ON THE RADIUS OF SPATIAL ANALYTICITY FOR THE KLEIN-GORDON-SCHRÖDINGER SYSTEM

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Abstract. In this paper, we study persistence of spatial analyticity of solutions for the Klein-Gordon-Schrödinger system with initial data that have uniform radius of analyticity. We start by establishing the local well-posedness in the space of the analytic functions, so called the Gevrey class. It shows that the initial radius of analyticity is maintained for a short time. We then extend it to arbitrary large time inductively by making use of a suitable almost conservation law which is obtained from certain delicate bilinear estimates in Gevrey-Bourgain spaces. It is shown that the uniform radius of spatial analyticity of solutions obeys an algebraic lower bound as time tends to infinity.

1. Introduction

In this paper we consider the Cauchy problem for the Klein-Gordon-Schrödinger system

\[ \begin{cases} 
  i\partial_t u + \Delta u = -un, & u(0, x) = u_0(x), \\
  \partial^2_t n + (1 - \Delta)n = |u|^2, & (n, \partial_t n)(0, x) = (n_0, n_1)(x),
\end{cases} \tag{1.1} \]

where \( u : \mathbb{R}^{1+d} \to \mathbb{C} \) and \( n : \mathbb{R}^{1+d} \to \mathbb{R} \) for \( d = 1, 2, 3 \). This system is a classical model which describes a system of complex scalar nucleon fields \( u \) interacting with neutral real scalar meson fields \( n \). The mass of the meson is normalized to be 1.

The well-posedness of this Cauchy problem with initial data in Sobolev spaces \( H^s(\mathbb{R}^d) \) has been intensively studied. The best known result is that (1.1) is globally well-posed for \((u_0, n_0, n_1) \in L^2(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)\) where \(-1/2 < s < 1/2\) for \( d = 1, -1/2 < s < 3/2\) for \( d = 2 \) and \(-1/2 < s \leq 1\) for \( d = 3\); see [18, 11]. For earlier studies, we refer the reader to [16, 7, 17].

Once we have the well-posedness, it is often of great interest whether spatial analyticity of the initial data persists at later times. More precisely, if the initial data are real-analytic and have a uniform radius of analyticity \( \sigma_0 > 0 \), so there is a holomorphic extension of the data to a complex strip \( S_{\sigma_0} = \{ x + iy : x, y \in \mathbb{R}, |y_1|, |y_2|, \cdots, |y_d| < \sigma_0 \} \), then we may ask whether or not and up to what degree the solution at some later time \( t \) preserves the initial analyticity; we would like to estimate the radius of analyticity of the solution at time \( t, \sigma(t) \), which is possibly shrinking. This type of question was first introduced by Kato and Masuda [14] in 1986 and there are plenty

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of works for nonlinear dispersive equations such as the KP equation [3], KdV type
equations [4 5 21 25 12 19 2], Schrödinger equations [6 24 1], and Klein-Gordon
equations [15].

Notwithstanding, when it comes to nonlinear dispersive system not much is known;
this is not so surprising as it is harder to show the uniform analyticity of multiple
variables, controlling all of them at the same time to the desired extent. The only
results known so far are rather recent ones for the Dirac-Klein-Gordon system [22 20]
which governs the physical system when nucleons are described by Dirac spinor fields
in the case of relativistic fields. Motivated thereby, we aim here to obtain the spatial
analyticity for the Klein-Gordon-Schrödinger system that works in the non-relativistic
regime.

A class of analytic functions suitable for the problem is the Gevrey class $G^{\sigma,s}(\mathbb{R}^d)$
introduced by Foias and Temam [10], which may be defined with the norm
\[ \|f\|_{G^{\sigma,s}} = \|e^{\sigma|D|}(D)^s f\|_{L^2} \]
for $\sigma \geq 0$ and $s \in \mathbb{R}$. Here, $D = -i\nabla$ with Fourier symbol $\xi$, $\|\xi\| = \sum_{i=1}^d |\xi_i|$ and
$\langle \xi \rangle = \sqrt{1 + |\xi|^2}$. According to the Paley-Wiener theorem (see e.g. [13], p. 209), a
function $f$ belongs to $G^{\sigma,s}$ with $\sigma > 0$ if and only if it is the restriction to the real
line of a function $F$ which is holomorphic in the strip $S_\sigma = \{x + iy : x, y \in \mathbb{R}^d, |y_1|, |y_2|, \ldots, |y_d| < \sigma \}$
and satisfies $\sup_{|y|<\sigma} \|F(x + iy)\|_{H^s_x} < \infty$. Therefore, every function in $G^{\sigma,s}$ with
$\sigma > 0$ has an analytic extension to the strip $S_\sigma$.

In view of this property of the Gevrey class, it is now natural to take initial data
in $G^{\sigma,s}$ for some initial radius $\sigma > 0$ and then try to extend it globally in time with
estimating the behavior of the radius of analyticity $\sigma(t)$ as time $t$ goes. Our result is
the following theorem.

**Theorem 1.1.** Let $d = 1, 2, 3$. Let $(u, n)$ be the global $C^\infty$ solution of the Cauchy
problem (1.1) with the initial data $(u_0, n_0, n_1) \in G^{\sigma_0,r}(\mathbb{R}^d) \times G^{\sigma_0,s}(\mathbb{R}^d) \times G^{\sigma_0,s-1}(\mathbb{R}^d)$
for some $\sigma_0 > 0$ and $r, s \in \mathbb{R}$. Then for all $t \in \mathbb{R}$
\[ (u, n, \partial_t n)(t) \in G^{\sigma(t),r}(\mathbb{R}^d) \times G^{\sigma(t),s}(\mathbb{R}^d) \times G^{\sigma(t),s-1}(\mathbb{R}^d), \]
where the radius of analyticity $\sigma(t)$ satisfies an asymptotic lower bound
\[ \sigma(t) \geq ct^{-p^+} \quad \text{as} \quad |t| \to \infty, \]
with $p = \max\{8/(4 - d), 4\}$ and a constant $c > 0$ depending on $\sigma_0, r, s$ and the norm
of the initial data.

Only when $d = 1, 2, 3$ does the existing well-posedness theory in $H^s$ guarantee the
existence of the global $C^\infty$ solution in the theorem, given initial data $(u_0, n_0, n_1) \in$
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$G^\sigma_0(\mathbb{R}^d) \times G^\sigma_0(\mathbb{R}^d) \times G^\sigma_0(\mathbb{R}^d)$ for any $\sigma_0 > 0$ and $r, s \in \mathbb{R}$. Indeed, observe first that $G^\sigma_0$ coincides with the Sobolev space $H^\sigma$ and the embeddings

$$G^\sigma_0 \subset G^\sigma_1$$

hold for all $0 \leq \sigma' < \sigma$ and $s, s' \in \mathbb{R}$. As a consequence of this embedding with $\sigma' = 0$ and the existing well-posedness theory in $H^s$, the Cauchy problem (1.1) has a unique global smooth solution.

The outline of this paper is as follows. In Section 2 we transform the system (1.1) into an equivalent system of first order in time and restate Theorem 1.1 accordingly. In Section 3 we introduce some analytic function spaces and their basic properties to be used in later sections. In Section 4 we list a couple of norm estimates employed in the proof of Theorem 1.1. In Section 5 we first show local-wellposedness in the Gevrey space in time by a contraction argument, and establish an almost conservation law to bound the growth of the solution in the time interval measured in the norm $G^{\sigma_0,s}$. Then we finish the proof of Theorem 1.1 by iterating the local result based on the approximate conservation law. The last section, Section 6, is devoted to proving the norm estimates given in Section 4.

Throughout this paper, we denote $A \lesssim B$ to mean $A \leq CB$ for some positive constant $C$, and $A \sim B$ to mean $A \lesssim B \lesssim A$.

2. Reformulation of the system

We shall transform the system (1.1) into an equivalent system of first order in $t$ in the usual way. We first let

$$n_\pm = n \pm i\langle D \rangle^{-1}\partial_t n.$$

Then we have

$$n = \frac{1}{2}(n_+ + n_-),$$

and the equivalent system is

$$\begin{cases}
    i\partial_t u + \Delta u = -u(n_+ + n_-)/2, & u(0) = u_0, \\
    i\partial_t n_+ - \langle D \rangle n_+ = -\langle D \rangle^{-1}|u|^2, & n_+(0) = \phi_+,
    \\
    i\partial_t n_- + \langle D \rangle n_- = \langle D \rangle^{-1}|u|^2, & n_-(0) = \phi_-.
\end{cases}$$

(2.1)

Notice that if

$$(u, n, \partial_t n) \in C([-T, T]; G^{\sigma,r}(\mathbb{R}^d) \times G^{\sigma,s}(\mathbb{R}^d) \times G^{\sigma,s-1}(\mathbb{R}^d))$$

is a solution of (1.1) with initial data $(u_0, n_0, n_1) \in G^{\sigma,r} \times G^{\sigma,s} \times G^{\sigma,s-1}$, then

$$(u, n_+, n_-) \in C([-T, T]; G^{\sigma,r}(\mathbb{R}^d) \times G^{\sigma,s}(\mathbb{R}^d) \times G^{\sigma,s}(\mathbb{R}^d))$$

is that of (2.1) with initial data $(u_0, \phi_+, \phi_-) \in G^{\sigma,r} \times G^{\sigma,s} \times G^{\sigma,s}$, and vice versa.

With this observation we can restate Theorem 1.1 as follows, and will prove the restatement in the remainder of the paper.
Theorem 2.1 (Theorem 1.1). Let \( d = 1, 2, 3 \). Let \((u, n_+, n_-)\) be the global \( C^\infty\) solution of the Cauchy problem \((2.1)\) with the initial data \((u_0, \phi_+, \phi_-) \in G^{\sigma_0, r}(\mathbb{R}^d) \times G^{\sigma_0, s}(\mathbb{R}^d) \times G^{\sigma_0, s}(\mathbb{R}^d)\) for some \( \sigma_0 > 0 \) and \( r, s \in \mathbb{R} \). Then for all \( t \in \mathbb{R} \)
\[(u, n_+, n_-)(t) \in G^{\sigma(t), r}(\mathbb{R}^d) \times G^{\sigma(t), s}(\mathbb{R}^d) \times G^{\sigma(t), s}(\mathbb{R}^d),\]
where the radius of analyticity \( \sigma(t) \) satisfies an asymptotic lower bound
\[\sigma(t) \geq ct^{-p^+} \text{ as } |t| \to \infty,\]
with \( p = \max\{8/(4 - d), 4\} \) and a constant \( c > 0 \) depending on \( \sigma_0, r, s \) and the norm of the initial data.

3. Preliminaries

In this section we introduce some function spaces and linear estimates in these spaces which will be used later for the proof of Theorem 2.1.

For \( s, b \in \mathbb{R} \) and some real valued polynomial \( h \), we use \( X^{s, b}_h = X^{s, b}_h(\mathbb{R}^{1+d}) \) to denote the Bourgain space defined by the norm
\[\|f\|_{X^{s, b}_h} = \|\langle \xi \rangle^{s - h(\xi)} \hat{f}(\tau, \xi)\|_{L^1_{r, t}},\]
where \( \hat{f} \) denotes the space-time Fourier transform given by
\[\hat{f}(\tau, \xi) = \int_{\mathbb{R}^{1+d}} e^{-i(\tau + x \cdot \xi)} f(t, x) \, dt \, dx.\]

For simplicity, we omit \( h \) in the notation \( X^{s, b}_h \) when \( h(\xi) = -|\xi|^2 \), and replace \( h \) with \( \pm \) when \( h(\xi) = \mp|\xi| \). Since \( \langle \tau \mp |\xi| \rangle \sim \tau \mp |\xi| \) just for technical reasons rather than \( \mp |\xi| \) for the Klein-Gordon evolution. The restriction of the Bourgain space, denoted \( X^{s, b}_h(\delta) \), to a time slab \((0, \delta) \times \mathbb{R}^d \) is a Banach space when equipped with the norm
\[\|f\|_{X^{s, b}_h(\delta)} = \inf \{\|g\|_{X^{s, b}_h} : g = f \text{ on } (0, \delta) \times \mathbb{R}^d\}.\]

We also introduce the Gevrey-Bourgain space \( X^{\sigma, s, b}_h = X^{\sigma, s, b}_h(\mathbb{R}^{1+d}) \) defined by the norm
\[\|f\|_{X^{\sigma, s, b}_h} = \|e^{\sigma D} f\|_{X^{s, b}_h},\]
which coincides with the Bourgain space \( X^{s, b}_h \) particularly when \( \sigma = 0 \). Its restriction \( X^{\sigma, s, b}_h(\delta) \) to a time slab \((0, \delta) \times \mathbb{R}^d \) is defined in a similar way as above. The Gevrey-modification of the Bourgain spaces was used already by Bourgain [3] to study spatial analyticity for the Kadomtsev-Petviashvili equation. He proved that the radius of analyticity remains positive as long as the solution exists. His argument is quite general and applies to a class of dispersive equations, but does not give any lower bound on the radius \( \sigma(t) \) as \(|t| \to \infty\).

The \( X^{\sigma, s, b}_h \)-estimates in Lemmas 3.1, 3.2, and 3.3 follow easily by substitution \( f \to e^{\sigma D} f \) using the properties of \( X^{s, b}_h \)-spaces and the restrictions thereof. The proofs of the first two lemmas (when \( \sigma = 0 \)) can be found in Section 2.6 of [23], and see Lemma 7 of [20] for the third lemma.
Lemma 3.1. Let $\sigma \geq 0$, $s \in \mathbb{R}$ and $b > 1/2$. Then $X^{\sigma,s,b}_h \subset C(\mathbb{R}, G^{\sigma,s})$ and
\[ \sup_{t \in \mathbb{R}} \| f(t) \|_{G^{\sigma,s}} \leq C \| f \|_{X^{\sigma,s,b}_h}, \]
where $C > 0$ is a constant depending only on $b$.

Lemma 3.2. Let $\sigma \geq 0$, $s \in \mathbb{R}$, $-1/2 < b < b' < 1/2$ and $\delta > 0$. Then
\[ \| f \|_{X^{\sigma,s,b}_h(\delta)} \leq C \delta^{b'-b} \| f \|_{X^{\sigma,s,b'}_h(\delta)}, \]
where the constant $C > 0$ depends only on $b$ and $b'$.

Lemma 3.3. Let $\sigma \geq 0$, $s \in \mathbb{R}$, $-1/2 < b < 1/2$ and $\delta > 0$. Then, for any time interval $I \subset [0, \delta]$,
\[ \| \chi_I f \|_{X^{\sigma,s,b}_h(\delta)} \leq C \| f \|_{X^{\sigma,s,b}_h(\delta)}, \]
where $\chi_I(t)$ is the characteristic function of $I$, and the constant $C > 0$ depends only on $b$.

Next, consider the Cauchy problem
\[
\begin{cases}
(i\partial_t - h(D))u = F(t,x), \\
u(0,x) = f(x),
\end{cases}
\]
for sufficiently regular $F(t,x)$ and $f(x)$. By Duhamel’s principle, the solution can be then written as
\[ u(t,x) = e^{-ith(D)}f(x) - i \int_0^t e^{-i(t-s)h(D)}F(s,\cdot)ds, \]
(3.1)
where the Fourier multiplier $e^{-ith(D)}$ with symbol $e^{-ith(\xi)}$ is given by
\[ e^{-ith(D)}f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\xi} e^{-ith(\xi)} \hat{f}(\xi) d\xi. \]

Then the following is the standard energy estimate in $X^{\sigma,s,b}_h(\delta)$-spaces (see Proposition 2.12 in [23]).

Lemma 3.4. Let $\sigma \geq 0$, $s \in \mathbb{R}$, $1/2 < b \leq 1$ and $0 < \delta \leq 1$. Then we have
\[ \| e^{-ith(D)}f \|_{X^{\sigma,s,b}_h(\delta)} \leq C \| f \|_{G^{\sigma,s}} \]
and
\[ \left\| \int_0^t e^{-i(t-s)h(D)}F(s,\cdot)ds \right\|_{X^{\sigma,s,b-1}_h(\delta)} \leq C \| F \|_{X^{\sigma,s,b-1}_h(\delta)}, \]
Here the constant $C > 0$ depends only on $b$.  

4. Estimates in Gevrey-Bourgain spaces

In this section we first present some bilinear estimates in Gevrey-Bourgain spaces, Lemma 4.1, which are proved in Section 6. This lemma plays a key role in estimating the product terms in the right side of the system (2.1) to ultimately obtain its local well-posedness in Gevrey spaces in the next section.

Lemma 4.1. Let $d = 1, 2, 3$. If $\sigma > 0$ and $s > -1/2$, then we have
\[
\|fg\|_{X^{\sigma,0,b'}} \lesssim \|f\|_{X^{\sigma,0,b}} \|g\|_{X^{\sigma,0,b}}
\] (4.1)
and
\[
\|\hat{f}g\|_{X^{\sigma,-0,b}} \lesssim \|f\|_{X^{\sigma,0,b}} \|g\|_{X^{\sigma,0,b}}
\] (4.2)
whenever $1/2 < b \leq b' < \min\{(6 + 2s - d)/4, 1, s + 1\}$.

Remark 4.2. As we will see later, uniform difference between $b$ and $b'$ when $s \leq 0$ in the lemma is needed to get a lower bound on the radius of spatial analyticity, and larger difference yields better results. For $d = 1$, it is known [11] that $b' - b$ can be (uniformly) as large as $(1/2)^{-}$, but for $d = 2, 3$, the largest possible difference we have (prior to this work) is essentially zero: see [10, 17, 18, 8]. The lemma allows us to have the significant difference for $d = 2, 3$ comparable to what we have for $d = 1$.

When deriving the global well-posedness from the local one, we need to control the growth of the Gevrey norm of the data. We will carry out this obtaining an approximate conservation law in the next section. In this step the bilinear operator
\[
F(v, m) := vm - e^{\sigma\|D\|}(me^{-\sigma\|D\|}v)
\] (4.3)
will appear and the following bilinear estimates thereof will play an important role:

Lemma 4.3. Let $d = 1, 2, 3$ and $1/2 < b \leq b' < \min\{(6 - d)/4, 1\}$. Then we have
\[
\|F(v, m)\|_{X^{0,b'}-1} \lesssim \sigma \|v\|_{X^{0,b}} \|m\|_{X^{\sigma,1,b}}
\]
for $v \in X^{0,b}$ and $m \in X^{\sigma,1,b}$.

Proof. We first take the space-time Fourier transform $\mathcal{F}$ of $\hat{F}$ to see
\[
\mathcal{F}\left[F(v, m)\right](\tau, \xi) = \int_{\mathbb{R}^{1+d}} (1 - e^{\sigma\|\xi\|-(\xi-\xi_1)})(\hat{v}(\tau_1 - \tau, \xi_1 - \xi)\hat{m}(\tau, \xi_1) - \tau_1 - \xi_1)\hat{d}\tau_1 d\xi_1
\]
\[
= \int_{\mathbb{R}^{1+d}} (1 - e^{\sigma\|\xi\|-(\xi-\xi_1)})(\hat{v}(\tau_1 - \tau, \xi_1 - \xi)\hat{m}(\tau, \xi_1))\hat{d}\tau_1 d\xi_1.
\]
Then we estimate
\[
|\mathcal{F}\left[F(v, m)\right](\tau, \xi)| \leq \int |(1 - e^{\sigma\|\xi\|-(\xi-\xi_1)})(\hat{v}(\tau_1 - \tau, \xi_1)\hat{m}(\tau, \xi_1))|\hat{d}\tau_1 d\xi_1
\]
\[
\leq \sigma \int |\hat{v}(\tau_1 - \tau, \xi_1)|\|\xi_1\|e^{\sigma\|\xi_1\|}\|\hat{m}(\tau, \xi_1)\|\hat{d}\tau_1 d\xi_1
\]
\[
= \sigma\mathcal{F}\left[|v|e^{\sigma\|D\|}\partial_x|m|\right](\tau, \xi)
\]
using
\[
|e^{\sigma\|\xi\|-(\xi-\xi_1)} - 1| \leq |e^{\sigma\|\xi\|} - 1| \leq \sigma\|\xi\|e^{\sigma\|\xi\|}.
\]
with $s = 0$, we obtain

$$
\|F[v, m]\|_{X^{0, \nu - 1}} \leq \sigma \|v e^{\nu \|D\| \partial_x}m\|_{X^{0, \nu - 1}} \\
\lesssim \sigma \|v\|_{X^{0, \nu}} \|e^{\nu \|D\| \partial_x}m\|_{X^{0, \nu}} \\
\leq \sigma \|v\|_{X^{0, \nu}} \|m\|_{X^{\sigma, \nu}}.
$$

5. Proof of Theorem 2.1

In this section we prove Theorem 2.1. In Subsection 5.1 we shall show that the radius of analyticity of the solution is positive locally in time, and then in Subsection 5.2 we establish the following local well-posedness in time interval $0 \leq t \leq \delta$, where $\delta > 0$ depends on the norm of the initial data.

**Theorem 5.1.** Let $d = 1, 2, 3$ and $\sigma > 0$. Then, for any initial data $(u_0, \phi_+, \phi_-) \in G^{\sigma, 0} \times G^{\sigma, 1} \times G^{\sigma, 1}$, there exist $\delta > 0$ and a unique solution

$$(u, n_+, n_-) \in C([0, \delta]; G^{\sigma, 0} \times G^{\sigma, 1} \times G^{\sigma, 1})$$

of the Cauchy problem (2.1). Here we may take

$$\delta = C(1 + \|u_0\|_{G^{\sigma, 0}} + \|\phi_+\|_{G^{\sigma, 1}} + \|\phi_-\|_{G^{\sigma, 1}})^{-q^+}$$

for $q = \max\{4/(4-d), 2\}$ and for some constant $C > 0$. Furthermore, for $b = (1/2)^+$,

$$\|u\|_{X^{\sigma, 0, b}(\delta)} \lesssim \|u_0\|_{G^{\sigma, 0}}$$

and

$$\|n_\pm\|_{X^{\sigma, 1, b}(\delta)} \lesssim \|\phi_\pm\|_{G^{\sigma, 1}} + \|u_0\|_{G^{\sigma, 0}}.$$  

**Proof.** Fix $\sigma > 0$ and $(u_0, \phi_+, \phi_-) \in G^{\sigma, 0} \times G^{\sigma, 1} \times G^{\sigma, 1}$. By Lemma 3.1 we shall employ an iteration argument in the space $X^{\sigma, 0, b}(\delta)$ and $X^{\sigma, 1, b}(\delta)$ instead of $G^{\sigma, 0}$ and $G^{\sigma, 1}$, respectively. Define the Picard iterates $\{(u^{(k)}, n_+^{(k)}, n_-^{(k)})\}_{k=0}^\infty$ by

$$
\begin{cases}
\partial_t u^{(0)} + \Delta u^{(0)} = 0, & u^{(0)}(0, x) = u_0(x), \\
\partial_t n_+^{(0)} - \langle D\rangle n_+^{(0)} = 0, & n_+^{(0)}(0, x) = \phi_+(x), \\
\partial_t n_-^{(0)} + \langle D\rangle n_-^{(0)} = 0, & n_-^{(0)}(0, x) = \phi_-(x),
\end{cases}
$$

\begin{thebibliography}{1}
\bibitem{1} The uniqueness is immediate since the solution is certainly $C^\infty$.
\end{thebibliography}
Lemma 3.4 to (5.4) implies and (5.8) and (5.9), respectively, to obtain

and

for \( k \in \mathbb{Z}^+ \). By (5.1), we first write

and

Now we show these sequences are Cauchy in Gevrey-Bourgain spaces. Applying Lemma 3.4 to (5.4) implies

and

while applying Lemmas 3.3 and 3.2 to (5.9) yields

and

with \( 1/2 < b < b' < 1 \). We then apply Lemma 4.1 with \( s = 0 \) to the last norms in (5.10) and (5.11), respectively, to obtain

and

(5.11)
for $1/2 < b < b' < \min\{(6 - d)/4, 1\}$. By induction together with (5.6), (5.7), (5.10) and (5.11), it follows that for all $k \geq 0$

$$\|u^{(k)}\|_{X^{\sigma,0}(\delta)} \leq C\|u_0\|_{G^{\sigma,0}}$$  \hspace{1cm} (5.12)

and

$$\|n^{(k)}\|_{X^{\sigma,1}(\delta)} \leq C\|\phi\|_{G^{\sigma,1}} + C\|u_0\|_{G^{\sigma,0}}$$  \hspace{1cm} (5.13)

with a choice of $\delta$ as

$$\delta^{b'-b} = \frac{1}{8C} (1 + \|u_0\|_{G^{\sigma,0}} + \|\phi\|_{G^{\sigma,1}} + \|\phi\|_{G^{\sigma,0}})^{-1}$$  \hspace{1cm} (5.14)

where $0 < b' - b < \min\{(4 - d)/4, 1/2\}$. Here we take $b' - b = (1/q)^-$ with $q$ as in (5.1). Similarly, we apply Lemmas 3.4 and 3.2 with the same $\delta$ to yield

$$\|u^{(k)} - u^{(k-1)}\|_{X^{\sigma,0}(\delta)}$$

$$\leq C\delta^{(1/q)^-} \|u^{(k-1)}\|_{n^{(k-1)}_+ - n^{(k-1)}_- - u^{(k-2)}\|_{n^{(k-2)}_+ - n^{(k-2)}_-}}$$

$$= C\delta^{(1/q)^-} \|u^{(k-1)} - u^{(k-2)}\|_{n^{(k-1)}_+ - n^{(k-1)}_- - \sum \|u^{(k-2)}\|_{n^{(k-2)}_+ - n^{(k-2)}_-}}$$

which can be in turn bounded by

$$C\delta^{(1/q)^-} \sum \|\phi\|_{G^{\sigma,1}} + \|u_0\|_{G^{\sigma,0}}\|u^{(k-1)} - u^{(k-2)}\|_{X^{\sigma,0}(\delta)}$$

$$+ C\delta^{(1/q)^-} \sum \|u_0\|_{G^{\sigma,0}}\|n^{(k-1)}_+ - n^{(k-2)}_-\|_{X^{\sigma,1}(\delta)}$$

using Lemma 4.1 together with (5.12) and (5.13). Consequently, we get

$$\|u^{(k)} - u^{(k-1)}\|_{X^{\sigma,0}(\delta)} \leq \frac{1}{4} \left(\|u^{(k-1)} - u^{(k-2)}\|_{X^{\sigma,0}(\delta)} + \sum \|n^{(k-1)}_+ - n^{(k-2)}_-\|_{X^{\sigma,1}(\delta)}\right)$$

by (5.14). Similarly, we have

$$\|n^{(k)}_+ - n^{(k-1)}_-\|_{X^{\sigma,1}(\delta)} \leq \frac{1}{8}\|u^{(k-1)} - u^{(k-2)}\|_{X^{\sigma,0}(\delta)}.$$
5.2. Almost conservation law. Now we would like to apply repeatedly the local result just obtained above to cover time intervals of arbitrary length. This, of course, requires some sort of control on the growth of the norm on which the local existence time depends. For a solution to the Cauchy problem (1.1) we have charge conservation:

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \equiv \|u(0)\|_{L^2(\mathbb{R}^d)}.$$ 

Let $M_\sigma(t) = \|u(t)\|_{G^{\sigma,0}}^2$ and $N_\sigma(t) = \|n_+(t)\|_{G^{\sigma,1}} + \|n_-(t)\|_{G^{\sigma,1}}$. Then one can easily see

$$M_0(t) \equiv M_0(0).$$

This quantity is approximately conservative in the sense that, although it fails to hold for $\sigma > 0$, the discrepancy between both sides is bounded as well as the quantity reduces to the conservation of mass and energy in the limit $\sigma \to 0$. We will also see that the growth for $N_\sigma(t)$, if there is any, is bounded in terms of $M_\sigma(0)$. Indeed, the following almost conservation will allow us in Subsection 5.3 to repeat the local result on successive short time intervals to reach any target time $T > 0$, by adjusting the strip width parameter $\sigma$ according to the size of $T$.

**Proposition 5.2.** Let $d = 1, 2, 3, b = (1/2)^+$ and $\delta$ be as in Theorem 5.1. Then there exists $C > 0$ such that for any $\sigma > 0$ and any solution $(u, n_+, n_-) \in X^{\sigma,0,b}(\delta) \times X^{\sigma,1,b}(\delta) \times X^{\sigma,1,b}(\delta)$ to the Cauchy problem (2.1) on the time interval $[0, \delta]$, we have

$$\sup_{t \in [0, \delta]} M_\sigma(t) \leq M_\sigma(0) + C\sigma\delta^{(1/q)}M_\sigma(0)(M_\sigma^{1/2}(0) + N_\sigma(0)) \quad (5.15)$$

and

$$\sup_{t \in [0, \delta]} N_\sigma(t) \leq N_\sigma(0) + C\delta^{(1/q)}M_\sigma(0) \quad (5.16)$$

where $q$ is as in Theorem 5.1.

**Proof.** Let $0 \leq \delta' \leq \delta$. Setting $v(t,x) = e^{\sigma\|D\|}u(t,x)$ and applying $e^{\sigma\|D\|}$ to the first equation in (2.1), we obtain

$$(i\partial_t + \Delta)v = -v\left(\frac{n_+ + n_-}{2}\right) + F(v, \frac{n_+ + n_-}{2})$$

where $F$ is as in (4.3). For brevity, here we shall simply write $F$ for $F\left(v, \frac{n_+ + n_-}{2}\right)$. Multiplying both sides by $\bar{v}$ and taking imaginary parts thereon, we see

$$\text{Re}(\bar{v}\partial_t v) + \text{Im}(\bar{v}\Delta v) = \text{Im}(\bar{v}F),$$

or equivalently

$$\partial_t |v|^2 + 2\text{Im}(\bar{v}\Delta v) = 2\text{Im}(\bar{v}F)$$

where we used the fact $\partial_t |v|^2 = 2\text{Re}(\bar{v}\partial_t v)$. Integrating in space and using integration by parts, we also see

$$\frac{d}{dt} \int_{\mathbb{R}^d} |v|^2 dx = 2\text{Im} \int_{\mathbb{R}^d} \bar{v}F dx,$$
where we may assume that $v$ and its all spatial derivatives decay to zero as $|x| \to \infty$.

Subsequently integrating in time over $[0, \delta']$, we now have

$$
\int_{\mathbb{R}^d} |v(\delta')|^2 dx = \int_{\mathbb{R}^d} |v(0)|^2 dx + 2\text{Im} \int_{\mathbb{R}^{1+d}} \chi_{[0, \delta']}(t)\bar{v} F dt dx.
$$

Using Hölder’s inequality, and then Lemmas 3.3 and 3.2 as before, the rightmost integral is bounded as,

$$
\left| \int_{\mathbb{R}^{1+d}} \chi_{[0, \delta']}(t)\bar{v} F dt dx \right| \leq \| \chi_{[0, \delta']} \|_{X^{0,b-1}} \| \chi_{[0, \delta']} \|_{X^{0,1-b}}
$$

which implies (5.15).

Next we show (5.16). By (3.1) we first note

$$
\| u(\delta') \|_{L_{\alpha,0}}^2 \leq \| u_0 \|_{L_{\alpha,0}}^2 + C\delta^{(1/q)} \| u \|_{L_{\alpha,0}}^2 + \| \phi \|_{L_{\alpha,1}} + \| \phi \|_{L_{\alpha,1}}
$$

from Lemma 4.3. By applying (5.2) and (5.3) we therefore get

$$
\| u(\delta') \|_{L_{\alpha,0}}^2 \leq \| u_0 \|_{L_{\alpha,0}}^2 + C\delta^{(1/q)} \| u \|_{L_{\alpha,0}}^2 + \| \phi \|_{L_{\alpha,1}} + \| \phi \|_{L_{\alpha,1}}
$$

which implies (5.16). By (3.4) we then see

$$
\| n_\pm(t) \|_{L_{\alpha,1}} \leq \| \phi \|_{L_{\alpha,1}} + C\delta^{(1/q)} \| u \|_{X_{\pm,0,b}^2(\delta)}
$$

while

$$
\| D^{-1} |u|^2 \|_{X_{\pm,1,b-1}^2(\delta)} \lesssim \delta^{(1/q)} \| u \|_{X_{\pm,0,b}^2(\delta)}
$$

by Lemma 3.2, 1.2 and 5.3. Summing up, we get for any $t \in [0, \delta]$

$$
\sum_{\pm} \| n_\pm(t) \|_{L_{\alpha,1}} \leq \sum_{\pm} \| \phi \|_{L_{\alpha,1}} + C\delta^{(1/q)} \| u_0 \|_{L_{\alpha,0}}^2
$$

which implies (5.16).
5.3. **Global extension and radius of analyticity.** The last step is to put it all together to complete the proof of Theorem 2.1. By the embedding (4.2), the general case \((r, s) \in \mathbb{R}^2\) reduces to \((r, s) = (0, 1)\) as shown at the end of the proof.

Let us now prove the case \((r, s) = (0, 1)\). Given \(\sigma_0 > 0\) and data such that \(\mathcal{M}_{\sigma_0}(0)\) and \(\mathcal{N}_{\sigma_0}(0)\) are finite, we must prove that for all large \(T\), the solution has a positive radius of analyticity
\[
\sigma(t) \geq ct^{-p^+} \quad \text{for all } t \in [0, T],
\]
where \(c > 0\) is a constant depending on the data norms \(\mathcal{M}_{\sigma_0}(0)\) and \(\mathcal{N}_{\sigma_0}(0)\). Now fix \(T > 1\) arbitrarily large and let \(A \gg 1\) denote a constant which may depend on \(\mathcal{M}_{\sigma_0}(0)\) and \(\mathcal{N}_{\sigma_0}(0)\); the choice of \(A\) will be made explicit below. Let \(q\) be as in Theorem 5.1.

It suffices to show that for all \(t \in [0, T]\)
\[
\mathcal{M}_{\sigma(t)}(t) \leq 4\mathcal{M}_{\sigma(t)}(0) \tag{5.18}
\]
and
\[
\mathcal{N}_{\sigma(t)}(t) \leq 2AT^q^+ \tag{5.19}
\]
with \(\sigma(t) \leq \sigma_0\) satisfying (5.17), which in turn implies \((u, n_+, n_-)(t) \in G^{p(t), 0} \times G^{p(t), 1} \times G^{p(t), 1}\) as desired. For brevity we denote \(\sigma = \sigma(t)\). To prove (5.18) and (5.19), we may first assume that
\[
\mathcal{M}_{\sigma_0}^{1/2}(0) + \mathcal{N}_{\sigma_0}(0) \leq T^{q^+ + c_0} \tag{5.20}
\]
for some small \(c_0 > 0\) since we are considering large \(T\). Then we let
\[
\delta = \frac{c_0}{AT^{q^+ + c_0}}, \tag{5.21}
\]
for some small \(c_0 > 0\) such that \(n := T/\delta\) is an integer. It suffices to show for any \(k \in \{1, 2, \cdots, n\}\) and for some small \(\varepsilon > 0\) that
\[
\sup_{t \in [0, k\delta]} \mathcal{M}_{\sigma}(t) \leq \mathcal{M}_{\sigma}(0) + kC\sigma\delta^{1/(q^+ - \varepsilon)}(4\mathcal{M}_{\sigma}(0))(4AT^{q^+ + c_0}), \tag{5.22}
\]
\[
\sup_{t \in [0, k\delta]} \mathcal{N}_{\sigma}(t) \leq \mathcal{N}_{\sigma}(0) + kC\delta^{1/(q^+ - \varepsilon)}(4\mathcal{M}_{\sigma}(0)), \tag{5.23}
\]
provided
\[
nC\sigma\delta^{1/(q^+ - \varepsilon)}(4\mathcal{M}_{\sigma}(0))(4AT^{q^+ + c_0}) \leq \mathcal{M}_{\sigma}(0), \tag{5.24}
\]
\[
nC\delta^{1/(q^+ - \varepsilon)}(4\mathcal{M}_{\sigma}(0)) \leq AT^{q^+ + c_0}. \tag{5.25}
\]
For this, we shall use induction. The case \(k = 1\) is immediate from (5.15), (5.16) and (5.20). Now assume (5.22) and (5.23) hold for some \(k \in \{1, 2, \cdots, n - 1\}\). By this assumption, (5.18) and (5.19) hold for \(t \in [0, (n - 1)\delta]\). Hence applying (5.15) with \(k\delta\) as the initial time we have
\[
\sup_{t \in [k\delta, (k + 1)\delta]} \mathcal{M}_{\sigma}(t) \leq \mathcal{M}_{\sigma}(k\delta) + C\sigma\delta^{1/(q^+ - \varepsilon)}\mathcal{M}_{\sigma}(k\delta)(4\mathcal{M}_{\sigma}(0) + \mathcal{N}_{\sigma}(k\delta)) \leq \mathcal{M}_{\sigma}(k\delta) + C\sigma\delta^{1/(q^+ - \varepsilon)}(4\mathcal{M}_{\sigma}(0))(2\mathcal{M}_{\sigma}^{1/2}(0) + 2AT^{q^+ + c_0}),
\]
which is in turn bounded by
\[
\mathcal{M}_{\sigma}(0) + (k + 1)C\sigma\delta^{1/(q^+ - \varepsilon)}\mathcal{M}_{\sigma}(0)(4AT^{q^+ + c_0}),
\]
using (5.20) and (5.22). In the same manner, we get
\[ \sup_{t \in [k\delta, (k+1)\delta]} \mathfrak{N}_\sigma(t) \leq \mathfrak{N}_\sigma(0) + (k + 1)C\delta^{1/q-\varepsilon}(4M_\sigma(0)). \]

We will show that (5.24) and (5.25) hold under (5.17). From \( T = n\delta \) and (5.21), we get \( n^{1/q-\varepsilon} = T^{1/(1-q)/(q-\varepsilon)} = c_1A^{(q-1)/q+\varepsilon}T^{1-(q+\varepsilon)(1-q)/(q-\varepsilon)} \) where \( c_1 \) is an absolute constant. We can choose \( \varepsilon > 0 \) such that \(- (q + \varepsilon_0)((1-q)/(q-\varepsilon)) = q - 1 + \varepsilon_0 \), which gives \( n^{1/q-\varepsilon} = c_1A^{(q-1)/q+\varepsilon}T^{q+\varepsilon} \); simple calculation shows that \( \varepsilon \to 0 \) as \( \varepsilon \to 0 \). Therefore (5.24) and (5.25) reduce to
\[ C\sigma A^{(q-1)/q+\varepsilon}T^{q+\varepsilon_0}(16AT^{q+\varepsilon_0}) \leq 1, \quad (5.26) \]
\[ C\sigma A^{(q-1)/q+\varepsilon}T^{q+\varepsilon_0}(4M_\sigma(0)) \leq AT^{q+\varepsilon_0}. \quad (5.27) \]

To satisfy (5.27) we choose \( A \) so large that
\[ C\sigma A^{1/q-\varepsilon}. \]

Finally, (5.20) is satisfied if \( \sigma = cT^{-2q-2\varepsilon_0} \) where \( c \) is a constant that may depend on \( M_{\sigma_0}(0) \) and \( \mathfrak{N}_{\sigma_0}(0) \). Since \( 2q = p \) and \( \varepsilon_0 \) can be arbitrarily small, we get the radius of analyticity (5.17).

Now we consider the general case \((r, s) \in \mathbb{R}^2\). Recall that
\[ G^{\sigma, s} \subset G^{\sigma', s'} \quad \text{for all} \quad \sigma > \sigma' \geq 0 \quad \text{and} \quad s, s' \in \mathbb{R}, \]
from which we see that for any \((r, s) \in \mathbb{R}^2\),
\[ (u_0, \phi_+, \phi_-) \in G^{\sigma, r}(\mathbb{R}^d) \times G^{\sigma_0, s}(\mathbb{R}^d) \times G^{\sigma_0, s}(\mathbb{R}^d) \]
\[ \subset G^{\sigma_0/2, 0}(\mathbb{R}^d) \times G^{\sigma_0/2, 1}(\mathbb{R}^d) \times G^{\sigma_0/2, 1}(\mathbb{R}^d). \]

From the local theory there is a \( \delta \) such that
\[ (u(t), n_+(t), n_-(t)) \in G^{\sigma_0/2, 0}(\mathbb{R}^d) \times G^{\sigma_0/2, 1}(\mathbb{R}^d) \times G^{\sigma_0/2, 1}(\mathbb{R}^d) \quad \text{for} \quad 0 \leq t \leq \delta. \]

As in the case \((r, s) = (0, 1)\), for fixed \( T \) greater than \( \delta \), we have \((u(t), n_+(t), n_-(t)) \in G^{\sigma', 0}(\mathbb{R}^d) \times G^{\sigma', 1}(\mathbb{R}^d) \times G^{\sigma', 1}(\mathbb{R}^d) \) for \( t \in [0, T] \) and \( \sigma' \geq cT^{-p} \) with the constant \( c > 0 \) depending on the data norms \( M_{\sigma_0/2}(0) \) and \( \mathfrak{N}_{\sigma_0/2}(0) \). Applying the embedding again, we conclude
\[ (u(t), n_+(t), n_-(t)) \in G^{\sigma, r}(\mathbb{R}^d) \times G^{\sigma', s}(\mathbb{R}^d) \times G^{\sigma, s}(\mathbb{R}^d) \quad \text{for} \quad t \in [0, T] \]
where \( \sigma = \sigma'/2. \) This concludes the proof. \( \square \)

6. Proof of Lemma 4.1

This final section is devoted to the proof of Lemma 4.1. Note first that
\[ \|fg\|_{X_{k}^{\sigma, r, b}} \leq \|(e^{\sigma\|D\|}f)(e^{\sigma\|D\|}g)\|_{X_{k}^{\sigma, r, b}} \]
by the definitions of the norms. From this observation, (4.1) and (4.2) reduce to showing the case \( \sigma = 0 \):
\[ \|fg\|_{X_{0, 0, -1}} \leq \|f\|_{X_{0, 0}} \|g\|_{X_{0, 0}^{r, b}} \]
The case approach due to the absence of boundedness. Now we begin the first part. the matter relatively simple. However, the second part requires a more delicate approach. Since the latter can be obtained in a similar manner, we shall only show (6.3). Using the Hölder inequality in $dξ_0 dτ_0$ and then in $dξ_1 dτ_1$, the left hand side of (6.3) is bounded as

$$\|f_0\|_{L^2_{0,\tau_0}} \bigg\| \int f_1(ξ_1, τ_1) f_2(ξ_2, τ_2) \frac{1}{(τ_0 - |ξ_0|^2)^{1-b}(τ_1 + |ξ_1|^2)^b(τ_2 ± |ξ_2|)^b} dξ_0 dτ_0 dξ_1 dτ_1 \bigg\|_{L^2_{0,\tau_0}}$$

and

$$\|f_0\|_{L^2_{0,\tau_0}} \bigg\| \int f_1(ξ_1, τ_1) f_2(ξ_2, τ_2) \frac{1}{(τ_0 - |ξ_0|^2)^{1-b}(τ_1 + |ξ_1|^2)^b(τ_2 ± |ξ_2|)^b} dξ_0 dτ_0 dξ_1 dτ_1 \bigg\|_{L^2_{0,\tau_0}}$$

For this, the following lemma will be used repeatedly.

**Lemma 6.1** ([11]). If $α > 1$ and $α ≥ β ≥ 0$, then

$$\int_ℝ \frac{dy}{(y-a)^α(y-b)^β} \lesssim (a-b)^{-β}$$

To show (6.1) and (6.2) we first break the integration region into two parts: $|ξ_1|, |ξ_2| \lesssim 1$ (and thus $|ξ_0| \lesssim 1$) and the rest. The bound $|ξ_1|, |ξ_2| \lesssim 1$ will make the matter relatively simple. However, the second part requires a more delicate approach due to the absence of boundedness. Now we begin the first part.
Now the estimate (6.3) is obtained since
\[
\iint \frac{\langle \tau_0 - |\xi_0|^2 \rangle^{2b'} - 2d\xi_1 d\tau_1}{\langle \tau_1 + |\xi_1|^2 \rangle^{2b}\langle \tau_0 + \tau_1 + |\xi_0 + \xi_1| \rangle^{2b}} = \iint \frac{\langle \tau_0 - |\xi_0|^2 \rangle^{2b'} - 2\langle \tau_1 + |\xi_1|^2 \rangle^{-2b}}{\langle \tau_0 + \tau_1 + |\xi_0 + \xi_1| \rangle^{2b}} d\xi_1 d\tau_1 \lesssim \int \frac{\langle \tau_0 + \tau_1 - |\xi_0|^2 + |\xi_1|^2 \rangle^{2b'} - 2\langle \tau_1 + |\xi_1|^2 \rangle^{-2b}}{\langle \tau_1 + |\xi_1| \rangle^{2b}} d\xi_1 d\tau_1 \lesssim \langle |\xi_0|^2 - |\xi_1|^2 \rangle^{2b'} \lesssim d\xi_1 < \infty
\]
uniformly in $\xi_0$ and $\tau_0$. This follows easily from using Lemma 6.1 together with $\langle a + b \rangle \lesssim \langle a \rangle \langle b \rangle$, $-2b < 2b' - 2 < 0$ and $|\xi_1| \lesssim 1$.

6.2. The case $\xi_1 \gg 1$ or $\xi_2 \gg 1$. Let $M_i$ be dyadic numbers and $f_i^{M_i} = \chi(\{\xi_i \sim M_i\}) f$ so that $f_i = \sum M_i f_i^{M_i}$. For simplicity, we drop the superscripts on $f_i^{M_i}$. Then the left hand side of (6.1) and (6.2) are dyadically decomposed as
\[
\sum_{M_0, M_1, M_2} \langle M_2 \rangle^{-s} \iint \iint \frac{f_0(\xi_0, \tau_0) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2)}{\langle \tau_0 - |\xi_0|^2 \rangle^{1-b}\langle \tau_1 + |\xi_1|^2 \rangle^b\langle \tau_2 \pm |\xi_2| \rangle^b} d\xi_1 d\xi_2 d\tau_1 d\tau_2 (6.4)
\]
and
\[
\sum_{M_0, M_1, M_2} \langle M_2 \rangle^{-s} \iint \iint \frac{f_0(\xi_0, \tau_0) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2)}{\langle \tau_0 - |\xi_0|^2 \rangle^{1-b}\langle \tau_1 + |\xi_1|^2 \rangle^b\langle \tau_2 \pm |\xi_2| \rangle^b} d\xi_1 d\xi_2 d\tau_1 d\tau_2. (6.5)
\]
Since $\tau_0 + \tau_1 + \tau_2 = 0$ and $\xi_0 + \xi_1 + \xi_2 = 0$, we also note
\[
\max\{|\tau_0 - |\xi_0|^2|, |\tau_1 + |\xi_1|^2|, |\tau_2 \pm |\xi_2||\} \geq |\tau_0 - |\xi_0|^2| + |\tau_1 + |\xi_1|^2| + |\tau_2 \pm |\xi_2||
\]
\[
= ||\xi_1 + \xi_2|^2 - |\xi_1|^2 \pm |\xi_2||
\]
\[
= 2|\xi_1||\xi_2||B|,
\]
with $\alpha_{12}$ being an angle between $\xi_1$ and $\xi_2$. Now we set
\[
M = \max\{|\tau_0 - |\xi_0|^2|, |\tau_1 + |\xi_1|^2|, |\tau_2 \pm |\xi_2||\}
\]
and then $M \gtrsim M_1 M_2 |B|$. Let $h(\xi) = \pm |\xi|^2$ or $\pm |\xi|$. Then we may bound
\[
\sum_{M_0, M_1, M_2} \langle M_2 \rangle^{-s} \iint \iint \frac{f_0^M(\xi_0, \tau_0) f_1^{M_1}(\xi_1, \tau_1) f_2^{M_2}(\xi_2, \tau_2)}{(M_2)^{1-b'}(\tau + h(\xi))^b(M_1)^{1-b'}(\tau + h(\xi))^b} d\xi_1 d\xi_2 d\tau_1 d\tau_2 (6.6)
\]
at one time for (6.4) and (6.5), where $\sigma$ is a permutation in $\{0, 1, 2\}$. Here we used the fact that $(M_1)^{1-b'}(\tau + h(\xi))^b \lesssim (M_1)^b(\tau + h(\xi))^{1-b'}$ since $1/2 < b < b'$. To bound the above, we will first estimate the dyadic pieces in the sum and then check dyadic summability thereof. Depending on which of the three becomes $M$ in (6.7), we have three cases to examine.
We shall first consider the case when $M = |\tau_0 - |\xi_0|^2|$. We have to bound
\[
\sum_{M_0, M_1, M_2} \langle M_2 \rangle^{-s} \left\langle \frac{\langle M_1 \rangle^{1-b} \langle \tau_1 + |\xi_1|^2 \rangle^b \langle \tau_2 + |\xi_2|^2 \rangle^b}{\langle M \rangle^{1-b} \langle \tau_1 + |\xi_1|^2 \rangle^b \langle \tau_2 + |\xi_2|^2 \rangle^b} \right\rangle d\xi_1 d\xi_2 d\tau_1 d\tau_2.
\] (6.8)

Recall $M \geq M_1 M_2 | B |$. If $|B| \geq 1$ then $M$ is bounded below by $M_1 M_2$. In this case we bound
\[
\sum_{M_0, M_1, M_2} \langle M_2 \rangle^{-s} \left\langle \frac{\langle M_1 M_2 \rangle^{-1-b} \langle \tau_1 + |\xi_1|^2 \rangle^b \langle \tau_2 + |\xi_2|^2 \rangle^b}{\langle M \rangle^{1-b} \langle \tau_1 + |\xi_1|^2 \rangle^b \langle \tau_2 + |\xi_2|^2 \rangle^b} \right\rangle d\xi_1 d\xi_2 d\tau_1 d\tau_2.
\] (6.9)

However, for $|B| \ll 1$, the absence of such a bound again requires us further dyadic decomposition for $|B|$. This finally leads to the following cases:

a. $|B| \geq 1$, $M_0 \lesssim M_1 \sim M_2$,

b. $|B| \geq 1$, $M_2 \ll M_0 \sim M_1$,

c. $|B| \geq 1$, $M_1 \ll M_0 \sim M_2$,

d. $|B| \ll 1$.

Case a. $|B| \geq 1$, $M_0 \lesssim M_1 \sim M_2$. We decompose the functions $f_1$ and $f_2$ as
\[
f_1 = \sum_{n \in \mathbb{Z}} f_1^n \quad \text{where} \quad f_1^n = \chi_{\{n - \frac{1}{2} \leq |\xi_1| < n + \frac{1}{2}\}} f_1 \quad \text{and}
\]
\[
f_2 = \sum_{n \in \mathbb{Z}} f_2^m \quad \text{where} \quad f_2^m = \chi_{\{m - \frac{1}{2} \leq |\xi_2| < m + \frac{1}{2}\}} f_2.
\]

Using these decompositions, the integral in (6.9) is bounded by
\[
\sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \langle n \rangle^{-b} \langle m \rangle^{-b} \iint \int \int f_0(-\xi_1 - \xi_2, |\xi_1|^2 \pm |\xi_2| - n - m - \theta_1 - \theta_2)
\times f_1^n(\xi_1, |\xi_1|^2 + n + \theta_1) f_2^m(\xi_2, |\xi_2| + m + \theta_2) d\xi_1 d\xi_2 d\theta_1 d\theta_2
\] (6.10)

by the change of variables $\tau_1 + |\xi_1|^2 = n + \theta_1$ and $\tau_2 + |\xi_2| = m + \theta_2$. To bound (6.10), we first decompose
\[
f_1^n_{Q_i} = \chi_{\{\xi_1 \in Q_i\}} f_1^n \quad \text{and} \quad f_2^m_{R_j} = \chi_{\{\xi_2 \in R_j\}} f_2^m,
\]

where $Q_i$ and $R_j$ are essentially disjoint $d$-dimensional cubes of side length $M_0(\lesssim M_1, M_2)$ so that $\{\xi_1 \sim M_1\} = \bigcup_i Q_i$ and $\{\xi_2 \sim M_2\} = \bigcup_j R_j$. Then we consider the inner $d\xi_1 d\xi_2$ integral in (6.10), for fixed $\theta_1, n$ and $m$, first change variables $u = -\xi_1 - \xi_2$ and $v = |\xi_1|^2 + |\xi_2| - n - m - \theta_1 - \theta_2$, replacing $\xi_1$ and one component of $\xi_2$, respectively. Let $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,d})$. Computing the determinant of the Jacobian matrix, we next see that
\[
du dv d\xi_{2,1} \cdots d\xi_{2,j-1} d\xi_{2,j+1} \cdots d\xi_{2,d} = \left| \begin{array}{c}
2\xi_{1,j} + \frac{\xi_{2,j}}{|\xi_2|}
\end{array} \right| d\xi_1 d\xi_2.
\] (6.11)
Since we may assume $M_1 \gg 1$ in this case, we have $|\xi_{1,j}| \sim M_1$ for some $j$. Hence, for fixed $j$, the determinant of the Jacobian is nonzero in the region where $|\xi_{1,j}| \sim M_1$. For this reason we divide the integration region in (6.10) into $d$ parts, $|\xi_{1,j}| \sim M_1$ for $j = 1, \ldots, d$. Without loss of generality, we may assume $j = 1$. The inner integral is then rephrased as

$$
\sum_{Q_1, (f_{2,2}, \ldots, f_{2,d}) \in \pi(R_{j(i)})} \int_{(f_{2,2}, \ldots, f_{2,d})} f_0(u, v) H_{Q_1}(u, v, \xi_{2,2}, \ldots, \xi_{2,d}) \left| 2\xi_{1,1} \pm \frac{\xi_{2,1}}{|\xi_{2}|} \right|^{-1} dud\xi_{2,2} \cdots d\xi_{2,d},
$$

where $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is the projection onto the last $d-1$ components and

$$
H_{Q_1}(u, v, \xi_{2,2}, \ldots, \xi_{2,d}) = f_1^{(0)}(\xi_1, -|\xi_1|^2 + n + \theta_1) f_2^{(m)}(\xi_2, \mp|\xi_2| + m + \theta_2).
$$

Here we used the fact that the cube $R_j = R_{j(i)}$ is essentially determined by the cube $Q_i$ since $\xi_0 + \xi_1 + \xi_2 = 0$ and $M_0 \lesssim M_1 \sim M_2$. More precisely, each region $Q_i$ could correspond to up to at most $3^d$ of the $R_j$ regions.

By using Hölder’s inequality twice, the above is bounded by

$$
M_1^{-1} \left\| f_0 \right\|_{L^2_{u,v}} \sum_{Q_i} \left\| \int_{(f_{2,2}, \ldots, f_{2,d}) \in \pi(R_{j(i)})} H_{Q_1}(u, v, \xi_{2,2}, \ldots, \xi_{2,d}) d\xi_{2,2} \cdots d\xi_{2,d} \right\|_{L^2_{u,v}} 
\lesssim M_1^{-1} \left\| f_0 \right\|_{L^2_{u,v}} M_0^{(d-1)/2} \sum_{Q_i} \left\| H_{Q_1}(u, v, \xi_{2,2}, \ldots, \xi_{2,d}) \right\|_{L^2_{u,v,\xi_{2,2}, \cdots, \xi_{2,d}}}.
$$

Changing variables back to $(\xi_1, \xi_2)$ in the $L^2_{u,v,\xi_{2,2}, \cdots, \xi_{2,d}}$ norm, we gain a factor of $M_1^{1/2}$ and observe that

$$
\sum_{Q_i} \left\| H_{Q_1}(u, v, \xi_{2,2}, \ldots, \xi_{2,d}) \right\|_{L^2_{u,v,\xi_{2,2}, \cdots, \xi_{2,d}}} 
= M_1^{1/2} \sum_{Q_i} \left\| f_1^{(0)}(\xi_1, -|\xi_1|^2 + n + \theta_1) \right\|_{L^1_{\xi_1}} \left\| f_2^{(m)}(\xi_2, \mp|\xi_2| + m + \theta_2) \right\|_{L^2_{\xi_2}} 
\lesssim M_1^{1/2} \left\| f_1^{(0)}(\xi_1, -|\xi_1|^2 + n + \theta_1) \right\|_{L^1_{\xi_1}} \left\| f_2^{(m)}(\xi_2, \mp|\xi_2| + m + \theta_2) \right\|_{L^2_{\xi_2}}.
$$

Here we used the Cauchy-Schwarz inequality in $Q_i$ and the disjointness of $Q_i$ for the last inequality. Thus (6.10), as a whole, is bounded by

$$
M_0^{(d-1)/2} M_1^{-1/2} \left\| f_0 \right\|_{L^2_{u,v}} \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \langle n \rangle^{-b} \langle m \rangle^{-b} 
\times \int_{-\frac{1}{2} \leq \theta_1 \leq \frac{1}{2}} \int_{-\frac{1}{2} \leq \theta_2 \leq \frac{1}{2}} \left\| f_1^{(0)}(\xi_1, -|\xi_1|^2 + n + \theta_1) \right\|_{L^1_{\xi_1}} \left\| f_2^{(m)}(\xi_2, \mp|\xi_2| + m + \theta_2) \right\|_{L^2_{\xi_2}} d\theta_1 d\theta_2.
$$

\footnote{We have already handled the case where all $M_i$ are small in the previous subsection.}
Using the Cauchy-Schwarz inequality in \( \theta_1 \) and \( \theta_2 \), and then in \( n \) and \( m \) with the fact that \( b > 1/2 \), the double sum on \( n, m \) here is bounded by

\[
\sum_{n \in \mathbb{Z}} (n)^{-b} \|f_1^n(\xi_1, -|\xi_1|^2 + n + \theta_1)\|_{L^2_{\xi_1, \theta_1}} (-\frac{1}{4} < \theta_1 < \frac{1}{4}) \times \sum_{m \in \mathbb{Z}} (m)^{-b} \|f_2^m(\xi_2, \mp |\xi_2| + m + \theta_2)\|_{L^2_{\xi_2, \theta_2}} (-\frac{1}{4} < \theta_2 < \frac{1}{4}) \leq \left( \sum_{n \in \mathbb{Z}} (n)^{-2b} \right)^{\frac{1}{2}} \left( \sum_{n \in \mathbb{Z}} \|f_1^n(\xi_1, -|\xi_1|^2 + n + \theta_1)\|_{L^2_{\xi_1, \theta_1}}^{2} (-\frac{1}{4} < \theta_1 < \frac{1}{4}) \right)^{\frac{1}{2}} \times \left( \sum_{m \in \mathbb{Z}} (m)^{-2b} \right)^{\frac{1}{2}} \left( \sum_{n \in \mathbb{Z}} \|f_2^m(\xi_2, \mp |\xi_2| + m + \theta_2)\|_{L^2_{\xi_2, \theta_2}}^{2} (-\frac{1}{4} < \theta_2 < \frac{1}{4}) \right)^{\frac{1}{2}} \lesssim \|f_1\|_{L^2_{\xi_1}} \|f_2\|_{L^2_{\xi_2}}.
\]

Therefore, (6.9) for Case \( a \) is bounded by

\[
\sum_{M_0 \leq M_1 \sim M_2} \langle M_2 \rangle^{-s} \langle M_1 M_2 \rangle^{b'-1} M_0^{\frac{d-1}{2}} M_1^{-\frac{1}{2}} \prod_{i=0}^{2} \|f_i\|_{L^2_{\xi_i, \tau_i}}.
\]

Since \( M_1 \gg 1 \) and \( b' < 1 \), the sum here is bounded as

\[
\sum_{M_0 \gg 1} M_0^{\frac{d-1}{2}} \sum_{M_1 \gg M_0} \langle M_1 \rangle^{-s} \langle M_1 \rangle^{2b' - 1} M_1^{-\frac{1}{2}} \lesssim \sum_{M_0 \gg 1} M_0^{\frac{d-1}{2}} \sum_{M_1 \gg M_0} M_1^{-s + 2(b' - 1) - \frac{1}{2}} + \sum_{M_0 \ll 1} M_0^{\frac{d-1}{2}} \sum_{M_1 \gg 1} M_1^{-s - \frac{1}{2}},
\]

which is in turn bounded by

\[
\sum_{M_0 \gg 1} M_0^{2b' - (2s + 6 - d)/4} + \sum_{M_0 \ll 1} M_0^{d-1}
\]

since \( s > -1/2 \). This is finally summable under the assumption \( b' < (2s + 6 - d)/4 \). \( \square \)

**Case b.** \( |B| \geq 1, M_2 \ll M_0 \sim M_1 \). Repeating the change of variables and the ensuing procedure as in Case \( a \), one can bound

\[
\int_{\xi_0 + \xi_1 + \xi_2 = 0} \langle \xi_1 \rangle^{b(\xi_1^2 + |\xi_2|^2 + 2\xi_2 + 2) + 2} d\xi_1 d\xi_2 d\tau_2 \lesssim M_2^{\frac{d+1}{2}} M_1^{-\frac{1}{2}} \prod_{i=0}^{2} \|f_i\|_{L^2_{\xi_i, \tau_i}}
\]

since we may assume \( M_1 \gg 1 \) as well in this case. The restriction \( M_1 \gg 1 \) is necessary to ensure that the Jacobian is nonzero. Furthermore, no decomposition of the integration regions into cubes is required here. The decomposition makes the projection on the integration region in \( \xi_2 \) onto any axis have measure at most \( \min\{M_0, M_1, M_2\} \), but it is automatically true when \( M_2 = \min\{M_0, M_1, M_2\} \). Now, (6.9) for Case \( b \) is bounded by

\[
\sum_{M_2 \ll M_0 \sim M_1} \langle M_2 \rangle^{-s} \langle M_1 M_2 \rangle^{b'-1} M_2^{\frac{d+1}{2}} M_1^{-\frac{1}{2}} \prod_{i=0}^{2} \|f_i\|_{L^2_{\xi_i, \tau_i}}
\]
where the sum is finite as before;

$$
\sum_{M_2} (M_2)^{-s} M_2^{\frac{d-1}{2}} \sum_{M_1 \gg M_2} (M_1 M_2)^{b'-1} M_1^{-\frac{b}{2}} \lesssim \sum_{M_2 \gg 1} (M_2)^{-s+\frac{d+1}{2}+2(b'-1)} \sum_{M_1 \gg M_2} M_1^{-\frac{b}{2}} + \sum_{M_2 \lesssim 1} (M_2)^{-s+2(b'-1)} M_2^{\frac{d-1}{2}} \sum_{M_1 \gg 1} M_1^{-\frac{b}{2}} \\
\lesssim \sum_{M_2 \gg 1} (M_2)^{2b'-\frac{(2s+6)\cdot b}{2}} + \sum_{M_2 \lesssim 1} M_2^{\frac{d-1}{2}} < \infty.
$$

Proof.

Case c. $|B| \gtrsim 1, M_1 \ll M_0 \sim M_2$. For the same reason we can proceed as in Case a when $M_1 \gg 1$. So we shall only consider the case $M_1 \lesssim 1$. Since we cannot guarantee that the Jacobian is nonzero any more, we shall use a similar argument as in Section 6.1. Using Hölder’s inequality, the integral in (6.9) is bounded as

Using Hölder’s inequality, the integral in (6.9) is bounded as

$$
\left\| \int_\mathbb{R} f_0(\xi_0, \tau_0) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2) d\xi_1 d\xi_2 d\tau_1 d\tau_2 \right\|_{L^2_{\xi_0, \tau_0}} \lesssim \left\| \int_\mathbb{R} f_0(\xi_0, \tau_0) \frac{f_1(\xi_1, \tau_1) f_2(-\xi_0 - \xi_1, -\tau_0 - \tau_1)}{(\tau_1 + |\xi_1|^b)(\tau_2 + |\xi_2|^b)} d\xi_1 d\tau_1 \right\|_{L^2_{\xi_0, \tau_0}}.
$$

Here,

$$
\sup_{\xi_0, \tau_0} \int_\mathbb{R} \frac{(\tau_1 - |\xi_1|^b)^{-2b} d\xi_1 d\tau_1}{(\tau_0 + \tau_1 + |\xi_0 + \xi_1|^b)} \lesssim \sup_{\xi_0, \tau_0} \int_\mathbb{R} (\tau_0 + |\xi_1|^b + |\xi_0 + \xi_1|)^{-2b} d\xi_1 \lesssim M_1^d
$$

using Lemma 6.1 with $\alpha = \beta = 2b$. Hence (6.9) is bounded by

$$
\sum_{M_1 \ll M_0 \sim M_2, M_1 \lesssim 1} (M_2)^{-s}(M_1 M_2)^{b'-1} M_1^{\frac{d}{2}} \prod_{i=0}^2 \|f_i\|_{L^2_{\xi_0, \tau_0}}.
$$

Using the simple inequality $\langle ab \rangle \gtrsim |a| \langle b \rangle$ for $a \lesssim 1$, we bound the sum as

$$
\sum_{M_1 \lesssim 1} M_1^{\frac{d}{2}} \sum_{M_2 \gg M_1} (M_2)^{-s}(M_1 M_2)^{b'-1} \lesssim \sum_{M_1 \lesssim 1} M_1^{\frac{d}{2}+b'-1} \sum_{M_2 \gg 1} (M_2)^{-s+b'-1}
$$

since we may assume $M_2 \gg 1$ (otherwise all $|\xi_j| \lesssim 1$, and this case was already addressed in Subsection 6.1). This is summable since $s > b'-1$ and $d \geq 2 > 2b'$. □
**Case d.** $|B| \ll 1$. Recall first from (6.6) that

$$B = \cos \alpha_{12} + \frac{|\xi_2| + 1}{2|\xi_1|}.$$  

Since $B \ll 1$, we then see $|\xi_2| \lesssim |\xi_1| \pm 1$ and hence $M_2 \lesssim M_0 \sim M_1$ in this case.

We may also assume that $|\xi_1| \gg 1$. Otherwise all $|\xi_j| \lesssim 1$, and this case was already addressed in Subsection 6.1. Recalling $M \gtrsim M_1 M_2 |B|$ and decomposing dyadically $|B| \sim v \ll 1$, we need to bound

$$\sum_{M_2 \lesssim M_0 \sim M_1} \frac{(M_2)^{-s}}{(M_1 M_2)^{1-b'}} \sum_{v \ll 1} \frac{1}{v^1 b'} \int \int \int \int \frac{f_0(\xi_0, \tau_0) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2)}{(\tau_1 + |\xi_1|^2)^b |\tau_2 + |\xi_2||} d\xi_1 d\xi_2 d\tau_1 d\tau_2$$

this time instead of (6.9). The integral here boils down to

$$\sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} (n)^{-b} (m)^{-b} \int \int \int \int \frac{f_0(-\xi_1 - \xi_2, |\xi_1|^2 + |\xi_2| - n - m - \theta_1 - \theta_2)}{-\frac{1}{2} \leq \theta_1, \theta_2 < \frac{1}{2}}$$

$$\times f_1^n(\xi_1, -|\xi_1|^2 + n + \theta_1) f_2^m(\xi_2, +|\xi_2| + m + \theta_2) d\xi_1 d\xi_2 d\theta_1 d\theta_2$$

(6.12)

similarly as in Case a (see (6.10)). For a fixed $\xi_1$, the curve $B = 0$ satisfies $|\xi_2|^2 + 2|\xi_1||\xi_2| \cos \alpha_{12} - |\xi_2| = 0$, which is a slightly distorted version of a circle of radius $|\xi_1|$ centered at $-\xi_1$ given by equation $|\xi_2|^2 + 2|\xi_1||\xi_2| \cos \alpha_{12} = 0$. The region of integration in $\xi_2$ is given by a shell centered on the curve $B = 0$, with thickness $\lesssim v M_1$ (see Figure 1). This holds since for a fixed $\xi_1$ and a fixed angle $\alpha_{12}$,

$$|\xi_2| \in \left[2|\xi_1|(v - \cos \alpha_{12}) \pm 1, 2|\xi_1|(2v - \cos \alpha_{12}) + 1\right]$$

which is an interval of length $2v|\xi_1|$. This follows from $|B| \in [v, 2v]$.

Now we decompose the annulus $\{\xi_1 : |\xi_1| \sim M_1\}$ into two parts, a set $S$ where $|\xi_{1,j}| \sim M_1$ for each $j$, and its complement. In two dimensions, this can be described
explicitly by taking
\[
S = \left\{ \xi_1 : |\xi_1| \sim M_1, \arg(\xi_1) \in \left[ \frac{\pi}{8}, \frac{3\pi}{8} \right] \cup \left[ \frac{5\pi}{8}, \frac{7\pi}{8} \right] \cup \left[ \frac{9\pi}{8}, \frac{11\pi}{8} \right] \cup \left[ \frac{13\pi}{8}, \frac{15\pi}{8} \right] \right\}.
\]

Notice that the complement of \( S \) is simply a \( \pi/4 \) radian rotation of \( S \) about the origin. In higher dimensions, the set \( S \) is similarly given; if we describe the space in hyperspherical coordinates, we require all \( d-1 \) angular variables to be bounded away from multiples of \( \pi/2 \)—specifically to fall within the intervals \([n\pi/8, (n+2)\pi/8]\) given above. The complement of \( S \) then consists of \( 2^{d-1} - 1 \) copies of \( S \), each of which can be obtained from \( S \) by a sequence of \( \pi/4 \) radian rotations.

Now we shall discuss the two-dimensional case in detail since there is no fundamental difference in higher dimensions. We perform a rotation so that \((6.12)\) can be written as a sum of two integrals over \( S \) as
\[
\sum_{k=0}^{1} \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} (m)^{-b} (n)^{-b} \int_{\frac{\pi}{8} \leq \theta_1 < \frac{\pi}{4}} f_0(R) d\theta_1 d\theta_2 \times f_1(R) d\theta_1 d\theta_2,
\]
where \( R_y \) denotes a rotation by \( y \) radians. Here we break the \( d\xi_1 d\xi_2 \) integration into two regions: one where for fixed \( \xi_1 \) and \( \xi_2 \), the projection of the integration region onto the \( \xi_2 \) axis is length \( \lesssim vM_1 \), and one where for fixed \( \xi_1 \) and \( \xi_2 \), the projection onto the \( \xi_1 \) axis is length \( \lesssim vM_1 \). We then use the change of variables \( u = -\xi_1 - \xi_2 \) and \( v = |\xi_1|^2 + |\xi_2|^2 - n - m - \theta_1 - \theta_2 \) once again from Case a. In the first region, changing variables to replace \( d\xi_1 d\xi_2 \) with \( dudv \), we have
\[
d\xi_1 d\xi_2 = \left| 2\xi_1,1 \pm \frac{\xi_1,1}{|\xi_2|} \right|^{-1} dudv \sim |\xi_1,1|^{-1} dudv.
\]
where \( |\xi_1,1| \sim M_1 \gg 1 \) which guarantees the nonzero determinant of the Jacobian. Similarly in the other region, we replace \( d\xi_1 d\xi_2 \) with \( dudv \).

Following exactly the same lines as in Case a for each \( k = 0, 1 \), one can bound the above sum by
\[
\left( \frac{vM_1}{M_1} \right)^{\frac{d}{2}} \prod_{i=0}^{2} \| f_i \|_{L^2_2}.
\]
But here, we note that
\[
\sum_{v \ll 1} \frac{1}{v^{1-b'}} \left( \frac{vM_1}{M_1} \right)^{\frac{d}{2}} \lesssim 1
\]
using the fact that \( b' > 1/2 \). In general dimensions, this bound becomes \( M_2^{\frac{d-2}{2}} \). What we wanted is therefore bounded by
\[
\sum_{M_2 \lesssim M_0 \sim M_1} \frac{\langle M_2 \rangle^{-a}}{(M_1 M_2)^{1-b'}} M_2^{\frac{d-2}{2}} \prod_{i=0}^{2} \| f_i \|_{L^2_2}.
\]
Since \( b' < 1 \), the sum here when \( M_2 \gg 1 \) is bounded as
\[
\sum_{M_2 \gg 1} \langle M_2 \rangle^{-s} M_2^{b'-1} M_2^{d-2} \sum_{M_1 \geq M_2} M_1^{b'-1} \lesssim \sum_{M_2 \gg 1} M_2^{b' - \frac{(2s+6)d-2}{2}}
\]
which is summable under the assumption \( b' < (2s+6-d)/4 \). But when \( M_2 \lesssim 1 \), we apply the same procedure as in Case b after reducing (6.8) to
\[
\sum_{M_2 \leq 1} \langle M_2 \rangle^{-s} \iint f_{0}(\xi_0, \tau_0) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2) d\xi_1 d\xi_2 d\tau_1 d\tau_2.
\]
Then we immediately arrive at
\[
\sum_{M_2 \leq 1} \langle M_2 \rangle^{-s} M_2^{\frac{d-1}{2}} M_0^\frac{1}{2} \lesssim \sum_{M_2 \leq 1} M_2^{\frac{d-1}{2}} \sum_{M_0 \geq 1} M_0^{-\frac{1}{2}} < \infty.
\]

Now the case \( M = |\tau_0 - |\xi_0|^2| \) is complete. The other cases where \( M = |\tau_1 + |\xi_1|^2| \) or \( M = |\tau_2 \pm |\xi_2|| \) are handled in the same manner. One can directly apply the proof above again to the former case, merely exchanging the roles of \((\xi_0, \tau_0)\) and \((\xi_1, \tau_1)\). The procedure for the latter one is also similar; one needs to bound
\[
\iint f_{0}(\xi_0, \tau_0) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2) d\xi_0 d\xi_1 d\tau_0 d\tau_1
\]
and proceed just as before to arrive at using the change of variables \( u = -\xi_0 - \xi_1 \) and \( v = -|\xi_0|^2 + |\xi_1|^2 - n - m - \theta_1 - \theta_2 \). The determinant of the Jacobian matrix here may differ but it is harmless to the process. For example,
\[
dudv d\xi_{1,1} \cdots d\xi_{1,j-1} d\xi_{1,j+1} \cdots d\xi_{1,d} = |2\xi_{0,j} + 2\xi_{1,j}| d\xi_0 d\xi_1 = |2\xi_{2,j}| d\xi_0 d\xi_1
\]
replaces (6.11).

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