Spreading speed of locally regulated population models in macroscopically heterogeneous environments

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Abstract

We consider a certain lattice branching random walk with on-site competition and in an environment which is heterogeneous at a macroscopic scale $1/\varepsilon$ in space and time. This can be seen as a model for the spatial dynamics of a biological population in a habitat which is heterogeneous at a large scale (mountains, temperature or precipitation gradient . . .). The model incorporates another parameter, $K$, which is a measure of the local population density. We study the model in the limit when first $\varepsilon \to 0$ and then $K \to \infty$. In this asymptotic regime, we show that the rescaled position of the front as a function of time converges to the solution of an explicit ODE. We further discuss the relation with another popular model of population dynamics, the Fisher-KPP equation, which arises in the limit $K \to \infty$. Combined with known results on the Fisher-KPP equation, our results show in particular that the limits $\varepsilon \to 0$ and $K \to \infty$ do not commute in general. We conjecture that an interpolating regime appears when $\log K$ and $1/\varepsilon$ are of the same order.

Contents

1 Introduction 3
   1.1 Model and main result .......................................................... 3
   1.2 Discussion and comparison with deterministic models .................. 5
   1.3 Explicit example and simulations ............................................. 7
   1.4 Relation with other stochastic models ...................................... 10

2 General bounds on the speed of invasion 10
   2.1 Generalization of the model .................................................... 10
   2.2 Results .............................................................................. 12
   2.3 Notation ............................................................................ 13
   2.4 Structure of the proof ............................................................. 13

3 A coupling lemma 13

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4 **Proof of Proposition 2.1: Upper bound on the propagation speed** 15
   4.1 An estimate on the branching random walk ................................. 15
   4.2 Invasion speed estimate: small time steps .................................. 17
   4.3 Comparison with the solution of \((C)\): proof of Proposition 2.1 .......... 22

5 **Proof of Proposition 2.2: Lower bound on the propagation speed** 24
   5.1 The rebooted process \(X^0\) ....................................................... 24
   5.2 Comparison with a branching random walk .................................... 25
   5.3 Comparison with the solution of \((C)\) ............................................ 29

A **Appendix: the branching random walk** 34
   A.1 Many-to-one lemma ................................................................. 34
   A.2 Regularity of the rate function ................................................. 34
   A.3 First and second moment of the maximum ..................................... 36

B **Appendix: stability of the solution of \((C)\) and convergence of the Euler scheme** 37
   B.1 Stability .................................................................................. 37
   B.2 Euler scheme .............................................................................. 38
1 Introduction

In this article, we are interested in the spatial propagation of a biological population in a heterogeneous environment, where the population lives on discrete sites or demes. Formally, the population is a system of interacting particles on the integers $\mathbb{Z}$ evolving in discrete time. At each generation, particles duplicate at a certain space- and time-depending probability, undergo a regulation step where $K$ particles at most survive at each site and jump (migrate) according to a discretized Gaussian distribution.

In this first section, we introduce our model, state the main result of this article and compare it with previous results from the PDE literature.

1.1 Model and main result

We consider a particle system evolving on the rescaled lattice $\Delta x \cdot \mathbb{Z}$ at discrete time steps $0, \Delta t, 2\Delta t, \ldots$, where $\Delta t, \Delta x > 0$ are small parameters. The system depends on the following parameters:

- $\varepsilon > 0$ a small constant (with $1/\varepsilon$ being the space- and time-scale of interest)
- $K > 0$ a large constant (the carrying capacity)
- $r : [0, \infty) \times \mathbb{R} \to (0, \infty)$ a function (the growth rate).

The function $r$ will be assumed to satisfy certain regularity conditions, which will be described later in this section. A function satisfying these assumptions will be referred to as a good growth rate function.

Formally, our model is a Markov chain $(n_k)_{k \in \mathbb{N}}$ taking values in $\mathbb{N}^\mathbb{Z}$, where $n_k(i)$ is interpreted as the number of particles on the site $i\Delta x$ at time $k\Delta t$. At each time step, the configuration $n_{k+1}$ is derived from $n_k$ through three consecutive steps: reproduction, competition and migration. These steps are defined in such a way that the process $n_k$ satisfies a monotonicity condition: for two copies $n^1$ and $n^2$ of the system, starting from two initial conditions such that $n^1_{0}(i) \geq n^2_{0}(i)$ for all $i \in \mathbb{Z}$, the processes $n^1$ and $n^2$ can be coupled in such a way that $n^1_{k}(i) \geq n^2_{k}(i)$ for all $k \in \mathbb{N}$ and $i \in \mathbb{Z}$. In this setting, we will need the following two definitions.

**Definition 1.** Let $\mu$ and $\nu$ be two probability distributions on $\mathbb{R}$. We say that $\nu$ stochastically dominates $\mu$ if $\nu([x, \infty)) \geq \mu([x, \infty))$ for all $x \in \mathbb{R}$.

**Definition 2.** A family of probability distributions $(\nu_r)$ on $\mathbb{R}$ is increasing with respect to $r$ if for all $r_1 < r_2$, $\nu_{r_2}$ stochastically dominates $\nu_{r_1}$.

We are now ready to define our interacting particle system. Suppose we are given a good growth rate function $r$ and an increasing sequence of reproduction laws (i.e. probability distributions on $\mathbb{N}$) denoted by $(\nu_r)$. Assume that $\nu_r(0) = 0$ for all $r > 0$. Additionally, we assume that the initial condition satisfies $n_0(i) = 0$ for $i > 0$ and $n_0(0) \geq 1$, i.e. the right-most particle is at the origin. The configuration $(n_{k+1}(i))_{i \in \mathbb{Z}}$ is obtained from $(n_k(i))_{i \in \mathbb{Z}}$ as follows:

1. **Reproduction step.** Each particle living on the $i$-th site at generation $k$ independently gives birth to a random number of children distributed according to $\nu_r(ck\Delta t, ci\Delta x)$.

2. **Competition step.** Only $K$ particles per site survive to the next generation. In other words, the number of particles on site $i$ after the competition step is given by the truncated sum

$$\left(\sum_{m=1}^{n_k(i)} Y_m \right) \land K, \quad (1.1)$$

where $(Y_m)$ is a sequence of i.i.d. random variables of law $\nu_r(ck\Delta t, ci\Delta x)$.
3. Migration step. A particle on the $i$-th site jumps to the site $i + j$ with probability $\mu(j)$, where the migration law $\mu$ is a discretized normal distribution:

$$
\mu(j) = \int_{(j-\frac{1}{2})\Delta x}^{(j+\frac{1}{2})\Delta x} \frac{1}{\sqrt{2\pi}\Delta t} e^{-\frac{x^2}{2\Delta t}} dx.
$$

(1.2)

All particles jump independently and simultaneously.

The resulting configuration is $n_{k+1}$.

We denote by $X_k^* = \max\{i \in \mathbb{Z} : n_k(i) > 0\} \cdot \Delta x$ the position of the rightmost particle at generation $k$ in this system. Note that $X_0^* = 0$ by assumption.

The main goal of this article is to investigate the long-time behaviour of the process $(\varepsilon X_k^*)_{k \in \mathbb{Z}}$. More precisely, we compare $(\varepsilon X_k^*)_{k \geq 0}$ with $(x(\varepsilon k \Delta t))_{k \geq 0}$, where $x$ is the solution of the Cauchy problem

$$
\begin{align*}
    x'(t) &= \sqrt{2r(t, x(t))} \\
    x(0) &= 0.
\end{align*}
$$

(1.3)

The result is the following.

**Theorem 1.** Assume that $r$ is a good growth rate function. Let $T > 0$ and $\delta > 0$. There exists $\Delta t_\delta > 0$ and $C_\delta > 0$ such that, if $\Delta t < \Delta t_\delta$ and $\Delta x < C_\delta \Delta t$, there exists $K_0$ such that, for all $K > K_0$

$$
\lim_{\varepsilon \to 0} \mathbb{P}\left( \max_{k \in [0, \frac{T}{\varepsilon \Delta t}]} \left| \varepsilon X_k^* - x(\varepsilon k \Delta t) \right| \leq \delta \right) = 1.
$$

We conclude this section by introducing our definition of a good growth rate function. This definition will ensure the existence and uniqueness of a solution to the Cauchy problem (1) defined on $[0, \infty)$.

As we shall see, it will be sufficient to prove our intermediate results for growth rate functions $r$ satisfying strong regularity conditions. In this case, we will rely on the following assumptions.

**Definition 3.** We say that $r$ is a smooth growth rate function if

(i) there exists $0 < \underline{r} < \bar{r}$ such that

$$
\forall (t, x) \in [0, \infty) \times \mathbb{R}, \quad \underline{r} \leq r(t, x) \leq \bar{r},
$$

(ii) we have

$$
r \in C^1((0, \infty) \times \mathbb{R}), \quad \text{and} \quad \exists M > 0 : \|\partial r/\partial t\|_\infty + \|\partial r/\partial x\|_\infty < M. \tag{1.3}
$$

These two conditions imply that, for all $(t_1, x_1), (t_2, x_2) \in [0, \infty) \times \mathbb{R},$

$$
|\sqrt{2r(t_1, x_1)} - \sqrt{2r(t_2, x_2)}| \leq L(|t_1 - t_2| + |x_1 - x_2|), \quad \text{with} \quad L := \frac{M}{\sqrt{2\underline{r}}}. \tag{1.4}
$$

and that (1) has a unique global solution.

**Definition 4.** We say that $r$ is a good growth rate function if there exists $0 < \underline{r} < \bar{r}$ and two sequences of smooth growth rate functions $(\theta_n)$ and $(\Theta_n)$ defined on $[0, \infty) \times \mathbb{R}$ such that the following holds.
(i) For all \( n \in \mathbb{N} \) and for all \( (t, x) \in [0, \infty) \times \mathbb{R} \),
\[
\gamma \leq \theta_n(t, x) \leq r(t, x) \leq \Theta_n(t, x) \leq \bar{r}.
\]

(ii) Let \( (\bar{x}_n, \bar{r}_n) \) denote the unique couple of global solutions to the Cauchy problems
\[
\bar{x}_n = \sqrt{2n(t, \bar{x}_n(t))}, \quad \bar{r}_n = \sqrt{2\Theta_n(t, \bar{x}_n(t))}, \quad \bar{x}_n(0) = \bar{x}_n(0) = x_0.
\]

We assume that for all \( T > 0 \),
\[
\sup_{t \in [0, T]} |\bar{x}_n(t) - \bar{x}_n(t)| \xrightarrow{n \to \infty} 0.
\]

Example 1. Smooth growth rate functions are good growth rate functions. Functions that only depend on one coordinate, and that are piecewise \( C^1 \) in this coordinate are good growth rate functions. This example will be further developed in the next section. The function \( r(t, x) = 1 + 1_{x \geq (\sqrt{2}+2)t/2} \) is not a good growth rate function: for \( x_0 = 0 \), \( \bar{x}_n(t) = \sqrt{2t} \), while \( \bar{x}_n(t) = 2t \), for any sequence of approximations \( \theta_n, \Theta_n \). If \( \phi : \mathbb{R} \to \mathbb{R}_+ \) denotes a smooth function supported on \( [-1-\sqrt{2}/2, 1-\sqrt{2}/2] \) and of integral 1, \( \bar{r}(t, x) := \int_\mathbb{R} \phi((x-y)/t) \bar{r}(t, y) dy \) is another example of a growth rate function that is not good (we also have \( \bar{x}_n(t) = \sqrt{2t} \) and \( \bar{x}_n(t) = 2t \) for \( x_0 = 0 \)), even though the only discontinuity point of \( \bar{r} \) is \((0, 0)\).

1.2 Discussion and comparison with deterministic models

In Theorem 1, we consider \( K \) large but fixed and let \( \varepsilon \to 0 \). A more classical large population asymptotics of individual-based models such as this one consists in letting first \( K \to \infty \), then \( \varepsilon \to 0 \). If we divide the population size by \( K \) and let \( K \to \infty \) (and let \( \Delta r, \Delta t \) tend to 0 in a controlled way), we obtain a (deterministic) PDE \((22, 17)\). This PDE is a reaction-diffusion equation of Fisher-KPP type governing the density of individuals \( u(\varepsilon, t, x) \):
\[
u(\varepsilon, t, x) = \frac{1}{2} u(\varepsilon, t, x) + r(\varepsilon, x) f(u(\varepsilon, t, x)), \quad (1.6)
\]
with \( f(u) = u 1_{u<1} \). The limiting behavior of \( u(\varepsilon) \) as \( \varepsilon \to 0 \) has been widely investigated in the PDE literature \([11, 21, 0]\] but also with probabilistic arguments \([24]\). Introducing the change of variables \( (t, x) \to (t/\varepsilon, x/\varepsilon) \) and the WKB ansatz
\[
u(\varepsilon, t, x) = e^{-\nu(\varepsilon, t/\varepsilon, x/\varepsilon)}, \quad (1.7)
\]
and assuming that, in a certain sense, \( u(\varepsilon, t, x) \to 1_{x=0} \) as \( \varepsilon \to 0 \), it can be shown that the function \( \nu(\varepsilon) \) converges, when \( \varepsilon \to 0 \), to the viscosity solution \( \nu \) of the following Hamilton-Jacobi equation (or, more precisely, a variational inequality) \([21]\):
\[
\begin{cases}
\min \left( v_0(t, x) + \frac{1}{2}(v_x(t, x))^2 + r(t, x), v(t, x) \right) = 0 \\
v(t, x) \to \infty 1_{x\neq 0}, t \to 0.
\end{cases}
\]

As a consequence, \( u(\varepsilon, t/\varepsilon, x/\varepsilon) \) converges to 1 (resp. 0) uniformly on compact subsets of \( \text{int}(I) \) (resp. \( \partial I \)), where
\[
I = \{(t, x) \in [0, \infty) \times \mathbb{R} : v(t, x) = 0\}.
\]

In particular, if \( x(\varepsilon, t) \) denotes the position of the front (for example, \( x(\varepsilon, t) = \sup\{x : u(\varepsilon, t, x) \geq 1/2\} \)), then, for fixed \( t \geq 0 \),
\[
x(\varepsilon, t) \to x^{HJ}(t) := \sup\{x : v(t, x) = 0\}, \quad \text{as} \ \varepsilon \to 0.
\]
This approach has been extensively employed so far to deal with different types of heterogeneous environments: periodic \([42, 41, 44]\), random \([0, 36]\), but does not provide an explicit propagation speed, except in very specific situations \([26]\). However, it is known that the following bound holds.

**Lemma 1.1.** Let \(x(\cdot)\) denote the solution of the Cauchy problem \([4]\). We always have

\[
x^{HJ}(t) \geq x(t) \quad \text{for all } t \geq 0.
\]

**Proof.** To see this, recall the variational representation of the function \(v\) \([21]\):

\[
v(t, x) = \sup_{\tau} \inf_{z} \left\{ \int_{0}^{\tau(z)} \frac{z'(s)^2}{2} - r(t - s, z(s)) \, ds \mid z(0) = x, z(t) = 0 \right\}. \tag{1.9}
\]

Here, the infimum is over all \(z \in H^1_{\text{loc}}([0, \infty); \mathbb{R})\) and the supremum is over all stopping times \(\tau\), i.e. maps \(\tau : H^1_{\text{loc}}([0, \infty); \mathbb{R}) \to [0, \infty)\) satisfying for all \(z, \tilde{z}\) and all \(s \geq 0:\)

\[
\text{if } z \equiv \tilde{z} \text{ on } [0, s] \text{ and } \tau(z) \leq s, \text{ then } \tau(\tilde{z}) = \tau(z).
\]

In order to show that \(x^{HJ}(t) \geq x(t)\), it suffices to show that \(v(t, x(t)) = 0\) for all \(t \geq 0\). Fix \(t \geq 0\). Define \(z(s) = x(t - s)\) for \(s \in [0, t]\). Then \(z(0) = x(t), z(t) = 0\) and for all \(s \in [0, t], (z'(s))^2/2 = r(t - s, z(s))\). Hence, for every stopping time \(\tau\), the integral in (1.9) equals 0. This shows that \(v(t, x(t)) \leq 0\) and thus \(v(t, x(t)) = 0\) by non-negativity of \(v\).

It is easy to construct examples where \(x^{HJ}(t) > x(t)\) for some or all \(t > 0\). This is for example the case when \(r(t, x) = r_0(x)\) for some (strictly) increasing function \(r_0\). Indeed, in this case, it is easy to construct an affine function \(z\) such that \(z(0) > x(t)\) and \(z(t) = 0\) and such that the integrand in (1.9) is negative for all \(s \in (0, t), \) whence \(v(t, z(0)) = 0\) and \(x^{HJ}(t) \geq z(0) > x(t)\). It is even possible to construct an example in which \(x^{HJ}\) has jumps: if we consider a function \(r\) such that, for some \(h > 0, r(x) = c_1 > 0\) for \(x < h\) and \(r(x) = c_2 > 2c_1\) if \(x \geq h > 0\), and an initial condition \(1_{(-\infty, 0]}\), we observe a jump in the wavefront at time \(T_0 := \frac{h}{c_0} \sqrt{2(c_2 - c_1)} < \frac{h}{\sqrt{2c_1}}\) (see Example 3 in \([23]\)). On the other hand, when \(r_0\) is non-increasing, then \(x^{HJ}(t) = x(t)\) for all \(t \geq 0\), see \([21, 23]\) for a detailed discussion and other sufficient conditions such that \(x^{HJ}(t) = x(t)\) for all \(t \geq 0\). In this case, one says that the Huygens principle is verified, in that the propagation of the front is described by a velocity field, see Freidlin \([24]\) for a discussion of this principle and its relation with the Hamilton-Jacobi limit, that he relates to geometric optics.

It has been observed previously that the viscosity solution method may be unsatisfactory, from a biological standpoint, in some situations \([25, 29]\). This has been dubbed the “tail problem” \([29]\): artifacts may be generated in the deterministic model by the infinite speed of propagation \([25]\) of the solutions of (1.6), where meaningless, exponentially small “populations” are sent to favourable regions by diffusion before the invasion front \(x(t)\), accelerating the speed of propagation and possibly causing jumps in the position of the invasion front. Some adjustments were suggested to “cut the tails” in the deterministic model. For instance, one can add a square root term with a survival threshold parameter in the F-KPP equation \([29, 33]\). Another correction suggested in \([25]\) consists in adding a strong Allee effect in Equation (1.6). Namely, they set the growth rate \(f\) to be negative at low densities, leading to a bistable reaction-diffusion equation. For such equations, the Huygens principle is verified, as shown by Freidlin \([24]\).

In this article, we propose to come back to the microscopic, or individual-based population model and study it under a double limit, where we let first the space-time scale \(1/\varepsilon\), then the carrying capacity \(K\) go to infinity.

---

\(^1\) In fact, general theory of variational inequalities (see e.g. \([3, \text{p.6}]\)) implies that for given \((t, x)\), the optimal stopping time in (1.6) is given by \(\tau_{1, x}(z) = \inf\{s \in [0, t] : v(t - s, z(s)) = 0\}\), but we don’t make use of this fact.
The discrete nature of our model has the effect of a “cutoff” which prevents the solution from being exponentially small in $1/\epsilon$. In terms of the function $v$, which arises in the limit after a hyperbolic scaling, the cutoff prevents the function $v$ from taking finite positive values and thus formally “pushes it up to $\infty$” whenever it is (strictly) positive. The main conceptual advantage of this approach compared to the PDE approach is that our model naturally satisfies the Huygens principle, without the need of ad-hoc modifications.

In order to determine which of the two models, with or without cutoff, is a better model for a given biological population, one might consider our microscopic model in the limit when $K$ and $1/\epsilon$ go to infinity together. Indeed, we conjecture that it is possible to interpolate between the two double limits in $K$ and $\epsilon$, when $K$ and $1/\epsilon$ go to infinity in such a way that $\log K$ is of the same order as $1/\epsilon$. This relation is indeed suggested by the hyperbolic scaling. It also appears in the proof of Theorem 1. In order to determine which of the two models, with or without cutoff, is a better model for a given biological population, one might consider our microscopic model in the limit when $1/\epsilon$ goes to infinity. Precisely, setting $\log K = \kappa / \epsilon$ for a constant $\kappa > 0$, we believe that this should not have an impact on what follows.

First, consider the Fisher-KPP equation (1.6), however, in order to be able to state results from the literature, assume that non-linearity appearing in (1.6) has the form $f(u) = u(1 - u)$ instead — we believe that this should not have an impact on what follows. For $\epsilon > 0$, denote by $c^*_\epsilon$ the speed of propagation, i.e. the smallest number such that a pulsating travelling front with speed $c \geq c^*_\epsilon$ exists for all $c \geq c^*_\epsilon$ [5]. It has been shown [33] that $c^*_\epsilon$ is nonincreasing with respect to $\epsilon$ and bounded. Therefore, it converges to some limit $c^*_0$ as $\epsilon \to 0$. An explicit formula for $c^*_0$ has been computed in [26] with the viscosity solution method. Under assumption (1.10), this expression is even more explicit and given by

$$c^*_0 = \frac{2\sqrt{\Delta}}{(\mu^+)^2 + (\mu^-)^2 + (\mu^+ - \mu^-)\sqrt{\Delta}}$$

where

$$\Delta = (\mu^+)^2 + (\mu^-)^2 - \mu^- \mu^+.$$

On the other hand, the limit speed of the ODE (11) is the harmonic mean between the two speeds $\sqrt{2\mu^+}$ and $\sqrt{2\mu^-}$:

$$c^*_{ODE} = \frac{2\sqrt{\mu^+ \mu^-}}{\sqrt{\mu^-} + \sqrt{\mu^+}}.$$
Therefore, as noted in [25], $c^{ODE}$ is strictly smaller than the quadratic mean $\sqrt{\mu^+ + \mu^-}$, which corresponds to the homogenization limit $\bar{c}^* := \lim_{\varepsilon \to \infty} c^*_\varepsilon$ [43]. Summarizing, we have, using again the fact that $c^*_\varepsilon$ is non-increasing with respect to $\varepsilon$,

$$c^*_0 \geq \bar{c}^* > c^{ODE}.$$  

We have simulated our particle system for

$$\nu_r = (1 - r \delta t) \delta_0 + r \Delta t \delta_1,$$

and $r$ as in (1.10), with $\mu^+ = 3$ and $\mu^- = 0.1$. In this case, $c^*_0 = 1.901..$ and $c^{ODE} = 0.756..$ The results of the simulations are shown in Figure 1. They illustrate the behaviour of the process under the limits $\varepsilon \to 0$ and $K \to \infty$. We observe that when $\varepsilon$ is fixed and $K$ goes to infinity, the position of the rightmost particle in a simulation of the process approaches the front of the solution of the PDE. On the other hand, when $K$ is fixed and $\varepsilon$ tends to 0, it tends to the solution of the ODE, in line with Theorem 1.
Figure 1: Rescaled position of the rightmost particle in simulations of the process defined in Section 1.1 (green line) for different values of \((K, \varepsilon)\). Left column: fixed \(\varepsilon\), increasing \(K\), right column: fixed \(K\), decreasing \(\varepsilon\). The growth rate \(r\) is a 1-periodic function of the form (1.10) with \(\mu^+ = 3\) and \(\mu^- = 0.1\), and the initial configuration is given by \(n_k(i) = 1_{x \leq 0}\) (which implies that \(X^*_n = 0\)). The orange dotted line is the graph of the solution of the ODE (1). The blue solid line is the position of the front \(x(t) = \sup\{x \in \mathbb{R} : u(t, x) > \frac{1}{N}\}\) for a solution to \(u_t(t, x) = \frac{1}{2}u_{xx}(t, x) + r(\varepsilon t, \varepsilon x)u(t, x)(1 - u(t, x))\), with initial condition \(\chi(x) = 1_{x < 0}\). Note that the orange dotted line lies below the blue one (see Lemma 1.1).
1.4 Relation with other stochastic models

The model we consider in this work is an example of a microscopic model for front propagation. Such models have seen considerable interest in the last two decades in mathematics, physics and biology. The prototypical model of front propagation is the Fisher-KPP equation, a semi-linear parabolic partial differential equation which admits so-called travelling waves, i.e. solutions which are stationary in shape and which travel at constant speed. Many microscopic models of front propagation (in homogeneous environments) can be seen as noisy versions of the Fisher-KPP equation, see e.g. the reviews [38, 30]. A rich theory originating in the work of Brunet, Derrida and co-authors [14, 15, 16] has put forward some universal asymptotic behavior when the local population density at equilibrium, $K$, goes to infinity. First, the speed of propagation of such systems admits a correction of the order $O((\log K)^{-2})$ compared to the limiting PDE. Second, the genealogy at the tip of the front is described by the Bolthausen–Sznitman coalescent over the time scale $(\log K)^3$, in stark contrast to mean-field models where the genealogy evolves over the much longer time scale $K$ and is described by Kingman’s coalescent. These facts have been proven rigorously for several models [4, 34, 7, 31, 37, 18].

Compared to the case of homogeneous environment, the model in heterogeneous environment considered in this paper has a different speed of propagation than its continuous limit, even in the limit of infinite population size. A similar situation happens in homogeneous environment when the displacement is heavy-tailed. Such a microscopic model, with branching, competition and displacement with polynomial tails, was considered in [9]. For their model, the authors show the existence of a phase transition in the tail exponent of the displacement law: when the exponent is sufficiently large, the model grows linearly, whereas it grows superlinearly when the exponent is small. On the other hand, the continuous limit of the model, a certain integro-differential equation, always grows exponentially fast regardless of the exponent. This example, as well as the one considered in this paper, show that microscopic probabilistic models of front propagation or of spatial population dynamics can exhibit quite different qualitative behavior than their continuous limits. We believe this to be an exciting direction for future research.

Another body of literature is concerned with the behavior of locally regulated population models at equilibrium, i.e. in the bulk. Basic questions like survival and ergodicity are often studied using two methods stemming from interacting particle systems: duality and/or comparison with simpler models such as directed percolation [19, 28, 13, 11, 8]. The genealogy of such systems is also of interest. Some related models from population genetics admit an explicit description of their genealogy in terms of coalescing, and sometimes branching random walks. Their behavior is therefore dimension-dependent, see e.g. [2] for a survey. For the one-dimensional model considered here, we expect the same to happen: the genealogy should be described by random walks coalescing when they meet at a rate proportional to $1/K$, where $K$ is the local population density at equilibrium. In particular, on the time-scale $K$, its scaling limit should be a system of Brownian motions which coalesce at a rate proportional to their intersection local time, whereas on a larger time-scale, corresponding to small population density, it should be described by the Brownian web. See [39, 20, 12] for recent results on related models.

Finally, we point out that our model has been defined in such a way that it is a monotone particle system. This property is crucial in order to compare the process to other, simpler processes. It is the analogue of the parabolic maximum principle for PDEs. Its absence causes significant technical difficulties, see for example [32] which studies (homogeneous) branching random walks with non-local competition.

2 General bounds on the speed of invasion

2.1 Generalization of the model

In this section, we introduce a version of the model defined in the introduction and state intermediate results under weaker hypotheses on the dynamics of the particle system.
As before, we consider a system of interacting particles $X$ on the rescaled lattice $\Delta x \cdot Z$, evolving in discrete time $(t_k)_{k \in \mathbb{N}}$, where $t_k = k\Delta t$. The state of the system at time $t_k$ (or equivalently at generation $k$) is described by its configuration $n_k : Z \to \mathbb{N}$, where the integer $n_k(i)$ counts the number of particles living on the site $x_i = i\Delta x$. At each time step, the particles give birth to a random number of children and die. After their birth, the offspring migrate independently. As before, we denote by $X_k^*$ the position of the right-most particle at time $t_k$.

The parameters of the model, $r$, $\varepsilon$, $K$, are the same as in the introduction. Furthermore, the migration step does not change: particles migrate according to the discretized normal distribution $\mu$ defined in (1.2). However, the reproduction and competition steps are generalized. To this end, we suppose that we are given a family $(\nu_{r,n,K})_{n,K}$ of probability distributions on $\mathbb{N}$. Reproduction and competition are then contracted into a single step as follows: if a site $x_i$ is inhabited by $n$ particles at time $t_k$, these particles are replaced by a random number of offspring distributed according to $\nu_{r(\varepsilon t_k,\varepsilon x_i),n_k(i),K}$, independently on all sites. Once the population is renewed on all sites, the particles migrate independently according to $\mu$. The resulting configuration is $n_{k+1}$.

The proof of Theorem 1 requires to establish an upper bound (Proposition 2.1) and a lower bound (Proposition 2.2) on the position of the rightmost particle $X^*$. The upper bound holds under relatively weak assumptions, which we now introduce.

**Assumption 1.** We assume that $\nu_{r,n,K}$ is stochastically dominated by $\tilde{\nu}_{n,r,K}$, where $(\tilde{\nu}_{n,r,K})$ is a family of reproduction laws satisfying the three following conditions:

1. $(\tilde{\nu}_{r,n,K})$ is increasing with respect to $r$ and with respect to $n$.
2. there exists a family of discrete probability distributions on $\mathbb{N}$, denoted by $(P_r)$, such that $(P_r)$ is increasing with respect to $r$, $\mathbb{E}[P_r] = 1 + r\Delta t$ and, for all $K > 0$, $P_r^n$ stochastically dominates $\tilde{\nu}_{r,n,K}$.
3. there exists a probability distribution on $\mathbb{N}$, denoted by $\bar{\nu}$, with finite expectation and such that $\bar{\nu}^K$ stochastically dominates $\tilde{\nu}_{r,n,K}$, for all $n \in \mathbb{N}$.

In words, the first point ensures that the particle system is stochastically dominated by a monotone system (though it is not itself monotone), and the third point that all reproduction laws are uniformly bounded by a law that only depends on the carrying capacity of the system.

The proof of the lower bound requires stronger assumptions:

**Assumption 2.** Let $(\nu_r)$ be a family of reproduction laws such that

1. $(\nu_r)$ is continuous and increasing with respect to $r$,
2. $\nu_r(0) = 0$,
3. $\sum k\nu_r(k) = 1 + r\Delta t$.

Let $(Y_i)_{i \in \mathbb{N}}$ be a sequence of i.i.d. random variables of law $\nu_r$ and let $\tilde{\nu}_{n,r,K}$ denote the law of the random variable $\left(\sum_{i=1}^{n} Y_i\right) \wedge K$.

We assume that $\nu_{r,n,K}$ stochastically dominates the $\tilde{\nu}_{n,r,K}$.

**Remark 1.** Note that under Assumption 2, the process cannot die out. This is a simplifying assumption without which the proof of Proposition 2.2 would be more involved.
Remark 2. While neither Assumption 7 nor 8 requires the interacting particle system to be monotone, both imply that the system is bounded by monotone systems. Note that Assumption 7 does not specify the precise form of the competition mechanism for the upper-bound system, in contrast to Assumption 8. In addition, the process can go extinct under Assumption 7, but not under Assumption 8.

As we will see, the proof of Proposition 2.2 relies on couplings with branching random walks. For the upper bound, these couplings only require Assumption 7. The lower bound is more delicate and Assumption 8 proves particularly useful. Indeed, under Assumption 8, the system behaves like a branching system as long as the population size at each site remains below $K$.

Remark 3. The system defined in Section 2.2 satisfies Assumption 7 and Assumption 8.

Example 2. Suppose $\Delta t \leq 1/\bar{r}$. The family of distributions $(\nu_r)$ with $\nu_r = r\Delta t\bar{\delta} + (1 - r\Delta t)\bar{\gamma}$ satisfies Assumptions 8, 1, 2 and 3.

Example 3. The family of Poisson distributions with parameters

$$\lambda_{n,r,K} = n \left(1 + r\Delta t \left(1 - \frac{r}{K}\right)\right)_+$$

forms a family of reproduction laws $(\nu_{r,n,K})$ which satisfies Assumption 7 but which does not satisfy Assumption 8. Consider the family of Poisson distributions with parameters $\tilde{\lambda}_{n,r,K} = n(1 + r\Delta t)\wedge (K(1 + r\Delta t)^2/(4\bar{r}\Delta t))$. One can easily check that this family of distribution satisfies the three conditions of Assumption 7 (point 2 is satisfied with $P_r$ the Poisson distribution of parameter $1 + r\Delta t$ and point 3 is satisfied with $\tilde{\nu}$ the Poisson distribution with parameter $(1 + r\Delta t)^2/(4\bar{r}\Delta t)$, and that it stochastically dominates $(\nu_{r,n,K})$. On the other hand, this family does not satisfy any of the points of Assumption 8.

If $(P_r)$ is as in Assumption 7, the family of reproduction laws $(P^n_{r,n})$ satisfies Assumption 8 but does not satisfy Assumption 7.

2.2 Results

Proposition 2.1 (Upper bound on the propagation speed). Assume that $r$ is a good growth rate function and that Assumption 7 holds. Assume that $X^*_0 = 0$. Let $T > 0$ and $\delta > 0$. There exist $\Delta t_0 > 0$, $C_0 > 0$ such that, if

$$\Delta t < \Delta t_0, \quad \Delta x < C_0\Delta t,$$

the following holds: there exists $\alpha > 0$ and $\varepsilon_0 > 0$ such that,

$$\forall \varepsilon < \varepsilon_0, \forall K \geq 1, \quad \mathbb{P}\left(\exists k \in \mathbb{N} : \frac{T}{\varepsilon\Delta t} \leq k \Rightarrow \varepsilon X^*_k < x(k\varepsilon\Delta t) + \delta\right) \leq Ke^{-\frac{\alpha}{\varepsilon}}. \quad (2.1)$$

Proposition 2.2 (Lower bound on the propagation speed). Assume that $r$ is a good growth rate function and that Assumption 8 holds. Assume that $X^*_0 = 0$. Let $\delta > 0$ and $T > 0$. There exist $\Delta t_0 > 0$, $C_0 > 0$ such that, if

$$\Delta t < \Delta t_0, \quad \Delta x < C_0\Delta t,$$

the following holds: there exists $K_0 > 0$ and $\varepsilon_0 > 0$ such that,

$$\forall K > K_0, \forall \varepsilon < \varepsilon_0, \quad \mathbb{P}\left(\exists k \in \mathbb{N} : \frac{T}{\varepsilon\Delta t} \leq k \Rightarrow \varepsilon X^*_k < x(k\varepsilon\Delta t) - \delta\right) \leq \sqrt{\varepsilon}.$$

Remark 4. We believe that Proposition 2.2 holds in fact for every $K \geq 1$. Indeed, choosing $\Delta x$ small has a similar effect than choosing $K$ large in that it increases the density of particles in a unit interval at equilibrium (which one expects to be of the order of $K/\Delta x$). We believe that one could extend the proof of Proposition 2.2 to this case, at the expense of more elaborate arguments, see Remark 10 below.
Remark 5. Note that in both theorems above, $\varepsilon_0$ is chosen small, depending on all the other parameters. This indeed corresponds to the limit where we first let $\varepsilon \to 0$, then $K \to \infty$ (or $\Delta x \to 0$, see Remark 4), as mentioned in Section 1.2.

Remark 6. Our results depend on the initial configuration only through the position of the rightmost particle.

2.3 Notation

We introduce some notation that will be used throughout the article. For each particle $u$ living in the process $X$, we denote by $X_u$ its position. The process $X$ will be compared to several branching random walks (BRW). Likewise, for a branching random walk $\Xi$, $\Xi_u$ refers to the position of the particle $u$. In both cases, we denote by $|u|$ the generation of $u$. We further define the following constants:

$$\gamma = \log(2) \quad \text{and} \quad C_0 = 16\gamma^{-\frac{3}{2}}. \quad (2.2)$$

2.4 Structure of the proof

In Section 3, we state a general coupling lemma allowing us to compare systems with different reproduction mechanisms. In particular, it allows us to compare the interacting particle system with branching random walks.

Section 4 contains the proof of Proposition 2.1 (upper bound). The proof uses a Trotter-Kato-type scheme and local comparisons with branching random walks. The proof of Proposition 2.2 (lower bound) is given in Section 5 and uses a martingale argument together with first and second moment estimates.

Appendix A recalls known results on branching random walks and provides explicit estimates on the rate functions of the branching random walks used in this article. Finally, Appendix B recalls known results on the Euler scheme for the solution $x$ of (4).

3 A coupling lemma

Let $S^1$ and $S^2$ be two systems of interacting particles on $\mathbb{Z}$ whose configurations, $(n^1_k)$ and $(n^2_k)$, evolve as follows. At time $t_k$, the particles of $S^1$ (resp. $S^2$) living on $x_i$, are replaced by a random number of offspring distributed according to $(p^1_l(n^1_k(i), x_i, t_k))_{l \in \mathbb{N}}$ (resp. $(p^2_l(n^2_k(i), x_i, t_k))_{l \in \mathbb{N}}$). Once the population is renewed on each site, the particles migrate independently according to $\mu$ in both processes. Furthermore, let $\tau$ be a stopping time for the process $S^2$ (which may be infinite). We say that $S^1$ dominates $S^2$ until time $\tau$ if

$$\mathbb{P}(\forall i \in \mathbb{Z}, \forall k \leq \tau : n^1_k(i) \geq n^2_k(i)) = 1.$$ 

If $\tau = +\infty$ almost surely, we simply say that $S^1$ dominates $S^2$.

The following lemma establishes a coupling ($\tilde{S}^1, \tilde{S}^2$) of $S^1$ and $S^2$ such $\tilde{S}^1$ dominates $\tilde{S}^2$, provided that the reproduction laws $p^1$ and $p^2$ and the initial conditions meet certain conditions.

Lemma 3.1. Assume that

1. The initial configurations satisfy $n^1_0(i) \geq n^2_0(i)$, for all $i \in \mathbb{Z}$.
2. The system $S^1$ is monotone: for all $(m, n) \in \mathbb{N}^2$ such that $n \geq m$,

$$\sum_{q \geq l} p^1_q(n, t_k, x_i) \geq \sum_{q \geq l} p^1_q(m, t_k, x_i), \quad \forall l \in \mathbb{N}. \quad (3.1)$$
3. Almost surely with respect to the process $S^2$, for every $k < \tau$, $i \in \mathbb{N}$,

$$
\sum_{q \geq 1} p^1_q(n^2_q(i), t_k, x_i) \geq \sum_{q \geq 1} p^2_q(n^2_q(i), t_k, x_i), \quad \forall l \in \mathbb{N}, \forall i \in \mathbb{Z}.
$$

(3.2)

Then, there exists two processes $\tilde{S}^1$ and $\tilde{S}^2$, distributed as $S^1$ and $S^2$, such that $\tilde{S}^1$ dominates $\tilde{S}^2$ until time $\tau$.

Proof. We first assume that $\tau = +\infty$. We construct a probability space supporting two processes $\tilde{S}^1$ and $\tilde{S}^2$, distributed as $S^1$ and $S^2$, such that $\tilde{S}^1$ dominates $\tilde{S}^2$. We first consider a set of particles organised according to $n^0_0(i)$. On each site $x_i$, $n^2_q(i)$ of these particles are coloured blue. The remaining individuals are coloured red. The initial population of the process $\tilde{S}^2$ (resp. $\tilde{S}^1$) is defined as the set of blue (resp. red and blue) particles.

We then construct the first generation $(k = 1)$ as follows. Consider a site $x_i$ such that $n^0_0(i) = \tilde{n}^0_0(i) \neq 0$: this site is inhabited by $k_1$ red particles, $k_2$ blue particles such that $k_1 + k_2 > 0$. Draw a uniform random variable $U$ on $[0, 1]$ and consider $l_1(\omega)$ and $l_2(\omega)$, two integers defined by

$$
l_1(\omega) = \max \left\{ n \in \mathbb{N} : U(\omega) \geq \sum_{q = 1}^{n-1} p^1_q(k_1 + k_2, t_1, x_i) \right\},
$$

and likewise,

$$
l_2(\omega) = \max \left\{ n \in \mathbb{N} : U(\omega) \geq \sum_{q = 1}^{n-1} p^2_q(k_2, t_1, x_i) \right\}.
$$

By definition of $l_1$ and $l_2$, if $l_2 \geq l_1 - 1$, Equations (3.2) and (3.1) imply that

$$
U(\omega) > \sum_{q = 1}^{l_2} p^2_q(k_2, t_1, x_i) \geq \sum_{q = 1}^{l_1-1} p^2_q(k_2, t_1, x_i) \geq \sum_{q = 1}^{l_1-1} p^1_q(k_1 + k_2, t_1, x_i)
$$

$$
\geq \sum_{q = 1}^{l_1-1} p^1_q(k_1 + k_2, t_1, x_i) \geq U(\omega).
$$

(3.3)

Thus, we deduce that $l_1(\omega) \geq l_2(\omega)$. We then generate $l_1(\omega)$ individuals on $x_i$ and $l_2(\omega)$ of them are coloured blue. The remaining ones are painted red. We repeat this construction until the population is renewed on each non-empty site $x_i$. Then, all the particles (red and blue ones) migrate independently according to $\mu$. After the migration phase, the first generation of $\tilde{S}^1$ (resp. $\tilde{S}^2$) is the set of blue (resp. red and blue) particles. The following generations are constructed similarly by induction on $k$.

If $\tau$ is an arbitrary stopping time, since it is a measurable function of the process $S^2$, it can be transferred to the probability space constructed above, to become a stopping time for the process $\tilde{S}^2$. The above chain of inequalities then still hold for every $k < \tau$ and the statement follows.

Remark 7. The same result holds when the roles of $S^1$ and $S^2$ are swapped. Indeed, a similar coupling can be constructed if $\tau$ is a stopping time for the process $S^1$, $S^2$ is monotone and (3.2) holds for any configuration of the system $S^1$. In this case, the proof goes along the same lines, with (3.3) replaced by the following inequalities:

$$
U(\omega) > \sum_{q = 1}^{l_2} p^2_q(k_2, t_1, x_i) \geq \sum_{q = 1}^{l_2} p^2_q(k_1 + k_2, t_1, x_i) \geq \sum_{q = 1}^{l_2} p^1_q(k_1 + k_2, t_1, x_i)
$$

$$
\geq \sum_{q = 1}^{l_1-1} p^1_q(k_1 + k_2, t_1, x_i) \geq U(\omega).
$$

14
4 Proof of Proposition 2.1: Upper bound on the propagation speed

In this section, we give an upper bound on the invasion speed of the process $X$ under Assumption 1. The idea of the proof of Proposition 2.1 is to first establish a coupling between $X$ and a process without competition. The absence of competition in this process then allows to compare it with several branching random walks, for which we can easily control the position of their rightmost particles (see Section 4.1).

4.1 An estimate on the branching random walk

Let $X_1$ be a random variable of law $\mu$. We define the function $\Lambda$ by

$$E[e^{\lambda X_1}] = e^{\Lambda(\lambda)}, \quad \forall \lambda \in \mathbb{R},$$

(4.1)

and denote by $I$ its convex conjugate:

$$I(y) = \sup_{\lambda \in \mathbb{R}} (\lambda y - \Lambda(\lambda)), \quad \forall y \in \mathbb{R}.$$  

(4.2)

Note that $\mu$ from (1.2) has super-exponential tails and that its support is unbounded both to the right and to the left, which implies that both $\Lambda$ and $I$ are finite and strictly convex on $\mathbb{R}$. Furthermore, $I$ has a minimum at $E[X_1] = 0$ (and $I(0) = 0$) so that $I$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$, as a consequence of strict convexity. We also define

$$\Lambda_0(\lambda) := \log \int \frac{1}{\sqrt{2\pi \Delta t}} e^{-\frac{x^2}{2\Delta t} + \lambda x} dx = \frac{\Delta t}{2} \lambda^2, \quad \forall \lambda \in \mathbb{R},$$

(4.3)

and remark that $I_0(y) := \sup_{\lambda \in \mathbb{R}} (\lambda y - \Lambda_0(\lambda))$ is given by

$$I_0(y) = \left(\frac{y}{\Delta t}\right) y - \Lambda_0 \left(\frac{y}{\Delta t}\right) = \frac{y^2}{2\Delta t}.$$  

(4.4)

Thus, for all $m > 1$, the equation $I_0(c) = \log(m)$ has a unique positive solution,

$$c_0 = \sqrt{2\Delta t \log m}.$$  

(4.5)

Since $I$ increases on $(0, \infty)$, the equation $I(c) = \log(m)$ also has a unique solution $c \in (0, \infty)$. In Appendix A.2, we state several results (Lemma A.2 to A.5) on the regularity of $I$ and $c$. These results lead to a first rough estimate on $c$.

Lemma 4.1. Let $\Delta t < \bar{r}^{-1}$ and $\Delta x < \frac{1}{16} \sqrt{2\gamma r \Delta t}$. Let $r \in [\underline{r}, \bar{r}]$ and $c$ be the unique positive solution of $I(x) = \log(1 + x \Delta t)$. Then,

$$\frac{1}{2} \sqrt{2\gamma \Delta t} < c < 2\sqrt{2r} \Delta t.$$  

(4.6)

Proof. By concavity of the logarithm function,

$$(1 - r \Delta t) \log(1 + r \Delta t) \log(2) = \gamma r \Delta t \leq \log(1 + r \Delta t).$$  

(4.7)

This implies that $\Delta x < \frac{1}{16} \sqrt{2\Delta t \log(1 + r \Delta t)}$ and, according to Lemma A.3 and Equation (4.2), that

$$\frac{1}{2} \sqrt{2\Delta t \log(1 + r \Delta t)} < c < 2\sqrt{2\Delta t \log(1 + r \Delta t)}.$$  

(4.8)
Finally, combining (4.7) and (4.8), we get that

\[
\frac{1}{2} \sqrt{2\gamma r} \Delta t \leq \frac{1}{2} \sqrt{2\gamma r} \Delta t \leq c \leq 2 \sqrt{2\gamma r} \Delta t \leq 2 \sqrt{2\gamma r} \Delta t. \tag{4.9}
\]

Lemma 4.2. Under the same assumptions as in Lemma 4.1,

\[
|c - \sqrt{2\Delta t \log(1 + r \Delta t)}| \leq a \Delta x,
\]

with \( a = 16\gamma^{-\frac{1}{2}} \left( \frac{c}{2} \right)^{\frac{1}{2}} \leq 16\gamma^{-\frac{1}{2}} \left( \frac{\bar{r}}{2} \right)^{\frac{1}{2}} \).

Proof. According to Lemma 4.1, \( c \) is located in a compact interval that does not depend on \( r \). Let \( \varepsilon_0 = \sqrt{2\Delta t \log(1 + r \Delta t)} \) and remark that the inequality (4.6) also holds when \( c \) is replaced by \( \varepsilon_0 \). Then, since \( \Delta x < \frac{1}{4} \left( \frac{1}{2} \sqrt{2\gamma r} \Delta t \right) \), Lemma A.5 applied with \( y = \frac{1}{2} \sqrt{2\gamma r} \Delta t \) implies that

\[
\frac{1}{8} \sqrt{2\gamma r} |c - \varepsilon_0| \leq |I(c) - I(\varepsilon_0)|. \tag{4.10}
\]

In addition, note that \( I(c) = I_0(\varepsilon_0) = \log(1 + r \Delta t) \) (see Equation (4.4)), so that

\[
|I(c) - I(\varepsilon_0)| = |I_0(\varepsilon_0) - I(\varepsilon_0)| \leq 2 \varepsilon_0 \frac{\Delta x}{\Delta t} \leq 2 \sqrt{2\gamma r} \Delta x, \tag{4.11}
\]

according to Lemma A.2. Then, combining Equations (4.10) and (4.11), we get that

\[
|c - \varepsilon_0| \leq a \Delta x. \tag{4.12}
\]

Proof. Let \( \eta > 0 \), \( A \geq 0 \). It will be enough to prove that

\[
P(\exists n \in \mathbb{N} : M_n > (1 + \eta)nx + A) \leq h(\eta) e^{-\frac{\sqrt{2\gamma r}}{2A} A},
\]

with \( h(\eta) \) defined by

\[
h(\eta) = \frac{e^{-\frac{2\gamma \Delta t}{2A} \eta}}{1 - e^{-\frac{2\gamma \Delta t}{2A} \eta}}, \quad \forall \eta > 0. \tag{4.14}
\]

Proof. Let \( \eta > 0 \), \( A \geq 0 \). It will be enough to prove that

\[
P(\exists n \in \mathbb{N} : M_n > (1 + \eta)nx + A) \leq g(\eta) e^{-\frac{\sqrt{2\gamma r}}{2A} A}, \tag{4.13}
\]

where \( c \) refers to the unique positive solution of \( I(c) = \log(1 + r \Delta t) \) and

\[
g(\eta) = \frac{e^{-\frac{c^2}{2\gamma r} \eta}}{1 - e^{-\frac{c^2}{2\gamma r} \eta}}. \tag{4.14}
\]
Indeed, according to Lemma 4.11, \( \varepsilon > \frac{1}{2} \sqrt{2 \gamma \Delta t} \), so that
\[
\frac{\varepsilon^2}{4\Delta t} \geq \frac{1}{8} \gamma \Delta t, \quad \text{and,} \quad \frac{\varepsilon}{4\Delta t} \geq \frac{1}{8} \sqrt{2 \gamma \Delta t}.
\]

We now prove (4.13). For a particle \( v \) living in the BRW, we denote by \( \Xi_v \) its position and define
\[ Z_n = \sum_{|v|=n} \mathbb{1}_{\Xi_v > (1+\eta)n\varepsilon + A}. \]  
Markov’s inequality implies that
\[
\mathbb{P}(M_n > (1+\eta)n\varepsilon + A) = \mathbb{P}(Z_n \geq 1) \leq \mathbb{E}[Z_n], \tag{4.15}
\]
and thanks to the many-to-one lemma (see Lemma A.1), we know that
\[
\mathbb{E}[Z_n] = (1 + r \Delta t)^n \mathbb{P}(\Xi_v > (1+\eta)n\varepsilon + A), \tag{4.16}
\]
for any particle \( v \) of the \( n \)-th generation. Besides, by Chernoff’s bound,
\[
\mathbb{P}(\Xi_v > (1+\eta)n\varepsilon + A) \leq e^{n(\Lambda(\theta) - \theta((1+\eta)n\varepsilon + A)/n)}, \quad \forall \theta \geq 0. \tag{4.17}
\]
Remark that for \( \theta < 0 \),
\[
\theta((1+\eta)n\varepsilon + A)/n - \Lambda((1+\eta)n\varepsilon + A/n) \leq -\Lambda((1+\eta)n\varepsilon + A/n) \leq I(0) = 0.
\]
Yet, \( I((1+\eta)n\varepsilon + A/n) \geq 0 \), so that \( I((1+\eta)n\varepsilon + A/n) = \sup_{\theta \geq 0} \theta((1+\eta)n\varepsilon + A/n) - \Lambda((1+\eta)n\varepsilon + A/n) \), and Equation (4.17) gives that
\[
\mathbb{P}(\Xi_v > (1+\eta)n\varepsilon + A) \leq e^{-nI((1+\eta)n\varepsilon + A)/n}. \tag{4.18}
\]
Moreover, thanks to Lemma 4.11, we know that \( \Delta x \leq \frac{\varepsilon}{8} \), therefore, according to Lemma A.3,
\[
I(\varepsilon) + \frac{\varepsilon^2}{4\Delta t} \left( \eta + \frac{A}{n\varepsilon} \right) \leq I((1+\eta + A/(n\varepsilon))\varepsilon), \quad \forall n \in \mathbb{N}.
\]
Thus, combining (4.15), (4.16) and (4.18), we get that
\[
\mathbb{E}[Z_n] \leq e^{-\frac{\varepsilon^2}{8\Delta t}(\eta + \frac{\Delta x}{n})},
\]
since \( I(\varepsilon) = \log(1 + \frac{\varepsilon^2}{\Delta t}) \). Finally, by a union bound,
\[
\mathbb{P}(\exists n \in \mathbb{N} : M_n > (1+\eta)n\varepsilon + A) \leq \sum_{n=1}^{\infty} \mathbb{E}[Z_n] \leq e^{-\frac{\varepsilon^2}{8\Delta t}} \sum_{n=1}^{\infty} e^{-\frac{\varepsilon^2}{8\Delta t}n} = \frac{e^{-\frac{\varepsilon^2}{8\Delta t}} - e^{-\frac{\varepsilon^2}{8\Delta t}n}}{1 - e^{-\frac{\varepsilon^2}{8\Delta t}n}}.
\]
This proves (4.13) and finishes the proof of the lemma. \( \square \)

4.2 Invasion speed estimate: small time steps

In this subsection, we bound the displacement of the rightmost particle in \( X \) after \( \lfloor \varepsilon^{-1} \rfloor \) generations, i.e. after time \( \Delta t \cdot \lfloor \varepsilon^{-1} \rfloor \).

**Proposition 4.1.** Assume that Assumption 1 holds. Suppose that \( r \) is a smooth growth rate function. Let \( \Delta t < \frac{1}{r} \) and \( \Delta x < \frac{1}{5} \sqrt{2 \gamma \Delta t} \). There exist two positive constants \( \alpha \) and \( \varepsilon_0 \) such that, for all \( K > 0, \varepsilon < \varepsilon_0, (k_0, i_0) \in \mathbb{N} \times \mathbb{Z} \) and \( k_1 = k_0 + \lfloor \varepsilon^{-1} \rfloor \),
\[
\mathbb{P} \left( \exists k \in [k_0, k_1] : \varepsilon X_\varepsilon > \varepsilon x_{i_0} + \sqrt{2r(x_{i_0})\varepsilon \Delta t}k - k_0 + A(\Delta x + \Delta t)^2 | X_{k_0} \leq x_{i_0} \right) \leq Ke^{-\alpha}, \tag{4.19}
\]
for some constant \( A > 0 \) that only depends on \( \bar{r}, \bar{z} \) and \( M \) (see (1.3)).
Thus, we have that

Using Equation (4.22), we have by Markov’s inequality and a union bound,

where we have used the Gaussian tail estimate

\[ \tilde{K} \]

\[ \left(4.23\right) \]

Remark 8. Note that Proposition \( \tilde{K} \) requires stronger regularity assumptions on \( r \) than Proposition \( \tilde{K} \). We will show in Section \( \tilde{K} \) that Theorem \( \tilde{K} \) can be derived from Proposition \( \tilde{K} \) using an approximation argument.

Remark 9. It is clear from Assumption \( \tilde{K} \) and Lemma \( \tilde{K} \) that it suffices to prove the result for the particle system with reproduction law \( \tilde{\nu}_{r,n,K} \).

Proof. Let \((k_0, i_0) \in \mathbb{Z} \times \mathbb{N}\). Throughout the proof, we will assume that we start the process at generation \( k_0 \) with a deterministic initial condition \( n_{k_0} \) such that \( X_{k_0}^* \leq x_{i_0} \). The estimates we obtain will not depend on this initial condition. Rewriting the statement slightly, it will therefore be enough to show the following:

\[ \mathbb{P} \left( \exists k \in [k_0 + 1, k_1]: X_k^* > a_k \right) \leq K e^{-\frac{\alpha}{2}}, \tag{4.20} \]

where \( a_k = x_{i_0} + \sqrt{2r(\varepsilon t_{k_0}, \varepsilon x_{i_0})} \Delta t(k - k_0) + A(\Delta x + \Delta t^2)/\varepsilon \) and \( \alpha, \varepsilon, 0, A \), to be defined later, are as in the statement of the proposition.

The proof is divided into two steps. In the first step, we let the process run for one time step, after which the expected local density of the process can be bounded by a constant multiple of \( K \), thanks to Assumption \( \tilde{K} \). In the second step, we control the displacement of the rightmost particle in \( X \) between generations \( k_0 + 1 \) and \( k_1 \), thanks to several couplings with processes without competition and distinguishing the particles according to the position of their ancestor at generation \( k_0 + 1 \).

Step 1: Control of the population at generation \( k_0 + 1 \). In this step, we control the number of particles on each site in the process \( X \) after one generation. Recall that \( n_k \) denotes the configuration of the process \( X \) at generation \( k \). We denote by \( N_i \) the number of individuals born on the site \( x_i \) during the first reproduction phase. In addition, for \( \ell \in [1, N_i] \), we denote by \( U_i^\ell \) the displacement of the \( \ell \)-th particle born on the site \( x_i \) during the first reproduction phase. Therefore, we have for every \( i \in \mathbb{Z} \)

\[ n_{k_0+1}(i) = \sum_{j \leq i_0} N_j \sum_{j \leq i_0} 1_{U_j^\ell = i-j}. \tag{4.21} \]

Recall that \( (U_i^\ell) \) is a sequence of i.i.d. random variables of law \( \mu \) and that the \( (N_j) \) are stochastically dominated by the sum of \( K \) i.i.d. random variables of law \( \tilde{\nu} \) of finite expectation \( m \), by Assumption \( \tilde{K} \).

Thus, we have that

\[ \mathbb{E}[n_{k_0+1}(i)] \leq mK \sum_{j \leq i_0} \mathbb{P}(U = i-j), \tag{4.22} \]

where \( U \) is a random variable of law \( \mu \). In particular, we get

\[ \mathbb{E}[n_{k_0+1}(i)] \leq mK. \tag{4.23} \]

We will also need a bound on the position of the maximal particle at generation \( k_0 + 1 \). Let \( k \in \mathbb{N}_0 \) and \( i \in \mathbb{Z} \) such that \( x_i - x_{i_0} \geq \frac{1}{2\sqrt{\varepsilon}^2} + (k + 1) \Delta x \). Then, we have that

\[ \sum_{j \leq i_0} \mathbb{P}(U = i-j) \leq \int_{(i-i_0-\frac{1}{2})\Delta x}^{\infty} \frac{1}{\sqrt{2\pi \Delta t}} e^{-\frac{x^2}{2\Delta t}} dx \leq \int_{\frac{1}{2\sqrt{\varepsilon}} + k \Delta x}^{\infty} \frac{1}{\sqrt{2\pi \Delta t}} e^{-\frac{x^2}{2\Delta t}} dx \]

\[ \leq e^{-\frac{\Delta t}{2\Delta t} \left( \frac{1}{2\sqrt{\varepsilon}} + k \Delta x \right)^2} \leq e^{-\frac{\Delta t}{2\Delta t} \varepsilon^{-1/2}}, \]

where we have used the Gaussian tail estimate \( \mathbb{P}(Z \geq x) \leq e^{-x^2/2} \) for a standard Gaussian r.v. \( Z \). Using Equation (4.22), we have by Markov’s inequality and a union bound,

\[ \mathbb{P} \left( X_{k_0+1}^* > x_{i_0} + \frac{1}{2\sqrt{\varepsilon}^2} + \Delta x \right) \leq mKe^{-\frac{1}{2\Delta t} \varepsilon^{-1/2}} \sum_{k=0}^{\infty} e^{-\frac{k\Delta x}{2\sqrt{\varepsilon}^2}}, \]

18
and therefore, as long as $\varepsilon$ is small enough,
\[
P \left( X_{k_0 + 1}^* > x_i + \frac{1}{\sqrt{\varepsilon}} \right) \leq 2mKe^{-\frac{1}{\sqrt{\varepsilon}}}.
\] (4.24)

**Step 2: Between generations $k_0 + 1$ and $k_1$.** As mentioned above, between generations $k_0 + 1$ and $k_1$, we control the process $X$ by another process without competition between particles. More precisely, we denote by $X^1$ the process defined as $X$, but where for every $k \in [k_0 + 1, k_1]$ the reproduction law on site $x_i$ at time $t_k$ is given by the probability distribution $P^{\nu_{k_0}(i)}_{\nu_{t_k}(x_i)}$ instead of $\nu_{t_k}(x_i)$. The position of its maximum at generation $k$ is analogously denoted by $X^1_k$. By the first part of Assumption 1 and Lemma 3.1, we can couple $X$ and $X^1$ such that $X^1$ dominates $X$. Hence, in what follows, it will be enough to prove (4.20) with $X^1$ instead of $X$. The advantage of working with $X^1$ instead of $X$ is the fact that $X^1$ satisfies the branching property, i.e. the descendants of different individuals from the same generation evolve independently.

We first make use of the estimates from Step 1. Conditioning on the process at generation $k_0 + 1$, and using a union bound over the particles from that generation, with the notation $P_{(\delta, k_0 + 1)}$ to mean that the process starts with one particle at site $x_i$ at generation $k_0 + 1$, we get for sufficiently small $\varepsilon$,
\[
P \left( \exists k \in [k_0 + 1, k_1] : X^1_k > a_k \right)
\leq \sum_{i \in Z; x_i \leq x_i + \frac{1}{\sqrt{\varepsilon}}} \mathbb{E} \left[ n_{k_0 + 1}(i) P_{(\delta, k_0 + 1)} \left( \exists k \in [k_0 + 1, k_1] : X^1_k > a_k \right) \right]
+ P \left( X_{k_0 + 1}^* > x_i + \frac{1}{\sqrt{\varepsilon}} \right)
\leq mK \sum_{i \in Z; x_i \leq x_i + \frac{1}{\sqrt{\varepsilon}}} P_{(\delta, k_0 + 1)} \left( \exists k \in [k_0 + 1, k_1] : X^1_k > a_k \right) + 2mKe^{-\frac{1}{\sqrt{\varepsilon}}}.
\] (4.25)

Here, we used (4.23) and (4.24) from Step 1 in the last line.

In what follows, we bound the probability appearing on the RHS of (4.25) for various values of $x_i$. The bound will depend on whether $x_i \geq x_i - R/\varepsilon$ or not, where
\[
R := 2\sqrt{2r}\Delta t.
\] (4.26)

We need a few more definitions. Define
\[
r = \max \left\{ r(\varepsilon, \varepsilon x) ; t_{k_0} \leq t \leq t_{k_1}, |x - x_i| \leq \frac{2R}{\varepsilon} \right\}.
\] (4.27)

and $\varepsilon$ the unique positive solution of
\[
I(\varepsilon) = \log(1 + r\Delta t).
\] (4.28)

We see from (4.24) that
\[
|\sqrt{2\varepsilon} - \sqrt{2r(\varepsilon t_{k_0}, x_i)}| \leq L(\varepsilon x_{k_1} - x_{k_0}) + 2R \leq L(1 + 4\sqrt{2r})\Delta t.
\] (4.29)

Denote by $\tilde{r}$ the function
\[
\tilde{r}(t, x) = \begin{cases} r & \text{if } |x - x_i| \leq \frac{2R}{\varepsilon} \\ \tilde{r} & \text{if } |x - x_i| > \frac{2R}{\varepsilon}. \end{cases}
\] (4.30)

Now introduce three more processes $X^2$, $X^3$ and $X^4$. These processes are defined as $X^1$, except that their reproduction law on site $x_i$ at time $t_k$, $k \in [k_0 + 1, k_1]$, is given by $P^{\nu_{k_0}(i)}_{\tilde{r}(t_k, x_i)}$, $P^{\nu_{k_0}(i)}_{\tilde{r}(t_k, x_i)}$ and $P^{\nu_{k_0}(i)}_{\tilde{r}(t_k, x_i)}$, respectively. In other words, $X^3$ and $X^4$ are BRW with reproduction laws $P^\nu$ and $P^\nu$, respectively.
From the definition of \( \tilde{r} \), we immediately get that \( \tilde{r} \geq r \). Therefore, according to Lemma 3.4 and Assumption \( \Pi \) there exists a coupling between \( X^1 \) and \( X^2 \) so that \( X^2 \) dominates \( X^1 \). Similarly, there exists a coupling between \( X^2 \) and \( X^3 \) so that \( X^3 \) dominates \( X^2 \). In order to construct a coupling between \( X^2 \) and \( X^3 \), define the stopping time \( \tau \) as the first time at which a particle from the process \( X^2 \) exits the interval \( [x_{i_0} - \frac{2R}{\varepsilon}, x_{i_0} + \frac{2R}{\varepsilon}] \) before time \( t_{k_i} \). By the definition of \( \tilde{r} \), there exists then a coupling between \( X^2 \) and \( X^3 \), such that \( X^3 \) dominates \( X^2 \) until the time \( \tau \).

Let \( i \in \mathbb{Z} \). As a consequence of the previous couplings, we have the following two bounds for the probability appearing on the RHS of (4.20). First,

\[
\mathbb{P}_{(\delta, k_0+1)} \left( \exists k \in [k_0 + 1, k_1] : X^1_k > a_k \right) \leq \mathbb{P}_{(\delta, k_0+1)} \left( \exists k \in [k_0 + 1, k_1] : X^2_k > a_k \right) \tag{4.31}
\]

This bound will be used for \( x_i \leq x_{i_0} - R/\varepsilon \). Second, denoting by \( \bar{X}^3 \) the position of the minimal particle in the process \( X^3 \) at generation \( k \), we have

\[
\mathbb{P}_{(\delta, k_0+1)} \left( \exists k \in [k_0 + 1, k_1] : X^1_k > a_k \right) \leq \mathbb{P}_{(\delta, k_0+1)} \left( \exists k \in [k_0 + 1, k_1] : X^3_k > a_k \right) + \mathbb{P}_{(\delta, k_0+1)} \left( \exists k \in [k_0 + 1, k_1] : X^3_k > x_{i_0} + 2R/\varepsilon \text{ or } X^3_k < x_{i_0} - 2R/\varepsilon \right)
\]

\[
=: T_1 + T_2. \tag{4.32}
\]

This bound will be used for \( i \) such that \( x_i \in (x_{i_0} - R/\varepsilon, x_{i_0} + 1/\sqrt{\varepsilon}) \).

**Step 2a: particles close to the maximum.** Let \( i \in \mathbb{Z} \) such that \( x_i \in (x_{i_0} - R/\varepsilon, x_{i_0} + 1/\sqrt{\varepsilon}) \). We bound the RHS of (4.32). Assume \( \varepsilon \) is small enough so that \( 1/\sqrt{\varepsilon} \leq R/\varepsilon \). Using first the assumption on \( x_i \) and then the symmetry and translational invariance of \( X^3 \), we have

\[
T_2 \leq \mathbb{P}_{(\delta, k_0+1)} \left( \exists k \in [k_0 + 1, k_1] : X^3_k > x_i + R/\varepsilon \text{ or } X^3_k < x_i - R/\varepsilon \right)
\]

\[
\leq 2\mathbb{P}_{(\delta, 0)} \left( \exists k \in [0, k_1 - (k_0 + 1)] : X^3_k > R/\varepsilon \right). \tag{4.33}
\]

We can now apply Lemma 4.3 with \( A = \frac{R}{\varepsilon} \) and \( \eta = \frac{1}{4} \). Indeed, since \( \Delta x < \frac{1}{5} \sqrt{2}\sqrt{r} \Delta t \), Lemma 4.2 applied to \( \varepsilon \) (see Equation (4.28)) gives that

\[
\varepsilon \leq \sqrt{2\Delta t \log(1 + r \Delta t)} + a \Delta x \leq \sqrt{2\Delta t} + \frac{1}{5} \sqrt{2r} \Delta t \leq \frac{6}{5} \sqrt{2r} \Delta t \leq \frac{3}{5} R
\]

so that, for \( k \leq k_1 \)

\[
(1 + \eta)(k - k_0 - 1)\varepsilon + A \leq \frac{5}{4\varepsilon} \varepsilon + A \leq \frac{3R}{4\varepsilon} + \frac{R}{\varepsilon} = \frac{R}{\varepsilon}.
\]

Lemma 4.3 now gives that

\[
T_2 \leq 2h(1/4)e^{-\frac{3R}{4\varepsilon}}, \tag{4.33}
\]

with \( h \) as in the statement of Lemma 4.3.

We now bound the term \( T_1 \) on the RHS of (4.32). Let \( i \in \mathbb{Z} \) such that \( x_i \in (x_{i_0} - R/\varepsilon, x_{i_0} + 1/\sqrt{\varepsilon}) \). We then have by the definitions of \( (a_k) \) and \( \tilde{r} \)

\[
T_1 \leq \mathbb{P}_{(\delta, k_0+1)} \left( \exists k \in [k_0 + 1, k_1] : X^3_k > x_{i_0} + \sqrt{2r} \Delta t(k - k_0) + \frac{A}{\varepsilon}(\Delta x + \Delta t^2) \right). \tag{4.34}
\]

Now, according to Lemma 4.3 we have, for some \( C > 0 \) not depending on \( \varepsilon \),

\[
\mathbb{P} \left( \exists k \in [k_0 + 1, k_1] : X^3_k > x_i + (1 + \Delta t)(k - k_0 - 1)\varepsilon + \frac{\Delta x^2}{\varepsilon} \right) \leq Ce^{-\frac{3R}{4\varepsilon} \Delta t^2}. \tag{4.35}
\]

Besides, recall from Lemma 4.2 that

\[
\varepsilon \leq \sqrt{2r} \Delta t + a \Delta x, \tag{4.36}
\]
with some \( a \leq 16\gamma^4 \left( \frac{1}{A_0} \right)^{1/4} \). Therefore, combining (4.35) and (4.36) and using that \( t_i \leq t_i + \frac{1}{r_0} \) and \( \Delta t \leq \bar{r}^{-1} \) and \( k - k_0 \leq \frac{1}{r} \) for \( k \leq k_1 \), we have

\[
P \left( \exists k \in [k_0 + 1, k_1] : X^*_k > x_{i_0} + \sqrt{2}\Delta t (k - k_0) + \frac{\sqrt{\varepsilon + \sqrt{2r}\varepsilon + (1 + \sqrt{2r})\Delta t^2 + a(1 + \bar{r}^{-1})\Delta x}}{\varepsilon} \right) \leq Ce^{-\frac{R\Delta t^2}{32}}. \tag{4.37}
\]

Combining (4.34), (4.38) and (4.29), it follows that for \( \bar{r} \) small enough, we get for \( \varepsilon \) small enough,

\[
\text{Combining (4.32), (4.33) and (4.39), we now get, using again that } 1/\sqrt{\varepsilon} \leq R/\varepsilon,
\]

\[
\sum_{i \in Z: x_i \in (x_{i_0} - R/\varepsilon, x_{i_0} + R/\varepsilon)} P_{(\delta_i, k_0 + 1)} \left( \exists k \in [k_0 + 1, k_1] : X^*_k > a_k \right) \leq \frac{2R}{\Delta x} \left( 2h(1/4)e^{-\frac{\sqrt{2r}\varepsilon}{32}} + Ce^{-\frac{\sqrt{2r}\varepsilon}{32} \frac{\Delta t^2}{2}} \right) \leq e^{-\alpha_1/\varepsilon}, \tag{4.40}
\]

for some \( \alpha_1 > 0 \) and for \( \varepsilon \) sufficiently small, and with \( A \) as above.

**Step 2b: particles far away from the maximum.** Let \( i \in Z \) such that \( x_i \leq x_{i_0} - R/\varepsilon \). We bound the RHS of (4.31). We have for every \( A \geq 0 \), using that \( a_k \geq x_{i_0} \) for every \( k \in [k_0 + 1, k_1] \),

\[
P_{(\delta_i, k_0 + 1)} \left( \exists k \in [k_0 + 1, k_1] : X^*_k > a_k \right) \leq P_{(\delta_i, k_0 + 1)} \left( \exists k \in [k_0 + 1, k_1] : X^*_k > x_{i_0} \right)
\]

\[
\leq P_{(\delta_i, 0)} \left( \exists k \leq [\varepsilon^{-1}] : X^*_k > x_{i_0} - x_i \right).
\tag{4.41}
\]

Denote by \( \bar{c} \) the unique positive solution of \( I(\bar{c}) = \log(1 + \bar{r}\Delta t) \). Following the same calculations as in Step 2a, we have \( \bar{c} \leq \frac{3}{2}R \). We then use Lemma 4.3 with \( \eta = 1/4 \) and \( A = x_{i_0} - x_i - \frac{3R}{4\varepsilon} \) to see that

\[
P_{(\delta_i, 0)} \left( \exists k \leq [\varepsilon^{-1}] : X^*_k > x_{i_0} - x_i \right) \leq h(1/4) \exp \left( -\frac{\sqrt{2\gamma r}}{8} \left( x_{i_0} - x_i - \frac{3R}{4\varepsilon} \right) \right). \tag{4.42}
\]

Combining (4.31) and (4.43), we now get

\[
\sum_{i \in Z: x_i \leq x_{i_0} - R/\varepsilon} P_{(\delta_i, k_0 + 1)} \left( \exists k \in [k_0 + 1, k_1] : X^*_k > a_k \right) \leq \sum_{j \geq 0} h(1/4) \exp \left( -\frac{\sqrt{2\gamma r}}{8} \left( R + (\Delta x)j \right) \right) \leq \exp(-\alpha_2/\varepsilon), \tag{4.44}
\]

for some \( \alpha_2 > 0 \) and for \( \varepsilon \) sufficiently small.

Combining (4.29), (4.40) and (4.44), and using the fact that \( X^1 \) dominates \( X \), we obtain (4.20) for some \( \alpha > 0 \) and for \( \varepsilon \) sufficiently small, with \( A \) as above, depending only on \( \bar{r}, \varepsilon \) and \( M \) (see (1.3)). This concludes the proof of the proposition.
4.3 Comparison with the solution of (C): proof of Proposition 2.1

Step 1. Let us first assume that the function \( r \) is as a smooth growth rate function. Let \( T > 0 \), \( N = \lfloor (T + 1)/\Delta t \rfloor \) and consider the sequence \((s_i)_{i=0}^{N}\) defined as

\[
s_i = i\varepsilon^{-1}\Delta t, \quad \forall i \in [0, N].
\]

We also denote by \( A_{\varepsilon} = \frac{A}{\varepsilon^{-1}} \), where \( A > 0 \) is the constant from Proposition 4.1 and consider the sequence \((\tilde{y}_j)_{i=1}^{N}\) such that

\[
\begin{cases}
\tilde{y}_0 = 0 \\
\tilde{y}_{j+1} = \tilde{y}_j + \left( \sqrt{2r(s_j, \tilde{y}_j)} + A_{\varepsilon} \left( \frac{\Delta x}{\Delta t} + \Delta t \right) \right) |\varepsilon^{-1}|\varepsilon \Delta t, \quad \forall j \in [1, N - 1].
\end{cases}
\]

(4.45)

For \( j \in \mathbb{N} \), we define \( \varphi(j) = j|\varepsilon^{-1}| \) and consider the following function:

\[
f(t) = \tilde{y}_j + (\tilde{y}_{j+1} - \tilde{y}_j) \frac{t/(\varepsilon \Delta t) - \varphi(j)}{\varphi(j+1) - \varphi(j)}, \quad \text{if } t \in [\varepsilon \varphi(j) \Delta t, \varepsilon \varphi(j+1) \Delta t].
\]

(4.46)

First, recall that \( X_0^* = 0 \) and note that the following three events coincide:

\[
B_0 := \left\{ \exists k \in \lfloor \varphi(0), \varphi(1) \rfloor : \varepsilon X_k^* > \tilde{y}_0 + (\tilde{y}_1 - \tilde{y}_0) \frac{k - \varphi(0)}{\varphi(1) - \varphi(0)} \right\}
\]

\[
= \left\{ \exists k \in \lfloor \varphi(0), \varphi(1) \rfloor : \varepsilon X_k^* > f(k \varepsilon \Delta t) \right\}
\]

\[
= \left\{ \exists k \in \lfloor \varphi(0), \varphi(1) \rfloor : \varepsilon X_k^* > \varepsilon X_0^* + \left( \sqrt{2r(0, \varepsilon X_0^*)} + A_{\varepsilon} \left( \frac{\Delta x}{\Delta t} + \Delta t \right) \right) \varepsilon \Delta t (k - \varphi(0)) \right\}.
\]

Then, for all \( j \in [0, N - 1] \), we define

\[
B_j = \left\{ \exists k \in \lfloor \varphi(j), \varphi(j+1) \rfloor : \varepsilon X_k^* > f(k \varepsilon \Delta t) \right\}.
\]

According to Proposition 4.1, there exists \( \alpha \) and \( \varepsilon_0 \), that does not depend on \( \tilde{y}_j \) nor on \( s_j \), such that, if \( \varepsilon < \varepsilon_0 \), \( K > 0 \),

\[
\mathbb{P}(B_j \cap B_0^* \cap \cdots \cap B_{j-1}^*) \leq \mathbb{P}(B_j \cap \{ X_{\varphi(j)}^* \leq \tilde{y}_j \}) \leq Ke^{-\frac{\alpha}{2}}, \quad \forall j \in [0, N - 1].
\]

Hence, we have

\[
\mathbb{P} \left( \bigcup_{j=0}^{N-1} B_j \right) = \sum_{j=0}^{N-1} \mathbb{P}(B_j \cap B_0^* \cap \cdots \cap B_{j-1}^*) \leq NKe^{-\frac{\alpha}{2}}.
\]

(4.47)

Then, let us consider the solution \( \hat{x} \) of

\[
\begin{cases}
\hat{x}(t) = \sqrt{2r(t, \hat{x}(t))} + A_{\varepsilon} \left( \frac{\Delta x}{\Delta t} + \Delta t \right) \\
\hat{x}(0) = 0.
\end{cases}
\]

We know from standard results on the Euler method (see Equation (B.1), Appendix B) that

\[
\max_{j \in [0, N - 1]} |\hat{x}(s_j) - \tilde{y}_j| \leq \frac{1}{2} \varepsilon L(T+1) \varepsilon^{-1} |\varepsilon^{-1}| \Delta t \leq \frac{1}{2} \varepsilon L(T+1) \Delta t,
\]

where \( L \) is as in (1.3). Thus, using this equation and the mean value theorem, we get that, for all \( j \in [0, N - 1] \) and \( t \in [s_j, s_{j+1}] \), we have

\[
|\hat{x}(t) - f(t)| \leq |\hat{x}(t) - \hat{x}(s_j)| + |\hat{x}(s_j) - \tilde{y}_j| + |f(t) - \tilde{y}_j| \leq 2 \left( \sqrt{2r} + A_{\varepsilon} \left( \frac{\Delta x}{\Delta t} + \Delta t \right) \right) \Delta t + \frac{1}{2} \varepsilon L(T+1) \Delta t.
\]

22
Let us now compare \( \hat{x} \) with the solution \( x \) of \((\mathcal{C})\) on \([0, T]\). According to Lemma B.1, we have

\[
\max_{t \in [0,T]} |x(t) - \hat{x}(t)| \leq A_\varepsilon \left( \frac{\Delta x}{\Delta t} + \Delta t \right) (T + 1)e^{LT}.
\]

Thus, there exists a constant \( B > 0 \) that only depends on \( \varepsilon, \tilde{r}, M \) (see (1.3)) and \( T \) such that, for all \( \varepsilon < 1 \)

\[
\sup_{t \in [0,T]} |x(t) - f(t)| \leq B \left( \Delta t + \frac{\Delta x}{\Delta t} \right).
\]  

(4.48)

Finally, remarking that for \( \Delta t < \frac{1}{2} \) and \( \varepsilon < \left( \varepsilon_0 \wedge \frac{2}{T+1} \right) \)

\[
\varphi(N) = \left\lfloor \frac{T + 1}{\Delta t} \right\rfloor \left( \varepsilon^{-1} - 1 \right) > \frac{T + \varepsilon(T + 1)}{\varepsilon \Delta t} \geq \frac{T}{\varepsilon \Delta t} \geq \left\lfloor \frac{T}{\Delta t} \right\rfloor,
\]

and combining Equations (4.47) and (4.48), we get that for all \( K \geq 1 \) and \( \varepsilon < \left( \varepsilon_0 \wedge \frac{T+1}{2} \right) \)

\[
P \left( \exists k \in [0, T/(\varepsilon \Delta t)] : \varepsilon X^*_k > x(k \varepsilon \Delta t) + B \left( \Delta t + \frac{\Delta x}{\Delta t} \right) \right) \leq P \left( \exists k \in [0, \varphi(N)] : \varepsilon X^*_k > x(k \varepsilon \Delta t) + B \left( \Delta t + \frac{\Delta x}{\Delta t} \right) \right) \leq \frac{T + 1}{\Delta t} K e^{-\alpha}.
\]

This proves Proposition 2.1 when \( r \) is a smooth growth rate functions, possibly with a different value of \( \alpha \).

**Step 2.** Suppose that \( r \) is a good growth rate function.

Let \( \delta > 0 \). Pick \( n \) large enough so that

\[
\sup_{t \in [0,T]} |x(t) - \bar{x}_n(t)| < \frac{\delta}{2},
\]  

(4.49)

where \( \bar{x}_n \) refers to the solution of the Cauchy problem \((\mathcal{C})\) with smooth growth rate function \( \Theta_n \) (see 1.5).

Note that \( \Theta_n \) satisfies the assumptions of Proposition 4.1. Moreover, it follows from Assumption 1 and Lemma 3.1 that the interacting particle system with reproduction laws \((\bar{\nu}_{\Theta_n,K})\) stochastically dominates the system with reproduction laws \((\nu_{r,K})\). Recalling Remark 9 that Proposition 4.1 was actually proved for the interacting particle system with reproduction laws \((\bar{\nu}_{r,K})\), Step 1 shows that there exists \( \varepsilon_0 \equiv \varepsilon_0(n, \Delta t, \Delta x), \alpha \equiv \alpha(n, \Delta t, \Delta x) \) and \( B \equiv B(n) \) such that

\[
\forall \varepsilon < \varepsilon_0, \quad \forall K \geq 1, \quad P \left( \exists k \in [0, T/(\varepsilon \Delta t)] : \varepsilon X^*_k > x_n(k \varepsilon \Delta t) + B \left( \Delta t + \frac{\Delta x}{\Delta t} \right) \right) \leq K e^{-\frac{\alpha}{2}}.
\]

We then fix \( \Delta t \) and \( \Delta x \) such that

\[
B \left( \Delta t + \frac{\Delta x}{\Delta t} \right) \leq \frac{\delta}{2},
\]

and combine this bound with (4.49) to get the result.
5 Proof of Proposition 2.2: Lower bound on the propagation speed

In this part, we establish a lower bound on the propagation speed of the process $X$ under Assumption 2.2. We will use the same series of reductions as in the proof of Proposition 2.1: (i) we will derive the lower bound on the invasion speed for the interacting particle system with reproduction laws $\tilde{\nu}_{r,n,K}$ (instead of $\nu_{r,n,K}$). The desired lower bound will then follow from Remark 7. (ii) we will assume that the good growth rate function $r$ satisfies additional regularity assumptions and then conclude using an approximation argument.

The idea of the proof of Proposition 2.2 is to construct a minimising process $X^0$ in which the effect of local competition is negligible (Section 5.1) so that it can be compared to a BRW (Section 5.2), and then to the solution of the ODE (5.3) (Section 5.3). In contrast to Section 4, we can no longer compare the process $X$ with several BRWs over $[\epsilon^{-1}]$ generations. That is why, we will consider smaller time intervals, of order $\log(K) \ll \lfloor \epsilon^{-1} \rfloor$, during which the population size does not grow too much. Note that the length of the time steps considered in Section 4.2 was only constrained by the scale of heterogeneity of the function $r$ and not by the carrying capacity of the environment.

In Section 5, we denote by $\bar{c}$ the unique positive solution of

$$I(\bar{c}) = \log(1 + \bar{r}\Delta t). \quad (5.1)$$

5.1 The rebooted process $X^0$

As explained above, this subsection is aimed at constructing a minimising process $X^0$ in which we can ignore the effect of local competition. By minimising process we mean a process that can be coupled with $X$ in such a way that it is dominated by $X$ in the sense of Section 3. We recall that $X^0_k$ denotes the position of the rightmost particle in the process $X^0$ at generation $k$.

The idea of the following construction is to "reboot" the process $X$ before its population size gets too large. Let $(\varphi(k))_{k \in \mathbb{N}}$ be a sequence of rebooting times i.e. an increasing sequence of integers. The process $X^0$ starts with a single particle at $X^0_0$ and has the same reproduction and migration laws as $X$. At generation $\varphi(1)$, all the particles in $X^0$ are killed, except one, located at $X^0_{\varphi(1)}$. The process $X^0$ then evolves as $X$ until the following rebooting time $\varphi(2)$. Similarly, $X^0$ is rebooted at each generation $\varphi(k)$ and is distributed as $X$ between generations $\varphi(k)$ and $\varphi(k + 1)$.

The goal of the following lemma is to show that for $K$ large enough, the population size of $X^0$ does not exceed $K$ with high probability for

$$\varphi(k) = k[\log(K)]. \quad (5.2)$$

Lemma 5.1. Let $\Delta t < \bar{r}^{-1}$ and $\Delta x > 0$. Let $K > 0$. Consider a branching random walk of reproduction law $\nu_r$ and displacement law $\mu$, starting with a single particle at 0. Let $\tau_0$ be the first generation during which the population size of the process exceeds $K$. Then,

$$P(\tau_0 \leq [\log(K)]) \leq K^{\log(1 + \bar{r}\Delta t) - 1}.$$ 

Proof. Let $N_k$ be the number of individuals alive during generation $k$ in the BRW. Recall that $(N_k)$ is a Galton-Watson process of reproduction law $\nu_r$. According to Assumption 2, the expectation of the reproduction law is equal to $1 + \bar{r}\Delta t$. Thanks to basic results on Galton-Watson processes, we know...
that \(((1 + \bar{r} \Delta t)^{-k} N_k)_{k \geq 0}\) is a positive martingale of mean one. Thus, Doob’s inequality implies that

\[
\mathbb{P}(\tau_0 \leq \lfloor \log(K) \rfloor) = \mathbb{P}\left(\max_{t \leq \lfloor \log(K) \rfloor} N_t \geq K\right)
\leq \mathbb{P}\left(\max_{t \leq \lfloor \log(K) \rfloor} \frac{N_t}{(1 + \bar{r} \Delta t)^t} \geq \frac{K}{(1 + \bar{r} \Delta t)^{\log(K)}}\right)
\leq \frac{(1 + \bar{r} \Delta t)^{\log K}}{K} = e^{(\log(1 + \bar{r} \Delta t) - 1) \log(K)}.
\]

\(\square\)

Note that there exists a coupling between \(X^0\) and a BRW of reproduction law \(\nu_r\), displacement law \(\mu\), starting with a single particle at \(X^0_{\phi(k)}\) on each time interval \([t_{\phi(k)}, t_{\phi(k+1)}]\). Thus, if we consider the sequence of rebooting generations \((\phi(k))_{k \in \mathbb{N}}\) given by Equation \((5.2)\), the probability that the population size of \(X^0\) exceeds \(K\) between generations \(\phi(k)\) and \(\phi(k+1)\) is bounded by \(K^{\log(1 + \bar{r} \Delta t) - 1}\), which tends to 0 as \(K\) tends to infinity as long as \(\Delta t < \bar{r}^{-1}\).

### 5.2 Comparison with a branching random walk

In this section, we bound the first (Lemma \((5.5)\) and \((5.3)\)) and the second moment (Lemma \((5.4)\)) of the increments of the process \((X^0_k)_{k \in \mathbb{N}}\) between generations \(\phi(k)\) and \(\phi(k+1)\), for \(\phi\) defined by \((5.2)\). Let \((\mathcal{F}_k)_{k \in \mathbb{N}}\) be its natural filtration.

In Lemma \((5.2)\) we state a result on some stopping times, that will be needed to construct a coupling between \(X^0\) and a BRW between generations \(\phi(k)\) and \(\phi(k+1)\). In what follows, we denote by \(h\) the function defined by

\[
\tilde{h}(x) = \frac{e^{-\frac{x}{4\sqrt{2}}} - e^{-\frac{3x}{4\sqrt{2}}}}{1 - e^{-\frac{x}{4\sqrt{2}}}}, \quad \forall x > 0.
\]  

(5.3)

Note that \(\tilde{h}(x) \to 0\) as \(x \to +\infty\).

**Lemma 5.2.** Let \(\Delta t < \bar{r}^{-1}\) and \(\Delta x < \frac{4}{\bar{r}} \sqrt{2\pi} \Delta t\). Let \(K > 0\) and \(\varepsilon \leq (4\sqrt{2} \bar{r} \varphi(1) \Delta t)^{-4}\). Let \(r \in [\bar{r}, \tilde{r}]\). Let \(\Xi\) be a branching random walk of reproduction law \(\nu_r\), displacement law \(\mu\), starting with a single particle at 0. Denote by \(N_k\) the size of this process at generation \(k\) and consider

\[
\tau_K = \inf \{k \in \mathbb{N} : N_k \geq K\} \quad \text{and} \quad \tau_\varepsilon = \inf \left\{k \in \mathbb{N} : \exists v : |v| = k, |\Xi_v| > \varepsilon^{-1/4}\right\}.
\]

Then,

\[
\mathbb{P}(\tau_K \leq \varphi(1)) \leq K^{\log(1 + \bar{r} \Delta t)^{-1}},
\]

and

\[
\mathbb{P}(\tau_\varepsilon \leq \varphi(1)) \leq 2\tilde{h}\left(\frac{\varepsilon^{-1/4}}{\varphi(1)}\right).
\]

**Proof.** The branching random walk \(\Xi\) can be coupled with a BRW of reproduction law \(\nu_r\), displacement law \(\mu\), starting with a single particle at 0. Thus, the estimate on \(\tau_K\) directly ensues from Lemma \((5.1)\) and it is sufficient to establish the result on \(\tau_\varepsilon\) for \(r = \bar{r}\).

Thanks to a similar argument to that of Equation \((4.18)\), one can prove that, for any particle \(v\) such that \(|v| = n\)

\[
\mathbb{P}(\Xi_v > \varepsilon^{-1/4}) \leq e^{-n\tilde{h}\left(\frac{\varepsilon^{-1/4}}{n}\right)}.
\]
Thus, by the many-to-one lemma (see Lemma [A.1]) and by symmetry of \( \mu \), we get that, for all \( n \leq \varphi(1) \),

\[
\mathbb{P}(\exists v : |v| = n, |\Xi_n| > \varepsilon^{-1/4}) \leq 2(1 + \bar{r}\Delta t)^n e^{-n(\varepsilon^{-1/4})} \leq 2e^{-n(\varepsilon^{-1/4}) - \log(1 + \bar{r}\Delta t)} \leq 2e^{-n(\varepsilon^{-1/4}) - \log(1 + \bar{r}\Delta t)} = 2e^{-n(\varepsilon^{-1/4}) - I(\bar{c})},
\]

where \( \bar{c} \) is as in [A.1]. Besides, \( I \) is convex, therefore \( I \left( \frac{\varepsilon^{-1/4}}{\varphi(1)} \right) \geq \frac{I(\bar{c})\varepsilon^{-1/4}}{\varphi(1)} \) as long as \( \varepsilon^{-1/4} \geq \bar{c}\varphi(1) \). Thus, if \( \varepsilon^{-1/4} \geq 2\bar{c}\varphi(1) \),

\[
I \left( \frac{\varepsilon^{-1/4}}{\varphi(1)} \right) - I(\bar{c}) \geq I(c) \left( \frac{\varepsilon^{-1/4}}{\varphi(1)} - 1 \right) \geq \frac{I(c)}{2\bar{c}\varphi(1)} \varepsilon^{-1/4} \geq \frac{\gamma \sqrt{\bar{r}} \varepsilon^{-1/4}}{4\sqrt{2} \varphi(1)} ,
\]

since \( I(\bar{c}) = \log(1 + \bar{r}\Delta t) \geq \gamma \bar{r}\Delta t \) and \( \bar{c} \leq 2\sqrt{2}\bar{r}\Delta t \) (see Lemma [A.1]). Hence, a union bound yields the inequality

\[
\mathbb{P}(\tau_c \leq \varphi(1)) = \mathbb{P} \left( \exists n \leq \varphi(1) : \exists v : |v| = n, |\Xi_n| > \varepsilon^{-1/4} \right) 
\leq 2\sum_{n=1}^{\varphi(1)} e^{-n(\varepsilon^{-1/4}) - \log(1 + \bar{r}\Delta t)} 
\leq 2\bar{h} \left( \frac{\varepsilon^{-1/4}}{\varphi(1)} \right) .
\]

\[\square\]

**Lemma 5.3** (Lower bound on the first moment). Assume that \( r \) is a smooth growth rate function. Let \( \Delta t < \bar{r}^{-1} \), \( \Delta x < \frac{1}{16\sqrt{2}\bar{r}\Delta t} \) and \( \eta > 0 \). There exists \( K_0 > 0 \) such that, for all \( K > K_0 \), there exists \( \varepsilon_0 > 0 \) such that, for all \( \varepsilon < \varepsilon_0 \),

\[
\mathbb{E} \left[ X_{\varphi(k+1)}^0 - X_{\varphi(k)}^0 | \mathcal{F}_{\varphi(k)} \right] \geq (c_{\varphi(k)} - \eta) \varphi(1),
\]

with \( c_{\varphi(k)} \) the unique positive solution of \( I(c_{\varphi(k)}) = \log(1 + r_{\varphi(k)}\Delta t) \) for

\[
r_{\varphi(k)} = \min \left\{ r(\varepsilon t, \varepsilon x), (t, x) \in [\varphi(k)\Delta t, \varphi(k+1)\Delta t] \times [\varphi(k)\Delta t, \varphi(k+1)\Delta t] \right\} .
\]

**Proof.** For the sake of simplicity, we assume that \( X^0 \) starts at generation \( \varphi(k) \) with a deterministic configuration \( n_{\varphi(k)}^0 = \delta X_{\varphi(k)}^0 \). The estimates we obtain will not depend on this initial condition. The proof of the lemma relies on a coupling argument.

Consider a branching random walk \( \Xi \) of reproduction law \( \nu_{r_{\varphi(k)}} \), displacement law \( \mu \), starting with a single particle at \( X_{\varphi(k)}^0 \) at time \( \varphi(k)\Delta t \). Denote by \( N_k \) the size of the BRW at generation \( k \) and define

\[
\tau_k^\ast = \inf \left\{ l \geq \varphi(k) : N_l \geq K \right\} \quad \text{and} \quad \tau_c^\ast = \inf \left\{ l \geq \varphi(k) : \exists v : |v| = l, |\Xi_v - X_{\varphi(k)}^0| > \varepsilon^{-1/4} \right\} .
\]

In addition, for \( n \geq \varphi(k) \), define

\[
M_n = \max \{ \Xi_v, |v| = n \} .
\]

26
According to Lemma 5.1 one can couple $X^0$ and $\Xi$ such that $X^0$ dominates $\Xi$ until generation $(\tau'_K \land \tau'_e)$. Besides, note that $\tau'_K - \varphi(k)$ (resp. $\tau'_e - \varphi(k)$) follows the same law as $\tau_K$ (resp. $\tau_e$) from Lemma 5.2 for $r = r_{\varphi(k)}$. Thus,

$$
\mathbb{E} \left[ X^0_{\varphi(k)} - X^{0*}_{\varphi(k)} \right] = \mathbb{E} \left[ (X^0_{\varphi(k)} - X^{0*}_{\varphi(k)}) \mathbb{1}_{\tau'_k \land \tau'_e > \varphi(k+1)} \right] + \mathbb{E} \left[ (X^0_{\varphi(k)} - X^{0*}_{\varphi(k)}) \mathbb{1}_{\tau'_k \land \tau'_e \leq \varphi(k+1)} \right] + \mathbb{E} \left[ (X^0_{\varphi(k)} - X^{0*}_{\varphi(k)}) \mathbb{1}_{\tau'_k \land \tau'_e > \varphi(k+1)} \right] + \mathbb{E} \left[ (X^0_{\varphi(k)} - X^{0*}_{\varphi(k)}) \mathbb{1}_{\tau'_k \land \tau'_e \leq \varphi(k+1)} \right]. \tag{5.7}
$$

We first bound the first term on the RHS of (5.7). We have

$$
\mathbb{E} \left[ (M_{\varphi(k)} - X^{0*}_{\varphi(k)}) \mathbb{1}_{\tau'_k \land \tau'_e > \varphi(k+1)} \right] = \mathbb{E} \left[ M_{\varphi(k)} - X^{0*}_{\varphi(k)} \right] - \mathbb{E} \left[ (M_{\varphi(k)} - X^{0*}_{\varphi(k)}) \mathbb{1}_{\tau'_k \land \tau'_e \leq \varphi(k+1)} \right], \tag{5.8}
$$

and by the Cauchy-Schwarz inequality,

$$
\mathbb{E} \left[ (M_{\varphi(k)} - X^{0*}_{\varphi(k)}) \mathbb{1}_{\tau'_k \land \tau'_e \leq \varphi(k+1)} \right] \leq \sqrt{\mathbb{E} \left[ (M_{\varphi(k)} - X^{0*}_{\varphi(k)})^2 \right]} \sqrt{\mathbb{P}(\tau_K \land \tau_e \leq \varphi(1))}. \tag{5.9}
$$

According to Lemma A.7 and Lemma A.8 there exists $K_0 > 0$, that does not depend on $r_{\varphi(k)}$ nor on $X^{0*}_{\varphi(k)}$ such that, if $K > K_0$,

$$
\mathbb{E} \left[ M_{\varphi(k)} - X^{0*}_{\varphi(k)} \right] \geq (c_{\varphi(k)} - \eta)\varphi(1), \tag{5.10}
$$

and

$$
\mathbb{E} \left[ (M_{\varphi(k)} - X^{0*}_{\varphi(k)})^2 \right] \leq 4c^2\varphi(1)^2. \tag{5.11}
$$

Hence, combining (5.8), (5.9), (5.10) and (5.11), we get that

$$
\mathbb{E} \left[ (M_{\varphi(k)} - X^{0*}_{\varphi(k)}) \mathbb{1}_{\tau'_k \land \tau'_e > \varphi(k+1)} \right] \geq (c_{\varphi(k)} - \eta)\varphi(1) - 2c\varphi(1) \sqrt{\mathbb{P}(\tau_K \land \tau_e \leq \varphi(1))}. \tag{5.12}
$$

As for the second term on the RHS of (5.7), we have again by the Cauchy-Schwarz inequality,

$$
\mathbb{E} \left[ (X^0_{\varphi(k)} - X^{0*}_{\varphi(k)}) \mathbb{1}_{\tau'_k \land \tau'_e \leq \varphi(k+1)} \right] \geq \mathbb{E} \left[ (X^0_{\varphi(k)} - X^{0*}_{\varphi(k)}) \mathbb{1}_{\tau'_k \land \tau'_e \leq \varphi(k+1)} \right] \geq -\sqrt{\mathbb{E} \left[ (X^0_{\varphi(k)} - X^{0*}_{\varphi(k)})^2 \right]} \sqrt{\mathbb{P}(\tau_K \land \tau_e \leq \varphi(1))}. \tag{5.13}
$$

In order to bound the first term on the RHS of (5.13), we note that by Assumption 2 there exists a particle $v$ at generation $\varphi(k+1)$, such that $X^0 - X^{0*}_{\varphi(k)}$ is equal in law to $S_{\varphi(1)}$, where $(S_n)_{n \geq 0}$ is a random walk with displacement distribution $\mu$ (heuristically, $v$ is obtained by choosing at each time step one of the children at random and iterating). It follows that

$$
\mathbb{E} \left[ (X^0_{\varphi(k)} - X^{0*}_{\varphi(k)})^2 \right] \leq \mathbb{E} \left[ (X^0_{\varphi(k)} - X^{0*}_{\varphi(k)})^2 \right] \leq \mathbb{E}[S^2_{\varphi(1)}] = \text{Var}(\mu)\varphi(1). \tag{5.14}
$$

Finally, remark that $c_{\varphi(k)} \leq \tilde{c}$ since $I$ is increasing on $(0, \infty)$ (see Equation (5.1)) and that, according to Lemma 5.2 there exists $K_1 > 0$ such that, for all $K > K_1$, there exists $\varepsilon_1 > 0$ such that for all $\varepsilon < \varepsilon_1$,

$$
\mathbb{P}(\tau_K \land \tau_e \leq \varphi(1)) \leq \frac{\eta^2}{(2\varepsilon + \text{Var}(\mu))^2}. \tag{5.15}
$$

27
Combining (5.7), (5.12), (5.13), (5.14) and (5.15), we get, for $K > \max(K_0, K_1)$ and $\varepsilon < \varepsilon_1$,

$$\mathbb{E} \left[ X_{\varphi(k+1)}^{0*} - X_{\varphi(k)}^{0*} \right] \geq (c_\varphi(k) - 2\eta)\varphi(1).$$

This proves the lemma. \(\square\)

Remark 10. Lemma 5.3 is where the assumption that $K$ is sufficiently large is crucial. However, one could replace it by the sole assumption that $\Delta x$ is sufficiently small. Indeed, in this case, the comparison with a branching random walk still holds until a certain time of order $\log(1/\Delta x)$, since one can show that particles do not meet until that time. For simplicity, we leave out the details.

Lemma 5.4 (Upper bound on the second moment). Assume that $r$ is a smooth growth rate function. Let $\Delta t < \bar{r}^{-1}$, $\Delta x < \sqrt{2 \gamma_2 \Delta t}$. There exists $K_0 > 0$ such that, for all $K > K_0$, for all $\varepsilon > 0$,

$$\mathbb{E} \left[ \left( X_{\varphi(k+1)}^{0*} - X_{\varphi(k)}^{0*} \right)^2 | \mathcal{F}_\varphi(k) \right] \leq 2\varepsilon^2 \varphi(1)^2.$$

Proof. By Lemma 3.1 one can couple $X^0$ and a BRW $\Xi$ of reproduction law $\nu_r$, displacement law $\mu$, starting from a single particle at $X_{\varphi(k)}^{0*}$ at time $\varphi(k)\Delta t$, such that $\Xi$ dominates $X^0$ until generation $\varphi(k+1)$. Then, if we denote by $M_n$ the position of the rightmost particle in $\Xi$ at generation $n$, we get that

$$\mathbb{E} \left[ \left( X_{\varphi(k+1)}^{0*} - X_{\varphi(k)}^{0*} \right)^2 | \mathcal{F}_\varphi(k) \right] \leq \mathbb{E} \left[ (M_{\varphi(k+1)} \lor 0)^2 | \mathcal{F}_\varphi(k) \right] \leq \mathbb{E} \left[ M_{\varphi(k+1)}^2 | \mathcal{F}_\varphi(k) \right] \leq \frac{3}{2} \varepsilon^2 \varphi(1)^2,$$

for $K$ large enough, according to Lemma A.8. As in the proof of Lemma 5.3 consider $(S_n)_{n \geq 0}$ a random walk whose increments are distributed as $\mu$. Then,

$$\mathbb{E} \left[ \left( X_{\varphi(k+1)}^{0*} - X_{\varphi(k)}^{0*} \right) \wedge 0 \right]^2 | \mathcal{F}_\varphi(k) \right] \leq \mathbb{E} \left[ (S_{\varphi(k+1)} \wedge 0)^2 \right] \leq \mathbb{E} \left[ S_{\varphi(k+1)}^2 \right] = \text{Var}(\mu)\varphi(1) \leq \frac{1}{2} \varepsilon^2 \varphi(1)^2,$$

for $K$ large enough. Combining the two previous inequalities yields the lemma. \(\square\)

Lemma 5.5 (Upper bound on the first moment). Assume that $r$ is a smooth growth rate function. Let $\Delta t < \bar{r}^{-1}$, $\Delta x < \sqrt{2 \gamma_2 \Delta t}$ and $\eta > 0$. There exists $K_0 > 0$ such that, for all $K > K_0$, there exists $\varepsilon_0$ such that, for all $\varepsilon < \varepsilon_0$,

$$\mathbb{E} \left[ X_{\varphi(k+1)}^{0*} - X_{\varphi(k)}^{0*} | \mathcal{F}_\varphi(k) \right] \leq (c_\varphi(k) + \eta)\varphi(1)$$

with $c_\varphi(k)$ the unique positive solution of $I(c_\varphi(k)) = \log(1 + r_\varphi(k)\Delta t)$ for

$$r_\varphi(k) = \max \left\{ r(\varepsilon t, \varepsilon x), \ (t, x) \in [\varphi(k)\Delta t, \varphi(k+1)\Delta t] \times [X_{\varphi(k)}^{0*} - \varepsilon^{-1/4}, X_{\varphi(k)}^{0*} + \varepsilon^{-1/4}] \right\}.$$

Proof. The proof is similar of the proof of Lemma 5.3 but some details are different, which is why we give a complete proof. Again, we assume that $X^0$ starts at generation $\varphi(k)$ with a deterministic configuration $n_{\varphi(k)}^0 = \delta_{X_{\varphi(k)}^{0*}}$ and the proof of the lemma relies on a coupling argument.

Consider a branching random walk $\tilde{\Xi}$ of reproduction law $\nu_{r_\varphi(k)}$, displacement law $\mu$, starting with a single particle at $X_{\varphi(k)}^{0*}$ at time $\varphi(k)\Delta t$. Denote by $\tilde{N}_k$ the size of the population in $\tilde{\Xi}$ at generation $k$ and define

$$\tau_\varepsilon'' = \inf \left\{ l \geq \varphi(k) : \exists v : |v| = l, |X_{\varphi(k)}^{0*} - X_{\varphi(k)}^{0*}v| > \varepsilon^{-1/4} \right\}.$$
In addition, for \( n \geq \varphi(k) \), define
\[
\hat{M}_n = \max\{\hat{\Xi}_v, |v| = n\}. \tag{5.16}
\]

According to Lemma 3.1, one can couple \( X^0 \) and \( \hat{\Xi} \) such that \( \hat{\Xi} \) dominates \( X^0 \) until generation \( r_{\varepsilon}'' \). Besides, according to Lemma 3.2, there exists \( K_0 > 0 \), that does not depend on \( r_{\varphi(k)} \) nor on \( X^0_{\varphi(k)} \), such that, for all \( K > K_0 \),
\[
\mathbb{E}\left[ (\hat{M}_{\varphi(k)} - X_{\varphi(k)}^0)^2 \right] \leq (c_{\varphi(k)} + \eta^2)\varphi(1)^2. \tag{5.18}
\]

Besides, according to Lemma 5.4, there exists \( K_1 > 0 \) such that for all \( K > K_1 \),
\[
\mathbb{E}\left[ \left( X_{\varphi(k)}^0 - X_{\varphi(k)}^0 \right)^2 \right] \leq 2\varepsilon^2\varphi(1)^2. \tag{5.19}
\]

Let us now assume that \( K > \max(K_0, K_1) \). According to Lemma 5.2, there exists \( \varepsilon_1 > 0 \) such that for all \( \varepsilon < \varepsilon_1 \),
\[
\mathbb{P} (\tau_{\varepsilon} \leq \varphi(1)) \leq \frac{\eta^2}{2\varepsilon^2}. \tag{5.20}
\]

Finally, combining Equations (5.17), (5.19), (5.19) and (5.20), we get that, for \( \varepsilon < \varepsilon_1 \),
\[
\mathbb{E}\left[ X_{\varphi(k)}^0 - X_{\varphi(k)}^0 \right] \leq (c_{\varphi(k)} + 2\eta)\varphi(1),
\]
which concludes the proof of the lemma. \bbox

### 5.3 Comparison with the solution of (C)

**Lemma 5.6.** Assume that \( r \) is a smooth growth rate function. Let \( \delta \in (0, 1) \).

*If \( a \) is as in Lemma 4.2 and*
\[
\Delta t < \delta r^{-1}, \quad \Delta x < \left( \sqrt{2r} \wedge \frac{\sqrt{2r} \varphi}{3d} \right) \Delta t, \tag{H_\delta}
\]
*then, for all \( K > 0 \), there exists \( \varepsilon_0 \) such that for all \( \varepsilon < \varepsilon_0 \),
\[
c_{\varphi(k)} \geq (1 - \delta) \sqrt{2r(\varepsilon t_{\varphi(k)}) \varepsilon X_{\varphi(k)}^0} \Delta t, \quad \forall k \in \mathbb{N},
\]
*and*
\[
c_{\varphi(k)} \leq (1 + \delta) \sqrt{2r(\varepsilon t_{\varphi(k)}) \varepsilon X_{\varphi(k)}^0} \Delta t, \quad \forall k \in \mathbb{N},
\]
*for \( c_{\varphi(k)} \) and \( c_{\varphi(k)} \) respectively defined in Lemmas 5.3 and 5.5.*
Proof. Let $\Delta t < \bar{r}^{-1}$ and $\Delta x < \sqrt{2\gamma\bar{r}\Delta t}$. Thanks to Lemma 4.2, we know that
\begin{equation}
|c_{\varphi(k)} - \sqrt{2\Delta t \log(1 + r_{\varphi(k)}\Delta t)}| \leq a\Delta x.
\end{equation}
(5.21)
Let $\delta \in (0, 1)$. Note that $\log(1 + x) \geq (1 - \delta/3)^2x$, for $x \leq \delta$. Indeed, $x \mapsto \log(1 + x) - (1 - \delta/3)^2x$ is concave, equal to 0 when $x = 0$, and
\[
\log(1 + \delta) - (1 - \delta/3)^2\delta = \log(1 + \delta) - (\delta - 2\delta^2/3 + \delta^3/9)
\]
\[
= \int_0^\delta \frac{1}{1 + y} - \left(1 - \frac{4}{3}y + \frac{1}{3}y^2\right) dy = \int_0^\delta \frac{1 + 3y - y^2}{3(1 + y)} dy \geq 0,
\]
since $1 + 3y - y^2 \geq 0$ for $y \in [0, 1]$. Let us now assume that $\Delta t < \delta\bar{r}^{-1}$. Equation (5.21) then gives
\begin{equation}
c_{\varphi(k)} \geq \sqrt{2\Delta t \log(1 + r_{\varphi(k)}\Delta t)} - a\Delta x \geq (1 - \delta/3)\sqrt{2r_{\varphi(k)}\Delta t} - a\Delta x.
\end{equation}
(5.22)
Besides, by definition of $r_{\varphi(k)}$ (see Equation (5.5)) and Equation (1.3) we have
\begin{equation}
|\sqrt{2r_{\varphi(k)}} - \sqrt{2r(\varepsilon t_{\varphi(k)}, \varepsilon X_{\varphi(k)}^{0*})}| \leq L(\varepsilon \varphi(1)\Delta t + \varepsilon^{3/4}),
\end{equation}
(5.23)
($L$ is as in Equation (1.4)). Combining (5.22) and (5.23) we get that
\begin{equation}
c_{\varphi(k)} \geq (1 - \delta/3)\sqrt{2r(\varepsilon t_{\varphi(k)}, \varepsilon X_{\varphi(k)}^{0*})}\Delta t - (1 - \delta/3)L(\varepsilon \varphi(1)\Delta t + \varepsilon^{3/4})\Delta t - a\Delta x
\end{equation}
Remark that Assumption $[H_3]$ implies that
\[
a\Delta x \leq \frac{1}{3} \sqrt{2\gamma \delta \Delta t} \leq \delta \sqrt{\frac{2}{3}r(\varepsilon t_{\varphi(k)}, \varepsilon X_{\varphi(k)}^{0*})}\Delta t,
\]
so that it is sufficient to choose $\varepsilon$ such that
\[
L(\varepsilon \varphi(1)\Delta t + \varepsilon^{3/4}) \leq \frac{2}{3}\delta,
\]
to get
\begin{equation}
c_{\varphi(k)} \geq (1 - \delta)\sqrt{2r(\varepsilon t_{\varphi(k)}, \varepsilon X_{\varphi(k)}^{0*})}\Delta t.
\end{equation}
Similarly, Equations (5.21) and (5.23) give that
\begin{equation}
c_{\varphi(k)} \leq \sqrt{2r(\varepsilon t_{\varphi(k)}, \varepsilon X_{\varphi(k)}^{0*})}\Delta t + L(\varepsilon \varphi(1)\Delta t + \varepsilon^{3/4})\Delta t + a\Delta x,
\end{equation}
and Assumption $[H_3]$ implies that $a\Delta x \leq \frac{1}{3} \sqrt{2\gamma \delta \Delta t}$, so that it is sufficient to choose $\varepsilon$ such that $L(\varepsilon \varphi(1)\Delta t + \varepsilon^{3/4})\Delta t \leq \delta$ to get the result. \hfill \Box

**Corollary 5.1.** Assume that $r$ is a smooth growth rate function. Let $\delta > 0$. Suppose $\Delta t$ and $\Delta x$ satisfy $[H_3]$. There exists $K_0 > 0$ such that, for all $K > K_0$, there exists $\varepsilon_0$ such that for all $\varepsilon < \varepsilon_0$,
\[
\left| \mathbb{E} \left[ X_{\varphi(k+1)}^{0*} - X_{\varphi(k)}^{0*} | \mathcal{F}_{\varphi(k)} \right] - \sqrt{2r(\varepsilon t_{\varphi(k)}, \varepsilon X_{\varphi(k)}^{0*})}\varphi(1)\Delta t \right| \leq 2\sqrt{2\gamma \delta \varphi(1)\Delta t}.
\]

**Proof.** We choose $\eta = \delta \sqrt{2\gamma \Delta t}$ in Lemmas 5.3 and 5.5. \hfill \Box
Proof of Proposition 2.2. It suffices to prove the statement for some \( K \), since by Assumption 2 and Lemma 3.1 the statement then holds for all larger \( K \).

Let \( \delta \in (0,1) \) and assume that \( \Delta t \) and \( \Delta x \) satisfy (11). Let \( T > 0 \) and \( N = \left\lfloor \frac{T}{\varepsilon \varphi(1) \Delta t} \right\rfloor \). Recall that \( X_0 = 0 \).

Step 1. Let us first assume that \( r \) is a smooth growth rate function. The Euler scheme associated to (3) on \([0, T]\), with time step \( \varepsilon \varphi(1) \Delta t \), is defined as the sequence \((y_k)_{k=0}^N\) such that \( y_0 = 0 \) and

\[
y_{k+1} = y_k + 2r(\varepsilon t_{\varphi(k)}, y_k)\varepsilon \varphi(1) \Delta t, \quad \forall i \in [0, N - 1].
\]

We also define the process \((y_k)_{k=0}^N\) by \( Y_k = \varepsilon X^{\alpha_k}_{\varphi(k)} \), and consider its Doob decomposition \( Y_k = Z_k + W_k \), where

\[
W_k = \sum_{j=0}^{k-1} \mathbb{E} [Y_{j+1} - Y_j | \mathcal{F}_{\varphi(j)}], \quad Z_k = Y_k - W_k, \quad \forall k \in [0, N].
\]

Since \((Z_k)\) is a martingale, we have

\[
\text{Var}(Z_k) = \sum_{j=0}^{k-1} \mathbb{E} [\text{Var}(Y_{j+1} - Y_j | \mathcal{F}_{\varphi(j)})], \quad \forall k \in [0, N]. \tag{5.24}
\]

Besides, Lemma 5.4 implies that there exists \( K_0 > 1 \) such that, for all \( K \geq K_0 \),

\[
\text{Var}(Y_{j+1} - Y_j | \mathcal{F}_{\varphi(j)}) \leq 2\varepsilon^2 \varphi(1)^2, \quad \forall j \in \mathbb{N}. \tag{5.25}
\]

Thus,

\[
\text{Var}(Z_N) \leq 2\varepsilon^2 \varphi(1)^2 N \leq 2\varepsilon^2 \varphi(1) \epsilon T.
\]

Moreover, let \( \mathcal{A}_\delta \) be the event

\[
\left\{ \max_{k=0, \ldots, N} |Z_k| \leq \delta \right\},
\]

By Doob’s inequality, we have that

\[
\mathbb{P}(\mathcal{A}_\delta) \leq \frac{\text{Var}(Z_N)}{\delta^2} \leq \frac{2\varepsilon^2 T \varphi(1)}{\delta^2} \epsilon. \tag{5.26}
\]

Besides, we know that on the event \( \mathcal{A}_\delta \), \(|Y_k - W_k| = |Z_k| \leq \delta\) for all \( k \in [0, N] \) and therefore, by the triangle inequality,

\[
\text{on } \mathcal{A}_\delta, \quad |Y_{k+1} - y_{k+1}| \leq |Y_{k+1} - W_{k+1}| + |W_{k+1} - y_{k+1}| \leq \delta + |W_{k+1} - y_{k+1}|, \tag{5.27}
\]

for all \( k \in [0, N - 1] \). Moreover, using the definition of \( W_k \) and \( y_k \), we get that

\[
|W_{k+1} - y_{k+1}| \leq |W_k - y_k| + \mathbb{E}[|Y_{k+1} - Y_k| | \mathcal{F}_{\varphi(k)}] - \sqrt{2r(\varepsilon t_{\varphi(k)}, y_k)\varepsilon \varphi(1) \Delta t} \leq |W_k - y_k| + \mathbb{E}[|Y_{k+1} - Y_k| | \mathcal{F}_{\varphi(k)}] - \sqrt{2r(\varepsilon t_{\varphi(k)}, y_k)\varepsilon \varphi(1) \Delta t}
\]

\[
+ \sqrt{2r(\varepsilon t_{\varphi(k)}, y_k)\varepsilon \varphi(1) \Delta t} - \sqrt{2r(\varepsilon t_{\varphi(k)}, y_k)\varepsilon \varphi(1) \Delta t}, \tag{5.28}
\]

According to Corollary 5.1 there exists \( K_1 > 0 \) such that for all \( K \geq K_1 \), there exists \( \varepsilon_1 > 0 \) such that, for all \( \varepsilon < \varepsilon_1 \) and \( k \in [0, N - 1] \),

\[
\left| \mathbb{E}[|Y_{k+1} - Y_k| | \mathcal{F}_{\varphi(k)}] - \sqrt{2r(\varepsilon t_{\varphi(k)}, y_k)\varepsilon \varphi(1) \Delta t} \right| \leq \sqrt{2\varepsilon_1 \delta \varphi(1) \Delta t}. \tag{5.29}
\]
Besides, recall from Equation \((1.34)\) that, for \(k \in [0, N - 1]\), we have
\[
\left| \sqrt{2r(e^{t_{(k)}}, Y_k)}|\varepsilon(1)\Delta t - \sqrt{2r(e^{t_{(k)}}, y_k)}|\varepsilon(1)\Delta t \right| \leq L|y_k - Y_k||\varepsilon(1)\Delta t. \tag{5.30}
\]
Let us now assume that \(K > \max(K_0, K_1)\) and \(\varepsilon < \varepsilon_1(K)\). Combining Equations \((5.29), (5.29)\) and \((5.30)\), we obtain that
\[
|y_{k+1} - W_{k+1}| \leq |y_k - W_k| + \varepsilon\varphi(1)\Delta t \left( \sqrt{2\delta} + L|y_k - Y_k| \right), \tag{5.31}
\]
for all \(k \in [0, N - 1]\). Thus, combining Equations \((5.27)\) and \((5.31)\), we get that for all \(k \in [0, N - 1]\),
\[
on A_\delta, \quad |y_{k+1} - W_{k+1}| \leq |y_k - W_k| + \varepsilon\varphi(1)\Delta t \left( \sqrt{2\delta} + L|y_k - W_k| + \delta \right)
\leq (1 + L\varepsilon\varphi(1)\Delta t)|y_k - W_k| + \varepsilon\delta\varphi(1)\Delta t \left( \sqrt{2\delta} + L \right)
\leq e^{L\varepsilon\varphi(1)\Delta t}|y_k - W_k| + \varepsilon\delta\varphi(1)\Delta t \left( \sqrt{2\delta} + L \right). \tag{5.32}
\]
Then, by induction, since \(Y_0 = y_0 = 0\) and \(W_0 = 0\), for all \(k \in [0, N - 1]\)
\[
on A_\delta, \quad |y_{k+1} - W_{k+1}| \leq \varepsilon\delta\varphi(1)\Delta t \left( \sqrt{2\delta} + L \right) \sum_{j=0}^{k} e^{jL\varepsilon\varphi(1)\Delta t}
\leq N\varepsilon\varphi(1)\Delta t \delta \sqrt{2\delta} + L e^{N\varepsilon\varphi(1)\Delta t}
\leq \left( \sqrt{2\delta} + L \right) \delta T e^{LT}. \tag{5.33}
\]
Combining Equations \((5.27)\) and \((5.32)\) and setting
\[
\alpha = 1 + \left( \sqrt{2\delta} + L \right) T e^{LT}, \tag{5.33}
\]
we get that
\[
Y_k \geq y_k - \alpha \delta, \quad \forall k \in [0, N], \quad \text{on } A_\delta. \tag{5.34}
\]
Moreover, note that
\[
P \left( \exists k \in [0, N - 1] : \exists l \in [\varphi(k), \varphi(k + 1)] : \varepsilon X^{0*}_l < y_k - 2\alpha \delta \right)
\leq P \left( \exists k \in [0, N - 1] : \exists l \in [\varphi(k), \varphi(k + 1)] : \varepsilon X^{0*}_l < y_k - 2\alpha \delta | A_\delta \right) + P(A^c_\delta). \tag{5.35}
\]
To show that the first term in \((5.35)\) is small, we prove that the probability of the event
\[
\left\{ \exists l \in [\varphi(k), \varphi(k + 1)] : X^{0*}_l - X^{0*}_{\varphi(k)} < -\frac{\alpha \delta}{\varepsilon} \right\}
\]
decays exponentially as \(\varepsilon\) goes to zero, for all \(k \in [0, N - 1]\). Let \(l \in [\varphi(k), \varphi(k + 1)]\) and consider a random walk \((S_i)_{i \geq 0}\) of step distribution \(\mu\). As in the proof of Lemma \(5.3\) we have
\[
P \left( X^{0*}_l - X^{0*}_{\varphi(k)} < -\frac{\alpha \delta}{\varepsilon} \right) \leq P \left( S_{\varphi(k)} < -\frac{\alpha \delta}{\varepsilon} \right)
= P \left( S_{\varphi(k)} > \frac{\alpha \delta}{\varepsilon} \right)
\leq e^{-\left( l - \varphi(k) \right) I \left( \frac{\alpha \delta}{\varepsilon} \right)}
\leq e^{-l \left( \frac{\alpha \delta}{\varepsilon} \right)}.
\]
Remark 7 that the particle system with reproduction law \( \nu \) exists \( \varepsilon \).

By definition, \( \bar{\theta} \) with reproduction law \( \nu \).

This proves the theorem when \( \varepsilon \leq \delta \).

In addition, according to Equation (5.34),

\[
P(\exists k \in [0, N - 1] : \exists l \in [\varphi(k), \varphi(k + 1)] : \varepsilon X^0_k < y_k - 2\alpha \delta |A_\delta)
\]

\[
\leq P(\exists k \in [0, N - 1] : \exists l \in [\varphi(k), \varphi(k + 1)] : \varepsilon X^0_k < \varepsilon X^0_{\bar{\varphi}(k)} - \varepsilon) ,
\]

and since the function \( x \) is the solution of (C), Equation (B.1) from Appendix B and the mean value theorem imply that

\[
|y_k - x(Iz)\Delta t) | \leq |y_k - x(k \varepsilon(1) \Delta t) | + |x(k \varepsilon(1) \Delta t) - x(Iz)\Delta t) |\]

\[
\leq e^{LT} \varepsilon \varepsilon(1) \Delta t + \sqrt{2e} e \varepsilon(1) \Delta t,
\]

for all \( k \in [0, N - 1] \) and \( l \in [\varphi(k), \varphi(k + 1)] \). Thus, if we choose \( \varepsilon_1 \) small enough so that, for all \( \varepsilon \leq \varepsilon_1 \),

\[
\varepsilon \varepsilon(1) \Delta t \sqrt{2e} + e^{LT} \leq \alpha \delta,
\]

\[
\frac{2e^2 T \varepsilon(1)}{\delta^2} \leq \sqrt{\varepsilon},
\]

\[
\frac{T}{\varepsilon \Delta t} e^{-\frac{\varepsilon(1)}{\nu(1)}} \leq \frac{\sqrt{\varepsilon}}{2},
\]

we get by combining (5.20), (5.35), (5.36), and (5.37) that

\[
P(\exists k \in [0, N - 1] : \exists l \in [\varphi(k), \varphi(k + 1)] : X^0_k < y_k - 2\alpha \delta) \leq \sqrt{\varepsilon},
\]

and finally, using Equation (5.38), we conclude that

\[
P(\exists n \in [0, N] : X^n_k < x(n \varepsilon \Delta t) - 3\alpha \delta) \leq \sqrt{\varepsilon}.
\]

Again, choosing \( T + 1 \) instead of \( T \) in the definition of \( N \), we get that for \( K = \max(K_0, K_1) \), there exists \( \varepsilon' > 0 \) such that for all \( \varepsilon < \varepsilon' \),

\[
P(\exists n \in [0, T] : X^n_k < x(n \varepsilon \Delta t) - 3\alpha \delta) \leq \sqrt{\varepsilon}.
\]

This proves the theorem when \( r \) is a smooth growth rate function.

**Step 2.** We now assume that \( r \) is a good growth rate function. Let \( \delta > 0 \). Pick \( n \) large enough so that

\[
\sup_{t \in [0, T]} |x(t) - \bar{x}_n(t)| < \delta,
\]

where \( \bar{x}_n \) refers to the solution of the Cauchy problem (C) with growth rate function \( \theta_n \) (see 2.15). By definition, \( \bar{\theta}_n \) is a smooth growth rate function. Moreover, it follows from Assumption 2 and Remark 7 that the particle system with reproduction law \( \nu_{r,n,K} \) stochastically dominates the system with reproduction law \( \nu_{\bar{\theta}_n,n,K} \). This observation, combined with Step 1, shows that there exists \( \varepsilon_0 \equiv \varepsilon_0(n, \Delta t, \Delta x, \delta) \), \( \alpha \equiv \alpha(n) \), and \( K^* \equiv K^*(n, \Delta t, \Delta x, \delta) \) such that

\[
\forall \varepsilon < \varepsilon_0, \forall K \geq K^*, \quad P(\exists k \in [0, T/(\varepsilon \Delta t)] : \varepsilon X^*_k < x(k \varepsilon \Delta t) + 3\alpha \delta + \delta) \leq 2\sqrt{\varepsilon}.
\]

It then remains to choose \( \delta \) small enough so that \( 3\alpha \delta < \delta \) to conclude the proof of Proposition 2.2.
A Appendix: the branching random walk

A branching random walk is a branching particle system $\Xi$ governed by a reproduction law $(p_k)_{k \in \mathbb{N}}$ and a displacement law $\mu$. The process starts with a single particle located at the origin. This particle is replaced by $N$ new particles located at positions $(\zeta_1, \ldots, \zeta_N)$, where $N$ is distributed according to $(p_k)_{k \in \mathbb{N}}$ and $(\zeta_i)$ is an i.i.d. sequence of random variables of law $\mu$, independent of $N$. These individuals constitute the first generation of the branching random walk. Similarly, the individuals of the $n$-th generation reproduce independently of each other according to $(p_k)_{k \in \mathbb{N}}$ and their offspring are independently distributed around the parental location according to $\mu$.

In this section, we assume that the displacement law $\mu$ is given by Equation (1.2) and that the reproduction law $(p_k)_{k \in \mathbb{N}}$ satisfies

$$1 < m := \sum_{k \in \mathbb{N}} kp_k < \infty.$$ 

The notation used below are defined in Section 2.3 and Section 4.1.

A.1 Many-to-one lemma

For each particle $u$ such that $|u| = n$, we denote by $u_0, \ldots, u_n$ the set of its ancestors in chronological order (basically, $u_0$ is the particle living at generation 0 and $u_n = u$).

Lemma A.1 (Many-to-one Lemma, see e.g. [40], Theorem 1.1). Let $n \geq 1$ and $g : \mathbb{R}^n \to [0, \infty)$ be a measurable function. Let $(Z_k)_{k \in \mathbb{N}}$ be a sequence of random variables such that $(Z_{k+1} - Z_k)$ is i.i.d. of law $\mu$. Then,

$$\mathbb{E} \left[ \sum_{u \in D_n} g(\Xi_u_1, \ldots, \Xi_u_n) \right] = m^n \mathbb{E} \left[ g(Z_1, \ldots, Z_n) \right].$$ (A.1)

A.2 Regularity of the rate function

In this subsection, we state several results on the function $I$ defined by Equation (4.2) in Section 4. All the notations are introduced in Section 4.1.

Lemma A.2. Let $y \geq 0$ and assume $\Delta x \leq 2y$. Then,

$$I_0 \left( y - \frac{\Delta x}{2} \right) \leq I(y) \leq I_0 \left( y + \frac{\Delta x}{2} \right).$$ (A.2)

Proof. First, remark that, for $\lambda \geq 0$

$$\Lambda(\lambda) = \log \sum_{i \in \mathbb{Z}} \left( \int_{(i - \frac{1}{2})\Delta x}^{(i + \frac{1}{2})\Delta x} \frac{1}{\sqrt{\Delta t}} \mu \left( \frac{z}{\sqrt{\Delta t}} \right) dz \right) e^{\lambda \frac{\Delta x}{2}} \geq \log \sum_{i \in \mathbb{Z}} \left( \int_{(i - \frac{1}{2})\Delta x}^{(i + \frac{1}{2})\Delta x} \frac{1}{\sqrt{\Delta t}} \mu \left( \frac{z}{\sqrt{\Delta t}} \right) e^{\lambda (z - \frac{1}{2} \Delta x)} dz \right) = \Lambda_0(\lambda) - \lambda \frac{\Delta x}{2}.$$ 

Similarly, one can obtain an upper bound on $\Lambda$ and get the following estimate for any $\lambda \geq 0$:

$$\Lambda_0(\lambda) - \lambda \frac{\Delta x}{2} \leq \Lambda(\lambda) \leq \Lambda_0(\lambda) + \lambda \frac{\Delta x}{2}. \quad (A.3)$$

Now note that we have for $y \geq 0$, since $\mu$ has expectation 0,

$$I(y) = \sup_{\lambda \geq 0} (\lambda y - \Lambda(\lambda)).$$
and similarly, for every \( z \geq 0 \),

\[
I_0(z) = \sup_{\lambda \geq 0} (\lambda z - \Lambda_0(\lambda)).
\]

Applying these with \( z = y \pm \Delta x/2 \) (note that \( y - \Delta x/2 \geq 0 \) by assumption) and using \( A.3 \), this proves the lemma. \( \square \)

**Lemma A.3.** Let \( \rho > 1 \) and assume \( \Delta x < \sqrt{2\Delta t \log(\rho)} \). Then,

\[
\frac{c_0}{2} < c < 2c_0,
\]

where \( c_0 \) is the unique solution of \( I_0(c) = \log(\rho) \).

**Proof.** Recall from (4.5) that \( c_0 = \sqrt{2\Delta t \log(\rho)} \), hence \( \Delta x < c_0 \leq 2c_0 \). By Lemma A.2 and the definitions of \( c \) and \( c_0 \), it follows that

\[
|c - c_0| \leq \frac{\Delta x}{2} < \frac{c_0}{2}.
\]

This yields the lemma. \( \square \)

**Lemma A.4.** Let \( y > 0 \). If \( \Delta x < \frac{y}{2} \), then for all \( \eta \geq 0 \),

\[
I(y) + \frac{y^2}{4\Delta t} \eta \leq I((1 + \eta)y).
\]

**Proof.** The function \( I \) is convex on \( \mathbb{R} \). Thus,

\[
I(y) = I\left(\frac{1}{1 + \eta} (1 + \eta)y\right)
\]

\[
\leq \frac{1}{1 + \eta} I((1 + \eta)y) + \frac{\eta}{1 + \eta} I(0) = \frac{1}{1 + \eta} I((1 + \eta)y),
\]

which is equivalent to \( I((1 + \eta)y) \geq I(y) + \eta I(y) \). Besides, if \( \Delta x < \frac{y}{2} \), Lemma A.2 implies that

\[
I(y) \geq I_0\left(y - \frac{\Delta x}{2}\right) = \left(\frac{y - \frac{\Delta x}{2}}{2\Delta t}\right)^2 \geq \frac{y^2 - y\Delta x}{2\Delta t} \geq \frac{y^2}{4\Delta t}.
\]

Combining (A.6) and (A.7), we get Equation (A.5). \( \square \)

**Lemma A.5.** Let \( y > 0 \). If \( \Delta x < \frac{y}{2} \), then

\[
\frac{y}{4\Delta t} |y_1 - y_2| \leq |I(y_1) - I(y_2)|, \quad \forall y_1, y_2 \geq y.
\]

**Proof.** Since \( I \) is convex on \( \mathbb{R} \), we have that

\[
\frac{I(y)}{y} - \frac{I(0)}{0} \leq \frac{I(y_1) - I(y)}{y_1 - y} \leq \frac{I(y_2) - I(y)}{y_2 - y}.
\]

Besides, as in (A.7), we have that

\[
I(y) \geq \frac{y^2}{4\Delta t},
\]

since \( \Delta x \leq \frac{y}{2} \). We conclude the proof by combining (A.9) and (A.10) and recalling that \( I \) is non-decreasing on \((0, \infty)\) so that \( \frac{I(y_2) - I(y_1)}{y_2 - y_1} = \frac{I(y_2) - I(y_1)}{|y_2 - y_1|} \). \( \square \)
A.3 First and second moment of the maximum

For all \( n \in \mathbb{N} \), we denote by \( M_n \) the position of the right-most particle in the branching random walk. In this section we study the asymptotic behaviour of the first and second moments of \( M_n \). These are used for the proof of the lower bound in Section 5. We recall that the reproduction law of the BRW is denoted by \((p_k)_{k \in \mathbb{N}}\).

Lemma A.6 (Biggins’ theorem [10]). Let \( \Delta t > 0 \) and \( \Delta x > 0 \). Assume that \( m > 1 \) and \( p_0 = 0 \). Let \( c \) be the unique positive solution of \( I(c) = \log(m) \). Then,

\[
\lim_{n \to 0} \mathbb{E} \left[ \frac{M_n}{n} \right] = c.
\]

In fact, Biggins [10] proves almost sure convergence of \( M_n/n \). Lemma A.6 follows easily from Liggett’s subadditive ergodic theorem as outlined in Zeitouni [15].

Lemma A.7. Let \( \Delta t > 0 \) and \( \Delta x > 0 \). Let \( 1 < \rho < \bar{\rho} \). Consider a family of reproduction laws \((p_m)_{m \geq 0}\) such that \( \sum_{i=1}^{\infty} kp_{m,k} = m \) and \( (p_m) \) is increasing with respect to \( m \) (with respect to stochastic domination). Uniformly in \( m \in [\rho, \bar{\rho}] \),

\[
\lim_{n \to +\infty} \mathbb{E} \left[ \frac{M_n}{n} \right] = c,
\]

where \( c \) is the unique positive solution of \( I(c) = \log(m) \).

Proof. Define \( f_n : \begin{cases} (1, \infty) &\to \mathbb{R} \\ m &\mapsto \mathbb{E} \left[ \frac{M_n}{n} \right] \end{cases} \). We claim that \((f_n)\) is a sequence of increasing functions. Indeed, for any \( 1 < m_1 < m_2 \), consider \( S^1 \) (resp. \( S^2 \)) a branching random walk of reproduction law \( p_{m_1} \) (resp. \( p_{m_2} \)) and of displacement law \( \mu \). According to Lemma 3.1, we can construct a coupling between \( S^1 \) and \( S^2 \), such that \( S^2 \) dominates \( S^1 \). Hence, \( M_n^{1} \leq M_n^{2} \), where \( M_n^i \) denotes the position of the maximal particle in the branching random walk \( S^i \), \( i = 1, 2 \). It follows that \( f_n \) is increasing on \( (1, \infty) \), for all \( n \in \mathbb{N} \).

Let us now consider the function \( c : (1, \infty) \to (0, \infty) \) that maps \( m \) to the unique positive solution of \( I(c) = \log(m) \). According to Lemma A.5, the function \( c \) is continuous on \( (1, \infty) \). Moreover, by Lemma A.6, \( f_n \to c \) pointwise as \( n \to \infty \). Using the monotonicity of \( f_n \), Dini’s theorem then yields uniform convergence on the compact sets of \([\rho, \bar{\rho}]\). \( \square \)

Lemma A.8. Let \( \Delta t > 0 \), \( 1 < \rho < \bar{\rho} \) and \( \Delta x < \sqrt{2\Delta t \log(\bar{\rho})} \). Consider a reproduction law \((p_k)_{k \in \mathbb{N}}\) such that \( \sum kp_k = m > 1 \) and \( c \) the unique positive solution or \( I(c) = \log(m) \). For any \( \eta > 0 \), uniformly in \( m \in [\rho, \bar{\rho}] \), there exists \( N \in \mathbb{N} \) such that

\[
\forall n \geq N, \quad \mathbb{E} \left[ \frac{M_n^2}{n^2} \right] \leq (c + \eta)^2.
\]

Proof. Let us first remark that, for all \( n \in \mathbb{N} \),

\[
\mathbb{E} \left[ \frac{1}{n^2} \left( \max_{|v|=n} \Xi_v \right)^2 \right] \leq \mathbb{E} \left[ \frac{1}{n^2} \left( \max_{|v|=n} \Xi_v \right)^2 \right].
\]

Then, we define \( \xi_n = \frac{1}{n} \max_{|v|=n} |\Xi_v| \), and write that, for all \( R > 0 \),

\[
\mathbb{E}[\xi_n^2] = \mathbb{E}[\xi_n^2 I_{\xi_n^2 < R^2}] + \mathbb{E}[\xi_n^2 I_{\xi_n^2 \geq R^2}]. \tag{A.11}
\]
Besides, $\mathbb{E}[\xi_n^2 1_{\xi > R}] \leq R^2$ and,

$$\mathbb{E}[\xi_n^2 1_{\xi > R}] = \frac{1}{R} \int_R^\infty 2u P(\xi_n \geq u) du. \hspace{1cm} (A.12)$$

Thanks to the many-to-one lemma (Lemma A.1) and Markov’s inequality, we know that

$$P(\xi_n \geq u) = \mathbb{P} \left( \frac{1}{n} \max_{|v| = n} |\Xi_v| \geq u \right) \leq \mathbb{E} \left[ \sum_{|v| = n} 1_{|\Xi_v| \geq u} \right] \leq m_n \mathbb{P} \left( \frac{|Z_n|}{n} \geq u \right), \hspace{1cm} (A.13)$$

where $(Z_n)_{n \geq 0}$ is a random walk whose increments are distributed as $\mu$. Moreover, by symmetry,

$$P \left( \frac{|\Xi_v|}{n} \geq u \right) = 2P \left( \frac{|\Xi_v|}{n} \geq u \right). \hspace{1cm} (A.14)$$

Equations (A.13) and (A.14), together with Chernoff’s bound then give

$$P(\xi_n \geq u) \leq 2m_n e^{-nI(u)} = 2e^{-nI(c)}(u-c), \hspace{1cm} (A.15)$$

According to Lemma A.3 and Equation (4.5), since $\Delta x < \sqrt{2\Delta t \log(\bar{\rho})}$,

$$\frac{1}{2} \sqrt{2\Delta t \log(\bar{\rho})} < \frac{1}{2} \sqrt{2\Delta t \log(m)} < c < 2\sqrt{2\Delta t \log(m)} < 2\sqrt{2\Delta t \log(m)} < 2\sqrt{2\Delta t \log(\bar{\rho})},$$

so that

$$0 < \alpha := \frac{I \left( \frac{1}{2} \sqrt{2\Delta t \log(\bar{\rho})} \right)}{2\sqrt{2\Delta t \log(\bar{\rho})}} < \frac{I(c)}{c}. \hspace{1cm} (A.16)$$

Let $\eta > 0$ and consider $R = c + \eta$. Equations (A.12), (A.13), (A.15) and (A.16) give that

$$\mathbb{E}[\xi_n^2 1_{\xi > R}] \leq 4 \int_{c+\eta}^\infty u e^{-n\alpha(u-c)} du = 4 \int_{\eta}^\infty (u+c)e^{-n\alpha u} du \leq 4 \int_{\eta}^\infty (u+\sqrt{2\Delta t \log(\bar{\rho})}) e^{-n\alpha u} du.$$

Remark that the last integral tends to $0$ as $n$ tends to infinity, so that there exists $N_\eta \in \mathbb{N}$ such that, for all $n \geq N_\eta$,

$$\mathbb{E}[\xi_n^2 1_{\xi > R}] \leq \eta,$$

and, thanks to Equation (A.11), for $n$ large enough,

$$\mathbb{E}[\xi_n^2] \leq (c + \eta)^2 + \eta.$$

Since $\eta > 0$ was arbitrary, this yields the result.

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### B Appendix: stability of the solution of (C) and convergence of the Euler scheme

#### B.1 Stability

**Lemma B.1.** Assume that $r$ is a smooth growth rate function. Let $\delta > 0$ and $T > 0$. Consider $x$ the solution of

$$\begin{cases}
\dot{x}(t) = \sqrt{2r(t, x(t))} \\
x(0) = 0,
\end{cases}$$

37
and \( \tilde{x} \) the solution of
\[
\begin{align*}
\dot{x}(t) &= \sqrt{2r(t, \tilde{x}(t)) + \delta} \\
\tilde{x}(0) &= 0.
\end{align*}
\]

Then,
\[
\sup_{t \in [0, T]} |x(t) - \tilde{x}(t)| \leq \delta(T + 1)e^{LT}.
\]

**Proof.** Let \( u(t) = \sqrt{(x(t) - \tilde{x}(t))^2 + \delta} \) for \( t \in [0, T] \). Note that
\[
|x(t) - \tilde{x}(t)| \leq u(t), \quad \forall t \in [0, T].
\]

Besides, the function \( u \) is differentiable on \([0, T]\) and,
\[
u'(t) = \frac{1}{2u(t)} \frac{d}{dt}(\dot{x}(t) - x(t))^2 + \delta = \frac{1}{u(t)} \left( \frac{d}{dt} x(t) - \frac{d}{dt} \dot{x}(t) \right) (x(t) - \tilde{x}(t))
= \frac{1}{u(t)} \left( \sqrt{2r(t, \tilde{x}(t))} - \sqrt{2r(t, x(t)) + \delta} \right) (\dot{x}(t) - x(t))
\leq \frac{1}{u(t)} (L|x(t) - \tilde{x}(t)| + \delta) |\dot{x}(t) - x(t)|
\leq Lu(t) + \delta,
\]
where the first inequality follows from (1.4). Then, by Grönwall’s inequality, we obtain
\[
u(t) \leq (\delta t + u(0))e^{LT} \leq \delta(T + 1)e^{LT}, \quad \forall t \in [0, T],
\]
which concludes the proof of the lemma. \( \square \)

### B.2 Euler scheme

Consider \( x \) the solution of (C). For any \( T > 0 \) and \( h > 0 \), we can define the Euler scheme of this solution on \([0, T]\) by considering the sequence \((y_i)\) defined by
\[
\begin{align*}
y_0 &= x(0) \\
y_{i+1} &= y_i + \sqrt{2r(ih, y_i)}h.
\end{align*}
\]

Recall the definition of \( L \) from (1.4). Thanks to standard convergence results on the Euler method (see Theorem 14.3 from [27]), we know that
\[
\max_{i \in [0, [T/h]]} |x(t_i) - y_i| \leq e^{LT} \frac{h}{2}.
\]

**Remark 11.** For any \( \delta > 0 \), Equation (B.7) still holds for the function \( \dot{x} \) solution of
\[
\dot{x}(t) = \sqrt{2r(t, x(t))} + \delta,
\]
and its Euler scheme \((\tilde{y}_i)\) on \([0, T]\):
\[
\begin{align*}
\tilde{y}_0 &= x(0) \\
\tilde{y}_{i+1} &= \tilde{y}_i + (\sqrt{2r(ih, \tilde{y}_i)} + \delta)h.
\end{align*}
\]

38
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References

[1] G. Barles, L. C. Evans, and P. E. Souganidis. Wavefront propagation for reaction-diffusion systems of PDE. Technical report, Brown University Providence RI Lefschetz Center for dynamical systems, 1989.

[2] N. H. Barton, A. Etheridge, and A. Véber. Modelling evolution in a spatial continuum. Journal of Statistical Mechanics: Theory and Experiment, (01):P01002, jan 2013.

[3] A. Bensoussan and J.-L. Lions. Applications of Variational Inequalities in Stochastic Control. Studies in mathematics and its applications 12. North-Holland, first edition, 1982.

[4] J. Bérard and J.-B. Gouéré. Brunet-Derrida behavior of branching-selection particle systems on the line. Communications in Mathematical Physics, 298(2):323–342, jun 2010.

[5] H. Berestycki, F. Hamel, and L. Roques. Analysis of the periodically fragmented environment model: II - biological invasions and pulsating travelling fronts. de mathématique sociales (cams) probabilités (latp). Journal de Mathématiques Pures et Appliquées, page 1101, 2005.

[6] H. Berestycki and G. Nadin. Asymptotic spreading for general heterogeneous Fisher-KPP type equations. preprint, 2015.

[7] J. Berestycki, N. Berestycki, and J. Schweinsberg. The genealogy of branching Brownian motion with absorption. The Annals of Probability, 41(2):527–618, mar 2013.

[8] N. Berestycki, A. Etheridge, and M. Hutzenthaler. Survival, extinction and ergodicity in a spatially continuous population model. Markov Processes and Related Fields, 2009.

[9] V. Bezborodov, L. Di Persio, T. Krueger, and P. Tkachov. Spatial growth processes with long range dispersion: Microscopics, mesoscopics and discrepancy in spread rate. Annals of Applied Probability, 30(3):1091–1129, jul 2020.

[10] J. D. Biggins. Chernoff’s theorem in the branching random walk. Journal of Applied Probability, 14(3):630–636, 1977.

[11] M. Birkner and A. Depperschmidt. Survival and complete convergence for a spatial branching system with local regulation. The Annals of Applied Probability, 17(5/6):1777–1807, oct 2007.

[12] M. Birkner, N. Gantert, and S. Steiber. Coalescing directed random walks on the backbone of a 1+1-dimensional oriented percolation cluster converge to the brownian web. Latin American Journal of Probability and Mathematical Statistics, 16:1029, 01 2019.

[13] J. Blath, A. Etheridge, and M. Meredith. Coexistence in locally regulated competing populations and survival of branching annihilating random walk. Annals of Applied Probability, 17(5-6):1474–1507, 2007.
[14] É. Brunet and B. Derrida. Shift in the velocity of a front due to a cutoff. *Physical Review E*, 56(3):2597–2604, sep 1997.

[15] É. Brunet, B. Derrida, A. Mueller, and S. Munier. Phenomenological theory giving the full statistics of the position of fluctuating pulled fronts. *Physical Review E*, 73(5):056126, may 2006.

[16] É. Brunet, B. Derrida, A. H. Mueller, and S. Munier. Noisy traveling waves: Effect of selection on genealogies. *Europhysics Letters (EPL)*, 76(1):1–7, oct 2006.

[17] N. Champagnat and S. Méléard. Invasion and adaptive evolution for individual-based spatially structured populations. *Journal of Mathematical Biology*, 55(2):147, Jun 2007.

[18] A. Cortines. The genealogy of a solvable population model under selection with dynamics related to directed polymers. *Bernoulli*, 22(4):2209–2236, 2016.

[19] A. Etheridge. Survival and extinction in a locally regulated population. *The Annals of Applied Probability*, 14(1):188–214, feb 2004.

[20] A. Etheridge, N. Freeman, and D. Straulino. The Brownian net and selection in the spatial A-Fleming-Viot process. *Electronic Journal of Probability*, 22:1–37, 2017.

[21] L. C. Evans and P. E. Souganidis. A pde approach to geometric optics for certain semilinear parabolic equations. *Indiana University mathematics journal*, 38(1):141–172, 1989.

[22] N. Fournier and S. Méléard. A microscopic probabilistic description of a locally regulated population and macroscopic approximations. *The Annals of Applied Probability*, 14(4):1880–1919, 2004.

[23] M. Freidlin. Limit Theorems for Large Deviations and Reaction-Diffusion Equations. *The Annals of Probability*, 13(3):639–675, aug 1985.

[24] M. Freidlin. Geometric Optics Approach To Reaction-Diffusion Equations. *SIAM Journal on Applied Mathematics*, 46(2):222–232, 1986.

[25] F. Hamel, J. Fayard, and L. Roques. Spreading speeds in slowly oscillating environments. *Bulletin of Mathematical Biology*, 72(5):1166–1191, 2010.

[26] F. Hamel, G. Nadin, and L. Roques. A viscosity solution method for the spreading speed formula in slowly varying media. *Indiana University Mathematics Journal*, pages 1229–1247, 2011.

[27] P. Henrici. Elements of numerical analysis, 1965.

[28] M. Hutzenthaler and A. Wakolbinger. Ergodic behavior of locally regulated branching populations. *The Annals of Applied Probability*, 17(2):474–501, apr 2007.

[29] P. E. Jabin. Small populations corrections for selection-mutation models. *arXiv preprint arXiv:1203.1423*, 2012.

[30] C. Kuehn. Travelling Waves in Monostable and Bistable Stochastic Partial Differential Equations. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 3:1–30, 2019.

[31] P. Maillard. Speed and fluctuations of N-particle branching Brownian motion with spatial selection. *Probability Theory and Related Fields*, 166(3):1061–1173, 2016.

[32] P. Maillard and S. Pennington. Branching random walk with non-local competition. *preprint*, 2021.

[33] S. Mirrahimi, G. Barles, B. Perthame, and P. E. Souganidis. A singular hamilton–jacobi equation modeling the tail problem. *SIAM Journal on Mathematical Analysis*, 44(6):4297–4319, 2012.
[34] C. Mueller, L. Mytnik, and J. Quastel. Effect of noise on front propagation in reaction-diffusion equations of KPP type. *Inventiones mathematicae*, 184(2):405–453, nov 2010.

[35] G. Nadin. The effect of the schwarz rearrangement on the periodic principal eigenvalue of a nonsymmetric operator. *SIAM journal on mathematical analysis*, 41(6):2388–2406, 2010.

[36] G. Nadin. How does the spreading speed associated with the fisher-kpp equation depend on random stationary diffusion and reaction terms? *arXiv preprint arXiv:1609.01441*, 2016.

[37] M. Pain. Velocity of the L-branching Brownian motion. *Electronic Journal of Probability*, 21:no. 28, 1–28, oct 2016.

[38] D. Panja. Effects of fluctuations on propagating fronts. *Physics Reports*, 393(2):87–174, mar 2004.

[39] E. Schertzer, R. Sun, and J. M. Swart. The Brownian web, the Brownian net, and their universality. *Advances in Disordered Systems, Random Processes and Some Applications*, pages 270–368, 2016.

[40] Z. Shi. *Branching random walks*, volume 2151 of *Lecture Notes in Mathematics*. Springer, Cham, 2015.

[41] N. Shigesada. Traveling periodic waves in heterogeneous environments. *Theor. Popul. Biol.*, 30:143–160, 1986.

[42] N. Shigesada and K. Kawasaki. *Biological invasions: theory and practice*. Oxford University Press, UK, 1997.

[43] M. E. Smaily, F. Hamel, and L. Roques. Homogenization and influence of fragmentation in a biological invasion model. *arXiv preprint arXiv:0907.4951*, 2009.

[44] X. Xin. Existence and stability of traveling waves in periodic media governed by a bistable nonlinearity. *Journal of Dynamics and Differential Equations*, 3(4):541–573, 1991.

[45] O. Zeitouni. Branching random walks and Gaussian fields. In *Probability and statistical physics in St. Petersburg*, volume 91 of *Proc. Sympos. Pure Math.*, pages 437–471. Amer. Math. Soc., Providence, RI, 2016.