On the Dependence of the Spectra of Fluctuations in Inflationary Cosmology on Trans-Planckian Physics

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We calculate the power spectrum of metric fluctuations in inflationary cosmology starting with initial conditions which are imposed mode by mode when the wavelength equals some critical length \( \ell_c \) corresponding to a new energy scale \( M_c \), at which trans-Planckian physics becomes important. In this case, the power spectrum can differ from what is calculated in the usual framework (which amounts to choosing the adiabatic vacuum state). The fractional difference in the results depends on the ratio \( \sigma_0 \) between the Hubble expansion rate \( H_{\text{inf}} \) during inflation and the new energy scale \( M_c \). We show how and why different choices of the initial vacuum state (stemming from different assumptions about trans-Planckian physics) lead to fractional differences which depend on different powers of \( \sigma_0 \). As we emphasize, the power in general also depends on whether one is calculating the power spectrum of density fluctuations or of gravitational waves.

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I. INTRODUCTION

The exponential expansion of space in inflationary cosmology leads to the stretching of modes which were in the trans-Planckian regime at the beginning of inflation into the observable range. This leads to the possibility, first raised explicitly in [1], that trans-Planckian physics might be observable today in the cosmic microwave background. In earlier work [2, 3] we addressed this issue in a simple toy model obtained by replacing the linear dispersion relation of the cosmological fluctuations by new dispersion relations which differ from the linear one on length scales smaller than the Planck length (the same dispersion relations had been used earlier [4, 5] in the context of an analysis of possible trans-Planckian effects on black hole radiation). We were able to construct dispersion relations which give rise to large (order one) corrections to the usual spectrum of fluctuations, but the price to pay is a fine-tuning of the parameters describing the model and/or a back-reaction problem. This question has been further analyzed in many papers (see for instance Refs. [4, 5, 6, 7, 8, 9, 10, 11]). It was found that in order to obtain measurable differences in the predictions, non-adiabatic evolution of the state on trans-Planckian scales is required.

In another line of approach to the trans-Planckian challenge to inflationary cosmology, the possibility of measurable effects of trans-Planckian physics on observables such as CMB anisotropies and power spectra of scalar and tensor metric fluctuations was studied [12, 13, 14, 15, 16, 17, 18] in models where the trans-Planckian physics is based on stringy space-time uncertainty relations. In particular, the authors of [14] found a spectrum with oscillations of amplitude \( \sigma_0 \equiv H_{\text{inf}}/M_c \), where \( H_{\text{inf}} \) is the Hubble parameter during inflation and \( M_c \), a characteristic scale at which the trans-Planckian physics shows up, superimposed on the usual scale-invariant spectrum, whereas the authors of [15] found only much smaller effects.

The trans-Planckian problem was also tackled in the framework of non-commutative geometry in Ref. [19]. It was found that the effect is of order \( (H_{\text{inf}}/M_c)^4 \). It was also shown in this article that non-commutative geometry implies the presence of a preferred direction which would result in a correlation between different multipoles \( C_\ell \) and \( C_{\ell+2} \).

In yet another approach to the trans-Planckian issue, Danielsson [20] (see also Ref. [21]) suggested to replace the unknown physics on trans-Planckian scales by assuming that the modes representing cosmological fluctuations are generated mode by mode at the time when the wavelength of the mode equals the Planck length, or more generally when it equals the length \( \ell_c \) associated with the energy scale \( M_c \) of the new physics which sets the initial conditions. There is a one-parameter family of vacuum states (\( \alpha \) vacua) of a free quantum field in de Sitter space which can be considered, and for nontrivial \( \alpha \) vacua Danielsson found effects of the choice of the initial state which are of linear order in the ratio \( \sigma_0 \), and such effects could be seen in observations [23] [2]. Sim-

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1 Recently, Ref. [22] has shown that effects of the order \( \sigma_0 \) also occur in models of power-law inflation.
2 There has been a concern that nontrivial \( \alpha \) vacua are problematic from the point of view of interacting quantum field theory [24].
ilar results were found by Easther et al. 28, whereas Niemeyer et al. 29 have argued that if the modes are set off in the initial adiabatic vacuum state when their wavelength equals \( \ell_c \), then the effects are of order \( \sigma_0^3 \) and hence (in usual models) completely negligible.

Using an effective field theory method, Kaloper et al. 30 have argued that the effects of trans-Planckian physics on cosmological fluctuations should be at most of the order \( \sigma_0^3 \), assuming that the modes are in the adiabatic vacuum state when the wavelength is equal to the Hubble radius (see Ref. 31 for a criticism of imposing initial conditions at Hubble radius crossing, and see Ref. 32 for counterexamples to the claims of Ref. 30).

In this paper, we re-consider the calculation of the spectrum of cosmological perturbation in the minimal trans-Planckian setting 20 when mode by mode the initial conditions for the mode are set when the wavelength equals the Planck length (or, more generally, the length scale of the new physics). We find that the overall amplitude of the correction terms (compared to the usual spectra) depends sensitively on the precise prescription of the initial state, it depends on whether one is studying power-law or slow-roll inflation, and it also depends on whether one is computing the spectrum of scalar metric fluctuations or of gravitational waves. Some of the “discrepancies” between the results of previous studies is due to the fact that different quantities were calculated in different models. We show that when the initial state is chosen to be the instantaneous Minkowski vacuum, then the deviations of the power spectrum from the usual result are of the order \( \sigma_0^3 \), in agreement with what was found in 20. In an arbitrary \( \alpha^- \) vacuum, the choice of the value of \( \alpha \) has an effect on the amplitude of the fluctuation spectrum which is not suppressed by any power of \( \sigma_0 \). However, if \( \alpha \) is independent of \( k \), the effect will not be distinguishable from a slight change in the underlying parameters of the background inflationary model. However, in general (and specifically in the choice of the vacuum made in 20), the amplitude of the correction term in the power spectrum will have a \( k \)-dependent (and hence observable) piece which is first order in \( \sigma_0 \), at least in the case of the spectrum of gravitational waves.

While this paper was being finalized, three preprints appeared which investigate related aspects of the trans-Planckian problem. In Ref. 33, the choice of various initial states was related to the minimization of different Hamiltonians. In Ref. 34, the predictions of inflation for the power spectrum of fluctuations was studied for a two parameter class of initial states, and the amplitude of the corrections compared to the usual results was seen to depend sensitively on which state is chosen. In Ref. 35, the fact that the definitions of the Bunch-Davies and of the adiabatic vacua have some intrinsic ambiguities in a Universe with a de Sitter phase of finite duration was analyzed.

II. COSMOLOGICAL PERTURBATIONS OF QUANTUM-MECHANICAL ORIGIN

A. General considerations

The line element for the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) background plus small perturbations can be written as 26:

\[
\text{d}s^2 = a^2(\eta)\{-(1-2\phi)d\eta^2 + 2(\partial_i B)dx^i d\eta + [(1-2\psi)\delta_{ij} + 2\partial_i \delta_{ij} E + h_{ij}]dx^i dx^j\},
\]

(1)

where the functions \( \phi, B, \psi \) and \( E \) represent the scalar sector whereas the traceless and transverse tensor \( h_{ij} \) \( (h_{ij} = h_{ij}^{\mu\nu} = 0) \), represents the tensor sector, i.e. the gravitational waves. The time \( \eta \) is the conformal time and is related to the cosmic time \( t \) by the relation \( dt = a(\eta)d\eta \). It is convenient to introduce the background quantity \( \gamma(\eta) \) defined by \( \gamma \equiv -\dot{H}/H^2 \), where a dot means differentiation with respect to cosmic time and \( H \) is the Hubble rate, \( H \equiv a/\dot{a} \). Using the conformal time we may rewrite \( \gamma \) as \( \gamma = (q+1)/q \), which vanishes in the particular case of the de Sitter space-time characterized by \( q = -1 \). The perturbed Einstein equations provide us with the equations of motions for the cosmological perturbations. At the linear level, each type of perturbations decouple and we can treat them separately.

In the tensor sector (which is automatically gauge-invariant) we define the quantity \( \mu_\tau \) for each mode \( k \) according to

\[
h_{ij}(\eta, x) = \frac{1}{a} \frac{1}{(2\pi)^{3/2}} \sum_{s=1}^{2} \int \text{d}k p_{ij}^s(k) \mu_\tau(\eta, k)e^{ikx},
\]

(2)

where \( p_{ij}^s(k) \) is the polarization tensor. The plane waves appear in the previous expression because the space-like sections are flat. At linear order, gravitational waves do not couple to matter and, as a consequence, the equation of motion is just given by the perturbed vacuum Einstein equation. Explicitly, it reads 37:

\[
\frac{\text{d}}{d\eta}\left( k^2 - \frac{a'(\theta)}{a} \right) \mu_\tau = 0.
\]

(3)

This equation is similar to a time-independent Schrödinger equation with an effective potential given by \( U_\tau(\eta) = a''/a \). It can also be viewed as the equation of a parametric oscillator whose time-dependent frequency is given by \( \omega_\tau = k^2 - a''/a \), see Ref. 38. In the vacuum
state, the two point correlation function of gravitational waves reads

\[ \langle 0| h_{ij}(\eta, x) h^{ij}(\eta, x + r)|0 \rangle = \int_0^{+\infty} \frac{dk}{k} \frac{\sin kr}{kr} k^3 P_h(k). \]  

(4)

The power spectrum \( k^3 P_h(k) \) which appears in the expression for the two point correlation function is given by

\[ k^3 P_h(k) = \frac{2k^3}{\pi^2} \left| \frac{\mu_c}{a(\eta)} \right|^2. \]  

(5)

This quantity is \textit{a priori} time and wavenumber dependent but for super-horizon modes, it turns out to be time-independent because the growing mode is given by \( \mu_c \propto a(\eta) \).

Let us now turn to scalar metric (density) perturbations. The two most important differences with the gravitational waves are that the scalar sector is gauge-dependent and that the scalar perturbations of the metric are coupled to the perturbations of the stress-energy tensor describing the matter. Scalar perturbations of the geometry can be characterized by the two gauge-invariant Bardeen potentials \( \Phi \equiv \phi + (1/a) [(B - E')a'] \) and \( \Psi \equiv \psi - \mathcal{H}(B - E') \). From now on, we restrict ourselves to the case where the matter is described by a scalar field: \( \phi = \varphi_0(\eta) + \varphi_1(\eta, x) \). Fluctuations in the scalar field are characterized by the gauge-invariant quantity \( \delta \varphi \equiv \varphi_1 + \varphi_0'(B - E') \). The full set of the scalar perturbed (gauge-invariant) Einstein equations is

\[ -3\mathcal{H}(\mathcal{H} \Phi + \Psi') + \partial_k \partial^k \Psi = \frac{\kappa}{2} \left[ - (\varphi_0')^2 \Phi + \varphi_0 \delta \varphi' + a^2 \frac{dV}{d\varphi} \delta \varphi \right], \quad \partial_i (\mathcal{H} \Phi + \Phi' \partial^i) = \frac{\kappa}{2} \varphi_0' \partial_i \delta \varphi, \]  

(6)

\[ \partial_i \partial^j (\Phi - \Psi) = 0, \quad (2\mathcal{H}' + \mathcal{H}^2) \Phi + \mathcal{H} \Phi' + \Psi' + 2H \Psi' - \frac{1}{3} \partial_i \partial^i (\Phi - \Psi) = \frac{\kappa}{2} \left[ - (\varphi_0')^2 \Phi + \varphi_0' \delta \varphi' - a^2 \frac{dV}{d\varphi} \delta \varphi \right]. \]  

(7)

where \( \kappa = 8\pi/m_p^2 \), \( m_p \) being the Planck mass. At this point, one has to be careful because the case \( \varphi_0' = 0 \), which corresponds to the de Sitter space-time, is particular. In this situation, the solution of the above system of equations is simply \( \Phi = 0 \), i.e. there are no density perturbations at all (it is also necessary to require that the Bardeen potential is finite at infinity). If \( \varphi_0' \neq 0 \), then everything can be reduced to the study of a single gauge-invariant variable (the so-called Mukhanov-Sasaki variable) defined by \( 40 \)

\[ v \equiv a \left( \delta \varphi + \frac{\varphi_0'}{\mathcal{H}} \right). \]  

(8)

It turns out to be more convenient to work with the variable \( \mu_s \) defined by \( \mu_s \equiv -\sqrt{2\kappa} v \). Its equation of motion is very similar to that of the primordial spectrum of this quantity, namely

\[ \frac{1}{a(\eta)} \int dk \mu_s(k) e^{ikx} = \frac{1}{(2\pi)^{3/2}} \]  

(10)

The fact that this quantity is meaningless for the de Sitter case, \( \gamma = 0 \), is obvious.

Let us now consider a different situation: there are no cosmological fluctuations anymore but just a test scalar field \( \chi(\eta, x) \) living in a FLRW universe. If we Fourier expand the scalar field according to

\[ \chi(\eta, x) = \frac{1}{a(\eta)} (2\pi)^{3/2} \int dk \mu(k) e^{i k x}, \]  

(11)

then the Klein-Gordon equation takes the form

\[ \mu'' + \left( k^2 - \frac{a''}{a} \right) \mu = 0. \]  

(12)

We recognize the equation of motion of gravitational waves. The two-point correlation function of the scalar field can be written as

\[ \langle 0| \chi(\eta, x) \chi(\eta, x + r)|0 \rangle = \int_0^{+\infty} \frac{dk}{k} \frac{\sin kr}{kr} k^3 P_\chi(k). \]  

(13)
of the scalar field and is given by
\[ k^3 P_\chi(k) = \frac{k^3}{2\pi^2} \left( \frac{\mu}{a} \right)^2. \] (14)

We also see that the equation of motion for density perturbations is equivalent to the equation for a scalar field if the function \( \gamma \) is a non-vanishing constant. This is why, for this class of models, the study of density perturbations is in fact equivalent to the study of a scalar field. However, there is an important exception to this prescription: the de Sitter space-time since, in this case, \( \gamma = 0 \) so that the equation of motion is not given by Eq. (9) as already mentioned above. Therefore, it is inconsistent to first assume that \( \gamma \) is a constant, then to use Eq. (10) in order to calculate the power spectrum and finally to particularize the result to the de Sitter case. This procedure can only lead to the determination of the spectrum of gravitational waves and/or of a test scalar field but not of density perturbations in the de Sitter space-time.

**B. Power-law inflation**

The case of power-law inflation, for which the scale factor can be written as \( a(\eta) = t_0(\eta)^3 \), is important because the equation of motion for the cosmological perturbations can be solved explicitly. The parameter \( t_0 \) is a constant since we have chosen to work with a dimensionful scale factor. Since the function \( \gamma(\eta) \) reduces to a constant, the effective potential for density perturbations simplifies to \( a''/a \) and becomes identical to the gravitational waves effective potential. Its explicit form reads \( U_s(\eta) = U_q(\eta) = q(\eta - 1)/\eta^2 \). Then, the exact solution for the variables \( \mu_s \), \( \mu_t \) and \( \mu \) reads
\[ \mu_{s,t} = (kn)^{1/2}[A_1(k)J_{q-1/2}(kn) + A_2(k)J_{-q+1/2}(kn)]. \] (15)

In the above expression, \( J_\nu \) is a Bessel function of order \( \nu \). The two \( k \)-dependent constants \( A_1(k) \) and \( A_2(k) \) are fixed by the initial conditions.

We now consider the standard calculation of inflationary cosmology. In this case, the initial conditions are fixed in the infinite past. When \( kn \to -\infty \), the mode function tends to a plane wave with positive and negative frequency. The usual procedure requires that
\[ \lim_{k/(aH)\to+\infty} \mu_{s,t} = \mp \frac{4\sqrt{\pi} e^{-ikn}}{m_{pl} \sqrt{2k}}. \] (16)

Let us notice that the origin of the minus sign for density perturbations is the minus sign in the relation \( \mu_s = -\sqrt{2}\kappa \), \( \kappa \) being the quantity which is canonically quantized, i.e. which behaves as \( e^{-ikn}/\sqrt{2k} \) in the ultraviolet limit. Then, the constants \( A_1(k) \) and \( A_2(k) \) are completely specified and are given by
\[ \frac{A_1(k)}{A_2(k)} = e^{i\pi(q-1/2)}, \] (17)

\[ A_2(k) = \mp \frac{2\pi}{m_{pl}} e^{-i\pi(q-1/2)/2} k^{-1/2}. \] (18)

This implies that the mode functions \( \mu_{s,t} \) can be expressed as
\[ \mu_{s,t}^{\text{stand}}(k, \eta) = \mp \frac{2\pi}{m_{pl}} (\eta)^{1/2} e^{-i\pi q/2} H^{(1)}_{1/2-q}(-kn), \] (19)

where \( H^{(1)} \) is the Hankel function of first kind. This result allows us to calculate the power spectra. For density perturbations and gravitational waves, one respectively finds, on super-Hubble scales
\[ k^3 P_\chi(k) = \frac{\ell_0^2}{\ell_0^2 \pi} f(q) k^{2q+2}, \] (21)

\[ k^3 P_h(k) = \frac{\ell_0^2}{\ell_0^2 \pi} f(q) k^{2q+2}, \] (22)

where \( \ell_0 = m_{pl}^{-1/2} \) is the Planck length and the function \( f(q) \) is given by
\[ f(q) = \frac{1}{\pi} \left[ \Gamma((1/2 - q)^2 \right]^2. \] (23)

In the above definition of the function \( f(q) \), \( \Gamma \) denotes Euler’s integral of the second kind. For the de Sitter case, one has \( f(q = -1) = 1 \). However, for this case the amplitude of density perturbations blows up since \( \gamma(q = -1) = 0 \) while the amplitude of the gravitational waves spectrum remains finite. The origin of the singular limit for density perturbations is again the factor \( \sqrt{\gamma} \) at the denominator of the expression for the scalar power spectrum. In fact, this case must be analyzed separately and one can show that density perturbations do not exist in de Sitter space-time. We see that power-law inflation leads to
\[ k^3 P_h(k) = A_s k^{n_s}, \] (24)

\[ k^3 P_\chi(k) = A_s k^{n_s} \] (25)

with \( n_s - 1 = n_q \). Explicitly, one has
\[ n_s = 2q + 3. \] (26)

The case \( q = -1 \) leads to a scale-invariant spectral index, namely \( n_s = 1 \) and \( n_q = 0 \).

The power-spectrum of the scalar field can be also calculated and the result reads (contrary to the previous power spectra, the power spectrum of the scalar field is a dimensionful quantity)
\[ k^3 P_\chi(k) = \frac{1}{(2\pi^2)^2 4\pi} f(q) k^{2q+2}. \] (27)

\(^3\) Using the asymptotic behavior of the Hankel function
\[ H^{(1)}_\nu(z) \to z^{-\nu+i\pi/4} \sqrt{\frac{-\pi}{iz}} e^{i(z-\pi\nu/2-\pi/4)}, \] (28)

one can easily check that the mode function \( \mu_{s,t}^{\text{stand}}(k, \eta) \) has indeed the required limit given by Eq. (19).
It is also convenient to express this power spectrum in terms of the Hubble parameter during inflation $H_{\text{int}} = H/a = -q/(\ell_0(-\eta)^{q+1})$. This permits to replace the scale $\ell_0$ in the above equation and leads to

$$k^3P_X(k) = \frac{H^2_{\text{int}}(-\eta)^{2q+1}}{(2\pi)^2 q^2} f(q) k^{2q+2}. \quad (27)$$

Of course, in spite of the fact that the time dependence now appears explicitly, the power spectrum remains a time independent quantity. In the de Sitter case, $q = -1$, one recovers the result often cited in the literature, namely $k^3P_X = [H_{\text{int}}/(2\pi)]^2$. The spectrum becomes scale-invariant and its amplitude remains finite. The situation, except for some unimportant numerical factors, is very similar to that of gravitational waves.

To conclude this subsection, let us emphasize that the previous discussion has shown that the analogy between density perturbations and a scalar field must be used cautiously. The spectral indices are similar even in the de Sitter limit but the amplitudes differ radically in this limit.

### C. Slow-roll inflation

The slow-roll method is an approximation scheme which allows us to go beyond the simple power-law solutions considered in the previous section. It permits to treat a more general class of inflaton potentials. At leading order, the approximation is controlled by the slow-roll parameters (see e.g. Ref. 42; for a new set of slow-roll parameters with a very nice interpretation, see Ref. 43) defined by:

$$\epsilon \equiv 3\frac{\dot{\varphi}^2}{2} \left( \frac{\varphi^2}{2} + V \right)^{-1} = -\frac{\dot{H}}{H^2} = 1 - \frac{\dot{H}}{H^2},$$

$$\delta \equiv -\frac{\ddot{\varphi}}{H\dot{\varphi}} = -\frac{\epsilon}{2H\epsilon} + \epsilon, \quad \xi \equiv \frac{\epsilon - \delta}{H}. \quad (28)$$

The quantity $V(\varphi)$ is the inflaton potential. We see that $\gamma = \epsilon$, where $\gamma$ is the function that has been introduced before. The slow-roll conditions are satisfied if $\epsilon$ and $\delta$ are much smaller than one and if $\xi = O(\epsilon^2, \delta^2, \epsilon\delta)$. Since the equations of motion for $\epsilon$ and $\delta$ can be written as:

$$\frac{\dot{\epsilon}}{H} = 2\epsilon(\epsilon - \delta), \quad \frac{\dot{\delta}}{H} = 2\epsilon(\epsilon - \delta) - \xi, \quad (29)$$

it is clear that this amounts to considering $\epsilon$ and $\delta$ as constants if one works at first order in the slow-roll parameters. This property turns out to be crucial for the calculation of the power spectra of cosmological perturbations. For power-law inflation the slow-roll parameters satisfy: $\epsilon = \delta < 1$, $\xi = 0$.

The slow-roll approximation can be viewed as a kind of expansion around the de Sitter space-time. Indeed, at leading order, one has $aH \approx -(1 + \epsilon)/\eta$ which implies that the scale factor behaves like $a(\eta) \propto (-\eta)^{-1-\epsilon}$. Interestingly enough, the effective power index at leading order depends on $\epsilon$ only. At this point, one should make the following remark. Using the relation $\gamma = (q + 1)/q$, the slow-scale factor of power-law inflation can be re-written as $a(\eta) \propto (-\eta)^{-1/(1-\gamma)}$. From this expression, one might be tempted to write that, in the slow-roll framework, $a(\eta) \propto (-\eta)^{-1/(1-\epsilon)}$ since $\gamma = \epsilon$ in this approximation. Of course, this expression is inconsistent because it contains an infinite number of power of $\epsilon$. What should be done is to expand the expression $-1/(1 - \epsilon)$ in terms of $\epsilon$ from which we recover that the scale factor is given by $a(\eta) \propto (-\eta)^{-1-\epsilon}$. As long as one decides to keep high order terms, the whole hierarchy of slow-roll parameters should enter the game. This shows that the slow-roll approximation does not only consist in naively expanding the power index of the scale factor in powers of $\epsilon$. This conclusion is reinforced by a study of the cosmological perturbations within this approximation.

The effective potential of density perturbations can be calculated exactly in terms of the slow-roll parameters. The result is:

$$U_s(\eta) = a^2H^2[2 - \epsilon + (\epsilon - \delta)(3 - \delta) + \xi]. \quad (30)$$

We have seen before that, in the slow-roll approximation, $a^2H^2 \approx \eta^{-2}(1 + 2\epsilon)$. This implies that the effective potential reduces to

$$U_s(\eta) \approx \frac{1}{\eta^2}(2 + 6\epsilon - 3\delta). \quad (31)$$

Since, at leading order, $\epsilon$ and $\delta$ must be seen as constants in the slow-roll approximation, the equation of motion is of the same type as in power-law inflation and the solution is expressed in terms of Bessel functions according to:

$$\mu_s = (k\eta)^{1/2}[B_1J_{q_s-1/2}(k\eta) + B_2J_{q_s+1/2}(k\eta)]. \quad (32)$$

The parameter $q_s$, appearing in the order of the Bessel function is given by

$$q_s = -1 - 2\epsilon + \delta. \quad (33)$$

A comment is in order here: The potential $U_s$ depends on the scale factor and its derivatives only and we have seen before that the scale factor behaves as $a(\eta) \propto (-\eta)^{-1-\epsilon}$. Therefore, one might think that $U_s$ should depend on $\epsilon$ only. This is not the case. The reason is that $U_s$ contains terms like $\dot{\epsilon}/\epsilon$ (for instance) which are linear in $\delta$. First one must calculate all derivatives, replace them with their expression in terms of $\epsilon$ and $\delta$, and only then consider that the slow-roll parameters are constant. For gravitational waves, the same lines of reasoning can be applied. The effective potential can be written as $U_T(\eta) = a^2H^2\left(2 - \epsilon\right)$ and gives in the slow-roll limit

$$U_T(\eta) \sim \frac{2 + 3\epsilon}{\eta^2}. \quad (34)$$
Therefore, the solution of $\mu_+$ is similar to the one given in Eq. (32), where the effective index of the Bessel function is now given by: $q_\epsilon = -1 - \epsilon$. This time, the spectral index only depends on $\epsilon$ as expected from the shape of the tensor effective potential. Fixing the initial conditions in the infinite past in the standard manner, we arrive at the following expressions for the power spectra

\[
\mathcal{P}_\zeta(k) = \frac{H^2}{\pi \epsilon m_{\text{Pl}}^2} \left[ 1 - 2(C + 1)\epsilon - 2C(\epsilon - \delta) - 2(2\epsilon - \delta) \ln \left( \frac{k}{k_*} \right) \right], \tag{35}
\]

\[
\mathcal{P}_\eta(k) = \frac{16H^2}{\pi m_{\text{Pl}}^2} \left[ 1 - 2(C + 1)\epsilon - 2\epsilon \ln \left( \frac{k}{k_*} \right) \right], \tag{36}
\]

where $C \equiv \gamma_k + \ln 2 - 2 \simeq -0.7296$, $\gamma_k \simeq 0.5772$ being the Euler constant. All quantities are evaluated at Hubble radius crossing. The scale $k_*$ is called the pivot scale, see Refs. [44] and [45]. One sees again that the de Sitter limit $\epsilon \to 0$ is ill-defined for scalar metric fluctuations, whereas it is well-defined for gravitational waves.

### III. MINIMAL TRANS-PLANCKIAN PHYSICS

#### A. General description

We now consider the cosmological perturbations of quantum-mechanical origin in the framework of the minimal trans-Planckian physics. The new ingredient consists in assuming that the Fourier modes never penetrate the trans-Planckian region. Rather, the main idea is that a Fourier mode is “created” when its wavelength becomes equal to a new fundamental characteristic scale $\ell_C = (2\pi)/M_C$. Then, the evolution proceeds as usual since the equations of motion of the mode functions $\mu_+ \mu_-$ are taken to be unmodified and still given by Eqs. (3) and (4). In this way, the changes are entirely encoded in the initial conditions and there is no need to postulate some ad-hoc trans-Planckian physics. The time of “creation” of the mode of comoving wavenumber $k$, $\eta_k$, can be computed from the condition

\[
\lambda(\eta_k) = \frac{2\pi}{k} a(\eta_k) = \frac{2\pi}{M_C} = \ell_C, \tag{37}
\]

which implies that $\eta_k$ is a function of $k$. This has to be compared with the standard calculations where the initial time is taken to be $\eta_k = -\infty$ for any Fourier mode $k$ (in a certain sense, this initial time does not depend on $k$). The situation is summarized in Fig. [11].

In the framework described above, a crucial question is in which state the Fourier mode is created at the time $\eta_k$. There is now no asymptotic region in the infinite past where the standard prescriptions can be applied. In this article, we consider the most general conditions, namely that the mode is placed in an $\alpha$-vacuum according to

\[
\mu_{S,T}(\eta_k) = \mp \frac{c_k + d_k - 4\sqrt{\pi}}{2\omega_{S,T}(\eta_k)} \frac{1}{m_{\text{Pl}}}, \tag{38}
\]

\[
\mu'_{S,T}(\eta_k) = \pm \frac{i}{2} \sqrt{\omega_{S,T}(\eta_k)} \frac{1}{m_{\text{Pl}}} (c_k - d_k), \tag{39}
\]

where $\omega_{S,T} \equiv \sqrt{k^2 - T_{S,T}^2}$ is the effective frequency for density perturbations and gravitational waves. The coefficients $c_k$ and $d_k$ are a priori two arbitrary complex numbers satisfying the condition $|c_k|^2 - |d_k|^2 = 1$. The instantaneous Minkowski state corresponds to $c_k = 1$ and $d_k = 0$. If, in addition, we take $\eta_k = -\infty$, we recover the standard choice, see Eq. (10) since $\omega_{S,T}(\eta_k = -\infty) = k$.

We are now in a position where we can compute the mode functions and the corresponding power spectra for density perturbations and gravitational waves.

#### B. Calculation of the mode function

In this section, we write the scale factor as $a(\eta) \propto (-\eta)^p$, where $p$ is a generalized index defined by

\[
p = \begin{cases} 
q = -1, & \text{power-law inflation}, \\
-1 - \epsilon, & \text{slow-roll inflation}. 
\end{cases} \tag{40}
\]

Since the initial conditions given by Eqs. (85) and (86) are different from the standard ones, we will obtain different mode functions. The new mode functions that we want to calculate can be expanded according to

\[
\mu_{S,T}(\eta) = \alpha_{S,T}(k) \mu_{S,T}^{\text{stand}}(\eta) + \beta_{S,T}(k) [\mu_{S,T}^{\text{stand}}(\eta)]^*, \tag{41}
\]

where $\mu_{S,T}^{\text{stand}}(\eta)$ denotes the mode functions obtained in the standard situation and given by the expression

\[
\mu_{S,T}^{\text{stand}}(k, \eta) = \pm \frac{2i\pi}{m_{\text{Pl}}} (-\eta)^{1/2} e^{-i\pi/2} H^{(1)}_{1/2-\epsilon}(\sqrt{\eta}k). \tag{42}
\]
FIG. 1: Sketch of the evolution of the Hubble radius vs time comparing how the initial conditions are fixed in the standard procedure and in the framework of the minimal trans-Planckian physics. In the standard procedure, the initial conditions are fixed on a surface of constant time $\eta = \eta_i$ with $\eta_i \to -\infty$. In this sense, the initial time does not depend on the wavenumber.

On the contrary, in the minimal trans-Planckian physics, the initial time depends on the wavenumber and the Fourier modes are “created” when their wavelength is equal to a characteristic length. They never penetrate into the trans-Planckian region. This causes a modification of the power spectrum which is $k$ dependent since the modification is not the same for all Fourier modes as is apparent from the figure.

In this equation, we have introduced another generalized index $\nu$ defined by

\[
\nu = \begin{cases} 
q = -\frac{1 - \gamma - \epsilon}{1}, & \text{power-law inflation}, \\
q_s = -1 - 2\epsilon + \delta, & \text{slow-roll inflation}, \\
q_T = -1 - \epsilon, & \text{slow-roll inflation}.
\end{cases} \tag{43}
\]

For power-law inflation the generalized index is the same for density perturbations and for gravitational waves (hence we do not need to distinguish them in the above definition). For slow-roll inflation, as already mentioned above, they differ.

The coefficients $\alpha_{s,T}(k)$ and $\beta_{s,T}(k)$ are readily obtained using the initial conditions given in Eqs. (38) and (39)

\[
\alpha_{s,T}(k) = \frac{1}{4} (c_k + d_k) e^{i\pi \nu/2} \sqrt{\frac{\pi}{-2\omega_{s,T}(\eta_k)\eta_k}} \left\{ k\eta_k \left[ H^{(2)}_{3/2 - \nu} - H^{(2)}_{1/2 - \nu} \right] + \left[ 1 + 2i \frac{c_k - d_k}{c_k + d_k} \omega_{s,T}(\eta_k)\eta_k \right] H^{(2)}_{1/2 - \nu} \right\}, \tag{44}
\]

\[
\beta_{s,T}(k) = \frac{1}{4} (c_k + d_k) e^{-i\pi \nu/2} \sqrt{\frac{\pi}{-2\omega_{s,T}(\eta_k)\eta_k}} \left\{ k\eta_k \left[ H^{(1)}_{3/2 - \nu} - H^{(1)}_{1/2 - \nu} \right] + \left[ 1 + 2i \frac{c_k - d_k}{c_k + d_k} \omega_{s,T}(\eta_k)\eta_k \right] H^{(1)}_{1/2 - \nu} \right\}, \tag{45}
\]

where the Hankel functions are evaluated at $-k\eta_k$. The knowledge of the coefficients $\alpha_{s,T}(k)$ and $\beta_{s,T}(k)$ is equivalent to the knowledge of the modified mode function, see Eq. (31). The quantity $-k\eta_k$ can be written as...
-k\eta_k = -p/\sigma_k$ where $\sigma_k \equiv H(\eta_k)/M_c$. The explicit expression of $\sigma_k$ can be easily derived. Writing that $H = p/(a\eta)$, we find that $\sigma_k = \sigma_0(\eta_k/\eta_0)^{-p-1}$ where the time $\eta_0$ is a given, a priori arbitrary, time during inflation which, and this is the important point, does not depend on $k$. The quantity $\sigma_0$ is defined by $\sigma_0 \equiv H_0/M_c$. Using that $\eta_k = p/(k\sigma_k)$ and $\eta_0 = p/(\sigma_0 a_0 M_c)$ we finally arrive at

$$\sigma_k = \sigma_0 \left( \frac{k}{a_0 M_c} \right)^{-1-1/p}.$$ (46)

In the case of slow-roll inflation, the previous expression reduces to $\sigma_k = \sigma_0[k/(a_0 M_c)]^{-\epsilon}$. For the de Sitter case, $p = -1$ and $\epsilon = 0$, the Hubble parameter and therefore $\sigma_k$ are constant. This means that $\eta_k \propto 1/k$ as expected. An important remark is that $\sigma_k$ only depends on $p$. This means that, in the slow-roll approximation, $\sigma_k$ only depends on the slow-roll parameter $\epsilon$ and not on the slow-roll parameter $\delta$. This is due to the fact that the calculation of $\sigma_k$ only depends on background quantities. The slow-roll parameter $\delta$ appears in the calculation through the generalized index $\nu$, i.e. at the perturbed level. We now discuss the spectrum obtained according to the initial state chosen.

### C. Instantaneous Minkowski vacuum

In this section, we assume that the initial state is such that $c_k = 1$ and $d_k = 0$. As a warm up, let us derive the spectrum in the particular case of de Sitter, $p = -1$. As mentioned above, this can only been done for gravitational waves since the de Sitter limit is singular for density perturbations. For $p = -1$, we have $\sigma_k = \sigma_0$ and $-k\eta_k = 1/\sigma_0$. The spectrum can be determined exactly because the Bessel functions in Eqs. (44) and (45) reduce to ordinary functions in the case $p = -1$. The result reads

$$\alpha_\tau(k) = \frac{1}{2} e^{-i\pi/2 - i/\sigma_0 (1 - 2\sigma_0^2)^{-1/4}} \left[ i + \sigma_0 - i\sigma_0^2 + (i + \sigma_0) \sqrt{1 - 2\sigma_0^2} \right],$$ (47)

$$\beta_\tau(k) = \frac{1}{2} e^{i\pi/2 + i/\sigma_0 (1 - 2\sigma_0^2)^{-1/4}} \left[ -i + \sigma_0 + i\sigma_0^2 + (i - \sigma_0) \sqrt{1 - 2\sigma_0^2} \right].$$ (48)

The modulus of the two previous expressions can be easily derived from the above relation

$$|\alpha_\tau| = \frac{1}{2} \left( 2 - 2\sigma_0^2 - \sigma_0^4 + 2 \right)^{1/2}, \quad |\beta_\tau| = \frac{1}{2} \left( 2 - 2\sigma_0^2 + \sigma_0^4 - 2 \right)^{1/2},$$ (49)

which coincides with the result of Ref. [29]. One can check that $|\alpha_\tau|^2 - |\beta_\tau|^2 = 1$. In addition, one notices that the result is independent on $k$ as expected for the de Sitter space-time. Then, the power spectrum on super-horizon scales is given by $k^3 P_h(k) = (2k^3/\pi^2) \left| \alpha_\tau \mu_\tau^{\text{stand}} + \beta_\tau \mu_\tau^{\text{stand}}/a \right|^2$ where the super-horizon standard mode function can be expressed as $\mu_\tau^{\text{stand}} \approx -4i\sqrt{(2k)^{-1/2}/(m_m k)}$. The result reads

$$k^3 P_h(k) = \frac{16H^2_{\text{inf}}}{\pi m_{\mu_\tau}^2} \left[ 1 - \sigma_0^2 - \frac{1}{2} \sigma_0^4 + \frac{1}{2} \sigma_0^3 \sin \left( \frac{2}{\sigma_0} \right) + \frac{1}{2} \sigma_0^3 \cos \left( \frac{2}{\sigma_0} \right) \right].$$ (50)

This spectrum is represented in Fig. 2. So far, no approximation has been made. If we now assume that $\sigma_0$ is a small quantity, we can expand the result in terms of this quantity. Let us also remark that the terms $\sin(2/\sigma_0)$ and $\cos(2/\sigma_0)$ are non-analytically for small values of $\sigma_0$. At leading order, we obtain that $|\alpha_\tau| \sim 1$ and $|\beta_\tau| \sim \sigma_0^3/2$ in agreement with Ref. [29]. This leads to the following spectrum

$$k^3 P_h(k) \approx \frac{16H^2_{\text{inf}}}{\pi m_{\mu_\tau}^2} \left[ 1 + \sigma_0^3 \sin \left( \frac{2}{\sigma_0} \right) \right].$$ (51)

Therefore, the corresponding effect in the power spectrum is of order $\sigma_0^3$, i.e. a tiny effect if $\sigma_0$ is small. Note, however, that it is not possible to distinguish this effect from a change in the parameter $H_{\text{inf}}$ of the inflationary background.

We now turn to the case of slow-roll inflation. We start with density perturbations. The first step consists in calculating the coefficients $\alpha_\delta$ and $\beta_\delta$. At first order in the slow-roll parameters and at leading order in the parameter $\sigma_0$ we have

$$\alpha_\delta \approx e^{ik\eta_k},$$ (52)

$$\beta_\delta \approx i \left( \frac{1}{2} - \frac{3}{4} \delta - \frac{3}{2} \epsilon \ln \frac{k}{\sigma_0 M_c} \right) \sigma_0^3 e^{-ik\eta_k}.$$ (53)
It is interesting to calculate the modulus of the coefficient $\beta_s$. Straightforward calculations lead to

$$ |\beta_s| \simeq \left( \frac{1}{2} - \frac{3}{4} \delta - \frac{3}{2} \epsilon \ln \frac{k}{a_0 M_C} \right) \sigma_0^3. \tag{54} $$

We see that this quantity depends on both slow-roll parameters $\epsilon$ and $\delta$, contrary to what was obtained in Ref. [29] where the coefficient $\beta_s$ was found not to depend on the slow-roll parameter $\delta$. The main reason for this discrepancy is that, in Ref. [29], the spectrum of a scalar field on an unperturbed background cosmological model (or, equivalently, the spectrum of gravitational waves) was calculated, and not the spectrum of scalar metric fluctuations, see the discussion after Eq. (14). If one uses the expression of the scale factor $a(\eta) \propto (-\eta)^{-1-\epsilon}$ and inserts this into the expression for the power spectrum for a scalar field on an unperturbed background, one does not obtain the correct expression for the power spectrum of density perturbations and one misses the terms proportional to the slow-roll parameter $\delta$. One only obtains either the power spectrum of gravitational waves or the power spectrum of density perturbations in the very particular case of power-law inflation for which we have $\epsilon = \delta$ (and for which, in fact, the exact solution is known). This is why putting $\epsilon = \delta$ in Eq. (54) reproduces the result found in Ref. [29].

Then, the power spectrum of the conserved quantity $\zeta$ is given by the following expression [to be compared with (35)]

$$ k^3 P_{\zeta} = \frac{H^2}{\pi \epsilon m_p^2} \left( 1 - 2(C+1)\epsilon - 2C(\epsilon - \delta) - 2(2\epsilon - \delta) \ln \frac{k}{k_s} + \sigma_0^3 \left\{ \sin \left[ \frac{2}{\sigma_0} \left( 1 + \epsilon + \epsilon \ln \frac{k}{a_0 M_C} \right) \right] \right. $$

$$ - \left[ \frac{3}{2} \delta + 3 \epsilon \ln \frac{k}{a_0 M_C} + 2(C+1)\epsilon + 2C(\epsilon - \delta) + 2(2\epsilon - \delta) \ln \frac{k}{k_s} \right] \sin \left[ \frac{2}{\sigma_0} \left( 1 + \epsilon + \epsilon \ln \frac{k}{a_0 M_C} \right) \right] $$

$$ - \pi(2\epsilon - \delta) \cos \left[ \frac{2}{\sigma_0} \left( 1 + \epsilon + \epsilon \ln \frac{k}{a_0 M_C} \right) \right] \left\} \right) \right). \tag{55} $$
Thus, the power spectrum has oscillations superimposed on the scale-invariant base. Hence, in this case the change in the spectrum is “in principle” physically measurable. However, the prospects for actually detecting the difference of order $\sigma_0^2$ between (32) and (33) are hopeless if the parameter $\sigma_0$ is small. Indeed if, for instance, we assume that $\sigma_0 = 10^{-2}$, a value consistent with string theory, and if we consider that $\varepsilon \approx \delta \approx 10^{-3}$ then the correction to the power spectrum is of order $\approx 10^{-7}$. There is no possibility to detect such a small effect even with the Planck satellite.

For completeness, let us mention the slow-roll result for gravitational waves. In this case the coefficients $\alpha_\tau$ and $\beta_\tau$ can be deduced from $\alpha_\phi$ and $\beta_\phi$ by putting $\epsilon = \delta$. Then, the power spectrum is given by

$$ k^3 P_h = \frac{16H^2}{\pi m_{\nu_1}^2} \left[ 1 - 2(C + 1)\epsilon - 2\epsilon \ln \frac{k}{k_*} + \sigma_0^3 \left\{ \sin \left[ \frac{2}{\sigma_0} \left( 1 + \epsilon + \epsilon \ln \frac{k}{a_0 M_\odot} \right) \right] - \left( \frac{3}{2} \epsilon + 3\epsilon \ln \frac{k}{a_0 M_\odot} + 2(C + 1)\epsilon \right) \right\} \right] - 2\epsilon \ln \frac{k}{k_*} \sin \left[ \frac{2}{\sigma_0} \left( 1 + \epsilon + \epsilon \ln \frac{k}{a_0 M_\odot} \right) \right] - \pi \epsilon \cos \left[ \frac{2}{\sigma_0} \left( 1 + \epsilon + \epsilon \ln \frac{k}{a_0 M_\odot} \right) \right] \right) . \tag{56} $$

We can also estimate how the consistency check of inflation is modified. At first order in the slow-roll parameters and at leading order in the parameter $\sigma_0$, this quantity is the same as in the standard case, namely

$$ R \equiv \frac{k^3 P_h}{k^3 P_c} = 16 \epsilon . \tag{57} $$

This is because at zeroth order in the slow-roll parameter, the leading contribution in $\sigma_0$ is the same for gravitational waves and density perturbations.

\[
\alpha_\tau(k) = \frac{1}{2} (c_k + d_k) e^{-i \pi / 2 - i \sigma_0 (1 - 2 \sigma_0^2)^{-1/4}} \left[ i + \sigma_0 - i \sigma_0^2 + z_k (i + \sigma_0) \sqrt{1 - 2 \sigma_0^2} \right], \tag{58}
\]

\[
\beta_\tau(k) = \frac{1}{2} (c_k + d_k) e^{i \pi / 2 + i \sigma_0 (1 - 2 \sigma_0^2)^{-1/4}} \left[ -i + \sigma_0 + i \sigma_0^2 + z_k (i - \sigma_0) \sqrt{1 - 2 \sigma_0^2} \right], \tag{59}
\]

where we have introduced the quantity $z_k$ defined by $z_k \equiv (c_k - d_k)/(c_k + d_k)$. On can easily check that for $c_k = 1$ and $d_k = 0$, i.e. $z_k = 1$, the above expressions reduce to Eqs. (47) and (48). One can also calculate the modulus of these Bogoliubov coefficients,

\[
|\alpha_\tau|^2 = \frac{|c_k + d_k|}{4} \left( (1 + z_k z_k^* - \sigma_0^2 (1 + z_k z_k^*) + \sigma_0^4 (1 - 2 z_k z_k^*) \right) \sqrt{1 - 2 \sigma_0^2}, \tag{60}
\]

\[
|\beta_\tau|^2 = \frac{|c_k + d_k|}{4} \left( (1 + z_k z_k^*) - \sigma_0^2 (1 - 2 z_k z_k^*) \right) \sqrt{1 - 2 \sigma_0^2}, \tag{61}
\]

and verify that $|\alpha_\tau|^2 - |\beta_\tau|^2 = |c_k|^2 - |d_k|^2 = 1$. From the two previous relations, one can determine the power spectrum exactly. The result reads

\[
k^3 P_h = \frac{16H^2}{\pi m_{\nu_1}^2} \sqrt{1 - 2 \sigma_0^2} \left( \frac{1}{2} (1 + z_k z_k^*) (1 - \sigma_0^2) + \frac{\sigma_0^3}{2} (1 - 2 z_k z_k^*) \left[ \sigma_0 - \sigma_0 \cos \left( \frac{2}{\sigma_0} \right) - 2 \sin \left( \frac{2}{\sigma_0} \right) \right] \right) + \frac{1}{2} (1 - z_k z_k^*) (3 \sigma_0^2 - 1) \cos \left( \frac{2}{\sigma_0} \right) + \frac{1}{2} (1 - z_k z_k^*) \sigma_0 \sin \left( \frac{2}{\sigma_0} \right) - i \left[ \frac{2}{2} (z_k - z_k^*) (1 - 2 \sigma_0^2)^{3/2} \sin \left( \frac{2}{\sigma_0} \right) \right] - \frac{i}{2} \sigma_0^3 (z_k^* - z_k) \sqrt{1 - 2 \sigma_0^2} + i \frac{1}{2} \sigma_0 (z_k^* - z_k) (\sigma_0^2 - 2) \sqrt{1 - 2 \sigma_0^2} \cos \left( \frac{2}{\sigma_0} \right) \right]. \tag{62}
\]

\[\text{D. Arbitrary } \alpha \text{ vacuum state}\]

We have established that, in the case of the instantaneous Minkowski state, the correction to the power spectra is of order $\sigma_0^2$. It is interesting to see whether this conclusion remains true for other initial states. For this reason, we now repeat the calculation of the Bogoliubov coefficients in the case of a de Sitter space-time with an arbitrary $\alpha$-vacuum state, characterized by a value of $c_k$ and $d_k$, as the initial state. One finds
As is apparent from the above expression, the correction in the power spectrum compared to the usual results is of order unity, i.e. not suppressed by any power of \( \sigma_0 \). If one argues that the Bogoliubov coefficients themselves should be of linear order in \( \sigma_0 \) then the correction terms would also be linear in \( \sigma_0 \). On the other hand, unless the Bogoliubov coefficients depend on \( k \), the effect is simply a change in the amplitude of the spectrum, and can hence be absorbed in a redefinition of the background. Thus, the effect is not physically measurable. On the other hand, if the coefficients depend on \( k \) (as discussed e.g. in the analysis of [21]) there is a large measurable effect on the power spectrum. For the instantaneous Minkowski state, one check easily that only the two first terms in the curly brackets survives and that the corresponding expression reduces to the one found previously. If we expand this spectrum in powers of \( \sigma_0 \), we find that

\[
k^3 P_h = \frac{16H^2_{\text{inf}}}{\pi m_{\text{Pl}}^2} |c_k + d_k|^2 \left\{ z_k \cos \left( \frac{2}{\sigma_0} \right) - i \sin \left( \frac{2}{\sigma_0} \right) \right\}^2 + \left\{ (1 - z_k z_k^*) \sin \left( \frac{2}{\sigma_0} \right) - i(z_k - z_k^*) \cos \left( \frac{2}{\sigma_0} \right) \right\} |\sigma_0 + \cdots| \quad \text{. (63)}
\]

We conclude that the result obtained in the previous subsection for the instantaneous Minkowski vacuum, namely that the corrections to the spectrum are suppressed by three powers of \( \sigma_0 \) appears to be very particular to the choice of that state.

However, the previous analysis does not cover all the possible cases. Indeed, as we are going to demonstrate, the proposal put forward in Ref. [20] corresponds in fact to a case where the complex number \( z_k \) can also depend on \( \sigma_0 \). The previous study assumed that \( c_k \) and \( d_k \) were pure complex numbers. Therefore, in Danielsson’s case [20], the analysis needs to be redone.

### E. Danielsson’s \( \alpha \)-vacuum state

We start our analysis with the simplest case, namely gravitational waves in a de Sitter background. We treat this case in some detail since then these calculations can be used to study more complicated situations, for instance slow-roll inflation. The Einstein-Hilbert action expanded to second order (since we are dealing with first order equations of motion) reads

\[
S_{\text{E-H}} = (16\pi G)^{-1} \int R \sqrt{-g} d^4x \quad \text{. (64)}
\]

We can check that this Lagrangian leads to the correct equation of motion: we have (we did not take into account the overall constant; here the bar over \( \mathcal{L} \) means that we are considering the Lagrangian in the Fourier space)

\[
\frac{\delta \mathcal{L}}{\delta \mu_i^s} = -2\mathcal{H} (\mu^s_i)^* + 2(\mathcal{H}^2 - k^2)(\mu^s_i)^* \quad \text{. (69)}
\]
\[
\frac{\delta \tilde{L}}{\delta \mu_T^*} = 2(\mu_T^*)' - 2H(\mu_T^*)'.
\] (70)

Therefore, the Euler-Lagrange equations given by the well-known expression
\[
\frac{d}{d\eta} \left[ \frac{\delta \tilde{L}}{\delta \mu_T^*} \right] - \frac{\delta \tilde{L}}{\delta \mu_T^*} = 0,
\] (71)
reproduce the correct equations of motion for the variables \((\mu_T^*)'\). Let us notice that we can also vary the Lagrangian with respect to \((\mu_T^*)'\) to obtain the equation of motion for \(\mu_T^*\).

From the Lagrangian formalism that we have just described, we can now pass to the Hamiltonian formalism. The conjugate momentum to \(\mu_T^*\) is defined by the formula
\[
p_T^* = \frac{\delta \tilde{L}}{\delta (\mu_T^*)'} = \frac{m_{\text{pl}}^2}{16\pi} \left[ (\mu_T^*)' - \frac{a'}{a} \mu_T^* \right].
\] (72)

In real space, the conjugate momentum \(\Pi^s\) is defined by the expression
\[
\Pi^s(\eta, x^k) = \frac{\delta \tilde{L}}{\delta (h^s)'} = \frac{m_{\text{pl}}^2}{16\pi} a^2 (h^s)'.
\] (73)

We can check that the two definitions are consistent by means of the relation
\[
\Pi^s(\eta, x^k) = \frac{a(\eta)}{(2\pi)^{3/2}} \int d\mathbf{k} p_T^s e^{i\mathbf{k} \cdot \mathbf{x}}.
\] (74)

Danielsson’s boundary condition consists in demanding that, at the time of creation \(\eta_k\), one has the usual relation characteristic of the standard vacuum state, namely
\[
\Pi^s(\eta_k, x^k) = -ik\frac{m_{\text{pl}}^2}{16\pi} a^2 (h^s)(\eta_k, x^k).
\] (75)

Using the Fourier decomposition of the scalar fields \(h^s\) and of their conjugate momenta, this last equation boils down to
\[
(\mu_T^*)' - \frac{a'}{a} \mu_T^* = -ik \mu_T^*.
\] (76)

This relation is to be satisfied at the time \(\eta = \eta_k\), and implies a link between the coefficients \(\alpha_\epsilon\) and \(\beta_\epsilon\). Since, in addition, \(|\alpha_\epsilon|^2 - |\beta_\epsilon|^2 = 1\), the coefficients are in fact completely fixed and they now depend on the parameter \(\sigma_0\). Explicitly, the link can be expressed as
\[
\frac{\beta_\epsilon}{\alpha_\epsilon} = -\frac{\mu_T^\text{stand}}{\mu_T^\text{stand}'} - \frac{\alpha'/(a')}{\mu_T^\text{stand} + i k \mu_T^\text{stand}'}.
\] (77)

So far no approximation has been made and the previous relation is general. We now restrict our study to the de Sitter case. Using the explicit form of the mode function in this case, one arrives at
\[
\frac{\beta_\epsilon}{\alpha_\epsilon} = \frac{i}{\epsilon + 2k\eta_k} e^{-2ik\eta_k},
\] (78)

which is exactly the relation found in Ref. [20]. From the normalization, we deduce that \(|\alpha_\epsilon|^2 = 1 + \sigma_0^2/4\). The exact power spectrum can now be determined since we explicitly know the coefficients \(\alpha_\epsilon\) and \(\beta_\epsilon\). Performing the standard calculation, one finds
\[
k^3 P_h = \frac{16H_{\text{inf}}^2}{\pi m_{\text{pl}}^2} \left[ 1 + \frac{\sigma_0^2}{2} - \sigma_0 \sin \left( \frac{2}{\sigma_0} \right) - \frac{\sigma_0^2}{2} \cos \left( \frac{2}{\sigma_0} \right) \right].
\] (79)

This expression should be compared with the corresponding equation found in the case of the Minkowski instantaneous state, see Eq. (60). In particular, expanding everything in terms of \(\sigma_0\), one obtains
\[
k^3 P_h = \frac{16H_{\text{inf}}^2}{\pi m_{\text{pl}}^2} \left[ 1 - \sigma_0 \sin \left( \frac{2}{\sigma_0} \right) \right],
\] (80)
i.e., a first order effect instead of a third order effect. However, once again note that this effect can be reproduced by redefining the background cosmological parameters, and hence it is not a physically measurable effect. The spectrum is represented in Fig. [3]

Let us now turn to the case of slow-roll inflation. The goal is simply to evaluate the ratio \(\beta_\epsilon/\alpha_\epsilon\) given by Eq. (77) to leading order in the parameter \(\sigma_0\) and to first order in the slow-roll parameter \(\epsilon\). One finds
\[
\frac{\beta_\epsilon}{\alpha_\epsilon} = -\frac{i\sigma_0}{2} e^{-2ik\eta_k} \left( 1 - \epsilon \ln \frac{k}{a_0 M_C} \right)\alpha_\epsilon.
\] (81)

Of course, for \(\epsilon = 0\), one recover the exact result obtained previously at leading order in \(\sigma_0\). Finally the power spectrum for gravitational waves reads
\[
k^3 P_h = \frac{16H_{\text{inf}}^2}{\pi m_{\text{pl}}^2} \left[ 1 - 2(2C + 1)\epsilon - 2\epsilon \ln \frac{k}{k_*} - \sigma_0 \left[ 1 - 2(2C + 1)\epsilon - 2\epsilon \ln \frac{k}{k_*} - \epsilon \ln \frac{k}{a_0 M_C} \right] \right] \times \sin \left( \frac{2}{\sigma_0} \left[ 1 + \epsilon \ln \frac{k}{a_0 M_C} \right] \right) + \sigma_0 \pi \epsilon \cos \left( \frac{2}{\sigma_0} \left[ 1 + \epsilon \ln \frac{k}{a_0 M_C} \right] \right).
\] (82)

Note that since in this case the correction terms lead to oscillations in the spectrum about a scale-invariant base spectrum, the effect of the correction terms is in principle
FIG. 3: Amplitude of the gravitational wave power spectrum as a function of the parameter \(\sigma_0\) in the case where the initial state is the one singled out by Danielsson’s condition. For small values of \(\sigma_0\), the leading order correction is linear in \(\sigma_0\).

measurable. Since the amplitude is only suppressed by one power of \(\sigma_0\), the prospects of being able to detect such effects in upcoming experiments are good.

We now explore the same mechanism but for scalar metric (density) perturbations. The action is expressed in terms of the variable \(\mu_s\) introduced before. It reads

\[
S_2 = \frac{1}{2} \int d^4x \left[ (\mu_s')^2 - \delta^{ij} \partial_i \mu_s \partial_j \mu_s + \frac{(a\sqrt{\gamma})''}{a\sqrt{\gamma}} \mu_s^2 \right],
\]

(83)

In Fourier space, the action reads

\[
S_2 = \frac{1}{2} \int d\eta \int d^3k \left\{ \mu_s' \mu_s'^* - \left[ k^2 - \frac{(a\sqrt{\gamma})''}{a\sqrt{\gamma}} \right] \mu_s \mu_s^* \right\}.
\]

(84)

The conjugate momentum to the variable \(\mu_s\) is given by \(\Pi_{\mu_s} = \mu_s'\) and therefore Danielsson’s boundary condition reads

\[
\mu_s' = -ik \mu_s,
\]

(85)

where this relation must be evaluated at the time \(\eta = \eta_k\). We note that the term \((a'/a)\mu_s\), which was present in the case of the scalar field, does not appear for density perturbations. This can be traced back to the fact that the two quantities that are quantized, \(\mu_s\), and \(\mu_s\) do not have the same action. This means that the link between \(\alpha_s\) and \(\beta_s\) for density perturbations is not the same as for gravitational waves. For scalar metric fluctuations, it reads

\[
\frac{\beta_k}{\alpha_k} = -\frac{(\mu_s^{\text{stand}}')'}{i k \mu_s^{\text{stand}} + i k \mu_s^{\text{stand}}}.
\]

(86)

This equation should be compared with Eq. (77). The absence of the terms \(a'/a\) has important consequences. In order to guess what the difference is, we can apply the previous equation to the de Sitter case, even if in principle this is not allowed (see the discussion above). One finds

\[
\frac{\beta_k}{\alpha_k} = \frac{i}{i + 2k\eta_k - 2ik^2 \eta_k^2 e^{-2ik\eta_k}}.
\]

(87)

Expanding the previous expression in \(\sigma_0\), we find that the first order terms cancel out and we are left with \(\beta_k/\alpha_k \simeq -(a_0^2/2)e^{2i/\sigma_0}\). Therefore, the result will be of order \(\sigma_0^2\) and not of order \(\sigma_0\) as it was the case for a scalar field and gravitational waves. We would have obtained a linear correction in the power spectrum if Danielsson’s condition had been

\[
\left( \frac{\mu_s}{a} \right)' = -i \frac{k}{\alpha_s} \mu_s,
\]

(88)
as it is for gravitational waves, instead of \(\mu_s' = -ik \mu_s\), see Eq. (85).

Let us now evaluate Eq. (86) consistently and rigorously in the slow-roll approximation. One finds

\[
\beta_s = -\frac{\sigma_0^2}{2} e^{-2ik\eta_k} \left( 1 + \epsilon - \frac{3}{2} \delta - 2\epsilon \ln \frac{k}{a_0 M_C} \right) \alpha_s.
\]

(89)
As expected, the result is quadratic in $\sigma_0$. Repeating the standard calculations, one can find the explicit expression for the power spectrum

\begin{equation}
\kappa^3 P_\zeta = \frac{H^2}{\pi e m_{\nu i}^2} \left\{ 1 - 2(C + 1)\epsilon - 2C(\epsilon - \delta) - 2(2\epsilon - \delta) \ln \frac{k}{k_*} + \sigma_0^2 \left[ 1 - 2(C + 1)\epsilon - 2C(\epsilon - \delta) + \epsilon - \frac{3}{2} \delta \right] - 2(2\epsilon - \delta) \ln \frac{k}{k_*} - 2\epsilon \ln \frac{k}{a_0 M_C} \cos \left[ \frac{2}{\sigma_0} \left( 1 + \epsilon + \epsilon \ln \frac{k}{a_0 M_C} \right) \right] + \sigma_0^2 \pi(2\epsilon - \delta) \sin \left[ \frac{2}{\sigma_0} \left( 1 + \epsilon + \epsilon \ln \frac{k}{a_0 M_C} \right) \right] \right\}.
\end{equation}

(90)

As in the case of gravitational waves, in principle this is a measurable effect since it corresponds to oscillations of the power spectrum. Typical power spectra are represented in Fig. 4. Since the correction is quadratic in $\sigma_0$, the correction turns out to be extremely small and the prospects for the detection of such an effect are not optimistic even with a high-accuracy experiment like Planck.

Thus, at least with this choice of the precise form of the action [46], Danielsson’s prescription leads to trans-Planckian corrections of different strengths for gravitational waves and for scalar fluctuations. Note that this section is that the equation determining the behavior of the mode function is modified. A typical example where this happens is for the dispersion relation $\omega_{\text{phys}}^2 = k^2_{\text{phys}} - 2b_{11}k^4_{\text{phys}} + 2b_{12}k^6_{\text{phys}}$, which represents the first terms of a systematic Taylor expansion. This dispersion relation is represented in Fig. 5.

The equation of motion for the mode function of a scalar field is now given by

\begin{equation}
\mu'' + \left[ \omega^2(k, \eta) - \frac{a''}{a} \right] \mu = 0,
\end{equation}

(92)

where $\omega$ is the comoving frequency given by $\omega(k, \eta) = a(\eta)\omega_{\text{phys}}[k/a(\eta)]$. The difference compared to the previous section is that the equation determining the behavior of the mode function is modified. The mode functions depend on whether $w^2 > a^2/a$ or $w^2 < a''/a$.

Let us consider an expanding cosmological background. There will be four time intervals (“regions”), see Fig. 6. As the Universe expands, the physical wavenumber will decrease. The evolution starts in Region I, where the WKB approximation is valid, the mode function and its derivative at the matching point have already been shown that the corresponding spectrum is linear in $\sigma_0$.

**IV. COMPARISON WITH ANALYSES USING MODIFIED DISPERSION RELATIONS**

In this section, we consider a case where the trans-Planckian physics is described by mean of a modified dispersion relation. Although, this phenomenological description is different from the one considered above, it has already been shown that the corresponding spectrum can also possess superimposed oscillations [2, 3, 4]. In this section, we calculate the spectrum of a scalar field (or gravitational waves) living in a de Sitter space-time and we compare this result with the corresponding result obtained in the framework described in the previous sections, see Eqs. (82) and (90). In particular, our goal is to determine which parameter controls the magnitude of the correction to the standard power spectrum in the case of a modified dispersion relation (for the case treated before, it was the parameter $\sigma_0$).

The main shortcomings of describing the trans-Planckian physics is described by mean of a modified dispersion relation is that we need to assume something about the physics beyond the Planck, contrary to the kind of modification envisaged above. Modifications in the power spectrum are obtained if the WKB approximation is violated in the trans-Planckian regime. A typical example where this happens is for the dispersion relation $\omega_{\text{phys}}^2 = k^2_{\text{phys}} - 2b_{11}k^4_{\text{phys}} + 2b_{12}k^6_{\text{phys}}$, which represents the first terms of a systematic Taylor expansion. This dispersion relation is represented in Fig. 5.
been “generated” in Region II. It is here that the trans-Planckian physics modifies the standard result (which predicts $\beta_k = 0$). The mode function can be written as

$$\mu_{\text{III}}(\eta) = \frac{\alpha_k}{\sqrt{2\omega(k, \eta)}} e^{-i \int_{\eta_1}^{\eta} \omega(k, \tau) d\tau} + \frac{\beta_k}{\sqrt{2\omega(k, \eta)}} e^{i \int_{\eta_1}^{\eta} \omega(k, \tau) d\tau}. \quad (95)$$

The coefficients $\alpha_k$ and $\beta_k$ can be found in terms of $C_\pm(k)$ by matching the solution (and its derivative) at $\eta_2(k)$ as previously. Finally, in Region IV, the WKB approximation is again violated. This region corresponds to the usual super-Hubble region. We have

$$\mu_{\text{IV}}(\eta) = D_+(k) a(\eta) + D_-(k) a(\eta) \int_{\eta_1}^{\eta} \frac{d\tau}{a^2(\tau)}. \quad (96)$$

The transition between Regions III and IV occurs at the time $\eta_3(k)$, the time of Hubble radius crossing. Our aim is to calculate the coefficients $D_\pm(k)$ which determines the spectrum of the growing mode.

Performing the matching at $\eta_3(k)$, we obtain

$$\mu_{\text{IV}}(\eta) = \frac{1}{\sqrt{2\omega_3}} \left( \alpha_k e^{-i\Omega_3} + \beta_k e^{i\Omega_3} \right) \frac{a(\eta)}{a_3} + \frac{1}{\sqrt{2\omega_3}} a(\eta) a_3 \left( \alpha_k \gamma_3 e^{-i\Omega_3} + \beta_k \gamma_3^* e^{i\Omega_3} \right) \int_{\eta_3}^{\eta} \frac{d\tau}{a^2(\tau)}, \quad (97)$$

where the subscript “3” means that the corresponding quantity is evaluated at $\eta_3(k)$. In the previous equation, we have introduced the notation $\gamma_k \equiv \omega'/(2\omega) + i\omega + \mathcal{H}$ and $\Omega_3 \equiv \int_{\eta_1}^{\eta_3} \omega(k, \tau) d\tau$. On the other hand, a lengthy but straightforward calculation shows that the coefficients $\alpha_k$ and $\beta_k$ are given by

$$\alpha_k = \frac{i}{\sqrt{4\omega_1 \omega_2}} \left[ a_2 \frac{\gamma_2^*}{a_1} - \frac{a_1 a_2 \gamma_1 \gamma_2^*}{a_1 a_2} \int_{\eta_1}^{\eta_2} \frac{d\tau}{a^2(\tau)} \right] e^{i(\Omega_3 - \Omega_1)}, \quad (98)$$

$$\beta_k = -\frac{i}{\sqrt{4\omega_1 \omega_2}} \left[ \frac{a_2}{a_1} \gamma_2 - \frac{a_1 a_2 \gamma_1 \gamma_2}{a_1 a_2} \int_{\eta_1}^{\eta_2} \frac{d\tau}{a^2(\tau)} \right] e^{-i(\Omega_3 + \Omega_1)}. \quad (99)$$
FIG. 5: Typical example of a dispersion relation which breaks the WKB approximation in the trans-Planckian regime but allows a non-ambiguous definition of the initial state.

These expressions are similar to those found in Refs. [9] and [10]. So far, no approximation has been made. To go further, one has to take into account the fact that the difference \( |\eta_2(k) - \eta_1(k)| \) cannot be too large. Otherwise, this would mean that particles production in region III is too important and, as a consequence, that the calculation is not valid due to this back-reaction problem. Therefore, we can write \( \eta_1 = \eta_2(1 + \Delta) \) and perform an expansion of the coefficients \( \alpha_k \) and \( \beta_k \) in terms of the parameter \( \Delta \). The result reads

\[
\alpha_k = e^{i(\Omega_2 - \Omega_1)} \left[ 1 + \frac{i}{2} \omega_2 \xi \left( 1 + \frac{Q_2}{\omega_2^2} - \frac{a_2''}{\omega_2 a_2} \right) \Delta \right] + O(\Delta^2), \quad \beta_k = \frac{i}{2} e^{-i(\Omega_2 + \Omega_1)} \omega_2 \xi \left[ 1 - \frac{Q_2}{\omega_2^2} + \frac{a_2''}{\omega_2 a_2} \right] \Delta + O(\Delta^2). \quad (100)
\]

In this expression, \( Q \) is the parameter which controls the accuracy of the WKB approximation. It is defined by \( Q \equiv -\omega''/(2\omega) + 3(\omega')^2/(4\omega^2) \) [38]. As expected, if we send the parameter \( \Delta \) to zero, then \( \beta_k \) vanishes and \( \alpha_k \) becomes unity. Inserting the above expressions for \( \alpha_k \) and \( \beta_k \) into Eq. (97), one arrives at

\[
|D_+(k)|^2 = \frac{1}{2\pi^2 \omega_2} \left\{ 1 - \omega_2 \xi \left[ 1 - \frac{Q_2}{\omega_2^2} + \frac{a_2''}{\omega_2 a_2} \right] \sin \left[ \int_{\eta_1}^{\eta_3} \omega(k, \tau) d\tau \right] \Delta + O(\Delta^2) \right\}. \quad (101)
\]

Notice that in order to establish the previous expression we have just assumed that \( \Delta \) is a small number but we have not assumed that \( k_c/H \) or \( k_\star/H \) are small. As expected, The correction to \( |D_+(k)|^2 \) is given by the Bogoliubov coefficient \( \beta_k \).

We now need to calculate explicitly the previous quantities. For this purpose, we introduce an approximate piecewise form of the dispersion relation considered before, namely

\[
\omega_{\text{phys}} = \begin{cases} 
  k_{\text{phys}} & k_{\text{phys}} < k_c, \\
  \alpha (k_c - k_{\text{phys}}) + k_c & k_c < k_{\text{phys}} < k_\star, \\
  sk_{\text{phys}} + (\alpha + 1)k_c - (\alpha + s)k_\star & k_{\text{phys}} > k_\star.
\end{cases} \quad (102)
\]

This piecewise dispersion relation is represented in Fig. [3] The parameter \( \alpha \) controls the slope of the dispersion relation in the region where the group velocity does not have the same sign as the phase velocity. The parameter
FIG. 6: Approximate form of the dispersion relation considered in Fig. [5] Region II is the region where the WKB approximation is not satisfied.

$s$ controls the slope in the region $k_{\text{phys}} > k_*$. Assuming that the space-time is de Sitter, one can also compute the comoving frequency $\omega = a\omega_{\text{phys}}(k/a)$

$$\omega(k, \eta) = \begin{cases} k, & k_{\text{phys}} < k_C, \\
-\alpha k - \frac{\alpha + 1}{\eta} \frac{k_C}{H}, & k_C < k_{\text{phys}} < k_*, \\
\frac{1}{k}(\alpha s + \frac{\alpha + 1}{\alpha} k_C) - \frac{1}{\eta} \frac{k_{\text{phys}}}{H} - (\alpha + 1) \frac{k_C}{H}, & k_{\text{phys}} > k*. \end{cases}$$ (103)

Obviously, one has to choose $k_C > \alpha k_* / (\alpha + 1)$ in order to insure that the frequency remains positive. The various times of matching can now be calculated very simply. They read

$$\eta_1(k) = -\frac{1}{k_s} \left[ \sqrt{2} + (\alpha + s) \frac{k_*}{H} - (\alpha + 1) \frac{k_C}{H} \right],$$ (104)

$$\eta_2(k) = -\frac{1}{k} \left( \frac{\sqrt{2}}{\alpha} + \frac{\alpha + 1}{\alpha} \frac{k_C}{H} \right),$$ (105)

$$\eta_3(k) = -\frac{1}{k}.$$ (106)

The times at which $\omega_{\text{phys}} = k_C$ and $\omega_{\text{phys}} = k_*$, i.e. $\eta_C(k)$ and $\eta_*(k)$ respectively (see Fig. [6]), can also be determined easily. They are given by $\eta_C(k) = -(1/k)(k_C/H)$ and $\eta_*(k) = -(1/k)(k_*/H)$.

We have based the calculation of $|D_+(k)|^2$ on the assumption that the parameter $\Delta$ is small in order to avoid a back-reaction problem. This means that the time spent by the modes of interest in the region where the WKB approximation is violated is small. In turn, this requires a link between the two scales $k_C$ and $k_*$ which characterizes the shape of the dispersion relation and the Hubble constant $H$ which characterizes the “velocity” with which a mode crosses the region where the WKB approximation is not valid. Using the expressions of $\eta_1(k)$ and $\eta_2(k)$, one finds that the link between $k_*$, $k_C$ and $\Delta$ can be expressed as

$$\frac{k_*}{H} = \left[ \frac{1}{\alpha} + \frac{s}{\alpha(\alpha + s)} \Delta \right] \left[ -\sqrt{2} + (\alpha + 1) \frac{k_C}{H} \right].$$ (107)

If $k_C \gg H$ and $k_* \gg H$, then one has $k_* \simeq (\alpha + 1) k_C / \alpha$, as expected. The final result can be expressed in terms of $k_C / H$ and $\Delta$ only.
We can now calculate each term present in Eq. (101). Straightforward calculations show that $Q/\omega^2 |_2 = -(\alpha + 1)k_c/(2\sqrt{2}H) + 3(\alpha + 1)^2k_c^2/(16H^2)$, $a''/\omega a^2 |_2 = 1$ and $\omega \eta |_2 = -\sqrt{2}$. In order to calculate the integral appearing in the argument of the sine function in Eq. (101), one has to cut it into several pieces and to use the corresponding form of the piecewise dispersion relation. One obtains

$$\int_{n_1}^{n_2} \omega(k, \tau) d\tau = \int_{n_2}^{n_3} \omega(k, \tau) d\tau - \sqrt{2}\Delta \quad (108)$$

As expected the result does not depend on $k$ since the scalar field lives in de Sitter space-time. It is worth noticing that we have not assumed anything about the ratio $k_c/H$ in order to obtain the previous expression (only the parameter $\Delta$ was supposed to be small). The previous expression is the main result of this section. It should be compared with Eqs. (50) and (79) obtained previously. We see that the magnitude of the correction is no longer controlled by any power of the ratio of two scales as it was before. The magnitude of the effect is determined by the time spent in the region where the WKB approximation is violated which in turn depends on the shape of the dispersion relation that was assumed.

Finally, let us return to the back-reaction problem. There is no back-reaction problem if $|\beta_k| \ll 1$. Using the expression of $\beta_k$ derived before and assuming that $k_c/H \gg 1$, we see that the parameter $\Delta$ must satisfy $\Delta \ll H^2/k_c^2$. This amounts to a severe fine-tuning of the scales $k_c$, $k_*$ and of the Hubble parameter $H$ during inflation in order to satisfy this condition.

V. CONCLUSIONS

We have analyzed the magnitude of correction terms to the power spectra of scalar metric (density) fluctuations and gravitational waves in inflationary cosmology under the assumption that fluctuation modes are generated when their physical length scale equals some critical length determined by the unknown Planck-scale physics, but without modifying the equations of motion for the fluctuations. The magnitude of the correction terms can then be expressed as a function of the dimensionless ratio $\sigma_0 = H_0/M_c$, where $H_0$ is the characteristic Hubble expansion rate during inflation, and $M_c$ is the mass scale at which the new physics sets in.

It is important to realize that the magnitude of the correction terms is in general different for gravitational waves and for scalar metric fluctuations - a point not realized in some papers on the “trans-Planckian problem” of inflationary cosmology. In addition, the magnitude of the correction terms depends sensitively on the initial state chosen. We have shown that for the local Minkowski vacuum state, the correction terms are of the order $\sigma_0^3$ (in agreement with the results of [29]), whereas for nontrivial $\alpha$-vacua the effects are much larger. If the Bogoliubov coefficients which describe the mode mixing do not depend on $\sigma_0$, then the correction terms can be of order unity. In the case of Danielsson’s $\alpha$-vacuum for which the Bogoliubov coefficients depend on $\sigma_0$, the corrections to the gravitational wave spectrum are suppressed by one power of $\sigma_0$ (in agreement with the results of [20]), whereas for nontrivial $\alpha$-vacua the effects are much larger. If the Bogoliubov coefficients which describe the mode mixing do not depend on $\sigma_0$, then the correction terms can be of order unity. In the case of Danielsson’s $\alpha$-vacuum for which the Bogoliubov coefficients depend on $\sigma_0$, the corrections to the gravitational wave spectrum are suppressed by one power of $\sigma_0$ (in agreement with the results of [20]), whereas for nontrivial $\alpha$-vacua the effects are much larger. If the Bogoliubov coefficients which describe the mode mixing do not depend on $\sigma_0$, then the correction terms can be of order unity.

In the final section of the paper we compared the results obtained in earlier sections with the results obtained by assuming that trans-Planckian physics leads to a modified dispersion relation. In this case, corrections to the usual power spectra of fluctuations can be obtained which are not suppressed by any small dimensionless combinations of energy scales in the problem. However, demanding that the back-reaction remains under control leads to severe fine-tuning requirements on such models.

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