A quantum mechanical derivation of the Schwarzschild radius and its quantum correction using a model density distribution: Skin of a black hole

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Using a single particle density distribution for a system of self-gravitating particles which ultimately forms a black hole, we from a condensed matter point of view derive the Schwarzschild radius and by including the quantum mechanical exchange energy we find a small correction to the Schwarzschild radius, which we designate as the skin of the black hole.

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I. INTRODUCTION

In the evolution of a star, when the radius of a star becomes less than a certain limit, called the Schwarzschild radius $R_{Sch} = 2GM/c^2$, where $M$ is the mass of the object; $G$, Newton’s universal gravitational constant, and $c$, velocity of light; neither light nor material particles can escape from the star. Thus, one finds from above, the Schwarzschild radius corresponds to the situation when the escape velocity is equal to the velocity of light, a particle (or a photon) coming from a large distance when passes near by the black hole, it is not only attracted towards it but also its orbit diverges from a straight line. It may so happen that if the particle goes too close to a black hole it is likely to be trapped and hence, it cannot escape to infinity. As a special case, if the particle travels straight to the center of the black hole, it falls inside it and is lost forever. An interesting problem that is associated with the formation of a black hole is the final collapse of a massive star. This happens when the nuclear fuel inside the central core of the star gets exhausted. At this stage, it is the dominance of the gravitational attraction among the particles within a star over the internal outward pressure which makes the star to collapse. A black hole, though may be formed from baryons, leptons etc; the exterior observer cannot have access to the details of the inside of the black hole. The observer probes the black hole mass $M$, electromagnetic charge $Q$ and angular momentum $J$. This is referred to as baldness of the black hole or as Wheeler describes, a black hole has no hair. But actually it has only three hairs, $M$, $Q$, $J$. We proceed to evaluate the total energy of a star, we choose a trial single-particle density to account for the distribution of particles within star. The form of our single-particle density is such that it has a singularity at the origin. Applying it to the case of a neutron star, we not only arrive at a compact expression for the radius of the neutron star, but also obtain an expression for the binding energy of the star which varies with the particle number $N$ as $N^{7/3}$, where $N$ is the particle number. Such a dependence with $N$ is in agreement with those of the earlier workers. The aim of the present work is to give a derivation of the so-called Schwarzschild radius even without using GTR and relativistic quantum mechanics. By accounting for the exchange effects due to the interparticle correlations to the total ground state energy of the system, we find a quantum correction to the Schwarzschild radius.

II. MATHEMATICAL FORMULATION

In order to describe a system of $N$ self-gravitating particles in absence of any source for radiation, we use a Hamiltonian of the form:

$$H = \sum_{i=1}^{N} -\frac{\hbar^2 \nabla_i^2}{2m} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, i \neq j}^{N} v(|\vec{X}_i - \vec{X}_j|),$$  \hspace{1cm} (1)

where $v(|\vec{X}_i - \vec{X}_j|) = -g^2/|\vec{X}_i - \vec{X}_j|$, is the interparticle interaction between pair of gravitating particles and $g^2 = Gm^2$, $m$ being the mass of particle and $G$ being the universal gravitational constant. In the present case we confine ourselves to the system of neutrons only. Since the wavefunction of a neutron star is not known, we proceed to evaluate the total kinetic energy of the system using a model density distribution function for...
particles (neutrons) within the neutron star. The particles being fermions, we use the Thomas-Fermi formula for calculating the total kinetic energy of the system. For an infinite many-fermion system, the average particle density and the fermi momentum of a particle are related to each other as:

\[ n = \frac{k_F^3}{3\pi^2} \]  

(2)

For a finite system like the star, since the density distribution is a function of radius vector \( \vec{r} \), the fermi energy of a particle is supposed to be dependent on \( r \). For an infinite many-fermion system, the total kinetic energy of the system is given as

\[ < KE >_{\infty} = \frac{3}{5} n \epsilon_F = \frac{3}{5} n (3\pi^2 n)^{2/3} \frac{\hbar^2}{2m} = \frac{3\hbar^2}{10m} (3\pi^2)^{2/3} n^{5/3}, \]

(3)

In analogy with Eq.(3), for a finite system, we write,

\[ < KE > = \frac{3\hbar^2}{10m} (3\pi^2)^{2/3} \int d\vec{r} \rho(\vec{r})^{5/3}, \]

(4)

The total energy \( E \) of the system is given as

\[ E = < H > = < KE > + < PE >, \]

(5)

where

\[ < PE > = \frac{y^2}{2} \int d\vec{r} d\vec{r}' \rho(\vec{r}) \rho(\vec{r}') |\vec{r} - \vec{r}'| \]

(6)

The expression given in Eq.(6) is written in the Hartree approximation. In order to find \( E \), we choose a trial single-particle density of the form

\[ \rho(\vec{r}) = A \frac{\exp\left[-\left(\frac{\vec{r}}{\lambda}\right)^{1/2}\right]}{\left(\frac{\vec{r}}{\lambda}\right)^{3/2}} \]

(7)

where \( A \) is the normalization constant, which is determined using the relation

\[ \int \rho(\vec{r}) d\vec{r} = N. \]

(8)

As one can see from Eq.(7), \( \rho(\vec{r}) \) is singular at \( r = 0 \). In general one could choose a single particle density of the form

\[ \rho(\vec{r}) = A \frac{\exp\left[-\left(\frac{\vec{r}}{\lambda}\right)^{\nu}\right]}{\left(\frac{\vec{r}}{\lambda}\right)^{3\nu}}, \]

(9)

where \( \nu = 1, 2, 3, 4... \) or \( \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ... \). Integer values of \( \nu \) are not permissible because they make the normalization constant infinite. Out of the fractional values, \( \nu = \frac{1}{2} \) is found to be most appropriate, because, as we shall see later, it gives the expected upper limit for the critical mass of a neutron star, beyond which black hole formation takes place. Any other value of \( \nu \) would give rise to a different value for the critical mass. Also because if \( \nu \) goes to zero (like \( 1/n, \ n \to \infty \)), \( \rho(r) \) would tend to the case of a constant density as found in an infinite many-fermion system. Having accepted the value \( \nu = \frac{1}{2} \), the parameter \( \lambda \) associated with \( \rho(r) \) is determined after minimizing \( E(\lambda) = < H > \) with respect to \( \lambda \). This is how, we are able to find the total energy of the system corresponding to its lowest energy state. After evaluating the integral shown in Eq.(6) and Eq.(9), we obtain

\[ E(\lambda) = \frac{\hbar^2}{m} \frac{12}{25 \pi} (3\pi N)^{2/3} \frac{1}{16} \frac{g^2 N^2}{\lambda^2} \]

(10)

Differentiating this with respect to \( \lambda \) and then equating it with zero, we obtain the value of \( \lambda \) at which the minimum occurs. This is found as:

\[ \lambda_0 = \frac{72}{25 m g^2} (3\pi)^{2/3} \frac{1}{16 N^{1/3}} \]

(11)

Here we are only concerned with the total kinetic energy of the system. At \( \lambda = \lambda_0 \), we have,

\[ < KE > = 0.015441 \frac{m g^4}{\hbar^2} N^{7/3} \]

(12)

The total energy \( E \) of the system is found to be just negative of this.

### III. DERIVATION OF SCHWARZSCHILD RADIUS

We now try to calculate the average velocity of a particle within the neutron star using Eq.(12). Let us denote it by \( < v^2 > \). If \( M \) denotes the total mass of the neutron star, one writes

\[ < KE > = \frac{1}{2} M < v^2 >, \]

(13)

where \( M = N m \). By comparing Eq.(13) with Eq.(12), one obtains,

\[ < v^2 > = 0.030882 \frac{g^4}{\hbar^2} N^{4/3}. \]

(14)

From the expression for the total kinetic energy of an infinite many-fermion system, one finds that the average velocity of a particle within the system is \( \sim 0.77 v_f \), \( v_f \) being the fermi velocity of the particle within the system, which is maximum velocity of that particle. From this, one clearly sees that the maximum velocity of a particle belonging to an infinite many-fermion system is greater than the average particle velocity. In view of this fact, we could write the maximum velocity of a particle within a neutron star as

\[ < v^2 >_{\text{max}} = \alpha < v^2 > = 0.030882 \alpha \frac{g^4}{\hbar^2} N^{4/3} \]

(15)
where \( \alpha \) is a constant whose value is to be greater than unity and it is to be calculated later. \( v_{\text{max}} \) can be identified as the escape velocity of a particle within a neutron star. According to special theory of relativity, \( v^2_{\text{max}} \) is to be less than \( c^2 \), \( c \) being the velocity of light. That is,

\[
0.0308820 \frac{g^4}{\hbar^2} N^{4/3} \leq c^2, \tag{16}
\]

From this it follows that

\[
N \leq \frac{13.574409 \left( \frac{\hbar c}{g^2} \right)^{3/2}}{\alpha^{3/4}} \quad (N_c \text{ (say)}), \tag{17}
\]

having \( g^2 = Gm^2 \). Substituting Eq. (17) in Eq. (11), one finds that,

\[
\lambda_0 \geq \lambda_c = \frac{Gm}{c^2} \alpha^{1/4}[0.8483718(\frac{\hbar c}{g^2})^{3/2}]. \tag{18}
\]

If we define the radius of a neutron star as \( R_0 = 2\lambda_0 \), we have the expression for the critical radius as,

\[
R_c = 2\lambda_c = 2\frac{Gm}{c^2} \alpha^{1/4}[0.8483718(\frac{\hbar c}{g^2})^{3/2}] = 2\frac{GM}{c^2}. \tag{19}
\]

Our identification about the radius \( R \) of the star with \( 2\lambda_0 \) is based on the use of so-called quantum mechanical tunneling effect. Classically, it is well known that a particle has a turning point where the potential energy becomes equal to the total energy. Since the kinetic energy and therefore the velocity are equal to zero at such a point, the classical particle is expected to be turned around or reflected by the potential barrier. From the present theory it is seen that the turning point occurs at a distance \( R = 2\lambda_0 \). This is the reason why we identify \( 2\lambda_0 \) with the radius of a star. For \( R > 2\lambda_0 \), a particle, belonging to the system, may have an access to the region beyond \( R > 2\lambda_0 \), because of quantum mechanical tunneling, but is forbidden by classical theory.

\( R_c \) as given in Eq. (19) is being identified as the so-called Schwarzschild radius which we have derived here by treating the system as a quantum many-body system. When \( R_0 \leq R_c \), the corresponding neutron star becomes a black hole. From Eq. (17), we therefore find that the lowest mass of the neutron star beyond which black hole formation takes place is given as

\[
M_c = mN_c = \frac{13.574409}{\alpha^{3/4}} m \left( \frac{\hbar c}{g^2} \right)^{3/2}. \tag{20}
\]

In order to determine \( \alpha \), we now try to evaluate the limiting mass of a neutron star following the general expression for the radius of a star. Beyond this mass, the black hole formation is likely to take place. For that, we consider the situation when

\[
(R_0 = 2\lambda_0) = (R_{\text{Sch}} = 2\frac{GM}{c^2}), \tag{21}
\]

where \( M = Nm \). From this, we arrive at

\[
N \geq \frac{1.696758}{(\frac{\hbar c}{g^2})^{3/2}} = N_c. \tag{22}
\]

Since the expression in the right hand side of Eq. (22) should be equal to the one given in right hand side of Eq. (17), we must have \( \alpha = 16 \). Under this situation, we have

\[
v^2_{\text{max}} = 0.494112 \frac{g^4}{\hbar^2} N^{4/3}, \tag{23}
\]

where \( N_c = 1.696758(\frac{\hbar c}{g^2})^{3/2} \), which, when evaluated, becomes \( 3.7390777 \times 10^{37} \). For a neutron star in which the number of neutrons exceeds \( N_c \) it has the tendency of forming a black hole. In that case, its mass must exceed \( M = M_c = mN_c = 3.12213 \, M_\odot \), \( M_\odot \) being the solar mass.

### IV. QUANTUM CORRECTION

So far we have been discussing about the quantum mechanical derivation of the Schwarzschild radius \( R_{\text{Sch}} \). The very form of \( R_{\text{Sch}} \) shows that it is a classical result, leaving aside the fact the number of particles \( N \) within a neutron star is to be less than \( N_c \) where \( N_c = 1.70(\frac{\hbar c}{Gm^2})^{3/2} \), which involves the Planck’s constant \( \hbar \). Now, in order to account for the quantum corrections to \( R_{\text{Sch}} \), we go beyond the Hartree contribution to the total energy of the system. That is the exchange correction or Hartree-Fock (HF) term over the Hartree result (direct correction). Since the HF-correction term is non-local we make use of the local density approximation to write it as,

\[
< PE >_{\text{ex}} = \frac{3}{2\pi} (3\pi^2)^{1/3} g^2 \int d\vec{r}[\rho(\vec{r})]^{4/3}. \tag{24}
\]

This when evaluated gives

\[
< PE >_{\text{ex}} = \frac{27}{4} \left( \frac{1}{16\pi} \right)^{4/3} (3\pi^2)^{1/3} g^2 N^{4/3}. \tag{25}
\]

With the inclusion of this extra term, the expression for \( E(\lambda) \), Eq. (10), is minimized with respect to \( \lambda \) and we arrive at

\[
\lambda'_0 = \frac{72}{25}, \frac{3\pi^2}{10} g^2 \frac{\hbar^2}{m g^2 N^{1/3}} \left[ 1 + \frac{1.8010}{N^{2/3}} \right]. \tag{26}
\]

Following the argument discussed earlier, we identify the radius of the neutron star by \( R_0 = 2\lambda'_0 \). As before writing \( v^2_{\text{max}} = 16 < v^2 > \) and using the condition that \( v^2_{\text{max}} \leq c^2 \), we obtain

\[
N \leq N'_c = 1.696758(\frac{\hbar c}{g^2})^{3/2} [1 + 0.474776(\frac{g^2}{\hbar c})]. \tag{27}
\]

Corresponding to \( N'_c \), the new expression for the critical radius \( R'_c \) becomes

\[
R'_c = 2\lambda'_c = 2\frac{GM_c}{c^2} \left[ 1 + 0.7912723(\frac{g^2}{\hbar c}) \right]. \tag{28}
\]
where $M_c = mN_c = 1.696758(\frac{\hbar c}{g^2})^{3/2}$. The above expression, Eq. (28) is obtained by keeping terms up to order $(\frac{g^2}{\hbar c})$ only in Eq. (20). For $N > N_c'$, the neutron star is likely to go over the black hole stage. From Eq. (28), we find that the second term within the square bracket, forms the quantum correction to the Schwarzschild radius. As expected, it involves the gravitational fine structure constant $(\frac{g^2}{\hbar c})$. Since it is of the order $10^{-39}$, obviously it makes an extremely small correction to $R_{Sch}$. It has been shown earlier that using the quantum field theoretic method and by including a single-closed-loop in the self energy, a quantum correction to the classical Schwarzschild solution of the order of $\sim G^2$ can be found. This comes from the gravity sector. The correction that we get is also of the order $\sim G^2$ but it comes from the exchange part of the matter-energy sector of the black hole.

V. CONCLUSION

We in this paper derive the Schwarzschild radius of a black hole from a condensed matter point of view by using a single particle density distribution for the many-body self-gravitating system which ultimately forms a black hole. By incorporating the quantum mechanical exchange interaction, we also find a thin correction to the Schwarzschild radius which we designate as the skin of the black hole.

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