The Ricci Flow on Complete Noncompact Kähler Manifolds

Xi-Ping Zhu
Department of Mathematics, Zhongshan University
Guangzhou 510275, P. R. China

Abstract
In this paper we survey the recent developments of the Ricci flows on complete noncompact Kähler manifolds and their applications in geometry.

1 Introduction

The classical uniformization theorem of Riemann surface states that a simply connected Riemann surface is biholomorphic to either the Riemann sphere, the complex line or the open unit disc. It gives the characterization for the standard complex structures of one-dimensional Kähler manifolds. Unfortunately, a direct analog of this beautiful result to higher dimension does not exist. For example, it is well-known that there is a vast variety of biholomorphically distinct complex structures on $\mathbb{R}^{2n}$ for $n > 1$. This says that the topological restrictions are no longer distinguishing the standard complex structures in higher dimensions. Thus, in order to characterize the standard complex structures for higher dimensional manifolds, one needs to impose geometric restrictions on the manifolds.

From the point of view of differential geometry, one consequence of the uniformization theorem is that a positively curved, compact or noncompact Riemann surface must be biholomorphic to the Riemann sphere or the complex line respectively. The higher dimensional version of the uniformization theorem is to search the similar characterization for complete Kähler manifolds with “positive curvature”. Such a characterization in the case of compact Kähler manifold is the famous Frankel conjecture which says that a compact Kähler manifold of positive holomorphic bisectional curvature is biholomorphic to a complex projective space. This conjecture was solved
by Andreotti-Frankel [11] and Mabuchi [19] in complex dimensions two and three respectively and the general case was then solved by Mori [22], and Siu-Yau [33] independently. Thus the further investigations are naturally led to complete noncompact Kähler manifolds of positive holomorphic bisectional curvature. The following conjecture provides the main impetus.

**Conjecture 1** (Greene-Wu [13], Yau [37]) A complete noncompact Kähler manifold of positive holomorphic bisectional curvature is biholomorphic to a complex Euclidean space.

Recall from Cheeger-Gromoll-Meyer [3], [14] that a complete noncompact Riemannian manifold with positive sectional curvature is diffeomorphic to a Euclidean space and recall from Greene-Wu [12] that a complete noncompact Kähler manifold with positive sectional curvature is Stein. Nonetheless, in the case of positive holomorphic bisectional curvature, very little is known about the topology and complex structure of these manifolds. For example, one does not even know whether a complete noncompact Kähler manifold of positive holomorphic bisectional curvature is simply connected. Moreover it is also unknown whether a complete noncompact Kähler manifold of positive holomorphic bisectional curvature is Stein, which is a conjecture of Siu [32]. The Ricci flow introduced by Hamilton [15] has been found to be an useful tool to understand these manifolds.

Let $M$ be a complete noncompact Kähler manifold with Kähler metric $g_{\alpha\overline{\beta}}$. The Ricci flow is the following evolution equation on the metric

$$
\begin{align*}
\frac{\partial g_{\alpha\overline{\beta}}(x,t)}{\partial t}(x,t) &= -R_{\alpha\overline{\beta}}(x,t), \quad x \in M, t > 0, \\
g_{\alpha\overline{\beta}}(x,0) &= g_{\alpha\overline{\beta}}(x), \quad x \in M,
\end{align*}
$$

(1.1)

where $R_{\alpha\overline{\beta}}(x,t)$ denotes the Ricci curvature tensor of the metric $g_{\alpha\overline{\beta}}(x,t)$.

The first result to the Ricci flow (1.1) on complete noncompact Kähler manifolds is the short time existence obtained by Shi [28]. The short time existence states that if the curvature of the initial metric $g_{\alpha\overline{\beta}}$ is bounded, the Ricci flow (1.1) has a maximal solution $g_{\alpha\overline{\beta}}(\cdot,t)$ on $[0, t_{\text{max}})$ with $t_{\text{max}} > 0$ and the curvature of $g_{\alpha\overline{\beta}}(\cdot,t)$ becomes unbounded as $t$ tends to $t_{\text{max}}$ if $t_{\text{max}} < +\infty$. It is also known (see [17] or [31]) that the Ricci flow (1.1) preserves the Kählerity of the metric and that if the initial metric has positive holomorphic bisectional curvature, the evolving metric still has positive holomorphic
bisectional curvature for all times. Thus to study the topological and complex structure of a complete noncompact Kähler manifold of positive holomorphic bisectional curvature, we can replace the Kähler metric by any one of the evolving metric of the Ricci flow (1.1). The most investigations of the Ricci flow (1.1) on complete noncompact Kähler manifolds are mainly devoted to understanding the topology, geometry at infinity and complex structures of such manifolds with positive holomorphic bisectional curvature.

In this paper we will describe the recent developments of the Ricci flow (1.1) on noncompact Kähler manifolds and will emphasize on its applications in geometry. Section 2 discusses the long time existence of the Ricci flow (1.1). In Section 3 we present the gap phenomena for complete noncompact Kähler manifolds of positive holomorphic bisectional curvature. Further the geometry of these manifolds at infinity is investigated in Section 4. Finally in Section 5 we report the recent progresses on the long standing conjecture of Greene-Wu and Yau.

2 Long Time Existence

Let \((M, g_{\alpha\bar{\beta}})\) be a complete noncompact Kähler manifold with bounded and positive holomorphic bisectional curvature. Consider the maximal solution \(g_{\alpha\bar{\beta}}(\cdot, t) (t \in [0, t_{\text{max}}])\) of the Ricci flow (1.1) with \(g_{\alpha\bar{\beta}}\) as the initial metric. By a direct calculation from the equation in (1.1), the scalar curvature \(R(x, t)\) of the metric \(g_{\alpha\bar{\beta}}(x, t)\) is evolved by

\[
\frac{\partial R}{\partial t}(x, t) = \Delta_{g(t)} R(x, t) + 2|R_{\alpha\bar{\beta}}(x, t)|^2_{g(t)}, \quad \text{on } M \times [0, t_{\text{max}}),
\]

where \(\Delta_{g(t)}\) is the Laplace-Beltrami operator with respect to the evolving metric \(g_{\alpha\bar{\beta}}(x, t)\). This is a nonlinear heat equation with superlinear growth.

At first sight, one thus believes that the scalar curvature will generally blow up in finite time.

In [31], Shi considered the volume element of the evolving metric and introduced the following function

\[
F(x, t) = \log \frac{\det(g_{\alpha\bar{\beta}}(x, t))}{\det(g_{\alpha\bar{\beta}}(x, 0))},
\]

(2.2)
on $M \times [0, t_{\text{max}})$. It can be easily obtained from (1.1) that

$$\frac{\partial F(x, t)}{\partial t} = -R(x, t). \quad (2.3)$$

Since the holomorphic bisectional curvature of $g_{\alpha\bar{\beta}}(\cdot, t)$ is positive, it follows that $F(\cdot, t)$ is nonincreasing in $t$ and $F(\cdot, 0) = 0$. And by the equation (1.1) we know that

$$g_{\alpha\bar{\beta}}(\cdot, t) \leq g_{\alpha\bar{\beta}}(\cdot, 0), \quad \text{on } M,$$

for any $t > 0$. Then we have

$$e^{F(x, t)} R(x, t) = g^{\alpha\bar{\beta}}(x, t) R_{\alpha\bar{\beta}}(x, t) \cdot \frac{\det(g_{\alpha\bar{\beta}}(x, t))}{\det(g_{\alpha\bar{\beta}}(x, 0))}$$

$$\geq g^{\alpha\bar{\beta}}(x, 0) R_{\alpha\bar{\beta}}(x, t)$$

$$= -\Delta g(0) F(x, t) + R(x, 0).$$

Combining with (2.3) we obtain

$$e^{F(x, t)} \frac{\partial F(x, t)}{\partial t} \geq \Delta g(0) F(x, t) - R(x, 0), \quad (2.5)$$

and

$$\Delta g(0) F(x, t) \leq R(x, 0), \quad (2.6)$$

on $M \times [0, t_{\text{max}})$.

In the PDE jargon, if the scalar curvature $R(x, 0)$ of the initial metric satisfies suitable decay conditions, the differential inequalities (2.5) and (2.6) will give two opposite estimates of $F$ by its average. Shi [31] observed that the combination of these two opposite estimates give the following a priori estimate for the function $F$.

**Lemma 2.1** (Shi [31]) Suppose $(M, g_{\alpha\bar{\beta}})$ is a complete noncompact Kähler manifold with bounded and positive holomorphic bisectional curvature. And suppose there exist positive constants $C_1, C_2$ and $0 < \theta < 2$ such that

(i) $R(x, 0) \leq C_1, x \in M$,

(ii) $\frac{1}{Vol(B_0(x_0, r))} \int_{B_0(x_0, r)} R(x, 0) dx \leq \frac{C_2}{(1 + r)^{\theta}}$. 

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for all \( x_0 \in M, 0 \leq r < +\infty \),

where \( B_0(x_0, r) \) is the geodesic ball of radius \( r \) and centered at \( x_0 \) with respect to the metric \( g_{\alpha \bar{\beta}}(x) \). Then the function \( F(x, t) \) satisfies the estimate

\[
F(x, t) \geq -C(t + 1)^{\frac{2-\theta}{\theta}}, \quad \text{on } M \times [0, t_{\max}),
\]

where \( C \) is a positive constant depending only on \( \theta, C_1, C_2, \) and the dimension.

The combination of the above lemma and (2.4) implies

\[
g_{\alpha \bar{\beta}}(x, 0) \geq g_{\alpha \bar{\beta}}(x, t) \geq e^{-C(t+1)^{\frac{2-\theta}{\theta}}} g_{\alpha \bar{\beta}}(x, 0), \quad \text{on } M \times [0, t_{\max}). \tag{2.7}
\]

Recall that the Ricci curvature is given by

\[
R_{\alpha \bar{\beta}}(x, t) = -\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \log \det(g_{\alpha \bar{\beta}}(x, t)).
\]

Thus the Ricci flow equation (1.1) is the parabolic version of the complex Monge-Amperé equation on the Kähler manifold. The inequality (2.7) is corresponding to the second order estimate for the Monge-Amperé equation. It is well-known that the third order and higher order estimates for the Monge-Amperé equation were developed by Calabi and Yau. Similarly, by adapting the Calabi and Yau’s arguments, Shi proved in [31] that the derivatives and higher order estimates for \( g_{\alpha \bar{\beta}}(x, t) \) are uniformly bounded on any finite time interval. Thus Shi obtained the following long time existence result.

**Theorem 2.2** (Shi [31]) Suppose \((M, g_{\alpha \bar{\beta}})\) is a complete noncompact Kähler manifold with bounded and positive holomorphic bisectional curvature. And suppose there exist positive constants \( C_2 \) and \( 0 < \theta < 2 \) such that

\[
(ii) \quad \frac{1}{Vol(B_0(x_0, r))} \int_{B_0(x_0, r)} R(x, 0) dx \leq \frac{C_2}{(1 + r)^\theta},
\]

for all \( x_0 \in M, 0 \leq r < +\infty \).

Then the Ricci flow (1.1) has a solution for all \( t \in [0, +\infty) \).

Later on, Ni-Tam [25] observed that when the scalar curvature of the initial metric decays faster than linear (i.e., \( \theta > 1 \) in the condition (ii)), the
above long time existence result can be deduced rather easily. In this case, one can solve the Poisson equation
\[ \Delta_{g(0)} u_0(x) = R(x, 0) \]
and then by a Bochner technique of Mok-Siu-Yau [21] the solution \( u(x) \) actually satisfies the Poincaré-Lelong equation
\[ \sqrt{-1} \partial \bar{\partial} u_0(x) = Ric(x, 0), \]
where \( Ric(x, 0) \) is the Ricci form of the initial metric \( g_{\alpha\bar{\beta}}(x) \). Set
\[ u(x, t) = u_0(x) - F(x, t) \]
where \( F(x, t) \) is defined in (2.2). Then one can easily check
\[ \sqrt{-1} \partial \bar{\partial} u(x, t) = Ric(x, t) \]
and
\[ \left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) u = 0 \]
on \( M \times [0, t_{\text{max}}) \). Moreover one can get the uniform bound on the gradient to \( u(x, t) \). Inspired by Hamilton [16] (see also Chow [9]), Ni-Tam [25] looked the quantity \( |\nabla u|_{g(t)}^2 + R(x, t) \) and verified
\[ (\Delta_{g(t)} - \frac{\partial}{\partial t})(|\nabla u|_{g(t)}^2 + R(x, t)) \geq 0, \]
which concludes from the maximum principle that the solution of (1.1) exists for all \( t \in [0, +\infty) \).

The above long time existence result of Shi suggests us to investigate what kind of decay estimate the curvature of positively curved Kähler manifolds may possess. The classical Bonnet-Myers theorem says that the Ricci curvature of a complete noncompact manifold can not be uniformly bounded from below by a positive constant. In [8] Chen and the author found that the curvature of a complete noncompact Kähler manifold actually satisfies the following decay estimate.

**Theorem 2.3** (Chen-Zhu [8]) Let \( M \) be a complete noncompact Kähler manifold with positive holomorphic bisectional curvature. Then for any \( x_0 \in M \), there exists a positive constant \( C \) such that
\[ \frac{1}{Vol(B(x_0, r))} \int_{B(x_0, r)} R(x)dx \leq \frac{C}{1 + r}, \quad \text{for all } 0 \leq r < +\infty, \]
where $R(x)$ is the scalar curvature of $M$.

We remark that the constant $C$ in the above decay estimate depending on the point $x_0 \in M$, while the assumption (ii) in Theorem 2.2 needs to be uniform for all $x_0 \in M$. Nevertheless, these two results motivate us to make the following conjecture.

**Conjecture 2** Let $(M, g_{\alpha\bar{\beta}})$ be a complete noncompact Kähler manifold with positive holomorphic bisectional curvature. Then the Ricci flow (1.1) with $g_{\alpha\bar{\beta}}$ as initial metric has a solution for all $t \in [0, +\infty)$.

In [5], Chen, Tang and the author obtained the following result to support the above conjecture.

**Proposition 2.4** (Chen-Tang-Zhu [5]) Let $(M, g_{\alpha\bar{\beta}})$ be a complex two-dimensional complete noncompact Kähler manifold with bounded and positive holomorphic bisectional curvature. Suppose the volume growth of $M$ is maximal, i.e.,

$$\text{Vol}(B(x_0, r)) \geq cr^d, \quad \text{for all } 0 \leq r < +\infty,$$

for some $x_0 \in M$ and some positive constant $c$. Then the Ricci flow (1.1) with $g_{\alpha\bar{\beta}}$ as the initial metric has a solution for all $t \in [0, +\infty)$.

The proof of this proposition is an indirect blow-up argument. It also uses some special features in dimension 2, such as the Gauss-Bonnet-Chern formula for the four-dimensional Riemannian manifolds.

### 3 Gap Theorems

From now on we discuss the applications of the Ricci flow (1.1) to the geometry of complete noncompact Kähler manifolds. In this section we are interested in the question that how much the curvature of a complete noncompact Kähler manifold of nonnegative holomorphic bisectional curvature could have near the infinities. The decay estimate of Theorem 2.3 tells us that there cannot too much at infinity. At first sight, there seems to be no restriction in the other direction, no limit on how less curvature can have. For example, it is easy to construct complete Kähler metrics on $\mathbb{C}$ from real surface of revolution such that their curvatures are zero outside some compact
set, nonnegative everywhere, and positive somewhere. But it is rather surprising that the corresponding situation can not occur for higher dimensions. In [21] Mok, Siu and Yau discovered the following isometrically embedding theorem.

**Theorem 3.1** (Mok-Siu-Yau [21], Mok [20]) Let $M$ be a complete noncompact Kähler manifold of nonnegative holomorphic bisectional curvature of complex dimension $n \geq 2$. Suppose for a fixed base point $x_0$,

(i) \[ \text{Vol}(B(x_0, r)) \geq C_1 r^{2n}, \quad 0 \leq r < +\infty, \]

(ii) \[ R(x) \leq \frac{C_2}{1 + d(x_0, x)^{2+\varepsilon}}, \quad x \in M, \]

for some $C_1, C_2 > 0$ and for any arbitrarily small positive constant $\varepsilon$. Then $M$ is isometrically biholomorphic to $\mathbb{C}^n$ with the standard flat metric.

This result shows that there is a gap between the flat metric and the other metrics of nonnegative curvature on $\mathbb{C}^n$. Their method is to consider the Poincaré-Lelong equation $\sqrt{-1} \partial \bar{\partial} u = \text{Ric}$. Under the condition (ii) that the curvature has faster than quadratic decay, they proved the existence of a bounded solution $u$ to the Poincaré-Lelong equation. By virtue of Yau’s Liouville theorem on complete manifolds with nonnegative Ricci curvature, this bounded plurisubharmonic function $u$ must be constant and hence the Ricci curvature must be identically zero. This implies that the Kähler metric is flat because of the nonnegativity of the holomorphic bisectional curvature. The maximal volume growth assumption (i) in the above Theorem 3.1 is made to solve the Poincaré-Lelong equation. This gap result was later generalized in [24] for non-parabolic manifolds. In [6], Chen and the author gave a further generalization as follows.

**Theorem 3.2** (Chen-Zhu [6]) Suppose $M$ is a complete noncompact Kähler manifold of complex dimension $n$ with bounded and nonnegative holomorphic bisectional curvature. Suppose there exists a positive function $\varepsilon : \mathbb{R} \to \mathbb{R}$ with $\lim_{r \to +\infty} \varepsilon(r) = 0$, such that for any $x_0$,

\[ \frac{1}{\text{Vol}(B(x_0, r))} \int_{B(x_0, r)} R(x) dx \leq \frac{\varepsilon(r)}{r^2}. \]

Then $M$ is a complete flat Kähler manifold.
The method was to use the Ricci flow. We considered the Kähler metric \( g_{\alpha\bar{\beta}}(x) \) in Theorem 3.2 as the initial metric and evolved it by the Ricci flow (1.1) to get a maximal solution \( g_{\alpha\bar{\beta}}(x, t) \) on \( M \times [0, t_{\text{max}}) \). By using the faster than quadratically decay condition in the theorem, we could prove that the solution exists for all times and the scalar curvature \( R(x, t) \) of the evolving metric \( g_{\alpha\bar{\beta}}(x, t) \) decays faster than linear in time, i.e.,

\[
\lim_{t \to +\infty} tR(x, t) = 0, \quad \text{for } x \in M. \tag{3.1}
\]

On the other hand, Cao [1] obtained the following Li-Yau type inequality for the scalar curvature of the solution of (1.1) with nonnegative bisectional curvature,

\[
\frac{\partial R}{\partial t} - 2\frac{\|\nabla R\|^2}{R} + \frac{R}{t} \geq 0, \quad \text{on } M \times [0, +\infty), \tag{3.2}
\]

which implies that the function \( tR(x, t) \) is nondecreasing in time. Thus the combination of (3.1) and (3.2) concludes that \( R(x, t) \equiv 0 \) on \( M \times [0, +\infty) \). Therefore \( (M, g_{\alpha\bar{\beta}}) \) must be flat.

Quite recently Ni-Tam [26] obtained an improved version of the above gap theorem which does not require the uniform decay and the boundedness of the curvature tensor.

**Theorem 3.3** (Ni-Tam [26]) Let \( M \) be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose that \( R(x) \leq C(d(x_0, x)^2 + 1) \) for some \( x_0 \in M \) and \( C > 0 \), and

\[
\int_0^r \left( \frac{s}{\text{Vol}(B(x_0, s))} \right) \int_{B(x_0, s)} R(x) dx ds = o(\log r).
\]

Then \( M \) must be flat.

### 4 Volume Growth and Curvature Decay

We continue to study the geometric properties of positively curved complete noncompact manifolds near the infinities. Let us first consider the volume growth of the manifolds. When an \( m \)-dimensional complete noncompact Riemannian manifold has nonnegative Ricci curvature, the classical Bishop volume comparison theorem implies that the volume growth of geodesic balls is at most as the Euclidean volume growth. On the other hand, Calabi
and Yau [36] showed that the volume growth of a complete noncompact $m-$dimensional Riemannian manifold with nonnegative Ricci curvature must be at least of linear, i.e.,

$$\text{Vol}(B(x_0, r)) \geq cr, \quad \text{for all } 1 \leq r < +\infty,$$

where $c$ is some positive constant depending on $x_0 \in M$ and the dimension. In [8], Chen and the author found that the volume growth estimate for Kähler manifolds of positive holomorphic bisectional curvature is at least of half of the real dimension (i.e., the complex dimension).

**Theorem 4.1** (Chen-Zhu [8]) Let $M$ be a complex $n-$dimensional complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose also its holomorphic bisectional curvature is positive at least at one point. Then the volume growth of $M$ satisfies

$$\text{Vol}(B(x_0, r)) \geq cr^n, \quad \text{for all } 1 \leq r < +\infty,$$

where $c$ is some positive constant depending on $x_0$ and the dimension $n$.

Note that Klembeck [18] and Cao [2] presented some complete Kähler metrics on $\mathbb{C}^n$ which have positive holomorphic bisectional curvature everywhere such that the volume of the geodesic ball $B(O, r)$ centered at the origin $O$ with respect to the Kähler metric grows like $r^n$. Thus the volume growth estimate of Theorem 4.1 is sharp. We also remark that the assumption that the holomorphic bisectional curvature is positive at least at one point is necessary. Indeed, let $M_1$ be a noncompact convex surface in $\mathbb{R}^3$ which is asymptotic to a cylinder at infinity. Clearly $M_1$ is a complete noncompact Riemann surface with positive curvature and has linear volume growth. And let $\mathbb{C}P^{n-1}$ be the complex projective space with the Fubini-Study metric. Then the product $M_1 \times \mathbb{C}P^{n-1}$ is a complex $n-$dimensional complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. But its volume growth is of linear.

By the way, in view of Theorem 4.1, it is naturally raised a question whether there is a similar volume growth estimate for Riemannian manifolds with positive sectional curvature. The answer is negative. More precisely, for each dimension $n \geq 2$, there exist $n-$dimensional Riemannian manifolds which have positive sectional curvature everywhere but have linear volume growth. In fact on any bounded, convex and smooth domain $\Omega$ in $\mathbb{R}^n$, we can choose a strictly convex function $u(x)$ defined over $\Omega$ which tends to
as $x$ approaches to the boundary $\partial \Omega$. The graph of the convex function $u(x)$ is a hypersurface in $\mathbb{R}^{n+1}$, denoted by $M^n$. Clearly the hypersurface $M^n$ is strictly convex, i.e., the second fundamental form is strictly positive definite. It then follows from the Gauss equation that $M^n$ has strictly positive sectional curvature. Since the domain $\Omega$ is bounded, the volume growth of $M^n$ must be of linear.

We further consider the relation between the volume growth and the curvature decay. Theorem 3.1 of the previous section due to Mok, Siu and Yau implies that the curvatures of complete noncompact Kähler manifolds with positive holomorphic bisectional curvature and maximal volume growth can not decay faster than quadratically. On the other hand, Yau predicted in [38] that if the volume growth is maximal, then the curvature must decay quadratically in certain average sense. By combining the decay estimate in Theorem 2.3 and Chen-Tang-Zhu [5], we had an affirmative answers to the case of complex two-dimensional Kähler manifolds. Later in [8], Chen and the author gave a further answer for higher dimensions under a more restricted curvature assumption. More precisely, we have

**Theorem 4.2** (Chen-Tang-Zhu [5], Chen-Zhu [8]) Let $M$ be a complex $n$-dimensional complete noncompact Kähler manifold with bounded curvature. Suppose the volume growth of $M$ is maximal, i.e.,

$$\text{Vol}(B(x_0, r)) \geq C_1 r^{2n}, \quad \text{for all } 0 \leq r < +\infty,$$

for some $x_0 \in M$ and some positive constant $C_1$. Suppose also one of followings holds:

(i) if $n = 2$, the holomorphic bisectional curvature of $M$ is positive;

(ii) if $n \geq 3$, the curvature operator of $M$ is nonnegative.

Then there exists a constant $C_2 > 0$ such that

$$\frac{1}{\text{Vol}(B(x, r))} \int_{B(x, r)} R(y) dy \leq C_2 \frac{\log(2 + r)}{r^2}, \quad \text{for all } x \in M \text{ and } r > 0.$$

Our method is not direct working on the Kähler metric. Instead we use the Kähler metric as initial data and evolve it by Ricci flow (1.1). By Theorem 2.3 we know that the curvature decays to zero at infinity in average sense. From the evolution equation of (1.1) we see that the deformation of
the metric at infinity is very small. In particular the maximal volume growth condition of the initial metric is preserved under the evolution of the Ricci flow (1.1), i.e.,

\[ \text{Vol}_t(B_t(x, r)) \geq C_1 r^{2n}, \quad \text{for all } r > 0, x \in M \text{ and } t \in [0, t_{\text{max}}), \quad (4.1) \]

with the same constant \( C_1 \) as in the initial metric, where \( B_t(x, r) \) is the geodesic ball of radius \( r \) centered at \( x \) with respect to the evolving metric \( g_{\alpha \bar{\beta}}(\cdot, t) \), and the volume \( \text{Vol}_t \) is also taken with respect to the metric \( g_{\alpha \bar{\beta}}(\cdot, t) \). By combining with the local injectivity radius estimate of Cheeger-Gromov-Taylor [4] we deduce

\[ \text{inj}(M, g_{\alpha \bar{\beta}}(\cdot, t)) \geq \frac{\beta}{\sqrt{R_{\text{max}}(t)}}, \quad \text{for } t \in [0, t_{\text{max}}), \quad (4.2) \]

for some positive constant \( \beta > 0 \), where \( R_{\text{max}}(t) = \sup \{ R(x, t) | x \in M \}, \quad t \in [0, t_{\text{max}}). \)

After obtaining the above injectivity estimate, we can use rescaling arguments to analysis the asymptotic behavior of the solution \( g_{\alpha \bar{\beta}}(\cdot, t) \) near the maximal time \( t_{\text{max}} \). According to Hamilton [17], the maximal solution \( g_{\alpha \bar{\beta}}(\cdot, t), t \in [0, t_{\text{max}}), \) of (1.1) is of either one of the following types:

**Type I:** \( t_{\text{max}} < +\infty \) and \( \sup(t_{\text{max}} - t)R_{\text{max}}(t) < +\infty \);

**Type II:** either \( t_{\text{max}} < +\infty \) and \( \sup(t_{\text{max}} - t)R_{\text{max}}(t) = +\infty \),

or \( t_{\text{max}} = +\infty \) and \( \sup tR_{\text{max}}(t) = +\infty \);

**Type III:** \( t_{\text{max}} = +\infty \) and \( \sup tR_{\text{max}}(t) < +\infty \).

Based on the injectivity radius estimate (4.2) we first blow up the maximal solution into a limiting model and we then blow down the limiting model at the infinity to get a rescaling limit which is splitted as the product of a nonflat solution of the Ricci flow with some Euclidean space. Since the maximal volume growth condition (4.1) is invariant under the rescaling, the nonflat factor must also have maximal volume growth. Thus we can repeat these rescaling arguments until deriving a complex one-dimensional nonflat solution of the Ricci flow with maximal volume growth. If the original maximal solution were of Type I or II, the rescaling limit should exist for all
$t \in (-\infty, 0]$. But only complex one-dimensional nonflat solution of the Ricci flow of nonnegative curvature on $(-\infty, 0]$ is either the standard round sphere or the cigar soliton. Since both the round sphere and the cigar soliton are not of maximal volume growth, we conclude that the maximal solution must be of type III, i.e., the solution exists for all $t \in [0, +\infty)$ and satisfies

$$0 \leq R(x, t) \leq \frac{C}{1+t}, \quad \text{on } M \times [0, +\infty),$$

for some positive constant $C$.

The scalar curvature $R(x, t)$ of the evolving metric $g_{\alpha\overline{\beta}}(x, t)$ satisfies the nonlinear heat equation (2.1). Intuitively, the Harnack inequality of heat equation bridges the time decay with the space decay. By using the time decay estimate (4.3) on the evolving curvature we are indeed able to prove that the curvature of the initial metric satisfies the quadratically decay estimate of Theorem 4.2.

5 Uniformization Theorems

The central question in the study of positively curved complete noncompact Kähler manifolds is the conjecture of Greene-Wu and Yau stated in Section 1. A weaker version of this conjecture is the following form.

**Conjecture 1’ (Greene-Wu [13], Yau [37])** A complete noncompact Kähler manifold of positive sectional curvature is biholomorphic to a complex Euclidean space.

Since the topology and differential structure of noncompact manifolds with positive sectional curvature is well understood from Cheeger-Gromoll-Meyer [3], [14], the above weaker version is more concentrative on the complex structure.

The first partial affirmative answer to Conjecture 1’ was given by Mok [20] for complex two-dimensional manifolds with maximal volume growth in the following theorem.

**Theorem 5.1 (Mok [20])** Let $M$ be a complete noncompact Kähler manifold of complex dimension $n$ with positive holomorphic bisectional curvature. Suppose there exist positive constants $C_1$ and $C_2$ such that for a fixed base
point $x_0$,

(i) $\Vol(B(x_0, r)) \geq C_1 r^{2n}, \quad 0 \leq r < +\infty,$

(ii) $R(x) \leq \frac{C_2}{1 + d(x_0, x)^2}, \quad \text{on } M.$

Then $M$ is biholomorphic to an affine algebraic variety. Moreover, if in addition the complex dimension $n = 2$ and

(iii) the sectional curvature of $M$ is positive,

then $M$ is holomorphic to $\mathbb{C}^2$.

Note that the conditions (i) and (ii) imply the integral $\int_M \text{Ric}^n$, the analytic Chern number $c_1(M)^n$, is finite. The following result of To [35] gave a generalization of the above Mok’s result to nonmaximal volume growth manifolds. It was proved in [35] that if $M$ is a complete noncompact Kähler manifold of positive holomorphic bisectional curvature and suppose for some base point $x_0 \in M$ that there exist positive $C_1, C_2$ and $p$ such that

(i) $\int_{B(x_0, r)} \frac{1}{(1 + d(x_0, x))^{np}} \, dx \leq C_1 \log(r + 2), \quad 0 \leq r < +\infty,$

(ii) $R(x) \leq \frac{C_2}{1 + d(x, x_0)^p}, \quad \text{on } M$

(iv) $c_1(M)^n = \int_M \text{Ric}^n < +\infty,$

then $M$ is quasi-projective. Moreover, if in addition the complex dimension $n = 2$ and

(iii) the sectional curvature of $M$ is positive,

then $M$ is biholomorphic to $\mathbb{C}^2$.

It is likely that the assumption (iv) is automatically satisfied for complete Kähler manifolds with positive sectional curvature. At least in the complex two-dimensional case, there holds the generalized Cohn-Vossen inequality

$$c_2(M) = \int_M \Theta \leq \chi(\mathbb{R}^4) < +\infty$$

where $\Theta$ is the Gauss-Bonnet-Chern integrand. In view of Miyaoka-Yau inequality on the Chern numbers, it is reasonable to believe that one can get
the finiteness of $c_1(M)^2$ from that of $c_2(M)$. Meanwhile in views of Demailly’s holomorphic Morse inequality [10] and the $L^2$-Riemann-Roch inequality of Nadel-Tsuji [23] (see also Tian [34]), the assumption (iv) is a natural condition for a complete Kähler manifold to be a quasi-projective manifold. However the assumptions on the curvature decay and the volume growth are more problematic since they demand the geometry of the Kähler manifold at infinity to be somewhat uniform. In the recent work [7], Chen and the author showed that the assumption (iv) alone is sufficient to give an affirmative answer to Conjecture 1’ for complex two-dimensional case.

**Theorem 5.2 (Chen-Zhu [7])** Let $M$ be a complex $n$–dimensional complete noncompact Kähler manifold with bounded and positive sectional curvature. Suppose

$$c_1(M)^n = \int_M \text{Ric}^n < +\infty.$$  

Then $M$ is biholomorphic to a quasi-projective variety, and in case of complex dimension $n = 2$, $M$ is biholomorphic to $\mathbb{C}^2$.

We now come to Conjecture 1. Let us recall a remark in [6]. In [29] and [30], Shi used the Ricci flow to construct a flat Kähler metric on $M$ and then claimed that $M$ is biholomorphic to $\mathbb{C}^n$. It was pointed out in [6] that from [29] it is unclear why the property of completeness is true. Thus one can only get biholomorphic embedding of $M$ into a domain of $\mathbb{C}^n$.

To the knowledge of the author, the only affirmative answer to Conjecture 1 on the level of holomorphic bisectional curvature is the following result.

**Theorem 5.3 (Chen-Tang-Zhu [5], Chen-Zhu [8])** Let $M$ be a complete noncompact complex two-dimensional Kähler manifold of bounded and positive holomorphic bisectional curvature. Suppose the volume growth of $M$ is maximal, i.e.,

$$\text{Vol}(B(x_0, r)) \geq cr^4, \quad \text{for all} \ 0 \leq r < +\infty,$$

for some point $x_0 \in M$ and some positive constant $c$. Then $M$ is biholomorphic to the complex Euclidean space $\mathbb{C}^2$.

We describe the proof of this theorem as follows. The argument there was divided into three parts. In the first part, we showed that $M$ is a Stein manifold homeomorphic to $\mathbb{R}^4$. For this, we evolved the Kähler metric on $M$ by the Ricci flow (1.1). By noting that the underlying complex structure,
Kählerness and positivity of holomorphic bisectional curvature are preserved under the Ricci flow, the Kähler metric in the theorem can be replaced by any one of the evolving metric. The advantage is that the properties of the evolving metric are improving during the flow. As seen in the proof of Theorem 4.2, the maximal volume growth condition is uncharged and the curvature of evolving metric decays linearly in time under the Ricci flow (1.1). These imply that the injectivity radius of the evolving metric is getting bigger and bigger and any geodesic ball with radius less than half of the injectivity radius is almost pseudoconvex. By a perturbation argument we were then able to modify these geodesic balls to a sequence of exhausting pseudoconvex domains of $M$ such that any two of them form a Runge pair. From this, it follows readily that $M$ is a Stein manifold homeomorphic to $\mathbb{R}^4$.

In the second part of the proof, we considered the algebra $P(M)$ of holomorphic functions of polynomial growth on $M$ and we would prove that its quotient field has transcendental degree two over $\mathbb{C}$. For this, we first needed to construct two algebraically independent holomorphic functions in the algebra $P(M)$. Using the $L^2$ estimate of $\bar{\partial}$ operator, it suffices to construct a strictly plurisubharmonic function of logarithmic growth on $M$. As already known in Mok-Siu-Yau [21], such a strictly plurisubharmonic function of logarithmic growth can be obtain by solving the Poincaré-Lelong equation when the curvature decays in space at least quadratically. Fortunately in Theorem 4.2 we had deduced that the curvature of the initial metric decays quadratically in space in the average sense and this turns out to be enough to insure the existence of a strictly plurisubharmonic function of logarithmic growth. Next, by using the time decay estimate and the injectivity radius estimate of the evolving metric, we proved that the dimension of the space of holomorphic functions in $P(M)$ of degree at most $p$ is bounded by a constant times $p^2$. Combining this with the existence of two algebraically independent holomorphic functions in $P(M)$ as above, we could prove that the quotient field $R(M)$ of $P(M)$ has transcendental degree over $\mathbb{C}$ by a classical argument of Poincaré-Siegel. In other words, $R(M)$ is a finite extension field of some $\mathbb{C}(f_1, f_2)$, where $f_1$, $f_2 \in P(M)$ are algebraically independent over $\mathbb{C}$. Then from the primitive element theorem, we deduced $R(M) = \mathbb{C}(f_1, f_2, g/h)$ for some $g, h \in P(M)$. Hence the mapping $F : M \to \mathbb{C}^4$ given by $F = (f_1, f_2, g, h)$ defines a birational map from $M$ into some irreducible affine algebraic subvariety $Z$ of $\mathbb{C}^4$.

In the last part of proof, we basically followed the approach of Mok in
[20] to establish a holomorphic map from $M$ onto a quasi-projective variety by desingularizing the map $F$. Our essential contribution in this part was to establish uniform estimates on the multiplicity and the number of irreducible components of the zero divisor of a holomorphic function in $P(M)$. Again the time decay estimate of the Ricci flow played a crucial role in the arguments. Based on these estimates, we could show that the mapping $F : M \to Z$ is almost surjective in the sense that it can miss only a finite number of subvarieties in $Z$, and can be desingularized by adjoining a finite number of holomorphic functions of polynomial growth. This shows that $M$ is a quasi-projective variety. Finally, by combining with the fact that $M$ is homeomorphic to $\mathbb{R}^4$, we conclude that $M$ is indeed biholomorphic to $\mathbb{C}^2$ by a theorem of Ramanujam [27] on algebraic surfaces.
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