Research Article
The Symmetric Versions of Rouché’s Theorem via $\overline{\partial}$-Calculus

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Let $(f, g)$ be a pair of holomorphic functions. In this expositional paper we apply the $\overline{\partial}$-calculus to prove the symmetric version “$|f + g| < |f| + |g|$” as well as the homotopic version of Rouché’s theorem for arbitrary planar compacta $K$. Using Eilenberg’s representation theorem we also give a converse to the homotopic version. Then we derive two analogs of Rouché’s theorem for continuous-holomorphic pairs (a symmetric and a nonsymmetric one). One of the rarely presented properties of the non-symmetric version is that in the fundamental boundary hypothesis, $|f + g| \leq |g|$, equality is allowed.

1. Introduction

The standard version of Rouché’s theorem that our students encounter reads as follows.

Let $\Omega$ be a domain (= open path-connected set) in the complex plane $\mathbb{C}$ and let $(f, g)$ be a pair of holomorphic functions in $\Omega$. Suppose that $\gamma$ is a simple closed curve in $\Omega$.

(i) If $|f| < |g|$ on $\gamma$, then $g$ and $f - g$ have the same number of zeros (counting multiplicities) inside the domain $D$ bounded by $\gamma$.

(ii) If $|f - g| < |g|$ on $\gamma$, then $f$ and $g$ have the same number of zeros on $D$.

They appear in most books giving an introduction to complex analysis; see for example [1, page 164], [2, page 229], [3, page 310], and [4, page 225].

One of the most important applications of Rouché’s theorem is that small (holomorphic) boundary-perturbations of holomorphic functions (in the sense that $|f - g| \leq \varepsilon < \min_{\gamma} |g|$) do not change the number of zeros.

A stronger result (and much easier to remember) is the following symmetric version of Rouché’s theorem:

(iii) if $|f + g| < |f| + |g|$ on $\gamma$, then $f$ and $g$ have the same number of zeros on $D$.

Although the proof is the same, this symmetric version is not so well-known among nonfunction theorists. Nevertheless, for special domains surrounded by simple closed curves, the symmetric case already came up in the second half of the twentieth century (see [5, page 156]). A version of it was reproved by Glicksberg [6]. Symmetric versions also appear in the classical monograph [7, page 265] by Burchel as well as in the books by Conway [8], Greene and Krantz [9], and Schmieder [10].

A natural question is whether $D$ can be replaced by arbitrary planar compacta. This is indeed the case. A proof (presenting the main ideas) of the unsymmetric version of Rouché’s theorem and based on contour integrals and surrounding cycles already appears in Bieberbach’s classical monograph [11]. A more detailed proof of the symmetric version on arbitrary compacta is given in [7, page 265]. Due to the quite technical construction of a null-homologous cycle in an open set $\Omega$ surrounding each point of $K, K \subseteq \Omega$, exactly once, this approach is, in our opinion, not so elegant. We think that there is a more enlightening and simple proof of this general version of Rouché’s theorem on compacta. It appears (in the nonsymmetric case) in Narasimhan’s monograph [12, page 105] and is based on $\overline{\partial}$-methods. This method entirely avoids curves and winding numbers; curves are
replaced by “thin” sets in $\mathbb{C}$ with positive planar Lebesgue measure. And contour integrals are replaced by integrals with respect to two-dimensional Lebesgue measure. We find this much easier to deal with, because one does not have to take care of curve orientation, curve indices, and rectifiability issues. So the common thread in our survey is the avoidance of contour integrals through the use of the $\mathcal{D}$-calculus. At the same time we therefore obtain nice applications of this $\mathcal{D}$-calculus in classical function theory, a method originally coming from complex analysis in several variables. Its biggest interest for the modern one-variable setting lies of course in Wolff’s proof of the corona-theorem and its siblings (see [13]).

Here is now the scheme of our expositional paper. In Section 1 we unveil a class of examples that shows that the symmetric version of Rouché’s theorem actually is a stronger result than the nonsymmetric one. In Section 2 we give a short introduction to the $\mathcal{D}$-calculus. In Section 3 we then present a short and elegant proof of the symmetric version of Rouché’s theorem on compacta (based on Narasimhan’s ideas). Special homotopies between $f$ and $g$ play an important role in the proof.

Therefore we also present a more abstract homotopic variant of Rouché’s theorem on compacta in Section 4. Here we need Runge’s approximation theorem; an elementary proof using $\mathcal{D}$-calculus is available (see [12]). Section 4 concludes with a proof of the converse of this variant. That proof is based on a very useful representation theorem for zero-free continuous functions (known as Eilenberg’s Theorem).

In Section 5 we reuse that method to generalize (to a certain extent) the symmetric version of Rouché’s theorem on compacta to continuous-holomorphic function pairs (for the unit disk in the unsymmetric case, see [14]). Further generalizations to continuous-continuous pairs will not be pursued in detail, since that direction leaves the realm of complex analysis and amounts in considering the Brouwer degree in nonlinear analysis, a topic going beyond the scope of our survey.

As a corollary, we are able to generalize another feature of Tsarpalias result [14] from the disk to arbitrary compacta. This tells us that in the unsymmetric version of Rouché’s theorem we may allow equality in the boundary condition: $|f + g| = |g|$.

2. Examples

Here we present natural classes of pairs $(f, g)$ of holomorphic self-maps of the unit disk $\mathbb{D}$ that will show that the symmetric version of Rouché’s theorem can handle cases where the unsymmetric version fails. Note that it is easy to come up with trivial examples: $f(z) = i$, $g(z) = 1$ or $f(z) = iz$, $g(z) = z$, and so forth. Figure 1 illustrates well that the condition $|f + g| < |f|$ in the unsymmetric version is very restrictive: for each $z$ on the boundary of $\mathbb{D}$, $g_1(z)$ stays in the disk of radius $|f(z)|$ and centered at $-f(z)$, whereas in the symmetric version $|f + g_2| < |f| + |g_2|$, $g_2(z)$ is allowed to move on the complement of the half ray starting at the origin and passing through $f(z)$.

Our first example (it originates in the problem of describing those holomorphic self-maps $u$ and $v$ of $\mathbb{D}$ for which $u + v$ is also a self-map) has been chosen so that the zeros in the specific example are known a priori. It also demonstrates that it can be very tough to verify that the hypothesis $|f + g| < |f| + |g|$ holds.

Example 1. Let $1/2 < a < 1$, $f_a(z) = (a - z)/(1 - az)$, and $g(z) = z$. Then, on $\partial \mathbb{D}$, $(f, g)$ satisfies

$$|f_a + g| < |f_a| + |g|,$$  

but none of the following inequalities hold:

$$|f_a + g| < |g|, \quad |f_a - g| < |g|,$$

$$|f_a + g| < |f_a|, \quad |f_a - g| < |f_a|.$$  

Proof. Since for $z \in \partial \mathbb{D}$ we have $|f_a(z)| = |g(z)| = 1$, it suffices to show that

$$1 < \max_{|z|=1} |f_a(z) + g(z)| < 2,$$

$$\max_{|z|=1} |f_a(z) - g(z)| = 2.$$  

Claim 1. $\max_{|z|=1} |f_a(z) + g(z)| = 2a$ (this holds for all $0 < a < 1$).

To see this, we note that $f_a(z) + g(z) = a(1 - z^2)/(1 - az)$. Now, if $z = e^{i\theta}$, then

$$|f_a(e^{i\theta}) + g(e^{i\theta})|^2 = a^2 \frac{(1 - \cos(2\theta))^2 + \sin^2(2\theta)}{(1 - a \cos \theta)^2 + a^2 \sin^2 \theta}$$

$$= 4a^2 \frac{\sin^2 \theta}{1 - 2a \cos \theta + a^2}. $$

Hence, we need to maximize for $x \in [-1, 1]$ the function $h$ given by

$$h(x) := 4a^2 \frac{1 - x^2}{1 + a^2 - 2ax}.$$
where \( x := \cos \theta \). But
\[
h'(x) = 8a^2 \frac{(ax-1)(x-a)}{(1+a^2-2ax)^2}.
\] (6)

Then \( h'(x) > 0 \) if \(-1 \leq x < a \) and \( h'(x) < 0 \) if \( a < x \leq 1 \). Since \( h(\pm 1) = 0 \) we see that \( h \) takes its global maximum in \([-1,1]\) at the point \( a \). Hence
\[
\max_{|z|=1} |f_a(z) + g(z)| = \sqrt{h(a)} = 2a.
\] (7)

**Claim 2.** \( \max_{|z|=1} |f_a(z) - g(z)| = 2. \) Just take \( z = 1 \); then
\[
2 = |f_a(1) - g(1)| \leq \max_{|z|=1} |f_a(z) - g(z)| \leq 2.
\] (8)

Thus the functions \( g(z) = z \) and \( f_a(z) = (a-z)/(1-az) \) with \( 1/2 < a < 1 \) are the items we were looking for.

Next we give a class of examples where one does not recognize, a priori, the number of zeros of \( f \) (or \( g \)).

**Example 2.** Let \( a > 0 \) and consider a function \( p \in A(D) \) satisfying
\[
\left\{ -\frac{a}{2}, \frac{3a}{2} \right\} \subseteq p(D) \subseteq \mathbb{C} \setminus [0,a].
\] (9)

Let \( f := p + g := a - p \). Then, on \( \partial D \), \((f,g)\) satisfies
\[
|f + g| < |f| + |g|,
\] (10)

but none of the following inequalities holds:
\[
|f + g| < |g| \quad \text{or} \quad |f + g| < |f|.
\] (11)

**Proof.** If \( |f(z_0) + g(z_0)| = |f(z_0)| + |g(z_0)| \) for some \( z_0 \in \partial D \), then there exists \( \lambda > 0 \) such that \( g(z_0) = \lambda f(z_0) \). Hence \( a - p(z_0) = \lambda p(z_0) \) and so \( p(z_0) = a/(\lambda + 1) \in [0,a] \), a contradiction to the choice of \( p \). Thus \((f,g)\) satisfies (10). Moreover, if \( p(z_1) = -a/2 \) and \( p(z_2) = 3a/2 \) for some \( z_1, z_2 \in \partial D \), then
\[
|f(z_1) + g(z_1)| = a > \frac{a}{2} = |f(z_1)|,
\]
\[
|f(z_2) + g(z_2)| = a > \frac{a}{2} = |g(z_2)|.
\] (12)

We note that by Theorem 11 (the symmetric version of Rouche’s theorem) or Theorem 15 (the homotopic version \((f_t := p - ta, 0 \leq t \leq 1)\) is a zero-free homotopy on \( \partial D \) between \( f \) and \(-g\)), \( f \) and \( g \) have the same number of zeros in \( D \).

The third example, in connection with Pisot numbers, circulates on the web and we would like to present it here, too.

**Example 3.** Let \( f(z) = z^n(z-2) - 1 \). Then \( f \) has \( n \) roots in \( \mathbb{D} \) and a real root \( s \) bigger than 1 (in other words, \( s \) is a Pisot number).

**Proof.** Let \( g(z) = -z^n(z-2) - 1 \). Then, on \( \partial D \), \( |g| = |z-2| \geq 2 - |z| = 1 \) and so \(|f + g| = 1 \leq |g|\) with equality at \( z = 1 \) (so the standard version of Rouche does not work). But for \( z \neq 1 \), \( |z-2| > 1 \). Thus, by noticing that \(|f(1)| = 2\), we obtain that on \( \partial D \)
\[
|f + g| < |g| + |f|.
\] (13)

Thus, the symmetric version of Rouche’s theorem applies and from the fact that \( f \) does not vanish at the roots of the derivative, we conclude that \( f \) has \( n \) distinct zeros in \( D \). Now \( f(1) = -2 \) and \( f(3) > 0 \) imply that there exists a real root \( s \) bigger than 1.

\section{The \( \overline{\partial} \)-Calculus}

Whereas in real analysis the partial derivatives \( f_{x_j} := \partial f/\partial x_j \) play a central role, their counterparts in complex analysis are the so-called Wirtinger derivatives \( \partial f = \partial f/\partial z \) and \( \overline{\partial f} = \partial f/\partial \overline{z} \).

So suppose that \( f : \Omega \to \mathbb{C} \) is \( \mathbb{R} \)-differentiable at \( z_0 = x_0 + iy_0 \in \Omega, \Omega \subseteq \mathbb{C} \) open. Then, by writing \( x-x_0 = (1/2)(z - z_0) + (\overline{z} - \overline{z_0}) \) and \( y-y_0 = (1/2)(z - z_0) - (\overline{z} - \overline{z_0}) \) we arrive at
\[
f(z) = f(z_0) + f_x(z_0)(x-x_0)
\]
\[
+f_y(z_0)(y-y_0) + \sigma(z-z_0)
\]
\[
= f(z_0) + \frac{1}{2} \left( f_x(z_0) - if_y(z_0) \right)(z-z_0)
\]
\[
+ \frac{1}{2} \left( f_x(z_0) + if_y(z_0) \right)(\overline{z} - \overline{z_0}) + \sigma(z-z_0).
\] (14)

The Wirtinger derivatives are now defined by
\[
f := \frac{\partial f}{\partial z} := \frac{1}{2} \left( f_x - if_y \right),
\]
\[
f := \frac{\partial f}{\partial \overline{z}} := \frac{1}{2} \left( f_x + if_y \right).
\] (15)

It is easy to see that the Cauchy-Riemann equations \( u_x = v_y, u_y = -v_x \) for \( f = u + iv \) take the form \( \overline{\partial f} = 0 \). Also, if \( f \) is holomorphic at \( z_0 \), then \( f(z_0) = f_x(z_0) = f_y(z_0) \).

The base of our proofs is the following representation theorem for smooth functions. It is a Cauchy-type representation (using two-dimensional integration with respect to planar Lebesgue measure \( \sigma_2 \) instead of integration on curves).

**Theorem 4.** Let \( f \) be continuously \( \mathbb{R} \)-differentiable in \( \mathbb{C} \) and suppose that \( f \) has compact support. Then
\[
f(z) = -\frac{1}{\pi} \int_{|z-\zeta|=\rho} \frac{\overline{\partial f}(\zeta)}{\zeta-z} d\sigma_2(\zeta).
\] (16)

A very nice proof can be found in Rudin’s book [4, page 389]. Another proof appears for example in [12, page 103]. Using Fubini’s theorem, the Gauss-Green-Stokes formula can easily be deduced (see [15, page 109]).
Theorem 5. Let $\Omega$ be an admissible domain; that is a bounded domain such that the boundary $\partial \Omega$ of $\Omega$ consists of finitely many closed, positively oriented, pairwise disjoint, piecewise-$C^1$ Jordan curves $\gamma_j$ ($j = 0, 1, \ldots, n$). Suppose that $f$ is continuously $\mathbb{R}$-differentiable in a neighborhood of $\overline{\Omega}$. Then

$$\int_{\partial \Omega} f(z) \, dz = 2i \int_{\Omega} \overline{f}(z) \, d\sigma_2(z). \quad (17)$$

The following result is an analog of the Cauchy-integral formula. As usual, $C_c^\infty(\mathbb{C})$ denotes the set of smooth functions with compact support in $\mathbb{C}$.

Proposition 6. Let $\psi \in C_c^\infty(\mathbb{C})$ and suppose that $f$ is holomorphic in $\Omega$, where $\Omega$ is an open neighborhood of the support of $\psi$. Then

$$\int_{\Omega} \overline{\psi}(z) f(z) \, d\sigma_2 = 0. \quad (18)$$

Of course we may replace the integration set $\Omega$ by $\mathbb{C}$ by extending $f$ to be constant zero outside $\Omega$.

Proof. Let $U$ be an open set such that supp $\psi \subseteq U \subseteq \overline{U} \subseteq \Omega$. Let $\phi \in C_c^\infty(\mathbb{C})$ satisfy $\phi = 1$ on $U$ and supp $\phi \subseteq \Omega$. Fix $a \in \mathbb{C}$ and define

$$F(z) = \begin{cases} \phi(z) f(z) (z-a) & \text{if } z \in \Omega \\ 0 & \text{if } z \notin \Omega \end{cases} \quad (19)$$

Then $F \in C_c^\infty(\mathbb{C})$. Since $F$ is holomorphic in $U \ni \text{supp}(\psi)$, we have $\overline{\psi}(F) = F \overline{\psi}$ on $U$. Thus, by Theorem 4,

$$-\frac{1}{\pi} \int_U \overline{\partial \psi}(z) F(z) \, d\sigma_2(z) = -\frac{1}{\pi} \int_U \overline{\partial \psi}(z) F(z) \, d\sigma_2(z)$$

$$= -\frac{1}{\pi} \int_U \overline{\partial \psi}(z) F(z) \, d\sigma_2(z)$$

$$= -\frac{1}{\pi} \int_U \overline{\partial \psi}(z) F(z) \, d\sigma_2(z)$$

$$= -\frac{1}{\pi} \int_U \overline{\partial \psi}(z) F(z) \, d\sigma_2(z)$$

$$= \psi(a) F(a) = 0. \quad (20)$$

Remark 7. We could also have used the Gauss-Green-Stokes formula (17) to prove Proposition 6: let $U$ and $\phi$ be as above and let

$$F^*(z) = \begin{cases} \phi(z) f(z) & \text{if } z \in \Omega \\ 0 & \text{if } z \notin \Omega \end{cases} \quad (21)$$

Then $F^* \in C_c^\infty(\mathbb{C})$. Since $F^*$ is holomorphic in $U \ni \text{supp} \psi \ni \text{supp} \overline{\psi}$, we have

$$I := \int_C \overline{\psi}(F^*) \, d\sigma_2 = \int_C \overline{\psi}(F^*) \, d\sigma_2$$

$$= \int_C \overline{\psi}(F^*) \, d\sigma_2 - \int_C \psi \overline{\partial F^*} \, d\sigma_2. \quad (22)$$

If supp $\psi$ is contained in the disk $D$, then $\overline{\partial} \psi F^* = 0$ on $\mathbb{C} \setminus D$. Hence, by Theorem 5,

$$\int_C \overline{\partial} \psi F^* \, d\sigma_2 = \frac{1}{2\pi} \int_{\partial D} \psi F^* \, (z) \, dz = 0. \quad (23)$$

Using $\overline{\partial} F^* = \overline{\partial} f = 0$ in the neighborhood $U$ of supp $\psi$, we are able to conclude that

$$I = -\int_C \psi \overline{\partial} F^* \, d\sigma_2 = -\int_{\text{supp} \psi} \psi \overline{\partial} F^* \, d\sigma_2 = 0. \quad (24)$$

Remark 8. It is interesting to note that if we merely assume that $f$ is holomorphic in a neighborhood of supp $\overline{\psi}$, then equality (18) does no longer hold, although the values of $f$ outside supp $\overline{\psi}$ do not play any role, as what is demonstrated by the following integral equality:

$$\int_C \overline{\partial \psi} f \, d\sigma_2 = \int_{\text{supp} \overline{\psi}} \overline{\partial \psi} \, f \, d\sigma_2. \quad (25)$$

Here is an example. Given $a \in \mathbb{C}$, choose $\psi \in C_c^\infty(\mathbb{C})$ so that

$$R := (\text{supp } \psi)^\circ \setminus \text{supp } \overline{\psi} \neq \emptyset, \quad (26)$$

$$a \in R \text{ and } \psi(a) = 1.$$

Let $U$ be an open neighborhood of supp $\overline{\psi}$ and $W$ an open set containing $a$ with $U \cap W = \emptyset$. Let $g \in C_c^\infty(\mathbb{C})$ be chosen so that $g = 0$ in a neighborhood of $\overline{W}$ and $g(z) = 1$ if $z \in U$. Define $f$ by

$$f(z) = \begin{cases} \frac{1}{z-a} g(z) & \text{if } z \in \mathbb{C} \setminus W \\ 0 & \text{if } z \in W \end{cases} \quad (27)$$

Then $f \in C_c^\infty(\mathbb{C})$ and $f$ is holomorphic in the neighborhood $U$ of supp $\overline{\psi}$. By Theorem 4,

$$-\frac{1}{\pi} \int_U \overline{\partial \psi} (\overline{\partial F^*})(z) \, d\sigma_2 = -\frac{1}{\pi} \int_{\text{supp} \overline{\psi}} \overline{\partial \psi} \, F^*(z) \, d\sigma_2$$

$$= -\frac{1}{\pi} \int_{\text{supp} \overline{\psi}} \overline{\partial \psi} \, F^*(z) \, d\sigma_2$$

$$= -\frac{1}{\pi} \int_{\text{supp} \overline{\psi}} \overline{\partial \psi} \, (z) \, d\sigma_2 = \psi(a) \neq 0. \quad (28)$$

Later we shall see that if $f = \log \vert h \vert / h$ is a logarithmic derivative that is holomorphic in a neighborhood of supp $\overline{\psi}$, then $\int_C (\overline{\partial} \psi) f \, d\sigma_2$ can be computed explicitly. This will yield an areal analog to the argument principle.

Remark 9. In view of Proposition 6 and the fact that $\int_C f(z) \, dz = 0$ whenever $f$ is the derivative of a holomorphic function and $\gamma$ a closed $C^1$-path in $\Omega$, it is tempting to conjecture that the following is true.

Desideratum. Let $\psi \in C_c^\infty(\mathbb{C})$ and $f \in C(\mathbb{C})$. Suppose that in a neighborhood $U$ of the support of $\overline{\psi}$, $f$ is the derivative of a holomorphic function $g$. Then $\int_C (\overline{\partial \psi}) f \, d\sigma_2 = 0.$
Unfortunately, the assertion above is not true. Here is an example. Note that functions built on $1/(z - a)$, as in Remark 8, cannot be taken, since they do not have primitives on annuli surrounding $a$.

Let $V$ be an open set containing $\text{supp} \tilde{\varphi}$ with smooth boundary and such that $\tilde{\varphi} g = 0$ and $\partial \varphi = f$ on $V \ni \nabla \ni V$. Then

$$\int_V \tilde{\varphi} g d\sigma_2 = \int_V \tilde{\varphi} d\sigma_2 = \int_V \tilde{\varphi} (-1) \frac{1}{z^2} dz = 2 \pi i \neq 0.$$

Now let $\varphi(z) = z$ for $|z| < 2$ and $\varphi = 0$ for $|z| \geq 3$. Then $\text{supp} \tilde{\varphi} \subseteq \{2 \leq |z| \leq 3\}.$

Choose $g \in C(\mathbb{C})$ such that $g(z) = 1/|z| \geq 1$ and $g(z) = \bar{z}$ for $|z| \leq 1$. Then $g$ is holomorphic on $|z| > 1$, a neighborhood of $\text{supp} \tilde{\varphi}$. As $V$ we may take

$$V = \left\{ \frac{3}{2} < |z| < \frac{7}{2} \right\}.$$

Then $V$ is a neighborhood of $\text{supp} \tilde{\varphi}$. But

$$\int_V \tilde{\varphi} g d\sigma = 0 - \int_{|z|=3/2} \frac{z}{z^2} d\sigma = 2 \pi i \neq 0.$$

### 4. Counting Zeros and Poles

The techniques above allow us to determine for meromorphic functions, $f$, the integer $n_K(f) - p_K(f)$, where $n_K(f)$ is the number of zeros and $p_K(f)$ the number of poles of $f$ on the compact set $K$ (each counted with the appropriate multiplicity). We follow the scheme and proofs given by Narasimhan [12, page 105].

For a function $f$, meromorphic on the open set $\Omega \subseteq \mathbb{C}$, let

$$Z(f) = \{z \in \Omega : f(z) = 0\},$$

$$P(f) = \{z \in \Omega : z \text{ is a pole of } f\}.$$

**Theorem 10.** Let $f$ be a meromorphic function on the open set $\Omega \subseteq \mathbb{C}$. We assume that $f$ is not identically zero on any connected component of $\Omega$. Let $K \subseteq \Omega$ be compact and suppose that $f$ has neither zeros nor poles on $\partial K$. Let $U$ be a bounded open set such that $\partial K \subseteq U \subseteq \Omega \subseteq \overline{U} \cap (Z(f) \cup P(f)) = \emptyset$. Then, for every $\psi \in C_0^\infty(\mathbb{C})$ with $\psi = 1$ in a neighborhood of $K$ and $\text{supp } \psi \subseteq K \cup U$ one has

$$n_K(f) - p_K(f) = -\frac{1}{\pi} \int_C \overline{\psi(\zeta)} \frac{f'(\zeta)}{f(\zeta)} d\sigma_2(\zeta).$$

**Proof.** The assumptions imply that the zeros and poles of $f$ form a discrete set in $\Omega$; accordingly, the sets $N(f) \cap K$ and $P(f) \cap K$ are finite (or void). Let $\{a_1, \ldots, a_m\}$ denote the zeros and $\{b_1, \ldots, b_p\}$ the poles of $f$ in $K$. Their associated multiplicities are given by $m_j$, respectively $p_j$. Using Riemann’s theorem on removable singularities and the fact that $m_j$, respectively $p_j$, is the residue of the meromorphic function $f'/f$ at $a_j$, respectively $b_j$, we see that for every $z \in K \cup \overline{U}$

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^m \frac{m_j}{z - a_j} - \sum_{j=1}^p \frac{p_j}{z - b_j} + h(z),$$

where $h$ is holomorphic in a neighborhood of the set $K \cup \overline{U}$. Note that this representation also holds if $m = 0$ or $p = 0$; that is in the case where $f$ does not have zeros or poles.

Since $h$ is holomorphic in the open neighborhood $K \cup U$ of the support of $\varphi$, we may apply Proposition 6 to conclude that

$$\int_C \overline{\psi(\zeta)} h(\zeta) d\sigma_2(\zeta) = 0.$$

From $a_j, b_j \in K$, we deduce from Theorem 4 that for $a \in \{a_1, \ldots, a_m\} \cup \{b_1, \ldots, b_p\}$

$$\int_C \overline{\psi(\zeta)} d\sigma_2(\zeta) = \psi(a) = 1.$$}

Hence

$$n_K(f) - p_K(f) = n_K(g) - p_K(g).$$

We recall the following definition. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and suppose that $S \subseteq \mathbb{C}$ is compact. Then $f$ and $g \in C(S)$ are said to be homotopic in $C(S, \mathbb{C}^*)$, if there exists a continuous map $H : [0, 1] \times S \rightarrow \mathbb{C}^*$ such that for every $z \in S$

$$H(0, z) = f(z), \quad H(1, z) = g(z).$$

For a compact set $K \subseteq \mathbb{C}$, let $A(K)$ denote the space of all functions continuous on $K$ and holomorphic in the interior of $K$.

**Theorem 11 (Rouché’s theorem).** Let $K \subseteq \mathbb{C}$ be compact and $f, g \in A(K)$. Suppose that on $\partial K$

$$|f + g| < |f| + |g|.$$}

Then $n_K(f) = n_K(g)$; that is the number of zeros of $f$ and $g$ on $K$ coincides (multiplicities included).

**Proof.** The hypothesis $|f + g| < |f| + |g|$ obviously implies that $f$ and $g$ are zero-free on $\partial K$. If $K^* = \emptyset$, then $K = \overline{\partial K}$ and $f$ and $g$ have no zeros. So let us assume that $K^* \neq \emptyset$.

Since the boundary of each connected component of $K^*$ is contained in $\partial K$, we first observe that $f$ and $g$ can not vanish identically on any of these components. Moreover, since the zeros inside a component $\Omega$ of $K^*$ of the holomorphic
functions \( f \) and \( g \) do not accumulate at points in \( \Omega \), we deduce that \( f \) and \( g \) can have at most a finite number of zeros in \( K \).

**Case 1.** We first assume that \( f \) and \( g \) are holomorphic in a neighborhood \( \Omega \) of \( K \). We consider the following homotopy \( f_t \) between \( f \) and \(-g\):

\[
f_t = (1-t)f - tg, \quad 0 \leq t \leq 1.
\]

Then \( f_t \) has no zeros on \( \partial K \) (since otherwise \( g \) would be a positive multiple of \( f \) at some point \( a \in \partial K \), a contradiction to the assumption that \( |f + g| < |f| + |g| \)).

Due to the uniform continuity of the map \([0,1] \times \partial K \to C, (t,z) \mapsto f_t(z)\), there is an open neighborhood \( U \subseteq \Omega \) of \( \partial K \), independent of \( t \), such that for every \( t \), \( f_t \) is zero-free on \( U \). Now choose \( \psi \in C_\infty^\infty(C) \) with \( \psi = 1 \) in an open neighborhood of \( K \) and supp \( \psi \subseteq K \cup U \). Then, by Theorem 10,

\[
n_K(f_t) = -\frac{1}{\pi} \int_C \overline{\partial \psi}(\zeta) \frac{f'_t(\zeta)}{f_t(\zeta)} \, d\sigma_2(\zeta). \quad (42)
\]

Since \( \text{supp} \overline{\partial \psi} \cap K = \emptyset \), the inclusion \( \text{supp} \overline{\partial \psi} \subseteq U \) implies that \( |f_t(z)| \geq \delta > 0 \) for every \( t \) and \( z \in \text{supp} \overline{\partial \psi} \). Hence the integral

\[
-\frac{1}{\pi} \int_{\text{supp} \overline{\partial \psi}} \overline{\partial \psi}(\zeta) \frac{f'_t(\zeta)}{f_t(\zeta)} \, d\sigma_2(\zeta)
\]

is a continuous function of \( t \). Since \( n_K(f_t) \) is integer-valued, it must be constant. Thus

\[
n_K(f) = n_K(f_0) = n_K(f_1) = n_K(-g) = n_K(g). \quad (44)
\]

**Case 2.** We shall now deduce the general case from the first case. In fact, let \( c \in K \) and let \( U, V \) be two open neighborhoods of \( \partial K \) such that \( c \notin U, V \subseteq U \subseteq U, \) and \( |f + g| < |f| + |g| \) on \( U \cap K \). Consider \( K' = K \setminus V \). Note that \( c \in K' \), hence \( K' \neq \emptyset \). Then \( K' \) is compact, \( K' \subseteq K \), and \( \partial K' \subseteq K \cap U \). Since \( f \) and \( g \) are holomorphic in the neighborhood \( K' \) of \( K \), the hypotheses of Case 1 above are satisfied. Thus \( n_{K'}(f) = n_{K'}(g) \). Since on \( U \cap K \subseteq V \cap K \) the functions \( f \) and \( g \) have no zeros, we deduce that \( n_K(f) = n_K(g) \).

**Remark 12.** The “usual” proof (using cycles, see [7, page 265]) of Case 1 would have been to consider the integrals

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{f'_t(z)}{f_t(z)} \, dz,
\]

where \( \Gamma \) is a cycle in \( \Omega \setminus K \) such that

1. \( n(\Gamma, z) \in \{0, 1\} \) for every \( z \in C \setminus \Gamma \);
2. \( n(\Gamma, z) = 0 \) for every \( z \in C \setminus \Omega \) (that is \( \Gamma \) is null-homologous with respect to \( \Omega \));
3. \( n(\Gamma, z) = 1 \) for every \( z \in K \).

Here we have replaced the curve \( \Gamma \), whose construction is very elaborate (although intuitively clear), by certain thin “blow-ups” of \( \Gamma \) (namely \( \text{supp} \overline{\partial \psi} \)) that have positive planar Lebesgue measure.

### 5. A Homotopic Variant of Rouché’s Theorem

Whereas the classical proof of the homotopic version of Rouché’s theorem (under the stronger hypothesis that there exists a “holomorphic homotopy” (that is a function \( H(t, z) : [0,1] \times K \to C \) with \( H(t, \cdot) \) holomorphic in a neighborhood of \( K \) and \( H(z, t) \) zero-free on \( \partial K \))) uses a version of the argument principle based again on the existence of a null-homologous cycle surrounding each point of \( K \), we present a proof depending on the following areal argument principle that is related to Theorem 10. The difference of Proposition 13 to Theorem 10 is that the current \( f \) is no longer holomorphic/meromorphic in a neighborhood of the entire support of \( \psi \), but only on \( \text{supp} \overline{\partial \psi} \).

**Proposition 13.** Let \( K \subseteq C \) be compact and let \( f \) be holomorphic and zero-free in an open neighborhood \( U \) of \( \partial K \). Then for every \( \psi \in C_\infty^\infty(C) \) with \( \psi = 1 \) in a neighborhood of \( K \) and supp \( \psi \subseteq K \cup U \) one has

\[
I := -\frac{1}{\pi} \int_C \overline{\partial \psi} \frac{f'}{f} \, d\sigma_2 \in \mathbb{Z}. \quad (46)
\]

**Proof.** Due to the fact that \( K \cup U \) is open and supp \( \psi \) is compact, we can find an open set \( V \) satisfying \( \partial K \subseteq V \subseteq \overline{V} \subseteq U \) and supp \( \overline{\partial \psi} \subseteq K \cup V \). Since supp \( \overline{\partial \psi} \cap K = \emptyset \), the former inclusion implies that supp \( \overline{\partial \psi} \subseteq V \). Since \( f \) is holomorphic in a neighborhood of \( \overline{V} \), we may use Runge’s approximation theorem (see [2, 4, 12] for three different proofs, the one in Narasimhan’s monograph being in the spirit of our note here) to uniformly approximate \( f \) on neighborhoods of \( \overline{V} \) by rational functions \( r_n \) having their poles and zeros outside \( \overline{V} \). Hence \( (r'_n) \) tends uniformly to \( f' \) on \( \overline{V} \). Thus the pairs \( (\psi, r_n) \) satisfy the hypotheses of Theorem 10, and so

\[
I_n := -\frac{1}{\pi} \int_C \overline{\partial \psi} \frac{r'_n}{r_n} \, d\sigma_2 \in \mathbb{Z}. \quad (47)
\]

Since \( f \) is bounded away from zero on \( \overline{V} \), we also have \( |r_n| \geq \delta > 0 \) on \( \overline{V} \) for almost all \( n \). Hence \( r'_n/r_n \) converges uniformly on \( \overline{V} \) (in particular on supp \( \overline{\partial \psi} \)) to \( f'/f \). Thus \( I_n \) tends to \( I \). We conclude that \( I \in \mathbb{Z} \).

We also need the following characterization of homotopic function pairs.

**Theorem 14.** Let \( S \subseteq C \) be compact. Then two functions \( f, g \in C(S, C^*) \) are homotopic in \( C(S, \mathbb{C}^*) \) if and only if there is \( h \in C(S) \) such that \( f = ge^h \). If additionally, \( f \) and \( g \) are holomorphic on an open subset \( U \) of \( S' \), then \( h \) is holomorphic on \( U \), too.

**Proof.** Since every homotopy between \( f \) and \( g \) induces a path that connects \( f \) to \( g \), we conclude that \( f \) and \( g \) belong to the same connected component of \( U \setminus (C(S)) \), the group of invertible elements in \( C(S) \). Hence, by a classical theorem in the theory of Banach algebras (see [16, page 268]), there is \( h \in C(S) \) such that \( f = ge^h \). The rest is clear by taking local logarithms.
A distinct (and very elementary) proof of Theorem 14 appears in [17].

**Theorem 15** (Rouché for homotopic maps). Let $K \subseteq C$ be compact and $f, g \in A(K)$. Suppose that $f$ and $g$ are homotopic in $C(\partial K, C^*)$. Then $n_K(f) = n_K(g)$.

**Proof.** Step 1. We first show that the homotopy can be extended to a neighborhood of $\partial K$.

So let $H_0 : [0, 1] \times \partial K \to C^*$ be a homotopy that connects $f$ with $g$. To describe this property, let us use the notation $H(t, z) = f(t, z) e^{\psi(t, z)}$. Hence $H(t, \partial K) = f(t, \partial K) e^{\psi(t, \partial K)}$.

Using Tietze’s theorem, we may extend $H_1$ continuously to a map $H_1 : [0, 1] \times U \to C$. Since $H_2$ is uniformly continuous, there is a closed neighborhood $W$ of $\partial K$, independent of $t$, such that $\partial K \subseteq W \subseteq U$ and $H_2|_{[0,1] \times W}$ is bounded away from zero. Hence $H_2$ is a homotopy in $C \setminus \{0\}$ that connects $f$ within the group $U_1(C(W))$ of invertible elements in $C(W)$; that is $f \sim g$.

**Step 2.** Let us assume that $f$ and $g$ are holomorphic in a neighborhood of $K$ (the general case will be done in Step 3).

Choose an open neighborhood $V$ of $\partial K$ such that

$$\partial K \subseteq V \subseteq \nabla \subseteq W^*$$

and such that $f$ and $g$ are holomorphic in a neighborhood of $\nabla$. Since $f \sim g$, we conclude from Theorem 14 that there is $h \in C(V)$ such that $f = g e^h$. Note that $h$ is holomorphic in $V$, too. Thus we have obtained a zero-free holomorphic homotopy

$$H(t, z) = g(z) e^{\psi(t, z)}$$

that connects $g$ and $f$ within $U_1(C(V))$.

Let $\psi \in C^\infty(C)$ satisfy $\psi = 1$ in a neighborhood of $K$ and $\text{supp } \psi \subseteq K \cup V$. Note that $K \cup V$ is open and that $\text{supp } \nabla \psi \subseteq V$. Let $H'$ denote the derivative of $H$ with respect to $z$. Noticing that $H'$ is holomorphic in a neighborhood of $\text{supp } \nabla \psi$, we obtain from Proposition 13 that

$$N(t) := -\frac{1}{\pi} \int_\nabla \nabla \psi(\xi) H'(t, \xi) \frac{H(t, \xi)}{H(t, \xi)} d\sigma_2(\xi) \in \mathbb{Z}. \quad (51)$$

Since $|H(t, \xi)| \geq \delta > 0$ for every $\xi \in V$ and $t \in [0, 1]$, we see that $N(t)$ is a continuous integer-valued function. Hence it must be constant.

But by Theorem 10, $N(t)$ also satisfies

$$N(t) = -\frac{1}{\pi} \int_\nabla \nabla \psi(\xi) \left( \frac{g'(\xi)}{g(\xi)} + t h'(\xi) \right) d\sigma_2(\xi)$$

$$= -\frac{1}{\pi} \int_\nabla \nabla \psi(\xi) \frac{g'(\xi)}{g(\xi)} d\sigma_2(\xi)$$

$$- t \frac{1}{\pi} \int_\nabla \nabla \psi(\xi) h'(\xi) d\sigma_2(\xi)$$

$$= n_K(g) - t \frac{1}{\pi} \int_\nabla \nabla \psi(\xi) h'(\xi) d\sigma_2(\xi). \quad (52)$$

This forces the integral $(1/\pi) \int_\nabla \nabla \psi(\xi) h'(\xi) d\sigma_2(\xi)$ to be zero. Hence $N(t) = n_K(g)$. Since $H(1, z) = f(z)$, we deduce from (51) and Theorem 10 that

$$N(1) = -\frac{1}{\pi} \int_\nabla \nabla \psi(\xi) \frac{f'(\xi)}{f(\xi)} d\sigma_2(\xi) = n_K(f). \quad (53)$$

Hence $n_K(f) = n_K(g)$.

**Step 3.** Let $f$ and $g$ satisfy the hypotheses of the theorem. Choose the set $W$ as in Step 1 and let $\Omega$ be an open set with

$$\partial K \subseteq \Omega \subseteq \overline{\Omega} \subseteq W.$$  \quad (54)

Let $K' = K \setminus \Omega$. Note that $K' \subseteq C$ and $\partial K' \subseteq K \cap W$. Since $f \sim g$, we also have $f \sim g$. Now the assumptions of Step 2 are satisfied. Hence $n_{K'}(f) = n_{K'}(g)$. Since $f$ and $g$ are zero-free on $K \cap W$, we obtain the assertion that $n_K(f) = n_K(g)$.

**Remark 16.** Our proof actually shows the stronger result that $n_C(f) = n_C(g)$ for every component $C$ of $K'$.

This homotopic version of Rouché’s theorem yields yet another proof of the classical Rouché theorem itself; just note that the condition $|f + g| < |f| + |g|$ on $\partial K$ implies that the function $H : [0, 1] \times \partial K \to C^*$ given by

$$H(t, z) = (1 - t) f(z) - t g(z) \quad (55)$$

is a homotopy that connects $f$ with $-g$ inside $C(\partial K, C^*)$.

It is now a natural question whether the converse of Theorem 15 holds, too. In [18] this was confirmed for Jordan domains with rectifiable boundary. We shall now give a characterization of those compacta in $C$ for which the converse is true. Our method is based on the following remarkable theorem of Eilenberg (see [17] for a proof based on $\beta$-calculus and [7, page 97] for the classical proof). We only present those parts that are relevant to our setting here.

**Theorem 17** (Eilenberg). Let $S \subseteq C$ be compact.

(1) If $a$ and $b$ belong to distinct components of $C \setminus S$, then the polynomials $z - a$ and $z - b$ are not homotopic in $C(S, C^*)$. 


(2) For each bounded component $C$ of $C \setminus S$, let $a_C \in C$. Suppose that $q \in C(S)$ is zero-free. Then there exist finitely many bounded components $C_j$ of $C \setminus S$, integers $s_j \in \mathbb{Z}$ ($j = 1, \ldots, n$), and $L \in C(S)$ such that for all $z \in S$

$$q(z) = \prod_{j=1}^{n} \left(z - a_C^j\right)^{s_j} e^{L(z)}. \quad (56)$$

The $C_j$ and $s_j$ are uniquely determined. In particular, there exists a rational function $r$ with poles and zeros outside $S$ such that within $C(S, C^*)$, $q$ is homotopic to $r$.

(3) If $C \setminus S$ is connected, then $q$ itself has a logarithm.

**Theorem 18.** Let $K \subseteq \mathbb{C}$ be compact. Consider the following assertions:

1. For all $f, g \in A(K)$: if $f$ is homotopic to $g$ in $C(\partial K, C^*)$, then $n_K(f) = n_K(g);$
2. For all $f, g \in A(K)$: if $n_K(f) = n_K(g)$ and $f$ and $g$ are zero-free on $\partial K$, then $f$ is homotopic to $g$ in $C(\partial K, C^*)$.

Then (1) holds for every $K$ and (2) holds if and only if $C \setminus K$ and $K^*$ are connected.

**Proof.** That (1) holds is the content of Theorem 15. Now suppose that $C \setminus K$ or $K^*$ is not connected. Let $a$ and $b$ be contained in distinct components of $C \setminus K$, respectively $K^*$. Consider on $K$ the polynomials $f(z) = z - a$ and $g(z) = z - b$. Then $n_K(f) = n_K(g) = 0$, respectively $n_K^* (f) = n_K^* (g) = 1$. Since $a$ and $b$ are contained in different components of $C \setminus K = (C \setminus K) \cup K^*$, we deduce from Theorem 17(1) that $f$ and $g$ are not homotopic in $C(\partial K, C^*)$.

Next we prove the converse. So let us suppose that $C \setminus K$ and $K^*$ are connected. We show that (2) holds. So let $f, g \in A(K), f$ and $g$ zero-free on $\partial K$ and $n_K (f) = n_K (g)$. If $K^* \neq \emptyset$, we fix a point $\alpha \in K^*$ (the other case will be done afterwards). By Eilenberg’s Theorem 17(2) applied to the compact set $S = \partial K$ (note that $C \setminus S$ has exactly one bounded component) and the function $q = f/g$, there is $s \in \mathbb{Z}$ such that $f(z)$ and $(z - \alpha)^s g(z)$ are homotopic in $C(\partial K, C^*)$. There is no loss of generality in supposing that $s \geq 0$ (otherwise interchange the roles of $f$ and $g$). By Theorem 15, $n_K^* (f) = s + n_K^* (g)$. The hypothesis $n_K^* (f) = n_K^* (g)$ now implies that $s = 0$. Hence $f \sim g$.

Now we consider the case where $K^* = \emptyset$. By our hypotheses, $C \setminus K$ is connected. Hence, by Theorem 17(3) applied to $S = \partial K = K$, $f/g$ has a continuous logarithm on $\partial K$. Thus, by Theorem 14, $f \sim g$. \qed

By the same methods, we can also prove the following result.

**Theorem 19.** Let $K \subseteq \mathbb{C}$ be a compact set with connected complement and let $f, g \in A(K)$ be zero-free on $\partial K$. Then $f$ and $g$ are homotopic in $C(\partial K, C^*)$ if and only if $n_C^* (f) = n_C^* (g)$ for every component $C$ of $K^*$.

### 6. Rouché for Continuous-Holomorphic Pairs

Whereas above we dealt with pairs of holomorphic functions, we shall now present two versions of Rouché’s theorem for continuous-holomorphic pairs.

**Theorem 20** (Rouché for a continuous-holomorphic pair). Let $K \subseteq \mathbb{C}$ be compact, $f \in C(K)$, and $g \in A(K)$. Suppose that on $\partial K$

$$|f + g| < |f| + |g|.$$ \quad (57)

Then $f$ has a zero on $K^*$ whenever $g$ has a zero on $K^*$. The converse does not hold, in general.

**Proof.** Since $|f + g| < |f| + |g|$ on $\partial K$, we see that $f$ and $g$ are zero-free on $\partial K$ and so

$$\left|\frac{f}{g} + 1\right| < \left|\frac{f}{g}\right| + 1.$$ \quad (58)

Thus the image of $\partial K$ under $f/g$ is contained in the slit plane $S := \mathbb{C} \setminus [0, \infty]$. Taking a continuous branch of $\log z$ on $S$, we see that on $\partial K$, $f/g$ can be written as $f/g = \exp L$ for some $L \in C(\partial K)$.

In particular

$$H(z, t) := g(z) e^{t\log z} \quad (59)$$

is a homotopy connecting $g$ to $f$ within $C(\partial K, C^*)$ (compare with the homotopy (55) given earlier).

Now suppose that $f$ is zero-free on $K$. Then, by Eilenberg’s Theorem 17(2), $f = r e^h$ for some rational function $r$ without poles and zeros on $K$ and $h \in C(K)$. Hence, on $\partial K$, $r = g \exp \left(L - h\right). \quad (60)$

Thus the map $H: [0, 1] \times \partial K$, given by

$$\tilde{H}(z, t) = g(z) e^{t(L(z) - (z - h(z)))},$$ \quad (61)

is a zero-free homotopy on $\partial K$ between $g$ and $r$. By Theorem 15, $n_K^* (r) = n_K^* (g)$. Since $n_K^* (r) = 0$, we deduce that $g$ has no zeros on $K$ either. Thus, by contraposition, $Z(f) \neq \emptyset$ whenever $Z(g) \neq \emptyset$.

Next we prove the remaining assertion that there exists a continuous-holomorphic pair $(f, g)$ satisfying $|f + g| < |f| + |g|$ on $\partial K$ such that $f$ has a zero but not $g$. In fact, let $U \subseteq K^*$ be a proper open subset of $K^*$ and $E \subseteq U$ a nonvoid closed set. Let $g \equiv 1$ on $K$ and let $f \in C(K)$ be chosen so that $f \equiv -1$ on $K \setminus U$ and $f \equiv 0$ on $E$. Then $g$ is holomorphic, $0 = |f + g| < |f| + |g| = 2$ on $\partial K$, $Z(f) \neq \emptyset$, but $Z(g) = \emptyset$. \qed

**Remark 21.** One of the reasons for this dissymmetry is that, in contrast to holomorphic functions, continuous functions can be arbitrarily changed on compacta contained in $K^*$ without perturbing the boundary values.

To conclude this section, we present a generalization to arbitrary compacta of Tsarpalias result [14] that in the unsymmetric version of Rouché’s theorem one can actually allow equality in the hypothesis $|f + g| \leq |g|$ to conclude that
for continuous-holomorphic pairs \((f, g)\) the zero set \(Z(f)\) is nonvoid whenever \(Z(g) \neq \emptyset\). In contrast to the previous versions of Rouché's theorem, the zero(s) of \(f\), though, may lie on the boundary of \(K\) (without \(g\) having zeros there). In fact, if \(K = \mathbb{D}\), let \(f(z) = 1 - z\) and \(g(z) = z\). Then \(|f(z) + g(z)| = 1 \leq |g(z)|\).

**Theorem 22.** Let \(K \subseteq \mathbb{C}\) be compact, \(f \in C(K)\), and \(g \in A(K)\). Suppose that on \(\partial K\)

\[|f + g| \leq |g|,\] 

(62)

Then \(f\) has a zero on \(K\) whenever \(g\) has a zero on \(K\).

**Proof.** If \(f\) has a zero on \(\partial K\), then nothing is to prove. If \(f\) is zero-free on \(\partial K\), then \(|f| \geq \delta > 0\) on \(\partial K\) and so

\[|f + g| \leq |g| < \delta + |g| \leq |f| + |g|,\] 

(63)

The assertion now follows from Theorem 20. \(\square\)

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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