SYMPLECTIC INSTANTON KNOT HOMOLOGY

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Abstract. There have been a number of constructions of Lagrangian Floer homology invariants for 3-manifolds defined in terms of symplectic character varieties arising from Heegaard splittings. With the aim of establishing an Atiyah-Floer counterpart of Kronheimer and Mrowka’s singular instanton homology, we generalize one of these, due to H. Horton, to produce a Lagrangian Floer invariant of a knot or link \( K \subset Y \) in a closed, oriented 3-manifold, which we call \textit{symplectic instanton knot homology} (SIK). We use a multi-pointed Heegaard diagram to parametrize the gluing together of a pair of handlebodies with properly embedded, trivial arcs to form \((Y, K)\). This specifies a pair of Lagrangian embeddings in the traceless SU(2)-character variety of a multiply punctured Heegaard surface, and we show that this has a well-defined Lagrangian Floer homology. Portions of the proof of its invariance are special cases of Wehrheim and Woodward’s results on the quilted Floer homology associated to compositions of so-called elementary tangles, while others generalize their work to certain non-elementary tangles.

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1. Introduction

We associate to a knot or link $K$ in a 3-manifold $Y$ a Lagrangian Floer homology invariant $\text{SIK}(Y, K)$, called symplectic instanton knot homology. By associating to the pair $(Y, K)$ a suitable multi-pointed Heegaard diagram, we obtain

- a smooth symplectic manifold in the form of the traceless $SU(2)$-character variety of a closed, oriented punctured surface;
- a pair of embedded monotone Lagrangian submanifolds parametrized by the Heegaard diagram.

By leveraging technical results of Oh [19] for the monotone case, we proceed to demonstrate that this gives rise to a well-defined Lagrangian Floer homology:

**Theorem A (Floer homology).** Let $Y$ be a connected, closed, oriented 3-manifold and $K \subset Y$ a knot or link. A choice of multi-pointed Heegaard diagram $\mathcal{H}$ compatible with the pair $(Y, K)$ determines a pair of Lagrangian submanifolds $L_\alpha, L_\beta$ in the traceless $SU(2)$-character variety of the punctured Heegaard surface. Their Lagrangian Floer homology

$$\text{SIK}(\mathcal{H}) := \text{HF}(L_\alpha, L_\beta)$$

is well defined with coefficients in $\mathbb{Z}$.

The Heegaard diagram is constructed similarly as for knot and link Floer homology (see [20] [23] [21]), having $k$ pairs of marked points, considered as punctures, on the genus-$g$ Heegaard surface $\Sigma_g$ where it meets $K$ (i.e. $K$ is in $k$-bridge position). However, a free basepoint is also taken and thickened into a triple of marked points. Our symplectic manifold is the traceless $SU(2)$-character variety of $\Sigma_g$ punctured at these marked points, denoted, $\mathcal{R}_{g,2k+3}$, with the extra triple of points serving to exclude reducible representations and thus ensure the character variety’s smoothness. The attaching curves of $\mathcal{H}$ determine two Lagrangian submanifolds $L_\alpha, L_\beta \subset \mathcal{R}_{g,2k+3}$, each of which is an embedding of the traceless character variety of one of the handlebodies in the Heegaard splitting, relative to $k$ properly embedded, trivial arcs where it meets $K$, along with three additional arcs forming a “tripod graph”. We proceed to show that $(L_\alpha, L_\beta)$ is a monotone and relatively spin Lagrangian pair and prove Theorem A via results of Oh regarding such pairs.

It is well known that any two Heegaard diagrams of the kind utilized here are related by a finite sequence of certain operations, called Heegaard moves. We study the impact of each of these moves on the Floer homology [11] using Wehrheim and Woodward’s quilted Floer homology [25] and find that any such move produces canonically isomorphic Floer homologies. As a consequence we conclude that the isomorphism class of the Lagrangian Floer homologies [11] is an invariant of the pair $(Y, K)$:

**Theorem B (Topological invariance).** Let $Y$ and $K$ be as in Theorem A and suppose that $\mathcal{H}$ and $\mathcal{H'}$ are two multi-pointed Heegaard diagrams compatible with the pair $(Y, K)$. Then the induced Lagrangian Floer homologies $\text{SIK}(\mathcal{H})$ and $\text{SIK}(\mathcal{H'})$ are canonically isomorphic as relatively graded abelian groups.
In light of Theorem B, we define the symplectic instanton knot homology as

$$\text{SIK}(Y, K) := \text{the isomorphism class of SIK}(\mathcal{H}),$$

for any choice of Heegaard diagram $\mathcal{H}$ compatible with $(Y, K)$.

1.1. Motivation and antecedents. The principal goal of this construction is to carry out an Atiyah-Floer-type exercise, viz. to form a Lagrangian Floer counterpart of Kronheimer and Mrowka’s singular instanton knot homology \cite{11, 10}. The latter comes in two variants: the undreduced $I^\sharp(Y, K)$ and the reduced $I^\natural(Y, K)$. By taking a certain auxiliary object $\theta$, called a theta graph, alongside $K$ as we construct a compatible Heegaard diagram, we obtain a Floer complex with generators which are identified via holonomy—at least before perturbations on either side—with those of $I^\sharp$. The theta graph can be thought of as analogous to the Hopf link, with an arc connecting its components, employed in defining $I^\sharp$. The computations in Section 5 show that SIK agrees with $I^\sharp$ for the empty link and for the unknot in the 3-sphere (see \cite[Lemma 8.3]{10}), and we hypothesize that the theories agree in general, as predicted by the Atiyah-Floer conjecture:

**Conjecture 1.1.1.** Let $\mathcal{H}$ be a compatible diagram for $(Y, K)$. Then there is an isomorphism $\text{SIK}(\mathcal{H}) \cong I^\sharp(Y, K)$ as $\mathbb{Z}/2$-graded abelian groups.

An Atiyah-Floer-type construction has already been undertaken for the reduced variant $I^\natural$ by Hedden, Herald and Kirk in the form of the their pillowcase homology \cite{6, 7}. Their construction decomposes $(Y, K)$ along a Conway sphere, producing an intersection of immersed Lagrangian curves in the pillowcase. A nice feature of this theory is the provision of explicit holonomy perturbations by which the authors obtain a minimal generating set for singular instanton homology. Such concrete perturbation methods are currently absent from the theory we develop here, and whether similar holonomy perturbations can be applied in the context of SIK is an interesting question for study.

Our construction generalizes one of symplectic instanton homology for closed 3-manifolds due to Horton \cite{8}, denoted SI$(Y)$. Indeed, SIK$(Y, \emptyset) = \text{SI}(Y)$, where $\emptyset$ denotes the empty link. In the first instance, Horton’s invariant is defined via Heegaard diagrams similarly as done here. An appealing aspect of his theory, however, is that it admits an alternative definition as the quilted Floer homology \cite{25} induced by a Cerf decomposition of the 3-manifold $Y$. It is by means of this characterization that Horton is able to demonstrate both a Künneth principle for connected sums and a surgery exact triangle. There is a natural analogue of this in the form of the quilted Floer homology associated to compositions of tangles. A quilted Floer theory for tangles with trivial 3-manifold topology has already been worked out in unpublished work of Wehrheim and Woodward \cite{24}. We make a slight generalization of some of their results in proving the topological invariance of SIK, with the expectation that a further generalization can be carried out, and that the formal properties of Horton’s SI can thereby be extended to SIK. In particular, the example of (Heegaard) knot Floer homology $HF_K$ \cite{20, 23} suggests that the surgery exact triangle for SI may be leveraged to demonstrate that SIK obeys a skein exact sequence. Such a property is
to be expected, given that instant knot homology, along with $HFK$, categorifies the classical Alexander polynomial.

1.2. **Organization of the paper.** In Section 2 we review material necessary for the proof of Theorem A, specifically the subjects of monotone Lagrangian Floer theory (Section 2.1) and of traceless character varieties of punctured surfaces (Section 2.2). The construction of our invariant $SIK$ is carried out in Section 3, wherein Theorem A is fully established. Section 4 addresses topological invariance, proving Theorem B. In 5 we compute $SIK$ for the empty link and for unlinks in the 3-sphere.

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2. **Preliminaries**

2.1. **Monotone Lagrangian Floer theory.** In [3] Floer introduced a homology theory associated to a pair of Lagrangian submanifolds $(L_0, L_1)$ in a symplectic manifold $(M, \omega)$, invariant up to Hamiltonian isotopy of the Lagrangians, called their Lagrangian Floer (or Lagrangian intersection) homology. In this initial formulation, strong constraints are placed on the topology of $M$, but subsequent elaborations of the theory have relaxed these considerably, at the expense of more or less complexity in the definition. A broad treatment of the subject can be found in [4, 5]. An especially convenient situation occurs when $(L_0, L_1)$ is a monotone pair. By work of Oh [15, 16], it is relatively straightforward to show under this condition that the obstructions to a well-defined Lagrangian Floer homology are averted.

The construction below produces a monotone Lagrangian pair, and our definition of symplectic instanton knot homology will rely on an easy corollary (Corollary 2.1.9) of Oh’s results. We recall the key results below, preceded by a brief exposition to establish some notation and important notions such as relative spin structures appearing in the suite. A thorough treatment can be found in Oh’s two volumes [18, 19], the second of which is the principal source for the material of this subsection. Overviews of relevant subjects are given in e.g. [1] and [22].

Let $(M, \omega)$ be a symplectic manifold. For simplicity we will assume that $M$ is closed and connected. Write $\mathcal{J}_\omega$ for the space of $\omega$-compatible almost complex structures on $M$, and let $c_1(M) \in H^2(M; \mathbb{Z})$ be the first Chern class of $M$, defined as the first Chern class $c_1(TM, J)$ of the tangent bundle for any choice of $J \in \mathcal{J}_\omega$, being that the latter space is contractible. Recall that $M$ is called monotone if $[\omega] \in H^2(M; \mathbb{R})$ is proportional to $c_1(M)$, as made precise in the following:
Definition 2.1.1 (Monotone symplectic manifold). A symplectic manifold \((M, \omega)\) is called \((\tau-)\textit{monotone}\) if there exists a positive real number \(\tau\), called the \textit{monotonicity constant}, such that
\[
[\omega] = \tau \cdot c_1(M).
\]

Remark 2.1.2. Both \(c_1(M), [\omega] : H_2(M) \to \mathbb{Z}, \mathbb{R}\) pull back to \(\pi_2(M)\) via the Hurewicz homomorphism, and we use the same notation for these pullbacks. Furthermore, because \(\omega|_L \equiv 0\) by definition, there is a well-defined restriction of \([\omega]\) to \(\pi_2(M,L)\).

Now let \(L \subset M\) be a Lagrangian submanifold. There is an associated map \(\mu_L : \pi_2(M,L) \to \mathbb{Z}\) called the \textit{Maslov homomorphism}, and \(L\) is called monotone when this and \([\omega]\) are proportional; more precisely:

Definition 2.1.3 (Monotone Lagrangian). A Lagrangian submanifold \(L\) in a symplectic manifold \((M, \omega)\) is called \textit{monotone} if there exists a positive real number \(\kappa\) such that
\[
2[\omega]|_{\pi_2(M,L)} = \kappa \cdot \mu_L.
\]

In general, if a Lagrangian \(L \subset M\) is monotone, then \(M\) must itself be monotone and their monotonicity constants equal, i.e. \(\kappa = \tau\) (cf. \[15, Remark 2.3\]). There is a converse for simply connected Lagrangians:

Proposition 2.1.4. If \(M\) is a \(\tau\)-monotone symplectic manifold, then any compact simply connected Lagrangian \(L \subset M\) is monotone with monotonicity constant \(\tau\).

Proof. One may cap off any \(u : (D^2, \partial D^2) \to (M, L)\) with a disk in \(L\) to form a sphere \(v \subset M\). By \[13, Appendix C\] the Maslov index is given in this case in terms of the first Chern class by \(\mu_L(u) = 2c_1(M)(v)\), which by the monotonicity of \(M\) and the fact that \(\omega|_L \equiv 0\) is equal to \(2\tau^{-1}[\omega](u)\), from which the result follows. \(\square\)

Denote the positive generator of \(\text{im}(c_1(M)) \subset \mathbb{Z}\) by \(N_M\), called the \textit{minimal Chern number}, and that of \(\text{im}(\mu_L) \subset \mathbb{Z}\) by \(N_L\), called the \textit{minimal Maslov number}. It is evident from the proof of Proposition 2.1.4 that
\[
N_L = 2N_M
\]
for every compact simply connected Lagrangian \(L \subset M\).

Along with monotonicity, a nice property of Lagrangians, and one which will be possessed by the ones constructed in the next section, is the following:

Definition 2.1.5 (Relatively spin). A tuple \((L_0, \ldots, L_m)\) of Lagrangian submanifolds in a symplectic manifold \(M\) is said to be \textit{relatively spin} if there exists a class \(b \in H^2(M; \mathbb{Z}/2)\), called the \textit{background class}, which restricts to the second Stiefel-Whitney class of each Lagrangian: \(b|_{L_i} = w_2(L_i)\) for \(i = 0, \ldots, m\).

Correspondingly, a \textit{relative spin structure} on \((L_0, \ldots, L_m)\) is a choice of background class \(b \in H^2(M; \mathbb{Z}/2)\) and of spin structures on \(T L_i \oplus E\) where \(E \to M\) is an orientable rank-2 vector bundle with \(w_2(E) = b\).
Remark 2.1.6. In particular, if the Lagrangians of the above definition are all spin manifolds, then they are clearly relatively spin, as one may take $b = 0$ (cf. [19, Sections 15.6.2-3]).

When this additional structure is present, a choice of orientations of the Lagrangians canonically induces a coherent system of orientations upon the various moduli spaces of pseudo-holomorphic disks with boundaries on the Lagrangians. As a result enumerative invariants derived from these moduli spaces may be counted with signs, taking values in $\mathbb{Z}$ rather than $\mathbb{Z}/2$; in particular, the Floer homology of a Lagrangian pair may be taken with integer coefficients.

We next establish one such enumerative invariant of monotone Lagrangians which will be important for our purposes, following [19, Sec. 16.3]. Let $L \subset M$ be a monotone Lagrangian with $N_L \geq 2$, and fix $\beta \in \pi_2(M, L)$ with $\mu_L(\beta) = 2$. Given $J \in \mathcal{J}_\omega$, denote by $\mathcal{M}_1(L, \beta, J)$ the moduli space of $J$-holomorphic disks $u : (D, \partial D) \to (M, L)$ with $[u(D)] = \beta$, modulo conformal automorphisms of the domain fixing $1 \in D \subset \mathbb{C}$.

Consider the evaluation map

$$\text{ev}_\beta : \mathcal{M}_1(L, \beta, J) \to L$$

for any $x \in L$, one may choose $J$ generically such that $\mathcal{M}_1(L, \beta, J)$ is a smooth compact manifold of dimension equal to that of $L$, and $\text{ev}_\beta$ is transverse to $x$. Consequently there is a well-defined count

$$\Phi(L, x) = \sum_{\beta \in \pi_2(M, L)} \# \text{ev}_\beta^{-1}(x),$$

taken in $\mathbb{Z}/2$ in general, and in $\mathbb{Z}$ if $L$ is relatively spin. Moreover, this count depends only on $[x] \in \pi_0(L)$; therefore, when $L$ is connected, we write simply $\Phi(L)$, called the **disk number** of $L$.

**Lemma 2.1.7.** Let $L \subset M$ be a connected monotone Lagrangian and $\phi : M \to M$ a symplectomorphism. Then $\Phi(\phi(L)) = \Phi(L)$.

**Proof.** For a suitably generic choice of almost complex structure $J$, the symplectomorphism $\phi$ induces a diffeomorphism $\mathcal{M}_1(L, \beta, J) \approx \mathcal{M}_1(\phi(L), \phi_*\beta, \phi_*J)$. Moreover, if $L$ is relatively spin and the disk numbers taken in $\mathbb{Z}$, one can choose relative spin structures compatibly, so that the induced diffeomorphism preserves the orientation systems on the moduli spaces. Finally, $\phi_* : \pi_2(M, L) \to \pi_2(M, \phi(L))$ is an isomorphism, hence the sums will be equal. \qed

The Lagrangians which will result from our construction below will turn out to have $N_L = 2$. Oh shows in [19, Sec. 16.4] that, for a pair of monotone Lagrangians $(L_0, L_1)$ with $N_L = 2$, the only potential obstruction to the Floer boundary operator’s squaring to zero is so-called disk bubbling along the Lagrangians. The following theorem provides conditions under which this is ruled out.

**Theorem 2.1.8 ([19, Cor. 16.4.8]).** Let $(L_0, L_1)$ be a pair of connected, compact monotone Lagrangian submanifolds in a symplectic manifold $(M, \omega)$ having minimal
Maslov number $N_L = 2$. For $i = 0, 1$, let $a_i$ be the positive generator of the group \{\omega(\beta) | \beta \in \pi_2(M, L_i)\}, and let $\Phi(L_i)$ be the disk number of $L_i$. Then $\partial \circ \partial = 0$ if and only if $\Phi(L_0) = \Phi(L_1)$ and $a_0 = a_1$.

When the conditions of the foregoing theorem hold, the Lagrangian Floer homology $HF(L_0, L_1)$ is well-defined. As explained in [19] Section 15.6, if $L_0, L_1$ are relatively spin, then the homology may be taken with coefficients in $\mathbb{Z}$.

We will be able to apply Theorem 2.1.8 in our case by virtue of the following special case:

**Corollary 2.1.9.** Let $(L_0, L_1)$ be as in Theorem 2.1.8. Suppose that there exists a symplectomorphism $\phi : (M, \omega) \rightarrow (M, \omega)$ such that $\phi(L_0) = L_1$. Then the Lagrangian Floer homology $HF(L_0, L_1)$ is well defined—with coefficients in $\mathbb{Z}/2$ generally and in $\mathbb{Z}$ if the pair $(L_0, L_1)$ is relatively spin.

**Proof.** Since $\phi$ induces an isomorphism on $\pi_2(M, L_0) \rightarrow \pi_2(M, L_1)$ and preserves $\omega$, we have $a_0 = a_1$. That $\Phi(L_0) = \Phi(L_1)$ follows from Lemma 2.1.7. \qed

2.2. Traceless character varieties and punctured surfaces. Let $M$ be a manifold with or without boundary and $G$ be a compact Lie group. When $M$ is connected, its $G$-character variety is the space $\chi_G(M) := \text{Hom}(\pi_1(M), G)/G$, where $G$ acts on representations by conjugation. When $M$ is disconnected with connected components $M_1, \ldots, M_n$, we define its character variety as the cartesian product $\Pi_{i=1}^n \chi_G(M_i)$. Now let $T \subset M$ be a properly embedded compact submanifold of codimension-2 having connected components $T_1, \ldots, T_m$. For each connected component $T_i$, one can choose a meridian $\mu_i$, the class of which is unique up to conjugacy in $\pi_1(M \setminus T)$. We obtain a subspace of the relative character variety $\chi_G(M \setminus T)$ by specifying conjugacy classes $C_1, \ldots, C_m$ in $G$ and requiring that representations take $[\mu_i]$ into $C_i$, which we write as

$$\chi_{G; C_1, \ldots, C_m}(M, T) := \{[\rho] \in \chi_G(M \setminus T) | \rho([\mu_i]) \in C_i \text{ for } i = 1, \ldots, m\}.$$  

Here we will concern ourselves only with the case where all of the $C_i$ are equal to some fixed conjugacy class $C \subset G$, for which we write just $\chi_{G; C}(M, T)$.

The types of pairs $(M, T)$ with which we will work are punctured surfaces and tangles. By a punctured surface we specifically mean that $M$ is a closed oriented surface, and $T$ is a finite collection of marked points, or punctures, on $M$ oriented by a map $\varepsilon : T \rightarrow \{\pm 1\}$. By a tangle we mean that $M$ is a connected compact oriented 3-manifold (possibly with boundary) and $T$ a properly embedded compact oriented 1-manifold. For our purposes the crucial result regarding the character varieties of such spaces is the following:

**Theorem 2.2.1** ([2, Cor. 4.5]). Let $G$ be a compact Lie group and $C \subset G$ an inversion-invariant conjugacy class. Let $(Y, T)$ be a tangle, as defined above, where $Y$ has the possibly disconnected boundary surface $\partial Y = \hat{S}$. Write $\partial T = p = (p_1, \ldots, p_n) \subset \hat{S}$, so that $(S, p)$ constitutes a punctured surface. The relative character variety $\chi_{G; C}(S, p)$ carries a natural symplectic structure on its smooth stratum.
Let \( r : \chi_{G,C}(Y, T) \to \chi_{G,C}(S, \mathbf{p}) \) be the canonical pullback map (i.e. the map induced by restriction to the boundary). Suppose that \( r \in \chi_{G,C}(Y, T) \) is a regular point of \( r \). Then there exists a neighborhood \( U \subset \chi_{G,C}(Y, T) \) of \( r \) such that the restriction \( r|_U : U \to \chi_{G,C}(S, \mathbf{p}) \) is a Lagrangian embedding. In particular, if all points of \( \chi_{G,C}(Y, T) \) are regular, then \( r \) is a Lagrangian immersion.

In this paper we restrict our attention to the case of \( G = SU(2) \), which we identify in the standard way with the group of unit quaternions:

\[
\begin{pmatrix}
\alpha & \beta \\
-\beta & \bar{\alpha}
\end{pmatrix} = \begin{pmatrix}
a + bi & c - di \\
-c - di & a - bi
\end{pmatrix} \mapsto \alpha + j\beta = a + bi + cj + dk,
\]

\[|\alpha|^2 + |\beta|^2 = a^2 + b^2 + c^2 + d^2 = 1.\]

Here the conjugacy classes are parametrized by the trace \( \text{tr}(Q) \), which is twice the real part of \( Q \in SU(2) \). Our choice of conjugacy class is that of traceless elements:

\[
C_0 := \{ Q \in SU(2) \mid \text{tr}(Q) = 0 \} = \{ \exp \left( \frac{\pi}{2} i \xi \right) \mid \xi \in \mathfrak{su}(2) \}.
\]

For these choices we adopt the following terminology and notation:

**Definition 2.2.2** (Traceless character variety). Let the pair \((M, T)\) be as above and \( C_0 \subset SU(2) \) be the conjugacy class of traceless elements. The **traceless character variety** of \( M \) relative to \( T \) is

\[
\mathcal{R}(M, T) := \chi_{SU(2),C_0}(M, T).
\]

Given a genus-\( g \) punctured surface \((\Sigma_g, \mathbf{p})\) with \( n \) punctures \( \mathbf{p} = (p_1, \ldots, p_n) \), fix meridians \( c_1, \ldots, c_n \) of the punctures with \( c_j \) oriented according to whether \( \varepsilon_j = \varepsilon(p_j) \) agrees with the canonical orientation of a point, and form a standard presentation

\[
\pi_1(\Sigma_g \setminus \mathbf{p}) = \left\{ a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n \mid \prod_{i=1}^{g} [a_i, b_i] \cdot \prod_{j=1}^{n} c_j^{\varepsilon_j} = 1 \right\},
\]

where \([a_i, b_i]\) denotes the commutator. Write \( \rho(a_i) = A_i \), \( \rho(b_i) = B_i \) and \( \rho(c_j) = C_j \) for each \([p] \in \mathcal{R}(\Sigma_g, \mathbf{p})\), and collect these into tuples \( A, B \in G^g \) and \( C \in C_0^n \) in the obvious fashion. In so doing we identify the traceless character variety with

\[
\mathcal{R}_{g,n} := \left\{ (A, B, C) \in G^g \times G^g \times C_0^n \mid \prod_{i=1}^{g} [A_i, B_i] \cdot \prod_{j=1}^{n} C_j^{\varepsilon_j} = 1 \right\} /\text{conj.,}
\]

elements of which we write \([A, B, C]\).

In general, \( \mathcal{R}_{g,n} \) is a stratified symplectic manifold carrying a symplectic form \( \omega_{g,n} \), called the Atiyah-Bott-Goldman form, on its top smooth stratum. Under the forgoing identification, the latter is precisely the symplectic form of Theorem 2.2.1. When \( n \) the number of punctures in \( \Sigma_g \) is odd, the situation is particularly convenient:

**Proposition 2.2.3** (\( \# \) Proposition 2.1). If \( n \) is odd, then \( (\mathcal{R}_{g,n}, \omega_{g,n}) \) is a smooth symplectic manifold of dimension \( 6g - 6 + 2n \). If \( n \) is even, then it has singularities.

Assume henceforth that \( n \) is odd. The following propositions list those symplectic and algebro-topological properties of \( \mathcal{R}_{g,n} \) which are needed here:
Proposition 2.2.4 ([14, Theorem 4.2]). The symplectic manifold \((\mathcal{R}_{g,n}, \omega_{g,n})\) is \(\frac{1}{4}\)-monotone.

Proposition 2.2.5 ([8, Proposition 2.1]). The minimal Chern number of \(\mathcal{R}_{g,n}\) is equal to 1.

Remark 2.2.6. It follows from Proposition 2.2.5 and Equation (5) that the minimal Maslov number \(N_L = 2\) for any compact, simply connected monotone Lagrangian \(L \subset (\mathcal{R}_{g,n}, \omega_{g,n})\).

3. Construction

Let \(Y\) be a connected, closed, oriented 3-manifold and \(K \subset Y\) be an \(\ell\)-component link. In summary, we will associate to the pair \((Y, K)\) a slightly modified balanced multi-pointed Heegaard diagram, the definition of which is reviewed at the outset of the construction below. This diagram determines a symplectic manifold, the traceless SU(2)-character variety of the Heegaard surface relative to its marked points, as well as a pair of Lagrangian submanifolds therein. The latter correspond to the traceless character varieties of the two handlebodies relative to their intersection with \(K\) and an additional datum \(\theta\). We conclude the section by showing that the resulting Lagrangian Floer homology is well defined as an abelian group with integer coefficients.

3.1. Compatible Heegaard diagrams.

Recall that a \((balanced)\) \((2k + m)\)-pointed Heegaard diagram is a tuple \(\mathcal{H} = (\Sigma_g, \alpha, \beta; w, z, q)\) satisfying the following:

(i) \(\Sigma_g\) is a closed, oriented surface of genus \(g\).
(ii) \(\alpha = (\alpha_1, \ldots, \alpha_{g+k+m-1})\) and \(\beta = (\beta_1, \ldots, \beta_{g+k+m-1})\) are each a collection of disjoint simple closed curves in \(\Sigma_g\), and each spans a \(g\)-dimensional lattice in \(H_1(\Sigma_g; \mathbb{Z})\).
(iii) \(w = (w_1, \ldots, w_k), z = (z_1, \ldots, z_k)\) and \(q = (q_1, \ldots, q_m)\) are mutually disjoint tuples of distinct points on \(\Sigma_g\). The points of \(q\) are called free basepoints.
(iv) For \(i = 1, \ldots, k + m\), let \(A_i\) denote the connected components of \(\Sigma_g \setminus \alpha\) and \(B_i\) those of \(\Sigma_g \setminus \beta\). Then for some permutation \(\sigma\) of \(\{1, \ldots, k\}\), we have \(w_i \in A_i \cap B_{\sigma(i)}\) and \(z_i \in A_i \cap B_{\sigma(i)}\) for \(i = 1, \ldots, k\), and \(q_j \in A_j \cap B_j\) for \(j = 1, \ldots, m\).

(This definition can be found e.g. in [12, Section 3.1] with different notation around free basepoints.) We will construct such a diagram for \((Y, K)\), with \(k \geq \ell\) the relative bridge number of \(K\) with respect to \(\Sigma_g\). An initial choice of a single free basepoint will be thickened into a triple, in order to arrive at an odd number of punctures.

In the following let \(\text{Crit}_i(f)\) denote the set of critical points of a Morse function \(f\) having Morse index \(i \in \mathbb{Z}_{\geq 0}\). To construct the desired Heegaard diagram, begin by fixing a Morse function \(f_K : K \to \mathbb{R}\) with \(\text{Crit}_0(f_K) = \{a_1, \ldots, a_k\}\) and \(\text{Crit}_1(f_K) = \{b_1, \ldots, b_k\}\) for \(k \geq \ell\). Extend \(f_K\) to a self-indexing Morse function \(f_{(Y,K)} : Y \to \mathbb{R}\) so that, for some pair of points \(a_0, b_0 \in Y \setminus K\),

(i) \(\text{Crit}_0(f_{(Y,K)}) = \text{Crit}_0(f_K) \cup \{a_0\}\);
(ii) \(\text{Crit}_3(f_{(Y,K)}) = \text{Crit}_1(f_K) \cup \{b_0\}\).
The function $f_{(Y,K)}$ determines the components of a balanced \((2k + 1)\)-pointed Heegaard diagram in the standard fashion: The Heegaard surface is the level set $\Sigma_g = f_{(Y,K)}^{-1} \left( \frac{3}{2} \right)$. The stable manifolds of the \(g + k\) index-1 critical points intersect $\Sigma_g$ in a \((g + k)\)-tuple of disjoint simple closed curves $\alpha = \{\alpha_1, \ldots, \alpha_{g+k}\}$, the $\alpha$-attaching circles, which determine a handlebody $U_\alpha$ with boundary $\Sigma_g$. Similarly the unstable manifolds of the index-2 critical points intersect $\Sigma_g$ in the $\beta$-attaching circles $\beta = \{\beta_1, \ldots, \beta_{g+k}\}$, determining the handlebody $U_\beta$. The intersection $\Sigma_g \cap K$ is a set of $2k$ points. By choosing an orientation of $K$, some ordering of its components and an arbitrary starting point on each, we obtain an indexing $w_1, z_1, \ldots, w_k, z_k$ of the points in $\Sigma_g \cap K$ as described above. Any choice of flowline connecting the critical points disjoint from $K$, viz. $a_0$ and $b_0$, intersects $\Sigma_g$ in the single free basepoint.

To suit our purposes, we modify this construction at the last step. Specifically, we choose three distinct flowlines connecting $a_0$ and $b_0$ rather than one, and collect their intersections with $\Sigma_g$ into the triple $q = (q_1, q_2, q_3)$. With regard to the Heegaard diagram above, one may think of this as thickening the single free basepoint into a triple, the auxiliary datum anticipated above. We will work with the \((2k + 3)\)-pointed Heegaard diagram $\mathcal{H} = (\Sigma_g, \alpha, \beta; w, z, q)$ and refer to any diagram produced in this manner as being compatible with \((Y,K)\). Denote the union of the three flowlines chosen above by $\theta \subset Y \setminus K$, referred to as a theta graph; see fig. 3.1 for a schematic illustration.

Note that this process of deriving a compatible \((2k + 3)\)-pointed Heegaard diagram from a link is reversible. We may assume that the indexing of the connected components in (iii) of the definition agrees with the one chosen for the points $w, z$. For $i = 1, \ldots, k$, draw a simple curve in the connected component $A_i \subset \Sigma_g$ connecting $w_i$ and $z_i$, and then push the interior of this arc into $U_\alpha$. Likewise connect $z_{\sigma^{-1}(i)}$ to $w_i$ by an arc in $B_i$ and push this off into $U_\beta$. The union of all the pushed-off arcs forms a link in $Y = U_\alpha \cup \Sigma_g \cup U_\beta$. Now choose a point in each of $A_{k+1}$ and $B_{k+1}$, and connect
the points of q to each of these by three simple arcs, disjoint except at the common endpoint. Pushing the triple of arcs in $A_{k+1}$ off into $U_{\alpha}$ and that in $B_{k+1}$ off into $U_{\beta}$ produces a theta graph $\theta \in Y$.

For brevity, fix the notation $K_{\square} = U_{\square} \cap K$ and $\theta_{\square} = U_{\square} \cap \theta$ for $\square = \alpha$ or $\beta$.

Remark 3.1.1. An important aspect of such a Heegaard splitting for our purposes is that the pairs $(U_{\square}, K_{\square} \cup \theta_{\square})$ are trivial: $K_{\square}$ consists of properly embedded arcs which are boundary parallel. The tripod $\theta_{\square}$ is similarly boundary parallel in the sense that there is a simultaneous isotopy of the closure of each of its three strands onto $\Sigma_g$.

3.2. Lagrangians from Heegaard diagrams. Since $\# w \cup z \cup q = 2k + 3$ is odd, the traceless SU(2)-character variety $R_{g, 2k + 3} = R(\Sigma_g, w \cup z \cup q)$ is a smooth symplectic manifold by Proposition 2.2.3. The attaching circles determine subvarieties in $R_{g, 2k + 3}$ in a natural way:

\begin{equation}
L_{\alpha} := \{ [\rho] \in R_{g, 2k + 3} \mid \rho([\alpha_i]) = 1, \text{ for } i = 1, \ldots, g + k \}
\end{equation}

and

\begin{equation}
L_{\beta} := \{ [\rho] \in R_{g, 2k + 3} \mid \rho([\beta_i]) = 1, \text{ for } i = 1, \ldots, g + k \}.
\end{equation}

Let $c_i, d_i$ be meridians of the punctures $w_i, z_i$, resp., and $m_1, m_2, m_3$ those of $q_1, q_2, q_3$, resp. For simplicity we use the same notation for their homotopy classes relative to some choice of basepoint. Fixing an orientation of $K$ determines an orientation of the points $w_i, z_i$, and this in turn orients their meridians. We can also fix a coherent orientation of $m_1, m_2, m_3$, so that we obtain the standard presentation of $\pi_1(\Sigma_g \setminus (w \cup z \cup q))$:

\begin{equation}
\langle a_1, b_1, \ldots, a_g, b_g, c_1, d_1, \ldots, c_k, d_k, \mid \prod_{i=1}^{g} [a_i, b_i] \cdot \left( \prod_{j=1}^{k} c_j \cdot d_j^{-1} \right) \cdot (m_1 \cdot m_2 \cdot m_3) = 1 \rangle.
\end{equation}

It is evident that, if a representation $\rho$ sends all of the $[\alpha_i]$ to 1 $\in$ SU(2), then $\rho(c_i) = -\rho(d_i)$ and $\rho(m_1)\rho(m_2)\rho(m_3) = 1$.

It will be useful to identify these subvarieties with the traceless character varieties of the pairs $(U_{\alpha}, K_{\alpha} \cup \theta_{\alpha})$ and $(U_{\beta}, K_{\beta} \cup \theta_{\beta})$, resp. We make this precise for $L_{\alpha}$ in the following; the argument for $L_{\beta}$ is essentially the same.

Lemma 3.2.1. Let $\tilde{U}_{\alpha} \subset U_{\alpha}$ be the 3-manifold with boundary $\partial \tilde{U}_{\alpha} = \Sigma_g \sqcup S$ where $S$ is the 2-sphere boundary of a small 3-ball neighborhood of the critical point $a_0$ (the vertex of the theta graph $\theta$ lying in $U_{\alpha}$), and write $\bar{\theta}_{\alpha} = \tilde{U}_{\alpha} \cap \theta$. Let

$r : R \left( \tilde{U}_{\alpha}, K_{\alpha} \cup \bar{\theta}_{\alpha} \right) \to R(\Sigma_g, w \cup z \cup q) \times R(S, S \cap \theta),$

be the pullback map. Then $r$ is injective, and its image in the first factor of the target is naturally identified with $L_{\alpha}$.

Proof. A computation shows that $R(S, S \cap \theta) = R_{0, 3}$ is just the point $\{ [i, j, -k] \}$ [Example 2.3], so the target of $r$ is naturally identified with $R_{g, 2k + 3}$. It is easy
to see that \( L_\alpha \) is composed of classes of precisely those traceless representations on \((\Sigma_g, w \cup z \cup q)\) which extend over \((\tilde{U}_\alpha, K_\alpha \cup \tilde{\theta}_\alpha)\). Because the canonical map \( \pi_1(\Sigma_g \setminus (w \cup z \cup q)) \to \pi_1(\tilde{U}_\alpha \setminus (K_\alpha \cup \tilde{\theta}_\alpha)) \) is surjective, injectivity follows ultimately from the left-exactness of the contravariant functor \( \text{Hom}(\cdot, \text{SU}(2)) \).

This characterization and Remark 3.1.1 lead to the conclusion that, for a fixed genus and relative bridge number, the subvarieties arising from a compatible Heegaard diagram as above are equivalent in the following sense.

**Lemma 3.2.2.** Let \( \mathcal{H} \) be an arbitrary balanced \((2k + 3)\)-pointed Heegaard diagram and \( L_\alpha, L_\beta \subset R_{g,2k+3} \) the subvarieties induced as above. Then there exists a symplectomorphism \( \phi : R_{g,2k+3} \to R_{g,2k+3} \) such that \( \phi(L_\alpha) = L_\beta \).

**Proof.** By Lemma 3.2.1 each of \( L_\alpha \) and \( L_\beta \) is identified with the images of \( \mathcal{R}(U_\alpha, K_\alpha \cup \theta_\alpha) \) and \( \mathcal{R}(U_\beta, K_\beta \cup \theta_\beta) \), resp., in \( R_{g,2k+3} \). As described in Remark 3.1.1, the pairs \((U_\alpha, K_\alpha \cup \theta_\alpha)\) and \((U_\beta, K_\beta \cup \theta_\beta)\) are trivial, and so there exists a diffeomorphism \((U_\beta, K_\beta \cup \theta_\beta) \to (U_\alpha, K_\alpha \cup \theta_\alpha)\) restricting to a pointed diffeomorphism \( \phi \) of their common boundary \( \Sigma_g \). Since mapping classes act by symplectomorphisms, this in turn induces a symplectomorphism \( \phi : R_{g,2k+3} \to R_{g,2k+3} \) which maps the image of \( \mathcal{R}(U_\alpha, K_\alpha \cup \theta_\alpha) \) to that of \( \mathcal{R}(U_\beta, K_\beta \cup \theta_\beta) \), i.e. \( \phi(L_\alpha) = L_\beta \). □

**Proposition 3.2.3.** The subvarieties \( L_\alpha, L_\beta \) are embedded Lagrangian submanifolds in \((R_{g,n}, \omega_{g,n})\).

**Proof.** By Theorem 2.2.1 the boundary restriction map \( r \) of Lemma 3.2.1 is a Lagrangian immersion at its regular points, which are precisely those mapped to the (classes of) irreducible representations in \( R_{g,2k+3} \). Since there are no reducibles in the latter, \( r \) is a global immersion. By Lemma 3.2.1 it is injective, hence its image, identified with \( L_\alpha \), is an embedded Lagrangian. □

We conclude the subsection by identifying the diffeomorphism type of \( L_\alpha \) and \( L_\beta \).

**Lemma 3.2.4.** The subvarieties \( L_\alpha, L_\beta \) are diffeomorphic to the product of spheres \( (S^3)^g \times (S^2)^k \).

**Proof.** By Lemma 3.2.2 it suffices to prove the result for \( L_\alpha \) where the \( \alpha \)-curves are standard in the following sense: The curves \( \alpha_1, \ldots, \alpha_g \) represent the generators \( a_1, \ldots, a_g \) appearing in the standard presentation of \( R_{g,2k+3} \) described in Section 2.2. Let \( c_j, d_j \) for \( j = 1, \ldots, k \) and \( m_1, m_2, m_3 \) be as above and collect the images \( \rho(c_j) \), \( \rho(d_j) \) and \( \rho(m_p) \) into tuples \( B \in G^g, C \in C_0^k \) and \( M \in C_0^3 \), resp., where \( G = \text{SU}(2) \). Then elements of \( L_\alpha \) may be expressed as \([1, B, C, M] \in G^g \times C_0^k \times C_0^3 / G \) subject to the constraint \( M_1 M_2 M_3 = 1 \).

Consider the map \( G^g \times C_0^k \to L \) defined by

\[
(B, C) \mapsto [1, B, C; (i, j, -k)].
\]

This map is injective since the common stabilizer of \((i, j, -k)\) is the center \( \{\pm 1\} \) of \( \text{SU}(2) \). On the other hand, for any \([1, B, C, M] \in L \), there is a unique \( R \in \text{SO}(3) \)
such that $RMR^{-1} = (i,j,-k)$, hence
\begin{equation}
[1,B,C,M] = [1, RBR^{-1}, CRC^{-1}, (i,j,-k)],
\end{equation}
is the image under the map (17) of $R(B,C)R^{-1}$, demonstrating that this map is surjective. Moreover, $R$ varies smoothly with $M$, and therefore (17) is a diffeomorphism. \hfill \Box

An easy consequence of Lemma 3.2.4 is the following:

**Corollary 3.2.5.** The Lagrangian submanifolds $L_\alpha, L_\beta$ are spin.

**Proof.** All spheres are spin manifolds, and the product of spin manifolds is again spin, as a consequence of the Whitney sum formula. \hfill \Box

### 3.3. Floer homology

In general the Lagrangian intersection $L_\alpha \cap L_\beta$ will not be transverse. However, standard arguments show that a Hamiltonian isotopy may be chosen generically after which transversality is achieved, and that such isotopies produce canonically isomorphic Lagrangian Floer homologies. It may in fact be possible to accomplish this by means of the *holonomy perturbations* defined in [2, Section 6.2], which are analogous to the perturbations employed by Kronheimer and Mrowka in [11] for singular instanton knot homology, although further study into this is needed.

In any event we assume henceforth that $L_\alpha$ and $L_\beta$ intersect transversely.

We now proceed to prove Theorem A of the introduction, recapitulated in the following:

**Theorem 3.3.1.** The Lagrangian Floer homology $HF(L_\alpha, L_\beta)$ is well-defined. Moreover, the pair $(L_\alpha, L_\beta)$ is relatively spin, so that the homology may be taken with coefficients in $\mathbb{Z}$.

The first step is to establish that we are in the monotone setting, so that Oh’s results for this case summarized in the previous section can be applied.

**Proposition 3.3.2.** The Lagrangians $L_\alpha, L_\beta$ are monotone in $(\mathcal{R}_{g,2k+3}, \omega_{g,2k+3})$ and have minimal Maslov number $N_L = 2$.

**Proof.** By Lemma 3.2.4 both $L_\alpha, L_\beta$ are compact and simply connected, and therefore monotone by Proposition 2.1.4. As for the second claim, recall from Proposition 2.2.5 that the minimal Chern number of $(\mathcal{R}_{g,n}, \omega_{g,n})$ is $N_M = 1$ for any $n$, hence $N_L = 2$ by Equation (5). \hfill \Box

**Proof of Theorem 3.3.1.** By Lemma 3.2.4 and Proposition 3.3.2, the pair $(L_\alpha, L_\beta)$ satisfy the conditions of Theorem 2.1.8. Since, as demonstrated in Lemma 3.2.2 there is a symplectomorphism of $\mathcal{R}_{g,2k+3}$ taking $L_\alpha$ to $L_\beta$, we can apply Corollary 2.1.9 to conclude that the Lagrangian Floer homology $HF(L_\alpha, L_\beta)$ is well defined. Being that $L_\alpha$ and $L_\beta$ are both spin by Corollary 3.2.5 they are trivially relatively spin in $\mathcal{R}_{g,2k+3}$ as a pair, hence their Floer homology may be taken with coefficients in $\mathbb{Z}$. \hfill \Box
3.3.1. \textit{Gradings.} For any compact monotone Lagrangians $L_0, L_1$ in a symplectic manifold $M$ with $N_L \geq 2$, the Lagrangian Floer homology $HF(L_0, L_1)$ is endowed with a relative $\mathbb{Z}/N_L$-grading. When $M, L_0, L_1$ are orientable, its reduction mod 2 is equivalent to the grading which results from fixing orientations of the Lagrangians and then assigning a degree in $\{\pm 1\}$ to each point $x \in L_0 \cap L_1$ according to whether the induced orientation agrees with the canonical orientation of the point $x \in M$. Thus $HF(L_0, L_1)$ categorifies the algebraic intersection number of $L_0, L_1 \subset M$.

Since $N_L = 2$ for our Lagrangians $L_\alpha, L_\beta$, and these are certainly orientable, the standard $\mathbb{Z}/N_L$-grading on $SIK(H) = HF(L_\alpha, L_\beta)$ obtained after fixing a base intersection point in grading 0 is exactly the $\mathbb{Z}/2$-grading induced by fixing orientations, up to an overall sign in the degrees. Finer $\mathbb{Z}/N$-gradings, where $N$ is divisible by $N_L = 2$, are possible \textit{a priori}. Defining such a grading requires an understanding of the space $\pi_2(L_\alpha, L_\beta)$ of topological Whitney disks with Lagrangian boundary conditions connecting intersection points, and we plan to visit this subject in future work on SIK.

4. Topological invariance

In this section we prove Theorem B of the introduction, viz. that the isomorphism class of the homology $SIK(H)$ constructed above is a topological invariant of the pair $(Y, K)$. To this end we make use of the fact that any two compatible diagrams $H$ and $H'$ are related by a sequence of multi-pointed Heegaard moves, as described by the following.

\textbf{Proposition 4.0.1} ([12, Proposition 3.9]). Let $H$ and $H'$ be two multi-pointed Heegaard diagrams compatible with the pair $(Y, K)$ (in the sense of Section 3.1 above). Then $H$ can be transformed into $H'$ by a sequence of the following moves.

(i) An \textit{isotopy} of an attaching circle supported in the complement of the punctures $w \cup z \cup q$;

(ii) A \textit{handleslide} among the $\alpha$- or among the $\beta$-attaching circles which is supported in the complement of $w \cup z \cup q$;

(iii) \textit{surface (de-)stabilization}: The new Heegaard diagram

\begin{equation}
H' = (\Sigma_{(g+1)}; \alpha \cup \alpha', \beta \cup \beta'; w, z, q)
\end{equation}

is formed by taking the connected sum of the Heegaard surface $\Sigma_g$ in $H$ with a torus along a disk which is disjoint from the $\alpha$- and $\beta$-attaching circles and from the marked points of $w \cup z \cup q$. On the torus are two new attaching curves $\alpha'$ and $\beta'$ which intersect transversely in a single point. (Resp. the inverse of this procedure, with the roles of $H$ and $H'$ exchanged.)

(iv) \textit{link (de-)stabilizations}: The new Heegaard diagram

\begin{equation}
H' = (\Sigma_g; \alpha \cup \alpha', \beta \cup \beta'; w \cup \{w'\}, z \cup \{z'\}, q)
\end{equation}

is formed by the addition to $H$ of a new pair of attaching circles $\alpha'$ and $\beta'$ and of two new marked points $w'$ and $z'$, satisfying the following: The circles $\alpha', \beta'$ are disjoint from the original ones of $\alpha, \beta$, resp., and intersect transversely in
two points. They bound disks \( A', B' \subset \Sigma_g \), resp., such that \( w' \in A' \cap B' \) while \( z' \in A' \setminus B' \) and there is precisely one of the original marked points \( z_0 \in z \) which lies in \( B' \setminus A' \). (Resp. the inverse of this procedure.)

Remark 4.0.2. What we have called surface stabilization in (iii) above is referred to as index-one/two stabilization in [12, Proposition 3.9]. Moreover, (iv) link stabilization is one of two kinds of index-zero/three stabilization enumerated there by the authors, along with free stabilization. The latter move serves to add a new free basepoint, of the kind which we have thickened into the triple \( q \). A well-defined Lagrangian Floer theory could result from certain free stabilizations, so long as the total number of marked points remains odd. For the moment we have nothing to motivate such moves, nor are they needed to obtain the equivalence classes of compatible Heegaard diagrams we have used in this construction, and we therefore exclude free stabilization from our consideration.

Note also that we have dispensed with the requirement that the Heegaard moves in question be admissible, in the sense given there by the authors, as this is not relevant in our context, and the weaker statement is clearly no less valid.

In light of Proposition 4.0.1, it will suffice to prove the following:

**Theorem 4.0.3.** Let \( \mathcal{H} \) and \( \mathcal{H}' \) be two multi-pointed Heegaard diagrams compatible with the pair \( (Y, K) \).

1. If \( \mathcal{H} \) and \( \mathcal{H}' \) are related by an isotopy or handleslide, then the respective Lagrangian pairs \((L_\alpha, L_\beta)\) and \((L'_\alpha, L'_\beta)\) determined by these diagrams (as in Section 3.2) are identical, hence \( \SIK(\mathcal{H}) = \SIK(\mathcal{H}') \).

2. If \( \mathcal{H} \) and \( \mathcal{H}' \) are related by surface or link stabilization, then \( \SIK(\mathcal{H}) \) and \( \SIK(\mathcal{H}') \) are canonically isomorphic as relatively \( \mathbb{Z}/2 \)-graded abelian groups.

The following subsection is devoted to proving the foregoing theorem. With this result in place, we will define \( \SIK(Y, K) \) to be the isomorphism class of the Lagrangian Floer homology \( \SIK(\mathcal{H}) \) for any choice of compatible diagram \( \mathcal{H} \).

4.1. **Invariance under multi-pointed Heegaard moves.** The next proposition proves statement (1) of Theorem 4.0.3.

**Proposition 4.1.1.** Let the Heegaard diagrams \( \mathcal{H} \) and \( \mathcal{H}' \) be related by an isotopy or handleslide and let \((L_\alpha, L_\beta)\) and \((L'_\alpha, L'_\beta)\) be their respective induced Lagrangian pairs, as defined in Section 3. Then the latter are identical Lagrangian pairs in \( \mathbb{R}_{g,n} \), i.e. \( L_\alpha = L'_\alpha \) and \( L_\beta = L'_\beta \), and consequently \( \SIK(\mathcal{H}) = \SIK(\mathcal{H}') \).

**Proof.** The case of an isotopy is trivial, as the image of a loop under an element of the character variety is determined by the former’s free homotopy class.

For a handleslide, we may assume without loss of generality that the curve \( \alpha_1 \) of \( \mathcal{H} \) is slid over \( \alpha_2 \) to produce \( \alpha'_1 \) in \( \mathcal{H}' \). By hypothesis these three curves bound a pair of pants in \( \Sigma_g \setminus (w \cup z \cup q) \), hence any \( SU(2) \)-representation taking two of them to 1 does likewise for the third. It follows that \( L_\alpha = L'_\alpha \). \( \square \)
In the remainder of this section we verify statement (2) of Theorem 4.0.3. We will establish the claimed the canonical isomorphism by associating to the Heegaard move in question a kind of Cerf decomposition of the triple $(Y, K, \theta)$, then deriving from this decomposition a generalized Lagrangian correspondence among traceless character varieties of punctured surfaces and finally carrying out computations on the quilted Floer homology \cite{24} of the correspondence. This largely follows the work of Wehrheim and Woodward \cite{24} on quilted-Floer invariants of tangles between punctured surfaces, as made precise in the following:

**Definition 4.1.2.** Let $(\Sigma_-, \mathbf{p}-)$ and $(\Sigma_+, \mathbf{p}+)$ be punctured surfaces, and let the decorations $\overline{\Sigma}_-$ and $\overline{\mathbf{p}}-$ indicate reversal of orientation. A tangle from $(\Sigma_-, \mathbf{p}-)$ to $(\Sigma_+, \mathbf{p}+)$ is a tangle $(X, T)$ together with an orientation-preserving diffeomorphism $\phi : \partial X \to \overline{\Sigma}_- \cup \Sigma_+$ which restricts to an orientation-preserving bijection $\phi|_{\partial T} : \partial T \to \overline{\mathbf{p}} \sqcup \mathbf{p}'$. A tangle $(X, T)$ is called

- **elementary** if there is an orientation-preserving diffeomorphism $X \to \Sigma_g \times [0, 1]$, where $\Sigma_g$ is an orientable genus-$g$ surface;
- **cylindrical** if there is an orientation-preserving diffeomorphism $(X, T) \to (\Sigma_g, \mathbf{p}) \times [0, 1]$, where $(\Sigma_g, \mathbf{p})$ is a punctured surface.

In the following we use a Morse function to obtain a decomposition of $(Y, K \cup \theta)$ into tangles $(Y_{i(i+1)}, Y_{i(i+1)} \cap (K \cup \theta))$ with oriented boundary surfaces $\partial Y_{i(i+1)} = \Sigma_i \cup \Sigma_{i+1}$ punctured where they intersect $K \cup \theta$. It will be convenient to suppress the intersections with $K \cup \theta$ from the notation by writing $Y_{i(i+1)}^*$ for the tangle and e.g. $\Sigma_2^*$ for the punctured surface.

Let $\mathcal{H}$ be a compatible Heegaard diagram induced by a Morse function $f$ and $\mathcal{H}'$ the result of surface or link stabilization on $\mathcal{H}$. Take $a_\pm = f(\Sigma_g) \pm \varepsilon$ for some $\varepsilon > 0$. When $\varepsilon$ is taken sufficiently small, the preimage $N = f^{-1}([a_-, a_+])$ is a normal neighborhood of the Heegaard surface $\Sigma_g$ in $\mathcal{H}$ which contains no critical points of $f$. The normalized gradient flow of $f$ serves to identify the punctured surfaces $\Sigma_i^* = f^{-1}(a_-) \cap (K \cup \theta)$ and $\Sigma_3^* = f^{-1}(a_+) \cap (K \cup \theta)$ each with $(\Sigma_g, \mathbf{w} \cup \mathbf{z} \cup \mathbf{q})$, so that we may consider their traceless character varieties to be identified with each other as $\mathcal{R}_{g,2k+3}$.

We may deform the restriction of $f$ to the interior of $N$ so as to produce a new Morse function $f'$ which induces the stabilized Heegaard diagram $\mathcal{H}'$. We write its punctured Heegaard surface as $\Sigma_2^*$. Then the function $f'$ induces the tangle decomposition

\begin{equation}
(Y, K \cup \theta) = B_- \cup \Sigma_0^* \cup \Sigma_1^* \cup \Sigma_2^* \cup \Sigma_3^* \cup Y_{34}^* \cup B_+,
\end{equation}

where each of $\Sigma_0^*$ and $\Sigma_4^*$ is a 2-sphere, disjoint from $K$ and meeting $\theta$ in three punctures, which bounds the small 3-ball neighborhood $B_-$ or $B_+$, resp., of a vertex of $\theta$.

The tangles of (21) bear the following descriptions. The “outer pieces” $B_- \cup Y_{01}^*$ and $Y_{34}^* \cup B_+$ can be identified in an obvious way with the pairs $(U_\alpha, K_\alpha \cup \theta_\alpha)$ and $(U_\beta, K_\beta \cup \theta_\beta)$, resp. Each of $Y_{01}, Y_{34}$ meets $\theta$ in three trivial arcs running from one boundary component to the other, and meets $K$ in $k$ trivial arcs with both endpoints on the same boundary component: on the outgoing boundary $\partial^+ Y_{01} = \Sigma_1$, where we
refer to the arcs as \textit{caps}, and on the incoming boundary \( \partial^- Y_{34} = \overline{\Sigma_3} \), where we refer to them as \textit{caps}.

The “inner piece” \( Y_{12}^* \cup Y_{23}^* \) is diffeomorphic to the cylindrical tangle \((\Sigma_g, w \cup z \cup q) \times [0, 1]\), containing \(2k + 3\) trivial arcs running from \(\Sigma_1\) to \(\Sigma_3\). The characterizations of the individual constituent tangles \( Y_{12}^* \) and \( Y_{23}^* \) depend on the type of stabilization performed. In the case of surface stabilization, they can be viewed as the result of taking the cylindrical tangle \( \Sigma_2^* \times [0, 1] \) and attaching a 2-handle along \( \alpha' \) or \( \beta' \), resp. In the case of link stabilization, they are both elementary and each contain \(2k + 3\) arcs running between the two boundary components. On the one hand, \( Y_{12}^* \) has in addition a cup connecting \( w', z' \in \Sigma_2 \), and one of the other arcs connects \( z_0 \in \Sigma_1 \) with \( z_0 \in \Sigma_2 \). On the other \( Y_{23}^* \) has a cap connecting \( z_0, w' \in \Sigma_2 \) and \( z' \in \Sigma_2 \) is connected by an arc to \( z_0 \in \Sigma_3 \).

For \( i = 0, \ldots, 4 \), let \( M_i = R(\Sigma_i^*) \) denote the traceless character variety of the punctured surface, and \( M_i^- \) the result of replacing the Atiyah-Bott-Goldman symplectic form by its negative. Recall that, for \( i = 1, \ldots, 4 \), the traceless character variety \( R(\partial Y_i^*\{(i-1)i\}) \) of the disconnected boundary, comprising the pair of punctured surfaces \( \Sigma_{i-1}^* \) and \( \Sigma_i^* \), is by definition the product \( M_{i-1} \times M_i \). Write \( r_{(i-1)i} : R(Y_{(i-1)i}^*) \to R(\partial Y_i^*\{(i-1)i\}) \) for the pullback (or boundary-restriction) map. Then \( L_{(i-1)i} := \operatorname{im}(r_{(i-1)i}) \subset M_{i-1}^- \times M_i \) is an immersed Lagrangian correspondence by Theorem \ref{thm:lagrangian_correspondence}. Moreover, since each \( M_i \) contains only classes of irreducible representations, these Lagrangian correspondences are embedded (cf. the proof of Proposition \ref{prop:embedded_lagrangian_correspondence}). The symplectic manifolds \( M_0 \) and \( M_4 \) can each be identified with the single-pointed space \( R_{0,3} = \{[i, j, -k]\} \). Altogether the decomposition \ref{eq:decomposition} induces a cyclic generalized Lagrangian correspondence \( (L_{01}, L_{12}, L_{23}, L_{34}) \) from \( M_0 = \{\text{pt.}\} \) to itself.

By comparing the descriptions of \( Y_{01}^* \) and \( Y_{34}^* \) above with the proof of Lemma \ref{lem:lagrangian_correspondence}, it is evident that the induced Lagrangian correspondences may be expressed as

\begin{equation}
L_{01} = \{ ([\rho], \text{pt.}) \mid [\rho] \in R_{g,2k+3}, \rho([\alpha_i]) = 1 \},
\end{equation}

and

\begin{equation}
L_{34} = \{ ([\rho], \text{pt.}) \mid [\rho] \in R_{g,2k+3}, \rho([\beta_i]) = 1 \},
\end{equation}

where the \( \alpha \)- and \( \beta \)-curves are induced by the Morse function in the usual manner, and that these may be identified with \( L_\alpha \) and \( L_\beta \), resp. As we shall see below, the Lagrangians \( L'_{\alpha}, L'_{\beta} \) induced after stabilization may be identified with two of the geometric compositions in our generalized correspondence, and the embedded composition theorem in quilted Floer homology will permit to relate the Floer homologies of these Lagrangian pairs. We will give descriptions of the remaining Lagrangian correspondences along with their geometric compositions below in the course of establishing these facts.

Recall that the \textit{geometric composition} \( L_{(i-1)i} \circ L_{i(i+1)} \subset M_i^- \times M_{i+1} \) is the image under the projection of the intersection

\begin{equation}
L_{(i-1)i} \times L_{i(i+1)} \cap M_i^- \times \Delta M_i \times M_{i+1},
\end{equation}
where $\Delta_{M_r} \subset M_r^c \times M_i$ is setwise the diagonal. When the intersection $[24]$ is transversal and its projection to $M_r^c \times M_i^{i+1}$ injective, the geometric composition is said to be embedded. The embedded composition theorem $[25]$ Theorem 5.4.1] states that, when the generalized Lagrangian correspondence $(L_{01}, \ldots, L_{(i-1)i}, L_{i(i+1)}, \ldots, L_{(r+1)r})$ satisfies certain technical conditions and if $L_{(i-1)i} \circ L_{i(i+1)}$ is an embedded geometric composition, then the quilted Floer homologies $\text{HF}(L_{01}, \ldots, L_{(i-1)i}, L_{i(i+1)}, \ldots, L_{(r+1)r})$ and $\text{HF}(L_{01}, \ldots, L_{(i-1)i} \circ L_{i(i+1)}, \ldots, L_{(r+1)r})$ are related by a canonical, grading-preserving isomorphism. We will apply this theorem to exhibit such an isomorphism between $\text{SIK}(\mathcal{H}) = \text{HF}(L_\alpha, L_\beta)$ and $\text{SIK}(\mathcal{H}') = \text{HF}(L'_\alpha, L'_\beta)$ by characterizing the three geometric compositions arising here according to the following:

**Lemma 4.1.3.** The “outer” geometric compositions $L_{01} \circ L_{12}$ and $L_{23} \circ L_{34}$ can be identified with the Lagrangians $L'_\alpha$ and $L'_\beta$, resp., in $\mathcal{R}(\Sigma_2^*)$. By considering $M_1 \times M_3$ as $\mathcal{R}_{g,2k+3} \times \mathcal{R}_{g,2k+3}$ the “inner” composition $L_{12} \circ L_{23}$ is equal to the diagonal $\Delta_{13} \subset M_1^c \times M_3$ in the case of either surface or link stabilization. Moreover, all three of these geometric compositions are embedded.

After showing that $(L_{01}, L_{12}, L_{23}, L_{34})$ meets the aforementioned technical conditions, we use Lemma 4.1.3 to arrive at the following computation on quilted Floer homologies:

\[(25) \quad \text{HF}(L_{01}, \Delta_{13}, L_{34}) = \text{HF}(L_{01}, L_{12} \circ L_{23}, L_{34})\]
\[(26) \quad \cong \text{HF}(L_{01}, L_{12}, L_{23}, L_{34})\]
\[(27) \quad \cong \text{HF}(L_{01} \circ L_{12}, L_{23} \circ L_{34}).\]

Since the first of these is equivalent to $\text{HF}(L_\alpha, L_\beta)$ and the last to $\text{HF}(L'_\alpha, L'_\beta)$, we conclude that $\text{SIK}(\mathcal{H})$ and $\text{SIK}(\mathcal{H}')$ are canonically isomorphic as relatively graded abelian groups, proving Theorem 4.0.3.

**Proof of Lemma 4.1.3.** In the following let $R(X^*)$ denote the $\text{SU}(2)$-representation variety of the tangle or punctured surface $X^*$ subject to the condition that meridians of the implicitly specified codimension-$2$ submanifold be taken to traceless elements—prior to taking the quotient by conjugation. In the absence of such a codimension-$2$ submanifold, this is just the representation variety.

**Case 1 (surface stabilization):** In topological terms the punctured surface $\Sigma_2^*$ is the connected sum of $\Sigma_1^*$ (or $\Sigma_3^*$) with a torus containing the new attaching curves $\alpha', \beta'$. Let $\gamma$ be the boundary of the disk along which this sum is formed, so that $\Sigma_2^*$ decomposes along $\gamma$ into two pieces: one the torus minus a disk, which we denote $S$, and the other $\Sigma_2^* \setminus S$ diffeomorphic to $\Sigma_1^* \setminus D$, the complement of a disk in $\Sigma_1^*$ disjoint from all punctures and attaching curves. By the Seifert–Van Kampen theorem, the traceless representation variety of $\Sigma_2^*$ takes the form of the twisted product $R(\Sigma_2^* \setminus S) \times_{R(A)} R(S)$, where $A$ is an annular neighborhood of $\gamma$. Since $\pi_1(S)$ is the free group $\langle [\alpha'] \rangle \ast \langle [\beta'] \rangle$, the representation variety $R(S)$ is just $\text{SU}(2) \times \text{SU}(2)$.

The curve $\gamma$ bounds a disk in $Y_{12}^*$ along which the tangle likewise splits into two pieces: one diffeomorphic to the cylindrical tangle $\Sigma_1^* \times [0, 1]$, and the other a solid torus $V$ with $S \subset \partial V$. We therefore have $R(Y_{12}^*) = R(\Sigma_1^* \times [0, 1]) \times R(V)$. Here we have
\( \pi_1(V) = \langle [\beta'] \rangle \), hence \( R(V) = SU(2) \). The pullback map \( R(Y_{12}^\ast) \to R(\Sigma_2^\ast) \) respects the product structures above, decomposing as the product of pullback maps \( R(\Sigma_1^\ast \times [0, 1]) \to R(\Sigma_2^\ast \backslash S) \) and \( R(V) \to R(S) \). The latter of these is the inclusion of the second factor \( SU(2) \to SU(2) \times SU(2) \), sending \( \rho|_V \) to its restriction \( \rho|_S \) satisfying \( \rho|_S([\alpha']) = 1 \) and \( \rho|_S([\beta']) = \rho|_V([\beta']) \). Since the inclusion of \( \Sigma_2^\ast \backslash S \) into the cylindrical part \( \Sigma_1^\ast \times [0, 1] \) induces a surjective map on fundamental groups, the pullback map \( R(\Sigma_1^\ast \times [0, 1]) \to R(\Sigma_2^\ast \backslash S) \) is likewise injective. Since any \( \rho|_{\Sigma_2^\ast \backslash S} \times \rho|_S \in \text{im}(R(Y_{12}^\ast) \to R(\Sigma_2^\ast)) \) is necessarily trivial on \( A \), it follows that \( \rho|_{\Sigma_2^\ast \backslash S} \) here extends uniquely over \( \Sigma_1^\ast \times [0, 1] \subset Y_{12}^\ast \). We therefore conclude that \( R(Y_{12}^\ast) \to R(\Sigma_2^\ast) \) is a smooth embedding, and by \( SU(2) \)-equivariance, so too is the projection \( L_{12} \to M_2 \) resulting from taking the quotient by conjugation.

Because \( \Sigma_1^\ast \) is a deformation retract of the cylindrical part \( \Sigma_1^\ast \times [0, 1] \) of \( Y_{12}^\ast \), we may view \( R(\Sigma_1^\ast) \) and \( R(\Sigma_1^\ast \times [0, 1]) \) as being canonically identified up to conjugacy. The pullback map \( R(Y_{12}^\ast) \to R(\Sigma_1^\ast) \) may then be interpreted as the projection from \( R(Y_{12}^\ast) = R(\Sigma_1^\ast \times [0, 1]) \to R(V) \) onto the first factor, so that \( L_{12} \to M_1^\ast \) is a smooth submersion. We then see that the restrictions \( \rho|_{\Sigma_2^\ast \backslash S} \) of traceless representations \( \rho \in \text{im}(R(Y_{12}^\ast) \to R(\Sigma_2^\ast)) \) are naturally identified with those on \( \Sigma_1^\ast \), and \( \rho \) is then uniquely determined by the choice of \( \rho([\beta']) \in SU(2) \). From this it is clear that the geometric composition \( L_01 \circ L_{12} \) may be identified with \( L'_0 \). The correspondence \( L_{23} \) admits a similar description, with \( \rho \in \text{im}(R(Y_{23}^\ast) \to R(\Sigma_2^\ast)) \) characterized by the condition \( \rho|_S([\beta']) = 1 \), while \( \rho|_S([\alpha']) \) may be chosen arbitrarily in \( SU(2) \), and it follows from the same line of reasoning that \( L_{23} \circ L_{34} \) may be identified with \( L'_\beta \).

The injectivity of the projection \( L_{01} \times_{M_1} L_{12} \to L_{01} \circ L_{12} \) is a consequence of the injectivity of \( R(Y_{12}^\ast) \to R(\Sigma_2^\ast) \) and the \( SU(2) \)-equivariance of the pullback maps. If \( (pt., [\rho_2]) \in L_{01} \circ L_{12} \), then \( \rho_2 \in R(\Sigma_2^\ast) \) extends uniquely over \( Y_1^\ast \), and this extension in turn corresponds uniquely with some \( \rho_1 \in R(\Sigma_1^\ast) \) determined by \( \rho_2|_{\Sigma_2^\ast \backslash S} \). Hence \( (pt., [\rho_1], [\rho_1], [\rho_2]) \in M_0 \times \Delta_{M_1} \times M_2 \) is the unique element in the preimage of the projection of \( (pt., [\rho_2]) \).

That \( L_{01} \times_{M_1} L_{12} \) is cut out transversally by the diagonal \( \Delta_{M_1} \) follows from \( SU(2) \)-equivariance and the fact that \( R(Y_{12}^\ast) \to R(\Sigma_1^\ast) \) is surjective, hence \( L_{12} \to M_1^\ast \) a smooth submersion. Take an arbitrary tangent vector \((0, \xi_1, \xi_1', \xi_2) \in T_{[\rho]}(pt. \times M_1 \times M_1 \times M_2) \). There can be found a vector of the form \((0, 0, \xi_1' - \xi_1, \xi_2') \in T_{[\rho]}(L_{01} \times L_{12}) \). Then summing with \((0, \xi_1, \xi_1, \xi_2 - \xi_2') \in T_{[\rho]}(M_0 \times \Delta_{M_1} \times M_2) \) gives the original tangent vector, and we conclude that the intersection is transverse. Therefore \( L_{01} \circ L_{12} \) is embedded. The arguments for the embeddedness of \( L_{23} \circ L_{34} \) are essentially the same.

Finally, we consider the composition \( L_{12} \circ L_{23} \). Representations \( \rho_1 \in \text{im}(R(Y_{12}^\ast) \to R(\Sigma_2^\ast)) \) and \( \rho_2 \in \text{im}(R(Y_{23}^\ast) \to R(\Sigma_2^\ast)) \) agree up to conjugation if and only if their restrictions to \( \Sigma_2^\ast \backslash S \) are conjugate and are both trivial along \( \gamma \). Two such representations correspond uniquely with representations in \( R(\Sigma_1^\ast) \) and \( R(\Sigma_3^\ast) \), resp., and these necessarily represent the same conjugacy class in \( \mathcal{A}_{g,2k+3} \). From this we conclude that the geometric composition \( L_{12} \circ L_{23} \) is set-theoretically the diagonal \( \Delta_{13} \subset M_1 \times M_3 = \mathcal{A}_{g,2k+3} \times \mathcal{A}_{g,2k+3} \). The uniqueness further implies that the projection from \( L_{12} \times M_2 \to L_{23} \) is injective.
The transversality of $L_{12} \times L_{23} \cap M_1 \times \Delta M_2 \times M_3$ can be deduced from the fact that the images of $R(Y_{12}^*)$ and $R(Y_{23}^*)$ intersect transversally in $R(\Sigma_2^*)$, as can be worked out simply in terms of the dimensions of the representation varieties. This clearly implies that the intersection of $\text{im}(R(Y_{12}^*)) \times \text{im}(R(Y_{23}^*)) \subset R(\Sigma_2^*) \times R(\Sigma_2^*)$ with the diagonal is transversal, and then transversality at the level of character varieties follows, since the quotient map induced by the conjugation action is a smooth submersion.

**Case 2 (link stabilization):** Recall that we now have $\alpha', \beta'$ bounding disks $A', B' \subset \Sigma_2^*$, resp., with two new punctures $w' \in A' \cap B'$ and $z' \in B'$ and exactly one of the original punctures $z_0 \in B'$ as well. Take a smooth curve $\gamma \subset \Sigma_2^*$ which is a slight pushoff of $\partial(A' \cup B')$. The gradient flow of $\gamma$ is diffeomorphic to the cylinder $S^1 \times [1,2]$ which splits $Y_{12}^*$ into two pieces—one cylindrical and one non-cylindrical tangle. The non-cylindrical component is equivalent to a disk times an interval, containing two arcs, which we write altogether as $(D \times [1,2])^*$. One end $D^*_1 = (D \times \{1\})^* \subset \Sigma_1^*$ is a disk with a single puncture identified with $z_0$. The other $D^*_2 = (D \times \{2\})^* \subset \Sigma_2^*$ is a disk with the three punctures $z_0, w', z'$. Within $(D \times [1,2])^*$ is one arc connecting $z_0 \times \{1\}$ with $z_0 \times \{2\}$ and a cup connecting $w'$ with $z'$. The gradient flow of the Morse function serves to identify the cylindrical component with $\Sigma_g^* \setminus D \times [1,2]$, where $D$ is a disk and the punctures in $\Sigma_g^*$ are identified with $w \cup z \setminus \{z_0\} \cup q$, so that $\Sigma_g^* \setminus D \times \{i\} = \Sigma_g^* \setminus D_i^*$ for $i = 1, 2$. By the Seifert–Van Kampen theorem we obtain the following expressions of the representation varieties as twisted products:

\begin{align}
R(\Sigma_2^*) &= R(\Sigma_g^* \setminus D) \times_{R(S^1)} R(D_2^*), \\
R(Y_{12}^*) &= R(\Sigma_g^* \setminus D \times [1,2]) \times_{R(S^1 \times [1,2])} R((D \times [1,2])^*), \\
R(\Sigma_1^*) &= R(\Sigma_g^* \setminus D) \times_{R(S^1)} R(D_1^*).
\end{align}

Note with regard to $R(\Sigma_1^*)$ that the restriction to $\partial D$ completely determines the restriction to $D_1^*$, hence $R(\Sigma_1^*)$ may be identified with its first factor $R(\Sigma_g^* \setminus D)$.

The pullback maps $R(Y_{12}^*) \to R(\Sigma_1^*), R(\Sigma_2^*)$ again respect the product structures. The restrictions to the first factor are induced by the inclusion of a deformation retract, hence we may again think of each of the first factors in the products above as being related by the identity map, and $R(Y_{12}^*) \to R(\Sigma_1^*)$ as the projection on the first factor. Since the map $\pi_1(D_2^*) \to \pi_1((D \times [1,2])^*)$ is surjective, the pullback map $R(Y_{12}^*) \to R(\Sigma_2^*)$ is a smooth embedding. Let $c_0, c_1, c_2 \in \pi_1(D_2^*)$ be the classes of standard meridians about $z_0, w', z'$, resp. By writing representations in $\text{im}(R(Y_{12}^*) \to R(\Sigma_2^*))$ in the form $\rho|_{\Sigma_g^* \setminus D} \times_{\gamma_1} \rho|_{D_2^*}$, we see that the image is cut out by the equation $\rho|_{D_2^*}(c_1) = -\rho|_{D_2^*}(c_2)$. This is equivalent to the condition that $\rho|_{D_2^*}(\alpha') = 1$. The image of $c_0$ on the other hand is arbitrary, and this together with $\rho|_{\Sigma_g^* \setminus D}$ uniquely determine a representation in $R(\Sigma_1^*)$. Hence we conclude as above that $L_{01} \circ L_{12}$ may be identified with $L_{01}'$. The embeddedness of this composition follows from precisely the same argument as in the case of surface stabilization above, and again analogous descriptions of the representation varieties lead to the same conclusions for $L_{23} \circ L_{34}$.

The image of $R(Y_{23}^*) \to R(\Sigma_2^*)$ is cut out by the equation $\rho|_{D_2^*}(c_0) = -\rho|_{D_2^*}(c_1)$. Representations $\rho_1 \in \text{im}(R(Y_{12}^*) \to R(\Sigma_2^*))$ and $\rho_2 \in \text{im}(R(Y_{23}^*) \to R(\Sigma_2^*))$ are conjugate
if and only if both satisfy $\rho_i(c_0) = -\rho_i(c_1) = \rho_i(c_2)$. Similarly as in the previous case, the account above shows that these representations correspond uniquely with ones on $R(\Sigma_1^i)$ and $R(\Sigma_3^i)$, resp., which necessarily represent the same class in $\mathcal{R}_{g,2k+3}$. Hence, after passing to the quotient, we again have that $L_{12} \circ L_{23}$ is equal to the diagonal $\Delta_{13}$ on $\mathcal{R}_{g,2k+3}$, with the projection from $L_{12} \times_{M_2} L_{23}$ being injective. Transversality follows by the same argument as given for surface stabilization.

To establish the validity of our application of the embedded composition theorem above, we now check the outstanding technical conditions regarding the Lagrangian correspondences’ monotonicity, relative spin structures and disk numbers.

**Monotonicity:** The embedded composition theorem requires that all of the symplectic manifolds $M_i$, for $i = 0, \ldots, 3$, be compact and monotone with the same monotonicity constant. The compactness condition is obviously satisfied, and that of monotonicity is established in Proposition 2.2.4. The theorem further requires that each of the Lagrangian correspondences $L_{i(i+1)}$ be a monotone Lagrangian submanifold of minimal Maslov number $N_\mathcal{L} \geq 2$. We have already seen this verified for $L_{01} = L_\alpha$ and $L_{34} = L_\beta$. The same holds for $L_{12}$ and $L_{23}$ by Proposition 2.1.4 and the subsequent remarks since both of these correspondences are likewise simply connected. From the descriptions in the proof of Lemma 4.1.3 it can be seen that, for instance, $R(Y_{i_2}^{i_2})$ is diffeomorphic to $R(\Sigma_1^i) \times SU(2)$ in the case of surface stabilization and to $R(\Sigma_1^i) \times C_0$ in that of link stabilization. Since all representations in $R(\Sigma_1^i)$ have central stabilizer, the quotient by conjugation $L_{12}$ has the structure of a fiber bundle over the simply connected space $M_1 = \mathcal{R}_{g,2k+3}$ with fiber $SU(2)$ or $C_0$, and is therefore simply connected. An analogous argument holds for $L_{23}$.

**Relative spin structures:** What is required here is that, for $i = 0, \ldots, 3$, there should be classes $b_i \in H^2(M_i; \mathbb{Z}/2)$ and $\tilde{b}_{i+1} \in H^2(M_{i+1}; \mathbb{Z}/2)$ such that the sum of pullbacks under the projections $\pi_i^* b_i + \pi_{i+1}^* b_{i+1} \in H^2(M_i \times M_{i+1}; \mathbb{Z}/2)$ constitutes a background class for a relative spin structure on $L_{i(i+1)}$. In [24, Lemma 3.19] Wehrheim and Woodward exhibit such background classes for Lagrangian correspondences associated to elementary tangles, in the general case of a compact Lie group $G$ and suitable choices of conjugacy classes. Their argument can be generalized to non-elementary tangles, but due to our choice of $G = SU(2)$ and the conjugacy class $C_0$ of traceless elements, we require only what amounts to a special case of Wehrheim and Woodward’s construction. Namely, we claim that all of our Lagrangian correspondences are embeddings of spin manifolds—as we have already seen of $L_{01} = L_\alpha$ and $L_{34} = L_\beta$—so that it suffices to take $b_i = 0$ for all $i$.

To see that $L_{12}$ and $L_{23}$ are spin, we start by observing that, for instance, $R(Y_{12}^*)$ is a regular level set of a smooth submersion $\Phi : X \to SU(2)$ where $X$ is a cartesian product of copies of $SU(2)$ and $C_0$. The normal bundle of a regular level set is trivial, being isomorphic to the pullback of the tangent space at a point. It follows that the second Stiefel-Whitney class $w_2(R(Y_{12}^*))$ is given by the restriction $w_2(X)|_{R(Y_{12}^*)}$. Since $SU(2)$ is equivariantly spin, the product $X$ has a unique $SU(2)$-equivariant spin structure, and the same must then be true of $R(Y_{12}^*)$. After passing to the quotient by conjugation, we obtain a spin structure on $L_{12}$, and similarly for $L_{23}$.
Disk numbers: The identifications of $L_{01} \circ L_{12}$ and $L_{23} \circ L_{34}$ with $L'_\alpha$ and $L'_\beta$, resp., imply that $\Phi(L_{01} \circ L_{12}) = \Phi(L'_\alpha) = \Phi(L'_\beta) = -\Phi(L_{23} \circ L_{34})$. The sign change in the right-most term is a result of the composition $L_{23} \circ L_{34}$ residing in $M_2^\perp \times M_0$ with the sign of the symplectic form switched on Floer homology in (27) that $\Phi(L_{01} \circ L_{12}) + \Phi(L_{23} \circ L_{34}) = 0$ is satisfied. As noted in the proof of [25, Theorem 5.4.1], it is enough to verify that the disk numbers of the Lagrangian correspondences sum to 0 for just one of the generalized correspondences appearing in (25). That this holds for the others may then be deduced \textit{a posteriori} from the embedded composition theorem.

This completes the proof of Theorem 4.0.3

4.2. Naturality. It would be preferable to have $\text{SIK}(Y, K)$ assign a concrete group to the pair $(Y, K)$, rather than an isomorphism class of groups, i.e. to be a \textit{natural} homology invariant. Only for natural invariants does it make sense to speak of e.g. isomomorphisms of the Floer homology induced by cobordisms.

Loosely speaking, given a pair of Heegaard diagrams $\mathcal{H}$ and $\mathcal{H}'$ as above, two different sequences of Heegaard moves $\mathcal{H} \rightarrow \mathcal{H}'$ correspond to a loop $\mathcal{H}_t$ in the space of compatible diagrams based at $\mathcal{H}$, and the obstruction to naturality is the monodromy of $\text{SIK}$ along $\mathcal{H}_t$. Juh\'asz, Thurston and Zemke [9] furnish a set of four conditions which being met imply that an algebraic object associated to pointed Heegaard diagrams has trivial monodromy and is therefore a natural invariant. Horton uses this with an application of quilted Floer homology, along the lines of the one used above, to demonstrate naturality for the 3-manifold symplectic instanton homology $\text{SI}(Y)$, and so it would seem that the same should hold here:

Conjecture 4.2.1 (Naturality). As a Lagrangian Floer homology invariant of a multi-pointed Heegaard diagram, $\text{SIK}(\mathcal{H})$ satisfies the four conditions of a strong Heegaard invariant, see [9, Definition 2.32]. There is therefore a canonical group representing its isomorphism class, which one may take to be the definition of $\text{SIK}(Y, K)$.

5. Examples and computations

Example 5.0.1 (Empty link). If $K = \emptyset$ is the empty link, then a Heegaard diagram compatible with $(Y, \emptyset)$ is just a triply-pointed diagram of the kind used to construct Horton’s symplectic instanton homology $\text{SI}(Y)$ of the 3-manifold. We may therefore simply define $\text{SIK}(Y, \emptyset) := \text{SI}(Y)$, which is a concrete homology group owing to the proven naturality of the latter theory [8, Cor. 5.6]. In particular $\text{SIK}(S^3, \emptyset) \cong \mathbb{Z}$ and $\text{SIK}(S^2 \times S^1, \emptyset) \cong H^*(S^3) \cong \mathbb{Z} \oplus \mathbb{Z}$ (see [8, Sec. 9]).

Example 5.0.2 (Unlinks in $S^3$). Let $Y = S^3$ and $K = U^\ell$ be the unlink with $\ell \geq 1$ components. For this we may choose a genus-0 Heegaard diagram $\mathcal{H}_0$ with respect to which each component of $U^\ell$ is in 1-bridge position. In this case it is easy to see that the induced Lagrangians $L_\alpha = L_\beta = L \cong (S^2)^{\times \ell}$ coincide in $\mathcal{R}_{0,2\ell+3}$, hence $\text{SIK}(\mathcal{H}_0) = \text{HF}(L)$ is the self-Floer homology of $L$.

Let $f : L \rightarrow \mathbb{R}$ be a $C^2$-small Morse function. This induces a Hamiltonian perturbation $L'$ of $L$ lying in a Weinstein neighborhood of the latter, with $\text{Crit}(f) = L \cap L'$. 



We may arrange that $f$ has a unique local minimum $x_0$ and take this as the base intersection point in grading $0 \mod \mathbb{Z}/2$, thereby fixing the relative $\mathbb{Z}/2$-grading on the self-Floer cochain complex $\text{CF}^*(L) = \text{CF}^*(L, L')$. In exhibiting his spectral sequence relating self-Floer cohomology to singular cohomology, Oh [17] demonstrates that $\text{CF}^*(L)$ can be identified with the Morse cochain complex $C^*_f(L)$ in such a way that the $\mathbb{Z}/N_L$-grading on the Floer complex aligns with usual grading of the Morse complex:

\[(31) \quad \text{CF}^i \mod N_L(L) = \bigoplus_{j \equiv i \mod N_L} C^j_f(L).\]

Since both the Floer and Morse coboundary maps raise degree by 1, and since $N_L$ is necessarily even, it is evident from (31) that, if $L$ admits a Morse function with all critical points residing in even degree, the self-Floer cohomology is isomorphic to the Morse cohomology, and thus to singular cohomology. Hence we obtain the self-Floer homology, which is the dual of and isomorphic to the Floer cohomology.

Because in our case $L$ is a product of 2-spheres, we may choose $f$ to be a perfect Morse function, with all critical points of even index. Therefore $\text{SIK}(\mathcal{H}_0) = \text{HF}(L) \cong H^*(\mathbb{S}^2 \times \mathbb{S}^2)$. 


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