An Element Weakly Primary to Another Element

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We introduce the concept "An element weakly primary to another element" and using this concept we have generalized some result proved by Manjarekar and Chavan (2004). It is shown that if \([b_\alpha]\) is a family of elements weakly primary to \(a\) in \(L\), then \(\vee a b_\alpha\) is weakly primary to \(a\).

1. Introduction

Multiplicative lattice is a complete lattice provided with commutative, associative, and joint distributive multiplication for which the largest element \(1\) acts as a multiplicative identity. A proper element \(p\) of \(L\) is called prime element if \(ab \leq p \Rightarrow a \leq p\) or \(b \leq p\) for \(a, b \in L\) and is called primary element if \(ab \leq p\) implies \(a \leq p\) or \(b^n \leq p\) for some \(n \in Z^+\). An element \(a\) of \(L\) is called compact if \(a \leq \vee X\), and \(X \subseteq L\) implies the existence of finite number of elements \(X_1, X_2, X_3, \ldots, X_n\) of \(L\) such that \(a \leq X_1 \vee X_2 \vee X_3 \vee \cdots \vee X_n\). Throughout this paper, \(L\) denotes compactly generated multiplicative lattice with \(1\) compact and every finite product of compact elements is compact. Let \(L_c\) be the set of all compact elements in \(L\). Also, \((a : b)\) is the greatest element \(c\) in \(L\) such that \(c b \leq a\). An element \(a\) of \(L\) is join principal if \(x \vee (y : a) = (xa \vee y) : a\) and meet principle if \(x \wedge ya = ((x : a) \wedge y)a\) for all \(x, y \in L\).

An element principle if it is both join and meet principle. For \(a \in L\), \(\sqrt{a} = \vee\{x \in L_\ast \mid x^n \leq a\) for some \(n \in Z_+\}\). An element \(a \in L\) is called semiprimary if \(\sqrt{a}\) is primary element. \(L\) is said to satisfy the condition \((\ast)\) if every semiprimary element is primary element.

An element \(a \in L\) is said to be strong join principle element if \(a\) is compact and join principle. An element \(a \in L\) is \(p\)-primary if \(a\) is primary and \(\sqrt{a} = p\) and \(a \in L\) is semiprime if \(\sqrt{a} = a\). An element \(a\) of \(L\) is called zero divisor if \(\exists 0 \neq b \in L\) such that \(ab = 0\), and if \(L\) has no zero divisor then \(L\) will be called lattice domain or simply a domain. \(L_\ast\) denotes the set of compact elements of \(L\).

The concept of weakly prime element is studied by Callialp et al. [1]. The concept of weakly primary element is introduced by Sachin and Vilas [2]. For other definitions and simple properties of multiplicative lattice, one can refer to Dilworth [3].

Definition 1. Weakly primary element is defined as follows.

An element \(q \in L\) is said to be a weakly primary element if for \(a, b \in L\ast, 0 \neq ab \leq q\) implies \(a \leq q\) or \(b^n \leq q\) for some \(n \in Z_+\).

Example 2. Lattice of ideals of ring \(R = \langle Z_{12}, +, \cdot \rangle\) (see Figure 1).

In the lattice of Example 2, an element \(a\) is weakly primary element. From Definition 1, it is clear that every weakly prime element is weakly primary element Converse need not be true. Since in Example 2\(a\) is weakly primary element but it is not weakly prime element. Further, if \(q\) is a weakly primary element, then \(\sqrt{q}\) is a weakly prime element. Because if for compact element \(x\) and \(y\) such that \(0 \neq xy \leq \sqrt{q}\) then \(x^n y^n = (xy)^n \leq q\) for some \(n \in Z_+\). As \(q\) is a weakly primary element, either \(x^n \leq q\) or \((y^n)^m = y^{mn} \leq q\) for some \(n \in Z_+\).

Consequently, \(x \leq \sqrt{q}\) or \(y \leq \sqrt{q}\). Thus, \(\sqrt{q}\) is a weakly prime element.
element. This implies that every weakly primary element is a weakly semiprimary element. It need not be true that a is always weakly prime or a is always weakly semiprimary. In Example 2, the least element 0 is not semiprimary as \( \sqrt{0} = b \) is not a weakly prime element. The concept of “An element primeto another element” is introduced in [4]. An element \( b \in L \) is primeto another element \( a \) if for any \( x \in L \), \( x \leq b \leq a \) implies \( x \leq a \).

Now, we define the following.

Definition 3. An element weakly prime to another element is defined as follows.

An element, \( b \in L \), is called weakly prime to an element \( a \in L \) if for any \( x \in L \), \( 0 \neq xb \leq a \) implies \( x \leq a \).

In Example 2, the element \( d \) is weakly prime to an element \( c \), but \( d \) is not weakly prime to any other element of \( L \). This follows directly from the fact that an element \( y \) is weakly prime to an element \( x \) if and only if \( \forall y : x = y \).

2. An Element Weakly Primary to Another Element

Now we introduce the following main concept which is a generalization of the concept introduced by Manjarekar and Chavan [5].

Definition 4. An element \( b \) is said to be weakly primary to another element \( a \) in \( L \) if \( \forall x \in L^* \), \( 0 \neq xb \leq a \) implies \( x^n \leq a \) for some \( n \in Z^+ \).

In Example 2, the element \( b \) is weakly primary to \( a \), but note that \( b \) is not weakly prime to \( a \). This follows directly from Corollary 9, and note that \( (a : b) = c = \sqrt{a} \) and \( a < c \). Evidently, if \( b \) is weakly prime to \( a \), then \( b \) is weakly primary to \( a \) in \( L \). Now if \( p \) is a weakly prime element and \( a \neq p \), then \( a \) is weakly prime to \( p \), and if \( q \) is a weakly primary element and a compact element \( a \neq q \), then \( a \) is a weakly primary element to \( q \).

Thus, from this, it is clear that elements weakly primary to another element exist in the lattice \( L \). Since \( L \) is compactly generated multiplicative lattice with 1 compact, weakly prime element and hence weakly primary element exists in \( L \). Hereafter, \( L \) will be a domain. We prove some interesting results including characterizations.

Theorem 5. No proper nonzero element is weakly prime or weakly primary to itself in \( L \).

Proof. If \( a \) is a proper nonzero element in \( L \) and \( a \) is weakly primary to \( a \) itself, then \( 0 \neq a \) implies \( 1 = a \), a contradiction. Therefore, no proper nonzero element is weakly prime or weakly primary to itself in \( L \).

Now we prove some characterizations of an element weakly primary to \( a \).

Theorem 6. Let \( a \in L \) be a semiprime element. Then \( 0 \neq b \) is weakly primary to \( a \) if and only if \( b \) is weakly prime to \( a \).

Proof. Assume that \( b \) is weakly primary to \( a \) semiprime element \( a \). Let \( 0 \neq xb \leq a \) for some \( x \in L^* \). Then \( x^2 \leq a \) for some \( n \in Z_+ \). Consequently, \( x \leq \sqrt{a} \). As \( a \) is semiprime, \( x \leq a \). Thus, \( b \) is weakly prime to \( a \). The converse part is obvious.

Theorem 7. Let \( L \) be a lattice domain. Let \( a, b \in L \); then \( 0 \neq b \) is weakly primary to \( a \) if and only if \( (a : b) \leq \sqrt{a} \).

Proof. Assume that \( 0 \neq b \) is weakly primary to \( a \). Let \( 0 \neq x \in L^* \) such that \( 0 \neq x \leq (a : b) \); then, \( 0 \neq xb \leq a \). As \( b \) is weakly primary to \( a \), \( x^n \leq a \) for some \( n \in Z_+ \). Hence, \( x \leq \sqrt{a} \). This shows that \( (a : b) \leq \sqrt{a} \).

Conversely, assume that \( (a : b) \leq \sqrt{a} \). Let \( 0 \neq xb \leq a \) for some \( x \in L^* \). Then, we have \( x \leq (a : b) \leq \sqrt{a} \). This implies that \( x^n \leq a \) for some \( n \in Z_+ \). Thus, \( b \) is weakly primary to \( a \).

Theorem 8. Let \( L \) be a lattice domain. Let \( a, b \in L \); then \( 0 \neq b \) is weakly prime to \( a \) if and only if \( (a : b) = a \).

Proof. Assume that \( 0 \neq b \) is weakly prime to \( a \) in \( L \). Let \( 0 \neq x \in L^* \) such that \( x \leq (a : b) \). Then \( 0 \neq xb \leq a \). As \( b \) is weakly prime to \( a \), we have \( x \leq a \). This shows that \( (a : b) \leq a \). But \( a \leq (a : b) \). Therefore, we get \( a = (a : b) \).

Conversely, assume that \( (a : b) = a \). Let \( 0 \neq xb \leq a \) for \( x \in L^* \). Then, we get \( x \leq (a : b) = a \). Thus \( b \) is weakly prime to \( a \).

Corollary 9. Let \( L \) be a lattice domain. Let \( a, b \in L \). Then, \( 0 \neq b \) is weakly primary to \( a \) but it is nonweakly prime to \( a \) if and only if \( a < (a : b) \leq \sqrt{a} \).

Proof. It follows from the fact that \( a \leq (a : b) \) and from Theorems 7 and 8.

Corollary 10. Let \( a, b \in L \). If \( a \) is a weakly semiprimary element and \( b \neq \sqrt{a} \), then \( 0 \neq b \) is weakly primary to \( a \).
Proof. Assume that $a \in L$ is a weakly semiprimary element and $b \not\in \sqrt{a}$. Let $x \in L^*$ such that $0 \neq xb \leq a$. Then $0 \neq xb \leq \sqrt{a}$. As $\sqrt{a}$ is a weakly prime element and $b \not\in \sqrt{a}$, we have $x \leq \sqrt{a}$. This implies that $x^n \leq a$ for some $n \in Z_+$. Thus, $b$ is weakly primary to $a$.

Theorem 11. Let $L$ be a lattice domain. Let $a, b \in L$. Then $0 \neq b$ is weakly prime to $a$ if and only if $b$ is weakly prime to $(a : x)$ for every $0 \neq x \in L$.

Proof. Assume that $b$ is weakly prime to $a$ in $L$. Let $0 \neq yb \leq (a : x)$ for some $y \in L$. Then, $0 \neq xyb \leq a$. As $b$ is weakly prime to $a$, $xy \leq a$. Consequently, $y \leq (a : x)$. Thus, $b$ is weakly prime to $(a : x)$ for every $0 \neq x \in L$. The converse is obvious, since, if $b$ is weakly prime to $(a : x)$ for every $0 \neq x \in L$, $b$ is weakly primary to $(a : 1) = a$.

Theorem 12. Let $a, b \in L$ and let $L$ be a lattice domain. If $b$ is weakly primary to $a$ and $a$ is a semiprime element in $L$, then $b$ is weakly primary to $(a : x)$ for every $x \in L$.

Proof. It follows from Theorems 6 and 11.

Theorem 13. Let $a, b \in L$. Then $0 \neq b$ is weakly primary to $a$ in $L$ if and only if each $x \geq a$ is weakly primary to $a$.

Proof. Assume that $0 \neq b$ is weakly primary to $a$ in $L$. Let $x \geq b$ and $xy \leq a$ for some $y \in L$. Then $0 \neq yb \leq a$. Therefore, by assumption, $y^n \leq a$ for some $n \in Z_+$. This shows that each $x \geq b$ is weakly prime to $a$. The converse part is obvious.

Theorem 14. If $\{b_\alpha\}$ is a family of $a$ elements weakly primary to $a$ in $L$, then $\bigvee_\alpha b_\alpha$ is weakly primary to $a$.

Proof. It follows from the fact that $b_\alpha \leq \bigvee_\alpha b_\alpha$ and from Theorem 13.

Theorem 15. Let $a, b \in L$. Then, $0 \neq b$ is nonweakly primary to $a$ if and only if each $x \leq b$ is nonweakly primary to $a$.

Proof. Assume that $b$ is nonweakly primary to $a$. Therefore, by Theorem 8, we have $a : b \not\in \sqrt{a}$. Let $x$ be an element of $L$ such that $x \leq b$. Then $(a : b) \leq (a : x)$. This shows that $(a : x) \not\in \sqrt{a}$. Thus, again by Theorem 8, each $x \leq b$ is nonweakly primary to $a$.

This lemma leads us to the following two obvious corollaries.

Corollary 16. If $\{b_\alpha\}$ is a family of elements nonweakly primary to $a$ in $L$, then $\bigwedge_\alpha b_\alpha$ is nonweakly primary to $a$.

Corollary 17. If $\bigvee_\alpha b_\alpha$ is nonweakly primary to $a$ in $L$, then each $b_\alpha$ is nonweakly primary to $a$.

Theorem 18. If $y$ is compact and $0 \neq xy$ is nonweakly primary to $a$, then either $x$ is nonweakly primary to $a$ or $y^n$ is nonweakly primary to $a$ for some $n \in Z_+$.

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