Perseus: A Simple High-Order Regularization Method for Variational Inequalities

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May 19, 2022

Abstract

This paper settles an open and challenging question pertaining to the design of simple high-order regularization methods for solving smooth and monotone variational inequalities (VIs). A VI involves finding $x^* \in X$ such that $\langle F(x), x - x^* \rangle \geq 0$ for all $x \in X$ and we consider the setting where $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is smooth with up to $(p - 1)^{th}$-order derivatives. For the case of $p = 2$, Nesterov [2006] extended the cubic regularized Newton’s method to VIs with a global rate of $O(\epsilon^{-1})$. Monteiro and Svaiter [2012] proposed another second-order method which achieved an improved rate of $O(\epsilon^{-2/3} \log(1/\epsilon))$, but this method required a nontrivial binary search procedure as an inner loop. High-order methods based on similar binary search procedures have been further developed and shown to achieve a rate of $O(\epsilon^{-2/(p+1)})$ [Bullins and Lai, 2020, Lin and Jordan, 2021b, Jiang and Mokhtari, 2022]. However, such search procedure can be computationally prohibitive in practice [Nesterov, 2018] and the problem of finding a simple high-order regularization methods remains as an open and challenging question in optimization theory. We propose a $p^{th}$-order method which does not require any binary search scheme and is guaranteed to converge to a weak solution with a global rate of $O(\epsilon^{-2/(p+1)})$. A version with restarting attains a global linear and local superlinear convergence rate for smooth and strongly monotone VIs. Further, our method achieves a global rate of $O(\epsilon^{-2/p})$ for solving smooth and non-monotone VIs satisfying the Minty condition; moreover, the restarted version again attains a global linear and local superlinear convergence rate if the strong Minty condition holds.

1 Introduction

Let $\mathbb{R}^d$ be a finite-dimensional Euclidean space and let $X \subseteq \mathbb{R}^d$ be a closed, convex and bounded set with a diameter $D > 0$. Given that $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a continuous operator, a basic assumption in optimization theory, generalizing convexity, is that $F$ is monotone:

$$\langle F(x) - F(x'), x - x' \rangle \geq 0, \text{ for all } x, x' \in \mathbb{R}^d.$$  \hspace{1cm} (1)

Another useful assumption in many settings is that $F$ is smooth; in particular, that it has with Lipschitz-continuous $(p - 1)^{th}$-order derivative $(p \geq 1)$ in the sense that there exists a constant $L > 0$ such that

$$\|\nabla^{(p-1)}F(x) - \nabla^{(p-1)}F(x')\| \leq L\|x - x'\|, \text{ for all } x, x' \in \mathbb{R}^d.$$  \hspace{1cm} (1)

With these assumptions, we can formulate the main problem of interest in this paper—the Minty variational inequality problem [Minty, 1962]. This consists in finding a point $x^* \in X$ such that

$$\langle F(x), x - x^* \rangle \geq 0, \text{ for all } x \in X.$$  \hspace{1cm} (2)
The solution to Eq. (2) is referred to as a weak solution to the variational inequality (VI) corresponding to $F$ and $\mathcal{X}$ [Facchinei and Pang, 2007]. By way of comparison, the Stampacchia variational inequality problem [Hartman and Stampacchia, 1966] consists in finding a point $x^* \in \mathcal{X}$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \text{for all } x \in \mathcal{X}, \quad (3)$$

and the solution to Eq. (3) is called a strong solution to the VI corresponding to $F$ and $\mathcal{X}$. In the setting where $F$ is continuous and monotone, the solution sets of Eq. (2) and Eq. (3) are equivalent. However, these two solution sets are different in general and a weak solution needs not exist even when a strong solution exists. In addition, computing an approximate strong solution involves a higher computational burden than finding an approximate weak solution [Monteiro and Svaiter, 2010, 2011, Chen et al., 2017].

VIs capture a wide range of problems in optimization theory and beyond, including saddle-point problems and models of equilibria in game-theoretic settings [Cottle et al., 1980, Kinderlehrer and Stampacchia, 2000, Trémolières et al., 2011]. Moreover, the challenge of designing solution methods for the VIs with provable worst-case complexity bounds has been a central topic during several past decades; see Harker and Pang [1990] and Facchinei and Pang [2007] and references therein. These foundations have been inspirational to machine learning researchers in recent years, where general saddle-point problems have found applications, including generative adversarial networks (GANs) [Goodfellow et al., 2014] and multi-agent learning in games [Cesa-Bianchi and Lugosi, 2006, Mertikopoulos and Zhou, 2019]. Some applications in ML induce a non-monotone structure, with examples including the training of robust neural networks [Madry et al., 2018] or robust classifiers [Sinha et al., 2018].

Building on seminal work in convex and nonconvex optimization [Baes, 2009, Birgin et al., 2017], we tackle the challenge of developing $p$th-order methods for VIs via inexact solutions of regularized subproblems based on a $(p-1)$th-order Taylor expansion of $F$. Accordingly, we make the following assumptions throughout this paper.

A1. $F: \mathbb{R}^d \mapsto \mathbb{R}^d$ is smooth with up to $(p-1)$th-order derivatives available to the algorithm.

A2. The subproblem based on a $(p-1)$th-order Taylor expansion of $F$ and a convex and bounded set $\mathcal{X}$ can be computed approximately in an efficient manner (see Section 3 for details).

For the first-order VI methods ($p=1$), Nemirovski [2004] has shown that the extragradient (EG) method [Korpelevich, 1976, Antipin, 1978] will converge to a weak solution with a global rate of $O(\epsilon^{-1})$ if $F$ is monotone and Eq. (1) holds. There are other methods that achieve the same rate, including the optimistic gradient method [Popov, 1980, Mokhtari et al., 2020, Kotsalis et al., 2020], the forward-backward splitting method [Tseng, 2000] and the dual extrapolation method [Nesterov, 2007]. All the aforementioned methods match the lower bound of [Ouyang and Xu, 2021] and are thus optimal.

Comparing to their first-order counterpart, the investigations of second-order and high-order methods ($p \geq 2$) are relatively rare, as exploiting the high-order derivative information is much more involved for VIs [Nesterov, 2006, Monteiro and Svaiter, 2012]. Aiming to fill this gap, recent work has generalized classical methods from first-order to high-order, including the extragradient method [Bullins and Lai, 2020, Lin and Jordan, 2021b] and the optimistic gradient method [Jiang and Mokhtari, 2022]. These generalizations are guaranteed to achieve a global rate of $O(\epsilon^{-2/(p+1)} \log(1/\epsilon))$ but all require a non-trivial binary search procedure at each iteration. This binary search procedure can be computationally expensive from a practical viewpoint. Thus, the problem of finding a simple high-order method with a rate of $O(\epsilon^{-2/(p+1)})$ remains open. Indeed, Nesterov [2018, Page 305] noted the difficulty of removing the binary search procedure without sacrificing the rate of convergence and highlighted this as an open and challenging question. We summarize the problem as follows:
Can we design a simple \( p \)-th-order method with a global rate of \( O(\epsilon^{-2/(p+1)}) \)?

In this paper we present an affirmative answer to this problem by identifying a \( p \)-th-order method that achieves a global rate of \( O(\epsilon^{-2/(p+1)}) \) while dispensing entirely with the binary search inner loop. The main idea of the proposed method is to incorporate a simple yet novel adaptive strategy into a high-order extension of the dual extrapolation method. Concurrently appearing on arXiv, Adil et al. [2022] established similar global convergence rate for solving smooth and monotone VIs without requiring a binary search procedure. Our results were derived independently from theirs.

More concretely, our contribution can be summarized as follows:

1. We present a new \( p \)-th-order method for solving smooth and monotone VIs where \( F \) is sufficiently smooth with Lipschitz continuous \((p-1)\)-th-order derivative and \( \mathcal{X} \) is convex and bounded. We prove that the number of calls of subproblem solvers required by our method to find an \( \epsilon \)-weak solution is bounded by
\[
O \left( \left( \frac{LD^{p+1}}{\epsilon} \right)^{\frac{2}{p+1}} \right),
\]
which shaves off a factor of \( \log(1/\epsilon) \) in the state-of-the-art rates achieved by existing \( p \)-th-order methods [Bullins and Lai, 2020, Lin and Jordan, 2021b, Jiang and Mokhtari, 2022]. Moreover, we present a restarted version of our method for solving smooth and strongly monotone VIs, and show that the number of calls of subproblem solvers required to find \( \hat{x} \in \mathcal{X} \) satisfying \( \|\hat{x} - x^*\| \leq \epsilon \) is bounded by
\[
O \left( \left( \frac{\kappa D^{p-1}}{\epsilon} \right)^{\frac{2}{p+1}} \log \left( \frac{D}{\epsilon} \right) \right),
\]
where \( \kappa = L/\mu \) refers to the generalized condition number of \( F \) (see Section 2). In addition, the restarted version of our method achieves local superlinear convergence for the case of \( p \geq 2 \).

2. We show how to modify our framework such that it can be used for solving smooth and non-monotone VIs satisfying the Minty condition. Again we note that a binary search procedure is not required. We prove that the number of calls of subproblem solvers to find an \( \epsilon \)-strong solution is bounded by
\[
O \left( \left( \frac{LD^{p+1}}{\epsilon} \right)^{\frac{2}{p}} \right).
\]
The restarted version is developed for solving smooth and non-monotone VIs satisfying the strong Minty condition and the number of calling subproblem solvers required to find \( \hat{x} \in \mathcal{X} \) satisfying \( \|\hat{x} - x^*\| \leq \epsilon \) is bounded by
\[
O \left( \max\{(\kappa_{\text{Minty}} D^{p-1})^{\frac{2}{p}}, (\kappa_{\text{Minty}} D^{p-1})^{\frac{2}{p+1}}\} \log \left( \frac{D}{\epsilon} \right) \right),
\]
where \( \kappa_{\text{Minty}} = L/\mu_{\text{Minty}} \) refers to the Minty condition number of \( F \) (see Section 2). In addition, the restarted version of our method achieves local superlinear convergence for the case of \( p \geq 2 \).
Further related work. In addition to the aforementioned work, we briefly review a line of related research on high-order methods for convex optimization, monotone variational inequalities, and monotone inclusion problems. We focus on the \( p \)-th-order methods for \( p \geq 2 \) and leave first-order methods out of our discussion.

To the best of our knowledge, the systematic investigation of the global convergence rate of second-order methods originates in work on the cubic regularization of Newton’s method (CRN) [Nesterov and Polyak, 2006] and its accelerated counterpart (ACRN) [Nesterov, 2008]. The ACRN method was then extended with a \( p \)-th-order regularization model, yielding an improved rate of \( O(\epsilon^{-1/(p+1)}) \) [Baes, 2009] while an adaptive \( p \)-th-order method was proposed in Jiang et al. [2020] with the same rate. This extension was recently revisited by Nesterov [2021b] and Grapiglia and Nesterov [2022] with a thorough discussion on an efficient implementation of a third-order method. Meanwhile, within the alternative accelerated Newton proximal extragradient (ANPE) framework [Monteiro and Svaiter, 2013], a \( p \)-th-order method was proposed by Gasnikov et al. [2019] with a global rate of \( O(\epsilon^{-2/(3p+1)} \log(1/\epsilon)) \) for minimizing a smooth and convex function whose \( p \)-th-order derivative is Lipschitz continuous. Their methods match a lower bound [Arjevani et al., 2019] up to log factors. Subsequently, the \( p \)-th-order ANPE framework was extended to minimizing a smooth and strongly convex function [Marques Alves, 2022] and achieved a global linear rate. Beyond the setting with Lipschitz continuous \( p \)-th-order derivatives, the aforementioned \( p \)-th-order methods have been adapted to a more general setting with Hölder continuous \( p \)-th-order derivatives [Grapiglia and Nesterov, 2017, 2019, 2020, Doikov and Nesterov, 2021, Song et al., 2021]. Some other settings include structured nonsmooth minimization [Bullins, 2020] and nonconvex smooth minimization [Cartis et al., 2010, 2011a,b, 2019, Birgin et al., 2016, 2017, Martínez, 2017]. We also acknowledge the existence of a complementary line of research, due to Nesterov [2021d], that studies favorable properties of lower-order methods in the setting of higher-order smoothness [Nesterov, 2021a,c, Doikov and Nesterov, 2022].

Finally, we also note the existence of high-order methods obtained via discretization of continuous-time dynamical systems [Wibisono et al., 2016, Lin and Jordan, 2021a]. In particular, Wibisono et al. [2016] showed that the ACRN method and its \( p \)-th-order variants can be derived from implicit discretization of an open-loop system without Hessian-driven damping. Lin and Jordan [2021a] have provided a control-theoretic perspective on \( p \)-order ANPE methods by recovering these methods from implicit discretization of a closed-loop system with Hessian-driven damping. Both of these works prove the convergence rate of \( p \)-order ACRN and ANPE methods via appeal to simple Lyapunov function arguments.

Organization. The remainder of this paper is organized as follows. In Section 2, we present the basic setup for variational inequality (VI) problems and provide definitions for the class of operators and optimality criteria we consider in this paper. In addition, we review the dual extrapolation method. In Section 3, we present our new method, its restarted version, and our main results on the global and local convergence guarantee for monotone and non-monotone VIs. In Section 4, we provide the proofs for our main results. In Section 5, we conclude the paper with a discussion on future research directions.

Notation. We use lower-case letters such as \( x \) to denote vectors and upper-case letters such as \( X \) to denote tensors. Let \( \mathbb{R}^d \) be a finite-dimensional Euclidean space (the dimension is \( d \in \{1,2,\ldots\} \)), endowed with the scalar product \( \langle \cdot, \cdot \rangle \). For \( x \in \mathbb{R}^d \), we let \( \|x\| \) denote its \( \ell_2 \)-norm. For a tensor
For a closed and convex set $X \subseteq \mathbb{R}^d$, we let $\mathcal{P}_X$ be the orthogonal projection onto $X$ and let $\text{dist}(x, \mathcal{X}) = \inf_{x' \in \mathcal{X}} \|x' - x\|$ denote the distance between $x$ and $\mathcal{X}$. Finally, $a = O(b(L, \mu, \epsilon))$ stands for an upper bound $a \leq C \cdot b(L, \mu, \epsilon)$, where $C > 0$ is independent of parameters $L, \mu$ and the tolerance $\epsilon \in (0, 1)$, and $a = \tilde{O}(b(L, \mu, \epsilon))$ indicates the same inequality where $C > 0$ depends on logarithmic factors of $1/\epsilon$.

2 Preliminaries and Technical Background

In this section, we present the basic formalism of variational inequality (VI) problems and provide definitions for the class of operators and optimality criteria considered in this paper. We further give a brief overview of Nesterov’s dual extrapolation method from which our new method originates.

2.1 Variational inequality problem

The regularity conditions that we consider for an operator $F : \mathbb{R}^d \mapsto \mathbb{R}^d$ are as follows.

Definition 2.1 An operator $F$ is $k^{\text{th}}$-order smooth if $\|\nabla^{(k)} F(x) - \nabla^{(k)} F(x')\| \leq L \|x - x'\|$ for all $x, x'$. 

Definition 2.2 An operator $F$ is $\mu$-strongly-monotone if $\langle F(x) - F(x'), x - x' \rangle \geq \mu \|x - x'\|^2$ for all $x, x'$. If $\mu = 0$, we recover the definition of monotonicity for a continuous operator.

With these definitions in mind, we formally state the assumptions that impose in addition to A1 and A2 in order to define assumption smooth variational inequality (VI) problems.

Assumption 2.3 The following statements hold true:

1. $F : \mathbb{R}^d \mapsto \mathbb{R}^d$ is $(p-1)^{\text{th}}$-order smooth with a Lipschitz constant $L > 0$.

2. $\mathcal{X}$ is closed, convex and bounded with a diameter $D > 0$.

The convergence of derivative-based optimization methods to a weak solution $x^* \in \mathcal{X}$ depends on the property of $F$ near this point; thus, some form of smoothness condition is needed. As for the boundedness condition for $\mathcal{X}$, it is standard in the VI literature [Facchinei and Pang, 2007]. This condition not only guarantees the validity of the most natural optimality criterion in the monotone setting—the gap function [Nemirovski, 2004, Nesterov, 2007]—but additionally it is satisfied in a wide range of real-world applications [Facchinei and Pang, 2007]. On the other hand, there is a line of works focusing on relaxing the boundedness condition via appeal to other notions of approximate solutions [Monteiro and Svaiter, 2010, 2011, 2012, Chen et al., 2017]. For simplicity, we retain the boundedness condition and leave the analysis for the cases with unbounded constraint sets to future work.
Monotone setting. For some of our results we focus on operators $F$ that are monotone in addition to Assumption 2.3. Under monotonicity, it is well known that any $\epsilon$-approximate strong solution is an $\epsilon$-approximate weak solution but the reverse does not hold in general. Accordingly, we define the approximate versions of the Minty VI in Eq. (2) and the Stampacchia VI in Eq. (3) as follows:

(Approximate weak solution) \[ \langle F(x), \hat{x} - x \rangle \leq \epsilon, \quad \text{for all } x \in \mathcal{X}, \]

(Approximate strong solution) \[ \langle F(\hat{x}), \hat{x} - x \rangle \leq \epsilon, \quad \text{for all } x \in \mathcal{X}, \]

These definitions motivate the use of a gap function, \( \text{GAP}(\cdot) : \mathcal{X} \mapsto \mathbb{R}_+ \), defined by

\[ \text{GAP}(\hat{x}) = \sup_{x \in \mathcal{X}} \langle F(x), \hat{x} - x \rangle, \quad \text{(8)} \]

with which we measure the optimality of a point $\hat{x} \in \mathcal{X}$ output by various iterative solution methods; see, e.g., Tseng [2000], Nemirovski [2004], Nesterov [2007], Mokhtari et al. [2020]. Note that the boundedness of $\mathcal{X}$ and the existence of a strong solution guarantees that the gap function is well defined. Formally, we have

**Definition 2.4** We say a point $\hat{x} \in \mathcal{X}$ is an $\epsilon$-weak solution to the monotone VI corresponding to $F : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\mathcal{X} \subseteq \mathbb{R}^d$ if we have $\text{GAP}(\hat{x}) \leq \epsilon$. If $\epsilon = 0$, then $\hat{x} \in \mathcal{X}$ is a weak solution.

In the strongly monotone setting, we let $\mu > 0$ denote the modulus of strong monotonicity for $F$. Under Assumption 2.3, we define $\kappa := L/\mu$ as the generalized condition number of $F$. It is worth mentioning that the condition number quantifies the difficulty of solving the optimization problem [Nesterov, 2018] and appears in the iteration complexity bound of derivative-based methods for optimizing a smooth and strongly convex function. Accordingly, the VI corresponding to $F$ and $\mathcal{X}$ is more and more computationally challenging as $\kappa > 0$ increases.

Structured non-monotone setting. We also study the case in which $F$ is non-monotone but satisfies the (strong) Minty condition. Imposing the (strong) Minty condition is important since the smoothness of $F$ is not sufficient to guarantee that the problem is computationally tractable. This has been shown by Daskalakis et al. [2021] who establish that even deciding whether an approximate min-max solution exists is NP-hard in smooth and nonconvex-nonconcave min-max optimization (which is a special instance of non-monotone VIs).

Recent work [Solodov and Svaiter, 1999, Dang and Lan, 2015, Iusem et al., 2017, Kannan and Shanbhag, 2019, Song et al., 2020, Liu et al., 2021, Diakonikolas et al., 2021] has shown that the non-monotone VI problem is computational tractable under the Minty condition. We thus make the following formal definition.

**Definition 2.5** We say the VI corresponding to $F : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\mathcal{X} \subseteq \mathbb{R}^d$ satisfies the Minty condition if there exists $x^* \in \mathcal{X}$ such that $\langle F(x), x - x^* \rangle \geq 0$ for all $x \in \mathcal{X}$.

We make some comments on the Minty condition. First, we note that it assumes the existence of at least one weak solution. Second, we see from Harker and Pang [1990, Theorem 3.1] that at least one strong solution to the VI since $F$ is continuous and $\mathcal{X}$ is closed and bounded. However, the set of weak solutions is only a subset of the set of strong solutions if $F$ is not necessarily monotone and the weak solution may not exist. From this perspective, the Minty condition gives a favorable structure. Finally, the Minty condition is a mild condition which makes the computation of an $\epsilon$-strong solution of
non-monotone VIs tractable; indeed, it is weaker than generalized monotone assumptions (e.g., pseudo-monotonicity or quasi-monotonicity) [Dang and Lan, 2015, Iusem et al., 2017, Kannan and Shanbhag, 2019].

Accordingly, we define the residue function \( \text{RES}() : \mathcal{X} \mapsto \mathbb{R}_+ \) given by

\[
\text{RES}(\hat{x}) = \sup_{x \in \mathcal{X}} \langle F(\hat{x}), \hat{x} - x \rangle, \tag{9}
\]

which measures the optimality of a point \( \hat{x} \in \mathcal{X} \) achieved by iterative solution methods; see, e.g., Dang and Lan [2015], Iusem et al. [2017], Kannan and Shanbhag [2019], Song et al. [2020]. Note that the boundedness of \( \mathcal{X} \) and the Minty condition guarantee that the residue function is well defined. Formally, we have

**Definition 2.6** We say a point \( \hat{x} \in \mathcal{X} \) is an \( \epsilon \)-strong solution to the non-monotone VI corresponding to \( F : \mathbb{R}^d \mapsto \mathbb{R}^d \) and \( \mathcal{X} \subseteq \mathbb{R}^d \) if we have \( \text{RES}(\hat{x}) \leq \epsilon \). If \( \epsilon = 0 \), then \( \hat{x} \in \mathcal{X} \) is a strong solution.

Proceeding a step further, we define the strong Minty condition and define \( \kappa_{Minty} := L/\mu_{Minty} \) to be the Minty condition number of \( F \) if the VI satisfies the \( \mu_{Minty} \)-strong Minty condition.

**Definition 2.7** We say the VI corresponding to \( F : \mathbb{R}^d \mapsto \mathbb{R}^d \) and \( \mathcal{X} \subseteq \mathbb{R}^d \) satisfies the \( \mu_{Minty} \)-strong Minty condition if there exists \( x^* \in \mathcal{X} \) such that \( \langle F(x), x - x^* \rangle \geq \mu_{Minty} \| x - x^* \|^2 \) for all \( x \in \mathcal{X} \).

There are many application problems that can be formulated as non-monotone VIs satisfying the (strong) Minty condition, including product pricing [Choi et al., 1990, Gallego and Hu, 2014, Ewerhart, 2014] and competitive exchange economies [Brighi and John, 2002]. A similar assumption (the Minty condition restricted to nonconvex optimization) has been adopted for analyzing the convergence of stochastic gradient descent for deep learning [Li and Yuan, 2017] and it has found real-world applications [Kleinberg et al., 2018].

### 2.2 Nesterov’s dual extrapolation method

Nesterov’s dual extrapolation method [Nesterov, 2007] has been shown to be optimal among all the first-order methods for computing the weak solution of the VI when \( F \) is zeroth-order smooth (see Definition 2.1) and monotone (see Definition 2.2) [Ouyang and Xu, 2021]. We recall the basic formulation in the setting of a VI defined via an operator \( F : \mathbb{R}^d \mapsto \mathbb{R}^d \) and a closed, convex and bounded set \( \mathcal{X} \subseteq \mathbb{R}^d \). Starting with the initial points \( x_0 \in \mathcal{X} \) and \( s_0 = 0 \in \mathbb{R}^d \), the \( k \)th iteration of the scheme is given by

\[
\begin{align*}
\text{Find } v_{k+1} & \in \mathcal{X} \text{ s.t. } v_{k+1} = \arg\max_{v \in \mathcal{X}} \langle s_k, v - x_0 \rangle - \frac{\beta}{2} \| v - x_0 \|^2, \\
\text{Find } x_{k+1} & \in \mathcal{X} \text{ s.t. } \langle F(v_{k+1}) + \beta(x_{k+1} - v_{k+1}), x - x_{k+1} \rangle \geq 0 \text{ for all } x \in \mathcal{X}, \\
s_{k+1} & = s_k - \lambda F(x_{k+1}).
\end{align*}
\]

Intuitively, the dual extrapolation method can be interpreted as an instance of the extragradient method in the dual space (we refer to \( s \in \mathbb{R}^d \) to as the dual variable). In particular, the rule which transforms a point \( s_k \) into the next point \( s_{k+1} \) at the \( k \)th iteration is called the dual extrapolation step. It has also been proven in Nesterov [2007, Theorem 2] that the dual extrapolation scheme, with \( \beta = L \) and \( \lambda = 1 \), generates a sequence \( \{x_k\}_{k \geq 0} \subseteq \mathcal{X} \) satisfying the condition that the average iterate \( \bar{x}_k = \frac{1}{k+1} \sum_{i=0}^{k} x_i \) is an \( \epsilon \)-weak solution (see Definition 2.4) after at most \( O(\epsilon^{-1}) \) iterations. Here \( L > 0 \) is the Lipschitz constant of \( F \); cf. Definition 2.1.
In this context, Nesterov [2006] also considered the setting where $F$ is monotone and first-order smooth with a Lipschitz constant $L > 0$ and proposed a second-order dual extrapolation method for computing the weak solution of the VI. Starting with the initial points $x_0 \in \mathcal{X}$ and $s_0 = 0 \in \mathbb{R}^d$, the $k^{th}$ iteration of the scheme is given by

Find $v_{k+1} \in \mathcal{X}$ s.t. $v_{k+1} = \arg\max_{v \in \mathcal{X}} \langle s_k, v - x_0 \rangle - \frac{\beta}{3} \|v - x_0\|^3$,

Find $x_{k+1} \in \mathcal{X}$ s.t. $F_{v_{k+1}}^1(x_{k+1}) + \frac{M}{2} \|x_{k+1} - v_{k+1}\| (x_{k+1} - v_{k+1}), x - x_{k+1}) \geq 0$

for all $x \in \mathcal{X}$,

$s_{k+1} = s_k - \lambda F(x_{k+1}),$

where $F_{v}^1(\cdot) : \mathbb{R}^d \mapsto \mathbb{R}^d$ is defined as first-order Taylor expansion of $F$ at a point $v \in \mathcal{X}$:

$$F_{v}^1(x) = F(v) + \nabla F(v)(x - v).$$

This scheme is based on the dual extrapolation step but with a different regularization and with a first-order Taylor expansion of $F$. This makes sense since we have zeroth-order and first-order derivative information available and hope to use both of them to accelerate convergence. Similar ideas have been adopted for convex optimization [Nesterov and Polyak, 2006], leading to a simple second-order method with faster global rate of convergence [Nesterov, 2008] than optimal first-order method [Nesterov, 1983]. Unfortunately, the above second-order dual extrapolation method with $\beta = 6L$, $M = 5L$ and $\lambda = 1$ is only guaranteed to achieve an iteration complexity of $O(\epsilon^{-1})$ [Nesterov, 2006, Theorem 4].

3 A Regularized High-Order Model and Algorithm

In this section, we present our algorithmic derivation of Perseus and provide a theoretical convergence guarantee for the method. We provide intuition into why Perseus and its restarted version yield fast rates of convergence for variational inequality (VI) problems. We present a full theoretical treatment of the global and local convergence of Perseus and its restarted version for both the monotone setting and the non-monotone setting under the Minty condition.

3.1 Algorithmic scheme

We summarize our general $p^{th}$-order method, which we denote formally as Perseus($p$, $x_0$, $L$, $T$, $\text{opt}$), in Algorithm 1 where $p \in \{1, 2, \ldots\}$ is the order, $x_0 \in \mathcal{X}$ is an initial point, $L > 0$ is a Lipschitz constant for the $(p - 1)^{th}$-order smoothness, $T$ is an iteration number and $\text{opt} \in \{0, 1, 2\}$ is the type of output. Our new method can be interpreted as a natural generalization of the dual extrapolation method [Nesterov, 2007] from first-order to general $p^{th}$-order.

The major novelty of our method lies in an adaptive strategy used for updating $\lambda_{k+1}$ (see Step 4). This modification is simple yet important. It serves as the key for obtaining a global rate of $O(\epsilon^{-2/(p+1)})$ (monotone) and $O(\epsilon^{-2/p})$ (non-monotone with the Minty condition) under Assumption 2.3. Focusing on the case of $p = 2$ and the monotone setting, our results improve existing global convergence rates of $O(\epsilon^{-1})$ [Nesterov, 2006] and $O(\epsilon^{-2/3} \log(1/\epsilon))$ [Monteiro and Svaiter, 2012] under Assumption 2.3 while not sacrificing the simplicity of the scheme. In addition, our methods allow the subproblem to be solved inexactly and we give options for choosing the type of outputs.
Algorithm 1 Perseus($p$, $x_0$, $L$, $T$, opt)

| Input: order $p$, initial point $x_0 \in \mathcal{X}$, parameter $L$, iteration number $T$ and $\text{opt} \in \{0, 1, 2\}$. |
|---------------------------------------------|
| Initialization: set $s_0 = 0_\mathbb{R}^d$. |
| for $k = 0, 1, 2, \ldots, T$ do |
|   STEP 1: If $x_k \in \mathcal{X}$ is a solution of the VI, then stop. |
|   STEP 2: Compute $v_{k+1} = \underset{v \in \mathcal{X}}{\text{argmax}} \{\langle s_k, v - x_0 \rangle - \frac{1}{2} \|v - x_0\|^2\}$. |
|   STEP 3: Compute $x_{k+1} \in \mathcal{X}$ such that Eq. (12) holds true. |
|   STEP 4: Compute $\lambda_{k+1} > 0$ such that $\frac{1}{20p-8} \leq \frac{\lambda_{k+1}L\|x_{k+1} - v_{k+1}\|^{p-1}}{p!} \leq \frac{1}{10p+2}$. |
|   STEP 5: Compute $s_{k+1} = s_k - \lambda_{k+1} F(x_{k+1})$. |
| end for |
| Output: $\hat{x} = \begin{cases} \sum_{k=1}^{p} x_{kT}, & \text{if } \text{opt} = 0, \\ x_{kT} & \text{for } kT = \arg\min_{1 \leq k \leq T} \|x_k - v_k\|, \quad \text{else if } \text{opt} = 1, \\ x_T, & \text{otherwise.} \end{cases}$ |

### Comments on adaptive strategies.

It is worth mentioning that the adaptive strategies for updating $\lambda_{k+1}$ are inspired by revisiting the reason why existing $p^{\text{th}}$-order methods all require a nontrivial binary search procedure. In particular, these existing methods compute a pair, $\lambda_{k+1} > 0, x_{k+1} \in \mathcal{X}$, that (approximately) solve the $x$-subproblem that contains $\lambda$ and the $\lambda$-subproblem that contains $x$. In particular, the conditions can be written as follows,

$$\alpha_- \leq \frac{\lambda_{k+1}L\|x_{k+1} - v_{k+1}\|^{p-1}}{p!} \leq \alpha_+ \quad \text{for proper choices of } \alpha_- \text{ and } \alpha_+, \quad \langle F_{v_{k+1}}(x_{k+1}) + \lambda_{k+1}\|x_{k+1} - v_{k+1}\|^{p-1}(x_{k+1} - v_{k+1}), x - x_{k+1} \rangle \geq 0 \text{ for all } x \in \mathcal{X},$$

where

$$F_v(x) = F(v) + \langle \nabla F(v), x - v \rangle + \ldots + \frac{1}{(p-1)!} \nabla^{(p-1)} F(v)[x - v]^{p-1} + \frac{L}{(p-1)!} \|x - v\|^{p-1}(x - v), \quad (10)$$

A key observation is that we can have some $x$-subproblems that do not need to refer to $\lambda$; e.g., the one used in Algorithm 1. Indeed, we compute $x_{k+1} \in \mathcal{X}$ that approximately satisfies the following condition:

$$\langle F_{v_{k+1}}(x_{k+1}), x - x_{k+1} \rangle \geq 0 \text{ for all } x \in \mathcal{X}.$$ 

It suffices to return $x_{k+1} \in \mathcal{X}$ with a good quality that further gives us $\lambda_{k+1} > 0$ using a simple update rule. Intuitively, such an adaptive strategy makes sense since $\lambda_{k+1}$ serves as the stepsize in the dual space and we need to be aggressive when the iterate $x_{k+1}$ approaches the set of optimal solutions to the VI. Meanwhile, the quantity $\|x_{k+1} - v_{k+1}\|$ can be used to measure the distance between $x_{k+1}$ and an optimal solution, and the power $p \in \{1, 2, \ldots\}$ quantifies the relationship between the closeness and the exploitation of high-order derivative information. In summary, $\lambda_{k+1}$ will be larger for a better iterate $x_{k+1} \in \mathcal{X}$ and such a choice leads to a faster global rate of convergence.

### Comments on inexact subproblem solving.

We remark that Step 3 resorts to the computation of an approximate strong solution to the VI in which we define the operator $F_{v_{k+1}}(x)$ as the sum of a high-order polynomial and a regularization term. Indeed, it is a modification of Eq. (10) as follows:

$$F_{v_{k+1}}(x) = F(v_{k+1}) + \langle \nabla F(v_{k+1}), x - v_{k+1} \rangle + \ldots + \frac{1}{(p-1)!} \nabla^{(p-1)} F(v_{k+1})[x - v_{k+1}]^{p-1} + \frac{5L}{(p-1)!} \|x - v_{k+1}\|^{p-1}(x - v_{k+1}),$$
where we write the VI of interest in the subproblem as follows:

$$\text{Find } x_{k+1} \in \mathcal{X} \text{ such that } \langle F_{v_{k+1}}(x_{k+1}), x - x_{k+1} \rangle \geq 0 \text{ for all } x \in \mathcal{X}. \quad (11)$$

Since $F_{v_{k+1}}$ is continuous and $\mathcal{X}$ is closed, convex and bounded, Harker and Pang [1990, Theorem 3.1] guarantees that a strong solution to the VI in Eq. (11) exists and the problem of finding an approximate strong solution is well defined.

In the monotone setting, we can show that the VI in Eq. (11) is monotone and thus computationally tractable [Dang and Lan, 2015], with the following approximation condition:

$$\sup_{x \in \mathcal{X}} \langle F_{v_{k+1}}(x_{k+1}), x_{k+1} - x \rangle \leq \frac{L}{p!} \| x_{k+1} - v_{k+1} \|^{p+1}. \quad (12)$$

Indeed, if $p = 1$, we have that $\nabla F_{v_{k+1}}(x) = L \cdot I_{d \times d}$ that is positive semidefinite for all $x \in \mathbb{R}^d$ where $I_{d \times d} \in \mathbb{R}^{d \times d}$ is an identity matrix. Otherwise, we have

$$\nabla F_{v_{k+1}}(x) = \nabla F(v_{k+1}) + \ldots + \frac{1}{(p-2)!} \nabla^{(p-1)} F(v_{k+1}) [x - v_{k+1}]^{p-2} + \frac{5L}{(p-1)!} \| x - v_{k+1} \|^{p-1} I_{d \times d} + \frac{5L}{(p-2)!} \| x - v_{k+1} \|^{p-2} (x - v_{k+1}) (x - v_{k+1})^\top.$$  

Under Assumption 2.3, it is clear that

$$\| \nabla F(x) - (\nabla F(v_{k+1}) + \ldots + \frac{1}{(p-2)!} \nabla^{(p-1)} F(v_{k+1}) [x - v_{k+1}]^{p-2} ) \| \leq \frac{L}{(p-1)!} \| x - v_{k+1} \|^{p-1}.$$  

This implies that

$$\nabla F_{v_{k+1}}(x) \geq \nabla F(x) + \frac{4L}{(p-1)!} \| x - v_{k+1} \|^{p-1} I_{d \times d} + \frac{L}{(p-2)!} \| x - v_{k+1} \|^{p-2} (x - v_{k+1}) (x - v_{k+1})^\top.$$  

Since $F$ is monotone, we have $\nabla F(x)$ is positive semidefinite for all $x \in \mathbb{R}^d$. This yields that $F_{v_{k+1}}$ is monotone and implies the desired result. As a consequence of our analysis, the VI in Eq. (11) admits relative strong monotonicity with respect to some reference function and we believe it can be solved by a modification of Bregman first-order methods [Lu et al., 2018, Grapiglia and Nesterov, 2021] with a linear rate of convergence.

In the non-monotone setting, the VI in Eq. (11) is not necessarily monotone and the computation of an approximate strong solution is intractable in general [Daskalakis et al., 2021]. However, we note that $F_{v_{k+1}}$ is the sum of a high-order polynomial and a regularization term; this special structure might lend itself to efficient numerical methods.

For example, we consider the setting where $F = \nabla f$ for a nonconvex and smooth function $f : \mathbb{R}^d \mapsto \mathbb{R}$ with a Lipschitz second-order derivative, $\mathcal{X} = \mathbb{R}^d$ and $p = 2$. Solving the VI in Eq. (11) is equivalent to solving the subproblem of the cubic regularization of Newton’s method [Nesterov and Polyak, 2006]:

$$x_{k+1} = \arg\min_{x \in \mathbb{R}^d} \langle \nabla f(v_{k+1}), x - v_{k+1} \rangle + \frac{1}{2} \langle x - v_{k+1}, \nabla^2 f(v_{k+1}) (x - v_{k+1}) \rangle + \frac{L}{3} \| x - v_{k+1} \|^3.$$  

This optimization problem is nonconvex but can be solved approximately in an efficient manner. Examples of cubic solvers include generalized conjugate gradient methods with Lanczos process [Gould et al., 1999, 2010] and a simple variant of gradient descent [Carmon and Duchi, 2019]. Can we generalize these solvers to handle the VI in Eq. (11), similar to what has been accomplished in the optimization setting? This topic is beyond the scope of this paper and we leave it to future work.
Restart version of Perseus. We summarize the restarted version of our $p$th-order method in Algorithm 2. This method, which we refer to as Perseus-restart($p, x_0, L, σ, D, T, \text{opt}$), combines Algorithm 1 with a restart scheme; see e.g., Nemirovskii and Nesterov [1985], Nesterov [2013], O’donoghue and Candes [2015], Nesterov [2018].

Intuitively speaking, the restart scheme stops an algorithm when a criterion is satisfied and then restarts the algorithm with a new input. It has been recognized as an important tool for designing linearly convergent algorithms when the objective function is strongly or uniformly convex [Nemirovskii and Nesterov, 1985, Nesterov, 2013, Ghadimi and Lan, 2013] or has other regularizing structure [Freund and Lu, 2018, Necoara et al., 2019, Renegar and Grimmer, 2022]. Note that the strong monotonicity and the strong Minty condition are generalizations of the aforementioned regularity conditions for the VIs. Thus, it is natural to consider a restarted version of our method, hoping to achieve linear convergence. In particular, at each iteration of Algorithm 2, we use $x_{k+1} = \text{Perseus}(p, x_k, L, t, \text{opt})$ as a subroutine. In other words, we simply restart Perseus every $t \geq 1$ iterations and take advantage of average iterates or best iterates to generate $x_{k+1}$ from $x_k$. In addition, it is worth mentioning that the choice of $t$ can be specialized to different settings and/or different type of convergence guarantees. Indeed, we set $\text{opt} = 0$ for the strong monotone setting, $\text{opt} = 1$ for the non-monotone setting satisfying the strong Minty condition and $\text{opt} = 2$ for a local convergence guarantee.

Moreover, several papers focus on the investigation of the adaptive restart schemes that can speed up convergence of first-order methods [Giselsson and Boyd, 2014, O’donoghue and Candes, 2015] and provide the theoretical guarantee for a general setting where the objective function is smooth and has Hölderian growth [Roulet and d’Aspremont, 2017, Fercoq and Qu, 2019]. However, most of these schemes are designed for a narrow family of first-order methods, and relies on learning appropriately-accurate approximations of problem parameters. To alleviate this issue, Renegar and Grimmer [2022] propose a simple restart scheme which makes no attempt to learn parameter values and only require the information that would be available in practice. Note that Algorithm 2 requires to choose $t \geq 1$ based on problem parameters and is thus not favorable in practice. Can we modify the existing adaptive restart schemes to speed up Perseus, similar to what was accomplished for first-order methods in optimization setting? We leave this topic to future works.
3.2 Main results

We provide our main results on the convergence rate for Algorithm 1 and 2 in terms of the number of calling the subproblem solvers. Note that Assumption 2.3 will be made throughout and we impose the (strong) Minty condition (see Definition 2.5 and 2.7) for the non-monotone setting.

Two of the following theorems give us the global convergence rate of Algorithm 1 and 2 for the smooth and (strongly) monotone VIs in terms of the number of calling the subproblem solvers.

**Theorem 3.1** Suppose that Assumption 2.3 holds true and $F : \mathbb{R}^d \mapsto \mathbb{R}^d$ is monotone and let $\epsilon \in (0, 1)$. There exists some $T > 0$ such that the output $\hat{x} = \text{Perseus}(p, x_0, L, T, 0)$ satisfies that $\text{gap}(\hat{x}) \leq \epsilon$ (i.e., $\hat{x}$ is an $\epsilon$-weak solution) and the total number of calling the subproblem solvers is bounded by

$$O \left( \left( \frac{LD^{p+1}}{\epsilon} \right)^{\frac{2}{p+1}} \right),$$

where $p \in \{1, 2, \ldots\}$ is an order, $L > 0$ is a Lipschitz constant for $(p-1)^{th}$-order smoothness of $F$ and $D > 0$ is the diameter of $\mathcal{X}$.

**Theorem 3.2** Suppose that Assumption 2.3 holds true and $F : \mathbb{R}^d \mapsto \mathbb{R}^d$ is $\mu$-strongly monotone and let $\epsilon \in (0, 1)$. There exists some $T > 0$ such that the output $\hat{x} = \text{Perseus-restart}(p, x_0, L, \mu, D, T, 0)$ satisfies that $\|\hat{x} - x^*\| \leq \epsilon$ and the total number of calling the subproblem solvers is bounded by

$$O \left( \left( \frac{\kappa D^{p-1}}{\epsilon} \right)^{\frac{2}{p+1}} \log_2 \left( \frac{D}{\epsilon} \right) \right),$$

where $p \in \{1, 2, \ldots\}$ is an order, $\kappa = L/\mu > 0$ is the generalized condition number of $F$, $D > 0$ is the diameter of $\mathcal{X}$ and $x^* \in \mathcal{X}$ is the unique weak solution of the VI.

**Remark 3.3** For the first-order methods (the case of $p = 1$), the convergence guarantee in Theorem 3.1 recovers a global rate of $O(L/\epsilon)$ achieved by the dual extrapolation method [Nesterov, 2007, Theorem 2]. The same rate was derived for other methods [Nemirovski, 2004, Monteiro and Svaiter, 2010, Mokhtari et al., 2020, Kotsalis et al., 2020] and was known to match the lower bound [Ouyang and Xu, 2021]. For the second-order and high-order methods (the case of $p \geq 2$), our results improve the state-of-the-art results [Monteiro and Svaiter, 2012, Bullins and Lai, 2020, Lin and Jordan, 2021b, Jiang and Mokhtari, 2022] by shaving off the log factors thanks to the simplicity of our methods.

**Remark 3.4** For the first-order methods, the convergence guarantee in Theorem 3.2 recovers a global linear convergence rate achieved by the dual extrapolation method and other methods and matches the lower bound [Zhang et al., 2021]. For the second-order and high-order methods, our results improve the results in Jiang and Mokhtari [2022] by shaving off the log factors and we believe that these bounds cannot be further improved despite the lack of lower bounds.

We consider the smooth and non-monotone VIs satisfying the (strong) Minty condition and present the global convergence rate of Algorithm 1 and 2 in terms of the number of calling the subproblem solvers.
**Theorem 3.5** Suppose that Assumption 2.3 and the Minty condition hold true and let \( \epsilon \in (0, 1) \). There exists some \( T > 0 \) such that the output \( \hat{x} = \text{Perseus}(p, x_0, L, T, 1) \) satisfies that \( \text{RES} (\hat{x}) \leq \epsilon \) (i.e., \( \hat{x}_T \) is an \( \epsilon \)-strong solution) and the total number of calling the subproblem solvers is bounded by

\[
O \left( \left( \frac{LD^{p+1}}{\epsilon} \right)^{\frac{2}{p}} \right),
\]

where \( p \in \{1, 2, \ldots \} \) is an order, \( L > 0 \) is the Lipschitz constant for \((p - 1)^{th}\)-order smoothness of \( F \) and \( D > 0 \) is the diameter of \( \mathcal{X} \).

**Theorem 3.6** Suppose that Assumption 2.3 and the \( \mu_{\text{Minty}} \)-strong Minty condition hold true and let \( \epsilon \in (0, 1) \). There exists some \( T > 0 \) such that the output \( \hat{x} = \text{Perseus-restart}(p, x_0, L, \mu_{\text{Minty}}, D, T, 1) \) satisfies that \( \|\hat{x} - x^*\| \leq \epsilon \) and the total number of calling the subproblem solvers is bounded by

\[
O \left( \max \{ (\kappa_{\text{Minty}}D^{p-1})^{\frac{2}{p}}, (\kappa_{\text{Minty}}D^{p-1})^{\frac{2}{p+1}} \} \log_2 \left( \frac{D}{\epsilon} \right) \right),
\]

where \( p \in \{1, 2, \ldots \} \) is an order, \( \kappa_{\text{Minty}} = L/\mu_{\text{Minty}} > 0 \) is the Minty condition number of \( F \), \( D > 0 \) is the diameter of \( \mathcal{X} \), and \( x^* \in \mathcal{X} \) is the weak solution of the VI.

**Remark 3.7** The convergence guarantee in Theorem 3.5 and 3.6 has been derived for other first-order methods [Dang and Lan, 2015, Song et al., 2020] for the case of \( p = 1 \) and are new for the case of \( p \geq 2 \) to our knowledge. In other words, we are the first to present the simple second-order and high-order methods for the smooth and non-monotone VIs satisfying the (strong) Minty condition with solid global convergence rate guarantee in the literature.

Finally, we proceed to the local convergence property of our methods for both the strongly monotone VIs and the non-monotone VIs satisfying the strong Minty condition.

**Theorem 3.8** Suppose that Assumption 2.3 holds true and \( F : \mathbb{R}^d \mapsto \mathbb{R}^d \) is \( \mu \)-strongly monotone and let \( \{x_k\}_{k \geq 0} \) be generated by \( \text{Perseus-restart}(p, x_0, L, \mu, D, T, 2) \). Then, the following statement holds true,

\[
\|x_{k+1} - x^*\| \leq \sqrt{\frac{2^p(5p-2)\kappa}{p!}} \|x_k - x^*\|^{\frac{p+1}{p}}, \quad \text{for all } k = 0, 1, 2, \ldots, T,
\]

where \( \kappa = L/\mu > 0 \) is the generalized condition number of the VI, \( D > 0 \) is the diameter of \( \mathcal{X} \) and \( x^* \) is the unique weak solution of the VI. As a consequence, if \( p \geq 2 \) and the following condition holds true,

\[
\|x_0 - x^*\| \leq \frac{1}{2} \left( \frac{p!}{2^p(5p-2)\kappa} \right)^{\frac{1}{p-1}},
\]

the iterates \( \{x_k\}_{k \geq 0} \) will converge to \( x^* \in \mathcal{X} \) in at least a superlinear rate.

**Theorem 3.9** Suppose that Assumption 2.3 and the \( \mu_{\text{Minty}} \)-strong Minty condition hold true and let \( \{x_k\}_{k \geq 0} \) be generated by \( \text{Perseus-restart}(p, x_0, L, \mu_{\text{Minty}}, D, T, 2) \). Then, the following statement holds true,

\[
\|x_{k+1} - x^*\| \leq \left( \frac{2^p(5p+1)\kappa_{\text{Minty}}}{p!} \right) \|x_k - x^*\|^p + \sqrt{\frac{2^p(5p+1)\kappa_{\text{Minty}}}{p!}} \|x_k - x^*\|^{\frac{p+1}{2}}.
\]
where $\kappa_{\text{Minty}} = L/\mu_{\text{Minty}} > 0$ is the Minty condition number of the VI, $D > 0$ is the diameter of $X$, and $x^*$ is the weak solution of the VI. As a consequence, if $p \geq 2$ and the following condition holds true,

$$\|x_0 - x^*\| \leq \frac{1}{2} \left( \frac{p!}{2^{p+2}(5p+1)\kappa_{\text{Minty}}} \right)^{\frac{1}{p-1}},$$

the iterates $\{x_k\}_{k \geq 0}$ will converge to $x^* \in X$ in at least a superlinear rate.

**Remark 3.10** The local convergence results in Theorem 3.8 and 3.9 are derived for the second-order and high-order methods (i.e., the case of $p \geq 2$) and is posted as their advantage over first-order method if we hope to pursue high-accurate solutions. In this context, Jiang and Mokhtari [2022] provided the same local convergence guarantee for the generalized optimistic gradient methods as our results in Theorem 3.8 but without counting the complexity bound of binary search procedure. The results in Theorem 3.9 seems new and shed light to the potential of our methods for solving smooth and non-monotone VIs.

### 4 Convergence Analysis

We present the convergence analysis for our $p^{th}$-order method (Algorithm 1) and its restarted version (Algorithm 2). In particular, we provide global and local convergence guarantee for the monotone setting (Theorem 3.1 and 3.2) and the non-monotone setting under the Minty condition (Theorems 3.5 and 3.6).

#### 4.1 Technical lemmas

We define a Lyapunov function for the iterates $\{x_k\}_{k \geq 0}$ generated by Algorithm 1 as follows:

$$E_k = \max_{v \in X} \langle s_k, v - x_0 \rangle - \frac{1}{2} \|v - x_0\|^2,$$  \hspace{1cm} (13)

This function will be used to prove technical results that pertain to the dynamics of Algorithm 1.

**Lemma 4.1** Suppose that Assumption 2.3 holds true. For every integer $T \geq 1$, we have

$$\sum_{k=1}^{T} \lambda_k \langle F(x_k), x_k - x \rangle \leq E_0 - E_T + \langle s_T, x - x_0 \rangle - \frac{1}{10} \left( \sum_{k=1}^{T} \|x_k - v_k\|^2 \right), \quad \text{for all } x \in X.$$  \hspace{1cm} (14)

**Proof.** By combining Eq. (13) and the definition of $v_{k+1}$, we have

$$E_k = \langle s_k, v_{k+1} - x_0 \rangle - \frac{1}{2} \|v_{k+1} - x_0\|^2.$$

Then, we have

$$E_{k+1} - E_k = \langle s_{k+1} - s_k, v_{k+1} - x_0 \rangle + \langle s_{k+1} - s_k, v_k - v_{k+1} \rangle - \frac{1}{2} \left( \|v_{k+2} - x_0\|^2 - \|v_{k+1} - x_0\|^2 \right).$$

By using the update formula for $v_{k+1}$ again, we have

$$\langle x - v_{k+1}, s_k - v_{k+1} + x_0 \rangle \leq 0, \quad \text{for all } x \in X.$$
Letting \( x = v_{k+2} \) in this inequality and using \( \langle a, b \rangle = \frac{1}{2} (\|a + b\|^2 - \|a\|^2 - \|b\|^2) \), we have

\[
\langle s_k, v_{k+2} - v_{k+1} \rangle \leq \langle v_{k+1} - x_0, v_{k+2} - v_{k+1} \rangle
= \frac{1}{2} (\|v_{k+2} - x_0\|^2 - \|v_{k+1} - x_0\|^2 - \|v_{k+2} - v_{k+1}\|^2).
\]

Plugging Eq. (15) into Eq. (14) and recalling the update formula of \( s_{k+1} \), we obtain:

\[
\begin{align*}
\mathcal{E}_{k+1} - \mathcal{E}_k \\
\text{Eq. (15)} \\
\leq \langle s_{k+1} - s_k, v_{k+1} - x_0 \rangle + \langle s_{k+1} - s_k, v_{k+2} - v_{k+1} \rangle - \frac{1}{2} \|v_{k+2} - v_{k+1}\|^2 \\
= \langle s_{k+1} - s_k, v_{k+2} - x_0 \rangle - \frac{1}{2} \|v_{k+2} - v_{k+1}\|^2 \\
\leq \lambda_{k+1} \langle F(x_{k+1}), x_0 - v_{k+2} \rangle - \frac{1}{2} \|v_{k+2} - v_{k+1}\|^2 \\
= \lambda_{k+1} \langle F(x_{k+1}), x_0 - x \rangle + \lambda_{k+1} \langle F(x_{k+1}), x - x_{k+1} \rangle \\
+ \lambda_{k+1} \langle F(x_{k+1}), x_{k+1} - v_{k+2} \rangle - \frac{1}{2} \|v_{k+2} - v_{k+1}\|^2,
\end{align*}
\]

for any \( x \in \mathcal{X} \). Summing up this inequality over \( k = 0, 1, \ldots, T - 1 \) and changing the counter \( k + 1 \) to \( k \) yields that

\[
\sum_{k=1}^{T} \lambda_k \langle F(x_k), x_k - x \rangle \leq \mathcal{E}_0 - \mathcal{E}_T + \sum_{k=1}^{T} \lambda_k \langle F(x_k), x_0 - x \rangle \\
+ \sum_{k=1}^{T} \lambda_k \langle F(x_k), x_k - v_{k+1} \rangle - \frac{1}{2} \|v_k - v_{k+1}\|^2.
\]

Using the update formula for \( s_{k+1} \) and letting \( s_0 = 0_d \in \mathbb{R}^d \), we have

\[
\mathbf{I} = \sum_{k=1}^{T} (\lambda_k F(x_k), x_0 - x) = \sum_{k=1}^{T} (s_{k-1} - s_k, x_0 - x) = \langle s_0 - s_T, x_0 - x \rangle = \langle s_T, x - x_0 \rangle.
\]

Since \( x_{k+1} \in \mathcal{X} \) satisfies Eq. (12), we have

\[
\langle F_{v_k}(x_k), x - x_k \rangle \geq - \frac{L}{p} \|x_k - v_k\|^{p+1}, \quad \text{for all } x \in \mathcal{X},
\]

where \( F_v(x) : \mathbb{R}^d \to \mathbb{R}^d \) is defined for any fixed \( v \in \mathcal{X} \) as follows:

\[
F_{v_k}(x) = F(v_k) + \langle \nabla F(v_k), x - v_k \rangle + \frac{1}{(p-1)!} \nabla^{(p-1)} F(v_k)(x - v_k)^{p-1} \\
+ \frac{5L}{(p-1)!} \|x - v_k\|^{p-1}(x - v_k).
\]

Under Assumption 2.3, we have

\[
\|F(x_k) - F_{v_k}(x_k) + \frac{5L}{(p-1)!} \|x_k - v_k\|^{p-1}(x_k - v_k)\| \leq \frac{L}{p} \|x_k - v_k\|^p.
\]

Letting \( x = v_{k+1} \) in Eq. (18), we have

\[
\langle F_{v_k}(x_k), x - v_{k+1} \rangle \leq \frac{L}{p} \|x_k - v_k\|^{p+1}.
\]
Inspired by Eq. (19) and Eq. (20), we perform a decomposition of \( \langle F(x_k), x_k - v_{k+1} \rangle \) as follows,

\[
\langle F(x_k), x_k - v_{k+1} \rangle = \langle F(x_k) - F_{x_k}(x_k) + \frac{5L}{(p-1)!} \| x_k - v_k \|^{p-1}(x_k - v_k), x_k - v_{k+1} \rangle + \langle F_{v_k}(x_k), x_k - v_{k+1} \rangle - \frac{5L}{(p-1)!} \| x_k - v_k \|^{p-1}(x_k - v_k, x_k - v_{k+1}) \]

\[
\leq \| F(x_k) - F_{v_k}(x_k) + \frac{5L}{(p-1)!} \| x_k - v_k \|^{p-1}(x_k - v_k) \| x_k - v_{k+1} \| + \langle F_{v_k}(x_k), x_k - v_{k+1} \rangle - \frac{5L}{(p-1)!} \| x_k - v_k \|^{p-1}(x_k - v_k, x_k - v_{k+1}) \]

Eq. (19) and Eq. (20)

\[
\leq \frac{5L}{p!} \| x_k - v_k \|^{p-1} \| x_k - v_{k+1} \| + \frac{5L}{p!} \| x_k - v_k \|^{p-1}(x_k - v_k, x_k - v_{k+1}) \]

\[
\leq \frac{2L}{p!} \| x_k - v_k \|^{p-1} + \frac{5L}{p!} \| x_k - v_k \|^{p-1} \| x_k - v_{k+1} \|
\]

\[- \frac{5L}{(p-1)!} \| x_k - v_k \|^{p-1}(x_k - v_k, x_k - v_{k+1}).\]

Note that we have

\[
\langle x_k - v_k, x_k - v_{k+1} \rangle = \| x_k - v_k \|^2 + \langle x_k - v_k, v_k - v_{k+1} \rangle \geq \| x_k - v_k \|^2 - \| x_k - v_k \| \| v_k - v_{k+1} \|.
\]

Putting these pieces together yields that

\[
\langle F(x_k), x_k - v_{k+1} \rangle \leq \frac{(5p+1)L}{p!} \| x_k - v_k \|^p \| v_k - v_{k+1} \| - \frac{(5p-2)L}{p!} \| x_k - v_k \|^{p+1}.
\]

Since \( \frac{1}{20p-8} \leq \frac{\lambda_k L \| x_k - v_k \|^{p-1}}{p!} \leq \frac{1}{10p+2} \) for all \( k \geq 1 \), we have

\[
\Pi \leq \sum_{k=1}^{T} \left( \frac{(5p+1)\lambda_k L}{p!} \| x_k - v_k \|^p \| v_k - v_{k+1} \| - \frac{1}{2} \| v_k - v_{k+1} \|^2 - \frac{(5p-2)\lambda_k L}{p!} \| x_k - v_k \|^{p+1} \right)
\]

\[
\leq \sum_{k=1}^{T} \left( \frac{1}{2} \| x_k - v_k \| \| v_k - v_{k+1} \| - \frac{1}{2} \| v_k - v_{k+1} \|^2 - \frac{1}{4} \| x_k - v_k \|^{2} \right)
\]

\[
\leq \sum_{k=1}^{T} \left( \max_{\eta \geq 0} \left\{ \frac{1}{2} \| x_k - v_k \| \eta - \frac{1}{2} \eta^2 \right\} - \frac{1}{4} \| x_k - v_k \|^{2} \right)
\]

\[
= -\frac{1}{8} \left( \sum_{k=1}^{T} \| x_k - v_k \|^{2} \right). \tag{21}
\]

Plugging Eq. (17) and Eq. (21) into Eq. (16) yields that

\[
\sum_{k=1}^{T} \lambda_k \langle F(x_k), x_k - x \rangle \leq \mathcal{E}_0 - \mathcal{E}_T + \langle s_T, x - x_0 \rangle - \frac{1}{8} \left( \sum_{k=1}^{T} \| x_k - v_k \|^2 \right).
\]

This completes the proof. \( \square \)
Lemma 4.2 Suppose that Assumption 2.3 and the Minty condition hold true and let \( x \in \mathcal{X} \). For every integer \( T \geq 1 \), we have
\[
\sum_{k=1}^{T} \lambda_k \langle F(x_k), x_k - x \rangle \leq \frac{1}{2} \| x - x_0 \|^2, \quad \sum_{k=1}^{T} \| x_k - v_k \|^2 \leq 4 \| x^* - x_0 \|^2,
\]
where \( x^* \in \mathcal{X} \) denotes the weak solution to the VI.

Proof. For any \( x \in \mathcal{X} \), we have
\[
\mathcal{E}_0 - \mathcal{E}_T + \langle s_T, x - x_0 \rangle = \mathcal{E}_0 - \left( \max_{v \in \mathcal{X}} \langle s_T, v - x_0 \rangle - \frac{1}{2} \| v - x_0 \|^2 \right) + \langle s_T, x - x_0 \rangle.
\]
Since \( s_0 = 0 \), we have \( \mathcal{E}_0 = 0 \) and
\[
\mathcal{E}_0 - \mathcal{E}_T + \langle s_T, x - x_0 \rangle \leq - \left( \langle s_T, x - x_0 \rangle - \frac{1}{2} \| x - x_0 \|^2 \right) + \langle s_T, x - x_0 \rangle = \frac{1}{2} \| x - x_0 \|^2.
\]
This together with Lemma 4.1 yields that
\[
\sum_{k=1}^{T} \lambda_k \langle F(x_k), x_k - x \rangle + \frac{1}{8} \left( \sum_{k=1}^{T} \| x_k - v_k \|^2 \right) \leq \frac{1}{2} \| x - x_0 \|^2, \quad \text{for all} \ x \in \mathcal{X},
\]
which implies the first inequality. Since the VI satisfies the Minty condition (see Definition 2.5), there exists \( x^* \in \mathcal{X} \) such that \( \langle F(x_k), x_k - x^* \rangle \geq 0 \) for all \( k \geq 1 \). Letting \( x = x^* \) in the above inequality yields the second inequality. \( \square \)

We provide a technical lemma establishing a lower bound for \( \sum_{k=1}^{T} \lambda_k \).

Lemma 4.3 Suppose that Assumption 2.3 and the Minty condition hold true. For every integer \( k \geq 1 \), we have
\[
\sum_{k=1}^{T} \lambda_k \geq \frac{\hat{p}!}{(20p-8)L} \left( \frac{1}{4 \| x^* - x_0 \|^2} \right)^{\frac{p-1}{2}} T^{\frac{k}{2}},
\]
where \( x^* \in \mathcal{X} \) denotes the weak solution to the VI.

Proof. Without loss of generality, we assume that \( x_0 \neq x^* \). For \( p = 1 \), we have \( \lambda_k = \frac{1}{12L} \) for all \( k \geq 1 \). For \( p \geq 2 \), we have
\[
\sum_{k=1}^{T} (\lambda_k)^{-\frac{2}{p-1}} (\frac{\hat{p}!}{(20p-8)L})^{\frac{2}{p-1}} \leq \sum_{k=1}^{T} (\lambda_k)^{-\frac{2}{p-1}} \| x_k - v_k \|^{p-1} \| x^* - x_0 \|^2 \leq \sum_{k=1}^{T} \| x_k - v_k \|^2 \leq 4 \| x^* - x_0 \|^2.
\]
By the Hölder inequality, we have
\[
\sum_{k=1}^{T} 1 = \sum_{k=1}^{T} \left( (\lambda_k)^{-\frac{2}{p-1}} \right)^{\frac{p-1}{p+1}} (\lambda_k)^{\frac{2}{p+1}} \leq \left( \sum_{k=1}^{T} (\lambda_k)^{-\frac{2}{p-1}} \right)^{\frac{p-1}{p+1}} \left( \sum_{k=1}^{T} \lambda_k \right)^{\frac{2}{p+1}}.
\]
Putting these pieces together yields that
\[ T \leq (4\|x^* - x_0\|_2^2)^{\frac{p-1}{p+1}} \left( \frac{(20p-8)L}{p!} \right)^{\frac{1}{p+1}} \left( \sum_{k=1}^{T} \lambda_k \right)^{\frac{2}{p+1}}, \]

Plugging this into the above inequality yields that
\[ \sum_{k=1}^{T} \lambda_k \geq \frac{p!}{(20p-8)L} \left( \frac{1}{4\|x^* - x_0\|_2^2} \right)^{\frac{p-1}{2}} T^{\frac{p+1}{2}}. \]

This completes the proof. \(\square\)

4.2 Proof of Theorem 3.1

We see from Harker and Pang [1990, Theorem 3.1] that at least one strong solution to the VI exists since \( F \) is continuous and \( \mathcal{X} \) is convex, closed and bounded. Since any strong solution is a weak solution if \( F \) is further assumed to be monotone, we obtain that the VI satisfies the Minty condition.

Letting \( x \in \mathcal{X} \), we derive from the monotonicity of \( F \) and the definition of \( \tilde{x}_T \) (Option I) that
\[ \langle F(x), \tilde{x}_T - x \rangle = \frac{1}{\sum_{k=1}^{T} \lambda_k} \left( \sum_{k=1}^{T} \lambda_k \langle F(x), x_k - x \rangle \right) \leq \frac{1}{\sum_{k=1}^{T} \lambda_k} \left( \sum_{k=1}^{T} \lambda_k \langle F(x_k), x_k - x \rangle \right). \]

Combining this inequality with the first inequality in Lemma 4.2 yields that
\[ \langle F(x), \tilde{x}_T - x \rangle \leq \frac{\|x - x_0\|^2}{2(\sum_{k=1}^{T} \lambda_k)}, \text{ for all } x \in \mathcal{X}. \]

Since \( x_0 \in \mathcal{X} \), we have \( \|x - x_0\| \leq D \) and hence
\[ \langle F(x), \tilde{x}_T - x \rangle \leq \frac{D^2}{2(\sum_{k=1}^{T} \lambda_k)}, \text{ for all } x \in \mathcal{X}. \]

Then, we combine Lemma 4.3 and the fact that \( \|x^* - x_0\| \leq D \) to obtain that
\[ \langle F(x), \tilde{x}_T - x \rangle \leq \frac{2p(5p-2)}{p!} L D^{p+1} T^{-\frac{p+1}{2}}, \text{ for all } x \in \mathcal{X}. \]

By the definition of a gap function (see Eq. (8)), we have
\[ \text{GAP}(\tilde{x}_T) = \sup_{x \in \mathcal{X}} \langle F(x), \tilde{x}_T - x \rangle \leq \frac{2p(5p-2)}{p!} L D^{p+1} T^{-\frac{p+1}{2}}. \tag{22} \]

Therefore, we conclude from Eq. (22) that there exists some \( T > 0 \) such that the output \( \hat{x} = \text{Perseus}(p, x_0, L, T, 0) \) satisfies that \( \text{GAP}(\hat{x}) \leq \epsilon \) and the total number of calls of the subproblem solvers is bounded by
\[ O \left( \left( \frac{LD^{p+1}}{\epsilon} \right)^{\frac{2}{p+1}} \right). \]

This completes the proof.
4.3 Proof of Theorem 3.2

In the strongly monotone setting with a convex, closed and bounded set, the solution $x^* \in X$ to the VI exists and is unique [Facchinei and Pang, 2007] and the VI satisfies the Minty condition.

We first consider the relationship between $\|\hat{x} - x^*\|$ and $\|x_0 - x^*\|$ where $\hat{x} = \text{Perseus}(p, x_0, L, t, 0)$. We derive from Jensen’s inequality and the definition of $\hat{x}_t$ that

$$\|\hat{x}_t - x^*\|^2 = \left|\frac{1}{\sum_{k=1}^{t} \lambda_k} \left( \sum_{k=1}^{t} \lambda_k x_k \right) - x^* \right|^2 \leq \frac{1}{\sum_{k=1}^{t} \lambda_k} \left( \sum_{k=1}^{t} \lambda_k \|x_k - x^*\|^2 \right).$$

Since $F$ is $\mu$-strongly monotone, we have

$$\|x_k - x^*\|^2 \leq \frac{1}{\mu} \langle F(x_k) - F(x^*), x_k - x^* \rangle \leq \frac{1}{\mu} \langle F(x_k), x_k - x^* \rangle.$$ 

Putting these pieces together yields that

$$\|\hat{x}_t - x^*\|^2 \leq \frac{1}{\mu(\sum_{k=1}^{t} \lambda_k)} \left( \sum_{k=1}^{t} \lambda_k \langle F(x_k), x_k - x^* \rangle \right).$$

This together with Lemma 4.2 and the fact that $\hat{x} = \hat{x}_t$ yields that

$$\|\hat{x} - x^*\|^2 \leq \frac{1}{2\mu(\sum_{k=1}^{t} \lambda_k)} \|x_0 - x^*\|^2.$$ 

Combining the first inequality in Lemma 4.2 with Eq. (23) yields that

$$\|\hat{x}_t - x^*\|^2 \leq \frac{1}{\mu(\sum_{k=1}^{t} \lambda_k)} \|x_0 - x^*\|^2.$$ 

This together with Lemma 4.3 and the fact that $\hat{x} = \hat{x}_t$ yields that

$$\|\hat{x} - x^*\|^2 \leq \left( \frac{1}{\sum_{k=1}^{t} \lambda_k} \right) \|x_0 - x^*\|^2.$$ 

Recalling that we have $x_{k+1} = \text{Perseus}(p, x_k, L, t, 0)$ in the scheme of Algorithm 2 and

$$t = \left( \frac{2^p(5p-2)}{p!} \frac{\mu L^{p-1}}{p+1} \right)^{\frac{1}{p+1}},$$

we have

$$\|x_{k+1} - x^*\|^2 \leq \frac{1}{2} \|x_k - x^*\|^2, \quad \text{for all } k = 0, 1, 2, \ldots, T.$$ 

Therefore, we conclude from Eq. (25) and Eq. (26) that there exists some $T > 0$ such that the output $\hat{x} = \text{Perseus-restart}(p, x_0, L, \mu, T, 0)$ satisfies that $\|\hat{x} - x^*\| \leq \epsilon$ and the total number of calls of the subproblem solvers is bounded by

$$O \left( \left( \frac{LD^{p-1}}{\mu} \right)^{\frac{1}{p+1}} \log_2 \left( \frac{D}{\epsilon} \right) \right).$$

This completes the proof.
4.4 Proof of Theorem 3.5

We see from the second inequality in Lemma 4.2 that
\[
\min_{1 \leq k \leq T} \|x_k - v_k\|^2 \leq \frac{1}{T} \sum_{k=1}^{T} \|x_k - v_k\|^2 \leq \frac{4\|x^* - x_0\|^2}{T}.
\]

By the definition of \(x_{k_T}\), we have
\[
\|x_{k_T} - v_{k_T}\|^2 \leq \frac{4\|x^* - x_0\|^2}{T}.
\]

Recalling that \(x_{k+1} \in \mathcal{X}\) satisfies Eq. (12), we have
\[
\langle F_{v_k}(x_k), x - x_k \rangle \geq -\frac{L}{p} \|x_k - v_k\|^p + 1,
\]
where \(F_v(x) : \mathbb{R}^d \to \mathbb{R}^d\) is defined for any fixed \(v \in \mathcal{X}\) as follows:
\[
F_{v_k}(x) = F(v_k) + \langle \nabla F(v_k), x - v_k \rangle + \frac{1}{(p-1)!} \nabla^{(p-1)}F(v_k)[x - v_k]^{p-1} + \frac{5L}{(p-1)!} \|x - v_k\|^{p-1}(x - v_k).
\]

Under Assumption 2.3, we have Eq. (19) which further leads to
\[
\|F(x_k) - F_{v_k}(x_k)\| \leq \frac{(5p+1)L}{p!} \|x_k - v_k\|^p.
\]

Putting these pieces together yields that
\[
\langle F(x_k), x - x_k \rangle = \langle F(x_k) - F_{v_k}(x_k), x - x_k \rangle + \langle F_{v_k}(x_k), x - x_k \rangle
\leq \|F(x_k) - F_{v_k}(x_k)\| \|x_k - x\| + \frac{L}{p} \|x_k - v_k\|^p
\leq \frac{L}{p} \|x_k - v_k\|^p \left(\frac{(5p+1)}{p!} \|x_k - x\| + \|x_k - v_k\|\right), \quad \text{for all } x \in \mathcal{X}.
\]

This implies that
\[
\langle F(x_k), x - x_k \rangle \leq \frac{(5p+1)L}{p!} \|x_k - v_k\|^p \|x_k - x\| + \frac{L}{p} \|x_k - v_k\|^p + 1, \quad \text{for all } x \in \mathcal{X}.
\]

Then, we derive from the fact that \(\|x_k - x\| \leq D\) and \(\|x_k - v_k\| \leq D\) that
\[
\langle F(x_k), x - x_k \rangle \leq \frac{(5p+2)LD}{p!} \|x_k - v_k\|^p, \quad \text{for all } x \in \mathcal{X}.
\]

By the definition of a residue function (see Eq. (9)), we have
\[
\text{RES}(x_{k_T}) = \sup_{x \in \mathcal{X}} \langle F(x_{k_T}), x_{k_T} - x \rangle \leq \frac{(5p+2)LD}{p!} \|x_{k_T} - v_{k_T}\|^p
\leq \frac{(5p+2)LD}{p!} \left(\frac{4\|x^* - x_0\|^2}{T}\right)^\frac{p}{2}.
\]

Since \(x_0, x^* \in \mathcal{X}\), we have \(\|x^* - x_0\| \leq D\) and hence
\[
\text{RES}(x_{k_T}) \leq \frac{2p(5p+2)LD}{p!} T^{-\frac{p}{2}}.
\]

Therefore, we conclude from Eq. (29) that there exists \(T > 0\) such that the output \(\hat{x} = \text{Perseus}(p, x_0, L, T, 1)\) satisfies that \(\text{RES}(\hat{x}) \leq \epsilon\) and the total number of calls of the subproblem solvers is bounded by
\[
O \left( \frac{LD^{p+1}}{\epsilon} \right)^\frac{p}{2}.
\]

This completes the proof.
4.5 Proof of Theorem 3.6

We first consider the relationship between \( \|\hat{x} - x^*\| \) and \( \|x_0 - x^*\| \) where \( \hat{x} = \text{Perseus}(p, x_0, L, t, 1) \). By the same argument as used in Theorem 3.5, we have (see Eq. (27)):

\[
\langle F(x_k), x_k - x \rangle \leq \frac{(5p+1)L}{p^t} \|x_k - v_k\|^p \|x_k - x\| + \frac{L}{p^t} \|x_k - v_k\|^{p+1}, \quad \text{for all } x \in \mathcal{X}.
\]

Suppose that \( x = x^* \in \mathcal{X} \) is a weak solution (the existence is guaranteed by the Minty condition). Since the VI satisfies the \( \mu_{\text{Minty}} \)-strong Minty condition, we have

\[
\|x_k - x^*\|^2 \leq \frac{1}{p_{\text{Minty}}} \langle F(x_k), x_k - x^* \rangle.
\]

Putting these pieces together yields that

\[
\|x_k - x^*\|^2 \leq \frac{L}{p_{\text{Minty}}} \left( \frac{5p+1}{p^t} \|x_k - v_k\|^p \|x_k - x^*\| + \frac{L}{p^t} \|x_k - v_k\|^{p+1} \right),
\]

which implies that

\[
\|x_k - x^*\| \leq \frac{5p+1}{p^t} \frac{L}{p_{\text{Minty}}} \|x_k - v_k\|^p + \sqrt{\frac{1}{p^t} \frac{L}{p_{\text{Minty}}} \|x_k - v_k\|^{p+1}},
\]

By the definition of \( x_k \), we have

\[
\|x_k - x^*\| \leq \frac{5p+1}{p^t} \frac{L}{p_{\text{Minty}}} \left( \frac{\sum_{k=1}^t \|x_k - v_k\|^2}{t} \right)^{\frac{p}{2}} + \sqrt{\frac{1}{p^t} \frac{L}{p_{\text{Minty}}} \left( \frac{\sum_{k=1}^t \|x_k - v_k\|^2}{t} \right)^{\frac{p+1}{2}}},
\]

This together with the second inequality in Lemma 4.2 yields that

\[
\|x_k - x^*\| \leq \frac{5p+1}{p^t} \frac{L}{p_{\text{Minty}}} \left( \frac{4\|x^* - x_0\|^2}{t} \right)^{\frac{p}{2}} + \sqrt{\frac{1}{p^t} \frac{L}{p_{\text{Minty}}} \left( \frac{4\|x^* - x_0\|^2}{t} \right)^{\frac{p+1}{2}}},
\]

which implies that

\[
\|x_k - x^*\| \leq \left( \frac{2p(5p+1)}{p^t} \frac{L}{p_{\text{Minty}}} t^{-\frac{p}{2}} \right) \|x_0 - x^*\|^p + \sqrt{\frac{2p+1}{p^t} \frac{L}{p_{\text{Minty}}} t^{-\frac{p+1}{2}} \|x_0 - x^*\|^{p+1}}. \quad (30)
\]

Recalling that we have \( x_{k+1} = \text{Perseus}(p, x_k, L, t, 1) \) in the scheme of Algorithm 2 and

\[
t = \left[ \left( \frac{2p+1}{p^t} \frac{L}{p_{\text{Minty}}} \right)^{\frac{2}{p}} + \left( \frac{2p+1}{p^t} \frac{L}{p_{\text{Minty}}} \right)^{\frac{2}{p+1}} \right], \quad (31)
\]

we have

\[
\|x_{k+1} - x^*\|^2 \leq \frac{1}{2} \|x_k - x^*\|^2, \quad \text{for all } k = 0, 1, 2, \ldots, T. \quad (32)
\]

Therefore, we conclude from Eq. (31) and Eq. (32) that there exists some \( T > 0 \) such that the output \( \hat{x} = \text{Perseus-restart}(p, x_0, L, p_{\text{Minty}}, T, 1) \) satisfies that \( \|\hat{x} - x^*\| \leq \epsilon \) and the total number of calls of the subproblem solvers is bounded by

\[
O \left( \max \left\{ \left( \frac{LD^{p-1}}{p_{\text{Minty}}} \right)^{\frac{2}{p}}, \left( \frac{LD^{p-1}}{p_{\text{Minty}}} \right)^{\frac{2}{p+1}} \right\} \log \left( \frac{D}{\epsilon} \right) \right).
\]

This completes the proof.
4.6 Proof of Theorem 3.8

We first consider the relationship between $\|\dot{x} - x^*\|$ and $\|x_0 - x^*\|$ where $\dot{x} = \text{Perseus}(p, x_0, L, t, 2)$. By the same argument as used in Theorem 3.2, we have (see Eq. (24))

$$\|\dot{x} - x^*\|^2 \leq \left(\frac{2p(5p-2)\kappa t}{p!}L^t - \frac{p+1}{2}\right) \|x_0 - x^*\|^{p+1}.$$

Recalling that we have $x_{k+1} = \text{Perseus}(p, x_k, L, t, 2)$ in the scheme of Algorithm 2 and $t = 1$, we have

$$\|x_{k+1} - x^*\|^2 \leq \left(\frac{2p(5p-2)\kappa}{p!}\right) \|x_k - x^*\|^{p+1}, \quad \text{for all } k = 0, 1, 2, \ldots, T,$$

which implies that

$$\|x_{k+1} - x^*\| \leq \sqrt{\frac{2p(5p-2)\kappa}{p!}} \|x_k - x^*\|^{\frac{p+1}{2}}, \quad \text{for all } k = 0, 1, 2, \ldots, T.$$

For the case of $p \geq 2$, we have $\frac{p+1}{2} \geq \frac{3}{2}$ and $p - 1 \geq 1$. If the following condition holds true,

$$\|x_0 - x^*\| \leq \frac{1}{2} \left(\frac{p!}{2p(5p-2)\kappa}\right)^{\frac{1}{p-1}},$$

we have

$$\left(\frac{2p(5p-2)\kappa}{p!}\right)^{\frac{1}{p-1}} \|x_{k+1} - x^*\| \leq \left(\frac{2p(5p-2)\kappa}{p!}\right)^{\frac{p+1}{2(p-1)}} \|x_k - x^*\|^{\frac{p+1}{2}}$$

$$= \left(\frac{2p(5p-2)\kappa}{p!}\right)^{\frac{1}{p-1}} \|x_k - x^*\|^{\frac{p+1}{2}} \leq \left(\frac{2p(5p-2)\kappa}{p!}\right)^{\frac{1}{p-1}} \|x_0 - x^*\| \left(\frac{2p(5p-2)\kappa}{p!}\right)^{\frac{p+1}{2}k+1}$$

$$\leq \left(\frac{1}{2}\right)^{(\frac{p+1}{2})k+1}.$$

This completes the proof.

4.7 Proof of Theorem 3.9

We first consider the relationship between $\|\dot{x} - x^*\|$ and $\|x_0 - x^*\|$ where $\dot{x} = \text{Perseus}(p, x_0, L, t, 2)$. By the same argument as used in Theorem 3.6, we have (see Eq. (30))

$$\|x_k - x^*\| \leq \left(\frac{2p(5p+1)\kappa L}{p!} + \sqrt{\frac{2p+1}{p!}L^t} \frac{\kappa M_{\text{Minty}}}{\mu_{\text{Minty}}} \right) \|x_0 - x^*\|^p + \sqrt{\frac{2p+1}{p!}L^t} \frac{\kappa M_{\text{Minty}}}{\mu_{\text{Minty}}} \|x_0 - x^*\|^{\frac{p+1}{2}}.$$

Recalling that we have $x_{k+1} = \text{Perseus}(p, x_k, L, t, 2)$ in the scheme of Algorithm 2 and $t = 1$, we have

$$\|x_{k+1} - x^*\| \leq \left(\frac{2p(5p+1)\kappa L}{p!} + \sqrt{\frac{2p+1}{p!}L} \frac{\kappa M_{\text{Minty}}}{\mu_{\text{Minty}}} \right) \|x_k - x^*\|^p + \sqrt{\frac{2p+1}{p!}L} \frac{\kappa M_{\text{Minty}}}{\mu_{\text{Minty}}} \|x_k - x^*\|^{\frac{p+1}{2}},$$

for all $k = 0, 1, 2, \ldots, T$, which implies that

$$\|x_{k+1} - x^*\| \leq \left(\frac{2p(5p+1)\kappa M_{\text{Minty}}}{p!} \frac{\kappa M_{\text{Minty}}}{\mu_{\text{Minty}}} \right) \|x_k - x^*\|^p + \sqrt{\frac{2p+1}{p!}L} \frac{\kappa M_{\text{Minty}}}{\mu_{\text{Minty}}} \|x_k - x^*\|^{\frac{p+1}{2}},$$

for all $k = 0, 1, 2, \ldots, T$. For the case of $p \geq 2$, we have $\frac{p+1}{2} \geq \frac{3}{2}$ and $p - 1 \geq 1$. If the following condition holds true:

$$\|x_0 - x^*\| \leq \frac{1}{2} \left(\frac{p!}{2^{p+2}p(5p+1)\kappa M_{\text{Minty}}}\right)^{\frac{1}{p-1}},$$

then
we claim that the following statement holds true,
\[ \|x_k - x^*\| \leq \frac{1}{2} \left( \frac{2^p (5p+1) \kappa_{\text{Minty}}}{\sqrt{p!}} \right)^{\frac{1}{p+1}}, \text{ for all } k = 0, 1, 2, \ldots, T + 1. \]
Indeed, the case of \( k = 0 \) holds true. Assuming that the case of \( k \geq 1 \) holds true, we have
\[
\|x_{k+1} - x^*\| \leq \left( \frac{2^p (5p+1) \kappa_{\text{Minty}}}{\sqrt{p!}} \right)^{\frac{1}{p+1}} \|x_k - x^*\|^p + \sqrt{\frac{2^p (5p+1) \kappa_{\text{Minty}}}{p!}} \|x_k - x^*\|^\frac{p+1}{2}
\leq \left( \frac{1}{2} \right)^{p+1} + \left( \frac{1}{2} \right)^{p+1} \|x_k - x^*\|.
\]
Since \( p \geq 2 \), we have \( \|x_{k+1} - x^*\| \leq \|x_k - x^*\| \) for all \( k = 0, 1, 2, \ldots, T \). By induction, we get the desired result. Then, we have
\[
\|x_{k+1} - x^*\| \leq 2\sqrt{\frac{2^p (5p+1) \kappa_{\text{Minty}}}{p!}} \|x_k - x^*\|^\frac{p+1}{2} = \sqrt{\frac{2^p (5p+1) \kappa_{\text{Minty}}}{p!}} \|x_k - x^*\|^\frac{p+1}{2},
\]
and
\[
\left( \frac{2^p (5p+1) \kappa_{\text{Minty}}}{p!} \right)^{\frac{1}{p+1}} \|x_{k+1} - x^*\| \leq \left( \frac{2^p (5p+1) \kappa_{\text{Minty}}}{p!} \right)^{\frac{1}{p+1}} \|x_k - x^*\|^\frac{p+1}{2}
\leq \left( \frac{2^p (5p+1) \kappa_{\text{Minty}}}{p!} \right)^{\frac{1}{p+1}} \|x_k - x^*\|^\frac{p+1}{2} \leq \left( \frac{2^p (5p+1) \kappa_{\text{Minty}}}{p!} \right)^{\frac{1}{p+1}} \|x_0 - x^*\|^\frac{p+1}{2} \leq \left( \frac{2^p (5p+1) \kappa_{\text{Minty}}}{p!} \right)^{\frac{1}{p+1}} \|x_0 - x^*\|^\frac{p+1}{2}.
\]
This completes the proof.

5 Conclusions

We have proposed and analyzed a new \( p \)-th-order method—Perseus—for finding a weak solution of smooth and monotone variational inequalities when \( F \) is smooth with up to \((p - 1)\)-th-order derivatives. Our results are based on the assumption that the subproblem arising from a \((p - 1)\)-th-order Taylor expansion of \( F \) can be computed approximately in an efficient way. For the case of \( p \geq 2 \), the best existing \( p \)-th-order methods can achieve a global rate of \( O(\epsilon^{-2/(p+1)} \log(1/\epsilon)) \) [Bullins and Lai, 2020, Lin and Jordan, 2021b, Jiang and Mokhtari, 2022] but require a nontrivial binary search procedure at each iteration. The open question has been whether it is possible to obtain a simple high-order regularization method which can achieve a global rate of \( O(\epsilon^{-2/(p+1)}) \) while dispensing with binary search.

Our results settle this open problem. Our method converges to a weak solution with a global rate of \( O(\epsilon^{-2/(p+1)}) \) and the restarted version can attain global linear and local superlinear convergence for smooth and strongly monotone VIs. Our method also achieves a global rate of \( O(\epsilon^{-2/p}) \) for solving a class of smooth and non-monotone VIs satisfying the Minty condition and its restarted version attains global linear and local superlinear convergence if the strong Minty condition holds true. Future research directions include the investigation of lower bounds for any \( p \)-th-order method in the case of \( p \geq 2 \) and the comparative study of lower-order methods in VI problems where higher-order smoothness is assumed; see Nesterov [2021a,c,d], Doikov and Nesterov [2022].

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Acknowledgments

This work was supported in part by the Mathematical Data Science program of the Office of Naval Research under grant number N00014-18-1-2764 and by the Vannevar Bush Faculty Fellowship program under grant number N00014-21-1-2941.

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