NOTES ON CONFORMAL PERTURBATION OF HEAT KERNELS

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ABSTRACT. Let \((M, g)\) be a smooth \(n\)-dimensional Riemannian manifold for \(n \geq 3\). Consider the conformal perturbation \(\tilde{g} = hg\) where \(h\) is a smooth bounded positive function on \(M\). Denote by \(\Delta\) the Laplace-Beltrami operator of manifold \((M, \tilde{g})\). In this paper, we derive the upper bounds of the heat kernels for \((-\Delta)^{\sigma}\) with \(0 < \sigma \leq 1\). Moreover, we also investigate the gradient estimates of the heat kernel for \(\Delta\).

1. INTRODUCTION

Let \((M, g)\) be a smooth \(n\)-dimensional Riemannian manifold with metric \(g\) for \(n \geq 2\). Moreover set the volume element \(d\mu\) and the Laplace-Beltrami operator \(\Delta\). Denote by \(d(x, y)\) the geodesic distance for \(x, y \in M\) and \(B(x, r)\) the geodesic ball centered at \(x\) with radius \(r > 0\). In local coordinates the Laplace-Beltrami operator can be expressed as

\[
\Delta = \frac{1}{\sqrt{G}} \partial_i (g^{ij} \sqrt{G} \partial_j),
\]

where \(G = \det(g_{ij})\), \((g^{ij}) = g^{-1}\) and we have used the Einstein summation convention. Denote the heat kernel on \(M\) by \(p_t(x, y)\) which is the integral kernel of semigroup \(e^{t\Delta}\) acting on \(L^2(M, d\mu)\).

On complete manifolds with non-negative Ricci curvature, Li and Yau(19) proved the upper and lower Gaussian bounds as well as the gradient estimates for the heat kernel as follows:

\[C_1 \frac{1}{V(x, \sqrt{t})} \exp \left( -\frac{d^2(x, y)}{c_1 t} \right) \leq p_t(x, y) \leq C_2 \frac{1}{V(x, \sqrt{t})} \exp \left( -\frac{d^2(x, y)}{c_2 t} \right),\]

and

\[|\nabla_x p_t(x, y)| \leq \frac{C}{\sqrt{t}V(x, \sqrt{t})} \exp \left( -\frac{d^2(x, y)}{ct} \right),\]

for all \(t > 0, x, y \in M\). Since then, much effort has been made to establish (LY) and (G) on more general settings. By [9][14], (LY) holds if and only if the manifolds satisfy the parabolic Harnack inequality. Moreover, it has been proved by [13] that the parabolic Harnack inequality is stable under rough isometries and hence (LY) holds under rough isometries. However, when considering such stability results for (G), the situation is more subtle. Recently, in [8] Devyver studied the gradient estimates under assumption on the negative part of the Ricci curvature. In [6], Coulhon et al proved several equivalent characterisations of (G). Among others, the \(L^\infty\)-reverse Hölder inequality for the gradients of harmonic functions is equivalent

2020 Mathematics Subject Classification. Primary 35K08.

Key words and phrases. Conformal Perturbation; Heat Kernel Estimates.
to (G) under some assumption on the underlying spaces. For more results, we refer the readers to [2, 7, 10, 11, 13, 16, 18, 23, 24] and references therein.

The aim of this paper is to study the upper bounds and gradient estimates of the heat kernel under conformal perturbation. This problem has applications in both mathematics and physics. We refer the readers to [1, 4, 8, 11, 15–17, 21] and references therein.

Now consider the conformal perturbation of the metric. Let \( \tilde{g} = h(x)g \) for \( \forall x \in M \) where \( h(x) \) is a positive smooth function on \( M \). Denote by \( d\tilde{\mu} , \bar{\Delta} , \bar{p}_t(x,y) \) the corresponding volume element, Laplace-Beltrami, heat kernel of \( e^{t\bar{\Delta}} \) defined on \( L^2(M,d\tilde{\mu}) \). Note that in local coordinates, we have

\[
\bar{\Delta} = \frac{1}{h(x)^{\frac{n}{2}}} \partial_i(h(x)^{\frac{n}{2}-1}g^{ij}\sqrt{G}\partial_j).
\]

Thus when \( n = 2 \), it follows \( \bar{\Delta} = h^{-1}\Delta \). In [21], Morpurgo used the Dyson series to study heat kernel \( \bar{p}_t(x,y) \) where the Dyson series are determined by \( p_t(x,y) \) as well as \( h(x) \) and hence one can get the upper bounds of \( \bar{p}_t(x,y) \) from that of \( p_t(x,y) \).

More generally, the methods works for the semigroup \( e^{th^{-1}\Delta} \) defined on \( L^2(M,h^2d\mu) \) of compact \( n \)-dimensional Riemannian manifold where \( h \) is a positive smooth function. However, as said by the author, the conformal-geometric significance of the results is only in dimension 2.

As is shown, \( \Delta = h^{-1}\Delta \) does not hold for \( n \geq 3 \). Indeed, the Laplace-Beltrami operator \( \bar{\Delta} \) is related to the weighted Laplace operator on \((M,g)\). Precisely,

\[
\bar{\Delta} = h^{-1}\frac{1}{h^{\frac{n}{2}}-1}\partial_i(h^{\frac{n}{2}-1}g^{ij}\sqrt{G}\partial_j) \triangleq h^{-1}\Delta,
\]

where \( \bar{\Delta} \) is the weighted Laplace operator on the weighted manifold \((M,g,d\bar{\mu})\) with \( d\bar{\mu} = h^{\frac{n}{2}-1}d\mu \). Denote the weighted heat kernel by \( \bar{p}_t(x,y) \) which is the integral kernel of \( e^{t\bar{\Delta}} \) acting on \( L^2(M,d\bar{\mu}) \) where \( d\bar{\mu} = h^{\frac{n}{2}-1}d\mu \). Thus according to the argument in [21], the Dyson series are determined by \( \bar{p}_t(x,y) \) on the weighted manifold \((M,g,d\bar{\mu})\) instead of \( p_t(x,y) \) on \((M,g,d\mu)\).

To proceed, we recall some facts about the weighted manifolds. Note that weighted manifolds have been extensively studied in recent years and found various applications in many areas such as geometric analysis, Markov diffusion theory. See for example [3, 11, 13, 17, 20, 25].

Now we recall some facts about the weighted manifolds. First we call the manifold satisfies the doubling condition, if there exits constant \( C > 0 \) such that

\[
V(x,2r) \leq CV(x,r), \quad \forall x \in M, r > 0,
\]

where \( V(x,r) \) is the volume of ball \( B(x,r) \) with respect to measure \( d\mu \).

Furthermore, we need some notations from [11, 13]. For a manifold \((M,g)\), fix a point \( o \in M \) and set

\[
|x| \triangleq d(x,o), \quad V(s) \triangleq V(o,s).
\]

The manifold is said to have relatively connected annuli (RCA) if there exists a constant \( K > 0 \) such that for all \( x,y \in M \) and large enough \( r \) with \( |x| = |y| = r \), there exists a continuous path \( \gamma : [0,1] \rightarrow M \) with \( \gamma(0) = x, \gamma(1) = y \) whose image is contained in \( B(o,Kr) \setminus B(o,K^{-1}r) \). Then the following results hold.

**Theorem 1.1.** Let \( M \) be a complete \( n \)-dimensional Riemannian manifold for \( n \geq 3 \) with non-negative Ricci curvature satisfying (RCA) with respect to \( o \in M \). Let \( h \)
be a positive bounded smooth function on \( M \). Then there exist constants \( \alpha, C_1 > 0, C_2 > 1 \) such that for \( \|\phi\|_\infty < C_2^{-1} \) and \( \forall t > 0, x, y \in M \),
\[
\tilde{p}_t(x, y) \leq \frac{C_1}{1 - C_2\|\phi\|_\infty} \frac{1}{V(x, \sqrt{at})} \exp\left(-\frac{d^2(x, y)}{at}\right),
\]
where \( \phi(x) = 1 - h(x) \).

Now we give some remarks about the assumptions on \( \phi \).

Remark 1.2. 1. Compared to the previous results [11, 15, 21, 24], our contributions are twofold. First we generalize the results in [21] to noncompact manifolds and hence provide an alternative approach to study the heat kernel under conformal perturbation. Second, the upper bounds here are expressed in terms of geodesic and volumes with respect to the origin metric \( g \).

2. Note that the results can be equivalently stated as
\[
\tilde{p}_t(x, y) \leq \frac{C_1}{1 - C_2\|\phi\|_\infty} \tilde{p}_{\alpha t}(x, y).
\]

The assumption \( \|\phi\|_\infty < C_2^{-1} \) implies that \( h \) is bounded. Next example shows that when \( h \) is unbounded, the results in Theorem 1.1 may not hold. Let \( M \) be the two dimension Euclidean spaces \( \mathbb{R}^2 \) with \( g_{ij} = \delta_{ij} \) and \( h(x) = (1 + |x|^2)^{\frac{1}{2}} \). According to [22, Theorem 1.2], the on diagonal estimates of \( \tilde{p} \) satisfies
\[
c_t t^{-1} (1 + |x|)^4 \leq \tilde{p}_t(x, x), \quad \forall 0 < t \leq c_2 (1 + |x|)^2.
\]

Meanwhile, when \( n = 2 \), \( \tilde{p}_t(x, y) = ct^{-1} e^{-\frac{|x-y|^2}{4t}} \) is the classical heat kernel of \( \mathbb{R}^2 \).

When \( V(x, r) \simeq r^n \) for \( \forall x \in M, r > 0 \), by the Bochners subordination principle, we obtain the following result.

Corollary 1.3. Let \( 0 < \sigma < 1 \). Denote by \( \tilde{p}_t^\sigma(x, y) \) the heat kernel for \( (-\Delta)^\sigma \).
Assume that \( V(x, r) \simeq r^n \) for \( \forall x \in M, r > 0 \). Then under the assumption of Theorem 1.1, there exists \( C > 0 \) such that
\[
\tilde{p}_t^\sigma(x, y) \leq C t^{-\frac{n\sigma}{2}} \wedge \frac{t}{d(x, y)^{n+2\sigma}}, \quad \forall x, y \in M, t > 0.
\]

To get the gradient estimates, we need some further assumptions. Let \( H(x) = h^{\frac{n-2}{2}}(x), W(x) = \frac{\Delta H(x)}{H(x)} \). Set
\[
(A) \quad \left| \frac{\nabla H(x)}{H(x)} \right|, |W(x)|, |\nabla W(x)|, |\Delta W(x)| \leq C, \quad \forall x \in M.
\]

Theorem 1.4. Let \( M \) be a complete \( n \)-dimensional Riemannian manifold for \( n \geq 3 \) with non-negative Ricci curvature satisfying (RCA) with respect to \( o \in M \). Let \( h \) be a positive bounded smooth function on \( M \). Suppose that \( H(x) \) satisfies (A).

Then there exist constants \( \gamma, C_1 > 0, C_2 > 1 \) such that for \( \|\phi\|_\infty < C_2^{-1} \) and \( \forall t > 0, x, y \in M \),
\[
|\nabla_x \tilde{p}_t(x, y)| \leq \frac{C_1}{1 - C_2\|\phi\|_\infty} \frac{1}{1 \wedge \sqrt{t} V(x, \sqrt{t})} \frac{1}{\sqrt{t}} \exp\left(-\frac{d^2(x, y)}{\gamma t}\right),
\]
where \( \phi(x) = 1 - h(x) \).
In this paper, we use the following notations. For positive functions $f(x)$ and $g(x)$ defined on $M$, we say $f \simeq g$ if there exists a constant $C > 0$ such that \( C^{-1} \leq \frac{f(x)}{g(x)} \leq C, \forall x \in M \). Set $f \wedge g(x) = \min\{f(x), g(x)\}$ for $\forall x \in M$ where $f(x), g(x)$ are two functions defined on $M$. The constants $c, C > 0$ may change from line to line, unless otherwise stated.

2. Preliminaries

In this section, we collect several estimates of the heat kernel $\bar{p}_t(x, y)$ on weighted manifolds which will be used in the sequel.

Note first that under the assumption of Theorem 1.1, 1.4, we always have $0 < 1 - C_2^{-1} < h(x) < 1 + C_2^{-1}$ for $\forall x \in M$. Thus by [13, p.861-863] we have $\bar{\mu}(B(x, r)) \simeq V(x, r)$ and hence $(M, d, d\bar{\mu})$ satisfies the doubling property. Moreover, by [11, 13], the following Li-Yau type estimates hold:

\[
\begin{align*}
C_1 \frac{1}{V(x, \sqrt{t})} \exp \left( -\frac{d^2(x, y)}{c_1 t} \right) & \leq \bar{p}_t(x, y) \leq C_2 \frac{1}{V(x, \sqrt{t})} \exp \left( -\frac{d^2(x, y)}{c_2 t} \right) \\
\end{align*}
\]

where $V(x, r) = \bar{\mu}(B(x, r))$. Moreover, by doubling property, we have for $0 < \alpha < \beta$

\[
\bar{p}_{\alpha t}(x, y) \leq C \bar{p}_{\beta t}(x, y), \quad \forall t > 0, x, y \in M,
\]

where $C$ is determined by $\alpha, \beta$. Moreover, we have the following results.

**Proposition 2.1.** Let $0 \leq \theta_0 < \frac{\pi}{2}$. Under the assumption of Theorem 1.1, there exist constants $C, c > 0$ such that

\[
|\bar{p}_z(x, y)| \leq C \bar{p}_{c|z|}(x, y), \quad \forall x, y \in M, |\arg z| \leq \theta_0,
\]

where $C, c$ are determined by $\theta_0$.

**Proof.** According to [5, Corollary 4.4], there exist constants $C, \delta > 0$ such that

\[
|\bar{p}_z(x, y)| \leq C \left( 1 + \Re \frac{d^2(x, y)}{|z|} \right)^\delta \exp \left( -c \Re \frac{d^2(x, y)}{|z|} \right) \frac{1}{(\cos \theta)^\delta}
\]

for all $x, y \in M, \Re z > 0$ where $\theta = \arg z$. Then by the doubling property of $(M, d, d\bar{\mu})$ and the fact $\bar{\mu}(B(x, r)) \simeq V(x, r)$, we have

\[
|\bar{p}_z(x, y)| \leq C \frac{1}{V(x, \sqrt{|z|}/\cos \theta)} \exp \left( -c \Re \frac{d^2(x, y)}{|z|} \right) \frac{1}{(\cos \theta)^\delta}.
\]

Note that, $\Re \frac{d^2}{|z|} = \frac{d^2 \cos \theta}{|z|}$ where $\theta = \arg z$. Finally by the doubling property of $(M, d, d\bar{\mu})$ and (2.2), the result holds. \( \square \)

**Proposition 2.2.** Under the assumption of Theorem 1.4, there exist constants $C, c > 0$ such that

\[
|\nabla_x \bar{p}_t(x, y)| \leq \frac{C}{1 \wedge \sqrt{t}} \bar{p}_{\alpha t}(x, y) \quad \forall t > 0, x, y \in M.
\]
Proof. By Doob transform, we have

\[ \bar{p}_t(x, y) = \frac{1}{H(x)H(y)}p_t^W(x, y) \]

where \( p_t^W(x, y) \) is the integral kernel of \( e^{t\Delta_W} \) defined on \( L^2(M, d\mu) \) and \( \Delta_W = \Delta - W \) with \( W(x) = \frac{\Delta H(x)}{H(x)} \). Therefore under the assumption (A), we obtain by [19, Theorem 3.1],

\[
|\nabla_x \bar{p}_t(x, y)| = \left| -\frac{\nabla_x H(x)}{H(x)} \bar{p}_t(x, y) + \frac{\nabla_x p_t^W(x, y)}{H(x)H(y)} \right| \\
\leq \frac{C}{1 + \sqrt{t}} \bar{p}_t(x, y) + \frac{H^{-1}(x)H^{-1}(y)}{C'} \sqrt{|\partial_t p_t^W(x, y)|p_t^W(x, y)} \\
\leq \frac{C'}{1 + \sqrt{t}} \bar{p}_{ct}(x, y).
\]

In the last inequality, we have used the equality \( \partial_t p_t^W(x, y) = H(x)H(y)\partial_t \bar{p}_t(x, y) \).

The time derivative estimates for \( \bar{p}_t(x, y) \) can be found in [11]. \( \square \)

3. Proof the main results

Now we are ready to prove our main results.

Proof of Theorem 1.1. First we give the Dyson series for \( \bar{p}_t(x, y) \). Now we claim that

\[(3.1) \quad \bar{p}_t(x, y) = \sum_{k=0}^{\infty} \bar{p}_t^{k} \beta_k(x, y), \quad \forall \ t > 0, x, y \in M.\]

Set \( \beta_k^0(x, y) = \bar{p}_t(x, y) \) and denote inductively

\[ \beta_k^k(x, y) = \int_0^1 \int_M \bar{p}_{(1-s)|z|}(x, w) \beta_{k-1}^{k-1}(w, y)\phi(w)d\bar{\mu}(w)dv, \]

where \( k \geq 1, \Re z > 0, \phi(w) = 1 - h(w) \) for all \( w \in M \).

By changing variable \( s = |z| \), we have

\[
|\beta_k^k(x, y)| = \left| \int_M \int_{|z|} e^{i\theta} \bar{p}_{(|z|)}(x, w)\beta_{k}^{k}(w, y)\phi(w)d\bar{\mu}d\theta \right| \\
\leq C \int_{|z|} \int_M \left| \bar{p}_{(|z|)}(x, w)\beta_{k}^{k}(w, y) \right| |\phi|_\infty d\bar{\mu}d\theta \\
\leq C |z| \left| \phi \right|_\infty \bar{\mu}_{|z|}(x, y).
\]

We have used (2.3) and the semigroup properties in the last inequality.
By induction for $k \geq 2$, we have for all $x, y \in M, \Re z > 0, \arg z \leq \frac{\pi}{4}$

$$|\beta^k_z(x,y)| \leq \int_0^{|z|} \int_M |\Bar{p}_{(|z|^{-s})(x,w)}\beta^{k-1}_s(w,y)||\phi||_\infty d\mu ds$$

$$\leq C^{k-1}|\phi|_\infty \frac{1}{(k-1)!} \int_0^{|z|} s^{k-1} ds \int_M \Bar{p}_{\alpha(|z|^{-s})}(x,w)\Bar{p}_{\alpha s}(w,y)d\mu$$

$$\leq C^k|\phi|_\infty \frac{|z|^k}{k!} \Bar{p}_{\alpha|z|}(x,y).$$

By the Cauchy’s integral formula and the above estimates we can give the estimates for $\partial^k_i \beta^k_t(x,y)$. To be precise, for any $t > 0$, consider the circle $\Gamma$ centered at $t$ with radius $t \sin \frac{\pi}{4}$. Thus we obtain by Cauchy’s integral formula

$$|\partial^k_i \beta^k_t(x,y)| \leq \frac{k!}{(t \sin \frac{\pi}{4})^k} \max_{z \in \Gamma} |\beta^k_t(x,y)| \leq 2^k C^k |\phi|_\infty \max_{z \in \Gamma} \frac{|z|^k}{t!} \Bar{p}_{\alpha|z|}(x,y).$$

Since $|z - t| = t \sin \frac{\pi}{4}$, it follows

$$|\partial^k_i \beta^k_t(x,y)| \leq 2^{k+1} C^k |\phi|_\infty \max_{z \in \Gamma} \Bar{p}_{\alpha|z|}(x,y) \leq C_1 (\sqrt{2} C^k |\phi|_\infty)^k \Bar{p}_{\alpha^*t}(x,y),$$

where we have used (2.1) in the last inequality. As a result, the right hand side of (3.1) converges for all $x, y \in M$. Moreover, it indicates

$$\left( \sum_{k=0}^{\infty} \partial^k_i \beta^k_t(x,y) \right) = \frac{C_1}{1 - \sqrt{2} C^k |\phi|_\infty} \Bar{p}_{\alpha^*t}(x,y).$$

We are only left to show the claim (3.1) holds. Indeed, by (3.2) the right hand side of (3.1) converges. Then we have

$$\sum_{k=0}^{\infty} \tilde{\Delta} \partial^k_i \beta^k_t(x,y) = \sum_{k=0}^{\infty} \partial^k_i [ \partial_t \beta^k_t - \phi \beta^{k-1}_t ] = (1 - \phi) \sum_{k=0}^{\infty} \partial^{k+1}_t \beta^k_t.$$

Thus the right hand side of (3.1) satisfies the equation $\partial_t u = \tilde{\Delta} u$. Moreover by (2.1) and (3.2), we have

$$\int_M \left( \sum_{k=0}^{\infty} \partial^k_i \beta^k_t(x,y) \right) h(x)d\mu < \infty,$$

and

$$\left| \sum_{k=0}^{\infty} \partial^k_i \beta^k_t(x,y) \right| \to 0 \quad \text{as} \quad t \to 0,$$

for $d(x, y) \geq \epsilon > 0$. Thus the right hand side of (3.1) tends to the Delta function as $t \to 0$. Then we have proved (3.1) and hence finished the proof. \(\square\)

**Proof of Corollary 1.3.** Under the assumption of Theorem 1.1, we have $h(x) \simeq 1$. Then by the fact $V(x, r) \simeq V(x, r) \simeq r^n$ and (2.1), we conclude

$$C_1 t^{-\frac{n}{2}} \exp \left( -\frac{d^2(x, y)}{c_1 t} \right) \leq \Bar{p}_t(x, y) \leq C_2 t^{-\frac{n}{2}} \exp \left( -\frac{d^2(x, y)}{c_2 t} \right).$$

Thus according to [12 Theorem 2.5], the result follows. \(\square\)
Proof of Theorem 1.4. By (3.1), it is sufficient to consider \( \nabla x \partial_k^t \beta_k^t \). By the above argument, we have
\[
|\nabla x \partial_k^t \beta_k^t(x, y)| = |\partial_k^t \nabla_x \beta_k^t(x, y)| \leq \frac{k!}{(t \sin \frac{\pi}{4})^k} \max_{z \in \Gamma} |\nabla_x \beta_k^t(x, y)|.
\]
Note that by Proposition 2.2, we have
\[
|\nabla_x \beta_k^t(x, y)| \leq \int_0^{\frac{|z|}{\sqrt{t}}} \int_M |\nabla_x \bar{p}_t(|z|-s)e^{\theta}(x, w)||\beta_k^{t-1}(w, y)||\phi||_\infty d\bar{\mu} ds
\]
\[
\leq C \frac{k-1||\phi||_\infty^k}{(k-1)!} \int_0^{\frac{|z|}{\sqrt{t}}} \frac{s^{k-1}}{1 + \sqrt{|z|-s}} ds \int_M \bar{p}_{\alpha_1(|z|-s)}(x, w) \bar{p}_{\alpha_2}(w, y) d\bar{\mu}
\]
\[
\leq C \frac{k-1||\phi||_\infty^k}{(k-1)!} \frac{|z|^k}{1 + \sqrt{|z|}} \bar{p}_{\alpha_3}(x, y),
\]
where \( \alpha = \max\{\alpha_1, \alpha_2\} \). As a result, we have
\[
|\nabla_x \bar{p}_t(x, y)| \leq \sum_{k \geq 0} |\partial_k^t \nabla_x \beta_k^t(x, y)| \leq \sum_{k \geq 0} C_1 (\sqrt{2C||\phi||_\infty})^k \frac{1}{1 + \sqrt{t}} \bar{p}_{\gamma t}(x, y).
\]
Since \( \bar{\nabla} = h^{-1}(x)\nabla \), we have proved the desired results. \( \square \)

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