Non-equilibrium dynamics of the Dicke model for mesoscopic aggregates: signatures of superradiance

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Abstract

The dynamics of the Dicke model, which describes the interaction of $N$ two-level atoms with a resonant field mode, is studied in the presence of dissipation and in strong non-equilibrium for a moderate number of atoms. Starting from a highly excited state, it is investigated to what extent signatures of superradiant phenomena known from the thermodynamic limit $N \to \infty$, namely, superradiant light emission and atom-field correlations characteristic for the superradiant phase, appear on transient time scales. Attention is also paid to subtleties of modeling dissipation in the weak coupling limit and to the dynamics in phase space. The latter allows phase space correlators to be defined as indicators for collective behavior on transient time scales. These findings may be of relevance for the realization of Dicke physics with superconducting circuits.

Keywords: Dicke-Hamiltonian, superradiance, cooperative phenomena, superradiant phase transition

(Some figures may appear in colour only in the online journal)

1. Introduction

The Dicke model, originally developed to describe a large number $N$ of two-level atoms interacting with a single-mode radiation field [1, 2], has regained substantial interest in the last decade [3–6]. Recent experimental realizations of Dicke physics have utilized various degrees of freedom of cold atom clouds, see e.g. [4, 7], and implementations with solid state devices have also been discussed [8].

In this context, theory has contributed to a variety of aspects of the model with most of the work considering the regime of a large atom number $N \gg 1$ (semiclassical limit) or the thermodynamic limit $N \to \infty$. In the former domain the focus has been on the dissipative dynamics, while in the latter properties of the bare system at and close to thermal equilibrium have been studied. In either case, characteristic phenomena have been identified associated with the term superradiance: on a transient time scale, the cooperativity of the atoms may lead to superradiant light emission (burst) while thermodynamically a phase transition occurs when the atom-field coupling exceeds a certain threshold. This threshold captures the changeover from a regime where energy eigenstates are product states of the atoms and the field mode, to a regime where collective behavior associated with strong atom-field correlations appears (superradiant phase). However, while both phenomena reflect strong quantum mechanical many-body correlations, the physical processes from which they originate are quite different. Superradiant light emission is based on the coherent superposition of atom dynamics, while the field can be eliminated. This is in contrast to the superradiant phase transition, which originates from mutual atom-field interactions at low temperatures.

What has gained much less attention is the transient coherent dynamics of the atom-field compound within the mesoscopic regime with only a moderate number of atoms.
It appears that this regime has gained experimental relevance recently as we will see below. In this context, relevant questions are these: (i) if, and if yes to what extent, signatures of superradiant light emission occur, (ii) what is the role of dissipation in the dynamics far from equilibrium, and (iii) can one identify phase space structures which may serve as indicators for the emergence of collective dynamics. To address these issues, a number of very powerful perturbative treatments which have been used previously such as the Holstein–Primakoff (HP) approximation or quasi-classical equations of motion cannot be applied as they are restricted either to the low or the high energy sector, respectively. Finite size effects in the regime \( N \gg 1 \) and close to the ground state have been addressed [9], but these findings do not cover the mesoscopic regime. Instead, here we think about situations where the system is initially prepared far from equilibrium, e.g. due to an initial quench, and then released to obey a relaxation dynamics at lower temperatures. The transient dynamics of excitation exchange between atoms and field, the transient buildup of atom-field correlations, and the changeover towards relaxation to a correlated ground state are then studied. We note in passing that some progress in this direction has been made recently in the limit of small atom numbers \( N \sim O(1) \) [10].

Technically, we follow the conventional quantum optical procedure and formulate the open quantum dynamics in terms of a master equation [11]. This will result in a rate description in the overdamped limit of the Dicke model, when field excitations will not act back on the atoms due to their quick decay (incoherent dynamics) and are consequently eliminated from the description. This limit is, for instance, relevant for various recent experimental realizations of superradiance scenarios involving excitons, see e.g. [12]. However, the quantum optical modeling is also associated with subtleties concerning the treatment of the dissipative mechanism, as we discuss in detail.

As already mentioned, this work has been motivated by recent experimental activities. Progress in fabrication techniques for tailored superconducting circuits allows multiple artificial atoms in the form of Cooper pair boxes to be integrated into micro-cavities. To maintain sufficiently long coherence times and tunability, the number of two-level systems in these devices must be kept moderate and far below the number of atoms in a cold atomic cloud. Furthermore, environmental degrees of freedom in the form of heat baths are always present in these solid state structures. Nevertheless, one may assume that signatures of Dicke physics are observable [5], at least in certain ranges of time and parameter space. Another line of research combines atomic ensembles with superconducting cavities to benefit from the long coherence times of the former and the fast tunability of the latter entities [13]. In this case, dissipation on the solid state side, i.e. the finite photon lifetime in the micro-cavity, is the dominant mechanism and must be taken into account.

Even conceptually, in these composite systems the Dicke model may not be realized on all time scales. For example, the atom-field interaction may include higher order terms in the field operator not captured in the model [8] or couplings to residual degrees of freedom may be present (e.g. trapping loss of atoms, charge-background fluctuations in solid state systems, etc) which may become relevant on longer time scales. The Dicke model as such may then be justified at least on transient time scales.

The paper is organized as follows. We start in section 2 with a brief discussion of the two characteristic phenomena, namely, the superradiant burst and the superradiant phase and then introduce the dissipative Dicke model explored in the sequel. In section 3.2, the dynamics of the photon population in the field mode (cavity) is studied in detail within the full model which is contrasted with its often used form based on a rotating wave approximation (RWA). Scaling properties of the initial radiation peak are analyzed. The issue of how to understand and monitor experimentally possible collective dynamics of atoms and cavity is addressed in section 4.

2. Preliminaries: superradiance in the Dicke model

In the context of the Dicke model the term ‘superradiance’ is associated with two quite different phenomena that we recall briefly below.

2.1. Light emission into free space: the superradiant burst

A single excited atom in free space will radiate its energy isotropically in an exponential decay process. If a large number of excited atoms are brought into close proximity their decay is not described by adding the light emission rates from all the atoms independently. Instead a cooperative emission process of all atoms is established, which results in a strong, short anisotropic burst of light. This superradiant free-space burst, predicted by Dicke in 1954 [1] and first experimentally observed by Skribanowitz [14], has since been extensively studied and its characteristic features are well understood (see e.g. [2] for a review).

For the purpose of this paper, we will consider as defining signature of superradiance the crossover from an exponential decay for a single atom to a peak in the time-dependence of the emission rate. This peak gradually becomes more pronounced when the effective number of atoms, \( N \), cooperating in the emission process increases (roughly speaking, \( N \) is the number of atoms within a volume, \( \lambda_0^3 \), where \( \lambda_0 \) is the wavelength of the emitted light). Due to the cooperative nature of the light emission, the peak’s height grows as \( N^2 \) while its temporal width shrinks as \( N^{-1} \), since the integrated emission is proportional to the total initial excitation energy and, hence, to \( N \).

The simplest theoretical description of such burst features relies on a rate model. The radiation of \( N \) excited indistinguishable two-level atoms is calculated by determining decay rates between the \( N + 1 \) levels of a spin-\( N/2 \) system (which results from adding the (pseudo)spins of two-level atoms, see below) from the corresponding dipole matrix elements [2].
2.2. Thermodynamics: the superradiant phase transition

A collection of \( N \) identical atoms in a single mode cavity can be described by the Dicke-Hamiltonian [1–3],

\[
\hat{H}_{\text{Dicke}} = \omega \hat{a}^\dagger \hat{a} + \epsilon \hat{J}^z + \frac{\lambda}{\sqrt{N}} (\hat{a}^\dagger + \hat{a})(\hat{J}^+ + \hat{J}^-),
\]

where the atoms are modeled as two-level systems with identical level spacing \( \epsilon \) for which collective (‘large’) spin operators are introduced, \( \hat{J}^z = 1/2 \sum_{j=1}^{N} \hat{\sigma}^z_j, \hat{J}^\pm = \sum_{j=1}^{N} \hat{\sigma}^\pm_j, \hat{b} = 1 \). The collective spin couples to a common, single electromagnetic mode of frequency \( \omega \) in the cavity described by bosonic operators \( \hat{a}, \hat{a}^\dagger \) with effective coupling strength \( \lambda/\sqrt{N} \). The latter one is conveniently introduced when the density of atoms per unit volume is fixed since then the bare coupling \( \lambda \) between any single atom and the atomic mode is effectively reduced with a growing number of atoms. If instead in a given experimental situation the actual density of atoms can be increased, \( \lambda \) grows accordingly.

The above modeling (see [3] for a recent review) relies on the assumption that all atoms couple to the mode with equal strength (see e.g. [15, 16] for inhomogeneous effects) and that a loss of atoms from the collective state (e.g. due to trapping losses or external noise sources, see [11]) can be neglected on the time scales of interest. Accordingly, the collective spin operators in (1) couple only states within one \( J \) manifold of the so-called Dicke states \( |J,M\rangle \). For example, these are states with maximum angular momentum \( J = N/2 \) when the atomic system is initialized in the atomic ground state \( |J = N/2, M = -J\rangle \) or the maximally inverted state \( |J = N/2, M = +J\rangle \). The validity of the Dicke-Hamiltonian for various experimental scenarios with (artificial) atoms has been widely discussed recently [8, 17]. Within an RWA, where only the coupling terms \( \omega \hat{a}^\dagger \hat{J} + \omega \hat{a} \hat{J}^- \) are kept, the Hamiltonian conserves in addition to the spin magnitude \( J \) also the total number of photonic and atomic excitations \( \hat{N}_a = \hat{a}^\dagger \hat{a} + \hat{J}^+ + \hat{J}^- \). The full (non-RWA) Hamiltonian (1) only conserves the parity \( \hat{\Pi} = \exp(\mp i \hat{N}_a) \).

Considering the ground state of the Dicke-Hamiltonian, it is immediately obvious, that there is a competition between the coupling term and the terms describing the energy of the uncoupled cavity and spin system, respectively. This competition depends on the strength of the coupling \( \lambda \) compared to the excitation energy for a cavity photon \( \omega \) or a single spin excitation \( \epsilon \). For weak coupling, the ground state of the composite system lies close to a product state of the individual ground states of isolated cavity and isolated spin, i.e. \( |G_{S_{a=0}}\rangle = |n = 0, M = -N/2\rangle \).

In the opposite regime of strong coupling, one has a finite expectation value for the operator \( (\hat{a}^\dagger + \hat{a})(\hat{J}^+ + \hat{J}^-) \) associated with a strongly correlated ground state with non-zero photon occupation and a finite number of spin excitations. In fact, in the thermodynamic limit \( N \to \infty \), the Dicke-Hamiltonian was shown [18, 19] to exhibit a phase transition between a superradiant phase with macroscopic occupations in the field and the atoms and a normal phase without excitations at zero temperature. This transition occurs at a critical coupling strength \( \lambda_c = \sqrt{\alpha}/2 \) [19] and at \( \lambda_{c,\text{RWA}} = \sqrt{\alpha} \) for the RWA version of the Dicke-Hamiltonian [18].

The most intuitive picture for the emergence of this transition and the nature of the respective ground states is provided by the HP approach [20]. The HP treatment of the Dicke-Hamiltonian essentially consists of (see [20] for details):

(i) Introducing bosonic operators \( \hat{b}, \hat{b}^\dagger \) to replace the large spin operator in a HP transformation

\[
\hat{J}^\pm = \hat{b}^\dagger \hat{b} - J, \quad \hat{J}^z = \hat{b}^\dagger \sqrt{2J - \hat{b}^\dagger \hat{b}}, \quad \hat{J}^- = \sqrt{2J - \hat{b}^\dagger \hat{b}}.
\]

(ii) Displacing the operators \( \hat{a}, \hat{b} \) by \( \pm \sqrt{\alpha}, \pm \sqrt{\beta} \), real-valued parameters to be determined self-consistently.

(iii) Performing the HP approximation (HPA), a lowest order expansion in the scaled number of excitations \( \hat{b}^\dagger \hat{b}/N \) assuming only small fluctuations around the mean field solutions. This leads to a Hamiltonian bilinear in the bosonic operators \( \hat{c}, \hat{d} \), while linear terms are eliminated by the self-consistent choice for the displacements (Bogoliubov transformation).

While for \( \lambda < \lambda_c \), there is only a trivial solution \( \alpha = 0 = \beta \), for \( \lambda > \lambda_c \) one finds replacements corresponding to macroscopic expectation values

\[
\langle \hat{\alpha}\hat{\alpha} \rangle_{\text{SP}} = \frac{\alpha}{J}, \quad \langle \hat{J}^z \rangle_{\text{SP}} = \frac{\beta J - 1 - \frac{\lambda^2}{\lambda_{c}^2}}{J}.
\]

(3)

with \( \langle \cdot \rangle_{\text{SP}} \) denoting expectation values in the superradiant phase.

The superradiant phase transition then corresponds to an abrupt change in the potential terms of the Hamiltonian, which is bilinear in the position degrees of freedom \( X_a/b \) reflecting the cavity and atomic degrees of freedom. The potential surface becomes unstable in one direction when \( \lambda \) exceeds the critical value \( \lambda_c \). Correspondingly, the HPA based on the assumption that fluctuations remain small around the fixed points \( \alpha = \beta = 0 \) fails. Instead, in the superradiant phase \( \lambda > \lambda_c \) the HPA approach leads to two quadratic potentials \( V^\pm \) centered around either one of the solutions, \( (X_a, X_b) = (\pm \sqrt{2\alpha}, \pm \sqrt{2\beta}) \).

In this latter regime, eigenstates of the model which respect the parity symmetry of the original Dicke-Hamiltonian, are constructed from (anti)symmetric superpositions of the individual states of \( H^\pm \). In the thermodynamic limit, the HPA becomes exact and corresponding energy eigenvalues are double degenerate. This degeneracy is lifted for any finite number \( N \) of atoms by quantum tunneling between the wells of \( V^\pm \), a non-locality which cannot be captured by the HP approach. Moreover, the Hamiltonians derived within the HP approach are only suited to study the low energy sector of superradiant and normal phase, respectively. They definitely fail to describe the non-equilibrium dynamics when the
system is initialized far from equilibrium, e.g. in a highly excited spin state, \(|n = 0, M = + N/2\).

3. Non-equilibrium dynamics of the dissipative Dicke model

The above discussion brings us to the question to what extent the two phenomena which are associated with the term ‘superradiance’ can appear within a single experimental setup. Here we argue that in a minimal setting it suffices to add dissipation in the field mode and then to consider the relaxation dynamics of the compound from a maximally excited atomic state. In this context, special attention must be paid to the modeling of the dissipative dynamics in terms of equations of motion for the reduced density operator as we will see below.

The limiting regimes of very strong and no damping have been addressed previously. In the former the level broadening in the cavity mode by far exceeds the coupling energy between atoms and cavity, so that cavity excitations decay quickly and the emission process resembles that of a light burst into free space. The density of states of the open cavity leads to slight modifications in the transition rates between different spin states as compared to the emission into free space though. The corresponding dynamics of the collective spin after elimination of the cavity degree of freedom has been studied in depth previously [21]. However, strong damping implies that the thermodynamics of the composite system may be strongly modified by the system-bath interaction so that no conclusions can be drawn from results found for the isolated system (e.g. the existence of a phase transition). In the opposite regime of vanishing dissipation, the dynamics of the Dicke model has been studied in the past with the focus on collapse-revival features, regularity, and quantum chaos [22]. This may give insight into collective properties of the compound but does not allow access to stationary state features for longer times.

3.1. Dissipation mechanism

In the following, we thus concentrate on finite but weak damping, where the dominant dissipative mechanism is due to a large but finite quality factor of the cavity. In actual realizations of the Dicke model, where trapped ensembles of cold atoms are brought into close proximity with a superconducting cavity, competing dissipation mechanisms like spontaneous emission, trap losses, processes involving additional states of a single atom, or dark states of the atomic ensemble are expected to be of minor relevance only, although they may have an impact in specific ranges of parameter space [11]. For circuit QED realizations with artificial atoms other noise sources must be considered, with decoherence most prominent in the Cooper pair boxes. We leave this issue for future work and, in the spirit of a minimal setting, model the decay of electromagnetic excitations of the cavity in a standard manner by adding to the Liouville–von Neumann equation for the density operator of the bare composite system a Lindblad-type dissipator similar as in [11]. Hence, we write for the time evolution of the reduced density operator of the cavity-atom aggregate

\[ \frac{d}{dt} \rho = \frac{1}{i} \left[ H_{\text{Dicke}}, \rho \right] + L_{\text{diss}}(\rho) \]

with the dissipator

\[ L_{\text{diss}}(\rho) = \kappa (\pi + 1) (2\hat{a}^{\dagger}\hat{a} - \hat{a}^{\dagger}\hat{a}\rho - \hat{a}\rho\hat{a}^{\dagger}) + \kappa \bar{n} (2\hat{a}^{\dagger}\hat{a} - \hat{a}^{\dagger}\hat{a}\rho - \hat{a}\rho\hat{a}^{\dagger}). \]

Here, \( \kappa \) is the damping rate and \( \bar{n}(\omega) = 1/(\exp(h\omega/k_B T) - 1) \) is the thermal Bose occupation factor for an environmental photon with cavity energy \( \omega \). Note, that we employ the Lindblad dissipators derived for radiation damping of the cavity without coupling to the spin.

It is instructive to compare this ad-hoc approach, commonly used in quantum optics, to the generic open-system approach, where the Lindblad dissipators are explicitly derived from a system-reservoir coupling Hamiltonian and spectral properties of the bath. The corresponding derivation of the dissipators as found in standard textbooks (see e.g. [23]) relies on a number of approximations: beside the Born–Markov approximation assuming weak coupling to a bath without memory, a RWA in the eigenstate basis of the system is employed based on \( \omega_{\text{res}} \gg \kappa \), where \( \omega_{\text{res}} \) are transition frequencies between bare system eigenstates. By construction this leads to a Liouvillian which describes relaxation of the system to the ground state of the bare system Hamiltonian (apart from cases with additional symmetries). However, this treatment is, generally speaking, not justified once eigenstates of the system Hamiltonian come close to degeneracy. In the Dicke case, this clearly poses serious problems both deep in the superradiant coupling regime and for a larger number of atoms also close to the transition, where the energy spectrum of \( H_{\text{Dicke}} \) becomes extremely dense. In this situation more elaborate non-perturbative techniques must be applied such as e.g. quantum Monte Carlo [24, 25] or stochastic approaches [26]. This is a formidable task though. Namely, apart from conceptual problems, these and related approaches necessitate a diagonalization of the full Dicke-Hamiltonian which becomes numerically prohibitive quickly with an increasing number \( N \) of atoms (see [10]), particularly, if the initial state is prepared far from equilibrium and/or stronger atom-field coupling.

To make progress, we employ the quantum optical approach described by (5), knowing, however, that it does not \( \textit{a priori} \) guarantee relaxation to the proper ground state. Indeed, for the Dicke-Hamiltonian for \( N = 2, 3 \), second-order correlation functions in the field mode were found [10] to differ substantially in the quantum optical and the open-system formulation. It turned out as well though that even for this limited number of atoms eigenspectra of the Dicke-Hamiltonian are dominated by (quasi-)level crossings if one leaves the regime of weak coupling between atoms and field mode.

Since this is a subtle point, we explore the validity of the quantum optical master equation in more detail. The situation in the subradiant regime is not problematic since then the
The impact of damping on the difference between the density $\rho$ of the relaxed system and the density $\bar{\rho}$ associated with the true ground state, quantified by the trace distance, is depicted in the top part of figure 1 and shrinks with decreasing damping as expected. Noticeable (though small) differences occur closer to the critical coupling strength, where energy differences between lowest and first excited state are small but finite and higher lying states mix in as well (inset), see also figure 2(b) below. In any case, energy expectation values are predicted very accurately with relative deviations being smaller than 1%.

What we learn from this analysis is that the quantum optical modeling, while it is not designed to drive the system into the true ground state asymptotically, captures at least certain properties, such as energies, also in the superradiant domain and for weak dissipation with sufficient accuracy if one starts initially in strong non-equilibrium. This statement may not apply to other observables, e.g. higher order correlation functions [10], and it is also not true if one starts close to the correct ground state in the superradiant domain. Under this caveat, we may conclude that (4) together with (5) describes the non-equilibrium dynamics from the initial transient period up to relatively long times fairly well, i.e. the regime we focus on in this work. Our results seem to indicate that this also applies to the situation close to the phase transition, with somewhat stronger deviations though. A detailed study of the asymptotic dynamics in the superradiant phase is certainly required, particularly the competition between coherent tunneling between the two energy minima and decoherence in the low energy sector, but this is beyond the scope of this work.

3.2. Dynamics of the cavity occupation

We now turn to a deeper analysis of the non-equilibrium dynamics and consider in this subsection the field mode. According to what we discussed above, we will focus on the cavity occupation for which (4) provides a quite accurate description over the relevant time range.

Specifically, we consider the situation where initially the compound is prepared far from its equilibrium state with all atoms in the excited state and the cavity in its ground state. We first concentrate on the zero temperature case and later on comment on finite temperature effects. As expected, the atom-cavity system starts to develop a coherent flow of excitations associated with an oscillatory pattern in typical observables such as the mean cavity occupation $\langle \hat{n} \rangle$ (see figure 2) and spin expectation values $\langle \hat{S}^{x,y,z} \rangle$ (not shown). Due to the leaky cavity these features are damped out so that the system finally relaxes to a stationary state. In figure 2(a) we contrast the dynamics of $\langle \hat{n} \rangle$ for couplings $\lambda$ below and above the critical coupling $\lambda_c$. Results for the RWA model are included as well. In the regime of weak coupling, i.e. sufficiently below $\lambda_c$, both the full Dicke model and its RWA version basically coincide and predict a decay towards $\langle \hat{n} \rangle = 0$. This agreement deteriorates substantially towards the critical coupling. Consequently, in the regime above $\lambda_c$, the full model approaches asymptotically a finite cavity occupation in accordance with

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{Relaxation dynamics of the damped Dicke model with Lindblad dissipator, equation (5), for $k_B T = 0$ and various damping rates $\kappa$. The main plots show relaxation towards the equilibrium for $\lambda = 2\lambda_c$ and $N = 5$ for both, the system’s state $\rho$ (top), quantified by the trace distance $T(\rho, \bar{\rho}) = \frac{1}{2}||\rho - \bar{\rho}||_1$ with the density $\bar{\rho}$ associated with the true ground state, and the system’s energy (bottom), quantified by $\delta E = (\bar{\rho} H_{\text{Dicke}}) (t) - E_0$$/E_0$ with $E_0$ being the true ground state energy. Well above the phase transition, the density operator $\bar{\rho}$ takes the form of an incoherent mixture (coherences turn out to be negligible) between the nearly degenerate ground $|0\rangle$ and first excited state $|1\rangle$ (corresponding to a mix of the two equivalent mean field solutions of the HPA), see text. Inset: small energy differences (bottom) between low lying states $E_0$ and $E_1$ (solid and dashed lines) are dangerous for any Lindblad-type approach. The Lindblad dissipator (5) does not resolve these energy differences, so that larger (but still reasonable small) deviations occur close to the critical coupling (crosses denote $\langle H_{\text{Dicke}} \rangle (t \to \infty)$ for $\kappa/\omega = 0.01$). Triangles indicate the trace distance (top) to an optimized mixed $\bar{\rho} = P |0\rangle \langle 0| + (1 - P) |1\rangle \langle 1|$, where $P = 1$ for $\lambda = 0.4\lambda_c$ and $P = \frac{1}{2}$ for $\lambda = 2\lambda_c$.}
\end{figure}
Insight into this failure of the RWA model can be obtained from the master-equation (4). One easily finds in the stationary limit \( \frac{d}{dt} \langle \hat{n} \rangle \big|_{t \rightarrow \infty} = \lambda \) that
\[
\langle \hat{n} \rangle = \bar{n} + \frac{i}{\sqrt{N \kappa}} \langle [\hat{J}, -\langle \hat{\mathcal{J}} \rangle] \rangle \big|_{t \rightarrow \infty},
\]
where the latter counter-rotating terms are absent if the RWA is employed. Hence, within RWA the Lindblad damping term used in (5) always drives the system into a thermal state of the bare cavity independent of the cavity-atom coupling. In contrast, in the full model \( \langle [\hat{\mathcal{J}}, \langle \hat{\mathcal{J}} \rangle - \langle \hat{\mathcal{J}} \rangle] \rangle \) plays the role of an order parameter which is zero below and takes finite values above the phase transition. Dissipation guarantees that the compound relaxes to a state shown above (see section 3.2 in the main text). (Parameters as above, but \( \bar{n} = 0.1 \) for bottom right panel.)

**Figure 2.** (a) Dynamics of the mean cavity occupation \( \langle \hat{n} \rangle \) for coupling strength above/below the critical coupling and with/without the RWA of the coupling term. Above the critical coupling and without RWA the system relaxes to a steady state with finite mean cavity occupation (for comparison the HP result is shown). Employing RWA (green dashed and blue dotted lines) the system does not approach a correlated ground state but reaches a steady state, where the cavity is empty. (Parameters: \( N = 10, \kappa / \omega = 1 / 20, \bar{n} = 0 \) and \( \lambda = 0.2 \lambda_c \)—normal, \( \lambda = 2.4 \lambda_c \)—superradiant.) (b) Occupation of eigenstates of \( \hat{H}_{\text{Dicke}} \) (1) (arranged in order of ascending energy) for the initial state \( \langle n \rangle = 0, \langle \mathcal{M} = \mathcal{M} \rangle \rangle \), an intermediate ‘burst’ time (when \( \langle \hat{n} \rangle \) is maximal), and in the long-time limit. At the burst many eigenstates are involved. The steady state occupation highlights the potential structure above/below the phase transition (see section 3.2 in the main text). (Parameters as above, but \( \bar{n} = 0.1 \) for bottom right panel.)

thermodynamic calculations (see [19, 20] and (3)) which cannot be captured by the RWA dynamics.
the RWA terms and the particular damping mechanism for the RWA together with the HP data are included as well\(^4\). While in the normal regime the low energy sector is dominantly occupied throughout the time evolution, this changes drastically for \(\lambda > \lambda_c\), particularly for times around the first emission peak. There, a large number of eigenstates participate substantially. Asymptotically, the probabilities \(P_{\pm}\) show a near-degeneracy of pairs of eigenstates which is reminiscent of the exact degeneracy in the thermodynamic limit according to the HP treatment. As discussed above, finite size effects induce correction terms to the HP Hamiltonians \(H^\pm\) and lift their degeneracy. How many pairs of near-degenerate eigenstates occur, is a measure of the height of the barrier between the potential minima at \((\pm \sqrt{2/\kappa}, \pm \sqrt{2/\beta})\). To illustrate the appearance of these near-degenerate states, in figure 2(b) data are shown at a slightly elevated temperature \(n = 0.1\). Note that one has also finite occupations in some higher lying eigenstates due to the small but finite value of the damping parameter \(\kappa\). Apparently, the data for the RWA model strongly deviate from these findings for \(\lambda > \lambda_c\). In particular, in the stationary limit the system is not relaxed to the lowest energy eigenstate but to a state close to the ground state of the uncoupled system.

3.3. Characteristics of the emission burst: is it superradiance?

In this subsection we will argue that the first pronounced peak in the cavity occupation of the damped Dicke model carries signatures of the superradiant free-space burst of emitted light and reflects its characteristic features. These are the crossover in the time dependence of the emission rate from an

\(^4\) Far below the critical coupling the probability distribution with and without RWA is obviously very similar.
exponential decay for a single atom to a peak, where for larger $N$ the height of the peak grows as $N^2$, its temporal width decreases as $N^{-1}$ and the integrated emission is proportional to the total initial excitation energy and, hence, to $N$. We emphasize that it is by no means clear that scaling properties of the observable $\langle \hat{n} \rangle$ are related to those of the actual energy loss $E_{\text{loss}} = d \langle \hat{H}_{\text{Dicke}} \rangle / dt$. In fact, asymptotically for any finite atom-field coupling one has $\langle \hat{n} \rangle \to 0$, while $E_{\text{loss}} \to 0$, a feature which always applies, independent of the way cavity dissipation is modeled in the non-equilibrium dynamics. Hence, one can only expect to find correspondences on a transient time scale, where the burst occurs and on which we concentrate here.

These scaling properties are modified when one considers a fixed value of the parameter $\lambda$ upon increasing $N$, the typical scenario for the analysis of the phase transition in the Dicke model. Then, the rescaled coupling constant between a single atomic excitation and the field mode is $g = \lambda / \sqrt{N}$ so that the emission rate due to a single excitation is reduced $\propto N^{-1}$ for a larger number of atoms. Accordingly, we expect the following characteristics of the light emission: (i) for $N$ independent atoms the radiation follows an exponential decay from a fixed (N-independent) value over a time scale growing linearly with $N$; (ii) for a superradiant burst of $N$ cooperating atoms a peak with height $\propto N$ and constant width $\propto N^0$ emerges.

Results for the mesoscopic dissipative Dicke model are depicted in figure 3, where we consider mean cavity occupation and the dissipative energy loss of the system for increasing number of atoms. Of course, according to what we discussed above, for long times the energy dissipated by radiation $E_{\text{loss}} \to 0$, while the field occupation $\langle \hat{n} \rangle$ approaches a finite value. These deviations appear on time scales of order $1/\kappa$, whereas on shorter times, particularly in the regime where the first peak appears, the mean occupation qualitatively reflects indeed the dynamics of the energy loss.

For a more detailed study of the relation between intra-cavity field and emission an input-output approach can be used, as done in [29] for the weak-excitation (HPA) domain. For both sub- and supercritical coupling, we find an increase of the height of the first peak with growing $N$, while its width remains basically constant. Quantitatively, for the $N$-dependence of peak height and width (defined here as the left-sided half width at half maximum value) the expected scaling with $N$ and $N^3$, respectively, is obtained (insets). These findings do not only apply to the resonant case, where the cavity energy equals the energy needed for atomic excitations, but also to the off-resonant situation as checked by us numerically. In fact, the discussed dynamical features are not very sensitive to weak detunings. This is in stark contrast to the coherent dynamics in absence of dissipation, where complex collapse-revival dynamics with domains of regularity and quantum chaos occur [22]. We enter this dynamical range for extremely weak system-bath coupling as well.

In conclusion, our findings strongly suggest that the physics of the free-space superradiant burst can indeed be recovered in the damped dynamics of the Dicke model even for a moderate number of atoms. The scaling characteristics of the peak will allow for a wide range of damping rates, where friction is strong enough to suppress coherent collapse-revival dynamics but weak enough to achieve prominent radiation bursts with substantial peak cavity occupations. This suggests that signatures of superradiance are observable in realizations with superconducting circuits which are operated in the mesoscopic regime with only moderate $N$.

4. Monitoring collective dynamics

4.1. Dicke dynamics in phase space

Our discussion of the dynamics of the dissipative Dicke model has so far been focused on a single observable, the mean cavity occupation $\langle \hat{n} \rangle$, which is certainly not the only way, to gain experimental insight into the dynamics. A more complete characterization of the cavity dynamics is offered for example by the (reduced) Wigner density of the cavity degree of freedom, which is routinely measured for stripline resonator systems (see [30] for some recent examples).

In particular, in the thermodynamic limit $N \to \infty$ of a nearly classical spin, we may expect the Wigner density, which has a simple classical interpretation, to be exceptionally suitable. One question of interest is to distinguish sub- and superradiant coupling without waiting for the relaxation of the system to its stationary state. That way, such a distinction would be immune to the danger that the long-time limit is influenced or covered by extraneous terms, not present in the Dicke-Hamiltonian (e.g. non-homogenous coupling [15, 16]) or additional or different damping mechanism (e.g. atom decay [11]). In fact, we find that the Wigner density dynamics shows clear signatures, reflecting the contrast between a unique mean field solution for the ground state below the critical coupling and two equivalent solutions above, already for short times and a small number of atoms.

In figure 4(a) we show the reduced Wigner density of the cavity (in dimensionless coordinates $(X_a, P_a)$) well above the critical coupling at various times. The Wigner function of the initial state at $t = 0$ is a Gaussian peak centered at $(X_a = 0, P_a = 0)$. It starts to spread out in two arms in the positive and negative $X_a$ direction and evolves into the structure shown in figure 4(aiii) at the time of the burst, $t = t_{\text{burst}}$. A rich fine structure of complex interference patterns remains for some time, while substantial weight is assembling in two peaks (see (a)iiii at $t = 2 t_{\text{burst}}$). These peaks rotate around and finally slowly relax towards two final positions in phase space (see (a)iv), which correspond to the two equivalent solutions of the HPA in the $N \to \infty$ limit.

The spin-dynamics can similarly be visualized by another quasi probability distribution, the Husimi function, relying on the concept of spin-coherent states. We show the final state’s reduced spin-Husimi function in the phase space spanned by angular variables $(\phi, \theta)$, which indicate the ‘direction’ of the spin vector, in figure 4(b) for the same superradiant coupling as in (a). As for the cavity degree of freedom, we find a two peak structure for spin values corresponding to the two equivalent solutions of the HP result, see (3).
Dynamics in phase space. (a) Reduced Wigner density of the cavity above the critical coupling ($\lambda = 1.2\lambda_c$): starting from a ground state, $i$, the spin system excites the cavity. Occupation oscillates around $ii$ at $t = t_{\text{burst}}$ and $iii$ at $t = 2t_{\text{burst}}$, and finally relaxes to two minima positions $iv$. (b) The Husimi function of the spin degree of freedom in the stationary state, which shows a similar relaxation towards two final peaks. The spin is then directed towards two opposite points on the southern hemisphere (see the equator indicated by the white circle, while the south pole ($\bar{\vartheta} = \pi - \theta \equiv 0$, corresponding to $|M = -J|$) is in the center of the plot. (c) The spin-reduced $M_z = -J$ component of the cavity Wigner density relaxes to one of the stationary state positions. Below the critical coupling ($\lambda = 0.2\lambda_c$) the cavity Wigner density starting from the ground state, see $a(i)$, undergoes a breathing-type oscillation, $d(i)$ at $t = t_{\text{burst}}$, and finally relaxes to a final peak close to the ground state $d(iii)$ (other parameters: $N = 6$, $\kappa/\omega = 1/10$, $\bar{n} = 0$).

For subcritical coupling the initial central Gaussian peak broadens (see figure 4(d)ii for $\lambda = 0.2\lambda_c$ and $t = t_{\text{burst}}$) and re-sharpens in a breathing-type oscillation, which is damped out to relax to a final Gaussian peak centered around the unique minimum $(0, 0)$ (see figure 4(d)ii).

The long-time behavior observable in figures 4(a)iii, iv and (b) can be intuitively understood in the HP picture as damped dynamics well within the two equivalent parabolic potentials in cavity and spin variables. For the short-time behavior, however, the HPA is not applicable. To explain the observed splitting of the central peak into two arms (figure 4(a)ii), one can employ the representation of the Hamiltonian in a $J^z$ basis yielding the shifted coupled parabolas of equation (7). Starting in the initial state, which is not a $J^z$ eigenstate, each $J^z$ component (which—as a cavity ground state—is a centered Gaussian in $x$-space) evolves in its correspondingly shifted parabola and starts to move into positive or negative $x$-direction. According to this picture, we expect, e.g. the $M_z = \pm J$ component to start to move into positive/negative $x$-direction. Indeed, this behavior is observed in the corresponding spin-projected Wigner density of the cavity in figure 4(c) (same coupling and time as in (a) ii). This argument has so far neglected the coupling between the motion in the various parabolas induced by the $J^z$ term, which causes transitions and interference structures, which become prominent, when the various component-wavepackets meet at a crossing of the parabolas.

To go beyond these limitations, an alternative formulation of the Dicke-Hamiltonian [31] may offer some insight. For this purpose, one introduces collective operators $\hat{a}^\dagger, \hat{a}$ with $\hat{a} = \hat{a} + 2\lambda/(\omega\sqrt{N})\hat{J}^1$ so that

$$\hat{H}_{\text{Dicke}} = \omega\hat{a}^\dagger \hat{a} - \frac{4\lambda^2}{N\omega} (\hat{J}^z)^2 + \epsilon \hat{J}^z. \quad (7)$$

In a $J^z$ eigenbasis, Dicke physics can then be visualized as governed by a set of shifted parabolas with nearest neighbor coupling described by the $\epsilon J^z$ term. Starting for instance from the highly excited $\hat{J}^z$ eigenstate $|n = 0, M = +N/2\rangle$, the corresponding quantum dynamics includes Landau–Zener-type of transitions through a multitude of avoided level crossings. Details of the corresponding time evolution will be discussed elsewhere.

4.2. Indicators of sub/superradiant coupling

The concept of a phase transition applies to a system in the thermodynamic limit ($N \to \infty$), which after a long time ($t \to \infty$) is driven to equilibrium by a vanishingly weak coupling to an environment ($\kappa \to 0$). For that case, the mean cavity occupation, $\langle \hat{n} \rangle / J$, is an order parameter for the Dicke model, i.e., an observation of its value clearly indicates whether the coupling is sub- or superradiant, $\lambda \gtrless \lambda_c$. In our case, however, we are interested in an observable distinguishing between the weak and strong coupling case for a mesoscopic ($N \gtrsim 10 - 100$) Dicke aggregate with finite
coupling to the environment on short to intermediate time scales. The latter requirement is due to the presence of additional physical processes in any actual experimental realization, which our description of the dynamics of the damped Dicke problem does not take properly into account as discussed in the introduction. It is, therefore, advisable to search for alternative observables, which may in this scenario be better indicators of sub- or superradiant coupling than other observables like the Wigner density. The investigation of the dynamics of the Wigner density has revealed one distinctive difference between the sub- and superradiant case already apparent after short times: namely, a substantial fragmentation of the Wigner density into two parts, oscillating and finally relaxing to the two equivalent minima of the superradiant case. This is, for instance, reflected in the curvature of the Wigner density (in $X_\alpha$-direction)

$$\frac{d^2}{dX^2} W(X_\alpha, P_\alpha)|_{X_\alpha=0} = -\frac{4}{\pi} \langle \hat{\Pi}_n \hat{P}^2 \rangle$$

at the center of the cavity phase space $X_\alpha = 0 = P_\alpha$, where the second expression containing the parity operator for the cavity degree of freedom, $\hat{\Pi}_n$, follows from the definition of the Wigner density.

Note, that this observable, which can for example be gained from a subset of measurements of the full Wigner density, is but one example of a higher order correlation function of cavity operators. Other combinations, also mixed cavity-spin correlators may be of similar or in other parameter regimes even of superior use for deducing the coupling strength from observations of the short-time dynamics. In figure 5, we demonstrate that the sign of the new observable can indeed give a clear indication of sub- or superradiant coupling strength after a short observation time of the order of the inverse damping rate, $\kappa t \sim 1$. In the right column the value of the observables at an intermediate measurement time, $\kappa t = 3$ (symbols), is contrasted to its stationary limit with ($\kappa/\omega = 1/10$—magenta), and without damping (black solid line). The order parameter observable, $\langle \hat{n} \rangle/J$ (upper right panel), while distinctive in the thermodynamic limit (black dashes), is less suited as an indicator of super/subradiant coupling than other observables like the Wigner curvature (lower right panel), which shows a distinctive sign change at (approximately) the critical coupling strength.

The latter requirement is due to the presence of additional physical processes in any actual experimental realization, which our description of the dynamics of the damped Dicke problem does not take properly into account as discussed in the introduction.

The left column shows the dynamics of the two observables for various values of the coupling ($\lambda/\lambda_c = 0.8, 0.96, 0.99, \lambda/\lambda_c = 1.12, \lambda/\lambda_c = 1.28$) for times on the order of the inverse damping rate, $\kappa t \sim 1$. In the right column the value of the observables at a.

Figure 5. Observables as indicators of normal or superradiant phase. The order parameter, $\langle \hat{n} \rangle/J$, of the thermodynamic phase transition does not indicate clearly if the coupling $\lambda$ is above or below the critical value $\lambda_c$ when the dynamics of the damped (mesoscopic, $N = 6$) Dicke model is observed on short time scales only (upper row). Other observables, like the curvature of the (cavity-)Wigner density, $\langle \hat{\Pi}_n \hat{P}^2 \rangle$, which quickly goes to positive (negative) values for coupling values below (above) the critical value, can give clearer signatures (lower row).
5. Conclusion

The non-equilibrium dynamics of the Dicke model for a moderate number of atoms has been studied starting from a highly excited state. For this purpose, we employed a quantum optical modeling of dissipation which was shown to provide a fairly accurate description also in the long time limit at least for certain observables. While improved descriptions either by means of perturbative or non-perturbative treatments are highly desirable, they are associated with problematic subtleties that arise already when the number of atoms grows moderately and/or when the initial state is prepared far from equilibrium. We show that within such a minimal model as employed here precursors of both phenomena related to collective behavior occur, namely, superradiant light emission and a superradiant phase. The former appears as a collective behavior of the atoms in the transient domain and can be identified by characteristic scaling properties. The latter emerges on somewhat longer time scales (but still far from the asymptotic regime) beyond the critical coupling as a specific feature in phase space which displays a local instability. In terms of energy surfaces, this instability may be interpreted as a saddle between the two energy minima which capture the ground state properties of the Dicke model in the thermodynamic limit. Accordingly, the detection of a proper phase space correlation gives access to the transition from individual to collective quantum dynamics on transient time scales beyond the short-time but below the asymptotic domain. We note that Wigner functions for complex quantum states consisting of a few up to a relatively large number of photons have been probed recently in experiments with superconducting devices [30]. Since the features we report here already exist for a number of atoms in the mesoscopic regime, our findings may be of relevance for these activities to study Dicke physics.

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