1. Introduction

In all attempts to put chiral gauge theories on the lattice, the chiral gauge symmetry is explicitly broken by the lattice theory. The reason for this is that for each fermion species its contribution to the anomaly has to emerge in the continuum limit (CL) (even if all fermions together transform in an anomaly-free representation). A consequence is that, on the lattice, the gauge degrees of freedom (gdofs) couple to the fermions, introducing dynamics into the lattice theory that is not present in the continuum target theory. The actual fermion spectrum can then be different from the naively expected one, and come out vectorlike rather than chiral. This is precisely what happens in many proposals which have been considered to date (for reviews, see [1]).

As an example, let us look again at the U(1) Smit-Swift model [2] defined by

\[ L_{SS}^v = \bar{\psi}(D(U)P_L + \bar{\psi})\psi - \frac{r}{2}\bar{\psi}\Delta\psi + L_{\text{gauge}}(U) - \kappa \sum_\mu (U_{\mu} + U_{\mu}^\dagger) \]

\[ L_{\text{gauge}}(U) = \bar{\psi}(D(U)P_L + \bar{\psi})\psi - \frac{r}{2}\bar{\psi}\Delta\psi + L_{\text{gauge}}(U) - \kappa \sum_\mu (U_{\mu} + U_{\mu}^\dagger) \]

where \( L_{SS}^v \) contains a left-handed (LH) fermion that couples to the gauge fields, and a neutral right-handed (RH) fermion, while a Wilson term is introduced to remove the doublers (at least in the classical CL). \( \bar{\psi} \) and \( D \) are the free and covariant nearest-neighbor anti-hermitian difference operators, and \( \Delta \) is the nearest neighbor lattice laplacian. Since gauge invariance is lost, we expect to need counterterms, of which a gauge-field mass term is the most important one. This is why we added the \( \kappa \)-term to the lagrangian (1).

We can make the gdofs explicit by performing a gauge rotation \( \psi_L \rightarrow \phi^L \psi_L, U_{\mu x} \rightarrow \phi^L_{\mu x} \phi_{x+\mu} \), which yields

\[ L_{SS}^h = \bar{\psi}(D(U)P_L + \bar{\psi})\psi - \frac{r}{2}\bar{\psi}\Delta\psi + \kappa \sum_\mu (\phi^L_{\mu x} \phi_{x+\mu} + \text{hc}), \]

We note several important points: 1) the field \( \phi \) is unphysical, and should decouple in the CL; 2) the Wilson parameter \( r \) has become a Yukawa-like coupling, and couples the gdofs to the fermions.

We now ask the following important question: if we set \( U_{\mu x} = 1 \) in eq. (2), does the reduced model so defined have a CL with a free charged LH, and a free neutral RH fermion? If this is the case, we may expect to obtain the desired
target theory when the gauge field is turned on again. (In this case the target theory would be anomalous, but this is not important for this talk; we can always add fermion species to make the model anomaly free.)

Much work has been done to answer this question ([3] and refs. therein). We have no space here to give a detailed review, or reproduce the phase diagram, but just state that the region relevant for taking the CL is the region around $\kappa = 0$. For small values of $r$ there is a symmetric phase (I), and for large values of $r$ there is a different symmetric phase (II), in both of which $\langle \phi \rangle = 0$. They are separated by a broken phase, in which $\langle \phi \rangle \neq 0$. In phase I, perturbation theory in $r$ can be used, and, at tree level, we see that setting $\phi = \langle \phi \rangle = 0$ in eq. (3) removes the Wilson term for the doublers, which therefore are massless and survive in the CL. In phase II, the strong Yukawa force produces a $\phi^\dagger \psi_L$ bound state, which is neutral under $U(1)$ and therefore screened from the gauge fields. There are no doublers, but the physical fermion (which is massless because of a shift symmetry [3]) decouples entirely. In the broken phase, $\langle \phi \rangle$ sets the scale for both the doubler and gauge-field masses, which also is not what we want. These phases are separated by second-order phase transitions, near which the field $\phi$ itself becomes physical. The conclusion is that everywhere in the phase diagram, the $\phi$-dynamics destroys the chiral nature of the fermion spectrum. This phenomenon is known as the “problem of rough gauge fields.” It was pointed out in [3] that it is a consequence of the Nielsen-Ninomiya theorem in a wide class of models.

It is clear that a new ingredient is needed in order to “tame” the rough gauge fields. Currently, two approaches are under investigation. One is the “two-cutoff” approach, in which separate cutoffs are introduced for the gauge fields and for the fermions. The hope is that this can be used to make lattice gauge fields appear smooth from the point of view of the fermions. For recent results in this direction reported at this conference, see [3]. Here we describe the other approach, based on gauge fixing. For related work at this conference, see [3]; for more details on our approach, see [3].

2. Gauge Fixing

Gauge fixing as a fundamental ingredient in the definition of lattice chiral gauge theories was first proposed in [3]. The idea is to take perturbation theory (PT) as a guide, and define the “target” theory as the BRST-invariant, gauge-fixed theory. On the lattice, the action can be written as

$$S = S_{\text{gauge}} + S_{\text{gaugefix}} + S_{\text{ghost}} + S_{\text{fermion}} + S_{\text{counterterms}},$$

where one chooses a renormalizable gauge, with

$$S_{\text{gaugefix}} \rightarrow \frac{1}{2\xi} \int d^4x (\partial \mu A_\mu)^2$$

in the classical CL. We will take $S_{\text{fermion}}$ and $S_{\text{gauge}}$ as in eq. (1), while $S_{\text{ghost}}$ is not needed for the $U(1)$ case considered here.

If indeed the lattice theory admits a perturbative expansion in $g$, the gauge coupling, then one expects to have the proper scaling behavior in the CL. This immediately raises two important questions, which were first addressed in [3]:

- What should $S_{\text{gaugefix}}$ be on the lattice?
- How does the addition of $S_{\text{gaugefix}}$ change the conclusions obtained in the Smit-Swift model?

The answer to the first question begins with the observation that one does not want to use the naive discretization of the r.h.s. of eq. (1):

$$S_{\text{gaugefix}}^{\text{naive}} \rightarrow \frac{1}{2\xi} g^2 \sum_x \left( \sum_\mu \left( V_{\mu x} - V_{\mu x - \mu} \right) \right)^2,$$

$$V_{\mu x} = \frac{1}{2i} (U_{\mu x} - U_{\mu x}^\dagger).$$

The reason is [3] that $S_{\text{gaugefix}}^{\text{naive}}$ leads to a proliferation of lattice Gribov copies. In particular, the vacuum $U_{\mu x} = 1$ has a dense set of Gribov copies, and standard PT, which corresponds to the expansion in small fluctuations around $U_{\mu x} = 1$, does not apply. This is related to a more general theorem, which implies that lattice Gribov copies occur in general for lattice actions with exact BRST symmetry [3].

What we want is a lattice gauge-fixing action $S_{\text{gaugefix}}(U)$ which has the configuration $U_{\mu x} = 1$...
as the unique global minimum, and which reduces in the classical CL to eq. (4). Such a functional does exist \[11\], a realization is given by

\[
S_{\text{gaugefix}} = \frac{1}{2g^2} \left( \sum_{xyz} \Box(U)_{xy} \Delta(U)_{yz} - \sum_x B_x^2 \right),
\]

\[
B_x = \frac{1}{4} \sum_{\mu} (V_{\mu x} - \mu + V_{\mu x})^2.
\]

Apart from satisfying eq. (4), this action has the following properties:

- There is no BRST symmetry even without fermions, and therefore counterterms \(S_{\text{counterterms}}\) will be needed in any case;

- \(U_{\mu x} = 1\) is the unique vacuum, and therefore we have standard PT;

- In continuum notation, when we restrict \(S_{\text{gaugefix}} \sim \frac{1}{2g^2} \int d^4x (\partial_\mu A_\mu)^2\) to the trivial orbit \(A_\mu = \frac{1}{g} \partial_\mu \lambda\), we end up with a higher-derivative action, \(\frac{1}{2g^2} (\square \lambda)^2\).

The latter item will play a key role in obtaining the desired fermion content in the reduced model.

Before we get to the fermions, let us briefly consider the phase diagram of the action

\[
S_{\text{gauge}}(U) + S_{\text{gaugefix}}(U) - \kappa \sum_{x \mu} (U_{\mu} + U_{\mu}^\dagger).
\]

Here we consider only the most important counterterm, which is the gauge-field mass term. For a discussion of other counterterms in relation to the phase diagram, see \[11\]. Expanding \(U_\mu = \exp(i g A_\mu)\) with \(A_\mu\) constant, we get for the classical potential (which, because we have PT, is the leading order approximation of the effective potential)

\[
V_{\text{cl}} = \frac{g^4}{4\xi} \sum_{\mu \nu} A_\mu^2 A_\nu^4 + \ldots + \kappa g^2 \left( \sum_\mu A_\mu^2 + \ldots \right).
\]

We see that for \(\kappa > 0\) \(V_{\text{cl}}\) has a minimum at \(A_\mu = 0\) while \(m_A^2 > 0\), corresponding to a phase with broken symmetry. For \(\kappa = 0 = \kappa_c\), the minimum is still at \(A_\mu = 0\), but \(m_A^2 = 0\), which corresponds to a critical point. For \(\kappa < 0\), the minimum shifts to

\[
A_\mu = \pm \left( \frac{\xi |\kappa|}{3g^2} \right)^{\frac{1}{4}} \text{ for all } \mu,
\]

implying a novel phase with broken rotational symmetry, in addition to broken U(1) symmetry \[11\]. We conclude that one recovers a massless gauge field by tuning \(\kappa\) to a critical value (which beyond tree level is not equal to zero \[12\]). The appearance of a phase transition to this unusual phase is just a consequence of working with a regulator that breaks gauge invariance.

### 3. Perturbation Theory

Now we would like to address the second question raised in the beginning of section 2. As explained in the introduction, in order to answer this question, one considers the reduced model, which one gets by performing a gauge transformation and then setting \(U_\mu = 1\). For our model with only the gauge-field mass counterterm, we obtain, from eqs. (1) and (4)

\[
\mathcal{L}_{\text{red}} = \overline{\psi} \gamma_\mu \psi - \frac{1}{2} [\overline{\psi}_{L} \phi \Box \psi_{R} + \text{hc}] + \frac{1}{2g^2} (\Box \phi)^2 - \kappa \phi \Box \phi,
\]

where now \(V_{\mu x} = (\phi_{x+\mu}^3 - \text{hc})/2i\). What we want to show is that this lattice model has a continuum limit with a free charged LH fermion and a free neutral RH fermion, while the \(g\text{dofs} (\phi \text{ field})\) decouple. We set up PT by expressing \(\phi\) in terms of a Goldstone field \(\theta\),

\[
\phi = \exp(i \sqrt{\xi} g \theta).
\]

Expanding in \(g\) leads to the usual Wilson propagator for the fermions, while the \(\theta\)-propagator \(G\) is given by

\[
G^{-1}(p) = \hat{p}^2 (\hat{p}^2 + m^2),
\]

\[
m^2 = 2\xi g^2 \kappa, \quad \hat{p}_\mu = 2 \sin(p_\mu/2).
\]

There are \(\theta^{2n}\) self-interaction vertices of order \(g^{2n-2}\), and \(\theta^n \overline{\psi} \psi\) vertices of order \(g^n\). This would make the perturbative expansion straightforward, if \(G\) were not infrared singular for \(m^2 \to 0\). Power
counting actually shows that diagrams with only \( \theta \) self-interaction vertices are infrared finite, because all \( \theta \)-lines on these vertices carry at least one derivative. However, this is not the case for \( \theta \)-fermion interactions, or for \( \theta \)-vertices arising from “composite” operators like \( \phi \).

Let us consider the expectation value \( \langle \phi \rangle \) as an example. Naively,

\[
\langle \phi \rangle = 1 - \frac{\xi g^2}{2} \int \frac{d^4k}{(2\pi)^4} G(k) + \ldots
\]

This diverges for \( m^2 \to 0 \), and, in order to obtain a finite result, we perform a resummation. To leading order, the integral in eq. (14) exponentiates, and we obtain

\[
\langle \phi \rangle = \exp \left( -\frac{\xi g^2}{2} \int \frac{d^4k}{(2\pi)^4} G(k) \right) \sim m^2 \eta, \quad (15)
\]

where \( m^2 \propto \kappa - \kappa_c \) (recall that \( \kappa_c = 0 \) at lowest order). For more details on \( \langle \phi \rangle \), including a numerical computation, see [4]. This result leads to a very important conclusion: the full \( U(1)_L \times U(1)_R \) symmetry of eq. (11) is restored at \( \kappa = \kappa_c \). This means that one can actually ask what the \( U(1)_L \) charge of a fermion is in the \( \kappa = \kappa_c \) theory (recall that \( U(1)_L \) is the gauge group of the full theory).

4. Fermions

This brings us to the fermion content of the reduced model. If it were not for the infrared divergences, the conclusion would be straightforward: at tree level, from eq. (11), we see that there are no doublers, and that the LH (RH) fermion have charge one (zero) under \( U(1)_L \). Also, all interactions are irrelevant (dimension \( > 4 \)), so these fermions are free in the CL, and we end up with the desired chiral fermion spectrum. (Note that \( \theta \) has mass dimension zero, because of the propagator \( \frac{1}{m^2} \).) The field \( \theta \) decouples, as it should.

What we need is an argument that shows this conclusion to be correct despite the singular infrared behavior of the \( \theta \)-propagator \( G \). Here we will first give a very simple, but heuristic argument. We will use continuum notation to make it transparent, but the same reasoning can be applied to the lattice model. We can improve the infrared behavior by performing a unitary field redefinition

\[
\psi^n_L = \phi^i \psi_L, \quad \psi^n_R = \psi_R. \quad (16)
\]

Note that \( \psi^n \) is neutral under \( U(1)_L \). Using also eq. (12), the fermionic part of \( \mathcal{L}_{\text{red}} \) in terms of the neutral fermion \( \psi^n \) is

\[
\mathcal{L}_F = \overline{\psi^n} \gamma^a \psi^n - \frac{1}{2} \overline{\psi^n} \Box \psi^n + i \sqrt{\xi} g \partial_{\mu} \theta \overline{\psi^n} \gamma_{\mu} \psi^n_L. \quad (17)
\]

In this formulation, all \( \theta \)'s, including those in fermion vertices, have derivatives, and PT is infrared finite. But now we only have a neutral fermion, which moreover is not free, since the interaction in eq. (17) has dimension four, and therefore is relevant!

To see what is going on, consider the low-energy effective lagrangian \( \mathcal{L}_{\text{eff}} \), which can be obtained by integrating out the high-frequency modes and dropping all irrelevant terms. Using the symmetries of \( \mathcal{L}_F \), in particular shift symmetry \( \xi \)

\[
\psi^n_R \to \psi^n_R + \epsilon_R, \quad (18)
\]

one can derive that

\[
\mathcal{L}_{\text{eff}} = \overline{\psi^n} \gamma^a \psi^n + i \sqrt{\xi} g \partial_{\mu} \theta \overline{\psi^n} \gamma_{\mu} \psi^n_L + \mathcal{L}_0. \quad (19)
\]

In particular, a fermion mass term or a coupling of \( \partial_{\mu} \theta \) to the right-handed fermion current are forbidden by eq. (18). Note that we use the same notation for bare and renormalized quantities.

The key step is now that this may be brought into a much simpler form by performing the unitary transformation (16) in reverse order. Defining \( \psi_L^n = \exp(i \sqrt{\xi} g \theta) \psi^n_L \), we obtain

\[
\mathcal{L}_{\text{eff}} = \overline{\psi^n_L} \gamma^a \psi^n_L + \overline{\psi^n_R} \gamma_{\mu} \psi^n_R + \mathcal{L}_0. \quad (20)
\]

We conclude that

- the reduced model contains charged LH and neutral RH free fermions;
- the \( gdofs (\theta) \) decouple.

Moreover, this result is

- confirmed by one-loop PT [3] and numerical results [13] (see below);
- not in conflict [7] with the no-go theorem of [1].
Let us look at numerical results for the neutral fermion propagator. Figs. 1a and 1b show the modulus of the RH and LH components of the propagator $S^n(p)$ in momentum space, for $\vec{p} = (0, 0, 0)$ (open triangles) and $\vec{p} = (\pi, 0, 0)$ (filled triangles) as a function of $p_4$. All data are at $\frac{1}{2\xi g^2} = 0.2$, $\kappa = 0.05$, $r = 1$, and on a volume of size $6^3 \times 24$. Dotted and solid lines represent tree-level and one-loop PT (for the RH component they fall on top of each other). The filled triangles show that there are no doublers. In order to study the relation between data and PT in more detail for small $p_4$, we replot the same data in figs. 2a and 2b, where now the vertical axes represent the ratio of the full lattice propagator and the tree-level Wilson propagator. We now show results for three different $\kappa$-values, 1 (squares), 0.3 (circles) and 0.05 (triangles), which are decreasing to the critical value for which $\langle \phi \rangle$ vanishes (recall that in the full theory, this corresponds to vanishing gauge-field mass). The solid lines denote one-loop PT, while (of course) the dotted horizontal lines denote tree-level PT.

For the RH component (Fig. 2a), we see that agreement with PT is very good, and that indeed the RH neutral fermion is free. For the LH component (Fig. 2b), agreement between data and PT is again very good. From the dip at $p_4 \sim 0$ we learn that the LH neutral fermion is not free. For $m^2 = 0$, explicit one-loop results [13] show that there is only a cut, and no isolated pole at $p = 0$, which can precisely be explained by the fact that $\psi^n_L = \phi^\dagger \psi^n_L$ only excites multi-“particle” states (of course, $\theta$-excitations are not physically healthy particles). Similar numerical and perturbative results have been obtained for the charged propagator [7,13], which also confirm our claims about the fermion spectrum discussed here.

5. Conclusion

We have shown that gauge fixing can be used to control the rough gauge fields that have hampered progress with the formulation of lattice chiral gauge theories for so long. We designed a local lattice action which can be studied systematically in PT, as well as, in principle, numerically. In the context of the reduced model, we showed that this approach is indeed successful in putting chiral fermions on the lattice.

We believe that the gauge-fixing approach is universal, in that it should work for all standard lattice-fermion formulations: Wilson fermions with either Dirac- or Majorana-Wilson terms, staggered fermions, and domain-wall fermions.
Of course, there are many open problems. Here we only list some of the most important ones.

First, there is the question of fermion-number violation \[14\] on which work is in progress – we believe that the problem is not one of principle, and can be solved satisfactorily within the framework described in this talk.

Then, at a more fundamental level, our approach implies a completely new nonperturbative formulation of lattice gauge theories, which is closer to what we understand gauge theories to be in the continuum, where gauge fixing is indispensable. It will be crucial to understand gauge fixing nonperturbatively in the nonabelian case. At this stage, it is simply not known whether the BRST formulation of nonabelian gauge theories makes sense outside of perturbation theory. (Note that the current approach is different from earlier attempts to address the same question in the so-called noncompact formulation.) But even if the result would be negative, that could be very interesting, since in a sense this lattice gauge-fixing approach forms a bridge between the continuum and the usual lattice formulations of gauge theories. Time will tell.

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