Experimental Identification of the Second-Order Non-Hermitian Skin Effect with Physics-Graph-Informed Machine Learning

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Topological phases of matter are conventionally characterized by the bulk-boundary correspondence in Hermitian systems. The topological invariant of the bulk in $d$ dimensions corresponds to the number of $(d - 1)$-dimensional boundary states. By extension, higher-order topological insulators reveal a bulk-edge-corner correspondence, such that $n$th order topological phases feature $(d - n)$-dimensional boundary states. The advent of non-Hermitian topological systems sheds new light on the emergence of the non-Hermitian skin effect (NHSE) with an extensive number of boundary modes under open boundary conditions. Still, the higher-order NHSE remains largely unexplored, particularly in the experiment. An unsupervised approach—physics-graph-informed machine learning (PGIML)—to enhance the data mining ability of machine learning with limited domain knowledge is introduced. Through PGIML, the second-order NHSE in a 2D non-Hermitian topoelectrical circuit is experimentally demonstrated. The admittance spectra of the circuit exhibit an extensive number of corner skin modes and extreme sensitivity of the spectral flow to the boundary conditions. The violation of the conventional bulk-boundary correspondence in the second-order NHSE implies that modification of the topological band theory is inevitable in higher dimensional non-Hermitian systems.

1. Introduction

Conceptual theories about topological phases of matter are at the forefront of contemporary research. In Hermitian systems, the guiding principle of topological insulators (TIs) is the bulk-boundary correspondence, stating that the topological invariants of the bulk determine the number of gapless boundary modes.[1–3] With progress in research, higher-order TIs have revealed a novel bulk-edge-corner correspondence, where $n$th order topological phases in $d$ dimensions feature $(d - n)$-dimensional boundary modes.[4–16] Building up on the categories of Hermitian systems, non-conservative systems without Hermiticity reveal a plethora of unconventional physical principles, phenomena, and applications. Among many others, this includes parity-time symmetry,[17–19] exceptional points,[20] exceptional Fermi arcs,[21] sensing,[22,23] and lasing.[24,25] Recently, the concept of non-Hermiticity has been intertwined with topological phases...
been observed in very recent research,[53,54] the unique features of non-Hermitian topological systems,[37–49] experimental explorations are still at an early stage.[50–54] To date, the first-order NHSE has been realized in photonic[55] and in circuitry[50–52] environments, whereas the experimental realization of the higher-order NHSE remains open. Although skin corner modes have been observed in very recent research,[53,54] the unique features of the higher-order NHSE need to be fully demonstrated, both the extensive number of boundary modes under open boundary conditions and the extreme sensitivity of the spectral flow to the boundary conditions. To analyze the spectral flow in higher dimensions, traditional methodologies are challenged by the large-scale data generated. The data size will grow exponentially with the dimension, and additional boundary conditions make it more difficult to analyze the outcome. Machine learning (ML) is a promising way to process large amounts of data.[56–58] The existing approaches, however, are unable to efficiently extract the fundamental physical laws and domain knowledge by teaching ML models the governing physical rules, which can, in turn, provide informative priors, that is, theoretical constraints and inductive model of the governing physical rules, which can, in turn, provide informative priors, that is, theoretical constraints and inductive understanding of the observable features. To this end, physics-informed ML, using informative priors for the phenomenological description of the world, can be leveraged to improve the performance of the learning algorithm.[59]

In this article, we report two significant advances: i) The methodology of physics-graph-informed machine learning (PGIML) is introduced to enforce identification of an unrevealed physical phenomenon by integrating physical principles, graph visualization of features, and ML. The informative priors provided by PGIML enable an analysis that remains robust even in the presence of imperfect data (such as missing values, outliers, and noise) to make accurate and physically consistent predictions of phenomenological parameters. ii) The second-order NHSE, characterized by skin corner modes and the violation of the conventional bulk-boundary correspondence, is realized in a 2D non-Hermitian topo electrical circuit. We demonstrate experimentally the extreme sensitivity of the spectral flow to (fully controlled) boundary conditions (PBCx-PBCy, PBCx-OBCy, OB Rx-PBCy, and OB Cy-OB Cy), where PBC (OB C) represents a periodic (open) boundary condition and x (y) represents direction, and observe corner skin modes under OB Cy-OB Cy as well as edge skin modes under PBCx-OB Cy. Prospectively, the powerful tool of PGIML can be applied more widely to solve digital twin problems,[60–62] thus bridging the physical and digital worlds by linking the flow of data/information between them.[63,64]

2. Results

2.1. Physics-Graph-Informed Machine Learning

The PGIML framework is implemented in the context of a circuit environment. In an electrical circuit, the scattering matrix (S-matrix) relates the voltage of the waves incident to ports to those of the waves reflected from ports (see Section S1, Supporting Information), providing a complete description of the circuit.[65] According to graph theory (network topology), an N-port electrical circuit can be converted into a matrix \( G = (P, S) \) of complex-weighted directed bipartite graphs \( G_{ab} = (P_{ab}, S_{ab}) \) with the matrix \( P \) of positions \( P_{ab} = (a, b) \) and the S-matrix \( S \) of scattering-parameters (S-parameters) \( S_{ab} \) where, \( a, b \in \{1, 2, ..., N\} \) denotes the ports.[66] We define the set of graphs as \( G = \{G_{a,b} | a, b \in \{1, 2, ..., N\}\} \) with the set of positions \( P \) and the set of S-parameters \( S \). To identify the characteristic features of the circuit, especially of a large circuit, cluster analysis can be used to detect graphs with similar properties. Here, a K-means clustering algorithm[67,68] is employed to partition \( G \) into K clusters \( G_k \) based on the value of the S-parameter, where \( G = \bigcup_{k=1}^{K} G_k \).

The axiom of choice[69] states that for every indexed \( G_k \) we can find a representative graph \( \hat{G}_k \) such that \( \hat{G}_k \in G_k \). In a digital twin scenario of simulation and experiment, the set of simulated graphs \( G_{sim} \) is generated to describe the numerical outcome that imitates the set of experimental graphs \( G_{exp} \). As \( G_{sim} \) and \( G_{exp} \) are isomorphic, the subsets \( G_{sim,k} \) and \( G_{exp,k} \) are isomorphic.[70] Therefore, PGIML can be understood in the teacher-student scenario in the sense that the teacher \( G_{sim} \) imparts informative priors \( G_{sim,k} \) to the student \( G_{exp,k} \).

We depict the PGIML framework in Figure 1: i) A lattice model that embeds the unrevealed physical phenomenon is generated and converted into a matrix of graphs \( G \). ii) The simulated S-matrix \( S_{sim} \) of the circuit is constructed and a learning set \( G_{sim} = (P_{sim,k}, S_{sim,k}) \) is accumulated. iii) The set of simulated positions \( P_{sim,k} \) and the set of simulated S-parameters \( S_{sim,k} \) are classified into clusters \( P_{sim,k} \) and \( S_{sim,k} \) using the K-means method (see Section S3, Supporting Information). iv) The graph-to-graph mapping \( \hat{G}_{sim,k} \rightarrow \hat{G}_{exp,k} \) is translated into a sampling mask that mirrors the clustering information. v) The representative experimental S-parameters \( \hat{S}_{exp,k} \) are measured in the circuit. vi) The S-matrix is encoded with the measured features \( \bigcup_{k=1}^{K} \{\hat{S}_{exp,k}\} \) and the reconstructed experimental S-matrix \( \hat{S}_{exp} \) is retrieved. The experimental S-matrix \( \hat{S}_{exp} \) is then given by

\[
\hat{S}_{exp} \sim \sum_{k=1}^{K} \sum_{(a,b) \in P_{sim,k}} E_{ab} S_{exp,k}
\]  

where \( E_{ab} \) is a single-entry matrix (element \( ab \) is one and the other elements are zero).[71] Compared to conventional measurements of \( N^2 \) elements, the PGIML method is \( N^2/K \) times faster, as it
filters out redundancies, especially efficient for circuits that are too complex for a human to process.

### 2.2. Second-Order Non-Hermitian Skin Effect

We are now set up to explore the second-order NHSE, which gives rise to new types of boundary modes as a result of higher-order non-Hermitian topology. In a $L \times L$ lattice model, a first-order TI has $\mathcal{O}(L)$ edge modes with a gapless edge spectrum in the $x$- and $y$-directions. A second-order TI has $\mathcal{O}(1)$ corner modes with a gapped edge spectrum in the $x$- and $y$-directions. The first-order NHSE features extensive $\mathcal{O}(L^2)$ edge skin modes with a gapless complex-valued edge spectrum in the $x$- and $y$-directions. Distinct from the Hermitian limit and the first-order NHSE, the second-order NHSE features $\mathcal{O}(L)$ corner skin modes with a gapless complex-valued edge spectrum in one direction and no edge spectrum in the other direction (see Section S2, Supporting Information). Schematic diagrams of these four situations are shown in Figure 2a. The explicit violation of the conventional bulk-boundary correspondence clearly demonstrates that modification of the topological band theory is inevitable in higher-dimensional non-Hermitian systems.

To realize the second-order NHSE experimentally, we design a topoelectrical circuit that represents a 2D non-Hermitian two-band model. The $10 \times 10$ circuitry lattice is shown in Figure 2b and the unit cell is shown in Figure 2c as photograph and in Figure 2d as scheme. The tight-binding analog of the circuit is shown in Figure 2e with intracell couplings $\gamma_y$, intercell couplings $\lambda_x$ in the $y$-direction, and intercell non-reciprocal couplings $\pm \lambda_y$ in the $x$-direction.

According to Kirchhoff’s laws, any circuit can be described by the block diagonal admittance matrix (circuit Laplacian) $J(\omega) = i\omega C + \frac{1}{j\omega} W$, where $C$ and $W$ are the Laplacian matrices of the capacitance and inverse inductance, respectively. For a given input current of frequency $\omega = 2\pi f$, we obtain the non-reciprocal two-band admittance matrix (see Section S1, Supporting Information)

$$J(k, \omega) = \begin{cases} \frac{i\lambda_x}{(1 + i\lambda_y)^2} & 2C_1 - C_2 + C_1 e^{-ik_x} \\ C_2 + C_1 e^{ik_x} & \frac{i\lambda_x}{(1 + i\lambda_y)^2} \end{cases} = C_1 e^{-ik_y}$$

![Figure 1. PGIML framework.](image)
where two pairs of capacitors and inductors, \((C_1, L_1)\) and \((C_2, L_2)\), with the same resonance frequency \(\omega_0 = 1/\sqrt{L_1C_1} = 1/\sqrt{L_2C_2}\) are used to couple the nodes. This implies

\[
J(k, \omega_0) = i\sqrt{C_1L_1} \left[ -i\lambda_x \sin k_x \sigma_0 + \lambda_x \cos k_x \sigma_0 + \lambda_y \sin k_y \sigma_y \right] \\
+ (\lambda_x + \lambda_y \cos k_y) \sigma_y
\]

(3)

For \(C_1 = 1000 \, \text{pF}, \, C_2 = 330 \, \text{pF}, \, L_1 = 33 \, \mu \text{F}, \) and \(L_2 = 100 \, \mu \text{F}, \) we arrive at \(\lambda_x = 1, \, \lambda_y = 1, \) and \(\gamma_y = 0.33.\) The eigenvalues of \(J(k, \omega_0)\) are given by

\[
j(k, \omega_0) = i\sqrt{C_1/L_1}(\pm \sqrt{\lambda_x^2 \cos^2 k_x + 2\lambda_x \gamma_y \cos k_y + \lambda_y^2 k_y^2 + \gamma_y^2 - 1\lambda_x \sin k_x})
\]

(4)

As the boundary connections can be customized, we can observe phase transitions through differences in the spectral flow, enabling the study of the topological modes at any choice of boundary conditions. The admittance eigenvalues and eigenstates are accessible by an S-parameter measurement using the PGIML framework. We address the circuit for PBCx-PBCY in Figure 3a–e, for PBCx-OBCY in Figure 3f–j, for OBCx-PBCY in Figure 3k–o, and for PBCx-OBCY in Figure 3p–t. Figure 3a,f,k,p shows the ports selected for measuring the representative S-parameters (see Section S4, Supporting Information). According to Figure 3b,g,l,q, the frequency response of \(|\tilde{S}_{\text{exp},s}|\) (outer circle) agrees well with that of \(|\tilde{S}_{\text{sim},s}|\) (inner circle). Figures 3c,h,m,r and 3d,i,n,s show the imaginary and real parts, respectively, of the simulated (top panel) and experimental (bottom panel) admitance spectra as functions of the driving frequency \(f\), weighted by the inverse participation ratio IPR = \(\sum |\Psi_n|^4 / (\sum |\Psi_n|^2)^2\), where \(\Psi_n\) is the \(n\)th eigenmode. A larger IPR corresponds to a more localized mode. For simplicity, the results are given in normalized units (nu) as multiplies of \(\sqrt{L_1/C_1} \, \Omega^{-1}\). Figure 3e,j,o,t shows the simulated and experimental admittance spectra in the complex plane at the resonance frequency \(\omega_0 = (2\times\sqrt{L_1/C_1})^{-1} \approx 0.876 \, \text{MHz}\). The system has trivial topology without NHSE for PBCx-PBCY and OBCx-PBCY, and non-trivial topology with NHSE for PBCx-OBCY and OBCx-OBx. In particular, Figure 3j shows skin edge modes for PBCx-OBCY. We observe in Figure 4a a localized mode distribution at the left/right boundary, in contrast to the delocalized bulk modes. Remarkably, the skin corner modes in Figure 3t form a circle in the complex-energy plane for OBCx-OBx, analytically given by \(j_{\text{sim}} = 0.33\sqrt{C_1/L_1}e^{\theta}, \, \theta \in \)
Figure 3. Comparison of experimental and simulated results for different boundary conditions. a,f,k,p) Representative $S$-parameters are measured between ports connected by red directional circles and lines for $\kappa = 1, 2, \ldots, 100$. b,g,l,q) Frequency response of $|\hat{S}_{\text{sim, } \kappa}|$ (inner circle) and $|\hat{S}_{\text{exp, } \kappa}|$ (outer circle) for each $\kappa$, showing excellent agreement. c,d,h,i,m,n,r,s) Imaginary and real parts of the admittance spectra $j_{\text{sim}}$ (top panel) and $j_{\text{exp}}$ (bottom panel) as functions of the driving frequency $f$, weighted by the IPR. e,j,o,t) Complex admittance spectra for the resonance frequency $f_0 \approx 0.876$ MHz.

[0,2\pi] (see Section S2, Supporting Information). They are localized at the corners while the bulk modes are delocalized, as can be seen in Figure 4b. A non-Bloch 2D winding number $v_{2D} = 1$ characterizes the higher-order NHSE (Experimental Section). For all the $L^2$ eigenmodes, the number of corner skin modes is $2L$ while the number of delocalized bulk modes is $L^2 - 2L$.

3. Conclusion

In times of digital research and measurement, many scientific disciplines produce large amounts of data that by far surpass conventional computational abilities for processing and analyzing. Hence, we develop the PGIML method by integrating physical principles, graph visualization of features, and ML to enforce the identification of an unrevealed physical phenomenon. At the example of a topoelectrical circuit, we embed the physical principles of the second-order NHSE into the circuit, observe the skin corner modes, demonstrate the violation of the conventional bulk-boundary correspondence, and reveal an intriguing interplay between higher-order topology and non-Hermiticity. Our results suggest that the PGIML method provides a paradigm shift in processing and analyzing data, opening new avenues to understanding complex systems in higher dimensions.
4. Experimental Section

Topological invariant: According to point-gap topology, a topological characterization of the NHSE was derived. A non-Hermitian Hamiltonian \( H \) had a point gap at a reference point \( E \in C \) if and only if its complex spectrum does not cross \( E \), that is, \( \det(H - E) \neq 0 \). The topological invariant is given by the winding number

\[
w(E) = \int_{0}^{2\pi} \frac{d\omega}{2\pi i} \log \det(H(k) - E)
\]

where \( H(k) \) is the non-Hermitian Bloch Hamiltonian. The second-order NHSE occurred when \( w(E) \neq 0 \). The non-Hermitian topology of \( H(k) \) could also be understood in terms of the extended Hermitian Hamiltonian

\[
\tilde{H}(k, E) = \begin{pmatrix} 0 & H(k) - E \\ H^\dagger(k) - E^* & 0 \end{pmatrix}
\]

which is topologically nontrivial with a finite energy gap if and only if \( H(k) \) is topologically nontrivial with a point gap at \( E \).

To clarify the topological property of the second-order NHSE, the extended Hermitian admittance Hamiltonian was defined as

\[
\tilde{j}(k, \omega_0) = \begin{pmatrix} 0 & \tilde{j}(k, \omega_0) - j \\ \tilde{j}^\dagger(k, \omega_0) + j^* & 0 \end{pmatrix}
\]

and the unitary transformation \( \tilde{U}(k, \omega_0) = U \tilde{j}(k, \omega_0) U^\dagger \) was performed using

\[
U = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

and obtained

\[
\tilde{j}(k, \omega_0) = \tilde{J}_x(k_x, \omega_0) \otimes \tau_z + \sigma_y \otimes \tilde{J}_y(k_y, \omega_0)
\]

with

\[
\tilde{J}_x(k_x, \omega_0) = -i \sqrt{c_{1x}} C_{1x} \left[ \lambda_x \cos k_x \sigma_x + (\lambda_x \sin k_x - E) \sigma_y \right]
\]

\[
\tilde{J}_y(k_y, \omega_0) = -i \sqrt{c_{1y}} C_{1y} \left[ \lambda_y \cos k_y + \lambda_y \sin k_y \tau_z \right]
\]

Both \( \tilde{J}_x(k_x, \omega_0) \) and \( \tilde{J}_y(k_y, \omega_0) \) had chiral symmetry corresponding to \( \sigma_x \) and \( \tau_z \), respectively. Since chirality and inversion symmetry here commute, the non-Hermitian topology of \( \tilde{j}(k, \omega_0) \) was characterized by the chiral symmetry \( C = \sigma_z \otimes \tau_z \). Thus, the second-order NHSE was characterized by the \( Z \) topological invariant

\[
v_{2D} = w_x w_y
\]

with the winding numbers

\[
w_x(j) = \int_{0}^{2\pi} \frac{d\omega}{2\pi i} \log \det[\tilde{j}_x(k_y, \omega_0) - j]
\]

where \( a = x, y \). Thus, \( w_x = 1 \) as \( E \in (- \lambda_x, \lambda_x) \), and \( w_y = 1 \) as \( \lambda_y / \gamma_y > 1 \). Hence, a nonzero topological invariant \( v_{2D} = 1 \) was obtained if and only if \( E \in (- \lambda_x, \lambda_x) \) and \( \lambda_y / \gamma_y > 1 \). \( v_{2D} \) changed when the edge and bulk modes close the gap, establishing the second-order non-Hermitian topology.

Experiment: Nonreciprocal couplings were realized by voltage feedback operational amplifiers (Texas Instruments, LM6171), which blocked the input current while maintaining the output current. To ensure small linewidths of the circuit Laplacian spectra, high-Q inductors (Murata, Q-factor \( > 40 \) with 5% component variation) were used. Additional elements were added to the circuit to increase the stability of the voltage feedback operational amplifiers, including a 5 \( \Omega \) resistor connected in series at the output and a 2000 \( \Omega \) resistor in shunt with a 100 pF capacitor connecting across the inverting input and output of the voltage feedback operational amplifier. The circuit Laplacian spectra were obtained by measuring the \( S \)-parameters of the circuit at 10 kHz frequency resolution. A vector network analyzer (Tektronix TTH5000) was employed and the \( S \)-matrix was transformed into the circuit Laplacian using the impedance matrix, that is, the inverse of the circuit Laplacian \( J^{-1} = Z_0(S + I)(I - S)^{-1} \), where \( S \) is the identity matrix and \( Z_0 \) is the characteristic impedance. In an \( S \)-parameter measurement between two ports, the other ports were connected with 50 \( \Omega \) load terminators to ensure zero reflection. Note that the impedance matrix obtained by this method was equivalent to that obtained by current probes, while the measurement was simplified dramatically and the experimental stability was improved.

Supporting Information

Supporting Information is available from the Wiley Online Library or from the author.

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graph visualization, machine learning, non-Hermitian circuit, skin effect, topology

Conflict of Interest
The authors declare no conflict of interest.

Author Contributions
C.S., S.L., and R.S. contributed equally to this work. C.S. and S.L. conceived the idea. C.S. performed the theoretical analyses. C.S., S.L., and R.S. designed the circuits and performed the experiments. C.S. and P.H. developed the machine learning method. C.H.L. and R.T. evaluated the experiments. R.S. designed the circuits and performed the experiments. C.S. and P.H. acknowledged the Singapore MOE Tier I grant WBS: R-144-000-435-133. R.T. acknowledges funding from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through Project-ID 258499086-SFB 1170 and the Würzburg-Dresden Cluster of Excellence on Complexity and Topology in Quantum Matter (ct.qmat Project-ID 390858490-EXC 2147). S.Z. acknowledges the Research Grants Council of Hong Kong (AoE/P-701/20 and 17390921).

Data Availability Statement
The data that support the findings of this study are available from the corresponding author upon reasonable request.
