A New Solution to the Star–Triangle Equation Based on $U_q(\mathfrak{sl}(2))$ at Roots of Unit

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ABSTRACT

We find new solutions to the Yang–Baxter equation in terms of the intertwiner matrix for semi-cyclic representations of the quantum group $U_q(\mathfrak{sl}(2))$ with $q = e^{2\pi i/N}$. These intertwiners serve to define the Boltzmann weights of a lattice model, which shares some similarities with the chiral Potts model. An alternative interpretation of these Boltzmann weights is as scattering matrices of solitonic structures whose kinematics is entirely governed by the quantum group. Finally, we consider the limit $N \to \infty$ where we find an infinite–dimensional representation of the braid group, which may give rise to an invariant of knots and links.
1. Introduction

The $N$–chiral Potts model [1, 2] is a solvable lattice model satisfying the star–triangle relation. Their Boltzmann weights are meromorphic functions on an algebraic curve of genus $N^3 - 2N^2 + 1$. These models are the first example of solutions to the Yang–Baxter equation with the spectral parameters living on a curve of genus greater than one. Recent developments [3, 4] strongly indicate that quantum groups at roots of unit [5] characterize the underlying symmetry of these models. More precisely, it was shown in reference [4] that the intertwiner $R$ matrix for cyclic representations of the affine Hopf algebra $U_q(\hat{sl}(2))$ at $q$ an $N$-th root of unit, admit a complete factorization in terms of the Boltzmann weights of the $N$–chiral Potts model. In this more abstract approach, the spectral parameters of the Potts model are represented in terms of the eigenvalues of the central Hopf subalgebra of $U_q(\hat{sl}(2))$ ($q = e^{2\pi i/N}$) with the algebraic curve fixed by the intertwining condition.

In a recent letter [6], we have discovered a new solution to the Yang–Baxter equation, with the spectral parameters living on an algebraic curve. This solution was obtained as the intertwiner for semi-cyclic representations of the Hopf algebra $U_q(sl(2))$ for $q$ a third root of unit and it shares some of the generic properties of factorizable $S$–matrices. In this paper, we generalize the results of [6] for $q^N = 1$, $N \geq 5$. The case $N = 4$ is special and is studied separately in [7]. Inspired by the chiral Potts model, we propose a solvable lattice model whose Boltzmann weights are identified with the $U_q(sl(2))$ intertwiners for semi-cyclic representations. Since we are working with $U_q(sl(2))$ and not its affine extension, and based on the very simple structure of the spectral manifold, we conjecture that the models we describe correspond to critical points.

The organization of the paper is as follows. In section 2 we review the chiral Potts model from the point of view of quantum groups. In section 3 we give the intertwiner for semi-cyclic representations of $U_q(sl(2))$ for $N \geq 3$ ($q^N = 1$). In section 4 we construct a solvable lattice model whose Boltzmann weights are given by the previous intertwiner. In section 5 we study the decomposition rules of tensor products of semi-cyclic representations, and finally in section 6 we consider the $N \to \infty$ limit of the intertwiners of semi-cyclic representations which leads to an infinite–dimensional representation of the braid group. This representation also satisfies the Turaev condition for defining an invariant of knots and links.

2. Chiral Potts as a model for affine $U_q(\hat{sl}(2))$ intertwiners

To define the $N$–chiral Potts model we assign to the sites of a square lattice two different kinds of state variables: a $Z_N$ variable $m (= 0, 1, \ldots, N - 1)$ and a neutral variable $\ast$. On
these variables one defines the action of the group $Z_N$ as $\sigma(m) = m + 1$ and $\sigma(*) = *$. The allowed configurations for two adjacent sites of the lattice are of the kind $(*, m)$, i.e. with one of the site variables the $Z_N$ neutral element *. The rapidities $p, q, \ldots$ are associated with each line of the dual lattice (figure 1).

Correspondingly, there are two types of Boltzmann weights represented graphically as

$$W_{pq}(m - n) =$$

$$\overline{W}_{pq}(m - n) =$$

The star–triangle relation of the model is

$$\sum_{d=1}^{N} \overline{W}_{qr}(b - d)W_{pr}(a - d)\overline{W}_{pq}(d - c) = R_{pqrs}W_{pq}(a - b)\overline{W}_{pr}(b - c)W_{qr}(a - c)$$

Representing the vector rapidity by $(a_p, b_p, c_p, d_p) \in C$, it is found that the star triangle relations of the model restrict the rapidities to lie on the intersection of two Fermat curves:

$$a_p^N + k'b_p^N = kd_p^N$$

$$k'a_p^N + b_p^N = k'e_p^N$$

$$k^2 + k'^2 = 1$$

In reference [8] the previous model has been interpreted as describing a scattering of kinks defined as follows. To each link of the lattice one associates a kink operator which
can be of the type $K_{*,n}(p)$ or $K_{n,*}(p)$. This kink operators can be interpreted, in the continuum, as configurations which interpolate between two extremal of some potential à la Landau–Ginzburg, and which move with a rapidity $p$.

The Boltzmann weights of the lattice model are used in this interpretation to define the $S$–matrix for these kinks:

$$
S |K_{n*}(q)K_{sm}(p)\rangle = W_{pq}(m-n) |K_{n*}(p)K_{sm}(q)\rangle \\
S |K_{sm}(q)K_{m*}(p)\rangle = \overline{W}_{pq}(m-n) |K_{sn}(p)K_{n*}(q)\rangle
$$

(4)

These models can be characterized very nicely in connection with the quantum affine extension of $\mathfrak{sl}(2)$, namely $U_\epsilon(\hat{\mathfrak{sl}}(2))$ with $\epsilon = e^{2\pi i/N}$. First of all, the finite–dimensional irreps of $U_\epsilon(\hat{\mathfrak{sl}}(2))$ are parametrized by the eigenvalues of the central subalgebra which is generated, in addition to the Casimir, by $E_i^{N'}, F_i^{N'}$ and $K_i^{N'}$, where $i = 0, 1$ and $N' = N$ if $N$ is odd, and $N' = N/2$ if $N$ is even. The representations where $x_i = E_i^{N'}, y_i = F_i^{N'}$ and $z_i = K_i^{N'}$ are all different from zero, have dimension $N'$ and are called cyclic [4] or periodic [9] representations. The intertwining condition for the tensor product of cyclic representations force these parameters $x_i, y_i, z_i$ to lie on an algebraic curve which factorizes into two copies of the curve (3). The corresponding intertwiner $R$–matrix admits a representation as the product of four Boltzmann weights of the chiral Potts model, actually two of them are chiral ($W$) and the other two antichiral ($\overline{W}$) (see figure 2 and reference [4] for details).

**Figure 1.2.** The $R$–matrix for $U_\epsilon(\hat{\mathfrak{sl}}(2))$ and its chiral Potts interpretation as a product of four Boltzmann weights.

The need of four Boltzmann weights can be intuitively understood comparing the structure of indices of a generic $R$–matrix $R_{r_1r_2}^{r_1'r_2'}(\xi_1, \xi_2)$ with the Boltzmann weights (1). In fact, using only four Boltzmann weights and therefore two independent rapidities one has enough degrees of freedom to match the indices of the affine $R$–matrix in terms of lattice variables. This is the formal reason for using the affine Hopf algebra to describe the chiral Potts model.

Returning to the kink interpretation, one is associating to each $N'$–irrep of $U_\epsilon(\hat{\mathfrak{sl}}(2))$ a two–kink state with two different rapidities.
3. Intertwiners for semi-cyclic irreps of $U_\epsilon(s\ell(2))$ with $\epsilon = e^{2\pi i/N}$, $N \geq 3$

In reference [6] the intertwiner for semi-cyclic representations in the case $N = 3$ was considered. We shall give now the result for $N \geq 3$. The case $N = 4$, i.e. $q^2 = 1$, has more structure and is analyzed separately [7].

Let us first fix notation, essentially as in [5]. We consider the Hopf algebra $U_\epsilon(s\ell(2))$ with $\epsilon = e^{2\pi i/N}$ (we use the letter $\epsilon$ instead of $q$ in order to distinguish the case of $q$ a root of unit), generated by $E$, $F$ and $K$ subject to the relations

\[
\begin{align*}
EF - e^2FE &= 1 - K^2 \\
KE &= e^{-2}EK \\
KF &= e^2FK
\end{align*}
\]

and co-multiplication

\[
\begin{align*}
\Delta E &= E \otimes 1 + K \otimes E \\
\Delta F &= F \otimes 1 + K \otimes F \\
\Delta K &= K \otimes K
\end{align*}
\]

Notice that we do not include in the commutator between $E$ and $F$ the usual denominator $1 - e^{-2}$.

When $\epsilon = e^{2\pi i/N}$ the central Hopf subalgebra $Z_\epsilon$ is generated by $x = E^N$, $z = F^N$ and $z = K^N$ where $N' = N$ ($N$ odd) or $N' = N/2$ ($N$ even). We shall be interested in the special class of representations for which $x = 0$ but $y$ and $z = \lambda^{N'} \neq \pm 1$ are arbitrary non-zero complex numbers. These are the so-called semi-cyclic or semi-periodic representations [6]. Denoting by $\xi$ the couple of values $(y, \lambda)$ which characterizes a semi-cyclic representation, then the problem is to find a matrix $R(\xi_1, \xi_2)$ which intertwines between the tensor products $\xi_1 \otimes \xi_2$ and $\xi_2 \otimes \xi_1$. Let $V_\xi$ be the representation space associated with the semi-cyclic representation $\xi$, which is spanned by a basis $\{e_r(\xi)\}_{r=0}^{N'-1}$, then the intertwiner $R$–matrix is an operator $R : V_{\xi_1} \otimes V_{\xi_2} \rightarrow V_{\xi_2} \otimes V_{\xi_1}$:

\[
R(\xi_1, \xi_2)e_{r_1}(\xi_1) \otimes e_{r_2}(\xi_2) = R_{r_1r'_2}^{r'_1r_2}(\xi_1) \otimes e_{r'_2}(\xi_1)
\]

which satisfies the equation

\[
R(\xi_1, \xi_2)\Delta_{\xi_1\xi_2}(g) = \Delta_{\xi_2\xi_1}(g)R(\xi_1, \xi_2) \quad [\forall g \in U_\epsilon(s\ell(2))]
\]

where $\Delta_{\xi_1\xi_2}(g)$ reflects the action of the quantum operator $g$ on $V_{\xi_1} \otimes V_{\xi_2}$:

\[
\Delta_{\xi_1\xi_2}(g)(e_{r_1}(\xi_1) \otimes e_{r_2}(\xi_2)) = \Delta_{\xi_1\xi_2}(g)r_{1r_2}^{r'_1r'}(\xi_1) \otimes e_{r'_2}(\xi_2)
\]
Equation (8) then reads explicitly as

\[ R^{s_1s_2}_{r_1r_2}(\xi_1, \xi_2) \Delta_{\xi_1\xi_2}(g) r^{r_1r_2}_{12} = \Delta_{\xi_2\xi_1}(g) \]  
\[ s_1s_2^{r_1r_2} R^{r_2r_1}_{12}(\xi_1, \xi_2) \]  

(10)

Hence our convention is that contracted indices are summed up in the SW–NE direction. An alternative way to write eq. (8) is in terms of the matrix \( R(\xi_1, \xi_2) = PR(\xi_1, \xi_2) \) where \( P : V_{\xi_1} \otimes V_{\xi_2} \to V_{\xi_2} \otimes V_{\xi_1} \) is the permutation map. Note that \( R^{r_1r_2}_{r_2r_1} = R^{r_2r_1}_{r_1r_2} \). Equation (8) reads then

\[ R(\xi_1, \xi_2) \Delta_{\xi_1\xi_2}(g) = (\sigma \circ \Delta_{\xi_1\xi_2}(g)) R(\xi_1, \xi_2) \]  

(11)

where \( \sigma \) is the permutation map \( \sigma(a \otimes b) = b \otimes a \) of the Hopf algebra. The universal form of (11) is in fact one of the defining relations of a quantum group, but as will be clear soon, we shall be working at the representation level, without assuming the existence of a universal \( R \)-matrix.

After these preliminaries, the first important result that one derives from (8), when applied to an element \( g \) belonging to the center \( Z_\epsilon \) of the algebra, is the following constraint on the possible values of \( \xi_1 \) and \( \xi_2 \):

\[ \frac{y_1}{1 - \lambda_1^{-N}} = \frac{y_2}{1 - \lambda_2^{-N}} = k \]  

(12)

with \( k \) an arbitrary complex number different from zero. When \( N \) is odd, an explicit form of a semi-cyclic representation satisfying \( y = k(1 - \lambda N) \) is given in the basis \( \{ e_r \}_{r=0}^{N-1} \) by

\[ F e_r = k^{r/N} (1 - \lambda^r)e_{r+1} \]
\[ E e_r = \frac{1}{k^{1/N}} [r] (1 + \lambda^{r-1})e_{r-1} \]
\[ K e_r = \lambda^{2r} e_r \]  

(13)

where \([r] = \frac{1 - \epsilon^{2r}}{1 - \epsilon^2}\) (14)

Notice that this representation is highest weight, with \( e_0 \) the highest weight vector.

* In reference [6] we used another basis \( \tilde{e}_r \) which is related to the one we use now by

\[ \tilde{e}_r = k^{r/N} \prod_{\ell=0}^{r-1} (1 - \lambda^\ell)e_r \]

where the generator \( F \) acts as \( F \tilde{e}_r = \tilde{e}_{r+1} \) for \( 0 \leq r \leq N - 2 \) and \( F \tilde{e}_{N-1} = k \tilde{e}_0 \). In going to the new basis \( e_r \), we want to exploit the cyclicity of the generator \( F \): in fact, we may identify \( e_r \) with \( e_{r+N} \).
It will be convenient for the rest of our computations to introduce the following numbers

\[
(\lambda)_{r_1, r_2} = \begin{cases} 
\prod_{\ell=1}^{r_2-1} (1 - \lambda \ell) & \text{if } r_1 < r_2 \\
\frac{r_1!}{r_2!} \prod_{\ell=r_2}^{r_1-1} (1 + \lambda \ell) & \text{if } r_1 > r_2 \\
1 & \text{if } r_1 = r_2 
\end{cases}
\] (15)

in terms of which

\[
F^n_{e_r} = (\lambda)_{r, r+n} e_{r+n} \\
E^n_{e_r} = (\lambda)_{r, r-n} e_{r-n}
\]

They satisfy the obvious relation

\[
(\lambda)_{r_1, r_2}(\lambda)_{r_2, r_3} = (\lambda)_{r_1, r_3} \quad r_1 \geq r_2 \geq r_3 \quad r_1 \leq r_2 \leq r_3
\]

and the less obvious one which reflects the quantum algebra (5):

\[
(\lambda)_{r, r+1}(\lambda)_{r+1, n} = e^{2(r+1-n)}(\lambda)_{r, n}(\lambda)_{n, n-1}(\lambda)_{n-1, n}
\]

\[
= \frac{[r - n + 1]}{[r + n + 1]} (\lambda)_{r, n}(\lambda)_{r+n+1}(\lambda)_{r+n, r+n+1}
\]

(0 \leq n \leq r) (18)

Using now the representation (13) we shall look for intertwiners $R(\xi_1, \xi_2)$ satisfying the Yang–Baxter equation

\[
(1 \otimes R(\xi_1, \xi_2))(R(\xi_1, \xi_3) \otimes 1)(1 \otimes R(\xi_2, \xi_3)) = (R(\xi_2, \xi_3) \otimes 1)(1 \otimes R(\xi_1, \xi_3))(R(\xi_1, \xi_2) \otimes 1)
\]

(19)

which guarantees a unique intertwiner between the representations $\xi_1 \otimes \xi_2 \otimes \xi_3$ and $\xi_3 \otimes \xi_2 \otimes \xi_1$.

In reference [6], it was found (when $N = 3$) that an intertwiner $R$ matrix satisfying the Yang–Baxter equation (19) exists and is unique, and that in addition it satisfies the following three properties:

i Normalization.

\[ R(\xi, \xi) = 1 \otimes 1 \] (20)

ii Unitarity.

\[ R(\xi_1, \xi_2)R(\xi_2, \xi_1) = 1 \otimes 1 \] (21)

iii Reflection symmetry

\[ R(\xi_1, \xi_2) = PR(\xi_2, \xi_1)P \] (22)

Among these properties, the last one is the most important and, as we shall see, it is the key to find solutions for $N \geq 3$ (the case $N = 4$, thus $N' = 2$, is singular in this crucial point, see [7]). Explicitly, condition (22) reads

\[ R_{r_1 r_2}^{r_1' r_2'}(\xi_1, \xi_2) = R_{r_2 r_1}^{r_2' r_1'}(\xi_2, \xi_1) \] (23)
The strategy for finding solutions to the Yang–Baxter equation (19) is to look for intertwiners satisfying (8) and having the reflection symmetry (22). We also normalize the $R$ matrix to be one when acting on the vector $e_0 \otimes e_0$ (i.e., $R^{00}_{00}(\xi_1, \xi_2) = 1$):

$$R(\xi_1, \xi_2)e_0(\xi_1) \otimes e_0(\xi_2) = e_0(\xi_2) \otimes e_0(\xi_1) = Pe_0(\xi_1) \otimes e_0(\xi_2)$$  (24)

The procedure we follow consists of finding, from the intertwiner condition (8) and the reflection symmetry (22), a set of recursive equations for the $R$–matrix which can be finally solved using the equation (24). Indeed, introducing (22) into (8), we obtain

$$R(\xi_1, \xi_2)P \Delta_{\xi_2\xi_1}(g)P = P \Delta_{\xi_1\xi_2}(g)PR(\xi_1, \xi_2)$$  (25)

Specializing (8) and (25) to the case $g = F$, we obtain a set of equations which can be solved to yield the following recursion formulae:

$$R(\xi_1, \xi_2)(F_1 \otimes 1) = [AR(\xi_1, \xi_2) - BR(\xi_1, \xi_2)(K_1 \otimes 1)] \frac{1}{1 - K_1 \otimes K_2}$$

$$R(\xi_1, \xi_2)(1 \otimes F_2) = [BR(\xi_1, \xi_2) - AR(\xi_1, \xi_2)(1 \otimes K_2)] \frac{1}{1 - K_1 \otimes K_2}$$  (26)

where $A$ and $B$ are two commuting operators given by

$$A = \Delta_{\xi_2\xi_1}(F) = F_2 \otimes 1 + K_2 \otimes F_1$$

$$B = (\sigma \circ \Delta)_{\xi_2\xi_1}(F) = F_2 \otimes K_1 + 1 \otimes F_1$$  (27)

The subindices in $F_1$, $F_2$, etc. are in fact unnecessary if we recall that the whole equation is defined acting on the space $V_{\xi_1} \otimes V_{\xi_2}$.

Iterating eqs. (26) we find

$$R(\xi_1, \xi_2)(F_1^{r_1} \otimes F_2^{r_2}) = \left( \sum_{s_1=0}^{r_1} (-1)^{s_1} s_1^{r_1-1} \begin{bmatrix} r_1 \\ s_1 \end{bmatrix} A^{r_1-s_1} B^{s_1} \right) \left( \sum_{s_2=0}^{r_2} (-1)^{s_2} s_2^{r_2-1} \begin{bmatrix} r_2 \\ s_2 \end{bmatrix} B^{r_2-s_2} A^{s_2} \right)$$

$$\times \frac{1}{\prod_{\ell=0}^{r_1+r_2-1} (1 - \varepsilon^2 K_1 \otimes K_2)}$$  (28)

The final result is obtained by applying the above equation to the vector $e_0(\xi_1) \otimes e_0(\xi_2)$ and making use of (24):

$$R(\xi_1, \xi_2)(F_1^{r_1} e_0(\xi_1) \otimes F_2^{r_2} e_0(\xi_2)) = \frac{1}{\prod_{\ell=0}^{r_1+r_2-1} (1 - \varepsilon^2 \lambda_1 \lambda_2)} \prod_{\ell_1=0}^{r_1-1} (A - \lambda_1 e^{2\ell_1} B) \prod_{\ell_2=0}^{r_2-1} (B - \lambda_2 e^{2\ell_2} A)e_0(\xi_2) \otimes e_0(\xi_1)$$  (29)
In deriving (29), we have used the Gauss binomial formula

\[
\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} z^\nu e^{\nu(\nu-1)} = \prod_{\nu=0}^{n-1} (1 - z e^{2\nu}) \quad (30)
\]

Letting \( r_1 \) or \( r_2 \) be \( N' \) in (29), one derives the intertwining conditions (12) which means that the spectral manifold for the solution (29) is given by the genus zero algebraic curve

\[
y = k (1 - \lambda^{N'}) \quad (31)
\]

The general solution (29) is valid for \( N \) odd or even and it coincides, up to a change of basis, with the one presented in [6] for the particular case \( \epsilon^3 = 1 \).

Expanding (29) one can find the entries \( R_{r,-n,n}^{r',0} (\xi_1, \xi_2) \) as functions of the spectral variables \( \xi_1 \) and \( \xi_2 \). Some of them are easy to compute since they are products of monomials, for example

\[
R_{r,0}^{r,-n,n} (\xi_1, \xi_2) = \frac{(\lambda_1)_r (\lambda_2)_0 (r-n)}{[r-n]!} \prod_{\ell=0}^{n-1} (\lambda_2 - \lambda_1 \epsilon^{2\ell}) \prod_{\ell=0}^{r-1} (1 - \lambda_1 \lambda_2 \epsilon^{2\ell}) \quad (32)
\]

but in general they have a polynomial structure as in

\[
R_{11}^{11} (\xi_1, \xi_2) = 1 - [2] \frac{(\lambda_1 - \lambda_2)^2}{(1 - \lambda_1 \lambda_2)(1 - \epsilon^2 \lambda_2 \lambda_2)} \quad (33)
\]

We shall give a general formula of \( R_{r_1,r_2}^{r_1',r_2'} \) when we consider the limit \( N \to \infty \).

Another property of the \( R \) matrix is the conservation of the quantum number \( r \) modulo \( N' \), i.e.

\[
R_{r_1,r_2}^{r_1',r_2'} = 0 \quad \text{unless} \quad r_1 + r_2 = r_1' + r_2' \mod N' \quad (34)
\]

The previous construction reveals that the intertwining condition, when supplemented with the reflection relation (22), are enough data to produce solutions to the Yang–Baxter equation. The general solution also satisfies the normalization (20) and unitarity conditions (21).

4. A solvable lattice model for \( U_\epsilon(s\ell(2)) \) intertwiners

Our next task will be to define a lattice model whose Boltzmann weights admit a direct interpretation in terms of intertwiners for semi-cyclic representations. For these models, the \( R \)–matrix given in (29) is the solution to the associated star–triangle relation. To define the model we associate to each site of the lattice a \( Z_N \) variable \( m \) such that \( \sigma(m) = m + 1 \) with
\[ W_{\xi_1 \xi_2}(m_1, m_2, m_3, m_4) = R_{r_1 r_2}^{r_1' r_2'}(\xi_1, \xi_2) = \]

\[ r_1 = m_1 - m_2, \quad r_2 = m_2 - m_3, \quad r_1' = m_1 - m_4, \quad r_2' = m_4 - m_3 \]

The \( R \)-matrix in (35) is the intertwiner for two semi-cyclic representations \( \xi_1, \xi_2 \). The rapidities are now two-vectors \( \xi = (y, \lambda) \) and they are forced to live on the algebraic curve (31). The definition (35) is manifestly \( Z_N \) invariant: \( m_i \to \sigma(m_i) \). From (35) it is also easy to see that the star–triangle relation for the \( W \)'s becomes the Yang–Baxter equation for the \( R \)'s.

The equivalent here to the kink interpretation can be easily done substituting the kinks by some solitonic–like structure. In fact, if we interpret the lattice variables \( m \) as labelling different vacua connected by \( Z_N \) transformations, we can interpret in the continuum each link as representing an interpolating configuration between two different vacua. \( e_r \) is identified with a soliton configuration connecting the two vacua \( m \) and \( n \) with \( r = m - n \). In these conditions the Boltzmann weights (35) become the scattering \( S \)-matrix for two of these configurations with rapidities \( \xi_1 \) and \( \xi_2 \) (see fig. 3).

**Figure 1.3.** \( S \)-matrix interpretation of the \( U_q(\mathfrak{sl}(2)) \) \( R \)-matrix (29).

Notice that the conditions we have used in the previous section for solving the Yang–Baxter equation are very natural in this solitonic interpretation. In fact, the reflection relations (22) simply mean that the scattering is invariant under parity transformations. The condition \( R_{00}^{00} = 1 \) can now be interpreted as some kind of vacuum stability. Notice
that this condition is very dependent on the highest weight vector nature of semi-cyclic representations. Moreover, the normalization condition (20) and the unitarity property (21) strongly support the $S$–matrix interpretation. The main difference between the model (35) and chiral Potts is that for (35) we allow for all sites in the same plaquette arbitrary $Z_N$ variables. In this way we lose the chiral difference between vertical and horizontal interactions ($W, \tilde{W}$). What we gain is the possibility to connect the $U_c(s\ell(2))$ intertwiners with Boltzmann weights without passing through the affine extension.

5. Decomposition rules for semi-cyclic representations: the bootstrap property

One of the main ingredients used in ref. [4] to obtain solutions of the Yang–Baxter equation was the fact that generic cyclic representations of $U_c(s\ell(2))$ are indecomposable. This is not the case for semi-cyclic representations of $U_c(s\ell(2))$ [6, 9]. The decomposition rules are given by

$$(\lambda_1, y_1) \otimes (\lambda_2, y_2) = \bigoplus_{\ell=0}^{N'-1} (\epsilon^{2\ell} \lambda_1 \lambda_2, y_1 + \lambda_1^{N'} y_2)$$

(36)

If we consider the tensor product in the reverse order,

$$(\lambda_2, y_2) \otimes (\lambda_1, y_1) = \bigoplus_{\ell=0}^{N'-1} (\epsilon^{2\ell} \lambda_1 \lambda_2, y_2 + \lambda_2^{N'} y_1)$$

(37)

we deduce that the irreps appearing in $\xi_1 \otimes \xi_2$ and $\xi_2 \otimes \xi_1$ are the same provided the spectral condition (12) is satisfied. This is of course related to the existence of an intertwiner between the two tensor products. We also observe that the tensor product of irreps on the algebraic variety (31) decomposes into irreps belonging to the same variety. To obtain these rules, we simply need to use the co-multiplication laws (6) and the relation

$$\Delta_{\xi_1 \xi_2}(g) K_{\xi_1 \xi_2}^{\xi} = K_{\xi_1 \xi_2}^{\xi} \rho_{\xi}(g) \quad \forall g \in U_c(s\ell(2))$$

(38)

with $K_{\xi_1 \xi_2}^{\xi}$ the Clebsch–Gordan projector $K_{\xi_1 \xi_2}^{\xi} : V_{\xi} \rightarrow V_{\xi_1} \otimes V_{\xi_2}$ which exists whenever $\xi \subset \xi_1 \otimes \xi_2$. Writing now

$$e_r(\xi) = K_{\xi_1 \xi_2}^{r_1 r_2, r_1}(\xi_1) \otimes e_{r_2}(\xi_2)$$

(39)

we obtain for the CG coefficients $K_{\xi_1 \xi_2}^{r_1 r_2, r_1}(\xi)$, which give the highest weight vector $e_0(\xi)$ of the irrep $\xi(\ell) = (2\ell \lambda_1 \lambda_2, y_{12})$ in the semi-cyclic basis (13), the following expression with $\ell = r_1 + r_2$:

$$K_{\xi_1 \xi_2}^{r_1 r_2, \ell}(\xi) = (-1)^{r_1} e_{r_1}(r_1-1) \left[ \frac{\ell}{r_1} \right] \lambda_1^{r_1} \prod_{r_1} (1 + \lambda_1 \epsilon^\nu) \prod_{r_2} (1 - \lambda_2 \epsilon^\nu)$$

(40)

$$= (-1)^{r_1} \frac{1}{[\ell]!} e_{r_1}(r_1-1) \lambda_1^{r_1} (\lambda_1)_{\ell, r_1} (\lambda_2)_{\ell, r_2}$$

(41)
Coming back to the $S$–matrix picture the decomposition rules (36) should be interpreted as reflecting some kind of bootstrap property. This naive interpretation is not quite correct. In fact, the decomposition rule (36) implies a strictly quantum composition of the rapidities determined by the Hopf algebra structure of the center $Z_\epsilon$ of $U_\epsilon(s\ell(2))$ at $\epsilon$ a root of unit. This is a new phenomenon derived from the fact that we are considering something that looks like a factorized $S$–matrix for particles, but where the kinematical properties of these particles, the rapidities, are quantum group eigenvalues and therefore their composition rules are not classical. Moreover, from the explicit expression of the Clebsch–Gordan (40) we observe another interesting phenomenon of mixing between what we should consider internal quantum numbers, those labelling the basis for the representation (i.e., the $r$’s) with the kinematical ones, i.e. the ones labelling the irreps (see fig. 4). If this physical picture is correct this is the first case where the quantum group appears not only at the level of internal symmetries but also determines the kinematics.

Figure 1.4. Decomposition rules derived from the Clebsch–Gordan coefficients (40) with the quantum group decomposition of rapidities.

From a strictly quantum group point of view we can, using the intertwiner solution (29) and the CG coefficients (40), check some of the standard results in representation theory of quantum groups. As an interesting example we consider the relation

$$R(\xi_1, \xi_2)K_\xi^{\xi_1\xi_2} = \phi(\xi_1, \xi_2, \xi)K_\xi^{\xi_2\xi_1}$$

(41)

which is true for regular representations of spin $j$ with $\phi(j_1, j_2, j) = (-1)^{j_1+j_2-j}\epsilon^{C_j-C_{j_1}-C_{j_2}}$ and $C_j = j(j+1)$ the classical Casimir. For semi-cyclic representations and the case $N = 3$ we have found that the factors $\phi(\xi_1, \xi_2, \xi)$ in (41) are equal to one, and we presume that this fact persists for all $N$.

6. The limit $N \to \infty$

In this section we shall study the limit $N \to \infty$ of the $R$–matrix found in section 3. The quantum deformation parameter $\epsilon$ goes in this limit to 1, so one could expect that the Hopf
algebra $U_\epsilon(sl(2))$ becomes the classical universal enveloping algebra of $sl(2)$. This is not however what is happening here as can be seen by taking $\epsilon \to 1$ in eqs. (5):

$$[E, F] = 1 - K^2 \quad [K, E] = [K, F] = 0$$

(42)

Let us recall that in the quantum group relations (5) we have not included the denominator $1 - \epsilon^{-2}$, hence there is no need to apply the L'Hôpital rule. What we obtain rather in the limit $N \to \infty$ is the Heisenberg algebra of a harmonic oscillator. Indeed, defining $a$ and $a^\dagger$ as

$$a = \frac{1}{1 + K} E \quad a^\dagger = \frac{1}{1 - K} F$$

(43)

we find that (42) amount to

$$[a, a^\dagger] = 1 \quad [K, a] = [K, a^\dagger] = 0$$

(44)

The role of the operator $K$ is therefore to produce non–trivial co-multiplications preserving the algebra (42) or (44), and this is why we may have non–trivial $R$–matrices. The representation spaces of the algebra (42) are now infinite–dimensional and are labelled by the value of $K$; we shall call them $\mathcal{H}_\lambda$. In a basis $\{e_r\}_{r=0}^\infty$ of $\mathcal{H}_\lambda$ we have

$$Fe_r = (1 - \lambda)e_{r+1}$$
$$Ee_r = r(1 + \lambda)e_{r-1}$$
$$Ke_r = \lambda e_r$$

(45)

Hence we see that $e_r$ can be identified with the $r$–th level of a harmonic oscillator. Of course the value $\lambda = 1$ in (45) has to be treated with care.

The $R$–matrix is now an operator $R(\lambda_1, \lambda_2) : \mathcal{H}_{\lambda_1} \otimes \mathcal{H}_{\lambda_2} \to \mathcal{H}_{\lambda_2} \otimes \mathcal{H}_{\lambda_1}$ which in the basis (45) has the following non–vanishing entries:

$$R_{r_1, r_2}^{r_1' + r_2' - \ell, \ell}(\lambda_1, \lambda_2) = \frac{1}{(1 - \lambda_1 \lambda_2)^{r_1 + r_2}} \sum_{\ell_1 + \ell_2 = \ell} \left( \begin{array}{c} r_1' \\ \ell_1 \end{array} \right) \left( \begin{array}{c} r_2' \\ \ell_2 \end{array} \right) [(1 + \lambda_1)(1 - \lambda_2)]^{r_1 - \ell_1} [(1 - \lambda_1)(1 + \lambda_2)]^{\ell_2} (\lambda_2 - \lambda_1)^{\ell_1} (\lambda_1 - \lambda_2)^{r_2 - \ell_2}$$

(46)

where $0 \leq \ell \leq r_1 + r_2$. This expression has been obtained from (29) taking the limit $\epsilon \to 1$ and using (45).

A first observation is that $R(\lambda_1, \lambda_2)$ depends only on the following harmonic ratio

$$\eta_{12} = \frac{z_{12} z_{34}^*}{z_{13} z_{24}^*} = 2 \frac{\lambda_1 - \lambda_2}{(1 + \lambda_1)(1 - \lambda_2)}$$

(47)
which corresponds to a sphere with 4 punctures at the points \( z_1 = \lambda_1, z_2 = \lambda_2, z_3 = -1 \) and \( z_4 = 1 \). In these new variables we have

\[
R^{r_1+r_2-\ell,\ell}(\eta_{12}) = \frac{1}{(1-\eta_{12}/2)^{r_1+r_2}} \sum_{\ell_1+\ell_2=\ell} (-1)^{\ell_1} \binom{r_1}{\ell_1} \binom{r_2}{\ell_2} (\eta_{12}/2)^{r_2-\ell_2+\ell_1}(1-\eta_{12})^{\ell_2}
\]

(48)

A reflection transformation \( \eta_{12} \rightarrow \frac{\eta_{12}}{\eta_{12}^{-1}} \) corresponds to a Möbius transformation.

It is however more convenient to use another variable \( u \) defined as

\[
u_{12} = \frac{\lambda_1 - \lambda_2}{1 - \lambda_1 \lambda_2} = \frac{\eta_{12}}{2 - \eta_{12}}
\]

(49)

which changes sign under a reflection symmetry. The \( R \)-matrix reads in the \( u \)-variable as

\[
R^{r_1+r_2-\ell,\ell}(u_{12}) = (1 + u_{12})^{r_1}u_{12}^{r_2} \sum_{\ell_1+\ell_2=\ell} (-1)^{\ell_1} \binom{r_1}{\ell_1} \binom{r_2}{\ell_2} \left( \frac{u_{12}}{1+u_{12}} \right)^{\ell_1} \left( \frac{1-u_{12}}{u_{12}} \right)^{\ell_2}
\]

(50)

Recalling the definition of the Jacobi polynomials \( P_{n}^{(\alpha,\beta)}(x) \) \((n = 0, 1, \ldots)\):

\[
P_{n}^{(\alpha,\beta)}(x) = \frac{1}{2^n} \sum_{m=0}^{n} \binom{n + \alpha}{m} \binom{n + \beta}{n-m} (x-1)^{n-m}(x+1)^{m}
\]

(51)

we see finally that \( R(u) \) can be written in the form

\[
R^{r_1+r_2-\ell,\ell}(u) = (1 + u)^{r_1-\ell}u^{r_2-\ell}P_{\ell}^{(r_2-\ell,r_1-\ell)}(1 - 2u^2)
\]

(52)

As an application of this formula we may derive the classical limit of eq. (32) knowing that

\[
P_{\ell}^{(-\ell,\ell)}(x) = \frac{(-1)^{\ell}}{2^{\ell}} \binom{r}{\ell} (1-x)^{\ell}
\]

(53)

The reflection symmetry of \( R \)

\[
R^{r_1+r_2-\ell,\ell}(u) = R_{r_2,r_1}^{\ell,r_2-\ell}(-u)
\]

(54)

implies the following identity between Jacobi polynomials:

\[
P_{n+\alpha+n}^{(-\alpha,-\beta)}(x) = \binom{x-1}{2}^{\alpha} \binom{x+1}{2}^{\beta} P_{n}^{(\alpha,\beta)}(x)
\]

(55)

whenever \( n + \alpha, n + \beta \) and \( n + \alpha + \beta \) are all non-negative integers. The Yang–Baxter equation of \( R(u) \) implies a cubic equation for the Jacobi polynomials that we shall not write down.

The fact that the \( R \)-matrix can be written entirely as a function of a single variable \( u \) has some interesting consequences. First of all, we see from the definition of \( u_{12} \) that
interpreting \( \lambda_i \) as the velocity of the \( i \)-th soliton, then \( u_{ij} \) is nothing but the relative speed of the \( i \)-th soliton with respect to the \( j \)-th soliton in a \( 1+1 \) relativistic world. Moreover, the YB equation (19) when written in the \( u \) variables has a relativistic look:

\[
(1 \otimes R(u)) \left( R \left( \frac{u + v}{1 + uv} \right) \otimes 1 \right) (1 \otimes R(v)) = (R(v) \otimes 1) \left( 1 \otimes R \left( \frac{u + v}{1 + uv} \right) \right) (R(u) \otimes 1)
\]

(56)

and in particular the scattering matrices \( R(u) \) of these relativistic solitons give us a representation of the braid group provided

\[
u = \frac{u + v}{1 + uv}
\]

(57)
a situation which happens when \( u = 0, \pm 1 \). The case \( u = 0 \) is trivial, since \( R(u = 0) = 1 \), but the other two yield the following braiding matrices:

\[
R^{r'_{1}r'_{2}}_{r_{1}r_{2}}(+) = \lim_{u \to 1} R^{r'_{1}r'_{2}}_{r_{1}r_{2}}(u) = \delta_{r_{1}+r_{2}} \delta_{r'_{1}+r'_{2}} (-1)^{r'_{2}r_{1}-r_{1}r'_{2}} (r_{1} \atop r'_{2})
\]

\[
R^{r'_{1}r'_{2}}_{r_{1}r_{2}}(-) = \lim_{u \to -1} R^{r'_{1}r'_{2}}_{r_{1}r_{2}}(u) = \delta_{r_{1}+r_{2}} \delta_{r'_{1}+r'_{2}} (-1)^{r'_{2}r_{1}-r_{1}r'_{2}} (r_{1} \atop r'_{2})
\]

(58)

which satisfy, in addition to the YB equation, the relation

\[
R(+)R(-) = 1
\]

(59)

which is a consequence of the unitarity condition (21).

This means that we have obtained an infinite–dimensional representation \( \pi \) of the braid group \( B_n \) given by

\[
\pi: B_n \to \text{End}(\mathcal{H}^{\otimes n})
\]

\[
\sigma_{i}^{\pm 1} \mapsto 1 \otimes \cdots \otimes R_{i,i+1}(\pm) \otimes \cdots 1
\]

(60)

where \( \mathcal{H} \) is isomorphic to the Hilbert space of a harmonic oscillator. In fact, using the universal matrix \( \mathcal{R} = PR \in \text{End}(\mathcal{H}^{\otimes 2}) \) we can write eqs. (58) as

\[
\mathcal{R}(+) = (e^{i\pi N} \otimes 1)e^{2a^\dagger a}\mathcal{R}(+) = (1 \otimes e^{i\pi N})e^{2a^\dagger a}
\]

(61)

where \( N = a^\dagger a \) is the number operator and

\[
a^\dagger e_r = e_{r+1} \quad ae_r = re_{r-1}
\]

(62)

The Yang–Baxter relation (56) in the limit \( u \to \pm 1 \) can be most easily proved in terms of the YB solution for the \( \mathcal{R} \) matrix which reads

\[
\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}
\]

(63)
where $R_{12} = (e^{i\pi N} \otimes 1 \otimes 1)e^{2a \otimes a^\dagger \otimes 1}$ etc.

We have thus seen that the limit $N \to \infty$ of our construction is well defined, and has some interesting structure. It describes essentially the scattering of relativistic solitons whose spectrum is that of a harmonic oscillator. Interestingly enough, in the limit of very deep scattering one obtains an infinite-dimensional representation of the braid group. One may wonder whether this representation provides us with new invariants of knots and links, just as the usual finite-dimensional $R$ matrices from quantum groups do.

With this in mind, we shall propose a slight generalization of the $R$ matrices (61) given by

$$R(x, y; +) = (x^N \otimes y^{-N})e^{(y-x)a \otimes a^\dagger}$$

$$R(x, y; -) = e^{(x-y)a^\dagger \otimes a}(y^N \otimes x^{-N})$$

where $x$ and $y$ are two independent complex numbers. It is easy to see that these new $R$ matrices also satisfy the YB equation, yielding a representation $\pi_{x,y}: B_n \to End(\mathcal{H} \otimes n)$ of the braid group. The previous case is recovered with $x = -1, y = 1$. The braiding matrices that follow from (64) are

$$R_{r_1 r_2}^{r_1' r_2'}(x, y) = \delta_{r_1+r_2, r_1'+r_2'}(r_1 y - r_2 x)^{r_1-r_1'}r_2' y^{r_2-r_2'}$$

$$R_{r_1 r_2}^{-1}^{r_1' r_2'}(x, y) = \delta_{r_1+r_2, r_1'+r_2'}(r_2 y - r_1 x)^{r_2-r_2'}y^{r_1-r_1'}$$

This braid group representation admits an extension à la Turaev [10], i.e. there exists an isomorphism $\mu: \mathcal{H} \to \mathcal{H}$ satisfying the following three conditions:

$$i) \quad (\mu_i \mu_j - \mu_k \mu_\ell) R_{ij}^{k\ell} = 0$$

$$ii) \quad \sum_j R_{ij}^{kj} \mu_j = \delta_i^k ab$$

$$iii) \quad \sum_j R_{ij}^{-1} R_{ij}^{kj} \mu_j = \delta_i^k a^{-1}b$$

for some constants $a$ and $b$. For the $R$ matrix (65), the Turaev conditions hold if

$$\mu = 1 \quad a = b^{-1} = \sqrt{(y/x)}$$

The invariant of knots and links that one would get is thus

$$T_{x,y}(\alpha) = (x/y)^{\frac{1}{2}|w(\alpha)-n|} \text{tr} \pi_{x,y}(\alpha)$$

where $\alpha \in B_n$ and $w(\alpha)$ is the writhe of $\alpha$. The trace in (68) is defined on the $n$-th tensor product of Hilbert spaces, therefore to make sense of $T_{x,y}(\alpha)$ one should regularize this trace.
without losing the invariance under the Markov moves. We leave the identification and proper definition of the invariant (68) for a future publication.

7. Final Comments

A possible framework where to study in more detail the physical meaning of our results could be the one recently developed by Zamolodchikov [11] in connection with the analysis of integrable deformations of conformal field theories. The moral we obtain from our analysis is that a unique mathematical structure, the quantum group, can describe at the same time conformal field theories [12] and integrable models. The dynamics that fixes what of the two kinds of physical systems is described is the way the central subalgebra \( Z_q \), for \( q \) a root of unit, is realized. In the conformal case, the central subalgebra is realized trivially with vanishing eigenvalues which correspond to the regular representations. The quantum group symmetry is defined in this case by the Hopf algebra quotiented by its central subalgebra. From the results in [3, 4] and the ones described here it seems that when the center is realized in a non–trivial way the system we describe is an integrable model. The star–triangle solution (29) for the lattice model we have defined in section 4 has good chances of describing a self–dual point. The heuristic reason for this is that the algebraic curve (31) on which the spectral parameters live is of genus zero.

We want to mention also the possible physical implications in this context of the recent mathematical results of reference [5]. In fact, these authors have defined a quantum co-adjoint action on the space of irreps. This action divides the finite–dimensional irreps into orbits each one containing a semi-cyclic representation. The co-adjoint action does not preserve the spectral manifold (31) but if we maintain the interpretation of the irrep labels as rapidities then it acts on them, opening in this way the door to new kinematics completely based on quantum group properties.

Finally, we summarize the non–trivial results we have obtained in the limit \( N \to \infty \). First of all, the intertwiner matrix for semi-cyclic representations can be interpreted as describing the scattering of solitons in a 1 + 1 relativistic world. The Hilbert space of these solitons is isomorphic to that of the harmonic oscillator. Moreover, in the limit when the solitons become relativistic, the scattering matrices provide us with an infinite–dimensional representation of the braid group. This representation seems to have a Markov trace which would allow us to find an invariant of links and knots.

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