Universal routes to spontaneous $\mathcal{PT}$-symmetry breaking in non-hermitian quantum systems

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$\mathcal{PT}$-symmetric systems can have a real spectrum even when their Hamiltonian is non-hermitian, but develop a complex spectrum when the degree of non-hermiticity increases. Here we utilize random-matrix theory to show that this spontaneous $\mathcal{PT}$-symmetry breaking can occur via two distinct mechanisms, whose predominance is associated to different universality classes. Present optical experiments fall into the orthogonal class, where symmetry-induced level crossings render the characteristic absorption rate independent of the coupling strength between the symmetry-related parts of the system.

Non-hermitian quantum systems generally have a complex energy spectrum, with imaginary parts of the energies related to decay or amplification rates. However, when loss and gain are in balance the spectrum can still be real. One intensely researched route to try and achieve such a balance is to couple two identical systems symmetrically and then induce opposite amounts of gain and loss into the two parts, as illustrated in Fig. 1(a) [1–13]. The Hamiltonian then possesses a combined parity ($\mathcal{P}$) and time-reversal ($\mathcal{T}$) symmetry, and its secular equation is real. However, this does not guarantee a real spectrum; as the level of non-hermiticity (loss and gain) is increased, pairs of complex-conjugate energy levels appear [14]. This phenomenon of spontaneous $\mathcal{PT}$-symmetry breaking has gained recent prominence because it leads to optical effects such as double refraction, solitons and non-reciprocal diffraction patterns, which provide mechanisms for the design of unidirectional couplers and left-right sensors [15, 8], concepts that are now being realized experimentally in a variety of optical settings [9].

Over the past months, these systems were proposed for at-threshold lasers [10] and laser-absorbers [11, 14]. In turn, these developments have instigated a deeper theoretical understanding of the role of the dynamics (such as the consequences of Anderson localization and wave chaos [12], as well as interactions [13]). In this paper, we establish distinct universality classes which directly affect the nature of spontaneous $\mathcal{PT}$-symmetry breaking.

To do so, we derive random-matrix ensembles where loss and gain in the two parts of the system are implemented by a uniform rate $\mu$, while coupling is established through $N$ channels with transmission probability $T$; the mean level spacing of the decoupled parts is $\Delta$ [15]. We find that the mechanism behind spontaneous $\mathcal{PT}$-symmetry breaking depends on whether the hermitian limit $\mu = 0$ is time-reversal symmetric or not, amounting to a predominance of symmetry-induced level crossings or level repulsion, as illustrated in Fig. 1(b). This results in different characteristic scales $\mu_{\mathcal{PT}}$ of amplification/absorption governing the transition from an essentially real to an essentially complex spectrum. Present optical experiments and theoretical studies concern systems without magneto-optical effect (the orthogonal symmetry class), in which the hermitian limit is time-reversal symmetric. In this case $\mu_{\mathcal{PT}} \sim \sqrt{N\Delta}/2\pi \equiv \mu_0$ becomes fully independent of the coupling strength $T_c \sim 1/N$, thereby exhibiting a level of universality that

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{PT.png}
\caption{(Color online) (a) Sketch of a nonhermitian $\mathcal{PT}$-symmetric system, where a region with absorption rate $\mu$ (and mean level spacing $\Delta$, left) is coupled symmetrically via a tunnel barrier (supporting $N$ channels with transmission probability $T$) to an amplifying region with a matching amplification rate (right). Below this, the scattering description of the system. (b) Two routes to spontaneous $\mathcal{PT}$-symmetry breaking, depending on whether the hermitian limit $\mu = 0$ is $\mathcal{T}$-symmetric (orthogonal class displaying level crossings, left) or not (unitary class displaying avoided crossings, right). Shown are real eigenvalues of a random Hamiltonian $\mathcal{H}$ [Eq. (4)] as function of $T$ for fixed $\mu = 0$ (left of dashed line), and then as a function of $\mu$ for fixed $T = 1$ (right of dashed line). Complex-valued levels (formed by level coalescence at $\mu > 0$) are not shown. Here $\mu_0 = \sqrt{N\Delta}/2\pi$, and we set $N = 10$.}
\end{figure}
the time-reversal operation describes the closed channels [16]. Adopting a basis where \( \mu \) in Fig. 2.

\[ \mu \] weak and strong coupling. These findings are illustrated with \( N \) levels within a range of energies over which the mean spacing \( \Delta \) can be assumed constant), for the orthogonal class (left) and the unitary class (right). In (a) and (b), \( \mu \) is scaled to \( \mu_0' = \mu_0/\sqrt{1 + 1/N\Delta} \) (c) and \( \mu'_\Gamma = \sqrt{\Gamma}/\mu_0 \) (d). Numerical results with \( N = 50 \).

goes beyond what is normally encountered in mesoscopic systems. For weak coupling \( (T < T_c) \), \( \mu_{PT} \sim \sqrt{N\Delta} \mu_0 \). Adding magneto-optical effects to the system essentially changes the nature of the transition. In this case (the unitary symmetry class), \( \mu_{PT} \sim \sqrt{\Gamma}/\mu_0 \) in the full range of weak and strong coupling. These findings are illustrated in Fig. 2.

**Random-matrix ensembles.**—To derive the appropriate random-matrix ensembles we formulate a quantization condition based on scattering theory [16, 17]. The \( N \times N \)-dimensional scattering matrix

\[
S_L(E; \mu) = 1 - 2iV^\dagger (E - i\mu - H + iVV^\dagger)^{-1}V \quad (1)
\]

of the left subsystem can be expressed in terms of an \( M \times M \)-dimensional Hamiltonian \( H \), which is real symmetric (a member of the standard Gaussian orthogonal ensemble, GOE) if the hermitian limit \( \mu = 0 \) is time-reversal symmetric, and complex hermitian (a member of the Gaussian unitary ensemble, GUE) if this is not the case [18]. We assume \( M \gg N \gg 1 \) and denote the mean level spacing in the energy range of interest by \( \Delta \). The \( M \times N \) coupling matrix \( V \) then fulfills \( VV^\dagger = \text{diag}(v_m) \), where \( N \) diagonal entries \( v_m = \Delta M/\pi \) correspond to fully transparent channels, and \( M - N \) entries \( v_m = 0 \) describe the closed channels [16]. Adopting a basis where the time-reversal operation \( \mathcal{T} \) is identical to complex conjugation, \( \mathcal{PT} \) -symmetry results in the relation [10]

\[
S_R(E; -\mu) = [S_L^{-1}(E^*; \mu)]^* = 1 - 2iV^\dagger (E + i\mu - H^* + iVV^\dagger)^{-1}V \quad (2)
\]

for the scattering matrix of the right subsystem. The tunnel barrier is described by reflection amplitudes \( r = -\sqrt{1 - T} \) and transmission amplitudes \( t = i\sqrt{T} \). As shown in the lower part of Fig. 1(a), these scattering matrices relate amplitudes of left and right propagating waves at the two interfaces of the tunnel barrier. The requirement of consistency of these relations results in the quantization condition

\[
\det \left[ \begin{pmatrix} r & t \\ t & -r \end{pmatrix} \begin{pmatrix} S_L & 0 \\ 0 & S_R \end{pmatrix} - \mathbb{1} \right] = 0, \quad (3)
\]

which can be rearranged into an eigenvalue problem \( \det (E - H) = 0 \) with effective Hamiltonian

\[
H = \begin{pmatrix} H - i\mu & \Gamma \\ \Gamma & H^* + i\mu \end{pmatrix}. \quad (4)
\]

The positive semi-definite coupling matrix \( \Gamma = \text{diag}(\gamma_m) \) now incorporates the finite transmission probability of the barrier; its \( N \) non-vanishing entries read \( \gamma_m = \sqrt{T/(1 + \sqrt{1 - T})}\Delta M/\pi \equiv \mathcal{\gamma} [19] \).

**Numerical evaluation.**—Before engaging in an analytical discussion of the different routes to spontaneous \( \mathcal{PT} \)-symmetry breaking [illustrated in Fig. 1(b)], we put forward numerical results which illustrate the physical consequences of the points to be made below. These results, presented as (color) gradient plots in Fig. 2 concern the fraction \( f(\mu, T) \) of complex-valued energy levels within a range where the mean level spacing can be assumed constant. We fix \( M = 1000 \), which ensures a large number of levels within the range in question, and set \( N = 50 \) [20].

Figure 2(a) shows results for the orthogonal ensemble, with \( \mu \) scaled to \( \mu_0 = \sqrt{N\Delta}/2\pi \). We see that above a small threshold \( T_c \) (to be determined below as \( T_c \sim 1/N \)), the transition from the real spectrum \( (f = 0, \text{obtained for } \mu = 0) \) to a spectrum which is partially real and partially complex spectrum \( (f \sim 1/2) \) indeed occurs on the scale \( \mu_{PT} \sim \mu_0 \), and then is independent of the value of \( T \). Only for \( T < T_c \), \( \mu_{PT} \sim \sqrt{N\Delta}/\mu_0 \equiv \mu_T \) is coupling-dependent. In order to get a unified view over both regimes, we plot in panel (c) the same data, but with \( \mu \) scaled to \( \mu_0' = \mu_0/\sqrt{1 + 1/N\Delta} \), which interpolates between \( \mu_T \) for \( T < T_c \) and \( \mu_0 \) for \( T \gg T_c \). The convergence of gradient lines for \( T \to 0 \) indicates that for weak coupling the transition becomes more abrupt.

Figure 2(b) shows the corresponding results for the unitary ensemble, where \( \mu \) is again scaled to \( \mu_0 \). Here we find that a systematic \( T \)-dependence persists across the full range of coupling strengths. As shown in panel (d), this dependence takes the form \( \mu_{PT} \sim \sqrt{\Gamma}/\mu_0 \equiv \mu'_T \).
Now, the only difference between the strong and weak coupling regimes is a factor of order $1$. In further contrast to the orthogonal case, for weak coupling the transition remains smooth; however, since $\mu'_{T} = \mu_{T}/\sqrt{N} \ll \mu_{T}$, it then occurs at a far smaller deviation from hermiticity.

**Underlying mechanisms.** We now show that the features reported above originate from two distinct mechanisms of spontaneous $\mathcal{PT}$-symmetry breaking. It is instructive to start in a regime which can be treated perturbatively. For $\mu = 0$, the effective Hamiltonian $\mathcal{H}$ [Eq. (4)] is hermitian and all its eigenvalues are real. For $T = 0$ ($\Gamma = 0$), on the other hand, the spectrum is a superposition of two level sequences $E_k = \varepsilon_k \pm i\mu$, which are all complex if $\mu \neq 0$; here $\varepsilon_k$ are the eigenvalues of $H$. Therefore, in regard to the question of how many levels are complex, the limits $T, \mu \to 0$ do not commute. Nonetheless, for $T = \mu = 0$ the spectrum reduces to the superposition of two degenerate level sequences $\varepsilon_k$, so that quasi-degenerate perturbation theory applies. Denote by $\psi^{(k)}_m$ the wave function of $H$ corresponding to eigenvalue $\varepsilon_k$; in random-matrix theory, this is a random-normalized vector with average $|\psi^{(k)}_m|^2 = 1/M$. Reduced to the symmetric and antisymmetric extension of this wavefunction across the whole system, the effective Hamiltonian takes the form

$$\mathcal{H}' = \begin{pmatrix} \varepsilon_k - i\mu & \sum_m |\psi^{(k)}_m|^2 \gamma_m \\ \sum_m |\psi^{(k)}_m|^2 \gamma_m & \varepsilon_k + i\mu \end{pmatrix},$$

whose eigenvalues become complex for $\mu_{PT} = \sqrt{\sum_m |\psi^{(k)}_m|^2 \gamma_m}$. Therefore, on average (and using $\gamma \sim \sqrt{T \Delta M}/2\pi$ for $T \ll 1$)

$$\mu_{PT} \sim \begin{cases} N \sqrt{T \Delta}/2\pi = \mu_T \quad \text{(orthogonal class)}, \\ \sqrt{N \Delta T}/2\pi = \mu'_T \quad \text{(unitary class)} \end{cases},$$

which recovers the numerical scales in the weak coupling regime.

Note that the two expressions for $\mu_{PT}$ differ by the parametrically large factor $\sim \sqrt{N}$. Mathematically, this arises because $\psi^{(k)}_m$ is real in the orthogonal class and complex in the unitary class; physically, it amounts to vastly different tunnel splittings. This difference signifies that in the orthogonal class, the levels of the originally degenerate sequence $\varepsilon_k$ from the two subsystems quickly cross as $T$ is increased. A second route to $\mathcal{PT}$-symmetry breaking then becomes available, which involves two energy levels that are non-degenerate for $T = 0$. In order to describe this case we reformulate the problem by starting with $\mu = 0$, and exploit the thus-emerging $\mathcal{PT}$-symmetry in the orthogonal class to transform the effective Hamiltonian to

$$\mathcal{H}_p = \begin{pmatrix} H + \Gamma & i\mu \\ i\mu & H - \Gamma \end{pmatrix}.$$  

We denote by $\varepsilon_{k}^{\pm}$ the two level sequences of $H \pm \Gamma$. Since $\Gamma$ is positive semidefinite these sequences arise from the sequence $\varepsilon_k$ by an opposite shift which is approximately rigid. From the resulting combined sequence, consider two levels $\varepsilon_{k}^+$ and $\varepsilon_{k}^-$ which lie adjacent to each other; the corresponding eigenvectors are $\psi^{(k+)}$ and $\psi^{(l-)}$. Finite $\mu$ mixes these levels, which is embodied in the reduced Hamiltonian

$$\mathcal{H}' = \begin{pmatrix} i\mu(\varepsilon^{(k+)}_k|\psi^{(k+)}_l) & i\mu(\varepsilon^{(k+)}_k|\psi^{(l-)}_l) \\ i\mu(\varepsilon^{(l-)}_k|\psi^{(k+)}_l) & i\mu(\varepsilon^{(l-)}_k|\psi^{(l-)}_l) \end{pmatrix}.$$  

Now, treating $2\Gamma$ as a perturbation which connects the + and − sequence, $|\varepsilon^{(k+)}_k|\psi^{(k+)}_l)$ $\approx \frac{\varepsilon^{(k+)}_k - \varepsilon^{(l-)}_l}{\varepsilon^{(k+)}_k - \varepsilon^{(l-)}_l}$. Because $|\varepsilon^{(k+)}_k - \varepsilon^{(l-)}_l| = O(\Delta)$ is small compared to the shift due to the coupling, the denominator can be estimated as $\varepsilon^{(k+)}_k - \varepsilon^{(l-)}_l$, implying that this mechanism becomes favorable around $|\varepsilon^{(k+)}_k - \varepsilon^{(l-)}_l|^2 \sim 1/N$, i.e., the mixing is small. As a result, the level pair in question becomes complex for $\mu^2|\varepsilon^{(k+)}_k|\psi^{(k+)}_l)^2 \sim (\varepsilon^{(k+)}_k - \varepsilon^{(l-)}_l)^2 \sim \Delta^2$, i.e.,

$$\mu_{PT} \sim \sqrt{N \Delta}/2\pi = \mu_0 \quad \text{(orthogonal class, } T \gg 1/N).$$

As indicated, comparison of this expression with Eq. (4) implies that this mechanism becomes favorable around $T = T_{c} \sim 1/N$.

This analysis of strong coupling does not apply to the unitary case, which does not display $\mathcal{PT}$-symmetry for $\mu = 0$. In the $\mathcal{P}$-basis, in place of Eq. (7) we then have

$$\mathcal{H}_p = \begin{pmatrix} \Re H + \Gamma & i\mu \Re \Im H + i\mu \\ i\mu \Re H + i\mu & \Im H - \Gamma \end{pmatrix}.$$  

Consequently, finite coupling not only results in a far reduced systematic shift of the levels, but also in a direct mixing of levels in the individual sequences. Therefore, instead of level crossings one encounters level repulsion. This difference is illustrated in Fig. (b), which shows the evolution of energy levels as $T$ is increased from $0$ to $1$ (while $\mu = 0$), and the subsequent fate of real levels as $\mu$ is increased from $0$ to $4\mu_0$ (while $T = 1$); pairwise coalescing levels become complex, and then are no longer shown. In the orthogonal class, such pairs trace back to well-separated levels $\varepsilon^{+}_k, \varepsilon^{-}_l$ from the two different sequences (which are distinguished by the opposite slopes of the levels for increasing coupling). In contrast, in the unitary class the coalescing levels trace back to originally closely spaced or degenerate levels, even when the coupling is strong.

Finally, we point to an alternative scenario where the coupling-independent $\mu_{PT} \sim \mu_0$ becomes relevant even in the unitary symmetry class. Observe that by definition, for a hermitian Hamiltonian the $\mathcal{T}$ operation (complex conjugation) is equivalent to transposition, an operation that we denote by $\mathcal{T}'$. However, for non-hermitian systems there is a physical difference: Absorption and amplification, taken by themselves, break $\mathcal{T}$-symmetry, but
preserve \( T' \)-symmetry; the latter is broken by magneto-optical effects. We find that a combined \( P T T' \)-symmetry still results in a spectrum with levels that are either real or occur in complex conjugate pairs. Compared to the case of \( P T \)-symmetry, experimental implementation simply requires to invert the magneto-optical effects in one part of the system. The effective Hamiltonian then takes the form

\[
\mathcal{H} = \begin{pmatrix}
H - i\mu & \Gamma \\
\Gamma & H + i\mu
\end{pmatrix},
\]

which differs from [4] when \( H \) is complex (i.e., in the unitary symmetry class). In the parity basis, the Hamiltonian takes the form of Eq. (7) even for unitary symmetry. Coupling now induces level crossings, and the transition to the complex spectrum is governed by the same characteristic scales \( \mu_T \) and \( \mu_0 \) as encountered in the orthogonal symmetry class of \( P T \)-symmetric systems.

Conclusions.—In summary, we identified two routes to the formation of complex energy levels in non-hermitian quantum systems with \( P T \)-symmetry (spontaneous \( P T \)-symmetry breaking). The predominant mechanism depends on whether or not the hermitian limit possesses time-reversal symmetry (orthogonal or unitary universality class, respectively). Present optical experiments fall into the orthogonal class, where level crossings result in a characteristic absorption/amplification rate \( \mu_{PT} \) which is independent of the coupling between the symmetry-related parts of the system (unless the coupling is very weak). The unitary class features strong level repulsion, which reduces \( \mu_{PT} \) and makes it coupling-dependent. While we employed random-matrix theory, these findings can be verified for individual systems by varying the coupling between their symmetry-related parts.

[1] C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80, 5243 (1998).
[2] M. Znojil, Phys. Lett. A 285, 7 (2001).
[3] C. M. Bender, D. C. Brody, and H. F. Jones, Am. J. Phys. 71, 1095 (2003).
[4] A. Mostafazadeh, J. Math. Phys. (N.Y.) 43, 205 (2002).
[5] C. M. Bender, Rep. Prog. Phys. 70, 947 (2007).
[6] M. V. Berry, J. Phys A 41, 244007 (2008); H. F. Jones, Phys. Rev. D 76, 125003 (2007); *ibid.* 78, 065032 (2008).
[7] R. El-Ganainy, K. G. Makris, D. N. Christodoulides, Z. H. Musslimani, Opt. Lett. 32, 2632 (2007); Z. H. Musslimani, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, Phys. Rev. Lett. 100, 030402 (2008); K. G. Makris, R. El-Ganainy, D. N. Christodoulides, Z. H. Musslimani, *ibid.* 100, 103904 (2008); Phys. Rev. A 81, 063807 (2010).
[8] S. Longhi, Phys. Rev. Lett. 103, 123601 (2009); Phys. Rev. B 80, 235102 (2009); Phys. Rev. Lett. 105, 013903 (2010).
[9] A. Guo et al., Phys. Rev. Lett. 103, 093902 (2009); C. E. Rüter et al., Nature Phys. 6, 192 (2010).
[10] H. Schomerus, Phys. Rev. Lett. 104, 233601 (2010).
[11] S. Longhi, Phys. Rev. A 82, 031801(R) (2010); Y. D. Chong, L. Ge, and A. D. Stone, [arXiv:1008.5156](http://arxiv.org/abs/1008.5156).
[12] O. Bendix, R. Fleishmann, T. Kottos, and B. Shapiro, Phys. Rev. Lett. 103, 030402 (2009); C. T. West, T. Kottos, and T. Prosen, *ibid.* 104, 054102 (2010).
[13] E. M. Graefe, H. J. Korsch, and A. E. Niederle, Phys. Rev. Lett. 101, 150408 (2008); Phys. Rev. A 82, 013629 (2010).
[14] Y. D. Chong, L. Ge, H. Cao, and A. D. Stone, Phys. Rev. Lett. 105, 053901 (2010).
[15] For pseudohermitian ensembles applying to other physical settings, see N. Hatano and D. R. Nelson, Phys. Rev. Lett. 77, 570 (1996); Z. Ahmed and S. R. Jain, J. Phys. A: Math. Gen. 36, 3349 (2003); S. R. Jain and S. C. L. Srivastava, Phys. Rev. E 78, 036213 (2008).
[16] C. W. J. Beenakker, Rev. Mod. Phys. 69, 731 (1997).
[17] K. M. Frahm, P. W. Brouwer, J. A. Melsen, and C. W. J. Beenakker, Phys. Rev. Lett. 76, 2981 (1996); J. A. Melsen, P. W. Brouwer, K. M. Frahm, and C. W. J. Beenakker, Europhys. Lett. 35, 7 (1996).
[18] M. L. Mehta, *Random Matrices*, 3rd ed. (Elsevier, New York, 2004).
[19] In Ref. [15], which concerns proximity-induced mesoscopic superconductivity (where particle-hole symmetry replaces \( P T \)-symmetry), the expression for \( \gamma \) differs because there a barrier of transmission probability \( T_N = 2\sqrt{T_0} / (1 + \sqrt{T_0}^2) \) is traversed twice, first by an electron and then by the Andreev-reflected hole. This is only a factor of definition since the relevant quantities (Thouless energy \( E_T \equiv \mu_T \) and dimensionless conductance \( G = N T / (2 - T_N)^2 \) change accordingly [16].
[20] We certified that results are robust when \( M \) is further increased. As long as \( 1 \leq N \leq M \), they only depend on the number of channels \( N \) inasmuch as this affects the border \( T_0 \) between strong and weak coupling.