ON OBLIQUE DOMAINS OF JANOWSKI FUNCTION

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Abstract. We investigate certain properties of tilted (oblique) domains, associated with the Janowski function \((1 + Az)/(1 + Bz)\), where \(A, B \in \mathbb{C}\) with \(A \neq B\) and \(|B| \leq 1\). We find several bounds for these oblique domains. Further, we extend the various known argument, radius, and subordination results involving Janowski functions with complex parameters.

1. Introduction

Let \(\mathcal{H}(\mathbb{D})\) denote the class of analytic functions defined on the open unit disk \(\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}\). Assume \(\mathcal{H}[a, n] := \{f \in \mathcal{H}(\mathbb{D}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}\), where \(n = 1, 2, \ldots\) with \(a \in \mathbb{C}\) and \(\mathcal{H}_1 := \mathcal{H}[1, 1]\). Let \(\mathcal{A}_n := \{f \in \mathcal{H}(\mathbb{D}) : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots\}\) and \(\mathcal{A} := \mathcal{A}_1\). A subclass of \(\mathcal{A}\) consisting of all univalent functions is denoted by \(\mathcal{S}\). We say, \(f\) is subordinate to \(g\), written as \(f \prec g\), if \(f(z) = g(\omega(z))\) where \(f, g\) are analytic functions and \(\omega(z)\) is a Schwarz function. Moreover, if \(g\) is univalent, then \(f \prec g\) if and only if \(f(\mathbb{D}) \subseteq g(\mathbb{D})\) and \(f(0) = g(0)\). For \(-\pi/2 < \lambda < \pi/2\), Wang in [24] introduced the tilted Carathéodory class by angle \(\lambda\) as:

\[
\mathcal{P}_\lambda := \left\{ p \in \mathcal{H}_1 : e^{i\lambda} p(z) < \frac{1 + z}{1 - z} \right\}.
\]

Here \(\mathcal{P}_0 = \mathcal{P}\), the well known Carathéodory class. A Janowski function is a bilinear transformation, which was first investigated by Janowski in [4]. He introduced the class \(\mathcal{P}(A, B)\), where \(-1 \leq B < A \leq 1\), which comprises of the set of all \(p\) in \(\mathcal{H}_1\) such that

\[
p(z) < \frac{1 + Az}{1 + Bz}.
\]

\(\mathcal{S}^*(A, B)\), the class of Janowski starlike functions and \(\mathcal{C}(A, B)\), the class of Janowski convex functions, consist of the set of all \(f \in \mathcal{S}\) satisfying

\[
\frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz} \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} < \frac{1 + Az}{1 + Bz},
\]

respectively. For \(0 \leq \alpha < 1\), \(\mathcal{S}^*(1 - 2\alpha, -1) = \mathcal{S}^*(\alpha)\), \(\mathcal{C}(1 - 2\alpha, -1) = \mathcal{C}(\alpha)\), the classes of starlike and convex functions of order \(\alpha\) respectively and \(\mathcal{S}^*(1, 2\alpha - 1) = \mathcal{RS}^*(\alpha)\), the class of starlike functions of reciprocal order \(\alpha\), which can also be stated as:

\[
\mathcal{RS}^*(\alpha) = \left\{ f \in \mathcal{S} : \text{Re} \frac{zf'(z)}{f(z)} > \alpha \right\}.
\]

Clearly, \(\mathcal{RS}^*(\alpha) \subset \mathcal{RS}^*(0) = \mathcal{S}^*(0) = \mathcal{S}^*\). And for \(\beta > 1\), \(\mathcal{S}^*(1 - 2\beta, -1) = \mathcal{M}(\beta)\), the class of Uralegaddi functions, which is stated as:

\[
\mathcal{M}(\beta) = \left\{ f \in \mathcal{S} : \text{Re} \frac{zf'(z)}{f(z)} < \beta \right\}.
\]

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For $-\pi/2 < \alpha < \pi/2$ and $0 \leq \beta < 1$, $S^*(e^{-i\alpha} (e^{-i\alpha} - 2\beta \cos \alpha), -1) = S^*_\alpha(\beta)$, the class of $\alpha$-spirallike of order $\beta$. From the above classes the range of $A$ in $\mathcal{P}_\lambda$, $M(\beta)$ and $S^*_\alpha(\beta)$ is not as given by Janowski. This motivates us to extend and study Janowski function with complex parameters. In the past many authors have studied $(1 + Az)/(1 + Bz)$, where $A, B \in \mathbb{C}$ with $|B| \leq 1$ and $A \neq B$. For instance see [1, 5, 6]. In the present paper we investigate the class $\mathcal{P}(A, B, \alpha)$, where $A, B \in \mathbb{C}$ with $|B| \leq 1$, $A \neq B$ and $0 < \alpha \leq 1$, which includes the set of all $p \in \mathcal{H}_1$ satisfying

$$p(z) < \left( \frac{1 + Az}{1 + Bz} \right)^\alpha.$$  

The class of Janowski strongly starlike functions of order $\alpha$, denoted by $SS^*(A, B, \alpha)$, is the set of all $f \in \mathcal{S}$ such that

$$zf'(z) < \left( \frac{1 + Az}{1 + Bz} \right)^\alpha.$$  

In particular, $SS^*(1, -1, \alpha) = SS^*(\alpha)$, the class of strongly starlike functions of order $\alpha$, which can also be stated as:

$$SS^*(\alpha) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} < \left( \frac{1 + Az}{1 + Bz} \right)^\alpha \right\}$$

and note that $SS^*(1, -1, 1) = SS^*(1) = S^*$. Apart from the above classes we also establish various subordination, radius, argument problems involving Janowski function with complex parameters. Also we point out many known results in this direction which are especial cases of our result.

2. Basic properties of the class $\mathcal{P}(A, B, \alpha)$

In the present section we discuss various geometric properties concerned with functions belonging to $\mathcal{P}(A, B, \alpha)$. We begin with the following bound estimate result.

**Theorem 2.1.** Let $h \in \mathcal{H}_1$ and $A, B \in \mathbb{C}$ with $A \neq B$ and $|B| \leq 1$. Further if $|A - B| \leq |1 - AB|$ and

$$h(z) < (1 - \gamma)\left( \frac{1 + Az}{1 + Bz} \right)^\alpha + \gamma,$$

for some $0 < \alpha \leq 1$, $\gamma \in \mathbb{C}\setminus\{1\}$, then for $|z| = r < 1$, we have

(i) $\left| \arg(h(z) - \gamma) - \alpha \tan^{-1} \frac{\text{Im}(AB)r^2}{\text{Re}(AB)r^2 - 1} \right| < \alpha \sin^{-1} \frac{|A - B|r^2}{|1 - AB|r^2}$

(ii) $\left( \frac{1 - ABr^2 - |A - B|r^2}{1 - |B|^2r^2} \right)^\alpha \leq \left| \frac{1}{1 - \gamma} \right| \leq \frac{\left| \frac{h(z) - \gamma}{1 - \gamma} \right|}{\left| \frac{1 - ABr^2 + |A - B|r^2}{1 - |B|^2r^2} \right|^\alpha}$

(iii) $M(t_1 + \pi) \cos(N(t_1 + \pi)) \leq \text{Re} \left( \frac{h(z) - \gamma}{1 - \gamma} \right) \leq M(t_1) \cos(N(t_1))$

(iv) $M(\tau - t_2) \sin(N(\tau - t_2)) \leq \text{Im} \left( \frac{h(z) - \gamma}{1 - \gamma} \right) \leq M(t_2) \sin(N(t_2))$,

where $M(t) = \left( \sqrt{(u(t))^2 + (v(t))^2} \right)^\alpha$ and $N(t) = \alpha \tan^{-1} \left( \frac{v(t)}{u(t)} \right)$, with $u(t) = \frac{1 - \text{Re}(AB)r^2 + |A - B|r \cos t}{1 - |B|^2r^2}$ and $v(t) = \frac{|A - B|r \sin t - \text{Im}(AB)r^2}{1 - |B|^2r^2}$. Further $t_1$ and $t_2$ are the roots of

$$\frac{u(t)u'(t) + v(t)v'(t)}{u(t)v'(t) - v(t)u'(t)} = \tan \left( \alpha \tan^{-1} \frac{v(t)}{u(t)} \right)$$

and

$$\frac{v(t)u'(t) - u(t)v'(t)}{u(t)u'(t) + v(t)v'(t)} = \tan \left( \alpha \tan^{-1} \frac{v(t)}{u(t)} \right)$$

respectively.
Proof. Let \( H(z) := ((h(z) - \gamma)/(1 - \gamma))^{\frac{1}{2}} \). Since \( h(z) \neq 0 \) and \( h(0) = 1 \), we have \( H \in \mathcal{H}_1 \) and
\[
H(z) \prec \frac{1 + Az}{1 + Bz}. \tag{2.4}
\]
As \( A, B \in \mathbb{C} \), clearly \( H(z) \) is contained in the oblique circle, shown in the Figure 1, whose radius and center are given by \( R := |A - B|r/(1 - |B|^2r^2) \) and \( C := 1 - A\overline{B}r^2/(1 - |B|^2r^2) \), respectively with angles \( \tau(r) := \tan^{-1}\left(\frac{\text{Im}(A\overline{B})r^2}{\text{Re}(A\overline{B})r^2 - 1}\right) \) and \( \zeta(r) := \sin^{-1}\left(|A - B|r^2/(|1 - A\overline{B}|r^2)\right) \).

By taking argument estimate of (2.4) and using the Figure 1 with the fact that circle is symmetric about the line passing through origin and center, we obtain (i). Let \( |z| = r < 1 \) and \( t \in [0, 2\pi) \), we have \( (h(re^{it}) - \gamma)/(1 - \gamma) \in \Omega := ((1 + Az)/(1 + Bz))^\alpha \), which implies \( \partial \Omega : (C + Re^{it})^\alpha := M(t)e^{iN(t)} \).

To find modulus, real and imaginary parts estimate, we need the critical points. By a simple computation, we obtain \( M'(t) = 0 \) at \( t = \tau(1) = \tau \) and \( \tau + \pi \), \( (M(t) \cos N(t))' = 0 \) at the roots \( t_1 \) and \( t_1 + \pi \) of the equation \( (2.2) \) and \( (M(t) \sin N(t))' = 0 \) at the roots \( t_2 \) and \( \tau - t_2 \) of the equation \( (2.3) \), all these values eventually yield (ii), (iii) and (iv) respectively.

![Figure 1. The image of \( \partial \mathbb{D} \) under \( (1 + Az)/(1 + Bz) \).](image)

Remark 2.2. 1. When \( \gamma = 0 \), we can obtain from Theorem 2.1 various bound estimates for functions in \( \mathcal{P}(A, B, \alpha) \).
2. By taking \( \alpha = 1/2, \gamma = 0, A = 1 \) and \( B = 0 \) in Theorem 2.1, we have \( h(z) \prec \sqrt{1 + z} \). Therefore the estimates are \( |\arg h(z)| \leq \sin^{-1} r/2 \) and \( \sqrt{1 - r} \leq |h(z)| \leq \sqrt{1 + r} \). If \( r = 1 \), we have \( t_1 = 0 \) and \( t_2 = 2\pi/3 \), which implies \( 0 < \text{Re } h(z) < \sqrt{2} \) and \( -0.5 < \text{Im } h(z) < 0.5 \).
3. Note that if \( w(z) = (1 + Az)/(1 + Bz) \), then \( w(0) = 1 \in w(\mathbb{D}) \). Thus the image domain \( w(\mathbb{D}) \) will always intersect real axis even if it is an oblique domain, non-symmetric with respect to real axis.

For \( \alpha = 1 \) and \( \gamma = 0 \), Theorem 2.1 reduces to the following sharp bounds:

Corollary 2.3. Let \( h \in \mathcal{H}_1 \) and \( A, B \in \mathbb{C} \) with \( A \neq B \), \( |A| \leq 1 \) and \( |B| \leq 1 \). Further if
\[
h(z) \prec \frac{1 + Az}{1 + Bz},
\]
then for \( |z| = r < 1 \), we have
is analytic, univalent and convex in an exterior or interior or boundary point of $H$

Clearly, $(1 + Az)$ given in (2.7) satisfy $(1 + Az)/|1 - bz|$ maps the unit disk onto

$$H(D) := \left\{ w \in \mathbb{C} : \left| w - \frac{1 - A\overline{B}}{1 - |B|^2} \right| < \frac{|A - B|}{1 - |B|^2} \right\}. \quad (2.5)$$

Clearly, $H(D)$ represents a disk when $b < 1$ and a half plane when $b = 1$. We see that $w = 0$ is an exterior or interior or boundary point of $H(D)$ is decided by the value of $a$ as $a < 1$ or $a > 1$ or $a = 1$ respectively. Therefore to investigate argument related problems of a Janowski function, we take $0 \leq a \leq 1$ or $|A - B| \leq |1 - A\overline{B}|$ so that $w = 0$ is not an interior point of $H(D)$. Following are the radius and center of the disk (2.5):

$$R := \frac{|A - B|}{1 - |B|^2} = \sqrt{a^2 + b^2 - 2ab \cos((n - m)\pi)} \frac{1}{1 - b^2},$$

$$C := \frac{1 - A\overline{B}}{1 - |B|^2} = \frac{1 - ab \cos((n - m)\pi) + iab \sin((n - m)\pi)}{1 - b^2}.$$

We observe that whenever the difference of $m$ and $n$ is same then the corresponding $R$ and $C$ also remain same. Therefore without lose of generality, we can fix $n = 1$. Accordingly we confine our study of Janowski function by considering

$$h(z) = \left( \frac{1 + e^{im\pi}z}{1 - z} \right)^\alpha, \quad (2.7)$$

where $A + b \neq 0$, $A \in \mathbb{C}$, $0 \leq b \leq 1$ and the class $\mathcal{P}(A, b) = \{p \in \mathcal{H}_1 : p(z) \prec (1 + Az)/(1 - bz)\}$. The corresponding class of Janowski strongly starlike functions of order $\alpha$ is denoted by $\mathcal{S}S^*(A, b, \alpha) = \{f \in \mathcal{S} : z^\alpha f'(z)/f(z) \prec ((1 + Az)/(1 - bz))^{\alpha} \}$. As a consequence of Theorem 2.1, the following result yields an equivalence relation between half plane Janowski sector whose boundary passes through origin and its argument bounds.

**Theorem 2.4.** For $0 < \alpha \leq 1$ and $-1 < m < 1$, then the function

$$h(z) = \left( \frac{1 + e^{im\pi}z}{1 - z} \right)^\alpha, \quad (2.7)$$

is analytic, univalent and convex in $D$ with

$$h(D) = \left\{ w \in \mathbb{C} : -\alpha(1 - m)\frac{\pi}{2} \leq \arg w \leq \alpha(1 + m)\frac{\pi}{2} \right\}. \quad (2.8)$$

**Proof.** By using Theorem 2.1, the function $h(z)$ given in (2.7) satisfy

$$\left| \arg h(z) - \alpha \tan^{-1} \left( \frac{\tan \frac{m\pi}{2}}{2} \right) \right| < \frac{\alpha\pi}{2},$$

which gives the desired result.
Remark 2.5. 1. From the domain (2.8) we have:

\[(h(\mathbb{D}))^{1/\alpha} = \{ w \in \mathbb{C} : \text{Re} e^{-im\pi/2}w > 0 \}. \quad (2.9)\]

2. For \(0 < \alpha_1 \leq 1\) and \(0 < \alpha_2 \leq 1\), if \(m = (\alpha_1 - \alpha_2)/(\alpha_1 + \alpha_2)\) and \(\alpha = (\alpha_1 + \alpha_2)/2\) in (2.7) then Theorem 2.4 reduces to [9, Lemma 3].

The following result generalizes Theorem 2.4.

**Theorem 2.6.** Let \(h \in \mathcal{H}_1\) and \(0 \leq b \leq 1\) with \(b + e^{im\pi} \neq 0\), where \(-1 \leq m \leq 1\). Also, if

\[h(z) < \frac{1 + e^{im\pi}z}{1 - bz}, \quad (2.10)\]

then

\[\text{Re } e^{-i\lambda}h(z) > 0, \quad (2.11)\]

where \(\lambda = \tan^{-1}\left(\frac{b\sin(m\pi)}{b\cos(m\pi) + 1}\right)\).

**Proof.** To obtain (2.11), it suffices to show that \(|\arg(e^{-i\lambda}w)| < \pi/2\) or \(|\arg w - \lambda| < \pi/2\), where \(w = h(z)\). By using Theorem 2.1 with \(\alpha = 1\), the function \(h(z)\) given in (2.10) satisfy

\[\left|\arg h(z) - \tan^{-1} \frac{b\sin(m\pi)}{b\cos(m\pi) + 1}\right| < \sin^{-1} \frac{|e^{im\pi} + b|}{|1 + be^{im\pi}|} = \frac{\pi}{2},\]

which leads to the desired result. \(\square\)

**Remark 2.7.**

1. When \(b = 1\) then Theorem 2.6 provides sufficient condition for functions to be in the class \(\mathcal{P}_{-\lambda}\).

2. Let \(|A| \leq 1\) and \(|B| \leq 1\) with \(A \neq B\). Assume \(a(\alpha) := \arg((1 + Az)/(1 + Bz))^\alpha\). Then from Theorem 2.1 we observe that \(\max a(\alpha_1) \leq \max a(\alpha_2)\) and \(\min a(\alpha_1) \geq \min a(\alpha_2)\), whenever \(0 < \alpha_1 \leq \alpha_2 \leq 1\). Therefore we have

\[\left(1 + Az\right)^{\alpha_1} \left(1 + Bz\right)^{\alpha_2} < 0 < \alpha_1 \leq \alpha_2 \leq 1.\]

**Theorem 2.8.** Let \(|A_j| \leq 1\) and \(|B_j| \leq 1\) \((j = 1, 2)\) with \(A_1 \neq B_1\) and \(A_2 \neq B_2\). Let \(C_j\) and \(R_j\) be the centres and radii of \((1 + A_jz)/(1 + B_jz) = \phi_j(z)\) \((j = 1, 2)\), respectively. Then (i) \(\mathcal{P}(A_1, B_1) \subseteq \mathcal{P}(A_2, B_2)\) if and only if the line segment \(C_1C_2\) lies entirely in the domain \(\phi_1(\mathbb{D})\), whenever \(|C_1 - C_2| \leq R_1\).

(ii) \(\mathcal{P}(A_1, B_1) \subseteq \mathcal{P}(A_2, B_2)\) if and only if the line segment \(C_1C_2\) does not lie entirely in the domain \(\phi_1(\mathbb{D})\), whenever \(|C_1 - C_2| \geq R_1\).

The proof is skipped here, as the result is evident from the Figure 2.
Figure 2. The image of $\mathbb{D}$ under $\phi_i(z)$ ($i = 1, 2$).

We note that if $\mathcal{P}(A_1, B_1) \subseteq \mathcal{P}(A_2, B_2)$, then $(1 + A_1z)/(1 + B_1z) < (1 + A_2z)/(1 + B_2z)$, thus from Theorem 2.8, we obtain the following result:

**Corollary 2.9.** If $\mathcal{P}(A_1, B_1) \subseteq \mathcal{P}(A_2, B_2)$, then for $0 < \alpha \leq 1$, we also have

$$
\left( \frac{1 + A_1z}{1 + B_1z} \right)^\alpha < \left( \frac{1 + A_2z}{1 + B_2z} \right)^\alpha.
$$

Note that the observations made in [20] are generalized in part 2 of Remark 2.7 and Corollary 2.9.

3. Argument related results

In this section, we generalized various subordination results using Theorem 2.4 and the following versions of the Jack’s Lemma:

**Lemma 1.** [12, p. 234-235] Let $h \in \mathcal{H}[1, n]$. If there exists a point $z_0 \in \mathbb{D}$, such that

$$
|\text{arg} \, p(z)| < |\text{arg} \, p(z_0)| = \frac{\pi \beta}{2}, \quad (|z| < |z_0|),
$$

for some $\beta > 0$, then we have

$$
\frac{z_0p'(z_0)}{p(z_0)} = \frac{2ik \text{arg} \, p(z_0)}{\pi},
$$

for some $k \geq n(a + a^{-1})/2 > n$, where $p(z_0)^{1/\beta} = \pm ia$, and $a > 0$.

**Lemma 2.** [14] Let $h(z)$ be analytic in $\mathbb{D}$, with $h(0) = 1$ and $h(z) \neq 0$. If there exist two points $z_1, z_2 \in \mathbb{D}$, such that

$$
-\frac{\alpha_1 \pi}{2} = \text{arg} \, h(z_1) < \text{arg} \, h(z) < \text{arg} \, h(z_2) = \frac{\alpha_2 \pi}{2}
$$

for $\alpha_1, \alpha_2 \in (-1, 1)$ and $|z| < |z_1| = |z_2|$, then we have

$$
\frac{z_1h'(z_1)}{h(z_1)} = -i \frac{\alpha_1 + \alpha_2}{2} k \quad \text{and} \quad \frac{z_2h'(z_2)}{h(z_2)} = i \frac{\alpha_1 + \alpha_2}{2} k,
$$

where

$$
k \geq \frac{1 - |a|}{1 + |a|} \quad \text{and} \quad a = i \tan \frac{\pi}{4} \left( \frac{\alpha_2 - \alpha_1}{\alpha_2 + \alpha_1} \right).
$$
The first result of this section is the generalization of [15, Theorem 1.6], which produces various corollaries and also generalizes Pommerenke’s result [18]: Let \( f \in \mathcal{A}, \ g \in \mathcal{C} \) and \( 0 < \alpha \leq 1 \), then
\[
|\arg \left( \frac{f'(z)}{g'(z)} \right) | < \frac{\alpha \pi}{2} \implies \arg \left( \frac{f(z)}{g(z)} \right) < \frac{\beta(\alpha)\pi}{2}.
\]

**Theorem 3.1.** Let \( f, g \in \mathcal{A} \) and \( 0 < \alpha \leq 1 \). For some \( m \in [-1, 1] \) and \( \beta \in (0, 1) \), let \( a = i \tan \frac{m\pi}{1} \) and \( |g(z)/(zg'(z))| > \beta \). If
\[
f'(z) < \left( \frac{1 + e^{i\mu z}}{1 - z} \right)^{\alpha(m + \mu_{2})/2},
\]
then
\[
f(z) < \left( \frac{1 + e^{i\mu z}}{1 - z} \right)^{\alpha},
\]
where \( \mu_j = 1 + (-1)^{j}m + \frac{2}{\alpha \pi} \tan^{-1} \frac{\alpha \beta (1 - |a|) \cos \left( \arg \left( \frac{g(z)}{zg'(z)} \right) \right)}{1 + |a| + (-1)^{j+1} \alpha \beta (1 - |a|) \sin \left( \arg \left( \frac{g(z)}{zg'(z)} \right) \right)} \) for \( j = 1, 2 \) and
\[
\mu = \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2}.
\]

**Proof.** Let \( p(z) := f(z)/g(z) \). Then in view of Theorem 2.4, to prove (3.2), it is sufficient to show
\[
-\alpha(1 - m)\pi/2 \leq \arg p(z) \leq \alpha(1 + m)\pi/2.
\]
on the contrary, if there exists two points \( z_1, z_2 \in \mathbb{D} \) such that
\[
-\alpha(1 - m)\pi/2 = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \alpha(1 + m)\pi/2
\]
for \( |z| < |z_1| = |z_2| \), then by Lemma 2, we have
\[
\frac{z_1p'(z_1)}{p(z_1)} = -iak \quad \text{and} \quad \frac{z_2p'(z_2)}{p(z_2)} = iak.
\]
Since \( p(z) = f(z)/g(z) \), thus we have
\[
f'(z) = p(z) \left( 1 + \frac{zp'(z)}{p(z)} \frac{g(z)}{zg'(z)} \right)
\]
and
\[
\arg \left( \frac{f'(z)}{g'(z)} \right) = \arg p(z) + \arg \left( 1 + \frac{zp'(z)}{p(z)} \frac{g(z)}{zg'(z)} \right).
\]
For \( z = z_1 \), we have
\[
\arg \left( \frac{f'(z_1)}{g'(z_1)} \right) \leq -\alpha(1 - m)\pi/2 + \arg \left( 1 - iak\beta \left( \cos \left( \arg \frac{g(z_1)}{z_1g'(z_1)} \right) + i \sin \left( \arg \frac{g(z_1)}{z_1g'(z_1)} \right) \right) \right)
\]
\[
\leq -\alpha(1 - m)\pi/2 + \tan^{-1} \left( \frac{\alpha \beta (|a|-1) \cos \left( \arg \frac{g(z_1)}{z_1g'(z_1)} \right)}{1 + |a| + \alpha \beta (1 - |a|) \sin \left( \arg \frac{g(z_1)}{z_1g'(z_1)} \right)} \right),
\]
\]
which contradicts (3.1). Similarly for \( z = z_2 \), we have
\[
\arg\left( \frac{f'(z_2)}{g'(z_2)} \right) \geq \alpha(1 + m) \frac{\pi}{2} + \arg\left( 1 + i \alpha m \beta \left( \cos \left( \arg \left( \frac{g(z_2)}{z_2 g'(z_2)} \right) + i \sin \left( \arg \left( \frac{g(z_2)}{z_2 g'(z_2)} \right) \right) \right) \right) \geq \alpha(1 + m) \frac{\pi}{2} + \tan^{-1} \left( \frac{\alpha \beta (1 - |a|) \cos \left( \arg \left( \frac{g(z_2)}{z_2 g'(z_2)} \right) \right)}{1 + |a| - \alpha \beta (1 - |a|) \sin \left( \arg \left( \frac{g(z_2)}{z_2 g'(z_2)} \right) \right)} \right),
\]
which again contradicts (3.1), that completes the proof.

If we choose \( g(z) = z \) in Theorem 3.1 it reduces to the following corollary:

**Corollary 3.2.** Let \( f \in A \) and \( 0 < \alpha \leq 1 \). For some \( m \in [-1, 1) \), let \( a = i \tan \frac{m \pi}{4} \). If
\[
f'(z) < \left( \frac{1 + e^{i \mu \pi z}}{1 - z} \right)^{\alpha(\mu_1 + \mu_2)/2},
\]
then
\[
\frac{f(z)}{z} < \left( \frac{1 + e^{im \pi z}}{1 - z} \right)^{\alpha},
\]
where \( \mu_j = 1 + (-1)^j m + \frac{2}{\alpha \pi} \tan^{-1} \frac{\alpha(1 - |a|)}{1 + |a|} \) (\( j = 1, 2 \)) and \( \mu = \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2} \).

Further, \( f \in S^*(\beta) \), where \( \beta = \alpha \pi + \tan^{-1} \frac{\alpha(1 - |a|)}{1 + |a|} \).

**Proof.** The subordination (3.3) holds, by using Theorem 3.1. Now just to prove \( f \in S^*(\beta) \). Since
\[
\arg \frac{zf'(z)}{f(z)} = \arg \frac{z}{f(z)} + \arg f'(z),
\]
using Theorem 2.4 we obtain
\[
-\alpha(1 + m + \mu_1) \frac{\pi}{2} < \arg \frac{zf'(z)}{f(z)} < \alpha(1 - m + \mu_2) \frac{\pi}{2},
\]
\[
-\alpha \pi - \tan^{-1} \frac{\alpha(1 - |a|)}{1 + |a|} < \arg \frac{zf'(z)}{f(z)} < \alpha \pi + \tan^{-1} \frac{\alpha(1 - |a|)}{1 + |a|},
\]
which implies \( f \in S^*(\beta) \).

**Remark 3.3.** If \( m = 0 \), then Corollary 3.2 reduces to [15, Theorem 1.7].

The next corollary is a generalization of [13, Theorem 1].

**Corollary 3.4.** Let \( f \in A \), \( g \in C \) and \( g \in RS^*(\beta) \), where \( 0 \leq \beta < 1 \). Suppose \( 0 < \alpha \leq 1 \) and for some \( m \in [-1, 1) \), let \( a = i \tan \frac{m \pi}{4} \). If
\[
\frac{f'(z)}{g'(z)} < \left( \frac{1 + e^{i \mu \pi z}}{1 - z} \right)^{(\alpha(\mu_1 + \mu_2))/2},
\]
then
\[
\frac{f(z)}{g(z)} < \left( \frac{1 + e^{im \pi z}}{1 - z} \right)^{\alpha},
\]
where \( \mu_j = 1 - m + \frac{2}{\alpha \pi} \tan^{-1} \frac{\alpha \beta (1 - |a|)}{1 + |a| + \alpha (1 - |a|)} \) (\( j = 1, 2 \)) and \( \mu = \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2} \).
Proof. Since $g \in \mathcal{C}$, from Marx-Strohhäcker’s theorem, we have $\text{Re}(zg'(z)/g(z)) > 1/2$, which implies that $|g(z)/(zg'(z))| < 1$. Thus we obtain
\[
\left| \text{Im} \left( \frac{g(z)}{zg'(z)} \right) \right| < 1 \quad \text{and} \quad \text{Re} \left( \frac{g(z)}{zg'(z)} \right) > \beta. \tag{3.4}
\]
Now using (3.4) and the methodology of Theorem 3.1, the result follows at once. \qed

Remark 3.5. When we take $m = 0$ in Corollary 3.4, then the result reduces to [13, Theorem 1].

**Theorem 3.6.** Let $p \in \mathcal{H}_1$ and $m \in [-1, 1)$. Also, for a fixed $\gamma \in [0, 1]$ and $\alpha > 0$, let $\beta > \beta_0(\geq 0)$, where $\beta_0$ is the solution of the equation
\[
\alpha \beta (1 - m) + \frac{2\gamma}{\pi} \tan^{-1} \eta = 0
\]
and for a suitable fixed $\eta \geq 0$, let $\lambda(z) : \mathbb{D} \to \mathbb{C}$ be a function satisfying
\[
\frac{\beta \text{Re} \lambda(z)}{1 + \beta |\text{Im} \lambda(z)|} \geq \eta. \tag{3.5}
\]
If
\[
(p(z))^a \left( 1 + \lambda(z) \frac{zp'(z)}{p(z)} \right)^\gamma < \left( \frac{1 + e^{i\mu z}}{1 - z} \right)^\delta, \tag{3.6}
\]
then
\[
p(z) < \left( \frac{1 + e^{im\pi z}}{1 - z} \right)^\beta, \tag{3.7}
\]
where $\mu_j = \alpha \beta (1 + (-1)^j m) + \frac{2\gamma}{\pi} \tan^{-1} \eta$ $(j = 1, 2)$, $\delta = \frac{\mu_1 + \mu_2}{2}$ and $\mu = \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2}$.

Proof. According to Theorem 2.4, to prove (3.7), it is sufficient to show that $-\beta (1 - m) \frac{\pi}{2} < \arg p(z) < \beta (1 + m) \frac{\pi}{2}$. On the contrary if there exists two points $z_1, z_2 \in \mathbb{D}$ such that
\[
-\beta (1 - m) \frac{\pi}{2} = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \beta (1 + m) \frac{\pi}{2}
\]
for $|z| = |z_1| = |z_2|$, then from Lemma 2 we have
\[
\frac{z_1 p'(z_1)}{p(z_1)} = -ik\beta \quad \text{and} \quad \frac{z_2 p'(z_2)}{p(z_2)} = ik\beta
\]
where, $k \geq \frac{1 - \mid a \mid}{1 + \mid a \mid}$ and $a = i \tan \frac{\pi m}{4}$. For $z = z_1$, using (3.5) we have
\[
\arg \left( (p(z_1))^a \left( 1 + \lambda(z_1) \frac{z_1 p'(z_1)}{p(z_1)} \right)^\gamma \right) 
\leq -\alpha \beta (1 - m) \frac{\pi}{2} - \gamma \tan^{-1} \frac{\beta m \text{Re} \lambda(z_1)}{1 + \beta m |\text{Im} \lambda(z_1)|}
\leq - \left( \alpha \beta (1 - m) \frac{\pi}{2} + \gamma \tan^{-1} \eta \right),
\]
which contradicts (3.6). Similarly, for $z = z_2$, we have
\[
\arg \left( (p(z_2))^a \left( 1 + \lambda(z_2) \frac{z_2 p'(z_2)}{p(z_2)} \right)^\gamma \right) 
\geq \alpha \beta (1 + m) \frac{\pi}{2} + k \tan^{-1} \frac{\beta \gamma \text{Re} \lambda(z_2)}{1 + \beta \gamma |\text{Im} \lambda(z_2)|}
\geq \alpha \beta (1 + m) \frac{\pi}{2} + \gamma \tan^{-1} \eta,
\]
which also contradicts (3.6). Hence the result follows. \qed

Remark 3.7. If we take $m = 0$ in Theorem 3.6 then the result reduces to [22, Theorem 1].
4. Radius results

This section aims to find the largest radius $R$ of a property $\mathcal{P}$ such that every function of a set $\mathcal{M}$ has the property $\mathcal{P}$ in the disk $\mathbb{D}_r$, where $r \leq R$. First result of this section generalizes [1, Theorem 2.1-2.2].

**Theorem 4.1.** Let $\gamma, \delta \in \mathbb{C}\{1\}$ and $A, B, C, D \in \mathbb{C}$ with $A \neq B$, $|B| \leq 1$, $D \neq C$ and $|D| \leq 1$. If
\[
f(z) \prec (1 - \delta) \left( \frac{1 + Cz}{1 + Dz} \right)^\beta + \delta,
\]
then
\[
f(z) \prec (1 - \gamma) \left( \frac{1 + Az}{1 + Bz} \right)^\alpha + \gamma,
\]
in $|z| \leq R$, where
\[
R = \min \left\{ \frac{1}{\alpha|(A - B)(1 - \gamma)\alpha|}, 1 \right\},
\]
\[
\frac{1}{|(C - D)\beta(1 - \gamma)\alpha^{-1}(\delta - 1)| + \beta|(1 - \gamma)\alpha^{-1}(AD(\gamma - 1) + B(C(1 - \delta) + D(\delta - \gamma)))|} \leq 1.
\]

**Proof.** Let $P$ and $Q$ be functions defined as
\[
P(z) = (1 - \gamma) \left( \frac{1 + Az}{1 + Bz} \right)^\alpha + \gamma \quad \text{and} \quad Q(z) = (1 - \delta) \left( \frac{1 + Cz}{1 + Dz} \right)^\beta + \delta.
\]
Further, define the function $M$ as
\[
M(z) = P^{-1}(Q(z)) = \frac{(1 - \delta)(1 + Cz)^\delta + (\delta - \gamma)(1 + Dz)^\beta}{\beta} - (1 + Dz)^\beta - (1 - \gamma)\alpha^{-1} \frac{1}{\beta}.
\]
\[
A(1 + Dz)^{\beta} \frac{1}{\alpha}(1 - \gamma)\alpha - B((1 - \delta)(1 + Cz)^\beta + (\delta - \gamma)(1 + Dz)^\beta)\alpha.
\]
Replacing $z$ by $Rz$ so that $|z| = R \leq 1$, we obtain
\[
|M(Rz)| \leq \frac{1}{\alpha|(A - B)(1 - \gamma)\alpha|} \leq 1
\]
for
\[
R \leq \frac{1}{\beta}(1 - \gamma)\alpha^{-1} \frac{1}{\alpha - \beta}(AD(\gamma - 1) + B(C(1 - \delta) + D(\delta - \gamma)))|R|
\]
Thus $f(z) \prec Q(z)$ for $|z| \leq \min\{R, 1\}$. Hence the result follows.

Taking $\gamma = \delta = 0$ in the Theorem 4.1, we obtain the following corollary.

**Corollary 4.2.** Let $A, B, C, D \in \mathbb{C}$ with $A \neq B$, $|B| \leq 1$, $D \neq C$ and $|D| \leq 1$. Then $S^*(C, D, \beta) \subseteq S^*(A, B, \alpha)$ if and only if
\[
\beta(|C - D| + |AD - BC|) \leq \alpha|A - B|.
\]

In particular,

(1) for $\beta_1 \in (0, 1]$, we have $SS^*(\beta_1) \subseteq S^*(A, B, \alpha)$ if and only if
\[
\beta(2 + |A + B|) \leq \alpha|A - B|.$
(2) for \( \alpha_1 \in (0, 1] \), we have \( S^*(C, D, \beta) \subseteq SS^*(\alpha_1) \) if and only if 
\[ \beta(|C + D| + |C - D|) \leq 2\alpha. \]

Remark 4.3. Let \( \alpha_1, \alpha_2 \in [0, 1) \), \( A = 1 - 2\alpha_1 \), \( C = 1 - 2\alpha_2 \), \( B = D = -1 \) and \( \alpha = \beta = 1 \), then the corollary \[4.2\] reduces to the well known fact that \( S^*(\alpha_2) \subseteq S^*(\alpha_1) \) if and only if \( \alpha_2 \leq \alpha_1 \).

Corollary 4.4. Let \( A, B \in \mathbb{C} \) with \( A \neq B \), \( |B| \leq 1 \) and \( \beta_2 > 0 \). If \( f \in S^*(A, B, \alpha) \), then

1. \( f \in M(\beta_2) \) in \(|z| \leq R_M(\beta_2) \) for \( \beta > 1 \), where
\[
R_M(\beta_2) = \min \left\{ \frac{a|A - B|}{2(\beta_2 + 1) + |A - B(2\beta - 1)|}, 1 \right\}. \tag{4.1}
\]

2. \( f \in RS^*(\beta_2) \) in \(|z| \leq R_{RS^*}(\beta_2) \) for \( 0 \leq \beta < 1 \), where
\[
R_{RS^*}(\beta_2) = \min \left\{ \frac{a|A - B|}{2\beta_2 + |A - B(2\beta - 1)|}, 1 \right\}. \tag{4.2}
\]

Theorem 4.5. Let \( f(z) \in A \) with \( f'(z) \neq 0 \) in \( \mathbb{D} \). Also let \( 0 < A \leq 1 \), \( -1 \leq B < 0 \), \( \alpha = 0.3834486 \cdots \) and \( \beta \geq 0.61655 \cdots \) which satisfy the conditions \( \tan^{-1} \alpha = (1 - 2\alpha)/2 \) and \( \tan^{-1} \alpha \leq (\beta - \alpha)\pi/2 \), respectively. If 
\[
f'(z) < \left( \frac{1 + Az}{1 + Bz} \right)^\beta, \tag{4.3}
\]
then \( f(z) \) is starlike in \(|z| \leq r_0 \), where
\[
r_0 = \frac{-(A - B) + \sqrt{(A - B)^2 + 4AB(\sin(\alpha + \frac{2\pi}{\pi} \tan^{-1} \alpha) \frac{\pi}{2\beta})^2}}{2AB \sin \left( \alpha + \frac{2\pi}{\pi} \tan^{-1} \alpha \frac{\pi}{2\beta} \right)}, \tag{4.4}
\]
is the smallest positive root of the equation
\[
\left( \alpha + \frac{2\pi}{\pi} \tan^{-1} \alpha \right)^\frac{\pi}{2} = \beta \sin^{-1} \left( \frac{(A - B)x}{1 - ABx^2} \right). \tag{4.5}
\]

Proof. By Theorem 2.1, the subordination [4.3] yields that 
\[
|\arg f'(z)| \leq \beta \sin^{-1} \left( \frac{(A - B)|z|}{1 - AB|z|^2} \right). \tag{4.6}
\]
Let \( p(z) = f(z)/z \), we suppose that there exists a point \( z_0, |z_0| = r_0 < 1 \), such that \( |\arg p(z)| < \alpha \pi/2 \) for \(|z| < r_0 \) and \( |\arg p(z_0)| = \alpha \pi/2 \) then by Lemma 1, we have
\[
\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2ik \arg p(z_0)}{\pi}, \tag{4.7}
\]
where \( k \geq m(a + a^{-1})/2 \geq 1 \), \( p(z_0) = \pm ia \) and \( a > 0 \). Also if \( \arg p(z_0) = \alpha \pi/2 \) then using (4.5), we have 
\[
\arg f'(z_0) = \arg(p(z_0) + zp'(z_0)) = \arg p(z_0) + \arg \left( 1 + \frac{z_0 p'(z_0)}{p(z_0)} \right) = \frac{\alpha \pi}{2} + \arg(1 + i\alpha k) \geq \frac{\alpha \pi}{2} + \tan^{-1} \alpha = \beta \sin^{-1} \left( \frac{(A - B)r_0}{1 - ABr_0^2} \right),
\]

which contradicts (4.6). Similar contrary conclusion will arrive for the case when \( \text{arg} \, p(z_0) = -\alpha \pi/2 \). Therefore we have

\[
|\text{arg} \, p(z_0)| = \left| \text{arg} \left( \frac{f(z)}{z} \right) \right| < \frac{\alpha \pi}{2}.
\]

(4.8)

Now using (4.5), (4.6) and (4.8), we have

\[
\begin{align*}
\left| \text{arg} \left( \frac{zf'(z)}{f(z)} \right) \right| & \leq |\text{arg} \, f'(z)| + \left| \text{arg} \left( \frac{z}{f(z)} \right) \right| \\
& < \left( 2\alpha + \frac{2}{\pi} \tan^{-1} \alpha \right) \frac{\pi}{2}.
\end{align*}
\]

For \( f \) to be starlike, we must have \( 2\alpha + (2 \tan^{-1} \alpha)/\pi = 1 \), which gives \( \alpha = 0.38344486 \cdots \) and since

\[
\left( \alpha + \frac{2}{\pi} \tan^{-1} \alpha \right) \frac{\pi}{2} = \beta \sin^{-1} \left( \frac{(A-B)r_0}{1-ABr_0^2} \right),
\]

we have \( f(z) \) is starlike in \(|z| < r_0\), where \( r_0 \) is given in (4.4), that completes the proof.

\[ \blacksquare \]

5. Subordination results

In this section we discuss subordination related results for the Janowski function. The book [11] provides numerous results pertaining to differential subordination and some of the results which we need in context of our study are listed below:

Lemma 3. [11] Theorem 3.1d,p.76] Let \( h \) be analytic and starlike univalent in \( \mathbb{D} \) with \( h(0) = 0 \). If \( g \) is analytic in \( \mathbb{D} \) and \( zg'(z) < h(z) \), then

\[
g(z) < g(0) + \int_0^z \frac{h(t)}{t} \, dt.
\]

Lemma 4. [11] Theorem 3.4h,p.132] Let \( g(z) \) be univalent in \( \mathbb{D} \), \( \Phi \) and \( \Theta \) be analytic in a domain \( \Omega \) containing \( g(\mathbb{D}) \) such that \( \Phi(w) \neq 0 \), when \( w \in g(\mathbb{D}) \). Now letting \( G(z) = zg'(z) \cdot \Phi(g(z)), h(z) = \Theta(g(z)) + G(z) \) and either \( h \) or \( g \) is convex. Further, if

\[
\text{Re} \left( \frac{zh'(z)}{G(z)} \right) = \text{Re} \left( \frac{\Phi'(g(z))}{\Phi(g(z))} + \frac{zg'(z)}{G(z)} \right) > 0.
\]

as well as \( p \) is analytic in \( \mathbb{D} \), with \( p(0) = g(0) \), \( p(\mathbb{D}) \subset \Omega \) and

\[
\Theta(p(z)) + zp'(z) \cdot \Phi(p(z)) < \Theta(g(z)) + zg'(z) \cdot \Phi(g(z)) := h(z)
\]

then \( p \prec g \), and \( g \) is the best dominant.

Antonino and Romaguera in [2] gave the following result:

Lemma 5. [2] Let \( p(z) \) be analytic in \( \mathbb{D} \), \( p(0) = c \), let \( q(z) \) be univalent and convex in \( \mathbb{D} \) and let \( g(z) \) be analytic in \( \mathbb{D} \) such that \( zg'(z)/g(z) \) is analytic and different from zero in \( \mathbb{D} \). Let \( \beta \) and \( \gamma \) be complex constants such that

\[
\text{Re} \left( \frac{\xi g'(\xi)}{g(\xi)} (\beta q(z) + \gamma) \right) > 0 \quad (z, \xi \in \mathbb{D}).
\]

then

\[
p(z) + \frac{g(z)}{g'(z)} \left( \frac{zp'(z)}{\beta p(z) + \gamma} \right) < q(z) \implies p(z) < q(z).
\]
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Several authors studied various combinations of \( z f'(z)/f(z) \) and \( 1 + z f''(z)/f'(z) \). One such combination is their quotient, which was first investigated by Silverman in [21]. Silverman considered the following class for \( \beta \in (0, 1] \),

\[
\mathcal{G}_\beta = \left\{ f \in A : \frac{1 + z f''(z)/f'(z)}{z f'(z)/f(z)} - 1 < \beta \right\}.
\]

**Theorem 5.1.** Let \( |A| \leq 1 \) and \( 0 \leq b \leq 1 \) with \( A + b \neq 0 \) and \( \beta(1 + b)\alpha^{-1}(1 + |A|)^{\alpha+1} \leq \alpha|A + b| \). If \( f \in \mathcal{G}_\beta \), then \( f \in \tilde{SS}^\ast(A, b, \alpha) \).

**Proof.** For \( f \in \mathcal{G}_\beta \), we define the function \( p(z) = z f'(z)/f(z) \). By standard calculations we obtain that

\[
\frac{1 + z f''(z)/f'(z)}{z f'(z)/f(z)} - 1 = \frac{z p'(z)}{p^2(z)}.
\]

For \( f \in \tilde{SS}^\ast(A, b, \alpha) \), it is enough to show \( p(z) < ((1 + A z)/(1 - b z))^\alpha := q(z) \). For if, there exist points \( z_0 \in \mathbb{D}, s_0 \in \partial \mathbb{D} \setminus \{-1/A, 1/b\} \) and \( m \geq 1 \) such that \( p(|z| < |z_0|) \subset q(\mathbb{D}) \) with \( p(z_0) = q(s_0) \) and \( z_0 p'(z_0) = m s_0 q'(s_0) \). If

\[
\frac{\alpha |A + b|}{(1 + |A|)^{\alpha+1}(1 + b)^{\alpha-1}} \geq \beta \implies \frac{\alpha s_0(A + b)}{(1 + A s_0)^{\alpha+1}(1 - b s_0)^{\alpha-1}} \geq \beta.
\]

Thus,

\[
\frac{|m s_0 q'(s_0)|}{q^2(s_0)} \geq \beta \quad \text{for all } m \geq 1.
\]

or, equivalently,

\[
\frac{z_0 p'(z_0)}{p^2(z_0)} \geq \beta,
\]

which contradicts \( f \in \mathcal{G}_\beta \). Thus \( p(z) < q(z) \) and that completes proof.

**Remark 5.2.** If \( \alpha = 1 \) and \(-1 \leq B < A \leq 1 \) in Theorem 5.1 then the result reduces to Theorem 3.2,p.9.

**Theorem 5.3.** If \( f \in \mathcal{G}_\beta \), then \( f \) is starlike of reciprocal order \( \alpha \).

**Proof.** For \( f \in \mathcal{G}_\beta \), we define the function \( p(z) \) by \( f(z)/(z f'(z)) = \alpha + (1 - \alpha)p(z) \). Clearly \( p \in \mathcal{H}_1 \) and \( \alpha + (1 - \alpha)p(z) \neq 0 \) for \( z \in \mathbb{D} \). Now by a simple computation we obtain that

\[
1 - \frac{1 + z f''(z)/f'(z)}{z f'(z)/f(z)} = (1 - \alpha)z p'(z).
\]

Since \( f \in \mathcal{G}_\beta \), therefore we have

\[
(1 - \alpha)z p'(z) < \beta z. \tag{5.1}
\]

Now using Lemma 3, (5.1) leads to

\[
p(z) < 1 + \frac{\beta}{1 - \alpha} z,
\]

which shows that

\[
|p(z) - 1| < \frac{\beta}{1 - \alpha}.
\]

Now using Theorem 2.1 we obtain that

\[
\left| \operatorname{Arg} \left( \frac{f(z)}{z f'(z)} - \alpha \right) \right| = |\operatorname{Arg} p(z)| < \sin^{-1} \frac{\beta}{1 - \alpha} = \frac{\delta \pi}{2}.
\]

Since \( \alpha \in [0, 1] \), and \( \beta \in (0, 1] \). Therefore \( \delta \in (0, 1] \), which completes our result.
In [19], authors have considered Janowski function with complex parameters and therefore the corresponding Janowski disk is non-symmetric with respect to real axis. However, their findings were based on the Janowski disk that are symmetric with respect to real axis, which is possible only if the parameters are real in Janowski function. Therefore, by eliminating this limitation, we obtain the following result as an extension of [19, Lemma 2.17].

**Theorem 5.4.** Let \( \alpha \in (0, 1] \), \( l, m \in [-1, 1] \) and \( a, b, c, d \in [0, 1] \) such that \( a e^{i\eta \pi} + b \neq 0 \) and \( c e^{im\pi} + d \neq 0 \). Let \( Q \in \mathcal{H}[1, n] \) satisfy

\[
Q(z) < \frac{1 + a e^{i\eta \pi} z}{1 - b z} \quad (5.2)
\]

and

\[
Q(z)p^\alpha(z) < \frac{1 + c e^{im\pi} z}{1 - d z}, \quad (5.3)
\]

for \( p \in \mathcal{H}_1 \). If

\[
\mu := \sin^{-1}\sqrt{\frac{c^2 + d^2 + 2cd \cos(m\pi)}{1 + c^2 d^2 + 2cd \cos(m\pi)} + \sin^{-1}\sqrt{\frac{a^2 + b^2 + 2ab \cos(l\pi)}{1 + a^2 b^2 + 2ab \cos(l\pi)}} \leq \frac{\alpha \pi}{2}, \quad (5.4)
\]

then

\[
\text{Re}(e^{-\gamma \pi/2}p(z)) > 0,
\]

where \( \gamma = (\tan^{-1} A - \tan^{-1} B)/\mu \) with \( A = \frac{cd \sin(m\pi)}{cd \cos(m\pi) + 1} \) and \( B = \frac{ab \sin(l\pi)}{ab \cos(l\pi) + 1} \).

**Proof.** From Theorem 2.1 (5.2) yields

\[
\left| \arg Q(z) - \tan^{-1}\left(\frac{ab \sin(l\pi)}{ab \cos(l\pi) + 1}\right) \right| < \sin^{-1}\sqrt{\frac{a^2 + b^2 + 2ab \cos(l\pi)}{1 + a^2 b^2 + 2ab \cos(l\pi)}}. \quad (5.5)
\]

Similarly, from (5.3), we obtain

\[
\left| \arg Q(z) + \alpha \arg p(z) - \tan^{-1}\left(\frac{cd \sin(m\pi)}{cd \cos(m\pi) + 1}\right) \right| < \sin^{-1}\sqrt{\frac{c^2 + d^2 + 2cd \cos(m\pi)}{1 + c^2 d^2 + 2cd \cos(m\pi)}}. \quad (5.6)
\]

After some computations using (5.5) and (5.6), we obtain

\[
-\frac{\pi}{2} \left(\frac{2}{\alpha \pi} (\mu - \gamma \mu)\right) \leq \arg p(z) \leq \frac{\pi}{2} \left(\frac{2}{\alpha \pi} (\mu + \gamma \mu)\right) \quad (5.7)
\]

which eventually yields

\[
p(z) < \frac{1 + e^{i\gamma \pi} z}{1 - z},
\]

that completes the proof.

**Theorem 5.5.** Let \( p(z) \in \mathcal{H}_1 \), \( A \in \mathbb{C} \) and \( 0 \leq b \leq 1 \) with \( |A| \leq 1 \), \( A + b \neq 0 \) and \( \text{Re}(1 + Ab) \geq |A + b| \). Further let \( \alpha, \gamma \) are two real parameters lying in \([0, 1]\) and \( \mu, \delta, \rho \) and \( \eta \) are complex parameters such that \( \text{Re}(\mu/\eta) > 0 \), \( \text{Re} \delta > 0 \) and \( \text{Re} \rho \geq 0 \). If

\[
\mu(p(z))^\alpha (\delta + \rho p(z)) + \eta \zeta p(z) p(z)^{\alpha - 1} < h(z),
\]

then \( p(z) \in \tilde{SS}^\times (A, b, \gamma) \), where

\[
h(z) = \left(\frac{1 + Az}{1 - bz}\right)^{\alpha \gamma} \left(\frac{\mu \delta + \mu p}{1 + bz}\right)^\gamma + \eta \zeta p(z) p(z)^{\alpha - 1}.
\]
Proof. Let us choose \( g(z) = ((1 + Az)/(1 - bz))^\gamma \), \( \Phi(w) = \eta w^{\alpha-1} \) and \( \Theta(w) = \mu w^\alpha (\delta + \rho w) \) then clearly \( g \in \mathcal{P} \), univalent and convex in \( \mathbb{D} \). \( \Phi, \Theta \) are analytic in a domain \( \Omega \) containing \( g(\mathbb{D}) \), with \( \Phi(w) \neq 0 \) when \( w \in g(\mathbb{D}) \). If \( G(z) = zg'(z)\Phi(g(z)) \) then
\[
\text{Re} \frac{zG'(z)}{G(z)} = \text{Re} \left( \frac{\alpha + 1}{1 - bz} - \frac{\alpha - 1}{1 + Az} - 1 \right) \geq \frac{\alpha + 1}{1 + b} - \frac{\alpha - 1}{1 + |A|} - 1 \geq 0
\]
and
\[
\text{Re} \frac{zh'(z)}{G(z)} = \text{Re} \left( \frac{\mu \alpha \delta}{\eta} + \frac{\mu (\alpha + 1) \rho g(z)}{\eta} + \frac{zG'(z)}{G(z)} \right) > 0.
\]
Thus by Lemma 4 we obtain that \( p \prec g \) and \( g \) is the best dominant.

By taking \( \mu = 1, \delta = 1 - \lambda, \rho = \lambda, \eta = \lambda \) and \( p(z) = zf'(z)/f(z) \) in Theorem 5.5 where \( \lambda \in [0, 1] \), we obtain the following more modified and simplified form of results in [5].

**Corollary 5.6.** Let \( f(z) \in \mathcal{A} \), such that \( f(z)/z \neq 0 \) in \( \mathbb{D} \), \( A \in \mathbb{C} \) and \( 0 \leq b \leq 1 \) with \( |A| \leq 1 \), \( A + b \neq 0 \) and \( \text{Re}(1 + Ab) \geq |A + b| \). Suppose also that the real parameters \( \lambda, \alpha \) and \( \gamma \) are such that they lie in \([0, 1]\). If
\[
\left( \frac{zf'(z)}{f(z)} \right)^\alpha \left( 1 + \lambda \frac{zf''(z)}{f'(z)} \right) \prec h(z),
\]
then \( f \in \mathcal{S}^{\ast}(A, b, \gamma) \), where
\[
h(z) = \left( \frac{1 + Az}{1 - bz} \right)^{\alpha \gamma - 1} \left( 1 - \lambda \right) \frac{1 + Az}{1 - bz} + \frac{\lambda(1 + Az)^{1 + \gamma}(1 - bz)^{1 - \gamma} + \lambda \gamma(A + bz)}{(1 - bz)^2}.
\]

By taking \( \delta = 1 \) and \( \rho = 0 \) in Theorem 5.5, we obtain the following result.

**Corollary 5.7.** Let \( p(z) \in \mathcal{H}_1 \) and \( 0 \leq b \leq 1 \) with \( b + e^{im\pi} \neq 0 \), where \(-1 \leq m \leq 1\). Also let \( \alpha \succ -1 \) and if
\[
\mu(p(z))^\alpha + \eta z p'(z)(p(z))^{\alpha - 1} \prec \left( \frac{1 + e^{im\pi} z}{1 - bz} \right)^{\alpha \gamma} \left( \mu + \eta \gamma (b + e^{im\pi}) \frac{(1 + e^{im\pi} z)}{(1 + e^{im\pi} z)(1 - bz)} \right) := h(z),
\]
for some \( \mu, \eta \in \mathbb{C} \) such that \( \text{Re}(\mu/\eta) \geq 0 \), then
\[
\text{Re} e^{-i\beta} (p(z))^{1/\gamma} > 0,
\]
where \( \beta = \tan^{-1} \frac{b \sin (m\pi)}{b \cos (m\pi) + 1} \). And the inequality is sharp for the function \( p(z) \) defined by
\[
p(z) = \left( 1 + e^{im\pi} z \right)^\gamma.
\]

**Proof.** By Theorem 5.5, we have \( p(z) \prec ((1 + e^{im\pi} z)/(1 - bz))^{\gamma} := g(z) \) and \( g \) is the best dominant. Further by Theorem 2.6, we obtain the desired conclusion.

By taking \( \mu = 1 - \lambda, \alpha = 1 \) and \( \eta = \lambda \) in Corollary 5.7, we obtain the following result.

**Corollary 5.8.** Let \( p(z) \in \mathcal{H}_1 \) and \( 0 \leq b \leq 1 \) with \( b + e^{im\pi} \neq 0 \), where \(-1 \leq m \leq 1\). Now if
\[
(1 - \lambda)p(z) + \lambda z p'(z) \prec \left( \frac{1 + e^{im\pi} z}{1 - bz} \right)^\gamma \left( 1 - \lambda + \lambda \gamma \frac{(b + e^{im\pi}) z}{(1 + e^{im\pi} z)(1 - bz)} \right) := h(z),
\]
for some \( 0 < \lambda \leq 1 \) and \( 0 < \gamma \leq 1 \), then
\[
\text{Re} e^{-i\beta} (p(z))^{1/\gamma} > 0,
\]
where \( \beta = \tan^{-1} \frac{b \sin(m \pi)}{b \cos(m \pi) + 1} \). Further the inequality is sharp for the function \( p(z) \) given by

\[
p(z) = \left( \frac{1 + e^{im \pi z}}{1 - bz} \right)^\gamma.
\]

**Remark 5.9.** When \( m = 0 \) and \( b = 1 \) then Corollary 5.8 reduces to the \[17\] Theorem 1.

In case when \( m = 0, \gamma = 1 \) and \( \lambda = 0.5 \), Corollary 5.8 yields:

**Corollary 5.10.** Let \( p(z) \in \mathcal{H}_1 \) and \( 0 \leq b \leq 1 \). Suppose

\[
p(z) + zp'(z) < \frac{1 + 2z - bz^2}{(1 - bz)^2},
\]

then

\[
p(z) < \frac{1 + z}{1 - bz}.
\]

**Theorem 5.11.** Let \( \alpha \in (0, 1], \beta, \gamma \in \mathbb{C} \) with \( \beta \neq 0, \Re(\beta + \gamma) > 0 \) and \( A, B \in \mathbb{C} \) with \( A \neq B, |B| \leq 1 \) and \( |A - B| \leq 1 - \Re(AB) \). Let \( \lambda(z) \) be Carathéodory and if \( p(z) \) is analytic such that

\[
p(z) + \lambda(z) \frac{zp'(z)}{\beta p(z) + \gamma} < \left( \frac{1 + Az}{1 + Bz} \right)^\alpha,
\]

then

\[
p(z) < q(z) := H(z) \left( \int_0^z \frac{H(t)g'(t)dt}{g(t)} \right)^{-1} < \left( \frac{1 + Az}{1 + Bz} \right)^\alpha,
\]

where

\[
g(z) = z \exp \int_0^z \frac{(\lambda(t))^{-1} - 1}{t} dt \quad \text{and} \quad H(z) = g(z) \exp \int_0^z \frac{g'(t)}{g(t)} \left( \frac{1 + Az}{1 + Bz} \right)^\alpha - 1 dt.
\]

The function \( q(z) \) is convex and is the best \((1, n)\)-dominant.

**Proof.** Let \( f \in \mathcal{A}_n \) with \( f(z)/z \neq 0 \) in \( \mathbb{D} \) and

\[
p(z) = \frac{F'(z)g(z)}{F(z)g'(z)},
\]

where

\[
F(z) = \left( \frac{\beta + \gamma}{g^\alpha(z)} \right) \int_0^z f^\beta(t)g^{\alpha-1}(t)g'(t)dt \right)^{\frac{1}{\beta}}.
\]

Since \( \Re(\beta + \gamma) > 0 \), by using an argument similar to that of \[11\] Lemma 1.2.c.p.11, we can show that the function \( S(z) \) defined by

\[
S(z) = \frac{z^{\beta+\gamma-1}}{z^{\beta+\gamma}g^{\gamma-1}f^{\beta}(z)} \int_0^z \left( \frac{f(t)}{t} \right)^\beta g^{\alpha-1}(t)g'(t)dt,
\]

is analytic in \( \mathbb{D} \) and \( S \in \mathcal{H}(\left[ \frac{1}{\beta+\gamma}, n \right]) \). From (5.9) and (5.10) we obtain

\[
\frac{F(z)}{z} = \left( (\beta + \gamma)S(z) \frac{zg'(z)}{g(z)} \right)^{\frac{1}{\beta}} \left( \frac{f(z)}{z} \right).
\]

Since both the expressions on the right are analytic and nonzero, therefore we conclude that \( F \in \mathcal{A}_n \) and \( F(z)/z \neq 0 \) for \( z \in \mathbb{D} \). Now by a simple computation using 5.8 and 5.9 we obtain

\[
p(z) + \lambda(z) \frac{zp'(z)}{\beta p(z) + \gamma} = \frac{f'(z)g(z)}{f(z)g'(z)} < \left( \frac{1 + Az}{1 + Bz} \right)^\alpha.
\]
Since $zg'(z)/g(z)$ and $((1 + Az)/(1 + Bz))^\alpha$ are Carathéodory, also $\text{Re}(\beta + \gamma) > 0$, thus we have

$$\text{Re}\left(\frac{zg'(z)}{g(z)} \left(\beta \left(\frac{1 + Az}{1 + Bz}\right)^\alpha + \gamma\right)\right) > 0.$$  

Therefore the result holds by Lemma 5.

Theorem 5.11 is a generalisation of [7, Theorem 2] and also we have the following result.

**Corollary 5.12.** Let $\alpha \in (0, 1], \beta, \gamma \in \mathbb{C}$ with $\text{Re}(\beta + \gamma) > 0$ and $A, B \in \mathbb{C}$ with $A \neq B$, $|B| \leq 1$ and $|A - B| \leq 1 - \text{Re}(AB)$. Suppose that $g(z) \in S^*$ and $f \in A$ satisfies

$$f'(z)g(z) < \left(\frac{1 + Az}{1 + Bz}\right)^\alpha.$$  

then the function $F(z)$ defined in (5.9) is in $A$, $F(z)/z \neq 0$, for $z \in \mathbb{D}$ and

$$F'(z)g(z) < \left(\frac{1 + Az}{1 + Bz}\right)^\alpha.$$  

As stated in (2.6), we can consider $B \in [-1, 0]$ and by taking $\lambda(z) \equiv 1$, $\beta = 1$ and $\gamma = 0$ in Theorem 5.11 and letting $p(z) = zf'(z)/f(z)$, for $f \in A$, we obtain the following subordination implication:

**Corollary 5.13.** If $\alpha \in (0, 1]$ and $A \in \mathbb{C}$, $b \in [0, 1]$ such that $A + b \neq 0$ and $|A + b| \leq 1 + \text{Re}(Ab)$. Also, if $f(z) \in A$ satisfies

$$1 + zf''(z) < \left(\frac{1 + Az}{1 - bz}\right)^\alpha$$  

then

$$zf'(z) < \frac{zK'(z)}{K(z)},$$  

where

$$K(z) = \begin{cases} 
\int_0^z \exp \left( \int_0^w \frac{(1 + At)^\alpha - 1}{t} \, dt \right) \, dw, & A \neq 0, b \neq 0 \\
\int_0^z \exp(\alpha tw[3F_2[1, 1 + \alpha, 2, 2, b, tw]]) \, dt, & A = 0, b \neq 0, \\
\int_0^z \exp(\alpha At[3F_2[1, 1 + \alpha, -\alpha, 2, 2, -At]]) \, dt, & A \neq 0, b = 0.
\end{cases}$$

Remark 5.14. When $\alpha = 1$, Corollary 5.13 reduces to [7, Corollary 3.2], for which

$$K(z) = \begin{cases} 
((1 - bz)^{-A/b} - 1)/A, & A \neq 0, b \neq 0, \\
-\log(1 - bz)/b, & A = 0, b \neq 0, \\
(e^{A} - 1)/A, & A \neq 0, b = 0.
\end{cases}$$

Moreover, for $A = 1 + 2\beta$ ($0 \leq \beta < 1$) and $b = 1$, with $\alpha = 1$ in Corollary 5.13 we obtain that, if $f(z) \in C(\beta)$ then $f(z) \in S^*(\gamma)$, where

$$\gamma = \gamma(\beta) = \begin{cases} 
n\frac{1 - 2\beta}{2(1 - 2\beta)}, & \beta \neq \frac{1}{2}, \\
n\frac{1}{2\log 2}, & \beta = \frac{1}{2}.
\end{cases}$$

Note that this relationship was proved by MacGregor in [10].

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