On the absolute constants in the Berry–Esseen type inequalities for identically distributed summands

Irina Shevtsova

Abstract

By a modification of the method that was applied in (Korolev and Shevtsova, 2010), here the inequalities

\[ \Delta_n \leq 0.3328 \cdot \frac{\beta_3 + 0.429}{\sqrt{n}}, \]

\[ \Delta_n \leq 0.33554 \cdot \frac{\beta_3 + 0.415}{\sqrt{n}}, \]

are proved for the uniform distance \( \Delta_n \) between the standard normal distribution function and the distribution function of the normalized sum of an arbitrary number \( n \geq 1 \) of independent identically distributed random variables with zero mean, unit variance and finite third absolute moment \( \beta_3 \). The first of these two inequalities improves one that was proved in (Korolev and Shevtsova, 2010), and as well sharpens the best known upper estimate for the absolute constant \( C_0 \) in the classical Berry–Esseen inequality to be \( C_0 < 0.4756 \), since \( 0.3328(\beta_3 + 0.429) \leq 0.3328 \cdot 1.429 \beta_3 < 0.4756 \beta_3 \) by virtue of the condition \( \beta_3 \geq 1 \). The second of these inequalities is also a structural improvement of the classical Berry–Esseen inequality, and as well sharpens the upper estimate for \( C_0 \) still more to be \( C_0 < 0.4748 \).

Keywords: central limit theorem, Berry–Esseen inequality, absolute constant, smoothing inequality, characteristic function

MSC classes: 60F05

1 Introduction and formulation of the main results

By \( \mathcal{F}_3 \) we will denote the set of distribution functions with zero mean, unit variance and finite third absolute moment \( \beta_3 \). Let \( X_1, X_2, \ldots \) be independent random variables with common distribution function \( F \in \mathcal{F}_3 \) defined on a probability space \((\Omega, \mathcal{A}, P)\). Throughout the paper by a distribution function we will mean its left-continuous version. Denote

\[ F_n(x) = F^{*n}(x\sqrt{n}) = P(X_1 + \ldots + X_n < x\sqrt{n}), \]

\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt, \quad x \in \mathbb{R}. \]

The classical Berry–Esseen theorem states that there exists a finite positive absolute constant \( C_0 \) which guarantees the validity of the inequality

\[ \Delta_n \equiv \sup_x |F_n(x) - \Phi(x)| \leq C_0 \beta_3 / \sqrt{n} \quad (1) \]
for all \( n \geq 1 \) and any \( F \in \mathcal{F}_3 \) (Berry, 1941), (Esseen, 1942). The problem of establishing the best value of the constant \( C_0 \) in inequality (1) has a long history and is very rich in deep and interesting results. A detailed history of the efforts to lower the upper estimates of \( C_0 \) from the original works of A. Berry (Berry, 1941) and C.-G. Esseen (Esseen, 1942) to the papers of I. Shiganov (Shiganov, 1982), (Shiganov, 1986) can be found in (Korolev and Shevtsova, 2009). Here we will give an outline of the recent history of the subject.

In 2006 I. Shevtsova improved Shiganov’s upper estimate \( C_0 \leq 0.7655 \) by approximately 0.06 and obtained the estimate \( C_0 \leq 0.7056 \) (Shevtsova, 2006). In 2008 she sharpened this estimate to \( C_0 \leq 0.7005 \) (Shevtsova, 2008). In 2009 the mutually beneficial competition for the best estimate of the constant became especially keen. On 8 June, 2009 I. Tyurin submitted his paper (Tyurin, 2010a) to the «Theory of Probability and Its Applications». That paper, along with other results, contained the estimate \( C_0 \leq 0.5894 \). Two days later the summary of those results was submitted to «Doklady Akademii Nauk» (translated into English as «Doklady Mathematics») (Tyurin, 2009b). Independently, on 14 September, 2009 V. Korolev and I. Shevtsova submitted their paper (Korolev and Shevtsova, 2009) to the «Theory of Probability and Its Applications». In that paper the inequality

\[
\Delta_n \leq 0.34445 \cdot \frac{\beta_3 + 0.489}{\sqrt{n}}, \quad n \geq 1,
\]

was proved which holds for any distribution \( F \in \mathcal{F}_3 \) yielding the estimate \( C_0 \leq 0.5129 \) by virtue of the condition \( \beta_3 \geq 1 \).

On 17 November, 2009 the paper (Tyurin, 2010c) was submitted to the «Russian Mathematical Surveys» (its English version (Tyurin, 2009d) appeared on 3 December, 2009 on arXiv:0912.0726). In this paper together with the other results the estimate \( C_0 \leq 0.4785 \) was proved.

By a modification of the method used in (Korolev and Shevtsova, 2009) with sharpened estimate of the difference of characteristic functions in the vicinity of zero the authors of the mentioned work proved the inequality

\[
\Delta_n \leq 0.33477 \cdot \frac{\beta_3 + 0.429}{\sqrt{n}} \tag{2}
\]

in the paper (Korolev and Shevtsova, 2010) submitted to «Scandinavian Actuarial Journal» on 16 March 2010 and published online on 04 June, 2010. Inequality (2) also improves the upper bound for the absolute constant \( C_0 \), since it implies the estimate \( C_0 \leq 0.33477 \cdot 1.429 < 0.4784 \).

In his oral communication at the 10th International Vilnius Conference on Probability Theory and Mathematical Statistics on 1 July, 2010 I. Tyurin announced the same estimate as in (Korolev and Shevtsova, 2010): \( C_0 \leq 0.4784 \). However, in the corresponding abstract (Tyurin, 2010) no concrete values of the constant \( C_0 \) were presented.

In his talk at the 3rd Northern Triangular Seminar organized by the Euler International Mathematical Institute (an international office of St.-Petersburg Department of Steklov Institute of Mathematics) on 11 April, 2011, the presentation (Tyurin, 2011) of which is available at [http://www.pdmi.ras.ru/EIMI/2011/NTS/presentations/tyurin.pdf](http://www.pdmi.ras.ru/EIMI/2011/NTS/presentations/tyurin.pdf) I. Tyurin announced the estimate \( C_0 \leq 0.4774 \). However, the proof of that result was not given in (Tyurin, 2011).
As concerns the lower estimates for \( C_0 \), in 1956 C.-G. Esseen found the bound: \( C_0 \geq C_E \) with
\[
C_E = \frac{\sqrt{10} + 3}{6\sqrt{2\pi}} = 0.409732\ldots
\]
(Eseen, 1956). In 1967 V. Zolotarev put forward the hypothesis that \( C_0 = C_E \) in (1) (Zolotarev, 1967a), (Zolotarev, 1967b). However, up till now this hypothesis has been neither proved nor rejected.

By a modification of the method used in (Korolev and Shevtsova, 2010), here we prove the following

**Theorem 1.** For all \( n \geq 1 \) and all \( F \in \mathcal{F}_3 \) we have
\[
\Delta_n \leq 0.3328 \cdot \frac{\beta_3 + 0.429}{\sqrt{n}}.
\]

Obviously, inequality (3) sharpens (2) for all \( F \in \mathcal{F}_3 \) and \( n \geq 1 \). Moreover, under the conditions imposed on the moments of the random variable \( X_1 \) we always have \( \beta_3 \geq 1 \). Therefore, \( 0.3328(\beta_3 + 0.429) \leq 0.3328 \cdot 1.429\beta_3 < 0.4756\beta_3 \), which slightly improves the estimate \( C_0 \leq 0.4774 \) announced in (Tyurin, 2011). However, the method proposed below allows to improve the upper bound for \( C_0 \) still more, if replace 0.429 by another constant. Namely, the following statement holds.

**Theorem 2.** For all \( n \geq 1 \) and all \( F \in \mathcal{F}_3 \) we have
\[
\Delta_n \leq 0.33554 \cdot \frac{\beta_3 + 0.415}{\sqrt{n}}.
\]

**Remark 1.** Inequality (4) is more precise than (3) for \( \beta_3 < 1.2854\ldots \). For other values of \( \beta_3 \) inequality (3) is better.

Since \( 0.33554(\beta_3 + 0.415) \leq 0.33554 \cdot 1.415\beta_3 < 0.4748\beta_3 \), we obtain

**Corollary 1.** The classical Berry–Esseen inequality (1) holds with \( C_0 = 0.4748 \).

**Remark 2.** Inequality (4) is sharper than the classical Berry–Esseen inequality (1) with the best known constant \( C_0 = 0.4748 \) presented in corollary 1 for all possible values of \( \beta_3 \geq 1 \). Corollary 1 also slightly improves the estimate of the absolute constant in the classical Berry–Esseen inequality announced in (Tyurin, 2011).

**Remark 3.** Even if the hypothesis of V. M. Zolotarev that \( C_0 = C_E = 0.4097\ldots \) in (1) (see (Zolotarev, 1967a), (Zolotarev, 1967b)) turns out to be true, then, due to that \( \beta_3 \geq 1 \), inequalities (3) and (4) will be sharper than the classical Berry–Esseen inequality (1) for \( \beta_3 \geq 1.86 \) and \( \beta_3 \geq 1.88 \) respectively.

## 2 Proofs

Here we use the approach proposed and developed by V. M. Zolotarev in his works (Zolotarev, 1965), (Zolotarev, 1966), (Zolotarev, 1967a), (Zolotarev, 1967b), modified in (Korolev and Shevtsova, 2010). Below we will point out only the main ideas which distinguish this work from the previous ones (for details see (Korolev and Shevtsova, 2010)).
Denote
\[ f(t) = \mathbb{E} e^{itX_1}, \quad f_n(t) = \left( f \left( \frac{t}{\sqrt{n}} \right) \right)^n, \quad r_n(t) = |f_n(t) - e^{-t^2/2}|, \quad t \in \mathbb{R}. \]

**Lemma 1** (Prawitz, 1972). For an arbitrary distribution function \( F \) and \( n \geq 1 \) for any \( 0 < t_0 \leq 1 \) and \( T > 0 \) we have the inequality
\[
\Delta_n \leq 2 \int_0^{t_0} |K(t)|r_n(Tt) \, dt + 2 \int_{t_0}^1 |K(t)| \cdot |f_n(Tt)| \, dt + \frac{i}{2} \int_0^{t_0} K(t) - i \frac{e^{-T^2t^2/2}}{2\pi} \, dt + \frac{1}{\pi} \int_{t_0}^\infty e^{-T^2t^2/2} \, dt,
\]
where
\[
K(t) = \frac{1}{2}(1 - |t|) + \frac{i}{2} \left( (1 - |t|) \cot \pi t + \frac{\text{sign} t}{\pi} \right), \quad -1 \leq t \leq 1.
\]

Now consider the estimates of the characteristic functions appearing in lemma 1.

**Lemma 2.** For all \( t \in \mathbb{R} \)
\[
r_n(t) \leq 2e^{-t^2/2} \int_0^{[t]} u e^{u^2/2} \sin \left( \frac{u \ell}{4} + \frac{\pi}{2} \right) \left| f \left( \frac{u}{\sqrt{n}} \right) \right|^{n-1} \, du,
\]
\[
r_n(t) \leq 2 \int_0^{[t]} u e^{u^2/(2n)} \sin \left( \frac{u \ell}{4} + \frac{\pi}{2} \right) \, du \cdot \frac{1}{n} \sum_{k=0}^{n-1} \left| f \left( \frac{t}{\sqrt{n}} \right) \right|^{n-k-1} \exp \left\{ - \frac{k t^2}{2n} \right\}.
\]

**Proof.** The first estimate is proved in (Korolev, Shevtsova, 2010), and the second one follows from the inequality
\[
|a^n - b^n| \leq |a - b| \sum_{k=0}^{n-1} |a|^k |b|^{n-k-1},
\]
which is valid for any complex numbers \( a \) and \( b \), where we put \( a = e^{-t^2/(2n)} \), \( b = f(t/\sqrt{n}) \), as well as the first estimate of the lemma with \( n = 1 \) being applied.

Now we proceed to the estimation of \( |f(t)| \). For \( \varepsilon > 0 \) and \( t \in \mathbb{R} \) set
\[
\psi(t, \varepsilon) = \begin{cases} \ell^2/2 - \varepsilon |t|^3, & |t| \leq \theta_0/\varepsilon, \\ (1 - \cos \varepsilon t)/\varepsilon^2, & \theta_0 < \varepsilon |t| \leq 2\pi, \\ 0, & |t| > 2\pi/\varepsilon, \end{cases}
\]
where \( \theta_0 = 3.99589567 \ldots \) is the unique root of the equation
\[
\theta^2 + 2\theta \sin \theta + 6(\cos \theta - 1) = 0, \quad \pi \leq \theta \leq 2\pi,
\]
\[
\varepsilon \equiv \sup_{x > 0} x^{-3} \left| \cos x - 1 + x^2/2 \right| = \theta_0^{-3} \left( \cos \theta_0 - 1 + \theta_0^2/2 \right) = 0.09916191 \ldots
\]
It can easily be made sure that the function \( \psi(t, \varepsilon) \) monotonically decreases in \( \varepsilon > 0 \) for any fixed \( t \in \mathbb{R} \). Denote the Lyapunov fraction by \( \ell = \beta_3/\sqrt{n} \).
Lemma 3. For any $F \in \mathcal{F}_3$, $n \geq 1$ the following estimates hold:

$$|f_n(t)| \leq \left[ 1 - 2\psi(t, \ell + 1/\sqrt{n})/n \right]^{n/2} \leq \exp\{-\psi(t, \ell + 1/\sqrt{n})\} \leq \exp\{-t^2/2 + \kappa(\ell + 1/\sqrt{n})|t|^3\}, \quad t \in \mathbb{R},$$

$$|f_n(t)| \leq \left[ \left( 1 - \frac{\psi(t, \ell)}{n} \right)^2 + \frac{t^2\ell^2/6}{36n^2} \right]^{n/2} \leq \exp\{-\psi(t, \ell) + \frac{\psi^2(t, \ell)}{2n} + \frac{t^2\ell^2}{72n}\}, \quad |t| \leq \frac{\pi}{2}\sqrt{n}.$$

Proof. For the first series of the estimates see (Prawitz, 1973) and (Korolev, Shevtsova, 2010). Let us prove the second one. Evidently,

$$|f_n(t)| = \left[ \left( \mathbb{E} \cos \frac{tX_1}{\sqrt{n}} \right)^2 + \left( \mathbb{E} \sin \frac{tX_1}{\sqrt{n}} \right)^2 \right]^{n/2}.$$

Since $\mathbb{E}X_1 = 0$, for all $t \in \mathbb{R}$ we have

$$\left( \mathbb{E} \sin tX_1 \right)^2 = \left( \mathbb{E} (\sin tX_1 - tX_1) \right)^2 \leq \left( \mathbb{E} |tX_1 - tX_1| \right)^2 \leq \left( \frac{|t|^3}{6} \mathbb{E} |X_1|^3 \right)^2 = \frac{\beta^2^3|t|^6}{36}.$$

In (Saković, 1965) it is proved that for any random variable $X$ with $\mathbb{E}X^2 = 1$ the inequality $\mathbb{E} \cos tX \geq 0$ holds for all $|t| \leq \pi/2$. On the other hand, from (Prawitz, 1973) it follows that for any random variable $X$ with $\mathbb{E}X^2 = 1$ and $\mathbb{E}|X|^3 < \infty$

$$\mathbb{E} \cos(tX) \leq 1 - \psi(t, \mathbb{E}|X|^3), \quad t \in \mathbb{R}.$$

Thus, for all $|t| \leq \pi \sqrt{n}/2$ we have

$$\left| \mathbb{E} \cos \frac{tX_1}{\sqrt{n}} \right| = \mathbb{E} \cos \frac{tX_1}{\sqrt{n}} \leq 1 - \psi(t/\sqrt{n}, \beta_3) = 1 - \frac{\psi(t, \ell)}{n},$$

which together with the estimate for $(\mathbb{E} \sin tX_1)^2$ leads to the desired result.

Finally, the process of computational optimization can be properly organized with the help of the following statements.

Lemma 4 (Bhattacharya and Ranga Rao, 1976). For any distribution $F$ with zero mean and unit variance we have

$$\rho(F, \Phi) \leq \sup_{x > 0} \{ \Phi(x) - x^2/(1 + x^2) \} = 0.54093654\ldots$$

Moreover, repeating the algorithms described in (Prawitz, 1975) and (Gaponova, Shevtsova, 2009) we conclude that inequality (3) holds true at least for $\ell \leq 0.0357$ and inequality (4) is true at least for $\ell \leq 0.0353$.

The lemmas presented above give the grounds for restricting the domain of the values of $\varepsilon = (\beta_3 + 0.429)/\sqrt{n} = \ell + 0.429/\sqrt{n}$ in theorem 1 and $\varepsilon = (\beta_3 + 0.415)/\sqrt{n} = \ell + 0.415/\sqrt{n}$ in theorem 2 by a bounded interval separated from zero. The supremum in $n$ is estimated by using the monotonically decreasing majorants for the characteristic functions for $n$ large enough. The fact that all the estimates used for the characteristic functions monotonically increase in $\varepsilon$
allows to estimate the supremum in $\varepsilon$ by computation of the quantity under consideration in a finite number of points as in the preceding works.

**Proof** of theorem 1. Using the algorithm described above we found one extremal point: $n = 4$, $\varepsilon \approx 0.8565$ ($\beta_3 \approx 1.284$, $t_0 \approx 0.398$, $T \approx 5.451$), the extremal value do not exceeding 0.3328, which proves theorem 1.

**Proof** of theorem 2. The computations carried out according to the algorithm described above show that there are two extremal points: $n = 4$, $\varepsilon \approx 0.838$ ($\beta_3 \approx 1.261$, $t_0 \approx 0.394$, $T \approx 5.513$) and $n = 6$, $\varepsilon \approx 0.5777$ ($\beta_3 \approx 1$, $t_0 \approx 0.317$, $T \approx 7.723$). Both extremal values do not exceed 0.33554, whence the assertion of theorem 1 follows.

**Acknowledgements.** The author has the pleasure to express her sincere gratitude to professor V. Yu. Korolev for permanent attention to this work.

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