HIGHER SPHERICAL ALGEBRAS

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Abstract. We introduce and study higher spherical algebras, an exotic family of finite-dimensional algebras over an algebraically closed field. We prove that every such an algebra is derived equivalent to a higher tetrahedral algebra studied in [7], and hence that it is a tame symmetric periodic algebra of period 4.

1. Introduction and main results

Throughout this paper, $K$ will denote a fixed algebraically closed field. By an algebra we mean an associative finite-dimensional $K$-algebra with an identity. For an algebra $A$, we denote by $\text{mod}
A$ the category of finite-dimensional right $A$-modules and by $D$ the standard duality $\text{Hom}_K(-,K)$ on $\text{mod}
A$. An algebra $A$ is called self-injective if $A_A$ is injective in $\text{mod}
A$, or equivalently, the projective modules in $\text{mod}
A$ are injective. A prominent class of self-injective algebras is formed by the symmetric algebras $A$ for which there exists an associative, non-degenerate symmetric $K$-bilinear form $(\cdot,\cdot) : A \times A \to K$. Classical examples of symmetric algebras are provided by the blocks of group algebras of finite groups and the Hecke algebras of finite Coxeter groups. In fact, any algebra $A$ is a quotient algebra of its trivial extension algebra $T(A) = A \rtimes D(A)$, which is a symmetric algebra.

For an algebra $A$, the module category $\text{mod}
\Omega
A$ of its enveloping algebra $\Omega
A = A^\text{op} \otimes_K A$ is the category of finite-dimensional $A$-$A$-bimodules. We denote by $\Omega
A$ the syzygy operator in $\text{mod}
\Omega
A$ which assigns to a module $M$ in $\text{mod}
\Omega
A$ the kernel $\Omega
A(M)$ of a minimal projective cover of $M$ in $\text{mod}
\Omega
A$. An algebra $A$ is called periodic if $\Omega
n A(A) \cong A$ in $\text{mod}
\Omega
A$ for some $n \geq 1$, and if so the minimal such $n$ is called the period of $A$. Periodic algebras are self-injective and have periodic Hochschild cohomology.

Finding or possibly classifying periodic algebras is an important problem. It is very interesting because of connections with group theory, topology, singularity theory and cluster algebras.

We are concerned with the classification of all periodic tame symmetric algebras. In [4] Dugas proved that every representation-finite self-injective algebra, without simple blocks, is a periodic algebra. The representation-infinite, indecomposable, periodic algebras of polynomial growth were classified by Bialkowski, Erdmann and Skowroński in [2]. It is conjectured in [6, Problem] that every indecomposable symmetric periodic tame algebra of non-polynomial growth is of period 4. Prominent classes of tame symmetric algebras of period 4 are provided by the weighted surface algebras and their deformations investigated in [5], [7], [8], [9].

In this article we introduce and study higher spherical algebras, which are “higher analogs” of the non-singular spherical algebras introduced in [9], and provide a new exotic family of tame symmetric periodic algebras of period 4.
Let $m \geq 1$ be a natural number and $\lambda \in K^\times$. We denote by $S(m, \lambda)$ the algebra given by the quiver $\Delta$ of the form

and the relations:

\begin{align*}
\beta \nu \delta &= \beta \gamma \sigma + \lambda (\beta \gamma \sigma \alpha)^{m-1} \beta \gamma \sigma, \\
\nu \delta \alpha &= \gamma \sigma \alpha + \lambda (\gamma \sigma \alpha \beta)^{m-1} \gamma \sigma \alpha, \\
\sigma \omega \gamma &= \alpha \beta \gamma + \lambda (\alpha \beta \gamma \sigma)^{m-1} \alpha \beta \gamma, \\
\gamma \sigma \theta &= \nu \delta \varrho, \\
(\alpha \beta \gamma \sigma)^m \alpha &= 0, \\
(\gamma \sigma \alpha \beta)^m \gamma &= 0.
\end{align*}

We call $S(m, \lambda)$ with $m \geq 2$ a higher spherical algebra. For $m = 1$, this is the non-singular spherical algebra $S(1 + \lambda)$ investigated in [9, Section 3]. The above quiver is its Gabriel quiver, and $S(1 + \lambda)$ is a surface algebra (in the sense of [9]) given by the following triangulation of the sphere $S^2$ in $\mathbb{R}^3$

with the coherent orientation of triangles: $(1 \ 2 \ 5), (2 \ 3 \ 5), (3 \ 4 \ 6), (4 \ 1 \ 6)$. We note that the non-singular spherical algebras in [9] appear since in the general setting for weighted surface algebras we allow ‘virtual’ arrows.

The following two theorems describe basic properties of higher spherical algebras.

**Theorem 1.** Let $S = S(m, \lambda)$ be a higher spherical algebra. Then $S$ is a finite-dimensional algebra with $\dim_K S = 36m + 4$.

**Theorem 2.** Let $S = S(m, \lambda)$ be a higher spherical algebra. Then the following statements hold:

(i) $S$ is a symmetric algebra.
(ii) $S$ is a periodic algebra of period 4.
(iii) $S$ is a tame algebra of non-polynomial growth.

It follows from the above theorems that the higher spherical algebras $S(m, \lambda)$, $m \geq 2$, $\lambda \in K^\times$, form an exotic family of algebras of generalized quaternion type (in the sense of [8]) whose Gabriel quiver is not 2-regular. The classification of
the Morita equivalence classes of all algebras of generalized quaternion type with 2-regular Gabriel quivers having at least three vertices has been established in [8, Main Theorem]. During the work on this, surprisingly, we discovered new algebras, which we call higher tetrahedral algebras \( \Lambda(m, \lambda) \), \( m \geq 2 \), \( \lambda \in K^* \). They are introduced and studied in [7] (see Section 3 for definition and properties).

The following theorem relates these two classes of algebras.

**Theorem 3.** Let \( m \geq 2 \) be a natural number and \( \lambda \in K^* \). Then the algebras \( S(m, \lambda) \) and \( \Lambda(m, \lambda) \) are derived equivalent.

Then Theorem 2 is the consequence of Theorem 3, by applying general theory as described in Theorems 2.3, 2.4, 2.5, and Theorem 3.1.

For general background on the relevant representation theory we refer to the books [1], [10], [15], [16].

2. Derived equivalences

In this section we collect some facts on derived equivalences of algebras which are needed in the proofs of Theorems 2 and 3.

Let \( A \) be an algebra over \( K \). We denote by \( D^b \text{mod} A \) the derived category of \( \text{mod} A \), which is the localization of the homotopy category \( K^b \text{mod} A \) of bounded complexes of modules from \( \text{mod} A \) with respect to quasi-isomorphisms. Moreover, let \( K^b(P_A) \) be the subcategory of \( K^b \text{mod} A \) given by the complexes of projective modules in \( \text{mod} A \). Two algebras \( A \) and \( B \) are called derived equivalent if the derived categories \( D^b \text{mod} A \) and \( D^b \text{mod} B \) are equivalent as triangulated categories.

The triangulated structure is induced by shift in degrees of complexes. Following J. Rickard [12], a complex \( T \) in \( K^b(P_A) \) is called a tilting complex if the following properties are satisfied:

1. \( \text{Hom}_{K^b(P_A)}(T, T[i]) = 0 \) for all \( i \neq 0 \) in \( \mathbb{Z} \),
2. the full subcategory \( \text{add}(T) \) of \( K^b(P_A) \) consisting of direct summands of direct sums of copies of \( T \) generates \( K^b(P_A) \) as a triangulated category.

Here, \([\ ]\) denotes the translation functor by shifting any complex one degree to the left.

The following theorem is due to J. Rickard [12, Theorem 6.4].

**Theorem 2.1.** Two algebras \( A \) and \( B \) are derived equivalent if and only if there is a tilting complex \( T \) in \( K^b(P_A) \) such that \( B \cong \text{End}_{K^b(P_A)}(T) \).

We will need the following special case of an alternating sum formula established by D. Happel in [10, Sections III.1.3 and III.1.4].

**Proposition 2.2.** Let \( A \) be an algebra and \( Q = (Q^r)_{r \in \mathbb{Z}} \), \( R = (R^s)_{s \in \mathbb{Z}} \) two complexes in \( K^b(P_A) \) such that \( \text{Hom}_{K^b(P_A)}(Q, R[i]) = 0 \) for any \( i \neq 0 \) in \( \mathbb{Z} \). Then

\[
\dim_K \text{Hom}_{K^b(P_A)}(Q, R) = \sum_{r,s} (-1)^{r-s} \dim_K \text{Hom}_A(Q^r, R^s).
\]

We note that the right-hand side of the above formula can easily be computed using the Cartan matrix of \( A \).

We end this section with the following collection of important results.

**Theorem 2.3.** Let \( A \) and \( B \) be derived equivalent algebras. Then \( A \) is symmetric if and only if \( B \) is symmetric.
Proof. This is [14, Corollary 5.3].

**Theorem 2.4.** Let $A$ and $B$ be derived equivalent algebras. Then $A$ is periodic if and only if $B$ is periodic. Moreover, if so, then they have the same period.

**Proof.** See [5, Theorem 2.9].

**Theorem 2.5.** Let $A$ and $B$ be derived equivalent selfinjective algebras. Then the following equivalences hold.

(i) $A$ is tame if and only if $B$ is tame.

(ii) $A$ is of polynomial growth if and only if $B$ is of polynomial growth.

**Proof.** It follows from the assumption and [13, Corollary 2.2] (see also [14, Corollary 5.3]) that the algebras $A$ and $B$ are stably equivalent. Then the equivalences (i) and (ii) hold by [3, Theorems 4.4 and 5.6] and [11, Corollary 2].

## 3. Higher tetrahedral algebras

In this section we recall some facts on higher tetrahedral algebras established in [7], which will be crucial in the proofs of Theorems 1 and 2.

Consider the tetrahedron

with the coherent orientation of triangles: $(1 \ 5 \ 4), (2 \ 5 \ 3), (2 \ 6 \ 4), (1 \ 6 \ 3)$. Then, following [6, Section 6], we have the associated triangulation quiver $(Q, f)$ of the form

where $f$ is the permutation of arrows of order 3 described by the four shaded 3-cycles. We denote by $g$ the permutation on the set of arrows of $Q$ whose $g$-orbits are the four white 3-cycles.

Let $m \geq 2$ be a natural number and $\lambda \in K^*$. Following [7], the (non-singular) tetrahedral algebra $\Lambda(m, \lambda)$ of degree $m$ is the algebra given by the above quiver $Q$.
and the relations:
\[
\gamma \delta = \beta \varepsilon + \lambda (\beta \omega)^{m-1} \beta \varepsilon, \quad \delta \eta = \nu \omega, \quad \eta \gamma = \xi \alpha, \quad \nu \mu = \delta \xi,
\]
\[
\omega \varepsilon = \varepsilon \omega = (\varepsilon \sigma)^{m-1} \varepsilon \eta, \quad \omega \beta = \mu \sigma, \quad \beta \omega = \varepsilon \omega, \quad \mu \alpha = \omega \gamma, \n\]
\[
\xi \sigma = \eta \beta + \lambda (\eta \gamma \delta)^{m-1} \eta \beta, \quad \sigma \varepsilon = \alpha \delta, \quad 
\varepsilon \xi = \omega \mu, \quad \alpha \nu = \sigma \theta,
\]
\[
(\theta f(\theta) f(\theta)^{2}(\theta))^{m-1} \theta f(\theta) g(\theta)^{2}(\theta) = 0 \text{ for any arrow } \theta \text{ in } Q.
\]

The following theorem follows from Theorems 1, 2, 3 proved in [7] and describes some basic properties of higher tetrahedral algebras.

**Theorem 3.1.** Let \( \Lambda = \Lambda(m, \lambda) \) be a higher tetrahedral algebra with \( m \geq 2 \) and \( \lambda \in K^* \). Then the following statements hold:

(i) \( \Lambda \) is a finite-dimensional algebra with \( \dim K \Lambda = 36m. \)

(ii) \( \Lambda \) is a symmetric algebra.

(iii) \( \Lambda \) is a periodic algebra of period 4.

(iv) \( \Lambda \) is a tame algebra of non-polynomial growth.

The following proposition follows from [7, Section 4].

**Proposition 3.2.** Let \( \Lambda = \Lambda(m, \lambda) \) be a higher tetrahedral algebra with \( m \geq 2 \) and \( \lambda \in K^* \). Then the Cartan matrix \( C_\Lambda \) of \( \Lambda \) is of the form

\[
\begin{pmatrix}
\begin{array}{cccccc}
m+1 & m-1 & m & m & m & m \\
m-1 & m+1 & m & m & m & m \\
m & m & m+1 & m-1 & m & m \\
m & m & m-1 & m+1 & m & m \\
m & m & m & m & m+1 & m-1 \\
m & m & m & m & m & m+1
\end{array}
\end{pmatrix}
\]

4. **Proof of Theorem**

In this section we describe the Cartan matrices of higher spherical algebras.

Let \( S = S(m, \lambda) \) be a higher spherical algebra with \( m \geq 2 \) and \( \lambda \in K^* \). We start by collecting further identities in \( S \), they follow directly from the relations defining \( S \).

**Lemma 4.1.** The following relations hold in \( S \):

(i) \( (\beta \gamma \sigma \alpha)^{m-1} \beta \gamma \sigma \alpha = 0 \) and \( (\alpha \beta \gamma \sigma \alpha)^{m-1} \sigma \alpha \beta \gamma = 0. \)

(ii) \( \omega \gamma \sigma \alpha (\beta \gamma \sigma \alpha)^{m-1} = 0 \) and \( \omega (\gamma \sigma \alpha \beta)^{m-1} = 0. \)

(iii) \( (\sigma \alpha \beta \gamma)^{m-1} \sigma \alpha \beta = 0 \) and \( (\gamma \sigma \alpha \beta)^{m-1} \sigma \alpha \beta = 0. \)

(iv) \( \delta \alpha \beta \gamma (\sigma \alpha \beta \gamma)^{m-1} = 0 \) and \( \delta (\alpha \beta \gamma \sigma)^{m-1} = 0. \)

(v) \( (\omega \nu \delta)^{r} = (\alpha \beta \gamma \sigma)^{r} \) and \( (\nu \delta \omega)^{r} = (\gamma \sigma \alpha \beta)^{r}, \) for \( 2 \leq r \leq m. \)

For example, consider part (i). We have \( \lambda (\beta \gamma \sigma \alpha)^{m-1} \beta \gamma \sigma = \beta \nu \delta - \beta \gamma \sigma. \) We postmultiply this with \( \rho \) and get zero since \( \gamma \sigma \rho = \nu \delta \rho. \) The second part follows, by rewriting the first part and premultiply with \( \alpha. \) For part (v), starting with

\[
\rho \omega \nu \delta = \alpha \beta \gamma \sigma + \lambda (\alpha \beta \gamma \sigma)^{m}
\]

and squaring, one gets \( (\rho \omega \nu \delta)^{2} = (\alpha \beta \gamma \sigma)^{2} \) and (v) follows by induction.

Using the relations, it is easy to write down bases for the indecomposable projective modules, and prove the following.
Proposition 4.2. The Cartan matrix $C_S$ of $S$ is of the form

$$
\begin{bmatrix}
m + 1 & m & m + 1 & m & m & m \\
m & m + 1 & m & m & m & -1 \\
m + 1 & m & m + 1 & m & m & m \\
m & m & m & m + 1 & m & -1 \\
m & m & m & m & m + 1 & m \\
m & m & m & m & -1 & m + 1 \\
m & m & m & m & m & m + 1
\end{bmatrix}.
$$

In particular, $\dim_K S = 36m + 4$.

5. Proof of Theorem [3]

Let $\Lambda = \Lambda(m, \lambda)$ for some fixed $m \geq 2$ and $\lambda \in K^*$. For each vertex $i$ of the quiver $Q$ defining $\Lambda$, we denote by $P_i = e_i \Lambda$ the associated indecomposable projective module in $\text{mod} \, \Lambda$. Moreover, for any arrow $\theta$ from $j$ to $k$, we denote by $\theta : P_k \to P_j$ the homomorphism in $\text{mod} \, \Lambda$ given by the left multiplication by $\theta$. We consider the following complexes in $K^b(\text{mod} \, \Lambda)$:

- $T_1 : 0 \to P_1 \to 0$,
- $T_2 : 0 \to P_5 \to 0$,
- $T_4 : 0 \to P_3 \to 0$,
- $T_5 : 0 \to P_4 \to 0$,
- $T_6 : 0 \to P_6 \to 0$,

concentrated in degree 0, and

- $T_3 : 0 \to P_2 \xrightarrow{\sigma^{-\beta}} P_3 \oplus P_4 \to 0$,

concentrated in degrees 1 and 0. Moreover, we set $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5 \oplus T_6$.

Lemma 5.1. $T$ is a tilting complex in $K^b(\text{mod} \, \Lambda)$.

Proof. It is sufficient to show the equalities

$$\text{Hom}_{K^b(\text{mod} \, \Lambda)}(T_3, T_r[1]) = 0, \quad \text{Hom}_{K^b(\text{mod} \, \Lambda)}(T_r, T_3[-1]) = 0,$$

for $r \in \{1, 2, \ldots, 6\}$. The first equalities hold, because any nonzero homomorphism $f : P_2 \to P_1$ with $i \neq 2$ factors through $P_2 \to P_3 \oplus P_4$. The second equalities hold, because for any nonzero $g : P_1 \to P_2$ with $i \neq 2$, the composition $P_2 \to P_3 \oplus P_4 \to 0$ is nonzero. \qed

We define $R = R(m, \lambda) = \text{End}_{K^b(\text{mod} \, \Lambda)}(T)$, and note that $P_i = \text{Hom}_{K^b(\text{mod} \, \Lambda)}(T, T_i)$, $i \in \{1, 2, \ldots, 6\}$, form a complete family of pairwise non-isomorphic indecomposable projective modules in $\text{mod} \, R$. We abbreviate $S = S(m, \lambda)$, and use the ordering $P_i = e_i S$, $i \in \{1, 2, \ldots, 6\}$, of the indecomposable projective modules in $\text{mod} \, S$ corresponding to the numbering of vertices of the quiver $\Delta$ defining $S$. In this notation, we have the following lemma.

Lemma 5.2. The Cartan matrices $C_R$ and $C_S$ coincide. In particular, the algebras $R$ and $S$ have the same dimension $36m + 4$. 

Proof. This follows by the computation of the dimensions \( \dim_K \text{Hom}_{K^b(P_\Lambda)}(T_i, T_j) \), \( i, j \in \{1, 2, \ldots, 6\} \), using Proposition 3.2 and the form of the Cartan matrix \( C_\Lambda \) of \( \Lambda \) presented in Proposition 3.2. For example, the first row of \( C_R \) is of the form \( [m+1 \quad m \quad m+1 \quad m \quad m \quad m \quad m] \), because

\[
\dim_K \text{Hom}_{K^b(P_\Lambda)}(T_1, T_1) = \dim_K \text{Hom}_{\Lambda}(P_1, P_1) = m + 1,
\]
\[
\dim_K \text{Hom}_{K^b(P_\Lambda)}(T_1, T_2) = \dim_K \text{Hom}_{\Lambda}(P_1, P_3) = m,
\]
\[
\dim_K \text{Hom}_{K^b(P_\Lambda)}(T_1, T_3) = \dim_K \text{Hom}_{\Lambda}(P_1, P_3) + \dim_K \text{Hom}_{\Lambda}(P_1, P_4)
\quad - \dim_K \text{Hom}_{\Lambda}(P_1, P_2)
\quad = m + m - (m - 1) = m + 1,
\]
\[
\dim_K \text{Hom}_{K^b(P_\Lambda)}(T_1, T_4) = \dim_K \text{Hom}_{\Lambda}(P_1, P_3) = m,
\]
\[
\dim_K \text{Hom}_{K^b(P_\Lambda)}(T_1, T_5) = \dim_K \text{Hom}_{\Lambda}(P_1, P_4) = m,
\]
\[
\dim_K \text{Hom}_{K^b(P_\Lambda)}(T_1, T_6) = \dim_K \text{Hom}_{\Lambda}(P_1, P_5) = m.
\]

We define now irreducible morphisms between the summands of \( T \) in \( K^b(P_\Lambda) \):

\[
\begin{align*}
\alpha : T_2 &\to T_1, \text{ given by } \delta : P_5 \to P_1, \\
\tilde{\beta} : T_3 &\to T_2, \text{ given by } [\xi \quad \eta + \lambda(\eta\gamma\sigma)^{-1}\eta] : P_3 \oplus P_4 \to P_5, \\
\tilde{\gamma} : T_4 &\to T_3, \text{ given by } [0] : P_5 \to P_3 \oplus P_4, \\
\tilde{\sigma} : T_1 &\to T_4, \text{ given by } \alpha : P_1 \to P_3, \\
\tilde{\delta} : T_1 &\to T_5, \text{ given by } \gamma : P_1 \to P_4, \\
\tilde{\nu} : T_6 &\to T_1, \text{ given by } \nu : P_6 \to P_1, \\
\tilde{\omega} : T_3 &\to T_6, \text{ given by } [\mu \quad \omega] : P_3 \oplus P_4 \to P_6, \\
\tilde{\nu} : T_5 &\to T_3, \text{ given by } [0] : P_4 \to P_3 \oplus P_4.
\end{align*}
\]

We obtain then the irreducible homomorphisms between the indecomposable projective modules in mod \( R \)

\[
\alpha = \text{Hom}_{K^b(P_\Lambda)}(T, \tilde{\alpha}) : \tilde{P}_2 \to \tilde{P}_1, \quad \beta = \text{Hom}_{K^b(P_\Lambda)}(T, \tilde{\beta}) : \tilde{P}_3 \to \tilde{P}_2, \\
\gamma = \text{Hom}_{K^b(P_\Lambda)}(T, \tilde{\gamma}) : \tilde{P}_4 \to \tilde{P}_3, \quad \sigma = \text{Hom}_{K^b(P_\Lambda)}(T, \tilde{\sigma}) : \tilde{P}_1 \to \tilde{P}_4, \\
\delta = \text{Hom}_{K^b(P_\Lambda)}(T, \tilde{\delta}) : \tilde{P}_1 \to \tilde{P}_4, \quad \varrho = \text{Hom}_{K^b(P_\Lambda)}(T, \tilde{\varrho}) : \tilde{P}_5 \to \tilde{P}_1, \\
\omega = \text{Hom}_{K^b(P_\Lambda)}(T, \tilde{\omega}) : \tilde{P}_3 \to \tilde{P}_6, \quad \nu = \text{Hom}_{K^b(P_\Lambda)}(T, \tilde{\nu}) : \tilde{P}_5 \to \tilde{P}_3,
\]

which are representatives of all irreducible homomorphisms between the modules \( \tilde{P}_i, i \in \{1, 2, \ldots, 6\} \), in mod \( R \). This shows that the Gabriel quiver \( Q_R \) of \( R \) is the quiver

![Gabriel quiver](image-url)
being the quiver $\Delta$ defining the algebra $S$.

**Theorem 5.3.** The algebras $R$ and $S$ are isomorphic.

**Proof.** We first prove that the following identities hold in $R$:

1. $\alpha \beta \gamma = \omega \gamma$,
2. $\sigma \alpha \beta = \sigma \omega$,
3. $\gamma \sigma \varrho = \nu \delta \varrho$,
4. $\omega \gamma \sigma = \omega \nu \delta$,
5. $\beta \nu \delta = \beta \gamma \sigma + \lambda (\beta \gamma \sigma \alpha)^{m-1} \beta \gamma \sigma$,
6. $\nu \delta \alpha = \gamma \sigma \alpha + \lambda (\gamma \sigma \alpha \beta)^{m-1} \gamma \sigma \alpha$,
7. $\delta \alpha \beta = \delta \varrho \omega + \lambda (\delta \varrho \omega \nu)^{m-1} \delta \varrho \omega$,
8. $\alpha \beta \nu = \varrho \omega \nu + \lambda (\varrho \omega \nu \delta)^{m-1} \varrho \omega \nu$,
9. $(\alpha \beta \gamma \sigma)^{m-1} \alpha \mu = 0$,
10. $(\alpha \beta \gamma \sigma)^{m-1} \alpha \nu = 0$.

For (1), it is enough to show that $\tilde{\alpha} \tilde{\beta} \tilde{\gamma} = \tilde{\omega} \tilde{\nu} \tilde{\gamma}$. We have $\tilde{\alpha} \tilde{\beta} \tilde{\gamma} = \delta \xi : P_3 \to P_1$ and $\tilde{\omega} \tilde{\nu} \tilde{\gamma} = \nu \mu : P_3 \to P_1$, with $\delta \xi = \nu \mu$ in $\Lambda$, and so the required equality holds.

For (2), it is enough to show that $\tilde{\sigma} \tilde{\alpha} \tilde{\beta} = \tilde{\sigma} \tilde{\delta} \tilde{\varrho}$. We have $\tilde{\sigma} \tilde{\alpha} \tilde{\beta} = [\alpha \delta \xi : \alpha \delta \eta + \lambda \alpha \delta (\eta \gamma \delta)^{m-1} \eta] : P_3 \oplus P_4 \to P_3$ and $\tilde{\sigma} \tilde{\delta} \tilde{\varrho} = [\alpha \nu \mu, \alpha \nu \omega] : P_3 \oplus P_4 \to P_3$. Moreover, we have in $\Lambda$ the equalities

$$\alpha \nu \omega = \sigma \varrho \omega = \sigma \varepsilon \eta + \lambda \sigma (\varepsilon \xi \sigma)^{m-1} \varepsilon \eta = \alpha \delta \eta + \lambda \alpha \delta (\xi \varepsilon \sigma)^{m-1} \eta$$

and $\xi \sigma \varepsilon = \xi \alpha \delta = \eta \gamma \delta$ in $\Lambda$. Hence the required equality holds.

For (3), we prove that $\tilde{\delta} \tilde{\varrho} \tilde{\varrho} - \tilde{\gamma} \tilde{\delta} \tilde{\nu} = 0$ in $K^0(\Lambda)$. We have

$$\tilde{\gamma} \tilde{\delta} \tilde{\nu} = \left[ \begin{array}{c} \alpha \\ \varepsilon \end{array} \right] : P_6 \to P_3 \oplus P_4 \quad \text{and} \quad \tilde{\delta} \tilde{\varrho} \tilde{\varrho} = \left[ \begin{array}{c} 0 \\ \gamma \nu \sigma \end{array} \right] : P_6 \to P_3 \oplus P_4.$$

Moreover, we have the following commutative diagram in mod $\Lambda$

$$\begin{array}{ccc}
P_2 & \xrightarrow{\gamma} & P_3 \\
\downarrow{\beta} & & \downarrow{\gamma} \\
\text{P}_3 \oplus \text{P}_4 & \xrightarrow{\sigma} & \text{P}_6
\end{array}$$

because $\alpha \nu = \sigma \varrho$ and $\gamma \nu = \beta \varrho$ in $\Lambda$. This proves the claim.

For (4), we note that $\tilde{\omega} \tilde{\gamma} \tilde{\sigma} = \mu \alpha : P_1 \to P_6$ and $\tilde{\omega} \tilde{\nu} \tilde{\delta} = \omega \gamma \sigma : P_1 \to P_6$, with $\mu \alpha = \omega \gamma$.

For (5), we prove equality $\tilde{\beta} \tilde{\nu} \tilde{\delta} = \tilde{\beta} \tilde{\gamma} \tilde{\sigma} + \lambda (\tilde{\beta} \tilde{\gamma} \tilde{\sigma} \tilde{\alpha})^{m-1} \tilde{\beta} \tilde{\gamma} \tilde{\sigma}$. Observe that,

$$\tilde{\beta} \tilde{\nu} \tilde{\delta} = \eta \gamma + \lambda (\eta \gamma \delta)^{m-1} \eta \gamma : P_1 \to P_5,$$
$$\tilde{\beta} \tilde{\gamma} \tilde{\sigma} = \xi \alpha = \eta \gamma : P_3 \to P_5,$$
$$\lambda (\tilde{\beta} \tilde{\gamma} \tilde{\sigma} \tilde{\alpha})^{m-1} \tilde{\beta} \tilde{\gamma} \tilde{\sigma} = \lambda (\xi \alpha \delta)^{m-1} \eta \gamma = \lambda (\eta \gamma \delta)^{m-1} \eta \gamma : P_1 \to P_5,$$

and hence the required equality holds.
Therefore, the required equality holds.

Moreover, \( \alpha \delta = \sigma \varepsilon \), and hence

\[
\lambda(\alpha \delta \xi)^{m-1} \alpha \delta = \lambda(\sigma \varepsilon \xi)^{m-1} \sigma \varepsilon = \lambda \sigma (\varepsilon \xi \sigma)^{m-1} \varepsilon.
\]

But then

\[
\tilde{\gamma} \tilde{\sigma} \tilde{\alpha} = \lambda(\tilde{\gamma} \tilde{\sigma} \tilde{\alpha} \tilde{\beta})^{m-1} \tilde{\gamma} \tilde{\sigma} \tilde{\alpha} = \left[\sigma \varepsilon + \lambda \sigma (\varepsilon \xi)^{m-1} \varepsilon\right] : P_5 \to P_3 \oplus P_4,
\]

Further, we have the following commutative diagram in mod \( \Lambda \)

\[
\begin{array}{ccc}
P_2 & \xrightarrow{\epsilon + \lambda \varepsilon \xi^{m-1}} & P_5 \\
\alpha \delta & \downarrow{\beta} & \downarrow{-\sigma} \\
P_3 \oplus P_4 & & P_3 \oplus P_4
\end{array}
\]

because \( \omega \beta = \mu \sigma = \varepsilon \sigma \varepsilon \) in \( \Lambda \). This shows that \( \tilde{\nu} \tilde{\delta} \tilde{\alpha} - \tilde{\gamma} \tilde{\sigma} \tilde{\alpha} - \lambda(\tilde{\gamma} \tilde{\sigma} \tilde{\alpha} \tilde{\beta})^{m-1} \tilde{\gamma} \tilde{\sigma} \tilde{\alpha} = 0 \) in \( K^h(P_\Lambda) \), and hence the required equality holds.

For \([3]\), we prove that \( \tilde{\delta} \tilde{\alpha} \tilde{\beta} = \tilde{\delta} \tilde{\omega} + \lambda(\tilde{\delta} \tilde{\omega} \tilde{\nu})^{m-1} \tilde{\delta} \tilde{\omega} \). We have

\[
\tilde{\delta} \tilde{\alpha} \tilde{\beta} = \left[\sigma \delta \xi \quad \gamma \delta \eta + \lambda \gamma \delta (\eta \gamma \delta)^{m-1} \eta\right] : P_3 \oplus P_4 \to P_4,
\]

\[
\tilde{\delta} \tilde{\omega} = \left[\sigma \nu \mu \quad \gamma \nu \omega\right] : P_3 \oplus P_4 \to P_4,
\]

and \( \delta \xi = \nu \mu \), \( \delta \eta = \nu \omega \) in \( \Lambda \). Furthermore,

\[
\tilde{\delta} \tilde{\delta} \tilde{\omega} = \gamma \nu \omega = \gamma \delta \eta : P_4 \to P_4.
\]

Hence we obtain

\[
\lambda(\tilde{\delta} \tilde{\omega} \tilde{\nu})^{m-1} \tilde{\delta} \tilde{\omega} = \left[\lambda(\gamma \delta \eta)^{m-1} \gamma \delta \xi \quad \lambda(\gamma \delta \eta)^{m-1} \gamma \delta \eta\right] : P_3 \oplus P_4 \to P_4.
\]

We note that

\[
\lambda(\gamma \delta \eta)^{m-1} \gamma \delta \xi = \lambda(\gamma f(\gamma) f^2(\gamma))^{m-1} \gamma f(\gamma) g(f(\gamma)) = 0.
\]

Therefore, the required equality holds.

For \([3]\), we have to show that \( \tilde{\alpha} \tilde{\beta} \tilde{\nu} = \tilde{\delta} \tilde{\omega} \tilde{\nu} + \lambda(\tilde{\delta} \tilde{\omega} \tilde{\nu} \tilde{\delta})^{m-1} \tilde{\delta} \tilde{\omega} \). We have

\[
\tilde{\alpha} \tilde{\beta} \tilde{\nu} = \delta \eta + \lambda \delta (\eta \gamma \delta)^{m-1} \eta = \delta \eta + \lambda(\delta \eta \gamma)^{m-1} \delta \eta : P_4 \to P_4,
\]

\[
\tilde{\delta} \tilde{\omega} \tilde{\nu} = \nu \omega = \delta \eta : P_4 \to P_1,
\]

\[
\tilde{\delta} \tilde{\omega} \tilde{\delta} = \delta \nu \omega = \delta \eta \gamma : P_1 \to P_1,
\]

and then

\[
\lambda(\tilde{\delta} \tilde{\omega} \tilde{\nu} \tilde{\delta})^{m-1} \tilde{\delta} \tilde{\omega} \tilde{\nu} = \lambda(\delta \eta \gamma)^{m-1} \delta \eta : P_4 \to P_1.
\]

Hence the required equality holds.

For \([3]\), we observe that

\[
(\tilde{\alpha} \tilde{\beta} \tilde{\gamma} \tilde{\alpha})^{m} = (\delta \xi \alpha)^{m} \delta : P_5 \to P_1,
\]

and this is zero, because in \( \Lambda \), the element \( (\delta \xi \alpha)^{m} \) belongs to the socle of \( P_1 \) (see \([2]\) Lemma 4.2)).
For \([10]\), we observe that \(\tilde{\delta}\tilde{\alpha}\tilde{\beta}\tilde{\gamma} = \alpha\delta\xi : P_3 \to P_3\), and therefore
\[
(\tilde{\gamma}\tilde{\delta}\tilde{\alpha}\tilde{\beta})^m\tilde{\gamma} = \left[\begin{smallmatrix} (\alpha\delta\xi)^m
\end{smallmatrix}\right] : P_3 \to P_3 \oplus P_4,
\]
Then, we have in \(\text{mod } \Lambda\) the commutative diagram
\[
\begin{array}{ccc}
P_2 & \xrightarrow{\beta} & P_3 \\
\downarrow{\sigma} & & \downarrow{\beta} \\
P_3 & & P_3 \oplus P_4
\end{array}
\]
because \(\beta\xi(\sigma\xi)^{m-1}\) is the path of length \(3m\) in \(Q\) from 4 to 3, and hence the zero path in \(\Lambda\), by \([7\), Lemma 4.5\]. Therefore, \((\tilde{\gamma}\tilde{\delta}\tilde{\alpha}\tilde{\beta})^m\tilde{\gamma} = 0\) in \(K^b(P_\Lambda)\), and equality \([10]\) holds.

We also observe that, in \(R\), we have by (1) and (4) that \(\alpha\beta\gamma\sigma = \varrho\omega\gamma\sigma = \varrho\omega\nu\delta\).

To obtain the defining relations for \(S\), we replace \(\varrho\) by \(\varrho^* = \varrho + \lambda(\alpha\beta\gamma\sigma)^{m-1}\varrho = \varrho + \lambda(\varrho\omega\delta)^{m-1}\varrho\). Then identities \((1\), \(2\), \(3\), \(7\), \(8\)) are replaced by the following identities:

\[
\begin{align*}
\tag{1*} \varrho^*\omega\gamma &= \varrho\omega\gamma + \lambda(\alpha\beta\gamma\sigma)^{m-1}\varrho\omega\gamma = \alpha\beta\gamma + \lambda(\alpha\beta\gamma\sigma)^{m-1}\alpha\beta\gamma, \\
\tag{2*} \sigma\varrho^*\omega &= \sigma\varrho\omega + \lambda(\alpha\beta\gamma\sigma)^{m-1}\varrho\omega = \sigma\alpha\beta + \lambda(\sigma\alpha\beta\gamma)^{m-1}\sigma\alpha\beta, \\
\tag{3*} \gamma\varrho\varrho^* &= \gamma\varrho\varrho + \lambda(\varrho\omega\delta)^{m-1}\varrho = \gamma\varrho\varrho + \lambda(\varrho\omega\delta)^{m-1}\varrho = \nu\delta\varrho + \lambda(\nu\delta\varrho)^{m-1}\varrho = \nu\delta\varrho^*, \\
\tag{7*} \delta\varrho^*\omega &= \delta(\varrho + \lambda(\varrho\omega\delta)^{m-1}\varrho)\omega = \delta\varrho\omega + \lambda(\delta\varrho\omega\delta)^{m-1}\varrho = \delta\varrho\omega + \lambda(\delta\varrho\omega\delta)^{m-1}\varrho = \delta\omega + \lambda(\delta\varrho\omega\delta)^{m-1}\delta\omega = \delta\omega + \lambda(\delta\varrho\omega\delta)^{m-1}\delta\omega = \delta\omega + \lambda(\delta\varrho\omega\delta)^{m-1}\delta\omega,
\end{align*}
\]

Therefore, after replacing \(\varrho\) by \(\varrho^*\), the relations defining the algebra \(S = S(m, \lambda)\) are satisfied. Then, applying Lemma \([5,2]\) we conclude that algebras \(R\) and \(S\) are isomorphic.

Summing up, Theorem \([3]\) follows from Theorems \([2,1]\) and \([5,3]\).

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