VIRTUAL COMPTON SCATTERING OFF THE NUCLEON IN
THE LINEAR SIGMA MODEL

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Virtual Compton scattering off the nucleon has been studied in the one-loop approximation of the linear sigma model. The three generalized scalar polarizabilities of the nucleon have been calculated and compared with the existing theoretical predictions. In particular, we find that only two of the three scalar polarizabilities are independent observables.

1 Introduction

At low energies, real Compton scattering (RCS) off the nucleon has been discussed for quite some time in terms of the electric (α) and magnetic (β) polarizabilities, which depend on the excitation spectrum of the nucleon. The complementary, purely electromagnetic reaction of virtual Compton scattering (VCS), γ∗ + N → γ + N, promises to provide even more information about the structure of the nucleon. First, the polarizabilities in VCS become functions of the four–momentum transfer Q², and second, the virtual photon carries also a longitudinal polarization.

The formalism of the generalized polarizabilities of the nucleon was developed by Guichon, Liu and Thomas. The same authors gave a first estimate of the generalized polarizabilities of the proton in the framework of a nonrelativistic constituent quark model (CQM), and investigated a more detailed version of the model including recoil effects. Using an effective lagrangian model (ELM) Vanderhaeghen determined the Q² behaviour of α and β of the proton. Recently, the slopes of the electric and magnetic polarizability were predicted by Hemmert et al. in heavy baryon chiral perturbation theory (HBChPT) to order p³. The first experiments to measure the Q² evolution of the electric polarizability of the proton will soon provide data.

We have performed a one–loop calculation of the scalar polarizabilities of both nucleons in the linear sigma model (LSM) in the limit of an infinite sigma mass. The sigma model contains all symmetries relevant for hadron physics at low energies. The model is Lorentz, gauge and chirally invariant, in particular it obeys the PCAC relation.

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2 Definitions and Generalized Polarizabilities

The initial (final) photon of the VCS reaction is characterized by the polarization vector \( \epsilon^\mu = (0, \hat{\epsilon}) \) (\( \epsilon'^\mu = (0, \hat{\epsilon}') \)) and the four–momentum \( q^\mu = (\omega, \vec{q}) \) (\( q'^\mu = (\omega', \vec{q}') \)). We define the momentum transfer as \( Q^2 = -q^2 = \bar{q}^2 - \omega^2 \), where \( \bar{q} = |\vec{q}| \) is the absolute value of the three–momentum of the virtual photon. In the cm frame, the Lorentz vector of the incoming (outgoing) nucleon reads \( p^\mu = (E, -\vec{q}) \) (\( p'^\mu = (E', -\vec{q}') \)). For convenience we introduce the two variables

\[
\omega_0 = \omega(\omega' = 0) = m_N - E = -\frac{\bar{q}^2}{2m_N} + \mathcal{O}(\bar{q}^4), \quad (1)
\]

\[
Q_0^2 = Q^2(\omega' = 0) = 2m_N \left[ \sqrt{\bar{q}^2 + m_N^2} - m_N \right]. \quad (2)
\]

The expansion in (1) is valid if \( \bar{q} \) is small in comparison to the nucleon mass.

The low energy theorem states that the leading order terms of the scattering amplitude, in an expansion in \( \omega' \), are completely determined by the Born amplitude, which depends only on ground state properties of the nucleon. The excitations of the nucleon as described by the generalized polarizabilities are of relative order \( \omega'^2 \). These polarizabilities are given by the multipoles

\[
H_{NB}^{(\rho' L', \rho L) S}(\omega', \vec{q}) \text{ of the non Born amplitude}. \quad (3)
\]

In this notation \( \rho(\rho') \) indicates the type of the initial (final) photon (\( \rho = 0 \) : charge, \( \rho = 1 \) : magnetic, \( \rho = 2 \) : electric), and \( L(L') \) characterizes its angular momentum. The quantum number \( S \) distinguishes between the no spin–flip (\( S = 0 \)) and the spin–flip (\( S = 1 \)) contributions. In the Siegert limit, the electromagnetic multipoles have a well–defined dependence on the momenta of the real or virtual photons. Therefore the generalized polarizabilities have been defined as

\[
P^{(\rho' L', \rho L) S}(\vec{q}) = \frac{1}{\omega' L' q' L} H_{NB}^{(\rho' L', \rho L) S}(\omega', \vec{q}) \bigg|_{\omega'=0} \quad (\rho, \rho' = 0, 1), \quad (3)
\]

\[
\hat{P}^{(\rho' L', L) S}(\vec{q}) = \frac{1}{\omega' L' q' L + T} \hat{H}_{NB}^{(\rho' L', L) S}(\omega', \vec{q}) \bigg|_{\omega'=0}. \quad (3)
\]

The generalized polarizabilities are functions of \( \bar{q} \), or alternatively functions of \( Q_0^2 \) (see eq. (2)). The different treatment of the electric multipole for the virtual photon is due to the fact that it is related to the charge multipole in the Siegert limit. Therefore, only the difference of these two multipoles is an independent quantity, the mixed polarizability \( \hat{P} \) in eq. (3).

In the following we will restrict the discussion to the three scalar polarizabilities (\( S = 0 \)). Two of them generalize the electric and magnetic polariz-
ability to the virtual photon case,

\[ \alpha(Q_0^2) = -\frac{e^2}{4\pi} \sqrt{\frac{3}{2}} P^{(01,01)}_0(Q_0^2), \quad \beta(Q_0^2) = -\frac{e^2}{4\pi} \sqrt{\frac{3}{8}} P^{(11,11)}_0(Q_0^2), \quad (4) \]

the third one is a mixed polarizability, \( \hat{P}^{(01,10)}_0 \). These polarizabilities define the spin–independent part of the non Born amplitude to lowest order in \( \omega' \), \( \sqrt{m_N E_T^{\text{long}}} = \omega' \omega_0 \alpha(Q_0^2) \hat{\epsilon} \cdot \hat{q} + O(\omega'^2) \), \( \sqrt{m_N E_T^{\text{trans}}} = \left[ \omega' \hat{q} \cos \beta(Q_0^2) + \omega' \omega_0 \alpha(Q_0^2) - \frac{3e^2}{8\pi} \omega' \hat{q}^2 \hat{\epsilon} P^{(01,10)}_0(Q_0^2) \right] \hat{\epsilon} \cdot \hat{\epsilon} \\
- \omega' \hat{q} \beta(Q_0^2) \hat{\epsilon} \times \hat{\epsilon} \cdot \hat{q}' + O(\omega'^2) \). 

3 Results

Using the defining eqs. (5) we have determined the scalar polarizabilities for arbitrary \( Q_0^2 \). As has been shown in Fig. 1, both \( \alpha \) and \( \beta \) are underestimated at the real photon point (\( Q_0^2 = 0 \)). The corresponding experimental values are, in units of \( 10^{-4} \text{fm}^3 \), \( \alpha_{p}^{\text{exp}}(0) = 12.1 \pm 1.0 \), \( \beta_{p}^{\text{exp}}(0) = 2.1 \mp 1.0 \), \( \alpha_{n}^{\text{exp}}(0) = 12.6 \pm 2.5 \), \( \beta_{n}^{\text{exp}}(0) = 3.2 \mp 2.5 \). In general, we achieve a better description for \( \alpha(0) \) than for \( \beta(0) \). The shortcoming of the magnetic polarizability is due to the neglect of the \( \Delta \) resonance in the LSM.

Fig. 1 also compares our results with the predictions of the CQM and the ELM. The various models rely on quite different pictures of the nucleon. While in the LSM the excitations of the nucleon are given by pion–nucleon scattering states, the CQM describes the excitation spectrum by a number of resonances of the nucleon. The ELM contains both degrees of freedom in the form of effective lagrangians. As a consequence the three models lead to a different \( Q^2 \) dependence of the polarizabilities. At low \( Q^2 \), the LSM predicts a rapid decrease (increase) of the electric (magnetic) polarizability. In contrast to this the CQM results in a rather smooth decrease of the polarizabilities close to the real photon point.

In the real photon limit, we are able to derive analytical results. Restricting ourselves to the proton, the polarizabilities may be expanded in the mass ratio \( \mu = m_\pi/m_N \),

\[ \alpha_p(0) = \frac{e^2 g_N^2}{192\pi^3 m_N^3} \left[ \frac{5\pi}{2\mu} + 18 \ln \mu + \frac{33}{2} + O(\mu) \right], \]
Figure 1: The electric and magnetic polarizabilities as function of the momentum transfer $Q_0^2$. Solid line: calculation with the LSM for the proton, dashed line: LSM result for the neutron, dash–dotted line: CQM (proton), dotted line: ELM (proton).

$$\beta_p(0) = \frac{e^2 g^2_{\pi N}}{192 \pi^3 m_N^4 N} \left[ \frac{\pi}{4\mu} + 18 \ln \mu + \frac{63}{2} + \mathcal{O}(\mu) \right], \quad (6)$$

$$\frac{e^2}{4\pi} \tilde{P}^{(01,1)0}_{p}(0) = \frac{e^2 g^2_{\pi N}}{192 \pi^3 m_N^4 N} \left[ -\frac{11 \pi}{12\mu} - 12 \ln \mu - 16 + \mathcal{O}(\mu) \right], \quad (7)$$


where $g_{\pi N} = 13.4$ denotes the coupling strength of the pseudoscalar pion–nucleon coupling. Our results for $\alpha(0)$ and $\beta(0)$ are in complete agreement with the predictions of a one–loop calculation in relativistic ChPT. The expansion of the derivatives takes the form

$$\frac{d}{dQ^2} \alpha_p(0) = \frac{e^2 g^2_{\pi N}}{3840 \pi^3 m_N^5 N} \left[ -\frac{7 \pi}{\mu^3} + \frac{68}{\mu^2} - \frac{319 \pi}{4\mu} - 445 \ln \mu - \frac{3655}{12} + \mathcal{O}(\mu) \right],$$

$$\frac{d}{dQ^2} \beta_p(0) = \frac{e^2 g^2_{\pi N}}{3840 \pi^3 m_N^5 N} \left[ \frac{16}{\mu^3} + \frac{40 \pi}{\mu} - 365 \ln \mu - \frac{4505}{12} + \mathcal{O}(\mu) \right], \quad (7)$$

$$\frac{e^2}{4\pi} \frac{d}{dQ^2} \tilde{P}^{(01,1)0}_{p}(0) = \frac{e^2 g^2_{\pi N}}{3840 \pi^3 m_N^5 N} \left[ \frac{28}{\mu^3} + \frac{89 \pi}{2\mu} + 330 \ln \mu + \frac{920}{3} + \mathcal{O}(\mu) \right].$$

Both the leading $m_{\pi}^{-1}$ term in $\alpha_p$ and the $m_{\pi}^{-3}$ contribution in $\tilde{P}^{(01,1)0}_p$ for the electric and magnetic polarizability coincide with the predictions of HBChPT to third order in the external momenta $Q_0^2$. These leading terms are completely fixed by the one–loop calculation, while the other terms of the expansion will be modified by higher loops and additional low energy constants.
4 Relation between the Scalar Polarizabilities

Our analytical results in (6) and (7) obey the equations

\[ \frac{e^2}{4\pi} \hat{\beta}^{(01,1)0}(0) = -\frac{1}{3m_N} \left[ \alpha(0) + \beta(0) \right], \]

\[ \frac{e^2}{4\pi} \frac{d}{dQ_0^2} \hat{\beta}^{(01,1)0}(0) = -\frac{1}{3m_N} \left[ \frac{d}{dQ_0^2} \alpha(0) + \frac{d}{dQ_0^2} \beta(0) \right] + \frac{1}{12m_N} \left[ \alpha(0) + \beta(0) \right], \]

which are valid to all orders in \( \mu \). Numerically we have also established the more general result

\[ \frac{e^2}{4\pi} \hat{\beta}^{(01,1)0}(Q_0^2) = 2\omega_0 \bar{q} \left[ \alpha(Q_0^2) + \beta(Q_0^2) \right], \]

i.e., the scalar polarizabilities are related for the whole range of \( Q_0^2 \). As a consequence the electric and the mixed polarizability may be eliminated from the transverse amplitude in eq. (5), i.e., the transverse amplitude is completely determined by the magnetic polarizability \( \beta \). This result depends, of course, on the fact that only terms up to \( \omega' \) have been considered in (5). The relation (9) between the polarizabilities leads to the even more surprising result

\[ H^{(21,21)0}(\omega', \bar{q}) = \frac{4\pi}{e^2} \sqrt{\frac{8}{3}} \omega_0 \beta(q_0^2) + \mathcal{O}(\omega'^2). \]

According to (10) the transverse electric multipole, to linear order in \( \omega' \), is determined by the magnetic polarizability. Since \( \omega_0 \) vanishes in the static limit, \( m_N \to \infty \), the right hand sides of both eq. (9) and (10) vanish in that limit, and the corresponding observables are recoil effects to that order.

It can be shown that (9) is not a peculiarity of the LSM but a model–independent result. The proof relies on a low energy expansion of the non Born amplitude. The spin–averaged Compton tensor \( T_{NB}^{\mu\nu} \) (\( T_{NB} = \epsilon_\mu \epsilon_\nu T_{NB}^{\mu\nu} \)) has the general structure

\[ T_{NB}^{\mu\nu} = \left[q^\mu q^\nu - q \cdot q' g^{\mu\nu}\right] f_1 + \left[P \cdot q' (P^\mu q'^\nu + q'^\mu P^\nu) - q \cdot q' P^{\mu\nu} - (P \cdot q')^2 g^{\mu\nu}\right] f_2 + \left[P \cdot q' q^2 g^{\mu\nu} - P \cdot q' q'^\mu q'^\nu - q'^2 q^{\mu\nu} P^{\mu\nu} + q \cdot q' q'^\mu P^{\mu\nu}\right] f_3, \]

where \( P = p + p' \) and \( f_i = f_i(Q^2, q' \cdot q, P \cdot q') \), \( i = 1, 2, 3 \). Due to charge conjugation symmetry and nucleon crossing the Compton tensor is even as function of \( P \),

\[ T_{NB}^{\mu\nu}(q, q', P) = T_{NB}^{\mu\nu}(q, q', -P). \]
Accordingly, $f_3$ is odd as function of $P \cdot q'$ and therefore at least linear in $\omega'$. Since the tensor structure in front of $f_3$ is also linear in $\omega'$, we may neglect the contribution of $f_3$ to the scattering amplitude if we are only interested in terms linear in $\omega'$. Hence, the transverse amplitude to that order is given by

$$T_{NB}^{\text{trans}} = \left[ -\omega' \bar{q} \cos \vartheta f_1(Q^2_0, 0, 0) + \omega' \omega_0 f_1(Q^2_0, 0, 0) \right] \tilde{\epsilon}^* \cdot \tilde{\epsilon} + \omega' \bar{q} f_1(Q^2_0, 0, 0) \tilde{\epsilon}^* \cdot \tilde{\epsilon} + O(\omega^2) .$$

Comparing this equation with the low energy expansion (5) leads to relation (9) between the scalar polarizabilities. This result is generally difficult to obtain in phenomenological calculations, because it requires not only Lorentz and gauge symmetry but also symmetry under charge conjugation and nucleon crossing.

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