BIRATIONAL BOUNDEDNESS OF RATIONALLY CONNECTED LOG CALABI–YAU PAIRS WITH FIXED INDEX

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Abstract. We show that the set of rationally connected projective varieties $X$ of a fixed dimension such that $(X, B)$ is klt, and $-l(K_X + B)$ is Cartier and nef for some fixed positive integer $l$, is bounded modulo flops.

1. Introduction

Throughout this paper, we work over an uncountable algebraically closed field of characteristic 0, for instance, the complex number field $\mathbb{C}$.

A normal projective variety $X$ is a Fano (resp. Calabi–Yau) variety if $-K_X$ is ample (resp. $K_X \sim_{\mathbb{Q}} 0$). According to the Minimal Model Program, Fano varieties and Calabi–Yau varieties form fundamental classes in birational geometry as building blocks of algebraic varieties. Hence, it is interesting to ask whether such kinds of varieties satisfy any finiteness or boundedness properties. For Fano varieties, Birkar [3, 4] showed that the set of $\epsilon$-lc Fano varieties of a fixed dimension forms a bounded family for a fixed $\epsilon > 0$, which is known as the Borisov–Alexeev–Borisov (BAB) Conjecture. For (log) Calabi–Yau varieties, things get more complicated. It is expected that certain Calabi–Yau varieties with special geometric properties, for example, Calabi–Yau manifolds or rationally connected Calabi–Yau varieties, form a bounded family, see [9, 8, 21, 6, 10, 22] for some recent progress.

In this paper, we focus on the following conjecture of MʻKernan and Prokhorov. It includes rationally connected Calabi–Yau varieties, and is a natural generalization of the BAB conjecture as any klt Fano variety is rationally connected [27, 13].

Conjecture 1.1 ([25, Conjecture 3.9]). Fix a positive integer $d$ and a positive real number $\epsilon$. Then the set of varieties $X$ such that

(1) $X$ is a normal projective variety of dimension $d$,
(2) $X$ is rationally connected,
(3) $(X, B)$ is $\epsilon$-lc for some $\mathbb{R}$-divisor $B \geq 0$, and
(4) $-(K_X + B)$ is nef,

is bounded.

Conjecture [14] was proved in dimension 2 by [1 Theorem 6.9]. When $\dim X = 3$, [6, Theorem 1.6] showed that $X$ is bounded modulo flops (see §2.5 for definition). Indeed this holds for any non-canonical klt pair $(X, B)$ when the coefficients of $B$ belong to a set satisfying the descending chain condition, without assuming $X$ being rationally connected, see [17, Theorem 1.9], [21, Theorem 1.6]. Even the case $K_X \equiv 0$ and $B = 0$ is sufficiently interesting and widely open in dimension at least 4 [8, Conjecture 1.3].

In this paper, we prove that $X$ is bounded modulo flops in Conjecture [14] under an additional assumption that the Cartier index of $K_X + B$ is fixed. The following is our main theorem.

Theorem 1.2. Fix positive integers $d$ and $l$. The set of varieties $X$ such that

(1) $X$ is a normal projective variety of dimension $d$,
(2) $X$ is rationally connected,

is bounded.

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is bounded modulo flops.

In particular, when \( l(K_X + B) \sim 0 \), Theorem 1.2 gives a simpler proof for [6, Theorem 1.4]. This special case is already interesting enough as it implies the birational boundedness of elliptic Calabi–Yau varieties with a rational section [6, Theorems 1.2 and 1.3].

Both the proof of [6, Theorem 1.4] and ours heavily rely on Birkar’s earlier work on log Calabi–Yau fibrations [2] to finish the induction on the dimension. The difference is that [6] applies towers of Fano fibrations introduced in [9] which makes the proof quite involved, and the main new ingredient of ours is to use generalized pairs (see §2.2) introduced in [7] to reduce the problem to log Calabi–Yau g-pairs, and then to apply the theory of complements to control the index of the moduli part of the canonical bundle formula. We prove the following theorem under this general setting which fits well with dimension induction.

**Theorem 1.3.** Fix positive integers \( d \) and \( l \). The set of varieties \( X \) such that

1. \( X \) is a normal projective variety of dimension \( d \),
2. \( X \) is rationally connected,
3. \( (X, B + M) \) is klt g-pair for some \( \mathbb{Q} \)-divisor \( B \geq 0 \) and b-\( \mathbb{Q} \)-divisor \( M \),
4. \( l(K_X + B + M_X) \sim 0 \), and
5. \( lM \) is b-Cartier,

is bounded modulo flops.

In fact, we can more generally prove the following relative version of Theorem 1.3. See §2 for definitions.

**Theorem 1.4.** Fix positive integers \( d, l, r \) and \( A \). The set of base-polarized fibrations \( \pi : X \to (Z, A) \) such that

1. \( X \) is a normal projective variety of dimension \( d \),
2. \( \pi : X \to Z \) is a rationally connected fibration,
3. \( (X, B + M) \) is a klt g-pair for some \( \mathbb{Q} \)-divisor \( B \geq 0 \) and b-\( \mathbb{Q} \)-divisor \( M \),
4. \( l(K_X + B + M_X) \sim \pi^*L \) for some Cartier divisor \( L \) on \( Z \),
5. \( lM \) is b-Cartier,
6. \( A \geq 0 \) is a very ample divisor on \( Z \) such that \( A^{\dim Z} \leq r \), and
7. \( A - \frac{1}{r}L \) is ample,

is bounded in codimension one. In particular, such \( X \) is bounded in codimension one.

Here if \( \dim Z = 0 \), then we always take \( L = A = A^{\dim Z} = 0 \) by default.

If \(-K_X\) is big over \( Z \), then Theorem 1.4 is a special case of [2, Theorem 2.2] (see Theorem 3.1). So here in Theorem 1.4 we treat the general setting that \( X \to Z \) is rationally connected, and we expect that even if the torsion index of \( K_X + B + M_X \) over \( Z \) and the b-Cartier index of \( M \) are not assumed to be bounded, \( X \to Z \) should still belong to a bounded family. Conjecture 1.5 is a generalization of [2, Theorems 1.2, 2.2], which could be regarded as the relative version of Conjecture 1.1.

**Conjecture 1.5.** Let \( d, r \) be positive integers, and \( \epsilon \) a positive real number. Then the set of base-polarized fibrations \( X \to (Z, A) \) such that there exists a \( (d, r, \epsilon) \)-logCY rationally connected fibration \( (X, B + M) \to (Z, A) \), is bounded.

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2. Preliminaries

A contraction or a fibration is a projective morphism $\pi : X \to Z$ between normal varieties such that $\pi_*\mathcal{O}_X = \mathcal{O}_Z$. In this case, for a closed point $z \in Z$, the fiber over $z$ is always denoted by $X_z$.

A base-polarized fibration $\pi : X \to (Z, A)$ consists of a fibration $\pi : X \to Z$ and a very ample divisor $A \geq 0$ on $Z$.

A birational contraction is a birational map $\phi : X \dashrightarrow X'$ between normal varieties such that $\phi^{-1}$ does not contract any divisors. A small birational map is a birational map $\phi : X \dashrightarrow X'$ between normal varieties which is isomorphic in codimension one.

A projective variety is said to be rationally connected if any two general points can be connected by the image of a rational curve. A fibration is said to be rationally connected if its general fibers are rationally connected.

2.1. Divisors and b-divisors. Let $\mathbb{K}$ be either the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let $X$ be a normal variety. A $\mathbb{K}$-divisor is a finite $\mathbb{K}$-linear combination $D = \sum_i d_i D_i$ of prime Weil divisors $D_i$, and $d_i$ denotes the coefficient of $D_i$ in $D$. A $\mathbb{K}$-Cartier divisor is a $\mathbb{K}$-linear combination of Cartier divisors.

**Definition 2.1** (cf. [26]). Let $X$ be a normal variety. Consider an infinite linear combination $D := \sum_D d_D D$, where $d_D \in \mathbb{K}$ and the infinite sum runs over all divisorial valuations of the function field of $X$. For any birational model $Y$ of $X$, the trace of $D$ on $Y$ is defined by $D_Y := \sum_{\mathrm{codim} D = 1} d_D D$. Such $D$ is called a $\mathbb{K}$-divisor (or $\mathbb{R}$-divisor for short when $\mathbb{K}$ is clear) if on each birational model $Y$ of $X$, the trace $D_Y$ is a $\mathbb{K}$-divisor, or equivalently, $D_Y$ is a finite sum.

For a $\mathbb{K}$-Cartier divisor $D$ on $X$, the Cartier closure of $D$ is the b-divisor $\overline{D}$ whose trace on every birational model $f : Y \to X$ is $f^*D$. A b-divisor $D$ is said to be b-Cartier if there is a birational model $X'$ over $X$ such that $D_{X'}$ is Cartier and $D = \overline{D_{X'}}$.

When $X$ is projective, a b-divisor $D$ is said to be b-nef if there is a birational model $X'$ over $X$ such that $D_{X'}$ is $\mathbb{K}$-Cartier and nef, and $D = \overline{D_{X'}}$.

2.2. Singularities of g-pairs.

**Definition 2.2.** A generalized pair (g-pair for short) $(X, B + M)$ consists of

- a normal projective variety $X$,
- an effective $\mathbb{R}$-divisor $B$ on $X$ called the boundary part, and
- a b-nef b-$\mathbb{R}$-divisor $M$ on $X$ called the moduli part, such that $K_X + B + M_X$ is $\mathbb{R}$-Cartier.

Let $(X, B + M)$ be a g-pair. For a prime divisor $E$ over $X$, take a resolution $f : W \to X$ such that $E$ is a divisor on $W$, then we may write

$$K_W + B_W + M_W \sim_{\mathbb{R}} f^*(K_X + B + M_X),$$

where $B_W$ is the unique $\mathbb{R}$-divisor on $W$ such that $f_*B_W = B$. The log discrepancy of $E$ with respect to $(X, B + M)$ is defined as

$$a(E, X, B + M) := 1 - \mult_E B_W.$$

**Definition 2.3.** Fix a non-negative real number $\epsilon$. A g-pair $(X, B + M)$ is said to be klt (resp. $\epsilon$-lc, lc), if $a(E, X, B + M) > 0$ (resp. $\geq \epsilon, \geq 0$) for any prime divisor $E$ over $X$.

An lc g-pair $(X, B + M)$ is said to be dlt if there is a closed subset $V \subset X$ such that

- $X \setminus V$ is smooth and $\mathrm{Supp} B|_{X \setminus V}$ is simple normal crossing, and
- if $a(E, X, B + M) = 0$ for some prime divisor $E$ over $X$, then $\mathrm{Center}_X(E)|_{X \setminus V} \neq \emptyset$ is an lc center of $(X \setminus V, B|_{X \setminus V})$.

If $M = 0$, then all definitions coincide with those of a usual pair. In this paper, we only consider (g)-pairs with projective ambient varieties. We refer the reader to [12] and references therein for recent progress on g-pairs.
2.3. Contractions of Fano type.

Definition 2.4 ([29]). Let $\pi : X \to Z$ be a contraction between normal varieties, $X$ is said to be of \textit{Fano type} over $Z$ if one of the following equivalent conditions holds:

1. there exists a klt pair $(X, B)$ such that $-(K_X + B)$ is ample over $Z$;
2. there exists a klt pair $(X, B')$ such that $K_X + B' \equiv_Z 0$ and $-K_X$ is big over $Z$;
3. there exists a klt $g$-pair $(X, B'' + M)$ such that $K_X + B'' + M \equiv_Z 0$ and $-K_X$ is big over $Z$.

Here for the equivalence of (3) and (1), we can use the proof of [2] Lemma 3.24.

It is well-known that if $X \to Z$ is a contraction of Fano type, then we can run the MMP for any $\mathbb{R}$-Cartier divisor on $X$ over $Z$, and the property of being Fano type is preserved by MMP and contractions, see for example [3, §2.10] for details.

2.4. Log Calabi–Yau fibrations.

Definition 2.5. A pair $(X, B)$ is called a \textit{log Calabi–Yau pair} (logCY pair for short) if $K_X + B \sim_{\mathbb{R}} 0$. A $g$-pair $(X, B + M)$ is called a \textit{log Calabi–Yau g-pair} (logCY g-pair for short) if $K_X + B + M \sim_{\mathbb{R}} 0$.

Definition 2.6. Fix positive integers $d, r$ and a positive real number $\epsilon$. A \textit{weak} $(d, r, \epsilon)$-logCY fibration $(X, B + M) \to (Z, A)$ consists of a $g$-pair $(X, B + M)$ and a base-polarized contraction $\pi : X \to (Z, A)$ satisfying the following conditions:

1. $(X, B + M)$ is $\epsilon$-lc of dimension $d$,
2. $K_X + B + M \sim_{\mathbb{R}} \pi^*L$ for some $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $L$ on $Z$,
3. $A^{\dim Z} \leq r$, and
4. $A - L$ is pseudo-effective.

We call $(X, B + M) \to (Z, A)$ a $(d, r, \epsilon)$-logCY fibration if further $A - L$ is ample.

We say that $(X, B + M) \to (Z, A)$ is of \textit{Fano type} if $\pi : X \to Z$ is of Fano type, or equivalently, $-K_X$ is big over $Z$ (cf. [2] Lemma 3.24).

We say that $(X, B + M) \to (Z, A)$ is \textit{rationally connected} if $\pi : X \to Z$ is rationally connected.

For example, in Theorem 1.4, $(X, B + M) \to (Z, A)$ is a $(d, r, \frac{1}{4})$-logCY rationally connected fibration. In [2] Definition 2.1, a $(d, r, \epsilon)$-logCY fibration of Fano type is called a \textit{generalized} $(d, r, \epsilon)$-Fano type fibration.

Lemma 2.7. Under the settings (1)-(2) of Definition 2.6, $L + d'A$ is ample for all $d' > 2d$. In particular, $Z$ is the log canonical model of $(X, B + d'\pi^*A + M_X)$.

Proof. It suffices to show that $L + 2dA$ is nef, or equivalently, $K_X + B + M_X + 2d\pi^*A$ is nef. This follows immediately from the length of extremal rays for g-pairs ([16, Proposition 3.13]).

Namely, if $K_X + B + M_X + 2d\pi^*A$ is not nef, then $K_X + B + M_X$ is not nef and there exists a curve $C$ such that $(K_X + B + M_X + 2d\pi^*A)\cdot C < 0$ and $0 > (K_X + B + M_X)\cdot C \geq -2d$, which implies that $\pi^*A \cdot C < 1$. As $K_X + B + M_X \equiv_Z 0$, $C$ is not contracted by $\pi$, so $\pi^*A \cdot C \geq 1$, a contradiction.

2.5. Boundedness. A collection $\mathcal{P}$ of projective varieties is said to be \textit{bounded} (resp. \textit{bounded in codimension one}) if there exists a projective morphism $h : \mathcal{X} \to S$ between schemes of finite type such that each $X \in \mathcal{P}$ is isomorphic (resp. isomorphic in codimension one) to $\mathcal{X}_s$ for some closed point $s \in S$. Here by taking a normalization of $\mathcal{X}$ and applying Noetherian induction, we may assume that each fiber $\mathcal{X}_s$ is normal.

Moreover, if $\mathcal{P}$ is a set of logCY pairs $(X, B)$ (resp. logCY g-pairs $(X, B + M)$), then the set of $X$ in $\mathcal{P}$ is said to be \textit{bounded modulo flops} if it is bounded in codimension one.

A collection $\mathcal{P}$ of projective log pairs is said to be \textit{log bounded} if there is a projective morphism $h : \mathcal{X} \to S$ between schemes of finite type and a reduced divisor $\mathcal{E}$ on $\mathcal{X}$ where $\mathcal{E}$ does not contain
any fiber of $h$, such that for every $(X, B) \in \mathcal{P}$, there is a closed point $s \in S$ and an isomorphism $f : X_s \to X$ such that $\mathcal{E}_s := \mathcal{E}^{|X_s}$ coincides with the support of $f^{-1}_*B$.

A collection $\mathcal{P}$ of base-polarized fibrations between projective varieties is said to be bounded in codimension one (resp. bounded) if there exist projective morphisms $\mathcal{X} \to \mathcal{Z} \to S$ between schemes of finite type and a Cartier divisor $A$ on $\mathcal{Z}$ ample over $S$ such that for every $(X \to (Z, A)) \in \mathcal{P}$, there exists a closed point $s \in S$ such that $\mathcal{X}_s \simeq Z, \mathcal{A}_s \simeq A$, and there is a small birational map (resp. isomorphism) $X_s \dasharrow X$, where all relations are assumed to commute.

We provide two easy but useful lemmas on boundedness of base-polarized fibrations.

**Lemma 2.8.** Let $\mathcal{P}$ be a set of base-polarized fibrations. Suppose that there exists a positive integer $r$ such that for each $\pi : X \to (Z, A)$ in $\mathcal{P}$, there exists a very ample divisor $H$ on $X$ such that $H^{\dim X} \leq r$ and $H - \pi^*A$ is pseudo-effective. Then $\mathcal{P}$ is bounded.

**Proof.** By the boundedness of Hilbert schemes, we may find a projective morphism $h : \mathcal{X} \to S$ between schemes of finite type and a Cartier divisor $\mathcal{L}$ on $\mathcal{X}$ such that for every $(\pi : X \to (Z, A)) \in \mathcal{P}$, there exists a point $s \in S$ such that $\mathcal{X}_s \simeq X$ and $\mathcal{L}_s \simeq \pi^*A$. By Noetherian induction and [13] Corollary 12.9, after decomposing $S$, we may assume that the stalk of $h_*\mathcal{L}$ at $s$ is isomorphic to $H^0(X_s, \mathcal{L}_s)$ for any $s \in S$. Moreover, since $\pi^*A$ is free, by possibly shrinking $S$, we may assume that $h^*h_*\mathcal{L} \to \mathcal{L}$ is surjective. So $\mathcal{L}$ induces a fibration $\Pi : \mathcal{X} \to \mathcal{Z}$ over $S$ with a Cartier divisor $A$ on $\mathcal{Z}$ ample over $S$ such that $\Pi^*A = \mathcal{L}$. Therefore, for $s \in S$ corresponding to $(\pi : X \to (Z, A)) \in \mathcal{P}$, $\mathcal{X}_s \to (\mathcal{Z}_s, \mathcal{A}_s)$ is isomorphic to $X \to (Z, A)$, which shows that $\mathcal{P}$ is bounded.

**Lemma 2.9.** Let $\mathcal{P}$ and $\mathcal{P}'$ be two sets of base-polarized fibrations. Denote $\mathcal{P}''$ to be the set of base-polarized fibrations $X \to (Z, H_Z)$ such that $X \to Z$ is induced by $X \to Y \to Z$ for some $(f : X \to (Y, H_Y)) \in \mathcal{P}$ and $(g : Y \to (Z, H_Z)) \in \mathcal{P}'$ with $H_Y - g^*H_Z$ pseudo-effective. If $\mathcal{P}'$ is bounded and $\mathcal{P}$ is bounded in codimension one (resp. bounded), then $\mathcal{P}''$ is bounded in codimension one (resp. bounded).

**Proof.** After replacing $X$ by a birational model, it suffices to prove the case that $\mathcal{P}$ is bounded. In this case, there exists a positive integer $r$ such that for every $(f : X \to (Y, H_Y)) \in \mathcal{P}$, there exists a very ample divisor $H_X$ on $X$ such that $H_X^{\dim X} \leq r$ and $H_X - f^*H_Y$ is ample. In particular, $H_X - f^*g^*H_Z$ is pseudo-effective. So $\mathcal{P}''$ is bounded by Lemma 2.8.

### 3. Proofs of main theorems

#### 3.1. Boundedness results

**Theorem 3.1** ([2], Theorem 2.2]. Fix positive integers $d, r$ and a positive real number $\epsilon$. Then the set of base-polarized fibrations $X \to (Z, A)$ such that there exists a $(d, r, \epsilon)$-logCY fibration of Fano type $(X, B + M) \to (Z, A)$ as in Definition 2.6, is bounded.

**Proof.** By applying [2, Theorem 2.3] to $P = \pi^*A$, there exists a positive real number $t$ depending only on $d, r$, and $\epsilon$ such that $(X, B + t\pi^*A + M)$ is klt where $B + t\pi^*A$ is the boundary part. By assumption, $A - L$ is ample. Here we may assume that $t < 1$, so $2A - \frac{t}{2}A - L$ is ample. Now $(X, B + \frac{t}{2}\pi^*A + M) \to (Z, 2A)$ is a $(d, 2^2, \frac{t}{2})$-logCY fibration of Fano type. By applying [2, Theorem 2.2] to $\Delta = \frac{t}{2}\pi^*A$, $(X, \frac{t}{2}\pi^*A)$ belongs to a log bounded family. Since coefficients of $t\pi^*A$ are less than 1, we have $\frac{t}{2}\pi^*A \leq \text{Supp} \pi^*A$, and hence there exists a positive integer $r'$ depending only on $d, r$ and $\epsilon$, and a very ample divisor $H$ on $X$ such that $H^d \leq r'$ and $H - \pi^*A$ is ample. Hence the conclusion follows from Lemma 2.8.

The following lemma is useful in order to show that if a set of base-polarized fibrations is bounded in codimension one, then the set of certain birational models of them remains bounded in codimension one.
Lemma 3.2. Fix a positive rational number $\epsilon$ and positive integers $d, r, l$. Let $\mathcal{P}$ be a set of weak $(d, r, \epsilon)$-logCY fibrations $(Y, C) \to (Z, A)$ with $lC$ integral and $M = 0$. Suppose that $\mathcal{P}$ is bounded in codimension one. Denote $\mathcal{P}'$ to be the set consisting of all weak $(d, r, \epsilon)$-logCY fibrations $(X, B) \to (Z, A)$ such that

- there exists a birational morphism $h : Y \to X$ with $h^*(K_X + B) = K_Y + C$,
- all $h$-exceptional divisors are in $\text{Supp} \, C$,
- $((Y, C) \to (Z, A)) \in \mathcal{P}$ where the fibration $Y \to Z$ is induced by $Y \to X \to Z$.

Then the set of base-polarized fibrations $X \to (Z, A)$ in $\mathcal{P}'$ is bounded in codimension one.

Proof. As $\mathcal{P}$ is bounded in codimension one, there are quasi-projective schemes $Y, Z$, a Cartier divisor $A$ on $Z$ ample over $S$, and projective morphisms $\varphi : Y \to Z \to S$, where $S$ is a disjoint union of finitely many varieties such that for every $((Y, C) \to (Z, A)) \in \mathcal{P}$, there is a closed point $s \in S$ and a small birational map $f_s : Y_s \to Y$ such that $Y_s$ is normal, $Z_s \simeq Z$, and $A_s \simeq A$. We may assume that the set of points $s$ corresponding to $((Y, C) \to (Z, A)) \in \mathcal{P}$ is dense in $S$. After decomposing $S$, we may also assume that $S$ is smooth and affine.

For a point $s$ corresponding to $((Y, C) \to (Z, A)) \in \mathcal{P}$, as $f_s$ is small and $Y_s$ is normal, there exist $\mu_s : (Y_s, f_s^{-1}C) \to (Z_s, A_s)$ is a weak $(d, r, \epsilon)$-logCY fibration. So $K_{Y_s} + f_s^{-1}C \sim_{\Q} \mu_s^*L_s$ for a $\Q$-Cartier $\Q$-divisor $L_s$ on $Z_s$ and $A_s \sim A$. Denote by $\mathcal{Y}_s$ the birational map $\varphi_s$ such that $B_s = f_s^{-1}(h_s^{-1}B)$ for some $(X_s, B_s) \to (Z_s, A_s) \in \mathcal{P}'$ corresponding to $(Y, C) \to (Z, A)$. As $\mathcal{Y}_s$ is bounded in codimension one, it is a set consisting of all weak $(d, r, \epsilon)$-logCY fibrations.

In particular, $(\mathcal{Y}_s, f_s^{-1}C)$ belongs to a log bounded family and the number of components of $\text{Supp}(f_s^{-1}C)$ is at most $2l$. Therefore, for a fixed $((Y, C) \to (Z, A)) \in \mathcal{P}$, the number of $\Q$-divisors $B_Y$ such that $B_Y = h_s^{-1}B$ for some $(X, B) \to (Z, A) \in \mathcal{P}'$ is at most $2l^2$.

So possibly after taking a finite base change of $S$, we may assume that there exist $\Q$-divisors $B$ and $D$ on $\mathcal{Y}$ which do not contain any fiber of $\mathcal{Y} \to S$ such that for each $(X, B) \to (Z, A) \in \mathcal{P}'$ and $(Y, C) \to (Z, A) \in \mathcal{P}$ as in the assumption, there exists a closed point $s \in S$ and a small birational map $f_s : Y_s \to Y$ such that $B_s = f_s^{-1}(h_s^{-1}B)$ and $D_s = f_s^{-1}(C - h_s^{-1}B)$.

Now consider a log resolution $g : Y' \to Y$. Denote by $B'$ the strict transform of $B$ and denote by $E'$ the sum of the strict transform of $\text{Supp} \, D$ and all $g$-exceptional prime divisors on $Y'$. After decomposing $S$ into finitely many locally closed subsets, we may assume that for every $s \in S$, $Y'_s$ is a log resolution of $(Y_s, B_s + D_s)$ and $E'_s$ is the sum of the strict transform of $\text{Supp} \, D_s$ and all exceptional prime divisors on $Y'_s$.

Fix an integer $d_0 > 2 \dim Y \geq 2d$. For a point $s \in S$ corresponding to $(X, B) \to (Z, A) \in \mathcal{P}'$ and $(Y, C) \to (Z, A) \in \mathcal{P}$ as in the assumption, we may write

$$K_{Y'} + C'_s + d_0 g_s^*\mu_s^*A_s := g_s^*(K_{Y_s} + f_s^{-1}C + d_0 \mu_s^*A_s) \sim_{\Q} g_s^*\mu_s^*(L_s + d_0 A_s)$$

where the coefficients of $C'_s$ are $\leq 1 - \epsilon$ and its support is contained in $\text{Supp} \, (B'_s + E'_s)$. Then

$$(K_{Y'} + B' + (1 - \frac{\epsilon}{2})E')|_{Y'_s} \sim_{\Q} g_s^*\mu_s^*(L_s + d_0 A_s) + B'_s + (1 - \frac{\epsilon}{2})E'_s - C'_s$$

Note that $B'_s + (1 - \frac{\epsilon}{2})E'_s - C'_s \geq 0$ and its support coincides with $E'_s$ which are precisely the divisors on $Y'_s$ exceptional over $X$. By Lemma 2.7, $Z_s$ is the log canonical model of $(X, B + d_0 \pi^*A_s)$ with

$$K_X + B + d_0 \pi^*A_s = \pi^*(L_s + d_0 A_s),$$

where $\pi : X \to Z_s$ is the natural fibration. So $(X, B + d_0 \pi^*A_s)$ is a good minimal model of $(Y_s, B'_s + (1 - \frac{\epsilon}{2})E'_s + d_0 g_s^*\mu_s^*A_s)$. By replacing $A$ by a general member $A'$ in its $\Q$-linear system, we may assume that $(Y', B' + (1 - \frac{\epsilon}{2})E' + d_0 g^*\mu^*A')$ is log smooth and klt over $S$. By [15, Theorem 1.2], $(Y', B' + (1 - \frac{\epsilon}{2})E' + d_0 g^*\mu^*A')$ has a good minimal model $\mathcal{Y}''$ with the log
canonical model $\mathcal{Y}'' \to Z''$ over $S$. In particular, $\mathcal{Y}'' \to Z''$ is also a good minimal model with the log canonical model for $(\mathcal{Y}'', B'' + (1 - \frac{d}{l})g^*E' + d_0g^*\mu^*A)$ over $S$. We may assume that this good minimal model is obtained by an MMP by \cite{LX} Corollary 2.9. By the choice of $d_0$ and the length of extremal rays \cite{LX} Theorem 1], this MMP is $g^*\mu^*A$-trivial. So there is a natural morphism $Z'' \to Z$. By Noetherian induction, after decomposing $S$ into finitely many locally closed subsets, we may assume that for every $s \in S$, $\mathcal{Y}''_s \to Z''_s$ is a good minimal model with the log canonical model of $(\mathcal{Y}'_s, B'_s + (1 - \frac{d}{l})E'_s + d_0g^*\mu^*A_s)$. In particular, for any point $s \in S$ corresponding to $((X, B) \to (Z, A)) \in \mathcal{P}''$ and $((Y, C) \to (Z, A)) \in \mathcal{P}$ as in the assumption, by \cite{LX} Theorem 3.5.2], $\mathcal{Y}'_s$ is isomorphic to $X$ in codimension one as they are both good minimal models of a same pair, and $Z''_s \simeq Z_s$ with $A''_s \simeq A_s$ where $A''$ is the pullback of $A$ on $\mathcal{Y}''$. Hence the family $\mathcal{Y}'' \to (Z'', A'') \to S$ shows that $X \to (Z, A)$ is bounded modulo flops. 

3.2. Descending logCY g-pairs of fixed index. The following proposition is a generalization of \cite{BCHM} Proposition 6.3] to the setting of generalized pairs.

Proposition 3.3 (see \cite{LX} Lemma 4.2], \cite{BCHM} Theorem 1.5], \cite{Fuj] Theorem 1.2]). Let $d, l$ be two positive integers. Then there exists a positive integer $l'$ depending only on $d$ and $l$ satisfying the following.

Assume that $(X, B + M)$ is a g-pair and $\pi : X \to Z$ is a contraction such that

1. $(X, B + M)$ is lc of dimension $d$, and $\dim Z > 0$,
2. $X$ is of Fano type over $Z$,
3. $lB$ is integral and $lM$ is b-Cartier, and
4. $K_X + B + M \sim_{Q, Z} 0$.

Then there is an lc g-pair $(Z, D + N)$ such that

$$l'(K_X + B + M_X) \sim l'\pi^*(K_Z + D + N_Z),$$

and $l'N$ is b-Cartier. Moreover, if $(X, B + M)$ is klt, then $(Z, D + N)$ is klt.

As a corollary, we generalize Proposition 3.3 to rationally connected fibrations of finite index instead of fibrations of Fano type.

Corollary 3.4. Let $d, l$ be two positive integers. Then there exists a positive integer $l'$ depending only on $d$ and $l$ which is divisible by $l$ and satisfies the following.

Assume that $(X, B + M)$ is a g-pair and $\pi : X \to Z$ is a contraction such that

1. $(X, B + M)$ is lc of dimension $d$, and $\dim Z > 0$,
2. $\pi$ is rationally connected, and
3. $lM$ is b-Cartier, and
4. $l(K_X + B + M_X) \sim_Z 0$.

Then there is an lc g-pair $(Z, D + N)$ such that

$$l'(K_X + B + M_X) \sim l'\pi^*(K_Z + D + N_Z),$$

and $l'N$ is b-Cartier. Moreover, if $(X, B + M)$ is klt, then $(Z, D + N)$ is klt.

Proof. Note that in the assumption $l(K_X + B + M_X)$ is Cartier and $lM$ is b-Cartier, so $lB$ is automatically integral.

We prove the statement by induction on $\dim X - \dim Z$. If $\dim X = \dim Z$, then $\pi$ is birational and we may take $D = \pi_*B$, $N = M$, and $l' = l$. From now on, suppose that $\dim X - \dim Z > 0$. By \cite{BCHM} Proposition 3.9], possibly replacing $(X, B + M)$ with its dlt model, we may assume that $(X, B + M)$ is $Q$-factorial dlt. In particular, $X$ is $Q$-factorial klt.

Case 1. $K_X$ is not pseudo-effective over $Z$.

We can run a $K_X$-MMP over $Z$ to get a Mori fiber space $\pi' : X' \to Z'$ over $Z$. Then by the negativity lemma, $(X', B' + M)$ is lc with $l((K_X + B' + M_X')) \sim_Z 0$. Here $(X', B' + M)$ is klt if
(X, B + M) is klt. By applying Proposition 3.3 to X' \to Z', there exists a constant l'' depending only on d, l and an lc g-pair (Z', D' + N'), such that
\[ l''(K_X + B' + M_X) \sim l''\pi^{*}(K_{Z'} + D' + N'_{Z'}), \]
and l''N' is b-Cartier. Here (Z', D' + N') is klt if (X', B' + M) is klt. We may assume that l divides l'', so l''(K_{Z'} + D' + N'_{Z'}) \sim_{Z} 0. Also Z' \to Z is rationally connected as its general fibers are dominated by those of \( \pi \). So we conclude the statement by applying the inductive hypothesis to (Z', D' + N') \to Z.

Case 2. K_X is pseudo-effective over Z.

In this case, by Lemma 3.5 there exists a prime divisor E_0 over X such that a(E_0, X, B) < 1 and E_0 dominates Z. By [21] Corollary 1.4.3, there is a projective birational morphism h : Y \to X extracting only E_0. Note that a = a(E_0, X, B + M) = a(E_0, X, B) < 1 whenever the equality is from Lemma 3.5(2), so we may write
\[ K_Y + B_Y + (1 - a)E_0 + M_Y = h^{*}(K_X + B + M_X), \]
where B_Y is the strict transform of B on Y. Here Y \to Z is rationally connected as its general fibers are birational to those of X \to Z. It is clear that (Y, B_Y + (1 - a)E_0 + M) \to Z satisfies all conditions of Corollary 3.4 and (Y, B_Y + (1 - a)E_0 + M) is klt if (X, B + M) is klt. As E_0 dominates Z, K_Y is not pseudo-effective over Z, and hence we may apply Case 1 to (Y, B_Y + (1 - a)E_0 + M) to conclude the statement.

Lemma 3.5. Let \( \pi : X \to Z \) be a rationally connected fibration and (X, B + M) a Q-factorial lc g-pair such that X is klt and K_X + B + M_X \equiv_{Z} 0. Suppose that K_X is pseudo-effective over Z. Then the following statements hold:

1. for a general fiber F of Z, B_F = 0 and K_F \equiv M_X|_F \equiv 0;
2. there exists a prime divisor E_0 over X such that a(E_0, X, B) < 1, E_0 dominates Z, and for any birational model h : Y \to X on which E_0 is a divisor, \( \text{mult}_{E_0}(h^{*}M_X - M_Y) = 0 \).

Proof. On a general fiber F of \( \pi \), K_F and M_X|_F are pseudo-effective and K_F + B|_F + M_X|_F \equiv 0. So B|_F = 0 and K_F \equiv M_X|_F \equiv 0.

Note that F is rationally connected and K_F \equiv 0, by [21] Lemma 6.3, F is not canonical. Hence by adjunction, (X, B) is not canonical over a non-empty open subset of Z. Thus there exists a prime divisor E_0 over X such that a(E_0, X, B) < 1 and E_0 dominates Z.

To show the last statement, we may assume that h : Y \to X is a sufficiently high model such that M_Y is nef. Then by the negativity lemma, M_Y + G = h^{*}M_X where \( G \geq 0 \) is h-exceptional. Restricting on a general fiber F_Y of Y \to Z, we have M_Y|_{F_Y} + G|_{F_Y} = h^{*}(M_X|_F) \equiv 0. This implies that M_Y|_{F_Y} \equiv 0 and G|_{F_Y} = 0. Therefore, mult_{E_0} G = 0 as E_0|_{F_Y} \neq 0.

3.3. Lifting morphisms via flops. When we consider a contraction X \to W where W belongs to a family which is bounded modulo flops, we use the following lemma to replace this contraction with a birational modification so that W could be assumed to be in a bounded family. The proof is well-known to experts (cf. [9] Lemma 3.3)).

Lemma 3.6. Let Z be a projective variety and let X and W be normal projective varieties over Z. Suppose the following conditions hold:

1. \( \pi : X \to W \) is a contraction of Fano type and W is Q-factorial;
2. (X, B + M) and (W, D + N) are klt g-pairs such that \( K_X + B + M_X \equiv_{Z} \pi^{*}(K_W + D + N_W) \equiv_{Z} 0 \);
3. there is a small birational map \( \phi_W : W \dashrightarrow W' \) to another normal projective variety W' over Z.

Then there exists a Q-factorial projective normal variety X' with a birational contraction \( \phi : X \dashrightarrow X' \) and a contraction \( \pi' : X' \to W' \) of Fano type such that \( \pi' \circ \phi = \phi_W \circ \pi \).
Proof. The following simplified proof is provided by the referee. Replacing $X$ by a small $\mathbb{Q}$-factorialization [5] Corollary 1.4.3], we may assume that $X$ is $\mathbb{Q}$-factorial. Pick an effective ample divisor $H'$ on $W'$ and denote by $H$ the strict transform of $H'$ on $W$. Then $H$ is $\mathbb{Q}$-Cartier as $W$ is $\mathbb{Q}$-factorial, and it is big as $W$ and $W'$ are isomorphic in codimension one.

Fix a sufficiently small positive real number $\delta$ such that $(X, B + \delta \pi^*H + M)$ is klt. We claim that $B + \delta \pi^*H + M_X$ is big over $Z$. In fact, as $\pi$ is a contraction of Fano type, $B + M_X$ is big over $W$. So we may write $B + M_X = A_X + E_X$ where $A_X$ is ample over $W$ and $E_X$ is an effective $\mathbb{R}$-divisor on $X$. On the other hand, we may write $H = A_W + E_W$ where $A_W$ is ample over $Z$ and $E_W$ is an effective $\mathbb{R}$-divisor on $W$. Then

$$B + \delta \pi^*H + M_X = (1 - \delta')(B + M_X) + (\delta' A_X + \delta \pi^*A_W) + \delta' E_X + \delta \pi^*E_W$$

is big over $Z$ because $\delta' A_X + \delta \pi^*A_W$ is ample over $Z$ for sufficiently small $\delta' > 0$.

Then by [7] Lemma 4.4], $(X, B + \delta \pi^*H + M)$ has a good minimal model $X'$ over $Z$ and $W'$ is just the ample model as $\phi_W$ is small and

$$K_X + B + \delta \pi^*H + M_X \equiv_Z \delta \pi^*H = \delta \pi^*\phi_W^{-1}H'.$$

Hence $X' \to W'$ are the desired models. Here note that $-K_X'$ is big over $W'$ as $-K_X$ is big over $W$ and $\phi_W$ is birational.

3.4. Proof of main theorems.

Lemma 3.7. Under the assumptions in Theorem 1.4, if there exists a birational contraction $X \to X'$ over $Z$ such that $X' \to (Z, A)$ is bounded in codimension one, then $X \to (Z, A)$ is bounded in codimension one.

Proof. After replacing $X'$ by its birational model, we may assume that $X' \to (Z, A)$ is bounded. Then there exists a positive integer $r_0$ independent of $X'$ and a very ample divisor $H'$ on $X'$ such that $H'^d \leq r_0$ and $H' - \eta^*A$ is ample, where $\eta : X' \to Z$ is the natural morphism.

Denote by $B'$ the strict transform of $B$ on $X'$. Then $l(K_X' + B' + M_{X'}) \sim \eta^*L$. By the negativity lemma, for any prime divisor $E$ on $X$ which is exceptional over $X'$, $a(E, X', B' + M) = a(E, X, B + M) \leq 1$. So by [2] Lemma 4.5], there is a projective birational morphism $g : X'' \to X'$ extracting exactly all prime divisors on $X$ which are exceptional over $X'$. In particular, $X''$ is isomorphic to $X$ in codimension one. We may write

$$K_{X''} + B'' + M_{X''} = g^*(K_{X'} + B' + M_{X'}).$$

Then $(X'', B'' + M)$ is $\frac{1}{l}$-lc and $X'' \to X'$ is of Fano type ([3] §2.13(7)). So $(X'', B'' + M) \to (X', H')$ is a $(d, r_0, \frac{1}{l})$-log CY fibration of Fano type. By applying Theorem 3.1, $X'' \to (X', H')$ belongs to a bounded family. Then $X'' \to (Z, A)$ belongs to a bounded family by Lemma 2.9 and the construction of $H'$, which shows that $X \to (Z, A)$ is bounded in codimension one.

Proof of Theorem 1.4. We prove the statement by induction on $\dim X - \dim Z$. If $\dim X = \dim Z$, then $\pi$ is birational and hence $-K_X$ is big over $Z$. So the statement follows from Theorem 3.1. Now suppose that $\dim X > \dim Z$.

Replacing $X$ by a small $\mathbb{Q}$-factorialization by [7] Lemma 4.5], we may assume that $X$ is $\mathbb{Q}$-factorial. In particular, $X$ is klt.

Case 1. $K_X$ is not pseudo-effective over $Z$.

In this case, we can run a $K_X$-MMP over $Z$ to get a Mori fiber space $X' \to Z'$ over $Z$ with fibrations $\pi' : X' \to Z'$ and $f : Z' \to Z$. Here $Z'$ is $\mathbb{Q}$-factorial as $X'$ is $\mathbb{Q}$-factorial. As $l(K_X + B + M_X) \sim \pi^*L$, $l(K_{X'} + B' + M_{X'}) \sim \pi'^*f^*L$ where $B'$ the strict transform of $B$ on $X'$, and $(X', B' + M)$ is klt by the negativity lemma.
Corollary 3.4, there exists a positive integer \( l \) such that there exists a klt g-pair \((Z', D' + N')\) with \( l'(K_{Z'} + D' + N'_{Z'}) \sim \frac{1}{l'} f^* L \) and \( l'N' \) is Cartier. Also \( \dim Z' < \dim X \) and \( Z' \to Z \) is rationally connected. Hence by induction on dimension, we may assume that \( Z' \to (Z, A) \) is bounded in codimension one. By Lemma 3.6 after replacing \( X' \) and \( Z' \) by their birational models, we may assume that

1. \( X \dashrightarrow X' \) is a birational contraction,
2. \( \pi' : X' \to Z' \) is of Fano type, and
3. \( f : Z' \to (Z, A) \) belongs to a bounded family.

Note that after the replacement, it remains true that \( l(K_{X'} + B' + M_{X'}) \sim \pi'^* f^* L \) and \( (X', B' + M) \) is klt.

As \( Z' \to (Z, A) \) is bounded, there exists a very ample divisor \( A' \geq 0 \) on \( Z' \) and a positive integer \( r' \) independent of \( Z' \) such that \( A'^\dim Z' \leq r' \) and \( A' - f^* A \) is ample. In particular, \( A' - \frac{1}{r'} f^* L \) is ample and \((X', B' + M) \to (Z', A')\) is a \((d, r', \frac{1}{r'})\)-log CY fibration of Fano type. So by Theorem 3.8 such \( X' \to (Z', A') \) belongs to a bounded family. Then \( X' \to (Z, A) \) belongs to a bounded family by Lemma 3.9 and the construction of \( A' \). Therefore, \( X \to (Z, A) \) is bounded in codimension one by Lemma 3.7.

**Case 2.** \( K_X \) is pseudo-effective over \( Z \).

In this case, by Lemma 3.5 \((X, B)\) is a klt pair with \( K_F + B|_F \equiv 0 \) for a general fiber \( F \) of \( \pi \). By [14, Theorem 2.12], \((X, B)\) has a good minimal model \( X' \to Z' \) over \( Z \), where \( Z' \) is the log canonical model. Hence \( K_{X'} + B' \sim_{q, Z'} 0 \) where \( B' \) is the strict transform of \( B \) on \( X' \). Note that \( Z' \) is birational to \( Z \) as \( K_F + B|_F \equiv 0 \) for a general fiber \( F \) of \( \pi \).

It is clear that \((X', B' + M) \to (Z, A)\) satisfies the same conditions as \((X, B + M) \to (Z, A)\). By Lemma 3.7 it suffices to show that \( X' \to (Z, A) \) is bounded in codimension one.

Denote the induced fibrations by \( \pi' : X' \to Z' \) and \( f : Z' \to Z \). As \( Z' \to Z \) is birational and \( \pi \) is rationally connected, \( \pi' \) is rationally connected. So by applying Corollary 3.4 to \( \pi' : X' \to Z' \), there exists a constant \( l' \) depending only on \( d, l \) and a klt g-pair \((Z', D' + N')\) such that

\[
l'(K_{X'} + B' + M_{X'}) \sim l' \pi'^* (K_{Z'} + D' + N'_{Z'}) \sim \frac{l'}{l} \pi'^* f^* L,
\]

and \( l'N' \) is Cartier. In particular, \((Z', D' + N') \to (Z, A)\) is a \((\dim Z', r', \frac{1}{r'})\)-log CY fibration. Moreover, \( Z' \) is of Fano type over \( Z \) as \(-K_{Z'}\) is big over \( Z \). So by Theorem 3.1 \( Z' \to (Z, A) \) belongs to a bounded family. Then there exists a positive integer \( r' \) independent of \( Z' \) and a very ample divisor \( A' \geq 0 \) on \( Z' \) such that \( A'^\dim Z' \leq r' \) and \( A' - f^* A \) is ample. Therefore, \((X', B' + M) \to (Z', A')\) is a \((d, r', \frac{1}{r'})\)-log CY fibration.
Now we have $K_{X'} + B' \sim_{Q} Z' \sim 0$ and $K_{X'} + B' + M_{X'} \sim_{Q} Z' \sim 0$, so $M_{X'} = \pi^{*}L'$ where $L'$ is a pseudo-effective $\mathbb{Q}$-divisor on $Z'$. Then $K_{X'} + B' \sim_{Q} \pi^{*}(\frac{1}{r}f^{*}L - L')$ and $A' - \frac{1}{r}f^{*}L + L'$ is pseudo-effective. In particular, $(X', B') \to (Z', A')$ is a weak $(d, r', \frac{1}{r})$-log CY fibration.

As $X'$ and $X$ are isomorphic over the generic point of $Z$, $K_{X'}$ is pseudo-effective over $Z'$, so by Lemma 3.3 there exists a prime divisor $E_{0}$ over $X'$ such that $a = a(E_{0}, X', B') < 1$ and $E_{0}$ dominates $Z'$. By [3] Corollary 1.4.3, there is a projective birational morphism $h : Y \to X'$ extracting only $E_{0}$. By Lemma 3.2 again, $M_{Y} = h^{*}M_{X'}$, so we may write

$$K_{Y} + B_{Y} + (1 - a)E_{0} = h^{*}(K_{X'} + B')$$

(3.1)

and

$$K_{Y} + B_{Y} + (1 - a)E_{0} + M_{Y} = h^{*}(K_{X'} + B' + M_{X'})$$

(3.2)

where $B_{Y}$ is the strict transform of $B'$ on $Y$. Here $Y \to Z'$ is rationally connected as its general fibers are birational to those of $X' \to Z'$. So by (3.2), $(Y, B_{Y} + (1 - a)E_{0} + M_{Y}) \to (Z', A')$ satisfies all conditions of Theorem 1.4 with integers $d, l, r'$. In particular, $(1 - a)E_{0} + B_{Y}$ is integral. As $E_{0}$ dominates $Z'$, $K_{Y}$ is not pseudo-effective over $Z'$. So we may apply Case 1 to $(Y, (1 - a)E_{0} + B_{Y} + M_{Y}) \to (Z', A')$ to conclude that $Y \to (Z', A')$ is bounded in codimension one. On the other hand, by (3.1), $(Y, (1 - a)E_{0} + B_{Y}) \to (Z', A')$ is a weak $(d, r', \frac{1}{r})$-log CY fibration with $(1 - a)E_{0} + B_{Y}$ integral. Hence we may apply Lemma 3.2 to conclude that $X' \to (Z', A')$ is bounded in codimension one. So by Lemma 2.9 and the construction of $A', X' \to (Z, A)$ is bounded in codimension one. Therefore, $X \to (Z, A)$ is bounded in codimension one by Lemma 2.7.

Proof of Theorem 1.3. This follows from Theorem 1.4 by taking $Z$ to be a point.

Proof of Theorem 1.2. This follows from Theorem 1.3 by taking $M = -(K_{X} + B)$ and $M = \overline{M}$.

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