Generalized Additive Entropies in Fully Developed Turbulence

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Abstract

We explore a possible application of the additive generalization of the Boltzmann-Gibbs-Shannon entropy proposed in [A. N. Gorban, I. V. Karlin, Phys. Rev. E, 67:016104 (2003)] to fully developed turbulence. The predicted probability distribution functions are compared with those obtained from Tsallis' entropy and with experimental data. Consequences of the existence of hidden subsystems are illustrated.

Key words: Nonextensive statistical mechanics, Fully developed turbulence

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1 Introduction

In the past years, a body of experimental data of stationary statistical distributions has been collected that are not well described by the usual Boltzmann-Gibbs distribution (see e.g. [1] and references therein). In many cases, the concept of Tsallis entropy [2] together with the maximum entropy principle has been found useful to describe these data more accurately [1]. The one-parametric family of Tsallis’ entropies is defined as

$$S_q = \frac{1 - \sum_i p_i^q}{q - 1},$$

where \( q > 0 \) is the so-called nonextensivity parameter. If a system consists of two statistically independent subsystems then Tsallis’ entropy of this system is not equal to the sum of the Tsallis’ entropies of the subsystems for \( q \neq 1 \). The additivity is recovered only in the limit \( q \to 1 \), where Tsallis’ entropy (1) reduces to the classical Boltzmann-Gibbs-Shannon entropy \( S_1 \),

$$S_1 = \lim_{q \to 1} S_q = -\sum_i p_i \ln p_i.$$  

Since Tsallis’ entropy is postulated rather than derived, this point is open to discussion [3,4].

Very recently, two of the present authors have derived a unique trace–form extension of the classical Boltzmann-Gibbs-Shannon entropy to finite systems that is still additive under joining statistically independent subsystems [5],

$$S_\alpha^* = -(1 - \alpha) \sum_i p_i \ln(p_i/p_i^0) + \alpha \sum_i \ln(p_i/p_i^0), \quad 0 \leq \alpha \leq 1.$$  

In the limit \( \alpha \to 0 \), the Boltzmann-Gibbs-Shannon entropy is recovered, \( S_1 = S_0^* \). In Eq. (3), \( p_i^0 \) denotes a general reference equilibrium state. In the sequel, we use equipartition as a reference, \( p_i^0 = \text{const} \). Using a different parameterization, Eq. (3) thus becomes

$$S_\alpha^* = -\sum_i p_i \ln p_i + \alpha \sum_i \ln p_i, \quad \alpha \geq 0$$

up to adding a constant and multiplying by a constant factor. It is shown in Ref. [5] that the maximum entropy principle applied to this extensive generalized entropy results in non-exponential tails of probability distributions which are not accessible within classical Boltzmann-Gibbs-Shannon entropy (see also Refs. [6] and [7]). The present paper provides an application of the additive
generalized entropies (4) to experimental data on fully developed turbulence, where non-exponential tails of probability distributions have been observed [8,9]. We also present consequences of incomplete description or hidden subsystems discussed in Ref. [5]. Limitations of such an approach are discussed in Ref. [10].

This paper is organized as follows. In Section 2, we briefly present the approach of Ref. [8] based on Tsallis’ entropy and contrast it to the alternative approach based on the additive entropies (4). In Section 3, we compare the results of the two approaches for describing the experimental data of Refs. [8,9].

2 Maximum Entropy and Generalized Probability Densities

The maximum entropy principle under suitable constraints can be used to derive many relevant statistical distributions in physics but also in a wide range of other problems such as image reconstruction and time series analysis (see e.g. A. R. Plastino in [1] and references therein). Using only conserved quantities as constraints, the corresponding maximum entropy distributions describe equilibrium situations while the use of non-conserved quantities as constraints offers access to time-dependent processes.

Consider a system with continuous state variable $u$, which is characterized by the energy $\epsilon(u)$. Let $p(u)$ denote the probability density of state $u$. Extremizing the entropy functional $S[p]$ subject to the constraint of fixed normalization $\int du p(u) = 1$ and fixed total energy $\int du p(u) \epsilon(u) = E$, leads to

$$\frac{\delta S[p]}{\delta p(u)} = \lambda_0 + \beta \epsilon(u),$$

(5)

where $\delta/\delta p$ denotes the Volterra functional derivative and $\lambda_0$ and $\beta$ are Lagrange multipliers that satisfy the constraints of fixed normalization and total energy, respectively. Thus, $\beta$ can be interpreted as a suitable inverse temperature. The solution to Eq. (5) gives the relevant or maximum entropy distribution consistent with the constraints.

In case of Tsallis’ entropy (1), the solution to Eq. (5) are the probability distributions [11]

$$p_q(u) = \frac{1}{Z_q} [1 + \beta(q - 1)\epsilon(u)]^{1/(q-1)},$$

(6)

where $Z_q$ denotes a normalization constant. In the limit $q \to 1$, Eq. (6) reduces to a Boltzmann factor $p_1(u) \propto e^{-\beta \epsilon(u)}$. 

3
If instead of Tsallis’ entropy (1) the family of additive entropy functions (4) is used, the maximum entropy condition (5) becomes [5,6]

\[ \ln p^*_\alpha(u) - \alpha/p^*_\alpha(u) = -\Lambda(u), \]  

(7)

where \( \Lambda(u) = 1 + \lambda_0 + \beta\varepsilon(u) \). Solving Eq. (7) for \( p^*_\alpha \) leads to

\[ p^*_\alpha(u) = \frac{\alpha}{\text{lm}(\alpha e^{\Lambda(u)})}. \]  

(8)

In Eq. (8), use has been made of the modified logarithm \( \text{lm} y \), that denotes the solution to the transcendent equation \( xe^x = y \). Note, that in the limit \( \alpha \to 0 \), the Boltzmann distribution is recovered from Eq. (8).

Due to their different analytical form, a direct comparison of the distribution functions (6) and (8) is difficult. In the regime \( \beta|\varepsilon| \ll 1 \), both, Eq. (6) and Eq. (8) reduce to \( p \propto (1 - \beta\varepsilon) \). For \( \beta|\varepsilon| \gg 1 \), however, the distribution functions \( p_q \) and \( p^*_\alpha \) in general show a different behavior. While \( p_q \propto (\beta\varepsilon)^{-1/(1-q)} \) depends on the value of \( q \) for \( \beta|\varepsilon| \gg 1 \), one finds a universal behavior \( p^*_\alpha \propto (\beta\varepsilon)^{-1} \), independent of the parameter \( \alpha \) [5]. In the following section, some comparisons of Eqs. (6) and (8) are presented for a special choice of \( \varepsilon \).

Before proceeding to a specific example, we briefly address the problem of incomplete description as presented in Sec. V. of Ref. [5]. Incomplete description in this context means, that in addition to \( p \) other components or hidden subsystems \( g(p) \) exist, whose entropy has to be taken into account. Define the two-parametric family of entropy functionals

\[ S^*_{\alpha,t}[p] = (1 - t)S^*_\alpha[p] + tS^*_\alpha[g(p)], \quad 0 \leq t \leq 1. \]  

(9)

In case of no hidden subsystems, \( t = 0 \), Eq. (9) reduces to (4), \( S^*_{\alpha,t=0} = S^*_\alpha \). Applying the maximum entropy principle to the extended family of entropy functionals (9), Eq. (7) generalizes to

\[ (1 - t)\{\ln p^*(u) - \alpha/p^*(u)\} + t\{\ln g(p^*(u)) - \alpha/g(p^*(u))\}J = -\Lambda_t(u), \]  

(10)

where \( \Lambda_t(u) = \Lambda(u) + t(J - 1) \) and \( J = \delta g(p^*)/\delta p(u) \). In particular, we consider the case

\[ g = 1 - \mu p, \quad 0 \leq \mu \leq 1. \]  

(11)

The Fermi-Dirac entropy, for example, corresponds to \( t = 1/2, \alpha = 0, \) and \( \mu = 1 \). Explicit solution of Eq. (10) for \( p^* \) is possible only for special cases.
Results for Fully Developed Turbulent Flows

The authors of Ref. [8] studied velocity differences of high Reynolds number flow in a Taylor-Couette apparatus. For sufficiently small distances, the experimentally observed probability distribution function of the velocity differences clearly shows non-exponential tails.

In order to apply Eqs. (6) and (8) to this system, the expression for the energy $\epsilon$ has to be specified appropriately. In Ref. [8], the energy $\epsilon(u)$ is assumed to be given by

$$\epsilon(u) = \frac{1}{2}|u|^{2\zeta} - c\sqrt{\tau}\gamma \text{sgn}(u)(|u|^\zeta - \frac{1}{3}|u|^{3\zeta}),$$

where the skewness $c$ and the intermittency parameter $\zeta$ are related to Tsallis’ $q$-parameter by $c\sqrt{\tau}=0.124(q-1)$ and $\zeta=2-q$, respectively. Thus, only one independent parameter $q$ is left. Beck offers in Ref. [12] some arguments in favor of Eq. (12) for the case $\zeta=1$, while the extension to $\zeta \neq 1$ in Ref. [8] is done in analogy to turbulence modeling.

For the distribution functions (8), the parameter $\alpha$ needs to be specified. Remember, that the classical Boltzmann-Gibbs-Shannon entropy is recovered for $q \to 1$ in case of Tsallis’ entropy (1) and for $\alpha \to 0$ in case of the additive generalized entropy (4). We suggest that the parameters $q$ and $\alpha$, describing the deviation from the classical Boltzmann-Gibbs-Shannon entropy are related by $\alpha = (q - 1)^{\nu}$ with some exponent $\nu$. Since the parameters $q$ and $\alpha$ describe the non-ergodicity of the phase space dynamics, this relation might be interpreted in terms of excluded volume in phase space. Below, we use this simple power law relation as a plausible mapping between the parameters of both the theories.

Fig. 1 shows a comparison of the probability distribution functions (6) and (8) with the energy given by (12). The values of the nonextensivity parameter $q$ are the same as used in Ref. [8]. For comparison with Eq. (8), exactly the same expressions for the energy, the parameters $\zeta$, $\beta$ and $c$ are used. We also choose exactly the same values for $q$ as done in Ref. [8]. Thus, the only parameter left is the exponent $\nu$, relating Tsallis’ nonextensivity parameter $q$ to $\alpha$. The same value $\nu = 2.25$ has been chosen in all cases to determine the parameter $\alpha$ in Eq. (8). On a linear scale, Fig. 1 (a), the curves $p_q$ and $p_\alpha^*$ are almost indistinguishable by the naked eye. On a logarithmic scale however, differences between these curves are seen to become important for $u \gtrsim 4$. As mentioned above, the asymptotic behavior of $p_\alpha^*$, Eq. (8), is independent of $\alpha$, while the decay of $p_q$, Eq. (6), can be varied by varying $q$. Thus, it appears the the distribution functions (6) describe the experimental data of Ref. [8]
better than Eq. (8). We like to mention, however, that we made no attempt to improve the agreement of the distribution functions (8) with Eq. (6) by varying the relations between the parameters $\zeta$, $\beta$, $c$ and $q$.

In Ref. [13], Beck provides a comparison of Eq. (6) to the experimental results of La Porta et al., [9]. In the latter experiment, the acceleration of a test particle in a fully developed turbulent flow was measured. If the acceleration $a$ is interpreted as velocity difference on the smallest time scale of the turbulent flow (Kolmogorov time scale), the previous consideration apply also to this experiment with the identification $u = a/\sqrt{\langle a^2 \rangle}$. Fig. 2 shows a comparison of the experimental results [9] to the formulas (6) and (8) with $q = 1.49$, $\zeta = 0.92$, $\beta = 4$ and $c = 0$, which are the values of the parameters proposed in [13]. As noted in Ref. [13], Eq. (6) with this choice of parameters provides a very good description of the experimental results. The distribution function (8) with the same values of parameters, however, overestimates the tails significantly already at $u \gtrsim 1$. The same value $\nu = 2.25$ as before was chosen. Fig. 2 also shows the result of numerical solutions to Eq. (10) for $t = \mu = 0.5$ where all other parameters remain unchanged. Fig. 2 demonstrates that inclusion of a single hidden subsystem decreases the tails of the probability distribution. In the present case, this decrease leads to an improved comparison with the experimental data in a range $|u| \lesssim 3$. By including more hidden subsystems, a systematic improvement in the description of the experimental data is possible.

4 Conclusions

We have presented an application of the additive generalization of the Boltzmann-Gibbs-Shannon entropy presented in Ref. [5] to experimental data in fully developed turbulence [8,9]. We found good agreement between the generalized distributions and the experimental data when compared on a linear scale. On a logarithmic scale, however, discrepancies in the tails of the distribution functions are evident. In particular, the generalized distributions overpredict the tails in comparison with experiments. Improved experimental results presented very recently in Ref. [14] seem to indicate that the tails of the probability distribution do not obey a power law behavior. Thus, it appears that the maximum entropy distributions obtained either from Tsallis’ or from the generalized additive entropy do not describe the tails of the distribution correctly, at least for the experimental results of [9,14].

It should be mentioned, that we used the same values of parameters that give very good agreement to the power-law distributions obtained from Tsallis’ entropy and made no attempt to optimize this choice for the new distribution functions obtained from the generalized additive entropies. Rather then trying to improve the fit of the experimental data by choosing different values of
the parameters, we illustrate the consequences of incomplete description. We found that the inclusion of the entropy of a single hidden subsystem helps to improve the comparison to the experimental data significantly. Systematic improvements by including more hidden subsystems is straightforward.

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Fig. 1. Probability distribution functions $p_q(u)$, Eq. (6), solid lines, and $p_\alpha^*(u)$, Eq. (8), dashed lines: (a) Linear plot, (b) logarithmic plot. The values of the nonextensivity parameter $q$ are from top to bottom: $q = 1.168, 1.150, 1.124, 1.105, 1.084, 1.065, 1.055$ and $1.038$, respectively. These are the same values that are used in Ref. [8] to describe experimental results of velocity differences. For all curves the value $\alpha$ in Eq. (8) has been chosen as $\alpha = (q - 1)^\nu$ with $\nu = 2.25$. 
Fig. 2. Experimentally measured probability distribution of normalized acceleration $u = a/\sqrt{\langle a^2 \rangle}$ in Lagrangean turbulence as measured by La Porta et al. [9], symbols, and comparison with functions $p_q(u)$, Eq. (6), solid line, and $p^*_\alpha(u)$, Eq. (8), dashed lines. The values of the parameters are $q = 1.49$, $\zeta = 0.92$, $c = 0$ for Eq. (6). The dashed lines correspond from top to bottom to Eq. (8) with $\nu = 2.25$, Eq. (10) with the same parameters and $t = \mu = 0.5$ and Eq. (10) with $q = 1.49$, $\zeta = 1.0$, $c = 0$, $\nu = 2.25$ and $t = \mu = 0.5$. 