Codegree Threshold for Tiling $k$-graphs with Two Edges Sharing Exactly $\ell$ Vertices

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Abstract  Given integer $k$ and a $k$-graph $F$, let $t_{k-1}(n, F)$ be the minimum integer $t$ such that every $k$-graph $H$ on $n$ vertices with codegree at least $t$ contains an $F$-factor. For integers $k \geq 3$ and $0 \leq \ell \leq k-1$, let $\mathcal{Y}_{k,\ell}$ be a $k$-graph with two edges that shares exactly $\ell$ vertices. Han and Zhao (J. Combin. Theory Ser. A, (2015)) asked the following question: For all $k \geq 3$, $0 \leq \ell \leq k-1$ and sufficiently large $n$ divisible by $2k-\ell$, determine the exact value of $t_{k-1}(n, \mathcal{Y}_{k,\ell})$. In this paper, we show that $t_{k-1}(n, \mathcal{Y}_{k,\ell}) = \frac{n}{2k-\ell}$ for $k \geq 3$ and $1 \leq \ell \leq k-2$, combining with two previously known results of Rödl, Ruciński and Szemerédi (J. Combin. Theory Ser. A, (2009)) and Gao, Han and Zhao (Combinatorics, Probability and Computing, (2019)), the question of Han and Zhao is solved completely.

Keywords  Hypergraph, codegree, factor

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1 Introduction

Given $k \geq 2$, a $k$-uniform hypergraph ($k$-graph for short) consists of a vertex set $V$ and an edge set $E \subseteq \binom{V}{k}$, where $\binom{V}{k}$ is the set of all $k$-element subsets of $V$. Let $H$ be a $k$-graph and let $S \subset V(H)$ with $|S| = d$ ($1 \leq d \leq k-1$). The degree of $S$, denoted by $\deg_H(S)$, is the number of edges containing $S$ (the subscript $H$ will be omitted if it is clear from the context). The minimum $d$-degree $\delta_d(H)$ of $H$ is the minimum of $\deg_H(S)$ over all $d$-element vertex sets $S$ in $H$. We refer to $\delta_1(H)$ and $\delta_{k-1}(H)$ as the minimum degree and codegree of $H$, respectively.

Given two hypergraphs $H$ and $F$, an $F$-tiling in $H$ is a collection of vertex-disjoint copies of $F$ in $H$. An $F$-tiling is called perfect if it covers all the vertices of $H$. Perfect $F$-tilings are also referred to as $F$-factors. Given a $k$-graph $F$ and integer $n$ divisible by $|F|$, let $t_{k-1}(n, F)$ be the minimum integer $t$ such that every $k$-graph $H$ on $n$ vertices with $\delta_{k-1}(H) \geq t$ contains an $F$-factor, we also call $t_{k-1}(n, F)$ the codegree threshold of $F$.

Given $k \geq 3$ and $0 \leq \ell \leq k-1$, let $\mathcal{Y}_{k,\ell}$ be a $k$-graph with two edges that shares exactly $\ell$ vertices. In this paper, we mainly concern the codegree threshold of $\mathcal{Y}_{k,\ell}$. Some special cases of $t_{k-1}(n, \mathcal{Y}_{k,\ell})$ have been obtained in literatures. Rödl, Ruciński and Szemerédi [8] determined...
the exact value of $t_{k-1}(n, \mathcal{Y}_{k,\ell})$ when $\ell = 0$, more precisely, they proved that for all $k \geq 3$ and sufficiently large $n$ divisible by $2k$, \begin{equation}
abla t_{k-1}(n, \mathcal{Y}_{k,0}) = \frac{n}{2} - k + c, \tag{1.1} \end{equation}
where $c \in \{2,3\}$; for $k = 3$ and $\ell = 2$, Kühn and Osthus [6] showed that $t_2(n, \mathcal{Y}_{3,2}) = n/4 + o(n)$, the exact value of $t_2(n, \mathcal{Y}_{3,2})$ was given by Czygrinow, DeBiasio and Nagle [1]; as a generalization, Gao, Han and Zhao [3] determined the exact value of $t_{k-1}(n, \mathcal{Y}_{k,\ell})$ for all $k \geq 3$ and $\ell = k - 1$, i.e. they proved that for any $k \geq 3$ and sufficiently large $n$ divisible by $k + 1$, \begin{equation}
abla t_{k-1}(n, \mathcal{Y}_{k,k-1}) = \frac{n}{k+1} + c, \tag{1.2} \end{equation}
where $c \in \{0,1\}$; for general $k \geq 3$ and $1 \leq \ell \leq k - 1$, by a result on tiling $k$-partite $k$-graphs given by Mycroft [7], we have $t_{k-1}(n, \mathcal{Y}_{k,\ell}) = \frac{n}{2k-\ell} + o(n)$, in [4], Han and Zhao constructed an extremal graph for $\mathcal{Y}_{k,\ell}$, which yields that $t_{k-1}(n, \mathcal{Y}_{k,\ell}) > \frac{n}{2k-\ell} - 1$, and in the same paper, the authors asked the following question.

**Question 1.1** ([4]) For all $k \geq 3$, $0 \leq \ell \leq k - 1$ and sufficiently large $n$ divisible by $2k - \ell$, determine the exact value of $t_{k-1}(n, \mathcal{Y}_{k,\ell})$.

In this paper, we give the exact value of $t_{k-1}(n, \mathcal{Y}_{k,\ell})$ for $k \geq 3$, $1 \leq \ell \leq k - 2$ and sufficiently large $n$, combining with (1.1) and (1.2), Question 1.1 is answered completely.

**Theorem 1.2** For all $k \geq 3$, $1 \leq \ell \leq k - 2$ and sufficient large $n$ divisible by $2k - \ell$, \begin{equation}
abla t_{k-1}(n, \mathcal{Y}_{k,\ell}) = \frac{n}{2k - \ell}. \end{equation}

**Construction 1.3** (Extremal graph, [4]) Let $H_0$ be a $k$-graph on $n \in (2k - \ell)\mathbb{N}$ vertices such that $V(H_0) = A \cup B$ with $|A| = \frac{n}{2k-\ell} - 1$, and $E(H_0)$ consists of all $k$-subsets of $A \cup B$ intersecting $A$ and some $k$-subsets of $B$ such that $H_0[B]$ contains no copy of $\mathcal{Y}_{k,\ell}$.

Clearly, $\delta_{k-1}(H_0) \geq \frac{n}{2k-\ell} - 1$. Since every copy of $\mathcal{Y}_{k,\ell}$ contains at least one vertex in $A$, there is no $\mathcal{Y}_{k,\ell}$-factor in $H_0$.

The proof of Theorem 1.2 follows the clue given by Han and Zhao in [4], that is we use the standard “absorbing method”, which has been widely used in study of tiling problems (see for example [1, 3–5, 8]). As pointed by Han and Zhao in [4], to determine the exact value of $t_{k-1}(n, \mathcal{Y}_{k,\ell})$, it suffices to prove an absorbing lemma and the extremal case. Fortunately, the absorbing lemma given in [3] does work here and so our main contribution in this paper is to deal with the extremal case.

We give more definitions and notation which will be used in the paper. Let $H$ be a $k$-graph, write $e(H)$ or $|H|$ for the size of $E(H)$. For a set $A \subseteq V(H)$, let $H[A]$ be the subgraph induced by $A$ and denote $e_H(A) = |H[A]|$, and $\overline{e}_H(A) = (|V(A)| - e_H(A))$. The subscript will be omitted if the underlying hypergraph is clear from the context. For two vertex sets $S, R \subseteq V(H)$ with $|S| < k$, let $N_H(S, R) = \{ T : T \subseteq R \text{ such that } S \cap T = \emptyset \text{ and } S \cup T \in E(H) \}$ and $\deg_H(S, R) = |N_H(S, R)|$. Define $\overline{\deg}_H(S, R) = (|R\setminus S| - \deg_H(S, R))$, the number of non-edges in $S \cup R$ that contain $S$. By the definitions here, $\deg_H(S) = \deg_H(S, V(H))$ and $\overline{\deg}_H(S) = \overline{\deg}_H(S, V(H))$. If $S = \{ v \}$, write $\deg_H(v, R)$ and $\overline{\deg}_H(v, R)$ for $\deg_H(\{ v \}, R)$ and $\overline{\deg}_H(\{ v \}, R)$, respectively. We say $H$ is $\xi$-extremal if there exists a set $B \subseteq V(H)$ of size
(1 - \frac{1}{2k-\ell})n such that \(e_H(B) \leq \xi \binom{|B|}{k}\). In the paper, for constants \(\alpha, \beta\), \(\alpha \ll \beta\) means \(\alpha\) is small enough compared to \(\beta\).

The rest of the paper is arranged as follows. In Section 2, we give lemmas and the proof of Theorem 1.2. The extremal case lemma will be proved in Section 3.

2 Proof of Theorem 1.2

To cope with the non-extremal case, we need an absorbing lemma and an almost tiling lemma for \(\mathcal{Y}_{k,\ell}\). In [3], Gao, Han and Zhao gave an absorbing lemma (Lemma 3.1) for general complete \(k\)-partite \(k\)-graphs, as a special case, we have the absorbing lemma for \(\mathcal{Y}_{k,\ell}\).

**Lemma 2.1** (Absorbing Lemma) Let \(k \geq 3\), \(1 \leq \ell \leq k - 2\), suppose \(0 < \alpha \ll \gamma \ll \frac{1}{2k-\ell}\) and \(n\) is sufficiently large. If \(H\) is an \(n\)-vertex \(k\)-graph such that \(\delta_{k-1}(H) \geq \frac{n}{2k-\ell}\), then there exists a vertex set \(W \subseteq V(H)\) with \(|W| \leq \gamma n\) and \(|W| \in (2k - \ell)\mathbb{N}\) such that for any vertex set \(U \subseteq V(H)\backslash W\) with \(|U| \leq \alpha n\) and \(|U| \in (2k - \ell)\mathbb{N}\), both \(H[W]\) and \(H[U \cup W]\) contain \(\mathcal{Y}_{k,\ell}\)-factors.

The almost tiling lemma used here also is a special case of the \(\mathcal{Y}_{k,\ell}\)-tiling lemma (Lemma 2.8) given by Han and Zhao in [4] and a special case of the almost tiling lemma for general \(k\)-partite \(k\)-graphs (Lemma 3.2) given by Gao, Han and Zhao in [3].

**Lemma 2.2** (Almost tiling Lemma) Let \(k \geq 3\), \(1 \leq \ell \leq k - 2\), for any \(\alpha, \gamma, \xi > 0\) such that \(\gamma \ll \xi\), there exists an integer \(n_0\) such that the following holds. If \(H\) is a \(k\)-graph on \(n > n_0\) vertices with \(\delta_{k-1}(H) \geq \left(\frac{1}{2k-\ell} - \gamma\right)n\), then \(H\) has a \(\mathcal{Y}_{k,\ell}\)-tiling that covers all but at most \(\alpha n\) vertices unless \(H\) is \(\xi\)-extremal.

Our contribution in the proof of Theorem 1.2 is to give the lemma of extremal case for \(\mathcal{Y}_{k,\ell}\). The proof will be given in the next section.

**Lemma 2.3** (Extremal case) Given \(k \geq 3\), \(1 \leq \ell \leq k - 2\), \(0 < \xi \ll \frac{1}{2k-\ell}\) and let \(n \in (2k - \ell)\mathbb{N}\) be sufficiently large. Suppose \(H\) is a \(k\)-graph on \(n\) vertices with \(\delta_{k-1}(H) \geq \frac{n}{2k-\ell}\). If \(H\) is \(\xi\)-extremal, then \(H\) contains a \(\mathcal{Y}_{k,\ell}\)-factor.

Now we give the proof of Theorem 1.2.

**Proof of Theorem 1.2** Let \(k \geq 3\), \(1 \leq \ell \leq k - 2\) and let \(n \in (2k - \ell)\mathbb{N}\) be sufficiently large. Choose \(\alpha\) and \(\gamma\) small enough such that \(0 < \alpha \ll \gamma \ll \xi \ll \frac{1}{2k-\ell}\). Suppose \(H\) is an \(n\)-vertex \(k\)-graph satisfying \(\delta_{k-1}(H) \geq \frac{n}{2k-\ell}\). If \(H\) is \(\xi\)-extremal, then \(H\) contains a \(\mathcal{Y}_{k,\ell}\)-factor by Lemma 2.3. Otherwise, we assume that \(H\) is not \(\xi\)-extremal, by Lemma 2.1, we find an absorbing set \(W\) in \(V(H)\) of size at most \(\gamma n\) which has the absorbing property. Let \(H' := H - W\) and \(n' = |V(H')| \geq (1 - \gamma)n\). If \(H'\) is \(\frac{\xi}{2}\)-extremal, then there exists a \(B' \subseteq V(H')\) of order \((1 - \frac{1}{2k-\ell})n'\) such that \(e_{H'}(B') \leq \frac{\xi}{2} \binom{|B'|}{k}\). Thus by adding to \(B'\) at most \(n - n' \leq \gamma n\) vertices, we get a set \(B\) of size precisely \((1 - \frac{1}{2k-\ell})n\) in \(V(H)\) with

\[
e_{H}(B) \leq e_{H'}(B') + \gamma n \binom{n - 1}{k - 1} \leq \frac{\xi}{2} \binom{|B'|}{k} + k\gamma \binom{n}{k} \leq \xi \binom{|B'|}{k},
\]

a contradiction to the assumption that \(H\) is not \(\xi\)-extremal. So we assume that \(H'\) is not \(\frac{\xi}{2}\)-extremal. Since

\[
\delta_{k-1}(H') \geq \frac{n}{2k-\ell} - \gamma n \geq \left(\frac{1}{2k-\ell} - \gamma\right)n',
\]
applying Lemma 2.2 on \( H' \) with \( \frac{5}{3} \), we obtain a \( Y_{k, \ell} \)-tiling \( Y \) that covers all but a set \( U \) of at most \( n \) vertices. Since both \( n \) and \( |W| \) are divisible by \( 2k - \ell \), \( |U| \in (2k - \ell)N \). By the absorbing property of \( W \), \( H[W \cup U] \) contains a \( Y_{k, \ell} \)-factor and together with the \( Y_{k, \ell} \)-tiling \( Y \) we obtain a \( Y_{k, \ell} \)-factor of \( H \). \( \square \)

3 Proof of Lemma 2.3

We need more definitions and notation in the proof. Given two disjoint sets \( X, Y \) and two integers \( i, j \geq 0 \), a set \( S \subseteq X \cup Y \) is called of type \( X^iY^j \) if \( |S \cap X| = i \) and \( |S \cap Y| = j \). If \( X \) and \( Y \) are two disjoint vertex subsets of a \( k \)-graph \( H \) and \( i + j = k \), denote by \( H(X^iY^j) \) the subgraph induced by all edges of type \( X^iY^j \) in \( H \) and let \( e_H(X^iY^j) = |H(X^iY^j)| \) and \( \overline{e}_H(X^iY^j) = \binom{|X|}{i}\binom{|Y|}{j} - e_H(X^iY^j) \) (the subscript may be omitted if it is clear from the context).

Given a set \( L \subseteq X \cup Y \) with \( |L \cap X| = l_1 \leq i \) and \( |L \cap Y| = l_2 \leq k - i \), define \( \deg(L, X^iY^{k-i}) \) be the degree of \( L \) in \( H(X^iY^{k-i}) \) and \( \deg(L, X^iY^{k-i}) = \binom{|X|}{i}\binom{|Y|}{k-i} - |\overline{e}_H(X^iY^{k-i})| - \deg(L, X^iY^{k-i}) \).

Given two \( k \)-graphs \( F \) and \( H \), we call \( H \) \( F \)-free if \( H \) does not contain \( F \) as a subgraph. The well-known Turán number \( ex(n, F) \) is the maximum number of edges in an \( F \)-free \( k \)-graph on \( n \) vertices. The following result was given by Frankl and Füredi [2].

**Lemma 3.1** ([2]) For \( k \geq 2, 0 \leq \ell \leq k - 1 \), there exists a constant \( d_k \) depending only on \( k \) such that \( ex(n, Y_{k, \ell}) \leq d_k n^{\max(\ell, k-\ell)} \).

The following lemma also is a special version of a result [3, Lemma 6.1] given by Gao, Han and Zhao.

**Lemma 3.2** ([3]) Given \( k \geq 3, 1 \leq \ell \leq k - 2 \). Let \( 0 < \rho < \frac{1}{2k-\ell} \) and let \( n \) be sufficiently large. Suppose \( H \) is a \( k \)-graph on \( n \in (2k - \ell)N \) vertices with a partition of \( V(H) = X \cup Y \) such that \( |Y| = (2k - \ell - 1)|X| \). Furthermore, assume that

(a) for every vertex \( v \in X \), \( \deg(v, Y) \leq \rho \binom{|Y|}{k-1} \),

(b) for every vertex \( u \in Y \), \( \deg(u, XY^{k-1}) \leq \rho \binom{|Y|}{k-1} \).

Then \( H \) contains a \( Y_{k, \ell} \)-factor.

**Proof of Lemma 2.3** Since \( H \) is \( \xi \)-extremal, there is a set \( B \subseteq V(H) \) such that \( |B| = (1 - \frac{1}{2k-\ell})n \) and \( e(B) \leq \xi \binom{|B|}{k} \). Let \( A = V(H) \setminus B \). Then \( |A| = \frac{n}{2k-\ell} \). Let \( \epsilon_1 \leq \xi \frac{1}{k} \), \( \epsilon_2 = 2\epsilon_1^2 = 2\xi \frac{1}{k} \). Define

\[
A' := \left\{ v \in V(H) : \deg(v, B) \geq (1 - \epsilon_1) \binom{|B|}{k-1} \right\},
\]

\[
B' := \left\{ v \in V(H) : \deg(v, B) \leq \epsilon_1 \binom{|B|}{k-1} \right\},
\]

and \( V_0 := V(H) \setminus (A' \cup B') \).

**Claim 3.3** Max\( \{ |A \setminus A'|, |B \setminus B'|, |A' \setminus A|, |B' \setminus B| \} \leq \epsilon_2 |B| \) and \( |V_0| \leq 2\epsilon_2 |B| \).

**Proof of Claim 3.3** First assume that \( |B \setminus B'| > \epsilon_2 |B| \). By the definition of \( B' \), we have

\[
e(B) > \frac{1}{k} \epsilon_2 |B| \cdot \epsilon_1 \binom{|B|}{k-1} > 2\xi \binom{|B|}{k},
\]

a contradiction to \( e(B) \leq \xi \binom{|B|}{k} \).
Second, assume that \( |A \setminus A'| > \epsilon_2 |B| \). By the definition of \( A' \), for any vertex \( v \notin A' \), we have \( \overline{\deg}(v, B) > \epsilon_1(\binom{|B|}{k-1}) \). So

\[
\overline{\deg}(AB^{k-1}) > \epsilon_2 |B| \cdot \epsilon_1 \left( \frac{|B|}{k-1} \right) = 2\xi |B| \left( \frac{|B|}{k-1} \right).
\]

Together with \( e(B) \leq \xi(\binom{|B|}{k}) \), we have

\[
\sum_{S \in \binom{B}{k-1}} \overline{\deg}(S) = k \cdot \overline{\deg}(B) + \overline{\deg}(AB^{k-1})
\]

\[
> k(1 - \xi) \left( \frac{|B|}{k} \right) + 2\xi |B| \left( \frac{|B|}{k-1} \right)
\]

\[
= (1 - \xi)(|B| - k + 1) + 2\xi |B| \left( \frac{|B|}{k-1} \right)
\]

\[
> |B| \left( \frac{|B|}{k-1} \right),
\]

where the last inequality holds because \( n \) is sufficiently large. By the pigeonhole principle, there exists a set \( S \in \binom{B}{k-1} \), such that \( \overline{\deg}(S) > |B| = (1 - \frac{1}{2k-\ell})n \), a contradiction to \( \delta_{k-1}(H) \geq \frac{n}{2k-\ell} \).

Consequently,

\[
|A' \setminus A| = |A' \cap B| \leq |B \setminus B'| \leq \epsilon_2 |B|,
\]

\[
|B' \setminus B| = |A \cap B'| \leq |A \setminus A'| \leq \epsilon_2 |B|,
\]

\[
|V_0| \leq |A \setminus A'| + |B \setminus B'| \leq \epsilon_2 |B| + \epsilon_2 |B| \leq 2\epsilon_2 |B|.
\]

By \( |B \setminus B'| \leq \epsilon_2 |B| \) and \( |B' \setminus B| \leq \epsilon_2 |B| \), for any vertex \( v \in V_0 \), we have

\[
\deg(v, B') \geq \deg(v, B) - |B \setminus B'| \left( \frac{|B|}{k-2} \right) \geq \frac{\epsilon_1}{2} \left( \frac{|B'|}{k-1} \right), \quad (3.1)
\]

for any vertex \( v \in A' \),

\[
\overline{\deg}(v, B') \leq \overline{\deg}(v, B) + |B' \setminus B| \left( \frac{|B'|}{k-2} \right) \leq 2\epsilon_1 \left( \frac{|B'|}{k-1} \right), \quad (3.2)
\]

and for any vertex \( v \in B' \),

\[
\deg(v, B') \leq \deg(v, B) + |B' \setminus B| \left( \frac{|B'|}{k-2} \right) \leq 2\epsilon_1 \left( \frac{|B'|}{k-1} \right). \quad (3.3)
\]

Moreover, for any \( (k-1) \)-set \( S \subseteq B' \), since \( \deg(S, A') + \deg(S, B') + \deg(S, V_0) \geq \delta_{k-1}(H) \) and \( \overline{\deg}(S, A') = |A'| - \deg(S, A') \), we have

\[
\overline{\deg}(S, A') \leq |A'| - \delta_{k-1}(H) + \deg(S, B') + \deg(S, V_0) \leq \deg(S, B') + 3\epsilon_2 |B|,
\]

where the last inequality holds since \( \deg(S, V_0) \leq |V_0| \leq 2\epsilon_2 |B| \), \( |A'| \leq \frac{n}{2k-\ell} + \epsilon_2 |B| \) and \( \delta_{k-1}(H) \geq \frac{n}{2k-\ell} \). Furthermore, by (3.3), for any \( v \in B' \), we have

\[
\sum_{S \in \binom{B'}{k-1}} \deg(S, B') = (k-1) \deg(v, B') \leq 2(k-1)\epsilon_1 \left( \frac{|B'|}{k-1} \right).
\]

Putting this together gives that for any \( v \in B' \),

\[
\overline{\deg}(v, A'(B')^{k-1}) = \sum_{S \in \binom{B'}{k-1}} \overline{\deg}(S, A')
\]
\[
\sum_{S: v \in S \setminus \bigcup_{i=1}^{k-1} Y_i} \deg(S, B') + 3\epsilon_2 |B| \left(\frac{|B'|}{k-2}\right) \\
\leq 2k\epsilon_1 \left(\frac{|B'|}{k-1}\right).
\]

(3.4)

So, if \(|B'| = (2k - \ell - 1)|A'|\) and \(|V_0| = 0\), then applying Lemma 3.2 we obtain a \(\mathcal{Y}_{k,\ell}\)-factor of \(H\).

Now we assume \(|B'| \neq (2k - \ell - 1)|A'|\) or \(|V_0| \neq 0\). Let \(q := |B'| - |B| = \frac{n}{2k-\ell} - |A'| - |V_0|\). Then \(-\epsilon_2 |B| \leq q \leq \epsilon_2 |B|\).

The first step, we find \(q\) vertex-disjoint copies of \(\mathcal{Y}_{k,\ell}\) in \(H[B']\) when \(q > 0\). We claim that we can greedily construct \(q\) vertex-disjoint copies of \(\mathcal{Y}_{k,\ell}\) in \(H[B']\). In fact, suppose that we have found \(i\) copies of \(\mathcal{Y}_{k,\ell}\) for some \(0 \leq i < q\) and let \(U\) be the set of the vertices of \(B'\) covered by these \(i\) copies of \(\mathcal{Y}_{k,\ell}\). Then \(|U| \leq (2k - \ell)(q - 1)|\). Since by (3.3), \(\deg(v, B') \leq 2\epsilon_1(|B'|)\) for any vertex \(v \in B'\) and \(\delta_{k-1}(H[B']) \geq q\),

\[\epsilon(B' \setminus U) \geq \frac{q}{k} \left|\frac{|B'|}{k-1}\right| - (2k - \ell)(q - 1)2\epsilon_1 \left(\frac{|B'|}{k-1}\right) > \epsilon_k (|B'| - |U|, \mathcal{Y}_{k,\ell}),\]

where the last inequality holds because \(n\) is sufficiently large and \(\epsilon_1\) is small enough. By Lemma 3.1, we can find a copy of \(\mathcal{Y}_{k,\ell}\) avoiding \(U\). The claim holds. Set \(Y_1\) be the \(q\) vertex-disjoint copies of \(\mathcal{Y}_{k,\ell}\) in \(H[B']\). If \(q \leq 0\), set \(Y_1 := \emptyset\).

The next step, we choose a \(\mathcal{Y}_{k,\ell}\)-tiling \(Y_2\) such that each copy of \(\mathcal{Y}_{k,\ell}\) contains one vertex in \(V_0\) and \(2k - \ell - 1\) vertices in \(B'\). Let \(V_0 = \{w_1, \ldots, w_{|V_0|}\}\). We claim that, for each \(w_i\), we can find a copy of \(\mathcal{Y}_{k-1,\ell-1}\) in the \((k-1)\)-graph \(N(w_i, B')\) such that these \(|V_0|\) copies of \(\mathcal{Y}_{k-1,\ell-1}\) are vertex disjoint and are also vertex disjoint from \(Y_1\). This is possible because the total number of vertices in \(B'\) that we need to avoid is at most

\[|V(Y_1)| + (2k - \ell - 1)|V_0| \leq \epsilon_2 |B|(2k - \ell) + (2k - \ell - 1)2\epsilon_2 |B| \leq 3(2k - \ell)\epsilon_2 |B|,\]

and so by (3.1), we have

\[|N(w_i, B')| - 3(2k - \ell)\epsilon_2 |B| \left(\frac{|B'|}{k-2}\right) \geq \epsilon_1 \left(\frac{|B'|}{k-1}\right)\]

By Lemma 3.1, \(N(w_i, B')\) contains a desired \(\mathcal{Y}_{k-1,\ell-1}\). Note that the copy of \(\mathcal{Y}_{k-1,\ell-1}\) union \(\{w_i\}\) spans a copy of \(\mathcal{Y}_{k,\ell}\) in \(H\). Therefore, the \(|V_0|\) copies of \(\mathcal{Y}_{k,\ell}\) form the desired \(Y_2\).

Now reset \(B_1\) to the set of vertices in \(B'\) not covered by \(Y_1 \cup Y_2\), \(A_1 = A'\) and \(V_1 = A_1 \cup B_1\). The third step to the \(\mathcal{Y}_{k,\ell}\)-tiling \(Y_3\) to adjust the sizes of \(A_1\) and \(B_1\) such that \(|B_1 \setminus V(Y_3)| = (2k - \ell - 1)|A_1 \setminus V(Y_3)|\). Let \(p = \frac{n}{2k-\ell} - |V_1| - |A_1|\). Note that \(|Y_1| = q\) if \(q > 0\) and 0 otherwise, \(|Y_2| = |V_0|\) and \(|V_1| = n - (2k - \ell)(|Y_1| + |V_0|)\). We have

\[p = \frac{n}{2k - \ell} - |Y_1| - |V_0| - |A_1| = q - |Y_1|\]

If \(q > 0\), then \(p = 0\). Thus \(|B_1| = |V_1| - |A_1| = (2k - \ell - 1)|A_1|\). Therefore, we choose \(Y_3 = \emptyset\) in this case. Now assume \(q \leq 0\). Then \(Y_1 = \emptyset\) and so \(p = q \geq -\epsilon_2 |B|\). We claim that we can pick \(-p\) vertex disjoint copies of \(\mathcal{Y}_{k,\ell}\) such that each of them contains two vertices in \(A_1\) and \(2k - \ell - 2\) vertices in \(B_1\). In fact, for any pair \(\{u_i, v_i\} \subseteq A_1\) \((i \leq -p)\), we show that we can find a copy of \(\mathcal{Y}_{k-1,\ell}\) in the \((k-1)\)-graph \(N(u_i, B_1) \cap N(v_i, B_1)\) such that these \(-p\) copies of \(\mathcal{Y}_{k-1,\ell}\)
are vertex disjoint. since

\[ |B_1| \geq |B'| - |V(Y_2)| \geq |B'| - 2\varepsilon_2|B|(2k - \ell) > (1 - \varepsilon_1)|B'|. \]

We have, by (3.2), for any \( v \in A_1, \)

\[ \overline{\deg}(v, B_1) \leq \deg(v, B') \leq 2\varepsilon_1 \left( \frac{|B'|}{k - 1} \right) < 3\varepsilon_1 \left( \frac{|B_1|}{k - 1} \right). \]

Thus

\[ |N(u_i, B_1) \cap N(v_i, B_1)| \geq (1 - 6\varepsilon_1) \left( \frac{|B_1|}{k - 1} \right). \]

Since the total number of vertices in \( B_1 \) that we need to avoid is at most \( (2k - \ell - 2)(-p) \leq 2k\varepsilon_2|B|, \) we have

\[ |N(u_i, B_1) \cap N(v_i, B_1)| - 2k\varepsilon_2|B| \left( \frac{|B_1|}{k - 2} \right) \geq \frac{1}{2} \left( \frac{|B_1|}{k - 1} \right). \]

By Lemma 3.1, we can greedily find \(-p\) desired copies of \( Y_{k-1, \ell}. \) Note that each copy of \( Y_{k-1, \ell} \) union its corresponding pair \( \{u_i, v_i\} \) spans a copy of \( Y_{k, \ell}. \) These \(-p\) copies of \( Y_{k, \ell} \) form the claimed \( Y_{k, \ell}\)-tiling, say \( Y_3. \) Let \( A_2 = A_1 \setminus V(Y_3) \) and \( B_2 = B_1 \setminus V(Y_3). \) Then

\[ |B_2| = |B_1| + (2k - \ell - 2)p = |V_1| - |A_1| + (2k - \ell)p - 2p = (2k - \ell)(p + |A_1|) + (2k - \ell)p - |A_2| = (2k - \ell)(|A_1| + 2p) - |A_2| = (2k - \ell - 1)|A_2|. \]

The last step, we show that \( H[A_2 \cup B_2] \) contains a \( Y_{k, \ell}\)-factor \( Y_4. \) Since \( |Y_3| \leq -p \leq \varepsilon_2|B|, \) we have

\[ |B_2| \geq |B'| - |V(Y_1) \cup V(Y_2) \cup V(Y_3)| \geq |B'| - 4\varepsilon_2|B|(2k - \ell) > (1 - \varepsilon_1)|B'|. \]

Hence, by (3.2), for every \( v \in A_2, \)

\[ \overline{\deg}(v, B_2) \leq \overline{\deg}(v, B') \leq 2\varepsilon_1 \left( \frac{|B'|}{k - 1} \right) \leq 2\varepsilon_1 \left( \frac{1}{k - 1} |B_2| \right) < 3\varepsilon_1 \left( \frac{|B_2|}{k - 1} \right), \]

and by (3.4), for every \( v \in B_2, \)

\[ \overline{\deg}(v, A_2B_2^{k-1}) \leq \overline{\deg}(v, A'(B')^{k-1}) \leq 2k\varepsilon_1 \left( \frac{|B'|}{k - 1} \right) \leq 3k\varepsilon_1 \left( \frac{|B_2|}{k - 1} \right). \]

Apply Lemma 3.2 to \( H[A_2 \cup B_2] \) with \( X = A_2, Y = B_2 \) and \( \rho = 3k\varepsilon_1, \) we get a \( Y_{k, \ell}\)-factor \( Y_4 \) of \( H[A_2 \cup B_2] \).

Finally, the union \( Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \) forms a \( Y_{k, \ell}\)-factor of \( H. \) This concludes the proof of Lemma 2.3. \( \square \)

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