Dynamics of a multi-strain malaria model with diffusion in a periodic environment

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\textbf{ABSTRACT}

This paper mainly explores the complex impacts of spatial heterogeneity, vector-bias effect, multiple strains, temperature-dependent extrinsic incubation period (EIP) and seasonality on malaria transmission. We propose a multi-strain malaria transmission model with diffusion and periodic delays and define the reproduction numbers $R_i$ and $\hat{R}_i$ ($i = 1, 2$). Quantitative analysis indicates that the disease-free $\omega$-periodic solution is globally attractive when $R_i < 1$, while if $R_i > 1 > R_j$ ($i \neq j, i, j = 1, 2$), then strain $i$ persists and strain $j$ dies out. More interestingly, when $R_1$ and $R_2$ are greater than 1, the competitive exclusion of the two strains also occurs. Additionally, in a heterogeneous environment, the coexistence conditions of the two strains are $\hat{R}_1 > 1$ and $\hat{R}_2 > 1$. Numerical simulations verify the analytical results and reveal that ignoring vector-bias effect or seasonality when studying malaria transmission will underestimate the risk of disease transmission.

\section{1. Introduction}

Malaria is a vector-borne disease that is spread from person to person owing to the bite of a female \textit{Anopheles} mosquito and is epidemic in more than 100 countries \cite{32}. According to available information, there are 219 million cases and approximately 3.3 billion people in the world are at risk of contracting this disease \cite{38}. The spread of malaria directly threatens public health and has a huge negative shock on the local economy. Hence, it is critical to investigate the transmission of malaria in the population. Ross \cite{28} first introduced the mathematical model of malaria transmission and then MacDonald \cite{23} extended it, which provides useful insights into the transmission dynamics \cite{41}. Whereas the Ross–Macdonald model is exceedingly simplistic and ignores many key elements of real-world biology and epidemiology \cite{29}. For example, the distribution of rural and urban areas in society. Therefore, in epidemiological models of vector-borne, spatial heterogeneity must be considered \cite{11,13}. People often use reaction–diffusion...
equation to understand the impact of the movements of human and mosquito populations in heterogeneous environments on disease transmission [2,21]. Furthermore, the following several important biological factors related to malaria transmission cannot be ignored.

A typical feature of malaria transmission is ‘vector-bias’ effect, which expounds the distinction in the probability of mosquitoes selecting humans. In 2005, Lacroix proved that mosquitoes are more likely to be attracted to infected persons, a phenomenon known as the vector-bias effect [18]. To understand this preference, Ref. [5] combined vector bias into the model, selecting humans based on the different probability that mosquitoes randomly reach humans and whether he is infected or not. Subsequently, people begin to show solicitude for this issue [5,34]. Assessing the effect of vector bias in diverse regions is of great significance for understanding the spread of malaria.

Trade-off mechanisms for coexistence of strains have been a long-standing question in the evolution of pathogens. It is well known that malaria is caused by *Plasmodium* parasites, and there exists five species of *Plasmodium* that can infect humans: (1) *P. vivax*, (2) *P. falciparum*, (3) *P. malariae*, (4) *P. ovale*, and (5) *P. knowlesi*. *P. falciparum* causes the highest mortality rate and is malignant, while the others are benign [12]. In fact, there are many clinical cases of malaria caused by the other four species of *Plasmodium*. In the mathematical model with multiple strains, the main outcome is competitive exclusion [3], but multiple mechanisms have been found to cause strain coexistence [30,46]. The WHO reports that both *P. falciparum* and *P. vivax* are endemic in countries, for example, Brazil, Cambodia, Guyana, Pakistan, Afghanistan, and Ethiopia, indicating that humans in these areas are at risk of infection with these two species [37]. In fact, there are many clinical cases of malaria caused by the other four species of *Plasmodium*. Whereas we find that most of the existing studies only considered transmission of a single strain of malaria between humans and mosquitoes. Simultaneously, different malaria parasites have variable infection and recovery rates [48]. It is necessary to study multi-strain malaria transmission models.

Temperature directly affects the survivability, lifespan, size, blood supply and reproductive capacity of mosquitoes [7], which is the main vector of malaria. More interestingly, there is considerable evidence that the EIP is exceedingly sensitive to seasonal changing temperature [9,22]. EIP means that mosquitoes may not transmit disease to humans for some time after picking up infected blood. The longevity of a mosquito is generally less than 100 days, and the EIP can arrive 30 days [36]. Understanding its role in malaria spread is critical on account of seasonal changing temperature [26], but few papers on malaria transmission think out the seasonality. Especially the combined effect of the aforesaid elements on the spatial spread of malaria seems to receive little attention.

This paper modifies the Ross model [28] to include vector-bias effect, two strains, temperature-dependent EIPs and diffusion in a heterogeneous space. This work is motivated by the following biological questions: (1) How do seasonal temperature changes and vector bias affect the spread of malaria? (2) Do mosquito and human movements have different effects on the spread of malaria in different regions? (3) Considering seasonal factors, can these two strains coexist in a heterogeneous space? If yes, what conditions need to be met? As these issues are strongly associated with the spread and control of malaria, further research is needed. The challenging point of this work
is to explore the asymptotic behaviour of a complex model that considers multiple factors that influence malaria transmission. In this paper, we extend the theoretical analysis method in Refs. [22,40]. Considering the interaction and coexistence conditions of the two strains in a spatially heterogeneous environment is what distinguishes us from other papers. Since two strains are considered, the system will have multiple boundary periodic solutions, which is different from previous models with only one strain, which leads to the complexity of proofs, especially the proof of uniform persistence.

The remaining parts of the paper are arranged. In Section 2, we develop a reaction–diffusion model in a heterogeneous space that differs from the previously mentioned models. The meaning of the parameters in the model is also explained in this section. Section 3 is dedicated to the well-posedness. In Section 4, we introduce \( R_i \), \( R_0 \) and \( \hat{R}_i \) \((i = 1, 2)\), and explore the solution maps of correlated linear reaction–diffusion subsystems with periodic delay. Section 5 establishes threshold dynamics based on \( R_i \) and \( \hat{R}_i \). Section 6 verifies the previous theoretical results through simulations. In addition, the impact of the diffusion of humans and mosquitoes on the prevention and control of malaria transmission is explored and, how to adjust the allocation of medical resources is analysed. The last is a brief conclusion.

2. The model

The objective of this section is to establish a model of malaria transmission in a spatially heterogeneous environment, incorporating temperature-dependent delays, vector-bias effect and two strains. One hypothesis is that susceptible individuals and mosquitoes can only be infected by one strain and that individuals become susceptible again after recovering, but the mosquitoes cannot recover due to their relatively short lifespan. Let \( N_h(t,x) \) be the total population consisting of three classes: susceptible population \( (S_h(t,x)) \), individuals infected with strain 1 \( (I_1(t,x)) \), and individuals infected with strain 2 \( (I_2(t,x)) \), where \( t \) denotes time and \( x \) stands for position. Then for \( t > 0 \), \( N_h(t,x) = S_h(t,x) + I_1(t,x) + I_2(t,x) \), obeys

\[
\frac{\partial N_h(t,x)}{\partial t} = D_h \Delta N_h(t,x) + b_h(x) - \mu_h(x)N_h(t,x),
\]

and \( x \in \Omega \), where \( \Omega \) is a bounded domain with smooth boundary \( \partial \Omega \); \( D_h \) means the diffusion coefficient of human, which is Hölder continuous and positive on \( \overline{\Omega} \); \( \Delta \) is the usual Laplacian operator. Assume that the form of diffusion is random, it means that individual walkers walk randomly on the solid line using a fixed step [4]. \( \mu_h(x) \) and \( b_h(x) \) are the natural death rate and the recruitment rate of human at position \( x \), respectively. The Neumann boundary condition means that no population flux crosses the boundary \( \partial \Omega \),

\[
\frac{\partial N_h(t,x)}{\partial \nu} = 0, \quad t > 0, \quad x \in \partial \Omega,
\]
where $\frac{\partial}{\partial v}$ is the normal derivative along the outward unit normal vector $v$ on $\partial \Omega$. We know that system (1) with (2) has a positive steady state $N(x) \in C(\Omega, \mathbb{R}_+)^{\times} \setminus \{0\}$ under appropriate assumptions which are globally attractive [45, Theorem 3.15 and Theorem 3.16], where $\mathbb{R}_+$ is the set of positive real numbers and $C(\Omega, \mathbb{R}_+)$ marks the Banach space of continuous functions from $\Omega$ to $\mathbb{R}_+$. For the sake of simplicity, the total number of human at time $t$ and location $x$ stabilizes at $N(x)$, namely, $\forall t \geq 0, x \in \Omega, N_h(t, x) \equiv N(x)$, which is motivated by Refs. [2,21,35]. Then $S_h(t, x) = N(x) - I_1(t, x) - I_2(t, x)$.

Similar to the discussion in Ref. [27], temperature-dependent EIPs are introduced. Let $S_v(t, x), E_v(t, x)$, and $I_v(t, x)$ ($i = 1, 2$) be the number of susceptible, exposed and infected mosquitoes, respectively. Here, the subscript $i$ indicates strain $i$ ($i = 1, 2$). For $t > 0$, we set $M(t, x) = S_v(t, x) + E_v1(t, x) + E_v2(t, x) + I_v1(t, x) + I_v2(t, x)$ to be the overall number of mosquitoes, and to obey

\[
\begin{align*}
\frac{\partial M(t, x)}{\partial t} &= D_v \Delta M(t, x) + \Lambda(t, x) - \mu_v(t, x)M(t, x), \quad x \in \Omega, \\
\frac{\partial M(t, x)}{\partial v} &= 0, \quad x \in \partial \Omega,
\end{align*}
\]

where $\Lambda(t, x) \neq 0$ and $\mu_v(t, x)$ denote the recruitment and the natural mortality rates of mosquitoes, respectively. Particularly, based on Ref. [40], we concentrate on the case where this disease has no discernible effect on the movement of humans and mosquitoes. Mathematically, it is assumed that all individuals have the identical diffusion coefficient $D_h > 0$, and all mosquitoes have the same diffusion coefficient $D_v > 0$. Mosquito biting rate $\beta(t, x)$ refers to the number of bites of a mosquito per unit time at $t$ and $x$. Furthermore, $\Lambda(t, x)$, $\mu_v(t, x)$ and $\beta(t, x)$ are Hölder continuous and nonnegative nontrivial on $\mathbb{R} \times \overline{\Omega}$ and is $\omega$-periodic in $t$. Following the approach in Refs. [2,5,36], at time $t$ and location $x$, the numbers of newly infectious humans and newly infected mosquitoes per unit, are

\[
\frac{c_i \beta(t, x) l [N(x) - I_1(t, x) - I_2(t, x)] I_v(t, x)}{p [I_1(t, x) + I_2(t, x)] + l [N(x) - I_1(t, x) - I_2(t, x)]}
\quad \text{and}
\]

\[
\frac{\alpha_i \beta(t, x) p I_i(t, x) S_v(t, x)}{p [I_1(t, x) + I_2(t, x)] + l [N(x) - I_1(t, x) - I_2(t, x)]},
\]

$i = 1, 2$, respectively, where $p$ and $l$ are defined as the probability that mosquitoes randomly arrive at humans and choose infect and susceptible humans, respectively [5]. Here $c_i$ is the transmission probability from mosquitoes infected with strain $i$ to susceptible humans per bite, and $\alpha_i$ is the transmission probability from humans infected with strain $i$ to susceptible mosquitoes per bite ($i = 1, 2$). The incubation period of humans is relatively short (about 20 days), but the infection period can last for several months, and the lifespan of humans is decades, so the incubation period of humans can be ignored here.
On the basis of the above discussions, one has

\[
\begin{aligned}
\frac{\partial l_1(t,x)}{\partial t} &= D_h \Delta l_1(t,x) + \frac{c_1 \beta(t,x) I_l(t,x)}{p[I_1(t,x) + I_2(t,x)] + \gamma N(x)} - (\mu_\alpha(x) + q_1(x)) l_1(t,x), \quad x \in \Omega, \\
\frac{\partial l_2(t,x)}{\partial t} &= D_h \Delta l_2(t,x) + \frac{c_2 \beta(t,x) I_l(t,x)}{p[I_1(t,x) + I_2(t,x)] + \gamma N(x)} - (\mu_\alpha(x) + q_2(x)) l_2(t,x), \quad x \in \Omega, \\
\frac{\partial s_i(t,x)}{\partial t} &= D_i \Delta s_i(t,x) + \Lambda(t,x) - (\alpha_2(t,x) p l_2(t,x) s_i(t,x)) - \mu_\nu(t,x) s_i(t,x), \quad x \in \Omega, \\
\frac{\partial e_i(t,x)}{\partial t} &= D_i \Delta e_i(t,x) + (\alpha_1(t,x) p l_1(t,x) s_i(t,x)) - \mu_\nu(t,x) e_i(t,x), \quad x \in \Omega,
\end{aligned}
\]

for \( t > 0 \), where \( M_{vi}(t,x) \) measures the number of newly occurred infected mosquitoes with strain \( i \) \((i = 1, 2)\). Further, \( q_i(x) \) represents the recovery rate of strain \( i \) \((i = 1, 2)\), which is Hölder continuous and positive on \( \Omega \).

The expression of \( M_{vi} \) is derived. Let \( \tau_i(t) \) denote the length of temperature-dependent EIP infected by strain \( i \) \((i = 1, 2)\), because it is assumed that the temperature \( T \) varies with time \( t \). Set \( q \) as the development level of infection, it is easy to see that \( q \) describes the completeness of the parasite during the developmental stage of the mosquito (in other words, the completeness of the incubation period), and \( \gamma(t) \) is a temperature-dependent rate increase in \( q \), i.e. \( \frac{dq}{dt} = \gamma(t) \). Let \( \rho(t,q,x) \) be the density of mosquitoes infected by strain 1, with the infection development level \( q \) at time \( t \) and position \( x \).

According to the discussions in Appendix 1 and Refs. [17,40], we can derive the expression of \( M_{vi} \) \((i = 1, 2)\) is

\[
M_{vi}(t,x) = (1 - \tau_i'(t)) \int_\Omega \Gamma(t, t - \tau_i(t), x, y) \frac{\alpha_1(t - \tau_i(t), y) p l_1(t - \tau_i(t), y) s_i(t - \tau_i(t), y)}{p[I_1(t - \tau_i(t), y) + I_2(t - \tau_i(t), y)] + \gamma N(y) - I_1(t - \tau_i(t), y) - I_2(t - \tau_i(t), y)} dy.
\]

\( \Gamma(t, t_0, x, y) \) represents Green function related to \( \frac{\partial u(t,x)}{\partial t} = D_h \Delta u(t,x) - \mu_\nu(t,x) u(t,x) \) obeys the Neumann boundary condition. \( \Gamma(t, t_0, x, y) \) is also biologically meaningful and represents the probability that a mosquito at position \( y \) at time \( t_0 \) will arrive at position \( x \)
after time $t - t_0$. Emphasise that $\Gamma(t + \omega, t_0 + \omega, x, y) = \Gamma(t, t_0, x, y)$ for $t > t_0 \geq 0$ and $x, y \in \Omega$ by virtue of $\mu_v(t, \cdot) = \mu_v(t + \omega, \cdot)$. Here, we assume that the natural death rate of mosquitoes infected with two strains is the same.

Substituting $M_v(t, x)$ ($i = 1, 2$) into system (4), one has

\[
\frac{\partial I_1(t, x)}{\partial t} = D_h \Delta I_1(t, x) + \frac{c_1 \beta(t, x) \left[ N(x) - I_1(t, x) - I_2(t, x) \right] I_1(t, x)}{p \left[ I_1(t, x) + I_2(t, x) \right] + I \left[ N(x) - I_1(t, x) - I_2(t, x) \right]} - (\mu_h(x) + \rho_1(x)) I_1(t, x), \quad x \in \Omega,
\]

\[
\frac{\partial I_2(t, x)}{\partial t} = D_h \Delta I_2(t, x) + \frac{c_2 \beta(t, x) \left[ N(x) - I_1(t, x) - I_2(t, x) \right] I_2(t, x)}{p \left[ I_1(t, x) + I_2(t, x) \right] + I \left[ N(x) - I_1(t, x) - I_2(t, x) \right]} - (\mu_h(x) + \rho_2(x)) I_2(t, x), \quad x \in \Omega,
\]

\[
\frac{\partial S_v(t, x)}{\partial t} = D_s \Delta S_v(t, x) + \Lambda(t, x)
\]

\[
- \frac{\alpha_1 \beta(t, x) p l_1(t, x) S_v(t, x)}{p \left[ I_1(t, x) + I_2(t, x) \right] + I \left[ N(x) - I_1(t, x) - I_2(t, x) \right]} - \mu_v(t, x) S_v(t, x), \quad x \in \Omega,
\]

\[
\frac{\partial E_{v1}(t, x)}{\partial t} = D_e \Delta E_{v1}(t, x) - \mu_v(t, x) E_{v1}(t, x)
\]

\[
+ \frac{\alpha_1 \beta(t, x) p l_1(t, x) S_v(t, x)}{p \left[ I_1(t, x) + I_2(t, x) \right] + I \left[ N(y) - I_1(t, x) - I_2(t, x) \right]} - (1 - \tau_1(t))
\]

\[
\int_\Omega \Gamma(t, t - \tau_1(t), x, y) \frac{\alpha_1 \beta(t - \tau_1(t), y) p l_1(t - \tau_1(t), y) S_v(t - \tau_1(t), y)}{p \left[ I_1(t - \tau_1(t), y) + I_2(t - \tau_1(t), y) \right] + I \left[ N(y) - I_1(t - \tau_1(t), y) - I_2(t - \tau_1(t), y) \right]} dy, \quad x \in \Omega,
\]

\[
\frac{\partial E_{v2}(t, x)}{\partial t} = D_e \Delta E_{v2}(t, x) - \mu_v(t, x) E_{v2}(t, x)
\]

\[
+ \frac{\alpha_2 \beta(t - \tau_2(t), y) p l_2(t - \tau_2(t), y) S_v(t - \tau_2(t), y)}{p \left[ I_1(t - \tau_2(t), y) + I_2(t - \tau_2(t), y) \right] + I \left[ N(y) - I_1(t - \tau_2(t), y) - I_2(t - \tau_2(t), y) \right]} - (1 - \tau_2(t))
\]

\[
\int_\Omega \Gamma(t, t - \tau_2(t), x, y) \frac{\alpha_2 \beta(t - \tau_2(t), y) p l_2(t - \tau_2(t), y) S_v(t - \tau_2(t), y)}{p \left[ I_1(t - \tau_2(t), y) + I_2(t - \tau_2(t), y) \right] + I \left[ N(y) - I_1(t - \tau_2(t), y) - I_2(t - \tau_2(t), y) \right]} dy, \quad x \in \Omega,
\]

\[
\frac{\partial I_{v1}(t, x)}{\partial t} = D_v \Delta I_{v1}(t, x) - \mu_v(t, x) I_{v1}(t, x) + (1 - \tau_1(t)) \int_\Omega \Gamma(t, t - \tau_1(t), x, y)
\]

\[
\frac{\partial I_{v2}(t, x)}{\partial t} = D_v \Delta I_{v2} - \mu_v(t, x) I_{v2}(t, x) + (1 - \tau_2(t)) \int_\Omega \Gamma(t, t - \tau_2(t), x, y)
\]
for $t > 0$, where all constant parameters are positive and $\tau_i(t)$ is positive, continuous and $\omega$-periodic function in $t$. Based on Ref. [40], we get that $1 - \tau'_i(t) = \frac{\gamma(t)}{\gamma(t - \tau_i(t))} > 0, (i = 1, 2)$. See Table A1 for the biological interpretations.

Since $E_{\nu}(t, x) \ (i = 1, 2)$ is decoupled from the other equations in system (5), the following system should be adequate,

$$
\begin{cases}
\frac{\partial u_1(t, x)}{\partial t} = D_h \Delta u_1(t, x) + \frac{c_1 \beta(t, x) l [N(x) - u_1(t, x) - u_2(t, x)] u_4(t, x)}{p [u_1(t, x) + u_2(t, x)] + l [N(x) - u_1(t, x) - u_2(t, x)]} \\
- (\mu_h(x) + \varrho_1(x)) u_1(t, x), \quad x \in \Omega,
\end{cases}
$$

$$
\begin{cases}
\frac{\partial u_2(t, x)}{\partial t} = D_h \Delta u_2(t, x) + \frac{c_2 \beta(t, x) l [N(x) - u_1(t, x) - u_2(t, x)] u_5(t, x)}{p [u_1(t, x) + u_2(t, x)] + l [N(x) - u_1(t, x) - u_2(t, x)]} \\
- (\mu_h(x) + \varrho_2(x)) u_2(t, x), \quad x \in \Omega,
\end{cases}
$$

$$
\begin{cases}
\frac{\partial u_3(t, x)}{\partial t} = D_v \Delta u_3(t, x) + \Lambda(t, x) \\
- \frac{\alpha_1 \beta(t, x) p u_1(t, x) u_3(t, x)}{p [u_1(t, x) + u_2(t, x)] + l [N(x) - u_1(t, x) - u_2(t, x)]} - \mu_v(t, x) u_3(t, x), \quad x \in \Omega,
\end{cases}
$$

$$
\begin{cases}
\frac{\partial u_4(t, x)}{\partial t} = D_v \Delta u_4(t, x) - \mu_v(t, x) u_4(t, x) + (1 - \tau'_1(t)) \int_{\Omega} \Gamma(t, t - \tau_1(t), x, y) \\
\cdot \frac{\alpha_1 \beta(t - \tau_1(t), y) p u_1(t - \tau_1(t), y) u_3(t - \tau_1(t), y)}{p [u_1(t - \tau_1(t), y) + u_2(t - \tau_1(t), y)] + l [N(y) - u_1(t - \tau_1(t), y) - u_2(t - \tau_1(t), y)]} dy, \quad x \in \Omega,
\end{cases}
$$

$$
\begin{cases}
\frac{\partial u_5(t, x)}{\partial t} = D_v \Delta u_5 - \mu_v(t, x) u_5(t, x) + (1 - \tau'_2(t)) \int_{\Omega} \Gamma(t, t - \tau_2(t), x, y) \\
\cdot \frac{\alpha_2 \beta(t - \tau_2(t), y) p u_2(t - \tau_2(t), y) u_4(t - \tau_2(t), y)}{p [u_1(t - \tau_2(t), y) + u_2(t - \tau_2(t), y)] + l [N(y) - u_1(t - \tau_2(t), y) - u_2(t - \tau_2(t), y)]} dy, \quad x \in \Omega,
\end{cases}
$$

$$
\begin{cases}
\frac{\partial u_1(t, x)}{\partial v} = \frac{\partial u_2(t, x)}{\partial v} = \frac{\partial u_3(t, x)}{\partial v} = \frac{\partial u_4(t, x)}{\partial v} = \frac{\partial u_5(t, x)}{\partial v} = 0, \quad x \in \partial \Omega,
\end{cases}
$$

(6)

for $t > 0$, where $(u_1(t, x), u_2(t, x), u_3(t, x), u_4(t, x), u_5(t, x)) \triangleq (I_1(t, x), I_2(t, x), S_v(t, x), I_{v_1}(t, x), I_{v_2}(t, x))$.

### 3. Well-posedness

Let $X_1 := C(\overline{\Omega}, \mathbb{R}^5)$ be the Banach space of continuous functions from $\overline{\Omega}$ to $\mathbb{R}^5$ with the supremum norm $\| \cdot \|_{X_1}$, and $X^*_1 := C(\overline{\Omega}, \mathbb{R}^5_+)$. Let $\hat{\tau}_1 := \max_{t \in [0, \omega]} \tau_1(t), \hat{\tau}_2 := \max_{t \in [0, \omega]} \tau_2(t)$ and $\hat{\tau} := \max \{ \hat{\tau}_1, \hat{\tau}_2 \}$. Define $X_1 := C([\hat{\tau}, 0], X_1)$ to be a Banach space, with the norm $\| \phi \| := \max_{\theta \in [\hat{\tau}, 0]} \| \phi(\theta) \|_{X_1}, \forall \phi \in X_1$, and $X^+_1 := C([\hat{\tau}, 0], X^+_1)$. Then $(X_1, X^+_1)$ and $(X_1, X^+_1)$ are ordered spaces. Considering a function $z : [\hat{\tau}, 0) \rightarrow X_1$ for
\(\sigma > 0,\) we define \(z_t \in \mathcal{X}_1\) by

\[
z_t(\theta) = (z_1(t + \theta), z_2(t + \theta), z_3(t + \theta), z_4(t + \theta), z_5(t + \theta)), \quad \forall \theta \in [-\hat{\tau}, 0],
\]

for any \(t \in [0, \sigma),\)

Set \(\mathcal{Y} := C(\overline{\Omega}, \mathbb{R}), \mathcal{Y}^+ := C(\overline{\Omega}, \mathbb{R}_+),\) and \(\mathcal{Y}_h := \{\varphi \in \mathcal{Y}^+: 0 \leq \varphi(x) \leq N(x), \forall x \in \overline{\Omega}\}.\)

Presume that \(T_j(t, s): \mathcal{Y} \rightarrow \mathcal{Y} (i = 1, 2)\) is the evolution operator related to

\[
\frac{\partial u_i(t, x)}{\partial t} = D_h \Delta u_i(t, x) - (\mu_h(x) + \varrho_i(x))u_i(t, x) := A_iu_i(t, x),
\]

and \(T_3(t, s): \mathcal{Y} \rightarrow \mathcal{Y}\) is the evolution operator related to

\[
\frac{\partial u_3(t, x)}{\partial t} = D_v \Delta u_3(t, x) - \mu_v(t, x)u_3(t, x) := A_3(t)u_3(t, x),
\]

obeys the Neumann boundary condition, respectively. Noting that \(T_j(t, s) = T_j(t - s)\) and \(\mu_v(t, \cdot)\) is \(\omega\)-periodic in \(t\), then \(T_j(t + \omega, s + \omega) = T_j(t, s) \ (j = 1, 2, 3)\) for any \((t, s) \in \mathbb{R}^2\) with \(t \geq s\) [16, Lemma 6.1]. Moreover, from Section 7.1 and Corollary 7.2.3 in Ref. [31], we know that \(T_j(t, s)\) is compact and strongly positive for \((t, s) \in \mathbb{R}^2\) with \(t > s\). Set \(T(t, s) = \text{diag}(T_1(t, s), T_2(t, s), T_3(t, s), T_3(t, s), T_3(t, s), T_3(t, s)) : \mathcal{X}_1 \rightarrow \mathcal{X}_1, t \geq 0,\) to be a strongly continuous semigroup, and \(A(t) = \text{diag}(A_1, A_2, A_3(t), A_3(t), A_3(t)).\)

Define \(A_i\)

\[
D(A_i) := \left\{ \tilde{\varphi} \in C^2(\overline{\Omega}): \frac{\partial \tilde{\varphi}}{\partial v} = 0 \text{ on } \partial \Omega \right\},
\]

\[
A_i\tilde{\varphi} = D_h \Delta \tilde{\varphi} - (\mu_h(x) + \varrho_i(x))\tilde{\varphi}, \quad \forall \tilde{\varphi} \in D(A_i), \quad i = 1, 2,
\]

and \(A_3(t)\) is defined by

\[
D(A_3(t)) := \left\{ \tilde{\varphi} \in C^2(\overline{\Omega}): \frac{\partial \tilde{\varphi}}{\partial v} = 0 \text{ on } \partial \Omega \right\},
\]

\[
A_3(t)\tilde{\varphi} = D_v \Delta \tilde{\varphi} - \mu_v(t, x)\tilde{\varphi}, \quad \forall \tilde{\varphi} \in D(A_3(t)).
\]

Then \(T(t, s): \mathcal{X}_1 \rightarrow \mathcal{X}_1\) is a semigroup generated by the operator \(A(t)\) defined on \(D(A(t)) = D(A_1) \times D(A_2) \times D(A_3(t)) \times D(A_3(t)) \times D(A_3(t)).\) Set \(\mathbb{W}_h := \{\phi \in \mathcal{X}_1^+: 0 \leq \phi_1(x) + \phi_2(x) \leq N(x), \forall x \in \overline{\Omega}\},\) and \(\mathcal{W}_h := C([-\hat{\tau}, 0], \mathbb{W}_h).\) We define \(F = (F_1, F_2,\)
\[ F_3, F_4, F_5 : [0, +\infty) \times W_h \rightarrow \mathbb{R}_1 \] by

\[
\begin{align*}
F_1(t, \phi) &= \frac{c_1 \beta(t, \cdot)[N(\cdot) - \phi_1(0, \cdot) - \phi_2(0, \cdot)]\phi_4(0, \cdot)}{p[\phi_1(0, \cdot) + \phi_2(0, \cdot)] + l[N(\cdot) - \phi_1(0, \cdot) - \phi_2(0, \cdot)]}, \\
F_2(t, \phi) &= \frac{c_2 \beta(t, \cdot)[N(\cdot) - \phi_1(0, \cdot) - \phi_2(0, \cdot)]\phi_5(0, \cdot)}{p[\phi_1(0, \cdot) + \phi_2(0, \cdot)] + l[N(\cdot) - \phi_1(0, \cdot) - \phi_2(0, \cdot)]}, \\
F_3(t, \phi) &= \Lambda(t, \cdot) - \frac{\alpha_1 \beta(t, \cdot)p\phi_1(0, \cdot)\phi_3(0, \cdot)}{p[\phi_1(0, \cdot) + \phi_2(0, \cdot)] + l[N(\cdot) - \phi_1(0, \cdot) - \phi_2(0, \cdot)]} \\
&\quad - \frac{\alpha_2 \beta(t, \cdot)p\phi_2(0, \cdot)\phi_3(0, \cdot)}{p[\phi_1(0, \cdot) + \phi_2(0, \cdot)] + l[N(\cdot) - \phi_1(0, \cdot) - \phi_2(0, \cdot)]}, \\
F_4(t, \phi) &= (1 - \tau_1(t)) \int_{\omega} \Gamma(t, t - \tau_1(t), \cdot, y) \\
&\quad \frac{\alpha_1 \beta(t - \tau_1(t), y)p\phi_1(-\tau_1(t), y)\phi_3(-\tau_1(t), y)}{p[\phi_1(-\tau_1(t), y) + \phi_2(-\tau_1(t), y)] + l[N(y) - \phi_1(-\tau_1(t), y) - \phi_2(-\tau_1(t), y)]} \, dy, \\
F_5(t, \phi) &= (1 - \tau_2(t)) \int_{\omega} \Gamma(t, t - \tau_2(t), \cdot, y) \\
&\quad \frac{\alpha_2 \beta(t - \tau_2(t), y)p\phi_2(-\tau_2(t), y)\phi_3(-\tau_2(t), y)}{p[\phi_1(-\tau_2(t), y) + \phi_2(-\tau_2(t), y)] + l[N(y) - \phi_1(-\tau_2(t), y) - \phi_2(-\tau_2(t), y)]} \, dy,
\end{align*}
\]

for \( \phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)^T \in W_h, t \geq 0 \) and \( x \in \overline{\omega} \). As a consequence, system (6) can be rewritten as the following abstract functional differential equation

\[
\begin{align*}
\frac{\partial u(t, x)}{\partial t} &= A(t)u(t, x) + F(t, u_t), \\
u(\theta, x) &= \phi(\theta, x),
\end{align*}
\]

where \( u(t, \cdot) = (u_1(t, \cdot), u_2(t, \cdot), u_3(t, \cdot), u_4(t, \cdot), u_5(t, \cdot)), t > 0, x \in \omega \) and \( \theta \in [-\bar{\tau}, 0] \). Owing to the \( \omega \)-periodicity of \( \beta(t, \cdot), \tau_i(t), \Lambda(t, \cdot), \tau'_i(t) \) and \( \mu_i(t, \cdot) \) \( (i = 1, 2) \), we know that \( A(t) \) and \( F(t, u_t) \) are \( \omega \)-periodic in \( t \). Then, from Corollary 4 in Ref. [25] and Corollary 8.1.3 in Ref. [39], for each \( \phi \in W_h \), a mild solution can be obtained as a continuous solution of the following integral equation

\[
u(t, \phi) = T(t, 0)\phi(0) + \int_0^t T(t - s)F(s, u_s)ds, \quad t > 0, \quad u_0 = \phi \in W_h.
\]

Using Corollary 7.3.2 of Ref. [31], we have the following proposition on the unique global solution that exists in the system (6).

**Lemma 3.1:** For \( \phi \in W_h \), system (6) admits the unique solution, remarked as \( z(t, \cdot, \phi) \) on its maximal existence interval \([0, \bar{t}_\phi] \) with \( z_0 = \phi \), where \( \bar{t}_\phi \leq \infty \). Additionally, for \( \forall t \in [0, \bar{t}_\phi] \), \( z(t, \cdot, \phi) \in \mathbb{R}_h \times \mathbb{R}_h \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \) and for all \( t > \bar{\tau} \), \( z(t, \cdot, \phi) \) is a classical solution to system (6).
We first return to system (5) for more observations. Considering the biological significance of $\tau_i(t)$ ($i = 1, 2$), the following compatibility condition are imposed

$$E_{vi}(0, \cdot) = \int_{-\tau_i(0)}^{0} T_3(0, s) \frac{\alpha_i \beta(s, \cdot) p I_i(s, \cdot) S_v(s, \cdot)}{p [I_1(s, \cdot) + I_2(s, \cdot)] + l [N(\cdot) - I_1(s, \cdot) - I_2(s, \cdot)]} \, ds. \tag{7}$$

Define

$$D := \left\{ \tilde{\psi} \in C([-\hat{\tau}, 0], C(\overline{\Omega}, \mathbb{R}^7)) : \tilde{\psi}_1(\theta, \cdot) + \tilde{\psi}_2(\theta, \cdot) \leq N(\cdot), \quad \forall \theta \in [-\hat{\tau}, 0], \right\}$$

where

$$\tilde{\psi}_4(0, \cdot) = \int_{-\tau_1(0)}^{0} T_3(0, s) \frac{\alpha_1 \beta(s, \cdot) p \tilde{\psi}_1(s, \cdot) \tilde{\psi}_3(s, \cdot)}{p [\tilde{\psi}_1(s, \cdot) + \tilde{\psi}_2(s, \cdot)] + l [N(\cdot) - \tilde{\psi}_1(s, \cdot) - \tilde{\psi}_2(s, \cdot)]} \, ds,$$

$$\tilde{\psi}_5(0, \cdot) = \int_{-\tau_2(0)}^{0} T_3(0, s) \frac{\alpha_2 \beta(s, \cdot) p \tilde{\psi}_2(s, \cdot) \tilde{\psi}_3(s, \cdot)}{p [\tilde{\psi}_1(s, \cdot) + \tilde{\psi}_2(s, \cdot)] + l [N(\cdot) - \tilde{\psi}_1(s, \cdot) - \tilde{\psi}_2(s, \cdot)]} \, ds.$$

Therefore, for any $\tilde{\psi} \in D$, system (5) admits a unique solution $U(t, \cdot, \tilde{\psi}) = (I_1(t, \cdot), I_2(t, \cdot), S_v(t, \cdot), E_{v1}(t, \cdot), E_{v2}(t, \cdot), I_1(t, \cdot), I_2(t, \cdot))$ satisfying $U_0 = \tilde{\psi}$. From [25, Corollary 4], we get $I_i(t, \cdot) \geq 0$, $S_v(t, \cdot) \geq 0$ and $I_{vi}(t, \cdot) \geq 0$ ($i = 1, 2$). According to the uniqueness of solution and the compatibility conditions (7), it follows that

$$E_{vi}(t, \cdot) = \int_{t-\tau_i(t)}^{t} T_3(t, s) \frac{\alpha_i \beta(s, \cdot) p I_i(s, \cdot) S_v(s, \cdot)}{p [I_1(s, \cdot) + I_2(s, \cdot)] + l [N(\cdot) - I_1(s, \cdot) - I_2(s, \cdot)]} \, ds, \quad i = 1, 2. \tag{8}$$

Hence, $E_{vi}(t, \cdot) \geq 0$.

Next, we proof the ultimate boundedness of the solution of system (6).

**Lemma 3.2:** There is a $G > 0$, such that any solution $(u_1(t, x), u_2(t, x), u_3(t, x), u_4(t, x), u_5(t, x))$ of (6) meets

$$0 \leq u_i(t, x) \leq G, \quad i = 1, 2, 3, 4, 5, \quad \forall t > 0, \quad x \in \overline{\Omega}. \tag{9}$$

**Proof:** Based on (1), one has

$$\frac{\partial N_h(t, x)}{\partial t} \leq D_h \Delta N_h(t, x) + \tilde{b}_h - \bar{\mu}_h N_h(t, x),$$

where $\tilde{b}_h = \max_{x \in \Omega} b_h(x)$ and $\bar{\mu}_h = \min_{x \in \Omega} \mu_h(x)$. Similarly, for (3), we also have

$$\frac{\partial M(t, x)}{\partial t} \leq D_v \Delta M(t, x) + \tilde{A} - \bar{\mu}_v M(t, x),$$

where $\tilde{A} = \max_{t \in [0, \infty], x \in \Omega} \Lambda(t, x)$ and $\bar{\mu}_v = \min_{t \in [0, \infty], x \in \Omega} \mu_v(t, x)$. Choose $G = \max \{ \tilde{b}_h / \bar{\mu}_h, \tilde{A} / \bar{\mu}_v \}$, therefore, (9) holds.

Based on the above discussion, the solution of (6) in $W_h$ exists globally on $[0, \infty)$ and is ultimately bounded. Through similar arguments with Lemma 2.6 in [15], Lemma 2.1 in [43], Theorem 2.1 and Theorem 7.3.1 of [31], together with Theorem 2.9 in [24], we have the existence of continuous semi-flow.

**Lemma 3.3:** For each $\phi \in W_h$, system (6) admits a unique solution $u(t, \cdot, \phi)$ on $[0, +\infty)$ with $u_0 = \phi$. Moreover, system (6) generates an $\omega$-periodic semi-flow $\Phi_t : W_h \rightarrow W_h$, defined by $\Phi_t := u_t(\phi), t \geq 0$, and $\Phi := \Phi_\omega$ has a strong global attractor in $W_h$. 


4. Reproduction numbers

An extremely important concept in epidemiology is basic reproduction number, which is generally defined as the average number of secondary infections produced by a type infected human when join a completely susceptible population during the entire infection period [8]. Furthermore, we also introduce invasion reproduction numbers, the average number of secondary infections in a population where an infected individual is susceptible to this strain, but another strain is already an endemic disease, which together with the basic reproduction numbers define the threshold behaviours for the epidemic model.

4.1. Basic reproduction numbers

We first explore two subsystems involving merely one strain, for \( t > 0 \),

\[
\begin{align*}
\frac{\partial \tilde{u}_1(t, x)}{\partial t} &= D_h \Delta \tilde{u}_1(t, x) + \frac{c_1 \beta(t, x)l[N(x) - \tilde{u}_1(t, x)]\tilde{u}_3(t, x)}{\rho \tilde{u}_1(t, x) + l[N(x) - \tilde{u}_1(t, x)]} \\
&\quad - (\mu_h(x) + \varphi_1(x))\tilde{u}_1(t, x), \quad x \in \Omega, \\
\frac{\partial \tilde{u}_2(t, x)}{\partial t} &= D_v \Delta \tilde{u}_2(t, x) + \Lambda(t, x) - \frac{\alpha_1 \beta(t, x)\rho \tilde{u}_1(t, x)\tilde{u}_2(t, x)}{\rho \tilde{u}_1(t, x) + l[N(x) - \tilde{u}_1(t, x)]} \\
&\quad - \mu_v(t, x)\tilde{u}_2(t, x), \quad x \in \Omega, \\
\frac{\partial \tilde{u}_3(t, x)}{\partial t} &= D_v \Delta \tilde{u}_3(t, x) - \mu_v(t, x)\tilde{u}_3(t, x) + (1 - \tau'_1(t)) \int_\Omega \Gamma(t, t - \tau_1(t), x, y) \cdot \\
&\quad \times \left[ \frac{\alpha_1 \beta(t - \tau_1(t), y)\rho \tilde{u}_1(t - \tau_1(t), y)\tilde{u}_2(t - \tau_1(t), y)}{\rho \tilde{u}_1(t - \tau_1(t), y) + l[N(y) - \tilde{u}_1(t - \tau_1(t), y)]} \right] dy, \quad x \in \Omega, \\
\frac{\partial \tilde{u}_1(t, x)}{\partial v} &= \frac{\partial \tilde{u}_2(t, x)}{\partial v} = \frac{\partial \tilde{u}_3(t, x)}{\partial v} = 0, \quad x \in \partial \Omega,
\end{align*}
\]

which is the subsystem of strain 1, where \((\tilde{u}_1(t, x), \tilde{u}_2(t, x), \tilde{u}_3(t, x)) \triangleq (u_1(t, x), u_3(t, x), u_4(t, x))\), and for \( t > 0 \),

\[
\begin{align*}
\frac{\partial \hat{u}_1(t, x)}{\partial t} &= D_h \Delta \hat{u}_1(t, x) + \frac{c_2 \beta(t, x)l[N(x) - \hat{u}_1(t, x)]\hat{u}_3(t, x)}{\rho \hat{u}_1(t, x) + l[N(x) - \hat{u}_1(t, x)]} \\
&\quad - (\mu_h(x) + \varphi_2(x))\hat{u}_1(t, x), \quad x \in \Omega, \\
\frac{\partial \hat{u}_2(t, x)}{\partial t} &= D_v \Delta \hat{u}_2(t, x) + \Lambda(t, x) - \frac{\alpha_2 \beta(t, x)\rho \hat{u}_1(t, x)\hat{u}_2(t, x)}{\rho \hat{u}_1(t, x) + l[N(x) - \hat{u}_1(t, x)]} \\
&\quad - \mu_v(t, x)\hat{u}_2(t, x), \quad x \in \Omega, \\
\frac{\partial \hat{u}_3(t, x)}{\partial t} &= D_v \Delta \hat{u}_3(t, x) - \mu_v(t, x)\hat{u}_3(t, x) + (1 - \tau'_2(t)) \int_\Omega \Gamma(t, t - \tau_2(t), x, y) \cdot \\
&\quad \times \left[ \frac{\alpha_2 \beta(t - \tau_2(t), y)\rho \hat{u}_1(t - \tau_2(t), y)\hat{u}_2(t - \tau_2(t), y)}{\rho \hat{u}_1(t - \tau_2(t), y) + l[N(y) - \hat{u}_1(t - \tau_2(t), y)]} \right] dy, \quad x \in \Omega, \\
\frac{\partial \hat{u}_1(t, x)}{\partial v} &= \frac{\partial \hat{u}_2(t, x)}{\partial v} = \frac{\partial \hat{u}_3(t, x)}{\partial v} = 0, \quad x \in \partial \Omega,
\end{align*}
\]
which is the subsystem of strain 2, where \((\hat{u}_1(t,x), \hat{u}_2(t,x), \hat{u}_3(t,x)) \triangleq (u_2(t,x), u_3(t,x), u_5(t,x))\). Notice that the positive \(\omega\)-periodic solution or disease-free \(\omega\)-periodic solution for each subsystem can be used as a boundary periodic solution of system (6). According to Ref. [10], \(E_0 = (0,0,u_3^0(t,\cdot),0,0)\), \(E_0^1 = (0,u_3^0(t,\cdot),0)\) and \(E_0^2 = (0,0,0,0)\) denote the disease-free \(\omega\)-periodic solution of system (6), (10) and (11), respectively. \(E_0^1 = (\hat{u}_1^1(t,\cdot), \hat{u}_2^1(t,\cdot), \hat{u}_3^1(t,\cdot))\), is the positive \(\omega\)-periodic solution of (10) and the boundary periodic solution of system (6). When \(E_0^1\) is the nontrivial boundary periodic solution of system (6), we call it the periodic solution of strain 1, represented by \(E_0^1 = (\hat{u}_1^1(t,\cdot), \hat{u}_2^1(t,\cdot), \hat{u}_3^1(t,\cdot),0)\). \(E_0^2 = (\hat{u}_2(t,\cdot), \hat{u}_3(t,\cdot), \hat{u}_5(t,\cdot))\) represents the positive \(\omega\)-periodic solution of (11) and the boundary periodic solution of system (6). When \(E_0^2\) is the nontrivial boundary periodic solution of system (6), we call it the periodic solution of strain 2, denoted by \(0, \hat{u}_2(t,\cdot), \hat{u}_3(t,\cdot),0, \hat{u}_5(t,\cdot))\).

Set \(R_i\) to be basic reproduction number of strain \(i\) without other types of strains \((i = 1, 2)\). The parameters are space-dependent and time-periodic, leading to the complexity of model analysis. We will make use of the results in Refs. [19,44] to define basic reproduction numbers \(R_1\) and \(R_2\) of subsystems (10) and (11). Let strain 1 as an exemplar, and the outcomes also apply to strain 2. The next generation generator of subsystem (10) is defined, whose spectral radius is \(R_1\).

Set \(\hat{u}_1(t,x) = \hat{u}_3(t,x) = 0\) in system (10), for \(t > 0\), we get

\[
\begin{cases}
\frac{\partial \hat{u}_2(t,x)}{\partial t} = D_v \Delta \hat{u}_2(t,x) + \Lambda(t,x) - \mu_v(t,x) \hat{u}_2(t,x), & x \in \Omega, \\
\frac{\partial \hat{u}_2(t,x)}{\partial v} = 0, & x \in \partial \Omega.
\end{cases}
\]

(12)

According to [43, Lemma 2.1], we can get the following lemma.

**Lemma 4.1**: System (12) has a unique globally attractive positive \(\omega\)-periodic solution \(u_3^\ast(t,\cdot)\) in \(\mathbb{R}^+\).

Linearizing system (10) at \(E_0^1\), and for \(t > 0\), considering the equations

\[
\begin{cases}
\frac{\partial v_1(t,x)}{\partial t} = D_h \Delta v_1(t,x) + c_1 \beta(t,x)v_2(t,x) - (\mu_h(x) + \phi_1(x))v_1(t,x), & x \in \Omega, \\
\frac{\partial v_2(t,x)}{\partial t} = D_v \Delta v_2(t,x) - \mu_v(t,x)v_2(t,x) + (1 - \tau_1'(t))
\begin{array}{c}
\int_\Omega \\
\Gamma (t,t - \tau_1(t),x,y)
\end{array}
\end{cases}
\]

\[
\begin{array}{c}
\alpha_1 \beta(t - \tau_1(t),y)pu_3^\ast(t - \tau_1(t),y)v_1(t - \tau_1(t),y)
\end{array}
\end{cases}
\]

\[
\frac{\partial v_1(t,x)}{\partial v} = \frac{\partial v_2(t,x)}{\partial v} = 0, & x \in \partial \Omega.
\]

(13)
Let \( \tilde{v} \triangleq (v_1(t,x), v_2(t,x)) = (\tilde{u}_1(t,x), \tilde{u}_3(t,x)) \). Similarly, linearizing (11) around \( E_0^2 \), and considering the equations of infective compartments

\[
\begin{align*}
\frac{\partial v_3(t,x)}{\partial t} &= D_h \Delta v_3(t,x) + c_2 \beta(t,x)v_4(t,x) - (\mu_h(x) + \varrho_2(x))v_3(t,x), \quad t > 0, \ x \in \Omega, \\
\frac{\partial v_4(t,x)}{\partial t} &= D_v \Delta v_4(t,x) - \mu_v(t,x)v_4(t,x) + (1 - \tau'_2(t)).
\end{align*}
\]

\[
\left\{ \begin{array}{ll}
\int_{\Omega} \Gamma (t, t - \tau_2(t), x, y) \\
\quad \alpha_2 \beta(t - \tau_2(t), y) \mu_3^*(t - \tau_2(t), y) v_3(t - \tau_2(t), y) dy, \\
\quad \text{for } t > 0, \ x \in \Omega, \\
\quad \frac{\partial v_3(t,x)}{\partial \nu} = \frac{\partial v_4(t,x)}{\partial \nu} = 0, \quad t > 0, \ x \in \partial \Omega.
\end{array} \right.
\]

(14)

Then \( \tilde{v} \triangleq (v_3(t,x), v_4(t,x)) = (\tilde{u}_1(t,x), \tilde{u}_3(t,x)) \).

Let \( X_2 := C(\Omega, \mathbb{R}^2), X_2^+ := C(\Omega, \mathbb{R}^3_+) \), and \( C_\omega(\mathbb{R}, X_2) \) be the Banach space consisting of all \( \omega \)-periodic and continuous functions from \( \mathbb{R} \) to \( X_2 \), where \( \| \psi \|_{C_\omega(\mathbb{R}, X_2)} := \max_{\theta \in [0, \omega]} \| \psi(\theta) \|_{X_2} \) for any \( \psi \in C_\omega(\mathbb{R}, X_2) \). Denote \( \mathcal{X}_2 := C([-\tilde{t}, 0], X_2) \) and \( \mathcal{X}_2^+ := C([-\tilde{t}, 0], X_2^+) \). Define \( \mathcal{F}_1(t) : \mathcal{X}_2 \to \mathcal{X}_2 \) by

\[
\mathcal{F}_1(t) \begin{pmatrix} \phi_1 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} c_1 \beta(t, \cdot) \phi_4(0, \cdot) \\ (1 - \tau'_1(t)) \int_{\Omega} \Gamma (t, t - \tau_1(t), \cdot, y) \\
\quad \alpha_1 \beta(t - \tau_1(t), y) \mu_3^*(t - \tau_1(t), y) \phi_1(-\tau_1(t), y) dy \\
\end{pmatrix},
\]

for \( t \in \mathbb{R}, (\phi_1, \phi_4) \in \mathcal{X}_2 \), and \( -V_1(t) \tilde{v} = \tilde{D} \Delta \tilde{v} - W_1(t) \tilde{v} \), where \( \tilde{D} = \text{diag}(D_h, D_v) \) and

\[
-W_1(t)(x) = \begin{pmatrix} -(\mu_h(x) + \varrho_1(x)) & 0 \\ 0 & -\mu_v(t, x) \end{pmatrix}, \quad x \in \Omega.
\]

System (14) can be marked as

\[
\frac{d\tilde{v}}{dt} = \mathcal{F}_1(t) \tilde{v}_t - V_1(t) \tilde{v}, \quad t \geq 0.
\]

Set \( \Psi_1(t,s) = \text{diag}(T_1(t,s), T_3(t,s)), t \geq s, \) to be the evolution operator, related to

\[
\frac{d\tilde{v}}{dt} = -V_1(t) \tilde{v},
\]

where \( T_1(t,s) \) and \( T_3(t,s) \) are defined in Section 3, and subject to the Neumann boundary condition. Then [33, Theorem 3.12] implies that \(-V_1(t)\) is resolvent positive.
Define the exponential growth bound of \( \Psi_1(t,s) \) as

\[
\hat{\omega}(\Psi_1) = \inf\{\hat{\omega} : \exists L \geq 1 \text{ such that } \|\Psi_1(t+s,s)\| \leq Le^{\hat{\omega} t}, \forall s \in \mathbb{R}, \ t \geq 0\}.
\]

By [33, Proposition A.2], we have

\[
\hat{\omega}(\Psi_1) = \frac{\ln r(\Psi_1(\omega,0))}{\omega} = \frac{\ln r(\Psi_1(\omega+s,s))}{\omega}, \quad s \in [0,\omega].
\]

Refer to Krein–Rutman theorem and [14, Lemma 14.2],

\[
0 < r(\Psi_1(\omega,0)) = \max\{r(T_1(\omega,0)), r(T_3(\omega,0))\} < 1,
\]

where \( r(\Psi_1(\omega,0)) \) is the spectral radius of \( \Psi_1(\omega,0) \). By [33, Proposition 5.6] with \( s = 0 \), we obtain \( \hat{\omega}(\Psi_1) < 0 \). Emphasize that \( \Psi_1(t,s) \) is a positive operator, in that sense, \( \Psi_1(t,s)X_2^+ \subseteq X_2^+ \) for all \( t \geq s \). It is easy to see that \( \mathcal{F}_1(t) \) maps \( X_2^+ \) into \( X_2^+ \), for any \( t \geq 0 \), \(-W_1(t)\) is cooperative and \( \hat{\omega}(\Psi_1) < 0 \).

Using [19,44], we can define \( R_1 \) for subsystem (10). Suppose that humans and mosquitoes are near \((0,u_2^+(t,x),0)\). Let \( \bar{v} \in C_{0\omega}(\mathbb{R},X_2) \) and \( \bar{v}(t) \) be the initial distribution of infectious humans and mosquitoes with strain 1 at \( t \in \mathbb{R} \). Note that for \( s \geq 0 \), \( \mathcal{F}_1(t-s)\bar{v}_{t-s} \) is the density distribution of newly infected humans and mosquitoes who were introduced over the interval \([t-s-\check{r},t-s]\). Thus \( \Psi_1(t,s)\mathcal{F}_1(t-s)\bar{v}_{t-s} \) denotes the distribution of those infected humans and mosquitoes who newly became infectious at time \( t-s \) and still remain in the infectious compartments at time \( t \). Thus the integral \( \int_0^{+\infty} \Psi_1(t,s)\mathcal{F}_1(t-s)\bar{v}_{t-s}ds = \int_0^{+\infty} \Psi_1(t)(t-s)\bar{v}(t-s + \cdot)ds \) represents the distribution of cumulative newly infected humans and mosquitoes at time \( t \) caused by all those infectious introduced at all prior time to \( t \). Define the next generation operator \( L_1 \) associated with subsystem (10) as

\[
[L_1\bar{v}](t) := \int_0^{+\infty} \Psi_1(t,s)\mathcal{F}_1(t-s)\bar{v}(t-s + \cdot)ds, \quad \forall t \in \mathbb{R}, \quad \bar{v} \in C_{0\omega}(\mathbb{R},X_2).
\]

Then \( L_1 \) is a positive and continuous operator, which maps \( \bar{v}(t) \) to the distribution of the total infected members generated among infection period.

Motivated by Refs. [8,33], the spectral radius of \( L_1 \) is defined as the basic reproduction number of (10),

\[
R_1 := r(L_1).
\]

\( R_2 \) is the same. Define the basic reproduction number of system (6) as \( R_0 = \max\{R_1,R_2\} \).

For any \( \psi \) in \( X_2^* \), let \( P_{t_1} \) be the solution map of (13) on \( X_2^* \), i.e. \( P_{t_1}(\psi) = \hat{v}_1(\psi), \ t \geq 0 \), where \( \hat{v}_1(\psi)(\theta)(x) = \hat{v}(t + \theta, x, \psi) = (v_1(t + \theta, x, \psi), v_2(t, x, \psi)), \forall \theta \in [-\check{r},0], \) and \( \hat{v}(t,x,\psi) \) is the unique solution of system (13) with \( \hat{v}(\theta,x) = \psi(x) \) for all \( \theta \in [-\check{r},0], \) \( x \in \overline{\Omega} \). Hence, \( P_{t_1} := P_{t_10} \) is the Poincaré map related to subsystem (13). Denote \( P_{2t} \) to be the solution map of the linear periodic Equation (14) on \( X_2 \), it means \( P_{2t}(\phi) = \hat{v}_1(\phi), \ t \geq 0 \), and \( \hat{v}_1(\phi)(\theta)(x) = \hat{v}(t + \theta, x, \phi) = (v_3(t + \theta, x, \phi), v_4(t, x, \phi)), \forall \theta \in [-\check{r},0] \) with \( \hat{v}(\theta,x) = \hat{\phi}(\theta,x) \) for all \( \theta \in [-\check{r},0], \) and \( \hat{\phi} \) in \( X_2 \). \( P_{2t} := P_{2t0} \) is the Poincaré map associated with subsystem (14). \( r(P_{t_1}) \) denotes the spectral radius of \( P_{t_1}, \ (i = 1,2) \).
Lemma 4.2: \( R_i - 1 \) and \( r(P_i) - 1 \) \((i = 1, 2)\) have the same sign.

Define \( X_3 := C(\Omega_1, \mathbb{R}^3), X_3^+ := C(\Omega_1, \mathbb{R}^3_+) \), then \((X_3, X_3^+)\) is an order Banach space. Set \( X_3 := C([-\bar{\tau}, 0], \mathbb{Y}_h \times \mathbb{Y}^+ \times \mathbb{Y}^+) \). Let \( Q_t(\phi) := \tilde{u}_t(\phi), \forall t \geq 0 \), is an \( \omega \)-periodic semi-flow of system (10) in \( X_3 \), and \( \tilde{Q} := \tilde{Q}_o \).

Lemma 4.3: For \( \phi \in X_3 \), system (10) has a unique solution \( \tilde{u}(t, \cdot, \phi) \) with \( \tilde{u}_0 = \phi \). Additionally, system (10) begets an \( \omega \)-periodic semi-flow \( \tilde{Q}_t(\phi) := \tilde{u}_t(\phi) \) in \( X_3 \), \( t \geq 0 \), and \( \tilde{Q} := \tilde{Q}_o \) has a strong attractor in \( X_3 \).

In order to investigate the dynamics behaviour of system (6), first explore the system (13) to generates a periodic semi-flow on the phase space \( \mathcal{E}_1 \) that is eventually strongly monotone,

\[
\mathcal{E}_1 := C([-\tau_1(0), 0], \mathbb{Y}) \times \mathbb{Y}.
\]

Set \( \mathcal{E}_1^+ := C([-\tau_1(0), 0], \mathbb{Y}^+) \times \mathbb{Y}^+ \), \( \mathcal{E}_2 := C([-\tau_2(0), 0], \mathbb{Y}) \times \mathbb{Y} \), and \( \mathcal{E}_2^+ := C([-\tau_2(0), 0], \mathbb{Y}^+) \times \mathbb{Y}^+ \). Thus, \( (\mathcal{E}_1, \mathcal{E}_1^+) \) and \( (\mathcal{E}_2, \mathcal{E}_2^+) \) are ordered Banach spaces. For \( \varphi \in \mathcal{E}_1 \), let \( \varpi(t, x, \varphi) = (\varpi_1(t, x, \varphi), \varpi_2(t, x, \varphi)) \) be the unique solution of (13) with \( \varpi_0(\varphi)(\theta, x) = \varphi(\theta, x) \) for all \( \theta \in [-\tau_1(0), 0] \), \( x \in \bar{\Omega} \), where

\[
\varpi_1(\varphi)(\theta, x) = \varpi(t + \theta, x, \varphi) = (\varpi_1(t + \theta, x, \varphi), \varpi_2(t + \theta, x, \varphi)), \quad \forall t \geq 0, \quad (\theta, x) \in [-\tau_1(0), 0] \times \bar{\Omega}. \tag{16}
\]

Lemma 4.4: For each \( \varphi \in \mathcal{E}_1^+ \), system (13) has a unique nonnegative solution \( \varpi(t, \cdot, \varphi) \) on \([0, +\infty)\) with \( \varpi_0 = \varphi \).

**Proof:** Set \( \bar{\tau}_1 = \min_{t \in [0, \omega]} \tau_1(t) \). For \( t \in [0, \bar{\tau}_1] \), since \( t - \tau_1(t) \) is strictly increasing in \( t \), one has

\[
-\tau_1(0) = 0 \quad -\tau_1(0) \leq t - \tau_1(t) \leq \bar{\tau}_1 - \tau_1(\bar{\tau}_1) \leq \bar{\tau}_1 - \bar{\tau}_1 = 0.
\]

Thus, \( \tilde{u}_1(t - \tau_1(t), \cdot) = \varphi_1(t - \tau_1(t), \cdot) \). Therefore, for any \( t \in [0, \bar{\tau}_1] \), system (13) becomes

\[
\begin{aligned}
\frac{\partial \tilde{u}_1(t, x)}{\partial t} &= D_h \Delta \tilde{u}_1(t, x) + c_1 \beta(t, x) \tilde{u}_3(t, x) - (\mu_h(x) + \varphi_1(x)) \tilde{u}_1(t, x), \quad x \in \Omega, \\
\frac{\partial \tilde{u}_3(t, x)}{\partial t} &= D_v \Delta \tilde{u}_3(t, x) - \mu_v(t, x) \tilde{u}_3(t, x) + (1 - \tau_1'(t)), \\
\frac{\int_{\Omega} \Gamma(t, t - \tau_1(t), x, y) \alpha_1 \beta(t - \tau_1(t), y) \rho u_3(t - \tau_1(t), y) \varphi_1(t - \tau_1(t), y) dy}{\ln(y)} \\
\frac{\partial \tilde{u}_1(t, x)}{\partial v} &= \frac{\partial \tilde{u}_3(t, x)}{\partial v} = 0, \quad x \in \partial \Omega.
\end{aligned}
\]

Fix \( \varphi \in \mathcal{E}_1^+ \), the solution \( (\tilde{u}_1(t, \cdot), \tilde{u}_3(t, \cdot)) \) of the above system uniquely exists for \( t \in [0, \bar{\tau}_1] \). In other words, \( \tilde{u}_1(\theta, \cdot) = \varpi_1(\theta, \cdot) \) for \( \theta \in [-\tau_1(0), \bar{\tau}_1] \) and \( \tilde{u}_2(\theta, \cdot) = \varpi_2(\theta, \cdot) \), for \( \theta \in [0, \bar{\tau}_1] \).
By repeating the above step to \( t \in [n\bar{\tau}_1, (n+1)\bar{\tau}_1], \ n = 1, 2, 3 \ldots \), we see that the solution \( \varpi(t, \cdot, \varphi) \) with initial date \( \varphi \in \mathcal{E}_1^+ \) exists uniquely for all \( t \geq 0 \). \hfill \blacksquare

**Remark 4.1:** The uniqueness of solutions stated in Lemmas 4.3 and 4.4, consequently, for \( \psi \in \mathcal{X}_2^+ \) and \( \varphi \in \mathcal{E}_1^+ \) with \( \psi_1(\theta, \cdot) = \varphi_1(\theta, \cdot) \), \( \forall \theta \in [-\tau_1(0), 0] \) and \( \forall \psi_2(\cdot) = \varphi_2(\cdot) \), then \( \nu(t, \cdot, \psi) = \varpi(t, \cdot, \varphi), t \geq 0 \), where \( \nu(t, \cdot, \psi) \) and \( \varpi(t, \cdot, \varphi) \) are solutions of system (13) meeting \( \nu_0 = \psi \) and \( \varpi_0 = \varphi \), respectively.

Set \( \bar{P}_{1t} \) to be the solution map of (13) on space \( \mathcal{E}_1 \), then \( \bar{P}_1 := \bar{P}_{1\omega} \) is corresponding Poincaré map, and \( r(\bar{P}_1) \) is the spectral radius of \( \bar{P}_1 \). The next lemma indicates that \( \bar{P}_{1t} \) is eventually strongly monotone motivated by Ref. [40] and [20, Lemma 3.5 and Lemma 3.6].

**Lemma 4.5:** For each \( \varphi \in \mathcal{E}_1^+ \) with \( \varphi \neq 0 \), the solution \( \varpi(t, \cdot) \) of (13) with \( \varpi_0 = \varphi \) satisfies \( \varpi(t, \cdot) > 0 \) for all \( t > 2\bar{\tau}_1 \), therefore, \( \forall t > 3\bar{\tau}_1, \bar{P}_{1t}\varphi > 0 \).

The proof of Lemma 4.5 is given in Appendix 2.

Fixed an integer \( n_0 \) satisfying \( n_0\omega > 3\bar{\tau}_1 \). Based on Lemma 4.5, one can easily see that \( \bar{P}_1^{n_0} \) is strongly monotone. In addition, through similar arguments in [15, Lemma 2.6], one can prove that \( \bar{P}_1^{n_0} \) is compact. Apply the Krein–Rutman theorem to \( \bar{P}_1^{n_0} \), and \( r(\bar{P}_1^{n_0}) = (r(\bar{P}_1))^{n_0} \), one has \( \lambda = r(\bar{P}_1) > 0 \), where \( \lambda \) is the simple eigenvalue of \( \bar{P}_1 \), and has a strongly positive eigenfunction \( \vartheta \in int(\mathcal{E}^+) \). Denote \( \bar{P}_{2t} \) to be the solution map of system (14) on space \( \mathcal{E}_2 \), and let \( \bar{P}_2 := \bar{P}_{2\omega} \) be the Poincaré map related to (14). \( r(\bar{P}_2) \) is the spectral radius of \( \bar{P}_2 \). According to [22, Lemma 3.8], one has the following lemma.

**Lemma 4.6:** Two Poincaré maps \( P_i : \mathcal{X}_2 \to \mathcal{X}_2 \) and \( \bar{P}_i := \mathcal{E}_i \to \mathcal{E}_i \) have the same spectral radius, i.e. \( r(P_i) = r(\bar{P}_i) \), \( i = 1, 2 \).

To explore long-term dynamics, we give the fact that the system has a special solution. The next argument is motivated by the treatment in [2, Lemma 5] and [42, Proposition 1.1].

**Lemma 4.7:** There exists a \( \bar{\nu}^*(t, x) \), which is positive \( \omega \)-periodic function, then \( e^{\mu_1 t} \bar{\nu}^*(t, x) \) is a solution of (13), where \( \mu_1 = \frac{\ln r(\bar{P}_1)}{\omega} \).

### 4.2. Invasion reproduction numbers

Use \( \hat{R}_i \) to denote the invasion reproduction number, which means the ability of strain \( i \) to intrude another strain \( i = 1, 2 \). When \( E_0^2 \) is regarded as the non-trivial boundary periodic solution of (6), \( \hat{u}_2(t, x) > 0 \) and \( \hat{u}_5(t, x) > 0 \) are non-zero. The periodic solution of strain
2 meets

\[
\begin{align*}
0 &= D_h \Delta \tilde{u}_2(t,x) + \frac{c_2 \beta(t,x) l [N(x) - \tilde{u}_2(t,x)] \tilde{u}_5(t,x)}{p \tilde{u}_2(t,x) + l [N(x) - \tilde{u}_2(t,x)]} \\
&\quad - (\mu_h(x) + \varphi_2(x)) \tilde{u}_2(t,x), \quad t > 0, \quad x \in \Omega, \\
0 &= D_v \Delta \tilde{u}_3(t,x) \tilde{u}_3(t,x) + \Lambda(t,x) - \frac{\alpha_2 \beta(t,x) p \tilde{u}_2(t,x) \tilde{u}_3(t,x)}{p \tilde{u}_2(t,x) + l [N(x) - u_2(t,x)]} \\
&\quad - \mu_v(t,x) \tilde{u}_3(t,x), \quad t > 0, \quad x \in \Omega, \\
0 &= D_v \Delta \tilde{u}_5(t,x) - \mu_v(t,x) \tilde{u}_5(t,x) + (1 - \tau'_2(t)) \int_\Omega \Gamma(t, t - \tau_2(t), x, y) \cdot \\
&\quad \frac{\alpha_2 \beta(t - \tau_2(t), y) p \tilde{u}_2(t - \tau_2(t), y) \tilde{u}_3(t - \tau_2(t), y)}{p \tilde{u}_2(t - \tau_2(t), y) + l [N(y) - \tilde{u}_2(t - \tau_2(t), y)]} dy, \quad t > 0, \quad x \in \Omega, \\
\frac{\partial \tilde{u}_2(t,x)}{\partial \nu} &= \frac{\partial \tilde{u}_3(t,x)}{\partial \nu} = \frac{\partial \tilde{u}_5(t,x)}{\partial \nu} = 0, \quad t > 0, \quad x \in \partial \Omega.
\end{align*}
\]

Based on Section 4.1, there always exists \( E_0^2 = (0, \tilde{u}_2(t,x), \tilde{u}_3(t,x), 0, \tilde{u}_5(t,x)) \) of (6). Let

\[
\begin{align*}
u_1(t,x) &= w_1(t,x), \quad \nu_2(t,x) = \tilde{u}_2(t,x) + w_2(t,x), \quad \nu_3(t,x) = \tilde{u}_3(t,x) + w_3(t,x), \\
\tilde{u}_4(t,x) &= w_4(t,x), \quad \nu_5(t,x) = \tilde{u}_5(t,x) + w_5(t,x).
\end{align*}
\]

Linearizing (6) at \((0, \tilde{u}_2(t,x), \tilde{u}_3(t,x), 0, \tilde{u}_5(t,x))\), we gain the system which contains only \(w_1\) and \(w_4\) decoupled from other equations

\[
\begin{align*}
\frac{\partial w_1(t,x)}{\partial t} &= D_h \Delta w_1(t,x) + \frac{c_1 \beta(t,x) l [N(x) - \tilde{u}_2(t,x)] w_4(t,x)}{p \tilde{u}_2(t,x) + l [N(x) - \tilde{u}_2(t,x)]} \\
&\quad - (\mu_h(x) + \varphi_1(x)) w_1(t,x), \quad x \in \Omega, \\
\frac{\partial w_4(t,x)}{\partial t} &= D_v \Delta w_4(t,x) - \mu_v(t,x) w_4(t,x) + (1 - \tau'_1(t)) \int_\Omega \Gamma(t, t - \tau_1(t), x, y) \cdot \\
&\quad \frac{\alpha_1 \beta(t - \tau_1(t), y) p \tilde{u}_3(t - \tau_1(t), y) w_1(t - \tau_1(t), y)}{p \tilde{u}_3(t - \tau_1(t), y) + l [N(y) - \tilde{u}_2(t - \tau_1(t), y)]} dy, \quad x \in \Omega, \\
\frac{\partial w_1(t,x)}{\partial \nu} &= \frac{\partial w_4(t,x)}{\partial \nu} = 0, \quad x \in \partial \Omega,
\end{align*}
\]

for \(t > 0\). Define \(\hat{F}_1(t) : X_2 \to X_2\) by

\[
\hat{F}_1(t) \begin{pmatrix} \phi_1 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_4 \end{pmatrix} = \begin{pmatrix}
\frac{c_1 \beta(t, \cdot) l [N(\cdot) - \tilde{u}_2(\cdot, \cdot)] \phi_4(0, \cdot)}{p \tilde{u}_2(\cdot, \cdot) + l [N(\cdot) - \tilde{u}_2(\cdot, \cdot)]} \\
(1 - \tau'_1(t)) \int_\Omega \Gamma(t, t - \tau_1(t), \cdot, y) \frac{\alpha_1 \beta(t - \tau_1(t), y) p \tilde{u}_3(t - \tau_1(t), y) \phi_1(-\tau_1(t), y)}{p \tilde{u}_3(t - \tau_1(t), y) + l [N(y) - \tilde{u}_2(t - \tau_1(t), y)]} dy
\end{pmatrix}.
\]
for \( t \in \mathbb{R}, \phi_1, \phi_4 \in E \). Then we have

\[
[\hat{L}_1 \hat{v}](t) := \int_0^\infty \Psi_1(t, t - s) \hat{F}_1(t - s) \hat{v}(t - s + \cdot) ds, \quad \forall t \in \mathbb{R}, \quad \hat{v} \in C_\omega(\mathbb{R}, \mathbb{R}_2),
\]

where the definition of \( \Psi_1(t, t - s) \) is consistent with that in Section 4.1 and \( \hat{v}(t) \) is the spatial distribution of infective humans and mosquitoes with strain 1 introduced at time \( t \in \mathbb{R} \). Obviously, \( \hat{L}_1 \) on \( C_\omega(\mathbb{R}, \mathbb{R}_2) \) is positive and bounded. From the Section 4.1, we know that

\[
\hat{R}_1 := r(\hat{L}_1).
\]

Let \( Q_{1t} \) be the solution map of (19) on \( \mathcal{X}_2 \) for \( t \geq 0 \), that is, \( Q_{1t}(\phi) = w_t(x, \phi) \), where \( w_t(\phi)(\theta, x) = (w_1(t + \theta, x, \phi), w_4(t, x, \phi)) \), \( \forall \theta \in [-\tau, 0] \) and \( w(t, \phi) \) is the unique solution of (19) with \( w(\theta, x) = \phi(\theta, x) \) for all \( \theta \in [-\tau, 0] \) and \( x \in \Omega \). Subsequently, \( Q_1 := Q_{1\omega} \) is Poncaré map related to (19), i.e. \( Q_1(\phi) = w_\omega(\phi) \). \( r(Q_1) \) denotes the spectral radius of \( Q_1 \).

Analogously, linearizing system (6) at \( (\bar{u}_1^1(t, x), 0, \bar{u}_1^2(t, x), \bar{u}_2(t, x), 0) \), one has the following system containing only \( w_2 \) and \( w_5 \) decoupled from other equations

\[
\begin{aligned}
\frac{\partial w_2(t, x)}{\partial t} &= D_h \Delta w_2(t, x) + \frac{c_2 \beta(t, x) I [N(x) - \bar{u}_1^1(t, x)] w_5(t, x)}{\bar{u}_1^1(t, x) + I [N(x) - \bar{u}_1^1(t, x)]} \\
&\quad - (\mu_h(x) + q_2(x)) w_2(t, x), \quad t > 0, \quad x \in \Omega, \\
\frac{\partial w_5(t, x)}{\partial t} &= D_v \Delta w_5(t, x) - \mu_v(t, x) w_5(t, x) + (1 - \tau_2^1(t)) \int_\Omega \Gamma(t, t - \tau_2(t), x, y) \\
&\quad \frac{\alpha_2 \beta(t - \tau_2(t), y) \bar{u}_1^2(t - \tau_2(t), y) w_2(t - \tau_2(t), y)}{\bar{u}_1^2(t - \tau_2(t), y)} dy, \quad t > 0, \quad x \in \Omega, \\
\frac{\partial w_2(t, x)}{\partial \nu} &= \frac{\partial w_5(t, x)}{\partial \nu} = 0, \quad t > 0, \quad x \in \partial \Omega.
\end{aligned}
\]

For \( t \geq 0 \), denote \( Q_{2t} \) to be the solution map of (21) on \( \mathcal{X}_2 \), that is, \( Q_{2t}(\phi) = w_t(\phi, x) \), where \( w_t(\phi)(\theta, x) = (w_2(t + \theta, x, \phi), w_5(t, x, \phi)) \), \( \forall \theta \in [-\tau, 0], x \in \Omega \) and \( w(t, x, \phi) \) is the unique solution of system (21) with \( w(\theta, x) = \phi(\theta, x) \) for all \( \theta \in [-\tau, 0] \). Let \( Q_2 := Q_{2\omega} \) be the Poncaré map associated with system (22), i.e. \( Q_2(\phi) = w_\omega(\phi, x) \), and \( r(Q_2) \) denote the spectral radius of \( Q_2 \). Use \( \hat{Q}_{1t} \) to be the solution map of (19) on space \( \mathcal{E}_1 \), then \( \hat{Q}_1 := \hat{Q}_{1\omega} \) be Poincaré map related to (19), and \( r(\hat{Q}_1) \) be the spectral radius of \( \hat{Q}_1 \). Set \( \hat{Q}_{2t} \) to be the solution map of (22), then \( \hat{Q}_2 := \hat{Q}_{2\omega} \) is Poincaré map related to (21) on space \( \mathcal{E}_2 \), and \( r(\hat{Q}_2) \) is the spectral radius of \( \hat{Q}_2 \). We know that there exists a positive linear operator \( \hat{L}_2 \) on \( C_\omega(\mathbb{R}, \mathbb{R}_2) \), defined by

\[
[\hat{L}_2 \hat{v}](t) := \int_0^\infty \Psi_2(t, t - s) \hat{F}_2(t - s) \hat{v}(t - s + \cdot) ds, \quad \forall t \in \mathbb{R}, \quad \hat{v} \in C_\omega(\mathbb{R}, \mathbb{R}_2),
\]
where \( \Psi_2(t, s) = \text{diag}(T_2(t, s), T_3(t, s)) \), \( \tilde{v}(t) \) is the spatial distribution of infected humans and mosquitoes with strain 2 introduced at time \( t \in \mathbb{R} \) and

\[
\hat{F}_2(t) \left( \frac{\phi_2}{\phi_5} \right) = \begin{pmatrix}
\frac{e_2 \beta(t, \cdot) l \left[ N(\cdot) - \bar{u}^2_i(t, \cdot) \right] \phi_5(0, \cdot)}{p \bar{u}^2_i(t, \cdot) + l \left[ N(\cdot) - \bar{u}_1^0(t, \cdot) \right]} \\
(1 - \tau_2(t)) \int_\Omega \Gamma(t, t - \tau_2(t), \cdot, y) \\
\frac{\alpha_2 \beta(t - \tau_2(t), y) p \bar{u}^2_i(t - \tau_2(t), y) \phi_2(-\tau_2(t), y)}{p \bar{u}^2_i(t - \tau_2(t), y) + l \left[ N(y) - \bar{u}_1^0(t - \tau_2(t), y) \right]} dy
\end{pmatrix}.
\]

Then

\[\hat{R}_2 := r(\hat{L}_2).\]

Similar to [43, Lemma 3.4] and [19, Theorem 3.7], one has the following result.

**Lemma 4.8:** \( \hat{R}_i \) and \( r(Q_i) - 1 \) (\( i = 1, 2 \)) have the same sign.

According to [22, Lemma 3.8], [2, Lemma 5] and [42, Proposition 1.1], the following results can be obtained.

**Lemma 4.9:** Two Poincaré maps \( Q_i : X_i \rightarrow X_i \) and \( \bar{Q}_i := E_i \rightarrow E_i \) have the same spectral radius, i.e. \( r(Q_i) = r(\bar{Q}_i) \) (\( i = 1, 2 \)).

We wish to explore the long-time behaviour of (6), based on the evidence that the system has a special solution.

**Lemma 4.10:** There is a \( \tilde{v}^*(t, x) \), which is positive \( \omega \)-periodic function, in a manner that \( e^{\mu_2 t} \tilde{v}^*(t, x) \) is a solution of (22), where \( \mu_2 = \frac{\ln r(Q_2)}{\omega} \).

According to [46, Proposition 3.11], we have the following lemma.

**Lemma 4.11:** If \( \hat{R}_i > 1 \), then \( R_i > 1 \) for \( i = 1, 2 \).

## 5. Threshold dynamics

The main focus of this position is to explore the threshold dynamics of (5) based on \( R_i \) and \( \hat{R}_i \) (\( i = 1, 2 \)). The stability analysis and persistent results contribute to a discernment of the interactions between the disease dynamics as well as possible coexistence of the two strains.

**Theorem 5.1:** If \( R_1 < 1 \) and \( R_2 < 1 \), then \( (0, 0, u^*_2(t, x), 0, 0) \) of (6) is globally attractive.

The conclusion of this theorem can be obtained directly from the conclusions of the following two theorems.

Let \( X_1 = C([-\tau_1(0), 0], Y_h) \times C([-\tau_1(0), 0], Y^+) \times Y^+ \) and \( X_2 = C([-\tau_2(0), 0], Y_h) \times C([-\tau_2(0), 0], Y^+) \times Y^+ \). In the light of Lemma 4.4, for any \( \bar{\phi} \in X_3 \) and \( \bar{\psi} \in X_4 \) with \( \bar{\phi}_1(\theta, \cdot) = \bar{\psi}_1(\theta, \cdot), \forall \theta \in [-\tau_1(0), 0], \bar{\phi}_2(\theta, \cdot) = \bar{\psi}_2(\theta, \cdot), \forall \theta \in [-\tau_1(0), 0], \) and \( \hat{\phi}_3(\cdot) = \hat{\psi}_3(\cdot) \).
\( \hat{\psi}_3(0, \cdot), \tilde{u}(t, \cdot, \hat{\phi}) = \hat{z}(t, \cdot, \hat{\psi}) \) holds, \( t \geq 0 \), where \( \tilde{u}(t, \cdot, \hat{\phi}) \) and \( \hat{z}(t, \cdot, \hat{\psi}) \) are solutions of system (10) meeting \( \hat{u}_0 = \hat{\phi} \) and \( \hat{z}_0 = \hat{\psi} \), respectively. Consequently, the solution of system (10) on \( \mathcal{X}_4 \) exists globally on \([0, +\infty)\) and ultimately bounded.

**Theorem 5.2:**
(a) As \( R_1 < 1, E_0^1 \) is globally attractive for (10);
(b) As \( R_1 > 1, \) system (10) has at least one positive \( \omega \)-periodic solution \( E_0^1 \), and there is a constant \( \gamma_1 > 0 \), making \( \forall \hat{\phi} \in \mathcal{X}_4 \) with \( \hat{\phi}_1(0, \cdot) \neq 0 \) and \( \hat{\phi}_3(\cdot) \neq 0 \), one has
\[
\liminf_{t \to \infty} \min_{x \in \Omega} \hat{u}_i(t, x, \hat{\phi}) \geq \gamma_1, \quad i = 1, 2, 3.
\]

Proof of Theorem 5.2 is given in Appendix 3. Similarly, the following conclusion holds.

**Theorem 5.3:**
(a) When \( R_2 < 1, E_0^2 \) is globally attractive for system (11);
(b) When \( R_2 > 1, \) system (11) has at least one positive \( \omega \)-periodic solution \( E_0^2 \), and there is a constant \( \gamma_5 > 0 \), such that, for any \( \hat{\psi} \in \mathcal{X}_5 \) with \( \hat{\psi}_1(0, \cdot) \neq 0 \) and \( \hat{\psi}_3(\cdot) \neq 0 \), one has
\[
\liminf_{t \to \infty} \min_{x \in \Omega} \hat{u}_i(t, x, \hat{\phi}) \geq \gamma_5, \quad i = 1, 2, 3.
\]

Furthermore, by [22, Lemma 3.5], [15] and [24, Theorem 2.9], the following result is given.

**Lemma 5.1:** Denote \( D_t \) to be the solution map of system (10) on \( \mathcal{X}_4 \), that is, for each \( \hat{\psi} \in \mathcal{X}_4 \), \( D_t(\hat{\psi}) = \hat{u}(\hat{\psi}) \), \( \forall t \geq 0 \). Let \( D_t \) be an \( \omega \)-periodic semi-flow on \( \mathcal{X}_4 \), in this sense, one has:
(i) \( D_0 = I \); (ii) \( \forall t \geq 0, D_{t+\omega} = D_t \circ D_\omega \) holds; and (iii) \( D_t(\hat{\psi}) \) is continuous in \( (t, \hat{\psi}) \in [0, \infty) \times \mathcal{X}_4 \). Additionally, in \( \mathcal{X}_4 \), \( D := D_\omega \) has a strong global attractor.

**Lemma 5.2:** Use \((\tilde{u}_1(t, \cdot, \hat{\psi}), \tilde{u}_2(t, \cdot, \hat{\psi}), \tilde{u}_3(t, \cdot, \hat{\psi}))\) to be the solution of (10) with initial data \( \hat{\psi} \in \mathcal{X}_4 \), if it exists \( t_1 \geq 0 \), such that \( \tilde{u}_i(t_1, x, \hat{\psi}) \neq 0, (i = 1, 3) \), in that way, the solution of (10) meets
\[
\tilde{u}_i(t, x, \hat{\psi}) > 0, \quad t > t_1, \quad x \in \Omega.
\]
Besides, for nontrivial data \( \hat{\psi} \in \mathcal{X}_4 \), one has \( \tilde{u}_2(t, x, \hat{\psi}) > 0, \forall t > 0, x \in \Omega \), and
\[
\liminf_{t \to \infty} \tilde{u}_2(t, x, \hat{\psi}) \geq \varepsilon_1, \quad \text{uniformly for } x \in \Omega,
\]
where \( \varepsilon_1 \) is a \( \hat{\psi} \)-independent positive constant.

The proof of Lemma 5.2 is given in Appendix 4.

### 5.1. Competitive exclusion and coexistence

Denote \( \bar{\tau} = \min\{\tau_1(0), \tau_2(0)\} \),
\[
\mathcal{X}_6 = C([-\bar{\tau}, 0], \mathbb{Y}_h) \times C([-\bar{\tau}, 0], \mathbb{Y}_h) \times C([-\bar{\tau}, 0], \mathbb{Y}_+) \times \mathbb{Y}_+ \times \mathbb{Y}_+.
\]

Next, we prove the competitive exclusion and coexistence of (6).
For each $\phi \in \mathcal{X}_1^+$ and $\tilde{\psi} \in \mathcal{X}_6$ with $\phi_1(\eta_1, \cdot) = \tilde{\psi}_1(\eta_1, \cdot)$, $\phi_2(\eta_1, \cdot) = \tilde{\psi}_2(\eta_1, \cdot)$, $\phi_3(\eta_1, \cdot) = \tilde{\psi}_3(\eta_1, \cdot)$, $\phi_4(0, \cdot) = \tilde{\psi}_4(0, \cdot)$, $\phi_5(0, \cdot) = \tilde{\psi}_5(0, \cdot)$, for $\eta_1 \in [-\bar{\tau}, 0]$. We have $u(t, \cdot, \phi) = \tilde{u}(t, \cdot, \psi)$, for $t \geq 0$, where $u(t, \cdot, \phi)$ and $\tilde{u}(t, \cdot, \psi)$ are solutions of system (6) satisfying $u_0 = \phi$ and $\tilde{u}_0 = \psi$, respectively. Consequently, the solution of system (6) on $\mathcal{X}_6$ exists globally on $[0, +\infty)$ and ultimately bounded. Furthermore, in view of those in [22, Lemma 3.5] and [15] together with [24, Theorem 2.9], we obtain the following result.

**Lemma 5.3:** Denote $\Phi_t$ to be the solution map of system (6) on $\mathcal{X}_6$, that is, for $\tilde{\psi} \in \mathcal{X}_6$, $\Phi_t(\tilde{\psi}) = u_t(\tilde{\psi})$, $\forall t \geq 0$. Then $\Phi_t$ is an $\omega$-periodic semi-flow on $\mathcal{X}_6$ in this sense, one has: (i) $\Phi_0 = I$; (ii) $\Phi_{t+\omega} = \Phi_t \circ \Phi_\omega$, $\forall t \geq 0$; and (iii) $\Phi_t(\tilde{\psi})$ is continuous in $(t, \tilde{\psi}) \in [0, \infty) \times \mathcal{X}_6$. Additionally, $\Phi := \Phi_\omega$ has a strong global attractor in $\mathcal{X}_6$.

Based on the comparison principle and Lemma 5.3, we see that the solution of system (6) is strictly positive.

**Lemma 5.4:** Use $(u_1(t, \cdot, \tilde{\psi}), u_2(t, \cdot, \tilde{\psi}), u_3(t, \cdot, \tilde{\psi}), u_4(t, \cdot, \tilde{\psi}), u_5(t, \cdot, \tilde{\psi}))$ to be the solution of (6) with the initial data $\tilde{\psi} \in \mathcal{X}_6$, if there is some $t_3 \geq 0$ such that $u_j(t_3, \cdot, \tilde{\psi}) \neq 0$, in that way, the solution of (6) meets

$$u_j(t, x, \tilde{\psi}) > 0, \quad \text{for any } t > t_3, \quad j = 1, 2, 4, 5, \quad x \in \overline{\Omega}.$$ 

Besides, for any nontrivial initial data $\tilde{\psi} \in \mathcal{X}_6$, one has $u_3(t, x, \tilde{\psi}) > 0$, for $t > 0$, $x \in \overline{\Omega}$, and

$$\liminf_{t \to \infty} u_3(t, x, \tilde{\psi}) \geq \varepsilon_2, \quad \text{uniformly for } x \in \overline{\Omega},$$

where $\varepsilon_2 > 0$ is a $\tilde{\psi}$-independent constant.

**Theorem 5.4:** Under the condition of $R_2 > 1 > R_1$, suppose $(u_1(t, x, \tilde{\psi}), u_2(t, x, \tilde{\psi}), u_3(t, x, \tilde{\psi}), u_4(t, x, \tilde{\psi}), u_5(t, x, \tilde{\psi}))$ is the solution of (6) with the initial data $\tilde{\psi} = (\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3, \tilde{\psi}_4, \tilde{\psi}_5) \in \mathcal{X}_6$. Then there is a $\mathcal{P}_1 > 0$, such that, for each $\tilde{\psi} \in \mathcal{X}_6$ with $\tilde{\psi}_2(0, \cdot) \neq 0$ and $\tilde{\psi}_5(\cdot) \neq 0$, one has $\lim_{t \to \infty} u_1(t, x, \tilde{\psi}) = 0$, $\lim_{t \to \infty} u_4(t, x, \tilde{\psi}) = 0$ and

$$\liminf_{t \to \infty} \min_{x \in \overline{\Omega}} u_2(t, x, \psi) \geq \mathcal{P}_1, \quad \liminf_{t \to \infty} \min_{x \in \overline{\Omega}} u_5(t, x, \psi) \geq \mathcal{P}_1.$$

**Proof:** As that in Theorem 5.3(a), based on $R_1 < 1$, we can prove $\lim_{t \to \infty} u_1(t, x, \tilde{\psi}) = 0$ and $\lim_{t \to \infty} u_4(t, x, \tilde{\psi}) = 0$. According to the proof of Theorem 5.3(b), one has $\liminf_{t \to \infty} \min_{x \in \overline{\Omega}} u_2(t, x, \tilde{\psi}) \geq \mathcal{P}_1$ and $\liminf_{t \to \infty} \min_{x \in \overline{\Omega}} u_5(t, x, \tilde{\psi}) \geq \mathcal{P}_1$ by virtue of $R_2 > 1$. 

**Theorem 5.5:** Suppose that $R_3 > 1 > R_2$, and $(u_1(t, x, \tilde{\psi}), u_2(t, x, \tilde{\psi}), u_3(t, x, \tilde{\psi}), u_4(t, x, \tilde{\psi}), u_5(t, x, \tilde{\psi}))$ is the solution of (6) with the initial data $\tilde{\psi} = (\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3, \tilde{\psi}_4, \tilde{\psi}_5) \in \mathcal{X}_6$. Then there is a $\mathcal{P}_2 > 0$ such that for each $\tilde{\psi} \in \mathcal{X}_6$ with $\tilde{\psi}_1(0, \cdot) \neq 0$ and $\tilde{\psi}_4(\cdot) \neq 0$, one has $\lim_{t \to \infty} u_2(t, x, \tilde{\psi}) = 0$, $\lim_{t \to \infty} u_5(t, x, \tilde{\psi}) = 0$ and

$$\liminf_{t \to \infty} \min_{x \in \overline{\Omega}} u_1(t, x, \tilde{\psi}) \geq \mathcal{P}_2, \quad \liminf_{t \to \infty} \min_{x \in \overline{\Omega}} u_4(t, x, \tilde{\psi}) \geq \mathcal{P}_2.$$
The persistence of (6) is shown. Given \( \tilde{\psi} \in X_6 \), set \( u(t, x, \tilde{\psi}) \) to be the unique solution of system (6) with \( u_0 = \tilde{\psi} \). The following result shows the results of (6) based on \( \hat{R}_i (i = 1, 2) \).

**Theorem 5.6:** Presume that \( R_1 > 1 \) and \( R_2 > 1 \), so that the boundary periodic solutions of system (6) exist.

(i) If \( \hat{R}_1 > 1 \), then \( (0, \tilde{u}_2(t, x), \tilde{u}_3(t, x), 0, \tilde{u}_5(t, x)) \) is unstable. There exists a \( P_3 \), then for any \( \tilde{\psi} \in X_6 \) with \( \tilde{\psi}_1(0, \cdot) \not\equiv 0 \) and \( \tilde{\psi}_4(\cdot) \not\equiv 0 \), one has \( \lim_{t \to \infty} u_2(t, x, \tilde{\psi}) = 0 \), \( \lim_{t \to \infty} u_5(t, x, \tilde{\psi}) = 0 \) and

\[
\liminf_{t \to \infty} \min_{x \in \Omega} u_1(t, x, \tilde{\psi}) \geq P_3, \quad \liminf_{t \to \infty} \min_{x \in \Omega} u_4(t, x, \tilde{\psi}) \geq P_3.
\]

(ii) If \( \hat{R}_2 > 1 \), then the \( \omega \)-periodic solution of strain 1, \( (\tilde{u}_1^1(t, x), 0, \tilde{u}_3^1(t, x), \tilde{u}_5^1(t, x), 0) \), is unstable. There is a \( P_4 \) such that for any \( \tilde{\psi} \in X_6 \) with \( \tilde{\psi}_2(0, \cdot) \not\equiv 0 \) and \( \tilde{\psi}_5(\cdot) \not\equiv 0 \), we have \( \lim_{t \to \infty} u_1(t, x, \tilde{\psi}) = 0 \), \( \lim_{t \to \infty} u_5(t, x, \tilde{\psi}) = 0 \) and

\[
\liminf_{t \to \infty} \min_{x \in \Omega} u_2(t, x, \tilde{\psi}) \geq P_4, \quad \liminf_{t \to \infty} \min_{x \in \Omega} u_5(t, x, \tilde{\psi}) \geq P_4.
\]

**Theorem 5.7:** As \( R_1 > 1 \), \( R_2 > 1 \), \( \hat{R}_1 > 1 \) and \( \hat{R}_2 > 1 \), system (6) admits at least one positive solution \( (\tilde{u}_1^s(t, x), \tilde{u}_2^s(t, x), \tilde{u}_3^s(t, x), \tilde{u}_4^s(t, x), \tilde{u}_5^s(t, x)) \), which is \( \omega \)-periodic, and there is a constant \( \tilde{\eta} > 0 \) such that, for any \( \tilde{\psi} = (\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3, \tilde{\psi}_4, \tilde{\psi}_5) \in X_6 \), meets \( \tilde{\psi}_1(0, \cdot) \not\equiv 0 \), \( \tilde{\psi}_2(0, \cdot) \not\equiv 0 \), \( \tilde{\psi}_4(\cdot) \not\equiv 0 \), \( \tilde{\psi}_5(\cdot) \not\equiv 0 \), we have

\[
\lim_{t \to \infty} \min_{x \in \Omega} u_i(t, x, \tilde{\psi}) \geq \tilde{\eta}, \quad i = 1, 2, 3, 4, 5.
\]

**Proof:** By virtue of Lemma 4.11 and \( \hat{R}_i > 1 \) \( (i = 1, 2) \), one has \( R_i > 1 \) \( (i = 1, 2) \). Let

\[
Z_0 := \left\{ \tilde{\psi} \in X_6 : \tilde{\psi}_1(0, \cdot) \not\equiv 0 \text{ and } \tilde{\psi}_2(0, \cdot) \not\equiv 0 \text{ and } \tilde{\psi}_4(\cdot) \not\equiv 0 \text{ and } \tilde{\psi}_5(\cdot) \not\equiv 0 \right\},
\]

and

\[
\partial Z_0 := X_6 \setminus Z_0 = \left\{ \tilde{\psi} \in X_6 : \tilde{\psi}_1(0, \cdot) \equiv 0 \text{ or } \tilde{\psi}_2(0, \cdot) \equiv 0 \text{ or } \tilde{\psi}_4(\cdot) \equiv 0 \text{ or } \tilde{\psi}_5(\cdot) \equiv 0 \right\}.
\]

Notice that for any \( \tilde{\psi} \in Z_0 \), Lemma 5.4 indicates that \( u_i(t, x, \tilde{\psi}) > 0 \) \( (i = 1, 2, 4, 5) \) \( \forall t > 0 \), \( x \in \Omega \). Thus, we get \( \Phi^n(Z_0) \subset Z_0 \), \( \forall n \in \mathbb{N} \). Lemma 5.3 reveals that \( \Phi \) has a strong global attractor in \( X_6 \).

Define

\[
Z_\partial := \left\{ \tilde{\psi} \in \partial Z_0 : \Phi^n(\tilde{\psi}) \in \partial Z_0, \forall n \in \mathbb{N} \right\},
\]

and \( \omega(\tilde{\psi}) \) to be the omega limit set and the corresponding orbit is \( \gamma^+(\tilde{\psi}) := \{ \Phi^n(\tilde{\psi}) : \forall n \in \mathbb{N} \} \). Denote \( M_1 = \{ (0, 0, \tilde{u}_3^s(t, \cdot), 0) \} \), \( M_2 = \{ (\tilde{u}_1^s(t, \cdot), 0, \tilde{u}_3^s(t, \cdot), \tilde{u}_5^s(t, \cdot), 0) \} \) and \( M_3 = \{ (0, \tilde{u}_2(t, \cdot), \tilde{u}_3^s(t, \cdot), 0, \tilde{u}_5^s(t, \cdot)) \} \), \( \forall t > 0 \), where \( \hat{\circ} \) represents the constant function identically zero, \( \hat{\circ}(\eta_1, \cdot) = 0, \eta_1 \in [-\tau, 0] \). Next, we show the following claims.

**Claim 1.** \( \bigcup_{\tilde{\phi} \in Z_\partial} \omega(\tilde{\phi}) = M_1 \cup M_2 \cup M_3 \), \( \forall \tilde{\phi} \in Z_\partial \), where \( \omega(\tilde{\phi}) \) is the omega limit set and the corresponding orbit is \( \gamma^+ = \{ \Phi^n(\tilde{\phi}) : \forall n \in \mathbb{N} \} \) of system (6) for \( \tilde{\phi} \in Z_\partial \).
According to the definition of \( \mathbb{Z}_\delta \), we obtain \( \Phi^n(\bar{\vartheta}) \in \partial \mathbb{Z}_0 \) for \( \forall \bar{\vartheta} \in \mathbb{Z}_\delta \). Then, either \( u_1(n\bar{\vartheta}, \cdot, \bar{\vartheta}) \equiv 0 \) or \( u_3(n\bar{\vartheta}, \cdot, \bar{\vartheta}) \equiv 0 \) or \( u_4(n\bar{\vartheta}, \cdot, \bar{\vartheta}) \equiv 0 \) or \( u_5(n\bar{\vartheta}, \cdot, \bar{\vartheta}) \equiv 0 \), for \( n \in \mathbb{N} \). What’s more, by contradiction and Lemma 5.4, it is evident that for \( t \geq 0 \), either \( u_1(t, \cdot, \bar{\vartheta}) \equiv 0 \) or \( u_2(t, \cdot, \bar{\vartheta}) \equiv 0 \) or \( u_4(t, \cdot, \bar{\vartheta}) \equiv 0 \) or \( u_5(t, \cdot, \bar{\vartheta}) \equiv 0 \). In the case where \( u_1(t, \cdot, \bar{\vartheta}) \equiv 0 \) for \( t \geq 0 \), we get that the forth equation in (6) satisfies
\[
\frac{\partial u_4(t,x, \bar{\vartheta})}{\partial t} \leq D \Delta u_4(t,x, \psi) - \bar{\mu}_v u_4(t,x, \bar{\vartheta}),
\]
where \( \bar{\mu}_v \) is defined in Section 3. Based on the comparison principle, one has \( \lim_{t \to \infty} u_4(t,x, \bar{\vartheta}) = 0 \) uniformly for \( x \in \overline{\mathbb{O}} \). In this instance, \( u_2(t,x, \bar{\vartheta}) \) and \( u_5(t,x, \bar{\vartheta}) \) have the following possibilities:

(i) If \( u_2(t, \cdot, \bar{\vartheta}) \equiv 0 \), then \( \lim_{t \to \infty} u_5(t,x, \bar{\vartheta}) = 0 \) uniformly for \( x \in \overline{\mathbb{O}} \). Consequently, \( u_3 \) equation abides by an autonomous system, which is asymptotic to the system (12).

Again, according to the theory of internal chain transitive sets [45], one can obtain that \( \lim_{t \to \infty} (u_3(t,x, \bar{\vartheta}) - u_3^\ast(t,x, \bar{\vartheta})) = 0 \) uniformly for \( x \in \overline{\mathbb{O}} \).

(ii) If there is some \( t_4 \geq 0 \), such that \( u_2(t_4, \cdot, \bar{\vartheta}) \neq 0 \), then \( u_2(t, \cdot, \bar{\vartheta}) > 0 \), \( \forall t \geq t_4 \), based on Lemma 5.4. Thus, we get \( u_5(t, \cdot, \bar{\vartheta}) \equiv 0 \) or \( u_5(t_5, \cdot, \bar{\vartheta}) \neq 0 \) for some \( t_5 \geq 0 \). Let’s discuss the following situations.

(a) (a) If \( u_5(t, \cdot, \bar{\vartheta}) \equiv 0 \), then we obtain \( \lim_{t \to \infty} u_2(t,x, \bar{\vartheta}) = 0 \) uniformly for \( x \in \overline{\mathbb{O}} \), which contracts \( u_2(t_4, \cdot, \bar{\vartheta}) \neq 0 \), \( \forall t \geq t_4 \).

(b) (b) If there is some \( t_5 \geq 0 \), such that \( u_5(t_5, \cdot, \bar{\vartheta}) \neq 0 \), then based on Lemma 5.4, one has \( u_5(t, \cdot, \bar{\vartheta}) > 0 \), for \( t \geq t_5 \). Then, equations \( u_2, u_3 \) and \( u_5 \) satisfy a nonautonomous system, which is asymptotic to system (17). Additionally, according to the theory of internal chain transitive sets [45], one has \( \lim_{t \to \infty} (u_2(t,x, \bar{\vartheta}), u_3(t,x, \bar{\vartheta}), u_5(t,x, \bar{\vartheta})) - (\bar{u}_2(t,x), \bar{u}_3(t,x), \bar{u}_5(t,x))) = 0 \), uniformly for \( x \in \overline{\mathbb{O}} \).

Under the above discussion, we get \( \omega(\bar{\vartheta}) = M_1 \cup M_3 \).

If there is some \( t_6 \geq 0 \), such that \( u_1(t_6, \cdot, \bar{\vartheta}) \neq 0 \), then according to Lemma 5.4, we obtain \( u_1(t, \cdot, \bar{\vartheta}) > 0, t \geq t_6 \). Consequently, one has \( u_2(t, \cdot, \bar{\vartheta}) \equiv 0 \) or \( u_4(t, \cdot, \bar{\vartheta}) \equiv 0 \) or \( u_5(t, \cdot, \bar{\vartheta}) \equiv 0 \).

If \( u_4(t, \cdot, \bar{\vartheta}) \equiv 0 \), from the fifth equation of system (6), we can get \( \lim_{t \to \infty} u_1(t,x, \bar{\vartheta}) = 0 \), which is contradict with the fact that \( u_1(t, \cdot, \bar{\vartheta}) > 0, t \geq t_6 \). Thus, there is \( t_7 \geq 0 \) such that \( u_4(t_7, \cdot, \bar{\vartheta}) \neq 0 \), then we get \( u_4(t, \cdot, \bar{\vartheta}) > 0 \) from Lemma 5.4 for \( t \geq t_7 \). Therefore, \( u_2(t, \cdot, \bar{\vartheta}) \equiv 0 \) or \( u_5(t, \cdot, \bar{\vartheta}) \equiv 0 \) must be established. Here we only discuss the case that \( u_2(t, \cdot, \bar{\vartheta}) \equiv 0 \), and the same is true for \( u_5(t, \cdot, \bar{\vartheta}) \equiv 0 \). If \( u_2(t, \cdot, \bar{\vartheta}) \equiv 0 \), for \( t \geq 0 \), according to comparison principle, one has \( \lim_{t \to \infty} u_5(t,x, \bar{\vartheta}) = 0 \) uniformly for \( x \in \overline{\mathbb{O}} \). So, \( u_3 \) satisfies a nonautonomous system, which is asymptotic to the second equation of periodic system (22).

Consequently, we get \( \omega(\bar{\vartheta}) = M_2 \). Thus, Claim 1 holds.

**Claim 2.** \( M_1 \) is a uniformly weak repeller for \( \mathbb{Z}_0 \), in this sense, on has
\[
\limsup_{t \to \infty} \|\Phi^n(\bar{\psi}) - M_1\| \geq \sigma_3, \quad \forall \bar{\psi} \in \mathbb{Z}_0,
\]
for \( \sigma_3 \) small enough.
Claim 3. $M_i$ is uniformly weak repellor for $\mathbb{Z}_0$ ($i = 2, 3$), in this sense, there is a sufficiently small $\sigma_4 > 0$ meeting

$$\lim_{t \to \infty} \sup \|\Phi^n(\tilde{\psi}) - M_i\| \geq \sigma_4, \quad \forall \tilde{\psi} \in \mathbb{Z}_0,$$

We only give the proof of $M_2$, the same proof step is also applicable to $M_3$. For $\tilde{\epsilon} > 0$, consider the following equation with parameter $\tilde{\epsilon}$

\[
\begin{align*}
\frac{\partial \tilde{v}_1(t, x)}{\partial t} &= D_h \Delta \tilde{v}_1(t, x) + \frac{c_2 \beta(t, x) l[N(x) - \tilde{u}_1(t, x) - 2\tilde{\epsilon}] \tilde{v}_1(t, x)}{\rho[\tilde{u}_1(t, x) + 2\tilde{\epsilon}] + l[N(x) - \tilde{u}_1(t, x) + \tilde{\epsilon}]} \\
&\quad - (\mu_h(x) + \varrho_2(x)) \tilde{v}_1(t, x), \quad x \in \Omega, \\
\frac{\partial \tilde{v}_2(t, x)}{\partial t} &= D_v \Delta \tilde{v}_2(t, x) - \mu_v(t, x) \tilde{v}_2(t, x) + (1 - \tau_2(t)) \cdot \tilde{v}_2(t, x) \\
\int_{\Omega} \int_{\Omega} \Gamma(t, t - \tau_2(t), x, y) \frac{\alpha_2 \beta(t - \tau_2(t), y) p[\tilde{u}_1(t - \tau_2(t), y) - \tilde{\epsilon}] \tilde{v}_1(t - \tau_2(t), y)}{\rho[\tilde{u}_1(t - \tau_2(t), y) + 2\tilde{\epsilon}] + l[N(y) - \tilde{u}_1(t - \tau_2(t), y) + \tilde{\epsilon}]} dy, \quad x \in \Omega, \\
\frac{\partial \tilde{v}_1(t, x)}{\partial \nu} &= \frac{\partial \tilde{v}_2(t, x)}{\partial \nu} = 0, \quad x \in \partial \Omega.
\end{align*}
\]

Set $\tilde{v}(t, x, \varphi) = (\tilde{v}_1(t, x, \varphi), \tilde{v}_2(t, x, \varphi))$ to be the unique solution of (22), for $\varphi \in \mathcal{E}_2$, with $v_0^\varphi(\theta_2, x) = (\varphi_1(\theta_2, x), \varphi_2(0, x))$ for all $\theta_2 \in [-\tau_2(0), 0]$, $x \in \Omega$, where

$$
\begin{align*}
\tilde{v}_1(\varphi)(\theta_2, x) &= \tilde{v}_1(t + \theta_2, x, \varphi) \\
&= (\tilde{v}_1(t + \theta_2, x, \varphi), \tilde{v}_2(t, x, \varphi)), \quad \forall t \geq 0, \quad (\theta_2, x) \in [-\tau_2(0), 0] \times \Omega.
\end{align*}
$$

Let $\tilde{Q}_{2\tilde{\epsilon}} : \mathcal{E}_2 \to \mathcal{E}_2$ be the Poincaré map of (22), i.e. $\tilde{Q}_{2\tilde{\epsilon}}(\varphi) = \tilde{v}_\omega(\varphi)$, $\forall \varphi \in \mathcal{E}_2$, and $r(\tilde{Q}_{2\tilde{\epsilon}})$ be the spectral radius of $\tilde{Q}_{2\tilde{\epsilon}}$. Since $\lim_{\tilde{\epsilon} \to 0} r(\tilde{Q}_{2\tilde{\epsilon}}) = r(\tilde{Q}) > 1$, fixing a sufficiently small constant $\tilde{\epsilon} > 0$, such that

$$\tilde{\epsilon} < \min \left\{ \min_{t \in [0, \omega], x \in \Omega} \tilde{u}_1^\varphi(t), x), \min_{t \in [0, \omega], x \in \Omega} \frac{N(x) - \tilde{u}_1^\varphi(t, x)}{2} \right\} \quad \text{and} \quad r(\tilde{Q}_{2\tilde{\epsilon}}) > 1.$$

Based on the continuous dependence of solutions on the initial data, for fixed $\tilde{\epsilon} > 0$, there is $\tilde{\epsilon}^* > 0$, then for $\tilde{\psi}$ with $\|\tilde{\psi} - M_2\| < \tilde{\epsilon}^*$, and $t \in [0, \omega]$, we arrive that $\|\Phi_t(\tilde{\psi}) - \Phi_t(M_2)\| < \tilde{\epsilon}$. Now, by contradiction, we state that $M_2$ is uniformly weak repellor for $\mathbb{Z}_0$.

Assuming contradictions, for some $\tilde{\psi}_0 \in \mathbb{Z}_0$, one has $\lim_{n \to \infty} \|\Phi^n(\tilde{\psi}_0) - M_2\| < \tilde{\epsilon}^*$. There is $n_2 \geq 1$, then $\|\Phi^n(\tilde{\psi}_0) - M_2\| < \tilde{\epsilon}^*$ for $n \geq n_2$. For each $t \geq n_2 \omega$, let $t = n \omega + t'$ with $n = \lfloor t/\omega \rfloor$ and $t \in [0, \omega)$, one has

$$\|\Phi_t(\tilde{\psi}_0) - \Phi_t(M_2)\| = \|\Phi_t'(\tilde{\psi}_0) - \Phi_t'(M_2)\| < \tilde{\epsilon}. \quad (23)$$

Following (23) and Lemma 5.4,
\[ \tilde{u}^1(t, x) - \xi < u_1(t, x) < \tilde{u}^1(t, x) + \xi, \quad 0 < u_2(t, x) < \xi, \]
\[ \tilde{u}^3(t, x) - \xi < u_3(t, x) < \tilde{u}^3(t, x) + \xi, \]
\[ \tilde{u}^4(t, x) - \xi < u_4(t, x) < \tilde{u}^4(t, x) + \xi, \quad 0 \leq u_5(t, x) < \xi, \] (24)

for each \( t \geq n_2\omega - \hat{\tau} \) and \( x \in \overline{\Omega} \). As a consequence, as \( t \geq n_2\omega, u_2(t, x, \tilde{\psi}_0) \) and \( u_5(t, x, \tilde{\psi}_0) \) meet

\[
\begin{cases}
\frac{\partial u_2(t, x)}{\partial t} \geq D_h \Delta u_2(t, x) + \frac{c_2 \beta(t, x) l[N(x) - \tilde{u}_1^1(t, x) - 2\xi]}{p[\tilde{u}_1^1(t, x) + 2\xi] + l[N(x) - \tilde{u}_1^1(t, x) + \xi]} u_5(t, x) \\
- (\mu_h(x) + \rho_2(x)) u_2(t, x), \quad x \in \Omega, \\
\frac{\partial u_5(t, x)}{\partial t} \geq D_v \Delta u_5(t, x) - \mu_v(t, x) u_5(t, x) + (1 - \tau_2^t(t)) \cdot \\
\int_{\Omega} \Gamma(t, t - \tau_2(t), x, y) \frac{\alpha_2 \beta(t - \tau_2(t), y) p[\tilde{u}_3^3(t - \tau_2(t), y) - \xi]}{p[\tilde{u}_3^3(t - \tau_2(t), y) + 2\xi] + l[N(y) - \tilde{u}_3^3(t - \tau_2(t), y) + \xi]} u_2(t - \tau_2(t), y) \, dy, \\
x \in \Omega, \\
\frac{\partial u_2(t, x)}{\partial v} = \frac{\partial u_5(t, x)}{\partial v} = 0, \quad x \in \partial \Omega.
\end{cases}
\] (25)

Based on \( u(t, x, \tilde{\psi}_0) \gg 0 \) for \( x \in \overline{\Omega} \) and \( t \geq 0 \), there must be a constant \( \sigma_5 > 0 \) then

\[
\left( u_2(t, x, \tilde{\psi}_0), u_5(t, x, \tilde{\psi}_0) \right) \geq \sigma_5 e^{\mu \tilde{\tau} t} \tilde{v}_x^*(t, x), \forall t \in [n_2\omega - \hat{\tau}, n_2\omega], \quad x \in \overline{\Omega},
\] (26)

where \( \tilde{v}_x^*(t, x) \) is a positive \( \omega \)-periodic function then \( e^{\mu \tilde{\tau} t} \tilde{v}_x^*(t, x) \) is a solution of system (23), where \( \mu_{2\tilde{\tau}} = \frac{\ln r(\tilde{\Omega}_x)}{\omega} \). It follows the comparison theorem, that

\[
\left( u_2(t, x, \tilde{\psi}_0), u_5(t, x, \tilde{\psi}_0) \right) \geq \sigma_5 e^{\mu \tilde{\tau} t} \tilde{v}_x^*(t, x), \forall t \geq n_2\omega, \quad x \in \overline{\Omega}.
\]

Since \( \mu_{2\tilde{\tau}} > 0 \), then \( u_2(t, \cdot, \tilde{\psi}_0) \to \infty \) and \( u_5(t, \cdot, \tilde{\psi}_0) \to \infty \) as \( t \to \infty \) can be obtained directly, which leads to a contradiction. As a consequence, Claim 3 holds.

With the above discussion, it is easy to obtain that \( F := M_1 \cup M_2 \cup M_3 \) is isolated and invariant for \( \Phi \) in \( X_6 \), and \( W^s(F) \cap Z_0 = \emptyset \), where \( W^s(F) \) is the stable set of \( \Gamma \) for \( \Phi \). It then follows from [45, Theorem 1.3.1 and Remark 1.3.1] that \( \Phi \) is uniformly persistent with respect to \( (Z_0, \partial Z_0) \), in this sense, there is an \( \hat{\eta} > 0 \),

\[
\liminf_{t \to \infty} d(\Phi(t, \tilde{\psi}), \partial Z_0) \geq \hat{\eta}, \quad \forall \tilde{\psi} \in \partial Z_0.
\]

(27)

Since \( \Phi^n := \Phi_{n\omega} \) is compact, for \( n \) with \( n\omega > \hat{\tau} \), then, \( \Phi \) is asymptotically smooth. Lemma 5.3 also indicates that \( \Phi \) has a global attractor on \( X_6 \). According to [24, Theorem 3.7], \( \Phi \) admits a global attractor \( \hat{A}_0 \) in \( Z_0 \).

Now we prove practical persistence required. Following \( \hat{A}_0 = \Phi(\hat{A}_0) \), we obtain that \( \check{\psi}_1(0, \cdot) > 0, \check{\psi}_2(0, \cdot) > 0, \check{\psi}_4(\cdot) > 0 \) and \( \check{\psi}_5(\cdot) > 0 \) for any \( \check{\psi} \in \hat{A}_0 \). Set \( \hat{B} := \bigcup_{\xi \in [0, \omega]} \Phi_\tau(\hat{A}_0) \). Obviously, we have \( \hat{B} \subset Z_0 \) and \( \lim_{t \to \infty} d(\Phi(t, \check{\psi}), \hat{B}) = 0 \), \( \forall \check{\psi} \in Z_0 \). A continuous function \( q_2 : X_6 \to \mathbb{R}_+ \) is defined as

\[
q_2(\check{\psi}) := \min \left\{ \min_{x \in \Omega} \check{\psi}_1(0, x), \min_{x \in \Omega} \check{\psi}_2(0, x), \min_{x \in \Omega} \check{\psi}_4(0, x), \min_{x \in \Omega} \check{\psi}_5(0, x) \right\},
\]
\forall \tilde{\psi} = (\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3, \tilde{\psi}_4, \tilde{\psi}_5) \in \mathcal{X}_6.

Owing to $\hat{B}$ is compact subset of $\mathbb{Z}_0$, then $\inf_{\tilde{\psi} \in \hat{B}} q_2(\tilde{\psi}) = \min_{\tilde{\psi} \in \hat{B}} q_2(\tilde{\psi}) > 0$. Thence, there is $\eta^* > 0$ such that

$$\liminf_{t \to \infty} q_2(\Phi_t(\tilde{\psi})) = \liminf_{t \to \infty} \min_{x \in \Omega} \left( \min_{x \in \Omega} u_1(t, x, \tilde{\psi}), \min_{x \in \Omega} u_2(t, x, \tilde{\psi}), \min_{x \in \Omega} u_4(t, x, \tilde{\psi}), \min_{x \in \Omega} u_5(t, x, \tilde{\psi}) \right) \geq \eta^*, \quad \forall \tilde{\psi} \in \mathbb{Z}_0.$$

Furthermore, according to Lemma 5.4, there must be an $\tilde{\eta} \in (0, \eta^*)$, then

$$\liminf_{t \to \infty} \min_{x \in \Omega} u_i(t, x, \tilde{\psi}) \geq \tilde{\eta}, \quad i = 1, 2, 3, 4, 5.$$  

The existence of a positive periodic steady state remains to be proved. Based on [2, Lemma 8] and [45, Theorem 3.5.1], one has that for $t > 0$, solution map $\Phi_t : W_h \to W_h$ of (6), defined in Lemma 3.3, is a $\kappa$-contraction. Define

$$Q_0 := \left\{ \tilde{\psi} \in W_h : \tilde{\psi}_1(0, \cdot) \neq 0 \text{ and } \tilde{\psi}_2(0, \cdot) \neq 0 \text{ and } \tilde{\psi}_4(\cdot) \neq 0 \text{ and } \tilde{\psi}_5(\cdot) \neq 0 \right\},$$

and

$$\partial Q_0 := W_h / Q_0 = \left\{ \tilde{\psi} \in W_h : \tilde{\psi}_1(0, \cdot) = 0 \text{ or } \tilde{\psi}_2(0, \cdot) = 0 \text{ or } \tilde{\psi}_4(\cdot) = 0 \text{ or } \tilde{\psi}_5(\cdot) = 0 \right\}.$$ 

Then $\Phi$ is $\bar{\rho}_2$-uniformly persistent with $\bar{\rho}_2(\tilde{\psi}) = d(\tilde{\psi}, \partial Q_0)$, $\forall \tilde{\psi} \in Q_0$, is easily attainable. According to [24, Theorem 4.5], system (6) has an $\omega$-periodic solution $(z_1^\omega(t, \cdot), z_2^\omega(t, \cdot), z_4^\omega(t, \cdot), z_5^\omega(t, \cdot))$ with $(z_1^\omega, z_2^\omega, z_4^\omega, z_5^\omega) \in Q_0$. Set $\tilde{u}_i^*(\eta_1, \cdot) = z_i^\omega(\eta_1, \cdot), (i = 1, 2, 3)$, and $\tilde{u}_j^*(0, \cdot) = z_j^\omega(0, \cdot), (j = 4, 5)$ with $\theta \in [-\bar{\tau}, 0]$. Again, due to the uniqueness of solutions, we get that $(u_1^\omega(t, \cdot, \tilde{\psi}), u_2^\omega(t, \cdot, \tilde{\psi}), u_3^\omega(t, \cdot, \tilde{\psi}), u_4^\omega(t, \cdot, \tilde{\psi}), u_5^\omega(t, \cdot, \tilde{\psi}))$ is a positive periodic solution of system (6).

Next, we prove the asymptotic behaviour of $E_{vi}(t, x)$ $(i = 1, 2)$ in system (5). When $R_1 < 1$ and $R_2 < 1$, we obtain

$$\lim_{t \to \infty} \left( I_1(t, x), I_2(t, x), S_\nu(t, x), I_{v1}(t, x), I_{v2}(t, x) - (0, 0, u_3^\omega(t, x), 0, 0) \right) = 0,$$

uniformly for $x \in \overline{\Omega}$. By (8),

$$\lim_{t \to \infty} E_{v1}(t, x) = \lim_{t \to \infty} E_{v2}(t, x) = 0, \quad \text{uniformly for } x \in \overline{\Omega}.$$

Presume that $R_1 > 1$ and $R_2 > 1$. 

\[\text{}``\]
(a) If \( \hat{R}_1 > 1 \), for each \( \tilde{\psi} \in \mathcal{X}_6 \) with \( \tilde{\psi}_1(0, \cdot) \neq 0 \) and \( \tilde{\psi}_4(\cdot) \neq 0 \), we have \( \lim_{t \to \infty} I_2(t, x, \tilde{\psi}) = 0 \), \( \lim_{t \to \infty} I_2(t, x, \tilde{\psi}) = 0 \) and
\[
\liminf_{t \to \infty} \min_{x \in \Omega} I_1(t, x, \tilde{\psi}) \geq \mathcal{P}_3, \quad \liminf_{t \to \infty} \min_{x \in \Omega} I_{v1}(t, x, \tilde{\psi}) \geq \mathcal{P}_3.
\]
Based on the integral forms of (8), there is \( \mathcal{P}_5 > 0 \),
\[
\liminf_{t \to \infty} \min_{x \in \Omega} E_{v1}(t, x) \geq \mathcal{P}_5, \quad \liminf_{t \to \infty} \min_{x \in \Omega} E_{v2}(t, x) = 0.
\]
(b) If \( \hat{R}_2 > 1 \), for each \( \tilde{\psi} \in \mathcal{X}_6 \) with \( \tilde{\psi}_2(0, \cdot) \neq 0 \) and \( \tilde{\psi}_5(\cdot) \neq 0 \), we have \( \lim_{t \to \infty} I_1(t, x, \tilde{\psi}) = 0 \), \( \lim_{t \to \infty} I_{v1}(t, x, \tilde{\psi}) = 0 \) and
\[
\liminf_{t \to \infty} \min_{x \in \Omega} I_2(t, x, \tilde{\psi}) \geq \mathcal{P}_4, \quad \liminf_{t \to \infty} \min_{x \in \Omega} I_{v2}(t, x, \tilde{\psi}) \geq \mathcal{P}_4.
\]
Based on the integral forms of (8), there is \( \mathcal{P}_6 > 0 \), such that
\[
\liminf_{t \to \infty} \min_{x \in \Omega} E_{v1}(t, x) = 0, \quad \liminf_{t \to \infty} \min_{x \in \Omega} E_{v2}(t, x) \geq \mathcal{P}_6.
\]
(c) As \( \hat{R}_1 > 1 \) and \( \hat{R}_2 > 1 \), for each \( \tilde{\psi} \in \mathcal{X}_6 \) with \( \tilde{\psi}_1(0, \cdot) \neq 0 \), \( i = 1, 2 \), and \( \tilde{\psi}_j(\cdot) \neq 0 \) \( j = 4, 5 \), we get
\[
\liminf_{t \to \infty} I_i(t, x, \tilde{\psi}) \geq \tilde{\eta}, \quad \liminf_{t \to \infty} S_v(t, x, \tilde{\psi}) \geq \tilde{\eta}, \quad \liminf_{t \to \infty} I_{v1}(t, x, \tilde{\psi}) \geq \tilde{\eta}, \quad i = 1, 2,
\]
uniformly for \( x \in \overline{\Omega} \). On the basis of the integral forms of (8), there exists a constant \( \eta_2 > 0 \), such that
\[
\liminf_{t \to \infty} \min_{x \in \Omega} E_{v1}(t, x) \geq \eta_2, \quad \liminf_{t \to \infty} \min_{x \in \Omega} E_{v2}(t, x) \geq \eta_2,
\]
with \( E_{v1}(0, x) \) and \( E_{v2}(0, x) \) meeting compatibility conditions (7). Besides, if \((I_1(t, x), I_2(t, x), S_v(t, x), I_{v1}(t, x), I_{v2}(t, x))\) is \( \omega \)-periodic in \( t \), then \( E_{v1}(t, x) \) and \( E_{v2}(t, x) \) are also \( \omega \)-periodic in \( t \).

6. Numerical simulations

In this position, numerical simulations exhibit how some epidemiological insights can be gained from our analytical results. Without loss of generality, we make use of one-dimensional domain \( \Omega = [0, \pi] \) to simulate the long-time behaviour which is inspired by Lou and Zhao [21], Bai et al. [2] and Wu et al. [40], and apply system (6) to the spread of malaria in Maputo Province, Mozambique. Set the periodic \( \omega = 12 \) months.

6.1. Numerical verification of theoretical analysis

First, we illustrate the theoretical results obtained in the previous sections. The reproduction numbers usually (but not always) control the outcome of strain competition. The basic reproduction numbers are calculated based on the theory provided by [19] and the expected results are summarized in Table 1.

In the next subsections, the theoretical results are verified one by one using the parameter values defined in Appendix 5.
Table 1. The potential dynamical outcomes of system (6).

| Case | Strain 1 | Strain 2 | Expected outcome | Comment |
|------|----------|----------|------------------|---------|
| 1    | $R_1 < 1$| $R_2 < 1$| $u_1 \to 0, u_2 \to 0, u_4 \to 0, u_5 \to 0$ | Theorem 5.1, Figure 1 |
| 2    | $R_1 < 1$| $R_2 > 1$| $u_1 \to 0, u_2$ persists, $u_4 \to 0, u_5$ persists | Theorem 5.4, Figure 2 |
| 3    | $R_1 > 1$| $R_2 < 1$| $u_1$ persists, $u_2 \to 0, u_4$ persists, $u_5 \to 0$ | Theorem 5.5 |
| 4    | $R_1 > 1$| $R_2 > 1$| (i) $u_1 \to 0, u_2$ persists, $u_4 \to 0, u_5$ persists | Theorem 5.6 (ii), Figure 3 |
|      |          |          | (ii) $u_1$ persists, $u_2 \to 0, u_4$ persists, $u_5 \to 0$ | Theorem 5.6 (i), Figure 4 |
|      |          |          | (iii) $u_1$ persists, $u_2$ persists, $u_4$ persists, $u_5$ persists | Theorem 5.7, Figure 5 |

6.1.1. Case 1 in Table 1

We choose $c_1 = 0.0151$, $c_2 = 0.01454$, $\alpha_1 = 0.0146$, $\alpha_2 = 0.012$. Let $\varrho_1 = a_1 \cdot (1.05 - \cos(2x))$ Month$^{-1}$ and $\varrho_2 = a_2 \cdot (1.05 - \cos(2x))$ Month$^{-1}$, where $a_1 = 0.085$, $a_2 = 0.081$, reflect the fact that people living in urban areas (around the centre) can enjoy added medical treatment (the number of physicians and hospitals, supply of medicines, state-of-the-art medical equipment) than people in rural area, resulting in a higher recovery

Figure 1. The evolution of infection compartments of humans and mosquitoes when $R_1 < 1$ and $R_2 < 1$. (a) The evolution of $u_1$. (b) Then evolution of $u_4$. (c) Then evolution of $u_2$. (d) Then evolution of $u_5$. 
rate around the centre, and maintaining the other parameters listed in Table A1. To compute $R_1$ and $R_2$, the numerical scheme recently proposed in [19, Lemma 2.5 and Remark 3.2] is used. For this set of parameters, we can numerically calculate $R_1 = 0.3644 < 1$, $R_2 = 0.3733 < 1$. Figure 1 shows numerical plots of infected compartments following the initial data

$$u(\theta, x) = \begin{pmatrix} 113465 - 40000 \cos 2x \\ 13465 - 5386 \cos 2x \\ 6807900 - 113465 \cos 2x \\ 890000 - 100000 \cos 2x \\ 17720 - 6158 \cos 2x \end{pmatrix}, \quad \forall \theta \in [-\hat{\tau}, 0], \, x \in [0, \pi].$$

Then Figure 1 implies that two strains die out, which is coincident with Theorem 5.1. Infectious humans and mosquitoes go to 0, indicating that the disease will be eliminated under this set of parameters.

![Figure 1](image1.png)

**Figure 2.** The evolution of infection compartments of humans and mosquitoes when $R_1 < 1$ and $R_2 > 1$. (a) The evolution of $u_1$. (b) The evolution of $u_4$. (c) The evolution of $u_2$. (d) The evolution of $u_5$. 
6.1.2. Case 2 and Case 3 in Table 1
Here, we simulate the result of Case 2 in Table 1, and then we can handle Case 3 similarly by adjusting the parameters. We choose \( \varrho_1 = a_1 \cdot (1.05 - \cos(2x)) \) Month\(^{-1} \), \( \varrho_2 = a_2 \cdot (1.05 - \cos(2x)) \) Month\(^{-1} \), where \( a_1 = 0.06 \), \( a_2 = 0.05 \), \( c_1 = 0.02 \), \( c_2 = 0.025 \), \( \alpha_1 = 0.03 \), \( \alpha_2 = 0.04 \), \( R_1 = 0.7045 < 1 \), \( R_2 = 1.1136 > 1 \). Figure 2 shows that if \( R_1 < 1 \) and \( R_2 > 1 \), strain 1 becomes extinct and strain 2 persists, which is consistent with Theorem 5.4. Note that we truncate the time interval by [30, 60]. Adjust the values of parameters befittingly to satisfy \( R_1 > 1 \) and \( R_2 < 1 \), and the conclusion of Case 3 in Table 1 is obtained. Details are not mentioned here.

6.1.3. Case 4 in Table 1
For Case 4 in Table 1, there are three results for \( R_1 > 1 \) and \( R_2 > 1 \). For the first conclusion, we choose \( \varrho_1 = a_1 \cdot (1.05 - \cos(2x)) \) Month\(^{-1} \), \( \varrho_2 = a_2 \cdot (1.05 - \cos(2x)) \) Month\(^{-1} \), where \( a_1 = 0.07 \), \( a_2 = 0.05 \), \( c_1 = 0.09 \), \( c_2 = 0.1 \), \( \alpha_1 = 0.04 \), \( \alpha_2 = 0.05 \), and maintain other parameters recorded in Table A1. Based on this set of parameters, numerically compute Figure 3.

Figure 3. The evolution of infection compartments of humans and mosquitoes when \( R_1 > 1 \) and \( R_2 > 1 \). (a) The evolution of \( u_1 \). (b) Then evolution of \( u_4 \). (c) Then evolution of \( u_2 \). (d) Then evolution of \( u_5 \).
Figure 4. The evolution of infection compartments of humans and mosquitoes when $R_1 > 1$ and $R_2 > 1$. (a) The evolution of $u_1$. (b) The evolution of $u_4$. (c) The evolution of $u_2$. (d) The evolution of $u_5$.

$R_1 = 1.6079 > 1$ and $R_2 = 2.2901 > 1$. Figure 3 indicates that strain 1 becomes extinct and strain 2 is persistent. Note that we truncate the time interval by $[30, 60]$. Because under this set of parameters, we assume that individuals and mosquitoes infected with strain 2 are more infectious and have a smaller recovery rate. Therefore, once this phenomenon occurs, there will be greater challenges to the control and eradication of malaria.

By adjusting the parameters, we can also get the result that strain 1 is persistent, and strain 2 is extinct under the condition of $R_1 > 1$ and $R_2 > 1$. We choose $\varrho_1 = a_1 \cdot (1.05 - \cos(2x)) \text{Month}^{-1}$, $\varrho_2 = a_2 \cdot (1.05 - \cos(2x)) \text{Month}^{-1}$, where $a_1 = 0.07$, $a_2 = 0.06$, $c_1 = 0.15$, $c_2 = 0.1$, $\alpha_1 = 0.055$, $\alpha_2 = 0.05$, and maintain other parameters recorded in Table A1. Then, compute $R_1 = 2.4342 > 1$ and $R_2 = 2.2901 > 1$. Figure 4 exhibits that strain 1 is persistent and strain 2 is extinct. Note that we truncate the time interval by $[30, 60]$.

Next, the coexistence of two strains under the conditions of $R_1 > 1$ and $R_2 > 1$ is simulated. We choose $\varrho_1 = a_1 \cdot (1.05 - \cos(2x)) \text{Month}^{-1}$, $\varrho_2 = a_2 \cdot (1.05 - \cos(2x)) \text{Month}^{-1}$, where $a_1 = 0.025$, $a_2 = 0.0505$, $c_1 = 0.025$, $c_2 = 0.04836$, $\alpha_1 = 0.035$, $\alpha_2 = 0.04$, and keep other parameters listed in Table A1. For this set of parameters, we can...
Figure 5. The evolution of infection compartments of humans and mosquitoes when $R_1 > 1$ and $R_2 > 1$. (a) The evolution of $u_1$. (b) The evolution of $u_4$. (c) The evolution of $u_2$. (d) The evolution of $u_5$.

compute $R_1 = 1.2759 > 1$ and $R_2 = 1.5417 > 1$. Figure 5 shows that two strains can coexist, which is consistent with Theorem 5.7. Note that we truncate the time interval $[50, 80]$ to demonstrate the existence of periodic solution, in which two strains coexist.

### 6.2. The impact of diffusion

Except for $D_h = 0$ and $D_v = 0$, the other parameters are in accordance with in Case 1. When $D_h = 0.1$ and $D_v = 0.0125$ in Case 1, then $R_1 = 0.3644$ and $R_2 = 0.3733$, and the disease dies out. Whereas, if $D_h = 0$ and $D_v = 0$, then $R_1 = 1.3858$, $R_2 = 1.4100$ are exceeding 1, and the disease persists. As a consequence, the result is shown in Figure 6.

Figure 6 shows some intriguing phenomena. In this case, malaria becomes extinct in urban areas, but persists in rural areas. It is caused by spatial heterogeneity. Because the urban environment is not apposite to mosquitoes to survive, there are relatively few mosquitoes, and there are more medical resources, the disease control in urban is relatively better. Compare with Figure 1, the movement will reduce disease spread risk, to a limited degree, which is consistent with the conclusion in [40]. The implications include:
Figure 6. The evolution of infection compartments of humans and mosquitoes when $D_h = 0$ and $D_v = 0$. (a) The evolution of $u_1$. (b) The evolution of $u_4$. (c) The evolution of $u_2$. (d) The evolution of $u_5$.

(i) The presence of spread suggests that infected humans in rural areas can enjoy better treatment in cities. (ii) As stated in Ref. [40], it is challenging for mosquitoes to collect blood meal in moving humans.

In order to understand the effect of human and mosquito movement on malaria transmission, under a set of the parameter values in Case 1, we do the following work:

(i) We assume that the diffusion coefficient of humans is $D_h = 0$, and the diffusion coefficient of mosquitoes is $D_v = 0.0125$ (see Figure 7). Through calculation, $R_1 = 1.1369$ and $R_2 = 1.1719$. Compared with the above $D_v = 0$ and $D_h = 0$ (Figure 6), the diffusion of mosquitoes reduces the risk of disease transmission to some extent. Biologically, this may be due to the fact that there are more mosquitoes in rural areas, and infected mosquitoes move to urban areas. With the same bite rate, city has better medical conditions and a greater recovery rate, so the risk of malaria transmission will be decreased.

(ii) We assume that the diffusion coefficient of humans is $D_h = 0.1$, but the diffusion coefficient of mosquitoes is $D_v = 0$ (see Figure 8). Then $R_1 = 0.3645$ and $R_2 = 0.3734$. Compared with the case of $D_h = 0$ and $D_v = 0$ (Figure 6), humans diffusion reduces the risk of disease transmission. Biologically, it can be explained that the diffusion of humans
gives them the opportunity to find a better medical environment, which will increase the recovery rate and reduce the risk of disease transmission. This may be due to the better medical conditions in the city, and the infected people will go to the city for treatment, which will increase the recovery rate and reduce the risk of disease transmission. Compared with $D_h = 0$ and $D_v = 0.0125$, humans diffusion has a greater impact on disease transmission. In summary, the risk of malaria transmission will be overestimated if the diffusion is ignored in the model analysis.

6.3. The impact of medical resources on disease transmission

In this position, the influence of medical resources on malaria transmission is analysed from two aspects: (i) The impact of the amount of medical resources; (ii) For fixed medical resources, the influence of different allocation measures on malaria transmission. In this subsection, we take strain 1 as an example and draw conclusions about strain 2.

Since there are more medical resources in urban areas than in rural areas, people can get better medical services, and therefore have higher recovery rates. We already know that
Figure 8. The evolution of infection compartments of humans and mosquitoes with \( D_h = 0.1 \) and \( D_v = 0 \). (a) The evolution of \( u_1 \). (b) The evolution of \( u_4 \). (c) The evolution of \( u_2 \). (d) The evolution of \( u_5 \).

\( a_1 \) varies between \([0.042, 0.57]\), here, take \( a_1 = 0.085, 0.15, 0.25, 0.35, 0.45 \) to get the spatial distribution of the recovery rate in Figure 9. The corresponding medical resources in the whole region are \( \int_0^T \varrho_1(x)dx = 0.08925, 0.1575, 0.2625, 0.3675, 0.4725 \). In this paper, we use the recovery rate of the entire region \( \int_0^T \varrho_1(x)dx \) to represent medical resources. Choose \( c_1 = 0.09, \alpha_1 = 0.04 \), and other parameters are consistent with those in Table 1. Under different medical resources, through calculation, the corresponding \( R_1 \) is 1.4724, 1.1484, 0.9347, 0.8251, 0.7556, respectively. By observation, in a heterogeneous environment, increasing medical resources will not have a great impact on improving the recovery rate in rural areas. However, the analysis in Subsection 6.2 shows that, due to human diffusion, some patients in rural areas will go to the cities for medical treatment and enjoy better medical care, so the overall transmission risk is reduced.

First of all, we hope to see whether maintaining a balanced distribution of urban and rural medical resources will help control the disease for a fixed recovery rate. When fixed medical resources, exploring the effect of different allocation on malaria spread. Fix \( a_1 = 0.085 \) and introduce a new parameter \( \sigma_6 \) into \( \varrho_1 \), that is, \( \varrho_1(x) = 0.085 \times (1.05 - \)
6.4. The impact of seasonal changes in temperature and vector-bias on disease transmission

Here, we are interested in the impact of temperature-dependent EIP and vector-bias on the spread of disease. Now take single-strain subsystem (10) as an example.

6.4.1. The role of vector-bias

To understand the influence of vector-bias on the size of malaria transmission, we make two comparison figures. The parameter value is consistent with the parameter value of Case 1, in this case, \( p = 0.8 \) and \( l = 0.6 \), i.e. \( l/p = 3/4 \). Figure 11(a,c), show the evolutions of infected mosquitoes and individuals at each position in \([0, \pi]\) as time evolves. Adjust the values of \( p = 0.8 \) and \( l = 0.8 \), so that \( l/p = 1 \), that is, there is no vector-bias effect, shown in
Figure 10. For fixed medical resources, the spatial distribution of $\varrho_1$ and the corresponding value of $R_1$.

Figure 11(b,d). Through observation and comparison, if vector-bias effect is not taken into account, the duration of disease extinction will be shortened, i.e. if there is no vector-bias effect, the risk of malaria transmission will be underestimated.

6.4.2. The impact of seasonal changes in temperature on disease transmission

To understand the influence of temperature-dependent EIP on malaria transmission, two comparative figures of evolution of infectious humans and mosquitoes are produced, $[\tau_1] := \frac{1}{\omega} \int_0^\omega \tau_1(t)\,dt = 17.25/30.4\,\text{Month}$, $[\beta] := \frac{1}{\omega} \int_0^\omega \beta(t)\,dt = 6.983\,\text{Month}^{-1}$, $[\mu_v] := \frac{1}{\omega} \int_0^\omega \mu_v(t)\,dt = 3.086\,\text{Month}^{-1}$, $[\Lambda] := \frac{1}{\omega} \int_0^\omega \Lambda(t)\,dt = \hat{k}\times[\beta]$. The parameters are constant, that is, they are not affected by temperature, and the other parameter values are consistent with Case 1.

Figure 12 reveals that if we do not consider seasonal changes in temperature when analysing the spread of malaria, the duration of disease extinction will be shortened, that is, the risk of disease transmission will be underestimated. This will lead to deviations in the prevention and control departments when formulating measures to control and eliminate malaria.

7. Discussion

This paper proposes and analyses a reaction–diffusion malaria model with vector-bias effect and multi-strains in a periodic environment. With the purpose of exploring the dynamics, we first use the existing theory to derive $R_i$, $R_0$, and then introduce $\tilde{R}_i\,((i = 1, 2))$. Using the comparison arguments and persistence theory of periodic semi-flow, we prove that malaria disease will be eliminated if $R_1 < 1$ and $R_2 < 1$, and the strain $i$ is extinct and the strain $j$ is persistent if $R_j > 1 > R_i\, (i, j = 1, 2, i \neq j)$. When $R_1 > 1$ and $R_2 > 1$, there are three cases: (i) strain 1 is die out and strain 2 persists; (ii) strain 1 is persistent and strain 2 is extinct; (iii) the two strains coexist under the condition of $\tilde{R}_1 > 1$, $\tilde{R}_2 > 1$ and exhibit spatial and seasonal fluctuations. This phenomenon is consistent with the multi-strain coexistence problem considered in Ref. [30,46], but they do not consider the periodic
Figure 11. The evolution of infection compartments of humans and mosquitoes with different vector-bias parameter. Among them, green indicates that there is a vector-bias effect \((l/p = 3/4)\), and red indicates that there is no vector-bias effect \((l/p = 1)\). (a) With vector-bias effect. (b) Without vector-bias effect. (c) With vector-bias effect. (d) Without vector-bias effect.

delay. Wu and Zhao [40] propose a reaction–diffusion model of vector-borne disease with periodic delays to investigate the multiple effects of the temperature sensitivity of incubation periods, spatial heterogeneity, and the seasonality on disease transmission. However, they did not consider the case of multiple strains.

The numerical analysis is divided into four parts. First, the simulation results support the theoretical analysis. Second, the impact of diffusion on malaria transmission in a heterogeneous space is explored. Biologically, we figure out this spatial heterogeneity to be the difference between rural and urban regions. Simulation results show that under the condition of uneven distribution of medical resources, ignoring the diffusion of humans and mosquitoes, the risk of malaria transmission will be overestimated, which is consistent with the result in Ref. [40,47]. The implications include: (i) Infected humans in rural areas can seek better treatment in cities. (ii) It is challenging for mosquitoes to pick up a
Figure 12. The evolution infection compartments of humans and mosquitoes with seasonal temperature changes and without seasonal temperature changes. Among them, red indicates that the parameters depend on temperature, and green indicates that the parameters do not depend on temperature.

blood meal with moving humans. (iii) The impact of medical resources allocation on the spread of malaria is explored from two aspects: (a) increasing medical resources; (b) fixed medical resources, different allocation measures. Third, through research, we see that in the case of uneven distribution of medical resources, increasing medical resources has little impact on the spread of disease in rural areas, but it will reduce the risk of disease transmission in the entire region. Besides, when medical resources is limited, reducing the gap in the allocation of medical resources between rural and urban areas can further decrease the malaria spread risk, which is consistent with the result in Ref. [40]. Therefore, when formulating measures to eliminate malaria, attention should be paid to the input of rural medical resources. Numerical results indicate that the risk of malaria may be underestimated if ignoring vector-bias effect or season. [1] established a single strain ordinary differential equation model to study the effect of vector-bias on malaria transmission. The result taken
by Abboubakar et al. [1] in Figure 3 shows that increasing vector-bias parameter decreases the equilibrium prevalence of infectious humans which is consistent with our conclusion.

In our study, there are some shortcomings that need to be improved. In the process of theoretical analysis, due to the complexity of periodic delays, we have not discussed all situations clearly. We have not solved the global stability of the positive periodic solution of this model. In malaria transmission model, few researchers have studied the influence of infection age and spatial diffusion on disease transmission. In fact, after malaria infection, the intensity of infectivity varies at different stages of infection. The time after infection is called the age of infection, affects the number of secondary infections. This important factor needs to be considered in the process of malaria transmission. These problems are very important for exploring the law of malaria transmission and formulating control measures. In future work, we hope to better explore them.

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Appendices

Appendix 1. Derivation of $M_{vi} (i = 1, 2)$

Once again, $\rho(t, q, x)\Delta q$ is the number of female mosquitoes of development level infection $(q, q + \Delta q)$ at time $t$ and location $x$. If we consider the rate of change of the number of female mosquitoes in a given development level of infection interval $(q, q + \Delta q)$, we may write

$$\frac{\partial}{\partial t} \left[ \int_\Omega \rho(t, q, x)\Delta q dV \right] = \begin{bmatrix} +\text{rate of entry at } q \\ -\text{rate of departure at } (q + \Delta q) \\ -\text{deaths} - \text{move out} \end{bmatrix},$$

or

$$\frac{\partial}{\partial t} \int_\Omega \rho(t, q, x)\Delta q dV = \int_\Omega f(t, q, x)\Delta q dV + \int_\Omega J_1(t, q, x)dV$$

$$- \int_\Omega J_1(t, q + \Delta q, x)dV - \int_{\partial\Omega} J_2(t, q, x)\Delta q dS,$$

where $f(t, q, x)$ represents the rate of the population change per unit level of infection development, $J_1(t, q, x)$ denotes the positive (left to right) ‘flux’ of individuals at the level of infection development.

Figure A13. A small interval of development level of infection.
$q$, time $t$ and position $x$, and $J_2(t, q, x)$ is the diffusive flux. $dS$ is the vector element of surface. Dividing by $\Delta q$ gives us

$$\frac{\partial}{\partial t} \int_{\Omega} \rho(t, q, x) dV = \int_{\Omega} f(t, q, x) dV - \int_{\Omega} \frac{J_1(t, q + \Delta q, x) - J_1(t, q, x)}{\Delta q} dV - \int_{\partial \Omega} J_2(t, q, x) dS.$$  

Use Gauss’s divergence theorem to transform the third integral into a volume integral to simplify this equation,

$$\int_{\partial \Omega} J_2(t, q, x) dS = \int_{\Omega} \nabla \cdot J_2(t, q, x) dV.$$  

Taking the limit as $\Delta q$ approaches zero produces a conservation law for the density of individuals

$$\frac{\partial}{\partial t} \int_{\Omega} \rho(t, q, x) dV = \int_{\Omega} f(t, q, x) dV - \int_{\Omega} \frac{J_1(t, q, x)}{\Delta q} dV - \int_{\Omega} \nabla \cdot J_2(t, q, x) dV. \quad (A1)$$  

This causes Equation (A1) reducing to

$$\int_{\Omega} \left[ \frac{\partial \rho(t, q, x)}{\partial t} - f(t, q, x) + \frac{\partial J_1(t, q, x)}{\partial q} + \nabla \cdot J_2(t, q, x) \right] dV = 0.$$  

Since it is for arbitrary $\Omega$, this integral is zero, then

$$\frac{\partial \rho(t, q, x)}{\partial t} - f(t, q, x) + \frac{\partial J_1(t, q, x)}{\partial q} + \nabla \cdot J_2(t, q, x) = 0. \quad (A2)$$  

For more clarity, the form of $f(t, q, x)$ needs to be discussed. According to Ref. [17], $f(t, q, x)$ depends on the independent variables, the level of infection development, time and space. However, this is usually achieved through the dependent variable, population density. Here, let $f(t, q, x) = -\nu_v(t, x) \rho(t, q, x)$. According to Fick’s law and the hypothesis of the mosquito diffusion coefficient $D_v$, write the diffusive flux as

$$J_2(t, q, x) = -D_v \nabla \rho(t, q, x).$$  

General balance Equation (A2) now reduces to

$$\frac{\partial \rho(t, q, x)}{\partial t} = -\nu_v(t, x) \rho(t, q, x) - \frac{\partial J_1(t, q, x)}{\partial q} + D_v \nabla^2 \rho(t, q, x), \quad (A3)$$

where $\nabla^2 \rho(t, q, x)$ is the Laplacian, i.e. $\nabla^2 \rho(t, q, x) = \Delta \rho(t, q, x)$. So (A3) can write as

$$\frac{\partial \rho(t, q, x)}{\partial t} = D_v \Delta \rho(t, q, x) - \frac{\partial J_1(t, q, x)}{\partial q} - \nu_v(t, x) \rho(t, q, x), \quad (A4)$$

To progress any more, we must say something about the flux of mosquitoes $J_1(t, x, q)$. This is not a flux in space but a development level of infection. All mosquito infection levels develop, and then

$$J_1(t, q, x) = \gamma(t) \rho(t, q, x).$$

Write Equation (A4) as

$$\frac{\partial \rho(t, q, x)}{\partial t} = D_v \Delta \rho(t, q, x) - \frac{\partial \gamma(t) \rho(t, q, x)}{\partial q} - \nu_v \rho(t, q, x). \quad (A5)$$

Then by the same arguments as in Ref. [40] and Ref. [17], we can derive the expression of $M_{vi}(i = 1, 2)$ as

$$M_{vi}(t, x) = (1 - \tau'_i(t)) \int_{\Omega} \Gamma(t, t - \tau(t), x, y) \alpha_i \beta(t - \tau'_i(t), y) p_i(t - \tau(t), y) S_v(t - \tau(t), y) \left[ I_1(t - \tau(t), y) + I_2(t - \tau(t), y) \right] dy.$$

$\Gamma(t, t_0, x, y)$ is the Green function associated with $\frac{\partial u(t, x)}{\partial t} = D_v \Delta u(t, x) - \nu_v(t, x) u(t, x)$, for $t \geq t_0 \geq 0$ and $x, y \in \Omega$, subject to the Neumann boundary condition.
Appendix 2. Proof of Lemma 4.5

Proof: Similar to the proof of Lemma 4.4, on interval \([n\bar{t}_1, (n + 1)\bar{t}_1]\), \(n \in \mathbb{N}\), one can easily obtain that \(\sigma_i(t, \cdot) \geq 0, (i = 1, 2)\) for all \(t \geq 0\). Choose a large number \(K > \max\{\tilde{\mu}_h + \tilde{\varrho}_1, \tilde{\mu}_v\}\), where \(\tilde{\mu}_h = \max_{x \in \Omega} \mu_h(x)\), \(\tilde{\mu}_v = \max_{t \in [0, \omega], x \in \Omega} \mu_v(t, x)\) and \(\tilde{\varrho}_1 = \max_{x \in \Omega} \varrho_1(x)\), such that for each \(t \in \mathbb{R}\), \(g_1(t, \cdot, \varpi_1) := - (\mu_h(\cdot) + \varrho_1(\cdot)) \varpi_1 + K \varpi_1\) is increasing in \(\varpi_1\), and \(g_2(t, \cdot, \varpi_2) := - \mu_v(t, \cdot) \varpi_2 + K \varpi_2\) is increasing in \(\varpi_2\). Then, both \(\sigma_1(t, x)\) and \(\sigma_2(t, x)\) satisfy the following system

\[
\begin{align*}
\frac{\partial \sigma_1(t, x)}{\partial t} &= D_h \Delta \sigma_1(t, x) + c_1 \beta(t, x) \sigma_2(t, x) + g_1(t, x, \sigma_1) - K \sigma_1(t, x), \quad x \in \Omega, \\
\frac{\partial \sigma_2(t, x)}{\partial t} &= D_v \Delta \sigma_2(t, x) + g_2(t, x, \sigma_2) - (1 - \tau_1(t)) \sigma_1(t, x) + \int_{\Omega} \Gamma(t, t - \tau_1(t), x, y) \frac{\alpha_1 \beta(t - \tau_1(t), y) p \varpi_1^*(t - \tau_1(t), y) \varpi_1(t - \tau_1(t), y)}{\ln(y)} \, dy, \quad x \in \Omega, \\
\frac{\partial \sigma_1(t, x)}{\partial \nu} &= \frac{\partial \sigma_2(t, x)}{\partial \nu} = 0, \quad x \in \partial \Omega.
\end{align*}
\]

Thus, for a given \(\xi \in \mathcal{E}_1^+\), one has

\[
\begin{align*}
\sigma_1(t, \cdot, \xi) &= \tilde{T}_1(t, 0) \xi(0) + \int_0^t \tilde{T}_1(t, s) g_1(s, \cdot, \sigma_1(s, \cdot)) \, ds + \int_0^t \tilde{T}_1(t, s) c_1 \beta(s, \cdot) \sigma_2(s, \cdot) \, ds, \\
\sigma_2(t, \cdot, \xi) &= \tilde{T}_2(t, 0) \xi_2(0) + \int_0^t \tilde{T}_2(t, s) g_2(s, \cdot, \sigma_2(s, \cdot)) \, ds + \int_0^t \tilde{T}_2(t, s)(1 - \tau_1(s)) \, ds \\
&\quad + \int_\Omega \Gamma(s, s - \tau_1(s), \cdot, y) \frac{\alpha_1 \beta(s - \tau_1(s), y) p \varpi_1^*(s - \tau_1(s), y) \varpi_1(s - \tau_1(s), y)}{\ln(y)} \, dy \, ds, \\
\end{align*}
\]

where \(\tilde{T}_1(t, s), \tilde{T}_2(t, s) := Y \rightarrow Y\) are the evolution operators associated with \(\frac{\partial \sigma_1}{\partial t} = D_h \Delta \sigma_1 - K \sigma_1\) and \(\frac{\partial \sigma_2}{\partial t} = D_v \Delta \sigma_2 - K \sigma_2\) subject to the Neumann boundary condition, respectively. Since \(\tilde{L}(t) := t - \tau_1(t)\) is increasing in \(t \in \mathbb{R}\), it easily follows that \([-\tau_1(0), 0] \subset \tilde{L}([0, \bar{t}_1])\). We assume that \(\varrho_1 > 0\). Then there exists an \((\theta, x_0) \in [-\tau_1(0), 0] \times \Omega\) such that \(\sigma_1(\theta, x_0) > 0\). According to the second equation of \((A6)\), we have \(\sigma_2(t, \cdot, \varrho) > 0\) for all \(t > \bar{t}_1\). Note that if \(s > 2\bar{t}_1\), then \(s - \tau_1(s) > 2\bar{t}_1 - \bar{t}_1 = \bar{t}_1\). It is easy to know that \(\sigma_1(t, \cdot, \varrho) > 0\) for all \(t > 2\bar{t}_1\) from the first equation of \((A6)\). This shows that \(\sigma_i(t, \cdot) > 0\) for all \(t > 2\bar{t}_1\), therefore, the solution map \(\tilde{P}_t\) is strongly positive whenever \(t > 3\bar{t}_1\). \(\blacksquare\)
Appendix 3. Proof of Theorem 5.2

Proof: (a) In this case of $R_1 < 1$, it follows from Lemmas 4.2 and 4.6 that $r(\hat{P}_\ell) < 1$, and therefore

$$\mu_1 = \frac{\ln r(\hat{P}_\ell)}{\omega} < 0.$$  

Discuss the following system with parameter $\epsilon > 0$

$$\begin{align*}
\begin{cases}
\frac{\partial \hat{w}_1(t,x)}{\partial t} &= D_h \Delta \hat{w}_1(t,x) + c_1 \beta(t,x) \hat{w}_2(t,x) - (\mu_h(x) + \phi_1(x)) \hat{w}_1(t,x), &t > 0, \quad x \in \Omega, \\
\frac{\partial \hat{w}_2(t,x)}{\partial t} &= D_v \Delta \hat{w}_2(t,x) - \mu_v(t,x) \hat{w}_2(t,x) + (1 - \tau'(t)), \\
\int_\Omega \Gamma(t, t - \tau_1(t), y) \frac{\alpha_1 \beta(t - \tau_1(t), y) p[u_1^*(t - \tau_1(t), y) + \epsilon] \hat{w}_1(t - \tau_1(t), y)}{ln(y)} dy, &t >, \\
x \in \Omega,
\end{cases}
\end{align*}$$

(A7)

Denote $\hat{w}(t, x, \varphi) = (\hat{w}_1(t, x, \varphi), \hat{w}_2(t, x, \varphi))$ to be the unique solution of system (A7) for any $\varphi \in \mathcal{E}_1$, with $\hat{w}_0(\varphi)(\theta, x) = \varphi(\theta, x)$ for all $\theta \in [-\tau_1(0), 0], x \in \overline{\Omega}$, where

$$\hat{w}_1(\varphi)(\theta, x) = \hat{w}(t + \theta, x, \varphi) = (\hat{w}_1(t + \theta, x, \varphi), \hat{w}_2(t, x, \varphi)), \; \forall t \geq 0, \; (\theta, x) \in [-\tau_1(0), 0] \times \overline{\Omega}.$$  

Let $\bar{P}_\ell := E_1 \rightarrow \mathcal{E}_1$ be the Poincaré map of system (A7), i.e. $\bar{P}_\ell(\varphi) = \hat{w}_0(\varphi), \forall \varphi \in \mathcal{E}_1$, and make $r(\bar{P}_\ell)$ to be the spectral radius of $\bar{P}_\ell$. From the continuity of the principle eigenvalue about parameters, we know that $\lim_{t \rightarrow 0} r(\bar{P}_\ell) = r(\bar{P}_1)$. Allow $\epsilon > 0$ to be small enough, one has $r(\bar{P}_\ell) < 1$. Based on Lemma 4.7, there exists a positive $\omega$-periodic function $\hat{w}^* (t, x)$ such that $\hat{w}(t, x) = e^{l(t,x)} \hat{w}^*(t, x)$ is a solution of system (A7), where $\mu_\epsilon = \frac{\ln r(\bar{P}_\ell)}{\omega} < 0$. For $\epsilon > 0$ fixed above, by the global attractivity of $u_1^*(t, x)$ of system (12) and the comparison principle, there must be an integer $n_1 > 0$ large enough such that $n_1 \omega > \hat{\tau}$ and

$$\hat{u}_2(t, x) \leq u_1^*(t, x) + \epsilon, \; \forall t \geq n_1 \omega - \hat{\tau}, \; x \in \overline{\Omega}.$$  

Thus, as $t > n_1 \omega$, one has

$$\begin{align*}
\begin{cases}
\frac{\partial \hat{u}_1(t,x)}{\partial t} &\leq D_h \Delta \hat{u}_1(t,x) + c_1 \beta(t,x) \hat{u}_3(t,x) - (\mu_h(x) + \phi_1(x)) \hat{u}_1(t,x), \quad x \in \Omega, \\
\frac{\partial \hat{u}_3(t,x)}{\partial t} &\leq D_v \Delta \hat{u}_3(t,x) - \mu_v(t,x) \hat{u}_3(t,x) + (1 - \tau'(t)), \\
\int_\Omega \Gamma(t, t - \tau_1(t), y) \frac{\alpha_1 \beta(t - \tau_1(t), y) p[u_1^*(t - \tau_1(t), y) + \epsilon] \hat{u}_1(t - \tau_1(t), y)}{ln(y)} dy, &x \in \Omega,
\end{cases}
\end{align*}$$

(A8)

There exists some $\sigma_1 > 0$, for any designated $\tilde{\varphi} \in \mathcal{X}_4$, such that

$$\left(\hat{u}_1(t, x, \tilde{\varphi}), \hat{u}_3(t, x, \tilde{\varphi})\right) \leq \sigma_1 \left(\hat{w}(t, x, \varphi), \hat{w}_2(t, x)\right), \; \forall t \in [n_1 \omega - \hat{\tau}, n_1 \omega], \; x \in \overline{\Omega}.$$  

Using (A7), (A8) and the comparison theorem for abstract functional differential equation [25, Proposition 1], we get

$$\left(\hat{u}_1(t, x, \tilde{\varphi}), \hat{u}_3(t, x, \tilde{\varphi})\right) \leq \sigma_1 e^{l(t,x)} \hat{w}^*(t, x), \; \forall t \geq n_1 \omega, \; x \in \overline{\Omega}.$$

Accordingly, $\lim_{t \rightarrow \infty} (\hat{u}_1(t, x, \tilde{\varphi}), \hat{u}_3(t, x, \tilde{\varphi})) = (0, 0)$ uniformly for $x \in \overline{\Omega}$.

Using the theory of internal chain transitive sets [45, Section 1.2], for $x \in \overline{\Omega}$, $\lim_{t \rightarrow \infty} (\hat{u}_2(t, x, \tilde{\varphi}) - u_1^*(t, x)) = 0$ uniformly, where $u_1^*(t, x)$ is a globally attractive solution of system (12). Based on the above argument, it is easy to see that $\hat{u}_2(t, x, \varphi)$ meets a nonautonomous system, which is asymptotic to the periodic system (12).
Clearly, $D_t$ and $D$ are solution map and Poincaré map associated with system (10) on $\mathcal{X}_4$, respectively. Let $\mathcal{J} = \omega(\bar{\varphi})$ be the omega limit set of $\bar{\varphi} = (\bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3) \in \mathcal{X}_4$ for $D$. Since $\lim_{t \to \infty} \bar{u}_1(t, x, \varphi) = 0$ and $\lim_{t \to \infty} \bar{u}_3(t, x, \varphi) = 0$ uniformly for $x \in \bar{\Omega}$, one has $\mathcal{J} = [\bar{0}] \times \mathcal{J} \times [0]$. Here, $\bar{0}$ represents a function which is identical to 0, i.e. $\bar{0}(\theta, \cdot) = 0$, $\forall \theta \in [-\tau_1(0), 0]$. Following Lemma 5.2, one has $\bar{0} \notin \mathcal{J}$.

For any $\zeta \in C([-\tau_1(0), 0], \mathbb{Y}^+)$, let $w(t, x, \zeta(0, \cdot))$ be the solution of (12) with initial data $w(0, x) = \zeta(0, x)$. A solution semi-flow of (12) on $C([-\tau_1(0), 0], \mathbb{Y}^+)$ described as

\[
\begin{aligned}
 w_t(\theta, x, \zeta) = \begin{cases}
 w(t + \theta, x, \zeta(0)), & \text{if } t + \theta > 0, \theta \in [-\tau_1(0), 0], t > 0, \\
 \zeta(t + \theta, x), & \text{if } t + \theta \leq 0, \theta \in [-\tau_1(0), 0], t > 0.
\end{cases}
\end{aligned}
\]

Set $\tilde{D}(\zeta) = w_\omega(\zeta)$. Following [45, Lemma 1.2.1], $\mathcal{J}$ is an internal chain transitive set for $D$, and hence $\tilde{f}$ is an internal chain transitive set for $\tilde{D}$. Define $u_0^d \in C([-\tau_1(0), 0], \mathbb{Y}^+)$ by $u_0^d(\theta, \cdot) = u^d_3(\theta, \cdot)$ for $\theta \in [-\tau_1(0), 0]$. Due to $\tilde{J} \neq [\bar{0}]$ and $u_0^d$ being globally attractive in $C([-\tau_1(0), 0], \mathbb{Y}^+) \setminus \{\bar{0}\}$, we know $\tilde{J} \cap W^s(u_0^d) = \emptyset$, where $W^s(u_0^d)$ is the stable set of $u_0^d$. By [45, Theorem 1.2.1], one has $\tilde{J} = \{u_0^d\}$, which leads to $\mathcal{J} = \{0, u_0^d, 0\}$, thence,

\[
\lim_{t \to \infty} \| (\bar{u}_1(t, \cdot, \bar{\varphi}), \bar{u}_2(t, \cdot, \bar{\varphi}), \bar{u}_3(t, \cdot, \bar{\varphi})) - (0, u_0^d(t, \cdot), 0) \|_{\mathcal{X}_3} = 0.
\]

(b) As $R_1 > 1$, Lemmas 4.2 and 4.6 indicate $r(\tilde{P}_1) > 1$, consequently, $\mu_1 = \frac{\ln r(\tilde{P}_1)}{\omega} > 0$. Let

\[
P_0 := \{\bar{\varphi} \in \mathcal{X}_4 : \bar{\varphi}_1(0, \cdot) \not\equiv 0 \text{ and } \bar{\varphi}_3(\cdot) \not\equiv 0\},
\]

and

\[
\partial P_0 := \mathcal{X}_4 \setminus P_0 = \{\bar{\varphi} \in \mathcal{X}_4 : \bar{\varphi}_1(0, \cdot) \equiv 0 \text{ or } \bar{\varphi}_3(\cdot) \equiv 0\}.
\]

For any $\bar{\varphi} \in P_0$, Lemma 5.2 reveals that $\bar{u}_1(t, x, \bar{\varphi}) > 0$ and $\bar{u}_3(t, x, \bar{\varphi}) > 0$, $\forall t > 0$, $x \in \bar{\Omega}$. Thus knowing $D^n(P_0) \subseteq P_0$, $\forall n \in \mathbb{N}$. Lemma 5.1 evidences that $D : \mathcal{X}_4 \to \mathcal{X}_4$ has a strong global attractor in $\mathcal{X}_4$.

Set

\[
M_3 := \{\bar{\varphi} \in \partial P_0 : D^n(\bar{\varphi}) \in \partial P_0, \forall n \in \mathbb{N}\},
\]

and $\omega(\bar{\varphi})$ to be the omega limit set of the orbit $\gamma^+ := \{D^n(\bar{\varphi}) : \forall n \in \mathbb{N}\}$. Denote $Q = (\bar{0}, u_0^d(t, x), 0)$. Next, we prove that $Q$ cannot form a cycle for $D$ in $\partial P_0$.

\textbf{Claim 1.} For any designated $\bar{\varphi} \in M_3$, $D^n(\bar{\varphi}) \in \partial P_0, \forall n \in \mathbb{N}$. Then, either $\bar{u}_1(n \omega_0, \cdot, \bar{\varphi}) \equiv 0$ or $\bar{u}_3(n \omega_0, \cdot, \bar{\varphi}) \equiv 0$, for each $n \in \mathbb{N}$. Furthermore, by contradiction and Lemma 5.2, it is obvious that for each $t \geq 0$, either $\bar{u}_1(t, \cdot, \bar{\varphi}) \equiv 0$ or $\bar{u}_3(t, \cdot, \bar{\varphi}) \equiv 0$. If $\bar{u}_1(t, \cdot, \bar{\varphi}) \equiv 0$ for all $t \geq 0$, then Lemma 4.1 guarantees $\lim_{t \to \infty} \bar{u}_2(t, x, \bar{\varphi}) = u_0^d(t, x)$ uniformly for $x \in \bar{\Omega}$. Thereby, the $\bar{u}_3$ equation in (10) meets

\[
\frac{\partial \bar{u}_3(t, x, \bar{\varphi})}{\partial t} \leq D_v \Delta \bar{u}_3(t, x, \bar{\varphi}) - \mu_v \bar{u}_3(t, x, \bar{\varphi}),
\]

where $\mu_v = \min_{t \in [0, \omega_0], x \in \bar{\Omega}} \mu_v(t, x)$. By the comparison principle, we have $\lim_{t \to \infty} \bar{u}_3(t, x) = 0$ uniformly for $x \in \bar{\Omega}$. Suppose that $\bar{u}_1(t_2, \cdot, \bar{\varphi}) \not\equiv 0$ for some $t_2 \geq 0$, in the light of Lemma 5.2, we obtain $\bar{u}_1(t_2, \cdot, \bar{\varphi}) > 0$, $\forall t \geq t_2$. Accordingly, $\bar{u}_3(t, x) \equiv 0$, $\forall t \geq t_2$. From the $\bar{u}_1$ equation in (10), one has $\lim_{t \to \infty} \bar{u}_1(t, x, \bar{\varphi}) = 0$ uniformly for $x \in \bar{\Omega}$. Consequently, the $\bar{u}_2$ equation in (10) abides by a nonautonomous system, which is asymptotic to the periodic system (12). The theory of asymptotically periodic system [45, Section 3.2] implies that $\lim_{t \to \infty} (\bar{u}_2(t, x) - u_0^d(t, x)) = 0$ uniformly for $x \in \bar{\Omega}$. Therefore, $\omega(\bar{\varphi}) = Q$ for any $\bar{\varphi} \in M_3$, and $Q$ cannot form a cycle for $D$ in $\partial P_0$.
Consider the following time-periodic parabolic system with parameter $\delta > 0$

$$
\frac{\partial v^1(t,x)}{\partial t} = \frac{D_h \Delta v^1(t,x) + c_1 \beta(t,x) [N(x) - \delta] v^1(t,x) - (\mu_h(x) + q_1(x)) v^1(t,x)}{p \delta + \ln N(x)}, \quad t > 0, \quad x \in \Omega,
$$

$$
\frac{\partial v^2(t,x)}{\partial t} = D_v \Delta v^2(t,x) - \mu_v(t,x) v^2(t,x) + (1 - \tau'_1(t)) \int_{\Omega} \Gamma(t, t - \tau_1(t), x, y) \cdot \frac{\alpha_1 \beta(t - \tau_1(t), y) p [u^*_2(t - \tau_1(t), y) - \delta] v^1(t - \tau_1(t), y)}{p \delta + \ln N(y)} dy, \quad t > 0, \quad x \in \Omega,
$$

$$
\frac{\partial v^1(t,x)}{\partial \nu} = \frac{\partial v^2(t,x)}{\partial \nu} = 0, \quad t > 0, \quad x \in \partial \Omega.
$$

(A10)

For any $\varphi \in \mathcal{E}_1$, set $v^0(t, x, \varphi) = (v^1(t, x, \varphi), v^2(t, x, \varphi))$ to be the unique solution of system (A10) with $v^0(\varphi)(\theta, x) = (\phi_1(\theta, x), \phi_2(0, x))$ for all $\theta \in [-\tau_1(0), 0], x \in \overline{\Omega}$, where

$$
v^0(\varphi)(\theta, x) = \frac{v^0(t + \theta, x, \varphi)}{v^0(t, x, \varphi)}, \quad \forall t \geq 0, (\theta, x) \in [-\tau_1(0), 0] \times \overline{\Omega}.
$$

Let $\tilde{P}_{1\delta} : \mathcal{E}_1 \to \mathcal{E}_1$ be the Poincaré map of system (A10), i.e., $\tilde{P}_{1\delta}(\varphi) = v^0(\varphi), \forall \varphi \in \mathcal{E}_1$. Make $r(\tilde{P}_{1\delta})$ to be the spectral radius of $\tilde{P}_{1\delta}$. Since $\lim_{\delta \to 0} r(\tilde{P}_{1\delta}) = r(\tilde{P}_1) > 1$, choose a sufficiently small $\delta > 0$ such that

$$
\delta < \min \left\{ \min_{x \in \Omega} u^*_2(t, x), \min_{x \in \Omega} N(x) \right\}, \quad \text{and} \quad r(\tilde{P}_{1\delta}) > 1.
$$

For the above fixed $\delta > 0$, according to the continuous dependence of the solution on the initial data, there must be a constant $\delta^* > 0$ such that for all $\bar{\varphi}$ with $\| \bar{\varphi} - Q \| < \delta^*$, which leads to $\| D_\tau(\bar{\varphi}) - D_\tau(Q) \| < \delta$ for all $t \in [0, \omega]$. We now prove the following claim.

Claim 2. For any $\bar{\varphi} \in \mathcal{P}_0$, $\lim_{n \to \infty} \| D^p(\bar{\varphi}) - Q \| \geq \delta^*$.

Assuming contradictions, for some $\bar{\varphi}_0 \in \mathcal{P}_0$, one has $\lim_{n \to \infty} \| D^p(\bar{\varphi}_0) - Q \| < \delta^*$. So there is $n_2 \geq 1$ such that $\| D^p(\bar{\varphi}_0) - Q \| < \delta^*$ for all $n \geq n_2$. For each $t \geq n_2 \omega$, setting $t = n \omega + t'$ with $n = \lfloor t/\omega \rfloor$ and $t' \in [0, \omega)$, one has

$$
\| D_\tau(\bar{\varphi}_0) - D_\tau(Q) \| = \| D_\tau(D^p(\bar{\varphi}_0)) - D_\tau(Q) \| < \delta.
$$

(A11)

Following (A11) and Lemma 5.2, we can receive

$$
u^*_i(t, x, \varphi_0) - \delta < \bar{u}_i(t, x, \bar{\varphi}_0) \quad \text{and} \quad 0 < \bar{u}_i(t, x, \bar{\varphi}_0) < \delta, \quad i = 1, 3,
$$

for each $t \geq n_2 \omega - \tilde{\tau}$, and $x \in \overline{\Omega}$. Thus, as $t \geq n_2 \omega$, $\bar{u}_1(t, x, \bar{\varphi}_0)$ and $\bar{u}_3(t, x, \bar{\varphi}_0)$ satisfy

$$
\frac{\partial \bar{u}_1(t,x)}{\partial t} \geq D_h \Delta \bar{u}_1(t,x) + \frac{c_1 \beta(t,x)[N(x) - \delta] \bar{u}_3(t,x)}{p \delta + \ln N(x)} - (\mu_h(x) + q_1(x)) \bar{u}_1(t,x), \quad x \in \Omega,
$$

$$
\frac{\partial \bar{u}_3(t,x)}{\partial t} \geq D_v \Delta \bar{u}_3(t,x) - \mu_v(t,x) \bar{u}_3(t,x) + (1 - \tau'_1(t)) \int_{\Omega} \Gamma(t, t - \tau_1(t), x, y) \cdot \frac{\alpha_1 \beta(t - \tau_1(t), y) p [\bar{u}_2^*(t - \tau_1(t), y) - \delta] \bar{u}_1(t - \tau_1(t), y)}{p \delta + \ln N(y)} dy, \quad x \in \Omega,
$$

$$
\frac{\partial \bar{u}_1(t,x)}{\partial \nu} = \frac{\partial \bar{u}_3(t,x)}{\partial \nu} = 0, \quad x \in \partial \Omega.
$$

(A12)

Based on $\bar{u}(t, x, \bar{\varphi}_0) \gg 0$ for all $t > 0$ and $x \in \overline{\Omega}$, there must be a constant $\sigma_2 > 0$ such that

$$
(\bar{u}_1(t, x, \bar{\varphi}_0), \bar{u}_3(t, x, \bar{\varphi}_0)) \geq \sigma_2 e^{\mu \tau(t)} v^*_\delta(t, x), \quad \forall t \in [n_2 \omega - \tilde{\tau}, n_2 \omega], \quad x \in \overline{\Omega},
$$

(A12)
where $\nu^\gamma(t,x)$ is a positive $\omega$-periodic function such that $e^{\mu_{15} t} \nu^\gamma(t,x)$ is a solution of system (A10), and $\mu_{15} = \frac{\ln n(\bar{P}_2)}{\omega} > 1$. It follows from (A12) and the comparison theorem that

$$(\bar{u}_1(t,x,\bar{\varphi}_0),\bar{u}_3(t,x,\bar{\varphi}_0)) \geq \sigma_2 e^{\mu_{15} t} \nu^\gamma(t,x), \quad \forall t \geq n_2 \omega, \quad x \in \overline{\Omega}.$$

Obviously, $\bar{u}_i(t,x,\bar{\varphi}_0) \to +\infty$ as $t \to +\infty$, $i = 1, 3$, which leads to a contradiction.

With the above claim, it is easy to obtain that $Q$ is an isolated invariant set for $D$ in $\mathcal{X}_4$, and $W^s(Q) \cap \mathbb{P}_0 = \emptyset$, where $W^s(Q)$ is the stable set of $Q$ for $D$. For any integer $n$ with $n \omega > \tau$, $D^n := D_{n \omega}$ is compact, it follows that $D$ is compact, then $D$ is asymptotically smooth on $\mathcal{X}_4$. Moreover, by Lemma 5.1, we obtain that $D$ has a global attractor on $\mathcal{X}_4$. According to [24, Theorem 3.7], $D$ has a global attractor $A_0$ in $\mathbb{P}_0$. Based on the acyclicity theorem on uniform persistence for maps [45, Theorem 1.3.1, Remark 1.3.1], then $D$ is uniformly persistent with respect to $(\mathbb{P}_0, \partial \mathbb{P}_0)$ in the sense that there exists an $\gamma_2 > 0$, such that

$$\liminf_{n \to \infty} d(D^n(\bar{\varphi}), \partial \mathbb{P}_0) \geq \gamma_2, \quad \forall \bar{\varphi} \in \mathbb{P}_0. \quad (A13)$$

Now we derive the practical persistence required. Following $A_0 = D(A_0)$, then $\bar{\varphi}_1(0,\cdot) > 0$ and $\bar{\varphi}_3(\cdot) > 0$ for any $\bar{\varphi} \in A_0$. Set $B := \bigcup_{t \in [0,\omega]} D_t(A_0)$. Obviously, $B \in \mathbb{P}_0$ and $\lim_{t \to \infty} d(D_t(\bar{\varphi}), B) = 0, \forall \bar{\varphi} \in \mathbb{P}_0$. Define a continuous function $q_1 : \mathcal{X}_4 \to \mathbb{R}_+$ by

$$q_1(\bar{\varphi}) = \min \left\{ \min_{x \in \overline{\Omega}} \bar{\varphi}_1(0,x), \min_{x \in \overline{\Omega}} \bar{\varphi}_3(x) \right\}, \forall \bar{\varphi} = (\bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3) \in \mathcal{X}_4.$$

Because $B$ is a compact subset of $\mathbb{P}_0$, we obtain that $\inf_{\bar{\varphi} \in B} q_1(\bar{\varphi}) = \min_{\bar{\varphi} \in B} q_1(\bar{\varphi}) > 0$. Consequently, there is an $\gamma_3 > 0$ such that

$$\liminf_{t \to \infty} q_1(D_t(\bar{\varphi})) = \liminf_{t \to \infty} \left( \min_{x \in \overline{\Omega}} \bar{u}_1(t,x,\bar{\varphi}), \min_{x \in \overline{\Omega}} \bar{u}_3(t,x,\bar{\varphi}) \right) \geq \gamma_3, \forall \bar{\varphi} \in \mathbb{P}_0.$$

Furthermore, according to Lemma 5.2, there must be an $\gamma_4 \in (0, \gamma_3)$ such that

$$\liminf_{t \to \infty} \min_{x \in \overline{\Omega}} \bar{u}_j(t,x,\bar{\varphi}) \geq \gamma_4, \quad \forall \bar{\varphi} \in \mathbb{P}_0 \quad (j = 1,2,3).$$

The existence of a positive periodic steady state remains to be proved. By [2, Lemma 8] and [45, Theorem 3.5.1], for each $t > 0$, the solution map $\tilde{Q}_t$ of system (10), is a $\kappa$-contraction with respect to an equivalent norm on $\mathcal{X}_3$, where $\kappa$ is Kuratowski measure of noncompactness. Define

$$\mathcal{W}_0 = \{ \bar{\varphi} \in \mathcal{X}_3 : \bar{\varphi}_1(0,\cdot) \neq 0 \text{ and } \bar{\varphi}_3(\cdot) \neq 0 \},$$

and

$$\partial \mathcal{W}_0 = \mathcal{X}_3 \setminus \mathcal{W}_0 = \{ \bar{\varphi} \in \mathcal{X}_3 : \bar{\varphi}_1(0,\cdot) \equiv 0 \text{ or } \bar{\varphi}_3(\cdot) \equiv 0 \}.$$

For any $\bar{\varphi} \in \mathcal{W}_0$, $\tilde{Q}$ is $\tilde{\rho}_1$-uniformly persistent with $\tilde{\rho}_1(\bar{\varphi}) = d(\bar{\varphi}, \partial \mathcal{W}_0)$ is easily attainable. It then follows from [24, Theorem 4.5.1], as applied to $\tilde{Q}$, that system (10) has an $\omega$-periodic solution $((z_1^\omega(t,\cdot), z_2^\omega(t,\cdot), z_3^\omega(t,\cdot)) \in \mathcal{W}_0$. Let $\tilde{u}_1^\omega(\theta,\cdot) = z_1^\omega(\theta,\cdot), \tilde{u}_2^\omega(\theta,\cdot) = z_2^\omega(\theta,\cdot)$ and $\tilde{u}_3^\omega(0,\cdot) = z_3^\omega(0,\cdot)$, where $\theta \in [-\tau_1(0), 0]$. Combing the uniqueness of solutions and Lemma 5.2, we have that $((\tilde{u}_1^\omega(t,\cdot), \tilde{u}_2^\omega(t,\cdot), \tilde{u}_3^\omega(t,\cdot)))$ is periodic solution of system (10), and it is also strictly positive.
Appendix 4. Proof of Lemma 5.2

Proof: According to the comparison principle for cooperative systems, we have \( \bar{u}_1(t, x, \hat{\psi}) \geq 0 \) and \( \bar{u}_3(t, x, \hat{\psi}) \geq 0 \) for all \( t > 0 \) and \( x \in \Omega \). Moreover, \( \bar{u}_1(t, x, \hat{\psi}) \) and \( \bar{u}_3(t, x, \hat{\psi}) \) satisfy

\[
\begin{align*}
\frac{\partial \bar{u}_1(t, x)}{\partial t} &\geq D_h \Delta \bar{u}_1(t, x) - (\tilde{\mu}_h + \tilde{\gamma}_1) \bar{u}_1(t, x), \quad t > 0, \quad x \in \Omega, \\
\frac{\partial \bar{u}_3(t, x)}{\partial t} &\geq D_v \Delta \bar{u}_3(t, x) - \tilde{\mu}_v \bar{u}_3(t, x), \quad t > 0, \quad x \in \Omega, \\
\frac{\partial \bar{u}_1(t, x)}{\partial \nu} &\leq \frac{\partial \bar{u}_3(t, x)}{\partial \nu} = 0, \quad t > 0, \quad x \in \partial \Omega,
\end{align*}
\]

where \( \tilde{\mu}_h, \tilde{\gamma}_1, \tilde{\mu}_v \) are defined in Section 3. If there exists \( t_3 \geq 0 \), such that \( \bar{u}_1(t_3, \cdot, \hat{\psi}) \neq 0 \) and \( \bar{u}_3(t_3, \cdot, \hat{\psi}) \neq 0 \), one has \( \bar{u}_1(t_3, \cdot, \hat{\psi}) > 0 \) and \( \bar{u}_3(t_3, \cdot, \hat{\psi}) > 0 \), \( \forall t > t_3 \), based on the strong maximum principle [14, Proposition 13.1]. Set \( \bar{n}(t, x, \hat{\psi}) \) to be the solution of

\[
\begin{align*}
\frac{\partial \bar{n}(t, x)}{\partial t} &= D_v \Delta \bar{n}(t, x) + \Lambda(t, x) - \left( \frac{\alpha_1 \beta(t, x)p}{l} + \mu_v(t, x) \right) \bar{n}(t, x), \quad t > 0, \quad x \in \Omega, \\
\frac{\partial \bar{n}(t, x)}{\partial \nu} &= 0, \quad t > 0, \quad x \in \partial \Omega.
\end{align*}
\]

(A14)

Clearly, \( \Lambda(t, x) \) is Hölder continuous and nonnegative nontrivial on \( \mathbb{R} \times \Omega \). By the comparison principle,

\( \bar{u}_2(t, x) \geq \bar{n}(t, x) > 0 \), \( t > 0 \), \( x \in \Omega \).

In addition, from [43, Lemma 2.1], system (A14) admits a unique positive \( \omega \)-periodic \( \bar{n}^*(t, \cdot) \), which is globally attractive. Then

\[
\liminf_{t \to \infty} \bar{u}_2(t, x, \hat{\psi}_3) \geq \bar{\varepsilon}_2 := \inf_{t \in [0, \omega], x \in \Omega} \bar{n}^*(t, x),
\]

uniformly for \( x \in \Omega \). The proof is complete.

Appendix 5. Parameter definition and value

Wang et al. [36] have studied the seasonal factors that affect malaria transmission, including seasonal forced bite rate

\[
\beta(t) = 6.983 - 1.993 \cos(\pi t/6) - 0.4247 \cos(\pi t/3) - 0.128 \cos(\pi t/2) - 0.04095 \cos(2\pi t/3) + 0.0005486 \cos(5\pi t/6) - 1.459 \sin(\pi t/6) - 0.007642 \sin(\pi t/3) - 0.0709 \sin(\pi t/2) + 0.05452 \sin(2\pi t/3) - 0.06235 \sin(5\pi t/6) \text{ Month}^{-1},
\]

(A15)

periodic mortality rate of mosquitoes

\[
\mu_v(t) = 3.086 + 0.04788 \cos(\pi t/6) + 0.01942 \cos(\pi t/3) + 0.007133 \cos(\pi t/2) + 0.0007665 \cos(2\pi t/3) - 0.001459 \cos(5\pi t/6) + 0.02655 \sin(\pi t/6) + 0.01819 \sin(\pi t/3) + 0.01135 \sin(\pi t/2) + 0.005687 \sin(2\pi t/3) + 0.003198 \sin(5\pi t/6) \text{ Month}^{-1},
\]

(A16)

periodic EIP

\[
\tau_1(t) = 1/30.4[17.25 + 8.369 \cos(\pi t/6) + 4.806 \sin(\pi t/6) + 3.27 \cos(\pi t/3) + 2.857 \sin(\pi t/3) + 1.197 \cos(\pi t/2) + 1.963 \sin(\pi t/2) + 0.03578 \cos(2\pi t/3) + 1.035 \sin(2\pi t/3)
\]

Figure A2. The parameters change with time. (a) Change of contact rate $\beta(t)$ with time. (b) Change of EIP $\tau_1(t)$ with time. (c) Change of the death rate of mosquitoes $\mu_v(t)$ with time.

Table A1. Definition and value of parameters.

| Parameter | Definition | Value (range) | Dimension | References |
|-----------|------------|---------------|-----------|------------|
| $N$       | Total human | 1205709       | Dimensionless | [35]        |
| $\mu_h$  | Human natural death rate | 0.00157 | Month$^{-1}$ | [35]        |
| $\rho_i$  | Recovery rate | (0.04258,0.568) | Month$^{-1}$ | [40]        |
| $l/p$     | Vector bias parameter | (0.1) | Dimensionless | [5]         |
| $c_i$     | Transmission probability from infectious mosquitoes to humans | (0.01,0.27) | Dimensionless | [6]         |
| $\alpha_i$ | Transmission probability from infectious humans to mosquitoes | (0.072,0.64) | Dimensionless | [6]         |
| $D_h$     | Human diffusion rate | 0.1 | km$^2$Month$^{-1}$ | [40]        |
| $D_v$     | Mosquito diffusion rate | 0.0125 | km$^2$Month$^{-1}$ | [40]        |
| $\beta$  | Mosquito biting rate | (A15) | Month$^{-1}$ | [35]        |
| $\mu_v$  | Mosquito death rate | (A16) | Month$^{-1}$ | [35]        |
| $\tau_1$ | EIP of mosquitoes infected with strain 1 | (A17) | Month | [35]        |
| $\tau_2$ | EIP of mosquitoes infected with strain 2 | (A18) | Month | [35]        |
| $\Lambda$ | Recruitment rate of adult female mosquitoes | (A19) | Month$^{-1}$ | [35]        |

\[
\tau_2(t) = 0.8/30.4[17.25 + 8.369 \cos(\pi t/6) + 4.806 \sin(\pi t/6) + 3.27 \cos(\pi t/3) + 2.857 \sin(\pi t/3) \\
+ 1.197 \cos(\pi t/2) + 1.963 \sin(\pi t/2) + 0.03578 \cos(2\pi t/3) + 1.035 \sin(2\pi t/3) \\
- 0.3505 \cos(5\pi t/6) + 0.6354 \sin(5\pi t/6) - 0.3257 \cos(\pi t) + 0 \sin(\pi t)] \text{ Month,} \quad (A17)
\]

and mosquitoes recruitment rate

\[
\Lambda(t) = \hat{k} \times \beta(t) (\text{km}^2 \text{Month})^{-1}, \quad \text{where } \hat{k} = 5 \times 1205709. \quad (A19)
\]

In addition, due to the assumption of vector-bias effect, and according to Ref. [35], in the following simulations, we take $p = 0.8$ and $l = 0.6$.

In order to better understand the periodicity and biological significance of the above parameters, we make the following figures. Looking at Figure A2, the distribution and seasonal dynamics of mosquito populations are affected by climate change. Analysing Figure A2(a), the peak of mosquito bites happens during warm or rainy seasons (May to September). In a year, the death rate may be lower in spring and summer when the humidity and temperature are suitable for species growth, and higher in winter due to low temperature and dry weather. In addition, higher temperature can shorten the duration of EIP.