Creating quanta with ‘annihilation’ operator

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Abstract
An asymmetric nature of the boson ‘destruction’ operator $\hat{a}$ and its ‘creation’ partner $\hat{a}^\dagger$ is made apparent by applying them to a quantum state $|\psi\rangle$ different from the Fock state $|n\rangle$. We show that it is possible to increase (by many times or by any quantity) the mean number of quanta in the new ‘photon-subtracted’ state $\hat{a}|\psi\rangle$. Moreover, for certain ‘hyper-Poissonian’ states $|\psi\rangle$ the mean number of quanta in the (normalized) state $\hat{a}|\psi\rangle$ can be much greater than in the ‘photon-added’ state $\hat{a}^\dagger|\psi\rangle$. The explanation of this ‘paradox’ is given and some examples elucidating the meaning of Mandel’s $q$-parameter and the exponential phase operators are considered.

1 Introduction
The non-Hermitian bosonic operators $\hat{a}$ and $\hat{a}^\dagger$ of the harmonic oscillator, satisfying the canonical commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, are usually called ‘annihilation’ [1] (or ‘destruction’ [2, 3]) and ‘creation’ operators (perhaps, only P.A.M Dirac, in his book [4], did not use these terms, introducing instead of $\hat{a}$ and $\hat{a}^\dagger$ the ‘complex dynamical variables’ $\bar{\eta}$ and $\eta$.) This is due to their action on the Fock (number) state $|n\rangle$ [5]

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle,$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.$$

Therefore, there is some belief that the operator $\hat{a}$ may ‘destroy’ quanta (photons) in an arbitrary state $|\psi\rangle$. This belief is reflected even in the name ‘photon-subtracted state’, sometimes used for the state $\hat{a}|\psi\rangle$ [6, 7, 8]. However, this concept, guided maybe more by intuition than by a sound proof, seems to be misleading when one deals not with single Fock states, but with their superpositions or quantum mixtures.

The better known counter-example is the harmonic oscillator coherent state $|\alpha\rangle$: since it is an eigenstate of $\hat{a}$, the state $\hat{a}|\alpha\rangle/|\alpha\rangle$ has the same mean number of photons as $|\alpha\rangle$ (see a similar remark in [9]). But as we will see below, this example is not the only one. In contradistinction, when operator $\hat{a}^\dagger$ is applied on an arbitrary state $|\psi\rangle$ it produces a new one, whose mean number of quanta is always greater than that of $|\psi\rangle$, therefore justifying its designation as ‘creation operator’. Our aim is to provide a deeper investigation of the problem of ‘photon subtraction’ or ‘photon addition’ for arbitrary quantum states, drawing readers’ attention to apparent ‘paradoxes’ and giving possible explanations.

The plan of the paper is as follows. In Section 2 we discuss the relations between the ‘photon excess’ in the ‘photon-subtracted’ states and Mandel’s parameter, illustrating them in the example of superposition of two Fock states. The concept of ‘hyper-Poissonian’ states is introduced in Section 3. The examples of such states include certain superpositions of the coherent and vacuum states and the negative binomial states. New families of ‘logarithmic’ states corresponding to the given values of the mean number of quanta and Mandel’s parameter are considered in Section 4. A distinguished role of the exponential phase operators and their eigenstates – the phase coherent states – is discussed in Section 5. Section 6 concludes the paper.

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2 ‘Photon-subtracted’ states, ‘photon excess’ and Mandel’s parameter

For any normalized state

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle,$$

$$\sum_{n=0}^{\infty} |c_n|^2 = 1,$$  (3)

the normalized ‘photon-subtracted state’ is given by

$$|\psi_-=\rangle = \frac{1}{\sqrt{\bar{n}}} \hat{a} |\psi\rangle = \frac{1}{\sqrt{\bar{n}}} \sum_{n=1}^{\infty} c_n \sqrt{n} |n-1\rangle = \frac{1}{\sqrt{\bar{n}}} \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1} |n\rangle,$$  (4)

where

$$\bar{n} = \langle \psi | \hat{n} |\psi\rangle = \sum_{n=1}^{\infty} n |c_n|^2$$  (5)

is the mean value of the quantum number operator $\hat{n} \equiv \hat{a}^\dagger \hat{a}$ in $|\psi\rangle$. (In this article we consider mainly pure states, with the only aim to simplify notation.) Designating the mean number of quanta in the state $|\psi_-=\rangle$ as

$$N_- = \langle \psi_- | \hat{n} |\psi_-\rangle = \frac{1}{\bar{n}} \sum_{n=1}^{\infty} n(n-1) |c_n|^2,$$  (6)

it is easy to see that the difference between $N_-$ and $\bar{n}$ (the ‘photon excess’) is nothing but Mandel’s $q$-parameter:

$$N_- - \bar{n} = \frac{n^2}{\bar{n}} - 1 - \bar{n} \equiv q.$$  (7)

Note that the transformation $|\psi\rangle \rightarrow |\psi_-=\rangle$ is not a pure mathematical exercise. It is a real consequence of conditional measurements on beam splitters [6, 10]. Moreover, according to the Srinivas–Davies theory of photodetection [11], the quantum states of field modes undergo the same transformation upon detection of one photon. In the last case, equation (6) and the related interpretation of the physical meaning of Mandel’s parameter were discussed in [12, 13].

As follows from (7), the ‘annihilation’ operator $\hat{a}$ diminishes effectively the mean number of quanta ($q < 0$) only for states having sub-Poissonian statistics (which have been always considered as ‘nonclassical’). There are many families of ‘sub-Poissonian’ states, besides the Fock ones. In particular, all ‘binomial’ states [14, 15] possess sub-Poissonian photon statistics. Other examples include the families of so-called ‘Barut–Girardello states’ [16, 17] or odd coherent states [18].

For any super-Poissonian state, the action of the ‘annihilation’ operator results in increase of the mean number of quanta. The explanation of such a ‘paradoxical’ behaviour stems from the first factor, $\sqrt{\bar{n}}$, in the right-hand side of equation (6): besides shifting the Fock state $|n\rangle$ by one quantum to the left, the operator $\hat{a}$ also increases the amplitude of the new state. Therefore, under certain conditions the relative weight of Fock states with larger numbers of quanta in state (4) can be greater than in the initial state (3), which results finally in the increase of the mean number of quanta (cf. [12]).

2.1 Superposition of two Fock states

Suppose the superposition of two different Fock states

$$|\psi\rangle_F = \sqrt{r} |n\rangle + \sqrt{1-r} |m\rangle,$$

$$0 < r < 1.$$  (8)

Then

$$\bar{n} = rn + (1-r)m,$$

$$N_- = \frac{rn(n-1) + (1-r)m(m-1)}{rn + (1-r)m},$$

and the condition $N_- > \bar{n}$ is equivalent to the inequality $r(1-r)(n-m)^2 > r(n-m) + m$, which can be satisfied for any $r$ (different from zero and 1) if the difference $n - m$ is sufficiently large.

Since there are no limitations on the positive values of Mandel’s parameter, applying the ‘annihilation’ operator to highly super-Poissonian states one can increase the mean number of quanta by any desired
Let us consider a superposition of the stronger than in the squeezed vacuum states. ‘Hyper-Poissonian’ states can be characterized as those for which the fluctuations of the photon number are right-hand side of equation (11) (it can be calculated, e.g., using the results of [22]). Consequently, the squeezed vacuum states ‘hyper-Poissonian’. Note that Mandel’s parameter of the state $|\psi_g\rangle$ is given exactly by the \textit{quantum superpositions, increasing the relative weight} of the higher-energy components.

3 ‘Hyper-Poissonian’ states

Acting on the state $|\psi\rangle$ by the operator $\hat{a}^\dagger$ we obtain the ‘photon-added state’ $|\psi_+\rangle = \frac{1}{\sqrt{1+n}} \hat{a}^\dagger |\psi\rangle = \frac{1}{\sqrt{1+n}} \sum_{n=0}^{\infty} c_n \sqrt{1+n} |n+1\rangle$.

The mean value of the quantum number operator in the state $|\psi_+\rangle$ equals

$$N_+ = \langle \psi_+ | \hat{n} |\psi_+\rangle = \frac{1}{1+n} \sum_{n=0}^{\infty} (n+1)^2 |c_n|^2 = \bar{n} + 1 + \frac{1}{2} \frac{(\Delta n)^2}{1+n}.$$ (10)

The difference $N_+ - \bar{n}$ is always not less than 1 (being equal to 1 only for the Fock states). Therefore the name ‘creation operator’ is justified for the operator $\hat{a}^\dagger$. Nevertheless, in certain cases one can add much more quanta by acting on state $|\psi\rangle$ not with the ‘creation’ operator, but with the ‘annihilation’ partner! Using the relation $(\Delta n)^2 = \bar{n}(1+q)$ together with equations (5) and (10) one can see that the condition $N_- > N_+$ is equivalent to

$$q > 1 + \frac{1}{2} \bar{n}.$$ (11)

The states possessing the property $q > \bar{n}$ were named in [20, 21] ‘super-chaotic’ or ‘super-random’ states (for such states, the second-order correlation parameter $g^{(2)}(0)$ is greater than its value for the thermal states $g^{(2)}_{\text{therm}}(0) = 2$). Therefore the states possessing the property (11) can be named ‘hyper-chaotic’ or ‘hyper-Poissonian’. Note that Mandel’s parameter of the \textit{squeezed vacuum states} is given exactly by the right-hand side of equation (11) (it can be calculated, e.g., using the results of [22]). Consequently, the ‘hyper-Poissonian’ states can be characterized as those for which the fluctuations of the photon number are stronger than in the squeezed vacuum states.

3.1 Superposition of coherent and vacuum states

Let us consider a superposition of the coherent state $|\alpha\rangle$ and the vacuum state $|0\rangle$

$$|\psi\rangle_\alpha = \sqrt{\eta} |\alpha\rangle + \xi |0\rangle, \quad \eta + |\xi|^2 + 2\sqrt{\eta} e^{-\alpha^2/2} \text{Re} \xi = 1.$$ (12)

Since the overall phases are not essential, we assume that $\eta$ and $\alpha$ are real positive numbers (whereas the coefficient $\xi$ may be complex; however, it does not enter the formulae which we are interested in). In this case the operator $\hat{a}^\dagger$ ‘annihilates’ the vacuum component of the initial superposition (12), transforming $|\psi\rangle_\alpha$ into the coherent state $|\psi_\alpha\rangle = |\alpha\rangle$. This transformation is accompanied by an increase of the mean number of quanta under certain conditions:

$$\bar{n} = \eta \alpha^2, \quad N_- = \alpha^2, \quad N_+ = \frac{\eta \alpha^4 + 3 \eta \alpha^2 + 1}{1 + \eta \alpha^2}.$$ (13)
The ratio $N_+/N_-$ equals

$$\frac{N_+}{N_-} = 1 - \frac{1 - \alpha^{-2} - 3\eta}{1 + \eta\alpha^2},$$

thus for $3\eta + \alpha^{-2} < 1$ we have $N_+ < N_-$. Moreover, if $\alpha > 1$, but $\eta\alpha^2 \ll 1$, then $\bar{n} \ll 1 < N_+ \ll N_-$. The probabilities of finding $n$ quanta in the superposition (12) are

$$p_n = 1 - \eta \left(1 - e^{-\alpha^2}\right), \quad p_n = \eta e^{-\alpha^2} \frac{\alpha^{2n}}{n!}.$$  \hspace{1cm} (14)

A special case of the state (12) without vacuum component ($\eta = 0$) was considered in [23] as an example of 'the most nonclassical' state (see also a comment in [24]).

### 3.2 Generating functions and negative binomial states

It is well known that the statistical properties of the probability distribution $\{p_n \equiv |c_n|^2\}$ can be calculated with the aid of the generating function

$$G(z) = \sum_{n=0}^{\infty} p_n z^n, \quad p_n = \frac{1}{n!} G^{(n)}(0), \quad G(1) = 1,$$

since its derivatives at $z = 1$ generate the factorial moments,

$$d^r G(z)/dz^r|_{z=1} = n(n-1)(n-2)...(n-r+1) \equiv n^{(r)}.$$  \hspace{1cm} (16)

If the function $G(z)$ for the state $|\psi\rangle$ is known, then the functions $G_{\pm}(z)$ for the states $|\psi_{\pm}\rangle$ are given by the formulae

$$G_-(z) = \frac{1}{\bar{n}} \frac{dG}{dz}, \quad G_+(z) = \frac{1}{1 + \bar{n}} \frac{d}{dz} [zG(z)].$$  \hspace{1cm} (17)

Applying these relations to the super-Poissonian negative binomial state [25, 26]

$$|\xi, \mu\rangle = \sum_{n=0}^{\infty} \left[ (1 - \xi)^\mu \frac{\Gamma(\mu + n)}{\Gamma(\mu) n!} \xi^n \right]^{1/2} |n\rangle, \quad \mu > 0, \quad 0 \le \xi < 1,$$

for which

$$q = \xi/(1 - \xi), \quad \bar{n} = \mu q, \quad G(z) = \left(\frac{1 - \xi}{1 - z\xi}\right)^\mu,$$

we find

$$G_-(z) = \left(\frac{1 - \xi}{1 - z\xi}\right)^{\mu+1}, \quad G_+(z) = z \left(\frac{1 - \xi}{1 - z\xi}\right)^{\mu+1} \frac{1 + z\xi(\mu - 1)}{1 + \xi(\mu - 1)}.$$  

Consequently, applying the operator $\hat{a}$ to the state (15) several times, one can add each time equal portions of quanta $q$ ($N_- = \mu q + \bar{n} = \bar{n} + q$) without changing Mandel’s parameter ($q_- = q$). On the other hand, after applying the operator $\hat{a}^\dagger$ on $|\psi\rangle$ the mean number of quanta is increased,

$$N_+ - \bar{n} = 1 + \frac{\xi}{1 - \xi} + \frac{\xi(\mu - 1)}{1 + \xi(\mu - 1)},$$

and one can easily verify that for $\mu < 1/2 < (1 - \mu)$ the action of the ‘annihilation’ operator adds more quanta than the action of the ‘creation’ operator.

### 4 Quantum states with given mean photon number and Mandel’s parameter

Suppose that we want to find a probability distribution $\{p_n\}$ with two given parameters: the mean number $\bar{n}$ and the ‘mean photon multiplication factor’ $\gamma \equiv N_-/\bar{n}$. Since $N_- = dG_-/dz|_{z=1}$, we can rewrite the
second condition, using the first of the relations in equation (17), as $G''(1) = \gamma \bar{n} G'(1)$. Assuming that this relation must hold not only for $z = 1$, but also in some interval of values of the auxiliary variable $z$, including the point $z = 1$, we obtain a simple equation, $G''(z) = \gamma \bar{n} G'(z)$, whose solution, satisfying the conditions $G(1) = 1$ and $G'(1) = \bar{n}$, reads

$$G(z) = 1 - \frac{1}{\gamma} + \frac{1}{\gamma} e^{\gamma \bar{n}(z-1)}. \tag{19}$$

The corresponding ‘photon number’ distribution is

$$p_0 = 1 - \frac{1}{\gamma} + \frac{1}{\gamma} e^{-\gamma \bar{n}}, \quad p_n = \frac{1}{\gamma} e^{-\gamma \bar{n}} \frac{(\gamma \bar{n})^n}{n!}, \quad n \geq 1. \tag{20}$$

A pure quantum state possessing the photon number probability distribution $\{p_n\}$ can be written as

$$|\psi\rangle = \sum_{n=0}^{\infty} e^{i\phi_n} \sqrt{p_n} |n\rangle, \tag{21}$$

where the set of phases $\{\phi_n\}$ may be quite arbitrary. In particular, one can adjust the phases in such a way that function (21) together with the weight set (20) become a special case of the superposition (12),

$$\{\bar{n} \} = \{ \psi \} = \sum_{n=0}^{\infty} e^{i\phi_n} \sqrt{p_n} |n\rangle,$$

with $|\psi\rangle = \sum_{n=0}^{\infty} e^{i\phi_n} \sqrt{p_n} |n\rangle$, whose solution, satisfying the conditions $G(1) = 1$ and $G'(1) = \bar{n}$, reads

$$G(z) = \cosh [\bar{n} \sqrt{\gamma} (z - 1)] + \gamma^{-1/2} \sinh [\bar{n} \sqrt{\gamma} (z - 1)]. \tag{22}$$

However, this function can be interpreted as generating function for nonnegative probabilities $\{p_n\}$ only under certain restrictions on the values of $\bar{n}$ and $\gamma$, because the coefficient $p_1$ in the Taylor expansion (13) of function (22) becomes negative for $\bar{n} \sqrt{\gamma} \gg 1$ and $\gamma > 1$, whereas for $p_0$ the same happens if $\bar{n} \sqrt{\gamma} \gg 1$ and $\gamma < 1$. This example shows the main difficulty of finding new distributions through differential equations for their generating functions: although one can write infinitely many differential equations for $G(z)$ which result in the necessary conditions at $z = 1$, only few of them can be solved analytically, and even much fewer functions thus obtained have all derivatives at $z = 0$ nonnegative. Nonetheless, several interesting distributions can be found in this way.

### 4.1 Generalized logarithmic states

For example, looking for probabilities resulting in a given value of the ‘photon excess’ (or Mandel’s parameter) $q$, we can rewrite equation (2) (or $n^2 - \bar{n}^2 = (q + 1)\bar{n}$) as (using relations (14))

$$G''(1) - |G'(1)|^2 - qG'(1) = 0. \tag{23}$$

One of many possible extensions of this equality to a finite interval of values of variable $z$ is the equation

$$G''(z) - |G'(z)|^2 - qG'(z) = 0. \tag{24}$$

Although being nonlinear this equation is easily solved by several methods, for example, multiplying it on the left by the integrating factor $e^{-G}$ one gets a linear equation $dF/dz = qF$ for $F = e^{-G}G'$. Taking into account the conditions $G(1) = 1$ and $G'(1) = \bar{n}$, we obtain

$$G(z) = 1 - \ln \left[ 1 + \frac{\bar{n}}{q} \left( 1 - e^{q(z-1)} \right) \right]. \tag{25}$$

The probability distribution for $q > 0$ is

$$p_0 = \ln \left[ \frac{qc}{q + \bar{n}(1 - e^{-q})} \right], \quad p_n = \frac{q^n}{n!} \sum_{k=1}^{\infty} k^{n-1} \left( \frac{\bar{n}e^{-q}}{q + \bar{n}} \right)^k, \quad n \geq 1. \tag{26}$$
Although all coefficients $p_n$ with $n \geq 1$ are positive for any positive $q$ and $\bar{n}$, the vacuum probability $p_0$ is nonnegative provided

$$\bar{n} \leq \frac{q (e-1)}{1-e^{-q}}. \tag{27}$$

For $q \gg 1$ and $q \gg \bar{n}$ the probabilities (26) become close to (20):

$$p_0 \approx 1 - \frac{\bar{n}}{q}, \quad p_n \approx \frac{\bar{n}}{q} \left(1 + \frac{\bar{n}}{q} \right)^n, \quad n = 1, 2, 3, \ldots. \tag{28}$$

Using equations (14) and (17) we obtain Mandel’s parameter $q_-$ for the state $|\psi_\rightarrow\rangle$ (4) in terms of the factorial moments of the initial arbitrary state $|\psi\rangle$

$$q_- = \bar{n}n^{(3)} - [n^{(2)}]^2 \overline{n^{(2)}}. \tag{29}$$

The peculiar feature of the state generated by function (25) is that for this state $q_- = \bar{n}$, i.e., while for state $|\psi\rangle$, $\bar{n}$ is the mean photon number and $q$ is Mandel’s parameter, for state $|\psi_\rightarrow\rangle$ the mean photon number is $\bar{n} + q$ and Mandel’s parameter is $q_- = \bar{n}$. And the ‘excess of photons’ of state $\hat{a}|\psi_\rightarrow\rangle$ is $\bar{n}$.

Going to the limit $q \to 0$ in (25) we obtain the generating function

$$G(z) = 1 - \ln \left[1 - \bar{n}(z-1)\right], \tag{30}$$

which yields the following probabilities and factorial moments (16):

$$p_0 = 1 - \ln (1 + \bar{n}), \quad p_n = \frac{1}{n} \left(\frac{\bar{n}}{1 + \bar{n}}\right)^n, \quad n \geq 1, \tag{31}$$

$$n^{(r)} = (r-1)! \bar{n}^r, \quad r = 1, 2, \ldots. \tag{32}$$

The probability distribution with the same factorial moments (32) was found in [27] practically by the same method. The difference is that the authors of [27] calculated the generating function not for the probabilities, but for the factorial moments, having obtained, in particular, the function

$$Q_1(x) \equiv \sum_{\nu=0}^{\infty} (-x)^\nu \langle \hat{a}^\dagger \nu \hat{a}^\nu \rangle / \nu! = (A-1)^{-1} \ln [(1-A)\bar{n}x + 1] + 1, \tag{33}$$

where the positive constant $A$ was considered as a measure of the relative deviation from the Poissonian case (the article [27] appeared before [1]):

$$\frac{\langle \hat{a} \hat{a} \rangle (\langle \hat{a}^\dagger \hat{a} \rangle - 1) - \langle \hat{a}^\dagger \hat{a} \rangle^2}{\langle \hat{a}^\dagger \hat{a} \rangle^2} = -A. \tag{34}$$

In the limit case $A = 0$ formula (33) yields the factorial moments in the form (32) (but for $A \neq 0$ and $q = -A\bar{n} \neq 0$ the functions (25) and (33) give different values of factorial moments).

The statistics of the probability distribution (31) (not found explicitly in [27]) is quite different from the Poissonian statistics, although this distribution gives Mandel’s factor $q = 0$ and the same first two factorial moments as in the coherent state with $|\alpha|^2 = \bar{n}$. The corresponding quantum state was named in [27] sub-coherent state (the same name was used in [17] for quite different states possessing sub-Poissonian statistics). We can note in this connection that one of many possible pure states corresponding to the distribution (31) is a special case of the family of so-called logarithmic states introduced in [28]

$$|\psi\rangle_{log} = |0\rangle + \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n}} |n\rangle, \quad |c|^2 = 1 + \ln (1 - |z|^2), \quad \bar{n} = \frac{|z|^2}{1 - |z|^2}. \tag{35}$$
5 A distinguished role of the exponential phase operator and phase coherent states

Acting on the ‘logarithmic’ state \([35]\) by the operator \(\hat{a}\) we arrive at its ‘subtracted’ partner

\[
|\psi_\text{log}\rangle = \sqrt{1-|z|^2} \sum_{n=0}^{\infty} z^n |n\rangle,
\]

(36)

known under the names coherent phase state \([29, 30, 31]\), harmonious state \([32]\), or pseudothermal state \([33]\) (see also \([34]\)).

The mean number of quanta in the ‘sub-coherent’ state \((35)\) cannot exceed a small value \(\epsilon - 1 \approx 1.72\). However, it is easy to show that there exist many sub-coherent states \((q = 0)\) with arbitrary values of \(\tilde{n}\). An example is the superposition of two Fock states \((8)\). Considering for simplicity the case of \(1-r \approx 1/n\) (when \(\tilde{n} \approx n\)).

The coherent phase state \((36)\) is the eigenstate of the exponential phase operator

\[
\hat{E}_- \equiv \sum_{n=1}^{\infty} |n-1\rangle \langle n| = (\hat{a}\hat{a}^\dagger)^{-1/2} \hat{a} \equiv (\tilde{n} + 1)^{-1/2} \hat{a},
\]

(37)

introduced in \([37]\) and discussed in \([38, 39, 40]\). Its Hermitian conjugated partner is

\[
\hat{E}_+ = \sum_{n=1}^{\infty} |n\rangle \langle n - 1| = \hat{a}^\dagger (\hat{a}\hat{a}^\dagger)^{-1/2} \equiv \hat{a}^\dagger (\tilde{n} + 1)^{-1/2},
\]

(38)

\[
[\hat{E}_-, \hat{E}_+] = |0\rangle \langle 0|,
\]

\[
\hat{E}_+ \hat{E}_- = \hat{1} - |0\rangle \langle 0|.
\]

Acting on the Fock states, the operators \(\hat{E}_\pm\) shift the number of quanta by \(\pm 1\) without changing the amplitude of the state vector:

\[
\hat{E}_- |n\rangle = (1 - \delta_{n0}) |n-1\rangle, \quad \hat{E}_+ |n\rangle = |n+1\rangle.
\]

(39)

Therefore, applying the operators \(\hat{E}_\pm\) to an arbitrary state \(|\psi\rangle\) \((3)\) we obtain the following normalized states \(|\tilde{\psi}_\pm\rangle \sim \hat{E}_\pm |\psi\rangle\):

\[
|\tilde{\psi}_-\rangle = \frac{1}{\sqrt{1 - |c_0|^2}} \sum_{n=1}^{\infty} c_{n+1} |n\rangle, \quad |\tilde{\psi}_+\rangle = \sum_{n=0}^{\infty} c_n |n + 1\rangle.
\]

(40)

Their quantum number generating functions are related to the initial generating function \(G(z)\) \((5)\) by simple relations

\[
\tilde{G}_-(z) = \frac{G(z) - G(0)}{z(1 - p_0)}, \quad \tilde{G}_+(z) = zG(z).
\]

(41)

Consequently, the mean numbers of quanta in the new states \((40)\) are connected with the mean number \(\tilde{n}\) in the initial state as follows:

\[
\tilde{N}_- \equiv \langle \tilde{\psi}_- | \hat{n} | \tilde{\psi}_- \rangle = \frac{\tilde{n}}{1 - p_0} - 1, \quad \tilde{N}_+ \equiv \langle \tilde{\psi}_+ | \hat{n} | \tilde{\psi}_+ \rangle = \tilde{n} + 1.
\]

(42)

We see that the operator \(\hat{E}_+\) adds exactly one quantum to any quantum state, whereas the operator \(\hat{E}_-\) removes exactly one quantum from any state which has no contribution of the vacuum state (i.e., if \(p_0 = 0\)). Therefore, perhaps, not operators \(\hat{a}\) and \(\hat{a}^\dagger\), but operators \(\hat{E}_-\) and \(\hat{E}_+\) should be named ‘annihilation’ and ‘creation’ operators in the literal meaning of these words! (See in this connection study \([37]\), where it was shown how the states \(\hat{E}_\pm |\psi\rangle\) could arise as a result of the interaction of the Jaynes–Cummings type between atoms and cavity fields or between motional and internal degrees of freedom of trapped ions. Shifted thermal states, which can be written as \(\rho_{th}^{(shift)} = \hat{E}_+ \rho_{th} \hat{E}_-^\dagger\), have been considered in \([38]\), whereas methods of generating such states in a micromaser were discussed in \([39]\).)
For the coherent states $p_0 = \exp(-\bar{n})$, and for $\bar{n} \ll 1$ we have $\bar{N}_- = \bar{n}/2$. For the coherent phase states (30) we have $\bar{N}_- \equiv \bar{n}$ for any $\bar{n} = |z|^2 / (1 - |z|^2)$. For many other states with $p_0 > 0$, one can always make the ‘modified photon excess’

$$ \bar{q} \equiv \bar{N}_- - \bar{n} = \frac{p_0 \bar{n}}{1 - p_0} - 1 $$

(43)

positive by increasing the value of $\bar{n}$, which is, in a generic case, a free parameter independent of $p_0$. This can be easily seen, e.g., in the example (31), when the quantity

$$(1 - p_0)\bar{q} = \bar{n} - \frac{\bar{n} + 1}{\gamma} (1 - e^{-\gamma \bar{n}})$$

is obviously positive for sufficiently large $\bar{n}$ and $\gamma > 1$. The same is true for the negative binomial states (18) with $\mu < 1$ and $\xi$ sufficiently close to 1. Moreover, if $\bar{n} > 2(1 - p_0)/p_0$, then $\bar{N}_- > \bar{N}_+$. Even using operators of the form $\hat{A} = f(\bar{n})\hat{a}$ we can find the states $|\psi\rangle$ for which state $\hat{A}|\psi\rangle$ has more quanta (in average) than the state $|\psi\rangle$, for arbitrary function $f(\bar{n})$. The simplest example is the ‘two-Fock’ state (8) with $m = 0$. In this case $\bar{n} = r\bar{n}$. Any operator $f(\bar{n})\hat{a}$ transforms this superposition into the single Fock state $|n - 1\rangle$, and obviously $(n - 1) > r\bar{n}$ if $(1 - r) > 1/n$.

6 Conclusion

Concluding, we have shown that the standard ‘annihilation’ boson operator diminishes the mean number of quanta only for ‘nonclassical’ states with negative values of Mandel’s parameter, whereas there are many states (all of them superpositions or mixtures of the Fock states), with the peculiarity that when acted by $\hat{a}$ produce new states whose mean number of quanta is increased. Moreover, we have shown that in certain cases the ‘annihilation’ operator $\hat{a}$ can increase the mean number of quanta much more effectively than its ‘creation’ partner $\hat{a}^\dagger$. We emphasized an interpretation (discovered in [12, 13] but still not well known) of Mandel’s $g$-factor as the ‘photon excess’ in the mean number of quanta arising as a result of action of operator $\hat{a}$ on the given quantum state. Besides, we have shown that maybe the pair of Hermitian conjugated exponential phase operators could be the best candidates for being called ‘annihilation-creation’ operators, although the ‘lowering’ operator from this pair also cannot diminish the mean number of quanta by exactly one unit, for quantum states with nonzero vacuum component. This gives a hint that, perhaps, in the quantum photocount formalism of continuous measurements, as for instance, that proposed by Srinivas and Davies [14] one should use instead of $\hat{a}$ and $\hat{a}^\dagger$ the operators $\hat{E}_-$ and $\hat{E}_+$, which would not violate the property V of the cited work (namely, the condition of boundness for the counting rate). However, this issue is outside the objective of this letter, and we shall report on it elsewhere.

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