R-mode oscillations of rapidly rotating barotropic stars in general relativity: Analysis by the relativistic Cowling approximation

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ABSTRACT

We develop a numerical scheme for obtaining the r-mode oscillations of rapidly rotating relativistic stars. In the present scheme, we neglect all metric perturbations and only take account of the dynamics of the fluid in the background spacetime of the unperturbed star (the relativistic Cowling approximation). We also assume the star is barotropic, i.e., neutrally stable against convection under the assumption of adiabatic oscillations. Our numerical scheme is based on the Yoshida-Eriguchi formulation for the analysis of the general relativistic f-mode oscillations in the Cowling approximation and a general relativistic generalization of the Karino-Yoshida-Yoshida-Eriguchi’s numerical scheme for obtaining oscillations of rapidly rotating Newtonian stars. By this new numerical scheme, frequencies of the r-mode oscillations are obtained as functions of the ratio of the rotational energy to the absolute value of the gravitational energy \( T/|W| \) along sequences of polytropic equilibrium stars whose ratio of the pressure to the total energy density at the center of the star and the polytropic index are kept constant. It is found that the dimensionless oscillation frequency \( \sigma/\Omega \) is a linearly decreasing function of \( T/|W| \), where \( \sigma \) and \( \Omega \) are the oscillation frequency and the angular velocity of the star measured in an inertial frame at spatial infinity. We also find that oscillation frequencies of the r-modes are highly dependent on the relativistic factor \( M/R \) of the star as already found in previous studies in which the slow rotation approximation has been used. Here \( M \) and \( R \) denote the mass and radius of the star, respectively.

Key words: relativity – stars: neutron – stars: oscillations – stars: rotation

1 INTRODUCTION

The inertial modes as well as the f-modes are basic oscillation modes of rotating stars in the sense that they exist even in incompressible stars (Bryan 1889; Lindblom & Ipser 1999). The inertial mode oscillations are mainly restored by the Coriolis force due to stellar rotation and their oscillation frequencies are therefore comparable to the stellar rotation frequencies (Papaloizou & Pringle 1978; Pringle, Berthomieu & Rocca 1981; Saio 1982; Unno et al. 1989). The r-mode oscillations belong to a sub-class of the inertial mode oscillations and are characterized by small radial displacement vectors of the eigenfunctions. They are unstable against gravitational radiation reactions for all rotating stars if the stars are inviscid (r-mode instability: Andersson 1998; Friedman & Morsink 1998). The r-mode instability is attributed to the so-called Chandrasekhar-Friedman-Schutz (CFS) mechanism, which makes various modes of oscillations of rotating stars unstable (Chandrasekhar 1970; Friedman & Schutz 1978). The r-mode oscillations excited in neutron stars due to the CFS mechanism may slow the rotation of newly born neutron stars as well as cold old neutron stars in low mass x-ray binary (LMXB) systems. Furthermore, there have been suggestions that the gravitational radiation emitted due to the excitation of the r-mode oscillations will be observed by the Laser Interferometric Gravitational Wave Observatory (LIGO) and other detectors (Lindblom, Owen...
& Morsink 1998; Andersson, Kokkotas & Schutz 1999; Owen et al. 1998; Bildsten 1998; Andersson, Kokkotas & Stergioulas 1999; See also the review by Andersson and Kokkotas 2001), though the non-linear saturation of the mode may inhibit the mechanism to work in newly born stars (Arras et al. 2003).

So far, most studies of the $r$-mode oscillations of neutron stars have been carried out within the slow rotation approximation and/or the weak gravitational field approximation (Newtonian dynamics). However, analyses under these approximations are not sufficient to investigate the $r$-mode oscillations of neutron stars because of the following reasons. First, the effect of general relativity on the $r$-mode oscillations of neutron stars would be significant because the relativistic factor of neutron stars, $M/R$, could be as large as $M/R \sim 0.2$, where $M$ and $R$ denote the mass and radius of the star, respectively. Since the effect of dragging of inertial frame on the angular velocity of the fluid measured by the locally non-rotating observer is in this order, the inertial mode should be also modified to this order (see Eq. (16) and the discussion there). Second, from time to time, the $r$-mode instability of nascent neutron stars has been investigated because this is one of the most important issues of the $r$-mode instability related to neutron stars. The spin rates of neutron stars at their birth are believed to be near the mass-shedding limit where the mass of the star begins to shed from the stellar surface of the equator. If nascent neutron stars are considered, therefore, the effect of deformation of the star due to rapid stellar rotation or of the centrifugal force on the $r$-mode oscillations should not be neglected. In short, both effects of general relativity and rapid rotation have to be taken into account in studies of the $r$-mode oscillations of neutron stars.

The $r$-mode oscillations of rapidly rotating compressible stars have been obtained for the first time by Yoshida et al. (2000) for uniformly rotating stars within the framework of Newtonian gravity. They have shown that the basic properties of the $r$-mode oscillations of slowly rotating stars do not change greatly and that extrapolating the results of the slow rotation approximation gives us good estimates for properties of the $r$-mode oscillations and the $r$-mode instability even for the rapidly rotating models (see, also, Karino et al. 2000). The Newtonian scheme has been applied to differentially and rapidly rotating stars by Karino, Yoshida & Eriguchi (2001).

As for general relativistic investigations of the $r$-mode oscillations, Kojima (1998) considered for the first time the axial parity perturbations of slowly rotating stars by assuming the oscillation frequencies to be proportional to the stellar angular velocities (low-frequency approximation) and derived a single second order ordinary differential equation (Kojima’s equation) for the $r$-mode-like oscillations. He found that the relativistic $r$-mode oscillations are absolutely different from Newtonian ones in the sense that Kojima’s equation allows a continuous spectrum for the eigenfrequency (see, also, Beyer & Kokkotas 1999). Lockitch, Andersson & Friedman (2001) noticed that Kojima’s equation is valid only if stars are non-barotropic and that, for barotropic stars, the polar parity perturbations have to be taken into account as well as the axial parity perturbations in order to obtain the $r$-mode-like oscillations. For barotropic stars, they then obtained a general relativistic counterpart of the $r$-mode oscillation in Newtonian stars. Their eigenfunctions are composed both of the polar and axial parity perturbations (see, also, Lockitch 1999; Lockitch, Friedman & Andersson 2003). Their analysis has been recently extended to superfluid stars by Yoshida & Lee (2003).

For the relativistic $r$-mode oscillations of non-barotropic stars, the situation is different from that of barotropic stars. For non-barotropic stars, as mentioned before, if we use the low-frequency approximation and the slow rotation approximation, basic equations becomes Kojima’s equation, which contains a singular point inside the star for some frequency range and yields a continuous spectrum and singular eigenfunctions (Kojima 1998). The main issue, which has not been fully solved yet, is whether there exist the $r$-mode-like solutions characterized by discrete eigenfrequencies and regular eigenfunctions in non-barotropic relativistic stars or not. In spite of various attempts to improve our understanding of the relativistic $r$-mode oscillations, we have not yet understood the $r$-mode oscillations of non-barotropic relativistic stars clearly (Kojima & Hosonuma 1999; Kojima & Hosonuma 2000; Lockitch, Andersson & Friedman 2001; Yoshida 2001; Ruoff & Kokkotas 2001; Yoshida & Futamase 2001; Ruoff & Kokkotas 2002; Yoshida & Lee 2002; Ruoff, Stavridis & Kokkotas 2003). The present study focuses upon the $r$-mode oscillations of barotropic stars. We will study non-barotropic stars in a forthcoming paper.

The main purpose of this paper is to improve our understanding of properties of the $r$-mode oscillations of rapidly rotating relativistic stars. Particularly, we want to explore both the effects of general relativity and rapid rotation on the $r$-mode oscillations. Up to now, there has been no prescription for managing pulsations of rapidly rotating relativistic stars because of difficulties due to gauge conditions, boundary conditions at spatial infinity and other factors, although there is a method to obtain zero frequency $f$-modes devised by Stergioulas & Friedman (1998). However, this method is not available for obtaining general oscillation modes of non-zero frequencies. In order to avoid these difficulties, we employ a relativistic version of the Cowling approximation in which all the metric perturbations are neglected (McDermott, Van Horn & Scholl 1983; Finn 1988; Ipser & Lindblom 1992; Yoshida & Eriguchi 1997). As discussed by Lindblom & Splinter (1990), Yoshida & Kojima (1997) and Yoshida & Eriguchi (1997), the relativistic Cowling approximation is a good approximation for determining frequencies of oscillation modes for which the fluid motions dominate vibrations of the spacetime. Therefore, the Cowling approximation would be suitable for analyzing the $r$-mode oscillations for which the fluid motions are dominating.

Making use of the relativistic Cowling approximation, we calculate the $r$-mode oscillations of rapidly rotating stars. In this paper, we assume oscillations to be adiabatic and consider only barotropic stars as mentioned before. In order to obtain the $r$-mode oscillations of rapidly rotating relativistic stars, we make use of the relativistic Cowling formulation for the rapidly rotating polytropes by Yoshida & Eriguchi (1997) and generalize to rapidly rotating general relativistic stars the Karino-Yoshida-Yoshida-Eriguchi numerical method (Karino et al. 2000) for obtaining the $r$-mode oscillations of rapidly rotating Newtonian stars. The generalization is straightforward. Since we employ the rel-
ativistic Cowling approximation, basic equations are those of fluid motions and very similar to those of the Newtonian stars (we do not need to solve the linearized Einstein equation). In \S 2 we present the basic equations and the numerical method employed in this paper for studying pulsations of rapidly rotating relativistic stars. Note that notations are not the same as those of Yoshida and Eriguchi (1997). Numerical results are given in \S 3, and \S 4 is devoted to discussion and summary. In this paper, we use units in which \( c = G = 1 \), where \( c \) and \( G \) denote the velocity of light and the gravitational constant, respectively.

2 FORMULATION

2.1 Equilibrium State

We consider oscillations of axisymmetric equilibrium configurations of uniformly rotating polytropic stars without meridional circulations. The spacetime for such configurations can be described by the line element:

\[
ds^2 = -e^{2\nu} dt^2 + e^{2\gamma} (dr^2 + \hat{r}^2 d\theta^2) + e^{2\sigma} r^2 \sin^2 \theta (d\phi - \omega dt)^2,
\]

where \( \nu, \alpha, \beta, \) and \( \omega \) are functions of \( r \) and \( \theta \) only. Here spherical coordinates \((r, \theta, \phi)\) are used. The four velocity of the fluid, \( u^\mu \), is given by

\[
u = r \sin \theta e^{3-\nu} (\Omega - \omega),
\]

where \( t^\mu \) and \( \varphi^\mu \) are, respectively, the time-like and rotational Killing vectors, and \( \Omega \) is the constant angular velocity of the star. The polytropic equation of state is defined by

\[
p = K \rho^{1+1/N}, \quad \varepsilon = \rho + N p,
\]

where \( p, \rho, \) and \( \varepsilon \) mean the pressure, the rest mass density, and the total energy density, respectively. Here, \( N \) and \( K \) are the polytropic index and the polytropic constant, respectively. In the present study, numerical solutions of the equilibrium states are constructed with the numerical method devised by Komatsu, Eriguchi & Hachisu (1989).

2.2 Pulsation Equations in the Cowling Approximation

In general relativity, not only the fluid but also the spacetime is dynamical. We therefore have to solve linearized Einstein equation in order to determine exact oscillation frequencies of relativistic stars. When we consider oscillations of rapidly rotating stars within the framework of general relativity, it is too difficult to solve the linearized Einstein equation. No one has succeeded in obtaining a general method for treating pulsations in rapidly rotating relativistic stars. In order to avoid this difficulty, we employ a relativistic version of the Cowling approximation for determining the r-mode oscillation frequencies of rapidly rotating stars. We therefore neglect all the metric perturbations and solve only the linearized continuity equation and the linearized Euler equations, assuming the perturbations to be adiabatic. The dynamics of the fluid can then be described in terms of the rest mass density perturbation, \( \delta \rho \), and the perturbations of the fluid velocity, \( \delta u^\mu \), where \( \delta \) denotes the Eulerian perturbation of a physical quantity \( q \). Since perturbations of axisymmetric stars in stationary states are considered, we assume that the perturbations have harmonic time-dependence and \( \varphi \)-dependence given by \( e^{-i\epsilon t + i\sigma \varphi} \), where \( m \) is an integer and \( \sigma \) is an oscillation frequency measured by an inertial observer at spatial infinity. The pulsation equations are given as follows.

For adiabatic perturbations,

\[
\frac{\delta p}{p} = \left( 1 + \frac{1}{N} \right) \frac{\delta \rho}{\rho},
\]

where \( \delta p \) is the Euler perturbation of the pressure and we have assumed a barotropic relation with the same index for both equilibrium states and for pulsations, i.e. the adiabatic index, \( \Gamma \), is taken to be \( \Gamma = 1 + m^{-1} \). Note that due to the barotropic equation of state, the star is neutrally stable against convection.

The perturbed continuity equation is expressed as

\[
u t^\mu (\varphi - m\Omega) \frac{\partial \delta \rho}{\partial t} - \frac{\partial \delta u^\mu}{\partial r} \frac{\partial \rho}{\partial t} - \left( \frac{2}{r} + \frac{1}{\rho} \frac{\partial \rho}{\partial r} + \frac{\partial}{\partial r} (2\alpha + \beta + \nu) \right) \frac{\delta u^r}{r} - \left( \cot \theta + \frac{1}{\rho} \frac{\partial \rho}{\partial \theta} + \frac{\partial}{\partial \theta} (2\alpha + \beta + \nu) \right) \frac{\delta u^\theta}{\rho} + \{m \sigma F - m \} \delta u^\varphi = 0,
\]

where \( \delta u^\mu = i \delta \nu^\mu \), \( \delta \nu^\mu = i \delta \theta^\mu \), and \( \delta u^\nu \), are the \( t \)-component of the four velocity, the Euler perturbations of the \( r \)-component, \( \theta \)-component, and \( \varphi \)-component of the four velocity, respectively. The perturbed equations for the \( r \)-component, \( \theta \)-component and \( \varphi \)-component of the equations of motion are:

\[
u \frac{1}{\epsilon + p} \frac{\partial \delta p}{\partial t} - \frac{1}{\epsilon + p} u^t \frac{\partial \delta \rho}{\partial r} - \frac{1}{\epsilon + p} u^t \frac{\partial \delta \rho}{\partial t} (\delta \varepsilon + \delta p) + (\sigma - m\Omega) \frac{\partial \delta \nu^t}{\partial r} = 0,
\]

\[
u \frac{1}{\epsilon + p} \frac{\partial \delta p}{\partial t} - \frac{1}{\epsilon + p} w^\varphi \frac{\partial \delta \rho}{\partial \varphi} - \frac{1}{\epsilon + p} w^\varphi \frac{\partial \delta \rho}{\partial \varphi} (\delta \varepsilon + \delta p) + (\sigma - m\Omega) \frac{\partial \delta \nu^\varphi}{\partial \varphi} = 0,
\]

\[
u \frac{1}{\epsilon + p} \frac{\partial \delta p}{\partial t} - \frac{1}{\epsilon + p} w^r \frac{\partial \delta \rho}{\partial r} - \frac{1}{\epsilon + p} w^r \frac{\partial \delta \rho}{\partial r} (\delta \varepsilon + \delta p) + (\sigma - m\Omega) \frac{\partial \delta \nu^r}{\partial r} = 0.
\]
\[ H \equiv \frac{1}{\Omega - \omega - \frac{v^2}{1 - v^2}}, \]  

where \( \delta \varepsilon \) is the Euler perturbation of the energy density.

Physically acceptable solutions of pulsation equations have to satisfy boundary conditions on the rotation axis and on the stellar surface. The regularity conditions are imposed on the rotation axis, i.e. all the eigenfunctions have to vanish on the rotation axis if non-axisymmetric perturbations are considered. The surface boundary condition is given by

\[-i\gamma(\sigma - m\Omega)\Delta p = -i\gamma(\sigma - m\Omega)\delta p + \delta u^\mu \frac{\partial p}{\partial x^\nu} = 0, \]

where \( \Delta q \) means the Lagrangian perturbation of a physical quantity \( q \), which is related to the Eulerian perturbation \( \delta q \) through the relation \( \Delta q = \delta q + \mathcal{L}\xi. \) Here \( \mathcal{L}\xi \) denotes the Lie derivative along the Lagrangian displacement vector \( \xi^\mu \).

As in the Newtonian models (see e.g. Karino et al. 2000), we introduce a surface-fitted coordinate system as follows:

\[ r_* \equiv \frac{r}{r_*(\theta)}, \quad \theta_* \equiv \theta, \]

where \( r = r_*(\theta) \) denotes the surface shape of the unperturbed configuration. In the actual numerical computations, we make use of equidistant mesh points both in the radial and angular directions (0 \( \leq r_* \leq 1; 0 \leq \theta_* \leq \pi/2 \)). Note that it is sufficient to consider the angular range 0 \( \leq \theta_* \leq \pi/2 \) to determine oscillation modes of rotating axisymmetric equilibrium stars, because eigenfunctions of the stars must have reflection symmetry or reflection anti-symmetry with respect to the equatorial plane. For the \( r \)-mode oscillations we consider, the eigenfunctions are reflection anti-symmetric. In the present investigation, we take the number of mesh points to be mostly \( (r_* \times \theta_*) = (101 \times 70) \) and sometimes \( (r_* \times \theta_*) = (121 \times 101) \). In order to obtain numerical solutions of the \( r \)-mode oscillations, pulsation equations described before are transformed to a new surface-fitted coordinate system and discretized on the mesh points, and are cast into coupled non-linear algebraic equations in terms of the oscillation frequency \( \sigma \) and the discretized eigenfunctions, which are solved with the Newton-Raphson iteration scheme. The numerical procedure we use is based on that of Yoshida and Eriguchi (1997) and is a generalization of that of Karino et al. (2000).

### 3 NUMERICAL RESULTS

Since the \( m = 2 \) \( r \)-mode is the most unstable against gravitational radiation reactions among the inertial modes (Lockitch & Friedman 1999; Yoshida & Lee 2000a; Yoshida & Lee 2000b), it would be the most important mode in the spin evolution of neutron stars. Therefore, in the present study, we concentrate on the \( r \)-mode oscillations associated with \( m = 2 \). There are neither overtone modes nor modes associated with \( l \neq |m| \) in the non-rotation limit, where \( l \) is an angular eigenvalue of the spherical harmonic function, for the \( r \)-mode oscillations of barotropic stars (Provost, Berthomieu & Rocca 1981; Lockitch & Friedman 1999). Thus, only fundamental \( r \)-modes with \( l = |m| \), whose eigenfunctions have no node in the radial direction, are calculated. By decreasing the axis ratio \( r_p/r_e \) of the equilibrium star from \( r_p/r_e = 1 \),

**Table 1.** Equilibrium sequences for which the \( r \)-mode oscillations have been analyzed. Here, the relativistic factors \( M/R \) shown in this table are evaluated in the non-rotation limit. Here \( M \) and \( R \) are the gravitational mass and the circumferential radius of the equilibrium models. Therefore, the values of \( M/R \) along the equilibrium sequences are not constant but their variations are not large (typically a few percent for \( N = 0.5 \) and 10\% for \( N = 1 \)). \( p_c \) and \( \varepsilon_c \) are the pressure and the energy density at the center of the star.

| Sequence | \( N \) | \( M/R \) | \( p_c/\varepsilon_c \) |
|----------|-------|---------|----------------|
| a        | 0.5   | 0.1     | 0.06180        |
| b        | 0.2   |         | 0.1744         |
| c        | 1.0   | 0.1     | 0.06614        |
| d        | 0.2   |         | 0.2016         |

**Table 2.** \( r \)-mode frequency in the limit of no rotation. The results of Yoshida and Lee (2002) in which the relativistic Cowling approximation is used are also shown. The results of the full problem are obtained by solving the full equations for the \( r \)-mode oscillations in the slow rotation approximation. The quantities \( \omega_c \) and \( \omega_{eq} \) are the values of the metric function \( \omega \) at the center and the surface, respectively.

| Method       | \( N \) | \( M/R \) | \( p_c/\varepsilon_c \) | \( \omega_{eq}/\Omega \) | \( \omega_c/\Omega \) | \( \sigma/\Omega \) |
|--------------|-------|---------|----------------|----------------|----------------|---------------|
| Present      | 0.5   | 0.1     | 0.06180        | 0.073           | 0.236          | 1.410         |
| Yoshida & Lee| 0.5   | 0.1     | —              | —              | —              | 1.410         |
| Full         | 0.5   | 0.1     | —              | —              | —              | 1.379         |
| Present      | 0.5   | 0.2     | 0.1744         | 0.167           | 0.478          | 1.502         |
| Yoshida & Lee| 0.5   | 0.2     | —              | —              | —              | 1.503         |
| Full         | 0.5   | 0.2     | —              | —              | —              | 1.443         |
| Present      | 1.0   | 0.1     | 0.06614        | 0.059           | 0.274          | 1.410         |
| Yoshida & Lee| 1.0   | 0.1     | —              | —              | —              | 1.410         |
| Full         | 1.0   | 0.1     | —              | —              | —              | 1.380         |
| Present      | 1.0   | 0.2     | 0.2016         | 0.139           | 0.556          | 1.505         |
| Yoshida & Lee| 1.0   | 0.2     | —              | —              | —              | 1.506         |
| Full         | 1.0   | 0.2     | —              | —              | —              | 1.453         |

where \( r_p \) and \( r_e \) are the polar radius and the equatorial radius of the star, respectively; we obtain equilibrium configurations from a nonrotating spherical star to a deformed star at the mass-shedding limit by the Komatsu, Eriguchi, and Hachisu numerical scheme (Komatsu, Eriguchi & Hachisu 1989, KEH). Along an equilibrium sequence, the ratio of the pressure to the total energy density at the center of the star, \( p_c/\varepsilon_c \), the polytropic index, \( N \), are kept constant. As another rotation parameter, in addition to \( r_p/r_e \), we use the ratio of the rotational energy (\( T \)) to the absolute value of the gravitational energy (\( W \)), \( T/|W| \) (for the definition of \( T/|W| \), see, Komatsu, Eriguchi & Hachisu 1989). Physical quantities characterizing the stars along an equilibrium sequence are tabulated in Table 1.

In order to check our numerical code, we have computed several \( r \)-mode oscillations for slowly rotating stars and extrapolated the \( r \)-mode frequencies to the limit of \( \Omega \rightarrow 0 \) from the obtained frequencies. The results in the non-rotating limit are compared with the solutions obtained by Yoshida.
and Lee’s numerical scheme, in which all oscillation modes of slowly rotating relativistic stars have been computed under the relativistic Cowling approximation (Yoshida & Lee 2002). The results summarized in Table 2 show that oscillation frequencies obtained by two different methods are in excellent agreement and that the numerical scheme developed in the present study works quite well. In this table, $\omega_c$ and $\omega_{\text{eq}}$ denote the values of the metric function $\omega$ at the center and the surface of the spherical star, respectively.

In Table 2, the results for the r-mode oscillations of the full equations including the metric perturbations are also shown. These results are obtained by using the formalism proposed by Lockitch et al. (2001) (see, also, Yoshida & Lee 2003). Although it was very difficult to obtain the smooth eigenfunction for the model with $N = 1.0$ and $M/R = 0.2$ in the full analysis, the obtained eigenfrequency for that model should not be far from the true value. As seen from this table, the relativistic Cowling approximation can give very good approximate values for the oscillation frequencies to within a few percent.

Stergioulas & Font (2001) have extracted an r-mode-like characteristic frequency from their general relativistic simulations. They performed hydrodynamical simulations using the relativistic Cowling approximation where the spacetime geometry was fixed during the time evolution of the fluid. Therefore, the present results can be directly compared with their results. We have obtained a frequency of $0.05\,\text{kHz}$ for the $r$-mode oscillation of an $N = 1$ rotating polytrope similar to that of Stergioulas & Font, whose gravitational mass, equatorial circumferential radius, and spin period are $M = 1.63M_\odot$, $R = 17.17\,\text{km}$, and $P = 1.25\,\text{ms}$, respectively. Our result is in good agreement with the oscillation frequency of the $r$-mode obtained by Stergioulas & Font, i.e., $1.03\,\text{kHz}$.

In Figures 1 and 2, the ratio of the $r$-mode frequency to the angular velocity, $\sigma/\Omega$, of rotating stars is shown as a function of $T/|W|$ for four equilibrium sequences listed in Table 1. Note that in the present study the frequency measured by an inertial observer at spatial infinity is shown; the corotating frequency $\sigma_c$ used in many papers is related to $\sigma$ by $\sigma_c = \sigma - m\Omega$. Figures 1 and 2 show results for polytropes with $N = 0.5$ (sequence a and sequence b) and $N = 1.0$ (sequence c and sequence d), respectively. From these figures, it is found that the scaled eigenfrequencies of the $r$-mode oscillations, $\sigma/\Omega$, are decreasing functions of $T/|W|$ and are given as nearly linear functions of $T/|W|$ except for configurations in the mass-shedding states. Similar features have been found in the Newtonian r-mode oscillations of rapidly rotating stars (Karino, Yoshida & Eriguchi 2001). It is important to note that this simple relation between $\sigma/\Omega$ and $T/|W|$ is preserved even for highly relativistic stars. Since the oscillation frequencies $\sigma/\Omega$ are nicely approximated by linear functions of $T/|W|$ except for the region near the mass-shedding limit, it is meaningful to derive approximation formulas for $\sigma/\Omega$ versus $T/|W|$, which are given by

$$\frac{\sigma}{\Omega} \approx \begin{cases} 
1.41 - 1.85 \frac{T}{|W|} & \text{for sequence a} \\
1.50 - 1.23 \frac{T}{|W|} & \text{for sequence b} \\
1.41 - 1.95 \frac{T}{|W|} & \text{for sequence c} \\
1.51 - 1.36 \frac{T}{|W|} & \text{for sequence d}.
\end{cases} \tag{15}$$

Here, after discarding data near the mass-shedding limit, we apply a least-square-fit to the scaled frequencies as functions of $T/|W|$. Comparing the coefficients appearing in these formulae for different sequences, we can see that the $r$-mode frequencies of relativistic stars do not strongly depend on the stiffness of the equation of state, but they are sensitive to the relativistic factor $M/R$ and to the rotation rate. Similar strong dependence of the relativistic factor $M/R$ on the oscillation frequencies of the $r$-mode oscillations has been already observed in the $r$-mode oscillations of slowly rotating relativistic stars (e.g., Kojima 1998; Lockitch, Andersson & Friedman 2001; Yoshida 2001; Ruoff & Kokkotas 2001; Yoshida & Lee 2002; Lockitch, Friedman & Andersson 2003). As discussed by Kojima (1998), frequencies of the $r$-mode oscillations in the limit of no rotation can be approximated by a certain average of the local oscillation frequency of the $r$-mode oscillation, namely

$$\frac{\sigma}{\Omega} = \frac{1}{|m| + 1} \left\{ (|m| + 2)(|m| - 1) + 2 \frac{\omega}{\Omega} \right\}. \tag{16}$$

In equation (16), the relativistic effect appears only in the second term in the braces, which is always positive for non-pathologically rotating stars. Therefore, the frequency of the $r$-mode oscillation tends to increase as the relativistic factor $M/R$ is increased because the dragging of inertial frame becomes significant for highly relativistic stars as seen from Table 2. This property is consistent with our numerical results as shown in Figures 1 and 2.

In Figures 3 and 4, typical distributions of velocity perturbations $\delta v_r$ and $\delta v_\theta$ are displayed for the $r$-modes of slowly and rapidly rotating stars, respectively. Here we define

$$\delta v_r \equiv \delta u^r, \tag{17}$$

$$\delta v_\theta \equiv \delta u^\theta. \tag{18}$$

The equilibrium models shown in the figures are taken from the sequence b in Table 1. The rotation parameters of the stars shown in Figures 3 and 4 are chosen as $r_p/r_e = 0.96$ ($T/|W| = 0.01248$) and $r_p/r_e = 0.70$ ($T/|W| = 0.1035$), respectively. In each figure, the amplitudes of the eigenfunctions are shown against the surface-fitted radial coordinate $r_*$ ($0 \leq r_* \leq 1$) for five different constant values of $\theta_*$, whose values are given in the figures. Figures 3 and 4 show that the relativistic $r$-mode oscillations of rotating stars are very similar to the corresponding Newtonian modes. Moreover, comparing Figure 3 with Figure 4, we observe that the basic properties of the $r$-mode oscillations of relativistic stars depend only weakly on the magnitude of the rotation. However, it is important to emphasize the following properties; the horizontal motion of the fluid due to the $r$-mode oscillations is more concentrated near the surface of the star for a rapidly rotating star than for a slowly rotating star. Similar behaviors are observed in the $r$-mode oscillations of Newtonian stars (Karino et al. 2000).

For slowly rotating stars, the $r$-mode oscillations of Newtonian stars do not excite the fluid motion in the radial direction, while in the relativistic star, the fluid element

1 Note that these are not exactly the components of 3 velocity on the orthonormal frame which is naturally defined on our space-time.
can move in this direction due to the effect of general relativity (Lockitch, Andersson & Friedman 2000). This property can be confirmed in Figure 3, which shows that the perturbations of the radial velocity $\delta v_r$ for a slowly rotating relativistic star have non-vanishing amplitudes. On the other hand, as already shown in Newtonian studies (Provest, Berthomieu & Rocca 1981; Saio 1982), the amplitude of $\delta v_r$ for the $r$-mode oscillations becomes large as a rotation parameter $T/|W|$ of the equilibrium star is increased. This fact is also observed in Figure 4. In summary, our results for modal properties are consistent with previous results that properties of pure-axial perturbations for the $r$-mode oscillations cease to appear for a highly relativistic star and/or for a rapidly rotating star.

4 DISCUSSION AND SUMMARY

4.1 Discussion

It is the most crucial thing to know how far we could rely on the results obtained by using the Cowling approximation. Since no one has succeeded in developing a formulation for the full analysis of $r$-mode oscillations of rapidly rotating stars in general relativity, what we can do is to estimate the accuracy by considering the results obtained for the Newtonian configurations and those for slowly rotating general relativistic stars.

The frequencies of $m = 2$ $r$-mode oscillations for the Newtonian polytropes are shown in Table 3. In this table, the ratios of the $r$-mode oscillation frequency to the angular velocity in the full problem and in the Cowling approximation are tabulated both for slowly rotating polytropes and for rapidly rotating polytropes with the polytropic indices $N = 0.5$ and 1. As seen from this table, the values of the Cowling approximation are to within a several percent from the exact values. Since there occurs no density perturbation in the $r$-mode oscillation of spherical Newtonian stars, the Cowling approximation gives very accurate values for slowly rotating Newtonian configurations irrespective of the polytropic indices. This kind of behavior obtained by using the Newtonian Cowling approximation is not directly applied to the general relativistic configurations because the basic equations in general relativistic oscillations are not the same as those in the Newtonian models. However, it should be noted that the results of the scaled eigenfrequencies $\sigma/\Omega$ obtained by the Cowling approximation are located within a several percent from those of the full problem. Together with the errors for general relativistic spherical stars discussed in the previous section, we may as well believe that the results obtained in this paper for the $r$-mode oscillations of rapidly rotating general relativistic polytropes would be also well approximated ones.

In general, the master equations for oscillations of a stationary and axisymmetric rotating star can be reduced to a single two-dimensional second-order partial differential equation as shown below. However, the type of the reduced second order partial differential equation cannot be known beforehand because it depends on the oscillation frequency of the mode (see, e.g., Balbinski 1985; Skinner & Lindblom 1996; Lindblom 1997). In other words, it implies that the type of the equation cannot be determined before some solutions of the modes are obtained. This is an important fact because the properness of the boundary value problem for the partial differential equations is deeply related to the type of the differential equation. If the equation is of the elliptic type, the boundary value problem is a properly posed one. However, if it is hyperbolic, one might need to devise a proper treatment of the boundary value problem for the hyperbolic equations.

In our relativistic Cowling problem, we can derive the following second order partial differential equation in terms of the pressure perturbation and consider its type. From the basic equations (7) and (11)-(13), we can obtain

$$A \frac{\partial^2 \delta p}{\partial r^2} + B \frac{\partial^2 \delta p}{\partial r \partial \theta} + C \frac{\partial^2 \delta p}{\partial \theta^2} + f \left( \frac{\partial \delta p}{\partial r}, \frac{\partial \delta p}{\partial \theta}, \delta p, r, \theta \right) = 0 , \quad (19)$$

$$A \equiv \frac{1}{\varepsilon + p} \frac{e^{-2\alpha}}{u^t} \frac{1}{D(\sigma - m\Omega)^2} \left\{ \frac{\sigma - m\Omega}{\Omega - \omega + \omega v^2} \right\} , \quad (20)$$

$$B \equiv \frac{1}{\varepsilon + p} \frac{e^{-2\alpha}}{u^t} \frac{1}{D(\sigma - m\Omega)^2} \left\{ \frac{\sigma - m\Omega}{\Omega - \omega + \omega v^2} \right\} , \quad (21)$$

$$C \equiv \frac{1}{\varepsilon + p} \frac{e^{-2\alpha}}{u^t} \frac{1}{D(\sigma - m\Omega)^2} \left\{ \frac{\sigma - m\Omega}{\Omega - \omega + \omega v^2} \right\} , \quad (22)$$

$$D \equiv \frac{1}{\sigma - m\Omega} \left\{ \frac{\sigma - m\Omega}{\Omega - \omega + \omega v^2} \right\} \right\} \left\{ \frac{\sigma - m\Omega}{\Omega - \omega + \omega v^2} \right\} , \quad (23)$$

Table 3. $R$-mode frequencies of uniformly rotating Newtonian polytropes. The results for the full problem and those obtained by the Cowling approximation are tabulated for $N = 0.5$ and 1 polytropes. Here $r_p/r_e$ is the axis ratio of the rotating configurations, where $r_p$ and $r_e$ are the polar radius and the equatorial radius, respectively.

| Method          | $N$ | $r_p/r_e$ | $\sigma/\Omega$ |
|-----------------|-----|-----------|-----------------|
| Full            | 0.5 | 0.98      | 1.320           |
| Cowling         | 0.5 | 0.98      | 1.319           |
| Full            | 0.5 | 0.67      | 1.099           |
| Cowling         | 0.5 | 0.67      | 1.050           |
| Full            | 1.0 | 0.98      | 1.323           |
| Cowling         | 1.0 | 0.67      | 1.159           |
| Cowling         | 1.0 | 0.67      | 1.116           |

where $f$ is a function of the first order derivative of $\delta p$, $\delta \sigma$ itself, and coordinates $(r, \theta)$ whose form is not essential for the following discussion (see, also, Ipser & Lindblom 1992).

The type of the above partial differential equation (21) at each mesh point can be determined by the following quan-
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For a uniformly rotating Newtonian barotropic star, the quantity $Q$ is given by

$$ Q \equiv B^2 - 4AC. $$

(24)

As for all general relativistic r-mode oscillations obtained in this paper, values of the quantity $Q$ are also positive at every point inside the star. It follows that the equation is again hyperbolic. Although there is no mathematical proof, therefore, it seems that solutions of the inertial modes are basically given as solutions of partial differential equations of the hyperbolic type. Note that the situation would be more complicated if non-barotropic stars were considered. In non-barotropic stars, the inertial modes behave like $g$-modes which are solutions of an elliptic equation when the rotational velocities of the stars are sufficiently slow (see, e.g., Yoshida & Lee 200b).

4.2 Summary

In this paper we have developed a numerical scheme for obtaining the r-mode oscillations of rapidly rotating relativistic stars. In the present scheme, we neglect all the metric perturbations and only take account of the dynamics of the fluid motion in the fixed background spacetime of the star (the relativistic Cowling approximation). We also assume stars to be barotropic under the assumption of adiabatic oscillations. Our numerical scheme is based on the Yoshida-Eriguchi formulation (Yoshida & Eriguchi 1997) and is a general relativistic extension of the Karino-Yoshida-Yoshida-Eriguchi numerical scheme for determining oscillations of rapidly rotating Newtonian stars (Karino et al. 2000). With this new numerical scheme, the frequencies of the r-mode oscillations are obtained as solutions of the ratio of the rotational energy to the absolute value of the gravitational energy $T/H$ along sequences of polytropic equilibrium stars whose ratios of the pressure to the total energy density at the center of the stars and the polytropic index are kept constant. It is found that scaled oscillation frequencies $\sigma/\Omega$ are decreasing functions of $T/H$ [and are given as nearly linear functions of $T/H$]. As already found in studies on the relativistic r-mode oscillations of slowly rotating stars, we observe that the r-mode oscillation frequencies are sensitive to change of the relativistic factor of the stars but not to the change of stiffness of the equation of state. The present results are consistent with the previous results of the r-mode oscillations both for rapidly rotating Newtonian stars and for slowly rotating relativistic stars.

It should be noted that three simplifications about the basic properties of the relativistic r-mode oscillations of rapidly rotating general relativistic stars have been employed in the present investigation; namely, the use of (1) the relativistic Cowling approximation, (2) the rigid rotation law, and (3) the barotropic equation of state. Although it is known that the Cowling approximation can produce approximate eigenfrequencies with acceptable errors and basic modal properties similar to the exact ones for such oscillation modes as the fluid motion is dominating, the exact treatment of full equations for obtaining oscillation modes of rapidly rotating relativistic stars is required for understanding accurate solutions for the oscillation modes and for exploring purely general relativistic properties such as damping times of oscillations due to gravitational radiation reactions. A great challenging problem to find a method for obtaining and solving exact pulsation equations for rapidly rotating relativistic stars has been left unsolved yet.

An extension of the present method to differentially rotating stars and/or to non-barotropic stars would be worthwhile in the following context: nascent neutron stars are expected to rotate differentially and the effect of differential rotation could strongly affect the frequencies and the modal properties of the r-mode oscillations (for Newtonian models, see e.g. Karino et al. 2001). In order to have a correct scenario of spin evolutions of young neutron stars due to the r-mode instability, therefore, we have to know how the r-mode oscillations would be changed under the effect of differential rotation. As for the r-mode oscillations of non-barotropic stars, as mentioned in Section 4, we do not yet know whether the r-mode oscillations similar to those of Newtonian stars, whose eigenfrequencies are discrete and eigenfunctions are regular, can exist in relativistic non-barotropic stars. Future investigations on the relativistic r-mode oscillations of rapidly rotating non-barotropic stars in the Cowling approximation might give us useful clues for understanding of the full r-mode oscillations of non-barotropic relativistic stars.

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Figure 1. Scaled frequencies \( \sigma/\Omega \) of the \( m = 2 \) \( r \)-mode oscillation of rotating stars with the polytropic index \( N = 0.5 \) (sequence \( a \) and sequence \( b \)) are plotted as functions of the ratio of the rotational energy to the absolute value of the gravitational energy, \( T/|W| \). The frequencies of the \( r \)-mode oscillations for the sequence \( a \) and the sequence \( b \) are shown by dashed and solid curve, respectively.

Figure 2. Same as Figure 1 but for sequences \( c \) (dashed curve) and \( d \) (solid curve).
Figure 3. Eigenfunctions $\delta v_r$ (top) and $\delta v_\theta$ (bottom) of the $m = 2$ $r$-mode oscillation for a slowly rotating star. The equilibrium configuration belongs to the sequence $b$ and its rotation parameter is taken to be $r_p/r_e = 0.96 (T/|W| = 0.01248)$. Distributions of the eigenfunctions for several values of $\theta_*$ are displayed as functions of the surface fitted radial coordinate $r_*$. Normalization of the eigenfunctions is taken so as to be $\delta v_\theta = 1$ at the stellar surface on the equator. Note that $\delta v_r$ for $\theta_* = \pi/10$ and $\pi/2$ are very small and cannot be seen in this figure.

Figure 4. Same as Figure 3 but for a rapidly rotating star whose rotation parameter is taken to be $r_p/r_e = 0.70 (T/|W| = 0.1035)$. 

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