Abstract. In the deBranges-Rovnyak functional model for contractions on Hilbert space, any completely non-coisometric (CNC) contraction is represented as the adjoint of the restriction of the backward shift to a deBranges-Rovnyak space, \( \mathcal{H}(b) \), associated to a contractive analytic operator-valued function, \( b \), on the open unit disk.

We extend this model to a large class of CNC row contractions of several copies of a Hilbert space into itself (including all CNC row contractions with commuting component operators). Namely, we completely characterize the set of all CNC row contractions, \( T \), which are unitarily equivalent to an extremal Gleason solution for a deBranges-Rovnyak space, \( \mathcal{H}(b_T) \), contractively contained in a vector-valued Drury-Arveson space of analytic functions on the open unit ball in several complex dimensions. Here, a Gleason solution is the appropriate several-variable analogue of the adjoint of the restricted backward shift and the characteristic function, \( b_T \), belongs to the several-variable Schur class of contractive multipliers between vector-valued Drury-Arveson spaces. The characteristic function, \( b_T \), is a unitary invariant, and we further characterize a natural sub-class of CNC row contractions for which it is a complete unitary invariant.

1. Introduction

The deBranges-Rovnyak and Sz.-Nagy-Foiaş functional models are two widely-used and powerful approaches to the representation theory of contractions on Hilbert space \[1, 2, 3\]. These two constructions provide equivalent models for completely non-unitary (CNU) contractions \[4, 5, 6, 7\]. In this paper we focus on the deBranges-Rovnyak model and its several-variable extension to the setting of row contractions from several copies of a Hilbert space into itself.

In the full deBranges-Rovnyak model for a CNU contraction on a Hilbert space, the model operator acts on a two-component reproducing kernel Hilbert space (RKHS), \( \mathcal{H}(\tilde{k}^b) \), associated to an operator-valued contractive analytic function \( b \) on the open unit disk, \( \mathbb{D} \), in the complex plane \[4, 7\]. Here the reproducing kernel \( \tilde{k}^b \) is given by

\[
\tilde{k}^b(z, w) := \begin{bmatrix} k^b(z, w) & b(z) - b(w) \\ \bar{b}(z) - b(w)^* & 1 - z\bar{w} \end{bmatrix} ; \quad z, w \in \mathbb{D},
\]

where the positive sesqui-analytic deBranges-Rovnyak kernel \( k^b \) is

\[
k^b(z, w) := \frac{I - b(z)b(w)^*}{1 - zw^*}.
\]

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In the above $z^* := \overline{z}$ denotes complex conjugation. The deBranges-Rovnyak space $\mathcal{H}(b) := \mathcal{H}(k^b)$ is the (unique) reproducing kernel Hilbert space (RKHS) of analytic functions associated to the deBranges-Rovnyak kernel $k^b$, and in the case where $b \equiv 0$ we recover the Szegö kernel for the classical (vector-valued) Hardy space of analytic functions in the unit disk.

In the case where $T$ is a completely non-coisometric (CNC) contraction, there is a contractive (operator-valued) analytic function, $b_T$, on the unit disk, $b_T(z) \in L(H, K)$ (i.e. a member of the operator-valued Schur class), so that $T$ is unitarily equivalent to $X$ where $X^* := S^*|_{\mathcal{H}(b_T)}$ is the restriction of the backward shift of the (vector-valued) Hardy space $H^2(D) \otimes K$ to the deBranges-Rovnyak space $\mathcal{H}(b_T)$ [7, 4]. Here, recall that any deBranges-Rovnyak space $\mathcal{H}(b_T)$ associated to a contractive (operator-valued) analytic function, $b_T$, on the disk is always contractively contained in (vector-valued) Hardy space and is always co-invariant for the shift [6]. This provides a natural model for this class of contractions as adjoints of restrictions of the backward shift to deBranges-Rovnyak reproducing kernel Hilbert spaces, and this is the model we extend to several variables in this paper.

A canonical several-variable extension of the Hardy space of the disk is the Drury-Arveson space, $H^2_d$, the unique RKHS of analytic functions on the open unit ball $B_d := (\mathbb{C}^d)^1$ corresponding to the several-variable Szegö kernel. The Schur classes of contractive (operator-valued) functions on the disk are promoted to the multi-variable Schur classes, $\mathcal{S}_d^1(\mathcal{B}, \mathcal{K})$, of contractive (operator-valued) multipliers between vector-valued Drury-Arveson spaces $H^2_d \otimes \mathcal{B}$, $H^2_d \otimes \mathcal{K}$, and the appropriate analogue of the adjoint of the restricted backward shift in this several-variable setting is a contractive solution to the Gleason problem in $\mathcal{H}(b)$. We will recall these basic definitions in the upcoming Subsection 1.4.

Given any contraction between Hilbert spaces, $T : \mathcal{H} \to \mathcal{K}$, recall that the defect operator, $D_T$ of $T$ is defined as

$$D_T := \sqrt{I - T^*T}.$$  

We say that a row contraction, $T = (T_1, ..., T_d) : \mathcal{H} \otimes \mathcal{C}^d \to \mathcal{K}$, obeys the commutative CNC condition, and we write: $T$ is CCNC if

$$\mathcal{H} = \bigvee_{z \in B_d} (I - Tz^*)^{-1} \text{Ran}(D_{Tz^*}).$$

Here, and throughout, $\bigvee$ denotes closed linear span. We will prove that any CCNC row contraction $T$ is automatically CNC (Corollary 2.4), and that any $d$-contraction (a row contraction with $d$ mutually commuting component operators) is CNC if and only if it is CCNC. One of the main results of this paper is the extension of the deBranges-Rovnyak model for CNC contractions to the class of all CCNC row contractions:

**Theorem.** (Theorem 4.12) A row contraction $T : \mathcal{H} \otimes \mathcal{C}^d \to \mathcal{K}$ is CCNC (obeys the commutative CNC condition) if and only if it is unitarily equivalent to an extremal (contractive) Gleason solution in a multi-variable deBranges-Rovnyak space $\mathcal{H}(b)$ for a Schur class multiplier $b \in S_d^1(\mathcal{B}, \mathcal{K})$.

If $T$ is unitarily equivalent to an extremal contractive Gleason solution $X^b$ for $\mathcal{H}(b)$, then the characteristic function $b := b_T$ is a unitary invariant for $T$: If $T_1, T_2$ are unitarily equivalent CCNC row contractions, then $b_{T_1}$ coincides weakly with $b_{T_2}$. 


In [8, 9], the concept of a quasi-extreme multiplier for Drury-Arveson space was introduced. This concept is a several-variable extension of a ‘Szegő approximation property’, the salient idea being that this property is equivalent to being an extreme point of the Schur class in the classical, single-variable, scalar-valued setting. Recently, it has been shown that quasi-extreme implies extreme in the scalar-valued, several-variable setting as well [10]. If b is a quasi-extreme Schur multiplier, then \( \mathcal{H}(b) \) has a unique contractive (and extremal) Gleason solution (see Theorem 5.3). We say that a CCNC row contraction \( T \) is quasi-extreme (QE) if and only if its characteristic function \( b_T \) is a quasi-extreme multiplier. Our second main result characterizes the class of all QE row contractions:

**Theorem.** (Theorem 5.10, Theorem 5.11) A row contraction \( T : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H} \) is QE if and only if it is CCNC and obeys the QE condition:

\[
\text{Ker}(T)^\perp \subseteq \bigvee_{z \in \mathbb{B}^d} z^* (I - Tz^*)^{-1} \text{Ran} (D_{T^*}).
\]

\( T \) is QE if and only if it is unitarily equivalent to the (unique) extremal contractive Gleason solution in a deBranges-Rovnyak space \( \mathcal{H}(b) \) for a quasi-extreme \( b \in \mathcal{S}_d(\mathcal{J}, \mathcal{K}) \). The QE characteristic function \( b_T := b \) of a QE row contraction \( T \) is a complete unitary invariant: two QE row contractions \( T_1, T_2 \) are unitarily equivalent if and only if their characteristic functions coincide weakly.

Previous work on functional models for row contractions include [11, 12, 13, 7, 14, 15]. The theory of Popescu [11, 12] for CNC row contractions constructs a Sz.-Nagy-Foiaş-type model by studying the structure of the space of the minimal row isometric dilation of the row contraction, and defines a non-commutative ‘characteristic function’ which is a complete unitary invariant. This theory is extended to arbitrary CNC row contractions by Ball-Vinnikov in [13] and is applied to d-contractions (commutative row contractions) by Ball and Bolotnikov in [7]. This characteristic function can be identified with an element of the non-commutative or free (operator-valued) Schur classes [16, 17].

The papers [14, 15] of Bhattacharyya-Eschmeier-Sarkar extend the classical Sz.-Nagy-Foiaş model and the definition of the Sz.-Nagy-Foiaş characteristic function to CNC d-contractions (row contractions with mutually commuting component operators). In this theory the characteristic function is an element of the (operator-valued, several-variable) Schur classes.

In comparison, our results construct a Schur class characteristic function and a commutative deBranges-Rovnyak functional model for any CCNC row contraction. The class of all CCNC row contractions includes (but is strictly larger than) the class of all CNC d-contractions. Our characteristic function, \( b_T \), is equivalent to the Sz.-Nagy-Foiaş-type characteristic function of \( T \) as constructed for CNC d-contractions in [14, 15] (see Proposition 4.13). In summary, given the following strict hierarchy of classes of row contractions on Hilbert space,

\[
\text{CNU} \supseteq \text{CNC} \supseteq \text{CCNC} \supseteq \text{QE},
\]

this paper constructs a commutative deBranges-Rovnyak model for the CCNC and QE classes. Note that in the classical case where \( d = 1 \), the concepts of QE, CCNC, and CNC contraction coincide.
1.1. Vector-valued RKHS. We will be using the theory of vector-valued reproducing kernel Hilbert spaces throughout this paper, as presented in e.g. [13]. Recall the following basic facts from RKHS theory:

Given a set $X \subset \mathbb{C}^d$, and an auxiliary Hilbert space $\mathcal{H}$, an $\mathcal{H}$-valued RKHS $\mathcal{K}$ on $X$ is a Hilbert space of $\mathcal{H}$-valued functions on $X$ so that for any $x \in X$ the linear maps $K_x^* \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ defined by

$$K_x^* F = F(x) \in \mathcal{H}; \quad F \in \mathcal{K},$$

are bounded. We write $K_x := (K_x^*)^* \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ for the Hilbert space adjoint. The operator-valued function $K : X \times X \to \mathcal{L}(\mathcal{H})$:

$$K(x, y) := K_x^* K_y \in \mathcal{L}(\mathcal{H}); \quad x, y \in X,$$

is called the reproducing kernel of $\mathcal{K}$. One usually writes $\mathcal{K} = \mathcal{H}(K)$. The reproducing kernel $K$ of any vector-valued RKHS on $X$ is a positive kernel function on $X$: A function $K : X \times X \to \mathcal{L}(\mathcal{H})$ is an operator-valued positive kernel function on $X$ if for any finite set $\{x_k\}_{k=1}^N \subset X$, the matrix

$$[K(x_i, x_j)] \in \mathcal{L}(\mathcal{H}) \otimes \mathbb{C}^{N \times N},$$

is non-negative. The vector-valued extension of the theory of RKHS developed by Aronszajn and Moore (see e.g. [13]) shows that there is a bijection between positive $\mathcal{L}(\mathcal{H})$-valued kernel functions on $X \times X$ and RKHS of $\mathcal{H}$-valued functions on $X$. Namely, given any positive kernel $K$ on $X$ there is a unique RKHS $\mathcal{K}$ on $X$ so that $K$ is its reproducing kernel, $\mathcal{K} = \mathcal{H}(K)$.

Any two RKHS $\mathcal{H}(k), \mathcal{H}(K)$ with $\mathcal{L}(\mathcal{I})$ and $\mathcal{L}(\mathcal{K})$-valued positive kernel functions $k, K$ on some set $X$, respectively, can be naturally equipped with a multiplier space:

$$\text{Mult}(\mathcal{H}(k), \mathcal{H}(K)) := \{F : X \to \mathcal{L}(\mathcal{H}, \mathcal{K})| Fh \in \mathcal{H}(K) \forall h \in \mathcal{H}(k)\}.$$ 

Here, and throughout $\mathcal{H}, \mathcal{I}, \mathcal{K}$ denote separable (or finite-dimensional) Hilbert spaces. That is, $\text{Mult}(\mathcal{H}(k), \mathcal{H}(K))$ is the space of all operator-valued functions which multiply elements of $\mathcal{H}(k)$ into $\mathcal{H}(K)$. Viewing multipliers, $F$, (elements of the multiplier space) as linear maps, $M_F$, from $\mathcal{H}(k)$ into $\mathcal{H}(K)$, standard functional analysis arguments show that any multiplier is a bounded linear map, $\text{Mult}(\mathcal{H}(k), \mathcal{H}(K)) \subset \mathcal{L}(\mathcal{H}(k), \mathcal{H}(K))$, and $\text{Mult}(\mathcal{H}(k), \mathcal{H}(K))$ is closed in the weak operator topology. In the particular case where $\mathcal{H}(k) = \mathcal{H}(K)$, $\text{Mult}(\mathcal{H}(K)) := \text{Mult}(\mathcal{H}(K), \mathcal{H}(K))$ is a unital WOT-closed algebra of bounded linear operators on $\mathcal{H}(K)$, the multiplier algebra.

We work in the setting of vector-valued Drury-Arveson space $H^2_d \otimes \mathcal{H}$, where $\mathcal{H}$ is finite dimensional or separable. This is the vector-valued reproducing kernel Hilbert space $\mathcal{H}(k)$ of $\mathcal{H}$-valued functions on the ball $B^d : \mathbb{C}^d$ corresponding to the several-variable, operator-valued Szegő kernel:

$$k(z, w) := \frac{1}{1 - zw^*}I_{\mathcal{H}}; \quad z, w \in B^d.$$

Here, $zw^* := (w, z)_{\mathbb{C}^d}$, all inner products are assumed to be conjugate linear in the first argument. The Drury-Arveson space is arguably the canonical several-variable generalization of the classical Hardy space $H^2 = H^2(\mathbb{D})$, at least from an operator-theoretic viewpoint [19] [20] [21].
We will use the notation $H^\infty_d \otimes \mathcal{L}(\mathcal{J}, \mathcal{K}) := \text{Mult}(H^2_d \otimes \mathcal{J}, H^2_d \otimes \mathcal{K})$ (the multiplier spaces are the closure of this algebraic tensor product in the weak operator topology). The Schur classes are the closed unit balls of these multiplier spaces: $\mathcal{S}_d(\mathcal{J}, \mathcal{K}) := [H^\infty_d \otimes \mathcal{L}(\mathcal{J}, \mathcal{K})]_1$. In the single variable ($d = 1$) and scalar-valued ($\mathcal{K} = \mathbb{C}$) setting we recover the classical Hardy space $H^2(\mathbb{D})$ and algebra $H^\infty(\mathbb{D})$ of analytic functions on the disk which embed isometrically into $L^2, L^\infty$ of the unit circle $\mathbb{T}$, respectively, by taking non-tangential boundary limits [22].

As in the single-variable case, given any Schur class $b \in \mathcal{S}_d(\mathcal{J}, \mathcal{K})$, one can construct a positive kernel function $k^b$ on $\mathbb{B}^d \times \mathbb{B}^d$, the deBranges-Rovnyak kernel:

$$k^b(z, w) := \frac{I - b(z)b(w)^*}{1 - zw^*} \in \mathcal{L}(\mathcal{K}); \quad z, w \in \mathbb{B}^d,$$

and the corresponding RKHS $\mathcal{H}(b) := \mathcal{H}(k^b)$ is called the deBranges-Rovnyak space associated to $b$. It is straightforward to show that a bounded analytic (operator-valued) function $b$ on $\mathbb{B}^d$ belongs to the Schur class if and only if the above formula defines a positive kernel on $\mathbb{B}^d$ [23 Theorem 2.1], and we will use this fact frequently in the sequel.

It is easy to check that $k - k^b$ (with $k$ the Szegö kernel of vector-valued Drury-Arveson space) is a positive kernel function so that standard vector-valued RKHS theory implies that $\mathcal{H}(b)$ is contractively contained in $H^2_d \otimes \mathcal{K}$ [18]. Recall that in the single-variable setting, multiplication by the independent variable $z$ defines an isometry $S$ on $H^2 = H^2(\mathbb{D})$ called the shift. The shift plays a central role in the classical theory of Hardy spaces [22 23 24]. The adjoint, $S^*$, of the shift is called the backward shift, and acts as the difference quotient

$$(S^* f)(z) = \frac{f(z) - f(0)}{z}; \quad f \in H^2, \ z \in \mathbb{D}.$$

In the single-variable setting, any deBranges-Rovnyak space $\mathcal{H}(b)$ is co-invariant for the shift [10]. The natural several-variable generalization of the shift is the Arveson $d$-shift, $S : H^2_d \otimes \mathbb{C}^d \to H^2_d$, a row partial isometry on $H^2_d$, also denoted by $S = (S_1, ..., S_d)$ [20]. The component operators of $S$ mutually commute ($S$ is a $d$-contraction) and act as multiplication by the independent variables on $H^2_d$:

$$(SF)(z) = z_1F_1(z) + ... + z_dF_d(z); \quad F = (F_1, ..., F_d) \in H^2_d \otimes \mathbb{C}^d.$$

In contrast to the classical case, multi-variable deBranges-Rovnyak spaces are generally not co-invariant for the component operators of the Arveson $d$-shift [26]. The appropriate several-variable analogue of the adjoint of the restricted backward shift will be a Gleason solution for $\mathcal{H}(b)$, and Gleason solutions will play the role of the model operator in our commutative model for CCNC row contractions.

1.2. Herglotz spaces. It will be convenient to define a second reproducing kernel Hilbert space $\mathcal{H}^+(H_b)$ associated to suitable ‘square’ $b \in \mathcal{S}_d(\mathcal{J}, \mathcal{J})$.

In general we will say that a contraction $T \in \mathcal{L}(\mathcal{J}, \mathcal{K})$ is pure or purely contractive if $\|Ty\| < \|y\|$ for all $y$ in $\mathcal{J}$, and that $T$ is strict or strictly contractive if $\|T\| < 1$. A similar argument to [3] Proposition 2.1, Chapter V] shows that any $b \in \mathcal{S}_d(\mathcal{K}, \mathcal{J})$ decomposes as $b = b_0 + b_1$ on $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$ where $b_0(z) := b(z)|_{\mathcal{K}_0}$ is purely contractive and $b_1$ is a constant isometry on $\mathbb{B}^d$ from $\mathcal{K}_1$ onto its range in $\mathcal{J}$. 


We say a Schur class \( b \in \mathcal{S}(\mathcal{H}) \) is purely contractive or strictly contractive if \( b(z) \) is a pure or strict contraction, respectively, for all \( z \in \mathbb{B}^d \). As discussed in [9, Section 1.8], \( b \) is strictly contractive if and only if \( b(0) \) is a strict contraction.

We say that \( b \in \mathcal{S}(\mathcal{H}) \) is non-unital if \( I - b(z) \) is invertible for all \( z \in \mathbb{B}^d \). Any strictly contractive \( b \in \mathcal{S}(\mathcal{H}) \) is certainly non-unital. The Herglotz-Schur class, \( \mathcal{S}^+(\mathcal{H}) \), is the set of all \( \mathcal{L}(\mathcal{H}) \)-valued analytic functions on \( \mathbb{B}^d \) that are compositional inverses and define bijections between \( \mathcal{S}^+(\mathcal{H}) \). If \( b \) is an operator-valued positive kernel function. Any Herglotz-Schur function necessarily has positive real part. In particular, if \( b \in \mathcal{S}(\mathcal{H}) \) is square and non-unital, and

\[
H_b(z) := (I - b(z))^{-1}(I + b(z)),
\]

then

\[
K^b(z, w) := K^{H_b}(z, w) = (I - b(z))^{-1}k^b(z, w)(I - b(w))^* - 1,
\]

is a positive kernel so that \( H_b \in \mathcal{S}^+(\mathcal{H}) \). As described in [9, Section 1.8], the maps

\[
b \mapsto H_b := (I - b)^{-1}(I + b); \quad \text{and} \quad H \mapsto b_H := (H + I)^{-1}(H - I),
\]

are compositional inverses and define bijections between \( \mathcal{S}^+(\mathcal{H}) \) and non-unital elements of \( \mathcal{S}(\mathcal{H}) \). If \( H = H_b \in \mathcal{S}^+(\mathcal{H}) \) we call \( \mathcal{H}^+(H_b) := \mathcal{H}(K_b) \), the Herglotz space of \( b \).

By standard vector-valued RKHS theory, the above relationship between the deBranges-Rovnyak and Herglotz kernels for non-unital \( b \in \mathcal{S}(\mathcal{H}) \) implies that there is a unitary multiplier \( U_b : \mathcal{H}(b) \to \mathcal{H}^+(H_b) \).

**Lemma 1.3.** The map \( U_b : \mathcal{H}(b) \to \mathcal{H}^+(H_b) \) defined by multiplication by

\[
(1.1) \quad U_b(z) := (I - b(z))^{-1},
\]

is an onto isometry. The action of \( U_b \) on point evaluation kernels is

\[
U_b K^b_z = K^b_z(I - b(z)^*).
\]

It will be useful to consider the natural row partial isometry \( V^b : \mathcal{H}^+(H_b) \otimes \mathbb{C}^d \to \mathcal{H}^+(H_b) \) on the Herglotz space of any square \( b \in \mathcal{S}(\mathcal{H}) \) defined by

\[
(1.2) \quad z^* K^b_z h := \begin{pmatrix} \tau_1 K^b_z h \\ \vdots \\ \tau_d K^b_z h \end{pmatrix} \overset{V^b}{\mapsto} (K^b_z - K^b_0) h; \quad h \in \mathcal{H}.
\]

Verifying that this defines a partial isometry with

\[
\text{Ker}(V^b) = \bigvee_{z \in \mathbb{B}^d} z^* K^b_z \mathcal{H}, \quad \text{and} \quad \text{Ran}(V^b) = \bigvee_{z \in \mathbb{B}^d} (K^b_z - K^b_0) \mathcal{H},
\]

is a straightforward computation using the Herglotz kernel [9 Section 2].
1.4. **Gleason solutions.** In contrast with the single variable situation, as soon as \( d > 1 \), the deBranges-Rovnyak spaces \( \mathcal{H}(b) \) for arbitrary \( b \in \mathcal{A}_d(\mathfrak{J}, \mathfrak{K}) \) are generally not co-invariant for the component operators \( S_j \) of the Arveson \( d \)-shift \[26\]. Instead, the appropriate replacement for the ‘adjoint of the restricted backward shift’ in the several-variable theory is a contractive solution to the **Gleason problem** \[27, 28, 26, 29, 7\]:

**Definition 1.5.** A row-operator \( X \in \mathcal{L}(\mathcal{H}(b) \otimes \mathbb{C}^d, \mathcal{H}(b)) \) solves the **Gleason problem** in \( \mathcal{H}(b) \) if

\[
z(X^* f)(z) := z_1(X_1^* f)(z) + \ldots + z_d(X_d^* f)(z) = f(z) - f(0); \quad \forall f \in \mathcal{H}(b).
\]

We say that a **Gleason solution** \( X \) is **contractive** if

\[
XX^* \leq I - k_b^h(k_0^h)^*,
\]

and we say that \( X \) is **extremal** if equality holds in the above.

It is easy to check that for any row contraction \( X \) on \( \mathcal{H}(b) \), the Gleason solution condition \((1.3)\) above is equivalent to:

\[
(I - X^*)^{-1} k_b^h = k_2^h; \quad \forall \ z \in \mathbb{B}^d,
\]

and this property will also be used frequently in the sequel. Given \( z \in \mathbb{B}^d \), and any Hilbert space \( \mathcal{H} \), we will often view \( z \) as a strict contraction from \( \mathcal{H} \otimes \mathbb{C}^d \) into \( \mathcal{H} \): Define \( z^* : \mathcal{H} \to \mathcal{H} \otimes \mathbb{C}^d \) by \( z^* h := (z_1 h, \ldots, z_d h)^T \in \mathcal{H} \otimes \mathbb{C}^d \) and for any \( h \in \mathcal{H} \otimes \mathbb{C}^d \), the adjoint map \( z := (z^*)^* \in \mathcal{L}(\mathcal{H} \otimes \mathbb{C}^d, \mathcal{H}) \) obeys:

\[
zh = z \begin{bmatrix} h_1 \\ \vdots \\ h_d \end{bmatrix} = z_1 h_1 + \ldots + z_d h_d,
\]

and \( \|z\|^2 = z z^* := (z, z)_{\mathbb{C}^d} < 1 \).

**Remark 1.6.** If \( X \) is any extremal contractive Gleason solution for \( \mathcal{H}(b) \) where \( b \in \mathcal{A}_d(\mathfrak{J}, \mathfrak{K}) \) obeys \( b(0) = 0 \), then

\[
XX^* = I - k_b^h(k_0^h)^*,
\]

is a projection so that \( X \) is a row partial isometry on \( \mathcal{H}(b) \). In general, \( P_0 := I - k_b^h(0, 0)^{-1}(k_0^h)^* \) is the projection onto the subspace of all functions \( f \in \mathcal{H}(b) \) such that \( f(0) = 0 \), and

\[
k_b^h(0, 0) = I - b(0) b(0)^*.
\]

so that an extremal contractive Gleason solution \( X \) for \( \mathcal{H}(b) \) is a row partial isometry if and only if \( b(0) = 0 \).

In the case where \( d = 1 \), the unique solution to equation \((1.3)\) is the adjoint of the restriction of the backward shift \( S^* \) to \( \mathcal{H}(b) \), so that adjoints of Gleason solutions are natural analogues of the restricted backward shift in the several-variable setting. Many references define a Gleason solution for \( \mathcal{H}(b) \) as the adjoint of our definition above, we prefer to view it as a row contraction from several copies of a
Hilbert space into itself. Contractive solutions $X$ to the Gleason problem in $\mathcal{H}(b)$ always exist, although they are in general non-unique [23]. Also note that the component operators of a contractive Gleason solution $X$ for $\mathcal{H}(b)$ are generally non-commuting. In fact, the existence of a commuting contractive Gleason solution $X$ for $\mathcal{H}(b)$ is equivalent to co-invariance of $\mathcal{H}(b)$ with respect to the component operators of the Arveson $d$-shift, and in this case $X = (S^*|_{\mathcal{H}(b)})^*$ is a commuting Gleason solution for $\mathcal{H}(b)$ [28 Theorem 3.5]. This happens, for example, if $b \in \mathcal{A}(\mathcal{J}, \mathcal{K})$ is an inner multiplier, i.e. multiplication by $b$, $M_b : H_2^2 \otimes \mathcal{J} \to H_2^2 \otimes \mathcal{K}$ is a partial isometry so that $\mathcal{H}(b) \subset H_2^2 \otimes \mathcal{K}$ is a co-invariant model subspace of vector-valued Drury-Arveson space [23, 14, 30, 31].

Similarly we define contractive Gleason solutions for any $b \in \mathcal{A}(\mathcal{J}, \mathcal{K})$:

**Definition 1.7.** A linear map $b \in \mathcal{L}(\mathcal{J}, \mathcal{H}(b) \otimes \mathbb{C}^d)$, $b = \begin{bmatrix} b_1 \\ \vdots \\ b_d \end{bmatrix}$, $b_j \in \mathcal{L}(\mathcal{J}, \mathcal{H}(b))$, $1 \leq j \leq d$, is a solution to the Gleason problem for $b \in \mathcal{A}(\mathcal{J}, \mathcal{K})$ provided that

$$b(z) - b(0) = z \cdot b(z) := \sum_{j=1}^{d} z_j b_j(z).$$

We say that $b$ is a **contractive** Gleason solution for $b$ if

$$b^*b \leq I - b(0)^*b(0),$$

and an **extremal** Gleason solution for $b$ if equality holds in the above.

There is a surjection $b \mapsto X(b)$ of contractive Gleason solutions for $b \in \mathcal{A}(\mathcal{J}, \mathcal{K})$ onto contractive Gleason solutions for $\mathcal{H}(b)$ given by the formula

$$X(b)^* k_w^b := w^* k_w^b - b^* b(w)^*,$$

this surjection is injective if and only if $\bigcap_{z \in \mathbb{C}^d} \ker(b(z)) = \{0\}$, and it preserves extremal Gleason solutions. This follows as in [9, Section 5] (which considers the case $\mathcal{J} = \mathcal{K}$).

If $b \in \mathcal{A}(\mathcal{H})$ is square and non-unital, there is a bijection between contractive extensions $D$ of the row partial isometry $V^b$ defined on the Herglottz space $\mathcal{H}^+ (H_b)$ (i.e. $D : \mathcal{H}^+(H_b) \otimes \mathbb{C}^d \to \mathcal{H}^+(H_b)$, and $D(V^b)^* V^b = V^b$) and contractive Gleason solutions $b = b[D]$ given by

$$b[D] := U_b^* D^* K_0^b (I - b(0)),$$

see [9 Section 5]. If $X = X(b[D])$, we will simply write $X = X[D]$.

2. Completely non-coisometric row contractions

In this section we characterize completely non-coisometric (CNC) row contractions and motivate the definition of CNC row contractions (contractions obeying the commutative CNC condition). As in the classical setting a row contraction $T : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H}$ is CNC if there is no non-trivial co-invariant subspace $\mathcal{H}' \subset \mathcal{H}$ (co-invariant for each component operator $T_k$ of $T$, $1 \leq k \leq d$) so that $T^*|_{\mathcal{H}'}$ is an isometry of $\mathcal{H}'$ into $\mathcal{H}' \otimes \mathbb{C}^d$. As observed in [17], a row contraction $T : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H}$ is CNC if and
only if \( \bigvee_{\alpha \in \mathbb{F}^d} T^\alpha \text{Ran} (D_{T^*}) = \mathcal{H} \), where, as before, the defect operator of any contraction \( T : \mathcal{J} \rightarrow \mathcal{K} \) is \( D_T := \sqrt{I - T^*T} \), and \( \mathbb{F}^d \) denotes the free monoid on \( d \) generators (we will shortly recall the definition). For convenience, we will provide a proof based on methods ultimately due to Kre˘ın.

Initially, we focus on the case of a (row) partial isometry, \( X : \mathcal{H} \otimes \mathbb{C}^d \rightarrow \mathcal{H} \), and we define the restricted range spaces,
\[
\mathcal{R} (X - z) := \text{Ran} \left( (X - z)X^*X \right); \quad z \in \mathbb{C}^d,
\]
as the range of \( X - z \) restricted to the initial space of \( X \). The orthogonal complement, \( \mathcal{R} (X - z)^\perp \), will be called a \( z \)-deficiency space or the \( z \)-defect space. More generally, as in \( \text{[32]} \), consider the non-commutative (NC) open unit ball:
\[
\Omega := \bigotimes_{n=1}^{\infty} \Omega_n; \quad \Omega_n := \left( \mathbb{C}^{n \times n} \otimes \mathbb{C}^d \right)_1.
\]
Elements \( Z \in \Omega_n \) are viewed as strict row contractions from \( \mathbb{C}^n \otimes \mathbb{C}^d \) into \( \mathbb{C}^n \):
\[
Z =: (Z_1, ..., Z_d); \quad Z_k \in \mathbb{C}^{n \times n}.
\]
In particular, note that \( \Omega_1 \simeq \mathbb{B}^d = (\mathbb{C}^d)_1 \) can be identified with the open unit ball of \( \mathbb{C}^d \).

**Definition 2.1.** For any \( Z \in \Omega_n \), \( n \in \mathbb{N} \), let
\[
\mathcal{R} (X - Z) := \text{Ran} \left( \left[ (X \otimes I_n) - (I_{\mathcal{H}} \otimes Z) \right] (X^*X \otimes I_n) \right).
\]
\[
= \left[ I_{\mathcal{H}} \otimes I_n - (I_{\mathcal{H}} \otimes Z)(X^* \otimes I_n) \right] \text{Ran} (X) \otimes \mathbb{C}^n
\]
\[
= : (I - ZX^*) \text{Ran} (X) \otimes \mathbb{C}^n,
\]
where
\[
(2.1) \quad ZX^* := X_1^* \otimes Z_1 + ... + X_d^* \otimes Z_d \in \left( \mathcal{L}(\mathcal{H} \otimes \mathbb{C}^n) \right)_1.
\]

The main results of this section will be:

**Theorem 2.2.** Let \( V : \mathcal{H} \otimes \mathbb{C}^d \rightarrow \mathcal{H} \) be a row partial isometry. The subspace
\[
\mathcal{H}' := \{ h \in \mathcal{H} \mid h \otimes \mathbb{C}^n \subseteq \mathcal{H}'_n; \quad n \in \mathbb{N} \}; \quad \mathcal{H}'_n := \bigcap_{Z \in \Omega_n} \mathcal{R} (V - Z),
\]
is the largest co-invariant subspace for \( V \) on which \( V^* \) acts isometrically.

In particular, \( V \) is CNC if and only if
\[
\mathcal{H} = \left( \mathcal{H}' \right)^\perp = \bigvee_{\lambda \in \mathbb{C}^n; \quad Z \in \Omega_n} \lambda (I - VZ^*)^{-1} \text{Ran} (V)^\perp \otimes \mathbb{C}^n
\]
\[
= \bigvee_{\alpha \in \mathbb{F}^d} V^\alpha \text{Ran} (V)^\perp.
\]

In the above, as in Subsection \( \text{[1.1]} \) we view any \( \lambda \in \mathbb{C}^n \) as a row operator, \( \lambda : \mathcal{H} \otimes \mathbb{C}^n \rightarrow \mathcal{H} \) (and \( V^\alpha \) for \( \alpha \in \mathbb{F}^d \) is defined below).

This theorem is a generalization of a characterization of CNU partial isometries due to Kre˘ın \( \text{[33]} \) Chapter 1, Theorem 2.1. Kre˘ın’s result is proven in the setting of unbounded symmetric operators.
This can be restated in terms of partial isometries using the Cayley transform, a fractional linear transformation that implements a bijection between partial isometries and symmetric linear transformations [34].

In the special case where $V$ is commutative, i.e. a $d$-contraction, this yields:

**Theorem 2.3.** Let $V : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H}$ be a $d$-partial isometry. The subspace

$$\mathcal{H}' := \bigcap_{z \in \mathbb{B}^d} \mathcal{R}(V - z),$$

is the largest co-invariant subspace for $V$ on which $V^*$ acts isometrically.

In particular, $V$ is CNC if and only if

$$\mathcal{H} = (\mathcal{H}')^\perp = \bigcup_{z \in \mathbb{B}^d = \Omega_1} (I - Vz^*)^{-1} \mathcal{R}(V)^\perp = \bigcup_{n \in \mathbb{N}^d} V^n \mathcal{R}(V)^\perp.$$

In the above, $\mathbb{N}^d$ is the additive monoid of $d$-tuples of non-negative integers (and $V^n$ is defined below).

A key corollary of these results is:

**Corollary 2.4.** A row contraction $T : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H}$ is CNC if and only if

$$\mathcal{H} = \bigcup_{\alpha \in \mathbb{B}^d} T^\alpha \mathcal{R}(D_T) .$$

A $d$-contraction $T : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H}$ is CNC if and only if

$$\mathcal{H} = \bigcup_{n \in \mathbb{N}^d} T^n \mathcal{R}(D_{T^*}) = \bigcup_{z \in \mathbb{B}^d} (I - Tz^*)^{-1} \mathcal{R}(D_{T^*}) .$$

In particular, if a (not necessarily commutative) row contraction $T$ on $\mathcal{H}$ obeys

$$\mathcal{H} = \bigcup_{z \in \mathbb{B}^d} (I - Tz^*)^{-1} \mathcal{R}(D_{T^*}) ,$$

then $T$ is CNC.

Motivated by the above corollary, we define:

**Definition 2.5.** A row contraction $T : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H}$ obeys the commutative CNC condition, and we write: $T$ is CCNC, if

$$\mathcal{H} = \bigcup_{z \in \mathbb{B}^d} (I - Tz^*)^{-1} \mathcal{R}(D_{T^*}) .$$

Corollary 2.4 implies that any CCNC row contraction is CNC, and that any CNC $d$–contraction is CCNC.

**Example 2.6.** (Gleason solutions are CCNC) Any extremal contractive Gleason solution, $X$, for a several-variable deBranges-Rovnyak space $\mathscr{H}(b), b \in \mathscr{S}_d(\mathcal{H}, \mathcal{H})$, is CNC. Since $X$ is extremal,

$$XX^* = I - k_0^b(k_0^b)^* ,$$
and since $D_X^2 = I - XX^*$, it follows that $\text{Ran} \ (D_X^* \cdot)$ is the range of the projection $k_0 b k_0^b (0,0)^{-1} (k_0^b)^*$. 

Since $X$ is a contractive Gleason solution, it follows that $\bigvee_{z \in \mathbb{B}^d} (I - X z^*)^{-1} \text{Ran} \ (D_X^\cdot) = (I - XX^*)^{-1} k_0^b \mathcal{H}$, so that $\text{Ran} \ (D_X^* \cdot)$ is the range of the projection $k_0^b \mathcal{H} = \mathcal{H}(b)$, (by equation (1.4))

so that $X$ is CCNC (and therefore CNC) by Corollary 2.4. It should be pointed out that extremal contractive Gleason solutions are generally not $d-$contractions, i.e. they are generally non-commutative.

In the above, recall that the free semigroup (or monoid), $\mathbb{F}^d$, on $d \in \mathbb{N}$ letters, is the multiplicative unital semigroup of all finite products or words in the $d$ letters $\{1, ..., d\}$. That is, given words $\alpha := i_1 \cdots i_n$, $\beta := j_1 \cdots j_m$, $i_k, j_l \in \{1, ..., d\};$ $1 \leq k \leq n$, $1 \leq l \leq m$, their product $\alpha \beta$ is defined by concatenation: $\alpha \beta = i_1 \cdots i_n j_1 \cdots j_m$, and the unit is the empty word, $\emptyset$, containing no letters. Given $\alpha = i_1 \cdots i_n$, we use the standard notation $|\alpha| = n$ for the length of the word $\alpha$. Let $\mathbb{N}^d$ be the unital additive semigroup or monoid of $d$-tuples of non-negative integers. By the universality property of the free unital semigroup $\mathbb{F}^d$, there is a unital semigroup epimorphism $\lambda : (\mathbb{F}^d, \cdot) \to (\mathbb{N}^d, +)$, the letter counting map which sends a given word $\alpha = i_1 \cdots i_n \in \mathbb{F}^d$ to $n = (n_1, ..., n_d) \in \mathbb{N}^d$ where $n_k$ is the number of times the letter $k$ appears in the word $\alpha$.

Given any $\alpha = \alpha_1 \cdots \alpha_k \in \mathbb{F}^d$, and any row contraction $T = (T_1, ..., T_d)$, we use the standard notation $T^\alpha := T_{\alpha_1} \cdots T_{\alpha_k}$, and for any $n \in \mathbb{N}^d$ we define the symmetrized monomial:

$$T^n := \sum_{\alpha; \ \lambda(\alpha) = n} T^\alpha.$$ 

Note, in particular, that if $T$ is commutative, i.e. $T$ is a $d$-contraction, that

$$\bigvee_{\alpha \in \mathbb{F}^d} T^\alpha \text{Ran} \ (D_T^\cdot) = \bigvee_{n \in \mathbb{N}^d} T^n \text{Ran} \ (D_T^\cdot),$$

and this second set can be re-expressed as follows:

**Lemma 2.7.** [8, Lemma 2.2] Given any row contraction $T$ on $\mathcal{H}$, 

$$\bigvee_{z \in \mathbb{B}^d} (I - T z^*)^{-1} = \bigvee_{n \in \mathbb{N}^d} T^n.$$ 

The next example below shows that not every CNC row contraction is CCNC:
Example 2.8. (The Arveson d-shift and the left free shift) The Arveson d-shift, \( S \), on the Drury-Arveson space, \( H_d^2 \), is a commutative row partial isometry. In particular \( D_{S^*} = k_0 k_0^* \) is a projection so that
\[
\bigvee_{z \in \mathbb{B}^d} (I - Sz^*)^{-1} \operatorname{Ran}(D_{S^*}) = \bigvee_{z \in \mathbb{B}^d} k_z = H_d^2,
\]
and \( S \) is CCNC. Observe that \( S \) is an extremal contractive Gleason solution for the deBranges-Rovnyak space \( H(b) = H_d^2 \) with \( b = 0 \in \mathcal{S}_d = \mathcal{S}_d(\mathbb{C}) \).

For an example of a CNC row contraction which is not CCNC, consider the left free shift \( L \) on the full Fock space \( F_d^2 \). Recall here that the full Fock space over \( C^d \), \( F_d^2 \), is the direct sum of all tensor powers of \( C^d \):
\[
F_d^2 := C \oplus (C^d \otimes C^d) \oplus (C^d \otimes C^d \otimes C^d) \oplus \cdots = \bigoplus_{k=0}^{\infty} (C^d)^{k\otimes}.
\]
Fix an orthonormal basis \( e_1, \ldots, e_d \) of \( C^d \). The left creation operators \( L_1, \ldots, L_d \) are the operators which act as tensoring on the left by these basis vectors:
\[
L_k f := e_k \otimes f; \quad f \in F_d^2.
\]
The left free shift is the row operator \( L := (L_1, \ldots, L_d) : F_d^2 \otimes C^d \to F_d^2 \). The left free shift, is, in fact, a row isometry: \( L^* L = I_{F_2} \otimes I_d \). The orthogonal complement of the range of \( L \) is the vacuum vector \( 1 \) which spans the subspace \( C := (C^d)^{0\otimes} \subset F_d^2 \). A canonical orthonormal basis for \( F_d^2 \) is then \( \{e_\alpha\}_{\alpha \in \mathbb{F}_d} \) where \( e_\alpha = L^\alpha 1 \) and \( \mathbb{F}_d \). It follows that \( L \) is CNC:
\[
\bigvee_{\alpha \in F_d^2} L^\alpha \operatorname{Ran}(D_{L^*}) = \bigvee_{\alpha \in F_d^2} L^\alpha 1 = F_d^2.
\]
However, \( L \) is not CCNC, since
\[
\bigvee_{n \in \mathbb{N}_d} L^n \operatorname{Ran}(D_{L^*}) = \bigvee_{n \in \mathbb{N}_d} L^n 1,
\]
is equal to symmetric or Bosonic Fock space, a proper subspace of \( F_d^2 \) (which is in fact canonically isomorphic to \( H_d^2 \), see e.g. [19, Section 4.5]).

It will be convenient to first establish several preliminary facts before proving Theorem 2.2 and its corollaries. Given a partial isometry \( V : \mathcal{H}_1 \to \mathcal{H}_2 \), and a contraction \( T : \mathcal{K}_1 \to \mathcal{K}_2 \) where \( \mathcal{K}_k \subset \mathcal{K}_k \), we say that \( T \) extends \( V \) and write \( V \subseteq T \) if \( TV^* V = V \).

Lemma 2.9. ([9 Lemma 2.3]) Let \( V : \mathcal{H}_1 \to \mathcal{H}_2 \) be a partial isometry and assume that \( \mathcal{H}_k \subseteq \mathcal{J}_k \) for Hilbert spaces \( \mathcal{H}_k, \mathcal{J}_k \), \( k = 1, 2 \). For any contraction \( T : \mathcal{J}_1 \to \mathcal{J}_2 \) the following are equivalent:

(i) \( T \) is a contractive extension of \( V \), \( V \subseteq T \).

(ii) \( T^* \) is a contractive extension of \( V^* \), \( V^* \subseteq T^* \).

(iii) There is a contraction \( C : \mathcal{J}_1 \to \mathcal{J}_2 \) such that \( \operatorname{Ker}(C) = \mathcal{J}_1 \cap \operatorname{Ker}(V) \), \( \operatorname{Ran}(C) \subseteq \mathcal{J}_2 \cap \operatorname{Ran}(V) \) and \( T = V - C \).
Lemma 2.10. Let $V$ be a row partial isometry on $\mathcal{H} \otimes \mathbb{C}^d$ and let $T \supseteq V$ be a row contractive extension to $\mathcal{K} \otimes \mathbb{C}^d$, $\mathcal{K} \supseteq \mathcal{H}$. Then for any $Z \in \Omega_n$,\[
(I - TZ^*)^{-1} : \mathcal{K} \otimes \mathbb{C}^n \ominus (\operatorname{Ran}(V) \otimes \mathbb{C}^n) \to \mathcal{K} \otimes \mathbb{C}^n \ominus \mathcal{R}(V - Z), \]is an isomorphism.

Proof. Since $Z$ is strictly contractive, $TZ^*$, as defined above in equation (2.1), is a strict contraction so that $(I - TZ^*)^{-1}$ is well-defined. Assume that $f \in \mathcal{K} \otimes \mathbb{C}^n \ominus \mathcal{R}(V - Z)$, and suppose $g \in \operatorname{Ran}(V) \otimes \mathbb{C}^n$ so that $g = (V \otimes I_n)G$ for some $G \in (\mathcal{K} \otimes \mathbb{C}^n) \otimes \mathbb{C}^d \ominus \ker(V \otimes I_n)$. Then,
\[
\langle (I - TZ^*)f, g \rangle = \langle f, (I - ZT^*)(V \otimes I_n)G \rangle = \langle f, (I - ZV^*)(V \otimes I_n)G \rangle = 0.
\]
This proves that
\[
(I - TZ^*)((\mathcal{K} \otimes \mathbb{C}^n) \ominus \mathcal{R}(V - Z)) \subseteq (\mathcal{K} \otimes \mathbb{C}^n) \ominus (\operatorname{Ran}(V) \otimes \mathbb{C}^n),
\]
so that
\[
(\mathcal{K} \otimes \mathbb{C}^n) \ominus \mathcal{R}(V - Z) \subseteq (I - TZ^*)^{-1}(\mathcal{K} \otimes \mathbb{C}^n \ominus \operatorname{Ran}(V) \otimes \mathbb{C}^n).
\]
For the reverse inclusion suppose that $f \in \mathcal{K} \otimes \mathbb{C}^n \ominus \operatorname{Ran}(V \otimes I_n)$. Also assume that $g \in \mathcal{R}(V - Z)$, $g = (I - ZV^*)(V \otimes I_n)G$. Then,
\[
\langle (I - TZ^*)^{-1}f, g \rangle = \langle f, (I - ZT^*)^{-1}(I - ZV^*)VG \rangle = \langle f, (I - ZT^*)^{-1}(I - ZT^*)VG \rangle = \langle f, VG \rangle = 0.
\]
\(\square\)

Lemma 2.11. Suppose that $\mathcal{L} \subseteq \mathcal{H}$ is co-invariant for $X$, and that $X^{|_{\mathcal{L}}}$ is an isometry. Then for any $Z \in \Omega_n$, $\mathcal{L} \otimes \mathbb{C}^n \subseteq \mathcal{R}(X - Z)$.

Proof.\[
\mathcal{R}(X - Z) \supseteq (I - ZX^*)X \otimes I_n(\mathcal{L} \otimes \mathbb{C}^n \otimes \mathbb{C}^d)
= (I - ZX^*)\mathcal{L} \otimes \mathbb{C}^n; \quad X \text{ is co-isometric on } \mathcal{L} \otimes \mathbb{C}^d, \text{ hence onto } \mathcal{L}
= \mathcal{L} \otimes \mathbb{C}^n; \quad (I - ZX^*) \text{ is invertible, hence onto.}
\] \(\square\)

Definition 2.12. For any $n \in \mathbb{N}$ define $\mathcal{H}'_n \subset \mathcal{H} \otimes \mathbb{C}^n$ by\[
\mathcal{H}'_n := \bigcap\limits_{Z \in \Omega_n} \mathcal{R}(X - Z).
\]
Also define $\mathcal{H}' \subseteq \mathcal{H}$ by

$$\mathcal{H}' := \{ h \in \mathcal{H} \mid h \otimes C^n \subseteq \mathcal{H}'', \forall n \in \mathbb{N} \}$$

$$= \{ h \in \mathcal{H} \mid \lambda^* h \in \mathcal{H}'', \forall \lambda \in C^n \forall n \in \mathbb{N} \}.$$  

In the above, as before, given any $\lambda \in C^n$, we view $\lambda : \mathcal{H} \otimes C^n \to \mathcal{H}$ as a row operator.

**Lemma 2.13.** $\mathcal{H}'$ is a closed subspace, and

$$\left(\mathcal{H}' \right)^\perp = \bigvee_{\lambda, \tau \in C^n; \ Z \in \Omega_n; \ n \in \mathbb{N}} \lambda(I - XZ^*)^{-1} \operatorname{Ran}(X)^\perp$$

$$= \bigvee_{\lambda \in C^n; \ Z \in \Omega_n; \ n \in \mathbb{N}} \lambda(I - XZ^*)^{-1} \operatorname{Ran}(X)^\perp \otimes C^n.$$  

**Proof.** Let $h_k$ be Cauchy in $\mathcal{H}'$ with limit $h \in \mathcal{H}$. Then for any fixed $n \in \mathbb{N}$ and $\tau \in C^n$, $\tau^* h_k$ is Cauchy in $\mathcal{H}' \otimes C^n$, and hence the limit $\tau^* h$ belongs to $\mathcal{H}_n'$, since $\mathcal{H}_n'$ is clearly closed. Since this holds for any $n$, we obtain that $h \in \mathcal{H}'$.  

For the proof of the second statement suppose that $h \in \mathcal{H}$ is orthogonal to any vector of the form

$$\lambda(I - XZ^*)^{-1} \tau^* \operatorname{Ran}(X)^\perp : \lambda, \tau \in C^n, \ Z \in \Omega_n.$$  

This happens if and only if:

$$h \perp \lambda(I - XZ^*)^{-1} \operatorname{Ran}(X)^\perp \otimes C^n; \quad \lambda \in C^n, \ Z \in \Omega_n, \ n \in \mathbb{N}$$

$$\Leftrightarrow h \otimes C^n \perp (I - XZ^*)^{-1} \operatorname{Ran}(X)^\perp \otimes C^n; \quad Z \in \Omega_n, \ n \in \mathbb{N}.$$  

This proves that $h$ has this property if and only if $h \otimes C^n \subseteq \mathcal{R}(X - Z)$ for any $Z \in \Omega_n$, i.e. if and only if $h \otimes C^n \subseteq \mathcal{H}_n'$ for any $n \in \mathbb{N}$ and therefore if and only if $h \in \mathcal{H}'$.  

**Lemma 2.14.** If $\mathcal{L} \subseteq \mathcal{H}$ is co-invariant for the row partial isometry $X$ and $X^*$ is isometric on $\mathcal{L}$ then $\mathcal{L} \subseteq \mathcal{H}'$.

**Proof.** By Lemma 2.11 $\mathcal{L} \otimes C^n \subseteq \mathcal{H}_n''$ for any $n \in \mathbb{N}$. In particular, given any $l \in \mathcal{L}$, $\tau^* l$ belongs to $\mathcal{H}_n''$ for any $\tau \in C^n$ and any $n \in \mathbb{N}$. Therefore $l \in \mathcal{H}'$ by definition and $\mathcal{L} \subseteq \mathcal{H}'$.  

The following technical fact provides a useful description of $(\mathcal{H}')^\perp$.

**Lemma 2.15.**

$$\bigvee_{\lambda, \tau \in C^n; \ Z \in \Omega_n; \ n \in \mathbb{N}} \lambda(I - XZ^*)^{-1} \tau^* = \bigvee_{\alpha \in \mathbb{P}_d} (X^*)^\alpha.$$  

**Proof.** For simplicity let $R := X^*$. First note that anything in the the left hand side (LHS) of the above equation is a linear combination of products of the $R_1, ..., R_d$, and so it follows that the left hand side is contained in the right hand side (RHS).
We will prove the converse inductively on the length, $N = |\alpha|$ of a word $\alpha \in \mathbb{F}^d$. First, taking $n = 1$, we have by Lemma 2.7 that $R^n$ belongs to the LHS for all $n \in \mathbb{N}^d$. In particular $R_k \in LHS$ for all $1 \leq k \leq d$ so that the inductive hypothesis holds for $N = 1$. Assume that the inductive hypothesis holds for all $\alpha \in \mathbb{F}^d$ of length less than or equal to $K \in \mathbb{N}$. That is, $|\alpha| \leq K$ implies that $R^\alpha$ belongs to the LHS. To complete the induction step we need to prove that given any $\beta \in \mathbb{F}^d$ of length $K + 1$ that $R^\beta$ belongs to the LHS.

Any such $\beta$ can be written $\beta = j\alpha$ where $j \in \{1, ..., d\}$ and $|\alpha| = K$. By the hypothesis $R^\alpha$ is the norm-limit of finite linear combinations of terms of the form

$$\lambda(I - ZX^*)^{-1}\tau^*,$$

where $\lambda, \tau \in \mathbb{C}^n$ and $Z \in \Omega_n$ for some $n \in \mathbb{N}$. It therefore suffices to prove that for any such term, and any fixed $1 \leq j \leq d$, we can find $W \in \Omega_m$ and $\Lambda, \Gamma \in \mathbb{C}^m$ so that

$$\Lambda(I - WR)^{-1}\Gamma^* = R_j \lambda(I - ZX^*)^{-1}\tau^*.$$

For simplicity fix $j = 1$. The other cases will follow from an analogous argument.

Choose $W = (W_1, \ldots, W_d) \in \Omega_{2n}$ as follows:

$$W_1 := \begin{bmatrix} Z_1 & 0_n \\ rI_n & 0_n \end{bmatrix}; \quad W_k := \begin{bmatrix} Z_k & 0 \\ 0 & 0 \end{bmatrix}; \quad 2 \leq k \leq d,$$

where if $\|Z\|^2 = s < 1$, choose $0 < r < 1$ small enough so that $1 > s(1 + r^2)$.

Then,

$$I - WW^* = \begin{bmatrix} I - ZZ^* & -rZ_1 \\ -rZ_1^* & I \end{bmatrix},$$

and by Schur complement theory [35, Appendix A.5.5], this is strictly positive if and only if $I - ZZ^* - r^2Z_1Z_1^* > 0$. By our choice of $r$, this is the case, and it follows that $W \in \Omega_{2n}$ is strictly contractive. Observe that

$$WR = \begin{bmatrix} ZR & 0 \\ rR_1 & 0 \end{bmatrix}; \quad \text{and} \quad (WR)^k = \begin{bmatrix} (ZR)^k & 0 \\ rR_1(ZR)^{k-1} & 0 \end{bmatrix}; \quad k \geq 2.$$

It follows that

$$(I - WR)^{-1} = \begin{bmatrix} (I - ZR)^{-1} & 0 \\ rR_1(I - ZR)^{-1} & I \end{bmatrix}.$$

Taking $\Lambda := (0_n, \lambda)$ and $\Gamma = (\tau, 0_n)$ then yields

$$\Lambda(I - WR)^{-1}\Gamma^* = rR_1\lambda(I - ZR)^{-1}\tau^*,$$

and the inductive step follows. \hfill \Box

We now have all the necessary ingredients to prove the main result of this section:

**Proof.** (of Theorem 2.2) By Lemma 2.11 any co-invariant subspace, $\mathcal{L}$, on which $V^*$ acts isometrically is contained in $\mathcal{H}'$. It remains to show that $\mathcal{H}'$ is co-invariant for $V$, and that $V^*|_{\mathcal{H}'}$ is an isometry.
Let \( U \) be a co-isometric extension (e.g. a Cuntz unitary dilation) of \( V \) on \( \mathcal{K} \supseteq \mathcal{H} \). If \( h \in \mathcal{H'} \), then \( \tau^* h \in \mathcal{H}'' \). Hence,
\[
\tau^* h \perp \mathcal{K} \otimes \mathbb{C}^n \ominus \mathcal{R}(V - Z) = (I - UZ^*)^{-1} ((\mathcal{K} \otimes \mathbb{C}^n) \ominus \mathcal{R}(V) \otimes \mathbb{C}^n).
\]
This happens if and only if
\[
(I - ZU^*)^{-1} \tau^* h \in \mathcal{R}(V) \otimes \mathbb{C}^n,
\]
and this implies
\[
\bigvee_{\lambda, \in \mathcal{C}^n} \lambda (I - ZU^*)^{-1} \tau^* h \in \mathcal{R}(V).
\]
By the last lemma this happens if and only if
\[
\bigvee_{\alpha \in \mathbb{F}^d} (U^*)^\alpha h \in \mathcal{R}(V).
\]
Given any \( VF \in \mathcal{R}(V) \), it is easy to see that by definition
\[
(I - ZU^*) \tau^* VF \in \mathcal{R}(V - Z).
\]
Hence,
\[
\mathcal{R}(V - Z) \supseteq (I - ZU^*) \bigvee_{h \in \mathcal{H'}} \tau^* (U^*)^\alpha h
\]}
\[
\supseteq (I - ZU^*) \bigvee_{h \in \mathcal{H'}', \alpha \in \mathbb{F}^d} \tau^* \lambda (I - WU^*)^{-1} \kappa^* (U^*)^\alpha h, \quad \text{(by Lemma 2.15)}
\]}
so that \( (U^*)^\alpha h \in \mathcal{H}' \) for all \( \alpha \in \mathbb{F}^d \) and all \( h \in \mathcal{H}' \).

Given any \( h \in \mathcal{H}' \subset \mathcal{R}(V) \), \( h = VH \) where \( V^* VH = H \) so that
\[
V^* h = V^* VH = U^* VH = U^* h.
\]
This proves that \( \mathcal{H}' \) is co-invariant for \( V \), and that \( V \) is co-isometric on \( \mathcal{H}' \). \( \square \)

Although Theorem 2.3 is an immediate consequence of Theorem 2.2 under the assumption that \( X \) is a \( d \)-partial isometry, the above proof can be modified to prove Theorem 2.3 directly.

**Lemma 2.16.** Let \( T : \mathcal{K} \otimes \mathcal{C}^d \rightarrow \mathcal{K} \) be a row contraction. Then \( \mathcal{K} \otimes \mathcal{C}^d \) decomposes as \( \mathcal{K} \otimes \mathcal{C}^d = \mathcal{K}_0 \oplus \mathcal{K}_1 \) and \( \mathcal{K} = \mathcal{K}_0' \oplus \mathcal{K}_1' \), where \( V := TP_{\mathcal{K}_0} \) is a row partial isometry with initial space \( \mathcal{K}_0 \subseteq \mathcal{K} \otimes \mathcal{C}^d \) and final space \( \mathcal{K}_0' \subseteq \mathcal{K} \). The row contraction \( C := -TP_{\mathcal{K}_1} \) is a pure row contraction, i.e., \( \| Ch \| < \| h \| \) for any \( h \in \mathcal{K}_1 \otimes \mathcal{C}^d \), with final space \( \mathcal{R}(C) \subseteq \mathcal{K}_1' = \mathcal{R}(V)^\perp \).

**Proof.** Let \( \mathcal{H}_0 := \mathcal{R}(D_T)^\perp = \mathcal{K}(D_T) \), where recall \( D_T = \sqrt{T - T^* T} \). It is clear that \( V|_{\mathcal{H}_0} \) is an isometry, and can be extended to a row partial isometry on \( \mathcal{H} \otimes \mathcal{C}^d \) with initial space \( \mathcal{H}_0 \). It follows
that $C := V - T$ is a pure contraction and by Lemma 2.9 the initial space of $C$ is contained in $\mathcal{H}_1 = \mathcal{H}_0^\perp$ and the final space of $C$ is contained in $\mathcal{H}_1' = (\mathcal{H}_0')^\perp$. □

**Definition 2.17.** The above decomposition $T = V - C$ of any row contraction on $\mathcal{H}$ into a row partial isometry, $V$, on $\mathcal{H} \otimes \mathbb{C}^d$ and a pure row contraction $C$ on $\mathcal{H} \otimes \mathbb{C}^d$ with $\text{Ker}(C)^\perp \subseteq \text{Ker}(V)$ and $\text{Ran}(C) \subseteq \text{Ran}(V)^\perp$, will be called the *isometric-pure decomposition* of $T$.

**Proof.** (of Corollary 2.4) By Lemma 2.16, it follows that $T$ is CNC if and only if its partial isometric part $V$ is CNC. Since $T \supseteq V$ is a row contractive extension of $V$ on $\mathcal{H}$, Lemma 2.10, Lemma 2.15 and Theorem 2.2 imply that $T$ is CNC if and only if

$$\mathcal{H} = \bigvee_{\lambda \in \mathbb{C}^n; \; Z \in \Omega_n} \lambda (I - V Z^*)^{-1} \text{Ran}(V)^\perp \otimes \mathbb{C}^n$$

$$= \bigvee_{\lambda \in \mathbb{C}^n; \; Z \in \Omega_n} \lambda (I - T Z^*)^{-1} \text{Ran}(D_{T^*}) \otimes \mathbb{C}^n$$

$$= \bigvee_{\alpha \in \mathbb{F}^d} T^\alpha \text{Ran}(D_{T^*}).$$

If $T$ is CCNC then it is clearly CNC, since in this case,

$$\bigvee_{\alpha \in \mathbb{F}^d} T^\alpha \text{Ran}(D_{T^*}) \supseteq \bigvee_{n \in \mathbb{N}^d} T^n \text{Ran}(D_{T^*}) = \mathcal{H}. □$$

We will sometimes refer to a CCNC row contraction $T : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H}$ as a *Gleason* row contraction for reasons that will be made clear by the results of this paper.

### 3. Model maps for CCNC row partial isometries

Let $V : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H}$ be a Gleason or CCNC row partial isometry. By definition,

$$\mathcal{H} = \bigvee_{z \in \mathbb{B}^d} \mathcal{R}(V - z)^\perp = \bigvee_{z \in \mathbb{B}^d} (I - V z^*)^{-1} \text{Ran}(V)^\perp,$$

and $V$ is CNC by the results of the previous section.

**Definition 3.1.** A *model triple*, $(\gamma, \mathcal{J}_\infty, \mathcal{J}_0)$ for a CCNC row partial isometry $V$ on $\mathcal{H}$ consists of auxiliary Hilbert spaces $\mathcal{J}_\infty, \mathcal{J}_0$ of dimension $\text{Ker}(V)$ and $\text{Ran}(V)^\perp$, respectively, and a *model map*, $\gamma$, defined on $\mathbb{B}^d \cup \{\infty\}$:

$$\gamma : \begin{cases} \mathbb{B}^d \to \mathcal{L}((\mathcal{J}_0, \mathcal{R}(V - z)^\perp)) \\ \{\infty\} \to \mathcal{L}(\mathcal{J}_\infty, \text{Ker}(V)) \end{cases},$$

such that $\gamma(z)$ is a linear isomorphism for any $z \in \mathbb{B}^d$ and $\gamma(0), \gamma(\infty)$ are onto isometries.

We say that $(\Gamma, \mathcal{J}_\infty, \mathcal{J}_0)$ is an *analytic model triple* and that $\Gamma$ is an *analytic model map* if $z \mapsto \Gamma(z)$ is anti-analytic for any $z \in \mathbb{B}^d$. 


Lemma 2.10 shows that any row contractive extension $T \supseteq V$ of a row partial isometry on $\mathcal{H}$ gives rise to an analytic model map. Let $\Gamma_T(0) : \mathcal{J}_0 \rightarrow \text{Ran}(V)^\perp$, $\Gamma_T(\infty) : \mathcal{J}_\infty \rightarrow \text{Ker}(V)$ be any two fixed onto isometries and then define

$$\Gamma_T(z) := (I - Tz^*)^{-1}\Gamma_T(0); \quad z \in \mathbb{B}^d,$$

an analytic model triple. Most simply, we are free to choose $T = V$.

Let $(\Gamma, \mathcal{J}_\infty, \mathcal{J}_0)$ be an analytic model triple for a CCNC row partial isometry $V$ on $\mathcal{H}$. For any $h \in \mathcal{H}$, define

$$\hat{h}\Gamma(z) := \Gamma(z)^*h,$$

an analytic $\mathcal{J}_0$-valued function on $\mathbb{B}^d$. (When it is clear from context, we will sometimes omit the superscript $\Gamma$.) Let $\hat{H}\Gamma$ be the vector space of all $\hat{h} = \hat{h}\Gamma$ for $h \in \mathcal{H}$. We can endow $\hat{H}\Gamma$ with an inner product:

$$\langle \hat{h}, \hat{g} \rangle_\Gamma := \langle h, g \rangle_H.$$

**Proposition 3.2.** The sesquilinear form $\langle \cdot, \cdot \rangle_\Gamma$ is an inner product on $\hat{H}\Gamma$, and $\hat{H}\Gamma$ is a Hilbert space with respect to this inner product.

The Hilbert space $\hat{H}\Gamma$ is the reproducing kernel Hilbert space of $\mathcal{J}_0$-valued analytic functions on $\mathbb{B}^d$ with reproducing kernel

$$\hat{K}_\Gamma(z, w) := \Gamma(z)^*\Gamma(w) \in \mathcal{L}(\mathcal{J}_0), \quad \hat{H}\Gamma = \mathcal{H}(\hat{K}_\Gamma).$$

The map $\hat{U}_\Gamma : \mathcal{H} \rightarrow \hat{H}\Gamma$ defined by $\hat{U}_\Gamma h = \hat{h}\Gamma$ is an onto isometry and

$$\hat{K}_z g = \hat{U}_\Gamma \Gamma(z) g = \hat{\Gamma}(z) g; \quad g \in \mathcal{J}_0.$$

**Proof.** This is all pretty easy to verify. For simplicity we omit the superscript $\Gamma$. In order to show that $\langle \hat{h}, \hat{g} \rangle_\Gamma := \langle h, g \rangle_\mathcal{H}$ is an inner product, the only non-immediate property to check is that there are no non-zero vectors of zero length with respect to this sesquilinear form, or equivalently that $\hat{h}(z) = 0$ for all $z \in \mathbb{B}^d$ implies that $h = 0$. This is clear since $\hat{h}(z) = \Gamma(z)^*h$, and since $V$ is CCNC,

$$\bigvee_{z \in \mathbb{B}^d} \text{Ran}(\Gamma(z)) = \bigvee_{z \in \mathbb{B}^d} \mathcal{R}(V - z)^\perp = \mathcal{H},$$

so that $\bigcap_{z \in \mathbb{B}^d} \text{Ker}(\Gamma(z)^*) = \{0\}$.

The map $\hat{U} : \mathcal{H} \rightarrow \hat{H}$ is an onto isometry by definition of the inner product in $\hat{H}$. For any $g \in \mathcal{J}_0$, compute

$$\langle \hat{U}\Gamma(z) g, \hat{h} \rangle_{\mathcal{J}_\mathcal{H}} = \langle \Gamma(z) g, h \rangle_{\mathcal{J}_\mathcal{H}}$$

$$= \langle g, \Gamma(z)^*h \rangle_{\mathcal{J}_0}$$

$$= \langle g, h(z) \rangle_{\mathcal{J}_0}$$

$$= \langle g, \hat{K}_z h \rangle_{\mathcal{J}_0}.$$

(3.1)
This proves simultaneously that $\hat{\mathcal{H}}$ is a RKHS of analytic $\mathcal{H}_0$-valued functions on $\mathbb{B}^d$ with point evaluation maps
\[ \hat{K}_z = \bar{U}\Gamma(z) \in \mathcal{L}(\mathcal{H}_0, \hat{\mathcal{H}}), \]
and reproducing kernel
\[ \hat{K}(z, w) = \hat{K}_z^*\hat{K}_w = \Gamma(z)^*\Gamma(w) \in \mathcal{L}(\mathcal{H}_0). \]

**Proposition 3.3.** Let $V : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H}$ be a CCNC row partial isometry with analytic model map $\Gamma$. The image, $\hat{V}^\Gamma := \bar{U}^\Gamma V(\bar{U}^\Gamma)^*$ of $V$ under the unitary map $\bar{U}^\Gamma$ onto the model RKHS $\hat{\mathcal{H}}^\Gamma$ has initial and final spaces
\[ \text{Ker}(\hat{V}^\Gamma) \perp \subseteq \{ \hat{h} \in \hat{\mathcal{H}}^\Gamma \otimes \mathbb{C}^d \mid z\hat{h}(z) = \hat{g}(z) \text{ for some } \hat{g} \in \hat{\mathcal{H}}^\Gamma \text{ with } \|\hat{h}\| = \|\hat{g}\| \}, \]
\[ \text{Ran}(\hat{V}^\Gamma) = \{ \hat{h} \in \hat{\mathcal{H}}^\Gamma \mid \hat{h}(0) = 0 \}, \]
and $\hat{V}^\Gamma$ acts as multiplication by $z = (z_1, ..., z_d)$ on its initial space: For any $\hat{h} = (\hat{h}_1, ..., \hat{h}_d)^T \in \text{Ker}(\hat{V}^\Gamma) \perp$,
\[ (\hat{V}^\Gamma \hat{h})(z) = \begin{pmatrix} \hat{h}_1 \\ \vdots \\ \hat{h}_d \end{pmatrix} \Gamma(z) = z_1\hat{h}_1(z) + ... + z_d\hat{h}_d(z) = z\hat{h}(z). \]

**Proof.** Again, we omit the superscript $\Gamma$ to simplify notation. First suppose that $\hat{h} \in \text{Ker}(\hat{V}) \perp$. Then,
\[ \hat{V} \hat{h} = (\bar{U}V\hat{h})(z) \]
\[ = \Gamma(z)^*(V\hat{h}) \]
\[ = \Gamma(z)^*((V - z)\hat{h} + z\hat{h}) \]
\[ = \Gamma(z)^*z\hat{h} \quad \text{(since } (V - z)\hat{h} = (V - z)V^*V\hat{h} \in \mathfrak{R}(V - z) = \text{Ran}(\Gamma(z))^\perp) \]
\[ = z(\Gamma(z)^* \otimes I_d)\hat{h} \]
\[ = z\hat{h}(z). \]
This proves that $\hat{V}$ acts as multiplication by $z$ on its initial space and that
\[ \text{Ker}(\hat{V}) \perp \subseteq \{ \hat{h} \in \hat{\mathcal{H}} \otimes \mathbb{C}^d \mid z\hat{h}(z) \in \hat{\mathcal{H}} \text{ and } \|z\hat{h}\| = \|\hat{h}\| \}, \]
since $\hat{V}$ is a partial isometry. The range statement is clear since $0 = \hat{h}(0) = \Gamma(0)^*\hat{h}$ if and only if $h \in \text{Ran}(V)$. □

Any model triple $(\gamma, \mathcal{H}_\infty, \mathcal{H}_0)$ for a CCNC row partial isometry $V$ can be used to define a characteristic function, $b_V^\gamma$, on $\mathbb{B}^d$ as follows: First consider
\[ D^\gamma(z) := \gamma(z)^*\gamma(0) = \hat{K}^\gamma(z, 0) \in \mathcal{L}(\mathcal{H}_0), \]
\[ N^\gamma(z) := (\gamma(z)^* \otimes I_d)\gamma(\infty) \in \mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_0 \otimes \mathbb{C}^d). \]
Then define 

$$b^\gamma_V(z) := D^\gamma(z)^{-1}zN^\gamma(z) \in \mathcal{L}(\mathcal{J}_0, \mathcal{J}_0).$$

The function $b^\gamma_V$ is called a (representative) characteristic function of the Gleason or CCNC row partial isometry $V$. Observe that if $\gamma = \Gamma$ is an analytic model map then $b^\gamma_V$ is analytic on $\mathbb{B}^d$ and that $D^\gamma(z)$ is invertible since $\gamma(z)$ is, by assumption, an isomorphism of $\mathcal{J}_0$ onto the defect space $\mathcal{R}(X - z)^{\perp}$ for any $z \in \mathbb{B}^d$.

**Definition 3.4.** Let $b_1, b_2$ be two analytic functions on $\mathbb{B}^d$ taking values in $\mathcal{L}(\mathcal{H}_1, \mathcal{J}_1)$ and $\mathcal{L}(\mathcal{H}_2, \mathcal{J}_2)$, respectively. We say that $b_1, b_2$ coincide if there are fixed unitary $R \in \mathcal{L}(\mathcal{J}_1, \mathcal{J}_2), Q \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ so that

$$Rb_1(z) = b_2(z)Q; \quad z \in \mathbb{B}^d.$$ 

This clearly defines an equivalence relation on such functions, and we call the corresponding equivalence classes coincidence classes.

**Lemma 3.5.** Let $V$ be a CCNC row partial isometry. Let $b^\gamma_V$ be a characteristic function for $V$ defined using any model triple $(\gamma, \mathcal{J}_\infty, \mathcal{J}_0)$. The coincidence class of $b^\gamma_V$ is invariant under the choice of model triple.

It follows, in particular, that any characteristic function $b^\gamma_V$ for $V$ is analytic whether or not $\gamma$ is an analytic model map, since $b^\gamma_V$ coincides with $b^\delta_V$ for any analytic model map $\Gamma$ (and analytic model maps always exist).

**Proof.** Let $(\gamma, \mathcal{J}_\infty, \mathcal{J}_0), (\delta, \mathcal{K}_0, \mathcal{K}_\infty)$ be any two choices of model triples for $V$. Since both $\mathcal{K}_0, \mathcal{J}_0$ are isomorphic to $\text{Ran}(V)^{\perp}$ and $\mathcal{K}_\infty, \mathcal{J}_\infty$ are isomorphic to $\text{Ker}(V)$, we can define onto isometries $R = \gamma(0)^*\delta(0) \in \mathcal{L}(\mathcal{K}_0, \mathcal{J}_0)$ and $Q := \gamma(\infty)^*\delta(\infty) \in \mathcal{L}(\mathcal{K}_\infty, \mathcal{J}_\infty)$. Moreover $C_z := \gamma(z)^*(\delta(z)\delta(z)^*)^{-1}\delta(z)$ is a linear isomorphism (bounded and invertible) of $\mathcal{K}_0$ onto $\mathcal{J}_0$ for any $z \in \mathbb{B}^d$. As before,

$$D^\gamma(z) := \gamma(z)^*\gamma(0); \quad N^\gamma(z) = (\gamma(z)^* \otimes I_d)\gamma(\infty),$$

$$b^\gamma_V(z) := D^\gamma(z)^{-1}zN^\gamma(z),$$

and $b^\delta_V$ is defined analogously. It follows that:

$$Rb^\gamma_V(z)Q = RD^\delta(z)^{-1}C_z^{-1}C_zzN^\delta(z)Q$$

$$= (C_zD^\delta(z)R^*)^{-1}z ((C_z \otimes I_d)N^\delta(z)Q).$$

In particular,

$$C_zD^\delta(z)R^* = \gamma(z)^*(\delta(z)\delta(z)^*)^{-1}\delta(z)\delta(z)^*\delta(0)\delta(0)^*\gamma(0)$$

$$= \gamma(z)^*\gamma(0)$$

$$= D^\gamma(z),$$

and similarly

$$(C_z \otimes I_d)N^\delta(z)Q = N^\gamma(z),$$
so that \( Rb_V^\gamma(z)Q = b^\gamma(z) \), and \( b_V^\gamma, b_V^\gamma \) belong to the same equivalence (coincidence) class.

\[ \text{Theorem 3.6.} \text{ Let } \hat{\mathcal{H}}^F \text{ be the abstract model RKHS on } \mathbb{B}^d \text{ defined using an analytic model triple } (\Gamma, \mathcal{J}_0, \mathcal{J}_\infty). \text{ The reproducing kernel for } \hat{\mathcal{H}}^F \text{ can be written:} \]

\[
\hat{K}^F(z, w) = \frac{D^F(z)D^F(w)^* - zN^F(z)N^F(w)^*w^*}{1 - zw^*} = D^F(z) \left( I - b_V^\gamma(z)b_V^\gamma(w)^* \right) \frac{D^F(w)^*}{1 - zw^*} \in \mathcal{L}(\mathcal{J}_0).
\]

It follows that \( b_V^\gamma \in \mathcal{S}_d(\mathcal{J}_\infty, \mathcal{J}_0) \) is Schur class, and that multiplication by \( D^F(z) \) is an isometry, \( M^F := M_{D^F} \), of \( \mathcal{H}(b_V^\gamma) \) onto \( \hat{\mathcal{H}}^F \).

\[ \text{Proof.} \text{ As before, we omit the superscript } \Gamma \text{ for the proof. Let } P_0 := P_{\text{Ran}(\mathcal{V})}^\perp = \hat{\Gamma}(0)\Gamma(0)^*\hat{U}^* \text{ and } P_\infty := P_{\text{Ker}(\mathcal{V})} = \hat{\Gamma}(\infty)\Gamma(\infty)^*\hat{U}^*. \text{ Then consider} \]

\[
\hat{K}_z^*z\hat{V}^*\hat{K}_w = \left( \hat{K}_w^*\hat{V}^*(I - P_\infty)z^*\hat{K}_z^* \right)^* = \left( \hat{K}_w^*w(I - P_\infty)z^*\hat{K}_z^* \right)^* = \hat{K}_z^*z(I - P_\infty)w^*\hat{K}_w = zw^*\hat{K}(z, w) - \hat{K}_z^*zP_\infty w^*\hat{K}_w.
\]

In the above we used that \( \hat{V} \) acts as multiplication by \( z \) on its initial space and \( I - P_\infty = P_{\text{Ker}(\mathcal{V})}^\perp \) is the projection onto this initial space.

The same expression can be evaluated differently:

\[
\hat{K}_z^*z\hat{V}^*\hat{K}_w = \hat{K}_z^*z\hat{V}^*(I - P_0)\hat{K}_w = \hat{K}_z^*\hat{V}^*(I - P_0)\hat{K}_w = \hat{K}_z^*(I - P_0)\hat{K}_w \text{ and } (I - P_0) = P_{\text{Ran}(\mathcal{V})} = \hat{V}^*.
\]

Again, in the above we used that \( \hat{V} \) acts as multiplication by \( z \) on its initial space, the range of \( \hat{V}^* \).

Equating these two expressions and solving for \( \hat{K}(z, w) \) yields:

\[
\hat{K}(z, w) = \frac{\hat{K}_z^*P_0\hat{K}_w - \hat{K}_z^*zP_\infty w^*\hat{K}_w}{1 - zw^*}. \]

Use that \( \hat{K}_z = \hat{\Gamma}(z), P_0 = \hat{\Gamma}(0)\Gamma(0)^*\hat{U}^*, \text{ and } P_\infty = \hat{\Gamma}(\infty)\Gamma(\infty)^*\hat{U}^*, \text{ and that} \]

\[
\hat{K}_z^*zP_\infty w^*\hat{K}_w = \Gamma(z)^*z\Gamma(\infty)^*w^*\Gamma(w) \]

to obtain

\[
\hat{K}(z, w) = \frac{D(z)D(w)^* - zN(z)N(w)^*w^*}{1 - zw^*}; \quad z, w \in \mathbb{B}^d.
\]

In particular it follows that \( b_V = D(z)^{-1}zN(z) \in \mathcal{S}(\mathcal{J}_\infty, \mathcal{J}_0) \) as claimed (by [23 Theorem 2.1]).

Given an analytic model map \( \Gamma \), let \( U^F : \mathcal{H} \to \mathcal{H}(b_V^\gamma) \) denote the canonical onto isometry \( U^F := (M^F)^{-1}\hat{U}^F \), where, recall, \((M^F)^{-1} = M_{D^F}^{-1} = M_{(D^F)^{-1}} = (M^F)^* \), since \( M^F = M_{D^F} \) is a unitary multiplier of \( \mathcal{H}(b_V^\gamma) \) onto \( \hat{\mathcal{H}}^F \).
Theorem 3.7. Let $V : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H}$ be a CCNC row partial isometry with analytic model triple $(\Gamma, \mathcal{J}_\infty, \mathcal{J}_0)$. The image, $X^\Gamma := U^T V(U^T)^* \otimes I_d$, of $V$ under the corresponding canonical unitary is an extremal contractive Gleason solution for $\mathcal{H}(b_V^r)$.

If $\Gamma(z) = \Gamma_T(z) := (I - Tz^*)^{-1} \Gamma(0)$ is an analytic model map corresponding to a contractive extension $T \supseteq V$ on $\mathcal{H}$, then $X^T := X(b_T^r)$ (as in Equation 1.6) is the unique contractive Gleason solution corresponding to the extremal contractive Gleason solution

$$b_T^r := U^T \Gamma_T(\infty),$$

for $b_{V_r}^r \in \mathcal{S}_d(\mathcal{J}_\infty, \mathcal{J}_0)$.

Proof. (As before we will omit superscripts.) Since multiplication by $D(z)^{-1}$ is an isometric multiplier of $\hat{\mathcal{H}}$ onto $\mathcal{H}(b_V)$, Proposition 3.3 implies that $X := U_2^\dagger VU_D$ acts as multiplication by $z$ on its initial space, and

$$\text{Ran}(X) = \{f \in \mathcal{H}(b_V) \mid f(0) = 0\}.$$

Hence, $XX^* = I - k^b_0(k^b_0)^*$ and

$$z(X^*f)(z) = (XX^*f)(z) = f(z) - k^b_0(z, 0)f(0) = f(z) - f(0),$$

since $b_{V_r}^r(0) = 0$. This proves that $X$ is a (contractive) extremal Gleason solution for $\mathcal{H}(b_V^r)$.

To see that $b := M_{D^{-1}} \hat{\Gamma}(\infty)$ is a contractive extremal Gleason solution for $b_{V_r}^r$, let $b := b_{V_r}^r$ and calculate:

$$zb(z) = (z^*k^b_0)^*b = \left(z^*\hat{U}^*(M_{D^{-1}})k^b_0\right)^* \Gamma(\infty) = \left(z^*\hat{U}^*k^b_0(0)^\ast\right)^* \Gamma(\infty) = \left(z^*\Gamma(z)(D(z)^{-1})^\ast\right)^* \Gamma(\infty) = D(z)^{-1}z(\Gamma(z)^* \otimes I_d)\Gamma(\infty) = D(z)^{-1}zN(z) = b(z).$$

This proves that $b^r = b$ is a Gleason solution (since $b_{V_r}^r(0) = 0$). Furthermore,

$$b^*b = \Gamma(\infty)^\ast \Gamma(\infty) = I_{\mathcal{J}_\infty},$$

so that $b^r = b$ is contractive and extremal.

To see that $X = X^r(\Gamma_T) = X(b^r_T)$, calculate the action of $X^* - w^*$ on point evaluation maps,

$$(X^* - w^*)k^b_w = \left(M_{D^{-1}}^\dagger \hat{U} \otimes I_d\right)(V^* - w^*)\hat{U}^*M_D k^b_w = (M_{D^{-1}}^\dagger \hat{U} \otimes I_d)(V^* - w^*)\Gamma(w)(D(w)^{-1})^*,$$
and compare this to

\[
\begin{align*}
b^* b(w)^* &= (M_D^{-1} \hat{U} \otimes I_d) \Gamma(\infty) N(w)^* w^* (D(w)^{-1})^* \\
&= (M_D^{-1} \hat{U} \otimes I_d) \Gamma(\infty) (\Gamma(w) \otimes I_d) w^* (D(w)^{-1})^* \\
&= (M_D^{-1} \hat{U} \otimes I_d) P_{Ker(V)} w^* \Gamma(w) (D(w)^{-1})^* \\
&= (M_D^{-1} \hat{U} \otimes I_d) (I - V^* V) w^* \Gamma(w) (D(w)^{-1})^*.
\end{align*}
\]

Under the assumption that \( \Gamma = \Gamma_T \) for a contractive extension \( T \supset V \) on \( \mathcal{K} \), recall that by Lemma \ref{lem:gamma_on_model_triple}, \( T = V - C \), where \( C \) is a pure row contraction with \( \text{Ker}(C)^\perp \subseteq \text{Ker}(V) \) and \( \text{Ran}(C) \subseteq \text{Ran}(V)^\perp \).

Applying that \( \Gamma_T(z) = (I - Tz^*)^{-1} \Gamma_T(0) \),

\[
\begin{align*}
b b(w)^* &= (M_D^{-1} \hat{U} \otimes I_d) (I - V^* (T + C)) w^* \Gamma_T(w) (D(w)^{-1})^* \\
&= (M_D^{-1} \hat{U} \otimes I_d) w^* \Gamma_T(w) (D(w)^{-1})^* \\
&\quad - (M_D^{-1} \hat{U} \otimes I_d) V^* (C w^* \Gamma_T(w) + \Gamma_T(w) - \Gamma_T(0)) (D(w)^{-1})^* \\
&= (M_D^{-1} \hat{U} \otimes I_d) (w^* - V^*) \Gamma_T(w) (D(w)^{-1})^*,
\end{align*}
\]

and it follows that \( X^* k_w^b = w^* b_w^b - b^F_T b^F_T(w)^* \), proving the claim. \( \square \)

**Remark 3.8.** Lemma \ref{lem:characteristic_function_invariant} shows that the coincidence class of any characteristic function \( b^F_T \) of a CCNC row partial isometry \( V \) is invariant under the choice of model triple \( (\gamma, \mathcal{J}_\infty, \mathcal{J}_0) \) and Theorem 3.6 shows that \( b^\gamma_T \in \mathcal{S}(\mathcal{J}_\infty, \mathcal{J}_0) \) belongs to the Schur class. It will also be useful to define weak coincidence of Schur class functions as in \cite{15} Definition 2.4:

**Definition 3.9.** The **support** of \( b \in \mathcal{S}(\mathcal{H}, \mathcal{J}) \) is

\[
\text{supp}(b) := \bigvee_{z \in \mathbb{B}^d} \text{Ran}(b(z)^*) = \mathcal{K} \ominus \bigcap_{z \in \mathbb{B}^d} \text{Ker}(b(z)).
\]

Schur class multipliers \( b_1 \in \mathcal{S}(\mathcal{H}, \mathcal{J}) \) and \( b_2 \in \mathcal{S}(\mathcal{H}', \mathcal{J}') \) coincide weakly if \( b'_1 := b_1|_{\text{supp}(b_1)} \) coincides with \( b'_2 := b_2|_{\text{supp}(b_2)} \).

By \cite{15} Lemma 2.5], \( b_1 \in \mathcal{S}(\mathcal{H}, \mathcal{J}) \) and \( b_2 \in \mathcal{S}(\mathcal{H}', \mathcal{J}') \) coincide weakly if and only if there is an onto isometry \( \mathcal{V}: \mathcal{J} \to \mathcal{J}' \) so that

\[
\mathcal{V} b_1(z)b_1(w)^* \mathcal{V} = b_2(z)b_2(w)^*; \quad z, w \in \mathbb{B}^d,
\]

i.e., if and only if \( \mathcal{H}(b_2) = \mathcal{H}(\mathcal{V} b_1) \).

Weak coincidence also defines an equivalence relation on Schur class functions, and we define:

**Definition 3.10.** The **characteristic function**, \( b^\gamma_T \), of a CCNC row partial isometry \( V \) is the weak coincidence class of any Schur class characteristic function \( b^\gamma_T \in \mathcal{S}(\mathcal{J}_\infty, \mathcal{J}_0) \) constructed using any model triple \( (\gamma, \mathcal{J}_\infty, \mathcal{J}_0) \) for \( V \). We will often abuse terminology and simply say that any \( b^\gamma_T \) is the characteristic function of \( V \).

Note that the characteristic function, \( b^\gamma_T \), of any CCNC row partial isometry always vanishes at 0, \( b_V(0) = 0 \), and (as discussed in Subsection \ref{subsection:characteristic_function}) this implies that \( b_V \) is strictly contractive on \( \mathbb{B}^d \). In the
single variable case when \( d = 1 \) and \( X \) has equal defect indices, the above definition of characteristic function reduces to that of the Livšic characteristic function of the partial isometry \( X \) [36, 37, 38].

**Remark 3.11.** There is an interesting alternative proof of Theorem 3.6 using the colligation or transfer function theory of [23, 39, 29]: Given Hilbert spaces \( \mathcal{H}, \mathcal{J}, \mathcal{K} \), any contractive linear map

\[
R := \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{H} \oplus \mathcal{J} \rightarrow \mathcal{H} \otimes \mathcal{C}^d \oplus \mathcal{K},
\]

is called a colligation. By [39, Theorem 1.3], an \( L(\mathcal{J}, \mathcal{K}) \)-valued function, \( b \), on \( B^d \) belongs to the Schur class \( \mathcal{S}_d(\mathcal{J}, \mathcal{K}) \) if and only if there is a Hilbert space \( \mathcal{H} \), and a colligation, \( R : \mathcal{H} \oplus \mathcal{J} \rightarrow \mathcal{H} \otimes \mathcal{C}^d \oplus \mathcal{K} \) as above so that \( b \) is the transfer function of \( R \):

\[
b(z) = D + C(I - zA)^{-1}zB.
\]

As shown in [40, Chapter 3],

**Theorem 3.12.** Let \( V \) be a CCNC row partial isometry on \( \mathcal{H} \), and let \( (\gamma, \mathcal{J}_\infty, \mathcal{J}_0) \) be any model triple for \( V \). Then

\[
\Xi \gamma := \begin{bmatrix} V^* & \gamma(\infty) \\ \gamma(0)^* & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{J}_\infty \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H} \otimes \mathcal{C}^d \\ \mathcal{J}_0 \end{bmatrix},
\]

is a unitary (onto isometry) colligation with transfer function equal to \( b_\gamma^V \).

**Example 3.13.** Given any Schur class \( b \in \mathcal{S}_d(\mathcal{J}, \mathcal{K}) \) such that \( b(0) = 0 \), let \( X \) be an extremal contractive Gleason solution for \( \mathcal{H}(b) \). By Remark 1.6 and Example 2.6, any such \( X \) is a CCNC row partial isometry.

To calculate the characteristic function of \( X \), use the analytic model map \( \Gamma = \Gamma_X \) with \( \Gamma(0) := k_b^0 \), an isometry of \( \mathcal{K} \) onto \( \text{Ran}(X)^\perp \) since \( b(0) = 0 \). Then,

\[
\Gamma(z) = (I - Xz^*)^{-1}k_b^0 = k_b^z,
\]

and

\[
D(z) = \Gamma(z)^*\Gamma(0) = (k_b^z)^*k_b^0 = k_b(z, 0) = I_{\mathcal{K}}.
\]

It then follows that \( zN(z) = b_X(z) \), and by equation (3.2) and Theorem 3.6 the reproducing kernel for the model space \( \mathcal{H}(b) \) is

\[
\hat{K}(z, w) = \Gamma(z)^*\Gamma(w) = k_b(z, w) = k_b^X(z, w); \quad z, w \in B^d.
\]

This proves that \( \mathcal{H}(b_\Gamma^\gamma_X) = \mathcal{H}(b) \) and [15, Lemma 2.5] implies that \( b_\Gamma^\gamma_X \) coincides weakly with \( b \).

**Corollary 3.14.** If \( X \) is an extremal contractive Gleason solution for any Schur class \( b \) such that \( b(0) = 0 \) then \( X \) acts as multiplication by \( z = (z_1, ..., z_d) \) on its initial space.

**Proof.** If we take \( \Gamma = \Gamma_X \) as our model map for \( X \), the above example shows that \( \mathcal{H}(b_\Gamma) = \mathcal{H}(b) \) and \( X = \hat{X} \), so that \( X \) acts as multiplication by \( z \) on its initial space by Proposition 3.3. \( \square \)

**Remark 3.15.** Theorem 3.6 and Example 3.13 show that the (weak coincidence class of any) characteristic function is a unitary invariant: If \( X, Y \) are two CCNC row partial isometries which are unitarily
equivalent then one can choose model triples for \( X, Y \) so that they have the same characteristic function. Conversely if the characteristic functions of \( X, Y \) coincide weakly, \( \mathcal{M}(b_X) = \mathcal{M}(Ub_Y) \) for a fixed unitary \( U \), then both \( X, Y \) are unitarily equivalent to extremal contractive Gleason solutions for \( \mathcal{M}(b_X) \) (it is easy to see that multiplication by \( U \) is a constant unitary multiplier of \( \mathcal{M}(b_X) \) onto \( \mathcal{M}(Ub_X) \) taking extremal Gleason solutions onto extremal Gleason solutions).

Unless \( d = 1 \), the characteristic function of a CCNC row partial isometry is not a complete unitary invariant. Indeed, if \( b \in \mathcal{S}_d(J, X) \), \( b(0) = 0 \), has two extremal contractive Gleason solutions \( X, X' \) which are not unitarily equivalent, the above results show that \( X, X' \) are two non-equivalent row partial isometries with the same characteristic function. The example below shows that such Schur class functions exist.

**Example 3.16.** Let \( b \in \mathcal{S}_d(\mathcal{H}) \) be any square Schur class function with \( b(0) = 0 \) and \( \text{supp}(b) = \mathcal{H} \) (also recall that since \( b(0) \) is a strict contraction, so is \( b(z) \) for any \( z \in \mathbb{B}^d \).

As described in Subsection 1.2, one can define a row partial isometry \( V^b \) on the Herglotz space \( \mathcal{H}^+(H_b) \) by \( V^b z^* K^b_z = K^b_z - K^b_0 \), and there is a bijection from row-contractive extensions \( D \) of \( V^b \) on \( \mathcal{H}^+(H_b) \) onto contractive Gleason solutions \( X[D] \) for \( \mathcal{M}(b) \) given by \( X[D]^* k^b_w := w^* k^b_w - b[D]^* b(w)^* \) where \( b[D] := U_b^* D^* K^b_0 \), and \( U_b : \mathcal{M}(b) \to \mathcal{H}^+(H_b) \) is the canonical unitary multiplier of Lemma 1.3.

As in the proof of [9, Corollary 4.7], it is not difficult to see that \( b[D] \) (and hence \( X[D] \)) is extremal if and only if \( D \) is a co-isometric extension of \( V^b \).

**Claim 3.17.** Suppose that \( b \in \mathcal{S}_d(\mathcal{H}) \) obeys \( b(0) = 0 \), and that \( \mathcal{M}(b) \) has two extremal contractive Gleason solutions \( X, Y \) which are unitarily equivalent, \( WX = YW \), for some unitary \( W \in \mathcal{L}(\mathcal{M}(b)) \). Then \( W \) is a constant unitary multiplier: There is a constant unitary \( R \in \mathcal{L}(\mathcal{H}) \) so that \( W k^b_z = k^b_z R \), \((WF)(z) = R^* F(z) \) for all \( F \in \mathcal{M}(b) \) and \( z \in \mathbb{B}^d \).

**Proof.** Suppose that \( X, Y \) are unitarily equivalent, \( WX = YX \) for some unitary \( W : \mathcal{M}(b) \to \mathcal{M}(b) \). Since both \( X, Y \) are extremal and \( b(0) = 0 \), they are both row partial isometries with the same range projection \( X X^* = I - k^b_0 (k^b_0)^* = YY^* \). It follows that the range of \( k^b_0 \) is a reducing subspace for \( W \) so that there is a unitary \( R \in \mathcal{L}(\mathcal{H}) \) such that

\[
W k^b_0 = k^b_0 R.
\]

Moreover, by the Property 1.3, given any \( z \in \mathbb{B}^d \),

\[
W k^b_z = W (I - X z^*)^{-1} k^b_0 = (I - Y z^*)^{-1} W k^b_0 = (I - Y z^*)^{-1} k^b_0 R = k^b_z R,
\]

so that \( W \) is multiplication by the fixed unitary operator \( R^* \in \mathcal{L}(\mathcal{H}) \).

**Corollary 3.18.** If \( b \in \mathcal{S}_d(\mathbb{C}) \) satisfies \( b(0) = 0 \) and \( \dim(\text{Ker}(V^b)) > \dim\left(\text{Ran}(V^b)^{-1}\right) \neq 0 \), then \( \mathcal{M}(b) \) has two extremal contractive Gleason solutions which are not unitarily equivalent.
Proof. In this case where $\mathcal{H} = \mathbb{C}$ the dimension of $\text{Ran}(V^b)^\perp$ is either 0 or 1. By assumption $\dim(\text{Ker}(V^b)) > 1 = \dim(\text{Ran}(V^b))$ so that we can define two co-isometric extensions $D, d$ of $V^b$ on $\mathcal{H}^+(H_b)$ so that

$$D^*K^b_0 \perp d^*K^b_0.$$ 

As discussed at the beginning of this example, these co-isometric extensions can be used to construct extremal contractive Gleason solutions $X[D], X[d]$ for $\mathcal{H}(b)$ (see Section [1.2] and Equation [1.6]).

Suppose that $X[D], X[d]$ are unitarily equivalent, $WX[d] = X[D]W$ for a unitary $W$ on $\mathcal{H}(b)$. By the previous claim, $W$ is a constant unitary multiplier by a unitary operator $R \in \mathcal{L}(\mathbb{C})$. That is, $W$ is simply multiplication by a unimodular constant $R = \alpha \in \mathbb{T}$, $W = \alpha I_{\mathcal{H}(b)}$. In particular, $X[D] = WX[d]W^* = X[d]$ so that $b[D]b(w)^* = b[d]b(w)^*$ and $b[D] = b[d]$. Equivalently,

$$D^*K^b_0 = d^*K^b_0,$$

a contradiction. \qed

Claim 3.19. If $b \in \mathcal{S}_d(\mathbb{C})$, $d \geq 2$, obeys $b(0) = 0$, $V^b$ is not a co-isometry and $\dim(\mathcal{H}(b)) \geq d + 2$, then $\dim(\text{Ker}(V^b)) > \dim(\text{Ran}(V^b)^\perp) = 1$.

To construct an example of a Schur class $b$ satisfying the conditions of this claim, let $b \in \mathcal{S}_d(\mathbb{C})$, $d \geq 2$ be any Schur class function obeying $b(0) = 0$ and such that $V^b$ is not a co-isometry. Such a Schur class function is called non quasi-extreme (see Section [5.1]), and it is not difficult to apply Theorem [5.3] below to show, for example, that if $b \in \mathcal{S}_d(\mathbb{C})$ is any Schur class function and $0 < r < 1$, $rb \in \mathcal{S}_d(\mathbb{C})$ is non quasi-extreme. Moreover, $\mathcal{H}(rb)$ is infinite dimensional (this is just $H^2_2$ with a new norm since $\mathcal{H}(rb) = \text{Ran}(\sqrt{D_{rM^r_2}})$ equipped with the norm that makes $D_{rM^r_2}$ a co-isometry onto its range as in [3]) so that $rb$ satisfies the above claim statement.

Proof. This is easily established using the argument in the proof of [3] Proposition 4.4]: Let $X = X(b)$ be a contractive Gleason solution for $\mathcal{H}(b)$, where $b$ is a contractive Gleason solution for $b$ (see Equation [1.5]). Then $b = (b_1, ..., b_d)^T \in \mathcal{H}(b) \otimes \mathbb{C}^d$, $d \geq 2$. Choose any two linearly independent and non-zero $F, f \in \mathcal{H}(b)$ orthogonal to the linear span of the $b_j$, $1 \leq j \leq d$ (we can do this since we assume that $\dim(\mathcal{H}(b)) \geq d + 2$). It follows that

$$\langle k^b_j, (X_jF) \rangle = (\tau_j k^b_j - b_j b(z)^*, f) = z_j F(z)_{\mathcal{H}},$$

and similarly for $f$. This proves that for any $1 \leq j \leq d$, $S_j F, S_j f \in \mathcal{H}(b)$. This in turn implies that $H_j := (I-b)^{-1}S_j F$, $h_j := (I-b)^{-1}S_j f \in \mathcal{H}^+(H_b)$ and if we define $H := (-H_2, H_1, 0, ..., 0)$, $h := (-h_2, h_1, 0, ..., 0) \in \mathcal{H}^+(H_b) \otimes \mathbb{C}^d$, we then have that

$$\langle z^* K^b_2, H \rangle = -z_1 H_2(z) + z_2 H_1(z) = (-z_1 z_2 + z_2 z_1)(I - b(z))^{-1} F(z) = 0,$$
and similarly for $h$. Since $H, h$ are linearly independent, it follows that $\text{Ker}(V^b)$ has dimension greater or equal to 2.

In summary, the above claim, along with Corollary 3.18 and Example 3.13 imply that there exist CCNC row partial isometries which are not unitarily equivalent and yet have the same characteristic function. This shows that the characteristic function of a CCNC row partial is not a complete unitary invariant. This is in contrast to the result of [15, Theorem 3.6], which shows that the Sz.-Nagy-Foiaş characteristic function of any CNC $d$-contraction (which is a CCNC row contraction by Corollary 2.4) is a complete unitary invariant for CNC $d$-contractions. We will also later prove in Proposition 4.13 that our characteristic function, $b_T$, of any CCNC row contraction coincides weakly with the Sz.-Nagy-Foiaş characteristic function of $T$.

We conclude this section with an example which will motivate an approach to extending our commutative deBranges-Rovnyak model for CCNC row partial isometries to arbitrary CCNC row contractions.

**Example 3.20.** (Frostman Shifts) As in Example 3.16 suppose that $X$ is any extremal Gleason solution for $\mathcal{H}(b)$, where $b \in \mathcal{A}(\beta, \mathcal{K})$ is strictly contractive on the ball (recall this equivalent to assuming $b(0)$ is a strict contraction), but we do not assume that $b(0) = 0$. In this case, $X$ is not a row partial isometry, but it is still CCNC by Example 2.6 so that we can consider the isometric-purely contractive decomposition $X = V - C$ of $X$, and we calculate the characteristic function, $b_V$, of the CCNC partial isometric part, $V$, of $X$.

Since $X$ is extremal,

$$D^2_{X^*} = I - XX^* = I - k^b_0(k^b_0)^*,$$

is not a projection, but it is clear that $\text{Ran}(D_{X^*})$ is equal to the range of the projection:

$$P_{\text{Ran}(D_{X^*})} = I - k^b_0 k^b(0, 0)^{-1}(k^b_0)^*,$$

so that

$$V^* = X^* P_{\text{Ran}(D_{X^*})} = X^* (I - k^b_0 k^b(0, 0)^{-1}(k^b_0)^*).$$

Observe that $k^b(0, 0) = D^2_{b(0)^*}$, so that

$$\Gamma(0) := k^b_0 D^{-1}_{b(0)^*} : \mathcal{H} \to \text{Ran} \left( V \right)^\perp,$$

is an onto isometry. Similarly, since $X$ is an extremal Gleason solution for $\mathcal{H}(b)$, as described in Section 1.4, there is an extremal Gleason solution $b$ for $b$ so that $X^* k^b_w = w^* k^b_w - b(b)^*$. Since $b$ is extremal, $b^* b = I - b(0)^* b(0) = D^2_{b(0)}$, so that $b D^{-1}_{b(0)}$ is also an isometry. Recall here that we are assuming that $b(0)$ is a strict contraction.

**Claim 3.21.** The isometry $\Gamma_0(\infty) := b D^{-1}_{b(0)}$ maps $\mathcal{K}$ into $\text{Ker}(V)$. 
Proof. Calculate on kernel maps:

\[ D_{b(0)} \Gamma_0(\infty)^* V^* k_w^b = b^* X^* (I - k_0^b k_b^b(0,0)^{-1}(k_0^b)^*) k_w^b \]
\[ = b^* (X^* k_w^b - X^* k_0^b k_b^b(0,0)^{-1} k_b^b(0,w)) \]
\[ = b^* (w^* k_w^b - bb(w)^* - bb(0)^* k_b^b(0,0)^{-1} k_b^b(0,w)) \]
\[ = b(w)^* - b(0)^* - (I - b(0)^* b(0)) (b(w)^* - b(0)^* (I - b(0) b(0)^{-1} (I - b(0)b(0)^*)) \]
\[ = b(w)^* - b(0)^* - (I - b(0)^* b(0)) b(w)^* + b(0)^* (I - b(0)b(0)^*) \]
\[ = 0. \]

It follows that we can choose an isometry \( \Gamma_1(\infty) : \mathcal{K}' \to \text{Ker}(V) \oplus \text{Ran}(\Gamma_0(\infty)) \) so that

\[ \Gamma(\infty) := \Gamma_0(\infty) \oplus \Gamma_1(\infty) : \mathcal{K} \oplus \mathcal{K}' \to \text{Ker}(V), \]

is an onto isometry. Since \( X \supset V \) is a contractive extension of the CCNC row partial isometry \( V \), we can then set

\[ \Gamma(z) := (I - X z^*)^{-1} \Gamma(0), \]

and this defines an analytic model triple, \((\Gamma, \mathcal{K} \oplus \mathcal{K}', \mathcal{K})\) for \( V \).

We can now calculate the characteristic function of \( V \) using this model triple:

\[ D^\Gamma(z) = \Gamma(z)^* \Gamma(0) \]
\[ = \Gamma(0)^* (I - z X^*)^{-1} \Gamma(0) \]
\[ = D_{b(0)}^{-1}(k_0^b)^*(I - z X^*)^{-1} k_b^b D_{b(0)}^{-1}, \]
\[ = D_{b(0)}^{-1}(k_z^b)^* k_0^b D_{b(0)}^{-1}, \]
\[ = D_{b(0)}^{-1}(I_{3b} - b(z)b(0)^*) D_{b(0)}^{-1}. \]

Note that even if \( b \) is only purely contractive on the ball that \( D^\Gamma(z) = \Gamma(z)^* \Gamma(0) \) is always a bounded, invertible operator for \( z \in B^d \). Similarly,

\[ z N^\Gamma(z) = z (\Gamma(z)^* \otimes I_d) \Gamma(\infty) \]
\[ = D_{b(0)}^{-1} z (k_z^b \otimes I_d)^* \left( b D_{b(0)}^{-1} \oplus \Gamma_1(\infty) \right) \]
\[ = D_{b(0)}^{-1} (b(z) - b(0)) D_{b(0)}^{-1} + D_{b(0)}^{-1}(k_z^b)^* z_1(\infty). \]

Claim 3.22. The range of \( \Gamma_1(\infty) \) is in the kernel of \((k_z^b)^* z\) for any \( z \in B^d \).

Proof. If \( h \in \text{Ran}(\Gamma_1(\infty)) \) then \( h \in \text{Ker}(V) \), where \( V = (I - k_0^b k_b^b(0,0)^{-1}(k_0^b)^*) X \) and \( h \in \text{Ker}(b^*) = \text{Ran}(b)^\perp \). Then calculate:

\[ z (k_z^b)^* h = (X^* k_z^b + bb(z)^*)^* h \]
\[ = (X^* k_z^b k_b^b(0,0)^{-1}(k_0^b)^*) k_z^b + V^* k_z^b)^* h \]
\[ = (bb(z)^* k_b^b(0,0)^{-1} k_b^b(0,z))^* h = 0. \]
This claim proves that $zN^T(z) = D^{-1}_{b(0)}(b(z) - b(0))D^{-1}_{b(0)} + 0_{\mathcal{H}}$, and we conclude that

$$b^T_V(z) = D^T(z)^{-1}zN^T(z)$$

$$= D_{b(0)}(I_{b_0} - b(z)b(0)^*)^{-1}D_{b(0)} D_{b(0)}^{-1}(b(z) - b(0))D_{b(0)}^{-1} + 0_{\mathcal{H}}.$$  

(3.3)

That is, $b^T_V|_{\mathcal{H}} = \Phi_{b(0)} \circ b$ (this coincides weakly with $b^T_V$), where for any strict contractions $\alpha, \beta \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, we define

$$\Phi_\alpha(\beta) := D_\alpha \cdot (I_{\mathcal{K}} - \beta \alpha^*)^{-1} (\beta - \alpha) D^{-1}_{\alpha} \in \mathcal{L}(\mathcal{H}, \mathcal{K}).$$  

As we will show in the following section, for any strict contraction $\alpha \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, the map $\Phi_\alpha : [\mathcal{L}(\mathcal{H}, \mathcal{K})]_1 \to [\mathcal{L}(\mathcal{H}, \mathcal{K})]_1$ is an automorphism (i.e. a bijection) of the unit ball of $\mathcal{L}(\mathcal{H}, \mathcal{K})$, and $\Phi_\alpha : \mathcal{K}(\mathcal{H}, \mathcal{K}) \to \mathcal{K}(\mathcal{H}, \mathcal{K})$ maps the Schur class onto itself. Given any strict contraction $\alpha \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $b \in \mathcal{K}(\mathcal{H}, \mathcal{K})$, $\Phi_\alpha \circ b$ is an operator-generalization of what is called a Frostman shift of the Schur class function $b$ in the classical case where $b$ is a contractive analytic function on the disk [11, 12, 25].

The above example shows that if $X$ is an extremal contractive Gleason solution for $\mathcal{H}(b)$ with partial isometric part $V$, then $b_V = \Phi_{b(0)} \circ b$ is the Frostman shift of $b$ vanishing at the origin.

4. Commutative deBranges-Rovnyak model for CCNC row contractions

The goal of this section is to show that if $T = V - C : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H}$ is any CCNC row contraction with purely isometric part $V$ and purely contractive part $C$, that $T$ is unitarily equivalent to an extremal contractive Gleason solution $X$ for $\mathcal{H}(b)$, where the (purely contractive) Schur class function $b = b_T$ is a certain Frostman shift of the characteristic function $b_V$ of $V$. As in the case where $T = V$ is a row partial isometry, the characteristic function $b_T$, of $T$, will be a unitary invariant for CCNC row contractions.

4.1. Automorphisms of the unit ball of $\mathcal{L}(\mathcal{H}, \mathcal{K})$. Recall that $b \in \mathcal{K}(\mathcal{H}, \mathcal{K})$ is said to be strictly contractive if $b(z)$ is a strict contract for all $z \in \mathbb{D}^d$ and purely contractive, if $b(z)$ is a pure contraction for all $z \in \mathbb{D}^d$, i.e., if $\|b(z)h\| < \|h\|$ for any $h \in \mathcal{H}$ and $z \in \mathbb{D}^d$. Further recall that $b$ is strictly contractive if and only if $b(0)$ is a strict contraction so that, in particular, $b_V$, is strictly contractive for any CCNC row partial isometry $b_V$ (since $b_V(0) = 0$).

Consider the closed unit ball $[\mathcal{L}(\mathcal{H}, \mathcal{K})]_1$, where $\mathcal{H}, \mathcal{K}$ are Hilbert spaces. Assume that $\alpha \in [\mathcal{L}(\mathcal{H}, \mathcal{K})]_1$ is a pure contraction so that the defect operators $D_\alpha, D_{\alpha^*}$ have closed, densely defined inverses. The $\alpha$-Frostman transformation, $\Phi_\alpha$, and the inverse $\alpha$—Frostman transformation, $\Phi_\alpha^{-1}$ are the maps defined on the open unit ball $(\mathcal{L}(\mathcal{H}, \mathcal{K})_1$ by

$$\Phi_\alpha(\beta) := D_\alpha \cdot (I_{\mathcal{K}} - \beta \alpha^*)^{-1} (\beta - \alpha) D^{-1}_{\alpha},$$

(4.1)

and

$$\Phi_\alpha^{-1}(\beta) := \Phi_{-\alpha^*}(\beta^*) = D^{-1}_{\alpha^*} (\beta + \alpha) (I_{\mathcal{K}} + \alpha^* \beta)^{-1} D_{\alpha}. $$
If $\alpha$ is actually a strict contraction, we extend these definitions of $\Phi_\alpha, \Phi_\alpha^{-1}$ to the closed unit ball.

**Lemma 4.2.** Let $\alpha \in [\mathcal{L}(\mathcal{H}, \mathcal{K})]_1$ be a pure contraction.

(i) The Frostman transformations $\Phi_\alpha, \Phi_\alpha^{-1}$ map $[\mathcal{L}(\mathcal{H}, \mathcal{K})]_1$ into pure contractions in $[\mathcal{L}(\mathcal{H}, \mathcal{K})]_1$.

(ii) If $\alpha$ is a strict contraction, $\Phi_\alpha, \Phi_\alpha^{-1}$ are compositional inverses and define bijections of $[\mathcal{L}(\mathcal{H}, \mathcal{K})]_1$ onto itself and $([\mathcal{L}(\mathcal{H}, \mathcal{K})]_1)$ onto itself.

(iii) Given any two strict contractions $\beta, \gamma$, $\Phi_\alpha$ and $\Phi_\alpha^{-1}$ obey the identities:

$$I - \Phi_\alpha(\beta)\Phi_\alpha(\gamma)^* = D_\alpha^*(I - \beta\alpha^*)^{-1}(I - \beta\gamma^*)(I - \alpha\gamma^*)^{-1}D_\alpha^*, \text{ and},$$

$$I - \Phi_\alpha^{-1}(\beta)\Phi_\alpha^{-1}(\gamma)^* = D_\alpha^*(I + \beta\alpha^*)^{-1}(I - \beta\gamma^*)(I + \alpha\gamma^*)^{-1}D_\alpha^*.$$

**Proof.** Assume that $\alpha$ is a pure contraction so that $D_\alpha := \sqrt{1 - \alpha^*\alpha}$ has dense range. Given any $h = D_\alpha g \in \text{Ran}(D_\alpha)$, and any strict contraction $\beta$, consider:

$$\|h\|^2 - \|\Phi_\alpha(\beta)\|h\| = (D_\alpha^2 g, g) - ((\beta^* - \alpha^*)(I - \alpha\beta^*)^{-1}D_\alpha^2)(I - \beta\alpha^*)^{-1}(\beta - \alpha)g, g).$$

(A)

Observe that

$$(I - \beta\alpha^*)^{-1}(\beta - \alpha) = \beta(I - \alpha\beta^*)^{-1}D_\alpha^2 - \alpha.$$

This identity is easily verified by multiplying both sides from the left by $(I - \beta\alpha^*)$. Substitute this identity into the expression (A) to obtain:

$$\text{(A)} = D_\alpha^2(I - \beta^*\alpha)^{-1}\beta^*D_\alpha^2\beta(I - \alpha^*\beta)^{-1}D_\alpha^2 + \alpha^*D_\alpha^2\alpha$$

$$- D_\alpha^2(I - \beta^*\alpha)^{-1}\beta^*D_\alpha^2\alpha - \alpha^*D_\alpha^2\beta(I - \alpha^*\beta)^{-1}D_\alpha^2$$

$$= D_\alpha^2(I - \beta^*\alpha)^{-1}\beta^*D_\alpha^2\beta(I - \alpha^*\beta)^{-1}D_\alpha^2 + \alpha^*D_\alpha^2\alpha$$

$$- D_\alpha^2(I - \beta^*\alpha)^{-1}\beta^*\alpha D_\alpha^2 - \alpha^*\beta(I - \alpha^*\beta)^{-1}D_\alpha^2.$$

Applying the identity $D_\alpha^2 - D_\alpha^2\alpha^*\alpha = D_\alpha^2$,

$$D_\alpha^2 - \text{(A)} = D_\alpha^2(I - \beta^*\alpha)^{-1}\beta^*D_\alpha^2\beta(I - \alpha^*\beta)^{-1}D_\alpha^2$$

$$+ D_\alpha^2(I - \beta^*\alpha)^{-1}\beta^*\alpha D_\alpha^2 + D_\alpha^2\alpha^*\beta(I - \alpha^*\beta)^{-1}D_\alpha^2.$$

$$= D_\alpha^2(I - \beta^*\alpha)^{-1}(I - \beta^*\alpha)(I - \alpha^*\beta) - \beta^*D_\alpha^2\beta$$

$$+ \beta^*\alpha(I - \alpha^*\beta) + (I - \beta^*\alpha)\alpha^*\beta(I - \alpha^*\beta)^{-1}D_\alpha^2$$

$$= D_\alpha^2(I - \beta^*\alpha)^{-1}(I - \beta^*\alpha)(I - \alpha^*\beta)^{-1}D_\alpha^2.$$

This proves that for any $h \in \text{Ran}(D_\alpha)$,

$$\|h\|^2 - \|\Phi_\alpha(\beta)h\|^2 = \langle D_\alpha(I - \beta^*\alpha)^{-1}(I - \beta^*\beta)(I - \alpha^*\beta)^{-1}D_\alpha h, h \rangle > 0,$$

and it follows that $\Phi_\alpha(\beta)$ is a pure contraction. If $\alpha$ is a strict contraction then $D_\alpha$ is bounded below, and it follows that $\Phi_\alpha(\beta)$ will be a strict contraction in this case.
Assuming that $\alpha, \beta$ are both strict contractions, the expression $\Phi^{-1}_\alpha(\Phi_\alpha(\beta))$ is well-defined. It remains to calculate:

$$
\Phi^{-1}_\alpha(\Phi_\alpha(\beta)) = D^{-1}_\alpha \left( \frac{\Phi_\alpha(\beta) + \alpha}{(N)} \cdot \left( I + \alpha^* \Phi_\alpha(\beta) \right) \right) D_\alpha.
$$

The denominator, $(D)$, evaluates to:

$$
(D) = D_\alpha \left( I - \alpha^* \beta \right)^{-1} \left( (I - \alpha^* \beta + \alpha^* \beta - \alpha^* \alpha) D^{-1}_\alpha \right)
= D_\alpha \left( I - \alpha^* \beta \right)^{-1},
$$

while the numerator, $(N)$, evaluates to

$$
(N) = D_\alpha^* \left( (I - \beta \alpha^*)^{-1} \left( \beta - \alpha + (I - \beta \alpha^*) D^{-1}_\alpha \alpha D_\alpha \right) \right) D^{-1}_\alpha
= D_\alpha \left( I - \beta \alpha^* \right)^{-1} \beta.
$$

It follows that the full expression is

$$
\Phi^{-1}_\alpha(\Phi_\alpha(\beta)) = (I - \beta \alpha^*)^{-1} \beta (I - \alpha^* \beta) = \beta.
$$

The remaining assertions are similarly easy to verify. \(\square\)

4.3. **Frostman shifts of Schur functions.** For any purely contractive $b \in \mathcal{S}(\mathcal{H}, \mathcal{K})$, Example 4.20 shows that

$$
b^{(0)}(z) := D_{b(0)} \cdot (I_{b_0} - b(z)b(0)^*)^{-1} (b(z) - b(0)) D_{b(0)}^{-1},
$$

belongs to $\mathcal{S}(\mathcal{H}, \mathcal{K})$ and vanishes at 0. In particular, it follows that $b^{(0)}$ is strictly contractive on the ball (since it vanishes at 0). Even though $b(z), b(0)$ are not strict contractions in general (since we only assume that $b$ is purely contractive), Lemma 4.12 shows that $\Phi^{-1}_b \circ b^{(0)}$ is well-defined since $b^{(0)}(z)$ is always a strict contraction and $b(0)$ is a pure contraction by assumption. Moreover, it is easy to check that $b = \Phi^{-1}_b \circ b^{(0)}$, and that formally $b^{(0)} = \Phi_{b(0)} \circ b$.

Conversely, given any pure contraction $\alpha \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, and any Schur class $b \in \mathcal{S}(\mathcal{H}, \mathcal{K})$ obeying $b(0) = 0$ (so that $b(z)$ is a strict contraction for all $z \in \mathbb{D}$), we can define $\Phi^{-1}_\alpha \circ b$, and this is a purely contractive analytic function on the ball by Lemma 4.2. More generally, given any purely contractive Schur class $b \in \mathcal{S}(\mathcal{H}, \mathcal{K})$ we define the $\alpha$-Frostman Shift of $b$ as

$$
b^{(\alpha)} := \Phi^{-1}_\alpha \circ \Phi_{b(0)} \circ b := \Phi^{-1}_\alpha \circ b^{(0)}.
$$

This is a well-defined purely contractive analytic function on the ball. Observe that, by definition, $b^{(0)} = (b^{(\alpha)})^{(0)}$, $b = b^{(\alpha)}$, and $b^{(\alpha)}(0) = \alpha$.

**Theorem 4.4.** Given any purely contractive Schur function $b \in \mathcal{S}(\mathcal{H}, \mathcal{K})$, and any pure contraction $\alpha \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, the $\alpha$-Frostman shift, $b^{(\alpha)} \in \mathcal{S}(\mathcal{H}, \mathcal{K})$ is also Schur class and purely contractive,

$$
b^{(\alpha)}(z) = D_{b^{(\alpha)}}^{-1} \left( \frac{b^{(0)}(z) + \alpha}{(D)} \cdot \left( I + \alpha^* b^{(0)}(z) \right) \right) D_\alpha.
$$
Multiplication by

\[ M^{(\alpha)}(z) := (I - b^{(\alpha)}(z)\alpha)D_{\alpha}^{-1} = D_{\alpha^*}(I + b^{(0)}(z)\alpha^*)^{-1}, \]
defines a unitary multiplier, \( M^{(\alpha)} \), of \( \mathcal{H}(b^{(0)}) \) onto \( \mathcal{H}(b^{(\alpha)}) \).

The unitary multiplier \( M^{(\alpha)} \) is a multivariable and operator analogue of a Crofoot multiplier or Crofoot transform [25, 43]. Since \( M^{(\alpha)} \) is a unitary multiplier, it follows, in particular, that \( M^{(\alpha)}(z) \) defines a bounded invertible operator for any \( z \in \mathbb{B}^d \).

**Proof.** By Example [3.20] given any pure contraction \( \alpha \in \mathcal{L}(\mathcal{K}, \mathcal{K}) \), \( b^{(0)} = \Phi_{b^{(0)}} \circ b^{(\alpha)} \in \mathcal{S}(\mathcal{K}, \mathcal{K}) \) is Schur class (and strictly contractive). By the identities of Lemma [4.2]

\[ I - b^{(\alpha)}(z)b^{(\alpha)}(w)^* = I - \Phi^{-1}_{\alpha}(b^{(0)}(z))\Phi^{-1}_{\alpha}(b^{(0)}(w))^* \]

\[ = D_{\alpha^*}(I + b^{(0)}(z)\alpha^*)^{-1} \left( I - b^{(0)}(z)b^{(0)}(w)^* \right) \left( I + ab^{(0)}(w)^* \right)^{-1}D_{\alpha^*}, \]

and, similarly,

\[ I - b^{(0)}(z)b^{(0)}(w)^* = D_{\alpha^*}(I - b^{(\alpha)}(z)\alpha^*)^{-1} \left( I - b^{(\alpha)}(z)b^{(\alpha)}(w)^* \right) \left( I - ab^{(\alpha)}(w)^* \right)^{-1}D_{\alpha^*}. \]

This proves simultaneously that \( k^{(\alpha)} := k^{b^{(\alpha)}} \) is a positive kernel so that \( b^{(\alpha)} \in \mathcal{S}(\mathcal{K}, \mathcal{K}) \) by Theorem 2.1, and that \( M^{(\alpha)}(z) \) as written above is a unitary multiplier of \( \mathcal{H}(b^{(0)}) \) onto \( \mathcal{H}(b^{(\alpha)}) \).

**Theorem 4.5.** The map,

\[ b^{(\alpha)} \mapsto b^{(0)} := (M^{(\alpha)} \otimes I) - 1 b^{(\alpha)} D_{\alpha}^{-1}, \]
defines a bijection from contractive Gleason solutions for \( b^{(\alpha)} \) onto contractive Gleason solutions for \( b^{(0)} \) which preserves extremal solutions.

**Proof.** Let \( b^{(\alpha)} \) be any contractive Gleason solution for \( b^{(\alpha)} \). Then,

\[ zb^{(0)}(z) = z(M^{(\alpha)}(z)^{-1} \otimes I)b^{(\alpha)}(z)D_{\alpha}^{-1} \]

\[ = M^{(\alpha)}(z)^{-1}(b^{(0)}(z) - \alpha)D_{\alpha}^{-1} \]

\[ = D_{\alpha^*}(I - b^{(\alpha)}(z)\alpha^*)^{-1}(b^{(\alpha)}(z) - \alpha)D_{\alpha}^{-1} \]

\[ = b^{(0)}(z), \]

and this equals \( b^{(0)}(z) - b^{(0)}(0) \) since \( b^{(0)}(0) = 0 \). This shows that \( b^{(0)} \) is a Gleason solution, and

\[ (b^{(0)})^*b^{(0)} = D_{\alpha}^{-1}(b^{(\alpha)})^*b^{(\alpha)}D_{\alpha}^{-1} \]

\[ \leq D_{\alpha}^{-1}(I - \alpha^*\alpha)D_{\alpha}^{-1} = I, \]

so that \( b^{(0)} \) is a contractive Gleason solution which is extremal if \( b^{(\alpha)} \) is. The converse follows similarly. \( \square \)

**Proposition 4.6.** Let \( b \in \mathcal{S}(\mathcal{K}, \mathcal{K}) \) be a purely contractive Schur class function, let \( X = X(b) \) be an extremal contractive Gleason solution for \( \mathcal{H}(b) \), and let \( b \) be the corresponding contractive extremal Gleason solution for \( b; X^*k_b = w^*k_w - bb(w)^* \).
If \( b^{(0)} = (M^{(b^{(0)})} \otimes I)^* \mathfrak{b} D_{b^{(0)}}^{-1}, \ X^{(0)} := X(b^{(0)}) \) are the corresponding extremal Gleason solutions for \( b^{(0)} \) and \( \mathscr{H}(b^{(0)}) \), and \( X = V - C \) is the isometric-purely contractive decomposition of \( X \), then
\[
V = M^{(b^{(0)})} X^{(0)} (M^{(b^{(0)})} \otimes I_d)^*.
\]

**Proof.** As in Example 5.20, we have that \( V^* = X^* P_{\text{Ran}(X)} \), where
\[
P_{\text{Ran}(X)} = I - k_0^b k_b(0, 0)^{-1}(k_0^b)^* = I - k_0^b D_{b(0)}^{-2} (k_0^b)^*.
\]
Let \( k^{(0)} = k_b^{(0)} \), we need to verify that \( M^{(b^{(0)})} X^{(0)} (M^{(b^{(0)})} \otimes I_d)^* = V \). To prove this, check the action on point evaluation maps:
\[
V^* k_w^b = w^* k_w^b - \mathfrak{b} (w)^* + \mathfrak{b}(0)^* D_{b(0)}^{-2} (I - b(0) b(w)^*)
\]
\[
= w^* k_w^b - \mathfrak{b} D_{b(0)}^{-2} ((I - b(0)^* b(0)) b(w)^* - b(0)^* (I - b(0) b(w)^*)
\]
\[
= w^* k_w^b - \mathfrak{b} D_{b(0)}^{-2} (b(w)^* - b(0)*).
\]
Compare this to:
\[
(M^{(b^{(0)})} \otimes I_d)(X^{(0)})^* (M^{(b^{(0)})})^* k_w^b
\]
\[
= (M^{(b^{(0)})} \otimes I_d)(X^{(0)})^* k_w^{b^{(0)}} M^{(b^{(0)})}(w)^*
\]
\[
= (M^{(b^{(0)})} \otimes I_d) \left( w^* k_w^{b^{(0)}} - \mathfrak{b}(0)^* b(0)^* (w)^* \right) M^{(b^{(0)})}(w)^*
\]
\[
= w^* k_w^{b^{(0)}} - \mathfrak{b} D_{b(0)}^{-2} b(0)^* M^{(b^{(0)})}(w)^*
\]
\[
= w^* k_w^{b^{(0)}} - \mathfrak{b} D_{b(0)}^{-2} (b(w)^* - b(0)^*).
\]

\[
\square
\]

### 4.7. Gleason solution model for CCNC row contractions

Let \( T \) be an arbitrary CCNC row contraction on \( \mathcal{H} \) with partial isometric-purely contractive decomposition \( T = V - C \).

**Lemma 4.8.** Let \( V \) be a CCNC row partial isometry on \( \mathcal{H} \), and let \( (\gamma, \mathcal{J}_\infty, \mathcal{J}_0) \) be a model triple for \( V \). Given any pure contraction \( \delta \in [\mathscr{L}(\mathcal{J}_\infty, \mathcal{J}_0)]_1 \), the map
\[
\delta \mapsto T_\delta := V - \gamma(0)^* \gamma(\infty)^*,
\]
is a bijection from pure contractions onto CCNC row contractions with partial isometric part \( V \).

**Proof.** Since \( \gamma(0) : \mathcal{J}_0 \rightarrow \text{Ran}(V)^\perp \) and \( \gamma(\infty) : \mathcal{J}_\infty \rightarrow \text{Ker}(V) \) are onto isometries, it is clear that \( \delta \mapsto -\gamma(0)^* \delta \gamma(\infty)* \) maps pure contractions \( \delta \in [\mathscr{L}(\mathcal{J}_\infty, \mathcal{J}_0)]_1 \) onto pure contractions in \( [\mathscr{L}(\text{Ker}(V), \text{Ran}(V))^\perp] \). It is also clear that \( T_\delta \) is CCNC if and only if \( V \) is, and that \( V \) is the partial isometric part of \( T_\delta \). Conversely, given any CCNC row contraction \( T \) on \( \mathcal{H} \) such that \( T = V - C \), we have that \( \delta := -\gamma(0)^* T \gamma(\infty) = \gamma(0)^* C \gamma(\infty) \). Since \( C \) is a pure row contraction, \( \delta \) is a pure contraction and \( T = T_\delta \).

\[
\square
\]

**Definition 4.9.** Let \( T : \mathcal{H} \otimes \mathbb{C}^d \rightarrow \mathcal{H} \) be a CCNC row contraction with partial isometric-purely contractive decomposition \( T = V - C \). For any fixed model triple \( (\gamma, \mathcal{J}_\infty, \mathcal{J}_0) \) of \( V \), define
\[
\delta_T^\gamma := -\gamma(0)^* T \gamma(\infty) = \gamma(0)^* C \gamma(\infty),
\]
the zero-point contraction of $T$. The characteristic function, $b_T$, of $T$, is then any Schur class function in the weak coincidence class of the $\delta^*_T$-Frostman shift of $b^*_\gamma$,

$$b^*_T := (b^*_\gamma)^{(\delta^*_T)} \in \mathcal{S}_d(\mathcal{J}_\infty, \mathcal{J}_0),$$

$$b^*_T(z) = D^{-1}_{(\delta^*_T),T}(b^*_\gamma(z)) \cdot (I + (\delta^*_T)^*b^*_\gamma(z))^{-1} D_{\delta^*_T}, \quad z \in \mathbb{B}^d.$$  

Since $C$ is a pure row contraction, it follows that $\delta^*_T$ is always a pure contraction, and that $b^*_T$ is purely contractive on the ball, by Lemma 4.2.

**Lemma 4.10.** The coincidence class of $b^*_T$ is invariant under the choice of model triple $(\gamma, \mathcal{J}_\infty, \mathcal{J}_0)$.

*Proof.* Let $(\gamma, \mathcal{J}_\infty, \mathcal{J}_0)$ and $(\varphi, \mathcal{K}_\infty, \mathcal{K}_0)$ be two model triples for $V$, $T = V - C$. By Lemma 3.5 we know that there are onto isometries $R : \mathcal{K}_\infty \to \mathcal{J}_\infty$ and $Q^* : \mathcal{K}_0 \to \mathcal{J}_0$ so that $\varphi(\infty) = \gamma(\infty)R$, $\varphi(0) = \gamma(0)Q^*$, and

$$b^*_\varphi = Qb^*_\gamma R.$$  

Similarly,

$$\delta^*_T = -\varphi(0)^*T \varphi(\infty) = Q\delta^*_T R.$$  

Finally, we calculate

$$b^*_T = (Rb^*_\gamma Q)^{(R\delta^*_T Q)} = D_{(R\delta^*_T Q),T}^{-1} \cdot (R(b^*_\gamma + \delta^*_T)Q) \cdot (I + Q^*(\delta^*_T)^*R^*)R^*b^*_\gamma Q)^{-1} D_{R\delta^*_T Q} = RD_{(\delta^*_T),T}^{-1} \cdot R^* \cdot R(b^*_\gamma + \delta^*_T) \cdot QQ^* \cdot (I + (\delta^*_T)^*b^*_\gamma)^{-1} QQ^* D_{\delta^*_T} Q = R(b^*_\gamma)^{(\delta^*_T)}Q = Rb^*_T Q.$$  

\[\square\]

We will refer to any Schur class function in the weak coincidence class of any $b^*_T$ as the characteristic function of $T$.

**Example 4.11.** (Extremal Gleason solutions) Given any purely contractive $b \in \mathcal{S}_d(\mathcal{H}, \mathcal{K})$, Example 2.6 proved that any contractive extremal Gleason solution, $X$, for $\mathcal{K}(b)$ is a CCNC row contraction. Using the model constructed as in Example 3.20 we will now show that the characteristic function, $b_X$, of $X$, coincides weakly with $b$.

As discussed in Subsection 1.4 $X = X(b)$ for an extremal contractive Gleason solution $b$ for $b$, where $X(b)$ is given by Formula (1.5). Let $b^{(0)} := (M(b^{(0)}) \otimes I_d)^*bD_{b^{(0)}}^{-1}$ be the extremal contractive Gleason solution for $b^{(0)}$ which corresponds uniquely to $b$ as in Lemma 3.5 and let $X = V - C$ be the isometric-pure decomposition of $X$. Proposition 4.7 then implies that if $X^{(0)} := X(b^{(0)})$ is the corresponding extremal Gleason solution for $\mathcal{K}(b^{(0)})$, that $V = M(b^{(0)})X^{(0)}(M(b^{(0)}) \otimes I_d)^*$.

As in Example 3.20 it then follows that we can define an analytic model triple for $V$ as follows. Let $\Gamma(0) := k^b b^{(0)} \Gamma(0) = k^b b^{(0)}$, $\Gamma(z) := (I - Xz^*)^{-1} \Gamma(0) = k^b b^{(0)}$, and $\Gamma(\infty) := bD_{b^{(0)}}^{-1} \Gamma(\infty)'$, where $\Gamma(\infty)' : \mathcal{K} \to \text{Ker}(V) \cap \text{Ran}(b)$ is an arbitrary onto isometry (that $b$ maps into $\text{Ker}(V)$ follows as in Example 3.20). As in Example 3.20 $(\Gamma, \mathcal{K} \oplus \mathcal{K}', \mathcal{K})$ is an analytic model triple for $V$, and we will
use this triple to compute the characteristic function $b^T_X$. Using the relationship between $V$ and $X^{(0)}$, it is easy to check that $b^T_V = b^{(0)} \oplus 0_{\mathcal{H}}$ as in Example 3.20 and it remains to check that $\delta^T_X = b(0) \oplus 0_{\mathcal{H}}$:

$$
\delta^T_X = -(k_0 D_{b(0)}^{-1} \ast X)(\infty) \\
= -(X^* k_0 D_{b(0)}^{-1})^* b D_{b(0)}^{-1} \oplus \Gamma(\infty)' \\
= D_{b(0)}^{-1} b(0) b^* \left( b D_{b(0)}^{-1} \oplus \Gamma(\infty)' \right) \\
= b(0) \oplus 0_{\mathcal{H}}.
$$

We conclude that $b^T_X := (b^T_V)^{(\delta^T_X)}$ coincides weakly with $b = (b^{(0)})^{(b(0))}$.

We are now sufficiently prepared to prove one of our main results:

**Theorem 4.12.** A row contraction $T : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H}$ is CCNC if and only if $T$ is unitarily equivalent to an extremal contractive Gleason solution $X^b$ acting on a multi-variable deBranges-Rovnyak space $\mathcal{H}(b)$ for a purely contractive Schur class $b \in \mathcal{S}_d(\partial, \mathcal{H})$.

If $T \simeq X^b$, the characteristic function, $b_T := b$ is a unitary invariant: if two CCNC row contractions $T_1, T_2$ are unitarily equivalent, then their characteristic functions $b_{T_1}, b_{T_2}$ coincide weakly.

In the above $\simeq$ denotes unitary equivalence. Recall that as shown in Example 5.10 the characteristic function of a CCNC row contraction is not a complete unitary invariant: there exist CCNC row contractions $T_1, T_2$ which have the same characteristic function but are not unitarily equivalent.

**Proof.** We have already proven that any extremal Gleason solution, $X$, for $\mathcal{H}(b)$, $b \in \mathcal{S}_d(\partial, \mathcal{H})$ is a CCNC row contraction with characteristic function coinciding weakly with $b$ in Examples 2.6 and 4.11 above. Conversely, let $T$ be a CCNC row contraction on $\mathcal{H}$ with isometric-contractive decomposition $T = V - C$.

Let $(\Gamma, \delta_\infty, \delta_0)$ be the analytic model triple $\Gamma = \Gamma_V$ for $V$. By Theorem 3.7 the unitary $U^\Gamma := (M^\Gamma)^{-1} U^T : \mathcal{H} \to \mathcal{H}(b^T_V)$, where recall $M^\Gamma = M_{D^\Gamma}$, a unitary multiplier, is such that $X := U^T V(U^\Gamma)^* = X(b^\Gamma)$ is an extremal contractive Gleason solution for $\mathcal{H}(b^\Gamma_V)$ corresponding to the extremal contractive Gleason solution $b^\Gamma = U^\Gamma \Gamma(\infty)$ for $b^T_V$. It is clear that $(U^\Gamma \Gamma, \delta_\infty, \delta_0)$ is then an analytic model triple for $X$.

By Theorem 4.5, since $b^T_V = \Phi_{b^\Gamma_V(0)}(b^\Gamma_T) = (b^\Gamma_T)^{(0)}$ is the 0-Frostman shift of $b_T$, there is a unique extremal contractive Gleason solution $b^T$ for $b_T = b^\Gamma_T$ so that

$$
b^\Gamma := (M^{(b^{\Gamma(0)})} \otimes I)^{-1} b^T D_{b^{\Gamma(0)}}^{-1},
$$

where, by definition, $b_T(0) = b^\Gamma_T(0) = \delta^T_T$. Proposition 4.6 then implies that if $X^T := X(b^T)$ is the corresponding extremal contractive Gleason solution for $b_T$ with isometric-pure decomposition $X^T = V^T - C^T$, then $\tilde{X} := (M^{(b^{\Gamma(0)})})^* X^T (M^{(b^{\Gamma(0)})}) \otimes I_d = X - \tilde{C}$ where $(M^{(b^{\Gamma(0)})})^* V^T (M^{(b^{\Gamma(0)})}) = X$.

Our goal is to prove that $T^T = U^T T(U^\Gamma)^* \otimes I_d := X - C'$ is equal to $\tilde{X} = (M^{(b^{\Gamma(0)})})^* X^T M^{(b^{\Gamma(0)})} \otimes I_d = X - \tilde{C}$ so that $T \simeq X^T$, i.e. $T$ is unitarily equivalent to the extremal contractive Gleason solution $X^T$. 

for $\mathcal{H}(b^T_{T'})$. By Lemma 4.8 it suffices to show that $\delta^{T'\Gamma}_{X} = \delta^{T'\Gamma}_{X}$. Let $b := b^T_{T}$, $b_{V} := b^T_{V}$, and calculate

$$
\delta^{T'\Gamma}_{X} = -\Gamma(0)^*(U^\Gamma)^* T'(U^\Gamma \otimes I_d) \Gamma(\infty)
$$

$$
= -\Gamma(0)^* T \Gamma(\infty) = \delta^{T}_{X} = b^T_{T}(0).
$$

Similarly, since $X = X(b^T)$ where $b^T = U^\Gamma \Gamma(\infty)$,

$$
\delta^{T'\Gamma}_{X} = -(U^\Gamma \Gamma(0))^* \tilde{X} b^T
$$

$$
= -(M_0^{b_{T'}})^* (M(b(0))) X^T (M(b(0)) \otimes I_d)(M(b(0)) \otimes I)^{-1} b^T D_{b(0)}^{-1}
$$

$$
= -(M(b(0)))^{-1} k_0^{b_{T'}} X^T b^T D_{b(0)}^{-1}
$$

$$
= (X^T)^* k_0^{b_{T'}} (M(b(0))) (0)^{-1} b(0) b(0)^* b(0) D_{b(0)}^{-1}
$$

$$
= D_{b(0)} (I - b(0) b(0)^*)^{-1} b(0) b(0) D_{b(0)}^{-1}
$$

$$
= b_T(0) = \delta^T_{X}.
$$

Finally, if $T_1, T_2$ are two unitarily equivalent row contractions, $T_2 = UT_1 U^*$ for a unitary $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, and $T_k = V_k - C_k$, then given any analytic model triple $(\Gamma, J, \delta_0)$ for $T_1$, $(UT, J, \delta_0)$ is an analytic model triple for $T_2$, and $b^T_{T_1} = b^T_{T_2}$.

As a final observation, our characteristic function, $b_T$, for any CCNC row contraction, $T$, coincides weakly with the Sz.-Nagy-Foiaş-type characteristic function of $T$, as defined for CNC $d$–contractions in [15]:

$$
\Theta_T(z) := (T + z D_T \Gamma(\infty)^{-1} D_T)|_{\text{Ran}(D_T) \subset \mathcal{L}(\text{Ran}(D_T))}, \quad z \in \mathbb{B}^d.
$$

**Proposition 4.13.** Let $T$ be a CCNC row contraction. The characteristic function, $b_T$, of $T$, coincides weakly with the Nagy–Foiaş characteristic function, $\Theta_T$.

**Proof.** It suffices to show that given any purely contractive $b \in \mathcal{S}(J, \mathcal{K})$, and any contractive, extremal Gleason solution, $X$, for $\mathcal{H}(b)$ that $b_X := b$ coincides weakly with $\Theta_X$. Let $b$ be a contractive extremal Gleason solution for $b$ so that $X = X(b)$.

As in Example 4.11 consider the analytic model triple $(\Gamma, \mathcal{K} \oplus \mathcal{K}', \mathcal{K})$ where $\Gamma(0) = k_0^{b} D_{b(0)}^{-1}$, $\Gamma(z) := (I - X z^*)^{-1} \Gamma(0)$, and $\Gamma(\infty) = b D_{b(0)}^{-1} \oplus \Gamma(\infty)'$, where if $X = V - C$ is the isometric-pure decomposition of $X$, then $\Gamma(\infty)' : \mathcal{K}' \rightarrow \text{Ker}(V) \oplus \text{Ran}(b)$ is an onto isometry. Since $\Gamma(0), \Gamma(\infty)$ are onto isometries, $\Theta_X$ coincides with

$$
-\Gamma(0)^* X \Gamma(\infty) + \Gamma(0)^* z D_X \Gamma(\infty) = b(0) \oplus 0_{\mathcal{K}'}, \quad \text{as in Example 4.11}
$$

Since $X$ is extremal, one can verify (by uniqueness of the positive square root) that

$$
D_X = \sqrt{k_0^{b}(k_0^{b})^*} = k_0^{b} D_{b(0)}^{-1} (k_0^{b})^*.
$$
and it follows that
\[
\Gamma(0)^* zD_{X^*}(I - zX^*)^{-1}D_{X}\Gamma(\infty) = D_{b(0)}^{-1}(b(0)^*b(0)) D_{b(0)}^{-1}(z^*k_{0}^b)^* \sqrt{I - X^*X}\Gamma(\infty) \\
= (z^*k_{0}^b)^* \sqrt{I - X^*X}\Gamma(\infty).
\]

Since \(\Gamma(\infty)\) is an isometry onto \(\text{Ker}(V)\), and
\[
X^*X = V^*V + C^*C,
\]

it follows that
\[
\sqrt{I - X^*X}\Gamma(\infty) = \sqrt{P_{\text{Ker}(V)} - P_{\text{Ker}(V)}C^*CP_{\text{Ker}(V)}\Gamma(\infty)}.
\]

Moreover, in Example 3.20 we calculated that
\[
V^* = X^*(I - k_{0}^b D_{b(0)}^{-1}(k_{0}^b)^*),
\]

and it follows from this that
\[
C^* = bD_{b(0)}^{-1}b(b(0))^*D_{b(0)}^{-1}(k_{0}^b)^*.
\]

In particular, it follows that
\[
(C^*C)^k = bD_{b(0)}^{-1}(b(0)^*b(0))^k D_{b(0)}^{-1}b^*;
\]

and the functional calculus then implies that
\[
\sqrt{I - X^*X}\Gamma(\infty) = bD_{b(0)}^{-1} \sqrt{I - b(0)^*b(0)D_{b(0)}^{-1}}b^* = bD_{b(0)}^{-1}b^*.
\]

In conclusion we obtain
\[
\Theta_X(z) \simeq b(0) \oplus 0_{\mathcal{X}'} + (z^*k_{0}^b)^*bD_{b(0)}^{-1}b^*bD_{b(0)}^{-1} \oplus 0_{\mathcal{X}'} = b(0) \oplus 0_{\mathcal{X}'} + (b(z) - b(0)) \oplus 0_{\mathcal{X}'} = b(z) \oplus 0_{\mathcal{X}'},
\]

and \(\Theta_X\) coincides weakly with \(b = b_X\).

\[\square\]

5. QE row contractions

In this section we focus on the sub-class of quasi-extreme (QE) row contractions. This is the set of all CCNC row contractions, \(T\), whose characteristic function \(b_T\) coincides weakly with a quasi-extreme Schur multiplier as defined and studied in \([8, 9, 10]\). We will see that the characteristic function is a complete unitary invariant for QE row contractions.

5.1. Quasi-extreme Schur multipliers. As discussed in the introduction, the concept of a quasi-extreme Schur class multiplier was introduced in \([8, 9]\), as a several-variable analogue of a ‘Szegö approximation property’ that is equivalent to being an extreme point of the Schur class in the single-variable, scalar-valued setting (see e.g. \([10]\)).

In \([9]\), the quasi-extreme property was defined for any non-unital and square \(b \in \mathcal{S}_d(\mathcal{H})\) (recall from Subsection 1.2 that the non-unital assumption is needed to ensure that the corresponding Herglotz-Schur function \(H_b\) takes values in bounded operators), but we will require the extension of this property to arbitrary purely contractive and ‘rectangular’ \(b \in \mathcal{S}_d(\mathcal{J}, \mathcal{K})\). For this purpose, it will be useful to define the square extension of any \(b \in \mathcal{S}_d(\mathcal{J}, \mathcal{K})\): Any such \(b\) coincides with an element \(b \in \mathcal{S}_d(\mathcal{J}', \mathcal{K}')\) where
\[ \mathcal{J}' \subseteq \mathcal{K}' \text{ or } \mathcal{K}' \subseteq \mathcal{J}' . \] Given any \( b \in \mathcal{A}(\mathcal{J}, \mathcal{K}) \), and assuming \( \mathcal{J} \subseteq \mathcal{K} \) or \( \mathcal{K} \subseteq \mathcal{J} \), the square extension, \( [b] \), of \( b \) is

\[
[b] := \begin{cases} 
\begin{bmatrix} b & 0_{\mathcal{K} \otimes \mathcal{J}, \mathcal{K}} \\ 0_{\mathcal{J} \otimes \mathcal{K}} \end{bmatrix} & \in \mathcal{A}(\mathcal{K}); \quad \mathcal{J} \subset \mathcal{K} \\
\begin{bmatrix} b & 0_{\mathcal{K} \otimes \mathcal{K}} \\ 0_{\mathcal{J} \otimes \mathcal{K}} \end{bmatrix} & \in \mathcal{A}(\mathcal{J}); \quad \mathcal{K} \subset \mathcal{J} .
\end{cases}
\]

**Remark 5.2.**

1. If \( \mathcal{J} \subseteq \mathcal{K} \), then it follows that \( \mathcal{H}(b) = \mathcal{H}([b]) \). In this case it follows that \( b \) is a contractive Gleason solution for \( b \) if and only if there is a contractive Gleason solution \([b]\) for \([b]\) so that \( b = [b] \mid_{\mathcal{J}} \). Moreover if \([b]\) is extremal, so is \( b = [b] \mid_{\mathcal{J}} \).
2. If \( \mathcal{K} \subseteq \mathcal{J} \) then \( \mathcal{H}'(\mathcal{J}) = \mathcal{H}'(b) \oplus H_{d}^{2} \otimes (\mathcal{J} \otimes \mathcal{K}) \). In this case \( b \) is a contractive Gleason solution for \( b \) if and only if \([b]:= b \oplus 0\) is a contractive Gleason solution for \( b \). Here \( 0 : \mathcal{J} \otimes \mathcal{K} \rightarrow H_{d}^{2} \otimes (\mathcal{J} \otimes \mathcal{K}) \) sends every vector to \( 0 \in H_{d}^{2} \otimes (\mathcal{J} \otimes \mathcal{K}) \). A contractive Gleason solution \([b]\) for \([b]\) is extremal if and only if \( b = \begin{bmatrix} I_{\mathcal{H}(b)} \otimes I_{d}, & 0 \end{bmatrix} [b] \) is a contractive and extremal Gleason solution for \( b \).

**Theorem 5.3.** Given any \( b \in \mathcal{A}(\mathcal{J}, \mathcal{K}) \) such that \([b]\) is non-unital, the following are equivalent:

1. \( b \) has a unique contractive Gleason solution and this solution is extremal.
2. \( \text{supp}(b) = \mathcal{J} \), \( \mathcal{H}'(b) \) has a unique contractive Gleason solution, and this solution is extremal.
3. There is no non-zero \( g \in \mathcal{J} \) so that \( bg \in \mathcal{H}'(b) \).
4. There is no non-zero \( \mathcal{J} \)-valued constant function \( F \equiv g \in \mathcal{H}'(H_{d}[b]), \ g \in \mathcal{J} \).
5. \( K_{0}^{[b]}(I - b(0)) \mathcal{J} \subseteq \text{Ran} (V^{[b]}) \).

Any Schur multiplier is said to be quasi-extreme if it obeys the assumptions and equivalent conditions of this theorem. If, for example, \( b \) is strictly contractive, then \([b]\) will be strictly contractive (and hence non-unital). For conditions (iv) and (v) of the above theorem we are assuming that either \( \mathcal{J} \subseteq \mathcal{K} \) or \( \mathcal{K} \subseteq \mathcal{J} \). There is no loss of generality with this assumption, since it is easy to see that \( b \in \mathcal{A}(\mathcal{J}, \mathcal{K}) \) is quasi-extreme if and only if every member of its coincidence class is quasi-extreme. In the particular case where \( \mathcal{J} = \mathcal{K} \) so that \( b = [b] \), items (iv) and (v) reduce to:

- (iv) \( \mathcal{H}'(H_{d}[b]) \) contains no constant functions.
- (v) \( V^{[b]} \) is a co-isometry,

see [9] Theorem 4.17. Since the proof and proof techniques of Theorem 5.3 are very similar to those of [9] Section 4, we will not include it here (a proof can be found in [10] Theorem 2.25). The equivalence of (iii) and (iv), for example, follows as in the proof of [9] Theorem 3.22. An arbitrary purely contractive \( b \in \mathcal{A}(\mathcal{J}, \mathcal{K}) \) may still not satisfy the assumptions of Theorem 5.3 i.e. \([b]\) may not be non-unital, and so we define:

**Definition 5.4.** A purely contractive \( b \in \mathcal{A}(\mathcal{J}, \mathcal{K}) \) is quasi-extreme if \( b \) has a unique contractive Gleason solution, and this solution is extremal.

In particular, the bijection between contractive Gleason solutions for \( b \) and (the strictly contractive) \( b^{(0)} \) of Lemma 4.5 implies:
Corollary 5.5. A purely contractive \( b \in \mathcal{S}_d(J, K) \) is quasi-extreme if and only if the \( \alpha \)-Frostman shift \( b^{(\alpha)} \) is quasi-extreme for any pure contraction \( \alpha \in \mathcal{L}(J, K) \).

In particular, \( b \) is quasi-extreme if and only if the strictly contractive \( b^{(0)} \) is quasi-extreme (so that \( b^{(0)} \) obeys the equivalent properties of Theorem 5.3).

Lemma 5.6. If \( b \in \mathcal{S}_d(J, K) \) is purely contractive and \( \text{supp}(b) = J \), then \( b \) is quasi-extreme if and only if the strictly contractive \( b^{(0)} \) is quasi-extreme (so that \( b^{(0)} \) obeys the equivalent properties of Theorem 5.3).

Proof. This follows from the Formula \( (1.5) \), as in \[9, Theorem 4.9, Theorem 4.4\].

The next result will yield an abstract characterization of CCNC row contractions with quasi-extreme characteristic functions.

Theorem 5.7. Let \( b \in \mathcal{S}_d(J, K) \) be a purely contractive Schur class multiplier such that \( \text{supp}(b) = J \). Then \( b \) is quasi-extreme if and only if there is an extremal contractive Gleason solution, \( X \), for \( \mathcal{H}(b) \) so that

\[
\ker(X) \subseteq \bigvee_{z \in B^d} z^* k_z^b \mathcal{K} = \bigvee_{z \in B^d} (I - X z)^{-1} \text{ran}(D z^*).
\]

Proof. We will first prove that any purely contractive \( b \) has this property if and only if \( b^{(0)} \) has this property. This will show that we can assume, without loss of generality, that \( b \) is strictly contractive so that the equivalent conditions of Theorem 5.3 apply.

Given a purely contractive \( b \in \mathcal{S}_d(J, K) \), with \( \text{supp}(b) = J \), let \( X = X(b) \) be a contractive and extremal Gleason solution for \( \mathcal{H}(b) \). Recall that \( X(b) \) is defined as in Formula \( (1.5) \), and that since \( \text{supp}(b) = J \), \( X(b) \) is extremal if and only if \( b \) is. Then

\[
\ker(X) = \text{ran}(X^*) = \bigvee_{z \in B^d} (z^* k_z^b - b z)^* \mathcal{K},
\]

and it follows that \( b \) will have the desired property if and only if

\[
\bigvee_{z \in B^d} b z^* \mathcal{K} = \bigvee_{w \in B^d} w^* k_w^b \mathcal{K}, \quad (\text{supp}(b) = J)
\]

By the bijection between Gleason solutions for \( b \) and \( b^{(0)} \), Lemma 4.5 \( b \) will have this property if and only if

\[
\bigvee_{z \in B^d} z^* k_z^{(0)} \mathcal{K} \supseteq (M^{(b^{(0)})} \otimes I_d)^* b_j^b; \quad k^{(0)} := k^{(b^{(0)})} = b^{(0)}_j,
\]

where \( b^{(0)} := (M^{(b^{(0)})} \otimes I_d)^* b D^{-1} b^{(0)}_b \) is a contractive and extremal Gleason solution for \( b^{(0)} \). As above it follows that this happens if and only if \( \ker(X^{(0)}) \subseteq \bigvee z^* k_z^{(0)} \mathcal{K} \), where \( X^{(0)} := X(b^{(0)}) \), and \( b \) has the desired property if and only if its Frostman shift \( b^{(0)} \) does. We can now assume without loss of generality that \( b \in \mathcal{S}_d(J, K) \) is strictly contractive and that \( b(0) = 0 \).
First suppose that $b$ is quasi-extreme (QE) so that $b$ has a unique contractive Gleason solution $\mathfrak{b}$ which is extremal, by Theorem $5.3$, and $X := X(\mathfrak{b})$ is the unique contractive and extremal Gleason solution for $\mathscr{H}(b)$. As in the first part of the proof, it follows that $b$ will have the desired property provided that $b|_{\mathfrak{J}} \subseteq \bigvee_{z \in B_d} z^* k_z^b \mathfrak{K}$. Assume without loss of generality that $\mathfrak{J} \subseteq \mathfrak{K}$ or $\mathfrak{K} \subseteq \mathfrak{J}$ and consider $a := [b]$, the (strictly contractive) square extension of $b$. As described in Subsection $1.4$, any contractive Gleason solution for $a$ is given by

$$a = U_a^* D^* K_0^a, \quad (a(0) = 0)$$

where $D \supseteq V^a$ is a contractive extension of $V^a$ on $\mathscr{H}^+(H_a)$, the Herglotz space of $a$, and $U_a : \mathscr{H}(a) \to \mathscr{H}^+(H_a)$ is the canonical unitary multiplier of Lemma $1.3$. Choose $D = V^a$. In the first case where $\mathfrak{J} \subseteq \mathfrak{K}$, uniqueness of $b$ implies that $b = a|_{\mathfrak{J}}$ so that

$$D^* K_0^a |_{\mathfrak{J}} \subseteq \bigvee_{z \in B_d} z^* k_z^b \mathfrak{K}.$$ 

Similarly, in the second case where $\mathfrak{K} \subseteq \mathfrak{J}$,

$$b|_{\mathfrak{J}} = \begin{bmatrix} I_{\mathscr{H}(b)} & 0 \end{bmatrix} a|_{\mathfrak{J}}$$

$$= \begin{bmatrix} I, & 0 \end{bmatrix} \bigvee_{z \in B_d} z^* k_z^b \mathfrak{J}$$

$$= \begin{bmatrix} z^* \left[ I, & 0 \right] \begin{bmatrix} k_z^b & 0 \\ 0 & k_z \otimes I_{\mathfrak{J} \otimes \mathfrak{K}} \end{bmatrix} \begin{bmatrix} \mathfrak{K} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} z^* k_z^b \mathfrak{K} \end{bmatrix}.$$ 

Conversely, suppose that $\mathscr{H}(b)$ has an extremal contractive Gleason solution, $X$, with the desired property. Then it follows, as above, that $X = X(\mathfrak{b})$, where $\mathfrak{b}$ is an extremal contractive Gleason solution for $b$ obeying $b|_{\mathfrak{J}} \subseteq \bigvee_{z \in B_d} z^* k_z^b \mathfrak{K}$. By Remark $5.2$, setting $a = [b]$, we have that there is a contractive Gleason solution $a$ for $a$ such that either $b = a|_{\mathfrak{J}}$ or $b = [I_{\mathscr{H}(b)} \otimes I_d, \ 0] a$. Also, again by Subsection $1.4$, there is a contractive extension $D \supseteq V^a$ so that $a = a[D]$. 

Consider the first case where $\mathfrak{J} \subseteq \mathfrak{K}$. It follows that $b$ has the form $b = a[D]|_{\mathfrak{J}}$ so that

$$D^* K_0^a |_{\mathfrak{J}} \subseteq \bigvee_{z \in B_d} z^* K_0^a \mathfrak{K} = \text{Ker}(V^a)^\perp.$$ 

Since $D^* = (V^a)^* + C^*$ where $C^* : \text{Ran}(V^a)^\perp \to \text{Ker}(V^a)$ (by Lemma $2.9$), it follows that

$$D^* K_0^a |_{\mathfrak{J}} = P_{\text{Ker}(V^a)} D^* K_0^a |_{\mathfrak{J}} = (V^a)^* K_0^a |_{\mathfrak{J}},$$
is contained in $\text{Ker}(V^a)$. Since we assume $X$ and hence $b$ are extremal,
\[ 0 = P_3(K_0^a)^*(I - V^a(V^a)^*)K_0^a P_3, \]
and it follows that
\[ K_0^a J \subseteq \text{Ran}(V^a), \]
so that $b$ is quasi-extreme by Theorem 5.3 (recall here that we assume $b(0) = 0 = a(0)$).

In the second case where $K \subseteq J$, we have that $a := b \oplus 0$ is a contractive (and extremal) Gleason solution for $\mathcal{H}(a)$ so that there is a $D \supseteq V^a$ such that $a = a[D]$. As before
\[ a[D] J \subseteq \bigvee_{z \in \mathbb{B}^d} z^* k \mathcal{K}, \quad \text{and} \quad D^* K_0^a J \subseteq \text{Ker}(V^a) \perp. \]
Again, the same argument as above implies that $b$ is QE. \qed

5.8. deBranges-Rovnyak model for quasi-extreme row contractions.

**Definition 5.9.** A CCNC row contraction $T : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H}$ with isometric-pure decomposition $T = V - C$ is said to be quasi-extreme (QE) if its characteristic function coincides weakly with a QE Schur multiplier.

We obtain a refined model for QE row contractions:

**Theorem 5.10.** A row contraction $T : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H}$ is QE if and only if $T$ is unitarily equivalent to the (unique) contractive and extremal Gleason solution $X$ in a multi-variable deBranges-Rovnyak space $\mathcal{H}(b)$ for a quasi-extreme and purely contractive Schur multiplier $b$.

In particular, any QE row contraction, $T$, is unitarily equivalent to $X^{br}$ where $b_T$ is any characteristic function for $T$. The characteristic function, $b_T$, of $T$, is a complete unitary invariant: Any two QE row contractions $T_1, T_2$ are unitarily equivalent if and only if their characteristic functions coincide weakly.

**Proof.** This follows from Theorem 4.12 under the added assumption that the characteristic function of $T$ is quasi-extreme. For the final statement simply note that if $b_1, b_2$ are quasi-extreme Schur functions that coincide weakly so that $\mathcal{H}(b_1) = \mathcal{H}(U b_2)$ for some unitary $U$, it is easy to see that $X^{b_1}$ is unitarily equivalent to $X^{b_2}$ (via a constant unitary multiplier), where $X^{b_1}, X^{b_2}$ are the unique, contractive, and extremal Gleason solutions for $\mathcal{H}(b_1)$ and $\mathcal{H}(b_2)$, respectively. \qed

We will conclude with an abstract characterization of the class of QE row contractions:

**Theorem 5.11.** A row contraction $T : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H}$ is QE if and only if
\[ \bigvee_{z \in \mathbb{B}^d} (I - T z^*)^{-1} \text{Ran}(D_{T^*}) = \mathcal{H}; \quad T \text{ is CCNC}, \]
and
\[ \text{Ker}(T) \subseteq \bigvee_{z \in \mathbb{B}^d} z^*(I - T z^*)^{-1} \text{Ran}(D_{T^*}). \quad (T \text{ obeys the QE condition.}) \]
Proof. Let $T$ be a QE row contraction on $\mathcal{H}$. By Theorem 5.10 $T$ is unitarily equivalent to the unique contractive and extremal Gleason solution, $X^T$ for $\mathcal{H}(b_T)$. We can assume that $b_T = b_T|_{\text{supp}(b_T)}$ so that $b_T$ is QE by Theorem 5.10. By Theorem 5.7

$$\text{Ker}(X^T)^\perp \subseteq \bigvee_{z \in \mathbb{B}^d} z^*(I - X^T z^*)^{-1}\text{Ran}(D_{(X^T)\ast}),$$

and it follows that $T \simeq X^T$ also obeys the QE condition.

Conversely suppose that $T$ is CCNC and $T$ obeys the QE condition. Then $T \simeq X^T$, an extremal Gleason solution in $\mathcal{H}(b_T)$. Again we can assume that $b_T = b_T|_{\text{supp}(b_T)}$, and since $T$ obeys the QE condition, so does $X^T$. Theorem 5.7 implies that $b_T$ is quasi-extreme so that $T$ is QE.

□

Remark 5.12. If $T$ is a QE row contraction on $\mathcal{H}$ with isometric-pure decomposition $T = V - C$, then its partial isometric part, $V$, is necessarily a QE row partial isometry: Since $\text{Ker}(T) \subseteq \text{Ker}(V)$ and $T \supseteq V$ is a QE contractive extension,

$$\text{Ker}(V)^\perp \subseteq \text{Ker}(T)^\perp \subseteq \bigvee_{z \in \mathbb{B}^d} z^*(I - T z^*)^{-1}\text{Ran}(D_{T\ast})$$

and $V$ also obeys the QE condition.

On the other hand,

Lemma 5.13. Let $b \in \mathcal{L}_d(\mathcal{J}, \mathbb{K})$, $b(0) = 0$ be a Schur multiplier and $\delta \in [\mathcal{L}(\mathcal{J}, \mathbb{K})]_1$ be any pure contraction obeying $\text{Ker}(\delta)^\perp \subseteq \text{supp}(b)$. If $b' := b|_{\text{supp}(b)}$ and $\delta' := \delta|_{\text{supp}(b)}$, then $\text{supp}(b) = \text{supp}(b^{(\delta)})$ and $b^{(\delta)}|_{\text{supp}(b)} = (b')^{(\delta')}$.\n
Proof. Let $\mathcal{H} := \text{supp}(b)$ Writing elements of $\mathcal{J} = \mathcal{H} \oplus (\mathcal{J} \ominus \mathcal{H})$ as two-component column vectors, let $\alpha := \delta|_{\mathcal{J}}$ and $a := b|_{\mathcal{J}}$. Recall that $b^\delta(z) = D_{\mathcal{J}^\ast}^{-1}((b(z) + \delta)(I_\mathcal{J} + \delta b(z))^{-1}D_\delta$. Writing

$$\delta = \begin{bmatrix} \alpha, & 0_{\mathcal{J} \ominus \mathcal{H}, \mathbb{K}} \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} a, & 0_{\mathcal{J} \ominus \mathcal{H}, \mathbb{K}} \end{bmatrix},$$

it is easy to check that

$$b^{(\delta)}(z)P_{\mathcal{J} \oplus \mathcal{H}} = D_{\mathcal{J}^\ast}^{-1} \begin{bmatrix} a(z) + \alpha, & 0_{\mathcal{J} \ominus \mathcal{H}, \mathbb{K}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{J} \ominus \mathcal{H}} \end{bmatrix} = 0,$$

proving that $\text{supp}(b^{(\delta)}) \subseteq \mathcal{H} = \text{supp}(b)$. The remaining assertions are similarly easy to verify. □

If $V$ is a QE row partial isometry, and $(\gamma, \delta, \mathbb{K})$ is any model triple for $V$, $b := (b(V))^\delta|_{\text{supp}(b(V))}$ is quasi-extreme. Given any purely contractive $\delta \in [\mathcal{L}(\mathcal{J}, \mathbb{K})]_1$, we can define, as in Lemma 4.8, the CCNC row contraction $T = T_\delta := V - \gamma(0)\delta\gamma(\infty)^*$, which, by definition, has the characteristic function $b_T^\gamma = (b(V)^\delta)$. Under the assumption that $\text{Ker}(\delta)^\perp \subseteq \text{supp}(b(V)^\delta)$, the above lemma proves that $b_T^\gamma$ coincides weakly with a Frostman shift of the quasi-extreme Schur class function $b$, so that $T$ is also QE.

If, however, $\text{Ker}(\delta)^\perp$ is not contained in $\text{supp}(b(V)^\delta)$, $T_\delta$ can fail to be QE. That is, as the following simple example shows, there exist CCNC row contractions $T$ with partial isometric part $V$ such that $V$ is QE but $T$ is not.
Example 5.14. Let \( b \in \mathcal{A}(\mathfrak{H}) \) be any purely contractive quasi-extreme multiplier and set

\[
B := \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{A}(\mathfrak{H} \oplus \mathbb{C}), \quad \delta := \begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix}; \quad 0 < r < 1.
\]

Then,

\[
B^{(\delta)}(z) = D_\delta^{-1}(B(z) + \delta)(I + \delta^*B(z))^{-1}D_\delta
= \begin{bmatrix} b(z) & 0 \\ 0 & r \end{bmatrix},
\]

which cannot be quasi-extreme since \( 0 < r < 1 \).

6. Outlook

Motivated by the characterization of CNC row contractions in Section 2, given any CNC row partial isometry, \( V \), on \( \mathfrak{H} \), it is natural to extend our definition of model triple and model map to the non-commutative setting of non-commutative function theory \[32, 44\].

Namely, recall that the non-commutative (NC) open unit ball is the disjoint union \( \Omega := \biguplus_{n=1}^\infty \Omega_n \), where \( \Omega_n := (\mathbb{C}^{n \times n} \otimes \mathbb{C}_d)^1 \), is viewed as the set of all strict row contractions (with \( d \) component operators) on \( \mathbb{C}^n \), and \( \Omega_1 \simeq \mathbb{B}_d \).

A natural extension of our concept of model triple to the NC unit ball, \( \Omega \), would be a triple \((\gamma, J_\infty, J_0)\) consisting of two Hilbert spaces \( J_\infty \simeq \text{Ker}(V) \), \( J_0 \simeq \text{Ran}(V)^\perp \), and a map \( \gamma \) on \( \Omega \cup \{\infty\} \),

\[
\gamma : \begin{cases} 
Z \in \Omega & \mapsto \gamma(Z) \in \mathcal{L}(J_0, \mathcal{R}(V-Z)^\perp) \\
\{\infty\} & \mapsto \gamma(\infty) \in \mathcal{L}(J_\infty, \text{Ker}(V))
\end{cases}
\]

where \( \gamma(Z) \) is an isomorphism for each \( Z \in \Omega \) and \( \gamma(0_n), \gamma(\infty) \) are onto isometries. We will call such a model map \( \gamma \) a non-commutative (NC) model map. In particular, as in Section 3, if \( T \supseteq V \) is any contractive extension of \( V \), and \( \Gamma_T(0) : J_0 \to \text{Ran}(V)^\perp \), \( \Gamma_T(\infty) : J_\infty \to \text{Ker}(V) \) are any fixed onto isometries, then

\[
\Gamma_T(Z) := (I - TZ^*)^{-1}(\Gamma_T(0) \otimes I_n); \quad Z \in \Omega_n, \quad TZ^* := T_1 \otimes Z_1^* + ... + T_d \otimes Z_d^*;
\]

defines an analytic NC model map for \( V \) (we expect \( \Gamma_T(Z) \) will be anti-analytic in the sense of non-commutative function theory \[32, \text{Section 7]\}). Moreover, as in Section 3 for any analytic NC model map \( \Gamma \), we expect that one can then define an abstract model space, \( \hat{\mathcal{H}}^\Gamma \) with non-commutative reproducing kernel

\[
\hat{K}^\Gamma(Z, W) = \Gamma(Z)^*\Gamma(W); \quad Z, W \in \Omega_n,
\]

and that this will be a non-commutative reproducing kernel Hilbert space (NC-RKHS) in the sense of \[44, 45\]. If this analogy continues to hold, it would be natural to use \( \Gamma \) to define a NC characteristic function, \( B_T(Z) \), on \( \Omega \), and one would expect this to be an element of the free (left or right) Schur class of contractive NC multipliers between vector-valued Fock spaces over \( \mathbb{C}^d \) \[16, 17\]. Ultimately, it would
be interesting to investigate whether such an extended theory will yield a NC deBranges-Rovnyak model for CNC row contractions as (adjoints of) the restriction of the adjoint of the left or right free shift on (vector-valued) full Fock space over $\mathbb{C}^d$ to the right or left non-commutative deBranges-Rovnyak spaces, $\mathcal{H}^L(B_T)$ or $\mathcal{H}^R(B_T)$ [16, 14, 13].

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