On Initial-Boundary Value Problem of Stochastic Heat Equation in a Lipschitz Cylinder

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Abstract

We consider the initial boundary value problem of non-homogeneous stochastic heat equation. The derivative of the solution with respect to time receives heavy random perturbation. The space boundary is Lipschitz and we impose non-zero cylinder condition. We prove a regularity result after finding suitable spaces for the solution and the pre-assigned datum in the problem. The tools from potential theory, harmonic analysis and probability are used. Some Lemmas are as important as the main Theorem.

Keywords: Stochastic heat equation, Lipschitz cylinder domain, Initial-boundary value problem, Anisotropic Besov space.

AMS 2000 subject classifications: primary ; 60H15, secondary; 35R60

1 Introduction

We study the following initial boundary value problem:

\[
\begin{aligned}
&du(t,x) = (\Delta u(t,x) + f(t,x))dt + g(t,x)dw_t, \quad (t,x) \in (0,T) \times D, \\
u(t,x) = b(t,x), \quad (t,x) \in (0,T) \times \partial D, \\
u(0,x) = u_0, \quad x \in D,
\end{aligned}
\]

(1.1)

where \(D\) is a bounded Lipschitz domain in \(\mathbb{R}^n\) and \(\{w_t(\omega) : t \geq 0, \omega \in \Omega\}\) is a one-dimensional Brownian motion with a probability space \(\Omega\). Any solution of (1.1) depends not only \((t,x)\), but also \(\omega\). We investigate the regularity of the solution of (1.1) in \((t,x)\) for each \(\omega\).

If \(g \equiv 0\), the problem is deterministic and the theory has been well-developed. For instance, [5] considered the problem when \(D\) is a bounded \(C^1\)-domain and [1] and [2] studied the problem when \(D\) is a bounded Lipschitz domain. Later, [6] developed a theory using anisotropic Besov spaces. However in our paper, as we let \(g \neq 0\), we deal with a stochastic heat equation. This job is nontrivial. Viewing the heat equation in (1.1) as \(u_t(t,x) = \Delta u(t,x) + f(t,x) + \hat{w}_tg(t,x)\), we notice that our equation includes an internal source/sink with the white noise coefficient. The (probabilistic) variance of the random noise \(\hat{w}_t\), \(t \in (0,T)\) is not bounded. Moreover \(\hat{w}_{t_1}\) and \(\hat{w}_{t_2}\)
are independent as long as \( t_1 \neq t_2 \). Thus, we do not expect good regularity in time direction since the solution keeps receiving the white noises along the time variable. An \( L_p \)-theory of the Cauchy problem \((D = \mathbb{R}^n)\) was established in [12] and since then the initial boundary value problem with zero boundary condition is studied by many authors (see, for instance, [14], [15], [11], [10], [17] and references therein). In this paper we allows the space domain to be Lipschitz and the boundary condition can be non-zero. Moreover, when we do not require the high regularity in \( x \), we consider the joint regularity in \((t, x)\) using anisotropic Besov spaces. The usage of anisotropic Besov spaces is natural with the deterministic heat equation.

Having said that, let us find a formal solution of (1.1): this will be a unique solution in an appropriate space. Firstly, extend \( u_0 \) on \( \mathbb{R}^n \), \( f \) and \( g \) on \((0, T) \times \mathbb{R}^n\) (see Section 3 for the mathematical details on these extensions). Let \( v \) be a solution of the Cauchy problem, i.e. \( D = \mathbb{R}^n \), consisting of (1.1) with the extended \( u_0 \) as the initial condition. Let \( \hat{h} \) denote the Fourier transform of a function \( h \) in \( \mathbb{R}^n \). Taking Fourier transform in space on the equation, we have a stochastic differential equation for each frequency \( \xi \in \mathbb{R}^n \),

\[
d\hat{v}(t, \xi) = (\xi^2 \hat{v}(t, \xi) + \hat{f}(t, \xi)) dt + \hat{g}(t, \xi) dw_t.
\]

Putting the terms with \( \hat{v} \) together in the left hand side, we get

\[
d\left( \hat{v}(t, \xi) e^{\xi^2 t} \right) = e^{\xi^2 t} \hat{f}(t, \xi) dt + e^{\xi^2 t} \hat{g}(t, \xi) dw_t
\]

and hence

\[
\hat{v}(t, \xi) = e^{-\xi^2 t} \hat{u}_0(\xi) + \int_0^t e^{-\xi^2 (t-s)} \hat{f}(s, \xi) ds + \int_0^t e^{-\xi^2 (t-s)} \hat{g}(s, \xi) dw_s.
\]

Taking the inverse Fourier transform, we obtain

\[
v(t, x) = (\Gamma(t, \cdot) *_x u_0)(x) + \int_0^t (\Gamma(t-s, \cdot) *_x f(s, \cdot))(x) ds
\]

\[
+ \int_0^t (\Gamma(t-s, \cdot) *_x g(s, \cdot))(x) dw_s, \quad t > 0, \ x \in \mathbb{R}^n,
\]

where \( \Gamma(t, x) := \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} I_{t>0} \) is the inverse Fourier transform of \( e^{-|\xi|^2 t} I_{t>0} \) and \(*_x\) denotes convolution on \( x \). We restrict \( v \) on \( \Omega \times (0, T) \times D \). Secondly, we find the solution \( h = h(\omega, t, x) \) of the following simple (stochastic) initial-boundary value problem:

\[
\begin{cases}
  h_t(\omega, t, x) = \Delta h(\omega, t, x), & (\omega, t, x) \in \Omega \times (0, T) \times D, \\
  h(\omega, t, x) = b(\omega, t, x) - v(\omega, t, x), & (\omega, t, x) \in \Omega \times (0, T) \times \partial D, \\
  h(\omega, 0, x) = 0, & \omega \in \Omega, \ x \in D.
\end{cases}
\]

Then one can easily check that \( u = v + h \) is indeed a solution of (1.1). Since information of \( h \) is well known, the estimations of three parts of \( v \) in [12] are important to us; especially the third one, the stochastic integral part.

We are to find a solution space for \( u \) and the spaces for \( f, g, b, u_0 \) so that the restriction of the three terms in the right hand side of (1.2) on \( \Omega \times (0, T) \times D \) and \( h \) belong to the solution space and
moreover $u$ is unique in it. We use two types of spaces in this paper; spaces of Bessel potentials and Besov spaces.

In this paper we let $n \geq 2$, $0 < T < \infty$, and $D$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Denote

$$D_T := (0,T) \times D, \quad \partial D_T := (0,T) \times \partial D, \quad \mathbb{R}^n_T := (0,T) \times \mathbb{R}^n.$$ 

Also, we assume $2 \leq p < \infty$ instead of the usual deterministic setup $1 < p < \infty$; this restriction is due to the stochastic part in (1.1) (see [13]). The main result in this paper is the following.

**Theorem 1.1.** Let $2 \leq p < \infty$ and $\frac{1}{p} < k < 1 + \frac{1}{p}$. Assume $f \in \mathbb{B}_{p,o}^{k-2,\frac{1}{2}(k-2)}(D_T)$, $g \in \mathbb{B}_{p,o}^{k-\frac{1}{2},\frac{1}{2}(k-\frac{1}{2})}(\partial D_T)$ and $u_0 \in L^p(\Omega,G_0,U_p^{k-\frac{1}{2}}(D))$. If $\frac{3}{p} < k < 1 + \frac{1}{p}$, we further assume the compatibility condition $u_0(\omega,x) = b(\omega,0,x)$ for $\omega \in \Omega$, $x \in \partial D$. Then

1. if $\frac{1}{p} < k < 1$, there is a unique solution $u \in \mathbb{B}_{p}^{k,\frac{1}{2}(k-1)}(D_T)$ of the initial boundary value problem (1.1) such that

$$\|u\|_{\mathbb{B}_{p}^{k,\frac{1}{2}(k-1)}(D_T)} \leq c\left(\|u_0\|_{L^p(\Omega,G_0,U_p^{k-\frac{1}{2}}(D))} + \|f\|_{\mathbb{B}_{p,o}^{k-2,\frac{1}{2}(k-2)}(D_T)} + \|g\|_{\mathbb{B}_{p,o}^{k-\frac{1}{2},\frac{1}{2}(k-\frac{1}{2})}(\partial D_T)} + \|b\|_{\mathbb{B}_{p}^{k-\frac{1}{2},\frac{1}{2}(k-\frac{1}{2})}(\partial D_T)}\right), \tag{1.4}$$

where $c$ depends only on $D,k,n,p,T$.

2. if $1 \leq k < 1 + \frac{1}{p}$, there is a unique solution $u \in \mathbb{B}_{p}^{k}(D_T)$ of the problem (1.1) such that

$$\|u\|_{\mathbb{B}_{p}^{k}(D_T)} \leq c\left(\|u_0\|_{L^p(\Omega,G_0,U_p^{k-\frac{1}{2}}(D))} + \|f\|_{\mathbb{B}_{p,o}^{k-2,\frac{1}{2}(k-2)}(D_T)} + \|g\|_{\mathbb{B}_{p,o}^{k-\frac{1}{2},\frac{1}{2}(k-\frac{1}{2})}(\partial D_T)} + \|b\|_{\mathbb{B}_{p}^{k-\frac{1}{2},\frac{1}{2}(k-\frac{1}{2})}(\partial D_T)}\right), \tag{1.5}$$

where $c$ depends only on $D,k,n,p,T$.

The explanation of spaces and notations appearing in Theorem 1.1 is placed in Section 2.

**Remark 1.2.** 1. In the part (1) of Theorem 1.1 we estimate the regularity of $u$ in $(t,x)$ simultaneously using anisotropic Besov norm whereas in part (2) we focus on the regularity in $x$. As we mentioned earlier, the regularity in time is limited while the one in space is not.

2. If $g \equiv 0$ and $u_0 \equiv 0$, then (1) of Theorem 1.1 coincides with [9].

We organized the paper in the following way. Section 2 explains spaces and notations. In Section 3 we place main lemmas and the proof of Theorem 1.1. The long proofs of some main lemmas are located in Section 4 and 5. Although this paper we denote $A \approx B$ when there are positive constants $c_1$ and $c_2$ such that $c_1A \leq B \leq c_2A$. Also, $A \lesssim B$ means that there is a positive constant $c$ such that $A \leq cB$. All such constants depend only on $n,k,p,T$ and the Lipschitz constant of $\partial D$. We use the notations $a \lor b = \max\{a,b\}$, $a \land b = \min\{a,b\}$.
2 Preliminaries

Throughout this paper we let \((\Omega, \mathcal{G}, \{\mathcal{G}_t\}, P)\) be a probability space, where \(\{\mathcal{G}_t \mid t \geq 0\}\) be a filtration of \(\sigma\)-fields \(\mathcal{G}_t \subset \mathcal{G}\) with \(\mathcal{G}_0\) containing all \(P\)-null subsets of \(\Omega\). Assume that a one-dimensional \(\{\mathcal{G}_t\}\)-adapted Wiener processes \(w\) is defined on \((\Omega, \mathcal{G}, P)\). We denote the mathematical expectation of a random variable \(X = X(\omega), \omega \in \Omega\) by \(E[X]\) or simply \(EX\); we suppress the argument \(\omega \in \Omega\) under the expectation \(E\).

For \(k \in \mathbb{R}\) let \(H^k_p(\mathbb{R}^n)\) be the space of Bessel potential and \(B^k_p(\mathbb{R}^n)\) be the Besov space (see, for instance, [3, 20]). For later purpose we place a definition of Beso \(v\) spaces. Let \(\hat{\mathcal{F}}\) denote the Fourier transform of \(f\). Then we define the Besov space \(B^k_p(\mathbb{R}^n)\) as the closure of \(B^k_p(\mathbb{R}^n)\).

Remark 2.1. Let \(k_0\) be a nonnegative integer. Then the followings hold.

(1)

\[
\|f\|_{H^k_p(\mathbb{R}^n)}^{p_p(D)} \approx \sum_{0 \leq |\beta| \leq k_0} \|D^\beta f\|_{L^p(D)}^{p},
\]

where \(D^\beta = D_{\xi_1}^{\beta_1} D_{\xi_2}^{\beta_2} \cdots D_{\xi_n}^{\beta_n}\) for \(\beta = (\beta_1, \beta_2, \cdots, \beta_n) \in (\{0\} \cup \mathbb{N})^n\).
(2) For $k \in (k_0, k_0 + 1)$

$$\|f\|^p_{B^k_p(D)} \approx \|f\|^p_{H^k_p(D)} + \sum_{|\beta| = k_0} \int_D \int_D \frac{|D^\beta f(x) - D^\beta f(y)|^p}{|x - y|^{n + p(k - k_0)}} \, dx \, dy.$$ 

The spaces $B^k_p(\partial D)$, $k \in (0, 1)$ are defined similarly.

(3) Let $k = k_0 + \theta$ with $\theta \in (0, 1)$. Then the space $B^k_p(D)$ satisfies the following real interpolation property (see Section 2 of [7]):

$$(H^{k_0}_p(D), H^{k_0+1}_p(D))_{\theta, p} = B^k_p(D).$$

(2.2)

When $k < 0$ we define $B^k_p(D)$ as the dual space of $B^{-k}_q(D)$ and $B^{k}_{p,o}(D)$ as the dual space of $B^{-k}_q(D)$, i.e., $B^k_p(D) = (B^{-k}_q(D))^*$, $B^{k}_{p,o}(D) = (B^{-k}_q(D))^*$ with $\frac{1}{p} + \frac{1}{q} = 1$.

We define $H^k_p(0,T)$, $B^k_p(0,T)$ and $B^{k}_{p,o}(0,T)$ similarly.

**Remark 2.2.** By the subscript $o$ in $B^k_{p,o}(D)$ ($k < 0$) we mean that the natural extension of any distribution in this space vanishes outside $D$ in the following sense. Let $h \in B^k_{p,o}(D) = (B^{-k}_q(D))^*$. We define the extension $\tilde{h} \in B^k_p(\mathbb{R}^n)$ of $h$ by

$$\langle \tilde{h}, \Phi \rangle := \langle h, \Phi |_D \rangle, \quad \Phi \in B^{-k}_q(\mathbb{R}^n);$$

note that by the very definition of $B^{-k}_q(D)$ we have $\Phi |_D \in B^{-k}_q(D)$ and $\langle h, \Phi |_D \rangle$ is well defined; here the condition that $D$ is Lipschitz is used. Then for any $\Phi$ with its support outside $D$, then $\langle \tilde{h}, \Phi \rangle = 0$. This means that $\tilde{h}$ vanishes outside $D$. A similar reasoning says that the extension of any distribution in $B^k_p(D)$ may not vanish outside $D$ and hence we do not add the subscript $o$.

For the initial condition $u_0$ we need

$$U^k_p(D) := \left\{ \begin{array}{ll} B^k_p(D), & k \geq 0, \\ B^k_{p,o}(D), & k < 0. \end{array} \right.$$  

(2.3)

### 2.2 Spaces for $D_T$, $\partial D_T$

For $k \geq 0$ we define the anisotropic Besov space $B^{k,\frac{1}{p}}_p(D_T)$ by

$$B^{k,\frac{1}{p}}_p(D_T) := L^p \left( (0, T); B^k_p(D) \right) \cap L^p \left( D; B^{\frac{1}{p}}_p((0, T)) \right)$$

(2.4)

with the norm

$$\|f\|_{B^k_p(D_T)} := \left( \int_0^T \|f(t, \cdot)\|^p_{B^k_p(D)} \, dt \right)^{\frac{1}{p}} + \left( \int_D \|f(\cdot, x)\|^p_{B^{\frac{1}{p}}_p((0, T))} \, dx \right)^{\frac{1}{p}},$$

(2.5)

where $B^{\frac{1}{p}}_p((0, T))$ is defined similarly as in Section 2.1 we also define

$$B^{k,\frac{1}{p}}_{p,o}(D_T) := L^p \left( (0, T); B^k_{p,o}(D) \right) \cap L^p \left( D; B^{\frac{1}{p}}_{p,o}((0, T)) \right)$$

with the same norm (2.5).

For $k < 0$ we define $B^{k,\frac{1}{p}}_p(D_T) = (B^{-k,\frac{1}{p}}_q(D_T))^*$ and $B^{k,\frac{1}{p}}_{p,o}(D_T) = (B^{-k,\frac{1}{p}}_q(D_T))^*$ with $\frac{1}{p} + \frac{1}{q} = 1$.

We define $B^{k,\frac{1}{p}}_p(\partial D_T)$, $B^{k,\frac{1}{p}}_{p,o}(\partial D_T)$, $k \in (0, 1)$ similarly.
2.3 Stochastic Banach spaces

The solution \( u \) and functions \( f, g, b, u_0 \) in (1.1) are all random. Using Section 2.1 and 2.2 we construct the spaces for them. We describe two types of spaces. The first type emphasizes the regularity in \( x \) whereas the second type does the regularity in \( t, x \) together. Again, let \( k \in \mathbb{R} \).

We can consider \( u, f, g, b \) as function space-valued stochastic processes and hence \( (\Omega \times (0, T), \mathcal{P}, P \otimes \ell((0, T])) \) is a suitable choice for their common domain, where \( \mathcal{P} \) is the predictable \( \sigma \)-field generated by \( \{ \mathcal{G}_t : t \geq 0 \} \) (see, for instance, pp. 84–85 of [12]) and \( \ell((0, T)) \) is the Lebesgue measure on \((0, T)\). We define

\[
\mathbb{H}^k_p(\mathbb{R}^n_T) = L^p(\Omega \times (0, T), \mathcal{P}, H^k_p(\mathbb{R}^n)), \quad \mathbb{B}^k_p(\mathbb{R}^n_T) = L^p(\Omega \times (0, T), \mathcal{P}, B^k_p(\mathbb{R}^n))
\]

and the norms

\[
\|f\|_{\mathbb{H}^k_p(\mathbb{R}^n_T)} = \left( E \int_0^T \|f(s, \cdot)\|^p_{H^k_p(\mathbb{R}^n)} ds \right)^{\frac{1}{p}}, \quad \|f\|_{\mathbb{B}^k_p(\mathbb{R}^n_T)} = \left( E \int_0^T \|f(s, \cdot)\|^p_{B^k_p(\mathbb{R}^n)} ds \right)^{\frac{1}{p}}
\]

; we suppress \( \omega \) in \( f \). Similarly we define

\[
\mathbb{H}^k_p(D_T) = L^p(\Omega \times (0, T), \mathcal{P}, H^k_p(D)), \quad \mathbb{B}^k_p(D_T) = L^p(\Omega \times (0, T), \mathcal{P}, B^k_p(D)),
\]

\[
\mathbb{B}^k_{p, o}(D_T) = L^p(\Omega \times (0, T), \mathcal{P}, B^k_{p, o}(D)).
\]

We also define the stochastic anisotropic Besov spaces

\[
\mathbb{B}^{k, \frac{1}{k}}_p(\partial D_T) = L^p(\Omega, \mathcal{G}, B^{k, \frac{1}{k}}_p(\partial D_T)), \quad \mathbb{B}^{k, \frac{1}{k}}_p(\partial D_T) = L^p(\Omega, \mathcal{G}, B^{k, \frac{1}{k}}_p(\partial D_T))
\]

with norms

\[
\|f\|_{\mathbb{B}^{k, \frac{1}{k}}_p(\partial D_T)} = \left( E\|f\|^p_{B^{k, \frac{1}{k}}_p(\partial D_T)} \right)^{\frac{1}{p}}, \quad \|f\|_{\mathbb{B}^{k, \frac{1}{k}}_p(\partial D_T)} = \left( E\|f\|^p_{B^{k, \frac{1}{k}}_p(\partial D_T)} \right)^{\frac{1}{p}}.
\]

Similarly we define \( \mathbb{B}^{k, \frac{1}{k}}_{p, o}(D_T) = L^p(\Omega, \mathcal{G}, B^{k, \frac{1}{k}}_{p, o}(D_T)) \).

3 Lemmas and Proof of Theorem 1.1

In this section we estimate the three terms of (1.2) and prove our main theorem.

For \( l < 0 \), if \( h \in B^l_{p, o}(D) = (B^{-l}_q(D))^* \), then we define \( \hat{h} \in B^l_p(\mathbb{R}^n) \) as the trivial extension of \( h \) by

\[
< \hat{h}, \phi > = < h, \phi |_D >, \quad \phi \in B^{-l}_q(\mathbb{R}^n), \tag{3.1}
\]

; note \( \|\hat{h}\|_{B^l_p(\mathbb{R}^n)} \approx \|h\|_{B^l_{p, o}(D)} \). For \( l \geq 0 \), if \( h \in B^l_p(D) \), then we define \( \hat{h} \in B^l_p(\mathbb{R}^n) \) as the Stein’s extension of \( h \) with \( \|\hat{h}\|_{B^l_p(\mathbb{R}^n)} \lesssim \|h\|_{B^l_p(D)} \) (see section 2 of [7] and Chapter 6 of [10]); this extension is possible since our space domain \( D \) is at least Lipschitz. Recall the definition of \( \mathcal{U}_p(D) \) in (2.3)
Lemma 3.1. Let $0 < k < 2$. We assume $u_0(\omega, \cdot) \in \mathcal{U}_p^{k,\frac{k}{2}}(D)$ for each $\omega \in \Omega$. Let $\tilde{u}_0$ denote the extension of $u_0$ (trivial or Stein’s). For each $(\omega, t, x) \in \Omega \times (0, T) \times \mathbb{R}^n$ define

$$v_1(\omega, t, x) := \begin{cases} < \tilde{u}_0(\omega, \cdot), \Gamma(t, x - \cdot) >, & \text{if } 0 \leq k < \frac{2}{p}, \\ \int_{\mathbb{R}^n} \Gamma(t, x - y)\tilde{u}_0(\omega, y) \, dy, & \text{if } \frac{2}{p} \leq k. \end{cases} \quad (3.2)$$

Then $v_1(\omega, \cdot, \cdot) \in B_p^{k,\frac{k}{p}}(\mathbb{R}_T^n)$ for each $\omega$ and

$$\|v_1(\omega, \cdot, \cdot)\|_{B_p^{k,\frac{k}{p}}(\mathbb{R}_T^n)} \leq c \|u_0(\omega, \cdot)\|_{\mathcal{U}_p^{k,\frac{k}{2}}(D)}, \quad \omega \in \Omega, \quad (3.3)$$

where $c$ is independent of $u_0$ and $\omega$.

; the proof is presented in Section 4.

For $0 < k < 2$ and $h = h(t, x) \in B_p^{k-2,\frac{k}{2}k^{-1}}(D_T)$ we define $\tilde{h} \in B_p^{k-2,\frac{k+1}{2}k^{-1}}(\mathbb{R}^{n+1})$ by

$$< \tilde{h}, \phi > := < h, \phi |_{D_T} >, \quad \phi \in B_q^{2-k,1-\frac{1}{k}}(\mathbb{R}^{n+1}). \quad (3.4)$$

In this case $\|\tilde{h}\|_{B_p^{k-2,\frac{k+1}{2}k^{-1}}(\mathbb{R}^{n+1})} \approx \|h\|_{B_p^{k-2,\frac{k+1}{2}k^{-1}}(D_T)}$.

Lemma 3.2. Let $0 < k < 2$ and $f \in B_p^{k,\frac{k}{p}}(\mathbb{R}_T^n)$. Define

$$v_2(\omega, t, x) := < f(\omega, \cdot, \cdot), \Gamma(t - \cdot, x - \cdot) >. \quad (3.5)$$

Then $v_2 \in B_p^{k,\frac{k}{p}}(\mathbb{R}_T^n)$ and

$$\|v_2\|_{B_p^{k,\frac{k}{p}}(\mathbb{R}_T^n)} \leq c \|f\|_{B_p^{k,\frac{k}{p}}(\mathbb{R}_T^n)}, \quad (3.6)$$

; the proof is in Section 5.

Before we estimate $v_3$ let us place the following lemma which is Exercise 5.8.6 in [3]:

Lemma 3.3. Assume that $A_0$ and $A_1$ are Banach spaces and that $1 \leq p < \infty$, $0 < \theta < 1$. Then

$$(L_p(A_0), L_p(A_1))_{\theta,p} = L_p((A_0, A_1)_{\theta,p}),$$

where $(\cdot, \cdot)_{\theta,p}$ is a real interpolation.

If $0 < k < 1$, then for $g = g(\omega, t, x) \in B_p^{k-1}(D_T)$ we define $\tilde{g} \in B_p^{k-1}(\mathbb{R}^{n+1})$ by

$$< \tilde{g}(\omega, t, \cdot), \phi > := < g(\omega, t, \cdot), \phi |_{D_T} >, \quad \phi \in B_q^{k-1}(\mathbb{R}^n) \quad (3.7)$$

and, if $k \geq 1$, we define $\tilde{g}(\omega, t, \cdot) \in B_p^{k-1}(\mathbb{R}^{n+1})$ by $\tilde{g}(\omega, t, x) = g(\omega, t, x)$ for $x \in D$ and $\tilde{g}(\omega, t, x) = 0$ for $x \in \mathbb{R}^n \setminus D$. Then we get $\|\tilde{g}\|_{B_p^{k-1}(\mathbb{R}^{n+1})} \approx \|g\|_{B_p^{k-1}(D_T)}$. 7
Lemma 3.4. Let $k > 0$ and $g \in \mathbb{B}^{k-1}_{p,\alpha}(D_T)$. Define
\[
v_3(t, x) := \begin{cases} 
\int_0^t \mathbb{1}_{\omega}(\cdot), \mathbb{G}(t-s, x-\cdot) > dw_s, & \text{if } 0 < k < 1, \\
\int_0^t \int_{\mathbb{R}^n} \mathbb{G}(t-s, x-y)\mathbb{G}(s, y)dy dw_s, & \text{if } 1 \leq k
\end{cases}
\]
(3.8)
we suppressed $\omega$. Then $v_3 \in \mathbb{B}^k_{p}(\mathbb{R}^n_T)$ with
\[
\|v_3\|_{\mathbb{B}^k_{p}(\mathbb{R}^n_T)} \leq c\|g\|_{\mathbb{B}^{k-1}_{p,\alpha}(D_T)}.
\]
(3.9)

Proof. Apply the result in [12] and Lemma 3.3. \hfill \Box

For $\epsilon \in (0, 1)$ we let $p_0 = \frac{1}{2} + \frac{1}{2} \epsilon$, $p_0' = \frac{1}{2} + \frac{1}{2} \epsilon$. We say that $(\frac{1}{p}, k) \in \mathcal{R}_\epsilon$ if $\alpha$ and $p$ are numbers satisfying one of the followings:
1. $p_0 < p < p_0'$ if $0 < k < 1$,
2. $1 < p \leq p_0$ if $\frac{2}{p} - 1 - \epsilon < k < 1$,
3. $p_0' \leq p < \infty$ if $0 < k < \frac{2}{p} + \epsilon$.

Lemma 3.5. There is a positive constant $\epsilon \in (0, 1)$ depending only on Lipschitz constant of $\partial D$ such that if $(\frac{1}{p}, k) \in \mathcal{R}_\epsilon$, then for all $b' \in \mathbb{B}^{k+\frac{1}{2}}_{p,\alpha}(\partial D_T)$ with $b'(\omega, 0, x) = 0$ for $\omega \in \Omega, x \in \partial D$ if $k > \frac{2}{p}$. Then there is a unique solution $h \in \mathbb{B}^{k+\frac{1}{2}+\frac{1}{2}k+\frac{1}{p}}_{p,\alpha}(D_T)$ of the problem (1.5) in $\Omega \times D_T$ with boundary value $b'$ in place of $b - v$ and $h(\omega, 0, x) = 0$ for $\omega \in \Omega, x \in D$ and it satisfies
\[
\|h\|_{\mathbb{B}^{k+\frac{1}{2}+\frac{1}{p}k+\frac{1}{p}}_{p,\alpha}(D_T)} \leq c\|b'\|_{\mathbb{B}^{k+\frac{1}{2}k}_{p,\alpha}(\partial D_T)}.
\]
(3.10)
If $D$ is a $C^1$-domain, then we can take $\epsilon = 1$.

Proof. Apply [1], [2] and [6] for each $\omega \in \Omega$. \hfill \Box

We need the following restriction theorem from [4]:

Lemma 3.6. Let $\frac{1}{p} < k < 1 + \frac{1}{p}$. Then for any $h = h(t, x) \in \mathbb{B}^{k+\frac{1}{2}k}_{p,\alpha}(\mathbb{R}^n_T)$, we have $h|_{\partial D_T} \in \mathbb{B}^{k-\frac{1}{2}k-\frac{1}{p}}_{p,\alpha}(\partial D_T)$.

The following lemma for the stochastic part $v_3$ in (1.2) is important and we elaborate the proof in Section 6.

Lemma 3.7. Assume $2 \leq p < \infty$.

1. Let $\frac{1}{p} < k < 1$ and $g \in \mathbb{B}^{k-1}_{p,\alpha}(D_T)$. Then $v_3$ defined for such $k$ in Lemma 3.4 belongs to $\mathbb{B}^{k-\frac{1}{2}k}_{p,\alpha}(\mathbb{R}^n_T)$ and
\[
\|v_3\|_{\mathbb{B}^{k-\frac{1}{2}k}_{p,\alpha}(\mathbb{R}^n_T)} \leq c\|g\|_{\mathbb{B}^{k-1}_{p,\alpha}(D_T)},
\]
(3.11)
(2) Let \( 1 \leq k < 1 + \frac{1}{p} \) and \( g \in \mathbb{B}_p^{k-1}(D_T) \). Then \( v_3 \) defined for such \( k \) in Lemma 3.4 satisfies

\[
\|v_3\|_{\partial D_T} \|_{\mathbb{B}_p^{k-\frac{1}{p}k-\frac{1}{p}}(\partial D_T)} \leq c\|g\|_{\mathbb{B}_p^{k-1}(D_T)}
\]  

(3.12)

By Lemma 3.1 - Lemma 3.7 the proof of Theorem 1.1 follows.

**Proof of Theorem 1.1** Recall the derivation of the solution \( u = v + h \) in Section 1

(1) By Lemma 3.1, 3.2 and Lemma 3.7 (1), the (random) function \( v := v_1 + v_2 + v_3 \) is in \( \mathbb{B}_p^{k-\frac{1}{p}k}(\mathbb{R}^n_T) \); note that the definition of \( u_1 \) in Lemma 3.1 is different by the cases \( k \in (\frac{1}{p}, 1) \) and \( k \in [\frac{2}{p}, 1) \). Moreover, we choose the definition of \( u_3 \) in Lemma 3.4 for \( k \in (\frac{1}{p}, 1) \). Now, using Lemma 3.6 for each \( \omega \in \Omega \), we have \( b' := b - v|_{\partial D_T} \in \mathbb{B}_p^{k-\frac{1}{p}k-\frac{1}{p}}(\partial D_T) \). Let \( u_4 \in \mathbb{B}_p^{k-\frac{1}{p}k}(D_T) \) be the unique solution of the problem (1.3) which does exist by Lemma 3.5. Then \( u := v + h \) is a solution of (1.1) and the estimate (1.4) follows (3.3), (3.6), (3.11) and (3.10). The uniqueness of such \( u \) follows the theory of deterministic heat equation.

(2) Set \( v \) as in (1) by choosing the appropriate definitions of \( v_1, v_3 \) when \( k \in [1, 1 + \frac{1}{p}) \). Then proof is similar to the case (1). However, this time we can not have \( v_3 \) in \( \mathbb{B}_p^{k-\frac{1}{p}k}(\mathbb{R}^n_T) \) although it is in \( \mathbb{B}_p^{k}(\mathbb{R}^n_T) \) by the Lemma 3.4. Hence, we have \( v \) is in \( \mathbb{B}_p^{k}(\mathbb{R}^n_T) \) as \( v_1, v_2 \) are trivially in \( \mathbb{B}_p^{k}(\mathbb{R}^n_T) \) (see 2.4). Nevertheless, by using Lemma 3.7 (2) we still have \( b' \in \mathbb{B}_p^{k-\frac{1}{p}k-\frac{1}{p}}(\partial D_T) \). By choosing \( v_4 \) as before in \( \mathbb{B}_p^{k-\frac{1}{p}k}(D_T) \) and hence \( \mathbb{B}_p^{k}(D_T) \), we have a solution of (1.1) in \( \mathbb{B}_p^{k}(\mathbb{R}^n_T) \) and the estimate (1.5) follows 3.3, 3.6, 3.12, 3.10 with 2.4. The solution is unique. \( \square \)

## 4 Proof of Lemma 3.1

We believe that one may find a proof of Lemma 3.1 is in the literature. However, we can not find the exact reference and, hence, we provide our own proof. We start with a lemma for multipliers.

**Lemma 4.1.** Let \( \Phi(\xi) = \hat{\phi}(2^{-1}\xi) + \hat{\phi}(\xi) + \hat{\phi}(2\xi) \) with \( \phi \) in the definition of Besov spaces, \( \Phi_j(\xi) = \Phi(2^{-j}\xi) \), and \( \rho_{ij}(\xi) = \Phi_j(\xi)e^{-\gamma|\xi|^2} \) for each integer \( j \). Then \( \rho_{ij}(\xi) \) is a \( L^p(\mathbb{R}^n) \)-multiplier with the finite norm \( M(t, j) \) for \( 1 < p < \infty \). Moreover for \( t > 0 \)

\[
M(t, j) \lesssim e^{-\frac{t^2}{4t^{2j}}} \sum_{0 \leq i \leq n} t^{2e^{2ij}} \lesssim e^{-\frac{t^2}{4t^{2j}}}. 
\]  

(4.1)

**Proof.** The \( L^p(\mathbb{R}^n) \)-multiplier norm \( M(t, j) \) of \( \rho_{ij}(\xi) \) is equal to the \( L^p(\mathbb{R}^n) \)-multiplier norm of \( \rho'_{ij}(\xi) := \Phi(\xi)e^{-\frac{t^2}{4t^{2j}}|\xi|^2} \) (see Theorem 6.1.3 in [3]). Now, we make use of the Theorem 4.6’ of [10]. We assume \( \beta_1, \beta_2, \ldots, \beta_l = 1 \) and \( \beta_i = 0 \) for \( l + 1 \leq i \leq n \), and set \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \). Since \( \text{supp} (\Phi) \subset \{ \xi \in \mathbb{R}^n \mid \frac{1}{2} < |\xi| < 4\} \), we have

\[
\left| D_\xi^\beta \rho_{ij}(\xi) \right| \lesssim \sum_{0 \leq i \leq n} t^{12+2ij}e^{-\frac{t^2}{4t^{2j}}} \chi_{4<|\xi|<4}(\xi),
\]

where \( \chi_A \) is the characteristic function on a set \( A \). Hence, for \( A = \prod_{1 \leq i \leq n} [2^{k_i}, 2^{k_i+1}] \) we receive

\[
\int_A \left| \frac{\partial^{|eta|}}{\partial \xi^\beta} \rho_{ij}(\xi) \right| d\xi \lesssim c \sum_{0 \leq i \leq n} t^{12+2ij}e^{-\frac{t^2}{4t^{2j}}}. 
\]
Below $\tilde{u}_0$ is the extension of $u_0$; note $\tilde{u}_0(\omega, \cdot) \in B_p^{k-\frac{2}{p}}(\mathbb{R}^n)$ for each $\omega \in \Omega$. The following lemma handles the case $k = 0$.

**Lemma 4.2.** We have

$$\|v_1(\omega)\|_{L^p(\mathbb{R}^n)} \leq c \|\tilde{u}_0(\omega)\|_{B_p^{k-\frac{2}{p}}(\mathbb{R}^n)}, \quad \omega \in \Omega,$$

(4.2)

where the constant $c$ is independent of $u_0$ and $\omega$.

**Proof.** We may assume that $\tilde{u}_0 \in C_0^\infty(\mathbb{R}^n)$ since $C_0^\infty(\mathbb{R}^n)$ is dense in $B_p^{k-\frac{2}{p}}(\mathbb{R}^n)$. We use the dyadic partition of unity $\hat{\psi}(\xi) + \sum_{j=1}^\infty \hat{\phi}(2^{-j}\xi) = 1$ for $\xi \in \mathbb{R}^n$, so that we can write

$$\hat{v}_1(t, \xi) = \hat{\psi}(\xi)e^{-t|\xi|^2}\tilde{u}_0(\xi) + \sum_{j=1}^\infty \hat{\phi}(2^{-j}\xi)e^{-t|\xi|^2}\tilde{u}_0(\xi).$$

For $t > 0$ we have

$$\begin{align*}
\int_{\mathbb{R}^n} |v_1(t, x)|^p dx &\leq \int_{\mathbb{R}^n} \left| \mathcal{F}^{-1} \left( e^{-t|\xi|^2} \hat{\psi}(\xi) \tilde{u}_0(\xi) \right) (x) \right|^p dx + \int_{\mathbb{R}^n} \left| \mathcal{F}^{-1} \left( \sum_{j=1}^\infty e^{-t|\xi|^2} \hat{\phi}_j(\xi) \tilde{u}_0(\xi) \right) (x) \right|^p dx. \\
&=: I_1(t) + I_2(t). \tag{4.3}
\end{align*}$$

The first term on the right-hand side of (4.3) is dominated by

$$\|\psi \ast \tilde{u}_0\|_{L^p(\mathbb{R}^n)}^p. \tag{4.4}$$

Now, we estimate the second term on the right-hand side of (4.3). We use the facts that $\hat{\phi}_j = \Phi_j \hat{\phi}_j$ for all $j$, where $\Phi_j$ is defined in Lemma 4.1. By Lemma 4.1, $\Phi_j(\xi)e^{-t|\xi|^2}$ are the $L^p(\mathbb{R}^n)$-Fourier multipliers with the norms $M(t, j)$. Then we divide the sum as

$$\begin{align*}
\int_{\mathbb{R}^n} &\left| \mathcal{F}^{-1} \left( \sum_{j=1}^\infty e^{-t|\xi|^2} \hat{\phi}_j(\xi) \tilde{u}_0(\xi) \right) (x) \right|^p dx \\
= &\int_{\mathbb{R}^n} \left| \mathcal{F}^{-1} \left( \sum_{j=1}^\infty \Phi_j(\xi)e^{-t|\xi|^2} \hat{\phi}_j(\xi) \tilde{u}_0(\xi) \right) (x) \right|^p dx \\
\leq &\left( \sum_{2^{2j} \leq t} M(t, j) \|\tilde{u}_0 \ast \phi_j\|_{L^p} \right)^p + \left( \sum_{2^{2j} \geq 1/t} M(t, j) \|\tilde{u}_0 \ast \phi_j\|_{L^p} \right)^p \\
=: &I_1(t) + I_2(t).
\end{align*}$$

By Lemma 4.1 we have $M(t, j) \leq c$ for $t2^{2j} \leq 1$. We take $a$ satisfying $-\frac{2}{p} < a < 0$ and then use
Hölder inequality to get
\[
\int_0^T I_1(t)^p \, dt \lesssim \int_0^T \left( \sum_{2^j \leq 1/t} 2^{-\frac{p}{2} m j} \right)^{p-1} \sum_{2^j \leq 1/t} 2^{p m j} \| \phi_j \ast \tilde{u}_0 \|_{L^p} \, dt \\
\lesssim \int_0^T t^{2 p m} \sum_{2^j \leq 1/t} 2^{p m j} \| \phi_j \ast \tilde{u}_0 \|_{L^p} \, dt \\
\lesssim \sum_{j=1}^\infty 2^{p m j} \| \phi_j \ast \tilde{u}_0 \|_{L^p} \int_0^{2^{-j}} t^{2 p m} \, dt \\
= c \sum_{j=1}^\infty 2^{-2 j} \| \phi_j \ast \tilde{u}_0 \|_{L^p}.
\]

By Lemma 4.1 again \( M(t, j) \leq c (t^{2 j})^{-m} \sum_{0 \leq i \leq n} (t^{2 j})^i \leq c 2^{(2 n - 2 m) j t^{n - m}} \) for \( t \cdot 2^j \geq 1 \) and \( m > 0 \).

We fix \( b > 0 \) and then choose \( m \) satisfying \( p (2 n - m) + \frac{1}{2} p b + 1 < 0 \), so that we obtain
\[
\int_0^T I_2(t)^p \, dt \lesssim \int_0^T \left( \sum_{2^j \geq 1/t} 2^{(2 n - 2 m) j t^{n - m}} \| \phi_j \ast \tilde{u}_0 \|_{L^p} \right)^p \, dt \\
\lesssim \int_0^\infty t^{p (n - m) + \frac{1}{2} p b} \sum_{2^j \geq 1/t} 2^{p b j} 2^{p (2 n - 2 m) j t^{n - m}} \| \phi_j \ast \tilde{u}_0 \|_{L^p} \, dt \\
\lesssim \sum_{j=1}^\infty 2^{p b j} 2^{p (2 n - 2 m) j} \| \phi_j \ast \tilde{u}_0 \|_{L^p} \int_2^{2^{-j}} t^{p (n - m) + \frac{1}{2} p b} \, dt \\
= c \sum_{j=1}^\infty 2^{-2 j} \| \phi_j \ast \tilde{u}_0 \|_{L^p}.
\]

□

**Proof of Lemma 3.1** The following is a classical result (see [16]):
\[
\int_0^T \| v_1(\omega, t, \cdot) \|_{H^k_p(\mathbb{R}^n)}^p \, dt + \int_{\mathbb{R}^n} \| v_1(\omega, \cdot, x) \|_{H^k_p((0, T))}^p \, dx \leq c \| \tilde{u}_0(\omega) \|_{B^k_{p, \frac{1}{2}}(\mathbb{R}^n)}^p, \quad \omega \in \Omega. \quad (4.5)
\]

Using (4.3), Lemma 4.2 and the following real interpolations
\[
(\ell^p(\mathbb{R}), H^2_p(\mathbb{R}^n))_{\frac{1}{2} p} = B^2_p(\mathbb{R}^n), \quad (\ell^p((0, T)), H^1_p((0, T)))_{\frac{p}{p}} = B^p_p((0, T)), \quad (B^k_p(\mathbb{R}^n), B^{\frac{1}{2} k - \frac{1}{2}}_p(\mathbb{R}^n))_{\frac{1}{2} p} = B^{k - \frac{1}{2}}_p(\mathbb{R}^n),
\]
we have
\[
\| v_1(\omega) \|^p_{B^k_{p, \frac{1}{2} k}(\mathbb{R}^n)} = \int_0^T \| v_1(\omega, t, \cdot) \|_{B^k_{p, \frac{1}{2} k}(\mathbb{R}^n)}^p \, dt + \int_{\mathbb{R}^n} \| v_1(\omega, \cdot, x) \|_{B^k_{p, \frac{1}{2} k}((0, T))}^p \, dx \leq c \| \tilde{u}_0(\omega) \|_{B^{k - \frac{1}{2}}_p(\mathbb{R}^n)}^p.
\]

This implies Lemma 3.1. □
5 Proof of Lemma 3.2

We need the space of the parabolic Bessel potentials. For \( l \in \mathbb{R} \) the parabolic Bessel potential \( \Pi_l \) is a distribution whose Fourier transform in \( \mathbb{R}^{n+1} \) is defined by

\[
\widehat{\Pi_l}(\tau, \xi) = c_l (1 + i \tau + |\xi|^2)^{-\frac{1}{2}}, \quad \tau \in \mathbb{R}, \ \xi \in \mathbb{R}^n.
\]

In particular, if \( l > 0 \), then

\[
\Pi_l(t, x) = \begin{cases} 
  c_l \frac{t^{l-n-2}}{n-2} e^{-t} e^{-\frac{|x|^2}{4t}} & \text{if } t > 0, \\
  0 & \text{if } t \leq 0
\end{cases}
\]

(5.1)

; see [3]. In particular, \( \Pi_2 = e^{-t} \Gamma \), where \( \Gamma \) is the heat kernel introduced in Section [1].

For \( 1 \leq p < \infty \) we define the space of the parabolic Bessel potentials, \( H_p^{l,\frac{l}{2}}(\mathbb{R}^{n+1}) \), by

\[
H_p^{l,\frac{l}{2}}(\mathbb{R}^{n+1}) = \{ f \in S'(\mathbb{R}^{n+1}) \mid \Pi_{-l} * f \in L^p(\mathbb{R}^{n+1}) \}
\]

with the norm

\[
\| f \|_{H_p^{l,\frac{l}{2}}(\mathbb{R}^{n+1})} = \| \Pi_{-l} * f \|_{L^p(\mathbb{R}^{n+1})},
\]

where \( * \) in this case is a convolution in \( \mathbb{R}^{n+1} \) and \( S'(\mathbb{R}^{n+1}) \) is the dual space of the Schwartz space \( S(\mathbb{R}^{n+1}) \). Note that if \( l \geq 0 \), we have

\[
H_p^{l,\frac{l}{2}}(\mathbb{R}^{n+1}) = L^p(\mathbb{R}; H_p^l(\mathbb{R}^n)) \cap L^p(\mathbb{R}^n; H_p^{\frac{1}{2}}(\mathbb{R}^n)).
\]

For \( l \geq 0 \) we define

\[
H_p^{l,\frac{l}{2}}(\mathbb{R}_T^n) := \{ f|_{\mathbb{R}_T^n} \mid f \in H_p^{l,\frac{l}{2}}(\mathbb{R}^{n+1}) \}
\]

and let \( H_p^{l,\frac{l}{2}}(\mathbb{R}_T^n) \) be the closure of \( C_c^\infty(\mathbb{R}_T^n) \) in \( H_p^{l,\frac{l}{2}}(\mathbb{R}_T^n) \).

For \( l < 0 \) we also define \( H_p^{l,\frac{l}{2}}(\mathbb{R}_T^n) \) and \( H_p^{l,\frac{l}{2}}(\mathbb{R}_T^n) \) as the dual spaces of \( H_q^{-l,-\frac{l}{2}}(\mathbb{R}_T^n) \) and \( H_q^{-l,-\frac{l}{2}}(\mathbb{R}_T^n) \) respectively with \( \frac{1}{p} + \frac{1}{q} = 1 \); \( H_q^{l,\frac{l}{2}}(\mathbb{R}_T^n) = (H_q^{-l,-\frac{l}{2}}(\mathbb{R}_T^n))^*, \ H_p^{l,\frac{l}{2}}(\mathbb{R}_T^n) = (H_q^{-l,-\frac{l}{2}}(\mathbb{R}_T^n))^* \).

Proof of Lemma 3.2 We assumed \( 0 < k < 2 \) and \( f \in B_{p.o}^{-2,\frac{1}{2}k-1}(D_T) \). Let \( \tilde{f} \) is the extension of \( f \) on \( \mathbb{R}^{n+1} \).

1. We just show the case \( k = 0 \)

\[
\| u_2(\omega) \|_{L^p(\mathbb{R}_T^n)} \leq c\| \tilde{f}(\omega) \|_{H_p^{-2,-1}(\mathbb{R}^{n+1})}, \quad \omega \in \Omega.
\]

(5.2)

Then the classical result ([16]):

\[
\| u_2(\omega) \|_{H_p^{2,1}(\mathbb{R}_T^n)} \leq c\| \tilde{f}(\omega) \|_{L^p(\mathbb{R}^{n+1})}, \quad \omega \in \Omega
\]

and the real interpolations

\[
(L^p(\mathbb{R}_T^n), H_p^{-2,1}(\mathbb{R}_T^n))_{\frac{1}{2},p} = B_{p.o}^{-k,\frac{k}{2}}(\mathbb{R}_T^n), \quad (H_p^{-2,-1}(\mathbb{R}^{n+1}), L^p(\mathbb{R}^{n+1}))_{\frac{1}{2},p} = B_{p.o}^{-k,\frac{1}{2}k-1}(\mathbb{R}^{n+1})
\]
lead us to
\[
\|u_2(\omega)\|_{L^{k,\frac{1}{2}}_p(\mathbb{R}^d)} \leq c\|\tilde{f}(\omega)\|_{L^{k-2,\frac{1}{2}}(\mathbb{R}^d)}, \quad \omega \in \Omega
\]
and follows.

2. Since \(C_c^\infty(\mathbb{R}^n)\) is dense in \(H_{p,0}^{l,\frac{1}{2}}(\mathbb{R}^d)\) even for \(l < 0\), we may assume \(\tilde{f}\) is in \(C_c^\infty(\mathbb{R}^n)\). In this case the representation

\[
u_2(\omega, t, x) = \int_0^t \int_{\mathbb{R}^n} \Gamma(t-s, x-y)\tilde{f}(\omega, s, y) \, dy \, ds
\]

is legal. Recalling \(\Pi_2(t, x) = e^{-t}\Gamma(t, x)\), we have

\[
u_2(\omega, t, x) = \int_0^t \int_{\mathbb{R}^n} e^{-s}\Pi_2(t-s, x-y)\tilde{f}(\omega, s, y) \, dy \, ds = e^t(\Pi_2 * g(\omega, t, x)),
\]

where \(g(\omega, s, y) = e^{-s}\tilde{f}(\omega, s, y)\). Hence,

\[
\int_0^T \int_{\mathbb{R}^n} \|\nu_2(\omega, t, x)\|_p^p \, dx \, dt = \int_0^T \int_{\mathbb{R}^n} e^p|\Pi_2 * g(\omega, t, x)|^p \, dx \, dt
\]

\[
\leq e^{pT} \int_0^\infty \int_{\mathbb{R}^n} |\Pi_2 * g(\omega, t, x)|^p \, dx \, dt
\]

\[
\leq e^{pT} \|g(\omega)\|^p_{H_{p,2}^{-1}(\mathbb{R}^{n+1})}
\]

\[
\leq e^{pT} \|\tilde{f}(\omega)\|^p_{H_{p,2}^{-1}(\mathbb{R}^{n+1})},
\]

where the last inequality follows by

\[
| \langle g(\omega), \phi \rangle | = | \langle \tilde{f}(\omega), e^{-t}\phi \rangle | \leq \|\tilde{f}(\omega)\|_{H_{p,2}^{-1}(\mathbb{R}^{n+1})} \|e^{-t}\phi\|_{H_{p,2}^{-1}(\mathbb{R}^{n+1})}, \quad \phi \in H_{p,2}^{-1}(\mathbb{R}^{n+1})
\]

and the fact \(\|e^{-t}\phi\|^2_{H_{p,2}^{-1}(\mathbb{R}^{n+1})} \leq \|\phi\|^2_{H_{p,2}^{-1}(\mathbb{R}^{n+1})}\). We have received \(6.1\) and the lemma is proved. □

6 Proof of Lemma 3.7 (1)

We only need to prove the case \(T = 1\):

\[
E \int_{\mathbb{R}^n} \int_0^1 \int_0^1 \frac{|v_3(t, x) - v_3(s, x)|^p}{|t-s|^{1+\frac{p}{2}}} \, ds \, dt \, dx \leq \|\tilde{g}\|^p_{L_p(\Omega \times (0,1), \mathcal{P}, B_{p,+}^{k-1}(\mathbb{R}^n))},
\]

where \(\tilde{g}\) is the extension of \(g\) and \(v_3\) is defined in 3.2 using \(\tilde{g}\). Then the general case follows a scaling argument with the fact that under the expectation we can use any Brownian motion in the definition of \(v_3\) and the observation that \(\tilde{w}_r := \frac{1}{\sqrt{r}}w\sqrt{r}, \ r \in [0, 1]\) is also a Brownian motion. Indeed, let \(\tilde{g}(\omega, r, y), \ \omega \in \Omega, \ r \in [0, T], \ y \in \mathbb{R}^n\) be given. Notice that we may assume that \(\tilde{g}\) is smooth in \(y\). In this case

\[
v_3(t, x) = \int_0^t \int_{\mathbb{R}^n} <\Gamma(t-r, x-\cdot), \tilde{g}(r, \cdot) > \, dw_r
\]

\[
= \int_0^t \int_{\mathbb{R}^n} \Gamma(t-r, x-y)\tilde{g}(r, y) \, dy \, dw_r, \quad t \in [0, T], \ x \in \mathbb{R}^n.
\]
Define $\bar{v}_3(t,x) = v_3(Tt,\sqrt{T}x)$, $t \in [0,1]$ and $\bar{g}(r,y) = \bar{g}(Tr,\sqrt{T}y)$, $r \in [0,1]$. Note
\[
\bar{v}_3(t,x) = \int_0^T \int_{\mathbb{R}^n} \Gamma(Tt - r, \sqrt{T}x - y) \bar{g}(r,y) dy \, dw_r
\]
\[
= \sqrt{T} \int_0^T \int_{\mathbb{R}^n} (\sqrt{T})^n \Gamma(Tt - Tr, \sqrt{T}x - \sqrt{T}y) \bar{g}(r,y) dy \, dw_r
\]
\[
= \sqrt{T} \int_0^T \int_{\mathbb{R}^n} \Gamma(t - r, x - y) \bar{g}(s,y) dy \, dw_r.
\]
By obvious scaling and (6.1) we receive
\[
E \int_{\mathbb{R}^n} \int_0^T \int_0^T \frac{|v_3(t,x) - v_3(s,x)|^p}{|t - s|^{1 + \frac{1}{k}}} \, ds \, dt \, dx
\]
\[
= T^{1 - \frac{1}{p} + \frac{2}{q}} E \int_{\mathbb{R}^n} \int_0^1 \int_0^1 \frac{|\bar{v}_3(t,x) - \bar{v}_3(s,x)|^p}{|t - s|^{1 + \frac{1}{k}}} \, ds \, dt \, dx
\]
\[
\leq T^{1 - \frac{1}{p} + \frac{2}{q}} \|\bar{g}\|^p_{L^p_p(\Omega \times (0,1), \mathcal{P}, B^{k-1}_{p}((\mathbb{R}^n)))}.
\]
(6.2)
To dominate (6.2) by $\|\bar{g}\|^p_{L^p_p(\Omega \times (0,T), \mathcal{P}, B^{k-1}_{p}((\mathbb{R}^n)))}$ we observe the following. Given a smooth function $f = f(y)$ define $f_{\sqrt{T}}(y) = f(\sqrt{T}y)$. Then for any $\phi \in C^\infty_0(\mathbb{R}^n)$, $\|\phi\|_{B^{k-1}_p(\mathbb{R}^n)} = 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,
\[
\int_{\mathbb{R}^n} f_{\sqrt{T}}(y) \phi(y) dy = T^{-\frac{2}{q}} \int_{\mathbb{R}^n} f(y) \phi_{\sqrt{T}}(y) dy
\]
\[
\leq T^{-\frac{2}{q}} \|f\|_{B^{k-1}_p(\mathbb{R}^n)} \|\phi_{\sqrt{T}}\|_{B^{k-1}_p(\mathbb{R}^n)}
\]
\[
\leq T^{-\frac{2}{q}} \|f\|_{B^{k-1}_p(\mathbb{R}^n)} \cdot T^{\frac{2}{q}} (1 \lor T^{-p(1-k)}) \|\phi\|_{B^{k-1}_p(\mathbb{R}^n)}
\]
\[
\leq (1 \lor T^{-p(1-k)}) \|f\|_{B^{k-1}_p(\mathbb{R}^n)};
\]
see Remark 2.4 (2) for the second inequality. Hence, $\|f_{\sqrt{T}}\|^p_{B^{k-1}_p(\mathbb{R}^n)} \leq (1 \lor T^{k-1}) \|f\|^p_{B^{k-1}_p(\mathbb{R}^n)}$. This and another simple scaling imply that (6.2) is indeed bounded by $c \|\bar{g}\|^p_{L^p_p(\Omega \times (0,T), \mathcal{P}, B^{k-1}_{p}((\mathbb{R}^n)))}$, where $c$ depends only on $p, n, k, T$. We need two more lemmas to prove Lemma 3.7 (1) with $T = 1$. The proof of the following lemmas are placed at the end of this section.

Lemma 6.1. Let $\frac{1}{p} < k < 1$, $p \geq 2$ and $\bar{g} \in L^p_p(\Omega \times (0,T), \mathcal{P}, B^{k-1}_{p}((\mathbb{R}^n)))$. Then for $i = -1, -2, \ldots$ we have
\[
E \int_{\mathbb{R}^n} \int_{4^i \leq |t - s| \leq 4^{i+1}} \frac{|v_3(t,x) - v_3(s,x)|^p}{|t - s|^{1 + \frac{1}{k}}} ds \, dt \, dx \leq \|\bar{g}\|^p_{L^p_p(\Omega \times (0,1), \mathcal{P}, B^{k-1}_{p}((\mathbb{R}^n)))}.
\]
(6.3)

Let $X_0$ and $X_1$ be a couple of Banach spaces continuously embedded in a topological vector space and let $Y_0$ and $Y_1$ be another such couple. We denote the real interpolation spaces
\[
X_{\theta q} := (X_0, X_1)_{\theta, q}, \text{ } Y_{\theta q} := (Y_0, Y_1)_{\theta, q}, \text{ } 0 < \theta < 1, \text{ } 1 \leq q \leq \infty
\]
and the following well known result (see Theorem 1.3 in [13]):

Lemma 6.2. Let $T = \sum_{i=1}^{\infty} T_i$, where $T_i : X_{\nu} \rightarrow Y_{\nu}$ are bounded linear operators with norms $M_{i, \nu}$ such that $M_{i, \nu} \leq c \omega^{i(\theta - \nu)}$, $\nu = 0, 1$, for some fixed $\omega \neq 1$ and $0 < \theta < 1$. Then $T : X_{\theta 1} \rightarrow Y_{\theta \infty}$ is a bounded linear operator.
Let us denote $Sg := v_3$.

**Proof of Lemma 3.7** (1) 1. As we discussed, it is enough to consider the case $T = 1$. Recall $1/p < k < 1$ and $p \geq 2$. Note that the extension $\tilde{g}$ of $g$ is in $L^p(\Omega \times (0, 1), \mathcal{P}, B_p^{k-1}(\mathbb{R}^n))$. Since the random function $S\tilde{g}$ belongs to $\mathbb{R}^k_p(\mathbb{R}^n)$ and satisfies (3.4) (Lemma 3.4), to prove (3.11) we only need to show

$$
E \int_{\mathbb{R}^n} \int_0^1 \frac{|S\tilde{g}(t, x) - S\tilde{g}(s, x)|^p}{|t - s|^{1+\frac{2k}{p}}} ds dt dx \lesssim \|\tilde{g}\|_{L^p(\Omega \times (0, 1), \mathcal{P}, B_p^{k-1} (\mathbb{R}^n))}^p \tag{6.5}
$$

; see (2.5) and the time version of Remark 2.1 (2). We follows the outline of (9).

2. Define the space $Y$ whose element $h : \Omega \times (0, 1)^2 \times \mathbb{R}^n \rightarrow \mathbb{C}$ satisfies

$$
\|h\|_Y^p := E \int_{\mathbb{R}^n} \int_0^1 \int_0^1 \frac{|h(t, s, x)|^p}{|t - s|} ds dt dx < \infty.
$$

Let $1/p < \alpha_1 < k < \alpha_2 < 1$. Denote

$$X_\nu = L^p(\Omega \times (0, 1), \mathcal{P}, H_p^{\alpha_2 - 1}(\mathbb{R}^n)), \quad Y_\nu = Y, \quad \nu = 1, 2
$$

and define the operators $T_i : X_\nu \rightarrow Y_\nu \quad (i = -1, -2, \ldots)$ by

$$T_i\tilde{g}(\varphi, t, s, x) = \left\{ \begin{array}{ll}
\frac{S\tilde{g}(\omega, t, x) - S\tilde{g}(\omega, s, x)}{|t - s|^{1 + \frac{2k}{p}}}, & \text{if } 4^i \leq |t - s| < 4^{i+1}, \\
0, & \text{otherwise.}
\end{array} \right.
$$

Then, using Lemma 6.1 we have

$$\|T_i\tilde{g}\|_{Y_\nu} \lesssim 2^{i(\alpha_\nu - k)} \|\tilde{g}\|_{X_\nu}, \quad \nu = 1, 2, \quad i = -1, -2, \ldots.
$$

As we take $\theta = \frac{k - \alpha_1}{\alpha_2 - \alpha_1}$ and $\gamma = 2^{\alpha_1 - \alpha_2}$, the norms $M_i, \mu$ of the map $T_i : X_\nu \rightarrow Y_\nu$ satisfy

$$M_i, \mu \lesssim 2^{i(\alpha_\nu - k)} = c_1, \gamma^i(\theta - \nu).
$$

Note that $Y_\theta = Y$. Hence, by Lemma 6.2 we have

$$E \int_{\mathbb{R}^n} \int \int_{|t - s| < 1} \frac{|S\tilde{g}(t, x) - S\tilde{g}(s, x)|^p}{|t - s|^{1 + \frac{2k}{p}}} ds dt dx \lesssim \|\tilde{g}\|_{X_\theta_1}^p, \tag{6.6}
$$

where

$$X_{\theta_1} := \left( L^p(\Omega \times (0, 1), \mathcal{P}, H_p^{\alpha_2 - 1}(\mathbb{R}^n)), L^p(\Omega \times (0, 1), \mathcal{P}, H_p^{\alpha_2 - 1}(\mathbb{R}^n)) \right)_{\theta_1}.
$$

3. Now, choose $k_1, k_2$ and set $\eta \in (0, 1)$ so that

$$\frac{1}{p} < \alpha_1 < k_1 < k < k_2 < \alpha_2 < 1, \quad k = (1 - \eta)k_1 + \eta k_2.
$$

Denote $\theta_\mu = \frac{k_2 - \alpha_1}{\alpha_2 - \alpha_1}, \mu = 1, 2$. Then (6.6) holds for the quadruples $(\alpha_1, k_1, \alpha_2, \theta_1)$ and $(\alpha_1, k_2, \alpha_2, \theta_2)$.

By Theorem 3.11.5 in (3) and lemma 3.3 we have

$$(X_{\theta_1}, X_{\theta_2})_{\eta, p} = (L^p(\Omega \times (0, 1), \mathcal{P}, H_p^{\alpha_2 - 1}(\mathbb{R}^n)), L^p(\Omega \times (0, 1), \mathcal{P}, H_p^{\alpha_2 - 1}(\mathbb{R}^n)))_{\theta, p} = L^p(\Omega \times (0, 1), \mathcal{P}, (H_p^{\alpha_2 - 1}(\mathbb{R}^n), H_p^{\alpha_2 - 1}(\mathbb{R}^n))_{\theta, p}$$

$$= L^p(\Omega \times (0, 1), \mathcal{P}, B_p^{k-1}(\mathbb{R}^n)).$$

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Lemma 6.3. Let \( \parallel F \parallel_m \) be defined by

\[
\begin{align*}
\text{Lemma 6.3.} & \quad \text{Let } \parallel F \parallel_m (2) \text{ We set } \\
\text{Then by Theorem 5.4.1 (Stein-Weiss interpolation theorem) in [3] we have} \\
\text{Hence, we receive (6.5). Lemma 6.1 (1) now follows.} \quad \square \\
\end{align*}
\]

We need the followings. Recall that \( S(\mathbb{R}^n) \) is dense in any \( B^k_p(\mathbb{R}^n) \).

Lemma 6.3. Let \( l < 0, 1 < q < \infty \) and \( g \in S(\mathbb{R}^n) \). Then the followings hold.

(1) For \( t > 0 \),

\[
\left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Gamma(t, x - y)g(y)dy \right)^{1/q} \lesssim (1 + t^\beta)\| g \|_{H^s_p(\mathbb{R}^n)}.
\]

(2) For \( t, h > 0 \),

\[
\left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \Gamma(t + h, x - y) - \Gamma(t, x - y) \right)g(y)dy \right)^{1/q} \lesssim h^{(t-1)}(1 + t^\beta)\| g \|_{H^s_p(\mathbb{R}^n)}.
\]

Proof. (1) Denote \( \mathcal{F}(h) = \hat{h} \), the spatial Fourier transform of \( h \). We observe that

\[
\mathcal{F}(\Gamma(t, \cdot) \ast g)(\xi) = (1 + |\xi|^{-l})e^{-t|\xi|^2} \cdot m(\xi)(1 + |\xi|^{2}) \frac{1}{\xi} \hat{g}(\xi),
\]

where \( m(\xi) = \frac{(1 + |\xi|^2)^{l/2} - l/2}{1 + |\xi|^2} \). We note that \( m \) is an \( L^\infty \)-Fourier multiplier, i.e., the operator \( T_m \) defined by \( T_m(f)(\xi) = m(\xi)f(\xi) \) is \( L^\infty \)-bounded. On the other hand we set

\[
\widehat{K}(\xi) = (1 + |\xi|^{-l})e^{-t|\xi|^2}.
\]

Since \( \| \mathcal{F}^{-1}(\hat{\phi}(\sqrt{\tau} \xi)) \|_1 = \| \phi \|_1 \), we obtain

\[
\| K \|_1 \leq \| \mathcal{F}^{-1}(e^{-t|\xi|^2}) \|_1 + t^\beta \| \mathcal{F}^{-1}(e^{-t|\xi|^2}) \|_1 \lesssim (1 + t^\beta).
\]

We have

\[
\Gamma(t, \cdot) \ast g = K \ast (T_m(I - \Delta)^{\frac{1}{2}}g).
\]

By Young’s inequality and the multiplier theorem, we conclude that for \( 1 < q < \infty \)

\[
\left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Gamma(t, x - y)g(y)dy \right)^{1/q} \lesssim (1 + t^\beta)\| g \|_{H^s_p(\mathbb{R}^n)}.
\]

(2) We set

\[
\mathcal{F}((\Gamma(t + h, \cdot) - \Gamma(t, \cdot)) \ast g)(\xi) = (-h|\xi|^2)(1 + |\xi|^{-l})e^{-t|\xi|^2} \cdot \frac{1 - e^{-h|\xi|^2}}{h|\xi|^2} \cdot m(\xi)(1 + |\xi|^{2}) \frac{1}{\xi} \hat{g}(\xi),
\]

where \( m(\xi) = \frac{(1 + |\xi|^2)^{l/2} - l/2}{1 + |\xi|^2} \). Note that \( \frac{1 - e^{-h|\xi|^2}}{h|\xi|^2} \) is the \( L^\infty \)-Fourier multiplier and the norm is independent of \( h \). Set

\[
\widehat{K}^{t,h}(\xi) = (-h|\xi|^2)(1 + |\xi|^{-l})e^{-t|\xi|^2}.
\]

Then we have \( \| K^{t,h} \|_1 \lesssim h^{(t-1)}(1 + t^\beta) \) and the rest is similar to the case (1). \( \square \)
Lemma 6.4. Let $\frac{1}{k} < k < 1$. Fix $i = -1, -2, \ldots$ and denote $D_i := \{(s,t) \in (0,1) \times (0,1) | 4^i \leq t - s < 4^{i+1}\}$. Consider the following operators $T_1, T_2, T_3$ which map function defined on $(0,1)$ to a function defined on $D_i$:

\[
(T_1f)(s,t) := \int_s^t (t-r)^{k-1} f(r)dr,
\]

\[
(T_2f)(s,t) := \int_{(s-4^i) \cup 0}^s (s-r)^{k-1} f(r)dr,
\]

\[
(T_3f)(s,t) := \int_0^{(s-4^i) \cup 0} (s-r)^{k-3} f(r)dr
\]

; note that $T_2f$ and $T_3$, in fact, are independent of $t$. Then for $1 \leq q < \infty$ we have

\[
\|T_m f\|_{L^q(D_i)} \leq c_m 4^{i(k+\frac{4}{q})} \|f\|_{L^q(0,1)}, \quad m = 1, 2 ; \quad \|T_3 f\|_{L^q(D_i)} \leq c_3 4^{i(k-2+\frac{4}{q})} \|f\|_{L^q(0,1)}, \quad (6.7)
\]

where $c_1, c_2, c_3$ are absolute constants.

Proof. 1. For $q = 1$ Fubini’s theorem gives us

\[
\|T_1 f\|_{L^1(D_i)} = \int_1^t \int_{(t-4^i) \cup 0}^1 \int_s^t (t-r)^{k-1} |f(r)|dr ds dt
\]

\[
\leq \int_1^t |f(r)| \left[ \int_r^{r+4^i+1} (t-r)^{k-1} \left( \int_{t-4^i+1}^r ds \right) dt \right] dr
\]

\[
\leq \frac{4^{k+1}}{k(k+1)} \cdot 4^{i(k+1)} \|f\|_{L^1(0,1)}.
\]

For $q = \infty$ we have

\[
\sup_{(s,t) \in D_i} |(T_1 f)(s,t)| \leq \|f\|_{L^\infty(0,1)} \cdot \sup_{(s,t) \in D_i} \int_s^t (t-r)^{k-1} dr \leq \frac{4^k}{k} \cdot 4^{i k} \|f\|_{L^\infty(0,1)}.
\]

Then, by the real interpolation theorem $(L_1, L_\infty)_{\theta,q} = L_\theta$ with the relation $\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{\infty} = \theta$, we get

\[
\|T_1 f\|_{L^\theta(D_i)} \leq c (4^{i(k+1)})^\theta (4^{i k})^{1-\theta} \|f\|_{L^\infty(0,1)} \leq c 4^{i(k+\frac{4}{q})} \|f\|_{L^\infty(0,1)}.
\]

hence, (6.7) for $T_1$ holds .

2. For $q = 1$ we have

\[
\|T_2 f\|_{L^1(D_i)} = \int_1^t \int_{(t-4^i) \cup 0}^1 \int_{(s-4^i) \cup 0}^s (s-r)^{k-1} |f(r)| dr ds dt
\]

\[
\leq \int_1^t |f(r)| \left[ \int_r^{r+4^i} (s-r)^{k-1} \left( \int_{t-4^i+1}^r ds \right) dr \right]
\]

\[
\leq \frac{3}{k} \cdot 4^{i(1+k)} \|f\|_{L^1(0,1)}
\]

and for $q = \infty$

\[
\sup_{(s,t) \in D_i} |(T_2 f)(s,t)| \leq \|f\|_{L^\infty(0,1)} \cdot \sup_{(s,t) \in D_i} \int_s^t (s-r)^{k-1} dr \leq \frac{1}{k} \cdot 4^{ki} \|f\|_{L^\infty(0,1)}.
\]
By the real interpolation theorem, for $T_2$ holds.

3. For $T_3$ the proof is similar. Observe that
\[
\|T_3f\|_{L^1(D_1)} = \int_0^1 \int_0^{t-4^i} \int_0^{s-4^i} (s-r)^{k-3} |f(r)| dr ds dt
\]
\[
\leq \int_0^1 |f(r)| \left( \int_0^{t-4^i} (s-r)^{k-3} \left( \int_0^{s+4^i} dt \right) ds \right) dr
\]
\[
\leq \frac{3}{2-k} \cdot 4^{i(k-1)} \|f\|_{L^1(0,1)}.
\]
and
\[
\sup_{(s,t) \in D_1} \|T_3f\|_{L^\infty(0,1)} \leq \sup_{(s,t) \in D_1} \int_0^{s-4^i} (s-r)^{k-3} dr \leq \frac{1}{2-k} \cdot 4^{i(k-2)} \|f\|_{L^\infty(0,1)}.
\]

By the real interpolation theorem, for $T_3$ holds.

Proof of Lemma 6.1

1. Fix $i = -1, -2, \ldots$. Since
\[
E \int_{\mathbb{R}^n} \int_0^1 \int_{4^i < |t-s| < 4^{i+1}} \frac{|v_3(t,x) - v_3(s,x)|^p}{|t-s|^{1+\frac{2k}{p}}} ds dx dt
\]
\[
\leq 2E \int_{\mathbb{R}^n} \int_0^1 \int_{4^i}^{t-4^i} \frac{|v_3(t,x) - v_3(s,x)|^p}{(t-s)^{1+\frac{2k}{p}}} ds dx dt,
\]
we assume $t > s$. Note that
\[
v_3(t,x) - v_3(s,x) = \int_s^t \int_{\mathbb{R}^n} \Gamma(t-r,x-y) \tilde{g}(r,y) dy dw_r
\]
\[
+ \int_s^t \int_{\mathbb{R}^n} (\Gamma(t-r,x-y) - \Gamma(s-r,x-y)) \tilde{g}(r,y) dy dw_r.
\]
The right-hand side of (6.9) is bounded by the sum of the following quantities (up to a constant multiple):

\[
I_1 = E \int_{\mathbb{R}^n} \int_0^1 \int_{4^i}^{t-4^i} \frac{|\int_s^t \int_{\mathbb{R}^n} \Gamma(t-r,x-y) \tilde{g}(r,y) dy dw_r|^p}{(t-s)^{1+\frac{2k}{p}}} ds dx dt,
\]
\[
I_2 = E \int_{\mathbb{R}^n} \int_0^1 \int_{4^i}^{t-4^i} \frac{|\int_s^t \int_{\mathbb{R}^n} (\Gamma(t-r,x-y) - \Gamma(s-r,x-y)) \tilde{g}(r,y) dy dw_r|^p}{(t-s)^{1+\frac{2k}{p}}} ds dx dt.
\]

2. Recall that we assume $\frac{1}{p} < k < 1$ and $p \geq 2$.

Estimation of $I_1$. By Burkholder-Davis-Gundy inequality (BDG) (see Section 2.7 in [12]) $I_1$ is dominated by, up to a constant multiple,
\[
E \int_{\mathbb{R}^n} \int_0^1 \int_{4^i}^{t-4^i} \frac{(\int_s^t \int_{\mathbb{R}^n} \Gamma(t-r,x-y) \tilde{g}(r,y) dy)^2 dr}{(t-s)^{1+\frac{2k}{p}}} ds dx dt.
\]

Next, by Minkowski’s inequality for integrals and Lemma 6.3 (1), the expression (6.10) is bounded
by, up to a constant multiple,
\[
E \int_4^1 \int_{(t-4')^1+1}^{t-4'} \left( \int_1^1 (t-r, x-y, y) \hat{y}(r, y) dy |p dx) \right)^{\frac{p}{2}} dsdt \\
\lesssim E \int_4^1 \int_{(t-4')^1+1}^{t-4'} \left( \int_1^1 (t-r, x-y, y) \hat{y}(r, y) dy |p dx) \right)^{\frac{p}{2}} dsdt \\
\lesssim 4^{-i(1+\frac{q}{2})} E \int_4^1 \int_{(t-4')^1+1}^{t-4'} \left( \int_1^1 (t-r, x-y, y) \hat{y}(r, y) dy |p dx) \right)^{\frac{p}{2}} dsdt.
\]
Applying Lemma 6.3 with the operator $T_1$ and $\frac{q}{2}$ in place of $q$, we receive
\[
I_1 \lesssim c \| \hat{y} \|^p_{L^p(\Omega \times (0, 1), \mathcal{H}^{k-1}(\mathbb{R}^n))}.
\]

**Estimation of $I_2$**

BDG inequality $I_2$ is dominated by, up to a constant multiple,
\[
E \int_{\mathbb{R}^n} \int_4^1 \int_{(t-4')^1+1}^{t-4'} \left( \int_1^1 |\frac{\int_{\mathbb{R}^n} (t-r, x-y, y) \hat{y}(r, y) dy |p dx) \right)^{\frac{p}{2}} dsdt \\
\lesssim E \int_{\mathbb{R}^n} \int_4^1 \int_{(t-4')^1+1}^{t-4'} \left( \int_1^1 (t-r, x-y, y) \hat{y}(r, y) dy |p dx) \right)^{\frac{p}{2}} dsdt \\
\lesssim + E \int_{\mathbb{R}^n} \int_4^1 \int_{(t-4')^1+1}^{t-4'} \left( \int_1^1 (t-r, x-y, y) \hat{y}(r, y) dy |p dx) \right)^{\frac{p}{2}} dsdt \\
= I_{21} + I_{22}.
\]

By Minkowski's inequality for integrals and Lemma 6.3 (1) the term $I_{21}$ is bounded by, up to a constant multiple,
\[
E \int_4^1 \int_{(t-4')^1+1}^{t-4'} \left( \int_1^1 (t-r)^{k-1} + (s-r)^{k-1} \| \hat{y}(r, \cdot) \|^2_{H^{k-1}(\mathbb{R}^n)} dr \right)^{\frac{p}{2}} dsdt \\
\lesssim 4^{-i(1+\frac{q}{2})} E \int_4^1 \int_{(t-4')^1+1}^{t-4'} \left( \int_1^1 (s-r)^{k-1} \| \hat{y}(r, \cdot) \|^2_{H^{k-1}(\mathbb{R}^n)} dr \right)^{\frac{p}{2}} dsdt
\]

; we used $k < 1$. Lemma 6.3 with the operator $T_1$ gives us
\[
I_{21} \lesssim c \| \hat{y} \|^p_{L^p(\Omega \times (0, 1), \mathcal{H}^{k-1}(\mathbb{R}^n))}.
\]

By Minkowski's inequality for integrals again and Lemma 6.3 (2) the term $I_{22}$ is dominated by, up to a constant multiple,
\[
E \int_4^1 \int_{(t-4')^1+1}^{t-4'} \left( \int_0^1 |\int_{\mathbb{R}^n} (t-r, x-y, y) \hat{y}(r, y) dy |p dx) \right)^{\frac{p}{2}} dsdt \\
\lesssim 4^{-i(1+\frac{q}{2})} E \int_4^1 \int_{(t-4')^1+1}^{t-4'} \left( \int_0^1 (t-r)^2 (s-r)^{k-3} \| \hat{y}(r, \cdot) \|^2_{H^{k-1}(\mathbb{R}^n)} dr \right)^{\frac{p}{2}} dsdt \\
\lesssim 4^{-i(1+\frac{q}{2})} \cdot 4^p \cdot E \int_4^1 \int_{(t-4')^1+1}^{t-4'} \left( \int_0^1 (s-r)^{k-3} \| \hat{y}(r, \cdot) \|^2_{H^{k-1}(\mathbb{R}^n)} dr \right)^{\frac{p}{2}} dsdt.
\]
Then Lemma 6.4 with the operator $T_3$ gives us

$$I_{22} \lesssim c \|\tilde{g}\|_{L^p(\Omega \times (0,1), \mathcal{P}, H^{k-1}_p(\mathbb{R}^n))^\cdot}^p.$$  

3. By the estimations of $I_1, I_2$ our claim follows. \hfill \Box

7 Proof of Lemma 3.7 (2)

Again, we just assume $T = 1$. We start with the following lemmas.

Lemma 7.1. For $0 < t, r < \infty$

$$\int_{\mathbb{R}^n} |\Gamma(t + r, y) - \Gamma(r, y)| dy \lesssim \begin{cases} \frac{t}{r}, & t < r, \\ 1, & t \geq r. \end{cases}$$

; this is almost obvious and the proof is omitted.

Lemma 7.2. Let $0 \leq \theta < 1, 1 < p < \infty$. Then for $g \in H^\theta_{p,o}(D)$,

$$\int_D \delta(y)^{-p\theta} |g(y)|^p dy \leq c \|g\|_{H^{\theta p,o}(D)}^p,$$

where $\delta(y) = \text{dist}(y, \partial D)$. The constant $c$ depends only on $p, n$.

Proof. We may assume $0 < \theta < 1$. We use complex interpolation of $L^p$-spaces of measures. Let $d\mu_0(y) = dy$ and $d\mu_1(y) = \delta^{-p}(y) dy$. The complex interpolation space between $L^p(d\mu_0)$ and $L^p(d\mu_1)$ with index $\theta$ is

$$(L^p(d\mu_0), L^p(d\mu_1))[\theta] = L^p(d\mu_\theta), \quad d\mu_\theta(y) := \delta^{-p\theta} dy$$

(see Theorem 5.5.3 in [3]). Note that using Hardy’s inequality, we obtain that for $g \in H^1_{p,o}(D)$

$$\left( \int_D \delta(y)^{-p} |g(y)|^p dy \right)^\frac{1}{p} \leq c \left( \int_D |\nabla g(y)|^p dy \right)^\frac{1}{p} = c \|g\|_{H^1_{p,o}(D)}.$$  

Since $(H^1_{p,o}(D), L^p(D))[\theta] = H^{\theta}_{p,o}(D)$ (see Proposition 2.1 in [7]), we get

$$\left( \int_D \delta^{-p\theta}(y) |g(y)|^p dy \right)^\frac{1}{p} \leq c \|g\|_{(H^{1}_{p,o}(D), L^p(D))[\theta]} = c \|g\|_{H^{\theta}_{p,o}(D)}.$$  

\hfill \Box

Proof of Lemma 3.7 (2) 1. Recall $1 \leq k < 1 + \frac{1}{p}$ and $p \geq 2$. For $g \in \mathbb{R}^{k-1}(D_T) = L^p(\Omega \times (0,1), \mathcal{P}, H^{k-1}_p(\mathbb{R}^n))$, we denote $\tilde{g} \in L^p(\Omega \times (0,1), \mathcal{P}, H^{k-1}_p(\mathbb{R}^n))$ by $\tilde{g}(\omega, t, x) = g(\omega, t, x)$ for $x \in D$ and $\tilde{g}(\omega, t, x) = 0$ for $x \in \mathbb{R}^n \setminus D$. Then by lemma 6.4 we have $v_3 \in L^p(\Omega \times (0,1), \mathcal{P}, H^{k}_p(\mathbb{R}^n))$, where

$$v_3(t, x) = \int_0^t \int_{\mathbb{R}^n} \Gamma(t - s, x - y)\tilde{g}(s, y) dy dw_s.$$
By the usual trace theorem (see [3]), we get \( v_3|_{\partial D} \in L^p(\Omega \times (0,1), P, B_{k-\frac{1}{p}}(\partial D)) \). Hence, it is sufficient to show that

\[
E \int_{\partial D} \int_0<t<1 \frac{|v_3(x,t) - v_3(x,s)|^p}{(t-s)^{1+\frac{\frac{1}{p}}{k-\frac{1}{p}}}} ds dt \leq \|g\|_{L^p(\{0,1\}, P, H_{k-1}^{k-\frac{1}{p}}(D))}.
\] (7.1)

Then, using real interpolation (see lemma 3.3), we complete the proof of lemma 3.7 (2).

2. The left-hand side of (7.1) is bounded by the sum of the following quantities (up to a constant multiple):

\[
J_1 = E \int_{\partial D} \int_0^t \int_0^t |\int_{\partial D} \Gamma(t-r,x-y)\tilde{g}(r,y)dydw_r|^p \frac{ds dt ds}{(t-s)^{1+\frac{\frac{1}{p}}{k-\frac{1}{p}}}} d\sigma(x),
\]

\[
J_2 = E \int_{\partial D} \int_0^t \int_0^t |\int_{\partial D} \Gamma(t-r,x-y) - \Gamma(s-r,x-y))\tilde{g}(r,y)dydw_r|^p \frac{ds dt ds}{(t-s)^{1+\frac{\frac{1}{p}}{k-\frac{1}{p}}}} d\sigma(x).
\]

**Estimation of \( J_1 \)**

By BDG’s inequality, \( J_1 \) is dominated by, up to a constant multiple,

\[
E \int_{\partial D} \int_0^t \int_0^t \left( \int_{\partial D} \Gamma(t-r,x-y)\tilde{g}(r,y)dydw_r \right)^\frac{p}{2} \frac{ds dt ds}{(t-s)^{1+\frac{\frac{1}{p}}{k-\frac{1}{p}}}} d\sigma(x).
\] (7.2)

Note

\[
\int_{\partial D} \left( \int_0^t |\int_{\partial D} \Gamma(t-r,x-y)\tilde{g}(r,y)dy|^p ds \right)^\frac{1}{p} d\sigma(x)
\]

\[
\lesssim \left( \int_0^t \left( \int_{\partial D} \Gamma(t-r,x-y)\tilde{g}(r,y)dy \right)^p ds \right)^\frac{1}{p} \frac{ds}{dr}
\]

\[
\lesssim \left( \int_0^t \left( \int_{\partial D} \Gamma(t-r,x-y)\tilde{g}(r,y)dy \right)^p ds \right)^\frac{1}{p} \frac{ds}{dr}
\]

\[
\lesssim \left( \int_0^t \left( \int_{\partial D} \Gamma(t-r,x-y)\tilde{g}(r,y)dy \right)^p ds \right)^\frac{1}{p} \frac{ds}{dr}
\]

where \( \frac{1}{p} + \frac{1}{p'} = 1 \). Note that for \( y \in D \) there is a \( x_y \in \partial D \) such that \( \delta(y) = |y - x_y| \), where \( \delta(y) = \text{dist}(y, \partial D) \). Since \( D \) is a bounded Lipschitz domain, there is \( r_0 > 0 \) independent of \( x_y \) such that \( |y - x| \approx \delta(y) + |x - x| \) for all \( |x - x| < r_0 \). We have

\[
\int_{\partial D} \Gamma(t-r,x-y)d\sigma(x)
\]

\[
\lesssim \int_{|x-x'|<r_0} \left( t-r - \frac{\varepsilon}{t-r} \right) e^{-\frac{\delta(y)^2+|x-y|^2}{t-r}} d\sigma(x) + \int_{|x-x'|\geq r_0} \left( t-r - \frac{\varepsilon}{t-r} \right) e^{-\frac{\delta(y)^2+|x-y|^2}{t-r}} d\sigma(x)
\]

\[
\lesssim \int_{|x'|<r_0, x' \in R^{n-1}} \left( t-r - \frac{\varepsilon}{t-r} \right) e^{-\frac{\delta(y)^2+|x|^2}{t-r}} d\sigma(x) + \int_{|x'|\geq r_0} \left( t-r - \frac{\varepsilon}{t-r} \right) e^{-\frac{\delta(y)^2+|x|^2}{t-r}} d\sigma(x)
\]

\[
\lesssim \left( t-r - \frac{\varepsilon}{t-r} \right) e^{-\frac{\delta(y)^2}{t-r}} \int_{R^{n-1}} e^{-\frac{\delta(y)^2}{t-r}} d\sigma(x) + \left( t-r - \frac{\varepsilon}{t-r} \right) e^{-\frac{\delta(y)^2}{t-r}} d\sigma(x)
\]

\[
\lesssim \left( t-r - \frac{\varepsilon}{t-r} \right) e^{-\frac{\delta(y)^2}{t-r}}
\]

(7.4)
By (7.3) and the Hölder inequality, the last term in (7.3) is bounded by, up to a constant multiple,
\[
\left( \int_s^t \left( \int_D |\tilde{g}(r, y)|^p (t - r)^{-\frac{1}{2} - \frac{p}{2} \cdot \frac{\delta(y)}{2}} dy \right)^\frac{2}{p} dr \right)^\frac{p}{2} \lesssim (t - s)^{\frac{p}{2} - 1} \int_s^t \int_D |\tilde{g}(r, y)|^p (t - r)^{-\frac{1}{2} - \frac{p}{2} \cdot \frac{\delta(y)}{2}} dy dr.
\]  

(7.5)

Hence, via Fubini’s Theorem, (7.2) is dominated by, up to a constant multiple,
\[
E \int_0^1 \int_D |\tilde{g}(r, y)|^p \left[ \int_r^t (t - s)^{-\frac{p}{2} \cdot \frac{\delta(y)}{2}} dsdt \right] dy dr
\lesssim E \int_0^1 \int_D |\tilde{g}(r, y)|^p \left[ \int_0^r e^{-c \delta(y)} \cdot (t - r)^{-\frac{p}{2} \cdot \frac{\delta(y)}{2}} dt \right] dy dr
= E \int_0^1 \int_D |\tilde{g}(r, y)|^p \left[ \int_0^1 e^{-c \delta(y)} \cdot t^{-\frac{p}{2} \cdot \frac{\delta(y)}{2}} dt \right] dy dr
= E \int_0^1 \int_D \delta^{-\frac{p}{2} \cdot \frac{\delta(y)}{2}} (r, y)) |\tilde{g}(r, y)|^p \left[ \int_0^\infty e^{-c t \cdot t^{-\frac{p}{2} \cdot \frac{\delta(y)}{2}} dt \right] dy dr
\lesssim E \int_0^1 \int_D \delta^{-\frac{p}{2} \cdot \frac{\delta(y)}{2}} (r, y)) |\tilde{g}(r, y)|^p dy dr
\lesssim E \int_0^1 \|g(\cdot, r)\|_{H_{p, o}^{k-1}(D)}^p dr
\]

: for the last inequality we used the assumption \( g \in H_{p, o}^{k-1}(D) \) and Lemma 7.2 with \( \theta = k - 1 \).

Estimation of \( J_2 \) By BDG’s inequality, \( J_2 \) is dominated by, up to a constant multiple,
\[
E \int_0^1 \int_D \int_0^t \left( \int_0^s |D(t - r, x - y) - \Gamma(s - r, x - y))|\tilde{g}(r, y)| dy dr \right)^2 dsdt dr dx
\]

(7.6)

Define \( A := A(t, s, r, x, y) = \Gamma(t - r, x - y) - \Gamma(s - r, x - y) \). If \( p > 2 \), using the Hölder inequality twice, we get
\[
\left( \int_0^s \left[ \int_D A \cdot \tilde{g}(r, y) dy \right]^2 dr \right)^\frac{p}{2} \leq \left( \int_0^s \left[ \int_D |A| dy \right]^{2(p-1)/p} \left[ \int_D |A||\tilde{g}(r, y)|^p dy \right]^{2/p} dr \right)^\frac{p}{2}
\leq \left( \int_0^s \left[ \int_D |A| dy \right]^{2(p-1)/p} dr \right)^{\frac{p}{2(p-1)}} \int_0^s \int_D |A||\tilde{g}(r, y)|^p dy dr.
\]

Next, by changing variable from \( r \) to \( s - r \) and Lemma 7.1
\[
\int_0^s \left( \int_D |A| dy \right)^{2(p-1)/p} dr = \int_0^s \left( \int_D |\Gamma(t - s + r, x - y) - \Gamma(r, x - y)| dy \right)^{2(p-1)/p} dr
\lesssim \left\{ \begin{array}{ll}
\int_0^s dr, & s < t - s \\
\int_0^s dr + \int_{s-t}^{s} (t - s)^{2(p-1)/p} dr, & s \geq t - s
\end{array} \right.
= \left\{ \begin{array}{ll}
t - s + (s - t)^{2(p-1)/p} (t - s)^{-\frac{p}{2} \cdot \frac{\delta(y)}{2}} - s^{-\frac{p}{2} \cdot \frac{\delta(y)}{2}}, & s \geq t - s
\end{array} \right.
\lesssim (t - s)
\]

and
\[
\left( \int_0^s \left[ \int_D A \cdot \tilde{g}(r, y) dy \right]^2 dr \right)^\frac{p}{2} \lesssim (t - s)^{\frac{p}{2} - 1} \int_0^s \int_D |A||\tilde{g}(r, y)|^p dy dr.
\]

(7.8)
If \( p = 2 \), Lemma \( \text{[7.4]} \) with \( p = 2 \) and Lemma \( \text{[7.4]} \) immediately yields \( \text{[7.8]} \). Hence, \( \text{[7.3]} \) is dominated by, up to a constant multiple,

\[
E \int_{\partial D} \int_{0}^{1} \int_{0}^{s} \int_{D} |A(t, s, r, x, y)||\tilde{g}(r, y)|^p dy dr \, (t - s)^{-\frac{p}{2} (k-1) - \frac{\gamma}{2}} ds dt ds \sigma(x)
\]

\[
\lesssim E \int_{D} |\tilde{g}(r, y)|^p \left[ \int_{0}^{t-s} \int_{0}^{t} (t - s)^{-\frac{p}{2} (k-1) - \frac{\gamma}{2}} ds dt \right] dy dr.
\]

We estimate the boundary \( (\partial D) \) integral part: Since \( s < t \), we have

\[
\int_{\partial D} |\Gamma(t, x - y) - \Gamma(s, x - y)| d\sigma(x)
\]

\[
\leq \left( \frac{1}{s^{\frac{n}{2}}} - \frac{1}{t^{\frac{n}{2}}} \right) \int_{\partial D} e^{-\frac{|x-y|^2}{a}} d\sigma(x) + t^{-\frac{n}{2}} \int_{\partial D} e^{-\frac{|x-y|^2}{a}} - e^{-\frac{|x-y|^2}{a}} d\sigma(x)
\]

\[
= K_1 + K_2.
\]

Applying \( \text{[7.4]} \) again,

\[
K_1 = \frac{t^{\frac{n}{2}} - s^{\frac{n}{2}}}{t^{\frac{n}{2}}s^{\frac{n}{2}}} \int_{\partial D} e^{-\frac{|x-y|^2}{a}} d\sigma(x) \leq \frac{t^{\frac{n}{2}} - s^{\frac{n}{2}}}{t^{\frac{n}{2}}} s^{-\frac{n}{2}} e^{-\frac{\delta(y)^2}{a}}
\]

\[
\leq \begin{cases} s^{-\frac{n}{2}} e^{-\frac{\delta(y)^2}{a}}, & 0 < s < \frac{t}{2}, \\ t^{-\frac{n}{2}} (t - s) e^{-\frac{\delta(y)^2}{a}}, & \frac{t}{2} \leq s < t. \end{cases}
\]

For \( K_2 \) we consider two cases. If \( 0 < s < \frac{1}{2} t \), using \( \text{[7.3]} \), we get

\[
K_2 \leq t^{-\frac{n}{2}} \int_{\partial D} e^{-\frac{|x-y|^2}{a}} d\sigma(x) \leq t^{-\frac{1}{2}} e^{-\frac{\delta(y)^2}{a}}.
\]

For \( \frac{1}{2} t < s < t \), using the Mean Value Theorem, there is a \( \eta \) satisfying \( s < \eta < t \) such that

\[
K_2 = t^{-\frac{n}{2}} \int_{\partial D} (t - s) \frac{|x-y|^2}{4\eta^2} e^{-\frac{|x-y|^2}{\eta}} d\sigma(x)
\]

and this leads to

\[
K_2 \lesssim t^{-\frac{n}{2}} \int_{\partial D} (t - s) \frac{|x-y|^2}{t^2} e^{-\frac{|x-y|^2}{a}} d\sigma(x)
\]

\[
\lesssim t^{-\frac{n}{2} - 2} (t - s) \int_{\partial D} (|x - y|^2 + \delta(y)^2) e^{-\frac{|x-y|^2 + \delta(y)^2}{a}} d\sigma(x)
\]

\[
\lesssim t^{-\frac{n}{2} - 2} (t - s)e^{-\frac{\delta(y)^2}{a}} (t^{\frac{n}{2}} + \delta^2(y)t^{\frac{n}{2}})
\]

\[
= (t - s)e^{-\frac{\delta(y)^2}{a}} (t^{\frac{n}{2}} + \delta(y)^2 t^{\frac{n}{2}}).
\]

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By these estimations, the bracket in (7.9) is bounded by, up to a constant multiple,
\[
\int_0^{1-r} t^{-\frac{k}{2} (k-1) - \frac{3}{2}} \left( \int_0^t \left( t^{-\frac{3}{2}} e^{-c\delta(y)^2 s} + s^{-\frac{1}{2}} e^{-\frac{c\delta(y)^2}{s}} \right) ds \right) dt + \int_0^{1-r} e^{-c\delta(y)^2 s} (t^{-\frac{3}{2}} + \delta(y) t^{-\frac{3}{2}}) \left( \int_0^t (t-s)^{-\frac{k}{2} (k-1) - \frac{3}{2}} ds \right) dt \\
\lesssim \int_0^{1-r} \left[ t^{-\frac{k}{2} (k-1) - \frac{3}{2}} e^{-c\delta(y)^2 s} + t^{-\frac{k}{2} (k-1) - \frac{3}{2}} \cdot \delta(y) \cdot \int_{2\delta(y)^2}^\infty s^{-\frac{1}{2}} e^{-c\delta(y)^2 s} ds + t^{-\frac{k}{2} (k-1) - \frac{3}{2}} \cdot e^{-c\delta(y)^2} \cdot \delta(y)^2 \right] dt =: L_1 + L_2 + L_3
\]
for the inequality we used the assumption \(k < 1 + \frac{1}{p}\). It is easy to see that the terms \(L_1\) and \(L_3\) are dominated by \(\delta(y)^{-p(k-1)}\). This is also true for \(L_2\); if \(2\delta(y)^2 \geq (1-r)\), then \(2\delta(y)^2 \geq t\) and
\[
L_2 \lesssim \delta(y) \int_0^{1-r} t^{-\frac{k}{2} (k-1) - \frac{3}{2}} \int_{2\delta(y)^2}^\infty s^{-\frac{1}{2}} e^{-c\delta(y)^2 s} ds dt \lesssim \delta(y) \int_0^{1-r} t^{-\frac{k}{2} (k-1) - \frac{3}{2}} e^{-c\delta(y)^2} dt \lesssim \delta(y)^{-p(k-1)}.
\]
If \(2\delta(y)^2 \leq 1-r\),
\[
L_2 \lesssim \delta(y) \int_0^{1-r} t^{-\frac{k}{2} (k-1) - \frac{3}{2}} \int_{2\delta(y)^2}^\infty s^{-\frac{1}{2}} e^{-c\delta(y)^2 s} ds dt \lesssim \delta(y) \int_0^{2\delta(y)^2} t^{-\frac{k}{2} (k-1) - \frac{3}{2}} e^{-c\delta(y)^2} dt + \int_{2\delta(y)^2}^{1-r} t^{-\frac{k}{2} (k-1) - \frac{3}{2}} dt \lesssim \delta(y)^{-p(k-1)}.
\]
After all, \(L_1\) (hence \(J_2\)) is bounded by, up to a constant multiple,
\[
E \int_0^1 \int_D \delta(y)^{-p(k-1)} |\tilde{g}(r,y)| dy dr \lesssim E \int_0^1 \|g(\cdot, r)\|_{H^{k-1}_p(D)}^p dr
\]
for the assumption \(g \in H^{k-1}_{p,\infty}(D)\) and Lemma 7.2.

3. The step 2 implies (7.11). The lemma is proved. \(\square\)

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