Amplitude equations for 3D double-diffusive convection interacted with a horizontal vortex

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Three dimensional roll-type double-diffusive convection in a horizontally infinite layer of an uncompressible liquid is considered in the neighborhood of Hopf bifurcation points. A system of amplitude equations for the variations of convective rolls amplitude is derived by multiple-scaled method. An attention is paid to an interaction of convection and horizontal vortex. Different cases of the derived equations are discussed.

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I. INTRODUCTION

Physical systems, in which double-diffusion induced convection plays an essential role, are often found in nature. There are two components with significantly different diffusion coefficients in such systems. It can be heat and salt in the sea water, heat and helium in stellar atmospheres, or two reagents in chemical reactors. As a result of various spatial distribution of these components in a gravitational field arises convection, which can have various forms and lead to a variety of phenomena. Widely known, for example, salt fingers are, arising in salted and warmed from above water. It is understandable that the results of double-diffusive convection, for example, in the ocean, can be applied to double-diffusive convection in astrophysical systems or in chemical reactor.

There are a number of works devoted to various theoretical models of systems with double-diffusive convection. In 80–90 years the formation of structures in the neighborhood of Hopf bifurcation points for the horizontally translation-invariant systems was actively studied in some works. The development of oscillations in such systems give rise to different types of waves (eg, standing, running, modulated, chaotic), which well described by a generalized Ginzburg–Landau equations. The equations of this type must be derived from the basic system of partial differential equations for the given physical system by asymptotic methods. However, a full and well-grounded derivation of amplitude equations for systems with double-diffusive convection (especially three-dimensional) is still poorly represented in the literature.

The purpose of this work is the derivation of amplitude equations for the three-dimensional double-diffusive system in the neighborhood of Hopf bifurcation points for the case of roll-type convection. This extends the idea of previous work, where a two-dimensional and three-dimensional convection in a square-cells was investigated by alike methods.

II. FORMULATION OF THE PROBLEM AND BASIC EQUATIONS

Consider 3D double-diffusive convection in a liquid layer of a width $h$, confined by two plane horizontal boundaries. The liquid layer is heated and salted from below. The governing equations in this case are hydrodynamical equations for a liquid mixture in the gravitational field:

$$
\begin{align*}
\partial_t \mathbf{v} + (\mathbf{v} \nabla) \mathbf{v} &= -\rho^{-1} \nabla p + \nu \Delta \mathbf{v} + \mathbf{g}, \\
\partial_t T + (\mathbf{v} \nabla) T &= \chi \Delta T, \\
\partial_t S + (\mathbf{v} \nabla) S &= D \Delta S, \\
\text{div} \mathbf{v} &= 0.
\end{align*}
$$

Where $\mathbf{v}(t, x, y, z)$ is the velocity field of liquid, $T(t, x, y, z)$ is the temperature, $S(t, x, y, z)$ is the salt concentration, $p(t, x, y, z)$ is the pressure, $\rho(t, x, y, z)$ is the density of liquid, $\mathbf{g}$ is the acceleration of gravity, $\nu$ is the kinematic viscosity of fluid, $\chi$ is the thermal diffusivity of the liquid, $D$ is the salt diffusivity. Cartesian frame with the horizontal $x$-axis and $y$-axis is used, while the $z$-axis is directed upward and $t$ is the time variable.

Distributed sources of heat and salt are absent. On the upper and lower boundaries of the layer the constant values of temperature and salinity are supported, higher at the lower boundary. The governing equations are transformed into dimensionless form with the use of Boussinesq approximation and following units for length, time, velocity, pressure, temperature and salinity respectively: $h, h^2/\chi, \chi/h, \rho_0 \chi^2/h^2, T_\Delta, S_\Delta$, where $T_\Delta$ and $S_\Delta$ are temperature and salinity differences across the layer.

The dimensionless governing equations for momentum...
and diffusion of temperature and salt are [3]:

\[ u_t + (uu_x + vv_y + wz_z) = -p_x + \sigma \Delta u, \]
\[ v_t + (uw_x + vv_y + wz_z) = -p_y + \sigma \Delta v, \]
\[ w_t + (uw_x + vv_y + wz_z) = -p_z + \sigma \Delta w + \sigma R_T \theta - \sigma R_S \xi, \] (1)

\[ \theta_t + (u\theta_x + v\theta_y + w\theta_z) - w = \Delta \theta, \]
\[ \xi_t + (u\xi_x + v\xi_y + w\xi_z) - w = \tau \Delta \xi, \]
\[ u_x + v_y + w_z = 0. \]

Where \( \sigma = \nu/\chi \) is the Prandtl number (\( \sigma \approx 7.0 \)), \( \tau = D/\chi \) is the Lewis number (0 < \( \tau < 1 \), usually \( \tau = 0.01 - 0.1 \)). \( R_T = (\alpha h^3/\nu)T_1 \) is the temperature Rayleigh number and \( R_S = (\gamma h^3/\nu)S_1 \) is the salinity Rayleigh number.

Free-slip boundary conditions are used for the dependent variables (the horizontal velocity component is undefined):

\[ u_z = v_z = w = \theta = \xi = 0 \text{ at } z = 0, 1. \]

It is believed that they are suitable to describe the convection in the inner layers of liquid and do not change significantly the convective instability occurrence criteria for the investigated class of systems [7].

III. DERIVATION OF AMPLITUDE EQUATIONS – GENERAL FRAME OF DECOMPOSITION

Consider the equations for double-diffusive convection in the vicinity of a bifurcation point, the temperature and salinity Rayleigh numbers for which are designated as \( R_T^* \) and \( R_S^* \) respectively. In this case the Rayleigh numbers can be represented as follows:

\[ R_T = R_T^*(1 + \varepsilon^2 r_T), \quad R_S = R_S^*(1 + \varepsilon^2 r_S). \]

Values of \( r_T \) and \( r_S \) are of unit order, and the small parameter \( \varepsilon \) shows how far from the bifurcation point the system is. To derive the amplitude equations we use the derivative-expansion method, which is a variant of multiple-scale method [8, 9]. Introduce the slow variables:

\[ T_1 = \varepsilon t, \quad T_2 = \varepsilon^2 t, \quad X_1 = \varepsilon x, \quad Y_1 = \sqrt{\varepsilon} y. \]

In accordance with the method chosen, we assume that the dependent variables now depend on \( t, T_1, T_2, x, y, z, X_1, Y_1 \), which are considered independent. Also replace the derivatives in equations (1) for the prolonged ones by the rules:

\[ \partial_t \rightarrow \partial_t + \varepsilon \partial_{T_1} + \varepsilon^2 \partial_{T_2}, \quad \partial_x \rightarrow \partial_x + \varepsilon \partial_{X_1}, \quad \partial_y \rightarrow \sqrt{\varepsilon} \partial_{Y_1}. \]

Then the equations (1) can be written as:

\[ u_t + (uu_x + wz_x) + p_x - \sigma(u_{xx} + u_{zz}) = -\sqrt{\varepsilon} uu_y, \]
\[ -\varepsilon[u_{T_1} + uu_{X_1} + p_{X_1} - 2\sigma u_{xX_1} - \sigma uu_{Y_1}], \]
\[ + \varepsilon^2[\sigma uu_{X_1}, u_{T_2}], \]
\[ v_t + (uw_x + wz_x) - \sigma(v_{xx} + v_{zz}) = -\sqrt{\varepsilon}(vv_y + pv_y), \]
\[ -\varepsilon[v_{T_1} + uw_{X_1} - 2\sigma v_{xX_1} - \sigma vv_{Y_1}], \]
\[ + \varepsilon^2[\sigma vv_{X_1}, u_{T_2}], \]
\[ w_t + (uw_x + wz_x) + p_x - \sigma(w_{xx} + w_{zz}) \]
\[ - \sigma R_T \theta - \sigma R_S \xi = -\sqrt{\varepsilon} vv_y, \]
\[ -\varepsilon[w_{T_1} + uw_{X_1} - 2\sigma w_{xX_1} - \sigma vv_{Y_1}], \]
\[ + \varepsilon^2[\sigma vv_{X_1}, u_{T_2}], \]
\[ \theta_t + (u\theta_x + v\theta_y + w\theta_z) - w = \Delta \theta, \]
\[ \xi_t + (u\xi_x + v\xi_y + w\xi_z) - w = \tau \Delta \xi, \]
\[ u_x + v_y + w_z = 0. \] (2)

We seek solutions of these equations in the form of asymptotic series in powers of small parameter \( \varepsilon \):

\[ u = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots, \]
\[ v = \varepsilon \varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \cdots, \]
\[ w = \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + \cdots, \]
\[ p = \varepsilon p_1 + \varepsilon^2 p_2 + \varepsilon^3 p_3 + \cdots, \]
\[ \theta = \varepsilon \theta_1 + \varepsilon^2 \theta_2 + \varepsilon^3 \theta_3 + \cdots, \]
\[ \xi = \varepsilon \xi_1 + \varepsilon^2 \xi_2 + \varepsilon^3 \xi_3 + \cdots. \] (3)

After their substitution in (2) and collection the terms at \( \varepsilon^n \) we obtain the systems of equations to determine the terms of the series (3).

IV. TERMS OF THE FIRST ORDER IN \( \varepsilon \)

At \( O(\varepsilon^1) \) we obtain the following system:

\[ u_{1t} + p_{1x} - \sigma(u_{1xx} + u_{1zz}) = 0, \]
\[ v_{1t} + p_{1Y_1} - \sigma(v_{1xx} + v_{1zz}) = 0, \]
\[ w_{1t} + p_{1z} - \sigma(w_{1xx} + w_{1zz}) - \sigma R_T \theta_1 + \sigma R_S \xi_1 = 0, \]
\[ \theta_{1t} - w_1 - (\theta_{1xx} + \theta_{1zz}) = 0, \]
\[ \xi_{1t} - w_1 - \tau(\xi_{1xx} + \xi_{1zz}) = 0, \]
\[ u_{1x} + w_{1z} = 0. \] (4)
Solutions in the form of normal modes (roll-type cells) are:
\begin{align*}
w_1 &= A e^{i k x} e^{\lambda t} \sin \pi z + \text{c.c.}, \\
\theta_1 &= \tilde{\theta}_1 e^{i k x} e^{\lambda t} \sin \pi z + \text{c.c.}, \\
\xi_1 &= \tilde{\xi}_1 e^{i k x} e^{\lambda t} \sin \pi z + \text{c.c.}, \\
p_1 &= \tilde{p}_1 e^{i k x} e^{\lambda t} \cos \pi z + \tilde{p}_1 + \text{c.c.}, \\
u_1 &= \tilde{u}_1 e^{i k x} e^{\lambda t} \cos \pi z + \tilde{u}_1 + \text{c.c.}, \\
v_1 &= \tilde{v}_1 e^{i k x} e^{\lambda t} \cos \pi z + \tilde{v}_1 + \text{c.c.}.
\end{align*}
Terms with the bars and hats depend only on the slow variables $X_1, Y_1, T_1, T_2$. Substituting the expressions in the equation \((5)\) gives relations for the amplitudes. Hereinafter $\kappa^2 = k^2 + \pi^2$ is the full wave number.

Parameters $\lambda, k, R_T, R_S$ are related by:
\begin{equation}
(\lambda + \sigma \kappa^2)(\lambda + \tau \kappa^2) + \sigma(k^2/\kappa^2)[R_S(\lambda + \kappa^2) - R_T(\lambda + \tau \kappa^2)] = 0.
\end{equation}
This equation has three roots, two of which can be complex conjugates. In the case of Hopf bifurcation these two roots acquire positive real part at some $R_T^*$ ($\omega$ is a frequency of convective waves):
\begin{align*}
R_T^* &= \frac{\sigma + \tau}{1 + \sigma} R_S^* + \frac{\kappa^2}{\sigma \kappa^2} (1 + \tau)(\tau + \sigma), \\
\omega^2 &= \frac{1 - \tau}{1 + \sigma} \frac{k^2}{\kappa^2} - \tau^2 \kappa^4.
\end{align*}

V. THE EQUATIONS AT $\varepsilon^2$ AND $\varepsilon^3$

The obtained systems can be written in general form as:
\[ \hat{L} \varphi_i = Q_i. \]
Here $\hat{L}$ is linear differential operator such that $\hat{L} \varphi_1 = 0$ corresponds to the system \((1)\), where $\varphi_i = (u_i, v_i, w_i, \theta_i, \xi_i)$. Functions $Q_i$ include terms, resonating with the left side of the equations, namely: $Q_i = Q^{(1)}_i + Q^{(2)}_i + Q^{(3)}_i$. Here $Q^{(1)}_i$ and $Q^{(2)}_i$ generate secular terms of the two types in solutions, and $Q^{(3)}_i$ does not generate secular terms of any kind. The condition of absence of secular terms of the first type consists in the requirement of orthogonality of functions $Q^{(1)}_i$ and the solution of the adjoint homogeneous equation $\hat{L}^* \varphi_i^* = 0$ \([3][5]\), which usually takes the form of an amplitude equation. Terms $Q^{(2)}_i$ are constants with respect to the fast variables, and to exclude the violation of the regularity of the expansions \([5]\), they should be equated to zero \([3]\). These conditions also have the form of amplitude equations.

As a result, the condition of absence of secular terms of the first type in solution of equations at $\varepsilon^2$ is as follows ($\alpha_0$ is defined in \((7)\) below):
\[ A_{T_1} + \alpha_0(2i k A_{X_1} + A_{Y_1} Y_1) + i k A_{Y_1} = 0. \]

To exclude secular terms of the second type, we introduce the horizontal stream function $\Omega$ such that:
\[ \tilde{u}_1 = \Omega_{Y_1}, \quad \tilde{v}_1 = -\Omega_{X_1}, \quad \Omega_{T_1} - \sigma \Omega_{Y_1} Y_1 = 0. \]

VI. $A\Omega$-SYSTEM OF AMPLITUDE EQUATIONS

We write the resulting amplitude equations for the system at $\varepsilon^3$:
\begin{align*}
\left\{ \begin{array}{l}
A_{T_2} = rA + \alpha_1 \left( \partial_{X_1} + \frac{1}{2ik} \partial_{Y_1}^2 \right) A - \alpha_0 A_{X_1} X_1 \\
+ \alpha_2 A |A|^2 + J(\Omega, A) + \hat{F}(\Omega, A), \\
(\Omega_{T_2} - \sigma \Omega_{X_1} X_1) X_1 = J(\Omega, \Omega_{X_1}) + \hat{G}(A).
\end{array} \right.
\end{align*}
Here the Jacobian $J(\Omega, f) = (\Omega_{X_1} f_{Y_1} - \Omega_{Y_1} f_{X_1})$ is introduced. Also operators $\hat{F}(\Omega, A)$ and $\hat{G}(A)$ are defined as follows:
\begin{align*}
\hat{F}(\Omega, A) &= \alpha_3 (ik A_{O_{X_1} X_1} Y_1 + A_{Y_1} \Omega_{Y_1} Y_1) + \alpha_4 A \Omega_{Y_1} Y_1, \\
\hat{G}(A) &= \frac{\pi^2}{k^4} |A_{Y_1}|^2 Y_1 + \frac{\pi^2}{k^4} \text{Re}(i k A_{Y_1} X_1). \end{align*}
FIG. 2: Re($\alpha_3(\omega)$) (left) and Im($\alpha_3(\omega)$) (right) with different $k$: $k = 1.75$ (dots), $k = \pi/\sqrt{2}$ (solid), $k = 3$ (dash). At $\sigma = 7$ and $\tau = 0.02$. At $\sigma = 7$ and $\tau = 0.02$.

FIG. 3: Re($\alpha_4(\omega)$) (left) and Im($\alpha_4(\omega)$) (right) with different $k$: $k = 1.75$ (dots), $k = \pi/\sqrt{2}$ (solid), $k = 3$ (dash). At $\sigma = 7$ and $\tau = 0.02$.

Here for the convenience and compactness of the formulas we introduced functions:

$$\beta = \frac{i\omega}{\omega^2} \left( \frac{\pi^2}{2k^2} - 1 \right) \left( 1 - \frac{\omega^4}{\omega^2} \frac{\tau + \sigma}{\omega^2} + \frac{\omega^4}{\omega^2} \frac{\tau + \sigma}{\omega^2} \right),$$

$$\beta_1 = \frac{(\tau + \sigma + \tau \sigma)(\omega + \tau \sigma \omega^2)}{i\omega + (1 + \tau + \sigma)^2 \omega^2},$$

$$\beta_2 = \frac{(i\omega + \omega^2)(i\omega + \tau \omega^2)}{2\omega(i\omega + (1 + \tau + \sigma)\omega^2)},$$

$$\beta_3 = \frac{x^4(1 + \tau + 2\sigma)\omega + (1 + \tau^2 + \tau \sigma + \sigma)\omega^2}{2(i\omega + \omega^2)(i\omega + \tau \omega^2)(i\omega + (1 + \tau + \sigma)\omega^2)},$$

It is easy to see that for $k = \pi/\sqrt{2}$, which corresponds to the fastest growing mode for not too high frequencies, the parameter $\beta$ vanishes, and the above formulas are
significantly simplified.

\begin{align*}
  r &= \left(\frac{\sigma}{6i\omega}\right) \frac{(2i\omega + 3\tau\pi^2)r_T - (2i\omega + 3\pi^2)r_S}{2\omega + 3(1 + \tau + \sigma)\pi^2}, \\
  \alpha_0 &= \frac{2i\omega}{3\pi^2}, \quad \alpha_1 = \frac{4i\omega}{3\pi^2} + \frac{3\pi^2}{i\omega}\beta_1, \quad \alpha_2 = \frac{3\pi^2}{8i\omega}, \\
  \alpha_3 &= \frac{i2\sqrt{2}}{3\pi}(1 + \beta_2), \quad \alpha_4 = \frac{i\sqrt{2}}{3\pi}(1 - \beta_2).
\end{align*}

VII. COMPLEX GINZBURG-LANDAU (CGL) EQUATION

Let us consider some special cases of the system (6) to which it reduces at various simplifying assumptions. The simplest form of the equation is obtained if we assume that we are considering: 1). neglect the interaction with the vortex, and 2). consider the dynamics on the single spatial variable, and 3). believe that the wave number corresponds to the first losing stability mode \( k = \frac{\pi}{\sqrt{2}} \).

Under these conditions, the system takes the following form:

\[ A_T^2 = rA + \alpha_5A_{X_1}X_1 + \alpha_2|A|^2. \quad (8) \]

Coefficients of the obtained CGL equation (consistent with [3, 10]) are:

\[ \alpha_5 = \frac{i\omega}{\sqrt{2}} + \frac{2\omega^2}{i\omega}\beta_1 = \frac{i\omega}{\sqrt{2}} + \frac{2\omega^2}{i\omega}(\tau + \sigma + \tau\sigma)i\omega + \tau\sigma\omega^2 \]

In the case when the wave number \( k \) is not restricted by any conditions the coefficient \( \alpha_5 \) is expressed in general terms as \( \alpha_5 = \alpha_1 - \alpha_0 \) (see fig. 4). In the limit of high frequencies \( \omega \) the resulting equation reduces to the nonlinear Schrödinger equation and has such solutions as “dark” solitons [3].

VIII. CONCLUSION

- The \( \Delta \Omega \)-system of amplitude equations [3] describing 3D double-diffusive roll-type convection interacting with horizontal vorticity field \( \Omega \) was derived.
- An approach to calculation of amplitude equation coefficients which allows to get relatively compact formulas such as (7) was developed.
- As a special case the complex Ginzburg-Landau equation [3] for the 2D double-diffusive convection for an arbitrary \( k \) (width of the convective cells) was derived.

The results can be used to describe the processes of heat and mass transfer, the formation of vortex structures in the ocean and the atmosphere by convection, and may also be the basis for constructing more advanced models of this kind.

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