IMPURITIES AND DEFECTS IN MOTT SYSTEMS

By

SHAO TANG

A Dissertation submitted to the
Department of Physics
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
# TABLE OF CONTENTS

List of Figures . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . iii

Abstract . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . vii

1 Introduction 1

2 Mottness Induced Healing in Strongly Correlated Superconductors 8
   2.1 Model and method .................................................. 9
   2.2 Healing .......................................................... 11
   2.3 Mottness-induced healing ........................................... 15
   2.4 A minimal model .................................................. 16
   2.5 Conclusions ......................................................... 17

3 Strong Correlations Generically Protect $d$-wave Superconductivity against Disorder 18
   3.1 Model and method .................................................. 18
   3.2 Analytical results .................................................. 21
      3.2.1 The gap equation in AG theory .................................. 21
      3.2.2 $T$-matrix for the spinons and the pair-breaking scattering rate .... 25
      3.2.3 $T$-matrix for the physical electrons and the normal state transport scattering rate ........................................... 28
   3.3 Discussions ......................................................... 30
      3.3.1 Pair breaking parameter .......................................... 30
      3.3.2 Transport property in correlated normal state ...................... 31
      3.3.3 Enhanced forward scattering ..................................... 32
   3.4 Conclusions ........................................................ 33

4 Shock Waves and Commutation Speed of Memristors 34
   4.1 Generalized Burgers’ equation ...................................... 34
   4.2 Model system ....................................................... 38
**LIST OF FIGURES**

| Figure | Description | Page |
|--------|-------------|------|
| 1.1    | Spatial variations of the normalized local paring amplitude in cases of with correlations (left panel) and without correlations (right panel) (A. Garg et al., 2008). | 2 |
| 1.2    | Three gap function maps are measured on samples with three different oxygen doping levels as shown in plots A,B and C respectively. The field of view size is identical and corresponds to 16,000 CuO$_2$ plaquettes approximately. Average gap magnitudes $\bar{\Delta}$ are shown at the top, together with values of $N$, which is the total number of dopant atoms shown as white circles. Curves in plot D represent the local density of states at different locations (K. McElroy et al., 2005). | 3 |
| 1.3    | Spatial fluctuations of electron density (a) and quasiparticle weight (b) vary as we approach the Mott transition critical point from top to bottom (E. C. Andrade et al., 2010). | 4 |
| 1.4    | Experimental hysteresis loops measured in (left) manganite and (right) cuprate devices (M. J. Rozenberg et al., 2010). | 5 |
| 2.1    | Spatial variations of normalized local SC gap $\Delta_i$ for three impurities (first column) and the corresponding power spectra $S(k)$, $S(k)_{loc}$ and $S(k)_{nonloc}$ (second to fourth columns) in the presence (top) and in the absence (bottom) of correlations for $x = 0.2$. The strong suppression of gap oscillations by correlations can be traced to the dominance of the local, spherically symmetric power spectrum $[S_{loc}(k)]$ over the non-local anisotropic part $[S_{nonloc}(k)]$. | 9 |
| 2.2    | Spatial variations of normalized local gap function $\frac{\Delta_i}{\Delta_0}$ for three impurities (first column) and the corresponding power spectra $S(k)$, $S(k)_{loc}$ and $S(k)_{nonloc}$ (second to fourth columns) for $x = 0.15$ (first row), $x = 0.25$ (second row), and $x = 0.3$ (third row). The corresponding healing factors are (a) $h = 0.23\%$, (b) $h = 1.77\%$, and (c) $h = 2.74\%$. | 13 |
| 2.3    | Left panel: the healing factor $h$ as a function of doping in the uncorrelated case (blue curve with squares), in the correlated case (red curve with circles), and in the correlated case without $\delta \chi_i$ fluctuations (green curve with diamonds). Right panel: doping dependence of the SC ($\xi_S$, red curve with circles) and normal state ($\xi_N$, blue curve with squares) healing lengths. The green curve with diamonds gives $\xi_S$ calculated within the minimal model (see text). | 14 |
| 2.4    | Local (left) and nonlocal (right) parts of the charge-fluctuation power spectra $N(k)_{loc}$ and $N(k)_{nonloc}$ in the presence of strong correlations for $x = 0.2$. | 17 |
| 3.1    | Feynman diagrams in superconducting state with impurities. (L.P. Gor’kov 2008, Chapter 5 of Karl Heinz Bennemann et al., 2008) | 22 |
3.2 The pair-breaking and transport scattering rates normalized by the non-correlated value $1/\tau_0$ as a function of the doping level. .......................................................... 32

3.3 The angular dependence of the renormalized $T$-matrices at $x = 0.15$ (blue) and $x = 0.3$ (red). .......................................................... 33

4.1 Schematic representation of the shock wave evolution. The orange region indicates the metallic electrode and the blue indicates the TMO dielectric. Small spheres denote the ionic defects (oxygen vacancies) whose density profile form a shock wave. It evolves through a highly resistive (Schottky) interface and eventually leaks over the more conductive bulk, producing the resistive change. Black arrows depict the strength of the local electric fields. .......................................................... 35

4.2 Time evolution of a generic density profile according to the generalized Burgers’ equation (4.1). Curves correspond to profiles at $t = 0, 2, 4$ and 6. The (arbitrary) initial profile is taken Gaussian and we adopt $f(u) \propto u^2$. The color dots track constant-$u$ points and show the formation of the universal steep shock wave profile. Inset: $t-x$ coordinates of the constant-$u$ tracking points (characteristics). The velocity of each point only depends on its $u$ value. The crossing of the characteristics signals the formation of the multivalued steep shock-wave front. .......................................................... 36

4.3 Top panel: Snap shots of the time evolution of the density profile $u(t,x)$ within the active interfacial region in a simulation of the VEOVM model (see Appendix E.1 and E.2 for details). The current used is $I[a.u.] = 71.5$, with $A_{SB}=1000$ and $A_B=1$. The time steps of the successive profiles can be read-off from the corresponding color dots in the inset. The initial state $u(0,x)$ (black line) exhibits a vacancy pile-up next to the electrode at $x=1$. The SB-bulk interface is denoted with a vertical dash-dot line at $x_{int}=100$. The large accumulation of vacancies on the right of $x_{int}$ (bulk side) results from the initial “forming” cycles performed on an originally uniform profile of density $u_0$. The result of the forming cycles is an approximately fixed background on top of which the density profile further evolves (see details in [64]). Inset: Resistance of the device as a function of time. Color dots indicate the value of $R(t)$ at the corresponding snapshots of the main panel. Bottom-left panel: evolution of the shock wave front position $x_s(t)$ for different currents ($I[a.u.] = 58.5, 71.5, 84.5$ and $97.5$). Dots are from numerical simulations and the solid lines are analytic fits from integration of Eq. (4.4). Inset: characteristic impact-time $\tau_1$ as a function of applied currents from the numerical simulation (circles) and analytic fit (dotted-line) in semi-log scale. Bottom-right panel: Shock wave parameters. .......................................................... 37

4.4 Time dependence of the resistive change $R(t)$ for various external current intensities. Top left: experimental data measured on an Ag/LPCMO/Ag memristor with $I = 37.5$ mA (black), 40 mA (green), 50 mA (blue), 80 mA (red) and 100 mA (cyan). $t_{eff}$ is the effective time duration of the applied currents (see Appendix E.4 for details). The initial state was reset by applying an intense negative polarity current of 350 mA. Note that the initial value of the resistance, 78 $\Omega$ is recovered within $\pm 1\Omega$. Top right: model simulations with applied current: $I[a.u.] = 58.5$ (black), 71.5 (green), 84.5 (blue) and
97.5 (red). Middle panels: experimental data for $I = 40$ mA (left) and simulations for $I[a.u.] = 71.5$ (right). $\tau_1$ is defined as the time interval from the beginning of the applied pulse until the resistance starts to drop. Bottom panels: Idem as middle panels in a semilog scale.

4.5 Scaled curves of the $R(t)$ data sets of Fig.4.4. The left panels show the collapsed experimental data and the right ones the numerical simulations. The time $\tau_0$ is an auxiliary scaling variable, which is proportional to the characteristic time $\tau_2$ (see Appendix E.5 for details on the scaling procedure). The scaled data were fitted (white dotted line) with a generalized version of Eq.4.8 (see Appendix E.5). The lower panels show the same data sets scaled with the shock wave impact time $\tau_1$ (the experimental curves show only three data sets for the lower current values. At higher currents our electronics could not resolve $\tau_1$). To achieve the scaling of the lower panels, we assumed for each plot the normalization value of $\Delta R$ determined from the previous scaling (top panels).

B.1 Power spectra of gap fluctuations $S(k)$, $S(k)_{loc}$ and $S(k)_{nonloc}$ (first to third columns) for $x = 0.2$ in the presence of correlations. The top figures were obtained from the full solution of the linearized Eqs. (A.18), whereas the bottom ones correspond to the minimal model (Eqs. (B.3)). Note that the healing factors are $h = 0.74\%$ (top) and $h = 0.69\%$ (bottom).

C.1 Spatial variations of normalized local density $\delta n_i/n_0$ in the normal state for three impurities (first column) and the corresponding power spectra $N(k)$, $N(k)_{loc}$ and $N(k)_{nonloc}$ (second to fourth columns), in the presence (top) and in the absence (bottom) of strong correlations for $x = 0.2$. The strong suppression of density oscillations by correlations is accompanied by the dominance of the spherically symmetric local power spectrum $[N_{loc}(k)]$ over the anisotropic non-local one $[N_{nonloc}(k)]$.

E.1 Top panels: top-left: Schematic diagram of the VEOVM model with a single active contact. The two regions SB and B correspond to the high resistance interface (Schottky Barrier) and the more conductive central bulk, respectively. The small cells within the channel indicate the domains. Top-right: vacancy distribution along the conductive channel before (black squares) and after (red circles) the forming process. The distribution shown after the forming process correspond to a Hi resistance state. Starting from a uniform distribution, during the forming process a characteristic distribution is obtained where vacancies initially at the interface migrates to the low-field bulk region generating a pile-up of vacancies close to the limit between the two regions. In the Hi state, part of these vacancies migrates back to the interfacial region and are accumulated in the vicinity of to the metal contact.Bottom-left (table): parameters used in the simulations. Bottom-right: Snapshots of the shock wave formation in the early stage of simulations (between the first and second snapshot of Fig. 4.3).
E.2 Evolution of the resistance during the switching phase for different applied currents according to simulations (dots) and theory (lines) from Eq. (4.6). The currents shown are $I = 58.5a.u., 71.5a.u., 84.5a.u.$ and $97.5a.u.$

E.3 (a) Diagram of the experimental set up. (b) Sketch of the pulsed current protocol used in the experiments. A high current Write pulse (blue) is followed by a low current Read pulse (red). The first pulse generates the RS while the second pulse measures the non-volatile resistance. (c) R vs t curves from a complete RS process. The system is first taken to a Hi resistance state under a current of -350mA (left panel). Then the Hi to Lo RS is measured under an applied current of 37.5mA (middle panel). Finally the system is taken back, ie “re-initialized”, to a Hi resistance state with the application of a -350mA current.

E.4 Fitting results for the evolution of the normalized resistance according to the Eq. (E.6).

E.5 Evolution of the characteristic times for different applied currents for both simulations (left panel) and experiments (right panel). It can be seen that both times follow an approximate exponential dependence with $I$, and that there exist a relative proportionality between them as predicted by our analysis.

E.6 Shock wave dynamics for two different applied currents, which differ by nearly an order of magnitude. The lower current case (right panel) corresponds to applied voltages comparable to the experimental ones, and the higher voltage cases (left panel, same as FIG. 2) show similar shock wave dynamics. The two lower-right panels demonstrate that (i) that the propagation of the shock wave front remains qualitatively the same as for higher applied currents, and (ii) the collapse of the snap-shots of successive profiles indicates the formation of the shock-wave.

F.1 The schematic plot for the oxygen shock wave front propagation. The green curve is the initial distribution where oxygen are piled up at $x < x_0$. 

vi
Disorder has intriguing consequences for correlated electronic materials, which include several families of high-temperature superconductors and resistive switching systems. In this dissertation, we study the effects of impurities intertwined with correlations.

First, we study impurity healing effects in models of strongly correlated superconductors. We show that in general both the range and the amplitude of the spatial variations caused by nonmagnetic impurities are significantly suppressed in the superconducting as well as in the normal states. We explicitly quantify the weights of the local and the non-local responses to inhomogeneities and show that the former are overwhelmingly dominant over the latter. We find that the local response is characterized by a well-defined healing length scale, which is restricted to only a few lattice spacings over a significant range of dopings in the vicinity of the Mott insulating state. We demonstrate that this healing effect is ultimately due to the suppression of charge fluctuations induced by Mottness. We also define and solve analytically a simplified yet accurate model of healing, within which we obtain simple expressions for quantities of direct experimental relevance.

Second, we address the question of why strongly correlated $d$-wave superconductors, such as the cuprates, prove to be surprisingly robust against the introduction of non-magnetic impurities. We show that, very generally, both the pair-breaking and the normal state transport scattering rates are significantly suppressed by strong correlations effects arising in the proximity to a Mott insulating state. We also show that the correlation-renormalized scattering amplitude is generically enhanced in the forward direction, an effect which was previously often ascribed to the specific scattering by charged impurities outside the copper-oxide planes.

Finally, we provide the theoretical insights for resistive switching systems and show how impurities and underlying correlations can play significant roles in practical devices. We report the striking result of a connection between the resistive switching and shock wave formation, a classic topic of non-linear dynamics. We argue that the profile of oxygen vacancies that migrate during the commutation forms a shock wave that propagates through a highly resistive region of the device. We validate the scenario by means of model simulations and experiments in a manganese-oxide based memristor device and we extend our theory to the case of binary oxides. The shock wave scenario
brings unprecedented physical insight and enables to rationalize the process of oxygen-vacancy-driven resistive change with direct implications for a key technological aspect – the commutation speed.
 CHAPTER 1

INTRODUCTION

Strongly correlated systems have become one of the central puzzles in condensed matter physics after the discovery of unconventional superconductivity in heavy fermion systems and copper oxide materials [1, 2, 3, 4]. The cuprates, which are widely considered as strongly correlated systems, have many important features, such as the breakdown of conventional Fermi-liquid state, quantum antiferromagnetism, pseudogap phase [5], spin-liquid phase, localization and the quantum critical point [6, 7], etc. Strong electronic correlations are believed to be essential for a complete understanding of many classes of unconventional superconductors including the cuprates [2, 3, 8, 9], heavy fermion superconductors [10], organic materials [11, 12] and iron pnictides [13].

Among the many puzzling features of these systems is their behavior in the presence of disorder [14, 15]. In the case of the cuprates, experiments have shown that these \( d \)-wave superconductors are quite robust against disorder as introduced by carrier doping [16, 9, 17]. In particular, there seems to be a “quantum protection” of the \( d \)-wave nodal points [18]. Typically, it has been confirmed that the low energy part of the density of state at nodal points shows the clear \( V \) shape and would not be smeared in presence of doping disorder. Other anomalies were found in the organics [19] and the pnictides [20]. Although it is controversial whether conventional theory is able to explain these features [21], strong electronic interactions can give rise to these impurity screening effects. Indeed, they have been captured numerically by the Gutzwiller-projected wave function [22, 23, 24], even though a deeper insight into the underlying mechanism is still lacking. Similar impurity screening phenomena have been found as a result of strong correlations in the metallic state of the Hubbard model [25].

In weakly interacting \( d \)-wave superconductors, Abrikosov-Gor’kov (AG) theory predicts that a tiny amount of non-magnetic impurities should bring the transition temperature \( T_c \) to zero as shown in Table 1.1. In the case of the cuprates, however, experiments have shown that these \( d \)-wave superconductors are very robust against disorder [16, 26, 17, 9]. This feature was frequently ascribed to scattering by charged off-plane impurities, which is mostly in the forward direction.
Table 1.1: Effects of potential and magnetic scattering for different types of superconductors are provided. “+” indicates the impurity is pair breaking while “−” implies that it is not. At high enough concentration of impurity, superconductivity would be suppressed in all cases.

|                      | s | p | d |
|----------------------|---|---|---|
| Magnetic scattering   | + | + | + |
| Non-magnetic scattering| −| + | + |

(see, e.g., [27]). The puzzle was partially clarified, however, once strong electronic interactions were shown to give rise to the impurity screening effects seen in these experiments, as captured by the Gutzwiller-projected wave function [22, 23, 28, 24, 29].

Numerically, exact diagonalization has been utilized to study the disorder effect in the superconducting state, while the main drawback is the inability to deal with a macroscopic system with a large amount of lattice sites. Additionally, solving Bogoliubov–de Gennes and Andreev (BdG) equations self-consistently has also been applied [30, 31]. However, recent findings showed a drastic contrast between the simple BdG method and the BdG plus correlations within the Gutzwiller approximation [24]. In this work, A. Garg et al. generalized the conventional theory including the correlation effects by projecting the ground state wave function to the restricted subspace, where double occupancy of electrons is not allowed on any lattice site. Remarkable results show that the impurity effects are largely screened in the $t−t′−J$ model as shown in Fig.(1.1). Comparing (a) with (b) of Fig.(1.1), they unambiguously show how impurity effects are screened when correlation effects are taken into account.

Figure 1.1: Spatial variations of the normalized local pairing amplitude in cases of with correlations (left panel) and without correlations (right panel) (A. Garg et al., 2008).
Figure 1.2: Three gap function maps are measured on samples with three different oxygen doping levels as shown in plots A, B and C respectively. The field of view size is identical and corresponds to 16,000 CuO$_2$ plaquettes approximately. Average gap magnitudes $\bar{\Delta}$ are shown at the top, together with values of $N$, which is the total number of dopant atoms shown as white circles. Curves in plot D represent the local density of states at different locations (K. McElroy et al., 2005).

Taking Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ as an example \cite{17}, the experimental results confirm that the $d$ wave superconductivity is indeed robust against the doping disorder. In (D) of Fig.(1.2), density of states curves clearly indicate the significant spectral weight redistribution and the strong coherence peak modification in high energy range. On the contrary, weak scatterings between quasi-particles are dominant in low energy scale seem to be protected from disorder effects. Similar phenomena are common in all nonstoichiometric oxygen-doped high-Tc cuprates and such energy-resolved inhomogeneity might be a robust and general feature of disordered Mott systems.

Analytically, it has been shown that, similar to the numerical results obtained from the $t-t'-J$ model, the impurity screening effect is also found in the Hubbard model. Following the Kotliar’s representation of four slave boson scheme \cite{32}, it has been shown that we can obtain analytical results regarding the spatial fluctuations for different physical quantities at the long wavelength limit.
Considering the Hubbard model on a square lattice at half-filling, there exists a metal-insulator transition when the interaction strength $U$ is above the critical value $U_c$. From Fig.(1.3), as we approach the critical point, it is clear that the density fluctuation is suppressed and the correlation length for quasiparticle weight $Z_i$ diverges. Starting from half-filling, the finite impurity potential tends to push the site occupancy away from 1 and the system becomes locally more metallic by increasing the quasiparticle weight $Z_i$ \[25\].

Despite this progress, it would be desirable to understand to what extent this disorder screening is due only to the presence of strong correlations or whether it is dependent on the details of the particular model or system. The main difference separating cuprates from other materials is due to the comparable large $J$ term ($J/t = 0.3$). Thus, we shall start from the $t - J$ model \[33\] which is then solved via a spatially inhomogeneous slave boson treatment quantitatively \[34, 32, 8, 1, 35\].

At this stage, we shall briefly introduce the concept and main usage of the slave boson method without technical details. This method is a useful tool for analyzing strong correlation effects. In the large-$N$ limit in which there is a controlled treatment, the bosonic field (the holon in this case) condenses at zero temperature. It is well known that, within this approach, the effects of strong correlations are determined by the condensed slave-boson and Lagrange multiplier fields. They
determine, through the constraint, the charge response of the system. In the non-homogeneous situation we are considering, the slave-boson and Lagrange multiplier fields show strong spatial fluctuations and, as a result, the charge response is also non-homogeneous and non-trivial.

Furthermore, the slave-boson treatment provides information on the superconducting gap and quasiparticle spectrum. As has been shown in many contexts (the Kondo effect, heavy fermions, the Mott transition) [8], a condensed holon/charge field (together with the Lagrange multiplier fields) signals the strong suppression of charge fluctuations, perhaps the dominant feature in strongly correlated systems. In the particular case of Mott systems, it is the single most important force leading to the Mott insulating state. As the pairing field and the single-particle Green’s function involve convolutions of the holon and the spinon fields, the fact that the former is condensed simply means that it comes into the convolution as a multiplicative factor. It is precisely this seemingly innocuous multiplicative factor, ultimately coming from the suppression of charge fluctuations, that is responsible for important effects in this theory, including the healing we describe and the Mott transition itself.

Finally, besides a variety of theoretical interests, there are many practical applications of correlated materials with impurities, which includes the resistive switching devices. The resistance of certain oxide films was surprisingly found to switch between low and high values upon the application of external voltage pulses and displays an interesting hysteresis loop as shown in Fig.(1.4). It turns
out that the impurity and correlation also play an important role here. Typically, these materials are required to have a rather high resistance in order to present a resistive switching phenomenon. Mysteriously, the resistance is so high that it even exceeds the “Mott-Ioffe-Regel limit” in many of these systems fabricated by the transition metal oxides. Such metals is often considered as the “bad metal”, which have the resistance scales like $T$ over a wide temperature range and thus indicates the importance of correlations in the underlying electronic state.

The information age we live in is made possible by a physical underlayer of electronic hardware, which originates in condensed matter physics research. Despite the mighty progress made in recent decades, the demand for faster and power efficient devices continues to grow. Progress of silicon based technology is nearing its physical limit, as minimum feature size of components is reaching a mere 10 nm. Thus, there is urgent need to identify novel materials and physical mechanisms for future electronic device applications. The resistive switching behavior of transition metal oxides and the associated memristor device is emerging as a competitive technology for next generation electronics. In this context, transition metal oxides (TMOs) are capturing a great deal of attention for non-volatile memory applications [36]. In particular, TMO are associated to the phenomenon of resistive switching (RS) [37] and the memristor device [38] that is emerging as a competitive technology for next generation electronics [36, 39, 40, 41, 42, 43, 44, 45]. The RS effect is a large, rapid, non-volatile, and reversible change of the resistance, which may be used to encode logic information. In the simplest case one may associate high and low resistance values to binary states, but multi-bit memory cells are also possible [46, 47]. Typical systems where RS is observed are two-terminal capacitor-like devices, where the dielectric might be a TMO and the electrodes are ordinary metals. The phenomenon occurs in a strikingly large variety of systems. Ranging from simple binary compounds, such as NiO, TiO$_2$, ZnO, Ta$_2$O$_5$, HfO$_2$ and CuO, to more complex perovskite structures, such as superconducting cuprates and colossal magneto resistive manganites [37, 40, 42, 41, 48, 49, 50]. From a conceptual point of view, the main challenges for a non-volatile memory are: (i) to change its resistance within nano seconds (required for modern electronics applications), (ii) to be able to retain the state for years (i.e. non-volatile), and (iii) to reliably commute the state hundreds of thousands of times. Through extensive experimental work in the past decade, a consensus has emerged around the notion that the change in resistance is due to migration of ionic species, including oxygen vacancies, across different regions of the device, affecting
the local transport properties of the oxide [51]. In particular, the important role of highly resistive interfaces, such as Schottky barriers, has also been pointed out [42, 43, 52]. In contrast with the experimental efforts, theoretical studies remain relatively scarce [53, 54, 55, 56, 57, 58, 59, 60, 61]. A few phenomenological models were proposed and numerically investigated, which captured different aspects of the observed effects [62, 38, 63, 64]. Significant progress has already been made in the past decade and devices are beginning to hit the market; however, it has been mainly the result of empirical trial and error. Hence, gaining theoretical insight is of essence.
CHAPTER 2

MOTTNESS INDUCED HEALING IN STRONGLY CORRELATED SUPERCONDUCTORS

Despite the progress, it would be desirable to understand to what extent this disorder screening is due only to the presence of strong correlations or whether it is dependent on the details of the particular model or system. For example, are the effects of the inter-site super-exchange, crucial to describe the cuprates, essential for this phenomenon? To address these issues, it would be fruitful to have an analytical treatment of the problem. We will describe in this Letter how an expansion in the disorder potential is able to provide important insights into these questions. In particular, we show that the “healing” of the impurities is a sheer consequence of the strong correlations and depend very little on the symmetry of the superconducting (SC) state or the inclusion of inter-site magnetic correlations.

We considered dilute nonmagnetic impurities in an otherwise homogenous, strongly correlated electronic state. We avoided complications related to the nucleation of possible different competing orders by the added impurities, such as fluctuating or static charge- and spin-density-waves [65, 66, 67, 68] or the formation of local moments [69]. Therefore, we focused only on how a given strongly correlated state readjusts itself in the presence of the impurities. We used a spatially inhomogeneous slave boson treatment [34, 32, 8, 1, 35], which allowed us to perform a complete quantitative calculation. We have allowed for either or both of $d$-wave SC and $s$-wave resonating valence bond (RVB) orders.

Our analytical and numerical results demonstrate that (i) for sufficiently weak correlations we recover the results of the conventional theory [21], in which the variations of the different fields induced by the impurities show oscillations with a long-ranged power-law envelope; (ii) for strong interactions and in several different broken symmetry states, the amplitude of the oscillations is strongly suppressed by a common pre-factor $x$, the deviation from half-filling; (iii) the spatial disturbances of the SC gap are healed over a precisely defined length scale, which does not exceed a
few lattice parameters around the impurities; and (iv) this “healing effect” is intrinsically tied to the proximity to the Mott insulating state, even though it survives up to around 30% doping.

2.1 Model and method

We study the $t-t'-J$ model on a cubic lattice in $d$ dimensions with dilute nonmagnetic impurities

$$H = -\sum_{ij\sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + J \sum_{ij} \mathbf{S}_i \cdot \mathbf{S}_j + \sum_{i}(\epsilon_i - \mu_0)n_i,$$

(2.1)

where $t_{ij}$ are the hopping matrix elements between nearest-neighbor ($t$) and second-nearest-neighbor ($t'$) sites, $c_{i\sigma}^\dagger$ ($c_{i\sigma}$) is the creation (annihilation) operator of an electron with spin projection $\sigma$ at site $i$, $J$ is the super-exchange coupling constant between nearest-neighbor sites, $n_i = \sum_\sigma c_{i\sigma}^\dagger c_{i\sigma}$ is the number operator, $\mu_0$ is the chemical potential and $\epsilon_i$ is the impurity potential. The no double occupancy constraint ($n_i \leq 1$) is implied. We set the nearest-neighbor hopping $t$ as the
energy unit and choose \( t' = -0.25t \). To treat this model, we employ the \( U(1) \) slave boson theory [34, 1, 8, 70]. Details can be found in [8] and we only describe it very briefly here. It starts with the replacement \( c_{i\sigma}^\dagger = f_{i\sigma}^\dagger b_i \), where \( f_{i\sigma}^\dagger \) and \( b_i \) are auxiliary fermionic (spinon) and bosonic fields, and the representation is faithful in the subspace \( n_i \leq 1 \) if the constraint \( \sum_{\sigma} f_{i\sigma}^\dagger f_{i\sigma} + b_i^\dagger b_i = 1 \) is enforced. This is implemented by a Lagrange multiplier \( \lambda_i \) on each site. The \( J \) term is then decoupled by Hubbard-Stratonovich fields in the particle-particle (\( \Delta_{ij} \)) and particle-hole (\( \chi_{ij} \)) channels.

The auxiliary bosonic fields are all treated in the saddle-point approximation: \( \langle b_i \rangle = r_i = \sqrt{Z_i} \) gives the quasiparticle residue, \( \langle \lambda_i \rangle \) renormalizes the site energies and \( \chi_{ij} = \sum_{\sigma} \langle f_{i\sigma}^\dagger f_{j\sigma} \rangle \) and \( \Delta_{ij} = \langle f_i^\dagger f_{j+} - f_i^+ f_j \rangle \) describe, respectively, the strength of a spinon singlet and the pairing amplitude across the corresponding bonds. Note that we do not assume these values are spatially uniform. This treatment is equivalent to the Gutzwiller approximation [3, 24]. In terms of Gorkov’s spinor notation [71] with \( \Psi_i(i\omega_n) = \left[ f_{i\uparrow}^\dagger(i\omega_n) \ f_{i\downarrow}(-i\omega_n) \right] \), where \( \omega_n \) is the fermionic Matsubara frequency, the spinon Green’s function is a \( 2 \times 2 \) matrix: \( [G_{ij}(i\omega_n)]_{ab} = -\langle \Psi_i(i\omega_n) \Psi_j^\dagger(i\omega_n) \rangle_{ab} \).

Defining \( h_{ij} \equiv -t_{ij} \), the saddle-point equations read as follows

\[
\begin{align*}
\chi_{ij} &= 2T \sum_n (G_{ij})_{11}, \quad (2.2) \\
\Delta_{ij} &= -2T \sum_n (G_{ij})_{12}, \quad (2.3) \\
(r_i^2 - 1) &= -2T \sum_n (G_{ii})_{11}, \quad (2.4) \\
\lambda_i r_i &= -2T \sum_{nl} h_{il} r_l (G_{il})_{11} = -\sum_l h_{il} r_l \chi_{il}. \quad (2.5)
\end{align*}
\]

Note that we used Eq. (2.2) in the second equality of Eq. (2.5). At \( T = 0 \) and in the clean limit \( \epsilon_i = 0 \), we have \( Z = Z_0 = x \). The Mott metal-insulator transition is signaled by the vanishing of the quasi-particle weight \( Z_0 \to 0 \) at half-filling. It will be interesting to compare the results of the above procedure with the ones obtained from solving only Eqs. (2.2-2.3) while setting \( Z_i = 1 \) and \( \lambda_i = 0 \). The two sets will be called correlated and non-correlated, respectively. In order to be able to compare them, we set \( J = t/3 \) in the correlated case and adjusted \( J \) in the non-correlated case in such a way that the two clean dimensionful SC gaps coincide, as discussed in reference [24].

10
2.2 Healing

Although the detailed solutions of Eqs. (2.2-2.5) can be straightforwardly obtained numerically, we will focus on the case of weak scattering by dilute impurities and expand those equations up to first order in \( \varepsilon \) around the homogeneous case. It has been shown and we confirm that disorder induces long-ranged oscillations in various physical quantities, specially near the nodal directions in the \( d \)-wave SC state [21]. The linear approximation we employ is quite accurate for these extended disturbances far from the impurities, since these are always small. Besides, it provides more analytical insight into the results.

In general, we can expand the spatial variations of the various order parameters in different symmetry channels through cubic harmonics:

\[
\delta \varphi_{ij} = \sum_g \delta \varphi_{i} \Gamma(g)_{ij} \quad \text{where} \quad \varphi_{ij} = \chi_{ij} \text{ or } \Delta_{ij} \text{ and } \Gamma(g)_{ij} \text{ are the basis functions for cubic harmonic } g \text{ of the square lattice}.
\]

In the current discussion, we choose \( \delta \chi_{ij} = \delta \chi_{i} \Gamma_{s}(s)_{ij} \) and \( \delta \Delta_{ij} = \delta \Delta_{i}(d_{x^2-y^2})_{ij} \), as we are interested in oscillations with the same symmetry as the ground state [34, 1, 8, 70]. We also assume there is no phase difference between order parameters on different bonds linked to same site. Then, we can define “local” spatial variations of the order parameters as

\[
\delta \chi_i \equiv \frac{1}{2t} \sum_j \delta \chi_{ij} \Gamma(s)_{ij} \quad \text{and} \quad \delta \Delta_i \equiv \frac{1}{2t} \sum_j \delta \Delta_{ij} \Gamma(d_{x^2-y^2})_{ij}.
\]

Details of the calculation can be found in Appendix A.

We find that both \( \delta \chi_{ij} \) and \( \delta \Delta_{ij} \), as well as the impurity-induced charge disturbance \( \delta n_i \), are proportional to \( Z_0 = x \), indicating the importance of strong correlations for the healing effect. Indeed, we can trace back this behavior to the readjustment of the \( r_i \) and \( \lambda_i \) fields, as encoded in Eqs. (2.4-2.5). Besides, this \( O(x) \) suppression is a generic consequence of the structure of the mean-field equations and holds for different broken symmetry states, such as the flux phase state, \( s \)-wave superconductivity, etc.

Let us focus in more detail on the spatial variations of the local pairing field \( \delta \Delta_i \). In the first column of Fig. 2.1 we show results for \( \delta \Delta_i \) for three identical impurities. The “cross-like” tails near the nodal directions [72] are conspicuous in the absence of correlations (bottom) but are strongly suppressed in their presence (top). While this suppression is further enhanced as the Mott metal-insulator transition is approached \((x \to 0)\), it is still quite significant even at optimal doping \((x = 0.2)\). This is the “healing” effect previously reported [22, 23, 24]. In order to gain insight into

\[s, d_{x^2-y^2}, d_{xy}, \text{ etc.}, \text{ with basis functions expressed as: } \cos k_x + \cos k_y, \cos k_x - \cos k_y, \text{ and } \sin k_x \sin k_y, \text{ etc.} \]

\[11\]
its underlying mechanism, we look at the spatial correlation function of local gap fluctuations

\[
\left\langle \frac{\delta \Delta_i \delta \Delta_j}{\Delta_0^2} \right\rangle_{\text{disorder}} = f(r_i - r_j),
\]

where the brackets denote an average over disorder, after which lattice translation invariance is recovered. The Fourier transform of \( f(r) \) can be written in the linear approximation as

\[
f(k) = \alpha W^2 S(k),
\]

where \( W \) is the disorder strength, \( \alpha \) depends on the detailed bare disorder distribution, and the “power spectrum” (PS) \( S(k) \) is related to gap linear response function \( M_\Delta(k) \) by \( S(k) = M_\Delta^2(k) \). The latter is defined by Fourier transforming the kernel in \( \delta \Delta_i = \Delta_0 \sum_j M_\Delta(r_i - r_j) \varepsilon_j \), which in turn can be easily obtained from the solution of the linearized equations in Appendix A. Inspired by the strongly localized gap fluctuations at the top left of Fig. 2.1, we define the local component of the PS \( S_{\text{loc}}(k) \equiv M_{\Delta,\text{loc}}^2(k) \), where \( M_{\Delta,\text{loc}}(k) \) is obtained by restricting the lattice sums up to the second nearest neighbor distance \( (\sqrt{2}a) \) in the linearized equations in Appendix A. We also define

\[
S_{\text{nonloc}}(k) = M_{\Delta,\text{nonloc}}^2(k) \equiv [M_\Delta(k) - M_{\Delta,\text{loc}}(k)]^2.
\]

In the last three columns of Fig. 2.1, we show, in this order, \( S(k) \), \( S_{\text{loc}}(k) \), and \( S_{\text{nonloc}}(k) \) for the correlated (top) and non-correlated (bottom) cases at \( x = 0.2 \). Clearly, in the presence of correlations the local PS is characterized by a smooth, spherically symmetric bell-shaped function, whereas the non-local part is highly anisotropic. Besides and more importantly, the non-local PS is negligibly small in the correlated case. The full PS is thus overwhelmingly dominated by the local part, unlike in the non-correlated case. Below, we extend the analysis to the underdoped and overdoped regimes, where very similar behavior is found, even up to dopings of \( x = 0.3 \).

The gap fluctuations \( \delta \Delta_i \) for three impurities and power spectra, for several dopings and in the presence of correlations, are shown in Fig. 2.2. The strong healing in the presence of correlations is conspicuous. It is important to note that this suppression of gap fluctuations is not restricted to small dopings and remains quite strong even at \( x = 0.3 \), where the healing factor does not exceed 3%. As explained in the main text, the healing effect originates in the dominance of the local spherically symmetric part (third column in Fig. 2.2) over the anisotropic non-local response (fourth column in Fig. 2.2).

In order to quantify the localized nature of the healing effect, we are led to a natural definition of a “healing factor” \( h \) in the \( d \)-wave SC state.
Figure 2.2: Spatial variations of normalized local gap function $\frac{\delta \Delta}{\Delta_0}$ for three impurities (first column) and the corresponding power spectra $S(k)$, $S(k)_{loc}$ and $S(k)_{nonloc}$ (second to fourth columns) for $x = 0.15$ (first row), $x = 0.25$ (second row), and $x = 0.3$ (third row). The corresponding healing factors are (a) $h = 0.23\%$, (b) $h = 1.77\%$, and (c) $h = 2.74\%$. 
Figure 2.3: Left panel: the healing factor $h$ as a function of doping in the uncorrelated case (blue curve with squares), in the correlated case (red curve with circles), and in the correlated case without $\delta \chi_i$ fluctuations (green curve with diamonds). Right panel: doping dependence of the SC ($\xi_S$, red curve with circles) and normal state ($\xi_N$, blue curve with squares) healing lengths. The green curve with diamonds gives $\xi_S$ calculated within the minimal model (see text).

$$h = \frac{\int S_{\text{nonloc}}(k) \, d^2k}{\int S_{\text{loc}}(k) \, d^2k}, \quad (2.8)$$

where the integration is over the first Brillouin zone. It measures the relative weight of non-local and local parts of the gap $PS$. The healing factor as a function of doping is shown on the left panel of Fig. 2.3 for the non-correlated (blue) and correlated (red) cases. The contrast is striking. When correlations are present, $h$ is extremely small up to 30% doping and the gap disturbance is restricted to a small area around the impurities. In contrast, without correlations significant pair fluctuations occur over quite a large area for all dopings shown. We conclude that the strong dominance of the local part over the highly anisotropic non-local contribution caused by correlations is the key feature behind the healing process.

The shape of $S_{\text{loc}}(k)$ shows that the gap disturbance created by an impurity is healed over a well-defined distance, the “healing length” $\xi_S$. This length scale can be obtained by expanding the inverse of $M_{\Delta,\text{loc}}(k)$ [or equivalently $M_{\Delta}(k)$] up to second order in $k^2$, thus defining a Lorentzian in $k$-space

$$M_{\Delta,\text{loc}}(k) \approx \frac{1}{A + Bk^2}. \quad (2.9)$$

The SC healing length is then given by $\xi_S = \sqrt{B/A}$. The $x$ dependence of $\xi_S$ is shown in red on the right panel of Fig. 2.3. It is of the order of one lattice spacing in the relevant range $0.15 < x < 0.3$. It should be noted that precisely the same length scale also governs the healing of charge fluctuations.
in the SC state, showing that this phenomenon is generic to the strongly correlated state. A similar
procedure can be carried out for the charge fluctuations in the normal state, thus defining a normal
state healing length $\xi_N$. The blue curve of the right panel of Fig. 2.3 shows the $x$ dependence of $\xi_N$,
which is also of the order of one lattice spacing.

2.3 Mottness-induced healing

The healing effect we have described comes almost exclusively from the $\delta r_i$ and $\delta \lambda_i$ fluctuations:
$h$ is hardly affected by the $\delta \chi_i$ field. If we suppress the $\delta \chi_i$ fluctuations completely, there is only
a tiny change in the results, as shown by the green curve of the left panel of Fig 2.3. The same
is not true, however, if we turn off either $\delta r_i$ or $\delta \lambda_i$ or both. We conclude that the healing effect
in the $d$-wave SC state originates from the strong correlation effects alone, rather than the spinon
correlations.

Within the linear approximation we are employing, all fluctuation fields ($\delta \Delta$, $\delta r$, etc.) are
proportional, in $k$-space, to the disorder potential $\varepsilon (k)$. Therefore, they are also proportional to
each other. In particular, given the centrality of the strong correlation fields, it is instructive to
write the gap fluctuations in terms of the slave boson fluctuations

$$\delta \Delta (k) = -2\chi_{pc}(k) r \delta r (k) = \chi_{pc}(k) \delta n (k).$$

In the last equality, we used $n_i = 1 - r_i^2$, which enables us to relate two physically transparent
quantities: the gap and the charge fluctuations. Indeed, this will provide crucial physical insight
into the healing process. By focusing on the linear charge response to the disorder potential $\delta n (k) =
n_0 M_n (k) \varepsilon (k)$, we can, in complete analogy with the gap fluctuations, define a PS for the spatial
charge fluctuations, $N (k) = M_n^2 (k)$. This PS can also be broken up into local $[N_{loc}(k) = M_{n,loc}^2 (k)]$
and non-local $\{N_{nonloc}(k) = [M_n (k) - M_{n,loc}(k)]^2\}$ parts, as was done for the gap-fluctuation PS.
These two contributions, obtained from the solution of the full linearized equations, are shown in
Fig. 2.4. The charge PS in the correlated $d$-wave SC state is also characterized by a smooth, almost
spherically symmetric local part and a negligibly small anisotropic non-local contribution. Note also
the strong similarity between the local PS for gap (top row of Fig. 2.1) and charge fluctuations. This
shows a strong connection between the gap and charge responses. Evidently, this is also reflected in
real space, where the charge disturbance is healed in the same strongly localized fashion as the gap

15
disturbance. In fact, the local part of the charge response function \( M_{n,\text{loc}}(k) \) can be shown to be well approximated by a Lorentzian and we can write for small \( k \)

\[
\delta \Delta_{\text{loc}}(k) \approx -\chi_{pc}(k = 0) \frac{8r^2/\lambda}{k^2 + \xi_S^{-2}} \varepsilon(k),
\]

where the SC healing length \( \xi_S \) can be expressed in terms of the Green’s functions of the clean system in Appendix A. The relations implied by Eqs. (2.10) and (2.11), as well as the doping dependence of the quantities in them, could be tested in STM studies and would constitute an important test of this theory.

Eqs. (2.10-2.11) allow us to obtain a clear physical picture of the healing mechanism. The spatial gap fluctuations can be viewed as being ultimately determined by the charge fluctuations. Furthermore, their ratio \( \chi_{pc}(k) \), which is essentially a pair-charge correlation function, is a rather smooth function of order unity, only weakly renormalized by interactions. Therefore, it is the strong suppression of charge fluctuations by “Mottness”, as signaled by the \( r^2 \) factor in Eq. (2.11), which is behind the healing of gap fluctuations. This elucidates the physics of healing previously found numerically [22, 23, 24]. It also suggests that the healing phenomenon is generic to Mott systems [25] and is not tied to the specifics of the cuprates.

2.4 A minimal model

Interestingly, the crucial role played by the strong correlation fields \( (r_i \text{ and } \lambda_i) \) suggests a “minimal model” (MM) for an accurate description of the healing process, which we define as follows: (i) the spatially fluctuating strong correlation fields \( r_i \text{ and } \lambda_i \) are first calculated for the self-consistently determined, fixed, uniform \( \Delta \text{ and } \chi \), and then (ii) the effects of their spatial readjustments are fed back into the gap equation (2.3) in order to find \( \delta \Delta_i \) as shown in Appendix B. The accuracy of this procedure can be ascertained by the behavior of the healing factor: it is numerically indistinguishable from the green curve of the left panel of Fig. 2.3. Furthermore, the value of \( \xi_S \) calculated within the MM differs from the one obtained from the solution of the full linearized equations by at most 20% (red and green curves on the right panel of Fig. 2.3). Besides its accuracy, the advantage of this MM description lies in the simplicity of the analytical expressions obtained. As shown in Appendix B, it provides simple expressions for the important quantities \( \chi_{pc}(k) \) and \( \xi_S \).
2.5 Conclusions

In this work, we have found an inextricable link between the healing of gap and charge disturbances in strongly correlated superconductors, suggesting that this phenomenon is generic to any system close to Mott localization. An important experimental test of this link would be provided by STM studies of the organic superconductors [19] and maybe the pnictides [20]. Whether it is also relevant for heavy fermion systems [73] is an open question left for future study.
In this part, we provide a route to understand both qualitatively and quantitatively whether disorder screening has any significant influence on $T_c$ as well as on the normal state transport properties. The transition temperature in the under-doped region of the hole-doped cuprates is believed to be influenced by phase fluctuations, various types of competing orders (such as charge- and spin-density waves), stripe formation, etc. Consequently, impurities act as nucleations centers, which complicates the analysis considerably. In the over-doped region, however, $T_c$ is dominated by the superconducting gap opening, thus offering a particularly favorable window into the interplay between disorder and interactions.

In the presence of impurities, the strongly correlated state readjusts itself and creates a renormalized disorder potential. In the dilute limit, the AG theory can be extended to describe the effect of this renormalized potential on $T_c$ degradation and transport properties. We will describe in this part how electronic interactions lead to a much slower decrease of $T_c$ as compared to the weak-coupling theory. Our results demonstrate that (i) this effect is intrinsically tied to the proximity to the Mott insulating state, although it is significant even above optimal doping; (ii) the doping dependence of normal state resistivity is different from that of the pair-breaking scattering rate, which governs $T_c$; and (iii) the softening of the disorder potential by interactions leads to a strong enhancement of the forward scattering amplitude.

3.1 Model and method

We start with the $t - J$ model on a cubic lattice in $d$ dimensions with dilute nonmagnetic impurities
\[ H = -t \sum_{\langle ij \rangle} c_{i\sigma}^\dagger c_{j\sigma} + J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + \sum_i (\epsilon_i - \mu_0) n_i, \]  

(3.1)

where \( c_{i\sigma}^\dagger (c_{i\sigma}) \) is the creation (annihilation) operator of an electron with spin projection \( \sigma \) on site \( i \), \( t \) is the hopping matrix element between nearest neighbors, \( J \) is the super-exchange coupling constant between nearest-neighbor sites, \( n_i = \sum_\sigma c_{i\sigma}^\dagger c_{i\sigma} \) is the number operator, \( \mu_0 \) is the chemical potential. The no-double-occupancy constraint \( n_i \leq 1 \) is implied. We work in units such that \( \hbar = k_B = a = 1 \), where \( a \) is the lattice spacing and the total number of lattice sites is \( V \). For definiteness, we will set \( J = t/3 \). The impurities are taken into account through a random on-site potential described by \( \epsilon_i \). We use a model of disorder in which we set the potential \( \epsilon_i = t \) and randomly place the impurities on lattice sites with \( n \) impurities per unit volume and no correlations between their positions. Note that this model assumes random non-magnetic scattering but does not describe the removal of magnetic ions. We will focus on the two-dimensional case relevant to the cuprates, but our results are easily generalizable to higher dimensions with few modifications.

We proceed with \( U(1) \) slave boson theory, details of which can be found in [34, 32, 8, 1, 35]. Briefly, it starts with the replacement \( c_{i\sigma}^\dagger \rightarrow f_{i\sigma}^\dagger b_i \), where \( f_{i\sigma}^\dagger \) and \( b_i \) are auxiliary fermionic (spinon) and bosonic (slave boson) fields. This substitution is faithful if the constraint \( n_i \leq 1 \) is replaced by \( \sum_\sigma f_{i\sigma}^\dagger f_{i\sigma} + b_i^\dagger b_i = 1 \). The latter is enforced through Lagrange multiplier fields \( \lambda_i \) on each site. The \( J \) term is then decoupled through additional Hubbard-Stratonovitch bosonic fields in the particle-particle (\( \Delta_{ij} \)) and particle-hole (\( \chi_{ij} \)) channels. The auxiliary bosonic fields are all treated in the saddle-point approximation, which here is \( \text{spatially inhomogeneous} \) due to the presence of disorder: \( \langle b_i \rangle = r_i \), which governs the local quasiparticle residue \( Z_i = r_i^2 \), \( \langle \lambda_i \rangle \) (we will denote it simply by \( \lambda_i \)) which renormalizes the site energies and \( \chi_{ij} = \sum_\sigma \langle f_{i\sigma}^\dagger f_{j\sigma} \rangle \) and \( \Delta_{ij} = \langle f_i^\uparrow f_j^\downarrow - f_i^\downarrow f_j^\uparrow \rangle \), which describe, respectively, the strength of a spinon singlet and the pairing amplitude across the corresponding bonds. We also made the change \( J \rightarrow \tilde{J} = \frac{3}{8} J \). This choice is made so that the saddle-point approximation of the above multi-channel Hubbard-Stratonovitch transformation coincides with the mean-field results [8]. We note, however, that the usual choice \( \tilde{J} = \frac{1}{4} J \) would give rise to hardly noticeable changes in the numerical results. We stress that the \( f \)-electrons mentioned throughout the text are only auxiliary fermions, usually called spinons, rather than the physical electrons. They are related at the saddle point by \( c_{i\sigma}^\dagger = r_i f_{i\sigma}^\dagger \). Note that the non-trivial effects of
this work come from the self-consistent spatial readjustments of the condensed fields to the disorder potential.

In the clean limit ($\epsilon_i = 0$) and in the saddle-point approximation, the bosonic fields are spatially uniform: $r_i = r_0$, $\lambda_i = \lambda_0$, $\chi_{ij} = \chi \Gamma_s(i,j)$ and $\Delta_{ij} = \Delta_0 \Gamma_d(i,j)$. Here, $\Gamma_{s,d}(i,j)$ are the real space cubic harmonics which, in $k$-space, are given by $\Gamma_s(k) = 2(\cos k_x + \cos k_y)$ and $\Gamma_d(k) = 2(\cos k_x - \cos k_y)$. As the doping level (measured with respect to half-filling) $x = 1 - \sum_i n_i/V = r_0^2$ is increased, the slave boson condensation temperature $T_b$ increases monotonically from zero whereas the $\Delta$ field condenses at a transition temperature $T_\Delta$ which decreases monotonically from a finite value at $x = 0$ to zero at an upper doping level $x_{\text{max}} [1, 8]$. The two curves meet at optimal doping $x_{\text{opt}}$. The dome below the two curves is the superconducting dome. Our focus in this paper is on the overdoped region $x > x_{\text{opt}}$, in which the superconducting transition temperature $T_c = T_\Delta < T_b$.

Within this spatially inhomogeneous theory, we are able to perform a complete quantitative calculation of the effective disorder potential. Details have been explained elsewhere [28]. Here, we will focus on the effects of disorder on the superconducting transition temperature $T_c$ and on transport properties in the correlated normal state for $T \geq T_c$ in the over-doped region. For this purpose, we can set $\Delta_{ij} = 0$. Moreover, in this range of temperature and dopings the other bosons, $r_i$, $\lambda_i$ and $\chi_{ij}$, are thoroughly condensed and therefore fairly insensitive to finite temperature effects. We are thus justified in approximating them by their zero-temperature values.

We will focus on the case of weak scattering by dilute impurities $n \ll 1$, where a linear response theory is sufficient. In other words, we calculate the spatial fluctuations of the various condensed fields to first order in the perturbing potential $\epsilon_i$ [28]. Extensive numerical calculations carried out both in the normal and in the superconducting states have shown that the crucial spatial fluctuations come from the $\lambda_i$ and $r_i$ fields whereas fluctuations of $\chi_{ij}$ play only a negligible role [28]. We will thus simply fix $\chi_{ij}$ at its clean limit value $\chi$ while allowing for the full self-consistent spatial adjustment of the $\lambda_i$ and $r_i$ fields to the disordered situation.

Given this setup, the superconducting transition at $T_c$ corresponds to the formation of the order parameter $\Delta_{ij} = \langle f_{i\uparrow} f_{j\downarrow} - f_{i\downarrow} f_{j\uparrow} \rangle$. The condensing $f$-electrons, on the other hand, are governed, in the clean limit by a dispersion relation renormalized by the slave boson fields $r_0$ and $\chi$, $\tilde{\epsilon}(k) \equiv - \left( tx + \tilde{J} \chi \right) \Gamma_s(k)$ and a renormalized chemical potential $\mu_0 - \lambda_0 \equiv -\nu_0$. This theory, therefore, describes a BCS-type condensation of the $f$-electrons. In the presence of disorder, the various fields
will readjust themselves. The effect of dilute identical non-magnetic impurities on \( T_c \) will therefore be captured within the Abrikosov-Gor’kov (AG) theory \cite{74}. In that theory, the only input needed is the scattering \( T \)-matrix due to a single impurity. For that purpose, we place a single impurity at the lattice origin \( \epsilon_i = t \delta_{i,0} \). Crucially, however, the \( \lambda_i \) and \( r_i \) fields will differ from their clean-limit value inside an extended region around the impurity, not only at the origin. The effective \( T \)-matrix will thus reflect this non-trivial rearrangement. As shown in reference \cite{28}, the impurity potential is “healed” within a length scale of a few lattice parameters, the so-called healing length. Furthermore, it was shown that the healing process/length is strongly influenced by electronic correlations and ‘Mottness’, even up to dopings \( x \approx 0.3 \). Therefore, as will be shown, the effective scattering will be strongly suppressed relative to the non-correlated case.

We also look at the transport properties in the normal state around \( T_c \). Again, the AG analysis can be straightforwardly applied in our case. The relevant input for the calculation of the resistivity is the physical electron scattering \( T \)-matrix for a single impurity.

### 3.2 Analytical results

#### 3.2.1 The gap equation in AG theory

Within the \( U(1) \) slave boson theory for the \( t-J \) model of the cuprates, the superconducting transition in the over-doped regime is described by the usual BCS equation for a \( d \)-wave superconductor in which it is the spinons (the auxiliary \( f \)-fermions) which pair to form the condensate \cite{8}. In other words, the transition is signaled by a non-zero value of the \( d \)-wave pairing order parameter \( \Delta_{ij} = \langle f_{i\uparrow} f_{j\downarrow} - f_{i\downarrow} f_{j\uparrow} \rangle \). We remind the reader that at the transition the other auxiliary fields (\( r_i, \lambda_i, \chi_{ij} \)) are well formed. The effects of non-magnetic impurities on \( T_c \) can then be treated by using the AG theory \cite{74}.

Below, we discuss briefly the derivation of the linearized gap equation following from the theory of superconducting alloys \cite{75}. As the pairing field \( \Delta_{ij} \) is assumed to be small near the transition temperature \( T_c \), we are justified to keep it up to the linear order throughout the calculation. In superconducting state, there are Green’s function and two anomalous Green’s functions involved. Therefore, the diagrammatic approach is well suited in this case to study the disorder effects.

With the help of diagrammatic tricks, it is straightforward to keep the leading order of the anomalous green’s functions in the Dyson equation according to (a) of Fig.(3.1). Furthermore, the
(a) Diagrammatic representation of the Dyson equations for the green's function and anomalous green's functions in superconducting state, where impurity scattering events are indicated by the dashed line.

(b) Diagrammatic form of corrections to each of the two green's functions (left) and “vertex” corrections (right)

Figure 3.1: Feynman diagrams in superconducting state with impurities. (L.P. Gor’kov 2008, Chapter 5 of Karl Heinz Bennemann et al., 2008)
main physical processes are summarized in (b) of Fig. (3.1), namely, (i) each of the green’s functions
for paring process are independently renormalized by all possible impurity scattering diagrams
which is non-crossing, (ii) the impurity scattering events happens between the two spinons which
pair to form the condensate.

Therefore, within standard AG theory, the linearized gap equation is written as [75]

\[
\Delta_0 = \frac{2\tilde{J}kT_c}{d} \sum_{\omega_n} \frac{d^2k}{(2\pi)^2} G^f(k, -i\omega_n) G^f(-k, i\omega_n) \Lambda(k, i\omega_n) \Gamma_d(k),
\]

(3.2)

where \(\Delta_0\) is the superconducting gap amplitude, \(\omega_n = (2n + 1)\pi T_c\), and \(\Lambda(k, i\omega_n)\) is the vertex
correction function, which satisfies

\[
\Lambda(k, i\omega_n) = \Delta_0 \Gamma_d(k) + n \int \frac{d^2k'}{(2\pi)^2} \left| \langle k | T^f | k' \rangle \right|^2 G^f(k', -i\omega_n) G^f(-k', i\omega_n) \Lambda(k', i\omega_n).
\]

(3.3)

In the last equation, \(\langle k | T^f | k' \rangle\) is the single-impurity scattering \(T\)-matrix for the spinons. The
order-averaged spinon Green’s function in Eqs. (3.2) and (3.3) is [74]

\[
G^f(k, i\omega_n) = \frac{1}{i\omega_n \left( 1 + \frac{1}{2\tau_{k|\omega_n|}} \right) - \tilde{h}(k) - \nu_0},
\]

(3.4)

where \(\tilde{h}(k) = -(tx + J\chi) \Gamma_s(k)\) is the renormalized dispersion, \(\nu_0 = \lambda_0 - \mu_0\) is the negative value
of the renormalized chemical potential, which controls the doping level, and

\[
\frac{1}{\tau_k} \equiv 2\pi n \int \frac{d^2k'}{(2\pi)^2} \left| \langle k | T^f | k' \rangle \right|^2 \delta \left( \tilde{h}(k) - \tilde{h}(k') \right),
\]

(3.5)

is the quasiparticle scattering rate.

Following the arguments in [74, 75], Eq. (3.3) can be solved to first order to give

\[
\Lambda(k, i\omega_n) = \Delta_0 \left[ \Gamma_d(k) + \frac{1}{2\tau^d_{k|\omega_n|}} \left( 1 + \frac{1}{2\tau_{k|\omega_n|}} \right) \right],
\]

(3.6)

where

\[
\frac{1}{\tau^d_k} \equiv 2\pi n \int \frac{d^2k'}{(2\pi)^2} \left| \langle k | T^f | k' \rangle \right|^2 \delta \left( \tilde{h}(k) - \tilde{h}(k') \right) \Gamma_d(k'),
\]

(3.7)

is a different scattering rate. We will show later that [see Eq. (3.34)], when calculated at the
approximately circular Fermi surface \(k = k_F \hat{k}\), the quasiparticle scattering time is essentially
isotropic \(\tau_k \approx \tau\), whereas

\[
\frac{1}{\tau^d_k} = \frac{1}{\tau^d} \Gamma_d(k).
\]

(3.8)

23
We can thus write
\[ \Lambda(k, i\omega_n) = \Delta_0 \Gamma_d(k) \left[ 1 + \frac{1}{2\tau^d |\omega_n| \left( 1 + \frac{1}{2\tau^d |\omega_n|} \right)} \right] \]
Eq. (3.9) can now be solved to all orders by noting that
\[ \Lambda(k, i\omega_n) = \Delta_0 \Gamma_d(k) \left\{ 1 + \frac{1}{2\tau^d |\omega_n| \left( 1 + \frac{1}{2\tau^d |\omega_n|} \right)} \right\}^2 + \ldots \]
\[ = \Delta_0 \Gamma_d(k) \left[ \frac{1 - \frac{1}{2\tau^d |\omega_n| \left( 1 + \frac{1}{2\tau^d |\omega_n|} \right)}}{1 - \frac{1}{2\tau^d |\omega_n| \left( 1 + \frac{1}{2\tau^d |\omega_n|} \right)}} \right]^{-1} \]
\[ = \Delta_0 \Gamma_d(k) \frac{|\omega_n| + \frac{1}{2\tau^d}}{|\omega_n| + \frac{1}{2\tau^d} - \frac{1}{2\tau^d}}. \] (3.10)
Plugging this result into Eq. (3.2), we find that \( T_c \) is determined by:
\[ 1 = \frac{2\tilde{J}kT_c}{d} \sum_{i\omega_n} \frac{d^2k}{(2\pi)^2} g^f(k, i\omega_n) g^f(-k, i\omega_n) \frac{|\omega_n| + \frac{1}{2\tau^d} - \frac{1}{2\tau^d}}{|\omega_n| + \frac{1}{2\tau^d} - \frac{1}{2\tau^d}} \Gamma_d^2(k). \] (3.11)
The momentum integral is, as usual, dominated by the region close to the renormalized Fermi surface, which we assume to be approximately circular in the over-doped region. We thus get, using polar coordinates in the \((k_x, k_y)\)-plane,
\[ 1 = \frac{\tilde{J}kT_c}{d} \sum_{i\omega_n} \int \frac{d\theta}{2\pi} \frac{1}{|\omega_n| + \frac{1}{2\tau^d} - \frac{1}{2\tau^d}} \Gamma_d^2(k_F \hat{k}). \] (3.12)
Using now \( \Gamma_d(k_F \hat{k}) = 2(\cos k_x - \cos k_y) \approx k_x^2 - k_y^2 \approx k_F^2 \cos(2\theta) \) we get
\[ 1 = \frac{\tilde{J}kT_c m^* k_F^2}{d} \sum_{n \geq 0} \frac{1}{\omega_n + \frac{1}{2\tau^d} - \frac{1}{2\tau^d}}. \] (3.13)
As usual, the integral is formally divergent, but by comparing with the equally divergent expression for the clean transition temperature \( T_{c0} \), we can get the ratio of clean \( (T_{c0}) \) to dirty \( (T_c) \) transition temperatures [74]
\[ \ln \frac{T_{c0}}{T_c} = \psi \left( \frac{1}{2} + \frac{\alpha}{2} \right) - \psi \left( \frac{1}{2} \right), \] \[ \alpha = \frac{1}{2\pi T_c} \left( \frac{1}{\tau^d} - \frac{1}{\tau} \right) \equiv \frac{1}{2\pi T_c \tau_{pd}}. \] The leading behavior is
\[ T_c = T_{c0} - \frac{\pi}{8\tau_{pb}}. \]  

The relevant scattering rates \( \tau \) and \( \tau_d \) will be calculated in the next Section.

### 3.2.2 T-matrix for the spinons and the pair-breaking scattering rate

As seen in Section 3.2.1, the suppression of the superconducting transition temperature \( T_c \) by disorder requires the determination of the scattering \( T \)-matrix of the \( f \) fermions. We will find it to first order in the disorder. In other words, the fields \((r_i, \lambda_i)\) will be calculated to \( O(\epsilon_i) \). We define the renormalized site energy for the \( f \) electrons as \( \nu_i \equiv \epsilon_i - \mu_0 + \lambda_i \), whose clean value limit is \( \nu_0 = \lambda_0 - \mu_0 \). Clean and disordered \( f \)-fermion Green’s functions are given by, respectively,

\[
\begin{align*}
G_f^{-1} &= i\omega_n 1 + r_0^2 \tau^s - v_0 1 + J_\chi \Gamma_s, \\
G_f^{-1} &= i\omega_n 1 - v + t r \tau^s r + J_\chi \Gamma_s,
\end{align*}
\]

where we have used boldface to denote matrices in the lattice site basis, whose elements are \( 1_{ij} = \delta_{ij} \), \( r_{ij} = r_i \delta_{ij} \), \( v_{ij} = v_i \delta_{ij} \), and \((\Gamma^s)_{ij}\) is equal to 1 if sites \( i \) and \( j \) are nearest neighbors and zero otherwise. We remind the reader that we are neglecting spatial fluctuations of the \( \chi_{ij} \) field. The spinon \( T \)-matrix is defined through

\[
G_f = G_f^0 + G_f^0 T_f G_f^0 = G_f^0 \left( 1 + T_f G_f^0 \right),
\]

from which we obtain

\[
G_f^{-1} = \left( 1 + T_f G_f^0 \right)^{-1} G_f^{-1},
\]

and, to first order in the disorder,

\[
G_f^{-1} \approx \left( 1 - T_f G_f^0 \right) G_f^{-1} = G_f^{-1} - T_f.
\]

Thus, again to first order,

\[
T_f = G_f^{-1} - G_f^{-1} = (\nu - v_0 1) - t r \Gamma_s r + r_0^2 \tau^s \\
= \delta \nu - t r_0 (\delta \tau^s + \Gamma^s \delta r),
\]

\[ (3.21) \]
where $\delta r \equiv (r - r_0) \mathbf{1}$ and $\delta v \equiv (v - v_0) \mathbf{1}$. Defining $\delta v_i \equiv \epsilon_i + \lambda_i - \lambda_0$ and $\delta r_i = r_i - r_0$, we have, in components,

$$T_{ij}^f = \delta v_i \delta_{ij} - r_0 (\delta r_i + \delta r_j) t_{ij}. \quad (3.22)$$

All we need now is to find $\delta r_i$ and $\delta v_i$ to first order in $\epsilon_i$. This was already obtained in reference [28] (see, in particular, the Supplemental Material). After setting in those equations the gap and its fluctuations to zero $\Delta = \delta \Delta = 0$ (normal state) and $\delta \chi = 0$ (as being negligible), we obtain in $\mathbf{k}$-space

$$\Pi^a(\mathbf{k}) \delta v(\mathbf{k}) + r_0 \left[ 1 + \Pi^b(\mathbf{k}) \right] \delta r(\mathbf{k}) = 0, \quad (3.23)$$

$$\left[ \lambda_0 - \frac{\lambda_0}{2d} \Gamma_s(\mathbf{k}) \right] \delta r(\mathbf{k}) + r_0 \delta v(\mathbf{k}) = r_0 \epsilon(\mathbf{k}), \quad (3.24)$$

where

$$\Pi^a(\mathbf{k}) = \frac{1}{V} \sum_{\mathbf{q}} \frac{f \left[ \tilde{h}(\mathbf{q} + \mathbf{k}) \right] - f \left[ \tilde{h}(\mathbf{q}) \right]}{h(\mathbf{q} + \mathbf{k}) - h(\mathbf{q})}, \quad (3.25)$$

$$\Pi^b(\mathbf{k}) = \frac{1}{V} \sum_{\mathbf{q}} \frac{f \left[ \tilde{h}(\mathbf{q} + \mathbf{k}) \right] - f \left[ \tilde{h}(\mathbf{q}) \right]}{h(\mathbf{q} + \mathbf{k}) - h(\mathbf{q})} \left[ h(\mathbf{q} + \mathbf{k}) + h(\mathbf{q}) \right], \quad (3.26)$$

where $d$ is the lattice dimension ($d = 2$, for our purposes), $f(x)$ is the Fermi-Dirac function, $\Gamma_s(\mathbf{k}) = 2(\cos k_x + \cos k_y)$, and $h(\mathbf{k}) = -t \Gamma_s(\mathbf{k})$ is the bare energy dispersion. Solving Eqs. (3.23)-(3.24) for $\delta r(\mathbf{k})$ and $\delta v(\mathbf{k})$

$$\delta v(\mathbf{k}) = -x \frac{\Pi(\mathbf{k}) \epsilon(\mathbf{k})}{\lambda_0 - \frac{\lambda_0}{2d} \Gamma_s(\mathbf{k}) - x \Pi(\mathbf{k})}, \quad (3.27)$$

$$\delta r(\mathbf{k}) = r_0 \frac{\epsilon(\mathbf{k})}{\lambda_0 - \frac{\lambda_0}{2d} \Gamma_s(\mathbf{k}) - x \Pi(\mathbf{k})}, \quad (3.28)$$

where we used $x = r_0^2$ and defined

$$\Pi(\mathbf{k}) \equiv \frac{1 + \Pi^b(\mathbf{k})}{\Pi^a(\mathbf{k})}. \quad (3.29)$$

Therefore
\begin{align}
\langle k | T^f | k' \rangle &= \delta v (k' - k) + r_0 [h(k) + h(k')] \delta r (k' - k) \\
&= x \left[ \frac{h(k) + h(k')}{{\lambda}_0 - \frac{{\lambda}_0}{2 \Gamma_s (k' - k) - x \Pi (k' - k)}} \right] \epsilon (k' - k).
\end{align}

The result in Eq. (3.31) is general. For the \( T_c \) calculation within the AG theory, we only need it for a single impurity [see Eq. (3.3)]. We therefore set \( \epsilon_i = t \delta \epsilon_{i0} \) or \( \epsilon(k) = t \). We must now plug Eq. (3.31) into Eqs. (3.5) and (3.7). Since the superconducting pairing is mostly affected by the scattering near Fermi surface, we can set \( k = k_F \hat{k} \) in Eq. (3.31). For computations, \( k_F \) is taken as the magnitude of the Fermi momentum averaged over the approximately circular Fermi surface. We can thus make the following simplifications:

\begin{align}
&h(k) + h(k') = 2 E_F, \\
&\Pi (k') = 2 E_F \Pi (k), \\
&\Pi (k) = [\Pi (k)]^{-1} + 2 E_F, \text{ and } \Pi (k - k') = -\rho^* g_L (y), \text{ where } y = \frac{k - k'}{2k_F} = | \sin \left( \frac{\varphi}{2} \right) |, \varphi = \theta - \theta' \text{ is the angle between } k \text{ and } k', \ E_F \text{ is the bare Fermi energy (obtained by solving the mean-field equations with the constraint of an electron filling of } 1 - x), \text{ the function } g_L (y) \text{ is defined as} [76]
\end{align}

\begin{equation}
g(y) \equiv \begin{cases} 1 & y \leq 1, \\
1 - \sqrt{1 - y^2} & y > 1,
\end{cases}
\end{equation}

and \( \rho^* = \frac{n^*}{2\pi} \) is the renormalized (spinon) density of states. Finally, defining

\begin{equation}
g(y) \equiv \frac{t^2}{\rho^* \lambda_0 k_F^2 y^2 g_L (y) + x [1 - 2 \rho^* E_F g_L (y)]^2},
\end{equation}

we obtain

\begin{align}
\frac{1}{\tau^d (\theta)} &= x^2 n m^* \frac{2\pi}{2\pi} \int_0^{2\pi} d\theta' g \left[ \left| \sin \left( \frac{\theta - \theta'}{2} \right) \right| \right] \cos (2\theta') \\
&= x^2 n m^* \frac{2\pi}{2\pi} \int_0^{2\pi} d\theta g \left[ \left| \sin \left( \frac{\theta}{2} \right) \right| \right] \cos 2\theta \\
&= \cos 2\theta \left[ x^2 n m^* \frac{2\pi}{2\pi} \int_0^{2\pi} d\theta g \left[ \left| \sin \left( \frac{\theta}{2} \right) \right| \right] \cos 2u \right] \\
&= \cos 2\theta \frac{1}{\tau^d} = \Gamma_d (k) \frac{1}{\tau_d}.
\end{align}

In the last step we dropped the term in \( \sin 2\theta \) since this term vanishes after integration in Eq.(3.2). Analogously,

\begin{equation}
\frac{1}{\tau} = x^2 n m^* \frac{2\pi}{2\pi} \int_0^{2\pi} d\theta g \left[ \left| \sin \left( \frac{\theta}{2} \right) \right| \right],
\end{equation}

27
and
\[
\frac{1}{\tau_{pb}} = x^2 \frac{nm^*}{2\pi} \int_0^{\frac{2\pi}{x}} d\theta g \left[ \sin \left( \frac{\theta}{2} \right) \right] \left( 1 - \cos 2\theta \right). \tag{3.36}
\]

### 3.2.3 T-matrix for the physical electrons and the normal state transport scattering rate

In order to describe transport in the normal state, we must analyze the physical electron scattering \( T \)-matrix. The calculation is analogous to the one in Section 3.2.2. The bare and renormalized Green’s functions for the physical electrons are given by
\[
G_0 = r_0^{-2} \left[ i\omega_n + r_0^2 \Gamma_s - v_0 \right], \tag{3.37}
\]
\[
G_t^{-1} = r^{-1} \left[ i\omega_n - v + t \Gamma_s r + J_\chi \Gamma_s \right]. \tag{3.38}
\]

Proceeding to first order in the disorder as before yields
\[
T^e = G_0^{-1} - G_t^{-1} = r_0^{-2} \delta v - 2v_0 r_0^{-3} \delta r + J_\chi \left( r_0^{-2} \Gamma_s - r^{-1} \Gamma_s r^{-1} \right) = x^{-1} \left[ \delta v - 2v_0 r_0^{-1} \delta r + J_\chi r_0^{-1} \left( \delta r \Gamma_s + \Gamma_s \delta r \right) \right]. \tag{3.39}
\]

\[
\langle k | T^e | k' \rangle = - \left\{ \Pi \left( k' - k \right) + \frac{2v_0}{x} + \frac{J_\chi}{t \epsilon} \left[ h(k) + h(k') \right] \right\} \left( \lambda_0 - \frac{\lambda_0}{2d} \Gamma_s \right) \left( k' - k \right) \] \tag{3.40}
\]

This scattering \( T \)-matrix can now be used to calculate the transport scattering rate that enters the expression for the conductivity in the normal state
\[
\frac{1}{\tau_k} \equiv 2\pi xn \int \frac{d^2k'}{(2\pi)^2} \left\langle |(k | T^e | k')|^2 \delta \left[ h(k) - h(k') \right] \right\rangle \left( 1 - \cos \varphi \right), \tag{3.41}
\]
where, as before, \( \varphi \) is the angle between \( k \) and \( k' \) and the \( x \) factor in front of the integral comes from the quasiparticle weight of the physical electron Green’s function. We now focus again on wave vectors close to the approximately circular Fermi surface and implement the same approximations used in the previous Section. We first note that the numerators of the fractions is Eqs. (3.31) and (3.40), though seemingly different, are actually the same, since
\[
\frac{2v_0}{x} + \frac{J_\chi}{t \epsilon} \left[ h(k) + h(k') \right] = -2 \frac{t \epsilon + J_\chi}{t \epsilon} E_F + 2 \frac{J_\chi}{t \epsilon} E_F = -2E_F, \tag{3.42}
\]
where the negative of the renormalized chemical potential \( v_0 = -\frac{m}{m_0} E_F = -\frac{tx + Jx}{t} E_F \). Like the quasiparticle scattering rate, the transport scattering rate does not depend on the wave vector direction within the assumed approximations and we get

\[
\frac{1}{\tau tr} = \frac{mn m^*}{2\pi} \int_0^{2\pi} d\theta g \left[ \sin \left( \frac{\theta}{2} \right) \right] (1 - \cos \theta).
\]

(3.43)

A straightforward calculation up to first order in the impurity potential gives the \( T \)-matrix in momentum space for \( f \) fermions and physical (\( e \)) electrons, respectively, as shown previously in Subsection 3.2.2 and 3.2.3

\[
\langle k | T^f | k' \rangle = x t \left[ \frac{h(k) + h(k') - \Pi(k' - k)}{\lambda_0 - \frac{2v_m}{\pi} \Gamma_s (k' - k) - x \Pi(k' - k)} \right],
\]

(3.44)

\[
\langle k | T^e | k' \rangle = -t \left\{ \frac{\Pi(k' - k) + \frac{2v_m}{\pi} + \frac{Jx}{t} [h(k) + h(k')]}{\lambda_0 - \frac{2v_m}{\pi} \Gamma_s (k' - k) - x \Pi(k' - k)} \right\},
\]

(3.45)

where \( h(k) = -t \Gamma_s(k) \) is the bare energy dispersion,

\[
\Pi(k) \equiv \frac{1 + \Pi^b(k)}{\Pi^r(k)},
\]

(3.46)

with

\[
\Pi^r(k) = \frac{1}{V} \sum_q f \left[ \frac{h(q + k)}{h(q + k) - h(q)} - f \frac{h(q)}{h(q) - h(q)} \right],
\]

\[
\Pi^b(k) = \frac{1}{V} \sum_q f \left[ \frac{h(q + k)}{h(q + k) - h(q)} - f \frac{h(q)}{h(q) - h(q)} \right] \frac{h(q + k) + h(q)}{h(q + k) + h(q)},
\]

and \( f(x) \) is the Fermi-Dirac function at \( T = 0 \).

In order to assess the role of electronic correlations we will compare our full results as described above with a corresponding non-correlated system in which \( J = 0 \). In the latter case, the \( T \)-matrix is given simply by the lattice Fourier transform of the bare disorder potential, \( \langle k | T_0 | k' \rangle = \epsilon(k' - k) = t \), and there is no distinction between auxiliary and physical fermions. The two sets of results will be called correlated and non-correlated, respectively. Even at this point, the renormalizations due to strong correlations are clear: the \( k \)-dependent factors in Eqs. (3.44) and (3.45), which reflect the spatial readjustments of the \( r_i \) and \( \lambda_i \) fields, make the bare potential “softer” and more non-local. Note also the extra \( x \) factor in Eq. (3.44) as compared to Eq. (3.45).
At low temperatures, only scattering very close to the Fermi level is relevant. We will thus calculate the $T$-matrices at the Fermi surface. Furthermore, we are interested in the over-doped region, where the Fermi surface anisotropy becomes increasingly less pronounced as the doping increases. Therefore, we will simplify the actual lattice dispersion in favor of an isotropic one corresponding to the continuum limit, $h(k) \approx -4t + tk^2$. This is equivalent to a bare effective mass $m = 1/2t$ and a renormalized one $m^* = 1/(2tx + 2\tilde{J}_X)$. Finally, we call $E_F$ and $k_F$ the Fermi energy and momentum for the bare dispersion $h(k)$, respectively, while $\tilde{E}_F = m^*E_F$ is the Fermi energy for the renormalized dispersion of the $f$-fermions.

3.3 Discussions

3.3.1 Pair breaking parameter

Once the scattering matrix has been determined, it is a trivial matter to write down the predictions of the Abrikosov-Gor’kov (AG) theory for the suppression of the superconducting transition temperature $T_c$ as shown in Subsection 3.2.1

$$\ln \frac{T_{c0}}{T_c} = \psi\left(\frac{1}{2} + \frac{\alpha}{2}\right) - \psi\left(\frac{1}{2}\right),$$

(3.47)

where $T_{c0}$ is the transition temperature in clean limit, $\alpha \equiv 1/(2\pi T_c \tau_{pb})$, and $\tau_{pb}$ is the pair breaking scattering time. The latter is given in the continuum limit by

$$\frac{1}{\tau_{pb}} = \frac{xnm^*}{2\pi} \int_0^{2\pi} d\theta g\left[\sin\left(\frac{\theta}{2}\right)\right] (1 - \cos 2\theta),$$

(3.48)

where

$$g(y) \equiv \frac{t^2}{\rho^*\lambda_0k_F^2y^2g_L(y) + x[1 - 2\rho^*E_Fg_L(y)]^2},$$

(3.49)

where $\rho^* = \frac{m^*}{2\pi}$ is the renormalized density of states and

$$g_L(y) \equiv \begin{cases} 1 & y \leq 1, \\ 1 - \sqrt{1 - y^2} & y > 1. \end{cases}$$

(3.50)

The factor of $1 - \cos 2\theta$ comes from the vertex corrections for $d$-wave pairing and can be generalized to other pairing symmetries by changing $\cos 2\theta$ to the corresponding lattice harmonic. The leading behavior for low impurity concentrations is
\[ T_c = T_{c0} - \pi / 8\tau_{pb}. \] (3.51)

Fig. 3.2 shows the ratio of the pair-breaking scattering rate \( 1 / \tau_{pb} \) in the correlated case to the non-correlated one. Note that, for the non-correlated case,

\[ \frac{1}{\tau_0} = \frac{nm}{2\pi} \int_0^{2\pi} d\theta t^2 (1 - \cos 2\theta) = nmt^2. \] (3.52)

Clearly, pair breaking is strongly suppressed by electronic correlations. While this suppression is enhanced as the density-driven Mott transition is approached \( (x \to 0) \), it is still quite significant up to dopings of \( x \approx 0.3 \). As a result, the \( T_c \) degradation is expected to be considerably slower in that case and we expect the \( d \)-wave superconductivity to be more robust than predicted by the weak coupling theory. Equivalently, the critical impurity concentration \( n_c \) at which \( T_c \) vanishes is enhanced when compared to the non-correlated case, 5 – 10 times in the range of dopings from 0.15 to 0.3. We note that this suppression of pair-breaking by the impurities is completely dominated by the \( x^2 \) dependence of Eq. (3.48). Indeed, in the whole range of dopings from \( \sim 0.01 \) to \( \sim 0.3 \), the product of the effective mass \( m^* \) and the angular integral in Eq. (3.48) varies very little (roughly from 5 to 3). Thus, in a manner very reminiscent of the strong healing of gap fluctuations found in reference [28], here the robustness of \( T_c \) can also be attributed to ‘Mottness’.

### 3.3.2 Transport property in correlated normal state

The normal state resistivity is governed by the impurity induced transport scattering rate, which can be evaluated straightforwardly via Eq. (3.45) to give:

\[ \frac{1}{\tau_{tr}} = \frac{xnm^*}{2\pi} \int_0^{2\pi} d\theta g \left[ \sin \left( \frac{\theta}{2} \right) \right] (1 - \cos \theta). \] (3.53)

The non-correlated transport scattering rate defined as

\[ \frac{1}{\tau_{tr0}} = \frac{nm}{2\pi} \int_0^{2\pi} d\theta t^2 (1 - \cos \theta), \] (3.54)

which coincides with the above \( 1 / \tau_0 \) for the bare isotropic scattering impurity potential we used. As shown in Fig. 3.2, the transport rate is also suppressed by electronic correlations and ‘Mottness’. In contrast to Eq. (3.48), however, the dependence is almost linear in \( x \). This is because, as before, the product of \( m^* \) and the angular integral in Eq. (3.53) is almost doping independent. As a result, as
Figure 3.2: The pair-breaking and transport scattering rates normalized by the non-correlated value $1/\tau_0$ as a function of the doping level.

seen in Fig. 3.2, for a wide range of doping levels the suppression of the pair-breaking scattering rate is much more significant than the transport one.

### 3.3.3 Enhanced forward scattering

The doping dependence of the scattering rates illustrated in Fig. 3.2 makes it clear that the dominant effect comes from the explicit $x$ dependence in Eqs. (3.48) ($\sim x^2$) and (3.53) ($\sim x$). The $x$-dependence coming from $m^*$ times the angular integrals over the scattering matrices is very weak. However, this does not mean that the angular dependence of the $T$-matrices is not affected by strong correlations, as we will now show.

In Fig. 3.3 we show, for two doping levels, the angular dependence of the function $g\left[\sin\left(\frac{\theta}{2}\right)\right]$ [defined in Eq. (3.49)], which is integrated over in Eqs. (3.48) and (3.53). This should be compared to the bare impurity result, which is $\sim t^2$ and thus $\theta$-independent. Clearly, there is a large enhancement of forward scattering, indicating a “softening” of the impurity scattering by correlations, even for point-like impurities in the plane.

This function is weighted by $1 - \cos 2\theta$ and $1 - \cos \theta$ in the integrations in Eqs. (3.48) and (3.53), respectively. These weight functions amplify the contributions from the regions $\theta \approx \pi/2$ and $\theta \approx \pi$, respectively.
respectively, which are, however, hardly affected by correlations. As a result, even with the softening of the impurity scattering, the angular integrals are not renormalized significantly in the range from \( \sim 0.15 \) to \( \sim 0.3 \), when compared to the non-correlated bare impurity result: \( \sim 0.8 - 1 \) in Eq. (3.48) and \( \sim 0.3 - 0.5 \) in Eq. (3.53). The conclusion, then, is that strong correlations enhance significantly the forward scattering region even for point-like in-plane impurities, but this is not the reason for the robustness of \( T_c \) or the resilience of the normal state conductivity.

### 3.4 Conclusions

We have shown how the weak-coupling AG theory of \( T_c \) suppression and normal state resistivity by dilute non-magnetic impurities is modified in a strongly correlated metal. Even though the renormalized scattering amplitude is strongly enhanced in the forward direction, the most significant effect comes from the suppression of the electron fluid compressibility by ‘Mottness’, which is effective even relatively far from the Mott insulating state. Given its simplicity, we suggest that this phenomenon is generic to other systems close to Mott localization.
CHAPTER 4

SHOCK WAVES AND COMMUTATION SPEED OF MEMRISTORS

In this context gaining theoretical insight is of essence. Thus, in the present work we shall address one of the key aspects of the RS phenomenon, namely, the issue of the commutation speed of the resistance change. Our first striking result is a connection between the RS phenomenon and shock wave formation, a classic topic of non-linear dynamics [77]. In fact, we shall argue that the profile of oxygen vacancies that migrate during the resistive change forms a shock wave that propagates through the Schottky barrier and leaks onto the bulk of the device, which we schematically illustrate in Fig.4.1. We further validate the scenario by means of numerical simulations on a successful model of RS and by novel experiments done on a manganese based memristor device. Both model calculations and experiments reveal a striking scaling behaviour as predicted by the shock wave scenario.

4.1 Generalized Burgers’ equation

When ions migrate through a conducting medium under the influence of strong applied voltage, they are likely to undergo a nonlinear diffusion process, as we explain in the following. The total ionic current \( j(t,x) = j_{\text{diffusion}} + j_{\text{drift}} \) can be expressed as the sum of a diffusion current \( j_{\text{diffusion}} = -D \nabla u \) and a drift current \( j_{\text{drift}} \), which is induced by the local electric field \( E \) and the local concentration \( u \). Together with the continuity equation \( \partial_t u + \nabla \cdot j(t,x) = 0 \), this immediately gives us a generalized diffusion equation of the Nernst-Planck type. This would represent a familiar drift-diffusion equation if the local electric field \( E \) was held constant, i.e. independent of the local ion concentration \( u \). In contrast, in (poorly) conducting media and under voltage pulses, the local electric field may strongly depend on the local ion concentration; this effect is the key source of nonlinearity causing the formation of shock waves and very sudden resistance switching [43, 44].

Since electrons move much faster than the ions, we can view the ions as static when considering the electronic current \( I \), which obeys a steady-state condition \( \nabla \cdot I = 0 \). The local electric field is
Figure 4.1: Schematic representation of the shock wave evolution. The orange region indicates the metallic electrode and the blue indicates the TMO dielectric. Small spheres denote the ionic defects (oxygen vacancies) whose density profile form a shock wave. It evolves through a highly resistive (Schottky) interface and eventually leaks over the more conductive bulk, producing the resistive change. Black arrows depict the strength of the local electric fields.

then simply determined, through Ohm’s law, by the local resistivity $E = \rho(u)I$, which may be a strong function of the local ion concentration $u$. In particular, in bad metals such as the transition metal oxides, the migrating ions (e.g. oxygen vacancies) act as scattering centers for the conduction electrons. In such situations, we expect $\rho(u)$ to be a monotonically increasing function of the local ion vacancy density $u(t,x)$. Therefore, the redistribution of the local ion density results in the change of local resistivities and, consequently, of the local electric fields, which further promotes the non-linear effect in the drift.

Under the experimentally-relevant case where the transverse currents may be neglected, the problem simplifies to a one-dimensional non-linear diffusion equation,

$$\partial_t u + f(u) \partial_x u = D \partial_{xx} u,$$  \hspace{1cm} (4.1)

where $f(u) \equiv \partial_u J_{\text{drift}}(u,I)$, and $I(t)$ is the magnitude of the electronic current. Equation (4.1) can be considered a generalization of the famous Burgers’ equation, which corresponds to the special case $f(u) \propto u$. Its most significant feature is the presence of a density-dependent drift term, which physically means that the “crest of the wave” experiences a stronger external force than the “trough”.

35
Figure 4.2: Time evolution of a generic density profile according to the generalized Burgers’ equation (4.1). Curves correspond to profiles at \( t = 0, 2, 4 \) and 6. The (arbitrary) initial profile is taken Gaussian and we adopt \( f(u) \propto u^2 \). The color dots track constant-\( u \) points and show the formation of the universal steep shock wave profile. Inset: \( t - x \) coordinates of the constant-\( u \) tracking points (characteristics). The velocity of each point only depends on its \( u \) value. The crossing of the characteristics signals the formation of the multivalued steep shock-wave front.

This generally leads to the formation of a sharply defined shock-wave front in the \( u(t,x) \) profile, which assumes a universal form at long times, completely independent of the - quickly “forgotten” - initial conditions. Although the process is driven by the drift term, the stability of the shock wave form is provided by the existence of the diffusion term which prevents the shock wave from self-breaking \([78, 77]\). Remarkably, the formation of shock waves proves to be robust in a much more general family of models with the nonlinear drift term specified by the function \( f(u) \), any monotonically increasing function of \( u \). The qualitative behaviour can be established by using the well-known method of characteristics \([77, 79]\), that we illustrate with an example in Fig. 4.2 (see Appendix D.1 for a more detailed explanation).

The drift current is generally given by the expression \( j_{\text{drift}} = u g(E) \). The form of the function \( g(E) \) is material-dependent, and here we envision two limiting situations. In homogeneous conductors, we should have simple “Ohmic” behavior as \( g(E) \sim E \) while in granular materials, we expect

\[ \text{While the drift current contains the nonlinear term that leads to a steep wave front, according to Fick’s law, the diffusion current flows in such a way as to reduce the steepness of the density profile. Therefore, the diffusion current stabilizes the propagation of the shock wave and it protects the solution from entering the self-breaking region (multi-valued solution). In the scenario of the standard Burgers’ equation, where } f(u) \sim u \text{ in Eq}(1), \text{ the equation can be explicitly solved via Cole-Hopf transformation (see references } [78, 77]. \text{ In that case we can check the effects of the diffusion current against nonlinear drift exactly.} \]
exponential dependence due to activated transport, corresponding to: \( g(E) \sim \sinh\left(\frac{E}{E_0}\right) \), where \( E_0 \) is a parameter describing the activation process.

Figure 4.3: Top panel: Snap shots of the time evolution of the density profile \( u(t,x) \) within the active interfacial region in a simulation of the VEOVM model (see Appendix E.1 and E.2 for details). The current used is \( I[a.u]=71.5 \), with \( A_{SB}=1000 \) and \( A_B=1 \). The time steps of the successive profiles can be read-off from the corresponding color dots in the inset. The initial state \( u(0,x) \) (black line) exhibits a vacancy pile-up next to the electrode at \( x=1 \). The SB-bulk interface is denoted with a vertical dash-dot line at \( x_{int}=100 \). The large accumulation of vacancies on the right of \( x_{int} \) (bulk side) results from the initial “forming” cycles performed on an originally uniform profile of density \( u_0 \). The result of the forming cycles is an approximately fixed background on top of which the density profile further evolves (see details in [64]). Inset: Resistance of the device as a function of time. Color dots indicate the value of \( R(t) \) at the corresponding snapshots of the main panel. Bottom-left panel: evolution of the shock wave front position \( x_s(t) \) for different currents \( (I[a.u]=58.5, 71.5, 84.5 \) and \( 97.5) \). Dots are from numerical simulations and the solid lines are analytic fits from integration of Eq. (4.4). Inset: characteristic impact-time \( \tau_1 \) as a function of applied currents from the numerical simulation (circles) and analytic fit (dotted-line) in semi-log scale. Bottom-right panel: Shock wave parameters.

Remarkably, these general ideas find an explicit realization in the context of RS in transition metal oxide memristors, such as manganites [52, 80]. In fact, their transport properties are very sensitively dependent on the oxygen stoichiometry, i.e. on the concentration of oxygen vacancies \([V_O]\). Thus, it is now widely accepted that the mechanism of the bipolar (i.e. polarity dependent)
RS in those systems is due to the induced changes in the spatial distribution of $[V_O(x)] = u(x)$ by means of externally applied strong electric stress [43, 44]. In particular, the accumulation of vacancies within highly resistive regions between the oxide and the metallic electrode, such as Schottky barrier (SB) interfaces, greatly increases the (two-terminal) resistance across the device [64]. This accumulation can be achieved by applying strong voltage pulses across the device, leading to the high resistance state $R_{HI}$. Abrupt resistance switching from such high-resistance state to a significantly lower resistance state can be accomplished by reversing the voltage applied, which removes a significant fraction of vacancies from the SB region. The precise characterization of this resistance switching process is the main subject of this paper.

We should mention that an important assumption is that the nonlinear drift term plays the dominant role as compared to the normal diffusion, i.e. we shall not be concerned with the resistive changes involving thermal effects [43, 44]. This restriction enables us to apply our analytical tools in a simple manner, allowing us to obtain a simplified mathematical description of the migration process, as we show in the following.

4.2 Model system

For concreteness, we adopt the voltage-enhanced oxygen-vacancy migration model [64] (VEOVM), which corresponds to granular materials with activated transport process and has been previously used for manganite devices [64]. Within the framework of this model, we shall perform numerical simulations to validate our shock-wave scenario. The VEOVM simply assumes that the local resistance of the cell at (discretized) position $x$ along the conductive path of the device is simply given as a linear function of the local vacancy concentration, namely,

$$r(x) = A_\alpha u(x)$$

with $\alpha = SB, B$, where $SB$ denotes the highly resistive (Schottky barrier) region and $B$ the more conductive bulk [52]. The values of these constants are taken $A_{SB} \gg A_B = 1$, which allows us to neglect the bulk resistance [64]. The discretized conducting path assumes the metal-electrode at $x=0$, and $x = x_{int}$ denotes the point within the dielectric where the SB meets the bulk region. Under the action of the external stress (electric current $I$), the local fields at each cell position $x$ are computed at every discrete time step $t$. The field-driven migration of vacancies is simulated
computing the local ionic migration rates from cell $x$ to $x + \Delta x$ as \[64\]

$$P(x, x + \Delta x) = u(x)[1 - u(x + \Delta x)]\exp\left(-\frac{V_0 + qIr(x)}{k_BT}\right), \quad (4.3)$$

where, for simplicity, we take the ionic charge $q=1$ and $k_BT=1$. The parameter $V_0$ denotes the activation energy for ionic diffusion. The new profile $u(t, x)$ is updated from the migration rates, and from (4.2) we get the new total (two point) $R(t)$ as the discrete $x$-integral of the local cell’s resistance $r(t, x)$. Here, for simplicity, we focus on a single active SB-bulk interface, while the more general situation with two barriers may be analyzed following a similar line of argument \[64\].

The applied external electric stress that we adopt is a constant current, in both, simulations and experiments (see below).

As described in Ref.\[64\], the initial vacancy concentration profile is assumed to be constant $[u(x)] = [u^0]$. The “forming” or initialization of the memory is done by first applying a few current loops of alternate polarity, $\pm I^0$, until the migration of vacancies evolves towards a limit cycle, with a well defined profile $u_0(x)$. After this, the system begins to repetitively switch between two values: $R_{HI}$ and $R_{LO}$. In the first, most of the vacancies reside within the high-resistance region SB, and in the second vacancies accumulate in the more conductive bulk. The $R_{HI}$ state with the vacancies piled up in the first cell, at $x=1$, defines the initial state for the shock wave propagation (see Fig.4.3).

### 4.3 Shock wave formation: the “propagation phase”

We apply an external field with polarity pointing from the SB to the bulk and observe the evolution of the vacancy profile as a function of the (simulation) time. As can be observed in Fig.4.3 there is a rapid evolution of the profile into a shock wave form with a sharply defined front. We also notice that the total resistance remains approximately constant during an initial phase, and suddenly starts to decrease after the front hits the internal SB-bulk interface at $x_{int}$ (inset of top panel of Fig.4.3). We shall analyze these key features in the following.

First, we focus on the propagation of the shock wave front position $x_s(t)$, as shown in Fig.4.3 (bottom-left panel) for different values of the electronic current $I$. We observe that the characteristic time $\tau_1$ for the shock wave to travel through the Schottky barrier and reach the SB-bulk point $x_{int}$ decreases exponentially with the magnitude of $I$. To obtain analytical insight for this behavior, we recall that the velocity of the shock wave front $dx_s/dt$ is very generally given by the Rankine–Hugoniot
conditions [77, 81], which express it as the ratio of the spatial discontinuity of the (vacancy) drift current, and the spatial discontinuity of the density profile across the shock viz. \( \frac{dx_s}{dt} = \frac{\Delta j}{\Delta u|_{x_s}} \).

Within the VEOVM model [64], we obtain the following *nonlinear rate equation* (see Appendix D.2 for details), which describes the dynamics of the shock wave front:

\[
\frac{dx_s}{dt} = \frac{2Du_- \sinh(IA_{SB}u_-) - 2Du_+ \sinh(IA_{SB}u_+)}{\Delta u},
\]

where \( D \) is a prefactor related to the activation energy for vacancy migration (Arrhenius factor) (see Eq.4.1 and Ref.[64]), and \( u_-/+ \) are the density of vacancies at the two sides of the shock wave front (see Fig.4.3). The density \( u_- \) depends on the shock wave front position via: \( u_- = Q/x_s + u_+ \), where \( Q \) is the total number of vacancies carried by the shock wave, which remains a constant parameter through the propagation phase \( (t < \tau_1) \) and \( u_+ \) is a constant background density which can be written as \( u_+ = Q_B/x_{int} \), \( Q_B \) standing for the total number of background vacancies.

Our description of the propagation phase is fully consistent with our numerical simulations. As shown in the inset of Fig.4.3 (top panel), the resistance remains essentially constant until the wave front reaches the SB-bulk interface after a (current dependent) time \( \tau_1 \), and then begins to drop. Moreover, we also achieved a good fit to the shock front velocity by using Eq.4.4, as is shown in Fig.4.3 (see Appendix E.3 for details).

### 4.4 Resistance switching: the “leakage phase”

After the shock front reaches the boundary point \( x_{int} \) the resistance begin to drop. To understand this behaviour, we note that the total resistance of the Schottky barrier is given by the *total number of vacancies* within the barrier region viz. from (4.2) \( R_{SB} = \int_{SB} dx A_{SB} u(x) \). As a result, the resistance drop per unit time is approximately given by the ionic vacancy-current passing through the SB-bulk interface at \( x_{int} \),

\[
\frac{dR(t)}{dt} = -A_{SB}j(x = x_{int})
\]

since \( R_{SB} \gg R_B \) as \( A_{SB} \gg A_B \). Notice that during the propagation phase the ionic current through the interface \( x_{int} \) is negligibly small. This is because the initial vacancy concentration there, and hence the local field, are also negligibly small. However, when the shock wave front eventually reaches the end of the SB region, after travelling for a time \( \tau_1 \), we do expect a sudden resistance drop as a large number of ionic vacancies begin to leak out into the bulk region.
Figure 4.4: Time dependence of the resistive change $R(t)$ for various external current intensities. Top left: experimental data measured on an Ag/LPCMO/Ag memristor with $I = 37.5$ mA (black), 40 mA (green), 50 mA (blue), 80 mA (red) and 100 mA (cyan). $t_{eff}$ is the effective time duration of the applied currents (see Appendix E.4 for details). The initial state was reset by applying an intense negative polarity current of 350 mA. Note that the initial value of the resistance, 78 $\Omega$ is recovered within $\pm 1\Omega$. Top right: model simulations with applied current: $I[\text{a.u.}] = 58.5$ (black), 71.5 (green), 84.5 (blue) and 97.5 (red). Middle panels: experimental data for $I = 40$ mA (left) and simulations for $I[\text{a.u.}] = 71.5$ (right). $\tau_1$ is defined as the time interval from the beginning of the applied pulse until the resistance starts to drop. Bottom panels: Idem as middle panels in a semilog scale.

We shall now focus on the detailed description of the resistive drop. In Fig.4.4 we show the systematic dependence of $R(t)$ as a function of the applied external (electronic) current. Along with the simulations of the VEOVM, we also present our experimental results measured on a manganite-based (La$_{0.325}$Pr$_{0.300}$Ca$_{0.375}$MnO$_3$) memristive device [82, 83, 84, 85, 86]. The experimental setup and device are described in detail in Appendix E.4 and in [87, 88]. The set of curves were obtained for applied current intensities just above the threshold for the onset of the resistance switch. The goal here was not to demonstrate the fast switching speed of the device, but rather on the contrary, achieve relatively slow switching speeds in order to access the different time scales. In addition, this also minimizes thermal heating effects [87]. We observe that in both, simulation and experiments, the resistance-change rapidly becomes larger and faster with the increase of the applied electric
stress intensity. We also observe an overall good qualitative agreement between experiments and model simulations. This is also highlighted by the semi-log plots, which clearly display the two-stage process involved in the resistive switch, before and after the impact time $\tau_1$.

Remarkably, within shock wave scenario, we may also obtain explicit expressions that quantify the resistance change during the leakage phase. Our analysis may be simplified by first noting, from general considerations of shock waves, that their shape at long times becomes "flat", i.e. the gradient of the local density rapidly decreases ($\partial_x u \to 0$) at all points that were overtaken by the shock wave front\[77, 79\]. Indeed, our data is fully consistent with this observation, as the vacancy density profile within the SB remains approximately "flat" (i.e. spatially constant $u(t, x) = u_{SB}(t)$) at all times after the shock front reaches the interface (see Fig.4.3). Then, within the VEOVM the SB resistance is simply proportional to the total vacancy concentration within the barrier and we have, $R(t) \approx R_{SB}(t) = A_{SB}x_{int}u_{SB}(t)$. Since the electronic current $I$ is held fixed, the vacancy (i.e. ionic) current through the interface depends only on the vacancy concentration $u_{SB}$ (cf Eq.4.3). Thus, within the VEOVM we obtain a nonlinear rate equation, describing the resistance drop during the "leakage phase":

$$\frac{dR}{dt} = -\frac{2DR}{x_{int}} \sinh\left(\frac{IR}{x_{int}}\right).$$

(4.6)

Similarly as we showed before for the propagation phase, this equation may be validated by a quantitative fit to the simulation results (see Appendix E.3). Note that due to the strong nonlinear form of this rate equation, the $R(t)$ response is significantly different from the simple exponential decay expected in the familiar linear case (e.g. in standard RC circuits). Therefore, within the short time scale associated with the initial fast drop of resistance and where the RS is significant ($IR/x_{int} \gg 1$), the present type of nonlinear system is dominated by the activated process and the approximation $\sinh\left(\frac{IR}{x_{int}}\right) \approx \frac{1}{2} \exp\left(\frac{IR}{x_{int}}\right)$ is valid. This enables the approximate solution of the Eq.4.6.

$$R = R_{HI} - \frac{x_{int}}{I} \ln\left(1 + \frac{t^*}{\tau_2(I)}\right),$$

(4.7)

where the time $t^*$ is measured from the “impact” time $\tau_1$ and $\tau_2(I) = \frac{x_{int}}{IR_{HI}} \exp\left(\frac{-IR_{HI}}{x_{int}}\right)$ (see Appendix E.5) is the current-dependent characteristic time for the resistance drop.
4.5 Resistivity scaling

An interesting consequence of Eq. 4.7 is that it suggests the scaling behaviour of the curves $R(t^*)$. In fact, one may define the normalized resistance drop $\delta R(t^*) = \frac{R - R(\tau_2)}{(R_{HI} - R(\tau_2))}$ and see from Eq. 4.7 that obeys it the scaling form:

$$\delta R(t^*) = 1 - \frac{\ln(1 + t^*/\tau_2)}{\ln(2)},$$

(4.8)

In Fig. 4.5 we demonstrate that this striking feature is indeed present in both, our experiments and simulations data. In the upper panels of the figure we show the excellent scaling that is achieved, where all the experimental and the simulation curves $R(t)$ from Fig. 4.4 were respectively collapsed onto a single one. Moreover, the collapsed data can also be fitted with a slightly more general form of Eq. 4.8, that we discuss in Appendix E.5. Remarkably, in the lower panels of Fig. 4.5 we show that a collapse of the data $R(t)$ can also be obtained using the impact time $\tau_1$ as the scaling variable. This is significant, because it shows that a single scaling behavior may include the two phases of the resistive switching process, namely before and after $\tau_1$.

We should mention that the scaling scenario was derived with the assumption of an ohmic behaviour in the $I$-$V$ characteristics. While this may not be the case in general [87], within the present set of experiments, which are performed near the current threshold of RS, our results indicate that this is a reasonable assumption or at least a valid approximation.

4.6 Conclusions

To conclude, from quite general considerations of migration of ionic defects under strong electric fields in solids, we have argued that the dynamics of the spatial profile of defect concentration should be governed by a Burgers'-type nonlinear equation and develop shock waves. We demonstrated that this scenario is indeed realized within a concrete realization, namely a ionic migration model that was previously applied to describe resistive switching phenomena in manganite based memristive devices. In those systems, a key role is played by the migration of oxygen vacancies, which are the ionic defects relevant to the electronic transport properties. We thus predicted a two-stage process for the resistive switch phenomenon. An initial one, where the oxygen-vacancy concentration profile develops a shock wave that propagates throughout a highly resistive (Schottky barrier) region near the electrode. During this phase the resistance essentially does not change. This is followed
Figure 4.5: Scaled curves of the $R(t)$ data sets of Fig.4.4. The left panels show the collapsed experimental data and the right ones the numerical simulations. The time $\tau_0$ is an auxiliary scaling variable, which is proportional to the characteristic time $\tau_2$ (see Appendix E.5 for details on the scaling procedure). The scaled data were fitted (white dotted line) with a generalized version of Eq.4.8 (see Appendix E.5). The lower panels show the same data sets scaled with the shock wave impact time $\tau_1$ (the experimental curves show only three data sets for the lower current values. At higher currents our electronics could not resolve $\tau_1$). To achieve the scaling of the lower panels, we assumed for each plot the normalization value of $\Delta R$ determined from the previous scaling (top panels).

by a second phase, where the shock wave emerges from the high resistive region and the ionic defects leak into the conductive bulk. Our scenario was further validated by novel experimental data on a manganite based memristor device. A remarkable results of our study is that both, the numerical simulations and the experimental curves, obeyed a scaling behaviour, providing decisive support to our theory. The present work provides novel insights on the physical mechanism behind the commutation speed on novel non-volatile electronic memories, unveiling an unexpected connection between a phenomenon of technological relevance and a classic theme of nonlinear dynamical systems. Furthermore, and from a practical point of view, our study unveils some of the key physical parameters that control the resistive switching properties of devices. Of foremost importance are the characteristic time scales, $\tau_1$ and $\tau_2$. Interestingly, we find that both depend on on the same combination of parameters, namely, $x_{int}^2/(DIR_{HI})$ and $x_{int}/(IR_{HI})$, which appears in
the exponential factor (see Eq. 4.7, Eq. E.7 and E.12 in Appendix E.5). While we may intuitively understand that a shorter interface or a stronger current pulse would increase the commutation speed, this expression also provides concrete guidance for the engineering effort of devices, which is less evident. For instance, it shows that shortening the length of the interface by a factor of three should at least gain a full order magnitude in speed. It also points towards adopting materials where the diffusion constant of oxygen vacancies is large. While this is a priori fixed by the specific chemistry of the material, one may also envision that the diffusion might be boosted by preparing samples with smaller grain sizes and with extended columnar defects that could be induced by heavy ion irradiation.

A final important point to consider regards the general validity of the present results for other type of resistive switching systems, such as for instance those based on HfO$_x$, TaO$_x$ and TiO$_{2-x}$, which are currently receiving a great deal of attention. Those compounds, in their stoichiometric form, are, unlike PCMO, very good insulators. Thus, in order to make RS devices one possibility is to perform an initial "electroforming" procedure, which creates metallic filaments that may subsequently be "burned" and regenerated, irrespective to the applied polarity of the electric pulses. That mechanism is not relevant for the present study. There is, however, another possibility to make RS devices with these compounds, which is to heavily dope very thin films (≈ tens of nm) with oxygen vacancies [89]. In this case, the systems become more conductive than their respective stoichiometric compound and display a qualitatively different type of resistive switching under electric pulsing, which is bipolar and relevant here. In fact, in these systems the mechanism relays on oxygen ionic migration under the strong electric fields, thus it is interesting to note that the shock wave formation may be at work, shaping the evolution of the spatial profile of oxygen density. Indeed, a local increase of oxygen concentration brings these systems (locally) closer to their respective stoichiometric composition, thus it locally increases the resistivity. Hence we realize that for such systems one may fulfill the conditions for onset of shock wave propagation. In fact, an analogous role to that of oxygen-vacancy density in PCMO (that renders the system more resistive), may be played in the simpler binary compounds by the oxygen density (see Appendix F for details and the simulation results of [59]). It would be very interesting indeed to explore the fast commuting behavior in thin films of TaO$_x$, HfO$_x$ and related compounds, to search for evidence of shock waves propagation. These are exciting directions that are now open for future investigations.
CHAPTER 5

CONCLUSIONS AND OUTLOOK

In chapters 2 and 3, we have found analytically an inextricable link between the healing of gap and charge disturbances in strongly correlated superconductors, suggesting that this phenomenon is generic to any system close to Mott localization. Furthermore, we have shown how the weak-coupling AG theory of $T_c$ suppression and normal state resistivity by dilute non-magnetic impurities is modified in a strongly correlated metal. Given its simplicity, we suggest that this phenomenon is generic to other systems close to Mott localization. An important experimental test of these findings would be provided by STM studies on correlated superconductors with impurities. Perhaps it is also interesting to check how the normal state conductivity and the superconducting transition temperature $T_c$ varies with the impurity concentration at different doping levels.

Consider systems with pairing symmetry other than the $d$-wave. We may check their robustness against non-magnetic disorders via similar techniques. Another interesting direction is to study the correlated systems where impurities are not dilute.

In chapter 4, we show the important role of defects in the correlated system for practical applications. Due to generality of the theory, we may study the resistive switching phenomena based on binary oxides, which are currently receiving a great deal of attention. When heavily doped with oxygen vacancies, the systems become more conductive than their respective stoichiometric compound and display the bipolar type of resistive switching under electric pulsing.

In fact, in these systems the mechanism relays on oxygen ionic migration under the strong electric fields, thus it is interesting to note that the shock wave formation may be at work, shaping the evolution of the spatial profile of oxygen density. It would be very interesting indeed to explore the fast commingling behavior in thin films of $\text{TaO}_x$, $\text{HfO}_x$ and related compounds, to search for evidence of shock waves propagation. These are exciting directions that are now open for future investigations.

As our results are obtained through the classical phenomenological model, it is crucial to establish underlying microscopic guidelines. Furthermore, we may predict whether the candidate material
is good for fabricating fast switching devices under certain situations. For example, as some of the materials are correlated bad metal, we may apply similar slave boson methodology discussed previously and try to capture the relevant physics.
APPENDIX A

THE LINEAR APPROXIMATION

A.1 $U(1)$ slave boson approach

Although many results have been obtained within the weak coupling framework, only few controlled methods for strongly correlated system are established. Among these theoretical tools, we pick the slave-boson method as our main tool, which can be treated in mean-field level [90, 91] with Gaussian fluctuations. There are several representations of the slave boson theory [92, 93, 32] and the key idea is to introduce auxiliary bosons. Certain constraints are further imposed in order to force the matrix elements of the Hamiltonian equal in the original and the enlarged Hilbert space. The physics contained in this method is equivalent to that included in the simple Gutzwiller approximation. The main advantage of the slave boson method is the inequality constraint, namely, the electron occupancy is less than one (away from half filling) can be treated easily in the path integral formalism with one Lagrange multiplier $\lambda_i$ on each site. The functional integral over $\lambda_i$ would reproduce the delta function of the constraint on each site. However, we ignore the imaginary time dependence of $\lambda_i$ and the constraint is obeyed in average sense. At mean-field level, we absorb its saddle point value to the chemical potential and neglect the thermal fluctuations of the $\lambda$ field which is a gauge field [94, 95].

Different representations are suitable for different problems. Kotliar’s representation [32] makes the Hubbard model simple to analyze, while in the large $U$ limit, Coleman’s representation [34] is much more convenient since it has already projected out the double occupancy. In the following discussion, we would focus on this so called $U(1)$ slave boson theory and introduce one bosonic field which is related directly to the hole doping density. While the $U(1)$ theory has a clearer relation between auxiliary field and physical quantity, the system we are interested in has the $SU(2)$ symmetry, especially in studying the $d$ wave superconducting state. Although the phase diagram obtained by $U(1)$ or $SU(2)$ slave boson theory does not fully agree with the experimental results, they do capture important features of the strongly correlated Fermi liquid state.
In the large $U$ limit of the standard Hubbard model, we can obtain the effective $t-J$ Hamiltonian which can is used to describe doped antiferromagnets [33] and the doped Mott insulator. In this model, we have the following Hamiltonian with $n_i \equiv \sum_\sigma c_{i\sigma}^\dagger c_{i\sigma}$:

$$H = P_s TP_s + J \sum_{ij} \mathbf{S}_i \cdot \mathbf{S}_j + \sum_i (\varepsilon_i - \mu_0) n_i,$$

(A.1)

where $T$ denotes the kinetic energy $-t \sum_{ij\sigma} c_{i\sigma}^\dagger c_{j\sigma} + h.c.$ with $h.c.$ as the abbreviation for “hermitian conjugate”, and $P_s$ is the projection operator for projecting out the double occupant states. We then obtain the explicit form [96]:

$$P_s TP_s = -t \sum_{ij\sigma} (1 - n_{i\bar{\sigma}}) c_{i\sigma}^\dagger c_{j\sigma} (1 - n_{j\bar{\sigma}}) + h.c..$$

(A.2)

Below, we discuss how various quantities transform within the slave boson approach. According to [8], we have

$$c_{i\sigma}^\dagger = f_{i\sigma}^\dagger b_i + \epsilon_{\sigma\sigma} f_{i\sigma} d_i^\dagger,$$

(A.3)

and the identity:

$$\sum_\sigma f_{i\sigma}^\dagger f_{i\sigma} + b_i^\dagger b_i + d_i^\dagger d_i = 1.$$

(A.4)

In the above equations, $f_{i\sigma}$ are the auxiliary fermions and $b, d$ are the auxiliary bosons. Combine Eq.(A.3) and Eq.(A.4), we can easily recover the anti-commutation algebra among $c$ operators. After projecting out the double occupant states, we obtain:

$$c_{i\sigma}^\dagger = f_{i\sigma}^\dagger b_i$$

(A.5)

$$\sum_\sigma f_{i\sigma}^\dagger f_{i\sigma} + b_i^\dagger b_i = 1$$

(A.6)

However, Eq.(A.5) is not an operator identity since we combine it with the Eq.(A.6), we can no longer recover the anti-commutation relations among $c$ operators. Although it is not an operator identity, we can safely use it in our Hamiltonian as the matrix elements are all faithful. Alternatively, we can utilize Eq.(A.3) and then perform the projection, both routes will provide us the same results.
Comparing the original and the enlarged Hilbert space, we have relations $c^\dagger_{i\sigma} c_{j\sigma} = f^\dagger_{i\sigma} f_{j\sigma} b^\dagger_i b_j$ and $c^\dagger_{i\sigma} c_{i\sigma} = f^\dagger_{i\sigma} f_{i\sigma}$.

In the mean-field level, we set $\lambda_i(\tau) = \lambda$ with auxiliary fields $\chi$ and $\Delta$ as order parameters in order to decouple the four fermion terms, i.e. $J \sum_{ij} S_i \cdot S_j$. As we require the saddle point approximation with Gaussian fluctuations around it to reproduce the mean-field theory, cautions should be paid to choose the parameters in the Lagrangian. In the following discussion, we set $\bar{J} = \frac{3}{8}J$ according to the fact we are using the multi-component Hubbard Stratonovich transformation [8].

In the homogenous state, we have $\langle b^\dagger_i b_j \rangle = 1 - n = x$ as a parameter keeping track of the hole density. The mean-field conditions as $\chi_{ij} = \sum_{\sigma} \langle f^\dagger_{i\sigma} f_{j\sigma} \rangle$ and $\Delta_{ij} = \langle f_{i\uparrow} f_{j\downarrow} - f_{i\downarrow} f_{j\uparrow} \rangle$ can be derived from $\frac{\delta Z}{\delta \chi_{ij}} = \frac{\delta Z}{\delta \Delta_{ij}} = 0$. The partition function can be expressed in the path integral formalism as follows,

$$Z = \int Df^\dagger Df Db^\dagger Db D\lambda D\chi D\Delta \exp(-S), \quad (A.7)$$

with action $S = \int_0^\beta L(\tau) d\tau$ and the Lagrangian in imaginary time formalism obtained from the Hamiltonian:

$$L(\tau) = \bar{J} \sum_{ij} (|\chi_{ij}|^2 + |\Delta_{ij}|^2) + \sum_{i\sigma} f^\dagger_{i\sigma} \partial_\tau f_{i\sigma} - \bar{J} \sum_{ij} \chi^*_{ij} \{ \sum_{\sigma} f^\dagger_{i\sigma} f_{j\sigma} \} + h.c.] + \bar{J} \sum_{ij} \Delta_{ij} \{ f^\dagger_{i\uparrow} f^\dagger_{j\downarrow} - f^\dagger_{i\downarrow} f^\dagger_{j\uparrow} \} + h.c.] + \sum_i b^\dagger_i \partial_\tau b_i$$

$$+ \sum_i \lambda_i \{ \sum_{\sigma} f^\dagger_{i\sigma} f_{i\sigma} + b^\dagger_i b_i - 1 \} + \sum_i (\varepsilon_i - \mu_f) f^\dagger_{i\sigma} f_{i\sigma},$$

where $\mu_f = \mu - d \bar{J}$. Below, we use the conventional parametrization of order parameters described in many contexts [97, 68]. Based on the Matsubara frequency representation we have: $f(\tau) = \frac{1}{\sqrt{\beta}} \sum_n f(\omega_n) e^{-i\omega_n \tau}$. In radial gauge, $b^*_{i\uparrow} = b_i = r_i$ which indicates that we only consider the amplitude of the bosonic field. Below, we first obtain the effective action by integrating out the fermion field, then derive the free energy and mean-field equations as follows [90, 91]:

50
In order to derive the mean-field equations, we only consider the imaginary time dependence of the fermion field which has been integrated over. Ignoring the imaginary time dependence of the slave boson, Lagrange multiplier and order parameters, we obtain the free energy as follows:

\[
S = \beta J \sum_{ij} (|\chi_{ij}|^2 + |\Delta_{ij}|^2) + \beta \sum_i \lambda_i (r_i^2 - 1)
\]

\[
+ \sum_{n\sigma} \sum_{ij} f_{i\sigma}^\dagger(\omega_n)(-i\omega_n + \varepsilon_i - \mu_f + \lambda_i)\delta_{ij} f_{j\sigma}(\omega_n)
\]

\[
- \sum_n \sum_{ij\sigma} f_{i\sigma}^\dagger(\omega_n)(t_{ij} + t_{ji}) r_i r_j f_{j\sigma}(\omega_n)
\]

\[
+ J \sum_n \sum_{ij} \left[ f_{i\uparrow}^\dagger(\omega_n) f_{i\downarrow}(\omega_n) \right] \left[ \begin{array}{cc}
-\chi_{ij}^* & \Delta_{ij}^* \\
\Delta_{ij} & \chi_{ij}
\end{array} \right] \left[ \begin{array}{c}
f_{j\uparrow}(\omega_n) \\
f_{j\downarrow}(\omega_n)
\end{array} \right]
\]

\[
+ J \sum_n \sum_{ij} \left[ f_{i\downarrow}^\dagger(\omega_n) - f_{i\uparrow}(\omega_n) \right] \left[ \begin{array}{cc}
-\chi_{ij}^* & \Delta_{ij}^* \\
\Delta_{ij} & \chi_{ij}
\end{array} \right] \left[ \begin{array}{c}
f_{j\downarrow}(\omega_n) \\
-f_{j\uparrow}(\omega_n)
\end{array} \right],
\]

\[
S = \beta J \sum_{ij} (|\chi_{ij}|^2 + |\Delta_{ij}|^2) + \beta \sum_i \lambda_i (r_i^2 - 1)
\]

\[
+ \sum_{n\sigma} \sum_{ij} f_{i\sigma}^\dagger(\omega_n)(-i\omega_n + \varepsilon_i - \mu_f + \lambda_i)\delta_{ij} f_{j\sigma}(\omega_n)
\]

\[
- \sum_n \sum_{ij\sigma} f_{i\sigma}^\dagger(\omega_n)(t_{ij} + t_{ji}) r_i r_j f_{j\sigma}(\omega_n)
\]

\[
+ \sum_n \sum_{ij} \left[ f_{i\uparrow}^\dagger(\omega_n) f_{i\downarrow}(\omega_n) \right] \left[ \begin{array}{cc}
-2J\chi_{ij}^* - (t_{ij} + t_{ji}) r_i r_j \\
2J\Delta_{ij}^* + (t_{ij} + t_{ji}) r_i r_j
\end{array} \right] \left[ \begin{array}{c}
f_{j\uparrow}(\omega_n) \\
f_{j\downarrow}(\omega_n)
\end{array} \right],
\]

\[
S = \beta J \sum_{ij} (|\chi_{ij}|^2 + |\Delta_{ij}|^2) + \beta \sum_i \lambda_i (r_i^2 - 1)
\]

\[
- \sum_n \sum_{ij} \left[ f_{i\uparrow}^\dagger(\omega_n) f_{i\downarrow}(\omega_n) \right] \left[ \begin{array}{cc}
[G^{-1}_{ij}]_{11} & [G^{-1}_{ij}]_{12} \\
[G^{-1}_{ij}]_{21} & [G^{-1}_{ij}]_{22}
\end{array} \right] \left[ \begin{array}{c}
f_{j\uparrow}(\omega_n) \\
f_{j\downarrow}(\omega_n)
\end{array} \right].
\]
\[
F = \bar{J} \sum_{ij} (|\chi_{ij}|^2 + |\Delta_{ij}|^2) + \sum_i \lambda_i (r_i^2 - 1) - kT \sum_n \text{tr} \ln -G^{-1},
\]
which then leads to the mean-field equations presented in the main text.

### A.2 Linearized equations

Our linear approximation approach consists of expanding the mean-field equations (2.2-2.5) of the main text to first order in the site energies \(\varepsilon_i\). Denoting linear deviations in the various fields by \(\delta\) we get

\[
\delta \chi_{ij} = 2kT \sum_{nl} (-g_{il}g_{lj} + G_{1li}G_{1lj}) (\delta \lambda_l + \varepsilon_l) + 2kT \sum_{nlm} (-g_{il}g_{mj} + G_{1li}G_{1mj}) (\delta r_l h_{lm} + \delta r_m h_{lm}) - 2kT \sum_{nlm} (g_{il}G_{1mj} + g_{mj}G_{1il}) (\bar{J} \delta \chi_{lm}) + 2kT \sum_{nlm} (g_{il}g_{mj} + G_{1il}G_{1jm}) (\bar{J} \delta \Delta_{lm}),
\]

(A.9)

\[
\delta \Delta_{ij} = -2kT \sum_{nl} (g_{il}G_{1lj} + g_{lj}G_{1il}) (\delta \lambda_l + \varepsilon_l) - 2kT \sum_{nlm} (g_{il}G_{1mj} + g_{mj}G_{1il}) (\delta r_l h_{lm} + \delta r_m h_{lm}) + 2kT \sum_{nlm} (g_{il}G_{2mj} + g_{mj}G_{ij}) (\bar{J} \delta \chi_{lm}) - 2kT \sum_{nlm} (G_{1il}G_{2mj} + g_{ij}g_{mj}) (\bar{J} \delta \Delta_{lm}),
\]

(A.10)

\[
-r \delta r_i = kT \sum_{nl} (-g_{il}g_{li} + G_{1il}G_{1li}) (\delta \lambda_l + \varepsilon_l) + kT \sum_{nlm} (-g_{il}g_{mi} + G_{1il}G_{1mi}) (\delta r_l h_{lm} + \delta r_m h_{lm}) - kT \sum_{nlm} (g_{il}g_{mi} + G_{1il}G_{1mi}) (\bar{J} \delta \chi_{lm}) + kT \sum_{nlm} (g_{il}G_{1mi} + g_{mi}G_{1il}) (\bar{J} \delta \Delta_{lm}),
\]

(A.11)

\[
\lambda \delta r_i + r \delta \lambda_i + \sum_l h_{il} \chi_{il} \delta r_l + r \sum_l h_{il} \delta \chi_{il} = 0,
\]

(A.12)

where \(G_{1ij} \equiv [G_{ij}]_{11}, G_{2ij} \equiv [G_{ij}]_{22}, g_{ij} \equiv [G_{ij}]_{12} = [G_{ij}]_{21}\) are the Green’s functions of the clean system, \(n\) is the fermionic Matsubara frequency index and \(\bar{J} = \frac{3}{8}J\). The latter choice is made, in the presence of correlations, so that the multi-channel Hubbard-Stratonovich transformation we used reproduces, at the saddle-point level, the mean-field results \(^1[8]\). In general, the clean Green’s

---

\(^1\)The usual choice \(\bar{J} = \frac{1}{2}J\) does not change the analytical results and would give rise to hardly noticeable changes in the numerical plots.
functions in $k$-space are given by

$$G_1(\omega_n, k) = \frac{i\omega_n + e(k)}{(i\omega_n)^2 - e^2(k) - J^2\Delta^2(k)}, \quad (A.13)$$

$$G_2(\omega_n, k) = \frac{i\omega_n - e(k)}{(i\omega_n)^2 - e^2(k) - J^2\Delta^2(k)}, \quad (A.14)$$

$$g(\omega_n, k) = \frac{J\Delta(k)}{(i\omega_n)^2 - e^2(k) - J^2\Delta^2(k)}, \quad (A.15)$$

where the renormalized dispersion is

$$e(k) = -2\left(\chi + \tilde{J}\right)\left[\cos(k_x a) + \cos(k_y a)\right] - 4x't'\cos(k_x a)\cos(k_y a) - \mu, \quad (A.16)$$

we have absorbed the clean $\lambda$ in the chemical potential, and

$$\Delta(k) = 2\Delta_0 \left[\cos(k_x a) - \cos(k_y a)\right]. \quad (A.17)$$

Notice that the dimensionful gap function is $\Delta_{\text{phys}}(k) = \tilde{J}\Delta(k)$. As we focus on the asymptotic long-range behavior of the different fields, their variations are dominated by the corresponding clean-limit symmetry channel. We therefore define local order parameters as

$$\delta\chi_i \equiv \frac{1}{2d} \sum_j \delta\chi_{ij} \Gamma(s)_{ij}, \quad \delta\Delta_i \equiv \frac{1}{2d} \sum_j \delta\Delta_{ij} \Gamma(d_{x^2-y^2})_{ij}.$$  

Thus, defining vectors and matrices in the lattice site basis with bold-face letters, Eqs. (A.9-A.12) can be recast as

$$\left(A + r^2B\right)\delta\Phi = r^2C, \quad (A.18)$$

where

$$A = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ 0 & 0 & \lambda I - \frac{\Delta}{2d} \Gamma(s) & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2dtI & 0 & 0 & 1 \end{pmatrix}, \quad \delta\Phi = \begin{pmatrix} \delta\chi \\ \delta\Delta \\ r\delta r \\ \delta\lambda \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \varepsilon \end{pmatrix}. \quad (A.19)$$

Here, the elements of (the vector) $\varepsilon$ are the disorder potential values $\varepsilon_i$, $1$ is the identity matrix, $\delta\lambda_i = \delta\lambda_i + \varepsilon_i$, and

$$M_{11ij} = -\delta_{ij} - \frac{\tilde{J}kT}{d} \sum_{nm} \Gamma(s)_{nt} (-g_{ij}g_{ml} + G_{1ij}G_{1ml}) \Gamma(s)_{jm}. \quad (A.20)$$
\[
M_{12ij} = \frac{\bar{J}kT}{d} \sum_{nml} \Gamma(s)_{il} (g_{ij}G_{1ml} + g_{ml}G_{1ij}) \Gamma(d_{x^2-y^2})_{jm} \tag{A.21}
\]

\[
M_{13ij} = \frac{kT}{d} \sum_{nml} \Gamma(s)_{il} (-g_{ij}g_{ml} + G_{1ij}G_{1ml} - g_{im}g_{jl} + G_{1im}G_{1jl}) h_{jm} \tag{A.22}
\]

\[
M_{14ij} = \frac{kT}{d} \sum_{nml} \Gamma(s)_{il} (-g_{ij}g_{ml} + G_{1ij}G_{1ml} - g_{im}g_{jl} + G_{1im}G_{1jl}) h_{jm} \tag{A.23}
\]

\[
M_{21ij} = -\frac{\bar{J}kT}{d} \sum_{nml} \Gamma(d_{x^2-y^2})_{il} (g_{ij}G_{1ml} + g_{ml}G_{1ij}) \Gamma(s)_{jm} \tag{A.24}
\]

\[
M_{22ij} = \delta_{ij} + \frac{\bar{J}kT}{d} \sum_{nl} \Gamma(d_{x^2-y^2})_{il} (G_{1ij}G_{2ml} + g_{ij}g_{ml}) \Gamma(d_{x^2-y^2})_{jm} \tag{A.25}
\]

\[
M_{23ij} = \frac{kT}{d} \sum_{nml} \Gamma(d_{x^2-y^2})_{il} (g_{ij}G_{1ml} + g_{ml}G_{1ij} + g_{im}G_{1jl} + g_{jl}G_{1im}) h_{jm} \tag{A.26}
\]

\[
M_{24ij} = \frac{kT}{d} \sum_{nl} \Gamma(d_{x^2-y^2})_{il} (g_{ij}G_{1jl} + g_{jl}G_{1ij}) \tag{A.27}
\]

\[
M_{31ij} = -\bar{J}kT \sum_{nm} (-g_{ij}g_{mi} + G_{1ij}G_{1mi}) \Gamma(s)_{jm} \tag{A.28}
\]

\[
M_{32ij} = \bar{J}kT \sum_{nm} (g_{ij}G_{1mi} + g_{mi}G_{1ij}) \Gamma(d_{x^2-y^2})_{jm} \tag{A.29}
\]

\[
M_{33ij} = \delta_{ij} + kT \sum_{nm} h_{jm} (-g_{ij}g_{mi} + G_{1ij}G_{1mi} - g_{im}g_{ji} + G_{1im}G_{1ji}) \tag{A.30}
\]

\[
M_{34ij} = kT \sum_{n} (-g_{ij}g_{ji} + G_{1ij}G_{1ji}) \tag{A.31}
\]
In writing Eqs. (A.18), we have made explicit the \( r \) dependence of Eqs. (A.9-A.12). We note, however, that there is also an implicit dependence on \( r \) through the dispersion (A.16) (where \( x = r^2 \)), which enters the various Green’s functions in Eqs. (A.13-A.15).

Since the matrix elements in Eqs. (A.20-A.31) are all calculated in the translation-invariant clean system, Eqs. (A.18) can be easily solved in \( k \)-space by matrix inversion. Normal state results are obtained by removing the second row and column and setting \( \Delta (k) \) to zero. Non-correlated results correspond to the absence of slave bosons and constraints, so we just remove the third and fourth rows and columns and set \( x = 1 \) and \( \lambda_i = 0 \). In every case, the clean limit is first solved self-consistently for \( \chi, \Delta, \lambda \) and \( \mu \), and then the fluctuations in the presence of impurities are obtained.

In discussing the solution to Eqs. (A.18), we rely on the fact that all quantities in Eqs. (A.20-A.31) are non-singular and finite as \( x \to 0 \). Thus, we can write their formal solution as

\[
\delta \Phi = r^2 \left( A + r^2 B \right)^{-1} C = r^2 A^{-1} C + \mathcal{O}(r^4).
\]  

(A.32)

It follows that \( \delta \chi_i, \delta \Delta_i, r \delta r_i, \) and \( \delta \lambda_i = \delta \lambda_i + \varepsilon_i \) are all of order \( r^2 = x \)
APPENDIX B

THE IRRELEVANCE OF SPINON FLUCTUATIONS AND THE “MINIMAL MODEL”

We can shed light on the strong healing effect by studying a simplified case obtained by “turning off” the $\delta \chi_i$ fluctuations. In this case, we need to solve the smaller set of equations

$$
\begin{pmatrix}
M_{22} & M_{23} & M_{24} \\
M_{32} & M_{33} & M_{34} \\
0 & \lambda 1 - \frac{\lambda_2}{2} \Gamma(s) & r^2 1
\end{pmatrix}
\begin{pmatrix}
\delta \Delta \\
r \delta \lambda \\
r^2 \varepsilon
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
r^2 \varepsilon
\end{pmatrix}. \tag{B.1}
$$

The healing factor obtained in this simplified model is almost identical to the full solution, as shown by the red and green curves of the left panel of Fig. 2 of the main text. This shows that the spinon field fluctuations are utterly irrelevant for the strong healing.

A further fruitful simplification is obtained by setting $M_{32}$ to zero in Eqs. (B.1). This defines what we called the “minimal model” (MM). In this case, the “strong-correlation sub-block” of $\delta r_i$ and $\delta \lambda_i$ fluctuations decouples and suffers no feed-back from the gap fluctuations. In fact, the MM corresponds to breaking up the solution to the problem into two parts: (i) the spatially fluctuating strong correlation fields $r_i$ and $\lambda_i$ are first calculated for fixed, uniform $\Delta$ and $\chi$, and then (ii) the effects of their spatial readjustments are fed back into the gap equation to find $\delta \Delta_i$.

Strikingly, the healing factor in this case is numerically indistinguishable from the one obtained from Eqs. (B.1) (green curve of the left panel of Fig. 2 of the main text). Furthermore, the full, local and non-local PS of gap fluctuations are also captured quite accurately by the MM, as seen in Fig. B.1. We conclude that the MM, which incorporates only the effects of strong correlations, is able to describe with very high accuracy the healing process in the $d$-wave SC state.

The MM also permits us to obtain simple and physically transparent expressions. In particular, it follows immediately that

$$
r \delta r(k) = \frac{r^2}{\lambda a(k) - r^2 M_{33}(k) / M_{34}(k)} \varepsilon(k), \tag{B.2}
$$

56
Figure B.1: Power spectra of gap fluctuations $S(k)$, $S(k)_{\text{loc}}$ and $S(k)_{\text{nonloc}}$ (first to third columns) for $x = 0.2$ in the presence of correlations. The top figures were obtained from the full solution of the linearized Eqs. (A.18), whereas the bottom ones correspond to the minimal model (Eqs. (B.3)). Note that the healing factors are $h = 0.74\%$ (top) and $h = 0.69\%$ (bottom).

where $a(k) = 1 - \Gamma_s(k)/4$, and we used the Fourier transform of $\Gamma(s)$, $\Gamma_s(k) = 2[\cos(k_xa) + \cos(k_ya)]$.

Moreover,

$$
\delta \Delta(k) = \frac{r^2 [M_{24}(k) M_{33}(k) - M_{23}(k) M_{34}(k)]}{M_{22}(k)} \delta n(k),
$$

(B.3)

$$
\delta \Delta_{\text{loc}}(k) \approx -\chi_{\text{pc}}^{MM}(0) \frac{8r^2/\lambda}{k^2 + \xi_S^2} \varepsilon(k),
$$

(B.7)
where

\[
\frac{1}{\xi_S} = \sqrt{-\frac{4r^2 M_{33}(k = 0)}{\lambda M_{34}(k = 0)}}.
\]  \hspace{1cm} (B.8)

Eqs. (B.6) and (B.8) give us the expressions for the pair-charge correlation function and the healing length within the MM.
It is instructive to analyze also the behavior of the charge fluctuations in the normal state. This can be achieved by suppressing the second row and column of Eqs. (A.18) and setting $\Delta(k)$, and thus $g(i\omega_n, k)$, to zero. Even after these simplifications, the full solution is long and cumbersome. However, accurate insight can be gained from a MM of the normal state, in which we also set the $\delta\chi_i$ to zero by hand. As before, the strong-correlation sub-block decouples and Eq. (B.2) is still valid (albeit with matrix elements calculated in the normal state). The local part of the charge response is given by an expression similar to Eq. (B.7)

$$\delta n_{loc}(k) \approx -\frac{8r^2/\lambda}{k^2 + \xi_N^{-2}} \epsilon(k),$$

(C.1)

where $\xi_N$ is given by Eq. (B.8), again with matrix elements calculated in the normal state. The behavior of $\xi_N$ as a function of doping is shown by the green curve of the right panel of Fig. 2 of the main text.

In addition, just like in the Coulomb gas, the density fluctuations also show Friedel-like oscillations coming from the singularity in the response function at $2k_F$. Thus, expanding Eq. (B.2) in $r^2$,

$$\delta n_{nonloc}(|k| \approx 2k_F) \approx -\frac{2r^2}{\lambda a(|k| \approx 2k_F)} \left[ 1 + \frac{r^2 M_{33}(|k| \approx 2k_F)/M_{34}(|k| \approx 2k_F)}{\lambda a(|k| \approx 2k_F)} \right] \epsilon(k).$$

(C.2)

Since

$$M_{34}(k) = \Pi(k),$$

(C.3)

$$M_{33}(k) = 1 + \Pi^b(k),$$

(C.4)

$$\Pi(k) = \frac{1}{V} \sum_q f \left[ \hat{h}(q+k) - \hat{h}(q) \right],$$

(C.5)

$$\Pi^b(k) = \frac{1}{V} \sum_q f \left[ \hat{h}(q+k) - \hat{h}(q) \right] [h(q+k) + h(q)],$$

(C.6)

$$h(k) = -t\Gamma_s(k) - 4t' \cos(k_x a) \cos(k_y a),$$

(C.7)
Figure C.1: Spatial variations of normalized local density $\delta n_i / n_0$ in the normal state for three impurities (first column) and the corresponding power spectra $N(k)$, $N(k)_{loc}$ and $N(k)_{nonloc}$ (second to fourth columns), in the presence (top) and in the absence (bottom) of strong correlations for $x = 0.2$. The strong suppression of density oscillations by correlations is accompanied by the dominance of the spherically symmetric local power spectrum $[N_{loc}(k)]$ over the anisotropic non-local one $[N_{nonloc}(k)]$.

The leading divergent behavior is

$$\frac{M_{33} (|k| \approx 2k_F)}{M_{34} (|k| \approx 2k_F)} \approx \frac{1}{\Pi (|k| \approx 2k_F)}.$$  \hfill (C.8)

The two contributions from Eqs. (C.1) and (C.2) together give, in real space,

$$\frac{\delta n_i}{n_0} = x \sum_j \left( c_1 \frac{e^{-r_{ij}/\xi}}{\xi^{(d-3)/2}(r_{ij})^{(d-1)/2}} + c_2 x \left[ \Pi^{-1} \right]_{ij} \right) \varepsilon_j, \hfill (C.9)$$

where $r_{ij}$ is the distance between sites $i$ and $j$, and $c_1$ and $c_2$ are constants that depend on $t, t', J$ and $x$.

We stress that in the full solution of the linearized equations in which $\delta \chi_i \neq 0$, the structure of Eq. (B.2) is still preserved, with the factor $M_{33}/M_{34}$ being replaced by a long combination of several $M_{ij}$ elements, which, however, has a finite negative $k \to 0$ limit and a singularity at $2k_F$. Therefore, the results of Eqs. (C.1), (C.2) and (C.9) remain valid in the general case. The spatial charge fluctuations for three impurities and the PS in the normal state in the full solution are shown.
in Fig. C.1 both in the absence and in the presence of strong correlations. Note how the non-local part is down by an additional factor of $x$ as compared to the local part [see Eqs. (C.2) and (C.9)].
APPENDIX D

SHOCK WAVE PROPERTIES

D.1 Method of characteristics

In this section, we demonstrate that shock waves are typically formed in presence of sufficient nonlinearity, and analyze the problem via the method of characteristics. Considering the vacancy migration in highly resistive barriers, where we can neglect the normal diffusion term $D\partial_{xx}u$, we obtain a first order, partial differential equation of the general form

$$\partial_t u + c(u, t)\partial_x u = 0.$$  \hspace{1cm} (D.1)

Here, the $u$ dependence in $c(u, t)$ gives rise to the nonlinearity effect for the onset of shock wave formation. In order to determine the density profile evolution, we need to track the motion of points corresponding to specific values of $u$ (see Fig.4.2, left panel). We can solve the trajectories (the so-called "characteristics") followed by such point in the $t-x$ plane by solving the following equation:

$$\frac{dx}{dt} = c(u, t).$$  \hspace{1cm} (D.2)

As different characteristics might intersect with each other in the $t-x$ plane (see the inset of Fig.4.2), the emergence of intersecting trajectories indicates that the solution $u(t, x)$ becomes multivalued, since the points can be traced back along each of the characteristics to different initial values of $u(t = 0, x)$. The intersection which happens chronologically first determines the formation of the shock wave, as shown in Fig.4.2. As an illustration, we provide an example with $c(u, t) = u^2$ and an initial Gaussian distribution $u(t = 0, x) = 0.5\exp\left[-(x - 5)^2/4\right]$. The time dependent wave profile is solved numerically with fixed boundary conditions, and characteristics are computed according to Eq.D.2.

We should stress, however, that this qualitative behavior is a robust feature of any such nonlinear diffusion equation, provided that the prefactor $c(u, t)$ is a monotonically increasing function of $u$ (i.e. $\partial_u c(u, t) > 0$). To see this, note that Eq. (D.1) takes the form of a simple wave equation, thus its

\footnote{It’s straightforward to see that $u$ is constant along the characteristics, since $du/dt = dx/dt\partial_x u + \partial_t u = 0.$}
solution is a traveling wave, with speed locally proportional to \(c(u,t)\). Therefore, at any given time, each point on the wave front moves with a different speed proportional to \(c(u,t)\) and the “crest” having the largest value of \(u\) moves fastest, which leads to a “kink” type shock. Alternatively, it is easy to check that if \(\partial_u c(u,t) \leq 0\), the shock wave will not form. In the case where \(c(u,t)\) is any function independent of \(u\), the characteristics are curves of the same shape but parallel to each other, and hence they would not have any intersections; the shock wave would again not form. Finally, it can be shown that if the diffusion term \(D\partial_{xx}u\), is non-zero but is parametrically small (as compared to the drift term), the diffusion will prevent the wave from topping over (to undergo 'self-breaking'), and the shock wave front will remain sharp as it propagates [77, 79].

**D.2 Dynamics of shock waves**

In this section, we provide a simple derivation of the Rankine–Hugoniot condition [77, 81] which determines the equation of motion of the shock wave front. An important feature of the shock wave are the spatial discontinuities of \(u\) and \(j\). Considering a shock wave front propagating on the interval \([0,d]\) in one dimension, we define \(u_+ \equiv u(t, x_s(t) + \epsilon)\), \(u_- \equiv u(t, x_s(t) - \epsilon)\), \(j_+ \equiv j(t, x_s(t) + \epsilon)\) and \(j_- \equiv j(t, x_s(t) - \epsilon)\), where \(x_s(t)\) is the coordinate of the shock wave front and \(\epsilon \to 0^+\) is a infinitesimal positive quantity. According to the continuity equation \(\partial_t u + \partial_x j = 0\), we can write:

\[
\frac{d}{dt} \left( \int_0^{x_s(t)} u dx + \int_{x_s(t)}^d u dx \right) = j(0) - j(d),
\]

\[\text{(D.3)}\]

\[
\frac{dx_s}{dt} u_- - \frac{dx_s}{dt} u_+ + \int_0^{x_s(t)} \partial_t u dx + \int_{x_s(t)}^d \partial_t u dx = j(0) - j(d).
\]

\[\text{(D.4)}\]

Using again the continuity equation, we can immediately determine the shock wave front velocity via the following equation:

\[
v_s = \frac{dx_s}{dt} = \frac{j_+ - j_-}{u_+ - u_-} = \frac{\Delta j}{\Delta u} \bigg|_{x_s},
\]

\[\text{(D.5)}\]

which is often called Rankine–Hugoniot condition.
APPENDIX E

ANALYSIS OF NUMERICAL AND EXPERIMENT RESULTS

E.1 VEOVM model

As it was mentioned in the main text, to simulate the vacancies dynamics we adopted the voltage enhanced oxygen vacancy migration model (VEOVM), which is a well validated model for the RS effect [64, 48]. Taking a capacitor-like device, this model considers a 1 dimensional conductive channel connecting the two contacts, along which oxygen vacancies can migrate through (see Fig. E.1). This channel is divided in small cells corresponding to physical nano-domains, and at the same time the whole device is divided in two regions: the active interfacial region close to the metal contact (i.e. high resistive Schottky Barrier) and the high conductive bulk region. A diagram of the model with a single active contact, as used in the simulations, is presented at the top panel on Fig. E.1. The resistance of each cell in the channel $r(x)$ is proportional to the local density of vacancies: $r(x) = u(x)A_\alpha$, where $u(x)$ is the density of vacancies on the cell located at $x$ and $A_\alpha$ is a proportionality constant whose magnitude depends on the region of the device: $\alpha = SB, B$, where $SB$ stands for Schottky barrier and $B$ for bulk, and where $A_{SB} \gg A_B$. Migration is assumed to occur only between neighbours cells, and in each step of the simulation the probability for vacancy migration from a cell at $x$ to its neighbour at $x + \Delta x$ is computed according to:

$$P(x, x + \Delta x) = u(x)(1 - u(x + \Delta x))\exp(\Delta V(x) - V_0), \quad (E.1)$$

where $\Delta V(x)$ is the drop of voltage (per unit length) at $x$ and $V_0$ is an activation constant for vacancy migration. In each step of the simulation the migration from cell to cell is computed and the new resistance of each cell is calculated. The total resistance of the device is calculated simply as the addition of all the cells in the channel. For the values used in the simulations, the middle term $(1 - u(x + \Delta x))$ can and will be neglected in the theoretical analysis explained in the following sections. As mentioned in the main text, this model corresponds to the case of granular materials with activated transport process. To put it in the context of the generalized Burgers' equation description made
in main text, it is easy to see that the drift current originated from this model satisfies the general form \( j_{\text{drift}} \sim \sinh(E) \). The net current of vacancies generated by the action of an external electric field \( E(x) = \partial V(x)/\partial x \), between a cell at \( x \) and its neighbour cell at \( x + \Delta x \) can be written simply as Fick’s-law for the migration probability: \( j_{\text{drift}} = \partial P/\partial x \approx [P(x, x + \Delta x) - P(x + \Delta x, x)]/\Delta x \), where in the last term the discrete character of the model has been introduced. In particular for our model we take \( \Delta x = 1 \). Using the definition of \( P \) from the model (neglecting its middle term), we get from here that \( j_{\text{drift}} = P(x, x + \Delta x) - P(x + \Delta x, x) = 2Du \sinh(\Delta V(x)) \), where \( D \) stands for the Arrhenius factor \( \exp(-V_0) \). If the external electric stress is a controlled current \( I \), then \( \Delta V(x) = Ir(x) = IA_\alpha u(x) \).

E.2 Simulations details

In order to analyze the existence of a shock wave scenario, the Hi to Lo process was simulated for different external currents. Previously, an initial forming process was performed at which current of different polarities was applied and well defined Hi and Lo states were obtained. In the top-right panel of Fig.E.1 the vacancy profile is shown before and after the forming process. The device starts with a uniform distribution along the channel, and by the end of the forming process the profile shows a characteristic distribution where vacancies migrated out from the interfacial region and accumulated in the neighboring low-field bulk region [64]. The figure shows the formed Hi resistance state where it can be seen a second accumulation of vacancies next to the left metal contact in the interfacial region. This second accumulation of vacancies constitutes the initial distribution for the Hi to Lo process and it will evolve to form the shock wave during switching as shown in the main text. To improve the stability of the simulations the currents were applied using a rise time (from 0 to their actual values) always negligible compared with the characteristic times of the process \( \tau_1 \) and \( \tau_2 \) (with rise times in the order of a few thousands of steps). The simulated device contained a total of 1000 cells with a 100 cells long interfacial region (left interface). For the initial distribution a uniform density of \( u_0 = 1 \times 10^{-4} \) per cell was used. The value of the activation constant \( V_0 \) was set to 16, which corresponds to the activation energy of 0.4\( eV \) reported in similar manganite systems [51] ( i.e. , 0.4\( eV/k_B T = 0.4/0.026 \) at room temperature). For the resistivity constants we used \( A_{SB} = 1000 \) and \( A_B = 1 \) (for a table with the parameters see Fig.E.1). For the analysis developed in this paper we solely consider the shock wave after it is formed. In the bottom panel of Fig.E.1
we show a standard formation process of the shock wave: the initial narrow accumulation against the metal contact rapidly evolves into the shock wave profile.

Figure E.1: Top panels: top-left: Schematic diagram of the VEOVM model with a single active contact. The two regions SB and B correspond to the high resistance interface (Schottky Barrier) and the more conductive central bulk, respectively. The small cells within the channel indicate the domains. Top-right: vacancy distribution along the conductive channel before (black squares) and after (red circles) the forming process. The distribution shown after the forming process correspond to a Hi resistance state. Starting from a uniform distribution, during the forming process a characteristic distribution is obtained where vacancies initially at the interface migrates to the low-field bulk region generating a pile-up of vacancies close to the limit between the two regions. In the Hi state, part of these vacancies migrates back to the interfacial region and are accumulated in the vicinity of to the metal contact. Bottom-left (table): parameters used in the simulations. Bottom-right: Snapshots of the shock wave formation in the early stage of simulations (between the first and second snapshot of Fig. 4.3).

### E.3 Fitting for VEOVM model

In this section, we explain the details of the comparison between simulation and theory and the procedures used for fitting. Consider the drift current within left interfacial region in VEOVM model:
\[ j(u, t, x) = P(x, x + \Delta x) - P(x + \Delta x, x) = 2Du \sinh (IA_{SB}u) \]  
(E.2)

where the term \(1 - u(x + \Delta x)\) in the probability has been neglected as explained before \(^1\). From Rankine–Hugoniot conditions an expression for the shock-wave front velocity \(\frac{dx_s}{dt}\) can be obtained, as provided in Eq. (4.4) in main text. Performing a redefinition of parameters (for practical reasons only) we can rewrite such equation as:

\[
\frac{dx}{dt} = \frac{2Dx_{int}}{x_{int}} \left[ \left(1 + \frac{x}{\alpha x_{int}}\right) \sinh \left(\frac{I}{\beta} + \frac{I}{\alpha x}\right) - \frac{x}{\beta} \sinh \left(\frac{I}{\beta}\right)\right],
\]  
(E.3)

where we used that \(u_+ = u_+ + \Delta u\), \(Q = \Delta u x_s\) is the total number of vacancies carried by shock wave, \(Q_B \equiv u_+ x_{int}\) is total number of background vacancies in the left region (being \(x_{int}\) the length of the interfacial region), \(x \equiv x_s/x_{int}\) is the normalized coordinate, and where we defined the new parameters \(\alpha \equiv x_{int}/A_{SB}Q\) and \(\beta \equiv x_{int}/A_{SB}Q_B\). From the previous definitions we can write the high resistance value as \(R_{HI} = A_{SB}(Q_B + Q)\) where \(R_{HI}\) is a constant determined by the vacancy concentration and independent of \(I\) \(^2\).

We can solve then this equation numerically with the material dependent parameters \(x_{int}, \alpha,\beta\) and \(D\). Considering the initial rise time for the current and the non-flatness of the shock wave, an accurate test of the \(x_s(t)\) prediction can be performed for \(x > x_0 \approx 0.4\) using the integral-form equation:

\[
t - t_0 = \int_{x_0}^{x} \frac{(x_{int}/D)dy}{2 \left(1 + \frac{\alpha y}{\beta y}\right) \sinh \left(\frac{I}{\beta} + \frac{I}{\alpha y}\right) - \frac{2\alpha y}{\beta} y \sinh \left(\frac{I}{\beta}\right)}
\]  
(E.4)

which is used for the fit of the simulation data for \(t < \tau_1\) in the lower left panel of Fig.4.3 in the main text.

On the other hand, we have also the rate equation for the switching (i.e. 'leakage') phase (cf. Eq.4.6 from main text). The numerical solution of this equation is used to fit the simulation data in the resistance switching phase for \(\tau_1 < t\), shown in Fig.E.2.

\(^1\)The typical value of \(u(x)\) in simulation varies between \(10^{-4}\) and \(10^{-3}\).

\(^2\)There is in fact a small migration of background vacancies into the bulk during the period of shock wave propagation which causes a small variation of the resistance during this phase.
The values of the parameters that enter the equations Eq. E.4 and Eq. 4.6 were extracted directly from the simulation results. We find $Q_B = 63.4u_0$, $Q = 15.3u_0$, $A_{SB} = 1000$, $R_{HI} = 7.9a.u.$ and $D = 1.12 \times 10^{-7}$. The very good fits of the simulation data shown in Fig. 4.3 and Fig.E.2 were achieved by slightly relaxing the value of the single parameter $x_{int}=100$ to the values 94.3 (Fig. 4.3) in the propagation phase and 82.2 (Fig.E.2) in the leakage phase.

The small discrepancy between the relaxed parameters and its actual value mainly comes from the non-flatness of the density profile both during the propagation of the shock wave and during the leaking phase, as well as from the fact that a small part of background vacancies leaks into the bulk during the propagating phase.

![Figure E.2: Evolution of the resistance during the switching phase for different applied currents according to simulations (dots) and theory (lines) from Eq. (4.6). The currents shown are $I = 58.5a.u., 71.5a.u., 84.5a.u.$ and 97.5a.u.](image)

### E.4 Experimental details

For the experimental validation of our theory, the RS phenomenon was studied in a bulk La$_{0.325}$Pr$_{0.300}$Ca$_{0.375}$MnO$_3$ (LPCMO) polycrystalline sample with hand-painted Ag contacts of approximately 1mm diameter. The device is a parallelepiped of $1 \times 1 \times 10\text{mm}^3$, with the contacts painted over the 10mm surface. The distance between contacts is also in the order of 1mm.
Figure E.3: (a) Diagram of the experimental set up. (b) Sketch of the pulsed current protocol used in the experiments. A high current Write pulse (blue) is followed by a low current Read pulse (red). The first pulse generates the RS while the second pulse measures the non-volatile resistance. (c) R vs t curves from a complete RS process. The system is first taken to a Hi resistance state under a current of -350mA (left panel). Then the Hi to Lo RS is measured under an applied current of 37.5mA (middle panel). Finally the system is taken back, ie “re-initialized”, to a Hi resistance state with the application of a -350mA current.

Measurements were performed using a three wire configuration in order to develop a single contact analysis.

To induce the RS effect an external constant current was used as the electric stimulus. To generate this current and to acquire the data a Keithley 2612 Sourcemeter was employed. The measurements were done using a three wire configuration in order to measure a single interface resistance (Fig.E.3.a). The Hi to Lo RS was studied for currents of different magnitude, as described in the main text. More information about the measurement technique and simultaneous contact analysis can be seen in ref. [82].

The application of current is done following a pulsed protocol: a high current pulse (Write) is followed by a low current pulse (Read). A schematic diagram of the pulsed protocol is shown in (Fig.E.3.b). Each pulse lasts 1ms. Between two pulses there is an interval of about 0.5s, meant to reduce possible heating effects. The time axis exhibited in the $R$ vs $t$ curves (both in Fig.4.4 and in
Fig.E.3.c, below) is the effective time elapsed during the actual application of the Write pulses, i.e. disregarding both the 0.5 s timeout and the Reading elapsed time. The Write pulses possess enough strength to change the resistive state of the system, while the Read pulse (low current) measures the remnant (stable, non-volatile) resistance of the device without affecting it. In the experiments shown in main text, the Write pulses are in the order of the mA while the Read pulses are in the order of the µA.

The complete measurement process is shown in Fig.E.3.c. To obtain the initial Hi state, pulses of -350mA were applied (Fig.E.3.c, left panel). Next, the desired accumulation experiment is performed, by applying positive pulses of constant amplitude which decrease the resistance (Fig.E.3.c, middle panel). The initial Hi -resistance state is recovered by applying -350 mA pulses (Fig.E.3.c, right panel). In every case the initial Hi-resistance state obtained was of the same magnitude within a range of about 1%.

### E.5 Scaling

In this section, we show that the scaling behaviour is a direct consequence of the strong non-linearity of the drift current $j(u,t,x)$, which depends on local electric field exponentially. In the time range where significant resistive switching occurs, we have $(IR/x_{int} \gg 1)$ and we may approximate $\sinh(I R/x_{int}) \approx \frac{1}{2} \exp (I R/x_{int})$. From Eq.(4.6), using a normalized resistance $\tilde{R} = R/R_{HI}$, we have:

$$\frac{d\tilde{R}}{dt} = -\frac{D}{x_{int}} \exp \left[ \frac{I R_{HI}\tilde{R}}{x_{int}} \right] \left( 1 + \lambda \right), \quad (E.5)$$

where $\lambda = \frac{x_{int}}{R_{HI}} \ln \tilde{R}/(I \tilde{R})$ is a small parameter as long as $\tilde{R}$ close is to 1 (i.e. at the beginning of the resistive change). For simplicity we consider the leading order as we set $\lambda = 0$:

$$\tilde{R} = 1 - \frac{x_{int}}{I R_{HI}} \ln \left( 1 + \frac{t^*}{\tau_2} \right), \quad (E.6)$$

where $t^* = t - \tau_1$ is the time measured from the impact time as explained in main text, and $\tau_2$ is a characteristic time for the resistance switch and is dominated by an exponential dependence of the applied current as follows:
\[ \tau_2 = \frac{x_{int}^2}{DIR_{HI}} \exp\left(-\frac{IR_{HI}}{x_{int}}\right). \]  

(E.7)

In Fig. E.4 we use this approximate expression to fit the \( R(t) \) simulation data. Comparison with the previous fit done with Eq. 4.6 and shown in Fig. E.2 allows us to check that this approximate solution is relatively accurate within the time domain we are interested in for fast switching devices.

![Figure E.4](image_url)

Figure E.4: Fitting results for the evolution of the normalized resistance according to the Eq. (E.6).

Now, using Eq. (E.6), we can show that the \( I \)-dependent family curves \( R(t) \) should obey scaling. We first consider the normalized resistive change \( \delta R(t) \) defined as

\[
\delta R(t) = \frac{R(t) - R(\tau_0)}{R_{HI} - R(\tau_0)}
\]

(E.8)

where \( \tau_0 \) is some yet unspecified time (and we drop the \( * \) from \( t^* \)). Then, replacing with Eq. E.6 we have,

\[
\delta R(t) = \frac{R_{HI} - \frac{x_{int}}{\tau_2} \ln(1 + \frac{t}{\tau_2}) - R_{HI} + \frac{x_{int}}{\tau_2} \ln(1 + \frac{\tau_0}{\tau_2})}{R_{HI} - R_{HI} + \frac{x_{int}}{\tau_2} \ln(1 + \frac{\tau_0}{\tau_2})}
\]

(E.9)

And rescaling the time by \( \tau_0 \),

\[
\delta R\left(t/\tau_0\right) = 1 - \frac{\ln(1 + \frac{\tau_0}{\tau_2} \frac{t}{\tau_0})}{\ln(1 + \frac{\tau_0}{\tau_2})}
\]

(E.10)

Notice that with the natural choice of simply setting \( \tau_0 \) as \( \tau_2 \), we obtain the scaling form that we presented in the main text.
However, in the experiment we do not know, a priori, how to determine the characteristic time scale $\tau_2$, which is a strong function of the current $I$ and other material parameters. So we adopt the following strategy. We use $\tau_0$ as a free scaling parameter, one for each $I$-dependent curve $R(t)$, that we rescale according to Eq. E.10. We vary the set of values $\tau_0[I]$ until we obtain a collapse of all the experimental and the simulation curves. The results of the successful collapse are shown in Fig.4.5 of the main text. In our experience, the collapse is unique. We found a sole way to properly collapse the whole set of curves, which gave us further confidence on the adopted procedure. A crucial point now is that the collapsed set of curves could be fitted with the expression

$$F(t) = 1 - \frac{\ln(1 + ct)}{\ln(1 + c)}.$$  

(E.11)

with $c$ a current independent constant. We find the values $c = 15.2$ for the experimental data and $c = 29.4$ for the simulation ones. Hence, in regard of Eqs.E.10 and E.11 we observe that the constant $c$ is nothing but the ratio between the analytically established characteristic time $\tau_2$ and the empirically determined $\tau_0$, which are simply proportional to one another.

In Fig.E.5 we plot the dependence of the scaling time as a function of the applied current $\tau_0(I)$. In the case of the simulation results, the data follow the same exponentially decreasing behaviour as deduced for $\tau_2(I)$ (see Eq.E.7). Moreover, we find the ratio $\tau_0(I)/\tau_2(I)$ in good agreement with the constant $c = 29.4$, which validates our practical scaling procedure for the determination of the characteristic switching time in the experimental case.

Under the same set of approximations that we have assumed in this section, plus the additional one of neglecting the effect of the background distribution of vacancies, one may also derive an explicit expression for the characteristic time $\tau_1$. We start from Eq. (E.3), and similarly as before, adopt the approximation $\sinh(\frac{IR}{x_{int}}) \approx \frac{1}{2} \exp(\frac{IR}{x_{int}})$. Then, making the substitution $y' = 1/y$ in Eq. (E.4), and assuming that the integral is dominated by the exponential factor (i.e. neglecting the change in lower order factors), the integral has an analytical solution, and we get for the case with no background vacancies (i.e. $1/\beta = 0$)

$$\tau_1 \approx C(t_0, x_0) + \frac{x_{int}^2}{D \tau_{HI}} \exp \left( \frac{-IR_{HI}}{x_{int}} \right).$$  

(E.12)

where the first term is the integration constant determined by the initial conditions $t_0$ and $x_0$. If this constant can be neglected (for example for fast forming shock waves) then we have the surprising result that $\tau_1 = \tau_2$.  

72
In the case of the model simulations, the results displayed in Fig.E.5 shows that, in fact, both characteristic times have the same $I$-dependent exponentially decaying behavior. Moreover, the ratio of the two characteristic times is approximately 25, which is close to the constant $c = 29.4$ quoted before, therefore our simulations also validate the equality between the two characteristic times $\tau_1$ and $\tau_2$ predicted by the shock wave scenario.

As may be expected, the experimental situation is only qualitatively consistent with the previous discussion. As seen in Fig.E.5, for the experimentally determined characteristic times, we observe that they have approximately a similar dependence with the applied current, however, it is less clear if the equality between them also holds.

We show and compare the evolution of both characteristic times under different applied current in Fig.E.5 for both simulations and experiments, observing a good agreement with the analysis presented here.

![Figure E.5: Evolution of the characteristic times for different applied currents for both simulations (left panel) and experiments (right panel). It can be seen that both times follow an approximate exponential dependence with $I$, and that there exist a relative proportionality between them as predicted by our analysis.](image)

E.6 Quantitative Comparison: Experiments & Simulations

The magnitudes of the main parameters used in the simulations can be related to actual experimental magnitudes. In our model all voltages are normalized by $k_B T/e$ and energies by $k_B T$. As experiments were performed at room temperature, we have $k_B T/e \approx 26$ mV and $k_B T \approx 0.026$ eV. In the simulations the activation energy constant was given a value of $V_0 = 16 \ k_B T \approx 0.4eV$, which
is consistent with experimental values reported for similar PCMO devices [51]. A correspondence can also be established for the voltages adopted in experiments and simulations. In the former the applied voltages used for the resistive-commutation were in the order of 1V. In units of $k_B T/e$ this leads to the dimensionless voltage values in the order of a few tens ($1V/26mV = 38.5$). However, in order to perform the extensive computational work required by our study within reasonable computational times, the values adopted in the simulations were higher by about one order of magnitude. For evident practical reasons (equipment and device limitations) such an increase of applied voltage cannot be performed in the experiments. Nevertheless, it should be clear from our theoretical discussion of the mathematical nature of the equations, that the same qualitative results, namely formation of shock waves, also remain valid at lower applied voltages. Despite the significantly higher computational cost (days versus hours), we present here a set of runs for voltages that are close to the experimental values. In figure E.6, we show the results for a simulation with an applied dimensionless current $I = 6.75$, which corresponds to a physical units $V = 1.4V$, since the initial (dimensionless) $R = 8$ and then $V = IRk_B T/e = 6.75 \times 8 \times 26mV \approx 1.4V$. For convenience, in the figure we also reproduce the corresponding results from the main text for a simulated voltage of $V = 14.87V$ ($I = 71.5 a.u.$).
Figure E.6: Shock wave dynamics for two different applied currents, which differ by nearly an order of magnitude. The lower current case (right panel) corresponds to applied voltages comparable to the experimental ones, and the higher voltage cases (left panel, same as FIG. 2) show similar shock wave dynamics. The two lower-right panels demonstrate that (i) the propagation of the shock wave front remains qualitatively the same as for higher applied currents, and (ii) the collapse of the snap-shots of successive profiles indicates the formation of the shock-wave.
APPENDIX F

SHOCK-WAVE FORMATION IN BINARY OXIDES

In this section we describe under what conditions one may expect the formation of shock waves (SW) to occur in binary insulating transition metal oxides such as HfO$_2$, TiO$_2$, TaO, which are of current interest. These insulating systems are qualitatively different from the doped manganite compound that we consider in the main text. Nevertheless, it is interesting to realize that they may also sustain the formation of a SW front according to our general formalism. A key point to realize is, however, that the SW may occur for the propagation of the oxygen density profile and not the oxygen vacancy as in the manganite case. This is because in the latter oxygen vacancies increase the local resistivity by means of creating defects in the oxygen-metal-oxygen bonds, hence in the conduction bands, while, in contrast in the former case of binary systems the oxygen vacancies dope electron carriers to an otherwise good insulator. Hence, the crucial features to realize that SW may occur in binary systems are the following: (i) the electroforming step renders the highly insulating system poorly conductive by creating a path with a massive production of oxygen vacancies (OV). In an extreme case, this path may become a purely metallic filament, which leads to the so called non-polar resistive switching mode. However, if this is not so drastic, one gets to the bi-polar switching mode with an intermediate density of OV. (ii) Within such a conductive path, oxygen ions move through the OV sites by means of the applied electric field. As local oxygen density increases, the system locally approaches the stoichiometric formula, hence, becoming locally more insulator with a local increase of resistivity. (iii) The local increase of resistivity leads to higher local voltage drops, hence higher local electric fields, which further promote a higher motion of oxygen ions, leading to the formation of the SW. Interestingly, this phenomenon may have occurred in the simulation study of Strukov et al. (cf. Figs. 2(f) and 2(h) of [59]).

We shall now describe how the expressions of our SW scenario derived for conductive manganites may be recast for insulating binary compounds. Let's consider $u_O + u_{OV} = n$ where $n$ is the integer for the oxygen atoms in the unit formula, and $u_O$ and $u_{OV}$ are the actual number of oxygen atoms and oxygen vacancies per unit formula in the sample respectively, hence they are proportional to
the respective local densities. In the main text we have derive the expressions in terms of \( u \), the oxygen vacancies that corresponds to \( u_{OV} \). As we said before we shall now focus on the oxygen density \( u_O \). Similarly as in the main text (MT), we have a continuity equation for the oxygen

\[
\partial_t u_O + \partial_x j_O(t, x) = 0.
\]

From the components of drift and diffusion currents we get

\[
\partial_t u_O + f(u_O) \partial_x u_O = D \partial_{xx} u_O, \tag{F.1}
\]

where \( f(u_O) \equiv \partial_{u_O} j_{O, drift}(u_O, I), j_{O, diffusion} = -D \partial_{xx} u_O \) and \( I(t) \) is the magnitude of the electronic current.

Notice that this equation is the generalized Burgers’ equation for the case of oxygen ion movement. The key point here is that, as pointed out in the MT, the function \( f(u_O) \) should be any monotonically increasing function of \( u_O \). Again, following the MT, \( j_{O, drift} = u_O g(E) \), where \( E \) is the magnitude of the local electric field and \( g \) clearly is an increasing function of it. The specific form of \( g(E) \) is material dependent but the important point already mentioned is that a local increase of \( u_O \) leads to a local increase of \( \rho(u_O) \), and hence of \( E \). Therefore, from analogous arguments as discussed in the MT a SW front may also occur in the binary insulating compounds. As we said before, they may occur for the case of systems in bi-polar mode RS. An interesting issue would be to investigate the temperature or current intensity dependence of the SW formation. A priori, temperature may play two competing roles. On one hand, higher currents or fields would lead to higher temperature, hence to higher oxygen ion mobility. This may favour the formation of a SW. However, temperature also decreases the local resistivity of an insulator, hence may prevent the local increase of the field intensity.

Having discussed and established under which general conditions one may expect the formation of a SW, we now present the specific formulas for the behavior of the resistance \( R(t) \) during the evolution of the SW front. Figure F.1 is a schematic representation of the time evolution of the SW and serves for the definition of the various relevant variables.

We assume that electroforming has created a path rich in oxygen vacancies, which shall dominate the conduction through the system. Thus there is a constant background of density of oxygen \( u^0_O = u_{O,+} \) on top of which the SW evolves. The initial state is a high resistive state with a pile-up of oxygen atom (at the l.h.s.) that brings the local density close to the stoichiometric value, and hence a highly insulating region. As a strong (positive) voltage is applied at the electrode at \( L \), the
The schematic plot for the oxygen shock wave front propagation. The green curve is the initial distribution where oxygen are piled up at \( x < x_0 \).

Oxygen ions will migrate rapidly forming a plateau of density \( u_{O,-} \) with a sharp front of magnitude \( u_{O,-} - u_{O,+} \) at the position \( x_s \). The front moves with velocity \( v_s \).

The total (two point) resistance of the system (we neglect the effect of the electrode contacts) is given by,

\[
R(t) = \int_{x=0}^{x=L} \rho(u_O(x)) \, dx, \tag{F.2}
\]

using the continuity equation for oxygen, we can describe the decrease rate of the resistance from the high resistive state \( R_{HI} \) to the low resistive state \( R_{LO} \) as,

\[
\frac{dR(t)}{dt} = - \int_0^{x_s} \left( \frac{d\rho}{du_O} \right) \partial_x j_O(u_O, t, x) \, dx - \int_{x_s}^L \left( \frac{d\rho}{du_O} \right) \partial_x j_O(u_O, t, x) \, dx + \rho(u_O(x_s - \varepsilon)) v_s - \rho(u_O(x_s + \varepsilon)) v_s, \tag{F.3}
\]

where the quantity \( \left( \frac{d\rho}{du_O} \right) \) is a function of \( u_O(x,t) \). The shock wave velocity \( v_s \) can be obtained via the Rankine–Hugoniot condition as described in the Appendix D.2. Since we assume a large initial pile-up distribution of oxygen density near \( x = 0 \), this creates a large local resistivity and thus large drift force, hence we expect the shock wave will form quickly. Therefore, the distribution of \( u_O(x,t) \) for \( x < x_s \) soon becomes approximately flat and will be denoted as \( u_O(t,x_s-\varepsilon) \approx u_{O,-}(t) \).
Using the boundary condition \( j_O(u_O, t, x = 0) = j_O(u_O, t, x = L) = 0 \), and defining \( j_O(u_O, t, x_s \pm \varepsilon) \equiv j_{O,\pm} \), \( \rho_{\pm} \equiv \rho(u_O(x_s \pm \varepsilon)) \), \( \rho'(u_O)_{\pm} \equiv \frac{d\rho}{du_O}|_{x_s \pm \varepsilon}, \) \( u_O(x_s \pm \varepsilon) \equiv u_{O,\pm} \), we have:

\[
\frac{dR(t)}{dt} = - \left[ \rho'(u_O)_- j_{O,\pm} - \rho'(u_O)_+ j_{O,\pm} \right] + [\rho_- - \rho_+] v_s, \tag{F.4}
\]

Recalling that we assumed the background oxygen density approximately constant, we define \( Q_O \) as the total number of active oxygen ions, which is a constant during lapse of motion of the shock wave:

\[
Q_O = [u_{O,-}(t) - u_{O,+}] x_s(t). \tag{F.5}
\]

In terms of the parameters that we defined we have,

\[
R(t) = \rho(u_{O,-}(t)) x_s(t) + \rho(u_{O,+}) (L - x_s(t)). \tag{F.6}
\]

Then taking the time derivative and comparing to Eq. F.4, we obtain

\[
\frac{du_{O,-}}{dt} = \frac{j_{O,+} \rho'(u_O)_+ / \rho'(u_O)_- - j_{O,-}}{x_s(t)}, \tag{F.7}
\]

where \( u_{O,+} \) is time independent and known and similarly for \( \rho'(u_O)_+, j_{O,+} \). To make further progress we now need the specific form of \( \rho(u_O) \) and \( j_O(u_O, t, x) \), which depend on the specific transport properties of the physical system. Eliminating \( x_s \) we could then obtain a differential equation for \( u_{O,-}(t) \):

\[
\frac{du_{O,-}}{dt} = \left( j_{O,+} \frac{\rho'(u_O)_+ / \rho'(u_O)_- - j_{O,-}}{Q_O} \right) (u_{O,-} - u_{O,+}) \tag{F.8}
\]

and then, finally, can obtain the expression for \( R(t) \) through equation Eq.F.6.

For a concrete illustration we may now assume a specific model for the conduction, where the local resistivity is a increasing function \( u_O \), as expected for the case of the binary insulators:

\[
\begin{align*}
    j_O &= 2Du_O \sinh[I\rho(u_O)] \\
    \rho(u_O) &= \frac{\rho_0}{1 + A_0 (n - u_O)} \tag{F.9}
\end{align*}
\]

This is motivated by the expected approximate linearity of the conductivity with the concentration of dopants, namely, oxygen vacancies, therefore we have, \( \sigma = 1/\rho \sim u_{OV} = (n - u_O) \). The parameter \( A_0 \) is a suitable constant and \( \rho_0 \) is the intrinsic resistivity of the stoichiometric system (i.e., for \( u_{OV}=0 \)).
From the schematic Fig.F.1, we have \( Q_O = |u_{O,-}(t = 0) - u_{O,+}|x_0 \) and we can solve Eq.F.8 using the explicit model defined in Eq.F.9. We may then solve (numerically) for the evolution of \( u_{O,-}(t) \), and finally obtain the resistance as,

\[
R(t = 0) = R_{HI} = \frac{\rho_0 x_0}{1 + A_0 (n - u_{O,-}(t = 0))} + \frac{\rho_0 (L - x_0)}{1 + A_0 (n - u_{O,+})}
\]

\[
R(t) = \frac{\rho_0 x_s}{1 + A_0 (n - u_{O,-}(t))} + \frac{\rho_0 (L - x_s)}{1 + A_0 (n - u_{O,+})}
\]

\[
R(t = t_{final}) = R_{LO} = \frac{\rho_0 L}{1 + A_0 (n - u_{O,-}(t_{final}))}.
\]

As a final remark, one may notice that from Eq.F.5 the resistance can be reparametrized as a function of the shock wave front position \( x_s \),

\[
R[x_s(t)] = \frac{\rho_0 x_s}{1 + A_0 (n - \frac{Q_O}{x_s} u_{O,+})} + \frac{\rho_0 (L - x_s)}{1 + A_0 (n - u_{O,+})},
\]

and \( R_{HI} = R[x_s(t) = x_0] \), \( R_{LO} = R[x_s(t) = L] \). It is straightforward to see that \( R[x_s(t)] \) is a monotonically decreasing function of \( x_s(t) \) and thus \( R_{LO} < R_{HI} \).
REFERENCES

[1] G. Kotliar and J. Liu. Superexchange mechanism and d-wave superconductivity. Physical Review B, 38(7):5142, 1988.

[2] P.W. Anderson. The resonating valence bond state in La$_2$CuO$_4$ and superconductivity. Science, 235(4793):1196–1198, 1987.

[3] PW Anderson, PA Lee, M Randeria, TM Rice, N Trivedi, and FC Zhang. The physics behind high-temperature superconducting cuprates: the 'plain vanilla' version of RVB. J. Phys.: Condens. Matter, 16(24):R755, 2004.

[4] P.W. Anderson et al. The Theory of Superconductivity in the High-Tc Cuprate Superconductors, volume 446. Princeton University Press, 1997.

[5] R. Daou, J. Chang, D. LeBoeuf, O. Cyr-Choinière, F. Laliberté, N. Doiron-Leyraud, BJ Ramshaw, R. Liang, DA Bonn, WN Hardy, et al. Broken rotational symmetry in the pseudogap phase of a high-Tc superconductor. Nature, 463(7280):519–522, 2010.

[6] S. Caprara, C. Castellani, C.D. Castro, M. Grilli, and A. Perali. Charge and spin inhomogeneity as a key to the physics of the high-Tc cuprates. Physica B: Condensed Matter, 280(1-4):196–200, 2000.

[7] J.L. Tallon and J.W. Loram. The doping dependence of $T^*$- what is the real high-$T_c$ phase diagram? Physica C: Superconductivity, 349(1):53–68, 2001.

[8] P.A. Lee, N. Nagaosa, and X.G. Wen. Doping a mott insulator: Physics of high-temperature superconductivity. Rev. Mod. Phys., 78(1):17, 2006.

[9] Elbio Dagotto. Complexity in strongly correlated electronic systems. Science, 309(5732):257–262, 2005.

[10] C. M. Varma. Valence fluctuations, heavy fermions and their superconductivity. Comments Solid State Phys., 11:221, 1985.

[11] B J Powell and Ross H McKenzie. Strong electronic correlations in superconducting organic charge transfer salts. J. Phys.: Condens. Matter, 18:R827, 2006. doi: 10.1088/0953-8984/18/45/R03.

[12] B J Powell and Ross H McKenzie. Quantum frustration in organic mott insulators: from spin liquids to unconventional superconductors. Rep. Prog. Phys., 74:056501, 2011. doi: 10.1088/0034-4885/74/5/056501.
[13] David C. Johnston. The puzzle of high temperature superconductivity in layered iron pnictides and chalcogenides. *Adv. Phys.*, 59:803, 2010. doi: 10.1080/00018732.2010.513480.

[14] Samiyeh Mahmoudian, Shao Tang, and Vladimir Dobrosavljević. Quantum criticality at the anderson transition: A typical medium theory perspective. *Phys. Rev. B*, 92:144202, Oct 2015.

[15] D. Rhodes, R. Schönemann, N. Aryal, Q. Zhou, Q. R. Zhang, E. Kampert, Y.-C. Chiu, Y. Lai, Y. Shimura, G. T. McCandless, J. Y. Chan, D. W. Paley, J. Lee, A. D. Finke, J. P. C. Ruff, S. Das, E. Manousakis, and L. Balicas. Bulk fermi surface of the weyl type-ii semimetallic candidate $\gamma$-mote$_2$. *Phys. Rev. B*, 96:165134, Oct 2017.

[16] Kazuhiro Fujita, Andrew R Schmidt, Eun-Ah Kim, Michael J Lawler, Dung Hai Lee, JC Davis, Hiroshi Eisaki, and Shin-ichi Uchida. Spectroscopic imaging scanning tunneling microscopy studies of electronic structure in the superconducting and pseudogap phases of cuprate high-Tc superconductors. *J. Phys. Soc. Jap.*, 81:1005, 2012.

[17] K. McElroy, J. Lee, JA Slezak, D.H. Lee, H. Eisaki, S. Uchida, and JC Davis. Atomic-scale sources and mechanism of nanoscale electronic disorder in disorder in Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$. *Science*, 309(5737):1048–1052, 2005.

[18] Philip W. Anderson. Sources of quantum protection in high-Tc superconductivity. *Science*, 288(5465):480–482, 2000. doi: 10.1126/science.288.5465.480.

[19] Effect of irradiation-induced disorder on the conductivity and critical temperature of the organic superconductor $\kappa$-(BEDT-TTF)$_2$Cu(SCN)$_2$. *Physical review letters*, 96(17):177002, 2006.

[20] J. Li, Y. F. Guo, S. B. Zhang, J. Yuan, Y. Tsujimoto, X. Wang, C. I. Sathish, Y. Sun, S. Yu, W. Yi, K. Yamura, E. Takayama-Muromachi, Y. Shirako, M. Akaogi, and H. Kontani. Superconductivity suppression of Ba$_{0.5}$K$_{0.5}$Fe$_{2-2x}$M$_{2x}$As$_2$ single crystals by substitution of transition metal (M = Mn, Ru, Co, Ni, Cu, and Zn). *Phys. Rev. B*, 85:214509, June 2012. doi: 10.1103/PhysRevB.85.214509.

[21] AV Balatsky, I. Vekhter, and J.X. Zhu. Impurity-induced states in conventional and unconventional superconductors. *Rev. Mod. Phys.*, 78:373, 2006.

[22] Noboru Fukushima, Chung-Pin Chou, and Ting Kuo Lee. Impurity scattering effects in STM studies of high-Tc superconductors. *J. Phys. Chem. Solids*, 69(12):3046, 2008.

[23] Noboru Fukushima, Chung-Pin Chou, and Ting Kuo Lee. Impurity potential renormalization by strong electron correlation. *Phys. Rev. B*, 79:184510, 2009.

[24] A. Garg, M. Randeria, and N. Trivedi. Strong correlations make high-temperature superconductors robust against disorder. *Nature Physics*, 4(10):762–765, 2008.
[25] E. C. Andrade, E. Miranda, and V. Dobrosavljević. Quantum ripples in strongly correlated metals. *Phys. Rev. Lett.*, 104:236401, June 2010. doi: 10.1103/PhysRevLett.104.236401.

[26] B. Keimer, S. A. Kivelson, M. R. Norman, S. Uchida, and J. Zaanen. From quantum matter to high-temperature superconductivity in copper oxides. *Nature*, 518:179, 2015. doi: 10.1038/nature14165.

[27] Seung Hwan Hong, Jin Mo Bok, Wentao Zhang, Junfeng He, X. J. Zhou, C. M. Varma, and Han-Yong Choi. Sharp low-energy feature in single-particle spectra due to forward scattering in *d*-wave cuprate superconductors. *Phys. Rev. Lett.*, 113:057001, 2014.

[28] S. Tang, E. Miranda, and V. Dobrosavljevic. Mottness-induced healing in strongly correlated superconductors. *Phys. Rev. B*, 91:020501, 2015. doi: 10.1103/PhysRevB.91.020501.

[29] Shao Tang, V. Dobrosavljević, and E. Miranda. Strong correlations generically protect *d*-wave superconductivity against disorder. *Phys. Rev. B*, 93:195109, May 2016.

[30] A. Paramekanti, M. Randeria, and N. Trivedi. Projected wave functions and high temperature superconductivity. *Physical review letters*, 87(21):217002, 2001.

[31] Arun Paramekanti, Mohit Randeria, and Nandini Trivedi. High-Tc superconductors:a variational theory of the superconducting state. *Phys. Rev. B*, 70:054504, August 2004. doi: 10.1103/PhysRevB.70.054504.

[32] G. Kotliar and A.E. Ruckenstein. New functional integral approach to strongly correlated fermi systems: The gutzwiller approximation as a saddle point. *Physical review letters*, 57(11):1362–1365, 1986.

[33] Efstratios Manousakis. The spin-1/2 heisenberg antiferromagnet on a square lattice and its application to the cuprous oxides. *Rev. Mod. Phys.*, 63:1–62, January 1991. doi: 10.1103/RevModPhys.63.1.

[34] P. Coleman. New approach to the mixed-valence problem. *Phys. Rev. B*, 29:3035, 1984.

[35] Patrick A Lee, Naoto Nagaosa, Tai-Kai Ng, and Xiao-Gang Wen. SU (2) formulation of the tJ model: Application to underdoped cuprates. *Phys. Rev. B*, 57(10):6003, 1998.

[36] *International technology road map for semiconductors*, 2011. URL http://www.itrs.net/.

[37] M. Rozenberg. Resistive switching. *Scholarpedia*, 6(4):11414, 2011.

[38] Dmitri B Strukov, Gregory S Snider, Duncan R Stewart, and R Stanley Williams. The missing memristor found. *nature*, 453(7191):80–83, 2008.
[39] J Joshua Yang, Matthew D Pickett, Xuema Li, Douglas A.A. Ohlberg, Duncan R Stewart, and R Stanley Williams. Memristive switching mechanism for metal/oxide/metal nanodevices. *Nature nanotechnology*, 3(7):429–433, 2008.

[40] Rainer Waser and Masakazu Aono. Nanoionics-based resistive switching memories. *Nature materials*, 6(11):833–840, 2007.

[41] Rainer Waser, Regina Dittmann, Georgi Staikov, and Kristof Szot. Redox-based resistive switching memories - nanionic mechanisms, prospects, and challenges. *Advanced Materials*, 21 (25-26):2632–2663, 2009. ISSN 1521-4095. doi: 10.1002/adma.200900375.

[42] Akihito Sawa. Resistive switching in transition metal oxides. *Materials Today*, 11(6):28–36, 2008. ISSN 1369-7021. doi: 10.1016/S1369-7021(08)70119-6.

[43] Satishchandra Balkrishna Ogale, T Venky Venkatesan, and Mark Blamire. *Functional Metal Oxides: New Science and Novel Applications*. John Wiley & Sons, 2013.

[44] J. Joshua Yang, Isao H. Inoue, Thomas Mikolajick, and Cheol Seong Hwang. Metal oxide memories based on thermochemical and valence change mechanisms. *MRS Bulletin*, 37:131–137, February 2012. ISSN 1938-1425.

[45] A. Baikalov, Y. Q. Wang, B. Shen, B. Lorenz, S. Tsui, Y. Y. Sun, Y. Y. Xue, and C. W. Chu. Field-driven hysteretic and reversible resistive switch at the Ag-Pr0.7Ca0.3MnO3 interface. *Applied Physics Letters*, 83(5), 2003.

[46] P Stoliar, Pablo Levy, Maria Jesus Sanchez, AG Leyva, CA Albornoz, F Gomez-Marlasca, Anibal Zanini, C Toro Salazar, N Ghenzi, and MJ Rozenberg. Nonvolatile multilevel resistive switching memory cell: a transition metal oxide-based circuit. *Circuits and Systems II: Express Briefs, IEEE Transactions on*, 61(1):21–25, 2014.

[47] Fabien Alibart, Ligang Gao, Brian D Hoskins, and Dmitri B Strukov. High precision tuning of state for memristive devices by adaptable variation-tolerant algorithm. *Nanotechnology*, 23(7):075201, 2012.

[48] N Ghenzi, MJ Sánchez, F Gomez-Marlasca, P Levy, and MJ Rozenberg. Hysteresis switching loops in ag-manganite memristive interfaces. *Journal of Applied Physics*, 107(9):093719, 2010.

[49] Federico Tesler, Shao Tang, Vladimir Dobrosavljević, and Marcelo Rozenberg. Shock waves in binary oxides memristors. In *Spintronics X*, volume 10357, page 103572L. International Society for Optics and Photonics, 2017.

[50] Qiong Zhou, D. Rhodes, Q. R. Zhang, S. Tang, R. Schönenmann, and L. Balicas. Hall effect within the colossal magnetoresistive semimetallic state of mote2. *Phys. Rev. B*, 94:121101, Sep 2016.
[51] YB Nian, J Strozier, NJ Wu, X Chen, and A Ignatiev. Evidence for an oxygen diffusion model for the electric pulse induced resistance change effect in transition-metal oxides. *Physical review letters*, 98(14):146403, 2007.

[52] X Chen, NJ Wu, J Strozier, and A Ignatiev. Direct resistance profile for an electrical pulse induced resistance change device. *Applied Physics Letters*, 87(23):233506, 2005.

[53] Stefano Larentis, Federico Nardi, Simone Balatti, David C Gilmer, and Daniele Ielmini. Resistive switching by voltage-driven ion migration in bipolar RRAM—Part II: Modeling. *Electron Devices, IEEE Transactions on*, 59(9):2468–2475, 2012.

[54] Stephan Menzel, Matthias Waters, Astrid Marchewka, Ulrich Böttger, Regina Dittmann, and Rainer Waser. Origin of the ultra-nonlinear switching kinetics in oxide-based resistive switches. *Advanced Functional Materials*, 21(23):4487–4492, 2011.

[55] Michael Bocquet, Damien Deleruyelle, Hassen Aziza, Candice Muller, J-M Portal, Thomas Cabout, and Eric Jalaguier. Robust compact model for bipolar oxide-based resistive switching memories. *Electron Devices, IEEE Transactions on*, 61(3):674–681, 2014.

[56] Ji Hyun Hur, Myoung-Jae Lee, Chang Bum Lee, Young-Bae Kim, and Chang-Jung Kim. Modeling for bipolar resistive memory switching in transition-metal oxides. *Physical Review B*, 82(15):155321, 2010.

[57] Peng Huang, Xiao Yan Liu, Bing Chen, Hai Tong Li, Yi Jiao Wang, Ye Xin Deng, Kang Liang Wei, Lang Zeng, Bin Gao, Gang Du, et al. A physics-based compact model of metal-oxide-based rram DC and AC operations. *Electron Devices, IEEE Transactions on*, 60(12):4090–4097, 2013.

[58] Mohammad Noman, Wenkan Jiang, Paul A Salvador, Marek Skowronski, and James A Bain. Computational investigations into the operating window for memristive devices based on homogeneous ionic motion. *Applied Physics A*, 102(4):877–883, 2011.

[59] Dmitri B Strukov, Julien L Borghetti, and R Stanley Williams. Coupled ionic and electronic transport model of thin-film semiconductor memristive behavior. *small*, 5(9):1058–1063, 2009.

[60] Jae Sung Lee, Shin Buhm Lee, Byungnam Kahng, and Tae Won Noh. Two opposite hysteresis curves in semiconductors with mobile dopants. *Applied Physics Letters*, 102(25):253503, 2013.

[61] Sungho Kim, ShinHyun Choi, and Wei Lu. Comprehensive physical model of dynamic resistive switching in an oxide memristor. *ACS nano*, 8(3):2369–2376, 2014.

[62] M. J. Rozenberg, I. H. Inoue, and M. J. Sánchez. Nonvolatile memory with multilevel switching: A basic model. *Phys. Rev. Lett.*, 92:178302, April 2004. doi: 10.1103/PhysRevLett.92.178302.
[63] Ugo Russo, Daniele Ielmini, Carlo Cagli, and Andrea L Lacaita. Filament conduction and reset mechanism in NiO-based resistive-switching memory (RRAM) devices. *Electron Devices, IEEE Transactions on*, 56(2):186–192, 2009.

[64] M. J. Rozenberg, M. J. Sánchez, R. Weht, C. Acha, F. Gomez-Marlasca, and P. Levy. Mechanism for bipolar resistive switching in transition-metal oxides. *Phys. Rev. B*, 81:115101, March 2010. doi: 10.1103/PhysRevB.81.115101.

[65] Eduardo Fradkin and Steven A Kivelson. High-temperature superconductivity: Ineluctable complexity. *Nature Phys.*, 8:864–866, 2012.

[66] G. Ghiringhelli, M. Le Tacon, M. Minola, S. Blanco-Canosa, C. Mazzoli, N. B. Brookes, G. M. De Luca, A. Frano, D. G. Hawthorn, F. He, T. Loew, M. Moretti Sala, D. C. Peets, M. Salluzzo, E. Schierle, R. Sutarto, G. A. Sawatzky, E. Weschke, B. Keimer, and L. Braicovich. Long-range incommensurate charge fluctuations in (Y, Nd)Ba$_2$Cu$_3$O$_{6+x}$. *Science*, 337:821, 2012.

[67] A. C. Fang, L. Capriotti, D. J. Scalapino, S. A. Kivelson, N. Kaneko, M. Greven, and A. Kapitulnik. Gap-inhomogeneity-induced electronic states in superconducting Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$. *Phys. Rev. Lett.*, 96:017007, 2006.

[68] M.U. Ubbens and P.A. Lee. Flux phases in the tJ model. *Phys. Rev. B*, 46(13):8434, 1992.

[69] H. Alloul, J. Bobroff, M. Gabay, and PJ Hirschfeld. Defects in correlated metals and superconductors. *Rev. Mod. Phys.*, 81(1):45, 2009.

[70] Andrei E. Ruckenstein, Peter J. Hirschfeld, and J. Appel. Mean-field theory of high-Tc superconductivity: The superexchange mechanism. *Phys. Rev. B*, 36:857–860, July 1987.

[71] A.A. Abrikosov, LP Gorkov, and IE Dzyaloshinski. *Methods of quantum field theory in statistical physics*. Courier Dover Publications, 1975.

[72] A. V. Balatsky and M. I. Salkola. Impurity states and the absence of quasiparticle localization in disordered d-wave superconductors. *Phys. Rev. Lett.*, 76:2386, 1996. doi: 10.1103/PhysRevLett.76.2386.

[73] Jeremy Figgins and Dirk K Morr. Defects in heavy-fermion materials: unveiling strong correlations in real space. *Physical review letters*, 107(6):066401, 2011.

[74] A. A. Abrikosov and L. P. Gor’kov. Contribution to the theory of superconducting alloys with paramagnetic impurities. *Sov. Phys. JETP*, 12:1243, 1961.

[75] Karl Heinz Bennemann and John Boyd Ketterson. *Superconductivity: Conventional and unconventional superconductors*, volume 1. Springer, 2008.
[76] T. Ando, A. B. Fowler, and F. Stern. Electronic properties of two-dimensional systems. *Rev. Mod. Phys.*, 54:437, 1982.

[77] L. Debnath. *Nonlinear partial differential equations for scientists and engineers*. Birkhauser, 2011.

[78] M.E. Taylor. Partial differential equations iii: Nonlinear equations. *Applied Mathematical Sciences*, 2011.

[79] Richard Courant and David Hilbert. *Methods of mathematical physics. Vol. II: Partial differential equations*. Interscience, New York, 1962.

[80] Hong Sub Lee, Sun Gyu Choi, Hyung-Ho Park, and MJ Rozenberg. A new route to the mott-hubbard metal-insulator transition: Strong correlations effects in Pr$_{0.7}$Ca$_{0.3}$MnO$_3$. *Scientific reports*, 3, 2013.

[81] EM Lifshitz LD Landau. *Fluid Mechanics: Volume 6 (Course Of Theoretical Physics)*. Butterworth-Heinemann, 1987.

[82] M Quintero, P Levy, AG Leyva, and MJ Rozenberg. Mechanism of electric-pulse-induced resistance switching in manganites. *Physical review letters*, 98(11):116601, 2007.

[83] Yu-ling Jin, Zhong-tang Xu, Kui-juan Jin, Xu He, Can Wang, and Hui-bin Lu. Enhancement of resistive switching effect in double-layered Pt/Pr$_{0.7}$Ca$_{0.3}$MnO$_3$/La$_{0.6}$Pr$_{0.4}$MnO$_3$/SrNb$_{0.01}$Ti$_{0.99}$O$_3$ heterostructure. *Physica B: Condensed Matter*, 449:52–56, 2014.

[84] M Scherff, J Hoffmann, B Meyer, Th Danz, and Ch Jooss. Interplay of cross-plane polaronic transport and resistive switching in Pt–Pr$_{0.67}$Ca$_{0.33}$MnO$_3$–Pt heterostructures. *New Journal of Physics*, 15(10):103008, 2013.

[85] Sangsu Park, Seungjae Jung, Manzar Siddik, Minseok Jo, Jubong Park, Seonghyun Kim, Wootae Lee, Jungho Shin, Daeseok Lee, Godeuni Choi, et al. Self-formed schottky barrier induced selector-less rram for cross-point memory applications. *physica status solidi (RRL)-Rapid Research Letters*, 6(11):454–456, 2012.

[86] Anja Herpers, Christian Lenser, Chanwoo Park, Francesco Offi, Francesco Borgatti, Giancarlo Panaccione, Stephan Menzel, Rainer Waser, and Regina Dittmann. Spectroscopic proof of the correlation between redox-state and charge-carrier transport at the interface of resistively switching ti/pcmo devices. *Advanced materials*, 26(17):2730–2735, 2014.

[87] F Gomez-Marlasca, N Ghenzi, AG Leyva, C Albornoz, D Rubi, P Stoliar, and P Levy. Modeling electronic transport mechanisms in metal-manganite memristive interfaces. *Journal of Applied Physics*, 113(14):144510, 2013.
[88] Shao Tang, Federico Tesler, Fernando Gomez Marlasca, Pablo Levy, V. Dobrosavljević, and Marcelo Rozenberg. Shock waves and commutation speed of memristors. *Phys. Rev. X*, 6:011028, Mar 2016. doi: 10.1103/PhysRevX.6.011028.

[89] N. Ghenzi, M. J. Rozenberg, R. Llopis, P. Levy, L. E. Hueso, and P. Stoliar. Tuning the resistive switching properties of TiO$_{2-x}$ films. *Applied Physics Letters*, 106(12):123509, 2015.

[90] N. Nagaosa. *Quantum field theory in condensed matter physics*. Springer Verlag, 1999.

[91] N. Nagaosa. *Quantum field theory in strongly correlated electronic systems*. Springer Verlag, 1999.

[92] G. Baskaran, Z. Zou, and PW Anderson. The resonating valence bond state and high-Tc superconductivity—A mean field theory. *Solid state communications*, 63:973, 1987.

[93] Tiago C. Ribeiro and Xiao-Gang Wen. New mean-field theory of the $tt't''J$ model applied to High-$T_c$ superconductors. *Phys. Rev. Lett.*, 95:057001, July 2005. doi: 10.1103/PhysRevLett.95.057001.

[94] X.G. Wen. *Quantum field theory of many-body systems: from the origin of sound to an origin of light and electrons*. Oxford University Press, USA, 2004.

[95] LB Ioffe, AI Larkin, et al. Gapless fermions and gauge fields in dielectrics. *Phys. Rev. B*, 39:8988, 1989.

[96] A. Altland and B. Simons. *Condensed matter field theory*. 2010.

[97] G.D. Mahan. *Many-particle physics*. Springer, 2000.