c q_n}
\]
more details in Lemma 4.2 in [18].

Following Lyapunov growth of the unbounded cocyle \( (\alpha, A) \)
\[
\max \{ \| \varphi_{q_n} \|, \| \varphi_{-q_n} \|, \| \varphi_{2q_n} \| \} \geq \frac{1}{4}.
\]

4. Generic full spectral/packing dimension

In this part, we fix \( \theta \) such that \( \delta(\alpha, \theta) = \tau_{\min} \cdot \beta(\alpha) \) as in (1.4). Let the subsequence \( q_n \) the given as in (3.1) and (3.6) satisfying 
\[
\delta(\alpha, \theta) > \frac{\tau_{\min} \ln q_{n+1}}{q_n} > \delta(\alpha, \theta) - \frac{\varepsilon}{4} \iff e^{q_n \tau_{\min} \cdot \beta} > q_{n+1}^\varepsilon > e^{q_n (\tau_{\min} \cdot \beta - \varepsilon/4)}.
\]

By repeating the argument in the proof of Lemma 3.1 and 3.2 one can prove a more general version of (3.9) for \( \theta \) such that \( \delta(\alpha, \theta) = \tau_{\min} \cdot \beta(\alpha) \). We restate the result here directly. Readers can find more details in Lemma 4.2 in [18].

Lemma 4.1. Let \( f \in \mathcal{F} \) be given as in (1.3) with \( \int_{\Theta} |f(\theta)| \, d\theta = 0 \). For \( \beta(\alpha), \varepsilon > 0 \), let the sequence \( q_n \to \infty \) be defined as in (1.4). There is a full Lebesgue measure set \( \Theta = \Theta(\alpha, \theta_1, \cdots, \theta_m) \) such that for any \( \theta \in \Theta \) and \( q_n \) large enough, \( f(\theta + \alpha) \) satisfies:

\[
\min_{|m| \leq e^{q_n/10}} |f(\theta + m\alpha)| > e^{-\varepsilon q_n}.
\]

Let \( D \) and \( D_n \) be defined as in (2.2), (2.3). By Lemma 4.2 for \( r \) large enough,
\[
\| D_r(\theta) \| \leq e^{(L(D) + \varepsilon)r}.
\]

Combine the definition (2.3) and the estimates (2.2), (3.6), direct computation shows the following Lyapunov growth of the unbounded cocyle \( (\alpha, A) \). For \( E \) in a compact set \( I \), let \( \Lambda = 2\varepsilon + \sup_{E \in I} L(E) \), we have

Lemma 4.2. For any \( \theta \in \Theta \) and \( q_n \) large enough and \( 1 \leq r \leq q_n \).
\[
\sup_{|m| \leq e^{q_n/10}} \| A_r(\theta + m\alpha) \| < e^{(L(E) + 2\varepsilon)q_n} = e^{\Lambda q_n}.
\]

Remark 4.1. The above lemma actually shows that the following lim inf
\[
\liminf_n \frac{1}{n} \ln \| A_n(E, \theta) \| \leq L(E)
\]

The sequence itself only depends on \( \beta(\alpha) \), while the largeness depends on \( \theta, \alpha, \beta, \varepsilon, \tau. \)
holds uniformly on a full measure set \( \tilde{\Theta} = \bigcup_{|m| \leq e^{\varepsilon qn/10}} (\Theta + m\alpha) \). It was showed in [12] that the lim sup in (4.5) holds uniformly on \( T \) for any continuous cocycle. The upper semi continuity was generalized in [22] to almost continuous bounded cocyles. In general, the lim sup in (4.5) shall not hold for unbounded cocyles. Lemma 4.2 can be viewed as a quantitative generalization of [12, 22] to the unbounded case.

On the other hand, by (3.9) and (4.2), \( A_n(\theta) \) has strong repetitions (\( \tau_{\min, \beta} \)-almost periodicity):

\[
\sup_{\theta} \| A(\theta + qn\alpha) - A(\theta) \| < \frac{\tilde{C}}{q_{\min}^{n+1}} < \tilde{C}q^{-n(\tau_{\min, \beta} - \varepsilon/4)}.
\]

(4.6)

As proved in [24], the growth of the transfer matrices (4.4) and the strong repetition (4.6) imply that there exists an absolute constant \( C_1 \) such that \( \mu_\gamma \) is \( \gamma \)-spectral continuous for all \( \gamma \leq 1 - \frac{C_1A}{\tau_{\min, \beta} - \varepsilon/4} \).

This completes the proof of Theorem 1.2 by the definition of spectral dimension in (1.11). We omit the proof here. For more details about the \( \gamma \)-spectral continuity, we refer readers to Theorem 1 and Theorem 6 in [24].

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REFERENCES

1. A. Avila and S. Jitomirskaya, The ten Martini problem, Ann. of Math. (2) 170 170 (2009), no. 1, 303-342.
2. A. Avila and S. Jitomirskaya, Almost localization and almost reducibility, J. Eur. Math. Soc. 12 (2010), no. 1, 93-111.
3. M. Berry, Incommensurability in an exactly-soluble quantal and classical model for a kicked rotator, Physica D: Nonlinear Phenomena 10 (1984), no. 3, 369-378.
4. K. Bjerklöv, Dynamics of the quasi-periodic Schrödinger cocycle at the lowest energy in the spectrum, Comm. Math. Phys. 272 (2007), no. 2, 397-442.
5. J. Blütschard, R. Lima and E. Scoppola, Localization in v-dimensional incommensurate structures, Comm. Math. Phys. 88 (1983), no. 4, 465-477.
6. M. Boshernitzan and D. Damanik, Generic continuous spectrum for ergodic Schrödinger operators, Comm. Math. Phys. 283 (2008), no. 3, 647-662.
7. H. Cycon, R. Froese, W. Kirsch and B. Simon, Schrödinger Operators with Application to Quantum Mechanics and Global Geometry, Texts and Monographs in Physics, Springer-Verlag, Berlin, (1987).
8. D. Damanik, Gordon-type arguments in the spectral theory of one-dimensional quasicrystals, Directions in mathematical quasicrystals, CRM Monogr., Ser., 13, Amer. Math. Soc., Providence, RI, 277305, (2000).
9. D. Damanik, A version of Gordon’s theorem for multi-dimensional Schrödinger operators, Trans. Amer. Math. Soc. 356 (2004), no. 2, 495-507.
10. R. del Rio, S. Jitomirskaya, Y. Last and B. Simon, Operators with singular continuous spectrum, IV. Hausdorff dimensions, rank one perturbations, and localization, J. Anal. Math. 69 (1996), 153-200.
11. K. Falconer, Techniques in Fractal Geometry, John Wiley & Sons, Ltd., Chichester, (1997).
12. A. Furman, On the multiplicative ergodic theorem for uniquely ergodic systems, Annales de l’Institut Henri Poincare (B) Probability and Statistics, 33 (1997), no. 6, 797-815.
13. A. L. Figotin and L. A. Pastur, An exactly solvable model of a multidimensional incommensurate structure, Comm. Math. Phys. 95 (1984), no. 4, 401-425.
14. S. Ganeshan, K. Kechedzhi and S. Das Sarma, Critical integer quantum hall topology and the integrable maryland model as a topological quantum critical point, Phys. Rev. B 90 (2014), no. 4, 041405.
15. D. Grempel, S. Fishman and R. Prange, Localization in an incommensurate potential: An exactly solvable model, Physical Review Letters 49 (1982) no.11, 833.
16. Y.A. Gordon, The point spectrum of the one-dimensional Schrödinger operator, Uspehi Mat. Nauk 31 (1976), 257-258.
17. Y.A. Gordon, V. Jaksic, S. Molchanov and B. Simon, Spectral properties of random Schrödinger operators with unbounded potentials, Comm. Math. Phys. 157 (1993), 230.
18. R. Han, F. Yang and S. Zhang, Spectral Dimension for $\beta$-almost periodic singular Jacobi operators and the extended Harper’s model, preprint, arXiv:1804.04322 (2018).
19. J. Janas, S. Naboko and G. Stolz, Decay Bounds on Eigenfunctions and the Singular Spectrum of Unbounded Jacobi Matrices, Int. Math. Res. Not. (2009), no. 4, 736-764.
20. S. Jitomirskaya and I. Kachkovskiy, All couplings localization for quasiperiodic operators with Lipschitz monotone potentials, to appear in J. Eur. Math. Soc (2017).
21. S. Jitomirskaya and W. Liu, Arithmetic spectral transitions for the Maryland model, Comm. Pure Appl. Math. 70 (2017), no. 6, 1025-1051.
22. S. Jitomirskaya and R. Mavi, Dynamical bounds for quasiperiodic Schrödinger operators with rough potentials, Int. Math. Res. Not. 1 (2017), 96-120.
23. S. Jitomirskaya and F. Yang, Singular continuous spectrum for singular potentials, Comm. Math. Phys. 351 (2017), no. 3, 1127-1135.
24. S. Jitomirskaya and S. Zhang, Quantitative continuity of singular continuous spectral measures and arithmetic criteria for quasiperiodic Schrödinger operators, preprint, arXiv:1510.07086 (2015).
25. W. Kirsch, S. A. Molchanov, and L. A. Pastur, The one-dimensional Schrödinger operator with unbounded potential: the pure point spectrum, Functional Analysis and its Applications, 24 (1990), 176-86.
26. W. Kirsch, S. A. Molchanov, and L. A. Pastur, One-dimensional Schrödinger operators with high potential barriers, Operator Theory: Advances and Applications 57 (1992), 163-70.
27. S. Klein, Anderson localization for the discrete one-dimensional quasi-periodic Schrödinger operator with potential defined by a Gevrey-class function, J. Funct. Anal. 218 (2005), no. 2, 255-292.
28. B. Simon and T. Spencer, Trace class perturbations and the absence of absolutely continuous spectra, Comm. Math. Phys. 125 (1989), no.1, 113-125.
29. B. Simon, Almost periodic Schrödinger operators. IV. The Maryland model, Ann. Physics 159 (1985), no.1, 157-183.
30. B. Simon, Equilibrium measures and capacities in spectral theory, Inverse Problems and Imaging, 1 (2007), 376-382.
31. Y. Wang, Z. Zhang, Cantor spectrum for a class of $C^2$ quasiperiodic Schrödinger operators, Int. Math. Res. Not. (2017), no. 8, 2300-2336.

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