Equivalence of Two Nonequilibrium Ensembles Based on Maximum Entropy Principle

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Abstract

The relation between two versions of so-called nonequilibrium statistical operator method (NESOM), NESOM-1 due to Zubarev (1961) and NESOM-2 due to Zubarev and Kalashnikov (1970), is considered. It is proved that, once the balance equations of NESOM-2 are satisfied, those of NESOM-1 will be satisfied with the same set of the macro-parameters. The proof uses the convexity-type inequalities, and does not involve any assumptions additional to the rationales behind NESOM. However, converse statement cannot be proved within this technique. An extension of the proof to overall equivalence of the two nonequilibrium ensembles is discussed.

Dedicated to the memory of Dmitrii Nikolaevich Zubarev
Preface

The methods of non-equilibrium statistical ensembles based on maximum entropy (MaxEnt for short) principle are nowadays well documented. One of them, developed independently by Zubarev [1] and McLennan [2] extended the Gibbs ensembles method to non-equilibrium. Later method, due to Zubarev and Kalashnikov [3, main text], [4, main text] made also use of the motion quasi-invariant construction [1] but applied it to the MaxEnt distribution rather than to the entropy itself. Both developments are referred to as Zubarev’s Nonequilibrium Statistical Operator Method (NESOM), while the notations NESOM-1 for the earlier and NESOM-2 for the later version of NESOM, respectively, were introduced to distinct them [6]. Other methods, which are close to NESOM, were published in the interim [3], [4], [5].

When NESOM-2 became a method of choice, the expected question of its equivalence to NESOM-1 was promptly addressed that time [6]. Other proofs of the equivalence, restricted to classical systems, were published later [7], [8]. More later, the present author and Kalashnikov showed that the above mentioned proofs used assumptions, not found among the NESOM rationales, and proposed another proof [9], which seems satisfactory from the viewpoint of rigor adopted in physics. Though available in English, Ref.[9] remained un-noted - the recent review [10] refers the proofs of Refs.[6], [8] to as the ultimate results on the problem. For that, and not only that, reason I decided to republish Ref.[9] on Internet. My motivation for this is better explained by the passage below.

Albeit much more applications were treated using NESOM-1 than NESOM-2, no working perturbation technique beyond lowest-order approximation (such as high-temperature or weak-interaction approximations) were developed in the former method. Moreover, because of rather involved statistical distribution structure in NESOM-1, such a development seems desperate. On the other hand, the statistical distribution construction in NESOM-2 is certainly analogous to the scattered wave function construction in the Gell-Mann and Goldberger form of scattering theory, the MaxEnt distribution being an analog of the incoming wave function. Due that analogy, the Gell-Mann-Goldberger integral equation based perturbation technique applies to NESOM-2 [3, main text]. So, once the equivalence of NESOM-1 and NESOM-2 is proved, there is no more need to deal with cumbersome and intractable perturbation series of NESOM-1.

Just within the NESOM-2 framework, on such important examples as plasma screening and Kondo effect at non-equilibrium, partial summation of the perturbation series was shown to be performable [11]. However, this trend was not focused on since then. As a result, in spite of its conceptual advantages (see review by Luzzi et al. [10]), NESOM
legs behind of regular methods (such as the Keldysh and Kadanoff-Baym ones) regarding the state-of-art respect. Nevertheless I believe that the potential of NESOM-2 (contrary to NESOM-1) has not yet been exhausted and that a proper diagrammatic technique development would benefit NESOM.

The present publication is rather a ‘remake’ than reproduction of Ref.[9]. The text of the paper is essentially revised but the references remained in the format of the original. (In Preface the citation of Russian papers and books, even cited in the paper text, done on their English translations, when available). Upon revision I did not aimed to make the proof as rigorous as the modern Mathematical Physics standard requires [12], since nowadays the level of rigor customary for Equilibrium Statistical Mechanics [13] still remains unreachable for most problems of Nonequilibrium Statistical Mechanics.

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1 Introduction

NESOM is formally based on special construction, which involves the so called quasi-equilibrium statistical operator (QESO)

\[ \rho_q(t, 0) = e^{-S(t, 0)}, \quad S(t, 0) = \Phi(t) + \sum_n F_n(t) P_n. \quad (1) \]

Here \( t \) is the time variable (which is dummy one unless dynamics enters the play), \( \{P_n\} \) is a set of gross variables, that is observables expectation values of which describe the non-equilibrium state of interest at instant \( t \). \( \Phi(t) \) is the Massieur-Planck function

\[ \Phi(t) = \ln \text{Tr} \left[ e^{-\sum_n F_n(t) P_n} \right] \quad (2) \]

and \( F_n(t) \) are the macro-parameters conjugated, in thermodynamical sense, to the gross variables averages \( \langle P_n \rangle_q^t \). The first set of the equations, connecting \( F_n(t) \) and \( \langle P_n \rangle_q^t \), follows from Eq.(2)

\[ \langle P_n \rangle_q^t = \text{Tr} [\rho_q(t, 0) P_n] = -\frac{\partial \Phi(t)}{\partial F_n(t)} \quad (3) \]

To arrive at the second set, define over the whole set of density matrices \( \rho \) the information entropy functional

\[ \mathcal{E}[\rho] = -\text{Tr}(\rho \ln \rho), \quad (4) \]

and the quasi-equilibrium entropy

\[ \Sigma(t) = \mathcal{E}[\rho_q] = \langle S(t, 0) \rangle_q^t = \Phi(t) + \sum_n F_n(t) \langle P_n \rangle_q^t; \quad (5) \]

thus \( S(t, 0) \) may be called the entropy operator. Then it is proven that

\[ F_n(t) = \frac{\partial \Sigma(t)}{\partial \langle P_n \rangle_q^t}. \quad (6) \]

Consider now the dependence of the operators on time. Any operator \( Q \) may explicitly depend on \( t \) that is notated by \( Q(t, 0) \) (this notation has already been used above for QESO and the entropy operator). Operator \( Q(t, 0) \) is said to be dynamics invariant (or integral) if it satisfies the Liouville equation

\[ \dot{Q}(t, 0) = \frac{\partial Q(t, 0)}{\partial t} + i\Sigma(t)Q(t, 0) = 0, \quad (7) \]
where $\mathcal{L}(t)$ is the Liouville super-operator, which action is given by the commutator (in classical case Poisson-bracket) with the Hamiltonian. In particular, any non-equilibrium statistical operator (NESO) satisfies Eq.(7) and QESO does not. One of the crucial concepts in NESOM is the dynamics quasi-invariant (or quasi-integral). Given an operator $Q(t,0)$ and a positive number $\varepsilon$, define the dynamics quasi-integral associated with $Q(t,0)$ by

$$\varepsilon \tilde{Q}(t,0) = \varepsilon \int_{-\infty}^{t} e^{\varepsilon(t_0-t)} Q(t_0,t_0-t) \, dt_0 = Q(t,0) - \int_{-\infty}^{t} e^{\varepsilon(t_0-t)} \dot{Q}(t_0,t_0-t) \, dt_0,$$

where $Q(t_1,t_1-t_2) = \mathfrak{U}(t_1,t_2) Q(t_1,0)$ and $\mathfrak{U}(t_1,t_2)$ is the dynamic evolution super-operator. It satisfies the couple of dual to each other equations

$$\frac{\partial}{\partial t_1} \mathfrak{U}(t_1,t_2) = \mathfrak{U}(t_1,t_2)i\mathcal{L}(t_1) \tag{9}$$

and

$$\frac{\partial}{\partial t_2} \mathfrak{U}(t_1,t_2) = -i\mathcal{L}(t_2) \mathfrak{U}(t_1,t_2) \tag{10}$$

with the initial conditions $\mathfrak{U}(t_2,t_2) = \mathfrak{U}(t_1,t_1) = \mathfrak{I}$, where and $\mathfrak{I}$ is the identity super-operator. For the time-independent Hamiltonians, Eqs.(9), (10) give $\mathfrak{U}(t_1,t_2) = e^{i(t_1-t_2)\mathcal{L}}$. As seen from Eqs.(7) and (8), $\varepsilon \tilde{Q}(t,0) = Q(t,0)$ if $Q(t,0)$ is the dynamics invariant. Moreover, $Q(t,0)$ satisfies the equation

$$\frac{\partial}{\partial t} \varepsilon \tilde{Q}(t,0) + i\mathcal{L}(t) \varepsilon \tilde{Q}(t,0) = \varepsilon \left[ Q(t,0) - \varepsilon \tilde{Q}(t,0) \right] \tag{11}$$

approaching to Eq.(7) at $\varepsilon \downarrow 0$ that explicates using the term ‘quasi-invariant’.

For a system occupying a finite domain $\Omega$ of volume $|\Omega|$, NESOM builds from QESO so-called quasi-non-equilibrium statistical operator (QNESO) using the procedure of Eq.(8). QNESO is denoted here by the notation $^{(\alpha)}\rho_\varepsilon (t,0)$, where the superscript $\alpha = 1,2$ points to the NESOM version. More historically earlier one (NESOM-1) was formulated in the papers of D.N. Zubarev [1] and McLennan [2] as the generalization of canonical ensemble method to non-equilibrium. In terms of Eq.(8) QNESO-1 is quasi-canonical distribution.
of the form
\[
(1)\rho_\varepsilon(t,0) = e^{\Psi_\varepsilon_t(t) - \frac{\varepsilon}{\varepsilon}S(t,0)}, \quad \Psi_\varepsilon_t(t) = -\ln \text{Tr} \left[ e^{-\frac{\varepsilon}{\varepsilon}S(t,0)} \right].
\] (12)

The later version (NESOM-2) was developed by D.N. Zubarev and V.P. Kalashnikov [3].

QNESO-2 is the dynamics quasi-integral built from \(\rho_q(t,0)\), that is
\[
(2)\rho_\varepsilon(t,0) = \frac{\varepsilon}{\varepsilon}\rho_q(t,0) = e^{-S(t,0)}. \quad (13)
\]

Eq.(12) includes additional normalization to the unity trace. The logarithm of the QNESO-1 trace normalization factor \(\Psi_\varepsilon(t)\) plays a crucial role in the present paper. In Eq.(13) the normalization holds automatically.

The equation for \((2)\rho_\varepsilon(t,0)\) results directly from Eq.(11):
\[
\left[ \frac{\partial}{\partial t} + i\mathfrak{L}(t) \right](2)\rho_\varepsilon(t,0) = \varepsilon \left[ \rho_q(t,0) - (2)\rho_\varepsilon(t,0) \right]. \quad (14)
\]

More involved equation for \((1)\rho_\varepsilon(t,0)\) is obtained by applying to the operator exponent in Eq.(12) the rules of time differentiation and of \(\mathfrak{L}(t)\) [4], and then Eq.(11). The result is
\[
\left[ \frac{\partial}{\partial t} + i\mathfrak{L}(t) \right](1)\rho_\varepsilon(t,0) = \varepsilon \int_0^1 (1)\rho_\varepsilon(t,0) e^{\lambda S(t,0)} \left\{ (1)\Delta_\varepsilon^t \left[ \frac{\varepsilon}{\varepsilon}S(t,0) - S(t,0) \right] \right\} e^{-\lambda S(t,0)} d\lambda, \quad (15)
\]

where
\[
(\alpha)\Delta_\varepsilon^t A \triangleq A - (\alpha)\langle A \rangle_\varepsilon^t \quad (16)
\]
by the definition and
\[
(\alpha)\langle A \rangle_\varepsilon^t \triangleq \text{Tr} \left[ (\alpha)\rho_\varepsilon(t,0) A \right]. \quad (17)
\]

The source-like terms emerged in the rhs of Eqs.(15), (14) break time-reversal invariance of the Liouville equation. Non-equilibrium may be described using expectation values of the type given by Eq.(17) while \(\varepsilon\) tends to zero. But, due the Bogoljubov’s idea of quasi-averages, any invariance-breaking perturbation is switched off only after performing
the thermodynamic limit (TL). So the quasi-averages, which would sustain the broken symmetry of time arrow towards future, may be defined by

\[
(\alpha)\langle A \rangle_t^\varepsilon = \lim_{\varepsilon \downarrow 0} \lim_{|\Omega| \to \infty} (\alpha)\langle A \rangle_t^\varepsilon.
\]

(18)

Of course, Eq.(18) makes any sense if \(A\) is an intensive observable (like density of an extensive observable) that may be assumed without any loss of generality. As was argued by D.N. Zubarev [4], such a subordination of the \(\varepsilon\) limit (\(\varepsilon\)L) and TL is crucial for the correct simulation of irreversibility.\(^1\)

Eqs.(3) and (6) by no means define the macro-parameters. The ultimate rationale of NESOM, which facilitates a closed description of non-equilibrium, is the balance equations (BE). BE may be presented in different equivalent forms. The basic one is the self-consistency form that reads

\[
(\alpha)\langle P_n \rangle_t^\varepsilon = \langle P_n \rangle_t^\varepsilon \triangleq \lim_{|\Omega| \to \infty} \langle P_n \rangle_t^\varepsilon,
\]

but actually the differential form of BE

\[
\frac{\partial}{\partial t} \langle P_n \rangle_t^\varepsilon = (\alpha)\left( \cdot P_n (t, 0) \right)_t^\varepsilon
\]

(20)

is exploited in the practice. Slightly different route consists of using the ‘pre-limit’ BE

\[
(\alpha)\langle P_n \rangle_t^\varepsilon = \langle P_n \rangle_t^\varepsilon
\]

(21)

and

\[
\frac{\partial}{\partial t} \langle P_n \rangle_t^\varepsilon = (\alpha)\left( \cdot P_n (t, 0) \right)_t^\varepsilon,
\]

(22)

at yet finite but large \(|\Omega|\) and small \(\varepsilon\), and then performing TL and \(\varepsilon\)L in due order (see D.N. Zubarev’s review in Ref.[4]). For \(\alpha = 2\), Eq.(20) and Eq.(22) exactly follow from Eq.(19) and Eq.(21), respectively, (see proof in e.g. [4]) while for \(\alpha = 1\) the question of relation between the two forms of BE is not so simple. A discussion on this issue is presented in Appendix I.

\(^1\)TL and \(\varepsilon\)L may be performed simultaneously, but anyhow \(\varepsilon |\Omega| \to \infty \) [1], [1, Preface]
As seen, BE are complicated (non-local) functional equations for defining $F_n(t)$. NESOM asserts that solutions to BE, \((a)F_n(t)\), do exist and finds them in physical approximations. In the first order with respect to the entropy production, QNESO-1 and QNESO-2 are the same [4]. Thus, NESOM-1 and NESOM-2 are equivalent in that approximation. This concerns both the coincidence of the BE solutions

\[ (1)F_n(t) = (2)F_n(t) \]  

and of the averages over the two NESOM ensembles

\[ (1)\langle A \rangle^t = (2)\langle A \rangle^t \]  

at least for some class of intensive observables. While Eq.(24) holds, Eq.(23) holds automatically, but not vice versa. Quite naturally, the general equivalence of NESOM ensembles is in question. Authors of Ref.[5] discussed a condition for Eq.(23) to hold. In Refs.[6] and [7, Preface] proofs of Eq.(24) were suggested. As shown in Appendix II, these proofs essentially used assumptions which, in fact, are not found among the basics behind NESOM. Moreover, it is shown that the condition of Ref.[6] and ‘asymptotic normalization’ condition of Ref.[7, Preface] can not be satisfied.

In this paper a proof of Eq.(23) is presented. This proof is based only on the NESOM rationales and on a number of assumptions, which seem to be tacitly adopted in NESOM and in other methods of reduced description of non-equilibrium (see e.g. [3, Preface] - [5, Preface]).

## 2 Pierls-Bogoljubov Inequality

The main tool in this paper is the Pierls-Bogoljubov inequality [8] (also [13, Preface]). This inequality is used in the two-sided form:

\[
\frac{\text{Tr} \left[ e^{-A} (A - B) \right]}{\text{Tr} (e^{-A})} \leq \ln \text{Tr} \left( e^{-B} \right) - \ln \text{Tr} \left( e^{-A} \right) \leq \frac{\text{Tr} \left[ e^{-B} (A - B) \right]}{\text{Tr} (e^{-B})},
\]  

(25)

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the equality being possible only at $A = B$. To begin with, put $A = -\ln \left[\left(\alpha\right)\rho_\varepsilon(t,0)\right]$ and $B = S(t,0)$. On account of the trace normalisation, the left side of Eq. (25) leads to the inequality

\[ \langle S(t,0)\rangle_\varepsilon^t > \mathcal{E} \left[\left(\alpha\right)\rho_\varepsilon\right] \triangleq \langle \left(\alpha\right)\mathcal{G}_\varepsilon(t) \rangle \quad (26) \]

that, upon satisfying BE in the form of Eq. (21), transforms to

\[ \Sigma(t) > \langle \left(\alpha\right)\mathcal{G}_\varepsilon(t) \rangle \quad (27) \]

where the lhs also becomes dependent on $\varepsilon$ (and generally on $\alpha$). This inequality expresses the MaxEnt principle at finite $|\Omega|$ and $\varepsilon$. The function $\langle \left(\alpha\right)\mathcal{G}_\varepsilon(t) \rangle$ may be called quasi-Gibbs entropy. Post TL and $\varepsilon$L form of the MaxEnt principle, the only feasible for the NESOM’s route with Eq. (19), holds for the specific entropies

\[ \sigma(t) = \langle \left(\alpha\right)\sigma(t) \rangle = \lim_{\varepsilon \downarrow 0} \langle \left(\alpha\right)\sigma_\varepsilon(t) \rangle > \lim_{\varepsilon \downarrow 0} \langle \left(\alpha\right)\mathcal{S}_\varepsilon(t) \rangle = \langle \left(\alpha\right)\mathcal{S}(t) \rangle \quad (28) \]

where

\[ \sigma(t) = \lim_{|\Omega| \to \infty} \frac{\Sigma(t)}{|\Omega|}, \quad \langle \left(\alpha\right)\sigma_\varepsilon(t) \rangle = \lim_{|\Omega| \to \infty} \frac{\langle \left(\alpha\right)\langle S(t,0)\rangle_\varepsilon^t \rangle}{|\Omega|} \quad (29) \]

are the specific quasi-equilibrium and pre-$\varepsilon$L non-equilibrium entropies, and

\[ \langle \left(\alpha\right)\mathcal{S}_\varepsilon(t) \rangle = \lim_{|\Omega| \to \infty} \frac{\langle \left(\alpha\right)\mathcal{G}_\varepsilon(t) \rangle}{|\Omega|} \quad (30) \]

is the specific pre-$\varepsilon$L quasi-Gibbs entropy. It is assumed that all limits in Eqs. (28) - (30) exist and BE are satisfied (see next section).

3 Handling TL and $\varepsilon$L

TL and $\varepsilon$L are inevitable ingredients of NESOM. In order to accurately resolve TL issue (but, at the same time, do not enter its subtleties [8], [13, Preface]) it is expedient to introduce several relevant axioms, which are necessary for the NESOM formulation to make any sense and seem quite satisfactory from the viewpoint of rigor adopted by physicists. As to $\varepsilon$L, it is treated here in a simplified manner (for a mathematically correct treatment, see [7, soon on Web]).
**Axiom 1** The index $n$ is discrete at finite $|\Omega|$ and continuous at $|\Omega| \to \infty$, while there exists a measure $\mu_n$ such that

$$\lim_{|\Omega| \to \infty} \frac{1}{|\Omega|} \sum_n g_n = \int g_n d\mu_n \quad (31)$$

for every bounded function $g_n$. The $F_n(t)$ by $\langle P_n \rangle^t$ and by $\lim_{|\Omega| \to \infty} \langle P_n \rangle^t_\varepsilon$ products are $\mu$-integrable.\(^2\)

**Axiom 2** $\Phi(t)$ is an extensive quasi-thermodynamic potential, in the sense that there exists the limit

$$\phi(t) = \lim_{|\Omega| \to \infty} \frac{\Phi(t)}{|\Omega|}, \quad (32)$$

which is an analog of the equilibrium specific grand-canonical potential [9]. The above axioms assure existence of the functions defined in Eqs.(28), (29). Firstly $\sigma(t)$, being an analog of the equilibrium specific entropy [9], is given by

$$\sigma(t) = \phi(t) + \int F_n(t) \langle P_n \rangle^t d\mu_n. \quad (33)$$

Post TL form of Eqs.(3) and (6) express $\langle P_n \rangle^t$ and $F_n(t)$ by *variational derivation* of $\phi(t)$ and $\sigma(t)$ in $F_n(t)$ and $\langle P_n \rangle^t$, respectively (keeping $t$ frozen), so that $\phi(t)$ and $\sigma(t)$ are connected to each other via Legendre transformation, e.g.

$$\phi(t) = \sigma(t) - \int \langle P_n \rangle^t \frac{\delta \sigma(t)}{\delta \langle P_n \rangle^t} d\mu_n. \quad (34)$$

Lastly $^{(\alpha)}\sigma_\varepsilon(t)$ and $^{(\alpha)}\sigma(t)$ are given by

$$^{(\alpha)}\sigma_\varepsilon(t) = \phi(t) + \int F_n(t) \lim_{|\Omega| \to \infty} ^{(\alpha)}\langle P_n \rangle^t_\varepsilon d\mu_n \quad (35)$$

and

$$^{(\alpha)}\sigma(t) = \phi(t) + \int F_n(t) ^{(\alpha)}\langle P_n \rangle^t d\mu_n, \quad (36)$$

respectively.

\(^2\)Although in $n$ there may be a continuous species before and remain a discrete species after TL, this axiom may be assumed without any loss of generality.
Axiom 3 \[ \lim_{|Ω| \to \infty} \langle P_n (t_0, t_0 - t_1) \rangle_q^{t_1} \] exists and (by physical reasons) is continuous bounded function of time arguments.\(^3\) The product \( F_n (t_0) = \lim_{|Ω| \to \infty} \langle P_n (t_0, t_0 - t_1) \rangle_q^{t_1} \) is \( \mu \)-integrable function of \( n \).

The following corollary ensues from these axioms and the rationales of NESOM.

Corollary 1 \[ \lim_{|Ω| \to \infty} (2) \langle P_n (t_0, t_0 - t) \rangle_t^t \] exists and the product \( F_n (t_0) = (2) \langle P_n (t_0, t_0 - t) \rangle_t^t \) is \( \mu \)-integrable function. In addition

\[ (2) \langle P_n (t_0, t_0 - t) \rangle_t^t = (2) \langle P_n \rangle_t^{t_0}. \] (37)

**Proof.** By the definition of Eq.(17) and the evolution super-operator property

\[ (2) \langle P_n (t_0, t_0 - t) \rangle_t^t = \text{Tr} \left\{ P_n \left[ \Upsilon (t, t_0) (2) \rho_\varepsilon (t, 0) \right] \right\}. \] (38)

The action of the super-operator \( e^{\varepsilon (t-t_0) \Upsilon (t, t_0)} \) on both sides of Eq.(14) and the integration of the resulting equation from \( t_0 \) to \( t \) leads to

\[ e^{\varepsilon (t-t_0) \Upsilon (t, t_0)} (2) \rho_\varepsilon (t, 0) = (2) \rho_\varepsilon (t_0, 0) + \varepsilon \int_{t_0}^{t} e^{\varepsilon (t_1-t_0) \Upsilon (t_1, t_0)} \rho_\varepsilon (t_1, 0) dt_1. \]

The use of this operator identity and Eq.(38) gives the following functional identity

\[ (2) \langle P_n (t_0, t_0 - t) \rangle_t^t = (2) \langle P_n \rangle_t^{t_0} e^{\varepsilon (t_0-t)} + \varepsilon \int_{t_0}^{t} (2) \langle P_n (t_0, t_0 - t_1) \rangle_t^{t_1} e^{\varepsilon (t_1-t)} dt_1. \] (39)

from which the lemma statement follows, including Eq.(37). Note that performing \( \varepsilon L \) in Eqs.(35) and (39) is quite rigorous within the NESOM rationales. ■

Though assuming existence of \( ^{(a)} \xi_\varepsilon (t) \) is physically reasonable, nothing assures this formally. The statements below connect that question for \( \alpha = 1 \) to the behavior of \( \Psi_\varepsilon (t) \) in TL.

\(^3\)E.g. for bounded (in the Hilbert space of the system) operators \( P_n \), which norms \( \|P_n\| \) are independent of \( V_Ω \), the absolute value of this function is bounded by \( \|P_n\| \).
Lemma 2  
(a) Existence of \((1)_\varepsilon s(t)\) is equivalent to extensivity of \(\Psi_\varepsilon(t)\), that is to existence of
\[
\psi_\varepsilon(t) = \lim_{|\Omega|\to\infty} \frac{\Psi_\varepsilon(t)}{|\Omega|}.
\] (40)

(b) For both \(\psi_\varepsilon(t)\) and \((1)_\varepsilon s(t)\) to exist it is sufficient that
\[
\lim_{|\Omega|\to\infty} (1)_\varepsilon \langle P_n(t_0, t_0 - t)\rangle_\varepsilon^t
\]
exist and the \(F_n(t_0)\) by
\[
\lim_{|\Omega|\to\infty} (1)_\varepsilon \langle P_n(t_0, t_0 - t)\rangle_\varepsilon^t
\]
products are \(\mu\)-integrable.

Proof. Differentiation of the definition of \(\Psi_\varepsilon(t)\) in Eq.(12) gives the relation
\[
\frac{\partial}{\partial t} \Psi_\varepsilon(t) = \varepsilon \left[ (1)_\varepsilon \langle S(t, 0)\rangle_\varepsilon^t - (1)_\varepsilon \langle \frac{\varepsilon}{\varepsilon} S(t, 0)\rangle_\varepsilon^t \right].
\] (41)

By the definition of Eq.(26) and Eq.(41)
\[
(1)_\varepsilon \sigma_\varepsilon(t) = -\Psi_\varepsilon(t) + (1)_\varepsilon \langle \frac{\varepsilon}{\varepsilon} S(t, 0)\rangle_\varepsilon^t = (1)_\varepsilon \langle S(t, 0)\rangle_\varepsilon^t - (1 + \varepsilon^{-1} \frac{\partial}{\partial t}) \Psi_\varepsilon(t).
\] (42)

This equation may be integrated to give
\[
\Psi_\varepsilon(t) = \varepsilon \int_{-\infty}^{t} e^{\varepsilon(t_0 - t)} \left[ (1)_\varepsilon \langle S(t_0, 0)\rangle_\varepsilon^{t_0} - (1)_\varepsilon \sigma_\varepsilon(t_0) \right] dt_0.
\] (43)

Due to Eqs.(26, 43) \(\Psi_\varepsilon(t) > 0\), the fact that will further be proved in other way. Eqs.(42) and (43), due to extensivity of \((1)_\varepsilon \langle S(t, 0)\rangle_\varepsilon^t\) expressed by Eq.(35), show that existence of \(\psi_\varepsilon(t)\) does provide that of \((1)_\varepsilon \sigma_\varepsilon(t)\) and vice versa. Eq.(8) and Axiom 1 give
\[
\lim_{|\Omega|\to\infty} \frac{1}{|\Omega|} (1)_\varepsilon \langle \frac{\varepsilon}{\varepsilon} S(t, 0)\rangle_\varepsilon^t = \varepsilon \int_{-\infty}^{t} e^{\varepsilon(t_0 - t)} \int F_n(t_0) (1)_\varepsilon \langle P_n(t_0, t_0 - t)\rangle_\varepsilon^t d\mu_n dt_0,
\] (44)

provided that the condition of (b) holds. As seen from the above consideration, Eq.(44) assures existence of both \(\psi_\varepsilon(t)\) and \((1)_\varepsilon \sigma_\varepsilon(t)\). On physical level of rigor, the condition of (b) may be adopted as being also necessary for (a). It is worth emphasizing that an attempt to prove the analog of Corollary 1 for \((1)_\varepsilon \langle P_n(t_0, t_0 - t)\rangle_\varepsilon^t\), making use of Axioms 1-3 alone, fails. ■

In TL Eq.(43) and the first equality in Eq.(42) transform to
\[
\psi_\varepsilon(t) = \varepsilon \int_{-\infty}^{t} e^{\varepsilon(t_0 - t)} \left[ (1)_\varepsilon \sigma_\varepsilon(t_0) - (1)_\varepsilon \sigma_\varepsilon(t_0) \right] dt_0
\] (45)
\[ (1) \mathbf{s}_\varepsilon (t) = \varepsilon \int_{-\infty}^{t} e^{\varepsilon (t_0 - t)} \sigma (t_0) dt_0 - \psi_\varepsilon (t) + \int_{-\infty}^{t} \int F_n (t_0) \frac{\delta \psi_\varepsilon (t)}{\delta F_n (t_0)} d\mu_n dt_0, \quad (46) \]

respectively. Unfortunately nothing may be inferred from Eqs. (45), (46) on existence of \((1) \mathbf{s} (t)\) and

\[ \psi (t) = \lim_{\varepsilon \downarrow 0} \psi_\varepsilon (t), \quad (47) \]

except the fact that these quantities, if exist, should be independent of time: \(\psi (t) = \psi\) and \((1) \mathbf{s} (t) = (1) \mathbf{s}\).

4 Equivalence of Two NESOM Ensembles

4.1 Generating Functional in NESOM-1

Let \(f = \{f_n\}\) is a set of functions independent of time, satisfying the same requirements as \(F_n (t)\). Define upon \(f\) the functional

\[ \Psi_\varepsilon (t; f) = - \ln \text{Tr} \left[ e^{-S_\varepsilon (t; f)} \right], \quad (48) \]

where

\[ S_\varepsilon (t; f) = S (t, 0) + \sum_n f_n P_n. \quad (49) \]

Formally, \(\Psi_\varepsilon (t; f)\) is the logarithm of trace normalisation factor for the statistical distribution with an ‘entropy’ defined by Eq. (48). While the use made of the last equality in Eq. (8), this ‘entropy’ is presented as

\[ S_\varepsilon (t; f) = S (t, 0) + \sum_n f_n P_n - \int_{-\infty}^{t} e^{\varepsilon (t_0 - t)} \mathbf{S}_\varepsilon (t_0, t_0 - t) dt_0 \]

that may be thought as an artificial perturbation, by the shifts \(F_n (t) \rightarrow F_n (t) + f_n\), concerning the QESO entropy part with no effect on the entropy production. The distribution under consideration is auxiliary one, which tends to ‘shifted’ QESO, while zeroing the entropy production. The Taylor series for \(\Psi_\varepsilon (t; f)\) has the form

\[ \Psi_\varepsilon (t; f) = \Psi_\varepsilon (t) + \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{n_1 \ldots n_l} K_{\varepsilon}^{(l)} (t; n_1, \ldots, n_l) \prod_{i=1}^{l} f_{n_i}, \quad (50) \]
where $\Psi_\varepsilon (t)$ was defined earlier and

$$K_\varepsilon^{(l)} (t; n_1, ..., n_l) = \frac{\partial^{l} \Psi_\varepsilon (t; \hat{f})}{\partial f_{n_1}...\partial f_{n_l}} \bigg|_{\hat{f}=0}, \ l \geq 1. \quad (51)$$

For particular $l = 1$,

$$K_\varepsilon^{(1)} (t; n) = \langle P_{n} \rangle^{t}_\varepsilon. \quad (52)$$

Higher derivatives of $\Psi_\varepsilon (t; f)$ at $\hat{f} = 0$ are connected with the cumulant correlators of gross variables over QNESO-1

$$K_\varepsilon^{(l)} (t; n_1, ..., n_l) = (-1)^{l-1} \langle P_{n_1} \cdot \cdot \cdot P_{n_l} \rangle^{t}_\varepsilon, \quad (53)$$

where

$$\langle (A_1 \cdot \cdot \cdot A_{l-1} \cdot A_l) \rangle^{t}_\varepsilon \equiv \int_{0}^{1} \cdots \int_{0}^{1} \left\langle [A_1 (\lambda_1)^t \cdot \cdot \cdot A_{l-1} (\lambda_{l-1})^{t>_{\text{irr}}} A_l]^{t} \prod_{i=1}^{l-1} d\lambda_i, \quad (54)$$

with

$$A (\lambda)^t_\varepsilon = e^{\lambda S(t,0)} A e^{-\lambda S(t,0)}, \quad (55)$$

and the ‘irreducible’ ordered products calculated via the following recursive relations

$$[A_1 (\lambda_1)^t \cdot \cdot \cdot A_m (\lambda_m)^t]^{>_{\text{irr}}} = [A_1 (\lambda_1)^t \cdot \cdot \cdot A_m (\lambda_m)^t]^{>_{\text{irr}}} - \langle [A_1 (\lambda_1)^t \cdot \cdot \cdot A_m (\lambda_m)^t]^{>_{\text{irr}}} \rangle^{t}_\varepsilon$$

$$- \sum_{k=1}^{m-1} \sum_{\pi^{(m)}_k} \left[ A_{i_1} (\lambda_{i_1})^t \cdot \cdot \cdot A_{i_k} (\lambda_{i_k})^{t>_{\text{irr}}} \right]^{(1)} \left\langle [A_{i_{k+1} (\lambda_{i_{k+1}})^t \cdot \cdot \cdot A_{i_m} (\lambda_{i_m})^{t>_{\text{irr}}}]^{t} \right\rangle^{t}_\varepsilon. \quad (56)$$

In these relation $\pi^{m}_k$ are all permutations of the type

$$\pi^{m}_k = \left( i_1...i_k \text{ } i_{k+1}...i_m \right); \ i_1 < ... < i_k, \ i_{k+1} < ... < i_m \quad (57)$$

and $[B_1 (\lambda_1) \cdot \cdot \cdot B_p (\lambda_p)]^{>}$ is the Dyson-ordered product, in which the factors with larger $\lambda$'s are stood to the left. Note that $^{(1)} (A \cdot B)^t_\varepsilon$ is a non-equilibrium analog of the Kubo-Duhamel correlator [8] (also [13, Preface]). In spite of apparent asymmetry relative to $A_l$ of $^{(1)} (A_1 \cdot \cdot \cdot A_{l-1} \cdot A_l)^t_\varepsilon$, as defined by Eq.(54), this correlator appears to be symmetric with respect to all permutations of its operator constituents.
Consider the limit
\[
\psi_\varepsilon (t; \mathbf{f}) = \lim_{|\Omega| \to \infty} \frac{\psi_\varepsilon (t; \mathbf{f})}{|\Omega|},
\]
Term-by-term analysis of the series in Eq.(50) based upon Axiom 1, Eqs.(52) and (53) leads to the conclusion that \(\psi_\varepsilon (t; \mathbf{f})\) would be represented by functional Taylor series
\[
\psi_\varepsilon (t; \mathbf{f}) = \psi_\varepsilon (t) + \sum_{l=1}^{\infty} \frac{1}{l!} \int \cdots \int \kappa_\varepsilon^{(l)} (t; n_1, \ldots, n_l) \prod_{i=1}^{l} f_{n_i} \, d\mu_{n_i},
\]
where existence of the first and second terms is provided by adopting Eq.(40) and by the rationale of NESOM-1, respectively. The higher-order terms would be made meaningful by assuming existence of limits
\[
^{(1)} [P_{n_1} \cdots P_{n_l}]_{\varepsilon} = \lim_{|\Omega| \to \infty} (^{(1)} (P_{n_1} \cdots P_{n_l})_{\varepsilon} | \Omega|^{l-1},
\]
so that
\[
\kappa_\varepsilon^{(1)} (t; n) = \lim_{|\Omega| \to \infty} (^{(1)} (P_n)_{\varepsilon}
\]
and
\[
\kappa_\varepsilon^{(l)} (t; n_1, \ldots, n_l) = (-1)^{l-1} (^{(1)} [P_{n_1} \cdots P_{n_l}]_{\varepsilon}, l \geq 2.
\]
Note that there results a correspondence between derivative and variational derivative with respect to \(f_n\) in TL:
\[
|\Omega| \frac{\partial}{\partial f_n} \to \frac{\delta}{\delta f_n}, \quad |\Omega| \to \infty,
\]
similar to that for \(F_n (t)\). Eq.(60) means, roughly speaking, that the \(l\)-th order cumulant correlator for the gross variables falls off as \(|\Omega|^{-l+1}\) at \(|\Omega| \to \infty\), which expresses the expected TL behavior of the thermodynamic fluctuations for the intensive observables.

It worth emphasizing that adopting Eqs.(40) and Eqs.(60) is by no means sufficient for existence of \(\psi_\varepsilon (t; \mathbf{f})\) since it does not guarantee the convergence of the functional Taylor series in Eq.(59)

Finally, consider \(\varepsilon L\) of \(\psi_\varepsilon (t; \mathbf{f})\)
\[
\psi (t; \mathbf{f}) = \lim_{\varepsilon \downarrow 0} \psi_\varepsilon (t; \mathbf{f}).
\]
Taking formally \( \varepsilon L \) in each term of the series in Eq.(59) with adopting Eq.(40) and using the limit

\[
\chi^{(1)} (t; n_1) = \lim_{\varepsilon \downarrow 0} \chi^{(1)}_\varepsilon (t; n_1) = (1) \langle P_{n_1} \rangle^t
\]  

(64)

that exists in NESOM-1 unconditionally, gives

\[
\psi (t; f) = \psi (t) + \sum_{l=1}^{\infty} \frac{1}{l!} \int \ldots \int \chi^{(l)} (t; n_1, \ldots, n_l) \prod_{i=1}^{l} f_{n_i} d\mu_{n_i},
\]

(65)

provided that the quotients of terms with \( l \geq 2 \)

\[
\chi^{(l)} (t; n_1, \ldots, n_l) = \lim_{\varepsilon \downarrow 0} \chi^{(l)}_\varepsilon (t; n_1, \ldots, n_l) = (-1)^{l-1} (1) [P_{n_1} \cdot \ldots \cdot P_{n_l}]^t,
\]

(66)

\[
[P_{n_1} \cdot \ldots \cdot P_{n_l}]^t = \lim_{\varepsilon \downarrow 0} (1) [P_{n_1} \cdot \ldots \cdot P_{n_l}]^t \varepsilon
\]

(67)

at least exist. Again, this condition is not sufficient for \( \psi (t; f) \) to exist. To proceed with the proof at goal, the following ultimate statement is postulated.

**Axiom 4** \( \psi (t; f) \) exists for a set of macro-parameters (including the both BE solutions at least) at non-trivial \( f \)'s in a vicinity of the point \( f = 0 \) and is twice functionally differentiable at that point.

**Remark 1** Thus, the functional \( \psi (t; f) \) is not required to be analytic at \( f = 0 \). It follows with necessity from Axiom 4 and Eqs.(64)-(67) that

\[
\frac{\delta \psi (t; \tilde{f})}{\delta f_n} \bigg|_{f=0} = (1) \langle P_n \rangle^t
\]

(68)

\[
\frac{\delta^2 \psi (t; \tilde{f})}{\delta f_n \delta f_m} \bigg|_{f=0} = - (1) [P_n \cdot P_m]^t,
\]

(69)

but existence of the higher correlators is not necessary for the forthcoming proof.

### 4.2 Proof

Returning now to Eq.(25), put \( A = S_\varepsilon (t; \tilde{f}) \) and \( B = S (t_0, t_0 - t) \) with an arbitrary \( t_0 \).

Then two inequalities result. The first is

\[
\Psi_\varepsilon (t; \tilde{f}) > \sum_n f_n \frac{\partial \Psi_\varepsilon (t; \tilde{f})}{\partial f_n} + \frac{\Tr \left\{ e^{-S_\varepsilon (t; \tilde{f})} \left[ \varepsilon \left( S (t, 0) - S (t_0, t_0 - t) \right) \right] \right\}}{\Tr [e^{-S_\varepsilon (t; \tilde{f})}]}
\]
Multiplication this inequality by $\varepsilon e^{\varepsilon (t_0-t)} > 0$ with subsequent integration over $t_0$ from $-\infty$ to $t$ gives

$$\Psi_\varepsilon (t; f) > \sum_n \frac{\partial \Psi_\varepsilon (t; f)}{\partial f_n} f_n. \quad (70)$$

The second inequality is

$$\Psi_\varepsilon (t; f) < \sum_n f_n \text{Tr} \left[ e^{-S(t_0,t_0-t)} P_n \right] + \text{Tr} \left[ e^{-S(t_0,t_0-t)} \frac{\varepsilon}{t_0} \right] - \langle S(t_0,0) \rangle_{t_0}^{t_0}.$$

Multiplication of this inequality by $\eta e^{\eta (t_0-t)}$, where $\eta > 0$ and is arbitrary in any other respect, with subsequent integration over $t_0$ from $-\infty$ to $t$ gives

$$\Psi_\varepsilon (t; f) - \sum_n (2) \langle P_n \rangle_{\eta}^t f_n < \varepsilon \int_{-\infty}^t (2) \langle S(t_0,0) \rangle_{\eta}^{t_0} e^{\varepsilon (t_0-t)} dt_0 - \sum \langle \sigma(t) \rangle_{\eta}.$$

$$+ \varepsilon \int_{-\infty}^t \sum_n F_n (t_0) \left[ (2) \langle P_n (t_0, t_0-t) \rangle_{\eta}^t - (2) \langle P_n \rangle_{\eta}^{t_0} \right] e^{\varepsilon (t_0-t)} dt_0. \quad (71)$$

From this point forward, the notation of quasi-invariant is applied to usual functions of time, for brevity.

Dividing by $|\Omega|$, perform TL in Eqs.(70) and (71) with the use of Axioms 1-3, Eqs.(33) - (35) and the above stated correspondence between derivatives. This results in

$$\psi_\varepsilon (t; f) \geq \int \frac{\delta \psi_\varepsilon (t; f)}{\delta f_n} f_n d\mu_n, \quad (72)$$

which does reconfirm non-negativity of $\psi_\varepsilon (t) = \psi_\varepsilon (t; 0)$ obtained above, and in

$$\psi_\varepsilon (t; f) \leq \int (2) \langle P_n \rangle_{\eta}^t f_n d\mu_n + \langle \sigma(t) \rangle_{\eta}.$$  

$$+ \varepsilon \int_{-\infty}^t e^{\varepsilon (t_0-t)} \left\{ \int F_n (t_0) \lim_{|\Omega| \to \infty} \left[ (2) \langle P_n (t_0, t_0-t) \rangle_{\eta}^t - (2) \langle P_n \rangle_{\eta}^{t_0} \right] d\mu_n \right\} dt_0.$$

When keeping in the latter inequality $\varepsilon$ finite, $\eta$ may freely be tended to zero in its rhs without affecting the lhs which does not depend on $\eta$ at all. Let $\eta$ to take values of any infinitesimal sequence, for which the sequence $\sigma(t)$ converge to its upper limit. This limit process, on account of Eq.(36) and Eq.(37), leads to the interim bound

$$\psi_\varepsilon (t; f) \leq \int (2) \langle P_n \rangle_{\eta}^t f_n d\mu_n + \langle \sigma(t) \rangle - \lim_{\eta \to 0} \eta \sigma(t). \quad (73)$$

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Let $\varepsilon$ takes values of any infinitesimal sequence, for which the sequence $(2)\tilde{\sigma}(t)$ converges to its upper limit. Then Eq.(73) results in

$$
\overline{\psi}(t;\varepsilon) = \lim_{\varepsilon\downarrow 0} \psi_{\varepsilon}(t;\varepsilon) \leq \int (2)\langle P_n \rangle^t f_n d\mu_n + \lim_{\varepsilon\downarrow 0} (2)\tilde{\sigma}(t) - \lim_{\varepsilon\downarrow 0} \varepsilon \tilde{\sigma}(t) \tag{74}
$$

Let now $F_n(t_0) = (2)F_n(t_0)$. Then, in the rhs of Eq.(74) $(2)\langle P_n \rangle^t = \langle P_n \rangle^t$, while the second and third terms cancel each other to give

$$
\overline{\psi}(t;\varepsilon) \leq \int \langle P_n \rangle^t f_n d\mu_n \tag{75}
$$

At $f=0$ Eq.(75) shows that

$$
\lim_{\varepsilon\downarrow 0} \psi_{\varepsilon}(t) = \overline{\psi}(t;0) \leq 0,
$$

while this limit should be non negative due to $\psi_{\varepsilon}(t) \geq 0$. It is thus proved that on the BE solutions of NESOM-2 $\lim_{\varepsilon\downarrow 0} \psi_{\varepsilon}(t) = 0$. So in this case $\psi$ does exist and equals zero. It is quite remarkable as existence of $\psi(t;f)$ has not yet been invoked, but for finishing the proof it cannot be avoided. When adopting Axiom 4, Eq.(75) acquires the form

$$
\chi(t;f) = \psi(t;f) - \int \langle P_n \rangle^t f_n d\mu_n \leq 0. \tag{76}
$$

and $\chi(t;0) = \psi = 0$. This means that, while BE of NESOM-2 are satisfied, the functional $\chi(t;f)$ becomes non-positive attaining at $f=0$ absolute maximum (equal zero). Then using the necessary condition of functional extremum and Eq.(68) gives

$$
0 = \frac{\delta \chi(t;f)}{\delta f_n} \bigg|_{f=0} = \frac{\delta \psi(t;f)}{\delta f_n} \bigg|_{f=0} - \langle P_n \rangle^t = (1)\langle P_n \rangle^t - \langle P_n \rangle^t, \tag{77}
$$

that is BE of NESOM-1. Thus every BE solution of NESOM-2 proves to satisfy also BE of NESOM-1, i.e. $(1)F_n(t) = (2)F_n(t)$, but not vice versa. That the above extremum is indeed maximum is guaranteed by the second variation of $\chi(t;f)$ at the extremum, i.e.

$$
\delta^2 \chi(t;\Delta f) = \int \int \frac{\delta^2 \chi^*(t;f)}{\delta f_n \delta f_m} \bigg|_{f=0} \delta f_n \delta f_m d\mu_n d\mu_m = - (1)[P(\delta f) \cdot P(\delta f)]^t < 0,
$$

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where Eq. (69) was used, and \( \mathcal{P} (\delta \tilde{f}) = \int P_n \delta f_n d\mu_n \). The negativity of \( \delta^{(2)} \chi (t; \delta \tilde{f}) \) holds due to unconditional positivity of the Kubo-Duhamel auto-correlator [8], [13, Preface]. This proves the equivalence of NESOM-1 and NESOM-2, if the BE solutions in both methods are unique.

The equivalence in the sense of Eq. (24) is also proven using the technique developed above, provided that \( A \in \{ A_p \} \), where \( A_p \) are some intensive variables (other than gross ones), which index \( p \) satisfies Axiom 1, and the functional

\[
\psi_A (t; g) = - \lim_{\varepsilon \downarrow 0} \lim_{|\Omega| \to \infty} \ln \left( \frac{1}{|\Omega|} \ln \text{Tr} \left[ e^{-\frac{\varepsilon}{\varepsilon L} \left( \tilde{S}(t, 0) - \sum_p g_p A_p \right)} \right] \right)
\]

on a set \( g = \{ g_p \} \) may be constructed to satisfy Axiom 4. In the present framework more definite description of variables, for which Eq. (24) holds, seems hardly possible.

5 Conclusion

To conclude, the non-pertubtative proof of equivalence between two non-equilibrium ensembles in NESOM, based on MaxEnt principle, is proposed. The proof is thought as an improvement upon previous ones [5], [6] and [7, Preface]. Because the rationales of NESOM was not clearly delineated, some natural assumptions concerning TL and, inherent to NESOM, \( \varepsilon L \) are introduced. The present proof, is not also free of some ‘extra’ assumption, namely existence and second-order differentiability of the generating functional \( \psi (t; \tilde{f}) \) However, the latter seems natural in the field-theoretical context and much more appropriate than the ‘time correlation weackening’ [5], [6], entropy-production series convergence [6] and ‘asymptotic trace normalisation’ [7, Preface] conditions, which conflict with the basics of NESOM.

Appendix I

Consider ‘pre-limit’ differential form of BE in NESOM-1. Multiplying both sides of
Eq.(15) by $P_n$ and taking the trace gives

$$\frac{\partial}{\partial t} \langle P_n \rangle_q^t - (1) \left\langle P_n (t,0) \right\rangle_\varepsilon^t = \mathcal{J}_{n,\varepsilon} (t), \quad (78)$$

where

$$\mathcal{J}_{n,\varepsilon} (t) = \varepsilon \left[ (1) (P_n \cdot S (t,0))^t_\varepsilon - (1) \left( P_n \cdot S (t,0) \right)_\varepsilon^t \right]$$

$$= \varepsilon \int_{-\infty}^{t} (1) \left( P_n \cdot \dot{S} (t_0, t_0 - t) \right)_\varepsilon^t \epsilon^{x(t_0-t)} dt_0. \quad (79)$$

Thus, for Eq.(22) to hold at $\alpha = 1$, it is necessary that $\mathcal{J}_{n,\varepsilon} (t) = 0$. However, any connection of the resulting from this projective-type equations with Eq.(21) can hardly be stated. Note that weaker condition

$$\lim_{\varepsilon \downarrow 0} \lim_{|\Omega| \rightarrow \infty} \mathcal{J}_{n,\varepsilon} (t) = 0 \quad (80)$$

appears in Ref.[5] as a condition for the equivalence of NESOM-1 and NESOM-2. With Eq.(80), Eq.(20) for $\alpha = 1$ holds trivially, but connection of Eq.(19) and that condition remains obscure. The key problem here, as seen from Eq.(79), is the time behaviour of either $\left( P_n \cdot P_m (t_0, t_0 - t) \right)_\varepsilon^t$ or $\left( P_n \cdot \dot{S} (t_0, t_0 - t) \right)_\varepsilon^t$, which should be at least bounded or fall off at $t_0 \rightarrow -\infty$, respectively. These conditions are not found among the rationales of NESOM.

**Appendix II**

This Appendix overviews previous treatments of the equivalence problem.

Consider first the proof of Ref.[5]. The ‘time correlation-weakening’ condition of Ref.[5] is essentially the same as Eq.(80) and implies that $\lim_{\varepsilon \downarrow 0} \lim_{|\Omega| \rightarrow \infty} (1) \left( P_n \cdot \dot{S} (t_0, t_0 - t) \right)_\varepsilon^t \rightarrow 0$ as $t_0 \rightarrow -\infty$. However, the NESOM perturbational practice evidences that such a behavior shows up only after BE were used to eliminate the terms $\propto F_m (t_0)$ off $\dot{S} (t_0, t_0 - t)$. Thus, the condition of Ref.[5] may be necessary, but by no means sufficient, for proving the equivalence. ■
Consider next the proof of Ref.[6] (also [8, Preface]) claimed for classical case. Here the approach of Ref.[6] is extended to quantum case. Let us obtain formal expansion of \((α)\rho_ε(t)\) with respect the entropy production. Making use of the last equality in Eq.(8), expand the operator exponent defining QNESO-1 and its trace to obtain

\[
(1)\rho_ε(t) = ρ_q(t) \left\{1 + \sum_{k=1}^{∞} \frac{1}{k!} \int_{−∞}^{0} \int_{−∞}^{0} \int_{0}^{1} \int_{0}^{1} [V_t(λ_1, τ_1) \ldots V_t(λ_k, τ_k)]_{irr}^> Π_{j=1}^{k} dλ_j e^{ετ_j} dτ_j \right\},
\]

where \(V_t(λ, τ) ≡ e^{λS(t,0)} \tilde{S} (t + τ, τ) e^{-λS(t,0)}\). Using the identity: \(S(t + τ_0, τ_0) = S(t,0) - \int_{τ_0}^{0} \tilde{S} (t + τ_1, τ_1) dτ_1\), analogous expansion for QNESO-2 is obtained

\[
(2)\rho_ε(t) = ρ_q(t) \left\{1 + \sum_{k=1}^{∞} \frac{1}{k!} ε \int_{−∞}^{0} \int_{−∞}^{0} \int_{0}^{1} \int_{0}^{1} [V_t(λ_1, τ_1) \ldots V_t(λ_k, τ_k)]_{irr}^> Π_{j=1}^{k} dλ_j dτ_j \right\}.
\]

As \(\int_{0}^{0} \int_{0}^{1} [V_t(λ_1, τ_1) \ldots V_t(λ_k, τ_k)]_{irr}^> Π_{j=1}^{k} dλ_j dτ_j\) is symmetric in variables \(τ_1, ..., τ_k\), the integration over them in Eqs.(81) and (82) may be changed to \(\int_{−∞}^{0} dτ_1 \int_{−∞}^{0} dτ_2 \ldots \int_{−∞}^{0} dτ_k\) and \(\int_{τ_0}^{0} dτ_k \int_{τ_k}^{0} dτ_{k-1} \ldots \int_{τ_2}^{0} dτ_1\), respectively, to times \(k!\). Using the latter transformation, each term of expansion in Eq.(82) can be integrated over \(τ_0\) by parts, after which there remains integration over the domain \(-∞ < τ_k < τ_{k-1} < ... < τ_1 ≤ 0\). Changing this integration order to the opposite one makes the integration domains in both Eq.(81) and Eq.(82) the same. This results in the unified expansion of QNESO

\[
(α)\rho_ε(t) = ρ_q(t) \left\{1 + \sum_{k=1}^{∞} \int_{−∞}^{τ_1} \int_{−∞}^{τ_1} \int_{−∞}^{τ_{k-1}} \int_{−∞}^{τ_{k-1}} (α)g_k \int_{−∞}^{τ_1} \int_{−∞}^{τ_{k-1}} [V_t(λ_1, τ_1) \ldots V_t(λ_k, τ_k)]_{irr}^> Π_{j=1}^{k} dλ_j dτ_j \right\},
\]

from which the \(εL\) quasi-averages expansion follows

\[
(α)\langle A \rangle_{ε}^t = \langle A \rangle^t + \sum_{k=1}^{∞} \int_{−∞}^{τ_1} \int_{−∞}^{τ_1} \int_{−∞}^{τ_{k-1}} \int_{−∞}^{τ_{k-1}} (α)g_k \int_{−∞}^{τ_1} \int_{−∞}^{τ_{k-1}} \langle [V_t(λ_1, τ_1) \ldots V_t(λ_k, τ_k)]_{irr}^> \rangle Π_{j=1}^{k} A_{irr}^t dλ_j dτ_j,
\]

(83)
where \( g_k = e^{\varepsilon (\tau_1 + \cdots + \tau_k)} \), \( g_k = e^{\varepsilon \tau_k} \). As \( \lim_{\varepsilon \to 0} g_k = 1 \), it was concluded in Ref.[6] (also [8, Preface]) that Eq.(24) holds unconditionally. In fact, this statement is valid only if the irreducible correlators \( \langle \left[ V_t (\lambda_1, \tau_1) \cdots V_t (\lambda_k, \tau_k) \right]_{\text{irr}}^\geq A \rangle^t \) are absolutely integrable in the domain \( 0 > \tau_1 > \cdots > \tau_k > -\infty \), otherwise counterexamples show that the above integrals with \( \alpha = 1 \) and \( \alpha = 2 \) many have quite different values. This is another form of the ‘time correlation-weakening’ condition. Again, referring to the practice of NESOM shows that the irreducible correlators, though satisfying a ‘spatial correlation-weakening’ at all \( F_n (t_0) \), don’t satisfy ‘time correlation-weakening’ unless the above-mentioned exclusion of ‘secular’ terms is made with the use of BE. Thus, without the assumption that BE are satisfied in advance, the proof of Ref.[6] is deceptive, while that assumption makes the proof unclosed. ■

At last, consider proof of Ref.[7, Preface]. The use is made of Jensen convexity inequality that gives
\[
\frac{\varepsilon}{e^{-S(t,0)}} > e^{\frac{-\varepsilon}{S(t,0)}}
\]
at each phase-space point. This inequality is consistent with \( \Psi_\varepsilon (t) > 0 \), being a particular case of Eq.(70) for \( f = 0 \). As a result, in classical case the following pointwise inequality between QNESO-2 and QNESO-1 is stated
\[
\langle (2) \rho_\varepsilon (t) \rangle - \langle (1) \rho_\varepsilon (t) \rangle e^{-\Psi_\varepsilon (t)} > 0.
\]

Using the identity
\[
\langle (2) A \rangle^t - \langle (1) A \rangle^t \equiv \int \left[ \langle (2) \rho_\varepsilon (t) \rangle - \langle (1) \rho_\varepsilon (t) \rangle e^{-\Psi_\varepsilon (t)} \right] A d\omega + \left[ e^{-\Psi_\varepsilon (t)} - 1 \right] \langle (1) A \rangle^t,
\]
where \( d\omega \) is the phase-space measure, and assuming the observable \( A \) to be bounded, one obtains the estimation
\[
\left| \langle (2) A \rangle^t - \langle (1) A \rangle^t \right| \leq \int \left| \langle (2) \rho_\varepsilon (t) \rangle - \langle (1) \rho_\varepsilon (t) \rangle e^{-\Psi_\varepsilon (t)} \right| |A| d\omega + \left[ 1 - e^{-\Psi_\varepsilon (t)} \right] \left| \langle (1) A \rangle^t \right| \\
\leq \|A\| \Omega \left\{ \int \left| \langle (2) \rho_\varepsilon (t) \rangle - \langle (1) \rho_\varepsilon (t) \rangle e^{-\Psi_\varepsilon (t)} \right| d\omega + \left[ 1 - e^{-\Psi_\varepsilon (t)} \right] \right\}.
\]
where $\|A\|_\Omega = \max |A|$, and $\Psi_\varepsilon(t) > 0$ is taken into account. Next, the use of Eq.(84) and the trace normalisation of QNESO gives

$$\left|\langle A \rangle_\varepsilon^t - \langle A \rangle_\varepsilon^1 \right| \leq 2 \|A\|_\Omega \left[1 - e^{-\Psi_\varepsilon(t)} \right]$$

(85)

Note that this consideration cannot be extended to quantum systems, since operator exponent is not convex operator function [10]. Proceeding with the above estimate, Bitensky [7, Preface] assumed two conditions: (i) independence of $\|A\|_\Omega$ on $\Omega$; (ii) the ‘asymptotic normalisation condition’

$$\lim_{\varepsilon \downarrow 0} \lim_{|\Omega| \to \infty} \Psi_\varepsilon(t) = 0, \text{ or } \lim_{\varepsilon \downarrow 0} \lim_{|\Omega| \to \infty} \Psi_\varepsilon(t) = 0.$$  

(86)

As seen from Eq.(85), (i) and (ii) lead to the equivalence of the NESOM ensembles in the sense of Eq.(24). Although (i) is by no means a restrictive condition, (ii) is in doubt. The latter condition is obviously invalid at general $F_n(t_0)$, because TL and $\varepsilon$L of $\Psi_\varepsilon(t) / |\Omega|$ (performed either sequentially or simultaneously) is strictly positive, and so $\Psi_\varepsilon(t) \to \infty$ unless BE are satisfied. For TL and $\varepsilon$L performed sequentially and $F_n(t_0) = (1)F_n(t_0)$, $\Psi_\varepsilon(t) \to \infty$, since $\psi_\varepsilon(t)$ remains positive at finite $\varepsilon$. Simultaneous TL and $\varepsilon$L of $\Psi_\varepsilon(t)$ is uncertain, since that for $\Psi_\varepsilon(t) / |\Omega|$ is zero. However, for non-stationary process Eq.(86) also proves invalid. Indeed, assume that $(1)s$ exists. Then, given small $\delta > 0$, large $\Delta > 0$ and arbitrary $\tau > 0$, there exist $\Omega_0$ and $\varepsilon_0$ satisfying $|\Omega_0| \varepsilon_0 > \Delta$ (see footnote 1), such that

$$(1)\langle S(t_0, 0) \rangle_\varepsilon^t - (1)\mathcal{E}_\varepsilon(t_0) > |\Omega| \left[\sigma(t_0) - (1)s - \delta \right] > 0$$

for every $\Omega \supset \Omega_0$, $\varepsilon < \varepsilon_0$ satisfying $|\Omega| \varepsilon > \Delta$, and $t_0 \in [t - \tau, t]$. Making use of Eq.(43) and the above inequality gives

$$\Psi_\varepsilon(t) > \Delta \tau \min_{t_0 \in [t - \tau, t]} \sigma(t_0) - (1)s - \delta,$$

so that $\Psi_\varepsilon(t) \to \infty$ as well. For a stationary state, this reasoning is useless, since $\sigma = (1)s$ at $F_n = (1)F_n$. In this case simultaneous TL and $\varepsilon$L of $\Psi_\varepsilon$ remains uncertain, but it cannot be arbitrarily fixed at zero value.
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