GENERATING PRE-FRACTALS TO APPROACH REAL IFS-ATTRACTORS WITH A FIXED HAUSDORFF DIMENSION

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ABSTRACT. In this paper, we explain how to generate adequate pre-fractals in order to properly approximate attractors of iterated function systems on the real line within a priori known Hausdorff dimension. To deal with, we have applied the classical Moran’s Theorem, so we have been focused on non-overlapping strict self-similar sets. This involves a quite significant hypothesis: the so-called open set condition. The main theoretical result contributed in this paper becomes quite interesting from a computational point of view, since in such a context, there is always a maximum level (of the natural fractal structure we apply in this work) that may be achieved.

1. Introduction. The study and analysis of fractals have become more and more important once they were first introduced in science by Mandelbrot in the late seventies (see, e.g., [20, 21]). Indeed, since then, they have been explored in some scientific areas, including physics, statistics, and economics [10, 11]. Interestingly, it turns out that fractal patterns have been also explored in social sciences recently (see [8] and references therein).

It is worth mentioning that the key tool to study the complexity of a given set is the fractal dimension, since this is its main invariant which throws quite useful information about the complexity that it presents when being examined with enough level of detail.

In this way, fractal dimension has been usually understood as the classical box dimension, mainly in the field of practical applications. Its popularity is due to the possibility of its effective calculation and empirical estimation. On the other hand, the Hausdorff dimension constitutes a powerful theoretical model which allows to “measure” the complexity of a given set. Thus, while they are defined for any metric
More specifically, the calculation of the fractal dimension for the attractor of an iterated function system (IFS-attractor, herein) arises as an interesting question in Fractal Theory which could be dealt with from the point of view of fractal structures. In this way, the classical result proved by Moran in 1946 (see [22, Theorem III]), allows to affirm that both the box-counting and the Hausdorff dimensions for any strict self-similar set can be fully determined through a straightforward expression only involving the corresponding similarity factors. However, to achieve such a result, the open set condition hypothesis (OSC, herein) is required to be satisfied by the similarities in that IFS. Recall that such a hypothesis is a significant restriction required to the pieces of an attractor to ensure that they do not overlap too much.

Additionally, the application of fractal structures allows to provide new models for a fractal dimension definition on any generalized-fractal space, and not only on the Euclidean ones (see, e.g., [13, 12, 14]). This extends the classical theory of fractal dimension to the more general context of generalized-fractal spaces.

Following the above, the structure of this paper remains as follows. In Section 2, all the necessary preliminaries, involving the basics on IFS-attractors, fractal structures (and in particular, the natural fractal structure on any Euclidean space), the classical models for fractal dimension (namely, both the box and the Hausdorff dimensions), the OSC, as well as the classical Moran’s Theorem, are provided to make this paper self-contained. Further, in Section 3, we explain how to generate suitable approaches to real IFS-attractors. To deal with, we will apply the classical Moran’s Theorem, as well as the concept of natural fractal structure which any Euclidean space can be equipped with. Overall, we explicitly indicate which is the necessary level on such a fractal structure that should be reached in order to generate a pre-fractal which properly approaches a given real IFS-attractor, in the sense that their Hausdorff dimensions are equal. This result becomes quite interesting from a computational point of view, since in such a context, there is always a maximum level of that fractal structure that could be achieved.

2. Preliminaries. Along this paper, \( I = \{1, \ldots, k\} \), will be a finite index set. In this section, we provide all the necessary concepts, definitions and theoretical results, in order to make the present work self-contained. Accordingly, the structure of this preliminary section is as follows. In Subsection 2.1, we recall the classical notion consisting of an IFS-attractor. Additionally, in Subsection 2.2, the concept of fractal structure is sketched from a topological point of view. It is worth mentioning that, also in such subsection, the notion of natural fractal structure which any IFS can be equipped with, is introduced. Moreover, Subsection 2.3 recalls the basics on classical models for fractal dimension, namely, both the box and the Hausdorff dimensions. To end this section, we recall the definition of the OSC, as well as the 1946 Moran’s Theorem. This is done in upcoming Subsections 2.4 and 2.5, respectively.

2.1. IFS-attractors. First, let \( f : X \rightarrow X \) be a self-map defined on a metric space \((X, \rho)\). Recall that \( f \) is said to be a Lipschitz self-map whenever it satisfies that \( \rho(f(x), f(y)) \leq c \rho(x, y) \), for all \( x, y \in X \), where \( c > 0 \) is called the Lipschitz constant associated with \( f \). In particular, if \( c < 1 \), then \( f \) is said to be a contraction, and we will refer to \( c \) as its contraction factor. Further, if the equality in the previous
expression is reached, namely, \( \rho(f(x), f(y)) = c \rho(x, y) \), for all \( x, y \in X \), then \( f \) is called a similarity, and its Lipschitz constant is also called as its similarity factor.

**Definition 2.1.** For a metric space \((X, \rho)\), let us define an IFS as a finite family \( F = \{ f_i : i \in I \} \), where \( f_i \) is a similarity, for all \( i \in I \). The unique compact set \( A \subset X \) which satisfies that \( A = \bigcup_{f \in F} f(A) \), is called the attractor of the IFS \( F \), or IFS-attractor, as well. It is also called a self-similar set.

It is a standard fact from Fractal Theory that there exists an attractor for any IFS on a complete metric space \( X \). Regarding this, additional details are provided in forthcoming Subsection 3.1.

### 2.2. Fractal structures

**The natural fractal structure on any Euclidean space.** The concept of fractal structure, which naturally appears in several asymmetric topological topics [25], was first contributed in [1] to characterize non-Archimedeanly quasi-metrizable spaces. Moreover, in [4], it was applied to deal with IFS-attractors. On the other hand, fractal structures constitute a powerful tool to develop new fractal dimension models which allow to calculate the fractal dimension for a wide range of (non-Euclidean) spaces and contexts (see, e.g., [15]).

Recall that a family \( \Gamma \) of subsets of a space \( X \) is called a covering if \( X = \bigcup \{ A : A \in \Gamma \} \). A fractal structure is a countable collection of coverings of a given subset which provides better approximations to the whole space as deeper stages are reached, which we will refer to as **levels** of the fractal structure.

Let \( \Gamma_1 \) and \( \Gamma_2 \) be any two coverings for \( X \). Thus, \( \Gamma_1 \prec \Gamma_2 \) means that \( \Gamma_1 \) is a refinement of \( \Gamma_2 \), namely, for all \( A \in \Gamma_1 \), there exists \( B \in \Gamma_2 \) such that \( A \subseteq B \). In addition to that, \( \Gamma_1 \prec \Gamma_2 \) denotes that \( \Gamma_1 \prec \Gamma_2 \), and also, that for all \( B \in \Gamma_2 \), \( B = \bigcup \{ A \in \Gamma_1 : A \subseteq B \} \). Hence, a fractal structure on a set \( X \), is a countable family of coverings of \( X \), \( \Gamma = \{ \Gamma_n : n \in \mathbb{N} \} \), such that \( \Gamma_{n+1} \prec \Gamma_n \), for all \( n \in \mathbb{N} \). It is worth mentioning that covering \( \Gamma_n \) is level \( n \) of the fractal structure \( \Gamma \).

To simplify the theory, the levels of any fractal structure \( \Gamma \) will not be coverings in the usual sense. Instead of this, we are going to allow that a set can appear twice or more in any level of \( \Gamma \). We would like also to point out that a fractal structure \( \Gamma \) is said to be finite if all levels \( \Gamma_n \) are finite coverings.

We would like to point out that any IFS-attractor can be always equipped with a natural fractal structure (which was first sketched in [5], and formally defined later in [4, Definition 4.4]). However, in this paper, we will make use of the natural fractal structure that any Euclidean space can be always equipped with. In fact, this is a locally finite and starbase fractal structure, which was firstly provided in [13, Definition 3.1]. Next, we recall the description of such a fractal structure, which becomes essential for upcoming sections.

**Definition 2.2.** The natural fractal structure on the Euclidean space \( \mathbb{R}^d \) is defined as the countable family of coverings \( \Gamma = \{ \Gamma_n : n \in \mathbb{N} \} \), whose levels are given by

\[
\Gamma_n = \left\{ \left[ \frac{k_1}{2^n}, \frac{k_1 + 1}{2^n} \right] \times \cdots \times \left[ \frac{k_d}{2^n}, \frac{k_d + 1}{2^n} \right] : k_1, \ldots, k_d \in \mathbb{Z} \right\},
\]

for each natural number \( n \).

### 2.3. Classical models for fractal dimension

Fractal dimension consists of a single quantity which yields valuable information about the complexity that a given set presents when it is explored with enough level of detail.
Next, we recall the definition of the standard box dimension, which is mainly used in empirical applications of fractal dimension due to the easiness of its empirical estimation. As [10, Subsection 3.6] points out, its origins become hard to trace, though it seems that it would have been considered firstly by the Hausdorff dimension pioneers, who rejected it due to its lack of theoretical properties. Anyway, the standard definition for box dimension that we recall next, was firstly provided in [24].

**Definition 2.3.** The (lower/upper) box dimension for any subset $F \subseteq \mathbb{R}^d$ is given as the following (lower/upper) limit:

$$\dim_B(F) = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta},$$

where $\delta$ is the scale, and $N_\delta(F)$ is the largest number of disjoint balls of radii $\delta$ having centres in $F$.

On the other hand, in 1919, Hausdorff used a method developed by Carathéodory some years earlier [9] in order to define the measures that now bear his name, and showed that the middle third Cantor set has positive and finite measure of dimension equal to $\log 2 / \log 3$ [18]. A detailed study regarding the analytical properties of both Hausdorff measure and dimension was mainly developed by Besicovitch and his pupils during the XXth century (see, e.g., [6, 7]).

Next, let us recall a standard analytical construction for Hausdorff dimension. Thus, let $(X, \rho)$ be a metric space, and let $\delta$ be a positive real number. For any subset $F$ of $X$, recall that a $\delta$-cover of $F$ is just a countable family of subsets $\{U_j\}_{j \in J}$ such that $F \subseteq \bigcup_{j \in J} U_j$, where $\text{diam}(U_j) \leq \delta$, for all $j \in J$. Hence, let us denote by $C_\delta(F)$ the collection of all $\delta$-covers of $F$. Moreover, let us consider the following quantity:

$$H^s_\delta(F) = \inf \left\{ \sum_{j \in J} \text{diam}(U_j)^s : \{U_j\}_{j \in J} \in C_\delta(F) \right\}.$$

We would like also to point out that the next limit always exists:

$$H^s_H(F) = \lim_{\delta \to 0} H^s_\delta(F),$$

which is called the $s$-dimensional Hausdorff measure of $F$. Hence, the Hausdorff dimension of $F$ is fully determined as the point $s$ where $H^s_H(F)$ “jumps” from $\infty$ to $0$, namely,

$$\dim_H(F) = \inf \{ s : H^s_H(F) = 0 \} = \sup \{ s : H^s_H(F) = \infty \}.$$  

2.4. **About the OSC.** The OSC is a hypothesis required to the similarities $f_i$ of an Euclidean IFS $F$, in order to guarantee that the pieces $f_i(K)$ of an IFS-attractor $K$ do not overlap too much. Technically, such a condition is satisfied if and only if there exists a non-empty bounded open subset $V \subset \mathbb{R}^d$, such that $\bigcup_{i \in J} f_i(V) \subset V$, where that union remains disjoint (see, e.g., [10, Section 9.2]). If, additionally, it is satisfied that $V \cap K \neq \emptyset$, then the similarities $f_i$ are said to verify the so-called strong open set condition (SOSC, for short). It is worth mentioning that, by [26, Theorem 2.2], both the OSC and the SOSC become equivalent for Euclidean spaces.
2.5. Regarding the classical Moran’s theorem. In [22, Theorem III] (or see also [10, Theorem 9.3]), it was provided a quite interesting result which allows the calculation of both the box and the Hausdorff dimensions for a certain class of Euclidean self-similar sets, as the solution of an easy equation only involving a finite number of quantities, namely, the similarity factors that give rise to their corresponding IFS-attractor. Such a classical result (1946), which becomes essential in forthcoming sections in this paper, is described next.

**Moran’s Theorem.** Let $F$ be an Euclidean IFS satisfying the OSC, whose associated IFS-attractor is $K$. Let us suppose that $c_i$ is the similarity factor associated with each similarity $f_i$. Then

$$\dim_H(K) = \dim_B(K) = s,$$

where

$$\sum_{i \in I} c_i^s = 1.$$

Moreover, for this value of $s$, it is also satisfied that $H_s(K) \in (0, \infty)$.

In [22, Theorem II], a weaker version for the above result under the additional assumption that all the similarities are equal, was also due to Moran.

3. Generating pre-fractals to properly approach real IFS-attractors. This section has a double purpose: on the one hand, in Subsection 3.2, we explain how to computationally generate a wide collection of real IFS-attractors whose Hausdorff dimensions are a priori known, making use of a classical result due to Moran. On the other hand, in Subsection 3.3, we consider the natural fractal structure on the real line in order to find out an appropriate pre-fractal which properly approximates a whole real IFS-attractor, as generated in Subsection 3.2. In addition to that, Subsection 3.1 recalls the standard Hutchinson’s procedure to construct self-similar sets.

3.1. **Hutchinson’s construction procedure for self-similar sets.** First, let us recall the construction of self-similar sets as provided by Hutchinson in [19, Subsections 3.1 (3) & 3.2]. In fact, let $(X, d)$ be a metric space. The hyperspace $K_0(X)$ is the family of non-empty compact subsets of $X$. The Hausdorff metric $d_H$ is defined as $d_H(A, B) < \varepsilon$, if $B \subseteq B_d(A, \varepsilon)$, and $A \subseteq B_d(B, \varepsilon)$, where $B_d(A, \varepsilon) = \{x \in X : d(x, a) < \varepsilon \text{ for some } a \in A\}$, denotes the ball centered in $A$ and whose radius is equal to $\varepsilon$, with respect to the metric $d$. Thus, $(K_0(X), d_H)$ is a complete metric space due to the Zeno-Morita’s Theorem (see [23, Theorem 1.5]).

Let $F = \{f_i : i \in I\}$ be a finite family of contractions from $X$ into itself. Hence, the Hutchinson’s operator is defined by $F(C) = \bigcup_{f \in F} f(C)$, for all compact subset $C$ of $X$. It is worth mentioning that $H$ becomes a contractive mapping on the hyperspace $K_0(X)$. Additionally, the Banach’s Fixed Point Theorem allows to affirm that there exists a unique non-empty compact subset $K$ of $X$, such that $K = \bigcup_{f \in F} f(K)$. Indeed, $K$ is the attractor of the IFS $F$, as it was stated previously in Definition 2.1. Moreover, if $E$ is a non-empty compact subset of $X$, then $F^n(E)$ converges to $K$ with respect to the Hausdorff metric.

3.2. **How to computationally generate real IFS-attractors.** In this subsection, we will be focused on non-overlapping strict self-similar sets. Recall that the OSC is a hypothesis required to an IFS $F = \{f_i : i \in I\}$, in order to guarantee that the self-similar copies $f_i(K)$ of the whole IFS-attractor $K$ do not overlap too much (see Subsection 2.4). Thus, the main goal in this subsection is to explain
how to computationally generate real strict self-similar sets whose Hausdorff dimensions are a priori known. In this way, the main conditions to be satisfied by such IFS-attractors are collected next.

**Construction conditions for real IFS-attractors.** Given both a real number $d \in (0,1)$, as well as a natural number $k \geq 2$ (the minimum number of similarities), our purpose is to construct an IFS-attractor $K$ satisfying the three following conditions:

1. $0, 1 \in K$, and $K \subseteq [0,1]$.
2. $K$ is the attractor of an Euclidean IFS $(\mathbb{R}, \mathcal{F} = \{f_i : i \in I\})$, which satisfies the hypothesis of Moran’s Theorem.
3. $\dim_H(K) = d$.

Next, we explain in detail how to mathematically deal with the generation of such a class of strict self-similar sets under the previous construction conditions (1)-(3). Firstly, it turns out that the similarities $f_i \in \mathcal{F}$ can be expressed as $f_i(x) = a_i + c_i x$, for all $i \in I$, where $c_i \in (0,1)$ is the similarity factor associated with each similarity $f_i$. Thus,

(i) the first step to fully determine the similarities $f_i$ is to generate the similarity factors $c_i$. To deal with, let us randomly choose $c'_i \in (0,1)$, for all $i \in I$, and define $c_i = c c'_i$, where $c \in (0,1)$ has been chosen so that $\sum_{i \in I} c^2_i = 1$. Equivalently,

$$c = \frac{1}{(\sum_{i \in I}(c'_i)^d)^{\frac{1}{d}}}.
$$

(ii) The next stage is to define the coefficients $a_i$. According to that, let $d_1 = 0$, and let us randomly choose $d_2 \in [0,1 - \sum_{i=1}^k c_i]$. In general, we randomly choose $d_{j+1} \in [0,1 - \sum_{i=1}^k c_i - \sum_{i=1}^j d_i]$, except in the case concerning $d_k$, which is defined as $d_k = 1 - \sum_{i=1}^k c_i - \sum_{i=1}^{k-1} d_i$. Finally, we set $a_1 = 0$, $a_2 = c_1 + d_2$, and, in general, $a_{j+1} = \sum_{i=1}^j c_i + \sum_{i=1}^{j+1} d_i$, for $j = 1, \ldots, k-1$.

This allows to fully determine the whole set $\mathcal{F}$, so let $K$ be the attractor corresponding to the IFS $(\mathbb{R}, \mathcal{F})$. Further, we affirm that the next two statements hold:

- $f_i(1) \leq f_{i+1}(0)$, for all $i = 1, \ldots, k-1$. Indeed, it becomes clear that $f_i(0) = a_i$, and $f_i(1) = a_i + c_i$. Hence, since $f_i(1) = a_i + c_i$, $f_{i+1}(0) = a_{i+1} = a_i + c_i + d_{i+1} = f_i(1) + d_{i+1}$, and $d_{i+1} \geq 0$, then $f_i(1) \leq f_{i+1}(0)$.
- $f_k(1) = 1$. Note that $f_k(1) = a_k + c_k = a_{k-1} + c_{k-1} + d_k + c_k = 1$, since $d_k = 1 - \sum_{i=1}^k c_i - \sum_{i=1}^{k-1} d_i$, $a_k = \sum_{i=1}^{k-1} c_i + \sum_{i=1}^k d_i$, and $a_k = f_{k-1}(1) + d_k$.

Following the above, let us check that the former similarity IFS-attractor $K$ satisfies the construction conditions (1)-(3). Indeed,

(1) observe that $0 = f_1(0) < f_1(1) \leq \ldots \leq f_i(0) < f_i(1) \leq f_{i+1}(0) < f_{i+1}(1) \leq \ldots \leq f_k(0) < f_k(1) = 1$. Hence, $f_i([0,1]) = [a_i, a_i + c_i] \subseteq [0,1]$, for all $i = 1, \ldots, k$, which leads to $F([0,1]) = \bigcup_{i=1}^k f_i([0,1]) \subseteq [0,1]$ (recall that $F$ is the Hutchinson’s operator, see Subsection 3.1). Therefore, we can recursively calculate $F^n([0,1]) \subseteq [0,1]$, for all natural number $n$. Accordingly, $K \subseteq [0,1]$, due to the Banach’s Fixed Point Theorem (recall that this gives $K = \lim_{n \to \infty} F^n([0,1])$). Moreover, since 0 is a fixed point of $f_1$, and 1 is a fixed point of $f_k$, then it holds that $0, 1 \in K$. 

(2) Let us verify that the IFS $\mathcal{F}$ satisfies the OSC. To show that, just take $G = (0, 1)$. Hence, the previous arguments lead to $f_i(G) \subseteq G$, for all $i = 1, \ldots, k$, where the pieces $f_i(G)$ remain disjoint.

(3) Hence, the hypothesis of Moran’s Theorem are satisfied. Accordingly, $\sum_{i=1}^{k} c_i^d = 1$, so $\dim_H(K) = d$.

3.3. Getting suitable approximations to real IFS-attractors through pre-fractals. Recall that the Banach’s contraction mapping theorem allows to affirm that the attractor $K$ of an IFS $\mathcal{F}$ becomes the unique non-empty compact real subset that remains fixed under the Hutchinson’s operator action, namely, $K = F(K) = \bigcup_{i \in I} f_i(K)$. Thus, if $E$ is any non-empty compact real subset, then $F^k(E)$ provides better and better approximations to the actual IFS-attractor $K$, as $k$ increases. This is the main reason for which such iterative approaches $F^k(E)$ are sometimes called as pre-fractals of $K$ (see [10, Chapter 9]). It is worth mentioning that though any fractal structure has a countable number of levels (by definition), in computer applications, we are always going to reach a maximum level $n_{\text{max}}$, which depends on the number of available data in each case. This is the main reason which leads us to look for an appropriate finite approach to the actual IFS-attractor $K$, via a properly chosen pre-fractal $K_l$ of $K$, where $l \leq n_{\text{max}}$. In other words, once an IFS $(\mathbb{R}, \mathcal{F})$ has been generated, we are interested in the number of iterations $l$ that allows to ensure that $K_l = F^l([0, 1])$ properly approximates the whole IFS-attractor $K$, provided that a maximum level $n_{\text{max}}$ has been a priori given. In this way, the following theoretical result deals with that interesting question making use of fractal structures.

**Theorem 3.1.** Let $\mathcal{F}$ be a real IFS whose associated IFS-attractor is $K$. In addition to that, let $c_i$ be the similarity factor associated with each similarity $f_i \in \mathcal{F}$, and let $\Gamma$ be the natural fractal structure on $\mathbb{R}$. Thus, if $K_0 = \{0, 1\}$, and $K_l$ is a pre-fractal for $K$ given by $K_l = F(K_{l-1}) = F^l(K_0) : l \in \mathbb{N}$, then the following statements hold:

(i) $K_0 \subseteq K_1 \subseteq \ldots \subseteq K_l \subseteq K$, for all $l \in \mathbb{N}$.

(ii) $K = \bigcup_{n \in \mathbb{N}} K_n$.

(iii) Given $n_{\text{max}}$, let $l = \left\lceil \frac{1}{\log_{\max_{i \in I} c_i} n_{\text{max}}} \right\rceil$, where $\lceil \cdot \rceil$ denotes the ceiling function. Then

$$K_l \cap A \neq \emptyset \iff K \cap A \neq \emptyset,$$

for all $A \in \Gamma_m$, with $m \leq n_{\text{max}}$.

**Proof.**

(i) First of all, it becomes clear that $K_0 \subseteq \bigcup_{i \in I} f_i(K_0) = F(K_0) = K_1$, since $f_1(0) = 0$, and $f_k(1) = 1$. Thus, let us suppose that $K_{l-2} \subseteq K_{l-1}$, and let us show that $K_{l-1} \subseteq K_l$. Indeed, $K_{l-1} = F(K_{l-2}) \subseteq F(K_{l-1}) = K_l$, by induction hypothesis. On the other hand, recall that $K_0 \subseteq K$. Thus, $K_1 = F(K_0) \subseteq F(K) = K$, since the attractor $K$ remains fixed under the action of the Hutchinson’s operator. Accordingly, if we assume that $K_{l-1} \subseteq K$, then $K_l = F(K_{l-1}) \subseteq F(K) = K$, for all $l \in \mathbb{N}$, which gives the result.

(ii) Since $K_{n+1} = F(K_n)$, for all $n \in \mathbb{N}$, then $\lim_{n \to \infty} d_H(K_n, K) = 0$. Let $\varepsilon$ be a fixed but arbitrarily chosen positive real number. Hence, there exists $n_0 \in \mathbb{N}$ such that $d_H(K_n, K) \leq \varepsilon$, for all $n \geq n_0$. Thus, the two following hold:

(a) $K \subseteq B(K_n, \varepsilon)$, which leads to $K \subseteq B(\bigcup_{n \in \mathbb{N}} K_n, \varepsilon)$. 
Proof. Following the hypotheses of Theorem 3.1, there exists Corollary 3.2.

Finally, note that both Theorem 3.1 (iiia) and (iiib) lead to $d_H(K, \bigcup_{n\in\mathbb{N}} K_n) < \varepsilon$, for all $\varepsilon > 0$. Thus, $d_H(K, \bigcup_{n\in\mathbb{N}} K_n) = 0$, so $K = \bigcup_{n\in\mathbb{N}} K_n$.

(iii) Given $n_{\text{max}}$, let $l = \left\lfloor \frac{1}{\log_2 \max_{i \in I} \{c_i\} \frac{1}{n_{\text{max}}}} \right\rfloor$. Next, we study separately the two following implications:

$\implies$) let us assume that $K_1 \cap A \neq \emptyset$. Thus, by Theorem 3.1 (i), $K_1 \subseteq K$, so it becomes clear that $K \cap A \neq \emptyset$.

$\Leftarrow$) On the other hand, let us suppose that $K \cap A \neq \emptyset$, where $A \in \Gamma_m$, and $m \leq n_{\text{max}}$. Therefore, due to [10, Theorem 9.1], we can write $K = \bigcap_{k=0}^{\infty} F^k([0, 1])$, since $[0, 1]$ is a compact real subset such that $f_i([0, 1]) \subset [0, 1]$, for all $i \in I$. Moreover, since $l \geq \frac{1}{\log_2 \max_{i \in I} \{c_i\} \frac{1}{n_{\text{max}}}}$ by hypothesis, then it holds that $(\max_{i \in I} \{c_i\})^l < 1/2^{n_{\text{max}}}$. Thus, if $x \in K \cap A$, then $x \in F^l([0, 1]) \cap A$. In particular, $x \in F^l([0, 1])$, so there exists $(i_1, \ldots, i_l) \in I^l$, such that $x \in f_{i_1, \ldots, i_l}(0, 1) = f_{i_1} \circ \ldots \circ f_{i_l}(0, 1) = [a, b]$, where $a = f_{i_1, \ldots, i_l}(0)$, and $b = f_{i_1, \ldots, i_l}(1)$. Notice also that $K_j = \{f_{i_1, \ldots, i_j}(0), f_{i_1, \ldots, i_j}(1) : (i_1, \ldots, i_j) \in I^j\}$, for all $j \leq n_{\text{max}}$, so $a, b \in K_1$. Accordingly, it is satisfied that $x \in [a, b] \cap A$, where $b - a < 1/2^{n_{\text{max}}}$, since $a = f_{i_1, \ldots, i_l}(0), b = f_{i_1, \ldots, i_l}(1)$, and the similarity factor of $f_{i_1, \ldots, i_l}$ is $c_{i_1} \cdots c_{i_l} \leq (\max_{i \in I} \{c_i\})^l < 1/2^{n_{\text{max}}}$. Hence, we have that either $a \in A$ or $b \in A$ (note that $A \in \Gamma_m$, so $A$ is an interval whose length is equal to $1/2^m \geq 1/2^{n_{\text{max}}}$, $x \in A$, $x \in [a, b]$, and $b - a < 1/2^{n_{\text{max}}}$). In any case, $A \cap K_1 \neq \emptyset$, as desired.

Finally, observe that previous Theorem 3.1 gives the result advanced above.

Corollary 3.2. Following the hypotheses of Theorem 3.1, there exists $l$ such that $\mathcal{H}_{n,m}^s(K) = \mathcal{H}_{n,m}^s(K_1)$, for each $n \leq m \leq n_{\text{max}}$.

Proof. In fact, note that Theorem 3.1 (iii) leads to $\mathcal{H}_{n,m}^s(K) = \mathcal{H}_{n,m}^s(K_1)$, for each $n \leq m \leq n_{\text{max}}$, where $l = \left\lfloor \frac{1}{\log_2 \max_{i \in I} \{c_i\} \frac{1}{n_{\text{max}}}} \right\rfloor$ (recall both Eqs. (2) & (3) in [17, Subsection 3.1]).

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