GALOIS ACTIONS ON HIGHER DESSINS D’ENFANTS

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ABSTRACT. Just as dessins d’enfants parameterise coverings of a two-sphere with three ramification points, higher dessins d’enfants parameterise such coverings with an arbitrary number of ramification points. We study operations of absolute Galois groups upon them which are constructed similarly to Grothendieck’s Galois operation on classical dessins d’enfants. It turns out that these operations truly depend, in quite a strong way, on choosing a complex structure on the covered sphere. Furthermore, a generalisation of Belyi’s theorem to arbitrary finite subsets of \( \mathbb{P}^1(\overline{\mathbb{Q}}) \) with at least three elements is proved, and we present a connection between the Galois action on 4-dessins d’enfants and complex multiplication theory.

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1. INTRODUCTION

Dessins d’enfants are certain coloured graphs embedded in surfaces that encode ramified coverings of two-spheres with three ramification points. They have been introduced by Alexander Grothendieck in [10] with the aim of studying a certain rather mysterious action upon them by the absolute Galois group of the rational number field. Since then they have proved relevant to many different areas of mathematics.

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The notion of dessins d’enfants can be generalised so as to parameterise also ramified coverings of two-spheres with an arbitrary finite number of ramification points; the corresponding graph-theoretical objects have been termed “n-dessins d’enfants” or “higher dessins d’enfants” in this work. This idea is not new; it has been used in Pervova–Petronio [24] to solve some topological problems. The novelty is that we begin to study Galois operations similar to Grothendieck’s one on these higher dessins d’enfants. We assume throughout the text that the reader is familiar with the classical theory of dessins d’enfants and only explain those aspects of the theory in detail which differ from the classical case.

The main differences are twofold. Firstly, there are two distinct notions of isomorphisms for ramified coverings $S \to \mathbb{S}^2$, depending on whether one only allows diffeomorphisms of $S$ or also diffeomorphisms of $\mathbb{S}^2$ to play their game. While the former leads to the most natural notion of isomorphisms for higher dessins d’enfants, the latter is in some other respects more natural and corresponds to taking orbits under a braid group action on higher dessins d’enfants. Secondly, there is essentially only one complex structure on a thrice-punctured sphere whereas there is an $(n-3)$-dimensional family of complex structures on an $n$-punctured sphere. Each of them gives rise to a different Galois operation on higher dessins d’enfants. This is the main technical result of this article, the precise formulation being in Theorems 6.7 and 6.8. Also each Galois action intersects the braid group action only in the obvious elements, see Theorem 6.6. These results are obtained from previously known corresponding statements for Galois actions on étale fundamental groups, using a theorem of Jarden [14] in group theory.

Besides the setup of the machinery needed to prove these results, we present a generalisation of Belyï’s theorem, see Theorem 5.3 and a more geometric interpretation of 4-dessins. These are closely related to origamis or square-tiled surfaces, but not quite the same. We discuss some links with complex multiplication theory in section 8. This gives a more geometric interpretation of some Galois actions on 4-dessins, with the involved fields being certain abelian extensions of imaginary quadratic number fields. This section includes some explicit evaluations of the modular $\lambda$-function that seem not to have appeared elsewhere.

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2. Higher Dessins d’Enfants

We begin by introducing the purely topological notion of $n$-dessins d’enfants, a natural generalisation of Grothendieck’s dessins d’enfants. Since these have appeared in the literature otherwise, e.g. in Pervova–Petronio [24], we only give a short summary of their properties.

Higher Dessins. We need to choose one particular model of an oriented two-sphere with $n$ marked points lying on a circle, enumerated cyclically by $\mathbb{Z}/n\mathbb{Z}$. We set

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \text{ and } \mathbb{S}^1 = \mathbb{S}^2 \cap \{z = 0\},$$
where $S^2$ obtains its orientation from the canonical one on the plane $\{z = 0\}$ by stereographic projection through the “south pole” $(0, 0, -1)$. The marked points are

$$P_\nu = P_\nu^{(n)} = \left( \cos \frac{2\pi \nu}{n}, \sin \frac{2\pi \nu}{n}, 0 \right) \in S^1 \text{ for } \nu \in \mathbb{Z}/n\mathbb{Z}.$$  

Further, we denote the set of all marked points by

$$\mathcal{P}^{(n)} = \{ P_0^{(n)}, P_1^{(n)}, \ldots, P_{n-1}^{(n)} \}$$

and the segments that $S^1$ is cut into by $\mathcal{P}^{(n)}$ by

$$\sigma_\nu = \sigma_\nu^{(n)} = \{ (\cos t, \sin t, 0) \mid \frac{2\pi \nu}{n} \leq t \leq \frac{2\pi (\nu + 1)}{n} \}.$$  

The complement $S^2 \setminus S^1$ has two components, the upper hemisphere $\mathbb{D}^+ = S^2 \cap \{ z > 0 \}$ and the lower hemisphere $\mathbb{D}^- = S^2 \cap \{ z < 0 \}$. The entire construction is pictured for $n = 6$ in Figure 1.

Now we are ready to introduce “higher dessins” that classify ramified coverings of $S^2$, unramified outside $\mathcal{P}^{(n)}$:

**Definition 2.1.** Let $n \geq 3$. An $n$-dessin d’enfant on a smooth connected closed oriented surface $S$ is an embedded finite graph (i.e., one-dimensional cell complex) $\Gamma \subset S$ together with a colouring of the vertices (i.e., 0-cells) of $\Gamma$, written $c: \mathcal{V}(\Gamma) \to \mathbb{Z}/n\mathbb{Z}$, such that the following axioms are satisfied:

(i) Each component of $S \setminus |\Gamma|$ is homeomorphic to a disc, and $\partial C$ contains precisely $n$ vertices.

(ii) For every vertex $v$, consider all vertices $w$ neighbouring $v$. For these, $c(w) \equiv c(v) \pm 1 \mod n$, with the sign alternating as we go through all $w$ in the cyclic ordering determined by the embedding into $S$.

The process just described gives, for every finite covering $f: S \to S^2$ unramified outside $\mathcal{P}^{(n)}$, an $n$-dessin d’enfant on $S$. Here we endow a point of $S$ above $P_\nu$ with the “colour” $\nu$. For the inverse construction, let $S$ be a closed oriented surface and let $\Gamma$ be an $n$-dessin on $S$, with colouring $c: \mathcal{V}(\Gamma) \to \mathbb{Z}/n\mathbb{Z}$. We define a map $f: S \to S^2$ stepwise by first setting $f(v) = P_{c(v)}^{(n)}$ for $v \in \mathcal{V}$. If $e \subset \Gamma$ is an edge, let $v$ and $w$ its two adjacent vertices; then up to exchanging...
classes of coverings

The above correspondence defines mutually inverse bijections between the set of isomorphism classes of coverings of \( S \) and \( |\Gamma| \) taking \( \Gamma \) to \( \Gamma' \), respecting the colourings.

Two coverings \( f: S \to S_2 \) and \( f': S' \to S_2 \) which are unramified outside \( \mathcal{P}^{(n)} \) are (strictly) isomorphic if there exists a diffeomorphism \( \varphi: S \to S' \) with \( f = f' \circ \varphi \).

The above correspondence defines mutually inverse bijections between the set of isomorphism classes of coverings \( f: S \to S_2 \) unramified outside \( \mathcal{P}^{(n)} \) and the set of strict isomorphism classes of \( n \)-dessins d’enfants.

Subgroups of Free Groups. We can also encode higher dessins or ramified coverings of spheres in terms of subgroups of free groups. This point of view will be important later on.

Assume that \( f: S \to S_2 \) is a covering unramified outside \( \mathcal{P}^{(n)} \). Then its restriction \( f': S' \to S_2 \setminus \mathcal{P}^{(n)} \), where \( S' = S \setminus f^{-1}(\mathcal{P}^{(n)}) \), is an unramified covering of finite degree. Vice versa, every unramified covering of finite degree \( S' \to \mathcal{P}^{(n)} \) can be completed to a possibly ramified covering of \( S_2 \) in a unique way. Next, we observe that finite unramified coverings of \( S_2 \setminus \mathcal{P}^{(n)} \) correspond to conjugacy classes of finite index subgroups of the fundamental group of \( S_2 \setminus \mathcal{P}^{(n)} \). This is a free group of rank \( n - 1 \). Denoting by \( F_r \) the abstract free group on \( r \) letters, we fix once and for all a base point \( s \in S_2 \setminus \mathcal{P}^{(n)} \) and an isomorphism \( F_{n-1} \to \pi_1(S_2 \setminus \mathcal{P}^{(n)}, s) \).

Hence isomorphism classes of \( n \)-dessins correspond bijectively to conjugacy classes of finite index subgroups of \( F_{n-1} \). Note that finite index subgroups themselves correspond to finite marked coverings, which are identified with \( n \)-dessins \((S, \Gamma, c)\) together with one marked component of \( S \setminus |\Gamma| \).

We note a simple finiteness statement about higher dessins d’enfants. Let us say that an \( n \)-dessin has degree \( d \) if the corresponding covering of \( S_2 \) has degree \( d \).

**Proposition 2.3.** For every \( n \geq 3 \) and every \( d \geq 1 \), the set \( \mathcal{D}_n^d \) of isomorphism classes of \( n \)-dessins d’enfants of degree \( d \) is finite.

As a consequence, the set \( \mathcal{D}_n = \bigcup_{d=1}^{\infty} \mathcal{D}_n^d \) of all isomorphism classes of \( n \)-dessins is countably infinite. \( \square \)
3-Dessins d’Enfants and Classical Dessins d’Enfants. For \( n = 3 \), Definition 2.1 gives not quite the usual notion of dessins d’enfants, but a notion which is easily seen to be equivalent. From a 3-dessin we pass to a dessin by erasing all vertices coloured with \( 0 \in \mathbb{Z}/3\mathbb{Z} \) and all edges that lead to such vertices. The reverse operation is well-defined up to isotopy. Up to this minor reformulation, Definition and Proposition 2.2 generalises Grothendieck’s correspondence between dessins d’enfants and coverings of a sphere with three ordered ramification points.

Bipartite Origamis. Just as dessins d’enfants correspond to triangulations, \( n \)-dessins d’enfants correspond to what might be called “complexes of \( n \)-gons”. We leave it to the reader to spell this out for general \( n \) and only give a detailed account for \( n = 4 \). We begin with a slightly different model for our sphere with four marked points: the pillowcase surface \( \Pi \). This is obtained by gluing the unit square \( Q = [0, 1] \times [0, 1] \subset \mathbb{R}^2 \) along its boundary to another copy of \( Q \), but with opposite orientation (denoted by \( -Q \)). The resulting surface is topologically an oriented sphere. It has four special points, the vertices of \( Q \); we enumerate them counter-clockwise by \( \mathbb{Z}/4\mathbb{Z} \) in such a way that \((0, 0)\) is labelled \( 0 \). Finally there is a canonical topological circle \( \partial Q \subset \Pi \) containing these four points.

There is an orientation-preserving homeomorphism \( S^2 \to \Pi \) taking \( S^1 \) to \( \partial Q \) and \( P^{(4)}_\nu \) to the vertex of \( Q \) labelled \( \nu \). This is unique up to isotopy preserving these constraints, and it maps the interior of \( \pm Q \) to \( \mathbb{D}^\pm \). Colouring \( Q \) white and \( -Q \) grey, we obtain Figure 2.

Then, via this homeomorphism, strict isomorphism classes of ramified coverings of \( (S^2, Q^{(4)}) \) (and therefore of 4-dessins d’enfants) correspond to strict isomorphism classes of ramified coverings of \( \Pi \), unramified outside the vertices of \( Q \). But if \( f : S \to \Pi \) is such a covering, the colouring of \( \Pi \) and its marked points lift to a decomposition of \( S \) into squares, coloured alternately grey and white, and with distinguished “upper”, “lower”, “left” and “right” edges, glued consistently. The precise notion we arrive at is this:

**Definition 2.4.** A bipartite origami \( \mathcal{O} \) consists of a finite number of unit squares coloured grey and white, with the set of all their edges partitioned into pairs, such that the following constraints are satisfied:

(i) If an edge of a square \( Q_1 \) is paired with an edge of \( Q_2 \), then \( Q_1 \) and \( Q_2 \) have different colours.

(ii) Every right / left / upper / lower edge of a white square is paired with a left / right / upper / lower (sic!) edge of a grey square, respectively.

(iii) Glueing each pair of edges in the unique orientation-preserving and isometric way, we obtain a connected surface \( S(\mathcal{O}) \).
There is an obvious notion of isomorphism for bipartite origamis, corresponding to isomorphism for 4-dessins: a bijection between the respective sets of squares, respecting all the additional data.

We can then easily pass between bipartite origamis and 4-dessins as follows: for a bipartite origami $\mathcal{O}$, the associated 4-dessin is $(S, \Gamma, c)$ where $S = S(\mathcal{O})$ as above, $\Gamma$ is the union of all edges of the squares of $\mathcal{O}$, $V(\Gamma)$ is the set of all vertices of the squares of $\mathcal{O}$ and the “colouring” $c: V(\Gamma) \to \mathbb{Z}/4\mathbb{Z}$ is as in Figure 2. This construction is easily seen to be invertible.

As examples that will be used later on, we present two bipartite origamis $\mathcal{O}$ and $\mathcal{O}'$ and the corresponding 4-dessins $\Gamma$ and $\Gamma'$ in Figure 3. The origamis are drawn in such a way that “left” edges are really on the left, and so on; some of the squares are already drawn as glued together, the remaining gluing pairs are indicated by letters $a, b, c, \ldots$. On the other side, the 4-dessins are drawn as lying in the one-point compactification of $\mathbb{R}^2$. The squares of $\mathcal{O}$ and the connected components of $S^2 \setminus |\Gamma|$ are labelled by $A, B, C, \ldots$ (similarly for $\mathcal{O}'$ and $\Gamma'$); this is only to indicate which square corresponds to which component and does not provide any additional structure on either side.

### 3. Braid Group Actions

The theory developed so far has the drawback that a ramified covering of an “abstract” sphere does not readily give a higher dessin d’enfant; to do so, we need to choose an identification of such an abstract sphere, together with the ramification points of the coverings, with $(S^2, \Psi^{(n)})$. Choosing different such identifications will give different graphs on the covering surface, and this ambiguity is precisely measured by a braid group action on higher dessins.
DEFINITION 3.1. Let $S$ be a closed oriented surface and $\mathfrak{P} \subset S$ a finite subset. The non-oriented pure mapping class group of $(S, \mathfrak{P})$ is the quotient
\[ \text{PMod}^{\pm}(S, \mathfrak{P}) = \text{Homeo}(S, \mathfrak{P}) / \text{Homeo}^0(S, \mathfrak{P}), \]
where $\text{Homeo}(S, \mathfrak{P})$ is the group of all homeomorphisms $S \to S$ fixing $\mathfrak{P}$ pointwise and where $\text{Homeo}^0(S, \mathfrak{P})$ is the subgroup of those homeomorphisms which are isotopic to the identity through an isotopy fixing $\mathfrak{P}$ pointwise.

It contains the index-two subgroup $\text{PMod}^+(S, \mathfrak{P})$ of orientation-preserving mapping classes which is usually just called the (pure) mapping class group of $(S, \mathfrak{P})$. For $(S, \mathfrak{P}) = (S^2, \mathfrak{P}^{(n)})$, this subgroup can be identified with the pure spherical braid group on $n$ strands; for details about this and general facts about mapping class groups see the textbooks by Birman [3] and Farb–Margalit [6].

THE ACTION ON HIGHER DESSINS. Now we describe a natural action of $\text{PMod}^\pm(S^2, \mathfrak{P}^{(n)})$ on isomorphism classes of $n$-dessins. Let $(S, \Gamma, c)$ be an $n$-dessin, and let $f : S \to S^2$ be the associated ramified covering. Let further $[\psi] \in \text{PMod}^\pm(S^2, \mathfrak{P}^{(n)})$ be represented by the homeomorphism $\psi : S^2 \to S^2$. Then $\psi \circ f : S \to S^2$ is again a covering unramified outside $\mathfrak{P}^{(n)}$ and therefore defines an isomorphism class of $n$-dessins; we let this class be $[\psi] \ast (S, \Gamma, c)$. To put this into some formalism, let $\mathcal{D}_n$ as above be the set of all isomorphism classes of $n$-dessins, and let $\mathfrak{S}(\mathcal{D}_n)$ be the group of all bijections $\mathcal{D}_n \to \mathcal{D}_n$. Then the action we have constructed can be thought of as a group homomorphism
\[ \varpi_n : \text{PMod}^\pm(S^2, \mathfrak{P}^{(n)}) \to \mathfrak{S}(\mathcal{D}_n). \] (1)
We shall see below that this homomorphism is injective. The correspondence between $n$-dessins and ramified coverings of $(S^2, \mathfrak{P}^{(n)})$ descends to another correspondence

DEFINITION AND PROPOSITION 3.2. Let $n \geq 3$. Two coverings $f : S \to S^2$ and $f' : S' \to S^2$ which are unramified outside $\mathfrak{P}^{(n)}$ are called weakly isomorphic if there exist orientation-preserving diffeomorphisms $\varphi : S \to S'$ and $\psi : S^2 \to S^2$ such that $\psi \circ f = f' \circ \varphi$.

The correspondence in Definition and Proposition 2.2 descends to a bijection between the quotient $\text{PMod}^+(S^2, \mathfrak{P}^{(n)}) \setminus \mathcal{D}_n$ and the set of weak isomorphism classes of coverings of $S^2$ unramified outside $\mathfrak{P}^{(n)}$. \hfill $\square$

We leave it to the reader to spell out a corresponding statement for orbits under the non-oriented pure mapping class group.

EXPLICIT FORMULATION FOR BIPARTITE ORIGAMIS. We shall examine more closely the mapping class group operation on higher dessins in the cases $n = 3$ and $n = 4$. To begin with, there is a canonical involution $\iota \in \text{PMod}^\pm(S^2, \mathfrak{P}^{(n)})$ providing a splitting of the exact sequence
\[ 1 \to \text{PMod}^+ \to \text{PMod}^\pm \to \{\pm 1\} \to 1. \]
It is represented by inversion at the circle $S^1 \subset S^2$. Since this map leaves $S^1$ fixed, $\varpi_n(\iota)$ simply sends each $n$-dessin to the $n$-dessin with the same underlying topological surface $S$ and the same underlying graph and colouring, but where the orientation of $S$ is reversed.

The fact that $\text{PMod}^\pm(S^2, \mu_3) = \{\text{id}, \iota\}$ explains why the mapping class group operation is not considered explicitly in articles about the classical case $n = 3$: it just does nothing complicated.
The picture becomes more interesting for \( n = 4 \), where it can be nicely described within the framework of bipartite origamis. Viewing \( \text{PMod}^+(S^2, \mathcal{P}^{(4)}) \) as the non-oriented pure mapping class group of the pillowcase \( \Pi \) relative to its four vertices, we consider the unique double cover \( \pi: T \to \Pi \) which is ramified precisely at these vertices. This corresponds to the following bipartite origami (opposite sides identified):

\[
\begin{pmatrix}
1 & 2 \\
0 & 1 \\
\end{pmatrix}
\]

hence \( T \) is canonically identified with the torus \( \mathbb{R}^2/2\mathbb{Z}^2 \), with \( \mathbb{R}^2 \) coloured like a chess-board with squares of size \( 1 \times 1 \) whose vertices are the elements of \( \mathbb{Z}^2 \). The congruence subgroup \( \Gamma(2) = \{ \gamma \in \text{SL}(2, \mathbb{Z}) \mid \gamma \equiv 1 \mod 2 \} \) operates on \( T \) simply by its tautological linear action on \( \mathbb{R}^2 \). This descends to an action of \( \Gamma(2) \) on \( \Pi \) which fixes the vertices of \( \Pi \), therefore a homomorphism \( \Gamma(2) \to \text{PMod}^+(S^2, \mathcal{P}^{(4)}) \).

In fact this map is surjective with kernel \( \{ \pm 1 \} \), whence an isomorphism

\[
\text{P}\Gamma(2) = \Gamma(2)/\{ \pm 1 \} \cong \text{PMod}^+(S^2, \mu_4).
\]

We denote elements of \( \text{P}\Gamma(2) \) by their preimages in \( \Gamma(2) \), hoping that no confusion will occur. Now it is well-known that \( \text{P}\Gamma(2) \) is freely generated by \( \delta_{\text{hor}} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \) and \( \delta_{\text{ver}} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \).

Hence we only need to investigate how \( \delta_{\text{hor}} \) and \( \delta_{\text{ver}} \) operate on bipartite origamis:

**Proposition 3.3.** Let \( \mathcal{O} \) be a bipartite origami, then \( \delta_{\text{hor}}(\mathcal{O}) \) is obtained from \( \mathcal{O} \) by the following replacements on the individual squares, afterwards deleting the dotted lines and restretching the parallelograms to squares:

\[
\begin{align*}
\delta_{\text{hor}}(\square) &= \begin{array}{c}
\text{\ }
\end{array} \quad \text{and} \quad \delta_{\text{hor}}(\blacksquare) &= \begin{array}{c}
\text{\ }
\end{array}.
\end{align*}
\]

Similarly for \( \delta_{\text{ver}} \):

\[
\begin{align*}
\delta_{\text{ver}}(\square) &= \begin{array}{c}
\text{\ }
\end{array} \quad \text{and} \quad \delta_{\text{ver}}(\blacksquare) &= \begin{array}{c}
\text{\ }
\end{array}.
\end{align*}
\]

**Proof.** Draw the corresponding pictures for (2). \( \square \)

As an example, one easily computes that \( \delta_{\text{hor}}(\mathcal{O}) = \mathcal{O}' \) with notation as in Figure 3 it is easy to see that \( \mathcal{O} \) and \( \mathcal{O}' \) are not isomorphic, therefore the mapping class group action is nontrivial in this case.

**The Action in Group-Theoretic Terms.** For later considerations, we need to encode the previous notions in terms of group theory. By functoriality of the fundamental group, the pure spherical braid group acts on the fundamental group of the punctured sphere. This sloppy remark is in fact too sloppy, and we need to say some words about the fundamental group.

The fundamental group \( \pi_1 \) is not a functor from topological spaces to groups; it is a functor from *pointed* topological spaces to groups. If we have a situation where we do not wish to fix a basepoint, there is another remedy: consider not the category of groups but the category \( \mathcal{G} \)
of “groups up to inner automorphisms”. This is the category with groups as objects and set of morphisms

\[ \text{Hom}_G(G, H) = \text{Hom}(G, H) / H, \]

where \( H \) acts by postcomposition with inner automorphisms, i.e. \((f.h)(g) = h f(g) h^{-1}\) for \( f : G \to H \) and \( g \in G, h \in H \). We also write \( \text{Hom}_G(G, H) \) for \( \text{Hom}_G(G, H) \), and similarly \( \text{Aut}_G(G) = \text{Aut}_G(G) \) and \( \text{Isom}_G(G) = \text{Isom}_G(G) \) for the set of automorphisms and isomorphisms up to inner automorphisms. Symbolically we denote an element of \( \text{Hom}_G(G, H) \) by \( f : G \to H \). We speak of outer homomorphisms, outer automorphisms and outer isomorphisms with the obvious meanings.

With these conventions, \( \pi_1 \) is a functor from topological spaces to \( \mathcal{G} \). In particular we can safely write \( \pi_1(X) \) whenever we are interested in it only as a group up to inner automorphisms, not as a group in the usual sense. It does not make sense to talk about elements or subgroups of \( \pi_1(X) \), but it does make sense to speak of conjugacy classes of elements or subgroups. The notation \( \text{Aut}_\pi_1(X) \) does not make sense, the notation \( \text{Aut}_\pi_1(X) \) does. By functoriality we get a homomorphism \( \text{Homeo}(S^2, P^{(n)}) \to \text{Aut}_\pi_1(S^2 \smallsetminus P^{(n)}) \); this is trivial on \( \text{Homeo}_0(S^2, P^{(n)}) \) and therefore descends to a homomorphism

\[ \text{PMod}^+(S^2, P^{(n)}) \to \text{Aut}_\pi_1(S^2 \smallsetminus P^{(n)}) = \text{Aut}_\pi_1 F_{n-1}. \] (3)

**Theorem 3.4 (Dehn–Nielsen–Baer).** The homomorphism (3) is injective. Its image consists precisely of those outer automorphisms of \( \pi_1(S^2 \smallsetminus P^{(n)}) \) which preserve the conjugacy class of the monodromy around \( P_n \), for each \( \nu \in \mathbb{Z}/n\mathbb{Z} \).

**Proof.** For a modern proof see e.g. Farb–Margalit [6, Theorem 8.8]. □

For us only the injectivity statement is important. Also note that this setup allows us to translate the mapping class group action on \( \mathcal{D} \) into purely algebraic abstract nonsense: if the \( n \)-dessin \( \Gamma \) corresponds to the conjugacy class of subgroups represented by \( G \leq F_{n-1} \) and \( \varphi \in \text{PMod}^+(S^2, P^{(n)}) \) is a mapping class, then \( \varphi * \Gamma \) is the dessin corresponding to the conjugacy class of subgroups represented by \( \varphi_0(G) \), where \( \varphi_0 \in \text{Aut} F_{n-1} \) is a lift of the image of \( \varphi \) in \( \text{Aut} F_{n-1} \). In particular, \( \omega_n \) can naturally be extended to an action of the much larger group \( \text{Aut} F_{n-1} \) on \( n \)-dessins d’enfants.

**Proposition 3.5.** The homomorphism (7) is injective. In fact, the extended homomorphism \( \text{Aut}_\pi F_{n-1} \to \mathcal{G}(\mathcal{D}) \) is injective.

**Proof.** We may assume that \( n \geq 4 \). Let \( \varphi \) be in the kernel of \( \omega_n \). We can lift \( \varphi \) to \( \varphi_0 \in \text{Aut} \pi_1(S^2 \smallsetminus P^{(n)}, s) \). If \( \Gamma \leq \pi_1(S^2 \smallsetminus P^{(n)}, s) \) is a subgroup of finite index, we find that \( \varphi_0(\Gamma) \) must be conjugate in \( \pi_1(S^2 \smallsetminus P^{(n)}, s) \) to \( \Gamma \). This is because \( \varphi \) operates trivially on conjugacy classes of subgroups, since \( \omega_n(\varphi) = \text{id} \).

This already implies that \( \varphi_0 \) is an inner automorphism (i.e., \( \varphi \) is trivial) by a result of Lubotzky [20]. More precisely, an automorphism \( \sigma \) of a group \( G \) is called finite-normal if \( \sigma(N) = N \) for every normal subgroup of finite index \( N \) of \( G \). Then Theorem 1 in op.cit. contains the statement that every finite-normal automorphism of a free group on at least three letters is inner. □
Note that we only used triviality of the operation on normal subgroups. Hence if we define a normal $n$-dessin to be an $n$-dessin corresponding to a normal covering, we can strengthen Proposition 1 to the statement that an element of $\text{Aut}_\text{ext} F_n$ which operates trivially on all normal $n$-dessins must be the identity. It even suffices by [20, Theorem 1] to assume that it operates trivially on all normal $n$-dessins whose degree is a power of a fixed prime number.

4. Algebraic Curves and Hurwitz Spaces

In this section we explain how to associate algebraic curves with higher dessins d’enfants. Recall how this works in the “classical” case: a dessin d’enfant defines a ramified covering $f: S \rightarrow R$ and there are three distinct points $x, y, z \in R$ such that $R$ is an oriented two-sphere and $f$ is unramified outside $\{x, y, z\}$. Now on $R$ there is a unique complex structure up to biholomorphism fixing $x, y, z$ pointwise, and we lift this complex structure to $S$ via $f$. But by the equivalence between compact Riemann surfaces and smooth projective algebraic curves over $\mathbb{C}$, these data define a morphism of such curves which then turns out to be defined over $\bar{\mathbb{Q}}$.

Now an $n$-dessin d’enfant defines a ramified covering $S \rightarrow R$ with $n$ distinct marked points $x_1, \ldots, x_n$ in $R$, and there is an $(n-3)$-dimensional family of distinct complex structures on $(R, x_1, \ldots, x_n)$. More formally, we consider the moduli space $M_{0,n}$ of $(n+1)$-tuples $(C, x_1, \ldots, x_n)$ where $C$ is a smooth projective curve and the $x_i$ are distinct points in $C$. This should be taken, for our purposes, as a variety over $\mathbb{Q}$. As such it is isomorphic to a complement of a hyperplane arrangement in affine space $A^{n-3}$: we identify $M_{0,n}$ with the complement of

$$D = \bigcup_{i=1}^{n-3} \{x_i = 0\} \cup \bigcup_{i=1}^{n-3} \{x_i = 1\} \cup \bigcup_{1 \leq i < j \leq n-3} \{x_i = x_j\} \subset A^{n-3}$$

via the isomorphism

$$A^{n-3} \setminus D \xrightarrow{\cong} M_{0,n}$$

sending $(x_1, \ldots, x_{n-3})$ to $(\mathbb{P}^1, \infty, 0, 1, x_1, \ldots, x_{n-3})$.

We are particularly interested in the complex and real points of this moduli space. The space of complex points $M_{0,n}(\mathbb{C})$ is a connected complex manifold of complex dimension $n-3$ whose fundamental group is naturally isomorphic to the mapping class group $\text{PMMod}^+(S^2, \mathfrak{P}^{(n)})$, and its subspace $M_{0,n}(\mathbb{R}) \subseteq M_{0,n}(\mathbb{C})$ is a real-analytic submanifold of real dimension $n-3$. It is not connected for $n > 3$ but has $(n-1)!/2$ components each of which is diffeomorphic to $\mathbb{R}^{n-3}$. The symmetric group $\mathfrak{S}_n$, via its tautological action on $M_{0,n}$, acts transitively on the set of these connected components. It is therefore no essential restriction if we choose one of these components and ignore the others; we take

$$M_{0,n}(\mathbb{R})^+ = \{(x_1, \ldots, x_{n-3}) \in \mathbb{R}^{n-3} \mid 1 < x_1 < x_2 < \cdots < x_{n-3}\}.$$

More generally for a subfield $K \subseteq \mathbb{C}$ we set

$$M_{0,n}(K)^+ = M_{0,n}(\mathbb{R})^+ \cap M_{0,n}(K) \subset K^{n-3}.$$

Now we include coverings into the picture. Fix integer parameters $n, d \geq 1$. The Hurwitz stack $\mathcal{H}_n^d$ is the stack over $\mathbb{Q}$ which classifies $(n+3)$-tuples $(X, C, p, x_1, \ldots, x_n)$, where $(C, x_1, \ldots, x_n)$ is an algebraic curve of type $(0, n)$, $X$ is an algebraic curve and $p: X \rightarrow C$ is a finite morphism of degree $d$, unramified outside the $x_i$. This is proved to be a smooth Deligne–Mumford stack over $\mathbb{Q}$ in Wewers [33]. It has an obvious forgetful morphism $\mathcal{H}_n^d \rightarrow M_{0,n}$.
forgetting, in the above notation, $X$ and $p$. Note that although $M_{0,n}$ is a variety, $\mathcal{H}_n^d$ may still be an honest stack: a covering $X \to C$ can have nontrivial deck transformations, giving a nontrivial automorphism group of the corresponding object in $\mathcal{H}_n^d$. However, the stack structure of $\mathcal{H}_n^d$ is quite simple, see below. We denote the coarse moduli scheme over $\mathbb{Q}$ associated with $\mathcal{H}_n^d$ by $H_n^d$.

The Hurwitz stack $\mathcal{H}_n^d$ need not be connected. In fact, its set of geometric components, i.e. connected components of $\mathcal{H}_n^d \otimes \mathbb{Q}$, can be identified with the set of $\text{PMod}^+$-equivalence classes of $n$-dessins d’enfants of degree $d$. This association works as follows. The connected components of $\mathcal{H}_n^d \otimes \mathbb{Q}$ are in canonical bijection with those of $\mathcal{H}_n^d \otimes \mathbb{C}$, and these in turn are in canonical bijection with those of $H_n^d(\mathbb{C})$. An element $h \in H_n^d(\mathbb{C})$ is represented by a marked Riemann surface $C$ of type $(0,n)$ together with a ramified covering $X \to C$ of degree $d$, unramified outside the marked points. We have seen before that this can be identified with a $\text{PMod}^+$-equivalence class of $n$-dessins d’enfants. But this equivalence class depends continuously on $h$, hence it is constant on every connected component of $H_n^d(\mathbb{C})$.

**Proposition 4.1.** The thus defined map

$$\pi_0(\mathcal{H}_n^d \otimes \mathbb{Q}) = \pi_0(H_n^d(\mathbb{C})) \to \text{PMod}^+(\mathbb{S}^2, \mathbb{P}(n)) \setminus \mathcal{D}_n^d$$

is a bijection between finite sets.

□

Now we can describe the stack structure of $\mathcal{H}_n^d \otimes \mathbb{Q}$: choose a connected component $\mathcal{H}$ of $\mathcal{H}_n^d \otimes \mathbb{Q}$; this corresponds to a ramified covering with finite deck transformation group $G$. Then if $H$ is the coarse moduli scheme associated with $\mathcal{H}$, we have an isomorphism of stacks $\mathcal{H} \simeq H//G$, where $G$ operates trivially on $H$; furthermore, $H \to M_{0,n} \otimes \mathbb{Q}$ is an étale covering.

We have found an algebraic interpretation of equivalence classes of $n$-dessins d’enfants under $\text{PMod}^+$, but it is a bit more involved to find an algebraic interpretation of $n$-dessins d’enfants themselves. We have seen that they correspond to choosing a pointed homeomorphism between the covered punctured sphere and $(\mathbb{S}^2, \mathbb{P}(n))$, defined up to isotopy — such a structure is generally known as a Teichmüller marking. Now Teichmüller markings are clearly transcendental objects. But for curves $(C, x_i) \in M_{0,n}(\mathbb{R})^+$ there is a canonical Teichmüller marking of $(C(\mathbb{C}), x_i)$: it is determined by the isotopy class of the circle $C(\mathbb{R}) \subset C(\mathbb{C})$. With this observation, it becomes natural to construct bijections between the following sets:

(i) the set $\mathcal{D}_n^d$ of $n$-dessins d’enfants of degree $d$ up to isomorphism;

(ii) the set $\mathcal{L}_n^d$ of lifts of the contractible submanifold $M_{0,n}(\mathbb{R})^+ \subset M_{0,n}(\mathbb{C})$ along the finite unramified covering $p: H_n^d(\mathbb{C}) \to M_{0,n}(\mathbb{C})$;

(iii) for every (fixed) $a \in M_{0,n}(\mathbb{R})^+$ the preimage $p^{-1}(a)$ under $p$ as in (ii).

First we define a bijection between (ii) and (iii): Since $p$ is a covering map and $M_{0,n}(\mathbb{R})^+$ is contractible, there exists through each point $y \in p^{-1}(a)$ a unique lift $\lambda$ of $M_{0,n}(\mathbb{R})^+$; the desired bijection is then $y \leftrightarrow \lambda$. Next we describe a map

$$\mathcal{L}_n^d \to \mathcal{D}_n^d,$$

for a lift $\lambda \in \mathcal{L}_n^d$, given as $\lambda: M_{0,n}(\mathbb{R})^+ \to H_n^d(\mathbb{C})$, choose some point $a = (a_1, \ldots, a_{n-3}) \in M_{0,n}(\mathbb{R})^+$. Then $\lambda(a)$ is a point in $H_n^d(\mathbb{C})$, corresponding to a ramified covering of Riemann surfaces $f: S \to \mathbb{P}^1(\mathbb{C})$ unramified outside

$$D_a = \{\infty, 0, 1, a_1, \ldots, a_{n-3}\} \subset \mathbb{P}^1(\mathbb{C}).$$
Choose an orientation-preserving homeomorphism $h : (S^2, P^{(n)}) \rightarrow (\mathbb{P}^1(\mathbb{C}), D_a)$ sending the circle $S^1$ to the circle $\mathbb{P}^1(\mathbb{R})$ and mapping $P^{(n)}$ to $D_a$ in the way indicated by their canonical orderings, i.e.
\[ h(P^{(n)}_0) = \infty, h(P^{(n)}_1) = 0, h(P^{(n)}_2) = 1, h(P^{(n)}_{\nu}) = a_{\nu-3}. \]

Then $h$ is unique up to isotopy preserving these constraints; therefore, so is $h^{-1} \circ f : S \rightarrow S^2$. This map defines an $n$-dessin d’enfant up to isomorphism. More directly, we can construct the associated $n$-dessin $(S, \Gamma, c)$ by $\Gamma = f^{-1}(\mathbb{P}^1(\mathbb{R})) \subset S$ and the colouring $c$ in the obvious way via the identification $D_a \cong \mathbb{P}^{(n)}$ given in (3). This $n$-dessin is the image of $\lambda$ under (4); by continuity it does not depend on the choice of $a$.

**Proposition 4.2.** The map (4) is a bijection of finite sets and it makes the diagram
\[
\begin{array}{ccc}
p^{-1}(a) & \longrightarrow & \mathcal{L}_n^d \\
\downarrow & & \downarrow \\
\pi_0(H^d_n(\mathbb{C})) & \longrightarrow & \text{PMod}^+(S^2, \mathbb{P}^{(n)}) \setminus \mathcal{L}_n^d
\end{array}
\] (6)
commute (here $a \in \mathcal{M}_{0,n}(\mathbb{R})^+$).

Here the vertical map on the left sends a point in $p^{-1}(a) \subset H^d_n(\mathbb{C})$ to the connected component it is contained in. \qed

5. **A Higher Belyi Theorem**

In the classical theory of dessins d’enfants, a crucial ingredient is the Theorem of Belyi [1]: it can be phrased as saying that every smooth projective curve over $\overline{\mathbb{Q}}$ is defined by some dessin d’enfant. We generalise this and introduce the following useful notion:

**Definition 5.1.** Let $K$ be an algebraically closed field, let $X$ be a smooth projective algebraic curve over $K$, and let $D \subset \mathbb{P}^1(K)$ be a finite subset. A nonconstant morphism of algebraic curves $\beta : X \rightarrow \mathbb{P}^1_K$ is called a $D$-Belyi map if it is unramified outside $D$; it is called an irreducible $D$-Belyi map if it is not a $D'$-Belyi map for a proper subset $D' \subset D$.

Two $D$-Belyi maps $\beta_1 : X_1 \rightarrow \mathbb{P}^1$ and $\beta_2 : X_2 \rightarrow \mathbb{P}^1$ are called isomorphic if there exists an isomorphism $f : X_1 \rightarrow X_2$ with $\beta_2 \circ f = \beta_1$.

For $D = \{\infty, 0, 1\}$ we regain the by now classical notion of a Belyi map; its importance lies in the celebrated theorem of Belyi:

**Theorem 5.2 (Belyi).** Let $X$ be a smooth projective algebraic curve over $\overline{\mathbb{Q}}$. Then there exists a $\{\infty, 0, 1\}$-Belyi map $X \rightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}}$.

The original reference for this result is Belyi [1]; a more detailed and elementary account can be found in Köck [15]. We generalise this theorem in the following way:

**Theorem 5.3.** Let $X$ be a smooth projective algebraic curve over $\overline{\mathbb{Q}}$, and let $D \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ be a finite subset with at least three elements. Then there exists an irreducible $D$-Belyi map $\beta : X \rightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}}$.
We employ the following lemma:

**Lemma 5.4.** Let $D \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ be finite with at least three elements. Then there exists an irreducible $D$-Belyǐ map $g: \mathbb{P}^1_\mathbb{Q} \rightarrow \mathbb{P}^1_\mathbb{Q}$ with $g(\{\infty, 0, 1\}) \subseteq D$.

**Proof of Lemma 5.4.** It suffices to produce an irreducible $D$-Belyǐ map $g: \mathbb{P}^1 \rightarrow \mathbb{P}^1$, since then $g^{-1}(D)$ has at least (in fact, more than) three elements, so we can precompose $g$ with a Möbius transformation in $\text{PSL}(2, \overline{\mathbb{Q}})$ that maps $\{\infty, 0, 1\}$ into $g^{-1}(D)$ to meet the second requirement.

Next we take a ramified covering $h: S^2 \rightarrow S^2$ which is precisely ramified at $\mathbb{P}(n)$ (formally, the induced covering of $S^2 \setminus \mathbb{P}(n)$ is unramified, and for each $\nu$ there exists an $s_\nu \in S$ above $P_\nu^{(n)} \in S^2$ such that the ramification degree of $f$ at $s_\nu$ is at least two). To prove that this exists, it is easy to cook up an explicit family of such for every $n$, but we can make life even easier: consider $n-1$ real numbers $\alpha_1, \ldots, \alpha_{n-1}$ which are algebraically independent over $\overline{\mathbb{Q}}$. Then the map $f: \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ given by the polynomial

$$f(z) = \int_0^z (\zeta - \alpha_1) \cdots (\zeta - \alpha_n) \, d\zeta$$

is a ramified covering with the $n$ distinct ramification points $f(\alpha_1), \ldots, f(\alpha_{n-1}), \infty$, hence for a suitable homeomorphism $m: S^2 \rightarrow \mathbb{P}^1(\mathbb{C})$ we can take $h = m^{-1} \circ f \circ m$.

Now returning to the setting of the lemma, choose some enumeration $D = \{x_1, \ldots, x_n\}$ of $D$ and an orientation-preserving diffeomorphism $\varphi: S^2 \rightarrow \mathbb{P}^1(\mathbb{C})$ that sends $\mathbb{P}(n)$ to $D$ in such a way that $\varphi(P_\nu^{(n)}) = x_\nu$. With $h$ as constructed in the preceding paragraph, set $g = \varphi \circ h: S^2 \rightarrow \mathbb{P}^1(\mathbb{C})$. There is a unique complex (and therefore algebraic) structure on $S^2$ that turns this into a holomorphic (and therefore algebraic) map, and since the $x_i$ are in $\overline{\mathbb{Q}}$, this algebraic map comes from a unique morphism of algebraic curves $g: \mathbb{P}^1_\mathbb{Q} \rightarrow \mathbb{P}^1_\mathbb{Q}$, which is an irreducible $D$-Belyǐ map, as desired. \hfill \Box

**Proof of Theorem 5.3.** By Theorem 5.2 there exists a $\{\infty, 0, 1\}$-Belyǐ map $f: X \rightarrow \mathbb{P}^1_\mathbb{Q}$. We may assume it is irreducible, for otherwise, by the Riemann-Hurwitz formula, we have $X \cong \mathbb{P}^1_\mathbb{Q}$, which admits, for instance, the irreducible $\{\infty, 0, 1\}$-Belyǐ map

$$\mathbb{P}^1 \rightarrow \mathbb{P}^1, \quad t \mapsto \frac{27}{4}(t^2 - t^3).$$

Now we choose $g: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ as in Lemma 5.4 and set $\beta = g \circ f$. \hfill \Box

6. Galois Actions on Higher Dessins d’Enfants

Having associated algebraic objects with higher dessins d’enfants, we can let Galois groups act upon them. As before the picture changes whether we consider $n$-dessins up to isomorphism or up to braid equivalence. But this time we begin with the former.

We have seen that an $n$-dessin d’enfant $(S, \Gamma, c)$ of degree $d$ determines not a single morphism of algebraic curves but rather a family of such: for every $a \in M_{0,n}(\mathbb{R})^+$ we obtain by composition of the upper horizontal isomorphisms in (6) an element of $p^{-1}(a)$; unraveling definitions, we find that this is an isomorphism class of $D_a$-Belyǐ maps, where

$$D_a = \{\infty, 0, 1, a_1, \ldots, a_{n-3}\}.$$ (7)
Vice versa for every $D_a$-Belyı́ map $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$, we get an associated dessin d’enfant $(S, \Gamma, c)$ which can be described more directly: $S = X(\mathbb{C})$, $\Gamma = \beta^{-1}(\mathbb{P}^1(\mathbb{R}))$ and the colouring $c$ is obtained by the bijection $\mathbb{Z}/n\mathbb{Z} \to D_a$ indicated by the ordering in $\mathcal{P}$.

**Definition 6.1.** Let $(S, \Gamma, c)$ be an $n$-dessin d’enfant and $a \in M_{0,n}(\mathbb{R})^+$. The complex algebraic curve $X$ such that $X(\mathbb{C}) = S$ together with the $D_a$-Belyı́ morphism $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ is called the algebraic interpretation of the $n$-dessin $(S, \Gamma)$ at the parameter $a$.

By a standard argument, the algebraic interpretation of a dessin at $a$ can be defined over a finite extension field of the subfield $\mathbb{Q}(a) = \mathbb{Q}(a_1, \ldots, a_{n-3})$ of $\mathbb{R}$ generated by the entries of $a$. In particular, it can be defined over $\overline{\mathbb{Q}(a)}$, the algebraic closure of $\mathbb{Q}(a)$ in $\mathbb{C}$.

**Proposition and Definition 6.2.** Use notation as in Definition 6.1. The unique model of $(X, g)$ over $K = \overline{\mathbb{Q}(a)}$ — i.e. the unique algebraic curve $X_0$ over $K$ such that $X_0(\mathbb{C}) = S$ together with the unique morphism of algebraic curves $\beta_0 : X_0 \to \mathbb{P}^1_K$ whose extension to $\mathbb{C}$ is $\beta$ — is then called the arithmetic interpretation of $(S, \Gamma)$ at the parameter $a$.

Since we will usually be only interested in arithmetic models and not algebraic models, we drop the superscripts 0 without fear of confusion — one of them is uniquely determined by the other. To summarise:

**Lemma 6.3.** The above construction defines, for every fixed algebraically closed field $K \subseteq \mathbb{C}$ and every fixed $a \in M_{0,n}(K)^+$, a bijection between $\mathcal{D}_n$ and the set of $D_a$-Belyı́ maps $\beta : X \to \mathbb{P}^1_K$ of degree $d$ up to isomorphism.

But on the latter the absolute Galois group $G_{\overline{\mathbb{Q}}(a)}$ operates! Here, and in the following, we write $G_K = \text{Gal}(\overline{K}/K)$, where $\overline{K}$ is the algebraic closure of $K$ in $\mathbb{C}$, for any subfield $K \subseteq \mathbb{C}$. So we have obtained our desired Galois action; we can write it as a group homomorphism

$$\varrho_a : G_{\overline{\mathbb{Q}}(a)} \to \mathfrak{S}(\mathcal{D}_n)$$

This group homomorphism is a bit difficult to handle because the occurring groups are very different in nature: the absolute Galois group is a profinite topological group, but the infinite symmetric group has no obvious profinite topology. Hence we make use of the fact that the image of $\varrho_a$ lies in a smaller group $\mathfrak{D}_n \subset \mathfrak{S}(\mathcal{D}_n)$ that has a natural profinite topology:

**Definition 6.4.** Let $n \geq 3$. The dessin group on $n$ points is the infinite product of finite groups

$$\mathfrak{D}_n = \prod_{d=1}^{\infty} \mathfrak{S}(\mathcal{D}_n^d),$$

endowed with the product topology.

Since $\mathcal{D}_n$ is the disjoint union of the $\mathcal{D}_n^d$, the dessin group is in a canonical way a subgroup of $\mathfrak{S}(\mathcal{D}_n)$. It consists of those permutations that preserve the degree; hence not only $\varrho_a$ factors through

$$\varrho_a : G_{\overline{\mathbb{Q}}(a)} \to \mathfrak{D}_n,$$

but we can also view $\varpi_n$ (even its extension to $\text{Aut}_{\text{text}} F_{n-1}$) from $\mathfrak{I}$ as having image in $\mathfrak{D}_n$. 

$$\text{(8)}$$
It seems a difficult question whether the map \( \mathcal{S} \) is always injective. However, for \( a \) algebraic over \( \mathbb{Q} \) we can prove injectivity, making essential use of Belyi’s theorem.

**Proposition 6.5.** For every \( n \geq 3 \), every number field \( K \subset \mathbb{C} \) and every \( a \in M_{0,n}(K)^+ \), the homomorphism \( \mathcal{S} \) is continuous and injective. It is a homeomorphism onto its image, which is closed in the dessin group; in particular we can reconstruct the Krull topology on the Galois group from the topology on \( \mathcal{D}_n \).

**Proof.** We first show injectivity. Let \( \sigma \) be a nontrivial element of the Galois group \( G_K \). There exists some algebraic curve \( Z \) over \( \overline{\mathbb{Q}} \) with \( \sigma(Z) \not\cong Z \) (for instance, choose an element \( t \in \overline{\mathbb{Q}} \) with \( \sigma(t) \neq t \) and take \( Z \) to be an elliptic curve with \( j \)-invariant \( t \)). By Proposition 5.3 there exists a ramified covering \( f: Z \rightarrow \mathbb{P}^1 \) which is unramified outside \( D_a \). In particular it defines an \( n \)-dessin on which \( \sigma \) has to act nontrivially.

Next we show continuity. By definition of the product topology this is equivalent to all the maps \( G_K \rightarrow \mathcal{S}(\mathcal{D}_n^d) \) being continuous. Fix one such \( d \); there exists a finite Galois extension \( L|K \) over which all the elements of \( \mathcal{D}_n^d \) are defined, and hence \( G_K \rightarrow \mathcal{S}(\mathcal{D}_n^d) \) factors over \( G_K \rightarrow \text{Gal}(L|K) \), where the latter is given the discrete topology.

Finally, since both groups are compact and Hausdorff, \( \varrho_a \) has to be a homeomorphism onto its image. \( \square \)

For \( n = 3 \), the sequence \( a \) has length zero, so the only choice is the empty sequence \( \varnothing \), and \( \mathbb{Q}(\varnothing) = \mathbb{Q} \), so we can choose \( K = \mathbb{Q} \). The map \( \varrho_\varnothing: \text{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \rightarrow \mathcal{D}_3 \), is of course the classical Galois action on classical dessins d’enfants. It is well-known to be injective, and this is actually one of the main motivations for studying dessins d’enfants.

Now we can state the central technical results of this article. The first one says that Galois and mapping class group actions on higher dessins d’enfants agree only in the obvious cases.

**Theorem 6.6.** Let \( n \geq 3 \), let \( K \subset \mathbb{C} \) be an algebraic number field and let \( a \in M_{0,n}(K)^+ \). Let \( \varrho_a: G_K \rightarrow \mathcal{D}_n \) be the homomorphism from \( \mathcal{S} \), and let \( \varpi_n: \text{PMod}^\pm(\mathcal{S}^2, \mathcal{P}(n)) \rightarrow \mathcal{D}_n \) be the homomorphism from \( \mathcal{L} \). Then the image of \( \varrho_a \) and the closure of the image of \( \varpi_n \) intersect precisely in a subgroup of cardinality two or one, depending on whether \( K \) is stable under complex conjugation or not. Its non-identity element in the first case sends every \( n \)-dessin to that with the same graph, colouring and underlying surface, but with a different orientation of the surface. It is the image of the involution \( \iota \in \text{PMod}^\pm \) and complex conjugation \( (z \mapsto \overline{z}) \in G_K \).

The closure of the image appearing in this theorem can also be understood as follows: let \( \Gamma \) be the profinite completion of \( \text{PMod}^\pm(\mathcal{S}^2, \mathcal{P}(n)) \), then \( \varpi_n \) extends to a continuous homomorphism \( \Gamma \rightarrow \mathcal{D}_n \); it is this image that we are interested in. Apparently it is not known whether this homeomorphism between finite groups is still injective.

Then, the Galois action on all \( n \)-dessins uniquely determines the parameters \( a \):
Theorem 6.7. Let $K \subset \mathbb{C}$ be a finitely generated subfield and let $a, b \in M_{0,n}(K)^+$. Assume that $\varphi_a$ and $\varphi_b$ agree as homomorphisms $G_K \to \mathfrak{D}_n$. Then $a = b$.

In the case where $K$ is a number field, we can even theoretically reconstruct $K$ from the image of $\varphi_a$:

Theorem 6.8. Let $n \geq 3$. Let $K$ and $L$ be number fields contained in $\mathbb{C}$, and let $a \in M_{0,n}(K)^+$ and $b \in M_{0,n}(L)^+$. Assume that the images $\varphi_a(G_K)$ and $\varphi_b(G_L)$ agree as subsets of $\mathfrak{D}_n$. Then $K = L$ and $a = b$.

These theorems will be proved in section 10.

In contrast to this, the picture becomes much simpler if we look at $n$-dessins up to the mapping class group action: for each $n$ and $d$ we obtain a natural operation of $G_Q$ on the finite set $\text{PMod}^+(S^2, \mathfrak{P}^{(n)}) \setminus \mathfrak{D}^d$ via the bijection in Proposition 6.5. We see from the respective constructions that the projection

$$\mathfrak{D}^d \to \text{PMod}^+ \setminus \mathfrak{D}^d$$

is equivariant for the Galois actions $\varphi_a: G_K \to \mathfrak{S}(\mathfrak{D}^d)$ on the left hand side and that of $G_Q$ on the right hand side for the inclusion $G_K \subseteq G_Q$. We can rephrase this as follows:

Proposition 6.9. Let $\Gamma$ and $\Gamma'$ be $n$-dessins d’enfants, let $K \subset \mathbb{C}$ be a number field, let $a \in M_{0,n}(K)^+$ and let $\sigma \in G_K$. If $\Gamma$ and $\Gamma'$ are equivalent under the action of $\text{PMod}^+(S^2, \mathfrak{P}^{(n)})$, then so are $\varphi_a(\Gamma)$ and $\varphi_a(\Gamma')$.

The thus defined action of $G_K$ on mapping class group orbits of $n$-dessins d’enfants is independent of the basepoint $a$ and even independent of the base field $K$ in the sense that it is the restriction of one particular action of $G_Q$ on such orbits. □

7. An Explicit Example

We illustrate the previous considerations by examining a particular component of the Hurwitz stack $\mathcal{H}_4^4$ and certain fibres of its forgetful map to $M_{0,4}$. It is, to our knowledge, the simplest example which is nontrivial with respect to both the braid group and the Galois actions.

Definition 7.1. For this section, let $\mathcal{H}$ be the substack of the Hurwitz stack $\mathcal{H}_4^4$ parameterising ramified coverings $f: X \to Y$ of degree four with four ordered marked ramification points in $Y$ such that both $X$ and $Y$ are rational and the partitions of $4 = \deg f$ determined by the ramification points are (in this order) $4, 2 + 1 + 1, 2 + 1 + 1, 2 + 1 + 1$.

To describe $\mathcal{H}$ explicitly, we use the isomorphism $M_{0,4} \simeq \mathbb{P}^1 \setminus \{\infty, 0, 1\}$ as in section 4. A covering occurring in $\mathcal{H}$ is isomorphic to one of the form $f: \mathbb{P}^1 \to \mathbb{P}^1$ where $f$ is a polynomial of degree four, having critical points $\infty, 0, 1, s$ for some $s \neq \infty, 0, 1$ with $f(\infty) = \infty$ (which is automatic), $f(0) = 0, f(1) = 1$ and $f(s) = a$ for some $a \neq \infty, 0, 1$. A little computation shows that $f$ is entirely determined by $s$, so we write $f = f_s$, and it must be equal to

$$f_s(x) = \frac{12}{2s - 1} \left( \frac{1}{4} x^4 - \frac{s + 1}{3} x^3 + \frac{s}{2} x^2 \right).$$

Hence we can use $s$ as a global coordinate on $\mathcal{H}$, in particular $\mathcal{H}$ is connected and defined over $\mathbb{Q}$. Also the deck groups of the coverings $f_s: \mathbb{P}^1 \to \mathbb{P}^1$ are trivial, hence $\mathcal{H}$ is a variety, and
we write $H$ instead. The values of $s$ for which some of the ramification points collide are also readily computed, and we find that the coordinate $s$ identifies $H$ with $\mathbb{P}^1 \setminus \{\infty, 0, 1, -1, 2, \frac{1}{2}\}$. The forgetful morphism to $M_{0,4}$ is

$$p: H \subset \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \supset M_{0,4}, \quad s \mapsto f_s(s) = \frac{(2 - s)s^3}{2s - 1}. \quad (9)$$

Finally, the universal family of coverings is

$$\mathbb{P}^1 \times H \longrightarrow \mathbb{P}^1 \times M_{0,4}, \quad (x, s) \mapsto \left( f_s(x), \frac{(2 - s)s^3}{2s - 1} \right).$$

Now we present the mapping class group orbit of 4-dessins that $H$ corresponds to. By definition of $H$, they are precisely those 4-dessins on the sphere that have degree four, one vertex coloured 0 of degree 8, and for each other “colour” one vertex of degree 4 and two vertices of degree 2. We draw such dessins in the plane with the point coloured by 0 imagined at infinity, and the edges adjacent to this point drawn with loose ends. They are given in Figure 4.

They correspond to lifts of $M_{0,4}(\mathbb{R})^+$ along $H(\mathbb{C}) \to M_{0,4}(\mathbb{C})$, so we need to enumerate these next. Recall that $M_{0,4}(\mathbb{R})^+$ is identified with the open interval $]1, \infty[ \subset \mathbb{P}^1 \setminus \{\infty, 0, 1\}$; this interval has four distinct preimages in $H(\mathbb{C}) \subset \mathbb{P}^1(\mathbb{C})$:

(i) one path $L_1$ in the upper half plane with endpoints $-1$ and $\infty$;
(ii) one path $L_2$ in the lower half plane with endpoints $-1$ and $\infty$ (the complex conjugate of $L_1$);
(iii) the open interval $L_3 = ]-\infty, -1[$;
(iv) the open interval $L_4 = ]\frac{1}{2}, 1[.$

These are enumerated in such a way that $L_j$ corresponds to the 4-dessin $\Gamma_j$ in Figure 4.

Our next step is to consider the resulting equations for $a = 2$ and $a = 3$ more closely. Consider first $a = 2$; the preimages of $a$ under $H(\mathbb{C}) \to M_{0,4}(\mathbb{C})$ are as follows:
(i) The preimage in $L_1$, i.e. the one corresponding to the dessin $\Gamma_1$, is the complex number

$$s_1 = \frac{1}{2} \left( 1 + i\sqrt{2}\sqrt{3} + \sqrt{3} \right).$$

(ii) The preimage in $L_2$ is

$$s_2 = \overline{s_1} = \frac{1}{2} \left( 1 - i\sqrt{2}\sqrt{3} + \sqrt{3} \right).$$

(iii) The preimage in $L_3$ is

$$s_3 = \frac{1}{2} \left( 1 - \sqrt{2}\sqrt{3} - \sqrt{3} \right).$$

(iv) The preimage in $L_4$ is

$$s_4 = \frac{1}{2} \left( 1 + \sqrt{2}\sqrt{3} - \sqrt{3} \right).$$

The fields $Q(s_3)$ and $Q(s_4)$ are both quartic, but it is more convenient to work with the larger field $L = Q(\sqrt{2}, \sqrt{3})$ that contains both of them.

**Lemma 7.2.** The subfields $Q(\sqrt{2})$ and $Q(\sqrt{3})$ of $L$ intersect precisely in the subfield $Q$. Hence they are linearly disjoint, and the field $L$ has degree eight over $Q$.

**Proof.** Assume they intersected in a larger field; this would have to be $Q(\sqrt{2})$. Hence we would have $Q(\sqrt{2}) \subset Q(\sqrt{3})$. But $Q(\sqrt{3})$ also contains the quadratic field $Q(\sqrt{3})$. Thus it would contain $Q(\sqrt{2}, \sqrt{3})$ which, as a composite of two quadratic fields, is also a number field of degree four. This would imply $Q(\sqrt{3}) = Q(\sqrt{2}, \sqrt{3})$ which is impossible since the latter is Galois over $Q$, but the former is not (it is a real field, and $\sqrt{3}$ has the non-real Galois conjugate $i\sqrt{3}$). Contradiction. \qed

Let $\tilde{L}$ be the Galois closure of $L$ over $Q$; this is simply the field $L(i)$ (which is quadratic over $L$ since $L$ is real). We find that the subfields $Q(\sqrt{2})$ and $Q(i, \sqrt{3})$, which are both Galois over $Q$, are also linearly disjoint.

**Lemma 7.3.** There exists an automorphism $\sigma$ of $\tilde{L}$ with $\sigma(\sqrt{2}) = -\sqrt{2}$ and $\sigma(\sqrt{3}) = i\sqrt{3}$. For this automorphism, $\sigma(s_3) = s_4$ and $\sigma(s_4) = s_3$.

The restriction of $\sigma$ to $L$ is uniquely determined by these properties, and we only have some freedom in choosing $\sigma(i)$.

**Proof.** Immediate from Lemma (7.2) and the explicit formulæ for $s_3$ and $s_4$. \qed

For $a = 3$ the formulæ become significantly more complicated; since we shall make no use of expressions of the roots in terms of radicals (and every decent computer algebra system gives you one instantly), we do not reproduce them here. But to have precise definitions, let $s'_j$ be the unique preimage of $a = 3$ in $L_j$ under the projection map $p$ from (9). Numerically we have $s'_4 = -1.50884449494\ldots$ and $s'_4 = 0.5379312192\ldots$. More important is the precise shape of the equation $p(s) = 3$: it is equivalent to

$$\varphi(s) = s^4 - 2s^3 + 6s - 3 = 0.$$
Lemma 7.4. The polynomial $\varphi(x) \in \mathbb{Q}[x]$ is irreducible, and its Galois group is isomorphic to the full permutation group $S_4$.

Proof. Since $\varphi(x - 1) = x^4 - 6x^3 + 12x^2 - 4x - 6$ is irreducible by Eisenstein’s criterion for the prime 2, we conclude that $\varphi(x)$ is irreducible. By the method described in Kappe–Warren [15] we find that the Galois group must be isomorphic to $S_4$. □

In particular the field $K = \mathbb{Q}(s_3')$, which has degree four over $\mathbb{Q}$, has no subfields save $\mathbb{Q}$ and $K$ itself (otherwise an intermediate field would have to be quadratic, and the Galois group of $\varphi$, which is the Galois group of the Galois closure of $K$, would have order at most eight).

Lemma 7.5. The fields $\breve{L}$ and $K$ are linearly disjoint.

Proof. As mentioned after the proof of Lemma 7.4, the only subfields of $K$ are $K$ itself and $\mathbb{Q}$. If the intersection $\breve{L} \cap K$ were larger than $\mathbb{Q}$ it therefore would have to be $K$, so that $K \subset \breve{L}$. But $\breve{L}$ is a Galois extension of $\mathbb{Q}$ of degree 16, and the Galois closure of $K$ has degree $4! = 24 > 16$, contradiction. □

Corollary 7.6. There exists a number field $M$ containing $s_3', s_4', s_3, s_4$ and an automorphism $\sigma$ of $M$ with $\sigma(s_3) = s_4$ and $\sigma(s_3') = s_3'$.

Proof. Since $\breve{L}$ and $K$ are linearly disjoint, the restriction map from $\text{Gal}(\breve{L}K|K)$ to $\text{Gal}(\breve{L}|\mathbb{Q})$ is an isomorphism. Hence if we let $M = \breve{L}K$, we find that there is an automorphism $\sigma$ of $M$ which acts as the identity on $K$, sends $\sqrt{2}$ to $-\sqrt{2}$ and $\sqrt{3}$ to $\sqrt{3}$. This automorphism satisfies the requirements. □

We can now extend $\sigma$ to an automorphism of $\hat{\mathbb{Q}}$, also denoted by $\sigma$.

Corollary 7.7. With notation as above, $\varrho_2(\sigma)(\Gamma_3) = \Gamma_4$, but $\varrho_3(\sigma)(\Gamma_3) = \Gamma_3$.

8. Complex Multiplication and Bipartite Origamis

The case $n = 4$ is closely related to the theory of origami curves as introduced in Lochak [19]; for a well-written survey paying special attention to connections with dessins d’enfants, see Herrlich–Schmithüsen [13]. We introduce the following notion which is more suitable to our needs:

Definition 8.1. Let $t > 0$ be a real number, and let $R_t = [0, 1] \times [0, t] \subset \mathbb{R}^2 = \mathbb{C}$ be a rectangle of width 1 and height $t$. A bipartite $t$-origami consists of finitely many copies of $R_t$, partitioned into two sets called “grey” and “white rectangles” for convenience, and glued together along their edges by isometries, satisfying the same rules as in Definition 2.4.

The simplest bipartite $t$-origami is of course obtained by gluing two copies of $R_t$, one grey and one white, along their edges by the identity; this is a closed surface of genus zero and called the $t$-pillowcase $P_t$. Every bipartite $t$-origami becomes a ramified covering of $P_t$ unramified outside the four vertices of $P_t$, and vice versa every such covering defines a bipartite $t$-origami. As with bipartite origamis we get an identification between bipartite $t$-origamis and 4-dessins. We spell out what the choice of $t$ means in this context.
The metric on $P_t$ defines a complex structure, and for this complex structure $P_t \cong \mathbb{P}^1(\mathbb{C})$ as Riemann surfaces. The isomorphism becomes unique when demanding in addition that the first three vertices go to $\infty, 0, 1$ (in that order). The fourth vertex is then mapped to some point in $]1, \infty[$. For symmetry reasons, the pillowcase’s seam will be mapped to $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C})$, and the white rectangle to the upper half plane, the grey rectangle to the lower half plane. Let $f : R_t \to \mathbb{H}$ be the unique biholomorphism extending continuously to the boundaries such that

$$f(0) = \infty, \quad f(1) = 0 \text{ and } f(1 + it) = 1;$$

the point $f(it)$ is then a real number greater than one, called the *accessory parameter* of the rectangle. Therefore we denote it by $ap(t)$. We shall now express it in terms of modular functions.

For $\tau \in \mathbb{H}$ let $\wp_\tau$ be the Weierstrass $\wp$-function associated with the lattice $\mathbb{Z} + \tau \mathbb{Z}$; with this notation, the modular $\lambda$-function is the function

$$\lambda(\tau) = \frac{\wp_\tau \left( \frac{1+\tau}{2} \right) - \wp_\tau \left( \frac{\tau}{2} \right)}{\wp_\tau \left( \frac{1}{2} \right) - \wp_\tau \left( \frac{\tau}{2} \right)}.$$ 

For us it is slightly more convenient to work with a different permutation of the half-period values, and thus a different cross-ratio: we set

$$\lambda^*(\tau) = \frac{1}{1 - \lambda(\tau)} = \frac{\wp_\tau \left( \frac{1}{2} \right) - \wp_\tau \left( \frac{1+\tau}{2} \right)}{\wp_\tau \left( \frac{1}{2} \right) - \wp_\tau \left( \frac{\tau}{2} \right)}. \quad (10)$$

**Proposition 8.2.** For every $t > 0$ we have the identity $ap(t) = \lambda^*(it)$.

**Proof.** Write $\tau = ti$. Consider the biholomorphic function $f : R_t \to \mathbb{H}$ as in the definition of $ap(t)$; then the function $z \mapsto f(2z)$ gives a biholomorphism from $\frac{1}{2} R_t$ to the upper half plane which maps 0 to $\infty$, $\frac{1}{2}$ to 0 and $\frac{1+t}{2}$ to 1. By continued Schwarz reflection along the sides of $R_t$, the sides of its mirror images etc., we extend $f(2z)$ to a holomorphic function $p(z)$ on $\mathbb{C}$ minus the lattice $\mathbb{Z} + \tau \mathbb{Z}$. Its singularities at the lattice points are non-essential (e.g. by Picard’s theorem), so $p(z)$ is a meromorphic function on $\mathbb{C}$; it is also $(\mathbb{Z} + \tau \mathbb{Z})$-periodic and even, therefore it is a rational function in $\wp_\tau(z)$. Its poles at the lattice points are poles of order two, as one sees by considering the image of a little loop around a lattice point. Since it has no other poles, it must be of the form

$$p(z) = c \wp_\tau(z) + d$$

with $c, d \in \mathbb{C}$, $c \neq 0$. A direct computation shows the desired result. \qed

We now describe how $\lambda^*$ can be expressed in terms of other modular functions and modular forms, which gives a way to compute special values of $\lambda^*$. Recall that Dedekind’s *eta function* is the holomorphic function on the upper half plane

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \text{ where } q = e^{2\pi i \tau} \text{ and } q^{\frac{1}{24}} = e^{\frac{\pi i}{12}}. \quad (11)$$

It is related to the modular discriminant $\Delta(\tau)$ by

$$\Delta(\tau) = (2\pi i)^{12} \eta(\tau)^{24} = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}. \quad (12)$$
In his famous *Lehrbuch der Algebra* [33], Weber introduces three modular functions $f, f_1$ and $f_2$ given by

$$f(\tau) = e^{-\frac{\pi i}{24}} \frac{\eta\left(\frac{\tau+1}{2}\right)}{\eta(\tau)} = q^{-1/48} \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}}),$$

$$f_1(\tau) = \frac{\eta\left(\frac{\tau}{2}\right)}{\eta(\tau)} = q^{-1/48} \prod_{n=1}^{\infty} (1 - q^{n-\frac{1}{2}}),$$

$$f_2(\tau) = \sqrt{2} \frac{\eta(2\tau)}{\eta(\tau)} = \sqrt{2} q^{1/24} \prod_{n=1}^{\infty} (1 + q^n).$$

They satisfy a plethora of beautiful relations, two of which are of importance for us (both stated as formula (10) in § 34 of Weber [33]):

$$f(\tau)f_1(\tau)f_2(\tau) = \sqrt{2}$$

and

$$f(\tau)^8 = f_1(\tau)^8 + f_2(\tau)^8.$$  \hfill (15)

Here, (14) is a direct consequence of the $q$-expansions, whereas (15) requires some work. Weber proves it by expressing his functions in turn by theta functions.

**Proposition 8.3.** The function $\lambda^*$ can be expressed as

$$\lambda^*(\tau) = \frac{f(\tau)^8}{f_1(\tau)^8} = e^{-\frac{\pi i}{24}} \frac{\eta\left(\frac{\tau+1}{2}\right)^8}{\eta\left(\frac{\tau}{2}\right)^8} = -\frac{\Delta\left(\frac{\tau+1}{2}\right) + 16\Delta(\tau)}{\Delta\left(\frac{\tau}{2}\right) + 16\Delta(\tau)}.$$  \hfill (16)

**Proof.** We follow Cox [5] in his proof for his Theorem 12.17 where the third root of $j(\tau)$ is expressed by Weber functions. Formula (12.18) on page 257 (in the proof of Theorem 12.17) says that

$$\wp\left(\frac{1}{2}\right) - \wp\left(\frac{\tau+1}{2}\right) = \pi^2 \eta(\tau)^4 f(\tau)^8$$

and

$$\wp\left(\frac{1}{2}\right) - \wp\left(\frac{\tau}{2}\right) = \pi^2 \eta(\tau)^4 f_1(\tau)^8.$$  \hfill (12)

From this and (10) we get the first equality in (16). The second equality is a direct consequence of the defining formula (13).

Part of the statement in Cox’ Theorem 12.17 is the equation

$$\frac{f(\tau)^{24} - 16}{f(\tau)^8} = \frac{f_1(\tau)^{24} + 16}{f_1(\tau)^8}$$  \hfill (17)

(both are equal to a third root of $j(\tau)$). From the first equality in (16) and (17) we obtain

$$\lambda^*(\tau) = \frac{f(\tau)^{24} - 16}{f_1(\tau)^{24} + 16} = \frac{-\eta\left(\frac{\tau+1}{2}\right)^{24}}{\eta(\tau)^{24}} - 16 = -\frac{\eta\left(\frac{\tau+1}{2}\right)^{24} + 16\eta(\tau)^{24}}{\eta\left(\frac{\tau}{2}\right)^{24} + 16\eta(\tau)^{24}},$$

and using (12) we get the last equality in (16). \hfill \square

\footnote{Note that Cox uses different notations for $e_1, e_2$ and $e_3$.}
Corollary 8.4. Writing \( q_2 = q^{1/2} = e^{\pi i \tau} \), the Fourier expansion of \( \lambda^*(\tau) \) is

\[
\lambda^*(\tau) = \left( \prod_{m=0}^{\infty} \frac{1+q_2^{2m+1}}{1-q_2^{2m+1}} \right)^8 \in \mathbb{Z}[q_2],
\]

so we get the power series expansion

\[
\lambda^*(\tau) = (1 + 2q_2 + 2q_2^2 + \ldots)^8.
\]

Being interested in number theory, we prefer those rectangles \( R_t \) where both \( t \) and \( \text{ap}(t) \) are algebraic. To decide when this is the case, we recall that \( \lambda \), and therefore also \( \lambda^* \), is intimately related to the \( j \)-invariant; more precisely we have\(^3\)

\[
j(\tau) = 256 \frac{(x^2 - x + 1)^3}{x^2(x-1)^2} \quad \text{with} \quad x = \lambda^*(\tau)
\]

for any \( \tau \in \mathbb{H} \).

Theorem 8.5. For \( t > 0 \) the following conditions are equivalent:

(i) Both \( t \) and \( \text{ap}(t) \) are algebraic numbers.

(ii) \( t^2 \) is rational.

Proof. Clearly \( t \) is algebraic if and only if \( \tau = \sqrt{-a} \) is algebraic. Also by equation (18), \( \lambda^*(\tau) \) is algebraic if and only if \( j(\tau) \) is algebraic. By a theorem of Schneider \([28]\), both \( \tau \) and \( j(\tau) \) are algebraic if and only if \( \tau \) is imaginary quadratic.

Assume (ii), say \( t^2 = a \), then \( t = \sqrt{-a} \) is imaginary quadratic, showing (i). For the other implication, note that if \( \tau = \sqrt{-a} \) is imaginary quadratic, we can write \( \tau = a + b\sqrt{-d} \) with \( a, b \in \mathbb{Q} \) and \( d \in \mathbb{N} \). This can only be purely imaginary if \( a = 0 \), and then \( t^2 = b^2d \) is rational. \(\square\)

Corollary 8.6. Let \( t > 0 \) be such that \( t^2 \) is rational, and let \( K = \mathbb{Q}(\text{ap}(t)) \). The action \( \varrho_{\text{ap}(t)} : G_K \to \mathfrak{D}_4 \) can be understood geometrically as follows: let \( \sigma \in G_K \) and let \( \Gamma, \Gamma' \) be 4-dessins such that \( \varrho_{\text{ap}(t)}(\sigma)(\Gamma) = \Gamma' \). Let \( \mathcal{O} \) and \( \mathcal{O}' \) be the bipartite \( t \)-origamis corresponding to \( \Gamma \) and \( \Gamma' \). Let then \( X \) and \( X' \) be the algebraic curves over \( \overline{\mathbb{Q}} \) determined by the conformal classes of the metrics of \( \mathcal{O} \) and \( \mathcal{O}' \), and let \( f : X \to \mathbb{P}^1 \) and \( f' : X' \to \mathbb{P}^1 \) be the morphisms of curves corresponding to the natural maps \( \mathcal{O} \to \mathcal{P}_t \) and \( \mathcal{O}' \to \mathcal{P}_t \). Then \( \sigma(X) = X' \) and \( \sigma(f) = f' \).

For this reason we now focus on computing \( \text{ap}(\sqrt{n}) = \lambda^*(\sqrt{-n}) \) for a few positive integers \( n \), in order to have explicit examples. Here it turns out that expressing \( \lambda^*(\tau) \) in terms of Weber functions is extremely helpful. The very reason they were introduced by Weber is precisely that they facilitate computations for singular values of the \( j \)-invariant.

Weber computes a great number of singular values of his functions in \([33]\). This in turn enables us to compute \( \text{ap}(\sqrt{n}) \), by the following method: assume first that \( f(\sqrt{-n})^8 \) is explicitly known.

---

\(^3\)This equation is well-known for \( x = \lambda(\tau) \) instead of \( \lambda^*(\tau) \), see e.g. \([7\) Satz 7.7]. Assuming this, we obtain the desired equation by just substituting \( \lambda = 1 - 1/\lambda^* \) and computing the result. Of course, the expression on the right hand side is just made in such a way that it is invariant under Möbius transformations which permute 0, 1 and \( \infty \).
Set \( u = f_1(\sqrt{-n})^8 \) and \( v = f_2(\sqrt{-n})^8 \). By (14) and (15), we find that
\[
\begin{align*}
  u + v &= f_1(\sqrt{-n})^8 \quad \text{and} \\
  uv &= 16 f_1(\sqrt{-n})^8.
\end{align*}
\]
That is, \( u \) and \( v \) are the solutions of the quadratic equation
\[
x^2 - f_1(\sqrt{-n})^8 x + \frac{16}{f_1(\sqrt{-n})^8} = 0.
\]
Computing \( u \) and \( v \) numerically with sufficiently high precision we find out which is which solution, and hence obtain an explicit expression for \( u = f_1(\sqrt{-n})^8 \). Since
\[
ap(\sqrt{n}) = \lambda^*(\sqrt{-n}) = \frac{f(\sqrt{-n})^8}{f_1(\sqrt{-n})^8},
\]
this gives an explicit expression for \( ap(\sqrt{n}) \).

Similarly, if \( f_1(\sqrt{-n}) \) is known, then \( f(\sqrt{-n}) \) is a solution of the quadratic equation
\[
x^2 - f_1(\sqrt{-n})^8 x - \frac{16}{f_1(\sqrt{-n})^8} = 0,
\]
and an analogous computation gives the value of \( ap(\sqrt{n}) \). Using Tabelle VI at the end of Weber [33] we computed all values \( ap(\sqrt{n}) \), where \( n \) runs through all positive integers with \( h(-4n) \leq 2 \) (here \( h \) is the class number — this constraint makes the formulæ more tractable). The result is given in Table 1, and may be read as follows: \( G_\mathbb{Q} \) operates upon bipartite \( \sqrt{2} \)- and \( 2 \)-origamis, \( G_{\mathbb{Q}(\sqrt{2})} \) operates upon bipartite \( \sqrt{2} \)- and \( 2 \)-origamis, and so on.

**Proposition 8.7.** Let \( \tau \in \mathbb{H} \) be imaginary-quadratic. Then \( 16\lambda^*(\tau) \) is an algebraic integer. In particular, for every rational \( t > 0 \), the number \( 16\text{ap}(\sqrt{t}) \) is an algebraic integer.

The examples \( t = 12, 16, 28 \) (cf. Table 1) show that the factor 16 cannot be replaced by a smaller one.

**Proof.** It is a classical result in the theory of complex multiplication that for \( \tau \) imaginary-quadratic, \( j(\tau) \) is an algebraic integer; see e.g. [3] Theorem 11.1. Recall equation (18) which links \( j(\tau) \) and \( \lambda^*(\tau) \). If we write for convenience \( y = 16\lambda^*(\tau) \), we deduce
\[
j(\tau) = \frac{(y^2 - 16y + 256)^3}{y^2(y - 16)^2}.
\]
This means that \( y \) satisfies a monic polynomial whose coefficients are algebraic integers, hence it is an algebraic integer itself. \( \square \)

### 9. Étale Fundamental Groups

To prove Theorems 6.6, 6.7 and 6.8 we translate them into statements about actions on étale fundamental groups. Recall that for a connected scheme \( X \) and a geometric point \( \bar{x} : \text{Spec } \Omega \to X \), the étale fundamental group is a profinite group \( \pi_1^{\text{ét}}(X, \bar{x}) \) which classifies pointed étale coverings of \((X, \bar{x})\), such that isomorphism classes of connected pointed étale coverings \((Y, \bar{y}) \to (X, \bar{x})\) are in one-to-one correspondence with open subgroups of \( \pi_1^{\text{ét}}(X, \bar{x}) \). For generalities about étale fundamental groups, see Grothendieck–Raynaud [9].
Table 1. Some values of \(ap(\sqrt{n})\).

| \(n\) | \(ap(\sqrt{n})\) | \(Q(ap(\sqrt{n}))\) |
|---|---|---|
| 1 | 2 | \(Q\) |
| 2 | \(\frac{1 + \sqrt{2}}{2}\) | \(Q(\sqrt{2})\) |
| 3 | \(8 - 4\sqrt{3}\) | \(Q(\sqrt{3})\) |
| 4 | \(\frac{1}{2} + \frac{3}{8}\sqrt{2}\) | \(Q(\sqrt{2})\) |
| 5 | \(18 + 8\sqrt{5} - (14 + 6\sqrt{5})\sqrt{\frac{1 + \sqrt{5}}{2}}\) | \(Q\left(\sqrt{\frac{1 + \sqrt{5}}{2}}\right)\) |
| 6 | \(\frac{1}{2} + \sqrt{3} - \frac{1}{2}\sqrt{6}\) | \(Q(\sqrt{2}, \sqrt{3})\) |
| 7 | \(128 - 48\sqrt{7}\) | \(Q(\sqrt{7})\) |
| 8 | \(\frac{1}{2} + \left(\frac{1}{4} + \frac{3}{8}\sqrt{2}\right)\sqrt{\sqrt{2} - 1}\) | \(Q(\sqrt[4]{2} - 1)\) |
| 9 | \(194 - 104\sqrt{12} + 56\sqrt{12} - 30\sqrt{12}\) | \(Q(\sqrt{12})\) |
| 10 | \(\frac{1}{2} + \frac{3}{2}\sqrt{10} - 3\sqrt{2}\) | \(Q(\sqrt{2}, \sqrt{5})\) |
| 12 | \(\frac{1}{2} - \frac{3}{16}\sqrt{2} + \frac{5}{16}\sqrt{6}\) | \(Q(\sqrt{2}, \sqrt{3})\) |
| 13 | \(1298 + 360\sqrt{13} - (714 + 198\sqrt{13})\sqrt{\frac{3 + \sqrt{13}}{2}}\) | \(Q\left(\sqrt{\frac{3 + \sqrt{13}}{2}}\right)\) |
| 15 | \(3008 - 1736\sqrt{3} + 1344\sqrt{5} - 776\sqrt{15}\) | \(Q(\sqrt{3}, \sqrt{5})\) |
| 16 | \(\frac{1}{2} - \frac{3}{8}\sqrt{2} + \frac{9}{16}\sqrt{2}\) | \(Q(\sqrt{2})\) |
| 18 | \(\frac{1}{2} + \frac{35}{2}\sqrt{2} - 14\sqrt{3}\) | \(Q(\sqrt{2}, \sqrt{3})\) |
| 22 | \(\frac{1}{2} + 15\sqrt{11} - \frac{21}{2}\sqrt{22}\) | \(Q(\sqrt{2}, \sqrt{11})\) |
| 25 | \(103682 - 69336\sqrt{5} + 46368\sqrt{5} - 31008\sqrt{5}\) | \(Q(\sqrt{5})\) |
| 28 | \(\frac{1}{2} + \left(\frac{129}{16} - 3\sqrt{7}\right)\sqrt{8 + 3\sqrt{7}}\) | \(Q(\sqrt{8 + 3\sqrt{7}})\) |
| 37 | \(\begin{cases} 3111698 + 5111560\sqrt{37} \\ -(895188 + 147168\sqrt{37})\sqrt{6 + \sqrt{37}} \end{cases}\) | \(Q(\sqrt{6 + \sqrt{37}})\) |
| 58 | \(\frac{1}{2} + \frac{1287}{2}\sqrt{58} - 3465\sqrt{2}\) | \(Q(\sqrt{2}, \sqrt{29})\) |
What we have said about the functorial nature of topological fundamental groups carries over to étale fundamental groups. For two basepoints in the same scheme, say \( \bar{x}_i: \text{Spec} \Omega_i \to X \) for \( i = 1, 2 \), there is a natural isomorphism \( \pi_1^{\text{ét}}(X, \bar{x}_1) \cong \pi_1^{\text{ét}}(X, \bar{x}_2) \), well-defined up to inner automorphisms.

Consider, in analogy to \( \mathcal{G} \) introduced before, the category \( \mathcal{G}^c \) whose objects are profinite groups, and where the set of morphisms between two profinite groups \( G \) and \( H \) is

\[
\text{Homext}(G, H) = \{ \text{continuous group homomorphisms } G \to H \}/H,
\]

where \( H \) acts by conjugation: \( (fh)(g) = h^{-1}f(g)h \). We call morphisms in this category continuous outer homomorphisms and automorphisms or isomorphisms in this category continuous outer automorphisms or continuous outer isomorphisms, respectively. For a continuous outer homomorphism between profinite groups \( G \) and \( H \), we use the symbol \( f: G \to H \), in order to distinguish it from an honest homomorphism.

Now the étale fundamental group can be interpreted as a functor from the category of connected schemes (with arbitrary scheme morphisms) to \( \mathcal{G}^c \). Nonetheless, we always write \( \text{Homext} \) and \( \text{Autext} \) for homomorphisms and automorphisms in the category \( \mathcal{G}^c \), reserving the notions \( \text{Hom} \) and \( \text{Aut} \) for group homomorphisms and automorphisms in the usual sense (or for something completely different, like schemes or sets).

**Geometric Fundamental Groups.** For a variety over a non-algebraically closed fields, the étale fundamental group of the variety itself is usually very complicated, so it is sensible to look rather at the following group:

**Definition 9.1.** Let \( K \) be a field of characteristic zero, and let \( \bar{K} \) be an algebraic closure of \( K \). Let \( X \) be a variety over \( K \) and let \( \bar{x} \) be a geometric point of \( X \). Then the geometric fundamental group of \( X \) with respect to \( \bar{x} \) is the profinite group

\[
\pi_1^\text{geom}(X, \bar{x}) = \pi_1^{\text{ét}}(X \times_{\text{Spec} K} \text{Spec} \bar{K}, \bar{x}).
\]

This terminology is justified by the Lefschetz principle for fundamental groups: if \( X \) is a variety over a field \( K \subseteq \mathbb{C} \) and we choose \( \bar{K} \) to be the algebraic closure of \( K \) in \( \mathbb{C} \), then \( \pi_1^\text{geom}(X, \bar{x}) \) is canonically isomorphic to the profinite completion of the topological fundamental group \( \pi_1(X(\mathbb{C}), \bar{x}) \).

**Galois Actions on Geometric Fundamental Groups.** For an algebraic variety \( X \) over \( K \), there is an “outer action” of the absolute Galois group \( G_K = \text{Gal}(\bar{K}/K) \) on \( \pi_1^\text{geom}(X) \), i.e. a homomorphism \( G_K \to \text{Aut}_\text{ext} \pi_1^\text{geom}(X) \). If \( \bar{x} \) is a \( K \)-rational geometric point in \( X \), this can even be lifted to an action in the classical sense \( G_K \to \text{Aut} \pi_1^\text{geom}(X, \bar{x}) \). The usual construction of these actions is by considering the sequence \( X_K \to X \to \text{Spec} \bar{K} \) as a scheme-theoretic analogue of a fibration and proving that we have a short exact sequence of fundamental groups; then by abstract group theory one proceeds to construct (outer) group homomorphisms. While such an approach is very useful in a variety of settings, we go a different way, first because it suits our needs, second because we think that it makes the Galois action much more transparent. It is not difficult to see that our construction gives the same actions as the usual one, hence we may apply all sorts of results to it.

The basic idea is very simple. An action on a fundamental group should come from an action on the associated space. Now for a variety \( X \) over \( K \), there is a natural action of \( G_K \) on the
“geometric variety” $X_R$ denoted by $\psi_X : G_K \to \text{Aut} X_R$. This works as follows: recall that $X_R$ is the fibre product $X \times_{\text{Spec} K} \text{Spec} \bar{K}$; for an element $\sigma \in G_K$, let $\psi_X(\sigma)$ be the automorphism of this fibre product that acts as the identity on the factor $X$ and as the scheme morphism $(\sigma^{-1})^\flat$ associated with the field isomorphism $\sigma^{-1}$ on the factor $\text{Spec} \bar{K}$. To visualise this action, note that it is the tautological (i.e., coordinate-wise) action on the closed points of $\mathbb{P}_K^n$ and commutes with restriction to subvarieties.

So, by functoriality of the fundamental group, we get a homomorphism

$$\varphi_X : G_K \xrightarrow{\psi_X} \text{Aut} X_{\bar{K}} \xrightarrow{\pi_1^{\text{geom}}(\cdot)} \text{Aut}_{\text{ext}} \pi_1^{\text{geom}}(X).$$

We call this the canonical outer Galois action on $\pi_1^{\text{geom}}(X)$. As it is defined by functoriality, we immediately see that it behaves well with respect to $K$-morphisms: if $X$ and $Y$ are $K$-varieties and $f : X \to Y$ is a $K$-morphism, then for every $\sigma \in G_K$ the following diagram commutes:

$$
\begin{array}{ccc}
X_K & \xrightarrow{f} & Y_K \\
\psi_X(\sigma) \downarrow & & \downarrow \psi_Y(\sigma) \\
X_R & \xrightarrow{f} & Y_{\bar{K}} \\
\end{array}
$$

Hence, also the following diagram of outer homomorphisms commutes (in terms of homomorphisms, this means that picking a representative for each outer homomorphism, the diagram commutes up to inner automorphisms):

$$
\begin{array}{c}
\pi_1^{\text{geom}}(X) \\
\varphi_X(\sigma) \\
\pi_1^{\text{geom}}(X) \\
\end{array}
\xrightarrow{f_*} 
\begin{array}{c}
\pi_1^{\text{geom}}(Y) \\
\varphi_Y(\sigma) \\
\pi_1^{\text{geom}}(Y) \\
\end{array}
$$

For later reference, we need to see how the outer Galois action behaves under base change. Since this is only needed for number fields, let us consider this case. So $K \subset \bar{Q}$ is a number field and $\sigma \in G_{\bar{Q}}$; set $L = \sigma(K) \subset \bar{Q}$. Taking $\bar{Q}$ as algebraic closure for both, we can view $G_K$ and $G_L$ as subgroups of both, satisfying $G_L = \sigma G_K \sigma^{-1}$. Let $X$ be a variety over $K$, then we get a variety $\sigma(X)$ over $L$; extending scalars in either case to $\bar{Q}$, we get varieties $X_{\bar{Q}}$ and $\sigma(X)_{\bar{Q}}$ over $\bar{Q}$. All these fit into a big commutative diagram
where the four vertical faces are cartesian squares, but the top and bottom squares are not. Also, all left-to-right maps are isomorphisms of schemes. We are interested in the outer isomorphism $\hat{m}_*: \pi_1^{\text{geom}}(\sigma(X)) \to \pi_1^{\text{geom}}(X)$.

**Lemma 9.2.** With the above notations, the outer isomorphism $\hat{m}_*: \pi_1^{\text{geom}}(\sigma(X)) \to \pi_1^{\text{geom}}(X)$ respects the Galois action up to conjugation with $\sigma$, i.e. for every $\gamma \in G_K$ we have

$$\hat{m}_* \circ \varphi_{\sigma(X)}(\sigma \gamma \sigma^{-1}) = \varphi_X(\gamma) \circ \hat{m}_*.$$  

(19)

**Proof.** We show that the equation (19) comes from a corresponding statement about morphisms between schemes, namely, the commutativity of the following diagram:

$$\begin{array}{ccc}
\sigma(X)_Q & \rightarrow & X_{\bar{Q}} \\
\downarrow \psi_{\sigma(X)}(\sigma \gamma \sigma^{-1}) & & \downarrow \hat{\psi}_X(\gamma) \\
\sigma(X)_{\bar{Q}} & \rightarrow & X_{\bar{Q}}
\end{array}$$

And this can be seen as follows: recalling that $X_{\bar{Q}} = X \times_{\text{Spec } K} \text{Spec } \bar{Q}$ and the morphism $\psi_X(\gamma) = \text{id}_X \times (\gamma^{-1})^0$ respects this fibre product structure, we only have to show the commutativity of the two “component diagrams”: the first component diagram is

$$\begin{array}{ccc}
\sigma(X)_Q & \rightarrow & X \\
\downarrow \psi_{\sigma(X)}(\sigma \gamma \sigma^{-1}) & & \downarrow \text{id}_X, \\
\sigma(X)_{\bar{Q}} & \rightarrow & X
\end{array}$$

which clearly commutes, and the second component diagram is

$$\begin{array}{ccc}
\sigma(X)_Q & \rightarrow & \text{Spec } \bar{Q} \\
\downarrow \psi_{\sigma(X)}(\sigma \gamma \sigma^{-1}) & & \downarrow \text{id}_{\text{Spec } \bar{Q}} \\
\sigma(X)_{\bar{Q}} & \rightarrow & \text{Spec } \bar{Q}
\end{array}$$

where $p: \sigma(X)_Q \to \text{Spec } \bar{Q}$ is the structure morphism. \qed

We now construct a Galois action on coverings and show that it is compatible with the Galois action on fundamental groups. Let $K$ be a field with algebraic closure $\bar{K}$, and let $X$ be a $K$-variety. Let $f: Z \to X_{\bar{K}}$ be an étale covering. We define a new étale covering $\sigma(f): \sigma(Z) \to X_{\bar{K}}$ by pullback along $\psi_X(\sigma^{-1})$. More precisely, consider the pullback diagram

$$\begin{array}{ccc}
\sigma(Z) & \rightarrow & Z \\
\downarrow f' & & \downarrow f \\
X_{\bar{K}} & \rightarrow & X_{\bar{K}}
\end{array}$$

the covering map $\sigma(f): \sigma(Z) \to X_{\bar{K}}$ is then the diagonal morphism in that square, i.e. $\sigma(f) = \psi_X(\sigma^{-1}) \circ f'$. This is obviously an étale covering again. Moreover, we have a canonical isomorphism of coverings $(\sigma \tau)(Z) \cong \sigma(\tau(Z))$, so we get a Galois action from the left on the set of isomorphism classes of coverings of $X_{\bar{K}}$. 

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Proposition 9.3. With the above constructions, if the covering $Z \to X_\bar{K}$ corresponds to the conjugacy class of subgroups represented by $U \leq \pi_1^{\text{geom}}(X, \bar{x})$, then the covering $\sigma(Z) \to X_\bar{K}$ corresponds to the conjugacy class of subgroups represented by $\sigma(U) = \varphi_X(\sigma)(U)$. □

10. Anabelian Geometry

We now describe a conjecture stated by Grothendieck in a famous letter to Faltings [11], now a theorem due to Mochizuki. With the notation from above, let $X$ and $Y$ be two varieties over $K$. Fix an algebraic closure $\bar{K}$ of $K$, and set $G_K = \text{Gal}(\bar{K}|K)$. A continuous outer isomorphism $f: \pi_1^{\text{geom}}(X) \to \pi_1^{\text{geom}}(Y)$ is called Galois-compatible if for every $\sigma \in G_K$, the diagram

$$
\begin{array}{ccc}
\pi_1^{\text{geom}}(X) & \to & \pi_1^{\text{geom}}(Y) \\
\varphi_X(\sigma) \downarrow & & \downarrow \varphi_Y(\sigma) \\
\pi_1^{\text{geom}}(X) & \to & \pi_1^{\text{geom}}(Y)
\end{array}
$$

commutes. Denote the set of conjugacy classes of continuous Galois-compatible isomorphisms $\pi_1^{\text{geom}}(X) \to \pi_1^{\text{et}}(Y)$ by

$$\text{Isomext}^{\text{Gal}}(\pi_1^{\text{geom}}(X), \pi_1^{\text{geom}}(Y)).$$

A hyperbolic curve over a field $K$ is a $K$-variety of the form $X = X' \smallsetminus D$, where $X'$ is a smooth projective curve of genus $g$ over $K$ and $D$ is a finite set of $K$-rational points of $X$, of cardinality $n$, such that $2g - 2 + n > 0$. The pair $(g, n)$ is called the type of $X$. In the above mentioned letter, Grothendieck conjectured the following statement which was later proved by Mochizuki in [22] (in greater generality, in fact):

Theorem 10.1 (Mochizuki). Let $K$ be a finitely generated field extension of $\mathbb{Q}$, and let $\bar{K}$ be an algebraic closure of $K$. Let $X$ and $Y$ be hyperbolic curves over $K$. Then the obvious map

$$\text{Isom}_K(X, Y) \longrightarrow \text{Isomext}^{\text{Gal}}(\pi_1^{\text{geom}}(X), \pi_1^{\text{geom}}(Y))$$

is a bijection. □

However, we only need this statement for rational hyperbolic curves, which has an earlier and easier proof by Tamagawa [31]. For an affine hyperbolic curve $X$ over a field $K$ of characteristic zero, the geometric fundamental group $\pi_1^{\text{geom}}(X, \bar{x})$ is a free profinite group; this can be seen by the application of the Lefschetz principle discussed above. The rank of $\pi_1^{\text{geom}}(X, \bar{x})$ is $2g + n - 1$ if $X$ is of type $(g, n)$. Then the outer Galois action on $\pi_1^{\text{geom}}(X, \bar{x})$ permutes the open normal subgroups, since these are stable under inner automorphisms. This action on normal subgroups uniquely determines the outer action we started with. To prove this, we use once again a statement that links actions on elements and subgroups of a group:

Theorem 10.2 (Jarden). Let $r \geq 2$ and let $F, F'$ be profinite completions of free groups on $r$ generators. Let $f, g \in \text{Isomext}(F, F')$ (continuous isomorphisms up to inner automorphisms on either side) and assume that for every open normal subgroup $N$ of $F$ one has $f(N) = g(N)$. Then $f = g$. 
Proof. The main theorem in Jarden [14] contains the following special case: a continuous group automorphism $\varphi$ of $F$ such that $\varphi(N) = N$ for every open normal subgroup $N$ must be an inner automorphism. Lift $f$ and $g$ to actual isomorphisms $f_0$ and $g_0$, and apply this to $f_0^{-1}g_0$. \qed

To prove Theorem 6.6 we combine Jarden’s result with Theorem 1.1 in Matsumoto–Tamagawa [21]:

**Theorem 10.3 (Matsumoto–Tamagawa).** Let $X$ be an affine hyperbolic curve defined over a number field $K \subset \mathbb{C}$. Then the image of the outer Galois representation $\varphi_X : G_K \to \text{Autext } \pi_1^{\text{geom}}(X)$ and the closure of the image of the canonical mapping class group action

$$ \text{PMod}^+(X(\mathbb{C})) \rightarrow \text{Autext } \pi_1(X(\mathbb{C})) \subset \text{Autext } \pi_1^{\text{geom}}(X) $$

intersect trivially. \qed

**Proof of Theorem 6.6.** Set $X = \mathbb{P}^1 \setminus D_\alpha$; we have a canonical identification of $\pi_1^{\text{geom}}(X)$ with the profinite completion $\hat{F}_{n-1}$ of the free group on $n - 1$ letters. With notations as in Theorem 6.6, note that both $\varphi_\alpha$ and $\varphi_\beta$ factor over the canonical map $\text{Autext } \pi_1^{\text{geom}}(X) = \text{Autext } \hat{F}_{n-1} \rightarrow \hat{S}_n$. This map is injective by Theorem 10.2, so the statement of Theorem 6.6 reduces to the statement of Theorem 10.3. \qed

Theorem 6.7 can be deduced in a similar way from Theorem 10.3.

**Proof of Theorem 6.7.** Let $X = \mathbb{P}^1_K \setminus D_\alpha$ and $Y = \mathbb{P}^1_K \setminus D_\beta$. There is an orientation-preserving homeomorphism

$$ g : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C}) $$

with $g(\mathbb{P}^1(\mathbb{R})) = \mathbb{P}^1(\mathbb{R})$ and $g(D_\alpha) = D_\beta$, inducing the identity on $\infty$, $0$, $1$ and sending $a_i$ to $b_i$. Since $g$ defines a homeomorphism $X(\mathbb{C}) \rightarrow Y(\mathbb{C})$, it induces an outer isomorphism between the respective topological fundamental groups, hence by profinite completion also an outer isomorphism of geometric fundamental groups:

$$ g_* : \pi_1^{\text{geom}}(X) \rightarrow \pi_1^{\text{geom}}(Y). $$

We claim that this is Galois-compatible. So let $\gamma \in G_K$ and let $\Gamma$ be an $n$-dessin corresponding to a normal covering. Interpreting it over $\alpha$ gives a normal $D_\alpha$-Belyi map and therefore an open normal subgroup $N \leq \pi_1^{\text{geom}}(X)$. The normal subgroup associated with $\varphi_\alpha(\gamma(\Gamma))$ is $\gamma(N) = \varphi_X(\gamma(\Gamma))$. On the other hand, interpreting it as an $n$-dessin over $b$, it corresponds to the normal open subgroup $g_\beta(N) \leq \pi_1^{\text{geom}}(Y)$, and therefore $\varphi_\beta(\gamma)(\Gamma)$ corresponds to the subgroup $\gamma(g_\beta(N)) \leq \pi_1^{\text{geom}}(Y)$. But since $\varphi_\beta(\gamma) = \varphi_\alpha(\gamma)$, this has to agree with $g_\beta(\gamma(N))$. So, to summarise this, for all open normal subgroups $N \leq \pi_1^{\text{geom}}(X)$ and all $\gamma \in G_K$ we have

$$ g_\beta(\gamma(N)) = \gamma(g_\beta(N)). $$

By Theorem 10.2 this implies that $g$ is Galois-compatible. Hence by Theorem 10.1 there exists a unique isomorphism $f : X \rightarrow Y$ inducing $g_\alpha$ on geometric fundamental groups. In particular there is an orientation-preserving homeomorphism (namely $f$ on complex points), which we also denote by $f : X(\mathbb{C}) \rightarrow Y(\mathbb{C})$, inducing $g_*$. Since $f$ can be extended to an automorphism of $\mathbb{P}^1$ and since $g_\alpha$ sends the monodromy around $\infty$, $0$, $1$ to itself (respectively), $f$ must be the identity and $X = Y$, whence $a = b$. \qed
To prove Theorem 6.8, we apply the same method again and in addition make use of the following central result of anabelian geometry, see Uchida [32] for the number field case and Pop [25], [26] for the general case:

**Theorem 10.4 (Neukirch–Uchida–Pop).** Let $K,L$ be fields which are finitely generated over $\mathbb{Q}$, and let $\bar{K}$ and $\bar{L}$ be algebraic closures. Let $\Phi: \text{Gal}(\bar{K}|K) \to \text{Gal}(\bar{L}|L)$ be a continuous group isomorphism. Then there exists an isomorphism $\varphi: \bar{K} \to \bar{L}$ with $\varphi(K) = L$ and such that for every $\sigma \in \text{Gal}(\bar{K}|K)$ one has $\Phi(\sigma) = \varphi^{-1} \circ \sigma \circ \varphi$. □

The crucial technical work for proving Theorem 6.8 is hidden in the following lemma, which reduces everything to the classical case of three punctures:

**Lemma 10.5.** Let $D \subset \mathbb{P}^1(\bar{\mathbb{Q}})$ be a finite subset containing $\{\infty, 0, 1\}$, and set $X = \mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus D$. Let further $\sigma \in G_{\mathbb{Q}}$. Let $\bar{m}: \sigma(X) \to X$ be the natural isomorphism of schemes from the definition of $\sigma(X)$ as pullback $X \times_{\text{Spec } K} \text{Spec } K$. Then $\bar{m}$ induces an outer isomorphism of étale fundamental groups: $\bar{m}_*: \pi_1^\text{ét}(\sigma(X)) \to \pi_1^\text{ét}(X)$. Assume further that $g: \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is an orientation-preserving homeomorphism which induces $\sigma^{-1}$ on $D$; in particular $g$ defines an orientation-preserving homeomorphism $\sigma(X)(\mathbb{C}) \to X(\mathbb{C})$. Assume finally that the two induced outer isomorphisms on geometric fundamental groups are equal:

$$\bar{m}_* = g_*: \pi_1^\text{ét}(\sigma(X)) \cong \pi_1^\text{ét}(X).$$

Then $\sigma = \text{id}$, and $g$ is isotopic rel. $D$ to the identity.

**Proof.** The idea is simple: we fill in some of the punctures and reduce to the case $D = \{\infty, 0, 1\}$, where the outer Galois representation is injective and the mapping class group is trivial.

So let $Y = \mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{\infty, 0, 1\} \subset \mathbb{P}^1_{\bar{\mathbb{Q}}}$; the practical thing about this variety is that it is stable under $\sigma$, i.e. $\sigma(Y) = Y$ as subschemes of $\mathbb{P}^1_{\bar{\mathbb{Q}}}$. Let $j: X \to Y$ be the inclusion; then the following diagram of schemes commutes:

$$\begin{array}{ccc}
\sigma(X) & \xrightarrow{\bar{m}} & X \\
\sigma(j) \downarrow & & \downarrow j \\
Y & \xleftarrow{\psi_Y(\sigma)} & Y \\
\end{array}$$

Hence also the diagram that we get when applying $\pi_1^\text{ét}$ commutes:

$$\begin{array}{ccc}
\sigma(X) & \xrightarrow{\bar{m}_*} & X \\
(\sigma(j))_* \downarrow & & \downarrow j_* \\
Y & \xleftarrow{\varphi_Y(\sigma)} & Y \\
\end{array}$$

Now the diagram of topological spaces

$$\begin{array}{ccc}
(\sigma(X))(\mathbb{C}) & \xrightarrow{g} & X(\mathbb{C}) \\
(\sigma(j)) \downarrow & & \downarrow j \\
Y(\mathbb{C}) & \xleftarrow{g} & Y(\mathbb{C}) \\
\end{array}$$
induces the same diagram on profinite fundamental groups, except possibly for the bottom line. But \( \sigma(j) \) and \( (\sigma(j))_* \) being onto, it is also forced that the induced outer homomorphisms on the bottom line agree. Now recall that \( g \) induces the identity on \( \{\infty, 0, 1\} \) and it is orientation-preserving; since the pure oriented mapping class group of a thrice-punctured sphere is trivial, the restriction of \( g \) to \( Y(\mathbb{C}) \) must be isotopic to the identity, therefore the induced map

\[
\varphi_Y(\sigma) = g_*: \pi_1^{\text{geom}}(\mathbb{P}^1_{\mathbb{Q}} \setminus \{\infty, 0, 1\}) \to \pi_1^{\text{geom}}(\mathbb{P}^1_{\mathbb{Q}} \setminus \{\infty, 0, 1\})
\]

must be the identity. But since the outer Galois representation on this fundamental group is faithful by Belyi’s theorem, this means \( \sigma = \text{id} \). As a consequence, \( g_* = \text{id} \), so \( g \) is isotopic rel. \( D \) to the identity. \( \square \)

**Proof of Theorem 6.8** The proof is similar to that of Theorem 6.7 but has some more twists in it. We therefore spell it out almost completely, at the danger of being redundant at times.

By Proposition 6.5, there is a unique topological isomorphism of profinite groups \( \Phi: G_K \to G_L \) such that \( \phi_a = \phi_b \circ \Phi \). By Pop’s theorem, \( \Phi \) must be conjugation in \( G_{\mathbb{Q}} \) with some \( \sigma \in G_{\mathbb{Q}} \). Thus, \( \sigma(K) = L \) and

\[
\phi_a(\gamma) = \phi_b(\sigma \gamma \sigma^{-1}). \tag{20}
\]

Now set \( X = \mathbb{P}^1_K \setminus D_a \) and \( Y = \mathbb{P}^1_L \setminus D_b \). We can then consider the geometric fundamental groups \( \pi_1^{\text{geom}}(X) \) with an outer action of \( G_K \) and \( \pi_1^{\text{geom}}(Y) \) with an outer action of \( G_L \).

As in the proof of Theorem 6.7 we find a homeomorphism

\[
g: \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})
\]

with \( g(D_a) = D_b \), inducing the identity on \( \infty, 0, 1 \) and sending \( a_i \) to \( b_i \), and an induced outer isomorphism between profinite groups:

\[
g_*: \pi_1^{\text{geom}}(X) \to \pi_1^{\text{geom}}(X).
\]

Again we need to investigate how this behaves with respect to Galois actions. Applying the same reasoning as in the proof of Theorem 6.7, we find that for all open normal subgroups \( N \leq \pi_1^{\text{geom}}(X) \) and all \( \gamma \in G_K \) we have

\[
g_*(\gamma(N)) = \sigma \gamma \sigma^{-1} g_*(N).
\]

By Theorem 10.2 this implies that \( g \) is “Galois-compatible up to a twist by \( \sigma \)”, i.e.

\[
g_* \circ \varphi_X(\gamma) = \varphi_Y(\sigma \gamma \sigma^{-1}) \circ g_* \quad \tag{21}
\]

To apply Mochizuki’s theorem, we need to get varieties over the same base field. So we consider the variety \( \sigma(X) \) over \( L \) and the outer isomorphism \( \bar{m}_*: \pi_1^{\text{geom}}(\sigma(X)) \to \pi_1^{\text{geom}}(X) \) from Lemma 9.2. By composition, we get an isomorphism

\[
F = \bar{m}_* \circ \pi_1^{\text{geom}}(\sigma(X)) \to \pi_1^{\text{geom}}(Y).
\]

This is Galois-compatible for the outer action of \( G_L \); for any \( \sigma \gamma \sigma^{-1} \in G_L \) we have

\[
F \circ \varphi_*(\sigma \gamma \sigma^{-1}) = \bar{m}_* \circ \pi_1^{\text{geom}}(\sigma(X)) (\sigma \gamma \sigma^{-1}) = \bar{m}_* \circ \varphi_X(\gamma) \circ \pi_1^{\text{geom}}(\sigma(X)) (\sigma \gamma \sigma^{-1}) = \varphi_Y(\sigma \gamma \sigma^{-1}) \circ g_* \circ \bar{m}_* = \varphi_Y(\sigma \gamma \sigma^{-1}) \circ F.
\]

Here the second equals sign comes from Lemma 9.2 and the third one from (21). Finally we can apply Theorem 10.1 to obtain an isomorphism of \( L \)-varieties \( f: \sigma(X) \to Y \) inducing \( F \) on geometric fundamental groups. In particular there is an orientation-preserving homeomorphism...
(namely \( f \) on complex points), which we also denote by \( f: \sigma(X)(\mathbb{C}) \to Y(\mathbb{C}) \), inducing \( F \). But then
\[
g^{-1} \circ f: \sigma(X)(\mathbb{C}) \to X(\mathbb{C})
\]
induces the outer homomorphism \( g^{-1} \circ f = \tilde{m} \), between fundamental groups \( \pi_1^{\text{geom}}(\sigma(X)) \to \pi_1^{\text{geom}}(X) \), and we conclude by Lemma 10.3 that \( \sigma = \text{id} \), in particular \( K = L \) and \( \tilde{m} = \text{id} \). But this means that \( f \) is an isomorphism of \( L \)-varieties, inducing the same outer isomorphism on fundamental groups as the homeomorphism \( g \). We conclude as in the proof of Theorem 6.7.

\[
\square
\]

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