SLIDING METHOD FOR THE SEMI-LINEAR ELLIPTIC EQUATIONS INVOLVING THE UNIFORMLY ELLIPTIC NONLOCAL OPERATORS

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(Communicated by Congming Li)

Abstract. In this paper, we consider the uniformly elliptic nonlocal operators

\[ A_\alpha u(x) = C_{n,\alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{a(x-y)(u(x)-u(y))}{|x-y|^{n+\alpha}} dy, \]

where \( a(x) \) is positively uniform bounded satisfying a cylindrical condition. We first establish the narrow region principle in the bounded domain. Then using the sliding method, we obtain the monotonicity of solutions for the semi-linear equation involving \( A_\alpha \) in both the bounded domain and the whole space. In addition, we establish the maximum principle in the unbounded domain and get the non-existence of solutions in the upper half space \( \mathbb{R}^n_+ \).

1. Introduction. In this paper, we consider some semi-linear equations involving the uniformly elliptic nonlocal operators

\[ A_\alpha u(x) = f(u), \quad (1.1) \]

with

\[ A_\alpha u(x) := C_{n,\alpha} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \frac{a(x-y)(u(x)-u(y))}{|x-y|^{n+\alpha}} dy \]

where \( 0 < \alpha < 2 \), and the weight function \( a : \mathbb{R}^n \to \mathbb{R} \), \( n \geq 3 \), satisfies the following conditions,

\( A_1 \): (Uniform boundedness) \( 0 < m \leq a(x) \leq M < \infty \), \( \forall x \in \mathbb{R}^n \).

\( A_2 \): (Cylindrical condition) For any \( \tau > 0 \) and \( x = (x',x_n), y = (y',x_n) \in \mathbb{R}^n \), if \( |x'| = |y'| \), then \( a(x',x_n + \tau) = a(y',x_n) \).

2020 Mathematics Subject Classification. Primary: 35R11; Secondary: 35J61.
Key words and phrases. Uniformly elliptic nonlocal operator, sliding method, monotonicity.

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Roughly speaking, the cylindrical condition \((A_2)\) is equivalent to say the function \(a\) has the same value on the surface of any cylinder centering the \(x_n\)-axis. If we assume that \(u \in C^{1,1}_{\text{loc}} \cap L_\alpha(\mathbb{R}^n)\), where \(L_\alpha\) is defined as
\[
L_\alpha(\mathbb{R}^n) = \left\{ u \in L^1_{\text{loc}} \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} dx < \infty \right\},
\]
then the above integral is well defined. Also, one can refer to Appendix for more details. The operator \(A_\alpha\) arises from the stochastic control problems and stochastic games (see \([7]\)). Using the compactness and perturbation method, Caffarelli and Silvestre \([8]\) obtained the \(C^{1,\alpha}_{\text{loc}}\) regularity result for the purely nonlocal Isaacs equations, which is similar to the equation (1.1), where \(a(x,y)\) satisfies \((A_1)\), \(\|
abla_y a\| \leq C|y|^{-1}\), and \(a(x,y)\) is continuous in \(x\) for a modulus of continuity independent of \(y\). In this paper, we want to study the semi-linear equation (1.1), under the conditions \((A_1)-(A_2)\), without the decay and regular properties on \(a(x)\).

If \(a(x) = 1\), then the operator \(A_\alpha\) becomes the traditional fractional Laplacian operator \((-\Delta)^{\frac{\alpha}{2}}\). The difficulty of \((-\Delta)^{\frac{\alpha}{2}}\) can be overcome by using the extension method introduced by Caffarelli and Silvestre \([6]\) that reduces the nonlocal problem into a local one in the higher dimensions. One can also use the integral equations method, like the method of moving planes in integral forms and regularity lifting to investigate some properties of the equation that involves the fractional Laplacian operator, see \([9, 12, 14, 15, 17]\) and the reference therein.

However, it seems that the extension method and the integral equations method don’t work for more general nonlocal operators like \(A_\alpha\), the fractional \(p\)-Laplacian, etc., see \([10, 16, 23]\). Recently, Chen-Li-Li \([13]\) systemically developed a so-called \textit{direct moving plane method} which can handle \((-\Delta)^{\frac{\alpha}{2}}\) directly. Heuristically, this method can be used to handle general nonlocal operators. A series of fruitful results have been obtained, please see \([10, 11, 19, 20]\) and the reference therein. Based on this method, Tang \([23]\) investigated the radial symmetry, the monotonicity and non-existence of the solutions of the equations involving \(A_\alpha\), where the weight function \(a(x)\) should be added in the radial and monotonic conditions,

\[
a(x) = a(|x|), \text{ for all } x \in \mathbb{R}; \quad a(x) \geq a(y), \text{ for any } |x| \leq |y|.
\]  

(1.2)

Note that the condition (1.2) is a crucial part in \([23]\), we need a new method to investigate the monotonicity and non-existence of the solutions to the equation (1.1) without the radial and monotonic conditions (1.2).

In this paper, we use a direct sliding method for the uniformly elliptic nonlocal operators. The sliding method was introduced by Berestycki and Nirenberg \([3, 4, 5]\) and has been used successfully to obtain the symmetry and monotonicity of solutions for several kind of second-order elliptic equations, see \([1, 2]\) and the reference therein. Recently, Wu-Chen \([24]\) developed the sliding method for the fractional Laplacian equation. They established the narrow region principle in the bounded domain and maximum principle in the unbounded domain, eventually obtaining the monotonicity of solutions of some equations involving the fractional Laplacian operator. This method is also applied to investigate equations involving the fractional \(p\)-Laplacian \([21, 22, 25]\) and the nonlocal Monge-Ampère operator \([18]\). Based on the sliding method in \([24]\), we investigate the monotonicity of the solutions for the semi-linear equation (1.1) involving the uniformly elliptic nonlocal operator \(A_\alpha\).

We mainly consider the monotonicity of the solutions to the equation (1.1) in two different regions, the bounded domain and the whole space.
The following narrow region principle in the bounded domain is a key ingredient in the sliding method and it provides a starting position to slide the domain for Theorem 1.2. Here, we denote

\[ x = (x', x_n) \]  
\[ u^\tau(x) = u(x', x_n + \tau), \quad \text{and} \quad w^\tau(x) = u^\tau(x) - u(x), \quad \tau \in \mathbb{R}. \]

**Theorem 1.1** (Narrow region principle). Let \( D \) be a bounded narrow region in \( \mathbb{R}^n \). Suppose that \( u \in L_0(\mathbb{R}^n) \cap C^{1,1}_{\text{loc}}(D) \), \( w^\tau \) is lower semi-continuous on \( D \), and satisfies

\[
\begin{align*}
A_\alpha w^\tau(x) + c(x)w^\tau(x) &\geq 0 \quad \text{in } D, \\
\left. w^\tau(x) \right|_{\partial D} &\geq 0 \quad \text{in } D^c,
\end{align*}
\]

with \( c(x) \) bounded from below in \( D \). Let \( d_n(D) \) be the width of \( D \) in the \( x_n \) direction, in which we assume that \( D \) in narrow, i.e. there exists \( C \), such that

\[
d_n(D) \leq \frac{C}{\| \inf_D c(x) \|^\frac{1}{1}}. \tag{1.4}
\]

Under the conditions \( 0 < \alpha < 2 \) and \((A_1)-(A_2)\), then

\[
w^\tau(x) \geq 0 \quad \text{in } D. \tag{1.5}
\]

Moreover, we have

\[
either \ w^\tau(x) > 0 \quad \text{in } D \quad \text{or} \quad w^\tau(x) \equiv 0 \quad \text{in } \mathbb{R}^n. \tag{1.6}
\]

Now we use the sliding method to get the following result.

**Theorem 1.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), which is convex in \( x_n \)-direction. Assume that \( u \in L_0(\mathbb{R}^n) \cap C^{1,1}_{\text{loc}}(\Omega) \) is a solution of

\[
\begin{align*}
A_\alpha u(x) &= f(u(x)), \quad x \in \Omega, \\
u(x) &= \varphi(x), \quad x \in \mathbb{R}^n \setminus \Omega,
\end{align*}
\]

where \( \varphi(x) \) satisfies:

For any three points \( x = (x', x_n), y = (x', y_n), z = (x', z_n) \) lying on a segment parallel to the \( x_n \)-axis, \( y_n < x_n < z_n \), with \( y, z \in \Omega^c \), we have

\[
\varphi(y) < u(x) < \varphi(z), \quad \text{if } x \in \Omega \tag{1.8}
\]

and

\[
\varphi(y) \leq \varphi(x) \leq \varphi(z), \quad \text{if } x \in \Omega^c. \tag{1.9}
\]

The function \( f \) is supposed to be Lipschitz continuous. Under the conditions \( 0 < \alpha < 2 \) and \((A_1)-(A_2)\), then \( u \) is monotone increasing with respect to \( x_n \) in \( \Omega \), i.e. for any \( \tau > 0 \),

\[
u(x', x_n + \tau) > u(x', x_n), \quad \text{for all } (x', x_n + \tau), \ (x', x_n) \in \Omega. \tag{1.10}
\]

To be continuous, as an application of the sliding method, we derive the monotonicity of solutions for the semi-linear equation (1.1) with the uniformly elliptic nonlocal operator \( A_\alpha \) in the whole space.

**Theorem 1.3.** Let \( u \in L_0(\mathbb{R}^n) \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n) \) be a solution of

\[
A_\alpha u(x) = f(u(x)), \quad x \in \mathbb{R}^n, \tag{1.11}
\]

\[
|u(x)| \leq 1,
\]

and

\[
\lim_{x_n \to \pm \infty} u(x', x_n) = \pm 1, \quad \text{uniformly in } x' \in \mathbb{R}^{n-1}. \tag{1.12}
\]
Assume that \(f(\cdot)\) is continuous in \([-1,1]\) and there exists \(\delta > 0\) such that
\[
f \text{ is nonincreasing on } [-1,-1+\delta] \text{ and on } [1-\delta,1]. \tag{1.13}
\]
If \(u \in C^\theta(\mathbb{R}^n)\), for some \(\theta \in (0,1)\), then \(u\) is monotone increasing with respect to \(x_n\), and furthermore, it depends on \(x_n\) only.

Theorem 1.3 is closely related to the De Giorgi conjecture, but it is in the classical Laplacian sense. For more concrete details, please refer to Remark 3 in [24].

Furthermore, we give the nonexistence of the solutions in the upper half space by establishing the so-called maximum principle.

**Theorem 1.4** (Maximum principle in unbounded domains). Let \(D\) be an open set in \(\mathbb{R}^n\), possibly unbounded and disconnected, suppose that
\[
\lim_{j \to \infty} \frac{|(B_{2^j+1}(x) \setminus B_{2^j}(x)) \cap D^c|}{|B_{2^j+1}(x) \setminus B_{2^j}(x)|} \geq c_0 \tag{1.14}
\]
for some \(c_0 > 0\). Suppose \(u \in L_\alpha(\mathbb{R}^n) \cap C^{1,1}_{loc}(\mathbb{R}^n) \cap D\), bounded from above, and satisfies
\[
\begin{cases}
A_\alpha u(x) + c(x)u(x) \leq 0, & x \in D, \text{ for some points } u(x) > 0 \\
u(x) \leq 0, & x \in \mathbb{R}^n \setminus D,
\end{cases} \tag{1.15}
\]
for some nonnegative function \(c(x)\), then we have
\[
u(x) \leq 0 \text{ in } D. \tag{1.16}
\]

It is easy to check that the half-space \(\mathbb{R}^n_+\) satisfy the condition (1.14). It turns out that Theorem 1.4 can be applied to derive the non-existence of the solution of the equation (1.1) in the half-space.

**Corollary 1.1.** Let \(u \in L_\alpha(\mathbb{R}^n) \cap C^{1,1}_{loc}(\mathbb{R}^n_+)\) be a non-negative solution of
\[
\begin{cases}
A_\alpha u = f(u), & \text{at the point in } \mathbb{R}^n_+ \text{ where } u(x) > 0 \\
u = 0, & x \notin \mathbb{R}^n_+,
\end{cases}
\]
where the function \(f(\cdot)\) is continuous, non-increasing and \(f(0) = 0\). Assume that
\[
\lim_{x_n \to +\infty} u(x', x_n) = 0, \text{ uniformly in } x' \in \mathbb{R}^{n-1},
\]
then we have that \(u \equiv 0\).

We would like to mention that the unbounded domain \(D\) founded in Theorem 1.4 has many different shapes. We here only list the following two simple examples, \(D_1 = \{x \in [2i < x_n < 2i + 1, i = 0, \pm 1, \pm 2, \cdots]\}\), \(D_2 = \{x \in [2i < |x| < 2i + 1, i = 0, 1, 2, \cdots]\}\). For more considerations, one can refer inspirations in [24] (Remark 2).

Finally, we remark that the method so called Poisson integral representation of subharmonic functions, which used in [24], can derive the maximum principle for the semi-linear equation with the fractional Laplacian operator. However, this method is not suitable for \(A_\alpha\), due to the complexity of the weighted function. To overcome this difficulty, using the strategy of Liu and Chen [21], we get the maximum principle for the semi-linear equation involving the uniformly elliptic nonlocal operator \(A_\alpha\), see the proof of Theorem 1.4.

**Organization of the paper:** In Section 2, we deal with the narrow region principle in the bounded domain, and prove Theorem 1.2 by the sliding method. In Section 3, applying the sliding method and compactness method, we derive the monotonicity of solutions and hence complete the proof of Theorem 1.3. Section 4 is devoted to the maximum principle in the unbounded domain and its application.
Last but not least, we use the constant $C$, or $c$ (among others) to denote a general positive constant which may depend only on $n, \alpha$ and whose value may differ from line to line.

2. Proof of Theorems 1.1 and 1.2. In this section, we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Suppose (1.5) is not valid, then the lower semi-continuity of $w^\tau(x)$ in $\overline{D}$ implies that there exists a point $x_0 \in D$ such that
\[
    w^\tau(x_0) = \min_{D} w^\tau(x) < 0.
\]
By (1.3) and the condition $(A_1)$, we have
\[
A_{\alpha}(w_\tau(x_0)) + c(x_0)w^\tau(x_0)
= C_{n, \alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{a(x_0 - y)(w^\tau(x_0) - w^\tau(y))}{|x_0 - y|^{n+\alpha}} dy + c(x_0)w^\tau(x_0)
\leq C_{n, \alpha} \int_{D^c} \frac{a(x_0 - y)(w^\tau(x_0) - w^\tau(y))}{|x_0 - y|^{n+\alpha}} dy + c(x_0)w^\tau(x_0)
\leq C_{n, \alpha} m w^\tau(x_0) \int_{D^c} \frac{1}{|x_0 - y|^{n+\alpha}} dy + \inf_{D} c(x)w^\tau(x_0)
\leq w^\tau(x_0) \left( \frac{C}{d_n(D)^\alpha} + \inf_{D} c(x) \right)
< 0,
\]
where $d_n(D)$ denotes the width of $D$ in the $x_n$ direction, and more details are added to the second inequality from [13]. The inequality (2.1) contradicts (1.3), and we thus conclude that
\[
w^\tau(x) \geq 0, \quad x \in D.
\]
If there exists some point $x \in D$, such that $w^\tau(x) = 0$, then $x$ is a minimum point of $w^\tau(x)$ in $D$. If $w^\tau(x) \neq 0$ in $\mathbb{R}^n$, then we have
\[
A_{\alpha}w^\tau(x) = C_{n, \alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{a(x - y)(w^\tau(x) - w^\tau(y))}{|x - y|^{n+\alpha}} dy
\leq m C_{n, \alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{-w^\tau(y)}{|x - y|^{n+\alpha}} dy
< 0.
\]
This contradicts
\[
A_{\alpha}w^\tau(x) + c(x)w^\tau(x) \geq 0.
\]
Therefore, we obtain (1.6), which completes the proof of Theorem 1.1. \qed

Applying Theorem 1.1, we get the monotonicity result in Theorem 1.2. To better understand how the sliding method works, we only present the proofs for $\Omega$ when it is an ellipsoid or a rectangle. If necessary, one can refer to Fig.1 and Fig.2 in [25] for more intuitive understanding. When $\Omega$ is an arbitrary bounded domain of $\mathbb{R}^n$ which is convex in $x_n$-direction, i.e. for any $(x', x_n), (x', \bar{x}_n) \in \Omega$ imply that $(x', tx_n + (1 - t)\bar{x}_n) \in \Omega$ for $0 < t < 1$, the proof is entirely similar.
Proof of Theorem 1.2. For $\tau \geq 0$, we denote 
$$u^\tau(x) = u(x^\tau) = u(x', x_n + \tau).$$

Let $\Omega^\tau = \Omega - \tau e_n$, where $e_n = (0, \cdots, 0, 1)$. Set 
$$D^\tau := \Omega^\tau \cap \Omega,$$
$$\hat{\tau} = \sup\{\tau|\tau > 0, D^\tau \neq \emptyset\},$$
and 
$$w^\tau(x) = u^\tau(x) - u(x), \ x \in D^\tau.$$ 

By the fact that $x^\tau - y^\tau = x - y$, and the cylindrical condition of $a$ in $(A_2)$, we have 
$$A_n(u^\tau(x)) = C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{a(x - y)(u^\tau(x) - u^\tau(y))}{|x - y|^{n+\alpha}} dy$$
$$= C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{a(x - y)(u(x^\tau) - u(y))}{|x^\tau - y^\tau|^{n+\alpha}} dy$$
$$= C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{a(x - y)(u(x^\tau) - u(y))}{|x^\tau - y|^{n+\alpha}} dy$$
$$= C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{a(x^\tau - y)(u(x^\tau) - u(y))}{|x^\tau - y|^{n+\alpha}} dy$$
$$= f(u^\tau(x)).$$ 

By (2.2) and $A_n(u(x)) = f(u(x)), x \in \Omega$, we derive that $w^\tau$ satisfies 
$$A_n(w^\tau(x)) = c^\tau(x) w^\tau(x) \text{ in } D^\tau, \quad (2.3)$$
where 
$$c^\tau(x) = \frac{f(u^\tau(x)) - f(u(x))}{u^\tau(x) - u(x)} \leq C, \ \forall x \in D^\tau.$$ 

**Step 1.** We show that $w^\tau(x) \geq 0$ for $\tau$ sufficiently close to $\hat{\tau}$ when $D^\tau$ is narrow. And this can be derived directly from Theorem 1.1.

**Step 2.** Step 1 provides a starting point, from which we can carry out the sliding. Now we decrease $\tau$ as long as inequality $w^\tau(x) \geq 0$ holds to its limiting position. Define 
$$\tau_0 = \inf\{\tau|w^\tau \geq 0, x \in D^\tau; 0 < \tau < \hat{\tau}\}.$$ 

Claim 
$$\tau_0 = 0.$$ 

Otherwise, suppose that $\tau_0 > 0$, we show that the domain $\Omega^\tau$ can slide upward a little bit more and still have 
$$w^\tau(x) \geq 0, \ x \in D^\tau, \ \text{for any } \tau_0 - \epsilon < \tau \leq \tau_0, \quad (2.4)$$
which contradicts the definition of $\tau_0$.

By the continuity of $w^\tau(x)$, we have $w^{\tau_0}(x) \geq 0, x \in D^{\tau_0}$, and $w^{\tau_0}(x) > 0, x \in \Omega \cap \partial D^{\tau_0}$, then 
$$w^{\tau_0}(x) \neq 0, \ x \in D^{\tau_0}.$$ 

If there exists a point $x$ such that $w^{\tau_0}(x) = 0$, then $x$ is a minimum point and 
$$A_n w^{\tau_0}(x) = C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{a(x - y)(w^{\tau_0}(x) - w^{\tau_0}(y))}{|x - y|^{n+\alpha}} dy < 0.$$ 
This contradicts 
$$A_n w^{\tau_0}(x) = f(w^{\tau_0}(x)) - f(u(x)) = 0.$$
Therefore,
\[ w^{\tau_0}(x) > 0, \quad x \in D^{\tau_0}. \quad (2.5) \]

Now we can carve out of \( D^{\tau_0} \) a closed set \( K \subset D^{\tau_0} \) such that \( D^{\tau_0} \setminus K \) is narrow.

By (2.5), we get
\[ w^{\tau_0}(x) \geq C_0 > 0 \text{ in } K. \]

From the continuity of \( w^\tau(x) \) in terms of \( \tau \), we have, for small \( \varepsilon > 0 \),
\[ w^{\tau_0 - \varepsilon}(x) \geq 0 \text{ in } K. \]

In addition, we obtain from (1.8) and (1.9) that
\[ w^{\tau_0 - \varepsilon}(x) \geq 0 \text{ in } (D^{\tau_0} - \varepsilon) \c. \]

Since \((D^{\tau_0} - \varepsilon) \c = (D^{\tau_0} - \varepsilon) \cup K\), then
\[
\begin{aligned}
A_\alpha w^{\tau_0 - \varepsilon}(x) + c^{\tau_0 - \varepsilon}(x) w^{\tau_0 - \varepsilon}(x) &\geq 0, \quad \text{in } D^{\tau_0 - \varepsilon} \setminus K, \\
w^{\tau_0 - \varepsilon}(x) &\geq 0, \quad \text{in } (D^{\tau_0 - \varepsilon}) \c.
\end{aligned}
\quad (2.6)
\]

It follows from Theorem 1.1 that (2.4) holds. Therefore, we have reached a contradiction and
\[ w^\tau(x) \geq 0, \quad x \in D^\tau, \quad \text{for any } 0 < \tau < \tilde\tau. \quad (2.7) \]

Since \( w^\tau(x) \not\equiv 0, x \in D^\tau, \text{ for any } 0 < \tau < \tilde\tau, \)
if there exists a point \( x^0 \) such that \( w^\tau(x^0) = 0 \), then \( x^0 \) is a minimum point and
\[ A_\alpha w^\tau(x^0) = C_{n, \alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{a(x^0 - y)(w^\tau(x^0) - w^\tau(y))}{|x^0 - y|^{n+\alpha}} dy < 0. \]

This contradicts
\[ A_\alpha w^\tau(x) = f(u^\tau(x)) - f(u(x)) = 0. \]

Hence, we arrived at
\[ w^\tau(x) > 0, \quad x \in D^\tau, \quad \text{for any } 0 < \tau < \tilde\tau, \]
which means that \( u \) is strictly increasing in the \( x_n \) direction. This completes the proof of Theorem 1.2.

3. Proof of Theorem 1.3.

Proof of Theorem 1.3. To begin with, we obtain from (1.12) that there exists a constant \( b > 0 \) such that
\[ u(x', x_n) \geq 1 - \delta, \quad \text{for } x_n \geq b \]
and
\[ u(x', x_n) \leq -1 + \delta, \quad \text{for } x_n \leq -b. \]

For any \( \tau \geq 2b \), no matter where \( x \) is, we have either
\[ u^\tau(x', x_n) \geq 1 - \delta, \quad \text{for } x_n \geq -b, \quad (3.1) \]
or
\[ u(x', x_n) \leq -1 + \delta, \quad \text{for } x_n \leq -b. \quad (3.2) \]

We divide the proof into three steps.

Step 1. Let
\[ W^\tau(x) = u(x) - u^\tau(x). \]

Claim
\[ W^\tau(x) \leq 0 \quad \text{in } \mathbb{R}^n, \quad (3.3) \]
for \( \tau \geq 2b \). Suppose (3.3) is not valid, then
\[
0 < A := \sup_{x \in \mathbb{R}^n} W^\tau(x),
\]
then there exists a sequence \( \{x^k\}_{k=1}^\infty \subset \mathbb{R}^n \) such that
\[
W^\tau(x^k) \to A, \quad \text{as } k \to \infty.
\] (3.4)
Denote \( x_n^k \) the \( n \)-th component of \( x^k \). We claim that \( \{x_n^k\}_{k=1}^\infty \) is bounded, i.e., there exists \( M > 0 \) such that
\[
|x_n^k| \leq M.
\]
In fact, suppose not, there exists a subsequence \( \{x_n^k\}_{k=1}^\infty \subset \mathbb{R}^n \) such that \( x_n^k \to +\infty \) or \( x_n^k \to -\infty \). And we have
\[
W^\tau(x^k) = u(x^k) - u^\tau(x^k) \to 0 \quad \text{as } x_n^k \to \pm\infty.
\] This contradicts (3.4). Thus, we get \( |x_n^k| \leq M \).
Let
\[
\Psi(x) = \begin{cases} 
\frac{1}{2^{1-1}}, & |x| < 1, \\
0, & |x| \geq 1.
\end{cases}
\] (5.5)
It is easy to check that \( \Psi(0) = \max_{x \in \mathbb{R}^n} \Psi(x) = 1 \).
Set
\[
\Psi_k(x) = \Psi(x - x^k).
\]
By (3.4), there exists a sequence \( \{\varepsilon_k\} \), with \( \varepsilon_k \to 0 \) such that
\[
W^\tau(x^k) + \varepsilon_k \Psi_k(x^k) > A.
\]
Since for any \( x \in \mathbb{R}^n \setminus B_1(x^k) \), \( W^\tau(x) \leq A \) and \( \Psi_k(x) = 0 \), hence
\[
W^\tau(x^k) + \varepsilon_k \Psi_k(x^k) > W^\tau(x) + \varepsilon_k \Psi_k(x), \quad \text{for any } x \in \mathbb{R}^n \setminus B_1(x^k).
\] (3.6)
It follows that there exists a point \( \tilde{x}^k \in B_1(x^k) \) such that
\[
W^\tau(\tilde{x}^k) + \varepsilon_k \Psi_k(\tilde{x}^k) = \max_{x \in B_1(x^k)} (W^\tau(x) + \varepsilon_k \Psi_k(x)).
\] (3.7)
Combining (3.6) and (3.7), then \( x^k \) is a maximum point of the function \( W^\tau(x) + \varepsilon_k \Psi_k(x) \) in \( \mathbb{R}^n \).
Since \( \tau \geq 2b \), no matter where \( \tilde{x}^k \) is, one of the points \( \tilde{x}^k \) and \( \tilde{x}^k + (0', \tau) \) is in the domain \( \{x : |x_n| \geq b\} \) where \( f(u(x)) \) is non-increasing. This can be seen from (3.1) and (3.2). Hence
\[
f(u(\tilde{x}^k)) - f(u^\tau(\tilde{x}^k)) \leq 0,
\]
since \( u(\tilde{x}^k) > u^\tau(\tilde{x}^k) \). Therefore, by Proposition 5.1, we obtain
\[
A_\alpha(W^\tau + \varepsilon_k \Psi_k)(\tilde{x}^k) = f(u(\tilde{x}^k)) - f(u^\tau(\tilde{x}^k)) + C\varepsilon_k \leq C\varepsilon_k.
\] (3.8)
It follows from (3.8) and the uniformly boundedness of \( a \) in (A1), we have
\[
C\varepsilon_k \geq A_\alpha(W^\tau + \varepsilon_k \Psi_k)(\tilde{x}^k)
= C_{n,\alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{a(\tilde{x}^k - y)(W^\tau(\tilde{x}^k) + \varepsilon_k \Psi_k(\tilde{x}^k) - W^\tau(y) - \varepsilon_k \Psi_k(y))}{|\tilde{x}^k - y|^{n+\alpha}} dy
\geq C \int_{B_\varepsilon(\tilde{x}^k)} \frac{W^\tau(\tilde{x}^k) + \varepsilon_k \Psi_k(\tilde{x}^k) - W^\tau(y)}{|\tilde{x}^k - y|^{n+\alpha}} dy
\geq C \int_{B_\varepsilon(\tilde{x}^k)} \frac{A - W^\tau(y)}{|\tilde{x}^k - y|^{n+\alpha}} dy
= C \int_{B_\varepsilon(0)} \frac{A - W^\tau(z + x^k)}{|z|^{n+\alpha}} dz,
\] (3.9)
where the last inequality holds due to the fact that $|x^k - y| \leq |x^k - x^k| + |x^k - y| \leq \frac{3}{2}|x^k - y|$.

Denote

$$u_k(x) = u(x + x^k)$$
and $W_k^k(x) = W^k(x + x^k)$.

Because of $u \in C^\theta(\mathbb{R}^n)$, for some $\theta \in (0, 1)$, for any compact set $K \subset \mathbb{R}^n$, by Arzelà-Ascoli theorem, there exists a subsequence, such that

$$u_k(x) \to u_\infty(x),$$
as $k \to \infty$, uniformly in $K$.

It follows

$$W_k^k(x) \to u_\infty(x) - u_\infty^*(x), \quad x \in B_2^n(0).$$

On the other hand, in (3.9), letting $k \to \infty$, one has

$$W_k^k(x) \to A, \quad x \in B_2^n(0).$$

By the equations (3.10) and (3.11), we have

$$u_\infty(x) - u_\infty^*(x) \equiv A, \quad x \in B_2^n(0).$$

Since the sequence $\{x_n^k\}_{k=1}^\infty$ is bounded, we obtain from (1.12) that

$$u_\infty(x', x_n) \xrightarrow{n \to \infty} \pm 1, \quad \text{uniformly in } \mathbb{R}^{n-1} \text{ with respect to } x'.$$

Therefore,

$$u_\infty(x', x_n) = A + u_\infty(x', x_n + \tau) = 2A + u_\infty(x', x_n + 2\tau) = \cdots = kA + u_\infty(x', x_n + k\tau),$$

for any $k \in \mathbb{N}$. Now take $x_n$ sufficiently negative and $k$ sufficiently large, hence $u_\infty(x', x_n)$ is close to $-1$ while $kA + u_\infty(x', x_n + k\tau)$ becomes sufficiently large, this is obviously impossible. Hence (3.3) holds.

**Step 2.** The inequality (3.3) provides a starting point, from which we can carry out the sliding. We decrease the index $\tau$, and show that for any $0 < \tau < 2b$,

$$W^\tau(x) \leq 0, \quad x \in \mathbb{R}^n.$$  \hspace{1cm} (3.14)

Define

$$\tau_0 = \inf \{ \tau \mid W^\tau(x) \leq 0, x \in \mathbb{R}^n, 0 < \tau < 2b \}.$$  \hspace{1cm} (3.15)

Claim

$$\tau_0 = 0.$$  \hspace{1cm} (3.15)

Otherwise, suppose that $\tau_0 > 0$. To derive a contradiction, we prove that $\tau_0$ can be decreased a little bit while the inequality (3.14) still holds.

We first claim that

$$\sup_{\mathbb{R}^{n-1} \times [-b, b]} W^{\tau_0}(x) < 0.$$  \hspace{1cm} (3.15)

If not, then

$$\sup_{\mathbb{R}^{n-1} \times [-b, b]} W^{\tau_0}(x) = 0,$$
and there exists a sequence $\{x^k\}_{k=1}^\infty \subset \mathbb{R}^{n-1} \times [-b, b]$ such that

$$W^{\tau_0}(x^k) \to 0, \quad k \to \infty.$$  \hspace{1cm} (3.15)

Let $\Psi_k(x) = \Psi(x - x^k)$, where the definition of $\Psi$ is similar to (3.5). Then there exists a sequence of $\varepsilon_k \to 0$ such that

$$W^{\tau_0}(x^k) + \varepsilon_k \Psi_k(x^k) > 0.$$  \hspace{1cm} (3.15)

For any $x \in \mathbb{R}^n \setminus B_1(x^k)$, notice that $W^{\tau_0}(x) \leq 0$ and $\Psi_k(x) = 0$, we have

$$W^{\tau_0}(x^k) + \varepsilon_k \Psi_k(x^k) > W^{\tau_0}(x) + \varepsilon_k \Psi_k(x), \quad \text{for any } x \in \mathbb{R}^n \setminus B_1(x^k).$$
It follows that there exists a point \( \bar{x}^k \in B_1(x^k) \) such that
\[
W^{\tau_0}(\bar{x}^k) + \varepsilon_k \Psi_k(\bar{x}^k) = \max_{x \in B_1(x)} (W^{\tau_0}(x) + \varepsilon_k \Psi_k(x)) > 0. \tag{3.16}
\]

On one hand, we have
\[
A_\alpha(W^{\tau_0} + \varepsilon_k \Psi_k)(\bar{x}^k) \leq f(u(\bar{x}^k)) - f(u^*(\bar{x}^k)) + C\varepsilon_k \to 0, k \to \infty. \tag{3.17}
\]

On the other hand, we obtain from (3.17) that
\[
A_\alpha(W^{\tau_0} + \varepsilon_k \Psi_k)(\bar{x}^k)
\]
\[= C_{n,\alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{a(\bar{x}^k - y)(W^{\tau_0}(\bar{x}^k) + \varepsilon_k \Psi_k(\bar{x}^k) - W^{\tau_0}(y) - \varepsilon_k \Psi_k(y))}{|\bar{x}^k - y|^{n+\alpha}} \, dy
\]
\[\geq C \int_{B_2^c(k)} \frac{|W^{\tau_0}(y)|}{|\bar{x}^k - y|^{n+\alpha}} \, dy
\]
\[= C \int_{B_2^c(0)} \frac{|W^{\tau_0}(z + \bar{x}^k)|}{|z|^{n+\alpha}} \, dz,
\]
where the last inequality holds due to the fact that \( |\bar{x}^k - y| \leq |\bar{x}^k - x^k| + |x^k - y| \leq 3/2 |x^k - y| \).

Denote
\[u_k(x) = u(x + x^k) \quad \text{and} \quad W^{\tau_0}_k(x) = W^{\tau_0}(x + x^k).
\]

Since \( u(x) \) is uniformly continuous, for any compact set \( K \subset \mathbb{R}^n \), by Arzelà-Ascoli theorem, there exists a subsequence, we have
\[u_k(x) \to u_\infty(x), \quad k \to \infty, \quad \text{uniformly in } K.
\]

Combining (3.17) and (3.18), and letting \( k \to \infty \), one has
\[W^{\tau_0}_k(x) \to 0, \quad k \to \infty, \quad x \in B_2^c(0).
\]

Therefore,
\[W^{\tau_0}_k(x) \to u_\infty(x) - u_\infty(x^0), \quad x \in B_2^c(0).
\]

Since \( \{x_n^k\}_{k=1}^{\infty} \) is bounded, we obtain from (1.12) that
\[u_\infty(x', x_n) \xrightarrow{x_n \to \pm \infty} \pm 1, \quad \text{uniformly in } \mathbb{R}^{n-1} \text{ (with respect to } x'). \tag{3.19}
\]

Therefore,
\[u_\infty(x', x_n) = u_\infty(x', x_n + \tau_0) = u_\infty(x', x_n + 2\tau_0) = \cdots = u_\infty(x', x_n + k\tau_0),
\]
for any \( k \in \mathbb{N} \). Now take \( x_n \) sufficiently negative and \( k \) sufficiently large, hence \( u_\infty(x', x_n) \) is close to \( -1 \) and \( u_\infty(x', x_n + k\tau_0) \) is sufficiently close to \( 1 \), this is impossible. Hence (3.15) must hold.

Next we prove that, there exists an \( \varepsilon > 0 \), such that
\[W^\tau(x) \leq 0, \quad \forall x \in \mathbb{R}^n, \quad \forall \tau \in (\tau_0 - \varepsilon, \tau_0). \tag{3.20}
\]

First, by (3.15), we derive that there exists a small \( \varepsilon > 0 \) such that
\[
\sup_{\mathbb{R}^{n-1} \times [-b, b]} W^\tau(x) < 0, \quad \forall \tau \in (\tau_0 - \varepsilon, \tau_0).
\]

Therefore, we only need to prove that
\[
\sup_{\mathbb{R}^n \setminus (\mathbb{R}^{n-1} \times [-b, b])} W^\tau(x) \leq 0, \quad \forall \tau \in (\tau_0 - \varepsilon, \tau_0).
\]
If not, then
\[
\sup_{\mathbb{R}^n \setminus (\mathbb{R}^n \setminus [-b,b])} W^\tau (x) := A > 0, \; \forall \tau \in (\tau_0 - \varepsilon, \tau_0).
\]

There exists a sequence of \( \{x^k\}_{k=1}^{\infty} \) such that \( W^\tau (x^k) \to A \). By the asymptotic condition (1.12), \( \{x^k\}_{k=1}^{\infty} \) is bounded, and we assume that \( |x^k| \leq M (M > b) \). By a similar argument, setting \( \Psi_k (x) = \Psi (x - x^k) \), we have that there exists \( \varepsilon_k \to 0, \) and \( \bar{x}^k \in B_1 (x^k) \) such that
\[
W^\tau (\bar{x}^k) + \varepsilon_k \Psi_k (\bar{x}^k) = \max_{\mathbb{R}^n} (W^\tau (x) + \varepsilon_k \Psi_k (x)) > 0,
\]
and
\[
A_\alpha (W^\tau + \varepsilon_k \Psi_k) (\bar{x}^k) \leq C \varepsilon_k. \tag{3.21}
\]

In addition, by the uniformly boundedness of \( a \) in (A1), we have
\[
A_\alpha (W^\tau + \varepsilon_k \Psi_k) (\bar{x}^k)
= C_{n, \alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{a(\bar{x}^k - y)(W^\tau (\bar{x}^k) + \varepsilon_k \Psi_k (\bar{x}^k) - W^\tau (y) - \varepsilon_k \Psi_k (y))}{|\bar{x}^k - y|^{n+\alpha}} \, dy
\geq C \int_{B_{\delta}(x^k)} \frac{A - W^\tau (y)}{|\bar{x}^k - y|^{n+\alpha}} \, dy
= C \int_{B_{\delta}(0)} \frac{A - W^\tau (z + x^k)}{|z|^{n+\alpha}} \, dz. \tag{3.22}
\]

Denote
\[
W^\tau_k (x) = W^\tau (x + x^k),
\]
combining (3.21) and (3.22), and letting \( k \to \infty \), we derive
\[
W^\tau_\infty (x) = \lim_{k \to \infty} W^\tau (x + x^k) = A > 0,
\]
which contradicts the fact that \( W^\tau_\infty (x) \) equals 0 at infinity. This proves (3.20) which contradicts the definition of \( \tau_0 \). Hence we obtain (3.14).

**Step 3.** In this step, we show that \( u(x) \) is strictly increasing with respect to \( x_n \) and \( u(x) \) depends on \( x_n \) only.

Combining Step 1 and Step 2, we obtain that
\[
W^\tau (x) \leq 0 \text{ in } \mathbb{R}^n, \text{ for any } \tau > 0.
\]

If there exists a point \( x^0 \in \mathbb{R}^n \), such that \( W^\tau (x^0) = 0 \), then \( x^0 \) is a maximum point of \( W^\tau (x) \) in \( \mathbb{R}^n \). Since \( W^\tau (x) \neq 0 \) in \( \mathbb{R}^n \), by a direct calculation, we have
\[
A_\alpha W^\tau (x^0)
= C_{n, \alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{a(x^0 - y)(W^\tau (x^0) - W^\tau (x))}{|x^0 - y|^{n+\alpha}} \, dy
\geq m C_{n, \alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{-W^\tau (x)}{|x^0 - y|^{n+\alpha}} \, dy
> 0.
\]

This contradicts
\[
A_\alpha W^\tau (x^0) = f(u^\tau (x^0)) - f(u(x^0)) = 0.
\]

Therefore, we have
\[
W^\tau (x) < 0 \text{ in } \mathbb{R}^n, \text{ for } \forall \tau > 0.
\]

Now, we show that \( u(x) \) depends on \( x_n \) only.
In fact, it can be seen from the above process that the argument still holds if we replace \( u(x) \) by \( u(x + \tau \nu) \), where \( \nu = (\nu_1, \cdots, \nu_n) \) with \( \nu_n > 0 \) is an arbitrary vector pointing upward. Applying the similar arguments as in Steps 1 and 2, we can derive that, for each of such \( \nu \),

\[
\begin{align*}
\quad u(x + \tau \nu) &> u(x), \quad \forall \tau > 0, \ x \in \mathbb{R}^n. \\
\end{align*}
\]

Letting \( \nu_n \to 0 \), from the continuity of \( u \), we deduce that

\[
\begin{align*}
\quad u(x + \tau \nu) &\geq u(x) \\
\end{align*}
\]

for arbitrary \( \nu \) with \( \nu_n = 0 \). By replacing \( \nu \) by \( -\nu \), we find that

\[
\begin{align*}
\quad u(x + \tau \nu) &= u(x) \\
\end{align*}
\]

for arbitrary \( \nu \) with \( \nu_n = 0 \), this means that \( u \) is independent of \( x' \), hence \( u(x) = u(x_n) \). This completes the proof of Theorem 1.3. \( \square \)

4. Proof of Theorem 1.4 and Corollary 1.5.

Proof of Theorem 1.4. Suppose on the contrary, there is some point such that \( u(x) > 0 \), then

\[
0 < A := \sup_{x \in \mathbb{R}^n} u(x) < \infty. \tag{4.1}
\]

There exist the sequence \( \{x^k\}_{k=1}^{\infty} \subset D \) and \( \gamma_k \to 1(\gamma_k \in (0, 1)) \) as \( k \to \infty \) such that

\[
\begin{align*}
\quad u(x^k) &\geq \gamma_k A. \tag{4.2}
\end{align*}
\]

Let

\[
\begin{align*}
\Phi(x) &= \begin{cases} 
\frac{c_n}{|x|^{\frac{n-1}{2}}}, & |x| < 2, \\
0, & |x| \geq 2.
\end{cases} \tag{4.3}
\end{align*}
\]

It is easy to check that \( \Phi(x) \) is radially decreasing from the origin, and is in \( C_1^\infty(\mathbb{R}^n) \).

Define

\[
\Phi_k(x) := \Phi(x - x^k). \tag{4.4}
\]

For any \( x \in B_2(x^k) \setminus B_1(x^k) \), we can take \( \varepsilon_k > 0 \) such that

\[
\begin{align*}
\quad u(x^k) + \varepsilon_k \Phi_k(x^k) &\geq A + \varepsilon_k \Phi_k(x^k) + \nu \geq u(x) + \varepsilon_k \Phi_k(x), \tag{4.5}
\end{align*}
\]

where \( \nu \) is any unit vector in \( \mathbb{R}^n \).

Therefore, there exists \( \bar{x}^k \in B_1(x^k) \) such that

\[
\begin{align*}
\quad u(\bar{x}^k) + \varepsilon_k \Phi_k(\bar{x}^k) &= \max_{x \in B_2(x^k)} [u(x) + \varepsilon_k \Phi_k(x)]. \tag{4.6}
\end{align*}
\]

As a consequence,

\[
\begin{align*}
\quad u(\bar{x}^k) + \varepsilon_k \Phi_k(\bar{x}^k) &\geq u(x^k) + \varepsilon_k \Phi_k(x^k),
\end{align*}
\]

which implies

\[
\begin{align*}
\quad u(\bar{x}^k) &\geq u(x^k) + \varepsilon_k \Phi_k(x^k) - \varepsilon_k \Phi_k(\bar{x}^k) \geq u(x^k).
\end{align*}
\]

It follows from (4.2) that

\[
\begin{align*}
\quad u(\bar{x}^k) &\geq \gamma_k A. \tag{4.7}
\end{align*}
\]

From (4.5) and (4.6), we deduce that

\[
\begin{align*}
\quad u(\bar{x}^k) + \varepsilon_k \Phi_k(\bar{x}^k) &\geq A \geq u(x), \ \forall x \in \mathbb{R}^n. \tag{4.8}
\end{align*}
\]

Hence \( \bar{x}^k \) is a maximum point of the function \( u(x) + \varepsilon_k \Phi_k(x) \) in \( \mathbb{R}^n \).
Calculating directly, we have

\[
A_\alpha (u + \varepsilon_k \Phi_k) (\bar{x}^k) \\
= C_{n, \alpha} P.V. \int_{\mathbb{R}^n} \frac{a(\bar{x}^k - y)(u(\bar{x}^k) + \varepsilon_k \Phi_k(\bar{x}^k) - u(y) - \varepsilon_k \Phi_k(y))}{|\bar{x}^k - y|^{n+\alpha}} dy \\
+ C_{n, \alpha} P.V. \int_{B_2(x^k)} \frac{a(\bar{x}^k - y)(u(\bar{x}^k) + \varepsilon_k \Phi_k(\bar{x}^k) - u(y) - \varepsilon_k \Phi_k(y))}{|\bar{x}^k - y|^{n+\alpha}} dy \\
= I_1 + I_2.
\]

For \( I_1 \), by (4.6), we notice that

\[ u(\bar{x}^k) + \varepsilon_k \Phi_k(\bar{x}^k) - u(y) - \varepsilon_k \Phi_k(y) \geq 0, \]

for any \( y \in B_2(x^k) \). Combining with \( a(x) > 0 \), we have

\[ I_1 \geq 0. \tag{4.10} \]

The next is to estimate \( I_2 \). It follows

\[
I_2 = C_{n, \alpha} \int_{\mathbb{R}^n \setminus B_2(x^k)} \frac{a(\bar{x}^k - y)(u(\bar{x}^k) + \varepsilon_k \Phi_k(\bar{x}^k) - u(y) - \varepsilon_k \Phi_k(y))}{|\bar{x}^k - y|^{n+\alpha}} dy \\
\geq m C_{n, \alpha} \int_{\mathbb{R}^n \setminus B_2(x^k)} \frac{u(\bar{x}^k) + \varepsilon_k \Phi_k(\bar{x}^k) - u(y)}{|\bar{x}^k - y|^{n+\alpha}} dy \\
\geq m C_{n, \alpha} \int_{(\mathbb{R}^n \setminus B_2(x^k)) \cap D^c} \frac{u(\bar{x}^k) + \varepsilon_k \Phi_k(\bar{x}^k) - u(y)}{|\bar{x}^k - y|^{n+\alpha}} dy \\
\geq m C_{n, \alpha} \int_{(\mathbb{R}^n \setminus B_2(x^k)) \cap D^c} \frac{1}{|\bar{x}^k - y|^{n+\alpha}} dy,
\]

where the last inequality holds due to the fact that \(|\bar{x}^k - y| \leq |\bar{x}^k - x^k| + |x^k - y| \leq \frac{3}{2} |x^k - y|\).

By (1.14), there exists a \( j_0 \geq 1 \), such that

\[ |(B_{2j+1} (x) \setminus B_{2j} (x)) \cap D^c| \geq \frac{c_0}{2} |B_{2j+1} (x) \setminus B_{2j} (x)|. \]

Then, we have

\[
I_2 \geq c_1 \int_{D^c \cap (B_{2j} (x^k))} \frac{1}{|x^k - y|^{n+\alpha}} dy \\
\geq c_1 \sum_{j=j_0}^{\infty} \int_{D^c \cap (B_{2j+1} (x^k) \setminus B_{2j} (x^k))} \frac{1}{|x^k - y|^{n+\alpha}} dy \\
\geq c_1 \sum_{j=j_0}^{\infty} \frac{|D^c \cap B_{2j+1} (x^k) \setminus B_{2j} (x^k)|}{(2j+1)^{n+\alpha}}.
\]
Proof. For any \(c\) such that \(A^c\), where

\[
\geq c_2 \sum_{j=j_0}^{\infty} \frac{1}{(2^{j+1})^\alpha} \geq c', \tag{4.11}
\]

where \(c'\) depends on \(c_0, c_1\) and \(j_0\).

On the other hand, by (1.15) and (4.7), we deduce that

\[
A_\alpha(u(\bar{x}^k)) \leq 0,
\]

which combining with (4.9), (4.10) and (4.11), yields

\[
A_\alpha(\varepsilon_k \Phi_k)(\bar{x}^k) \geq A_\alpha(u(\bar{x}^k)) + A_\alpha(\varepsilon_k \Phi_k)(\bar{x}^k) \geq c'.
\]

By Proposition 5.1, we derive that \(A_\alpha(\varepsilon_k \Phi_k)(\bar{x}^k) \leq C \varepsilon_k\). Then, we arrive at

\[
C \varepsilon_k \geq c'. \tag{4.12}
\]

Since the left hand side of (4.12) must go to zero as \(\varepsilon_k \to 0 (k \to \infty)\), which contradicts the right hand side of (4.12). Therefore, (1.16) must be true. \(\square\)

In the following, we aim to prove Corollary 1.1.

Proof of Corollary 1.1. Let \(D = \mathbb{R}^n\), \(u^\tau(x) = u(x', x_n + \tau)\) and \(w^\tau(x) = u(x) - u^\tau(x)\). If \(w^\tau(x) > 0\), then \(u(x) > u^\tau(x) \geq 0\). When \(u(x) > u^\tau(x) > 0\), we have

\[
A_\alpha w^\tau(x) = f(u(x)) - f(u^\tau(x)) \leq 0,
\]

because of the non-increasing property of \(f(u)\). When \(u(x) > u^\tau(x) = 0\), we obtain

\[
A_\alpha w^\tau(x) = f(u(x)) - f(0) \leq 0,
\]

because of the non-increasing property of \(f(u)\).

Therefore,

\[
A_\alpha w^\tau(x) \leq 0 \text{ in } D \text{ where } w^\tau(x) > 0.
\]

In addition,

\[
w^\tau(x) \leq 0, \ x \notin \mathbb{R}_+^n.
\]

It follows from Theorem 1.4 that

\[
w^\tau(x) \leq 0, \ x \in \mathbb{R}^n, \ \text{for any } \tau > 0.
\]

This means that \(u\) is monotone increasing. By \(\lim_{x_n \to +\infty} u(x', x_n) = 0\), uniformly in \(x' \in \mathbb{R}^{n-1}\), we can derive \(u \equiv 0\). \(\square\)

5. Appendix.

Proposition 5.1. Assume that \(u \in C^{1,1}_{loc} \cap L_\alpha(\mathbb{R}^n)\), then there exists a constant \(C\), such that \(A_\alpha u(x) \leq C\).

Proof. For any \(x \in \mathbb{R}^n\), we divide the integral into two parts.

\[
A_\alpha u(x) = C_{n, \alpha} P.V. \int_{\mathbb{R}^n} \frac{a(x-y)(u(x) - u(y))}{|x-y|^{n+\alpha}} dy
\]

\[
= C_{n, \alpha} P.V. \int_{B_1(x)} \frac{a(x-y)(u(x) - u(y))}{|x-y|^{n+\alpha}} dy + C_{n, \alpha} \int_{\mathbb{R}^n \setminus B_1(x)} \frac{a(x-y)(u(x) - u(y))}{|x-y|^{n+\alpha}} dy
\]

\[
= I_1 + I_2.
\]

For \(I_1\), since \(u \in C^{1,1}_{loc} \cap L_\alpha(\mathbb{R}^n)\), by Taylor expansion, we have

\[
u(x) - u(y) = \nabla v(x) \cdot (x - y) + O(|x-y|^2).
\]
By the anti-symmetry of $\nabla u(x) \cdot (x - y)$ for $y \in B_1(x)$ and the cylindrical condition of $a$ in (A2), we obtain

$$
P.V. \int_{B_1(x)} \frac{a(x-y) \nabla u(x) \cdot (x - y)}{|x-y|^{n+\alpha}} dy = P.V. \int_{B_1(x)} \frac{a(x-y) \sum_{i=1}^{n-1} \frac{\partial u(x)}{\partial x_i} (x_i - y_i)}{|x-y|^{n+\alpha}} dy + P.V. \int_{B_1(x)} \frac{a(x-y) \frac{\partial u(x)}{\partial x_n} (x_n - y_n)}{|x-y|^{n+\alpha}} dy = 0.
$$

(5.1)

Hence by (5.1), we have

$$
I_1 = C_{n,\alpha} P.V. \int_{B_1(x)} \frac{a(x-y)(u(x) - u(y))}{|x-y|^{n+\alpha}} dy 
\leq C_{n,\alpha} P.V. \int_{B_1(x)} \frac{a(x-y)O(|x-y|^2)}{|x-y|^{n+\alpha}} dy 
\leq C'' \int_{B_1(x)} \frac{|x-y|^2}{|x-y|^{n+\alpha}} dy = C.
$$

For $I_2$, let

$$
u(x) - u(y) = \int_0^1 \nabla u(y + t(x - y)) dt.
$$

(5.2)

Because $u \in C^{1,1}_{\text{loc}}$, then (5.2) is bounded. Therefore we have

$$
I_2 = C_{n,\alpha} \int_{\mathbb{R}^n \setminus B_1(x)} \frac{a(x-y)(u(x) - u(y))}{|x-y|^{n+\alpha}} dy 
\leq CM \int_{\mathbb{R}^n \setminus B_1(x)} \frac{1}{|x-y|^{n+\alpha}} dy 
\leq C'.
$$

Thus, we eventually complete the proof of this proposition. \qed

Acknowledgments. The authors would like to thank Professor Yanyan Li for his helpful discussions and suggestions. The authors would like to thank the anonymous referee for her/his useful comments and valuable suggestions which improved and clarified the paper. This work was supported by National Natural Science Foundation of China (11871096, 11771389, 11931010).

REFERENCES

[1] H. Berestycki, L. A. Caffarelli and L. Nirenberg, Symmetry for elliptic equations in a half space, in Boundary Value Problems for Partial Differential Equations and Applications, RMA Res. Notes Appl. Math., Masson, Paris, 29 (1993), 27–42.
[2] H. Berestycki, F. Hamel and R. Monneau, One-dimensional symmetry of bounded entire solutions of some elliptic equations, Duke Math. J., 103 (2000), 375–396.
[3] H. Berestycki and L. Nirenberg, Monotonicity, symmetry and antisymmetry of solutions of semilinear elliptic equations, J. Geom. Phys., 5 (1988), 237–275.
[4] H. Berestycki and L. Nirenberg, Some qualitative properties of solutions of semilinear elliptic equations in cylindrical domains, in \textit{Analysis, et Cetera}, Academic Press, Boston, MA, (1990), 115–164.

[5] H. Berestycki and L. Nirenberg, On the method of moving planes and the sliding method, \textit{Bol. Soc. Brasil. Mat. (N.S.)}, 22 (1991), 1–37.

[6] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, \textit{Comm. Partial Differential Equations}, 32 (2007), 1245–1260.

[7] L. Caffarelli and L. Silvestre, Regularity theory for fully nonlinear integro-differential equations, \textit{Comm. Pure Appl. Math.}, 62 (2009), 597–638.

[8] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, \textit{Comm. Partial Differential Equations}, 32 (2007), 1245–1260.

[9] L. Caffarelli and L. Silvestre, Regularity results for nonlocal equations by approximation, \textit{Arch. Ration. Mech. Anal.}, 200 (2011), 59–88.

[10] W. Chen, Y. Fang and R. Yang, Liouville theorems involving the fractional Laplacian on a half space, \textit{Adv. Math.}, 274 (2015), 167–198.

[11] W. Chen and C. Li, Maximum principles for the fractional $p$-Laplacian and symmetry of solutions, \textit{Adv. Math.}, 335 (2018), 735–758.

[12] W. Chen, C. Li and G. Li, Maximum principles for a fully nonlinear fractional order equation and symmetry of solutions, \textit{Calc. Var. Partial Differential Equations}, 56 (2017), Paper No. 29, 18 pp.

[13] W. Chen, C. Li and Y. Li, A direct blowing-up and rescaling argument on nonlocal elliptic equations, \textit{Internat. J. Math.}, 27 (2016), 1650064, 20 pp.

[14] W. Chen, C. Li and Y. Li, A direct method of moving planes for the fractional Laplacian, \textit{Adv. Math.}, 308 (2017), 404–437.

[15] W. Chen, C. Li and B. Ou, Qualitative properties of solutions for an integral equation, \textit{Discrete Contin. Dyn. Syst.,} 12 (2005), 347–354.

[16] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, \textit{Comm. Pure Appl. Math.,} 59 (2006), 330–343.

[17] W. Chen and S. Qi, Direct methods on fractional equations, \textit{Discrete Contin. Dyn. Syst.,} 39 (2019), 1269–1310.

[18] W. Chen and J. Zhu, Indefinite fractional elliptic problem and Liouville theorems, \textit{J. Differential Equations}, 260 (2016), 4758–4785.

[19] X. Chen, G. Bao and G. Li, The sliding method for the nonlocal Monge–Ampère operator, \textit{Nonlinear Anal.,} 196 (2020), 111786, 13 pp.

[20] T. Cheng, G. Huang and C. Li, The maximum principles for fractional Laplacian equations and their applications, \textit{Commun. Contemp. Math.,} 19 (2017), 1750018, 12.

[21] C. Li, Z. Wu and H. Xu, Maximum principles and Böcher type theorems, \textit{Proc. Natl. Acad. Sci. USA,} 115 (2018), 6976–6979.

[22] Z. Liu, Maximum principles and monotonicity of solutions for fractional $p$-equations in unbounded domains, \textit{J. Differential Equations}, 270 (2021), 1043–1078. \texttt{arXiv:1905.06493}.

[23] L. Ma and Z. Zhang, Monotonicity of positive solutions for fractional $p$-systems in unbounded Lipschitz domains, \textit{Nonlinear Anal.,} 198 (2020), 111892, 18 pp.

[24] D. Tang, Positive solutions to semilinear elliptic equations involving a weighted fractional Laplacian, \textit{Math. Methods Appl. Sci.,} 40 (2017), 2596–2609.

[25] L. Wu and W. Chen, Monotonicity of solutions for fractional equations with De Giorgi type nonlinearities, (in chinese), \textit{Sci. Sin. Math.,} (2020), to appear.

Received June 2020; revised August 2020.

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