A CONSTRUCTION FOR R-MATRICES WITHOUT
DIFFERENCE PROPERTY IN THE SPECTRAL PARAMETER

Jon Links

Centre for Mathematical Physics,
Department of Mathematics,
The University of Queensland, 4072
Australia
e-mail jrl@maths.uq.edu.au

Report No. UQCMP-99-2

Abstract

A new construction is given for obtaining $R$-matrices which solve the McGuire-Yang-Baxter equation in such a way that the spectral parameters do not possess the difference property. A discussion of the derivation of the supersymmetric $U$ model for correlated electrons is given in this context, such that applied chemical potential and magnetic field terms can be coupled arbitrarily. As a limiting case the Bariev model is obtained.
1. Introduction

The study of integrable models which are constructed through the Quantum Inverse Scattering Method (QISM) makes available many significant and non-perturbative results as these models are generally solvable through Bethe ansatz methods. The quantum algebras \[ 1, 2 \] and their natural \( \mathbb{Z}_2 \)-graded analogues quantum superalgebras \[ 3, 4 \] provide us with the principle examples of quasi-triangular Hopf (super)algebras which allow us to systematically construct solutions of the McGuire-Yang-Baxter (MYB) equation which are central to the QISM approach. The solutions of the MYB equation are referred to as \( R \)-matrices. The approach of the QISM is to first build a family of commuting transfer matrices from the \( R \)-matrices and then define the Hamiltonian operator as the logarithmic derivative of the transfer matrix evaluated at the shift point. The shift point is that value of the spectral parameter such that the \( R \)-matrix behaves as the permutation operator. Those \( R \)-matrices that do have a shift point are called regular and it is this property which ensures that this definition gives a global Hamiltonian which is a sum of local two-site Hamiltonians.

The structure of the affine quantum (super)algebras is such that solutions of the MYB equation obtained automatically possess the difference property in the spectral parameter as a result of solving Jimbo’s equations \[ 5, 6 \]. On the other hand, there exist other solutions of the MYB equation which do not have the difference property such as Shastry’s solution for the Hubbard model \[ 7 \]. Another model which has been derived through the QISM is the Bariev model \[ 8 \] which was introduced as a solvable generalization of Hirsch’s hole superconductivity model \[ 9 \]. In independent works by Zhou \[ 10 \] and Shiroishi and Wadati \[ 11 \] integrability of the Bariev model was established by obtaining an appropriate solution of the MYB equation. Though the solutions of \[ 10 \] and \[ 11 \] are different, they both share the feature that they do not have the difference property. Energy spectra obtained through the algebraic Bethe ansatz have been studied in \[ 12, 13 \].

Yet another model which is of interest in the study of correlated electron systems is the supersymmetric \( U \) model \[ 14 \] and importantly in the context of the work discussed here its anisotropic generalization \[ 15, 16 \] which can be derived in the framework of the QISM using an \( R \)-matrix associated with one-parameter family of minimal typical representations of the quantum superalgebra \( U_q(gl(2|1)) \). As such, the \( R \)-matrix used to derive this model does have the difference property. In a particular limit, the anisotropic supersymmetric \( U \) model reduces to the Bariev model with the addition of a divergent chemical potential term. The fact that these two models are related yet have been individually derived from \( R \)-matrices of differing character with respect to their spectral parameter dependence has been somewhat mysterious.

In this paper a construction which relates models which differ only by field terms such as the chemical potential will be given and a connection between \( R \)-matrices of the difference property type and those without the difference property will be established. The connection lies in the use of twisting of the algebraic \( R \)-matrices by a suitable twistor which retains the quasi-triangular Hopf-algebra. The twisting construction was developed in \[ 17 \] and gave rise to the more general notion of quasi-triangular quasi-Hopf algebras. However, provided the twistor satisfies appropriate properties known as the twisted 2-
cocycle condition, the twisted structure can also be of the Hopf algebra type. Such twisting operations have lead to some significant developments such as the construction of elliptic solutions of the MYB equation \[18, 19\] and the construction of multiparametric integrable systems \[20\]. The latter were based on a particular type of twistor due to Reshetikhin \[21\]. Here, a generalized form will be used which was given by Engeldinger and Kempf \[22\]. By a certain parametrization, the regularity property of the $R$-matrix will be shown to still hold which allows the construction of an integrable one-dimensional model with two-site interactions. This type of parametrization is adopted from the method used in \[23\] to obtain an extended region of integrability for the supersymmetric $U$ model.

As an example, the anisotropic supersymmetric $U$ model will be constructed with arbitrary chemical potential and magnetic field term. In an appropriate limit the Bariev model is recovered. However, an attempt to obtain an $R$-matrix solution for the Bariev model proves fruitless and still leaves open the question of the origins of the solutions obtained in \[10, 11\] in terms of an underlying algebraic structure.

Finally, it worthwhile to observe that the construction employed here may be understood in terms of representations of coloured quantum algebras \[24\]. (These are not to be confused with colour algebras in the sense of \[25\].) In fact, the relation between coloured quantum algebras and twisting procedures has been discussed by Chakrabarti and Jagannathan \[26\], however it appears that their role in relation to integrable systems has not yet been addressed in the literature.

2. Quantum Inverse Scattering Method

Let $R(u), \overline{R}(u, v) \in \text{End } V \otimes V$ give a solution of the MYB equation

$$\overline{R}_{12}(u, v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)\overline{R}_{12}(u, v). \quad (1)$$

For full generality, we consider the cases when $V$ denotes a $\mathbb{Z}_2$-graded vector space. In such instances it is necessary to impose the following rule for the tensor product multiplication of matrices:

$$(a \otimes b)(c \otimes d) = (-1)^{(b)(c)}ac \otimes bd \quad (2)$$

for matrices $a, b, c, d$ of homogeneous degree. The symbol $(a) \in \mathbb{Z}_2$ denotes the degree of the matrix $a$. The transfer matrix is defined

$$t(u) = \text{str}_0 \left( R_{0N}(u)R_{N0}(N-1)(u)\ldots R_{01}(u) \right)$$

which from (1) can be shown to satisfy

$$[t(u), t(v)] = 0, \quad \forall u, v \in \mathbb{C}.$$ 

Above $\text{str}_0$ denotes the supertrace taken over the auxiliary space which is labelled by 0.

In the usual manner the Hamiltonian associated with the transfer matrix is defined by the relation

$$H = t^{-1}(u) \left. \frac{d}{du} t(u) \right|_{u=0}.$$
Assuming regularity of the $R$-matrix; i.e. 

$$R(0) = P$$

where $P$ is the ($\mathbb{Z}_2$-graded) permutation operator, yields

$$H_{\text{global}} = \sum_{i=1}^{N-1} H_{i(i+1)} + H_{N1}$$

where the local two site Hamiltonians are given by

$$H = \frac{d}{du} P R(u) \bigg|_{u=0} .$$

2. Solutions arising from quantum superalgebras

Let $G$ denote a simple Lie (super)algebra of rank $r$ with generators $\{e_l, f_l, h_l\}_{l=0}^r$ and let $\alpha_l$ be its simple roots. Here we adopt the convention that in the distinguished root basis $\alpha_0$ labels the unique odd simple root. The quantum (super)algebra $U_q(G)$ can be defined with the structure of a ($\mathbb{Z}_2$-graded) quasi-triangular Hopf algebra [27]. We will not give the full defining relations of $U_q(G)$ here (see e.g [3, 4]) but mention that $U_q(G)$ has a coproduct structure given by

$$\Delta(h_l) = I \otimes h_l + h_l \otimes I , \quad \Delta(a) = a \otimes q^{-h_l/2} + q^{h_l/2} \otimes a , \quad a = e_l, f_l .$$

Let $\pi$ be an irreducible representation (irrep) of $U_q(G)$ afforded by the irreducible module $V(\Lambda)$ where $\Lambda$ denotes the highest weight. Assume that the irrep $\pi$ is affinizable, i.e. it can be extended to an irrep of the corresponding quantum affine (super)algebra $U_q(\hat{G})$. Consider an operator $R(x) \in \text{End}(V \otimes V)$, where $x \in \mathbb{C}$ is the multiplicative spectral parameter. It has been shown by Jimbo [5] and also Zhang et. al. [6] for the $\mathbb{Z}_2$-graded case that a solution to the linear equations

$$R(x)\Delta(a) = \Delta(a)R(x) , \quad \forall a \in U_q(G) ,$$

$$R(x) (x\pi(e_\Psi) \otimes \pi(q^{-h_\Psi/2}) + \pi(q^{h_\Psi/2}) \otimes \pi(e_\Psi))$$

$$= (x\pi(e_\Psi) \otimes \pi(q^{h_\Psi/2}) + \pi(q^{-h_\Psi/2}) \otimes \pi(e_\Psi)) R(x)$$

satisfies the Yang-Baxter equation in the tensor product module $V \otimes V \otimes V$:

$$R_{12}(x)R_{13}(x y)R_{23}(y) = R_{23}(y)R_{13}(x y)R_{12}(x) .$$

In the above, $\bar{\Delta} = T \cdot \Delta$, with $T$ the twist map defined by

$$T(a \otimes b) = (-1)^{[a][b]} b \otimes a , \quad \forall a, b \in U_q(G)$$

and $\pi(e_\Psi), \pi(h_\Psi)$ are matrices for operators associated with the highest root $\Psi$ such that the representation extends to a loop representation of the untwisted affine extension of
\( U_q(G) \). The multiplicative spectral parameter \( x \) can be transformed into an additive spectral parameter \( u \) by \( x = \exp(u) \). The structure of the Jimbo equations is such that the solutions obtained must necessarily have the difference property; i.e.

\[
\overline{R}(u, v) = R(u - v).
\]

Explicit solutions of the Jimbo equations can be computed by tensor product graph methods as discussed in [28, 29, 30].

As consequence of the Jimbo equations, for any Cartan element \( h \) the operator \( \sum_i h_i \) on the periodic lattice commutes with the global Hamiltonian (3) and so we may arbitrarily add such terms to the Hamiltonian and still yield a solvable model. In the next section, it will be described how the resultant model may be derived directly from an \( R \)-matrix without the difference property.

### 3. Twisting construction

Let \((A, \Delta, R)\) denote a quasitriangular Hopf (super)algebra where \( \Delta \) and \( R \) denote the co-product and \( R \)-matrix respectively. Suppose that there exists an element \( F \in A \otimes A \) such that

\[
(\Delta \otimes I)(F) = F_{13}F_{23},
\]

\[
(I \otimes \Delta)(F) = F_{13}F_{12},
\]

\[
F_{12}F_{13}F_{23} = F_{23}F_{13}F_{12}.
\]

Then \((A, \Delta^F, R^F)\) is also a quasitriangular Hopf (super)algebra with co-product and \( R \)-matrix respectively given by

\[
\Delta^F = F_{12}\Delta F^{-1}_{12}, \quad R^F = F_{21}RF^{-1}_{12}.
\]

Throughout we refer to \( F \) as a twistor.

The result stated above is a little more general than that originally proposed by Reshetikhin and is due to Engeldinger and Kempf [22]. In the original work [21] Reshetikhin imposed the additional constraint

\[
F_{12}F_{21} = I \otimes I
\]

and in the case that \((A, \Delta, R)\) is an affine quantum (super)algebra Reshetikhin gave the example that \( F \) can be chosen to be

\[
F = \exp \sum_{i < j} (h_i \otimes h_j - h_j \otimes h_i) \phi_{ij}
\]

where \( \{h_i\} \) is a basis for the Cartan subalgebra of the affine quantum (super)algebra and the \( \phi_{ij}, i < j \) are arbitrary complex parameters. However following the construction of Engeldinger and Kempf it is possible to choose

\[
F = \exp \sum_{i,j} (H_i \otimes H_j) \phi_{ij}
\]
which obviously gives a twistor dependent on more free parameters. For our purposes either approach may be adopted but we will choose the latter for convenience. Note also that it is also possible (and for the construction below essential) to extend the Cartan subalgebra by an additional central extension (not the usual central charge) $\hbar$ which will act as a scalar multiple of the identity operator in any representation.

It is worth observing here that the class of twistors (11) above also qualify as Hopf algebra preserving twistors of the type described by Andrews and Cornwell [31] in their work on relating non-standard quantum algebras to standard ones. In notation as above, suppose there exists $F \in A \otimes A$ which satisfies the following relations

$$
F_{12}F_{23} = F_{23}F_{12}
$$

$$(\Delta \otimes I) F = F_{23}F_{13}
$$

$$(I \otimes \Delta) F = F_{12}F_{13}.
$$

Then $(A, \Delta^F, R^F)$ is also a quasi-triangular Hopf algebra with $\Delta^F, R^F$ given by eq. (11) above.

For a given element $h$ of the Cartan subalgebra we now choose a twistor in the particular form

$$
F = \exp(\eta h \otimes \hbar + \tau \hbar \otimes h)
$$

where $\eta, \tau$ are arbitrary scalars. This gives the twisted algebraic $R$-matrix the form

$$
R^F = \exp(\eta \hbar \otimes h + \tau h \otimes \hbar)R \exp(-\eta h \otimes \hbar - \tau \hbar \otimes h)
$$

which still satisfies the MYB equation

$$
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.
$$

Suppose again that $\pi$ extends to a loop representation of the affine quantum super-algebra. We let $\pi \otimes \pi \otimes \pi$ act on the above equation but let the central element $\hbar$ take different values $\beta, \gamma, \delta$ in each of the tensor product spaces. Letting $R^F(u, \beta, \gamma)$ denote the image of $R^F$ under $\pi \otimes \pi \otimes \pi$ where $h_0$ acts as $\beta$ in the first space and $\gamma$ in the second, this then yields the matrix solution

$$
R_{12}(u-v, \beta, \gamma)R_{13}(u, \beta, \delta)R_{23}(v, \gamma, \delta) = R_{23}(v, \gamma, \delta)R_{13}(u, \beta, \delta)R_{12}(u-v, \beta, \gamma)
$$

(12)

From this solution we may construct the transfer matrix

$$
t(u, \beta, \delta) = \text{str}_0 \left( R_{0L}(u, \beta, \delta) \ldots R_{02}(u, \beta, \delta)R_{01}(u, \beta, \delta) \right)
$$

(13)

which as a result of (12) forms a commuting family in two variables

$$
[t(u, \beta, \delta), t(v, \gamma, \delta)] = 0.
$$

(14)

It then follows that one may diagonalize $t(u, \beta, \delta)$ independently of both $u$ and $\beta$. Setting $\exp h = M$ and $R(u)$ as the image of $R$ under $\pi \otimes \pi$ we get

$$
R^F(u, \beta, \delta) = M_1^{\tau \delta} M_2^{\eta \beta} R(u) M_1^{-\eta \delta} M_2^{-\tau \beta}.
$$
The above solution $R^F(u, \beta, \delta)$ can be made regular by setting $\delta = 0, \beta = u$. The local two-site Hamiltonian operator associated with this solution is given by

\[
H^F = \left. \frac{d}{du} PR^F(u, \beta = u, \delta = 0) \right|_{u=0} = H + \eta h_1 - \tau h_2
\]

with $H$ given by (14) above. It immediately follows that the global Hamiltonians are related by

\[
H_{global}^F = H_{global} + (\eta - \tau) \sum_{i=1}^{L} h_i.
\]

5. The supersymmetric $U$ model

The quantum superalgebra $U_q(sl(2|1))$ has simple generators $\{e_0, f_0, h_0, e_1, f_1, h_1\}$ associated with the Cartan matrix

\[
A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}.
\]

This algebra admits a non-trivial one parameter family of four-dimensional representations which we label $\pi$ given by

\[
\begin{align*}
\pi(e_0) &= \sqrt{[\alpha]} e_2^1 + \sqrt{[\alpha + 1]} e_4^3 \\
\pi(f_0) &= \sqrt{[\alpha]} e_2^1 + \sqrt{[\alpha + 1]} e_3^4 \\
\pi(h_0) &= \alpha(e_1^1 + e_2^2) + (\alpha + 1)(e_3^3 + e_4^4) \\
\pi(e_1) &= -e_3^2 \\
\pi(f_1) &= -e_2^3 \\
\pi(h_1) &= e_2^2 - e_3^3.
\end{align*}
\]

Above the indices of the elementary matrices $e_j^i$ carry the $\mathbb{Z}_2$-grading $(1) = (4) = 0, (2) = (3) = 1$ and we employ the notation

\[
[x] = \frac{q^x - q^{-x}}{q - q^{-1}}.
\]

Associated with this representation there is a solution of the MYB equation with the difference property which is obtained by solving Jimbo’s equations. The problem of obtaining this solution was considered in [16, 32] (see also [33]). Adopting the prescription detailed in the previous section, the following solution of (12) is obtained with the choice

\[
h = (\alpha + 1) I - h_0 - kh_1
\]

where $k$ is a free variable.
\[ R(u, \beta, \delta) = \]

\[
\begin{pmatrix}
R_{11}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & R_{12}^{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & R_{13}^{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & R_{14}^{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & R_{21}^{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & R_{22}^{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & R_{23}^{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & R_{24}^{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_{31}^{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_{32}^{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_{33}^{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_{41}^{41} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_{42}^{42} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_{43}^{43} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_{44}^{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

where the non-zero entries are given by

\[
R_{11}^{11} = \exp[(\eta - \tau)(\beta - \delta)]\frac{[u - \alpha][u - \alpha - 1]}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{12}^{12} = \exp[(\eta - \tau)((1 - k)\beta - \delta)]\frac{[u][u - \alpha - 1]}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{13}^{13} = \exp[(\eta - \tau)(k\beta - \delta)]\frac{[u][u - \alpha - 1]}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{14}^{14} = \exp[-(\eta - \tau)\delta]\frac{[u][u - 1]}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{21}^{21} = \exp[((1 - k)\eta - \tau)(\beta - \delta)]q^u [\alpha][u - \alpha - 1]\frac{[\alpha][u - \alpha - 1]}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{31}^{31} = \exp[(k\eta - \tau)(\beta - \delta)]q^u [\alpha][u - \alpha - 1]\frac{[\alpha][u - \alpha - 1]}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{41}^{41} = \exp[-(\eta - \tau)(\beta - \delta)]q^{2u} [\alpha][\alpha + 1]\frac{[\alpha][\alpha + 1]}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{23}^{23} = \exp[-k\tau(\beta + ((k - 1)\eta + \tau))\delta]q^{u-1/2} [\alpha]^{1/2}[\alpha + 1]^{1/2}[u]\frac{[\alpha]^{1/2}[\alpha + 1]^{1/2}[u]}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{32}^{32} = -\exp[(k - 1)\tau\beta + (\tau - k\eta)\delta]q^{u+1/2} [\alpha]^{1/2}[\alpha + 1]^{1/2}[u]\frac{[\alpha]^{1/2}[\alpha + 1]^{1/2}[u]}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{21}^{21} = \exp[(\eta - \tau)(\beta + (k - 1)\delta)]\frac{[u][u - \alpha - 1]}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{22}^{22} = \exp[(1 - k)(\eta - \tau)(\beta - \delta)]\frac{[u - \alpha - 1]}{[u + \alpha + 1]}
\]
\[
R_{23}^{23} = \exp[(\eta - \tau)(k\beta + (k - 1)\delta)][u]^{2}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{24}^{21} = \exp[(k - 1)(\eta - \tau)\delta][u][u + \alpha + 1]
\]
\[
R_{12}^{21} = -\exp[(\eta + (k - 1)\tau)(\beta - \delta)]q^{-u}\frac{[\alpha][u - \alpha - 1]}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{32}^{23} = \exp[(k\eta + (k - 1)\tau)(\beta - \delta)]\frac{2q - q^{2\alpha + 1} - q^{-2\alpha - 1} - q^{2u + 1} + q^{2u - 1}}{(q^{u + \alpha} - q^{u - \alpha})(q^{u + \alpha + 1} - q^{u - \alpha - 1})}
\]
\[
R_{42}^{21} = \exp[\tau(k - 1)(\beta - \delta)]q^{u}\frac{[\alpha + 1]}{[u + \alpha + 1]}
\]
\[
R_{14}^{23} = -\exp[k\eta\beta + ((1 - k)\tau - \eta)\delta]q^{-u + 1/2}\frac{[\alpha]^{1/2}[\alpha + 1]^{1/2}[u]}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{41}^{23} = -\exp[(k\eta - \tau)\beta + (1 - k)\tau\delta]q^{u + 1/2}\frac{[\alpha]^{1/2}[\alpha + 1]^{1/2}[u]}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{31}^{31} = \exp[(\eta - \tau)(\beta - k\delta)]\frac{[u][u - \alpha - 1]}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{32}^{32} = \exp[(\eta - \tau)((1 - k)\beta - k\delta)]\frac{[u][u - \alpha - 1]}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{33}^{33} = \exp[k(\eta - \tau)(\beta - \delta)]\frac{[u - \alpha - 1]}{[u + \alpha + 1]}
\]
\[
R_{34}^{34} = \exp[-k(\eta - \tau)\delta]\frac{[u]}{[u + \alpha + 1]}
\]
\[
R_{13}^{31} = -\exp[(\eta - k\tau)(\beta - \delta)]q^{-u}\frac{[\alpha][u - \alpha - 1]}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{23}^{23} = \exp[(1 - k)\eta - k\tau)(\beta - \delta)]\frac{2q^{-1} - q^{2\alpha + 1} - q^{-2\alpha - 1} + q^{-2u + 1} - q^{-2u - 1}}{(q^{u + \alpha} - q^{u - \alpha})(q^{u + \alpha + 1} - q^{u - \alpha - 1})}
\]
\[
R_{43}^{41} = \exp[-k\tau(\beta - \delta)]q^{u}\frac{[\alpha + 1]}{[u + \alpha + 1]}
\]
\[
R_{14}^{32} = \exp[(1 - k)\eta\beta - (\eta - k\tau)\delta]q^{-u - 1/2}\frac{[\alpha]^{1/2}[\alpha + 1]^{1/2}[u]}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{41}^{32} = \exp([(1 - k)\eta - \tau)\beta + k\tau\delta]q^{-u - 1/2}\frac{[\alpha]^{1/2}[\alpha + 1]^{1/2}[u]}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{41}^{41} = \exp[(\eta - \tau)\beta]\frac{[u][u - 1]}{[u + \alpha][u + \alpha + 1]}
\]
\[
R_{42}^{42} = \exp[(1 - k)(\eta - \tau)\beta]\frac{[u]}{[u + \alpha + 1]}
\]
\[
R_{43}^{43} = \exp[k(\eta - \tau)\beta]\frac{[u]}{[u + \alpha + 1]}
\]
\[
R_{44}^{44} = 1
\]
\[ R^{41}_{14} = \exp[\eta(\beta - \delta)]q^{-2u} \frac{[\alpha][\alpha + 1]}{[u][u + \alpha + 1]} \]

\[ *R^{12}_{24} = - \exp[\eta(1 - k)(\beta - \delta)]q^{-u} \frac{[\alpha + 1]}{[u + \alpha + 1]} \]

\[ *R^{23}_{34} = - \exp[k\eta(\beta - \delta)]q^{-u} \frac{[\alpha + 1]}{[u + \alpha + 1]} \]

\[ *R^{41}_{23} = \exp[(\eta - k\tau)\beta + (k - 1)\eta\delta]q^{-u-1/2} \frac{[\alpha]^{1/2}[\alpha + 1]^{1/2}[u]}{[u + \alpha][u + \alpha + 1]} \]

\[ *R^{32}_{41} = - \exp[(\eta + (k - 1)\tau)\beta - k\eta\delta]q^{-u+1/2} \frac{[\alpha]^{1/2}[\alpha + 1]^{1/2}[u]}{[u + \alpha][u + \alpha + 1]} \]

We remark again that the above \( R \)-matrix solves the MYB equation subject to the rule (3) which is a natural consequence of the superalgebra structure underlying the \( R \)-matrix. However, a change of signs in those matrix elements with a \( * \) gives a solution which satisfies the Yang-Baxter equation in the usual matrix form.

The four dimensional space of states associated with the \( U_q(sl(2|1)) \) representation may be identified with the electronic states

\[ |0\rangle, \ |\uparrow\rangle, \ |\downarrow\rangle, \ |\uparrow\downarrow\rangle. \]

Taking the logarithmic derivative of the transfer matrix yields the following global Hamiltonian on a periodic lattice (with convenient normalization)

\[ H = - \sum_{i,\sigma}(c_{i\sigma}^\dagger c_{(i+1)\sigma} + h.c.) \exp[-1/2(\eta - \sigma\epsilon)n_{i\sigma} - 1/2(\eta + \sigma\epsilon)n_{(i+1)\sigma}] + U \sum_i n_{i\uparrow}n_{i\downarrow} + \left( q^{\alpha+1} + q^{-\alpha-1} - \frac{(\eta - \tau)(q^{\alpha+1} - q^{-\alpha-1})}{4\ln q} \right) \sum_i n_i + \frac{(\eta - \tau)(1 - 2k)(q^{\alpha+1} - q^{-\alpha-1})}{2\ln q} \sum_i S_i^z \]

where

\[ \exp \epsilon = q^{-1}, \quad \exp(-\eta) = \frac{[\alpha + 1]}{[\alpha]}, \quad U = 2[\alpha]^{-1} \]

and the standard notation has been used for the Fermi and spin operators. The above model is the supersymmetric \( U \) model with chemical potential and applied magnetic field whose couplings may be chosen arbitrarily through the parameters \( \eta - \tau \) and \( k \).
6. The Bariev model

The Bariev model on a one-dimensional periodic lattice has the Hamiltonian

\[
H = - \sum_{i,\sigma} \left( c_{i\sigma}^\dagger c_{(i+1)\sigma} + \text{h.c.} \right) \exp \left[ -1/2(\eta - \sigma \eta) n_{i(-\sigma)} - 1/2(\eta + \sigma \eta) n_{(i+1)(-\sigma)} \right].
\]

It is apparent that the Bariev model is obtainable from the supersymmetric $U$ model above in the limit $\alpha \to \infty$ by choosing $\eta - \tau = 4 \ln \eta$ and $k = 1/2$. It would thus appear reasonable to suspect that an $R$-matrix solution for the Bariev model is obtainable from the solution for the supersymmetric $U$ model. However, if one is to naively take the limit $\alpha \to \infty$ one finds

\[
\lim_{\alpha \to \infty} R(u, \beta, \delta) = M_2^{\eta(\beta-\delta)} P
\]

and as such the Bariev model is not obtainable in this fashion.

Alternatively, we may make the following rescaling of the parameters

\[
\begin{align*}
  u &\to \alpha u, & v &\to \alpha v, \\
  \eta &\to 2\eta \ln q, & \tau &\to 2\tau \ln q, \\
  q &\to q^{1/\alpha}
\end{align*}
\]

and set

\[
\beta = u, \quad \delta = v, \quad k = 1/2.
\]

In this manner we obtain a solution of the MYB equation

\[
R_{12}(u, v)R_{13}(u, w)R_{23}(v, w) = R_{23}(v, w)R_{13}(u, w)R_{12}(u, v)
\]

in the limit $\alpha \to \infty$ where the matrix elements of $R(u, v)$ now read

\[
\begin{align*}
R_{11}^{11} &= q^{2(\eta-\tau)(u-v)} \frac{[u-v-1]^2}{[u-v+1]^2} \\
R_{12}^{12} &= q^{(\eta-\tau)(u-2v)} \frac{[u-v][u-v-1]}{[u-v+1]^2} \\
R_{13}^{13} &= q^{(\eta-\tau)(u-2v)} \frac{[u-v][u-v-1]}{[u-v+1]^2} \\
R_{14}^{14} &= q^{-2(\eta-\tau)v} \frac{[u-v]^2}{[u-v+1]^2} \\
* R_{21}^{12} &= q^{(\eta-2\tau+1)(u-v)} \frac{[u-v-1]}{[u-v+1]^2} \\
* R_{31}^{13} &= q^{(\eta-2\tau+1)(u-v)} \frac{[u-v-1]}{[u-v+1]^2} \\
R_{41}^{14} &= q^{(2-2\tau)(u-v)} \frac{1}{[u-v+1]^2} \\
* R_{23}^{14} &= q^{(1-\tau)u+(-\eta+2\tau-1)v} \frac{[u-v]}{[u-v+1]^2}
\end{align*}
\]
\[* R_{32}^{14} = -q^{(1-\tau)u+(2\tau-\eta-1)v} \frac{[u-v]}{[u-v+1]^2} \]

\[ R_{21} = q^{(\eta-\tau)(2u-v)} \frac{[u-v][u-v-1]}{[u-v+1]^2} \]

\[* R_{22}^{22} = q^{(\eta-\tau)(u-v)} \frac{[u-v-1]}{[u-v+1]^2} \]

\[ R_{23} = q^{(\eta-\tau)(u-v)} \frac{[u-v]^2}{[u-v+1]^2} \]

\[ R_{24} = q^{-(\eta-\tau)v} \frac{[u-v]}{[u-v+1]} \]

\[ R_{21}^{12} = -q^{(2\eta-\tau-1)(u-v)} \frac{[u-v-1]}{[u-v+1]^2} \]

\[* R_{32}^{32} = -q^{(\eta-\tau)(u-v)} \frac{1}{[u-v+1]^2} \]

\[ R_{42} = q^{(1-\tau)(u-v)} \frac{1}{[u-v+1]} \]

\[ R_{31}^{31} = q^{(\eta-\tau)(2u-v)} \frac{[u-v][u-v-1]}{[u-v+1]^2} \]

\[* R_{32}^{32} = q^{(\eta-\tau)(u-v)} \frac{[u-v]^2}{[u-v+1]^2} \]

\[ R_{33} = q^{(\eta-\tau)(u-v)} \frac{[u-v-1]}{[u-v+1]} \]

\[ R_{34} = q^{-(\eta-\tau)v} \frac{[u-v]}{[u-v+1]} \]

\[ R_{13}^{31} = -q^{(2\eta-\tau-1)(u-v)} \frac{[u-v-1]}{[u-v+1]^2} \]

\[* R_{23}^{23} = -q^{(\eta-\tau)(u-v)} \frac{1}{[u-v+1]^2} \]

\[ R_{43} = q^{(1-\tau)(u-v)} \frac{1}{[u-v+1]} \]

\[ R_{14}^{14} = q^{(\eta-1)u+(\tau-2\eta+1)v} \frac{[u-v]}{[u-v+1]^2} \]

\[ R_{41}^{41} = q^{(1+\eta-2\tau)u+(\tau-1)v} \frac{[u-v]}{[u-v+1]^2} \]

\[ R_{41} = q^{2(\eta-\tau)u} \frac{[u-v]^2}{[u-v+1]^2} \]
$$R_{42}^{42} = q^{(\eta-\tau)u} \frac{[u-v]}{[u-v+1]}$$

$$R_{43}^{43} = q^{(\eta-\tau)u} \frac{[u-v]}{[u-v+1]}$$

$$R_{44}^{44} = 1$$

$$R_{14}^{41} = q^{(2\eta-2)(u-v)} \frac{1}{[u-v+1]^2}$$

$$\ast R_{24}^{42} = -q^{(\eta-1)(u-v)} \frac{1}{[u-v+1]}$$

$$\ast R_{34}^{43} = -q^{(\eta-1)(u-v)} \frac{1}{[u-v+1]}$$

$$\ast R_{23}^{41} = q^{(2\eta-\tau-1)u+(1-\eta)v} \frac{[u-v]}{[u-v+1]^2}$$

$$\ast R_{32}^{41} = -q^{(2\eta-\tau-1)u+(1-\eta)v} \frac{[u-v]}{[u-v+1]^2}.$$ 

However, calculating the Hamiltonian associated with this solution one finds

$$H = -\sum_{i,\sigma} (c_{i\sigma}^\dagger c_{i+1\sigma} + h.c.) + \left[1/2(q - q^{-1})(\tau - \eta) + (q + q^{-1})\right] \sum_i n_i$$

which of course is simply a free fermion model with chemical potential and corresponds to a particular case of the Bariev model.

7. Conclusions

Here it has been demonstrated how $R$-matrices which satisfy the MYB equation such that they do not have the difference property may be obtained. As an example, the anisotropic supersymmetric $U$ model with arbitrary chemical potential and magnetic field terms was derived, which, in a particular limit, reproduces the Bariev model. Regrettably, this approach does not shine a light on the connection with the $R$-matrix solutions for the Bariev model computed in [10, 11] without the difference property. It is of course possible that a different limiting procedure may yield an $R$-matrix solution for the Bariev model but its form is not apparent at present.

An open problem for future work is to investigate whether Shastry’s solution [4] and the recently introduced $SU(N)$ Hubbard models [34, 35] may be related to an $R$-matrix with the difference property via a twisting procedure.

Acknowledgements

This work is supported by the Australian Research Council. Many thanks to A. Foerster for assistance, M.J. Martins for discussions which motivated me to look at this problem and Y.-Z. Zhang for proofreading the manuscript.
References

[1] M. Jimbo 1985 Lett. Math. Phys. 10 63

[2] V.G. Drinfeld 1986 Quantum Groups in Proceedings of the International Congress of Mathematicians, ed. A.M. Gleason (American Mathematical Society, Providence) 798

[3] H. Yamane 1991 Proc. Jap. Acad. A67 108; 1994 RIMS Pub. 30 15

[4] M. Scheunert 1992 Lett. Math. Phys. 24 173; 1993 J. Math. Phys. 34 3780

[5] M. Jimbo 1986 Commun. Math. Phys. 102 537

[6] A.J. Bracken, M.D. Gould and R.B. Zhang 1990 Mod. Phys. Lett. A5 831

[7] B.S. Shastry 1986 Phys. Rev. Lett. 56 1529, 2453; 1988 J. Stat. Phys. 50 57

[8] R.Z. Bariev 1991 J. Phys. A: Math. Gen. 24 L549

[9] J.E. Hirsch 1989 Physica 158C 326

[10] H.-Q. Zhou 1996 Phys. Lett. A221 104; 1996 J. Phys. A: Math. Gen. 29 5509

[11] M. Shiroishi and M. Wadati 1997 J. Phys. A: Math. Gen. 30 1115

[12] H.-Q. Zhou 1997 J. Phys. A: Math. Gen. 30 L423

[13] M.J. Martins and P.B. Ramos 1998 Nucl. Phys. B522 413

[14] A.J. Bracken, M.D. Gould, J.R. Links and Y.-Z. Zhang 1995 Phys. Rev. Lett. 74 2768

[15] R.Z. Bariev, A. Klümper and J. Zittartz 1995 Europhys. Lett. 32 85

[16] M.D. Gould, K.E. Hibberd, J.R. Links and Y.-Z. Zhang 1996 Phys. Lett. A212 156

[17] V.G. Drinfeld 1990 Leningrad Math. J. 1 1419

[18] M. Jimbo, H. Konno, S. Odake and J. Shiraishi 1999 Commun. Math. Phys. 199 605

[19] Y.-Z. Zhang and M.D. Gould Quasi-Hopf algebras and elliptic quantum supergroups [math/9809156]

[20] A. Foerster, J. Links and I. Roditi 1998 J. Phys. A: Math. Gen. 31 687

[21] N.Y. Reshetikhin 1990 Lett. Math. Phys. 20 331

[22] R.A. Engeldinger and A. Kempf 1994 J. Math. Phys. 35 1931

[23] J. Links Extended integrability regime for the supersymmetric U model J. Phys. A: Math. Gen., to appear
[24] C. Quesne 1997 *Coloured Hopf algebras* in Quantum Group Symposium at Group 21. Eds. H.-D. Doebner and V.K. Dobrev (Heron Press, Sofia) 219

[25] D.S. McAnally 1995 Lett. Math. Phys. **33** 249

[26] R. Chakrabarti and R. Jagannathan 1994 J. Phys. A: Math. Gen. **27** 2023

[27] M.D. Gould, R.B. Zhang and A.J. Bracken 1993 Bull. Aust. Math. Soc. **47** 353

[28] R.B. Zhang, M.D. Gould and A.J. Bracken 1991 Nucl. Phys. **B354** 625

[29] G.W. Delius, M.D. Gould and Y.-Z. Zhang 1994 Nucl. Phys. **B432** 377

[30] G.W. Delius, M.D. Gould, J.R. Links and Y.-Z. Zhang 1995 Int. J. Mod. Phys. **A10** 3259

[31] A.D. Jacobs and J.F. Cornwell 1997 J. Math. Phys. **38** 5383

[32] A.J. Bracken, G.W. Delius, M.D. Gould and Y.-Z. Zhang 1994 J. Phys. A: Math. Gen. **27** 6551

[33] D. Arnaudon 1997 J. High Energy Phys. **12** 6

[34] Z. Maassarani 1998 Mod. Phys. Lett. **B12** 51

[35] M.J. Martins 1998 Phys. Lett. **A247** 218