The 6-Functor Formalism for $\mathbb{Z}_\ell$- and $\mathbb{Q}_\ell$-Sheaves on Diamonds

Lucas Mann

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For every nuclear $\mathbb{Z}_\ell$-algebra $\Lambda$ and every small $v$-stack $X$ we construct an $\infty$-category $\mathcal{D}_{\text{nuc}}(X, \Lambda)$ of nuclear $\Lambda$-modules on $X$. We then construct a full 6-functor formalism for these sheaves, generalizing the étale 6-functor formalism for $\Lambda = \mathbb{F}_\ell$. Prominent choices for $\Lambda$ are $\mathbb{Z}_\ell$, $\mathbb{Q}_\ell$ and $\mathbb{O}_\ell$ and especially in the latter two cases, no satisfying 6-functor formalism has been found before. Applied to classifying stacks we obtain a theory of nuclear representations, i.e. continuous representations on filtered colimits of Banach spaces.

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1 Introduction

Fix two primes $\ell \neq p$. In this paper we construct and study a full 6-functor formalism for certain sheaves of modules over nuclear $\mathbb{Z}_\ell$-algebras $\Lambda$ like $\mathbb{Z}_\ell$, $\mathbb{F}_\ell$, $\mathbb{Q}_\ell$, $\mathbb{Q}_\ell^\times$ or $\mathbb{C}_\ell$. The main difference to previous attempts at such a 6-functor formalism is that we never impose any finiteness conditions or pass to any isogeny categories – everything is completely natural and embeds fully faithfully into the category of pro-étale sheaves (even on the derived level). For simplicity this whole paper will work with diamonds and small v-stacks over $\mathbb{Z}_p$ [15], i.e. in the realm of rigid-analytic geometry. We expect that a similar 6-functor formalism can be constructed on schemes, albeit somewhat harder to implement.

Recall that for torsion coefficients, e.g. $\Lambda = \mathbb{F}_\ell$, a full 6-functor formalism for étale sheaves on diamonds has been worked out in [15] and has been used extensively in [3] to gain new insights into the $\ell$-adic geometric Langlands program. The present paper generalizes this étale 6-functor formalism to non-discrete nuclear $\mathbb{Z}_\ell$-algebras like the ones mentioned above. Note that even in the case that $\Lambda = \mathbb{Z}_\ell$ we allow non-complete $\mathbb{Z}_\ell$-sheaves in our 6-functor formalism (unlike the somewhat ad-hoc definition in [15, §26]), which becomes especially important when working with non-qcqs maps, see Example 1.4 below.

Without further ado, let us state the main definitions and results. In the following, we say that a spatial diamond (e.g. qcqs rigid-analytic variety) is $\ell$-bounded if it has finite cohomological dimension for étale $\mathbb{F}_\ell$-sheaves. We can then define the following category of nuclear sheaves:

**Definition 1.1** (cf. Definition 3.1.(b)). Let $X$ be a small v-stack. A nuclear $\mathbb{Z}_\ell$-sheaf on $X$ is a (derived) v-sheaf $M \in D(X_v, \mathbb{Z}_\ell)$ with the following property: For every map $f: Y \to X$ from an $\ell$-bounded spatial diamond $Y$, the sheaf $f^* M \in D(Y_v, \mathbb{Z}_\ell)$ is a filtered colimit of $\ell$-adically complete sheaves, all of which are étale modulo $\ell$. We denote by

$$D_{nuc}(X, \mathbb{Z}_\ell) \subseteq D(X_v, \mathbb{Z}_\ell)$$

the full subcategory of nuclear $\mathbb{Z}_\ell$-sheaves. For any nuclear $\mathbb{Z}_\ell$-algebra $\Lambda$ we denote by $D_{nuc}(X, \Lambda) \subseteq D(X_v, \Lambda)$ the full subcategory of those $\Lambda$-sheaves whose underlying $\mathbb{Z}_\ell$-sheaf is nuclear.

If $X$ is a geometric point then $D_{nuc}(X, \mathbb{Z}_\ell) = D_{nuc}(\mathbb{Z}_\ell) \subseteq D_c(\mathbb{Z}_\ell)$ recovers the $\infty$-category of nuclear $\mathbb{Z}_\ell$-modules as defined in [1, §8] and [16, Definition 13.10]. In particular, a nuclear $\mathbb{Z}_\ell$-algebra is a condensed $\mathbb{Z}_\ell$-algebra whose underlying $\mathbb{Z}_\ell$-module is a filtered colimit of Banach $\mathbb{Z}_\ell$-algebras. If $\Lambda$ is discrete then $D_{nuc}(X, \Lambda) = D_{et}(X, \Lambda)$ (see Proposition 3.20), so in this case we recover the classical étale theory.

Perhaps somewhat surprisingly, the $\infty$-category of nuclear sheaves satisfies v-descent. This observation lies at the heart of this paper.

**Theorem 1.2** (cf. Theorem 3.9.(ii) and Proposition 3.16.(i)). The $\infty$-category of nuclear $\Lambda$-modules is stable under pullback and satisfies v-descent. More precisely, the assignment

$$X \mapsto D_{nuc}(X, \Lambda)$$

defines a hypercomplete sheaf of $\infty$-categories on the v-site of all small v-stacks.

Let us now come to the 6-functor formalism for nuclear $\Lambda$-modules. The six functors can roughly be constructed as follows. Fix a map $f: Y \to X$ of small v-stacks; then:
1. The $\infty$-category $\mathcal{D}_{\text{nuc}}(X, \Lambda)$ comes naturally equipped with a symmetric monoidal structure

$$ - \otimes - : \mathcal{D}_{\text{nuc}}(X, \Lambda) \times \mathcal{D}_{\text{nuc}}(X, \Lambda) \to \mathcal{D}_{\text{nuc}}(X, \Lambda), $$

which agrees with the solid tensor product from [3, Proposition VII.2.2] (note that clearly $\mathcal{D}_{\text{nuc}}(X, \Lambda) \subseteq \mathcal{D}_\square(X, \Lambda)$).

2. Since $\mathcal{D}_{\text{nuc}}(X, \Lambda)$ is presentable, the symmetric monoidal structure is closed and hence allows an internal hom

$$ \text{Hom}(-, -) : \mathcal{D}_{\text{nuc}}(X, \Lambda)^{\text{op}} \times \mathcal{D}_{\text{nuc}}(X, \Lambda) \to \mathcal{D}_{\text{nuc}}(X, \Lambda) $$

given as the right adjoint of $\otimes$.

3. The pullback functor

$$ f^* : \mathcal{D}_{\text{nuc}}(X, \Lambda) \to \mathcal{D}_{\text{nuc}}(Y, \Lambda) $$

is simply the pullback of v-sheaves (it preserves nuclear sheaves by definition).

4. The pushforward functor

$$ f_* : \mathcal{D}_{\text{nuc}}(Y, \Lambda) \to \mathcal{D}_{\text{nuc}}(X, \Lambda) $$

is the right adjoint of the pullback functor. In general it does not agree with the v-pushforward, but if $f$ is qcqs then this is true under mild assumptions (see Section 4).

5. If $f$ is “\(\ell\)-fine” (see Definition 5.8) then we can construct a lower shriek functor

$$ f_! : \mathcal{D}_{\text{nuc}}(Y, \Lambda) \to \mathcal{D}_{\text{nuc}}(X, \Lambda) $$

as follows. If $f$ is proper then $f_! = f_*$. If $f$ is étale then $f_!$ is the left adjoint of $f^*$ (this agrees with the functor $f_!$ on v-sheaves, see Lemma 5.2). If $f$ is compactifiable and qcqs then we can define $f_!$ by composing the previous two examples along a relative compactification of $f$. If $f$ is only locally compactifiable then we can filter $Y$ by qcqs open subsets on which $f$ becomes compactifiable and then define $f_!$ as the colimit over this filtration. Finally, one can extend the construction of $f_!$ even to certain “stacky” maps. For more details on the construction of $f_!$ see Remark 5.13.

6. If $f$ is $\ell$-fine then we define

$$ f^! : \mathcal{D}_{\text{nuc}}(X, \Lambda) \to \mathcal{D}_{\text{nuc}}(Y, \Lambda) $$

as the right adjoint of $f_!$.

**Theorem 1.3** (cf. Theorem 5.11). The above six functors $\otimes$, $\text{Hom}$, $f^*$, $f_*$, $f_!$ and $f^!$ provide a 6-functor formalism. In particular, they satisfy the following properties:

(i) (Functoriality) For composable maps $f, g$ of small v-stacks we have natural isomorphisms

$$(f \circ g)^* = g^* \circ f^*$$

and

$$(f \circ g)_* = f_* \circ g_*.$$

If $f$ and $g$ are $\ell$-fine then also $(f \circ g)_! = f_! \circ g_!$ and $(f \circ g)^! = g^! \circ f^!$. 

(ii) (Special Cases) If \( j: U \to X \) is an étale map of small \( v \)-stacks then \( j^! = j^* \). If \( f: Y \to X \) is a proper \( \ell \)-fine map of small \( v \)-stacks then \( f_! = f_* \).

(iii) (Projection Formula) Let \( f: Y \to X \) be an \( \ell \)-fine map of small \( v \)-stacks. Then for all \( \mathcal{M} \in \mathcal{D}_{\text{nuc}}(X, \Lambda) \) and \( \mathcal{N} \in \mathcal{D}_{\text{nuc}}(Y, \Lambda) \) there is a natural isomorphism

\[
f_!(\mathcal{N} \otimes f^* \mathcal{M}) = (f_! \mathcal{N}) \otimes \mathcal{M}.
\]

(iv) (Proper Base-Change) Let

\[
\begin{align*}
Y' & \xrightarrow{g'} Y \\
X' & \xrightarrow{g} X
\end{align*}
\]

be a cartesian diagram of small \( v \)-stacks such that \( f \) is \( \ell \)-fine. Then there is a natural equivalence

\[
g^* f_! = f'_! g^*
\]

of functors \( \mathcal{D}_{\text{nuc}}(Y, \Lambda) \to \mathcal{D}_{\text{nuc}}(X', \Lambda) \).

Before we continue, let us briefly mention how the just constructed 6-functor formalism behaves in the classical rigid-analytic world.

**Example 1.4.** Suppose \( f: Y \to X \) is a map of rigid-analytic varieties over some fixed non-archimedean base field \( K \). Then \( f \) is \( \ell \)-fine. The pushforward functor \( f_*: \mathcal{D}_{\text{nuc}}(Y, \Lambda) \to \mathcal{D}_{\text{nuc}}(X, \Lambda) \) computes the nuclearized derived pushforward of pro-étale \( \Lambda \)-sheaves; in particular if \( X = \text{Spa} K \) then \( f_* \) computes nuclearized pro-étale cohomology. If \( f \) is qcqs then \( f_* \) preserves all small colimits, so for example we have \( f_* \mathbb{Q}_\ell = (\lim_n f_* (\mathbb{Z}/\ell^n \mathbb{Z}))[1/\ell] \); this is not true if \( f \) is not qcqs.

With \( f \) as before, let us explain how to compute \( f_! \): \( \mathcal{D}_{\text{nuc}}(Y, \Lambda) \to \mathcal{D}_{\text{nuc}}(X, \Lambda) \). By the projection formula \( f_! \) commutes with a change of coefficients, e.g. we have \( f_! \mathbb{Q}_\ell = (f'_! \mathbb{Z}_\ell)[1/\ell] \). This often reduces the computation of \( f_! \) to the case of \( \mathbb{Z}_\ell \)-sheaves. If \( f \) is qcqs then \( f_! \) preserves \( \ell \)-adically complete sheaves and so in this case we have \( f_! \mathbb{Z}_\ell = \lim_n f_!(\mathbb{Z}/\ell^n \mathbb{Z}) \), where each \( f_!(\mathbb{Z}/\ell^n \mathbb{Z}) \) is the usual lower shriek on étale sheaves. If \( f \) is not qcqs then this is not true; instead one can then filter \( Y \) by qcqs open subsets and compute \( f_! \) as the colimit of the lower shrieks on these subsets. In particular \( f_! \) agrees with previous definitions of “\( \mathbb{Z}_\ell \)-cohomology with compact support”, cf. [14, Remark 5.5].

With our full 6-functor formalism at hand, we study so-called relatively dualizable sheaves in this 6-functor formalism. This concept is not new and is known as universally locally acyclic sheaves in the case of discrete \( \Lambda \). With the power of the magical 2-category found by Lu-Zheng [7] we show that relatively dualizable sheaves satisfy all the expected properties and that this is completely formal. As an important special case of this theory we deduce that the following definition of \( \ell \)-cohomologically smooth maps is sensible:

**Definition 1.5** (cf. Definition [8.1]). An \( \ell \)-fine map \( f: Y \to X \) is \( \ell \)-cohomologically smooth if \( f^! \mathbb{F}_\ell \in \mathcal{D}_{\text{et}}(Y, \mathbb{F}_\ell) \) is invertible and its formation commutes with base-change along \( f \).

We emphasize that our definition of \( \ell \)-cohomological smoothness only depends on the étale theory with \( \mathbb{F}_\ell \)-coefficients and is therefore compatible with previously defined notions of \( \ell \)-cohomological smoothness in [13, 8]. This may seem very surprising, in particular because we
still obtain the expected Poincaré duality (see below). This fact was independently observed by Bogdan Zavyalov, whose suggestions led to our pursuit of these ideas and which will also be used in Zavyalov’s upcoming work on $p$-adic Poincaré duality in rigid-analytic geometry [19].

**Proposition 1.6** (cf. Proposition 8.5.(i)). Let $f: Y \to X$ be an $\ell$-cohomologically smooth map of small $v$-stacks. Then $f^!\Lambda \in D_{\text{nucl}}(Y, \Lambda)$ is invertible and the natural morphism

$$f^!\Lambda \otimes f^* \xrightarrow{\sim} f^!$$

is an isomorphism of functors $D_{\text{nucl}}(X, \Lambda) \to D_{\text{nucl}}(Y, \Lambda)$.

Of course, $\ell$-cohomologically smooth maps also satisfy all the other expected properties for nuclear sheaves, see Section 8, here the magic of Lu-Zheng’s construction really shines. We can use similar ideas to get a robust notion of $\ell$-cohomologically proper maps (which are more subtle to define than one might think at first), which we study in Section 9.

In the final part of this paper we apply the 6-functor formalism to classifying stacks, in which case we recover a category of representations. More concretely, suppose that $G$ is a locally profinite group which has locally finite $\ell$-cohomological dimension (e.g. $G$ is locally pro-$p$). Then there is a natural equivalence

$$D_{\text{nucl}}(*/G, \Lambda) = D_{\text{nucl}}(\Lambda)^{BG},$$

where the right-hand side denotes the $\infty$-category of nuclear $G$-representations, i.e. continuous $G$-representations on nuclear $\Lambda$-modules. If $\Lambda$ is concentrated in degree 0 (which is probably always true in practice) then $D_{\text{nucl}}(\Lambda)^{BG}$ is the derived $\infty$-category of its heart. For example, if $\Lambda = \mathbb{Q}_\ell$ then this heart is the abelian category of continuous $G$-representations on filtered colimits of $\mathbb{Q}_\ell$-Banach spaces.

We get the following result on the interaction of the 6-functor formalism with classifying stacks:

**Theorem 1.7** (cf. Theorem 10.13). Let $G$ be a locally pro-$p$ group. Then the map $*/G \to *$ is $\ell$-fine and $\ell$-cohomologically smooth. If $G$ is pro-$p$ then this map is additionally $\ell$-cohomologically proper.

Note that the pushforward along $*/G \to *$ computes nuclearized continuous $G$-cohomology, so the above theorem implies a Poincaré duality statement for continuous $G$-cohomology on nuclear $\Lambda$-modules. With this knowledge at hand, one can also introduce a good notion of admissible nuclear $G$-representations (see Definition 10.14) and deduce some basic properties for them (see Proposition 10.15). If $\Lambda$ is discrete then this recovers the usual notion of admissible representations. If $\Lambda$ is not discrete then the notion of admissible representations seems less common in the literature and has probably not been defined in the generality presented here.

**Background and Motivation.** In recent years, 6-functor formalisms have proven to be an extremely powerful tool in the study of various cohomology theories. With recent advances in the geometric Langlands program, 6-functor formalisms have also found tremendous success in applications to representation theory. In the case of rigid-analytic geometry, there have been introduced two 6-functor formalisms: A 6-functor formalism for étale $\Lambda$-sheaves for any discrete ring $\Lambda$ which is killed by some power of $\ell$ [15] and an analog for mod-$p$ coefficients [10]. These 6-functor formalisms have then been extended to certain stacky maps in [4, 5], which makes them applicable to representation theory. However, as far as we know there has not yet been introduced a satisfying 6-functor formalism for sheaves of modules over non-discrete rings like $\mathbb{Z}_\ell$, $\mathbb{Q}_\ell$ or $\overline{\mathbb{Q}}_\ell$. There have been the following attempts:
1. The most naive way of defining a 6-functor formalism for $\mathbb{Z}_\ell$-sheaves is by formally taking $\ell$-adic completions, i.e. associating to every small v-stack $X$ the $\infty$-category $D_{\text{naive}}(X, \mathbb{Z}_\ell) := \lim_{\leftarrow n} D_{\text{et}}(X, \mathbb{Z}/\ell^n\mathbb{Z})$ (cf. [15, §26]). This works well for most purposes: The $\infty$-category $D_{\text{naive}}(X, \mathbb{Z}_\ell)$ satisfies v-descent and embeds fully faithfully into $D_{\text{nuc}}(X, \mathbb{Z}_\ell)$. One also gets a full 6-functor formalism for these sheaves. In the case of qcqs spaces this 6-functor formalism mostly recovers the nuclear 6-functor formalism. For non-qcqs spaces and maps, the naive 6-functor formalism forces $\ell$-adic completions and thus loses information compared to the nuclear version. This is especially relevant in rigid-analytic geometry, where one is often interested in cohomologies of non-qcqs spaces like those appearing in the Drinfeld tower.

The main issue with the naive 6-functor formalism for $\mathbb{Z}_\ell$-sheaves is that it does not produce a good theory for $\mathbb{Q}_\ell$-sheaves. One can attempt to define $D_{\text{naive}}(X, \mathbb{Q}_\ell) := D_{\text{naive}}(X, \mathbb{Z}_\ell) \otimes \mathbb{Q}$, i.e. by formally inverting $\ell$. This roughly amounts to studying those $\mathbb{Q}_\ell$-sheaves which admit a $\mathbb{Z}_\ell$-model, but that is not precisely correct. In fact, $D_{\text{naive}}(X, \mathbb{Q}_\ell)$ does not even embed into $D(X_v, \mathbb{Q}_\ell)$!

2. In [3, §VII] an $\infty$-category $D_{\text{c}}(X, \mathbb{Z}_\ell) \subseteq D(X_v, \mathbb{Z}_\ell)$ of solid $\mathbb{Z}_\ell$-sheaves on $X$ is defined. This $\infty$-category satisfies v-descent and one can easily define $D_{\text{c}}(X, \Lambda)$ for every solid $\mathbb{Z}_\ell$-algebra $\Lambda$, so in particular for every nuclear $\mathbb{Z}_\ell$-algebra $\Lambda$. The theory of solid sheaves has excellent geometric properties; in fact it forms the foundation upon which we build our nuclear theory. Fargues-Scholze provide a “5-functor formalism” for $D_{\text{c}}(X, \Lambda)$, which in many applications is sufficiently strong to replace an actual 6-functor formalism. Still, $D_{\text{c}}(X, \Lambda)$ does not have a 6-functor formalism (it is far too big for that) and hence does not generalize the étale theory for discrete $\Lambda$. See below for more information on the comparison between the solid 5-functor formalism and our nuclear 6-functor formalism.

3. In [3, §VII.6] a full subcategory $D_{\text{lis}}(X, \Lambda) \subseteq D_{\text{c}}(X, \Lambda)$ is defined in order to remedy some of the downsides of $D_{\text{c}}(X, \Lambda)$. While $D_{\text{lis}}(X, \Lambda)$ seems to work very well with regards to the Langlands conjecture, it feels rather unnatural from a purely geometric point of view. Some of the main issues are that it is not stable under various operations and does not satisfy v-descent.

The nuclear 6-functor formalism has all the expected formal properties of a 6-functor formalism and generalizes the étale version for discrete coefficients. We therefore believe it to be the “right” way of working with sheaves over nuclear $\mathbb{Z}_\ell$-algebras. While the existence of the nuclear 6-functor formalism is a very neat abstract result and has many formal consequences (e.g. Poincaré duality for $\mathbb{Q}_\ell$-local systems), we do not provide any actual applications of the 6-functor formalism in our paper. From our point of view, the main scientific value of this paper comes from the following motivations:

(a) We found the definition of $D_{\text{lis}}$ rather unsatisfying and were therefore looking for different ways of understanding it. It seems plausible that one can describe $D_{\text{lis}} \subseteq D_{\text{nuc}}$ in terms of some abstract properties, like Ind-compact or Ind-perfect objects.

(b) With $D_{\text{nuc}}$ appearing so naturally in geometry, we would be surprised if it does not play some role in the local Langlands program. For example, one could hope that the categorical Langlands conjecture generalizes to a conjecture describing $D_{\text{nuc}}(\text{Bun}_G, \Lambda)$. So far $\ell$-adic Banach representations have not received much attention in the Langlands program, but in [18] it was shown that they seem to have some connection to Langlands.
(c) We hope that the nuclear 6-functor formalism sheds some light on the $p$-adic Langlands program. In fact, we have previously been trying to generalize our mod-$p$ 6-functor formalism to a $\mathbb{Z}_p$- or $\mathbb{Q}_p$-version without much success. The nuclear $\ell$-adic 6-functor formalism provides a lot of insight on how its $p$-adic analog should work.

(d) A large part of this paper is rather formal. One can therefore use this paper as a general recipe on how to construct a 6-functor formalism and study its basic properties like smoothness and representations.

**Why Nuclear Sheaves.** Our definition of nuclear sheaves, while being rather explicit, comes somewhat out of nowhere. Let us therefore explain why one should expect such a definition to appear in a 6-functor formalism for $\mathbb{Z}_\ell$-sheaves. There are two major motivations:

(i) In condensed mathematics, the $\infty$-category $\mathcal{D}^{\text{nuc}}(\mathbb{Z}_\ell)$ of nuclear $\mathbb{Z}_\ell$-modules should be seen as an analog of “discrete” modules over $\mathbb{Z}_\ell$ (since $\mathbb{Z}_\ell$ is equipped with a topology, the actual category of discrete $\mathbb{Z}_\ell$-modules is rather small). Since étale $\mathbb{F}_\ell$-sheaves on a small v-stack $X$ are a relative version of discrete $\mathbb{F}_\ell$-modules, it is believable that the correct $\mathbb{Z}_\ell$-analog should be a relative version of nuclear $\mathbb{Z}_\ell$-modules.

(ii) One of the main reasons why the solid 5-functor formalism cannot be upgraded to a 6-functor formalism is that for proper maps $f : Y \to X$ the functor $f_* : \mathcal{D}_c(Y, \Lambda) \to \mathcal{D}_c(X, \Lambda)$ often does not satisfy the projection formula. A geometric example of this phenomenon is given in [3, Warning VII.2.5]. Let us discuss a different example, which also demonstrates where the nuclearity condition comes from: Suppose $X = \text{Spa} C$ is a geometric point and $Y = \mathcal{S}$ is the pro-étale space over $X$ given by some profinite set $S$. Then $\mathcal{D}_c(X, \Lambda) = \mathcal{D}_c(\Lambda)$ is the $\infty$-category of solid condensed $\Lambda$-modules. If $f_*$ satisfied the projection formula then in particular for all solid $\Lambda$-modules $M$ we would have

$$f_* f^* M = f_* \Lambda \otimes_\Lambda M,$$

where the tensor product on the right is the solid tensor product. Note that $f_* \Lambda = C(S, \Lambda) = \Lambda_{\mathcal{S}}[S]^\vee$ and $f_* f^* M = \text{Hom}(\Lambda_{\mathcal{S}}[S], M)$. Thus for fixed $M$ the above condition (for all $S$) amounts precisely to the condition that $M$ is nuclear (as defined in [16, Definition 13.10]).

**Relation to the Solid 5-Functor Formalism.** In [3, §VII] Fargues-Scholze define a “5-functor formalism” for the $\infty$-category $\mathcal{D}_c(X, \Lambda)$ of solid $\Lambda$-sheaves on any small v-stack $X$. Apart from the usual four functors $\otimes$, $\text{Hom}$, $f^*$ and $f_*$ they introduce a functor $f_!$ defined to be the left adjoint of $f^*$. The goal of the solid 5-functor formalism is to approximate a 6-functor formalism for $\Lambda$-sheaves, but there has been some confusion as to what extent this is possible. With the nuclear 6-functor formalism at hand we can shed some light on this and in particular try to answer the questions raised in [3, Remark VII.3.6].

Namely, in Proposition 8.10 we show the following: Suppose $f : Y \to X$ is an $\ell$-fine and $\ell$-cohomologically smooth map of small v-stacks which is representable in locally spatial diamonds (this latter condition can be relaxed, cf. Remark 8.11). Then $f_* : \mathcal{D}_c(Y, \Lambda) \to \mathcal{D}_c(X, \Lambda)$ preserves nuclear sheaves and there is a natural isomorphism

$$f_* = f_! (- \otimes f^! \Lambda)$$
of functors \( D_{\text{nuc}}(Y, \Lambda) \to D_{\text{nuc}}(X, \Lambda) \). In other words, along \( \ell \)-cohomologically smooth maps \( f \), \( f_{\#} \) computes \( f \) up to a twist. In particular this allows one to simulate \( f \) by \( f_{\#} \) whenever one is in a geometric situation that can be built out of smooth maps, which explains why the 5-functor formalism is so useful in [3]. Note that it also follows that \( D_{\text{lis}}(X, \Lambda) \subseteq D_{\text{nuc}}(X, \Lambda) \).

Of course, for non-smooth \( f \) the functors \( f_{\#} \) and \( f ! \) are very different and \( f_{\#} \) does usually not preserve nuclear sheaves.

**Structure of the Paper.** This paper is roughly structured into four parts:

- **Part I** consists of Sections 2 and 3 and introduces the \( \infty \)-category of nuclear sheaves. These sections are very specific to the concrete setting at hand. In Section 2 we introduce \( \omega_1 \)-solid sheaves as a particularly nice full subcategory of all solid sheaves. Their main advantage is that it is easier to control their compact objects (which is crucial for studying nuclear sheaves later on) while still maintaining most of the nice properties of solid sheaves. In Section 3 we then introduce nuclear sheaves. We provide different characterizations of nuclearity (thereby also motivating the terminology) and show that they satisfy \( \nu \)-descent.

- **Part II** consists of Sections 4 and 5 and constructs the 6-functor formalism for nuclear sheaves. While not being fully formal, most of the ideas in these sections should be applicable to many different geometric settings. It can therefore be used as a general recipe for constructing 6-functor formalisms.

- **Part III** consists of Sections 6 to 9 and is almost completely formal. Here we introduce dualizable and relatively dualizable sheaves (the latter being also known as universally locally acyclic sheaves in the étale context) and use them to define cohomologically smooth and proper maps. The main insight is that by using the magic of Lu-Zheng’s ideas one can show that cohomological smoothness is a condition that only depends on very little data of the 6-functor formalism (one only needs to understand its behavior on the monoidal unit). Since all of the sections in part III are very formal, they should apply to every 6-functor formalism.

- **Part IV** consists of Section 10, where we apply the nuclear 6-functor formalism to classifying stacks in order to obtain a 6-functor formalism for representation theory. The main result here is that under mild assumptions on the locally profinite group \( G \) the classifying stack \( * / G \) is \( \ell \)-cohomologically smooth.

**Notation and Conventions.** The whole paper is written in the modern language of \( \infty \)-categories, as this is the most natural and clean way of describing our ideas (for example \( D_{\text{nuc}}(X, \Lambda) \) does not have a \( t \)-structure and is therefore not that easily accessible with conventional methods; also the 6-functor formalism for stacky maps requires an \( \infty \)-categorical framework). In particular every functor, sheaf, ring, module, representation etc. is always assumed to be derived if not explicitly specified differently. In the presence of a \( t \)-structure on a stable \( \infty \)-category we usually denote by \( \pi_n M = H^{-n}(M) \) the homology objects of \( M \). We say that \( M \) is static if it is concentrated in degree 0, i.e. has no derived structure. For a pro-étale map \( j: U \to X \) of diamonds and a pro-étale sheaf \( \mathcal{M} \) on \( U \) we denote \( \mathcal{M}[U] := j_* \mathcal{M} \) (in the sense of sites), which is a pro-étale sheaf on \( X \). All sheaves will always be considered as part of the derived \( \infty \)-category of \( \nu \)-sheaves. For example by an étale sheaf we mean a sheaf in \( D_{\text{et}}(X, \mathbb{Z}) \subseteq D(X, \mathbb{Z}) \). We denote by \( (-)_{\text{et}}: D(X, \mathbb{Z}) \to D_{\text{et}}(X, \mathbb{Z}) \) the right adjoint to the inclusion. For a \( \mathbb{Z}_\ell \)-module \( M \) we denote...
\(M/\ell^n M := \text{cofib}(M \xrightarrow{\ell^n} M)\) and similarly for sheaves of \(\mathbb{Z}_\ell\)-modules. We warn the reader that by \(\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)\) we denote the \(\infty\)-category of those pro-étale \(\mathbb{Z}_\ell\)-sheaves on \(X\) whose underlying sheaf of abelian groups is étale (this is different from the notation in \([15] \S 26\)).

We are aware of the fact that \(\infty\)-categories can be intimidating to the uninitiated, so we refer the reader to \([10] \S 1.5\) for a quick down-to-earth introduction to \(\infty\)-categories with a special focus on the terminology used in the algebraic setting.

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### 2 \(\omega_1\)-Solid Sheaves

Fix a prime \(\ell \neq p\). The \(\infty\)-category \(\mathcal{D}_\infty(X, \mathbb{Z}_\ell)\) constructed in \([3] \S VII\) lacks some formal properties which we need for our construction of nuclear sheaves, most prominently the unit object is usually not compact. We will remedy this by considering a much smaller subcategory \(\mathcal{D}_\omega(X, \mathbb{Z}_\ell)_{\omega_1} \subseteq \mathcal{D}_\infty(X, \mathbb{Z}_\ell)\) of \(\omega_1\)-solid sheaves. This subcategory is generated under small colimits by countable limits of qcqs étale sheaves. Since countable limits have finite cohomological dimension, this gives us a lot of control. In particular, the \(\infty\)-category of \(\omega_1\)-solid sheaves has similar properties as \(\mathcal{D}_\infty(X, \mathbb{Z}_\ell)\), but under mild assumptions on \(X\) the unit is compact. This “mild assumption” on \(X\) is the following:

**Definition 2.1.** A spatial diamond \(X\) is called \(\ell\)-bounded if there is some integer \(d\) such that for all static étale \(\mathbb{F}_\ell\)-modules \(\mathcal{M} \in \mathcal{D}_{\text{et}}(X, \mathbb{F}_\ell)\) on \(X\) we have \(H^k(X, \mathcal{M}) = 0\) for \(k > d\).

**Definition 2.2.** Let \(X\) be a spatial diamond. A quasi-pro-étale map \(U \to X\) is called basic if it can be written as a cofiltered limit \(U = \varprojlim_i U_i\) such that all \(U_i \to X\) are étale, quasicompact and separated.

**Remark 2.3.** Suppose that \(X\) is an \(\ell\)-bounded spatial diamond and let \(\mathcal{M} \in \mathcal{D}_{\text{et}}(X, \mathbb{Z})\) be any static étale sheaf on \(X\) which is killed by some power of \(\ell\). Then for the same \(d\) as in the definition of \(\ell\)-boundedness and all basic \(U \in X_{\text{proet}}\) we have \(H^k(U, \mathcal{M}) = 0\) for \(k > d + 1\). Namely, if \(U = X\) then this follows by writing \(\mathcal{M} = \varprojlim_n \mathcal{M}/\ell^n \mathcal{M}\) and noting that for each \(\mathcal{M}/\ell^n \mathcal{M}\) the result follows by repeatedly applying fiber sequences of the form \(\mathcal{M}/\ell^{n-1} \mathcal{M} \to \mathcal{M}/\ell^n \mathcal{M} \to \mathcal{M}/\ell \mathcal{M}\). If \(U\) is étale over \(X\) then the result follows from the observation that the étale pushforward along \(U \to X\) is \(t\)-exact (this can be checked after pullback to a strictly totally disconnected space, in which case \(U\) is also strictly totally disconnected and therefore has vanishing étale cohomology). For general \(U = \varprojlim_i U_i\) use that \(H^k(U, \mathcal{M}) = \varprojlim_i H^k(U_i, \mathcal{M})\). In the following we will implicitly make use of these facts.

Let us now come to the definition of \(\omega_1\)-solid sheaves. Here by \(\omega_1\) we mean the first uncountable cardinal.

**Definition 2.4.** Let \(X\) be an \(\ell\)-bounded spatial diamond. A solid \(\mathbb{Z}_\ell\)-module \(\mathcal{M} \in \mathcal{D}_\omega(X, \mathbb{Z}_\ell)\) on \(X\) is called \(\omega_1\)-solid if for every \(\omega_1\)-cofiltered limit \(U = \varprojlim_i U_i\) of basic objects in \(X_{\text{proet}}\) the
natural map
\[ \lim_i \Gamma(U_i, \mathcal{M}) \rightarrow \Gamma(U, \mathcal{M}) \]
is an isomorphism. We denote by
\[ \mathcal{D}_\omega(X, \mathbb{Z}_\ell) \subseteq \mathcal{D}_\omega(X, \mathbb{Z}_\ell) \]
the full subcategory spanned by the \(\mathbb{Z}_\ell\)-solid sheaves.

The basic properties of \(\mathcal{D}_\omega(X, \mathbb{Z}_\ell)\) are summarized by the following result, which appeared to be surprisingly subtle.

**Proposition 2.5.** Let \(X\) be an \(\ell\)-bounded spatial diamond.

(i) Let \(\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell) \subseteq \mathcal{D}_\omega(X, \mathbb{Z}_\ell)\) denote the full subcategory spanned by those sheaves whose underlying abelian sheaf lies in \(\mathcal{D}_{\text{et}}(X, \mathbb{Z})\). Then \(\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)\) is compactly generated and the compact objects are generated under finite (co)limits and retracts by the objects \(\mathbb{F}_\ell[U]\) for quasicompact separated \(U \in X_{\text{et}}\). Moreover, the t-structure on \(\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)\) restricts to a t-structure on \(\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)^\omega\).

(ii) \(\mathcal{D}_\omega(X, \mathbb{Z}_\ell)\) is compactly generated. The compact objects are \(\ell\)-adically complete and generated under finite (co)limits and retracts by the objects
\[
\mathbb{Z}_{\ell,\omega}[U] = \lim_n \mathbb{Z}[U_n] = \lim_n (\mathbb{Z}/\ell^n \mathbb{Z})[U_n]
\]
for sequential limits \(U = \lim_n U_n\) with all \(U_n \rightarrow X\) being étale, quasicompact and separated. This identifies \(\mathcal{D}_\omega(X, \mathbb{Z}_\ell)\omega\) with a full subcategory
\[
\mathcal{D}_\omega(X, \mathbb{Z}_\ell)^\omega \subseteq \text{Pro}(\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)^\omega).
\]

(iii) \(\mathcal{D}_\omega(X, \mathbb{Z}_\ell)\) is stable under all colimits and countable limits in \(\mathcal{D}_\omega(X, \mathbb{Z}_\ell)\) and contains \(\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)\). It admits a complete t-structure and a symmetric monoidal structure by restricting the ones on \(\mathcal{D}_\omega(X, \mathbb{Z}_\ell)\). Moreover, the compact objects in \(\mathcal{D}_\omega(X, \mathbb{Z}_\ell)\) are stable under tensor product.

**Proof.** We first prove (i). We claim that the forgetful functor \(\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell) \rightarrow \mathcal{D}_{\text{et}}(X, \mathbb{Z})\) is fully faithful. Indeed, the right adjoint of the forgetful functor \(\mathcal{D}(X_{\text{proet}}, \mathbb{Z}_\ell) \rightarrow \mathcal{D}(X_{\text{proet}}, \mathbb{Z})\) is given by \(\text{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell, -)\) and one observes that this functor preserves étale sheaves; in fact, if \(\mathcal{M} \in \mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)\) then \(\text{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell, -) = \lim_n \text{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}/\ell^n \mathbb{Z}, \mathcal{M})\) by the usual Breen resolution argument (cf. the proof of [3, Proposition VII.1.12]). One deduces from the same formula that if \(\mathcal{M} \in \mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)\) then \(\text{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell, \mathcal{M}) = \mathcal{M}\), proving the desired fully faithfulness.

The embedding \(\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell) \rightarrow \mathcal{D}_{\text{et}}(X, \mathbb{Z})\) is t-exact. Moreover, if \(\mathcal{P} \in \mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)\) is perfect constructible as an object in \(\mathcal{D}_{\text{et}}(X, \mathbb{Z})\) then it is compact as an object in \(\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)\). By the proof of [15, Proposition 20.17] this reduces to showing that \(\text{Hom}([U], -) = \Gamma(U, -)\) preserves small colimits of objects in \(\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)\), which follows from \(\ell\)-boundedness of \(X\) (using the fact that every \(\mathcal{M} \in \mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)\) can be written as \(\mathcal{M} = \lim_n \text{Hom}(\mathbb{Z}/\ell^n \mathbb{Z}, \mathcal{M})\) by the previous paragraph).

We now prove that conversely every compact object in \(\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)\) is perfect constructible as an object in \(\mathcal{D}_{\text{et}}(X, \mathbb{Z})\). First observe that for every quasicompact separated \(U \in X_{\text{et}}\), \(\mathbb{F}_\ell[U] \in \mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)\).
$\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)$ is compact because it is perfect constructible in $\mathcal{D}_{\text{et}}(X, \mathbb{Z})$. We now claim that the objects $\mathcal{F}_i[U]$ generate $\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)$. Note that these objects generate $(\mathbb{Z}/\ell^n\mathbb{Z})[U][k]$ for all integers $n \geq 1$ and $k$; by abstract nonsense it is therefore enough to show that the family of functors $\text{Hom}(\mathbb{Z}/\ell^n\mathbb{Z})[U][k], -)$ is conservative. We can ignore the shifts by $k$ by instead taking spectrum-enriched Hom’s. We now claim that for every $\mathcal{M} \in \mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)$ the natural map

$$\lim_{n} \text{Hom}((\mathbb{Z}/\ell^n\mathbb{Z})[U], \mathcal{M}) \xrightarrow{\sim} \text{Hom}(\mathbb{Z}_\ell[U], \mathcal{M}) = \Gamma(U, \mathcal{M}),$$

is an isomorphism of spectra. Using the $\ell$-boundedness of $X$, a standard Postnikov limit argument reduces this claim to the case that $\mathcal{M}$ is left-bounded. Since both sides commute with colimits in $\mathcal{M}$ we can further reduce to the case that $\mathcal{M}$ is static. But then the usual Breen-Deligne resolution works (cf. the proof of [3, Proposition VII.1.12]). We are thus reduced to showing that the family of functors $\Gamma(U, -)$ is conservative; but this is clear. In particular we deduce the claimed description of compact objects in $\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)$ and identify them as precisely those objects which are perfect constructible in $\mathcal{D}_{\text{et}}(X, \mathbb{Z})$. By [15, Proposition 20.12] the $t$-structure on $\mathcal{D}_{\text{et}}(X, \mathbb{Z})$ restricts to a $t$-structure on perfect constructible sheaves; this finishes the proof of (i).

We now prove (ii) and (iii). The fact that $\mathcal{D}_{\omega}(X, \mathbb{Z}_\ell)_{\omega_1}$ is stable under countable limits in $\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)$ follows immediately from the definition by using the fact that countable limits commute with $\omega_1$-filtered colimits in spectra. Also, it is obvious that $\mathcal{D}_{\omega}(X, \mathbb{Z}_\ell)_{\omega_1}$ contains $\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)$ because étale sheaves $\mathcal{M}$ on $X$ satisfy $\Gamma(U, \mathcal{M}) = \lim_{n} \Gamma(U_i, \mathcal{M})$ for all cofiltered limits $U = \lim_{n} U_i$ (this holds for unbounded $\mathcal{M}$ by the $\ell$-boundedness of $X$).

Let now $\bar{U} = \lim_{n} U_n$ be given as in (ii). We first check that $\mathbb{Z}_\ell[U]$ is $\omega_1$-solid. Since $\omega_1$-solid sheaves are stable under countable limits we reduce to showing this for $\mathbb{Z}_\ell[U_n]$ for all $n$. Since étale sheaves are $\omega_1$-solid, we further reduce to showing that the natural map $\mathbb{Z}_\ell[U_n] \xrightarrow{\sim} \lim_{n} (\mathbb{Z}/\ell^n\mathbb{Z})[U_n]$ is an isomorphism. Both sides of this claimed isomorphism are static, so we can check this by applying Yoneda in the heart, i.e. we need to see that for every $\mathcal{M} \in \mathcal{D}_{\omega}(X, \mathbb{Z}_\ell)$ the natural map

$$\text{Hom}(\lim_{n} (\mathbb{Z}/\ell^n\mathbb{Z})[U_n], \mathcal{M}) \xrightarrow{\sim} \text{Hom}(\mathbb{Z}_\ell[U_n], \mathcal{M})$$

is an isomorphism. Since both $\mathbb{Z}_\ell[U_n]$ and $\mathbb{Z}_\ell/\ell^n\mathbb{Z}[U_n]$ are compact objects in the heart (by the description of finitely presented, i.e. compact, objects in [3, Theorem VII.1.3]), we can reduce to $\mathcal{M}$ being of the form $\mathcal{M} = \lim_{i} \mathcal{M}_i$ for qcqs étale sheaves $\mathcal{M}_i$. Pulling out limits from both sides of the claimed Hom-identity, we reduce to the case that $\mathcal{M}$ is étale. But then by the proof of [3, Theorem VII.1.3] we have

$$\text{Hom}(\lim_{n} (\mathbb{Z}/\ell^n\mathbb{Z})[U_n], \mathcal{M}) = \lim_{n} \text{Hom}(\mathbb{Z}/\ell^n\mathbb{Z})[U_n], \mathcal{M}) = \lim_{n} \text{Hom}(\mathbb{Z}/\ell^n\mathbb{Z}, \mathcal{M}[U]) = \text{Hom}(\mathbb{Z}_\ell, \mathcal{M}[U]) = \text{Hom}(\mathbb{Z}_\ell[U], \mathcal{M}),$$

as desired. This finishes the proof that $\mathbb{Z}_\ell[U]$ is $\omega_1$-solid. Now let $\mathcal{C} \subseteq \mathcal{D}_{\omega}(X, \mathbb{Z}_\ell)_{\omega_1}$ be the full subcategory generated under finite (co)limits and retracts by the objects $\mathbb{Z}_\ell[U]$ for $U = \lim_{n} U_n$ as in (ii). This induces a natural functor

$$\alpha: \text{Ind}(\mathcal{C}) \to \mathcal{D}_{\omega}(X, \mathbb{Z}_\ell).$$

Our goal will be to show that $\alpha$ induces an equivalence of $\text{Ind}(\mathcal{C})$ and $\mathcal{D}_{\omega}(X, \mathbb{Z}_\ell)_{\omega_1}$. As a first step towards seeing this, let us prove that the $t$-structure on $\mathcal{D}_{\omega}(X, \mathbb{Z}_\ell)$ restricts to a $t$-structure
on the essential image of $\alpha$. Namely, this reduces to showing that for every $P \in \mathcal{C}$, all $\pi_k P$ lie in the essential image of $\alpha$. By [3, Theorem VII.1.3], $\pi_k P$ is finitely presented and can be written as a countable limit $\pi_k P = \lim_n P_{n,k}$ of qcqs étale sheaves. The latter property implies that $\pi_k P$ is $\omega_1$-solid, which in turn implies that the sections of $\pi_k P$ on any basic $U \in X_{proet}$ are determined by the sections of $\pi_k P$ with all $U_i$ as in (ii). But note that basic $U$ form a basis of $X_{proet}$ (this is clear if $X$ is a perfectoid space and follows from [15, Proposition 11.24, 11.23.(iii)] in general), hence there is a surjection of the form $\bigoplus_i \mathbb{Z}_{\ell,\mathbb{C}}[U_i] \to \pi_k P$ with all $U_i$ as in (ii). Note that this direct sum is still $\omega_1$-solid, hence so is the kernel of the map to $\pi_k P$ so that we get a two-term resolution $\bigoplus_j \mathbb{Z}_{\ell,\mathbb{C}}[U_j] \to \bigoplus_i \mathbb{Z}_{\ell,\mathbb{C}}[U_i] \to \pi_k P$. By writing this map of infinite direct sums as a filtered colimit of maps of finite direct sums and using that $\pi_k P$ is finitely presented, we deduce that one arrange that both direct sums are finite, i.e. lie in $\mathcal{C}$. Continuing this argument, we obtain a resolution of $\pi_k P$ by objects in $\mathcal{C}$ which implies that $\pi_k P$ is a geometric resolution of objects in $\mathcal{C}$ and in particular a filtered colimit of objects in $\mathcal{C}$ (as every finite step of the geometric resolution lies in $\mathcal{C}$). This implies that $\pi_k P$ lies in the essential image of $\alpha$, as desired.

Next we claim that for $d$ as in the definition of $\ell$-boundedness, for every $\mathcal{M}$ in the heart of the essential image of $\alpha$ and for every basic $U$ we have $H^k(U, \mathcal{M}) = 0$ for $k > d + 2$. Indeed, since $H^k(U, \mathcal{M})$ commutes with filtered colimits of static $\mathcal{M}$, we can reduce to the case that $\mathcal{M} = \pi_n P$ for some $P \in \mathcal{C}$. As seen in the previous paragraph this implies that $\mathcal{M}$ is a sequential limit $\mathcal{M} = \lim_n \mathcal{M}_n$ for qcqs static $\mathcal{M}_n \in \mathcal{D}_{et}(X, \mathbb{Z}_\ell)$. By factoring the countable limit out of $H^k(U, -)$ and noting limits have cohomological dimension 1 in spectra, we reduce to showing that $H^k(U, \mathcal{M}_n) = 0$ for $k > d$, which follows immediately from the definition of $\ell$-boundedness.

We are now ready to prove that $\alpha$ is fully faithful. This amounts to showing the following: Given $U$ as in (ii) and any filtered diagram $(P_i)$, of objects in $\mathcal{C}$, the natural map

$$\lim_i \text{Hom}(\mathbb{Z}_{\ell,\mathbb{C}}[U], P_i) \xrightarrow{\sim} \text{Hom}(\mathbb{Z}_{\ell,\mathbb{C}}[U], \lim_i P_i)$$

is an isomorphism (here the colimit $\lim_i P_i$ is formed in $\mathcal{D}_{et}(X, \mathbb{Z}_\ell)$). Using the identification $\text{Hom}(\mathbb{Z}_{\ell,\mathbb{C}}[U], -) = \Gamma(U, -)$ and hence the fact that this functor has finite cohomological dimension on the essential image of $\alpha$, we can employ a standard Postnikov limit argument to reduce to the claim that $\Gamma(U, -)$ preserves filtered colimits of uniformly left-bounded sheaves.

We have established that $\alpha: \text{Ind}(\mathcal{C}) \hookrightarrow \mathcal{D}_{et}(X, \mathbb{Z}_\ell)$ is an embedding. We now show that the essential image is precisely $\mathcal{D}_{et}(X, \mathbb{Z}_\ell)_{\omega_1}$. First observe that the essential image of $\alpha$ is contained in $\mathcal{D}_{et}(X, \mathbb{Z}_\ell)_{\omega_1}$. Indeed, this follows from the fact that every $P \in \mathcal{C}$ is $\omega_1$-solid and that for all basic $U$ the functor $\Gamma(U, -)$ preserves filtered colimits in the essential image of $\alpha$ because it has finite cohomological dimension on that image (as used in the previous paragraph). It remains to see that every $\omega_1$-solid $\mathbb{Z}_\ell$-module on $X$ lies in the essential image of $\alpha$. By standard arguments (e.g. looking at the right adjoint to the embedding $\text{Ind}(\mathcal{C}) \hookrightarrow \mathcal{D}_{et}(X, \mathbb{Z}_\ell)_{\omega_1}$) this reduces to showing that the family of functors $\text{Hom}(\mathbb{Z}_{\ell,\mathbb{C}}[U], -) = \Gamma(U, -)$, for $U$ as in (ii), is conservative on $\mathcal{D}_{et}(X, \mathbb{Z}_\ell)_{\omega_1}$. But by definition of $\omega_1$-solid objects, these functors determine $\Gamma(U, -)$ for all basic $U \in X_{proet}$, and as the basic $U$ form a basis of $X_{proet}$ (as noted above), this family of functors is indeed conservative.

We have finally shown the equivalence $\text{Ind}(\mathcal{C}) \cong \mathcal{D}_{et}(X, \mathbb{Z}_\ell)_{\omega_1}$. The rest of (ii) goes as follows: Everything is clear except that the compact objects of $\mathcal{D}_{et}(X, \mathbb{Z}_\ell)_{\omega_1}$ are $\ell$-adically complete and embed into $\text{Pro}(\mathcal{D}_{et}(X, \mathbb{Z}_\ell)^{\omega_1})$. The $\ell$-adic completeness was shown for $\mathbb{Z}_\ell[U]$ with qcqs $U \in X_{et}$ above and follows immediately for all compact objects because $\ell$-adic completeness is stable under limits. To get the embedding into the Pro-category, we use the following more general statement:
Equip \( \text{Pro}(\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)^{\omega}) \) with the natural \( t \)-structure induced from the one on \( \mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)^{\omega} \) (cf. [9, Lemma C.2.4.3]); then the natural functor
\[
\text{Pro}(\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)^{\omega}) \hookrightarrow \mathcal{D}_{\text{c}}(X, \mathbb{Z}_\ell)
\]
is fully faithful. Namely, the functor \( \text{Pro}(\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)^{\omega}) \to \mathcal{D}_{\text{c}}(X, \mathbb{Z}_\ell) \) is \( t \)-exact by [3, Proposition VII.1.6] and so the claimed fully faithfulness reduces to showing that \( \text{Hom}(\lim_{\to} \mathcal{P}_i, \mathcal{Q}) = \lim_{\to} \text{Hom}(\mathcal{P}_i, \mathcal{Q}) \) for compact static \( \mathcal{P}_i, \mathcal{Q} \in \mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell) \); this follows from the usual Breen resolution argument (as in [3, Proposition VII.1.12]).

The rest of (iii) is easy: We have already seen that \( \mathcal{D}_{\text{c}}(X, \mathbb{Z}_\ell)_{\omega_1} \) is stable under countable limits and contains all étale sheaves. By the proof of (ii) it follows that it is also stable under all small colimits and that the \( t \)-structure on \( \mathcal{D}_{\text{c}}(X, \mathbb{Z}_\ell) \) restricts to a \( t \)-structure on \( \mathcal{D}_{\text{c}}(X, \mathbb{Z}_\ell)_{\omega_1} \). It remains to see that \( \mathcal{D}_{\text{c}}(X, \mathbb{Z}_\ell)_{\omega_1} \) is stable under tensor product: This can be checked on compact generators and thus follows from the observation \( \mathbb{Z}_{\text{et},c}[U] \otimes \mathbb{Z}_{\text{et},c}[V] = \mathbb{Z}_{\text{et},c}[U \times V] \); this also proves that compact objects are stable under tensor product. \(\)

In order to work with \( \omega_1 \)-solid sheaves it is important to understand how they behave under a change of spatial diamond:

**Proposition 2.6.** Let \( f: Y \to X \) be a map of \( \ell \)-bounded spatial diamonds.

(i) The pullback \( \mathcal{D}(X, \mathbb{Z}_\ell) \to \mathcal{D}(Y, \mathbb{Z}_\ell) \) restricts to a \( t \)-exact symmetric monoidal functor
\[
f^*: \mathcal{D}_{\text{c}}(X, \mathbb{Z}_\ell)_{\omega_1} \to \mathcal{D}_{\text{c}}(Y, \mathbb{Z}_\ell)_{\omega_1}
\]
which preserves all small colimits and all countable limits.

(ii) The pushforward \( f_{\ast*}: \mathcal{D}(Y, \mathbb{Z}_\ell) \to \mathcal{D}(X, \mathbb{Z}_\ell) \) restricts to a functor
\[
f_{\ast*}: \mathcal{D}_{\text{c}}(Y, \mathbb{Z}_\ell)_{\omega_1} \to \mathcal{D}_{\text{c}}(X, \mathbb{Z}_\ell)_{\omega_1}
\]
which has finite cohomological dimension, is right adjoint to \( f^* \) and preserves all small limits and colimits.

**Proof.** Since the pullback on the \( v \)-site is \( t \)-exact and preserves all small limits and colimits, part (i) follows immediately from Proposition 2.5 and the fact that \( f^* \mathbb{Z}_\ell[U] = \mathbb{Z}_\ell[U \times X] \) for every quasicompact separated \( U \in \text{Spec}(\mathbb{Z}_\ell) \) (for the proof that \( f^* \) is symmetric monoidal see e.g. [3, Proposition VII.2.2]). This proves (i).

To prove (ii), first note that the \( v \)-pushforward has finite cohomological dimension when restricted to \( \mathcal{D}_{\text{c}}(Y, \mathbb{Z}_\ell)_{\omega_1} \): Recall that for a static \( v \)-sheaf \( \mathcal{M} \) on \( X \), \( H^k(f_\ast \mathcal{M}) \) is the sheafification of the presheaf \( U \mapsto H^k(U \times X, \mathcal{M}) \); but if \( \mathcal{M} \) is \( \omega_1 \)-solid then \( H^k(U, \mathcal{M}) = 0 \) for \( k > d+1 \) with \( d \) as in the definition of \( \ell \)-boundedness for \( Y \) (see the proof of Proposition 2.5). It follows that \( f_{\ast*} \) preserves small colimits on \( \mathcal{D}_{\text{c}}(Y, \mathbb{Z}_\ell)_{\omega_1} \), hence to show that it maps this \( \infty \)-category to \( \mathcal{D}_{\text{c}}(X, \mathbb{Z}_\ell)_{\omega_1} \) we are reduced to the compact generators, i.e. we need to show that \( f_{\ast*} \mathbb{Z}_\ell[\mathcal{U}] \in \mathcal{D}_{\text{c}}(X, \mathbb{Z}_\ell)_{\omega_1} \) for all \( \mathcal{U} = \lim_{\to} U_n \) as in Proposition 2.5(ii). But \( f_{\ast*} \mathbb{Z}_\ell[\mathcal{U}] = \lim_{\to} f_{\ast*} \mathbb{Z}/(\ell^n \mathbb{Z})[U_n] \) so since \( \mathcal{D}_{\text{c}}(X, \mathbb{Z}_\ell)_{\omega_1} \) is stable under countable limits it is enough to show that \( f_{\ast*} \mathbb{Z}/(\ell^n \mathbb{Z})[U_n] \) is étale – this follows from [15, Corollary 16.8.(ii)]. \(\)

**Corollary 2.7.** The assignment
\[
X \mapsto \mathcal{D}_{\text{c}}(X, \mathbb{Z}_\ell)_{\omega_1}
\]
defines a hypercomplete sheaf of symmetric monoidal \( \infty \)-categories on the \( v \)-site of \( \ell \)-bounded spatial diamonds.
Proof. By using the embedding \( \mathcal{D}_{\kappa}(X, \mathbb{Z}_\ell) \subseteq \mathcal{D}(X, \mathbb{Z}_\ell) \) and using that the right-hand category satisfies hypercomplete v-descent for formal reasons, we are left with showing that for any v-cover \( f: Y \to X \) of \( \ell \)-bounded spatial diamonds and any \( M \in \mathcal{D}(X, \mathbb{Z}_\ell) \) we have \( f^* M \in \mathcal{D}_{\kappa}(Y, \mathbb{Z}_\ell)_{\omega_1} \) if and only if \( M \in \mathcal{D}_{\kappa}(X, \mathbb{Z}_\ell)_{\omega_1} \). The “if” part follows immediately from Proposition 2.6(i). It remains to prove the “only if” part, so assume that \( f^* M \) is \( \omega_1 \)-solid. Extend \( f \) to a v-hypercover \( f: Y \to X \) such that all \( Y_n \) are \( \ell \)-bounded spatial diamonds. Then by v-hyperdescent for v-sheaves we have \( M = \lim_{n \in \Delta} f_{n!*} f_n^* M \), which we want to show to be \( \omega_1 \)-solid. Since \( \omega_1 \)-solid sheaves are stable under countable limits by Proposition 2.5(iii) it is enough to show that each \( f_{n!*} f_n^* M \) is \( \omega_1 \)-solid. But this follows immediately from Proposition 2.6 since \( f_{\beta!} M \) is \( \omega_1 \)-solid by assumption.

It is convenient to know that the tensor product of \( \ell \)-adically complete sheaves is again \( \ell \)-adically complete:

**Proposition 2.8.** Let \( X \) be an \( \ell \)-bounded spatial diamond and let \( M, N \in \mathcal{D}^{-}_\kappa(X, \mathbb{Z}_\ell) \) be \( \ell \)-adically complete. Then \( M \otimes N \) is \( \ell \)-adically complete.

Proof. By choosing projective resolutions we can write \( M \) and \( N \) as geometric realizations of direct sums of copies of the compact projective generators \( \mathbb{Z}_{\ell,\kappa}[U] \) for w-contractible \( U \in X_{\text{proet}} \). Since the \( \ell \)-adic completion functor on \( \mathcal{D}_{\kappa}(X, \mathbb{Z}_\ell) \) has homological dimension 1 and therefore preserves geometric realizations of uniformly left-bounded sheaves, we can reduce to the case that \( M \) and \( N \) are completed direct sums of copies of \( \mathbb{Z}_{\ell,\kappa}[U] \) for w-contractible \( U \in X_{\text{proet}} \); in particular both \( M \) and \( N \) are static (note that \( \mathbb{Z}_{\ell,\kappa}[U] \) is \( \ell \)-adically complete by the proof of Proposition 2.5). Since \( \ell \)-adic completion commutes with \( \omega_1 \)-filtered colimits, we can further reduce to the case of countable completed direct sums, i.e. we have \( M = \bigoplus_n \mathbb{Z}_{\ell,\kappa}[U_n] \) and \( N = \bigoplus_m \mathbb{Z}_{\ell,\kappa}[V_m] \) for w-contractible \( U_n, V_m \in X_{\text{proet}} \). Note that we can write \( M \) as a union of subsheaves of the form \( M_\alpha = \prod_n \ell_{\alpha_n} \mathbb{Z}_{\ell,\kappa}[U_n] \) for all sequences \( \alpha_n \in \mathbb{Z}_{\geq 0} \) converging to \( \infty \) (and similarly for \( N \) in terms of \( N_\beta \) for sequences \( \beta \)). Using [3, Proposition VII.2.3] we deduce

\[
M \otimes N = \lim_{\alpha, \beta} M_\alpha \otimes N_\beta = \lim_{\alpha, \beta} \prod_{m,n} (\ell_{\alpha_n} \mathbb{Z}_{\ell,\kappa}[U_n] \otimes \ell_{\beta_m} \mathbb{Z}_{\ell,\kappa}[V_m]) = \lim_{\alpha, \beta} \prod_{m,n} \ell_{\alpha_n + \beta_m} \mathbb{Z}_{\ell,\kappa}[U_n \times X V_m].
\]

By a simple cofinality argument one checks that this is just the completed direct sum of \( \mathbb{Z}_{\ell,\kappa}[U_n \times X V_m] \) for all \( n, m > 0 \). In particular this sheaf is \( \ell \)-adically complete, as desired. \( \square \)

### 3 Nuclear Sheaves

Fix the prime \( \ell \neq p \). In the following we will define a full subcategory \( \mathcal{D}_{\text{nu}}(X, \mathbb{Z}_\ell) \subseteq \mathcal{D}_{\kappa}(X, \mathbb{Z}_\ell) \) of nuclear sheaves on any small v-stack \( X \). In general this \( \infty \)-category will be defined by v-descent, but on \( \ell \)-bounded spatial diamonds it admits a much more explicit description:

**Definition 3.1.** Let \( X \) be an \( \ell \)-bounded spatial diamond and let \( M \in \mathcal{D}_{\kappa}(X, \mathbb{Z}_\ell) \) be a solid \( \mathbb{Z}_\ell \)-sheaf on \( X \).

(a) \( M \) is called a **Banach sheaf** if \( M \) is \( \ell \)-adically complete and \( M/\ell M \) is étale, i.e. lies in \( \mathcal{D}_{\text{et}}(X, \mathbb{F}_\ell) \).
(b) $\mathcal{M}$ is called nuclear if it is a filtered colimit of Banach sheaves. We denote by $\mathcal{D}_{\text{nuc}}(X, \mathbb{Z}_\ell) \subseteq \mathcal{D}_c(X, \mathbb{Z}_\ell)$ the full subcategory spanned by the nuclear sheaves.

**Remarks 3.2.**

(i) It follows immediately from Proposition [2.5.(iii)] that every nuclear $\mathbb{Z}_\ell$-sheaf on an $\ell$-bounded spatial diamond $X$ is $\omega_1$-solid, i.e. we have $\mathcal{D}_{\text{nuc}}(X, \mathbb{Z}_\ell) \subseteq \mathcal{D}_c(X, \mathbb{Z}_\ell)_{\omega_1}$.

(ii) The $t$-structure on $\mathcal{D}_c(X, \mathbb{Z}_\ell)$ does usually not restrict to a $t$-structure on $\mathcal{D}_{\text{nuc}}(X, \mathbb{Z}_\ell)$. The problem is that for a Banach sheaf $\mathcal{M}$ it is generally not true that $\tau_{\geq 0} \mathcal{M}$ is still a Banach sheaf: While it is true that $\tau_{\geq 0} \mathcal{M}$ is still $\ell$-adically complete, it may happen that it is not étale mod $\ell$ anymore. As a counterexample, suppose that $X = \mathbb{Z}_\ell$ (viewed as a pro-finite étale space over some algebraically closed non-archimedean field) and let $\mathcal{M}$ be the $\ell$-adic completion of the following étale sheaf $\mathcal{N}$: $\mathcal{N}$ is the étale product of sheaves $\mathcal{N}_n$ for $n \geq 1$, where $\mathcal{N}_n$ is the constant sheaf $\mathbb{Z}/\ell^n \mathbb{Z}$ supported on $X \setminus \ell^n \mathbb{Z}$. Then $\mathcal{M}$ is a Banach sheaf concentrated in homological degrees 0 and 1, but by a direct computation one checks that $\pi_1 M/\ell \pi_1 M$ is not étale: It does not have any étale sections, but it has a non-trivial value on the pro-étale map $\{0\} \to X$.

**Warning 3.3.** In [1, Definition 8.5] there is a very general definition of nuclear objects in a compactly generated symmetric monoidal $\infty$-category. However, our Definition 3.1.(b) differs from that general definition. In fact, if we apply the general definition from [1] to $\mathcal{D}_c(X, \mathbb{Z}_\ell)_{\omega_1}$ then we end up with a subcategory $\mathcal{C} \subseteq \mathcal{D}_{\text{nuc}}(X, \mathbb{Z}_\ell)$ spanned only by the overconvergent sheaves. There are still good reasons to call our sheaves nuclear, as we will see in the following.

The above definition of nuclear sheaves may seem a bit ad-hoc and it is not at all clear from this definition why the $\infty$-category of nuclear sheaves satisfies $v$-descent. Our first goal will therefore be to provide an equivalent definition in terms of trace-class maps. The general definition of trace-class maps provided in [1] §8 does not work in our setting, as it is far too restrictive. We will therefore provide our own definition tailored to the specific setting at hand:

**Definition 3.4.** Let $X$ be an $\ell$-bounded spatial diamond.

(a) We denote by $\text{Hom}_c(-, -)$ the internal hom functor in the symmetric monoidal $\infty$-category $\mathcal{D}_c(X, \mathbb{Z}_\ell)_{\omega_1}$ (this exists because this $\infty$-category is presentable).

(b) We define a functor

$$\text{Hom}^\text{tr}_c(-, -) : \mathcal{D}_c(X, \mathbb{Z}_\ell)^{\omega_1\text{op}} \times \mathcal{D}_c(X, \mathbb{Z}_\ell)_{\omega_1} \to \mathcal{D}_c(X, \mathbb{Z}_\ell)_{\omega_1}$$

as follows: This functor will preserve limits in the first argument and for compact first argument it will preserve colimits in the second argument, so that it is enough to construct its restriction to $\mathcal{D}_c(X, \mathbb{Z}_\ell)^{\omega_1\text{op}} \times \mathcal{D}_c(X, \mathbb{Z}_\ell)^{\omega_1}$. In this case we define it as the $\ell$-adic completion

$$\text{Hom}^\text{tr}_c(-, -) = \overline{\text{Hom}^\prime_c(-, -)},$$

where $\text{Hom}^\prime_c(-, -)$ is the following composition of functors:

$$\mathcal{D}_c(X, \mathbb{Z}_\ell)^{\omega_1\text{op}} \times \mathcal{D}_c(X, \mathbb{Z}_\ell)^{\omega_1} \hookrightarrow \text{Ind}(\mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)^{\omega_1\text{op}}) \times \mathcal{D}_c(X, \mathbb{Z}_\ell)^{\omega_1} \to \mathcal{D}_c(X, \mathbb{Z}_\ell)^{\omega_1}.$$

Here the first functor is induced by Proposition 2.5.(iii) and the second functor is the unique colimit-preserving functor which restricts to $\text{Hom}_c(\mathcal{P}, \mathcal{Q})$ for $\mathcal{P} \in \mathcal{D}_{\text{et}}(X, \mathbb{Z}_\ell)^{\omega_1}$ and $\mathcal{Q} \in \mathcal{D}_c(X, \mathbb{Z}_\ell)^{\omega_1}$.
(c) There is a natural transformation
\[ \text{Hom}^{\text{tr}}(-, -) \to \text{Hom}_c(-, -) \]
of functors \( D_\omega(X, \mathbb{Z}_\ell)^{\text{op}} \times D_\omega(X, \mathbb{Z}_\ell) \to D_\omega(X, \mathbb{Z}_\ell) \). Namely, it is enough to construct this on compact arguments, where it boils down to a natural transformation \( \text{Hom}' \to \text{Hom}_c \), which comes from the fact that \( \text{Hom}' \) is a left Kan extension of the restriction of \( \text{Hom}_c \) to \( D_{\text{et}}(X, \mathbb{Z}_\ell)^{\text{op}} \) in the first argument.

(d) For all \( \mathcal{M}, \mathcal{N} \in D_\omega(X, \mathbb{Z}_\ell) \) we denote
\[ \text{Hom}^{\text{tr}}(\mathcal{M}, \mathcal{N}) := \Gamma(X, \text{Hom}^{\text{tr}}(\mathcal{M}, \mathcal{N})) \].

A map \( \mathcal{M} \to \mathcal{N} \) is called trace-class if it lies in the image of the map \( \text{Hom}^{\text{tr}}(\mathcal{M}, \mathcal{N}) \to \text{Hom}(\mathcal{M}, \mathcal{N}) \) induced by the natural transformation above.

**Remark 3.5.** The definition of \( \text{Hom}^{\text{tr}}(-, -) \) may seem a bit intimidating, so let us provide a more intuitive (albeit less formal) description: Pick any compact objects \( P, Q \in D_\omega(X, \mathbb{Z}_\ell) \). By Proposition 2.5.(ii) we can write \( P = \lim_{\leftarrow n} P_n \) for compact étale sheaves \( P_n \in D_{\text{et}}(X, \mathbb{Z}_\ell)\omega \). We then define
\[ \text{Hom}^{\text{tr}}(P, Q) = (\lim_{\leftarrow n} \text{Hom}(P_n, Q))^- \],
where \( ^- \) denotes \( \ell \)-adic completion. For general \( \mathcal{M}, \mathcal{N} \in D_\omega(X, \mathbb{Z}_\ell) \) we write \( \mathcal{M} = \lim_{\leftarrow i} P_i \) and \( \mathcal{N} = \lim_{\rightarrow j} Q_j \) for compact \( P_i \) and \( Q_j \) and then define
\[ \text{Hom}^{\text{tr}}(\mathcal{M}, \mathcal{N}) = \lim_{\leftarrow i} \lim_{\rightarrow j} \text{Hom}^{\text{tr}}(P_i, Q_j). \]

Here the limit is taken in \( D_\omega(X, \mathbb{Z}_\ell) \).

Before getting back to nuclear sheaves, let us study some of the properties of trace-class maps. It turns out that they enjoy almost the same formal properties as the abstract trace-class maps studied in [1, §8]:

**Lemma 3.6.** Let \( X \) be an \( \ell \)-bounded spatial diamond.

(i) Let \( f : \mathcal{M} \to \mathcal{N}, g : \mathcal{M}' \to \mathcal{M} \) and \( h : \mathcal{N} \to \mathcal{N}' \) be maps in \( D_\omega(X, \mathbb{Z}_\ell) \). If \( f \) is trace-class then so is \( h \circ f \circ g \).

(ii) If \( f : \mathcal{M} \to \mathcal{N} \) and \( f' : \mathcal{M}' \to \mathcal{N}' \) are trace-class maps in \( D_\omega(X, \mathbb{Z}_\ell) \), then so is \( f \otimes f' : \mathcal{M} \otimes \mathcal{M}' \to \mathcal{N} \otimes \mathcal{N}' \).

(iii) Let \( f : \mathcal{M} \to \mathcal{N} \) be a trace-class map in \( D_\omega(X, \mathbb{Z}_\ell) \). Then for every \( \mathcal{L} \in D_\omega(X, \mathbb{Z}_\ell) \) the commutative square
\[
\begin{array}{ccc}
\text{Hom}^{\text{tr}}(\mathcal{L}, \mathcal{M}) & \longrightarrow & \text{Hom}^{\text{tr}}(\mathcal{L}, \mathcal{N}) \\
\downarrow & & \downarrow \\
\text{Hom}_c(\mathcal{L}, \mathcal{M}) & \longrightarrow & \text{Hom}_c(\mathcal{L}, \mathcal{N})
\end{array}
\]
admits a diagonal map \( \text{Hom}_c(\mathcal{L}, \mathcal{M}) \to \text{Hom}^{\text{tr}}(\mathcal{L}, \mathcal{N}) \) making both triangles commute.
(iv) Let $f : \mathcal{P} \to \mathcal{M}$ be a trace-class map in $\mathcal{D}_{\mathcal{Q}}(X, \mathbb{Z}_\ell)_{\omega_1}$ with $\mathcal{P}$ compact. Then there is a compact object $Q$ in $\mathcal{D}_{\mathcal{Q}}(X, \mathbb{Z}_\ell)_{\omega_1}$ such that $f$ factors as $\mathcal{P} \to Q \to \mathcal{M}$, where $\mathcal{P} \to Q$ is also trace-class.

**Proof.** We first make the following general observation: For any $\mathcal{M}, \mathcal{N}, \mathcal{L} \in \mathcal{D}_{\mathcal{Q}}(X, \mathbb{Z}_\ell)_{\omega_1}$ the natural maps

$$
\text{Hom}_* (\mathcal{M}, \mathcal{N}) \otimes \text{Hom}_*^\text{tr} (\mathcal{N}, \mathcal{L}) \to \text{Hom}_* (\mathcal{M}, \mathcal{L}),
$$

$$
\text{Hom}_*^\text{tr} (\mathcal{M}, \mathcal{N}) \otimes \text{Hom}_* (\mathcal{N}, \mathcal{L}) \to \text{Hom}_* (\mathcal{M}, \mathcal{L})
$$

factor over $\text{Hom}_*^\text{tr} (\mathcal{M}, \mathcal{L})$. To see this, consider the $\infty$-category $\mathcal{C}$ of morphisms $g : \mathcal{N} \to \mathcal{N}'$ in $\mathcal{D}_{\mathcal{Q}}(X, \mathbb{Z}_\ell)_{\omega_1}$, where morphisms $g_1 \to g_2$ in $\mathcal{C}$ are given by commuting squares

$$
\begin{array}{ccc}
N_1 & \xrightarrow{g_1} & N'_1 \\
\downarrow & & \uparrow \\
N_2 & \xrightarrow{g_2} & N'_2
\end{array}
$$

(This can easily be constructed as simplicial sets.) Then we can view all of the above expressions as functors in $\mathcal{M}$, $g : \mathcal{N} \to \mathcal{N}'$ and $\mathcal{L}$ (where we replace the second appearing $\mathcal{N}$ in the above expression by $\mathcal{N}'$) and the above morphisms as natural transformations of such functors. Since both $\text{Hom}_* (\mathcal{M}, \mathcal{L})$ and $\text{Hom}_*^\text{tr} (\mathcal{M}, \mathcal{L})$ transform colimits in $\mathcal{M}$ into limits, we can formally reduce the construction of the desired natural transformation to the case that $\mathcal{M}$ is compact (use that right Kan extension is a right adjoint functor). Then all functors preserve colimits in $\mathcal{N}$, so we can reduce to the case that $\mathcal{N}$ is compact as well. By factoring out colimits in $\mathcal{N}'$ (like for $\mathcal{M}$) we can reduce to the case that $\mathcal{N}'$ is compact. Then we can finally also factor out colimits in $\mathcal{L}$ to reduce to the case that all of $\mathcal{M}, \mathcal{N}, \mathcal{N}'$ and $\mathcal{L}$ are compact. In this case $\text{Hom}_*^\text{tr} (\mathcal{M}, \mathcal{L})$ is $\ell$-adically complete, so we can ignore $\ell$-adic completions. We therefore end up with constructing natural transformations

$$
\text{Hom}_* (\mathcal{M}, \mathcal{N}) \otimes \text{Hom}_*^\text{tr} (\mathcal{N}', \mathcal{L}) \to \text{Hom}_*^\text{tr} (\mathcal{M}, \mathcal{L}),
$$

$$
\text{Hom}_*^\text{tr} (\mathcal{M}, \mathcal{N}) \otimes \text{Hom}_* (\mathcal{N}', \mathcal{L}) \to \text{Hom}_*^\text{tr} (\mathcal{M}, \mathcal{L}),
$$

functorial in $\mathcal{M}$, $g : \mathcal{N} \to \mathcal{N}'$ and $\mathcal{L}$. For the second natural transformation, we can write $\mathcal{M} = \varprojlim_n \mathcal{M}_n$ for qcqs étale $\mathcal{M}_n$ and pull out this limit on both sides (by definition of $\text{Hom}_*^\text{tr}$), transforming it into colimits. This reduces the construction of the natural transformation to the case that $\mathcal{M}$ is qcqs étale, in which case $\text{Hom}_*^\text{tr} (\mathcal{M}, -) = \text{Hom}_* (\mathcal{M}, -)$, so we are done. For the first natural transformation we can similarly reduce to the case that $\mathcal{N}'$ is qcqs étale. Then if we write $\mathcal{N} = \varprojlim_n \mathcal{N}_n$ for qcqs étale $n$ we know that $g$ factors over some $\mathcal{N}_n$ and is thus a limit of maps $\mathcal{N}_{n'} \to \mathcal{N}_n$ in $\mathcal{C}$. By the same argument as above we can pull out this limit and thus assume that $\mathcal{N}$ is also qcqs étale. But then we can further pull out a limit in $\mathcal{M}$ to reduce to the case that $\mathcal{M}$ is qcqs étale, in which case the desired natural transformation is evident.

With the above preparations at hand, let us now prove the actual claims. Part (i) follows immediately from the just constructed natural transformations by applying $\Gamma(X, -)$. To prove (iii), note that the given map $f$ provides a map $\mathbb{Z}_\ell \to \text{Hom}_*^\text{tr} (\mathcal{M}, N)$, so that the desired diagonal map can be constructed as follows:

$$
\text{Hom}_* (\mathcal{L}, \mathcal{M}) \to \text{Hom}_* (\mathcal{L}, \mathcal{M}) \otimes \text{Hom}_*^\text{tr} (\mathcal{M}, \mathcal{N}) \to \text{Hom}_*^\text{tr} (\mathcal{L}, \mathcal{N}),
$$

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where the second map is the one constructed above. To prove (ii) we need to show that the natural map

\[ \text{Hom}^\text{fr}_!(\mathcal{M}, \mathcal{N}) \otimes \text{Hom}^\text{fr}_!(\mathcal{M'}, \mathcal{N'}) \to \text{Hom}_!(\mathcal{M} \otimes \mathcal{M}', \mathcal{N} \otimes \mathcal{N'}) \]

factors over \( \text{Hom}^\text{fr}_!(\mathcal{M} \otimes \mathcal{M}', \mathcal{N} \otimes \mathcal{N'}) \). This can be done in a similar manner as above by reducing to the case that all sheaves are qcqs étale; we leave the details to the reader. To prove (iv), write \( \mathcal{M} \) as a filtered colimit \( \mathcal{M} = \varinjlim_i \mathcal{Q}_i \) with compact \( \mathcal{Q}_i \) and note that \( \text{Hom}^\text{fr}_!(\mathcal{P}, \mathcal{M}) = \varprojlim_i \text{Hom}^\text{fr}_!(\mathcal{P}, \mathcal{Q}_i) \), which easily implies the claim.

In order to relate trace-class maps to nuclear sheaves (as defined in Definition 3.1.(b)) we need further properties of \( \text{Hom}^\text{fr}_! \):

**Lemma 3.7.** Let \( X \) be an \( \ell \)-bounded spatial diamond and let \( \mathcal{P} \in \mathcal{D}_c(X, \mathbb{Z}_\ell)_{\omega_1} \) be compact.

(i) The functor

\[ \text{Hom}^\text{fr}_!(\mathcal{P}, -) : \mathcal{D}_c(X, \mathbb{Z}_\ell)_{\omega_1} \to \mathcal{D}_c(X, \mathbb{Z}_\ell)_{\omega_1} \]

is bounded, i.e. there are integers \( a \leq b \) such that for every static \( \mathcal{M} \in \mathcal{D}_c(X, \mathbb{Z}_\ell)_{\omega_1} \) the sheaf \( \text{Hom}^\text{fr}_!(\mathcal{P}, \mathcal{M}) \) is concentrated in homological degrees \( [a, b] \).

(ii) For every \( \ell \)-adically complete \( \mathcal{M} \in \mathcal{D}_c(X, \mathbb{Z}_\ell)_{\omega_1} \) the sheaf \( \text{Hom}^\text{fr}_!(\mathcal{P}, \mathcal{M}) \) is \( \ell \)-adically complete.

**Proof.** By Proposition 2.5.(ii) we can assume that \( \mathcal{P} = \mathbb{Z}_{\ell, \text{ad}}[U] \) for some basic \( U = \varprojlim_n U_n \in X_{\text{proet}} \). We first prove (i) in the case that \( \mathcal{M} = \mathbb{Z}_{\ell, \text{ad}}[V] \) for some basic \( \varprojlim_m V_m \in X_{\text{proet}} \). Then we have

\[ \text{Hom}^\text{fr}_!(\mathcal{P}, \mathcal{M}) = (\varprojlim_n \varprojlim_m \text{Hom}_!(\mathbb{Z}/\ell^m \mathbb{Z}[U_n], \mathbb{Z}_\ell[V_m]))^\sim. \]

Thus the desired boundedness boils down to showing that \( \text{Hom}_!(\mathbb{Z}/\ell^m \mathbb{Z}[U_n], \mathbb{Z}_\ell[V_m]) \) is bounded independent of \( V_m \) (since countable limits are bounded by 1), which follows immediately from the \( \ell \)-boundedness of \( X \). We can now deduce that the functor \( \text{Hom}^\text{fr}_!(\mathcal{P}, -) \) is right-bounded by writing any static \( \mathcal{M} \) as a sifted colimit of static compact objects (i.e. choosing a resolution by direct sums of compact objects) and using that \( \text{Hom}^\text{fr}_!(\mathcal{P}, -) \) preserves this colimit.

Now let \( \mathcal{M} \) be general (and still static). By writing \( \mathcal{M} \) as a filtered colimit of finitely presented static objects, we can reduce (i) to the case that \( \mathcal{M} \) is finitely presented, i.e. a countable limit \( \mathcal{M} = \varinjlim_n \mathcal{M}_n \) of qcqs étale sheaves \( \mathcal{M}_n \in \mathcal{D}_\text{et}(X, \mathbb{Z}_\ell)_{\omega} \). Then \( \text{Hom}^\text{fr}_!(\mathcal{P}, \mathcal{M}) \) is \( \ell \)-adically complete: Since \( \ell \)-adic completion is stable under uniformly right-bounded geometric realizations and \( \text{Hom}^\text{fr}_!(\mathcal{P}, \mathcal{M}) \) is right-bounded by the above, we can pick a resolution of \( \mathcal{M} \) by compact objects in order to reduce to the case that \( \mathcal{M} \) is compact – then the \( \ell \)-adic completeness is clear by definition. It now follows that the natural map

\[ (\varprojlim_n \text{Hom}_!(\mathbb{Z}/\ell^m \mathbb{Z}[U_n], \mathcal{M}))^\sim \to \text{Hom}^\text{fr}_!(\mathcal{P}, \mathcal{M}) \]

is an isomorphism (by again passing to geometric realizations in \( \mathcal{M} \)). In the same way as for compact \( \mathcal{M} \) we deduce that \( \text{Hom}^\text{fr}_!(\mathcal{P}, \mathcal{M}) \) is bounded independent of \( \mathcal{M} \). This finishes the proof of (i).
We now prove (ii), so let the $\ell$-adically complete sheaf $\mathcal{M}$ be given. By the boundedness of $\text{Hom}^{tr}_\mathfrak{c}(\mathcal{P}, -)$ this functor commutes with Postnikov limits, so we can assume that $\mathcal{M}$ is left-bounded. Moreover, since $\ell$-adic completeness can be checked on homotopy sheaves and $\text{Hom}^{tr}_\mathfrak{c}(\mathcal{P}, -)$ is bounded, we can use $\mathcal{M} = \varinjlim_{n} \tau_{\geq n}\mathcal{M}$ in order to reduce to the case that $\mathcal{M}$ is bounded. We can then even assume that $\mathcal{M}$ is static. By choosing a resolution of $\mathcal{M}$ in terms of direct sums of copies of $\mathbb{Z}_{\ell,\omega}[V]$ for varying basic $V = \varprojlim_{n} V_n \in X_{\text{proet}}$ and using that both $\text{Hom}^{tr}_\mathfrak{c}(\mathcal{P}, -)$ and $\ell$-adic completion commute with uniformly right-bounded geometric realizations we can reduce to the case that $\mathcal{M}$ is a completed direct sum of copies of $\mathbb{Z}_{\ell,\omega}[V]$. Since $\ell$-adic completion commutes with $\omega_1$-filtered colimits we can further reduce to a countable such sum, i.e. we have $\mathcal{M} = \bigoplus_{k} \mathbb{Z}_{\ell,\omega}[V_k]$ for various basic $V = \varprojlim_{n} V_{k,n} \in X_{\text{proet}}$ and integers $k > 0$. To abbreviate notation let us denote $Q_k := \mathbb{Z}_{\ell,\omega}[V_k]$. For every sequence of integers $\alpha_m \geq 0$ converging to $\infty$ we write $\mathcal{M}_\alpha := \prod_{k} \ell^{\alpha_k} Q_k$, so that $\mathcal{M} = \varinjlim_{\alpha} \mathcal{M}_\alpha$. We now claim that the natural map

$$
\text{Hom}^{tr}_\mathfrak{c}(\mathcal{P}, \mathcal{M}) = \varprojlim_{\alpha} \text{Hom}^{tr}_\mathfrak{c}(\mathcal{P}, \mathcal{M}_\alpha) \simeq \varprojlim_{\alpha} \prod_{k} \ell^{\alpha_k} \text{Hom}^{tr}_\mathfrak{c}(\mathcal{P}, Q_k)
$$

is an isomorphism. Here by $\prod_{k} \ell^{\alpha_k} \text{Hom}^{tr}_\mathfrak{c}(\ldots)$ we mean the object $\prod_{k} \text{Hom}^{tr}_\mathfrak{c}(\ldots)$ but where the $\ell^{\alpha_k}$ determine the transitions maps in the filtered colimit. To prove the claimed isomorphism of the colimits over $\alpha$, it is enough to find suitable sections. Concretely, fix any sequence $\alpha$ and choose another sequence $\alpha'$ such that $\alpha' \leq \alpha$ and the sequence $\alpha - \alpha'$ still converges to $\infty$. Then we get natural maps

$$
\prod_{k} \ell^{\alpha_k} \text{Hom}^{tr}_\mathfrak{c}(\mathcal{P}, Q_k) \to \bigoplus_{k} \ell^{\alpha_k} \text{Hom}^{tr}_\mathfrak{c}(\mathcal{P}, Q_k) \to \text{Hom}^{tr}_\mathfrak{c}(\mathcal{P}, \prod_{k} \ell^{\alpha_k} Q_k).
$$

The first map exists because $\alpha - \alpha'$ converges to $\infty$; it is the one which multiplies the $k$-th part of the product/sum by $\ell^{\alpha_k - \alpha_k}$. The second map exists because $\text{Hom}^{tr}_\mathfrak{c}(\mathcal{P}, \prod_{k} \ell^{\alpha_k} Q_k)$ is $\ell$-adically complete (by the same argument as in the proof of (i) using that $\prod_{k} Q_k$ is static and finitely presented), so that we can replace the completed direct sum by an ordinary direct sum for the construction. One checks that the thus constructed map is indeed the desired section of the map of filtered systems, proving the above isomorphism. To finish the proof that $\text{Hom}^{tr}_\mathfrak{c}(\mathcal{P}, \mathcal{M})$ is $\ell$-adically complete, it remains to see that the natural map

$$
\varprojlim_{\alpha} \prod_{k} \ell^{\alpha_k} \text{Hom}^{tr}_\mathfrak{c}(\mathcal{P}, Q_k) \simeq \bigoplus_{k} \text{Hom}^{tr}_\mathfrak{c}(\mathcal{P}, Q_k)
$$

is an isomorphism, or equivalently that the source of this map is $\ell$-adically complete. This can be checked on homotopy sheaves and by the usual arguments involving geometric realizations we reduce to showing this statement after replacing each $\text{Hom}^{tr}_\mathfrak{c}(\mathcal{P}, Q_k)$ by a completed direct sum of static compact objects. Then everything is static and the claimed isomorphism can easily be checked on sections $\Gamma(U, -)$ for $w$-contractible $U$ (reducing the problem to an easy problem about classical abelian groups).

We are finally in the position to provide the promised characterization of nuclear sheaves in terms of trace-class maps. This will immediately enable us to prove $v$-descent as well, leading to our first main result on nuclear sheaves in this paper.

**Definition 3.8.** Let $X$ be an $\ell$-bounded spatial diamond. A sheaf $\mathcal{N} \in \mathcal{D}_{\mathfrak{c}}(X, \mathbb{Z}_{\ell})_{\omega_1}$ is called **basic nuclear** if it can be written as a sequential colimit $\mathcal{N} = \varinjlim_{n} \mathcal{P}_n$ such that all $\mathcal{P}_n$ are compact and all transition maps $\mathcal{P}_n \to \mathcal{P}_{n+1}$ are trace-class.
Theorem 3.9. (i) Let $X$ be an $\ell$-bounded spatial diamond. Then for an $\omega_1$-solid sheaf $\mathcal{M}$ on $X$ the following are equivalent:

(a) $\mathcal{M}$ is nuclear.

(b) $\mathcal{M}$ is an $\omega_1$-filtered colimit of basic nuclear sheaves.

(c) For every compact $\mathcal{P} \in D_c(X, \mathbb{Z}_\ell)_{\omega_1}$ the natural map $\text{Hom}^{tr}(\mathcal{P}, \mathcal{M}) \to \text{Hom}(\mathcal{P}, \mathcal{M})$ is an isomorphism.

The $\infty$-category $D_{\text{nuc}}(X, \mathbb{Z}_\ell)$ is $\omega_1$-compactly generated with the $\omega_1$-compact objects being precisely the basic nuclear sheaves. Moreover, this $\infty$-category contains all étale sheaves and is stable under all small colimits and under the tensor product in $D_c(X, \mathbb{Z}_\ell)_{\omega_1}$.

(ii) The assignment

$$X \mapsto D_{\text{nuc}}(X, \mathbb{Z}_\ell)$$

defines a hypercomplete sheaf of presentable symmetric monoidal $\infty$-categories on the $v$-site of $\ell$-bounded spatial diamonds.

Proof. We first prove (i), so let $X$ and $\mathcal{M}$ be given. The equivalence of (b) and (c) follows formally from Lemma 3.6, see [1, Theorem 8.6]. Now assume that $\mathcal{M}$ is nuclear; we will show that it satisfies condition (c). Since (c) is stable under colimits in $\mathcal{M}$ we can assume that $\mathcal{M}$ is a Banach module. Then by Lemma 3.7.(ii) both $\text{Hom}^{tr}(\mathcal{P}, \mathcal{M})$ and $\text{Hom}(\mathcal{P}, \mathcal{M})$ are $\ell$-adically complete, so it is enough to show that $N := \mathcal{M}/\ell \mathcal{M}$ satisfies (c). Now $N$ is étale, so by Proposition 2.5.(i) we can write it as a colimit of copies of $\mathbb{F}_\ell[U]$ for varying basic $U \in X_{\text{et}}$. It is therefore enough to show that $\mathbb{F}_\ell[U]$ satisfies (c), which follows by explicit computation (using that if we write $\mathcal{P} = \varprojlim_n \mathcal{P}_n$ for qcs étale sheaves $\mathcal{P}_n$ then $\text{Hom}(\mathcal{P}, \mathbb{F}_\ell[U]) = \varprojlim_n \text{Hom}(\mathcal{P}_n, \mathbb{F}_\ell[U])$).

To finish the equivalence of (a), (b) and (c) it remains to show that (b) implies (a), so from now on assume that $\mathcal{M} = \varprojlim_n \mathcal{P}_n$ is basic nuclear; we need to show that $\mathcal{M}$ is nuclear. This follows if we can show that every map $\mathcal{P}_n \to \mathcal{P}_{n+1}$ factors over some Banach sheaf because then $\mathcal{M}$ is the colimit of these Banach sheaves. Let $\mathcal{P}'_{n+1}$ be the $\ell$-adic completion of the pushforward of $\mathcal{P}_{n+1}$ to the étale site. We claim that the natural map

$$\text{Hom}(\mathcal{P}_n, \mathcal{P}'_{n+1}) = \text{Hom}^{tr}(\mathcal{P}_n, \mathcal{P}'_{n+1}) \xrightarrow{\sim} \text{Hom}^{tr}(\mathcal{P}_n, \mathcal{P}_{n+1})$$

is an isomorphism (here the first identity follows from (c) because $\mathcal{P}'_{n+1}$ is clearly nuclear), which provides the desired factorization $\mathcal{P}_n \to \mathcal{P}'_{n+1} \to \mathcal{P}_{n+1}$. To prove this isomorphism, note that by Lemma 3.7.(ii) both sides of the claimed isomorphism are $\ell$-adically complete, so we can check the isomorphism modulo $\ell$. Then it boils down to the following claim: Let $\mathcal{M} (\mathcal{M}'_{n+1}/\ell \mathcal{M}')$ be an $\mathbb{F}_\ell$-module in $D_c(X, \mathbb{Z}_\ell)_{\omega_1}$ and let $\mathcal{M}_{\text{et}}$ denote its pushforward to the étale site. Then the natural map

$$\text{Hom}(\mathcal{P}_n, \mathcal{M}_{\text{et}}) = \text{Hom}^{tr}(\mathcal{P}_n, \mathcal{M}_{\text{et}}) \xrightarrow{\sim} \text{Hom}^{tr}(\mathcal{P}_n, \mathcal{M})$$

is an isomorphism (here the first identity follows because $\mathcal{M}_{\text{et}}$ is nuclear and hence satisfies (c) by the above). Both sides of the claimed isomorphism preserve all small colimits (for the functor $\mathcal{M} \mapsto \mathcal{M}_{\text{et}}$ this follows from the $\ell$-boundedness of $X$, as this implies that $\Gamma(U, -)$ is cohomologically bounded for all basic $U \in X_{\text{proet}}$), so we can assume that $\mathcal{M}$ is compact. Then we have

$$\text{Hom}^{tr}(\mathcal{P}_n, \mathcal{M}) = \varprojlim_k \text{Hom}(\mathcal{P}_{n,k}, \mathcal{M}) = \varprojlim_k \text{Hom}(\mathcal{P}_{n,k}, \mathcal{M}_{\text{et}}) = \text{Hom}(\mathcal{P}_n, \mathcal{M}_{\text{et}})$$

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for any representation $P_n = \lim_{\llcorner k} P_{n,k}$ with all $P_{n,k}$ being qcqs étale (in the first identity there is no $\ell$-adic completion required because all terms are killed by $\ell$). This finishes the proof that (a), (b) and (c) are equivalent.

We finish the proof of (i): It is formal that $D_{\text{nuc}}(X,\mathbb{Z}_\ell)$ is $\omega_1$-compactly generated with the $\omega_1$-compact generators being precisely the basic nuclear objects (see [1, Theorem 8.6]). We have already shown that all étale sheaves are nuclear, and from (c) it follows immediately that nuclear sheaves are stable under all small colimits in $D_{\omega_1}(X,\mathbb{Z}_\ell)$. To show that nuclear sheaves are also stable under tensor products, it is now enough to see this for basic nuclear objects, where it follows from Lemma 3.6.(ii) (to get the require right-boundedness, note that the above identity is false, because by definition $\text{Hom}_{\text{nuc}}^\tau(P,\mathbb{Z}_\ell)$ transforms colimits in $P$ into limits, whereas $\text{Hom}_\omega(P,\mathbb{Z}_\ell) \otimes -$ does not. This does not affect the notion of nuclear sheaves, because this only every uses $\text{Hom}_{\text{nuc}}^\tau(P,\mathbb{Z}_\ell)$ for compact $P$ (some subtleties arise if compact objects are not stable under tensor products, though).

While many natural constructions of sheaves preserve the nuclear category, some (like internal hom) do not. We therefore need a way of “nuclearizing” sheaves, which can be done as follows.
**Definition 3.11.** Let $X$ be an $\ell$-bounded spatial diamond. By the adjoint functor theorem the inclusion $\mathcal{D}_{\text{nuc}}(X,\mathbb{Z}_\ell) \hookrightarrow \mathcal{D}(X,\mathbb{Z}_\ell)_{\omega_1}$ admits a right adjoint

$$(-)_{\text{nuc}} : \mathcal{D}(X,\mathbb{Z}_\ell)_{\omega_1} \rightarrow \mathcal{D}_{\text{nuc}}(X,\mathbb{Z}_\ell), \quad M \mapsto M_{\text{nuc}}.$$  

For every $M \in \mathcal{D}(X,\mathbb{Z}_\ell)_{\omega_1}$ we call $M_{\text{nuc}}$ the nuclearization of $M$.

**Proposition 3.12.** Let $X$ be an $\ell$-bounded spatial diamond. Then the nuclearization functor on $X$ is characterized by the following properties:

(i) The functor $(-)_{\text{nuc}} : \mathcal{D}(X,\mathbb{Z}_\ell)_{\omega_1} \rightarrow \mathcal{D}_{\text{nuc}}(X,\mathbb{Z}_\ell)$ preserves all small colimits.

(ii) If $M \in \mathcal{D}(X,\mathbb{Z}_\ell)_{\omega_1}$ is $\ell$-adically complete, then $M_{\text{nuc}}$ is the $\ell$-adic completion of $M_{\text{et}}$. In particular, if $M$ is killed by some power of $\ell$ then $M_{\text{nuc}} = M_{\text{et}}$.

(iii) The functor $(-)_{\text{nuc}}$ is bounded, i.e. there are integers $a \leq b$ such that for every static $M \in \mathcal{D}(X,\mathbb{Z}_\ell)_{\omega_1}$ the sheaf $M_{\text{nuc}}$ lies in homological degrees $[a,b]$ (with respect to the solid $t$-structure).

**Proof.** We construct a colimit-preserving functor

$$F : \mathcal{D}(X,\mathbb{Z}_\ell)_{\omega_1} \rightarrow \mathcal{D}_{\text{nuc}}(X,\mathbb{Z}_\ell)$$

as follows: It is enough to construct it on compact objects and for compact $Q \in \mathcal{D}(X,\mathbb{Z}_\ell)_{\omega_1}$ we define $F(Q)$ to be the $\ell$-adic completion of $Q_{\text{et}}$. After composing $F$ with the inclusion $\iota : \mathcal{D}_{\text{nuc}}(X,\mathbb{Z}_\ell) \rightarrow \mathcal{D}(X,\mathbb{Z}_\ell)_{\omega_1}$ we get a natural map $\iota F \rightarrow \text{id}$. It follows that there is a natural map $F \rightarrow (-)_{\text{nuc}}$. We claim that this is an isomorphism. This can be checked on sections from basic nuclear objects, i.e. for every $M \in \mathcal{D}(X,\mathbb{Z}_\ell)_{\omega_1}$ and every sequential colimit $N = \lim_n P_n$ of compact $P_n \in \mathcal{D}(X,\mathbb{Z}_\ell)_{\omega_1}$ with trace-class transition maps, we need to verify that the natural map

$$\text{Hom}(N, F(M)) \xrightarrow{\sim} \text{Hom}(N, M_{\text{nuc}})$$

is an isomorphism. Let us first compute the left-hand side. We claim that for any $L \in \mathcal{D}(X,\mathbb{Z}_\ell)_{\omega_1}$, the natural map

$$\text{Hom}(L, \iota F(M)) = \text{Hom}^\text{tr}(L, \iota F(M)) \xrightarrow{\sim} \text{Hom}^\text{tr}(L, M)$$

is an isomorphism. This can be checked for compact $L$; then both sides commute with colimits in $\mathcal{M}$, so we can also reduce to the case that $M$ is compact. Factoring out the $\ell$-adic completion on both sides, we can assume that $M$ is killed by $\ell$. But then both sides evaluate to $\lim_n \text{Hom}(L_n, M)$ for any representation $L = \lim_n L_n$ with qcqs étale $L_n$.

Let us get back to the claimed isomorphism of $F$ and $(-)_{\text{nuc}}$, so let $M$ and $N = \lim_n P_n$ be as above. The claim now reduces to showing that the natural map

$$\lim_n \text{Hom}^\text{tr}(P_n, M) = \text{Hom}^\text{tr}(N, M) = \text{Hom}(N, F(M))$$

$$\xrightarrow{\sim} \text{Hom}(N, M_{\text{nuc}}) = \text{Hom}(N, M) = \lim_n \text{Hom}(P_n, M)$$

is an isomorphism. This follows by constructing sections $\text{Hom}(P_{n+1}, M) \rightarrow \text{Hom}^\text{tr}(P_n, M)$, which exist by Lemma 3.6(i).
Having established the isomorphism $F = (−)_{\text{nuc}}$, we can now easily prove the claims (i)–(iii). Part (i) is obvious since $F$ preserves all small colimits by construction. For part (ii) note that if $\mathcal{M}$ is killed by $\ell$ then clearly $F(\mathcal{M}) = \mathcal{M}_{\text{et}}$. Since $(−)_{\text{nuc}}$ preserves limits (as it is a right adjoint functor) we deduce that $(−)_{\text{nuc}}$ preserves $\ell$-adically complete objects. For part (iii), we can immediately reduce to the case that $\mathcal{M}$ is static and finitely presented and hence $\ell$-adically complete. Then the claim follows from the fact that both $\ell$-adic completion and pushforward to the étale site are bounded.

We now have a good understanding of nuclear $\mathbb{Z}_\ell$-sheaves on $\ell$-bounded spatial diamonds. It is formal to extend this notion to all small $v$-stacks and to modules over any nuclear $\mathbb{Z}_\ell$-algebra $\Lambda$, where by “nuclear $\mathbb{Z}_\ell$-algebra” we mean an (animated condensed) $\mathbb{Z}_\ell$-algebra whose underlying $\mathbb{Z}_\ell$-module is nuclear. Prominent examples of nuclear $\mathbb{Z}_\ell$-algebras are $\mathbb{F}_\ell$, $\mathbb{Z}_\ell$, $\mathbb{Q}_\ell$, $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}_\ell$.

**Definition 3.13.** For any small $v$-stack $X$ we define

$$\mathcal{D}_{\text{nuc}}(X, \mathbb{Z}_\ell) \subseteq \mathcal{D}_{\omega_1}(X, \mathbb{Z}_\ell) \subseteq \mathcal{D}(X, \mathbb{Z}_\ell)$$

by descent from the case of $\ell$-bounded spatial diamonds using Corollary 2.7 and Theorem 3.9.(ii) (note that every strictly totally disconnected space is an $\ell$-bounded spatial diamond, so that $\ell$-bounded spatial diamonds form a basis for the $v$-site). In other words, a $\mathbb{Z}_\ell$-sheaf $\mathcal{M} \in \mathcal{D}(X, \mathbb{Z}_\ell)$ lies in $\mathcal{D}_{\text{nuc}}(X, \mathbb{Z}_\ell)$ resp. $\mathcal{D}_{\omega_1}(X, \mathbb{Z}_\ell)$ if and only if this is true after pullback to every $\ell$-bounded spatial diamond.

**Definition 3.14.** Let $\Lambda$ be a nuclear $\mathbb{Z}_\ell$-algebra.

(a) By pullback $\Lambda$ defines a v-sheaf of connective $E_\infty$-rings on the big $v$-site of all small $v$-stacks, which we still denote $\Lambda$. In particular, for every small $v$-stack $X$, we get a stable $\infty$-category $\mathcal{D}(X, \Lambda)$ of $\Lambda$-modules on $X$. This $\infty$-category admits a complete $t$-structure and the forgetful functor $\mathcal{D}(X, \Lambda) \to \mathcal{D}(X, \mathbb{Z}_\ell)$ is $t$-exact, conservative and preserves all small limits and colimits. Moreover, for any map of small $v$-stacks, the pullback and pushforward functors on $\mathbb{Z}_\ell$-modules can be enhanced to functors on $\Lambda$-modules.

(b) Let $X$ be a small $v$-stack. We define

$$\mathcal{D}_{\text{nuc}}(X, \Lambda) \subseteq \mathcal{D}_{\omega_1}(X, \Lambda) \subseteq \mathcal{D}(X, \Lambda)$$

to be the full subcategories spanned by those $\Lambda$-modules which lie in $\mathcal{D}_{\text{nuc}}(X, \mathbb{Z}_\ell)$ resp. $\mathcal{D}_{\omega_1}(X, \mathbb{Z}_\ell)$ after applying the forgetful functor. Equivalently we can view $\Lambda$ as an $E_\infty$-algebra in $\mathcal{D}_{\text{nuc}}(X, \mathbb{Z}_\ell)$ and define $\mathcal{D}_{\text{nuc}}(X, \Lambda)$ as the $\infty$-category of $\Lambda$-modules in $\mathcal{D}_{\text{nuc}}(X, \mathbb{Z}_\ell)$. The objects of $\mathcal{D}_{\text{nuc}}(X, \Lambda)$ are called the nuclear $\Lambda$-modules on $X$ and the objects of $\mathcal{D}_{\omega_1}(X, \Lambda)$ are called the $\omega_1$-solid $\Lambda$-modules on $X$.

We can formally extend most of the results on nuclear and $\omega_1$-solid $\mathbb{Z}_\ell$-modules on $\ell$-bounded spatial diamonds to nuclear and $\omega_1$-solid $\Lambda$-modules on small $v$-stacks:

**Proposition 3.15.** Let $\Lambda$ be a nuclear $\mathbb{Z}_\ell$-algebra.

(i) The assignment

$$X \mapsto \mathcal{D}_{\omega_1}(X, \Lambda)$$

defines a hypercomplete sheaf of presentable symmetric monoidal stable $\infty$-categories with complete $t$-structure on the $v$-site of all small $v$-stacks.
(ii) Let $X$ be a small $v$-stack. The inclusion $\mathcal{D}_c(X, \Lambda)_{\omega_1} \hookrightarrow \mathcal{D}(X_v, \Lambda)$ is $t$-exact and commutes with all pullbacks, colimits and countable limits.

(iii) Let $X$ be an $\ell$-bounded spatial diamond. Then $\mathcal{D}_c(X, \Lambda)_{\omega_1}$ is compactly generated. The compact objects are generated under finite (co)limits and retracts by the objects $P \otimes_{\mathbb{Z}_\ell} \Lambda$ for compact $P \in \mathcal{D}_c(X, \mathbb{Z}_\ell)_{\omega_1}$.

Proof. Part (i) is clear by definition for $\Lambda = \mathbb{Z}_\ell$ (for presentability use that presentable $\infty$-categories are stable under limits and use (iii)). For general $\Lambda$ it follows by repeating the argument in Corollary 2.7 and noting that everything commutes with the forgetful functor along $\mathbb{Z}_\ell \to \Lambda$. For part (ii) we can similarly reduce to the case $\Lambda = \mathbb{Z}_\ell$. Then the commutation with pullbacks is true by design and the commutation with colimits and countable limits follows from Proposition 2.5.(iii) because everything can be reduced to $\ell$-bounded spatial diamonds (using that pullbacks preserve limits and colimits). Part (iii) follows easily from Proposition 2.5.(ii).

Proposition 3.16. Let $\Lambda$ be a nuclear $\mathbb{Z}_\ell$-algebra.

(i) The assignment

$$X \mapsto \mathcal{D}_{\text{nuc}}(X, \Lambda)$$

defines a hypercomplete sheaf of presentable symmetric monoidal $\infty$-categories on the $v$-site of all small $v$-stacks.

(ii) Let $X$ be a small $v$-stack. The inclusion $\mathcal{D}_c(X, \Lambda)_{\text{nuc}} \hookrightarrow \mathcal{D}_v(X, \Lambda)_{\omega_1}$ is symmetric monoidal and commutes with all pullbacks and colimits.

(iii) Let $X$ be an $\ell$-bounded spatial diamond. Then $\mathcal{D}_{\text{nuc}}(X, \Lambda)$ is $\omega_1$-compactly generated. The $\omega_1$-compact objects are generated under countable colimits by the objects $\mathcal{N} \otimes_{\mathbb{Z}_\ell} \Lambda$ for basic nuclear $\mathcal{N} \in \mathcal{D}_{\text{nuc}}(X, \mathbb{Z}_\ell)$.

Proof. We can argue as in Proposition 3.15 to reduce everything to Theorem 3.9.

Proposition 3.17. Let $\Lambda \to \Lambda'$ be a map of nuclear $\mathbb{Z}_\ell$-algebras and let $X$ be a small $v$-stack.

(i) There is a natural pair of adjoint functors

$$- \otimes_\Lambda \Lambda': \mathcal{D}_{\text{nuc}}(X, \Lambda) \rightleftarrows \mathcal{D}_{\text{nuc}}(X, \Lambda'): \text{Forget},$$

both of which commute with all pullbacks and colimits. Moreover, the functor $- \otimes_\Lambda \Lambda'$ is symmetric monoidal and the forgetful functor is conservative and preserves all limits.

The same is true for the $\infty$-categories of $\omega_1$-solid sheaves instead of nuclear sheaves, in which case the forgetful functor is additionally $t$-exact.

(ii) Both $- \otimes_\Lambda \Lambda'$ and the forgetful functor commute with the inclusion of nuclear sheaves into $\omega_1$-solid sheaves.

Proof. This follows immediately from the fact that $\mathcal{D}_{\text{nuc}}(X, \Lambda)$ is the $\infty$-category of $\Lambda$-modules in $\mathcal{D}_{\text{nuc}}(X, \mathbb{Z}_\ell)$ (and similarly for $\mathcal{D}_c(X, \Lambda)_{\omega_1}$) and the fact that pullback functors and the inclusion of nuclear sheaves into $\omega_1$-solid sheaves are symmetric monoidal.
It is convenient to also define the nuclearization functor in the general setting of nuclear $\Lambda$-modules on small v-stacks, even though on general small v-stacks it will not have the same nice properties.

**Definition 3.18.** Let $\Lambda$ be a nuclear $\mathbb{Z}_\ell$-algebra and $X$ a small v-stack. We define the *nuclearization* functor

$$(-)_{\text{nuc}} : \mathcal{D}(X, \Lambda)_{\omega_1} \to \mathcal{D}_{\text{nuc}}(X, \Lambda)$$

to be the right adjoint of the inclusion.

**Proposition 3.19.** Let $\Lambda$ be a nuclear $\mathbb{Z}_\ell$-algebra and let $X$ be a small v-stack.

(i) If $X$ is an $\ell$-bounded spatial diamond then the functor $(-)_{\text{nuc}} : \mathcal{D}(X, \Lambda)_{\omega_1} \to \mathcal{D}_{\text{nuc}}(X, \Lambda)$ preserves all small colimits and is bounded with respect to the $t$-structure on $\mathcal{D}(X, \Lambda)_{\omega_1}$.

(ii) If $\Lambda \to \Lambda'$ is a map of nuclear $\mathbb{Z}_\ell$-algebras then nuclearization on $X$ commutes with the forgetful functor along $\Lambda \to \Lambda'$.

**Proof.** Part (ii) follows immediately from Proposition 3.17.(ii) by passing to right adjoints. Part (i) follows from (ii) and Proposition 3.12 (use the forgetful functor along $\mathbb{Z}_\ell \to \Lambda$).

In the case that the nuclear $\mathbb{Z}_\ell$-algebra $\Lambda$ is discrete (e.g. $\Lambda = \mathbb{F}_\ell$), we recover the classical theory of étale $\Lambda$-modules:

**Proposition 3.20.** Let $\Lambda$ be a discrete $\mathbb{Z}_\ell$-algebra and $X$ a small v-stack. Then we have

$$\mathcal{D}_{\text{nuc}}(X, \Lambda) = \mathcal{D}_{\text{et}}(X, \Lambda)$$

as full subcategories of $\mathcal{D}(X_v, \Lambda)$.

**Proof.** Since both $\infty$-categories are defined by descent, we can assume that $X$ is an $\ell$-bounded spatial diamond. By Theorem 3.9.(i) every étale $\Lambda$-module is nuclear (for any nuclear $\mathbb{Z}_\ell$-algebra $\Lambda$), so we have $\mathcal{D}_{\text{et}}(X, \Lambda) \subseteq \mathcal{D}_{\text{nuc}}(X, \Lambda)$. To get the other inclusion, note that the discreteness of $\Lambda$ implies that $\text{Hom}(\mathbb{Z}_\ell, \Lambda) = \lim_{\to} \text{Hom}(\mathbb{Z}/\ell^n\mathbb{Z}, \Lambda)$ (a priori this holds in the $\infty$-category of $\mathbb{Z}_\ell$-modules, but then also follows in the $\infty$-category of rings). Applying this to the structure map $\mathbb{Z}_\ell \to \Lambda$ we deduce that $\Lambda$ is a $\mathbb{Z}/\ell^n\mathbb{Z}$-algebra for some $n$. In particular every $\mathcal{M} \in \mathcal{D}_{\text{nuc}}(X, \Lambda)$ is killed by $\ell^n$, which implies that $\mathcal{M}$ is étale by Proposition 3.12.(ii) □

## 4 Pushforward and Base-Change

Fix a prime $\ell \neq p$ and a nuclear $\mathbb{Z}_\ell$-algebra $\Lambda$. In the previous section we have constructed the $\infty$-category $\mathcal{D}_{\text{nuc}}(X, \Lambda)$ of nuclear $\Lambda$-modules on every small v-stack $X$. We will now introduce the pushforward of nuclear sheaves and study its behavior.

**Definition 4.1.** Let $f : Y \to X$ be a map of small v-stacks.

(a) We denote the pullback functor on nuclear sheaves by

$$f^* : \mathcal{D}_{\text{nuc}}(X, \Lambda) \to \mathcal{D}_{\text{nuc}}(Y, \Lambda).$$

It is symmetric monoidal, preserves all small colimits and coincides with the v-pullback by design.
(b) We denote the nuclear pushforward functor by

\[ f_\ast : \mathcal{D}_{\text{nuc}}(Y, \Lambda) \to \mathcal{D}_{\text{nuc}}(X, \Lambda). \]

It is defined to be the right adjoint of \( f^\ast \), which exists by the adjoint functor theorem. It may occasionally be necessary to also consider pushforward functors on \( \omega_1 \)-solid sheaves and on all v-sheaves (defined to be the right adjoints of the pullback functor on the respective \( \infty \)-categories), which we will denote \( f_{\square \ast} \) and \( f_{v \ast} \), respectively.

In general we cannot expect the nuclear pushforward to coincide with the v-pushforward because the latter will in general not preserve nuclear sheaves. However, this usually works under a qcqs assumption, as follows.

**Definition 4.2.** Let \( X \) be a small v-stack.

(a) We denote by

\[ \mathcal{D}^b_{\text{nuc}}(X, \Lambda), \mathcal{D}^+_{\text{nuc}}(X, \Lambda), \mathcal{D}^-_{\text{nuc}}(X, \Lambda) \subseteq \mathcal{D}_{\text{nuc}}(X, \Lambda) \]

the full subcategories of those v-sheaves whose pullback to every \( \ell \)-bounded spatial diamond is bounded, resp. left bounded, resp. right bounded with respect to the \( t \)-structure on \( \omega_1 \)-solid sheaves. The elements of these subcategories are called the locally bounded, resp. locally left-bounded, resp. locally right-bounded nuclear \( \Lambda \)-modules on \( X \).

(b) Similar definitions as in (a) apply to \( \omega_1 \)-solid sheaves and to v-sheaves in place of nuclear sheaves.

**Lemma 4.3.** For any \( ? \in \{ b, +, - \} \) the assignment \( X \mapsto \mathcal{D}^?_{\text{nuc}}(X, \Lambda) \) defines a hypercomplete sheaf of stable \( \infty \)-categories on the v-site of all small v-stacks.

**Proof.** This follows immediately from the fact that pullbacks are \( t \)-exact and that \( \mathcal{D}_{\text{nuc}}(-, \Lambda) \) is a hypercomplete v-sheaf. \( \square \)

**Remark 4.4.** We warn the reader that there is no \( t \)-structure on \( \mathcal{D}_{\text{nuc}}(X, \Lambda) \) (unless \( \Lambda \) is discrete, see Proposition [3.20]), but it still makes sense to speak of (left/right) bounded nuclear sheaves.

We can now prove the following characterizations of the nuclear pushforward, generalizing the ones in [15, Proposition 17.6].

**Proposition 4.5.** Let \( f : Y \to X \) be a qcqs map of small v-stacks.

(i) The v-pushforward \( f_{v \ast} : \mathcal{D}(Y, \Lambda) \to \mathcal{D}(X, \Lambda) \) preserves \( \omega_1 \)-solid sheaves, i.e. we have \( f_{v \ast} = f_{\square \ast} \).

(ii) The v-pushforward preserves locally left-bounded nuclear sheaves, so that we have \( f_{v \ast} = f_\ast \) on locally left-bounded sheaves.

(iii) Suppose that the functor \( f_{\square \ast} : \mathcal{D}_{\square}(Y, \mathbb{Z}_\ell)_{\omega_1} \to \mathcal{D}_{\square}(X, \mathbb{Z}_\ell)_{\omega_1} \) has finite cohomological dimension. Then (ii) also hold for unbounded sheaves. Moreover, in this case both \( f_{\square \ast} \) and \( f_\ast \) preserve all small colimits.
Proof. All statements can be checked after pullback to any v-cover, so we can assume that X
is an ℓ-bounded spatial diamond; in particular Y is qcqs. Moreover, since the forgetful functor
along \( \mathbb{Z}_\ell \to \Lambda \) commutes with pushforwards we can further reduce to the case that \( \Lambda = \mathbb{Z}_\ell \). We
fix a hypercover \( g_* : Y_* \to Y \) by ℓ-bounded spatial diamonds, so that \( f_{v*} \) is computed as the
totalization \( f_{v*} = \lim_{n \in \Delta} f_{v*} g^*_n \).

Part (i) now follows from that fact that all \( f_{n*} \) preserve \( \omega_1 \)-solid sheaves (see Proposition 2.6.(ii)
and that \( \omega_1 \)-solid sheaves on \( X \) are stable under countable limits (see Proposition 2.5.(ii i)). Part
(ii) follows by the same argument as in the proof of Theorem 3.9.(ii) by exploiting the fact
that \( \text{Hom}^{\text{tr}}(P, -) \) preserves uniformly left-bounded totalizations. Part (iii) follows similarly by
additionally taking limits over Postnikov truncations (the fact that \( f_{v*} \) and \( f_* \) commute with
colimits follows easily from the facts that \( f \) is qcqs and has bounded cohomological dimension
by the usual Postnikov limit argument).

Corollary 4.6. Let

\[
\begin{array}{ccc}
Y' & \xrightarrow{g'} & Y \\
\downarrow^f & & \downarrow^f \\
X' & \xrightarrow{g} & X
\end{array}
\]

be a cartesian diagram of small v-stacks and assume that \( f \) is qcqs.

(i) The natural morphism \( g^* f_{\boxtimes} \simeq f'_{\boxtimes} g'^* \) is an isomorphism of functors \( \mathcal{D}_{\boxtimes}(Y, \Lambda)_{\omega_1} \to \mathcal{D}_{\boxtimes}(X', \Lambda)_{\omega_1} \).

(ii) The natural morphism \( g^* f_* \simeq f'_{*} g'^* \) is an isomorphism of functors \( \mathcal{D}^+_{\text{nuc}}(Y, \Lambda) \to \mathcal{D}^+_{\text{nuc}}(X', \Lambda) \).

(iii) Suppose that the functor \( f_{\boxtimes} : \mathcal{D}_{\boxtimes}(Y, \mathbb{Z}_\ell)_{\omega_1} \to \mathcal{D}_{\boxtimes}(X, \mathbb{Z}_\ell)_{\omega_1} \) has finite cohomological dimension. Then (ii) also holds for unbounded sheaves.

Proof. All claims follows immediately from Proposition 4.5 by using the fact that v-pushforward
satisfies arbitrary base-change (for (iii) note additionally that the functor \( f'_{v*} \) preserves nuclear
sheaves because so does \( g^* f_{v*} \)).

The condition that \( f_{\boxtimes} \) has finite cohomological dimension may look a bit hard to grasp at first,
but it turns out that this condition is usually satisfied in practice:

Proposition 4.7. Let \( f : Y \to X \) be a map of small v-stacks which is locally compactifiable, representable
in spatial diamonds and has locally finite \( \text{dim} \text{trg} \). Then \( f_{\boxtimes} : \mathcal{D}_{\boxtimes}(Y, \Lambda)_{\omega_1} \to \mathcal{D}_{\boxtimes}(X, \Lambda)_{\omega_1} \) has finite cohomological dimension.

Proof. By Corollary 4.6.(i) we can assume that X is an ℓ-bounded spatial diamond; in particular
Y is also a spatial diamond. By passing to a finite cover of Y we can assume that \( f \) is compactifiable. Then [15, Theorem 22.5] implies that \( f_* : \mathcal{D}_{\text{et}}(Y, \mathbb{F}_\ell) \to \mathcal{D}_{\text{et}}(X, \mathbb{F}_\ell) \) has finite cohomological dimension, which implies that Y is ℓ-bounded. We conclude by Proposition 2.6.(ii)

5 The 6-Functor Formalism

Fix a prime \( \ell \neq p \) and a nuclear \( \mathbb{Z}_\ell \)-algebra \( \Lambda \). We will finally construct the 6-functor formalism
for nuclear \( \Lambda \)-sheaves. In the previous subsection we have already introduced the pullback and
pushforward functors. The next pair of functors are tensor product and internal Hom, which are
easily defined:
Definition 5.1. Let $X$ be a small $v$-stack. We denote by $\otimes_\Lambda$ the symmetric monoidal structure on $\mathcal{D}_{\text{nuc}}(X,\Lambda)$ and by $\text{Hom}_\Lambda(-,-)$ the associated internal hom functor (which exists because $\mathcal{D}_{\text{nuc}}(X,\Lambda)$ is presentable). We often drop the subscript $\Lambda$ if there is no room for confusion.

The last pair of functors – the shriek functors – are the hardest to construct. The general idea is as follows: Given a nice compactifiable map $f: Y \to X$ of small $v$-stacks we want to define the functor $f!: \mathcal{D}_{\text{nuc}}(Y,\Lambda) \to \mathcal{D}_{\text{nuc}}(X,\Lambda)$ as the composition $f! = g^* \circ j!$ for any decomposition of $f$ into an open immersion $j$ and a proper map $g$; here $j!$ will be left adjoint of $j^*$.

As a first step towards constructing $f!$, let us show that $j!$ exists and satisfies the expected properties:

Lemma 5.2. For every étale map $j: U \to X$ of small $v$-stacks the pullback $j^*: \mathcal{D}_{\text{c}}(X,\Lambda)_{\omega_1} \to \mathcal{D}_{\text{c}}(U,\Lambda)_{\omega_1}$ admits a left adjoint $j!: \mathcal{D}_{\text{c}}(U,\Lambda)_{\omega_1} \to \mathcal{D}_{\text{c}}(X,\Lambda)_{\omega_1}$ with the following properties:

(i) For every map $g: X' \to X$ of small $v$-stacks with base-change $g': U' := U \times_X X' \to U$, $j': U' \to X'$ the natural morphism

$$j'g^* \sim g^* j!$$

is an isomorphism of functors $\mathcal{D}_{\text{c}}(U,\Lambda)_{\omega_1} \to \mathcal{D}_{\text{c}}(X',\Lambda)_{\omega_1}$.

(ii) For all $M \in \mathcal{D}_{\text{c}}(X,\Lambda)_{\omega_1}$ and $N \in \mathcal{D}_{\text{c}}(U,\Lambda)_{\omega_1}$ the natural map

$$j!(N \otimes j^* M) \sim j! N \otimes j! M$$

is an isomorphism.

(iii) $j!$ commutes with the forgetful functor and the base-change functor along any map $\Lambda \to \Lambda'$ of nuclear $\mathbb{Z}_\ell$-algebras.

(iv) If $j$ is quasicompact then $j!$ is $t$-exact and preserves $\ell$-adically complete sheaves.

(v) $j!$ preserves nuclear sheaves and hence restricts to a functor

$$j!: \mathcal{D}_{\text{nuc}}(U,\Lambda) \to \mathcal{D}_{\text{nuc}}(X,\Lambda)$$

which is left adjoint to $j^*$. 

Proof. By [3] Proposition VII.3.1] the pullback functor $j^*: \mathcal{D}_{\text{c}}(X,\Lambda) \to \mathcal{D}_{\text{c}}(U,\Lambda)$ admits a left adjoint $j_!$ which satisfies the analogous properties (i), (ii) and (iii) (the result in loc. cit. is only stated for static $\Lambda$ but the proof works in general; in fact, the proof is completely formal). We now show that $j_!$ preserves $\omega_1$-solid sheaves and hence restricts to the desired functor $j!$. Since $j_!$ commutes with the forgetful functor along $\mathbb{Z}_\ell \to \Lambda$, we can assume that $\Lambda = \mathbb{Z}_\ell$. Moreover, since $j_!$ commutes with any base-change, we can assume that $X$ is a strictly totally disconnected space. Then $U$ is a perfectoid space and hence admits a basis given by open subsets which are quasicompact and separated over $X$ (e.g. take affinoid perfectoid open subsets). It is formal that $j_!$ is computed as the colimit over this basis, so we can assume that $j$ is quasicompact and separated. In particular $U$ is strictly totally disconnected and hence an $\ell$-bounded spatial
diamond. Since \( j_2 \) preserves all small colimits, it is now enough to verify that for every compact \( P \in D_c(U, \mathbb{Z}_\ell)_\omega_1 \) we have \( j_2 P \in D_c(X, \mathbb{Z}_\ell)_\omega_1 \). By Proposition 2.5.(ii) we can assume that \( P = \mathbb{Z}_\ell[\mathbb{V}] \) for some countable basic \( V = \lim_n V_n \in X_{proet} \). But then for all \( M \in D_c(X, \mathbb{Z}_\ell) \) we have

\[
\text{Hom}(j_2 \mathbb{Z}_\ell[\mathbb{V}], \mathcal{M}) = \text{Hom}(\mathbb{Z}_\ell[\mathbb{V}], j^* \mathcal{M}) = \Gamma(V, j^* \mathcal{M}) = \Gamma(V, \mathcal{M}),
\]

hence \( j_2 \mathbb{Z}_\ell[\mathbb{V}] = \mathbb{Z}_\ell[\mathbb{V}] \) (where on the right-hand side we define it as a sheaf on \( X \)), which is indeed \( \omega_1 \)-solid.

We have now shown the existence of \( j_1 \) by restricting \( j_2 \) to \( \omega_1 \)-solid sheaves. The claims (i), (ii) and (iii) now follow immediately from the analogous properties of \( j_2 \). It remains to prove (iv) and (v), for which we can assume that \( \Lambda = \mathbb{Z}_\ell \) (by (iii)), that \( X \) is strictly totally disconnected (by (i)) and that \( j \) is quasicompact and separated (by passing to an open cover as above).

We first observe that \( j_1 \) preserves étale sheaves: By passing to right adjoints, this reduces to the observation that \( j^* \) commutes with the pushforward to the étale site, which follows immediately from the fact that \( j \) is étale. Therefore claim (v) reduces to claim (iv), because every nuclear \( \mathbb{Z}_\ell \)-sheaf on \( U \) is a colimit of Banach sheaves. To prove (iv) we need to make one more computation: Suppose \( P \in D_c(U, \mathbb{Z}_\ell)_\omega_1 \) is static and can be written as a sequential limit \( P = \lim_n P_n \) with static qcqs étale \( P_n \in D_{et}(U, \mathbb{Z}_\ell)^{\omega} \); then

\[
j_1 P = \lim_n j_1 P_n. \tag{5.2.1}
\]

To prove this, pick a surjective map \( Q_0 \rightarrow P \) for some static compact \( \omega_1 \)-solid \( \mathbb{Z}_\ell \)-module \( Q_0 \) on \( U \). We can write \( Q_0 = \lim_n Q_0,n \) for some qcqs étale sheaves \( Q_0,n \) and by the usual Breen resolution argument the map \( Q_0 \rightarrow P \) can be obtained from a map of Pro-systems \( (Q_0,n)_n \rightarrow (P)_n \). In particular the kernel of the map \( Q_0 \rightarrow P \) is again a limit of qcqs étale sheaves, so we can iterate the process in order to obtain a resolution of the Pro-system \( (P)_n \) in terms of Pro-systems \( (Q_k,n)_n \) of static qcqs étale sheaves such that each \( Q_k := \lim_n Q_k,n \) is compact. By passing to the associated simplicial objects (whose geometric realization is \( P \)) and using that both \( j_1 \) and countable limits commute with geometric realizations, we reduce the claim Eq. (5.2.1) to the case that \( (P)_n = (Q_k,n)_n \) for some \( k \). But since \( Q_k \) is compact, the claim follows from the above computation of \( j_1 \) on compact generators.

With Eq. (5.2.1) at hand, claim (iv) is now straightforward: Note that it follows immediately that \( j_1 \) is \( t \)-exact, because every static \( \omega_1 \)-solid sheaf can be written as a filtered colimit of objects of the form \( P \) in Eq. (5.2.1) (use also [3 Proposition VII.1.6]). Thus to show that \( j_1 \) preserves \( \ell \)-adically complete sheaves, we can immediately reduce to the static case. From now on let \( \mathcal{M} \in D_c(U, \mathbb{Z}_\ell)_\omega_1 \) be static and \( \ell \)-adically complete. To show that \( j_1 \mathcal{M} \) is \( \ell \)-adically complete, we can as in the proof of Proposition 2.8 pass to a geometric realization of \( \mathcal{M} \) in order to reduce to the case that \( \mathcal{M} = \bigoplus_k \mathbb{Z}_\ell[\mathbb{V}] \) for countably many basic \( V = \lim_n V_{k,n} \in U_{proet} \) (here we use that \( j_1 \) preserves colimits and is right \( t \)-exact). Now filter \( \mathcal{M} \) by the subsheaves \( \mathcal{M}_\alpha = \bigcap_k \mathbb{Z}_\ell[\mathbb{V}] \) for sequences of integers \( \alpha_k \geq 0 \) converging to \( \infty \). By Eq. (5.2.1) we get

\[
j_1 \mathcal{M} = \lim_{\alpha} j_1 \prod_k \mathbb{Z}_\ell[\mathbb{V}] = \lim_{\alpha} \prod_k j_1(\mathbb{Z}_\ell[\mathbb{V}]) = \bigoplus_k \mathbb{Z}_\ell[\mathbb{V}],
\]

as desired.

Having a good understanding of \( j_1 \) for étale \( j \), it remains to study \( g_* \) for nice proper maps \( g \). We already know base-change for \( g_* \) by Corollary 4.6.(iii). It remains to check the projection formula. Also, in order to show that the construction \( f_1 = g_* \circ j_1 \) is independent of the factorization, we need a compatibility of \( g_* \) and \( j_1 \):
Lemma 5.3. Let \( g: Y \to X \) be a proper map of small \( \nu \)-stacks which has locally bounded dimension (see \cite[Definition 3.5.3]{10}) and assume that the functor \( g_{!\bullet}: D_{\nu}(Y, \mathbb{Z}_\ell) \to D_{\nu}(X, \mathbb{Z}_\ell) \) has finite cohomological dimension. Then:

(i) For all \( M \in D_{\text{nuc}}(X, \Lambda) \) and \( N \in D_{\text{nuc}}(Y, \Lambda) \) the natural map
\[
g_* N \otimes M \simto g_*(N \otimes g^* M)
\]
is an isomorphism.

(ii) Let \( j: U \to X \) be an open immersion with base-change \( j': V \to Y \) and \( g': V \to U \). Then the natural map
\[
j g'_* \simto g_! j'_!
\]
is an isomorphism of functors \( D_{\text{nuc}}(V, \Lambda) \to D_{\text{nuc}}(X, \Lambda) \).

Proof. Part (ii) follows formally from part (i) together with Lemma \ref{5.2} and Corollary \ref{4.6.(iii)} see \cite[Lemma 3.6.8]{10}. To prove (i), we note that both sides of the claimed isomorphism commute with base-change (by Corollary \ref{4.6.(iii)}), so we can assume that \( X \) is a strictly totally disconnected space. Moreover, both sides of the claimed isomorphism also commute with colimits in \( M \), so we can assume \( M = M' \otimes_{\mathbb{Z}_\ell} \Lambda \) for some basic nuclear \( M' \in D_{\text{nuc}}(X, \mathbb{Z}_\ell) \) (see Proposition \ref{3.16.(iii)}). But then after applying the forgetful functor along \( \mathbb{Z}_\ell \to \Lambda \) (and using that \( g_* \) commutes with this forgetful functor) we get
\[
g_* N \otimes_{\Lambda} M = g_* N \otimes_{\Lambda} (\Lambda \otimes_{\mathbb{Z}_\ell} M') = g_* (N \otimes_{\mathbb{Z}_\ell} g^* M'),
\]
\[
g_*(N \otimes_{\Lambda} g^* M) = g_*(N \otimes_{\Lambda} (\Lambda \otimes_{\mathbb{Z}_\ell} g^* M')) = g_*(N \otimes_{\mathbb{Z}_\ell} g^* M').
\]

We can therefore assume that \( \Lambda = \mathbb{Z}_\ell \). Now write \( M = \lim_i \mathcal{P}_i \) for compact objects \( \mathcal{P}_i \in D_{\nu}(X, \mathbb{Z}_\ell)_{\omega_1} \). Then by Proposition \ref{3.12} we have \( M = \lim_i (\mathcal{P}_i)_{\text{nuc}} \) and by the other claims in Proposition \ref{3.12} each \( (\mathcal{P}_i)_{\text{nuc}} \) is a bounded Banach sheaf. By pulling out colimits on both sides of the claimed isomorphism we can thus reduce to the case that \( M \) is a bounded Banach sheaf.

We will first prove the claim in the case that \( Y \) is a spatial diamond (it is automatically \( \ell \)-bounded by \cite[Theorem 22.5]{15}). Then, since both sides of the claimed isomorphism commute with colimits in \( \mathcal{N} \) (for \( g_* \) this was shown in Proposition \ref{4.5.(iii)}), we can use the same strategy as for \( M \) in order to reduce to the case that \( \mathcal{N} \) is also a bounded Banach sheaf. Now by Proposition \ref{2.8} both sides of the claimed isomorphism are \( \ell \)-adically complete, hence we can check the isomorphism after reducing modulo \( \ell \). But then the claim follows immediately from \cite[Proposition 22.11]{15}.

To prove the claim for general \( Y \), pick a hypercover \( h_\bullet: Y_\bullet \to Y \) such that all \( Y_n \) are spatial diamonds and all maps \( g_n: Y_n \to X \) are proper and of finite dim.trg (e.g. start with any hypercover of \( Y \) by affinoid perfectoid spaces of finite dim.trg over \( X \) and then take relative compactifications). Note that the functor
\[
F: D_{\nu}(Y, \mathbb{Z}_\ell)_{\omega_1} \to D_{\nu}(X, \mathbb{Z}_\ell)_{\omega_1}, \quad N' \mapsto g_* N'^{\text{nuc}} \otimes M
\]
is bounded. Indeed, it follows formally from adjunctions that \( g_* N'^{\text{nuc}} = (g_{!\bullet} N')_{\text{nuc}}, \) so it is enough to see that the functors \( g_{!\bullet}, (-)_{\text{nuc}} \) (on \( X \)) and \( - \otimes M \) are bounded. For the first functor this holds by assumption, for the second functor this was shown in Proposition \ref{3.12.(iii)} and for the third functor it follows from the boundedness of \( M \) together with \cite[Proposition VII.2.3]{3}. By a
similar argument one sees that the composition of functors $F \circ h_{n \geq} h_n^*$ is bounded (this boils down to the functor $g_{n \geq}$ having finite cohomological dimension). Thus by taking Postnikov limits and commuting them with totalizations, we deduce that $F(\varprojlim_{n \in \Delta} h_{n \geq} h_n^* N') = \varprojlim_{n \in \Delta} F(h_{n \geq} h_n^* N')$ for all $N'$. Taking $N' = N$ we get

$$g_* N \otimes M = F(N) = \lim_{n \in \Delta} (g_n h_n^* N \otimes M),$$

and using the fact that we already know that the projection formula holds for each $g_n$ in place of $g$ by the above argument,

$$= \lim_{n \in \Delta} g_n (h_n^* N \otimes g_n^* M) = g_{n \geq} \lim_{n \in \Delta} h_{n \geq} h_n^* (N \otimes g_n^* M).$$

But by $v$-descent for $\omega_1$-solid sheaves we have $\lim_{n \in \Delta} h_{n \geq} h_n^* N' = N'$ for every $N' \in D_\omega(Y, \mathbb{Z}_\ell)_{\omega_1}$. By applying this to $N' = N \otimes g_n^* M$ we conclude

$$g_* N \otimes M = g_{n \geq} \lim_{n \in \Delta} h_{n \geq} h_n^* (N \otimes g_n^* M) = g_*(N \otimes g_n^* M),$$

as desired. \qed

With the above results at hand, it is now formal to construct the 6-functor formalism. We first construct it for maps which are representable in locally spatial diamonds and afterwards extend it to certain “stacky” maps. In the case of locally spatial diamonds, we will define shriek functors for the following class of maps:

**Definition 5.4.** A map $f: Y \to X$ of small v-stacks is called $\text{fdcs}$\footnote{The name comes from “Finite Dimension, Compactifiable and Spatial”} if it is locally compactifiable (i.e. there is some analytic cover of $Y$ on which $f$ is compactifiable) and representable in locally spatial diamonds and has locally finite dim. trg.

**Lemma 5.5.** (i) The property of being $\text{fdcs}$ is analytically local on both source and target.

(ii) $\text{fdcs}$ maps are stable under composition and base-change.

(iii) Every étale map is $\text{fdcs}$.

(iv) Let $f: Y \to X$ and $g: Z \to Y$ be maps of small v-stacks. If $f$ and $f \circ g$ are $\text{fdcs}$ then so is $g$.

*Proof.* Claims (i), (ii) and (iii) are obvious (using [15, Proposition 22.3] to handle the compactifiability condition). For (iv) we can argue in the same way as for the similar bdcs condition, see [10, Lemma 3.6.10.(iv)]. \qed

The next result constructs the 6-functor formalism for $\text{fdcs}$ maps. The result will freely make use of the theory of abstract 6-functor formalisms developed in [10 §A.5], so the reader is advised to take a look at that. In particular, recall the definition of the $\infty$-operad $\text{Corr}(C)_{E,\text{all}}$ of correspondences and of 6-functor formalisms, see [10 Definitions A.5.2.(b), A.5.4, A.5.7]. Also, in the following we will denote by vStack the 2-category of small v-stacks.

1. The name comes from “Finite Dimension, Compactifiable and Spatial”.

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Proposition 5.6. There is a 6-functor formalism

$$\mathcal{D}_{\text{nucl}}(-, \Lambda): \text{Corr} (\text{vStack})_{\text{fdcs,all}} \to \text{Cat}_{\infty}$$

with the following properties:

(i) Restricted to the symmetric monoidal subcategory $\text{vStack}^{\text{op}}$ (equipped with the coproduct monoidal structure), $\mathcal{D}_{\text{nucl}}(-, \Lambda)$ coincides with the functor constructed in Proposition 3.16.(i).

(ii) For every fdcs map $f: Y \to X$ the functor

$$f^! := \mathcal{D}_{\text{nucl}}([Y \leftarrow \text{id} Y \to X], \Lambda): \mathcal{D}_{\text{nucl}}(Y, \Lambda) \to \mathcal{D}_{\text{nucl}}(X, \Lambda)$$

preserves all small colimits. If $f = j$ is étale then $j_!$ is left adjoint to $j^*$ and if $f$ is proper then $f_! = f^*$.

Proof. The construction is very similar to [10, Theorem 3.6.12]; in fact, we copied most of the proof with only minor adjustments. The most prominent difference is that we make use of the notion of prespatial diamonds from [4] to get a replacement for $p$-boundedness.

We first construct the 6-functor formalism for morphisms in the collection $f_{\text{dcqc}}$ of those maps $f: Y \to X$ which are representable in prespatial diamonds (see [4, Definition 3.3]), compactifiable and quasicompact and have finite dim.trg. One checks that the class of maps $f_{\text{dcqc}}$ satisfies the “2-out-of-3 property” (i.e. the analog of Lemma 5.5.(iv)) by a similar argument as in the proof of Lemma 5.5.(iv) (note that it follows easily from the definition of prespatial diamonds that a quasicompact subdiamond of a prespatial diamond is prespatial). We denote by $P \subseteq f_{\text{dcqc}}$ the subclass of those maps which are additionally proper and by $I \subseteq f_{\text{dcqc}}$ the class of quasicompact open immersions. It follows easily from the 2-out-of-3 property for $f_{\text{dcqc}}$ that both $P$ and $I$ also have the 2-out-of-3 property. Thus by [4, Proposition 3.6] the pair $I, P \subseteq f_{\text{dcqc}}$ is a suitable decomposition in the sense of [10, Definition A.5.9]. We can therefore apply [10, Proposition A.5.10]: Condition (a) follows from Lemma 5.2, condition (b) follows from Corollary 4.6.(iii) and Lemma 5.3.(i) using [4, Proposition 3.7] (applied as in Proposition 4.7), condition (c) follows from Lemma 5.3.(ii), conditions (d) and (e) are clear because $\mathcal{D}_{\text{nucl}}(X, \Lambda)$ is presentable and condition (f) reduces to the observation that for $f \in P$ the functor $f_*$ preserves small colimits by Proposition 4.5.(iii). We obtain a 6-functor formalism

$$\mathcal{D}_{\text{qc}}: \text{Corr} (\text{vStack})_{f_{\text{dcqc},\text{all}}} \to \text{Cat}_{\infty}$$

mapping $X \in \text{vStack}$ to $\mathcal{D}_{\text{nucl}}(X, \Lambda)$ with the correct pullback functors. We will now extend $\mathcal{D}_{\text{qc}}$ from $f_{\text{dcqc}}$ to $f_{\text{dcs}}$ by using the extension results in [10, §A.5].

1. In the first step, we first restrict $\mathcal{D}_{\text{qc}}$ to $\text{Corr}(\text{vStack})_{f_{\text{dcqs},\text{all}}}$, where $f_{\text{dcqs}} \subseteq f_{\text{dcqc}}$ is the subset of those edges which are representable in spatial diamonds. We now wish to extend $\mathcal{D}_{\text{qc}}$ from $f_{\text{dcqs}}$ to the class of edges $f_{\text{dcs}}$ consisting of those fdcs maps which are separated. By [10, Proposition A.5.16] and Proposition 3.16.(i) this extension can be performed on the full subcategory $\mathcal{C}_1 \subseteq \text{vStack}$ consisting of separated locally spatial diamonds, i.e. we need to extend the $\infty$-operad map

$$\mathcal{D}_{\text{qc}}: \text{Corr}(\mathcal{C}_1)_{f_{\text{dcqs},\text{all}}} \to \text{Cat}_{\infty}$$

to an $\infty$-operad map

$$\mathcal{D}_s: \text{Corr}(\mathcal{C}_1)_{f_{\text{dcs},\text{all}}} \to \text{Cat}_{\infty}.$$
We first apply \cite{10} Proposition A.5.12, which allows us to extend $D_{qc}$ from $fdcso$ to the collection $E_1$ of edges of the form $\bigsqcup_i Y_i \to X$, where each $Y_i \to X$ lies in $fdcso$; let us denote the new 6-functor formalism by $D_{qc}$. We now apply \cite{10} Proposition A.5.14 to extend $D = D'_qc$ from $E_1$ to $E'_1 := fdcss$. Here we use the collection $S_1 \subseteq E_1$ of edges of the form $\bigsqcup_i U_i \to X$ for covers $X = \bigsqcup_i U_i$ by quasicompact open immersions $U_i \hookrightarrow X$. Then for $j \in S_1$ we have $j^! = j^*$, hence condition (b) of loc. cit. follows from the sheafiness of $D(\cdot)$. Condition (c) amounts to saying that every separated fdc map $Y \to X$ of separated locally spatial diamonds admits a cover $Y = \bigsqcup_i V_i$ by quasicompact open immersions $V_i \hookrightarrow Y$ such that each map $V_i \to X$ is quasicompact (it is automatically compactifiable by \cite{15} Proposition 22.3.(v)); but this is easily satisfied, e.g. pick the $V_i$ to be any open cover of $Y$ by quasicompact open subsets (then the maps $V_i \to Y$ and $V_i \to X$ are quasicompact because both $X$ and $Y$ are separated). Finally, condition (d) follows easily from the fact that all the spaces in $C_1$ are separated. This finishes the construction of the 6-functor formalism

$$D_s: \text{Corr}(\text{vStack})_{fdcss,all} \to \text{Cat}_\infty$$

(where we implicitly used \cite{10} Proposition A.5.16 to extend from $C_1$ to vStack).

2. In the second extension step we extend $D_s$ to the desired $\infty$-operad map

$$D: \text{Corr}(\text{vStack})_{fdcss,all} \to \text{Cat}_\infty.$$ 

This extension is similar to the previous one, albeit somewhat simpler: We can perform the extension directly on $C_2 = \text{vStack}$ by applying \cite{10} Proposition A.5.14 to $E_2 = fdcss$ and $E'_2 = fdc$ with $S_2 \subseteq E_2$ being the collection of all open immersions.

We have now constructed a 6-functor formalism $D(\cdot, \Lambda) = D$ on fdc maps. It remains to verify that it satisfies claims (i) and (ii). Claim (i) is obvious from the construction. For claim (ii), it follows from the definition of 6-functor formalisms that $f_!$ preserves all small colimits and it follows immediately from the construction that $f_! = f_*$ for proper $f$. It remains to see that for every étale map $j: U \to X$ of small v-stacks the just constructed functor $j_!$ is left adjoint to $j^*$ and thus coincides with the functor $j_!$ from Lemma 5.2. To see this, we first apply \cite{10} Proposition A.5.10 to the case that $E = I = et$ is the collection of étale maps in vStack and $P$ consists only of degenerate edges; then conditions (b) and (c) are vacuous and condition (a) is satisfied by Lemma 5.2. We thus obtain a 6-functor formalism

$$D_{et}: \text{Corr}(\text{vStack})_{et,all} \to \text{Cat}_\infty, \quad X \mapsto D_{nuc}(X, \Lambda).$$

We need to show that $D_{et}$ is equivalent to the restriction of $D$ to Corr(vStack)$_{et,all}$. By the uniqueness of the extension results \cite{10} Proposition A.5.12, A.5.14, A.5.16 we can show this equivalence on the full subcategory $C \subseteq vStack$ consisting of locally spatial diamonds and we can then restrict to the subset $etsqc \subseteq et$ of separated quasicompact étale maps. We can now further reduce to the full subcategory $C' \subseteq C$ consisting of strictly totally disconnected spaces. But note that every map of strictly totally disconnected spaces which lies in $etsqc$ is of the form $\bigsqcup_{i=1}^n U_i \to X$ for quasicompact open immersions $U_i \hookrightarrow X$, so we can further replace etsqc by the collection of quasicompact open immersions. But in this case $D_{et}$ and $D$ agree by construction.

\[\square\]
We now want to extend the 6-functor formalism from Proposition 5.6 to certain “stacky” maps. The relevant definitions are as follows:

**Definition 5.7.** We say that an fdcs map \( f: Y \to X \) of small v-stacks admits universal \( \ell \)-codescent if it satisfies the following property: Given any small v-stack \( X' \) which admits a map to some strictly totally disconnected space and given any map \( X' \to X \) with base-change \( f': Y' \to X' \) and Čech nerve \( Y'_n \to X' \) the natural functor

\[
D_{\text{nuc}}^!(X', \mathbb{Z}_\ell) \xrightarrow{\sim} \lim_{n \in \Delta} D_{\text{nuc}}^!(Y'_n, \mathbb{Z}_\ell)
\]

is an equivalence. Here \( D_{\text{nuc}}(-, \Lambda) \) denotes the functor \( Z \mapsto D_{\text{nuc}}(Z, \Lambda), h \mapsto h^! \) obtained from the 6-functor formalism in Proposition 5.6.

**Definition 5.8.** A map \( f: Y \to X \) of small v-stacks is called \( \ell \)-fine if there is a map \( g: Z \to Y \) such that \( g \) and \( f \circ g \) are fdcs and \( g \) admits universal \( \ell \)-codescent.

**Remark 5.9.** There is a definition of fine maps in [4, Definition 1.3.i] which is closely related to our definition of \( \ell \)-fine maps. In fact, every fine map is \( \ell \)-fine, as follows from Corollary 8.9 below. Moreover, we introduced a similar notion of \( p \)-fine maps in [5, Definition 2.4] which only takes into account the universal codescent for \( p \)-torsion coefficients – this seems to be a mistake, as it is probably not enough to imply universal codescent for \( p \)-adic non-torsion coefficients (once the corresponding 6-functor formalism has been worked out).

**Lemma 5.10.**

(i) The condition of being \( \ell \)-fine is étale local on source and target.

(ii) The collection of \( \ell \)-fine maps is stable under composition and base-change.

(iii) Every fdcs map is \( \ell \)-fine.

(iv) Let \( f: Y \to X \) and \( g: Z \to Y \) be maps of small v-stacks. If \( f \) and \( f \circ g \) are \( \ell \)-fine then so is \( g \).

**Proof.** This is formal, see [5, Lemma 2.5] for a \( p \)-torsion analog (and use Lemma 5.5.(iv) in place of [10, Lemma 3.6.10.(iv)]).

In Sections 8 and 10 we will provide many examples of \( \ell \)-fine maps which are not fdcs (and usually not 0-truncated). We can finally formulate the main result of this paper:

**Theorem 5.11.** There is a 6-functor formalism

\[
D_{\text{nuc}}(-, \Lambda): \text{Corr(vStack)}_{\text{fine,alt}} \to \text{Cat}_\infty
\]

with the following properties:

(i) Restricted to the symmetric monoidal subcategory vStack\text{OP} (equipped with the coproduct monoidal structure), \( D_{\text{nuc}}(-, \Lambda) \) coincides with the functor constructed in Proposition 3.16.(i).

(ii) For every \( \ell \)-fine map \( f: Y \to X \) the functor

\[
f_* := D_{\text{nuc}}([Y \leftarrow \text{id} Y \to X], \Lambda): D_{\text{nuc}}(Y, \Lambda) \to D_{\text{nuc}}(X, \Lambda)
\]

preserves all small colimits. If \( f = j \) is étale then \( j_! \) is left adjoint to \( j^* \) and if \( f \) is proper then \( f_* = f_* \).
Proof. We first prove the following claim: Let \( f: Y \to X \) be a map of small v-stacks with Čech nerve \( f_*: Y \to X \) and assume that \( f \) admits universal \( \ell \)-codescent and \( X \) admits a map to some strictly totally disconnected space; then the natural functor

\[
\left( f^{\mathbf{!}} \right)_* : \mathcal{D}_{\text{nuc}}^l(X, \Lambda) \xrightarrow{\sim} \lim_{n \in \Delta} \mathcal{D}_{\text{nuc}}^l(Y_n, \Lambda)
\]

is an equivalence. By definition of universal \( \ell \)-codescent this holds for \( \Lambda = \mathbb{Z}_\ell \), so we only need to reduce the general case to this case. Note that the functor \( f^{\mathbf{!}} \) has a left adjoint

\[
f_* : \lim_{n \in \Delta} \mathcal{D}_{\text{nuc}}^l(Y_n, \Lambda) \to \mathcal{D}_{\text{nuc}}^l(X, \Lambda), \quad (\mathcal{M}_n)_n \mapsto \lim_{n \in \Delta} f_n! \mathcal{M}_n.
\]

Thus in order to prove the desired equivalence it is enough to show that the natural maps \( f_* f^{\mathbf{!}} \sim \text{id} \) and \( \text{id} \sim f^{\mathbf{!}} f_* \) are isomorphisms. This follows immediately from the \( \mathbb{Z}_\ell \)-case together with the fact that both functors commute with the forgetful functor along \( \mathbb{Z}_\ell \to \Lambda \), the proof of which we delay to the end of this section (see Corollary 5.15 below).

With the above claim at hand, the construction of the desired 6-functor formalism for \( \ell \)-fine maps is now completely formal (cf. the proof of [5, Proposition 2.6]): Our goal is to show that the 6-functor formalism from Proposition 5.6 extends uniquely to a 6-functor formalism for all \( \ell \)-fine maps. To do that, let \( \mathcal{C} \subseteq \text{vStack} \) denote the full subcategory spanned by those small v-stacks which admit a map to some strictly totally disconnected space. Then \( \mathcal{C} \) is a basis of \( \text{vStack} \), hence by [10, Proposition A.5.16] it is enough to construct the desired extension of the 6-functor formalism on \( \mathcal{C} \), i.e. we need to construct a 6-functor formalism \( \mathcal{D}(\cdot, \Lambda) : \text{Corr}(\mathcal{C})_{lfine,all} \to \text{Cat}_{\infty} \) extending the one from Proposition 5.6. We now apply [10, Proposition A.5.14] with \( E = \text{fdes} \), \( E' = l\text{fine} \) and \( S \subseteq E \) being the subset of those maps which admit universal \( \ell \)-codescent. Condition (a) is clear, condition (b) was proved above, condition (c) holds by definition of \( \ell \)-fine maps and condition (d) follows from Lemma 5.5 (see [10, Remark A.5.15.(ii)]).

Let us extract the shriek functors from Theorem 5.11, thereby completing the collection of six functors for nuclear sheaves:

**Definition 5.12.** Let \( f: Y \to X \) be an \( \ell \)-fine map of small v-stacks.

(a) We define \( f_!: \mathcal{D}_{\text{nuc}}(Y, \Lambda) \to \mathcal{D}_{\text{nuc}}(X, \Lambda) \) to be the functor \( f_! := \mathcal{D}_{\text{nuc}}([Y \leftarrow Y \to X], \Lambda) \), where \( \mathcal{D}(\cdot, \Lambda) \) is the 6-functor formalism from Theorem 5.11.

(b) We define \( f^!: \mathcal{D}_{\text{nuc}}(X, \Lambda) \to \mathcal{D}_{\text{nuc}}(Y, \Lambda) \) to be the right adjoint of \( f_! \).

**Remark 5.13.** The construction of the functor \( f_! \) in Theorem 5.11 is not very explicit, so we provide a more direct description:

1. Suppose that \( f: Y \to X \) is quasicompact, compactifiable and representable in spatial diamonds with finite dim.trg. Then we define

\[
f_! := (\mathcal{J}^{\mathcal{Y}/X})_* \circ j_!,
\]

where \( j \) denotes the open immersion \( j: Y \to \mathcal{Y}^{\mathcal{Y}/X} \) and \( j_! \) is the functor from Lemma 5.2.
2. Suppose that \( f : Y \to X \) is an fdcms map of locally spatial diamonds. Let \( \mathcal{I} \) be the category of open subsets \( V \subseteq Y \) which are quasicompact and compactifiable over \( X \). Then we have \( D_{\text{nuc}}(Y, \Lambda) = \lim_{\longrightarrow V \in \mathcal{I}} D_{\text{nuc},!}(V, \Lambda) \) in the \( \infty \)-category of presentable \( \infty \)-categories and colimit preserving functors, where \( D_{\text{nuc},!} \) is the functor mapping an inclusion \( j : V \to V' \) to \( j_! \). We thus need to define \( f_! \) as the functor
\[
 f_! M := \lim_{V \in \mathcal{I}} f_{V!}(M|_V),
\]
where \( f_{V!} : V \to X \) denotes the composition \( V \leftarrow Y \to X \). The functors \( f_{V!} \) were defined in the previous step.

3. Suppose that \( f : Y \to X \) is an arbitrary fdcms map of small v-stacks. Choose a hypercover \( X_\bullet \to X \) such that all \( X_n \) are locally spatial diamonds and let \( f_\bullet : Y_\bullet \to X_\bullet \) be the base-change of \( f \). Then we have \( D_{\text{nuc}}(X, \Lambda) = \lim_{\longrightarrow n \in \Delta} D_{\text{nuc}}(X_n, \Lambda) \) and \( D_{\text{nuc}}(Y, \Lambda) = \lim_{\longrightarrow n \in \Delta} D_{\text{nuc}}(Y_n, \Lambda) \) and with this representation we define
\[
 f_!(M)_n := (f_! M)_n,
\]
where each \( f_! \) was defined in the previous step. This definition is possible because all \( f_! \) satisfy arbitrary base-change.

4. Suppose that \( f : Y \to X \) is an \( \ell \)-fine map such that \( X \) admits a map to some strictly totally disconnected space. Pick an fdcms map \( g : Z \to Y \) which admits universal \( \ell \)-codescent such that \( f \circ g \) is fdcms and let \( g_\bullet : Z_\bullet \to Y \) denote the Čech nerve of \( g \). Then for every \( \mathcal{M} \in D_{\text{nuc}}(Y, \Lambda) \) we have \( \mathcal{M} = \lim_{\longrightarrow n \in \Delta} g_! g_\bullet^! \mathcal{M} \). Hence we must define
\[
 f_!(\mathcal{M}) := \lim_{\longrightarrow n \in \Delta} (f \circ g)_! g_\bullet^! \mathcal{M},
\]
where the functors \((f \circ g)_! \) and \( g_\bullet^! \) were defined in the previous step.

5. Suppose that \( f : Y \to X \) is a general \( \ell \)-fine map of small v-stacks. Then we can use the same descent technique as in step 3 to reduce the definition of \( f_! \) to the previous step.

Note that one could attempt to carry out the above construction directly instead of relying on the theory of abstract 6-functor formalisms from [10, §A.5]. This is possible to some extent, e.g. steps 1, 2 and 3 were carried out in [15, §22]. This has two downsides though: Firstly, the direct construction will not provide all the higher homotopies one expects the shriek functors to satisfy, so that thorough proofs involving the shriek functors require a lot of diagram checking; secondly, it seems very hard to carry out steps 4 and 5 using this approach.

We have constructed the full 6-functor formalism for nuclear sheaves over a fixed nuclear \( \mathbb{Z}_\ell \)-algebra \( \Lambda \). Sometimes it is useful to also understand how this 6-functor formalism behaves under a change of \( \Lambda \). For the following result we denote by \( \text{Ring}_{\mathbb{Z}_\ell, \text{nuc}} \) the \( \infty \)-category of nuclear \( \mathbb{Z}_\ell \)-algebras.

**Proposition 5.14.** There is a 6-functor formalism
\[
 D_{\text{nuc}} : \text{Corr}(\text{vStack} \times \text{Ring}_{\mathbb{Z}_\ell, \text{nuc}}^{\op})_{\text{fine,alt}} \to \text{Cat}_\infty, \quad (X, \Lambda) \mapsto D_{\text{nuc}}(X, \Lambda),
\]
where \( \text{fine} \) denotes the class of those maps \( (Y, \Lambda') \to (X, \Lambda) \) where the map \( Y \to X \) is an \( \ell \)-fine map of small v-stacks and the map \( \Lambda \to \Lambda' \) is an isomorphism. This 6-functor formalism has the following properties:
(i) For every small v-stack $X$ and any map $\Lambda \to \Lambda'$ of nuclear $\mathbb{Z}_\ell$-algebras, the induced pullback functor $\mathcal{D}_{\text{nuc}}(X, \Lambda) \to \mathcal{D}_{\text{nuc}}(X, \Lambda')$ is $- \otimes_{\Lambda} \Lambda'$.

(ii) For every nuclear $\mathbb{Z}_\ell$-algebra $\Lambda$, the restriction of $\mathcal{D}_{\text{nuc}}$ to $\text{Corr}(\text{vStack} \times \{\Lambda\})_{\text{fine, all}}$ coincides with the 6-functor formalism $\mathcal{D}_{\text{nuc}}(-, \Lambda)$ from Theorem 5.11.

Proof. Using the functor $\mathcal{D}_{\text{nuc}}(-, \mathbb{Z}_\ell): \text{vStack} \to \text{CAt}_\infty$ and abstract nonsense involving the usual straightening and unstraightening techniques and the generalized $\infty$-operad $\text{Mod}(\mathcal{C})^\otimes$ from [8, Definition 4.5.1.1] we can construct a functor

$$\mathcal{D}_{\text{nuc}}: \text{vStack} \times \text{Ring}_{\mathbb{Z}_\ell, \text{nuc}}^{\text{op}} \to \text{CAt}_\infty,$$

which satisfies (i) and restricts to the functor from Proposition 3.16.(i) for fixed $\Lambda$ (here $\text{CAt}_\infty$ denotes the $\infty$-category of symmetric monoidal $\infty$-categories). Now the construction of the desired 6-functor formalism can be carried out in the same way as in Theorem 5.11. The only difference occurs at the very beginning of the construction in Proposition 5.6 where we additionally need to verify the compatibility of proper pushforward and étale lower shriek with the base-change $- \otimes_{\Lambda} \Lambda'$; this follows immediately from the projection formula.

Corollary 5.15. Let $\Lambda \to \Lambda'$ be a map of nuclear $\mathbb{Z}_\ell$-algebras and $f: Y \to X$ an $\ell$-fine map of small v-stacks.

(i) The functor $f_!$ commutes naturally with the base-change and the forgetful functor along $\Lambda \to \Lambda'$.

(ii) The functor $f^!$ commutes naturally with the forgetful functor along $\Lambda \to \Lambda'$.

Proof. The fact that $f_!$ commutes with the base-change functor $- \otimes_{\Lambda} \Lambda'$ follows immediately from proper base-change in the 6-functor formalism from Proposition 5.14 for the cartesian diagram

$$
\begin{array}{ccc}
(Y, \Lambda') & \longrightarrow & (Y, \Lambda) \\
\downarrow & & \downarrow \\
(X, \Lambda') & \longrightarrow & (X, \Lambda)
\end{array}
$$

Note that it is really necessary to use Proposition 5.14 here even though on underlying $\Lambda$-modules the desired commutation of functors reduces to the projection formula. Namely, without the 6-functor formalism from Proposition 5.14 we do not even get a natural morphism between the functors $f_!(\_ \otimes_{\Lambda} \Lambda')$ and $f_!(- \otimes_{\Lambda} \Lambda')$ from $\mathcal{D}_{\text{nuc}}(Y, \Lambda)$ to $\mathcal{D}_{\text{nuc}}(X, \Lambda')$ (only as functors to $\mathcal{D}_{\text{nuc}}(X, \Lambda)$, but that is not enough). Claim (ii) follows immediately by passing to right adjoints.

It remains to see that $f!$ commutes with the forgetful functor along $\Lambda \to \Lambda'$, which by the commutation of $f_!$ with $- \otimes_{\Lambda} \Lambda'$ is now a condition rather than an additional datum. Therefore this can be checked along all the steps in the direct computation of $f_!$ in Remark 5.13 which ultimately reduces us to the case that either $f$ is an open immersion or proper. In the former case we apply Lemma 5.2.(iii). In the latter case we have $f_! = f_*$, so that the claim follows by passing to left adjoints and using Proposition 3.17.(i).

We finish this section by recording the following formal consequences of any 6-functor formalism, which are useful statements by themselves.

Lemma 5.16. Let $f: Y \to X$ be an $\ell$-fine map of small v-stacks.
(i) For all $\mathcal{M} \in \mathcal{D}_{\text{nuc}}(X, \Lambda)$ and $\mathcal{N} \in \mathcal{D}_{\text{nuc}}(Y, \Lambda)$ there is a natural isomorphism
\[ \text{Hom}(f_! \mathcal{N}, \mathcal{M}) = f_* \text{Hom}(\mathcal{N}, f^! \mathcal{M}). \]

(ii) For all $\mathcal{M}, \mathcal{N} \in \mathcal{D}_{\text{nuc}}(X, \Lambda)$ there is a natural isomorphism
\[ f^! \text{Hom}(\mathcal{N}, \mathcal{M}) = \text{Hom}(f^* \mathcal{N}, f^! \mathcal{M}). \]

Proof. This is formal, see e.g. [15, Proposition 23.3].

6 Perfect, Dualizable and Overconvergent Sheaves

Fix a prime $\ell \neq p$ and a nuclear $\mathbb{Z}_\ell$-algebra $\Lambda$. Having developed a full 6-functor formalism for nuclear $\Lambda$-sheaves on small v-stacks, we now want to study some particular special cases of nuclear sheaves.

We start with dualizable sheaves. The notion of dualizable objects can be defined in any symmetric monoidal $\infty$-category:

**Definition 6.1.** Let $\mathcal{C}$ be a symmetric monoidal $\infty$-category. An object $P \in \mathcal{C}$ is called dualizable if there are an object $P^*$, called the dual of $P$, and morphisms
\[ \text{ev}_P : P^* \otimes P \to 1, \quad i_P : 1 \to P \otimes P^*, \]

called the evaluation map and coevaluation map respectively, such that there are homotopy coherent diagrams

\[
\begin{array}{ccc}
P & \xrightarrow{i_P \otimes \text{id}} & P \otimes P^* \otimes P \\
\downarrow{\text{id}} & & \downarrow{\text{id} \otimes \text{ev}_P} \\
P & \xrightarrow{\text{id} \otimes i_P} & P^* \otimes P \otimes P^*
\end{array}
\]
\[
\begin{array}{ccc}
P^* & \xrightarrow{\text{id} \otimes i_P} & P^* \otimes P \otimes P^* \\
\downarrow{\text{id}} & & \downarrow{\text{ev}_P \otimes \text{id}} \\
P^* & \xrightarrow{\text{id}} & P^*
\end{array}
\]

We denote by $\mathcal{C}_{\text{dlb}} \subseteq \mathcal{C}$ the full subcategory spanned by the dualizable objects in $\mathcal{C}$.

It is easy to see that if $P$ is dualizable then the dual $P^*$ is unique up to unique isomorphism. We will use that fact without further mention below.

Note that by design the notion of dualizability in a symmetric monoidal $\infty$-category $\mathcal{C}$ only depends on the underlying symmetric monoidal 1-category. In this case one of the first instances where it appears in the literature is in [2, §1]. If $\mathcal{C}$ is closed, i.e. admits an internal hom functor $\text{Hom}$, then there is a different way of defining the dual of an object $P \in \mathcal{C}$ by letting $P^\vee := \text{Hom}(P, 1)$, where $1 \in \mathcal{C}$ is the monoidal unit. We say that $P$ is reflexive if the natural map $P \xrightarrow{\sim} P^{\vee \vee}$ is an isomorphism. A natural question is how this notion of duals behaves with respect to dualizability. Here is the answer:

**Lemma 6.2.** Let $\mathcal{C}$ be a closed symmetric monoidal $\infty$-category and $P \in \mathcal{C}$ an object. Then the following are equivalent:

(i) $P$ is dualizable.

(ii) $P$ is reflexive and the natural composed map $P \otimes P^\vee \to P^{\vee \vee} \otimes P^\vee \to (P \otimes P^\vee)^\vee$ is an isomorphism.

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(iii) The natural map $P \otimes P^\vee \xrightarrow{\sim} \text{Hom}(P,P)$ is an isomorphism.

(iv) For all $M, N \in C$ the natural map

$$\pi_0 \text{Hom}(M, N \otimes P^\vee) \xrightarrow{\sim} \pi_0 \text{Hom}(M \otimes P, N),$$

given by sending $f: M \to N \otimes P^\vee$ into the composition

$$M \otimes P \xrightarrow{f \otimes \text{id}} N \otimes P^\vee \otimes P \xrightarrow{\text{id} \otimes \text{ev}} N,$$

is an isomorphism.

If this is the case then $P^\vee$ is the dual of $P$ in the sense of Definition 6.1 and for all $M \in C$ the natural map

$$P^\vee \otimes M \xrightarrow{\sim} \text{Hom}(P, M)$$

is an isomorphism.

Proof. Note that all the statements only depend on the underlying 1-category of $C$, so we can assume that $C$ is a 1-category. Then the equivalence of (i), (ii) and (iv) is shown in [2, Theorem 1.3]. It goes as follows: The equivalence of (i) and (iv) follows easily by observing that both statements are equivalent to the fact that the functors $- \otimes P$ and $- \otimes P^*$ (resp. $- \otimes P^\vee$) are adjoint with the obvious counit. This also proves that necessarily $P^* = P^\vee$ in (i). Note that this implies that condition (iv) equivalently holds with the roles of $P$ and $P^\vee$ swapped. With this known, it is straightforward to see that (iv) implies (ii). To prove that (ii) implies (i), note that we always have the evaluation map $\text{ev}: P \otimes P^\vee \to 1$ and if the isomorphism in (ii) holds then we also get a coevaluation map $i: 1 = 1^\vee \to (P \otimes P^\vee)^\vee = P \otimes P^\vee$. To show that the necessary triangles commute, we consider the following diagram in $C$:

Here the vertical isomorphisms are induced by the isomorphisms in (ii) (where in the middle column the maps act as the identity on the right-hand $P$). The map $\alpha$ is the natural map induced by the dualized evaluation map $1 = 1^\vee \to ((P \otimes P^\vee)^\vee$ (intuitively this singles out the element $A \in (P \otimes P^\vee)^\vee$ given as the map $P \otimes P^\vee \to 1$, $(x, f) \mapsto f(x)$), so that the left-hand triangle of the diagram commutes by definition of $i$. Also clearly the upper right square commutes. The map $\beta$ is the map which by adjunction corresponds to the map

$$(P \otimes P^\vee)^\vee = \text{Hom}(P \otimes P^\vee, 1) \xrightarrow{\sim} \text{Hom}(P, P^\vee)$$

(Intuitively it sends a map $A: P \times P^\vee \to 1$ and an element $x \in P$ to the element $f \mapsto A(x, f)$ in $P^\vee \otimes P$.) One checks immediately that the lower right triangle commutes. It follows that the whole diagram commutes. Moreover, we see that $\beta \circ \alpha$ is the natural map $P \to P^\vee \otimes P$. By the commutativity of the diagram and the fact that $P \xrightarrow{\sim} P^\vee \otimes P$ is an isomorphism, we deduce that the
composition of the upper two horizontal maps is the identity, as desired. The second diagram in the definition of dualizable maps can be verified similarly (or note that everything is symmetric in $P$ and $P^\vee$).

We have now shown that (i), (ii) and (iv) are all equivalent. To show that (ii) implies (iii), we simply note that if (ii) holds then

$$P \otimes P^\vee = (P^\vee \otimes P)^\vee = \text{Hom}(P, P^\vee) = \text{Hom}(P, P),$$

as desired. Finally, assume that (iii) is satisfied. Then additionally to the natural evaluation map $\text{ev}: P \otimes P^\vee \to 1$ there is a coevaluation map $1 \to \text{Hom}(P, P) = P \otimes P^\vee$ given by the identity on $P$. One checks that these two maps exhibit $P$ as dualizable with dual $P^\vee$: The left-hand triangle is obviously commutative; for the right-hand triangle we use the fact that for every $M, N \in \mathcal{C}$ the evaluation map $M^\vee \otimes M \otimes N^\vee \to N^\vee$ factors over $M^\vee \otimes \text{Hom}(N, M)$, where the map from this object to $N^\vee$ is a special case of the general map

$$\text{Hom}(X, Y) \otimes \text{Hom}(Y, Z) \to \text{Hom}(X, Z)$$

for $X, Y, Z \in \mathcal{C}$. Now apply this to $N = M = P$ to conclude.

We proved the equivalence of (i), (ii), (iii) and (iv). The final claim follows immediately from (iv).

**Remark 6.3.** We put a lot of care in the proof of Lemma 6.2 even though the proof is rather straightforward and the results are not new. The reasons for this are twofold:

1. We found it surprisingly hard to collect these results from the literature, and we were unable to find a reference where all displayed characterizations of dualizability are stated as cleanly as here.

2. We have repeatedly been confused about what is true and what not. For example, we found a reference where it is suggested that the isomorphism $P \otimes P^\vee \sim \text{Hom}(P, P)$ is not enough to deduce dualizability, but it certainly is. Moreover, when characterizing dualizability via adjoint functors, one has to be careful. In particular, criterion (iv) of [10, Lemma 3.7.4] seems to be false in general.

We also get the following abstract properties of dualizable objects. The first property will help us prove $\nu$-descent, while the second property is a sanity check.

**Lemma 6.4.** Let $F: \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor between symmetric monoidal $\infty$-categories and let $P \in \mathcal{C}$ be dualizable. Then $F(P)$ is dualizable. Moreover, if both $\mathcal{C}$ and $\mathcal{D}$ are closed then for all $M \in \mathcal{C}$ the natural map

$$F(\text{Hom}(P, M)) \sim \text{Hom}(F(P), F(M))$$

is an isomorphism.

**Proof.** It follows immediately from the definitions that $F(P)$ is dualizable. The second claim follows because by Lemma 6.2 both sides can be identified with $F(P)^\vee \otimes F(M)$.

**Lemma 6.5.** Let $\mathcal{C}$ be a closed symmetric monoidal stable $\infty$-category. Then the subcategory $\mathcal{C}_{\text{dub}}$ of dualizable objects is stable under finite (co)limits and retracts.
Proof. This follows easily from Lemma 6.2 by noting that both $P \otimes Q$ and $\text{Hom}(Q, P)$ commute with finite (co)limits and retracts in $P$ and $Q$. \qed

Let us now come to the geometric setting. In the setting of categories of sheaves on geometric objects, one can usually characterize dualizability in terms of some “perfectness” condition. The same happens for our $\infty$-category of nuclear sheaves, so let us introduce the relevant definitions.

We start with the notion of overconvergence, which naturally comes up when studying perfect sheaves:

**Definition 6.6.** Let $X$ be a strictly totally disconnected space.

(a) We define

$$D_{\text{nuc}}(\pi_0(X), \Lambda) \subseteq D_{\square}(\pi_0(X), \Lambda) \subseteq D(\pi_0(X)_{\text{proet}}, \Lambda)$$

in the obvious sense (e.g. view $\pi_0(X)$ as a strictly totally disconnected space by realizing it over some Spa $C$).

(b) We denote by

$$\pi : X_{\text{proet}} \to \pi_0(X)_{\text{proet}}$$

the natural map of sites.

(c) We denote $\mathbb{Z}_\ell(X) = \mathcal{C}(X, \mathbb{Z}_\ell)$, which is a static $\mathbb{Z}_\ell$-Banach algebra. We further denote $\Lambda(X) := \Lambda \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(X)$, which is a nuclear $\mathbb{Z}_\ell$-algebra.

**Lemma 6.7.** Let $X$ be a strictly totally disconnected space.

(i) The functor $\pi_* : D_{\square}(\pi_0(X), \Lambda) \omega_1 \to D_{\square}(\pi_0(X), \Lambda)_{\omega_1}$ is bounded and preserves all small limits and colimits.

(ii) The functor $\pi^* : D_{\square}(\pi_0(X), \Lambda)_{\omega_1} \hookrightarrow D_{\square}(\pi_0(X), \Lambda)_{\omega_1}$ is $t$-exact and fully faithful and preserves all small colimits and countable limits.

(iii) Both $\pi_*$ and $\pi^*$ preserve nuclear sheaves.

(iv) There is a natural equivalence of symmetric monoidal $\infty$-categories

$$D_{\text{nuc}}(\pi_0(X), \Lambda) = D_{\text{nuc}}(\Lambda(X)),$$

where $D_{\text{nuc}}(\Lambda(X))$ denotes the $\infty$-category of nuclear $\Lambda(X)$-modules on $\ast_{\text{proet}}$.

Proof. All of the claims commute with the forgetful functor along $\mathbb{Z}_\ell \to \Lambda$, so that we can assume $\Lambda = \mathbb{Z}_\ell$. The claims (i) and (ii) are easy, see e.g. [11, Lemma 5.7] and the arguments in Proposition 2.6.(ii). Claim (iii) follows immediately by checking on generators.

We now prove (iv). Consider the natural morphism $\alpha : (\pi_0(X)_{\text{proet}}, \mathbb{Z}_\ell) \to (\ast_{\text{proet}}, \mathbb{Z}_\ell(X))$ of ringed sites. Again, it is easy to see that $\alpha_*$ preserves colimits and countable limits of $\omega_1$-solid sheaves and hence restricts to a functor $\alpha_* : D_{\square}(\pi_0(X), \mathbb{Z}_\ell)_{\omega_1} \to D_{\square}(\mathbb{Z}_\ell(X))_{\omega_1}$. This restricted functor is $t$-exact (check on countable limits of qcqs étale sheaves, reducing to the case of étale sheaves, where it is clear) and preserves all small limits (this can be checked after applying the forgetful functor to $D_{\square}(\mathbb{Z}_\ell)_{\omega_1}$, where it is clear). It thus admits a left adjoint $\alpha^* : D_{\square}(\mathbb{Z}_\ell(X))_{\omega_1} \to$
\(D_{\mathcal{C}}(\pi_0(Y), Z_{\ell})_{\omega_1}\). Note that precomposing \(\alpha^*\) with the functor \(- \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(X)\) gives the functor
\[s^*: D_{\mathcal{C}}(\mathbb{Z}_\ell)_{\omega_1} \to D_{\mathcal{C}}(\pi_0(X), \mathbb{Z}_\ell)_{\omega_1},\]
where \(s: \pi_0(X)_{\text{proet}} \to *_{\text{proet}}\) is the natural projection; this allows us to compute \(\alpha^*\). First of all it implies that \(\alpha^*\) maps discrete \(\Lambda(X)\)-modules to étale \(\mathbb{Z}_\ell\)-sheaves on \(\pi_0(X)\), because the discrete \(\Lambda\)-modules are generated under colimits by the objects \(M \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(X)\) for discrete \(\mathbb{Z}_\ell\)-modules \(M\). We can similarly show that \(\alpha^*\) preserves right-bounded \(\ell\)-adically complete modules: We know that \(\alpha^*\) is right \(t\)-exact, so by the usual arguments (see the proof of Lemma \[3.7.(ii)\]) we can reduce this claim to countable products of a set of compact generators. But compact generators can be chosen as \(\prod_I \mathbb{Z}_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(X)\) and they get sent via \(\alpha^*\) to \(s^* \prod_I \mathbb{Z}_\ell = \prod_I \mathbb{Z}_\ell\) (where on the right hand side we view it as a sheaf on \(\pi_0(X)\)).

We now claim that \(\alpha^*\) preserves nuclearity and is fully faithful on nuclear modules. The first claim follows from the fact that for every basic nuclear \(\mathbb{Z}_\ell\)-module \(\alpha^*\) is right-bounded \(\ell\)-adically complete. But on discrete objects it is easy to see that \(\alpha_s^* \alpha^* = \text{id}\) (use again the trick via \(s^*\)). We have now shown that \(\alpha^*\) induces an embedding
\[\alpha^*: D_{\text{nuc}}(\mathbb{Z}_\ell(X)) \hookrightarrow D_{\text{nuc}}(\pi_0(X), \mathbb{Z}_\ell)\].

It remains to show that \(\alpha^*\) is essentially surjective, i.e. contains every \(M \in D_{\text{nuc}}(\pi_0(X), \mathbb{Z}_\ell)\). By writing \(M\) as the colimit of the nuclearizations of compact objects (see Proposition \[3.12\]) we can assume that \(M\) is bounded and \(\ell\)-adically complete. But \(\alpha^*\) is clearly essentially surjective onto étale sheaves, hence every \(\mathcal{M}/\ell^n\mathcal{M}\) lies in the image of \(\alpha^*\), i.e. \(\mathcal{M}/\ell^n\mathcal{M} = \alpha^* \alpha_s(M/\ell^n\mathcal{M})\). Since \(\alpha^*\) preserves right-bounded \(\ell\)-adically complete objects we deduce \(\mathcal{M} = \alpha^* \alpha_s \mathcal{M}\), as desired.

**Definition 6.8.** Let \(X\) be a small \(v\)-stack and \(\mathcal{M} \in D_{\text{nuc}}(X, \Lambda)\). We say that \(\mathcal{M}\) is overconvergent if for every map \(f: Y \to X\) from a strictly totally disconnected space \(Y\) the pullback \(f^* \mathcal{M}\) lies in the essential image of the embedding
\[\pi^* D_{\text{nuc}}(\Lambda(Y)) \hookrightarrow D_{\text{nuc}}(Y, \Lambda)\]
from Lemma \[5.7\]. We denote by
\[D_{\text{nuc}}(X, \Lambda)^{\text{oc}} \subseteq D_{\text{nuc}}(X, \Lambda)\]
the full subcategory spanned by the overconvergent sheaves.

**Lemma 6.9.** (i) Let \(X\) be a small \(v\)-stack. Then \(D_{\text{nuc}}(X, \Lambda)^{\text{oc}}\) is stable under the symmetric monoidal structure, all colimits, all pullbacks, all \(\ell\)-adic completions and under the forgetful and base-change functors for any map \(\Lambda \to \Lambda'\) of nuclear \(\mathbb{Z}_\ell\)-algebras.

(ii) The assignment \(X \mapsto D_{\text{nuc}}(X, \Lambda)^{\text{oc}}\) defines a hypercomplete sheaf of \(\infty\)-category on the \(v\)-site of small \(v\)-stacks.

(iii) Let \(X\) be a small \(v\)-stack and \(\mathcal{M} \in D_{\text{nuc}}(X, \Lambda)\). Then \(\mathcal{M}\) is overconvergent if and only if for every pro-étale map \(Y' \to Y\) of strictly totally disconnected spaces over \(X\) such that \(\pi_0(Y') \sim \pi_0(Y)\) is an isomorphism, the induced map \(\Gamma(Y, \mathcal{M}) \sim \Gamma(Y', \mathcal{M})\) is an isomorphism.
(iv) Let $X$ be an $\ell$-bounded spatial diamond and $P \in \mathcal{D}_\omega(X, \Lambda)$ compact. Then $\text{Hom}_\omega(P, \Lambda)$ is nuclear and overconvergent.

Proof. Part (i) follows easily from Lemma \ref{lem:basic}. Part (ii) can be checked on the v-site of strictly totally disconnected spaces, where it follows from the v-descent of nuclear sheaves (see Proposition \ref{prop:desc}).

We now prove (iii), so let $X$ and $M$ be given. By (i) we can assume that $\Lambda = \mathbb{Z}_\ell$. We first show that the given condition on $\Gamma(Y, M) \to \Gamma(Y', M)$ implies overconvergence, so assume that $M$ satisfies this condition. We immediately reduce to the case that $X$ is strictly totally disconnected, in which case we want to show that the natural map $\pi^* \pi_* M \to M$ is an isomorphism. Note that for every pro-étale $U \to X$ we have

$$\Gamma(U, \pi^* \pi_* M) = \Gamma(\pi_0(U), \pi_* M) = \Gamma(X \times_{\pi_0(X)} \pi_0(U), M)$$

(the first identity holds for all $\omega_1$-solid sheaves $N$ in place of $M$, which can be checked on compact generators and thus on qcqs étale sheaves, where it is obvious). But the map $U \to X \times_{\pi_0(X)} \pi_0(U)$ is clearly an isomorphism on $\pi_0$, so we deduce $\Gamma(U, \pi^* \pi_* M) = \Gamma(U, M)$, which implies $\pi^* \pi_* M = M$. By reversing the argument, we see that if $M$ is overconvergent then $\Gamma(U, M)$ depends only on $\pi_0(U)$. This proves (iii).

It remains to prove (iv), so let $X$ and $P$ be given. By Propositions \ref{prop:nuc} and \ref{prop:nuc2} we can assume that $P = \mathbb{Z}_\ell \to U \otimes \Lambda$ for some basic $U = \lim_n U_n$ in $X_{\text{proet}}$. If we denote $f : U \to X$ the structure map then $P = f_* \Lambda$, hence $\text{Hom}_\omega(P, \Lambda) = f_* \Lambda$. By Proposition \ref{prop:nuc3} this is indeed nuclear. It remains to show that this sheaf is overconvergent, for which we can w.l.o.g. assume that $X$ is strictly totally disconnected. Then for every affinoid pro-étale $V \to X$ we have

$$\Gamma(V, f_* \Lambda) = \Gamma(V \times_X U, \Lambda) = \Gamma(\pi_0(V \times_X U), \Lambda) = \Gamma(\pi_0(V) \times_{\pi_0(X)} \pi_0(U), \Lambda)$$

(where in the last identity we use e.g. \cite{[1]} Lemma 5.8). By (iii) this implies that $f_* \Lambda$ is indeed overconvergent, as desired.

With a good understanding of overconvergent sheaves at hand, we can now come to perfect and dualizable sheaves. We get the following:

**Definition 6.10.** Let $X$ be a small v-stack. A sheaf $P \in \mathcal{D}_{\text{nuc}}(X, \Lambda)$ is called **perfect** if there is a v-cover $(f_i : Y_i \to X)_i$ by strictly totally disconnected spaces $Y_i$ and for each $Y_i$ a dualizable $\Lambda(Y_i)$-module $P_i \in \mathcal{D}_{\text{nuc}}(\Lambda(Y_i))$ such that $f_i^* P \cong \pi_i^* P_i$, where $\pi_i : Y_i \to \pi_0(Y_i)$ is the map from Definition \ref{def:locally}. We denote by

$$\mathcal{D}_{\text{nuc}}(X, \Lambda)_{\text{perf}} \subseteq \mathcal{D}_{\text{nuc}}(X, \Lambda)^{\text{perf}}$$

the full subcategory spanned by the perfect sheaves.

**Remark 6.11.** We do not know if a dualizable $\Lambda(Y_i)$-module is automatically perfect in the sense that it is generated under retracts and finite colimits from $\Lambda(Y_i)$. By \cite{[1]} Proposition 9.3 being dualizable is equivalent to being compact in $\mathcal{D}_{\text{nuc}}(\Lambda(Y_i))$ and by the other results in \cite{[1]} §9 one should be able to show that this is equivalent to being perfect in all cases of practical interest. For example, it holds if $\Lambda$ is ind-compact as a module over some $\ell$-adically complete nuclear $\mathbb{Z}_\ell$-algebra $\Lambda_0$: First reduce to the case $\Lambda = \Lambda_0$, in which case $\text{Hom}(\Lambda, -)$ is easily seen to be conservative on compact objects – because these are $\ell$-adically complete – which implies that all compact objects are perfect.
Proposition 6.12.  (i) On every small \( v \)-stack \( X \) we have \( D_{\text{nuc}}(X, \Lambda)_{\text{perf}} = D_{\text{nuc}}(X, \Lambda)_{\text{dlb}} \), i.e. the perfect sheaves are precisely the dualizable objects.

(ii) The assignment \( X \mapsto D_{\text{nuc}}(X, \Lambda)_{\text{perf}} \) defines a hypercomplete sheaf of \( \infty \)-categories on the \( v \)-site of small \( v \)-stacks.

Proof. Let us first argue why the assignment \( X \mapsto D_{\text{nuc}}(X, \Lambda)_{\text{dlb}} \) is a hypercomplete \( v \)-sheaf. To see this, let \( f_* : Y_\bullet \to X \) be any \( v \)-hypercover of small \( v \)-stacks. First observe that dualizable objects are stable under pullback by Lemma 6.4. Now suppose we have some \( \mathcal{P} \in D_{\text{nuc}}(X, \Lambda) \) such that all \( f_n^* \mathcal{P} \) are dualizable; we need to show that \( \mathcal{P} \) is dualizable. Given any \( \text{coCartesian section} \ M_\bullet \in D_{\text{nuc}}(Y_\bullet, \Lambda) \), it follows immediately from Lemma 6.4 that \( \operatorname{Hom}(\mathcal{P}_\bullet, M_\bullet) \) is again a \( \text{coCartesian section} \) in \( D_{\text{nuc}}(Y_\bullet, \Lambda) \). If \( M \in D_{\text{nuc}}(X, \Lambda) \) is the sheaf corresponding to \( M_\bullet \), it follows that \( \operatorname{Hom}(\mathcal{P}, M) \) is the sheaf corresponding to \( \operatorname{Hom}(\mathcal{P}_\bullet, M_\bullet) \). In particular, \( f_0^* \operatorname{Hom}(\mathcal{P}, M) = \operatorname{Hom}(f_0^* \mathcal{P}, f_0^* M) \). Now Lemma 6.2 lets us easily deduce that \( \mathcal{P} \) is dualizable as desired.

Now suppose that \( X \) is a strictly totally disconnected space and fix any \( \mathcal{P} \in D_{\text{nuc}}(X, \Lambda) \). Assume that \( \mathcal{P} \) is dualizable. Then \( \operatorname{Hom}(\mathcal{P}, -) = \Gamma(X, \operatorname{Hom}(\mathcal{P}, -)) = \Gamma(X, \mathcal{P}^\vee \otimes -) \) by Lemma 6.2 which preserves all small colimits. Hence \( \mathcal{P} \) is compact in \( D_{\text{nuc}}(X, \Lambda) \) and since the nuclearization functor preserves small colimits, \( \mathcal{P} \) is also compact in \( D_{\text{nuc}}(X, \Lambda)_{\omega_1} \). Applying the same argument to \( \mathcal{P}^\vee \) and using that \( \mathcal{P} = \mathcal{P}^{\vee \vee} \) we deduce from Lemma 6.9.(iv) that \( \mathcal{P} \) and \( \mathcal{P}^\vee \) are overconvergent and hence lie in the essential image of \( \pi^* \). Since \( \pi^* \) is fully faithful, we deduce immediately from the definition of dualizable objects that \( \mathcal{P} \) is dualizable as an object of \( D_{\text{nuc}}(\Lambda(X)) \). Altogether we see that if \( \mathcal{P} \) is dualizable then it is of the form \( \mathcal{P} = \pi^* \mathcal{P} \) for some dualizable nuclear \( \Lambda(X) \)-module \( \mathcal{P} \). The converse of this statement is obviously true as well. By combining this observation with \( v \)-descent for nuclear modules we easily deduce (i) and (ii).\[\square\]

7 Relatively Dualizable Sheaves

Fix a prime \( \ell \neq p \) and a nuclear \( \mathbb{Z}_\ell \)-algebra \( \Lambda \). In the previous section we studied dualizable sheaves and identified them with the perfect ones. We now want to introduce a more general version of \textit{relatively dualizable} sheaves. This concept is not new: It is known as universally locally acyclic sheaves in [3] (for discrete \( \Lambda \)) and has also previously been studied in the realm of algebraic geometry. Although the original motivation and intuition for these objects comes from geometry, Lu-Zheng [7] have recently found an abstract way of describing them, which was also adopted in [3, Theorem IV.2.23]. As we will see in the following, this abstract definition can be carried out with great results in any 6-functor formalism and in particular produces an extremely powerful tool to study smoothness (see Section 8). On top of that, in the realm of representation theory the relatively dualizable sheaves will be precisely the admissible representations (see Proposition 10.15) which makes them very interesting for applications to the Langlands program.

Without further ado, let us start with the definition of relatively dualizable sheaves. It relies on the following magical 2-category (cf. [3, §IV.2.3.3]):

Definition 7.1. Given any small \( v \)-stack \( S \) we denote by \( \mathcal{C}_S \) the following 2-category: The objects of \( \mathcal{C}_S \) are the \( \ell \)-fine maps \( X \to S \). For any two objects \( X, Y \to S \) in \( \mathcal{C}_S \) we define the category \( \operatorname{Fun}_{\mathcal{C}_S}(X, Y) \) as

\[
\operatorname{Fun}_{\mathcal{C}_S}(X, Y) := D_{\text{nuc}}(Y \times_S X, \Lambda),
\]
where on the right-hand side we implicitly take the underlying 1-category of \( \mathcal{D}_{\text{nuc}} \). Given three objects \( X, Y, Z \to S \) in \( \mathcal{C}_S \), the composition functor

\[
\text{Fun}_{\mathcal{C}_S}(Y, Z) \times \text{Fun}_{\mathcal{C}_S}(X, Y) \to \text{Fun}_{\mathcal{C}_S}(X, Z)
\]

is defined to be the functor

\[
\mathcal{D}_{\text{nuc}}(Z \times_S Y, \Lambda) \times \mathcal{D}_{\text{nuc}}(Y \times_S X, \Lambda) \to \mathcal{D}_{\text{nuc}}(Z \times_S X, \Lambda),
\]

\[
(N, M) \mapsto N \star M := \pi_{13}^* \Lambda \otimes \pi_{23}^* \Lambda,
\]

where \( \pi_{ij} \) denote the various projections of \( Z \times_S Y \times_S X \). It follows from the projection formula that \( \mathcal{C}_S \) is indeed a 2-category. For every \( X \to S \) in \( \mathcal{C}_S \) the identity functor on \( X \) is given by \( \Delta^! \Lambda \), where \( \Delta : X \to X \times_S X \) is the diagonal.

We also recall the definition of adjoint morphisms in a 2-category. Applied to the 2-category of categories, this recovers the usual notion of adjoint functors.

**Definition 7.2.** Let \( \mathcal{C} \) be a 2-category. Then a morphism \( f : X \to Y \) in \( \mathcal{C} \) is left adjoint to a morphism \( g : Y \to X \) if there are a unit \( \varepsilon : \text{id}_X \to gf \) and a counit \( \eta : fg \to \text{id}_Y \) such that the composites

\[
f \xrightarrow{\varepsilon} fgf \xrightarrow{\eta f} f, \quad g \xrightarrow{\varepsilon g} gfg \xrightarrow{g \eta} g
\]

are the identity. In this case \( g \) is uniquely determined up to unique isomorphism.

**Example 7.3.** The following example is a very enlightening special case of what is to come: Let \( \mathcal{D} \) be a symmetric monoidal \( \infty \)-category. We associate to it the 2-category \( \mathcal{C} \) which has only one object \( * \) such that \( \text{Fun}_C(*,*) = \mathcal{D} \) (viewed as the underlying 1-category) and such that the composition is given by the tensor product. Then an object \( P \in \mathcal{D} \) is dualizable if and only if it is a left adjoint when viewed as a morphism \( * \to * \) in \( \mathcal{C} \). In this case the dual of \( P \) is its right adjoint. In fact, the evaluation and coevaluation map for \( P \) translate directly to the counit and unit map for the corresponding adjunction.

**Remark 7.4.** It may seem a bit awkward that in the definition of \( \mathcal{C}_S \) we only work with the 1-categorical version of \( \mathcal{D}_{\text{nuc}} \), thereby throwing away the \( \infty \)-enrichment (and in particular any sensible notion of limits and colimits). One may also attempt to construct \( \mathcal{C}_S \) as an \((\infty, 2)\)-category, but this seems hard to do. However, similar to how dualizable objects in a symmetric monoidal \( \infty \)-category only depend on the underlying 1-category (and still satisfy nice properties in the \( \infty \)-enrichment, see Lemma 6.5), for all our applications it is completely sufficient to work with the 2-category (instead of a potential \((\infty, 2)\)-category) \( \mathcal{C}_S \). This is further illustrated by Example 7.3.

With the above preparations at hand, we can finally come to the definition of relatively dualizable nuclear sheaves:

**Definition 7.5.** Let \( f : X \to S \) be an \( \ell \)-fine map of small v-stacks. A sheaf \( P \in \mathcal{D}_{\text{nuc}}(X, \Lambda) \) is called \( f \)-dualizable if it is a left adjoint when viewed as a morphism \( X \to S \) in \( \mathcal{C}_S \). We denote by

\[
\mathcal{D}_{\text{nuc}}(X, \Lambda)_{\text{dlb}_f} \subseteq \mathcal{D}_{\text{nuc}}(X, \Lambda)
\]

the full subcategory spanned by the \( f \)-dualizable sheaves.
For example, if $X = S$ and $f = \text{id}_S$ then by Example 7.3 an id-dualizable sheaf on $S$ is the same as a dualizable sheaf on $S$. Somewhat surprisingly, it turns out that relatively dualizable objects have very similar formal properties as dualizable objects, which we collect in the following. The results tend to get rather convoluted due to many pullback and shriek functors flying around – we encourage the reader in each statement to first try to understand the case that all geometric maps are the identity, in which case one always obtains a basic property of dualizable sheaves like the ones proved in Section 6.

**Definition 7.6.** Let $f: X \to S$ be an $\ell$-fine map of small v-stacks. Then for every $\mathcal{M} \in \mathcal{D}_{\text{nuc}}(X, \Lambda)$ we denote

$$D_f(\mathcal{M}) := \text{Hom}(\mathcal{M}, f^! \Lambda)$$

and call it the $f$-dual of $\mathcal{M}$.

**Proposition 7.7.** Let $f: X \to S$ be an $\ell$-fine map of small v-stacks and $\mathcal{P} \in \mathcal{D}_{\text{nuc}}(X, \Lambda)$. Then the following are equivalent:

(i) $\mathcal{P}$ is $f$-dualizable.

(ii) The natural map

$$\pi_1^* D_f(\mathcal{P}) \otimes \pi_2^* \mathcal{P} \xrightarrow{\sim} \text{Hom}(\pi_1^* \mathcal{P}, \pi_2^* \mathcal{P})$$

is an isomorphism, where $\pi_1, \pi_2: X \times_S X \to X$ are the two projections.

(iii) For every map $g: S' \to S$ of small v-stacks with associated pullback square

$$\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
S' & \xrightarrow{g} & S
\end{array}$$

and base-change $\mathcal{P}' := g'^* \mathcal{P}$ the following natural maps of functors are isomorphisms:

$$D_{f'}(\mathcal{P}') \otimes f'^* \xrightarrow{\sim} \text{Hom}(\mathcal{P}', f'^! \mathcal{P})$$

$$g'^* \text{Hom}(\mathcal{P}, f^! \mathcal{P}) \xrightarrow{\sim} \text{Hom}(\mathcal{P}', f'^! g^* \mathcal{P}).$$

In particular the natural map $g'^* D_f(\mathcal{P}) \xrightarrow{\sim} D_{f'}(g'^* \mathcal{P})$ is an isomorphism.

If this is the case then also $D_f(\mathcal{P})$ is $f$-dualizable and the natural map $\mathcal{P} \xrightarrow{\sim} D_f(D_f(\mathcal{P}))$ is an isomorphism.

**Proof.** We first prove that (ii) implies (i), which is a more elaborate version of the observation that if $\mathcal{P} \otimes \mathcal{P}^\vee \xrightarrow{\sim} \text{Hom}(\mathcal{P}, \mathcal{P})$ is an isomorphism for some object $\mathcal{P}$ in a closed symmetric monoidal $\infty$-category, then $\mathcal{P}$ is dualizable (see Lemma 6.2). Namely, as in the proof of Lemma 6.2 we explicitly construct the counit (the analog of the evaluation map) and unit (the analog of the coevaluation map) for an adjunction between $\mathcal{P}$ and $D_f(\mathcal{P})$. With this analog in mind, it is not surprising that the counit is easy to construct and does not require (ii); it is the map

$$\mathcal{P} \star D_f(\mathcal{P}) = f_!(D_f(\mathcal{P}) \otimes \mathcal{P}) \to \Lambda$$
which is adjoint to the canonical pairing $D_f(\mathcal{P}) \otimes \mathcal{P} \to f^! \Lambda$. On the other hand, the unit is obtained by inverting the isomorphism in (ii), namely it is the map

$$\Delta^! : \Lambda \to D_f(\mathcal{P}) \ast \mathcal{P} = \pi_1^* D_f(\mathcal{P}) \otimes \pi_2^* \mathcal{P} = \text{Hom}(\pi_1^* \mathcal{P}, \pi_2^* \mathcal{P})$$

which is given via adjunction by the canonical map (see Lemma 5.16.(i))

$$\Lambda \to \Delta^! \text{Hom}(\pi_1^* \mathcal{P}, \pi_2^* \mathcal{P}) = \text{Hom}(\Delta^* \pi_1^* \mathcal{P}, \Delta^! \pi_2^* \mathcal{P}) = \text{Hom}(\mathcal{P}, \mathcal{P})$$

induced by the identity on $\mathcal{P}$; here $\Delta : X \to X \times_S X$ denotes the diagonal. To prove that these maps indeed define an adjunction one can argue similar to Lemma 6.2; we leave the details to the reader.

We now prove that (i) implies (iii). First of all, note that condition (i) is stable under every base-change: Given a map $g : S' \to S$, consider the functor of 2-categories $\mathcal{C}_S \to \mathcal{C}_{S'}$ given by mapping $X \to S$ to $X' := X \times_S S' \to S'$ and acting via pullbacks on the morphisms. Since functors of 2-categories obviously preserve adjoint functors, it follows immediately that condition (i) is indeed stable under base-change. Therefore for proving the first isomorphism in (iii) we can from now on assume $S' = S$. Then the claim is an analog of the observation that if $\mathcal{P}$ is a dualizable object in a closed symmetric monoidal $\infty$-category then the map $P^! \otimes - \xrightarrow{\sim} \text{Hom}(\mathcal{P}, -)$ is an isomorphism of functors (see Lemma 6.2). In fact, we can apply a similar proof strategy: Consider the functor from $\mathcal{C}_S$ to the 2-category of stable $\infty$-categories (viewed as 1-categories by forgetting the $\infty$-enhancement) which maps $X$ to $D_{\text{hom}}(X, \Lambda)$ and $M \in \text{Fun}_{\mathcal{C}_S}(X, Y)$ to the functor $\pi_2^*(M \otimes \pi_1^*)$. By assumption $\mathcal{P}$ is left adjoint to some object $Q \in D_{\text{hom}}(X, \Lambda)$ which by the just constructed functor of 2-categories results in the fact that the functor $f_!(\mathcal{P} \otimes -)$ is left adjoint to the functor $Q \otimes f^*$. But the right adjoint of the former functor is also given by $\text{Hom}(\mathcal{P}, f^!)$, which produces a uniquely determined 1-categorical isomorphism of functors

$$Q \otimes f^* \cong \text{Hom}(\mathcal{P}, f^!).$$

Plugging in $\Lambda$ yields $Q \cong D_f(\mathcal{P})$. Note furthermore that the adjunction is induced by a map $\mathcal{P} \otimes Q = f_!(\mathcal{P} \otimes Q) \to \Lambda$, i.e. a pairing $\mathcal{P} \otimes Q \to f^! \Lambda$. It follows from this observation that under the identification $Q \cong D_f(\mathcal{P})$ the above 1-categorical isomorphism of functors is the first one in (iii) (and in particular it is an $\infty$-categorical isomorphism). To prove the second isomorphism, note that by applying the first isomorphism on both sides we end up with proving that the natural map $g^* D_f(\mathcal{P}) \xrightarrow{\sim} D_f(g^* \mathcal{P})$ is an isomorphism. But this follows easily from the fact that the pullback functor $\mathcal{C}_S \to \mathcal{C}_{S'}$ preserves right adjoints and that these right adjoints are uniquely determined up to unique isomorphism.

We now prove that (iii) implies (ii), so assume that (iii) satisfied. Then after base-change along $X \to S$ we deduce that the natural map of functors

$$D_{\pi_2}(\pi_1^* \mathcal{P}) \otimes \pi_2^* \xrightarrow{\sim} \text{Hom}(\pi_1^* \mathcal{P}, \pi_2^*)$$

is an isomorphism. By evaluating both sides on $\mathcal{P}$ and using that $D_{\pi_2}(\pi_1^* \mathcal{P}) = \pi_1^* D_f(\mathcal{P})$ (by the second isomorphism in (iii)) we get the desired identity in (ii).

The final claim follows from the above identification of $D_f(\mathcal{P})$ as the right adjoint of $\mathcal{P}$ in $\mathcal{C}_S$ and the fact that by the equivalence $\mathcal{C}_S \cong \mathcal{C}_S^{\text{op}}$ it is also a left adjoint of $\mathcal{P}$. 

Using Proposition 7.7.(iii) we can show that the notion of relative dualizability satisfies v-descent, in the following sense.
Corollary 7.8. Let

\[
\begin{array}{ccc}
  X' & \xrightarrow{g'} & X \\
  \downarrow{f'} & & \downarrow{f} \\
  S' & \xrightarrow{g} & S
\end{array}
\]

be a cartesian square of small v-stacks such that \(f\) is \(\ell\)-fine and let \(\mathcal{P} \in \mathcal{D}_{\text{nuc}}(X, \Lambda)\) be given.

(i) If \(\mathcal{P}\) is \(f\)-dualizable then \(g'^*\mathcal{P}\) is \(f'\)-dualizable.

(ii) If \(g'^*\mathcal{P}\) is \(f'\)-dualizable and \(g\) is a v-cover then \(\mathcal{P}\) is \(f\)-dualizable.

Proof. Part (i) is easy, e.g. use the pullback functor \(\mathcal{C}_S \rightarrow \mathcal{C}_{S'}\) constructed in the proof of Proposition 7.7. The proof of (ii) is very similar to the proof of the descent of dualizable objects in Proposition 6.12. Assume that \(g\) is a v-cover and that \(\mathcal{P}' := g'^*\mathcal{P}\) is \(f'\)-dualizable. Fix any v-hypercover \(g_\bullet: S'_\bullet \rightarrow S\) extending \(g\) and let \(g'_\bullet: X'_\bullet \rightarrow X\) be the base-change. Let \(\mathcal{P}'_\bullet \in \mathcal{D}_{\text{nuc}}(X'_\bullet, \Lambda)\) denote the coCartesian section given by \(\mathcal{P}\) (i.e. \(\mathcal{P}'_n = g'^*_n\mathcal{P}\)) and let \(f'_n: X'_n \rightarrow S'_n\) denote the base-change of \(f\). Then for all \(n\), \(\mathcal{P}'_n\) is \(f'_n\)-dualizable by (i), hence from the second isomorphism of functors in Proposition 7.7.(iii) the functor \(\mathcal{D}_{\text{nuc}}(S'_\bullet, \Lambda) \rightarrow \mathcal{D}_{\text{nuc}}(X'_\bullet, \Lambda), \mathcal{M}_\bullet \mapsto \text{Hom}(\mathcal{P}'_\bullet, f'_n^*\mathcal{M}_\bullet)\) preserves coCartesian sections and hence restricts to a functor \(\mathcal{D}_{\text{nuc}}(S, \Lambda) \rightarrow \mathcal{D}_{\text{nuc}}(X, \Lambda)\) which necessarily coincides with the functor \(\mathcal{M} \mapsto \text{Hom}(\mathcal{P}, f^!\mathcal{M})\) (look at the left adjoints). It follows that the natural morphism of functors

\[
g'^*\text{Hom}(\mathcal{P}, f^!) \sim \text{Hom}(\mathcal{P}', f'^!g^*)
\]

is an isomorphism. We can deduce that also the natural morphism \(D_f(\mathcal{P}) \otimes f^* \sim \text{Hom}(\mathcal{P}, f^!)\) is an isomorphism, because this can be checked after applying \(g'^*\) where by the just proved isomorphism of functors it transforms to the corresponding statement for \(\mathcal{P}'\) and \(f'\), which follows from Proposition 7.7. But note that the whole argument still works after any base-change, so we deduce that \(\mathcal{P}\) satisfies Proposition 7.7 and is therefore \(f\)-dualizable.

We also get the following analog of Lemma 6.5 showing that the notion of \(f\)-dualizable sheaves is compatible with the \(\infty\)-categorical enhancement on \(\mathcal{D}_{\text{nuc}}\) (even though we ignored this enhancement in the definition of \(f\)-dualizability).

Corollary 7.9. Let \(f: X \rightarrow S\) be an \(\ell\)-fine map of small v-stacks. Then the subcategory \(\mathcal{D}_{\text{nuc}}(X, \Lambda)_{\text{dlb}} \subseteq \mathcal{D}_{\text{nuc}}(X, \Lambda)\) of \(f\)-dualizable sheaves is stable under retracts and finite (co)limits.

Proof. This follows immediately from Proposition 7.7.(iii) because all the involved functors commute with retracts and finite (co)limits in \(\mathcal{P}\).

We have the following base-change result, generalizing smooth base-change to a relatively dualizable version:

Proposition 7.10. Let
be a cartesian square of small v-stacks and assume that \( g \) is \( \ell \)-fine. Then for every \( g \)-dualizable \( \mathcal{P} \in \mathcal{D}_{\text{nuc}}(S', \Lambda) \) the natural morphism of functors

\[
\mathcal{P} \otimes g^* f_* \xrightarrow{\sim} f'_*(f^* \mathcal{P} \otimes g'^*)
\]

is an isomorphism.

**Proof.** We apply the first isomorphism in Proposition 7.7.(iii) for \( D_g(\mathcal{P}) \) and use the reflexivity \( \mathcal{P} = D_g(D_g(\mathcal{P})) \) in order to obtain the natural isomorphism

\[
\mathcal{P} \otimes g^* f_* = \text{Hom}(D_g(\mathcal{P}), g^1 f_*).
\]

By passing to right adjoints in proper base-change we obtain \( g^1 f_* = f'_* g'^* \), so that we can transform the right-hand side to

\[
\text{Hom}(D_g(\mathcal{P}), f'_* g'^*) = f'_* \text{Hom}(f^* D_g(\mathcal{P}), g'^*).
\]

Apply the isomorphisms in Proposition 7.7.(iii) again to arrive at the desired identity. \( \square \)

The next statement tells us that relatively dualizable sheaves are stable under “relatively dualizable pullback”. An important special case is that relatively dualizable sheaves are stable under smooth pullback (see Proposition 8.6.(i)).

**Proposition 7.11.** Let \( f: X \to S \) and \( g: Y \to X \) be \( \ell \)-fine maps of small v-stacks and let \( \mathcal{P} \in \mathcal{D}_{\text{nuc}}(X, \Lambda) \) be \( f \)-dualizable and \( \mathcal{Q} \in \mathcal{D}_{\text{nuc}}(Y, \Lambda) \) be \( g \)-dualizable. Then \( g^* \mathcal{P} \otimes \mathcal{Q} \) is \((f \circ g)\)-dualizable and the natural map

\[
g^* D_f(\mathcal{P}) \otimes D_g(\mathcal{Q}) \xrightarrow{\sim} D_{f \circ g}(g^* \mathcal{P} \otimes \mathcal{Q})
\]

is an isomorphism.

**Proof.** There is a functor \( \mathcal{C}_X \to \mathcal{C}_S \) which sends \([Z \to X] \in \mathcal{C}_X\) to \([Z \to S] \in \mathcal{C}_S\) and a morphism \( \mathcal{M} \in \mathcal{D}_{\text{nuc}}(Z' \times_X Z, \Lambda) = \text{Fun}_{\mathcal{C}_X}(Z, Z')\) to \( i^! \mathcal{M} \in \mathcal{D}_{\text{nuc}}(Z' \times_S Z, \Lambda) = \text{Fun}_{\mathcal{C}_S}(Z, Z')\), where \( i: Z' \times_X Z \to Z' \times_S Z \) is the obvious map: This follows easily from repeated application of the projection formula; we leave the details to the reader. With this functor at hand, the claim is now easy: By assumption \( \mathcal{Q} \) is a left adjoint in \( \mathcal{C}_X \), hence so is its image in \( \mathcal{C}_S \) (as a morphism from \( Y \) to \( X \) over \( S \)). If we denote \( i: Y \to X \times_S Y \) the canonical map then this image of \( \mathcal{Q} \) is \( i^! \mathcal{Q} \). Denoting \( \pi_X \) and \( \pi_Y \) the two projections on \( X \times_S Y \) we can compute the composition of \( i^! \mathcal{Q} \) with \( \mathcal{P} \) in \( \mathcal{C}_S \) as

\[
\mathcal{P} \star i^! \mathcal{Q} = \pi_Y !(i^! \mathcal{Q} \otimes \pi_X^! \mathcal{P}) = \pi_Y !(ii^* \pi_Y^! \mathcal{Q} \otimes \pi_X^! \mathcal{P}) = \pi_Y !(\pi_Y^* \mathcal{Q} \otimes i_! \Lambda \otimes \pi_X^! \mathcal{P}) = Q \otimes \pi_Y !(a^* \pi_X^! \mathcal{P}) = Q \otimes g'^* \mathcal{P}.
\]

This proves that \( g^* \mathcal{P} \otimes \mathcal{Q} \) is a left adjoint in \( \mathcal{C}_S \) and hence \((f \circ g)\)-dualizable. The claim about the duals follows from the symmetry of the situation and the fact that adjoints are unique up to unique isomorphism. \( \square \)
Next up we want to prove that relatively dualizable sheaves are stable under proper push-forward. In fact this holds for pushforward along maps which are only cohomologically proper, which roughly means that \( f_! = f_* \). Defining this notion thoroughly requires some effort and will be carried out in Section 9 (see in particular Proposition 9.10). For now, we will work with an even more general version of maps which are “cohomologically proper up to a twist”.

**Definition 7.12.** Let \( f \colon X \to S \) be an \( \ell \)-fine map of small v-stacks. We say that a sheaf \( \mathcal{P} \in D_\text{nuc}(X, \Lambda) \) is \( f \)-proper if it is a right adjoint when viewed as a morphism from \( X \) to \( S \) in \( C_\mathcal{S} \). The \( f \)-proper dual \( f^! \mathcal{P} \in D_\text{nuc}(X, \Lambda) \) is the corresponding left adjoint.

**Proposition 7.13.** Let \( f \colon X \to S \) and \( g \colon Y \to X \) be \( \ell \)-fine maps of small v-stacks. Suppose that \( \mathcal{P} \in D_\text{nuc}(Y, \Lambda) \) is \((f \circ g)\)-dualizable and that \( \mathcal{Q} \in D_\text{nuc}(Y, \Lambda) \) is \( g \)-proper. Then \( g_* \mathcal{Hom}(\mathcal{Q}, \mathcal{P}) \) is \( f \)-dualizable.

**Proof.** Let \( i \colon Y \to Y \times_S X \) be the natural map. Then the functor \( C_X \to C_S \) from Proposition 7.11 sends \( \mathcal{Q} \) to \( i_* \mathcal{Q} \) and \( P_g(\mathcal{Q}) \) to \( i_* P_g(\mathcal{Q}) \). By assumption \( P_g(\mathcal{Q}) \) is a left adjoint in \( C_X \) when viewed as a morphism from \( X \) to \( Y \). Consequently the same is still true in \( C_S \). Since left adjoints are stable under composition, we deduce that the following morphism from \( X \) to \( S \) is a left adjoint:

\[
\mathcal{P} \star i_* P_g(\mathcal{Q}) = \pi_X!(\pi_Y^! \mathcal{P} \otimes i_* P_g(\mathcal{Q})) = \pi_X! i_* (P_g(\mathcal{Q}) \otimes \pi_Y^! \mathcal{P}) = g_!(P_g(\mathcal{Q}) \otimes \mathcal{P}).
\]

Now consider the functor from \( C_X \) to the category of (underlying 1-categories of) stable \( \infty \)-categories mapping \( \mathcal{M} \) to \( \pi_2(\mathcal{M} \otimes \pi_Y^! \mathcal{P}) \) (as considered in the proof of Proposition 7.7). This functor preserves adjoint morphisms so that we deduce that the morphism \( \mathcal{g}_!(\mathcal{Q} \otimes -) \) is 1-categorically right adjoint to the functor \( P_g(\mathcal{Q}) \otimes g^* \). But we know that the latter functor also has the right adjoint \( \mathcal{g}_* \mathcal{Hom}(P_g(\mathcal{Q}), -) \), which produces a 1-categorical isomorphism of functors \( \mathcal{g}_* \mathcal{Hom}(P_g(\mathcal{Q}), -) \cong \mathcal{g}_!(\mathcal{Q} \otimes -) \). Reversing the roles of \( \mathcal{Q} \) and \( P_g(\mathcal{Q}) \) via the natural equivalence \( C_X \cong C_X^{\text{op}} \) we also deduce that there is a 1-categorical isomorphism of functors \( \mathcal{g}_* \mathcal{Hom}(\mathcal{Q}, -) \cong \mathcal{g}_!(P_g(\mathcal{Q}) \otimes -) \). In particular it follows that \( \mathcal{P} \star i_* P_g(\mathcal{Q}) \cong \mathcal{g}_* \mathcal{Hom}(\mathcal{Q}, \mathcal{P}) \), as desired.

Let us also discuss how the notion of relatively dualizable sheaves behaves under a change of the nuclear \( \mathbb{Z}_\ell \)-algebra \( \Lambda \).

**Proposition 7.14.** Let \( f \colon X \to S \) be an \( \ell \)-fine map of small v-stacks and \( \mathcal{P} \in D_\text{nuc}(X, \Lambda) \).

(i) If \( \mathcal{P} \) is \( f \)-dualizable then \( \mathcal{P} \otimes_{\Lambda} \Lambda' \in D_\text{nuc}(X, \Lambda') \) is also \( f \)-dualizable.

(ii) Suppose that \( \Lambda = \mathbb{Z}_\ell \) and that \( \mathcal{P} \) is locally bounded and \( \ell \)-adically complete. Then \( \mathcal{P} \) is \( f \)-dualizable if and only if \( \mathcal{P}/\mathcal{F}_\ell \mathcal{P} \in D_\text{nuc}(X, \mathbb{F}_\ell) \) is \( f \)-dualizable.

**Proof.** Part (i) follows immediately from the fact that base-change along \( \Lambda \to \Lambda' \) invokes a functor on the respective versions of \( C_\mathcal{S} \). This also implies the “only if” part of (ii), so it remains to prove the “if” part. We therefore assume that \( \Lambda = \mathbb{Z}_\ell \) and that \( \mathcal{P} \) is locally bounded and \( \ell \)-adically complete with \( \mathcal{P}/\mathcal{F}_\ell \mathcal{P} \) being \( f \)-dualizable as an \( \mathbb{F}_\ell \)-module. Note that pullback and upper shriek functors preserve \( \ell \)-adically complete sheaves and that if \( \mathcal{M} \) is \( \ell \)-adically complete then so is \( \mathcal{Hom}(\mathcal{N}, \mathcal{M}) \) for any \( \mathcal{N} \). Moreover, tensoring with a fixed \( \ell \)-adically complete and locally bounded sheaf preserves \( \ell \)-adically complete sheaves (this follows from Proposition 2.8 by observing that the statement also holds if one of the sheaves is bounded and the other unbounded because the tensor product has finite Tor dimension by [3, Proposition VII.2.3]). Altogether this implies that
both $\pi_1^*D_f(\mathcal{P}) \otimes \pi_2^*\mathcal{P}$ and $\text{Hom}(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P})$ are $\ell$-adically complete, where $\pi_i$ denote the projections from $X \times_S X$. By Proposition 7.7 we need to check that the natural morphism $\pi_1^*D_f(\mathcal{P}) \otimes \pi_2^*\mathcal{P} \rightarrow \text{Hom}(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P})$ is an isomorphism, which by $\ell$-adic completeness can be checked modulo $\ell$. But one checks immediately that this reduces the claim to the similar statement for $\mathcal{P}/\ell\mathcal{P}$ as an $\mathcal{F}_\ell$-module, where the claim holds by assumption and Proposition 7.7.

8 Cohomological Smoothness

Fix a prime $\ell \neq p$ and a nuclear $\mathbb{Z}_\ell$-algebra $\Lambda$. We now introduce $\ell$-cohomologically smooth maps of small v-stacks, which are those that satisfy a strong form of Poincaré duality. With the magic of relatively dualizable sheaves, the definition of cohomologically smooth maps is rather simple:

**Definition 8.1.** An $\ell$-fine map $f: Y \rightarrow X$ of small v-stacks is called $\ell$-cohomologically smooth if the constant sheaf $\mathbb{F}_\ell \in D_{et}(Y, \mathbb{F}_\ell)$ is $f$-dualizable and its $f$-dual $D_f(\mathbb{F}_\ell)$ is invertible.

Our definition of cohomological smoothness is a bit unorthodox, but we believe it to be the “right” one from a formal standpoint. In the specific 6-functor formalism at hand, we recover the previously defined notion of $\ell$-cohomologically smooth maps:

**Lemma 8.2.** Let $f: Y \rightarrow X$ be an $\ell$-fine map of small v-stacks. Then the following are equivalent:

(i) $f$ is $\ell$-cohomologically smooth.

(ii) The sheaf $f^!\mathbb{F}_\ell \in D_{et}(Y, \mathbb{F}_\ell)$ is invertible and its formation commutes with base-change along $f$.

If $f$ is fdcs then these conditions are also equivalent to:

(iii) For every map $X' \rightarrow X$ from a strictly totally disconnected space $X'$ with base-change $f': Y' \rightarrow X'$, the sheaf $f'^!\mathbb{F}_\ell \in D_{et}(Y', \mathbb{F}_\ell)$ is invertible and the natural transformation of functors $f'^!\mathbb{F}_\ell \otimes f'^* \rightarrow f^!\mathbb{F}_\ell$ is an isomorphism.

**Proof.** The equivalence of (i) and (ii) follows immediately from Proposition 7.7 by observing that $\pi_1^*D_f(\mathbb{F}_\ell) \otimes \pi_2^*\mathbb{F}_\ell = \pi_1^*f^!\mathbb{F}_\ell$ and $\text{Hom}(\pi_1^*\mathbb{F}_\ell, \pi_2^*\mathbb{F}_\ell) = \pi_1^*\mathbb{F}_\ell$, so that the equivalence of these two sheaves amounts precisely to the condition that the formation of $f^!\mathbb{F}_\ell$ commutes with base-change along $f$.

The implication that (i) implies (iii) follows immediately from Proposition 7.7.(iii) (here we do not need $f$ to be fdcs). For the converse, assume that $f$ is fdcs and satisfies (iii). We can assume that $f$ is separated. Then $f$ is $\ell$-cohomologically smooth in the sense of [15, Definition 23.8] and so by the results in [15, §23] we see that $f$ satisfies Proposition 7.7.(iii) for $\mathcal{P} = \mathbb{F}_\ell$.

**Remark 8.3.** In [15, Definition 23.8] a notion of $\ell$-cohomological smoothness for separated fdcs maps is introduced. It follows from Lemma 8.2 that this notion coincides with the one in Definition 8.1. It follows that our definition of $\ell$-cohomological smoothness is also equivalent to the one in [4] and to the definition of $\ell$-cohomological smooth maps of Artin v-stacks in [3] (all maps in the references for which $\ell$-cohomological smoothness is defined are $\ell$-fine in our sense; the converse is probably false). In particular all results in [15, 3] which show that certain maps of small v-stacks are $\ell$-cohomologically smooth also apply in our setting.
Remark 8.4. We were unable to show that the last criterion in Lemma 8.2 implies $\ell$-cohomological smoothness without the fdcs assumption. In particular, this criterion does not seem to capture the correct notion of smoothness in the abstract setup – there is always a non-formal argument specific to the setting required to make this definition work.

With the ambiguity of definitions out of the way, we can now deduce all expected properties of nuclear $\Lambda$-modules along smooth maps. Note that everything is completely formal.

Proposition 8.5. (i) Let $f: Y \to X$ be an $\ell$-fine and $\ell$-cohomologically smooth map of small $v$-stacks. Then $f^!\Lambda$ is invertible and the natural morphism

$$f^!\Lambda \otimes f^* \xrightarrow{\sim} f^!$$

is an isomorphism of functors $D_{\text{nuc}}(X, \Lambda) \to D_{\text{nuc}}(Y, \Lambda)$.

(ii) Let

$$
\begin{array}{ccc}
Y' & \xrightarrow{g'} & Y \\
\downarrow f' & & \downarrow f \\
X' & \xrightarrow{g} & X
\end{array}
$$

be a cartesian square of small $v$-stacks.

(a) Assume that $f$ is $\ell$-fine and that either $f$ or $g$ is $\ell$-fine and $\ell$-cohomologically smooth. Then the natural morphism

$$g'^*f^! \xrightarrow{\sim} f'^!g'^*$$

is an isomorphism of functors $D_{\text{nuc}}(X, \Lambda) \to D_{\text{nuc}}(Y', \Lambda)$.

(b) Assume that $g$ is $\ell$-fine and $\ell$-cohomologically smooth. Then the natural morphism

$$g'^!f_* \xrightarrow{\sim} f'_*g'^!$$

is an isomorphism of functors $D_{\text{nuc}}(Y, \Lambda) \to D_{\text{nuc}}(X', \Lambda)$.

(iii) Let $f: Y \to X$ be an $\ell$-fine and $\ell$-cohomologically smooth map of small $v$-stacks. Then for all $\mathcal{M}, \mathcal{N} \in D_{\text{nuc}}(X, \Lambda)$ the natural map

$$f^* \text{Hom}(\mathcal{M}, \mathcal{N}) \xrightarrow{\sim} \text{Hom}(f^*\mathcal{M}, f^*\mathcal{N})$$

is an isomorphism.

Proof. Suppose that $f: Y \to X$ is an $\ell$-fine and $\ell$-cohomologically smooth map of small $v$-stacks. Then it follows from Proposition 7.14 that the constant sheaf $\Lambda \in D_{\text{nuc}}(Y, \Lambda)$ is $f$-dualizable. Also, $f^!\Lambda$ is invertible: In the case that $\Lambda = \mathbb{Z}_\ell$ this easily reduces to the mod-$\ell$ case, because $f^!$ commutes with $\ell$-adic completions. For general $\Lambda$ we now deduce that $f^!\Lambda = \Lambda \otimes_{\mathbb{Z}_\ell} f^!\mathbb{Z}_\ell$ from Proposition 7.7.(iii) so that $f^!\Lambda$ is indeed invertible.

By the previous paragraph, all claims reduce to similar claims about relatively dualizable sheaves or follow formally from them. Namely, (i) is a special case of Proposition 7.7.(iii). The case in (ii).(a) where $f$ is $\ell$-cohomologically smooth also follows from Corollary 5.15. The case where $g$ is $\ell$-cohomologically smooth can formally be reduced to that case, see [10, Proposition 3.8.6.(iii)]. Part (ii).(b) is a special case of Proposition 7.10 (it also follows easily from (ii).(a)). Part (iii) follows formally from (i), see [10, Proposition 3.8.7].
The following result shows that the notion of relatively dualizable sheaves is \( \ell \)-cohomologically smooth local on the source.

**Proposition 8.6.** Let \( f : X \to S \) and \( g : Y \to X \) be \( \ell \)-fine maps of small \( v \)-stacks and let \( \mathcal{P} \in \mathcal{D}_{\text{loc}}(X, \Lambda) \). Assume that \( g \) is \( \ell \)-cohomologically smooth.

(i) If \( \mathcal{P} \) is \( f \)-dualizable then \( g^* \mathcal{P} \) is \( (f \circ g) \)-dualizable.

(ii) If \( g^* \mathcal{P} \) is \( (f \circ g) \)-dualizable and \( g \) is surjective then \( \mathcal{P} \) is \( f \)-dualizable.

**Proof.** Part (i) is a special case of Proposition 7.11. We now prove (ii), so assume that \( g \) is surjective and that \( g^* \mathcal{P} \) is \( (f \circ g) \)-dualizable. Let us denote by \( h : Y \times_S Y \to X \times_S X \) the obvious map and for \( i = 1, 2 \) let \( \pi_i : X \times_S X \to X \) and \( \pi'_i : Y \times_S Y \to Y \) denote the \( i \)th projection. One checks easily that \( \ell \)-cohomologically smooth maps are stable under base-change and composition (the latter is actually a special case of (i)), which implies that \( h \) is \( \ell \)-cohomologically smooth. Using also that invertible objects can be pulled out of upper shriek and internal Hom functors (which is easily checked with Yoneda) we deduce

\[
h^*(\pi_1^*D_f(\mathcal{P}) \otimes \pi_2^* \mathcal{P}) = \pi_1^h(g^*D_f(\mathcal{P})) \otimes \pi_2^h(g^* \mathcal{P}) = (\pi_1^*g^1\Lambda)^{-1} \otimes \pi_2^*(g^1D_f(\mathcal{P})) \otimes \pi_2^h(g^* \mathcal{P})
\]

\[
= \pi_1^*D_{f \circ g}(g^* \mathcal{P}) \otimes \pi_2^h(g^* \mathcal{P}) \otimes (\pi_1^*g^1\Lambda)^{-1},
\]

and using the fact that \( g^* \mathcal{P} \) is \( (f \circ g) \)-dualizable and Proposition 7.7.

\[
= \text{Hom}(\pi_1^h g^* \mathcal{P}, \pi_2^h g^* \mathcal{P}) \otimes (\pi_1^*g^1\Lambda)^{-1} = \text{Hom}(\pi_1^h g^1\mathcal{P}, \pi_2^h g^* \mathcal{P}) \otimes (\pi_1^*g^1\Lambda \otimes \pi_2^*g^1\Lambda)^{-1},
\]

and by an easy computation via factoring \( h \) as \( Y \times_S Y \to X \times_S X \to X \times_S X \) we have \( (\pi_1^*g^1\Lambda \otimes \pi_2^*g^1\Lambda)^{-1} = h^!\Lambda \) and therefore

\[
= \text{Hom}(h^*\pi_1^* \mathcal{P}, h^*\pi_2^* \mathcal{P}) \otimes (h^!\Lambda)^{-1} = h^*(\text{Hom}(\pi_1^* \mathcal{P}, \pi_2^* \mathcal{P}) \otimes (h^!\Lambda)^{-1}
\]

\[
= h^*\text{Hom}(\pi_1^* \mathcal{P}, \pi_2^* \mathcal{P}).
\]

All in all we see that the natural map \( \pi_1^*D_f(\mathcal{P}) \otimes \pi_2^* \mathcal{P} \to \text{Hom}(\pi_1^* \mathcal{P}, \pi_2^* \mathcal{P}) \) becomes an isomorphism after applying \( h^* \). Since \( h \) is surjective, this implies that \( \mathcal{P} \) is indeed \( f \)-dualizable by Proposition 7.7.

We also get the following stability properties of \( \ell \)-cohomologically smooth maps. Again, all of them are formal:

**Lemma 8.7.** (i) The condition of being \( \ell \)-cohomologically smooth is étale local on both source and target.

(ii) Among \( \ell \)-fine maps the condition of being \( \ell \)-cohomologically smooth is \( v \)-local on the target and \( \ell \)-cohomologically smooth local on the source.

(iii) \( \ell \)-fine and \( \ell \)-cohomologically smooth maps are stable under composition and base-change.

(iv) Every étale map is \( \ell \)-cohomologically smooth.

**Proof.** Part (iv) is obvious and part (i) is a special case of (ii). By Corollary 7.8 \( \ell \)-cohomological smoothness is \( v \)-local on the target and by Proposition 8.6.(ii) it is \( \ell \)-cohomologically smooth local on the source; this proves (ii). The claim about base-change in (iii) is a special case of (ii) and the claim about compositions follows immediately from Proposition 8.6.(i).
It is similarly formal that \( \ell \)-cohomologically smooth maps satisfy universal \( \ell \)-codescent, which provides us with a lot of stacky \( \ell \)-fine maps.

**Lemma 8.8.** Let \( f : Y \rightarrow X \) be an \( \ell \)-fine and \( \ell \)-cohomologically smooth cover of small \( v \)-stacks. Then the natural functor

\[
\mathcal{D}^1_{\text{nuc}}(X, \Lambda) \xrightarrow{\sim} \lim_{n \in \Delta} \mathcal{D}^1_{\text{nuc}}(Y_n, \Lambda)
\]

is an equivalence. Here \( \mathcal{D}^1_{\text{nuc}} \) denotes the functor whose transition maps are given by upper shriek functors.

**Proof.** The same argument as in [5, Lemma 2.8] applies. \( \Box \)

**Corollary 8.9.** Every fdcs and \( \ell \)-cohomologically smooth map of small \( v \)-stacks admits universal \( \ell \)-codescent.

**Proof.** This is a special case of Lemma 8.8. \( \Box \)

With the formalism of \( \ell \)-cohomologically smooth maps at hand, we can answer the question how the solid 5-functor formalism from [3, §VII] relates to the nuclear 6-functor formalism. The following result generalizes [3, Proposition VII.3.5]:

**Proposition 8.10.** Let \( f : Y \rightarrow X \) be an fdcs and \( \ell \)-cohomologically smooth map of small \( v \)-stacks. Then \( f^\#_! : \mathcal{D}_!(Y, \Lambda) \rightarrow \mathcal{D}_!(X, \Lambda) \) preserves nuclear sheaves and the natural morphism

\[
f^\#_! \sim f^!(- \otimes f^!\Lambda)
\]

is an isomorphism of functors \( \mathcal{D}_{\text{nuc}}(Y, \Lambda) \rightarrow \mathcal{D}_{\text{nuc}}(X, \Lambda) \).

**Proof.** It is enough to show that \( f^\#_! \) preserves nuclear sheaves, then the claimed isomorphism of functors follows easily from the fact that \( f^\#_! \) is left adjoint to \( f^* \) (as functors on nuclear sheaves) and \( f_! \) is left adjoint to \( f^! = f^* \otimes f^!\Lambda \). Since \( f^\#_! \) commutes with the forgetful functor along the map \( \mathbb{Z}_\ell \rightarrow \Lambda \) and satisfies arbitrary base-change (see [3, Proposition VII.3.1]) we can formally reduce to the case that \( X \) is a strictly totally disconnected perfectoid space and \( \Lambda = \mathbb{Z}_\ell \). Then \( Y \) is a locally spatial diamond, and by passing to an open cover of \( Y \) we can assume that \( Y \) is a spatial diamond (note that \( f^\#_! \) is computed as the colimit along an open cover). Note that \( Y \) is \( \ell \)-bounded because it has finite \( \dim \text{trg} \) over the strictly totally disconnected space \( X \).

We now need to show that for every nuclear \( M \in \mathcal{D}_{\text{nuc}}(Y, \mathbb{Z}_\ell) \) the natural map \( f^\#_!(\mathcal{M} \otimes f^!\mathbb{Z}_\ell) \) is an isomorphism in \( \mathcal{D}_!(X, \mathbb{Z}_\ell) \). Since both sides commute with colimits in \( M \), we can assume that \( M \) is a right-bounded Banach sheaf. Note that \( f^\#_! \) preserves right-bounded \( \ell \)-adically complete objects: It preserves compact solid sheaves (i.e. finite (co)limits and retracts of objects of the form \( \mathbb{Z}_\ell, [U] \) for w-contractible \( U \in Y_{\text{proet}} \)) and since the compact solid sheaves are \( \ell \)-adically complete one can argue similarly to the proof of Lemma 3.7.(ii). Clearly also \( f^! \) preserves \( \ell \)-adically complete sheaves (by computing it as the composition of an \( \text{étale} \) lower shriek and a pushforward), hence both \( f^\#_! \) and \( f^!(\mathcal{M} \otimes f^!\mathbb{Z}_\ell) \) are \( \ell \)-adically complete. Therefore the desired isomorphism can be checked modulo \( \ell \), so from now on we can replace \( \Lambda = \mathbb{Z}_\ell \) by \( \Lambda = \mathbb{F}_\ell \) and in particular assume that \( M \) is \( \text{étale} \).

Using that both \( f^\#_! \) and \( f^! \) preserve colimits, we can now reduce to the case that \( M = \mathbb{F}_\ell[U] \) for some \( U \in Y_{\text{ét}} \). Then \( M = j_!\mathbb{F}_\ell = j_!^\#\mathbb{F}_\ell \), where \( j : U \rightarrow Y \) is the structure map. Hence by
replacing $Y$ by $U$ we can further reduce to the case that $\mathcal{M} = \mathbb{F}_\ell$. We now end up with the claim that the natural map

$$f_2^!\mathbb{F}_\ell \xrightarrow{\sim} f_1^!\mathbb{F}_\ell$$

is an isomorphism in $\mathcal{D}_c(X, \mathbb{F}_\ell)$. Note that $f_1^!\mathbb{F}_\ell$ is compact in $\mathcal{D}_{et}(Y, \mathbb{F}_\ell)$ (because it is invertible), hence $f_1^!\mathbb{F}_\ell$ is compact in $\mathcal{D}_{et}(X, \mathbb{F}_\ell)$. In particular it is pseudocoherent in $\mathcal{D}_c(X, \mathbb{F}_\ell)$, i.e. $\text{Hom}(f_1^!\mathbb{F}_\ell, -)$ preserves uniformly left-bounded filtered colimits in $\mathcal{D}_c(X, \mathbb{F}_\ell)$ (this follows because $(-)_{et}: \mathcal{D}_c(X, \mathbb{F}_\ell) \to \mathcal{D}_{et}(X, \mathbb{F}_\ell)$ preserves uniformly left-bounded filtered colimits). Similarly $f_2^!\mathbb{F}_\ell$ is a pseudocoherent object of $\mathcal{D}_c(X, \mathbb{F}_\ell)$ because $\mathbb{F}_\ell$ is a pseudocoherent object of $\mathcal{D}_c(Y, \mathbb{F}_\ell)$. To prove the above isomorphism, we now need to show that for all $\mathcal{N} \in \mathcal{D}_c(X, \mathbb{F}_\ell)$ the natural map

$$\text{Hom}(f_2^!\mathbb{F}_\ell, \mathcal{N}) \xleftarrow{\sim} \text{Hom}(f_1^!\mathbb{F}_\ell, \mathcal{N})$$

is an isomorphism of spectra. Via Postnikov limits we can assume that $\mathcal{N}$ is left-bounded. Writing $\mathcal{N}$ as a filtered colimit of its right truncations and using pseudocoherence of $f_2^!\mathbb{F}_\ell$ and $f_1^!\mathbb{F}_\ell$ to pull out this colimit, we can further reduce to the case that $\mathcal{N}$ is bounded, which further reduces to the case that $\mathcal{N}$ is static. Now write $\mathcal{N}$ as a filtered colimit of static finitely presented sheaves in $\mathcal{D}_c(X, \mathbb{F}_\ell)$ to reduce to the case that $\mathcal{N}$ is static and finitely presented, i.e. of the form $\mathcal{N} = \varprojlim N_i$ for some qcqs étale sheaves $N_i \in \mathcal{D}_{et}(X, \mathbb{F}_\ell)$ (see [3, Theorem VII.1.3]). By pulling out this limit (which is automatically a derived limit by [3, Proposition VII.1.6]) we can further reduce to the case that $\mathcal{N}$ is qcqs étale. But then

$$\text{Hom}(f_1^!\mathbb{F}_\ell, \mathcal{N}) = \text{Hom}(f_1^!\mathbb{F}_\ell, f_1^*\mathcal{N}) = \text{Hom}(f_1^!\mathbb{F}_\ell, f_1^*\mathcal{N} \otimes f_1^!\mathbb{F}_\ell) = \text{Hom}(\mathbb{F}_\ell, f_1^*\mathcal{N}) = \text{Hom}(f_2^!\mathbb{F}_\ell, \mathcal{N}),$$

as desired. □

**Remark 8.11.** One can generalize Proposition 8.10 to many $\ell$-fine and $\ell$-cohomologically smooth maps $f: Y \to X$ which are not necessarily fdc. Here are two examples:

(a) If there is an fdc and $\ell$-cohomologically smooth cover $g: Z \to Y$ such that the composition $Z \to X$ is fdc and $\ell$-cohomologically smooth, then the conclusion of Proposition 8.10 holds for $f$. Namely, in this case $f_2$ is computed as the colimit of the functors $(f \circ g_n)_2g_n$, where $g_n: Z_n \to Y$ is the Čech nerve of $g$.

(b) If $G$ is a virtually $\ell$-Poincaré group (see Definition 10.10.(b)) then the conclusion of Proposition 8.10 holds for $f: */G \to *$. Namely, by the proof of Proposition 10.6 $*/G$ behaves very similarly to an $\ell$-bounded spatial diamond, so the proof of Proposition 8.10 applies.

From these two examples we see that Proposition 8.10 generalizes to all $\ell$-fine and $\ell$-cohomologically smooth maps that appear in practice. However, we do not know if it holds for all $\ell$-fine and $\ell$-cohomologically smooth maps because we cannot control maps admitting universal $\ell$-codescent well enough.

### 9 Cohomological Properness

Fix a prime $\ell \neq p$ and a nuclear $\mathbb{Z}_\ell$-algebra $\Lambda$. We now define a notion of $\ell$-cohomologically proper maps of small v-stacks which roughly requires that lower shriek and pushforward along
this map agree. Similar to the case of cohomologically smooth maps we will make use of the magical 2-category $\mathcal{C}_S$ constructed in Definition 7.1 in order to reduce cohomological properties to a condition on the unit object.

For simplicity we will only study cohomological properness in the context where the diagonal is proper. One can extend the notion of cohomological properness to more general maps by mimicking the definition of smoothness: A map $f: X \to S$ can be called $\ell$-cohomologically proper if $Z_\ell$ is $f$-proper and $P_f(Z_\ell)$ is invertible (cf. Definition 7.12 for this terminology). We will not pursue this idea further as we see no useful application for it.

**Definition 9.1.** A map $f: Y \to X$ of small v-stacks is called 1-separated if the diagonal $\Delta_f: Y \to Y \times_X Y$ is proper.

**Lemma 9.2.** (i) 1-separated morphisms of small v-stacks are stable under composition and base-change.

(ii) The notion of 1-separatedness is v-local on the target.

(iii) Let $f: Y \to X$ and $g: Z \to Y$ be maps of small v-stacks. If $f$ and $f \circ g$ are 1-separated then so is $g$.

**Proof.** These results are all formal and follow in the same way as for algebraic stacks. More concretely, for (i) see [17, Lemma 050K, 050F], for (ii) see [17, Lemma 06TZ] and [15, Proposition 10.11.(ii)] and for (iii) see [17, Lemma 050M].

The notion of 1-separatedness is useful because it automatically gives us a comparison map between lower shriek and pushforward:

**Lemma 9.3.** Let $f: Y \to X$ be an $\ell$-fine 1-separated map of small v-stacks with diagonal $\Delta$. Then the equivalence $\Delta_* = \Delta^*$ induces a morphism

$$f_! \to f_*$$

of functors $\mathcal{D}_{\text{nuc}}(Y, \Lambda) \to \mathcal{D}_{\text{nuc}}(X, \Lambda)$.

**Proof.** The desired morphism of functors is adjoint to a morphism $f^* f_! \to \text{id}$ which can be constructed as follows: For $i = 1, 2$ let $\pi_i: Y \times_X Y \to Y$ denote the $i$th projection. Using proper base-change and the natural map $\text{id} \to \Delta_* \Delta^* = \Delta^! \Delta^*$ we get the map

$$f^* f_! = \pi_2^! \pi_1^* \to \pi_2^! \Delta^! \Delta_* \pi_1^* = \text{id},$$

as desired.

One can now define a 1-separated $\ell$-fine map $f: Y \to X$ to be $\ell$-cohomologically proper if the map of functors $f_! \to f_*$ is an isomorphism. However, this makes it hard to check cohomological properness in practice and it is also unclear why this notion is stable under base-change, so we prefer to work a little harder and use the magic of the 2-category from Definition 7.1 to come up with a simpler definition. The crucial observation is the following (recall the notion of $f$-proper sheaves from Definition 7.12):

**Lemma 9.4.** Let $f: Y \to X$ be a 1-separated $\ell$-fine map of small v-stacks and let $\mathcal{P}, \mathcal{Q} \in \mathcal{D}_{\text{nuc}}(Y, \Lambda)$ be given. Suppose that the natural map $f_!(\mathcal{P} \otimes \mathcal{Q}) \to f_*(\mathcal{P} \otimes \mathcal{Q})$ is an isomorphism. Then $\mathcal{P}$ is $f$-proper with $f$-proper dual $\mathcal{Q}$ if and only if $\mathcal{P}$ is dualizable with dual $\mathcal{Q}$.
Proof. The property of $\mathcal{P}$ being $f$-proper with dual $\mathcal{Q}$ is captured by the existence of a unit and a counit map satisfying certain commuting diagrams. A similar description holds for the property of $\mathcal{P}$ being dualizable with dual $\mathcal{Q}$, so all we need to do is to show that these data are equivalent. Let us investigate $f$-properness: The unit of an adjunction between $\mathcal{P}$ and $\mathcal{Q}$ is a map

$$\varepsilon: \Lambda \to \mathcal{P} \star \mathcal{Q} = f_!(\mathcal{P} \otimes \mathcal{Q}) = f_*(\mathcal{P} \otimes \mathcal{Q}),$$

which by adjunction of $f_*$ and $f^*$ is the same as a map $i: \Lambda \to \mathcal{P} \otimes \mathcal{Q}$, i.e. the same as a coevaluation map. Similarly, the counit of an adjunction between $\mathcal{P}$ and $\mathcal{Q}$ is a map

$$\eta: \mathcal{Q} \star \mathcal{P} = \pi_1^*\mathcal{Q} \otimes \pi_2^*\mathcal{P} \to \Delta_!\Lambda = \Delta_*\Lambda,$$

which by adjunction between $\Delta_*$ and $\Delta^*$ is the same as a map $\text{ev}: \mathcal{Q} \otimes \mathcal{P} = \Delta^*(\pi_1^*\mathcal{Q} \otimes \pi_2^*\mathcal{P}) \to \Lambda$, i.e. the same as an evaluation map. It remains to see that the required compatibilities are the same for $f$-properness and dualizability. For $f$-properness, the first requirement is that the following solid map is the identity:

$$\begin{array}{ccc}
\mathcal{P} & \xrightarrow{f^*\Lambda \otimes \mathcal{P}} & f^*\mathcal{P} \otimes \mathcal{Q} \\
& \downarrow{\delta} & \uparrow{\beta} \\
& f^*f_*(\mathcal{P} \otimes \mathcal{Q}) \otimes \mathcal{P} & \xrightarrow{\alpha} & \pi_2!(\pi_1^*\mathcal{P} \otimes \pi_1^*\mathcal{Q} \otimes \pi_2^*\mathcal{P}) \\
& & \downarrow{\pi_2!(\id \otimes \eta)} & \pi_2!(\pi_1^*\mathcal{P} \otimes \Delta_!\Lambda) \\
& f^*f_*(\mathcal{P} \otimes \mathcal{Q}) \otimes \mathcal{P} & \xrightarrow{\gamma} & \mathcal{P} \otimes \mathcal{Q} \otimes \mathcal{P} & \xrightarrow{\id \otimes \text{ev}} & \mathcal{P}
\end{array}$$

Note that there is a dashed map $\alpha$ induced by the map $f^*f_! \to \id$ and one checks easily that the right-hand square commutes. Moreover, there is a dashed isomorphism $\beta$ induced by the map $f_1 \to f_*$ and a dashed map $\gamma$ induced by the counit $f^*f_* \to \id$ such that the triangle consisting of $\alpha$, $\beta$ and $\gamma$ also commutes. The dashed map $\delta$ is chosen such that the diagram commutes. One checks easily that $\gamma \circ \delta = i \otimes \id$. This shows that the first $f$-properness requirement for $\mathcal{P}$ and $\mathcal{Q}$ is indeed the same as the first dualizability requirement for these sheaves. One argues similarly with the second requirement. \(\square\)

We now get a simple yet effective definition of $\ell$-cohomologically proper maps with all the expected properties.

**Definition 9.5.** We say that an $\ell$-fine 1-separated map $f: Y \to X$ is $\ell$-cohomologically proper if the induced map $f_!\mathcal{O}_\ell \simto f_*\mathcal{O}_\ell$ is an isomorphism.

**Lemma 9.6.** Let $f: Y \to X$ be an $\ell$-fine 1-separated map. Then $f$ is $\ell$-cohomologically proper if and only if $\mathcal{O}_\ell \in \mathcal{D}_{nuc}(Y, \mathcal{O}_\ell)$ is $f$-proper with invertible $f$-proper dual. If this is the case then also $\Lambda \in \mathcal{D}_{nuc}(Y, \Lambda)$ is $f$-proper and its $f$-proper dual is $\Lambda$.

**Proof.** If $f$ is $\ell$-cohomologically proper then by Lemma 9.4, $\mathcal{O}_\ell$ is indeed $f$-proper and is its $f$-proper dual. It is clear that the same then holds for $\Lambda$ by considering the base-change functor along $\mathcal{O}_\ell \to \Lambda$ between the associated versions of $\mathcal{C}_X$.

Conversely assume that $\mathcal{O}_\ell$ is $f$-proper and $P_f(\mathcal{O}_\ell)$ is invertible. Consider the functor from $\mathcal{C}_X$ to the category of (underlying 1-categories of) stable ∞-categories sending $\mathcal{M}$ to $\pi_2!(\mathcal{M} \otimes \pi_1^!)$ (as in the proof of Proposition [1.1]). This functor preserves adjoint functors, which implies that the functor $f_! = f_!(\mathcal{O}_\ell \otimes -)$ is 1-categorically right adjoint to the functor $P_f(\mathcal{O}_\ell) \otimes f^*$. We deduce that
there is a 1-categorical isomorphism of functors \( f_!(\mathbb{Z}_\ell \otimes -) \cong f_* \mathbb{Hom}(P_f(\mathbb{Z}_\ell), -) \). By the magic of \( \mathcal{C}_X \) the same still holds after pullback along \( f \), i.e. if \( \pi_i : Y \times_X Y \to Y \) denote the two projections then there is a 1-categorical isomorphism of functors \( \pi_2_* \mathbb{Hom}(\pi_1^* P_f(\mathbb{Z}_\ell), -) \cong \pi_2!(\pi_1^* \mathbb{Z}_\ell \otimes -) \). Plugging in \( \Delta_f \mathbb{Z}_\ell \) yields

\[
\mathbb{Z}_\ell = \pi_2!(\pi_1^* \mathbb{Z}_\ell \otimes \Delta_f \mathbb{Z}_\ell) \cong \pi_2_* \mathbb{Hom}(\pi_1^* P_f(\mathbb{Z}_\ell), \Delta_f \mathbb{Z}_\ell) = \pi_2_* \mathbb{Hom}(\pi_1^* P_f(\mathbb{Z}_\ell), \mathbb{Z}_\ell) = \mathbb{Hom}(P_f(\mathbb{Z}_\ell), \mathbb{Z}_\ell).
\]

Since \( P_f(\mathbb{Z}_\ell) \) is invertible, this implies \( P_f(\mathbb{Z}_\ell) \cong \mathbb{Z}_\ell \). In particular we obtain a 1-categorical isomorphism of functors \( f_! \cong f_* \mathbb{Hom}(P_f(\mathbb{Z}_\ell), -) \cong f_* \). Both isomorphisms are induced from the adjunction of \( \mathbb{Z}_\ell \) with \( P_f(\mathbb{Z}_\ell) \) and the second isomorphism additionally used the fact \( \Delta_f = \Delta_* \) to get the isomorphism \( P_f(\mathbb{Z}_\ell) \cong \mathbb{Z}_\ell \). One checks that these isomorphisms “cancel”, so that the obtained isomorphism \( f_! \cong f_* \) is indeed given by the natural map \( f_! \to f_* \). In particular it is an isomorphism of \( \infty \)-functors and plugging in \( \mathbb{Z}_\ell \) we obtain that \( f \) is \( \ell \)-cohomologically proper. \( \square \)

In the following, when we require a map to be \( \ell \)-cohomologically proper then we always assume that it is additionally \( \ell \)-fine and 1-separated.

**Proposition 9.7.** Let \( f : Y \to X \) be an \( \ell \)-cohomologically proper map of small v-stacks. Then the natural map \( f_! \cong f_* \) is an isomorphism of functors \( \mathcal{D}_{\text{nuc}}(Y, \Lambda) \to \mathcal{D}_{\text{nuc}}(X, \Lambda) \).

**Proof.** This was part of the proof of Lemma 9.6 \( \square \)

**Lemma 9.8.**

(i) Every \( \ell \)-fine proper map of small v-stacks is \( \ell \)-cohomologically proper.

(ii) \( \ell \)-cohomologically proper maps are stable under composition and base-change.

(iii) Among \( \ell \)-fine maps, the condition of being \( \ell \)-cohomologically proper is v-local on the target.

(iv) Let \( f : Y \to X \) and \( g : Z \to Y \) be maps of small v-stacks. If \( f \) and \( f \circ g \) are \( \ell \)-cohomologically proper then so is \( g \).

**Proof.** Note that all of the stabilities are satisfied by 1-separated and by \( \ell \)-fine maps by Lemmas 5.10 and 9.2, which we will use without mention. Part (i) is clear. In part (ii), stability under base-change follows from Lemma 9.6 because \( f \)-proper sheaves are certainly stable under base-change (same as for \( f \)-dualizable sheaves). Stability under composition follows immediately from the definition. In part (iii), if a map is \( \ell \)-cohomologically proper on a v-cover, then the pushforward along the base-changes of this map preserves coCartesian edges in the associated Čech cover and therefore the pushforward along the map commutes with base-change; then it follows immediately that the map is \( \ell \)-cohomologically proper. To prove (iv) assume that \( f \) and \( g \) are given as in the claim and that both \( f \) and \( f \circ g \) are \( \ell \)-cohomologically proper. Then we can factor \( g \) as \( Z \to Z \times_X Y \to Y \). The second map is \( \ell \)-cohomologically proper because it is a base-change of \( f \circ g \), so it remains to see that the first map is \( \ell \)-cohomologically proper. We can thus assume that \( f \circ g \) is id, i.e. \( g : X \to Y \) is a section of the \( \ell \)-cohomologically proper map \( f : Y \to X \). We have a cartesian square of small v-stacks (cf. the proof of [17, Lemma 050H])

\[
\begin{array}{ccc}
X = X \times_Y Y & \longrightarrow & Y = X \times_X Y \\
\downarrow & & \downarrow \\
Y & \overset{\Delta_f}{\longrightarrow} & Y \times_X Y
\end{array}
\]
The top map is evidently \( g \) and the bottom map is proper by 1-separatedness of \( f \). Thus \( g \) is proper and in particular \( \ell \)-cohomologically proper.

**Remark 9.9.** We see that the magical 2-category \( C_X \) allows us to do two things: Firstly we can check cohomological properness on the unit object and secondly we immediately see that cohomological properness is stable under base-change, which is not at all obvious.

With a good notion of \( \ell \)-cohomologically proper maps at hand, we can now prove that relatively dualizable sheaves are stable under proper pushforward:

**Proposition 9.10.** Let \( f : Y \to X \) and \( g : Z \to Y \) be \( \ell \)-fine maps and \( P \in \mathcal{D}_{\text{nuc}}(Z, \Lambda) \) such that \( g \) is \( \ell \)-cohomologically proper and \( P \) is \((f \circ g)\)-dualizable. Then \( g_* P \) is \( f \)-dualizable.

**Proof.** Combine Proposition 7.13 with Lemma 9.6.

---

### 10 Classifying Stacks and Representations

Fix a prime \( \ell \neq p \) and a nuclear \( \mathbb{Z}_\ell \)-algebra \( \Lambda \). Throughout this section we work with small \( v \)-stacks over \( \text{Spec} \mathbb{F}_p \). Our goal is to apply the above theory of nuclear sheaves to the classifying stack of a locally profinite group \( G \). This will give us an \( \infty \)-category of nuclear \( G \)-representations together with a robust notion of admissible representations and a full 6-functor formalism.

Before we can start studying classifying stacks, we need to get one technicality out of the way: The final object \(*\) is not representable in perfectoid spaces. We can use the same arguments as in the case of discrete coefficients to compute sheaves on \(*\):

**Proposition 10.1.** Let \( X \to S \leftarrow S' \) be a diagram of small \( v \)-stacks and assume that \( S \) and \( S' \) satisfy one of the following conditions:

(i) Both \( S \) and \( S' \) are spectra of some algebraically closed discrete field of characteristic \( p \).

(ii) \( S \) is the spectrum of some algebraically closed discrete field of characteristic \( p \) and \( S' = \text{Spa}(C', C'^+) \) for some algebraically closed non-archimedean field \( C' \) and an open and bounded valuation subring \( C'^+ \).

(iii) \( S = \text{Spa}(C, C^+) \) and \( S' = \text{Spa}(C', C'^+) \) for algebraically closed non-archimedean fields and open and bounded valuation subrings \( C^+ \) and \( C'^+ \).

Let \( X' := X \times_S S' \). Then the pullback functor

\[
\mathcal{D}_{\mathbb{C}}(X, \Lambda) \hookrightarrow \mathcal{D}_{\mathbb{C}}(X', \Lambda)
\]

is fully faithful and hence also induces fully faithful pullback functors on nuclear and on \( \omega_1 \)-solid sheaves.

**Proof.** This is a generalization of [15, Theorem 19.5] and for a large part we can argue very similarly. We start with some general observations which are valid in all cases. By taking a hypercover of \( X \) in terms of disjoint unions of strictly totally disconnected spaces and using the fact that fully faithfulness is preserves under limits of \( \infty \)-categories, we can reduce to the case that \( X = \text{Spa}(A, A^+) \) is strictly totally disconnected. Now let \( f : X' \to X \) denote the natural map; we need to show that for all \( \mathcal{M} \in \mathcal{D}_{\mathbb{C}}(X, \Lambda) \) the natural morphism \( \mathcal{M} \to f_{v*} f^* \mathcal{M} \) is an
isomorphism. We can assume $\Lambda = \mathbb{Z}_\ell$ and by the usual Postnikov argument we can also assume that $\mathcal{M}$ is static.

We now come to the specific proofs of each claim. First we note that (i) follows from (ii) and (iii). Moreover, part (iii) is rather easy with the above preparations: Now $f$ is qcqs and hence $f_{v*}$ preserves uniformly left-bounded filtered colimits. By writing $\mathcal{M}$ as a filtered colimit of finitely presented static solid sheaves, we can assume that $\mathcal{M}$ is finitely presented and hence a cofiltered limit of qcqs étale sheaves. Since both $f^*$ and $f_{v*}$ preserve limits of solid sheaves, we end up with the case of a qcqs étale sheaf in $D_{\square}(X, \mathbb{Z}_\ell)$. This sheaf is killed by some power of $\ell$, so the claim follows from [15, Theorem 19.5.(iii)] (more concretely from the final part of [15, Theorem 16.1]).

It remains to prove (ii). By (iii) it is enough to prove it in the case that $C$ is the completed algebraic closure of $k(( t ))$, where $S = \text{Spec } k$. Fix a pseudouniformizer $\pi \in A$, so that $X'$ can be written as the increasing union of the affinoid perfectoid subspaces

$$X'_n = \{|t| \leq |\pi| \leq |t^{1/n}|\} \subseteq X'.$$

Now $f_{v*}f^*$ is the limit over the functors $f_{v*n_*}f^*_n$, where $f_n: X'_n \to X$ is the natural map. It is therefore enough to show that for all $n$ the morphism $\mathcal{M} \sim \rightarrow f_{v*n_*}f^*_n\mathcal{M}$ is an isomorphism. But $f_n$ is qcqs, so by the same reasoning as in the proof of (iii) we can reduce to the case that $\mathcal{M}$ is étale and thus reduce to [15, Theorem 19.5.(ii)].

**Corollary 10.2.** Let $S$ be a locally profinite set and let $S$ denote the associated small $v$-stack (so that if $S$ is a point then $S = \ast$). Then there is a natural equivalence of $\infty$-categories

$$D_{\square}(S, \Lambda) = D_{\square}(S, \Lambda),$$

where on the right-hand side we mean solid sheaves on the pro-étale site of $S$. The same is true for nuclear and $\omega_1$-solid $\Lambda$-sheaves.

**Proof.** Fix any algebraically closed non-archimedean field $C$. Then $D_{\square}(S, \Lambda) = D_{\square}(S \times \text{Spa } C, \Lambda)$, so by Proposition [10.1] the pullback functor induces an embedding $D_{\square}(S, \Lambda) \hookrightarrow D_{\square}(S, \Lambda)$. But this embedding has a section obtained by pullback along the map of sites $S_\text{v} \to S_\text{proet}$. 

With the technicalities involving the final object $\ast$ out of the way, we can now come to the representation theory. Fix a locally profinite group $G$. In [10, §3.4] we constructed the $\infty$-category $D_{\square}(\mathbb{Z}_\ell)^{BG}$ of continuous $G$-representations on solid $\mathbb{Z}_\ell$-modules. It can be identified with the derived $\infty$-category of solid $\mathbb{Z}_\ell, \square [G]$-modules. By passing to $\Lambda$-modules we similarly obtain the $\infty$-category $D_{\square}(\Lambda, \mathbb{Z}_\ell)^{BG}$ of continuous $G$-representations on $(\Lambda, \mathbb{Z}_\ell)_\square$-modules (we write $(\Lambda, \mathbb{Z}_\ell)_\square$ here to denote the solid structure induced from $\mathbb{Z}_\ell$; this is the analog of $D_{\square}(X, \Lambda)$ for small $v$-stacks $X$ by somewhat sloppy notation in the latter case). Also recall the definition of $\ell$-cohomological dimension of locally profinite groups in [10, Definition 3.4.20] (see also [10, Proposition 3.4.22]). We get the following interpretation of sheaves on classifying stacks:

**Lemma 10.3.** Let $G$ be a locally profinite group. Then there is a natural equivalence of $\infty$-categories

$$D_{\square}(\ast / G, \Lambda) = D_{\square}(\Lambda, \mathbb{Z}_\ell)^{BG}.$$

Moreover, we have the following:
(i) Under the above equivalence, the pushforward functor along the natural projection \(*/G \to \ast\) corresponds to the functor \(\Gamma(G, -) : \mathcal{D}_c(\Lambda, \mathcal{Z}_\ell)^{BG} \to \mathcal{D}_c(\Lambda, \mathcal{Z}_\ell)\) computing (continuous) group cohomology.

(ii) Suppose that \(G\) is profinite and \(\text{cd}_G \ell < \infty\). Then the functor \(\Gamma(G, -)\) has cohomological dimension \(\leq \text{cd}_G \ell + 1\) and thus preserves all small colimits.

Proof. For all claims we can assume \(\Lambda = \mathcal{Z}_\ell\). To prove the claimed equivalence of \(\infty\)-categories we note that both \(\infty\)-categories admit natural left-complete \(t\)-structures and all the pullback functors in the Čech nerve of the cover \(* \to */G\) are \(t\)-exact, hence by [10] Proposition A.1.2.(ii) the claim can be verified on the hearts. The heart of the left-hand category admits a simple description in terms of descent data (cf. [10] Proposition A.1.2.(i)), which one easily verifies to be the same as a continuous \(G\)-representation (here we implicitly use Corollary [10.2].

It remains to prove claims (i) and (ii). Claim (i) follows immediately by comparing the associated left adjoints. For claim (ii) we note that if \(G\) is profinite then \(*/G\) is qcqs, hence \(\Gamma(G, -)\) commutes with filtered colimits of static objects. Thus the cohomological dimension of \(\Gamma(G, -)\) can be checked on \(\ell\)-adically complete objects (recall that \(\mathcal{D}_c(\mathcal{Z}_\ell)^{BG}\) is compactly generated by the \(\ell\)-adically complete objects \(\mathcal{Z}_\ell, [G]\)), where we can pull the limit \(\lim_{\leftarrow n} \mathcal{M}/\ell^n\mathcal{M}\) out of the cohomology, so that the claim follows from the fact that countable limits have cohomological dimension 1 and that mod \(\ell^n\) the claim follows inductively from the definition and the standard cofiber sequences \(M/\ell^{n-1}M \to M/\ell^nM \to M/\ell M\).

With Lemma [10.3] at hand we now study the full subcategory spanned by the nuclear sheaves. The corresponding representations are the continuous representations on nuclear modules:

**Definition 10.4.** Let \(G\) be a locally profinite group. We denote by \(\mathcal{D}_\text{nuc}(\Lambda)^{BG} \subseteq \mathcal{D}_c(\Lambda, \mathcal{Z}_\ell)^{BG}\) the full subcategory spanned by those \(G\)-representations whose underlying \(\Lambda\)-module is nuclear. The objects of \(\mathcal{D}_\text{nuc}(\Lambda)^{BG}\) are called the nuclear \(G\)-representations over \(\Lambda\).

**Lemma 10.5.** Let \(G\) be a locally profinite group. Then there is a natural equivalence of \(\infty\)-categories

\[
\mathcal{D}_\text{nuc}(*/G, \Lambda) = \mathcal{D}_\text{nuc}(\Lambda)^{BG}.
\]

Moreover, the \(t\)-structure on \(\mathcal{D}_c(\Lambda, \mathcal{Z}_\ell)^{BG}\) restricts to a complete \(t\)-structure on \(\mathcal{D}_\text{nuc}(\Lambda)^{BG}\). The heart \(\mathcal{A} := (\mathcal{D}_\text{nuc}(\Lambda)^{BG})^\heartsuit\) is a Grothendieck abelian category which is stable under kernels, cokernels and extensions in the heart of \(\mathcal{D}_c(\Lambda, \mathcal{Z}_\ell)^{BG}\). If \(\Lambda\) is static then there is a natural equivalence \(\mathcal{D}_\text{nuc}^+(\Lambda)^{BG} = \mathcal{D}_+^+(\mathcal{A})\).

Proof. The first claim follows immediately from Lemma [10.3] because on both sides of the claimed equivalence the nuclear sheaves are characterized by those solid sheaves which become nuclear after pullback to \(*\) (respectively, after forgetting the \(G\)-action).

That the \(t\)-structure on \(\mathcal{D}_c(\Lambda, \mathcal{Z}_\ell)^{BG}\) restricts to a \(t\)-structure on nuclear representations can be checked on underlying \(\Lambda\)-modules, i.e. we need to see that the \(t\)-structure on \(\mathcal{D}_c(\Lambda, \mathcal{Z}_\ell)\) restricts to a \(t\)-structure on \(\mathcal{D}_\text{nuc}(\Lambda)\). We can further assume \(\Lambda = \mathcal{Z}_\ell\). Then the claim boils down to the observation that truncations of Banach \(\mathcal{Z}_\ell\)-modules are again Banach \(\mathcal{Z}_\ell\)-modules, which follows from the fact that \(\ell\)-adically complete objects are stable under truncations and that discreteness mod \(\ell\) is stable under truncations because discretization is \(t\)-exact (the latter property fails on spatial diamonds, already on profinite sets). The completeness of the \(t\)-structure follows immediately from the completeness of the \(t\)-structure of the ambient \(\infty\)-category \(\mathcal{D}_c(\Lambda, \mathcal{Z}_\ell)^{BG}\).
Now let $\mathcal{A}$ denote the heart of $\mathcal{D}_{nuc}(\Lambda)^{BG}$. It is clear that $\mathcal{A}$ is stable under kernels, cokernels and extensions because all of these can be constructed using finite (co)limits and truncations in $\mathcal{D}_{\omega}(\Lambda, \mathbb{Z}_\ell)^{BG}$. It is also clearly stable under filtered colimits, hence filtered colimits in $\mathcal{A}$ are exact. It is formal that $\mathcal{A}$ is a Grothendieck abelian category: By descent, $\mathcal{D}_{nuc}(\Lambda)^{BG}$ is presentable and clearly its $t$-structure is accessible in the sense of [8, Proposition 1.4.13]; thus the claim follows from [8, Remark 1.3.5.23].

Now assume that $\Lambda$ is static. Then for all profinite sets $S$ the solid $G$-representation $(\Lambda, \mathbb{Z}_\ell)_S[\mathbb{G} \times S] = \Lambda \otimes_{\mathbb{Z}_\ell, S} \mathbb{Z}_\ell, [\mathbb{G} \times S]$ is static because $\mathbb{Z}_\ell, [\mathbb{G} \times S]$ is flat over $\mathbb{Z}_\ell, [\mathbb{G}]$ (by Proposition 2.8 this flatness reduces to the flatness of $\mathbb{F}_\ell, [\mathbb{G} \times S]$ over $\mathbb{F}_\ell, [\mathbb{G}]$ which can be deduced easily from the fact that compact $\mathbb{F}_\ell$-modules are stable under kernels and cokernels, cf. the proof of [10, Lemma 2.9.35]). But $\mathcal{D}_{\omega}(\Lambda, \mathbb{Z}_\ell)^{BG}$ is the $\infty$-category of modules over the associative analytic ring $(\Lambda, \mathbb{Z}_\ell)^{\omega}$ and since all the compact projective generators of this ring are static it follows that $\mathcal{D}_{\omega}(\Lambda, \mathbb{Z}_\ell)^{BG}$ is the derived $\infty$-category of its heart. Now consider the natural functor

$$F: \mathcal{D}^{+}(\Lambda) \rightarrow \mathcal{D}^{+}_{\omega}(\Lambda, \mathbb{Z}_\ell)^{BG}.$$ 

It admits a right adjoint $RG$ which is the right derived functor of the nuclearization functor $G$ on the hearts. We claim that the unit id $\sim \rightarrow RG \circ F$ is an isomorphism. This can be checked on static representations, i.e. for $M \in \mathcal{A}$ we need to see that $M \rightarrow RG(\mathcal{F}(M))$ is an isomorphism. Let $M \rightarrow I^\bullet$ be an injective resolution of $M = F(M)$ in the heart of $\mathcal{D}_{\omega}(\Lambda, \mathbb{Z}_\ell)^{BG}$. Then $RG(\mathcal{F}(M))$ is represented by the complex $G(I^\bullet)$. On the other hand, this complex clearly also computes $M_{\text{nuc}}$ in $\mathcal{D}_{\omega}(\Lambda, \mathbb{Z}_\ell)^{BG}$. Since $M$ is nuclear this implies that this complex is indeed isomorphic to $M$ in either derived $\infty$-category. This proves that $F$ is fully faithful. The essential image is stable under finite and filtered colimits and contains $\mathcal{A}$ and is therefore precisely $\mathcal{D}^{+}_{\text{nuc}}(\Lambda)^{BG}$. 

**Proposition 10.6.** Let $G$ be a locally profinite group with locally finite $\ell$-cohomological dimension.

(i) If $\Lambda$ is $\ell$-adically complete and $G$ is profinite then $\mathcal{D}_{\text{nuc}}(\Lambda)^{BG}$ is generated under filtered colimits by $\ell$-adically complete nuclear $G$-representations.

(ii) If $G$ is profinite and $\text{cd}_\ell G < \infty$ then the $\nu$-pushforward along $\ast/G \rightarrow \ast$ preserves nuclear sheaves and thus restricts to a colimit-preserving functor $\mathcal{D}_{\text{nuc}}(\ast/G, \Lambda) \rightarrow \mathcal{D}_{\text{nuc}}(\Lambda)$.

(iii) If $\Lambda$ is static then $\mathcal{D}_{\text{nuc}}(\Lambda)^{BG}$ is the derived $\infty$-category of its heart.

**Proof.** We first prove (i), so assume that $G$ is profinite. Let us furthermore assume that $\text{cd}_\ell G < \infty$. Then by Lemma [10.3.(ii)] the $\etale$ site of the stack $\ast/G$ behaves in a very similar way as it does for an $\ell$-bounded spatial diamond. In particular by the same arguments as in Section 3 we see that $\mathcal{D}_{\text{nuc}}(\Lambda)^{BG}$ is $\omega_1$-compactly generated by the basic nuclear sheaves, which themselves are sequential colimits of $\ell$-adically complete sheaves. We also get a good description of the $\omega_1$-solid sheaves on $\ast/G$ in a similar fashion as in Section 2 and the nuclearization functor $(-)_{\text{nuc}}: \mathcal{D}_{\omega}(\Lambda, \mathbb{Z}_\ell)_{\omega_1} \rightarrow \mathcal{D}_{\text{nuc}}(\Lambda)^{BG}$ preserves all small colimits and is bounded. In particular we deduce that every nuclear $G$-representation is a filtered colimit of $\ell$-adically complete nuclear $G$-representations.

To finish the proof of (i) we still need to treat the case that $\text{cd}_\ell G = \infty$. But by assumption on $G$ there is an open compact subgroup $H \subseteq G$ such that $\text{cd}_\ell H < \infty$. The conservative pullback along $\ast/H \rightarrow \ast/G$ has a left adjoint (the lower shriek functor, see Lemma [5.2]) on nuclear sheaves. Therefore, since $\mathcal{D}_{\text{nuc}}(\Lambda)^{BH}$ is $\omega_1$-compactly generated, the same follows for
Lemma 10.5 we have finite cohomological dimension in $D$ of group cohomology, cf. [10, Proposition 3.4.6]. Of a discrete representation is discrete. This can for example be checked by a direct computation of group cohomology, cf. [10, Proposition 3.4.6].

We now prove (ii) so assume that $G$ is as in the claim. We can assume $\Lambda = \mathbb{Z}_\ell$. By Lemma [10.3.(ii)] the $v$-pushforward along $*/G \to */G$ preserves small colimits of solid sheaves, hence by (i) we only need to show that this $v$-pushforward preserves Banach sheaves. This in turn reduces to showing that it preserves étale sheaves, i.e. that the continuous group cohomology of a discrete representation is discrete. This can for example be checked by a direct computation of group cohomology, cf. [10, Proposition 3.4.6].

We now prove (ii), so assume that $\Lambda$ is static and let $A$ be the heart of $D_{\text{nuc}}(\Lambda)^{BG}$. By Lemma [10.5] we have $D^+(A) = D_{\text{nuc}}(\Lambda)^{BG}$ and $D_{\text{nuc}}(\Lambda)^{BG}$ is left-complete. It is therefore enough to show that $D(A)$ is left-complete. For this it is enough to show that countable products have finite cohomological dimension in $D(A)$ (e.g. by adapting the proof of [8, Proposition 1.2.1.19]). Equivalently we need to show that countable products in $D_{\text{nuc}}(\Lambda)^{BG}$ have finite cohomological dimension. This can be checked after pullback along any étale cover of $*/G$ (such a pullback is $t$-exact and preserves limits of nuclear sheaves by the existence of the left adjoint lower shriek functor), so we can replace $G$ by any open subgroup. In particular we can assume that $G$ is profinite and $\text{cd}_\ell G < \infty$. Limits in $D_{\text{nuc}}(\Lambda)^{BG}$ can be computed as the composition of the limit in $D_{\text{nuc}}(\Lambda, \mathbb{Z}_\ell)^{BG}$ and the nuclearization functor. By the proof of (i) nuclearization has finite cohomological dimension, hence so do countable products in $D_{\text{nuc}}(\Lambda)^{BG}$. □

We have acquired a clear understanding of the connection of nuclear sheaves on classifying stacks and nuclear representations. We now study the geometry of these classifying stacks from an $\ell$-cohomological viewpoint. Our first goal is to show that essentially all maps of classifying stacks that appear in practice are $\ell$-fine.

Proposition 10.7. Let $G$ be a profinite group and let $M, N \in D_{\text{nuc}}(\mathbb{Z}_\ell)^{BG}$ be static nuclear $G$-representations such that $N$ is $\ell$-adically complete. Then

$$\text{Ext}^k(M, N) = 0 \quad \text{for all } k > \text{cd}_\ell G + 3.$$ 

Proof. We can assume that $\text{cd}_\ell G < \infty$ because otherwise there is nothing to prove. Let $\text{Hom}(M, N)$ denote the internal hom of $M$ and $N$ in $D_{\text{nuc}}(\mathbb{Z}_\ell)^{BG}$. Then for the spectra-enriched Hom from $M$ to $N$ we have $\text{Hom}(M, N) = \Gamma(G, \text{Hom}(M, N))$. By Lemma [10.3.(ii)] it is therefore enough to show that $\text{Ext}^k(M, N) = 0$ for $k > 2$. Note that the solid internal hom is computed on the underlying $\mathbb{Z}_{\ell,\text{c}}$-modules (it agrees with the pro-étale internal hom and is thus preserved under pullback along the pro-étale map $* \to */G$). We can therefore ignore the group action from now on and simply assume that $M, N \in D_{\text{nuc}}(\mathbb{Z}_\ell)$ with $N$ being $\ell$-adically complete. By pulling out the limit $N = \lim \leftarrow_n N/\ell^n N$ and using the fact that countable limits have cohomological dimension 1, we reduce to showing that $\text{Ext}^k(M, N/\ell^n N) = 0$ for $i > 1$. In other words, from now on we can assume that $N$ is a discrete $\mathbb{Z}/\ell^n \mathbb{Z}$-module. Then $\text{Hom}(M, N) = \text{Hom}_{\mathbb{Z}/\ell^n \mathbb{Z}}(M/\ell^n M, N)$. Since $M/\ell^n M$ is discrete, we can equivalently write it as $M/\ell^n M = M_0/\ell^n M_0$, where $M_0$ is the underlying discrete abelian group of $M$. We can pick a short exact sequence $0 \to \bigoplus_I \mathbb{Z} \to \bigoplus_J \mathbb{Z} \to M_0 \to 0$ for some sets $I$ and $J$. Thus $M_0/\ell^n M_0$ admits a resolution of length 2 in terms of direct sums of copies of $\mathbb{Z}/\ell^n \mathbb{Z}$. Since products are exact in $D_{\text{nuc}}(\mathbb{Z}/\ell^n \mathbb{Z})$ it follows immediately that $\text{Hom}_{\mathbb{Z}/\ell^n \mathbb{Z}}(M_0/\ell^n M_0, N)$ is concentrated in cohomological degrees 0 and 1, as desired. □

Lemma 10.8. Let $G$ be a profinite group with $\text{cd}_\ell G < \infty$. Then the map $* \to */G$ is fdc and admits universal $\ell$-codescent.
Proof. It is clear that the map \(* \to \ast / G \)** is fdcsc because it is proper and pro-étale. To prove universal \(\ell\)-codescend we follow our argument in [5 Lemma 3.11], so the reader is encouraged to have a look at loc. cit. for more details. Pick any small \(v\)-stack \(X\) with a map \(X \to \ast / G \)** and let \(Y := X \times_{\ast / G \ast} \ast \). We denote \(q: Y \to X\) the base-change of \(* \to \ast / G \)** and \(q_\ast: Y_\ast \to X\) the associated \(\check{\text{C}}\)ech nerve. Then we need to show that the natural functor

\[
\mathcal{D}_{\text{nuc}}^1(X, \mathbb{Z}_\ell) \xrightarrow{\sim} \lim_{n \in \Delta} \mathcal{D}_{\text{nuc}}^1(Y_n, \mathbb{Z}_\ell)
\]

is an equivalence. By employing Lurie’s Beck-Chevalley condition (see [8 Corollary 4.7.5.3]) this reduces to the following claim:

(a) The functor \(q^!: \mathcal{D}_{\text{nuc}}(X, \mathbb{Z}_\ell) \to \mathcal{D}_{\text{nuc}}(Y, \mathbb{Z}_\ell)\) is conservative and preserves geometric realizations of \(q^!\)-split simplicial objects in \(\mathcal{D}_{\text{nuc}}(X, \mathbb{Z}_\ell)\).

To prove this we apply ideas of Mathew [12]. We first note that it is enough to show the following claim:

(b) Let \(\langle q_*q^! \rangle \subseteq \text{Fun}(\mathcal{D}_{\text{nuc}}(X, \mathbb{Z}_\ell), \mathcal{D}_{\text{nuc}}(X, \mathbb{Z}_\ell))\) be the full subcategory generated by \(q_*q^!\) under finite (co)limits, compositions and retracts; then \(\langle q_*q^! \rangle\) contains the identity functor.

One checks easily that (b) implies (a): It follows easily from (b) that the functor \(q_*q^!\) is conservative, hence so is \(q^!\). Moreover, if \(\mathcal{M}_\ast\) is any \(q^!\)-split simplicial object in \(\mathcal{D}_{\text{nuc}}(X, \Lambda)\) then it is also \(q_*q^!\)-split and by (b) it follows that it is split; then of course its geometric realization commutes with \(q^!\).

It remains to prove (b). We compute \(q_*q^! = \text{Hom}(q_*Z_\ell, -)\), so (b) reduces to the claim that \(Z_\ell \in \mathcal{D}_{\text{nuc}}(X, \mathbb{Z}_\ell)\) can be generated using finite (co)limits, retracts and tensor products from \(q_*Z_\ell\). In other words, using Mathew’s notion of descendable algebras (see [12 Definition 3.18, Proposition 3.20]) the claim (b) boils down to:

(c) The algebra object \(q_*Z_\ell \in \mathcal{D}_{\text{nuc}}(X, \mathbb{Z}_\ell)\) admits descent.

Let us denote \(q_0: \ast \to \ast / G\) the canonical projection, so that \(q\) is a base-change of \(q_0\) along the map \(f: X \to \ast / G\). The pullback functor \(f^*: \mathcal{D}_{\text{nuc}}(\ast / G, \mathbb{Z}_\ell) \to \mathcal{D}_{\text{nuc}}(X, \mathbb{Z}_\ell)\) is symmetric monoidal and sends \(q_0_*Z_\ell\) to \(q_*Z_\ell\) (by proper base-change), so it is enough to show (c) for \(q_0\). Thus from now on we assume \(X = \ast / G\). By Proposition [10.6] the proof of (c) reduces to the following claim:

(d) The algebra object \(\mathcal{C}(G, Z_\ell) \in \mathcal{D}_{\text{nuc}}(\mathbb{Z}_\ell)^{BG}\) admits descent.

Let \(d := \text{cd}_G + 3\) and consider the cofiber sequence

\[
\mathbb{Z}_\ell \to \text{Tot}_d(\mathcal{C}(G, \mathbb{Z}_\ell)^{\otimes^* + 1}) \to X
\]

in \(\mathcal{D}_{\text{nuc}}(\mathbb{Z}_\ell)^{BG}\), where \(\text{Tot}_d\) denotes the \(d\)-truncated totalization of a cosimplicial object. By descent we have \(\text{Tot}(\mathcal{C}(G, \mathbb{Z}_\ell)^{\otimes^* + 1})\) and thus \(X\) is concentrated in cohomological degrees \(\geq d\). But then it follows from Proposition [10.7] that the connecting map \(X \to \mathbb{Z}_\ell[1]\) must be zero, which implies that \(\mathbb{Z}_\ell\) is a retract of \(\text{Tot}_d(\mathcal{C}(G, \mathbb{Z}_\ell)^{\otimes^* + 1})\), as desired.

**Proposition 10.9.** Let \(G\) be a locally profinite group which has locally finite \(\ell\)-cohomological dimension. Then:
(i) The natural projection \( \ast/G \to \ast \) is \( \ell \)-fine.

(ii) If \( G \) is profinite and has finite \( \ell \)-cohomological dimension then the map \( \ast/G \to \ast \) is \( \ell \)-cohomologically proper.

Proof. For (i) we can replace \( G \) by any compact open subgroup because the property of being \( \ell \)-fine is étale local on the source (see Lemma 5.10). We can thus assume that \( G \) is profinite with \( cd_\ell G < \infty \). In this case the map \( \ast/G \to \ast \) is covered by the map \( \ast \to \ast/G \) which is fdc and admits universal \( \ell \)-codescent by Lemma 10.8, so that \( \ast \to \ast/G \) is \( \ell \)-fine by definition.

It remains to prove (ii) so assume that \( G \) is as in the claim. First note that the diagonal of \( f : \ast/G \to \ast \) has fiber \( G \) and is thus proper, so that \( f \) is 1-separated. By Proposition 10.6.(ii) \( f \ast \) preserves all small colimits, so we can argue as in [5, Corollary 3.12] to deduce that the natural map \( f_! : \ast \to f_* \ast \) is an isomorphism, implying \( \ell \)-cohomological properness of \( f \).

We have established the fact that for nice enough locally profinite groups \( G \) the classifying stack \( \ast/G \) is \( \ell \)-fine, so in particular we have the full 6-functor formalism for nuclear sheaves on classifying stacks at our disposal. In order to compute shriek functors, it is very useful to know that in practice the classifying stack \( \ast/G \) is even \( \ell \)-cohomologically smooth. We have already worked this out for \( p \)-adic sheaves in [5, §3.2] and it works the same way \( \ell \)-adically. For the convenience of the reader we present the main definitions and results.

Definition 10.10. (a) Let \( G \) be a profinite group with \( cd_\ell G < \infty \). We say that \( G \) is \( \ell \)-Poincaré of dimension \( d \) if it satisfies the following conditions:

(i) \( H^k(G, M) \) is finite for all \( k \geq 0 \) and all \( G \)-representations \( M \) on finite \( \mathbb{F}_\ell \)-vector spaces.

(ii) The solid \( \mathbb{F}_\ell \)-vector space \( \Gamma(G, \mathbb{F}_\ell, [-]) \) is invertible and concentrated in cohomological degree \( d \).

(b) Let \( G \) be a locally profinite group with locally finite \( \ell \)-cohomological dimension. We say that \( G \) is virtually \( \ell \)-Poincaré of dimension \( d \) if there is some compact open subgroup \( H \subseteq G \) such that \( cd_\ell H < \infty \) and \( H \) is \( \ell \)-Poincaré of dimension \( d \).

Examples 10.11. Let \( G \) be a locally profinite group.

(a) If \( G \) is an \( \ell \)-adic Lie group of dimension \( d \) then \( G \) is virtually \( \ell \)-Poincaré of dimension \( d \). This follows from results of Lazard [6], see [5, Theorem 3.18].

(b) If \( G \) is locally pro-\( p \) then it is virtually \( \ell \)-Poincaré of dimension 0. Indeed, if \( G \) is pro-\( p \) then \( cd_\ell G = 0 \) by [13] Corollary III.3.3.7, i.e. \( \Gamma(G, -) \) is acyclic on \( \ell \)-torsion representations of \( G \); it follows easily that \( G \) is \( \ell \)-Poincaré of dimension 0.

In the following when we say that a locally profinite group \( G \) is virtually \( \ell \)-Poincaré then we implicitly always mean that \( G \) has locally finite \( \ell \)-cohomological dimension.

Lemma 10.12. Let \( G \) be a profinite group with \( cd_\ell G < \infty \). Then \( G \) satisfies condition (i) of Definition 10.10.(a) if and only if \( \mathbb{F}_\ell \in D_{\text{et}}(\ast/G, \mathbb{F}_\ell) \) is dualizable over \( \ast \).

Proof. We have a cartesian square

\[
\begin{array}{ccc}
\ast/(G \times G) & \xrightarrow{\pi_1} & \ast/G \\
\downarrow{\pi_2} & & \downarrow{f} \\
\ast/G & \xrightarrow{f} & \ast
\end{array}
\]
By Proposition [7.7] the constant sheaf $\mathbb{F}_\ell \in \mathcal{D}_{et}(*/G, \mathbb{F}_\ell)$ is $f$-dualizable if and only if the natural map $\pi_1^H f^! \mathbb{F}_\ell \to \pi_2^H \mathbb{F}_\ell$ is an isomorphism. For every compact open subgroup $H \subseteq G$ let $g_H : */(H \times H) \to */(G \times G)$ be the natural maps. Then the collection of functors $g_H, h_H^*: \mathcal{D}_{et}(*/(G \times G), \mathbb{F}_\ell) \to \mathcal{D}_{et}(*, \mathbb{F}_\ell)$ is conservative (this is a reformulation of the fact that a smooth $(G \times G)$-representation is determined by its $(H \times H)$-invariants for all $H$).

Consequently $g_H^* h_H^* \pi_1^H f^! \mathbb{F}_\ell \to g_H^* h_H^* \pi_2^H \mathbb{F}_\ell$ is an isomorphism. We compute both sides in the case that $\pi_1^H f^! \mathbb{F}_\ell = \pi_2^H \mathbb{F}_\ell$.

Then the map $\ell$ virtually $\mathcal{H}$ we deduce that Eq. (10.12.1) is an isomorphism (for at least state some easy formal consequences of the definition:

Let $\Lambda$ is not discrete then it seems to be a notion which has not been studied much in the literature yet. We do not attempt a full study of admissible nuclear representations, but we can at least state some easy formal consequences of the definition:

**Theorem 10.13.** Let $G$ be a locally profinite group with locally finite $\ell$-cohomological dimension. Then the map $*/G \to *$ is $\ell$-cohomologically smooth (of pure dimension $d/2$) if and only if $G$ is virtually $\ell$-Poincaré (of dimension $d$).

**Proof.** By Proposition [10.9.(i)] the map $f: */G \rightarrow *$ is $\ell$-fine so it makes sense to speak of $\ell$-cohomological smoothness. Since $\ell$-cohomological smoothness is étale local on the source we can assume that $G$ is profinite with $\ell \text{d} G < \infty$. By Lemma [10.12] we can assume that $G$ satisfies condition (i) of Definition [10.10.(a)] and we only need to show that then $G$ satisfies condition (ii) of that definition if and only if $f^! \mathbb{F}_\ell$ is invertible. This is a formal computation for which we refer the reader to our argument in [5] Proposition 3.14.(ii)].

With a good understanding of the classifying stacks at hand, it now makes sense to introduce admissible representations in the following form:

**Definition 10.14.** Let $G$ be a locally profinite group with locally finite $\ell$-cohomological dimension. A nuclear $G$-representation $M \in \mathcal{D}_{nuc}(\Lambda)^{BG}$ is called admissible if it is relatively dualizable over $*$ (when viewed as a nuclear $\Lambda$-module on $*/G$).
Proposition 10.15. Let $G$ be a locally profinite virtually $\ell$-Poincaré group and let $M \in \mathcal{D}_{nuc}(\Lambda)^{BG}$ be a nuclear $G$-representation.

(i) Let $i_1, i_2 : \mathcal{D}_{nuc}(\Lambda)^{BG} \to \mathcal{D}_{nuc}(\Lambda)^{B(G \times G)}$ denote the two inflation operators. Then $M$ is admissible if and only if the natural map

$$i_1(M^\vee) \otimes i_2(M) \to \text{Hom}(i_1(M), i_2(M))$$

is an isomorphism of nuclear $(G \times G)$-representations.

(ii) If $M$ is admissible then it is reflexive, i.e. the map $M \rightarrow M^{\vee\vee}$ is an isomorphism.

(iii) Let $H \subseteq G$ be an open subgroup. Then $M$ is admissible as a $G$-representation if and only if it is admissible as an $H$-representation.

(iv) If $M$ is admissible then $\Gamma(H, M)$ is dualizable for every compact open subgroup $H \subseteq G$ with $cd\ell H < \infty$. If $\Lambda$ is discrete then the converse of this statement holds.

(v) Let $\Lambda \to \Lambda'$ be a map of nuclear $\mathbb{Z}_\ell$-algebras. If $M$ is admissible then $M \otimes_{\Lambda} \Lambda' \in \mathcal{D}_{nuc}(\Lambda')^{BG}$ is admissible.

(vi) Suppose that $\Lambda = \mathbb{Z}_\ell$ and that $M$ is bounded and $\ell$-adically complete. Then $M$ is admissible if and only if $M/\ell M \in \mathcal{D}_{nuc}(\Lambda/\ell\Lambda)^{BG}$ is admissible.

Proof. For (i), note that $i_1$ and $i_2$ are just the pullbacks along the two projections $*/(G \times G) \to */G$. Using also the fact that both are $\ell$-cohomologically smooth by Theorem 10.13 we easily reduce the claim to Proposition 7.7. Part (ii) is a special case of the last part of Proposition 7.7. For (iii), note that $*/H \to */G$ is an étale and hence $\ell$-cohomologically smooth cover, hence the claim is a special case of Proposition 8.6. For (iv), by Proposition 10.9.(ii) the stack $*/H$ is $\ell$-cohomologically proper over $*$, hence the first part of the claim is a special case of Proposition 9.10. If $\Lambda$ is discrete then the family of functors $\Gamma(H, -) : \mathcal{D}_{nuc}(\Lambda)^{BG} \to \mathcal{D}_{nuc}(\Lambda)$ is conservative, so one can argue as in the proof of Lemma 10.12 to get the second part of (iv). Claims (v) and (vi) are special cases of Proposition 7.14.

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