UNIFORM ESTIMATES FOR THE CONSTANT MEAN CURVATURE SOLUTIONS OF THE VACUUM EINSTEIN CONSTRAINT EQUATIONS ON COMPACT MANIFOLDS

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Abstract. We give some uniform estimates for constant mean curvature solutions of the conformal vacuum Einstein constraint equations on compact manifolds. Existence of those solutions was given in [4].

1. Introduction

Let $M^3$ be a 3-dimensional Riemannian manifold with metric $\gamma$, let $K$ be a symmetric $(0,2)$ tensor on $M$. It was shown by Y. Choquet-Bruhat [3] that any initial data set $(M^3, \gamma, K)$ can be viewed as an embedded hypersurface in a spacetime with $\gamma$ as the induced metric and $K$ as its second fundamental form, provided it satisfies the following Einstein constraint equations

\begin{align*}
R^M + (\text{Tr}_\gamma K)^2 - \|K\|^2 &= 16\pi T_{nm}, \\
K_{ij} - K_{ji} &= 8\pi T_{ni},
\end{align*}

where $R^M$ is the scalar curvature of $M$ and $T_{ab}$ is the stress-energy tensor that describes the matter content of the ambient spacetime. Therefore, constructing an initial data set that satisfies those constraint equations is the first step in understanding the global evolution of the spacetime.

The most widely used approach to the constraint equations in the vacuum case ($T_{ab} = 0$) is the conformal method [2], which divides the initial data on $M^3$ into the “Free (conformal) Data” (a Riemannian metric $\lambda_{ij}$, a divergence-free, trace-free symmetric tensor $\sigma_{ij}$, and a scalar function $\tau$), and the “Determined Data” (a strictly positive scalar function $\phi$ and a vector field $W^i$). Given a choice of the conformal data, if one can find a determined data satisfying the determined elliptic PDE system

\begin{align*}
\nabla_i (LW)_j^i &= \frac{2}{3} \phi^6 \nabla_j \tau, \\
\Delta \phi &= \frac{1}{8} R\phi - \frac{1}{8} (\sigma^{ij} + LW_{ij})(\sigma_{ij} + LW_{ij})\phi^{-7} + \frac{1}{12} \tau^2 \phi^5,
\end{align*}

then the two sets of data can be combined to produce an initial data set which satisfies the constraint equations by

\begin{align*}
\gamma_{ij} &= \phi^4 \lambda_{ij}, \\
K_{ij} &= \phi^{-2} (\sigma_{ij} + LW_{ij}) + \frac{1}{3} \phi^4 \lambda_{ij} \tau.
\end{align*}

This conformal method has been very successful in finding constant mean curvature solutions ($\tau = \text{constant}$) of the vacuum constraint equations. With $\tau$ constant, [3] implies...
$LW = 0$, and consequently (4) becomes the Lichnerowicz equation

\begin{equation}
\Delta \phi = \frac{1}{8}R \phi - \frac{1}{8}\|\sigma\|^2 \phi^{-7} + \frac{1}{12}\tau^2 \phi^5.
\end{equation}

This equation is conformally invariant in the sense that it has a solution with respect to conformal data $(\lambda, \sigma)$ if and only if it has a solution with respect to conformal data $(\psi^4 \lambda, \psi^{-2}\sigma, \tau)$ for some function $\psi > 0$.

The Yamabe invariant of $\lambda$ is defined as

\[ Y(M, \lambda) = \inf \left\{ \frac{\int_M R \lambda dv_\lambda}{(\int_M dv_\lambda)^{\frac{n-2}{n}}} : \bar{\lambda} = \psi(x)^{\frac{4}{n-2}} \lambda, \psi(x) > 0, \psi \in W^{1,2}(M) \right\} \]

By the Yamabe Theorem ([1], [7], [8]), $Y(M, \lambda)$ being positive, zero, and negative implies that $\lambda$ is conformal to a metric of positive constant, zero, and negative constant scalar curvature, respectively. Therefore, all Riemannian metrics on $M^3$ can be divided into three classes by their Yamabe invariant: Yamabe positive $Y^+(M)$, Yamabe zero $Y^0(M)$, and Yamabe negative $Y^-(M)$. Since we can also divide $\sigma$ and $\tau$ into $\|\sigma\| \neq 0, \|\sigma\| \equiv 0$ and $\tau \neq 0, \tau = 0$, there are totally twelve classes of conformal data to consider. In [4] Isenberg gave a theorem which completely determines for which of the twelve classes equation (7) can be solved by the conformal method. His theorem can be organized into the following table

| $\|\sigma\|^2 \equiv 0, \tau = 0$ | $\|\sigma\|^2 \equiv 0, \tau \neq 0$ | $\|\sigma\|^2 \neq 0, \tau = 0$ | $\|\sigma\|^2 \neq 0, \tau \neq 0$ |
|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| $Y^+(M)$                    | No                          | No                          | Yes                         |
| $Y^0(M)$                    | Yes                         | No                          | No                          |
| $Y^-(M)$                    | No                          | Yes                         | No                          |

In the class $(Y^0(M), \|\sigma\|^2 \equiv 0, \tau = 0)$, any constant is a solution. For data in all other classes for which solutions exist, the solution is unique.

Since the solvability of (7) has been completely determined in the constant mean curvature case, the next question is whether the set of solutions has an interesting and useful mathematical structure [5]. We study this question in this paper. Since any constant is a solution in the class $(Y^0(M), \|\sigma\|^2 \equiv 0, \tau = 0)$, we cannot expect to have any compactness of solutions. In fact, let $c$ be any positive constant, from (7) we know that the function $c\phi$ satisfies

\[ \Delta (c\phi) = \frac{1}{8}R(c\phi) - \frac{1}{8}\|c^4\sigma\|^2 (c\phi)^{-7} + \frac{1}{12}(c^{-2}\tau)^2 (c\phi)^5. \]

When $c \to 0$, $\|c^4\sigma\|^2 \to 0$ and $(c^{-2}\tau)^2 \to \infty$; when $c \to \infty$, $\|c^4\sigma\|^2 \to \infty$ and $(c^{-2}\tau)^2 \to 0$. So in order to get uniform $C^0$ estimates for $\phi$, we will need to control $\|\sigma\|^2$ and $\tau^2$ from above and below as well. The following theorem shows that with an additional $C^{0,\alpha}$ bound on $\|\sigma\|^2$, in the remaining five classes we do have compactness of solutions to (7) with respect to $\sigma$ and $\tau$. For simplicity in the rest of the paper we denote $\|\sigma\|^2$ as $\sigma^2$.

**Theorem 1.1.** Let $(\lambda_{ij}, \sigma_{ij}, \tau)$ be a determined data on $M^3$, where $\tau$ is a constant. Let $\phi$ be a positive solution of the Lichnerowicz equation (7).
• In classes \( (Y^+, \sigma^2 \neq 0, \tau \neq 0), (Y^0, \sigma^2 \neq 0, \tau \neq 0), \) and \( (Y^-, \sigma^2 \neq 0, \tau \neq 0), \) assume \( C_1^{-1} < \sigma^2 < C_1, ||\sigma^2||_{C^{0, \alpha}(M)} < C_1, \) and \( C_2^{-1} < \tau^2 < C_2 \) for some constants \( C_1, C_2 > 0 \) and \( 0 < \alpha < 1. \) Then there exists a constant \( C = C(C_1, C_2, \lambda) > 0 \) such that \( C^{-1} \leq \phi \leq C \) on \( M. \)

• In class \( (Y^+, \sigma^2 \neq 0, \tau = 0), \) assume \( C_1^{-1} < \sigma^2 < C_1 \) and \( ||\sigma^2||_{C^{0, \alpha}(M)} < C_1 \) for some constants \( C_1 > 0 \) and \( 0 < \alpha < 1. \) Then there exists a constant \( C = C(C_1, \lambda) > 0 \) such that \( C^{-1} \leq \phi \leq C \) on \( M. \)

• In class \( (Y^-, \sigma^2 \equiv 0, \tau \neq 0), \) assume \( C_2^{-1} < \tau^2 < C_2 \) for some constant \( C_2 > 0. \) Then there exists a constant \( C = C(C_2, \lambda) > 0 \) such that \( C^{-1} \leq \phi \leq C \) on \( M. \)

By a bootstrap argument which will be carried out in detail in the proof of this theorem, these estimates imply that \( \phi \) has a uniform \( C^3 \) norm bound and therefore a sequence of solutions will produce another solution in its limit. We prove these estimates in the next three sections.

2. THE CLASSES \( (Y^+, \sigma^2 \neq 0, \tau \neq 0), (Y^0, \sigma^2 \neq 0, \tau \neq 0), \) AND \( (Y^-, \sigma^2 \neq 0, \tau \neq 0) \)

First, the lower bound.

Consider the 1-variable function \( f(t) = \frac{1}{12} t^2 t^3 + \frac{1}{8} R t^2 - \frac{1}{8} \sigma^2. \) Since \( f(0) = -\frac{1}{8} \sigma^2 < -\frac{1}{8} C_1^{-1} \) and \( f \) is continuous, there exists a constant \( \epsilon = \epsilon(C_1, C_2, R(\lambda)) > 0 \) such that if \( |t| < \epsilon \) then \( f(t) < 0. \) Therefore if \( 0 < \phi < \epsilon^\frac{1}{4}, \) then \( \frac{1}{12} \tau^2 \phi^{12} + \frac{1}{8} R \phi^8 - \frac{1}{8} \sigma^2 < 0, \) which implies that \( \Delta \phi = \frac{1}{8} R \phi - \frac{1}{8} \sigma^2 \phi^{-7} + \frac{1}{12} \tau^2 \phi^5 < 0. \) Since \( \Delta \phi \) cannot be negative at the minimum point of \( \phi, \) we then know that the minimum of \( \phi \) cannot be smaller than \( \epsilon^\frac{1}{4}. \) This proves the lower bound \( \phi \geq \epsilon^\frac{1}{4}. \)

Next, the upper bound.

Suppose there is no uniform upper bound on \( \phi, \) then we can find sequences \( \{\phi_i\}, \{\sigma_i\}, \{\tau_i\}, \) and \( \{x_i\}, \) such that

\[
\Delta \phi_i = \frac{1}{8} R \phi_i - \frac{1}{8} \sigma_i^2 \phi_i^{-7} + \frac{1}{12} \tau_i^2 \phi_i^5,
\]

where \( \phi_i > 0, C_i^{-1} < \sigma_i^2 < C_i, ||\sigma_i^2||_{C^{0, \alpha}(M)} < C_i, C_i^{-1} < \tau_i^2 < C_i, \) and \( \max_M \phi_i = \phi_i(x_i) \to \infty. \)

Let \( x = (x^1, x^2, x^3) \) be the geodesic normal coordinates centered at each of the points \( x_i, \) and let \( y = \phi_i^2(x_i) x. \) Define

\[
u_i(y) = \frac{\phi_i\left(\frac{y}{\phi_i^2(x_i)}\right)}{\phi_i(x_i)}.
\]

Then \( u_i \) satisfies

\[
\Delta \lambda^{(i)}(y) u_i(y) = \frac{1}{8} R \left(\frac{y}{\phi_i^2(x_i)}\right) u_i(y) \phi_i^{-4}(x_i) - \frac{1}{8} \sigma_i^2 \left(\frac{y}{\phi_i^2(x_i)}\right) \phi_i^{-7} \left(\frac{y}{\phi_i^2(x_i)}\right) \phi_i^{-5}(x_i) + \frac{1}{12} \tau_i^2 u_i^5(y),
\]

where the metric \( \lambda^{(i)}(y) = \sum_{k,l=1}^3 \lambda_{kl} \left(\frac{y}{\phi_i^2(x_i)}\right) dy^k dy^l. \) By the definition we also know that \( 0 < u_i \leq 1 \) and \( u_i(0) = 1. \) On any compact subset \( \Omega \) of the \( y \)-plane, since we have proved that \( \phi \) has a positive lower bound, the right hand side of (8) has a uniformly bounded \( L^p \)
norm for arbitrary \( p > 1 \). Then by standard elliptic estimates \( u_i \) has a uniform \( W^{2,p} \) norm bound, which by the Sobolev embedding theorem implies that \( u_i \) is uniformly bounded in the \( C^{1,\alpha} \) norm when \( p \) is large enough such that \( 1 - \frac{2}{p} \geq \alpha \).

Next we show that \(-\frac{1}{8}\sigma^2_i \left( \frac{y}{\phi_i^5(x_i)} \right) \phi_i^{-7} \left( \frac{y}{\phi_i^5(x_i)} \right) \phi_i^{-5}(x_i)\), i.e. the second term on the right hand side of (8), has bounded \( C^{0,\alpha} \) norm on \( \Omega \). Given the assumption on \( \sigma_i^2 \) and the fact that \( \phi_i \) is bounded below and \( \phi_i(x_i) \to \infty \), we only need to obtain a \( C^{0,\alpha} \) bound on \( \phi_i^{-7} \left( \frac{y}{\phi_i^5(x_i)} \right) \phi_i^{-5}(x_i) \). Let \( y_1 \) and \( y_2 \) be any two points on \( \Omega \), by the definition of \( u_i \)

\[
\left| \frac{\phi_i^{-7} \left( \frac{y_1}{\phi_i^5(x_i)} \right) - \phi_i^{-7} \left( \frac{y_2}{\phi_i^5(x_i)} \right)}{\phi_i^5(x_i)|y_1 - y_2|^\alpha} \right| = \frac{|\phi_i^{-7}(x_i)(u_i^{-7}(y_1) - u_i^{-7}(y_2))|}{\phi_i^5(x_i)|y_1 - y_2|^\alpha} = \frac{|u_i^{-7}(y_1) - u_i^{-7}(y_2)|}{\phi_i^{12}(x_i)|y_1 - y_2|^\alpha}.
\]

(9)

Note that

\[
|u_i^{-7}(y_1) - u_i^{-7}(y_2)| = \left| \int_{u_i(y_2)}^{u_i(y_1)} \frac{d}{dt}(t^{-7})dt \right| = 7 \int_{u_i(y_2)}^{u_i(y_1)} t^{-8}dt \leq 7 \left( \max_{y \in \Omega} u_i^{-8}(y) \right) |u_i(y_1) - u_i(y_2)| = 7 \left( \max_{y \in \Omega} \phi_i^{-8} \left( \frac{y}{\phi_i^5(x_i)} \right) \phi_i^8(x_i) \right) |u_i(y_1) - u_i(y_2)| \leq C\phi_i^8(x_i)|u_i(y_1) - u_i(y_2)|
\]

for some constant \( C \), where the last inequality has used the positive lower bound on \( \phi \). Therefore

\[
\left| \frac{u_i^{-7}(y_1) - u_i^{-7}(y_2)}{\phi_i^{12}(x_i)|y_1 - y_2|^\alpha} \right| \leq \frac{C\phi_i^8(x_i)|u_i(y_1) - u_i(y_2)|}{\phi_i^{12}(x_i)|y_1 - y_2|^\alpha} \leq \frac{C\|u_i\|_{C^{0,\alpha}(\Omega)}}{\phi_i^5(x_i)} \leq C(C_1, C_2, \Omega).
\]

By (9) this gives a uniform bound on the \( C^{0,\alpha} \) norm of \( \phi_i^{-7} \left( \frac{y}{\phi_i^5(x_i)} \right) \phi_i^{-5}(x_i) \) on \( \Omega \). As we explained earlier, this then implies a uniform \( C^{0,\alpha} \) bound on the second term of the right hand side of (8). Then since the other two terms on that side both have \( C^{1,\alpha} \) bound, by the Schauder estimates we have uniform \( C^{2,\alpha} \) bound on \( u_i \) on the compact set \( \Omega \). This implies that a sequence of \( u_i \) converges in \( C^2 \) norm to some function \( u \) on \( \Omega \) where \( u \) satisfies \( 0 \leq u \leq 1 \) and \( u(0) = 1 \). By the assumptions on \( \sigma_i^2 \) and \( \tau_i^2 \) we also know that on \( \Omega \), passing to subsequences \( \{\sigma_i^2\} \) converges to some function \( C_1^{-1} \leq \sigma^2 \leq C_1 \) and \( \{\tau_i^2\} \) converges to some constant \( C_2^{-1} \leq \tau^2 \leq C_2 \). Additionally, the metrics \( \lambda^{(i)} \) converge to the Euclidean metric on \( \Omega \). Then we let \( i \to \infty \) on both sides of (8) for \( y \in \Omega \). Since
φ_i(x_i) → ∞ and R and φ_i^{-7} are bounded above, in the limit the equation becomes

(10) \[ \Delta_\delta u = \frac{1}{12} \tau^2 u^5 \]

where \( \Delta_\delta \) denotes the Euclidean Laplacian. Since \( \Omega \) is arbitrary, we thus have obtained a function 0 ≤ u ≤ 1, u(0) = 1 which satisfies (10) on \( \mathbb{R}^3(y) \). This is in fact impossible due to the following lemma, and therefore we have completed the proof in these three classes.

**Lemma 2.1.** There does not exist any function 0 ≤ u ≤ 1, u(0) = 1 which satisfies equation (10) on \( \mathbb{R}^3 \).

**Proof:** To simplify the proof we assume the constant \( \frac{1}{12} \tau^2 = 1 \). We will use the moving plane method as in [6] by C.S. Lin, but in our case the proof is much easier with the extra assumptions on u.

Suppose such a function u exists. Then for any \( t ∈ \mathbb{R} \) and any \( y = (y_1, y_2, y_3) \), denote \( y_t = (2t - y_1, y_2, y_3) \) and define \( u_t(y) = u(y_t) \). Then \( u_t \) also satisfies \( \Delta_\delta u_t = u_t^5 \). We claim that for any \( t \), \( u_t(y) ≥ u(y) \) whenever \( y_1 < t \).

Suppose this is not true. Define the function \( w_t(y) = u_t(y) - u(y) \), then for some \( t \) we have

\[ \inf_{\{y : y^1 < t\}} w_t(y) < 0. \]

Now let

\[ g_t(y) = \ln \left( (t - y_1^1 + 2)^2 + (y_2^2 + y_3^2) \right) + \ln \left( (t - y_1^1 + 2)^2 + (y_3^2)^2 \right) \]

for \( y ∈ \mathbb{R}^3 \) with \( y_1 < t \). By this definition we know that

\[ g_t(y) > \ln 2^2 > 1, \quad \lim_{|y| → ∞} g_t(y) = ∞, \quad \text{and} \quad \Delta_\delta g_t = 0. \]

Define

\[ \bar{w}_t(y) = \frac{w_t(y)}{g_t(y)}, \]

then \( \inf_{\{y : y^1 < t\}} \bar{w}_t(y) < 0 \) because \( \inf_{\{y : y^1 < t\}} w_t(y) < 0 \) and \( g_t(y) > 1 \). Because \( \lim_{|y| → ∞} g_t(y) = ∞ \) and \( w_t \) is bounded, we have \( \lim_{|y| → ∞} \bar{w}_t(y) = 0 \). Therefore \( \inf_{\{y : y^1 < t\}} \bar{w}_t(y) \) is achieved at some point \( y_0 \), and \( \bar{w}_t(y_0) < 0 \).

On one hand, from \( g_t(y_0) > 1 \) and \( \bar{w}_t(y_0) < 0 \) we know that \( w_t(y_0) < 0 \), i.e. \( u_t(y_0) < u(y_0) \). This implies that

\[ \Delta_\delta w_t(y_0) = \Delta_\delta u_t(y_0) - \Delta_\delta u(y_0) = u_t^5(y_0) - u^5(y_0) < 0. \]

On the other hand, since \( y_0 \) is a minimum point for \( \bar{w}_t \), we have that \( \nabla_\delta \bar{w}_t(y_0) = 0 \) and \( \Delta_\delta \bar{w}_t(y_0) ≥ 0 \). Combined with \( g_t > 1 \) and \( \Delta_\delta g_t = 0 \), this leads to

\[ \Delta_\delta w_t(y_0) = \Delta_\delta (\bar{w}_t g_t)(y_0) = g_t(y_0) \Delta_\delta \bar{w}_t(y_0) + \bar{w}_t(y_0) \Delta_\delta g_t(y_0) + 2 \nabla_\delta \bar{w}_t(y_0) \cdot \nabla_\delta g_t(y_0) \geq 0. \]

Thus we have reached a contradiction. Therefore \( u_t(y) ≥ u(y) \) for any \( t ∈ \mathbb{R} \) and \( y_1 < t \).
Then because \( u(0) = 1 \) is the maximum, \( u \) must be identically equally to 1 on the positive \( y^1 \)-axis. Since the equation \( \Delta_3 u = u^5 \) is invariant under rotations about the origin, this implies that \( u \) is identically equal to 1 on any half line emanating from the origin. Therefore we know that \( u \equiv 1 \). However, this contradicts the equation \( \Delta_3 u = u^5 \), and the proof of this lemma is finished.

\[ \blacksquare \]

3. The Class \((V^+, \sigma^2 \neq 0, \tau = 0)\)

In this class the Lichnerowicz equation becomes

\[ \Delta \phi = \frac{1}{8} R \phi - \frac{1}{8} \sigma^2 \phi^7. \]

Due to the conformal invariant property of this equation, we can assume \( R \) to be a positive constant.

First the lower bound. If \( \phi < \left( \frac{\sigma^2}{R} \right)^{\frac{1}{8}} \), then \( \Delta \phi = \frac{1}{8} R \phi - \frac{1}{8} \sigma^2 \phi^{-7} < 0 \), hence \( \phi \) cannot reach a minimum. Therefore \( \phi \geq \left( \frac{\sigma^2}{R} \right)^{\frac{1}{8}} \geq \left( C_1^{-1} R^{-1} \right)^{\frac{1}{8}} \).

Next the upper bound. Suppose there is no uniform upper bound on \( \phi \), then we can find sequences \( \{ \phi_i \}, \{ \sigma_i \}, \{ x_i \} \), such that

\[ \Delta \phi_i = \frac{1}{8} R \phi_i - \frac{1}{8} \sigma_i^2 \phi_i^{-7}, \]

where \( \phi_i > 0, C_1^{-1} < \sigma_i^2 < C_1, \| \sigma_i^2 \|_{C^0(M)} < C_1 \), and \( \max_{M} \phi_i = \phi_i(x_i) \to \infty \).

Now define another sequence of functions \( \{ v_i \} \) on \( M \) by

\[ v_i(x) = \frac{\phi_i(x)}{\phi_i(x_i)}. \]

Then \( 0 < v_i(x) \leq 1, v_i(x_i) = 1 \), and

\[ \Delta v_i = \frac{1}{8} R v_i - \frac{1}{8} \sigma_i^2 \phi_i^{-7} \phi_i^{-1}(x_i). \]

Since we already proved the positive lower bound on \( \phi_i \), the right hand side of (11) has uniformly bounded \( L^p(M) \) norm for any \( p > 1 \). Then by standard elliptic estimates \( v_i \) has bounded \( W^{2,p} \) norm, which leads to uniform \( C^{1,\alpha} \) bound when \( p \) is large enough. Next we show that \( \phi_i^{-7} \phi_i^{-1}(x_i) \) has uniform \( C^{0,\alpha} \) bound as follows.

For any two points \( p_1, p_2 \in M \), denote \( d_\Lambda(p_1, p_2) \) as \( |p_1 - p_2| \). Then
Due to the conformal invariant property of this equation, we can assume
\[ \Delta \equiv \text{constant}. \]
Then since \( \phi \) where \( C \) hand side of (11) has bounded
\[ \text{proof of the upper bound.} \]
Therefore \( \phi \) and \( \{ \phi_i \} \) are both bounded above, we can take the limits of both sides of (11) as \( i \to \infty \). However, since \( \phi \) is compact, there exists a point \( x_0 \in M \) such that passing to a subsequence \( x_0 = \lim_{i \to \infty} x_i \). Then \( v(x_0) = \lim_{i \to \infty} v(x_i) = 1 \). This contradiction finishes the proof of the upper bound.

4. The Class \( (\mathcal{V}^{-}, \sigma^2 \equiv 0, \tau \neq 0) \)

In this class the Lichnerowicz equation becomes
\[ \Delta \phi = \frac{1}{8} R \phi + \frac{1}{12} \tau^2 \phi^5. \]
Due to the conformal invariant property of this equation, we can assume \( R \) to be a negative constant.

First the lower bound.
If \( \phi < (\frac{-3R}{2\tau^2})^{\frac{1}{4}} \), then \( \Delta \phi = \frac{1}{8} R \phi + \frac{1}{12} \tau^2 \phi^5 < 0 \), hence \( \phi \) cannot reach a minimum. Therefore \( \phi \geq (\frac{-3R}{2\tau^2})^{\frac{1}{4}} \geq (\Delta \phi)^{\frac{1}{2}} (2^{-1} C_2^{-1})^{\frac{1}{4}}. \)

Next the upper bound.
Suppose there is no uniform upper bound on \( \phi \), then we can find sequences \( \{ \phi_i \}, \{ \tau_i \}, \) and \( \{ x_i \} \), such that
\[ \Delta \phi_i = \frac{1}{8} R \phi_i + \frac{1}{12} \tau_i^2 \phi_i^5, \]
where \( \phi_i > 0, C_2^{-1} < \tau_i^2 < C_2, \) and \( \max_M \phi_i = \phi_i(x_i) \to \infty. \)
Let \( x = (x^1, x^2, x^3) \) be the geodesic normal coordinates centered at each of the points \( x_i \), and let \( y = \phi_i^2(x_i)x \). Define \( u_i(y) = \frac{\phi_i(y)}{\phi_i(x_i)} \). Then by the same argument as in Section
we can show that on any compact subset of $\mathbb{R}^3(y)$, a subsequence of $\{u_i\}$ converges in $C^2$ norm to a function $u$ which satisfies $0 \leq u \leq 1$, $u(0) = 1$ and
\[
\Delta u = \frac{1}{12} \tau^2 u^5
\]
on $\mathbb{R}^3$, where $\Delta$ is the Euclidean Laplacian and $\tau^2 = \lim_{i \to \infty} \tau_i^2$. In fact, the proof of this convergence is easier in this class than in Section 2 because here we do not have the term $\frac{1}{8} \sigma_i^2 \phi_i^{-7}$ in the Lichnerowicz equations. Finally, by Lemma 2.1 such a function $u$ does not exist, and this contradiction completes the proof.

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