Wilson Renormalization Group Analysis of Theories with Scalars and Fermions

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Abstract

The continuous block spin (Wilson) renormalization group equation governing the scale dependence of the action is constructed for theories containing scalars and fermions. A locally approximated form of this equation detailing the structure of a generalized effective potential is numerically analyzed. The role of the irrelevant operators in the nonperturbative renormalization group running is elucidated and a comparison with the 1-loop perturbative results is drawn. Focusing on the spontaneously broken phase of a model possessing a discrete symmetry forbidding an explicit fermion mass term, mass bounds on both the scalar and fermion degrees of freedom are established. The effect of the generalized Yukawa coupling on the scalar mass upper bound is emphasized.
1 Introduction

A basic, generally unstated, tenet used when describing elementary particle interactions is that the physics at one distance scale follows uniquely from the dynamics specified on a more finely grained distance scale. As such, it is crucial to understand the short distance dynamics of the standard model of elementary particle interactions. Since the non-Abelian gauge interactions are asymptotically free, their short distance behavior is amenable to a renormalization group improved perturbative analysis. On the other hand, the scalar self interactions and the scalar-fermion Yukawa interactions are characterized by couplings which are not asymptotically free. Consequently any conclusions drawn from a perturbative analysis of their short distance structure is at best indicative and require confirmation using nonperturbative techniques.

For instance, a study of the one-loop effective potential in the self coupled $\lambda\phi^4$ scalar model produces a Landau singularity\cite{1} which has in turn been used to place a perturbative triviality bound\cite{2} on the scalar mass. The qualitative nature of these conclusions has been largely substantiated by various nonperturbative studies.\cite{3,4} Here it is found that a nontrivial self coupled scalar model can only be consistently defined as a cutoff theory; that is, as an effective description of the dynamics below some momentum scale $\Lambda$. The only consistent $\Lambda \to \infty$ limit of this model is one characterized by a vanishing renormalized coupling. This free field behavior is a reflection of the Landau singularity already observed in renormalization group improved perturbation theory.

Within the perturbative framework, given the value of the four scalar self coupling at the scale $\Lambda$, $\lambda(\Lambda)$, which in general is such that $\lambda(\Lambda) > 1$,
then the coupling at smaller momentum scales follows from the one-loop renormalization group equation as

$$\frac{1}{\lambda(m_H)} = \frac{1}{\lambda(\Lambda)} + \frac{3}{2\pi^2} \ln \frac{\Lambda}{m_H},$$

(1.1)

where $m_H$ is the scalar mass. In general, this equation is inadequate as it is being applied in a range outside the domain of validity of perturbation theory. If however, $m_H \ll \Lambda$, then the low energy value $\lambda(m_H)$ is insensitive to the initial value $\lambda(\Lambda)$ due to the renormalization group trajectory focusing effects of the trivial (Gaussian) infrared fixed point for $\lambda$. On the other hand, when discussing upper bounds on the scalar mass, one must consider values of $m_H$ which are not much smaller than $\Lambda$ itself. For instance for $\Lambda \sim 1$ Tev, the upper bound on $m_H$ using perturbation theory\cite{2} is found to be $m_H \sim 800$ GeV. Certainly these quantitative results require substantiation in a nonperturbative framework. In fact it turns out that various such estimates\cite{3-4} have been obtained yielding a maximum scalar mass of $m_H \sim 650 - 800$ GeV again for $\Lambda \sim 1$ TeV. This rough quantitative agreement of these nonperturbative predictions with the perturbative estimate is certainly not a priori anticipated. In addition to the fact that the scalar self coupling is sufficiently large to render the 1-loop perturbative result suspect, it also follows that such a coupling induces additional irrelevant operators whose effects are not necessarily insignificant for scales not too far removed from $\Lambda$. In general, the coefficients of these operators are suppressed by powers of $m_H/\Lambda$. While for $m_H \ll \Lambda$ such terms are clearly quite small, this need not be the case as $m_H$ approaches $\Lambda$.

Of the various nonperturbative studies which have been performed for these scalar self coupled systems, most have employed lattice Monte Carlo simulations.\cite{3} A somewhat different approach\cite{4} to the study of these systems
employs the continuous Wilson renormalization group.\cite{5-7} In this case, one specifies the model at the ultraviolet (UV) cutoff scale $\Lambda$ and, by demanding the cutoff independence of the renormalized Green functions, one constructs a functional differential equation for the action at the longer distance scale. This action contains not only the relevant and marginal operators but also the irrelevant operators with the coefficient of each operator fixed in terms of the initial action defined at scale $\Lambda$. Moreover as the construction is independent of the strength of the initial coupling, it constitutes a nonperturbative procedure. Using a local action approximation\cite{8} to these exact functional equations, Hasenfratz and Nager\cite{4} extracted an upper bound for the scalar mass consistent with the other nonperturbative estimates. Recently, the sensitivity of this bound to the choice of action at scale $\Lambda$ has also been investigated.\cite{9} The resulting maximum scalar mass was found to be insensitive to the initial action.

The purpose of this paper is to extend the Wilson renormalization group approach to the construction of renormalized effective Lagrangians in theories containing fermions in addition to scalars. Once again, a consistent model of this sort requires the introduction of an ultraviolet cutoff $\Lambda$. Provided that a huge hierarchy exists between $\Lambda$ and the scalar and fermion masses (generally accomplished by a fine tuning of parameters), then the infrared stable quasi fixed point\cite{10} of the Yukawa and scalar self couplings again focuses the renormalization group trajectories to a low energy value basically insensitive to the exact nature of the short distance couplings. Moreover the position of the infrared quasi fixed point can be determined using the perturbative 1-loop renormalization group equations for these couplings. This is precisely the situation which has been exploited in the top quark condensate
On the other hand, for scalar and fermion masses not too far removed from the UV cutoff, some nonperturbative tools must be employed. In the next section, we use the continuous Wilson renormalization group to construct a functional differential equation for the action functional which can serve as just such a nonperturbative procedure. To obtain this equation, one varies the UV cutoff and demands the resultant action functional reproduces the same physics (i.e., renormalized Green functions) on all momentum scales less than the new (lower) cutoff. That is, the change in cutoff is compensated by changing the coupling constants of a complete set of local operators. This set includes irrelevant operators in addition to the relevant and marginal operators. Thus the various action functionals satisfying this equation correspond to points on a particular (Wilson) renormalization group flow each producing the same physics on momentum scales less than the UV cutoff used in its definition.

This exact Wilson renormalization group equation is tantamount to an infinite number of coupled consistency equations for the various coupling constants of the complete set of operators. As such, its analysis and extraction of physical results from it requires additional approximations. In section 3, we employ a local action approximation in terms of which the functional differential equation reduces to a partial differential equation for a generalized effective potential in a model containing one scalar and one fermion. In the limit of small scalar self coupling and Yukawa coupling at the UV cutoff scale, this equation reduces to the familiar 1-loop perturbative renormalization group equations for these couplings. For larger couplings at the UV cutoff scale, however, the equation for the generalized effective poten-
tial requires a numerical evaluation. We perform such an analysis under the further assumption that only terms up to bilinear in the fermion fields are retained in the generalized potential. The resulting numerical solutions are then analyzed for various choices of the initial Yukawa and scalar self couplings and for different initial cutoffs. In particular, the role of the induced irrelevant operators in driving the theory toward the trivial (Gaussian) fixed point is emphasized. The exact numerical solutions are also contrasted with the analytic solutions of the one loop perturbative equations extrapolated well outside their domain of validity.

By examining the numerical solutions resulting from different values of the scalar self coupling and Yukawa coupling chosen at various cutoff scales and resulting in nontrivial scalar vacuum expectation values, we obtain in section 4 the triviality and vacuum stability bounds on both the scalar and fermion masses. Our results indicate that the scalar mass bound increases in the presence of the Yukawa coupling. The amount of this increase depends on the strength of the initial scalar self coupling; larger self couplings yielding smaller percentage increases. For the range of couplings we considered, the percent increase in the scalar mass bound varied from roughly 15-20\% to 6-10\%. Similar conclusions have also recently been reached from lattice Monte Carlo simulations of scalar-Yukawa theories.\cite{12} In addition, using the values of the mass upper bounds as a function of the cutoff, we determine the allowed ranges of scalar and fermion masses for a specific choice of the cutoff.
2 Wilson Renormalization Group Equation

Consider a general Euclidean quantum field theory containing the fields $\Omega_i$. The subscript $i$ delineates bosonic or fermionic fields, as well as labels any Lorentz and internal symmetry indices, e.g. $\Omega_i \in \{\varphi, \psi, \bar{\psi}, A_\mu, \ldots\}$. The theory is regulated so that the contributions to the Schwinger functions (the Euclidean space Green functions) of the degrees of freedom with momentum above $\Lambda$ are strongly damped, and it is renormalized so that the Schwinger functions are independent of the cutoff $\Lambda$. Thus the generating functional for the renormalized Schwinger functions, $Z[J]$, is cutoff independent: $\frac{d}{d\Lambda} Z[J] = 0$. In general, $Z[J]$ may be represented by the path integral

$$Z[J] = \int [d\Omega] e^{-S[\Omega; \Lambda]} + \int d^4x J_i \Omega_i.$$  \hspace{1cm} (2.1)

Here $S[\Omega; \Lambda]$ is the Euclidean action, assumed known at scale $\Lambda$ where the coupling constants are specified while the sources $J_i$ are of complimentary character to the fields $\Omega_i$ and have support only on momentum scales below $\Lambda$. More specifically, the fields depend on the chosen regulation scheme, $\Omega_i = \Omega_i(p; \Lambda)$, while the action has the form

$$S[\Omega; \Lambda] \equiv \sum_n \frac{1}{n!} \int \frac{d^4p_1}{(2\pi)^4} \cdots \frac{d^4p_n}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 + \cdots + p_n) g_{i_1 \cdots i_n}(p_1, \ldots, p_n; \Lambda) \Omega_{i_1}(-p_1; \Lambda) \cdots \Omega_{i_n}(-p_n; \Lambda),$$  \hspace{1cm} (2.2)

with given coupling constants $g(p; \Lambda)$. The detailed question of regulator dependence of $S$ will be addressed in future work. For simplicity, in the following, we consistently employ a hard cutoff regulator in momentum space. Thus the degrees of freedom with momentum above $\Lambda$ are taken to be zero and we can write $\Omega(p; \Lambda) \equiv \theta(\Lambda - |p|)\Omega(p)$. The action then becomes
\[ S[\Omega; \Lambda] = \sum_n \frac{1}{n!} \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_n}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 + \cdots + p_n) \theta(\Lambda - |p_1|) \]
\[
\cdots \theta(\Lambda - |p_n|) g_{i_1 \cdots i_n}(p_1, \ldots, p_n; \Lambda) \Omega_{i_1}(p_1) \cdots \Omega_{i_n}(p_n),
\]

while the normalized functional measure in Eq. (2.1) contains only those degrees of freedom with momentum less than \( \Lambda \): \[ [d\Omega] = \prod_{|p| \leq \Lambda} d\Omega(p). \]

To derive the continuous Wilson renormalization group equation, we allow the cutoff to vary and consider the response of the theory subject to the constraint that the Schwinger functions remain unaltered. Parametrizing the cutoff as \( \Lambda(t) = e^{-t}\Lambda \), the dependence of the action on \( \Lambda(t) \), or equivalently on \( t \), can be determined by integrating out the degrees of freedom with momentum lying between the differentially close cutoff values, \( \Lambda(t = 0) = \Lambda \) and \( \Lambda(\delta t) = e^{-\delta t}\Lambda = \Lambda - \delta t\Lambda \). Towards this end, we expand the action around the degrees of freedom with momentum below \( \Lambda(\delta t) \) retaining terms up to linear in \( \delta t \). This requires retaining only terms no higher than quadratic in the fields with momentum in the differential shell \( e^{-\delta t}\Lambda \leq |p| \leq \Lambda \) since each momentum integral over the momentum shell introduces a differential factor of \( \delta t \) (save for the one momentum integration over the overall energy-momentum conserving delta function). This Gaussian integral over the fields with momentum in the shell can be performed yielding an action corresponding to the cutoff \( \Lambda(\delta t) \), which can then be used in the path integral for the remaining lower momentum fields. So doing we find

\[ Z[J] = \int_{|p| \leq \Lambda(\delta t)} [d\Omega] e^{-S[\Omega; \Lambda(\delta t)]} + \int \frac{d^4 p}{(2\pi)^4} J(p) \Omega_i(-p). \]

The new action defined at the lower cutoff \( \Lambda(\delta t) \), \( S[\Omega; \Lambda(\delta t)] \), is thus obtained as
\[ S[\Omega; \Lambda(\delta t)] = S[\Omega; \Lambda(0)] + \frac{1}{2} \int_{\text{shell}} \frac{d^4 p}{(2\pi)^4} \text{str} \ln \left( \frac{M(p, -p)}{\Lambda^2} \right) \]
\[ -\frac{1}{2} \int_{\text{shell}} \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} (-)^F_i \frac{\delta S[\Omega; \Lambda]}{\delta \Omega_i(p)} \left| M^{-1}_{ij}(p, q) \frac{\delta S[\Omega; \Lambda]}{\delta \Omega_j(q)} \right|, \]

(2.5)

where \( F_i \) is the Grassmann number (even for bosons, odd for fermions) of the field \( \Omega_i \) and the vertical lines appearing on the right hand side indicate that each such term is to be evaluated setting the fields having momentum greater than \( e^{-\delta t} \Lambda \) to zero. We have further defined the inverse propagator, \( M_{ij}(p, q) \), as the second order expansion coefficient of the action so that

\[ M_{ij}(p, q) \equiv (-)^{\Delta_{ij}} \left( \frac{\delta^2 S[\Omega; \Lambda]}{\delta \Omega_j(q) \delta \Omega_i(p)} \right), \]

(2.6)

where \( \Delta_{ij} \) is unity if both the field \( \Omega_i \) is an even element of the Grassmann algebra and the field \( \Omega_j \) is an odd element of the Grassmann algebra and vanishes otherwise. More explicitly, \( M \) has the form

\[ M = \begin{pmatrix} M_{BB} & M_{BF} \\ M_{FB} & M_{FF} \end{pmatrix}, \]

(2.7)

where \( M_{BB} \) (\( M_{FF} \)) is the submatrix of bosonic character obtained by differentiating \( S \) with respect to two bosonic (fermionic) fields, while the submatrices \( M_{BF} \) and \( M_{FB} \) are of fermionic character and result from one bosonic and one fermionic field derivative. The superdeterminant of \( M \) is defined in terms of the supertrace of the above component matrices (see the Appendix for a review of manipulations with Grassmann valued matrices) yielding

\[ \text{sdet} M = \frac{\det M_{BB}}{\det N_{FF}}, \]

(2.8)
with
\[
N_{FF} = M_{FF} - M_{FB} M_{BB}^{-1} M_{BF} ,
\]  
and hence the result
\[
\text{str } \ln M = \text{tr } \ln M_{BB} - \text{tr } \ln N_{FF} .
\]

The action \( S[\Omega; \Lambda] \) is obtained from \( S[\Omega; \Lambda] \) by setting all modes with momentum above \( \Lambda(\delta t) \) to zero and thus can be expanded as
\[
S[\Omega; \Lambda] = \sum_n \frac{1}{n!} \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_n}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 + \cdots + p_n) \\
\theta(\Lambda(\delta t) - |p_1|) \cdots \theta(\Lambda(\delta t) - |p_n|) g_{i_1 \cdots i_n}(p_1, \ldots, p_n; \Lambda) \\
\Omega_{i_1}(-p_1) \cdots \Omega_{i_n}(-p_n) .
\]

(Note that the coupling constants are still those originally defined at \( \Lambda \).) The new action appearing on the left hand side of Eq. (2.3), on the other hand, involves the lower momentum fields as well as the coupling constants defined at the lower cutoff and can be written as \( \Lambda(\delta t) \)
\[
S[\Omega; \Lambda(\delta t)] = \sum_n \frac{1}{n!} \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_n}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 + \cdots + p_n) \\
\theta(\Lambda(\delta t) - |p_1|) \cdots \theta(\Lambda(\delta t) - |p_n|) g_{i_1 \cdots i_n}(p_1, \ldots, p_n; \Lambda(\delta t)) \\
\Omega_{i_1}(-p_1) \cdots \Omega_{i_n}(-p_n) .
\]

Their difference just results in the variation of the coupling constants when the cutoff is lowered so that
\[
S[\Omega; \Lambda(\delta t)] - S[\Omega; \Lambda] = \sum_n \frac{1}{n!} \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_n}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 + \cdots + p_n) \\
\theta(e^{-\delta t} \Lambda - |p_1|) \cdots \theta(e^{-\delta t} \Lambda - |p_n|) [g(p_1, \ldots, p_n; e^{-\delta t} \Lambda) \\
- g(p_1, \ldots, p_n; \Lambda)] \Omega_{i_1}(-p_1) \cdots \Omega_{i_n}(-p_n) .
\]
\[
\begin{align*}
&= -\delta t \sum_n \frac{1}{n!} \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_n}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 + \cdots + p_n) \\
&\quad \theta(\Lambda - |p_1|) \cdots \theta(\Lambda - |p_n|) \\
&\quad \left[ \Lambda \frac{\partial}{\partial \Lambda} g(p_1, \ldots, p_n; \Lambda) \right] \Omega_i(-p_1) \cdots \Omega_i(-p_n),
\end{align*}
\]

(2.13)

where the overall factor of \(\delta t\) allows \(t\) to be set to zero inside the integrand in the last line on the right hand side (we consistently retain only terms through \(\delta t\)).

In order to obtain a differential equation for the action functional \(S\) itself, the \(\Lambda\)-derivative must be brought past the regulated fields (i.e. the \(\theta\)-functions). This can be achieved by rescaling all the momentum vectors to unit vectors

\[p^\mu = \Lambda \hat{p}^\mu\]

(2.14)

so that \(\theta(\Lambda - |p|) = \theta(1 - |\hat{p}|)\) and by rescaling all the fields to dimensionless ones as

\[\Omega_i(p) = C^{-1}_{\Omega_i}(\Lambda) \hat{\Omega}_i(\hat{p}),\]

(2.15)

(i.e. for the regulated fields, \(\Omega_i(p; \Lambda) = C^{-1}_{\Omega_i}(\Lambda) \hat{\Omega}_i(\hat{p}; 1)\)). It follows that the coupling constant variation now includes a momentum dependence since

\[
\Lambda \frac{\partial}{\partial \Lambda} g(p; \Lambda) = \Lambda \frac{d}{d\Lambda} g(\Lambda \hat{p}; \Lambda) - \sum_{i=1}^n \hat{p}_i^\mu \frac{\partial}{\partial \hat{p}_i^\mu} g(\Lambda \hat{p}; \Lambda).
\]

(2.16)

Substituting these variable changes into Eq. (2.13), and recalling that \(\Lambda(t) = e^{-t} \Lambda\) so that a differential logarithmic change in \(\Lambda\) corresponds to a negative differential change in \(t\), we obtain
\[ \Lambda \frac{d}{d\Lambda} S[\Omega; \Lambda(\delta t)] \equiv -\frac{d}{dt} \hat{S}[\hat{\Omega}; t] \]
\[ = -\frac{1}{\delta t} \left\{ S[\Omega; \Lambda(\delta t)] - S[\Omega; \Lambda] \right\} \]
\[ - \left[ 4 - \sum_{\hat{\Omega}_i} \int \frac{d^4 \hat{p}_i}{(2\pi)^4} \hat{\Omega}_i(-\hat{p}) \right. \]
\[ \left. \left( d_{\hat{\Omega}_i} - \gamma_{\hat{\Omega}_i} - \hat{p}_\mu \frac{\partial'}{\partial \hat{p}_\mu} \right) \frac{\delta}{\delta \hat{\Omega}_i(\hat{p})} \right] \hat{S}[\hat{\Omega}; \Lambda], \]
\[ (2.17) \]

where \( \hat{S} \) is the same action as \( S \), but expressed in terms of the rescaled quantities as

\[ \hat{S}[\hat{\Omega}; t] = S[\Omega; \Lambda(t)] \]
\[ = \sum_n \frac{1}{n!} \int \frac{d^4 \hat{p}_1}{(2\pi)^4} \cdots \frac{d^4 \hat{p}_n}{(2\pi)^4} (2\pi)^4 \delta^4(\hat{p}_1 + \cdots + \hat{p}_n) \]
\[ \theta(1 - |\hat{p}_1|) \cdots \theta(1 - |\hat{p}_n|) \]
\[ \hat{g}_{i_1 \cdots i_n}(\hat{p}_1, \ldots, \hat{p}_n; \Lambda) \hat{\Omega}_{i_1}(-\hat{p}_1) \cdots \hat{\Omega}_{i_n}(-\hat{p}_n). \]
\[ (2.18) \]

with the dimensionless coupling constants defined according to

\[ \hat{g}_{i_1 \cdots i_n}(\hat{p}_1, \ldots, \hat{p}_n; \Lambda) \equiv \left[ \Lambda^{4n-4} \prod_{j=1}^n C_{\Omega_j}^{-1}(\Lambda) \right] g_{i_1 \cdots i_n}(\Lambda \hat{p}_1, \ldots, \Lambda \hat{p}_n; \Lambda). \]
\[ (2.19) \]

In addition, the anomalous dimension, \( \gamma_{\Omega_i} \), of each field is defined by

\[ \Lambda \frac{d}{d\Lambda} \ln C_{\Omega_i}(\Lambda) = 4 - d_{\Omega_i} + \gamma_{\Omega_i}, \]
\[ (2.20) \]

with \( d_{\Omega_i} \) the naive field dimension; e.g. \( d_{\Omega_i} = 1 \) for bosons, and \( d_{\Omega_i} = 3/2 \) for fermions. The prime on the momentum derivative in Eq. (2.17) indicates
that it does not act on the overall energy-momentum conservation Dirac delta function or $\theta$-functions but only on the dimensionless couplings in the expression for $\hat{S}$.

Finally, substituting Eq. (2.15) with $M$ expressed in terms of the rescaled momenta and fields into Eq. (2.17), we obtain the continuous Wilson renormalization group equation for the action:

$$\frac{d}{dt}S[\Omega; t] = \frac{1}{2} \frac{1}{\delta t} \int_{e^{-\delta t} \leq |p| \leq 1} \frac{d^4p}{(2\pi)^4} \text{str} \ln M(p, -p)$$

$$- \frac{1}{2} \frac{1}{\delta t} \int_{e^{-\delta t} \leq |p|, |q| \leq 1} \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} (-)^F \frac{\delta S[\Omega; \Lambda]}{\delta \Omega_i(p)} M_{ij}^{-1}(p, q) \frac{\delta S[\Omega; \Lambda]}{\delta \Omega_j(q)}$$

$$+ \left[ - \sum_{\Omega_i} \int_{|p| \leq 1} \frac{d^4p}{(2\pi)^4} \Omega_i(-p) \left( d\Omega_i - \gamma \Omega_i + p^\mu \frac{\partial}{\partial p^\mu} \right) \frac{\delta}{\delta \Omega_i(p)} \right] S[\Omega; \Lambda].$$

(2.21)

Here and in what follows we drop the carat notation on all quantities and work only with dimensionless fields and unit momentum vectors.
3 The Local Action Approximation

The Wilson renormalization group equation (2.24) is an exact equation determining the action at scale $\Lambda(t)$ given its initial short-distance form at $\Lambda = \Lambda(0)$. Unfortunately its complexity is such that it can only be solved approximately at the present time. Towards this end, we ignore all interactions involving derivative couplings and furthermore set the wave function renormalizations to unity. This is clearly a drastic approximation but does not leave the equation completely devoid of content. Actually, in the small coupling constant regime, this approximation reproduces the one-loop perturbative renormalization group equations with anomalous dimensions neglected. The locally approximated action, $S[\Omega; \Lambda]$, takes the general form

$$S[\Omega; \Lambda] = \int d^4x \frac{1}{2} \Omega_{i_1}(x) K_{i_1 i_2} (\partial_\mu) \Omega_{i_2}(x)$$

$$+ \sum_n \frac{1}{n!} v^{(n)}_{i_1 \cdots i_n} \int d^4x \Omega_{i_1}(x) \cdots \Omega_{i_n}(x)$$

$$= \frac{1}{2} \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \delta^4(p_1 + p_2) \Omega_{i_1}(-p_1) K_{i_1 i_2} (p_1) \Omega_{i_2}(-p_2)$$

$$+ \sum_n \frac{1}{n!} v^{(n)}_{i_1 \cdots i_n} \int \frac{d^4p_1}{(2\pi)^4} \cdots \frac{d^4p_n}{(2\pi)^4} \delta^4(p_1 + \cdots + p_n)$$

$$\Omega_{i_1}(-p_1) \cdots \Omega_{i_n}(-p_n),$$

where the coefficients $v^{(n)}_{i_1 \cdots i_n}$ are momentum independent and it is understood that the Euclidean momentum integrals are over unit vectors.

When expanding the fields about space-time independent values, $\Omega_i(x) = \Omega_i$, (or in momentum space $\Omega_i(p) = (2\pi)^4 \delta^4(p) \Omega_i$), it is convenient to introduce the “generalized potential” $U(\Omega; \Lambda)$ as

$$U(\Omega; \Lambda) \equiv \sum_n \frac{1}{n!} v^{(n)}_{i_1 \cdots i_n} \Omega_{i_1} \cdots \Omega_{i_n};$$

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so that, when evaluated at constant fields, and using $K_{ij}(0) = 0$, the action simply takes the form

$$S[\Omega = \text{constant}; \Lambda] = (2\pi)^4 \delta^4(0) U(\Omega; \Lambda).$$  \hspace{1cm} (3.3)

Thus for constant fields, the Wilson renormalization group equation (2.21) in this approximation reduces to a differential equation for this generalized potential given by

$$\frac{\partial}{\partial t} U(\Omega, t) = \frac{1}{2} \frac{1}{\delta t} \int_{1-\delta t \leq |p| \leq 1} \frac{d^4 p}{(2\pi)^4} \text{str} \ln \left[ K_{ij}(p) + \frac{\partial^2}{\partial \Omega_j \partial \Omega_i} U(\Omega, t) \right]$$

$$+ \left[ 4 - \sum_{\Omega_i} d_{\Omega_i} \Omega_i \frac{\partial}{\partial \Omega_i} \right] U(\Omega, t).$$  \hspace{1cm} (3.4)

In obtaining this result, we have used the fact that the “tree” graph contributions to Eq. (2.21) (the penultimate term on the right hand side) vanish in the constant field expansion since it sets the “propagator”, $M^{-1}$, momentum to zero, which is far from the momentum shell $1 - \delta t \leq |p| \leq 1$. In addition, the approximation of neglecting the anomalous dimensions, $\gamma_{\Omega_i} = 0$, is consistent with the assumption of no wavefunction renormalization.

In order to proceed further, a particular choice of fields and action must be made. For simplicity, we will study the field theory of one real scalar field $\varphi$ and one Dirac spinor fermion field $\psi$ and its conjugate $\bar{\psi}$. The local approximation action then has the form

$$S[\varphi, \psi, \bar{\psi}, t] = -\frac{1}{2} \int d^4x \varphi \partial^2 \varphi + \int d^4x \bar{\psi} \gamma \cdot \partial \psi$$

$$+ \sum_{l,m} \frac{1}{l! m!} \nu^{(l,m)}(t) \int d^4x \phi^l(\bar{\psi} \psi)^m,$$  \hspace{1cm} (3.5)

where for the possible fermion interaction terms we have assumed only products involving scalar bilinears enter. Thus only functions of $\bar{\psi} \psi$ occur in $U$.  

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If the discrete $\gamma_5$-symmetry,

$$
\begin{align*}
\varphi' &= -\varphi \\
\psi' &= \gamma_5\psi \\
\bar{\psi}' &= -\bar{\psi}\gamma_5,
\end{align*}
$$

(3.6)

is also imposed, the interaction terms involve monomials with $l + m$ equal an even integer and so explicit fermion mass terms are forbidden. Hence the generalized potential is simply

$$
U(\varphi, \psi, \bar{\psi}, t) = \sum_{l,m} \frac{1}{l!m!} v^{(l,m)}(t) \int d^4x \phi^l(\bar{\psi}\psi)^m,
$$

(3.7)

and the kinetic energy matrix in $\{\varphi, \psi, \bar{\psi}\}$ space is

$$
K(p) = \begin{pmatrix}
  p^2 & 0 & 0 \\
  0 & 0 & -i\gamma^T \cdot p \\
  0 & -i\gamma \cdot p & 0
\end{pmatrix}.
$$

(3.8)

Here the Hermitian Euclidean space $\gamma$-matrices are defined to satisfy

$$
\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad \text{with} \quad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4.
$$

Inserting these expressions in the renormalization group equation (3.4) and performing some straightforward, but lengthy, algebra we find the locally approximated form of the continuous Wilson renormalization group equation for the generalized potential given by

$$
\begin{align*}
\frac{\partial}{\partial t} U(\varphi, \sigma; t) &= 4U(\varphi, \sigma; t) - \varphi U_\varphi(\varphi, \sigma; t) - 3\sigma U_\sigma(\varphi, \sigma; t) \\
&\quad + \frac{1}{16\pi^2} \ln \left(1 + U_{\varphi\varphi}(\varphi, \sigma; t)\right) - \frac{1}{4\pi^2} \ln \left(1 + U^2(\varphi, \sigma; t)\right) \\
&\quad + \frac{1}{16\pi^2} \ln \left(1 + \Sigma(\varphi, \sigma; t)\right),
\end{align*}
$$

(3.9)
where we have introduced the $\sigma$ field as $\sigma \equiv \bar{\psi}\psi$ and have defined the combination
\[
\Sigma \equiv 2 \frac{\sigma U_{\sigma}}{1 + U_{\sigma}^2} \left[ U_{\sigma\sigma} - U_{\sigma\phi} \left( \frac{1}{1 + U_{\phi\phi}} \right) U_{\phi\sigma} \right].
\] (3.10)

In addition, we have denoted the derivatives of the potential with respect to the field variables by subscripts on the generalized potential, so that, for example, $U_{\phi\sigma} = \frac{\partial^2}{\partial \phi \partial \sigma} U$. Finally, we shall assume that the model at the UV cutoff scale is simply given by the canonical kinetic matrix $K$ of Eq. (3.8) and the initial generalized potential function
\[
U(\varphi, \sigma; t = 0) = \frac{1}{2} \mu^2(0) \varphi^2 + \frac{1}{4} \lambda(0) \varphi^4 + g(0) \varphi \sigma,
\] (3.11)

which is invariant under the discrete $\gamma_5$-symmetry of Eq. (3.6). Note again that this symmetry forbids the appearance of an explicit fermion mass term or equivalently a term linear in $\sigma$.

We seek solutions for the generalized potential in which this discrete $\gamma_5$-symmetry is spontaneously broken. This is achieved by choosing the parameters of the model at the cutoff scale $\Lambda$ such that the vacuum energy is minimized by the scalar field $\varphi$ acquiring a nonzero vacuum expectation value. At the same time, we assume that no fermion bound states or condensates form. As such, the vacuum energy is identified with the potential
\[
V(\varphi; t) = U(\varphi; \sigma = 0; t).
\] (3.12)

That is, the vacuum is characterized by $\varphi \neq 0$ but with $\sigma = 0$.

In order to proceed with the numerical solution of the partial differential equation, we make the further truncation of retaining terms in the generalized
potential only through those linear in $\sigma$ (i.e. $\bar{\psi}\psi$) so that

$$U(\varphi, \sigma; t) = V(\varphi; t) + \sigma G(\varphi; t). \quad (3.13)$$

Here $V(\varphi; t)$ is the effective potential function defined in Eq. (3.12), while $G(\varphi; t)$ is a generalized Yukawa coupling. This approximation, made for technical reasons only, neglects those irrelevant operators containing four or more fermion fields. Introducing the function

$$F(\varphi; t) = V_{\varphi}(\varphi; t) \quad (3.14)$$

and recalling that $\sigma$ vanishes in the vacuum, then the partial differential equation for $U$ reduces to the two coupled partial differential equations for $F$ and $G$ given by

$$\frac{\partial F}{\partial t} = 3F - \varphi F_{\varphi} + \frac{1}{16\pi^2} \frac{F_{\varphi\varphi}}{1 + F_{\varphi}} - \frac{1}{2\pi^2} \frac{G G_{\varphi}}{1 + G^2},$$

$$\frac{\partial G}{\partial t} = G - \varphi G_{\varphi} + \frac{1}{16\pi^2} \frac{G_{\varphi\varphi}}{1 + F_{\varphi}} - \frac{1}{8\pi^2} \frac{G G_{\varphi}^2}{(1 + G^2)(1 + F_{\varphi})}. \quad (3.15)$$

From the functions $F$ and $G$, the running scalar self coupling, $\lambda(t)$, and running Yukawa coupling, $g(t)$, are defined as

$$\lambda(t) = \frac{1}{6} F_{\varphi\varphi\varphi}(0; t)$$

$$g(t) = G_{\varphi}(0; t) \quad (3.16)$$

and are plotted in Figures 1-2 for various choices of their initial values at the UV cutoff $\Lambda$ ($t = 0$).

\footnote{Since $\bar{\psi}\psi \equiv \sigma$ is a nontrivial even element of the Grassmann algebra and since no fermion condensate forms, then the general expansion for $U$ terminates after terms of order $(\bar{\psi}\psi)^4 \equiv \sigma^4$.}
Figure 1: Scalar self coupling $\lambda(t)$ as a function of $t$ for various initial Yukawa couplings and cutoff to scalar vacuum expectation values.

Figure 2: Yukawa coupling $g(t)$ as a function of $t$ for various initial scalar self couplings and cutoff to scalar vacuum expectation values.
Using Eqs. (3.11), (3.15) and (3.16), it is readily seen that $g(t)$ decreases as $t$ increases. On the other hand, $\lambda(t)$ can either initially increase or decrease depending on whether $g(0)$ is greater than or less than the value, $g_c(0)$, which is a function of $\lambda(0)$ and $\mu^2(0)$ and produces a zero initial $\lambda$ slope: $\frac{d}{dt}\lambda(0) = 0$. This value is readily computed as

$$g_c^2(0) = \frac{3}{2}\frac{\lambda(0)}{1 + \mu^2(0)}.$$  (3.17)

For $g(0) > (<)g_c(0)$, $\lambda(t)$ initially increases (decreases) as is readily observed in Fig. 1. In either case, the effect of the initially large (nonperturbative) $\lambda(0)$ is such as to induce non-negligible higher dimensional (irrelevant) operators in the action and consequently to drive the generalized effective potential away from the surface spanned by the relevant and marginal operators only as was initially the case (cf. Eq. (3.11)). These induced operators play an important role in the dynamics for small to moderate $t$ values.

As a measure of their importance, we display in Fig. 3 the evolution of the dimension 6 operator coupling

$$\chi(t) \equiv \frac{1}{5!} e^{-2t} F_{\phi\phi\phi\phi\phi}(0; t).$$  (3.18)

Here we have included the explicit scale factor of $e^{-2t} \sim \frac{1}{\Lambda^2}$ accompanying the dimension 6 operator in the definition of the coupling rather than in the operator itself. As seen in Fig. 3, this induced coupling rises rapidly to a peak value before beginning to tail off as $t$ is increased. For small to moderate $t$ values, this coupling can be quite large and is instrumental in helping to drive the system towards the Gaussian fixed point at the origin of coupling constant space. That is, the effect of this and the other higher dimensional operators is to dampen the $\lambda(t)$ coupling so that eventually the
Figure 3: The $t$-evolution of the coupling $\chi(t)$ of a dimension 6 operator for various ratios of the cutoff to scalar vacuum expectation value when the initial couplings are $\lambda(0) = 5$ and $g(0) = 1$.

theory evolves into a region where it can accurately be described using a perturbative expansion for the generalized effective potential.

It is clear from Fig. 1 that there is a range of $t$ values greater than a value $t^*$ for which $\lambda(t)$ can be well approximated by a linear function of $t$. Over this range, one can therefore write

$$U(\varphi, \sigma; t) = U(\varphi, \sigma; t^*) + (t - t^*) \frac{\partial}{\partial t} U(\varphi, \sigma; t^*).$$  \hspace{1cm} (3.19)

Moreover, the coefficient of $(t - t^*)$ can be directly gleaned using the locally approximated Wilson renormalization group equation (3.9) for the generalized effective potential. So doing, one readily computes for this range of $t$

\footnote{A more precise definition of $t^*$ will be provided in section 4.}
values that
\[
\lambda(t) = \lambda(t^*) + (t - t^*) \left[ \frac{1}{96\pi^2} F_{\phi\phi\phi\phi}(0; t^*) - \frac{108\lambda^2(t^*)}{(1 + F_\varphi(0; t^*))^2} \right] + (t - t^*) \frac{1}{6\pi^2} \left[ 3g^4(t^*) - 2g(t^*)G_{\phi\phi\phi}(0; t^*) \right].
\]  
(3.20)

This same result for \(\lambda(t)\) can also be secured by computing the generalized effective potential for this range of \(t > t^*\) values using the 1-loop perturbative result and then retaining terms linear in \(t - t^*\). The 1-loop contribution to the generalized effective potential is given by
\[
U(\varphi, \sigma; t) = U(\varphi, \sigma; t^*) + \frac{1}{2} \int_{e^{-(t-t^*)} \leq |p| \leq 1} \frac{d^4p}{(2\pi)^4} \ln \left[ K_{ij} + \partial^2 \partial_{\Omega_j} \partial_{\Omega_i} U(\varphi, \sigma; t^*) \right].
\]  
(3.21)

Once again if we retain only the terms through linear in \(\sigma\) and decompose \(U(\varphi, \sigma; t)\) as in Eq. (3.13), we isolate
\[
V(\varphi; t) = V(\varphi; t^*) + \frac{1}{2} \int_{e^{-(t-t^*)} \leq |p| \leq 1} \frac{d^4p}{(2\pi)^4} \ln \left[ p^2 + V_{\phi\phi}(\varphi; t^*) \right] - 2 \int_{e^{-(t-t^*)} \leq |p| \leq 1} \frac{d^4p}{(2\pi)^4} \ln \left[ p^2 + G^2(\varphi; t^*) \right]
\]  
(3.22)

\[
G(\varphi; t) = G(\varphi; t^*) + \frac{1}{2} G_{\phi\phi}(\varphi; t^*) \int_{e^{-(t-t^*)} \leq |p| \leq 1} \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + V_{\phi\phi}(\varphi; t^*)} - G(\varphi; t^*)G_{\varphi\varphi}(\varphi; t^*) \int_{e^{-(t-t^*)} \leq |p| \leq 1} \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + G^2(\varphi; t^*))(p^2 + V_{\phi\phi}(\varphi; t^*))}.
\]  
(3.23)

The momentum integrations are standard and can be straightforwardly performed. Retaining only terms linear in \((t - t^*)\), which constitutes a reasonable
approximation for the $t$ range under consideration ($0 << t - t^* < 1$), the expression for $\lambda(t)$ is found to be identical with that of Eq. (3.20). It is important to note that the 1-loop perturbative result includes the effects of the higher dimensional operators. Their presence is crucial for allowing a smooth joining with the numerically generated solution of the locally approximated Wilson renormalization group equation.

Of course, as $t$ continues to increase to even larger values, the effects of all the higher dimensional operators will eventually decouple and can be completely ignored. (They are truly irrelevant!) In addition, the marginal operators will have couplings well in the perturbative region. The onset of this behavior is more rapidly achieved the larger the value of the initial cutoff $\Lambda$. For this range of $t$ values, the evolution of the model can be accurately parametrized by summing the leading logarithms of the 1-loop graphs thus obtaining the 1-loop perturbative renormalization group running for $\lambda(t)$. 

To perform the perturbative analysis, we reconsider the partial differential equations (3.15) this time for small values of the initial couplings. Expanding $F$ and $G$ as

$$ F(\varphi; t) = \mu^2(t) + \lambda(t)\varphi^3 + ... $$

$$ G(\varphi; t) = g(t)\varphi + ... \quad (3.24) $$

and retaining only the lowest order terms, the differential equations reduce to

$$ 16\pi^2 \frac{d\lambda(t)}{dt} = -18\lambda^2(t) + 8g^4(t) $$

$$ 16\pi^2 \frac{dg(t)}{dt} = -2g^3(t) \quad (3.25) $$

which are precisely the one-loop perturbative renormalization group equations in the approximation in which the anomalous dimensions are neglected.
In obtaining these equations, we have assumed that the parameter $\mu^2(0)$ has been tuned to be close to its critical value, $\mu^2_{\text{crit}} = -\lambda(0)(v^2/\Lambda^2)$, which is proportional to $\lambda(0)$ and hence small. Here we denoted the $\varphi$ vacuum expectation value by $v/\Lambda$. The analytic solution to these equations is readily secured as

\[
\frac{1}{g^2(t)} = \frac{1}{g^2(0)} + \frac{1}{4\pi^2}t,
\]

\[
\lambda(t) = 4g^2(t) \left[ \frac{a(t) + 1}{\sqrt{37}(a(t) - 1) - (a(t) + 1)} \right],
\]

(3.26)

where

\[
a(t) = \left( \frac{g^2(0)}{g^2(t)} \right)^{\frac{1}{37}} \left[ \sqrt{37} + 1 + 4 \frac{g^2(0)}{\lambda(0)} \right] \left[ \sqrt{37} - 1 - 4 \frac{g^2(0)}{\lambda(0)} \right].
\]

(3.27)

Extrapolating these solutions well outside their domain of validity, we display these perturbative results for the running couplings as the dashed lines in Figs. 4-5 using the same initial inputs as used in the numerical solution to the nonperturbative locally approximated Wilson renormalization group equation.

As can be seen in the $g(0) = 5$, $\lambda(0) = 1$ case, the perturbative renormalization group running of $\lambda(t)$ and $g(t)$ exhibits the same qualitative features as the nonperturbative evolution, but are quantitatively quite different. The large Yukawa coupling again initially causes $\lambda(t)$ to increase. This time, the growth is quite dramatic, as depicted by the dashed line curve in Fig. 4, since the higher dimensional operators or even higher loop self coupling effects which provide the additional damping are now absent. This rise in $\lambda(t)$ continues until $g(t)$ has decreased and $\lambda(t)$ has increased sufficiently so that $g^2(t) = \frac{2}{3}\lambda(t)$. At this point the slope of the curve is zero. Subsequently, the pure scalar self coupling term in Eq. (3.25) drives $\lambda(t)$ slowly towards zero.
Figure 4: Comparison of the numerically integrated locally approximated Wilson (solid line) and 1-loop perturbative (dashed line) renormalization group equation solutions for the running of the scalar self coupling for various initial Yukawa couplings and cutoff to scalar vacuum expectation value ratios.

For the case of a smaller initial Yukawa coupling constant and larger initial scalar self coupling, \( g(0) = 1, \lambda(0) = 5 \), \( g(t) \) is seen in Fig. 5 to be completely described by its perturbative renormalization group evolution, while Fig. 4 shows \( \lambda(t) \) monotonically decreasing from its initial value since \( g^2(0) \leq \frac{15}{2(1+\mu^2(0))} \). This is again in crude qualitative agreement with the exact solution, although the decrease in \( \lambda(t) \) in the exact solution is far more rapid due to the effects of the induced higher dimensional operators (cf. Fig. 1). On the other hand, for the larger cutoff, \( \Lambda = 38.6v \), the effects of the higher dimensional operators are far less significant and the \( t \)-evolution of \( \lambda(t) \) runs essentially as in 1-loop renormalization group improved perturbation theory.
Figure 5: Comparison of the numerically integrated locally approximated Wilson (solid line) and 1-loop perturbative (dashed line) renormalization group equation solutions for the running of the Yukawa coupling for various initial scalar self couplings and cutoff to scalar vacuum expectation value ratios.
4 Scalar and Fermion Mass Bounds

The numerical solution to the locally approximated Wilson renormalization group equation can also be used to obtain bounds on the domain of allowed scalar and fermion masses. Working in the broken $\gamma_5$-symmetry phase, we define the $\varphi$ and $\psi$ masses as the zero momentum value of the one particle irreducible scalar and fermion two point Schwinger functions, $\Gamma^{(2,0,0)}(p)$ and $\Gamma^{(0,1,1)}(p)$, respectively, so that

$$\frac{M_s^2}{\Lambda^2} \equiv \Gamma^{(2,0,0)}(p = 0)$$  \hspace{1cm} (4.1)

$$\frac{m_f}{\Lambda} \equiv \frac{1}{4} \text{tr} \ \Gamma^{(0,1,1)}(p = 0).$$ \hspace{1cm} (4.2)

Since $Z = 1$ in the local action approximation we are employing, these masses are also the location of the poles of the propagators when analytically continued to Minkowski space.

The short distance contributions to the Schwinger functions are evaluated by numerically integrating the Wilson renormalization group equation from the initial cutoff $\Lambda$, where the generalized potential is specified, down to a lower momentum scale, $\Lambda^* = e^{-t^*} \Lambda$, below which the dynamics can be accurately described perturbatively. The contributions to the zero momentum Schwinger functions from degrees of freedom with momentum below $\Lambda^*$ (infrared contributions) are then included as (one-loop) perturbative corrections to the generalized potential $U(\varphi, \sigma; t^*)$ obtained from the numerical integration. In this sense, the Wilson renormalization group equation provides a systematic procedure of generating the effective Lagrangian valid on one distance scale from that of a more finely grained distance scale. Integrating all the way down to zero momentum ($t \to \infty$), the effective generalized
potential is secured as (cf. Eqs. (3.21-3.23)) as

\[ U_{\text{eff}}(\varphi, \sigma) \equiv U(\varphi, \sigma; t^*) + \frac{1}{2} \int_{0 \leq |p| \leq 1} \frac{d^4 p}{(2\pi)^4} \text{str} \ln \left[ K_{ij} + \frac{\partial^2 U(\varphi, \sigma; t^*)}{\partial \Omega_j \partial \Omega_i} \right] \]

\[ = V_{\text{eff}}(\varphi) + \sigma G_{\text{eff}}(\varphi), \quad (4.3) \]

where

\[ V_{\text{eff}}(\varphi) = V(\varphi; t^*) + \frac{1}{64\pi^2} \left[ (1 - V_{\varphi \varphi}^2(\varphi; t^*)) \ln (1 + V_{\varphi \varphi}(\varphi; t^*)) + V_{\varphi \varphi}(\varphi; t^*) + V_{\varphi \varphi}^2(\varphi; t^*) \ln V_{\varphi \varphi}(\varphi; t^*) \right] \]

\[ - \frac{1}{16\pi^2} \left[ (1 - G^4(\varphi; t^*)) \ln (1 + G^2(\varphi; t^*)) + G^2(\varphi; t^*) + G^4(\varphi; t^*) \ln G^2(\varphi; t^*) \right] \quad (4.4) \]

\[ G_{\text{eff}}(\varphi) = G(\varphi; t^*) + \frac{1}{32\pi^2} \left[ G_{\varphi \varphi}(\varphi; t^*) + \frac{2G(\varphi; t^*)G_{\varphi \varphi}^2(\varphi; t^*)}{(V_{\varphi \varphi}(\varphi; t^*) - G^2(\varphi; t^*))} \right] \]

\[ \left[ 1 + V_{\varphi \varphi}(\varphi; t^*) \ln \frac{V_{\varphi \varphi}(\varphi; t^*)}{(1 + V_{\varphi \varphi}(\varphi; t^*))} \right] \]

\[ - \frac{1}{16\pi^2} \frac{G(\varphi; t^*)G_{\varphi \varphi}^2(\varphi; t^*)}{(V_{\varphi \varphi}(\varphi; t^*) - G^2(\varphi; t^*))} \]

\[ \left[ 1 + G^2(\varphi; t^*) \ln \frac{G^2(\varphi; t^*)}{(1 + G^2(\varphi; t^*))} \right]. \quad (4.5) \]

To proceed with the numerical integration of Eq. (3.15), we first determine the critical value of the mass parameter, \( \mu_{\text{crit}}^2 \), for all possible choices of the initial coupling constants \( \lambda(0) \) and \( g(0) \). For each particular choice of initial \( \lambda(0) \) and \( g(0) \) parameters, \( \mu_{\text{crit}}^2 \) is defined so as to separate the discrete \( \gamma_5 \)-symmetric phase \( (\mu^2(0) > \mu_{\text{crit}}^2) \) from the spontaneously broken \( \gamma_5 \)-symmetry phase \( (\mu^2(0) < \mu_{\text{crit}}^2) \). The broken symmetry ground state is determined by the location of the zeros of \( F(\varphi; t) \). The value of \( \mu_{\text{crit}}^2 \) is the maximum of
\(\mu^2(0)\), for a given \(\lambda(0)\) and \(g(0)\), which results in a nontrivial zero of \(F(\varphi; t)\) as \(t\) increases into the infrared. This is determined by evaluating \(F(\varphi; t)\) for \(\mu^2(0)\) well into the broken phase and then increasing \(\mu^2(0)\) until the vacuum value decreases, rather than increases, as the model is evolved in \(t\). The transition value of \(\mu^2(0)\) delineating the increasing and decreasing evolution of the vacuum value is \(\mu^2_{\text{crit}}\). Performing this analysis, one also finds that the model has only a trivial (Gaussian) infrared fixed point in accord with the conclusions based on lattice simulations. \([12]\)

Once the critical line is determined, a choice of initial parameters \(\lambda(0)\), \(g(0)\) and \(\mu^2(0) < \mu^2_{\text{crit}}\) is made and the \(t\) dependence of the generalized potential terms \(F(\varphi; t)\) and \(G(\varphi; t)\) is extracted. This nonperturbative evolution is continued until the the scale \(t^*\) is reached at which point the dynamics of the system can be accurately described by perturbation theory and the generalized potential is given via Eqs. (3.21-3.23). The scalar and fermion masses are then extracted using the (one-loop) perturbative expressions for the corresponding zero momentum Schwinger functions. Thus the scalar mass is given by the curvature of the effective potential at its minimum via

\[
\frac{M_s^2}{\Lambda^2} = \Gamma^{(2,0,0)}(p = 0) = V_{\varphi\varphi}^{\text{eff}}(\frac{v}{\Lambda}) \tag{4.6}
\]

and the fermion mass is simply the effective Yukawa coupling constant also evaluated at the minimum of the effective potential:

\[
\frac{m_f}{\Lambda} = \frac{1}{4} \text{tr} \Gamma^{(0,1,1)}(p = 0) = G_{\text{eff}}^{(0,1,1)}(\frac{v}{\Lambda}). \tag{4.7}
\]

The vacuum expectation value, \(v/\Lambda\), is given, through one loop, by the
location of the minimum of the effective potential as

\[ V_{\phi}^{\text{eff}}(\frac{v}{\Lambda}) = 0. \] (4.8)

Letting \( v_0 \) denote the vacuum expectation value of the tree level potential \( V(\varphi; t^*) \), then the one loop shift in expectation value, \( \delta v \), is just

\[
\frac{\delta v}{\Lambda} = \frac{v}{\Lambda} - \frac{v_0}{\Lambda} = \left\{ \begin{array}{c}
- \frac{1}{32\pi^2} \frac{V_{\phi\phi\phi}}{V_{\phi\phi}} \left[ 1 + V_{\phi\phi} \ln \frac{V_{\phi\phi}}{1 + V_{\phi\phi}} \right] \\
+ \frac{1}{4\pi^2} \frac{GG_{\phi}}{V_{\phi\phi}} \left[ 1 + G^2 \ln \frac{G^2}{1 + G^2} \right]
\end{array} \right\}_{\varphi = v_0/\Lambda, \ t = t^*}. \] (4.9)

Using these results in Eqs. (4.4-4.7), the scalar and fermion masses are secured in terms of the effective potential and effective generalized Yukawa coupling evaluated at \( t^* \) and the tree vacuum value \( v_0 \) as:

\[
\frac{M^2_s}{\Lambda^2} = \left\{ V_{\phi\phi} + \frac{1}{32\pi^2} \left[ V_{\phi\phi\phi\phi} - \frac{V_{\phi\phi\phi\phi}}{V_{\phi\phi}(1 + V_{\phi\phi})} \right] + \frac{V_{\phi\phi} V_{\phi\phi\phi\phi} \ln \frac{V_{\phi\phi}}{1 + V_{\phi\phi}}}{V_{\phi\phi}} \right\}_{\phi = v_0/\Lambda, \ t = t^*}

\[
\frac{m_f}{\Lambda} = \left\{ G - \frac{1}{32\pi^2} \left[ \frac{G_{\phi} V_{\phi\phi}}{V_{\phi\phi}} + G_{\phi\phi} + \frac{2G_{\phi} G_{\phi}}{V_{\phi\phi} - G^2} \right] \left[ 1 + V_{\phi\phi} \ln \frac{V_{\phi\phi}}{1 + V_{\phi\phi}} \right] \right\}_{\phi = v_0/\Lambda, \ t = t^*}
\]
The value of $t^*$ is ascertained by the desired numerical accuracy of these masses determinations. As the system evolves into the perturbative regime, the mass values obtained remain stable as $t^*$ is varied. We choose for $t^*$ that particular value such that the computed mass is stable to within an error the size of the $t$-grid spacing: $10^{-5} < \delta t < 10^{-4}$.

Using the above procedure, the ratios $M_*/\Lambda$, $m_f/\Lambda$ and $v/\Lambda$ are determined as the initial couplings $\lambda(0)$ and $g(0)$ are varied. Taking appropriate products of these ratios, we plot in Fig. 6 the allowed $M_*/v$ values as a function of $\Lambda/M_*$ for various different choices of $\lambda(0)$ and $g(0)$. Note that the plotted data is further restricted to satisfy the physically imposed constraint that $\Lambda/M_* \geq 1$. This follows since no masses are allowed to become larger than the initial UV cutoff $\Lambda$. An upper bound on the $M_*/v$ values as a function of $\Lambda/v$ is secured by noting that a dynamical envelope is formed as the initial $\lambda(0)$ and $g(0)$ couplings are varied. That is, for fixed $\Lambda/v$, the different $M_*/v$ values obtained appear to converge to an upper limit, at least for initial Yukawa couplings $\leq 10$.  

Focusing on the dependence of $M_*/v$ on the initial Yukawa coupling, we note that for $\lambda(0) = 5$, a change in $g(0)$ from 0 to 7 leads to roughly a 15-20% increase in $M_*/v$. On the other hand, for the larger initial value of $\lambda(0) = 15$, the same change in $g(0)$ from 0 to 7 results in only an 6-10% increase in $M_*/v$. Thus the effect of the initial Yukawa coupling on $M_*/v$.

Due to limitations in the presently used numerical integration technique, we do not establish the dynamical envelope for larger values of the initial Yukawa coupling $g(0)$. 

\[ + \frac{1}{4\pi^2} \left[ \frac{G G^2_\varphi}{V_\varphi} + \frac{1}{4} \frac{G G^2_\varphi}{V_\varphi (G^2 - G^2)} \right] \left[ 1 + G^2 \ln \frac{G^2}{(1 + G^2)} \right] \bigg|_{\varphi = v_0/\Lambda, \ t = t^*}. \quad (4.11) \]
Figure 6: $M_s/v$ as a function of $\Lambda/M_s$ for different initial $\lambda(0)$ and $g(0)$ values.
diminishes with increasing initial λ(0).

For fixed λ(0) and g(0), note that all the curves in Fig. 6 except the one denoted with the crosses (×), Λ/v increases as Λ/M_s increases. The contrary situation persists for the curve with λ(0) = 0, g(0) = 7 where the maximum scalar mass also corresponds to the largest Λ/v. Furthermore, the curve corresponding to λ(0) = 5 and g(0) = 7 has the behavior that M_s/v initially increases with Λ/v before reaching its maximum and then subsequently deceases. Except for these two curves, which are characterized by having g(0) > λ(0), all the other curves exhibit a maximum scalar mass when Λ/v ratio is at its minimum. The reason for the behavior of these two curves is that the initially large g(0) value requires a longer evolution time (or equivalently a larger initial cutoff) to indirectly feed into an increase in the scalar mass.

In addition, we can extract the allowed domain of scalar and fermion masses for a fixed value of the ultraviolet cutoff to scalar vacuum expectation value ratio Λ/v. Figure 7 is just such a plot [13] corresponding to the specific choice Λ/v = 5. The allowed values are those lying interior to the various boundaries displayed. We now discuss the origin of each of the boundary restrictions in turn. The upper portion of the boundary, denoted by the cross (×) marks, is the scalar mass triviality bound discussed above. Note that the absolute scalar mass upper bound, for this value of Λ/v, occurs for the larger fermion masses and is simply given by M_s = Λ = 5v. Next, the square markings (□) correspond to the triviality bounds on m_f. These points are obtained by finding the largest g(0) values for a given λ(0) which produce an envelope in the m_f/v versus Λ/m_f plane. Finally, by demanding that the scalar potential, V(φ; t), be bounded from below on all scales,
we further restrict the allowed masses. This vacuum stability requirement, which corresponds to the condition that \( \lim_{|\phi| \to \infty} F(\phi; t) > 0 \) for all \( t \), is implemented by setting \( \lambda(0) = 0 \) and allowing \( g(0) \) to vary. This results in the points on the boundary in Fig. 7 represented by the diamonds (\( \bigcirc \)). Note that for larger fermion masses, the fermion mass triviality condition provides a more stringent bound than that of vacuum stability. This is not the case for very large \( \Lambda/v \) ratios where the vacuum stability constraint is always more stringent \(^{13}\) than that due to the fermion triviality.
5 Conclusions

The Wilson renormalization group equation provides a systematic nonperturbative tool for constructing the action at a particular distance scale given the form of the action at a more finely grained scale. The resultant action contains operators of all mass dimension. In particular, if one defines the form of the action at some UV cutoff scale $\Lambda$, then for slightly lower scales, the irrelevant operators (those of mass dimension greater than the dimensionality of space-time) can play a highly nontrivial role in the ensuing dynamics.

We constructed the Wilson renormalization group equation for a theory involving a scalar and a fermion and numerically solved the equation in a local approximation. The resulting renormalization group trajectories were analyzed and we established triviality and vacuum stability mass bounds on both the scalar and fermion degrees of freedom. Clearly, it is desirable to extend this study beyond the local approximation to include the anomalous dimensions (nontrivial wavefunction renormalizations) for the scalar and fermion as well as higher dimension derivative couplings. At the present time, the complications limiting such an extension are purely of a numerical nature and we plan to address these issues in future studies. In addition, the analysis can also be enlarged to allow for the more realistic case of scalars and chiral fermions transforming nontrivially under an internal non-Abelian Lie group and also via the inclusion of gauge bosons. This later step can prove technically challenging since the hard cutoff we employ here breaks the manifest gauge invariance. Finally, the model can also be made supersymmetric and an analysis similar to that performed here can yield SUSY mass triviality bounds.
Appendix:

Supermatrices, Supertraces and Superdeterminants

When discussing theories containing both bosons and fermions, it often proves convenient to combine terms possessing different Grassmann algebra character into a single supermatrix structure. In this appendix, we review some of the properties of the supermatrix

\[ M = \begin{pmatrix} M_{BB} & M_{BF} \\ M_{FB} & M_{FF} \end{pmatrix}, \quad (A.1) \]

where the square submatrices \( M_{BB} \) and \( M_{FF} \) contain only even elements of a Grassmann algebra while \( M_{BF} \) and \( M_{FB} \) contain odd elements of the Grassmann algebra. In particular, \( M_{BB} \) and \( M_{FF} \) are so defined such that the components of \( M_{BB} \) are composed of some ordinary c-numbers (Grassmann number zero) as well as possibly some higher even elements of the algebra, while the components of \( M_{FF} \) are of Grassmann number two or higher.

We define the supertrace \((\text{str})\) of the supermatrix \( M \) via

\[ \text{str} \ M = \text{tr} \ M_{BB} - \text{tr} \ M_{FF}, \quad (A.2) \]

where \( \text{tr} \) denotes the ordinary trace. This definition is chosen so that the familiar trace property

\[ \text{str} \ (MN) = \text{str} \ (NM) \quad (A.3) \]

is still guaranteed to hold. Here the multiplication law for supermatrices is defined precisely the same as for ordinary matrices. Using the definition of the \( \text{str} \) operation, we next define the superdeterminant \((\text{sdet})\) by

\[ \text{sdet} \ M = e^{\text{str} \ \ln \ M}. \quad (A.4) \]
Using the str property (A.3), it is straightforward to establish that

\[
\delta(\text{sdet } M) = \text{sdet } M \cdot \text{str } (M^{-1}\delta M) \tag{A.5}
\]

and that

\[
\text{sdet } (MN) = \text{sdet } M \cdot \text{sdet } N. \tag{A.6}
\]

Now consider decomposing the supermatrix \( M \) as

\[
M = \begin{pmatrix} M_{BB} & 0 \\ M_{FB} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & M_{BB}^{-1}M_{BF} \\ 0 & N_{FF} \end{pmatrix}, \tag{A.7}
\]

where

\[
N_{FF} = M_{FF} - M_{FB}M_{BB}^{-1}M_{BF}. \tag{A.8}
\]

From property (A.6), it follows that

\[
\text{sdet } M = \text{sdet } \begin{pmatrix} M_{BB} & 0 \\ M_{FB} & 1 \end{pmatrix} \cdot \text{sdet } \begin{pmatrix} 1 & M_{BB}^{-1}M_{BF} \\ 0 & N_{FF} \end{pmatrix}. \tag{A.9}
\]

Because each of the supermatrices appearing on the right hand side in the decomposition (A.7) has zeroes in either the upper right or lower left off diagonal blocks, their respective sdet can be directly evaluated using the definition (A.4) and the str property (A.2). For example,

\[
\text{sdet } \begin{pmatrix} M_{BB} & 0 \\ M_{FB} & 1 \end{pmatrix} = \exp \left\{ \text{str } \ln \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} M_{BB}^{-1} & 0 \\ M_{FB} & 0 \end{pmatrix} \right] \right\} = \exp \left\{ \text{str} \sum_{n=1}^{\infty} (-)^{n-1} \frac{1}{n} \begin{pmatrix} M_{BB}^{-1} & 0 \\ M_{FB} & 0 \end{pmatrix} \right\} = \exp \left\{ \text{tr} \sum_{n=1}^{\infty} (-)^{n-1} (M_{BB} - 1)^n \right\} = \exp \{ \text{tr } \ln M_{BB} \} = \det M_{BB}, \tag{A.10}
\]
where det is the ordinary determinant. Similarly, one finds
\[
sdet \begin{pmatrix}
1 & M_{BB}^{-1}M_{BF} \\
0 & N_{FF}
\end{pmatrix} = \exp \{(-\text{tr \ ln } N_{FF})\} = (\text{det } N_{FF})^{-1}. \tag{A.11}
\]

Here the negative sign appearing in the exponential, which in turn leads to the inverse determinant, is a direct consequence of the supertrace definition, Eq. (A.2). Combining terms, we secure
\[
sdet M = \frac{\text{det } M_{BB}}{\text{det } N_{FF}}. \tag{A.12}
\]

Alternately, we could have chosen to decompose \( M \) as
\[
M = \begin{pmatrix}
N_{BB} & M_{BF}M_{FF}^{-1} \\
0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
1 & 0 \\
M_{FB} & M_{FF}
\end{pmatrix}, \tag{A.13}
\]
with
\[
N_{BB} = M_{BB} - M_{BF}M_{FF}^{-1}M_{FB}. \tag{A.14}
\]

So doing, one finds
\[
\text{det } M = \frac{\text{det } N_{BB}}{\text{det } M_{FF}}. \tag{A.15}
\]

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