Hilbert Irreducibility above algebraic groups

Umberto Zannier

ABSTRACT. This paper concerns Hilbert Irreducibility for covers of algebraic groups, with results which appear to be difficult to treat by existing techniques. The present method works by first studying irreducibility above ‘torsion’ specializations (e.g., over cyclotomic extensions) and then descending the field (by Chebotarev Theorem). Among the results we offer an irreducibility theorem for the fibers, above a cyclic dense subgroup, of a cover of $\mathbb{G}_m^n$ (Thm. 1) and of a power $E^n$ of an elliptic curve without CM (Thm. 2); this had not been treated before for $n > 1$. As a further application, in the function field context, we obtain a kind of Bertini’s theorem for algebraic subgroups of $\mathbb{G}_m^n$ in place of linear spaces (Thm. 3). Along the way we shall prove other results, as a general lifting theorem above tori (Thm. 3.1).

§1. INTRODUCTION.

This paper is in the context of the Hilbert Irreducibility Theorem (HIT in the sequel); we offer some results on the lifting of rational points above algebraic groups, which appear to be difficult to treat by existing techniques.

In the paper by ‘cover’ we mean a dominant rational map $\pi : Y \to X$ of finite degree between irreducible varieties. (We remark at once that by shrinking $Y$ and $X$ to Zariski-open subsets of them, we may actually assume for most purposes that $\pi$ is a morphism, and even that it is étale, i.e. that there are exactly $\text{deg} \pi$ points above every point of $X$.)

Consider a cover $\pi : Y \to X$, defined over a number field $k$. Basic questions in Diophantine Geometry can be formulated in terms of the lifting of rational points $x \in X(k)$ to $Y$: when does it happen that $\pi^{-1}(x)$ contains a point in $Y(k)$, or is $k$-irreducible? The classical HIT states that if $X = \mathbb{A}^n$ then one may find $x \in X(k)$ such that this last possibility happens (even considering simultaneously finitely many covers of $X$).

Now, it is of interest, also for the applications, to obtain such ‘good’ points $x$ in restricted sets of rational points. Situations which are obviously relevant occur when $X$ is an algebraic group, because these are the fundamental varieties where we are able to generate systematically rational points. Here we shall study the lifting of points in a Zariski dense cyclic subgroup $\Omega \subset X(k)$, for $X$ either a multiplicative torus or a power of an elliptic curve: under a necessary geometrical condition on $Y$ (see Def. below) we prove that $\pi^{-1}(x)$ is $k$-irreducible for each $x$ in a suitable coset of finite index in $\Omega$.

We stress that, though the literature is rich of many versions of HIT (see e.g. [BG, §9.6], [FJ, Ch. 11], [Sch, §4.4], [Se1, Ch. 9], [Se2, Ch. 3]), the said basic situations do not appear to fall into existing methods. After the work of Faltings, Vojta and others, much is known for subvarieties of commutative algebraic groups, but for covers of them the situation is still unsatisfactory in dimension $> 1$, even in very simply-stated cases.

(1) The original case of HIT is no exception: $\mathbb{A}^n = \mathbb{G}_a^n$ as a variety.
Some results (dealing basically with linear recurrences) came implicitly from the papers [Z], [FZ], whose conclusions were applied in [C] to a HIT over linear algebraic groups. In fact, these papers contain primordial ad hoc versions of the present method; here, in addition to further results, this is developed in a more systematic way, also in view of future possible applications.

In particular, here we consider the context of abelian varieties (not touched in [Z], [FZ], [C]) from this viewpoint, an issue which is explicitly mentioned in the discussion in Serre’s [Se2], §5.4.

Below we focus on the case $X = E^n$. This case can be better treated because of results on the Galois action on torsion points which are better known than otherwise, but is not an a priori limitation of the method.

The method consists of two main stages and may be very roughly described as follows:

(A) To use a suitable (explicit) HIT over a big (cyclotomic) field, of infinite degree over $\mathbb{Q}$.

(B) To transfer the irreducibility to points over a number field.

A relevant issue here is that a kind of HIT may be proved directly over the big field, actually for explicit specializations at torsion points: for $\mathbb{G}_m^n$ this ingredient has been essentially done in [DZ] (see Theorem [DZ] below) hence (A) applies. In the case of abelian varieties step (A) is obtained below in a different way. Then the transfer (B) leads to the sought explicit versions of HIT over number fields. This step involves $v$-adic approximation to the torsion points coming from step (A), and Chebotarev Theorem (which may be seen as a 0-dimensional version of HIT).

In all of this, it turns out that the group structure of torsion points is especially relevant for the location of ‘good’ specializations in algebraic groups.

Before stating some conclusions, we introduce a simple geometrical condition on the covers, which shall turn out to be necessary and sufficient for our purposes.

For $X$ a commutative connected algebraic group, we let $[m] : X \to X$ denote the multiplication map. By ‘irreducible’ we mean throughout ‘$\bar{k}$-irreducible’ (supposed for all involved $X,Y$).

**Definition:** We say that the cover $\pi : Y \to X$ satisfies condition (PB) (‘pull-back’) if for any integer $m > 0$ the pull-back $[m]^*Y := X \times_{[m],X} Y$ is irreducible.

For instance: if $X = \mathbb{G}_m^n$ and $Y : f(x_1, \ldots, x_r, y) = 0$, (PB) means that $f(x_1^m, \ldots, x_r^m, y)$ is irreducible for all $m > 0$. For $X = \mathbb{G}_a^n$, one instead finds that (PB) is always trivially verified.

**Remarks.**

(i) Note that this condition is unavoidable for our lifting issues. In fact, suppose that $[m]^*Y$ is reducible, equal to the union $U \cup V$ of proper closed subsets. Let $\Omega$ be any finitely generated (dense) subgroup of $X(k)$. By enlarging $k$ to a finite extension, assume that $U, V$ are defined over $k$ and that $\Omega \subset [m](X(k))$. Let $x \in \Omega$ and write $x = [m]x'$ for $x' \in X(k)$. If $\pi(y) = x$, the pair $(x', y)$ is in $[m]^*Y(k) = U(k) \cup V(k)$. For ‘general’ $x$, this yields a nontrivial splitting of the fiber $\pi^{-1}(x)$ into two subsets defined over $k$, so $\pi^{-1}(x)$ cannot be $k$-irreducible.

(ii) In Proposition 2.1 we shall prove in a simple way two equivalences for this condition. First, we shall see that it holds if and only if it holds for $m = \deg \pi$ (so it is a ‘computable’ condition). Secondly, we shall prove that it holds if and only if the map $\pi$ has no nontrivial isogeny factors, which shows the relevance of ramification. (Note that when $\pi$ itself is an isogeny up to birationality, for large $k$ we have $\Omega \subset \pi(Y(k))$, so if $\deg \pi > 1$ the irreducibility of $\pi^{-1}(x)$ badly fails.)
Let us now give some statements, starting with the case \( X = \mathbb{G}_m^n \times \mathbb{G}_a \); we stress that the crux is represented by the \( \mathbb{G}_m^n \)-component (the \( \mathbb{G}_a \) being included for completeness).

**Theorem 1.** For \( i = 1, \ldots, h \), let \( \pi_i : Y_i \to X := \mathbb{G}_m^n \times \mathbb{G}_a \) be a cover satisfying (PB). Then, if \( \Omega \) is a cyclic Zariski-dense subgroup of \( X(k) \), there exists a coset \( C \) of finite index in \( \Omega \) such that for all \( x \in C \) and for all \( i = 1, \ldots, h \) the fiber \( \pi_i^{-1}(x) \) is irreducible over \( k \).

This result, derived from the more general Proposition 3.1, immediately implies a sharp form of the so-called Pisot \( d \)-th root conjecture on linear recurrences (proved in [Z]). It also implies the more general results on recurrences of [FZ], which have been used in [C] to derive an elegant version of HIT over linear algebraic groups \( X \); in this last paper it is proved in particular ([C], Cor. 7.15) that if \( Y \) is smooth and \( \pi : Y \to X \) is finite, then either it is unramified or any Zariski-dense semigroup \( \Omega \subset X(k) \) contains ‘good’ points; from [C] (which works also with reducible \( Y \)) with some work one may derive a weak version of Theorem 1 in which \( C \) is just an infinite set.\(^{(2)}\)

In the context of abelian varieties, we have the following analogue for powers of an elliptic curve \( E \) without CM:

**Theorem 2.** For \( i = 1, \ldots, h \), let \( \pi_i : Y_i \to E^n \) be a cover satisfying (PB). Then, if \( \Omega \) is a cyclic Zariski-dense subgroup of \( E^n(k) \), there exists a coset \( C \) of finite index in \( \Omega \) such that for all \( x \in C \) and for all \( i = 1, \ldots, h \) the fiber \( \pi_i^{-1}(x) \) is irreducible over \( k \).

The case \( n = 1 \) follows in stronger ‘finiteness’ form from Faltings’ solution of Mordell’s conjecture (see [Se2], §5.4), but for \( n > 1 \) even the weaker assertion in which \( C \) is just an infinite subset of \( \Omega \) had not been treated before.

Theorem 1 looks similar, but can be obtained more rapidly, due to our results for cyclotomic fields for which we have no counterpart for fields generated by torsion points of abelian varieties (see §2). So, Theorem 2 requires additional arguments, and we treat it separately.

Theorems 1, 2 concern irreducibility of the fiber \( \pi^{-1}(x) \). Another question is whether a fibre contains rational points. This easily reduces to the former; an explicit statement is Theorem 4 in §4, where (PB) is replaced by the weaker condition of being not birationally equivalent to an isogeny.

Our next result is a simple application of the method in the function field context: we offer a toric analogue of Bertini Theorem, where algebraic subgroups of \( \mathbb{G}_m^n \) replace linear subspaces. For this result we denote by \( \kappa \) an algebraically closed field of characteristic zero, and by \( \theta G \) the translate of the algebraic subgroup \( G \) by the torsion point \( \theta \).

**Theorem 3.** Let \( \pi : Y \to \mathbb{G}_m^n \) be a cover defined over \( \kappa \), satisfying (PB). Then there is a finite union \( \mathcal{E} \) of proper connected algebraic subgroups of \( \mathbb{G}_m^n \) such that if a connected algebraic subgroup \( G \) is not contained in \( \mathcal{E} \), then \( \pi^{-1}(\theta G) \) is irreducible (over \( \kappa \)) for every torsion point \( \theta \).

The Bertini Theorem may be seen as a version of a similar statement for \( \mathbb{G}_a^n \); a main difference is that in the present case the algebraic subgroups form a ‘discrete’, rather than algebraic, family.

\(^{(2)}\) The paper [C] reduces to \( \mathbb{G}_m \) and \( \mathbb{G}_a \) by considering subgroups generated by a single matrix; the component of the identity in the closure of such a subgroup is isomorphic to \( \mathbb{G}_m^e \times \mathbb{G}_a^e, e = 0, 1 \).
with degrees tending to infinity. We note that also here condition (PB) cannot be omitted. By specialization the theorem may be readily reduced to the crucial case $\kappa = \mathbb{Q}$.

Here is a polynomial version of the theorem: Let $f \in \kappa[x_1, \ldots, x_n, y]$ be of degree $d > 0$ in $y$ and such that $f(x_1^d, \ldots, x_n^d, y)$ is irreducible. Then there is a finite union $\mathcal{H}_f$ of proper subgroups of $\mathbb{Z}^n$ such that if $(a_1, \ldots, a_n) \in \mathbb{Z}^n \setminus \mathcal{H}_f$, then $f(\theta_1 t^{a_1}, \ldots, \theta_n t^{a_n}, y) \in \kappa[t, t^{-1}, y]$ is irreducible for all roots of unity $\theta_1, \ldots, \theta_n$.

In particular, the Kronecker’s substitution $(x_1, \ldots, x_n) \mapsto (t, t^m, \ldots, t^{m^{n-1}})$ preserves the irreducibility over $\mathbb{Q}$ of a polynomial $f$ as above, for all integers $m$ large enough in terms of $f$. (We wonder whether it suffices that $m > M_0(\deg f)$.) For results in the same direction, but only over $\mathbb{Q}$, not $\mathbb{Q}$, and with an additional assumption on $f$ (not to be self-inversive in the $x_i$), see [Sch3].

We have stated some applications of the method, but along the way we shall obtain other results, and the organization of this paper is as follows. We shall soon conclude this introduction with a few further examples and remarks on the above theorems. In §2 we shall discuss condition (PB) and state (relying on [DZ]) a basic ingredient (Theorem 2.1), an explicit HIT over cyclotomic fields; an analogue for abelian varieties seems not free of interest but cannot be obtained with the same methods; in this direction we shall formulate a conjecture, related to the Manin-Mumford conjecture. In §3 we shall prove a Theorem 3.1 on the lifting of rational points in $v$-adic neighborhoods of torsion points, leading to Theorem 1. In §4 we shall obtain Theorem 2, which shall be distinctly more delicate than the toric case. In §5 we shall present a brief deduction of Theorem 3 from Theorem 2.1 and also a different proof of a (weaker but more laborious) version of this for $E^n$ in place of $G_m^n$.

Further remarks and examples. The case of $G_a$ of Theorem 1 (i.e. $r = 0$) reduces to a refined version of the classical HIT, obtained first by Schinzel [Sch2], who proved in any dimension the existence of whole arithmetical progressions of ‘good’ integer specializations. For $G_a$, the fundamental case occurs in dimension 1, since the closure of a cyclic subgroup of $G_m^2$ is a line. (Note also that (PB) is always true for covers of $G_m^2$.)

On the contrary, the case of $G_m^2$ does not admit such a reduction to curves, because a cyclic group $\Omega = \xi^Z$ may well be Zariski-dense in $G_m^n$: this happens when the coordinates $\xi_1, \ldots, \xi_n$ of $\xi$ are multiplicatively independent. In practice, the situation of Theorem 1 without $G_a$ components leads to diophantine equations of the shape $f(\xi_1^n, \ldots, \xi_n^n, y) = 0$, $n \in \mathbb{Z}, y \in k$. For $r = 1$ one can use Siegel’s theorem or other strong results on curves (see [D] or [CZ]) to prove even finiteness in the ramified case. But for $r > 1$ finiteness is known only in special cases. See [CZ], Theorem 2, for a proof of finiteness on the so-called dominant root assumption; this is satisfied ‘often’, but not generally.

Here is an amusing example-problem: Take $Y : \{y^2 = x_1 + x_2 + 1\}$, and $X := G_m^2$, and let $\pi(x_1, x_2, y) = (x_1, x_2)$ be the projection $(x_1, x_2, y) \mapsto (x_1, x_2)$. Take also $\Omega = (2 + i, 2 - i)^\mathbb{Z}$ where $i^2 = -1$. I do not know of any method to prove finiteness of solutions $(n, y) \in \mathbb{Z} \times k$ of $y^2 = (2 + i)^n + (2 - i)^n + 1$. Theorem 1 yields a whole progression of integers $n$ such that $(2 + i)^n + (2 - i)^n + 1$ is not a square in $k$.

It may be worthwhile to point out why in these cases we indeed expect only finitely many solutions. Consider more generally the points of $Y$ with $S$-units $x_1, x_2 \in \mathcal{O}_{k,S}$ and $y \in \mathcal{O}_{k,S}$. We
may embed \( X = \mathbb{G}_m^2 \) in \( \mathbb{P}_2 \) and take the closure \( \overline{Y} \) of \( Y \) in \( \mathbb{P}_2 \times \mathbb{P}_1 \). We are seeking the integral points on \( \overline{Y} \) with respect to the complement \( D = \overline{Y} \setminus Y \), which is the pull-back by \( \pi \) of the three lines \( \mathbb{P}_2 \setminus \mathbb{G}_m^2 \); it turns out that \( K_{\overline{Y}} + D \) is the class of the ramification divisor, i.e. the closure of \( x_1 + x_2 + 1 = 0 \). This is ‘big’, so a conjecture of Bombieri-Lang-Vojta (see Cor. 4.2, p. 223 of [L2], or Conj. 14.3.2, p. 483 of [BG]) predicts that the integral points all lie on a curve. Given this, it is then easy to derive finiteness of our actual solutions from Siegel’s theorem. (One can also use probabilistic considerations. Further evidence is provided by the function-field case: see [CZ3], Thm. 1.1.)

As to the abelian context of Theorem 2, our proofs are more laborious and represent one of the main points of this paper. They involve, in place of cyclotomic fields, a kind of weak form of step (A) for fields generated by torsion points of elliptic curves. In principle, these techniques seem to be extendable to other abelian varieties, provided the Galois properties of the corresponding torsion points are sufficiently well known.

We have treated cyclic groups for simplicity, but the method extends to finitely generated ones, with some complication of details but with no conceptual difference with the cyclic case.

As to Theorem 3, the cover of \( \mathbb{G}_m^2 \) given by \( y^2 = 1 + 2x_1 + x_2 \) shows that \( \mathcal{E} \) cannot be generally taken \( \{(1, 1)\} \). By standard arguments one can deduce that, on the same assumptions, if \( G \not\subset \mathcal{E} \) the set \( \{ c \in \mathbb{G}_m^n(\bar{k}) : \pi^{-1}(cG) \) is reducible \} is closed and proper in \( \mathbb{G}_m^n \). However the irreducibility of \( \pi^{-1}(cG) \) for arbitrary \( c \) is more delicate and to our knowledge not yet completely clarified (see [CZ2], §5, for the case \( n = 2 \)).

I owe to a referee that Theorem 3 is related to a theorem of Kleiman (see [K], Thm. 2), which implies in particular that \( \pi^{-1}(gG) \) is regular, for \( g \) in an open dense \( U = U_G \). Now, if we also know that \( \pi^{-1}(gG) \) is connected, we can deduce irreducibility (as is done in some proofs of Bertini’s theorem). It is also to be noted that the set \( U_G \) depends on \( G \), whereas the above \( \mathcal{E} \) does not.

All arguments of this paper are effective (except that for the application of Serre’s theorem on the Galois image we need an effective version of it).

As a related topic for the interested reader, we mention the paper [CS]: it deals with Hilbert Irreducibility in the context of linear algebraic groups, but with an entirely different viewpoint, focusing on the role of the ground field.

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**§2. The condition (PB) and a cyclotomic HIT.**

In this section we prove some simple properties of the condition (PB), and then deduce a cyclotomic version of HIT from results in [DZ]. As above, \( X \) denotes a commutative algebraic group. We assume it is defined over a number field \( k \), and that it is connected over \( \bar{k} \).
Proposition 2.1. Let $\pi : Y \to X$ be a cover of degree $d$, defined over $\bar{k}$. Then it satisfies (PB) if and only if $[d]^*Y$ is irreducible. The map $\pi$ factors as $\lambda \circ \rho$, where $\lambda : Z \to X$ is an isogeny of algebraic groups and $\rho : Y \to Z$ is a rational map satisfying (PB).

Proof. We shall work with varieties and maps defined over $\bar{k}$. Let us consider a decomposition $[m]^*Y = U_1 \cup \ldots \cup U_s$ into irreducible components, where $m \geq 1$. The kernel $T$ of $[m]$ operates by translation on $[m]^*Y$, as $t : (x, y) \mapsto (x + t, y)$, for $t \in T$. Hence $T$ permutes the $U_i$; let $T_1$ be the stabilizer of $U_1$, and set $X_1 := X/T_1$. Note that $X_1$ is an algebraic group isogenous to $X$, and $[m]$ factors as $\lambda \circ \tau$, where $\lambda : X \to X_1$ is the projection. Also, the degree of $[m]^*Y \to Y$ is deg$[m]$, whereas the $U_i$ are covers of $Y$ of degree $|T_i|$ and we have deg$[m] = s \cdot |T_1|$. Let now $(x, y) \in U_1$; the class $x + T_1$ depends only on $y$, so we have a rational map $\eta : y \mapsto x + T_1 = \tau(x) \in X_1$. Hence $\eta(y) = \tau(x)$, so $\pi(y) = [m]x = \lambda \circ \tau(x) = \lambda \circ \eta(y)$.

Note that deg $\lambda = |T_1/|T_1| = s$, and this divides $d = \deg \pi = \deg \eta \cdot \deg \lambda$. Hence the map $[d]$ on $X$ factors through $X_1$, and we may write $[d] = \lambda \circ \tilde{\tau}$, where $\tilde{\tau} : X \to X_1$ is another isogeny.

Now, let $u \in \ker \lambda$ and consider the map $\eta_u = \eta + u$ from $Y$ to $X_1$. We have $\pi = \lambda \circ \eta_u$ for each $u$, and $[d]^*Y = \cup_{u \in \ker \lambda} X_1 \times_{\tilde{\tau}, \eta_u, Y} Y$. This is a decomposition into the union of $s$ closed proper subsets, proving that if $s > 1$, then already $[d]^*Y$ is reducible.

Finally, take a factorization as in the statement with deg $\rho$ as small as possible. If this cover $\rho$ does not satisfy (PB) then the above argument shows that $\rho$ factors nontrivially as $\tilde{\lambda} \circ \tilde{\eta}$, for a rational map $\tilde{\eta} : Y \to Z_1$ and an isogeny $\tilde{\lambda} : Z_1 \to Z$; we have $\pi = (\lambda \circ \tilde{\lambda}) \circ \tilde{\eta}$, contradicting the minimality of deg $\rho$. 

From now on we shall tacitly use the content of this proposition. We go on to state a HIT over cyclotomic fields, with explicit specializations at the set $\mathcal{T}_r$ of torsion points of $G_m^r$. We denote by $k^c$ the field generated over $k$ by all roots of unity. By torsion coset of $G_m^r$ we mean a translate of an irreducible algebraic subgroup by a torsion point (see [BG, Ch. 3] for the simple theory).

Theorem 2.1. Let $\pi : Y \to G_m^r$ be a cover defined over $k^c$ and satisfying (PB). Then there exists a finite union $\mathcal{E}$ of proper torsion cosets such that if $\zeta \in \mathcal{T}_r \setminus \mathcal{E}$ then $\zeta \in \pi(Y)$ and if $\pi(u) = \zeta$, then $[k^c(u) : k^c] = \deg \pi$ and $\pi^{-1}(\zeta)$ is $k^c$-irreducible.

- The argument in Remark(i) to the Definition shows that (PB) is necessary for the conclusion.

Proof. The crux of the proof lies in the following result, obtained in [DZ] (see Theorem 1 therein):

Theorem [DZ]. Let $Y$ be a $k^c$-irreducible variety and let $\pi : Y \to G_m^r$ be a cover defined over $k$. Suppose that $\pi(Y(k^c)) \cap \mathcal{T}_r$ is Zariski-dense in $G_m^r$. Then there exists an isogeny $\rho : G_m^r \to G_m^r$ (over $k^c$) and a birational map $\psi : Y \to G_m^r$ such that $\pi = \rho \circ \psi$.

We proceed with the proof of Theorem 2.1, denoting $X := G_m^r$. We can freely replace the field $k$ with a finite extension and we suppose to have done this so that the finitely many varieties which appear are irreducible over $\bar{k}$ and defined over $k$. Let $\tilde{Y}$ be a Galois closure of $Y \to X$. Then $\tilde{Y}$ is an irreducible variety and a Galois cover of $X$, with group $G$, say. Let $x \in X$; if $x$ lies out of a fixed proper subvariety $W$ of $X$, the fiber $\pi^{-1}(x)$ will have $d := \deg \pi$ points in $Y$; let this fiber be $\{y_1, \ldots, y_d\}$. Then $G$ acts as a transitive permutation group on such fiber. Note that the fiber above $x$ in $\tilde{Y}$ may be thought as a set of orderings of $\{y_1, \ldots, y_d\}$, precisely the $G$-orbit of one such ordering, say $(y_1, \ldots, y_d)$. Let now $\zeta \in X \setminus W$ be a torsion point and let
(u_1, \ldots, u_d) \in \hat{Y} be an ordering of the fiber above \( \zeta \) in \( Y \); since \( \hat{Y} \) is defined over \( k \), the Galois group \( H_\zeta \) of \( k^e(u_1, \ldots, u_d)/k^e \) sends this ordering of \{u_1, \ldots, u_d\} in orderings which must correspond to points of \( \hat{Y} \). In turn, an ordering corresponds to a point of \( \hat{Y} \) if and only if it lies in the \( G \)-orbit of \{u_1, \ldots, u_d\}. Therefore \( H_\zeta \) acts as a (decomposition) subgroup of \( G \).

For any subgroup \( H \) of \( G \), let now \( Y_H := \hat{Y}/H \), i.e., the space of \( H \)-orbits of points of \( \hat{Y} \);
this is a variety whose function field \( k^e(Y_H) \) is the fixed field of \( H \) in \( k^e(\hat{Y}) \). Note that we have a natural map \( \pi_H: Y_H \to X \) induced by \( \pi \). Then, by the above, \( \zeta \) lifts to a \( k^e \)-rational point of \( Y_{H_\zeta} \) (namely, the image of \( (u_1, \ldots, u_d) \) in \( Y_{H_\zeta} \)). Let us now fix \( H \) and consider those \( \zeta \) with \( H_\zeta = H \).

If \( H \) acts transitively on \{u_1, \ldots, u_d\}, then the degree over \( k^e \) of any point \( u_i \) in the fiber is \( d = \deg \pi \), and we have the conclusion. Hence, let us suppose in the sequel that \( H \) is not transitive as a permutation group on \{u_1, \ldots, u_d\}. Note that this implies that the fiber product \( Y_H \times_{(\pi_H, \pi)} Y \) is \( k^e \)-reducible.

By Theorem [DZ] applied to \( Y_H, \pi_H \), and since we are assuming that \( \zeta \) lifts to a \( k^e \)-rational point of \( Y_H \), either there is a proper subvariety \( W_H \) of \( X \) containing all of these \( \zeta \) or there exists an isogeny \( \rho = \rho_H: X \to X \) and a birational map \( \psi = \psi_H: Y_H \to X \) such that \( \pi_H = \rho \circ \psi \). But in this second case the fiber product \( X \times_{(\rho_H, \pi)} \hat{Y} \) would be reducible, like \( Y_H \times_{(\pi_H, \pi)} Y \). But since any isogeny is a factor of some multiplication map, we deduce that \( Y \) would not satisfy (PB), contrary to the assumptions. Hence this case cannot occur, and the exceptional torsion points \( \zeta \) are confined in the proper subvariety \( (\bigcup_H W_H) \cup W \). But the Zariski-closure of a set of torsion points in \( X \) is anyway a finite union of torsion cosets (see [BG, Thm. 4.2.2, p. 95]). This proves the result, with \( E \) equal to such Zariski-closure.

An extension of Theorem [DZ], and its consequence Theorem 2.1, to abelian varieties in place of \( G^e_m \) would be desirable. We explicitly make the following:

**Conjecture.** Let \( A/k \) be an abelian variety and \( T \) be its set of torsion points, generating the field \( k(T) \) over \( k \). Let \( \pi: Y \to A \) be a cover, and suppose that \( \pi(Y(k(T))) \cap T \) is Zariski-dense in \( A \). Then there exist an isogeny \( \rho: B \to A \) and a birational map \( \psi: Y \to B \) such that \( \pi = \rho \circ \psi \).

It turns out that the same arguments of the proof of Theorem 1 in [DZ] do not work in this abelian case, already for elliptic curves.

As another motivation for this conjecture, we sketch how it implies the Manin-Mumford conjecture (proved by Raynaud in 1983) that a curve \( C \) of genus \( g \geq 2 \), embedded in its Jacobian \( J \), contains only finitely many torsion points of \( J \). To deduce this from the conjecture, let \( \pi: C^g \to J \) be the map \( (P_1, \ldots, P_g) \mapsto P_1 + \ldots + P_g \); it is a surjective (ramified) cover of degree \( g! \). If \( C \) contains infinitely many torsion points, then \( C^g \) has a Zariski-dense set of points defined over \( k(T) \), sent to \( T \) by \( \pi \). So the assumptions of the conjecture are verified, and then let us assume its conclusion. Then the birational map \( \psi: C^g \to B \) would be a morphism ([BG], Cor. 8.2.22, p. 238) and \( \pi \) would be unramified, a contradiction for \( g \geq 2 \).

### §3. A LIFTING THEOREM AND APPLICATIONS TO HIT FOR COVERS OF ALGEBRAIC TORI.

We now present a lifting theorem, crucial for Theorem 1. The proof illustrates the combination of parts (A), (B) of the method. To state this result, we denote by \( | \cdot |_v \) the sup-norm with respect to a place \( v \) and by \( \mathcal{T}_v \) the set of torsion points of \( G^e_m \):
Theorem 3.1. Let \( \pi: Y \to G_m^r \) be a cover defined over \( k \), of degree \( d := \deg \pi > 1 \) and satisfying (PB). Then there is a finite union \( E \subseteq G_m^r \) of proper torsion cosets with the following property: if \( \zeta \in \mathcal{T} \setminus E \) there exists a set of positive Dirichlet density of places \( w \) of \( k(\zeta) \), of residual degree 1 above \( \mathbb{Q} \), such that \( \pi(Y(k(\zeta))) \) does not intersect the set \( \{x \in k(\zeta)^* : |x - \zeta|_w < 1\} \).

Remarks. (i) Note that the prime \( l := w|\mathbb{Q} \) splits completely in \( k(\zeta) \), because its residual degree there is 1. In particular, the set \( \{x \in (\mathbb{Q}^*)^r : |x - \zeta|_w < 1\} \) contains a whole residue class in \( \mathbb{Z}^r/l\mathbb{Z}^r \).

(ii) Inspection shows that given \( Y, \pi \), one may calculate: equations for the set \( E \), roots of unity \( \zeta \) and places \( w|l \) with the relevant properties.

Proof of Theorem 3.1. We may apply Theorem 2.1 to \( Y, \pi \), so let \( E \) be the finite union of proper torsion cosets mentioned there. There is a proper subvariety \( E' \) of \( G_m^r \) such that the fiber of \( \pi \) outside \( E' \) has exactly \( d \)-elements (even in a projective closure of \( Y \)). The Zariski-closure of the set of torsion points in \( E' \) is another finite union of torsion cosets, and by enlarging \( E \) we may suppose it is contained in \( E' \). Now, for a torsion point \( \zeta \not\in E \) let \( u \in Y(k) \) be such that \( \pi(u) = \zeta \). The conclusion of Theorem 2.1 guarantees that \( u \) exists and we have \( [k^c(u) : k^c] = d \).

In the sequel, we shall tacitly assume that this is the case for the \( \zeta \) in question. Let \( H = H_\zeta \) be the Galois group of the normal closure \( K = K_\zeta \) of \( k(\zeta, u)/k(\zeta) \). (Note that \( K \) depends in fact only on \( \zeta \), not on \( u \) because \( [k^c(u) : k^c] = d \), and we have \( K = k(\zeta, u_1, \ldots, u_d) \) where \( u_i \) are the elements of \( \pi^{-1}(\zeta) \).

It is a well-known simple fact (attributed to Jordan - see [Se3]) that \( H \) cannot be the union of conjugates of a proper subgroup.\(^3\) Therefore, since \( k(\zeta, u) \neq k(\zeta) \), there exists an element \( g = g_{\zeta, u} \in H \) such that \( u^\tau \neq u^\tau \) for all \( \tau \in H \).

We now apply the theorem of Chebotarëv to the normal closure \( K' \) of \( K/\mathbb{Q} \). There exists an element \( \sigma \in \Gamma := \Gal(K'/\mathbb{Q}) \) which restricts to \( g \) on \( K \). In particular, \( \sigma \) fixes \( k(\zeta) \) pointwise. We obtain the existence of infinitely many places \( l \) of \( \mathbb{Q} \) (in fact a set of positive density), unramified in \( K' \) and such that the Frobenius class of \( l \) in \( \Gamma \) is the class of \( \sigma \). Let then \( v \) be a place of \( K' \) above \( l \) with \( \text{Frob}(v|l) = \sigma \), and denote by \( w \) the place of \( k(\zeta) \) below \( v \). We let \( \{u_1, \ldots, u_d\} \) be the fiber of \( \pi \) above \( \zeta \) and we choose \( l \) large enough so that \( u_1, \ldots, u_d \) are defined and remain distinct modulo \( v \) (recall that they are distinct) and so that \( Y, \pi \) have good reduction at \( v \).\(^4\)

Since \( \sigma \) fixes \( k(\zeta) \), the residual degree of \( w|l \) is 1. Let \( a \in G_m^r(k(\zeta)) \) be such that \( |\zeta - a|_w < 1 \) and consider the fiber of \( \pi \) above \( a \). Suppose that there is an element \( b \in Y(k(\zeta)) \) so that \( \pi(b) = a \). We have \( \pi(b) \equiv \zeta \pmod{v} \); hence the reduction of \( b \) at \( v \) is defined and \( b \equiv u_i \pmod{v} \) for some \( i \). In fact, otherwise the fiber above the reduction of \( \zeta \), in a projective closure of \( Y \), would contain more than \( d \) elements and the same would be true for the fiber above \( \zeta \) (e.g. by Hensel.

\(^3\) A subgroup \( B \) of a finite group \( H \) has at most \(|H : B|\) conjugates, all of which contain the origin. Hence if \( B \neq H \) their union contains \(<|H : B| > |B| = |H| \) elements.

\(^4\) We need just a simple concept of ‘good reduction’, i.e. we suppose that the reduction of \( Y, \pi \) is defined and has still the same degree. Easy inspection of the proof shows that an estimate \( l \geq c^m \), where \( c = c(Y, \pi) \) and \( m \) is the order of \( \zeta \), suffices.
lifting, or simply by good reduction, on taking \( l \) large enough so that \( \zeta \) does not lie modulo \( v \) in the ‘exceptional’ variety \( \mathcal{E}' \) mentioned in the opening argument).

Now, \( b^p = b \), whence \( u_i^p \equiv u_1 \pmod{v} \), because \( \sigma \) fixes \( v \). However any \( u_i \) is a conjugate over \( k(\zeta) \) of \( u \), so of the shape \( u^\tau \) for a \( \tau \in H \). Hence \( \sigma \) does not fix any of the \( u_i \) and permutes them, so we would have \( u_i \equiv u_j \pmod{v} \) for some \( i \neq j \), a contradiction which proves that \( b \) cannot be defined over \( k(\zeta) \), proving the sought conclusion.

There are several variations on Theorem 3.1, and we mention one of them, useful for our applications:

**Refinement.** Under the same assumptions, let \( F \) be a number field, Galois over \( k(\zeta) \) and such that \([F : k(\zeta)]\) is not divisible by any prime smaller than \( d = \deg \pi \). Then we may further prescribe arbitrarily the Frobenius class in \( \text{Gal}(F/k(\zeta)) \) of the relevant places \( \wp|l \).

**Proof.** With respect to the above arguments, it suffices to observe that \( F \) and \( K \) are linearly disjoint over \( k(\zeta) \), because the degree \([K : k(\zeta)]\) divides \( d! \) whereas \([F : k(\zeta)]\) is divisible only by primes > \( d \). Hence the Galois group \( H^* := \text{Gal}(FK/k(\zeta)) \) is the product \( \text{Gal}(F/k(\zeta)) \times H \) and we may find an automorphism in \( H^* \) which restricts in a prescribed arbitrary way to \( F/k(\zeta) \) and to \( K \). Then the proof works as before, prescribing the action of \( \sigma \) also on \( F \).

We now proceed to the proof of Theorem 1; we argue through two other statements, which yield further conclusions.

**Proposition 3.1.** Let \( \pi : Y \to \mathbb{G}_m^{r} \times \mathbb{G}_a \) have degree > 1 and satisfy (PB) and suppose that the cyclic group \( \Omega \) generated by \( \omega := (\xi, \tau) := (\xi_1, \ldots, \xi_r, \tau) \in \mathbb{G}_m(k)^r \times \mathbb{G}_a(k) \) is Zariski-dense in \( \mathbb{G}_m^r \times \mathbb{G}_a \). Then for all large primes \( p \) there exist infinitely many primes \( l \equiv 1 \pmod{p} \) such that, for a class \( t_0 \mod{p} \) and any integers \( b_1, \ldots, b_r \) coprime to \( p \) and any \( b \in t_0 \), \((\xi_1^{b+1}, \ldots, \xi_r^{b+1}, b\tau)\) does not lie in \( \pi(Y(k)) \).

In particular, there exists a coset of \( \Omega/[l(l-1)]\Omega \) disjoint from \( \pi(Y(k)) \).

**Proof.** In order to apply Theorem 3.1, we view \( \pi \) as a cover of \( \mathbb{G}_m^{r+1} \); however this might not satisfy (PB); a simple remedy is to define a modified map by replacing \( \pi \) with \( \pi' := \lambda \cdot \pi \), for a fixed \( \lambda \in \mathbb{G}_m^r(k) \times \mathbb{G}_a(k) \) (where the dot refers to the group law), supposing that \( \pi' \) is not ramified above the whole \( \mathbb{G}_m^r \times \{0\} \); all but finitely many choices of the last coordinate of \( \lambda \) shall do. To check (PB) for \( \pi' \), by factoring \([m]\) on \( \mathbb{G}_m^{r+1} \) through \([m]\) on the first \( r \) components and then \([m]\) on the last one, and using that \( \pi \) satisfies (PB), we are reduced to show that if a cover of \( \mathbb{G}_m^{r+1} \) is not ramified above the whole \( \mathbb{G}_m^r \times \{0\} \), then pull-back by \([m]\) on the last coordinate leaves it irreducible. Now, if not, then the cover map would factor through a nontrivial isogeny on the last \( \mathbb{G}_m \)-coordinate (proof of Prop. 2.1). However this isogeny would be a multiplication by a divisor of \( m \), and thus would be totally ramified above \( \mathbb{G}_m^r \times \{0\} \), leading to a contradiction.

Note now that the denseness of \( \Omega \) amounts to the \( \xi_i \) being multiplicatively independent elements of \( k \) and \( \tau \neq 0 \). We choose \( \lambda \) of the shape \((1, \ldots, 1, a\tau)\), with an integer \( a \), large enough so that the previous argument applies; hence we assume that \( \pi' : Y \to \mathbb{G}_m^{r+1} \) satisfies the assumptions of Theorem 3.1. For a large prime \( p \), let us choose a torsion point \( \zeta \in \mathbb{G}_m^{r+1} \) of exact order \( p \), satisfying the corresponding conclusion: note that for large \( p \) we may choose it out of the proper subset \( \mathcal{E} \) (relative to \( \pi' \)).
By the hypothesis on \( \omega \), we have that for all large enough \( p, \tau \) is a unit at each place above \( p \) and the coordinates \( \xi_1, \ldots, \xi_r \) have multiplicatively independent classes in \( k(\zeta)^*/(k(\zeta)^*)^p \). This independence is not difficult to check: if a product \( \xi_1^{b_1} \cdots \xi_r^{b_r} \) is nontrivially a \( p \)-th power in \( k(\zeta)^* \) then it is a \( p \)-th power in \( k^* \) (for \( |k(\zeta) : k| \) divides \( p - 1 \)). For a given \( \varepsilon > 0 \), find now with a well-known Dirichlet Lemma an integer \( q = q(\varepsilon) \) so that the \( qa_i \) have not all zero residues \( b_i \mod p \) satisfying \( |b_i| < \varepsilon p \); then \( \xi_1^{b_1} \cdots \xi_r^{b_r} \) is also a \( p \)-th power, say \( \eta^p, \eta \in k^* \), but has height \( < n\varepsilon \max h(\xi) \), so \( h(\eta) < n\varepsilon \max h(\xi(i)) \). For large enough \( p \) one can take an arbitrarily small \( \varepsilon \), which eventually forces \( \eta \) and \( \xi_1^{b_1} \cdots \xi_r^{b_r} \) to be roots of 1, contrary to the independence assumption. (See also [Z], Lemma 2.)

Now, the Refinement applies to \( F = k(\zeta, \zeta^{2/p}) \). Note that, by multiplicative independence modulo \( p \)-th powers, Kummer Theory shows that \( F/k(\zeta) \) is Galois, abelian of degree \( p^r \). We thus may find infinitely many primes \( l \) and extensions \( w \) of \( l \) to \( k(\zeta) \) such that:

(i) The prime \( l \) splits completely in \( k(\zeta) \).

(ii) The image \( \pi'(Y'(k(\zeta))) \) does not intersect the set \( \{ x \in k(\zeta)^{r+1} : |x - \zeta|_w < 1 \} \).

(iii) The Frobenius of \( w \) in \( F/k(\zeta) \) equals a prescribed element of \( \text{Gal}(F/k(\zeta)) \).

Now, this Frobenius is an automorphism \( g \) fixing \( k(\zeta) \) and such that \( g(\zeta^{2/p}) = \theta g_0 \zeta^{2/p} \), for some integers \( h_i \), where \( \theta \) is a primitive \( p \)-th root of unity; by multiplicative independence modulo \( p \)-th powers, Kummer Theory again shows that all choices of \( h_i \) are possible: if \( \zeta = (\theta c_1, \ldots, \theta c_r, \theta c_0) \) and if \( a_i \) are integers coprime to \( p \), we choose \( h_i = a_i c_i \). Now, by (i) we have \( \zeta^{2/p} \equiv g(\zeta^{2/p}) \) (mod \( v \)), where \( v \) is a place of \( F \) above \( w \) with Frobenius \( g \), so by our choice we have \( \zeta^{2/p} \equiv \theta c_i \) (mod \( v \)), where \( b_i \) is any inverse to \( a_i \) modulo \( p \). Hence, this congruence holds for the place \( w \) of \( k(\zeta) \) below \( v \). Also, for large \( l \) both \( \tau \) and \( \zeta \) reduce to \( \mathbb{F}_l \) modulo \( w \), so for \( b \) in a whole progression \( t_0 + Zl \) we may prescribe that \( (b + a)\tau \equiv \theta c_i \) (mod \( w \)).

Hence, we have \( (\xi_1^{b_1^{(i-1)/p}}, \ldots, \xi_r^{b_r^{(i-1)/p}}, (b + a)\tau) \equiv \zeta \) (mod \( w \)) so by (ii) we conclude that \( (\xi_1^{b_1^{(i-1)/p}}, \ldots, \xi_r^{b_r^{(i-1)/p}}, (b + a)\tau) \) does not lie in \( \pi'(Y'/(k(\zeta))) \), i.e. that \( (\zeta_1^{b_1^{(i-1)/p}}, \ldots, \zeta_r^{b_r^{(i-1)/p}}, b\tau) \) does not lie in \( \pi(Y'(k(\zeta))) \). This proves the first part.

We now let \( b_0 \) be an integer coprime to \( p \) and find an integer \( u_0 \) such that \( u_0 \equiv b_0(l - 1)/p \) (mod \( p \)) and \( u_0(l - 1) \equiv pt_0 \) (mod \( l \)). Putting \( b_1 = u_0 + m_1p, b = u_0 + ml \) for arbitrary integers \( m_1, \ldots, m_r, m \), we obtain that \( (\xi_1^{u_0^{(i-1)/p} + m_1(l - 1)}, \ldots, \xi_r^{u_0^{(i-1)/p} + m_r(l - 1)}, (u_0^{1/p} + ml)\tau) \) does not lie in \( \pi(Y'(k(\zeta))) \). In turn, for \( m_i = nl, m = n(l - 1) \), we conclude that \( |u_0^{1/p} + l(l - 1)\zeta|_w \) is disjoint from \( \pi(Y'(k(\zeta))) \), as required.

\[ \square \]

**Corollary.** Suppose that for \( i = 1, \ldots, h, \pi_i : Y_i \to \mathbb{G}_m^r \times \mathbb{G}_a \) is a cover of degree \( > 1 \) satisfying (PB) and suppose that the cyclic group \( \Omega \subseteq \mathbb{G}_m(k)^r \times \mathbb{G}_a(k) \) is Zariski-dense in \( \mathbb{G}_m^r \times \mathbb{G}_a \). Then there exists a coset \( C \) of finite index in \( \Omega \) and disjoint from \( \cup_{i=1}^h \pi_i (Y_i(k)) \).

**Proof.** We argue by induction on \( h \), for \( h = 0 \) the assertion being empty. Suppose it proved up to \( h - 1 \), and let \( [a + q\zeta|_w \omega \) be the corresponding coset involving the first \( h - 1 \) maps, where \( \omega \) generates \( \Omega \). We now apply the last assertion of Proposition 3.1, with \( Y : = Y_r, \pi : = (-a\omega) \cdot \pi_r \) (this still verifies (PB)) and \( [q|_w \omega \) in place of \( \omega \); we obtain \( a', q' \) such that for \( m \in \mathbb{Z} \), the point \( [a + (a' + mq')q|_w \omega \) does not lift under \( \pi_r \) to \( Y_r(k) \). This completes the induction, with \( C = [a + a'q + qq'] \Omega \). \[ \square \]
Proof of Theorem 1. Note that this Corollary is in fact a weak version of Theorem 1; we are going to use a rather standard method for the converse deduction.

A first point is to ensure (PB) for the Galois closures of our covers; let us drop for a moment the index $i$ and let us consider a Galois closure $\hat{Y}$ of $\pi : Y \to X := \mathbb{G}_m^r \times \mathbb{G}_n$. (In all of this we can freely enlarge $k$ to a finite extension.) Now let us take the pullbacks $[B]^{*}Y, [B]^{*}\hat{Y}$ by a multiplication map $[B]$ on $X$, where $B$ is divisible by $[\hat{Y} : X]$. Now, $[B]^{*}Y$ is irreducible by assumption, but $[B]^{*}\hat{Y}$ may become reducible, and let $V$ be a component, noting it is a Galois closure of $[B]^{*}Y$ over $X$. The natural map $\pi : \hat{Y} \to X$ has degree dividing $B$, hence by the last assertion of Proposition 2.1, applied to $\hat{Y} \to X$, we deduce that $V$ satisfies (PB).

Letting now $\omega$ be a generator of $\Omega$, let us replace $Y$ by $[B]^{*}Y$ and $\omega$ by $\omega^{*}$, where $[B]\omega^{*} = \omega$, extending $k$ so that every point in $[B]^{-1}(\omega)$ is defined over $k$; since $\pi$ lifts to a map also denoted $\pi$ from $[B]^{*}Y$ to $X$, of the same degree, we are then reduced to prove the assertion with $[B]^{*}Y$ in place of $Y$ and $\omega^{*}$ in place of $\omega$. (Note that we can choose a single $B$ which works for all the original covers $Y_{i}$.) Since $V$ is related to $[B]^{*}Y$ as $\hat{Y}$ is related to $Y$, we conclude that we may work under the assumption that $\hat{Y}_{i}$ satisfies (PB), for $i = 1, \ldots, h$, as we suppose from now on.

Now, dropping again the index, suppose that a point $[n]\omega$ lifts to a point on $Y$ of degree $< \deg \pi$ over $k$. Then (as in the proof of Theorem 2.1) $[n]\omega$ lifts to a rational point of some $Y_{H} := \hat{Y} / H$, intermediate inside $\hat{Y} \to X$, where $H$ is some subgroup of $\text{Gal}(\hat{Y} / X)$, intransitive on the fiber on $Y$ of a generic point of $X$; this intransitivity ensures that the degree $[Y_{H} : X] > 1$. But $\hat{Y} / X$ satisfies (PB), and since (PB) clearly transfers to any intermediate cover, we deduce that $Y_{H} / X$ satisfies (PB) as well. Hence, finally, it suffices to apply the last Corollary to the larger number of varieties $Y_{i,H}$ so obtained, to deduce that for a whole arithmetical progression of $n$ this does not happen.

Remarks. The theorem of Chebotarev in Theorem 3.1 allows to transfer the information from a torsion point $\zeta$, over the big field $k^{c}$, to a point $x$ over $k$, near to $\zeta$ with respect to a suitable place.

Inspection shows that Chebotarev’s theorem implicitly appears in [Sch2], and was also developed independently by Fried [Fr] for function fields over finite fields.

§4. Proof of the elliptic HIT Theorem 2.

In this section we shall prove Theorem 2. The general principles are analogous to the proof of Theorem 1 just given. However in the elliptic context we miss Theorem [DZ] and its consequence Theorem 2.1. Hence, step (A) has to be carried out differently. For this we shall now adopt ideas from [Z], which however need several new ingredients for the present situation; fortunately, this still suffices to provide a partial substitute. We pause for a brief sketch before the details.

The approach uses the well-known Lang-Weil estimate for points of varieties over finite fields (derived from Weil’s Riemann-Hypothesis for curves). We recall this in the following form (see also [Se1], p. 184 or [Se2], p. 30): Let $Z/k$ be an absolutely irreducible variety of dimension $n$. For a prime $p$, let $v|p$ be a place of $k$ with residue field contained in the finite field $\mathbb{F}_q$. Then, as $p \to \infty$, the number $|Z_{\nu}(\mathbb{F}_q)|$ of points of the reduction $Z_{\nu}$ of $Z$ satisfies $|Z_{\nu}(\mathbb{F}_q)| = q^n + O(q^{n-\frac{1}{2}})$.

Let us now go back to our setting of a map $\pi : Y \to \mathbb{G}_m^r$, and let $\hat{Y}$ be a Galois closure over $k$, assuming it to be irreducible over $k$. As in a method introduced by Eichler, Fried and by S.D.
Lemma 1. There exists a torsion point \( \zeta \) in \( X \), lying above \( p \), such that \( \zeta \) does not lift to \( \hat{k} \) of good reduction, lying above \( p \). Then, by reduction one may easily check that \( \zeta \) does not lie in \( \pi(Y(k(\zeta))) \). At this point we have an information similar to (although weaker than) the conclusion of Theorem 2, and this provides a starting point for the step (B). As to step (B), in principle it is entirely similar to what is carried out in Theorem 3.1. For the present application, additional difficulties come in when dealing with several covers simultaneously and with the Kummer Theory which appears in Proposition 3.1; it is here that we use the special abelian variety \( E^n \), for which we have Serre’s results on the Galois action \([Se4]\). However, taking for granted the appropriate results from Galois action and Kummer Theory the method could work generally.

Let us now go on with the details of the proof of Theorem 2; we shall argue through auxiliary facts. Let from now on \( E \) be an elliptic curve defined over \( k \), without CM, and set \( X := E^n \). One begins by a step exactly analogue to the reduction of Theorem 1 to the Corollary to Proposition 3.1: precisely the same argument given for the proof of Theorem 1 after the Corollary (which works for general commutative algebraic groups) shows that we only need to prove the following statement:

**Proposition 4.1.** Suppose that for \( i = 1, \ldots, h \), \( \pi_i : Y_i \rightarrow X \) is a cover of degree \( > 1 \), such that a Galois closure \( \hat{Y}_i/X \) of \( Y_i/X \) satisfies (PB). Suppose also that the cyclic group \( \Omega \subset X(k) \) is Zariski-dense in \( X \). Then there exists a coset \( C \) of finite index in \( \Omega \) and disjoint from \( \cup_{i=1}^h \pi_i(Y_i(k)) \).

**Proof of Proposition 4.1.** The easy inductive argument given above for the Corollary to Proposition 3.1 allows us to reduce to the case \( h = 1 \). Thus from now on we drop the index \( i \). Also, by enlarging \( k \) to a finite extension if necessary, we suppose in the sequel that all the varieties which appear are \( \bar{k} \)-irreducible and defined over \( k \) and we shall indicate by \( c_1, c_2, \ldots \) positive numbers (integers if necessary), depending only on \( k, E, \pi \).

As usual, \( E[m] \) denotes the kernel of \( [m] \) on \( E \), whereas a tilde shall denote reduction modulo a place. We shall refer by ST to the already cited Serre’s theorem \([Se4]\) that the Galois group of the field generated over \( k \) by all the \( E[m] \) has finite index in \( \prod_{i} GL_2(\mathbb{Z}_l) \). In particular, we may choose \( c_1 \) such that if \( m \) has all prime factors \( \geq c_1 \) then \( \text{Gal}(k(E[m])/k) \cong GL_2(\mathbb{Z}/m) \).

A first step is to obtain a certain torsion point \( \zeta \in X \) such that any lift to \( Y \) has the ‘correct’ degree \( \deg \pi \) over \( k(\zeta) \). To gain this irreducibility of \( \pi^{-1}(\zeta) \) over \( k(\zeta) \), we may use again the ‘standard’ argument of Theorem 2; we may reduce to show that \( \zeta \) has no lift over \( k(\zeta) \), to any among the finitely many \( Y_H \), defined as the quotients \( \hat{Y}/H \), where \( H \) is an intransitive subgroup of \( \text{Gal}(\hat{Y}/X) \). Note that these varieties are subcovers of \( \hat{Y}/X \) and thus satisfy (PB) (because \( \hat{Y}/X \) does, by the present assumptions). This motivates us to pause by proving the following:

**Lemma 1.** There exists a torsion point \( \zeta = (\zeta_1, \ldots, \zeta_n) \) of \( E^n \) of order divisible only by primes \( > c_1 \), such that it does not lift to \( Y_H(k(\zeta)) \), for any of the finitely many \( H \) in question.

**Proof of Lemma 1.** We might argue as in the opening sketch, by reduction to a finite field \( \mathbb{F}_q \), applying the Lang-Weil theorem individually for each of the varieties \( Y_H \). But the problem is that...
we need a point \( x \in E^n(\mathbb{F}_q) \) which is ‘good’ for all the \( \tilde{Y}_H \) simultaneously, i.e. does not lift to any \( \tilde{Y}_H(\mathbb{F}_q) \), and such a common point \( x \) need not exist over a finite field. (Using several primes, one for each \( H \), does not work since we successively need to lift \( x \) to a torsion point \( \zeta \in E^n(\overline{\mathbb{Q}}) \).)

We overcome this serious obstacle in some steps as follows: we shall choose good points \( x_H \in \tilde{E}^n \) (relative to \( Y_H \)) over a finite field, lift them to torsion points \( \zeta_H \overline{\mathbb{Q}} \) of a same order, independent of \( H \) (which is crucial), and at this stage we conjugate the \( \zeta_H \) over \( k \) to obtain a same point \( \zeta \), simultaneously suitable for all the \( Y_H \).

With this program in mind, it shall be convenient (for the moment and for this task only!) to go to the case of curves, i.e. to \( n = 1 \), by restricting the cover \( Y \rightarrow E^n \) above a suitable copy of \( E \) inside \( E^n \). In doing this we want to preserve our irreducibility assumptions. Suppose that \( n > 1 \) and that for generic \( x \in E \), \( Y \) becomes reducible above \( \{x\} \times E^{n-1} \). Then the product \( Y \times_{E^n} V \) is reducible in \( t > 1 \) components, where \( V = C \times E^{n-1} \) and \( C \) is a suitable smooth curve with a nonconstant map \( C \rightarrow E \) of degree \( t \). If this map is unramified, then it is a factor of an isogeny \( E \rightarrow E \), against our assumptions (PB) on \( Y \). So the map is ramified, say above \( x_0 \in E \).

But then, since two distinct components of \( Y \times_{E^n} V \) merge above any point in the branch locus of \( V \rightarrow E^n \), the branch locus of \( \pi \) contains \( \{x_0\} \times E^{n-1} \). By an automorphism of \( E^n \) we may assume this is not the case, so \( \pi^{-1}(\{x\} \times E^{n-1}) \) is not generically reducible, and so it is irreducible except for finitely many \( x \in E \). Continuing in this way, we may choose inductively torsion points \( \zeta_1, \ldots, \zeta_{n-1} \in E \), of large but fixed prime orders \( l_1, \ldots, l_{n-1} \), so that \( \pi^{-1}(\{(\zeta_1, \ldots, \zeta_{n-1})\} \times E) \) is irreducible. In fact, we may assume that the \( \zeta_i \) work simultaneously for all the finitely many \( Y_H \) in place of \( Y \) and, denoting by \( W_H \) the irreducible curve obtained by restricting the cover \( Y_H \) above \( \{(\zeta_1, \ldots, \zeta_{n-1})\} \times E \), the same method, applied to \( [B]^*Y_H \) in place of \( Y_H \) for a suitable fixed \( B \) (the same for all \( H \)), shows that actually we may prescribe that \( W_H \) satisfies (PB).

Let us go on by choosing a sufficiently large prime \( p > l_1 \cdots l_{n-1} \), which shall be the characteristic of the finite field to work with. We choose \( p \) splitting completely in \( k_1 := k(E[l_1 \cdots l_{n-1}]) \), denoting by \( v \) a fixed place of \( k_1 \) above \( p \). This place has residual degree 1 above \( p \), so the reduction \( \tilde{E} \) of \( E \) modulo \( v \) is defined over \( \mathbb{F}_p \) and the points in \( E[l_1] \) have reductions in \( \tilde{E}(\mathbb{F}_p) \) as well. In particular, \( l_1 \) divides \( |\tilde{E}(\mathbb{F}_p)| \) so \( p \) does not divide \( |\tilde{E}(\mathbb{F}_p)| \) (which is \( < 2p \)).

As in the remarks at the beginning of this proof, we shall need points \( x_H \in \tilde{E} \) having a same order, i.e. independent of \( H \). To keep control on the order and prove the existence of such points we shall need that \( |\tilde{E}(\mathbb{F}_p)| \) is not divisible by ‘high’ powers of many ‘small’ primes \( l \) (which shall be measured through a certain product \( \prod (1 - l^{-1}) \)). Possibly this may be achieved for some \( p \), by using Analytic Number Theory, but a delicate quantification would anyway be involved; instead, it is possible, and seems simpler, to work over a finite field \( \mathbb{F}_q \) in place of \( \mathbb{F}_p \), \( q = p^m \), keeping \( p \) fixed and letting \( m \) vary.

We start with some technical choices, whose motivation shall be clearer later.

Write \( \gamma_m = |\tilde{E}(\mathbb{F}_q)| \), so \( \gamma_m = (1 - \alpha^m)(1 - \beta^m) \), where \( \alpha, \beta \) are the Frobenius eigenvalues (see [Si]); note that since \( \gamma_1 \) is prime to \( p \), \( \gamma_m \) is also prime to \( p \) for all \( m \) in a suitable arithmetical progression \( P \). In fact, let \( p' \) be a place of \( \mathbb{Q}(\alpha) \) above \( p \); since \( \alpha \beta = p \) and \( \alpha, \beta \) are conjugate over \( \mathbb{Q} \), \( p' \) has residual degree 1. Hence \( \alpha^p \equiv \alpha \), \( \beta^p \equiv \beta \) (mod \( p' \)), so \( \gamma_{m+p-1} \equiv \gamma_m \) (mod \( p \)) for \( m > 0 \) and we can take \( P = 1 + (p - 1)\mathbb{Z} \).

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By a similar argument, using that \( \alpha^{l^2 h} \) and \( \beta^{l^2 h} \) are constant modulo \( l^u \) for a prime \( l' \) above \( l \) in \( \mathbb{Q}(\alpha) \) and \( h > h(e) \), we may also assume that, for all \( m \in P \), \( \gamma_m \) is not divisible by a large enough fixed power \( l_i \) of any \( l_i \) or any power \( l' \), \( e > c_2 \), of any other prime \( < c_1 \); it suffices to take \( P = 1 + M\mathbb{Z} \) with \( M \) divisible by \( p - 1 \) and, for each of the said primes \( l \), by \( (l^2 - 1)b \) where \( h \) is large enough. We tacitly let \( m \) run through \( P \) in what follows.

Thinking of \( p \) and \( P \) as fixed, we denote by \( m_0 \) an integer in \( P \), to be chosen sufficiently large so to satisfy certain properties that we are going to explain; in doing this we shall use asymptotic formulas which have to be understood as holding for \( m_0 \to \infty \). We set \( q_0 = p^m_0 \) and we let

\[
B := N! \leq \frac{m_0}{q_0} = p^{m_0} < (N + 1)!
\]

For fixed \( p \) and large \( m_0 \) we have

\[
N \log N \sim \left( \frac{\log p}{10} \right) \cdot m_0, \quad N \sim \left( \frac{\log p}{10} \right) \cdot \frac{m_0}{\log m_0}.
\]

We write \( B \) as a product \( \prod_{l \leq N} l^u \) of prime powers \( l^u \), where \( l \leq N \) and \( u = u_{l,N} \geq N/2l \).

Suppose that such a prime-power \( l^u \) divides three numbers \( \gamma_{\mu_i} \) with \( \mu_1 < \mu_2 < \mu_3 \) integers in \( P \) and in a certain interval \( J = [m_0, m_1] \). Let \( l' \) be a place above \( l \) in \( \mathbb{Q}(\alpha) \). Then, by recalling \( \gamma_m = (1 - \alpha^m)(1 - \beta^m) \), we find that \( \operatorname{ord}_{l'}(1 - \alpha^m) + \operatorname{ord}_{l'}(1 - \beta^m) \geq u \). By symmetry, we may assume that for two indices \( i < j \in \{1, 2, 3\} \), the maximum order on the left is attained by the \( \alpha \)-term, so \( \operatorname{ord}_{l'}(\alpha^{\mu_j} - 1) \geq u/2 \). Taking the norm \( N_{\mathbb{Q}(\alpha)}(\alpha^{\mu_j} - 1) = \gamma_{\mu_j} \), we get that \( \operatorname{ord}_{m_j}(\gamma_{\mu_j} - \gamma_{\mu_i}) \geq u/2 \), whence \( l^u \leq \gamma_{\mu_j} - \gamma_{\mu_i} \leq (1 + \sqrt{p})^{(\mu_j - \mu_i)} \leq (4p)^2|J| \), where \( |J| \) denotes the length of \( J \).

Therefore \( |J| \geq u \log l/2 \log(4p) \geq N \log l/4l \log(4p) \geq N/4 \log(4p) \), where the last inequality follows from \( l \leq N \). Let us then suppose in the sequel that

\[
\frac{\log N}{8 \log(4p)} < |J| = m_1 - m_0 < \frac{\log N}{4 \log(4p)}.
\]

With this choice we have proved that for each of the said prime-powers \( l^u \), the interval \( J \) may contain at most two integers \( m \in P \) such that \( \gamma_m \) is divisible by \( l^u \).

For an \( m \in J \) let us now define \( \phi_m = \prod_{|\gamma_m| > \frac{1}{l - 1}} (1 - l^{-1}) \). We have just checked that a power \( l^u \) may divide at most two integers \( \gamma_m \) for \( m \in J \cap P \), so a prime \( l \leq N \) contributes to at most two of the \( \phi_m \), for \( m \in J \cap P \); hence

\[
\prod_{m \in J \cap P} \phi_m \geq \prod_{l \leq N} (1 - l^{-1})^2 \gg (\log N)^{-2},
\]

where the right-hand estimate comes from Mertens’s theorem in elementary Analytic Number Theory (see [I], Thm. 7, p. 22).

We thus obtain \( |J \cap P| \log \left( \max_{m \in J \cap P} \phi_m \right) \geq -2 \log \log N + O(1) \). Then, taking into account the choice of \( |J| \), it immediately follows that for fixed \( p \), \( P \) and for every given \( \delta > 0 \), if \( m_0 \) is large
enough we have $\max_{m \in J \cap P} \phi_m \geq 1 - \delta$. We choose therefore an $m \in J \cap P$ such that this estimate
$\phi_m \geq 1 - \delta$ is verified, with a $\delta$ small enough, in terms of deg $\pi$, to justify the coming arguments.

We now let $W \to E$ denote one of the above covers $W_H \to E$; its degree $[W_H : E]$ is
$\leq [Y_H : X] \leq [Y : X]!$. Note that any fixed cover remains irreducible under reduction modulo a
place of large enough norm. Then, since $W_H$ satisfies (PB), we may choose $p$ large enough such
that a fixed set of pull-backs of $W_H$ remains irreducible modulo $v$; hence by Proposition 2.1 if
$p$ is large enough the reduction of $W_H$ at $v$ satisfies (PB). So, denoting $W(B) := [B]^*W$ be the
pullback by the multiplication-map $[B] : E \to E$, the reduction of this cover is irreducible.

We denote with a tilde such a reduction modulo $v$, and we choose $m$ in $P$ and in an interval
$J$ as above, so that $\phi_m$ is maximal and thus $\geq 1 - \delta$. As above we put $q := p^m$.

We have $m_0 \leq m \leq m_0 + |J|$, so the above displayed inequalities yield

$$m = m_0 + O(\log m_0).$$

We now apply the Lang-Weil theorem, as in [Se1], pp. 184/185 or [Se2], Thm. 3.6.2 (see also
the above sketch) to the reduction of $W(B)$ and its Galois closure over $E$. More precisely, taking
into account that $B$ is varying, we shall apply the Weil Theorem (Riemann Hypothesis) for curves,
in place of Lang-Weil, so to have a uniform control of the error term.\(^{(5)}\) This error term depends
on the genera of the involved curves. Now, note that the genera of $\tilde{W}$ and its Galois closure are
bounded, for $W$ running through the $W_H$ and varying $p$. So, by the Riemann-Hurwitz formula,
the genera of $\tilde{W}(B)$ and its Galois closure are bounded by $\ll \deg_E[B] = B^2 \leq q_0^{1/5}$, because any
isogeny is unramified and so $W(B)$ is unramified above $W$.

Let $\Lambda$ be the set of points in $E(\mathbb{F}_q)$ which do not lift to $\tilde{W}(B)(\mathbb{F}_q)$ (under the natural map $\pi$
on $W(B)$); i.e., $\Lambda$ is the set of $x \in E(\mathbb{F}_q)$ such that $[B]x$ does not lift to $\tilde{W}(\mathbb{F}_q)$. Then by the Weil
Theorem (applied as in the books referred to above, as recalled in the opening sketch) we deduce,
taking into account our estimate for the genera that

$$|\Lambda| \geq c_3 q.$$ 

Here this estimate works for all large $m$, where $c_3$ is a positive constant depending only on deg $\pi$.

Now, the group $\tilde{E}(\mathbb{F}_q)$ is of the shape $\langle \mathbb{Z}/a \rangle \oplus \langle \mathbb{Z}/b \rangle$ for integers $a, b$ with $a|b$ and $ab = \gamma_m$.
Let $a_1 = a/\gcd(a, B)$, $b_1 = b/\gcd(b, B)$. The map $[B] : \tilde{E}(\mathbb{F}_q) \to \tilde{E}(\mathbb{F}_q)$ has kernel $K$ isomorphic
to $\langle \mathbb{Z}/\gcd(a, B) \rangle \oplus \langle \mathbb{Z}/\gcd(b, B) \rangle$ and image isomorphic to $\langle \mathbb{Z}/a_1 \rangle \oplus \langle \mathbb{Z}/b_1 \rangle$.

Note that, by the previous choices, the primes $l \leq N$ which may divide $a_1 b_1$ are such that
the corresponding product $\prod_{l \leq N, l|a_1 b_1} (1 - l^{-1})$ of $1 - l^{-1}$ is at least $1 - \delta$: in fact if $l|a_1 b_1$ then $l$ must
divide $ab$ to a power superior to the power $l^a$, with which it divides $B$; so, a fortiori, $l^a | \gamma_m$ and
the above estimate $\phi_m \geq 1 - \delta$ applies.

Let us now estimate from below the product $\prod_{N < l | \gamma_m} (1 - l^{-1})$, of $1 - l^{-1}$ over the primes
$l > N$ which divide $\gamma_m$. If there are exactly $h$ such primes, and if $N < p_1 < p_2 < \ldots < p_h$ are the

\(^{(5)}\) Here we could argue differently, without reducing to curves, by combining bounds of Deligne
and Bombieri, as noted in [Se2], §3.6. This could be crucial in dealing with simple abelian varieties
in place of $E^n$, but here the present method is simpler.
first \( h \) primes greater than \( N \), we clearly have

\[
\prod_{N < l | \gamma_m} (1 - l^{-1}) \geq \prod_{i=1}^{h}(1 - p_i^{-1}).
\]

On the other hand, \( \prod_{N < l | \gamma_m} l \leq \gamma_m \), so \( \prod_{i=1}^{h} p_i \leq \gamma_m \), whence \( \sum_{i=1}^{h} \log p_i \leq \log \gamma_m \leq 2m \log p \).

Hence, by Chebyshev’s elementary estimates in Prime Number Theory (see [I], Ch. 1, §§4,5) we have \( p_h \ll \sum_{\text{prime } l \leq p_h} \log l = \sum_{\text{prime } l \leq N} \log l + \sum_{i=1}^{h} \log p_i \ll N + 2m \log p \ll m_0 \log p \), the implied constants being absolute. Therefore, using again Mertens’s theorem we obtain

\[
\prod_{i=1}^{h}(1 - p_i^{-1}) \geq \frac{\log N}{\log p_h} (1 + o(1)) \geq \frac{\log N}{\log m_0} (1 + o(1)) \geq 1 + o(1),
\]

where the terms \( o(1) \) tend to 0 as \( m_0 \) grows to \( \infty \) (recall that we are working with a fixed \( p \)).

Hence, for large enough \( m_0 \) we find that \( \prod_{N < l | \gamma_m} (1 - l^{-1}) \) is also at least \( 1 - \delta \).

In conclusion, we have shown in particular that, denoting by \( \phi \) the Euler’s function,

\[
\phi(b_1) \geq (1 - \delta)^2 b_1.
\]

Note that since \( W(B) \) is a pull-back by the map \( [B] \), the set \( \Lambda \) is invariant by addition of \( K \), i.e. \( \Lambda + K = \Lambda \). If every element \( (t, u) \) of \( \Lambda \) (we refer here to the above direct sum representation) is such that \( [B](t, u) \) has entries \( (t_1, u_1) \in \mathbb{Z}/a_1 \oplus \mathbb{Z}/b_1 \) such that \( \gcd(b_1, u_1) > 1 \), then

\[
|\Lambda| \leq |K|(a_1b_1 - a_1\phi(b_1)) \leq (a/a_1)(b/b_1)a_1(1 - (1 - \delta)^2)b_1 \leq 2\delta ab.
\]

However \( ab = \gamma_m = |\tilde{E}(\mathbb{F}_q)| \leq 2q \) whereas, by a previous displayed inequality, \( |\Lambda| \geq c_3q \), a contradiction for small enough \( \delta \), e.g. for \( \delta = c_3/8 \).

We then reach the crucial conclusion that there exist \( (t_1, u_1) \) as above with \( \gcd(u_1, b_1) = 1 \). This corresponds to a point \( \tau \in \tilde{E}(\mathbb{F}_q) \) such that \( \tau = (t_1, u_1) \) in a basis as above, and such that \( \tau = [B]x \) for an \( x \in \Lambda \). So \( \tau \) has exact order \( b_1 \) and does not lift to \( \tilde{W}(\mathbb{F}_q) \).

We have chosen \( m \in P \) so that \( p \) does not divide \( \gamma_m = ab \), so the torsion points in \( \tilde{E}(\mathbb{F}_q) \) may be lifted to \( \overline{\mathbb{Q}} \), and we get a torsion point \( \theta \in E[b_1] \) reducing to \( \tau \) modulo some place \( v' \) of \( k_1(E[b_1]) \) above \( v \), and such that \( \theta \) does not lift to \( W(k_1(\theta)) \): if this last fact was untrue, we could reduce modulo \( v' \) and obtain a contradiction. (In fact, the restriction \( v'' \) of \( v' \) to \( k_1(\theta) \) has residue field \( \mathbb{F}_q \) over \( v \), because the \( m \)-th power of the Frobenius of \( v'|v \) fixes \( k_1(\theta) \). Therefore the reduction of \( \theta \), i.e. \( \tau \), would lift to \( \tilde{W}(\mathbb{F}_q) \).

We can now conjugate over \( k_1 \) and obtain that all such conjugates \( \theta^g \) have the same properties as \( \theta \): namely, \( \theta^g \) does not lift to \( W(k_1(\theta^g)) \).

Now, recall again that we have chosen \( m \in P \), so that \( \gamma_m \) is not divisible by a fixed large power \( l^e, e \geq c_2 \), of a prime \( l \leq c_1 \) or \( l = l_i \); in particular, \( \tilde{E}(\mathbb{F}_q) \) does not contain points of order \( l^e \) for such \( l \). As a consequence, we have that, if \( m_0 \), and hence \( N \), is large enough, \( b_1 \) is coprime to such \( l \), for otherwise the \( l \)-part of \( B \) would divide \( b \), and \( \tilde{E}(\mathbb{F}_q) \) would have a point of order \( l^{u+1} \), where \( l^u || B \); but \( u \geq N/2l \), so this cannot hold if \( N > 4c_1c_2 \max(l_1, \ldots, l_{n-1}) \).
But then, by ST, for large enough $c_1$ the set of such conjugates $\theta^c$ over $k_1$ is the whole set of all torsion points of exact order $b_1$. Hence, since $q, B$ were chosen independently of the groups $H$, we may choose the point $\theta$ independently of $W$ among the $W_H$.\(^{(6)}\)

Taking into account the definition of the $W_H$, and setting $\zeta_n := \theta$, the proof of Lemma 1 is thus concluded.

This lemma immediately implies another conclusion:

**Lemma 2.** For $\zeta$ as in Lemma 1, any point $\rho \in Y$ in the fiber $\pi^{-1}(\zeta)$ has degree $\deg \pi > 1$ over $k(\zeta)$. Also, there is an automorphism $g \in \text{Gal}(\overline{k}/k(\zeta))$ which does not fix any point in such fiber.

**Proof.** For the proof of the first assertion, it suffices to take into account the properties of the $Y_H$, as recalled just before the statement of Lemma 1. For the second assertion, just recall Jordan’s observation (as in the proof of Theorem 3.1) that any finite transitive permutation group not reduced to the identity has an element without fixed points.

**End of the proof of Proposition 4.1.** We let $m$ be the order of $\zeta$ and $\rho_1, \ldots, \rho_d$ be the points in $\pi^{-1}(\zeta) \subset Y$; we put

$$K := k(\zeta, \rho_1, \ldots, \rho_d)$$

and we let $\tau \in \text{Gal}(K/k(\zeta))$ be the restriction of a $g$ as in the conclusion of Lemma 2, so $\tau$ does not fix any $\rho_i$.

Now, the strategy will be analogous to that for Proposition 3.1: first, to find primes $l$ such that $\zeta$ does not lift to $T$ modulo some place above $l$, and, second, such that the reduction of $\zeta$ lies in $\Omega$. This will require some preliminaries.

First, letting $\xi = (\xi_1, \ldots, \xi_n)$ be a generator of $\Omega$, we note that the $\xi_i$ are $\mathbb{Z}$-linearly independent points because $\Omega$ is Zariski dense. Therefore for large $c_1$ the $\xi_i$ are independent also modulo $[m]E(k)$ (actually modulo $[l]E(k)$, for every prime $l > c_1$): a simple proof of this known fact is as in Proposition 3.1, for the multiplicative case, using Néron-Tate heights in place of Weil heights.

We now pick algebraic points $\eta_1, \ldots, \eta_n \in E(k)$ such that $[m]\eta_i = \xi_i$ and we put $\eta := (\eta_1, \ldots, \eta_n)$. We also put $\zeta = (\zeta_1, \ldots, \zeta_n)$ and set $Z := \mathbb{Z}\zeta_1 + \ldots + \mathbb{Z}\zeta_n$, a subgroup of $E[m]$, of exact exponent $m$. We may find a basis $t, t'$ of $E[m]$ such that $Z$ is generated by $mt, mt'$ for a certain divisor $a > 0$ of $m$: $m = ab$, say. Note that $Z$ contains $E[l]$ and we have $|Z| = mb$.

Let us look at the subgroup $H$ of $GL_2(\mathbb{Z}/m^2)$ fixing pointwise $Z$. This subgroup corresponds by ST to the extension $k(E[m^2])/k(\zeta)$. In matrix representation (with respect to the basis $t, t'$), $H$ consists of the $2 \times 2$-matrices $I + M$ over $\mathbb{Z}/m^2$, invertible mod $m$ and such that the first column of $M$ is divisible by $m$ and the second one by $b$. The determinant map taken modulo $m$ gives a homomorphism of $H$ into $(\mathbb{Z}/m)^*$ whose kernel is easily checked to have cardinality $m^3 a$. Thus any possible intermediate field $F$ between $k(\zeta)$ and $k(E[m^2])$, and whose degree over $k(\zeta)$ is $< c_1$, must be contained in the field corresponding to this kernel. The Galois group of this last field over $k(\zeta)$ is isomorphic to a subgroup of $(\mathbb{Z}/m)^*$, namely to the image $H$ (modulo $m$) of $H$ by the determinant homomorphism; in turn, this image is the group of classes mod $m$ coprime to $m$ and

\(^{(6)}\) Note that this conclusion works for the points over $\overline{\mathbb{Q}}$; we couldn’t have achieved it over $\mathbb{F}_q$. 

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If \( m' \) is the part of \( m \) made up with the primes coprime to \( b \) (possibly \( m' = 1 \)), then the subgroup of \( \det H \) corresponding to \( F \) is identified with a subgroup of index \( [F : k(\zeta)] \) in \( (\mathbb{Z}/m')^* \). Let now \( r \) be a class modulo \( ma \), coprime to \( m \) and such that \( 1 + br \) is coprime to \( m \) and a primitive root modulo each prime dividing \( m' \) (recall \( \gcd(m', b) = 1 \)). The Chinese Remainder Theorem delivers such an \( r \). Define now the matrix \( \mu \) over \( \mathbb{Z}/m^2 \) by

\[
\mu := I + b\theta, \quad \theta := \begin{pmatrix} a & 0 \\ 0 & r \end{pmatrix} \in \text{Mat}_2(\mathbb{Z}/ma).
\] (4.1)

As above, by ST, \( \mu \) corresponds to an element of \( H \cong \text{Gal}(k(E[m^2])/k(\zeta)) \); by our choice of \( r \), the group generated by this element has index in \( H \) divisible only by primes \( > c_1 \), and thus cannot fix any field \( F \) as above.

Consider now the matrix \( \hat{\theta} := \begin{pmatrix} r & 0 \\ 0 & a \end{pmatrix} \in \text{Mat}_2(\mathbb{Z}/ma) \). It satisfies \( \hat{\theta} + \theta = \text{Tr}(\theta) \cdot I \) and \( \theta \hat{\theta} = \begin{pmatrix} ar & 0 \\ 0 & ar \end{pmatrix} = \text{det}(\theta) \cdot I \) modulo \( ma \). Also, in the said representation of \( E[m^2] \) with basis \( t, t' \), the image \( \theta(E[m]) \) is well defined (namely, even if \( \hat{\theta} \) is defined only modulo \( ma \) rather than modulo \( m^2 \)) and equals precisely \( Z \). Hence there are \( z_1, \ldots, z_n \in E[m] \) such that

\[
\hat{\theta}(z_i) = \zeta_i.
\] (4.2)

Now we recall some facts on elliptic Kummer Theory, relying crucially on ST, for which we refer e.g. to the paper [Be] and to [L], V (§5: ‘Bashmakov’s Theorem’). (In [Be] the case of prime order is treated; we also mention papers of Bashmakov, quoted in [Be], [L].)

Since \( \xi_1, \ldots, \xi_n \) are independent modulo \( [m]E(k) \), we may find, for every choice of points \( z_1, \ldots, z_n \) in \( E[m] \), an automorphism \( \sigma \in \text{Gal}(\mathbb{K}/k(E[m])) \) such that

\[
\eta^\sigma_i = \eta_i + z_i, \quad i = 1, \ldots, n.
\] (4.3)

Also, since the Kummer Theory continues to be valid by replacing \( m \) with \( m^2 \), the extensions \( k(E[m^2])/k(\zeta) \) and \( k(\eta)/k(\zeta) \) are linearly disjoint. In fact, the degree of \( \eta \) over \( k(E[m^2]) \) continues to be \( m^{2n} \), like the degree over \( k(E[m]) \). This degree is divisible only by primes \( > c_1 \), whereas \( [K : k(\zeta)] \) divides \( \text{deg } \pi! = d! \). So for large \( c_1 \) we have linear disjointness of \( k(\eta)/k(\zeta) \) and \( K(E[m^2])/k(\zeta) \). Since, by the above choice and remarks, the automorphism corresponding to \( \mu \) does not fix in particular any nontrivial subextension of \( (K \cap k(E[m^2]))/k(\zeta) \), there exists \( g \in \text{Gal}(\mathbb{K}/k(\zeta)) \) such that

\[
g|_{k(E[m^2])} = \mu = 1 + b\theta \in \text{Gal}(k(E[m^2])/k(\zeta)), \quad g|_K = \tau|_K, \quad \eta^g = \eta^\sigma.
\] (4.4)

Now, we perform a number of choices and deductions:

1. Let us choose a large prime \( l \) (coprime to \( m \) and of good reduction for \( E \)) so that its Frobenius with respect to some place \( v|l \) of the normal closure \( L \) of \( K(\eta, E[m^2])/k \) is \( g \). This is possible by Chebotarev Theorem. Note in particular that by reduction modulo \( v \), \( \zeta \) reduces inside \( E^n(\mathbb{F}_l) \), since \( g \) fixes \( k(\zeta) \).
2. Denote as above by $\tilde{E}$ the reduction of $E$ modulo $v$, and let $\varphi$ be the Frobenius on $\tilde{E}$. Since by n. 1 the point $\zeta$ reduces to a point defined over $\mathbb{F}_l$, the reduction of the group $Z$ is contained in the kernel of $\varphi - 1$, so in particular the kernel of $[b] : \tilde{E} \to \tilde{E}$ is contained in the kernel of $\varphi - 1$. Hence we may write $\varphi = 1 + b\psi$, for some endomorphism $\psi$ of $\tilde{E}$. (See [Si], Cor. 4.11, p. 77.)

3. By n. 1 and (4.4), the Frobenius of $v|l$ acts as $1 + b\theta$ on $E[m^2]$, so $b\psi$ and $b\theta$ have the same action on $E[m^2]$, i.e., $b(\psi - \theta)E[m^2] = 0$, so in particular $(\psi - \theta)bE[m^2] = 0$ and $\psi = \theta$ on $E[m]$.

4. By reduction modulo $v$ of (4.3), by n. 1 and (4.4), we have on $\tilde{E}$, $(\varphi - 1)\eta_i = z_i$, so by n. 2, $b\psi(\eta_i) = z_i$, (where we have denoted the reduced points by the same letters).

5. Let us apply to the last equations the dual endomorphism $\hat{\psi}$ of $\psi$, to obtain (recalling $\hat{\psi} \cdot \psi = \deg(\psi)$) that $b\deg(\psi)(\eta_i) = \hat{\psi}(z_i)$, $i = 1, \ldots, n$.

6. Now, by n. 3, $\psi$ acts as $\theta$ on $E[m]$, and so (by general theory) the same is true of $\hat{\psi}$ and $\hat{\theta}$, so $\hat{\psi}(z_i) = \zeta_i$ by (4.2), and the equations at n. 5 become $b\deg(\psi)\eta_i \equiv \zeta_i \pmod{v}$. Finally, $\deg(\psi)$ is divisible by $a$ (by (4.1) and n. 3), so $b\deg(\psi) = qm$ where $q \in \mathbb{Z}$ and we get

$$q\xi \equiv \zeta \pmod{v}. \quad (4.5)$$

(The fact that $\zeta$ may be represented by some multiple of $\xi$ modulo $v$ is here crucial and seems to be not entirely free of independent interest.)

Now we can conclude as follows. Let $\gamma = [q]\xi$, $\Omega' = [qm]\Omega = [qm\mathbb{Z}]\xi$. Then, by the congruence (4.5) we have $x \equiv \zeta \pmod{\gamma + \Omega'}$. We suppose that $l$ has been chosen large enough so that $\gamma$ has good reduction at $v$. Then, if $x$ lifts to a rational point $y \in Y(k)$, we may reduce modulo $v$ and obtain that $\zeta$ lifts to $\tilde{Y}(\mathbb{F}_l)$, so the reduction of some $\rho_i$ lies in $\mathbb{F}_l$. This is however impossible because by (4.4) and n. 1 above the Frobenius of $v|l$ acts as $g$ on $K$ and hence it moves each $\rho_i$.

This proves the case $h = 1$ of Proposition 4.1 (with $C = \gamma + \Omega'$) and, as remarked in the opening argument, this suffices for the general case.

**Remarks.**

(i) One may argue similarly with CM curves (see [L]), which we leave to the interested reader, and with non cyclic finitely generated $\Omega$, with more complicated, but conceptually similar, arguments. (The cyclic case is the most basic here, also because the main purpose in Hilbert Irreducibility is to find ‘good’ elements in ‘small’ sets.)

(ii) As already remarked, the method of proof in principle applies to more general abelian varieties, provided one has the suitable Kummer Theory and torsion-Galois action at disposal. This seems not to be yet available in the most general case, but probably one can deal with other special cases. All of this goes in the direction of the problem stated at p. 53, §5.4 of [Se2].

(iii) On weakening assumption (PB) one may obtain corresponding versions of Theorems 1,2 in which ‘irreducible’ is replaced by ‘irrational’. More precisely, denote by $X$ either $\mathbb{G}_m^r \times \mathbb{G}_a$ or $E^n$. We have:

**Theorem 4.** For $i = 1, \ldots, h$, let $\pi_i : Y_i \to X$ be a cover, not birationally equivalent to an isogeny. Let $\Omega \subset X(k)$ be a Zariski dense cyclic subgroup. Then there exists a coset $C$ of finite index in $\Omega$ and disjoint from $\bigcup_{i=1}^h \pi_i(Y_i(k))$. 

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This is easily deduced from Theorem 1 or 2 (depending on $X$). In fact, by Proposition 2.1, each map $\pi_i$ factors as $\lambda_i \circ \rho_i$ where $\lambda_i$ is an isogeny and $\rho_i$ satisfies (PB). Since none of the covers is birationally equivalent to an isogeny, each $\lambda_i$ has degree $> 1$. The isogeny $\lambda_i$ is a factor of a multiplication map $[m_i] = \lambda_i \circ \hat{\lambda}_i : X \to X$, so, replacing $Y_i$ by its pullback $\lambda_i^* Y_i$, with $\pi_i$ replaced by the natural map to $X$ induced by $\rho_i$, we may suppose that $Y_i$ satisfies (PB). Now, by Theorems 1, 2 we deduce the existence of the coset $C$ such that each lifting of a point of $C$ by $\rho_i$ has degree $> 1$, proving the claim. (The liftings by the $[m_i]$ do not matter, since they occur over a fixed number field, which can be supposed to be $k$.)

§5. THEOREM 3 AND AN ELLIPTIC ANALOGUE.

We start by proving Theorem 3. We recall that $k^c$ denotes the extension of $k$ obtained by adding to it all roots of unity.

Proof of Theorem 3. We first easily reduce to the case $\kappa = \overline{Q}$ by means of a specialization argument. First, up to birationality we may assume that $\pi$ is finite. Let $\overline{Q} \subset \kappa_0 \subset \kappa$ be a field of definition for $Y$, $\kappa_0$ finitely generated over $\overline{Q}$. We may view $Y$ as a finite cover, defined over $\overline{Q}$, of $V \times \mathbb{G}^n_m$, where $V$ is an affine variety with function field $\kappa_0$. For $\xi \in V(\overline{Q})$ we have a specialized cover $Y_\xi \to \mathbb{G}^n_m$. Since $Y$ is irreducible over $\kappa$, which is algebraically closed, it is well known that $Y_\xi$ is irreducible over $\overline{Q}$ for all $\xi$ in a Zariski-open $U \subset V$ (thinking of $Y$ as defined by a polynomial, this follows e.g. from [Sch], Thm. 32 and Cor. 2, pp. 201-202). By the same argument applied to the pullback cover $[d] Y$, we may shrink $U$ and suppose that $Y_\xi$ satisfies (PB) and that the degree $[Y_\xi : \mathbb{G}^n_m]$ equals $d = \deg \pi$. Pick now $\xi \in U$, let $k$ be a number field of definition for $Y_\xi$ and apply the conclusion of Theorem 3 to $Y_\xi$, obtaining a set $E$ as therein. We contend that this set works for $Y$ as well. In fact, let $G$ be a connected algebraic subgroup of $\mathbb{G}^n_m$, $G \not\subset E$, and $\theta$ a torsion point. Suppose that $\pi^{-1}(\theta G)$ is reducible over $\kappa$. Then, since $\pi$ is finite, there is an affine variety $V_1$ with a finite map $\rho : V_1 \to V$ such that the pullback $\rho^* Y$, as a cover of $V_1 \times G$, is reducible, say as a union $Z \cup W$ of two covers $Z, W$ of degree $< \deg \pi$. Now, since $\rho$ is finite, there is a $\xi_1 \in V_1(\overline{Q})$ with $\rho(\xi_1) = \xi$. It follows that $Y_\xi \cap \pi^{-1}(\theta G)$ is the union $Z_{\xi_1} \cup W_{\xi_1}$, and thus reducible (the fact that $Y_\xi \to \mathbb{G}^n_m$ has degree $\deg \pi$ ensures that $Z_{\xi_1} \cup W_{\xi_1}$ is a nontrivial decomposition), a contradiction.

Let us now prove the theorem in the crucial case $\kappa = \overline{Q}$. Let $k$ be a number field of definition for $Y$ and $\pi$, and let us apply, as we may, Theorem 2.1 to our cover $Y \to \mathbb{G}^n_m$, obtaining a finite union $E_1$ of torsion cosets as therein. By applying that conclusion to torsion points $\zeta \in \theta G \setminus E_1$ and recalling that torsion points are Zariski-dense in $G$, we obtain that if $\theta G \not\subset E_1$, then $\pi^{-1}(\theta G)$ is irreducible over $k^c$ (for otherwise $\pi^{-1}(\zeta)$ would be a fortiori reducible over $k^c$ for the Zariski-dense set of torsion points $\zeta \in \theta G \setminus E_1$).

The point is now to go from $k^c$ to $\bar{k}$, and for this we consider the cover $W := Y \times Y \to \mathbb{G}^{2n}_m \cong \mathbb{G}^{n}_m \times \mathbb{G}^{n}_m$, by the map $\pi_2 := \pi \times \pi$ of degree $d^2$ where $d := \deg \pi$. Since $Y$ satisfies (PB), the same is true of $W$, as a cover of $\mathbb{G}^{2n}_m$. Hence by Theorem 2.1 applied this time to $W, \pi_2$ we deduce that there is a finite union $E_2$ of proper torsion cosets of $\mathbb{G}^{2n}_m$ such that for $\zeta_1 \times \zeta_2$ a torsion point in $\mathbb{G}^{2n}_m \setminus E_2$ the fiber $\pi_2^{-1}(\zeta_1 \times \zeta_2)$ is $k^c$-irreducible.
Denote \( Z := \pi^{-1}(\theta G) \) and suppose that \( Z \) is reducible over \( \bar{k} \). If \( \theta G \not\in \mathcal{E}_1 \), we have observed that \( Z \) is irreducible over \( k^c \) and then the function field extension \( k^c(Z)/k^c(G) \) contains a nontrivial finite ‘constant’ extension \( L/k^c \). But then \( Z \times Z \) is reducible over \( k^c \), and in fact each \( k^c \)-component \( Z_2 \) satisfies \([k^c(Z_2) : k^c(G \times G)] \leq [k^c(Z) : k^c(G)]^2/[L : k^c] \). Hence the fiber in \( Z \times Z \) above a torsion point \( \zeta_1 \times \zeta_2 \in \theta G \times \theta G \) has at least \([L : k^c]\) components irreducible over \( k^c \). We conclude that \( \theta G \times \theta G \) is contained in \( \mathcal{E}_2 \).

Thus if \( \pi^{-1}(\theta G) \) is reducible, we obtain that either \( \theta G \subset \mathcal{E}_1 \) or \( \theta G \times \theta G \subset \mathcal{E}_2 \). From this we easily deduce that \( G \) is anyway contained in a certain finite union \( \mathcal{E} \) of proper connected algebraic subgroups of \( G_n \), concluding the argument.

We end this section with a result in the elliptic context, similar but weaker (in that the subgroup is restricted to a special shape) than Theorem 3; in this situation we lack Theorem 2.1, so we cannot argue as above. We let \( E/k \) be an elliptic curve without CM, \( r \in \mathbb{N} \) and put \( A = E^r \).

**Theorem 5.1.** Let \( \pi : Y \to A^n \) be a cover satisfying (PB). Then there are integers \( a_1, \ldots, a_n \neq 0 \) such that the restriction of \( Y \) above the subgroup \( B = \langle \{ [a_1]x, \ldots, [a_n]x : x \in A \} \rangle \) is irreducible. Moreover, we can choose the \( a_i \) so that \( B \) is not contained in any prescribed finite union \( \mathcal{E} \) of proper torsion cosets in \( A^n \).

**Proof.** For \( s = 1, \ldots, n \), we prove inductively on \( n, s \) that there are integers \( a_1, \ldots, a_s \neq 0 \) such that \( Y \) is irreducible above \( \langle \{ [a_1]x, \ldots, [a_s]x : x \in A \} \rangle \times A^{n-s} \), so that moreover this is not a subset of \( \mathcal{E} \). On taking a pullback by \( [\gcd(a_i)]|_{A^n} \times \text{Id}|_{A^n} \) we may assume that \( \gcd(a_i) = 1 \), in which case this variety is isomorphic to \( A^{n-s+1} \). The assertion is trivial for \( s = 1 \), any \( n \), and by induction on \( n \) we reduce to \( s = 2 \). Suppose that for generic \( x \in A^{n-1} \), \( Y \) becomes reducible above \( \{ x \} \times A \). Then the product \( Y \times A^n \) is reducible in \( t > 1 \) components, where \( V = C \times A \) and \( C \) is a suitable finite normal cover of \( A^{n-1} \) of degree \( t \). If \( C \to A^{n-1} \) is unramified, then the cover is birationally equivalent to an isogeny of algebraic varieties; but the reducibility then violates our assumptions on \( Y \). Therefore it is branched, say at \( x_0 \in A^{n-1} \). But two distinct components of \( Y \times A^n \) merge above any point in the branch locus of \( V \to A^n \), so the branch locus of \( \pi \) contains \( \{ x_0 \} \times A \). By an automorphism of \( A^n \) induced by a unimodular \( n \times n \) matrix of integers (which does not affect the result) we may assume this is not the case, so \( Y_x := \pi^{-1}(\{ x \} \times A) \) is generically irreducible, and so it is irreducible except for a proper closed subset of \( x \in A^{n-1} \). Arguing similarly with a pullback \( [d]^* Y \) in place of \( Y \), we conclude that \( Y_x \to A \) remains generically irreducible under pullback by \( [d] \) and so (by Prop. 2.1) satisfies (PB).

Put now \( B_m := \langle \{ x_1, \ldots, x_n \} \in A^n : x_n = [m]x_1 \rangle \) and \( Y(m) = \pi^{-1}(B_m) \). Suppose that \( Y(m) \) is reducible over \( \bar{k} \), but irreducible over \( k \). Then its function field contains a nontrivial finite extension \( L/k \), necessarily of degree \( \leq d \).

On the other hand, if \( B_m \) is not contained in the branch locus of \( \pi \), we contend that the discriminant of \( L/Q \) can be divisible only by primes in a finite set independent of \( m \) (but dependent only on \( Y, \pi, k \)). In fact, note first that \( Y(m) \to B_m \) is unramified above a generic point and thus

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(7) Note that we cannot directly work over \( L \), which a priori might depend on \( G \). See next theorem for an alternative argument.

(8) This argument also appears in the proof of Proposition 4.1 above.

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its reduction at a place \( v \) of \( k \) may be generically ramified only if the reduction of \( B_m \) is contained in the branch locus of the reduction of \( \pi \). Now, by reducing modulo \( v \) a nontrivial algebraic equation valid on the branch locus of \( \pi \), this yields a fixed algebraic relation modulo \( v \) between \( x_1, [m]x_1 \). So, since \( \deg[m] \to \infty \), for large \( m \) this relation must be trivial modulo \( v \), and hence \( v \) must lie in a finite set independent of the integers \( m \) in question.

Since \( [L : \mathbb{Q}] \leq d[k : \mathbb{Q}] \), we conclude that the discriminant of \( L \) is also bounded (see [Se5], p. 67, Remarque), and thus by a well-known result of Hermite \( L \) has only finitely many possibilities independent of \( m \). Let then \( k_1 \) be the number field generated by all such possible fields \( L \). We have proved that in any case either (i) \( Y(m) \) is irreducible, or (ii) \( B_m \) is contained in the branch locus of \( \pi \), or (iii) \( Y(m) \) is reducible over a number field \( k_1 \) independent of \( m \).

Finally, choose \( x_1 \in A(k_1) \) so that \( \mathbb{Z}x_1 \) is Zariski-dense in \( A \) (it exists if we enlarge \( k_1 \)). Replacing \( x_1 \) with a multiple and choosing other points \( x_2, \ldots, x_{n-1} \in A(k_1) \) and setting \( z_0 := (x_1, \ldots, x_{n-1}) \in A^{n-1} \), we have seen that we may assume that the variety \( Y_{z_0} \) and its pull-back by \( [d] \) are irreducible. Then, by Theorem 2 applied to \( Y_{z_0} \to A \) there exist infinitely many \( m \in \mathbb{N} \) such that the fiber above \( [m]x_1 \) in \( Y_{z_0} \), namely, \( \pi^{-1}((z_0, [m]x_1)) \) is \( k_1 \)-irreducible. But \( (z_0, [m]x_1) \in B_m \), so \( Y(m) \) must be itself \( k_1 \)-irreducible. Hence alternative (iii) is not verified, and the same holds for (ii) if \( m \) is large. Therefore \( Y(m) \) is irreducible for infinitely many \( m \). On the other hand, it is easily checked that \( B_m \) can be contained in \( E \) only for finitely many \( m \), which proves the result.

**Remark.** We could also have applied Thm. 2 to a product \( Y_{z_0} \times Y_{z_1} \), with the argument for Thm. 3 in place of exploiting ramification. Also, for \( r = 1 \) one can use the function field version of the Mordell Conjecture in place of Thm. 2.

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Umberto Zannier
Scuola Normale Superiore
Piazza dei Cavalieri, 7
56126 Pisa - ITALY
e-mail: u.zannier@sns.it

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