METRICS ON TILING SPACES, LOCAL ISOMORPHISM AND AN APPLICATION OF BROWN’S LEMMA.

RUI PACHECO AND HELDER VILARINHO

Abstract. We give an application of a topological dynamics version of multidimensional Brown’s lemma to tiling theory: given a tiling of an Euclidean space and a finite geometric pattern of points $F$, one can find a patch such that, for each scale factor $\lambda$, there is a vector $t$ so that copies of this patch appear in the tiling “nearly” centered on $\lambda F + t$ up to “bounded perturbations”. Furthermore, we introduce a new unifying setting for the study of tiling spaces which allows rather general group “actions” and we discuss the local isomorphism property of tilings within this setting.

1. Introduction

The main idea of Ramsey theory is that arbitrarily large sets cannot avoid a certain degree of “regularity”. This is exemplarily illustrated by Gallai’s theorem, a multidimensional version of the seminal van der Waerden’s theorem, which asserts that, given a finite coloration of $\mathbb{Z}^n$, any finite subset $F$ of $\mathbb{Z}^n$ has a monochromatic homothetic copy $\lambda F + t$. As shown recently by de la Llave and Windsor [6], this result has an interesting consequence in tiling theory. Roughly speaking, given a tiling $y$ of $\mathbb{R}^n$ and a finite geometric pattern $F \subset \mathbb{R}^n$ of points, one can find a patch $y'$ of $y$ so that copies of $y'$ appear in $y$ “nearly” centered on some homothetic version of the pattern. Hence, even if some sets of tiles tile the plane only non-periodically (perhaps the Penrose tiles [4, 8] are the most famous sets of tiles in such conditions), any tiling must exhibit some kind of “approximate periodicity”. The proof uses Furstenberg’s topological multiple recurrence theorem for commuting homeomorphisms [3] (which is a topological dynamic version of Gallai’s theorem) applied to certain tiling spaces $Y$ equipped with suitable metrics $d$.

Starting with a finite set $F$ of “prototiles”, three distinguished cases were considered by de la Llave and Windsor: the tiles of $y \in Y$ are obtained by taking translated (resp. direct isometric) copies of the prototiles, each $y \in Y$ exhibits finite local complexity, which is a property that, roughly speaking, does not allow two tiles to slide along their common boundary, and the distance $d$ makes two tilings close if they agree in a large ball about the origin up to a small translation (resp. direct isometry); thirdly, the tiles of $y \in Y$ are obtained by taking direct isometric copies of the prototiles, and $d$ makes two tilings close if they agree in a large ball about

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the origin up to small direct isometries (rigid motions) of each individual tile. In all these three cases, the metric spaces \((Y,d)\) are compact and the group of translations acts continuously on it.

A not so famous Ramsey-type result is the so called Brown’s lemma \([1, 2]\). Observe that Gallai’s theorem does not say nothing, apart its existence, about the scale factor \(\lambda\). On the other hand, the multidimensional version of Brown’s lemma asserts, roughly speaking, that one can take any \(\lambda\) once we allow “bounded perturbations” in the structure of the homothetic copies of \(F\). In this paper, we give an application of a topological dynamics version of this result (Lemma 3, Section 4) to tiling theory (Theorem 3, Section 5). In order to avoid to treat separately the three distinguished cases considered by de la Llave and Windsor, we start by establishing, in Section 2, a new unifying setting for the study of tiling spaces which allows rather general group “actions”.

The local isomorphism is a property that a tiling of an Euclidean space might have which also expresses a certain “regularity”. Quite surprisingly, this is not an unusual property. For example, all Penrose tilings satisfy it \([4]\). More recently, Radin and Wolff \([9]\) proved that any nonempty tiling space \(Y\) with finite local complexity under direct isometries must admit a tiling satisfying the local isomorphism property. In Section 3, we give a generalization of the results on the local isomorphism property by Radin and Wolff \([9]\) within our setting.

2. Metrics on tiling spaces

Roughly speaking, a tiling of \(\mathbb{R}^n\) is an arrangement of tiles that covers \(\mathbb{R}^n\) without overlapping. Typically one starts with a fixed finite set \(F\) of “prototiles” and each tile is an isometric copy of some prototile \([4, 6, 9, 10, 11]\). Denote by \(Y_F\) the set of all tilings of \(\mathbb{R}^n\) obtained in this way from a given prototile set \(F\). There is natural metric on \(Y_F\), \(d_F\), with respect to which two tilings are close if their skeletons are close (with respect to the usual Hausdorff distance between compact sets of \(\mathbb{R}^n\)) on a large ball about the origin \([9]\). However, it is possible to provide \(Y_F\) with alternative (but equivalent under certain conditions) metrics which carry much more geometric information. A familiar way to do this is by making two tilings close if they agree in large ball about the origin up to a small translation \([6, 10, 11]\). Instead of translations, one could consider in this definition any subgroup of the group of rigid motions group or even consider piecewise rigid motions, that is, rigid motions of each individual tile \([6, 11]\). In this section, we pretend to establish a new setting which unifies and generalizes the previous cases.

Consider \(\mathbb{R}^n\) with its usual Euclidean norm \(\| \cdot \|\) and write \(B_r = \{ \vec{v} \in \mathbb{R}^n : \|\vec{v}\| \leq r\}\). A set \(D \subset \mathbb{R}^n\) is called a tile if it is compact, connected and equal to the closure of its interior. A tiling of a subset \(S \subset \mathbb{R}^n\) is a collection \(x = \{D_i\}_{i \in I}\) of tiles such that:

\[
\begin{align*}
(T_1) \quad S &= \bigcup_{i \in I} D_i; \\
(T_2) \quad D_i \cap D_j &= \emptyset, \text{ for all } i, j \in I \text{ with } i \neq j.
\end{align*}
\]

In this case we say that \(S\) is the support of \(x\) and write \(S = \text{supp}(x)\). We denote by \(X(S)\) the set of all tilings of \(S\) and define \(\mathcal{X} := \bigcup_S X(S)\). If \(x, x' \in \mathcal{X}\) and \(x' \subseteq x\), then \(x'\) is called a patch of \(x\).
Let \( \mathcal{Y} \) be a subset of \( \mathcal{X} \) satisfying:

1. (\( \mathcal{Y}_1 \)) for all \( x \in \mathcal{Y} \) and \( x' \subseteq x \), we have \( x' \in \mathcal{Y} \);
2. (\( \mathcal{Y}_2 \)) given \( x \in \mathcal{X} \), if \( x' \in \mathcal{Y} \) for all \( x' \subseteq x \) with bounded support, then \( x \in \mathcal{Y} \).

If \( K \subseteq \mathbb{R}^n \) is compact, we denote by \( x[[K]] \) the set of all patches \( x' \) of \( x \in \mathcal{Y} \) with bounded support satisfying \( K \subseteq \text{supp}(x') \). Clearly, if \( x' \in x[[K']] \) and \( x'' \in x[[K'']] \) then \( x' \cap x'' \in x[[K' \cap K'']] \).

Consider the auxiliary set \( \Theta \) of functions \( \theta : \sqrt{2}, \infty[\times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \) satisfying:

1. (\( \Theta_1 \)) for each \( s > \sqrt{2} \) and \( b \geq 0 \), the function \( \theta_s(\cdot) := \theta(s, \cdot) \) is strictly increasing and the function \( \theta^b(\cdot) := \theta(\cdot, b) \) is increasing;
2. (\( \Theta_2 \)) \( \theta(s, a + b) \leq \theta(s, a) + \theta(s, b) \), for all \( s > \sqrt{2} \), and \( a, b \geq 0 \);
3. (\( \Theta_3 \)) \( \theta \) is continuous and \( \theta(s, 0) = 0 \), for all \( s > \sqrt{2} \).

Now, fix an equivalence relation on \( \mathcal{Y} \) and denote by \([x]\) the equivalence class of \( x \in \mathcal{Y} \). Associate to each equivalence class \([x]\) a metric group \((G[x], d_{G[x]})\) and to each ordered pair \((y, z)\) of patches in \([x]\) a non-empty subset \( \gamma(y, z) \) of \( G[x] \) such that:

1. (\( \Gamma_1 \)) \( \gamma(y, z) = \gamma(z, y)^{-1} \);
2. (\( \Gamma_2 \)) \( \gamma(z, w) \gamma(y, z) = \gamma(y, w) \);
3. (\( \Gamma_3 \)) \( \gamma(x, y) \cap \gamma(x, z) = \emptyset \) if \( y \neq z \).

If \( g \in \gamma(x, y) \), we write \( y = g(x) \). Observe that \( \text{Id}_{G[x]} \in \gamma(y, y) \) for all \( y \in [x] \). Define
\[
\gamma(x, [x]) := \bigcup_{y \in [x]} \gamma(x, y) \subseteq G[x].
\]

Assume that:

1. (\( G_1 \)) \( d_{G[x]} \) is right invariant, that is, \( d_{G[x]}(fg, hg) = d_{G[x]}(f, h) \) for all \( g, f, h \in G[x] \);
2. (\( G_2 \)) for each pair of equivalence classes admitting representatives \( x' \subseteq x \), there exists a homomorphism \( \iota_{[x,x']} : G[x] \rightarrow G[x'] \) such that, if \( g \in \gamma(x, [x]) \), then \( \iota_{[x,x']}(g) \in \gamma(x', [x']) \) and \( \iota_{[x,x']}(g)(x') \subseteq g(x) \);
3. (\( G_3 \)) if \( g \in G[x] \) and \( x' \subseteq x \), then \( \|\iota_{[x,x']}(g)\|_{G[x']} \leq \|g\|_{G[x]} \), where \( \|g\|_{G[x]} = d_{G[x]}(g, \text{Id}_{G[x]}) \);
4. (\( G_4 \)) given \( x, y \in \mathcal{Y}, G[x, y] := \{g \in \gamma(x, [x]) : g(x) \subseteq y\} \) is closed in \( \gamma(x, [x]) \);
5. (\( G_5 \)) \( \theta \in \Theta \) is a function such that, for all \( x \in \mathcal{Y}, s > \sqrt{2}, g \in \gamma(x, [x]) \) with \( \theta(s, \|g\|_{G[x]}) < \sqrt{2}/2 \),
   - (i) if \( B_s \subseteq \text{supp}(x) \cap \text{supp}(g(x)) \), then, for all \( \sqrt{2} < s' \leq s \) and \( x' \subseteq x \) with \( B_{s'} \subseteq \text{supp}(x') \), we have \( B_{s'-\theta(s',\|g\|_{G[x]})} \subseteq \text{supp}(\iota_{[x,x']}(g)(x')) \);
   - (ii) if \( \text{supp}(x) \subseteq \mathbb{R}^n \setminus B_s \), then \( \text{supp}(g(x)) \subseteq \mathbb{R}^n \setminus B_{s-\theta(s,\|g\|_{G[x]})} \);
   - (iii) if \( \text{supp}(x) \subseteq B_s \), then \( \text{supp}(g(x)) \subseteq B_{s+\theta(s,\|g\|_{G[x]})} \).

For notational convenience, we shall denote \( \iota_{[x,x']}(g) \) by \( g \) whenever it is clear which equivalence classes we are dealing with. We emphasize that \( \iota_{[x,x']} \) only depends on the equivalence classes \([x]\) and \([x']\).

**Lemma 1.** For all \( x \in \mathcal{Y}, g, h \in G[x] \), we have:

1. (a) \( \|g\|_{G[x]} = \|g^{-1}\|_{G[x]} \);
particular cases and find corresponding functions

Proof. The items (a) and (b) follow from the right invariance of \(d_{G[x]}\). For instance,
\[
\|hg\|_{G[x]} = d_{G[x]}(hg, \text{Id}_{G[x]}) \leq d_{G[x]}(h, g) + d_{G[x]}(g, \text{Id}_{G[x]}) = \|h\|_{G[x]} + \|g\|_{G[x]}.
\]
Item (c) is a direct consequence of \((G_3)\). 

Example 1. Let \(G\) be a group equipped with a right invariant metric \(d_G\). Suppose that we have a continuous group action of \(G\) on \(\mathbb{R}^n\). Define an equivalence relation \(\sim\) on \(\mathcal{Y}\) as follows:

\((E_1)\) given two elements \(x = \{D_j\}_{j \in J}\) and \(x' = \{D'_k\}_{k \in K}\) of \(\mathcal{Y}\), write \(x \sim x'\) if there exist a bijection \(\alpha : J \rightarrow K\) and \(g \in G\) such that, for each \(j \in J\), \(D'_k = g(D_j)\).

Let \(\gamma(x, x')\) be the set of all such elements of \(G\), and for each equivalence class \([x]\) put \(G[x] = G\).
If \(x' \subseteq x\), set \(\nu_{[x,x']}(g) = g\). Clearly, these choices satisfy \((G_1)-(G_4)\). Let us consider two particular cases and find corresponding functions \(\theta_G \in \Theta\) satisfying \((G_5)\):

(a) \(G\) is the group of rigid motions (direct isometries) of \(\mathbb{R}^n\):
\[
\mathcal{I} = \{g : g(\vec{v}) = R(\vec{v}) + \vec{g}, \text{ with } R \in SO(n) \text{ and } \vec{g} \in \mathbb{R}^n\}.
\]
We can define on \(\mathcal{I}\) a right invariant metric \(d_\mathcal{I}\) by
\[
d_\mathcal{I}(g, h) = \max_{||\vec{v}|| \leq 1} ||g^{-1}(\vec{v}) - h^{-1}(\vec{v})||.
\]

Given \(g \in \mathcal{I}\) with \(g(\vec{v}) = R(\vec{v}) + \vec{g}\), we have \(\|g\|_\mathcal{I} \geq \|\vec{g}\|\), and \(B_{s-\|g\|_\mathcal{I}} \subseteq B_{s-\|\vec{g}\|} \subseteq g(B_s)\).

Hence \((G_5)\) holds for \(\theta_\mathcal{I}(s, t) = t\).

(b) \(G\) is the group of homothetic transformations of \(\mathbb{R}^n\):
\[
\mathcal{H} = \{g : g(\vec{v}) = \lambda \vec{v} + \vec{g}, \lambda > 0 \text{ and } \vec{g} \in \mathbb{R}^n\}.
\]
Consider the metric on \(\mathcal{H}\) defined by: if \(g(\vec{v}) = \lambda \vec{v} + \vec{g}\) and \(h(\vec{v}) = \mu \vec{v} + \vec{h}\), then \(d_\mathcal{H}(g, h) = \max \{|\ln(\lambda/\mu)|, ||\vec{g} - \vec{h}||\}\). This equips \(\mathcal{H}\) with a structure of topological group with respect to which the standard action of \(\mathcal{H}\) on \(\mathbb{R}^n\) is a continuous action. Although \(\hat{d}_\mathcal{H}\) is not right invariant, we can construct a right invariant metric \(d_\mathcal{H}\) on \(\mathcal{H}\), topologically equivalent to \(\hat{d}_\mathcal{H}\), as follows: consider the continuous function \(F : \mathcal{H} \rightarrow [0, 1]\) defined by
\[
F(g) = \max\{1 - \hat{d}_\mathcal{H}(\text{Id}_\mathcal{H}, g), 0\}\]
and set
\[
d_\mathcal{H}(g, h) = \sup_{f \in \mathcal{H}} |F(gf) - F(hf)|.
\]
This construction is motivated by the standard proof of Birkhoff-Kakutani theorem (see [7] for details). Now, it follows from the definition \([1]\) that
\[
\|g\|_\mathcal{H} \geq \hat{d}_\mathcal{H}(\text{Id}_\mathcal{H}, g) = \max\{|\ln(\lambda)|, ||\vec{g}||\}
\]
if \(\hat{d}_\mathcal{H}(\text{Id}_\mathcal{H}, g) < 1\). In this case, it is clear that
\[
B_{s-\|g\|_\mathcal{H}} \subseteq B_{s-\|\ln(\lambda)|+||\vec{g}||} \subseteq B_{\lambda s-\|\vec{g}||} \subseteq g(B_s).
\]
Proposition 1. If \( \tilde{d}_\mathcal{H}(Id_\mathcal{H}, g) \geq 1 \), we have \( \|g\|_\mathcal{H} = 1 \), and \( B_{s-(\|g\|_\mathcal{H}+\|\hat{g}\|_\mathcal{H})} = \emptyset \). It is now easy to check that \( (G_5) \) holds for \( \theta_\mathcal{H}(s, t) = st + t \).

Example 2. Again, let \( G \) be a group equipped with a right invariant metric \( d_G \). Suppose that we have a continuous group action of \( G \) on \( \mathbb{R}^n \). Define an equivalence relation \( \sim \) on \( \mathcal{Y} \) as follows:

(\( E_2 \)) given two elements \( x = \{D_j\}_{j \in J} \) and \( x' = \{D'_k\}_{k \in K} \) of \( \mathcal{Y} \), write \( x \sim x' \) if there exist a bijection \( \alpha : J \to K \) and a collection \( \{g_j\}_{j \in J} \) of elements in \( G \), with \( j \in J \), such that \( D'_j = g_j(D_j) \) for each \( j \in J \), \( \sup_{j \in J} \|g_j\|_G < \infty \).

Of course, if \( x \) and \( x' \) satisfy \( (E_1) \) then they also satisfy \( (E_2) \). Associate to \( x = \{D_j\}_{j \in J} \) the group \( G[x] \subseteq \prod_{j \in J} G \) of all maps \( g : J \to G \) such that \( \sup_{j \in J} \|g_j\|_G < \infty \), where \( g_j = g(j) \).

The group multiplication is given in the usual way: \( gh(j) = g_j h_j \). In this case, \( \gamma(x, x') \) is the set of all \( g \in G[x] \) satisfying \( (E_2) \). We can define on \( G[x] \) a right invariant metric \( d_{G[x]} \) by

\[
d_{G[x]}(g, h) = \sup_{j \in J} d_G(g_j, h_j). \tag{2}
\]

For \( x' \subseteq x \), with \( x' = \{D_j\}_{j \in J'} \) and \( J' \subseteq J \), set \( u_{x,x'}(g) = g_{j'} \). These choices satisfy \( (G_1)-(G_4) \).

We claim that \( (G_5) \) holds for \( \theta(s, t) = \theta_G(s, t) \). In fact, take \( \sqrt{2} < s' \leq s \) and a patch \( x' \) of \( x \) such that \( B_s \subseteq \text{supp}(x) \cap \text{supp}(g(x)) \) and \( B_{s'} \subseteq \text{supp}(x') \). Set \( t = \|g\|_{G[x]} \). Suppose that \( B_{s'-\theta_G(s', t)} \) is not contained in \( \text{supp}(g(x')) \). In this case, since \( B_s \subseteq \text{supp}(x) \cap \text{supp}(g(x)) \), there must exists a tile \( D_1 \in \mathbb{R}^n \setminus B_{s'} \) in \( x \setminus x' \) such that \( g_1(D_1) \cap B_{s'-\theta_G(s', t)} \neq \emptyset \). But this contradicts the property \( (G_5) \) of \( \theta_G \) in Example \( \square \).

Given two elements \( x', y' \in \mathcal{Y} \) with bounded support, set

\[
\Delta(x', y') = \min\{s > 0 : \text{supp}(x') \cup \text{supp}(y') \subseteq B_s\}.
\]

Denote by \( Y \) the set of all tilings of \( \mathbb{R}^n \) in \( \mathcal{Y} \) and, for \( x, y \in Y \), define

\[
d(x, y) = \inf \left\{ \left\{ \sqrt{2}/2 \right\} \cup \{0 < r < \sqrt{2}/2 : \text{exist } x' \in x[[B_{1/r}]], y' \in y[[B_{1/r}]], \text{ and } g \in G[x'] \text{ with } \theta(\Delta(x', y'), \|g\|_{G[x']}) \leq r \} \right\}. \tag{3}
\]

Proposition 1. \( (Y, d) \) is a metric space.

Proof. Clearly \( d \) is non-negative and symmetric. Let us prove that the triangle inequality holds. Take \( x, y, z \in Y \) with \( 0 < d(x, y) \leq d(y, z) \) and \( d(x, y) + d(y, z) < \sqrt{2}/2 \). Choose \( \epsilon > 0 \) such that \( \epsilon + d(x, y) + d(y, z) < \sqrt{2}/2 \), and put \( a = d(x, y) + \epsilon/2 \) and \( b = d(y, z) + \epsilon/2 \). Then, by definition of \( d \), there are \( x' \in x[[B_{1/a}]], y' \in y[[B_{1/a}]] \) and \( g \in G[x'] \) with \( \theta(\Delta(x', y'), \|g\|_{G[x']}) \leq a \) such that \( g(x') = y' \). Similarly, there are \( y'' \in y[[B_{1/b}]], z'' \in z[[B_{1/b}]], h \in G[y''], \theta(\Delta(y'', z''), \|h\|_{G[y'']}) \leq b \) such that \( h(y'') = z'' \).

Let \( y_0 = y' \cap y'' \in y[[B_{1/b}]] \). By \( (G_2) \), we can define \( x_0 = g^{-1}(y_0), z_0 = h(y_0) \), and we have \( x_0 \subseteq x' \) and \( z_0 \subseteq z'' \). Put \( c = a + b \). Since \( 0 < c < \sqrt{2}/2 \), and taking account \( (\Theta_1) \), it turns out...
that
\[ \frac{1}{b} \geq \frac{1}{c} + a \geq \frac{1}{c} + \theta(\Delta(x', y'), \|g\|_{C[\gamma']}) \geq \frac{1}{c} + \theta(1/b, \|g\|_{C[\gamma']}). \] (4)

Then, by \((G_5)\) and (a) of Lemma 1 it follows from (1) that \(x_0 \in x[[B_{1/c}]].\) On the other hand, if \(y'' \setminus y_0 \neq \emptyset,\) we must have \(\text{supp}(y'' \setminus y_0) \subset \mathbb{R}^n \setminus B_{1/a}\) and, by \((\Theta_1)\) and \((G_3),\)
\[ \frac{1}{a} \geq \frac{1}{c} + b \geq \frac{1}{c} + \theta(\Delta(y'', z''), \|h\|_{\tilde{C}[\gamma'' \setminus \omega_0]}) \geq \frac{1}{c} + \theta(1/a, \|h\|_{\tilde{C}[\gamma'' \setminus \omega_0]}). \] (5)

Then, by \((G_5),\) it follows from (5) that \(\text{supp}(h(y'' \setminus y_0)) \subset \mathbb{R}^n \setminus B_{1/c}.\) Hence, since \(h(y'') \in z[[B_{1/c}]] \subset z[[B_{1/c}]],\) we see that \(h(y_0) \in z[[B_{1/c}]].\)

Since \(hg(x_0) = z_0\) and, by (b) of Lemma 1 \(\|hg\|_{\tilde{C}[z_0]} \leq \|g\|_{\tilde{C}[x_0]} + \|h\|_{\tilde{C}[x_0]},\) then, by \((\Theta_1)\) and \((\Theta_2),\)
\[ \theta(\Delta(x_0, y_0), \|hg\|_{\tilde{C}[z_0]}) \leq \theta(\Delta(x', y'), \|g\|_{\tilde{C}[\gamma']}) + \theta(\Delta(y'', z''), \|h\|_{\tilde{C}[\gamma']} \leq a + b, \]
and thus \(d(x, z) \leq a + b = d(x, y) + d(y, z) + \epsilon,\) with \(\epsilon > 0\) arbitrarily small. Hence the triangle inequality holds.

Finally, we want to prove that \(d(x, y) = 0\) if, and only if, \(x = y.\) Clearly, if \(x = y\) then \(d(x, y) = 0.\) Assume now, for a contradiction, that \(x \neq y\) and \(d(x, y) = 0.\) Take a patch \(x' \subset x,\) with \(\text{supp}(x') \subset B_{1/r}\) for some \(r > 0,\) such that \(x' \not\subset y.\) On the other hand, by definition of \(d,\) for each \(\delta > 0\) there are \(x_\delta \in x[[B_{1/\delta}]], y_\delta \in y[[B_{1/\delta}]]\) and \(g_\delta \in G[x_\delta],\) with \(\theta(\Delta(x_\delta, y_\delta), \|g\|_{\tilde{C}[x_\delta]}) < \delta,\)

such that \(y_\delta = g_\delta(x_\delta).\) Take \(\delta < r.\) In this case, since \(x' \subset x_\delta,\) by \((G_2)\) we have \(g_\delta(x') \subset y_\delta \subset y.\) Hence \(g_\delta \in G[x', y].\) At the same time, by \((\Theta_1)\) and \((G_3),\) we have, for any \(\sqrt{2} < s < \Delta(x_\delta, y_\delta),\)
\[ \theta(s, \|g_\delta\|_{\tilde{C}[\gamma' \setminus \omega_0]}) \leq \theta(s, \|g_\delta\|_{\tilde{C}[x_\delta]}) \leq \theta(\Delta(x_\delta, y_\delta), \|g_\delta\|_{\tilde{C}[x_\delta]}) < \delta, \]
which, by \((\Theta_3),\) implies that \(g_\delta \to Id_{G[x']} \) as \(\delta \to 0.\) By \((G_4),\) this would imply that \(Id_{G[x']} \in G[x', y],\) that is, \(x' \subset y,\) which is a contradiction. \(\square\)

**Proposition 2.** \((Y, d)\) is complete if, for all \(x \in \mathcal{Y}\) with bounded support, the following conditions hold:

\begin{enumerate}
  \item \((C_1)\) \(\gamma(x, [x]) \subset G[x]\) is complete with respect to the restriction of \(d_{G[x]};\)
  \item \((C_2)\) given a sequence \(\{g_n\}_{n \in \mathbb{N}}\) of elements in \(\gamma(x, [x]) \subset G[x]\) and a decreasing sequence of positive numbers \(\{r_n\}_{n \in \mathbb{N}}\) such that \(B_{r_n} \subset \text{supp}(g_n(x)),\) \(g_n \to g \in \gamma(x, [x]) \subset G[x]\) and \(r_n \to r,\) then \(B_r \subset \text{supp}(g(x));\)
  \item \((C_3)\) if \(\sum \theta(\alpha_n, \gamma_n)\) is convergent and \(\lim \alpha_n - \beta_n = 0,\) then \(\sum \theta(\beta_n, \gamma_n)\) is convergent.
\end{enumerate}

**Proof.** Consider a Cauchy sequence \(\{x_n\}_{n \in \mathbb{N}}\) of tilings of \(\mathbb{R}^n\) in \(Y\) and consider a sequence \(\{s_n\}_{n \in \mathbb{N}}\) such that \(d(x_n, x_{n+1}) < s_n.\) By definition of \(d,\) for each \(n\) there are \(x'_n \in x_n[[B_{1/s_n}]]\) and \(g_n \in G[x'_n]\) such that \(\theta(\Delta(x'_n, g_n(x'_n)), \|g_n\|_{\tilde{C}[x'_n]}) \leq s_n\) and \(g_n(x'_n) \in x_{n+1}[[B_{1/s_n}]].\) Without loss of generality, by passing to a subsequence if necessary, we can assume that the sequence \(\{s_n\}_{n \in \mathbb{N}}\) is rapidly decreasing so that \(\sum s_n < \infty\) and \(\Delta(x'_n, g_n(x'_n)) \leq 1/s_{n+1}.\) In particular, we have \(g_n(x'_n) \subset x'_{n+1}.\)
Given $n < m$ we define $h_{n,m} = g_m g_{m-1} \ldots g_{n+1} g_n \in \gamma(x'_n, [x'_n]) \subseteq G[x'_n]$. Of course, here we are denoting $t_{[x'_{n+k}, x'_{n+k}]}(g_{n+k})$ by $g_{n+k}$. Since, for $n < m' < m$, 

$$d_{G[x'_n]}(h_{n,m}, h_{n,m'}) = d_{G[x'_n]}(g_m g_{m-1} \ldots g_{n+1} g_n, g_{m'} g_{m'-1} \ldots g_{n+1} g_n)$$

$$= d_{G[x'_n]}(g_m g_{m-1} \ldots g_{m' + 1}, Id_{G[x'_n]})$$

$$\leq \|g_m g_{m-1} \ldots g_{m' + 1}\|_{G[x'_n]}$$

$$\leq s_m + s_{m-1} + \ldots + s_{m'} \to 0, \text{ as } m, m' \to \infty,$$

we see that $\{h_{n,m}\}_{m \in \mathbb{N}}$ is a Cauchy sequence. Hence, by the completeness of $\gamma(x'_n, [x'_n])$, $\{h_{n,m}\}_{m \in \mathbb{N}}$ is convergent to a certain element $h_n \in \gamma(x'_n, [x'_n])$.

Since $d_{G[x'_n]}$ is right invariant, the right multiplication by an element is continuous in $G[x'_n]$. At the same time, by (c) of Lemma 1, $t_{[x'_{n+k}, x'_{n+k}]}$ is a continuous homomorphism. Hence $h_n = h_{n+1} g_n$. Consequently, $h_n(x'_n) = h_{n+1} g_n(x'_n) \subseteq h_{n+1}(x'_{n+1})$. This implies that $\{h_n(x'_n)\}_{n \in \mathbb{N}}$ is an increasing sequence in $Y$, so, by $(Y_2)$, we can define a tiling $x = \bigcup_n h_n(x'_n) \in Y$. Next we prove that $\{x_n\}_{n \in \mathbb{N}}$ converges to $x$.

By $(G_5)$, we know that $h_{n,m}(x'_n) \in x_{m+1}[[B_{t_{n,m}}]]$ with

$$t_{n,m} = 1/s_n - \theta(1/s_n, \|g_n\|_{G[x'_n]}) - \theta(1/s_{n+1}, \|g_n + 1\|_{G[x'_{n+1}]}) - \ldots - \theta(1/s_m, \|g_m\|_{G[x'_m]}).$$

This defines a bounded decreasing sequence in $m$, hence $\{t_{n,m}\}_{m \in \mathbb{N}}$ converges to some $t_n$ as $m \to \infty$. Hence, by $(C_2)$, $h_n(x'_n) \in x[[B_{t_n}]]$. Now,

$$\theta(1/s_n, \|g_n\|_{G[x'_n]}) + \theta(1/s_{n+1}, \|g_n + 1\|_{G[x'_{n+1}]}) + \ldots + \theta(1/s_m, \|g_m\|_{G[x'_m]})$$

$$\leq s_n + \ldots + s_m,$$

then $t_{n,m} \geq 1/s_n - (s_n + \ldots + s_m)$. Taking the limit $m \to \infty$, we see that

$$t_n \geq 1/s_n - \sum_{i=n}^{\infty} s_i \to \infty, \text{ as } n \to \infty. \quad (6)$$

On the other hand, by $(G_5)$, we also have

$$\Delta(x'_n, h_{n,m}(x'_n)) \leq \Delta (x'_n, g_n(x'_n)) + \theta (\Delta (x'_{n+1}, g_{n+1}(x'_{n+1})), \|g_{n+1}\|_{G[x'_{n+1}]})$$

$$+ \ldots + \theta (\Delta (x'_m, g_m(x'_m)), \|g_m\|_{G[x'_m]})$$

$$\leq \Delta (x'_n, g_n(x'_n)) + s_{n+1} + \ldots + s_m.$$

Together with $(\Theta_1)$, $(\Theta_2)$ and $(\Theta_3)$, this gives

$$\theta (\Delta (x'_n, h_{n,m}(x'_n)), \|h_{n,m}\|_{G[x'_n]}) \leq \sum_{i=n}^{m} \theta (\Delta (x'_n, g_n(x'_n)) + s_{n+1} + \ldots + s_m, \|g_i\|_{G[x'_i]})$$

$$\to \sum_{i=n}^{\infty} \theta (\Delta (x'_n, g_n(x'_n)) + \sum_{j=n+1}^{\infty} s_j, \|g_i\|_{G[x'_i]}), \text{ as } m \to \infty,$$
and, consequently, since
\[ \sum_{i=n}^{\infty} \theta(\Delta(x'_n, g_n(x'_n)), \|g_i\|_{C[\mathbb{I}]}) \leq \sum_{i=n}^{\infty} s_i < \infty, \]
by (C3) we have
\[ \lim_{n \to \infty} \theta(\Delta(x'_n, h_n(x'_n)), \|h_n\|_{C[\mathbb{I}]})) = 0. \] (7)
Finally, from (6) and (7) we conclude that
\[ d(x, x_n) \leq \max \left\{ 1/t_n, \theta(\Delta(x'_n, h_n(x'_n)), \|h_n\|_{C[\mathbb{I}]})) \right\} \to 0, \quad \text{as } n \to \infty. \]

Now, associate to each equivalence class \([x]\) of elements in \(Y\) an element
\[ g_{[x]} \in \bigcap_{z \in [x]} \gamma(z, [x]) \subseteq G[x] \]
in such a way that, if \(x' \subseteq x, y' \subseteq y\) and \([x'] = [y']\), then \(\iota_{[x,x']}(g_{[x]}) = \iota_{[y,y']}(g_{[y]})\). This defines a map
\[ g : Y \to Y, \quad g(z) = g_{[x]}(z) \text{ if } z \in [x], \] (8)
and we have:

**Proposition 3.** Suppose that, for each \(x \in Y\), the left multiplication by \(g_{[x]} \in G[x]\) is continuous in \((G[x], d_{G[x]})\). Moreover, assume that, for each \(x \in Y\), there exists \(\epsilon < 1\) such that \(\delta \theta(1/\delta, \|g_{[x]}\|_{G[x]}) < \epsilon\) for all sufficiently small \(\delta > 0\). Then the map \(g : (Y, d) \to (Y, d)\) defined by (8) is continuous.

**Proof.** Consider a sequence \(\{x_n\}_{n \in \mathbb{N}}\) of elements in \(Y\) convergent to \(y \in Y\). This means that, for each \(n\) sufficiently large, there exist \(s_n > 0, x'_n \in x_n[[B_{1/s_n}]], y'_n \in y[[B_{1/s_n}]]\) and \(g_n \in G[y''_n]\) such that \(lim s_n = 0, y''_n = g_n(x''_n)\) and \(\theta(\Delta(x''_n, y''_n), \|g_n\|_{G[y''_n]}) \leq s_n\). Observe that, by (Θ3), we must have \(\|g_n\|_{G[y''_n]} = 0\).

Given \(\delta > 0\), take \(N_{\delta}, n_{\delta} > 0\) such that, for all \(n > n_{\delta}\), we have \(1/s_n > N_{\delta}\) (hence \(y''_n \in y[[B_{N_{\delta}}]]\)) and
\[ 1/\delta < N_{\delta} - \theta(\Delta(x''_n, y''_n), \|g_n\|_{G[y''_n]}). \]
Set \(y'_\delta = \bigcap_{n>n_{\delta}} y''_n\) and \(x'_\delta = g_n^{-1}(y'_\delta)\). Since \([x'_n] = [y'_\delta]\), we can define
\[ h_{\delta} := \iota_{x_n, x'_n}(g_{[x_n]}) = \iota_{y, y'_\delta}(g_{[y]}). \]
By (G5), \(x'_n \in x_n[[B_{1/\delta}]]\) and, for some \(\delta_1 > 0\),
\[ \supp(y'_\delta) \cup \supp(h_{\delta}(y'_\delta)) \cup \bigcup_{n>n_{\delta}} (\supp(x'_n) \cup \supp(h_{\delta}(x'_n))) \subseteq B_{1/\delta_1}. \]
Observe that $h_\delta g_n h_\delta^{-1}(h_\delta(x_n')) = h_\delta(y'_n)$ and, by $(G_2)$ and $(G_5)$, $h_\delta(x_n') \in g(x_n)[[B_{1/\delta'}]]$ and $h_\delta(y'_n) \in g(y)[[B_{1/\delta'}]]$, with

$$
\delta' = \frac{\delta}{1 - \delta \theta(1/\delta', \|h_\delta\|_{C[y'_n]})} \leq \frac{\delta}{1 - \delta \theta(1/\delta, \|g(y)\|_{C[y]})} \leq \frac{\delta}{1 - \varepsilon},
$$

for all sufficiently small $\delta > 0$. Then

$$
d(g(x_n), g(y)) \leq \max \left\{ \delta', \theta \left( \|h_\delta(x_n'), h_\delta(y'_n)\|, \|h_\delta g_n h_\delta^{-1}\|_{C[y'_n]} \right) \right\}
\leq \max \left\{ \delta', \theta \left( 1/\delta_1, \|h_\delta g_n h_\delta^{-1}\|_{C[y'_n]} \right) \right\}.
$$

(9)

Since $\lim \|g_n\|_{C[y'_n]} = 0$ and $\|g_n\|_{C[y'_n]} \leq \|g_n\|_{C[y_n]}$, we also have $\lim \|g_n\|_{C[y'_n]} = 0$ and, consequently, $\lim g_n = \text{Id}_{C[y'_n]}$. On the other hand, left multiplication by $h_\delta$ is continuous and, by the right invariance of $G[y'_n]$, right multiplication by any element of $G[y'_n]$ is also continuous. Then $\lim h_\delta g_n h_\delta^{-1} \rightarrow \text{Id}_{C[y'_n]}$. Hence, for $n$ sufficiently large, from inequality (9) we see that $d(g(x_n), g(y)) \leq \delta'$. But $\delta' \rightarrow 0$ as $\delta \rightarrow 0$. Hence $\{g(x_n)\}_{n \in \mathbb{N}}$ converges to $g(y) \in Y$. \(\square\)

Consider the setting of Example 2. Assume that $g(x) \in Y$ whenever $x \in Y$ and $g \in G$. When $G = \mathcal{I}$, it follows directly from the previous proposition that any isometry $g \in \mathcal{I}$ defines a continuous transformation on $Y$. On the other hand, when $G = \mathcal{H}$, it is also clear that any homothetic transformation $g \in \mathcal{H}$ with $\|g\|_\mathcal{H} < 1$ defines a continuous transformation on $Y$. The conditions of the proposition are not fulfilled if $\|g\|_\mathcal{H} = 1$. However, since $g$ can always be written as the product of a finite number of homothetic transformations, $g = g_1 \ldots g_n$ with $\|g_i\|_\mathcal{H} < 1$, we conclude that the corresponding transformation on $Y$, as the composition of a finite number of continuous transformations, is also continuous. Similarly, in the setting of Example 2, each element of $G$ defines a continuous transformation on $Y$, whether $G = \mathcal{I}$ or $G = \mathcal{H}$.

Next we shall be concerned with the compactness of $(Y, d)$. Given a compact $K \subset \mathbb{R}^n$, we say that $x' \in \mathcal{Y}$ is $K$-minimal if: $K \subseteq \text{supp}(x')$; given $x'' \in \mathcal{Y}$ such that $K \subseteq \text{supp}(x'')$ and $x'' \subseteq x'$, then $x'' = x'$. We denote by $\mathcal{Y}_{\text{min}}(K)$ the subset of all $K$-minimal elements of $\mathcal{Y}$, which is obviously nonempty.

**Proposition 4.** Suppose that $\mathcal{Y}$ satisfies:

(C4) for all compact $K \subset \mathbb{R}^n$, there exists a finite subset $\mathcal{A}_K \subset \mathcal{Y}$ such that, for all $x' \in \mathcal{Y}_{\text{min}}(K)$, there are $y' \in \mathcal{A}_K$ and $g \in G[y']$ with $x' = g(y')$;

(C5) for all compact $K \subset \mathbb{R}^n$ and $x' \in \mathcal{Y}_{\text{min}}(K)$,

$$
G_{x'}(K) = \{g \in \gamma(x', [x']): K \subseteq \text{supp}(g(x'))\}
$$

is compact.

Then the metric space $(Y, d)$ is compact if it is complete.

**Proof.** Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in $Y$. We want to extract a subsequence convergent to an element $x \in Y$. Fix an increasing sequence of positive real numbers $\{r_n\}_{n \in \mathbb{N}}$ such that $\lim r_n = \infty$. Denote by $x'_n(r_k)$ the unique $B_{h_k}$-minimal patch of $x_n$ and set $R_{n,k} = \min\{s > 0 :
Following [6, 10], in this case we say that \( \text{supp}(x'_n(r_k)) \subseteq B_n \). By (C4), since \( A_{B_n} \) is finite, there is an infinite subset \( I_1 \subseteq \mathbb{N} \), with \( m_1 = \min I_1 \), such that, for each \( n \in I_1 \), there is \( g_{1,n} \in G_{x'_m(r_1)}(B_{r_1}) \) satisfying \( x'_n(r_1) = g_{1,n}(x'_m(r_1)) \). By (G5), we have

\[
R_{n,1} \leq R_{m,1} + \theta(R_{m,1}, \|g_{1,n}\|_{G[x'_m(r_1)]}).
\]

However, by (G5), we can assume, by taking a subsequence if necessary, that the sequence \( \{g_{1,n}\}_{n \in I_1} \) converges to some \( g_1 \in G_{x'_m(r_1)}(B_{r_1}) \). Hence, there exist \( R(r_1) > 0 \) and \( n_1 \in I_1 \) such that \( R_{n,1} \leq R(r_1) \) and, for \( n > n_1 \) and \( n \in I_1 \), we have

\[
\theta(\Delta(x'_n(r_1), x'_n(r_1)), \|g_{1,n}g_{1,n}^{-1}\|_{G[x'_m(r_1)]}) \leq \theta(R(r_1), \|g_{1,n}g_{1,n}^{-1}\|_{G[x'_m(r_1)]}) \leq 1/r_1.
\]

We can now proceed recursively in order to obtain, for each \( k \in \mathbb{N} \), an infinite set \( I_k \), with \( I_k \subseteq I_{k-1} \subseteq \ldots \subseteq I_2 \subseteq I_1 \subseteq \mathbb{N} \), such that, for each \( n \in I_k \), there is \( g_{k,n} \in G_{x'_m(r_k)}(B_{r_k}) \) with \( x'_n(r_k) = g_{k,n}(x'_m(r_k)) \), where \( m_k = \min I_k \). Again, without loss of generality, we can assume that the sequence \( \{g_{k,n}\}_{n \in I_k} \) converges to some \( g_k \in G_{x'_m(r_k)}(B_{r_k}) \), which means that there exist \( R(r_k) > 0 \) and \( n_k \in I_k \), with \( n_k > n_{k-1} > \ldots > n_2 > n_1 \), such that, for \( n > n_k \) and \( n \in I_k \), we have

\[
\theta(\Delta(x'_n(r_k), x'_n(r_k)), \|g_{k,n}g_{k,n}^{-1}\|_{G[x'_m(r_k)]}) \leq \theta(R(r_k), \|g_{k,n}g_{k,n}^{-1}\|_{G[x'_m(r_k)]}) \leq 1/r_k.
\]

Observe that \( x'_{n_{k+1}}(r_k) = g_{k,n_{k+1}}g_{k,n_k}^{-1}(x'_n(r_k)) \). Then

\[
\begin{align*}
\text{d}(x_{n_k}, x_{n_{k+1}}) \leq \max \left\{ \frac{1}{r_k}, \theta(\Delta(x'_{n_k}(r_k), x'_{n_{k+1}}(r_k)), \|g_{k,n_{k+1}}g_{k,n_k}^{-1}\|_{G[x'_m(r_k)]}) \right\} \\
\leq \max \left\{ \frac{1}{r_k}, \theta(R(r_k), \|g_{k,n_{k+1}}g_{k,n_k}^{-1}\|_{G[x'_m(r_k)]}) \right\} \leq 1/r_k. \quad (10)
\end{align*}
\]

Consider the infinite subset \( I' = \{n_1, n_2, \ldots \} \subseteq \mathbb{N} \). From (10) we see that that \( \{x_n\}_{n \in I'} \) is a Cauchy subsequence of \( \{x_n\}_{n \in \mathbb{N}} \). Since \( Y \) is complete, \( \{x_n\}_{n \in I'} \) admits a subsequence that converges to some \( x \in Y \). \( \square \)

**Example 3.** Let \( F \) be a finite set of tiles and fix \( G = I \). Given a subset \( S \subseteq \mathbb{R}^n \), denote by \( X_F(S) \) the set of all tilings of \( S \) by direct isometric copies of tiles in \( F \). Take \( Y = \bigcup_S X_F(S) \), which satisfies (Y1) and (Y2). Assume that, with respect to the choices of Example \( \square \), the number \( \#F_{E_2} \) of equivalence classes \([x]\), with \( x \in Y \) composed by two tiles in \( F \), is finite. Following \( \square \), in this case we say that \( Y \) has finite local complexity under isometries. For all \( x \in Y \), \( \gamma(x, [x]) = I \), which is complete with respect to \( d_I \). Conditions (C2) and (C3) follow from the continuity of the group action of \( I \) on \( \mathbb{R}^n \). Clearly \( \theta_I(s, t) = t \) satisfies (C3). Finally, (C4) is a consequence of the finite local complexity under isometries property. Then \( (Y, d) \) is compact and the action of \( I \) on \( Y \) is continuous. The space of classical Penrose kite and dart tilings of \( \mathbb{R}^2 \) fits in this case. On the other hand, with respect to the choices of Example \( \square \), \( \#F_{E_2} \) is automatically finite, hence (C4) is satisfied. For each \( x \in Y \) with bounded support, the completeness of \( \gamma(x, [x]) \) follows from the continuity of the group action of \( I \) on \( \mathbb{R}^n \) and from the completeness of \( I[x] \) with respect to the metric \( d_I[x] \) given by \( \square \). The remain conditions hold by the same reasons as above. Then, \( (Y, d) \) is compact and the action of \( I \) on \( Y \) is continuous.
Example 4. Let \( \mathcal{F} \) be a finite set of tiles and fix \( G = \mathcal{H} \). Given a subset \( S \subseteq \mathbb{R}^n \), denote by \( X_F(S) \) the set of all tilings of \( S \) by homothetic copies of tiles in \( \mathcal{F} \) with scale factor \( \lambda \in [\frac{1}{2}, 1] \). Take \( \mathcal{Y} = \bigcup_S X_F(S) \), which satisfies \( (\mathcal{Y}_1) \) and \( (\mathcal{Y}_2) \). Consider the setting of Example 2. For each \( x \in \mathcal{Y} \) with bounded support, the completeness of \( \gamma(x, [x]) \) follows from the continuity of the group action of \( \mathcal{H} \) on \( \mathbb{R}^n \) and from the completeness of \( \mathcal{H}[x] \) with respect to the metric \( d_{\mathcal{H}[x]} \) given by \( (2) \). Clearly \( \theta_{\mathcal{H}}(s, t) = st + t \) satisfies \( (C_3) \) and, since \( \mathcal{F} \) is finite and \( \lambda \) takes values in a compact interval, \( (C_4) \) holds. Finally, \( (C_2) \) and \( (C_5) \) also follows from the continuity of the group action of \( \mathcal{H} \) on \( \mathbb{R}^n \). Hence \( (Y, d) \) is compact and the action of the translation group on \( Y \) is continuous.

Example 5. Let \( Q \) and \( P \) be the hypercubes in \( \mathbb{R}^n \), centered at the origin, with edges 1 and 1/3, respectively. Let \( \mathcal{T} \) be the group of translations. If \( T \in \mathcal{T} \) is the translation by the vector \( \vec{t} \), then \( \|T\|_\mathcal{T} = \|\vec{t}\| \). Given a subset \( S \subseteq \mathbb{R}^n \), let \( \mathcal{X}_{Q, P}(S) \) the set of all tilings of \( S \) by two equivalent classes of tiles: tiles of the form \( T(P) \), with \( T \in \mathcal{T} \); tiles of the form \( T(Q) \setminus T'(P) \), with \( \|T - T'\|_\mathcal{T} \leq 1/6 \). Set \( \mathcal{Y} = \bigcup_S \mathcal{X}_{Q, P}(S) \), \( G[\mathcal{P}] = \mathcal{T} \) and \( G[Q \setminus \mathcal{P}] = \mathcal{T} \times \mathcal{T} \). Consider on \( G[Q \setminus \mathcal{P}] \) the distance

\[
d_{\mathcal{T} \times \mathcal{T}}((T, T'), (T_1, T'_1)) = \max\{d_{\mathcal{T}}(T, T_1), d_{\mathcal{T}}(T', T'_1)\}.
\]

The elements of \( \mathcal{T} \) act on the tiles of the first class in the natural way, and an element \( (T_1, T'_1) \in \mathcal{T} \times \mathcal{T} \) acts on a tile \( T(Q) \setminus T'(P) \) of the second class by \( (T_1, T'_1)(T(Q) \setminus T'(P)) = T_1T(Q) \setminus T'_1T'(P) \). In particular,

\[
\gamma(T(Q) \setminus T'(P), [T(Q) \setminus T'(P)]) = \{(T_1, T'_1) \in \mathcal{T} \times \mathcal{T} : \|T_1T - T'_1T'\|_\mathcal{T} \leq 1/6\}.
\]

Extend in the natural way this equivalence relation to all \( \mathcal{Y} \) (see Figure 5), similarly to (E2). In particular, given \( x = \{D_i\}_{i \in I} \) and \( x' = \{D'_j\}_{j \in J} \) in \( \mathcal{Y} \), then \( x \sim x' \) if there exists \( \alpha : I \to J \) such that, for each \( i \in I \), \( D'_{\alpha(i)} = g_i(D_i) \) for some \( g_i \in G[D_i] \), with \( \sup_{i \in I} \|g_i\|_{G[D_i]} < \infty \). Thus, \( G[x] = \prod_{i \in I} G[D_i] \) is the subgroup of all such collections \( g = \{g_i\}_{i \in I} \) provided with the bi-invariant distance \( d_{G[x]}(g, g') = \sup_{i \in I} d_{G[D_i]}(g_i, g'_i) \). These choices satisfy \( (G_1)-(G_4) \) and \( (G_5) \) holds for \( \theta(s, t) = t \). Fix on \( Y \) the distance \( d \) defined by \( (3) \). Clearly \( \mathcal{T} \) acts continuously on \((Y, d)\). The completeness of \( (Y, d) \) follows from the continuity of the action of \( \mathcal{T} \) on \( \mathbb{R}^n \) and from the closed restriction \( \|T - T'\|_\mathcal{T} \leq 1/6 \). The remaining conditions \( (C_2)-(C_5) \) hold by the same reasons as above. Hence \((Y, d)\) is compact.

3. Local isomorphism

The local isomorphism is a property that a tiling of an Euclidean space might have which expresses a certain “regularity”. Quite surprisingly, this is not an unusual property. For example, all Penrose tilings, which are known to be aperiodic, satisfy the local isomorphism property \( [4] \). More recently, Radin and Wolff \( [9] \) proved that any nonempty tiling space \( Y \) with finite local complexity under isometries must admit a tiling satisfying the local isomorphism property. In order to generalize this result, we define:

Definition 1. Let \((Y, d)\) be the metric space of all tilings of \( \mathbb{R}^n \) in \( \mathcal{Y} \), where \( d \) is defined by \( (3) \). The tiling \( y \in Y \) is said to satisfy the local isomorphism property if for every patch \( y' \) of
We claim that if \( y \) is also compact, we can take a finite set \( X \) such that \( \text{supp}(g(y')) \subseteq B \) and \( g(y') \subseteq y \).

Observe that in the definitions of local isomorphism property given in [4, 9] only isometric copies of \( y' \) are allowed. Let \( T \) be the group of the translations in \( \mathbb{R}^n \). Assume that \( T \subseteq \gamma(y', [y']) \) for all \( y' \in Y \) and that \( T \) acts continuously on \( (Y, d) \).

**Theorem 1.** If \( (Y, d) \) is compact, then there exists a tiling \( y \in Y \) satisfying the local isomorphism property.

**Proof.** We follow closely the proof of the main theorem of [9]. Let \( X \) be a minimal invariant subset of \( Y \), that is, \( X \) is nonempty, closed, invariant under \( T \) and there is no closed proper subset of \( X \) which is also invariant under \( T \). Given a nonempty open subset \( U \) of \( X \), then due to the continuity of the action of \( T \) and the minimality of \( X \), we have \( X = \bigcup_{T \in T} T(U) \). Since \( X \) is also compact, we can take a finite set \( \{T_i\}_{i \in I} \) of translations such that \( X = \bigcup_{i \in I} T_i(U) \).

We claim that if \( y \in X \) then \( y \) satisfies the local isomorphism property. Indeed, let \( y' \) be a patch of \( y \) with bounded support. We fix \( \epsilon > 0 \) and define

\[
Y_{\epsilon, y'} = \{ z \in Y : g(y') \subset z, \text{ for some } g \in \gamma(y', [y']) \text{ with } \|g\|_{G[y']} < \epsilon \}.
\]

This is an open set of \( (Y, d) \). In fact, take \( z_0 \in Y_{\epsilon, y'} \). By definition, this means that there exist \( \epsilon_1 < \epsilon \) and \( g \in \gamma(y', [y']) \) such that \( g(y') \subset z_0 \) and \( \|g\|_{G[y']} = \epsilon_1 \). On the other hand, if \( z \in Y \) is such that \( d(z, z_0) < \delta \), then \( z' \in z_0[[B_{1/\delta}]] \) and \( h \in G[z'_0] \) such that \( h(z'_0) \in z[[B_{1/\delta}]] \) and \( \theta(\Delta(z'_0, h(z'_0)), \|h\|_{G[z'_0]}) \leq \delta \). We can choose \( \delta > 0 \) sufficiently small in order to \( \text{supp}(g(y')) \subseteq B_{1/\delta} \) and, taking account the properties of \( \theta \in \Theta \), \( \|h\|_{G[z'_0]} < \epsilon - \epsilon_1 \). Hence \( h g(y') \subset z \) and \( \|h g\|_{G[y']} \leq \delta + \|g\|_{G[y']} < \epsilon \), that is \( z \in Y_{\epsilon, y'} \), which shows that the open ball \( \{ z \in Y : d(z, z_0) < \delta \} \) is contained in \( Y_{\epsilon, y'} \). Consequently, \( Y_{\epsilon, y'} \) is open and, as we have seen before, we can take a finite set \( \{T_i\}_{i \in I} \) of translations such that \( X \subseteq \bigcup_{i \in I} T_i(Y_{\epsilon, y'}) \).

Choose \( r(y') > 0 \) such that \( \text{supp}(T_i g(y')) \subseteq B_{r(y')} \) for all \( i \) and \( g \in \gamma(y', [y']) \) with \( \|g\|_{G[y']} < \epsilon \). Each ball \( B \) with radius \( r(y') \) defines a translation \( T \) such that \( T(B) = B_{r(y')} \). Since \( T(y) \in X \), \( T(y) \) contains some patch of the form \( T_i g(y') \), which means that \( T^{-1} T_i g(y') \subset y \). This establishes our claim. \( \square \)
Recall that if \( G \) is a group acting continuously on a compact topological space \( X \), then \( X \) is said to be {**uniquely ergodic**} if admits one and only one \( G \)-invariant Borel probability measure. The support of a uniquely ergodic measure is minimal invariant. Hence, under the unique ergodicity condition, almost all tilings in \( Y \) satisfy the local isomorphism property:

**Corollary 1.** In the conditions of Theorem 1, if \((Y,d)\) is uniquely ergodic with unique \( T \)-invariant measure \( \mu \), then all tilings in \( Y \) have the local isomorphism property except for a subset of \( \mu \)-measure zero. Furthermore, the radius \( r(y') \) is independent of the tiling.

4. **Brown’s lemma and its topological dynamics version**

The main idea of Ramsey theory is that arbitrarily large sets cannot avoid a certain degree of "regularity". This is exemplarily illustrated by Gallai’s Theorem, a multidimensional version of the seminal van der Waerden theorem. In some sense, the results on local isomorphism property that we have seen in the previous section agree with this idea, which makes one expect deeper connections between Ramsey theory and tiling theory. In this direction, de la Llave and Windsor [6] exploited an application of the Furstenberg’s topological multiple recurrence theorem (which is a topological dynamics version of the multidimensional version of van der Waerden’s theorem) to tilings. A not so famous Ramsey-type result is the so called Brown’s lemma [1, 2], which asserts that any finite coloration of the natural numbers admits a monochromatic piecewise syndetic set. In Section 5 we shall give an application of this lemma to tiling Theory. Before that, let us fix a suitable statement of Brown’s lemma and establish its topological dynamics version.

Recall the following notions of largeness of subsets of a topological semigroup \( G \):

(a) a subset \( S \) of \( G \) is {**syndetic**} if there exists a compact \( K \subseteq G \) so that for any \( g \in G \), there exists \( k \in K \) with \( gk \in S \);

(b) a subset \( T \) of a topological semigroup \( G \) is {**thick**} if for any compact set \( K \subseteq G \) there exists \( g \in G \) with \( gK \subseteq T \);

(c) a subset of \( G \) is {**piecewise syndetic**} if it is the intersection of a syndetic set and a thick set.

When \( G = \mathbb{N} \), this means that \( S \) is piecewise syndetic if \( S \) contains arbitrarily long intervals (\( S \) is thick) with bounded gaps (\( S \) is syndetic).

Brown [1][2] proved that any finite coloration of the natural numbers admits a monochromatic piecewise syndetic set (Brown’s lemma). More recently, Hindman and Strauss (see Theorem 4.40 in [5]) proved that a subset of a discrete semigroup \( S \) is piecewise syndetic if and only if its closure intersects the smallest ideal of the Stone-Čech compactification of \( S \), which is never empty. Hence, any finite coloration of \( S \) admits a monochromatic piecewise syndetic set. In the \( \mathbb{Z}^n \) case, we have:

**Lemma 2** (Multidimensional Brown’s lemma). Given a finite coloration of the integer lattice \( \mathbb{Z}^n \), there exists \( q \in \mathbb{N} \) satisfying: for any finite subset \( F = \{P_i\}_{i \in I} \) of \( \mathbb{Z}^n \) and any \( \lambda \in \mathbb{N} \), there
exist $\bar{t} \in \mathbb{Z}^n$ and a collection $\{\bar{v}_i\}_{i \in I}$, with $\bar{v}_i \in Q^q_i = \{\bar{u} = (u_1, \ldots, u_n) \in \mathbb{Z}^n : -q \leq u_j \leq q\}$, such that $\{\lambda P_i + \bar{t} + \bar{v}_i\}_{i \in I}$ is monochromatic.

Let us compare this lemma with the well known (see [3]) multidimensional version of the van der Waerden’s Theorem (also known as Gallai’s Theorem), which asserts that, given a finite coloration of $\mathbb{Z}^n$, any finite subset $F$ of $\mathbb{Z}^n$ has a monochromatic homothetic copy $\lambda F + \bar{t}$. However, it says nothing, apart its existence, about the scale factor $\lambda$. On the other hand, Lemma 2 states that we can take any $\lambda$ once we allow “bounded perturbations” ($q$ only depends on the coloration) in the structure of the homothetic copies of $F$.

The Furstenberg’s topological multiple recurrence theorem [3] is a topological dynamics version of Gallai’s theorem. The following is a topological dynamics version of Lemma 2.

**Lemma 3** (Topological dynamics multidimensional Brown’s lemma). Let $(X, d)$ be a compact metric space and $T_1, \ldots, T_l$ commuting homeomorphism of $X$. Given $\epsilon > 0$, there exists $q \in \mathbb{N}$ satisfying: for each $k \in \mathbb{N}$, there exist $x \in X$ and a collection $\{\bar{u}_i\}_{i \in \{1, \ldots, l\}}$, with $\bar{u}_i = (u_i^1, \ldots, u_i^l) \in Q^q_i$, such that

$$d(x, T_{i_1}^{u_1} \cdot \ldots \cdot T_{i_l}^{u_l} x) < \epsilon$$

for all $i \in \{1, \ldots, l\}$.

**Proof.** This is analogous to the standard proof of Furstenberg’s topological multiple recurrence theorem from Gallai’s theorem. Let $U_1, \ldots, U_r$ be a covering of $X$ by pairwise disjoint sets of less than $\epsilon$ diameter. Choose $y \in X$ and consider the coloration $\mathbb{Z}^l = \bigcup_{i=1}^r C_i$ defined as follows: $(a_1, \ldots, a_l) \in C_i$ if $T_{i_1}^{a_1} \cdot \ldots \cdot T_{i_l}^{a_l} y \in U_i$. According to Lemma 2 we can fix $q/2 \in \mathbb{N}$ satisfying: for each $k \in \mathbb{N}$, there exists $\bar{t} = (t_1, \ldots, t_l) \in \mathbb{Z}^l$ and a collection $\{\bar{v}_i\}_{i \in \{0, \ldots, l\}}$, with $\bar{v}_i = (v_i^1, \ldots, v_i^l) \in Q^q_{q/2}$, such that one of the cells $C_i$ contains the homothetic “$q/2$-distorted” copy $\{kP_i + \bar{t} + \bar{v}_i : P_i \in F\}$ of

$$F = \{P_0 = (0, \ldots, 0), P_1 = (1, 0, \ldots, 0), P_2 = (0, 1, 0, \ldots, 0), \ldots, P_l = (0, \ldots, 0, 1)\}.$$

That is,

$$\{(t_1 + v_1^0, \ldots, t_i + v_i^0), (k + t_1 + v_1^1, t_2 + v_2^1, \ldots, t_l + v_l^1), \ldots, (t_1 + v_1^l, t_2 + v_2^l, \ldots, k + t_l + v_l^l)\}$$

is a subset of $C_i$. Let $x = T_{i_1}^{t_1} \cdot \ldots \cdot T_{i_l}^{t_l} y$. We then have, with $\bar{u}_i = \bar{v}_i - \bar{v}_0 \in Q^q_i$,

$$\{x, T_{i_1}^{u_1} \cdot \ldots \cdot T_{i_l}^{u_l} x, T_{i_1}^{k}T_{i_1}^{u_1} \cdot \ldots \cdot T_{i_l}^{u_l} x, \ldots, T_{i_l}^{k}T_{i_1}^{u_1} \cdot \ldots \cdot T_{i_l}^{u_l} x\} \subseteq U_i.$$

The result follows now from the fact that the diameter of $U_i$ is less than $\epsilon$. \qed

5. **An Application of Brown’s Lemma to Tiling**

In [3], the authors proved, by using the well known Furstenberg’s multiple recurrence theorem, that given a tiling $y$ of $\mathbb{R}^n$ and a finite geometric pattern $F \subset \mathbb{R}^n$ of points, one can find a patch $y'$ of $y$ so that copies of $y'$ appear in $y$ “nearly” centered on some scaled and translated version of the pattern. Three distinguished cases were considered (Theorems 2, 3 and 4 of [3]): when $y$ exhibits finite local complexity under translation; when $y$ exhibits finite local complexity under
isometries; finally, for tilings $y$ without finite local complexity. Taking account the general setting we have developed in Section $2$, next we present a unified and generalized reformulation of these results (Theorem $3$). Furthermore, we give an application of Lemma $3$ to tiling theory (Theorem $3'$).

Let $(Y, d)$ the metric space of all tilings of $\mathbb{R}^n$ in $\mathcal{Y}$, where $d$ is defined by $[3]$. Let $\mathcal{T}$ be the group of translations in $\mathbb{R}^n$ and denote by $T_\theta$ the translation by the vector $\theta \in \mathbb{R}^n$. Suppose that $\mathcal{T} \subseteq \gamma([y], [y'])$ for all $y' \in \mathcal{Y}$ and that each $g \in \mathcal{T}$ induces a map $g : (Y, d) \to (Y, d)$ in the conditions of Proposition $3$. In particular, $\mathcal{T}$ acts continuously on $(Y, d)$.

**Theorem 2.** Assume that $(Y, d)$ is compact. Given $y \in Y$, $\epsilon > 0$ and a finite subset $F = \{\bar{v}_1, \ldots, \bar{v}_l\} \subset \mathbb{R}^n$, there exist $k \in \mathbb{N}$ and a patch $y' \in y[[B_1/\epsilon]]$ satisfying: for each $\bar{v}_i \in F$ there exists $g_i \in G[y']$, with $\|g_i\|_{C[y']} < \epsilon$, such that $T_{\bar{v}_i}^k g_i y' \subset y$.

Again, this theorem says nothing about the scale factor $k$. The following theorem shows that we can take any $k$ once we allow “bounded perturbations” in the structure of the hypothetic copies of the geometric pattern $F$. The proof of Theorem $2'$, which we omit here, is a straightforward adaptation of the arguments used in $[6]$ and in the proof of Theorem $3$.

**Theorem 3.** Assume that $(Y, d)$ is compact. Given $y \in Y$, $\epsilon > 0$ and a finite subset $F = \{\bar{v}_1, \ldots, \bar{v}_l\} \subset \mathbb{R}^n$, there exist $q \in \mathbb{N}$ and a patch $y' \in y[[B_1/\epsilon]]$ satisfying: for each $\bar{v}_i \in F$ and $k \in \mathbb{N}$, there exists $g_{k,i} \in G[y']$, with $\|g_{k,i}\|_{C[y']} < \epsilon$, such that

$$T_{\bar{w}_{k,i}} T_{\bar{v}_i}^k g_{k,i} y' \subset y,$$

for some $\bar{w}_{k,i} \in Q_q(F) = \{\bar{w} \in \mathbb{R}^n : \bar{w} = \sum_{i=1}^l \alpha_i \bar{v}_i, -q \leq \alpha_i \leq q, \alpha_i \in \mathbb{Z}\}$.

**Proof.** Consider $y \in Y$ and $Y_0 = \text{closure}((Ty) \subseteq Y$. Clearly, $(Y_0, d)$ is compact and invariant under the action of $\mathcal{T}$. Let $F = (\bar{v}_1, \ldots, \bar{v}_l)$ and consider the $l$ commuting homeomorphisms of $Y_0$ given by $T_i = T_{-\bar{v}_i}$.

By Lemma $3$ for each $\epsilon' > 0$ there exists $q \in \mathbb{N}$ satisfying: for each $k \in \mathbb{N}$ there exist $x \in Y_0$ and a collection $\{\bar{u}_i\}_{i \in \{1, \ldots, l\}}$, with $\bar{u}_i \in Q_q$ and $\bar{u}_i = (u_{i1}, \ldots, u_{il})$, such that

$$d(x, T_i^k T_{u_{i1}} \ldots T_{u_{il}} x) < \epsilon'$$

for all $i \in \{1, \ldots, l\}$. Since $x \in Y_0$ is either a translation of $y$ or the limit of translations of $y$, we can find $\bar{v} \in \mathbb{R}^n$ such that

$$d(T_{\bar{v}} y, T_i^k T_{u_{i1}} \ldots T_{u_{il}} T_{\bar{v}} y) < \epsilon',$$

for all $i \in \{1, \ldots, l\}$. By the definition of the metric $d$, there exist

$$z_i' \in T_i^k T_{u_{i1}} \ldots T_{u_{il}} T_{\bar{v}} y[[B_1/\epsilon']], z_i'' \in T_{\bar{v}} y[[B_1/\epsilon']]$$

and $h_i \in G[z_i']$, with $\theta(\Delta(z_i', z_i''), \|h_i\|_{C[z_i']}) \leq \epsilon'$, such that $h_i(z_i') = z_i''$. Now consider $z''$ to be the connected component of $\bigcap_{i=1}^l z_i''$ whose support contains the ball $B_{1/\epsilon'}$. Set $y_k' = T_{-\bar{v}} z''$. By construction,
$y'_k \subset T_{-\vec{v}_i} T_{\vec{v}^i_1} \ldots T_{\vec{v}^i_l} y$, that is
\[ T_{-\vec{v}^i_1} \ldots T_{-\vec{v}^i_l} T_{-\vec{v}_i} T_{\vec{v}^i_1} \ldots T_{\vec{v}^i_l} y_k \subset y. \]
Write $g_{k,i} = T_{-\vec{v}^i_1} \ldots T_{-\vec{v}^i_l} T_{-\vec{v}_i} T_{\vec{v}^i_1} \ldots T_{\vec{v}^i_l} y_k$. By the continuity of the left and right multiplication by translations, and by the properties of $\theta \in \Theta$, we can choose $\epsilon' < \epsilon$ such that $B_{1/\epsilon} \subseteq \text{supp}(y'_k)$ and $\|g_{k,i}\|_{G[y'_k]} < \epsilon$. The result holds with $y' = \bigcap_k y'_k$ and $\vec{w}_{k,i} = \sum_{j=1}^l u^i_j \vec{v}_j$. □

Let us give an informal pictorial illustration of Theorem 3. Suppose that we have a tiling $y$ and a finite subset $F$ of $\mathbb{R}^n$, for example the set represented in Figure 2.

Given $\epsilon > 0$, there exists $q \in \mathbb{N}$ and a patch $y' \in y_1^{1/\epsilon}[[B_{1/\epsilon}]]$ such that, for each scale factor $\lambda$, one can find and a vector $\vec{t}$ so that copies of $y'$ appear in the tiling “nearly” centered (in the sense that, for each $\vec{v}_i \in F$, $\|g_{k,i}\|_{G[y']} < \epsilon$) on $\lambda F + \vec{t}$ up to “bounded perturbations” (in the sense that $q$, and consequently $Q_q(F)$, does not depend on $\lambda$), as represented in Figure 3.

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