SHORT-TIME EXISTENCE THEOREM FOR THE CR TORSION FLOW

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Abstract. In this paper, we study the torsion flow which is served as the CR analogue of the Ricci flow in a closed pseudohermitian manifold. We show that there exists a unique smooth solution to the CR torsion flow in a small time interval with the CR pluriharmonic function as an initial data. In spirit, it is the CR analogue of the Cauchy-Kovalevskaya local existence and uniqueness theorem for analytic partial differential equations associated with Cauchy initial value problems.

1. Introduction

One of the goals for differential geometry and geometric analysis is to understand and classify the singularity models of a nonlinear geometric evolution equation, and to connect it to the existence problem of geometric structures on manifolds. For instance in 1982, R. Hamilton ([H3]) introduced the Ricci flow. Then by studying the singularity models ([H2], [Pe1], [Pe2], [Pe3]) of Ricci flow, R. Hamilton and G. Perelman solved the Thurston geometrization conjecture and Poincare conjecture for a closed 3-manifold in 2002.

It is natural then to investigate a corresponding geometrization problem of closed CR manifolds by finding a CR analogue of the Ricci flow. More precisely, let us recall that a strictly pseudoconvex CR structure on a pseudohermitian $(2n + 1)$-manifold $(M, \xi, J, \theta)$ is given by a co-oriented plane field $\xi = \ker \theta$, where $\theta$ is a contact form, together with a

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\end{itemize}
compatible complex structure $J$. Given this data, there is a natural connection, the so-called Tanaka-Webster connection or pseudohermitian connection. We denote the torsion of this connection by $A_{J,\theta}$, and the Tanaka-Webster curvature by $W$. Then, following this direction, we consider the following so-called torsion flow on $(M, \xi, J, \theta) \times [0, T)$:

$$
\begin{align*}
\frac{\partial J}{\partial t} &= 2A_{J,\theta}, \\
\frac{\partial \theta}{\partial t} &= -2W\theta, \\
\theta &= e^{2\lambda}\theta_0
\end{align*}
$$

(1.1)

It seems to us that the torsion flow (1.1) is the right CR analogue of the Ricci flow. It is the negative gradient flow of CR Einstein-Hilbert functional (1.3) in view of (1.4). More precisely, let $\{T, Z_{\alpha}, Z_{\bar{\beta}}\}$ be a frame of $TM \otimes \mathbb{C}$ with $\xi \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$, where $\{Z_{\alpha}\}$ is any local frame of $T^{1,0}$, and $Z_{\bar{\beta}} = \overline{Z_{\beta}} \in T^{0,1}$. Then $\{\theta, \theta^\alpha, \theta^\bar{\beta}\}$, the coframe dual to $\{T, Z_{\alpha}, Z_{\bar{\beta}}\}$, satisfies

$$
d\theta = i h_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^\bar{\beta},
$$

where $h_{\alpha\bar{\beta}}$ is a positive definite Levi metric. By the Gram-Schmidt process we can always choose $Z_{\alpha}$ such that $h_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}$; throughout this paper, we shall take such a frame. Then we write $J = i\theta^\alpha \otimes Z_{\alpha} - i\theta^\bar{\beta} \otimes Z_{\bar{\beta}}$. Define

$$
A_{J,\theta} := A_{\alpha\bar{\beta}}Z_{\bar{\beta}} \otimes \theta^\alpha + A_{\beta\bar{\alpha}}Z_{\beta} \otimes \theta^\bar{\alpha},
$$

and

$$
E = E_{\alpha\bar{\beta}}\theta^\alpha \otimes Z_{\bar{\beta}} + E_{\beta\bar{\alpha}}\theta^\bar{\alpha} \otimes Z_{\beta},
$$

and consider the general CR flow on $(M, \xi, J, \theta) \times [0, T)$ given by

$$
\begin{align*}
\frac{\partial J(t)}{\partial t} &= 2E(t), \\
\frac{\partial \theta}{\partial t} &= 2\eta(t)\theta(t).
\end{align*}
$$

(1.2)

The CR Einstein-Hilbert functional is defined by

$$
\mathcal{E}(J(t), \theta(t)) = \int_M W(t)d\mu.
$$

(1.3)
Here $d\mu = \theta \wedge d\theta^n$ is the volume form. Since \( \frac{\partial}{\partial t} d\mu = 4\eta(t) d\mu \), it follows that

\[
\frac{d}{dt} \mathcal{E}(J(t), \theta(t)) = -\int_M \{(A^\alpha_\beta E^\beta_\alpha + A^\alpha_\beta E^\beta_\alpha) - 2\eta W\} d\mu
\]

(1.4)

\[
= -2\int_M \|A_{J,\theta}\|^2 d\mu - 2\int_M W^2 d\mu
\]

\[
\leq 0
\]

if we put

\[ E = A_{J,\theta} \quad \text{and} \quad d\eta(t) = -W(t). \]

Unlike the Ricci flow, it is not clear that one can apply the so-called DeTurck’s trick \([\text{De}]\) to prove the short-time solution of the torsion flow. It is therefore natural to consider the following version of CR torsion flow (see section 4 for details)

\[
\begin{aligned}
\frac{\partial J}{\partial t} &= 2A_{J,\theta}, \\
\frac{\partial \theta}{\partial t} &= -2W\theta, \\
\theta &= e^{2\gamma_0} \theta_0, \quad \gamma(0) = \gamma_0; \quad P_{\beta}^{\gamma}(0) = 0 = W^\perp(x, 0).
\end{aligned}
\]

(1.5)

on $M \times [0, T)$ in which the initial Tanaka-Webster scalar curvature $W(x, 0)$ w.r.t. $e^{\gamma_0} \theta_0$ and $\gamma_0$ are pluriharmonic with respect to $\theta_0$ (see section 3) and

\[ W^\perp(x, t) := W(x, t) - W^{\text{ker}}(x, t). \]

Here $W^{\text{ker}}$ is the CR-pluriharmonic function with respect to $\theta_0$ and $\xi = \ker \theta_0$. In fact, if $M$ is a hypersurface in $\mathbb{C}^{n+1}$, i.e. $M = \partial \Omega$ for a strictly pseudoconvex bounded domain $\Omega$ in $\mathbb{C}^{n+1}$, then for any pluriharmonic function $u : U \to \mathbb{R}$ ($\partial \bar{\partial}u = 0$) with $f := u|_M$, it follows that $f$ is a CR pluriharmonic function (see Definition 3.15). Moreover, for a simply connected $U \subset \overline{\Omega}$, there exists a holomorphic function $w$ in $U$ such that $u = \text{Re}(w)$.

In this paper, we show that there exists a unique smooth solution to the CR torsion flow (1.5) in a small time interval with the CR pluriharmonic function as an initial data. In the above spirit, it is the CR analogue of the Cauchy-Kovalevskaya local existence and uniqueness theorem for analytic partial differential equations associated with Cauchy initial value problems.
We first show that there always exist such $\theta_0$ in $\xi$ and $\gamma_0$ so that

\begin{equation}
\tag{1.6}
\mathcal{P}_\beta(\gamma_0) = 0 = W^\perp(x, 0).
\end{equation}

It is well-known, for the CR Yamabe problem on $(M, \xi)$, that there always exists a contact form $\tilde{\theta}_0$ of constant Tanaka-Webster scalar curvature in a contact class $[\tilde{\theta}_0]$ with $\xi = \text{ker} \tilde{\theta}_0$. Furthermore, we can write

$$
\tilde{\theta}_0 = e^{2\lambda_0} \tilde{\theta}_0
$$

for some function $\lambda_0$. Rewrite

$$
\tilde{\theta}_0 = e^{2\lambda_0} \tilde{\theta}_0 = e^{2(\lambda_0)^\text{ker}} (e^{2(\lambda_0)^\perp} \tilde{\theta}_0),
$$

we can choose the particular background contact form $\theta_0$ as

\begin{equation}
\tag{1.7}
\theta_0 := e^{2(\lambda_0)^\perp} \tilde{\theta}_0 \quad \text{and} \quad \gamma_0 := (\lambda_0)^\text{ker}.
\end{equation}

It follows that the initial contact form

$$
\theta(0) = \tilde{\theta}_0 = e^{2\gamma_0} \theta_0
$$

satisfies (1.6). Then we are in the situation as in (1.5).

In this paper, we first follow Hamilton’s original ideas ([H1]) to prove the following short-time existence result for the CR torsion flow (1.5) in a closed pseudohermitian $(2n+1)$-manifold $(M, \xi, \theta_0)$. In the appendix, we also give an alternative proof from the point view of Lax-Milgram theorem ([GT]) for $n = 1$.

**Theorem 1.1.** Let $(M, \xi)$ be a closed CR $(2n+1)$-manifold. Let $J_0$ be any $C^\infty$ smooth oriented CR structure compatible with $\xi$. If there exist $\gamma_0$ and $C^\infty$ smooth oriented contact form $\theta_0$ with $\xi = \text{ker} \theta_0$ so that

\begin{equation}
\tag{1.6}
\mathcal{P}_\beta(\gamma_0) = 0 \quad \text{and} \quad W^\perp(x, 0) = 0.
\end{equation}

Then there exists $\delta > 0$ and a unique smooth solution $(J(t), \theta(t))$ to the CR torsion flow (1.5) on $(M, \xi, J_0, \theta_0) \times [0, \delta)$ such that $(J(0), \theta(0)) = (J_0, e^{2\gamma_0} \theta_0)$. 


In particular, if we choose the special forms for $\gamma_0$ and $\theta_0$ as in (1.7), then we have the following short-time existence result.

**Corollary 1.1.** Let $(M, \xi)$ be a closed CR $(2n + 1)$-manifold. Let $J_0$ be any $C^\infty$ smooth oriented CR structure compatible with $\xi$ and choose a $C^\infty$ smooth oriented contact form $\theta_0$ and a smooth function $\gamma_0$ as in (1.7). Then there exists $\delta > 0$ and a unique smooth solution $(J(t), \theta(t))$ to the CR torsion flow (1.5) on $(M, \xi, J_0, \theta_0) \times [0, \delta)$ such that $(J(0), \theta(0)) = (J_0, e^{2\gamma_0}\theta_0)$ with

$$P_\beta(\gamma_0) = 0 \quad \text{and} \quad W^\perp(x, 0) = 0.$$

**Remark 1.1.** Let $(M, \xi)$ be a closed CR 3-manifold with a pseudo-Einstein contact form $\bar{\theta}_0$ of the pluriharmonic scalar curvature (in particular, the constant scalar curvature). Then there exists a pluriharmonic function $\gamma_0$ such that $\bar{\theta}_0 = e^{\gamma_0}\theta_0$ for some pseudo-Einstein contact form $\theta_0$ due to J. Lee ([L1]) and K. Hirachi ([Hi]). We will study the problem of asymptotic convergence of solutions of the CR torsion flow (1.5) on this special situation for the forthcoming topic.

The torsion flow greatly simplifies if the torsion vanishes. This only happens in very special setups. Indeed, CR 3-manifolds with vanishing torsion are $K$-contact, meaning that the Reeb vector field is a Killing vector field for the contact metric $g = \frac{1}{2}d\theta + \theta^2$. In general, one can still hope that the torsion flow improves properties of the contact manifold underlying the CR-manifold. It is the case in a closed homogeneous pseudohermitian 3-manifold whose Lie algebra is isomorphic to $su(2)$. The torsion flow reduces to an ODE if we start with some appropriate initial conditions and we are able to obtain the long-time existence and asymptotic convergence of solutions for the (normalized) torsion flow (1.5) in this special case. More precisely, for any choice of homogeneous complex structure $J$ on $SU(2)$, the solution of the normalized torsion flow exists for all times and converges to the unique standard CR-structure $J_{can}$. These computations illustrate the behavior of the torsion flow in special
cases, and in these cases the torsion flow behaves as can be expected from a Ricci-like flow. We refer to our previous paper [CKW] for more details.

In general, unlike the Hamilton Ricci flow, the problem of asymptotic convergence of solutions of the CR torsion flow is widely open in closed CR 3-manifolds. The structure of CR torsion solitons may be a necessary step in understanding the asymptotic convergence of solutions of the CR torsion flow. Indeed, one expects CR torsion solitons to model singularity formations of the CR torsion flow. In the joint works with H.-D. Cao and C.-W. Chen ([CaCC1], [CaCC2]), we investigate the geometry and classification of closed three-dimensional CR Yamabe and Torsion solitons. We obtain a classification theorem of complete three-dimensional CR Yamabe solitons of vanishing torsion which are the same as torsion solitons. Furthermore, by deriving the CR Hamilton’s type Harnack quantity, we are able to show that any closed three-dimensional CR torsion soliton must be the standard Sasakian space form ([T]).

We conclude this introduction with a brief plan of the paper. In Section 2, we survey basic notions in CR geometry. In section 3, we give some preliminary results concerning the short time existence of the torsion flow, in particular the evolution equation of the general CR flow ([1.2]) and the invariance property for the CR Paneitz operator as in Lemma 3.1. In section 4, We prove the main Theorem 1.1 as well as Corollary 1.1. In the appendix, we give an alternative proof via the Lions-Lax-Milgram theorem.

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2. Preliminaries

In this section, we introduce some basic notions from pseudohermitian geometry as in [L1] and [L2], and refer to these papers for proofs and more references.
**Definition 2.1.** Let $M$ be a smooth manifold and $\xi \subset TM$ a subbundle. A **CR structure** on $\xi$ consists of an endomorphism $J : \xi \to \xi$ with $J^2 = -\text{id}$ such that the following integrability condition holds:

(i) If $X, Y \in \xi$, then so is $[JX, Y] + [X, JY]$.

(ii) $J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y]$.

The CR structure $J$ can be extended to $\xi \otimes \mathbb{C}$, which we can then decompose into the direct sum of eigenspaces of $J$. The eigenvalues of $J$ are $i$ and $-i$, and the corresponding eigenspaces will be denoted by $T^{1,0}$ and $T^{0,1}$, respectively. The integrability condition can then be reformulated as

$$X, Y \in T^{1,0} \text{ implies } [X, Y] \in T^{1,0}.$$ 

Now consider a closed $2n + 1$-manifold $M$ with a cooriented contact structure $\xi = \ker \theta$. This means that $\theta \wedge d\theta^n \neq 0$. The **Reeb vector field** of $\theta$ is the vector field $T$ uniquely determined by the equations

$$\theta(T) = 1, \quad \text{and} \quad d\theta(T, \cdot) = 0.$$ 

A **pseudohermitian manifold** is a triple $(M^{2n+1}, \theta, J)$, where $\theta$ is a contact form on $M$ and $J$ is a CR structure on $\ker \theta$. The **Levi form** $\langle \cdot, \cdot \rangle$ is the Hermitian form on $T^{1,0}$ defined by

$$H(Z, W) = \langle Z, W \rangle = -i \langle d\theta, Z \wedge \overline{W} \rangle.$$ 

We can extend this Hermitian form $\langle \cdot, \cdot \rangle$ to $T^{0,1}$ by defining $\langle \overline{Z}, \overline{W} \rangle = \overline{\langle Z, W \rangle}$ for all $Z, W \in T^{1,0}$. Furthermore, the Levi form naturally induces a Hermitian form on the dual bundle of $T^{1,0}$, and hence on all induced tensor bundles.

We now restrict ourselves to strictly pseudoconvex manifolds, or in other words compatible complex structures $J$. This means that the Levi form induces a Hermitian metric $\langle \cdot, \cdot \rangle_{J, \theta}$ by

$$\langle V, U \rangle_{J, \theta} = d\theta(V, JU).$$
The associated norm is defined as usual: \( |V|^2_{J,\theta} = \langle V, V \rangle_{J,\theta} \). It follows that \( H \) also gives rise to a Hermitian metric for \( T^{1,0} \), and hence we obtain Hermitian metrics on all induced tensor bundles. By integrating this Hermitian metric over \( M \) with respect to the volume form \( d\mu = \theta \wedge d\theta^n \), we get an \( L^2 \)-inner product on the space of sections of each tensor bundle.

The **pseudohermitian connection** or **Tanaka-Webster connection** of \((J, \theta)\) is the connection \( \nabla \) on \( TM \otimes \mathbb{C} \) (and extended to tensors) given in terms of a local frame \( \{Z_\alpha\} \) for \( T^{1,0} \) by

\[
\nabla Z_\alpha = \omega_\alpha^\beta Z_\beta, \quad \nabla \bar{Z}_\alpha = \bar{\omega}_\alpha^\bar{\beta} \otimes \bar{Z}_{\bar{\beta}}, \quad \nabla T = 0,
\]

where \( \omega_\alpha^\beta \) is the 1-form uniquely determined by the following equations:

\[
d\theta^\beta = \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau^\beta, \\
\tau_\alpha \wedge \theta^\alpha = 0, \\
\omega_\alpha^\beta + \omega_{\bar{\beta}}^\bar{\alpha} = 0.
\]

Here \( \tau^\alpha \) is called the **pseudohermitian torsion**, which we can also write as

\[
\tau_\alpha = A_{\alpha\beta} \theta^\beta.
\]

The components \( A_{\alpha\beta} \) satisfy

\[
A_{\alpha\beta} = A_{\beta\alpha}.
\]

We often consider the **torsion tensor** given by

\[
A_{J,\theta} = A_{\alpha\beta} Z_\alpha \otimes \theta^\beta + A_{\alpha\bar{\beta}} \bar{Z}_{\alpha} \otimes \theta^\beta.
\]

We now consider the curvature of the Tanaka-Webster connection in terms of the coframe \( \{\theta = \theta^0, \theta^\alpha, \theta^\beta\} \). The second structure equation gives

\[
\Omega_{\beta}^\alpha = \Omega_{\bar{\beta}}^\alpha = d\omega_\beta^\alpha - \omega_\alpha^\gamma \wedge \omega_{\bar{\gamma}}^\alpha, \\
\Omega_0^\alpha = \Omega_{\alpha}^0 = \Omega_0^\beta = \Omega_{\bar{\beta}}^0 = \Omega_0^0 = 0.
\]
In [We Formulas 1.33 and 1.35], Webster showed that the curvature $\Omega_{\beta}^\alpha$ can be written as

\begin{equation}
\Omega_{\beta}^\alpha = R_{\beta}^\alpha \rho \theta^\rho \wedge \theta^\alpha + W_{\beta}^\alpha \rho \theta^\rho \wedge \theta - W_{\beta}^{\alpha \rho} \theta^\rho \wedge \theta + i\theta_{\beta} \wedge \tau^{\alpha} - i\tau_{\beta} \wedge \theta^\alpha,
\end{equation}

where the coefficients satisfy

\[ R_{\beta \alpha \rho \sigma} = R_{\alpha \beta \rho \sigma} = R_{\rho \alpha \beta \sigma} = R_{\rho \alpha \beta \sigma}, \quad W_{\beta}^\alpha \rho = W_{\alpha}^\beta \rho. \]

In addition, by [L2 (2.4)] the coefficients $W_{\alpha}^\beta \rho$ are determined by the torsion,

\[ W_{\alpha}^\beta \rho = A_{\alpha \rho \beta}. \]

Contraction of (2.1) yields

\[ \Omega_{\alpha}^\alpha = d\omega_{\alpha}^\alpha = R_{\alpha \rho} \theta^\rho \wedge \theta^\alpha + W_{\alpha}^\rho \theta^\rho \wedge \theta - W_{\alpha}^{\rho \sigma} \theta^\rho \wedge \theta - A_{\alpha \rho \sigma} \theta^\sigma \wedge \theta - A_{\alpha \rho \sigma} \theta^\sigma \wedge \theta \]

We will denote components of covariant derivatives by indices preceded by a comma. For instance, we write $A_{\alpha \beta \gamma}$ Here the indices $\{0, \alpha, \bar{\beta}\}$ indicate derivatives with respect to $\{T, Z_{\alpha}, Z_{\bar{\beta}}\}$. For derivatives of a scalar function, we will often omit the comma. For example, $\varphi_{\alpha} = Z_{\alpha} \varphi$, $\varphi_{\alpha \bar{\beta}} = Z_{\bar{\beta}} Z_{\alpha} \varphi - \omega_{\alpha}^\gamma (Z_{\bar{\beta}}) Z_{\gamma} \varphi$, $\varphi_0 = T \varphi$ for a (smooth) function $\varphi$.

For a real-valued function $\varphi$, the subgradient $\nabla_{\bar{b}} \varphi$ is defined as the unique vector field $\nabla_{\bar{b}} \varphi \in \xi$ such that $\langle Z, \nabla_{\bar{b}} \varphi \rangle = d\varphi(Z)$ for all vector fields $Z$ tangent to the contact distribution $\xi$. Locally $\nabla_{\bar{b}} \varphi = \varphi_{\alpha} Z_{\alpha} + \varphi_{\bar{\alpha}} Z_{\bar{\alpha}}$. Define the sublaplacian $\Delta_{\bar{b}}$ by

\[ \Delta_{\bar{b}} \varphi = \varphi_{\alpha}^\alpha + \varphi_{\bar{\alpha}}^{\bar{\alpha}}. \]

To consider smoothness for functions on strongly pseudo-convex manifolds, we recall below what the Folland-Stein space is $S_{k,p}$. Let $D$ denote a differential operator acting on functions. We say $D$ has weight $m$, denoted $w(D) = m$, if $m$ is the smallest integer such that $D$ can be
locally expressed as a polynomial of degree $m$ in vector fields tangent to the contact bundle $\xi$. We define the Folland-Stein space $S_{k,p}$ of functions on $M$ by

$$S_{k,p} = \{ f \in L^p \mid Df \in L^p \text{ whenever } w(D) \leq k \}. \tag{2.2}$$

We define the $L^p$ norm of $\nabla_b f, \nabla_b^2 f, \ldots$ to be $\left( \int |\nabla_b f|^p \theta \wedge d\theta^\alpha \right)^{1/p}, \left( \int |\nabla_b^2 f|^p \theta \wedge d\theta^\alpha \right)^{1/p}, \ldots$, respectively, as usual. So it is natural to define the $S_{k,p}$-norm $||f||_{S_{k,p}}$ of $f \in S_{k,p}$ as follows:

$$||f||_{S_{k,p}} \equiv \left( \sum_{0 \leq j \leq k} ||\nabla_b^j f||_{L^p}^p \right)^{1/p}.$$

In particular, we denote $S_k := S_{k,2}$ for $p = 2$. The function space $S_{k,p}$ with the above norm is a Banach space for $k \geq 0, 1 < p < \infty$. There are also embedding theorems of Sobolev type. For instance, $S_{2,2} \subset S_{1,4}$ (for $\dim M = 3$). We refer the reader to, for instance, [FS] for more discussions on these spaces. Finally, we also need commutation relations ([L2, Equation 2.15]), sometimes called Ricci identities. Let $\varphi$ be a scalar function and $\sigma = \sigma_\alpha \theta^\alpha$ be a $(1,0)$ form, then we have

\[
\begin{align*}
\varphi_{\alpha \beta} &= \varphi_{\beta \alpha}, \\
\varphi_{\alpha \beta} - \varphi_{\beta \alpha} &= i h_{\alpha \beta} \varphi_0, \\
\varphi_{0 \alpha} - \varphi_{\alpha 0} &= A_{\alpha \beta} \varphi_{\beta}, \\
\sigma_{\alpha,0 \beta} - \sigma_{\alpha,\beta 0} &= \sigma_{\alpha,\gamma} A_{\gamma \beta} - \sigma_{\gamma} A_{\alpha \beta,\gamma}, \\
\sigma_{\alpha,0 \beta} - \sigma_{\alpha,\beta 0} &= \sigma_{\alpha,\gamma} A_{\gamma \beta} + \sigma_{\gamma} A_{\alpha \beta,\gamma},
\end{align*}
\]

and

\[
\begin{align*}
\sigma_{\alpha,\beta \gamma} - \sigma_{\alpha,\gamma \beta} &= i A_{\alpha \gamma} \sigma_{\beta} - i A_{\alpha \beta} \sigma_{\gamma}, \\
\sigma_{\alpha,\beta \gamma} - \sigma_{\alpha,\gamma \beta} &= i h_{\alpha \beta} A_{\gamma \rho} \sigma_{\rho} - i h_{\alpha \gamma} A_{\beta \rho} \sigma_{\rho}, \\
\sigma_{\alpha,\beta \gamma} - \sigma_{\alpha,\gamma \beta} &= i h_{\beta \gamma} \sigma_{\alpha,0} + R_{\alpha \beta,\gamma} \sigma_{\rho},
\end{align*}
\]
We also mention some curvature identities. The ones that are relevant for us are the contracted CR Bianchi identities.

\[
R_{\rho\bar{\gamma},\gamma} - R_{\gamma\bar{\delta},\rho} = iA_{\alpha\gamma} h_{\rho\bar{\delta}} - iA_{\alpha\rho} h_{\gamma\bar{\delta}} ,
\]
\[
W_{\gamma} - R_{\gamma\bar{\delta}} = i(n-1)A_{\alpha\gamma} ,
\]
\[
R_{\rho\bar{\delta},0} = A_{\alpha\rho} \bar{\sigma} + A_{\beta\bar{\delta}} \bar{\beta} ,
\]
\[
W,0 = A_{\alpha\rho} \rho + A_{\beta\bar{\delta}} \bar{\beta} .
\]

3. Evolution Equations under the CR Curvature Flow

In this section, we give some preliminary results concerning the short time existence of the torsion flow. Let \( \theta(t) \) be a family of smooth contact forms and \( J(t) \) be a family of CR structures on \((M, J, \theta)\). We consider the following general CR flow on a closed pseudohermitian \((2n+1)\)-manifold \((M, J, \theta) \times [0, T)\) :

\[
\begin{aligned}
\frac{\partial J}{\partial t} &= 2E , \\
\frac{\partial \theta}{\partial t} &= 2\eta(t)\theta(t) .
\end{aligned}
\]

Here \( J = i\theta^\alpha \otimes Z_\alpha - i\theta^{\bar{\alpha}} \otimes Z_{\bar{\alpha}} \) and \( E = E_{\alpha} \bar{\theta}^\alpha \otimes Z_{\bar{\alpha}} + E_{\bar{\alpha}} \theta^{\bar{\alpha}} \otimes Z_\alpha \).

We start by deriving some evolution equations under the general flow (3.1) before specifying to the torsion flow, for which \( E = A_J \) (the torsion tensor), and \( \eta = -W \) (the Webster curvature). All computations will be done in a local frame. Fix a unit-length local frame \( \{Z_\alpha\} \) and let \( \{\theta^\alpha\} \) be its dual admissible 1-form. Let \( Z_{\alpha(t)}, \theta^\alpha(t) \) denote a unit-length frame and dual admissible 1-form with respect to \((J(t), \theta(t))\). Since \( \theta^\alpha(Z_{\beta(t)}) \) is a positive real function, we can write \( \dot{Z}_\alpha = F_{\alpha}^{\beta} Z_\beta + G_{\alpha}^{\bar{\alpha}} Z_{\bar{\alpha}} \) where \( F_{\alpha}^{\beta} \) are real and \( G_{\alpha}^{\bar{\alpha}} \) are complex. The fact that \( Z_{\alpha(t)} \) is an orthonormal frame means that

\[
\delta_{\alpha\beta} = -id\theta(t)(Z_{\alpha(t)} \wedge Z_{\beta(t)}) .
\]
By differentiating and substituting the above expression for $\dot{Z}_\alpha$, we obtain $F_\alpha^\beta = -\eta \delta_\alpha^\beta$. By differentiating $J(t)Z_\alpha(t) = iZ_\alpha(t)$ we find

$$0 = \dot{J}Z_\alpha + J\dot{Z}_\alpha - i\dot{Z}_\alpha = 2F_\alpha^\beta Z_\beta - 2iG_\alpha^\beta Z_\beta,$$

so

$$\dot{Z}_\alpha = -\eta Z_\alpha - iE_\alpha^\beta Z_\beta.$$

Now differentiate the identities

$$d\theta = i\eta_{\alpha\beta} \theta^\alpha \wedge \theta^\beta, \quad \theta^\alpha(Z_\beta(t)) = \delta^\alpha_\beta, \quad \text{and} \quad \theta^\alpha(Z_\beta(t)) = 0,$$

to deduce that

$$\dot{\theta}^\alpha = 2i\eta^\alpha \theta + \eta \theta^\alpha - iE_\alpha^\beta \theta^\beta.$$

Now we differentiate (3.2) to obtain

$$d\dot{\theta}^\alpha = \dot{\theta}^\gamma \wedge \omega^\alpha_\gamma + \theta^\gamma \wedge \dot{\omega}^\alpha_\gamma + \dot{A}_\alpha \theta \wedge \theta^\gamma + A_{\alpha \gamma} \dot{\theta} \wedge \theta^\gamma + A_{\alpha \gamma} \theta \wedge \dot{\theta}^\gamma.$$

Since we will derive an identity involving tensors, we will take an adapted frame satisfying $\omega^\alpha_\gamma = 0$ at a point. Plug in (3.3) and consider the $\theta \wedge \theta^\gamma$ terms to obtain

$$\dot{A}_{\alpha \gamma} = -2(i \eta_{\alpha \gamma} + \eta A_{\alpha \gamma}) - iE_{\alpha \gamma \alpha \gamma}.$$

On the other hand, contracting (3.3) with $Z_\beta$ and then contracting with $h^\gamma$, computing modulo $\theta^\gamma$ yields

$$\dot{\omega}^\alpha_\gamma = i(A^\alpha_\gamma E_\alpha^\gamma + A_{\alpha \gamma} E^\gamma_\alpha + \eta_{\alpha \gamma}) \theta - [(n + 2) \eta_{\alpha \gamma} + iE_{\alpha \gamma \alpha \gamma}] \theta^\gamma \mod \theta^\gamma.$$

Since $\dot{\omega}^\alpha_\gamma$ is pure imaginary, we have

$$\dot{\omega}^\alpha_\gamma = i(A^\alpha_\gamma E_\alpha^\gamma + A_{\alpha \gamma} E^\gamma_\alpha + \Delta \eta) \theta$$

$$+ [(n + 2) \eta_{\alpha \gamma} - iE_{\alpha \gamma \alpha \gamma}] \theta^\alpha - [(n + 2) \eta_{\alpha \gamma} + iE_{\alpha \gamma \alpha \gamma}] \theta^\gamma.$$

Since $\dot{A}_{\alpha \gamma} = 0$ at a point
Differentiate the structure equation with respect to $t$ and consider only the $\theta^\rho \wedge \theta^\sigma$ terms. This gives

$$\dot{R}_{\rho\sigma} = -(A_\alpha \nabla E_{\gamma}^\alpha + A_\pi \nabla E_{\pi}^\pi + \Delta_b \eta) h_{\rho\sigma} - 2\eta R_{\rho\sigma}$$

$$-\left[ (n + 2) \eta - i E_{\gamma\beta}, \pi \right] - \left[ (n + 2) \eta + i E_{\pi\gamma}, \pi \right].$$

After contracting with $h^{\rho\sigma}$ we get

$$\dot{W} = i(E_{\gamma\alpha}, \gamma^\alpha - E_{\pi\pi}, \pi) - n(A_\alpha \nabla E_{\gamma}^\alpha + A_\pi \nabla E_{\gamma}^\pi)$$

$$-\left[ 2(n + 1) \Delta_b \eta + 2W \eta \right]$$

$$= 2\text{ Re} \left( iE_{\gamma\alpha}, \gamma^\alpha - nA_\alpha \nabla E_{\gamma}^\alpha \right) - \left[ 2(n + 1) \Delta_b \eta + 2W \eta \right].$$

Recall that the transformation law of the connection under a change of pseudohermitian structure was computed in [L1, Sec. 5]. Let $\hat{\theta} = e^{2f} \theta$ be another pseudohermitian structure. Then we can define an admissible coframe by $\hat{\theta}^\alpha = e^{f} (\theta^\alpha + 2if^\alpha \theta)$. With respect to this coframe, the connection 1-form and the pseudohermitian torsion are given by

$$\hat{\omega}_{\beta}^\alpha = \omega_{\beta}^\alpha + 2(f_{\beta} \theta^\alpha - f^\alpha \theta_{\beta}) + \delta_{\beta}^\alpha (f_{\gamma} \theta^\gamma - f^\gamma \theta_{\gamma})$$

$$+ i(f_{\alpha \beta} + f_{\beta} \theta^\alpha + 4\delta_{\beta}^\alpha f_{\gamma} \theta^\gamma) \theta,$$

and

$$\hat{A}_{\alpha\beta} = e^{-2f} \left( A_{\alpha\beta} + 2if_{\alpha\beta} - 4if_{\alpha} f_{\beta} \right),$$

respectively. Thus the Webster curvature transforms as

$$\hat{W} = e^{-2f} \left( W - 2(n + 1) \Delta_b f - 4n(n + 1) f_{\gamma} f^\gamma \right).$$

Here covariant derivatives on the right side are taken with respect to the pseudohermitian structure $\theta$ and an admissible coframe $\theta^\alpha$. Note also that the dual frame of $\{\hat{\theta}, \hat{\theta}^\alpha, \hat{\theta}^{\pi}\}$ is given by $\{\hat{T}, \hat{Z}_\alpha, \hat{Z}_{\pi}\}$, where

$$\hat{T} = e^{-2f} (T + 2if^{\pi} Z_{\pi} - 2i f_{\gamma} Z_{\gamma}), \quad \hat{Z}_\alpha = e^{-f} Z_{\alpha}.$$
Next we will derive the invariance property for the CR Paneitz operator. Let us first recall the CR Paneitz operator as following:

**Definition 3.1.** Let $(M, \xi, \theta)$ be a closed pseudohermitian $(2n + 1)$-manifold. Define

$$P \varphi = \sum_{\alpha=1}^{n} (\varphi \overline{\alpha} \beta + i n A_{\alpha\beta} \varphi^\alpha) \theta^\beta = (P_\beta \varphi) \theta^\beta, \quad \beta = 1, 2, \ldots, n$$

which is an operator that characterizes CR-pluriharmonic functions ([L2] for $n = 1$ and [GL] for $n \geq 2$). Here $P_\beta \varphi = \sum_{\alpha=1}^{n} (\varphi \overline{\alpha} \beta + i n A_{\alpha\beta} \varphi^\alpha)$ and $\overline{P} \varphi = (\overline{P_\beta} \varphi) \theta^\beta$, the conjugate of $P$.

Moreover, we define

$$(3.11) \quad P_0 \varphi = \delta_b (P \varphi) + \overline{\delta_b (P \varphi)}$$

which is the so-called CR Paneitz operator $P_0$. Here $\delta_b$ is the divergence operator that takes $(1,0)$-forms to functions by $\delta_b (\sigma^\alpha_{\alpha}) = \sigma^\alpha_{\alpha}$. Hence $P_0$ is a real and symmetric operator and

$$\int_M \langle P \varphi + \overline{P} \varphi, d_b \varphi \rangle_{\theta^\beta} d\mu = - \int_M (P_0 \varphi) \varphi d\mu.$$ 

We observe that ([GL])

$$(3.12) \quad P_0 \varphi = \frac{1}{4} [\Box_b \overline{\Box_b} \varphi - 2ni (A_{\overline{\alpha} \beta})_{\beta}]$$

$$(3.13) \quad = \frac{1}{4} [(\Delta_b^2 + n^2 T^2) \varphi - 2n \text{Re}(i (A_{\overline{\alpha} \beta})_{\beta})],$$

for $\Box_b \varphi = (\overline{\partial}_b \overline{\partial}_b + \overline{\partial}_b \overline{\partial}_b) \varphi = (-\Delta_b + inT) \varphi = -2 \varphi_{\alpha\alpha}$. 

**Remark 3.1.** 1. The space of kernel of the CR Paneitz operator $P_0$ is infinite dimensional, containing all CR-pluriharmonic functions. However, for a closed pseudohermitian $(2n + 1)$-manifold $(M, \xi, \theta)$ with $n \geq 2$, it was shown ([GL]) that 

$$\ker P_\beta = \ker P_0.$$

For a closed pseudohermitian 3-manifold of transverse symmetry (i.e. vanishing torsion), we have ([Hi])

$$\ker P_1 = \ker P_0.$$
2. ([H]) Let \((M^3, \xi, \theta)\) be a pseudohermitian 3-manifold. Then, for rescaled contact form \(\tilde{\theta} = e^{2g}\theta\), we have

\[
(3.14) \quad \tilde{P}_1 = e^{-3g}P_1 \quad \text{and} \quad \tilde{P}_0 = e^{-4g}P_0.
\]

However, the above invariance property for the CR Paneitz holds up to lower-order in dimensions five and higher. See Corollary 3.1 for details.

Now we derive the following invariance property for the CR Paneitz operator.

**Lemma 3.1.** Let \(\theta\) and \(\hat{\theta}\) be contact forms on a \((2n+1)\)-dimensional pseudohermitian manifold \((M, \xi)\). If \(\hat{\theta} = e^{2f}\theta\), then we have

\[
(3.15) \quad \hat{W}_\alpha - in\hat{A}_{\alpha\beta},^\beta = e^{-3f}[W_\alpha - iA_{\alpha\beta},^\beta - 2(n + 2)P_\alpha f]
\]

\[
+ 2ne^{-2f}(\hat{R}_{\alpha\beta} - \frac{\hat{W}}{n}h_{\alpha\beta})f^\beta
\]

and

\[
(3.16) \quad (\hat{W}_\alpha - in\hat{A}_{\alpha\beta},^\beta),^\alpha = e^{-4f}[(W_\alpha - inA_{\alpha\beta},^\beta),^\alpha - 2(n + 2)P_0 f]
\]

\[
+ 2n[e^{-2f}(\hat{R}_{\alpha\beta} - \frac{\hat{W}}{n}h_{\alpha\beta})f^\beta],^\alpha.
\]

In particular for \(n = 1\), we have \(\hat{R}_{1\Gamma} - \hat{W}h_{1\Gamma} = 0\). Then

\[
(3.17) \quad \hat{W}_1 - in\hat{A}_{11},^1 = e^{-3f}[W_1 - iA_{11},^1 - 6P_1 f]
\]

and

\[
(3.18) \quad \hat{W}_1^1 - i\hat{A}_{11},^{11} = e^{-4f}[W_1^1 - iA_{11},^{11} - 6P_0 f].
\]

**Proof.** By the contracted Bianchi identity, we have

\[
\frac{n-1}{n}(W_\alpha - inA_{\alpha\beta},^\beta) = (R_{\alpha\beta} - \frac{\hat{W}}{n}h_{\alpha\beta}),^\beta.
\]

Also by [L2, P 172]

\[
(3.19) \quad (R_{\alpha\beta} - \frac{\hat{W}}{n}h_{\alpha\beta}) - 2(n + 2)(f_{\alpha\beta} - \frac{1}{n}f^\gamma h_{\alpha\beta}) = \hat{R}_{\alpha\beta} - \frac{\hat{W}}{n}h_{\alpha\beta}.
\]
Following the same computation as the proof of Lemma 5.4 in [Hi], by using (3.8), (3.9) and (3.10), we compute

\[
\hat{W}_\alpha = \hat{Z}_\alpha \hat{W} = e^{-f} Z_\alpha e^{-2f}(W - 2(n + 1)\Delta_b f - 2n(n + 1)|\nabla_b f|^2)
\]

\[
= e^{-3f}[W_\alpha - 2W f_\alpha + 4(n + 1)(\Delta_b f + n|\nabla_b f|^2)f_\alpha
\]

\[-2(n + 1)(f_\gamma \gamma_\alpha + f_{\bar{\gamma}} \bar{\gamma}_\alpha) - 4n(n + 1)(f_\gamma \alpha f_{\gamma} + f_{\bar{\gamma}} f_{\bar{\gamma}})],
\]

\[
i\hat{A}_{\alpha \beta} = i(\hat{Z}_\alpha \hat{A}_{\alpha \beta} - \hat{\omega}_\alpha^l \hat{A}_{\beta l} - \hat{\omega}_\beta^l \hat{A}_{\alpha l})
\]

\[
= ie^{-f}[(Z_\tau + 2f_\tau)\hat{\alpha}_{\beta} + 2(\delta_{\alpha \gamma} \hat{A}_{\beta l} + \delta_{\beta \gamma} \hat{A}_{\alpha l})f^l]
\]

\[
= ie^{-f}(Z_\tau + 2f_\tau)e^{-2f}(A_{\alpha \beta} + 2if_\alpha \beta - 4if_\alpha f_\beta)
\]

\[+ 2e^{-3f}[\delta_{\beta \gamma}(iA_{\alpha l} - 2f_\alpha f_\beta) + \delta_{\alpha \gamma}(iA_{\beta l} - 2f_\beta f_\alpha)]f^l
\]

\[
e^{-3f}[iA_{\alpha \beta} \tau - 2f_\alpha \beta + 4(f_\alpha \tau f_\beta + f_\alpha f_\beta)]
\]

\[+ 2e^{-3f}[\delta_{\beta \gamma}(iA_{\alpha l} - 2f_\alpha f_\beta) + \delta_{\alpha \gamma}(iA_{\beta l} - 2f_\beta f_\alpha)]f^l.
\]

Contracting the second equation with respect to the Levi metric \(\hat{h}_{\gamma \bar{\beta}} = h_{\gamma \bar{\beta}}\) yields

\[
i\hat{A}_{\alpha \beta} = e^{-3f}[iA_{\alpha \beta} - 2f_\alpha \beta + 4(f_\alpha \beta f_\beta + f_\alpha f_\beta)]
\]

\[+ 2(n + 1)(iA_{\alpha \beta} - 2f_\alpha \beta + 4f_\alpha f_\beta)f^\beta].
\]

Thus

\[
\hat{W}_\alpha - i\hat{A}_{\alpha \beta} = e^{-3f}[W_\alpha - iA_{\alpha \beta} - 2(n + 1)(f_{\beta} \alpha f_\beta + f_\beta \alpha) + 2n f_\alpha \beta
\]

\[-2W f_\alpha - 2n(n + 1)iA_{\alpha \beta} f^\beta + 4(n + 1)(f_\beta \alpha + f_\beta \bar{\beta})f_\alpha
\]

\[-4n(n + 1)f_\alpha \beta f_\beta - 4n(f_\alpha \beta f_\beta + f_\beta \alpha)],
\]

By using the commutation relations ([L2, Lemma 2.3])

\[ -2(n + 1)f_{\beta} \alpha + 2n f_\alpha \beta = -2f_\beta \alpha + 2n R_{\alpha \beta} \bar{\beta} - 2iA_{\alpha \beta} f^\beta,
\]

and

\[ f_{\alpha \beta} - f_{\beta} \alpha = ih_{\alpha \beta} f_0,
\]
and by (3.19)

\[ [(R_{\alpha \beta} - \frac{W}{n} h_{\alpha \beta}) - 2(n + 2)(f_{\alpha \beta} - \frac{1}{n} f_{\gamma} h_{\alpha \beta})] f^\beta = e^f (\hat{R}_{\alpha \beta} - \frac{W}{n} \hat{h}_{\alpha \beta}) f^\beta, \]

we obtain the following transformation law

\[
\hat{W}_\alpha - in \hat{A}_{\alpha \beta, \beta} - 2n e^{-2f} (\hat{R}_{\alpha \beta} - \frac{W}{n} \hat{h}_{\alpha \beta}) f^\beta
\]

\[ = e^{-3f} [W_\alpha - in A_{\alpha \beta, \beta} - 2(n + 2)(f_{\alpha \beta} - \frac{1}{n} f_{\gamma} h_{\alpha \beta})]
\]

\[ = e^{-3f} [W_\alpha - in A_{\alpha \beta, \beta} - 2(n + 2) P_\alpha f]. \]

Then (3.15) and (3.16) follow easily. \hfill \Box

By applying (3.15) and the method of [Hi, Lemma 4.7], we have

**Corollary 3.1.** Let \( \theta \) and \( \hat{\theta} \) be contact forms on a \((2n + 1)\)-dimensional pseudohermitian manifold \((M, \xi)\). If \( \hat{\theta} = e^{2f} \theta \), then

\[
\hat{P}_\alpha f = e^{-3f} P_\alpha f
\]

(3.20)

\[ + \frac{n}{2(n+1)} (1 - e^{-2f}) (\hat{R}_{\alpha \beta} - \frac{W}{n} \hat{h}_{\alpha \beta}) f^\beta
\]

\[ - \frac{n(n+2)}{(n+1)} (1 - e^{-2f}) \hat{B}(f)_{\alpha \overline{\beta}} f^\overline{\beta}. \]

Here \( \hat{B}(f)_{\alpha \overline{\beta}} = f_{\alpha \overline{\beta}} - \frac{1}{n} f_{\gamma} \hat{h}_{\alpha \overline{\beta}} \). In particular for \( n = 1 \), we have \( \hat{B}(f)_{\alpha \overline{\beta}} = 0 \) and

(3.21) \[ \hat{P}_1 f = e^{-3f} P_1 f \quad \text{and} \quad \hat{P}_0 f = e^{-4f} P_0 f. \]

**4. The Proof of Main Theorem**

Let \((M, \xi, J_0, \theta_0)\) be a closed pseudohermitian \((2n + 1)\)-manifold. As in (4.15) below, we do not know the subellipticity for the highest weight operator \( L_{J, \theta}(G) \) of the linearization \( \delta(2Q(J, \theta)) \) for the torsion flow (1.1). Instead we consider the CR torsion flow (1.5), with the particular choice of the contact form \( \theta_0 \) in \( \xi \) and \( \gamma_0 \) as in (1.6), in which the highest weight operator \( L_{J, \theta}(G) \) of the linearization \( \delta(2Q(J, \theta)) \) as in (4.19) is subelliptic. We refer to the **step I** of the proof of Theorem (1.1) below for details.
In the following, we first set up the CR torsion flow (4.1) from the point view of finding its
gauge-fixing condition or integrability condition which is a crucial step for the subellipticity
of the linearization (4.19). More precisely, we consider the following flow

\[
\begin{align*}
\frac{\partial J}{\partial t} &= 2F_{J,\theta}, \\
\frac{\partial \theta}{\partial t} &= 2\eta \theta,
\end{align*}
\]

on \((M, \xi, J, \theta, \varphi) \times [0, T)\) with

\[
F_{J,\theta} := F_{\alpha}^{\beta} \theta^\alpha \otimes Z_{\beta} + F_{\alpha}^{\beta} \theta^\beta \otimes Z_{\beta}
\]

and

\[
\eta := -e^{\varphi}[W + (n + 1)\Delta_\theta \varphi - n(n + 1)\varphi_\alpha \varphi^\alpha].
\]

Here \(J = i\theta^\alpha \otimes Z_\alpha - i\theta^\beta \otimes Z_\beta\) and \(F_{\alpha}^{\beta} := e^{\varphi}[A_{\alpha}^{\beta} - i\varphi_\alpha^{\beta} - i\varphi_\alpha \varphi^\beta]\). We come out with the following flow

\[
\begin{align*}
\frac{\partial J}{\partial t} &= 2e^{\varphi}(A_{\alpha}^{\beta} - i\varphi_\alpha^{\beta} - i\varphi_\alpha \varphi^\beta)\theta^\alpha \otimes Z_{\beta} + \text{Conj}, \\
\frac{\partial \theta}{\partial t} &= -2e^{\varphi}[W + (n + 1)\Delta_\theta \varphi - n(n + 1)\varphi_\alpha \varphi^\alpha] \theta,
\end{align*}
\]

on \((M, \xi, J, \theta) \times [0, T)\). Thus, for \(\theta = e^{2\lambda} \theta_0\), we denote

\[
\lambda^\perp := \lambda - \lambda^{\ker}
\]

with

\[
\lambda^{\ker} \in \ker \begin{pmatrix} 0 \\ \theta_0 \end{pmatrix} \text{ i.e. } \begin{pmatrix} 0 \\ \theta_0 \end{pmatrix}(\lambda^{\ker}) = 0.
\]

Then, by choosing \(\varphi = 2\lambda^\perp\), the torsion \(\tilde{A}_{\alpha}^{\beta}\) and scalar curvature \(\tilde{W}\) with respect to

\[
\tilde{\theta} = e^{-\varphi} \theta = e^{-\varphi} e^{2\lambda} \theta_0 = e^{2\lambda^{\ker}} \theta_0
\]

will be

\[
\begin{align*}
\tilde{A}_{\alpha}^{\beta} &= e^{\varphi}(A_{\alpha}^{\beta} - i\varphi_\alpha^{\beta} - i\varphi_\alpha \varphi^\beta), \\
\tilde{W} &= e^{\varphi}(W + (n + 1)\Delta_\theta \varphi - n(n + 1)\varphi_\alpha \varphi^\alpha).
\end{align*}
\]
It follows that (4.2) implies

\[
\begin{align*}
\frac{\partial J}{\partial t} &= 2 \tilde{A}_{J, \tilde{\theta}}, \\
\frac{\partial \tilde{\theta}}{\partial t} &= -2 \tilde{W} \tilde{\theta}, \\
\tilde{\theta} &= e^{2\lambda_{\operatorname{ker}} \theta_0}, \quad \theta = e^{2\lambda_0}. 
\end{align*}
\]

In particular

\[
\frac{\partial \lambda_{\operatorname{ker}}}{\partial t} = -\tilde{W} \lambda_{\operatorname{ker}} \quad \text{and} \quad \frac{\partial \lambda_{\perp}}{\partial t} = -\tilde{W} \perp.
\]

Furthermore, we compute for \( \tilde{\theta} = e^{-\varphi \theta} \)

\[
\begin{align*}
\frac{\partial \tilde{\theta}}{\partial t} &= -2(\tilde{W} + \frac{1}{2} \varphi_t) \tilde{\theta} \\
&= -2(\tilde{W} + \frac{\partial \lambda_{\perp}}{\partial t}) \tilde{\theta} \\
&= -2(\tilde{W} - \tilde{W} \perp) \tilde{\theta}.
\end{align*}
\]

Hence, with the choice of \( \varphi(x, t) = 2\lambda_{\perp}(x, t) \), (4.2) is equivalent to the following modified torsion flow on \( M \times [0, T] \) with the maximal time \( T \) :

\[
\begin{align*}
\frac{\partial J}{\partial t} &= 2 \tilde{A}_{J, \tilde{\theta}}, \\
\frac{\partial \tilde{\theta}}{\partial t} &= -2(\tilde{W} - \tilde{W} \perp) \tilde{\theta}, \\
\tilde{\theta} &= e^{2\lambda_{\operatorname{ker}} \theta_0}
\end{align*}
\]

coupled with

\[
\frac{\partial \lambda_{\perp}}{\partial t} = -\tilde{W} \perp.
\]

According to (4.4) which is the deformation with respect to \( e^{2\lambda_{\operatorname{ker}} \theta_0} \) only, then we in particular consider the original modified torsion flow (4.2) with the special choice

\[
\varphi(x, t) = \lambda_{\perp}(x, t) \quad \text{and} \quad \lambda_{\perp}(x, 0) = \tilde{W} \perp(x, 0).
\]

Then if the initial Tanaka-Webster scalar curvature \( \tilde{W}(x, 0) \) w.r.t. \( e^{\gamma_0} \theta_0 \) is pluriharmonic, i.e.

\[
\tilde{W} \perp(x, 0) = 0,
\]
we have
\[ \tilde{W}^{-1}(x, t) = 0 = \lambda^{-1}(x, t) \]
as long as the solution exists for all \( t \in [0, T] \). In this case, it follows that (4.2) is equivalent to the following torsion flow on \( M \times [0, T] \) with the maximal time \( T \):

\[
\begin{align*}
\frac{\partial J}{\partial t} &= 2 \tilde{A}_{J, \tilde{\theta}}, \\
\frac{\partial \tilde{\theta}}{\partial t} &= -2 \tilde{W} \tilde{\theta}, \\
\tilde{\theta} &= e^{2\gamma} \theta_0; \quad P_\beta(\gamma_0) = 0 = \tilde{W}^{-1}(0).
\end{align*}
\]

Finally by replacing \( \tilde{\theta} \) by \( \theta \) without ambiguity, we rewrite (4.7) as the following torsion flow on \( (M, \xi, J, \theta) \times [0, T] \):

\[
\begin{align*}
\frac{\partial J}{\partial t} &= 2 A_{J, \theta}, \\
\frac{\partial \theta}{\partial t} &= -2 W \theta, \\
\theta &= e^{2\gamma} \theta_0; \quad P_\beta(\gamma_0) = 0 = W^{-1}(0).
\end{align*}
\]

Note that
\[ W^{-1}(x, t) = 0 = \gamma^{-1}(x, t) \]
as long as the solution exists for all \( t \in [0, T] \).

Next for \( \theta = e^{2\lambda} \theta_0 \), it follows from Lemma 3.1, Corollary 3.1 and commutation relations, we compute

\[
\eta_\beta + i n F_\beta \overline{\alpha,n} = -W_\beta + i n A_\beta \overline{\alpha,n} \\
= -e^{-3\gamma}[ \overline{0} W_\beta - i n A_\beta \overline{0} + \overline{0} - 2(n+2) P_\gamma] \\
-2ne^{-2\gamma}(R_{\alpha\beta} - \overline{W_{\alpha\beta}}) \gamma_{\beta} \\
= -e^{-3\gamma}[ \overline{0} W_\beta - i n A_\beta \overline{0} ] \\
-2ne^{-2\gamma}(R_{\alpha\beta} - \overline{W_{\alpha\beta}}) \gamma_{\beta}.
\]

Hence

\[
\eta_\beta + i n F_\beta \overline{\alpha,n} = l.o.t. \text{in } \gamma.
\]
We observe that (4.9) will be the gauge-fixing condition or integrability condition for the flow (4.8) below.

We are ready to prove the existence of short-time solution for the torsion flow (1.5) on $(M, \xi, J, \theta) \times [0, T)$.

Proof of Theorem 1.1. Step I. We first find the linearization of the flow (1.5):

For $\theta = e^{2\gamma} \theta_0$ with $\mathcal{P}_\beta(\gamma) = 0$, we can rewrite the flow (1.5) as

$$\frac{\partial}{\partial t}(J \oplus \theta) = 2Q(J, \theta).$$

Here $Q(J, \theta) := F_{J, \theta} \oplus \eta \theta$ with

$$F_{J, \theta} := A_{J, \theta},$$
$$\eta := -W.$$

with $\mathcal{P}_\beta(W) = 0$. We use $\delta J, \delta \theta$ to denote the variations of $J$ and $\theta$, respectively. Set

$$\delta J = 2E \quad \text{and} \quad \delta \theta = 2h \theta$$

with $\mathcal{P}_\beta(h) = 0$. Here $E$ is an endomorphism: $\xi \rightarrow \xi$ satisfying $J \circ E + E \circ J = 0$ and $h$ is a smooth function. Let $\delta J$ and $\delta \theta$ denote the linearization operators with respect to $J$ and $\theta$. From section 3, we see that

$$\delta_J A_{\alpha\beta} = i(E_{\alpha\beta} \cdot 0)$$
$$\delta_\theta A_{\alpha\beta} = 2(ih_{\alpha\beta}) + \text{l.o.t.}$$

and

$$\delta_J W = i(E_{\gamma\alpha}, \gamma^\alpha - E_{\gamma\alpha}, \gamma^\alpha) + \text{l.o.t.}$$
$$\delta_\theta W = -2(n + 1)(\Delta h) + \text{l.o.t.}$$

with $\mathcal{P}_\beta(\gamma) = 0$ and $\mathcal{P}_\beta(h) = 0$. We first compute the linearization of $2A_{J, \theta}$. Let $O_m$ denote an operator of weight $\leq m$ (see [CL1, page 234]), and define the total variation as $\delta = \delta_J + \delta_\theta$. Put

$$G := E \oplus h \theta.$$
With (4.11) we compute the variation of the torsion as

\[ \delta(2A_{J,\theta}) = 2 \text{Re} \left( [2iE_{\alpha,\alpha} + 4ih_{\alpha}] \theta^\alpha \otimes Z_{\bar{\alpha}} \right) + \mathcal{O}_1(G). \]

The linearization of \(-2W\theta\) can be computed with (4.12),

\[ \delta(-2W \theta) = \{ 2i(E_{\gamma,\alpha} - E_{\gamma,\alpha}) \theta^\alpha \]
\[ + 4(n + 1)(\Delta_b h) + \mathcal{O}_1(G) \} \theta \]

With these linearizations, we write the highest weight operator \(L_{J,\theta}(G)\) of the linearization \(\delta(2Q(J, \theta))\) as

\[ L_{J,\theta}(G) = 2 \text{Re} \left( \{ [2iE_{\alpha,\alpha} + 4ih_{\alpha}] \theta^\alpha \otimes Z_{\bar{\alpha}} \right) \]
\[ \oplus \{ 2i(E_{\gamma,\alpha} - E_{\gamma,\alpha}) \theta^\alpha \} + 4(n + 1)\Delta_b h \} \theta. \]

Next we define the linear operator \(H(J, \theta)\) by

\[ H(J, \theta)(G) := (h_\alpha + iE_{\alpha,\beta}) - \mathcal{O}_0(G) \]

with \(G = E \oplus h\theta\) and \(h \in \ker(P_\beta)\). Here \(\mathcal{O}_0(G)\) is the lower-order terms as in (4.9).

Thus \(Q(J, \theta)\) satisfies the condition as in (4.9):

\[ H(J, \theta)Q(J, \theta) = (\eta_\alpha + iF_{\alpha,\gamma}) - \mathcal{O}_0(G) = 0. \]

Now from the following commutation relations

\[ E_{\alpha,\gamma} - E_{\alpha,\gamma} = i(n - 1)E_{\alpha,\gamma,0} + R_{\alpha,\sigma}E_{\gamma,\sigma} - R_{\alpha,\gamma,\sigma}E_{\gamma,\sigma}, \]
\[ E_{\alpha,\gamma,\beta} - E_{\alpha,\gamma,\beta} = inE_{\alpha,\gamma,0} + R_{\gamma,\sigma}E_{\alpha,\sigma} + R_{\alpha,\sigma}E_{\gamma,\sigma}. \]

We derive that on the subspace

\[ \ker H(J, \theta) = \{ G = E \oplus h\theta \mid (h_\alpha + iE_{\alpha,\beta}) - \mathcal{O}_0(G) = 0 \} \]
the following identities hold
\begin{align}
(\Delta_b h) &= \text{in}(E_{\alpha\beta})_\beta - E_{\alpha\beta} + O_1(G), \\
(\Delta_b E_{\alpha\beta}) &= [i(2 - n)E_{\alpha\beta} + \frac{2}{n}ih_{\alpha\beta}] + O_1(G).
\end{align}

Here we have used the gauge-fixing condition
\[(h_{\alpha} + \text{in}E_{\alpha\beta}) = O_0(G).\]

Therefore it follows from (4.15) that the highest weight operator \(L_{J,\theta}(G)\) of the linearization \(2\delta(Q(J, \theta))\) is
\begin{align}
L_{J,\theta}(E \oplus h\theta) &= \{4n \text{ Re}[(\mathcal{L}_{\alpha}E_{\alpha\beta})_\beta \otimes Z^\beta] \oplus 2(2n + 2 + \frac{1}{n})(\Delta_b h)\theta\} \\
&= \{4n \text{ Re}[(\mathcal{L}_{\alpha}E_{\alpha\beta})_\beta \otimes Z^\beta] \oplus 2(2n + 2 + \frac{1}{n})(\Delta_b h)\theta\} \\
&= \{4n \text{ Re}[(\mathcal{L}_{\alpha}E_{\alpha\beta})_\beta \otimes Z^\beta] \oplus 2(2n + 2 + \frac{1}{n})(\Delta_b h)\theta\}
\end{align}
on \text{ker} H(J, \theta)(E \oplus h\theta) with
\begin{align}
\mathcal{L}_{\alpha} = \Delta_b - i\alpha T, \quad \alpha = -\frac{(n - 1)^2}{n} = -(n - 2 + \frac{1}{n}).
\end{align}

\textbf{Step 2. The sub-ellipticity and short-time existence :}

It follows from (4.20) that the Folland-Stein operator \(\mathcal{L}_{\alpha}\) with this particular value
\[\alpha = -(n - 2 + \frac{1}{n})\]
is subelliptic (see [FS]). In particular \(\alpha = 0\) if \(n = 1\) and then \(\mathcal{L}_0 = \Delta_b\) which is subelliptic. Moreover, it follows from (4.9) and (A.2), (A.3) below that we may then use (see remark 4.1)
\begin{align}
H(J, \theta)(G) = (h_{\alpha} + \text{in}E_{\alpha\beta}) - O_0(G) = 0.
\end{align}
as the gauge-fixing condition (see appendix ) for the torsion flow (1.3). It follows that \(L\) as shown in (4.19) maps \text{ker} H(J, \theta) into itself modulo terms of lower weight. Since \(L\) is subelliptic on \text{ker} H(J, \theta), by comparing [CL1], Lemma 3.3], once the highest weight operator
$L_{J,\theta}(G)$ of the linearization $2\delta(Q(J,\theta))$ on $\ker H(J,\theta)$ is subelliptic, then the short-time existence of the torsion flow (1.5) follows easily from the linear theory of the Folland-Stein space and section 5 as in [CL1] under the gauge-fixing condition (4.21). For completeness, we will discuss the gauge-fixing condition more details as in Appendix for $n = 1$. □

To make the proof more clear, we make a remark as following:

**Remark 4.1.** As shown in the appendix (A.2), $\ker \tilde{B}_J$ is the natural gauge-fixing condition for

$$\begin{aligned}
\frac{\partial J}{\partial t} &= 2A_{J,\theta}, \\
\frac{\partial \theta}{\partial t} &= -2W\theta, \\
\theta &= e^{2\gamma\theta_0}.
\end{aligned}$$

However, we do not know the subellipticity for the highest weight operator $L_{J,\theta}(G)$ of the linearization $\delta(2Q(J,\theta))$ as in (4.15). Instead, it follows from (4.9) that (4.21) is the right gauge-fixing condition for the current type CR torsion flow

$$\begin{aligned}
\frac{\partial J}{\partial t} &= 2A_{J,\theta}, \\
\frac{\partial \theta}{\partial t} &= -2W\theta, \\
\theta &= e^{2\gamma\theta_0}; \quad P_{\delta}(\gamma_0) = 0 = W^\perp(0).
\end{aligned}$$

As we have proved that the highest weight operator $L_{J,\theta}(G)$ of the linearization $\delta(2Q(J,\theta))$ as in (4.19) is subelliptic.

**Appendix A. Alternative Proof via the Lions-Lax-Milgram Theorem**

**A. Finding the gauge-fixing condition for $n = 1$:**

For simplicity, let $(M, \xi, J_0, \theta_0)$ be a closed pseudohermitian 3-manifold. We first observe that the highest weight operator $L_{J,\theta}(G)$ is self-adjoint, i.e., $L^* = L$ with respect to the inner
product:

\[(A.1) \quad < E + h\theta, F + k\theta > = \int_M [E_{1\bar{1}}F_{\bar{1}1} + F_{1\bar{1}}E_{\bar{1}1} + 2hk]\theta \wedge d\theta.\]

Let \(X_f\) be the contact vector field for the real-valued function \(f \in C^2(M)\). Then one has

\[\mathcal{L}_{X_f}\theta = -(Tf)\theta,\]

and

\[L_{X_f}J = 2B'_f f := 2(f^T + iA_1^T)\theta^1 \otimes Z_\tau + 2(f^1 - iA_{1\bar{1}})\theta^\tau \otimes Z_1\]

See for instance [?]. In particular, we have \(X_f = T\) for \(f = 1\), and the above equation reduces to

\[\mathcal{L}_T J = 2JA_{1\bar{1}}\theta.\]

Recall that \(X_f\) is an infinitesimal contact diffeomorphism if and only if \(L_{X_f}\theta = \mu\theta\) for some function \(\mu\). So an infinitesimal contact orbit reads

\[L_{X_f}J + L_{X_f}\theta = 2B'_f f - (Tf)\theta := 2\tilde{B}_f f.\]

An orthogonal infinitesimal slice is described by \(\tilde{B}_f(G)\) where \(\tilde{B}_f\) is the adjoint of \(\tilde{B}'_f\) with respect to the inner product \((A.1)\). By a direct computation ([CL2 Lemma 3.4]), one obtains

\[\tilde{B}_f(G) = B_f(E) + h_0\]

\[= E_{1\bar{1},\bar{1}} + E_{\bar{1}1,1} + h_{0\bar{1}} + iA_{11}E_{\bar{1}\bar{1}} - iA_{\bar{1}\bar{1}}E_{11}\]

\[= E_{1\bar{1},\bar{1}} + E_{\bar{1}1,1} + h_{0\bar{1}} + \mathcal{O}_1(E, h).\]

Observe that \(F_{,\eta} \oplus \eta\theta\) satisfies

\[(\eta_1 + iF_1^T, \tau) - \mathcal{O}_0(F, \eta) = 0\]
and then

\begin{equation} \label{A.3}
\tilde{B}_J(Q)
= \tilde{B}_J(F_{I,\theta} \oplus \eta_\theta)
= F_{11,11} + F_{11,11} + \eta_{,0} + \mathcal{O}_1(F, \eta) = 0.
\end{equation}

by the CR Bianchi identity. On the other hand, it follows from \eqref{4.17} that

\[(E_{11,11} + E_{11,11} + h_{,0}) + \mathcal{O}_1(E, h) = 0\]

on \(\ker H(J, \theta)\). It follows from \eqref{4.9} that we may then use (see remark \ref{4.1})

\[H(J, \theta)(G) = ((h_1 + inE_{11,11}) - \mathcal{O}_0(G) = 0.\]

as the gauge-fixing condition for the torsion flow \eqref{1.5}.

**B. Alternative proof of the short-time existence via the Lions-Lax-Milgram theorem for \(n = 1\):**

*Proof.* We claim that \(L\) is subelliptic on \(\ker H(J, \theta)(G)\). Note that \(L^*L = L^2\). We denote that

\begin{equation} \label{A.4}
L^*L(E + h\theta) = F + k\theta
\end{equation}

with

\begin{equation} \label{A.5}
F = 4(\Delta_b^2 E_{11}\theta^1 \otimes Z_1 + \Delta_b^2 E_{11}\theta^I \otimes Z_1) := 4\Delta_b^2 E
\end{equation}

and

\begin{equation} \label{A.6}
k = 100\Delta_b^2 h.
\end{equation}
Substituting (A.5) and (A.6) into (A.4), we compute

\[ \|L(E + \theta h)\|_{L^2}^2 = \langle L^* L(E + \theta h), E + \theta h \rangle \]

(A.7)

\[ = \langle 4\Delta_b^2 E, E \rangle + 200 \int_M (\Delta_b^2 h) \theta h \wedge d\theta \]

\[ = \langle 4\Delta_b^2 E, E \rangle + 200 \|\Delta_b h\|_{L^2}^2. \]

Since \(\Delta_b\) is subelliptic, we conclude that

\[ \|E\|_{S^2}^2 + \|h\|_{S^2}^2 \leq C(\|L(E + \theta h)\|_{L^2}^2 + \|E + \theta h\|_{L^2}^2) \]

(A.8)

for some constant \(C > 0\), where \(\|\cdot\|_{S^2}\) denotes the norm on the Folland-Stein space \(S^2 := \{ (E, h) \in L^2 \mid P(E, h) \in L^2 \text{ whenever } w(P) \leq 2 \}\) as in (2.2). Finally, we can obtain estimates of higher order derivatives from (A.8) and interpolation inequalities in the Folland-Stein spaces \(S_k\) by observing that \([L, \nabla Z_1]\) and \([L, \nabla Z_{-1}]\) are operators of weight 2:

\[ \|E\|_{S_k}^2 + \|h\|_{S_k}^2 \leq C(\|L(E + \theta h)\|_{S_k}^2 + \|E + \theta h\|_{S_k}^2) \]

(A.9)

for \(E, h\) satisfying the gauge-fixing condition.

On the other hand

\[ \langle L(E + \theta h), E + \theta h \rangle = \langle 2\Delta_b E, E \rangle + 20 \int_M (\Delta_b h) \theta h \wedge d\theta \]

(A.10)

\[ = -2 \int_M |\nabla_b E|^2 \theta \wedge d\theta - 20 \int_M |\nabla_b h|^2 \theta \wedge d\theta \]

\[ = -2 \|\nabla_b E\|_{L^2}^2 - 20 \|\nabla_b h\|_{L^2}^2. \]

It follows from (A.10) and the Poincare-type inequality (II) that

\[ - \langle L(G), G \rangle \geq C(\|\nabla_b G\|_{L^2}^2 + \|G\|_{L^2}^2) \]

(A.11)

on \(\ker H(J, \theta)(G)\). Hence \(\langle L(\cdot), \cdot \rangle\) is coercive (III).

Note that Theorem 21.1 from [FS] gives the regularity in space and Theorem 4.6 in [CL1] gives the regularity in time.

Let \(J_0\) be any \(C^\infty\) smooth oriented \(CR\) structure compatible with \(\xi\) and \(\theta_0\) be any \(C^\infty\) smooth oriented contact form for \(\xi\). Based on the subelliptic estimates (A.9), the coercive property (A.11) plus the implicit function theorem, we have, for any integer \(m\), there exists
$\delta > 0$ and a unique $C^m$-solution $J(t)$ and $\theta(t)$ to the torsion flow (1.5) on the interval $[0, \delta)$ such that $(J(0), \theta(0)) = (J_0, e^{2\gamma(0)}\theta_0)$. This is the so-called Lions-Lax-Milgram theorem as in [GT, Theorem 5.8], [Tre], [Y], [LM] for elliptic-type and Theorem 4.1. in chapter IV of [S] and a generalization due to J. L. Lions of the Lax-Milgram Theorem of [Tre, Lemma 41.2] for parabolic-type.

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