THE BIERI-NEUMANN-STREBEL INVARIANTS
FOR GRAPH GROUPS

JOHN MEIER AND LEONARD VANWYK

Abstract

Given a finite simplicial graph $\mathcal{G}$, the graph group $G\mathcal{G}$ is the group with generators in one-to-one correspondence with the vertices of $\mathcal{G}$ and with relations stating two generators commute if their associated vertices are adjacent in $\mathcal{G}$. The Bieri-Neumann-Strebel invariant can be explicitly described in terms of the original graph $\mathcal{G}$ and hence there is an explicit description of the distribution of finitely generated normal subgroups of $G\mathcal{G}$ with abelian quotient. We construct Eilenberg-MacLane spaces for graph groups and find partial extensions of this work to the higher dimensional invariants.

Introduction

Let $G$ denote a finitely generated group. In several papers a collection of geometric invariants $\Sigma^k(G)$ ($k$ a positive integer) was developed, each of which is a subset of the character sphere for $G$. These Bieri-Neumann-Strebel invariants have proven to be quite rich. For instance if $G \cong \pi_1(M)$ for a smooth compact manifold $M$, then $\Sigma^1(G)$ yields information of the existence of circle fibrations of $M$, and if in addition $M$ is a 3-manifold then $\Sigma^1(G)$ can be described in terms of the Thurston norm. If $N$ is a normal subgroup of $G$ with $G/N$ abelian, then whether $N$ has the finiteness property $F_k$ is measured by $\Sigma^k(G)$. In particular $\Sigma^1(G)$ measures the finite generation of normal subgroups of $G$ with abelian quotient. (See [1], [2], [3] and the references cited there. Further background will be given in section 1.) Regretably, “$\Sigma^1(G)$ is, in general, very difficult to compute.” [3]

Free partially commutative (FPC) monoids were first introduced by P. Cartier and D. Foata [5] in order to study combinatorial problems involving rearrangements of words. In the last ten years, they have been studied by both computer scientists and mathematicians. The corresponding FPC groups are known as graph groups. A finite simplicial graph $\mathcal{G}$ induces the following presentation of a group $G\mathcal{G}$:

$$\langle V(\mathcal{G}) \mid xy = yx \forall x, y \in V(\mathcal{G}) \text{ such that } x \text{ and } y \text{ are adjacent} \rangle,$$

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where \( V(\mathcal{G}) \) denotes the vertex set of \( \mathcal{G} \). A group \( G \) is called a graph group provided there exists some finite simplicial graph \( \mathcal{G} \) such that \( G \simeq G\mathcal{G} \). Given any graph group \( G\mathcal{G} \) we will often assume the above presentation and we will identify the set of generators of \( G\mathcal{G} \) with the vertex set \( V(\mathcal{G}) \). Notice that if \( \mathcal{G} \) is completely disconnected, then \( G\mathcal{G} \) is the free group on \( V(\mathcal{G}) \), while if \( \mathcal{G} \) is a complete graph, then \( G\mathcal{G} \) is the free abelian group on \( V(\mathcal{G}) \). If \( Y \subseteq V(\mathcal{G}) \) and all \( x, y \in Y \) are adjacent, then \( Y \) is called a (commuting) clique.

Much is already known about graph groups. For instance, the word problem for graph groups was shown to be solvable by C. Wrathall [15]. Further, graph groups have been shown to satisfy quadratic isoperimetric and linear isodiametric inequalities [11]. The conjugacy problem was solved independently by C. Wrathall and H. Servatius. C. Droms, H. Servatius, and B. Servatius have studied various subgroups of graph groups ([6], [7], [12]). Graph groups were shown to be biautomatic by both authors: this was proven by the first author in a joint paper with S. Hermiller using the more general notion of a “graph product” of groups [10], and independently by the second author [14].

Sections 2 through 5 of this paper establish a description of the invariant \( \Sigma^1(G\mathcal{G}) \) in terms of the defining graph \( \mathcal{G} \) (Theorem 5.1). In particular we give an explicit decomposition of \( \Sigma^1(G\mathcal{G}) \) into open simplices in a standard simplicial decomposition of the \( n \)-sphere. (The number \( n \) depends on the number of vertices in \( \mathcal{G} \).)

In section 6 we use this decomposition of \( \Sigma^1(G\mathcal{G}) \) to describe the finitely generated normal subgroups of \( G\mathcal{G} \) with abelian quotient. In particular, let \( \chi \) be a rational character of a graph group \( G\mathcal{G} \), that is, \( \chi \) is a map from \( G\mathcal{G} \) onto an infinite cyclic group. Let \( \mathcal{L}(\chi) \) be the full subgraph of \( G\mathcal{G} \) generated by vertices of \( G\mathcal{G} \) which do not map to 0 under \( \chi \). Recall that a subgraph \( \mathcal{L} \) of \( \mathcal{G} \) is dominating if for every vertex \( v \in \mathcal{G}\setminus\mathcal{L} \), \( d(v, \mathcal{L}) = 1 \).

**Theorem 6.1.** Let \( \chi \) be a rational character of a graph group \( G\mathcal{G} \). The kernel of \( \chi \) is finitely generated if and only if \( \mathcal{L}(\chi) \) is a connected and dominating subgraph of \( \mathcal{G} \).

In section 7 we describe how to construct Eilenberg-MacLane spaces for graph groups using “non-positive curvature” techniques of Gromov. In the final section we use these Eilenberg-MacLane spaces to give partial descriptions of some of the higher Bieri-Neumann-Strebel invariants.

There is not currently a specific description of the higher invariants for general graph groups. However we establish that in certain cases the invariants have the same sort of stability as is present for 1-relator groups and three manifold groups.
Theorem 8.3. Let $G$ be a graph group which can be expressed as the direct product of a non-trivial free abelian group and a graph group with disconnected graph. Then $\Sigma_1(G) = \Sigma^k(G)$ for all positive integers $k$. \hfill \qed

It is known that Theorem 8.3 is not true for arbitrary graph groups.

We thank Cliff Reiter for providing figures 1 and 2 in section 4 and Walter Neumann for an excellent talk in the CUNY group theory seminar which inspired us to place this work within the framework of the Bieri-Neumann-Strebel invariants. Initial work on this project was done by the second author as part of his doctoral dissertation under the direction of the late Craig C. Squier.

1. Bieri-Neumann-Strebel invariants

The reader is advised to read [1], [2] or [3]; we will only briefly review the main definitions and relevant theorems.

Any non-zero map from a finitely generated group $G$ to the additive group of the reals is a character of $G$. This is slightly non-standard in that the zero map is usually considered to be a character. However using the standard terminology would require continual use of the phrase “non-zero character”.

The set of all characters is the complement of the zero map in the real vector space $\text{Hom}(G,\mathbb{R})$. For any character $\chi$ let $[\chi] = \{ r\chi \mid 0 < r \in \mathbb{R} \}$ be a ray in $\text{Hom}(G,\mathbb{R})$; the set of all such rays is denoted $S(G)$. Since any character of $G$ must factor through the abelianization of $G$, if $n$ is the integral rank of $G^{ab}$ then $S(G) \simeq S^{n-1}$. In particular for a graph group $G$, $S(G) \simeq S^{\left|\mathcal{V}(G)\right|-1}$.

Any character $\chi$ whose corresponding ray $[\chi]$ intersects an integral point of $\text{Hom}(G,\mathbb{R})$ is a rational character. It is easy to check that the image of a rational character is an infinite cyclic group.

The Bieri-Neumann-Strebel invariants $\Sigma_k(G)$ ($k$ an integer greater than zero) are open subsets of $S(G)$ where $\Sigma_1(G) \supseteq \Sigma_2(G) \supseteq \cdots \supseteq \Sigma^n(G) \supseteq \cdots$. We simplify the terminology and call these sets the BNS-invariants.

The invariant $\Sigma_1(G)$ has a fairly simple description. Choose a finite generating set for $G$ and let $\mathcal{C}$ be the corresponding Cayley graph. Then any character $G \xrightarrow{\chi} \mathbb{R}$ extends to a $G$-equivariant map $\mathcal{C} \xrightarrow{\tilde{\chi}} \mathbb{R}$ coinciding with $\chi$ on the vertices and is extended linearly over the edges of $\mathcal{C}$. Let $\mathcal{C}_+(\chi)$ be the maximal subgraph of $\mathcal{C}$ contained in $\tilde{\chi}^{-1}([0,\infty))$.

Definition. A character $G \xrightarrow{\chi} \mathbb{R}$ represents a point in $\Sigma_1(G)$ if and only if $\mathcal{C}_+(\chi)$ is connected.

Although it is not apparent, this definition is independent of choice of finite generating set.
The higher BNS-invariants are more difficult to define, but we will not need the full definition for the results in this paper. Instead we present a characterization which implies that a character $\chi$ represents a point in $\Sigma^k(G)$ for $k > 1$.

Assume that $G$ admits a $K(G,1)$ with finitely many cells in each dimension and let $\tilde{K}$ be such a $K(G,1)$ which, without loss of generality, we assume has a single vertex. Then as in the Cayley graph example, every map $G \rightarrow R$ extends to a $G$-equivariant map $\tilde{K} \rightarrow R$ which is defined by $\chi$ on the vertices and extended linearly over the cells of $K$. Let $\tilde{K}_+(\chi)$ be the maximal subcomplex of $\tilde{K}$ contain in $\tilde{\chi}^{-1}([0,\infty))$.

**Partial Description.** If $\tilde{K}_+(\chi)$ is $(n-1)$-connected, then $\chi$ represents a point in $\Sigma^n(G)$.

The higher BNS-invariants measure higher dimensional topological properties of the kernels of maps to free abelian groups. Recall the following definition due to C.T.C. Wall:

**Definition.** A group $G$ has property $F_n$ if and only if there exists a $K(G,1)$ which has finite $n$-skeleton. Thus $F_1$ is equivalent to finite generation, $F_2$ is equivalent to finite presentation, and a group is $F_\infty$ if it is $F_n$ for all $n$. For readers more versed in homological properties, $F_n$ implies $FP_n$ for each $n$. For background on the $FP_n$ properties see [4].

**Theorem 1.1.** (Bieri, Neumann, Renz, Strebel) Let $H$ be a normal subgroup of $G$ with $G/H$ an abelian group of integral rank $n$. Define $S(G,H) = \{[\chi] | \chi(H) = 0\} \simeq S^{n-1}$. Then $H$ is $F_k$ if and only if $S(G,H) \subseteq \Sigma^k(G)$.

**Corollary 1.2.** Let $\chi$ be a rational character of the group $G$. Then the kernel of $\chi$ is $F_k$ if and only if $[\chi]$ and $[-\chi]$ are both contained in $\Sigma^k(G)$.

The proof of the above theorem and corollary when $k = 1$ is due to Bieri, Neumann and Strebel and may be found in [1]. For the higher invariants the proof is contained in the thesis of Renz, and a good discussion may be found in either [2] or [3].

Since the finiteness properties $F_k$ are preserved when passing to subgroups of finite index, we state most of our results in terms of characters, not normal subgroups with abelian quotient. In most cases we leave the interested reader the task of translating the theorems into the language of normal subgroups.

2. Presentations of graph groups

The presentation of a graph group given in the introduction, with generators corresponding to the vertices of a graph $\mathcal{G}$ and relations corresponding to the
edges, we call the standard presentation. It is intuitively the natural presentation to use; however, it is not always the optimal presentation for specific problems. In [14], the second author presented biautomatic structures and finite complete rewriting systems for graph groups, which use an alternate system of generators and relations. (A similar system of generators and relations was used in [10] in the more general case of a graph product of groups.) We give this presentation as part of a finite complete rewriting system below.

If \( G \) is a group generated by a finite set \( X \), adjoin a set of formal inverses to obtain a set of monoid generators \( \mathcal{B} = X \cup \overline{X} \) for \( G \). In general, if \( \mathcal{S} \) is any set, \( \mathcal{S}^* \) denotes the free monoid on \( \mathcal{S} \). The inclusion of \( \mathcal{B} \) into \( G \) extends to a monoid homomorphism \( \mu : \mathcal{B}^* \to \mathcal{G} \). We will often regard \( \mathcal{B} \) as a subset of \( \mathcal{G} \), suppressing the homomorphism \( \mu \). We denote the length of a word \( w \) in the free monoid \( \mathcal{B}^* \) by \( l(w) \).

Let \( \mathcal{C} = \mathcal{C}(G, X) \) denote the Cayley graph of \( G \) with respect to the generating set \( X \). Then each word \( w = x_1x_2 \ldots x_n \in \mathcal{B}^* \), can be identified with an associated path in \( \mathcal{C} \), \([0, \infty) \xrightarrow{w} \mathcal{C} \) with \( w(0) = 1 \), \( w(i) = \mu(a_1 \ldots a_i) \) for each integer \( i \leq n \) and \( w(i) = w(n) \) for \( i > n \), where \([i, i+1]\) is mapped to the edge connecting \( w(i) \) to \( w(i+1) \).

Given a generating set \( X \) for \( G \), a set of normal forms for \( G \) with respect to \( X \) is a subset of \( \mathcal{B}^* \) which bijects under \( \mu \) to \( \mathcal{G} \). In other words, a set of normal forms defines a canonical way of representing each group element in terms of the generators and their inverses.

A finite complete rewriting system is essentially a finite set of rules which converts any given expression of a group element in terms of the generators and their inverses into the normal form for the group element. One simply replaces any occurrence of the left-hand side of a rule in a given word \( w \) with the right side of the rule. If this set of rules is a finite complete rewriting system, the above process will terminate in a unique normal form. The set of rules gives a set of relations defining the group. Since we do not use detailed information about finite complete rewriting systems, the reader unfamiliar with them is directed to [8].

The normal forms for a graph group \( \mathcal{G} \) used in this paper are induced by those in [14] and [10]. In order to define the normal forms we need to define an alternate set of generators for a graph group. The set

\[
\mathcal{D} = \{ \{x_1, x_2, \ldots, x_n\} \mid \forall i, j, x_i \in X \cup \overline{X}, x_i x_j = x_j x_i, x_i \neq \overline{x_j} \},
\]

consists of a generator for each collection of standard generators (and their inverses) which correspond to a clique in the graph \( \mathcal{G} \), with the restriction that a generator and its inverse do not both occur. If \( X \) is the set of standard generators, and \( \mathcal{B} = X \cup \overline{X} \), then we can regard \( \mathcal{B} \subseteq \mathcal{D} \). Further there is a map \( \zeta \) such that \( \mathcal{D}^* \xrightarrow{\zeta} \mathcal{B}^* \xrightarrow{\mu} \mathcal{GG} \), where \( \zeta \) is determined by a total ordering of \( X \): if \( x_{i+1} < x_i \),
then \( \zeta([\{x_1, x_2, \ldots, x_n\}]) = x_nx_{n-1}\ldots x_1 \). Since \( \mathcal{B} \) is contained in the image of \( \mathcal{D} \) under \( \zeta \), the composition \( \mu \circ \zeta \) maps \( \mathcal{D}^* \) onto \( \mathcal{G} \).

Let \( w_1, w_2 \in \mathcal{D}^* \), \( x \in X \cup \overline{X} \), and + denote disjoint union. The rewriting rules for elements of \( \mathcal{D}^* \) are:

1. \( w_1[u][v + \{x\}]w_2 \rightarrow w_1[u + \{x\}][v]w_2 \) provided \( \forall y \in u, x \) and \( y \) commute, \( x \neq y \), and \( x \neq \overline{y} \).
2. \( w_1[u + \{x\}][v + \{x\}]w_2 \rightarrow w_1[u][v]w_2 \).
3. \( w_1[\emptyset]w_2 \rightarrow w_1w_2 \).

This is a “left greedy” reduction system; we simply move an element of \( \mathcal{B} \) to the left whenever possible, cancelling if necessary (as in (2)). That this is a finite complete rewriting system for the graph group is shown in [14]. The presentation of \( \mathcal{G} \) with generating set \( \mathcal{D} \) and relations given by the rewriting system is the clique presentation.

We prefer to work with the standard presentation. However, the rewriting system as defined only gives normal forms for group elements in terms of the clique generators. Using the map \( \zeta \) these normal forms can be easily converted to normal forms in \( \mathcal{B}^* \). If \( w \) is any word in \( \mathcal{B}^* \), then \( w \) can be viewed also as a word in \( \mathcal{D}^* \) consisting of singletons. Applying the rewriting rules to \( w \) then gives an element \([u_1][u_2][u_k] \in \mathcal{D}^* \) in normal form. This induces a normal form \( \zeta([u_1])\zeta([u_2])\ldots\zeta([u_n]) \in \mathcal{B}^* \). It is this set of normal forms which will be used.

If \( w \) and \( w' \) are elements of \( \mathcal{B}^* \) we use \( w \rightarrow^* w' \) to indicate that one can get from \( w \) to \( w' \) by applying rewriting rules to \( w \) (thought of as a word in \( \mathcal{D}^* \)) and then using the map \( \zeta \) to convert the rewritten word into a word in \( \mathcal{B}^* \).

3. Normal forms and characters

In determining the structure of \( \Sigma^k(\mathcal{G}) \) it suffices to discuss only those characters which map the standard generators to the intersection of \( S(\mathcal{G}) \) with the cone formed by the positive coordinate axes. Specifically we have the following result.

**Proposition 3.1.** Let \( \Sigma^k_p(\mathcal{G}) \subseteq S(\mathcal{G}) \simeq S^{n-1} \) be the intersection of \( \Sigma^k(\mathcal{G}) \) with the closed cone formed by the positive coordinate axes. Let \( C \) be the finite Coxeter group generated by reflections in the \( n \) hyperplanes determined by each linearly independent set of \( n - 1 \) coordinate axes. Then the image of \( \Sigma^k_p(\mathcal{G}) \) under the action of \( C \) is \( \Sigma^k(\mathcal{G}) \).

**Proof.** Choose any point in \( S(\mathcal{G}) \) and let \( \mathcal{G} \xrightarrow{\chi} R \) be a character representing the point. Using the standard presentation of \( \mathcal{G} \), \( \chi \) is determined by the images of the group elements corresponding to the vertices of \( \mathcal{G} \). Let \( \mathcal{G} \xrightarrow{\chi'} R \)
be the map which takes \( v \) to \(-\chi(v)\) for each generator \( v \) with \( \chi(v) < 0 \), and which agrees with \( \chi \) on all other generators.

Changing from \( \chi \) to \( \chi' \) is essentially the same as replacing \( \{ v \mid \chi(v) < 0 \} \) in the generating set by \( \{ \overline{v} \mid \chi(v) < 0 \} \). But changing a generator from \( v \) to \( \overline{v} \), and similarly changing all the appearances of \( v \) in the relations, gives an isomorphic presentation of \( \mathcal{G} \). Because of this symmetry in the defining relations, the sets \( \tilde{K}_+(\chi) \) and \( \tilde{K}_+(\chi') \) are isomorphic, so \( [\chi'] \in \Sigma^k(\mathcal{G}) \) if and only if \( [\chi] \in \Sigma^k(\mathcal{G}) \). Since each of the generating reflections of \( C \) simply maps a single generator \( v \) to \( \overline{v} \), there exists \( c \in C \) such that \( c[\chi'] = [\chi] \). Hence \( \Sigma^k(\mathcal{G}) = C \cdot \Sigma^k_p(\mathcal{G}) \).

**Definition.** If \( \chi \) is a character of a graph group \( \mathcal{G} \) then \( \mathcal{L}(\chi) \) is the full subgraph of \( \mathcal{G} \) generated by the vertices corresponding to generators which are not mapped to 0 by \( \chi \). We refer to this as the “living” subgraph of \( \mathcal{G} \). In view of Proposition 3.1., we may assume \( \chi(x) > 0 \) for each generator \( x \) corresponding to a vertex in \( \mathcal{L}(\chi) \).

In section 4 we discuss the connectedness of \( \mathcal{C}_+(\chi) \). We use the set of normal forms described in section 2 to give some control over the form of paths in \( \mathcal{C}_+(\chi) \). In particular we will need to refer to the following two lemmas.

**Lemma 3.2.** Let \( \chi \) be a character for a graph group \( \mathcal{G} \). Let \( w \) be a path in \( \mathcal{C}_+(\chi) \) of minimal length from 1 to \( \mu(w) \) and assume \( \mu(w) \) has normal form \( z \). If \( x \) is any generator which occurs in \( w \) but not in \( z \), then \( x \) corresponds to a vertex in \( \mathcal{L}(\chi) \) and we can write \( w = pxqxr \).

*Proof.* Choose a sequence of reductions \( w \to^* z \). Since \( x \) doesn’t occur in \( z \), either \( w = pxqxr \) or \( w = p\overline{x}qxr \), where these “visible” \( x \)'s cancel somewhere in this sequence. Let \( w' = pqr \). Then \( w' \to^* z \) via the same sequence of reductions except for the steps involving these \( x \)'s. So \( \mu(w) = \mu(w') \).

Suppose \( x \) is a generator corresponding to a vertex \( v \not\in \mathcal{L}(\chi) \). Then for any decomposition of \( q \) as \( q = q'q'' \), \( \chi(pq') = \chi(pxq') = \chi(p\overline{x}q') \geq 0 \). Similarly, if \( r = r'r'' \), then \( \chi(pqr') = \chi(pxqxr') = \chi(p\overline{x}qxr') \geq 0 \). Thus every initial segment of \( w' \) is contained in \( \mathcal{C}_+(\chi) \), and hence the word \( w' \) is a path in \( \mathcal{C}_+(\chi) \) from 1 to \( \mu(w) \). But \( w' \) is shorter than \( w \) which contradicts the minimality of \( w \). Hence, \( x \) corresponds to a vertex in \( \mathcal{L}(\chi) \).

Now suppose \( w = p\overline{x}qxr \) with \( \chi(x) > 0 \). If \( q = q'q'' \), then \( \chi(pq') > \chi(p\overline{x}q') \geq 0 \) and if \( r = r'r'' \), then \( \chi(pqr') = \chi(p\overline{x}qxr') \geq 0 \). So again \( w' \) is a path in \( \mathcal{C}_+(\chi) \) from 1 to \( \mu(w) \), a contradiction. \( \square \)

**Lemma 3.3.** Let \( w \) be as in the previous lemma. Assume \( w \to^* w' \to^* z \) where \( z \) is the normal form for \( \mu(w) \) and \( l(w') > l(z) \). Then there exits a generator \( x \), corresponding to a vertex in \( \mathcal{L}(\chi) \), such that \( w' = p'xq'x' \) where \( x \) commutes with all the letters in \( q' \).
Proof. Since $l(w') > l(z)$ and $w' \to^* z$, a length-reducing rule must occur. Hence, there exists $x \in X$ and words $p$, $q$, $r$, with either $w' = p'xq'r$ or $w' = p'xq'r'$, where $x$ commutes with all the letters in $q'$. It remains to show that $\chi(x) > 0$ and that $x$ occurs before $\overline{p}$.

Since $w \to^* w'$, there exist words $p$, $q$, and $r$ with either $w = pxq'r$ or $w = p'xq'r$ (x does not necessarily commute with all the letters in $q$). As in the proof of Lemma 3.2, if $x$ does not correspond to a vertex in $L(\chi)$ or if $w = pxq'r$, then $pqr$ defines a path in $C_+(\chi)$ from 1 to $\mu(w)$, a contradiction. \qed

4. $\Sigma^1(G\mathcal{G})$

Whether or not a character $\chi$ of a graph group represents a point in the BNS-invariant $\Sigma^1(G\mathcal{G})$ is determined by the living subgraph $L(\chi)$ of $\mathcal{G}$. Recall that a subgraph $L$ of $\mathcal{G}$ is dominating if for every $v \in \mathcal{G}\setminus L$, $d(v, L) = 1$.

THEOREM 4.1. Let $\chi$ be a character of a graph group $G\mathcal{G}$. The subgraph $C_+(\chi)$ of $\mathcal{C}$ is connected if and only if $L(\chi)$ is a connected dominating subgraph of $\mathcal{G}$.

Proof. ($\Rightarrow$): Assume $A = \{a_1, a_2, ..., a_n\}$ is the subset of the generating set $X$, corresponding to the vertices in $L(\chi)$. Assume also that there is a chosen total ordering on $X$ where $a_i < a_{i+1}$ and for convenience, $\chi(a_1) \leq \chi(a_i)$ for all $i$. We first show that $L(\chi)$ is a dominating subgraph of $\mathcal{G}$. Let $b$ be a generator with $\chi(b) = 0$, let $a_k \in A$, and let $w$ be a path of minimal length in $C_+(\chi)$ from 1 to $\mu(\overline{a_k}ba_k)$. If $l(w) = 1$, then $a_k$ and $b$ commute, and hence $d(b, L(\chi)) = 1$. So assume $l(w) > 1$.

Since $a_k$ and $b$ don’t commute, $w$ isn’t in normal form (the normal form for $w$ is $\overline{a_k}ba_k$, which is not in $C_+(\chi)$ as a path). By repeated use of Lemma 3.3, we can define a sequence of words $w = w_1, w_2, w_3, ..., w_n = \overline{a_k}ba_k$, where for each $j$, $w_j = p_ja_{i_j}q_j\overline{a_{i_j}}r_j$ and the rewriting $w_j \to^* w_{j+1}$ consists of commuting letters within $p_j, q_j$ and $r_j$ along with cancelling the $a_{i_j}$ and $\overline{a_{i_j}}$ terms. Further, it can be assumed that $a_{i_j}$ commutes with all the letters in $q_j$.

By hypothesis $\chi(b) = 0$, so there is only one occurrence of $b$ in $w$ by minimality. Suppose $b$ occurs in none of the $q_j$. Then we can write $w = ubv$, where $\overline{a_k}$ occurs in $u$ and all the other letters in $u$ occur in inverse pairs. Thus $\chi(u) = -\chi(a_k) < 0$, which contradicts the assumption that $w$ is a path in $C_+(\chi)$.

It follows that there is a $j$ such that $b$ occurs in $q_j$, and therefore $b$ commutes with $a_{i_j}$. Since the distance from $b$ to $L(\chi)$ is 1, $L(\chi)$ is a dominating subgraph of $\mathcal{G}$.

It remains to show that $L(\chi)$ is connected. For each $i \neq 1$, let $w(i)$ be a path of minimal length in $C_+(\chi)$ from 1 to $\mu(\overline{a_i}a_i)$. The vertex $\mu(\overline{a_i}a_i)$ is contained
in $\mathcal{C}_+(\chi)$ since we have assumed that $\chi(a_1) \leq \chi(a_i)$ for all $i$. Note that the normal form for $w(i)$ is $\overline{a_i}$, which doesn’t lie in $\mathcal{C}_+(\chi)$ as a path since we may assume that $\chi$ maps the generators in $A$ to positive real numbers. The proof of the following claim then concludes the proof of (⇒).

**Claim:** For each $i$, there exists a path in $\mathcal{G}$ from $a_1$ to $a_i$.

**Proof of Claim.** We proceed by induction on the length of $w(i)$.

**Base:** $l(w(i)) = 2$. Then we must have $w(i) = a_i\overline{a_1}$, so $a_1$ and $a_i$ commute. Hence $a_1$ and $a_i$ are adjacent in $\mathcal{G}$.

**Inductive Step:** $l(w(i)) > 2$. In this case $a_1$ and $a_i$ do not commute. Write $w(i) = ua_i\nu v$, where this is the first occurrence of $a_i$. Then $v$ must be non-empty since otherwise $\chi(u) = -\chi(a_1) < 0$, which contradicts the assumption that $w$ lies in $\mathcal{C}_+(\chi)$.

By Lemma 3.2, if there is an $a_k$ which occurs in $v$, then so does $\overline{a_k}$, and $v = pa_kqakr$. Suppose such is the case, and that $a_k$ is the last element of $A$ in $v$, i.e., $p, q, r \in (A)^*$. Let $w' = ua_i\nu pq r$ and notice that $\mu(w') = \mu(w)$. As in the proofs of Lemmas 3.2 and 3.3 it can be shown that $w'$ is also contained in $\mathcal{C}_+(\chi)$ and is of shorter length than $w$, contradicting the minimality of $w$.

We may then assume that $w(i) = ua_i\nu v$ where $v \in (A)^*$. It follows that each element of $\overline{A}$ which occurs in $v$ commutes with $a_i$. Let $v = v_1v_2v_3$. Then $\mu(ua_i v_1 a_j) = \mu(uv_1 a_i v_2 a_j)$ and $\mu(uv_1) = \mu(\overline{a_1}a_j)$. Also, $\chi(uv_1) = \chi(a_j) - \chi(a_1) \geq 0$. Thus for each decomposition $v_1 = v_1' v_1''$, $\chi(uv_1') + \chi(v_1'') \geq \chi(uv_1) \geq 0$.

Hence $uv_1$ is a path in $\mathcal{C}_+(\chi)$ from $1$ to $\mu(\overline{a_1}a_j)$ of length $l(w(i)) - 2$. By definition, $l(w(j)) \leq l(uv_1)$, so by the inductive hypothesis there exists a path in $\mathcal{G}$ from $a_1$ to $a_j$. Since $a_i$ and $a_j$ commute ($\overline{a_j}$ occurred in $v$), we have a path from $a_1$ to $a_i$ in $\mathcal{G}$.

($\Leftarrow$): Let $g$ be a vertex in $\mathcal{C}_+(\chi)$; we will construct a path in $\mathcal{C}_+(\chi)$ from $1$ to $g$. First, we need some notation. We will be blatantly indentifying the set of standard generators with $V(\mathcal{G})$ and at times we will use the symbol “$\chi$” to denote the map induced on the free monoid $B^*$ by the character $\chi$ in $\text{Hom}(\mathcal{G}G, R)$. By Proposition 3.1, we can assume $\chi(x) > 0$ for all standard generators $x$.

Let $y_1, y_2 \in V(\mathcal{G}) \cup \overline{V(\mathcal{G})}$. Since $\mathcal{L}(\chi)$ is connected and dominating, there exists a path $y_1, x_1, x_2, \ldots, x_k, y_2$ from $y_1$ to $y_2$ in $\mathcal{G}$ with $\chi(x_i) > 0$ for all $i$. For positive integers $r$ and $s$, define

$$p(y_1^r, y_2^s) = x_1^{n_1} y_1^r x_2^{n_2} x_1^{n_3} x_2^{n_3} x_2^{n_2} \ldots x_k^{n_k-1} x_{k-1}^{n_{k-1}} x_k^{n_k-2} x_{k-1}^{n_{k-1}} x_k^{n_k-1} y_2^s$$

where the positive integers $n_i$ are minimal such that $\chi(x_i^{n_i}) \geq |\chi(y_1^r)|$, $\chi(x_i^{n_i}) \geq \chi(x_i^{n_{i-1}})$ for $2 \leq i \leq n - 1$, and $\chi(x_k^{n_k}) \geq \chi(x_k^{n_{k-1}}) + \chi(y_2^s)$. The net effect of these choices of $n_i$ is that the path defined by $p(y_1^r, y_2^s)$ starting at 1 in the
Cayley graph is contained in \( C_+ (\chi) \). In addition, define \( z(y_1^r, y_2^s) = \bar\pi_k \). The path \( p(y_1^r, y_2^s) \) is chosen so that much collapsing occurs, in particular \( \mu(y_1^r y_2^s) = \mu(p(y_1^r, y_2^s) z(y_1^r, y_2^s)). \)

Now, let \( w_0 \in (V(G) \cup \overline{V(G)})^* \). Then \( w_0 = uv_1 v_2 \ldots v_m \) where \( v_i \in V(G) \cup \overline{V(G)} \), and \( \mu(uv_1) \notin C_+ (\chi) \). (If \( w = u \), then we are done.) Using the machinery in the previous paragraph, replace \( v_1 v_2 \) by \( p(v_1, v_2) z(v_1, v_2) = p_1 z_1 \), forming \( w_1 = up_1 z_1 v_3 v_4 \ldots v_m \). Then \( \mu(w_1) = \mu(w_0) \) and \( up_1 \subseteq C_+ (\chi) \). Similarly, replace \( z_1 v_3 \) by \( p(z_1, v_3) z(z_1, v_3) = p_2 z_2 \), forming \( w_2 = up_1 p_2 z_2 v_4 \ldots v_m \), so that \( \mu(w_2) = \mu(w_1) = \mu(w_0) \) and \( up_1 p_2 \subseteq C_+ (\chi) \). Continuing in this manner, we arrive at the word \( w_m = up_1 p_2 \ldots p_m z_m \), where \( \mu(w_m) = \mu(w_0) \) and \( up_1 p_2 \ldots p_m \subseteq C_+ (\chi) \).

By construction, \( z_m = y^\alpha \) for some \( y \in V(G) \cup \overline{V(G)} \) and some non-negative integer \( \alpha \). Since both \( \mu(w_m) \) and \( \mu(up_1 p_2 \ldots p_m) \) are vertices in \( C_+ (\chi) \), if \( \chi(y) \geq 0 \), then \( \chi(up_1 p_2 \ldots p_m y^i) \geq 0 \) for \( 1 \leq i \leq \alpha \), and if \( \chi(y) < 0 \), then \( \chi(up_1 p_2 \ldots p_m y^i) = \chi(w_m y^{\alpha - i}) \geq 0 \) for \( 1 \leq i \leq \alpha \). It follows that \( w_m \) is a path in \( C_+ (\chi) \) from 1 to \( g \), and hence \( C_+ (\chi) \) is connected.

**Example 1:** The free group on two generators can be thought of as the graph group corresponding to the null graph with two vertices. If \( \chi \) maps each generator to \( 1 \in \mathbb{Z} \), then \( L(\chi) \) is disconnected, hence by Theorem 4.1 the subgraph \( C_+ (\chi) \) is disconnected.

In the figure below, the gray scale indicates the parts of the Cayley graph which are in \( C_+ (\chi) \), and it is apparent that \( C_+ (\chi) \) is disconnected.

**FIG. 1**

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Example 2: If we add an additional vertex and two edges to the graph in example 1, forming a graph $G = x - y - z$, then $G'G$ is the direct product of an infinite cyclic group with a free group. Let $\chi$ be the character sending each generator to $1 \in \mathbb{Z}$. (Thus $\chi$ restricted to the subgroup generated by $x$ and $z$ is the same map as was discussed in example 1.)

By Theorem 4.1, $C_+ (\chi)$ should be connected. Below is part of the Cayley graph, with the $x$ and $z$ generators forming the tree in front and the positive $y$-direction going back into the page. Once again the gray scale indicates the subgraph $C_+ (\chi)$.

**Fig. 2**

5. The simplicial structure of $\Sigma^1 (G'G)$

The unit sphere $S^{n-1}$ admits a natural simplicial structure where the vertices correspond to the intersection of the axes with $S^{n-1}$, two vertices are joined by an edge if they are a distance $\frac{\pi}{2}$ apart and any complete graph with $m$ vertices is filled by an $(m-1)$-simplex. Equivalently, it is the complex formed by stating that any $m$ vertices corresponding to linearly independent (positive or negative) coordinate axes are the vertices of an $(m-1)$-simplex.

**Notation.** Let $SP (G'G)$ denote the sphere $S(G'G)$ with the simplicial structure sketched above.

As was mentioned in section 1, for any finitely presented group $G$ the set
$\Sigma^1(G)$ is always an open subset of $S(G)$. By the previous theorem it follows that $\Sigma^1(\mathcal{G})$ actually is the union of open simplices in the simplicial decomposition $\mathcal{SP}(\mathcal{G})$ (see Theorem 5.1 below). The vertices of $\mathcal{SP}(\mathcal{G})$ correspond to characters of the graph group $\mathcal{G}$ whose living subgraph consists of a single vertex. Thus a vertex of $\mathcal{SP}(\mathcal{G})$ is included in $\Sigma^1(\mathcal{G})$ if and only if the corresponding vertex in $\mathcal{G}$ dominates $\mathcal{G}$. Similarly the open edges of $\mathcal{SP}(\mathcal{G})$ correspond to pairs of vertices in $\mathcal{G}$, and the entire open edge is included in $\Sigma^1(\mathcal{G})$ if and only if the living subgraph generated by the pair of vertices is connected and dominating. The conditions for the higher dimensional open simplices are similar.

**Definition.** Since any open simplex of $\mathcal{SP}(\mathcal{G})$ is defined by a collection of coordinate axes, each open simplex defines a collection of vertices of $\mathcal{G}$. In an abuse of terminology, if $\sigma$ is an open simplex of $\mathcal{SP}(\mathcal{G})$ then the *living subgraph* $L(\sigma)$ is the full subgraph of $\mathcal{G}$ generated by the vertices of $\mathcal{G}$ which correspond to the coordinate axes defining $\sigma$.

**Theorem 5.1.** For $\mathcal{G}$ a graph group, the BNS-invariant $\Sigma^1(\mathcal{G})$ is a union of open simplices in $\mathcal{SP}(\mathcal{G})$. In particular, an open simplex $\sigma$ of $\mathcal{SP}(\mathcal{G})$ is contained in $\Sigma^1(\mathcal{G})$ if and only if the corresponding living subgraph $L(\sigma)$ is connected and dominating.

**Proof.** If $\chi$ and $\chi'$ are two characters defining points in the same simplex $\sigma$ of $\mathcal{SP}(\mathcal{G})$, then $L(\chi) = L(\chi') = L(\sigma)$. Thus all the points in $\sigma$ have the same living subgraph of $\mathcal{G}$, hence either the entire open simplex $\sigma$ is contained in $\Sigma^1(\mathcal{G})$ or $\sigma \cap \Sigma^1(\mathcal{G}) = \emptyset$. □

**Corollary 5.2.** For every graph group $\mathcal{G}$, $\Sigma^1(\mathcal{G})$ is rational polyhedral. If $\mathcal{G}$ is disconnected then $\Sigma^1(\mathcal{G})$ is empty, and if $\mathcal{G}$ is connected then the closure of $\Sigma^1(\mathcal{G})$ is $S(\mathcal{G})$.

**Proof.** That $\Sigma^1(\mathcal{G})$ is rational polyhedral — that is, can be defined by finitely many inequalities with integer coefficients — follows directly from Theorem 5.1 and the description of $\mathcal{SP}(\mathcal{G})$.

If $\mathcal{G}$ is disconnected then there exist no connected dominating subgraphs, hence no simplex of $\mathcal{SP}(\mathcal{G})$ can be contained in $\Sigma^1(\mathcal{G})$.

If $\mathcal{G}$ is connected, then the graph $\mathcal{G}$ is a connected dominating subgraph, so every maximal simplex in $\mathcal{SP}(\mathcal{G})$ will be contained in $\Sigma^1(\mathcal{G})$. The result follows since the closure of the maximal simplices of $\mathcal{SP}(\mathcal{G})$ is $S(\mathcal{G})$. □

It is an open question whether the invariant $\Sigma^1(G)$ is rational polyhedral for all finitely generated groups $G$.

**Proposition 5.3.** Let $\tau$ be an open simplex of $\mathcal{SP}(\mathcal{G})$ whose closure contains an open simplex $\sigma$ which is contained in $\Sigma^1(\mathcal{G})$. Then $\tau$ is contained in $\Sigma^1(\mathcal{G})$. 

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Proof. The living subgraph \( L(\sigma) \) is connected and dominating by hypothesis. Since \( L(\tau) \) only adds vertices to the graph \( L(\sigma) \) it is clear that \( L(\tau) \) is also dominating. Further, since \( L(\sigma) \) is dominating, every vertex \( v \in L(\tau) \setminus L(\sigma) \) is a distance 1 from \( L(\sigma) \). Hence the full subgraph generated by adding these vertices to \( L(\sigma) \) is also connected. \( \square \)

**Corollary 5.4.** For a graph group \( \mathcal{G} \), \( \Sigma_1(\mathcal{G}) \simeq S_{|V(\mathcal{G})| - 1} \) if and only if \( \mathcal{G} \) is a complete graph.

**Proof.** By the previous proposition, \( \Sigma_1(\mathcal{G}) \simeq S_{|V(\mathcal{G})| - 1} \) if and only if the vertices of \( \mathcal{SP}(\mathcal{G}) \) are contained in \( \Sigma_1(\mathcal{G}) \). The vertices are contained in \( \Sigma_1(\mathcal{G}) \) if and only if each corresponding vertex in \( \mathcal{G} \) is dominating. The corollary follows since each vertex dominating is equivalent to the graph being a complete graph. \( \square \)

**Example:** If \( \mathcal{G} \) is given by \( x \rightarrow y \rightarrow z \), then \( \Sigma_1(\mathcal{G}) \) is a 2-sphere less the great circle determined by \( x \) and \( z \). It follows from Theorem 8.3 that \( \Sigma^k(\mathcal{G}) \) is a 2-sphere with a great circle removed for all \( k \).

6. **Free abelian quotients**

Because of the symmetry discussed in Proposition 3.1, \([\chi] \in \Sigma_1(\mathcal{G})\) if and only if \([-\chi] \in \Sigma_1(\mathcal{G})\). Thus Theorem 6.1 follows from Corollary 1.2 and Theorem 4.1.

**Theorem 6.1.** Let \( \chi \) be a rational character of a graph group \( \mathcal{G} \). The kernel of \( \chi \) is finitely generated if and only if \( L(\chi) \) is a connected and dominating subgraph of \( \mathcal{G} \). \( \square \)

If \( \mathcal{G} \rightarrow \mathbb{Z}^n \) with \( n > 1 \) then the situation is more complex. Let \( H \) be the kernel of \( \chi \). Then \( S(\mathcal{G}, H) \) is naturally identified with an \((n - 1)\)-sphere as was mentioned in Theorem 1.1. The sphere arises since any non-zero map to a free abelian group can be extended to a collection of characters via the elements of \( \text{Hom}(\mathbb{Z}^n, \mathbb{R}) \) and the non-zero elements of \( \text{Hom}(\mathbb{Z}^n, \mathbb{R}) \), scaled by the positive reals, is naturally identified with a copy of \( S^{n-1} \).

There is the following partial result. Recall that a graph \( \mathcal{G} \) is \( m \)-connected if there are \( m \) vertices whose removal yields a disconnected graph.

**Proposition 6.2.** If the graph \( \mathcal{G} \) is \( m \)-connected, then \( \mathcal{G} \) admits no map onto \( \mathbb{Z}^n \) for \( n > m \) with finitely generated kernel.

**Proof.** Let \( \chi \) map \( \mathcal{G} \) onto \( \mathbb{Z}^n \) for some \( n > m \). Let \( v_1, v_2, \ldots, v_m \) be the vertices whose removal disconnects \( \mathcal{G} \), and let \( x_1, x_2, \ldots, x_m \) be the corresponding generators of \( \mathcal{G} \). Since \( \chi \) is surjective, we can choose a non-zero element \( \phi \in \)
Hom($Z^n, Z$) such that $(\phi \circ \chi)(x_i) = 0$ for $i = 1, 2, \ldots, m$. Since each $v_i \notin \mathcal{L}(\phi \circ \chi)$, $\mathcal{L}(\phi \circ \chi)$ is not connected, so $[\phi \circ \chi] \notin \Sigma^1(\mathcal{G})$. Clearly $[\phi \circ \chi] \in S(\mathcal{G}, \text{Ker}(\chi))$, so $S(\mathcal{G}, \text{Ker}(\chi)) \not\subseteq \Sigma^1(\mathcal{G})$ and thus $\text{Ker}(\chi)$ is not finitely generated by Theorem 1.1.

By Theorem 5.1 it is clear that an open simplex of $\mathcal{SP}(\mathcal{G})$ is contained in $\Sigma^1(\mathcal{G})$ if and only if some rational point in the simplex is contained in $\Sigma^1(\mathcal{G})$. Thus it suffices to consider characters defined by composing $\chi$ with maps in $\text{Hom}(Z^n, Z)$.

**Proposition 6.3.** A map $\mathcal{G} \xrightarrow{\chi} Z^n$ has a finitely generated kernel if and only if for every nonzero $\phi \in \text{Hom}(Z^n, Z)$, $\mathcal{L}(\phi \circ \chi)$ is connected and dominating.

**Proof.** If there exists nonzero $\phi$ with $\mathcal{L}(\phi \circ \chi)$ non-connected or non-dominating, then it follows as in the proof of Proposition 6.2 that $\text{Ker}(\chi)$ is not finitely generated. Conversely, every representative of a rational map $\psi$ in $S(\mathcal{G}, \text{Ker}(\chi))$ factors as $\psi = \phi \circ \chi$ for some $\phi$. Since $[\phi \circ \chi] \in \Sigma^1(\mathcal{G})$ by hypothesis, $S(\mathcal{G}, \text{Ker}(\chi)) \subseteq \Sigma^1(\mathcal{G})$, whence $\text{Ker}(\chi)$ is finitely generated by Theorem 1.1.

The collection $\text{Hom}(Z^n, Z)$ is obviously a large set of maps, even modulo scalar multiplication. It is not necessary, however, to check infinitely many maps to establish that the kernel of a map onto a free abelian group is finitely generated.

**Definition.** Given a map $\mathcal{G} \xrightarrow{\chi} Z^n$ from a graph group to a free abelian group, we call a set of standard generators $\{x_1, \ldots, x_n\}$ $\chi$-linearly independent if for each subset $I$ of $\{1, 2, \ldots, n\}$ and each $j \in \{1, 2, \ldots, n\}\setminus I$, the abelian group $\langle \chi(x_i) \mid i \in I \rangle$ is of infinite index in $\langle \chi(x_i) \mid i \in I \cup \{j\} \rangle$. A map $\phi \in \text{Hom}(Z^n, Z)$ is $\chi$-oriented provided $n - 1$ $\chi$-linearly independent standard generators of $\mathcal{G}$ are contained in the kernel of $\phi \circ \chi$.

**Theorem 6.4.** A map $\mathcal{G} \xrightarrow{\chi} Z^n$ has a finitely generated kernel if and only if for every $\chi$-oriented map $\phi \in \text{Hom}(Z^n, Z)$, $\mathcal{L}(\phi \circ \chi)$ is connected and dominating.

**Proof.** If the kernel is finitely generated, then the result follows from 6.3. Conversely, if $\phi \in \text{Hom}(Z^n, Z)$, then there exists a $\chi$-oriented map $\phi'$ with $\mathcal{L}(\phi' \circ \chi) \subseteq \mathcal{L}(\phi \circ \chi)$. Since $\mathcal{L}(\phi' \circ \chi)$ is connected and dominating, so is $\mathcal{L}(\phi \circ \chi)$, and again the result follows from 6.3.

Theorem 6.4 yields some interesting, and even easy to apply, corollaries.

**Corollary 6.5.** Let $\mathcal{G} \xrightarrow{\chi} Z^n$ and assume $V(\mathcal{G})$ is $\chi$-linearly independent. Then $\text{Ker}(\chi)$ is finitely generated if and only if $\mathcal{G}$ is a complete graph. In particular, the commutator subgroup of a graph group is finitely generated if and only if $\mathcal{G}$ is a complete graph.
Proof. If $\mathcal{G}$ is a complete graph, then $\mathcal{L}(\phi \circ \chi)$ is connected and dominating for all nonzero $\phi : Z^n \rightarrow Z$. Conversely, since $\chi(V(\mathcal{G}))$ is linearly independent, for each $v \in V(\mathcal{G})$ there exists $\phi_v : Z^n \rightarrow Z$ which projects onto the maximal cyclic subgroup containing $\chi(v)$, so that $\mathcal{L}(\phi_v \circ \chi) = \{v\}$. Since $\text{Ker}(\chi)$ is finitely generated, $\mathcal{L}(\phi_v \circ \chi)$ must dominate $\mathcal{G}$ for each $v \in V(\mathcal{G})$ and therefore $\mathcal{G}$ must be complete. The last sentence is immediate since under the map defined by abelianization $V(\mathcal{G})$ is a linearly independent set.

Corollary 6.6. If $\mathcal{G}$ admits $n$ disjoint connected dominating sets, then there is a map from $\mathcal{G}$ onto $Z^n$ with finitely generated kernel.

Proof. If $A_1, A_2, \ldots, A_n$ are the connected dominating sets, define $\mathcal{G} \xrightarrow{\chi} Z^n$ by sending each vertex in $A_i$ to the standard unit vector $e_i$. Clearly for any $\phi : Z^n \rightarrow Z$, $\mathcal{L}(\phi \circ \chi)$ is connected and dominating.

Example. The converse of Corollary 6.6 is false. For instance, let $\mathcal{G}$ be the graph with five vertices $v_1, \ldots, v_5$ and an edge joining $v_i$ to $v_{i+1}$ for each $i$ (indices taken modulo 5). Let $x_i$ denote the standard generator corresponding to the vertex $v_i$. Using Theorem 6.4 it can be checked that the map $\chi$ from $\mathcal{G}$ onto the free abelian group of rank 2 defined by $\chi(x_1) = \chi(x_2) = (1, 0)$, $\chi(x_3) = \chi(x_4) = (0, 1)$ and $\chi(x_5) = (1, 1)$ has a finitely generated kernel. However $\mathcal{G}$ does not contain two disjoint connected dominating subgraphs.

7. Eilenberg-MacLane complexes

To extend the discussion of the topological properties of kernels of maps to free abelian groups, we need to construct $K(\mathcal{G}, 1)$ complexes in order to apply Theorem 1.1. The universal covers of the complexes we construct can be thought of as piecewise Euclidean cubical complexes, that is, as complexes where each cell is given the metric structure of a Euclidean $n$-cube of unit side length.

Let $C$ be such a cubical complex. If $v$ is any vertex of $C$, then the sphere of radius 1 about $v$ inherits a natural simplicial structure. Call this sphere the link of $v$. 

Fig. 3
**Definition.** A simplicial complex $\mathcal{X}$ is a *flag complex* if given any complete graph $C$ contained in $\mathcal{X}^{(1)}$, there is some simplex $S$ in $\mathcal{X}$ with $S^{(1)} = C$.

The proof of the following theorem is based on the piecewise Euclidean metric structure for cubical complexes. If the links of all the vertices are flag complexes, then the cubical complex is “nonpositively curved”, and hence is a unique geodesic space. For details see section 4.2 of [9].

**Theorem 7.1.** (Gromov) *Let $C$ be a 1-connected cubical complex. If the link of each vertex is a flag complex, then $C$ is contractible.*

Let $K \mathcal{G}^{(2)}$ be the standard 2-complex for the standard presentation of $G \mathcal{G}$. The universal cover of $K \mathcal{G}^{(2)}$ is a cubical 2-complex which we denote $\tilde{K} \mathcal{G}^{(2)}$. Let $Q$ be a set of $n$ vertices in $\mathcal{G}$ which are the vertices of a clique in $\mathcal{G}$. Let $KQ$ be the corresponding set of edges in $K \mathcal{G}^{(2)}$. Then $KQ$ is simply the quotient of the 1-skeleton of a Euclidean $n$-cube after identifying all opposite faces. In other words, $KQ$ is the 1-skeleton of the $n$-torus.

Let $T^n$ be the $n$-torus constructed as above by identifying opposite faces of the Euclidean $n$-cube. Thus $T^n$ is a subcomplex of $T^m$ if $n < m$. For each clique $Q_i$ in $\mathcal{G}$ let $T_i$ be a copy of $T^n$ where $n = |V(Q_i)|$. If two cliques $Q_i$ and $Q_j$ intersect in a (possibly trivial) sub-clique with $k$ vertices, then identify the corresponding copies of $T^k$ in $T_i$ and $T_j$. In particular, if the cliques $Q_i$ and $Q_j$ do not intersect, $T_i$ and $T_j$ are joined only at the single vertex. Call the resulting cell complex $K \mathcal{G}$.

**Theorem 7.2.** *The cell complex $K \mathcal{G}$ is a $K(G \mathcal{G}, 1)$.*

**Proof.** The 2-skeleton of $K \mathcal{G}$ is simply the original standard 2-complex for the standard presentation of $G \mathcal{G}$ which we suggestively called $K \mathcal{G}^{(2)}$ before. Thus $\pi_1(K \mathcal{G}) \simeq G \mathcal{G}$. It remains then to show that the universal cover $\tilde{K} \mathcal{G}$ is contractible.

Since the universal cover is a 1-connected cubical complex, by Gromov’s theorem it suffices to show that the links of the vertices in $\tilde{K} \mathcal{G}$ are flag complexes. To this end, let $v$ be a vertex in $\tilde{K} \mathcal{G}$ and let $C$ be a complete graph contained in the 1-skeleton of the link of $v$. Each vertex of $C$ corresponds to an edge in $\tilde{K} \mathcal{G}$ — the edge connecting the vertex to $v$ — so the vertices of $C$ correspond to the vertices of a subgraph $\mathcal{G}_C$ of the defining graph $\mathcal{G}$. In order to avoid confusion about which kind of “vertex” is being discussed, we use the phrase *link vertex* for any vertex in $\mathcal{C}$ which is contained in the link of $v$ and *graph vertex* for any vertex in $\mathcal{G}_C$.

Because $\mathcal{C}$ is a complete graph, each pair of link vertices has an edge joining them. Since this edge corresponds to a 2-cell in $\tilde{K} \mathcal{G}$, the corresponding graph vertices must be joined by an edge. Thus $\mathcal{G}_C$ is also a complete graph (which is
actually isomorphic to $C$). But by the construction of $\widetilde{K}\mathbb{G}$, $C$ must be embedded in a cube in $\widetilde{K}\mathbb{G}$; hence $C$ is the 1-skeleton of a simplex in the link of $v$.

8. Stability of higher invariants

For many classes of groups it is known that $\Sigma^1(G) = \Sigma^n(G)$ for all $n$. For example, this stability of the BNS-invariants is true for 3-manifold groups and 1-relator groups [3]. It is known that this stability does not hold for arbitrary groups, or even for arbitrary graph groups. For instance the direct product of two free groups $F(a, b)$ and $F(c, d)$ is a graph group based on a circuit of four edges. If $\chi$ is the character sending $a, b$ and $c$ to 1 and $d$ to 0, then the kernel of $\chi$ is finitely generated but not finitely presented [13]. Thus by 1.1, $S(G\mathbb{G}, \text{Ker}(\chi)) \subseteq \Sigma^1(G\mathbb{G})$ while $S(G\mathbb{G}, \text{Ker}(\chi)) \not\subseteq \Sigma^2(G\mathbb{G})$, so $\Sigma^1(G\mathbb{G}) \neq \Sigma^2(G\mathbb{G})$. There are, however, classes of graph groups for which a certain amount of stability holds.

Recall that to discuss the higher invariants for graph groups we must work with the maximal subcomplex $\widetilde{K}\mathbb{G}_+(\chi)$ of $\widetilde{K}\mathbb{G}$ contained in $\chi^{-1}([0, \infty))$.

**Definition.** A simplicial graph is **chordal** if and only if no full subgraph generated by more than three vertices is a circuit.

In [6], C. Droms shows that graph groups based on chordal graphs are coherent, i.e., every finitely generated subgroup is finitely presented. For a rational character this immediately implies the following proposition, and for non-rational characters the proposition can be proven using essentially the same argument as appears in [6].

**Proposition 8.1.** (Droms) If $\mathcal{G}$ is chordal, then $\Sigma^1(G\mathbb{G}) = \Sigma^2(G\mathbb{G})$.

Preliminary computations indicate that for a chordal graph $\mathcal{G}$ it may be the case that $\Sigma^1(G\mathbb{G}) = \Sigma^k(G\mathbb{G})$ for all $k$, but we have been unable to prove this.

There are other situations where one has this amount of stability in the BNS-invariants. The following lemma is proven using the "$\Sigma^k$-Criteria" in appendix B of [3]. Since [3] is not widely available, we write out a complete proof.

**Lemma 8.2.** Let $\mathcal{G}$ be a simplicial graph with a vertex $v$ connected to every other vertex in $\mathcal{G}$. Let $x$ be the generator corresponding to $v$ and let $\chi$ be a character with $\chi(x) \neq 0$ (so $L(\chi)$ is connected and dominating). Then $\chi$ defines a point in $\Sigma^k(G\mathbb{G})$ for all $k$.

**Proof.** Let $v$ and $x$ be as in the statement of the lemma. Without loss of generality we may assume $\chi(x) > 0$. For each integer $n$, let $\widetilde{K}\mathbb{G} \xrightarrow{f_n} \widetilde{K}\mathbb{G}$ be defined by $f_n(p) = x^n \cdot p$, where $\cdot$ denotes the action of $G\mathbb{G}$ on $\widetilde{K}\mathbb{G}$. Extend this map linearly to define for each real number $t$ a function $f_t$. Notice for all $p \in \widetilde{K}\mathbb{G}$
and for all $t$, $\tilde{\chi}(f_t(p)) = \tilde{\chi}(p) + t \cdot \chi(x)$. An immediate consequence is that for all $p \in \tilde{K}G$, there exists $s \geq 0$ such that $f_t(p) \in \tilde{K}G_+^\chi$ for all $t \geq s$.

We need to establish that $\tilde{K}G_+^\chi$ is $n$-connected for each $n$. Since $L(\chi)$ is connected and dominating, $\tilde{K}G_+^\chi$ is connected by 4.1. Let $n \geq 1$ and $S^n \xrightarrow{\phi} \tilde{K}G_+^\chi$ be an embedding of the $n$-sphere into $\tilde{K}G_+^\chi$. Since $\tilde{K}G$ is contractible, there is a map $D^n \xrightarrow{\Phi} \tilde{K}G$ extending $\phi$ to a map of a disk into $\tilde{K}G$. Let $N(\Phi)$ be the minimum value of $(\tilde{\chi} \circ \Phi)(D^n)$. While it is not necessarily true that $\tilde{K}G_+^\chi = \tilde{\chi}^{-1}([e, \infty))$, we may choose $\epsilon > 0$ such that $\tilde{\chi}^{-1}((\epsilon, \infty)) \subseteq \tilde{K}G_+^\chi$.

For $0 \leq t \leq 1$, the family of maps $\phi_t : S^n \longrightarrow \tilde{K}G_+^\chi$ given by $\phi_t = f_{K+t} \circ \phi$ is a homotopy from $\phi$ to $f_K \circ \phi$. The map $f_K \circ \phi$ extends to the map $f_K \circ \Phi : D^n \longrightarrow \tilde{K}G$; because the constant $K$ was chosen sufficiently large, the image of $f_K \circ \Phi$ is actually contained in $\tilde{K}G_+^\chi$. It follows that $\tilde{K}G_+^\chi$ is $n$-connected.

\textbf{Theorem 8.3.} Let $G\tilde{G}$ be a graph group which can be expressed as the direct product of a non-trivial free abelian group and a graph group with disconnected graph. Then $\Sigma^1(G\tilde{G}) = \Sigma^k(G\tilde{G})$ for all integers $k$.

\textbf{Proof.} The underlying graph of such a graph group decomposes into a complete graph $C$ corresponding to the free abelian group, and pairwise disjoint connected graphs $G_i$, with the only additional edges connecting each vertex of $C$ to each vertex of $G_i$ for all $i$.

Let $\chi$ represent a point in $\Sigma^1(G\tilde{G})$. By Theorem 4.1, $L(\chi)$ is connected and dominating, so $L(\chi)$ must contain a vertex of $C$. By Lemma 8.2, $\chi$ represents a point in $\Sigma^k(G\tilde{G})$ for all $k$. Since this is true for all characters with finitely generated kernels and since $\Sigma^1(G) \supseteq \Sigma^k(G)$ for any finitely generated group $G$, it follows that $\Sigma^1(G\tilde{G}) = \Sigma^k(G\tilde{G})$ for all $k$. \hfill $\Box$

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Department of Mathematics
Lafayette College
Easton
Pennsylvania 18042
USA
meierj@lafvax.lafayette.edu

Department of Mathematics
Binghamton University
Binghamton
New York 13902
USA
vanwyk@math.binghamton.edu