Abstract. We present an extension of $f(T)$ gravity, allowing for a general coupling of the torsion scalar $T$ with the trace of the matter energy-momentum tensor $\mathcal{T}$. The resulting $f(T, \mathcal{T})$ theory is a new modified gravity, since it is different from all the existing torsion or curvature based constructions. Applied to a cosmological framework, it leads to interesting phenomenology. In particular, one can obtain a unified description of the initial inflationary phase, the subsequent non-accelerating, matter-dominated expansion, and then the transition to a late-time accelerating phase. Additionally, the effective dark energy sector can be quintessence or phantom-like, or exhibit the phantom-divide crossing during the evolution. Moreover, in the far future the universe results either to a de Sitter exponential expansion, or to eternal power-law accelerated expansions. Finally, a detailed study of the scalar perturbations at the linear level reveals that $f(T, \mathcal{T})$ cosmology can be free of ghosts and instabilities for a wide class of ansatzes and model parameters.

Keywords: modified gravity, gravity

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1 Introduction

The verification of the late-time acceleration of the universe (see [1] for a detailed discussion of the recent astronomical observations) has led to extensive research towards its explanation. This result is based on fitting a Friedman-Robertson-Walker type geometry, together with the corresponding cosmology, to the existing astronomical data. However, strictly speaking, taking into account the present day astronomical observational information, the only model-independent conclusion that we can infer at this stage is that the observations do not favor the pressureless Einstein-de Sitter model.

In general, there are two main ways to achieve the goal of the theoretical explanation of the accelerated expansion of the universe. The first direction consists in modifying the universe content, by introducing a dark energy sector, starting either with a canonical scalar field, a phantom field, or the combination of both fields in a unified model, and proceeding to more complicated constructions (for reviews see [2, 3] and references therein). The second direction is to modify the gravitational sector itself (see [4–7] for reviews and references therein). However, we mention that, up to physical interpretation issues, one can transform, completely or partially, from one approach to the other, since the important issue is the number of extra degrees of freedom (for such a unified point of view see [8]). Thus, one could also have combinations of both directions, in scenarios with various couplings between gravitational and non-gravitational sectors.

In modified gravitational theories one usually generalizes the Einstein-Hilbert action of General Relativity, that is, one starts from the curvature description of gravity. However, a different and interesting class of modified gravity arises when one extends the action of the equivalent formulation of GR based on torsion. As it is known, Einstein constructed also
the “Teleparallel Equivalent of General Relativity” (TEGR) in which the gravitational field is described by the torsion tensor and not by the curvature one [9–15] (technically this is achieved by using the Weitzenböck connection instead of the torsion-less Levi-Civita one). Then, the corresponding Lagrangian given by the torsion scalar $T$, results from contractions of the torsion tensor, like the Einstein-Hilbert Lagrangian $R$ results from contractions of the curvature (Riemann) tensor. Thus, instead of starting from GR, one can start from TEGR and construct the $f(T)$ modified gravity, by extending $T$ to an arbitrary function in the Lagrangian [17–19]. The interesting feature is that although TEGR is completely equivalent with General Relativity at the level of equations, $f(T)$ is different than $f(R)$ gravity, that is they form different gravitational modifications. Hence, $f(T)$ gravity has novel and interesting cosmological implications [19–55]. Additionally, note that if one starts from TEGR, but instead of the $f(R)$ is inspired by higher-curvature modifications of General Relativity, one can construct higher-order torsion gravity, such as the $f(T,T_G)$ paradigm [56, 57], which also presents interesting cosmological behavior. Finally, another modification of TEGR is to extend it inserting the Weitzenböck condition in a Weyl-Cartan geometry via a Lagrange multiplier, with interesting cosmological implications [58, 59].

Nevertheless, in usual General Relativity one could proceed to modifications in which the geometric part of the action is coupled to the non-geometric sector. The simplest models are those with non-minimally coupled [60–63] and non-minimal-derivatively coupled [64–68] scalar fields, but one could further use arbitrary functions of the kinetic and potential parts such as in K-essence [69], resulting in the general Horndeski [70] and generalized Galileon theories [71–73]. However, since there is no theoretical reason against couplings between the gravitational sector and the standard matter one, one can consider modified theories where the matter Lagrangian is coupled to functions of the Ricci scalar [74–77], and extend the theory to arbitrary functions $(R, \mathcal{L}_m)$ [78–82]. Alternatively, one can consider models where the Ricci scalar is coupled with the trace of the energy momentum tensor $T$ and extend to arbitrary functions, such as in $f(R, T)$ theory [83–87], or even consider terms of the form $R_{\mu\nu}T^{\mu\nu}$ [88, 89]. We stress that the above modifications, in which one handles the gravitational and matter sectors on equal footing, do not present any problem at the theoretical level, and one would only obtain observational constraints due to non-geodesic motion.

Having these in mind, one could try to construct the above extended coupled scalar-field and coupled-matter modified gravities, starting not from GR but from TEGR. The incorporation of non-minimally coupled scalar-torsion theories was performed in [90–98], where a scalar field couples non-minimally to the torsion scalar $T$. Similarly, in [99] non-minimally matter-torsion theories were constructed, where the matter Lagrangian is coupled to a second $f(T)$ function. We mention that both these scenarios are different than the corresponding curvature ones, despite the fact that uncoupled GR coincides with TEGR. They correspond to novel modified theories, with a novel cosmological behavior.

In the present work, we are interested in constructing $f(T, T)$ gravity, that is, allowing for arbitrary functions of both the torsion scalar $T$ and the trace of the energy-momentum tensor $T$. We emphasize that the resulting theory differs from $f(R, T)$ gravity, in that it is a novel modified gravitational theory, with no curvature-equivalent, and its cosmological implications prove to be very interesting. Similar work has also been explored in [100], where the stability of the specific de Sitter solution, when subjected to homogeneous perturbations, was analyzed. Furthermore, the constraints imposed by the energy conditions were considered, and the parameter ranges of the proposed model were found to be consistent with the above stability conditions. In this work, we consider more general cases. In particular, we find late-
time accelerated solutions, as well as initial inflationary phases, followed by non-accelerating matter-dominated expansions, resulting to a late-time accelerating evolution.

The plan of the manuscript is outlined as follows: in section 2, we review the $f(T)$ gravitational modification. In section 3, we construct $f(T, T)$ gravity, and we apply it in a cosmological framework. In section 5, we analyze the cosmological implications of two specific examples. Finally, section 6 is devoted to the conclusions.

2 $f(T)$ gravity and cosmology

We start with a brief review of $f(T)$ gravity. Throughout the manuscript, we use Greek indices to span the coordinate space-time and Latin indices to span the tangent space-time. The fundamental field is the vierbein $e_A(x^\mu)$, which at each point $x^\mu$ of the space-time forms an orthonormal basis for the tangent space, namely $e_A \cdot e_B = \eta_{AB}$, where $\eta_{AB} = \text{diag}(1, -1, -1, -1)$. Furthermore, in the coordinate basis we can express it in terms of components as $e_A = \epsilon^A_\mu \partial_\mu$. Thus, the metric tensor can be expressed as

$$g_{\mu\nu}(x) = \eta_{AB} e^A_\mu(x) e^B_\nu(x).$$  \hspace{1cm} (2.1)

In the teleparallel gravitational formulation (the vierbein components at different points are “parallelized” and this is what is represented by the appellation “teleparallel”) one uses the Weitzenböck connection $\Gamma^\lambda_{\nu\mu} \equiv e^\lambda_A \partial_\mu e^A_\nu$ \cite{101} which leads to zero curvature, and not the Levi-Civita one which leads to zero torsion. Hence, the gravitational field is described by the torsion tensor

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu} = e^\lambda_A \left( \partial_\mu e^A_\nu - \partial_\nu e^A_\mu \right).$$  \hspace{1cm} (2.2)

Additionally, we introduce the contorsion tensor $K^{\mu\nu\rho} \equiv -\frac{1}{2} (T^{\mu\nu\rho} - T^{\nu\mu\rho} - T^{\rho\mu\nu})$, and the tensor $S^{\mu\nu} \equiv \frac{1}{2} \left( K^{\mu\nu\rho} + \delta^\mu_\rho T^{\alpha\nu\rho} - \delta^\nu_\rho T^{\alpha\mu\rho} \right)$. From the torsion tensor, one constructs the torsion scalar and the respective teleparallel Lagrangian \cite{10–15}

$$T = \frac{1}{4} T^{\rho\mu\nu} T_{\rho\mu\nu} + \frac{1}{2} T^{\rho\mu\nu} T_{\rho\mu\nu} - T_{\rho\mu\nu} T_{\rho\mu\nu}. \hspace{1cm} (2.3)$$

Thus, if $T$ is used in an action and one performs variation in terms of the vierbeins, one extracts the same equations as with General Relativity. That is why Einstein dubbed this theory “Teleparallel Equivalent of General Relativity” (TEGR).

One can start from TEGR in order to construct various gravitational modifications. In particular, one can extend $T$ to $T + f(T)$, resulting to the so-called $f(T)$ gravity, where the action is given

$$S = \frac{1}{16\pi G} \int d^4x \left[ T + f(T) \right], \hspace{1cm} (2.4)$$

with $\epsilon = \text{det} \left( e^A_\mu \right) = \sqrt{-g}$, $G$ the Newton’s constant, and setting the speed of light to one. It is clear that TEGR and thus General Relativity is obtained when $f(T) = 0$. However, note that $f(T)$ differs from $f(R)$ gravity, despite the fact that TEGR coincides with General Relativity at the level of the equations.

The cosmological applications of $f(T)$ gravity can be investigated incorporating the matter sector in the action. Thus, the latter is finally given by

$$S = \frac{1}{16\pi G} \int d^4x \left[ T + f(T) + \mathcal{L}_m \right], \hspace{1cm} (2.5)$$
where the matter Lagrangian is considered to correspond to a perfect fluid with energy density and pressure \( \rho_m \) and \( p_m \), respectively (one could include the radiation sector too). Variation of the action (2.5) with respect to the vierbein leads to the field equations

\[
(1 + f') \left[ e^{-1} \partial_\mu (e e_A^A S_\rho^\nu \rho) - e_A^A T_\mu^\nu S_\rho^\mu \right] + e_A^A S_\rho^\nu \partial_\mu T f'' + \frac{1}{4} e_A^A [T + f] = 4 \pi G e_A^A T_\rho^\nu,
\]

where we denote \( f' = \partial f / \partial T \) and \( f'' = \partial^2 f / \partial T^2 \), while \( e_A^A T_\rho^\nu \) stands for the usual energy-momentum tensor.

Additionally, in order to obtain a flat Friedmann-Robertson-Walker (FRW) universe

\[
ds^2 = dt^2 - a^2(t) \delta_{ij} dx^i dx^j,
\]

where \( a(t) \) is the scale factor, we consider

\[
e_\mu^A = \text{diag}(1, a(t), a(t), a(t)).
\]

Thus, with this vierbein ansatz, the equations of motion (2.6) give rise to the modified Friedmann equations

\[
H^2 = \frac{8 \pi G}{3} \rho_m - \frac{f}{6} - 2H^2 f'
\]

\[
\dot{H} = - \frac{4 \pi G (\rho_m + p_m)}{1 + f' - 12H^2 f''},
\]

respectively, where \( H \equiv \dot{a}/a \) is the Hubble parameter, and the overdot denote the \( t \)-derivatives.

We mention that we have incorporated the useful relation

\[
T = -6H^2,
\]

which holds for an FRW geometry, and which is determined from eq. (2.3) using eq. (2.8).

### 3 \( f(T,T) \) gravity and cosmology

In this section, we present a novel theory of gravitational modification, extending the previously described \( f(T) \) gravity. In particular, apart from an arbitrary function of the torsion scalar, we will also allow for an arbitrary function of the trace of the energy momentum tensor. Thus, we consider the action

\[
S = \frac{1}{16 \pi G} \int d^4 x \ e \ [T + f(T, T)] + \int d^4 x \ e \ L_m,
\]

where \( f(T, T) \) is an arbitrary function of the torsion scalar \( T \) and of the trace \( T \) of the matter energy-momentum tensor \( T_\rho^\nu \), and \( L_m \) is the matter Lagrangian density. Hereinafter, and following the standard approach, we assume that \( L_m \) depends only on the vierbein and not on its derivatives.

Varying the action, given by eq. (3.1), with respect to the vierbeins yields the field equations

\[
(1 + f_T) \left[ e^{-1} \partial_\mu (e e_A^A S^\rho_\mu) - e_A^A T_\mu^\nu \partial_\rho \right] + (f_T T_\rho^\nu \partial_\mu T + f_T T_\rho^\nu \partial_\mu T) e_A^A S^\rho_\mu

\[
+ e_A^A \left( f + \frac{T}{4} \right) - f_T \left( \frac{e_A^A T_\rho^\nu + p e_A^A}{2} \right) = 4 \pi G e_A^A T_\rho^\nu,
\]

where \( f_T = \partial f / \partial T \) and \( f_{TT} = \partial^2 f / \partial T^2 \).
In order to apply the above theory in a cosmological framework, we insert as usual the flat FRW vierbein ansatz (2.8) into the field equations (3.2), obtaining the modified Friedmann equations:

\[ H^2 = \frac{8\pi G}{3} \rho_m - \frac{1}{6} (f + 12 H^2 f_T) + f_T \left( \frac{\rho_m + p_m}{3} \right), \]  

\[ \dot{H} = -4\pi G (\rho_m + p_m) - H \left( f_T - 12 H^2 f_{TT} \right) - H \left( \dot{\rho}_m - 3 \dot{p}_m \right) f_{TT} - f_T \left( \frac{\rho_m + p_m}{2} \right). \]  

(3.3)

(3.4)

We mention that in the above expressions we have used that \( T = \rho_m - 3 p_m \), which holds in the case of a perfect matter fluid.

Proceeding, we assume that the matter component of the Universe satisfies a barotropic equation of state of the form \( p_m = p_m(\rho_m) \), with \( w_m = p_m/\rho_m \) its equation-of-state parameter, and \( c_s^2 = dp_m/d\rho_m \) the sound speed. Note that due to homogeneity and isotropy, both \( \rho_m \) and \( p_m \) are function of \( t \) only, and thus of the Hubble parameter \( H \). Thus, eq. (3.4) can be re-written as

\[ \dot{H} = -\frac{4\pi G (1 + f_T/8\pi G) (\rho_m + p_m)}{1 + f_T - 12 H^2 f_{TT} + H (d\rho_m/dH)(1 - 3 c_s^2) f_{TT}}. \]  

(3.5)

By defining the energy density and pressure of the effective dark energy sector as

\[ \rho_{DE} = -\frac{1}{16\pi G} \left[ f + 12 f_T H^2 - 2 f_T (\rho_m + p_m) \right], \]  

(3.6)

\[ p_{DE} = (\rho_m + p_m) \left[ \frac{1 + f_T/8\pi G}{1 + f_T - 12 H^2 f_{TT} + H (d\rho_m/dH)(1 - 3 c_s^2) f_{TT}} - 1 \right] + \frac{1}{16\pi G} \left[ f + 12 H^2 f_T - 2 f_T (\rho_m + p_m) \right], \]  

(3.7)

respectively, the cosmological field equations of the \( f(T, T) \) theory are rewritten in the usual form

\[ H^2 = \frac{8\pi G}{3} (\rho_{DE} + \rho_m), \]  

(3.8)

\[ \dot{H} = -4\pi G (\rho_{DE} + p_{DE} + \rho_m + p_m). \]  

(3.9)

Furthermore, we define the dark energy equation-of-state parameter as

\[ w_{DE} = \frac{p_{DE}}{\rho_{DE}}, \]  

(3.10)

and it proves convenient to introduce also the total equation-of-state parameter \( w \), given by

\[ w = \frac{p_{DE} + p_m}{\rho_{DE} + \rho_m}. \]  

(3.11)

Note that in the case of the dust universe, with \( p_m = 0 \), we have \( w = w_{DE}/(1 + \rho_m/\rho_{DE}) \).

As we can see from eqs. (3.8), the matter energy density and pressure, and the effective dark energy density and pressure, satisfy the conservation equation

\[ \dot{\rho}_{DE} + \dot{\rho}_m + 3H (\rho_m + \rho_{DE} + p_m + p_{DE}) = 0. \]  

(3.12)
Thus, one obtains an effective interaction between the dark energy and matter sectors, which is usual in modified matter coupling theories [78–83]. Therefore, in the present model the effective dark energy is not conserved alone, and there is an effective coupling between dark energy and normal matter, with the possibility of energy transfer from one component to another. The dark energy alone satisfies the “conservation” equation

$$\dot{\rho}_{\text{DE}} + 3H (\rho_{\text{DE}} + p_{\text{DE}}) = -Q (\rho_m, p_m),$$

where the effective dark energy “source” function $Q (\rho_m, p_m)$ is

$$Q (\rho_m, p_m) = \dot{\rho}_m + 3H (\rho_m + p_m).$$

Hence, in the present model it is allowed to have an energy transfer from ordinary matter to dark energy (which, even geometric in its origin, contains a matter contribution), and this process may be interpreted in triggering the accelerating expansion of the universe.

Finally, as an indicator of the accelerating dynamics of the Universe we use the deceleration parameter $q$, defined as

$$q = -\frac{\dot{H}}{H^2} - 1.$$  

Positive values of $q$ correspond to decelerating evolution, while negative values indicates accelerating behavior.

4 Scalar perturbations and stability analysis

One of the most important tests in every gravitational theory is the investigation of the perturbations [102]. Firstly, such a study reveals the stability behavior of the theory. Secondly, it allows the correlation of the gravitational perturbations with the growth of matter overdensities, and thus one can use growth-index data in order to constrain the parameters of the scenario. In this section, we examine the scalar perturbations of $f(T, T)$ gravity at the linear level. Specifically, we extract the set of gravitational and energy-momentum-tensor perturbations and using them we examine the stability. Additionally, we extract the equation for the growth of matter overdensities.

4.1 Matter and scalar perturbations

Let us perform a perturbation of the theory. As usual in theories where the fundamental field is the vierbein, we impose a vierbein perturbation, which will then lead to the perturbed metric. Without loss of generality we perform the calculations in the Newtonian gauge.

Denoting the perturbed vierbein with $e^A_{\mu}$ and the unperturbed one with $\bar{e}^A_{\mu}$, the scalar perturbations, keeping up to first-order terms, write as

$$e^A_{\mu} = \bar{e}^A_{\mu} + t^A_{\mu},$$

with

$$\bar{e}^0_{\mu} = \delta^0_{\mu}, \quad \bar{e}^a_{\mu} = \delta^a_{\mu} \alpha, \quad \bar{e}^\mu_0 = \delta^\mu_0, \quad \bar{e}^\mu_a = \frac{\delta^\mu_a}{a},$$

$$t^0_{\mu} = \delta^0_{\mu} \psi, \quad t^a_{\mu} = -\delta^a_{\mu} \phi, \quad t^\mu_0 = -\delta^\mu_0 \psi, \quad t^\mu_a = \frac{\delta^\mu_a}{a} \phi.$$
Note that we have made a simplifying assumption, namely that the scalar perturbations $t_\mu^A$ are diagonal, which is sufficient in order to study the stability. Furthermore, in this section subscripts zero and one denote zeroth and linear order values respectively. In the above expressions we have introduced the scalar modes $\psi$ and $\phi$, which depend $x$ and $t$. The various coefficients have been considered in a way that the induced metric perturbation to have the usual form in the Newtonian gauge, that is

$$ds^2 = (1 + 2\psi)dt^2 - a^2(1 - 2\phi)\delta_{ij}dx^idx^j.$$ (4.4)

Let us now calculate the various perturbed quantities under the perturbations (4.2) and (4.3). Firstly, the vierbein determinant reads

$$e = \det \left(e_\mu^A\right) = a^3(1 + \psi - 3\phi).$$ (4.5)

Similarly, the torsion tensor $T^\lambda_{\mu\nu}$ and the auxiliary tensor $S^\lambda_{\mu\nu}$ read (indices are not summed over):

$$T^0_{\mu\nu} = \partial_\mu \psi \delta^0_v - \partial_\nu \psi \delta^0_\mu, \quad T^i_{0i} = H - \dot{\phi}$$

$$S^0_{0i} = \frac{\partial_i \phi}{a^2}, \quad S^i_{0i} = -H + \dot{\phi} + 2H\psi$$

$$T^i_{ij} = \partial_j \phi, \quad S^i_{ij} = \frac{1}{2a^2}\partial_j(\phi - \psi).$$ (4.6)

Thus, the torsion scalar can be straightforwardly calculated using (2.3), leading to

$$T = T_0 + \delta T,$$ (4.7)

where

$$T_0 = -6H^2$$

$$\delta T = 12H(\dot{\phi} + H\psi)$$ (4.9)

are respectively the zeroth and first order results.

Having performed the perturbations of the gravitational sector we proceed to the perturbations of the energy-momentum tensor. As usual they are expressed as

$$\delta \em T^0_0 = \delta \rho_m$$

$$\delta \em T^i_0 = (\rho_m + p_m)\dot{\phi} \delta \dot{v}$$

$$\delta \em T^i_i = \frac{a^2(\rho_m + p_m)}{\partial_i}\delta \dot{v}$$

$$\delta \em T^i_j = -\delta^i_j \delta p_m - \partial_i \partial^j \pi^S,$$ (4.13)

where $\delta \rho_m$, $\delta p_m$, $\delta \dot{v}$ are respectively the fluctuations of energy density, pressure and fluid velocity, while $\pi^S$ is the scalar component of the anisotropic stress. Additionally, since $\em T = \em T^{\mu}_\mu = \em T^0_0 + \em T^i_i$, we conclude that

$$\em T = \em T_0 + \em \delta \em T,$$ (4.14)
where
\[ T_0 = \rho_m - 3p_m \] (4.15)
\[ \delta T = \delta\rho_m - 3\delta p_m - \nabla^2 \pi^S. \] (4.16)

Moreover, we have defined \( \nabla^2 = \sum_i \partial_i \partial^i \).

Finally, we can express the variations of the various \( f \)-derivatives that appear in the background equations of motion as:
\[ \delta f = f_T \delta T + f_T \delta T^T \]
\[ \delta f_T = f_T \delta T + f_T \delta T^T \]
\[ \delta f_T T = f_T T \delta T + f_T T \delta T^T \]
\[ \delta f_T T^T = f_T T \delta T + f_T T \delta T^T, \] (4.17)

where the various \( f \)-derivatives are calculated at the background values \( T_0 \) and \( T_0 \), for instance \( f_T = \frac{df}{dT} \bigg|_{T=T_0,T=T} \).

Inserting everything in the equations of motion (3.2), we acquire the scalar perturbation equations:
\[
(1 + f_T) \left[ \frac{\nabla^2 \phi}{a^2} - 6H \left( \dot{\phi} + H\phi \right) \right] + \left[ 3H^2 f_T T + \frac{1 + f_T}{4} \left( \rho_m + p_m \right) f_T \right] \left[ 12H \left( \dot{\phi} + H\phi \right) \right] \\
+ \left[ 3H^2 f_T T + \frac{f_T}{4} \right] \left( \rho_m + p_m \right) f_T \left[ \delta\rho_m - 3\delta p_m - \nabla^2 \pi^S \right] \\
- \frac{f_T}{2} (\delta\rho_m + \delta p_m) = 4\pi G \delta\rho_m, \tag{4.18}
\]
\[
- (1 + f_T) \partial_i \left[ \frac{\nabla^2 \phi}{a^2} - 6H \left( \dot{\phi} + H\phi \right) \right] + \left[ 12H \dot{\phi} f_T T - \left( \rho_m - 3\dot{\rho}_m \right) f_T \right] \partial_i \phi \\
- \frac{a^2 f_T}{2} \left( \rho_m + p_m \right) \partial_i \delta v = 4\pi G a^2 \left( \rho_m + p_m \right) \partial_i \delta v, \tag{4.19}
\]
\[
- (1 + f_T) \partial_i \left[ \dot{\phi} + H\phi \right] + H \partial_i \left[ 12H f_T T \left( \dot{\phi} + H\phi \right) + f_T \delta\rho_m - 3\delta p_m - \nabla^2 \pi^S \right] \\
- \frac{a^2 f_T}{2} \left( \rho_m + p_m \right) \partial_i \delta v = 4\pi G a^2 \left( \rho_m + p_m \right) \partial_i \delta v, \tag{4.20}
\]
\[
(1 + f_T) \left[ -H \left( \dot{\phi} + 6\phi \right) - 2\phi \left( 3H^2 + \dot{H} \right) - \phi + \frac{\nabla^2 (\phi - \psi)}{3a^2} \right] \\
+ 12H f_T T \left[ H \left( \dot{\phi} + H\phi \right) + H \left( \dot{H} + \dot{\phi} + H\psi \right) \right] + H f_T \left[ \delta\rho_m - 3\delta p_m - \nabla^2 \pi^S \right] \\
+ \left[ 12H \left( \ddot{\phi} + H\dot{\phi} \right) \right] \left\{ f_T T \left( 3H^2 + \dot{H} \right) - H \left[ 12H H f_T T - f_T T \right] \left( \rho_m - 3\dot{\rho}_m \right) \right\} + \frac{f_T}{4} \left( \rho_m - 3\dot{\rho}_m \right) \\
+ \left( \dot{\phi} + 2H\psi \right) \left[ 12H \ddot{H} f_T T - f_T \left( \rho_m - 3\dot{\rho}_m \right) \right] + \frac{f_T}{6} \frac{\nabla^2 \pi^S}{\pi^2} = -4\pi G \left( \delta p_m + \frac{\nabla^2 \pi^S}{3} \right), \tag{4.21}
\]
and
\[
(1 + f_T) (\psi - \phi) = -8\pi G a^2 \left( 1 + \frac{f_T}{8\pi G} \right) \pi^S, \tag{4.22}
\]
respectively.
4.2 Stability analysis

Since we have extracted the linear perturbation equations, we can examine the basic stability requirement by extracting the dispersion relation for the gravitational perturbations. As usual, for simplicity we will consider zero anisotropic stress ($\pi^S = 0$), and in this case equation (4.22) allows us to replace $\psi$ by $\phi$, and thus remaining with only one gravitational perturbative degree of freedom. We transform it in the Fourier space as

$$\phi(t, x) = \int \frac{d^3k}{(2\pi)^3} \tilde{\phi}_k(t)e^{ik\cdot x},$$

and therefore $\nabla^2\phi = -k^2\tilde{\phi}_k$.

Inserting this decomposition into (4.21), and using the other perturbative equations in order to eliminate variables, after some algebra we obtain the following equation of motion for the modes of the gravitational potential $\phi$:

$$\ddot{\tilde{\phi}}_k + \Gamma \dot{\tilde{\phi}}_k + \mu^2 \tilde{\phi}_k + c_s^2 \frac{k^2}{a^2} \tilde{\phi}_k = D.$$ (4.24)

The functions $\Gamma$, $\mu^2$ and $c_s^2$ are respectively the frictional term, the effective mass, and the sound speed parameter for the gravitational potential $\phi$, and along with the term $D$ are given in the appendix. Clearly, in order for our theory to be stable at the linear scalar perturbation level, we require $\mu^2 \geq 0$ and $c_s^2 \geq 0$.

Due to the complexity of the coefficients $\mu^2$ and $c_s^2$, we cannot extract analytical relations for the stability conditions. This is usual in complicated modified gravity models, for instance in generalized Galileon theory [73, 103], in Hořava-Lifshitz gravity [104, 105], in cosmology with non-minimal derivative coupling [106], etc. Furthermore, although in almost all modified gravity models one can, at first stage, perform the perturbations neglecting the matter sector, in the scenario at hand this cannot be done, and this is an additional complexity, since in that case one would kill the extra information of the model (which comes from the matter sector itself) remaining with the usual $f(T)$ gravity. A significant simplification arises if we consider as usual the matter to be dust, that is $p_m = \delta p_m = 0$, but still one needs to resort to numerical elaboration of equation (4.24) in order to ensure if a given $f(T, T)$ cosmological model is free of instabilities. However, we mention that since the simple $f(T)$ gravity is free of instabilities for a large class of $f(T)$ ansatzes [20, 21], we deduce that at least for $f(T, T)$ models that are small deviations from the corresponding $f(T)$ ones, the stability requirements $\mu^2 \geq 0$ and $c_s^2 \geq 0$ are expected to be satisfied.

5 Cosmological behavior

In this section, we investigate the cosmological implications of $f(T, T)$ gravity, focusing on specific examples. For convenience, we use the natural system of units with $8\pi G = c = 1$. From the analysis of the previous section we saw that the basic equations describing the cosmological dynamics are the two Friedmann equations (3.3) and (3.4). These can be re-written as

$$\rho_m = \frac{3H^2 + \left(f + 12H^2 f_T|_{T \rightarrow -6H^2}\right)/2 - f_T p_m}{1 + f_T},$$

and

$$\dot{H} = -\frac{(1 + f_T)(\rho_m + p_m)/2 + H(\rho_m - 3p_m) f_{TT}|_{T \rightarrow -6H^2}}{1 + f_T|_{T \rightarrow -6H^2} - 12H^2 f_{TT}|_{T \rightarrow -6H^2}},$$
respectively. Equations (5.1) and (5.2) compose a system of two differential equations for three unknown functions, namely \((H, \rho_m, p_m)\). In order to close the system of equations we need to impose the matter equation of state \(p_m = p_m(\rho_m)\). In this work, we restrict our study to the case of dust matter, that is \(p_m = 0\), and thus \(T = \rho_m\).

In the following, we investigate two specific \(f(T, T)\) models, corresponding to simple non-trivial extensions of TEGR, that is of GR. However, although simple, these models reveal the new features and the capabilities of the theory.

In order to relate our model with cosmological observations we will present the results of the numerical computations for the Hubble function, matter energy density, deceleration parameter and the parameter of the dark energy equation of state as functions of the cosmological redshift \(z\), defined as

\[
z = \frac{a_0}{a} - 1,
\]

where \(a_0\) is the present day value of the scale factor, which we take as one; that is, we choose \(a_0 = 1\). In terms of the redshift the derivatives with respect to time are expressed as

\[
\frac{d}{dt} = -(1 + z)H(z)\frac{d}{dz}.
\]

In particular for the deceleration parameter we obtain

\[
q(z) = \frac{1}{H(z)}\frac{dH}{dz} - 1.
\]

In order to numerically integrate the gravitational field equations we need to fix the value of the Hubble function at \(z = 0\), \(H(0) = H_0\). The present value of the Hubble function is of the order of \(H_0 \approx 2.3 \times 10^{-18}\) s\(^{-1}\) [107].

5.1 Model A: \(f(T, T) = \alpha T^n + \Lambda\)

A first model describing a simple departure from General Relativity is the one with \(f(T, T) = \alpha T^n + \Lambda\), where \(\alpha, n \neq 0\) and \(\Lambda\) are arbitrary constants. For this ansatz, we straightforwardly obtain \(f = \alpha (-6H^2)^n + \Lambda, f_T = n\alpha \rho_m (-6H^2)^{n-1}, f_{TT} = n\alpha(n-1)(-6H^2)^{n-2}, f_{TT} = \alpha n(-6H^2)^{n-1}\), and \(f_T = \alpha (-6H^2)^n\). Hence, inserting these into eq. (5.1) we can obtain the matter energy density as a function of the Hubble function as

\[
\rho_m = \frac{3H^2 + \Lambda/2}{1 + \alpha(n + 1/2)(-6H^2)^n}.
\]

Differentiating eq. (5.6) we acquire the useful relation

\[
\dot{\rho}_m = \frac{2\dot{H}}{H[2n+1]}\frac{[12H^2 - (2n+1)\alpha 6^n (-H^2)^n(6(n-1)H^2 + \Lambda n)]}{[6(2n+1)\alpha 6^n (-H^2)^n + 2]^2}.
\]

Thus, inserting the above expressions into eqs. (3.10), (3.15) and (5.2) we extract respectively the time-variation of the Hubble function, the deceleration parameter and the dark-energy equation-of-state parameter, as functions of \(H\), namely

\[
\dot{H} = \frac{1}{a^2 36^n (2n+1)(-H^2)^n [6(n+1)H^2 + \Lambda n] - \alpha \Lambda 2n+3^3 (-H^2)^n [6(n-2)(2n+1)H^2 + \Lambda n (2n-1)] + 24H^2},
\]

\[
q = \frac{3(6H^2 + \Lambda) [6^n (-H^2)^n + 1] [6^n (2n+1) (-H^2)^n + 2]}{a^2 36^n (2n+1)(-H^2)^n [6(n+1)H^2 + \Lambda n] - \alpha \Lambda 2n+3^3 (-H^2)^n [6(n-2)(2n+1)H^2 + \Lambda n (2n-1)] + 24H^2} - 1.
\]
and
\[
\begin{align*}
\omega_{\text{DE}} &= -\frac{3H^2[\alpha^6(n^2+1)(-H^2)^{n+2}]\left\{\alpha_1\alpha_3(-H^2)^nH^2+\alpha_4-\alpha_2(-H^2)^{2n}[6(n-1)H^2+\Lambda(n-2)]+4\Lambda\right\}}{\left\{1\right.}
\end{align*}
\]
respectively, where for convenience we have defined the parameters \(\alpha_1 = \alpha^2 n^2+3n\), \(\alpha_2 = \alpha^2 36n(2n+1)\), \(\alpha_3 = 6[n(2n-1)+1]\), \(\alpha_4 = \Lambda(2n^2+n+3)\), \(\alpha_5 = 6(n-2)(2n+1)\), and \(\alpha_6 = \Lambda n(2n-1)\).

### 5.1.1 The case \(n = 1\)

A first model describing the simplest departure from General Relativity is the one obtained for \(n = 1\) in the general scenario previously introduced, that is with \(f(T,T) = \alpha TT = \alpha T \rho_m + \Lambda\). For this ansatz, we straightforwardly obtain \(f = -6\rho_m H^2 + \Lambda\), \(f_T = \alpha \rho_m\), \(f_{TT} = 0\), \(f_{TT} = \alpha\), and \(f_T = \alpha T = -6\alpha H^2\). Thus, eq. (5.6) reduces to
\[
\rho_m = \frac{3H^2 + \Lambda/2}{1 - 9\alpha H^2},
\]
while from eqs. (5.8)–(5.10) we obtain
\[
\begin{align*}
\dot{H} &= -\frac{2(6\alpha H^2 - 1)(9\alpha H^2 - 1)(6H^2 + \Lambda)}{2[\alpha \Lambda + 9\alpha H^2(\alpha \Lambda + 12\alpha H^2 - 2) + 2]}, \\
q &= \frac{2H^2[\alpha \Lambda + 9\alpha H^2(\alpha \Lambda + 12\alpha H^2 - 2) + 2] - 1},
\end{align*}
\]
and
\[
\begin{align*}
\omega_{\text{DE}} &= \frac{2(9\alpha H^2 - 1)[9\alpha(3\alpha \Lambda - 4)H^4 - 18\alpha \Lambda H^2 + \Lambda]}{(54\alpha H^4 + \Lambda)[\alpha \Lambda + 9\alpha H^2(\alpha \Lambda + 12\alpha H^2 - 2) + 2]},
\end{align*}
\]
respectively. Note that relations (5.11)-(5.13) hold for every \(\alpha\), including \(\alpha = 0\) (in which case we obtain the GR expressions), while (5.14) holds for \(\alpha \neq 0\), since for \(\alpha = 0\) the effective dark energy sector does not exist at all (both \(\rho_{\text{DE}}\) and \(p_{\text{DE}}\) are zero).

As we may observe from eq. (5.13), the scenario at hand can give rise to both acceleration and deceleration phases, according to the values of the model parameters \(\alpha\) and \(\Lambda\). However, the most interesting feature that is clear from eq. (5.14) is that the dark energy equation-of-state parameter can be quintessence-like or phantom-like, or even experience the phantom-divide crossing during the evolution, depending on the choice of the parameter range. This feature is an additional advantage, since such behaviors are difficult to be obtained in dark energy constructions.

In order to present the above features in a more transparent way, we proceed to a detailed numerical elaboration for various parameter choices. We introduce the redshift \(z\) as the independent variables, and we rescale the parameters as
\[
\begin{align*}
H(z) &= H_0 h(z), \\
\rho_m(z) &= r_m(z) H_0^2, \\
\Lambda &= \lambda H_0^2, \\
\alpha &= \frac{\alpha_0}{H_0^2},
\end{align*}
\]
where \(H_0\) is the present value of the Hubble function, and \((r_m, \lambda, \alpha_0)\) represent the dimensionless matter density, and the dimensionless model parameters. Therefore eqs. (5.11)–(5.14)
Figure 1. Variation of the dimensionless Hubble function \( h(z) \) as a function of the redshift \( z \) for the model \( f(T, T) = \alpha TT + \Lambda \), for five different choices of the parameters \( \alpha_0 \) and \( \lambda \): \( \alpha_0 = -0.01, \lambda = -3 \) (solid curve), \( \alpha_0 = -0.02, \lambda = -3.5 \) (dotted curve), \( \alpha_0 = -0.03, \lambda = -4 \) (short-dashed curve), \( \alpha_0 = -0.04, \lambda = -4.5 \) (dashed curve), and \( \alpha_0 = -0.05, \lambda = -5 \) (long-dashed curve), respectively.

Eq. \((5.17)\) must be integrated with the initial condition \( h(0) = 1 \). In figures 1–5, we depict the corresponding results, namely the redshift-variation of the Hubble function, of the matter energy density, of the deceleration parameter, of the parameter of the dark energy equation of state, and of the total equation of state, respectively. We mention that for all these evolutions, we have numerically verified that the stability conditions extracted in section 4 are satisfied.

As one can see from the figures, depending on the values of the parameters \( \alpha \) and \( \Lambda \), the Universe can exhibit a very interesting dynamics. The Hubble function, presented in figure 1, is a monotonically decreasing function of time (monotonically increasing function of the redshift) during the entire evolution of the considered redshift range of the Universe. The scale factor is an increasing function of time, and the matter energy density, plotted in figure 2, tends to zero in the large-time limit. As one can see from figure 3, the dust filled Universe starts its evolution at the redshift \( z = 2 \) from a decelerating state, with \( q \approx 0.5 - 0.8 > 0 \). At around \( z \approx 0.5 \), \( q \approx 0 \), and the Universe enters in an accelerating phase, with \( q \) tending towards \(-1\) at around \( z = 0 \). This evolution is in agreement with the observed behavior of the recent Universe, namely a first decelerating matter dominated stage, a transition to accelerating expansion, and then the transition to late-time accelerating phase. Note that at asymptotically large times the Universe ends in a de Sitter expansion.

\[
(1 + z) \frac{dh}{dz} = \frac{(6\alpha_0 h^2 - 1) (9\alpha_0 h^2 - 1) (6\lambda^2 + \lambda)}{2 [\alpha_0 \lambda + 9\alpha_0 h^2 (\alpha_0 \lambda + 12\alpha_0 h^2 - 2) + 2]}. \tag{5.17}
\]

\[
r_m = \frac{3h^2 + \lambda/2}{1 - 9\alpha_0 h^2}, \tag{5.16}
\]

\[
w_{DE} = \frac{2 (9\alpha_0 h^2 - 1) [9\alpha_0 (3\alpha_0 \lambda - 4) h^4 - 18\alpha_0 \lambda h^2 + \lambda]}{(54\alpha_0 h^4 + \lambda) [\alpha_0 \lambda + 9\alpha_0 h^2 (\alpha \lambda + 12\alpha_0 h^2 - 2) + 2]}. \tag{5.19}
\]
Figure 2. Variation of the dimensionless matter energy density $r_m(z)$ as a function of the redshift $z$ for the model $f(T, T) = \alpha TT + \Lambda$, for five different choices of the parameters $\alpha_0$, and $\lambda$: $\alpha_0 = -0.01$, $\lambda = -3$ (solid curve), $\alpha_0 = -0.02$, $\lambda = -3.5$ (dotted curve), $\alpha_0 = -0.03$, $\lambda = -4$ (short-dashed curve), $\alpha_0 = -0.04$, $\lambda = -4.5$ (dashed curve), and $\alpha_0 = -0.05$, $\lambda = -5$ (long-dashed curve), respectively.

Figure 3. Variation of the deceleration parameter $q(z)$ as a function of the redshift $z$ for the model $f(T, T) = \alpha TT + \Lambda$, for five different choices of the parameters $\alpha_0$, and $\lambda$: $\alpha_0 = -0.01$, $\lambda = -3$ (solid curve), $\alpha_0 = -0.02$, $\lambda = -3.5$ (dotted curve), $\alpha_0 = -0.03$, $\lambda = -4$ (short-dashed curve), $\alpha_0 = -0.04$, $\lambda = -4.5$ (dashed curve), and $\alpha_0 = -0.05$, $\lambda = -5$ (long-dashed curve), respectively.

Figure 4. Variation of the parameter of the dark energy equation of state $w_{DE}(z)$ as a function of $z$ for the model $f(T, T) = \alpha TT + \Lambda$, for five different choices of the parameters $\alpha_0$, and $\lambda$: $\alpha_0 = -0.01$, $\lambda = -3$ (solid curve), $\alpha_0 = -0.02$, $\lambda = -3.5$ (dotted curve), $\alpha_0 = -0.03$, $\lambda = -4$ (short-dashed curve), $\alpha_0 = -0.04$, $\lambda = -4.5$ (dashed curve), and $\alpha_0 = -0.05$, $\lambda = -5$ (long-dashed curve), respectively.
The parameter $w_{\text{DE}}$ of the dark energy equation of state, presented in figure 4, shows a similar evolution, tending towards minus one at $z = 0$, when the Universe enters in a de Sitter phase, with its dynamics dominated by the effective dark energy component, mimicking a cosmological constant. Additionally, in figure 5 we present the total equation-of-state parameter $w = w_{\text{DE}}/(1 + \rho_m/\rho_{\text{DE}})$, and we can observe a dynamics similar to $w_{\text{DE}}$. Finally, for these specific parameter choices both the dark energy equation-of-state parameter, as well as the total one, lie in the quintessence regime, approaching the cosmological constant value $-1$ at large times (as $\rho_{\text{DE}}$ becomes larger and larger comparing to $\rho_m$, $w$ tends to coincide with $w_{\text{DE}}$).

We close this analysis by examining the limiting behavior of the model. In the limit $\alpha H^2 \ll 1$ and $\alpha \Lambda \ll 1$, eqs. (5.11) and (5.12) become

$$\rho_m = 3H^2 + \frac{\Lambda}{2},$$

$$\dot{H} = -\frac{3}{2}H^2 + \frac{\Lambda}{4}.\tag{5.21}$$

The above relationships, in the large-time limit and for $\Lambda < 0$, provide the standard de Sitter cosmological evolution, with $q = -1$, $H = H_0 = \sqrt{\Lambda/6}$ and $a \propto \exp(H_0 t)$. Note that this limit is valid independently of the $\alpha$-value. However, for $\alpha > 0$ the positivity of the matter energy density constraints the $\alpha$-values in the region that leads to $9\alpha H^2 < 1$.

On the other hand, for $\alpha H^2 \gg 1$ the matter energy density tends to

$$\rho_m = \frac{1}{3\alpha} + \frac{\Lambda}{18\alpha H^2},$$

while the dynamics of the Hubble function is determined by the equation

$$\dot{H} = -\frac{3}{2}H^2 + \frac{\Lambda}{4}.\tag{5.23}$$

Thus, the general solution given by

$$H(t) = \sqrt{\frac{\Lambda}{6}} \tanh \left[ \frac{\sqrt{6\Lambda}}{4} (t - 4C_1) \right],$$

where $C_1$ is an arbitrary constant of integration.
Figure 6. Variation of the dimensionless Hubble function $h(z)$ as a function of the redshift $z$ in the $f(T, T)$ gravity theory with $f(T, T) = \alpha \rho_m T^n + \Lambda$, for $\alpha_0 = -0.0011$, $\lambda = -5.5$, and for five different values of $n$: $n = 1$ (solid curve), $n = 2$ (dotted curve), $n = 3$ (short-dashed curve), $n = 4$ (dashed curve), and $n = 5$, respectively.

Figure 7. Variation of the dimensionless matter energy density $\rho_m(z)$ as a function of the redshift $z$ in the $f(T, T)$ gravity theory with $f(T, T) = \alpha \rho_m T^n + \Lambda$, for $\alpha_0 = -0.0011$, $\lambda = -5.5$, and for five different values of $n$: $n = 1$ (solid curve), $n = 2$ (dotted curve), $n = 3$ (short-dashed curve), $n = 4$ (dashed curve), and $n = 5$, respectively.

5.1.2 The case $n \neq 1$

Let us now investigate the effect of $n$ in the function $f(T, T) = \alpha T^n T + \lambda = \alpha T^n \rho_m + \Lambda$, on the cosmological evolution. In order to do so, we fix the values of $\alpha_0$ and $\lambda$ as $\alpha_0 = -0.0011$ and $\lambda = -5.5$, and we consider numerical solutions of eqs. (5.6) and (5.8) for different values of $n$, by adopting the redshift $z$ as the independent variable. In figures 6)–(10 we present the variations with the redshift of the Hubble function, of the matter energy density, of the deceleration parameter, of the dark energy equation-of-state parameter $w_{DE}$, and of the total equation-of-state parameter $w$, respectively, for $n = 1, 2, 3, 4, 5$.

Interestingly enough, we observe that even while in the behavior of the Hubble function, of the scale factor and of the matter energy density there are no major differences between all models with $n \in (1, 5)$, the dynamics of the Universe is very different for different values of $n$, as can be revealed by the behavior of the deceleration parameter. In particular, while for $n = 1$ the Universe starts its evolution from a decelerating phase, followed by an accelerating
Figure 8. Variation of the deceleration parameter $q(z)$ as a function of the redshift $z$ in the $f(T, T)$ gravity theory with $f(T, T) = \alpha \rho_m T^n + \Lambda$, for $\alpha_0 = -0.0011$, $\lambda = -5.5$, and for five different values of $n$: $n = 1$ (solid curve), $n = 2$ (dotted curve), $n = 3$ (short-dashed curve), $n = 4$ (dashed curve), and $n = 5$, respectively.

Figure 9. Variation of the parameter of the dark energy equation of state $w_{DE}(z)$ as a function of $z$ in the $f(T, T)$ gravity theory with $f(T, T) = \alpha \rho_m T^n + \Lambda$, for $\alpha_0 = -0.0011$, $\lambda = -5.5$, and for five different values of $n$: $n = 1$ (solid curve), $n = 2$ (dotted curve), $n = 3$ (short-dashed curve), $n = 4$ (dashed curve), and $n = 5$, respectively.

Figure 10. Variation of the total equation-of-state parameter $w$ as a function of $z$ in the $f(T, T)$ gravity theory with $f(T, T) = \alpha \rho_m T^n + \Lambda$, for $\alpha_0 = -0.0011$, $\lambda = -5.5$, and for five different values of $n$: $n = 1$ (solid curve), $n = 2$ (dotted curve), $n = 3$ (short-dashed curve), $n = 4$ (dashed curve), and $n = 5$, respectively.
begin their evolution in an accelerating phase, with \( q < 0 \) at \( z = 2 \), before entering in a de Sitter exponential expansion (\( q = -1 \)) at \( z = 0 \). However, the models with \( n > 1 \) exhibit a radical difference in the behavior of the dark energy sector, which is visible in the evolution of \( w_{\text{DE}} \). Specifically, \( w_{\text{DE}} \) can lie in the quintessence or phantom regime, depending on the value of \( n \). Thus, models that present a similar behavior in the global dynamics, can be distinguished by the behavior of the dark energy sector. Nevertheless, note that at late times \( w_{\text{DE}} \rightarrow -1 \) independently of the value of \( n \), and thus in order to distinguish the various models one should use \( w_{\text{DE}} \) at large redshifts. We mention that, as can be deduced from eqs. (5.6) and (5.8), independently of \( n \), once the condition

\[
\alpha \left( \frac{n + 1}{2} \right) \left( -6 H^2 \right)^n \ll 1
\]

is satisfied, for \( \Lambda \neq 0 \) the Universe results in the de Sitter accelerating stage, while for \( \Lambda = 0 \) its evolution ends in the Einstein-de Sitter, matter-dominated decelerating phase. Furthermore, from figure (9) and figure (10) notice the interesting behavior that the total equation-of-state parameter \( w \) and \( w_{\text{DE}} \) can be either quintessence-like or phantom-like, in all combinations. This is easily explained by recalling that \( w = w_{\text{DE}} / (1 + \rho_m / \rho_{\text{DE}}) \), and thus according to the signs of \( \rho_{\text{DE}} \) and \( p_{\text{DE}} \) all combinations are possible. Finally, the very similar behaviors that \( w_{\text{DE}} \) and \( w \) present in some subcases, result from the fact that in these subcases \( \rho_m \ll \rho_{\text{DE}} \), that is the universe is dark-energy dominated.

### 5.2 Model B: \( f(T, T) = \alpha T + \gamma T^2 \)

As a second model describing a simple departure from General Relativity in the framework of \( f(T, T) \) gravity we consider the case \( f(T, T) = \alpha T + \gamma T^2 = \alpha \rho_m + \gamma T^2 = \alpha \rho_m + \beta H^4 \), where \( \alpha \) and \( \beta = 36 \gamma \) are constants. In this case we obtain \( f_T = \beta T/18 = -\beta H^2/3 \), \( f_{TT} = \beta/18 \), \( f_T = \alpha \), and \( f_{TT} = 0 \), respectively. Thus, the matter energy density (5.1) becomes

\[
\rho_m = \frac{3 \left( 1 - \beta H^2/2 \right)}{1 + \alpha/2} H^2,
\]

while the time variation of the Hubble function (5.2) yields

\[
\dot{H} = \frac{3 (1 + \alpha) \left( 1 - \beta H^2/2 \right)}{\alpha + 2} \frac{H^2}{1 - \beta H^2},
\]

and therefore, the deceleration parameter (3.15) is given by

\[
q = \frac{3 (1 + \alpha) \left( 1 - \beta H^2/2 \right)}{\alpha + 2} \frac{1}{1 - \beta H^2} - 1.
\]

Additionally, the effective dark energy density and pressure, given by eqs. (3.6) and (3.7), respectively, can be obtained as

\[
\rho_{\text{DE}} = \frac{3 H^2 (\alpha + \beta H^2)}{\alpha + 2},
\]

\[
p_{\text{DE}} = -\frac{3 H^2 (\alpha + \beta H^2)}{(\alpha + 2) (\beta H^2 - 1)},
\]

resulting in the following dark energy equation-of-state parameter

\[
w_{\text{DE}} = \frac{1}{1 - \beta H^2}.
\]
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Figure 11. Variation of the dimensionless Hubble function $h(z)$ as a function of the redshift $z$ for the model $f(T, T') = \alpha T + \gamma T^2 = \alpha\rho_m + \beta T^2/36$, with $\beta = 36\gamma$, for $\alpha = -0.15$ and for five different choices of the parameter $\beta_0$: $\beta_0 = -0.10$ (solid curve), $\beta_0 = -0.15$ (dotted curve), $\beta_0 = -0.20$, (short-dashed curve), $\beta_0 = -0.20$ (dashed curve), and $\beta_0 = -0.25$ (long-dashed curve), respectively.

In order to examine the behavior of the above observables in a clearer way, we perform a numerical elaboration of the scenario at hand. We change the independent variable from the time $t$ to the redshift $z$, and we introduce a set of dimensionless variables $(h(z), r_m(z), \beta_0)$, defined as
\[ H(z) = h(z)H_0, \quad \rho_m(z) = r_m(z)H_0^2, \quad \beta = \frac{\beta_0}{H_0^2}. \] (5.31)

Therefore the basic equations describing the cosmological evolution of the model are
\[ r_m = \frac{3 (1 - \beta_0h^2/2) h^2}{1 + \alpha/2}, \] (5.32)
\[ (1 + z)\frac{dh}{dz} = \frac{3 (1 + \alpha) \left(1 - \beta_0h^2/2\right) h^2}{\alpha + 2 \left(1 - \beta_0h^2\right)}, \] (5.33)
\[ q = \frac{3 (1 + \alpha) \left(1 - \beta_0h^2/2\right)}{\alpha + 2 \left(1 - \beta_0h^2\right)} - 1, \] (5.34)
\[ w_{DE} = \frac{1}{1 - \beta_0h^2}. \] (5.35)

As before, eq. (5.33) must be integrated with the initial condition $h(0) = 1$.

In figures 11–15, we present the evolution of the Hubble function, of the matter energy density, of the deceleration parameter, of the dark-energy equation-of-state parameter, and of the total equation of state parameter, respectively. We mention that for all these evolutions, we have numerically verified that the stability conditions extracted in section 4 are satisfied.

The Hubble function, shown in figure 11, is monotonically decreasing in time (and monotonically increases with the redshift). As a result, the matter energy density depicted in figure 12, decreases monotonically in time. However, the deceleration parameter $q$, presented in figure 13, exhibits a a large variety of behaviors, depending on the values of $\alpha$ and $\beta$. In particular, the Universe can be purely accelerating or purely decelerating, or experience the transition from deceleration to acceleration. Finally, the evolution of $w_{DE}$, presented in figure 14, shows that during the entire cosmological evolution $w_{DE} > 0$, tending to 1 in the large-time limit. A significant difference can be observed in the behavior of the parameter of
Figure 12. Variation of the matter energy density $\rho_m(z)$ as a function of the redshift $z$ for the model $f(T,T) = \alpha T + \gamma T^2 = \alpha \rho_m + \beta T^2/36$, with $\beta = 36\gamma$, for $\alpha = -0.15$ and for five different choices of the parameter $\beta_0$: $\beta_0 = -0.10$ (solid curve), $\beta_0 = -0.15$ (dotted curve), $\beta_0 = -0.20$, (short-dashed curve), $\beta_0 = -0.20$ (dashed curve), and $\beta_0 = -0.25$ (long-dashed curve), respectively.

Figure 13. Variation of the deceleration parameter $q(z)$ as a function of the redshift $z$ for the model $f(T,T) = \alpha T + \gamma T^2 = \alpha \rho_m + \beta T^2/36$, with $\beta = 36\gamma$, for $\alpha = -0.15$ and for five different choices of the parameter $\beta_0$: $\beta_0 = -0.10$ (solid curve), $\beta_0 = -0.15$ (dotted curve), $\beta_0 = -0.20$, (short-dashed curve), $\beta_0 = -0.20$ (dashed curve), and $\beta_0 = -0.25$ (long-dashed curve), respectively.

Figure 14. Variation of the dark energy equation-of-state parameter $w_{DE}(z)$ as a function of the redshift $z$ for the model $f(T,T) = \alpha T + \gamma T^2 = \alpha \rho_m + \beta T^2/36$, with $\beta = 36\gamma$, for $\alpha = -0.15$ and for five different choices of the parameter $\beta_0$: $\beta_0 = -0.10$ (solid curve), $\beta_0 = -0.15$ (dotted curve), $\beta_0 = -0.20$, (short-dashed curve), $\beta_0 = -0.20$ (dashed curve), and $\beta_0 = -0.25$ (long-dashed curve), respectively.
Figure 15. Variation of the parameter $w$ of the total equation-of-state as a function of the redshift $z$ for the model $f(T, T) = \alpha T + \gamma T^2 = \alpha \rho_m + \beta T^2 / 36$, with $\beta = 36\gamma$, for $\alpha = -0.15$ and for five different choices of the parameter $\beta$: $\beta_0 = -0.10$ (solid curve), $\beta_0 = -0.15$ (dotted curve), $\beta_0 = -0.20$ (short-dashed curve), $\beta_0 = -0.20$ (dashed curve), and $\beta_0 = -0.25$ (long-dashed curve), respectively.

the total equation of state $w$ in figure 15, which has an opposite sign as compared to $w_{DE}$, although still in the quintessence regime. This can be explained by our particular choice of the parameters $\alpha < 0$ and $\beta < 0$ in eq. (5.28), which renders the dark energy density negative during the considered cosmological evolution period. As a result, the ratio $\rho_m/\rho_{DE} < 0$, and thus $w < 0$. On the other hand $w_{DE}$ is positive for the considered cosmological model, since it is the ratio of two negative quantities.

We close this subsection by referring to the limiting behavior of the model at hand. First of all, the positivity of the matter energy-density implies that for positive values of $\alpha$ and $\beta$ we must have $\beta H^2 / 2 < 1$. Additionally, for $\alpha < -2$ no negative values of $\beta$ are allowed, and the Hubble function must satisfy the constraint $\beta H^2 / 2 \geq 1$. For small $H$, that is at late times, and in particular for the time interval of the cosmological evolution for which $\beta H^2 / 2 \ll 1$, the Hubble function satisfies the equation $\dot{H} \approx -3 (1 + \alpha) H^2 / (\alpha + 2)$, giving $H = [(\alpha + 2) / 3 (1 + \alpha)] (1/t)$, $a \propto t^{(\alpha+2)/3(1+\alpha)}$, and $q \approx (1 + 2\alpha) / (\alpha + 2)$. Thus, the deceleration parameter is negative for $\alpha \in (-2, -1/2)$, however the accelerating phase is not of a de Sitter type, but it is described by a simple power-law expansion.

6 Conclusions

In the present paper, we have introduced a generalization of the $f(T)$ gravitational theory by allowing a general non-minimal coupling between the torsion scalar $T$ and the trace of the matter energy-momentum tensor $T$. The resulting $f(T, T)$ theory is different from $f(T)$ gravity, from the curvature-based $f(R, T)$ gravity [83], as well as from the recently constructed nonminimally torsion-matter coupled theory where $T$ is coupled to the matter Lagrangian $L_m$ instead of $T$ [99]. Therefore, it is a novel modified gravitational theory. Note that the only restriction imposed on $f$ is the requirement that it is an analytic function, that is, $f(T, T)$ is a real function that is locally given by a convergent power series, and it is infinitely differentiable.

In investigating the physical implications of the theory, in the present paper we focused on its cosmological implications. The cosmological equations, obtained for a flat Friedmann-Robertson-Walker type geometry, are a generalization of both the standard Friedmann equations of General Relativity, as well as of those of simple $f(T)$ gravity. The coupling between
the torsion scalar and the trace of the matter energy-momentum tensor contributes with new terms in the effective dark energy density pressure. More specifically, supplementary terms, proportional to the derivatives of $f$ with respect to $T$ and $\mathcal{T}$ appear in the cosmological field equations. The important feature is that the effective dark energy sector acquires a contribution from both the $f(T)$ terms, as well as from the matter energy density and pressure. Due to the extra freedom in the imposed Lagrangian, $f(T, \mathcal{T})$ cosmology allows for a very wide class of scenarios and behaviors. Finally, a detailed study of the scalar perturbations at the linear level reveals that $f(T, \mathcal{T})$ cosmology can be free of ghosts and instabilities for a wide class of ansatzes and model parameters. 

As applications, we investigated two specific $f(T, \mathcal{T})$ models, corresponding to simple departures from General Relativity. In particular, we examined the case where $f$ is chosen to be proportional to the product of the energy-momentum trace and the torsion scalar at some power, and the case where $f$ is the sum of the trace of the energy-momentum tensor and the square of the torsion scalar. We focused on expanding evolutions, bearing in mind that contracting or bouncing solutions can be also acquired.

We found a large variety of interesting cosmological behaviors, depending on the model parameters. For instance, we found specifically evolutions experiencing a transition from a decelerating to an accelerating state, capable of describing the late-time cosmic acceleration and the dark energy epoch. Additionally, we found evolutions where an initial accelerating phase is followed by a decelerating one, with a subsequent transition to a final acceleration at late times, a behavior in agreement with the observed thermal history of the Universe, namely a first inflationary stage, a transition to non-accelerating, matter-dominated expansion, and then the transition to late-time accelerating phase. Thus, $f(T, \mathcal{T})$ cosmology offers a unified description of the universe evolution.

An additional advantage of the scenario at hand, revealing its capabilities, is that the dark energy equation-of-state parameter can lie in the quintessence or phantom regime. Moreover, for models with similar expansion features, $w_{\text{DE}}$ may behave very differently, offering a way to distinguish them. Finally, at late times the universe results either to a de Sitter exponential expansion, or to eternal power-law accelerated expansions, with zero matter density, namely, with a complete effective dark-energy domination.

We close this work by mentioning that the present work is just a first presentation of $f(T, \mathcal{T})$ gravity and cosmology. In order for this theory to be a candidate for the description of Nature, many relevant investigations are necessary. In particular one should perform a detailed comparison with cosmological observations (for instance using data from Type Ia Supernovae (SNIa), Baryon Acoustic Oscillations (BAO), and Cosmic Microwave Background (CMB), along with requirements of Big Bang Nucleosynthesis (BBN)), which could constrain the allowed ansatzes and parameter ranges. Furthermore, after extracting the spherically symmetric solutions, one could confront $f(T, \mathcal{T})$ gravity with Solar System data. Additionally, one could use the scalar perturbation equations extracted in the present work in order to perform a detailed confrontation with the growth-index data. Moreover, one could extend the perturbation analysis to the vector and tensor modes, and use them in order to predict the inflationary induced tensor-to-scalar ratio, especially under the recent BICEP2 measurements that can exclude a large class of models [108]. These necessary studies lie beyond the scope of the present work, and are left for a separate project.
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A Coefficients of the stability equation

In this appendix, we give the coefficients $\Gamma$, $\mu^2$, $c_s^2$ and $D$ of the perturbation equation (4.24):

$$\ddot{\phi}_k + \Gamma \dot{\phi}_k + \mu^2 \phi_k + c_s^2 \frac{K^2}{a^2} \ddot{\phi}_k = D. \quad (A.1)$$

Concerning the effective mass we have

$$\mu^2 = \mu_{(1)}^2 + \mu_{(2)}^2 + \mu_{(3)}^2 + \mu_{(4)}^2 + \mu_{(5)}^2 + \mu_{(6)}^2, \quad (A.2)$$

with

$$\mu_{(1)}^2 = 2\dot{H}(f_T + 1) \frac{B}{E} + 2H(\dot{\rho}_m - 3\dot{\rho}_m) f_{TTT} \frac{B^2}{F}$$

$$+ H^2 \left[ 4\dot{H} \left( 3Af_{TT} - 5Bf_{TTT} \right) \frac{3B}{F} + Af_T + Bf_T + B \right], \quad (A.3)$$

$$\mu_{(2)}^2 = \frac{12H^3}{F} \left\{ 8\dot{\rho}_m f_{TT} (Bf_{TT} - Af_{TT}) + (\dot{\rho}_m - 3\dot{\rho}_m) \left[ 2p_m f_{TT} (Bf_{TTT} - Af_{TTT}) + Bf_{TT} - 3Bf_T \right] \right\}, \quad (A.4)$$

$$\mu_{(3)}^2 = \frac{12H^4}{F} \left\{ 4f_{TT}^2 \left[ 9A\dot{H} + B(p_m + \rho_m) \right] + f_{TT} \left[ 2B(f_T + 1) + AB \right. \right.$$

$$\left. + 2A\dot{H} \left( (p_m + \rho_m)(Af_{TTT} - Bf_{TTT}) - 3Bf_{TT} \right) \right\}$$

$$\left. - 3B \left[ 4\dot{H}(Af_{TT} - Bf_{TTT}) + f_{TT}(f_T + B) \right] \right\}, \quad (A.5)$$

$$\mu_{(4)}^2 = \frac{144H^5}{F} \left\{ 8\dot{p}_m f_{TT} f_{TTT} f_{TT} + (\dot{\rho}_m - 3\dot{\rho}_m) \left\{ f_{TTT} \left( Af_{TTT} - Bf_{TTT} \right. \right.$$

$$\left. + f_{TT} \left[ 2(p_m + \rho_m)f_{TTT} + 3f_{TT} \right] \right\} - Bf_{TTT} f_{TT} \right\}, \quad (A.6)$$

$$\mu_{(5)}^2 = -\frac{432H^6}{F} \left\{ f_{TT} \left[ 4\dot{H}(Af_{TT} - Bf_{TTT}) + f_{TT} \left[ 8\dot{H}(p_m + \rho_m)f_{TTT} + B \right] \right. \right.$$

$$\left. - 4B\dot{H} f_{TTT} f_{TTT} + 12\dot{H} f_{TT} f_{TT} \right\}, \quad (A.7)$$
and
\[ \mu^2_{(6)} = \frac{1728}{F} H^T f_{TTT} [12H f_{TTT} + (3\dot{p}_m - \dot{\rho}_m) f_{TTT}], \] 
respectively.

Concerning the sound speed, we have
\[ c_s^2 = c_{s(1)}^2 + c_{s(2)}^2 + c_{s(3)}^2 + c_{s(4)}^2, \]
with the following relations
\[ c_{s(1)}^2 = \frac{(f_T + 1)}{E} \left( 4H f_{TTT} + f_T \right), \] 
\[ c_{s(2)}^2 = \frac{4H}{f_T} \left\{ B(\dot{\rho}_m - 3\dot{p}_m) (f_T + 1)f_{TTT} + f_{TTT} \{ B(\dot{\rho}_m - 3\dot{p}_m) f_{TTT} \\
+ (f_T + 1) [(\dot{\rho}_m - 3\dot{p}_m) f_{TTT} + 2(p_m + \rho_m)(3\dot{p}_m - \dot{\rho}_m)f_{TTT}] \} \right\}, \] 
\[ c_{s(3)}^2 = \frac{4H^2}{f_T} \left\{ -12BH [f_{TTT} (f_T + 1)f_{TTT}] \\
+ f_{TTT}(f_T + 1) \left\{ 12\dot{H} [2(p_m + \rho_m)f_{TTT} + 3f_{TTT}] + B \right\} \right\}, \] 
\[ c_{s(4)}^2 = -\frac{48H^3}{f_T} (3\dot{p}_m - \dot{\rho}_m) (f_T + 1)f_{TTT} f_{TTT}, \] 
respectively.

Concerning the frictional coefficient we have
\[ \Gamma = \Gamma^{(1)} + \Gamma^{(2)} + \Gamma^{(3)} + \Gamma^{(4)} + \Gamma^{(5)} + \Gamma^{(6)}, \] 
with
\[ \Gamma^{(1)} = \frac{f_{TTT}}{F} \left[ 20736 H^T \dot{H} f_{TTT} f_{TTT} - (3\dot{p}_m - \dot{\rho}_m) (B^2 - 1728H^6 f_{TTT} f_{TTT}) \right], \] 
\[ \Gamma^{(2)} = \frac{H}{E} \left\{ 4 \left[ \dot{H} (6Af_{TT} - 9Bf_{TT}) + Bf_T + B \right] + 3Af_T + \frac{4k^2}{a^2} (f_T + 1) f_{TTT} \right\}, \] 
\[ \Gamma^{(3)} = \frac{\mu_{(2)}^2}{H}, \] 
\[ \Gamma^{(4)} = \frac{12H^3}{F} \left\{ 36\dot{H} f_{TT}^2 + 4ABf_{TT} - B[12H(Af_{TTT} - Bf_{TTT}) \\
+ f_{TT} (3f_T + 4B)] - 12\dot{H} f_{TTT} [5Bf_{TT} + 2(p_m + \rho_m)(Bf_{TTT} - Af_{TTT})] \right\}, \] 
\[ \Gamma^{(5)} = \frac{\mu_{(4)}^2}{H}, \] 
and
\[ \Gamma^{(6)} = \frac{576H^5}{F} \left\{ 3\dot{H} f_{TTT} f_{TTT} - 9\dot{H} f_{TTT} f_{TTT} - f_{TTT} [3\dot{H}(Af_{TTT} - Bf_{TTT}) \\
+ 6\dot{H}(p_m + \rho_m)f_{TTT} f_{TTT} + Bf_{TTT}] \right\}, \] 
respectively.
The coefficient \( D \) of the right-hand side of (A.1) is given by

\[
D = -D_1 \delta \tilde{p}_m^k - D_2 \delta \tilde{\rho}_m, 
\]

where

\[
D_1 = \frac{H f_{TT}}{E} (I + 36H^2 f_{TT}), 
\]

and

\[
D_2 = D_2^{(1)} + D_2^{(2)} + D_2^{(3)} + D_2^{(4)} + D_2^{(5)} + D_2^{(6)},
\]

with

\[
D_2^{(1)} = \frac{1}{4E} \left[ (4H f_{TT} + f_T)I - 16\pi GB \right],
\]

\[
D_2^{(2)} = -\frac{H}{F} \{ \hat{\rho}_m \{ f_{TT} (3I - 8B) + 2I(p_m + \rho_m) f_{TT} - B f_{TT} \} \\
- \hat{p}_m \{ f_{TT} (I - 24B) + 6I(p_m + \rho_m) f_{TT} - 3B f_{TT} \} \},
\]

\[
D_2^{(3)} = \frac{3H^2}{F} \left\{ 4H \left[ f_{TT}^2 (3I - 5B) - 1B f_{TT} \right] + f_{TT} \{ 8f_T \left[ 5\dot{H}(p_m + \rho_m) f_{TT} + B \right] \\
+ 3 \left[ B - 2(p_m + \rho_m) f_{TT} \right] \left[ 8\dot{H}(p_m + \rho_m) f_{TT} + B \right] \} \right\},
\]

\[
D_2^{(4)} = \frac{12H^3 f_{TT}}{F} \left\{ (\hat{\rho}_m - 3\hat{p}_m) f_{TT} (I + 3B) + 3f_{TT} \left[ (\hat{p}_m - 3\hat{p}_m) f_{TT} \\
+ 2(p_m + \rho_m)(3\hat{p}_m - \hat{\rho}_m) f_{TT} \right] \right\},
\]

\[
D_2^{(5)} = \frac{36H^4 f_{TT}}{F} \left\{ 3f_{TT} B - 4\dot{H} f_{TT} (I + 3B) + 3f_{TT} \left\{ 4\dot{H} \left[ 2(p_m + \rho_m) f_{TT} + 3f_{TT} \right] \right\} \right\},
\]

and

\[
D_2^{(6)} = -\frac{432H^5 f_{TT}^2}{F} \left\{ 12\dot{H} \dot{f}_{TT} + (3\hat{p}_m - \hat{\rho}_m) f_{TT} \right\},
\]

respectively.

Finally, in all the above expressions we have introduced the coefficients

\[
A \equiv 2(p_m + \rho_m) f_{TT} + f_T + 1,
\]

\[
B \equiv 2 \left[ 8\pi G + (p_m + \rho_m) f_{TT} - 6H^2 f_{TT} \right] + f_T,
\]

\[
E \equiv 12H^2 \left[ A f_{TT} - f_{TT} (12H^2 f_{TT} + B) \right] + B (f_T + 1),
\]

\[
I \equiv -6(p_m + \rho_m) f_{TT} + 5f_T + 3B,
\]

\[
F \equiv BE,
\]

respectively.
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