A Note On The Weighted Fourier Transforms on $p$ – Integrable Function Spaces

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Abstract. We have known that the Fourier transform is well defined on $L^1$ space. In this paper, we shall introduce generalization of the Fourier transform on $L^p$ spaces with $p \geq 1$ which is called by weighted Fourier Transform, and discuss its properties. The invers transform is studied and we also obtain a result which describe how the weighted Fourier transform is applied to solve generalization of the heat equation problem.

1. Introduction

Fourier Analysis studies how to analyse a range of function by expanding it to a series or integral of a certain function (which properties has been well known, as polynomial function or trigonometric function) ([1]). Fourier Analysis is a powerful tool to solve a various problems particularly in partial differential equation that has been existed in pure and applied science. Fourier Analysis also can be used to analyse digital signals and images as in [2], [3] and [4].

The theory in Fourier analysis which has been mostly used to solve problems in applied science is Fourier transform. One of its benefit is to find a solution of classical differential equation—heat equation in $\mathbb{R}$. The Fourier transform is only well defined on $L^1$ ([1]). One questions rise: can we find a transform which is a generalization of the Fourier transform and it is well defined not only on $L^1$ but also well defined on $L^p$ for $p \geq 1$? This is a basic problem in this paper. In this paper, by using Hölder’s inequnality [5], we will construct a new transform which is generalisation of the Fourier transform, and also found many interesting results, which we will use to solve generalization of the heat equation problem.

Throughout the paper, the researchers declared the definition of $L^p = L^p(\mathbb{R})$ spaces as in [5]. For $1 \leq p < \infty$,

$$L^p = \left\{ f : \int_{-\infty}^{\infty} |f(x)|^p dx < \infty \right\},$$

and for $p = \infty$,

$$L^\infty = \left\{ f : \text{ess sup}\{|f(x)|: x \in \mathbb{R}\} < \infty \right\}.$$

where $\text{ess sup}\{|f(x)|: x \in \mathbb{R}\}$ is “essential” supremum of $|f|$. In other word, $L^\infty$ is the space all of functions which are bounded almost everywhere.
2. Weighted Fourier Transformation For $1 \leq p < \infty$

In this section, we shall construct the generalized Fourier transform. To define the transform on $L^p$, we have to use weights (look at [6] and [7]). Let us choose $w \in L^p$ where $w(x) > 0$ for all $x \in \mathbb{R}$ when $1 \leq p < \infty$. Next, it was defined the mapping $\mathcal{F}_{p,w}$ which maps a function $f \in L^p$ for $1 \leq p < \infty$ to

$$\mathcal{F}_{p,w}f(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} |w(x)|^{p-1} f(x) dx.$$ (1)

Throughout this paper, $\mathcal{F}_{w}$ is called weighted Fourier transform of $f$.

The mapping is well defined for $f \in L^p$ because for all $x \in \mathbb{R}$, we have

$$\left| e^{-i\xi x} (w(x))^{p-1} f(x) \right| = \left| (w(x))^{p-1} f(x) \right|$$

which is integrable.

It follows from Hölder’s inequality [5] that we get this proposition.

**Proposition 2.1.** If $f \in L^p$ and $w \in L^p$ with $w(x) > 0$ for all $x \in \mathbb{R}$, then $\mathcal{F}_{p,w}f \in L^\infty$.

**Proof.** The first, using Hölder’s inequality, we obtained $w^{p-1} f \in L^1$. Indeed,

For $1 < p < \infty$,

$$\int_{-\infty}^{\infty} \left| (w(x))^{p-1} f(x) \right| dx \leq \left( \int_{-\infty}^{\infty} |w(x)|^{p-1} dx \right)^{\frac{p}{p-1}} \left( \int_{-\infty}^{\infty} |f(x)|^p dx \right)^{\frac{1}{p}}$$

$$= \left( \int_{-\infty}^{\infty} |w(x)|^p dx \right)^{\frac{p-1}{p}} \left( \int_{-\infty}^{\infty} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty,$$

for $p = 1$,

$$\int_{-\infty}^{\infty} \left| (w(x))^{p-1} f(x) \right| dx = \int_{-\infty}^{\infty} \left| w(x) \right|^0 f(x) dx = \int_{-\infty}^{\infty} \left| f(x) \right| dx < \infty.$$

So, $w^{p-1} f \in L^1$.

Now, it could be investigated that the weighted Fourier transform was well defined in $L^p$. For every $f \in L^p$ with $p \geq 1$, we get

$$\left| \mathcal{F}_{p,w}f(\xi) \right| = \left| \int_{-\infty}^{\infty} e^{-i\xi x} |w(x)|^{p-1} f(x) dx \right|$$

$$\leq \int_{-\infty}^{\infty} |e^{-i\xi x} |w(x)|^{p-1} f(x)| dx$$

$$= \int_{-\infty}^{\infty} \left| e^{-i\xi x} \right| \left| (w(x))^{p-1} f(x) \right| dx$$

$$= \int_{-\infty}^{\infty} \left| (w(x))^{p-1} f(x) \right| dx < \infty \quad \text{(since } w^{p-1} f \in L^1 \text{)}$$

Since $\left| \mathcal{F}_{p,w}f(\xi) \right| < \infty$ for all $\xi \in \mathbb{R}$. So, $\mathcal{F}_{p,w}f \in L^\infty$. ■
Remark 2.2. The transform on (1) was called by the weighted Fourier Transform because $F_{p,w} f$ is a generalization of the Fourier transform which was defined by

$$F(f)(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x) dx$$

(2)

where $f \in L^1$. The Fourier transform is well defined only on $L^1$, however, the weighted Fourier transform was well defined on $L^p$ for $1 \leq p < \infty$, According to Proposition 2.1. In addition, if $p = 1$, then $F_{p,w} f = F f$.

The proposition below show that the weighted Fourier transform of function on $L^p$ is continuous on $\mathbb{R}$ ([8])

**Proposition 2.3.** If $f \in L^p$ and $w \in L^p$ with $w(x) > 0$ for all $x \in \mathbb{R}$, then $F_{p,w} f$ is continuous on $\mathbb{R}$.

**Proof.** For all $\xi$ and $h$ in $\mathbb{R}$

$$F_{p,w} f(\xi + h) - F_{p,w} f(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} (e^{-ihx} - 1) |w(x)|^{p-1} f(x) dx,$$

Thus,

$$|F_{p,w} f(\xi + h) - F_{p,w} f(\xi)| \leq \int_{-\infty}^{\infty} |e^{-ihx} - 1| |w(x)|^{p-1} f(x) dx.$$

The integrand in the right side is dominated by $2|w^{p-1} f(x)|$ and went to 0 as $h \to 0$. So, according to the Lebesgue Dominated Convergence Theorem (look at [9]), the right side must converges to 0 as $h \to 0$. Consequently, the left side also went to 0 as $h \to 0$. $\blacksquare$

Some propositions below are basic properties of the weighted Fourier Transform.

**Proposition 2.4.** If $f \in L^p$ and $w \in L^p$ with $w(x) > 0$ for all $x \in \mathbb{R}$, then $\lim_{|\xi| \to \infty} F_{p,w} f(\xi) = 0$

**Proof.** It was observed that

$$-F_{p,w} f(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} \frac{2}{\xi} |w(x)|^{p-1} f(x) dx = \int_{-\infty}^{\infty} e^{-i\xi x} \left| w \left( x - \frac{\pi}{\xi} \right) \right|^{p-1} f \left( x - \frac{\pi}{\xi} \right) dx.$$

Hence,

$$2F_{p,w} f(\xi) = F_{p,w} f(\xi) - (-F_{p,w} f(\xi))$$

$$= \int_{-\infty}^{\infty} e^{-i\xi x} \left| w \left( x - \frac{\pi}{\xi} \right) \right|^{p-1} f \left( x - \frac{\pi}{\xi} \right) dx$$

Thus

$$2|F_{p,w} f(\xi)| \leq \int_{-\infty}^{\infty} \left| w \left( x - \frac{\pi}{\xi} \right) \right|^{p-1} f \left( x - \frac{\pi}{\xi} \right) dx.$$

Since $w^{p-1} f \in L^1$ (look at the proof of proposition 2.1), then (using the continuity of norm in $L^1$) the right side went to 0 if $|\xi| \to \infty$. Consequently, the left side also went to 0 if $|\xi| \to \infty$. $\blacksquare$

**Corollary 2.5.** Let $p \geq 1$ and $w \in L^p$ with $w(x) > 0$ for all $x \in \mathbb{R}$. The weighted Fourier transform maps a function $f \in L^p$ to a function $\hat{f}_{w}$ in $C_0(\mathbb{R})$. In other words, $F_{p,w} f$ is continuous and bounded.

Note. $C_0(\mathbb{R})$ is the space of functions that are continuous and bounded in $\mathbb{R}$ with zero limit in $\pm \infty$. 


Proposition 2.6. If \( f, g \in L^p \) for \( 1 \leq p < \infty \), \( \alpha \) and \( \beta \) are (real) scalars, and \( w \in L^p \) with \( w(x) > 0 \) for all \( x \in \mathbb{R} \), then \( \mathcal{F}_{p,w}(\alpha f + \beta g) = \alpha \mathcal{F}_{p,w}f + \beta \mathcal{F}_{p,w}g \).

Proof. Since integral is linear. □

Next, researchers would discuss a special case for the weighted Fourier transform properties. In this case, we take a function \( w \) as
\[
w_0(x) = \sum_{i=0}^{\infty} \lambda_{i-1,i}(x) \min_{x \in [i-1,i]} (e^{-|x|}) + \lambda_{i+1,i}(x) \min_{x \in [i+1,i]} (e^{-|x|}), \quad \forall x \in \mathbb{R}.
\]
It could be observed that \( w_0 \in L^p \) for any \( p \geq 1 \) because \( w_0(x) \leq e^{-|x|} \) and it was known that the function \( k \) with \( k(x) = e^{-|x|} \) is in \( L^p \). It could also be investigated that \( w'_0 = 0 \) almost everywhere.

Proposition 2.7. If \( f \) is a differentiable function on \( \mathbb{R} \), both \( f \) and its derivative \( f' \) are in \( L^p \) for \( p \geq 1 \), and
\[
w_0(x) = \sum_{i=0}^{\infty} \lambda_{i-1,i}(x) \min_{x \in [i-1,i]} (e^{-|x|}) + \lambda_{i+1,i}(x) \min_{x \in [i+1,i]} (e^{-|x|}), \quad \forall x \in \mathbb{R},
\]
then the weighted Fourier transform of the derivative is given by
\[
\mathcal{F}_{p,w_0}f'(\xi) = i\xi \mathcal{F}_{p,w_0}f(\xi), \quad \forall \xi \in \mathbb{R}.
\]

Proof. By integrating of parts, it was obtained
\[
\mathcal{F}_{p,w_0}f'(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} |w_0(x)|^{p-1} f'(x) dx = \int_{-\infty}^{\infty} e^{-i\xi x} w_0(x) \left| f'(x) \right| dx = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \left| w_0(x) \right|^{p-1} dx = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \left| w_0(x) \right|^{p-1} dx.
\]

Corollary 2.8. If \( f \in L^p \) for \( p \geq 1 \), \( f \) has second derivative, both its derivative \( f' \) and its second derivative \( f'' \) are in \( L^p \), and
\[
w_0(x) = \sum_{i=0}^{\infty} \lambda_{i-1,i}(x) \min_{x \in [i-1,i]} (e^{-|x|}) + \lambda_{i+1,i}(x) \min_{x \in [i+1,i]} (e^{-|x|}), \quad \forall x \in \mathbb{R},
\]
then the weighted Fourier transform of the derivative is given by
\[
\mathcal{F}_{p,w_0}f''(\xi) = -\xi^2 \mathcal{F}_{p,w_0}f(\xi), \quad \forall \xi \in \mathbb{R}.
\]

Proof. It followed that
\[
\mathcal{F}_{p,w_0}f''(\xi) = \mathcal{F}_{p,w_0}(f'(\xi)) = i\xi \mathcal{F}_{p,w_0}f'(\xi) = i\xi \left( i\xi \mathcal{F}_{p,w_0}f(\xi) \right) = -\xi^2 \mathcal{F}_{p,w_0}f(\xi),
\]
for every \( \xi \in \mathbb{R} \). □

3. Inverse Weighted Fourier Transform

Throughout this section, we will discuss inversion formula of the weighted Fourier transform. When we revisit the Fourier Transform on \( L^1 \), we shall find Fourier Inversion Theorem (at [1]) below:

“If \( f \in L^1 \) is piecewise continuous, and \( \mathcal{F}(f) \in L^1 \) then \( f \) is continuous and
\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(f)(\xi)e^{i\xi x} \, d\xi.
\]
for all \( x \in \mathbb{R} \).

Now, it will be generalised that the Fourier Inversion Theorem has inversion formula of the weighted Fourier Transform on \( L^p \).

Next, one of the main result of this section: Inversion formula of the weighted Fourier Transform on \( L^p \).

**Theorem 3.1. (Weighted Fourier Inversion Theorem)** If \( f \in L^p \) for \( 1 \leq p < \infty \), \( w \in L^p \) where \( w(x) > 0 \) for all \( x \in \mathbb{R} \), and \( f \) and \( w \) are piece wise continuous, and \( \mathcal{F}_{p,w} f \in L^1 \) then \( f \) is continuous and

\[
    f(x) = \frac{1}{2\pi|w(x)|} \int_{-\infty}^{\infty} \mathcal{F}_{p,w}(\xi) e^{i\xi x} d\xi
\]

for all \( x \in \mathbb{R} \).

**Proof.** Since \( f \in L^p \) and \( w \in L^p \), then according to Hölder’s inequation, we obtained \( |w|^{p-1} f \in L^1 \).

Base on hypothesis, \( f \) was piece wise continuous and \( \mathcal{F}(|w|^{p-1} f) = \mathcal{F}_{p,w} f \in L^1 \) and also the Fourier Inversion Theorem, we obtained

\[
    |w|^{p-1} f = \frac{1}{2\pi \int_{-\infty}^{\infty} \mathcal{F}(|w|^{p-1} f) e^{i\xi x} d\xi} = \frac{1}{2\pi \int_{-\infty}^{\infty} \mathcal{F}_{p,w} f(x) e^{i\xi x} d\xi}.
\]

Consequently,

\[
    f(x) = \frac{1}{2\pi |w(x)|^{p-1}} \int_{-\infty}^{\infty} \mathcal{F}_{p,w} f(\xi) e^{i\xi x} d\xi
\]

for every \( x \in \mathbb{R} \), and \( w^{p-1} f \) was continuous in \( \mathbb{R} \). Moreover, since \( w \) was piece wise continuous, then \( f \) was also wise continuous in \( \mathbb{R} \).

**Remark 3.2.** The Weighted Fourier Inversion Theorem show us that weighted Fourier transform has an invers which is called by invers weighted Fourier transform. If the weighted Fourir transform is denoted by \( \mathcal{F}_{p,w} \), then the invers weighted Fourier transform is denoted by \( \mathcal{F}_{p,w}^{-1} \). So, if \( \mathcal{F}_{p,w} f(x) = \frac{1}{2\pi |w(x)|^{p-1}} \int_{-\infty}^{\infty} f(\xi) e^{i\xi x} d\xi \), then \( f = \mathcal{F}_{p,w}^{-1}(\mathcal{F}_{p,w} f) \).

**Corollary 3.3.** If \( f \in L^p \) for \( 1 \leq p < \infty \) and \( w \in L^p \) where \( w(x) > 0 \) for all \( x \in \mathbb{R} \), and \( f \) is piece wise smooth, then

\[
    \lim_{T \to \infty} \frac{1}{2\pi |w(x)|^{p-1}} \int_{-T}^{T} \mathcal{F}_{p,w} f(\xi) e^{i\xi x} d\xi = \frac{1}{2} \left[ f(x^+) + f(x^-) \right],
\]

for every \( x \in \mathbb{R} \).

4. Further Result

Now, we will apply the weighted Fourier transform on classical differential equation—heat equation in \( \mathbb{R} \), that is

\[
    u_t = ku_{xx}, \quad \text{for almost every } x \in \mathbb{R}
\]

with the initial condition \( u(x,0) = f(x) \) where \( f \in L^p \) for \( p \geq 1 \) and the boundary condition \( u(x, t) \to 0 \) and \( f(x) \to 0 \) as \( x \to \pm \infty \).

By applying Weighted Fourier transform on both sides of the equation (both sides consider respect to \( x \)) with

\[
    w_0(x) = \sum_{i=0}^{\infty} \lambda_{(i-1,i)}(x) \min_{x \in (i-1,i]} e^{-|x|} + \lambda_{(i+1,i)}(x) \min_{x \in [i+1,i)} e^{-|x|}, \quad \forall x \in \mathbb{R}.
\]
we obtained

$$\frac{\partial \mathcal{F}_{p,w_0} u}{\partial t} (\xi, t) = -\xi^2 k \mathcal{F}_{p,w_0} u(\xi, t)$$  \hspace{1cm} (3)$$

$$\mathcal{F}_{p,w_0} u(\xi, 0) = \mathcal{F}_{p,w_0} w(\xi).$$

The solution of this differential equation is

$$\mathcal{F}_{p,w_0} u(\xi, t) = \mathcal{F}_{p,w_0} f(\xi) e^{-\xi^2 kt}.$$  

By using the Weighted Fourier Inversion Theorem, we get the following solution of the heat equations

$$u(x, t) = \frac{1}{2\pi |w(x)|^{p-1}} \int_{-\infty}^{\infty} \mathcal{F}_{p,w_0} f(\xi) e^{-\xi^2 kt} e^{i\xi x} \, d\xi$$

with $w \in L^p$ and

$$w_0(x) = \sum_{l=0}^{\infty} \lambda_{l-1-1} \left( e^{-|x|} \right) + \lambda_{l,1-1} \left( e^{-|x|} \right), \quad \forall x \in \mathbb{R}$$

for every $x \in \mathbb{R}$.

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