Hyers–Ulam–Rassias Stability of Derivations on Hilbert $C^*$-Modules

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Abstract. Consider the functional equation $E_1(f) = E_2(f)$ in a certain framework. We say a function $f_0$ is an approximate solution of $(E)$ if $E_1(f_0)$ and $E_2(f_0)$ are close in some sense. The stability problem is whether or not there is an exact solution of $(E)$ near $f_0$.

In this paper, the stability of derivations on Hilbert $C^*$-modules is investigated in the spirit of Hyers–Ulam–Rassias.

1. Introduction

One of the interesting questions in the theory of functional equations is the following (see [GRU]):

When is it true that a function which approximately satisfies a functional equation $E$ must be close to an exact solution of $E$?

If there exists an affirmative answer we say that the equation $E$ is stable.

The first stability problem was raised by S. M. Ulam during his talk before a Mathematical Colloquium at the University of Wisconsin in 1940 [ULA]:

Given a group $G_1$, a metric group $(G_2, d)$ and a positive number $\epsilon$, does there exist a number $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$ then there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(x), T(x)) < \epsilon$ for all $x \in G_1$?

Ulam’s problem was partially solved by D. H. Hyers in 1941 in the context of Banach spaces with $\delta = \epsilon$ in the following form [HYE]:

Suppose that $X_1$ and $X_2$ are Banach spaces and $f : X_1 \to X_2$ satisfies the following condition: If there is $\epsilon > 0$ such that $\|f(x + y) - f(x) - f(y)\| < \epsilon$ for all $x, y \in X_1$, then there is a unique additive mapping $T : X_1 \to X_2$ defined by $T(x) = \lim_{n \to \infty} f\left(\frac{2^n x}{2^n}\right)$ such that $\|f(x) - T(x)\| < \epsilon$ for all $x \in X_1$.

Th. M. Rassias [RAS] extended Hyers’ theorem in the following form where Cauchy difference is allowed to be unbounded:

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Assume that $X_1$ and $X_2$ are real normed spaces with $X_2$ complete, $f : X_1 \rightarrow X_2$ is a mapping such that for each fixed $x \in X_1$ the mapping $t \mapsto f(tx)$ is continuous on $\mathbb{R}$, and let there exist $\varepsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon (\|x\|^p + \|y\|^p)$$

for all $x, y \in X_1$. Then there exists a unique linear mapping $T : X_1 \rightarrow X_2$ such that

$$\|f(x) - T(x)\| \leq \frac{\varepsilon \|x\|^p}{1 - 2p^{-1}}$$

for all $x \in X_1$.

This result is still valid in the case where $p < 0$ by the same approach given in [RAS1] if we assume that $\|0\|^p = \infty$. In 1990, Th. M. Rassias during the 27th International Symposium on Functional Equations asked the question whether his theorem can be proved for $p \geq 1$. In 1991, Z. Gajda [GAJ] following the same approach as in [RAS1] provided an affirmative solution to this question for $p > 1$. Using Hyers’ method, indeed, $T(x)$ is defined by $\lim_{n \to \infty} 2^{-n}f(2^n x)$ if $p < 1$, and $\lim_{n \to \infty} 2f(2^{-n}x)$ if $p > 1$. It is shown that there is no analogue of Th. M. Rassias’ result for $p = 1$ (see [GAJ] and [R-S]). This phenomenon of stability that was introduced by Th. M. Rassias [RAS1] is called Hyers–Ulam–Rassias stability. Thus the Hyers–Ulam stability will be regarded as a special case of the Hyers–Ulam–Rassias stability. A number of Rassias type results related to the stability of various functional equations are presented in [RAS2, RAS3].

In 1992, a generalization of Rassias’ theorem was obtained by Gavruta as follows [GAV].

Suppose $(\mathcal{G}, +)$ is an abelian group, $X$ is a Banach space and the so-called admissible control function $\varphi : \mathcal{G} \times \mathcal{G} \rightarrow [0, \infty)$ satisfies

$$\varphi(x, y) := \frac{1}{2} \sum_{n=0}^{\infty} 2^{-n} \varphi(2^n x, 2^n y) < \infty$$

for all $x, y \in \mathcal{G}$. If $f : \mathcal{G} \rightarrow X$ is a mapping with

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in \mathcal{G}$, then there exists a unique mapping $T : \mathcal{G} \rightarrow X$ such that $T(x + y) = T(x) + T(y)$ and $\|f(x) - T(x)\| \leq \varphi(x, x)$ for all $x, y \in \mathcal{G}$.

There are four methods in the study of stability of functional equations. The first method is the direct method in which one uses an iteration process producing the so-called Hyers type sequences [HYE]. Another method is based on sandwich theorems which are generalizations of the Hahn-Banach separation theorems; cf. [PAL]. The third technique focuses on using invariant means; cf. [SZE], and the foundation of the forth method is fixed point techniques; cf. [C-R, I-R]. The reader is referred to [CZE, H-I-R, JUN] and references therein for further information on stability.

The notion of Hilbert $C^*$-module is a generalization of the notion of Hilbert space.

Let $\mathcal{A}$ be a $C^*$-algebra and $\mathcal{M}$ be a linear space which is a left $\mathcal{A}$-module with a scalar multiplication satisfying $\lambda(xa) = x(\lambda a) = (\lambda x)a$ for $x \in \mathcal{M}$, $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$. The space $\mathcal{M}$ is called a pre-Hilbert $\mathcal{A}$-module or inner product $\mathcal{A}$-module if there exists an inner product $(\cdot, \cdot) : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{A}$ with the following properties:

(i) $(x, x) \geq 0$; and $(x, x) = 0$ iff $x = 0$;
(ii) $(\lambda x + y, z) = \lambda(x, y) + (y, z)$;
(iii) $(ax, y) = a(x, y)$;
(iv) $\langle x, y \rangle^* = \langle y, x \rangle$.

$M$ is called a (left) $H$ Hilbert $A$-module if it is complete with respect to the norm $\| x \| = \| \langle x, x \rangle \|^{1/2}$.

(i) Every inner product space is a left Hilbert $C$-module.

(ii) Let $A$ be a $C^*$-algebra Then every closed left ideal $I$ of $A$ is a Hilbert $A$-module if one defines $\langle a, b \rangle = ab^*$ $\langle a, b \in I \rangle$.

Assume that $N$ is another Hilbert $A$-module. Recall that a mapping $T : M \to N$ is said to be adjointable if there exists a mapping $S : N \to M$ such that $\langle Tx, y \rangle = \langle x, Sy \rangle$ for all $x \in M, y \in N$. The mapping $S$ is denoted by $T^*$ and called the adjoint of $T$. If $T$ is adjointable, then it is $A$-linear and automatically continuous; cf. [LAN].

Following [L-X] a linear mapping $d : M \to M$ is called a derivation on the Hilbert $C^*$-module $M$ if it satisfies the condition $d(\langle x, y \rangle z) = \langle d(x), y \rangle z + \langle x, d(y) \rangle z + \langle x, y \rangle d(z)$ for every $x, y, z \in M$. It is clear that every adjointable mapping $T$ satisfying $T^* = -T$ is a derivation. The converse is not true in general. For example, let $u_0$ be a bounded linear operator acting on a Hilbert space $H$ of dimension greater than one such that $u_0^* = -u_0$ and $u_0$ is not an element of the center of $B(H)$ (e.g. consider a fixed vector $\xi \in H$ and put $u_0(\xi) = 2i(\xi, \xi)\xi$, $(\xi \in H)$). Obviously, the mapping $d : B(H) \to B(H)$ defined by $d(v) = u_0v - v u_0$ is an ordinary $*$-derivation on $B(H)$ and so it is a derivation on $B(H)$ regarded as a Hilbert $C^*$-module over itself. If $d$ were a $B(H)$-module map, in the sense that $d(vw) = vd(w)$ for all $v, w \in B(H)$, then $u_0(vw) - (vw)u_0 = v(u_0w - wu_0)$ and so $u_0vw = v u_0$ for each $v, w \in B(H)$. In particular, $u_0v = vu_0$ for all $v \in B(H)$ which is a contradiction. Since adjointable mappings are module map, we infer that $d$ is not adjointable.

Recently, several extended notions of derivations such as $(\sigma - \tau)$-derivations and generalized derivations have been treated in the Banach algebra theory (see [B-V]). In addition, the stability of these derivations in the spirit of Hyers–Ulam–Rassias has extensively studied by many mathematicians; see [MOS1] [MOS2] [PAR1] [PAR2]. In this paper we establish the stability of derivations on Hilbert $C^*$-modules. Throughout the paper, $M$ denotes a Hilbert module over a $C^*$-algebra $A$.

2. Main results

Recently, the stability of several mappings on Hilbert $C^*$-modules was investigated (see [AMY] [MOS3]). Using some ideas from [C-R] we investigate the stability of derivations on Hilbert $C^*$-modules. Our results may be regarded as an extension of those of [PAR2] when we consider a unital $C^*$-algebra $A$ as an $A$-bimodule via its multiplication. We start our work with a known fixed point theorem.

**THEOREM 1** (The alternative of fixed point). Suppose $(S, d)$ is a complete generalized metric space and $J : S \to S$ is a strictly contractive mapping with the Lipschitz constant $L$. Then, for each given element $x \in S$, either (A1) $d(J^n x, J^{n+1} x) = \infty$

for all $n \geq 0$, or

(A2) There exists a natural number $n_0$ such that:

(A20) $d(J^n x, J^{n+1} x) < \infty$, for all $n \geq n_0$;

(A21) The sequence $\{J^n x\}$ is convergent to a fixed point $y^*$ of $J$;
(#A22) $y^*$ is the unique fixed point of $J$ in the set $U = \{y \in S : d(J^{n_0}x, y) < \infty\};$
(#A23) $d(y, y^*) \leq \frac{1}{1-L}d(y, Jy)$ for all $y \in U$.

The following lemma gives us a useful strictly contractive mapping.

**Lemma 2.** Suppose that $X$ is a Banach space, $0 \leq L < 1$ and $\lambda \geq 0$ are given numbers and $\psi(x) : X \to [0, \infty)$ has the property
$$\psi(x) \leq \lambda L \psi\left(\frac{x}{\lambda}\right),$$
for all $x \in X$. Assume that $S := \{g : X \to X : g(0) = 0\}$ and the generalized metric $d$ on $S$ is defined by
$$d(g, h) = \inf\{c \in (0, \infty) : \|g(x) - h(x)\| \leq c\psi(x), \forall x \in X\}.$$
Then the mapping $J : S \to S$ given by $(Jg)(x) := \frac{1}{\lambda}g(\lambda x)$ is an strictly contractive mapping.

**Proof.** It is easy to see that $(S, d)$ is complete. For arbitrary elements $g, h \in S$ we have
$$d(g, h) < c \Rightarrow \|g(x) - h(x)\| \leq c\psi(x), \quad x \in X$$
$$\Rightarrow \left\|\frac{1}{\lambda}g(\lambda x) - \frac{1}{\lambda}h(\lambda x)\right\| \leq \frac{1}{\lambda}c\psi(x), \quad x \in X$$
$$\Rightarrow \left\|\frac{1}{\lambda}g(\lambda x) - \frac{1}{\lambda}h(\lambda x)\right\| \leq Lc\psi(x), \quad x \in X$$
$$\Rightarrow d(Jg, Jh) \leq Lc.$$
Therefore
$$d(Jg, Jh) \leq Ld(g, h), \quad g, h \in S.$$ Hence $J$ is a strictly contractive mapping on $S$ with the Lipschitz constant $L$. □

**Theorem 3.** Let $\varphi : M^5 \to [0, \infty)$ be a control function with the property
$$\lim_{n \to \infty} \frac{\varphi(2^n x, 2^n y, 2^n u, 2^n v, 2^n w)}{2^n} = 0,$$
for all $x, y, u, v, w \in M$. Suppose that $f : M \to M$ is a mapping satisfying $f(0) = 0$ and
$$\|f(\mu x + y) - \mu f(x) - f(y) + f(\langle u, v \rangle w) - \langle f(u), v \rangle w - \langle u, f(v) \rangle w - \langle u, v \rangle f(w)\| \leq \varphi(x, y, u, v, w),$$
for all $\mu \in T = \{z \in \mathbb{C} : |z| = 1\}$ and $x, y, u, v, w \in M$. Assume that there exists $0 \leq L_0 < 1$ such that the mapping $\psi(x) = \varphi\left(\frac{x}{2}, \frac{x}{2}, 0, 0, 0\right)$ has the property
$$\psi(x) \leq 2L_0\psi\left(\frac{x}{2}\right),$$
for all $x \in M$.
Then there exists a unique derivation $T_0 : M \to M$ such that
$$\|f(x) - T_0(x)\| \leq \frac{L_0}{1-L_0}\psi(x),$$
for all $x \in M$. 
PROOF. Setting $\mu = 1, y = x$ and $u = v = w = 0$ in (2.1) we obtain

\[(2.3) \quad \|f(2x) - 2f(x)\| \leq \varphi(x, x, 0, 0, 0).\]

It follows from (2.2) and (2.3) that

\[\|\frac{1}{2}f(2x) - f(x)\| \leq \frac{1}{2} \varphi(2x) \leq L_0 \psi(x).\]

Hence $d(f, J_0 f) \leq L_0 < \infty$, where the mapping $J_0$ is defined on $S := \{g : \mathcal{M} \to \mathcal{M} : g(0) = 0\}$ by $(J_0 g)(x) := \frac{1}{2} g(2x)$ as in Lemma 2. Applying the fixed point alternative we deduce the existence of a mapping $T_0 : \mathcal{M} \to \mathcal{M}$ such that $T_0$ is a fixed point of $J_0$ that is $T_0(2x) = 2T_0(x)$ for all $x \in \mathcal{M}$. Since $\lim_{n \to \infty} d(J_0^n f, T_0) = 0$ we easily conclude that

\[\lim_{n \to \infty} \frac{f(2^n x)}{2^n} = T_0(x),\]

for all $x \in \mathcal{M}$. Note that the sequence $\{\frac{f(2^n x)}{2^n}\}$ is the Hyers sequence when one use the direct method in establishing stability.

The mapping $T_0$ is the unique fixed point of $J_0$ in the set $U = \{g \in S : d(f, g) < \infty\}$. Hence $T_0$ is the unique fixed point of $J_0$ such that $\|f(x) - T_0(x)\| \leq K \varphi(x)$ for some $K > 0$ and for all $x \in \mathcal{M}$. Again, by applying the fixed point alternative theorem we obtain

\[d(f, T_0) \leq \frac{1}{1 - L_0} d(f, J_0 f) \leq \frac{L_0}{1 - L_0},\]

and so

\[\|f(x) - T_0(x)\| \leq \frac{L_0}{1 - L_0} \varphi\left(\frac{x}{2}, \frac{x}{2}, 0, 0, 0\right),\]

for all $x \in \mathcal{M}$.

It follows from (2.1) that

\[(2.4) \quad \|f(\mu x + y) - \mu f(x) - f(y)\| \leq \varphi(x, y, 0, 0, 0).\]

Let us replace $x$ and $y$ in (2.4) by $2^n x$ and $2^n y$, respectively, and divide the both sides by $2^n$. Passing the limit as $n \to \infty$ we get

\[T_0(\mu x + y) = \mu T_0(x) + T_0(y),\]

for all $\mu \in \mathbb{D}$ and all $x, y \in \mathcal{M}$.

Next, let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) and let $K$ be a natural number greater than $4|\lambda|$. Then $|\frac{1}{K^n}| < \frac{1}{3} < 1 - \frac{2}{3} = 1/3$. By Theorem 1 of [K-P], there exist three numbers $\mu_1, \mu_2, \mu_3 \in \mathbb{T}$ such that $3\frac{\lambda}{K^n} = \mu_1 + \mu_2 + \mu_3$. By the additivity of $T_0$ we get $T_0(\frac{1}{3} x) = \frac{1}{3} T_0(x)$ for all $x \in \mathcal{M}$. Therefore,

\[T_0(\lambda x) = T_0\left(\frac{K}{3} \cdot 3 \cdot \frac{\lambda}{K} x\right) = \frac{K}{3} T_0\left(3 \cdot \frac{\lambda}{K} x\right)\]

\[= \frac{K}{3} T_0(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{K}{3} (T_0(\mu_1 x) + T_0(\mu_2 x) + T_0(\mu_3 x))\]

\[= \frac{K}{3} (\mu_1 + \mu_2 + \mu_3) T_0(x)\]

\[= \lambda T_0(x),\]

for all $x \in \mathcal{M}$. So that $T_0$ is $\mathbb{C}$-linear.
Then there exists a unique derivation
\[ T(x) = \frac{\varphi(0,0,2^n u, 2^n v, 2^n w)}{2^n - 1} \]
for all \( x \in M \). Hence \( T_0 \) is a derivation on \( M \).

As a consequence of Theorem 3, we show the Rassias stability of derivations on Hilbert \( C^* \)-modules.

**Corollary 4.** Suppose that \( p \in [0, 1) \), \( \alpha, \beta, \gamma > 0 \) and \( f : M \to M \) is a mapping satisfying \( f(0) = 0 \) and
\[
\| f(mx + y) - \mu f(x) - f(y) + f(u, v)w - \langle u, f(v)w \rangle - f(\langle u, v \rangle w) - \langle u, f(v)w \rangle - \langle u, v \rangle f(w) \| \leq \alpha + \beta \| x \|^p + \| y \|^p + \| u \|^p + \| v \|^p + \| w \|^p + \gamma \| x \|^p + \| y \|^p + \| u \|^p + \| v \|^p + \| w \|^p,
\]
for all \( \mu \in \mathbb{T} \) and all \( x, y, u, v, w \in M \). Then there exists a unique derivation \( T_0 : M \to M \) such that
\[
\| f(x) - T_0(x) \| \leq \frac{\alpha + \beta 2^{1-p} \| x \|^p + \gamma 2^{-p} \| x \|^p}{2^{1-p} - 1},
\]
for all \( x \in M \).

**Proof.** Put \( \varphi(x, y, u, v, w) = \alpha + \beta \| x \|^p + \| y \|^p + \| u \|^p + \| v \|^p + \| w \|^p + \gamma \| x \|^p + \| y \|^p + \| u \|^p + \| v \|^p + \| w \|^p \), and let \( L_0 = \frac{1}{2^{1-p}} \) in Theorem 3. Then \( \psi(x) = \alpha + 2^{1-p} \| x \|^p + 2^{-p} \gamma \| x \|^p \) and there exists a derivation \( T_0 \) with the required property. \( \square \)

**Theorem 5.** Let \( \varphi : M^5 \to [0, \infty) \) be a control function with the property
\[
\lim_{n \to \infty} 2^n \varphi(2^{-n} x, 2^{-n} y, 2^{-n} u, 2^{-n} v, 2^{-n} w) = 0,
\]
for all \( x, y, u, v, w \in M \). Suppose that the mapping \( f : M \to M \) is a mapping satisfying \( f(0) = 0 \) and
\[
\| f(mx + y) - \mu f(x) - f(y) + f(\langle u, v \rangle w) - \langle f(u), v \rangle w - \langle u, f(v)w \rangle - \langle u, v \rangle f(w) \| \leq \varphi(x, y, u, v, w),
\]
for all \( \mu \in \mathbb{T} \) and \( x, y, u, v, w \in M \). Assume that there exists \( 0 \leq L_1 < 1 \) such that the mapping \( \psi(x) = \varphi \left( \frac{x}{2}, \frac{y}{2}, 0, 0, 0 \right) \) has the property
\[
\psi(x) \leq \frac{1}{2} L_1 \psi(2x),
\]
for all \( x \in M \).

Then there exists a unique derivation \( T_1 : M \to M \) such that
\[
\| f(x) - T_1(x) \| \leq \frac{1}{1 - L_1} \psi(x),
\]
for all \( x \in M \).
Hence \( d(\text{fixed point of } J) : M \rightarrow \) for all \( x \in \{ \infty \} \) and so the alternative we deduce the existence of a mapping we easily conclude that \( \psi \) for all \( x \) property.

The mapping \( T_1 \) is the unique fixed point of \( J_1 \) in the set \( U = \{ g \in S : d(f,g) < \infty \} \). Hence \( T_1 \) is the unique fixed point of \( J_1 \) such that \( \| f(x) - T_1(x) \| \leq K \psi(x) \) for some \( K > 0 \) and for all \( x \in M \). Again, by applying the fixed point alternative theorem we obtain

\[
d(f,T_1) \leq \frac{1}{1-L_1} d(f,J_1f) \leq \frac{1}{1-L_1},
\]

and so

\[
\| f(x) - T_1(x) \| \leq \frac{1}{1-L_1} \varphi\left(\frac{x}{2} \cdot \frac{x}{2}, 0, 0, 0\right),
\]

for all \( x \in M \). The rest is similar to the proof of Theorem \( \text{III} \).

The following corollary is similar to Corollary \( \text{IV} \) for the case where \( p > 1 \).

**Corollary 6.** Suppose that \( p > 1 \), \( \beta, \gamma > 0 \) and \( f : M \rightarrow M \) is a mapping satisfying \( f(0) = 0 \) and

\[
\| f(\mu x + y) - \mu f(x) - f(y) + f(\langle u,v \rangle w) - \langle f(u),v \rangle w - \langle u,f(v) \rangle w \| \leq \beta(\| x \|^p + \| y \|^p + \| u \|^p + \| v \|^p + \| w \|^p) + \gamma \| x \|^{p/2} \| y \|^{p/2} \| u \|^{p/2} \| v \|^{p/2} \| w \|^{p/2},
\]

for all \( \mu \in \mathbb{T} \) and all \( x,y,u,v,w \in M \). Then there exists a unique derivation \( T_1 : M \rightarrow M \) such that

\[
\| f(x) - T_1(x) \| \leq \frac{2^{p-1} \beta}{2^{p-1} - 1} \| x \|^p,
\]

for all \( x \in M \).

**Proof.** Put \( \varphi(x,y,u,v,w) = \alpha + \beta(\| x \|^p + \| y \|^p + \| u \|^p + \| v \|^p + \| w \|^p) + \gamma \| x \|^{p/2} \| y \|^{p/2} \| u \|^{p/2} \| v \|^{p/2} \| w \|^{p/2}, \) and let \( L_1 = \frac{1}{\alpha} \) in Theorem \( \text{V} \). Then \( \psi(x) = 2^{1-p} \beta \| x \|^p + 2^{-p} \gamma \| x \|^p \) and there exists a derivation \( T_1 \) with the required property.

**Remark 7.** The case \( p = 1 \) remains unsolved.
References

[ACZ] J. Aczél, *A short course on functional equations*, D. Reidel Publ. Co., Dordrecht, 1987.

[AMY] M. Amyari, *Stability of C*-inner Products*, to appear in J. Math. Anal. Appl.

[B-M] C. Baak and M. S. Moslehian, *On the stability of J*-homomorphisms*, Nonlinear Analysis (TAM), 63 (2005), 42–48.

[B-V] M. Brešar and A. R. Villena, *The noncommutative Singer-Wermer conjecture and C*-derivations*, J. London Math. Soc. (2) 66 (2002), no. 3, 710–720.

[C-R] L. Cădariu and V. Radu, *Fixed points and the stability of Jensen’s functional equation*, J. Inequal. Pure Appl. Math. 4 (2003), no. 1, Article 4, 7 pp.

[CZE] S. Czerwik (ed.), *Stability of Functional equations of Ulam–Hyers–Rassias Type*, Hadronic Press, 2003.

[GAV] P. Gavruta, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. 184 (1994), 431–436.

[GRU] P. M. Gruber, *Stability of isometries*, Trans. Amer. Math. Soc., 245 (1978), 263–277.

[HYE] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. 27 (1941), 222–224.

[HI-R] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, Basel, Berlin, 1998.

[I-R] G. Isac and Th. M. Rassias, *Stability of Ψ-additive mappings: applications to nonlinear analysis*, Internat. J. Math. Math. Sci. 19 (1996), no. 2, 219-228.

[JUN] S.-M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, 2001.

[K-P] R. V. Kadison and G. K. Pedersen, *Means and convex combinations of unitary operators*, Math. Scan. 57 (1985), 249–266.

[LAN] E. C. Lance, *Hilbert C*-modules*, LMS Lecture Note Series 210, Cambridge University Press, 1995.

[L-X] X. Liu and T. Z. Xu, *Automatic continuity of derivations of Hilbert C*-modules*, J. Baoji College Arts Sci. Nat. Sci. 1995, no. 2, 14-17.

[M-M] M. Mirzavaziri and M. S. Moslehian, *Automatic continuity of σ-derivations in C*-algebras*, Proc. Amer. Math. Soc., in press, arXiv: math.FA/0508028

[MOS1] M. S. Moslehian, *Approximate (σ−r)-contractibility*, to appear in Nonlinear Funct. Anal. Appl., arXiv: math.FA/0501012

[MOS2] M. S. Moslehian, *Hyers–Ulam–Rassias stability of generalized derivations*, preprint, arXiv: math.FA/0501012

[MOS3] M. S. Moslehian, *Stability of adjointable mappings in Hilbert C*-modules*, preprint, arXiv: math.FA/0501139

[PAR1] C. Park, *Linear derivations on Banach algebras*, Nonlinear Funct. Anal. Appl. 9 (2004), 359–368.

[PAR2] C. Park *Homomorphisms between C*-algebras and linear derivations on C*-algebras*, to appear in Math. Inequ. Appl.

[RAS1] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. 72 (1978), 297–300.

[RAS2] Th. M. Rassias (ed.), *Functional Equations and Inequalities*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.

[RAS3] Th. M. Rassias(ed.), *Functional Equations, Inequalities and Applications* Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.

[R-S] Th. M. Rassias and P. Šemrl, *On the behavior of mappings which do not satisfy Hyers–Ulam stability*, Proc. Amer. Math. Soc. 114 (1992), 989–993.

[SZE] L. Székelyhidi, *Note on a stability theorem*, Canad. Math. Bull. 25 (1982), 500–501.

[ULA] S. M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1960.
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