On the origin of the large scale structures of the universe

David H. Oaknin

Department of Physics and Astronomy,
University of British Columbia,
Vancouver V6T 1Z1, Canada

e-mail: doaknin@physics.ubc.ca

We revise the statistical properties of the primordial cosmological density anisotropies that, at the time of matter radiation equality, seeded the gravitational development of large scale structures in the, otherwise, homogeneous and isotropic Friedmann-Robertson-Walker flat universe. Our analysis shows that random fluctuations of the density field at the same instant of equality and with comoving wavelength shorter than the causal horizon at that time can naturally account, when globally constrained to conserve the total mass (energy) of the system, for the observed scale invariance of the anisotropies over cosmologically large comoving volumes. Statistical systems with similar features are generically known as glass-like or lattice-like. Obviously, these conclusions conflict with the widely accepted understanding of the primordial structures reported in the literature, which requires an epoch of inflationary cosmology to precede the standard expansion of the universe. The origin of the conflict must be found in the widespread, but unjustified, claim that scale invariant mass (energy) anisotropies at the instant of equality over comoving volumes of cosmological size, larger than the causal horizon at the time, must be generated by fluctuations in the density field with comparably large comoving wavelength.

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I. INTRODUCTION

All current cosmological models work within the paradigm in which the present distribution of matter in the universe is the result of mostly gravitational evolution operating since the time of matter-radiation equality on some seed of initial and very small quantum
fluctuations in the mass (energy) density field of an, otherwise, homogeneous, isotropic and flat Friedmann-Robertson-Walker universe. On the large comoving scales of cosmological interest this seed of initial density perturbations at the time of equality is well described by the Harrison-Zeldovich spectrum of gaussian fluctuations [1].

The most characteristic feature of the spectrum of primordial anisotropies is the linear dependence, \( (\Delta M_V)^2 \propto S \), of the variance of mass (energy) fluctuations in restricted spatial sub-volumes of comoving cosmological size \( L \) on the area \( S \sim L^2 \) of the surface that bounds the sub-volume, rather than on its volume \( V \sim L^3 \). This specific feature \( (\Delta M_V)^2 \propto L^2 \) is commonly known in the literature as scale invariance of primordial mass (energy) anisotropies because they produce a gravitational potential \( \Phi \sim G\Delta M_V / L \sim G \) that does not depend on the comoving scale \( L \). The scale invariance of the seed of primordial cosmological anisotropies at the time of equality was first observationally tested ten years ago by the COBE measurements [2] of the temperature anisotropies in the cosmic microwave background radiation (CMBR) and has been more recently confirmed by the high-precision data of the WMAP collaboration [3] and others [4].

In the context of standard cosmology, nonetheless, comoving scales \( L \) of cosmological interest, \( H^{-1}(t_{eq}) < L \lesssim H^{-1}(t_0) \), are much longer, by orders of magnitude, than the comoving causal scale \( H^{-1}(t_{eq}) \) at the time of matter radiation equality: \( H^{-1}(t_0) / H^{-1}(t_{eq}) \sim 10^4 \), where \( H^{-1}(t_0) \) is the comoving size of the present causal horizon. A central problem in contemporary theoretical cosmology is the quantitative explanation of the mechanism that produced by the time of equality scale invariant anisotropies \( (\Delta M_V)^2 \sim L^2 \) over cosmological comoving scales \( L \), which are much larger than the causal horizon at that time. The problem arises from the assumption that scale invariant mass (energy) anisotropies in comoving volumes of cosmological size \( L \) must have been produced by fluctuations in Fourier modes of the density field with comoving wavelength of comparable size \( \lambda \sim L \gg H^{-1}(t_{eq}) \). We label this assumption as \([HP]\) for easier reference later on. If this assumption were correct, then it would be justified to claim that in the context of standard cosmology we lack a causal explanation of the mechanism that generated, at \( t_{eq} \), scale invariant mass anisotropies in comoving volumes much larger than the causal horizon at that time. This enigma is usually called the origin of structures problem of standard cosmology. The most outstanding of the mechanisms that have been invented to solve this problem is inflation, which roughly speaking proposes that the whole observable present universe was once causally connected in
the remote past and the density inhomogeneities were already imprinted at that early time before the universe underwent a finite period of exponential expansion which stretched it to its current huge size \textsuperscript{[3]}. This mechanism would explain the presence at the time of equality of fluctuation modes in the density field with comoving wavelength larger than the standard causal horizon at that instant and with the appropriate power spectrum. Obviously, the logic of the inflationary explanation of primordial structures also relies on assumption \textsuperscript{[HP]}. The main purpose of this work is to bring into attention and revisit the assumption \textsuperscript{[HP]} under which the origin of structures problem has been raised in the context of standard cosmology.

Assumption \textsuperscript{[HP]} is formally expressed through the fast estimation \textsuperscript{(11)} of integral \textsuperscript{(4)} for the variance of mass (energy) anisotropies over cosmologically large comoving volumes (see, for example, eq. 9.1,9.2 and 9.3 in \textsuperscript{[6]}). According to this estimation mass (energy) anisotropies over volumes \(V\) of cosmologically large comoving size \(L\) are expected to be generated by the dominant contribution to integral \textsuperscript{(4)} from random fluctuations of the stochastic density field \textsuperscript{(1)} with comparably large comoving wavelength \(k \sim 1/L\), while the contributions to this integral from the ultraviolet modes \(k \gg 1/L\) seemingly cancel out and are expected to be sub-dominant or negligible. Therefore, under assumption \textsuperscript{[HP]} only a power spectrum with spectral index \(n = 1\) over the range of cosmologically short comoving momenta \(k \ll H(t_{eq})\) would be able to generate scale invariant mass (energy) anisotropies \((\Delta M_V)^2 \sim L^2\) over cosmologically large comoving volumes \(L \gg H^{-1}(t_{eq})\), see \textsuperscript{(12)}. This statement is reported in textbooks and research papers in cosmology \textsuperscript{[7]}, even though the fast estimation \textsuperscript{(11)} is since long ago known in statistical mechanics to be incorrect. The correct estimation of integral \textsuperscript{(4)} over cosmologically large comoving volumes \(V\) for a power-law power spectrum with spectral index \(n\) is given in \textsuperscript{(13),(14),(15)}, for all possible values of the spectral index. The new estimation shows that any power-law spectrum with spectral index \(n \geq 1\) would, in fact, generate a pattern of scale invariant cosmological mass (energy) anisotropies.

Scale invariant anisotropies \textsuperscript{(15)} correspond to stochastic density fields \textsuperscript{(1)} whose random fluctuations have been globally constrained to conserve the total mass (energy) of the system. The reason why fast estimation \textsuperscript{(11)} fails to predict the correct behaviour \textsuperscript{(15)} is precisely the fact that in these globally constrained systems the largest contribution to integral \textsuperscript{(4)} for mass (energy) anisotropies over macroscopically large volumes of comoving size \(L\) does not
come from modes with comparably large comoving wavelength $k \sim 1/L$, whose contribution is suppressed by the global constrain, but from the ultraviolet modes $k \gtrsim H(t_{eq})$ with comoving wavelength within the horizon at the time of equality. The anisotropies are produced by a mechanism of local rearrangement of matter in the uniform universe that can be visualized through the following metaphor: people moving short distances (i.e. density fluctuations with short comoving wavelength) from town to town at different sides of a border can generate fluctuations in the total number of inhabitants in each of the countries separated by that border (i.e. mass anisotropies over cosmologically large volumes), even though the typical size of those countries can be much larger than the distance between the two towns. Obviously, the longer the common border between the two neighbour countries, the larger can be the size of the surface fluctuations in the number of inhabitants in each of the countries. Of course, the displacement of population does not change the total number of inhabitants in the two countries together. This mechanism is well understood in statistical mechanics since long ago, but for some reason it has not been clearly noticed before in cosmology. Statistical systems that show scale invariant anisotropies are known in the literature as glass-like or lattice-like. In quantum Hall effect in condensed matter physics, for example, low-energy excitations are known to be associated to the borders, as a direct consequence of the global constrain imposed by the incompressibility of the gas of electrons confined in two spatial dimensions.

At the light of these comments we realize that all stochastic density fields characterized by the same average density $\rho_0$ and constrained to globally conserve the total mass (energy) of the system produce the same pattern of scale invariant mass (energy) anisotropies over cosmologically large volumes, even though they can be defined by very different power spectra. Hence, all power spectra with indexes $n \geq 1$ are, in principle, macroscopically indistinguishable. This degeneracy has been ignored for many years in the literature, but it is absolutely crucial to understand the statistical properties of primordial cosmological mass (energy) anisotropies at the time of matter radiation equality.

In next sections we will revisit in detail the arguments and concepts that we have briefly introduced in this section and will explicitly show that random fluctuations in the density field at the same instant of matter radiation equality with comoving wavelength shorter than the horizon at that time $H^{-1}(t_{eq})$ can account, when they are constrained to conserve the total mass (energy) in the whole system, for the scale invariant primordial mass (energy)
anisotropies over cosmologically large comoving volumes.

In the context of this discussion we feel it is necessary for historical reasons to comment and clarify a mechanism of local rearrangement of matter at the same instant of equality first considered, an discarded, long ago by Y.B. Zeldovich as responsible for the mass (energy) anisotropies over cosmologically large comoving volumes. The apparent incapability of this mechanism to generate the anisotropies is cited in textbooks in cosmology as a proof of the impossibility in the standard cosmology to generate prior to the time of equality scale invariant anisotropies over cosmologically large comoving volumes through causally connected physics. The argument is also presented as a motivation for the inflationary explanation of the origin of primordial structures. We will show, on the contrary, that Zeldovich’s mechanism does indeed generate scale invariant cosmological anisotropies and was discarded only on the ground of a misled analysis based on the wrong estimation.

To model and reanalyze Zeldovich’s proposal we consider a free scalar hamiltonian density, which is quadratic in the fundamental scalar field and its conjugate momentum. Zeldovich nicely noticed that random fluctuations of the fundamental fields with comoving wavelength within the horizon can, through the quadratic coupling, generate density fluctuations with comoving wavelength much larger than the horizon

\[ k^{-1} \gg H^{-1}(t_{eq}). \]

This is a trivial property of Fourier analysis. This observation, although correct, was obviously motivated by the mistaken assumption: the quadratic coupling offered a causal mechanism to generate at the same instant of equality a density power spectrum over the range of cosmologically short momenta and, thus, according to fast estimation, mass (energy) anisotropies over cosmologically large volumes. On the other hand, a brief and old argument that we reproduce in Section V, provided also by Zeldovich, assures that the density power spectrum over the range of cosmologically short momenta

\[ k \ll H(t_{eq}) \]

generated through the quadratic coupling of causally connected modes must have a spectral index \( n \geq 4 \) and, therefore, it was thought that the variance of mass (energy) anisotropies over cosmologically large volumes should decrease, at least, as \( (\Delta M)^2 \sim 1/L \) with the comoving size \( L \) of the considered volume. Hence, it was concluded that local rearrangement of matter, although able to generate anisotropies over cosmologically large volumes, is unable to generate scale invariant cosmological anisotropies \( (\Delta M)^2 \sim L^2 \). This conclusion, based on the wrong assumption, is fatally incorrect as we will show: the variance of mass (energy) anisotropies generated through local rearrangement of matter over cosmologically large volumes are indeed
scale invariant $(\Delta M_V)^2 \sim L^2$ (15), as we should have expected because the fluctuations are globally constrained to conserve the total mass (energy) of the system. Zeldovich’s analysis of his own proposal got wrong because, biased by incorrect fast estimation (11), it focused on the ability to produce a non-zero density power spectrum over the range of cosmologically short comoving momenta through the quadratic coupling in the hamiltonian density and missed the fact that the largest, scale invariant, contribution $(\Delta M_V)^2 \sim L^2$ to cosmological anisotropies comes from random fluctuations of the density modes with the shortest wavelength (15), while the contribution of density modes with cosmologically large comoving wavelength is sub-dominant, $(\Delta M_V)^2 \sim 1/L$. In other words, in this model the largest contribution to mass (energy) cosmological anisotropies does come from causally connected random fluctuations of the fundamental fields when they couple to produce density fluctuations with comoving wavelength still within the horizon, and not by modes of the fundamental fields that couple to produce density fluctuations with cosmologically large comoving wavelength.

The irrelevance of the quadratic coupling is evident when, in the same framework set up to model Zeldovich’s proposal, we consider the density field (1) to be proportional to the conjugate momentum field, instead of being given by the quadratic hamiltonian density, and study spatial mass (energy) over cosmologically large comoving volumes. The momentum field can only fluctuate, in this example, in modes with comoving wavelength shorter than the horizon and, in absence of quadratic coupling, there is no way to generate density fluctuations with wavelength longer than the horizon. Yet, an analytical evaluation of integral (24) over cosmological volumes show that the anisotropies over volumes of comoving size larger than the horizon are still scale invariant. The reason (24) is the same as above: the conjugate momentum field operator is associated to a global charge of the system, which forces total mass (energy) conservation. In conclusion, it is not necessary to generate density fluctuations with cosmologically large comoving wavelength in order to produce scale invariant anisotropies over cosmologically large volumes. Any mechanism of local rearrangement of matter can successfully generate scale invariant anisotropies over cosmologically large volumes. Zeldovich’s old example, properly analyzed, is only a particular example.

The analysis that we present here obviously lifts the obstacle that has long stood in the way to provide a causal explanation for the origin of the large scale structures of the universe in the context of standard cosmology (we mean standard cosmological model, without the
appendix of a preceding inflationary expansion) and provides an attractive alternative to inflation to solve the origin of structures problem. In principle, this alternative mechanism should probably involve only physics at the scale of matter radiation equality, $T_{eq} \sim 1 \text{ eV}$, instead of physics at the very high energy scales up to the Planck scale that are usually summoned in inflationary mechanisms. Of course, this does not exclude that new physics could be relevant at the instant of equality to explain the origin of primordial structures. Moreover, the generic problem of tuning the initial cosmological conditions in inflationary models in order to produce a continued period of exponential expansion simply vanishes in the context of the alternative scenario that we suggest: in this alternative scenario it is demanded in a naturally simple way that the fluctuating density field is in its ground state at the instant of equality.

The paper is organized in nine sections. In Section II we review the concepts and tools needed to describe the statistical properties of density anisotropies at the time of matter radiation equality in the homogeneous, isotropic and flat FRW universe. In Section III we characterize the power spectra of gaussian fluctuations of the density field that produce scale invariant mass (energy) anisotropies over cosmologically large comoving volumes. It is explicitly shown that any power-law spectrum $P(k) \sim A k^n$ with spectral index $n \geq 1$ produces an scale invariant spectrum of fluctuations $(\Delta M_V)^2 \sim L^2$ in volumes of cosmologically large comoving size, $L \gtrsim H^{-1}(t_{eq})$. This claim is the main reason why the analysis performed in this paper disagrees with the main conclusions that are found in the literature, where it is claimed that only a power-law spectrum with spectral index close to $n = 1$ can produce scale invariant mass (energy) anisotropies in cosmologically large comoving volumes. The motives of our disagreement are extensively discussed in this section III. The calculations presented also prove the central claim of this paper, that random fluctuations at the same instant of equality in the Fourier modes of the density field with comoving wavelength shorter than the horizon at that time can account for the observed scale invariant primordial anisotropies over cosmologically large comoving volumes. We go on in Section IV to discuss the characterization of scale invariant mass (energy) random anisotropies in terms of the two points correlation function of the fluctuating density field. We can advance here that we find that scale invariant anisotropies correspond to a certain class of short range correlation functions that result when the global mass (energy) contained in the system is conserved, and therefore, not allowed to fluctuate. The aim of section V is mainly to clarify the arguments that
historically laid the so-called origin of structures problem [HP] of the standard cosmology. We discuss these arguments in the setup of a quantum field theory and describe the mechanism of generation of scale invariant primordial cosmological mass (energy) anisotropies by local rearrangement of matter as random quantum fluctuations of the density field in the homogeneous and isotropic ground state of the system at the same instant of equality. Notwithstanding these first five sections of the paper constitute a self-contained body, which justifies and proves the claims that appear in the abstract, introduction (Section I) and conclusions (Section IX) of this paper, we have judged appropriated to include three additional sections where we introduce concepts that shall be relevant to set up a complete quantum theory of density anisotropies coupled to metric perturbations of the homogeneous and isotropic FRW space-time background. In Section VI we discuss technical concepts on the renormalization of ultraviolet divergent expressions in the theoretical setup to conclude that, in general, the size of the scale invariant spatial anisotropies can only be defined as an external renormalizable parameter of the theory, like masses or coupling constants. In Section VIII we obtain the renormalization equations that describe the dependence of this parameter on the resolution scale at which the system is probed. In Section VII we review, at the light of our findings, the linearized gauge invariant formalism of density anisotropies coupled to metric perturbations of the FRW expanding space-time background. Section IX summarizes the conclusions of this paper.

II. DENSITY ANISOTROPIES IN HOMOGENEOUS AND ISOTROPIC STATISTICAL SYSTEMS: CONCEPTS AND DEFINITIONS.

We review in this section some of the concepts and tools of statistical mechanics that are widely used to describe the cosmological density anisotropies in the early universe at the time $t_{eq}$ of matter radiation equality. Our aim is to gather some important results from statistical mechanics upon which we will develop our discussion. This review will also help us to fix the notation that we use in the rest of the paper. For the sake of simplicity we work in comoving coordinates and fix the scale factor due to the expansion of the universe to be equal to one at the time of equality, $a(t_{eq}) = 1$.

The statistical system is described by an homogeneous and isotropic density field $\rho(\vec{x})$ in 3D flat space $\Omega \equiv R^3$. 
\[ \rho(\vec{x}) = \rho_0 + \rho_0 \int \frac{d^3 \vec{k}}{(2\pi)^3} \delta_\vec{k} e^{-i\vec{k} \cdot \vec{x}}. \]  

(1)

If the fluctuations are small the coefficients \( \delta_\vec{k} = \delta^*_{-\vec{k}} \) are independent random complex variables with normal distribution, \( \langle \delta^*_{\vec{k}_1} \delta_{\vec{k}_2} \rangle = (2\pi)^3 \mathcal{P}(\vec{k}_1) \delta^3(\vec{k}_1 - \vec{k}_2) \), and zero expectation value, \( \langle \delta_\vec{k} \rangle = 0 \), so that each local variable \( \rho(\vec{x}) \) is real and has expectation value \( \langle \rho(\vec{x}) \rangle = \rho_0 \) independent of the position \( \vec{x} \). This average value is assumed to be non zero, \( \rho_0 \neq 0 \). The central limit theorem implies that the fluctuations of the density field \( \rho(\vec{x}) \) are gaussian.

The function \( \mathcal{P}(\vec{k}) \geq 0 \), which gives the variance of the fluctuations of each of the random Fourier modes \( \delta_\vec{k} \), is usually called the power spectrum of the statistical fluctuations and it is the most common statistical tool used to describe cosmological models. If the system is isotropic, the power spectrum \( \mathcal{P}(\vec{k}) = \mathcal{P}(k) \) depends only on the modulus of the momentum that labels each mode, \( k \equiv |\vec{k}| \).

Statistical fluctuations of the density field (1) in macroscopic, but finite, sub-volumes \( V \subset \Omega \) are usually described in terms of the statistical magnitude 

\[ \sigma^2(V) \equiv \frac{1}{V^2} \langle \left( \int_V d^3 \vec{x} \frac{\rho(\vec{x}) - \rho_0}{\rho_0} \right)^2 \rangle. \]

This magnitude is sometimes called the squared density contrast over the volume \( V \) and denoted by \( \left( \frac{\delta \rho}{\rho} \right)^2_V \). In terms of the power spectrum of the fluctuations the squared density contrast can be expressed as:

\[ \sigma^2(V) = \frac{1}{V^2} \int \frac{d^3 \vec{k}}{(2\pi)^3} \mathcal{P}(k) |F_V(\vec{k})|^2, \]  

(2)

where the geometric factor \( F_V(\vec{k}) \) is given by the expression

\[ F_V(\vec{k}) = \int_V d^3 \vec{x} e^{-i\vec{k} \cdot \vec{x}}. \]  

(3)

We wish to notice that this geometric factor \( F_V(k) \) is the restricted integral over the sub-volume \( V \) of the Fourier modes of expansion (1) in flat 3D space.

Step by step the proof of relation (2) proceeds as follows:

\[ \frac{1}{V^2} \langle \left( \int_V d^3 \vec{x} \frac{\rho(\vec{x}) - \rho_0}{\rho_0} \right)^2 \rangle = \frac{1}{V^2} \int_V d^3 \vec{x} \int_V d^3 \vec{y} \int \frac{d^3 \vec{k}_1}{(2\pi)^3} \int \frac{d^3 \vec{k}_2}{(2\pi)^3} \langle \delta^*_{\vec{k}_1} \delta_{\vec{k}_2} \rangle e^{+i\vec{k}_1 \cdot \vec{x}} e^{-i\vec{k}_2 \cdot \vec{y}} = \frac{1}{V^2} \int \frac{d^3 \vec{k}_1}{(2\pi)^3} \int d^3 \vec{k}_2 \mathcal{P}(k_1) \delta^3(\vec{k}_1 - \vec{k}_2) F^*_V(\vec{k}_1) F_V(\vec{k}_2) = \frac{1}{V^2} \int \frac{d^3 \vec{k}}{(2\pi)^3} \mathcal{P}(k) |F_V(\vec{k})|^2. \]

The statistical variable \( M(V) = \int_V d^3 \vec{x} \rho(\vec{x}) \) describes the total mass (energy) contained in the sub-volume \( V \). This is the mathematical object that describes the macro-
scopic properties of the statistical anisotropies in the density field \( \langle M(V) \rangle = \rho_0 V \) is proportional to the volume \( V \) of the considered region. Its average value \( \langle \Delta M(V) \rangle \equiv \langle [M(V) - \langle M(V) \rangle] \rangle = \langle M^2(V) \rangle - \langle M(V) \rangle^2 \) \( = \langle [\int_V d^3\vec{x} (\rho(\vec{x}) - \rho_0)]^2 \rangle \) measures the typical size of its gaussian fluctuations. Combining the previous expressions it is straightforward to obtain:

\[
(\Delta M_V)^2 = \rho_0^2 \int \frac{d^3\vec{k}}{(2\pi)^3} \mathcal{P}(k) |F_V(\vec{k})|^2.
\] 

(4)

The last tool we want to introduce in this section is the two points correlation function of the density field, \( F(\vec{x}, \vec{y}) \equiv \langle \rho(\vec{x}) \rho(\vec{y}) \rangle - \rho_0^2 \). If the system is homogeneous the two points function must depend only on their relative position \( \vec{r} = \vec{x} - \vec{y} \). If the system is also isotropic then the two points function \( F(r) \) can depend only on the distance between the two points. In terms of the power spectrum, we can write

\[
F(r) = \rho_0^2 \int \frac{d^3\vec{k}}{(2\pi)^3} \mathcal{P}(k) e^{+i\vec{k} \cdot \vec{r}}.
\] 

(5)

The proof is straightforward:

\[
F(\vec{x}, \vec{y}) = \rho_0^2 \int \frac{d^3\vec{k}_1}{(2\pi)^3} \int \frac{d^3\vec{k}_2}{(2\pi)^3} \left( \delta_{\vec{k}_1} \delta_{\vec{k}_2} \right) e^{+i\vec{k}_1 \cdot \vec{x}} e^{-i\vec{k}_2 \cdot \vec{y}} = \\
\rho_0^2 \int \frac{d^3\vec{k}_1}{(2\pi)^3} \int d^3\vec{k}_2 \mathcal{P}(k_1) \mathcal{P}(k_2) e^{+i\vec{k}_1 \cdot \vec{x}} e^{-i\vec{k}_2 \cdot \vec{y}} = \rho_0^2 \int \frac{d^3\vec{k}}{(2\pi)^3} \mathcal{P}(k) e^{+i\vec{k} \cdot (\vec{x} - \vec{y})}
\]

The variance of mass fluctuations \( \langle \Delta M \rangle \) can then be conveniently expressed in terms of the two point function:

\[
(\Delta M_V)^2 = \int_V d^3\vec{x} \int_V d^3\vec{y} F(\vec{x}, \vec{y}).
\] 

(6)

Let us remark at this point that \( \mathcal{P}(k) \) must be integrable over the whole 3D momentum space,

\[
\int d^3\vec{k} \mathcal{P}(k) < \infty,
\] 

(7)

if the two points function \( F(\vec{r}) \) is well defined at \( \vec{r} = 0 \) and the squared density contrast \( \langle \delta^2 \rangle \) vanishes when integrated over very large volumes, \( \lim_{V \to \Omega} \sigma^2(V) = 0 \). This condition \( \sigma^2 \)
allows the spectrum to diverge at $k = 0$, with the only condition that, if it diverges, it does slower than $1/k^3$,

$$\lim_{k \to 0} k^3 \mathcal{P}(k) = 0.$$

The discussion in this paper will focus on the statistical properties, namely power spectra and two points correlation functions, of scale invariant systems, in which the variance of mass (energy) fluctuations in restricted sub-volumes $V$ is proportional to the area of the surface $S$ that bounds the sub-volume, $(\Delta M_V)^2 \propto S$, rather than to its volume $V$. If the considered sub-volume is, in particular, an sphere of radius $L$ those statistical systems produce mass fluctuations

$$(\Delta M_V)^2 \propto S \sim L^2.$$  

Fluctuations in the density field $\rho(\vec{x})$ seed in turn fluctuations in the gravitational potential through the Poisson’s equation $\vec{\nabla}^2 \Phi(\vec{x}) = 4\pi G (\rho(\vec{x}) - \rho_0)$. When the mass (energy) fluctuations in cosmologically large comoving volumes are scale invariant the fluctuations in the gravitational potential

$$\Delta \Phi_V \propto \frac{G \Delta M_V}{L} \sim G,$$

do not depend on the scale $L$. Statistical systems with this remarkable feature are named after Harrison and Zeldovich, who first identified them.

### III. POWER SPECTRUM OF SCALE INVARIANT DENSITY ANISOTROPIES.

Our aim in this section is to characterize in terms of their power spectrum $\mathcal{P}(k)$ the statistical systems whose mass (energy) fluctuations over cosmologically large comoving volumes have scale invariant variance. The two magnitudes are directly related by equation (4).

A fast, but incorrect, estimation of relation (4) which, nevertheless, appears in many references in the literature (see, for example, eq. 9.1,9.2 and 9.3 in [6]) gives:

$$(\Delta M_V)^2 = \rho_0^2 \int \frac{d^3k}{(2\pi)^3} \mathcal{P}(k)|F_V(\vec{k})|^2 \sim \rho_0^2 \frac{1}{V} \mathcal{P}(k \sim 1/L)|F_V(0)|^2 =$$
\[ V_p(k \sim 1/L) \propto \rho_0^2 \frac{A}{L^n} L^3 = \rho_0^2 A L^{3-n}, \]

where \( L \) is the linear size of the considered volume of integration. This estimation is derived after noticing that the geometric factor \( F_V(k) \) evaluated at \( k = 0 \) measures the volume \( V \sim L^3 \) of the considered spatial region, \( F_V(k = 0) = V \), and it decreases fast to zero for values of the momentum \( k \) larger than the inverse \( 1/L \) of the typical linear size of the considered volume, \( F_V(k \sim 1/L) \sim 0 \).

According to this estimation a power-law spectrum with spectral index \( n, P(k) \sim A k^n \), over the range of comoving momenta \( k \leq H(t_{eq}) \) produces mass (energy) fluctuations

\[ (\Delta M_V)^2 \sim \rho_0^2 \frac{A}{L^n} L^3 = \rho_0^2 A L^{3-n}, \]

in cosmologically large comoving volumes, of size \( L \gtrsim H^{-1}(t_{eq}) \). Therefore, a power-law spectrum with spectral index \( n = 1, P(k) \sim A k \), over the whole range of comoving momenta \( k \leq H(t_{eq}) \) seems to be a necessary and sufficient condition to generate a pattern of scale invariant mass (energy) anisotropies over cosmologically large comoving volumes.

Notwithstanding this result \( (12) \) is cited in textbooks and reviews on the subject \([6, 7]\), we choose to check it, both analytically and numerically, integrating equation \( (11) \). It will turn out that \( (12) \) gives a reliable estimation only when the spectral index \( n \) is in the range, \( -3 < n < 1 \), but the estimation gets fatally wrong for larger values, \( n \geq 1 \). Notice, in particular, that \( (12) \) predicts an scaling law \( (\Delta M_V)^2 \sim L^\alpha \), with \( \alpha < 2 \) for values of the spectral index larger than one, \( n > 1 \). For example, for \( n = 4 \) it predicts \( (\Delta M_V)^2 \sim L^{-1} \).

Clearly, this prediction must be incorrect because it has been rigorously proved that \( \alpha \geq 2 \) \([11]\), the scaling power \( \alpha \) is necessarily equal or larger than 2 for any homogeneous and isotropic statistical system. The fast estimation \( (12) \) misses that for values of the spectral index \( n \geq 1 \), in spite of the rapid decrease to zero of the geometric factor \( |F_V(k)|^2 \) for values of comoving momenta \( k \gtrsim 1/L \), integral \( (11) \) is still dominated by the contribution of the very large momenta \( k \gg 1/L \), rather than by the modes \( k \sim 1/L \). This ultraviolet divergence has been noticed before in the literature on cosmological density anisotropies \([12]\). Once this contribution is taken into account it is found the correct behaviour of the variance of mass fluctuations,

\[ \text{If} \quad n \in (-3, 0], \quad (\Delta M_V)^2 \propto V^{1-n/3} \sim L^{3-n}, \]
If \( n \in (0, +1] \), \( (\Delta M_V)^2 \propto S^{3/2-n/2} \sim L^{3-n} \), \( (\Delta M_V)^2 \propto S \sim L^2 \).

These correct results \( (13, 14, 15) \), obviously, respect the bound \( \alpha \geq 2 \). We wish to remark that these results are not new neither unexpected. They were previously reported, for example, in \([13]\) in the context of a discussion of the spectrum of primordial cosmological anisotropies. They are also very well-known in condensed matter physics in the description of glass-like systems. We rederive them here only with the aim of convincing the reader about their validity. As we will show, the disagreement between the old estimation \( (12) \) and the correct result \( (15) \) listed above is the only reason why the conclusions of this paper do not agree with those widely accepted and reported in the literature.

In order to justify the result \( (15) \) we now consider as volume \( V \) of integration an sphere of comoving radius \( L \). The rotationally symmetric geometry of the considered spherical sub-volume does not change the qualitative features of the results that we want to prove, but it allows to obtain the geometric factor \( (3) \) analytically:

\[
F_V(k) = 4\pi L^3 \frac{1}{(kL)^3} (\sin(kL) - (kL)\cos(kL)).
\]

When this expression is introduced in \( (14) \) we obtain the relation:

\[
(\Delta M_V)^2 = 8\rho_0^2 L^3 \int d(kL) \frac{\mathcal{P}(k)}{(kL)^4} (\sin(kL) - (kL)\cos(kL))^2.
\]

It is important to notice that \( \mathcal{I}(w) = \frac{1}{w^4} (\sin(w) - w\cos(w))^2 \), where \( w = kL \), is not singular at \( w = 0 \) as the factor \( \frac{1}{w^4} \) could induce to think, because \( (\sin(w) - w\cos(w))^2 = \mathcal{O}(w^6) \) when \( w \to 0 \).

For the power spectrum we now assume a power-law dependence \( \mathcal{P}(k) = A \frac{k^n}{k_c^n} \) over a large, but finite, range of comoving momenta \( 0 \leq k \lesssim k_c \), and beyond which the power spectrum is cutoff by some supression factor. Later on we will specify the scale \( k_c \) of the cutoff procedure and will characterize it as a regularization step of an ultraviolet divergent expression in the context of the renormalization programme of a physical parameter. For the moment it is enough to say that this scale can be somewhat of the order of the Hubble horizon at the time of equality, \( k_c \approx H(t_{eq}) \). Notice also that coefficient \( A \) is dimensionless and can be fixed independently of the spectral index \( n \). For the sake of simplicity we will assume in the analytical proof that we present below that the cutoff at \( k_c \) in the power
spectrum is sharp, \( P(k \geq k_c) = 0 \). More realistic exponential or polynomial cutoffs are evaluated numerically. The results are presented in Fig. 1. They show the same behaviour \( [13,14,15] \) that we observe from the analytical discussion of the simpler case with a sharp cutoff.

We now discuss separately the four different cases when the spectral index \( n \in \mathbb{R} \) in the power-law is: case I) \( n > 1 \); case II) \( n = 1 \); case III) \( 0 < n < 1 \); and, finally, case IV) \( -3 < n \leq 0 \). The spectral index \( n \in \mathbb{R} \) is restricted by condition [8] to be larger than \( n > -3 \).

In case I), \( n > 1 \), the variance \( [17] \) can be estimated with very good accuracy by the analytical expression

\[
(\Delta M_V)^2 \sim 4 \rho_0^2 L^3 \int_0^{k_c L} d(kL) \frac{P(k)}{(kL)^2},
\]

if \( L \gg k_c^{-1} \). The reader can visualize the condition for this estimation after noticing that the factor \( \frac{1}{(kL)^2} (\sin(kL) - (kL) \cos(kL))^2 \) in the integrand of \( [17] \) approximately halves the area left under the curve \( \frac{P(k)}{(kL)^2} \). When the last integral is performed we obtain

\[
(\Delta M_V)^2 \sim 4 \rho_0^2 L^3 \int_0^{k_c L} d(kL) \frac{A(k^n/k_c^{n+3})}{(kL)^2} = 4 \rho_0^2 \frac{A}{(n-1)} \frac{1}{k_c^4 L^2} \equiv \rho_0^2 \frac{A'}{k_c^4} L^2,
\]

where \( A' \) is a dimensionless factor that fixes the absolute amplitude of the fluctuations. Notice the promised linear dependence of the variance on the area \( S \propto L^2 \) of the surface that bounds the sphere of integration. It is also interesting to realize that integral \( [17] \) is convergent at \( kL \to 0 \), but it is divergent in the modes with very large momenta \( k \sim k_c \gg 1/L \) close to the cutoff. The explicit way how the cutoff in \( P(k) \) regularize this expression is not qualitatively relevant in this discussion, as it is shown in Fig. 1, but the need for regularization proves that the largest contribution to the variance comes from fluctuations of the Fourier modes \( \delta_k \) of the density field \( [1] \) in the ultraviolet range \( k \sim k_c \). We also can understand this result by noticing that, according to \( [12] \), the contribution to integral \( [4] \) over comoving volumes of cosmological size \( L \) coming from modes with comparable wavelength \( k \sim 1/L \) decreases for \( n > 1 \) as \( \sim L^{3-n} \), which is subdominant to the contribution \( [15] \) \( \sim L^2 \) from the ultraviolet modes close to the cutoff \( k \sim k_c \).

It is also important to notice how the ultraviolet cutoff \( k_c \) enters expression \( [19] \): this regularization cutoff only modifies the absolute size of the mass (energy) fluctuations, but
it does not modify the scale invariant linear dependence of their variance on the area $S$ of the surface that bounds the considered sub-volume. We will discuss this point in more detail in next Section VI and Section VIII.

Before going on to cases II, III) and IV) let us further clarify the result we have just proved. We have shown that fluctuations in the modes $\delta_k$ of the density field with the shortest comoving wavelength $k \gtrsim H(t_{eq})$, can generate scale invariant mass (energy) anisotropies over much larger volumes of comoving cosmological size $L \gg H^{-1}(t_{eq})$. We have been asked where is the 'non-linear' mechanism that leads to this result? The answer is in the integral expression for the variance of mass anisotropies: the reader can clearly see that fluctuations in all the modes $\delta_k$ over the whole range $0 \leq k \leq k_c$ add up their positive contribution $dk \ k^2 \mathcal{P}(k) |F_V(\vec{k})|^2 > 0$ to the total variance. It is commonly assumed that for a given volume $V$ of cosmological comoving size $L \gg k_c^{-1}$, the overwhelmingly largest contribution to this integral comes from the modes $k \sim L^{-1}$ (see, for example, eq. (9.3) in [6]). This is the assumption [HP], which leads to the 'linear' relationship between the variance of mass (energy) anisotropies in a volume of comoving size $L$ and fluctuations $\mathcal{P}(k \sim L^{-1})$ of the modes $\delta_k$ with comparable comoving wavelength $L \sim k^{-1}$. We have explicitly shown above that this 'linear' relationship does not hold in case I), when fluctuations of the ultraviolet modes $k \sim k_c$ produce the largest contribution to the global integral and generate scale invariant mass (energy) anisotropies. The 'non-linear' connection between mass (energy) anisotropies in cosmologically large comoving volumes and fluctuations in the Fourier modes of the density field with much shorter comoving wavelength is intrinsically wired into the integral expression, which stands as the core hardware of the formal description of the macroscopic properties of gaussian fluctuations in the density field. This 'non-linear' connection is relevant in case I), when spatial anisotropies are scale invariant, but has been carelessly disregarded in the literature on the origin of cosmological primordial density anisotropies.

In case II), $n = 1$, the integral is apparently divergent not only at $kL \gg 1$ but also at $kL \to 0$. Now it is important to remember that, as $\mathcal{P}(0) = 0$ for any $n > 0$, the integrand $\mathcal{P}(k)\mathcal{I}(kL)$ in (17), of which (18) is only an estimate, is regular at $kL \to 0$. The variance is still dominated by the ultraviolet modes $k \sim k_c \gg 1/L$, but now they contribute only a sub-leading logarithmic divergence,
(ΔM_V)^2 \sim 4\rho_0^2A \frac{1}{k_c^4} L^2 \ln(k_cL). \tag{20}

To leading order the variance (ΔM_V)^2 \sim L^2 shows the same linear dependence on the area of the boundary surface that was obtained for the variance in case I).

In case III), 0 < n < 1, the integrand in (17) decreases fast enough in the large k modes to render the integral over these modes convergent. A proper estimation shows that now the result agrees with the estimation in (12), (ΔM_V)^2 \sim \rho_0^2B' \frac{1}{k_c^{n+3}} L^{3-n}, because the integral now is dominated by the ‘linear’ modes k_cL \sim 1.

Finally, we consider the case of negative values of the spectral index, −3 < n ≤ 0. In this case, the integrand in the l.h.s of (17) decreases even faster to zero in the ultraviolet regime k \sim k_c \gg 1/L and the integral is dominated by the modes kL \sim 1. The fast estimation presented in (12) is, therefore, also valid in this case. In fact we obtain,

(ΔM_V)^2 \sim \rho_0^2B'' \frac{1}{k_c^{n+3}} L^{3-n}. \tag{21}

This completes our proof of (13,14,15). These results show that any power-law spectrum over a finite range of comoving momenta 0 \leq k \lesssim k_c, that vanishes at k = 0, i.e. \mathcal{P}(0) = 0, and whose first derivative at the origin also vanishes, i.e. \mathcal{P}'(0) = 0, produces a pattern of scale invariant mass (energy) fluctuations in any volume whose comoving size is much larger than the comoving length scale of the cutoff, k_cL \gg 1:

(ΔM_V)^2 \propto S \sim L^2 \quad \Leftrightarrow \quad (\mathcal{P}(0) = 0 \quad \lor \quad \mathcal{P}'(0) = 0). \tag{22}

If only the power spectrum vanishes at zero, \mathcal{P}(0) = 0, but the first derivative does not, \mathcal{P}'(0) \neq 0, we still have surface dependance (14) for the variance of mass (energy) anisotropies, (ΔM_V)^2 \propto S^{\gamma}, although the polynomial growth is faster than linear 1 < \gamma \leq 1.5. On the other hand any power spectrum which does not vanish at k = 0, i.e \mathcal{P}(0) \neq 0, produces in spatial regions whose typical comoving size is much larger than the comoving length scale of the cutoff k_cL \gg 1 a pattern of mass (energy) anisotropies (ΔM_V)^2 \propto V^\beta, with 1 \leq \beta < 2, proportional to some power of the volume. We will further comment on this point in next section. Now we do not want to derail from the main discussion of this section that shall focus on power-law spectra with spectral index n > 1 or any linear combination of them, which produce strict scale invariant anisotropies (20) over
cosmologically large comoving volumes through the dominant contribution of the Fourier modes with comoving wavelength shorter than the horizon $k \gtrsim H(t_{eq})$.

The result that we report in equation (15) has further implications that we want to emphasize: power spectra with different spectral indices $n > 1$ produce gaussian fluctuations in the macroscopic variable $M(V) \equiv \int f_V \rho(\vec{x})$ with the same average value $\rho_0 V$ and the same characteristic variance $(\Delta M_V)^2 = \rho_0^2 A' k_c^{-4} L^2$ in any macroscopic volume $V$, for their amplitude can independently be fixed through the dimensionless prefactor $A'$. Therefore, we must conclude that power spectra with different spectral indices $n > 1$, or linear combinations of them, are macroscopically indistinguishable in the scale invariant mass (energy) anisotropies $\Delta M_V$ that they produce in cosmologically large spatial sub-volumes.

If all different power spectra $P(k) \sim k^n$, with index $n > 1$, produce mass (energy) anisotropies in macroscopic volumes with the same variance, which scales linearly with the boundary area $\delta$, all of them produce scale invariant gravitational potentials $\delta$. Hence, there is no way to formally distinguish between power-law spectra with different $n > 1$ through the macroscopic mass fluctuations they produce, neither through their gravitational effect on large macroscopic distances.

Does this discussion mean that power spectra with different spectral index $n > 1$ are physically indistinguishable? Not exactly, in principle. Different power spectra would produce different two points correlation functions $F(\vec{r})$, but the differences are hidden when the macroscopic magnitudes $(\Delta M_V)^2$, given by (6), are compared. We will come back to this point later in Section VI and Section VIII.

We finish this section with an estimation of the density contrast $(\delta \rho/\rho)_L \equiv \sqrt{\sigma_L^2}$ in spherical volumes of comoving size $L$ associated to scale invariant mass (energy) anisotropies (10). We have:

$$\sigma_L^2 = \frac{(\Delta M_V)^2}{(M_V)^2} \sim \frac{\rho_0^2 A' \frac{1}{8\pi} L^2}{\rho_0^2 V^2} = \frac{A'}{(4\pi/3)^2 (k_c L)^4}.$$  \hspace{1cm} (23)

Therefore, $(\delta \rho/\rho)_L \sim \frac{A'}{(4\pi/3)^2 (k_c L)^4} \sim 0.25 \sqrt{A'} (k_c L)^{-1}$. Present bounds on the density contrast over cosmological scales constrain $(\delta \rho/\rho) \lesssim 10^{-5}$. Therefore, we can constrain the scale $k_c$ of momentum cutoff to be such that $k_c L \gtrsim 100 (A')^{1/4}$ for all cosmologically large comoving scales, $H(t_{eq}) \gg 1$. Both expressions together demand that $A' < (k_c/H(t_{eq}))^4$, so that $k_c$ can be naturally larger than $H(t_{eq})$. 
IV. SCALE INVARIANT DENSITY ANISOTROPIES AND TOTAL MASS CONSERVATION.

We want to obtain in this section the conditions that characterize the two points correlation function (5) of statistical systems which show scale invariant mass (energy) fluctuations (9) over macroscopically large, but finite, spatial sub-volumes \( V \subset \Omega \) of the whole system. These conditions will be then used to prove that random fluctuations of the density field (1) produce scale invariant mass (energy) anisotropies, \((\Delta M_V)^2 \propto S\), in cosmologically large comoving volumes if an additional constrain of total mass (energy) conservation is imposed:

\[
(\Delta M_\Omega)^2 = 0 \quad \Rightarrow \quad (\Delta M_V)^2 \propto S. \quad (24)
\]

Let us start then characterizing the two points function of scale invariant density fluctuations. Equation (5) can be inverted and written as

\[
P(\vec{k}) = \frac{1}{\rho_0^2} \int_\Omega d^3\vec{r} e^{-i\vec{k}\cdot\vec{r}} F(\vec{r}) = \frac{4\pi}{\rho_0^2} \int_0^\infty dr r^2 \frac{\sin(kr)}{kr} F(r). \quad (25)
\]

The power spectrum is by definition a positive function, \(P(k) \geq 0\). This condition constrains any possible choice for the two points correlation function \(F(r)\). Yet the most interesting constrain is derived from the condition that appears in the right hand side of the logical equivalence (22): \(P(k)_{k=0} = 0\) and \(\frac{dP(k)}{dk}|_{k=0} = 0\).

First, we notice

\[
P(0) = 0 \quad \Leftrightarrow \quad \frac{1}{\rho_0^2} \int_0^\infty dr r^2 F(r) = 0. \quad (26)
\]

This is the only non-trivial constrain that we need to request on the two points correlation function \(F(r)\) in order to get scale invariance mass (energy) anisotropies according to (22), because the condition on the first derivative at the origin \(k = 0\) is trivially fulfilled if the linear operators \(\frac{d}{dk}\) and \(\int_0^\infty dr\) commute when acting on the function \(g(k, r) = r^2 \frac{\sin(kr)}{kr} F(r)\):

\[
P'(0) = \frac{4\pi}{\rho_0^2} \frac{d}{dk} \left( \int_0^\infty dr r^2 \frac{\sin(kr)}{kr} F(r) \right)_{k=0} = \frac{4\pi}{\rho_0^2} \int_0^\infty dr r^2 \frac{d}{dk} \left( \frac{\sin(kr)}{kr} \right)_{k=0} F(r) = 0, \quad (27)
\]

We will not discuss here the formal mathematical requirements needed to get this technical condition (27) satisfied and will assume, for simplicity, that it is trivially fulfilled in the
setup we have defined. In fact, in all the standard definitions introduced in Section II it was assumed that linear operators in real $\vec{r}$-space and momentum $\vec{k}$-space do commute. Of course, a further consideration of these formal aspects would be interesting.

The logical equivalence (22), therefore, implies that condition $\frac{1}{\rho_0^2} \int_\Omega d\vec{r} F(r) = 0$ in the r.h.s. of (26) completely characterizes the two points function $F(r)$ of statistical systems which produce scale invariant mass (energy) anisotropies. This condition on the two points function is known in statistical mechanics to characterize glass-like systems. In quantum Hall effect in condensed matter physics a similar constrain on low-energy excitations is associated to the incompressibility of the gas of electrons confined in 2D spatial dimensions. These excitations are known to be associated to the borders [8].

The condition above can be written in a different way as

$$
\frac{1}{\rho_0^2} \int_\Omega d\vec{r} F(r) = \frac{1}{\rho_0^2} \int_\Omega d\vec{r} (\langle \rho(\vec{x}) \rho(\vec{x} + \vec{r}) \rangle - \langle \rho(\vec{x}) \rangle \langle \rho(\vec{x} + \vec{r}) \rangle) = \frac{1}{\rho_0^2} (\langle \rho(\vec{x}) M(\Omega) \rangle - \langle \rho(\vec{x}) \rangle \langle M(\Omega) \rangle) = 0,
$$

where $M(\Omega) = \int_\Omega d\vec{x} \rho(\vec{x})$ is the total mass in the whole space $\Omega$. The condition of scale invariance (26), therefore, is satisfied if and only if the total mass (energy) $M(\Omega)$ is not correlated to the specific value of the density field at any point $\vec{x}$:

$$
\langle \rho(\vec{x}) M(\Omega) \rangle = \langle \rho(\vec{x}) \rangle \langle M(\Omega) \rangle.
$$

Furthermore, if we integrate the previous expression over a finite volume $V$ we obtain that in systems that show scale invariant random mass (energy) anisotropies (9),

$$
\langle M(V) M(\Omega) \rangle = \langle M(V) \rangle \langle M(\Omega) \rangle,
$$

the total mass (energy) contained in the system, $M(\Omega)$, is not correlated to the mass (energy) contained in any finite sub-volume $V \subset \Omega$ inside it. Of course this condition is easily fulfilled if the total mass $M(\Omega)$ is constrained to a constant value and cannot fluctuate. Hence, we conclude that, in the context of standard cosmology (again, without inflation) random fluctuations of the density field [11] are necessarily scale invariant [12] if the total mass of the system is constrained and it is not allowed to fluctuate [14]. This statement is expressed through the logical implication (24).
The surface dependence of the variance of mass (energy) fluctuations in a system whose global energy does not fluctuate is easily understandable: mass (energy) fluctuations in any two complementary closed sub-volumes $V$ and $\tilde{V} = d\Omega - V$ must be correlated, such that $(\Delta M_V)^2 = (\Delta M_{\tilde{V}})^2$, because $M_\Omega = M_V + M_{\tilde{V}}$ is an extensive magnitude. This correlation implies that the variance of mass fluctuations in sub-volume $V$ must be only a function $(\Delta M_V)^2 = f(S)$ of the area of its common boundary with $\tilde{V}$. This is what we found in (14) and (15): when $n > 0$ global mass (energy) condition (29) is satisfied because $P(0) = 0$. Furthermore, technical condition (27) guarantees the linear dependence (15) once this condition (29) is fulfilled.

V. COSMOLOGICAL DENSITY ANISOTROPIES FROM SHORT WAVELENGTH REARRANGEMENTS IN THE DISTRIBUTION OF MATTER.

At this point of our discussion we feel necessary to revisit and comment an old argument first discussed by Y.B. Zeldovich [9] and, after him, by many other authors in textbooks [7] and research papers [15]. The argument discusses, under assumption [HP], the possibility that causally connected processes at the instant of matter radiation equality could account in the standard cosmology (again, without inflation) for the cosmological mass (energy) anisotropies at that time.

According to mistaken estimation (11), derived from assumption [HP], the mechanism responsible for scale invariant mass (energy) anisotropies over cosmologically large comoving volumes at the instant of equality would need to generate a density power spectrum $P(k) \sim k$ over the range of cosmologically short comoving momenta. That is, the mechanism would need to justify: first, the presence at the instant of equality of fluctuating modes in the density field with comoving wavelength much larger than the causal horizon at that time; second, the linear growth with the momentum scale of the power stored in these modes. The first obstacle seems easily superable: trivial Fourier analysis is enough to notice that fluctuations at the same instant of equality in modes of weakly interacting fields with comoving wavelength shorter than the horizon can couple through the quadratic kinetic terms of the hamiltonian density to produce density fluctuations with cosmologically large comoving wavelength.

On the other hand, the second requirement seems unattainable. The power spectrum
that quadratic coupling creates over the range of cosmologically short comoving momenta, $k \ll H(t_{eq})$, is not linear, but quartic, in the momentum of the Fourier modes: $\mathcal{P}(k) \sim k^4$. The explicit calculation can be found in the textbooks. We reproduce it below in this section. For now, the following fast estimation will suffice:

$$\delta_k \simeq \int_V d^3\vec{x} \frac{\rho(\vec{x}) - \rho_0}{\rho_0} e^{+i\vec{k} \cdot \vec{x}} \simeq \int_V d^3\vec{x} \frac{\rho(\vec{x}) - \rho_0}{\rho_0} + i \int_V d^3\vec{x} (\vec{k} \cdot \vec{x}) \rho(\vec{x}) - \rho_0 + O(k^2).$$

(30)

If the total mass is conserved, then the first, $k$-independent, term of the expansion cancels out, which implies $\delta_k \sim O(k)$ that the power spectrum of the fluctuations $\mathcal{P}(k) = |\delta_k|^2 \simeq O(k^2)$ and, therefore, $\mathcal{P}(0) = 0$. This is in full agreement with our results (24) and (22) of the previous section. In addition it is also noticed in the literature that the linear term in $\vec{k}$, second in the above expansion, must also vanishes if the center of mass of the distribution does not fluctuate. In other words, conservation laws of the mass distribution demand $\delta_k \sim O(k^2)$. In consequence, the power spectrum of mass (energy) fluctuations generated locally grows, at least, as the quartic power $n = 4$ of the momentum, $\mathcal{P}(k) = |\delta_k|^2 \sim k^4$, over the whole range of cosmologically short momenta, $k \ll H(t_{eq})$.

Introducing this quartic power spectrum in estimation (12), derived from assumption [HP] through eq. (11), we obtain the clear prediction $(\Delta M_V)^2 \sim 1/L$, that the variance of mass (energy) cosmological anisotropies generated by local modes of a fundamental field that couple quadratically in the density field seem to decrease at least inversely proportional to the comoving size of the considered volume. On the ground of this analysis it is usually claimed in cosmology the impossibility to causally generate in the standard cosmology scale invariant mass (energy) anisotropies $(\Delta M_V)^2 \sim L^2$ over cosmologically large comoving volumes. This conclusion is known as the origin of structures problem of standard cosmology. We wish the reader to realize how heavily this conclusion relies on estimation (11), (12), which was derived from assumption [HP].

Let us now show the inconsistency of the previous analysis, highlighting the use of assumption [HP] and the assessment (30) for the power spectrum in Zeldovich’s proposal, $\mathcal{P}(k) \sim k^4$ over the range of cosmologically short momenta. We have learned in Section III, case I, that estimation [HP], written down in eq. (12), for mass (energy) anisotropies with a quartic, $n = 4$, power spectrum $\mathcal{P}(k) \sim k^4$, gets absolutely wrong because it does not prop-
erly take into account the contribution from density fluctuations with very short comoving wavelength. In fact, according to the correct estimation [15] the anisotropies generated by the power spectrum with spectral index \( n = 4 \) [30] are indeed scale invariant, as we should have expected since the very beginning of this section from the global constrain on total mass (energy) conservation [24].

Zeldovich’s proposal is only a particular example of a more general mechanism of generating scale invariant mass (energy) anisotropies over cosmologically large comoving volumes through the causally connected local rearrangement of matter at the time of equality in a previously uniform universe. How does this mechanism work? Displacement of masses through the border that bounds the considered sub-volume changes the total masses contained both inside and outside that volume, although the total mass (the sum of the mass inside plus the mass outside) is conserved and, therefore, the cosmological mass anisotropies it generates must be scale invariant [24].

This mechanism works even when the density field is considered to be linear, instead of quadratic, in the fundamental fields, as we explicitly show below in this section. In fact, the capability in Zeldovich’s original proposal to generate density fluctuations with cosmologically large comoving wavelength through the quadratic coupling of fundamental fields is only a secondary feature without relevance, because the contribution of these very long modes to integral (4) for the variance of mass (energy) cosmological anisotropies is negligible compared to the contribution of modes of the density field with short comoving wavelength within the horizon.

In order to make these arguments explicit and absolutely clear we want to show how Zeldovich’s mechanism works in a toy-model and point out where the analysis of the mechanism gets wrong in the literature, under assumption [11]. We consider a massless and free quantum scalar field \( \phi(x_{\nu}) \) in the homogeneous, isotropic and flat FRW space-time background

\[
d s^2 = a^2(x_0) \left( dx_0^2 - \sum_{i=1,2,3} dx_i^2 \right),
\]

where \( x^\nu \) are comoving coordinates and \( a(x_0) \) is the scale factor of the expansion, which we fixed to be equal to one at the instant of equality, \( a(x_{eq}^0) = 1 \). Let us state clearly that the scalar field \( \phi \) we use here is not the inflaton, neither it is any new particle that we are inventing. Action (31) is simply a toy-model of the mechanism proposed and discussed by
Y.B. Zeldovich long ago. We could have chosen to show this mechanism in a system of fermions or gauge bosons.

We are interested in describing the spectrum of energy (mass) anisotropies over cosmologically large comoving volumes due to local displacements of matter within the causal horizon at equality, which we describe as quantum fluctuations in the Fourier modes of the scalar field

\[ \phi(\vec{x}) = \int_{k_{IR}} \frac{d^3k}{(2\pi)^3} \frac{1}{|k|} Q_k e^{+i\vec{k} \cdot \vec{x}} \]  

(32)

with comoving wavelength shorter than the causal horizon, \( k^{-1} \lesssim H^{-1}(t_{eq}) \). Therefore, in our toy-model we introduce an infrared cutoff in momentum \( k_{IR} \simeq H(t_{eq}) \) in the phase space of the scalar quantum field. The way how this infrared cutoff is introduced is not relevant, it can be a sharp cutoff or a mass term in the action (31) of the scalar field. The important point is that we want to center the attention on the role of the under-horizon modes at the time of equality in the generation of mass (energy) anisotropies over cosmologically large comoving volumes. In Section VII this toy-model is embedded into a physically motivated description of the anisotropies, the linearized gauge invariant formalism \[6\]. There we will comment on the role of super-horizon modes of the fluctuating field (32).

The dynamics of Fourier modes of (32) with comoving wavelength within the horizon is not affected by the expansion of the universe and can be described by a free hamiltonian in flat Minkowski space-time,

\[ H = \int_{\Omega} d^3\vec{x} \ h(\vec{x}) = \int_{\Omega} d^3\vec{x} \ \left( \rho_0 + : \pi^*(\vec{x})\pi(\vec{x}) + \partial_i \phi^*(\vec{x})\partial_i \phi(\vec{x}) : \right) \]  

(33)

where \( \pi^*(\vec{x}) \) is the conjugate momentum of the scalar field:

\[ \pi^*(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \sqrt{|\vec{k}|} P_k^* e^{-i\vec{k} \cdot \vec{x}} \]  

(34)

and \( \rho_0 \neq 0 \) is the average energy density. Only for the sake of elegance we have separated the average energy density as a free parameter in (33) and canceled out the expectation value of the second term, enclosed between signs : -- : of normal ordering.

The conjugate operators \( Q_{\vec{k}_1} \) and \( P_{\vec{k}_2}^* \) obey canonical commutation relations \([Q_{\vec{k}_1}, P_{\vec{k}_2}^*] = i (2\pi)^3 \delta^3(\vec{k}_1 - \vec{k}_2)\). Introducing the Fourier expansions (32) and (34) in (33) we can describe
the dynamics of the scalar field as a collection $\mathcal{H} = \mathcal{H}_0 + \int_{k_{IR}}^k \frac{d^3k}{(2\pi)^3} |\vec{k}| [P_k^* P_k + Q_k^* Q_k]$ of free harmonic oscillators. The zero mode is $\mathcal{H}_0 \equiv \int_0^\infty d^3\vec{x} \rho_0$.

If the expansion of the space-time background preceding the time of matter radiation equality is perfectly adiabatic, the scalar quantum field reaches that instant in its fundamental state $|0\rangle$. In other words, we assume here that there is not cosmological production of real $\phi$-particles. The ground state of the scalar field is the tensorial product of the vacua of the decoupled harmonic modes. Fourier modes do fluctuate quantum mechanically in their ground state $\rho_0 |0\rangle = i \int \rho_0 |\vec{k}| P_{\vec{k}} |0\rangle$, with normal distribution of covariance $\langle \rho_0 \rangle = (2\pi)^3 \delta^3(\vec{k}_1 - \vec{k}_2)$ and can generate gaussian fluctuations in the macroscopic restricted observables $\mathcal{H}_V = \int_V d^3\vec{x} \mathcal{H}(\vec{x})$ that describe the energy (mass) contained in macroscopic spatial sub-volumes $V$,

$$\mathcal{H}_V = \rho_0 V + \int_V d^3\vec{x} \int_{k_{IR}}^{k_{UV}} \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \left( \sqrt{|k_1||k_2|} P_{k_1}^* P_{k_2} + \frac{k_1 \cdot k_2}{\sqrt{|k_1||k_2|}} Q_{k_1}^* Q_{k_2} \right) e^{-i(\vec{k}_1 - \vec{k}_2) \cdot \vec{x}}.$$

because the ground state $|0\rangle$ of the quantum field is not an eigenstate of these macroscopic bulk operators.

The typical size of the macroscopic fluctuations is estimated by the variance $(\Delta E_V)^2 \equiv \langle |\mathcal{H}_V|^2 \rangle - \langle |\mathcal{H}_V| \rangle^2$. A formal expansion of this expression was obtained in [16] and rederived in [17],

$$(\Delta E_V)^2 = \frac{1}{4(2\pi)^6} \int_{k_{IR}}^{k_{UV}} d^3\vec{k}_1 d^3\vec{k}_2 |F_V(\vec{k}_1 - \vec{k}_2)|^2 |\vec{k}_1||\vec{k}_2|(\cos(\theta) - 1)^2.$$

(35)

where $k_{UV} \geq H(t_{eq})$ is some cutoff that regularizes the theory in the ultraviolet and $\cos(\theta) \equiv \frac{\vec{k}_1 \cdot \vec{k}_2}{|k_1||k_2|}$ is the cosine of the angle opened between the two vectors $\vec{k}_1$ and $\vec{k}_2$. The need for a regularization ultraviolet cutoff to render the variance (35) finite was already encountered in our discussion in Section III, case I. In next section, we will discuss this regularization procedure in the context of a renormalization program of the physical parameter that measures the size of the mass anisotropies.

Comparing (35) with the general expression (4)

$$(\Delta E_V)^2 = \rho_0^2 \int_0^{2k_{UV}} \frac{d^3\vec{\zeta}}{(2\pi)^3} \mathcal{P}(|\vec{\zeta}|) |F_V(\vec{\zeta})|^2,$$

(36)
where the geometric factor $F_V(\vec{\zeta})$ is given in expression (3), we can easily obtain the power spectrum of vacuum fluctuations:

$$P(|\vec{\zeta}|) = \frac{1}{\rho_0} \frac{1}{32(2\pi)^3} \int_{d(\vec{\zeta})} d^3\bar{\mu} |\vec{k}_1||\vec{k}_2| (\cos(\theta) - 1)^2,$$

in terms of the new variables $\vec{k}_1 = \frac{1}{2} (\bar{\mu} + \vec{\zeta})$ and $\vec{k}_2 = \frac{1}{2} (\bar{\mu} - \vec{\zeta})$. The domain of integration in momentum space is defined by the condition $d(\vec{\zeta}) \equiv \{ \bar{\mu} \in \mathbb{R}^3 : k_{IR} \leq \frac{1}{2}|\bar{\mu} \pm \vec{\zeta}| \leq k_{UV} \}$. This domain is the intersection region of two similar annulus with inner radius $2k_{IR}$ and outer radius $2k_{UV}$ and centered, respectively, in $\pm \vec{\zeta}$.

It is easy to check that at $\zeta = 0$, that is $\vec{k}_1 = \vec{k}_2$, the power spectrum vanishes

$$P(\zeta = 0) = \lim_{\zeta \to 0} P(\zeta) = 0.$$ (38)

This result is a direct consequence of the fact that the quantum field is in its ground state, which is an eigenstate of the hamiltonian (33) and, therefore, the total energy $E(\Omega)$ in the whole system $\Omega$ does not fluctuate quantum mechanically, $(\Delta E(\Omega))^2 = 0$. This condition simply constrains the fluctuations of the density field to preserve the total energy (mass) in the system and, according to (24), it is sufficient to force the energy (mass) anisotropies (35) to be scale invariant (9) over cosmologically large covariant sub-volumes [14], $(\Delta E_V)^2 \propto S$.

In fact, the scale invariance of anisotropies (35) over cosmologically large comoving volumes was proved numerically in [16]. Additional numerical examples that beatufilly prove the claim were presented in [17]. In a first example, the variance of energy fluctuations of a free scalar field in $2 + 1$ Minkowski space-time is calculated in the spatial region within a wrinkled surface. The interesting aspect of this example comes from the fact that the total volume contained within the surface does not change when the boundary surface wrinkles, although the area of the boundary grows monotonically. The results presented there show that the variance of energy fluctuations does grow linearly with the growing boundary surface. In a second example, the variance of energy fluctuations is calculated within an annulus of inner radius $r_1$ and outer radius $r_2$. When the radius $r_1$ gets larger, but still smaller than $r_2$, the total volume of the annulus decreases. But the area of the boundary surface grows, and so also does the variance of energy fluctuations.

When this mechanism of local rearrangement is discussed in the literature the attention
is focused on the quadratic coupling in the hamiltonian density \((33)\) of Fourier modes of the fields \(\phi(\vec{x})\) or \(\pi^*(\vec{x})\) with close covariant momenta \(\vec{k}_1 \simeq \vec{k}_2\), which produces fluctuations of the density field \(\mathcal{H}(\vec{x})\) with covariant wavelength \(\lambda = 2\pi/|\vec{\zeta}| = 2\pi/|\vec{k}_1 - \vec{k}_2|\) that can be much longer than the horizon, even if both covariant wavelengths \(2\pi/|\vec{k}_1|\) and \(2\pi/|\vec{k}_2|\) are much shorter than the horizon. This is a very simple property of the coupling of Fourier modes that led to Y.B. Zeldovich, and many others after him, to notice many years ago [9] that local fluctuations of the scalar field \((32)\) with comoving wavelength shorter than the horizon can produce a non zero power spectrum \((37)\)

\[
\mathcal{P}(|\vec{\zeta}|) \sim \frac{1}{\rho_0^2} \frac{1}{2(2\pi)^2} k_{UV} |\vec{\zeta}|^4
\]  

(39)

over cosmologically short comoving momenta \(\zeta \sim H(t_{eq})\).

The interest in the literature on this aspect of the mechanism is motivated, nevertheless, by the erroneous assumption \((12)\), which wrongly asserts that mass (energy) anisotropies \((36)\) over cosmologically large comoving volumes of size \(L \gg H^{-1}(t_{eq})\) are generated by the contribution \(d\zeta \zeta^2 \mathcal{P}(\zeta)|F_V(\zeta)|^2\) at \(\zeta \sim L^{-1} \ll H(t_{eq})\). This erroneous assumption \((12)\) has misled the analysis of the mechanism of local rearrangement of matter as the origin of cosmological anisotropies since the early years when it was first studied. As we explained in Section III, case I), the contribution at \(\zeta \sim L^{-1}\) to \((36)\) is negligible compared to the ultraviolet contribution, \(d\zeta \zeta^2 \mathcal{P}(\zeta)|F_V(\zeta)|^2\) at \(\zeta \sim k_{UV}\), in statistical systems globally constrained by the condition \((\Delta M_\Omega)^2 = 0\). This can be checked directly on \((35)\) by noticing that the largest contribution comes from \(|\vec{k}_{1,2}| \sim k_{UV}\) when they are oppositely oriented, \(\vec{k}_1 \sim -\vec{k}_2\), so that \(\cos \theta = -1\), which contribute to the power spectrum \(\mathcal{P}(\zeta \gg 1/L)\) in the ultraviolet range, and not by the aligned modes \(\vec{k}_1 \sim \vec{k}_2\), which form an angle \(\cos \theta \sim +1\) and contribute to \(\mathcal{P}(\zeta \sim 1/L \ll H(t_{eq}))\). When the dominant ultraviolet contribution is taken into account \((15)\) the resulting anisotropies \([17]\) over cosmologically large volumes are proven numerically to be scale invariant.

The irrelevance of the quadratic dependence of the density field \((33)\) in the fundamental fields \((32)\) and \((34)\) for generating scale invariant anisotropies over cosmologically large volumes out of fluctuations in Fourier modes of the latter with very short comoving wavelength is manifest in the next example.

Let us consider the same theoretical framework laid in \((31), (33)\), but assume now that the matter density field is proportional to \((34)\) the conjugate momentum field. Obviously,
the density field \( \langle 0|\pi(\vec{x})|0 \rangle = 0 \). \( (40) \)

But, as we know, this constrain cannot prevent the appearance of random quantum anisotropies in the spatial energy (mass) density distribution, because the ground state \(|0\rangle\) is not an eigenstate of the restricted operators

\[
M_V \propto \int_V d^3\vec{x} \pi(\vec{x}) + h.c.,
\]

if the volume of integration \( V \) is not the whole 3D space \( \Omega \).

We find the variance of energy (mass) anisotropies in finite comoving volumes of cosmological size to be

\[
(\Delta M_V)^2 \propto \int_V \int_V d^3\vec{x} d^3\vec{y} \langle 0|\pi^*(\vec{x})\pi(\vec{y})|0 \rangle = \int_{k_{IR}}^{k_{UV}} \frac{d^3\vec{k}}{(2\pi)^3} k |F_V(\vec{k})|^2,
\]

given that \( \langle 0|\pi^*(\vec{x})\pi(\vec{y})|0 \rangle = \int \frac{d^3\vec{k}}{(2\pi)^3} k e^{i\vec{k} \cdot (\vec{x} - \vec{y})} \). Now, the density field depends linearly in the fundamental fields, which are allowed only to randomly fluctuate in modes with comoving wavelength shorter than the horizon. We read from the expression above the power spectrum of the fluctuations in this example, \( P(k) \sim k \) for \( k_{IR} \leq k \leq k_{UV} \) and is zero otherwise.

The ultraviolet cutoff \( k_{UV} \) is necessary to render expression \( (42) \) finite. Once we introduce the cutoff we immediately confirm, following previous discussion in Section III, case II):

\[
(\Delta M_V)^2 \sim L^2 \quad (43)
\]

quantum anisotropies of the density field \( (41) \) are scale invariant. This result was foreseeable on the ground \( (\Delta M_\Omega)^2 = 0 \), simply expressing that also in this second example total energy (mass) is conserved and not allowed to fluctuate quantum mechanically. The need for an ultraviolet cutoff in \( (42) \) proves that these anisotropies are also mainly generated by the contribution from the modes with the shortest comoving wavelength.

The two examples discussed in this section basically consist of random fluctuations that can displace the carriers of mass (energy), \( i.e. \) fluctuations of the fundamental fields \( (32) \) and \( (34) \), only over causally connected distances. They both generate scale invariant mass (energy) anisotropies over cosmologically large comoving volumes. On the other hand, the two examples are defined by completely different density power spectra. The power spectrum
associated to (35) density fluctuations of the quadratic operator $\mathcal{H}_V$ is $\mathcal{P}(k) \sim k^4$ over the whole range of cosmologically short momenta $k \lesssim k_{UV}$, with $k_{UV} \gtrsim H(t_{eq})$, while the power spectrum associated to (12) fluctuations of the momentum density operator $M_V$ in the second example is $\mathcal{P}(k) \sim k$ in the range of momenta within the horizon $H(t_{eq}) \lesssim k \lesssim k_{UV}$. We already found this degeneracy in the power spectrum of scale invariant anisotropies in our previous analysis in Section III, case I).

We close this section with a short comment on the cutoff procedures we have introduced above in this section. First, we realize that for the issues we have been discussing here the infrared cutoff $k_{IR}$ is irrelevant in both examples, as the contribution of the long wavelength modes to the total variance (14) is sub-dominant. The dominant contribution comes in both examples from the short wavelength modes of the fundamental fields. In the first example, (35), the dominant contribution is obtained from the quadratic coupling of oppositely oriented ultraviolet modes $|k_{1,2}| \sim k_{UV}$, $\cos \theta \simeq -1$. In the second example, the largest contribution to (12) comes also from the ultraviolet modes of the momentum field. Both contributions are divergent and needs to be regularized with ultraviolet cutoffs. Now it is important to notice from (19) that the ultraviolet regularization procedure only affects the absolute size of the fluctuations, $(\Delta M_V)^2 \sim \rho_0^2 \mu S$, but not the linear dependence on the area $S$ of the surface that bounds the sub-volume. It is natural then to understand this procedure as an intermediate regularization step in the renormalization programme of the physical parameter $\mu = \frac{A' k_{UV}^n}{k_0^{n+3}}$. This comment introduces us to the issues addressed in next Section.

VI. SCALE INVARIANT DENSITY ANISOTROPIES IN QUANTUM FIELD THEORIES.

The scalar quantum field theory (31) of last section is obviouslly only a toy-model designed to confront some widespread, but wrong, beliefs about the dynamics of primordial structure formation. The model has also been a satisfactory bench to discuss some important concepts that shall be very relevant in the development of a complete quantum field theory of primordial cosmological density perturbations.

Within the setup of a QFT the energy (mass) density field at the time of equality (1) is described by a density operator
\[ h(\vec{x}; t_{eq}) = h_0 + \langle h_0 \rangle \int \frac{d^3 \vec{k}}{(2\pi)^3} \delta_\vec{k} e^{-i\vec{k} \cdot \vec{x}}, \quad (44) \]

analogous to the density operator in (38), acting on the quantum state of the system \(|0\rangle\). This state is required to be invariant under spatial translations and rotations. Homogeneity and isotropy demand \(\langle 0| h(\vec{x}) |0 \rangle = \langle 0| h_0 |0 \rangle\), which implies

\[ \langle 0| \delta_\vec{k} |0 \rangle = 0, \quad (45) \]

and, therefore, the average value of \(M(V) \equiv \int_V d^3 \vec{x} \ h(\vec{x})\), the energy (mass) contained within a spatial macroscopic sub-volume \(V\), is proportional to the volume of the region, \(\langle 0| M(V) |0 \rangle = V \times \langle 0| h_0 |0 \rangle\). The magnitude \(M(V)\) is said to be an extensive properties of the system. We assume here, as we did in Section III, that \(\langle h_0 \rangle \equiv \langle 0| h_0 |0 \rangle \neq 0\).

Condition (45) does not prevent, nevertheless, the appearance of random spatial energy (mass) density anisotropies due to quantum fluctuations, with variance

\[ (\Delta E_V)^2 \equiv \langle 0| [M(V) - \langle M(V) \rangle]^2 |0 \rangle = \langle h_0 \rangle^2 \int \frac{d^3 \vec{k}}{(2\pi)^3} \mathcal{P}(k) |F_V(\vec{k})|^2, \quad (46) \]

where the power spectrum \(\mathcal{P}(\vec{k})\) is defined, as usual, by

\[ \langle 0| \delta_{\vec{k}_1}^* \delta_{\vec{k}_2} |0 \rangle = (2\pi)^3 \mathcal{P}(\vec{k}_1) \delta^3(\vec{k}_1 - \vec{k}_2). \quad (47) \]

In a wider context, (44) can be the density field operator of any global Noether charge

\[ \mathcal{H} = \int_\Omega h(\vec{x}) = \int_\Omega h_0. \quad (48) \]

The physical observable \(M(V)\) describes then the net value of that charge within a finite macroscopic comoving sub-volume of cosmological size. Condition (45) implies that charge \(\mathcal{H}\) is, on average, homogeneously and isotropically distributed in space, while (46) reminds us that quantum fluctuations can, nevertheless, produce random anisotropies in its spatial distribution.

According to the central limit theorem, for a complete statistical description of random anisotropies in the spatial distribution of the extensive charge is enough to provide the value of their variance (46) over comoving volumes of arbitrary cosmological size. The calculation of the variance of random spatial anisotropies in the distribution of a Noether charge is a
well-posed physical question that should get an unambiguous, finite answer in any properly formulated theoretical framework. Of course, the theoretical calculation must produce the clear result \((\Delta E_V)^2 = 0\) over any macroscopically large comoving volume if and only if the system does not generate random anisotropies.

Hence, it is intriguing to notice that within the setup that we are considering we are allowed to modify the definition of the charge density field operator (44), for example by adding a total three-derivative, without changing the definition of the global charge (48). Such 'symmetry' transformations of the charge density field (44) must preserve the definition of the zero-mode \(h_0\) and they must also respect the expectation values (45) of the non-zero Fourier modes \(\delta \vec{k}, \vec{k} \neq 0\) in order to preserve the average homogeneity and isotropy of the system, but they can alter the power spectrum \(\mathcal{P}(k)\) at \(k \neq 0\), defined in (47). Therefore, they can alter the variance of quantum mechanically generated spatial charge anisotropies in finite macroscopic sub-volumes, defined in (46). This degeneracy implies, in principle, that these two concepts (47) and (46) cannot be uniquely defined in the considered theoretical framework and raises questions on the capability of the setup to properly describe quantum mechanically generated spatial anisotropies. In the paragraphs below we investigate and clarify this point.

We need first to identify the invariant features of the power spectrum (47) and variance (46) under the 'symmetry' transformations of the density field (44): as the zero mode \(h_0\) is not modified by the permitted transformations of the density field operator, its statistical momenta \(\langle 0 | (h_0 - \langle h_0 \rangle)^p | 0 \rangle, p = 1, 2, 3, \ldots\), are invariant features. The first moment \(p = 1\) is zero, by definition. The second and higher momenta, on the other hand, are zero only if the quantum state \(|0\rangle\) is an eigenstate of the global charge (48). In particular, this implies that energy (mass) is globally conserved,

\[
(\Delta E_{\Omega})^2 = 0,
\]

by quantum random fluctuations of the density field (44).

This condition (49) is naturally satisfied if \(|0\rangle\) is the ground state of the system and (44) is a conserved Noether charge. According to our result (24) in section IV such constrain implies that quantum mechanically generated anisotropies (46) in the spatial distribution of charge \(\mathcal{H}\) must be scale invariant (3) over comoving volumes of cosmological size. Thus, the linear dependence of the variance of primordial cosmological energy (mass) anisotropies on
the area of the surface that bounds the considered comoving volume,

\[ Ln (\Delta E_{V_1})^2 = Ln (\Delta E_{V_2})^2 + Ln \left( \frac{S_1}{S_2} \right), \]

(50)

where \( V_1 \) and \( V_2 \) are arbitrary macroscopic sub-volumes and \( S_1, S_2 \) are, respectively, the areas of the surfaces that bounds each of them, is an invariant feature under 'symmetry' transformations of the density field operator in the ground state of the system [14]. This is nothing but an example of the basic feature that we already noticed in our analysis of Section III, case I), that power spectra with different spectral index \( n > 1 \) cannot be distinguished through the energy (mass) density anisotropies they produce over macroscopically large comoving volumes: 'symmetry' transformations of the density field (44) simply modify the power spectrum (47) of spatial anisotropies \( P(k) \rightarrow P(k) + \Delta P(k) \) by a regular term \( \Delta P(k) = o(k) \), which does not alter the macroscopic scale invariance (9) of the anisotropies.

The transformations of the density field (44) can modify, on the other hand the size of the spatial charge anisotropies over macroscopically large spatial sub-volumes, parameterized by the factor of proportionality \( \mu \),

\[ (\Delta E_V)^2 \simeq \langle h_0 \rangle^2 \mu S. \]

(51)

This parameter \( \mu \) is not an invariant feature under 'symmetry' transformations in the definition of the charge density and we should conclude that it describes a physical property that is not properly defined in this setup. We explain below the origin of this problem and argue, using usual statistical/QFT renormalization concepts, that this parameter \( \mu \) must be considered as an additional free parameter of the theory, like masses or coupling constants, which fixes the size of quantum mechanical random spatial charge anisotropies in the ground state of the system.

In Section III, case I), we learned that scale invariant energy (mass) anisotropies over macroscopically large volumes are dominated by the contribution of the ultraviolet modes to the integral expression (17). To render this expression finite we introduced by hand an ultraviolet cutoff, which fixes the size of the mass (energy) anisotropies in cosmologically large comoving volumes. We can now understand this step in Section III as a regularization procedure of the infinite parameter \( \mu \). The 'symmetry' transformations of the density field (44) can be reabsorbed within this infinite parameter. Or, in a different point of view,
the infinite parameter $\mu$ can be regularized by adding an appropriate counter-term in the density field. This procedure do not modify the finite relationship (50), which is a physical prediction of the setup. In conclusion, scale invariance of vacuum anisotropies, expressed through the finite relationship (50), must be respected through the renormalization program. The price we pay is the normal price in QFT/statistical mechanics: we cannot predict from first principles within this setup the value of the ’dressed’ parameter $\mu$, in the same way that we cannot predict the physical value of $\alpha_{em}$ in QED.

Only after fixing, normally through measurement, the variance $$(\Delta E_{V_1})^2 = \Delta^2,$$ of random anisotropies over a macroscopic volume of reference $V_1$, we have a meaningful physical prediction

$$(\Delta E_{V_2})^2 = \frac{S_2}{S_1} \Delta^2,$$ (52)

for the variance of mass (energy) anisotropies over any other comoving volume $V_2$ of cosmological size. This relationship is equivalent to the renormalization group equation that predicts physical values for the running of the coupling constant $\alpha_{em}(p^2)$ at arbitrary energy scales only after an initial value $\alpha_{em}(p_0^2)$ at a certain renormalization energy scale has been fixed by experiment.

Please notice that the incapability of the QFT setup to predict the ’dressed’ value of the physical parameter $\mu$ due to common ultraviolet divergences and the impossibility to single out in the setup the local definition of the energy (mass) density operator, reaches also the widely accepted predictions of the inflationary scenario. We can freely add a three-divergence to the hamiltonian density, which will dress the scale invariant anisotropies predicted by inflation. As we have said, the additional terms respect the scale invariance of the anisotropies (50), but make the parameter $\mu$ in (51) formally infinite. Let us say it in different words: under mistaken assumption (11), upon which the calculations of the cosmological anisotropies in inflationary cosmology are carried out, there are no divergences in the theoretical calculation of the anisotropies, but just because the infinities appear only after noticing the degeneracy (15) in the power spectrum of scale invariant mass (energy) cosmological anisotropies. Ordinary predictions of inflationary cosmology of primordial structure does not suffer from the problem of infinite expressions, only because in these calculations the infinities are thrown away, using approximation (11), without justification. A correct, not based on mistaken assumption (11), complete calculation of inflationary predictions will
also lead to finite relationship (50), but only after a regularization procedure of the infinite expression for the parameter $\mu$.

It is very interesting to notice that the infinite contribution from ultraviolet modes of the density field to the physical parameter $\mu$ in (51) is, according to (24), directly associated to the fact that total mass (energy) is globally conserved $(\Delta E_\Omega)^2 = 0$. Scale invariant anisotropies of extensive magnitudes defined in the bulk of a spatial volume $V$ are associated to the surface that bounds the sub-volume. If these anisotropies can be *holographically* described by a theory defined on the surface, as suggested in [18], [17], we interestingly find a connection between the infinite, renormalizable parameter $\mu$ of the *holographic* theory defined on the closed surface, and the globally conserved extensive magnitude defined in the interior and exterior volumes to that surface. In quantum Hall effect of strongly correlated electrons confined in 2D spatial dimensions by a perpendicular magnetic field in condensed matter physics it is known that is possible to obtain an *holographic* 1D description of the edge excitations over the boundary border [8].

The problem of determining the size $\mu$ of scale invariant energy (mass) anisotropies (51) is not very different from the problem of determining the cosmological constant in the context of relativistic QFTs. The average value $\langle M(V) \rangle = V \times \langle h_0 \rangle$ is, in general, ultraviolet divergent. We cannot predict within the QFT setup the value of $\Lambda \equiv \langle h_0 \rangle$, but once the theory is regularized we find that $\langle E_{V_1} \rangle = \frac{V_1}{V_2} \langle E_{V_2} \rangle$, the energy (mass) contained within a macroscopic sub-volume is an extensive magnitude. This is a finite relationship that must be respected whatever the unknown mechanism that renormalizes the value of $\Lambda$ could be. In the same way we cannot predict the physical value of the parameter $\mu$, but we can predict the scale invariance (50) of the variance of spatial charge anisotropies. Both of these problems become physically meaningful only when the energy (mass) density field gets coupled to gravity and, therefore, they can only get a final answer in the more fundamental setup that includes the quantum description of both matter components and dynamics of the space-time metric.
VII. SCALE INVARIANT ANISOTROPIES IN THE GAUGE INVARIANT FORMALISM OF LINEARIZED DENSITY PERTURBATIONS.

Spatial anisotropies in the energy-momentum density field necessarily perturbate the homogeneous and isotropic FRW expanding space-time background, which in turn can feedback the dynamics of the density anisotropies. A complete analysis of the locally Lorentz invariant equations of general relativity that describe the coupled dynamics of density and metric perturbations is technically very demanding even at the classical level because it must carefully take into account unphysical gauge degrees of freedom. Fortunately, a first approximation of the coupled dynamics at linear order in the perturbations has been formulated in a very simple gauge invariant framework. A compiling detailed report of this approach can be found in [6]. The linear approach is justified in the literature on the argument that cosmological primordial mass (energy) perturbations are tiny, $\delta \rho/\rho \lesssim 10^{-5}$. In the previous section we have found, nevertheless, that in the standard statistical setup the physical parameter $\mu$, which measures the size of the fluctuations, $\delta \rho \propto \sqrt{\mu \mathcal{S}}$, is infinite and needs to be renormalized. This brings us to face another usual procedure in perturbative QFT: the perturbative expansion must be carried out in powers of the dressed physical value of the parameter $\delta \rho/\rho$, which is really tiny, and not in terms of the infinite, or very large, undressed parameter. We will not further discuss here how to carry a renormalization programme. In this section we only explore within the gauge invariant framework at linear order the concepts introduced in the previous sections. This section VII, like Section VI, report only first results of research in progress.

First, let us very briefly review the basics of the linearized gauge invariant formalism, which start from the usual Einstein-Hilbert action for the coupled dynamics of the space-time metric and energy-momentum matter tensors. Both tensor field are then decomposed as a sum of two terms: a first homogeneous and isotropic FRW background term plus a second perturbative term describing the anisotropies. Using the equations of motion the action is rewritten in linear approximation keeping up to second order in the perturbative terms. After some straightforward, but lengthy, calculations the general action of coupled density and metric perturbations during the radiation dominated epoch of a FRW background is written in comoving coordinates in terms of a single gauge invariant scalar field potential $v(\vec{x})$, whose dynamics is described by the hackneyed hamiltonian of a free scalar field in flat
Minkowski space-time

\[ \mathcal{H} = \int_{\Omega} \mathcal{H}(\vec{x}) = \int_{\Omega} d^3\vec{x} \left( \pi(\vec{x}) \pi(\vec{x}) + \frac{1}{3} \vec{\nabla} v(\vec{x}) \vec{\nabla} v(\vec{x}) \right). \]  

(53)

Field operator \( v(\vec{x}) \) and its conjugate momentum \( \pi(\vec{y}) \) obey canonical commutation relations \([v(\vec{x}), \pi(\vec{y})] = i \delta^3(\vec{x} - \vec{y})\).

During the radiation dominated stage of the standard cosmology there is not cosmological particle production and, therefore, the gauge invariant scalar field remains in its ground state \(|0\rangle\) when the temperature of the universe cools down to the onset of matter radiation equality.

Furthermore, every physical observable can be expressed in this formalism in terms of this pair of conjugate fields, thus eliminating the problem of spurious gauge degrees of freedom. In particular, the Fourier modes of the energy (mass) density field operator \((44)\) with comoving wavelength shorter that the horizon are proportional in this formalism to the Fourier modes of the momentum density operator:

\[ \frac{1}{\langle h_0 \rangle} (h(\vec{x}) - h_0) \simeq \frac{1}{T_{eq}^2} (\pi(\vec{x}) + \text{h.c.}) + \partial_i J^i(\vec{x}), \]  

(54)

where \( T_{eq} \) is the cosmic temperature at the time of equality and \( \partial_i J^i(\vec{x}) \) is an undefined arbitrary three divergence according to our discussion in the previous section. Formally, this density field is the second of the two examples we discussed in the toy-model of Section V. We found there that in the ground state of the system \(|0\rangle\) its spatial anisotropies are scale invariant \((42)\).

We consider important to remark that the density field operator in the linearized gauge invariant formalism is proportional to the momentum density \((54)\) only over causally connected distances. Over super-horizon distances they do not necessarily coincide and the right hand side of \((54)\) gets correction terms. This is not really important to us, as we know that the largest contribution to scale invariant cosmological anisotropies, globally constrained to preserve the total energy (mass) of the system, comes from fluctuations in under-horizon modes of the density field \((19)\). These modes are described by \((54)\).

The unsolved situation regarding the theoretical prediction of the parameter \( \mu \) in \((51)\) has not improved in this formalism of linearized density anisotropies coupled to metric perturbations. A direct estimation of the variance of energy (mass) anisotropies from \((54)\) produces
\[(\Delta E_V)^2 \simeq \langle h_0 \rangle^2 \frac{1}{T_{eq}^4} S. \quad (55)\]

Over a sphere of comoving radius \(L\) we then obtain

\[\left(\frac{\delta \rho}{\rho}\right)_V^2 = \frac{(\Delta E_V)^2}{\langle h_0 \rangle^2 V^2} \simeq \frac{9}{4\pi} \left(\frac{M_P}{T_{eq}}\right)^4. \quad (56)\]

A rapid estimation for \(L \sim 10^3 H_{eq}^{-1} \sim 10^3 \frac{T_{eq}}{M_P}\) gives for the \textit{undressed} dimensionless parameter \(\left(\frac{\delta \rho}{\rho}\right)_V^\ast \sim \frac{9}{4\pi} 10^{-6} \left(\frac{M_P}{T_{eq}}\right)^4 \sim 10^{100}\). Obviously, this is much larger than the observed physical \textit{dressed} value \(\left(\frac{\delta \rho}{\rho}\right)_V \lesssim 10^{-5}\). The renormalization can be carried out through counter-terms introduced by the three-divergence in the definition of the density operator \(\langle 54 \rangle\).

For a deeper treatment of this problem it is necessary to go beyond the linear order of the density and metric perturbations. We guess that the final quantum theory of gravity has a say in fixing the size of scale invariant primordial anisotropies. Dynamics of structure formation could thus offer a bench for phenomenologically testing that final theory at the low energies of matter-radiation equality.

**VIII. RENORMALIZATION EQUATIONS OF PHYSICAL DENSITY ANISOTROPIES.**

We wish to explore in this section how the renormalized \textit{(dressed)} mass \textit{(energy)} anisotropies depend on the resolution scale at which the physical density field is probed and obtain, within the theoretical setup laid in section VI, the renormalization group equations that describe this dependence.

Let us say that \(\rho_{\text{phy}}(\bar{x}; l_0)\) is the physical density field probed with arbitrary length resolution \(l_0 = M_0^{-1}\). When the system is probed at blunter length resolution \(l_1 = M_1^{-1} \gtrsim l_0\) (or, in energy scale \(M_1 \lesssim M_0\)), the observable field

\[\rho_{\text{phy}}(\bar{x}; l_1) = \int d^3 \bar{y} \rho_{\text{phy}}(\bar{y}; l_0) W(\bar{y} - \bar{x}; l_1, l_0) \quad (57)\]

looks further smoothed because its Fourier modes with wavelength \(\lambda \lesssim l_1\) shorter than the size of the new probe are cut off by an appropriate convolution kernel \textit{(window function)} \(W(\bar{y} - \bar{x}; l_1, l_0)\) extended around the point of observation \(\bar{x}\) over a certain domain of typical size \(l_1\) and normalized to
\[
\int_\Omega d^3\vec{y} \, W(\vec{y} - \vec{x}; l_1, l_0) = 1. \tag{58}
\]

Expression (57) simply means that measuring a local observable at any point \( \vec{x} \in \Omega \) with a probe of length \( l_1 \) does in fact return an average weighted value of the field over a vicinity of the size of the probe around the tested point. This is typically the shortest scale that that probe can test.

Obviously, the physical density field (57) resolved to lengths of the order \( l_1 \) cannot depend on the arbitrary shorter scale \( l_0 \) used in the definition above. This scale \( l_0 \) plays here a role similar to the arbitrary renormalization scale in renormalization group equations. The condition of independence of the physical observable field (57) resolved at length \( l_1 \) on the renormalization scale \( l_0 \) fixes the functional dependence of the convolution kernel

\[
\int_\Omega d^3\vec{z} \, W(\vec{y} - \vec{z}; l, l_0) \, W(\vec{z} - \vec{x}; l_1, l) = W(\vec{y} - \vec{x}; l_1, l_0). \tag{59}
\]

The simplest solution to this convolution equation is a Dirac delta function

\[
W(\vec{x} - \vec{y}; l_a, l_b) = \delta^3(\vec{x} - \vec{y})
\]

for whatever two resolution length scales \( l_a, l_b \) we choose to compare, but it must be understood that this choice is rather unphysical as it assumes that we can solve the density field with infinitely sharp length resolution. A more realistic solution to (59) is a normalized gaussian kernel:

\[
W(\vec{x} - \vec{y}; l_a, l_b) = \mathcal{N} \exp\left[ -\frac{|\vec{x} - \vec{y}|^2}{l_a^2 - l_b^2} \right] = \mathcal{N} \exp\left[ -\frac{M_b^2 M_a^2}{M_b^2 - M_a^2} |\vec{x} - \vec{y}|^2 \right], \tag{60}
\]

where

\[
\mathcal{N}^{-1} = \int_\Omega d^3\vec{y} \, \exp\left[ -\frac{M_b^2 M_a^2}{M_b^2 - M_a^2} |\vec{x} - \vec{y}|^2 \right]. \tag{61}
\]

Naturally, when \( M_a \ll M_b \) the convolution kernel is

\[
W(\vec{x} - \vec{y}; l_a, l_b) \simeq \mathcal{N} \exp[-M_a^2 |\vec{x} - \vec{y}|^2].
\]

On the other hand, when we compare the observed physical density fields at two very close resolutions scales \( l_b \simeq l_a \), the convolution kernel is roughly \( W(\vec{x} - \vec{y}; l_a, l_b) \simeq \delta^3(\vec{x} - \vec{y}) \), as it should be.

It is also useful to compare in momentum space the physical density fields resolved at different length scales. Expanding \( \rho_{\text{phy}}(\vec{x}; l_1) \) and \( \rho_{\text{phy}}(\vec{x}; l_0) \) in Fourier modes, as in eq. (11),

\[
\rho_{\text{phy}}(\vec{x}; l_1) = \rho_0 + \rho_0 \int \frac{d^3k}{(2\pi)^3} \, \delta_{\vec{k}}[M_1] \, e^{-i\vec{k} \cdot \vec{x}},
\]

where

\[
\rho_{\text{phy}}(\vec{x}; l_0) = \rho_0 + \rho_0 \int \frac{d^3k}{(2\pi)^3} \, \delta_{\vec{k}}[M_0] \, e^{-i\vec{k} \cdot \vec{x}}.
\]
Thus, we obtain by comparison with (62) the relationship

$$\rho_{\text{phy}}(\vec{x}; l_0) = \rho_0 + \rho_0 \int \frac{d^3\vec{k}}{(2\pi)^3} \delta_k[M_0] e^{-i\vec{k} \cdot \vec{x}}$$

(63)

and then introducing the second expansion (63) in eq. (57)

$$\rho_{\text{phy}}(\vec{x}; l_1) = \int_\Omega d^3\vec{y} \left( \rho_0 + \rho_0 \int \frac{d^3\vec{k}}{(2\pi)^3} \delta_k[M_0] e^{-i\vec{k} \cdot \vec{y}} \right) W(\vec{y} - \vec{x}; l_1, l_0) = \rho_0 + \rho_0 \int \frac{d^3\vec{k}}{(2\pi)^3} \delta_k[M_0] \left( \int_\Omega d^3\vec{y} e^{-i\vec{k} \cdot (\vec{y} - \vec{x})} W(\vec{y} - \vec{x}; l_1, l_0) \right) e^{-i\vec{k} \cdot \vec{x}}.$$ 

Thus, we obtain by comparison with (62) the relationship

$$\delta_k[M_1] = \widetilde{W}_k[M_1, M_0] \delta_k[M_0] = e^{-\frac{M_0^2 - M_1^2}{4} \frac{M_0^2 M_1^2}{4} k^2} \delta_k[M_0],$$

(64)

where

$$\widetilde{W}_k[M_1, M_0] = \int_\Omega d^3\vec{y} e^{-i\vec{k} \cdot (\vec{y} - \vec{x})} W(\vec{y} - \vec{x}; l_1, l_0) = e^{-\frac{M_0^2 - M_1^2}{4} \frac{M_0^2 M_1^2}{4} k^2},$$

(65)

is the Fourier transform of the convolution kernel. We find (64) that, as a result of changing the scale at which we resolve the density field, the stochastic modes $\delta_k$ get renormalized by the window function (63). Therefore, the power spectrum associated to these stochastic modes, defined by $(2\pi)^3 P(\vec{k}) \delta^3(\vec{k} - \vec{k}_2) = \langle \delta_{k_1} \delta_{k_2} \rangle$, gets renormalized as

$$P_{\text{phy}}(\vec{k}_1; M_1) = |\widetilde{W}_{\vec{k}_1}[M_1, M_0]|^2 P_{\text{phy}}(\vec{k}_1; M_0) = e^{-\frac{M_0^2 - M_1^2}{2} \frac{M_0^2 M_1^2}{2} \vec{k}^2} P_{\text{phy}}(\vec{k}_1; M_0).$$

(66)

If we keep the arbitrary scale $l_0$ fixed for comparison and let the resolution scale $l_1 = M_1^{-1} \gg l_0$ to run, $P_{\text{phy}}(\vec{k}_1; M_1) \simeq e^{-\frac{M_0^2 - M_1^2}{2} \frac{M_0^2 M_1^2}{2} \vec{k}^2} P_{\text{phy}}(\vec{k}_1; M_0)$, we clearly see that the resolution scale simply introduces a running physical ultraviolet cutoff $M_1$, beyond which the power spectrum of the density fluctuations is exponentially suppressed.

With these tools at hand we can obtain the variance [4] of physical density anisotropies $(\Delta M_V[M])^2 = \int \frac{d^3\vec{k}}{(2\pi)^3} P_{\text{phy}}(\vec{k}; M)(F_V(\vec{k}))^2$ as a function of the scale $l = M^{-1}$ at which the density field (57) is resolved. The important feature to be noticed is that, for whatever two resolution scales $M_1, M_0$ that we choose to compare, the normalization condition (58) fixes $\widetilde{W}_{\vec{k}=0}[M_1, M_0] = 1$ at the origin $k = 0$ in momentum space, see eq. (65). Hence, the power spectrum at the origin is invariant $P_{\text{phy}}(k = 0; M_1) = \widetilde{W}_{\vec{k}=0}[M_1, M_0] P_{\text{phy}}(k = 0; M_0) = P_{\text{phy}}(k = 0; M_0)$ under running of the resolution scale $M$. In particular,
\[ \mathcal{P}_{\text{phy}}(k = 0; M_1) = 0 \quad \Leftrightarrow \quad \mathcal{P}_{\text{phy}}(k = 0; M_0) = 0. \] (67)

According to eq. (22), \( \mathcal{P}(k = 0) = 0 \) is a necessary and sufficient condition for the variance of mass (energy) anisotropies in a finite sub-volume \( V \) to be scale invariant. Let us remind that \( d\mathcal{P}(k)/dk|_{k=0} = 0 \) is necessarily fulfilled if \( \mathcal{P}(k = 0) = 0 \), up to some technical considerations. Therefore, we must conclude that scale invariance (33) of mass (energy) anisotropies in finite macroscopic sub-volumes is an invariant feature of the statistical density anisotropies under transformations of the scale \( l \) at which we resolve the density field (1), as long as this scale is typically shorter than the size \( L \) of the considered sub-volume \( l \lesssim L \).

In eq. (24) we related the scale invariance of mass (energy) anisotropies in macroscopic sub-volumes to statistical fluctuations of the density field (57) constrained to conserve the total mass (energy), \( (\Delta M_\Omega)^2 = 0 \). Equation (67) above simply expresses that this global constrain should remain unchanged when we change the resolution scale at which we test the density field.

It is interesting at this point to bring back our analysis of Section III, case I. There we found a general approximate analytic expression for the variance of scale invariant mass (energy) anisotropies in finite macroscopic sub-volumes of size \( L \), after introducing by hand a regularization scale \( k_c \):

\[ (\Delta M_V)^2 \sim 4\rho_0^2 L^3 \int_0^{k_cL} d(kL) \frac{\mathcal{P}(k)}{(kL)^2} = 4\rho_0^2 L^2 \int_0^{k_c} dk \frac{\mathcal{P}(k)}{k^2}, \] (68)

which we wrote in (51) as \( (\Delta M_V)^2 = \rho_0^2 \mu S \), with \( \mu = 4 \int_0^{k_c} dk \frac{\mathcal{P}(k)}{k^2} \). We found that the parameter \( \mu \) is formally infinite when we take the regularization scale \( k_c \) to be infinitely large and noticed in Section VI that this divergent parameter can be renormalized to its physical value following an appropriate procedure. It is also worth to comment that in our discussion in Section III the regularization scale \( k_c \) was introduced as a sharp ultraviolet cutoff in the integral above, but we remarked there and in Fig. 1 that similar results are obtained when we use, instead, exponential or polynomial cutoff procedures.

We now find exactly the same formal expression for the renormalized (physical) variance of mass (energy) anisotropies in finite sub-volumes \( V \) of typical comoving size \( L \) when the fluctuating density field is probed to a length scale \( l = M^{-1} \).
\[(\Delta M_V)^2[M] \sim 4\rho_0^2 L^3 \int_{M}^{M_L} d(kL) \frac{\mathcal{P}_{\text{phy}}(k)}{(kL)^2} = 4\rho_0^2 L^2 \int_0^M dk \frac{\mathcal{P}_{\text{phy}}(k)}{k^2}, \quad (69)\]

In this expression the upper limit in the integral is not a sharp ultraviolet cutoff, but it simply means that the resolution scale introduces a physical exponential cutoff \((66)\) at \(k > \sim M\). This expression is obviously finite. It describes a physically observable magnitude (a dressed magnitude in the common vocabulary of renormalization group equations). It is also clear from this expression that the effect of changing the resolution at which we probe the mass (energy) anisotropies can be reabsorbed in the renormalizable running parameter \(\mu_{\text{phy}}(M) = 4 \int_0^M dk \frac{\mathcal{P}_{\text{phy}}(k)}{k^2}\), which measures the typical size of the scale invariant anisotropies at that resolution scale \([22]\), \((\Delta M_V)^2[M] \simeq \rho_0^2 \mu_{\text{phy}}(M) S(V)\), such that

\[\frac{d\mu_{\text{phy}}}{dl}|_l = -M^2 \frac{d\mu_{\text{phy}}}{dM}|_M = -4 \mathcal{P}_{\text{phy}}(M). \quad (70)\]

This equation is a renormalization equation in the most common sense, but it deserves an explanation. We have discussed in Section III that all power spectra with index \(n \geq 1\) are physically indistinguishable through the scale invariant anisotropies \((\Delta M_V)^2[M]\) they produce over finite, but macroscopically large, volumes \(V\) because, we said, the different indices \(n\) can be reabsorbed in the renormalized parameter \(\mu_{\text{phy}}\). The analysis in this section corroborates this conclusion and it offers us a deeper insight. At a given resolution scale all the information contained in any power spectra with index \(n \geq 1\) is hidden in the renormalized parameter \(\mu_{\text{phy}}\), which measures the overall size of the scale invariant physical anisotropies. As we cannot predict the physical value of this parameter in the theoretical setup that we have laid, different power spectra with \(n > 1\) are physically indistinguishable at this point. On the other hand, the spectral index \(n\) can be physically observable through the renormalization equation \((70)\), that is, through the change in the overall size of the scale invariant mass (energy) anisotropies when we let the resolution scale \(M\) to run. We must conclude that, even though all power spectra with index \(n \geq 1\) produce scale invariant mass (energy) anisotropies over macroscopically large volumes, different spectral indices are physically distinguishable if we can reliably measure the change \(d\mu_{\text{phy}}/dl\) in the overall size of the anisotropies over a given macroscopic volume when we change the resolution scale \(l\). This is the reason why in this section we have labelled the power spectrum \(\mathcal{P}_{\text{phy}}(k)\) as a physically observable magnitude.
An additional note. In the preceding paragraphs we have discussed how the resolution scale at which the random density field is probed affects the theoretical predictions on physically observable mass (energy) anisotropies. The analysis led us to the renormalization equation. Quite naturally we found that the resolution scale introduces an exponential physical cutoff that suppresses all fluctuating modes with comoving wavelength shorter than the chosen scale of resolution. If this comoving scale is conveniently shorter than the comoving size of cosmological volumes the observed mass (energy) anisotropies are necessarily scale invariant.

Before closing the discussion on this issue we want to comment on a different approach about how to manage the dependence of physical observables on the finite resolution scale of any real probe. This second approach is widely discussed in the literature on primordial anisotropies, but it is somehow artificial.

Let $\rho_{\text{phy}}(\vec{x})$ be the physical density field resolved at a certain scale $l_0$. The mass in a macroscopic sub-volume $V$ of typical size $L \gg l_0$ (which we assume for simplicity centered at the origin $\vec{x} = 0$) is defined in this second approach as $M_{\text{gauss}}(V) = \mathcal{N} \int_{\Omega} d^3 \vec{x} \rho_{\text{phy}}(\vec{x}) e^{-|\vec{x}|^2/L^2}$, where $\mathcal{N}$ normalizes the gaussian weight, instead of the definition $M(V) = \int_V d^3 \vec{x} \rho_{\text{phy}}(\vec{x})$ we introduced in Section II and have used throughout the paper. The new operator $M_{\text{gauss}}(V)$ is commonly referred in the literature as the mass function with a normalized gaussian window, while $M(V)$ is commonly referred as a top-hat mass function.

In Section VIII we have discussed how to properly accomodate a finite resolution scale within the standard definition of a top-hat mass function. The use of a mass function with a gaussian window $M_{\text{gauss}}(V)$ follows a different, and more artificial, approach: it averages the density field over the whole 3D space using a weighting gaussian function $\mathcal{N} e^{-x^2/L^2}$ with a typical opening width of the size $L$ of the considered volume $V$. According to eq. (57) the gaussian mass function $M_{\text{gauss}}(V) \propto V \rho_{\text{phy}}(\vec{x} = 0; L)$ is nothing but the average mass in volumes of size $L$ when the physical density field is probed with a resolution scale of the same size of the volume $V$ of integration, $l \sim L$. The aim behind the definition $M_{\text{gauss}}(V)$ is to introduce in the theoretical expression for the variance of mass anisotropies in a sub-volume of typical size $L$ an ultraviolet cutoff in momentum space of the order $k \sim 1/L$ and artificially justify the estimation. It is important to notice that within this gaussian approach the volume $V$ is not treated as a macroscopic volume, but as a volume of the minimal size that the probe can test. And we must be aware that a theory endowed with
an ultraviolet cutoff in real space of size \( l \) can lead to contrived predictions when applied to physical sub-systems whose typical size is \( L \lesssim l \), in the same sense that the effective Fermi four-point vertex fails to reproduce the observational features of weak interactions at energies close or higher than the mass of the \( W \) and \( Z \) bosons. In both cases, we might be trying to apply the theory beyond the scales it can reach. In the case of a power spectrum with index \( n = 1 \) the result of using the gaussian mass function \( M_{\text{gauss}}(V) \) does not modify the scale invariance \((50)\), but it artificially render the divergent parameter \( \mu \) in \((51)\) finite. In the case of a power spectrum with index \( n > 1 \) the use of a gaussian mass function artificially produce the wrong results \((12)\), instead of the scale invariant \((50)\).

IX. DISCUSSION.

We have revised in this paper the standard theoretical framework commonly used to discuss the origin of scale invariant cosmological mass (energy) anisotropies in the homogeneous and isotropic FRW background at the time of matter radiation equality to point out a crucial fault \((12)\) in the argument that led to wrongly assert the so-called origin of structures problem of standard cosmology. We have shown that the correct evaluation \((15)\) offers a natural and elegant explanation of the scale invariance of the primordial cosmological anisotropies as a consequence of a conservation law \((24)\) of the total mass (energy) of the fluctuating density field. Surprisingly, the correct estimation \((15)\) that we present here is well-known since long ago in statistical mechanics and condensed matter physics, but was never properly noticed before in cosmology.

According to our analysis primordial cosmological mass (energy) anisotropies in the, otherwise, homogeneous, isotropic and flat FRW universe at the time of matter radiation equality could happen to be simply quantum vacuum fluctuations of the density field generated at the same instant of equality with comoving wavelength shorter than the causal horizon at that time. In other words, the mass (energy) primordial anisotropies over cosmologically large volumes would happen randomly as a result of local rearrangement of matter at the instant of equality through the surface that bounds the considered spatial regions, thus explaining why their variance is proportional to the area of the boundary surface.

Obviously, this explanation of the origin of primordial cosmological anisotropies at the time of matter radiation equality in the context of standard cosmology does not require
any previous epoch of inflationary expansion. In the context of this alternative scenario the fluctuating density field is naturally required to be in its ground state, thus avoiding also the problematic issues on the exceptional initial conditions of the universe that can produce a period of inflationary expansion. Furthermore, the physics involved in the generation of the anisotropies shall probably be physics at the scale $T_{eq} \sim 1 \text{ eV}$ of equality, instead of the very high energy scales $T \gtrsim 10^{15} \text{ GeV}$ summoned in inflationary mechanism. Yet, this alternative mechanism to the origin of large scale cosmological structures needs a further elaboration of the calculations presented in the last section VII on the interplay of the surface mass (energy) anisotropies with the theory of gravitational structure formation beyond the linear approximation.

Finally, we must remark that the correct estimation, which lies at the foundation of all the arguments in this paper, implies a continuous degeneracy in the power spectrum of statistical systems that produce scale invariant mass (energy) anisotropies, which we expressed in our discussion through the impossibility to distinguish spectra with indexes $n > 1$ over cosmologically large comoving volumes. The degeneracy is associated to the possibility of transforming the charge density field operator of a globally conserved Noether charge without actually modifying the charge and it has far reaching consequences. Such transformations do not modify the scale invariance of the spatial quantum anisotropies in the distribution of the charge, but they can actually change the formal expression of the infinite parameter that measures the size of these anisotropies. Or, from a different perspective these transformations can be reabsorbed in the definition of an infinite renormalizable parameter of the theory. From this point of view we understand the scale invariance statement in the sense of a renormalization group equation and conclude that the physical dressed parameter, i.e. the absolute size of the anisotropies, cannot be predicted in the available setup of quantum field theories. In the same sense that we cannot predict the actual value of coupling constants or masses, or the cosmological constant.

The picture of the origin of primordial cosmological fluctuations as vacuum fluctuations at the same instant of matter radiation equality would not be complete without a discussion of how vacuum fluctuations can decohere. Halliwell, in advanced that in the context of the decoherent histories approach to quantum mechanics fluctuations of local densities (momentum, energy) are more prone to decohere. More recently we have developed a new formalism to address the same issue: in this formalism the ground state of a bosonic or
fermionic system is described as a linear combination of randomly distributed pseudoclassical incoherent paths, which allows a description of the fluctuations of collective operators like the energy in a sub-volume in terms of classical stochastic concepts.

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[22] We are assuming at this point that the considered sub-volume $V$ is bounded by a smooth enough surface, such that its area $S(V)$ does not change with the resolution scale at which we probe the 3D space.
FIG. 1: The points on the plot represent the numerically evaluated normalized ratio \( \frac{(\Delta M_L)^2}{L^2} \) of the variance of mass (energy) fluctuations in spherical 3D volumes of comoving radius \( L \) to the area of its surface \( S = 4\pi L^2 \). Two different power-law spectra \( \mathcal{P}(k) = a k^n \) with positive spectral indices are considered: \( n = 1 \) (dashed lines) and \( n = 4 \) (straight lines). The normalized data clearly show that in both cases the ratio remains constant over a very large range (100 ←→ 10000) of values of the comoving radius \( L \), measured in arbitrary units of length. The cutoff scale has been chosen arbitrarily at \( k_c^{-1} \approx 0.1 \) units of length, so that the dimensionless parameter \( k_c L \gg 1 \) ranges from \( 10^3 \) to \( 10^5 \). Two different setups to cutoff the power spectrum in the ultraviolet modes has been considered in this graph: an exponential cutoff \( \mathcal{P}(k)\text{Exp}(-k/k_c) \) (squares); and a polynomial cutoff \( \mathcal{P}(k) \left( \frac{1}{1+(k/k_c)^4} \right) \) (circles).