A NOTE ON INTERIOR $W^{2,1+\varepsilon}$ ESTIMATES FOR THE
MONGE-AMPRÈ EQUATION

G. DE PHILIPPIS, A. FIGALLI, AND O. SAVIN

Abstract. By a variant of the techniques introduced by the first two authors
in [DF] to prove that second derivatives of solutions to the Monge-Ampère
equation are locally in $L \log L$, we obtain interior $W^{2,1+\varepsilon}$ estimates.

1. Introduction

Interior $W^{2,p}$ estimates for solutions to the Monge-Ampère equation with bounded
right hand side
\begin{equation}
\det D^2 u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega, \quad 0 < \lambda \leq f \leq \Lambda,
\end{equation}
were obtained by Caffarelli in [C] under the assumption that $|f - 1| \leq \varepsilon(p)$ locally.
In particular $u \in W^{2,p}_{\text{loc}}$ for any $p < \infty$ if $f$ is continuous.
Whenever $f$ has large oscillation, $W^{2,p}$ estimates are not expected to hold for
large values of $p$. Indeed Wang showed in [W] that for any $p > 1$ there are homoge-
nous solutions to (1.1) of the type
\[u(tx, t^\alpha y) = t^{1+\alpha}u(x, y) \quad \text{for } t > 0,\]
which are not in $W^{2,p}$.

Recently the first two authors, motivated by a problem arising from the semi-
geostrophic equation [ACDF], showed that interior $W^{2,1}$ estimates hold for the
equation (1.1) [DF]. In fact they proved higher integrability in the sense that
\[\|D^2 u\| (\log \|D^2 u\|)^k \in L^1_{\text{loc}} \quad \forall k \geq 0.\]

In this short note we obtain interior $W^{2,1+\varepsilon}$ estimates for some small $\varepsilon =
\varepsilon(n, \lambda, \Lambda) > 0$. In view of the examples in [W] this result is optimal. We use
the same ideas as in [DF], which mainly consist in looking to the $L^1$ norm of $\|D^2 u\|
over the sections of $u$ itself and prove some decay estimates. Below we give the
precise statement.

Theorem 1.1. Let $u : \Omega \to \mathbb{R}$,
\[u = 0 \quad \text{on } \partial \Omega, \quad B_1 \subset \Omega \subset B_n,\]
be a continuous convex solution to the Monge-AMPRÈ equation
\begin{equation}
\det D^2 u = f(x) \quad \text{in } \Omega, \quad 0 < \lambda \leq f \leq \Lambda,
\end{equation}
for some positive constants $\lambda$, $\Lambda$. Then
\[\|u\|_{W^{2,1+\varepsilon}(\Omega')} \leq C, \quad \text{with } \Omega' := \{u < -\|u\|_{L^{\infty}}/2\},\]
where $\varepsilon, C > 0$ are universal constants depending on $n$, $\lambda$, and $\Lambda$ only.

\[\text{A. Figalli was partially supported by NSF grant 0969962. O. Savin was partially supported by NSF grant 0701037.} \]
By a standard covering argument (see for instance [DF, Proof of (3.1)]), this implies that $u \in W^{2,1+\varepsilon}_{{\text{loc}}}(|\Omega|)$.

Theorem 1.1 follows by slightly modifying the strategy in [DF]: We use a covering lemma that is better localized (see Lemma 3.1) to obtain a geometric decay of the “truncated” $L^1$ energy for $\|D^2u\|$ (see Lemma 3.3).

We also give a second proof of Theorem 1.1 based on the following observation:

In view of [DF] the $L^1$ norm of $\|D^2u\|$ decays on sets of small measure:

$$|\{|\|D^2u\| \geq M\}| \leq \frac{C}{M \log M},$$

for an appropriate universal constant $C > 0$ and for any $M$ large. In particular, choosing first $M$ sufficiently large and then taking $\varepsilon > 0$ small enough, we deduce (a localized version of) the bound

$$|\{|\|D^2u\| \geq M\}| \leq \frac{1}{M^{1+\varepsilon}}|\{|\|D^2u\| \geq 1\}|$$

Applying this estimate at all scales (together with a covering lemma) leads to the local $W^{2,1+\varepsilon}_{{\text{loc}}}$ integrability for $\|D^2u\|$.

We believe that both approaches are of interest, and for this reason we include both. In particular, the first approach gives a direct proof of the $W^{2,1+\varepsilon}_{{\text{loc}}}$ regularity without passing through the $L \log L$ estimate.

We remark that the estimate of Theorem 1.1 holds under slightly weaker assumptions on the right hand side. Precisely if

$$\det D^2u = \mu$$

with $\mu$ being a finite combination of measures which are bounded between two multiples of a nonnegative polynomial, then the $W^{2,1+\varepsilon}_{{\text{loc}}}$ regularity still holds (see Theorem 3.7 for a precise statement).

The paper is organized as follows. In section 2 we introduce the notation and some basic properties of solution to the Monge-Ampère equation with bounded right hand side. Then, in section 3 we show both proofs of Theorem 1.1 together with the extension to polynomial right hand sides.

After the writing of this paper was completed, we learned that Schmidt [S] had just obtained the same result with related but somehow different techniques.

2. Notation and Preliminaries

Notation. Given a convex function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ bounded and convex, we define its section $S_h(x_0)$, centered at $x_0$ at height $h$ as

$$S_h(x_0) = \{x \in \Omega \mid u(x) < u(x_0) + \nabla u(x_0) \cdot (x - x_0) + h\}.$$  

We also denote by $\overline{S}_h(x_0)$ the closure of $S_h(x_0)$.

The norm $\|A\|$ of an $n \times n$ matrix $A$ is defined as

$$\|A\| := \sup_{|x| \leq 1} Ax.$$  

We denote by $|F|$ the Lebesgue measure of a measurable set $F$.

Positive constants depending on $n$, $\lambda$, $\Lambda$ are called universal constants. In general we denote them by $c$, $C$, $c_i$, $C_i$.  

Next we state some basic properties of solutions to (1.2).

2.1. Scaling properties. If $S_h(x_0) \subset \Omega$, then (see [C] for example) there exists an affine transformation $A$ i.e., $\det A = 1$, such that
\begin{equation}
\sigma B_{\sqrt{\pi}} \subset A(S_h(x_0) - x_0) \subset \sigma^{-1} B_{\sqrt{\pi}},
\end{equation}
for some $\sigma > 0$, small universal.

**Definition 2.1.** We say that $S_h(x_0)$ has *normalized size* $\alpha$ if
\begin{equation*}
\alpha := \|A\|^2
\end{equation*}
for some matrix $A$ that satisfies the properties above. (Notice that, although $A$ may not be unique, this definition fixes the value of $\alpha$ up to multiplicative universal constants.)

It is not difficult to check that if $u$ is $C^2$ in a neighborhood of $x_0$, then $S_h(x_0)$ has normalized size $\|D^2 u(x_0)\|$ for all small $h > 0$ (if necessary we need to lower the value of $\sigma$).

Given a transformation $A$ as in (2.1), we define $\tilde{u}$ to be the rescaling of $u$
\begin{equation}
\tilde{u}(\tilde{x}) = h^{-1} u(x), \quad \tilde{x} = Tx := h^{-1/2} A (x - x_0).
\end{equation}
Then $\tilde{u}$ solves an equation in the same class
\begin{equation*}
\det D^2 \tilde{u} = f, \quad \text{with} \quad \tilde{f}(\tilde{x}) := f(x), \quad \lambda \leq \tilde{f} \leq \Lambda,
\end{equation*}
and the section $\tilde{S}_1(0)$ of $\tilde{u}$ at height 1 is normalized i.e
\begin{equation*}
\sigma B_1 \subset \tilde{S}_1(0) \subset \sigma^{-1} B_1, \quad \tilde{S}_1(0) = T(S_h(x_0)).
\end{equation*}
Also
\begin{equation*}
D^2 u(x) = A^T D^2 \tilde{u}(\tilde{x}) A,
\end{equation*}
hence
\begin{equation}
\|D^2 u(x)\| \leq \|A\|^2 \|D^2 \tilde{u}(\tilde{x})\|,
\end{equation}
and
\begin{equation}
\gamma_1 I \leq D^2 \tilde{u}(\tilde{x}) \leq \gamma_2 I \quad \Rightarrow \quad \gamma_1 \|A\|^2 \leq \|D^2 u(x)\| \leq \gamma_2 \|A\|^2.
\end{equation}

2.2. Properties of sections. Caffarelli and Gutiérrez showed in [CG] that sections $S_h(x)$ which are compactly included in $\Omega$ have engulfing properties similar to the engulfing properties of balls. In particular we can find $\delta > 0$ small universal such that:

1) If $h_1 \leq h_2$ and $S_{\delta h_2}(x_1) \cap S_{\delta h_2}(x_2) \neq \emptyset$ then
   \begin{equation*}
   S_{\delta h_1}(x_1) \subset S_{h_2}(x_2).
   \end{equation*}

2) If $h_1 \leq h_2$ and $x_1 \in \overline{S_{h_2}(x_2)}$ then we can find a point $z$ such that
   \begin{equation*}
   S_{\delta h_1}(z) \subset S_{h_1}(x_1) \cap S_{h_2}(x_2).
   \end{equation*}

3) If $x_1 \in \overline{S_{h_2}(x_2)}$ then
   \begin{equation*}
   S_{\delta h_2}(x_1) \subset S_{2h_2}(x_2).
   \end{equation*}
Now we also state a covering lemma for sections.
Lemma 2.2 (Vitali covering). Let D be a compact set in Ω and assume that to each x ∈ D we associate a corresponding section S_h(x) ⊂⊂ Ω. We can find a finite number of these sections S_h_i(x_i), i = 1, . . . , m such that

\[ D \subset \bigcup_{i=1}^{m} S_{h_i}(x_i), \quad \text{with } S_{\delta h_i}(x_i) \text{ disjoint.} \]

The proof follows as in the standard case: we first select a finite number of sections S_{\delta h_i}(x_i) which cover D, and then choose a maximal disjoint set from these sections.

3. Proof of Theorem 1.1

We assume throughout that u is a normalized solution in S_1(0) in the sense that

\[ \det D^2 u = f \quad \text{in } \Omega, \quad \lambda \leq f \leq \Lambda, \]

and

\[ S_2(0) \subset⊂ \Omega, \quad \sigma B_1 \subset S_1(0) \subset \sigma^{-1} B_1. \]

In this section we show that

\[ \int_{S_1(0)} \|D^2 u\|^1 + \varepsilon dx \leq C, \]

for some universal constants ε > 0 small and C large. Then Theorem 1.1 easily follows from this estimate and a covering argument based on the engulfing properties of sections. Without loss of generality we may assume that u ∈ C^2, since the general case follows by approximation.

3.1. A direct proof of Theorem 1.1

In this section we give a self-contained proof of Theorem 1.1. As already mentioned in the introduction, the idea is to get a geometric decay for \[ \int_{\{\|D^2 u\| \geq M\}} \|D^2 u\|. \]

Lemma 3.1. Assume 0 ∈ S_t(y) ⊂⊂ Ω for some t ≥ 1 and y ∈ Ω. Then

\[ \int_{S_1(0)} \|D^2 u\|dx \leq C_0 \left| \left\{ C_0^{-1} I \leq D^2 u \leq C_0 I \right\} \cap S_\delta(0) \cap S_t(y) \right|, \]

for some C_0 large universal.

Proof. We have

\[ \int_{S_1(0)} \|D^2 u\|dx \leq \int_{S_1(0)} \Delta u dx \leq \int_{\partial S_1(0)} u_\nu \leq C_1, \]

where the last inequality follows from the interior Lipschitz estimate of u in S_2(0).

The second property given in Subsection 2.2 gives

\[ S_\delta(0) \cap S_t(y) \supset S_\delta^2(z) \]

for some point z, which implies that

\[ |S_\delta(0) \cap S_t(y)| \geq c_1 \]

for some c_1 > 0 universal. The last two inequalities show that the set

\[ \{\|D^2 u\| \leq 2C_1 c_1^{-1}\} \]

has at least measure c_1/2 in S_\delta(0) ∩ S_h(y).
Finally, the lower bound on $\det D^2 u$ implies that 

$$C_0^{-1} I \leq D^2 u \leq C_0 I \quad \text{inside } \{ \|D^2 u\| \leq 2C_1 c_1^{-1} \},$$

and the conclusion follows provided that we choose $C_0$ sufficiently large. \(\square\)

By rescaling we obtain:

**Lemma 3.2.** Assume $S_{2n}(x_0) \subset \Omega$, and $x_0 \in S_t(y)$ for some $t \geq h$. If $S_h(x_0)$ has normalized size $\alpha$, then

$$\int_{S_h(x_0)} \|D^2 u\| \, dx \leq C_0 \alpha \left| \left\{ C_0^{-1} \alpha \leq \|D^2 u\| \leq C_0 \alpha \right\} \cap S_{\delta h}(x_0) \cap S_t(y) \right| .$$

**Proof.** The lemma follows by applying Lemma 3.1 to the rescaling $\tilde{u}$ defined in Section 2 (see (2.2)). More precisely, we notice first that by (2.3) we have

$$\|D^2 u(x)\| \leq \alpha \|D^2 \tilde{u}(\tilde{x})\|, \quad \tilde{x} = T x,$$

hence

$$| \det T | \int_{S_h(x_0)} \|D^2 u\| \, dx \leq \alpha \int_{\tilde{S}_1(0)} \|D^2 \tilde{u}\| \, d\tilde{x} .$$

Also, by (2.4) we obtain

$$\left\{ C_0^{-1} I \leq D^2 \tilde{u} \leq C_0 I \right\} \subset T \left( \left\{ C_0^{-1} \alpha \leq \|D^2 u\| \leq C_0 \alpha \right\} \right) ,$$

which together with

$$\tilde{S}_h(0) = T(S_{\delta h}), \quad \tilde{S}_{\delta/h}(\tilde{y}) = T(S_t(y)),$$

implies that

$$\left| \left\{ C_0^{-1} I \leq D^2 \tilde{u} \leq C_0 \alpha \right\} \cap \tilde{S}_h(0) \cap \tilde{S}_{\delta/h}(\tilde{y}) \right|$$

is bounded above by

$$| \det T | \left| \left\{ C_0^{-1} \alpha \leq \|D^2 u\| \leq C_0 \alpha \right\} \cap S_{\delta h}(x_0) \cap S_t(y) \right| .$$

The conclusion follows now by applying Lemma 3.1 to $\tilde{u}$. \(\square\)

Next we denote by $D_k$ the closed sets

$$(3.2) \quad D_k := \left\{ x \in S_1(0) : \|D^2 u(x)\| \geq M^k \right\} ,$$

for some large $M$. As we show now, Lemma 3.2 combined with a covering argument gives a geometric decay for $\int_{D_k} \|D^2 u\|$. 

**Lemma 3.3.** If $M = C_2$, with $C_2$ a large universal constant, then

$$\int_{D_{k+1}} \|D^2 u\| \, dx \leq (1 - \tau) \int_{D_k} \|D^2 u\| \, dx ,$$

for some small universal constant $\tau > 0$.

**Proof.** Let $M \gg C_0$ (to be fixed later), and for each $x \in D_{k+1}$ consider a section $S_h(x)$ of normalized size $\alpha = C_0 M^k$, which is compactly included in $S_{\delta}(0)$. This is possible since for $h \to 0$ the normalized size of $S_h(x)$ converges to $\|D^2 u(x)\|$ (recall that $u \in C^2$) which is greater than $M^{k+1} > \alpha$, whereas if $h = \delta$ the normalized size is bounded above by a universal constant and therefore by $\alpha$. 

Now we choose a Vitali cover for $D_{k+1}$ with sections $S_h(x_i)$, $i = 1, \ldots, m$. Then by Lemma 3.2, for each $i$,

$$
\int_{S_h(x_i)} \|D^2u\| dx \leq C_0^2 M^K \{M^k \leq \|D^2u\| \leq C_0^2 M^K \} \cap S_{\delta h}(x_i) \cap S_1(t).
$$

Adding these inequalities and using \eqref{eq:lemma3.2} by Lemma 3.2, for each universal constants $C, \varepsilon > 0$, we easily deduce that there exist $C, \varepsilon > 0$ universal such that

$$(3.1) \text{ could also be easily deduced by applying the L estimate from [DF] inside every section, and then doing a covering argument.}
$$

First, any $K > 0$ we introduce the notation

$$
F_K := \{\|D^2u\| \geq 2K\} \cap S_1(t).
$$

**Lemma 3.4.** Suppose $u$ satisfies the assumptions of Lemma 3.1. Then there exist universal constants $C_0$ and $C_1$ such that, for all $K \geq 2$,

$$
|F_K| \leq \frac{C_1}{K \log(K)} \{C_0^{-1} I \leq D^2u \leq C_0 I\} \cap S_\delta(0) \cap S_t(y).\n$$

Indeed, from the proof of Lemma 3.1 the measure of the set appearing on the right hand side is bounded below by a small universal constant $c_1/2$, while by [DF]

$|F_K| \leq C/K \log(K)$ for all $K \geq 2$, hence

$$
|F_K| \leq \frac{2C}{c_1 K \log(K)} \{C_0^{-1} I \leq D^2u \leq C_0 I\} \cap S_\delta(0) \cap S_t(y).
$$

Exactly as in the proof of Lemma 3.2 by rescaling we obtain:

**Lemma 3.5.** Suppose $u$ satisfies the assumptions of Lemma 3.2. Then,

$$
|\{\|D^2u\| \geq \alpha K\} \cap S_h(x_0)\} \leq \frac{C_1}{K \log(K)} \{C_0^{-1} \alpha \leq \|D^2u\|\} \cap S_{\delta h}(x_0) \cap S_t(y),
$$

for all $K \geq 2$. 

Finally, as proved in the next lemma, a covering argument shows that the measure of the sets $D_k$ defined in (3.2) decays as $M^{-(1+2\varepsilon)k}$, which gives (5.1).

**Lemma 3.6.** There exist universal constants $M$ large and $\varepsilon > 0$ small such that

$$|D_{k+1}| \leq M^{-1-2\varepsilon}|D_k|.$$

**Proof.** As in the proof of Lemma 3.3 we use a Vitali covering of the set $D_{k+1}$ with sections $S_h(x)$ of eccentricity $\alpha = C_0M^k$, i.e.

$$D_{k+1} \subset \bigcup S_{h_i}(x_i), \quad S_{\delta h_i}(x_i) \text{ disjoint sets}.$$

Apply Lemma 3.5 above for $K := C_0^{-1}M$, hence $\alpha K = M^{k+1}$, and find that for each $i$

$$|D_{k+1} \cap S_{h_i}(x_i)| \leq \frac{2C_0}{M \log(M)}|D_k \cap S_{h_i}(x_i)|,$$

provided that $M \gg C_0$. Summing over $i$ and choosing $M \geq e^{4C_0}$ we get

$$|D_{k+1}| \leq \frac{2C_0}{M \log(M)}|D_k| \leq \frac{1}{2M}|D_k|,$$

and the lemma is proved by choosing $\varepsilon = \log(2)/\log(M)$. \hfill \Box

### 3.3. More general measures.

It is not difficult to check that our proof applies to more general right hand sides. Precisely we can replace $f$ by any measure $\mu$ of the form

$$\mu = \sum_{i=1}^{N} g_i(x)|P_i(x)|^{\alpha_i} \, dx, \quad 0 < \lambda \leq g_i \leq \Lambda, \quad P_i \text{ polynomial, } \alpha_i \geq 0.$$

We state the precise estimate below.

**Theorem 3.7.** Let $u : \overline{\Omega} \to \mathbb{R}$,

$$u = 0 \quad \text{on } \partial \Omega, \quad B_1 \subset \Omega \subset B_n,$$

be a continuous convex solution to the Monge-Ampère equation

$$\det D^2 u = \mu \quad \text{in } \Omega, \quad \mu(\Omega) \leq 1,$$

with $\mu$ as in (3.3). Then

$$\|u\|_{W^{2,1+\varepsilon}(\Omega')} \leq C, \quad \text{with } \Omega' := \{u < -\|u\|_{L^\infty}/2\},$$

where $\varepsilon, C > 0$ are universal constants.

The proof follows as before, based on the fact that for $\mu$ as above one can prove the existence of constants $\beta \geq 1$ and $\gamma > 0$, such that, for all convex sets $S$,

$$\frac{\mu(E)}{\mu(S)} \geq \gamma \left(\frac{|E|}{|S|}\right)^\beta \quad \forall E \subset S,$$

\footnote{Although this will not be used here, we point out for completeness that (3.4) is equivalent to the so-called $\textbf{Condition (}\mu(\infty)\textbf{)}$, first introduced by Caffarelli and Gutierrez in \cite{CG}. Indeed this follows by using (3.3) with $E = S \setminus F$ to show that if $|F|/|S| \ll 1$ then $\mu(F)/\mu(S) \leq 1 - \gamma/2$, and then use an iteration and covering argument in the spirit of \cite{CG} Theorem 6.}
In this general situation, we need to write the scaling properties of \( u \) with respect to the measure \( \mu \). More precisely, the scaling inclusion (2.1) becomes
\[
\sigma^{-1/\alpha} \mu(S_h(x_0))^{-\alpha} B_1 \subset A(S_h(x_0) - x_0) \subset \sigma^{-1} h \mu(S_h(x_0))^{-\alpha} B_1,
\]
and
\[
T x := h^{-1} \mu(S_h(x_0))^{1/\alpha} A(x - x_0).
\]
Also we define the normalized size \( \alpha \) of \( S_h(x_0) \) (relative to the measure \( \mu \)) as
\[
\alpha := \frac{1}{h} \mu(S_h(x_0))^{1/2} \| A \|^2.
\]
With this notation, the statements of the lemmas in Section 3 apply as before.

Indeed, first of all we observe that (3.4) implies that \( \mu \) is doubling, so all properties of sections stated in Section 2.2 still hold.

Then, in the proof of Lemma 3.1, we simply apply (3.4) with \( S = S_1(0) \) and \( E = \{ \det(D^2 u) \leq c_2 \} \) \((c_2 > 0 \text{ small})\) to deduce that
\[
\gamma |E|^\beta \leq C \mu(E) = C \int_E \det(D^2 u) \leq c_2 |E|.
\]
This implies that, if \( c_2 > 0 \) is sufficiently small, the set
\[
\{ \|D^2 u\| \leq 2C_1c_1^{-1} \} \cap \{ \det(D^2 u) \leq c_2 \}
\]
has at least measure \( c_1/4 \), and the result follows as before.

Moreover, since (3.4) is affinely invariant, Lemma 3.2 follows again from Lemma 3.1 by rescaling. Finally, the proof of Lemma 3.3 is identical.

REFERENCES

[ACDF] Ambrosio L., Colombo M., De Philippis G., Figalli A., Existence of Eulerian solutions to the semigeostrophic equations in physical space: the 2-dimensional periodic case, Comm. Partial Differential Equations, to appear.

[C] Caffarelli L., Interior \( W^{2,p} \) estimates for solutions of Monge-Ampère equation, Ann. of Math. 131 (1990), 135-150.

[CG] Caffarelli L., Gutierrez C., Properties of solutions of the linearized Monge-Ampère equations, Amer. J. Math. 119 (1997), 423-465.

[DF] De Philippis G., Figalli A., \( W^{2,1} \) regularity for solutions of the Monge-Ampère equation, Preprint, arXiv:1111.7207.

[S] Schmidt T., \( W^{2,1+\varepsilon} \) estimates for the Monge-Ampère equation. Preprint 2012.

[W] Wang X.-J., Regularity for Monge-Ampère equation near the boundary, Analysis 16 (1996) 101-107.

Scuola Normale Superiore di Pisa, 56126 Pisa, Italy
E-mail address: guidodephilippis@sns.it

Department of Mathematics, The University of Texas at Austin, Austin, TX 78712 USA
E-mail address: figalli@math.utexas.edu

Department of Mathematics, Columbia University, New York, NY 10027 USA
E-mail address: savin@math.columbia.edu