Force on a neutron quantised vortex pinned to proton fluxoids in the superfluid core of cold neutron stars

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ABSTRACT

The superfluid and superconducting core of a cold rotating neutron star is expected to be threaded by a tremendous number of neutron quantised vortices and proton fluxoids. Their interactions are unavoidable and may have important astrophysical implications. In this paper, the various contributions to the force acting on a single vortex to which fluxoids are pinned are clarified. The general expression of the force is derived by applying the variational multifluid formalism developed by Carter and collaborators. Pinning to fluxoids leads to an additional Magnus type force due to proton circulation around the vortex. Pinning in the core of a neutron star may thus have a dramatic impact on the vortex dynamics, and therefore on the magneto-rotational evolution of the star.

Key words: stars: interiors, stars: neutron

1 INTRODUCTION

Even before their actual discovery, neutron stars (NSs) were expected to be so dense that neutrons and protons in their interior may be in a superfluid state (see, e.g., Chamel (2017) and references therein). This theoretical prediction was later confirmed by the very long relaxation times following the first detections of pulsar sudden spin-ups so-called frequency ‘glitches’ (Haskell & Melatos 2015). Nucleon superfluidity in the core of NSs has recently found additional support from the direct monitoring of the rapid cooling of the young NS in Cassiopeia A (Page et al. 2011; Shternin et al. 2011). How-ever, the interpretation of these observations remains contro-versial (Posselt & Pavlov 2018; Wijngaarden et al. 2019).

Because NSs are rotating, their interior is threaded by a huge number of neutron quantised vortices, each carrying a quantum $\kappa_n = h/(2 m_n) \sim 2 \times 10^{-3}$ cm$^2$ s$^{-1}$ of circulation, where $h$ is the Planck constant and $m_n$ is the neutron rest mass. The mean surface density of vortices is proportional to the angular frequency $\Omega$ and is given by (Chamel 2017)

$$ N_n = 4 m_n \Omega / h \sim 6 \times 10^5 / P_{10} \text{ cm}^{-2}, \quad (1) $$

where $P_{10} = P/10$ ms is the observed rotation period of the neutron star. Assuming that protons in NS cores form a type-II superconductor (Baym et al. 1969), the magnetic flux penetrates the stellar interior only via fluxoids, each carrying a quantum magnetic flux $\phi_0 = h c/(2 e) \sim 2 \times 10^{-7}$ G cm$^2$, where $c$ is the speed of light and $e$ denotes the proton electric charge. For typical NS magnetic fields, the number of fluxoids is considerably larger than that of vortices, their mean surface density being given by (Chamel 2017)

$$ N_p = B / \phi_0 \sim 5 \times 10^{18} B_{12} \text{ cm}^{-2}, \quad (2) $$

where $B_{12} = B/10^{12}$ G is the stellar internal magnetic field. Interactions between neutron vortices and proton fluxoids are therefore unavoidable, and are pivotal in the magneto-rotational evolution of NSs. In particular, vortices may pin to fluxoids (Muslimov & Tsygan 1985; Sauls 1989; Srinivasan et al. 1990; Ruderman et al. 1998) (see also Alpar (2017) for a recent review), and this may have important implications for various astrophysical phenomena, such as precession (Sedrakian et al. 1999; Link 2006; Glampedakis et al. 2008), r-mode instability (Haskell et al. 2009, 2014) and pulsar glitches (Sedrakian et al. 1995; Sidery & Alpar 2009; Glampedakis & Andersson 2009; Haskell et al. 2013; Haskell & Melatos 2015; Gügercinoğlu 2017; Sourie et al. 2017; Haskell et al. 2018; Graber et al. 2018). However, the detailed force acting on individual vortices to which fluxoids are pinned remains poorly understood. In particular, the contribution associated with the proton circulation induced by pinned fluxoids has been generally overlooked or treated phenomenologically (see, e.g., Glampedakis & Andersson (2011)).

Building on the recent study of Gusakov (2019), who determined the force acting on a single fluxoid and clarified the role of degenerate electrons, we derive in this paper the
general expression for the force per unit length acting on a neutron vortex to which \( N_p \) proton fluxoids are pinned. To this end, we follow a general approach originally developed by Carter et al. (2002) in the relativistic framework, and later adapted to the Newtonian context by Carter & Chamel (2005a). The general expression of the vortex velocity is calculated and the role of pinning on the vortex dynamics is discussed. The implications for pulsar glitches are studied in an accompanying paper (Sourie & Chamel 2020).

2 FORCE ACTING ON A SINGLE VORTEX PINNED TO FLUXOIDS

2.1 General definition

Let us consider a rigid and infinitely long straight neutron superfluid vortex to which \( N_p \) proton fluxoids are pinned. The medium in which the vortex is embedded is assumed to be asymptotically uniform, stationary and longitudinally invariant, along say the \( z \) axis. The force density \( \mathbf{f} \) acting on a matter element is defined by the divergence of the momentum-flux tensor \( \Pi^{ij} \) (\( i, j \) denoting space coordinate indices),

\[
f_i \equiv \nabla_i \Pi^{ij}.
\]

The force \( \mathbf{dF} \) exerted on a vortex segment of length \( dz \) by a matter element whose volume is delimited by a closed contour \( C \) encircling the vortex and the pinned fluxoids, as represented on Fig. 1, is thus given by

\[
dF_i = - \iiint f_i \, dV = - \iiint \nabla_i \Pi^{ij} \, dV
\]

\[
= \int_{S(C)} \Pi^{ij}_{(\zeta)} \xi_j \, dS - \int_{S(C)} \Pi^{ij}_{(\zeta + dz)} \xi_j \, dS + dz \int_C \Pi^{ij} \mathbf{a}_j \, d\ell,
\]

where we have made use of Stokes’ theorem, and \( \mathbf{a} \) is a unit vector perpendicular to both the vortex line and the contour \( C \), and is oriented inside the contour. Longitudinal invariance along the vortex line implies that \( \Pi^{ij} \) is independent of \( z \). The two surface integrals in the second line of Eq. (4) thus cancel each other. The force per unit length acting on the vortex and the pinned fluxoids can be finally expressed as

\[
\mathbf{f}_i \equiv \frac{dF_i}{dz} = \oint_C \Pi^{ij} \mathbf{a}_j \, d\ell.
\]

The force (5) is well-defined provided the contour integral is evaluated at sufficiently large distances from the vortex where the force density vanishes, \( f_i = 0 \). Indeed, considering two different contours \( C_1 \) and \( C_2 \), we have

\[
\mathbf{f}_i(C_1) - \mathbf{f}_i(C_2) = \oint_{S(C_2) \setminus S(C_1)} f_i \, dS = 0,
\]

where the integration is carried out over the surface area \( S(C_2) \setminus S(C_1) \) delimited by the two contours (see Fig. 2).

Considering distances sufficiently far from the vortex for the first-order perturbation theory to hold, the momentum-flux tensor can be decomposed as

\[
\Pi^{ij} = \Pi^{ij}_0 + \delta\Pi^{ij}.
\]

Figure 1. Schematic picture illustrating the fluid element contributing to the force per unit length acting on the vortex directed along the \( z \) axis. The parallel sections \( S(C) \) lie in the plane perpendicular to this axis. See text for details.

Figure 2. Schematic picture illustrating the surface \( S(C_2) \setminus S(C_1) \) (shaded area) delimited by two different contours \( C_1 \) and \( C_2 \) around the vortex region.

\[\delta\Pi^{ij} \] denotes a small disturbance of the uniform background momentum-flux tensor \( \Pi^{ij}_0 \). Similarly, any quantity \( y \) will be expanded to first order as \( y = \bar{y} + \delta y \), where \( \delta y \) denotes a small disturbance of the uniform background quantity \( \bar{y} \). Since the force for the unperturbed uniform background flows must evidently vanish by symmetry, the corresponding force in the presence of the vortex (5) will be given to first order by

\[
\mathbf{f}_i = \oint_C \delta\Pi^{ij} \mathbf{a}_j \, d\ell,
\]

see also Carter et al. (2002); Carter & Chamel (2005a).

2.2 Momentum-flux tensor for npe-matter

Let us assume that the vortex and the \( N_p \) fluxoids pinned to it are evolving in a cold\(^1\) mixture of superfluid neutrons in which the vortex and the fluxoids are pinned. To this end, we follow a general approach originally developed by Carter et al. (2002) in the relativistic framework, and later adapted to the Newtonian context by Carter & Chamel (2005a). The general expression of the vortex velocity is calculated and the role of pinning on the vortex dynamics is discussed. The implications for pulsar glitches are studied in an accompanying paper (Sourie & Chamel 2020).

\(^1\) The temperature in mature neutron stars is expected to be low enough for thermal excitations to be negligible, see,
trons, superconducting protons and degenerate electrons. Such conditions are expected to be met in the outer core of neutron stars. The basic model of such a three-component superconducting-superfluid mixture we adopt here has been described by Carter & Langlois (1998). Although developed in the relativistic approach, their covariant approach remains formally applicable to the Newtonian spacetime since it is based on Cartan’s exterior calculus (see also Carter & Chamel (2002, 2005a,b) for the fully 4D covariant nonrelativistic formulation and a discussion of the specificity of the Newtonian spacetime). The explicit hydrodynamic equations in the usual 3+1 spacetime decomposition and based on a similar convective variational action principle have been derived by Prix (2004, 2005).

The momentum-flux tensor can be decomposed as
\[
\Pi_{ij} = \Pi_{ij}^{(nuc)} + \Pi_{ij}^{(e)} + \Pi_{ij}^{(em)},
\]
where \(\Pi_{ij}^{(nuc)}\), \(\Pi_{ij}^{(e)}\) and \(\Pi_{ij}^{(em)}\) denote the nucleon, electron and electromagnetic contributions, respectively. The nucleon part reads (Carter & Langlois 1998)
\[
\Pi_{ij}^{(nuc)(j)} = n_i^p \pi_p^j + n_i^n \pi_n^j \left( \pi_p^k - \frac{\epsilon}{c} A_j \right) + \Psi \delta_i^j,
\]
where \(n_i^{n,p} = n_i v_i^{n,p}\) and \(\pi_i^{n,p} = n_i v_i^{n,p}\) denote the neutron and proton currents respectively, \(\pi_i^n\) and \(\pi_i^p\) stand for the associated generalised momenta per particle, \(A_i\) is the magnetic potential vector, \(\Psi\) is the generalised pressure of the nucleon mixture, and \(\delta_i^j\) is the Kronecker symbol. The generalised momenta \(\pi_i^n\) and \(\pi_i^p\) are related to the purely nuclear momenta \(p_i^n\) and \(p_i^p\) (as obtained in the absence of electromagnetic fields) by the following relations (Carter & Langlois 1998)
\[
\pi_i^n = p_i^n, \quad \pi_i^p = p_i^p + \frac{\epsilon}{c} A_i.
\]

As stressed by Carter (1989), the distinction between momenta and currents is crucial. Because neutrons and protons are strongly interacting, they are mutually coupled by nondissipative entrainment effects of the kind originally discussed in the context of superfluid \(^3\)He–\(^4\)He mixtures by Andreev & Bashkin (1976) such that the nucleon momenta are expressible as (Carter & Langlois 1998)
\[
p_i^n = n_i \left( K_i^{nn} n_i^n + K_i^{np} n_i^p \right), \quad p_i^p = n_i \left( K_i^{pn} n_i^n + K_i^{pp} n_i^p \right),
\]
where \(K_i^{nn}\) denotes the space metric. The entrainment coefficients \(K_i^{nn}, K_i^{np} = K_i^{pn}, K_i^{pp}\) are not all independent since Galilean invariance imposes the following relations:
\[
K_i^{nn} n_i + K_i^{np} n_i^p = n_i, \quad K_i^{pn} n_i + K_i^{pp} n_i^p = n_i^p.
\]

The entrainment coefficients depend on the baryon number density and on the composition (see, e.g., Gusakov & Haensel (2005); Chamel (2008); Kheto & Bandyopadhyay (2014); Sourie et al. (2016)). They may also depend on the relative nucleon currents so that the relations (12) between nucleon momenta and currents are not necessarily linear (Leinson 2017, 2018).

The electromagnetic momentum-flux tensor is given by the usual expression (in Gaussian cgs units)
\[
\Pi_{ij}^{(em)(j)} = -\frac{1}{4\pi} \left( E_i E_j + B_i B_j - \frac{1}{2} E^2 E^j + B^2 B^j - \frac{1}{2} B^k B_k \delta_i^j \right),
\]
where \(E^i\) and \(B^i\) denote the electric and magnetic fields.

At the vortex scale, electrons do not form a fluid but are in a ballistic regime following classical trajectories. Instead of following the purely hydrodynamic treatment of Carter & Langlois (1998) for the electron momentum-flux tensor \(\Pi_{ij}^{(e)}\), we adopt here the expression given by Eq. (23) of Gusakov (2019).

The total force (8) experienced by the vortex can thus be decomposed as
\[
\mathcal{F}_i = \mathcal{F}_i^{(nuc)} + \mathcal{F}_i^{(e)} + \mathcal{F}_i^{(em)},
\]
where the separate contributions
\[
\mathcal{F}_i^{(nuc)} = \int_C \delta \Pi_{ji}^{(nuc)} a_j \, dt,
\]
\[
\mathcal{F}_i^{(e)} = \int_C \delta \Pi_{ji}^{(e)} a_j \, dt,
\]
and
\[
\mathcal{F}_i^{(em)} = \int_C \delta \Pi_{ji}^{(em)} a_j \, dt,
\]
are evaluated in Secs. 2.3, 2.4 and 2.5, respectively.

### 2.3 Nucleon contribution

The first-order perturbation in the nucleon momentum-flux tensor (10) reads
\[
\delta \Pi_{ji}^{(nuc)} = \delta \pi_i^n \pi_j^n + \delta \pi_i^p \pi_j^p + \delta \pi_i^n \pi_j^p + \frac{\epsilon}{c} A_j \delta \pi_i^n - \frac{\epsilon}{c} A_i \delta \pi_j^n + \delta \Psi \delta_i^j,
\]
using a gauge such that \(\Lambda_i = 0\) (Carter et al. 2002). In the 4D covariant formulations of Carter et al. (2002) and Carter & Chamel (2005a), the first-order perturbation of the nucleon pressure can be expressed as (see Eq. (A9) of Carter & Chamel (2005b))
\[
\delta \Psi = -\hat{n}_i \delta p_{0i}^n - \hat{n}_i \delta p_{0i}^p - \hat{n}_i^p \delta p_{0i}^n - \hat{n}_i^p \delta p_{0i}^p,
\]
where \(\hat{p}_i^n\) and \(\hat{p}_i^p\) correspond to the time-components of the neutron and proton 4-momenta. These time components can be more explicitly written as (see Eq. (34) of Prix (2005))
\[
\hat{p}_i^x = -\mu x + m_x v_i^x / 2 - v_i^x p_i^x,
\]
with \(\mu x\) denoting the chemical potential of nucleon species \(x \in \{n,p\}\). Introducing the time components of the generalised momenta (Carter & Langlois 1998)
\[
\pi_i^0 = \hat{p}_i^0, \quad \pi_i^n = \hat{p}_i^n + e A_i,
\]
e.g., Potekhin et al. (2015) for a review and Beloin et al. (2018) for recent neutron-star cooling simulations.
where $-A_0$ denotes the electric scalar potential, the first-order perturbation in the nucleon momentum-flux tensor (19) can be written as

$$\delta \Pi_i^{(nuc)} = \delta \Pi_0^{(nuc)} + \delta \Pi_1^{(nuc)} + \delta \Pi_2^{(nuc)} = \sum_{j} (\delta \Pi_0^{(nuc)} + \delta \Pi_1^{(nuc)} + \delta \Pi_2^{(nuc)}) \delta X_i^j.$$  

(23)

The nucleon force (16) acting on the vortex can thus be decomposed as

$$\mathcal{F}_i^{(nuc)} = \mathcal{F}_{E,i} + \mathcal{F}_{Mn,i} + \mathcal{F}_{Mp,i} + \mathcal{F}_{\Phi,i},$$

(24)

where the different force terms are given by

$$\mathcal{F}_{E,i} = -\int_C \left[ \hat{a}_i \delta \pi_0^{P^\perp} + \bar{a}_i \delta \pi_0^{P_0} - \bar{a}_i \delta \pi_0 \right] \alpha_i \, d\ell,$$

(25)

$$\mathcal{F}_{Mn,i} = \int_C \left[ \hat{a}_i \delta \pi_0^{P^\perp} - \bar{a}_i \delta \pi_0^{P_0} - \bar{a}_i \delta \pi_0 \right] \alpha_i \, d\ell,$$

(27)

$$\mathcal{F}_{Mp,i} = \int_C \left[ \hat{a}_i \delta \pi_0^{P^\perp} - \bar{a}_i \delta \pi_0^{P_0} - \bar{a}_i \delta \pi_0 \right] \alpha_i \, d\ell,$$

(28)

and

$$\mathcal{F}_{\Phi,i} = -\frac{e}{c} \int_C \left[ \bar{a}_i \delta A_0 - \bar{a}_i \delta \pi_0 \right] \alpha_i \, d\ell.$$  

(29)

Let us first focus on $\mathcal{F}_{E,i}$. The equations of motion for the neutron superfluid and for the proton superconductor, as expressed as the vanishing of a suitably generalised vorticity tensor, thus take a very simple form in the 4D covariant approach (see Eqs. (15), (17) and (18) of Carter & Langlois (1998), or Eqs. (161) and (171) of Carter & Chamel (2004)). In the usual spacetime decomposition, the stationary limit of these equations reduces to $\nabla_i \pi_0^{P^\perp} = \nabla_i \pi_0^{P_0} = 0$. As shown in Appendix A, this result can also be obtained from Eqs. (26)-(29) of Pric (2005) or from Eqs. (B4) and (B5) of Gusakov (2019) in the particular case in which mutual neutron-proton entrainment effects are neglected. The above equations imply that both $\pi_0^{P^\perp}$ and $\pi_0^{P_0}$ are uniform, i.e., $\pi_0^{P^\perp} = \pi_0^{P_0}$ and $\pi_0^{P_0} = \pi_0^{P_0}$, or equivalently, $\delta \pi_0^{P^\perp} = \delta \pi_0^{P_0} = 0$. Therefore, we get

$$\mathcal{F}_{E,i} = \bar{a}_i \delta A_0 \epsilon \int_C \alpha_i \, d\ell,$$

(30)

using the fact that $\delta A_0 = A_0 - \tilde{A}_0 = A_0$ since the uniform background value vanishes (in an appropriate gauge). Equation (30) can thus be interpreted as the (opposite of the) force acting on charged protons due to the electric field.

Introducing the coefficients $D_n$ and $D_p$ as

$$D_n = \int_C \bar{n}_i^{\ell} \alpha_i \, d\ell \quad \text{and} \quad D_p = \int_C \bar{n}_i^{\ell} \alpha_i \, d\ell,$$

(31)

and recalling that $\bar{a}_i^{\ell} = \tilde{a}_i^{\ell}$, the force term $\mathcal{F}_{E,i}$ can be rewritten as

$$\mathcal{F}_{E,i} = \delta D_n \tilde{a}_i^{\ell} + \delta D_p \tilde{a}_i^{\ell},$$

(32)

where $\delta D_n = D_n - D_n$ and $\delta D_p = D_p - D_p$. Using Stokes’ theorem, $D_n$ and $D_p$ can be equivalently expressed as

$$D_n = -\int_{S(C)} \nabla_k \pi_0^{P_0} \, dS \quad \text{and} \quad D_p = -\int_{S(C)} \nabla_k \pi_0^{P_0} \, dS,$$

(33)

where the integrals are over the surface $S(C)$ delimited by the contour $C$ and $dS$ is the corresponding surface element. Therefore, the background values vanish and $\delta D_n = 0 = \delta D_p$. The force $\mathcal{F}_{E,i}$ is thus associated with transverse processes, whereby particles of different species are converted into each other by nuclear reactions (Carter & Chamel 2005b). If each species is separately conserved, $\nabla_k \pi_0^{P_0} = 0 = \nabla_k \pi_0^{P_0}$ must hold everywhere throughout the fluids. In such a case, we deduce that $D_n = D_p = 0$, which leads to $\mathcal{F}_{E,i} = 0$.

Neutron and proton superflows far from the vortex must obey the irrotationality condition (see Eqs. (15), (17) and (18) of Carter & Langlois (1998) or Eqs. (161) and (171) of Carter & Chamel (2004))

$$\epsilon^{ijk} \nabla_j \pi_0^{P_0} = 0 \quad \text{and} \quad \epsilon^{ijk} \nabla_j \pi_0^{P_0} = 0,$$

(34)

where $\epsilon_{ijk}$ is the Levi-Civita symbol. The longitudinal invariance along the vortex, in association with the previous irrotationality conditions, lead to $\delta \pi_0^{P_0} = 0$ and $\delta \pi_0^{P_0} = 0$ (Carter & Chamel 2005a). Defining $\lambda_i^{\ell}$ as the operator of projection orthogonal to the vortex, i.e., $\lambda_i^{\ell} = \delta_i^{\ell} - \bar{\delta}_i^{\ell} z_j$, we thus have

$$\lambda_i^{\ell} \delta \pi_0^{P_0} = \delta \pi_0^{P_0}, \quad \lambda_i^{\ell} \delta \pi_0^{P_0} = \delta \pi_0^{P_0}.$$  

(35)

The force term $\mathcal{F}_{Mn,i}$ (27) can therefore be recast as

$$\mathcal{F}_{Mn,i} = \int_C \bar{a}_i^{\ell} \delta \pi_0^{P_0} \left[ \lambda_i^{\ell} \alpha_j^{\ell} - \lambda_j^{\ell} \alpha_i^{\ell} \right] \, d\ell.$$  

(36)

Introducing the unit vector $\beta^i$ along the contour such that $\alpha_i = -\epsilon_{ijk} \beta^j \beta^k$ as illustrated in Fig. 2, and making use of the identity

$$\lambda_i^{\ell} \epsilon_{ijm} + \beta_j^{\ell} \epsilon_{ijm} = -\lambda_j^{\ell} \epsilon_{ijm},$$

(37)

the force $\mathcal{F}_{Mn,i}$ can be equivalently written as

$$\mathcal{F}_{Mn,i} = -\epsilon_{ijk} \hat{z}^k \bar{a}_i^{\ell} \delta C^n,$$

(38)

where the neutron momentum integral $C^n$ is given by

$$C^n = \int_C \bar{a}_i^{\ell} d\ell,$$

(39)

with $d\ell = d\ell \beta^k$. Since the background value $\tilde{C}^n$ vanishes and using the fact that the neutron momentum circulation integral is quantised, we have

$$\delta C^n = C^n - m_n \kappa_n,$$

(40)

in the presence of a single vortex line. The force term $\mathcal{F}_{Mn,i}$ finally reads

$$\mathcal{F}_{Mn,i} = -\tilde{p}_n \kappa_n \epsilon_{ijk} \hat{z}^k \bar{a}_i^{\ell},$$

(41)

where $\tilde{p}_n = m_n \kappa_n$. This force can thus be recognized as the Magnus force induced by the (quantised) momentum circulation of the neutron superfluid around the vortex. Similar arguments also apply to the proton superconductor, with the only difference that the proton momentum circulation integral reads

$$C^p = \int_C \bar{a}_i^{\ell} d\ell = m_p N_p \kappa_p,$$

(42)
with $k_p = \hbar/(2m_p)$, if $N_p$ fluxoids are enclosed inside the contour. The force term $f^{(\text{MP})}_i$ thus reads

$$f^{(\text{MP})}_i = -\bar{p}_p N_p k_p \epsilon_{ijk} \hat{z}^j \hat{v}_k^p. \quad (43)$$

Finally, since the magnetic field $B^i = \epsilon^{ijk} \nabla_j A_k$ carried by the vortex-fluxoid configuration is directed along $\hat{z}$, the $z$-component of the vector potential $A_z$ must vanish, i.e., $A_z = 0$ (see, e.g., Eqs. (7) and (9) of Gusakov (2019)). As a consequence, we have $\mathcal{L}_c^i = \mathcal{L}_p^i$. Following a procedure similar to the one used to derive $f^{(\text{MP})}_i$ and $f^{(\text{MP})}_i$, we obtain

$$f^{(\text{MC})}_i = \bar{n}_p \epsilon^i_c \Phi \epsilon_{ijk} \hat{z}^j \hat{v}_k^p, \quad (44)$$

where

$$\Phi = \oint_C A_k \text{d}l^k \quad (45)$$

is the magnetic flux through the surface $S(C)$ delimitated by the contour $C$, and using the fact that the uniform background flux $\Phi$ necessarily vanishes. Equation (44) can thus be interpreted as the (opposite of the) force acting on charged protons due to the magnetic field.

Collecting terms in Eq. (24), the nucleon force contribution is finally expressible as

$$f^{(\text{MC})}_i = -\bar{n}_p k_n \epsilon_{ijk} \hat{z}^j \hat{v}_k^n + \bar{n}_p N_p k_p \epsilon_{ijk} \hat{z}^j \hat{v}_k^p + \bar{n}_p \epsilon_c \oint_C A_0 \text{d}l^f + \bar{n}_p \epsilon^i_c \Phi \epsilon_{ijk} \hat{z}^j \hat{v}_k^p. \quad (46)$$

Let us recall that this expression only holds in the absence of transverse fields, i.e., assuming that each species is separately conserved.

### 2.4 Electron contribution

Since electrons are in a ballistic regime at the scale of interest here, the calculation of $f^{(e)}_i$ needs a specific treatment, see, e.g., Gusakov (2019). Although Gusakov (2019) mainly focused on a single proton fluxoid, several conclusions of this work are actually very general and can be readily transposed to the vortex-fluxoid configuration under consideration. In particular, the electron force term (17) can be decomposed into two parts, i.e.,

$$f^{(e)}_i = f_i^{(e,\text{sc})} + f_i^{(e,\text{ind})}, \quad (47)$$

see Eq. (B27) of Gusakov (2019), where

$$f_i^{(e,\text{sc})} = 2 \oint_C \delta \Pi^{(e,\text{sc})}_i a_j \text{d}l^j \quad (48)$$

is associated with the scattering of electrons off the vortex-fluxoid system and

$$f_i^{(e,\text{ind})} = \oint_C \delta \Pi^{(e,\text{ind})}_i a_j \text{d}l^j \quad (49)$$

is an “induced” contribution related to the fact that the scattered electrons carry a charge and thus generate a weak electric field far from the vortex. Using (B15) of Gusakov (2019), this induced contribution reads

$$f_i^{(e,\text{ind})} = -\epsilon_c \bar{n}_e \oint_C A_0 a_j \text{d}l^j. \quad (50)$$

Besides, the force term $f_i^{(e,\text{sc})}$ is found to be expressible as

$$f_i^{(e,\text{sc})} = D_e \bar{v}_e + D'_e \epsilon_{ijk} \hat{z}^j \hat{v}_k^e, \quad (51)$$

see Eqs. (29) and (18) of Gusakov (2019), where $\bar{v}_e^c$ denotes the asymptotically uniform electron velocity (in the frame where the vortex is at rest). Furthermore, Gusakov (2019) derived the expressions for the coefficients $D_e$ and $D'_e$ in the particular context where $f_i^{(e,\text{sc})}$ is governed by the scattering of electrons off the magnetic field carried by a single flux-oid. His expression for $D'_e$ (see Eq. (61) of Gusakov (2019)), i.e.,

$$D'_e = -\epsilon_c \bar{n}_e \Phi, \quad (52)$$

where $\Phi$ is given by Eq. (45), happens to be very general in the sense that it does not depend on the detailed structure of the quantised lines carrying the magnetic flux (as long as electrons follow classical trajectories). Therefore, Eq. (52) remains valid in the present case where electrons are scattered off the magnetic field carried by the vortex and the pinned fluxoids$^4$. On the other hand, the expression (60) of Gusakov (2019) for the drag coefficient $D_e$ is not applicable here since it depends strongly on the configuration of the quantised line(s) carrying the flux $\Phi$. The determination of $D_e$ would thus require (i) to know the exact geometry and structure of the vortex and the $N_p$ fluxoids pinned to it, and (ii) to generalise the microscopic scattering calculations carried out by Gusakov (2019) to the present case.

From the previous considerations, the electron force (17) finally reads

$$f_i^{(e)} = -\bar{n}_p \epsilon_c \oint_C A_0 a_j \text{d}l^j + D_e \bar{v}_p i + \bar{n}_e \epsilon^i_c \Phi \epsilon_{ijk} \hat{z}^j \hat{v}_k^e, \quad (53)$$

where we have made use of the electric charge neutrality condition $\bar{n}_e = \bar{n}_p$ and the so-called screening condition $\bar{v}_e = \bar{v}_p$, see, e.g., Glampedakis et al. (2011); Gusakov & Dommes (2016).

### 2.5 Electromagnetic contribution

As can be seen from Eq. (14), the first-order perturbation in the electromagnetic stress tensor $\delta \Pi^{(em)}_i$ only involves terms of the kind $\delta E_i \hat{E}_j$ and $\delta B^i \hat{B}_j$. However, the asymptotically uniform magnetic and electric fields must vanish in view of the Meissner effect, i.e., $\hat{B}_i = 0$ and $\hat{E}_i = 0$ (Carter et al. 2002; Gusakov 2019). From $\delta \Pi^{(em)}_i = 0$, we conclude that

$$f_i^{(em)} = 0. \quad (54)$$

### 2.6 Final expression and comparison with previous studies

Combining Eqs. (46), (53) and (54), the total force per unit length (15) acting on a neutron vortex to which $N_p$ proton

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$^4$ Each proton fluxoid pinned to the vortex carries a quantum of magnetic flux $\phi_0$. Besides, the neutron vortex itself is magnetised due to neutron-proton entrainment effects, and thus carries a fractional quantum of magnetic flux $\phi_n = -\epsilon_p \phi_0$, where $\epsilon_p$ characterizes the importance of entrainment effects (see, e.g., Sedrakian & Shakhbasian (1980); Alpar et al. (1984)).
fluctuations are pinned is finally expressible as
\[
\mathbf{T}_f = -\tilde{p}_n \kappa_n \epsilon_{ijk} \dot{z}^j \bar{\epsilon}^i_k - \tilde{p}_n N_p \kappa_p \epsilon_{ijk} \dot{z}^j \bar{\epsilon}^i_k + D_c \dot{v}_{p,i}.
\]
(55)

Note that the electron force proportional to \(D'_e\) derived by Gusakov (2019), i.e., the last term in Eq. (53), is exactly cancelled by the proton force \(T_{q,i}\). Let us remark that Eq. (55) is also applicable to determine the force acting on a neutron-proton vortex cluster of the kind proposed by Sedrakian & Sedrakian (1995).

In the absence of pinning (\(N_p = 0\)), a situation considered by Alpar et al. (1984), a neutron vortex still experiences a drag force due to the scattering of electrons off the magnetic field induced by the circulation of entrained protons. By setting \(N_p = 1\) and \(\kappa_n = 0\), our expression (55) reduces to that obtained by Gusakov (2019) for the force acting on a single fluxoid. For NSs with a rotation period \(P_{\text{rot}} \approx 1\) and a typical magnetic field \(B_{L,2} \approx 1\), the number of pinned fluxoids may potentially be as large as \(N_p \sim N_p/N_m \sim 10^{13}\) so that the second term in Eq. (55) may have a very strong impact on the vortex dynamics. To illustrate the relative importance of the different force terms on the vortex motion, we give the expression of the vortex velocity in the next section.

### 2.7 Vortex motion

Once expressed in a frame where the neutron vortex moves at the velocity \(v'_L\), the vortex motion can be obtained from the force balance equation \(T_{f} = 0\) (neglecting the masses of the different quantised lines as in previous studies). Following the classical approach of Hall & Vinen (1956) and considering velocities orthogonal to \(\hat{z}\), the vortex velocity is found to be given by
\[
v'_L = \hat{v}_p + B \hat{\epsilon}_{ijk} \dot{z}_j \tilde{w}_{pn} k + (1 - \beta') \hat{\epsilon}_{ijk} \dot{z}_j \dot{\epsilon}_{klm} \hat{z}_l \tilde{w}_{pn}^m
\]
(56)

where \(\tilde{w}_{pn} = \hat{v}_p - v'_L\) denotes the relative velocity far from the quantised lines, see Appendix B for details (the coefficients usually denoted by \(B\) and \(\beta\) in the neutron-star literature were indicated by \(a\) and \(a'\) in the standard textbook of Donnelly (2005)). In this expression, the coefficients \(B\) and \(\beta'\) are expressible as
\[
B = \frac{\xi}{\bar{\xi}^2 + (1 + X)^2} \quad \text{and} \quad 1 - \beta' = \frac{1 + X}{\bar{\xi}^2 + (1 + X)^2},
\]
(57)

where the drag-to-lift ratio \(\xi\) and the momentum circulation ratio \(X\) read
\[
\xi = \frac{D_c}{\tilde{p}_n \kappa_n} \quad \text{and} \quad X = \frac{\tilde{p}_n}{\kappa_n} N_p.
\]
(58)

Note that, regardless of the actual values of \(\xi\) and \(X\), the following inequalities hold
\[
B \leq 1/2 \quad \text{and} \quad \beta' \leq 1.
\]
(59)

In the absence of pinning (\(N_p = 0\)), Eq. (57) reduces to well-known expressions (see, e.g., Carter (2001)). In particular, the vortex velocity \(v'_L\) coincides with \(\tilde{v}_n\) and \(\hat{v}_p\) in the weak (\(\xi \ll 1\)) and strong (\(\xi \gg 1\)) drag regimes, respectively.

The motion of a vortex is more complicated if proton fluxoids are pinned to it. In particular, the vortex will move with velocity \(v'_{p,i}\) even in the weak drag limit if the number of pinned fluxoids \(N_p\) is large enough such that \(X \gg 1\) and \(X \gg \xi\). However, it should be stressed that the drag-to-lift \(\xi\) itself depends on \(N_p\) and may thus be also very large (Ding et al. 1993; Sedrakian & Sedrakian 1995). Because pinning may lead to a dramatic reduction of the coefficient \(B\), it may also have important implications for the onset of superfluid turbulence, which is thought to be governed by the parameter \(q = B/(1 - \beta')\) (Finne et al. 2003).

### 3 CONCLUSIONS

Following an approach originally proposed by Carter and collaborators (Carter et al. 2002; Carter & Chamel 2005a), we have derived the expression for the force per unit length acting on a quantised neutron vortex to which \(N_p\) proton fluxoids are attached, see Eq. (55). Our expression is very general and can be applied to describe various situations. In particular, Eq. (55) extends the expression recently obtained by Gusakov (2019) for the force per unit length acting on a single fluxoid.

By clarifying the different contributions to the force, we have shown that the proton-momentum circulation around the vortex induced by the presence of pinned fluxoids gives rise to a Magnus type force. Due to mutual entrainment effects, the distinction between momenta (usually improperly introduced in terms of “superfluid velocities”) and currents is crucial to obtain the correct expression of the force. Unlike the drag force, this Magnus force does not depend on the microscopic arrangement of pinned fluxoids, as a consequence of the quantisation of the proton circulation.

Although the vortex velocity takes a similar form as in the absence of pinning, see Eq. (56), the friction coefficients \(B\) and \(\beta'\) are found to depend on the dimensionless ratio \(N_p \times \tilde{p}_n/k_n\) in addition to the drag-to-lift ratio \(\xi\). Because \(N_p\) may be potentially as large as \(\sim 10^{13}\), pinning may have a dramatic impact on the vortex motion and the onset of superfluid turbulence. A major complication comes from the fact that \(N_p\) thereby the coefficients \(B\) and \(\beta'\) may vary along the vortex trajectory: \(N_p\) may increase as the vortex moves and encounters more and more fluxoids, but \(N_p\) may also decrease as fluxoids get unpinned. The evolution of \(N_p\) will generally depend on the spatial distribution of fluxoids in the outer core of a NS, which in turn reflects the geometry of the internal magnetic field. Pinning of neutron vortices to proton fluxoids should thus be taken into account in the modelling of the magneto-rotational evolution of NSs.

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As shown in the accompanying Letter (Sourie & Chamel 2020), the coefficient \(B\) (\(\beta'\)) is associated with the dissipative (conservative) contribution in the smooth-averaged mutual-friction force arising at scales large compared to the intervortex spacing.
APPENDIX A: STATIONARY EQUATIONS OF MOTION FOR THE NEUTRON SUPERFLUID AND FOR THE PROTON SUPERCONDUCTOR

In this appendix, we show how the stationary equations of motion for the neutron superfluid and for the proton superconductor, i.e.,

$$\nabla_i \pi_n^0 = 0 \quad \text{and} \quad \nabla_i \pi_p^0 = 0,$$

(A1)
can be derived from Eqs. (26)–(29) of Priš (2005) and from Eqs. (B4) and (B5) of Gusakov (2019), although these two studies rely on different approaches.

A1 Derivation from Priš (2005)

Using Eqs. (26)–(29) of Priš (2005), the canonical force densities acting on the neutron superfluid and the proton superconductor are respectively given by

$$f_i^n = n_n \left( \partial_t p_n^0 - \nabla_i p_n^0 \right) - \epsilon_{ijk} n_n v_k^n e^{klm} \nabla_l p_m^n + n_n m_n \nabla_i \varphi + \Gamma_n \pi_i^n,$$

(A2)

and

$$f_i^p = n_p \left( \partial_t p_p^0 - \nabla_i p_p^0 \right) - \epsilon_{ijk} n_p v_k^p e^{klm} \nabla_l p_m^p + n_p m_p \nabla_i \varphi + \Gamma_p \pi_i^p,$$

(A3)

where $\varphi$ denotes the gravitational gauge field and we have used the shorthand notations

$$\Gamma_n = \partial_t n_n + \nabla_i \left( n_n v^n_i \right) \quad \text{and} \quad \Gamma_p = \partial_t n_p + \nabla_i \left( n_p v^p_i \right).$$

(A4)

In the absence of external forces, as characterised by $f_i^n = f_i^p = 0$, and ignoring any transverse processes, i.e., $\Gamma_n = \Gamma_p = 0$, the stationary equations of motion reduce to

$$0 = - \nabla_i p_n^0 - \epsilon_{ijk} v_k^n e^{klm} \nabla_l p_m^n,$$

(A5)

and

$$0 = - \nabla_i p_p^0 - \epsilon_{ijk} v_k^p e^{klm} \nabla_l p_m^p - \epsilon \left( E_i + \frac{1}{c} \epsilon_{ijk} v_j^p B^k \right),$$

(A6)

where we have neglected the small variations of the gravitational field $\varphi$ on the scales of interest. Making use of the irrotationality conditions (34), rewritten as

$$\epsilon_{ijk} v_{jk}^n p_n^0 = 0 \quad \text{and} \quad \epsilon_{ijk} v_{jk}^p p_p^0 + \frac{c}{B} B^0 = 0,$$

(A7)

in combination with the results (22) for $\pi_n^0$ and $\pi_p^0$, Eqs. (A5) and (A6) reduce to (A1), as expected.

A2 Derivation from Gusakov (2019)

In the alternative approach followed by Gusakov (2019), the conservation equations for the neutron and proton momenta read

$$\partial_t G_n = - \nabla_k \left( m_n n_n v^n_k v^n_i \right) - n_n \nabla_i \mu^n,$$

(A8)

$$\partial_t G_p = - \nabla_k \left( m_p n_p v^p_k v^p_i \right) - n_p \nabla_i \mu^p + n_p \epsilon \left( E_i + \frac{1}{c} \epsilon_{ijk} v_j^p B^k \right),$$

(A9)

where the variations of the gravitational field have been neglected, see Eqs. (B5) and (B4) of Gusakov (2019). The quantities $G_n = n_n p^n_0$ and $G_p = n_p p^p_0$ denote the neutron and proton momentum densities, respectively, with $p^n_0 = m_n v^n_i$ and $p^p_0 = m_p v^p_i$ in the absence of mutual neutron-proton entrainment effects (see Eqs. (12) and (13) with $K^{np} = 0$), as considered in Gusakov (2019). Ignoring transverse processes and focusing on stationary situations only, Eqs. (A8) and (A9) reduce to

$$0 = - m_n v^n_k v^n_i - \nabla_k \mu^n,$$

(A10)

and

$$0 = - m_p v^p_k v^p_i - \nabla_k \mu^p + \epsilon \left( E_i + \frac{1}{c} \epsilon_{ijk} v_j^p B^k \right).$$

(A11)

Making use of the relation

$$A_i \nabla_j A_j = \frac{1}{2} \nabla_i \left( A_i A_j \right) - \epsilon_{ijk} A_j e^{klm} \nabla_l A_m,$$

(A12)

valid for any vector field $A_i$, these equations can then be rewritten as

$$0 = - \nabla_i \left( \frac{1}{2} \epsilon_{ijk} p^n_k + \mu^n \right) + \epsilon_{ijk} v_j^n e^{klm} \nabla_l p_m^n,$$

(A13)

and

$$0 = - \nabla_i \left( \frac{1}{2} \epsilon_{ijk} p^p_k + \mu^p \right) + \epsilon_{ijk} v_j^p e^{klm} \nabla_l p_m^p + \epsilon \left( E_i + \frac{1}{c} \epsilon_{ijk} v_j^p B^k \right).$$

(A14)

Using Eq. (21), which can be recast as

$$p^n_k = - \mu^n - m_n v^n_k / 2 + \epsilon \left( v^n_k v^n_l - \delta^n_l \right) / 2,$$

(A15)

in the absence of entrainment effects, Eqs. (A13) and (A14) are found to be equivalent to the (opposite of) Eqs. (A5) and (A6). The irrotationality conditions (34) therefore lead to Eq. (A1), as in Section A1.

APPENDIX B: VORTEX VELOCITY

In a frame where the neutron vortex and the $N_p$ fluxoids pinned to it move at the velocity $v_i$, the total force per unit length (55) acting on the quantised lines reads

$$F_i = - (\rho_n e^{\iota_j} (v^n_k - v^n_l) - \rho_p N_p e^{\iota_j} (v^p_k - v^p_l) + \delta_j^L (v^n_i - v^n_j).$$

(B1)

Neglecting the masses of the quantised lines (as in previous studies), the force balance equation $F_i = 0$ leads to

$$v_i = v_i - \frac{1}{c} \epsilon_{ijk} j^k (v^n_k - v^n_l) - \frac{X}{\xi} \epsilon_{ijk} j^k (v^p_k - v^p_l).$$

(B2)

where $\xi$ and $X$ are given by Eq. (58). Projecting this latter equation along $\epsilon_{ijk} j^k$ yields

$$v_i = \epsilon_{ijk} j^k (v^n_k - v^n_l) - \frac{X}{\xi} \epsilon_{ijk} j^k (v^p_k - v^p_l)$$

(B3)

$$+ \frac{1}{\xi} \epsilon_{ijk} j^k (v^n_i - v^n_j) + \frac{X}{\xi} \epsilon_{ijk} j^k (v^p_i - v^p_j),$$

(B4)

where we have used the fact that $\epsilon_{ijk} j^k = \delta_i^j \delta^k_l - \delta_i^k \delta^j_l$, $\delta^j_l$ being the Kronecker delta, and we have only considered velocities orthogonal to $\hat{z}$. Using Eq. (B4) in the right-hand side of Eq. (B2) now leads to

$$v^L_i \left[ 1 + \frac{X}{\xi} \right] = v^n_i \left[ 1 + \frac{X}{\xi} \right] + \frac{X}{\xi} \left( v^n_i - v^n_j \right)$$

(B5)

which can be eventually recast in the equivalent forms (56).
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