ON SLANT MAGNETIC CURVES IN S-MANIFOLDS

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Abstract. We consider slant normal magnetic curves in \((2n+1)\)-dimensional S-manifolds. We prove that \(\gamma\) is a slant normal magnetic curve in an S-manifold \((M^{2n+1}, \varphi, \xi, \eta, g)\) if and only if it belongs to a list of slant \(\varphi\)-curves satisfying some special curvature equations. This list consists of some specific geodesics, slant circles, Legendre and slant helices of order 3. We construct slant normal magnetic curves in \(\mathbb{R}^{2n+1}(-3s)\) and give the parametric equations of these curves.

1. Introduction

Let \((M, g)\) be a Riemannian manifold, \(F\) a closed 2-form and let us denote the Lorentz force on \(M\) by \(\Phi\). If \(F\) is associated by the relation
\[ g(\Phi X, Y) = F(X, Y), \quad \forall X, Y \in \chi(M), \tag{1.1} \]
then it is called a magnetic field [1, 4 and 9]. Let \(\nabla\) be the Riemannian connection associated to the metric \(g\) and \(\gamma : I \rightarrow M\) a smooth curve. If \(\gamma\) satisfies the Lorentz equation
\[ \nabla_{\gamma'(t)}\gamma'(t) = \Phi(\gamma'(t)), \tag{1.2} \]
then it is called a magnetic curve or a trajectory for the magnetic field \(F\). The Lorentz equation is a generalization of the equation for geodesics. A curve which satisfies the Lorentz equation is called magnetic trajectory. Magnetic trajectories have constant speed. If the speed of the magnetic curve \(\gamma\) is equal to 1, then it is called a normal magnetic curve [10].

In [1], Adachi studied curvature bound and trajectories for magnetic fields on a Hadamard surface. He showed that every normal trajectory is unbounded in both directions in a 2-dimensional complete simply connected Riemannian manifold satisfying some special curvature conditions. In [5], Baikoussis and Blair considered Legendre curves in contact 3-manifolds and they proved that the torsion of a Legendre curve in a 3-dimensional Sasakian manifold is equal to 1. Moreover, in [8], Cho, Inoguchi and Lee proved that a non-geodesic curve in a Sasakian 3-manifold is a slant curve if and only if the ratio of \((\tau \pm 1)\) and \(\kappa\) is constant, where \(\tau\) is the geodesic torsion and \(\kappa\) is the geodesic curvature. Cabrera, Fernandez and Gomez gave a nice geometric construction of an almost contact metric structure compatible with an assigned metric on a 3-dimensional oriented Riemannian manifold in [6]. In the paper [10], Drută-Romaniuc, Inoguchi, Munteanu and Nistor studied the magnetic trajectories of the contact magnetic field \(F_q = q\Omega\) on a Sasakian \((2n + 1)\)-manifold \((M^{2n+1}, \varphi, \xi, \eta, g)\), where \(\Omega\) is the fundamental 2-form. The main objective of [11] is the study of trajectories for particles moving under the

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influence of a contact magnetic curve in a cosymplectic manifold. The paper [14] is concerned with closed magnetic trajectories on 3-dimensional Berger spheres. In [15], the authors studied magnetic trajectories in an almost contact metric manifold. They proved that normal magnetic curves are helices of maximum order 5. Moreover, in [16], Jleli and Munteanu worked in the context of a para-Kaehler manifold, showing that spacelike and timelike normal magnetic curves corresponding to the para-Kaehler 2-forms are circles. In [17], the authors gave a complete classification of Killing magnetic curves with unit speed. Furthermore, in [18], the same authors proved that a normal magnetic curve on the Sasakian sphere $S^{2n+1}$ lies on a totally geodesic sphere $S^3$. They also considered two particular magnetic fields on three-dimensional torus obtained from two different contact forms on the Euclidean space $E^3$ and studied their closed normal magnetic trajectories in their recent paper [19]. In [20], Nakagawa introduced the notion of framed $f$-structures, which is a generalization of almost contact structures. Vanzura studied almost $r$-structures in [21]. A differentiable manifold with this structure is the same as a framed $f$-manifold as defined by Nakagawa. On the other hand, Hasegawa, Okuyama and Abe defined a $p$th Sasakian manifold and gave some typical examples in [13].

Motivated by the above studies, in the present paper, we consider slant normal magnetic curves in $(2n+s)$-dimensional $S$-manifolds. In Section 2, we give brief information on $S$-manifolds and magnetic curves. In Section 3, we prove that $\gamma$ is a slant normal magnetic curve in an $S$-manifold $(M^{2n+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ if and only if it belongs to a list of slant $\varphi$-curves. This list consists of some specific geodesics, slant circles, Legendre and slant helices of order 3. Finally, in Section 4, we construct slant normal magnetic curves in $\mathbb{R}^{2n+s}(-3s)$ and give the parametric equations of these curves in two cases.

2. Preliminaries

In this section, we give brief information on $S$-manifolds and magnetic curves. Let $(M^{2n+s}, g)$ be a differentiable manifold, $\varphi$ a $(1, 1)$-type tensor field, $\eta^\alpha$ 1-forms, $\xi_\alpha$ vector fields for $\alpha = 1, \ldots, s$, satisfying

$$\varphi^2 = -I + \sum_{\alpha=1}^{s} \eta^\alpha \otimes \xi_\alpha, \tag{2.1}$$

$$\eta^\alpha (\xi_\beta) = \delta^\alpha_\beta, \quad \varphi \xi_\alpha = 0, \quad \eta^\alpha (\varphi X) = 0, \quad \eta^\alpha (X) = g(X, \xi_\alpha),$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{\alpha=1}^{s} \eta^\alpha(X)\eta^\alpha(Y), \tag{2.2}$$

$$d\eta^\alpha (X, Y) = -d\eta^\alpha (Y, X) = g(X, \varphi Y),$$

where $X, Y \in TM$. Then $(\varphi, \xi_\alpha, \eta^\alpha, g)$ is called framed $\varphi$-structure and $(M^{2n+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ is called framed $\varphi$-manifold [20]. $(M^{2n+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ is also called framed metric
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If the Nijenhuis tensor of $\varphi$ is equal to $-2d\eta^\alpha \otimes \xi_\alpha$, then $(M^{2n+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ is called an $S$-manifold [3]. For $s = 1$, an $S$-structure becomes a Sasakian structure. For an $S$-structure, the following properties are satisfied [3]:

$$(\nabla_X \varphi)Y = \sum_{\alpha=1}^{s} \{g(\varphi X, \varphi Y)\xi_\alpha + \eta^\alpha(Y)\varphi^2X\},$$

(2.3)

$$\nabla \xi_\alpha = -\varphi, \ \alpha \in \{1, \ldots, s\}.$$  

(2.4)

Let $M^{2n+s} = (M^{2n+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an $S$-manifold and $\Omega$ the fundamental 2-form of $M^{2n+s}$ defined by

$$\Omega(X, Y) = g(X, \varphi Y),$$

(2.5)

(see [20] and [24]). From the definition of framed $\varphi$-structure, we have $\Omega = d\eta^\alpha$. Hence, the fundamental 2-form $\Omega$ on $M^{2n+s}$ is closed. The magnetic field $F_q$ on $M^{2n+s}$ can be defined by

$$F_q(X, Y) = q\Omega(X, Y),$$

where $X$ and $Y$ are vector fields on $M^{2n+s}$ and $q$ is a real constant. $F_q$ is called the contact magnetic field with strength $q$ [15]. If $q = 0$ then the magnetic curves are geodesics of $M^{2n+s}$. Because of this reason we shall consider $q \neq 0$ (see [6] and [10]).

From (1.1) and (2.5), the Lorentz force $\Phi$ associated to the contact magnetic field $F_q$ can be written as

$$\Phi_q = -q\varphi.$$  

So the Lorentz equation (1.2) can be written as

$$\nabla_T T = -q\varphi T,$$

(2.6)

where $\gamma : I \subseteq \mathbb{R} \rightarrow M^{2n+s}$ is a smooth unit-speed curve and $T = \gamma'$ (see [10] and [15]).

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Let $(M^n, g)$ be a Riemannian manifold. A unit-speed curve $\gamma : I \rightarrow M$ is said to be a Frenet curve of osculating order $r$, if there exists positive functions $\kappa_1, \ldots, \kappa_{r-1}$ on $I$ satisfying

$$T = v_1 = \gamma',$$

$$\nabla_T T = k_1v_2,$$

$$\nabla_T v_2 = -k_1T + k_2v_3,$$

$$\ldots$$

$$\nabla_T v_r = -k_{r-1}v_{r-1},$$

(3.1)

where $1 \leq r \leq n$ and $T, v_2, \ldots, v_r$ are a $g$-orthonormal vector fields along the curve. The positive functions $\kappa_1, \ldots, \kappa_{r-1}$ are called curvature functions and $\{T, v_2, \ldots, v_r\}$ is called the Frenet frame field. A geodesic is a Frenet curve of osculating order $r = 1$. A circle is a Frenet curve of osculating order $r = 2$ with a constant curvature function $\kappa_1$. A helix of order $r$ is a Frenet curve of osculating order $r$ with constant curvature functions $\kappa_1, \ldots, \kappa_{r-1}$. A helix of order 3 is simply called a helix.
Let \((M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)\) be an \(S\)-manifold. For a unit-speed curve \(\gamma : I \to M\), if
\[ \eta^\alpha(T) = 0, \]
for all \(\alpha = 1, \ldots, s\), then \(\gamma\) is called a Legendre curve of \(M\) [21]. More generally, if there exists a constant angle \(\theta\) such that
\[ \eta^\alpha(T) = \cos \theta, \]
for all \(\alpha = 1, \ldots, s\), then \(\gamma\) is called a slant curve and \(\theta\) is called the contact angle of \(\gamma\), where \(|\cos \theta| \leq 1/\sqrt{s}\) [12].

Let \((M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)\) be an \(S\)-manifold. A Frenet curve of osculating order \(r \geq 3\) is called a \(\varphi\)-curve in \(M\) if its Frenet vector fields \(T, v_2, \ldots, v_r\) span a \(\varphi\)-invariant space. A \(\varphi\)-curve of osculating order \(r\) with constant curvature functions \(\kappa_1, \ldots, \kappa_{r-1}\) is called a \(\varphi\)-helix of order \(r\). A curve of osculating order 2 is called a \(\varphi\)-curve if
\[ sp \left\{ T, v_2, \sum_{\alpha=1}^{s} \xi_\alpha \right\} \]
is a \(\varphi\)-invariant space.

Throughout the paper, when we state ”slant magnetic curve”, we mean ”slant curves which satisfy equation (2.6)”. For magnetic curves, \(\eta^\alpha(T) = \cos \theta^\alpha\) does not have to be equal for all \(\alpha = 1, \ldots, s\). By taking the curve as slant, we only study the equality case of the slant angles \(\theta^\alpha\) in the present paper. The complete classification of magnetic curves in \(S\)-manifolds is still an open problem.

Firstly, we state the following theorem:

**Theorem 1.** Let \((M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)\) be an \(S\)-manifold and consider the contact magnetic field \(F_q\) for \(q \neq 0\). Then \(\gamma\) is a slant normal magnetic curve associated to \(F_q\) in \(M^{2m+s}\) if and only if \(\gamma\) belongs to the following list:

a) geodesics obtained as integral curves of \(\left( \pm \frac{1}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_\alpha \right)\);

b) non-geodesic slant circles with the curvature \(\kappa_1 = \sqrt{q^2 - s}\), having the contact angle \(\theta = \arccos \left( \frac{q}{s} \right)\) and the Frenet frame field
\[ \left\{ T, -\frac{\text{sgn}(q)\varphi T}{\sqrt{1 - s \cos^2 \theta}} \right\}, \]
where \(|q| > \sqrt{s}\);

c) Legendre helices with curvatures \(\kappa_1 = |q|\) and \(\kappa_2 = \sqrt{s}\), having the Frenet frame field
\[ \left\{ T, -\text{sgn}(q)\varphi T, -\frac{\text{sgn}(q)}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_\alpha \right\}; \]
i.e., a class of 1-dimensional integral submanifolds of the contact distribution;

d) slant helices with curvatures \(\kappa_1 = |q| \sqrt{1 - s \cos^2 \theta}\) and \(\kappa_2 = \sqrt{s} |1 - q \cos \theta|\), having the Frenet frame field
\[ \left\{ T, -\frac{\text{sgn}(q)\varphi T}{\sqrt{1 - s \cos^2 \theta}}, -\frac{\text{sgn}(q)}{\sqrt{s}} \sqrt{\frac{1 - s \cos^2 \theta}{1 - q \cos \theta}} \left( -s \cos \theta T + \sum_{\alpha=1}^{s} \xi_\alpha \right) \right\}, \]
where \(\theta \neq \frac{\pi}{2}\) is the contact angle satisfying \(|\cos \theta| < \frac{1}{\sqrt{s}}\) and \(\varepsilon = \text{sgn}(1 - q \cos \theta)\).
Proof. Let \( \gamma \) be a normal magnetic curve. If the magnetic curve is a geodesic, then
\[
\nabla_T T = 0 = -q\varphi T
\]
gives us
\[
T \in \text{sp} \{ \xi_1, ..., \xi_s \}.
\]
If \( \gamma \) is slant, then we can write
\[
T = \cos \theta \sum_{\alpha=1}^{s} \xi_\alpha.
\]
Since \( \gamma \) is unit speed, we have \( \cos \theta = \pm \frac{1}{\sqrt{s}} \). So the proof of a) is complete.

From now on, we suppose that \( \gamma \) is a non-geodesic Frenet curve of osculating order \( r > 1 \). Let us choose an \( \alpha \in \{1, ..., s\} \). Applying \( \xi_\alpha \) to \( \nabla_T T = -q\varphi T \), we obtain
\[
0 = g(-q\varphi T, \xi_\alpha) = g(\nabla_T T, \xi_\alpha) = \frac{d}{dt} g(T, \xi_\alpha) - g(T, \nabla_T \xi_\alpha). \quad (3.2)
\]
From (2.4), we also have
\[
\nabla_T \xi_\alpha = -\varphi T. \quad (3.3)
\]
Using equations (3.2) and (3.3), we find
\[
\frac{d}{dt} g(T, \xi_\alpha) = 0,
\]
that is,
\[
\eta^\alpha (T) = \cos \theta_\alpha = \text{constant}.
\]
Let us assume \( \theta_\alpha = \theta \) for all \( \alpha = 1, ..., s \), i.e., \( \gamma \) is slant. So, we have
\[
\eta^\alpha (T) = \cos \theta. \quad (3.4)
\]
Equations (2.10) and (3.1) give us
\[
\nabla_T T = \kappa_1 v_2 = -q\varphi T. \quad (3.5)
\]
Then we get
\[
\kappa_1 = |q| \| \varphi T \| = |q| \sqrt{1 - s \cos^2 \theta}. \quad (3.6)
\]
If we write (3.6) in (3.5), we find
\[
-q\varphi T = \kappa_1 v_2 = |q| \sqrt{1 - s \cos^2 \theta} v_2,
\]
which gives us
\[
\varphi T = \frac{|q|}{q} \sqrt{1 - s \cos^2 \theta} v_2 = -\text{sgn}(q) \sqrt{1 - s \cos^2 \theta} v_2. \quad (3.7)
\]
If \( \kappa_2 = 0 \), then the magnetic curve is a Frenet curve of osculating order \( r = 2 \). Since \( \kappa_1 \) is a constant, \( \gamma \) is a circle. From (3.7), we have
\[
\eta^\alpha (\varphi T) = 0 = -\text{sgn}(q) \sqrt{1 - s \cos^2 \theta} \eta^\alpha (v_2),
\]
that is,
\[
\eta^\alpha (v_2) = 0.
\]
If we differentiate the last equation along the curve \( \gamma \), we obtain
\[
\nabla_T \eta^\alpha (v_2) = 0 = g(\nabla_T v_2, \xi_\alpha) + g(v_2, \nabla_T \xi_\alpha). \quad (3.8)
\]
So, we calculate
\[
g(-\kappa_1 T, \xi_\alpha) + g(v_2, \text{sgn}(q) \sqrt{1 - s \cos^2 \theta} v_2) = 0.
\]
Since $r = 2$, we find

$$-\kappa_1 \cos \theta + \text{sgn}(q) \sqrt{1 - s \cos^2 \theta} = 0.$$  \hfill (3.6)

Using equation (3.6) in the last equation, it is easy to see that

$$|q| \sqrt{1 - s \cos^2 \theta} \left(-\cos \theta + \frac{1}{q}\right) = 0.$$  \hfill (3.7)

Since $\gamma$ is non-geodesic, we have

$$\cos \theta = \frac{1}{q}.$$  \hfill (3.8)

Then equation (3.6) becomes

$$\kappa_1 = |q| \sqrt{1 - s \cos^2 \theta} = \sqrt{q^2 - s},$$

where $|q| > \sqrt{s}$. So the proof of b) is complete.

Let $\kappa_2 \neq 0$. From (2.1) and (3.4), we find

$$\phi^2 T = -T + \cos \theta \sum_{a=1}^{s} \xi_{\alpha}.$$  \hfill (3.9)

Using (2.3) and (3.4), we have

$$\left(\nabla_T \phi\right) T = -s \cos \theta T + \sum_{a=1}^{s} \xi_{\alpha},$$

which gives us

$$\nabla_T \phi T = (\nabla_T \phi) T + \phi \nabla_T T$$

$$= -s \cos \theta T + \sum_{a=1}^{s} \xi_{\alpha} + \phi(-q \phi T)$$

$$= -s \cos \theta T + \sum_{a=1}^{s} \xi_{\alpha} - q \left(-T + \cos \theta \sum_{a=1}^{s} \xi_{\alpha}\right).$$  \hfill (3.10)

Differentiating (3.7), we also find

$$\nabla_T \phi T = -\text{sgn}(q) \sqrt{1 - s \cos^2 \theta} \left(-\kappa_1 T + \kappa_2 v_3\right).$$  \hfill (3.11)

By the use of (3.6), (3.10) and (3.11), after some calculations, we obtain

$$(1 - q \cos \theta) \left(-s \cos \theta T + \sum_{a=1}^{s} \xi_{\alpha}\right) = -\text{sgn}(q) \sqrt{1 - s \cos^2 \theta} \kappa_2 v_3.$$  \hfill (3.12)

If we find the norm of both sides in (3.12), we get

$$\kappa_2 = \sqrt{s} \left|1 - q \cos \theta\right|.$$  \hfill (3.13)

Let us denote $\varepsilon = \text{sgn}(1 - q \cos \theta)$. If we write (3.13) in (3.12), we obtain

$$\sum_{a=1}^{s} \xi_{\alpha} = s \cos \theta T - \varepsilon \sqrt{s} \sqrt{1 - s \cos^2 \theta} v_3.$$  \hfill (3.14)

Applying $\phi$ to (3.14), we find

$$\phi v_3 = -\varepsilon \sqrt{s} \cos \theta v_2.$$
If we apply \( \varphi \) to (3.7) and then use equations (3.4) and (3.14) together, we have
\[
\varphi v_2 = sgn(q) \sqrt{1 - s \cos^2 \theta T} + \varepsilon \cos \theta \sqrt{s} v_3.
\] (3.15)

Let us choose a \( \beta \in \{1, ..., s\} \). From (3.15), we calculate
\[
\eta^\beta(v_3) = -\varepsilon sgn(q) \frac{\sqrt{1 - s \cos^2 \theta T}}{\sqrt{s}}.
\]

If we differentiate (3.14) along the curve \( \gamma \), we get
\[
\sum_{\alpha=1}^{s} \nabla_T \xi_\alpha = s \cos \theta \nabla_T T - \varepsilon sgn(q) \sqrt{s} \sqrt{1 - s \cos^2 \theta T} \nabla_T v_3,
\]
which gives us
\[
-s (1 - q \cos \theta) \varphi T = -\varepsilon sgn(q) \sqrt{s} \sqrt{1 - s \cos^2 \theta T} (-\kappa_2 v_2 + \kappa_3 v_4).
\]

Since \( \varphi T \parallel v_2 \), we find \( \kappa_3 = 0 \). This proves d) of the theorem.

Let us examine Legendre case separately, that is, \( \theta = \frac{\pi}{2} \). Then we have \( \varepsilon = 1 \), \( \kappa_1 = |q| \), \( \kappa_2 = \sqrt{s} \), \( \kappa_3 = 0 \) and equation (3.14) gives us
\[
v_3 = -\frac{sgn(q)}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_\alpha.
\]

This completes the proof of c).

Conversely, let \( \gamma \) satisfy one of a), b), c) or d). Using the Frenet frame field and Frenet equations, it is straightforward to show that \( \nabla_T T = -q \varphi T \), i.e., \( \gamma \) is a slant normal magnetic curve. \( \square \)

The above theorem is a generalization of Theorem 3.1 of [10] (by Simona Luiza Druta-Romaniuc et al.) for \( S \)-manifolds. If we choose \( s = 1 \), since an \( S \)-manifold becomes a Sasakian manifold, we find their results.

**Remark.** The order of a slant magnetic curve in an \( S \)-manifold is still \( r \leq 3 \), as in the case of a magnetic curve of a Sasakian manifold, which was considered in [10].

Now, let us remove the slant condition from the hypothesis and show that the osculating order is still \( r \leq 3 \).

**Theorem 2.** Let \((M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)\) be an \( S \)-manifold and consider the contact magnetic field \( F_q \) for \( q \neq 0 \). If \( \gamma \) is a normal magnetic curve associated to \( F_q \) in \( M^{2m+s} \), then the osculating order \( r \leq 3 \).

**Proof.** Let \( \gamma \) be a normal magnetic curve. Then, the Lorentz equation (2.6) gives us
\[
\eta^\alpha(T) = \cos \theta_\alpha, \quad \alpha = 1, ..., s.
\]

If we differentiate this equation along the curve, we have
\[
\eta^\alpha(E_2) = 0
\]
for all \( \alpha = 1, ..., s \). From the Frenet equations (3.1), we obtain
\[
-q \varphi T = \kappa_3 v_2.
\]

From the definition of framed \( \varphi \)-structure, we calculate
\[
g(\varphi T, \varphi T) = 1 - A,
\]
where we denote
\[ A = \sum_{\alpha=1}^{s} \cos^2 \theta_\alpha. \]
Then, we have
\[ \| \varphi T \| = \sqrt{1 - A} \]
and
\[ \kappa_1 = |q| \sqrt{1 - A}. \]
Thus, \( \varphi T \) can be rewritten as
\[ \varphi T = -\text{sgn}(q) \sqrt{1 - A} v_2. \tag{3.16} \]
Again, from the definition of framed \( \varphi \)-structure, we have
\[ \varphi^2 T = -T + V, \]
where we denote
\[ V = \sum_{\alpha=1}^{s} \cos \theta_\alpha \xi_\alpha. \]
After some calculations, we get
\[ \nabla_T \varphi T = (q - B)T + (1 - A) \sum_{\alpha=1}^{s} \xi_\alpha + (-q + B)V, \]
which corresponds to equation (3.10). Here, we denote
\[ B = \sum_{\alpha=1}^{s} \cos \theta_\alpha. \]
From equation (3.10), we also find
\[ \nabla_T \varphi T = -\text{sgn}(q) \sqrt{1 - A}(\kappa_1 T + \kappa_2 v_3), \]
which corresponds to equation (3.11). In this last equation, we can replace \( \kappa_1 = |q| \sqrt{1 - A} \). Finally, we have
\[ -\text{sgn}(q) \sqrt{1 - A} \kappa_2 v_3 = (1 - A) \sum_{\alpha=1}^{s} \xi_\alpha + (-q + B) V \tag{3.17} \]
\[ + (qA - B)T. \]
So, if we denote the norm of the right hand side of equation (3.17) by \( C \), we find
\[ C = \sqrt{(1 - A)(Aq^2 - As + B^2 - 2Bq + s)}, \]
which is a constant. Hence, we obtain
\[ \kappa_2 = \frac{C}{\sqrt{1 - A}} = \sqrt{Aq^2 - As + B^2 - 2Bq + s} = \text{constant}. \]
From equation (3.17), we also have \( v_3 \in \text{span} \{ T, \xi_1, ..., \xi_s \} \). The angles between \( v_3 \) and \( T, \xi_1, ..., \xi_s \) are all constants since all the coefficients in equation (3.17) are constants. Then, we can write
\[ v_3 = c_0 T + c_1 \xi_1 + ... + c_s \xi_s \tag{3.18} \]
for some constants \( c_0, ..., c_s \). If we differentiate equation (3.18), we get
\[ -\kappa_2 v_2 + \kappa_3 v_4 = c_0 \kappa_1 v_2 - c_1 \varphi T - ... - c_s \varphi T. \]
Since $\varphi T$ is parallel to $v_2$, if we take the inner product of the last equation with $v_4$, we find $\kappa_3 = 0$. This proves the theorem. \hfill $\square$

In particular, if $\gamma$ is slant, i.e. $\theta_\alpha = \theta$ for all $\alpha = 1, \ldots, s$, then we obtain the following corollary:

**Corollary 1.** If $\theta_\alpha = \theta$, for all $\alpha = 1, \ldots, s$, then

$$A = s \cos^2 \theta, \ B = s \cos \theta, \ V = \cos \theta \sum_{\alpha=1}^{s} \xi_\alpha,$$

$$C = \sqrt{(1 - s \cos^2 \theta) s(1 - q \cos \theta)^2}$$

and $\kappa_2 = \sqrt{s} |1 - q \cos \theta|.$

Now, let us state the following proposition:

**Proposition 1.** Let $\gamma$ be a slant $\varphi$-helix of order 3 in an $S$-manifold $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ with contact angle $\theta$. Then

$$\sum_{\alpha=1}^{s} \xi_\alpha = s \cos \theta T + \rho v_3,$$

where $\rho = g \left( v_3, \sum_{\alpha=1}^{s} \xi_\alpha \right) = s \eta^\alpha(v_3)$ is a real constant such that $\rho^2 = s - s^2 \cos^2 \theta$.

Hence, $\gamma$ has the Frenet frame field

$$\left\{ T, \pm \varphi T/\sqrt{1 - s \cos^2 \theta}, \pm 1/\sqrt{s} \sqrt{1 - s \cos^2 \theta} \left( -s \cos \theta T + \sum_{\alpha=1}^{s} \xi_\alpha \right) \right\}.$$

**Proof.** From the assumption, the Frenet frame field $\{T, v_2, v_3\}$ is $\varphi$-invariant and $\eta^\alpha(T) = \cos \theta$.

Differentiating the last equation along the curve, it is easy to see that

$$\eta^\alpha(v_2) = 0.$$ (3.20)

If we differentiate once again, we have

$$g(\varphi T, v_2) = -\kappa_1 \cos \theta + \kappa_2 \eta^\alpha(v_3),$$ (3.21)

which means the value of $\eta^\alpha(v_3)$ does not depend on $\alpha$.

Firstly, let us assume that $\theta \neq \frac{\pi}{2}$. Since the space spanned by the Frenet frame field is $\varphi$-invariant, then $\varphi^2 T$ is in the set. Using (3.8) and (3.20), we can write

$$\sum_{\alpha=1}^{s} \xi_\alpha \in sp \{T, v_3\},$$

that is,

$$\sum_{\alpha=1}^{s} \xi_\alpha = s \cos \theta T + \rho v_3. \quad (3.22)$$

If we take the norm of both sides, we find $\rho^2 = s - s^2 \cos^2 \theta$. Since the value of $\eta^\alpha(v_3)$ does not depend on $\alpha$, we obtain

$$\rho = g \left( v_3, \sum_{\alpha=1}^{s} \xi_\alpha \right) = s \eta^\alpha(v_3).$$
If we apply $\varphi T$ to (3.22), we get $g(\varphi T, v_3) = 0$. Since $\varphi T \perp T$, $\varphi T \perp v_3$ and $sp \{ T, v_2, v_3 \}$ is $\varphi$-invariant, we have $\varphi T \parallel v_2$. As a result, we find

$$v_2 = \frac{\pm \varphi T}{\|\varphi T\|} = \frac{\pm \varphi T}{\sqrt{1 - s \cos^2 \theta}}.$$  

Now let us consider the Legendre case, i.e., $\theta = \frac{\pi}{2}$. From (3.21), we find

$$\nabla_T g(\varphi T, v_2) = -\kappa_2 g(\varphi T, v_3).$$  

(3.23)

Using (2.1) and (2.3), we calculate

$$\nabla_T \varphi T = (\nabla_T \varphi) T + \varphi \nabla_T T$$  

$$= \sum_{\alpha=1}^{s} \xi_{\alpha} + \kappa_1 \varphi v_2.$$  

(3.24)

Equations (3.23) and (3.25) give us $g(\varphi T, v_3) = 0$, that is, $\varphi T \parallel v_2$. Thus, we have $\varphi T = \pm v_2$. Consequently, the Frenet frame field becomes $\{ T, \pm \varphi T, v_3 \}$. Now, we must show that $v_3$ is parallel to $\sum_{\alpha=1}^{s} \xi_{\alpha}$. Since the space spanned by the Frenet frame field is $\varphi$-invariant, from orthonormal expansion, we can write

$$\varphi v_3 = g(\varphi v_3, T) T + g(\varphi v_3, \pm \varphi T) \varphi T + g(\varphi v_3, v_3) v_3,$$

which reduces to

$$\varphi v_3 = g(\varphi v_3, T) T.$$  

(3.26)

If we apply $\varphi$ to equation (3.26) and use (2.1), we find

$$-v_3 + \sum_{\alpha=1}^{s} \eta^\alpha(v_3) \xi_{\alpha} = g(\varphi v_3, T) \varphi T.$$  

(3.27)

Applying $\varphi T$ to (3.27) and using the Frenet frame field, we have $g(\varphi v_3, T) = 0$. As a result, we get $\varphi v_3 = 0$ and equation (3.27) becomes

$$-v_3 + \sum_{\alpha=1}^{s} \eta^\alpha(v_3) \xi_{\alpha} = 0.$$  

We have already shown that the value of $\eta^\alpha(v_3)$ does not depend on $\alpha$; so, we can write

$$v_3 = \eta^\alpha(v_3) \sum_{\alpha=1}^{s} \xi_{\alpha}.$$  

(3.28)

Since $v_3$ and $\xi_{\alpha}$ are unit for all $\alpha = 1, \ldots, s$, we find $\eta^\alpha(v_3) = \frac{1}{\sqrt{s}}$. Finally, for $\theta = \frac{\pi}{2}$, we have $\rho^2 = s$ and $\sum_{\alpha=1}^{s} \xi_{\alpha} = \rho v_3$, which completes the proof. \hfill \Box

**Corollary 2.** Let $\gamma$ be a Legendre $\varphi$-helix of order 3 in an $S$-manifold $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$. Then $\kappa_2 = \sqrt{s}$, $v_2 = \pm \varphi T$ and $v_3 = \pm \frac{1}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_{\alpha}$. 

Proof. From equation (3.28), we already have

\[ v_3 = \pm \frac{1}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_{\alpha}. \]

If we differentiate this equation and use (3.1), we obtain

\[- \kappa_2 v_2 = \pm \frac{1}{\sqrt{s}} \sum_{\alpha=1}^{s} \nabla_T \xi_{\alpha}. \quad (3.29)\]

Using equations (2.4) and (3.29), we find that \( \kappa_2 = \sqrt{s} \) and \( v_2 = \pm \varphi T. \)

Finally, we can give the following theorem:

**Theorem 3.** Let \( \gamma \) be a slant \( \varphi \)-helix of order \( r \leq 3 \) on an \( S \)-manifold \((M^{2m+s}, \varphi, \xi_{\alpha}, \eta^\alpha, g)\).

Let \( \theta \) denote the contact angle of \( \gamma \). Then we have

i. If \( \cos \theta = \pm \frac{1}{\sqrt{s}} \), then \( \gamma \) is an integral curve of \( \pm \frac{1}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_{\alpha} \), hence it is a normal magnetic curve for \( F_q \) with an arbitrary \( q \).

ii. If \( \cos \theta = 0 \) and \( \kappa_1 \neq 0 \) (i.e. \( \gamma \) is a non-geodesic Legendre curve), then \( \gamma \) is a magnetic curve for \( F_{1,\kappa_1} \).

iii. If \( \cos \theta = \frac{\kappa_1}{\sqrt{\kappa_1^2 + 2}} \), then \( \gamma \) is a magnetic curve for \( F_{\epsilon \sqrt{\kappa_1^2 + 2}} \), where \( \epsilon = -\text{sgn}(g(\varphi T, v_2)) \). In this case, \( \gamma \) is a slant \( \varphi \)-circle.

iv. If \( \cos \theta = \frac{\kappa_1}{\sqrt{\kappa_1^2 + (\sqrt{\kappa_1^2 + 2})^2}} \), then \( \gamma \) is a magnetic curve for \( F_{\frac{\kappa_1}{\sqrt{\kappa_1^2 + (\sqrt{\kappa_1^2 + 2})^2}}} \),

where \( \epsilon = -\text{sgn}(g(\varphi T, v_2)) \) and the sign \( \pm \) corresponds to the sign of \( \eta^\alpha(v_3) \).

v. Except the above cases, \( \gamma \) is not a magnetic curve for any \( F_q \).

Proof. Let \( \cos \theta = \pm \frac{1}{\sqrt{s}} \). Then we have

\[ T = \pm \frac{1}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_{\alpha}, \]

which gives us \( \nabla_T T = 0 \). We also have \( \varphi T = 0 \). So \( \gamma \) satisfies \( \nabla_T T = -q \varphi T \) for any \( q \), which proves i.

Now let \( \cos \theta = 0 \) and \( \kappa_1 \neq 0 \). Using Corollary 2 we have

\[ \nabla_T T = \kappa_1 (\pm \varphi T) = -q \varphi T, \]

which gives us \( q = \pm \kappa_1 \). This completes the proof of ii.

From Proposition 1, we have the Frenet frame field

\[ \left\{ T, \frac{\pm \varphi T}{\sqrt{1 - s \cos^2 \theta}}, \frac{\pm 1}{\sqrt{s} \sqrt{1 - s \cos^2 \theta}} \left( -s \cos \theta T + \sum_{\alpha=1}^{s} \xi_{\alpha} \right) \right\} \]

when \( r = 3 \) and

\[ \left\{ T, \frac{\pm \varphi T}{\sqrt{1 - s \cos^2 \theta}} \right\} \]

when \( r = 2 \). If we differentiate \( v_2 \) along the curve, after some calculations, in both cases, we find

\[ \left( 1 \pm \frac{\kappa_1 \cos \theta}{\sqrt{1 - s \cos^2 \theta}} \right) \left( -s \cos \theta T + \sum_{\alpha=1}^{s} \xi_{\alpha} \right) = \pm \kappa_2 \sqrt{1 - s \cos^2 \theta} v_3, \quad (3.30) \]

(taking \( \kappa_2 = 0 \), when \( r = 2 \)).
Next, let us assume \( \cos \theta = \frac{\varepsilon}{\sqrt{\kappa_1^2 + s}} \), where we denote \( \varepsilon = -\text{sgn}(g(\varphi T, v_2)) \).

Then the left side of equation (3.30) vanishes. Thus we get \( \kappa_2 = 0 \). From the assumption, we also have \( \kappa_1 \) constant, that is, \( \gamma \) is a slant \( \varphi \)-circle. Using the Frenet frame field, we find \( \nabla_T T = -q\varphi T = \kappa_1 v_2 \), where \( q = \varepsilon \sqrt{\kappa_1^2 + s} \). So, we have just completed the proof of iii.

Finally, let us assume \( \cos \theta = \frac{\varepsilon \sqrt{\kappa_1^2 + \kappa_2}}{\sqrt{\kappa_1^2 + (\varepsilon \sqrt{\kappa_1^2 + \kappa_2})^2}} \), where \( \varepsilon = -\text{sgn}(g(\varphi T, v_2)) \) and the sign \( \pm \) corresponds to the sign of \( \eta^\alpha(v_3) \). In this case, let us take \( \kappa_2 \neq 0 \), since we have already investigated order \( r = 2 \). Using the Frenet frame field, after some calculations, we obtain \( \nabla_T T = -q\varphi T = \kappa_1 v_2 \), where \( q = \varepsilon \sqrt{\kappa_1^2 + (\varepsilon \sqrt{\kappa_1^2 + \kappa_2})^2} \).

Hence, the proof of iv is complete.

Since we have considered all cases, we can state that there exist no other slant magnetic \( \varphi \)-helices in \( M \).

From the proof of Theorem 1, we can give the following proposition:

**Proposition 2.** Let \((M^{2n+s}, \varphi, \xi, \eta^\alpha, g)\) be an \( S \)-manifold. There exist no non-geodesic slant \( \varphi \)-circles as magnetic curves corresponding to \( F_q \) for \( 0 < |q| \leq \sqrt{s} \).

Theorem 3 and Proposition 2 generalize Theorem 3.2 and Proposition 3.2 in [10] to \( S \)-manifolds, respectively. Under the condition \( s = 1 \), we obtain their results.

**4. Construction of Slant Normal Magnetic Curves in \( \mathbb{R}^{2n+s}(-3s) \)**

In this section, we find parametric equations of slant normal magnetic curves in \( \mathbb{R}^{2n+s}(-3s) \). As a start, we recall structures defined on this \( S \)-manifold. Let us take \( M = \mathbb{R}^{2n+s} \) with coordinate functions \( \{x_1, ..., x_n, y_1, ..., y_n, z_1, ..., z_s\} \) and define

\[
\xi_\alpha = 2 \frac{\partial}{\partial z_\alpha}, \quad \alpha = 1, ..., s,
\]

\[
\eta^\alpha = \frac{1}{2} \left( dz_\alpha - \sum_{i=1}^{n} y_i dx_i \right), \quad \alpha = 1, ..., s,
\]

\[
\varphi X = \sum_{i=1}^{n} Y_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} X_i \frac{\partial}{\partial y_i} + \left( \sum_{i=1}^{n} Y_i y_i \right) \left( \sum_{\alpha=1}^{s} \frac{\partial}{\partial z_\alpha} \right),
\]

\[
g = \sum_{\alpha=1}^{s} \eta^\alpha \otimes \eta^\alpha + \frac{1}{4} \sum_{i=1}^{n} (dx_i \otimes dx_i + dy_i \otimes dy_i),
\]

where

\[
X = \sum_{i=1}^{n} \left( X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i} \right) + \sum_{\alpha=1}^{s} \left( Z_\alpha \frac{\partial}{\partial z_\alpha} \right) \in \chi(M).
\]

It is well-known that \((\mathbb{R}^{2n+s}, \varphi, \xi, \eta^\alpha, g)\) is an \( S \)-space form with constant \( \varphi \)-sectional curvature \(-3s \). Hence it is denoted by \( \mathbb{R}^{2n+s}(-3s) \) [13]. The following vector fields

\[
X_i = 2 \frac{\partial}{\partial y_i}, \quad X_{n+i} = \varphi X_i = 2(\frac{\partial}{\partial x_i} + y_i \sum_{\alpha=1}^{s} \frac{\partial}{\partial z_\alpha}), \quad \xi_\alpha = 2 \frac{\partial}{\partial z_\alpha}.
\]
form a $g$-orthonormal basis and the Levi-Civita connection is

$$\nabla_{X_j} X_i = \nabla_{X_{n+j}} X_{n+i} = 0, \nabla_{X_i} X_{n+j} = \delta_{ij} \sum_{\alpha=1}^{s} \xi_{\alpha}, \nabla_{X_{n+i}} X_j = -\delta_{ij} \sum_{\alpha=1}^{s} \xi_{\alpha},$$

$$\nabla_{X_j} \xi_{\alpha} = \nabla_{\xi_{\alpha}} X_i = -X_{n+i}, \nabla_{X_{n+i}} \xi_{\alpha} = \nabla_{\xi_{\alpha}} X_{n+i} = X_i.$$

(see [13]). Let $\gamma : I \to \mathbb{R}^{2n+s}(-3s)$ be a unit-speed slant curve with contact angle $\theta$. Let us denote

$$\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t), \gamma_{n+1}(t), \ldots, \gamma_{2n}(t), \gamma_{2n+1}(t), \ldots, \gamma_{2n+s}(t)),$$

where $t$ is the arc-length parameter. Then $\gamma$ has the tangent vector field

$$T = \gamma'_1 \frac{\partial}{\partial x_1} + \ldots + \gamma'_{n} \frac{\partial}{\partial x_n} + \gamma'_{n+1} \frac{\partial}{\partial y_1} + \ldots + \gamma'_{2n} \frac{\partial}{\partial y_{n}} + \gamma'_{2n+1} \frac{\partial}{\partial z_1} + \ldots + \gamma'_{2n+s} \frac{\partial}{\partial z_{s}},$$

which can be written as

$$T = \frac{1}{2} \left[ \gamma'_{n+1} X_1 + \ldots + \gamma'_{2n} X_n + \gamma'_{2n+1} X_{n+1} + \ldots + \gamma'_{2n+s} X_{2n} + \left( \gamma''_{2n+1} - \gamma'_1 \gamma_{n+1} - \ldots - \gamma'_n \gamma_{2n} \right) \xi_1 + \ldots 
+ \left( \gamma''_{2n+s} - \gamma'_1 \gamma_{n+1} - \ldots - \gamma'_n \gamma_{2n} \right) \xi_s \right].$$

Since $\gamma$ is slant curve, we have

$$\eta^\alpha(T) = \frac{1}{2} \left( \gamma'_{2n+1} - \gamma'_1 \gamma_{n+1} - \ldots - \gamma'_n \gamma_{2n} \right) = \cos \theta$$

for all $\alpha = 1, \ldots, s$. So, we obtain

$$\gamma''_{2n+1} = \ldots = \gamma''_{2n+s} = 2 \cos \theta + \gamma'_1 \gamma_{n+1} + \ldots + \gamma'_n \gamma_{2n}. \quad (4.1)$$

Since $\gamma$ is a unit-speed, we can write

$$(\gamma'_1)^2 + \ldots + (\gamma'_{2n})^2 = 4 \left( 1 - s \cos^2 \theta \right). \quad (4.2)$$

These equations were obtained in our paper [12].

Now, our aim is to find parametric equations for slant normal magnetic curves. So, let us assume that $\gamma : I \to \mathbb{R}^{2n+s}(-3s)$ is a normal magnetic curvature. From the Lorentz equation, we have

$$\nabla_{\gamma} T = -q\varphi T, \quad (4.3)$$

where $q \neq 0$ is a constant. Using the Levi-Civita connection, we calculate

$$\nabla_{\gamma} T = \frac{1}{2} \left\{ (\gamma''_{n+1} + 2s \cos \theta \gamma'_1) X_1 + \ldots + (\gamma''_{2n} + 2s \cos \theta \gamma'_n) X_n \right\} \quad (4.4)$$

and

$$\varphi T = \frac{1}{2} \left\{ -\gamma'_1 X_1 - \ldots - \gamma'_n X_n + \gamma'_{n+1} X_{n+1} + \ldots + \gamma'_{2n} X_{2n} \right\} \quad (4.5)$$

From equations [13], (4.4) and (4.5), we have

$$\frac{\gamma''_{n+1} + 2s \cos \theta \gamma'_1}{-\gamma'_1} = \ldots = \frac{\gamma''_{2n} + 2s \cos \theta \gamma'_n}{-\gamma'_n} = \frac{\gamma''_{2n+1}}{-\gamma'_1} = \ldots = \frac{\gamma''_{2n+s}}{-\gamma'_n} = -q,$$

$$\frac{-\gamma'_1}{\gamma'_{n+1}} = \ldots = \frac{-\gamma'_n}{\gamma'_{2n}} = -q,$$
which is equivalent to
\[
\frac{\gamma''_{n+1}}{-\gamma_1} = \ldots = \frac{\gamma''_2}{-\gamma_n} = \frac{\gamma''_1}{\gamma'_{n+1}} = \ldots = \frac{\gamma''_n}{\gamma'_{2n}} = \lambda, \tag{4.6}
\]
where we denote \( \lambda = -q + 2s \cos \theta \). Firstly, let us assume \( \lambda \neq 0 \). From equation \( \text{[4.6]} \), if we select pairs
\[
\frac{\gamma''_{n+1}}{-\gamma_1} = \ldots = \frac{\gamma''_2}{-\gamma_n} = \gamma''_1 \gamma'_n + 1 = \gamma''_n \gamma'_2, \]
solving ODEs, we have
\[
(\gamma'_1)^2 + (\gamma'_{n+1})^2 = c_1^2, \ldots, (\gamma'_n)^2 + (\gamma'_{2n})^2 = c_n^2,
\]
where \( c_1, \ldots, c_n \) are arbitrary constants. Thus, we can write
\[
\gamma'_1 = c_1 \cos f_1, \ldots, \gamma'_n = c_n \cos f_n, \tag{4.7}
\]
where \( f_1, \ldots, f_n \) are differentiable functions on \( I \). From \( \text{[4.6]} \) and \( \text{[4.7]} \), we find
\[
f'_1 = \ldots = f'_n = -\lambda,
\]
which gives us
\[
f_i = -\lambda t + a_i, \quad i = 1, 2, \ldots, n
\]
where \( a_1, \ldots, a_n \) are arbitrary constants. Now, if we integrate \( \text{[4.7]} \), we have
\[
\begin{align*}
\gamma_1 &= c_1 \sin f_1 + b_1, \ldots, \\
\gamma_n &= c_n \sin f_n + b_n, \\
\gamma_{n+1} &= c_1 \cos f_1 + d_1, \ldots, \\
\gamma_{2n} &= c_n \cos f_n + d_n,
\end{align*}
\]
where \( b_i \) and \( d_i \) are arbitrary constants \( (i = 1, \ldots, n) \). Thus, we get
\[
\gamma'_1 \gamma_{n+1} + \ldots + \gamma'_n \gamma_{2n} = \sum_{i=1}^{n} \left( \frac{c_i^2}{\lambda} \cos^2 f_i + c_i d_i \cos f_i \right).
\]
Using the last equation with \( \text{[4.13]} \), we obtain
\[
\gamma'_{2n+\alpha} = 2 \cos \theta + \sum_{i=1}^{n} \left( \frac{c_i^2}{\lambda} \cos^2 f_i + c_i d_i \cos f_i \right),
\]
where \( \alpha = 1, \ldots, s \). If we integrate this last equation, we find
\[
\gamma_{2n+\alpha} = 2t \cos \theta - \sum_{i=1}^{n} \left\{ \frac{c_i^2}{4 \lambda^2} \left[ \sin (2f_i) + 2f_i \right] + c_i d_i \lambda \sin f_i \right\} + h_\alpha,
\]
for \( \alpha = 1, \ldots, s \) and \( h_1, \ldots, h_s \) are arbitrary constants. Moreover, from \( \text{[4.2]} \) and \( \text{[4.7]} \), we have
\[
c_1^2 + \ldots + c_n^2 = 4 \left( 1 - s \cos^2 \theta \right). \tag{4.8}
\]
Thus, we have just finished the case \( \lambda \neq 0 \).

Secondly, let \( \lambda = 0 \). In this case, we have
\[
\gamma''_1 = \gamma''_2 = \ldots = \gamma''_{2n} = 0,
\]
which gives us
\[
\gamma_i = c_i t + d_i,
\]
for \( i = 1, \ldots, 2n \), where \( c_i \) and \( d_i \) are arbitrary constants. Using the last equation, we calculate
\[
\gamma_1' \gamma_{n+1} + \cdots + \gamma_n' \gamma_{2n} = \sum_{i=1}^{n} c_i (c_{n+i} t + d_{n+i}).
\]
So, equation (4.1) becomes
\[
\gamma_{2n+\alpha}' = 2 \cos \theta + \sum_{i=1}^{n} c_i (c_{n+i} t^2 + d_{n+i}),
\]
which gives us
\[
\gamma_{2n+\alpha}' = 2 t \cos \theta + \sum_{i=1}^{n} c_i \left( \frac{c_{n+i}}{2} t^2 + d_{n+i} \right) + h_\alpha,
\]
where \( h_\alpha \) are arbitrary constants for \( \alpha = 1, \ldots, s \). Since \( \gamma \) is unit-speed, from (4.2), we have
\[
c_i^2 + \cdots + c_{2n}^2 = 4 (1 - s \cos^2 \theta).
\]
To sum up, we give the following Theorem:

**Theorem 4.** The slant normal magnetic curves on \( \mathbb{R}^{2n}+(-3s) \) satisfying the Lorentz equation \( \nabla_T T = -q \phi T \) have the parametric equations

a) \[
\begin{align*}
\gamma_i(t) &= \frac{c_i}{\lambda} \sin f_i(t) + b_i, \\
\gamma_{n+i}(t) &= \frac{c_i}{\lambda} \cos f_i(t) + d_i, \\
\gamma_{2n+\alpha}(t) &= 2 t \cos \theta - \sum_{i=1}^{n} \left\{ \frac{c_i^2}{4 \lambda^2} [\sin (2 f_i(t)) + 2 f_i(t)] + \frac{c_i d_i}{\lambda} \sin f_i(t) \right\} + h_\alpha,
\end{align*}
\]

where \( f_i(t) = -\lambda t + a_i \), \( \alpha = 1, \ldots, s \), \( i = 1, 2, \ldots, n \), \( \lambda = -q + 2s \cos \theta \neq 0 \)

where \( a_i, b_i, c_i, d_i \) and \( h_\alpha \) are arbitrary constants such that \( c_i \) satisfies
\[
c_i^2 + \cdots + c_{2n}^2 = 4 (1 - s \cos^2 \theta);
\]

or

b) \[
\begin{align*}
\gamma_i(t) &= c_i t + d_i, \\
\gamma_{2n+\alpha}(t) &= 2 t \cos \theta + \sum_{i=1}^{n} c_i \left( \frac{c_{n+i}}{2} t^2 + d_{n+i} \right) + h_\alpha,
\end{align*}
\]

where \( c_i, d_i \) and \( h_\alpha \) are arbitrary constants such that \( c_i \) satisfies
\[
c_i^2 + \cdots + c_{2n}^2 = 4 (1 - s \cos^2 \theta).
\]

In both cases, \( q \neq 0 \) is a constant and \( \theta \) denotes the constant contact angle satisfying \( |\cos \theta| \leq \frac{1}{\sqrt{s}} \).

In particular, if \( s = 1 \), we obtain Theorem 3.5 in [10].

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