Canonical geometrically ruled surfaces

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We prove the existence of canonical scrolls; that is, scrolls playing the role of canonical curves. First of all, they provide the geometrical version of Riemann Roch Theorem: any special scroll is the projection of a canonical curve. We also prove that the generic canonical scroll is projectively normal except in the hyperelliptic case and for very particular cases in the nonhyperelliptic situation.

1 Introduction

A geometrically ruled surface, or simply a ruled surface, is a $\mathbb{P}^1$-bundle over a smooth curve $X$ of genus $g > 0$. It will be denoted by $\pi : S = \mathbb{P}(\mathcal{E}_0) \to X$, where $\mathcal{E}_0$ is the associated normalized locally free sheaf of rank 2. Let $X_0$ be the minimum self-intersection curve of the ruled surface. Let $H \sim X_0 + bf$, $b \in \text{Div}(X)$ be an unisecant divisor such that the complete linear system $|H|$ provides a regular morphism $\phi_H : \mathbb{P}(\mathcal{E}_0) \to R \subset \mathbb{P}^N$, $n = \dim(|H|)$. We call $R$ a linearly normal scroll. We refer to [2] for a systematic development of the projective theory of scrolls and ruled surfaces.

We define the speciality of a scroll $R$ as the superabundance of $|H|$; that is, $i(R) = s(O_S(H)) = h^1(O_S(H))$. The scroll $R$ is called special if $i(R) > 0$. If $W \subset \mathbb{P}^N$ is a subspace and $\pi_W : R - (W \cap R) \to R' \subset \mathbb{P}^N$ is the projection of $R$ from $W \cap R$, it is well-known that $i(R') - i(R) = \deg(W \cap R) - (\dim((W \cap R)) + 1)$, (see [2], §5), and the speciality of $R$ grows exactly the number of unassigned base points in the linear system of hyperplane sections containing $W \cap R$.

In the case of curves, the Riemann-Roch Theorem gives a nice geometrically interpretation of the speciality of a nonhyperelliptic curve. Any linearly normal special curve $C$ is the projection of a canonical curve $C_K$ from a set of points $A$. The canonical curve $C_K$ is projectively normal and it has speciality 1. The speciality of $C$ is $\deg(A) - \dim(A)$.

In this paper we prove the existence of canonical scrolls; that is, the existence of scrolls playing the role of canonical curves and, in particular, providing the geometrical version of Riemann-Roch Theorem for ruled surfaces. Moreover, we prove that in general they are projectively normal when the base curve is not hyperelliptic.

Note that the study of special scrolls is equivalent to the study of special locally free sheaves of rank 2 over a smooth curve. The projection of a scroll corresponds to the elementary transformation of a locally free sheaf of rank 2 (see [2], §4 and [8], §1). In this way we will prove the existence of locally free sheaves of rank 2 and speciality 1 such that any special locally free sheaf of rank 2 is obtained from them by elementary transformations.
The start point is a nice result mentioned by C. Segre in [11]: If \( R \subset \mathbb{P}^n \) is a linearly normal scroll and \( C \subset R \) is a bisecant curve which has no double points out of the singular locus of \( R \), then \( C \) is linearly normal and the speciality of \( C \) is equal to the speciality of \( R \). We call \( C \) a proper bisecant curve.

This theorem suggests the definition of canonical scroll: Let \( X \) be a smooth curve of genus \( g > 0 \) and let \( C \) be a smooth curve of genus \( \pi \) such that there exists an involution \( \gamma : C \to X \); that is, a finite morphism of degree 2. Let us suppose that \( C \) is not hyperelliptic and it has genus \( \pi \geq 3 \). Let \( \varphi_K : C \to C_K \subset \mathbb{P}^{\pi - 1} \) be the canonical map. Then \( C_K \) is a bisecant curve in the scroll \( S = \bigcup_{x \in X} (\gamma^{-1}(x)) \). By Segre’s result, \( S \) has speciality 1 and it contains a canonical curve as a proper bisecant curve. We call \( S \) a canonical scroll.

Note that the existence of canonical scrolls is related to the existence of canonical curves having an involution or finite morphism of degree 2, \( \gamma : C \to X \). The results in §6 about the projective normality of the canonical scroll will allow to give a nice characterization of the ideal of these canonical curves in [3].

The paper is organized in the following way:

In §2, we study the double covers of a smooth variety. We see how we can build a double cover \( \gamma : C \to X \) of a smooth variety \( X \). We characterize when the variety \( C \) is smooth. Moreover we study the ruled variety generated by the involution on \( C \). Although we will apply these results to the case of curves we will work over smooth varieties of arbitrary dimension to give them more generality.

In §3, we use the results of the first section to prove the Segre Theorem and give the geometrical model of the canonical scrolls. Given a nonspecial divisor \( b \) of degree \( \pi - 1 = \deg(b) \geq 2g - 2 \), we call canonical geometrically ruled surface to the ruled surface \( P(\mathcal{E}_b) \), \( \mathcal{E}_b = \mathcal{O}_X \oplus \mathcal{O}_X(K - b) \), such that the generic curve \( C \) of \( |2X_1| \) is smooth. When the curve \( C \) is nonhyperelliptic the image of \( P(\mathcal{E}_b) \) by the map defined by the linear system \( |X_0 + bf| \) is a canonical scroll. From this, we conclude that any special scroll has a special directrix curve. This last result was proved by Segre in [11] with a condition over the degree of the scroll. Furthermore, in this section we see how a ruled surface is transformed by projecting from a point of a bisecant curve.

In §4, we study when the smooth curve \( C \subset |2X_1| \) is not hyperelliptic and the complete linear system defined by \( H \sim X_0 + bf \sim X_1 + Kf \) is base-point-free. This clarifies the equivalence between both concepts: canonical scroll and canonical geometrically ruled surface. Furthermore, we characterize the hyperelliptic double cover of smooth curves.

In §5 we prove the existence of canonical geometrically ruled surfaces over a smooth curve \( X \) of genus \( g \geq 1 \).

First of all, Proposition 3.3 characterizes the nonspecial divisors \( b \) such that \( P(\mathcal{E}_b) \) is canonical. Such divisors satisfy the semicontinuity property that: if \( P \in X \) is a base point of the linear system \( |2(b - K)| \), \( P \) is not a base point of \( |2(b - K) - P| \). Then for any nonspecial divisor \( b \) such that \( \deg(2b) \geq 6(g - 1) + 1 \), the geometrically ruled surface \( P(\mathcal{E}_b) \) is canonical. The proof of the existence is reduced to the range \( \deg(b) \leq 3(g - 1) \).

In the range A: \( 5(g - 1) \leq \deg(2b) \leq 6(g - 1) \); if \( b \) is a generic nonspecial divisor, then \( P(\mathcal{E}_b) \) is a canonical ruled surface, \( |X_1| \) consists of a unique curve and \( 2(b - K) \) is nonspecial. Moreover if \( b \) and \( P \in X \) are generic in the range \( 5(g - 1) + 1 \leq \deg(2b) \leq 6(g - 1) \); the elementary transformation of \( P(\mathcal{E}_b) \) at the point \( X_b \cap P \) is the canonical ruled surface \( P(\mathcal{E}_{b - P}) \) in the case \( \deg(b) - 1 \). In the range B: \( 4(g - 1) \leq \deg(2b) \leq 5(g - 1) \); Proposition 3.3 implies that the linear system \( |2(b - K)| \) consists of a unique divisor formed by different points, and \( b \) cannot be a generic nonspecial divisor. Moreover, if \( a := \deg(2(b - K)) \), let \( Z_a = \{ b \in X^b / P(\mathcal{E}_b) \) is canonical \} \subset X^b \). For \( a < g - 1 \), if \( b \in Z_a \) is generic, \( 2b - 2K \sim P_1 + \ldots + P_a \), where all \( P_i \) are different. Then, \( P(\mathcal{E}_b) \) is the elementary transformation of \( P(\mathcal{E}_{b + P}) \) but, since \( 2b + 2P - 2K \sim P_1 + \ldots + P_a + 2P \), \( P(\mathcal{E}_{b + P}) \) is not canonical. That is, in range B, the generic component corresponding to canonical geometrically ruled surfaces in case \( \deg(b) \) does not dominate the generic component in the case \( \deg(b) - 1 \).

This result makes interesting to classify canonical geometrically ruled surfaces in range B. By the existence theorem this classification is equivalent to the existence of nonspecial curves that are not projectively generic, see Remark 5.1. We hope to study their geometrical characterization and classification in a future paper.

Finally, in §6 we study the analogous of Noether Theorem about the projective normality of the canonical scrolls. Theorem 6.11 says that the canonical scroll \( R \) is projectively normal iff the directrix curves \( X_0 \) and \( X_1 \) are projectively normal. Moreover, the speciality respect to hypersurfaces of degree \( m \) of \( R \) is the sum of the corresponding specialities respect to hypersurfaces of degree \( m \) of the two minimal directrix curves \( X_0 \) and \( X_1 \) (except when \( m \geq 3 \), and \( g = 2 \), \( \deg(b) = 3 \) or \( g = 3 \), \( \deg(b) = 4 \)).

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Therefore, in the hyperelliptic case, the canonical scroll is not projectively normal because Noether Theorem for canonical curves; but in the nonhyperelliptic case is projectively normal. In particular, if deg(b) ≥ 2g + 1, Castelnuovo-Mumford Lemma allows to assert that the canonical scroll is projectively normal. In cases deg(b) = 2g + 1 − k, k = 1, 2, 3, the results due to Green-Lazarsfeld [5] conclude Theorem 6.16 about the projective normality of the canonical scroll.

2 Double covers of a smooth variety

Definition 2.1 Let C, X be smooth varieties. Let γ : C → X be a surjective finite 2 : 1 map. We say that γ is a double cover of X.

Let E be a locally free sheaf of rank 2 over a smooth variety X. Let π : V = P(E) → X be the corresponding ruled variety and H the hyperplane section such that O(1) ≅ O_V(H). The fibers of π are isomorphic to P^1. We say that a divisor C on V is a bisecant divisor if C ∼ π^(-1)(x) = 2.

Proposition 2.2 Let j : C → V be a nonsingular bisecant irreducible divisor in V. Let γ : C → X be the map γ = p ∘ j. Then E ≅ γ_*O_C(H), j^*O_V(H) ≅ O_C(H) and H^i(E) ≅ H^i(O_C(H)), for i ≥ 0.

Proof. Consider the exact sequence

0 → O_V(H - C) → O_V(H) → j_*O_C(H) → 0.

Taking π*, we have

0 → π_*O_V(H - C) → π_*O_V(H) → π_*j_*O_C(H) → R^1π_*O_V(H - C).

Let x ∈ X and let V_x ≅ P^1 be the fiber of π over x. Then

h^i(V_x, O_V(H - C)|_{V_x}) = h^i(O_{P^1}(-1)) = 0 when i ≥ 0.

By the Theorem of Grauert ([6], III, 12.9), R^iπ_*O_V(H - C) = 0 when i ≥ 0 and then

E ≅ π_*O_V(H) ≅ π_*j_*O_C(H) = γ_*O_C(H).

Furthermore, the fiber of γ has dimension 0 so h^i(C_x, O_C(H)) = 0 when i > 0 and R^iγ_*O_C(H) = 0. By ([6], III, Ex 8.1), it follows that H^i(O_C(H)) ≅ H^i(γ_*O_C(H)) for all i ≥ 0.

Remark 2.3 It is well-known that to construct a smooth double cover γ : C → X is equivalent to give a divisor E in X such that -2E is linearly equivalent to a smooth divisor B corresponding to the branch divisor. Then γ_*O_C(H) ≅ O_X ⊕ O_X(E). The decomposable ruled variety P(O_X ⊕ O_X(E)) has two canonical sections corresponding to the surjections

O_X ⊕ O_X(E) → O_X(E) → 0, O_X ⊕ O_X(E) → O_X → 0.

We will denote them by X_0 and X_1 respectively. Moreover, X_1 ∼ X_0 − π^*E. In the following results we will give a detailed description of this construction.

Proposition 2.4 Let γ : C → X be a double cover of X. Let E_0 = γ_*O_C, then:

1. E_0 is a locally free sheaf on X of rank 2. From this, π : P(E_0) → X is a geometrically ruled variety.
2. E_0 is decomposable. Moreover, E_0 ≅ O_X ⊕ O_X(E)/E ∈ Div(X).
3. There is a closed immersion j : C → P(E_0) verifying π ∘ j = γ.

Proof. For 1. and 2. see [10]. We have the natural surjective morphism γ^*γ_*O_C → O_C. By Proposition 7.2.6, this is equivalent to have a map j : C → P(E_0) verifying π ∘ j = γ. We have to prove that it is a closed immersion. It is sufficient to check it in each fibre. But, the fibres are the two points over each point of X and O_C is very ample on these points.
**Theorem 2.5** Let $\gamma : C \to X$ be a double cover. Let $E \cong \gamma_\ast \omega_C$. Then:

1. $E \cong \omega_X \oplus \mathcal{O}_X(B) \cong \gamma_\ast \mathcal{O}_C \oplus \mathcal{O}_X(B)$, with $B \sim K_X - E$. From this, $\mathbb{P}(E) \cong \mathbb{P}(\mathcal{E}_0)$.

2. $C \sim 2X_1$ in $\mathbb{P}(E)$.

3. $K_C \sim \gamma^\ast B$ and $E$ is a divisor verifying $-2E \sim \beta$, where $\beta$ is the branch divisor of $\gamma$.

**Proof.** See [10].

Let $\pi : V = \mathbb{P}(\mathcal{E}) \to X$ be a decomposable ruled variety over a smooth variety $X$, s.t. $\mathcal{E} \cong \mathcal{O}_X \oplus \mathcal{O}_X(E)$, with $E \not\sim 0$. Consider the order 2 automorphism of $\mathcal{E}$ induced by $Id$ in the first factor and $-Id$ in the second factor. This induces a nontrivial involution $\varphi : V \to V$ on the ruled surface. The unique unisecant divisors invariant by $\varphi$ are $X_0$ and $X_1$. Moreover the involution acts on the line bundles of $V$. They can be decomposed into the +1 and -1 eigenspaces. In particular:

**Lemma 2.6** Let $D \sim X_0 + \pi^\ast A$ be an unisecant divisor on $V$. Then $|D|$ has exactly two spaces of base points by the involution $\varphi|_D$:

$W_0 = \{X_0 + \pi^\ast C/C \sim A\}, \quad W_1 = \{X_1 + \pi^\ast C/C \sim A + E\}$.

**Lemma 2.7** The linear system $|2X_1|$ has exactly two spaces of fixed divisors by the involution $\varphi$:

$F_0 = \langle F_0', 2X_1 \rangle$, where $F_0' = \{2X_0 + \pi^\ast \beta/\beta \sim -2E\}$,

$F_1 = \{X_0 + X_1 + \pi^\ast C/C \sim -E\}$.

**Lemma 2.8** Let $\beta$ be a divisor of $X_1$ such that $\beta \sim -2E$. Let $L$ be the pencil $\langle 2X_0 + \pi^\ast \beta, 2X_1 \rangle$. If the generic divisor of this pencil is smooth then it is irreducible.

**Proof.** It is sufficient to see that every divisor $C$ in $|2X_1|$ is connected. Consider the exact sequence:

$0 \longrightarrow H^0(\mathcal{O}_V(-2X_1)) \longrightarrow H^0(\mathcal{O}_V) \longrightarrow H^0(\mathcal{O}_C) \longrightarrow H^1(\mathcal{O}_V(-2X_1))$

where $H^0(\mathcal{O}_V(-2X_1)) = 0$ and by duality $H^1(\mathcal{O}_V(-2X_1)) \cong H^{\dim(X)}(\mathcal{O}_V(\pi^\ast(K_X - E))) \cong H^0(\mathcal{O}_X(E))$. This is 0 because $E \not\sim 0$. From this $h^0(\mathcal{O}_C) = h^0(\mathcal{O}_V) = 1$ and $C$ is connected.

**Theorem 2.9** Let $\beta$ be a divisor of $X_1$ such that $\beta \sim -2E$. Then, there is a pencil of bisecant divisors in $|2X_1|$ which are invariant by the involution and meet $X_1$ at $\beta$. Moreover the generic divisor of the pencil is irreducible and smooth if and only if $\beta$ is smooth.

**Proof.** By Lemma 2.7, we only have to prove that the generic curve of the pencil $L = \langle 2X_0 + \pi^\ast \beta, 2X_1 \rangle$ is irreducible and smooth.

By Bertini’s Theorem the generic element of the pencil is smooth away from the base locus. The base locus of the pencil are the points of the divisor $\beta$ on $X_1$. Let $D$ be the generic divisor of $L$. We saw that if $D$ is smooth then it is irreducible.

If $D$ has a singular point, then $D \cdot X_1 = \beta$ must have a singular point.

On the contrary, suppose that $D$ is smooth, and let $x \in \beta$ be a base point of the pencil. If we consider the trace of the pencil over the generator $\pi^{-1}(x)$, we see that any divisor of the pencil meets $\pi^{-1}(x)$ at $x$ with multiplicity 2. Since $D$ is smooth, $D$ is tangent to the generator $\pi^{-1}(x)$ and it meets $X_1$ transversally. From this, $x$ is a smooth point of $R$.

**Remark 2.10** We have excluded the case $E \sim 0$. Consider the ruled variety $\mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X)$. The linear system $|X_1| = |X_0|$ is a pencil of irreducible unisecant varieties isomorphic to $X$. Then, all the divisors of the linear system $|2X_1|$ are reducible pairs $D_1 + D_2$ with $D_i \sim X_0$. In fact this case corresponds to the trivial cover of $X$ by two copies of itself: $\gamma : X \times X \to X$.

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3 Canonical scrolls and canonical geometrically ruled surfaces

Let $\pi : \mathbb{P}(E_0) \rightarrow X$ be a geometrically ruled surface over a smooth curve $X$ of genus $g \geq 0$. Let $H \sim X_0 + bf$ be a unisecant divisor such that the complete linear system $|H|$ provides a birational morphism $\varphi_H : \mathbb{P}(E_0) \rightarrow S \subset \mathbb{P}^n$, $\dim(|H|) = n$. The study of the scroll $S$ is equivalent to study the polarized geometrically ruled surface $(\mathbb{P}(E_0), \mathcal{O}_{\mathbb{P}(E_0)}(H))$ and equivalent to the study of the locally free sheaf $\mathcal{E} \cong \pi_*\mathcal{O}_{\mathbb{P}(E_0)}(H)$ over $X$.

Applying Proposition 2.2 to the case of curves we recover the following result due to Corrado Segre and mentioned in the introduction.

**Theorem 3.1** Let $(\mathbb{P}(E_0), \mathcal{O}_{\mathbb{P}(E_0)}(H))$ be the polarized geometrically ruled surface $(\mathbb{P}(E_0), \mathcal{O}_{\mathbb{P}(E_0)}(H))$. If $C \sim 2X_0 + bf$ is a bisecant curve, then

$$H^0(\mathcal{O}_{\mathbb{P}(E_0)}(H)) \simeq H^0(\mathcal{O}_C(H)), \quad H^1(\mathcal{O}_{\mathbb{P}(E_0)}(H)) \simeq H^1(\mathcal{O}_C(H)).$$

By using this result we can give the following definition.

**Definition 3.2** Let $X$ be a smooth curve of genus $g \geq 0$ and let $C$ be a nonhyperelliptic smooth curve of genus $\pi \geq 4$. Let $C_K \subset \mathbb{P}^{\pi-1}$ be the canonical model of $C$. Suppose that $C$ has an involution over $X$. Then we call canonical scroll the variety obtained by joining the points of $C_K$ related by the involution.

In a similar way, from the results obtained in the section above we can give the following definition.

**Proposition 3.3** Given a smooth curve $X$ of genus $g$ and a geometrically ruled surface $\pi : \mathbb{P}(E) \rightarrow X$ the following conditions are equivalent:

1. There is a smooth irreducible bisecant curve on $\mathbb{P}(E)$, $j : C \rightarrow \mathbb{P}(E)$, such that $\mathbb{P}(E) \cong \mathbb{P}(\gamma_*\mathcal{O}_C)$, where $\gamma = \pi \circ j$.
2. There is a smooth irreducible curve $C$ and a double cover $\gamma : C \rightarrow X$ such that $\mathbb{P}(E) \cong \mathbb{P}(\gamma_*\mathcal{O}_C)$.
3. $\mathbb{P}(E) \cong \mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(K - b))$ where $b$ is a nonspecial divisor of degree $\deg(b) \geq 2g - 2$ and the generic element $C \in [2X_1]$ is smooth.
4. $\mathbb{P}(E) \cong \mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(K - b))$ where $b$ is a nonspecial divisor of degree $\deg(b) \geq 2g - 2$ and $2(b - K)$ is a smooth divisor.

**Definition 3.4** A geometrically ruled surfaced verifying any of these conditions is called a canonical geometrically ruled surface. We will denote it by $S_b = \mathbb{P}(E_0)$ where $E_0 \cong \mathcal{O}_X \oplus \mathcal{O}_X(K - b)$ and $b$ is a nonspecial divisor of degree $\deg(b) \geq 2g - 2$ verifying that $2(b - K)$ is a smooth divisor.

Thus, a canonical scroll $R$ is the image by the complete linear system $|H| = |X_0 + bf|$ of a canonical geometrically ruled surface $\mathbb{P}(E_b)$, where the bisecant curve $C \in [2X_1]$ is not hyperelliptic. In the next section, we will see that in this case the linear system $|H|$ is base-point-free and it defines a birational map. From this we have the following geometrical description of a canonical scroll:

**Theorem 3.5** Let $R \subset \mathbb{P}^{\pi-1}$ be a canonical scroll of genus $g \geq 0$. $R$ is generated by a correspondence between a canonical curve of genus $g$ and a nonspecial curve of genus $g$. They are linearly normal in disjoint spaces that generate $\mathbb{P}^{\pi-1}$. Let $\mathcal{O}_X(K)$ and $\mathcal{O}_X(b)$ the invertible sheaves of these curves. Then $\deg(b) \geq 2g - 2$.

Consider the normalized geometrically ruled surface $\mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(\epsilon))$ where $\epsilon \sim K - b$. Let $X_0$ and $X_1 \sim X_0 - \epsilon f$ be the minimal sections. $R$ is the image of

$$\phi_H : \mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(\epsilon)) \longrightarrow R \subset \mathbb{P}^{\pi-1}$$

where $H \sim X_0 + bf \sim X_1 + Kf$. Moreover, the restriction maps to the sections $X_0$ and $X_1$ are the morphisms defined by $\mathcal{O}_X(K)$ and $\mathcal{O}_X(b)$ respectively.

The support of the singular locus is at most $X_0 \cup X_1$. Given the involution $\gamma : C_K \rightarrow X$, the branch divisor $\beta$ satisfies: $\beta \sim 2(b - K)$ and $\beta \sim 2X_1$.

C. Segre gives in [11] a condition over the degree of an special scroll to have a special directrix curve. Now, we can see that the condition over the degree is not necessary.

**Theorem 3.6** A linearly normal special scroll $R \subset \mathbb{P}^N$ is the projection of a canonical scroll.

**Proof.** Suppose that $R$ is defined by the ruled surface $S$ and the unisecant linear system $|H|$ on $S$. We can take a smooth bisecant curve $C$ on $S$, verifying that the linear system $H \cap C$ on $C$ is very ample (for example

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we can take $C \in |2H|$. Therefore, we have a double cover $\gamma : C \rightarrow X$. Let $\overline{C}$ be the image of $C$ by the map defined by the linear system $|H|$ on $C$.

The speciality of $R$ is, by definition, $h^1(\mathcal{O}_S(H))$ and by Theorem 2.2, it coincides with the speciality of $\overline{C}$. Moreover, the curve $C$ is not hyperelliptic, because an special divisor over an hyperelliptic curve is never very ample. Thus, $\overline{C}$ is the projection of a canonical curve $C_K$ of genus $g$. The secant lines of $C_K$ joining points $P$ and $\gamma(P)$ are projected into the lines of $R$. From this, the scroll $R$ is the projection of the canonical scroll $R_b$ defined by $\gamma$ over $C_K$.

**Corollary 3.7** A linearly normal special scroll $R \subset \mathbb{P}^N$ always has an special directrix curve.

**Proof.** We saw that a canonical scroll $R_b$ has a canonical special directrix curve. This curve goes to an special curve for any projection of $R_b$. Now, it is sufficient to apply the above theorem. 

Finally, we see how a ruled surface is transformed by projecting from a point of a bisecant curve.

**Proposition 3.8** Let $\pi : S \rightarrow X$ be a ruled surface. Let $C \subset S$ be a bisecant smooth curve and $\gamma : C \rightarrow X$ be the corresponding double cover. Let $b$ be a divisor on $C$ and $H$ be an unisecant divisor on $S$ such that $\gamma_*\mathcal{O}_C(a) \cong \pi_*\mathcal{O}_S(H)$, or equivalently, $\mathcal{O}_C(H) \cong \mathcal{O}_C(a)$. Then,

1. If $b$ is a divisor on $X$ it holds:
   \[ \gamma_*\mathcal{O}_C(a + \gamma^*b) \cong \pi_*\mathcal{O}_S(H + bf). \]

2. If $x$ is a point of $C$ such that $\gamma(x) = P$, it holds:
   \[ \gamma_*\mathcal{O}_C(a - x) \cong \pi'_*\mathcal{O}_S(\nu^*(H) - Pf), \]
   where $\nu : S' \rightarrow S$ is the elementary transform of $S$ in $x$.

3. If $x$ is a point of $C$, such that $\gamma(x) = P$, it holds:
   \[ \gamma_*\mathcal{O}_C(a + x) \cong \pi'_*\mathcal{O}_S(\nu^*(H)) , \]
   where $\nu : S' \rightarrow S$ is the elementary transform of $S$ in $\gamma^*(P) - x$.

**Proof.** 1. Let $b$ be a divisor on $X$. We use the projection formula:
   \[ \gamma_*\mathcal{O}_C(a + \gamma^*b) \cong \gamma_*\mathcal{O}_C(a) \otimes \gamma^*\mathcal{O}_X(b) \cong \gamma_*\mathcal{O}_C(a) \otimes \mathcal{O}_X(b) \cong \pi_*\mathcal{O}_S(H + bf). \]

2. Let $x$ be a point of $C$ such that $\gamma(x) = P$. Consider the exact sequence:
   \[ 0 \rightarrow \mathcal{O}_C(a - x) \rightarrow \mathcal{O}_C(a) \rightarrow \mathcal{O}_x \rightarrow 0. \]
   Applying $\gamma_*$ and because $\mathcal{R}^1\gamma_*\mathcal{O}_C(a - x) = 0$, we obtain
   \[ 0 \rightarrow \gamma_*\mathcal{O}_C(a - x) \rightarrow \gamma_*\mathcal{O}_C(a) \rightarrow \mathcal{O}_P \rightarrow 0. \]
   In this way, we see that $\mathbf{P}(\gamma_*\mathcal{O}_C(a - x)) = S'$ is the elementary transform of $S$ in the point $x$. Moreover, if we denote by $\nu$ the elementary transformation,
   \[ \gamma_*\mathcal{O}_C(a - x) \cong \pi'_*\mathcal{O}_S(\nu^*(H) - Pf). \]

3. Let $x$ be a point of $C$ such that $\gamma(x) = P$. Let $y = \gamma^*(P) - x$. We have that
   \[ \gamma_*\mathcal{O}_C(a + x) \cong \gamma_*\mathcal{O}_C(a + \gamma^*(P) - y). \]
   But, applying the case 1 of this proposition, it follows that
   \[ \gamma_*\mathcal{O}_C(a + \gamma^*(P)) \cong \pi_*\mathcal{O}_S(H + Pf), \]
   and now, by using the above case, if $\nu : S' \rightarrow S$ the elementary transformation of $S$ at the point $y$, we see that:
   \[ \gamma_*\mathcal{O}_C(a + \gamma^*(P) - y) \cong \pi'_*\mathcal{O}_S(\nu^*(H + Pf) - Pf) \cong \pi'_*\mathcal{O}_S(\nu^*(H)). \]

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4 The elliptic and the hyperelliptic cases

Proposition 4.1 Let \( C \) be an hyperelliptic curve of genus \( \pi \geq 2 \) and let \( \gamma : C \to X \) be an involution of genus \( g \geq 1 \). Then \( X \) is elliptic or hyperelliptic. Moreover, the \( g_2^1 \) of \( X \) makes the following diagram commutative

\[
\begin{array}{ccc}
C & \xrightarrow{\pi_2^1} & \mathbb{P}^1 \\
\downarrow{\gamma} & & \downarrow{\varphi} \\
X & \xrightarrow{g_2^1} & \mathbb{P}^1
\end{array}
\]

where \( \varphi \) is a \( 2:1 \) morphism and \( \pi_2^1 \) is the pencil of degree \( 2 \) of \( C \).

Proof. Let \( \mu : C \to C \) be the automorphism defined by the involution \( \gamma \). Because \( C \) is hyperelliptic, it has a unique \( \pi_2^1 \). Therefore, there exist an isomorphism \( \varphi : \mathbb{P}^1 \to \mathbb{P}^1 \) verifying \( \pi_2^1 \circ \mu = \varphi \circ \pi_2^1 \).

\( \varphi \) induces a \( 2:1 \) morphism \( \varphi : \mathbb{P}^1 \to \mathbb{P}^1 \) parameterizing the points related by \( \varphi \), that is, \( \varphi(x) = \varphi(y) \Leftrightarrow \varphi(x) = \varphi(y) \Rightarrow \varphi(x) = \varphi(y) \).

Then, we have

\[
\begin{array}{ccc}
C & \xrightarrow{\pi_2^1} & \mathbb{P}^1 \\
\downarrow{\gamma} & & \downarrow{\varphi} \\
X & \xrightarrow{g_2^1} & \mathbb{P}^1
\end{array}
\]

where \( g_2^1(P) = \varphi(\pi_2^1(\gamma^{-1}(P))) \) and it is well defined, because

\[
\gamma^{-1}(P) = \{Q, \mu(Q)\} \quad \text{and} \quad \varphi(\pi_2^1(Q)) = \pi_2^1(\mu(Q)) \Rightarrow \varphi(\pi_2^1(\mu(Q))) = \varphi(\pi_2^1(Q)).
\]

Thus \( X \) has a \( g_2^1 \) and is elliptic or hyperelliptic.

Corollary 4.2 If \( X \) is neither elliptic nor hyperelliptic the linear system \( H \sim X_0 + bf \) is base-point-free and it defines a birational map.

Proof. Because \( |K| \) is very ample, it is sufficient to see that the linear system \( |b| \) is base-point-free:

1. If \( \deg(b) \geq 2g \), the divisor \( b \) is always base-point-free.
2. If \( \deg(b) = 2g - 1 \) and \( b \) has a base point, then \( b \sim K + P \). Since \( P(\mathcal{E}_b) \) is a canonical ruled surface, \( 2(b - K) \sim 2P \) is linearly equivalent to two different points. Then \( X \) has a \( g_2^1 \), but this is false by hypothesis.
3. If \( \deg(b) = 2g - 2 \) and \( b \) has a base point, then \( b - P \sim K - Q \). But \( 2(b - K) \sim 2P - 2Q \) must be linearly equivalent to divisor 0. Then \( X \) has a \( g_2^1 \), and this is false by hypothesis.

Proposition 4.3 A non-hyperelliptic curve \( C \) of genus \( \pi = 3 \) does not have involutions of genus 2.

Proof. Let \( \gamma : C \to X \) be an involution of genus \( 2 \) of \( C \). We know that \( C \in |2X_1| \subset S \) with \( S = P(\mathcal{E}_b) \) and \( b \) is a nonspecial divisor of degree 2. Thus, if \( b = P + Q \) then \( h^0(\mathcal{O}_X(b)) = 1 = h^0(\mathcal{O}_X(b - P)) \). From this it follows

\[
h^0(\mathcal{O}_C(K - \gamma^*(P))) = h^0(\mathcal{O}_S(H - P f)) = h^0(\mathcal{O}_S(H)) - 1 = h^0(\mathcal{O}_C(K)) - 1
\]

and so the canonical divisor of \( C \) is not very ample and \( C \) is hyperelliptic.

Proposition 4.4 Let \( X \) be an elliptic curve and \( S = P(\mathcal{E}_b) \) be a canonical ruled surface with \( \deg(b) = 2 \). Let \( C \) be a smooth irreducible curve of genus 3. We have a double cover \( \gamma : C \to X \) with branch divisor \( B = a_1 + a_2 + a_3 + a_4 \). Then \( C \) is hyperelliptic if and only if \( a_1 + a_2 \sim a_3 + a_4 \sim b \).

Proof. Let \( R = A_1 + A_2 + A_3 + A_4 \) be the ramification divisor such that \( \gamma(A_i) = a_i \). We have that \( K_C \sim \gamma^*K_X + R \sim R \).
Suppose that $C$ is hyperelliptic. Then we have that $A_1 + A_2 \sim A_3 + A_4 \sim \pi_2^1$. Applying $\gamma$ we see that $a_1 + a_2 \sim a_3 + a_4 \sim \gamma \pi_2^1$. Moreover,

\[
\begin{align*}
 h^0(\mathcal{O}_S(H - \gamma \pi_2^1 f)) &= h^0(\mathcal{O}_C(K_C - \gamma \pi_2^1)) = 1, \\
 h^0(\mathcal{O}_S(H - \gamma \pi_2^1 f)) &= h^0(\mathcal{O}_X(-\gamma \pi_2^1)) + h^0(\mathcal{O}_X(b - \gamma \pi_2^1)),
\end{align*}
\]

and it follows that $\gamma \pi_2^1 \sim b$.

On the contrary suppose that $a_1 + a_2 \sim a_3 + a_4 \sim b$. Then $2A_1 + 2A_2 \sim \gamma^*(a_1 + a_2) \sim \gamma^*b \sim K_C \sim A_1 + A_2 + A_3 + A_4$. From this $A_3 + A_4 \sim A_1 + A_2$ and $C$ has a $\pi_2^1$.

**Proposition 4.5** Let $S = \mathcal{P}(\mathcal{E}_b)$ be a canonical ruled surface and let $C \sim 2X_1$ be a smooth curve. Then $C$ is hyperelliptic if and only if:
1. $X$ is elliptic and $\deg(b) = 1$ or $\deg(b) = 2$ and the branch divisor $\beta = a_1 + a_2 + a_3 + a_4$ verifies $a_1 + a_2 \sim a_3 + a_4 \sim b$.
2. $X$ is hyperelliptic and $h^0(\mathcal{O}_X(b - g_2^1)) = h^0(\mathcal{O}_X(b)) - 1$.

Moreover, $C$ is elliptic if and only if $X$ is elliptic and $\deg(b) = 0$.

**Proof.** Let us suppose that $C$ is hyperelliptic. By Proposition 4.1 we know that $X$ is elliptic or hyperelliptic and we have that $\gamma^*(g_2^1) \sim 2\pi_2^1$. From this it follows

$$
\mathcal{O}_C(H - g_2^1 f) \sim \mathcal{O}_C(K_C - 2\pi_2^1).
$$

By Theorem 3.1 we have

$$
h^0(\mathcal{O}_S(H - g_2^1 f)) = h^0(\mathcal{O}_C(K_C - 2\pi_2^1)) = h^0(\mathcal{O}_C(K_C) - 2 = h^0(\mathcal{O}_S(H)) - 2.
$$

But,

\[
\begin{align*}
 h^0(\mathcal{O}_S(H - g_2^1 f)) &= h^0(\mathcal{O}_X(b - g_2^1)) + h^0(\mathcal{O}_X(K - g_2^1)) \\
 &= h^0(\mathcal{O}_X(b - g_2^1)) + h^0(\mathcal{O}_X(K)) - 1, \\
 h^0(\mathcal{O}_S(H)) &= h^0(\mathcal{O}_X(b)) + h^0(\mathcal{O}_X(K))
\end{align*}
\]

and we obtain $h^0(\mathcal{O}_X(b - g_2^1)) = h^0(\mathcal{O}_X(b)) - 1$. If $X$ is elliptic this happens when $\deg(b) = 1, 2$. Moreover, if $\deg(b) = 2$ the conditions of Proposition 4.3 hold.

Conversely, if we suppose that $X$ is hyperelliptic and $h^0(\mathcal{O}_X(b - g_2^1)) = h^0(\mathcal{O}_X(b)) - 1$, we can check that

$$
h^0(\mathcal{O}_C(K_C - \gamma^*(g_2^1))) = h^0(\mathcal{O}_S(H - g_2^1 f)) = h^0(\mathcal{O}_S(H)) - 2 = h^0(\mathcal{O}_C(K_C)) - 2.
$$

Thus, $C$ has a $g_2^1$. If the genus of $C$ is $\pi \geq 4$ then $C$ is hyperelliptic. Moreover, we have seen that a curve of genus $3$ with an involution of genus $2$ is hyperelliptic (Proposition 4.3).

If $X$ is elliptic and $\deg(b) = 1$ then the genus $\pi$ of $C$ is $2$ and $C$ is hyperelliptic. If $\deg(b) = 2$ we apply Lemma 4.3.

Finally, by Hurwitz’s formula, $C$ is elliptic if and only if $X$ is elliptic and $\deg(b) = 0$.

**Theorem 4.6** Let $S = \mathcal{P}(\mathcal{E}_b)$ be a canonical ruled surface and let $C \sim 2X_1$ be a smooth curve. Then $C$ is hyperelliptic if and only if:

1. $X$ is hyperelliptic and one of the following conditions holds:
   (a) $b \sim K + g_2^1$.
   (b) $b \sim K + P$, with $2P \sim g_2^1$.
   (c) $b \sim \sum_i g_i^2 - 2 g_1^1 + P + Q$, with $2P \sim 2Q \sim g_1^1$.

2. $X$ is elliptic and $\deg(b) = 1$ or $\deg(b) = 2$ and the branch divisor $\beta = a_1 + a_2 + a_3 + a_4$ verifies $a_1 + a_2 \sim a_3 + a_4 \sim b$.

Moreover, the cases (b), (c) or $\deg(b) = 0$, 1 are the unique cases where $b$ has base-points.
Proof. Suppose that $C$ is hyperelliptic. We apply Proposition 4.5. Let us study the hyperelliptic case. If $\deg(b) \geq 2g + 1$ then $h^0(\mathcal{O}_X(b - g^1_2)) = h^0(\mathcal{O}_X(b)) - 2$. It remains to check the following cases:

1. If $\deg(b) = 2g$ and $h^0(\mathcal{O}_X(b - g^1_2)) = h^0(\mathcal{O}_X(b)) - 1$ then $b - g^1_2$ is special of degree $2g - 2$, that is, $b \sim K + g^1_2$.

2. If $\deg(b) = 2g - 1$ and $h^0(\mathcal{O}_X(b - g^1_2)) = h^0(\mathcal{O}_X(b)) - 1$ then $b - g^1_2$ is special of degree $2g - 3$, that is, $b - g^1_2 \sim K - P'$, so $b \sim K + g^1_2 - P' \sim K + P$ with $P + P' \sim g^1_2$. Since $b$ defines a canonical ruled surface, $2(b - K) \sim 2P'$ must be linearly equivalent to two different points, so $2P' \sim g^1_2$.

3. If $\deg(b) = 2g - 2$ and $h^0(\mathcal{O}_X(b - g^1_2)) = h^0(\mathcal{O}_X(b)) - 1$ then $b - g^1_2$ has speciality $1$ and degree $2g - 4$, that is, $b - g^1_2 \sim K - P - Q$ with $P + Q \neq g^1_2$. Because $2(b - K)$ must be effective, $2P + 2Q \sim 2g^1_2$ or $2P \sim 2Q \sim g^1_2$ and from this $b \sim \sum_{i=1}^{g^1 - 2} g^1_2 + P + Q$.

Conversely, if $X$ is hyperelliptic and one of the three conditions holds, then $h^0(\mathcal{O}_X(b - g^1_2)) = h^0(\mathcal{O}_X(b)) - 1$ and by Proposition 4.5 the curve $C$ is hyperelliptic.

Finally, let us suppose $b$ has a base point. Then $\deg(b) \leq 2g - 1$ and by Corollary 4.2, $X$ is elliptic or hyperelliptic.

Corollary 4.7 Let $S = \mathcal{P}(\mathcal{E}_b)$ be a canonical ruled surface and $H \sim X_0 + b f$. The linear system $H$ defines a canonical scroll except when:

1. $X$ is hyperelliptic and one of the following conditions holds:
   - (a) $b \sim K + g^1_2$.
   - (b) $b \sim K + P$, with $2P \sim g^1_2$.
   - (c) $b \sim \sum_{i=1}^{g - 2} g^1_2 + P + Q$, with $2P \sim 2Q \sim g^1_2$.

2. $X$ is elliptic and $\deg(b) = 0, 1, 2$.

Corollary 4.8 The unique possible involutions of an hyperelliptic curve of genus $\pi$ are of genus $\frac{\pi - 1}{2}, \frac{\pi}{2}$, or $\frac{\pi + 1}{2}$.

Corollary 4.9 If $\mathcal{P}(\mathcal{E}_b)$ is a canonical ruled surface and $H \sim X_0 + b f$ defines a canonical scroll then $|H|$ is base-point-free and it defines a birational map.

Proof. If $X$ is nonhyperelliptic it is Corollary 4.2.

If $X$ elliptic or hyperelliptic and $b$ defines a canonical scroll. We saw at Proposition 4.5 that $h^0(\mathcal{O}_X(b - g^1_2)) = h^0(\mathcal{O}_X(b)) - 2$. $|K|$ and $|b|$ are base-point-free. Moreover, $|K - P|$ and $|b - P|$ don’t have common base points for all $P \in X$. From this $|H|$ is base-point-free and it defines a birational map.

We finish this section by studying the map defined by the linear system $|b|$ when $X$ is hyperelliptic.

Proposition 4.10 Let $b$ be a nonspecial divisor of degree $b$ over an hyperelliptic curve $X$ of genus $g$, with $g \geq 2$ and $b \geq 2g - 2$. Suppose that $b$ defines a canonical scroll. Let $\phi_b : X \to \mathcal{P}^{b - g}$ be the map defined by the linear system $|b|$. We have:

1. $\phi_b$ is birational except when $g = 2$ and $b = 3$ or $g = 3$ and $b = 4$.

2. $\phi_b$ is an isomorphism (or a very ample) except when:
   - (a) $b = 2g$ and
     - i. $g = 2, 3$ and $b \sim K + P_1 + P_2$.
     - ii. $g > 3$ and $b \sim K + P_1 + P_2$, with $P_1, P_2$ ramification points of the $g^1_2$.
   - (b) $b = 2g - 1$ and
     - i. $g = 2, 3$, or
     - ii. $g > 3$ and $b \sim \sum_{i=1}^{g - 2} g^1_2 + P_1 + P_2 + P_3$, with $P_1, P_2, P_3$ ramification points of the $g^1_2$. $\phi_b$ has a triple point which is its unique singular point.
   - (c) $b = 2g - 2$ and
     - i. $g = 3$, or
     - ii. $g > 3$ and $b \sim \sum_{i=1}^{g - 3} g^1_2 + P_1 + P_2 + P_3 + P_4$, with $P_1, P_2, P_3, P_4$ ramification points of $g^1_2$. $\phi_b$ has a quadruple point which is its unique singular point.
Proof. By Riemann-Roch Theorem we know that b is very ample when \( b \geq 2g + 1 \). Moreover, \( \phi_b \) fails to be an isomorphism if there are two points \( P_1, P_2 \) satisfying \( b - P_1 - P_2 \) is an especial divisor with speciality 1.

If \( b = 2g \) then \( \deg(b - P_1 - P_2) = 2g - 2 \). From this, if \( b - P_1 - P_2 \) is special, \( b \sim K + P_1 + P_2 \). Because \( b \) defines a canonical ruled surface \( P_1 + P_2 \not\sim g_4^1 \) and \( 2b + 2K \) is linearly equivalent to four different points. If \( g = 2, 3 \) this always happens. If \( g > 3 \) the unique \( g_1^1 \) or \( g_3^1 \) are composed of \( g_2^2 \). Since \( P_1 + P_2 \not\sim g_1^1 \), necessary \( 2P_1 \sim 2P_2 \not\sim g_1^1 \).

If \( b = 2g - 1 \) then \( \deg(b - P_1 - P_2) = 2g - 3 \). Now, if \( b - P_1 - P_2 \) is special, \( b \sim K + P_1 + P_2 - P_3 \). Because \( b \) defines a canonical ruled surface \( P_1 + P_2 \not\sim g_2^1 \) and \( 2b + 2K \) is linearly equivalent to two different points. If \( g = 2, \phi_b \) is a \( 3:1 \) morphism over a line. If \( g = 3, \phi_b \) projects \( X \) into a plane curve of degree 5 and genus 3 so it has a triple singular point. If \( g > 3 \) the unique \( g_1^1 \) or \( g_3^1 \) are composed of \( g_2^1 \). Since \( P_1 + P_2 \not\sim g_2^1 \), necessary \( 2P_1 \sim 2P_2 \sim 2P_3 \sim g_2^1 \), so \( b \sim K + P_1 + P_2 - P_3 \sim \sum_{i=1}^{g-2} g_3^1 + P_1 + P_2 + P_3 \) and \( \phi_b \) has a triple point which is its unique singular point.

If \( b = 2g - 2 \) then \( \deg(b - P_1 - P_2) = 2g - 4 \). If \( b - P_1 - P_2 \) is special with speciality 1, \( b \sim K + P_1 + P_2 - P_3 - P_4 \) with \( P_3 + P_4 \not\sim g_1^1 \). Because \( b \) defines a canonical ruled surface \( P_1 + P_2 \not\sim g_2^1 \) and \( 2b + 2K \) is linearly equivalent to 0. In this case \( g \neq 2 \) because a nonspecial divisor of degree 2 over a curve of genus \( X \) has base points. If \( g = 3 \), then \( \phi_b \) is a \( 4:1 \) morphism over a line. If \( g > 3 \) the unique \( g_1^1 \) or \( g_3^1 \) are composed of \( g_1^2 \). Since \( P_1 + P_2 \not\sim g_1^1 \), necessary \( 2P_1 \sim 2P_2 \sim 2P_3 \sim g_3^2 \), so \( b \sim K + P_1 + P_2 - P_3 - P_4 \sim \sum_{i=1}^{g-3} g_3^1 + P_1 + P_2 + P_3 + P_4 \) and \( \phi_b \) has a quadruple point which is its unique singular point.

5 The existence theorem

Given a smooth curve \( X \) of genus \( g \), let us study the divisors \( b \) defining a canonical ruled surface. We see that \( \mathbb{P}(E_b) \) is a canonical ruled surface if and only if \( 2(b - K) \) is a smooth divisor, that is, if \( P \) is a base point of \( 2(b - K) \), \( P \) is not a base point of \( 2(b - K) - P \). In particular \( 2(b - K) \) is an effective divisor.

Remark 5.1 Note that a necessary condition for \( b \) to define a canonical ruled surface is that the divisor \( 2(b - K) \) must be an effective divisor.

Let \( a = 2b - 2K \) and \( a = \deg(a) \). If \( a \geq g \) this divisor is always an effective divisor.

If \( a < g \), the generic divisor of degree \( a \) is not an effective divisor. In this case, we will see that the generic divisor \( b \) does not define a canonical geometrically ruled surface. Moreover, if \( b \leq 3g - 4 \) the points of \( a \) are contained in a hyperplane section of \( X_b \). If \( a < g \) this inequality holds, so we have \( b \sim a + c \), where \( c \) is an effective divisor. Now, we see that

\[ a \sim 2b - 2K \implies b \sim 2K + a - b \sim 2K - c \]

and

\[ 2K - 2c \sim 2K - (2K - b) \sim 2(b - K) \text{ is effective by hypothesis.} \]

We conclude that the hyperplane sections of \( X_b \) are the residual points of the system of quadrics sections of the canonical curve of genus \( g \) passing through a divisor \( c \), such that the \( c \)'s are sets of contact points of a quadric with the canonical curve.

Since \( h^0(O_X(2K)) = 3(g - 1) \), if \( 2 \deg(c) \geq 3(g - 1) \) the number of conditions imposed by \( 2c \) is greater than the dimension of the linear system \([2K]\), so the points of \( c \) cannot be generic. This happens when \( 2 \deg(b) \leq 5(g - 1) \), so in this range we hope that the generic divisor \( b \) does not define a canonical ruled surface.

Proposition 5.2 Let \( X^a \) be the family of effective divisors of degree \( a \) and let \( U^a \) be the open subset \( \{ a \in X^a \mid a \text{ is smooth} \} \). Then:

1. If \( a \geq 2g - 1 \) then \( U^a = X^a \) and any divisor is base-point-free and nonspecial.
2. If \( g + 1 \leq a \leq 2g - 2 \) then \( U^a \neq \emptyset \) and the generic divisor of \( U^a \) is base-point-free and nonspecial.
3. If \( 0 \leq a \leq g \) then \( U^a \neq \emptyset \) and the generic divisor \( a \) of \( U^a \) is formed by different points and it has speciality \( h^1(a) = g - a \).

Proof. It follows from the semicontinuity of the cohomology and the Riemann-Roch Theorem. \( \square \)

Given a nonspecial divisor \( b \), we will apply the results 3.3, 5.2 to study when it defines a canonical ruled surface. We will denote by \( O_X(b) \) the invertible sheaves of \( \text{Pic}^b(X) \), where \( b \) is a divisor of degree \( b \). Although
we will characterize the invertible sheaf $\mathcal{O}_X(b)$, the condition is given over the divisor $2b$ so we will use the following lemma:

**Lemma 5.3** Let $X$ be a smooth curve of genus $g$ and $b \geq 0$. Consider the map

$$\varphi^b : \text{Pic}^b(X) \longrightarrow \text{Pic}^{2b}(X)$$

$$\mathcal{L} \longrightarrow \mathcal{L}^2$$

Then $\varphi^b$ is a surjection with finite fibre (nontrivial when $g > 0$).

**Proof.** Given two invertible sheaves $\mathcal{L}, \mathcal{L}' \in \text{Pic}^b(X)$, $\varphi^b(\mathcal{L}) = \varphi^b(\mathcal{L}')$ iff $(\mathcal{L}^{-1} \otimes \mathcal{L}')^2 \cong \mathcal{O}_X$, that is, the fibre of $\varphi^b$ is isomorphic to the kernel of $\varphi^0$. But $\ker(\varphi^0) = H^1_\alpha(X, \mathbb{Z}_2)$ which is a finite group (but nontrivial if $g > 0$).

From this lemma and Proposition 5.2 we obtain directly the dimension of the family of invertible sheaves $\mathcal{O}_X(b)$ defining a canonical ruled surface:

**Theorem 5.4** Let $X$ be a smooth nonhyperelliptic curve of genus $g \geq 1$. Let $\text{Pic}^b(X)$ be the family of invertible sheaves $\mathcal{O}_X(b)$ of degree $b \geq 2g - 2$. Then:

1. If $b \geq 3(g - 1) + 1$, any invertible sheaf of degree $b$ defines a canonical ruled surface.
2. If $(3(g - 1) + 1)/2 \leq b \leq 3(g - 1)$, the generic invertible sheaf of degree $b$ defines a canonical ruled surface.
3. If $2(g - 1) \leq b \leq 5(g - 1)/2$, the generic invertible sheaf of degree $b$ does not define a canonical ruled surface, but there is a family of codimension $5(g - 1) + 1 - 2b$ of invertible sheaves of degree $b$ such that the generic element defines a canonical ruled surface.

## 6 Projective normality of the canonical scroll

In this section we will study the projective normality of the canonical scroll. Let us first remember some definitions and results about normality of curves and decomposable ruled surfaces:

**Definition 6.1** Let $V$ be a projective variety. Let $\mathcal{F}_i$, $i = 1, \ldots, s$, be coherent sheaves on $V$. We call $s(\mathcal{F}_1, \ldots, \mathcal{F}_s)$ the cokernel of the map

$$H^0(\mathcal{F}_1) \otimes \ldots \otimes H^0(\mathcal{F}_s) \longrightarrow H^0(\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_s).$$

If $\mathcal{F}_i$ are invertible sheaves $\mathcal{O}_V(D_i)$ where $D_i$ are divisors on $V$, we will write $s(D_1, \ldots, D_s)$.

**Lemma 6.2** If $s(\mathcal{F}_1, \mathcal{F}_2) = 0$, then

$$s(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots, \mathcal{F}_s) = s(\mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{F}_3, \ldots, \mathcal{F}_s).$$

**Proof.** It is sufficient to note that $s(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots, \mathcal{F}_s)$ is the cokernel of the composition

$$H^0(\mathcal{F}_1) \otimes \ldots \otimes H^0(\mathcal{F}_s) \longrightarrow H^0(\mathcal{F}_1 \otimes \mathcal{F}_2) \otimes \ldots \otimes H^0(\mathcal{F}_s) \longrightarrow H^0(\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_s).$$

**Definition 6.3** Let $V$ be a projective variety and let $|H|$ be a complete unisecant base-point-free linear system defining a birational map

$$\phi_H : V \longrightarrow \overline{V} \subset \mathbb{P}^N.$$

We say that $(V, H)$ is *projectively normal* or $\mathcal{O}_V(H)$ is *normally generated* or $\overline{V}$ is *projectively normal* if and only if the natural maps

$$\text{Sym}_k(H^0(\mathcal{O}_V(H))) \longrightarrow H^0(\mathcal{O}_V(kH))$$

are surjective for all $k \geq 1$. 
Remark 6.4 Because $H^0(\mathcal{O}_{P^N}(k)) \cong \text{Sim}_k(H^0(\mathcal{O}_{P^N}(1)))$ and $H^0(\mathcal{O}_{P^N}(1)) \cong H^0(\mathcal{O}_V(H))$, the following diagram is commutative:

\[
\begin{array}{ccc}
0 & \rightarrow & H^0(\mathcal{I}_{\mathcal{T},P^N}(k)) \\
\alpha_1 & & \alpha_2 \\
H^0(\mathcal{I}_{\mathcal{T},P^N}(k)) & \rightarrow & H^0(\mathcal{O}_{P^N}(k)) \\
\alpha_3 & & \alpha_4 \\
& & H^0(\mathcal{O}_V(kH))
\end{array}
\]

Since $\alpha_1$ is surjective, $\text{coker}(\alpha_2) = \text{coker}(\alpha_3)$. Thus, $(V, H)$ is projectively normal if and only if $s(H, k, H) = 0$, for all $k \geq 2$.

Moreover, we have the following formula to compute the hypersurfaces of degree $k$ containing $V$:

$$h^0(\mathcal{I}_{\mathcal{T},P^N}(k)) = h^0(\mathcal{O}_{P^N}(k)) - h^0(\mathcal{O}_V(kH)) + \dim(s(H, k, H)).$$

We will say that $\dim(s(H, k, H))$ is the speciality of $V$ with respect to hypersurfaces of degree $k$.

If the linear system $H$ is very ample, $V$ and $\mathcal{T}$ are isomorphic and

$$H^1(\mathcal{I}_{\mathcal{T},P^N}(k)) \cong s(H, k, H).$$

Lemma 6.5 (Green) Let $a, b$ be effective divisors on $X$. Let $b$ be base-point-free. If $h^1(\mathcal{O}_X(a - b)) \leq h^0(\mathcal{O}_X(b)) - 2$ then $s(a, b) = 0$.

**Proof.** It is a particular case of $H^0$-lemma, [4].

Lemma 6.6 Let $b_1, b_2$ be divisors on $X$. Let $a = a_1 + \ldots + a_d$ be an effective divisor such that

1. $h^0(\mathcal{O}_X(b_1 - a)) = h^0(\mathcal{O}_X(b_1)) - d$,
2. $h^0(\mathcal{O}_X(b_2 - a_i)) = h^0(\mathcal{O}_X(b_2)) - 1$, for $i = 1, \ldots, d$.
3. $s(b_1 - a, b_2) = 0$.

Then, $s(b_1, b_2) = 0$.

**Proof.** See [7].

Lemma 6.7 Let $X$ be a smooth curve, $\mathcal{L}$ be an invertible sheaf on $X$ and $\mathcal{F}$ be a torsion-free $\mathcal{O}_X$-module. Let $s_1$ and $s_2$ be linearly independent sections of $\mathcal{L}$ and denote by $V$ the subspace of $H^0(X, \mathcal{L})$ generated by them. Then the kernel of the cup-product map

$$V \otimes H^0(X, \mathcal{F}) \rightarrow H^0(X, \mathcal{F} \otimes \mathcal{L})$$

is isomorphic to $H^0(X, \mathcal{F} \otimes \mathcal{L}^{-1}(B))$ where $B$ is the base locus of the pencil spanned by $s_1$ and $s_2$.

**Proof.** See [1, p. 126, Base-point-free pencil trick].

Corollary 6.8 Let $X$ be a smooth curve. Let $a$ be a nonspecial divisor such that $|a|$ is a base-point-free pencil. Then $s(a, K) = 0$.

**Proof.** Consider the map

$$\ker(\alpha) \rightarrow H^0(\mathcal{O}_X(a)) \otimes H^0(\mathcal{O}_X(K)) \rightarrow H^0(\mathcal{O}_X(K + a)).$$

By applying Lemma 6.7 to $\mathcal{L} = \mathcal{O}_X(a)$, $V = H^0(X, \mathcal{L})$ and $\mathcal{F} = \mathcal{O}_X(K)$, we obtain that

$$\ker(\alpha) = h^0(\mathcal{O}_X(K - a)) = h^1(\mathcal{O}_X(a)) = 0.$$
Moreover,
\begin{align*}
\dim(H^0(\mathcal{O}_X(a))) \otimes H^0(\mathcal{O}_X(\mathcal{K})) &= 2h^0(\mathcal{O}_X(\mathcal{K})) = 2g
\end{align*}
and
\begin{align*}
h^0(\mathcal{O}_X(\mathcal{K} + a)) &= a + 2g - 2 - g + 1 = 2g.
\end{align*}

Then we see that \(\alpha\) is surjective (in fact, an isomorphism) and the result follows.

Lemma 6.9 Let \(X\) be a smooth curve of genus \(g\). Let \(\mathcal{K}\) be the canonical divisor and let \(b\) be a base-point-free nonspecial divisor of degree \(b \geq 2g - 2\) defining a birational morphism. Then \(s(\mathcal{K} - b) = 0\).

Proof. Let \(a = a_1 + \ldots + a_{b-g-1}\) be a generic effective divisor of degree \(b - g - 1\), such that \(|b - a|\) is a nonspecial base-point-free linear system. We always can obtain this divisor when \(b\) defines a birational map, by applying the general position theorem.

We will apply Lemma 6.6:
\begin{align*}
h^0(\mathcal{O}_X(b - a)) &= \deg(b - a) - g + 1 = h^0(\mathcal{O}_X(b)) - \deg(a),
\end{align*}
\begin{align*}
h^0(\mathcal{O}_X(\mathcal{K} - a_i)) &= h^0(\mathcal{O}_X(\mathcal{K})) - 1 \text{ because } \mathcal{K} \text{ is base-point-free}.
\end{align*}

Thus, we only have to prove that \(s(\mathcal{K} - b) = 0\). But this is Corollary 6.8.

Lemma 6.10 Let \(S = \mathbb{P}(\mathcal{E}_0)\) be a decomposable ruled surface over a smooth curve \(X\), with \(\mathcal{E}_0 \cong \mathcal{O}_X \oplus \mathcal{O}_X(e)\). Let \(|H| = |X_0 + bf|\) be a linear system on \(\mathbb{P}(\mathcal{E}_0)\). Then
\begin{align*}
s(H, k, H) &\cong s(b, k, b) \oplus s(b + e, k, b) \oplus \ldots \oplus s(b + e, k, b + e) .
\end{align*}

Proof. We know that \(s(H, k, H)\) is the cokernel of the map
\begin{align*}
h^0(\mathcal{O}_S(H)) \otimes \ldots \otimes h^0(\mathcal{O}_S(H)) \rightarrow h^0(\mathcal{O}_S(kH)).
\end{align*}

Because \(\mathbb{P}(\mathcal{E}_0)\) is decomposable, we have the natural isomorphisms
\begin{align*}
h^0(\mathcal{O}_S(H)) &\cong h^0(\mathcal{O}_X(b)) \oplus h^0(\mathcal{O}_X(b + e))
\end{align*}
and
\begin{align*}
h^0(\mathcal{O}_S(kH)) &\cong \bigoplus_{i=0}^{k} h^0(\mathcal{O}_X(kb + ie)).
\end{align*}

We see that \(\alpha\) factorizes through the maps \(\alpha_i:\)
\begin{align*}
\begin{array}{c}
\underbrace{h^0(\mathcal{O}_X(b)) \otimes \ldots \otimes h^0(\mathcal{O}_X(b))} \otimes \underbrace{h^0(\mathcal{O}_X(b + e)) \otimes \ldots \otimes h^0(\mathcal{O}_X(b + e))} \\
\alpha_i
\end{array}
\end{align*}
\begin{align*}
H^0(\mathcal{O}_X(kb + ie))
\end{align*}

In this way, \(\text{coker}(\alpha) = \text{coker}(\alpha_1) \oplus \ldots \oplus \text{coker}(\alpha_k)\).

We will prove the following theorem:

Theorem 6.11 Let \(S_b = \mathbb{P}(\mathcal{E}_b)\) be a canonical ruled surface, such that \(H \sim X_0 + bf\) defines a canonical scroll. Then,
\begin{enumerate}
\item \(s(H, H) \cong s(\mathcal{K}, \mathcal{K}) \oplus s(b, b)\).
\item \(s(H, \ldots, H) \cong s(\mathcal{K}, \ldots, \mathcal{K}) \oplus s(b, \ldots, b)\) for all \(m \geq 3\), except when \(g = 3\) and \(b = 4\) or \(g = 2\) and \(b = 3\).
\end{enumerate}
Proof. We reduce the proof to check that \( s(b, \mathcal{K}) = 0 \), and then we prove this fact in Lemma 6.12.

1. If \( k = 2 \), by Lemma 6.10 we know that

\[
s(H, H) \cong s(\mathcal{K}, \mathcal{K}) \oplus s(\mathcal{K}, b) \oplus s(b, b)
\]

so from Lemma 6.12 the result follows.

2. If \( k > 2 \), by Lemma 6.10, we have to proof that

\[
s(b, \ldots, b, \mathcal{K}, \ldots, \mathcal{K}) = 0
\]

for any \( i, 1 \leq i \leq m - 1 \).

Let us see that \( s(b, \mathcal{K} + \mu \mathcal{b}) = 0 \), when \( \mu \geq 0 \). If \( \mu = 0 \), it is Lemma 6.12. If \( \mu > 0 \) we apply Lemma 6.5:

\[
h_1^1(\mathcal{O}_X (\mathcal{K} + \mu \mathcal{b} - \mathcal{b})) \leq 1 \leq h_0^0(\mathcal{O}_X (\mathcal{b})) - 2 \quad \text{when} \quad b - g \geq 2.
\]

Because we have supposed that \( b \) defines a canonical scroll, the condition \( b - g \geq 2 \) holds except when \( g = 2 \) and \( b = 3 \) or \( g = 3 \) and \( b = 4 \).

Applying Lemma 6.2 we deduce that:

\[
s(b, \ldots, b, \mathcal{K}, \ldots, \mathcal{K}) = s(ib + \mathcal{K}, \mathcal{K}, \ldots, \mathcal{K})
\]

when \( 1 \leq i \leq m - 2 \) and

\[
s(b, \ldots, b, \mathcal{K}) = s(b, (m - 2)b + \mathcal{K}) = 0.
\]

Now, let us see that \( s(\mathcal{K}, \lambda \mathcal{K} + \mu \mathcal{b}) = 0 \), when \( \lambda \geq 0 \) and \( \mu \geq 1 \). We apply Lemma 6.6, with \( b_1 = \lambda \mathcal{K} + \mu \mathcal{b} \), \( b_2 = \mathcal{K} \) and \( a = \lambda \mathcal{K} + (\mu - 1)b \):

\[
h_0^0(\mathcal{O}_X (\lambda \mathcal{K} + \mu \mathcal{b})) = h_0^0(\mathcal{O}_X (\mathcal{b})) + \deg(\lambda \mathcal{K} + (\mu - 1)b),
\]

\[
h_0^0(\mathcal{O}_X (\mathcal{K} - x)) = h_0^0(\mathcal{O}_X (\mathcal{K})) - 1 \quad \text{for all} \quad x \in \lambda \mathcal{K} + (\mu - 1)b,
\]

\[
s(b, \mathcal{K}) = 0.
\]

Finally, applying Lemma 6.2, we obtain

\[
s(ib + \mathcal{K}, \mathcal{K}, \ldots, \mathcal{K}) = s(ib + (m - i - 2)\mathcal{K}, \mathcal{K}) = 0
\]

when \( 1 \leq i \leq m - 2 \).

\[\square\]

**Lemma 6.12** Let \( X \) be a smooth curve of genus \( g \). Let \( \mathcal{K} \) be the canonical divisor and let \( b \) a divisor of degree \( b \geq 2g - 2 \) defining a canonical scroll. Then \( s(b, \mathcal{K}) = 0 \).

**Proof.** If \( X \) is elliptic, \( \mathcal{K} \sim 0 \) so it is clear that \( s(b, \mathcal{K}) = 0 \). Now, we distinguish the hyperelliptic and the nonhyperelliptic cases:

Case 1: \( X \) is hyperelliptic.

If \( |b| \) defines a birational map it is sufficient to apply Lemma 6.9.

By Proposition 4.10 \( |b| \) does not define a birational map when \( g = 2 \) and \( b = 3 \) or \( g = 3 \) and \( b = 4 \). But in these cases \( |b| \) is a base-point-free pencil, so we can apply the Corollary 6.8.

Case 2: \( X \) is nonhyperelliptic.

We can apply Lemma 6.9 when \( |b| \) defines a birational map. But in this case if \( \deg(b) = 2g - 2 \), \( |b| \) could not verify this condition.
Thus, we will use a different strategy, which will be valid for any divisor \( b \) defining a canonical scroll over a nonhyperelliptic curve. We will prove that the following map is a surjection:

\[
H^0(\mathcal{O}_{S_b}(H - X_1)) \otimes H^0(\mathcal{O}_{S_b}(H)) \xrightarrow{\alpha_0} H^0(\mathcal{O}_{S_b}(2H - X_1)).
\]

It holds that

\[
H^0(\mathcal{O}_{S_b}(H - X_1)) \cong H^0(\mathcal{O}_{S_b}(Kf)) \cong H^0(\mathcal{O}_X(K)),
\]

\[
H^0(\mathcal{O}_{S_b}(H)) \cong H^0(\mathcal{O}_X(b)) \oplus H^0(\mathcal{O}_X(K)),
\]

\[
H^0(\mathcal{O}_{S_b}(2H - X_1)) \cong H^0(\mathcal{O}_X(b + K)) \oplus H^0(\mathcal{O}_X(2K)),
\]

so \( \alpha_0 \) is a surjection when \( s(b, K) = 0 \) and \( s(K, K) = 0 \).

We have the following commutative diagram:

\[
\begin{array}{ccc}
H^0(\mathcal{O}_{S_b}(H - X_1)) & \otimes & H^0(\mathcal{O}_{S_b}(H)) \\
\downarrow \alpha_0 & & \downarrow \alpha_1 \\
H^0(\mathcal{O}_{S_b}(2H - X_1)) & & H^0(I_{Pb-s, pN}(2))
\end{array}
\]

where \( P^N \cong \mathcal{P}(H^0(\mathcal{O}_{S_b}(H)))^* \). Since \( \alpha_1 \) is a surjection, we only have to prove that \( \alpha_2 \) is a surjection. Consider the following exact sequence:

\[
0 \longrightarrow \mathcal{O}_{S_b}(-C) \longrightarrow \mathcal{O}_{S_b} \longrightarrow \mathcal{O}_C \longrightarrow 0.
\]

Taking the tensor product with \( \mathcal{O}_{S_b}(2H - X_1) \) and applying cohomology we obtain:

\[
0 \longrightarrow H^0(\mathcal{O}_{S_b}(2H-C-X_1)) \longrightarrow H^0(\mathcal{O}_{S_b}(2H-X_1))
\]

\[
\longrightarrow H^0(\mathcal{O}_C(2H-X_1)) \longrightarrow H^1(\mathcal{O}_{S_b}(2H-C-X_1))
\]

but \( 2H-C-X_1 \) is a \((-1)\)-secant divisor, that is, \( \mathcal{O}_{S_b}(2H-C-X_1)_x \cong \mathcal{O}_x(-1) \), for all \( x \in X \). Because \( H^i(\mathcal{O}_x(-1)) = 0 \) for any \( i \geq 0 \), by the Theorem of Grauert, we deduce that \( H^i(\mathcal{O}_{S_b}(2H-C-X_1)) = 0 \) for any \( i \geq 0 \). Moreover \( \mathcal{O}_C(2H-X_1) \cong \mathcal{O}_C(2K-R) \), so we see that

\[
H^0(\mathcal{O}_{S_b}(2H-X_1)) \cong H^0(\mathcal{O}_C(2K-R)).
\]

In this way to see that \( \alpha_1 \) is a surjection it is sufficient to check that the following map is a surjection:

\[
H^0(I_{Pb-s, pN}(2)) \xrightarrow{\alpha_3} H^0(\mathcal{O}_C(2K-R)).
\]

Now, consider the following commutative map

\[
\begin{array}{ccc}
H^0(I_{Pb-s, pN}(2)) & \longrightarrow & H^0(I_{Pb-s, pN}(2)) \\
& \downarrow \alpha & \\
H^0(I_{C, pN}(2)) & \longrightarrow & H^0(\mathcal{O}_pN(2)) \\
& \downarrow \alpha & \\
H^0(I_{R, pb-s}(2)) & \longrightarrow & H^0(\mathcal{O}_{p-s}(2)) \\
& \downarrow \alpha & \\
H^0(\mathcal{O}_R(2)) & \longrightarrow & H^0(\mathcal{O}_R(2))
\end{array}
\]

If we prove that \( \alpha \) is a surjection then \( \alpha_3 \) is a surjective map. We will see this in the next Lemma.

\[\square\]
Lemma 6.13 If $X$ is neither elliptic nor hyperelliptic, then the map

$$H^0(I_{C_X}(2)) \xrightarrow{\alpha} H^0(I_{R,P^{b-g}}(2))$$

is surjective.

Proof. We know that $C_X \in \langle 2X_0 + R, 2X_1 \rangle$.

Let $Q_1$ be a quadric of $P^{b-g}$ containing the ramification points $R$.

1. If $X_1 \subset Q_1$, we can take the quadric cone over $Q_1$ of vertex $P^{g-1}$, $Q = \langle P^{g-1}, Q_1 \rangle$. This cone contains the scroll $R$, so it contains $C_X$. Moreover, $Q \cap P^{b-g} = Q_1$ so $\alpha(Q) = Q_1$.

2. If $X_1 \not\subset Q_1$, $R \subset Q_1$, then $\overline{Q_1} \cap Q_1 = \epsilon + R$ where $\epsilon \sim 2K$. Let $\epsilon'$ be the set of points corresponding to the divisor $\epsilon$ over $X_0$. Because $X_0$ is not hyperelliptic, it is projectively normal and we can take a quadric $Q_0$ meeting $X_0$ at $\epsilon'$.

Let us take the cones over $Q_0$ and $Q_1$, and with vertex $P^{b-g}$ and $P^{g-1}$ respectively:

$$\text{Cone}_0 = \langle P^{g-1}, Q_1 \rangle,$$

$$\text{Cone}_1 = \langle P^{b-g}, Q_0 \rangle.$$

Consider the pencil of quadrics generated by them. Note that any quadric in this pencil contains the quadric $Q_1$.

Let us study the trace of this pencil on the canonical ruled surface

$$\langle \text{Cone}_0, \text{Cone}_1 \rangle \longmapsto H^0(\mathcal{O}_{S_b}(2)),$$

$$\text{Cone}_0 \longmapsto 2X_1 + cf,$$

$$\text{Cone}_1 \longmapsto 2X_0 + Rf + cf.$$

Since $C_X \in \langle 2X_0 + R, 2X_1 \rangle$, there exists a quadric $Q$ in the pencil containing $C_X$ and such that $Q \cap P^{b-g} = Q_1$. Therefore $\alpha(Q) = Q_1$.

Remark 6.14 Let us see that in the cases $g = 2, b = 3$ and $g = 3, b = 4$ the theorem is not true. Since $s(b, K) = 0$, we have $s(K + b, K) = s(b + K, b)$. Furthermore, $h^0(\mathcal{O}_X(b)) = 2$, so we can compute $\dim(s(b + K, b))$ by using “Base-Point-Free Pencil Trick” (Lemma 6.7):

$$\dim(s(b + K, b)) = h^0(\mathcal{O}_X(2b + K)) - 2h^0(\mathcal{O}_X(b + K)) + h^0(\mathcal{O}_X(b + K - b)) = 1.$$

Then we have

$$\dim(s(H, H, H)) = \dim(s(K, K, K)) + \dim(s(b, b, b)) + 1.$$

Theorem 6.11 relates the projective normality of the canonical scroll to the projective normality of the directrix curves $\overline{X_0}$ and $\overline{X_1}$. If a pair $(X, b)$ is projectively normal and $b$ is ample, then the linear system must be very ample (see [9]). Then let us see when the divisor $b$ defining a canonical scroll is very ample.

Proposition 6.15 Let $b$ be a nonspecial divisor of degree $b \geq 2g - 2$ over a nonhyperelliptic smooth curve $X$ of genus $g \geq 5$ such that $\mathcal{P}(X)$ is a canonical scroll. Then:

1. If $b \geq 2g + 1$, $|b|$ is very ample.
2. If $b = 2g$ and $g \geq 3$, the generic divisor defining a canonical scroll is very ample.
3. If $b = 2g - 1$ and $g \geq 5$, the generic divisor defining a canonical scroll is very ample.
4. If $\text{Cliff}(X) \geq 3$, $|b|$ is very ample.

Furthermore, if $X$ is a generic nonhyperelliptic smooth curve of genus $g > 6$, any divisor $b$ defining a canonical ruled surface is very ample.

Proof. If $b \geq 2g + 1$ any divisor of degree $b$ is very ample by Riemann-Roch Theorem.

Let $b$ be a divisor of degree $b = 2g$. Suppose that $|b|$ is not very ample. Then the divisor $b$ must be $b \sim K + P + Q$. Since $b$ defines a canonical ruled surface, $2P + 2Q \sim 2b - 2K$ must be four different points; that is, $2P + 2Q \sim P_1 + P_2 + P_3 + P_4$. So there is a $g^1_4$. 

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Let $b$ be a divisor of degree $b = 2g - 1$. If $|b|$ is not very ample, $b - P - Q \sim K - T$ and $2b - 2K \sim 2P + 2Q - 2T$. Since $2b - 2K$ must be two different points, $2P + 2Q \sim 2T + P_1 + P_2$, so there is a $g_1^1$.

Finally, let $b$ be a divisor of degree $b = 2g - 2$. If $|b|$ is not very ample, $b - P - Q \sim K - T - S$ and $2b - 2K \sim 2P + 2Q - 2T - 2S$. Since $2b - 2K$ must be the divisor $0, 2P + 2Q \sim 2T + 2S$ and we deduce that there is a $g_1^1$.

Note that a divisor $b$ which is not very ample is built from $K$ by using the ramification points of a $g_1^1$. Then there are a finite number of non very ample divisors $b$ defining a canonical scroll, for each $g_1^1$ of $X$.

From this, if the curve $X$ is a generic curve of genus $g > 6$ or $\text{Cliff}(X) \geq 3$ then $X$ has not a $g_1^1$, so any divisor $b$ defining a canonical ruled surface is very ample.

Suppose that $g \geq 5$, by Martens Theorem ([11], IV, §5), every component of $W_4^1$ has dimension at most equal to 1. Therefore, there is at most a one-dimension family of $g_1^1$. If $b = 2g$ (resp. $b = 2g - 1$), by Theorem 5.4, the family of divisors $b$ defining a canonical ruled surface has dimension 4 (resp. 2), so the generic divisor is very ample.

Finally, if $g = 3$ (resp. $g = 4$) and $b = 2g$, there is a 3-dimensional (resp. 4-dimensional) family of divisors defining a canonical scroll. Moreover, there is at most a 2-dimensional family of divisors which are not very ample. Therefore, the generic divisor defining a canonical scroll is very ample. \hfill $\Box$

**Theorem 6.16** Let $X$ be a smooth curve of genus $g$. Let $\mathbf{P}(\mathcal{E}_b)$ be the canonical geometrically ruled surface and let $R_b$ be the corresponding canonical scroll. Then:

1. If $X$ is not hyperelliptic, then:
   (a) If $\deg(b) \geq 2g + 1$, $R_b$ is projectively normal.
   (b) If $\deg(b) = 2g$, $g \geq 3$, $b$ is generic then $R_b$ is projectively normal.
   (c) If $\deg(b) = 2g - 1$, $g \geq 5$ and $b$ is generic, $X$ is not trigonal and $X$ is not of genus 6 then $R_b$ is projectively normal.
   (d) If $\text{Cliff}(X) \geq 3$ then $R_b$ is projectively normal.

Furthermore, if $X$ is generic and $g > 6$, $R_b$ is projectively normal.

2. If $X$ is hyperelliptic, $R_b$ is not projectively normal.

3. If $X$ is elliptic, $R_b$ is projectively normal.

**Proof** We apply Theorem 6.11.

1. Suppose that $X$ is not hyperelliptic.

Because $X_0$ is projectively normal, it is sufficient to check that $\overline{X_1}$ is projectively normal.

We can suppose that $|b|$ is very ample in the mentioned cases by Proposition 6.15. Our claim follows from Theorem 1 in [5] and its Corollaries 1.4 and 1.6.

2. If $X$ is hyperelliptic, then, since $\overline{X_0}$ is not projectively normal, $R_b$ is not projectively normal.

3. If $X$ is elliptic, then $\deg(b) \geq 3$, so $\overline{X_0}, \overline{X_1}$ and $R_b$ are projectively normal. \hfill $\Box$

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