Discrete Laplace Method and Truncation Error of Gauss Continued Fraction

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Abstract

The leading asymptotics of the truncation error for Gauss’s continued fraction is determined exactly. Not only for this purpose but also for wider applicability elsewhere the discrete analogue of Laplace’s method for hypergeometric series containing a large parameter, which was developed in a previous paper, is generalized in two directions.

1 Introduction

In 1813 Gauss introduced a general continued fraction

\[ K \sum_{n=0}^{\infty} \frac{R(n)}{1} = \frac{R(0)}{1} + \frac{R(1)}{1} + \frac{R(2)}{1} + \cdots, \tag{1} \]

known today as Gauss’s continued fraction (GCF for short), where \( R(0) := 1 \) and

\[ R(2m + 1) := -\frac{(m + b)(m + c - a)z}{(2m + c)(2m + c + 1)}, \]

\[ R(2m + 2) := -\frac{(m + a + 1)(m + c - b + 1)z}{(2m + c + 1)(2m + c + 2)}, \quad m \in \mathbb{Z}_{\geq 0}, \]

where \( a, b, c \) and \( z \) are complex parameters, with \( z \) being referred to as the independent variable.

For non-vanishing of the numerators and denominators of \( R(n), n \in \mathbb{Z}_{\geq 1} \), we assume that

\[ a, c - b \not\in \mathbb{Z}_{\leq -1}; \quad b, c, c - a \not\in \mathbb{Z}_{\leq 0}. \tag{2} \]

It is well known that for \( z \in \mathbb{C}_z \setminus [1, \infty) \) the continued fraction (1) converges to the ratio

\[ _2F_1(a + 1, b; c + 1; z). \]

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where \( _2F_1(a; b; c; z) \) represents Gauss’s hypergeometric series as well as its analytic continuation to the cut plane \( \mathbb{C} \setminus [1, \infty) \); see e.g. Jones and Thron \[7, \text{Theorem 6.1}\].

Let \( a := (a, b; c), \) \( k := (1, 0; 1) \) and \( p := k + \sigma(k) = (1, 1; 2) \), where \( \sigma : (a, b; c) \rightarrow (b, a; c) \) exchanges the upper parameters \( a \) and \( b \). Notice that \( _2F_1(a; z) \) is invariant under the involution \( \sigma \). Continued fraction \( (1) \) is associated with three-term contiguous relations

\[
_2F_1(a; z) = _2F_1(a + k; z) - \frac{b(c - a)z}{c(c + 1)} _2F_1(a + p; z), 
\tag{3a}
\]

\[
_2F_1(a + k; z) = _2F_1(a + p; z) - \frac{(a + 1)(c - b + 1)z}{(c + 1)(c + 2)} _2F_1(a + p + k; z), 
\tag{3b}
\]

where \( (3a) \) can be found in Andrews et al. \[11, \text{formula (2.5.11)}\], while \( (3b) \) is obtained from \( (3a) \) by applying \( \sigma \) and replacing \( a \) with \( a + k \). For \( m \in \mathbb{Z}_{\geq 0} \) let

\[
F(2m) := _2F_1(a + mp; z), \quad F(2m + 1) := _2F_1(a + mp + k; z).
\]

Taking shifts \( a \mapsto a + mp \) in \( (3) \) induces a three-term recurrence relation

\[
F(n) = F(n + 1) + R(n + 1) F(n + 2), \quad n \in \mathbb{Z}_{\geq 0}, 
\tag{4}
\]

where \( n \) is either \( 2m \) or \( 2m + 1 \). Continued fraction \( (1) \) then follows from \( (4) \) formally.

We are interested in the truncation error of Gauss’s continued fraction,

\[
\mathcal{E}_n(a; z) := \frac{2F_1(a + 1, b; c + 1; z)}{2F_1(a, b; c; z)} - \frac{R(j)}{1}, 
\tag{5}
\]

It is also interesting to consider the specialization of letting \( a \rightarrow 0 \) followed by the substitution \( c \mapsto c - 1 \). The truncation error \( (5) \) then turns into

\[
\mathcal{E}_n^*(b; z) := \frac{2F_1(1, b; c; z)}{1} - \frac{R^*(j)}{1}, \quad b := (b; c),
\]

where \( R^*(0) := 1 \) and \( R^*(n) \) with \( n = 2m + 1 \) or \( 2m + 2 \) is given by

\[
R^*(2m + 1) := -\frac{(m + b)(m + c - 1)z}{(2m + c - 1)(2m + c)},
\]

\[
R^*(2m + 2) := -\frac{(m + 1)(m + c - b)z}{(2m + c)(2m + c + 1)}, \quad m \in \mathbb{Z}_{\geq 0}.
\]

In order for \( R^*(n) \), \( n \in \mathbb{Z}_{\geq 1} \), not to be indefinite, we assume that

\[
b, \ c, \ c - b \notin \mathbb{Z}_{\leq 0}.
\tag{6}
\]

J. Borwein et al. \[24, \text{Theorem 4}\] gave the following estimate in a special case of Gauss’s continued fraction: If \( (b, z) \) satisfies \( 2 \leq b, \ b + 1 \leq c \leq 2b \) and \( -1 \leq z < 0 \), then

\[
|\mathcal{E}_n^*(b; z)| \leq \frac{\Gamma(m + 1)(m + b)\Gamma(m + c - b)\Gamma(b)\Gamma(c)}{\Gamma(m + b)\Gamma(m + c)b\Gamma(c - b)} \left\{ \frac{2b}{(c - 2)(1 - \frac{z}{2}) + (2b - c)} \right\}^n,
\]
where \( m := \lfloor n/2 \rfloor \) is the largest integer not exceeding \( n/2 \). As another topic, based on Gauss’s continued fraction and other means, Colman et al. \[ [3] \] developed an efficient algorithm for the validated high-precision computation of certain special \( _2F_1 \) functions.

The purpose of this article is to determine the leading asymptotics of the truncation error \( E_n(a; z) \) as \( n \to \infty \) for general \( a = (a, b; c) \in \mathbb{C}^3 \) and \( z \in (-\infty, 1) \). Given two sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \), we mean by \( \alpha_n \sim \beta_n \) that their ratio behaves like \( \alpha_n/\beta_n = 1 + O(n^{-\frac{1}{2}}) \) as \( n \to \infty \). Then our main result is stated in the following manner.

**Theorem 1.1** If \((a; z)\) satisfies condition \((2)\), \( z \in (-\infty, 1) \) and \( _2F_1(a; z) \neq 0 \), then

\[
E_n(a; z) \sim \frac{2\pi}{2F_1(a; z)^2} \cdot \frac{\Gamma(c)\Gamma(c+1)}{\Gamma(a+1)\Gamma(b)\Gamma(c-a)\Gamma(c-b+1)} \times \frac{z(1-z)^{c-a-b}}{(1+\sqrt{1-z})^2(c+1)} \left\{ \frac{z}{(1+\sqrt{1-z})^2} \right\}^n.
\]

The relation \( \sim \) in \((7)\) is compatible with the specialization and we have the following.

**Corollary 1.2** If \((b; z)\) satisfies condition \((3)\) and \( z \in (-\infty, 1) \), then

\[
E'_n(b; z) \sim \frac{2\pi^*\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot \frac{z(1-z)^{c-b-1}}{(1+\sqrt{1-z})^{2c}} \left\{ \frac{z}{(1+\sqrt{1-z})^2} \right\}^n.
\]

For every \( z \in (-\infty, 1) \) the dilation constant \( z(1+\sqrt{1-z})^{-2} \) in \((7)\) and \((8)\) is smaller than 1 in its absolute value, so that \( E_n(a; z) \) and \( E'_n(b; z) \) decay exponentially as \( n \to \infty \).

Besides its intrinsic interest, the error estimate of Gauss’s continued fraction is instructive as a testing ground for our discrete analogue of Laplace’s method for general hypergeometric series containing a large parameter. The latter content is expected to have many applications to hypergeometric series, especially to those of higher order. Indeed, an earlier version of it has already had an interesting application to \( _3F_2(1) \) continued fractions in \[ [5] \].

In general a continued fraction is associated with a three-term recurrence relation and the truncation error of the former can be controlled by the ratio of a recessive solution to a dominant one of the latter. For a hypergeometric continued fraction the associated recurrence relation comes from a contiguous relation. For an efficient treatment of recessive and dominant solutions the contiguous relation should be rescaled in an appropriate sense. This is the theme of “simultaneous contiguous relations” in \[ [2] \]. In accordance with this rescaling, the rescaled Gauss continued fraction (rGCF for short) is introduced and its relation with the original GCF is established in \[ [3] \]. Then the recurrence relation associated with the rGCF is considered. An asymptotic representation of a recessive solution to it is given in \[ [4] \].

To deal with dominant solutions, we turn our attention to the general theory of discrete Laplace method. In \[ [5] \] two improvements of the earlier version in \[ [2] \] are made to facilitate its broader applicability. This generalization is illustrated by a couple of examples in \[ [6] \] which are chosen in anticipation of a later application to the rGCF. The assumption imposed in \[ [3] \] is not always fulfilled by a general hypergeometric series. To cope with this situation one has to cut the series into several pieces and manipulate them so that the desired assumption is recovered for each component. The recipe for this procedure is given in \[ [7] \]. In \[ [8] \] we return to the situation of rGCF and derive asymptotic formulas for two dominant solutions to the associated recurrence. In \[ [9] \] after calculating the Casoratian of recessive and dominant solutions, we establish Theorem \[ [1] \] and Corollary \[ [2] \] by putting all the discussions together.
2 Simultaneous Contiguous Relations

Consider a rescaled version of Gauss’s hypergeometric series

\[ 2f_1(a; z) := \sum_{k=0}^{\infty} \frac{\Gamma(a + k)\Gamma(b + k)}{\Gamma(1 + k)\Gamma(c + k)} z^k = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} 2F_1(a; z). \]  

(9)

For generic values of the parameters \( a = (a, b, c) \in \mathbb{C}^3 \) we also consider the rescaled version of Frobenius solutions to the Gauss hypergeometric equation,

\[ f_1^{(0)}(a; z) := 2f_1(a, b; c; z), \]  

(11E1)

\[ f_2^{(0)}(a; z) := z^{1-c} 2f_1(a - c + 1, b - c + 1; 2 - c; z), \]  

(11E17)

\[ f_1^{(1)}(a; z) := 2f_1(a, b; a + b - c + 1; 1 - z), \]  

(11E5)

\[ f_2^{(1)}(a; z) := (1 - z)^{c-a-b} 2f_1(c - a, c - b; c - a - b + 1; 1 - z), \]  

(11E21)

\[ f_1^{(\infty)}(a; z) := (-z)^{-a} 2f_1(a, a - c + 1; a - b + 1; z^{-1}), \]  

(11E9)

\[ f_2^{(\infty)}(a; z) := (-z)^{-b} 2f_1(b - c + 1, b; b - a + 1; z^{-1}). \]  

(11E13)

where for example \([11E17]\) indicates that the original non-rescaled solution appears as formula (17) in Erdélyi et al. [6, Chap. II, §2.8]. It will be more convenient to take a further rescaling

\[ y_1^{(0)}(a; z) := f_1^{(0)}(a; z), \quad y_2^{(0)}(a; z) := f_2^{(0)}(a; z), \]  

(11.1a)

\[ y_1^{(1)}(a; z) := \chi(a) f_1^{(1)}(a; z), \quad y_2^{(1)}(a; z) := \chi(a) f_2^{(1)}(a; z), \]  

(11.1b)

\[ y_1^{(\infty)}(a; z) := \frac{f_1^{(\infty)}(a; z)}{\sin \pi(c - b)}, \quad y_2^{(\infty)}(a; z) := \frac{f_2^{(\infty)}(a; z)}{\sin \pi(c - a)}, \]  

(11.1c)

where the multiplicative factor \( \chi(a) \) in (11.1b) is given by

\[ \chi(a) := \frac{\pi \sin \pi c}{\sin \pi(c - a) \cdot \sin \pi(c - b)} \cdot \frac{1}{\Gamma(e - a) \Gamma(e - b)}. \]  

(11.11)

The connection formulas for the rescaled Frobenius solutions (11.1) are given by

\[ y_1^{(1)}(a; z) = y_1^{(0)}(a; z) - y_2^{(0)}(a; z), \]  

(11.35)

\[ y_2^{(1)}(a; z) = \frac{\sin \pi a \cdot \sin \pi b}{\sin \pi(c - a) \cdot \sin \pi(c - b)} y_1^{(0)}(a; z) - y_2^{(0)}(a; z), \]  

(11.43)

\[ y_1^{(\infty)}(a; z) = \frac{\sin \pi b}{\sin \pi c \cdot \sin \pi(c - b)} y_1^{(0)}(a; z) + \frac{e^{i\pi c}}{\sin \pi c} y_2^{(0)}(a; z), \]  

(11.47)

\[ y_2^{(\infty)}(a; z) = \frac{\sin \pi a}{\sin \pi c \cdot \sin \pi(c - a)} y_1^{(0)}(a; z) + \frac{e^{i\pi c}}{\sin \pi c} y_2^{(0)}(a; z), \]  

(11.49)

where for example \([11E43]\) indicates that the original non-rescaled version can be found in formula (43) of Erdélyi et al. [6, Chap. II, §2.8]. It is remarkable that all of the rescaled connection coefficients are \( \mathbb{Z}^3 \)-periodic, that is, invariant under the translation of \( a \) by any integer vector.
Continued fractions (1) and (15) are equivalent up to a constant multiple, more precisely,

\[ y(a; z) = u(a; z) y(a + k; z) + v(a; z) y(a + p; z). \]  

An equation of this sort is called a contiguous relation. An argument in [1] §2 (which deals with \( _3F_2(1) \) but remains valid for \( _2F_1 \)) shows that the other rescaled Frobenius solution \( y_2^{(0)}(a; z) \) at the origin satisfies the same contiguous relation \((12)\). It then follows from the connection formulas mentioned above, especially from the \( \mathbb{Z}^3 \)-periodicity of the connection coefficients, that contiguous relation \((12)\) is satisfied by all the six rescaled Frobenius solutions \((10)\). We refer to this property as the simultaneousness of contiguous relations.

### 3 Rescaled Gauss Continued Fraction

The simultaneous contiguous relations corresponding to \((35)\) and \((35)\) are given by

\[
\begin{align*}
y(a; z) &= \frac{c}{a} y(a + k; z) + \frac{(a - c)z}{a} y(a + p; z), \quad (13a) \\
y(a + k; z) &= \frac{c + 1}{b} y(a + p; z) + \frac{(b - c - 1)z}{b} y(a + p + k; z), \quad (13b)
\end{align*}
\]

where \( y(a; z) \) is any member of the six functions in \((10)\) and \( k := (1, 0; 1), p := k + \sigma(k) = (1, 1; 2) \) as in \(11\). For \( m \in \mathbb{Z}_{\geq 0} \) let \( y(2m) := y(a + mp; z), y(2m + 1) := y(a + mp + k; z) \) and

\[
\begin{align*}
q(2m) &= \frac{2m + c}{m + a}, \quad r(2m + 1) := -\frac{(m + c - a)z}{m + a} \\
q(2m + 1) &= \frac{2m + c + 1}{m + b}, \quad r(2m + 2) := -\frac{(m + c - b + 1)z}{m + b}.
\end{align*}
\]

Taking shifts \( a \mapsto a + mp, m \in \mathbb{Z}_{\geq 0} \) in \((13)\) leads to a three-term recurrence relation

\[ y(n) = q(n) y(n + 1) + r(n + 1) y(n + 2), \quad n \in \mathbb{Z}_{\geq 0}. \]

where \( n \) is either \( 2m \) or \( 2m + 1 \). If \( y(a; z) \) is \( y_i^{(*)}(a; z) \) in \((10)\) then \( y(n) \) is denoted by \( y_i^{(*)}(n) \).

**Remark 3.1** Recall that there are two transformation formulas called Pfaff’s transformations,

\[ _2F_1(a, b; c; z) = (1 - z)^{-a} _2F_1(a, c - b; c; z/(z - 1)) = (1 - z)^{-b} _2F_1(c - a, b; c; z/(z - 1)), \]

together with their composite called Euler’s transformation (see e.g. [1] Theorem 2.2.5)). We can then speak of the rescaled version of these transformations for \( y_i^{(*)}(a; z) \) and \( y_i^{(*)}(n) \).

Recurrence relation \((14)\) formally induces a rescaled version of Gauss’s continued fraction

\[
\sum_{n=0}^{\infty} \frac{r(n)}{q(n)} = \frac{r(0)}{q(0)} + \frac{r(1)}{q(1)} + \frac{r(2)}{q(2)} + \cdots \quad \text{with} \quad r(0) := 1.
\]

Continued fractions \((1)\) and \((15)\) are equivalent up to a constant multiple, more precisely,

\[
\sum_{j=0}^{n} \frac{R(j)}{1} = \frac{c}{a} \frac{\sum_{j=0}^{n} \frac{r(j)}{q(j)}}{n} \quad \text{with} \quad r(0) := 1.
\]

\[ R(j) \]

\[ j = 0, \ldots, n \]

\[ a \]

\[ n \in \mathbb{Z}_{\geq 0}. \]
It will turn out that if \( y_1^{(0)}(a; z) = 2f_1(a; z) \) is chosen for \( y(a; z) \), then the corresponding sequence \( f(n) := y_1^{(0)}(n) \) is a recessive solution to the recurrence equation \( (14) \). So Pincherle’s theorem \([12, \text{Theorem 5.7}]\) implies that continued fraction \( (15) \) converges to the ratio \( f(1)/f(0) = 2f_1(a + k; z)/2f_1(a; z) \). We are interested in the asymptotic behavior of the truncation error

\[
\varepsilon_n(a; z) := \frac{2f_1(a + k; z)}{2f_1(a; z)} - \frac{n}{K} \sum_{j=0}^{n-1} \frac{r(j)}{q(j)} = \frac{c}{a} \varepsilon_n(a; z).
\]

(17)

where the second equality follows from definitions \([15] \) and \([20] \) and relation \([16] \).

If \( g(n) \) is a dominant solution to \( (14) \) then the error estimate in \([5, \text{§3.1, formula (29)}] \) reads

\[
\varepsilon_n(a; z) = \frac{\omega(0) \cdot h(n)}{f(0)^2} \left\{ 1 + O\left( \frac{g(0) \cdot h(n)}{f(0)} \right) \right\} \quad \text{as} \quad n \to \infty,
\]

(18)

where \( h(n) := f(n + 2)/g(n + 2) \) is the ratio of the recessive solution to the dominant one, while \( \omega(n) := f(n) \cdot g(n + 1) - f(n + 1) \cdot g(n) \) is the Casoratian of \( f(n) \) and \( g(n) \). We remark that Landau’s symbol in \([18] \) is locally uniform with respect a parameter contained in it, so that even if \( f(0), g(0) \) and/or \( h(n) \) are individually singular at some value of the parameter, it remains valid as far as the expression \( g(0) \cdot h(n)/f(0) \) is regular in total.

4 Recessive Solution

Using the usual (continuous) Laplace method we shall find the asymptotic behavior of the sequence \( y_1^{(0)}(n) \), which will serve as a recessive solution to the recurrence equation \( (14) \). Given two sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \), we mean by \( \alpha_n \approx \beta_n \) that \( \alpha_n/\beta_n = 1 + O(n^{-1}) \) as \( n \to \infty \).

Proposition 4.1 For any \( z \in (-\infty, 1) \) there exists an asymptotic representation

\[
y_1^{(0)}(n) \approx \frac{2\sqrt{\pi}(2\sqrt{1-z})^{c-a-b-\frac{1}{2}}}{n^{c-a-b+\frac{1}{2}}(1+\sqrt{1-z})^{n+c-1}}.
\]

(19)

Proof. For \( \Re a > 0 \) and \( \Re(c - a) > 0 \), Euler’s integral representation reads

\[
2f_1(a; z) = \frac{\Gamma(b)}{\Gamma(c - a)} \int_0^1 u(x) \, dx, \quad u(x) := x^{a-1}(1-x)^{c-a-1}(1-zx)^{-b}.
\]

Thus for \( m \in \mathbb{Z}_{\geq 0} \), \( \Re a > -m \) and \( \Re(c - a) > -m \), we have

\[
2f_1(a + mp; z) = \frac{\Gamma(m + b)}{\Gamma(m + c - a)} \int_0^1 \Phi(x)^m u(x) \, dx, \quad \Phi(x) := \frac{x(1-x)}{1-zx}.
\]

The gamma factor behaves like \( \Gamma(m + b)/\Gamma(m + c - a) \approx m^{a+b-c} \) due to Stirling’s formula. Define a function \( \phi(x) \) by \( \Phi(x) = e^{-\phi(x)} \) and observe that

\[
\phi'(x) = \frac{\phi_1(x)}{x(1-x)(1-zx)} \quad \text{with} \quad \phi_1(x) := -zx^2 + 2x - 1.
\]
The quadratic equation \( \phi(x) = 0 \) has a unique root \( x_0 := (1 + \sqrt{1 - z})^{-1} \) such that \( 0 < x_0 < 1 \). Some calculations show that \( \Phi(x_0) = (1 + \sqrt{1 - z})^{-2} \) and
\[
\phi''(x_0) = \frac{2(1 + \sqrt{1 - z})^2}{\sqrt{1 - z}} > 0, \quad u(x_0) = \frac{1 - z^{c-a-b-1}}{(1 + \sqrt{1 - z})^{c-2}}.
\]

A standard argument in Laplace’s asymptotic evaluation yields
\[
2 f_1(a + mp; z) \approx m^{a+b-c} \cdot \sqrt{2\pi} \cdot \frac{u(x_0)}{\Phi(x_0)^m} \cdot m^{-\frac{1}{2}} = \frac{2\sqrt{\pi} \cdot (2\sqrt{1 - z})^{c-a-b-\frac{1}{2}}}{(2m)^{c-a-b+\frac{1}{2}} \cdot (1 + \sqrt{1 - z})^{2m+c-1}}.
\]

For \( n = 2m \) even, since \( y_1^{(0)}(n) = 2 f_1(a + mp; z) \), formula (19) is a direct consequence of (19). For \( n = 2m + 1 \) odd, since \( y_1^{(0)}(n) = 2 f_1(a + mp + k; z) \), formula (19) is obtained from (19) by replacing \( a \) with \( a + k \). Hence the proposition is proved.

\[\Box\]

5 Discrete Laplace Method

In [5, 5] we developed a discrete analogue of Laplace’s method for a class of hypergeometric sums with a large parameter \( n \). The assumptions imposed there were unnecessarily too restrictive. We are able to relax them to some extent without essential changes in the proofs so that the improved results should have broader applicability. Consider a sum of the form
\[
g(n) = \sum_{k=|\mathbb{N}|}^{[r_1 n] - 1} G(k; n) z^k, \quad G(k; n) := \frac{\prod_{i \in I} \Gamma(\sigma_i k + \lambda_i n + \alpha_i)}{\prod_{j \in J} \Gamma(\tau_j k + \mu_j n + \beta_j)},
\]
with an independent variable \( z \), where \( 0 \leq r_0 < r_1 \leq +\infty \); \( \sigma_i, \tau_j \in \mathbb{R}^\times; \lambda_i, \mu_j \in \mathbb{R}; \alpha_i, \beta_j \in \mathbb{C} \), with \( I, J \) being finite sets of indices. The cardinality of \( I \) is denoted by \( |I| \). Put
\[
\rho := z \prod_{i \in I} \frac{\sigma_i}{|\tau_j|^\sigma_i} \prod_{j \in J} \frac{\tau_j}{|\tau_j|^\sigma_j}, \quad \nu := \sum_{i \in I} \lambda_i - \sum_{j \in J} \mu_j, \quad \gamma := \sum_{i \in I} \alpha_i - \sum_{j \in J} \beta_j + \frac{|J| - |I|}{2}.
\]

**Assumption 5.1** Suppose that \( z > 0 \) and the following four conditions are satisfied.

1. Balancedness: \( \sigma = (\sigma_i) \) and \( \tau = (\tau_j) \) are balanced to the effect that
\[
\sum_{i \in I} \sigma_i = \sum_{j \in J} \tau_j.
\]
2. Positivity: all gamma factors in \( G(k; n) \) are positive to the effect that
\[
l_i(x) := \sigma_i x + \lambda_i > 0, \quad m_j(x) := \tau_j x + \mu_j > 0, \quad r_0 < x < r_1.
\]

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(3) Genericness of parameters: \( \alpha := (\alpha_i) \times (\beta_j) \) is generic to the effect that

\[
\delta_\ast(n; \alpha) := \min \left\{ 1, \prod_{i \in I_\ast} \mathrm{dist} \left( \alpha_i^{(s)}(n), \mathbb{Z}_{\leq 0} + |\sigma_i|\mathbb{Z}_{\leq -s} \right) \right\} > 0, \quad * = 0, 1,
\]

where \( \mathrm{dist}(z, Z) \) stands for the distance of a point \( z \in \mathbb{C} \) from a set \( Z \subset \mathbb{C} \) and

\[
I_\ast := \{ i \in I : l_i(r_\ast) = 0 \}, \quad \alpha_i^{(s)}(n) := \alpha_i + \sigma_i([r_\ast n] - r_\ast n).
\]

(4) Convergence: when \( r_1 = +\infty \), the infinite series \( g(n) \) is absolutely convergent for every \( n \gg 1 \), which is the case if and only if one of the following conditions is satisfied:

\[
(i) \ 0 < \rho < 1; \quad (ii) \ \rho = 1, \ \nu < 0; \quad (iii) \ \rho = 1, \ \nu = 0, \ \Re \gamma < -1. \quad (21)
\]

**Remark 5.2** Three remarks are in order about Assumption 5.1.

1. Balancedness of \( \lambda = (\lambda_i) \) and \( \mu = (\mu_j) \), that is, the nullity of \( \nu \) was assumed in \([5, \S 5]\), but this condition is not essential and hence removed in this article. Another improvement is to allow the existence of an independent variable \( z \), which was fixed to be one in \([5, \S 5]\).

2. If \( r_\ast \) is an integer then \( \alpha_i^{(s)}(n) = \alpha_i \) and so \( \delta_\ast(n; \alpha) \) is independent of \( n \), in which case \( \delta_\ast(n; \alpha) \) is simply denoted by \( \delta_\ast(\alpha) \). This will often be the case in practical applications.

3. If \( r_1 = +\infty \) then the positivity (2) forces \( \sigma_i \) and \( \tau_j \) to be positive. Stirling’s formula gives

\[
G(k; n) z^k \approx \text{const.} \ k^{\nu n + \gamma} \rho^k \quad \text{as} \quad k \to +\infty,
\]

where const. is independent of \( k \) (but may depend on \( n \)). This asymptotics readily leads to the convergence conditions (i), (ii), (iii) in item (4) of Assumption 5.1.

The multiplicative phase function \( \Phi(x) \) and the amplitude function \( u(x) \) are defined by

\[
\Phi(x) := z^x \prod_{i \in I} l_i(x)^{l_i(x)} \prod_{j \in J} m_j(x)^{m_j(x)}, \quad u(x) := (2\pi)^{\frac{|\mu|+|\nu|}{2}} \prod_{i \in I} l_i(x)^{\alpha_i - \frac{\gamma}{2}} \prod_{j \in J} m_j(x)^{\beta_j - \frac{\gamma}{2}}, \quad r_0 < x < r_1.
\]

If \( r_1 \) is finite then \( \Phi(x) \) extends to a positive continuous function on the bounded closed interval \([r_0, r_1]\). If \( r_1 = +\infty \) then the function \( \Phi(x) \) admits an asymptotic behavior

\[
\Phi(x) = c \cdot x^\rho \{ 1 + O \left( x^{-1} \right) \} \quad \text{as} \quad x \to +\infty, \quad c := \prod_{i \in I} |\sigma_i|^{\lambda_i} / \prod_{j \in J} |\tau_j|^{\mu_j} > 0,
\]

so one can put \( \Phi(+\infty) := 0 \) if \( 0 < \rho < 1 \) and \( \Phi(+\infty) := c > 0 \) if \( \rho = 1 \). Thus under convergence condition (4) in Assumption 5.1 \( \Phi(x) \) extends to a continuous function on \([r_0, +\infty]\), which is positive on \([r_0, +\infty]\). In either case \( \Phi(x) \) attains a maximum value on \([r_0, r_1]\). Let

\[
\Phi_{\max} := \max_{r_0 \leq x \leq r_1} \Phi(x) > 0, \quad \text{Max} := \{ x \in [r_0, r_1] : \Phi(x) = \Phi_{\max} \}.
\]
The additive phase function $\phi(x)$ is defined by $\Phi(x) = e^{-\phi(x)}$. A little calculation shows

$$
\phi'(x) = \log \prod_{j \in J} m_j(x)^{r_j} z \prod_{i \in I} l_i(x)^{\sigma_i}, \quad \phi''(x) = \sum_{j \in J} \frac{r_j^2}{m_j(x)} - \sum_{i \in I} \frac{\sigma_i^2}{l_i(x)}.
$$

Note that any $x_0 \in \Max \cap (r_0, r_1)$ is a solution to the equation $\phi'(x) = 0$ or equivalently,

$$
\prod_{j \in J} m_j(x)^{r_j} - z \prod_{i \in I} l_i(x)^{\sigma_i} = 0.
$$

We are able to generalize [5, Theorem 5.2 and Proposition 5.14] in the following manner.

**Theorem 5.3** Suppose that $\Phi(r_*) < \Phi_{\max}$ for $* = 0, 1$ and that each maximum point $x_0 \in \Max$ is non-degenerate to the effect that $\phi''(x_0) > 0$. Then $g(n)$ can be expressed as

$$
g(n) = n^{\gamma + \frac{1}{2}} \left( \frac{n}{e} \right)^v \Phi_{\max}^n \{ C + \Omega(n) \}, \quad C := \sqrt{2\pi} \sum_{x_0 \in \Max} \frac{u(x_0)}{\phi''(x_0)}, \quad (22)
$$

and there exist constants $K > 0$, $\lambda > 1$ and $N \in \mathbb{N}$ such that the error term $\Omega(n)$ satisfies

$$
|\Omega(n)| \leq K \left\{ n^{-\frac{1}{2}} + \lambda^{-n} \left( \delta_0(n; \alpha)^{-1} + \delta_1(n; \alpha)^{-1} \right) \right\}, \quad \forall n \geq N. \quad (23)
$$

**Proposition 5.4** For any $\Psi > \Phi_{\max}$ there exist $K > 0$ and $N \in \mathbb{N}$ such that

$$
|g(n)| \leq K (n/e)^v \Psi^n \{ \delta_0(n; \alpha)^{-1} + \delta_1(n; \alpha)^{-1} \}, \quad \forall n \geq N. \quad (24)
$$

**Remark 5.5** The constants $K$ and $N$ in (23) and (24) can be taken uniformly with respect to the parameters $\alpha$ in any bounded subset of $\mathbb{C}^l \times \mathbb{C}^l$ (satisfying $\Re \gamma \leq -1 - \varepsilon$ with a fixed $\varepsilon > 0$ if $r_1 = +\infty$ and case (iii) occurs in (21)). This remark continues to (1) of Remark 8.3.

Note that if $r_1 = +\infty$ then $I_1 = \emptyset$ and hence $\delta_1(\alpha) = 1$ in estimates (23) and (24). What is new in Theorem 5.3 and Proposition 5.4 is the occurrence of the factor $(n/e)^n$ in formulas (22) and (24). The proofs of them are practically the same as those of [5, Theorem 5.2 and Proposition 5.14]. The only difference lies in the manipulation of the function

$$
H(x; n) z^k := \prod_{i \in I} \frac{\Gamma(l_i(x)n + \alpha_i)}{\prod_{j \in J} \Gamma(m_j(x)n + \beta_j)} z^k, \quad r_0 < x < r_1.
$$

Indeed an application of Stirling’s formula to $H(x; n)$ shows that as $n \to \infty$,

$$
H(x; n) z^k \approx n^\gamma u(x) \Phi(x)^n (n/e)^{\sum_{i \in I} l_i(x) - \sum_{j \in J} m_j(x)} n = n^\gamma u(x) \Phi(x)^n (n/e)^v n,
$$

by the balancedness (1) in Assumption 5.1 and the definition of $\nu$. See the proof of [5, Lemma 5.3], where $\lambda$ and $\mu$ are balanced, i.e., $\nu = 0$, so the factor $(n/e)^v n$ does not occur.
6 Some Examples

We illustrate Theorem 5.3 and Proposition 5.4 by a couple of examples. They will be applied to asymptotic analysis of the truncation error for Gauss's continued fraction in [8] and [9]. In this section, \( \rho, \nu \) and \( \gamma \) are the ones defined in [20] and other notations in [4] are also retained.

**Example 6.1** For \( a, b, c \in \mathbb{C} \) and \( z > 0 \) we consider the infinite sum

\[
g_1(n; a, b, c; z) := \sum_{k=0}^{\infty} \frac{\Gamma(k + n + a)\Gamma(k + n + b)}{\Gamma(k + 1)\Gamma(k + c)} z^k. \tag{25}
\]

Note that \( r_0 = 0, r_1 = +\infty, \nu = 2, \gamma = a + b - c - 1, l_1(x) = l_2(x) = x + 1, m_1(x) = m_2(x) = x, \)

\[
\Phi(x) = \frac{z^x (x + 1)^{2x+1}}{x^2 x^{1-x}}, \quad u(x) = \frac{(x + 1)^{a+b-1}}{x^c},
\]

\[
\phi'(x) = 2 \log \frac{x}{\sqrt{z(x + 1)}}, \quad \phi''(x) = \frac{2}{x(x + 1)} > 0, \quad x \in (0, +\infty).
\]

Since \( \rho = z \), the convergence condition is just \( 0 < z < 1 \). Under this condition the equation \( \phi'(x) = 0 \) has a unique solution \( x_0 = \sqrt{z}/(1 - \sqrt{z}) \) in \((0, +\infty)\). Observe that

\[
\Phi(x_0) = (1 - \sqrt{z})^{-2}, \quad u(x_0) = z^{-\frac{c}{2}} (1 - \sqrt{z})^{c-a-b+1},
\]

\[
\phi''(x_0) = 2z^{-\frac{c}{2}} (1 - \sqrt{z})^2, \quad \delta_0(\alpha) = \delta_1(\alpha) = 1.
\]

Hence for \( z \in (0, 1) \) Theorem 5.3 leads to an asymptotic representation

\[
g_1(n; a, b, c; z) \sim \frac{\sqrt{\pi}}{\sqrt{e^{1/2}}} \frac{\frac{n}{e} 2^n (1 - \sqrt{z})^{c-a-b-2n}}{n^{c-a-b+1/2}}. \tag{26}
\]

**Example 6.2** For \( a, b, c \in \mathbb{C} \) and \( z > 0 \) we consider the sum

\[
g_2(n; a, b, c; z) := \sum_{k=0}^{n-1} \frac{\Gamma(k + n + a)\Gamma(k + n + b)}{\Gamma(k + 1)\Gamma(k + c)} z^k.
\]

Note that \( r_0 = 0, r_1 = 1, \nu = 0, \gamma = a - b - c, l_1(x) = x + 1, m_1(x) = m_2(x) = x, m_3(x) = 1 - x, \)

\[
\Phi(x) = \frac{z^x (x + 1)^{x+1}}{x^{2x} (1 - x)^{1-x}}, \quad u(x) = \frac{1}{2\pi} \cdot \frac{(x + 1)^{a-b}}{x^c (1 - x)^{b-\frac{c}{2}}},
\]

\[
\phi'(x) = \log \frac{x^2}{z(1 - x^2)}, \quad \phi''(x) = \frac{2}{x(1 - x^2)} > 0, \quad x \in (0, 1).
\]

The equation \( \phi'(x) = 0 \) has a unique solution \( x_0 = \sqrt{z(1 + z)^{-1}} \) in \((0, 1)\). Observe that

\[
\Phi(x_0) = (\sqrt{z} + \sqrt{z + 1})^2, \quad u(x_0) = (2\pi)^{-1}z^{-\frac{c}{2}} (z + 1)^{1+b-a} (\sqrt{z} + \sqrt{z + 1})^{a+b-1},
\]

\[
\phi''(x_0) = 2z^{-\frac{c}{2}} (z + 1)^{\frac{c}{2}}, \quad \delta_0(\alpha) = \delta_1(\alpha) = 1.
\]

Hence for \( z > 0 \) Theorem 5.3 leads to an asymptotic representation

\[
g_2(n; a, b, c; z) \sim \frac{\sqrt{\pi}}{2\sqrt{\pi} e^{1/2}} \cdot \frac{(\sqrt{z} + \sqrt{z + 1})^{2n+a+b-1}}{n^{b+c-a-1/2}}. \tag{27}
\]
Example 6.3 For $a, b, c, d \in \mathbb{C}$ and $z > 0$ we consider the infinite series

$$g_3(n; a, b, c, d; z) := \sum_{k=n}^{\infty} \frac{\Gamma(2k + 2n + a) \Gamma(2k - 2n + b)}{\Gamma(2k + c) \Gamma(2k + d)} \cdot z^k.$$  

Note that $r_0 = 1$, $r_1 = +\infty$, $\rho = z$, $\nu = 0$ and $\gamma = a + b - c - d$, so the convergence condition is either $0 < z < 1$ or $z = 1$, $\text{Re} \gamma < -1$, which is assumed from now on. Since $l_1(x) = 2(x + 1)$, $l_2(x) = 2(x - 1)$, $m_1(x) = m_2(x) = 2x$, we have

$$\Phi(x) = \frac{z^x}{(x + 1)^2(1 - x)^2} \cdot \frac{u(x)}{x^4}, \quad u(x) = \frac{2^{a+b-c-d}(x + 1)\frac{a}{2}(1 - x)^{b - \frac{1}{2}}}{x^{c+d-1}}.$$  

Thus $\Phi(x)$ is strictly decreasing in $[1, +\infty)$ with maximum $\Phi(1) = 16z$. Observe that

$$\delta_0(\alpha) = \min\{1, \text{dist}(b, \mathbb{Z}_{\leq 0})\}, \quad \delta_1(\alpha) = 1.$$  

Hence for $0 < z < 1$ or $z = 1$, $\text{Re} \gamma < -1$ Proposition 5.3 implies that for any $\Psi > 16z$ there exist a constant $K > 0$ and an integer $N \in \mathbb{N}$ such that

$$|g_3(n; a, b, c, d; z)| < \frac{K \cdot \Psi^n}{\min\{1, \text{dist}(b, \mathbb{Z}_{\leq 0})\}}, \quad \forall n \geq N. \quad (28)$$

7 Decomposition and Sign Changes

The positivity condition (2) in Assumption 5.4 is not always satisfied by a general hypergeometric series. To cope with this situation we have to discuss how to recover the condition.

Consider an infinite series of the form

$$g(n) = \sum_{k=0}^{\infty} G(k; n) \cdot z^k, \quad G(k; n) := \prod_{i \in I} \frac{\Gamma(\sigma_i k + n \lambda_i + \alpha_i)}{\prod_{j \in J} \Gamma(\tau_j k + n \mu_j + \beta_j)}.$$  

where $\sigma_i, \tau_j, \lambda_i, \mu_j \in \mathbb{Z}$ with $\sigma_i \neq 0$ and $\tau_j \neq 0$ for $i \in I$ and $j \in J$. Let $r_1 < r_2 < \cdots < r_m$ be the distinct positive roots of the product $\prod_{i \in I} l_i(x) \prod_{j \in J} m_j(x)$ and put $r_0 := 0$ and $r_{m+1} := +\infty$ by convention. We decompose the series $g(n)$ into $m + 1$ components

$$g(n) = \sum_{s=0}^{m} g_s(n), \quad g_s(n) := \sum_{k=[r_{s+1}n]}^{[r_{s+1}n] - 1} G(k; n) \cdot z^k. \quad (30)$$

Since each of the linear functions $l_i(x)$ and $m_j(x)$ is either positive everywhere or negative everywhere on each interval $\Delta_s := (r_s, r_{s+1})$, one can define index subsets

$$I_s^\pm := \{ i \in I : l_i(x) \gtrless 0 \text{ on } \Delta_s \}, \quad J_s^\pm := \{ j \in J : m_j(x) \gtrless 0 \text{ on } \Delta_s \}.$$  

The corresponding gamma factors in $G(k; n)$ are said to be positive or negative on $\Delta_s$.  

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Applying Euler’s reflection formula $\Gamma(x) \Gamma(1-x) = \pi/\sin \pi x$ to each negative gamma factor of $G(k; n)$ and taking the assumption $\sigma_i, \tau_j, \lambda_i, \mu_j \in \mathbb{Z}$ into account, we have

$$g_s(n) = \pi |I_s^+|^{-|I_s^-|} \prod_{i \in I_s^-} \sin \pi \beta_j \prod_{i \in I_s^+} \sin \pi \alpha_i \cdot (-1)^{\nu^-} n \sum_{k=|I_s^-|}^{[\nu^- + 1]n - 1} G_s(k; n) z_s^k, \quad z_s := (-1)^{\nu^-} z,$$

where $\nu^- := \sum_{i \in I_s^-} \lambda_i - \sum_{j \in I_s^+} \mu_j \in \mathbb{Z}$, $\theta^- := \sum_{i \in I_s^-} \sigma_i - \sum_{j \in I_s^+} \tau_j \in \mathbb{Z}$, and

$$G_s(k; n) := \prod_{i \in I_s^+} \Gamma(\sigma_i k + \lambda_i n + \alpha_i) \prod_{j \in I_s^-} \Gamma(-\tau_j k - \mu_j n + 1 - \beta_j).$$

Notice that all gamma factors in $G_s(k; n)$ are positive on $\Delta_s$, as desired. Proceeding from (29) to (31) via (30) is referred to as the procedure of decomposition and sign changes.

Let $\kappa := |I_s^+| + |I_s^-| - |I_s^-| - |I_s^+|$. If $z_s$ is positive then the multiplicative phase function $\Phi_s(x)$ and the amplitude function $u_s(x)$ for the sum in (31) have representations

$$\Phi_s(x) = z_s^x \prod_{i \in I_s^-} |l_i(x)|^{\kappa_i} \prod_{j \in I_s^+} |l_j(x)|^{\kappa_j}, \quad u_s(x) = (2\pi)^{\nu^- x} \prod_{i \in I_s^-} |l_i(x)|^{\alpha_i - \frac{1}{2}} \prod_{j \in I_s^+} |l_j(x)|^{\beta_j - \frac{1}{2}}, \quad x \in \Delta_s,$$

which are independent of $s$ up the the first factors on the right-hand sides. When $z_s$ is negative, we should make a sign change by dividing the sum in (31) into its even and odd components, where the former is the sum over even $k$’s while the latter is the sum over odd $k$’s, so that $z_s^2 = z^2$ becomes a new independent variable that is positive. This procedure is called the even-odd decomposition. Here is an example illustrating these procedures.

Example 7.1 For $a, b, c \in \mathbb{C}$ and $z < 0$ we consider the infinite sum

$$g_4(n; a, b, c; z) := \sum_{k=0}^{\infty} \frac{\Gamma(k + n + a) \Gamma(k - n + b)}{\Gamma(k + 1) \Gamma(k + c)} z^k.$$

It is absolutely convergent if and only if either $-1 < z < 0$ or $z = -1$, Re$(c - a - b) > 0$, which is assumed from now on. The sum decomposes into two components corresponding to $0 \leq k \leq n - 1$ and $n \leq k < \infty$. After the procedure of decomposition and sign changes we have

$$g_4(n; a, b, c; z) = \frac{\pi (-1)^n}{\sin \pi b} \cdot g_2(n; a, 1 - b, c; -z) + h(n; a, b, c; z),$$

where $g_2(n; a, b, c; z)$ is defined in Examples 6.2 while

$$h(n; a, b, c; z) := \sum_{k=2n}^{\infty} \frac{\Gamma(k + n + a) \Gamma(k - n + b)}{\Gamma(k + 1) \Gamma(k + c)} z^k.$$

The result (27) in Example 6.2 shows that

$$g_2(n; a, 1 - b, c; -z) \sim \frac{(2\sqrt{1 - z})^{c-a-b+\frac{1}{2}}}{\sqrt{\pi} \sqrt{-z}^{c-b}} \cdot \frac{(\sqrt{-z} + \sqrt{1 - z})^{2n+a-b}}{(2n)^{c-a-b+\frac{1}{2}}}.$$
According to whether \( n = 2m \) or \( n = 2m+1 \) the even-odd decomposition of \( h(n; a, b, c; z) \) reads
\[
h(2m; a, b, c; z) = g_3(m; a, b, 1, c; z^2) + z g_3(m; a + 1, b + 1, 2, 1; z^2),
\]
\[
h(2m + 1; a, b, c; z) = z g_3(m; a + 2, b, 2, c + 1; z^2) + z^2 g_3(m; a + 3, b + 1, 3, 2; z^2),
\]
where \( g_3(n; a, b, c; z) \) is defined in Example \( 6.3 \). So the result \( 28 \) in this example shows that for any \( \Psi > 4(-z) \) there exists a constant \( K > 0 \) and an integer \( N > 0 \) such that
\[
|h(n; a, b, c; z)| \leq \frac{K \cdot \Psi^n}{\min\{1, \ \text{dist}(b, z_{\leq 0})\}}, \quad n \geq N,
\]
whether \( n \) is even or odd. Taking \( \Psi \) so that \( 4(-z) < \Psi < (\sqrt{-z} + \sqrt{1-z})^2 \), we have
\[
g_4(n; a, b, c; z) \sim \frac{\sqrt{\pi} (-1)^n}{\sin \pi b} \cdot \frac{(2\sqrt{1-z})^{c-a-b+\frac{1}{2}}}{\sqrt{-z}^{c-b}} \cdot \frac{(\sqrt{-z} + \sqrt{1-z})^{2n+a-b}}{(2n)^{c-a-b+\frac{3}{2}}}. \quad (32)
\]

### 8 Dominant Solutions

According to whether \( z \in (0, 1) \) or \( z \in (-\infty, 0) \), we take different kinds of dominant solutions to the recurrence equation \( 14 \), that is, the solution \( y_1^{(1)}(n) \) in the former case and a Pfaff transformation of \( y_1^{(\infty)}(n) \) in the latter case respectively; see Remark \( 5.1 \) for Pfaff’s transformations.

**Lemma 8.1** For any \( z \in (0, 1) \) we have
\[
y_1^{(1)}(n) \sim \frac{\sqrt{\pi} \sin \pi c}{\sin \pi (c-a) \sin \pi (c-b)} \cdot \frac{(2\sqrt{1-z})^{c-a-b+\frac{1}{2}}}{n^{c-a-b+\frac{3}{2}}} \cdot \frac{(1 + \sqrt{1-z})^{n+c-1}}{z^{n+c-1}}. \quad (33)
\]

**Proof.** From definitions \( 10.1 \) and \( 25 \) we have
\[
y_1^{(1)}(a + mp; z) = \chi(a + mp) g_1(m; a, b, a + b - c + 1; 1 - z),
\]
where definition \( 11.1 \) and Stirling’s formula yields
\[
\chi(a + mp) \approx \frac{\sin \pi c}{2 \sin \pi (c-a) \sin \pi (c-b)} \cdot m^{a+b-2c+1} \left( \frac{m}{c} \right)^{-2m}. \quad (34)
\]
This together with formula \( 26 \) in Example 6.1 leads to
\[
y_1^{(1)}(a + mp) \sim \frac{\sqrt{\pi} \sin \pi c}{\sin \pi (c-a) \sin \pi (c-b)} \cdot \frac{(2\sqrt{1-z})^{c-a-b+\frac{1}{2}}}{(2m)^{c-a-b+\frac{3}{2}}} \cdot \frac{(1 + \sqrt{1-z})^{2m+c-1}}{z^{2m+c-1}}. \quad (33)
\]
When \( n = 2m \) is even, since \( y_1^{(1)}(n) = y_1^{(1)}(a + mp; z) \), formula \( 33 \) directly follows from \( 33 \).
When \( n = 2m + 1 \) is odd, in view of \( y_1^{(1)}(n) = y_1^{(1)}(a + mp + k; z) \), formula \( 33 \) is obtained from \( 33 \) by replacing \( a \) with \( a + k \). Thus the lemma is proved. \( \square \)

**Lemma 8.2** For any \( z \in (-\infty, 0) \) we have
\[
y_1^{(\infty)}(n) \sim \frac{\sqrt{\pi} (-1)^n}{\sin (c-a) \sin (c-b)} \cdot \frac{(2\sqrt{1-z})^{c-a-b+\frac{1}{2}}}{n^{c-a-b+\frac{3}{2}}} \cdot \frac{(1 + \sqrt{1-z})^{n+c-1}}{z^{n+c-1}}. \quad (35)
\]
Proof. Recall that \( y_1^{(\infty)}(a; z) \) is defined by (10\(c \)) with (20\(a \)). A Pfaff transformation of it reads
\[
y_1^{(\infty)}(a; z) = \frac{\chi(a)}{\sin \pi c} (1 - z)^{-a} f_1(a, c - b; a - b + 1; (1 - z)^{-1}),
\]
which corresponds to formula (11) in [6, Chap. II, §2.8], where \( \chi(a) \) is defined in (11). So
\[
y_1^{(\infty)}(a + mp; z) = \frac{\chi(a + mp)}{\sin \pi c} (1 - z)^{-a} f_1(m; a, c - b; a - b + 1; (1 - z)^{-1}),
\]
where \( g_1(n; a, b, c; z) \) is defined in [25]. If \( z \in (-\infty, 0) \) then \( (1 - z)^{-1} \in (0, 1) \), so the result (26) in Example 6.1 is applicable. It follows from formulas (26) and (34) that
\[
y_1^{(\infty)}(a + mp; z) \sim \frac{\sqrt{\pi}}{\sin \pi(c - a) \sin \pi(c - b)} \cdot \frac{(2\sqrt{1 - z})^{c - a - b - \frac{1}{2}}}{(2m)^{c - a - b + \frac{1}{2}}} \left( \frac{1 + \sqrt{1 - z}}{z} \right)^{2m + c - 1}.
\]
When \( n = 2m \) is even, since \( y_1^{(\infty)}(n) = y_1^{(\infty)}(a + mp; z) \), formula (35) directly follows from (35). When \( n = 2m + 1 \) is odd, in view of \( y_1^{(\infty)}(n) = y_1^{(\infty)}(a + mp + k; z) \), formula (35) is obtained from (35\(b \)) by replacing \( a \) with \( a + k \). Thus the lemma is proved. ~

Remark 8.3 Two remarks are in order about Lemmas 8.1 and 8.2.

1. Due to Remark 5.5 the relations \( \sim \) in (33) and (35) are compatible with the specialization procedure of letting \( a \rightarrow 0 \) followed by the substitution \( c \rightarrow c - 1 \).
2. We wonder whether in the proof of Lemma 8.1 a Pfaff transformation of \( y_1^{(1)}(n) \) could be employed instead of itself. A Pfaff transformation of \( y_1^{(1)}(a; z) \) in (10\(b \)) reads
\[
y_1^{(1)}(a; z) = \frac{\sin \pi c}{\sin \pi(c - b)} \cdot \frac{\Gamma(b)}{\Gamma(c - b)} z^{-a} f_1(a, a - c + 1; a + b - c + 1; 1 - z^{-1}),
\]
which is a rescaled version of formula (7) in Erdélyi [6, Chap. II, §2.8]. So we have
\[
y_1^{(1)}(a + mp; z) = \frac{(-1)^m \sin \pi c}{\sin \pi(c - b)} \cdot \frac{\Gamma(m + b)}{\Gamma(m + c - b)} \times z^{-a} g_1(m; a, a - c + 1, a + b - c + 1; 1 - z^{-1}),
\]
where \( g_1(n; a, b, c; z) \) is defined in Example 7.1. Note that \( \Gamma(m + b)/\Gamma(m + c - b) \approx m^{2b-c} \) by Stirling’s formula. Thus the result (32) in Example 7.1 implies formula (35\(a \)), but unfortunately it is valid only for \( z \in (1/2, 1) \) not for all \( z \in (0, 1) \). Similarly, in the proof of Lemma 8.2 the use of \( y_1^{(\infty)}(n) \) itself in stead of its Pfaff transformation leads to formula (35\(b \)), but it is valid only for \( z \in (-\infty, -1) \) not for all \( z \in (-\infty, 0) \).

9 Casoratian and Error Estimates

To use error estimate (18) we have to evaluate the Casoratian \( \omega(0) \). Let \( k := (1, 0; 1) \) and
\[
\omega^{(0)}(a; z) := y_1^{(0)}(a; z) y_2^{(0)}(a + k; z) - y_1^{(0)}(a + k; z) y_2^{(0)}(a; z),
\]
\[
\omega^{(1)}(a; z) := y_1^{(0)}(a; z) y_1^{(1)}(a + k; z) - y_1^{(0)}(a + k; z) y_1^{(1)}(a; z),
\]
\[
\omega^{(\infty)}(a; z) := y_1^{(0)}(a; z) y_1^{(\infty)}(a + k; z) - y_1^{(0)}(a + k; z) y_1^{(\infty)}(a; z).
\]
Lemma 9.1 We have \( \omega^{(0)}(a; z) = -\omega^{(1)}(a; z) \) and

\[
\omega^{(1)}(a; z) = \frac{\pi \sin \pi c}{\sin \pi(c - a) \cdot \sin \pi(c - b)} \cdot \frac{\Gamma(a) \Gamma(b)}{\Gamma(c - a) \Gamma(c - b + 1)} \cdot z^{-c(1 - z)^{c-a-b}},
\]

\[
\omega^{(\infty)}(a; z) = -\frac{\pi}{\sin \pi(c - a) \cdot \sin \pi(c - b)} \cdot \frac{\Gamma(a) \Gamma(b)}{\Gamma(c - a) \Gamma(c - b + 1)} \cdot (-z)^{-c(1 - z)^{c-a-b}}.
\]

Proof. It follows from connection formula (37) that \(-\omega^{(1)}(a; z) = \omega^{(0)}(a; z)\). Let

\[
W(a; z) := y^{(0)}_1(a; z) y^{(0)}_1(a + 1; z) - y^{(0)}_1(a; z) y^{(0)}_1(a + 1; z), \quad 1 := (1, 1, 1).
\]

As in the proof of Lemma 2.1, formula (17c) for \( _3F_2(1) \), we can show

\[
\frac{d}{dz} y^{(i)}_i(a; z) = y^{(i)}_i(a + 1; z), \quad i = 1, 2,
\]

so that \( W(a; z) \) is the Wronskian of \( y^{(i)}_i(a; z) \) and \( y^{(0)}_i(a; z) \). A simple calculation yields

\[
W(a; z) = \frac{\pi \sin \pi c}{\sin \pi(c - a) \cdot \sin \pi(c - b)} \cdot \frac{\Gamma(a) \Gamma(b)}{\Gamma(c - a) \Gamma(c - b + 1)} \cdot z^{-c(1 - z)^{c-a-b-1}}.
\]

There is a simultaneous contiguous relations for the six functions in [10],

\[
y(a + k; z) = \frac{a}{c - b} y(a; z) - \frac{1 - z}{c - b} y(a + 1; z).
\]

Using this relation for \( y(a; z) = y^{(i)}_i(a; z), i = 1, 2 \), we have

\[
-\omega^{(1)}(a; z) = \omega^{(0)}(a; z) = -\frac{1 - z}{c - b} W(a; z).
\]

This together with formula (37) proves the lemma. \(\square\)

Now we are in a position to establish Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1 For \( z = 0 \) formula (7) is trivial and there is nothing to discuss.

For \( z \in (0, 1) \) we apply the general estimate (13) to \( f(n) = y^{(0)}_1(n) \) and \( g(n) = y^{(1)}_1(n) \). Note that \( f(0) = f^{(1)}_1(a; z), g(0) = \chi(a) f^{(1)}_1(a; z) \) and \( \omega(0) = \omega^{(1)}(a; z) \). If we put \( w := z(1 + \sqrt{1 - z})^{-2} > 0 \), then it follows from Proposition 4.1 and Lemma 8.1 that

\[
h(n) = \frac{y^{(0)}_1(n + 2)}{y^{(1)}_1(n + 2)} = 2 \frac{\sin \pi(c - a) \sin \pi(c - b)}{\sin \pi c} \cdot w^{n+c+1}.
\]

Using this formula, various definitions in §2 the first formula in Lemma 9.1 as well as the recursion formula for the gamma function, we obtain

\[
c \cdot \frac{\omega(0) \cdot h(n)}{f(0)^2} \sim \frac{2\pi}{2F_1(a; z)^2} \cdot \frac{\Gamma(c) \Gamma(c + 1)}{\Gamma(a + 1) \Gamma(b) \Gamma(c - a) \Gamma(c - b + 1)} \cdot \frac{z(1 - z)^{c-a-b} w^n}{(1 + \sqrt{1 - z})^{2(c+1)}}, \tag{38a}
\]

\[
g(0) \cdot h(n) \sim \frac{2\pi}{2F_1(a; z)} \cdot \frac{\Gamma(c) \Gamma(c + 1)}{\Gamma(a + 1) \Gamma(b) \Gamma(c - a) \Gamma(c - b + 1)} \cdot \frac{2\pi \Gamma(c) w^{n+c+1}}{\Gamma(c - a) \Gamma(c - b)}, \tag{38b}
\]

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where the left-hand side of (38c) is regular except at the poles of \( \Gamma(c) \) and the zeros of \( 2F_1(a; z) \).

Formula (7) is then derived by combining (17), (18) and (38).

For \( z \in (-\infty, 0) \) we apply estimate (18) to the case where \( f(n) = y_1^{(0)}(n) \) and \( g(n) = y_1^{(\infty)}(a; z) \) given by (36) and \( \omega(0) = \omega^{(\infty)}(a; z) \). Again we put \( w := z(1 + \sqrt{1-z})^{-2} \), which is negative this time. It follows from Proposition 4.1 and Lemma 8.2 that

\[
\frac{y_1^{(0)}(n+2)}{y_1^{(\infty)}(n+2)} \sim 2 \sin \pi(c-a) \sin \pi(c-b) \cdot (-w)^{c+1}w^n.
\]

Using this formula, various definitions in §2, the second formula in Lemma 9.1 as well as the recursion formula for the gamma function, we obtain the same formula as (38a) and

\[
\frac{g(0) \cdot h(n)}{f(0)} \sim \frac{2F_1(a, c-b; a-b+1; (1-z)^{-1})}{2F_1(a; z) \Gamma(a-b+1)} \cdot \frac{2\pi \Gamma(c)}{\Gamma(b) \Gamma(c-a)} \cdot \frac{(-w)^{c+1}w^n}{(1-z)^a},
\]

where the left-hand side of (38a) is regular except at the poles of \( \Gamma(c) \) and the zeros of \( 2F_1(a; z) \). Formula (7) is then derived from (17), (18), (38a) and (38b) as well as the reflection formula for the gamma function.

Proof of Corollary 1.2. By item (1) of Remark 8.3 the three relations \( \sim \) in (38) are compatible with the specialization procedure of letting \( a \to 0 \) followed by the substitution \( c \mapsto c - 1 \). Through this procedure the right-hand sides of (38) change in the following manner.

- RHS of (38a) \( \mapsto \frac{2\pi \Gamma(c)}{\Gamma(b) \Gamma(c-b)} \cdot \frac{z(1-z)^{c-b-1}}{(1+\sqrt{1-z})^2} \cdot w^n \),
- RHS of (38b) \( \mapsto -2 \sin \pi(c-b) \cdot w^n \),
- RHS of (38c) \( \mapsto 2 \sin \pi b \cdot (-w)^cw^n \).

Note that the latter two expressions are regular in \( b = (b; c) \) and hence cause no trouble in applying formulas (17) and (18). Now formula (8) follows from (7) readily.

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