Principal Minor Ideals and Rank Restrictions on their Vanishing Sets

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Abstract

All matrices we consider have entries in a fixed algebraically closed field \( K \). A minor of a square matrix is principal means it is defined by the same row and column indices. We study the ideal generated by size \( t \) principal minors of a generic matrix, and restrict our attention to locally closed subsets of its vanishing set, given by matrices of a fixed rank. The main result is a computation of the dimension of the locally closed set of \( n \times n \) rank \( n - 2 \) matrices whose size \( n - 2 \) principal minors vanish; this set has dimension \( n^2 - n - 4 \).

1. Introduction

Given a generic \( n \times n \) matrix \( X \), and \( K[X] \) the polynomial ring in entries \( x_{ij} \) of \( X \) over some algebraically closed field \( K \), we study the ideals \( \mathfrak{B}_t = \mathfrak{B}_t(X) \), generated by the size \( t \) principal minors of \( X \). Historically, various ideals defined using generic matrices have been of great interest to algebraists – such examples include the determinantal ideals (see [4–6,9,19,23]), due to their connection to invariant theory (as in [2]) and the Pfaffian ideals (see [3, 10–12,18]), whose study is often inspired by the result from [1], as well as their connection to invariant theory. In developing their generalized version of the Principal Minor Theorem, Kodiyalam, Lam, and Swan ([14]) reveal a contrast between the principal minor ideals and the Pfaffian ideals: while the Pfaffian ideals, like the determinantal ideals, satisfy a chain condition according to rank, the principal minor ideals do not. Furthermore, in [24] it is shown that, unlike determinantal ideals and Pfaffian ideals, principal minor ideals are not, in general, Cohen-Macaulay.

Principal minors arise in many other contexts – see, for example, [8,15,16, 21,22]. The most direct study of principal minors is in [24]. There, it is shown the algebraic set \( V(\mathfrak{B}_{n-1}) \) has two components: one given by the determinantal ideal \( I_{n-1} \) and the other given by a height \( n \) ideal, \( Q_{n-1} \), that is the contraction...
to $K[X]$ of the kernel of the map

$$K[X] \left[ \frac{1}{\det X} \right] \to K[X] \left[ \frac{1}{\det X} \right] / \mathfrak{B}_1$$

$$X \to X^{-1}.$$  

When $n = 4$, $\mathfrak{B}_{n-1}$ is reduced and as a consequence, $I_{n-1}$ and $\Omega_{n-1}$ are linked in that case. Identifying the components for $\mathcal{V}(\mathfrak{B}_{n-1})$ relies on another result within that paper, that if $\mathcal{Y}_{n,r,t}$ denotes the locally closed set of $\mathcal{V}(\mathfrak{B}_t)$ consisting of rank $r$ matrices, then for all $n, t$,

$$\mathcal{Y}_{n,n,t} \cong \mathcal{Y}_{n,n,n-t}$$

as schemes.

This paper is organized as follows: Section 2 gives the necessary preliminaries for the remainder of the paper. We focus on the components of $\mathcal{V}(\mathfrak{B}_t)$ by restricting to the locally closed subsets $\mathcal{Y}_{n,r,t}$, consisting of matrices of rank exactly $r$. Our main result, given in Section 3, is a computation of the dimension of $\mathcal{Y}_{n,n-2,n-2}$.

**Theorem** (3, Section 3.3). The locally closed set of $n \times n$ rank $n-2$ matrices in $\text{Spec } K[X]$, whose size $n-2$ principal minors vanish, has dimension $n^2 - 4 - n$.

In studying the components of $\mathcal{Y}_{n,n-2,n-2}$ we define a bundle map $\Theta$ (see Equation (1), Section 2) that reduces the problem to studying pairs of subsets in $\text{Grass}(n-2, n)$. The technique in proving Theorem 3 is as follows: Given a point in the Grassmannian, we encode exactly which of its Plücker coordinates do and do not vanish in a simple graph. Such graphs are called permissible (see Section 3.2). We then define the notion of a permissible subvariety of the Grassmannian, along with its corresponding graph. We prove and then use the properties of permissible graphs to compute the dimension of $\mathcal{Y}_{n,n-2,n-2}$.

In Section 3.4 we suggest a natural extension of the techniques from Section 3.2 to the locally closed sets $\mathcal{Y}_{n,n-3,n-3}$. More generally, the structure of $\mathcal{Y}_{n,t,t}$ turns out to be of great interest in its own right, in fact leading to questions that are NP-hard (see [7] and the result cited therein from [20]), though such questions are beyond the scope of this paper.

Finally, in Section 4 we go through the results of Section 3.3 for the case where $n = 5$.

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2. Preliminaries

As a tool in studying the components of $\mathcal{V}(\mathfrak{B}_t)$, we study the components of the locally closed sets $\mathcal{Y}_{n,r,t} \subset \mathcal{V}(\mathfrak{B}_t)$, the $n \times n$ matrices of rank $r$ whose size $t$ principal minors vanish. As $r$ ranges, the irreducible components of $\mathcal{Y}_{n,r,t}$ cover
$V(\mathcal{B}_t)$, hence so do their closures. There are finitely many such closures; whenever any closed set is a finite union of closed varieties, we can find its irreducible components by making the union irredundant. In other words, the components of $V(\mathcal{B}_t)$ can be found by first finding the closures of all the components of the $\mathcal{Y}_{n,r,t}$, as $r$ varies, and then omitting the ones that are not maximal in the family.

To study $\mathcal{Y}_{n,r,t}$ we observe and use its relationship to Grassmann varieties in the following way: For any matrix $A$, let $\text{col } A$ and $\text{row } A$ denote, respectively, the column space and row space of $A$. The Grassmann variety, or Grassmannian, $\text{Grass}(r,n) \subset \mathbb{P}^{(n^2-n)/2}$, is the projective variety whose points are in bijection with the $r$-dimensional vector spaces of $K^n$. As shorthand, we shall put $G = \text{Grass}(r,n)$ for fixed $r \leq n$. Under the Plücker embedding, write $g = [g_{\{1,\ldots,r\}} : \cdots : g_{\{n-r+1,\ldots,n\}}]$ to denote a point in $G$, so the Plücker coordinates are indexed by sets $i \subset \{1,\ldots,n\}$.

Suppose $\text{col } B \in G$ for some $n \times r$ matrix $B$ of full rank. Letting $B(i)$ denote the submatrix of $B$ consisting of the rows indexed by elements in $i \subset \{1,\ldots,n\}$, there exists such an $i = \{i_1,\ldots,i_r\}$ satisfying $\det B(i) \neq 0$. Thus we may perform row operations on $B$ to get a unique matrix $B'$, such that $\text{col } B' = \text{col } B$ and $B'(i) = I_{r \times r}$, the $r \times r$ identity matrix. We shall call $B'$ the normalized form of $B$, or normalization, with respect to $i$. Analogously, if row $C \in G$ for some $r \times n$ matrix $C$, then we define the normalized form $C'$ of $C$ with respect to a set of column indices, $j = \{j_1,\ldots,j_r\}$, provided the submatrix, $C(j)$ of $C$, consisting of the columns indexed by elements in $j$, is non-singular. The reason for considering both $n \times r$ and $r \times n$ representatives of $\text{Grass}(r,n)$ is motivated by the following important observation:

**Proposition 1.** Let $Z_{n,r} \subset \text{Spec } K[X]$ denote the set of $n \times n$ matrices of rank exactly $r$. Then

\[
\Theta : Z_{n,r} \rightarrow \mathcal{G} \times \mathcal{G}
\]

\[
A \mapsto (\text{col } A, \text{row } A).
\]

is a bundle map whose fibres are each isomorphic to $\text{GL}(r,K)$.

**Proof.** The sets where a specified Plücker coordinate does not vanish give an affine open cover of $\mathcal{G}$, and hence of $\mathcal{G} \times \mathcal{G}$. Explicitly, $\mathcal{G}$ is covered by the open sets

\[
\mathcal{G}_i = \left\{ [\cdots : g_i : \cdots ] \in \mathcal{G} | g_i \neq 0 \right\} \cong A^{(n-r)}.
\]

We will show, for each open set $\mathcal{G}_i \times \mathcal{G}_j$, that the diagram

\[
\Theta^{-1} (\mathcal{G}_i \times \mathcal{G}_j) \xrightarrow{\cong} \mathcal{G}_i \times \mathcal{G}_j \times \text{GL}(r,K)
\]

\[
\downarrow \quad \pi
\]

\[
\mathcal{G}_i \times \mathcal{G}_j \xrightarrow{=} \mathcal{G}_i \times \mathcal{G}_j
\]
commutes, where $\pi$ is the projection map. The preimage of $\mathcal{G}_i \times \mathcal{G}_j$ consists of matrices $A \in \mathbb{Z}_{n,r}$ that factor
\[
A = BC = B' A(i;j) C',
\]
where $B'$ is the normalization with respect to $i$ of the $n \times r$ matrix $B$, $C'$ is the normalization with respect to $j$ of the $r \times n$ matrix $C$, and $A(i;j)$ is the submatrix of $A$ consisting of its $i$-rows and $j$-columns. By uniqueness of the normalizations, given fixed $i, j$, such pairs of matrices $(B', C')$ are in bijection with points in $\mathcal{G}_i \times \mathcal{G}_j$. For any fixed pair $(B', C')$, the set of all possibilities for $A(i;j)$ that satisfy Equation (2) is in bijection with $\text{GL}(r, K)$. The maps are clearly regular.

2.1. Irreducible Components of $\mathcal{Y}_{n,t,t}$

Fix $r = t$ and $\mathcal{G} = \text{Grass}(t, n)$. Suppose we factor $A \in \mathcal{Y}_{n,t,t}$ as in Equation (2). Then we must have $i \neq j$. Furthermore, requiring the $t$ principal minors of $A$ to vanish means, equivalently, the diagonal entries of the exterior power matrix $\wedge^t A$ must vanish. Write
\[
\wedge^t A = (\wedge^t B) \cdot (\wedge^t A(i;j)) \cdot (\wedge^t C).
\]
Each of the factors $\wedge^t B, \wedge^t C$ are, respectively, column and row vectors, while $\wedge^t A(i;j)$ is a (non-zero) scalar.

Up to sign, the Plücker coordinates of the column space of any $n \times t$ matrix of full rank, $t \leq n$, are the coordinates of the exterior product of the columns with respect to the basis $\{e_{i_1} \wedge \cdots \wedge e_{i_t} \mid 1 \leq i_1 < \cdots < i_t \leq n\}$, where for $i \in \{1, \ldots, n\}$, $e_i$ is the standard basis vector in $K^n$ given by the $i$th row of the identity matrix; the analogous statement holds for the row space of any $t \times n$ matrix of full rank. It follows that the principal $t$-minors of a matrix $A \in \mathbb{Z}_{n,t}$ vanish if and only if the component-wise product of $\wedge^t B$ and $\wedge^t C$ is zero.

The inclusion $\mathcal{Y}_{n,t,t} \hookrightarrow \mathbb{Z}_{n,t}$ induces, via Equation (1), a bundle map:
\[
\begin{array}{ccc}
\mathcal{Y}_{n,t,t} & \hookrightarrow & \mathbb{Z}_{n,t} \\
\downarrow & & \downarrow \\
\Theta(\mathcal{Y}_{n,t,t}) & \hookrightarrow & \mathcal{G} \times \mathcal{G}
\end{array}
\]
From (3), let $\mathcal{H} \subseteq \mathcal{G} \times \mathcal{G}$ denote the closed set consisting of pairs $(g, h)$ where for each index $i$, either $g_i$ or $h_i$ vanishes. Then $\mathcal{Y}_{n,t,t} = \Theta^{-1}(\mathcal{H})$, and since $\Theta$ is a bundle map, the components of $\mathcal{Y}_{n,t,t}$ correspond bijectively to the components of $\mathcal{H}$.

To get an irreducible component of $\mathcal{H}$, we must partition the set of indices for the Plücker coordinates into two sets, $\mathcal{I}, \mathcal{J}$. Let $\mathcal{V}(\mathcal{I}), \mathcal{V}(\mathcal{J})$ denote the respective closed subsets of $\mathcal{G}$ defined by the vanishing of Plücker coordinates respectively indexed by $\mathcal{I}, \mathcal{J}$. Each component of $\mathcal{H}$ must be a component of $\mathcal{V}(\mathcal{I}) \times \mathcal{V}(\mathcal{J})$ for some partition $\mathcal{I} \cup \mathcal{J}$ and every component of $\mathcal{V}(\mathcal{I}) \times \mathcal{V}(\mathcal{J})$ arises as the product
of a component of \( V(I) \) and a component of \( V(J) \). Our goal is to consider all such partitions \( I, J \), and then for each component \( C \) of \( V(I) \) and each component \( D \) of \( V(J) \), we shall consider the irreducible set \( C \times D \). The components of \( \mathcal{H} \) are the maximal such sets \( C \times D \), and their inverse images under the bundle map \( \Theta \) give the irreducible components of \( Y_{n,t,t} \).

2.2. Group Actions on \( \mathcal{B}_t \)

In order to justify some of the arguments of preserving generality in what follows, we briefly remark on the group actions on \( \mathcal{B}_t \) which leave it invariant. The most useful is given by the symmetric group of degree \( n \). Suppose \( \tau \) is a size \( n \) permutation matrix, \( \tau^T \) its transpose. The action \( X \mapsto \tau X \tau^T \) performs the same permutation on the rows of \( X \) as it does the columns, hence preserves \( \mathcal{B}_t \). The obvious action given by \( X \mapsto X^T \) preserves \( \mathcal{B}_t \) as well. Finally, \( \mathcal{B}_t \) is unaffected by scalar multiplication, i.e., the action of \( \text{GL}(1,K) \cong K^* \) on each row and each column of \( X \).

3. Principal \((n-2)\)-Minors Case

The problem of understanding all the components of the various sets \( V(I) \), as described in Section 2.1, is known to be extremely hard (as mentioned in [7,20]). However, we shall be able to understand the situation completely when \( r = t = n - 2 \). Since an \((n-2)\)-minor of an \( n \times (n-2) \) matrix is determined by the two rows that are not used we can alternatively describe \( I \) using a collection of 2 element subsets of \( \{1,\ldots,n\} \), and then encode that data into a simple graph as follows: label the vertices \( \{v_1,\ldots,v_n\} \) and draw an edge between two vertices \( v_{i_1}, v_{i_2} \) if and only if \( \{1,\ldots,n\} \setminus \{i_1,i_2\} \) is not in any set in \( I \). We can associate an analogous graph to a set \( J \) of indices corresponding to minors of an \((n-2) \times n \) matrix.

We shall see that a component of a closed set of the form \( V(I) \) can also be encoded into a graph, and we will give a condition on its graph that is equivalent to irreducibility. We will then classify the minimal pairs of graphs that together cover the complete graph of order \( n \) and such that each graph corresponds to an irreducible closed set in \( \text{Grass}(n-2,n) \). In Section 4 we work through an explicit case, \( n = 5 \). Throughout this section \( \mathcal{G} \) shall denote the Grassmann variety, \( \text{Grass}(n-2,n) \), under the Plücker embedding. We also assert \( n \geq 3 \).

3.1. Basic Graph Theory Definitions

Before proceeding, we recall some basic graph theory notions: A graph is order \( n \) means it has \( n \) vertices. Two vertices in a graph are adjacent means there is an edge joining them. The degree of a vertex \( v \) is the number of edges incident to it, where a loop, an edge joining \( v \) to itself, counts as two edges. A vertex is isolated means it has no edges. A graph is simple means every edge joins exactly two vertices (i.e., there are no loops) and any two vertices are joined by at most one edge (i.e., there are no parallel edges). All graphs to which we refer from now on are simple.
Suppose $G$ is a graph of order $n$. A vertex in $G$ is dominating means it has degree $n - 1$, i.e., it is adjacent to every other vertex in $G$. $G$ is complete means all of its vertices are dominating. For any $a$, we use $K_a$ to denote the complete graph of order $a$. A clique is a complete subgraph $K_a \subseteq G$. A maximal clique of order $a$ is a subgraph $K_a \subseteq G$ that is not properly contained in any clique. A collection of simple graphs of order $n$ is a cover (or covering) means the union of their edges gives $K_n$. The complement $H$ of $G$ is the unique graph that, with $G$, gives a cover.

### 3.2. Plücker Coordinates to Graphs

Let $G = \text{Graph}(g)$, for a fixed point $g \in \mathcal{G}$, where we construct $\text{Graph}(g)$ as follows: $G$ consists of vertices $v_1, \ldots, v_n$, where $v_{i_1}$ and $v_{i_2}$ are adjacent if and only if the Plücker coordinate $g_{i_1}$, where $i_1 = \{1, \ldots, n\} \setminus \{i_1, i_2\}$, vanishes. If $A$ is a matrix whose column span (resp., row span) is $g$, then we write $\text{Graph}(A) = \text{Graph}(\text{col} A)$ (resp., $\text{Graph}(A) = \text{Graph}(\text{row} A)$).

**Definition 1.** A graph $G$ of order $n$ is permissible means

(a) $G$ has at most ${n \choose n-2} - 1$ edges, i.e., $G$ is not complete, and

(b) its subgraph obtained by omitting all dominating vertices is a disjoint union of maximal cliques.

Figure 1 shows examples of graphs which are permissible and Figure 2 shows examples of graphs which are not permissible. Condition (b) in Definition 1 gives a way to construct permissible graphs; the following is an equivalent condition that is useful in identifying permissible graphs.

**Proposition 2.** Suppose $G$ is a non-complete graph of order $n$. Then the following are equivalent:

(i) The subgraph of $G$ obtained by omitting all dominating vertices is a disjoint union of cliques.

(ii) A vertex $v \in G$ has degree $d \neq n - 1$ if and only if $v$ is part of a maximal clique of order $d$.

**Proof.** We shall show (ii) implies (i). The reverse implication is even more immediate. We also note if $G$ has no dominating vertices then (i) immediately follows from (ii). Therefore, we shall assert $G$ has $m \geq 1$ dominating vertices. Let $G' \subset G$ denote the subgraph obtained by omitting them.

Choose a vertex $v \in G$ with degree $d \neq n - 1$ and let $V \subseteq G$ denote the order $d$ maximal clique containing it. The subgraph $V' = V \cap G'$ is also a maximal clique in $G'$, of order $d - m$. Assume, without loss of generality, that $v$ is adjacent to a vertex $w \in G' \setminus V'$, so that $V'$ does not comprise a connected component of $G'$. Then since $w$ is not dominating in $G$, nor is $w \in V$, the degree of $v$ must be at least $d + 1$, a contradiction.

**Proposition 3.** The graph $\text{Graph}(g)$ associated to a point $g \in \mathcal{G}$ is permissible.
Proof. At least one Plücker coordinate \( g_i \) does not vanish, so \( G \) is not a complete graph. We will show (ii) from Proposition 2 holds for \( G \), as well.

Without loss of generality, because of the group actions described in Section 2.2, we can assume the Plücker coordinate \( g_{\{1,\ldots,n-2\}} \) is non-zero. We can write a matrix

\[
A = \begin{pmatrix}
I_{(n-2)\times(n-2)} \\
\alpha_{11} & \cdots & \alpha_{1,n-2} \\
\alpha_{21} & \cdots & \alpha_{2,n-2}
\end{pmatrix}
\]

so that \( \text{col} \, A = g \), and two vertices \( v_i, v_j \in G \) are adjacent if and only if the submatrix of \( A \) obtained by omitting the \( i \)th and \( j \)th rows is singular. For example, it is obvious that \( v_{n-1} \) and \( v_n \) are not adjacent. We also note how every Plücker coordinate corresponds to either an entry or a 2-minor of the submatrix

\[
A' = \begin{pmatrix}
\alpha_{11} & \cdots & \alpha_{1,n-2} \\
\alpha_{21} & \cdots & \alpha_{2,n-2}
\end{pmatrix}
\]

We now, again without loss of generality, consider the respective degrees of the vertices \( v_1 \) and \( v_n \).

First consider \( v_i \), where \( i \in \{1, \ldots, n-2\} \). If \( v_i \) is not adjacent to either of \( v_{n-1}, v_n \) then in \( A' \) some collection of 2-minors, all involving the \( i \)th column, must vanish. But the two ways this happens are either \( \alpha_{1i} = \alpha_{2i} = 0 \) and equivalently, \( v_i \) is dominating, contradicting the hypothesis \( v_i \) is not adjacent to \( v_{n-1} \) or \( v_n \); or, all Plücker coordinates involving entries from columns whose indices give vertices adjacent to \( v_i \) vanish, in which case, such vertices, along with \( v_i \), form a maximal clique.

Suppose, on the other hand, \( v_i \) is adjacent to, say, \( v_{n-1} \). We already know \( v_n \) is not adjacent to \( v_{n-1} \), and we have

\[
\alpha_{2i} = \det A(\{1, \ldots, n\} \setminus \{i, n-1\}) = 0.
\]

If \( v_i \) is also adjacent to \( v_n \) then \( \alpha_{1i} = 0 \), in which case the \( i \)th column of \( A' \) vanishes and equivalently, \( v_i \) is a dominating vertex. Suppose then, that \( v_i \) is adjacent to \( v_{n-1} \) but not to \( v_n \). We claim the vertices adjacent to \( v_{n-1} \), along with \( v_{n-1} \), must form a clique. If two such entries \( \alpha_{2i}, \alpha_{2j} \) vanish, i.e., \( v_i, v_j \) are both adjacent to \( v_{n-1} \), then so does the 2-minor \( \alpha_{11}\alpha_{2j} - \alpha_{1j}\alpha_{21} \); this is exactly the condition for \( v_i \) and \( v_j \) to be adjacent. Finally, we point out all of the above analysis for \( v_{n-1} \) applies analogously to \( v_n \). It follows that all of the vertices in \( G \) are either dominating or part of a clique.

It is clear the set of order \( n \) permissible graphs injects into Grass\((n-2,n)\).

Given a permissible graph \( G \), the set of points in \( \mathcal{S} \) with that graph is locally closed. Its closure is all points whose graph contains \( G \).
Figure 1: Permissible graphs of orders 3, 4, 5.

Figure 2: These graphs are not permissible.
**Definition 2.** A subvariety $S \subseteq \mathcal{G}$ is **permissible** means it is the closure of the set of all points with the same fixed permissible graph, which we denote $\text{Graph}(S)$.

Recall, given a collection $\mathcal{I}$ of $(n-2)$-element subsets of $\{1, \ldots, n\}$, we let $\mathcal{V}(\mathcal{I})$ denote the vanishing set of the Plücker coordinates indexed by the sets in $\mathcal{I}$. Using a known observation about ideals generated by minors of a $2 \times s$ matrix, we show, with $s = n-2$, that the components of $\mathcal{V}(\mathcal{I})$ are permissible. The observation is as follows: if two overlapping 2-minors of a generic $2 \times s$ matrix vanish, then either the two entries in the overlap vanish or else the third minor among the three columns in question must vanish.

**Proposition 4.** In $\mathcal{G} = \text{Grass}(n-2, n)$, the irreducible components of $\mathcal{V}(\mathcal{I})$, where $\mathcal{I} \subset \{1, \ldots, n\}$ has cardinality $n-2$, are permissible.

**Proof.** Recall, $\mathcal{G}$ is covered by open affine sets $\mathcal{G}_i$, where the $i$th Plücker coordinate does not vanish. We shall prove the result in the affine case, i.e., for $\mathcal{V} = \mathcal{V}(\mathcal{I}) \cap \mathcal{G}_i$. We can choose matrices normalized with respect to $i$ whose column spaces represent points in $\mathcal{V}$. For each class of sets $\mathcal{I}$ not containing $i$, we construct $G = \text{Graph}(\text{col} A)$, having supposed $\text{col} A \in \mathcal{V}(\mathcal{I})$.

We claim the components of $\mathcal{V}$ are in bijection with the minimal possible graphs associated to its points, in the sense that removing any edge from such a graph gives a graph to which no point in $\mathcal{V}$ is associated. Let $I$ denote the homogeneous defining ideal for $\mathcal{V}$, generated by the Plücker variables with indices in $\mathcal{I}$.

Write $\mathcal{I} = \{1, \ldots, n\} \setminus \{i, j\}$ and choose $g \in \mathcal{V}$. The Plücker coordinates for $g$ are, up to a scalar, the minors of the submatrix $A(\{i, j\}; \{1, \ldots, n-2\})$. Let

$$A' = A(\{i, j\}; \{1, \ldots, n-2\}) = \begin{pmatrix} a_{i1} & \cdots & a_{i,n-2} \\ a_{j1} & \cdots & a_{j,n-2} \end{pmatrix}.$$  

We now look for instances where such a matrix $A$, i.e., with $\text{col} A = g$, has a minimal graph. If two algebraically independent Plücker coordinates vanish then $\text{Graph}(A)$ does not require any more edges than the two given. We must consider the cases where non-algebraically independent Plücker coordinates vanish, since the Plücker relations may require the vanishing of additional Plücker coordinates.

Suppose two overlapping 2-minors of $A'$ vanish. This happens if either the third 2-minor in the three involved columns vanishes, or the two entries in the overlapping column vanish. This first case corresponds to a triangle (3-cycle) in $\text{Graph}(A)$. In the other case the vertex whose index matches the vanishing column is dominating. On the other hand, two non-overlapping 2-minors of $A'$ are algebraically independent of each other, so the homogeneous ideal generated by their respective Plücker coordinates is prime.

The next case we consider is when some collection of entries of $A'$ vanish. If two entries in the same column vanish then all 2-minors involving that column
vanish and the corresponding vertex in Graph(A) is dominating. If two entries in the same row vanish then the 2-minor involving those entries vanishes and this is reflected in Graph(A) as a triangle. Note, the entries of $A'$ themselves are algebraically independent of each other. If there are no other generators for $I$, then $V(I)$ is a permissible subvariety.

The final case to consider is when we suppose an entry $a$ of $A'$ and a 2-minor $\mu$ of $A'$, containing $a$, vanish. Write

$$\mu = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$ 

We lose no generality, as the position of $a$ only determines a name change of the other entries in order to get the same formula, $ad - bc$, for $\mu$. It follows either $b$ or $c$ must also vanish. If $c$ vanishes, then all 2-minors involving the column $(^i_2)$ vanish, and the vertex with the same index as that column is dominating. If $b = 0$ then in Graph(A) we get a triangle. As we can see, vanishing of any collection of Plücker coordinates can only cause other Plücker coordinates to vanish. We conclude the minimal primes for $V(I)$ are generated by Plücker coordinates, and thus, each has a unique corresponding graph.

3.3. Consequences

It is not hard to see any permissible graph $G$ will contain either isolated vertices, or dominating vertices, or neither, but not both. Let $G_{\text{triv}}$ denote this set of vertices. We will say $G_{\text{triv}}$ is dominating to mean its vertices are dominating, or isolated, to mean its vertices are isolated. $G_{\text{triv}}$ may be empty, and $|G_{\text{triv}}| \neq n$. By permissibility of $G$, the set $G \setminus G_{\text{triv}}$ is a disjoint union of $c$ cliques of respective orders $a_1, \ldots, a_c$.

**Theorem 1.** Suppose $S \subset \mathcal{G} = \text{Grass}(n-2, n)$ is a permissible subvariety with graph $G = \text{Graph}(S)$. Let $a_1, \ldots, a_c$ denote the respective orders of the cliques in $G \setminus G_{\text{triv}}$. Put $m = |G_{\text{triv}}|$ and $l = \sum_{j=1}^{c}(a_j - 1)$. Then the codimension of $S$ in $\mathcal{G}$ is

$$\text{codim } S = \begin{cases} n - c + m = 2m + l & \text{if } G_{\text{triv}} \neq \emptyset, \text{ dominating} \\ n - c - m = l & \text{otherwise.} \end{cases}$$

**Proof.** An isolated vertex contributes nothing to the codimension. Say $A$ is an $n \times (n-2)$ matrix, normalized with respect to some set of indices $\overline{i} = \{1, \ldots, n\} \setminus \{i, j\}$, and such that $\text{Graph}(A) = G$. Let

$$A' = \begin{pmatrix} a_{i1} & \cdots & a_{i,n-2} \\ a_{j1} & \cdots & a_{j,n-2} \end{pmatrix},$$

the complementary submatrix to the order $n-2$ identity submatrix of $A$. A dominating vertex $v \in G$ indicates that two entries on a column of $A'$ vanish, contributing 2 to the codimension. All other edges joined to $v$ correspond to 2-minors of $A'$ involving that vanishing column and thus contribute nothing to the codimension.
Put $G' = G \setminus G_{\text{triv}}$. Columns of $A'$ involved in a given clique of $G'$ are independent of those involved in minors corresponding to edges joined to dominating vertices. Finally, there are two cases left to consider. The first case is where a clique of order $a \geq 2$ indicates a collection of $a - 1$ entries from the same row of $A'$ vanish, so contributes $a - 1$ to the codimension. The other case is when a clique of order $a \leq 2$ indicates all 2-minors involving some set of columns in $A'$ vanish. The condition is equivalent to the condition that a generic $2 \times a$ matrix has rank 1. It is known (see [5]) that a generic rank 1 matrix of size $2 \times a$ defines an ideal of height $a - 1$.

Given a permissible graph $G$, let $H$ denote its complement. In understanding components of $\Theta(Y_{n,n-2,n-2})$ in Equation (1), we wish to minimally enlarge $H$ to a permissible graph, $\tilde{H}$, and then take a minimal permissible subgraph $\tilde{G} \subseteq G$ such that together, $\tilde{G}, \tilde{H}$ cover $K_n$. Specifically, $\tilde{H}$ should not properly contain any permissible subgraph containing $H$, and $\tilde{G}$ should not contain any permissible subgraph that, with $\tilde{H}$, forms a covering. Upon finding such a pair $(\tilde{G}, \tilde{H})$, we let $(S, T)$ denote the pair of permissible subvarieties with the respective graphs.

**Theorem 2.** A minimal pair, up to permutation, of permissible subvarieties $(S, T)$ whose associated graphs form a covering must satisfy:

(a) $\text{Graph}(S)$ consists of a clique of order $a$ with the remaining vertices isolated, and

(b) $\text{Graph}(T)$ is its complement, an order $n-a$ graph with $n-a$ dominating vertices.

These pairs completely describe the components of $Y_{n,n-2,n-2}$. The number of cliques in $\text{Graph}(S)$ may take on any value $2 \leq a \leq n - 1$.

**Proof.** We now give an algorithm for producing such a pair from a fixed permissible graph $G$. Let $G_{\text{triv}}$ denote the (possibly empty) set of isolated or dominating vertices of $G$, let $G' = G \setminus G_{\text{triv}}$ and let $H'$ denote the complement of $G'$. Let $a_1, \ldots, a_c$ denote the sizes of the respective cliques in $G'$.

Suppose $G_{\text{triv}}$ is non-empty and consists of dominating vertices. If $H'$ is permissible and $a_j = 1$ for all $j = 1, \ldots, c$ then let $H$ denote the union of $H'$ and the vertices from $G_{\text{triv}}$, so that $G, H$ give respective graphs for permissible subvarieties $S, T$ and we are done. If, on the other hand, $H'$ is permissible but $a_j > 1$ for some $j$ then $a_i = 1$ for all $i \neq j$ and we have two ways to enlarge $H'$:

A) Complete $H'$, then let $\tilde{H}$ denote its union with the vertices from $G_{\text{triv}}$.

Then let $\tilde{G} \subseteq G$ denote the complement of $\tilde{H}$. $\tilde{G}$ is permissible because it consists of the edges incident to vertices in $G_{\text{triv}}$.

B) There is at least one isolated vertices in $G'$. To construct $\tilde{H}$ make the isolated vertices from $G'$ into dominating vertices. Remove their edges from $G$ to get a subgraph $\tilde{G}$.

If $H'$ is not permissible, and $G_{\text{triv}}$ is non-empty and consists of dominating vertices, then we can either do A), as above, or we can do the following: choose a clique $B$ from $G'$ of order $a_j \geq 2$. In constructing $\tilde{H}$, make all vertices in
$G \setminus B$ dominating. The complement, $\tilde{G}$, of $\tilde{H}$ is a clique of order $a_j + m$, with the remaining vertices isolated.

To finish the proof, now suppose $G_{\text{triv}}$ is either empty or consists of isolated vertices. If $H'$ is permissible then adding the vertices from $G_{\text{triv}}$ to $H'$ and making them dominating does not change permissibility and we are done. If $H'$ is not permissible then let $H$ denote $H'$, together with the vertices from $G_{\text{triv}}$ as dominating vertices. Choose $j$ such that $a_j > 1$ and enlarge $H$ by making all vertices not in that clique, call it $B$, dominating. The complement $\tilde{G}$ is exactly the clique $B$. 

**Theorem 3.** The locally closed set $\mathcal{Y}_{n,n-2,n-2}$, of $n \times n$ matrices of rank $n-2$ whose size $n-2$ principal minors vanish, has dimension $n^2 - 4 - n$.

**Proof.** A matrix $A \in \mathcal{Y}_{n,n-2,n-2}$ has a normalized factorization given by $2(n-2) + (n-2)^2 + 2(n-2) = n^2 - 4$ parameters. Now subtract the minimal codimension of a component of $\mathcal{Y}_{n,n-2,n-2}$: Theorem 2 characterizes such components as products of permissible subvarieties $S \times T \subset \text{Grass}(n-2,n)$, where $\text{Graph}(S)$ is a clique of order $a$ plus $n-a$ isolated vertices and $\text{Graph}(T)$ is the complement of $\text{Graph}(S)$. For each $a = 2, \ldots, n-1$ use Theorem 1 to calculate the codimension of $S \times T$. The minimal codimension is $n$. 

In Section 4 we illustrate these results for the case $n = 5$. In the meantime, however, we digress briefly to discuss an attempt to extend the results to $t = n-3$.

3.4. Components of $\mathcal{Y}_{n,n-3,n-3}$

As in the $r = t = n-2$ case a matrix $A \in \mathcal{Y}_{n,n-3,n-3}$ factors so that we may identify it with a pair of points in $\text{Grass}(n-3,n)$. In the spirit of Section 3.2, to any set of Plücker coordinates we may associate a simplicial complex that is a union of 2-simplices. There is a notion of permissibility; permissible 2-complexes are the ones that actually come from a matrix. We can then ask the following:

**Question 1.** Given a permissible 2-complex, is the closure of the algebraic set defined by it irreducible?

**Question 2.** Is every algebraic set defined by vanishing of Plücker coordinates a union of 2-permissible ones which are irreducible?

The problem reduces to finding the conditions for a set of minors of a generic $3 \times (n-3)$ matrix to define a prime ideal.

**Example 1.** We used Macaulay2 (M2) for $n = 8$ and $K = \mathbb{Z}/101\mathbb{Z}$ to compute
the minimal primes for various collections of Plücker variables. Letting

\[
U = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
u_{61} & u_{62} & u_{63} & u_{64} & u_{65} \\
u_{71} & u_{72} & u_{73} & u_{74} & u_{75} \\
u_{81} & u_{82} & u_{83} & u_{84} & u_{85}
\end{pmatrix}
\]

parametrize a matrix whose column space is a point in Grass(5, 8), we considered ideals in \(\mathbb{Z}/101\mathbb{Z}[\wedge^5 U]\).

In Figure 3 we associate a 2-simplex to the ideal \((u_{61}u_{72} - u_{62}u_{71}, u_{62}u_{73} - u_{63}u_{72})\).

For simplicity, the vertices are labelled only by their numerical subscripts and axes are drawn to give perspective to the configuration. According to M2 the minimal primes are

\[
P_1 = (u_{61}u_{72} - u_{62}u_{71}, u_{62}u_{73} - u_{63}u_{72}, u_{61}u_{73} - u_{63}u_{71})
\]

and

\[
P_2 = (u_{62}, u_{72}),
\]

both of which are also given by Plücker variables. Their corresponding 2-simplices give a picture of what a permissible 2-simplex “ought” to look like. Different colors are used for different sized minors.

In Proposition 5 we show that an ideal generated by a 3 \(\times\) 3 minor and one of its nested 2 \(\times\) 2 minors has two minimal primes, also generated by minors.

**Example 2.** Suppose \(X = (x_{ij})\) is a 3 \(\times\) 3 generic matrix and let

\[
I = (\det X, x_{11}x_{22} - x_{12}x_{21}) \subset K[X].
\]

By Proposition 5, the minimal primes for \(I\) are

\[
(x_{11}x_{22} - x_{12}x_{21}, x_{11}x_{23} - x_{13}x_{21}, x_{12}x_{23} - x_{13}x_{22}) \quad \text{and}
\]

\[
(x_{11}x_{22} - x_{12}x_{21}, x_{11}x_{32} - x_{12}x_{31}, x_{21}x_{32} - x_{22}x_{31}),
\]

given by the highlighted minors:

\[
\begin{pmatrix}
\times_{11} & \times_{12} & \times_{13} \\
\times_{21} & \times_{22} & \times_{23} \\
\times_{31} & \times_{32} & \times_{33}
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\times_{11} & \times_{12} & \times_{13} \\
\times_{21} & \times_{22} & \times_{23} \\
\times_{31} & \times_{32} & \times_{33}
\end{pmatrix}
\]
Proposition 5. For a \( 3 \times 3 \) generic matrix \( X \) the ideal in \( K[X] \) generated by 
\[
\det X = x_{11} x_{22} x_{33} - x_{12} x_{23} x_{31} + x_{13} x_{21} x_{32} - x_{12} x_{31} x_{23} + x_{13} x_{22} x_{31} - x_{11} x_{23} x_{32}
\] has two minimal primes:
\[
(x_{11} x_{22} x_{33} - x_{12} x_{23} x_{31}, x_{13} x_{21} x_{32} - x_{11} x_{23} x_{32}) \quad \text{and} \quad (x_{11} x_{22} x_{33} - x_{12} x_{23} x_{31}, x_{13} x_{21} x_{32} - x_{11} x_{23} x_{32}).
\]

Proof. Write \( X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \). For the proof we simplify the notation; let
\[
\mu = x_{11} x_{22} x_{33} - x_{12} x_{23} x_{31}, \quad \Delta = \det X, \quad I = (\Delta, \mu) \subset K[X], \quad R = K[X]/I.
\]

\[
\Delta_{rs} = x_{rt} x_{su} - x_{ru} x_{st}, \quad \text{for } r, s, t, u \in \{1, 2, 3\}
\]

(\text{so that } \mu = \Delta_{i_1 j_2} \text{ for } i_1, j_2 \in \{1, 2, 3\}).

Translating to the new notation, we wish to show the minimal primes for \( I \) are
\[
(\mu, \Delta_{i_1 j_2}, \Delta_{j_2 j_3}) \quad \text{and} \quad (\mu, \Delta_{i_1 j_2}, \Delta_{j_1 j_3}).
\]

We shall localize at \( x = x_{i_1 j_2} \). We first show \( x \) is not a zerodivisor on \( R \) by
showing \( J = I + \langle x \rangle \) has height 3 in \( K[X] \). (The ideal \( I \) has two generators, so
its height is at most 2.) In $K[X]/J$,

\[0 = \Delta\]

\[
= \pm x_{i_3 j_1} \Delta_{i_2 j_2}^{j_2 j_3} \mp x \Delta_{i_2 j_2}^{j_2 j_3} \pm x_{i_3 j_2} \mu
\]

(note, the signs are actually ambiguous, but in this case, are also irrelevant)

\[
= \pm x_{i_3 j_2} \Delta_{i_2 j_2}^{j_2 j_3}
\]

implies we can decompose

\[
\mathcal{V}(J) = \mathcal{V}(\mu, x, x_{i_3 j_1}) \cup \mathcal{V}(\mu, x, \Delta_{i_2 j_2}^{j_2 j_3})
\]

\[
= \mathcal{V}(\mu, x, x_{i_3 j_1}) \cup \mathcal{V}(\mu, \Delta_{i_1 j_2}^{i_2 j_2} \Delta_{i_2 j_2}^{j_2 j_3} x, x) \cup \mathcal{V}(x_{i_1 j_2}, x_{i_2 j_2}, x).
\]

since $(\mu, \Delta_{i_2 j_2}^{j_2 j_3}) = (\mu, \Delta_{i_1 j_2}^{i_2 j_2}, \Delta_{i_2 j_2}^{j_2 j_3}) \cap (x_{i_1 j_2}, x_{i_2 j_2})$.

Thus $J$ has height 3 and it follows we can invert the element $x$.

Over the localized ring $R_x = R[\frac{1}{x}]$ we can clear the remaining entries in row $i_3$ of $X$; let $X' = (x'_{ij})$ denote the resulting matrix. We shall index its minors the same way we did for $X$, only replacing $\Delta$ with $\delta$, e.g., we let $\delta_{st}$ denote the minor $x'^{s} x'^{t} - x'^{ru} x'^{st}$ for $r, s, t, u \in \{1, 2, 3\}$. The entries for $X'$ are

\[
x'_{i_1 j_1} = x_{i_1 j_1} - \frac{x_{i_1 j_1} x_{i_1 j_2}}{x} x_{i_1 j_2},
\]

\[
x'_{i_2 j_1} = x_{i_2 j_1} - \frac{x_{i_2 j_1} x_{i_2 j_2}}{x} x_{i_2 j_2},
\]

\[
x'_{i_3 j_1} = 0
\]

Expanding along the $i_3$th row (again, the signs are ambiguous, but irrelevant),

\[
\Delta = \delta = \pm 0 \cdot \delta_{i_1 j_2}^{j_2 j_3} \mp 1 \cdot \delta_{i_1 j_2}^{j_2 j_3} \pm 0 \cdot \delta_{i_1 j_2}^{j_2 j_3},
\]

and $\mu = \delta_{i_1 j_2}^{j_2 j_3}$. Thus $IR_x$ has the decomposition

\[
IR_x = (\delta_{i_1 j_2}^{j_2 j_3}, \delta_{i_1 j_2}^{j_2 j_3}) R_x = (\delta_{i_1 j_2}^{j_2 j_3}, \delta_{i_1 j_2}^{j_2 j_3}, \delta_{i_1 j_2}^{j_2 j_3}) R_x \cap (x'_{i_1 j_1}, x'_{i_2 j_1}) R_x.
\]

The respective contractions to $R$ are the minimal primes for $I$. For the first ideal,

\[
R \cap (\delta_{i_1 j_2}^{j_2 j_3}, \delta_{i_1 j_2}^{j_2 j_3}, \delta_{i_1 j_2}^{j_2 j_3}) R_x = (\mu, \Delta_{i_1 j_2}^{j_2 j_3}, \delta_{i_1 j_2}^{j_2 j_3}, \delta_{i_1 j_2}^{j_2 j_3}, \delta_{i_1 j_2}^{j_2 j_3}) x^\infty
\]

\[
= (\mu, \Delta_{i_1 j_2}^{j_2 j_3}, \delta_{i_1 j_2}^{j_2 j_3}) x^\infty
\]

\[
= (\mu, \Delta_{i_1 j_2}^{j_2 j_3}, \delta_{i_1 j_2}^{j_2 j_3}),
\]

since a prime ideal is already saturated. For the other prime, since, by hypothesis, $\mu \in (x'_{i_1 j_1}, x'_{i_2 j_1}) R_x$, we can write

\[
R \cap (x'_{i_1 j_1}, x'_{i_2 j_1}) R_x = (\mu, \frac{1}{x} \Delta_{i_1 j_2}^{j_2 j_3}, \frac{1}{x} \Delta_{i_1 j_2}^{j_2 j_3}) x^\infty
\]

\[
= (\mu, \Delta_{i_1 j_2}^{j_2 j_3}, \delta_{i_1 j_2}^{j_2 j_3}),
\]

as desired. \qed
3.4.1. Generalizing Further

For \( r = t < n - 3 \) or, equivalently, \( r = t > 3 \), the combinatorial approach described above becomes too difficult. However, these cases always reduce to studying pairs of closed sets in a Grassmannian. Ford ([7]) describes a particular type of projective subvariety of the Grassmannian called a matroid variety. Matroids are a type of combinatorial data used to describe many seemingly unrelated objects in mathematics, including graphs, transversals, vector spaces, and networks. See [17] for a recent survey. Ford computes the codimension of a specific matroid variety called a positroid variety. Positroid varieties are particularly special; Knutson, Lam and Speyer ([13]) show positroid varieties not only have defining ideals given exactly by Plücker variables, but such varieties are Cohen-Macaulay, normal, and have rational singularities. Thus it is worthwhile to ask, are there any conditions where a component of \( Y_{n,t,t} \) is a product of positroid varieties?

4. Explicit Case: Theorem 3 for \( n = 5 \)

We explain Theorems 1-3 by focusing on the first non-trivial case, \( n = 5 \). A matrix \( A \in Y_{5,3,3} \) has rank 3 and its size 3 principal minors vanish. We have the identification:

\[
Y_{5,3,3} \rightarrow \text{Grass}(3, 5) \times \text{Grass}(3, 5)
\]

\[
A \mapsto (\text{col } A, \text{row } A)
\]

Without loss of generality, say \( i = \{1, 2, 3\} \) and \( j = \{1, 2, 4\} \) index the respective Plücker coordinates of \( (\text{col } A, \text{row } A) \) which do not vanish. The factorization

\[
A = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
b_{41} & b_{42} & b_{43}
b_{51} & b_{52} & b_{53}
\end{pmatrix}
\cdot A(\{1, 2, 3\}; \{1, 2, 4\})
\cdot \begin{pmatrix}
1 & 0 & c_{13} & 0 & c_{15} \\
0 & 1 & c_{23} & 0 & c_{25} \\
0 & 0 & c_{33} & 1 & c_{35}
\end{pmatrix}
\]

shows \((2 \times 3) + (3 \times 3) + (2 \times 3) = 21 = 25 - 4\) parameters, not yet considering the requirement that the size 3 principal minors of \( A \) vanish. Now, the principal 3-minors of \( A \) vanish if and only if the diagonal entries of \( \wedge^3 A \) vanish, if and only if for each \( i = 1, \ldots, 10 \), the \( i \)-th entry of either the column vector \( \wedge^3 B \) or the row vector \( \wedge^3 C \) vanishes.

**Example 3.** We give a quick example of a possible point \( A \in Y_{5,3,3} \). Put \( A \) as in (4), where we set the colored expressions from Equation (5) equal to 0.
Figure 4: What are the minimal pairs of permissible graphs that cover $n = 5$ vertices? We begin with a permissible red graph, $G$. The green graph to its right is its complement, $H$. The arrows point to minimal ways to enlarge $H$ to make it permissible; in (1), $H$ is already permissible. After enlarging $H$ to $\tilde{H}$, we remove as many edges from $G$ to obtain $\tilde{G}$, such that $\tilde{G}, \tilde{H}$ still form a covering. It turns out $\tilde{G}$ will always be permissible, and furthermore, will always be the complement of $\tilde{H}$. 
Figure 5: The vertices are labelled only with indices to simplify notation. In the graphs, an edge joining vertices \( v \) and \( v' \) is drawn if and only if the Plücker coordinate with index \( \{1, \ldots, 5\} \setminus \{v, v'\} \) vanishes. The dotted lines indicate Plücker coordinates which vanish as a consequence of the solid ones vanishing. On the left, either the red or the blue dotted line is necessary.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & c_{15} & c_{16} \\
0 & 1 & c_{25} & c_{26} \\
0 & 0 & c_{35} & c_{36}
\end{bmatrix}
\]

The solution is shown as the circled Plücker coordinates in Figure 5. Notice how the highlighted solution in (5) implies the vanishing of additional Plücker coordinates, as described in the proof of Proposition 4; in particular, we have

\[
\begin{bmatrix}
1(c_{33}) \\
b_{43}(1) \\
b_{53}c_{35} \\
-b_{42}c_{23} \\
-b_{52}(c_{23}c_{35} - c_{25}c_{33}) \\
(b_{42}b_{53} - b_{43}b_{52})(-c_{25}) \\
b_{41}(-c_{13}) \\
b_{51}(-c_{13}c_{35} + c_{15}c_{33}) \\
(-b_{41}b_{53} + b_{43}b_{51})c_{15} \\
(-b_{41}b_{52} + b_{42}b_{51})(c_{13}c_{25} - c_{15}c_{23})
\end{bmatrix} = 0. \quad (5)
\]

The Plücker indices for the chosen expressions comprise \( I, J \):

\[
I = \{\{1, 2, 4\}, \{1, 2, 5\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\}
\]

\[
J = \{\{1, 2, 3\}, \{1, 3, 4\}, \{1, 3, 5\}, \{2, 3, 4\}\}
\]

The solution is shown as the circled Plücker coordinates in Figure 5. Notice how the highlighted solution in (5) implies the vanishing of additional Plücker coordinates, as described in the proof of Proposition 4; in particular, we have

\[
\begin{bmatrix}
-b_{41}b_{52} + b_{42}b_{51} \\
b_{51}
\end{bmatrix} = 0
\]

implies either \( b_{41} = 0 \) or \( b_{52} = 0 \) and

\[
\begin{bmatrix}
c_{33} \\
c_{23} \\
-c_{13}
\end{bmatrix} = 0
\]
Figure 6: Characterization of the permissible subvarieties $S \times T \subset \text{Grass}(3,5) \times \text{Grass}(3,5)$ that give components of $9_{5,3,3}$. 

Codim($S \times T$) = 7

Codim($S \times T$) = 6

Codim($S \times T$) = 5
implies both $-c_{13}c_{35} + c_{15}c_{23} = 0$ and $c_{13}c_{25} - c_{15}c_{23} = 0$. In Figure 5 the dotted lines indicate other Plücker coordinates which vanish as a consequence, making the respective graphs for col $A$, row $A$ permissible. The different colored dotted lines in the lefthand matrix and graph reflect the condition that only one of $b_{11}$ or $b_{52}$ is required to vanish.

For any $A \in \mathcal{Y}_{3,3,3}$, we wish to find minimal pairs of permissible subvarieties whose respective graphs cover $K_5$. Figure 4 shows examples of how to construct a pair of permissible graphs which cover $K_5$, given an arbitrary partition of the Plücker coordinates, i.e., a covering of $K_5$ using a permissible graph and its complement. Figure 6 shows the types of configurations that give a minimally permissible pair of subvarieties.

5. Conclusion

Describing the minimal primes for $\mathcal{B}_{n-2}$ remains incomplete until we have analyzed the components of $\mathcal{Y}_{n,n-1,n,n-2}$. A complete description of $\mathcal{B}_{n-2}$ would be useful particularly for $n = 5$, because we would have a complete understanding of another example of an ideal $\mathcal{B}_3$. Then by Theorem 3 in [24], a natural next step would be to begin analysis of the ideals $\mathcal{B}_{n-3}$. We anticipate the difficulty will be in studying the locally closed sets $\mathcal{Y}_{n,n-1,n,n-3}$ and $\mathcal{Y}_{n,n-2,n,n-3}$. A possible strategy would be to apply the bundle map from Proposition 1, restricted to those sets.

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