Weakly symmetry of a class of $g$-natural metrics on tangent bundles

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Abstract

Considering the class $G$ of $g$-natural metrics on the tangent bundle of a Riemannian manifold $(M, g)$, it is shown that the flatness for $g$ is a necessary and sufficient condition of weakly symmetry (recurrent or pseudo-symmetry) of $G$. In particular, the cases of weakly symmetric Sasakian lift metric studied by Bejan and Crasmareanu and recurrent or pseudo-symmetric Sasakian lift metric studied by Binh and Tamásy are obtained.

Keywords: $g$-natural metric, Weakly symmetric Riemannian manifold.

1 Introduction

In [8], Tamásy and Binh introduced the notion of weakly symmetric Riemannian manifold which is a stronger variant of recurrent and pseudo-symmetric manifolds. Then they studied the weak symmetries of Einstein and Sasakian manifolds in [9]. Recent studies show that the notion of weakly symmetry has an important role in Riemannian geometry [2]-[10].

In [2], Bejan and Crasmareanu considered the Sasakian lift $g^*$ to the tangent bundle of a Riemannian manifold $(M, g)$ and proved that the weakly symmetry of $g^*$ is equivalent to the flatness for $g$ and $g^*$. Indeed, they extended the result obtained by Tamásy and Binh [3] for recurrent and pseudo-symmetric manifolds. Moreover, in [2] the authors provided the following open problem: to extend the present result to other classes of metrics on tangent bundles. To solving of this open problem, we consider the metric $G = ag^* + bg^* + cg^*$ ($a, b, c$ are constants) which is a class of $g$-natural metrics introduced by Abbassi and Sarih in [1] and we show that $(TM, G)$ is weakly symmetric (recurrent or pseudo-symmetric) Riemannian manifold if and only if $(M, g)$ is flat.

2 Preliminaries

Let $(M, g)$ be a Riemannian manifold with dimension $n \geq 3$ and $TM$ its tangent bundle. If we consider coordinate system $x = (x^i)$ on the base manifold $M$ and
corresponding coordinates \((x, y) = (x^i, y^i)\) on \(TM\), then the metric \(g\) has the local coefficients \(g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\). Let \(\nabla\) be a Riemannian connection on \(M\) with coefficients \(\Gamma^k_{ij}\) where \(1 \leq i, j, k \leq n\). The Riemannian curvature tensor is defined by

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z, \quad \forall X, Y, Z \in \mathfrak{X}(M).
\]

Let \(\pi\) the natural projection from \(TM\) to \(M\). Consider \(\pi^*: TTM \to TM\) and put

\[
\ker\pi^*_v = \{z \in TTM| \pi^*_v(z) = 0\}, \quad \forall v \in TM.
\]

Then the vertical vector bundle on \(M\) is defined by \(V TM = \bigcup_v \ker\pi^*_v\). A horizontal distribution on \(TTM\) is a complementary distribution \(HTM\) for \(VTM\) on \(TTM\). It is clear that \(HTM\) is a horizontal vector bundle. By definition, we have the decomposition

\[
TTM = VTM \oplus HTM. \tag{2.1}
\]

Using the induced coordinates \((x^i, y^i)\) on \(TM\), we can choose a local field of frames \(\{\delta_{x^i}, \frac{\partial}{\partial y^i}\}\) adapted to the above decomposition namely \(\delta_{x^i} \in X(HTM)\) and \(\frac{\partial}{\partial y^i} \in X(VTM)\) are sections of horizontal and vertical sub-bundles \(HTM\) and \(VTM\), defined by

\[
\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - y^a \Gamma^j_{ai} \frac{\partial}{\partial y^j}.
\]

According to (2.1), every vector field \(\tilde{X}\) on \(TM\) has the decomposition \(\tilde{X} = h\tilde{X} + v\tilde{X}\). Moreover, a vector field \(X = X^i \frac{\partial}{\partial x^i}\) on \(M\) has the vertical lift \(X^v = X^i \frac{\partial}{\partial y^i}\) and the horizontal lift \(X^h = X^i \frac{\delta}{\delta x^i}\).

### 2.1 A class of \(g\)-natural metrics on tangent bundle

Let \(g\) be a Riemannian metric on a manifold \(M\). The Sasaki lift \(g^s\) of \(g\) is defined by

\[
g^s(x, y) = \begin{cases} 
    g(x, y) & \text{if } X^v = X^i \frac{\partial}{\partial y^i}, \\
    g(x, y) & \text{if } X^h = X^i \frac{\delta}{\delta x^i}.
\end{cases}
\]

Also, the horizontal lift \(g^h\) and the vertical lift \(g^v\) of \(g\) are defined as follows

\[
g^h(x, y) = \begin{cases} 
    g(x, y) & \text{if } X^v = X^i \frac{\partial}{\partial y^i}, \\
    g(x, y) & \text{if } X^h = X^i \frac{\delta}{\delta x^i}.
\end{cases}
\]

2
Now we consider the metric $G = ag^* + bg^h + cg^v$, where $a$, $b$, $c$ are constants. Indeed we can present $G$ as follows

$$
\begin{align*}
G_{(x,y)}(X^h, Y^h) &= (a + c)g_x(X,Y), & G_{(x,y)}(X^h, Y^v) &= bg_x(X,Y), \\
G_{(x,y)}(X^v, Y^h) &= bg_x(X,Y), & G_{(x,y)}(X^v, Y^v) &= ag_x(X,Y).
\end{align*}
$$

(2.3)

This metric is a class of $g$-natural metrics and it is Riemannian if and only if $a > 0$ and $\alpha = a(a + c) - b^2 > 0$ hold. Also, for $a = 1$ and $b = c = 0$, the metric $G$ reduces to the Sasaki lift metric (See [1]). Let $\nabla$ be the Levi-Civita connection of $G$. Then it is characterized by [1]

$$
\begin{align*}
\{\nabla_X Y\}^h_t &= (\nabla_X Y)^h |_t + (A(t, X, Y))^h + (B(t, X, Y))^v, \\
\{\nabla_X Y\}^v_t &= (\nabla_X Y)^v |_t + (C(t, X, Y))^h + (D(t, X, Y))^v, \\
\{\nabla_X Y\}^h_t &= (C(t, Y, X))^h + (D(t, Y, X))^v, \\
\{\nabla_X Y\}^v_t &= 0,
\end{align*}
$$

for all vector fields $X, Y$ on $M$, where $A, B, C, D$ are the tensor fields of type $(1, 2)$ on $M$ defined by

$$
\begin{align*}
A(t, X, Y) &= -\frac{ab}{2\alpha} [R(X, t)Y + R(Y, t)X], \\
B(t, X, Y) &= \frac{b^2}{\alpha} R(X, t)Y - \frac{a(a + c)}{2\alpha} R(X, Y)t, \\
C(t, X, Y) &= -\frac{a^2}{2\alpha} R(Y, t)X, & D(t, X, Y) &= \frac{ab}{2\alpha} R(Y, t)X,
\end{align*}
$$

where $t$ is thought as a vector field on $M$ with local expression $t = y^i \frac{\partial}{\partial x^i}$. Moreover, $t^v = y^i \frac{\partial}{\partial y^i}$ is the Liouville vector field and $t^h = y^i \frac{\partial}{\partial z^i}$ is the geodesic spray of the metric $g$.

**Theorem 1.** Let $(M, g)$ be a Riemannian manifold and $G$ be the Riemannian metric given by (2.3) on $TM$. Then the Riemannian curvature tensor $\tilde{R}$ of $(TM, G)$ is completely determined by

$$
\tilde{R}(X^v, Y^v)Z^v = 0,
$$

$$
\tilde{R}(X^v, Y^v)Z^h = \left(\frac{a^2}{\alpha} R(X, Y)Z + \frac{a^4}{4\alpha^2} [R(X, t)R(Y, t)Z - R(Y, t)R(X, t)Z]\right)^h + \left\{\frac{ab}{\alpha} R(Y, t)X + \frac{a^3b}{4\alpha^2} [R(Y, t)R(X, t)Z - R(X, t)R(Y, t)Z]\right]^v,
$$

$$
\tilde{R}(X^h, Y^v)Z^v = \left\{\frac{a^2}{2\alpha} R(Z, Y)X - \frac{a^4}{4\alpha^2} R(Y, t)R(Z, t)X\right\}^h + \left\{\frac{a^3b}{4\alpha^2} R(Y, t)R(Z, t)X - \frac{ab}{2\alpha} R(Z, Y)X\right\}^v,
$$

where $\tilde{R}$ is the induced connection of $\nabla$. Moreover, $\tilde{R}$ is completely determined by the coefficients $A, B, C, D$ of the vector fields $t, t^v, t^h$.
\[\bar{R}(X^h, Y^h)Z^v = \left\{ \frac{\alpha^2}{2\alpha} \left[ (\nabla_Y R)(Z, t)X - (\nabla_X R)(Z, t)Y \right] \right. \]
\[+ \frac{\alpha^3b}{4\alpha^2} [R(X, t)R(Z, t)Y - R(Y, t)R(Z, t)X] \right)^h \]
\[+ \left\{ R(X, Y)Z + \frac{ab}{2\alpha} [R(Y, R(Z, t)X) - (\nabla_Y R)(Z, t)X] \right\} \]
\[+ \left\{ \frac{\alpha^2}{4\alpha} [R(X, R(Z, t)Y) - R(Y, R(Z, t)X)] \right\} \]
\[+ \left\{ \frac{\alpha^2b^2}{4\alpha^2} [R(Y, t)R(Z, t)X - R(X, t)R(Z, t)Y] \right\} \right)^v, \]

\[\bar{R}(X^h, Y^h)Z^h = \left\{ R(X, Y)Z + \frac{ab}{2\alpha} [2(\nabla Y R)(X, Y)Z - (\nabla Z R)(X, Y)t] \right. \]
\[+ \frac{\alpha^2}{4\alpha} [R(R(Y, Z), t)X - R(R(X, Z), t)Y] \right\} \]
\[+ \frac{\alpha^2b^2}{4\alpha^2} [R(X, t)R(Y, t)Z + R(X, t)R(Z, t)Y \right. \]
\[- R(Y, t)R(X, t)Z - R(Y, t)R(Z, t)X] \right\} \]
\[+ \left\{ \frac{\alpha^2}{2\alpha} [R(Y, t)Z - R(Y, t)R(Z, t)X] \right\} \]
\[+ \frac{\alpha^2b(a + c)}{4\alpha^2} [R(X, R(Y, t)Z) + R(X, R(Z, t)Y)] \right\} \]
\[+ \frac{ab}{2\alpha} [R(R(Y, t)Z, t)X + R(R(X, Z)t, t)Y] \right\} \]
\[+ \frac{ab}{2\alpha} R(R(X, Y)t, t)Z \right\} v, \]

\[\bar{R}(X^h, Y^v)Z^h = \left\{ -\frac{\alpha^2}{2\alpha} (\nabla_X R)(Y, t)Z + \frac{\alpha^3b}{4\alpha^2} [R(X, t)R(Y, t)Z \right. \]
\[+ R(Y, t)R(Z, t)X - R(Y, t)R(X, t)Z] \right\} \]
\[+ \frac{ab}{2\alpha} [R(Y, R(Z, t)X) - (\nabla_Y R)(Z, t)X] \right\} \]
\[+ \left\{ \frac{\alpha^2b^2}{4\alpha^2} [R(X, t)R(Y, t)Z - R(Y, t)R(Z, t)X] \right\} \]
\[+ \left\{ \frac{\alpha^2}{2\alpha} [R(Y, t)Z - R(Y, t)R(Z, t)X] \right\} \]
\[+ \frac{ab}{2\alpha} [R(R(Y, t)Z, t)X + R(R(X, Z)t, t)Y] \right\} \]
\[+ \frac{ab}{2\alpha} R(R(X, Y)t, t)Z \right\} v. \]

Proof. The proof is an special case of the proof of Proposition 2.9 of [1].
3 Weakly symmetric Riemannian manifold \((TM, G)\)

Let \((M, g)\) be a Riemannian manifold. If there exist 1-forms \(\alpha_1, \alpha_2, \alpha_3, \alpha_4\) and a vector field \(A\) on \(M\) such that

\[
(\nabla_W R)(X, Y, Z) = \alpha_1(W)R(X, Y)Z + \alpha_2(X)R(W, Y)Z + \alpha_3(Y)R(X, W)Z + \alpha_4(Z)R(X, Y)W + g(R(X, Y)Z, W)A,
\]

then \((M, g)\) is called weakly symmetric. In [4], the authors proved that the relations \(\alpha_2 = \alpha_3 = \alpha_4\) and \(A_2 = (\alpha_2)^2\) are necessary conditions to weakly symmetry of \(g\). Thus a weakly symmetric manifold \((M, g)\) is characterized by:

\[
(\nabla_W R)(X, Y, Z) = \alpha_1(W)R(X, Y)Z + \alpha_2(X)R(W, Y)Z + \alpha_2(Y)R(X, W)Z + \alpha_2(Z)R(X, Y)W + g(R(X, Y)Z, W)(\alpha_2)^2. \tag{3.4}
\]

**Theorem 2.** Let \((M, g)\) be a Riemannian manifold and \(TM\) be its tangent bundle with Riemannian metric \(G\) given by [2, 3]. Then \((TM, G)\) is weakly symmetric if and only if \((M, g)\) is flat. Hence \((TM, G)\) is flat.

**Proof.** If \(R = 0\), then from Theorem [1] we conclude that \(\tilde{R} = 0\) and so we have [3, 4]. Now let \((TM, G)\) be a weakly symmetric manifold. Then we have [3, 4] for all vector fields \(\tilde{X}, \tilde{Y}, \tilde{Z}\) and \(\tilde{W}\) on \(TM\). If we suppose \(\tilde{X} = X^h, \tilde{Y} = Y^v, \tilde{Z} = Z^v\) and \(\tilde{W} = W^h\), then the right side of (3.4) has the following vertical part

\[
v\{\alpha_1(W^h)\tilde{R}(X^h, Y^v)Z^v + \alpha_2(X^h)\tilde{R}(W^h, Y^v)Z^v
+ \alpha_2(Y^v)\tilde{R}(X^h, W^h)Z^v + \alpha_2(Z^v)\tilde{R}(X^h, Y^v)W^h
+ G(R(X^h, Y^v)Z^v, W^h)(\alpha_2)^2\} = \{\alpha_1(W^h)\} \frac{a^3b}{4\alpha^2} R(Y, t)R(Z, t)X
- \frac{ab}{2\alpha}(R(Z, X)Y) + \alpha_2(X^h) \frac{a^3b}{4\alpha^2} R(Y, t)R(Z, t)W - \frac{ab}{2\alpha} R(Z, Y)W
+ \alpha_2(Y^v) \{R(X, W)Z + \frac{ab}{2\alpha}[(\nabla_X R)(Z, t)W - (\nabla_W R)(Z, t)X]
+ \frac{a^2}{4\alpha} [R(X, R(Z, t)W)t - R(W, R(Z, t)X)t] + \frac{a^2b^2}{4\alpha^2} [R(W, t)R(Z, t)X
- R(X, t)R(Z, t)W]\} + \alpha_2(Z^v) \frac{ab}{2\alpha} (\nabla_X R)(Y, t)W - \frac{b^2}{\alpha} R(X, Y)W
- \frac{a^2b^2}{4\alpha^2} [R(X, t)R(Y, t)W - R(Y, t)R(W, t)X - R(Y, t)R(X, t)W]
+ \frac{a^2}{4\alpha} R(X, R(Y, t)W)t + \frac{a(a + c)}{2\alpha} R(X, W)Y
+ (a + c) \left[ - \frac{a^4}{4\alpha^2} g(R(Y, t)R(Z, t)X, W) + \frac{a^2}{2\alpha} g(R(Z, Y)X, W)\right] \alpha_2^2
+ b^2 \frac{a^3b}{4\alpha^2} g(R(Y, u)R(Z, u)X, W) - \frac{ab}{2\alpha} g(R(Z, Y)X, W)\right] \alpha_2^2 \} v. \tag{3.5}
\]
Now, we compute the vertical part of the left side of (3.4). Using Theorem 1 we obtain

\[
v(\nabla_{W^h} \tilde{R}(X^h, Y^v) Z^v) = \left\{ \begin{array}{l}
\frac{a^3 b}{4\alpha^2} R(Y(t, t) R(Z(t, t) X) - \frac{ab}{2\alpha} \nabla W R(Z, Y) X \\
+ \frac{a^3 (a + c)}{8\alpha^3} R(W, R(t, t) R(Z, t) X) t + \frac{a^3 (a + c)}{4\alpha^2} R(W, R(Z, Y) X) t \\
- \frac{a^3 b^2}{4\alpha^2} R(W, t) R(Z, t) X - \frac{a^2 b^2}{2\alpha^2} R(W, t) R(Z, Y) X \\
+ \frac{a^3 b^2}{8\alpha^3} R(R(t, t) R(Z(t) X, t) W - \frac{a^2 b^2}{4\alpha^2} R(R(Z, Y) X, t) W \right\}^v,
\end{array} \right. (3.6)
\]

\[
v(\tilde{R}(\nabla_{W^h} X^h, Y^v) Z^v) = \left\{ \begin{array}{l}
\frac{a^3 b}{4\alpha^2} R(Y(t, t) R(Z(t, t) X) - \frac{ab}{2\alpha} R(Z, Y) \nabla W X \\
+ \frac{a^3}{4\alpha^2} R(t, t) R(Z, t) t A(t, W, X) - \frac{ab}{2\alpha} R(Z, t) A(t, W, X) \right\}^v,
\end{array} \right. (3.7)
\]

\[
v(\tilde{R}(X^h, \nabla_{W^h} Y^v) Z^v) = \left\{ \begin{array}{l}
\frac{ab}{2\alpha} [(\nabla_X R)(Z(t, W, Y) - (\nabla C_{W, t}) R)(Z(t, X) \\
+ \frac{a^3 b}{4\alpha^2} R(\nabla_{W Y} R)(Z(t, t) X) - \frac{ab}{2\alpha} R(Z, \nabla_{W Y} X + R(X, C(t, W, Y)) \right) X \\
+ \frac{a^2 b^2}{4\alpha^2} [R(C(t, W, Y), t) R(Z(t) X - R(X, t) R(Z, t) C(t, W, Y)] \\
+ \frac{a^3}{4\alpha^2} [R(X, R(Z(t), t) C(t, W, Y)) t - R(C(t, W, Y), R(Z(t) X)] \\
+ \frac{a^3 b^2}{4\alpha^2} R(D(t, W, Y), t) R(Z(t) X - \frac{ab}{2\alpha} R(Z, D(t, W) Y) X \right\}^v,
\end{array} \right. (3.8)
\]

\[
v(\tilde{R}(X^h, Y^v) \nabla_{W^h} Z^v) = \left\{ \begin{array}{l}
\frac{a^3 b}{4\alpha^2} R(Y(t, t) R(\nabla_{W Z} t, X) - \frac{ab}{2\alpha} R(\nabla_{W Z} Y) X \\
+ \frac{a^3 b}{4\alpha^2} R(Y(t, t) R(D(t, W, Z), t) X + \frac{ab}{2\alpha} (\nabla_X R)(Y(t, t) C(u, W, Z) \\
- \frac{ab}{2\alpha} R(D(t, W, Z), Y) X + \frac{a^2}{4\alpha} R(X, R(Y(t, t) C(t, W, Z))] t \\
- \frac{b^2}{\alpha} R(X, Y) C(t, W, Z) + \frac{a(a + c)}{2\alpha} R(X, C(t, W, Z))] Y \\
- \frac{a^2 b^2}{4\alpha^2} [R(X(t), t) R(Y(t, t) C(t, W, Z) - R(Y(t) R(C(t, W, Z), t) \]] X \\
- R(Y(t) R(t, t) C(t, W, Z))] \right\}^v.
\end{array} \right. (3.9)
\]

Using (3.6) - (3.9) we have \(v((\nabla_{W^h} \tilde{R}) (X^H, Y^V) Z^V)\). Now we consider the following

\[
v(\tilde{R}(\nabla_{W^h} \tilde{R})(X^H, Y^V) Z^V) = (3.10)
\]
Setting $Y = t$ in the above equation implies

$$\begin{align*}
&- \frac{ab}{2\alpha} \alpha_1 (W^h)R(Z, t)X - \frac{ab}{2\alpha} \alpha_2 (X^h)R(Z, t)W + \alpha_2 (t^v) \{ R(X, W) \} Z \\
&+ \frac{ab}{2\alpha} \left[ (\nabla X) R(Z, t)W - (\nabla W) R(Z, t)X \right] + \frac{a^2}{4\alpha} [R(X, R(Z, t))W] t \\
&- R(W, R(Z, t)) \{ t \} + \frac{a^2 b^2}{4\alpha^2} [R(W, t)R(Z, t)X - R(X, t)R(Z, t)W] \\
&+ \alpha_2 (Z^v) \{ \frac{b(a + c)}{2\alpha} R(X, W) t - \frac{c^2}{\alpha} R(X, t)W \} + \frac{a}{2} g(R(Z, t)X, W) \alpha_1^2 \\
&= \frac{a^2 b^2}{2\alpha^2} R(W, t)R(Z, t)X - \frac{a^3 (a + c)}{4\alpha^2} R(W, R(Z, t))X t \\
&- \frac{ab}{2\alpha} \left[ (\nabla W) R(Z, t)X \right] - \frac{a^2 b^2}{4\alpha^2} [R(X, t)R(Y, t)W - R(Y, t)R(W, t)X] \\
&+ \alpha_2 (Z^v) \{ \frac{b(a + c)}{2\alpha} R(X, C(t, W, Z)) \} t \\
&+ \frac{b^2}{\alpha} R(X, t)C(t, W, Z). \\
\end{align*} \quad (3.11)$$

Similarly, setting $Z = t$ in $(3.10)$ gives us

$$\begin{align*}
&- \frac{ab}{2\alpha} \alpha_1 (W^h)R(U, Y)X - \frac{ab}{2\alpha} \alpha_2 (X^h)R(t, Y)W + \alpha_2 (Y^v) \{ R(X, W) \} t \\
&+ \alpha_2 (t^v) \{ \frac{ab}{2\alpha} (\nabla X) R(Y, t)W - \frac{a^2 b^2}{4\alpha^2} [R(X, t)R(Y, t)W - R(Y, t)R(W, t)X] \\
&- R(Y, t)R(X, t)W] + \frac{a^2}{4\alpha} R(X, R(Y, t))W t + \frac{a(a + c)}{2\alpha} R(X, W)Y \\
&- \frac{b^2}{\alpha} R(X, Y)W \} + \frac{a}{2} g(R(t, Y)X, W) \alpha_1^2 \\
&= \frac{a^2 b^2}{2\alpha^2} R(W, t)R(t, Y)X - \frac{a^3 (a + c)}{4\alpha^2} R(W, R(t, Y))X t \\
&- \frac{ab}{2\alpha} \left[ (\nabla W) R(t, Y)X \right] - \frac{a^2 b^2}{4\alpha^2} [R(R(t, Y)X, t)W - \frac{ab}{2\alpha} [R(t, Y)A(t, W, X)] \\
&+ \frac{ab}{2\alpha} R(t, D(t, W, Y))X - R(X, C(t, W, Y)) t. \\
\end{align*} \quad (3.12)$$

Setting $Y = Z$ in the above equation and then summing it with $(3.11)$ derive
that

\[
\alpha_2(Z^v)\{-\frac{b^2}{\alpha}R(X, t)W + \frac{a(a + c) + 2\alpha}{2\alpha}R(X, W)t\}
+ \alpha_2(t^v)\{R(X, W)Z + \frac{ab}{2\alpha}[2(\nabla_X R)(Z, t)W - (\nabla_W R)(Z, t)X]
+ \frac{a^2}{4\alpha}[2R(X, R(Z, t)W)t - R(W, R(Z, t)X)t]
+ \frac{a^2b^2}{4\alpha^2}[R(W, t)R(Z, t)X - 2R(X, t)R(Z, t)W + R(Z, t)R(W, t)X
+ R(Z, t)R(t, W)W] - \frac{b^2}{\alpha}R(X, Z)W + \frac{a(a + c)}{2\alpha}R(X, W)Z\}
= \frac{b^2}{\alpha}R(X, t)C(t, W, Z) - \frac{a(a + c) + 2\alpha}{2\alpha}R(X, C(t, W, Z))t. \tag{3.13}
\]

Putting \(Z = t\) in the above equation we get

\[
\alpha_2(t^v)\frac{b^2}{\alpha}R(t, X)W + \frac{a(a + c) + 2\alpha}{2\alpha}R(X, W)t = 0. \tag{3.14}
\]

Interchanging \(X\) and \(W\) in the above equation yields

\[
\alpha_2(t^v)\frac{b^2}{\alpha}R(t, W)X + \frac{a(a + c) + 2\alpha}{2\alpha}R(W, X)t = 0.
\]

By subtracting the above equation from (3.14) we get

\[
\alpha_2(t^v)\{\frac{b^2}{\alpha}[R(t, X)W + R(W, t)X] + \frac{a(a + c) + 2\alpha}{\alpha}R(X, W)t\} = 0.
\]

Using Bianchi identity in above relation gives us

\[
\alpha_2(t^v)R(X, W)t = 0.
\]

If \(\alpha_2(t^v) \neq 0\) we have the conclusion. Now let \(\alpha_2(t^v) = 0\), then from (3.13) we have

\[
\alpha_2(Z^v)\{\frac{a(a + c) + 2\alpha}{2\alpha}R(X, W)t - \frac{b^2}{\alpha}R(X, t)W\}
= \frac{a(a + c) + 2\alpha}{2\alpha}R(X, C(t, W, Z))t + \frac{b^2}{\alpha}R(X, t)C(t, W, Z). \tag{3.15}
\]

Exchanging \(X\) and \(W\) in the above equation we obtain

\[
\alpha_2(Z^v)\{\frac{a(a + c) + 2\alpha}{2\alpha}R(W, X)t - \frac{b^2}{\alpha}R(W, t)X\}
= -\frac{a(a + c) + 2\alpha}{2\alpha}R(W, C(t, X, Z))t + \frac{b^2}{\alpha}R(W, t)C(t, X, Z).
\]
By subtracting the above equation from (3.15) we get
\[
\begin{align*}
\alpha_2(Z^v)\left\{ \frac{a(a+c)+2\alpha}{\alpha} & R(X, W)t + \frac{b^2}{\alpha}[R(W, t)X - R(X, t)W] \right\} \\
= & \frac{a(a+c)+2\alpha}{2\alpha}[R(W, C(t, X, Z))t - R(X, C(t, W, Z))t] \\
+ & \frac{b^2}{\alpha}[R(X, t)C(t, W, Z) - R(W, t)C(t, X, Z)]. \quad (3.16)
\end{align*}
\]

Using Bianchi identity in the above relation we conclude
\[
3\alpha_2(Z^v)R(X, W)t = \frac{a(a+c)+2\alpha}{2\alpha}[R(W, C(t, X, Z))t \\
- R(X, C(t, W, Z))t] + \frac{b^2}{\alpha}[R(X, t)C(t, W, Z) \\
- R(W, t)C(t, X, Z)]. \quad (3.17)
\]

Now, we take the $g$–product with $t$
\[
0 = \frac{a(a+c)+2\alpha}{2\alpha}[g(R(W, C(t, X, Z))t, t) - g(R(X, C(t, W, Z))t, t)] \\
+ \frac{b^2}{\alpha}[g(R(X, t)C(t, W, Z), t) - g(R(W, t)C(t, X, Z), t)] \\
= -\frac{a^2b^2}{2\alpha^2}[g(R(X, t)R(Z, t)W, t) - g(R(W, t)R(Z, t)X, t)]. \quad (3.18)
\]

Setting $W = t$ and $Z = X$ in the above equation we obtain
\[
0 = \frac{a^2b^2}{2\alpha^2}g(R(X, t)t, R(X, t)t)
\]

If $b \neq 0$ then the above equation yields $R(X, t)t = 0$ which gives us $R = 0$. Now let $b = 0$. In this case we have $\alpha = a(a+c)$ and then from (3.17) we get
\[
\alpha_2(Z^v)R(X, W)t = -R(X, C(t, W, Z))t.
\]

Setting $W = X$ in the above equation gives us
\[
R(X, C(t, W, Z))t = 0,
\]
and consequently
\[
\frac{a^2}{2\alpha}R(X, R(t, Z)X)t = 0.
\]

Taking the $g$–product with $Z$ we have, $g(R(X, R(t, Z)X)t, Z)$ which gives us
\[
R(t, Z)X = 0.
\]

Thus $R = 0$, i.e. $(M, g)$ is flat. \qed
For $\alpha_2 = 0$ respectively $\alpha_1 = 2\alpha_2$ in [3,4] we get the following result

**Corollary 1.** Let $(M, g)$ be a Riemannian manifold and $TM$ be its tangent bundle with Riemannian metric $G$ given by (2.3). Then $(TM, G)$ is recurrent or pseudo-symmetric or locally symmetric if and only if $(M, g)$ is flat. Hence $(TM, G)$ is flat.

Considering $a = 1$ and $b = c = 0$ in (2.3) we get the results of [2], [3] for the Sasakian lift metric $g^s$.

**Corollary 2.** $(TM, g^s)$ is weakly symmetric (recurrent or pseudo-symmetric or locally symmetric) Riemannian manifold if and only if the base manifold $(M, g)$ is flat. Hence $(TM, G)$ is flat.

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