A TRILINEAR APPROACH TO SQUARE FUNCTION AND LOCAL SMOOTHING ESTIMATES FOR THE WAVE OPERATOR

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Abstract. The purpose of this paper is to improve the known estimates for Mockenhaupt’s square function in $\mathbb{R}^3$ and for Sogge’s local smoothing in $\mathbb{R}^{2+1}$ spacetime. For this we use the trilinear approach of S. Lee and A. Vargas for the cone multiplier with some trilinear estimates obtained from the $\ell^2$ decoupling theorem and multilinear restriction theorem.

1. Introduction

Let $\Gamma = \{(\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R} : \tau = |\xi|, 1 \leq \tau \leq 2\}$ be a truncated light cone in $\mathbb{R}^3$. For given small $0 < \delta < 1$, let $\Gamma_\delta$ denote the $\delta$-neighborhood of $\Gamma$. Let $f$ be a function on $\mathbb{R}^3$ whose Fourier transform is supported in $\Gamma_\delta$. We partition $\Gamma_\delta$ into $O(\delta^{-1/2})$ sectors $\Theta = \{(\xi, \tau) \in \Gamma_\delta : \xi/|\xi| \in \theta\}$ corresponding to an arc $\theta$ of angular length $O(\delta^{-1/2})$ in the unit circle, and let $\Pi_\delta$ denote the collection of such sectors. We take a collection of Schwartz functions $\Xi_\Theta$ so that its Fourier transform $\hat{\Xi}_\Theta$ is supported on a neighborhood of $\Theta$ and $\{\hat{\Xi}_\Theta\}_{\Theta \in \Pi_\delta}$ forms a partition of unity of $\Gamma_\delta$. The square function $S_\delta f$ is defined as

$$S_\delta f = \left( \sum_{\Theta \in \Pi_\delta} |f_\Theta|^2 \right)^{1/2}$$

where $f_\Theta = f * \Xi_\Theta$. For $1 \leq p \leq \infty$, we say that the square function estimate $S\mathcal{Q}(p \to p; \alpha)$ holds if the estimate

$$\|f\|_p \leq C_\epsilon \delta^{-\alpha-\epsilon} \|S_\delta f\|_p$$

holds for all $\epsilon > 0$ and all functions $f$ having Fourier support in $\Gamma_\delta$, where $C_\epsilon$ is a positive constant depending on $\epsilon$ but not on $\delta$.

It was conjectured that the square function estimate $S\mathcal{Q}(p \to p; \alpha)$ holds for $p > 2$ and $\alpha \geq \max(0, 1/2 - 2/p)$, see [20]. Mockenhaupt [14] first considered it, and proved the estimate $S\mathcal{Q}(4 \to 4; 1/8 = 0.125)$. It was observed by Bourgain [3] that the exponent $\alpha$ could be less than 1/8, and Tao and Vargas [20] gave an explicit exponent $\alpha$ by combining their bilinear cone restriction estimates with Bourgain’s arguments. After that, the sharp bilinear cone restriction estimate was obtained by Wolff [24], and the estimate $S\mathcal{Q}(4 \to 4; 5/44 = 0.11363)$ immediately followed by a theorem in [20].

Garrigós and Seeger [6] have studied $\ell^p$ decoupling estimates (called Wolff-type inequalities [23]) for cones, and they further improved the exponent $\alpha$ by combining $\ell^p$ decoupling estimates with bilinear arguments in [20]. In [23], Wolff introduced an important type of estimate related to the above square function which have become known as $\ell^p$ decoupling inequalities. Decoupling inequalities will play an important role in this paper and will be discussed in detail in section 3. Recently, the sharp $\ell^p$ decoupling theorem for the cone was proved by Bourgain and Demeter [4].

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using the multilinear restriction theorem due to Bennett, Carbery and Tao [1]. So, by results in [6] the estimate \( SQ(4 \to 4; 3/32 = 0.09375) \) was obtained. Our first result is to make a further progress on the exponent \( \alpha \).

**Theorem 1.1.** The estimate \( SQ(4 \to 4; 1/16 = 0.0625) \) holds.

The approach to Theorem 1.1 is based on trilinear methods. S. Lee and Vargas [12] already employed a trilinear approach to square function estimates by adapting the multilinear arguments of Bourgain and Guth [5], and obtained the sharp estimate \( SQ(3 \to 3; 0) \). In [12], it was observed that trilinear square function estimates for the cone are essentially equivalent to linear ones. To get a trilinear square function estimate, the multilinear restriction theorem of Bennett, Carbery and Tao [1] will be utilized as in [12]. However, to lift the \( L_3 \) estimate to the \( L_4 \) estimate we will combine this with the sharp \( \ell_2 \) decoupling theorem due to Bourgain and Demeter [4]. Also, we will adapt the induction-on-scales argument of Bourgain and Demeter [4]. However, since their arguments take advantage of some properties of decoupling norm not derived from the square function, we cannot formulate an iteration as strong as in [4]. Nevertheless, it is enough to obtain Theorem 1.1.

The square function estimate is related to several deep questions in harmonic analysis such as the cone multiplier, local smoothing conjecture and the \( L^p \) regularity conjecture for convolution operator with the helix. In particular, these conjectures follow from the sharp estimate \( SQ(4 \to 4; 0) \), see for example [20], [6]. Theorem 1.1 implies the following partial results on these problems.

**Corollary 1.2.** (i) If \( \alpha > 1/16 \) then the local smoothing estimate

\[
\left( \int_1^2 \| e^{it\sqrt{-\Delta}} f \|_{L^4_\alpha(R^2)}^4 \frac{dt}{t} \right)^{1/4} \leq C_\alpha \| f \|_{L^4_\alpha(R^2)}
\]

holds, where \( L^p_\alpha \) is the \( L^p \)-Sobolev space of order \( \alpha \).

(ii) If \( \alpha > 1/16 \) then the cone multiplier operator \( T_\alpha \) defined by \( \hat{T}_\alpha f(\xi, \tau) = \hat{\rho}(\tau)(1 - |\xi|^2/\tau^2)^{\alpha} \hat{f}(\xi) \) is bounded on \( L^4 \), where \( \rho \) is a bump function on \([1, 2]\).

(iii) If \( \alpha < 5/24 \) then the convolution operator \( T \) defined by

\[
Tf(x) = \int f(x_1 - \cos t, x_2 - \sin t, x_3 - t) \phi(t) dt
\]

maps \( L^4 \) to \( L^{4}_\alpha \), where \( \phi \) is a bump function.

We note that the sharp estimate \( L^p \to L^{p}_{1/p}, \ p > 4 \), for the averaging operator \( T \) may be obtained by combining the theorem due to Pramanik and Seeger [17] and the Bourgain–Demeter decoupling estimates.

The proof of Corollary 1.2 is well known, and we will not reproduce here, see for example [20]. For other related problems, see [6], [4].

We are further concerned with \( L^p \to L^q \) type local smoothing estimates

\[
\left( \int_1^2 \| e^{it\sqrt{-\Delta}} f \|_{L^q(R^2)}^q \frac{dt}{t} \right)^{1/q} \leq C_{p, q, \alpha} \| f \|_{L^p_\alpha(R^2)}.
\]

It is conjectured that this local smoothing estimate holds if

\[
\frac{1}{p} + \frac{3}{q} = 1, \quad \alpha \geq \frac{1}{p} - \frac{3}{q} + \frac{1}{2}.
\]
see [18,20]. Indeed, the necessity of condition $p \leq q$ follows from translation invariance, see [9]. From the focusing example, Knapp example and delta function, one has three necessary conditions

$$\alpha \geq \frac{1}{p} - \frac{3}{q} + \frac{1}{2},$$

(1.3)

$$\alpha \geq \frac{3}{2p} - \frac{3}{2q},$$

(1.4)

$$\alpha \geq \frac{2}{p} - \frac{1}{q} - \frac{1}{2},$$

(1.5)

respectively, see [20] for details. Let $I_1 = (1, 1; 1/2 + \varepsilon)$, $I_2 = (2, 2; 0)$, $I_\infty = (\infty, \infty; 1/2 + \varepsilon)$, $I_{1,\infty} = (1, \infty; 3/2 + \varepsilon)$ where $\varepsilon > 0$ is arbitrary. When $(p,q;\alpha) = I_1,I_2,I_\infty$ and $I_{1,\infty}$, one can obtain (1.1) from the fixed-time estimates due to Miyachi [13] and Peral [16]. First, in case that (1.5) is dominant, the reciprocal range $(1/p, 1/q)$ is the triangular shape with vertices $(1,1), (1/2,1/2)$ and $(1,0)$. In this case, by interpolation, the estimates (1.1) for such triangular shape range follow from the estimates for $I_1, I_2$ and $I_{1,\infty}$. We see that the conjecture (1.2) satisfies both (1.3) and (1.4). If we have the conjecture, by interpolating between (1.2) and $I_\infty$ the estimates (1.1) are obtained when (1.3) is dominant, and analogously the interpolation between (1.2) and $I_2$ gives the estimates (1.1) when (1.4) is dominant. For an endpoint $(p,q;\alpha) = (4,4;0)$, it is known that the local smoothing estimate does not hold, see [22]. But, for $q > 4$, $\frac{1}{p} + \frac{3}{q} = 1$ and $\alpha = \frac{1}{p} - \frac{3}{q} + \frac{1}{2}$, it is not known whether the local smoothing estimate holds or not.

The critical $L_\alpha^4 \to L^4$ estimate has been considered in Corollary 1.2. We continue to study a sharp $L_\alpha^p \to L^q$ estimate when $p < q$. From Strichartz’ estimate $L_{1/2}^p \to L^q$, this conjecture follows for $q \geq 6$. Schlag and Sogge [18] first improved this to $q \geq 5$, and Tao and Vargas [20] made further progress by using bilinear approach. By the sharp bilinear cone restriction estimate due to Wolff [24] and the results in [20], the conjecture was improved to $q \geq 14/3 = 4.6$, and the $\varepsilon$-loss of $\alpha$ was removed by S. Lee [11]. Our second result is to obtain an improved sharp local smoothing estimate.

**Theorem 1.3.** The estimate (1.1) holds for $q \geq 30/7 = 4.28571\frac{4}{4}$ and $p, \alpha$ satisfying the conditions in (1.2) except the endpoint $(p,q;\alpha) = (10/3,30/7;1/10)$.

Theorem 1.3 will be proved through the trilinear approach too. The proof is simpler than Theorem 1.1. We will reduce this linear estimate to a trilinear one, and the desired trilinear estimate will be obtained from interpolating between two trilinear estimates deduced from the multilinear restriction theorem [1] and the $L^2$ decoupling theorem [4].

Throughout this paper, we write $A \lesssim B$ or $A = O(B)$ if $A \sim CB$ for some constant $C > 0$ which may depend on $p$, $q$ but not on $\delta$, $R$ and $N$, and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. The constants $C$, $C_\varepsilon$, $C_\alpha$, $C_\gamma$, and the implicit constants in $\lesssim$ and $\sim$ will be adjusted numerous times throughout the paper. For any finite set $A$, we use $\# A$ to denote its cardinality, and if $A$ is a measurable set, we use $|A|$ to denote its Lebesgue measure. If $R$ is a rectangular box or an ellipsoid and $k$ is a positive real number, we use $kR$ to denote the $k$-dilation of $R$ with center of dilation at the center of $R$.

2. Reduction to a trilinear estimate

In this section, we will show that the linear square function estimate is equivalent to a trilinear one. The arguments of this section are a small modification of arguments found in [12]. Specifically, we replace $L^3$ arguments by $L^p$ ones for $p \geq 2$. 

For an arc \( \Omega \subset S^1 \) we define a sector \( \Gamma_{\Omega} \) and a \( \delta \)-fattened sector \( \Gamma_{\Omega}^{\delta} \) by
\[
\Gamma_{\Omega} = \{(\xi, \tau) \in \Gamma : \xi/|\xi| \in \Omega\}, \quad \Gamma_{\Omega}^{\delta} = \{(\xi, \tau) \in \Gamma : \xi/|\xi| \in \Omega\}.
\]
Let \( \Omega_1, \Omega_2, \Omega_3 \subset S^1 \) be arcs whose lengths are comparable to each other. We say that \( \Gamma_{\Omega_1}, \Gamma_{\Omega_2}, \Gamma_{\Omega_3} \) are \( \nu \)-transverse if for any unit normal vector \( n_i \) to \( \Gamma_{\Omega_i}, i = 1, 2, 3 \), the parallelepiped formed by \( n_1, n_2, n_3 \) has volume \( \geq \nu \), see Figure 1. A key geometric property of the cone \( \Gamma \) is that \( \Gamma_{\Omega_1}, \Gamma_{\Omega_2}, \Gamma_{\Omega_3} \) are \( \nu \)-transverse if and only if \( \Omega_1, \Omega_2, \Omega_3 \) are mutually separated by a distance \( \gtrsim \nu^{1/3} \), see [12].

Let \( \Omega \) be small positive numbers. We define \( \Xi_\Omega \) whose Fourier transform \( \hat{\Xi}_\Omega \) is a bump function supported on a neighborhood of \( \Gamma_{\Omega}^{\delta} \). The following is due to S. Lee and Vargas [12, equation (23)].

**Lemma 2.1** (Lee–Vargas [12] equation (23)). Suppose that \( f \) has Fourier support in \( \Gamma_{\delta} \) and let \( 0 < \gamma_2 < \gamma_1 < 1 \). Then for any \( x \in \mathbb{R}^3 \),
\[
|f(x)| \leq \max_{\Omega \in \Omega(\gamma_1)} |f_{\Omega}(x)| + \gamma_1^{-1} \max_{\Omega \in \Omega(\gamma_2)} |f_{\Omega}(x)| + \gamma_2^{-50} \max_{\Omega_1, \Omega_2, \Omega_3 \in \Omega(\gamma_2); \text{dist}(\Omega_i, \Omega_\delta) \geq \gamma_2, i \neq j} \left( \prod_{i=1}^3 |f_{\Omega_i}(x)| \right)^{1/3}
\tag{2.1}
\]
where \( f_{\Omega} = f \ast \Xi_\Omega \).

To obtain the above lemma, S. Lee and Vargas adapted the arguments of Bourgain and Guth [5] who made progress on the restriction conjecture by using a multilinear approach.

Using Lemma 2.1 we can establish the following relation between the linear and trilinear square function estimates.

**Proposition 2.2.** Let \( p \geq 2 \) and \( \alpha \geq 0 \). Suppose that \( \mathcal{S}Q(p \times p \times p \to p; \alpha) \) holds. Then \( \mathcal{S}Q(p \to p; \alpha) \) is valid.

**Proof.** Let \( \epsilon > 0 \) be given. We assume that \( \beta \geq 0 \) is the best exponent for which
\[
\|f\|_p \leq C\delta^{-\beta-\epsilon}\|S_\delta f\|_p
\tag{2.2}
\]
holds for all \( f \) with \( \text{supp} \hat{f} \subset \Gamma_{\delta} \), i.e.,

\[
\beta = \inf_{\delta > 0} \left( \log_{1/\delta} \sup_{f : \text{supp} f \subset \Gamma_{\delta}} \frac{\|f\|}{\|S_{\delta} f\|} \right) - \epsilon.
\]

It suffices to show that for any small \( 0 < \epsilon_0 < 1 \),

\[
\beta \leq \alpha + O(\epsilon_1) + \log_{1/\delta} C_{\epsilon, \epsilon_1},
\]

since if we choose a sufficiently small \( \epsilon_1 \) then \( O(\epsilon_1) \) is bounded by \( \epsilon \), which can be absorbed in an \( \epsilon \)-loss in the estimate \( SQ(p \to p; \alpha) \). The dependence on \( \epsilon \) and \( \epsilon_1 \) of the constant \( C_{\epsilon, \epsilon_1} \) in the above inequality comes from employing \( SQ(p \times p \times p \to p; \alpha) \). Especially \( \epsilon_1 \) is related to the transversality of trilinear estimates below.

We may assume that \( \delta > 0 \) is sufficiently small, say \( 0 < \delta \leq \delta_0 \), because the desired estimate is trivially obtained, otherwise, where \( \delta_0 \) is a small parameter to be fixed later in the proof. Let \( 1 > \gamma_1 > \gamma_2 \geq \delta_0^{1/2} \) be dyadic multiples of \( \delta^{1/2} \), the value of which is to be fixed later in the argument. By Lemma 2.1 and the embedding \( \ell^p \subset \ell^\infty \),

\[
\|f\|_p \lesssim \sum_{\Omega_1 \in \Omega(\gamma_1)} \|f_{\Omega_1}\|_p + \gamma_1^{-p} \sum_{\Omega_2 \in \Omega(\gamma_2)} \|f_{\Omega_2}\|_p
\]

\[
\quad + \gamma_2^{50p} \sum_{\Omega_1, \Omega_2, \Omega_3 \in \Omega(\gamma_3) : \text{dist}(\Omega_1, \Omega_j) \geq \gamma_2, \forall j \neq j} \left\| \left( \prod_{i=1}^3 \|f_{\Omega_i}\| \right)^{1/3} \right\|_p
\]

where \( \Omega_j \) is taken such that if \( \theta \) intersects the interior of \( \Omega_j \) then \( \theta \subset \Omega_j \) for \( j = 1, 2 \).

Consider the first and second summation in the right-hand side of (2.4). For convenience we denote by \( \Omega = \Omega_j \) and \( \gamma = \gamma_j \). Using Lorentz rescaling we will show

\[
\|f_{\Omega}\|_p \leq C_\epsilon (\delta/\gamma^2)^{-\beta-\epsilon} \|S_{\delta} f_\Omega\|_p.
\]

By rotating the unit circle we may assume that \( \Omega \) is centered at \( (1, 0) \). Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be a linear transformation so that

\[
T(\epsilon_1, 1) = (\epsilon_1, 1), \quad T(-\epsilon_1, 1) = \gamma^2 (-\epsilon_1, 1), \quad T(\epsilon_2, 0) = \gamma (\epsilon_2, 0)
\]

where \( \{\epsilon_1, \epsilon_2\} \) is a standard basis in \( \mathbb{R}^2 \). Then \( f_{\Omega} \circ T \) is supported in \( \Gamma_{\delta/\gamma^2} \). From the equation \( f_{\Omega} \circ T^{-t} = |\det T| f_{\Omega} \circ T \), it follows that \( f_{\Omega} \circ T^{-t} \) has support in \( \Gamma_{\delta/\gamma^2} \) where \( T^{-t} \) is the inverse transpose of \( T \). Since \( \gamma \geq \delta^{1/2} \), by (2.2) it follows that

\[
\|f_{\Omega} \circ T^{-t}\|_p \lesssim (\delta/\gamma^2)^{-\beta-\epsilon} \|S_{\delta/\gamma^2} (f_{\Omega} \circ T^{-t})\|_p.
\]

By definition,

\[
S_{\delta/\gamma^2} (f_{\Omega} \circ T^{-t}) = \left( \sum_{\gamma \in \Pi_{\delta/\gamma^2}} |(f_{\Omega} \circ T^{-t}) \ast \Xi_T|^2 \right)^{1/2}.
\]

From \( \Xi_T \circ T^{-1} = \hat{\Delta}_{\gamma}(\gamma) \), it follows that \( (f_{\Omega} \circ T^{-t}) \ast \Xi_T \approx |\det T| (f_{\Omega} \circ T) \hat{\Delta}_{\gamma} = |\det T| (f_{\Omega} \hat{\Delta}_{\gamma}(\gamma) \circ T) \).

Thus, by taking the inverse Fourier transform,

\[
(f_{\Omega} \circ T^{-t}) \ast \Xi_T = (f_{\Omega} \ast \Xi_T(\gamma)) \circ T^{-t}.
\]

Since \( f_{\Omega} \ast \Xi_T(\gamma) \) has Fourier support in \( T(\gamma) \) which is a sector of size \( 1 \times \delta \times C\delta^{1/2} \) in \( \Gamma_{\delta} \), we have

\[
S_{\delta/\gamma^2} (f_{\Omega} \circ T^{-t}) = \left( \sum_{\gamma \in \Pi_{\delta/\gamma^2}} |(f_{\Omega} \ast \Xi_T(\gamma)) \circ T^{-t}|^2 \right)^{1/2} = (S_{\delta} f_{\Omega}) \circ T^{-t}.
\]
We substitute this in (2.6) and remove $T^{-\ell}$ by changing variables. Then we obtain (2.5).

By (2.3) we have
\[ \sum_{\Omega \in \Omega(\gamma)} \| f_{\Omega} \|_{p}^{p} \leq C_{\epsilon} (\delta / \gamma^{2})^{-p\beta - p\epsilon} \sum_{\Omega \in \Omega(\gamma)} \| S_{\delta} f_{\Omega} \|_{p}^{p}. \]

Since we can decompose $f_{\Omega} = \sum_{\Theta \in \Pi_{s} : \theta \subset \Omega} f * \Xi_{\Theta}$, we have that for $p \geq 2$,
\[ \sum_{\Omega \in \Omega(\gamma)} \| S_{\delta} f_{\Omega} \|_{p}^{p} = \sum_{\Omega \in \Omega(\gamma)} \left( \sum_{\Theta \in \Pi_{s} : \theta \subset \Omega} \| f * \Xi_{\Theta} \|^{2} \right)^{p/2} \]
\[ \leq \int \left( \sum_{\Omega \in \Omega(\gamma)} \sum_{\Theta \in \Pi_{s} : \theta \subset \Omega} \| f * \Xi_{\Theta} \|^{2} \right)^{p/2} \]
\[ \leq \| S_{\delta} f \|_{p}^{p}. \]

Inserting this into the previous estimate, we obtain
\[ \sum_{\Omega \in \Omega(\gamma)} \| f_{\Omega} \|_{p}^{p} \leq C_{\epsilon} (\delta / \gamma^{2})^{-p\beta - p\epsilon} \| S_{\delta} f \|_{p}^{p}. \] (2.7)

Consider the trilinear part in (2.4). By applying $S \mathcal{Q}(p \times p \rightarrow p ; \alpha)$,
\[ \sum_{\Omega_{1}, \Omega_{2}, \Omega_{3} : \Theta \subset \Omega_{\gamma_{2}}} \left\| \left( \prod_{i=1}^{3} f_{\Omega_{i}} \right)^{1/3} \right\|_{p}^{p} \leq C_{\epsilon, \gamma_{2}} \gamma_{2}^{-3} \delta^{-p\alpha - p\epsilon} \| S_{\delta} f \|_{p}^{p}. \] (2.8)

We substitute (2.7) and (2.8) in (2.4). Then,
\[ \| f \|_{p} \leq (C_{\epsilon} \gamma_{1}^{2(\beta + \epsilon)} \delta^{-\beta - \epsilon} + C_{\epsilon} \gamma_{1}^{-1} \gamma_{2}^{2(\beta + \epsilon)} \delta^{-\beta - \epsilon} + C_{\epsilon, \gamma_{2}} \gamma_{2}^{-60} \delta^{-\alpha - \epsilon}) \| S_{\delta} f \|_{p}. \]

So, by the assumption for $\beta$,
\[ \delta^{-\beta} \leq (C_{\epsilon} \gamma_{1}^{2(\beta + \epsilon)} + C_{\epsilon} \gamma_{1}^{-1} \gamma_{2}^{2(\beta + \epsilon)} \delta^{-\beta} + C_{\epsilon, \gamma_{2}} \gamma_{2}^{-60} \delta^{-\alpha}). \]

We now choose $\gamma_{1}, \gamma_{2}$ and $\delta_{0}$ so that $C_{\epsilon} \gamma_{1}^{2(\beta + \epsilon)} \leq 1/4$, $C_{\epsilon} \gamma_{1}^{-1} \gamma_{2}^{2(\beta + \epsilon)} \leq 1/4$ and $1 > \gamma_{1} > \gamma_{2} \geq \delta_{0}^{-1/2}$. Then $\delta^{-\beta} \leq C_{\epsilon, \gamma_{2}} \gamma_{2}^{-60} \delta^{-\alpha} \leq C_{\epsilon, \gamma_{2}} \delta^{-30\epsilon_{1} - \alpha}$, which means (2.3). \hfill $\Box$

3. Decoupling norms

In this section, we will show that the decoupling norm for the cone essentially satisfies the reverse Hölder inequality, and apply this to the interpolation between decoupling estimates. In fact, our interpolation lemmas can be obtained by using known interpolation theorems, so our proof is an alternative one (which is actually weaker). This section is obtained by modifying the arguments for paraboloid decoupling in [4, section 3]. For further discussion for decoupling, see [23, 11, 8, 4].

Let $f$ be a function having Fourier support in $\Gamma_{\beta}$. For such functions, the norm $\| \cdot \|_{p, \delta}$, $1 \leq p \leq \infty$ is defined by
\[ \| f \|_{p, \delta} := \left( \sum_{\Theta \in \Pi_{s}} \| f_{\Theta} \|_{p}^{2} \right)^{1/2}. \]

It is easy to see that if $m$ is a positive real number then $\| f \|_{p, m\delta} \leq C_{m} \| f \|_{p, \delta}$ by Minkowski’s inequality.

We first introduce a wave packet decomposition, which is a fundamental tool for studying Fourier restriction type problems. To decompose $f$ both in frequency space and in spatial space, we define standard bump functions. Let $\phi(x) := (1 + |x|^{2})^{-M/2}$ where $M$ is a sufficiently large
exponent. Let \(\psi : \mathbb{R}^3 \to \mathbb{R}\) be a nonnegative Schwartz function such that \(\psi\) is strictly positive in the unit ball \(B(0, 1)\), Fourier supported in a ball \(B(0, 1/4)\) and \(\sum_{k \in \mathbb{Z}^3} \psi(x - k) = 1\). For an ellipsoid \(E\), we define \(a_E\) to be an affine map from the unit ball \(B(0, 1)\) to \(E\). Let \(\phi_E = \phi \circ a_E^{-1}\) and \(\psi_E = \psi \circ a_E^{-1}\).

**Lemma 3.1.** Suppose that \(f\) is Fourier supported in \(\Gamma_\delta\). Then there exists a decomposition

\[
 f(x) = \sum_{\Theta \in \Pi_\delta} \sum_{\pi \in P_\Theta} h_\pi f_\pi(x),
\]

where \(P_\Theta = P_\Theta(f)\) is a family of separated rectangles \(\pi\) of size \(\delta^{-1} \times \delta^{-1/2} \times 1\) with its dual \(\pi^* = \Theta\), such that the coefficients \(h_\pi > 0\) have the property that

\[
 \left( \sum_{\Theta \in \Pi_\delta} \left( \sum_{\pi \in P_\Theta} |\pi|h_\pi^p \right)^{2/p} \right)^{1/2} \sim \|f\|_{p, \delta}
\]

for all \(1 \leq p < \infty\) and

\[
 \left( \sum_{\Theta \in \Pi_\delta} \sup_{\pi \in P_\Theta} h_\pi^2 \right)^{1/2} \sim \|f\|_{\infty, \delta},
\]

and the functions \(f_\pi\) obey

\[
 \text{supp } f_\pi \subset 4\Theta
\]

and

\[
 |f_\pi(x)| \lesssim \phi_\pi(x).
\]

**Proof.** For each \(\Theta \in \Pi_\delta\), we partition \(\mathbb{R}^3\) into the dual rectangles \(\pi\) of \(\Theta\). For each \(\pi\), we define a coefficient \(h_\pi\) and a function \(f_\pi\) by

\[
 h_\pi = \frac{1}{|\pi|} \int |f_\Theta(x)| \psi_\pi(x) dx \quad \text{and} \quad f_\pi(x) = h_\pi^{-1} \psi_\pi(x) f_\Theta(x).
\]

Then, (3.4) immediately follows, and some direct calculating gives (3.1). By Bernstein’s inequality,

\[
 |\psi_\pi(x) f_\Theta(x)| \lesssim h_\pi,
\]

so we have \(|f_\pi(x)| \lesssim |\psi_\pi(x)|\). This implies (3.5).

By Hölder’s inequality we have \(h_\pi \lesssim \left( \frac{1}{|\pi|} \int |f_\Theta(x)|^p \psi_\pi(x) dx \right)^{1/p}\), and using Bernstein’s lemma we can see that \(\left( \frac{1}{|\pi|} \int |f_\Theta(x)|^p \psi_\pi(x) dx \right)^{1/p} \lesssim h_\pi\). So, we have

\[
 \sum_{\pi \in P_\Theta} |\pi|h_\pi^p \sim \sum_{\pi \in P_\Theta} \int |f_\Theta|^p \psi_\pi = \|f_\Theta\|^p_{p},
\]

from which (3.2) follows. Similarly, we have that \(h_\pi \sim \sup_{x \in \pi} |f_\Theta(x)|\) and that \(\sup_{\Theta \in P_\Theta} h_\pi \sim \|f_\Theta\|_{\infty}\). Thus (3.3) follows. \(\square\)

Now we study the reverse Hölder inequality for the decoupling norm. We say that \(f\) is a balanced function if \(f\) is a function of the form (3.1) with \(h_\pi = 1\) such that \(f\) satisfies (3.1), (3.5) and a property that for any \(\Theta, \Theta' \in \Pi_\delta\), the nonempty \(P_\Theta(f), P_{\Theta'}(f)\) have comparable cardinality. These kinds of functions were first explicitly used by Wolff [23].
Lemma 3.2. Suppose that $1 \leq p, q, r \leq \infty$ and that for some $\theta \in (0, 1)$,
\[
\frac{1}{r} = \frac{1 - \theta}{q} + \frac{\theta}{p}.
\]
Then
\[
\|f\|_{r,\delta} \sim \|f\|_{q,\delta}^{1 - \theta} \|f\|_{p,\delta}^\theta,
\]
for all balanced function $f$.

Proof. Since $f$ is a balanced function, there is a number $\kappa > 0$ such that every nonempty $P_\Theta(f)$ has cardinality comparable to $\kappa$. Let $\nu$ be the number of nonempty $P_\Theta(f)$. Then by (3.2) and (3.3), one has
\[
\|f\|_{r,\delta} \sim \nu^{1/2} \kappa^{1/r} |\pi|^{1/r} = \nu^{\frac{1 - \theta}{2} \frac{\theta}{q}} |\pi|^{\frac{1 - \theta}{q}} \nu^{\frac{\theta}{2} \kappa^2 |\pi|^2} \sim \|f\|_{q,\delta}^{1 - \theta} \|f\|_{p,\delta}^\theta.
\]

As an application we have the following interpolation lemma.

Lemma 3.3. Let $2 \leq p_1, p_2, q_1, q_2 \leq \infty$. Assume that
\[
\|f\|_{q_1} \leq A_1 \|f\|_{p_1,\delta}, \quad \|f\|_{q_2} \leq A_2 \|f\|_{p_2,\delta}
\]
for all $f$ with supp $\hat{f} \subset \Gamma_\delta$. Suppose that for some $\theta \in (0, 1)$,
\[
\frac{1}{q} = \frac{1 - \theta}{q_1} + \frac{\theta}{q_2}, \quad \frac{1}{p} = \frac{1 - \theta}{p_1} + \frac{\theta}{p_2},
\]
and $2 \leq p \leq q \leq \infty$. Then
\[
\|f\|_q \leq \delta^{-\varepsilon} A_1^{1 - \theta} A_2^\theta \|f\|_{p,\delta}
\]
for all $f$ with supp $\hat{f} \subset \Gamma_\delta$ and all $\varepsilon > 0$.

Proof. For localization we decompose $f = \sum_{k \in \mathbb{Z}^3} \psi_k f$ where $\psi_k := \psi(\delta(x - k))$. Then,
\[
\|f\|_q^q \leq \sum_{k} \left\| \sum_{k \sim k'} \psi_k f \right\|_{L^q(B(k, 2\delta^{-1}) \setminus B(k', 2\delta^{-1} - \varepsilon))}^q.
\]
Since $\psi_k$ has rapid decay outside $B(k, \delta^{-1 - \varepsilon})$, we have that if $x \in B(k', 2\delta^{-1})$ then
\[
\left| \sum_{k \in \mathbb{Z}^3 \setminus B(k', 2\delta^{-1} - \varepsilon)} \psi_k(x) \right| \leq C_K \delta^K
\]
for all $K > 0$. Using this and a rough estimate $\|f\|_q \lesssim \delta^{-C} \|f\|_{p,\delta}$, we have that for any $\varepsilon > 0$ and $K > 0$,
\[
\|f\|_q^q \leq \delta^{-3\varepsilon q} \left( \sum_{k \sim k'} \left\| \sum_{k \sim k'} \psi_k f \right\|_{L^q(B(k, 2\delta^{-1} - \varepsilon))}^q + C_K \delta^K \|f\|_{p,\delta}^q \right) + C_K \delta^K \|f\|_{p,\delta}^q,
\]
where $k \sim k'$ means that $k \in B(k', 2\delta^{-1} - \varepsilon) \cap \mathbb{Z}^3$. Since the number of $k \in \mathbb{Z}^3$ contained in $B(k', 2\delta^{-1} - \varepsilon)$ is $O(\delta^{3\varepsilon})$, we have
\[
\|f\|_q^q \lesssim \delta^{-3\varepsilon q} \left( \sum_{k' \sim k'} \left\| \psi_{k'} f \right\|_{L^q(B(k, 2\delta^{-1} - \varepsilon))}^q + C_K \delta^K \|f\|_{p,\delta}^q \right) + C_K \delta^K \|f\|_{p,\delta}^q
\]
\[
\lesssim \delta^{-3\varepsilon q} \left( \sum_{k' \sim k'} \left\| \psi_{k'} f \right\|_q^q + C_K \delta^K \|f\|_{p,\delta}^q \right) + C_K \delta^K \|f\|_{p,\delta}^q \]
\[
\lesssim \delta^{-3\varepsilon q - 3\varepsilon} \left( \sum_k \left\| \psi_k f \right\|_q^q + C_K \delta^K \|f\|_{p,\delta}^q \right).
\]
Since \( p \leq q \), we have that for any \( \varepsilon > 0 \) and any \( K > 0 \),
\[
\|f\|_q \lesssim \delta^{-\varepsilon} \left( \sum_k \|\psi_k f\|_q^p \right)^{1/p} + C_K \delta^K \|f\|_{p,\delta}.
\]

On the other hand, by Minkowski’s inequality and \( p \geq 2 \) it follows that
\[
\left( \sum_k \|\psi_k f\|_{p,2\delta}^p \right)^{1/p} \leq \|f\|_{p,2\delta} \lesssim \|f\|_{p,\delta}.
\]

Thus, by the above two estimates the proof of (3.7) is reduced to showing
\[
\|\psi_k f\|_q \lesssim \delta^{-\varepsilon} A_1^{1-\theta} A_2^\theta \|\psi_k f\|_{p,2\delta}.
\]

By translation invariance it is enough to consider \( \psi_0 f \). Let \( g := \psi_0 f \). By normalization we may assume that \( \|g\|_{p,2\delta} = 1 \). Then it is reduced to showing
\[
\|g\|_q \lesssim \delta^{-\varepsilon} A_1^{1-\theta} A_2^\theta.
\]

Since \( \psi_0 \) has fast decay outside \( B(0, C\delta^{-1}) \), we have \( \|g\|_q \leq \|g\|_{L^p(B(0,\delta^{-1}-\varepsilon))} + C_K \delta^K \) for all \( \varepsilon > 0 \) and \( K > 0 \). Since \( \psi_0 \) has Fourier support in \( B(0, \delta/2) \), \( \hat{g} \) is supported in \( \Gamma_{2\delta} \). By Lemma 3.1 it is decomposed into
\[
g(x) = \sum_{\Theta \in \Psi_{2\delta}} \sum_{\pi \in \mathcal{P}_\Theta} h_\pi g_\pi(x).
\]

We first remove some minor \( \pi \)'s. By (3.5), we can eliminate \( \pi \) that is disjoint from \( B(0, C\delta^{-1-\varepsilon}) \). Let \( \mathcal{P} \) be the collection of \( \pi \) intersecting \( B(0, C\delta^{-1-\varepsilon}) \). Then \( \# \mathcal{P} \lesssim \delta^{-2-3\varepsilon} \). The rectangles \( \pi \) with \( h_\pi = O(\delta^{500}) \) can be also eliminated, since
\[
\left\| \sum_{\pi \in \mathcal{P}, 0 < h_\pi \leq \delta^{500}} h_\pi g_\pi \right\|_q \lesssim \delta^{500} \# \mathcal{P} \lesssim \delta^{400}.
\]

We group the rectangles \( \pi \) by value of coefficients \( h_\pi \). Since \( \|g\|_{p,2\delta} = 1 \), from (3.2) we can see that \( h_\pi \gtrsim 1 \). For any dyadic number \( \delta^{500} \lesssim h \lesssim 1 \) we define \( \mathcal{P}_h := \{ \pi \in \mathcal{P} : h \leq h_\pi < 2h \} \). It is classified into \( \mathcal{P}_{h,\Theta} := \mathcal{P}_h \cap \mathcal{P}_\Theta \), and let
\[
\mathcal{P}_k := \bigcup_{k \leq \# \mathcal{P}_{h,\Theta} < 2k} \mathcal{P}_{h,\Theta}
\]
for dyadic numbers \( 1 \leq k \lesssim \delta^{-2} \). Since there are \( O(\log \delta^{-1}) \) dyadic numbers \( \delta^{500} \lesssim h \lesssim 1 \) and \( 1 \leq k \lesssim \delta^{-2} \), by pigeonholing there exist \( h \) and \( k \) so that
\[
\left\| \sum_{\delta^{500} \leq h \leq 1} h \sum_{1 \leq k \lesssim \delta^{-2}} \sum_{\pi \in \mathcal{P}_h} g_\pi \right\|_q \lesssim (\log \delta^{-1})^2 h \left\| \sum_{\pi \in \mathcal{P}_h} g_\pi \right\|_q.
\]

Let \( \tilde{g} := \sum_{\pi \in \mathcal{P}_k} g_\pi \). Then from these estimates, one has
\[
\|g\|_q \lesssim \delta^{-\varepsilon} h \|\tilde{g}\|_q + \delta^{400}.
\]

Since \( \tilde{g} \) is a balanced function, from Hölder’s inequality, (3.6) and Lemma (3.2) it follows that
\[
\|\tilde{g}\|_q \leq \|\tilde{g}\|_{q_1}^{1-\theta} \|\tilde{g}\|_{q_2}^\theta \leq A_1^{1-\theta} A_2^\theta \|\tilde{g}\|_{p_1,2\delta}^{1-\theta} \|\tilde{g}\|_{p_2,2\delta}^\theta \lesssim A_1^{1-\theta} A_2^\theta \|\tilde{g}\|_{p,2\delta},
\]
and by (3.2),
\[
h \|\tilde{g}\|_{p,2\delta} \lesssim \|g\|_{p,2\delta}.
\]

Therefore, by combining these estimates we obtain (3.8). \( \square \)
Remark 3.4. By using known interpolation theorems we can obtain Lemma 3.3 without \(\varepsilon\)-losses. Indeed, since \(f\) in Lemma 3.3 has the Fourier support condition, we are not able to apply interpolation theorems directly. To avoid this, we define a linear operator \(T\) by

\[
Tf = \sum_{j \in J} f_j * \Xi_{\Theta_j}
\]

for \(f = \{f_j\}_{j \in J}\), where \(J\) is an index set of \(\Pi_\delta\). Then the inequality \(\|f\|_q \leq A\|f\|_{p,\delta}\) in Lemma 3.3 is equivalent to \(\|Tf\|_q \leq A\|f\|_{\ell^2(L^p)}\), where \(\ell^2(L^p)\) is the space of \(L^p\)-valued \(\ell^2\)-sequences. Since the functions \(\{f_j\}_{j \in J}\) are not subject to the Fourier support condition, by applying the complex interpolation theorem we get Lemma 3.3 without \(\varepsilon\)-losses.

To prove Theorem 1.1 we need a trilinear interpolation lemma. Before stating the lemma let us define a notation \(\prod\), which will be repeatedly used in the remaining parts of this paper. For \(A_1, A_2, A_3 \in \mathbb{C}\), let \(\prod A_i\) denote the geometric mean of their absolute values; that is,

\[
\prod A_i := \left(\prod_{i=1}^3 |A_i|\right)^{1/3}.
\]

From simple calculations it is easy to see the followings. If \(A, A_i, B_i\) are complex numbers for \(i = 1, 2, 3\), then

\[
\prod CA_i = C \prod A_i \quad \text{for } C \geq 0,
\]

\[
\prod (A_i B_i) = \prod A_i \prod B_i,
\]

\[
\prod A_i^\alpha = \left(\prod A_i\right)^\alpha \quad \text{for } \alpha \in \mathbb{R}.
\]

Also, if all \(A_i, \Delta \in \mathbb{C}\) and \(f_i \in L^p\), then by Hölder’s inequality it follows that for \(1 \leq p \leq \infty\),

\[
\left(\sum_\Delta \prod A_{i,\Delta}^p\right)^{1/p} \leq \prod \left(\sum_\Delta |A_{i,\Delta}|^p\right)^{1/p}, \quad (3.9)
\]

\[
\left\|\prod f_i\right\|_p \leq \prod \left\|f_i\right\|_p. \quad (3.10)
\]

Now we state our trilinear interpolation lemma.

**Lemma 3.5.** Let \(2 \leq p_1, p_2, q_1, q_2 \leq \infty\). Assume that

\[
\left\|\prod f_i\right\|_{q_1} \leq A_1 \prod \left\|f_i\right\|_{p_1,\delta}, \quad \left\|\prod f_i\right\|_{q_2} \leq A_2 \prod \left\|f_i\right\|_{p_2,\delta} \quad (3.11)
\]

for all \(f_i, i = 1, 2, 3\), with \(\hat{f}_i \subset \Gamma_\delta\). Suppose that for some \(\theta \in (0, 1)\),

\[
\frac{1}{q} = \frac{1 - \theta}{q_1} + \frac{\theta}{q_2}, \quad \frac{1}{p} = \frac{1 - \theta}{p_1} + \frac{\theta}{p_2}
\]

and \(2 \leq p \leq q \leq \infty\). Then

\[
\left\|\prod f_i\right\|_q \lesssim \delta^{-\varepsilon} A_1^{1-\theta} A_2^\theta \prod \left\|f_i\right\|_{p,\delta}
\]

for all \(f_i, i = 1, 2, 3\), with \(\hat{f}_i \subset \Gamma_\delta\) and all \(\varepsilon > 0\).
Proof. The proof is similar to Lemma 3.3. We decompose \( \prod f_i = \sum_{k \in \delta^{-1} \mathbb{Z}} \psi_k \prod f_i \) where \( \psi_k := \psi(\delta(x - k)) \). We can reduce it in an analogous manner to the proof of Lemma 3.3. By localization, it suffices to show that
\[
\left\| \prod g_i \right\|_q \lesssim \delta^{-\varepsilon} A_1^{1-\theta} A_2^\theta
\] (3.12)
for all \( g_i := \psi_0 f_i \) with \( \|g_i\|_{p,25} = 1 \). Some minor portions can be removed as in the proof of Lemma 3.3. Since \( \psi_0 \) decays rapidly outside \( B(0, C\delta^{-1}) \), we have \( \|g_i\|_q \leq \|\prod g_i\|_{L^q(B(0,\delta^{-1} - \varepsilon))} + C_K \delta^K \) for all \( \varepsilon > 0 \) and \( K > 0 \). Since \( g_i \) is Fourier supported in \( \Gamma_{25} \), by Lemma 3.1
\[
g_i(x) = \sum_{\Theta_i \in \Pi_{h_i \pi_i \in \mathcal{P}_{\Theta_i}(h_i)}} h_{\pi_i} g_{\pi_i}(x).
\]
By (3.5), we can eliminate \( \pi_i \) that is disjoint from \( B(0, C\delta^{-1} - \varepsilon) \), so we can restrict \( \mathcal{P}_i \) to the collection \( \mathcal{P}_i \) of \( \pi_i \) intersecting \( B(0, C\delta^{-1} - \varepsilon) \). We can also remove \( \pi_i \) with \( 0 < h_{\pi_i} \leq \delta^{500} \).

For dyadic \( \delta^{500} \leq h_i \leq 1 \), we define \( \mathcal{P}_{h_i} := \{ \pi \in \mathcal{P}_i : h_i \leq h_{\pi} < 2h_i \} \). Let \( \mathcal{P}_{\Theta_i}(h_i) := \mathcal{P}_{h_i} \cap \mathcal{P}_{\Theta_i} \), and for any dyadic number \( 1 \leq k_i \leq \delta^{-2} \) we define
\[
\mathcal{P}_i(h_i, k_i) = \bigcup_{k_i \leq \# \mathcal{P}_{\Theta_i}(h_i) < 2k_i} \mathcal{P}_{\Theta_i}(h_i).
\]
Then, we have
\[
\left\| \prod g_i \right\|_q \lesssim \prod_{\delta^{500} \leq h_i \leq 1} \left( \sum_{1 \leq k_i \leq \delta^{-2}} \sum_{\pi \in \mathcal{P}_i(h_i, k_i)} h_{\pi} g_{\pi} \right) + \delta^{100}.
\]
We write as
\[
\prod_{i=1}^3 \left( \sum_{h_i} \sum_{k_i} \sum_{\pi \in \mathcal{P}_i(h_i, k_i)} h_{\pi} g_{\pi} \right) = \sum_{h_{1,2,3}} \sum_{k_{1,2,3}} \prod_{i=1}^3 \left( \sum_{\pi \in \mathcal{P}_i(h_i, k_i)} h_{\pi} g_{\pi} \right).
\]
By dyadic pigeonholing, there exist dyadic numbers \( h_i \) and \( k_i, i = 1, 2, 3 \), so that
\[
\left\| \prod_{\delta^{500} \leq h_i \leq 1} \left( \sum_{1 \leq k_i \leq \delta^{-2}} \sum_{\pi \in \mathcal{P}_i(h_i, k_i)} h_{\pi} g_{\pi} \right) \right\|_q \lesssim (\log \delta^{-1})^2 \left( \prod_{h_i} \right) \left( \prod_{\pi \in \mathcal{P}_i(h_i, k_i)} h_{\pi} g_{\pi} \right) \right\|_q.
\]
Let \( \tilde{g}_i := \sum_{\pi \in \mathcal{P}_i(h_i, k_i)} h_{\pi} g_{\pi} \). Then from these estimates we have
\[
\left\| \prod \tilde{g}_i \right\|_q \lesssim \delta^{-\varepsilon} \left( \prod h_i \right) \left\| \prod \tilde{g}_i \right\|_q + \delta^{100}.
\]
Since \( \tilde{g}_i \) are balanced functions, from Hölder’s inequality, (3.11) and Lemma 3.2 it follows that
\[
\left\| \prod \tilde{g}_i \right\|_q \leq \left\| \prod \tilde{g}_i \right\|_{q_1}^{1-\theta} \left\| \prod \tilde{g}_i \right\|_{q_2}^\theta \leq A_1^{1-\theta} A_2^\theta \prod_{h_i} \left\| \tilde{g}_i \right\|_{p,25} \left\| \tilde{g}_i \right\|_{p,25}^{1-\theta} \left\| \tilde{g}_i \right\|_{p,25}^\theta \leq A_1^{1-\theta} A_2^\theta \delta^{-\varepsilon} \prod_{h_i} \left\| \tilde{g}_i \right\|_{p,25}
\]
and by (5.2),
\[
|h_i| \left\| \tilde{g}_i \right\|_{p,25} \lesssim \left\| g_i \right\|_{p,25}.
\]
Therefore, these estimates yield (3.12). \( \square \)

Remark 3.6. By using analogous methods to Remark 3.4, we can obtain Lemma 3.5 without \( \varepsilon \)-losses by known multilinear interpolation theorems, see, e.g., [2].
4. Proof of Theorem 1.1

This section is devoted to the proof of \( \mathcal{S}Q(4 \to 4; 1/16) \). By Proposition 2.2 this follows from the trilinear square function estimate \( \mathcal{S}Q(4 \times 4 \times 4 \to 4; 1/16) \). To prove this we will utilize the following two theorems. The first one is the multilinear restriction theorem due to Bennet, Carbery and Tao [1].

**Theorem 4.1** (Bennet–Carbery–Tao [1]). Let \( f_i, i = 1, 2, 3 \), be supported in \( \Gamma \). Suppose that \( \Gamma_{\Omega}, \Gamma_{\Omega'}, \Gamma_{\Omega''} \) are \( \nu \)-transverse. If \( R \gg \nu^{-1} \) then for any \( \epsilon > 0 \) and any ball \( Q_R \) of radius \( R \),

\[
\left\| \prod_i f_i d\sigma_i \right\|_{L^1(Q_R)} \leq C \epsilon R^\epsilon \prod_i \| f_i \|_2, \tag{4.1}
\]

where \( d\sigma_i \) is the induced Lebesgue measure on \( \Gamma_i \).

Note that if the restriction operator \( \mathcal{R} \) is defined as the restriction \( \mathcal{R}f = \hat{f}|_{\Gamma} \) to \( \Gamma \) of the Fourier transform \( \hat{f} \), then the extension operator \( \hat{f} d\sigma \) is its adjoint operator \( \mathcal{R}^* f \).

The second one is the \( \ell^2 \) decoupling theorem due to Bourgain and Demeter [1].

**Theorem 4.2** (Bourgain–Demeter [1]). Suppose that the Fourier support of \( f \) is contained in \( \Gamma_\delta \). Then for any \( \epsilon > 0 \),

\[
\| f \|_6 \leq C \delta^{-\epsilon} \left( \sum_{\Theta \in \Pi_\delta} \| f_{\Theta} \|_6^2 \right)^{1/2}. \tag{4.2}
\]

To deal with local estimates we define local norms as follows:

\[
\| f \|_{L^p(\psi_{B})} := \| f \psi_{B} \|_p.
\]

and for any functions \( f \) with \( \text{supp} \hat{f} \subset \Gamma_\delta \),

\[
\| f \|_{p,\delta,B} := \left( \sum_{\Theta \in \Pi_\delta} \| f_{\Theta} \|_{L^p(\psi_{B})}^2 \right)^{1/2}.
\]

Note that if \( B \) is a ball of radius \( \geq 2/\sqrt{\delta} \) then for \( p \geq 2 \),

\[
\| f \psi_{B} \|_{p,\delta} \lesssim \| f \|_{p,\delta,B}. \tag{4.3}
\]

Indeed, we decompose the Fourier transform of \( (f \psi_{B})*\Xi_{\Theta} \) as follows:

\[
(\hat{f} * \hat{\psi}_{B}) \hat{\Xi}_{\Theta} = ((\hat{f} \hat{\Xi}_{C\Theta})*\hat{\psi}_{B}) \hat{\Xi}_{\Theta} + ((\hat{f}(1-\hat{\Xi}_{C\Theta}))*\hat{\psi}_{B}) \hat{\Xi}_{\Theta}.
\]

Consider the last term of the above equation. We write as

\[
((\hat{f}(1-\hat{\Xi}_{C\Theta}))*\hat{\psi}_{B})(x) \hat{\Xi}_{\Theta}(x) = \int \hat{f}(y)(1-\hat{\Xi}_{C\Theta})(y)\hat{\psi}_{B}(x-y) \hat{\Xi}_{\Theta}(x) dy.
\]

For \( y \in \Gamma_\delta \setminus C\Theta \) and \( x \in \Theta \) we have \( |x-y| \geq \sqrt{\delta} \), and \( \hat{\psi}_{B} \) is supported in a ball of radius \( \leq \sqrt{\delta}/2 \) with center 0. By considering supports we can see that the above equation is zero. Thus, by Fourier inversion,

\[
(f \psi_{B})*\Xi_{\Theta} = ((f*\Xi_{C\Theta})\psi_{B})*\Xi_{\Theta}.
\]

By this equation, Young’s inequality and the triangle inequality, we have

\[
\| (f \psi_{B})*\Xi_{\Theta} \|_p \lesssim \| (f*\Xi_{C\Theta})\psi_{B} \|_p \lesssim \sum_{\Theta' \subset C\Theta} \| (f*\Xi_{\Theta'})\psi_{B} \|_p.
\]

From this we can obtain (4.3).
4.1. We will deduce a trilinear decoupling estimate from Theorem 4.1 and Theorem 4.2. By combining Theorem 4.1 with a localization argument and a slicing argument, it follows that

\[ \left\| \prod_{\Theta \in \Pi_4} f_{\Theta} \right\|_3 \leq C_{\epsilon} \delta^{1/2-\epsilon} \prod_{\Theta \in \Pi_4} \| f_{\Theta} \|_2 \]

for all \( f_{\Theta} \) with \( \text{supp} \hat{f}_{\Theta} \subset \Gamma_{\delta}^{\Theta} \), (for the details, see [11], [12], [21]). By orthogonality, if \( f \) is a function with \( \text{supp} \hat{f} \subset \Gamma_{\delta} \), then

\[ \| f \|_2 \sim \left( \sum_{\Theta \in \Pi_4} \| f_{\Theta} \|_2^2 \right)^{1/2} = \| f \|_{2,\delta} \].

Thus, we have

\[ \left\| \prod_{\Theta \in \Pi_4} f_{\Theta} \right\|_3 \leq C_{\epsilon} \delta^{1/2-\epsilon} \prod_{\Theta \in \Pi_4} \| f_{\Theta} \|_{2,\delta}. \]

On the other hand, from (4.2) and Hölder’s inequality we have

\[ \left\| \prod_{\Theta \in \Pi_4} f_{\Theta} \right\|_6 \leq C_{\epsilon} \delta^{-\epsilon} \prod_{\Theta \in \Pi_4} \| f_{\Theta} \|_{6,\delta}. \]

We interpolate these two estimates by Lemma 3.3. Then,

\[ \left\| \prod_{\Theta \in \Pi_4} f_{\Theta} \right\|_4 \leq C_{\epsilon} \delta^{1/4-\epsilon} \prod_{\Theta \in \Pi_4} \| f_{\Theta} \|_{3,\delta}. \]

By Hölder’s inequality one has \( \| f_{\Theta} \|_{3,\delta} \leq \| f_{\Theta} \|_{4,\delta}^{2/3} \| f_{\Theta} \|_{2,\delta}^{1/3} \). Inserting this into the above we obtain

\[ \left\| \prod_{\Theta \in \Pi_4} f_{\Theta} \right\|_4 \leq C_{\epsilon} \delta^{1/4-\epsilon} \left( \prod_{\Theta \in \Pi_4} \| f_{\Theta} \|_{4,\delta} \right)^{2/3} \left( \prod_{\Theta \in \Pi_4} \| f_{\Theta} \|_{2,\delta} \right)^{1/3}. \tag{4.4} \]

4.2. Set \( R = \delta^{-1} \). We take a covering \( \{ \Delta \} \) of \( \mathbb{R}^3 \) by finitely overlapping \( 2R^{1/2} \)-balls. We apply the estimate (4.4) to \( f_i \psi_\Delta \). Since the Fourier support of \( f_i \psi_\Delta \) is in \( \Gamma_{2\sqrt{\delta}} \), by (4.4) and (4.3) we obtain

\[ \left\| \prod_{\Theta \in \Pi_4} f_{\Theta} \right\|_{L^4(\Delta)} \leq C_{\epsilon} R^{-1/8+\epsilon/2} \left( \prod_{\Theta \in \Pi_4} \| f_{\Theta} \|_{4,\sqrt{\delta},\Delta} \right)^{2/3} \left( \prod_{\Theta \in \Pi_4} \| f_{\Theta} \|_{2,\sqrt{\delta},\Delta} \right)^{1/3}. \]

After taking the 4th power in the above, we sum over \( \Delta \), and apply Hölder’s inequality. Then,

\[ \sum_{\Delta} \left\| \prod_{\Theta \in \Pi_4} f_{\Theta} \right\|_{L^4(\Delta)}^4 \leq C_{\epsilon} R^{-1/8+\epsilon/2} \left( \prod_{\Delta} \| f_i \|_{4,4,\sqrt{\delta},\Delta} \right)^{2/3} \left( \sum_{\Delta} \| f_i \|_{2,4,\sqrt{\delta},\Delta} \right)^{1/3}. \]

After taking the 4th root in the above, we apply (3.9) to the right-hand sums. Then,

\[ \left( \sum_{\Delta} \left\| \prod_{\Theta \in \Pi_4} f_{\Theta} \right\|_{L^4(\Delta)}^4 \right)^{1/4} \leq C_{\epsilon} R^{-1/8+\epsilon/2} \left( \prod_{\Delta} \left( \sum_{\Theta \in \Pi_4} \| f_{\Theta} \|_{4,4,\sqrt{\delta},\Delta} \right)^{4/4} \right)^{1/4} \left( \prod_{\Delta} \left( \sum_{\Theta \in \Pi_4} \| f_{\Theta} \|_{2,4,\sqrt{\delta},\Delta} \right)^{4/4} \right)^{1/4}. \]

We have \( \left( \sum_{\Delta} \| f_i \|_{4,4,\sqrt{\delta},\Delta} \right)^{4/4} \leq \| f_i \|_{4,\sqrt{\delta}} \) by Minkowski’s inequality. Thus, from the above estimate it follows that

\[ \left\| \prod_{\Theta \in \Pi_4} f_{\Theta} \right\|_4 \leq C_{\epsilon} R^{-1/8+\epsilon/2} \left( \prod_{\Delta} A_i \right)^{2/3} \left( \prod_{\Delta} B_i \right)^{1/3}, \tag{4.5} \]

where

\[ A_i := \| f_i \|_{4,\sqrt{\delta}}, \quad B_i := \left( \sum_{\Delta} \| f_i \|^4_{2,4,\sqrt{\delta},\Delta} \right)^{1/4}. \]
4.3. We will show that

\[ B_i \lesssim R^{3/8} \|S_\delta f_i\|_4. \]  

(4.6)

By definition we write \( \|f_i\|^2_{2,\sqrt{\Delta}} = \sum_{\gamma \in \Pi, \sigma} \|f_i,\gamma\|^2_{L^2(\psi_\Delta)}. \) Since \( f_i,\gamma \) is decomposed as \( f_i,\gamma = \sum_{\sigma \in \Pi_i, \sigma \subseteq 2\gamma} f_i,\sigma, \) we have

\[ \|f_i\|^2_{2,\sqrt{\Delta}} = \sum_{\gamma \in \Pi, \sigma} \left| \sum_{\sigma \in \Pi_i, \sigma \subseteq 2\gamma} f_i,\sigma \psi_\Delta \right|^2. \]

We see that the Fourier support of \( f_i,\sigma \psi_\Delta \) is contained in the \( \delta^{1/2} \)-neighborhood of \( \Theta \) which is a rectangular box of size \( C \delta^{1/2} \times C \delta^{1/2} \times C \) for some constant \( C > 1 \). So, by orthogonality it follows that

\[ \|f_i\|^2_{2,\sqrt{\Delta}} \lesssim \sum_{\gamma \in \Pi, \sigma} \sum_{\sigma \in \Pi_i, \sigma \subseteq 2\gamma} \left( \sum_{\sigma \in \Pi_i} |f_i,\sigma|^2 \right)^{1/2} \lesssim \sum_{\Theta \in \Pi, \sigma} \left( \sum_{\sigma \in \Pi_i} |f_i,\sigma|^2 \right)^{1/2}. \]

Since \( \sum_{\sigma \in \Pi_i} |f_i,\sigma|^2 = \left( \sum_{\Theta \in \Pi, \sigma} |f_i,\sigma|^2 \right)^{1/2} \lesssim \|S_\delta f_i\|_{L^2(\psi_\Delta)}, \) the above estimate may be written as

\[ \|f_i\|^2_{2,\sqrt{\Delta}} \lesssim \|S_\delta f_i\|_{L^2(\psi_\Delta)}. \]

By using this estimate and Hölder’s inequality,

\[ B_i \lesssim \left( \sum_{\Delta} \|S_\delta f_i\|_{L^2(\psi_\Delta)}^4 \right)^{1/4} \lesssim R^{3/4(\frac{1}{2} - \frac{1}{4})} \left( \sum_{\Delta} \|S_\delta f_i\|_{L^4(\psi_\Delta)}^4 \right)^{1/4} \lesssim R^{3/8} \|S_\delta f_i\|_4. \]

Thus we obtain (4.6).

4.4. Let \( \alpha \geq 0 \) be the best constant such that \( S\mathcal{Q}(4 \times 4 \times 4 \to 4; \alpha) \), i.e.,

\[ \alpha = \inf_{\delta > 0} \left( \log_{4/\delta} \sup_{f_i, \supp f_i \subset 4^{-\delta}_I} \prod_{f_i} \frac{\|f_i\|_4}{\prod_{f_i} \|S_\delta f_i\|_4} \right). \]

To prove \( S\mathcal{Q}(4 \times 4 \times 4 \to 4; 1/16) \) it is enough to show that for any \( \epsilon > 0, \)

\[ \alpha \leq \frac{1}{16} + C \epsilon. \]

By Hölder’s inequality,

\[ A_i \lesssim R^{\frac{3}{4}(\frac{1}{2} - \frac{1}{4})} \left( \sum_{\gamma \in \Pi, \sigma} \|f_i,\gamma\|_4^4 \right)^{1/4}. \]

By the definition of \( \alpha \) and Proposition 2.2 one has \( S\mathcal{Q}(4 \to 4; \alpha) \). By Lorentz rescaling, as in 2.3,

\[ \|f_i,\gamma\|_4 \leq C \epsilon R^{\alpha/2 + \epsilon} \|S_\delta f_i,\gamma\|_4. \]

So, we have

\[ A_i \leq C \epsilon R^{\alpha/2 + \epsilon} R^{\frac{3}{4}(\frac{1}{2} - \frac{1}{4})} \left( \sum_{\gamma \in \Pi, \sigma} \|S_\delta f_i,\gamma\|_4^4 \right)^{1/4}. \]
Since
\[
\sum_{\gamma \in \Pi_\sigma} \|S_\delta f_\gamma\|^4 \lesssim \sum_{\gamma \in \Pi_\sigma} \int (\sum_{\theta \in \Pi_\nu} \sum_{\theta \in \Omega_2 \gamma} |f_\gamma \theta|^2)^2 \lesssim \int (\sum_{\theta \in \Pi_\nu} \sum_{\theta \in \Omega_2 \gamma} |f_\gamma \theta|^2)^2 \lesssim \|S_\delta f_\gamma\|^4,
\]
we obtain
\[
A_1 \leq C \epsilon R^{1/16 + \alpha/2 + \epsilon} \|S f\|_4. \tag{4.7}
\]
Now we insert (4.7) and (4.6) into (4.3). Then,
\[
\|\prod f_j\|_{L^4(Q_R)} \leq C \epsilon R^{1/24 + \alpha/3 + C \epsilon} \prod \|S f\|_4.
\]
Since \(\alpha\) is the best constant holding \(S^2Q(4 \times 4 \times 4 \rightarrow 4; \alpha)\), we have \(\alpha \leq \frac{1}{24} + \frac{3}{16} + C \epsilon\). Therefore, \(\alpha \leq \frac{1}{32} + C \epsilon\). This completes the proof.

5. Proof of Theorem 1.3

In this section, Theorem 1.3 will be proved by using a corresponding trilinear estimate. Let us define an operator \(U_N\) by
\[
U_N f(x, t) = \hat{\eta}_N * e^{it \sqrt{-\Delta}} f(x)
\]
where \(\eta_N\) is a bump function supported in \(\{ \xi \in \mathbb{R}^2 : |\xi| \sim N \}\) and \(\hat{\eta}_N\) is the inverse Fourier transform of \(\eta_N\). By the Littlewood–Paley decomposition, to prove Theorem 1.3 it suffices to show that the estimate
\[
\|U_N f\|_{L^{30/7}(\mathbb{R}^2 \times [1, 2])} \leq C \epsilon N^{1/10 + \epsilon} \|f\|_{10/3}
\]
holds for all \(\epsilon > 0\), all \(N \geq 1\) and all \(f \in L^{10/3}(\mathbb{R}^2)\).

For convenience of rescaling we reform \(U_N f\) as follows. By a linear transformation \(J : (\xi_1, \xi_2, \xi_3) \mapsto (\zeta_1, \zeta_2, \zeta_3) = (\zeta_1 + \xi_1, \zeta_2 + \xi_2, \zeta_3 + \xi_3)\) which maps the cone \(\{(\xi_1, \xi_2, \pm \sqrt{x_1^2 + x_2^2})\}\) to the leaned cone \(\{(\zeta_1, \zeta_2, \zeta_3/\xi_1)\}\), we redefine \(U_N f\) by
\[
U_N f(x, t) = \int e^{2\pi i (x \cdot \xi + \xi_3/\xi_1)} \hat{f}(\eta_N) \varphi(\xi_2/\xi_1) d\xi, \quad \xi = (\zeta_1, \zeta_2),
\]
where \(\varphi\) is a bump function supported in the unit interval. Then, \(U_N f\) has Fourier support in
\[
\Gamma(N) := \{(\zeta_1, \zeta_2, \zeta_3/\xi_1) : |\zeta_1| \sim N, |\zeta_2/\xi_1| \leq 1\}.
\]
The leaned cone \((\zeta_1, \zeta_2, \zeta_3/\xi_1)\) is written as \(\xi_1(1, \theta, \theta^2)\) where \(\theta = \xi_2/\xi_1\). So one may identify \(\theta\) with an angular variable of the cone.

We say that the local smoothing estimate \(LS(p \to q; \alpha)\) holds if
\[
\|U_N f\|_{L^p(\mathbb{R}^2 \times [1, 2])} \leq C \epsilon N^\alpha \|f\|_p \tag{5.2}
\]
holds for all \(\epsilon > 0\), all \(N > 1\) and all \(f \in L^p(\mathbb{R}^2)\). To prove Theorem 1.3 it suffices to show
\[
LS(10/3 \to 30/7; 1/10).
\]
For given \(1 \leq p < q \leq \infty\) and \(\frac{1}{p} + \frac{3}{q} = 1\), we define
\[
\alpha = \alpha(p, q) \geq \frac{1}{p} - \frac{3}{q} + \frac{1}{2} \tag{5.3}
\]
to be the best exponent for which the estimate (5.2) holds for all $N > 1$ and all $f \in L^p(\mathbb{R}^2)$, i.e.,

$$\alpha(p, q) = \inf_{N > 1} \left( \log_N \sup_{f \in L^p(\mathbb{R}^2)} \frac{||U_N f||_{L^q(3^2 \times [1, 2])}}{||f||_p} \right).$$

Then it is enough to show that for all $\epsilon, \epsilon_1 > 0$,

$$\alpha \left( \frac{10}{3}, \frac{30}{7} \right) \leq \frac{1}{10} + C \epsilon_1 + \log_N C_{\epsilon, \epsilon_1},$$

since we may take $\epsilon = \epsilon_1$, which can be absorbed in an $\epsilon$-loss in (5.2).

5.1. Let an arbitrary small $\epsilon_1 > 0$ be given. Let $N \geq N_0$ and $1 > \gamma_1 > \gamma_2 \geq N_0^{-\epsilon_1/2}$. Later, $\gamma_1$, $\gamma_2$ and $N_0$ will be chosen. By rescaling and (a minor variant of) Lemma 2.1 one has that for any $(x, t) \in \mathbb{R}^2 \times [1, 2]$,

$$||U_N f(x, t)|| \leq \max_{\Omega \in \Omega(\gamma_1)} ||U_N^\Omega f(x, t)|| + \gamma^{-1}_1 \max_{\Omega \in \Omega(\gamma_2)} ||U_N^\Omega f(x, t)||$$

$$+ \gamma^{-50}_2 \max_{\Omega_1, \Omega_2, \Omega_3 \in \Omega(\gamma_2)} \left| \left( \frac{3}{||U_N^\Omega f(x, t)||} \right)^{1/3} \right|,$$

where $U_N^\Omega$ is defined as (5.1) with $\gamma \in \gamma(1)$ and (5.4) will be chosen. By rescaling and (a minor variant of) Lemma 2.1 one has that for any $(x, t) \in \mathbb{R}^2 \times [1, 2]$,

$$||U_N f||_{L^q(\mathbb{R}^2 \times I)} \leq \left( \sum_{\Omega \in \Omega(\gamma_1)} ||U_N^\Omega f||_{L^q(\mathbb{R}^2 \times I)} \right)^{1/q} + \gamma^{-1}_1 \left( \sum_{\Omega \in \Omega(\gamma_2)} ||U_N^\Omega f||_{L^q(\mathbb{R}^2 \times I)} \right)^{1/q}$$

$$+ \gamma^{-50}_2 \left( \sum_{\Omega_1, \Omega_2, \Omega_3 \in \Omega(\gamma_2)} ||U_N^{\Omega_i} f_i||_{L^q(\mathbb{R}^2 \times I)} \right)^{1/q},$$

where $I = [1, 2]$.

We consider the first and second summation in the right-hand side of (5.5). From rescaling and the definition of $\alpha$ it follows that

$$||U_N^\Omega f||_{L^q(\mathbb{R}^2 \times I)} \leq C\gamma_1^{3\left(\frac{1}{q} - \frac{1}{p}\right)} (\gamma_N^2 N)^{\alpha + \epsilon} ||f||_p.$$  (5.6)

More specifically, by rotating we may assume that $\Omega$ is centered at $0$. Then we may write $U_N^{\Omega_i} f$ as

$$U_N^{\Omega_i} f(x, t) = \int e^{2\pi i (x \cdot \xi + \xi^3 / \xi_1)} \hat{f}(\xi) \eta_N(\xi_1) \varphi(\gamma_1^{-1} \xi_2 / \xi_1) d\xi.$$  

Let $\sigma(x_1, x_2, t) = (\gamma_1^2 x_1, \gamma_1 x_2, t)$ and $\sigma_i(x_1, x_2) = (\gamma_i^2 x_1, \gamma_i x_2)$. Then, we have $U_N^{\Omega_i} f \circ \sigma = U_N^{\sigma_i} f$. Thus, using (5.2) and this relation we have (5.6).

If we define $f_0$ by

$$\hat{f_0}(\xi_1, \xi_2) = \hat{f}(\xi_1, \xi_2) \chi_{\{\xi_1 < N\}}(\xi_1) \chi_{\{\xi_2 / \xi_1\}},$$

then we may replace $U_N^{\Omega_i} f$ with $U_N^{\Omega_i} f_0$, where $\chi$ denotes a characteristic function. By (5.6),

$$\left( \sum_{\Omega_i \in \Omega(\gamma_1)} ||U_N^{\Omega_i} f_0||_q \right)^{1/q} \leq C\gamma_1^{3\left(\frac{1}{q} - \frac{1}{p}\right)} (\gamma_N^2 N)^{\alpha + \epsilon} \left( \sum_{\Omega \in \Omega(\gamma_1)} ||f_0||_p \right)^{1/q}. \tag{5.7}$$

We recall the following lemma from [20].
Lemma 5.1 ([20, Lemma 7.1]). Let \( R_k \) be a collection of rectangles such that the dilates \( 2R_k \) are almost disjoint, and suppose that \( f_k \) are supported on \( R_k \).

Then for all \( 1 \leq p \leq \infty \) we have

\[
\left( \sum_k \| f_k \|_p^{p^*} \right)^{1/p^*} \lesssim \left( \sum_k \| f_k \|_p \right)^{1/p^*} \lesssim \left( \sum_k \| f_k \|_{p^*} \right)^{1/p},
\]

where \( p^* = \min(p, p') \), \( p^* = \max(p, p') \).

It is remarked that Lemma 5.1 is elementary, and simply a consequence of interpolation between Plancherel’s theorem and Minkowski’s inequality for the \( L^\infty \) space.

After embedding \( \ell^p \subset \ell^q \) in the right-hand side of (5.5), we apply Lemma 5.1. Then we obtain

\[
\left( \sum_{\Omega_i \in \Omega(\gamma_1)} \| U_{\Omega_i}^\alpha f \|_{L^q(\mathbb{R}^2 \times I)} \right)^{1/q} \leq C \gamma_1^{3(\frac{1}{q} - \frac{1}{p})} (\gamma_1^2 N)^{\alpha + \epsilon} \| f \|_p. \tag{5.8}
\]

5.2. We consider the last summation in the right-hand side of (5.5). We will show that for any \( \epsilon > 0 \),

\[
\left\| \prod_{\Omega_i \in \Omega(\gamma_1)} U_{\Omega_i}^\alpha f \right\|_{L_{30/7}(\mathbb{R}^2 \times I)} \leq C \epsilon N^{1/10 + \epsilon} \| f \|_{10/3}. \tag{5.9}
\]

First we prove a corresponding local estimate.

Lemma 5.2. Let \( B \) be a unit ball. Then, for any \( \epsilon > 0 \),

\[
\left\| \prod_{\Omega_i \in \Omega(\gamma_1)} U_{\Omega_i}^\alpha f_i \right\|_{L_{30/7}(B \times I)} \leq C \epsilon N^{1/10 + \epsilon} \prod_{i} \| f_i \|_{10/3}. \tag{5.10}
\]

Proof. By interpolation it suffices to show

\[
\left\| \prod_{\Omega_i \in \Omega(\gamma_1)} U_{\Omega_i}^\alpha f_i \right\|_{L^6(B \times I)} \leq C \epsilon N^{1/6 + \epsilon} \prod \| f_i \|_6; \tag{5.11}
\]

\[
\left\| \prod_{\Omega_i \in \Omega(\gamma_1)} U_{\Omega_i}^\alpha f_i \right\|_{L^3(B \times I)} \leq C \epsilon N^{\epsilon} \prod \| f_i \|_2. \tag{5.12}
\]

Consider \(5.11\). By Hölder’s inequality it is enough to show

\[
\| U_N f \|_{L^6(B \times I)} \leq C \epsilon N^{1/6 + \epsilon} \| f \|_6. \tag{5.13}
\]

Since \( \psi(t) U_N f(x, t) \) has Fourier support in a \( C \)-neighborhood of \( \Gamma(N) \), from Theorem 1.2 and rescaling it follows that

\[
\| U_N f \|_{L^6(B \times I)} \leq C \epsilon N^{\epsilon} \left( \sum_{\Theta} \| (\psi U_N f) * (\Xi_\Theta) \|_6 \right)^{1/2},
\]

where \( \Theta \) is a sector of size \( CN^{1/2} \times CN \times C \). By Hölder’s inequality, this is bounded by

\[
\leq C \epsilon N^{1/6 + \epsilon} \left( \sum_{\Theta} \| (\psi U_N f) * (\Xi_\Theta) \|_6 \right)^{1/6}.
\]

It is well known (see, e.g., [23, Lemma 6.1], [19, XI: 4.13], [15]) that for \( p \geq 2 \),

\[
\left( \sum_{\Theta} \| (\psi U_N f) * (\Xi_\Theta) \|_p \right)^{1/p} \lesssim \| f \|_p.
\]

Thus, we obtain \(5.13\).
Consider (5.12). In (4.1), the restriction operator $\tilde{f}_j d\sigma_j$ can be replaced with $U_1^{1j} f$ where $\tilde{f}$ denotes the inverse Fourier transform of $f$. Thus, from Theorem 4.1 and Plancherel’s theorem it follows that

$$\left\| \prod U_1^{1\Omega_i} f_i \right\|_{L^1(Q_N)} \leq C_N \epsilon \prod \|f_i\|_2.$$  

If $s(x, t) = N^{-1} (x, t)$ and $\tilde{s}(x) = N^{-1} x$, then $U_1^{1\Omega} f \circ s = U_1^{1\Omega} (f \circ \tilde{s})$. So, by changing variables and translation invariance, the above estimate gives (5.12). □

We now prove that (5.11) implies (5.9). This immediately follows from the next localization lemma.

**Lemma 5.3.** Suppose that the local estimate

$$\left\| \prod U_N^{1\Omega_i} f_i \right\|_{L^p(B \times I)} \leq A(N) \prod \|f_i\|_p$$  

(5.14)

holds for all unit cubes $B$ and all $f_i \in L^p(\mathbb{R}^2)$. If $p \leq q$ then the estimate

$$\left\| \prod U_N^{1\Omega_i} f_i \right\|_{L^q(\mathbb{R}^2 \times I)} \leq C N^\epsilon A(N) \prod \|f_i\|_p$$  

(5.15)

holds for all $\epsilon > 0$ and all $f_i \in L^p(\mathbb{R}^2)$.

**Proof.** We write as

$$U_N f(x, t) = (K_N(t) * f)(x)$$

where

$$K_N(t)(x) = K_N(x, t) := \int e^{2\pi i (x \cdot \xi + t \xi_1^2 / \xi_1)} \eta_N(\xi_1) \varphi(\xi_2 / \xi_1) d\xi.$$  

By using a stationary phase method, it follows that for $(x, t) \in \mathbb{R}^2 \times I$,

$$|K_N(t)(x)| \leq C_M N^2 (1 + |x|)^{-M} \quad \forall M > 0.$$  

Thus, for $(x, t) \in \mathbb{R}^2 \times I$,

$$|U_N f(x, t)| \leq C_M (a_N * |f|)(x), \quad \forall M > 0,$$

(5.16)

where $a_N(x) = N^2 (1 + |x|)^{-M}$.

If a unit lattice square $B \subset \mathbb{R}^2$ is given, then we decompose

$$|U_N f| \chi_{B \times I} \lesssim |U_N (f \chi_{N^2 B})| \chi_{B \times I} + C_M |\mathcal{E}_{B^*} f| \chi_{B \times I},$$

(5.17)

where

$$\mathcal{E}_{B^*} f := a_N * (|f| \chi_{\mathbb{R}^2 \setminus N^2 B}).$$

Consider $|\mathcal{E}_{B^*} f| \chi_{B \times I}$. If $|x - y| \gtrsim N^\epsilon$ then one has $a_N(x - y) \lesssim N^2 N^{-\epsilon M} \lesssim N^{-2000C}$. So, we have

$$\chi_B(x)(a_N * (|f| \chi_{\mathbb{R}^2 \setminus N^2 B}))(x) = \chi_B(x) \int a_N(x - y) \chi_{\mathbb{R}^2 \setminus N^2 B}(y) |f(y)| dy$$

$$\lesssim N^{-1000C} \chi_B(x) \int a_N^{1/2}(x - y) |f(y)| dy$$

$$\lesssim N^{-1000C} \chi_B(x)(a_N^{1/2} * |f|)(x).$$

Thus, by Young’s inequality we obtain

$$\left( \sum_B \|\mathcal{E}_{B^*} f\|_{L^q(B)}^q \right)^{1/q} \lesssim N^{-900C} \|f\|_p,$$

(5.18)
On the other hand, by some rough estimates (cf. Young’s inequality) we see that \( \|U_N f\|_{L^q(B \times I)} \lesssim N^C \|f\|_p \). So, by embedding \( \ell^p \subset \ell^q \), we have

\[
\left( \sum_B \|U_N(f \chi_{N \cdot B})\|_{L^q(B \times I)}^q \right)^{1/q} \lesssim N^C \left( \sum_B \|f\|_{L^p(N \cdot B)}^q \right)^{1/q} \lesssim N^{2C} \|f\|_p. \tag{5.19}
\]

Now, we consider the estimate \( (5.15) \) by using \( (5.18) \) and \( (5.19) \). We define \( f_{\Omega_i} \) as

\[
\hat{f}_{\Omega_i}(\xi_1, \xi_2) = \hat{f}_i(\xi)\eta_N(\xi_1)\varphi_N(\xi_2/\xi_1).
\]

Then we may replace \( U_N^{\Omega_i} f_i \) with \( U_N f_{\Omega_i} \). By \( (5.17) \),

\[
\prod U_N f_{\Omega_i \cdot B \times I} \lesssim \prod \left( \|U_N(f_{\Omega_i \cdot N \cdot B})\|_{L^q(N \cdot B \times I)} + C_M (E_{B^*} f_{\Omega_i}) \right)
\lesssim \prod \left( \|U_N(f_{\Omega_i \cdot N \cdot B})\|_{L^q(N \cdot B \times I)} + C_M \mathcal{E}(f_{\Omega_1}, f_{\Omega_2}, f_{\Omega_3}) \chi_{B \times I} \right), \tag{5.20}
\]

where

\[
\mathcal{E}(f_{\Omega_1}, f_{\Omega_2}, f_{\Omega_3}) := \sum_{i,j,k \in \{1,2,3\}} \mathcal{E}_{B^*} f_{\Omega_i} \left( \|U_N(f_{\Omega_j \cdot N \cdot B})\|_{L^q(N \cdot B \times I)} \right)^{1/3} + \prod \mathcal{E}_{B^*} f_{\Omega_i}.
\]

By Minkowski’s inequality,

\[
\left( \sum_B \|\mathcal{E}(f_{\Omega_1}, f_{\Omega_2}, f_{\Omega_3})\|_{L^q(B \times I)}^q \right)^{1/q} \lesssim \max_{i,j,k} \left( \sum_B \|\mathcal{E}_{B^*} f_{\Omega_i} \left( \left\|U_N(f_{\Omega_j \cdot N \cdot B})\|_{L^q(N \cdot B \times I)} \right\|_{L^q(B \times I)} \right)^{1/3} \right)^{1/q} \leq \left( \sum_B \|\mathcal{E}_{B^*} f_{\Omega_i}\|_{L^q(B \times I)}^q \right)^{1/q} + \prod \mathcal{E}_{B^*} f_{\Omega_i}. \tag{5.21}
\]

Consider the right-hand side of \( (5.21) \). By Hölder’s inequality,

\[
\left( \sum_B \left( \|\mathcal{E}_{B^*} f_{\Omega_i} \left( \left\|U_N(f_{\Omega_j \cdot N \cdot B})\|_{L^q(N \cdot B \times I)} \right\|_{L^q(B \times I)} \right)^{1/3} \right)^q \right)^{1/q} \leq \left( \sum_B \|\mathcal{E}_{B^*} f_{\Omega_i}\|_{L^q(B \times I)}^q \right)^{1/3q} \left( \sum_B \left\|U_N(f_{\Omega_j \cdot N \cdot B})\|_{L^q(B \times I)} \right\|_{L^q(B \times I)} \right)^{1/3q} \times \left( \sum_B \left\|U_N(f_{\Omega_k \cdot N \cdot B})\|_{L^q(B \times I)} \right\|_{L^q(B \times I)} \right)^{1/3q}.
\]

Thus, by \( (5.18) \) and \( (5.19) \) it is bounded by

\[
\lesssim N^{-200C} \prod \|f_{\Omega_i}\|_p.
\]

The second and third summations in the right-hand side of \( (5.21) \) are estimated by an analogous method. Thus,

\[
\left( \sum_B \|\mathcal{E}(f_{\Omega_1}, f_{\Omega_2}, f_{\Omega_3})\|_{L^q(B \times I)}^q \right)^{1/q} \lesssim N^{-200C} \prod \|f_{\Omega_i}\|_p. \tag{5.22}
\]
By \((5.20)\)
\[
\left\| \prod U N f_{\Omega_{1}} \right\|_{L^q(\mathbb{R}^2 \times I)} = \left( \sum_B \left\| \prod U N f_{\Omega_{1}} \right\|_{L^q(\mathbb{R}^2 \times I)}^q \right)^{1/q}
\lesssim \left( \sum_B \left\| \prod U N (f_{\Omega_{1}} \chi_{N^* B}) \right\|_{L^q(\mathbb{R}^2 \times I)}^q \right)^{1/q} + \left( \sum_B \left\| \mathcal{E}(f_{\Omega_{1}}, f_{\Omega_{2}}, f_{\Omega_{3}}) \right\|_{L^q(\mathbb{R}^2 \times I)}^q \right)^{1/q}.
\]

By \((5.14), (5.22)\) and embedding \(\ell^p \subset \ell^q\), it follows that
\[
\left\| \prod U N f_{\Omega_{1}} \right\|_{L^q(\mathbb{R}^2 \times I)} \lesssim (N/e A(N) + N^{-200C}) \prod_B \left\| f_{\Omega_{1}} \right\|_p.
\]
Since \(\left\| f_{\Omega_{1}} \right\|_p \lesssim \left\| f_1 \right\|_p\) by Young’s inequality, we obtain \((5.15)\) \qed

5.3. Last of all, we will show \((5.4)\). We substitute \((5.8)\) and \((5.9)\) in \((5.5)\) with \((p, q) = (10/3, 30/7)\). Then, it follows that
\[
\left\| U_N f \right\|_{L^{30/7}(I \times \mathbb{R}^d)} \lesssim (\gamma_1^{2\alpha - \frac{1}{2} + 2\epsilon} N^{\alpha + \epsilon} + \gamma_1^{-1} + \gamma_2^{2\alpha - \frac{1}{2} + 2\epsilon} N^{\alpha + \epsilon} + C_{\epsilon, \ell} \gamma_2^{-60} N^{\frac{1}{10} - \epsilon}) \left\| f \right\|_{10/3},
\]
So, by the assumption that \(\alpha\) is a best exponent,
\[
N^\alpha \leq C (\gamma_1^{2\alpha - \frac{1}{2} + 2\epsilon} + \gamma_2^{2\alpha - \frac{1}{2} + 2\epsilon}) N^\alpha + C_{\epsilon, \ell} \gamma_2^{-60} N^{1/10}.
\]
Observe that \(2\alpha - \frac{1}{2} \geq 0\) by \((5.3)\). We now choose \(\gamma_1, \gamma_2\) and \(N_0\) so that \(C_{\ell} \gamma_1^{2\alpha - \frac{1}{2} + 2\epsilon} \leq 1/4, C_{\ell} \gamma_2^{2\alpha - \frac{1}{2} + 2\epsilon} \leq 1/4\) and \(1 > \gamma_1 > \gamma_2 \geq N_0^{-\epsilon/2}\). Then \(N^\alpha \leq C_{\epsilon, \ell} N^{1/10 + 30\epsilon/1}\). Thus we obtain \((5.4)\).

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