ON THE CONVERGENCE OF MULTIPLE FOURIER SERIES OF FUNCTIONS OF BOUNDED PARTIAL GENERALIZED VARIATION

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Abstract. The convergence of multiple Fourier series of functions of bounded partial \(\Lambda\)-variation is investigated. The sufficient and necessary conditions on the sequence \(\Lambda = \{\lambda_n\}\) are found for the convergence of multiple Fourier series of functions of bounded partial \(\Lambda\)-variation.

1. Classes of Functions of Bounded Generalized Variation

In 1881 Jordan [10] introduced a class of functions of bounded variation and applied it to the theory of Fourier series. Hereafter this notion was generalized by many authors (quadratic variation, \(\Phi\)-variation, \(\Lambda\)-variation etc., see [10, 15, 14, 11]). In two dimensional case the class BV of functions of bounded variation was introduced by Hardy [9].

Let \(T := [0, 2\pi]\) and \(J^k = (a^k, b^k) \subset T, \quad k = 1, 2, \ldots, d\).

Consider a measurable function \(f(x)\) defined on \(R^d\) and \(2\pi\)-periodic with respect to each variable. For \(d = 1\) we set

\[
f(J^1) := f(b^1) - f(a^1).
\]

If for any function of \(d - 1\) variables the expression \(f(I^1 \times \cdots \times I^{d-1})\) is already defined, then for a function of \(d\) variables the mixed difference is defined as follows:

\[
f(J^1 \times \cdots \times J^d) := f(J^1 \times \cdots \times J^{d-1}, b^d) - f(J^1 \times \cdots \times J^{d-1}, a^d).
\]

Let \(E = \{I_k\}\) be a collection of nonoverlapping intervals from \(T\) ordered in arbitrary way and let \(\Omega\) be the set of all such collections \(E\). We denote by \(\Omega_n\) the set of all collections of \(n\) nonoverlapping intervals \(I_k \subset T\).

For sequences of positive numbers \(\Lambda^j = \{\lambda^j_n\}_{n=1}^\infty, \quad j = 1, 2, \ldots, d\), the \((\Lambda^1, \ldots, \Lambda^d)\)-variation of \(f\) with respect to index set \(D := \{1, 2, \ldots, d\}\) is
defined as follows:

\[ V_{\Lambda^1, \ldots, \Lambda^d}^D(f) := \sup_{\{I_{i,j}\}_{i,j=1}^{s} \in \Omega} \sum_{i_1, \ldots, i_d} \frac{|f(I_{i_1}^1 \times \cdots \times I_{i_d}^d)|}{\lambda_{i_1} \cdots \lambda_{i_d}}. \]

For an index set \( \alpha = \{j_1, \ldots, j_p\} \subset D \) and any \( x = (x^1, \ldots, x^d) \in \mathbb{R}^d \) we set \( \bar{x} := D \setminus \alpha \) and denote by \( x^\alpha \) the vector of \( \mathbb{R}^p \) consisting of components \( x^j, j \in \alpha \), i.e.

\[ x^\alpha = (x^{j_1}, \ldots, x^{j_p}) \in \mathbb{R}^p. \]

By \( V_{\Lambda^1, \ldots, \Lambda^p}^\alpha(f, x^\bar{x}) \) and \( f(I_{i,j_1}^1 \times \cdots \times I_{i,j_p}^p, x^\bar{x}) \) we denote respectively the \( (\Lambda^1, \ldots, \Lambda^p) \)-variation and the mixed difference of \( f \) as a function of variables \( x^{j_1}, \ldots, x^{j_p} \) over the \( p \)-dimensional cube \( T^p \) with fixed values \( x^\bar{x} \) of other variables. The \( (\Lambda^1, \ldots, \Lambda^p) \)-variation of \( f \) with respect to index set \( \alpha \) is defined as follows:

\[ V_{\Lambda^1, \ldots, \Lambda^p}^\alpha(f) := \sup_{x^{\bar{x}} \in T^{d-p}} V_{\Lambda^1, \ldots, \Lambda^p}^\alpha(f, x^\bar{x}). \]

**Definition 1.** We say that the function \( f \) has total Bounded \( (\Lambda^1, \ldots, \Lambda^d) \)-variation on \( T^d = [0, 2\pi]^d \) and write \( f \in BV_{\Lambda^1, \ldots, \Lambda^d} \), if

\[ V_{\Lambda^1, \ldots, \Lambda^d}(f) := \sum_{\alpha \subset D} V_{\Lambda^1, \ldots, \Lambda^p}^\alpha(f) < \infty. \]

**Definition 2.** We say that the function \( f \) is continuous in \( (\Lambda^1, \ldots, \Lambda^d) \)-variation on \( T^d = [0, 2\pi]^d \) and write \( f \in CV_{\Lambda^1, \ldots, \Lambda^d} \), if

\[ \lim_{n \to \infty} V_{\Lambda^1, \ldots, \Lambda^{k-1}, \Lambda_n^{k+1}, \Lambda_n^{k+1}, \ldots, \Lambda^p}^\alpha(f) = 0, \quad k = 1, 2, \ldots, p \]

for any \( \alpha \subset D, \alpha = \{j_1, \ldots, j_p\} \), where \( \Lambda_n^{j_k} := \{\lambda_n^{j_k}\}_{s=n}^{\infty} \).

**Definition 3.** We say that the function \( f \) has Bounded Partial \( (\Lambda^1, \ldots, \Lambda^d) \)-variation and write \( f \in PBV_{\Lambda^1, \ldots, \Lambda^d} \) if

\[ PV_{\Lambda^1, \ldots, \Lambda^d}(f) := \sum_{i=1}^{d} V_{\Lambda^i}^{(i)}(f) < \infty. \]

In the case \( \Lambda^1 = \cdots = \Lambda^d = \Lambda \) we denote

\[ BV_{\Lambda} := BV_{\Lambda^1, \ldots, \Lambda^d}, \quad CV_{\Lambda} := CV_{\Lambda^1, \ldots, \Lambda^d}, \quad PBV_{\Lambda} := PBV_{\Lambda^1, \ldots, \Lambda^d} \]

and

\[ CV_{\Lambda} := V_{\Lambda^1, \ldots, \Lambda^d} \cdot CV_{\Lambda}, \quad PV_{\Lambda}(f) := PV_{\Lambda^1, \ldots, \Lambda^d}(f). \]

If \( \lambda_n \equiv 1 \) (or if \( 0 < c < \lambda_n < C < \infty, \quad n = 1, 2, \ldots \)) the classes \( BV_{\Lambda} \) and \( PBV_{\Lambda} \) coincide with the Hardy class \( BV \) and \( PBV \) respectively. Hence it is reasonable to assume that \( \lambda_n \to \infty \) and since the intervals in \( E = \{I_i\} \) are...
ordered arbitrarily, we suppose, without loss of generality, that the sequence \( \{\lambda_n\} \) is increasing. Thus,

\[
1 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lim_{n \to \infty} \lambda_n = \infty.
\]

When \( \lambda_n = n \) for all \( n = 1, 2, \ldots \) we say Harmonic Variation instead of \( \Lambda \)-variation and write \( H \) instead of \( \Lambda \) (\( BV_H, PBV_H, CV_H \), etc).

Remark 1. The notion of \( \Lambda \)-variation was introduced by Waterman [14] in one dimensional case, by Sahakian [13] in two dimensional case and by Sablin [12] in the case of higher dimensions. The notion of bounded partial variation (class \( PBV \)) was introduced by Goginava in [6, 7]. These classes of functions of generalized bounded variation play an important role in the theory Fourier series.

Observe, that the number of variations in Definition 1 of total variation is \( 2d - 1 \), while the number of variations in Definition 2 of partial variation is only \( d \).

The statements of the following theorem are known.

Theorem A. 1) (Dragoshanski [5]) If \( d = 2 \), then \( BV_H = CV_H \).

2) (Bakhvalov [1]) \( CV_H = \bigcup \Gamma BV_\Gamma \) for any \( d \), where the union is taken over all sequences \( \Gamma = \{\gamma_n\}_{n=1}^\infty \) with \( \gamma_n = o(n) \) as \( n \to \infty \).

3) (Goginava, Sahakian [8]) If \( d = 2 \), then \( PBV_\Lambda \subset BV_H \), provided that

\[
\lambda_n \frac{1}{n^2} < \infty, \quad \lambda_n \frac{1}{n^2} \downarrow 0 \quad \text{and} \quad \sum_{n=1}^\infty \frac{\lambda_n}{n^2} < \infty.
\]

Using the third statement of Theorem A, we have proved in [8] the convergence of double Fourier series of functions of any class \( PBV_\Lambda \) with (2).

To obtain similar result for higher dimensions we need stronger result, since the inclusion \( PBV_\Lambda \subset BV_H \) is not enough in this case (see next section for details).

Theorem 1. Let \( \Lambda = \{\lambda_n\}_{n=1}^\infty \) and \( d \geq 2 \). If

\[
\frac{\lambda_n}{n} \downarrow 0 \quad \text{and} \quad \sum_{n=1}^\infty \frac{\lambda_n}{n^2} \frac{\log^{d-2} n}{n^2} < \infty,
\]

then there exists a sequence \( \Gamma = \{\gamma_n\}_{n=1}^\infty \) with

\[
\gamma_n = o(n) \quad \text{as} \quad n \to \infty,
\]

such that \( PBV_\Lambda \subset BV_\Gamma \).

Proof. Choosing the sequence \( \{A_n\}_{n=1}^\infty \) such that

\[
A_n \uparrow \infty, \quad \frac{\lambda_n A_n}{n} \downarrow 0, \quad \sum_{n=1}^\infty \frac{\lambda_n}{n^2} \frac{\log^{d-2} n A_n^d}{n^2} < \infty,
\]

we set

\[
\gamma_n = \frac{n}{A_n}, \quad n = 1, 2, \ldots
\]
We prove that there is a constant $C > 0$ such that

$$
\sum_{i_1, \ldots, i_p} \left| \frac{f(I^{1}_{i_1} \times \cdots \times I^{p}_{i_p}, x^\alpha)}{\gamma_{i_1} \cdots \gamma_{i_p}} \right| < C \cdot PV_\Lambda(f),
$$

for any $f \in PBV_\Lambda$ and $\alpha := \{i_1, \ldots, i_p\} \subset D$, $\{I^j_{i_j} \}_{j=1}^{k_j} \in \Omega$.

To prove (7) observe, that

$$
\sum_{\sigma} \sum_{i_{s(1)} \leq \cdots \leq i_{s(p)}} \left| \frac{f(I^{1}_{i_1} \times \cdots \times I^{p}_{i_p}, x^\alpha)}{\gamma_{i_1} \cdots \gamma_{i_p}} \right| < \infty,
$$

where the sum is taken over all rearrangements $\sigma = \{\sigma(k)\}_{k=1}^{p}$ of the set $\{1, 2, \ldots, p\}$.

Denoting $M = PV_\Lambda(f)$ and using (6), (5) and (3) we obtain:

$$
\sum_{i_1 \leq i_2 \leq \cdots \leq i_p} \left| \frac{f(I^{1}_{i_1} \times \cdots \times I^{p}_{i_p}, x^\alpha)}{\gamma_{i_1} \cdots \gamma_{i_p}} \right| = \sum_{i_1 \leq i_2 \leq \cdots \leq i_{p-1}} A_{i_1} \cdots A_{i_{p-1}} \sum_{i_p \geq i_{p-1}} \left| \frac{f(I^{1}_{i_1} \times \cdots \times I^{p}_{i_p}, x^\alpha)}{\lambda_{i_p}} \right| \frac{\lambda_{i_p} A_{i_p}}{i_p} \leq M \sum_{i_{p-1}=1}^{\infty} \frac{A_{i_{p-1}}^p \lambda_{i_{p-1}}}{i_{p-1}^2} \frac{1}{i_1 \cdots i_{p-2}} \leq M \sum_{i_{p-1}=1}^{\infty} \frac{A_{i_{p-1}}^p \lambda_{i_{p-1}}}{i_{p-1}^2} \left( \frac{i_{p-1}}{i_1} \right)^{\frac{p-2}{2}} \frac{1}{i_1} \leq M \sum_{n=1}^{\infty} A_n^p \lambda_n \log^{d-2} n \frac{n}{n^2} < \infty.
$$

Similarly we can prove that all other summands in the right hand side of (8) are finite. Theorem $\square$ is proved.

In view of Theorem $\mathcal{A}$, Theorem $\square$ implies

**Corollary 1.** If the sequence $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ satisfies $\mathcal{B}$, then $PBV_\Lambda \subset CV_H$.

**Definition 4.** The partial modulus of variation $v_i(n, f)$, $i = 1, \ldots, d$ of a function $f$ are defined by

$$
v_i(n, f) := \sup_{x^\beta} \sup_{\{I_j\} \in \Omega_n} \sum_{j=1}^{n} \left| f(I_j, x^\beta) \right|, \quad \beta = D \setminus \{i\}, \quad n = 1, 2, \ldots.
$$
For functions of one variable the concept of modulus of variation was introduced by Chanturia [2].

**Theorem 2.** Let \( f \) be defined on \( T^d \) and

\[
\sum_{j=1}^{\infty} \frac{\sqrt{v_i (2^j, f)}}{2^{j/d}} < \infty, \quad i = 1, \ldots, d.
\]

Then there exists a sequence \( \Delta = \{\delta_n\}_{n=1}^{\infty} \) with

\[
\delta_n = o(n) \quad \text{as} \quad n \to \infty,
\]

such that \( f \in BV_\Delta \).

**Proof.** We use induction on dimension \( d \). We have proved in [8], that in the case \( d = 2 \) the condition (9) implies \( f \in BV_H \), which combined with Theorem A proves Theorem 2 for \( d = 2 \).

Supposing Theorem 2 is true if the dimension is less than \( d \), we prove it for the dimension \( d > 2 \).

According to induction hypothesis it is enough to prove that there exists a sequence \( \delta_n = o(n) \) such that

\[
\sup_{\{I_{ij}^k\}_{i,j=1}^{k} \in \Omega} \sum_{i_1, \ldots, i_d} \left| f \left( I_1^{i_1} \times \cdots \times I_d^{i_d} \right) \right| \frac{\delta_{i_1} \cdots \delta_{i_d}}{|\delta_{i_1} \cdots \delta_{i_d}|} < \infty.
\]

Let the sequence \( \{B_{2^j}\}_{j=1}^{\infty} \) be chosen so that

\[
B_{2^j} \uparrow \infty, \quad \sum_{j=1}^{\infty} B_{2^j} \sqrt{v_i (2^j, f)} \frac{2^{j/d}}{2^{j/d}} < \infty, \quad i = 1, \ldots, d.
\]

Defining

\[
B_n = B_{2^N}, \quad \text{for} \quad 2^N \leq n < 2^{N+1}, \quad N = 0, 1, \ldots,
\]

we set

\[
\delta_n = \frac{n}{B_n}, \quad n = 1, 2, \ldots.
\]

Then we can write

\[
\sum_{i_1, \ldots, i_d} \left| f \left( I_1^{i_1} \times \cdots \times I_d^{i_d} \right) \right| \frac{\delta_{i_1} \cdots \delta_{i_d}}{|\delta_{i_1} \cdots \delta_{i_d}|} = \sum_{i_1, \ldots, i_d} B_{i_1} \cdots B_{i_d} \left| f \left( I_1^{i_1} \times \cdots \times I_d^{i_d} \right) \right| \frac{\delta_{i_1} \cdots \delta_{i_d}}{|\delta_{i_1} \cdots \delta_{i_d}|}
\]

\[
= \sum_{r_1=0}^{\infty} \cdots \sum_{r_d=0}^{\infty} \frac{B_{2^{r_1}}}{2^{r_1}} \cdots \frac{B_{2^{r_d}}}{2^{r_d}} \sum_{i_1=2^{r_1}}^{2^{r_1+1}-1} \sum_{i_d=2^{r_d}}^{2^{r_d+1}-1} \left| f \left( I_1^{i_1} \times \cdots \times I_d^{i_d} \right) \right|.
\]
It is easy to show that
\[
\sum_{i_1=2^{r_1}}^{2^{r_1}+1-1} \ldots \sum_{i_d=2^{r_d}}^{2^{r_d}+1-1} \left| f \left( I_{i_1} \times \ldots \times I_{i_d} \right) \right|
\leq c(d) \prod_{k=1}^{d} 2^{r_k} \sup_{x^\beta} \sup_{I_{i_k}^k \in \Omega_{2^{r_k}}} \sum_{i_k=2^{r_k}}^{2^{r_k}+1-1} \left| f \left( I_{i_k}^k, x^\beta \right) \right|
\]
where \( \beta := D \setminus \{ k \}, k = 1, \ldots, d. \)
Consequently,
\[
\sum_{i_1=2^{r_1}}^{2^{r_1}+1-1} \ldots \sum_{i_d=2^{r_d}}^{2^{r_d}+1-1} \left| f \left( I_{i_1} \times \ldots \times I_{i_d} \right) \right| 
= \left[ \left( \sum_{i_1=2^{r_1}}^{2^{r_1}+1-1} \ldots \sum_{i_d=2^{r_d}}^{2^{r_d}+1-1} \left| f \left( I_{i_1} \times \ldots \times I_{i_d} \right) \right| \right)^{1/d} \right] \prod_{k=1}^{d} 2^{r_k} (1 - 1/d) \left( \sup_{x^\beta} \sup_{I_{i_k}^k \in \Omega_{2^{r_k}}} \sum_{i_k=2^{r_k}}^{2^{r_k}+1-1} \left| f \left( I_{i_k}^k, x^\beta \right) \right| \right)^{1/d}
\]
\[
= c(d) \prod_{k=1}^{d} 2^{r_k} (1-1/d) \mathcal{V} \left( v_k (2^{r_k}, f) \right).
\]
Combining (11) and (12) we obtain

\[
\sum_{i_1, \ldots, i_d} \frac{\left| f \left( I_{i_1} \times \ldots \times I_{i_d} \right) \right|}{\delta_{i_1} \cdots \delta_{i_d}} 
\leq c(d) \sum_{r_1=0}^{\infty} \ldots \sum_{r_d=0}^{\infty} \frac{B_{2^{r_1}, v_1 (2^{r_1}, f)}}{2^{r_1/d}} \cdots \frac{B_{2^{r_d}, v_d (2^{r_d}, f)}}{2^{r_d/d}} < \infty.
\]

Theorem 2 is proved. \( \square \)

2. Convergence of Multiple Fourier Series

The Fourier series of function \( f \in L^1 (T^d) \) with respect to the trigonometric system is the series

\[
S[f] := \sum_{n_1, \ldots, n_d = -\infty}^{+\infty} \widehat{f} (n_1, \ldots, n_d) e^{i(n_1 x + \ldots + n_d x_d)},
\]
where

\[
\widehat{f} (n_1, \ldots, n_d) = \frac{1}{(2\pi)^d} \int_{T^d} f(x^1, \ldots, x^d) e^{-i(n_1 x^1 + \ldots + n_d x_d)} dx^1 \cdots dx^d.
\]
are the Fourier coefficients of $f$. The rectangular partial sums are defined as follows:

$$S_{N_1,\ldots,N_d}(f; x^1, \ldots, x^d) := \sum_{n_1=-N_1}^{N_1} \cdots \sum_{n_d=-N_d}^{N_d} \hat{f}(n_1, \ldots, n_d) e^{i(n_1 x^1 + \cdots + n_d x^d)}$$

$$= \frac{1}{\pi^d} \int_{T^d} f(x_1, \ldots, x_d) \prod_{s=1}^{d} D_{N_s}(x_s) \, dx_1 \cdots dx_d,$$

where $D_N(t) = \frac{\sin(N+\frac{1}{2})t}{2\sin\frac{t}{2}}$ is the Dirichlet kernel.

In this paper we consider convergence of only rectangular partial sums (convergence in the sense of Pringsheim) of $d$-dimensional Fourier series. We denote by $C(T^d)$ the space of continuous and $2\pi$-periodic with respect to each variable functions with the norm

$$\|f\|_C := \sup_{(x^1, \ldots, x^d) \in T^d} |f(x^1, \ldots, x^d)|.$$

We say that the point $x := (x^1, \ldots, x^d)$ is a regular point of function $f$ if the following limits exist

$$f(x^1 \pm 0, \ldots, x^d \pm 0) := \lim_{t^1, \ldots, t^d \downarrow 0} f(x^1 \pm t^1, \ldots, x^d \pm t^d).$$

For the regular point $x := (x^1, \ldots, x^d)$ we denote

$$f^*(x^1, \ldots, x^d) := \frac{1}{2^d} \sum f(x^1 \pm 0, \ldots, x^d \pm 0).$$

**Definition 5.** We say that the class of functions $V \subset L^1(T^d)$ is a class of convergence on $T^d$, if for any function $f \in V$

1) the Fourier series of $f$ converges to $f^*(x)$ at any regular point $x \in T^d$,

2) the convergence is uniform on any compact $K \subset T^d$, if $f$ is continuous on the neighborhood of $K$.

The well known Dirichlet-Jordan theorem (see [16]) states that the Fourier series of a function $f(x)$, $x \in T$ of bounded variation converges at every point $x$ to the value $[f(x+0) + f(x-0)]/2$. If $f$ is in addition continuous on $T$, the Fourier series converges uniformly on $T$.

Hardy [9] generalized the Dirichlet-Jordan theorem to the double Fourier series and proved that $BV$ is a class of convergence on $T^2$.

The following theorem was proved by Waterman (for $d = 1$) and Sahakian (for $d = 2$).

**Theorem WS (Waterman [14], Sahakian [13]).** If $d = 1$ or $d = 2$, then the class $BV_H$ is a class of convergence on $T^d$. 


In [1] Bakhvalov showed that the class $BV_H$ is not a class of convergence on $T^d$, if $d > 2$. On the other hand, he proved the following

**Theorem B** (Bakhvalov [1]). The class $CV_H$ is a class of convergence on $T^d$ for any $d = 1, 2, \ldots$

Convergence of spherical and other partial sums of double Fourier series of functions of bounded $\Lambda$-variation was investigated in details by Dyachenko [3, 4].

The main result of this paper is the following theorem, that we have proved in [8] for $d = 2$.

**Theorem 3.** Let $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ and $d \geq 2$.

a) If

$$
\sum_{n=1}^{\infty} \frac{\lambda_n \log^{d-2} n}{n^2} < \infty,
$$

then $PBV_{\Lambda}$ is a class of convergence on $T^d$.

b) If

$$
\frac{\lambda_n}{n} = O\left(\frac{\lambda_{[n^\delta]}}{n^\delta}\right)
$$

for some $\delta > 1$, and

$$
\sum_{n=1}^{\infty} \frac{\lambda_n \log^{d-2} n}{n^2} = \infty,
$$

then there exists a continuous function $f \in PBV_{\Lambda}$, the Fourier series of which diverges at $(0, \ldots, 0)$.

**Proof of Theorem 2.** Part a) immediately follows from Corollary [1] and Theorem B.

To prove part b) we denote

$$
A_{i_1, \ldots, i_d} := \left[ \frac{\pi i_1}{N + 1/2}, \frac{\pi (i_1 + 1)}{N + 1/2} \right] \times \cdots \times \left[ \frac{\pi i_d}{N + 1/2}, \frac{\pi (i_d + 1)}{N + 1/2} \right],
$$

$$
W := \{(i_1, \ldots, i_d) : i_d < i_s < i_d + m_i, 1 \leq s < d, 1 \leq i_d \leq N_\delta\},
$$

$$
N_\delta = \left[ \left( \frac{N}{2} \right)^{\frac{1}{d}} \right], \quad t_j := \left( \sum_{i=1}^{m_j} \frac{1}{\lambda_i} \right)^{-1}, \quad m_j := \left[ j^{\delta} \right],
$$

where $[x]$ is the integer part of $x$.

It is not hard to see, that for any sequence $\Lambda = \{\lambda_n\}$ satisfying (11) the class $C(T^d) \cap PBV_{\Lambda}$ is a Banach space with the norm

$$
\|f\|_{PBV_{\Lambda}} := \|f\|_C + PV_{\Lambda}(f).
$$
Consider the following function

\[
f_N(x_1, \ldots, x_d) := \sum_{(i_1, \ldots, i_d) \in W} t_{i_d} 1_A(x_1, \ldots, x_d) \prod_{s=1}^d \sin \left( (N + 1/2) x_s \right),
\]

where \(1_A(x_1, \ldots, x_d)\) is the characteristic function of the set \(A \subset T^d\).

Let \((i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_d)\) be fixed \((k = 1, \ldots, d-1)\). Then it is easy to show that

\[
V_k^\Lambda (f_N) \leq C \cdot t_{i_d} \left( \sum_{i_k=i_d+1}^{i_d+m_{i_d}} \frac{1}{\lambda_{i_k-i_d}} \right) \leq C \cdot t_{i_d} \left( \sum_{i_k=1}^{m_{i_d}} \frac{1}{\lambda_{i_k}} \right) \leq C < \infty.
\]

If \((i_1, \ldots, i_{d-1})\) is fixed, the condition \((i_1, \ldots, i_d) \in W\) implies

\[
\max \{i_d(i_s) : 1 \leq s \leq d-1\} < i_d < \min \{i_s : 1 \leq s \leq d-1\},
\]

where

\[
i_d(i_s) := \min \{i_d : i_d + m_{i_d} > i_s\}.
\]

Consequently, by the definition of the function \(f_N\) we obtain that for any \(s = 1, \ldots, d-1\)

\[
V_d^\Lambda (f_N) \leq C \sum_{i_d=i_d(i_s)+1}^{i_s} \frac{t_{i_d}}{\lambda_{i_d-i_d(i_s)}} \leq C \cdot t_{i_d} \left( \sum_{i_k=1}^{i_d(i_s)-i_d(i_k)} \frac{1}{\lambda_{i_k}} \right) \leq C \cdot t_{i_d(i_s)} \sum_{i_d=1}^{m_{i_d(i_s)}} \frac{1}{\lambda_{i_d}} = C < \infty.
\]

Hence \(f_N \in \text{PBV}_\Lambda\) and

\[
\|f_N\|_{\text{PV}_\Lambda} \leq C, \quad N = 1, 2, \ldots.
\]

Observe, that by (15) we have

\[
\frac{1}{t_j} = \sum_{i=1}^{m_j} \frac{1}{\lambda_i} = \sum_{i=1}^{m_j} \frac{i}{\lambda_i} \leq C \frac{m_j}{\lambda_{m_j}} \log m_j \leq C \frac{j \log j}{\lambda_j}.
\]

Hence

\[
t_j \log j \geq c \frac{\lambda_j}{j}.
\]
Consequently,

\begin{equation}
\pi^d S_N,\ldots, N \left( f_N; 0, \ldots, 0 \right) = \int_{T^d} f_N \left( x^1, \ldots, x^d \right) \prod_{s=1}^{d} D_N \left( x^s \right) dx^1 \cdots dx^d
\end{equation}

\begin{align*}
&= \sum_{(i_1, \ldots, i_d) \in W} t_{i_d} \int_{A_{i_1, \ldots, i_d}} \prod_{s=1}^{d} \sin^2 \left( \frac{N + 1/2}{2} x^s \right) \frac{x^1 \cdots x^d}{2 \sin \left( \frac{N x^s}{2} \right)} dx^1 \cdots dx^d \\
&\geq c \sum_{(i_1, \ldots, i_d) \in W} t_{i_d} \frac{1}{i_1 \cdots i_d} \\
&\geq c \sum_{i_d = 1}^{N} \sum_{i_1 = i_d}^{i_{d+m_i}} \sum_{i_{d-1} = i_d}^{i_{d+m_i}} \frac{1}{i_1 \cdots i_{d-1}} \\
&\geq c \sum_{i_d = 1}^{N} t_{i_d} \log^{d-1} \left( \frac{i_d + m_i}{i_d} \right) \\
&\geq c (\delta - 1)^{d-1} \sum_{i_d = 1}^{N} \frac{t_{i_d} \log i_d \log^{d-2} i_d}{i_d}
\end{align*}

as \( N \to \infty \), according to \( (16) \).

By Banach-Steinhaus Theorem, \( (17) \) and \( (18) \) imply the existence of a continuous function \( f \in PBV_\Lambda \) such that

\[ \sup_N |S_{N, \ldots, N}[f, (0, \ldots, 0)]| = \infty. \]

\[ \square \]

**Corollary 2.** a) If \( \Lambda = \{ \lambda_n \}_{n=1}^{\infty} \) with

\[ \lambda_n = \frac{n}{\log^{d-1+\varepsilon} n}, \quad n = 1, 2, \ldots \]

for some \( \varepsilon > 0 \), then the class \( PBV_\Lambda \) is a class of convergence on \( T^d \).

b) If \( \Lambda = \{ \lambda_n \}_{n=1}^{\infty} \) with

\[ \lambda_n = \frac{n}{\log^{d-1} n}, \quad n = 1, 2, \ldots, \]

then the class \( PBV_\Lambda \) is not a class of convergence on \( T^d \).

The second part of Theorem \( 2 \) and Corollary \( 1 \) imply

**Corollary 3.** If the sequence \( \Lambda = \{ \lambda_n \}_{n=1}^{\infty} \) satisfies \( (17) \) and \( (16) \), then \( PBV_\Lambda \nsubseteq CV_H \).
Theorem 2 and Theorems A and B imply Theorem 4. The set of functions

$$\left\{ f : \sum_{j=0}^{\infty} \frac{d}{2^j/d} v_i (2^j, f) < \infty, \; i = 1, \ldots, d \right\}$$

is a class of convergence on $T^d$.

Corollary 4. The set of functions

$$\left\{ f : v_i (n, f) = O \left( n^{\alpha} \right), \; i = 1, \ldots, d \right\}$$

is a class of convergence on $T^d$ for any $\alpha \in (0, 1)$.

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