Avoiding Dense and Dynamic Obstacles in Enclosed Spaces: Application to Moving in Crowds

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Abstract—This paper presents a closed-form approach to constraining a flow within a given volume and around objects. The flow is guaranteed to converge and to stop at a single fixed point. The obstacle avoidance problem is inverted to enforce that the flow remains enclosed within a volume defined by a polygonal surface. We formally guarantee that such a flow will never contact the boundaries of the enclosing volume or obstacles. It asymptotically converges towards an attractor. We further create smooth motion fields around obstacles with edges (e.g., tables). Both obstacles and enclosures may be time-varying, i.e., moving, expanding, and shrinking. This technique enables a robot to navigate within enclosed corridors while avoiding static and moving obstacles. It was applied on an autonomous robot (QOLO) in a static complex indoor environment and tested in simulations with dense crowds. The final proof of concept was performed in an outdoor environment in Lausanne. The QOLO-robot successfully traversed a marketplace in the center of town in the presence of a diverse crowd with a non-uniform motion pattern.

I. INTRODUCTION

Robots navigating in human-inhabited environments will encounter disturbances constantly, for instance, when autonomous delivery robots drive around pedestrians. The robot must have a flexible control scheme to avoid collisions. As the number of obstacles increases and their motion becomes less predictable, the robot needs to reevaluate its path within milliseconds to prevent a crash while moving actively towards its goal. Control using dynamical systems (DS) is ideal for addressing such situations. In contrast to classical path planning, the control law is closed-form, hence requires no replanning, and can ensure impenetrability of obstacles [1]. [2]. DS offer stability and convergence guarantees in addition to the desired on-the-fly reactivity.

A. Related Work

1) Sampling Based Exploration: Sampling algorithms such as probabilistic road map (PRM) [3] or the rapidly exploring random trees (RRT) [4] can find a path in cluttered environments. They are computationally intensive, and early approaches were limited to static environments. In [5], online (partial) replanning and elastic-band methods deform the path locally. This allowed adapting to dynamic environments. Reinforcement learning allowed the authors in [6] to learn from previously explored PRM-paths and efficiently adapt to dynamic obstacles. However, the switching often comes with the theoretical loss of global convergence [7]. Recent work uses customized circuitry on a chip for onboard, fast global sampling and evaluation [8]. This method allows fast replanning, but a customized chip design is required for each robot configuration.

2) Real-Time Optimization: With improvements in hardware and computational speed, optimization algorithms such as model predictive control (MPC) have become feasible for onboard use in dynamic path planning, and obstacle avoidance [9]. MPC has been used to control non-holonomic robots in environments with multiple convex obstacles [10]. Most optimization methods cannot guarantee convergence to a feasible solution at runtime. Power diagrams were used to identify the robot’s collision-free, convex neighborhood, and an associated, well-known convex optimization problem generates a continuous flow [11]. This method is limited to convergence for convex obstacles with almost spherical curvature. Control barrier functions (CBFs) and Control Lyapunov Functions (CLFs) were united through the use of quadratic programming (QP) to create collision-free paths in [12]. The convergence constraint was softened to ensure the feasibility of the optimization problem. As a result, full convergence cannot be guaranteed anymore. In [13], local minima were overcome by introducing a (virtual) orientation state. This orientation state introduces a dependence on the history of the QP problem. Additionally, the authors do not address the challenge of finding an appropriate Lyapunov-candidate. In [14] a diffeomorphic transformation to a sphere-world is used, which is based on [15]. The method introduces a dependence on its history through virtual obstacle positions and radii. Furthermore, the construction of the diffeomorphism requires full knowledge about the space.

3) Learning Based Approaches: With the rising popularity of machine learning in the past years, these algorithms have been applied to sensor data to infer data-driven control [16] but this method cannot ensure impenetrability. Other approaches use neural networks on a circular representation of crowds to create steering laws but cannot guarantee convergence [17].

4) Velocity Obstacles: Velocity obstacles extend the obstacle shape by its potential future positions based on the current velocity [18]. Velocity obstacles allow safe navigation in dynamic environments. They were successfully applied in multi-agent scenarios [19] and extended to include acceleration, and non-holonomic constraints of the agent [20]. The velocity obstacle approach often conservatively limits the workspace.
5) Artificial Potential Fields and Navigation Functions: Artificial potential fields were used to create collision-free trajectories [21], but they are prone to local minima. In [22] artificial potential fields for sphere-worlds were designed to have only a global minimum. A diffeomorphic transformation was introduced to map star-worlds to sphere-worlds [15], and extended to include trees-of-stars [23]. The tuning of critical parameters needs the knowledge of the whole space. Hence, in practice, full convergence is difficult to achieve.

More recent approaches introduce artificial potential fields with only the global minimum for more general environments [24, 25]. Automated approximation of the tuning parameter has been proposed [26], but it does not generalize easily to dynamic environment. In [27], full convergence is ensured around ellipse obstacles through quadratic potential functions. Learning methods were used to tune the hyper-parameters of potential fields to obtain human-inspired behavior for obstacle avoidance [28].

Danger-fields were used in [29] to ensure collision avoidance for robot arms through repulsion from dynamic obstacles. The approach was extended by guiding the motion through an artificial potential field in [30]. The design of the artificial potential field remains a challenge for this method. Dynamic reference points help to reduce the probability of converging to a local minimum for potential fields [31].

6) Harmonic Potential Fields: Harmonic potential functions are interesting as they guarantee that no topologically critical points arise in free space. In [32], the harmonic potential functions are evaluated numerically to overcome the challenge of finding them analytically. Closed-form harmonic potential functions can be generated by approximating the obstacles through linear panels [33]. This linear approximation applies to concave obstacles but is limited to two-dimensional environments [1]. In [34], known harmonic potential functions are interpolated to navigate in more complex environments. A closed-form solution to harmonic potential flow around simple obstacles was presented in [35]. The work allows to avoid moving obstacles but is limited to convex. In [36], the approach was extended to concave obstacles by using a discrete, sensor-based representation. Closed-form approaches using harmonic potential fields are often simplified to a circular world or require high (close to circular) curvature. In [37], a dynamic reference point was introduced to ensure convergence for star-shaped environments. However, the approach was not able to handle boundaries or deforming obstacles.

B. Contributions

This paper addresses the need for a reactive (closed-form) obstacle avoidance approach with formal guarantees of implementability to handle highly dynamic environments and realistic obstacles, such as obstacles with sharp edges. To this end, this paper extends our previous work [37], in which we presented a closed-form obstacle avoidance approach guaranteed to not penetrate smooth concave, albeit star-shaped, obstacles. We present three novel theoretical contributions:

1) We invert the obstacle description to ensure that the robot moves within the enclosed space defined by the boundary while preserving convergences guarantees towards an attractor. This boundary may represent walls, furniture, or even joint limits of manipulators (Sec. III).

2) We extend the approach to handle nonsmooth surfaces, i.e., obstacles with sharp edges. The novelty comes from creating a smooth dynamical system around an obstacle without approximation of the curvature (Sec. IV).

3) We show that the approach can be extended to tackle dynamic environments, with obstacles that have deforming shapes (Sec. V).

1) Implementation and Technical Contributions: We validate these contributions with a wheelchair robot moving in a simulated crowd of pedestrians and an office environment with real furniture (Sec. VIII).

For the real-world implementation, we present three technical contributions:

T.1) We extend the modulation parameters of the obstacle avoidance with surface friction and repulsive value (Sec. II). They allow an agent to move slower and further away from obstacles, respectively. The extensions result in more cautious behavior.

T.2) We propose an approach on how to use the dynamical obstacle avoidance in combination with a robot arm (Sec. VI). The safe joint velocity is evaluated based on the proximity function combined with the obstacle avoidance algorithm.

T.3) We introduce a general directional weighting for (ensured) non-trivial summing of vector fields and corresponding gradient descent (Sec. A).

C. Notation

The state variable $\xi \in \mathbb{R}^d$ defines the state of a robotic system. If not mentioned otherwise, $\xi$ will refer to the Cartesian position of the agent.

The variable $\pi$ is used as the circle constant throughout the paper.

Superscripts are used to denote the name of variables and subscripts for enumeration.

Bold-face Latin characters describe vectors and matrices. Vectors defined by Greek characters are explicitly defined.

The brackets $\langle \cdot, \cdot \rangle$ denote the dot product of two vectors.

The $\times$-operator indicates the cross product.

The square brackets indicate elements of a vector, e.g., $\xi_{[1]}$ denotes the first element of $\xi$. Double dots within the brackets indicate a sub-vector up to the specified number, e.g., $\xi_{[1:2]}$ is a vector of the first two elements of $\xi$.

II. OBSTACLE AVOIDANCE FORMULATION

Dynamical system based obstacle avoidance has been proposed by the authors in [37]. We restate previous definitions, and introduce two new extensions: a friction parameter (Sec. II-E) and a repulsion parameter (Sec. II-F).

A. Dynamical Systems

This work focuses on motion towards a goal of an autonomous dynamical system, i.e. $\lim_{t \to \infty} f(\xi_t) = 0$. The most
direct dynamics towards the attractor is a linear dynamical system of the form:

\[ f(\xi) = -k(\xi - \xi^a) \]  

where \( k \in \mathbb{R} \) is a scaling parameter. The attractor \( \xi^a \) is visualized throughout this work as a star: * and with set \( k = 1 \).

**B. Obstacle Description**

Each obstacle has a continuous distance function \( \Gamma(\xi) : \mathbb{R}^d \rightarrow \mathbb{R}_{\geq 0} \), which allows to distinguish three regions:

- Free points: \( \mathcal{X}^f = \{ \xi \in \mathbb{R}^d : \Gamma(\xi) > 1 \} \)
- Boundary points: \( \mathcal{X}^b = \{ \xi \in \mathbb{R}^d : \Gamma(\xi) = 1 \} \)  
- Interior points: \( \mathcal{X}^i = \{ \xi \in \mathbb{R}^d \setminus (\mathcal{X}^f \cup \mathcal{X}^b) \} \)

1) **Reference Point**: For each obstacle \( i \) a reference point is chosen such that it lies within the kernel of the obstacle: \( \xi^r_i \in \mathcal{X}^a_i \). The reference direction is defined as:

\[ r_i(\xi) = (\xi - \xi^r_i)/\|\xi - \xi^r_i\| \quad \forall \xi \in \mathbb{R}^d \setminus \xi^r_i \]  

The reference point is visualized throughout this work as a cross +.

2) **Distance Function**: By construction, the distance function \( \Gamma(\cdot) \) increases monotonically in radial direction and has a continuous first-order partial derivative (\( C^1 \) smoothness). Here, we define the general distance function as:

\[ \Gamma^r(\xi) = (\|\xi - \xi^r\|/R(\xi))^2 \quad \forall \xi \in \mathbb{R}^d \setminus \xi^r \]  

with the power coefficient \( p \in \mathbb{N}_+ \). The local radius \( R(\xi) = \|\xi^b - \xi^r\| \) is a function of the local boundary point, which is defined as:

\[ \xi^b = b r(\xi) + \xi^r \quad \text{such that} \quad b > 0, \quad \xi^b \in \mathcal{X}^b \]  

**C. Obstacle Avoidance through Modulation**

Real-time obstacle avoidance is obtained by applying a dynamic modulation system to a dynamical system \( f(\xi) \):

\[ \dot{\xi} = M(\xi)f(\xi) \quad \text{with} \quad M(\xi) = E(\xi)D(\xi)E(\xi)^{-1} \]  

The modulation matrix is composed of the basis matrix:

\[ E(\xi) = [r(\xi), e_1(\xi), \ldots, e_{d-1}(\xi)] \]  

which has the orthonormal tangent vectors \( e_i(\xi) \) evaluated at the boundary point \( \xi^b \) given in (5). The diagonal eigenvalue matrix is given as:

\[ D(\xi) = \text{diag}(\lambda^r(\xi), \lambda^e(\xi), \ldots, \lambda^e(\xi)) \]  

We set the eigenvalues to

\[ \lambda^r(\xi) = 1 - 1/\Gamma(\xi)^1/\rho \quad \lambda^e(\xi) = 1 + 1/\Gamma(\xi)^1/\rho \]  

with the reactivity factor \( \rho \in \mathbb{R}_{>0} \) and the distance function \( \Gamma(\xi) \) from (4). In this work, we simply choose \( \rho = 1 \).

\[ \lambda^r(\xi) = \begin{cases} 1 - (c^{\text{rep}}/\Gamma(\xi))^{1/\rho} & \text{if } f(\xi), r < 0 \\ 1 & \text{otherwise} \end{cases} \]  

with the repulsive coefficient \( c^{\text{rep}} > 1 \). A repulsive coefficient \( c^{\text{rep}} = 1 \) corresponds to no repulsion. Note, that the repulsive
III. INVERTED OBSTACLE AVOIDANCE

An autonomous robot often encounters scenarios where it has boundaries that it cannot pass. This might be a wall for a wheeled robot or the joint limits for a robot arm. These obstacles are represented by limits of the free space.

A. Distance Inversion

The distance function $\Gamma(\xi)$ from $\mathcal{X}^o$ can be evaluated within the obstacle $\xi \in \mathcal{X}^o$. For interior points, our boundary function is monotonically decreasing along the radial direction, i.e. the Lie derivative with respect to the reference direction is positive. Furthermore it is bounded. This can be written as:

$$L_r \Gamma = \langle \frac{\partial \Gamma(\xi)}{\partial r(\xi)}, r(\xi) \rangle > 0, \quad \Gamma(\xi) \in [0, 1[ \quad \forall \mathcal{X}^o \setminus \xi^r$$

We consider the obstacle boundary $\mathcal{X}^b$ as the description of an enclosing hull. It follows that the interior points of the classical obstacle become points of free space of the enclosing hull and vice versa. Boundary points stay boundary points. We define the distance function of wall-obstacles as the inverse of the classical distance function:

$$\Gamma^w(\xi) = 1/\Gamma^w = (R(\xi)/||\xi_0 - \xi^r||)^2p \quad \forall \mathcal{R}^d \setminus \xi^r$$

In the classic obstacle avoidance case, this is of no use, since theoretically the obstacle does never reach the boundary. However, practically an emergency control has to be applied in this case.

This new distance function fulfills the condition for the three regions as given in $\mathcal{X}^o$. The distance function $\Gamma^w$ is now monotonically decreasing along radial direction and reaches infinity at the reference point, i.e., $\lim_{\xi \to \xi^w} \Gamma^w(\xi) \to \infty$ (see Fig. 3).

B. Modulation Matrix

The modulation matrix, defined in (6), consists of the eigenvalue $D(\xi)$ and basis matrix $E(\xi)$. For inverted obstacles, the diagonal eigenvalue matrix from (8) is a function of the inverted distance function $\Gamma^w(\xi)$.

Conversely, the basis matrix is constant along the radial direction. It can be evaluated everywhere (including the interior of a boundary) except at the reference point $\xi^w = (\xi^w)$. Since the reference direction from $\xi^w$ is a zero vector, no orthogonal basis is defined. However, the distance value $\Gamma^w(\xi^w)$ reaches infinity, hence from (8) we get that the diagonal matrix is equal to the identity matrix:

$$\Gamma^w(\xi^w) \to \infty \Rightarrow D(\xi^w) = I$$

Using (9), it follows for the modulated dynamics at the reference point:

$$M(\xi^w) = E(\xi^w)D(\xi^w)E(\xi^w)^{-1} = E(\xi^w)IE(\xi^w)^{-1} = I \Rightarrow \dot{\xi}^w = f(\xi^w)$$

The influence of modulation approaches zero when reaching the reference point. Hence the dynamical modulation is continuously defined.

**Theorem 1** Consider a star-shaped enclosing wall in $\mathcal{R}^d$ with respect to a reference point inside the obstacle $\xi^w$, and a boundary $\Gamma^w(\xi) = 1$ as in (12). Any trajectory $\{\xi_t\}_{t \geq 0}$, that starts within the free space of an enclosing wall, i.e. $\Gamma(\{\xi_t\}) > 1$ and evolves on a smooth path according to (4), will never reach the wall, i.e., $\Gamma(\{\xi_t\}) > 1, t = 0, \infty$ and converges towards an attractor $\tilde{\xi}^w \in \mathcal{R}^d$, i.e., $\lim_{t \to \infty} \xi_t \to \tilde{\xi}^w$.

**Proof:** see Appendix B.

C. Guiding Reference Point to Pass Wall Gaps

In many practical scenarios, a hull entails gaps or holes through which the agent enters or exits the space (e.g., door in a room). The modulation-based avoidance slows the agent
down when approaching the boundary and does not let it pass through such an exit. We introduce a guiding reference point \( \xi^* \) for boundary obstacles to counter this effect. We assume convex walls, and the robot size being smaller than the gap width. It is assumed that gap is a priori known from a map. In \( \{14\} \), it was shown that at the center of an inverted obstacle, the influence of the modulation vanishes.

Let us define the gap region \( \mathcal{X}^* \) enclosed by the lines connecting the gap edges and the reference point \( \xi^* \) (see Fig. 5). The guiding reference point is designed in the following manner: close to the gap, the guiding reference point is equal to the position of the evaluation, hence no influence of the modulation. Far away from the gap, the guiding reference point is equal to the reference point; hence the wall has no influence. In between the two regions, the guiding reference point is projected onto the gap region \( \mathcal{X}^* \). This can be written as:

\[
\xi^* = \begin{cases} 
\xi & \text{if } \xi \in \mathcal{X}^* \\
\arg\min_{\xi^* \in \mathcal{X}^*} \|\xi - \xi^*\| & \text{else if } \frac{\|\xi - \xi^*\|}{\|\xi - \xi^*\|} > 1 \\
\xi^* & \text{otherwise}
\end{cases}
\]

with \( \xi^* \) the center point of the gap (see Fig. 5).

IV. NONSMOOTH SURFACES

Human-designed environments often contain obstacles and enclosing walls with nonsmooth surfaces, e.g., a table with edges or a building with corners. An approximation with a high gradient of these surfaces can lead to undesired results. On the one hand, a smoothing of the edges increases the risk of colliding with them. On the other hand, an increased, smooth hull conservatively increases the boundary region, and certain parts of the space are not reachable with such a controller.

Moreover, the obstacle avoidance algorithm applied to a surface with a high gradient can lead to a fast change of the flow. Even though the trajectories are smooth, the curvature of the flow can be high and induce fast accelerations. This can result in dangerous behavior in the presence of humans or simply exceed the robot’s torque limits. We propose an approach to avoid obstacles with nonsmooth surfaces without smoothing the boundary.

A polygonal obstacle consists of \( i = 1...N^* \) individually smooth surface planes which form a star shape in \( d - 1 \) such that:

\[
\mathcal{X}^*_i = \{\xi, \hat{\xi} \in \mathbb{R}^b, \exists n_i : \langle n_i, (\xi - \hat{\xi}) \rangle = 0\} \quad (15)
\]

A. Pseudo Normal Vector

The normal to the surface of the obstacle is not defined continuously. As a result, the modulated flow would not be smooth.

We create a smoothly defined pseudo normal \( \hat{n}(\xi) \). It is equal to the normal on the surface of the obstacle. While far away from the obstacle, the pseudo normal approaches the reference direction, i.e.:

\[
\hat{n}(\xi) = n_i(\xi) \quad \forall \xi \in \mathcal{X}^*_i \quad \text{and} \quad \lim_{\|\xi - \xi^*\| \to \infty} \hat{n}(\xi) = r(\xi) \quad (16)
\]

The pseudo normal is the weighted sum of the normals, with the weights being evaluated as follows. At first the vector from the closest edge point \( p_i^e \) (Fig. 6) to the agent’s state is created:

\[
v_i^e(\xi) = \xi - p_i^e \quad \text{with} \quad p_i^e = \arg\min_{\xi^* \in \mathcal{X}^*_i} \|\xi - \xi^*\| \quad (17)
\]
with $\mathcal{F}_i$ the set of all points at the edge of a surface tile $i$. This vector is further projected onto the surface plane:

$$\hat{e}_i(\xi) = \left( \mathbf{v}_i^r(\xi) - (\mathbf{n}_i(\xi), \mathbf{v}_i^r(\xi)) \mathbf{n}_i(\xi) \right) \text{sign}(\mathbf{v}_i^r(\xi), \xi - \xi^r)^t$$

The angle to the plane (Fig. 5) in the range $[0, \pi]$ is evaluated as:

$$\varphi_i^w(\xi) = \arccos \left( \frac{\langle \hat{e}_i(\xi), \mathbf{v}_i^r(\xi) \rangle}{\|\hat{e}_i(\xi)\| \|\mathbf{v}_i^r(\xi)\|} \right) \text{sign}(\mathbf{n}_i(\xi), \mathbf{v}_i^r(\xi))$$

The edge-weight is evaluated as:

$$\tilde{w}_i^j(\xi) = \left\{ \begin{array}{ll} \pi \varphi_i^w(\xi)^p - 1 & \text{if } \varphi_i^w(\xi) \in [0, \pi] \\ 0 & \text{otherwise} \end{array} \right.$$  \hspace{1cm} (19)

with the the weight power $p \in \mathbb{R}$, a free parameter. A lower weight power $p$ results in an increased importance for the closest polygon face compared to the other faces, we simply choose $p = 3$. The final step is the normalization of the weights:

$$w_i^j(\xi) = \frac{\tilde{w}_i^j(\xi)}{\sum_{j} \tilde{w}_i^j(\xi)} \text{ if } \xi \in \mathbb{R}^d \setminus \mathcal{F}_i$$  \hspace{1cm} (20)

The pseudo normal is evaluated as the directional weighted mean (see Sec. A) of normal vectors of the surface tiles $\mathbf{n}_i(\xi)$, the weights $w_i^j$, and with respect to the reference direction $\mathbf{r}(\xi)$.

The basis matrix from $\mathbf{C}$ is redefined for nonsmooth surfaces as:

$$\mathbf{E}(\xi) = [\mathbf{r}(\xi) \ \hat{e}_1(\xi) \cdots \hat{e}_{d-1}(\xi)]$$  \hspace{1cm} (21)

with $\hat{e}_i(\xi)$ the orthonormal basis to $\hat{\mathbf{n}}(\xi)$. The resulting smooth vectorfield can be observed in Fig. 7.

### B. Inverted Obstacles

For an inverted obstacle (Sec. III) the pseudo normal is evaluated at the mirrored position $\xi^\text{mir}$. It is obtained by flipping the current robot state $\xi$ along the reference direction $\mathbf{r}(\xi) = \xi - \xi^r$ onto the other side of the boundary $\mathbb{A}$.

$$\xi^\text{mir} = \Gamma(\xi)^2 (\xi - \xi^r) + \xi^r$$  \hspace{1cm} (22)

The mirrored position allows the evaluation of the distance function as described in Sec. IV-A. Further, the inverted obstacle is treated as described in Sec. III. This allows avoiding nonsmooth obstacles and boundaries in Fig. 4.

**Theorem 2** Consider a polygon composed of $N^d$ surfaces as given in $\mathbb{A}$ or alternatively an inverted polygon as described Sec. IV-B. Any trajectory $\{\xi\}_t$, that starts in free space, i.e. $\Gamma(\{\xi\}_t_0) > 1$ and evolves on a smooth path according to $\mathbb{B}$, will never reach the surface, i.e. $\Gamma(\{\xi\}_t) > 1, t = 0, \infty$ and will converge towards the attractor as long as it is placed outside all obstacles, i.e. $\lim_{t \to \infty} \xi \to \xi^a \in \mathbb{X}^f$.

**Proof:** see Appendix C.

### C. Implementation

Pseudo normals are useful for surroundings with smooth boundaries, for example, pieces of furniture with edges or maps of buildings with sharp corners. These cases consist of polygons with a small number of faces. They allow the evaluation of the algorithm in real-time. It is designed for scenarios with obstacles known from previously learned libraries or a known map retrieved at runtime, such as the tracker of $\text{ao2021spinnet}$.

### V. Dynamic Environments

In changing environments with moving or deforming obstacles the system is modulated with respect to the relative velocity as:

$$\dot{\xi} = \mathbf{M}(\xi) \left( \mathbf{f}(\xi) - \dot{\xi}^\text{tot} \right) + \dot{\xi}^\text{tot}$$  \hspace{1cm} (23)

The local, relative velocity is summed up over all obstacles

$$\dot{\xi}^\text{tot} = \sum_{o=1}^{N_o} \mathbf{w}_o \dot{\xi}_o$$  \hspace{1cm} (24)

with the dynamic weight being a function of the distance $\Gamma(\xi)$:

$$w_o = \sum_o \tilde{w}_o$$  \hspace{1cm} (25)

The relative velocity consists of the obstacle’s velocity $\dot{\xi}_o^a$ and deformation $\dot{\xi}_o^d$:

$$\dot{\xi}_o = \dot{\xi}_o^a + \dot{\xi}_o^d$$  \hspace{1cm} (26)

Note that avoiding dynamic obstacles is not only a modulation of the DS, i.e., a matrix multiplication. This can result in the velocity at the attractor being non-zero, even though the initial dynamical system has zero value there: $\mathbf{f}(\xi^a) = \mathbf{0}$.

For the rest of this section, we will assume the application to each obstacle implicitly without using the subscript $(\cdot)_o$. 
A. Moving Obstacles

The relative velocity of a moving obstacle is obtained similarly to [40]:
\[
\dot{\xi}^v = \dot{\xi}^L_v + \dot{\xi}^R_v \times \xi
\]  
(26)

the linear velocity \( \dot{\xi}^L_v \) and angular velocity \( \dot{\xi}^R_v \) are with respect to the center point of the obstacle \( \xi^c \). The relative position is \( \xi = \xi - \xi^c \).

B. Deforming Obstacle

Obstacles and hulls can not only move, but they can also change their shape with respect to time, e.g. breathing body for a surgery robot. Conversely, the deformation of the perceived obstacle can be the result of uncertainties in real-time obstacle detection and position estimation.

The deformation velocity of an obstacle is evaluated as:
\[
\dot{\xi}^d = \dot{\xi}^L_d + \dot{\xi}^R_d \times \xi^d
\]  
(27)

Where the linear velocity \( \dot{\xi}^L_d \) and angular velocity \( \dot{\xi}^R_d \) are evaluated on the surface position in reference direction, and \( \xi^d = \xi - \xi^b \) is the relative position with respect to the boundary point (see Fig. 8).

The surface deformation should be explicitly given to the algorithm whenever it is known. Alternatively, it can be estimated from sensor readings, such as Lidar or camera.

1) Repulsive Mode: In many scenarios, the consideration of the obstacle’s deformation is only of importance when it reduces the robot’s workspace, and puts the robot at risk. It is sufficient to consider the deformation only along positive normal direction. For example for a circular object, we have:
\[
\dot{\xi}^d = \dot{\xi}^L_d = \begin{cases}
\dot{r} \mathbf{n}(\xi) & \dot{r} > 0 \\
0 & \text{otherwise}
\end{cases}
\]  
(28)

where \( \dot{r} \) is the rate of change of the circle radius.

C. Impenetrability with Respect to Maximum Velocity

Many agents have a maximum velocity they can move with, further referred to as \( v^{\text{max}} \). This limits the motion of obstacles which an agent can avoid to:

\[
v^o = \langle \dot{\xi}, \mathbf{n}(\xi) \rangle < v^{\text{max}} \quad \text{as} \quad \Gamma(\xi) \to 1
\]  
(29)

When close to an obstacle, the agent must prioritize moving away from the obstacle over following the desired motion. Since the modulated velocity \( \dot{\xi} \), see (6), does not take into account the maximal velocity, smart cropping needs to be applied. We propose the following method, which prioritizes avoidance in critical situations but sticks to the initial DS as when safe:

\[
\dot{\xi}^s = \begin{cases}
v^o \mathbf{n}(\xi) + v^o \mathbf{e}(\xi) & \text{if} \quad \langle \frac{\dot{\xi}}{\|\dot{\xi}\|}, \mathbf{n}(\xi) \rangle < \frac{v^o}{v^{\text{max}}} \\
\frac{v^{\text{max}}}{\|\dot{\xi}\|} \dot{\xi} & \text{else} \quad \|\dot{\xi}\| > v^{\text{max}} \\
\dot{\xi} & \text{otherwise}
\end{cases}
\]  
(30)

D. Reference Point Placement

1) Dynamic Extension of Hull: Clusters of more than two convex obstacles do often not form a star-shape. In such cases, we propose to extend the hull of each obstacle such that they all include a common reference point. The new hull is designed to be convex for each obstacle, and hence the cluster is star-shaped. The extended hull creates a cone that is tangent to the obstacle’s surface and has the reference point at its tip (Fig. 9). Since the clusters are star-shapes, there is a global convergence of the vector field towards the attractor.

The extension of the surfaces can be done dynamically, as a collision-free trajectory is ensured around deforming obstacles (Sec. V-B).

2) Clustered Environments with Obstacles and Boundaries: For obstacles which intersect with the boundary, the reference point has to be placed inside the wall, i.e., \( \Gamma(\xi) \to 1 \) (see Fig. 10). This enforces all trajectories to avoid the intersecting obstacles by moving away from the wall (counterclockwise in this example). The boundary modulates them in the same direction. Hence, there is full convergence of all trajectories towards the attractor. This is true for a boundary-obstacle with a positive (local) curvature:

\[
c^o(\xi) > 0 \quad \forall \xi
\]  
(31)

with the curvature given as:

\[
c^{(1)}(\xi) = \lim_{\Delta \xi \to 0} \frac{R(\xi) - R(\xi + \Delta \xi)}{\Delta \xi} \quad \forall \xi \in \mathbb{R}^b, \langle \Delta \xi, \mathbf{n}(\xi) \rangle = 0
\]  
(32)
The algorithm has so far been described for a point mass. It is straightforward to extend this to control robots, which can be approximated by a circle (e.g., drones, wheel-based platforms) by creating a margin around all obstacles wide enough to account for the shape of the robot. The method can also be extended to higher dimensions and multiple degrees of freedom robot arms by describing and evaluating the system (robot + obstacle) in joint-space. This, however, requires representing the obstacle in configuration space which is not always easy, especially when the obstacle moves.

Alternatively, in the rest of this section, we introduce a weighted evaluation of the desired dynamics along a robot arm’s links to obtain a collision-free trajectory towards the desired goal in Cartesian space.

**VI. OBSTACLE AVOIDANCE WITH ROBOTS**

The algorithm has so far been described for a point mass. It is straightforward to extend this to control robots, which can be approximated by a circle (e.g., drones, wheel-based platforms) by creating a margin around all obstacles wide enough to account for the shape of the robot. The method can also be extended to higher dimensions and multiple degrees of freedom robot arms by describing and evaluating the system (robot + obstacle) in joint-space. This, however, requires representing the obstacle in configuration space which is not always easy, especially when the obstacle moves.

Alternatively, in the rest of this section, we introduce a weighted evaluation of the desired dynamics along a robot arm’s links to obtain a collision-free trajectory towards the desired goal in Cartesian space.
Fig. 12: The resulting linear velocity \(v^L\) and angular velocity \(\omega^L\) for a rigid body with to sections points.

2) **Joint Modulation:** The desired joint avoidance command is obtained through inverse kinematics as:

\[
\dot{q}^m = J^\dagger(q, \xi_{S/0}^L) \left[ v^L \right]
\]

(36)

where \(J(q, \xi_{S/0}^L)\) is the Jacobian up the root of the link \(l\), i.e., the position of the section point \(\xi_{S/0}^L\).

### C. Evaluation Along the Robot Arm

The evaluation is performed by iterating over all \(N^L\) links, starting from to base of the robot arm. At each iteration, the joint control \(\dot{q}^c\) of all links which are part of the kinematic chain are updated, i.e., if the evaluation is performed for link \(l\), the joint control are updated for \(\dot{q}^c\) \(\forall i \leq l\).

The evaluation is weighted along with the link to ensure collision avoidance of all links while trying to follow the goal command \(\dot{q}^g\). The link weights \(w_i^L\) are further described in Sec. F.2.

1) **Initial Joint Control:** The joint control is initialized based on the goal control from (35) as:

\[
\dot{q}^c \leftarrow \left( 1 - \sum_{i=1}^{N^L} w_i^L \right) \dot{q}^g
\]

(37)

The first element of the joint command is then updated based on the avoidance velocity obtained in (36) as:

\[
\dot{q}^c_{l+1} \leftarrow \dot{q}^c_{l+1} + w_i^L \dot{q}^m_{l+1}
\]

(38)

2) **Joint Control Correction:** As an effect of influence the obstacle avoidance of each link, the obtained control command \(\dot{q}^c\) differs from the ideal goal command \(\dot{q}^g\). This difference is obtained at link \(l\) as:

\[
v^A = J_l(q) \left[ \dot{q}^c_{l+1} \right]
\]

(39)

where \(J_l(q)\) is the current control command, evaluated up to link \(l-1\), and \(J_l(q)\) is the arm Jacobian with respect to position up to link \(l\).

The joint speed of link \(l\) which would best account for this difference can be evaluated as:

\[
\dot{q}^A = (\omega^A \times \mathbf{I}^L\times) \quad \text{with} \quad \omega^A = v^A \times \mathbf{I}^L
\]

(40)

where \(\mathbf{I}^\omega\) is the direction of rotation of the joint actuating link \(l\) and \(\mathbf{I}^L\) is the direction along with the link, i.e., pointing from one joint to the next. The variables can be observed in Fig. 13.

\[\text{Algorithm 1 Joint Control Command for a Robot Arm}\]

\[\text{Input: } N^L, N^S, f(\xi), \text{ obstacle-environment}\]

\[\text{Output: } \dot{q}^c\]

1: \(\xi_{l+1}^c \forall o \in 1..N^\text{obs}\) \{update dynamic obstacles as in Sec. V\}

2: \(\dot{q}^c \leftarrow (J(q))^\dagger \xi \) \{compute goal command as in (33)\}

3: for \(l = 1 \to N^L\) do

4: for \(s = 1 \to N^S\) do

5: \(\Gamma(\xi_{l+1})\) \{compute danger-field as in (34)\}

6: \(w^A\) \{compute danger-weight as in (55)\}

7: end for

8: \(w_i^L\) \{compute link weight as in (57)\}

9: end for

10: \(\dot{q}^c \leftarrow (1 - \sum_i w_i^L \dot{q}^g)\) \{initialize control command\}

11: \(\dot{q}^c_{l+1} \leftarrow \dot{q}^c_{l+1} + w_i^L \dot{q}^m_{l+1}\)

12: for \(l = 2 \to N^L\) do

13: if \(w_i^L > 0\) then

14: for \(s = 1 \to N^S\) do

15: \(w_i^S\) \{compute section weight as in (58)\}

16: end for

17: \(v^A\), \(\omega^L\) \{compute avoidance velocities as in (35)\}

18: \(\dot{q}^m_l\) \{compute modulation command as in (36)\}

19: \(\dot{q}^c_{l+1} \leftarrow \dot{q}^c_{l+1} + w_i^L \dot{q}^m_{l+1}\)

20: end if

21: \(v^A \leftarrow J_l(q) \left[ \dot{q}^c_{l+1} \right]\)

22: \(\omega^A \leftarrow v^A \times \mathbf{I}^L\)

23: \(\dot{q}^A \leftarrow (\omega^A \times \mathbf{I}^L)\)

24: \(\dot{q}^c_{l+1} \leftarrow \dot{q}^c_{l+1} + \dot{q}^A \sum_i w_i^L\)

25: end for

### D. Validation in Simulation

We applied the algorithm to two scenarios with planar robotic arms (Fig. 14). The scenario in Fig. 14a was chosen similar to the one presented in (30). Our approach can avoid obstacles in a similar setup without a navigation function. The scenario in Fig. 14b includes a more complex robot with three links. Additionally, the skew placement of the obstacle and the position of the start end goal point require the robot to actively go around the obstacle guided by the reference point (see Fig. 11 for the full DS).
Fig. 13: A robot arm with three links ($N_l = 3$), which each containing three sections points ($N_s = 3$) is surrounded by two obstacles. The blue arrows are the goal velocity obtained (only evaluated at the joints), and the red arrows are the weighted avoidance velocities. The direction of joint rotation $I_l^0$ is pointing out of the plane.

VII. COMPARISON ALGORITHMS

A. Qualitative Comparison

We have selected multiple time-invariant, local obstacle avoidance algorithms for qualitative comparison with the presented method Table II. The Navigation Functions, Lyapunov QP, Sphere World QP, and Danger Fields all rely on critical tuning parameters. This is a Lyapunov function for the Lyapunov QP, and parameters for the diffeomorphic transformation or navigation function for the other three methods. While the parameters and functions can be obtained through manual tuning, no solution exists to set them in real-time automatically. Hence, these methods cannot be easily applied to dynamic environments.

Navigation functions and the diffeomorphic transform are defined globally. Their tuning parameter depends on the distribution of the obstacles all across space. Two obstacles close together far from an agent will determine the possible tuning parameters and influence the avoidance behavior. As a result, the methods cannot be transferred easily to clustered dynamic environments.

While other methods have already allowed navigation inside walls, this has not been done in combination with proven star-shape world convergence and dynamic surroundings.

B. Quantitative Comparison

For a quantitative comparison, we chose algorithms that can function in clustered, dynamic environments and can handle external hulls (see Table II). The method for the modulation algorithm in dynamic environments is presented in this paper (referred as Dynamic during this section). It is compared to [35], which uses modulation matrix based on an orthogonal decomposition matrix $E(\xi)$ (referred as Orthogonal) and the potential field algorithm [21] (referred to as Repulsion).

The comparison is made in a simulated environment (Fig. 15). Two ellipse-shaped obstacles randomly change shape, and the movement is inspired by random walk. The combined maximum expansion and obstacle’s velocity are lower than the maximum speed of the agents of $v_{\text{max}} = 1$ m/s. The three algorithms are given the same attractor as a goal. They start at the same time and encounter the same environment. The Dynamic algorithm is observed to have the highest rate of convergence (Tab. II) resulting from the increased environment information through the reference point. The Repulsion has a preferable behavior on avoiding collisions because of its conservative behavior around obstacles (moving far away and only approaching them slowly). This influences the distance traveled and the time needed to reach a goal (Tab. III). The mean of the velocity is lower for the Dynamic algorithm. This is a result of no tail-effect behind the obstacles (see Sec. II-F). The variance of the velocity is similar for the three algorithms.

C. Computational Complexity

The presented algorithm is closed-form, whereas the matrix inverse is the most complex computation. Since it is applied to all obstacles, the complexity follows as $O(d^2N_{\text{obs}})$, a function of the number of dimensions $d$ and the number of obstacles $N_{\text{obs}}$. It had an average time of 3.48 ms on a computer with 8 Intel Core i7-6700 CPU @ 3.40GHz.

This is more complex than the Orthogonal, where the matrix inverse is not required.

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5The source code of the implementation is available on https://github.com/epfl-lasa/dynamic_obstacle_avoidance
TABLE I: The proposed dynamic obstacle avoidance is compared to different state-of-the-art methods. The last three items refer to the main contributions of this work.

|                          | Dynamic (presented) | Reference [57] | Orthog. [55] | Repulsion [21] | Navigation Functions [15], [22], [23] | Lyapunov QP [13] | Sphere World QP | Danger Fields |
|--------------------------|---------------------|----------------|--------------|---------------|----------------------------------------|-------------------|----------------|--------------|
| Time invariant           | ✓                   | ✓              | ✓            | ✓             | ✓                                      | ✓                 | ✓              | ✓            |
| History invariant        | ✓                   | ✓              | ✓            | ✓             | ✓                                      | ✓                 | ✓              | ✓            |
| Convergence: convex      | ✓                   | ✓              | ✓            | ✓             | ✓                                      | ✓                 | ✓              | ✓            |
| Convergence: star-shape  | ✓                   | ✓              | ✓            | ✓             | ✓                                      | ✓                 | ✓              | ✓            |
| No critical tuning parameter (incl. Lyapunov function) | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Avoidance behavior independent of global distribution | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Closed from solution (no optimization) | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Considering initial dynamics (not goal position only) | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Considering initial dynamics (not goal position only) | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Navigation inside walls | ✓                   | ✓              | ✓            | ✓             | ✓                                      | ✓                 | ✓              | ✓            |
| Smooth motion around corners | ✓            | ✓              | ✓            | ✓             | ✓                                      | ✓                 | ✓              | ✓            |
| Clustered dynamic environment | ✓          | ✓              | ✓            | ✓             | ✓                                      | ✓                 | ✓              | ✓            |

TABLE II: The outcome of the 300 trials runs were either full convergence, collision with an obstacle or getting stuck in a local minimum.

|                          | Converged | Collided | Minimum |
|--------------------------|-----------|----------|---------|
| Dynamic                  | Dynamic   | Orthogonal | Repulsion |
|                          | 77%       | 20%      | 39%     |
|                          | 23%       | 23%      | 1%      |
|                          | 0%        | 57%      | 60%     |

TABLE III: The mean and the standard deviation (after the ±) are compared for the three algorithms from the 54 trials where all three agents converged. The metrics of distance (d), duration of the run (t), the mean velocity (v) and the standard deviation of the velocity (σ_{v}) are listed.

|                          | d [m]     | t [s]    | v [m/s]  | σ_{v} [m/s] |
|--------------------------|-----------|----------|----------|-------------|
| Dynamic                  | 9.69 ± 1.19 | 1.03 ± 0.13 | 0.61 ± 0.05 | 0.34 ± 0.02 |
| Orthogonal               | 10.06 ± 1.53 | 1.17 ± 0.21 | 0.55 ± 0.05 | 0.34 ± 0.02 |
| Repulsion                | 9.8 ± 1.29   | 1.39 ± 0.2   | 0.46 ± 0.03 | 0.21 ± 0.02 |

multiplication is the most complex operation. The complexity follows as \( O(d^2N_{\text{obs}}) \). This reflects in the slightly faster evaluation time of 2.84 ms.

The potential field is the simplest of the three algorithms, with the norm being the most complex operation and a complexity \( O(dN_{\text{obs}}) \). It has the lowest evaluation time of 1.75 ms.

The search for the optimal reference point is the computationally most extensive calculation because it requires the (iterative) closest-distance evaluation between obstacles. The Python library Shapely6 was used for the implementation, and an evaluation time of 5.14 ms was obtained.

VIII. EMPIRICAL VALIDATION

The empirical validation is performed with the mobile robot QOLO [44], see Fig. 16, first in simulation and then on a real robot platform. QOLO, a semi-autonomous wheelchair, is designed to navigate in pedestrian environments and indoors. This platform is hence suited to test our algorithm’s ability to avoid moving obstacles (pedestrians) and non-convex obstacles (walls, indoor furniture) containing sharp edges (tables, shelves). In all our experiments, we assume that QOLO has information about the goal, i.e., the attractor \( \xi_a \) of our nominal DS. A video of the experiments is available at 7

A. Static Environment

We task QOLO to navigate in an office-like environment. The room is a square (5 m x 5 m), modeled as a boundary

\[ 4\text{https://github.com/Toblerity/Shapely} \]

\[ 5\text{www.To Be Uploaded.com} \]
obstacle. Further, two tables are located in the room, one at the side and one at the center. The robot starts from the bottom left, and the attractor $\xi_a$ is placed at the opposite side of the room (illustrated with a star). All objects, including the wall, are static and known a priori, and the localization is performed using the SLAM algorithm. The robot evaluates the modulated avoidance in real-time. We run the following two scenarios:

1) QOLO is in the room, and there are two possible paths to go around the center table. The dynamical system is split by the obstacle at the center (Fig. 17a). The robot chooses its preferred trajectory at runtime.

2) Additionally there is a (static) person in the room, which blocks the center passage. The reference point of the obstacles is automatically placed inside the wall. The robot finds its path around the obstacles (Fig. 17b).

B. Dense Crowd (Simulation)

The robot is navigating in a corridor within a dense, simulated crowd. The motion of the crowd is created according to [45]. Two hundred people are moving uniformly along the 6 m wide corridor, either in the same or opposite direction as the robot. QOLO is tasked to travel from one end of the corridor to the other. It is guided by the attractor of the nominal DS. All pedestrians are modeled as circular obstacles with radius of 0.6 m (Fig. 19a).

At each timestep, the problem is reduced to avoiding a subset of the pedestrians. Due to the crowd’s density, the robot could realistically perceive only a subset of the pedestrians in real-time. The number of perceived people is set to $N^c = 10$. The rest of the people is hidden behind a virtual, circular wall. The center of the circular wall $\xi_{c,w}$ is displaced from the position of the robot $\xi_Q$ based on the remaining obstacles:

$$\xi_{c,w} = \xi_Q + \sum_{i=N^c+1}^{N_{obs}} \frac{\xi_i - \xi_Q}{\|\xi_i - \xi_Q\|^2} - (\|\xi_i - \xi_Q\|^2 - r_p^2 - r_Q^2)$$

where $i$ is iterating over the list of the obstacle which are ordered based on their distance to the robot. The displacement factor is with respect to the radius of each pedestrian, $r_p^2 = 0.6$ m, and the robot radius, $r_Q^2 = 0.5$ m.

The radius of the hull is chosen such that the next closest obstacle $N^c + 1$ is fully within the hull. The resulting environment has a dynamic hull with changing center-position and radius (Fig. 19). Reducing the environment to only sphere obstacles decreases the computational time since there is a closed-form solution for the closest distance between two spheres. The evaluation on ROS1 and Python 2.7 run at around 200 Hz on a Up Board: Intel Celeron N3350 with 2.4 GHz (CPU) and 8 GB of RAM. The number of nearby obstacles (including the wall) was eleven, while the wall remained far from the agent, it helped guide the robot around the local crowd (see Sec. V-D2). Further, fast contraction of the boundary can happen when the local crowd density is high. This forces the obstacle to stay away from surrounding obstacles.

1) Quantitative Analysis: We evaluate the effect of the crowd size on the time it takes for the robot to travel through

For a real-world implementation sensory distance measurements in the horizontal plane can be used to create the virtual circular wall and its displacement, since the detection of people is still a time intensive task.
Fig. 20: The QOLO agent is moving in parallel (red) and opposite direction (blue) to the crowd. When the robot moves with the crowd, the density of the crowd has a negligible effect. When the robot moves in the opposite direction, the denser the crowd, the larger the cumulative distance (D) and mean velocity (V), as well as time (T), needed to reach the end of the corridor. The standard deviation of the velocity (V Variance) increases for dense crowds with counterflow.

We assess the time, speed, and distance traveled by the robot when moving with and opposite to the flow, see Fig. 20. When moving with the flow (parallel-flow), the crowd has no significant effect on the distance traveled by the agent or the velocity. When moving against the flow of the crowd, a decrease of the robot’s velocity can be observed for crowds denser than 20 agents per 1000 square meters. (No effect on the crowd was observed since only a single robot was moving against a large crowd.) The distance traveled increases significantly for densities above 100 agents per 1000 square meters. As a result, the average time needed to reach the goal more than doubles for a crowd size of 100 people compared to 2 people.

In counterflow scenarios, the standard deviation of the flow increases. This results from situations where the robot has to slow down or stop to avoid the upcoming agents.

C. Proof of concept: Outdoor Environment

A qualitative proof of concept was performed in an outdoor environment. We brought the QOLO robot to the center of Lausanne, Switzerland 10. Approval was obtained from the EPFL Ethics board and the police of Lausanne city. A driver was on board the robot during the experiment. He could start and stop at all times. A second experimenter was watching the scene and verified the output of the tracker. This was necessary in case the detector/tracker dis-functioned.

Fig. 21: The desired path of the robot in the outdoor environment is the direct line from the initial position of the robot (right) to the target position on the left.

We see this as a successful proof-of-concept of the obstacle avoidance algorithm in a real crowd scenario. The crowd motion was more complex than the streamline simulation, as people would come from all directions and would not group in a uniform flow. Moreover, the crowd included diverse pedestrians, from families with small children to elderly people.

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IX. DISCUSSION

In this work, we have introduced a Dynamical System-based obstacle avoidance algorithm. The modulation-based approach has a theoretical proof of convergence and can be applied to higher-dimensional space. The implementations presented in
The proposed method adds value to the current state-of-the-art, not for global path planning but for reactive obstacle avoidance with convergence proofs. The modulation based avoidance algorithm fully converges in (local) star-shaped environments towards the desired attractor $\xi^*$. The behavior is similar to approaches using potential artificial fields. However, the presented approach does not require finding and deriving an artificial potential function, but the modulation directly outputs the desired velocity.

Compared to other closed-form and QP-based obstacle avoidance algorithms, our method does not require any tuning of critical parameters for its convergence, nor the finding of a Lyapunov candidate function. All parameters presented in this paper can be chosen within the defined range, and theoretical convergence is ensured.

In the presented paper (and initially introduced in [37]), we presented several methods of how to place the reference point (i.e., tune its position). Even though its position is critical for convergence, it is known for star-worlds:

- **Convex Obstacles:** The reference point can be placed anywhere within the obstacle, i.e., $\xi^* \in \mathcal{X}^i$.
- **Star-shapes:** The reference point has to be placed within the kernel of the obstacle. Most of the time, this is just the geometrical center of the obstacle. For polygons, an algorithm to find the kernel has been described in [38].
- **Intersecting Convex Obstacles** If several convex obstacles intersect and form a star-shape, the reference point can be placed anywhere at the center of all intersecting obstacles $o$, i.e., $\xi^* \in \mathcal{X}^i_o \forall o$.

The placement of the reference point only becomes challenging when obstacles merge or separate dynamically. While we propose approximations for many cases (see Sec. A2 and Sec. A3), we do not provide a solution for all scenarios. To the best of the authors’ knowledge, there currently exists no closed-form method to generate a flow around merging and dividing obstacles, which has convergence guarantees. The placement of the reference point for such dynamic scenarios is ongoing research.
C. Future Work

Future work can extend the proposed work in the following areas:

- **Low-level controller:** The low-level controller used in crowd navigation displaced the evaluation point away from the center of the robot. This resulted in an increased (conservative) margin around the robot. The way the shape of the robot is considered in the algorithm should be improved.

- **Environment recognition:** The update rate of the (deep-learning-based) tracker was approximately 5 Hz, while the present avoidance algorithm ran at a frequency of 50 Hz to 100 Hz. The obstacle avoidance was often evaluated with old environment information (but updated robot position). An intermediate estimator could predict how the crowds move in-between.

- **High-level planning:** The combination of the fast obstacle avoidance controller with slower planning algorithms could allow to handle more complex environments, i.e., including the avoidance of surrounding (non-star-shaped) environments.

- **Evaluation in high dimensional space:** The experimental implementation is executed on an autonomous wheelchair (a two-dimensional scenario). However, this work provides a theoretical solution, which can be applied to three and higher dimensional spaces. The next step will be the implementation and evaluation in a higher-dimensional space.

X. Conclusion

A dynamical system-based algorithm for local navigation under convergence constraint is presented. The work provides and tests the solution for local crowd navigation. It ensures certain convergence constraints to not only safely navigate but also reach the goal in local scenarios. The advantage of the method comes from the low complexity and speed of the algorithm. Furthermore, tuning free convergence is obtained in star-world scenarios. This will allow to scale to higher dimensions and transfer to various scenarios.

XI. Acknowledgement

We would like to thank the support of Diego Paez-Granados and David Gonon for the running of the experiments. Their effort and insight have contributed to the implementation using QOLO and the qualitative analysis of the results.

### Appendix

#### A. Directional Weighted Mean

The weighted summation of vectors can result in a zero-sum (e.g., two vectors opposing each other with equal weight) and lead to undesired local-minima of dynamical systems.

We extend here the directional weighted summing introduced in [37] for more general application. It ensures that the summed vector field is free of local minima.

\[ \{ \mathbf{v} \in \mathbb{R}^d : ||\mathbf{v}|| = 1 \} \]  

(a) Three initial vectors.  

(b) Direction-space

Fig. 24: Various directions (here three) are described with respect to a basis direction \( \mathbf{r}^0 \) (a). The directions are transformed to the direction-space \( \mathcal{H} \) (b) where the weighted mean, \( \bar{k} \), is obtained.

The transformation into direction space \( \mathcal{H} \) is given by:

\[ \mathcal{H} = \{ \mathbf{k} \in \mathbb{R}^{d-1} : ||\mathbf{k}|| < \pi \} \]  

The direction space is with respect to a base vector \( \mathbf{b} \) which is the first column of the orthonormal transformation matrix \( \mathbf{B} \). This allows the transformation into the new basis:

\[ \mathbf{v}_i = \mathbf{B}^T \hat{v}_i \]  

The magnitude of the transformed vector in direction space is equal to the angle between the original vector and the reference vector. The transformation of the initial vector \( \mathbf{v}_i \) in the direction-space is:

\[ \kappa_i(\mathbf{b}) = \mathbf{k}(\mathbf{v}_i, \mathbf{b}) = \begin{cases} \arccos(\hat{\mathbf{v}}_i[1]), & 0 \leq \kappa_i[1] \leq 1 \\ 0, & \kappa_i[1] = 1 \end{cases} \]  

The mean is evaluated as a function of the weight \( w_i \) of all \( N^r \) vectors:

\[ \bar{k} = \sum_{i=1}^{N^r} w_i \kappa_i \]  

The mapping into original space is evaluated as:

\[ \bar{\mathbf{v}}(\bar{k}) = \begin{cases} \mathbf{B} \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T, & ||\bar{k}|| = 0 \\ \mathbf{B} \begin{bmatrix} \cos(||\bar{k}||) & \sin(||\bar{k}||) \end{bmatrix}^T, & \text{otherwise} \end{cases} \]  

1) Intuition: In the two-dimensional case, this hyperplane is a line that represents the angle between the initial DS \( \hat{\mathbf{x}}(\hat{\mathbf{z}}) \) and the modulated DS \( \hat{\mathbf{z}} \). It has a magnitude strictly smaller than \( \pi \), the directional space is a vector space, where the weighted mean is taken (see for the three dimensional case in Fig. 24).

**Theorem A** Consider a unit vector \( \mathbf{b} \) as the basis for the projection given in (47) and the corresponding reconstruction function defined in (49). The resulting transformation of unit vector \( \mathbf{k}(\mathbf{v}, \mathbf{b}) : \{ \mathbf{v} \in \mathbb{R}^d : ||\mathbf{v}|| = 1 \} \to \mathcal{H} \) defined in (45) is a bijection and the basis vector projects to the origin, i.e., \( \mathbf{b} \to 0 \).
Proof: The proof is divided into three parts: (I) showing that transformation and reconstruction are the inverse functions of each other, (II) any unit vector is transformed to the direction space $X$, and (III) is reconstructed to a unit vector.

(I) Inverse Functions: To be the inverse functions, applying one after the other onto a vector, must result in the original vector. (Vectors $b$ will be treated separately below).

Let us apply the forward transformation based on (46) and (47) to a unit vector $v_1$. It can be summarized to:

$$\kappa(b) = \arccos \left( \langle b, v_1 \rangle \right) \frac{\hat{B}^T v_1}{\|Bv_1\|}$$

with $\hat{B}$ the matrix $B$ without the first row.

We apply it to a single vector, hence the corresponding weight is $w_1 = 1$. From (48), we get $\mathbf{k} = \mathbf{k}_1$. The reconstruction follows with (49):

$$\bar{v} = B \left[ \begin{array}{c} \cos \left( \frac{\|\hat{b}\|}{\sin \left( \frac{\|\hat{b}\|}{\|b\|} \right) \kappa} \right) \\ \sin \left( \frac{\|\hat{b}\|}{\sin \left( \frac{\|\hat{b}\|}{\|b\|} \right) \kappa} \right) \end{array} \right] = B \left[ \frac{\cos (\mathbf{k}_1) \langle \mathbf{v}_1 \rangle}{\sin (\mathbf{k}_1) \langle \mathbf{v}_1 \rangle} \right] = B \left( B \right)^T v_1 = v_1$$

by using

$$\sin (\arccos (\langle b, v_1 \rangle)) = \sqrt{1 - (b, v_1)^2} \frac{\|Bv_1\|^2 - (b, v_1)^2}{\|\hat{B}^T v_1\|}$$

For the case that $v_1 = b$, we get from (47) that $\mathbf{k} = \mathbf{k}_1 = 0$. And with (49) we get $\mathbf{v}(\mathbf{k}) = b$, i.e., the original vector. Hence for all cases, the transformation is bijective.

(II) Transformation Domain From definition of the transform in (47), the maximum magnitude is the arccos, which is $\|\mathbf{k}\| < \pi$. Hence it lies in the domain of (45). (The magnitude $\pi$ is only reached for a vector $-b$, which is excluded from the transform).

(III) Reconstruction Domain From the inverse transform (49), we have that norm of any transformed vector $\|\mathbf{v}(\mathbf{k})\| = 1$, this follows from the fact that $B$ is orthonormal. As a result they all lie in the domain (44).

2) Pairwise Closest Distance in Direction Space: The directional space can be used for gradient descent to find the closest distance between two (convex) obstacles. This is done by moving along the surface of the obstacle in direction space. Since the direction space is of dimension $d - 1$, finding the closest point of each obstacle has only $2(d - 1)$ degrees of freedom (instead of $2d$ in the Cartesian space). The problem converges to the global minimum when the points start on the line which connects the two center points.

The optimization problem is given as:

$$\min \mathbf{f}^b(\Phi) \quad \text{with} \quad \mathbf{f}^b(\Phi) = \|\xi^b_1(\phi_1) - \xi^b_2(\phi_2)\|, \quad \Phi = \left[ \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right]$$

where the $\xi^b_i(\phi_i)$ denotes the boundary point in the direction $\phi_i$ for the obstacle $i$ with respect to its reference point $\xi_i^r$.

Fig. 25: The minimum distance problem for a rectangular object (with margin) and an ellipsoid. The boundary-reference-point which corresponds to the closes point is marked red (a) and the corresponding gradient descent problem in direction space (b).

Fig. 26: The individual $\Gamma$-function from the obstacle in (a) and (b) are summed as described in (A3). The resulting value function (c) is used for the angle based gradient-descent.

The direction space of each obstacle is created such that the null-direction points towards the other obstacles center. An example is visualized in Fig. 25.

3) Intersecting Obstacle Descent: Two intersecting obstacles need to share one reference point, which lies within both obstacles. The simplification of the problem to surface points only is not of use anymore. Hence, the optimization problem is evaluated in Cartesian space to find a common point that lies inside of the two obstacles’ boundaries:

$$\min_{\xi \in \partial \mathcal{X}^b \cap \partial \mathcal{X}^2} \mathbf{f}(\xi) \quad \text{with} \quad \mathbf{f}(\xi) = \frac{\Gamma^b}{\Gamma^b - \Gamma^o_{1}(\xi)} + \frac{\Gamma^b}{\Gamma^b - \Gamma^o_{2}(\xi)}$$

with $\Gamma^b$ the base distance value, i.e. where the value function reaches infinity. It is chosen slightly larger than the $\Gamma$-value on the surface, we simply choose $\Gamma^b = 1.1$. The step size can be optimized based on the gradient. The optimization problem is convex, and points starting within the intersection region will stay inside due to the infinite repulsion at the boundary as $\Gamma \to 1$. The value function of two ellipses can be found in Fig. 26.

4) Closest Distance for Mixed Environments: The above method for gradient descent in directional space to find the closest distance between two objects, can be applied to an object-boundary pair, if obstacle’s curvature $c^o$ is larger than boundary’s curvature $c^b$ at any position:

$$c^o(\xi_1) < c^b(\xi_2) \quad \forall \xi_1, \xi_2$$ (51)
with the local curvature being defined in (32).

Note that for non-circular obstacles, the condition might locally not hold mainly if the space contains polygon obstacles with local flat regions (\(c = 0\)). This can lead to a locally non-optimal solution when placing the reference point based on the closest distance.

B. Proof of Theorem 1

1) Applicability of General Proofs: In [37] convergence has been proven for star-shaped obstacles. The proof was developed based on the distance function \(\Gamma(\xi)\). Due to inverting the distance function for enclosing wall obstacles and a continuous definition of the modulation, the proof of star-shaped obstacles applies to the case of enclosing walls.

2) Continuity Across Reference Point: In Sec. III-A the Inverted distance function \(\Gamma^*(\xi)\) was not defined at the reference point, as it reaches an infinite value. The continuous definition for the eigenvalue is a unit value, i.e. \(\lambda(\xi^*) = \lambda^*(\xi^*) = 1\), it follows that the diagonal matrix is equal to the identity matrix \(D(\xi^*) = I\). As shown in (1423), we get:

\[ \xi = \xi^* \rightarrow \dot{\xi} = \mathbf{E} \mathbf{D} \mathbf{E}^{-1} \mathbf{f}(\xi^*) = \mathbf{E} \mathbf{I} \mathbf{E}^{-1} \mathbf{f}(\xi^*) = \mathbf{f}(\xi^*) \quad (52) \]

i.e. no modulation of the initial DS. In fact, this is equivalent to the case far away for a classical obstacle with \(\lim_{\|\xi - \xi^*\| \to \infty} \Gamma^*(\xi) \to \infty\).

Even though the basis matrix \(\mathbf{E}(\xi)\) is not defined at \(\xi^*\), the DS is continuously defined across this point since the modulation has no effect.

The trajectory that traverses the reference point \(\xi^*\) of the inverted obstacle corresponds to the trajectory that gets stuck in a saddle point for a common obstacle. As a result, there is full convergence for the inverted obstacles.

C. Proof of Theorem 2

We show first that the modulation has full rank and hence that the dynamics does not vanish outside the attractor and that it is smooth.

1) Full Rank: The basis matrix from (21) has full rank everywhere outside of the obstacle, if the following condition holds:

\[ \arccos(\langle \mathbf{r}(\xi), \hat{\mathbf{n}}(\xi) \rangle) < \pi/2 \quad (53) \]

The angle between the normal to each surface i.e. \(\mathbf{n}_i(\xi)\) and the reference direction \(\mathbf{r}(\xi)\) can be evaluated by defining an vector \(\mathbf{p}_i(\xi) = \mathbf{n}_i(\xi) + \sum_{j=1}^{d_i} k_j e_j(\xi)\) with \(\langle e_j(\xi), \mathbf{r}(\xi) \rangle = 0\), \(k_j \in \mathbb{R}\) such that \(\mathbf{p}_i(\xi)\) and \(\hat{\mathbf{n}}_i(\xi)\), colored in blue in Fig. 27, Using the associative law of the dot product, the geometry constraint of the blue triangle and (19), the maximum angle results in:

\[ \langle \mathbf{n}_i, \mathbf{r} \rangle = (\mathbf{n}_i, \mathbf{r}) \geq \langle \xi - \mathbf{p}_i, \mathbf{r} \rangle \geq 0 \quad \forall w_i(\xi) > 0 \quad (54) \]

Hence the directional transformation of (47) results in \(\|\mathbf{k}_i\| < \pi/2\), \(\forall w_i(\xi) > 0\). Using additionally the triangle equality for vectors: \(\|\mathbf{k}_1 + \mathbf{k}_2\| \leq \|\mathbf{k}_1\| + \|\mathbf{k}_2\|\) applied to all surface directions, it follows with (48) that:

\[ \mathbf{k}_i = \|w_i\| \mathbf{k}_i = \sum_{i=1}^{N} w_i \mathbf{k}_i \leq \sum_{i=1}^{N} w_i \mathbf{k}_i \parallel \max_{\ell \text{ with } w_i > 0} \mathbf{k}_i = \frac{\pi}{2} \]

Since the basis vector of the directional mean is \(\mathbf{r}(\xi)\), with (49) condition (53) holds true.

2) Smooth Vector Field: The continuous extension across the reference point is defined in Appendix B2 and applicable, too.

The reference direction \(\mathbf{r}(\xi)\) and the distance function \(\Gamma(\xi)\) does not have any other discontinuity.

The pseudo normal \(\hat{\mathbf{n}}(\xi)\) is smoothly defined across space. Even in the case when the edge point with the minimum is switching (17), no discontinuity occurs since the angle will stay the same due to the flat surface.

3) Applicability of General Proofs: Since we have a smooth field of normal vectors \(\mathbf{n}(\xi)\), we further need to define any smooth distance function which decreases its value with increasing distance. The two properties are sufficient to comply with the proof of Sec. B.

D. Proof Theorem 3

As an agent approaches the surface of an obstacle, the importance weight from this obstacle is approaching one, i.e., \(\lim_{r(\xi) \to 1} w_\beta = 1\), see (V). It follows with (24) that the relative velocity is \(\dot{\xi} = \dot{\xi}_\text{tot} = \dot{\xi}_\beta\).

As a result it is sufficient to analyze impenetrability for each obstacle individually. The next step is to ensure impenetrability of the three cases in (30):

1) Evaluation in Moving Frame: \(\dot{\xi} = \dot{\xi}\)

The simplest case comes with no stretching, but the evaluation in the local frame of the moving boundary of the obstacle. It follows that the Neuman-boundary condition for impenetrability holds (see also (37)).

2) Contraction within Margin: \(\dot{\xi} = \dot{\xi}\max ||\xi||\)

This contraction is only performed, if it results in a normal velocity which is larger than the velocity of the obstacle \(\dot{\xi}\), i.e. the evaluation in the moving frame results \(\langle (\xi - \dot{\xi} \mathbf{n}(\xi) \rangle = 0\) and ensures impenetrability.
3) Contraction in Tangent Direction:

\[ \dot{\xi} = v^n \mathbf{n}(\xi) + \sqrt{(v^{max})^2 - ||\dot{\xi} n||^2} \mathbf{e}(\xi) \]

This limited contraction along the normal direction ensures that the velocity in normal direction remains equal to the obstacles’ velocity. The evaluation of the Neuman boundary condition in the moving frame leads to:

\[ \langle (v^n \mathbf{n}(\xi) + \sqrt{(v^{max})^2 - ||\dot{\xi} n||^2} \mathbf{e}(\xi)) - \hat{\xi}, \mathbf{n}(\xi) \rangle = v^n - v^n = 0 \]

using the definition of (29) and the fact that the normal \( \mathbf{n}(\xi) \) and the tangent \( \mathbf{e}(\xi) \) are orthogonal.

Hence, we have impenetrability for multiple obstacles.

E. Proof Theorem 3

As an agent approaches the surface of an obstacle, the importance weight from this obstacle is approaching one, i.e., \( \lim_{\xi(\xi) \to 0} w_o = 1 \), see (V). It follows with (23) that the relative velocity is \( \dot{\xi} = \dot{\xi}_{tot} = \dot{\xi}_o \).

As a result it is sufficient to analyze impenetrability for each obstacle individually. The next step is to ensure impenetrability of the three cases in (30):

1) Evaluation in Moving Boundary: \( \dot{\xi} = \dot{\xi}_o \)

The simplest case comes without stretching, but the evaluation in the local frame of the moving boundary of the obstacle. It follows that the Neuman-boundary condition for impenetrability holds (see also (37)).

2) Contraction within Margin: \( \dot{\xi} = v^{max} \|\dot{\xi}\| \hat{\xi} \)

This contraction is only performed, if it results in a normal velocity which is larger than the velocity of the obstacle \( \dot{\xi}_o \), i.e., the evaluation in the moving frame results \( \langle (\dot{\xi} - \hat{\xi}_o, \mathbf{n}(\xi)) \rangle \geq 0 \), hence ensuring impenetrability.

3) Contraction in Tangent Direction:

\[ \dot{\xi} = v^n \mathbf{n}(\xi) + \sqrt{(v^{max})^2 - ||\dot{\xi} n||^2} \mathbf{e}(\xi) \]

This limited contraction along the normal direction ensures that the velocity in normal direction remains equal to the obstacles’ velocity. The evaluation of the Neuman boundary condition in the moving frame leads to:

\[ \langle (v^n \mathbf{n}(\xi) + \sqrt{(v^{max})^2 - ||\dot{\xi} n||^2} \mathbf{e}(\xi)) - \hat{\xi}, \mathbf{n}(\xi) \rangle = v^n - v^n = 0 \]

using the definition of (29) and the fact that the normal \( \mathbf{n}(\xi) \) and the tangent \( \mathbf{e}(\xi) \) are orthogonal.

Hence, we have impenetrability for multiple obstacles.

F. Evaluation Weights for a Robot Arm

We introduce link weights \( w_l^L \) for each link \( l \in [1..N_L] \), and section weights \( w_s^L \) for each section point \( s \) with \( s \in [1..N_s] \). The weights determine how much the obstacle avoidance command influences the control, i.e., zero weights indicate following the goal velocity only, whereas a weight of 1 means full avoidance at this specific position. These weights are designed, such that their product is smaller than one, i.e.:

\[ 0 \leq \sum_l \left( w_l^L \sum_s w_s^L \right) \leq 1 \]

Furthermore, if a section point approaches the surface of an obstacle, it should dominate, i.e.:

\[ \Gamma d(\xi^L_s, l) \to 1 \quad \Rightarrow \quad w_l^L w_s^L \to 1 \]

And when the robot arm is far away from any surface, all weights should go to zero:

\[ \min_{l,s} (\Gamma(\xi^L_s)) \to \infty \quad \Rightarrow \quad \sum_l \left( w_l^L \sum_s w_s^L \right) \to 0 \]

1) Base Weight: For each link \( l \) and section point \( s \), we define a \( \Gamma \)-danger weight \( w_l^L(\xi^L_s) : \{1, \infty \to [0, \infty] \} \to \) to represent the danger of colliding for a specific point: the higher the weight, the greater the chance of collision. It is defined as:

\[ w_l^L(\xi^L_s) = \begin{cases} \Gamma^d(\xi^L_s) - \Gamma^c & \text{if } \Gamma^d(\xi^L_s) < \Gamma^c \\ 0 & \text{otherwise} \end{cases} \quad \text{for } l \in [1..N_L] \]

where \( \Gamma^min = 1 \) is the lower bound, \( \Gamma^c > 1 \) the cutoff value and \( \Gamma^d(\xi^L_s) \) defined in (34).

2) Link Weights: The influence of each link \( l \) is evaluated as:

\[ w_l^L = c^L_\text{Link} \left[ \xi^L_s \right] \max_{x \in [1..N^L]} w_l^F(\xi^L_s, x) \quad \forall l \in [1..N_L] \]

where \( c^L_\text{Link} \in \mathbb{R}_{>0} \) is a constant link-weight factor and \( \xi^L_s \) the \( l \)-th element of the goal vector defined in (33). The \( l \)-factor in the equation gives an increasing importance for links closer to the end-effector.

The link weights are obtained through normalization:

\[ w_l^L = \begin{cases} \frac{\hat{w}_l^L}{\sum_l w_l^L} & \text{if } \hat{w}_l^L > 1 \\ \frac{\hat{w}_l^L}{\sum_l w_l^L} & \text{otherwise} \end{cases} \quad \text{with } \hat{w}_l^L = \sum_l w_l^L \]

3) Section Weights: For all links \( l \) with \( w_l^L > 0 \), the influence weight of each section point is evaluated as:

\[ w_s^L = \frac{\hat{w}_s^L}{\sum_s \hat{w}_s^L} \quad \text{with } \hat{w}_s^L = \frac{s}{N_s} w_l^F(\xi^L_s) \quad \forall s \in [1..N_s] \]

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