Algebraic structure of Gravity with Torsion

Otmar MORITSCH*, Manfred SCHWEDA,

and Silvio P. SORELLA†

Institut für Theoretische Physik
Technische Universität Wien
Wiedner Hauptstraße 8-10
A-1040 Wien (Austria)

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Abstract

The BRS transformations for gravity with torsion are discussed by using the Maurer-Cartan horizontality conditions. With the help of an operator $\delta$ which allows to decompose the exterior space-time derivative as a BRS commutator we solve the Wess-Zumino consistency condition corresponding to invariant Lagrangians and anomalies.

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1 Introduction

It is well-known that the search of the invariant Lagrangians and of the anomalies corresponding to a given set of field transformations can be done in a purely algebraic way by solving the BRS consistency condition in the space of the integrated local field polynomials.

This amounts to study the nontrivial solutions of the equation

$$s \Delta = 0 ,$$

where $s$ is the nilpotent BRS operator and $\Delta$ is an integrated local field polynomial. Setting $\Delta = f \mathcal{A}$, condition \((1.1)\) translates into the local equation

$$s \mathcal{A} + d Q = 0 ,$$

where $Q$ is some local polynomial and $d = dx^\mu \partial_\mu$ denotes the exterior space-time derivative which, together with the BRS operator $s$, obeys to:

$$s^2 = d^2 = sd + ds = 0 .$$

(1.3)

$\mathcal{A}$ is said nontrivial if

$$\mathcal{A} \neq s \hat{\mathcal{A}} + d \hat{Q} ,$$

with $\hat{\mathcal{A}}$ and $\hat{Q}$ local polynomials. In this case the integral of $\mathcal{A}$ on space-time, $\int \mathcal{A}$, identifies a cohomology class of the BRS operator $s$ and, according to its ghost number, it corresponds to an invariant Lagrangian (ghost number zero) or to an anomaly (ghost number one).

The local equation \((1.2)\), due to the relations \((1.3)\) and to the algebraic Poincaré Lemma \([1, 2]\), is easily seen to generate a tower of descent equations

$$s Q + d Q^1 = 0$$
$$s Q^1 + d Q^2 = 0$$

.....

$$s Q^{k-1} + d Q^k = 0$$
$$s Q^k = 0 ,$$

(1.5)

with $Q^i$ local field polynomials.

As it is well-known since several years, these equations can be solved by using a transgression procedure based on the so-called Russian formula \([3, 4, 5, 6, 7, 8, 9, 10, 11]\). More recently an alternative way of finding nontrivial solutions of the ladder \((1.3)\) has been proposed by one of the authors and successfully applied to the study of the Yang-Mills gauge anomalies \([12]\). The method is based on the
introduction of an operator $\delta$ which allows to express the exterior derivative $d$ as a BRS commutator, i.e.:

$$d = -[s, \delta].$$

(1.6)

One easily verifies that, once the decomposition (1.6) has been found, successive applications of the operator $\delta$ on the polynomial $Q^k$ which solves the last equation of the tower (1.5) give an explicit nontrivial solution for the higher cocycles $Q^{k-1}, \ldots, Q^{1}, Q, \text{and } A$.

Let us mention that the decomposition (1.6) represents one of the most interesting features of the topological field theories [13, 14] and of the bosonic string in the Beltrami parametrization [15]. We remark also that solving the last equation of the tower (1.5) is a problem of local BRS cohomology instead of a modulo-$d$ one. One sees then that, due to the operator $\delta$, the study of the cohomology of $s$ modulo $d$ is essentially reduced to the study of the local cohomology of $s$ which, in turn, can be systematically analyzed by using the powerful technique of the spectral sequences [16]. Actually, as proven by [17], the solutions obtained by making use of the decomposition (1.6) turn out to be completely equivalent to that based on the Russian formula, i.e. they differ only by trivial cocycles.

The aim of this paper is twofold. First, we prove that the decomposition (1.6) extends to the gravitational case and that it holds also in the presence of torsion. Second, we show that the operator $\delta$ gives an elegant and straightforward way of classifying the cohomology classes of the gravitational BRS operator in any space-time dimension. In particular, we shall see that eq. (1.6) will allow for a cohomological interpretation of the cosmological constant, of the Einstein and the generalized torsion Lagrangians as well as of the Chern-Simons terms and the gravitational anomalies.

The paper is a continuation of a previous work [18], where the decomposition (1.6) was shown to hold in the case of pure Lorentz transformations involving only the spin connection $\omega$ and the Riemann tensor $R$ and without taking into account the explicit presence of the vielbein $e$ and of the torsion $T$.

In the following we will make use of the geometrical formalism introduced by L. Baulieu and J. Thierry-Mieg [4, 6] which allows to reinterpret the BRS transformations as a Maurer-Cartan horizontality condition. In particular, this formalism turns out to be very useful in the case of gravity [4, 6] since it naturally includes the torsion. In addition, it allows to formulate the diffeomorphism transformations as local translations in the tangent space by means of the introduction of the ghost field $\eta^a = \xi^\mu e_\mu^a$ where $\xi^\mu$ denotes the usual diffeomorphism ghost and $e_\mu^a$ is the vielbein$^1$. This step, as we shall see in details, will allow to introduce the decomposition (1.6) in a very simple way. Moreover, the explicit presence of the torsion $T$ and of the translation ghost $\eta^a$ gives us the possibility of introducing an algebraic BRS setup which turns out to be completely different from that of a recent work of Brandt et

$^1$As usual, Latin and Greek indices refer to the tangent space and to the euclidean space-time.
al. [11] where similar techniques has been used.

Let us finish this introduction by remarking that, as done in [18], we always refer to the gravitational fields, i.e. to the vielbein $e$, the spin connection $\omega$, the Riemann tensor $R$, and the torsion $T$ as unquantized classical fields which, when coupled to some matter fields (scalars or fermions), give rise to an effective action whose quantum expansion reduces to the one-loop order.

The paper is organized as follows. In Section 2 we briefly recall the Maurer-Cartan horizontality condition and we derive the BRS transformations for the local Lorentz rotations and diffeomorphisms. In Section 3 we introduce the operator $\delta$ and we show how it can be used to solve the descent equations (1.5). Section 4 is devoted to the study of some explicit examples and Section 5 deals with the geometrical meaning of the decomposition (1.6). Some detailed calculations are given in the final Appendices.

2 The Maurer-Cartan horizontality condition

The aim of this section is to derive the gravitational BRS transformations from the Maurer-Cartan geometrical formalism [4, 6]. For a better understanding of the so-called horizontality condition let us begin by considering the simpler case of a nonabelian Yang-Mills theory.

2.1 The Yang-Mills case

Denoting with $A^a = A^a_{\mu} dx^\mu$ and $c^a$ the one form gauge connection and the zero form ghost field, for the BRS transformations one has:

$$sA^a = dc^a + f^{abc} c^b A^c$$
$$sc^a = \frac{1}{2} f^{abc} c^b c^c$$
$$s^2 = 0,$$

where $f^{abc}$ are the structure constants of the corresponding gauge group. As usual, the adopted graduation is given by the sum of the form degree and of the ghost number. The fields $A^a$ and $c^a$ are both of degree one, their ghost number being respectively zero and one. A $p$-form with ghost number $q$ will be denoted by $\Omega^a_p$, its total graduation being $(p + q)$. The two form field strength $F^a$ is given by

$$F^a = \frac{1}{2} F^a_{\mu\nu} dx^\mu dx^\nu = dA^a + \frac{1}{2} f^{abc} A^b A^c,$$

and

$$dF^a = f^{abc} F^b A^c,$$
is its Bianchi identity. In order to reinterpret the BRS transformations (2.1) as a Maurer-Cartan horizontality condition we introduce the combined gauge-ghost field
\[ \tilde{A}^a = A^a + c^a, \tag{2.4} \]
and the generalized nilpotent differential operator
\[ \tilde{d} = d - s, \quad \tilde{d}^2 = 0. \tag{2.5} \]
Notice that both \( \tilde{A}^a \) and \( \tilde{d} \) have degree one. Let us introduce also the degree-two field strength \( \tilde{F}^a \):
\[ \tilde{F}^a = \tilde{d} \tilde{A}^a + \frac{1}{2} f^{abc} \tilde{A}^b \tilde{A}^c, \tag{2.6} \]
which, from eq.(2.5), obeys the generalized Bianchi identity
\[ \tilde{d} \tilde{F}^a = f^{abc} \tilde{F}^b \tilde{A}^c. \tag{2.7} \]
The Maurer-Cartan horizontality condition \[4, 6\] reads then
\[ \tilde{F}^a = F^a. \tag{2.8} \]
It is very easy now to check that the BRS transformations (2.1) can be obtained from the horizontality condition (2.8) by simply expanding \( \tilde{F}^a \) in terms of the elementary fields \( A^a \) and \( c^a \) and collecting the terms with the same form degree and ghost number. In addition, let us remark also the equality
\[ \tilde{d} \tilde{F}^a - f^{abc} \tilde{F}^b \tilde{A}^c = dF^a - f^{abc} F^b A^c. \tag{2.9} \]

2.2 The gravitational case

In order to generalize the horizontality condition (2.8) to the gravitational case let us first specify the functional space the BRS operator \( s \) acts upon. The latter is chosen to be the space of local polynomials which depend on the one forms \((e^a, \omega^a)\), \( e^a \) and \( \omega^a \) being respectively the vielbein and the spin connection
\[ e^a = e^a_\mu dx^\mu, \]
\[ \omega^a_b = \omega^a_{\mu b} dx^\mu, \tag{2.10} \]
and on the two forms \((T^a, R^a_b)\), \( T^a \) and \( R^a_b \) denoting the torsion and the Riemann tensor
\[ T^a = \frac{1}{2} T^a_{\mu \nu} dx^\mu dx^\nu = de^a + \omega^a_b e^b = De^a, \]
\[ R^a_b = \frac{1}{2} R^a_{b \mu \nu} dx^\mu dx^\nu = d\omega^a_b + \omega^a_c \omega^c_b, \tag{2.11} \]
where
\[ D = d + \omega \] (2.12)
is the covariant derivative. The tangent space indices \((a, b, c, \ldots)\) in eqs. (2.10) and (2.11) are referred to the group \(SO(N)\), \(N\) being the dimension of the euclidean space-time.

Applying the exterior derivative \(d\) to both sides of eq. (2.11) one gets the Bianchi identities
\[ DT^a = dT^a + \omega^a_b T^b = R^a_b \epsilon^b, \]
\[ DR^a_b = dR^a_b + \omega^a_c R^c_b - \omega^c_b R^a_c = 0. \] (2.13)

To write down the gravitational Maurer-Cartan horizontality condition let us introduce, as done in [4, 6], the local Lorentz ghost \(\theta^a_b\) and a further ghost \(\eta^a\) with indices in the tangent space. Both \(\theta^a_b\) and \(\eta^a\) have ghost number one.

As we shall see later, the introduction of the ghost \(\eta^a\) turns out to be quite useful since it allows to express the diffeomorphism transformations as local translations in the flat tangent space, the transition between the two formulations being realized by the relation
\[ \xi^\mu = E^\mu_a \eta^a, \quad \eta^a = \xi^\mu e^a_{\mu}, \] (2.14)
where \(\xi^\mu\) is the usual ghost for the diffeomorphisms and \(E^\mu_a\) denotes the inverse of the vielbein \(e^a_{\mu}\), i.e.
\[ e^a_{\mu} E^\mu_b = \delta^a_b, \]
\[ e^a_{\mu} E^\nu_a = \delta^\nu_{\mu}. \] (2.15)

Proceeding now as in the previous section, one defines the nilpotent differential operator \(\tilde{d}\) of degree one:
\[ \tilde{d} = d - s, \] (2.16)
and the generalized vielbein-ghost field \(\tilde{e}^a\) and the extended spin connection \(\tilde{\omega}^a_b\)
\[ \tilde{e}^a = e^a + \eta^a, \]
\[ \tilde{\omega}^a_b = \omega^a_b + \theta^a_b. \] (2.17)

where, following [3], \(\tilde{\omega}^a_b\) is given by
\[ \tilde{\omega}^a_b = \omega^a_{bm} \tilde{e}^m = \omega^a_b + \omega^a_{bm} \eta^m, \] (2.18)
with the zero form \(\omega^a_{bm}\) defined by the expansion of the one form connection \(\omega^a_{b\mu}\) in terms of the vielbein \(e^a_{\mu}\), i.e.:
\[ \omega^a_{b\mu} = \omega^a_{bm} e^m_{\mu}. \] (2.19)

\[ \underline{2}\]We remark that the zero form \(\omega^a_{bm}\) does not possess any symmetric or antisymmetric property with respect to the lower indices \((bm)\).
As it is well-known, this last formula stems from the fact that the vielbein formalism allows to transform locally the space-time indices of an arbitrary tensor $N_{\mu\nu\rho\sigma\ldots}$ into flat tangent space indices $N_{abcd\ldots}$ by means of the expansion

$$N_{\mu\nu\rho\sigma\ldots} = N_{abcd\ldots} e^a_\mu e^b_\nu e^c_\rho e^d_\sigma\ldots. \hspace{1cm} (2.20)$$

Vice versa

$$N_{abcd\ldots} = N_{\mu\nu\rho\sigma\ldots} E^a_\mu E^b_\nu E^c_\rho E^d_\sigma\ldots. \hspace{1cm} (2.21)$$

According to the definition (2.11), the generalized Riemann tensor and torsion field are given by

$$\tilde{T}^a = \tilde{d}e^a + \tilde{\omega}^a_b e^b = \tilde{D}e^a, \hspace{1cm} \tilde{R}^a_b = \tilde{d}\tilde{\omega}^a_b + \tilde{\omega}^a_c \tilde{\omega}^c_b, \hspace{1cm} (2.22)$$

and are easily seen to obey the generalized Bianchi identities

$$\tilde{D}\tilde{T}^a = \tilde{d}\tilde{T}^a + \tilde{\omega}^a_c \tilde{e}^c = \tilde{R}^a_b e^b, \hspace{1cm} \tilde{D}\tilde{R}^a_b = \tilde{d}\tilde{R}^a_b + \tilde{\omega}^a_c \tilde{R}^c_b - \tilde{\omega}^a_c \tilde{R}^c_b = 0, \hspace{1cm} (2.23)$$

with

$$\tilde{D} = \tilde{d} + \tilde{\omega} \hspace{1cm} (2.24)$$

the generalized covariant derivative.

We are now ready to formulate the Maurer-Cartan equations for gravity. Following [1], these conditions state that $\tilde{e}$ and all its generalized Lorentz covariant exterior differentials can be expanded over $\tilde{e}$ with classical coefficients, i.e.:

$$\tilde{e}^a = \delta^a_b \tilde{e}^b \equiv \text{horizontal}, \hspace{1cm} (2.25)$$

$$\tilde{T}^a(\tilde{e}, \tilde{\omega}) = \frac{1}{2} T^a_{mn}(e, \omega)e^m e^n \equiv \text{horizontal}, \hspace{1cm} (2.26)$$

$$\tilde{R}^a_b(\tilde{\omega}) = \frac{1}{2} R^a_{bmn}(\omega)e^m e^n \equiv \text{horizontal}, \hspace{1cm} (2.27)$$

where, from eq.(2.20), the zero forms $T^a_{mn}$ and $R^a_{bmn}$ are defined by the vielbein expansion of the two forms torsion and Riemann tensor of eq.(2.11),

$$T^a = \frac{1}{2} T^a_{mn} e^m e^n, \hspace{1cm} R^a_b = \frac{1}{2} R^a_{bmn} e^m e^n. \hspace{1cm} (2.28)$$

Notice also that eq.(2.18) is nothing but the horizontality condition for the spin connection expressing the fact that $\tilde{\omega}$ itself can be expanded over $\tilde{e}$.

Eqs.(2.25)-(2.27) define the Maurer-Cartan horizonality conditions for the gravitational case and, when expanded in terms of the elementary fields $(e^a, \omega^a_b, \eta^a, \theta^a_b)$,
give the nilpotent BRS transformations corresponding to the local Lorentz rotations and to the diffeomorphism transformations.

For a better understanding of this point let us discuss in details the horizontality condition (2.26) for the torsion. Making use of eqs. (2.16), (2.17), (2.18) and of the definition (2.22), one verifies that eq. (2.26) gives

\[
de e^a - s e^a + d\eta^a - s\eta^a + \omega^a_b e^b + \theta^a_{\mu} e^b + \omega^a_{bm} e^m e^b + \omega^a_{bm} \eta^m \eta^b + \frac{1}{2} T^a_{mn} e^m e^n + \frac{1}{2} T^a_{mn} \eta^m \eta^n ,
\]

from which, collecting the terms with the same form degree and ghost number, one easily obtains the BRS transformations for the vielbein \( e^a \) and for the ghost \( \eta^a \):

\[
se^a = d\eta^a + \omega^a_b \eta^b + \theta^a_{\mu} e^b + \omega^a_{bm} \eta^m e^b - T^a_{mn} e^m \eta^n ,
\]

\[
s\eta^a = \theta^a_{\mu} \eta^b + \omega^a_{bm} \eta^m \eta^b - \frac{1}{2} T^a_{mn} \eta^m \eta^n .
\]

These equations, when rewritten in terms of the variable \( \xi^\mu \) of eq. (2.14), take the more familiar form

\[
se^a_\mu = \theta^a_{\mu} e^b + \mathcal{L}_\xi e^a_\mu ,
\]

\[
s\xi^\mu = -\xi^\lambda \partial_\lambda \xi^\mu ,
\]

where \( \mathcal{L}_\xi \) denotes the ordinary Lie derivative along the direction \( \xi^\mu \), i.e.

\[
\mathcal{L}_\xi e^a_\mu = -\xi^\lambda \partial_\lambda e^a_\mu - (\partial_\mu \xi^\lambda) e^a_\lambda .
\]

It is apparent now that eq. (2.30) represents the tangent space formulation of the usual BRS transformations corresponding to local Lorentz rotations and diffeomorphisms.

One sees then that the Maurer-Cartan horizontality conditions (2.25)-(2.27) together with eq. (2.23) carry in a very simple and compact way all the informations relative to the gravitational gauge algebra. It is easy indeed to expand eqs. (2.25)-(2.27) in terms of \( e^a \) and \( \eta^a \) and work out the BRS transformations of the remaining fields (\( \omega^a_b, R^a_{bm}, T^a_{mn}, ... \)).

However, in view of the fact that we will use as fundamental variables the zero forms (\( \omega^a_{bm}, R^a_{bmn}, T^a_{mn} \)) rather than the one form spin connection \( \omega^a_b \) and the two forms \( R^a_{b} \) and \( T^a \), let us proceed by introducing the partial derivative \( \partial_a \) with indices in the tangent space. According to the formulas (2.20) and (2.21), the latter is defined by

\[
\partial_a \equiv E^a_\mu \partial_\mu ,
\]

and

\[
\partial_\mu \equiv e^a_\mu \partial_a ,
\]
so that the intrinsic exterior differential $d$ becomes
\[ d = dx^\mu \partial_\mu = e^a \partial_a . \] (2.35)

Let us emphasize that the introduction of the operator $\partial_a$ and the use of the zero forms $(\omega^a_{bm}, R^a_{bmn}, T^a_{mn})$ allows for a complete tangent space formulation of the gravitational gauge algebra. This step, as we shall see later, turns out to be very useful in the analysis of the corresponding BRS cohomology. Moreover, as one can easily understand, the knowledge of the BRS transformations of the zero form sector $(\omega^a_{bm}, R^a_{bmn}, T^a_{mn})$ together with the expansions (2.19), (2.28) and the equation (2.30) completely characterize the transformation law of the forms $(\omega^a_b, R^a_b, T^a)$.

Let us remark however that, contrary to the case of the usual space-time derivative $\partial_\mu$, the operator $\partial_a$ does not commute with the BRS operator due to the explicit presence of the vielbein (see Appendix A for the detailed calculations). One has:
\[ [s, \partial_m] = (\partial_m \eta^k - \theta^k_m - T^k_{mn} \eta^n - \omega^k_{mn} \eta^n + \omega^k_{nm} \eta^n) \partial_k , \] (2.36)
and
\[ [\partial_m, \partial_n] = -(T^k_{mn} + \omega^k_{mn} - \omega^k_{nm}) \partial_k . \] (2.37)

Nevertheless, taking into account the vielbein transformation (2.30), one consistently verifies that
\[ \{s, d\} = 0 , \quad d^2 = 0 . \] (2.38)

### 2.3 BRS transformations and Bianchi identities

Let us finish this chapter by giving, for the convenience of the reader, the BRS transformations and the Bianchi identities which come out from the Maurer-Cartan horizontality conditions (2.25)-(2.27) and from eqs. (2.22), (2.23) for each form sector and ghost number.

- **Form sector two** $(R^a_b, T^a)$

\[
sR^a_b = \theta^a_c R^c_b - \theta^c_b R^a_c + \omega^a_{ck} \eta^k R^c_b - \omega^c_{bk} \eta^k R^a_c \\
+ \omega^a_{c} R^c_{bmn} e^m \eta^n - \omega^c_{b} R^a_{c} R^e_{mn} e^m \eta^n + (dR^a_{bmn}) e^m \eta^n \\
+ R^a_{bkn} T^m \eta^n - R^a_{bkn} \omega^k \eta^m - R^a_{bmn} e^m d\eta^n ,
\]

\[
sT^a = \theta^a_b T^b + \omega^a_{bk} \eta^k T^b - R^a_b \eta^b \\
+ \omega^a_{b} T^b e^m \eta^n - R^a_{bmn} e^b e^m \eta^n + (dT^a_{bmn}) e^m \eta^n \\
- T^a_{bmn} e^m d\eta^n + T^a_{mn} T^m \eta^n - T^a_{khn} \omega^k \eta^m e^m \eta^n . \] (2.39)

For the Bianchi identities one has
\[
dR^a_b + \omega^a_{c} R^c_b - \omega^c_{b} R^a_c = 0 , \\
dT^a + \omega^a_{b} T^b = R^a_b e^b . \] (2.40)
• Form sector one \((\omega^a_b, e^a)\)

\[
\begin{align*}
\omega^a_b &= \frac{d\theta^a_b + \theta^a_c \omega^c_b + \omega^a_c \theta^c_b + (d \omega^a_{bm}) \eta^m + \omega^a_b d \eta^m}{\eta^m} \\
&+ \omega^a \epsilon_{bn} \eta^m + \omega^a_{cm} \eta^m \omega^b_c - R^a_{bmn} e^m \eta^n, \\
\omega^a_b &= \frac{d \eta^a + \omega^a_b \eta^b + \theta^a_b c^b + \omega^a_{bn} \eta^m e^b - T^a_{mn} e^m \eta^n}{\eta^m}. \\
\end{align*}
\]

(2.41)

• Form sector zero, ghost number zero \((\omega^a_{bn}, R^a_{bmn}, T^a_{mn})\)

\[
\begin{align*}
\omega^a_{bn} &= -\partial_m \omega^a_b + \omega^a_m \omega^c_{bm} - \theta^a_b \omega^m_{cm} - \theta^k_m \omega^a_{bk} - \eta^k \partial_k \omega^a_{bm}, \\
R^a_{bmn} &= \omega^a_{cm} R^c_{bmn} - \omega^a_{bn} R^a_{bmn} - \theta^k_{mn} R^a_{bkn} - \partial_k R^a_{bmn}, \\
T^a_{mn} &= \omega^a_{mn} T^a_{mn} - \theta^k_{mn} T^a_{kn} - \partial_k T^a_{mn}.
\end{align*}
\]

(2.42)

The Bianchi identities \(2.40\) are projected on the zero form curvature \(R^a_{bmn}\) and on the zero form torsion \(T^a_{mn}\) to give

\[
\begin{align*}
dR^a_{bmn} &= (\partial_l T^a_{bmn}) e^l, \\
&= (-\omega^a_{cm} R^c_{bmn} - \omega^a_{cn} R^c_{bln} - \omega^a_{cn} R^c_{blm}) \\
&+ \omega^a_{bl} R^a_{clm} - \omega^a_{bm} R^a_{clm} + \omega^a_{bn} R^a_{clm} \\
&+ R^a_{bkn} \omega^k_{lm} - R^a_{bkn} \omega^k_{lm} + R^a_{bkl} \omega^k_{mn} \\
&- \partial_m R^a_{bmn} - \partial_n R^a_{bmn} e^l, \\
\end{align*}
\]

(2.43)

\[
\begin{align*}
dT^a_{mn} &= (\partial_l T^a_{mn}) e^l, \\
&= (R^a_{lmn} + R^a_{mnl} + R^a_{nlm}) \\
&- \omega^a_{bl} T^b_{mn} - \omega^a_{bm} T^b_{nl} - \omega^a_{bn} T^b_{lm} \\
&+ T^a_{km} T^k_{ln} + T^a_{km} T^k_{ln} + T^a_{kl} T^k_{mn} \\
&- \partial_m T^a_{nl} - \partial_n T^a_{lm} e^l. \\
\end{align*}
\]

(2.44)

One has also the equation

\[
\begin{align*}
d\omega^a_{bn} &= (\partial_m \omega^a_{bn}) e^n, \\
&= (-R^a_{bmn} + \omega^a_{cm} \omega^c_{bn} - \omega^a_{cn} \omega^c_{bm}) \\
&+ \omega^a_{bk} T^k_{mn} - \omega^a_{bk} \omega^k_{mn} + \omega^a_{bk} \omega^k_{mn} + \partial_m \omega^a_{bn}) e^n.
\end{align*}
\]

• Form sector zero, ghost number one \((\theta^a_b, \eta^a)\)

\[
\begin{align*}
\theta^a_b &= \theta^a_c \theta^c_b - \eta^k \partial_k \theta^a_b, \\
\eta^a &= \omega^a_{bn} \eta^m \eta^b + \theta^a_b \eta^b - \frac{1}{2} T^a_{mn} \eta^m \eta^n.
\end{align*}
\]

(2.45)
• Algebra between $s$ and $d$

From the above transformations it follows:

\[ s^2 = 0 \ , \quad d^2 = 0 \ , \quad (2.46) \]

and

\[ \{s, d\} = 0 \ . \quad (2.47) \]

3 Decomposition of the exterior derivative

In this section we introduce the decomposition (1.6) and we show how it can be used to solve the ladder (1.5). To this purpose let us introduce the operator $\delta$ defined as

\[ \delta \eta^a = -e^a, \]
\[ \delta \varphi = 0 \quad \text{for} \quad \varphi = (\omega, e, R, T, \theta) \ . \quad (3.1) \]

It is easy to verify that $\delta$ is of degree zero and that, together with the BRS operator $s$, it obeys the following algebraic relations:

\[ [s, \delta] = -d \ , \quad (3.2) \]

and

\[ [d, \delta] = 0 \ . \quad (3.3) \]

One sees from eq.(3.2) that the operator $\delta$ allows to decompose the exterior derivative $d$ as a BRS commutator. This property, as already shown in [12], gives an elegant and simple procedure for solving the equations (1.5).

Let us consider indeed the tower of descent equations which originates from a local field polynomials $\Omega^G_N$ in the variables $(e^a, e^a_{bm}, R^a_{bmn}, T^a_{mn}, \theta^a_{b}, \eta^a)$ and their derivatives with ghost number $G$ and form degree $N$, $N$ being the dimension of the space-time,

\[ s\Omega^G_N + d\Omega^{G+1}_N = 0 \]
\[ s\Omega^{G+1}_{N-1} + d\Omega^{G+2}_{N-2} = 0 \]
\[ \ldots \]
\[ s\Omega^{G+N-1}_1 + d\Omega^{G+N}_0 = 0 \]
\[ s\Omega^{G+N}_0 = 0 \ , \quad (3.4) \]

with $(\Omega^{G+1}_{N-1}, \ldots, \Omega^{G+N-1}_1, \Omega^{G+N}_0)$ local polynomials which, without loss of generality, will be always considered as irreducible elements, i.e. they cannot be expressed as the product of several factorized terms. In particular, the ghost numbers $G = (0, 1)$ correspond respectively to an invariant gravitational Lagrangian and to an anomaly.
Thanks to the operator $\delta$ and to the algebraic relations (3.2)-(3.3), in order to find a solution of the ladder (3.4) it is sufficient to solve only the last equation for the zero form $\Omega_0^{G+N}$. It is easy to check that, once a nontrivial solution for $\Omega_0^{G+N}$ is known, the higher cocycles $\Omega_q^{G+N-q}$, $(q = 1, \ldots, N)$ are obtained by repeated applications of the operator $\delta$ on $\Omega_0^{G+N}$, i.e.

$$\Omega_q^{G+N-q} = \frac{\delta^q}{q!} \Omega_0^{G+N} \ , \ q = 1, \ldots, N \ , \ G = (0,1) \ .$$

(3.5)

Let us emphasize also that solving the last equation of the tower (3.4) is a problem of **local** BRS cohomology instead of a modulo-$d$ one. One sees then that, by means of the decomposition (3.2), the study of the cohomology of $s$ modulo $d$ is reduced to the study of the local cohomology of $s$. It is well-known indeed that, once a particular solution of the descent equations (3.4) has been obtained, i.e. eq.(3.5), the search of the most general solution becomes essentially a problem of local BRS cohomology.

Let us conclude this section by remarking that actually, also if a fully characterization of the local cohomology of the BRS operator in eqs.(2.39)-(2.45) has not yet been obtained [19], it is rather simple to produce some interesting examples. This is the aim of the next chapter.

4 Some examples

In this section we apply the previous algebraic setup, eqs.(3.4)-(3.5), to discuss some explicit examples. In particular we will focus on the cohomological origin of the cosmological constant, of the Einstein and torsion Lagrangians as well as of the Chern-Simons terms and of the gravitational anomalies. The analysis will be carried out for any space-time dimension, i.e. the Lorentz group will be assumed to be $SO(N)$ with $N$ arbitrary.

4.1 The cosmological constant

The simplest local BRS invariant polynomial which one can define is

$$\Omega_0^N = \frac{1}{N!} \varepsilon_{a_1a_2\ldots a_N} \eta^{a_1} \eta^{a_2} \ldots \eta^{a_N} .$$

(4.1)

with $\varepsilon_{a_1a_2\ldots a_N}$ the totally antisymmetric invariant tensor of $SO(N)$. Taking into account that in a $N$-dimensional space-time the product of $(N + 1)$ ghost fields $\eta^a$ automatically vanishes, it is easily checked that $\Omega_0^N$ identifies a cohomology class of the BRS operator, i.e.

$$s\Omega_0^N = 0 \ , \ \Omega_0^N \neq s\tilde{\Omega}_0^{N-1} .$$

(4.2)
The cocycle (4.1) corresponds to the case $G = 0$ (see eqs.(3.4)) and gives rise to the invariant Lagrangian $\Omega_0^N$

$$\Omega_0^N = \frac{\delta^N}{N!} \Omega^N_0 = \frac{(-1)^N}{N!} \epsilon_{a_1a_2,...,a_N} e^{a_1} e^{a_2} ..... e^{a_N},$$  

which is easily recognized to coincide with the $SO(N)$ cosmological constant. One sees thus that the cohomological origin of the cosmological constant (4.3) relies on the cocycles (4.1).

### 4.2 Einstein Lagrangians

In this case, using the zero form curvature $R^{ab}_{mn}$, for the cocycle $\Omega_0^N$ ($N > 2$) one gets

$$\Omega_0^N = \frac{1}{2} \frac{1}{(N-2)!} \epsilon_{a_1a_2,...,a_N} R^{a_1a_2}_{mn} \eta^m \eta^n \eta^{a_3} ..... \eta^{a_N},$$  

to which it corresponds the term

$$\Omega_0^N = \frac{\delta^N}{N!} \Omega_0^N$$

$$= \frac{1}{2} \frac{(-1)^N}{(N-2)!} \epsilon_{a_1a_2,...,a_N} R^{a_1a_2}_{mn} e^m e^n e^{a_3} ..... e^{a_N}$$

$$= \frac{(-1)^N}{(N-2)!} \epsilon_{a_1a_2,...,a_N} R^{a_1a_2} e^{a_3} ..... e^{a_N}. \quad (4.5)$$

Expression (4.5) is nothing but the Einstein Lagrangian for the case of $SO(N)$.

Notice also that for the case of $SO(2)$ the zero form cocycle $\Omega_0^2$

$$\Omega_0^2 = \frac{1}{2} \epsilon_{ab} R^{ab}_{mn} \eta^m \eta^n$$  

turns out to be BRS-exact:

$$\Omega_0^2 = - s(\epsilon_{ab} \omega_m^{ab} \eta^m + \epsilon_{ab} \theta^{ab}) \quad (4.7)$$

As it is well-known, this implies that the two dimensional Einstein Lagrangian

$$\Omega_0^2 = \epsilon_{ab} R^{ab} \quad (4.8)$$

is $d$-exact, i.e.

$$\Omega_0^2 = d(\epsilon_{ab} \omega^{ab}). \quad (4.9)$$
4.3 Generalized curvature Lagrangians

Replacing in the Einstein Lagrangians (4.5) any pair of vielbeins with the two form \( R^{ab} \), we get another set of gravitational Lagrangians containing higher powers of the Riemann tensor.

To give an example, let us consider the zero form cocycle

\[
\Omega^0_{2N} = \frac{1}{(2N)!} \frac{1}{2^N} (\varepsilon_{a_1 a_2 a_3 a_4 \ldots a_{2(N-1)} a_{2N}) R^{a_1 a_2}_{b_1 b_2} R^{a_3 a_4}_{b_3 b_4} \ldots R^{a_{2(N-1)} a_{2N}}{b_{(2N-1)} b_{(2N)}}) \\
\times (\eta^{b_1} \eta^{b_2} \eta^{b_3} \eta^{b_4} \ldots \eta^{b_{(2N-1)}} \eta^{b_{(2N)}}).
\]

(4.10)

Using eq. (3.5), for the corresponding invariant Lagrangian one gets

\[
\Omega^0_{2N} = \frac{\delta^{(2N)}}{(2N)!} \Omega^0_{2N} \\
= \frac{1}{(2N)!} \frac{1}{2^N} (\varepsilon_{a_1 a_2 a_3 a_4 \ldots a_{2(N-1)} a_{2N}) R^{a_1 a_2}_{b_1 b_2} R^{a_3 a_4}_{b_3 b_4} \ldots R^{a_{2(N-1)} a_{2N}}{b_{(2N-1)} b_{(2N)}}) \\
\times (e^{b_1} e^{b_2} e^{b_3} e^{b_4} \ldots e^{b_{(2N-1)}} e^{b_{(2N)}}) \\
= \frac{1}{(2N)!} (\varepsilon_{a_1 a_2 a_3 a_4 \ldots a_{2(N-1)} a_{2N}) R^{a_1 a_2} R^{a_3 a_4} \ldots R^{a_{2(N-1)} a_{2N}}).
\]

(4.11)

4.4 Lagrangians with torsion

It is known that \([4]\), for special values of the space-time dimension \( N \), i.e. \( N = (4M-1) \) with \( M \geq 1 \), there is the possibility of defining nontrivial invariant Lagrangians which explicitly contain the torsion.

Let us begin by considering first the simpler case of \( SO(3) \) (\( M = 1 \)). By making use of the zero form \( T^{a}_{mn} \) for the cocycle \( \Omega^{3}_{0} \) one has\(^3\)

\[
\Omega^{3}_{0} = \frac{1}{2} T^{a}_{mn} \eta^{m} \eta^{n} \eta^{a} ,
\]

(4.12)

from which one gets the three dimensional torsion Lagrangian

\[
\Omega^{3}_{3} = \frac{\delta^{3}}{3!} \Omega^{3}_{0} = -\frac{1}{2} T^{a}_{mn} e^{m} \eta^{n} e^{a} = -T^{a} e_{a} .
\]

(4.13)

Generalizing to the case of \( SO(4M-1) \) with \( (M > 1) \), one finds

\[
\Omega^{4M-1}_{0} = \frac{1}{2^{(2M-1)}} (T^{k}_{m_1 m_2} R^{k}_{a_1 m_3 m_4} R^{a_2}_{m_5 m_6} \ldots R^{a_{(2M-3)}}{m_{(4M-3)} m_{(4M-2)}}) \\
\times (\eta^{m_1} \eta^{m_2} \ldots \eta^{m_{(4M-3)}} \eta^{m_{(4M-2)}} \eta^{n_{(2M-2)}}),
\]

(4.14)

\(^3\)Tangent space indices are rised and lowered with the flat metric \( g_{ab} \), \( \eta_{a} = g_{ab} \eta^{b} \).
which yields the following torsion Lagrangians

\[
\Omega_{4M-1}^0 = -\frac{1}{2(2M-1)}\left( T_{k_{m_1m_2}}^k R_{a_1m_3m_4}^{a_1} R_{a_2m_5m_6}^{a_2} \ldots R_{a_{2M-2}m_{4M-3}m_{4M-2}}^{a_{2M-2}} \right) \\
\times \left( e^{m_1} e^{m_2} \ldots e^{m_{4M-3}} e^{m_{4M-2}} e^{a_{2M-2}} \right) \\
= -T_k R_{a_1}^k R_{a_2}^{a_1} \ldots R_{a_{2M-2}}^{a_{2M-2}} e^{a_{2M-2}} e^{a_{2M-2}}. \tag{4.15}
\]

Let us mention also the possibility of defining invariant torsion terms which are polynomial in \(T_{mn}^a\). These Lagrangians exist in any space-time dimension and are easily obtained from the \(SO(N)\) zero form cocycle

\[
\Omega_0^N = \frac{1}{N!} \varepsilon_{a_1a_2\ldots a_N} \eta^{a_1} \eta^{a_2} \ldots \eta^{a_N} \mathcal{P}(T), \tag{4.16}
\]

with \(\mathcal{P}(T)\) a scalar polynomial in the torsion as, for instance \([20]\) (see also \([21]\) for generalization),

\[
\mathcal{P}(T) = T_{mn}^a T_{mn}^a. \tag{4.17}
\]

The corresponding invariant torsion Lagrangians are given then by

\[
\Omega_0^N = \frac{\delta^N}{N!} \Omega_0^N = \frac{(-1)^N}{N!} \varepsilon_{a_1a_2\ldots a_N} e^{a_1} e^{a_2} \ldots e^{a_N} \mathcal{P}(T). \tag{4.18}
\]

### 4.5 Chern-Simons terms and anomalies

For what concerns the Chern-Simons terms and the Lorentz and diffeomorphism anomalies (see also Appendix B for the so-called first family diffeomorphism cocycles \([9, 22, 24]\) we recall that, as mentioned in the introduction, an algebraic analysis based on the decomposition \((3.2)\) has been recently carried out by \([18]\).

Let us remark, however, that the decomposition found in \([18]\) gives rise to a commutation relation between the operators \(\delta\) and \(d\) which contrary to the present case (see eq.\((3.3)\)) does not vanish. This implies the existence of a further operator \(\mathcal{G}\) of degree one which has to be taken into account in order to solve the ladder \((3.4)\).

Actually, the existence of the operator \(\mathcal{G}\) relies on the fact that the decomposition of the exterior differential \(d\) found in \([18]\) does not take into account the explicit presence of the vielbein \(e^a\) and of the torsion \(T^a\). It holds for a functional space whose basic elements are built only with the spin connection \(\omega^a_b\) and the Riemann tensor \(R^a_{bc}\), this choice being sufficient to characterize all known Lorentz anomalies and related second family diffeomorphism cocycles \([9, 22]\).

It is remarkable then to observe that the algebra between \(s, \delta,\) and \(d\) gets simpler only when the vielbein \(e^a\) and the torsion \(T^a\) are naturally present. Let us emphasize indeed that the particular elementary form of the operator \(\delta\) in eq.\((3.1)\) is due to the
use of the tangent space ghost $\eta^a$ whose introduction requires explicitly the presence of the vielbein $e^a$.

For the sake of clarity and to make contact with the results obtained in [18], let us discuss in details the construction of the $SO(3)$ Chern-Simons term. In this case the tower (3.4) takes the form

\begin{align}
 s\Omega^0_3 + d\Omega^1_2 &= 0 \\
 s\Omega^1_2 + d\Omega^2_1 &= 0 \\
 s\Omega^2_1 + d\Omega^3_0 &= 0 \\
 s\Omega^3_0 &= 0,
\end{align}

(4.19)

where, according to eq.(3.3),

\begin{align}
 \Omega^2_1 &= \delta \Omega^3_0 \\
 \Omega^1_2 &= \frac{\delta^2}{2!}\Omega^3_0 \\
 \Omega^0_3 &= \frac{\delta^3}{3!}\Omega^3_0.
\end{align}

(4.20)

In order to find a solution for $\Omega^3_0$ we use the redefined Lorentz ghost

$$
\hat{\theta}^a_b = \omega^a_{bm}\eta^m + \theta^a_b,
$$

(4.21)

which, from eq.(3.1), transforms as

$$
\delta \hat{\theta}^a_b = -\omega^a_b.
$$

(4.22)

For the cocycle $\Omega^3_0$ one gets then

$$
\Omega^3_0 = \frac{1}{3!}\hat{\theta}^a_b\hat{\theta}^b_c\hat{\theta}^c_a - \frac{1}{4}R^a_{bmn}\eta^m\eta^n\hat{\theta}^b_a,
$$

(4.23)

from which $\Omega^2_1$, $\Omega^1_2$, and $\Omega^0_3$ are computed to be

\begin{align}
 \Omega^2_1 &= -\frac{1}{2}\omega^a_b\hat{\theta}^b_c\hat{\theta}^c_a + \frac{1}{2}R^a_{bma}\eta^m\eta^n\hat{\theta}^b_a + \frac{1}{4}R^a_{bmn}\eta^m\eta^n\omega^b_a, \\
 \Omega^1_2 &= \frac{1}{2}(\omega^a_b\omega^b_c\hat{\theta}^c_a - R^a_b\hat{\theta}^b_a - R^a_{bmn}\eta^m\eta^n\omega^b_a), \\
 \Omega^0_3 &= \frac{1}{2}(R^a_b\omega^b_a - \frac{1}{3}\omega^a_b\omega^b_c\omega^c_a).
\end{align}

(4.24)

In particular, expression (4.20) gives the familiar $SO(3)$ Chern-Simons gravitational term. Finally, let us remark that the cocycle $\Omega^1_2$ of eq.(4.25), when referred to $SO(2)$, reduces to the expression

$$
-\frac{1}{2}(d\omega^a_b)\theta^b_a
$$

(4.27)

which directly gives the two dimensional Lorentz anomaly.
5 The geometrical meaning of the operator $\delta$

Having discussed the role of the operator $\delta$ in finding explicit solutions of the descent equations (3.4), let us turn now to the study of its geometrical meaning. As we shall see, this operator turns out to possess a quite simple geometrical interpretation which will reveal an unexpected and so far unnoticed elementary structure of the ladder (3.4).

Let us begin by observing that all the cocycles $\Omega_p^{G+N-p}$ ($p = 0, ..., N$) entering the descent equations (3.4) are of the same degree (i.e. $(G + N)$), the latter being given by the sum of the ghost number and of the form degree.

We can collect then, following [17], all the $\Omega_p^{G+N-p}$ into a unique cocycle $\hat{\Omega}$ of degree $(G + N)$ defined as

$$\hat{\Omega} = \sum_{p=0}^{N} \Omega_p^{G+N-p}. \quad (5.1)$$

This expression, using eq.(3.5), becomes

$$\hat{\Omega} = \sum_{p=0}^{N} \delta^p \Omega_0^{G+N}, \quad (5.2)$$

where the cocycle $\Omega_0^{G+N}$, according to its form degree, depends only on the set of zero form variables $(\omega^a_{bm}, R^a_{bmn}, T^a_{mn}, \theta^a_b, \eta^a)$ and their tangent space derivatives $\partial_m$. Taking into account that under the action of the operator $\delta$ the form degree and the ghost number are respectively raised and lowered by one unit and that in a space-time of dimension $N$ a $(N + 1)$-form identically vanishes, it follows that eq.(5.2) can be rewritten in a more suggestive way as

$$\hat{\Omega} = e^{\delta} \Omega_0^{G+N}(\eta^a, \theta^a_b, \omega^a_{bm}, R^a_{bmn}, T^a_{mn}). \quad (5.3)$$

Let us make now the following elementary but important remark. As one can see from eq.(3.1), the operator $\delta$ acts as a translation on the ghost $\eta^a$ with an amount given by $(-e^a)$. Therefore $e^{\delta}$ has the simple effect of shifting $\eta^a$ into $(\eta^a - e^a)$. This implies that the cocycle (5.3) takes the form

$$\hat{\Omega} = \Omega_0^{G+N}(\eta^a - e^a, \theta^a_b, \omega^a_{bm}, R^a_{bmn}, T^a_{mn}). \quad (5.4)$$

This formula collects in a very elegant and simple expression the solution of the descent equations (3.4).

In particular, it states the important result that:

To find a nontrivial solution of the ladder (3.4) it is sufficient to replace the variable $\eta^a$ with $(\eta^a - e^a)$ in the zero form cocycle $\Omega_0^{G+N}$ which belongs to the local cohomology of the BRS operator $s$. The expansion of
\[ \Omega_0^{G+N}(\eta^a - e^a, \theta^a, \omega^a_{\ b m}, R^a_{\ b mn}, T^a_{\ mn}) \text{ in powers of the one form vielbein } e^a \text{ yields then all the searched cocycles } \Omega_p^{G+N-p} \.\]

It is a simple exercise to check now that all the invariant Lagrangians and Chern-Simons terms computed in the previous section are indeed recovered by simply expanding the corresponding zero form cocycles \( \Omega_0^{G+N} \) taken as functions of \( (\eta^a - e^a) \).

Let us conclude by remarking that, up to our knowledge, expression (5.4) represents a deeper understanding of the algebraic properties of the gravitational ladder (3.4) and of the role played by the vielbein \( e^a \) and the associated ghost \( \eta^a \).

**Conclusion**

The algebraic structure of gravity with torsion has been analyzed in the context of the Maurer-Cartan horizontality formalism by introducing an operator \( \delta \) which allows to decompose the exterior space-time derivative as a BRS commutator. Such a decomposition gives a simple and elegant way of solving the Wess-Zumino consistency condition corresponding to invariant Lagrangians and anomalies. The same technique can be applied to the study of the gravitational coupling of Yang-Mills gauge theories as well as to the characterization of the Weyl anomalies [19].

**Appendices**

Appendix A is devoted to the computation of some commutators involving the tangent space derivative \( \partial_a \) introduced in Section 2.

In the Appendix B we show how the so-called first family diffeomorphism anomalies can be recovered with the help of the decomposition (3.2).

**A Commutation relations**

In order to find the commutator involving two tangent space derivatives \( \partial_a \), we make use of the fact that the usual space-time derivatives \( \partial_\mu \) have vanishing commutator:

\[ [\partial_\mu, \partial_\nu] = 0 \]  

(A.1)

From

\[ \partial_\mu = e^m_\mu \partial_m \]  

(A.2)
one gets

\[
[\partial_\mu, \partial_\nu] = 0 = \left[ e_\mu^m \partial_m, e_\nu^n \partial_n \right] \\
= e_\mu^m e_\nu^n [\partial_m, \partial_n] + e_\mu^m (\partial_m e_\nu^n) \partial_n - e_\nu^n (\partial_n e_\mu^m) \partial_m \\
= e_\mu^m e_\nu^n [\partial_m, \partial_n] + (\partial_\mu e_\nu^k - \partial_c e_\mu^k) \partial_k \\
= e_\mu^m e_\nu^n [\partial_m, \partial_n] + (T^k_{\mu\nu} - \omega^k_{nm} e_k^m) \partial_k \\
= e_\mu^m e_\nu^n \left\{ [\partial_m, \partial_n] + (T^k_{mn} - \omega^k_{mn} e_k^m) \partial_k \right\},
\]
(A.3)

so that

\[
[\partial_m, \partial_n] = -(T^k_{mn} + \omega^k_{mn} - \omega^k_{nm}) \partial_k.
\]
(A.4)

For the commutator of \(d\) and \(\partial_m\) we get

\[
[d, \partial_m] = [e^n \partial_n, \partial_m] \\
= -(\partial_m e^k) \partial_k - e^n [\partial_m, \partial_n] \\
= -(\partial_m e^k) \partial_k + e^n (T^k_{mn} + \omega^k_{mn} - \omega^k_{nm}) \partial_k.
\]
(A.5)

Analogously, from

\[
[s, \partial_\mu] = 0
\]
(A.6)

one easily finds

\[
[s, \partial_m] = (\partial_m \eta^k - \theta^k_m) \partial_k + \eta^n [\partial_m, \partial_n] \\
= (\partial_m \eta^k - \theta^k_m - T^k_{mm} \eta^n - \omega^k_{mn} \eta^n + \omega^k_{nm} \eta^n) \partial_k.
\]
(A.7)

B First family diffeomorphism anomalies

In this Appendix we give a brief discussion on the first family diffeomorphism anomalies.

As it is well-known, the diffeomorphism cocycles can be splitted in two groups, usually referred as first and second family anomalies. The latters are defined by the following result, valid for any space-time dimension:

Diffeomorphism anomalies \[9, 22\]

On the space of local polynomials in the connection \(\omega\), the vielbein \(e\) and its inverse \(E\), the Riemann tensor \(R\) and the torsion \(T\), the most general diffeomorphism anomaly \(A_{\text{diff}}\) has the form

\[
A_{\text{diff}} = \int d^N x (b_\sigma^\mu \partial_\mu \xi^\sigma + b \partial_\mu \xi^\mu),
\]
(B.1)
where $b^{\mu\nu}_{\sigma}$ is a tensor under linear $GL(N)$-transformations and $b$ is a scalar density which cannot be written as a total derivative, i.e.

$$b = e\mathcal{M}, \quad b \neq \partial_{\mu}b^{\mu}, \quad (B.2)$$

with $\mathcal{M}$ a scalar quantity

$$s\mathcal{M} = -\xi^{\lambda}\partial_{\lambda}\mathcal{M}. \quad (B.3)$$

The variables $\xi^{\mu}$ and $e$ in eqs. (B.1), (B.2) denote respectively the diffeomorphism ghost of eq. (2.14) and the determinant of the vielbein $e^{a}_{\mu}$

$$e = \text{det} e^{a}_{\mu} = \frac{1}{N!} \varepsilon^{a_1a_2...a_N} e^{a_1}_{\mu_1}e^{a_2}_{\mu_2}...e^{a_N}_{\mu_N},$$

$$se = -\partial_{\lambda}(\xi^{\lambda}e). \quad (B.4)$$

Let us remark also that

$$\xi^{\lambda}\partial_{\lambda} = \eta^{m}\partial_{m}, \quad (B.5)$$

so that eq. (B.3) becomes

$$s\mathcal{M} = -\eta^{m}\partial_{m}\mathcal{M}. \quad (B.6)$$

The coefficients $b$ and $b^{\mu\nu}_{\sigma}$ in eq. (B.1) define the so-called first and second family diffeomorphism anomalies and are related, through a nonpolynomial Wess-Zumino action [23], respectively to Weyl and Lorentz anomalies.

Since the second family diffeomorphism anomalies have been already extensively discussed in [18], let us focus here on the first family cocycles. Actually, it is not difficult to verify that the cohomological origin of the first family diffeomorphism anomalies relies on the zero form

$$\Omega^{N+1}_{0} = \frac{1}{N!} \varepsilon^{a_1a_2...a_N} \eta^{a_1}\eta^{a_2}...\eta^{a_N}(\partial_{m}\eta^{m})\mathcal{M}, \quad (B.7)$$

which is easily seen to be BRS invariant.

Indeed, according to the general procedure, i.e. equation (3.4), the corresponding anomaly is given by

$$\Omega^{1}_{N} = \frac{1}{N!} \delta^{N}\Omega^{N+1}_{0}$$

$$= \frac{(-1)^{N}}{N!} \varepsilon^{a_1a_2...a_N} e^{a_1}e^{a_2}...e^{a_N}(\partial_{m}\eta^{m})\mathcal{M}$$

$$+ \frac{(-1)^{N}}{(N-1)!} \varepsilon^{a_1a_2...a_N} \eta^{a_1}e^{a_2}...e^{a_N}(\partial_{m}e^{m})\mathcal{M}. \quad (B.8)$$
Expressing the tangent space ghost $\eta^a$ in terms of the diffeomorphism ghost $\xi^\mu$ (see eq. (2.14)) and using the identity

$$e^1 \cdots e^N = \frac{1}{N!} \varepsilon_{a_1 \cdots a_N} e^{a_1} \cdots e^{a_N}$$

$$= \frac{1}{N!} \varepsilon_{a_1 \cdots a_N} e^{a_1} \cdots e^{a_N} \epsilon^{\mu_1} \cdots \epsilon^{\mu_N} dx^1 \cdots dx^N$$

$$= \frac{1}{N!} \varepsilon_{a_1 \cdots a_N} \epsilon^{a_1} \cdots \epsilon^{a_N} \epsilon^{\mu_1} \cdots \epsilon^{\mu_N} dx^1 \cdots dx^N$$

$$= ed^N x , \quad (B.9)$$

for $\Omega^1_N$ one gets

$$\Omega^1_N = e \mathcal{M}(\partial_\lambda \xi^\lambda) d^N x . \quad (B.10)$$

We have thus recovered the first family anomalies by means of the operator $\delta$. Finally, as already mentioned, let us recall that the cocycle (B.10), also if cohomologically nontrivial in the space of polynomials, can be mapped into a Weyl anomaly [22, 24, 25, 26] by means of a nonpolynomial action. Indeed, using the logarithm of the determinant of the vielbein as the Goldstone boson field [23], one easily checks that

$$\int d^N x \ e \mathcal{M}(\partial_\lambda \xi^\lambda) = -s \int d^N x \ e \mathcal{M} (\log e) . \quad (B.11)$$

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