(Weak) $G_2$ holonomy from self-duality, flux and supersymmetry

Adel Bilal$^{1,2}$, Jean-Pierre Derendinger$^1$ and Konstadinos Sfetsos$^{1,3}$

$^1$ Institut de Physique, Université de Neuchâtel
CH–2000 Neuchâtel, Switzerland

$^2$ Laboratoire de Physique Théorique, Ecole Normale Supérieure
24 rue Lhomond, 75231 Paris Cedex 05, France

$^3$ Department of Engineering Sciences, University of Patras
26110 Patras, Greece

Abstract

The aim of this paper is two-fold. First, we provide a simple and pedagogical discussion of how compactifications of M-theory or supergravity preserving some four-dimensional supersymmetry naturally lead to reduced holonomy or its generalization, reduced weak $G_2$ holonomy. We relate the existence of a (conformal) Killing spinor to the existence of certain closed and co-closed $p$-forms, and to the metric being Ricci flat or Einstein.

Then, for seven-dimensional manifolds, we show that octonionic self-duality conditions on the spin connection are equivalent to $G_2$ holonomy and certain generalized self-duality conditions to weak $G_2$ holonomy. The latter lift to self-duality conditions for cohomogeneity-one spin(7) metrics. To illustrate the power of this approach, we present several examples where the self-duality condition largely simplifies the derivation of a $G_2$ or weak $G_2$ metric.

e-mail: adel.bilal, jean-pierre.derendinger, konstadinos.sfetsos@unine.ch
1 Introduction

Compactifications to four dimensions of M-theory or string theory, in their simplest setting (no background flux or gauge fields turned on), require the compactification manifold \( K \) to admit at least one covariantly constant spinor in order to preserve some four-dimensional supersymmetry. The existence of a covariantly constant spinor in turn implies that the manifold \( K \) does have reduced holonomy: \( SU(3) \) instead of \( SU(4) \simeq SO(6) \) in string theory and \( G_2 \) instead of \( SO(7) \) in M-theory/eleven-dimensional supergravity. It also implies that \( K \) is Ricci flat and hence the metric field equations are automatically satisfied in the compact directions. For these reasons, 6-dimensional manifolds with \( SU(3) \) holonomy (Calabi-Yau spaces) have been abundantly studied over the last decade and a half. Similarly, seven-dimensional manifolds with \( G_2 \) holonomy have attracted much attention more recently. Mathematically quite similar to the latter are also eight-dimensional manifolds of \( Spin(7) \) holonomy, and both cases can often be studied in parallel, but the real physical interest of course is in seven-dimensional manifolds.

One might ask what happens in a less simple setting when background fields are turned on. Let us now concentrate on M-theory, or on its low-energy limit, eleven-dimensional supergravity. Physically, the background fields induce a non-vanishing energy-momentum tensor and the metric is no longer Ricci flat. The next best thing we can hope for is that \( K \) is an Einstein space, i.e. \( R_{ij} \sim g_{ij} \). Conditions for unbroken four-dimensional supersymmetry now are the existence of a conformal Killing spinor \( \eta \) on \( K \) satisfying \( D_i \eta \sim \gamma_i \eta \). This indeed implies that \( K \) is an Einstein manifold. Being no longer Ricci flat, it cannot have \( G_2 \) holonomy. However, we will see that the appropriate notion in this setting is that of weak \( G_2 \) holonomy, a concept originally introduced by Gray [1]. While \( G_2 \) holonomy is equivalent to the existence of a 3-form \( \phi \) obeying \( d\phi = 0 \) and \( d*\phi = 0 \), weak \( G_2 \) holonomy is equivalent to the existence of a 3-form \( \phi \) obeying \( d\phi = \lambda * \phi \) and then also \( d*\phi = 0 \). Any weak \( G_2 \) holonomy manifold is an Einstein space with \( R_{ij} \sim \lambda^2 g_{ij} \), and we will see how the condition of weak \( G_2 \) holonomy is related to the conformal Killing spinor equation \( D_i \eta \sim \lambda \gamma_i \eta \). In this sense, one may view \( G_2 \) holonomy as the \( \lambda \to 0 \) limit of weak \( G_2 \) holonomy.

Many metrics of \( G_2 \) holonomy and some metrics of weak \( G_2 \) holonomy have been constructed over the past (for some examples, see e.g. [2, 3, 4, 5, 6, 7]), not always
without considerable effort. In the present paper, we will show how a simple self-duality condition for the spin connection $\omega^{ab}$ allows for a quick and easy proof of all desired properties. The basic ingredient to define self-duality in seven dimensions is the $G_2$ invariant 4-index tensor which is dual to the octonionic structure constants. (Recall that $G_2$ is the automorphism group of the octonion algebra.) A similar method in eight dimensions was first employed some time ago in [8] to find an Eguchi-Hanson type metric of spin(7) holonomy which had previously been found in [3] by solving second order differential equations. Also refs. [9, 10] use a self-duality in seven dimensions to get $G_2$ holonomy. In a way this is natural, since using self-duality of $\omega^{ab}$ is a well-known technique in four dimensions to generate self-dual and hence Ricci-flat curvatures. Actually, it was first realised in [11] that imposing an octonionic self-duality condition on the curvature $R^{ab}$ in eight dimensions leads to 7 relations between the 28 components and hence implies that the holonomy group must be spin(7). By doing trivial dimensional reduction, the same argument implied that self-dual curvature in seven dimensions must correspond to $G_2$ holonomy, in six dimensions to SU(3) holonomy etc.

In four dimensions the Eguchi-Hanson metric [14] does follow from imposing self-duality of $\omega^{ab}$ within a certain ansatz for the metric, but the Atiyah-Hitchin metric [15] does not obviously seem to fall into this class. However, it can be obtained from adding an appropriate inhomogeneous term to the self-duality condition. This motivated us to study generalized self-duality conditions for the seven-dimensional spin connection, allowing also for certain inhomogeneous terms. While it was known [3, 10] that in seven dimensions self-duality implies $G_2$ holonomy, we show that some appropriately generalized self-duality condition with an inhomogeneous term implies weak $G_2$ holonomy. Although self-duality of $\omega^{ab}$ is a “gauge”-dependent condition, in four dimensions it is known (see e.g. [10]) that a self-dual curvature always follows from an $\omega^{ab}$ that is self-dual up to a local Lorentz transformation. We prove the analogous statement for $G_2$ and weak $G_2$: for any manifold of (weak) $G_2$ holonomy, the spin connection is (generalized) self-dual, up to a local SO(7) transformation. For most of the $G_2$ and

\[1\] The relation between self-dual curvature and self-dual spin connection was also briefly mentioned in ref. [11]. A different approach was outlined in ref. [12] and used in ref. [13].

\[2\] It can actually be shown that in the case of the Atiyah-Hitchin metric a “non-canonical” choice of the 4-bein, related by some local Lorentz transformation to the obvious one, is equivalent to the inhomogeneous term mentioned above. Such an equivalence will obviously not be possible between $G_2$ and weak $G_2$ holonomy.
weak $G_2$ holonomy metrics that are known in the literature, the spin connection actually does satisfy the self-duality criterion without any need to perform a local Lorentz transformation. A standard example for weak $G_2$ holonomy is the metric based on the Aloff-Walach spaces $N(k,l)$ (cosets $SU(3)/U(1)$) and, with the obvious choice, its spin connection satisfies our generalized self-duality condition. (This corresponds to an analogous statement made in [17] but can also be easily checked directly.)

With hindsight, most of these results could have been expected since weak $G_2$ holonomy metrics can be generated from cohomogeneity-one spin(7) metrics [18], and indeed our inhomogeneously modified self-duality condition for weak $G_2$ holonomy is related to the standard self-duality of spin(7). On the other hand, we are not aware of a proof that weak $G_2$ holonomy implies the existence of a (generalized) self-dual $\omega^{ab}$.

Ref. [11] e.g. only considers trivial reductions from 8 to 7 dimensions resulting in Ricci flat $G_2$, not weak $G_2$, manifolds. Thus, although pieces of information are scattered throughout the literature, to the best of our knowledge, they have not been all linked together. At the risk of repeating one or the other fact well-known to the experts, we try to give a self-contained presentation in which self-duality relations play a central rôle.

The outline of the paper is the following. In the next section, we first review some well-known facts on covariantly constant spinors and reduced holonomy. In section 3, we introduce the self-duality condition for $\omega^{ab}$ and show equivalence with holonomy contained in $G_2$, implying self-duality of the curvature and Ricci flatness. To underscore the power of this approach, we show how to get in a few lines the differential equations for the general six-function ansatz for a $G_2$ metric obtained in [3]. In section 4, we turn to the modified self-duality condition and show how it implies weak $G_2$ holonomy, a modified version of self-duality of the curvature, and the Ricci tensor being Einstein. We also prove the converse, namely that for any weak $G_2$ holonomy manifold we can always find a spin connection satisfying the generalized self-duality condition. We discuss the relation with cohomogeneity-one spin(7) metrics which provides non-trivial examples of weak $G_2$ holonomy satisfying the generalized self-duality conditions. In section 5, we return to the physics of M-theory, resp. eleven-dimensional supergravity compactified on a seven-manifold and in particular rederive that unbroken four-dimensional supersymmetry only allows a four-form flux in space-time, namely $F_{\mu\nu\rho\sigma} = f\epsilon_{\mu\nu\rho\sigma}$, while the internal components $F_{ijkl}$ all must vanish [19, 20]. The
seven-manifold then has weak $G_2$ holonomy, and the flux strength $f$ turns out to be proportional to the weak $G_2$ holonomy parameter $\lambda$, \textit{i.e.} $d\phi \sim f * \phi$. This is the precise statement how weak $G_2$ holonomy is related to supersymmetric compactification. Finally, we conclude in section 6. Since the octonionic structure constants and their duals play a prominent rôle in all our paper, an appendix is devoted to their properties and collects useful identities.

2 (Conformal) Killing spinors and reduced (weak) holonomy

2.1 Killing spinors, reduced holonomy and Ricci flatness

We begin by reviewing a few well-known facts about reduced holonomy. As discussed in the introduction, in the absence of background fields besides the metric tensor, a compactification of supergravity will preserve some supersymmetry if the compactification manifold $K$ admits a Killing spinor, \textit{i.e.} a covariantly constant (non-zero) spinor $\eta$. This is reviewed in some detail in section 5. So assume

\[ D_i \eta = 0, \quad i = 1, \ldots, \dim K. \]  

(2.1)

Applying another $D_j$ and antisymmetrising, this leads to

\[ R_{abij} \gamma^{ab} \eta = 0. \]  

(2.2)

Our conventions are that $a, b, c, \ldots$ are flat and $i, j, k, \ldots$ are curved indices on $K$. The viel-bein one-forms will be denoted $e^a = e^a_i dx^i$. But $R_{abij} \gamma^{ab}$ are the generators of the holonomy group and eq. (2.2) precisely tells us that the holonomy group cannot be the maximal $SO(\dim K)$ but must be a subgroup of it: holonomy has been reduced.

In seven dimensions, the generic spinor transforms as representation $8$ of $SO(7)$. If one spinor solves eq. (2.1), one expects (at least) the decomposition $8 \rightarrow 7 + 1$ under the holonomy group and this is the decomposition under $G_2 \subset SO(7)$. Conversely, if the holonomy is $G_2 \subset SO(7)$, then the spinor splits as $8 \rightarrow 7 + 1$ and the $1$ corresponds to a covariantly constant (actually constant and $G_2$ invariant) spinor. In general, the existence of a covariantly constant spinor always implies reduced holonomy, but the converse is not always true. Actually the converse can only be true if the space is Ricci flat, since (2.1) always implies Ricci flatness as we now recall.
The (torsion-free) curvature \( R^{ab} = \frac{1}{2} R^{ab}_{\; cd} e^c \land e^d \) satisfies \( R_{abcd} = R_{cdab} = -R_{abdc} \) as well as \( R_{a[bcde]} = 0 \) and the Ricci tensor is \( R_{ij} = R_{iaj} \). Now multiply (2.2) by \( \gamma^j \) to conclude that

\[
R_{ij} \gamma^j \eta = 0, \tag{2.3}
\]

which in turn implies

\[
R_{ij} = 0. \tag{2.4}
\]

Another criterion for reduced holonomy more often used in the mathematical literature is the existence of certain closed \( p \)-forms, e.g. a closed 4-form \( \Phi \) for \( \text{spin}(7) \) and a closed 3-form \( \phi \) with its dual 4-form \( \ast \phi \) also closed for \( G_2 \) holonomy. This can be easily understood from the existence of a covariantly constant spinor. It is well-known that from a (commuting) spinor one can make \( p \)-forms as

\[
\phi(p) = \frac{(-i)^p}{p!} \eta \gamma_{i_1 \ldots i_p} \eta \; dx^{i_1} \land \ldots \land dx^{i_p}. \tag{2.5}
\]

In the absence of torsion,

\[
d \phi(p) = 0 \tag{2.6}
\]

is equivalent to \( D_{ij} \left( \eta \gamma_{i_1 \ldots i_p} \eta \right) = 0 \) and we see that (2.1) implies eq. (2.6). To have equivalence, both equations must contain the same number of equations. Consider the example of \( G_2 \). Then \( \eta \) is an 8-component (real) spinor and one can only make a non-vanishing 3-form \( \phi \equiv \phi(3) \) and of course its dual 4-form \( \ast \phi \). Imposing then

\[
G_2 : \quad d \phi = 0 \quad \text{and} \quad d \ast \phi = 0 \tag{2.7}
\]

corresponds to \( \binom{7}{4} + \binom{7}{5} = 56 \) conditions. But \( D_i \eta = 0 \) also corresponds to \( 7 \times 8 = 56 \) conditions. Hence, conditions (2.7) are equivalent to (2.1), and thus a necessary and sufficient condition for the holonomy to be contained in \( G_2 \). Actually, if we have more than one Killing spinor the holonomy may be further reduced. Consider the example of the seven-dimensional product manifold \( S^1 \times CY \) where the holonomy is \( \text{SU}(3) \subset G_2 \). Then \( 8 \rightarrow (3 + \overline{3} + 1) + 1 \). If we want to exclude such cases in order to have exactly \( G_2 \) holonomy, not some subgroup of it, we should require the existence of one and only one (real) Killing spinor.
2.2 Conformal Killing spinors, reduced weak holonomy and Einstein spaces

Similarly, it is easy to see that in general

\[ D_j \eta = i \tilde{\lambda} \gamma_j \eta \]  

(2.8)

implies that \( d\phi(p) \) is proportional to \( \phi(p+1) \). Specifically, for \( SO(7) \) spinors the relation is

\[ d\phi(p) = [1 - (-1)^p](p + 1) \tilde{\lambda} \phi(p+1), \quad p = 0, \ldots, 6, \]  

(2.9)

(where \( \phi(1) = \phi(2) = 0 \) and \( \phi(7-p) = *\phi(p) \)). With \( p = 3 \), \( \phi(3) \equiv \phi \) and \( \phi(4) = *\phi(3) \equiv *\phi \), this reads

\[ d\phi = \lambda \ast \phi \quad \text{with} \quad \lambda = 8 \tilde{\lambda}, \]  

(2.10)

and we see that (2.8) implies weak \( G_2 \) holonomy. Since (2.10) obviously implies \( d\ast\phi = 0 \), the count of independent conditions is unchanged and (2.10) and (2.8) are actually equivalent.

Acting on (2.8) with \( D_i \) and antisymmetrising, one gets

\[ R_{ij} \gamma^j \eta = 8 \tilde{\lambda}^2 \gamma_{ij} \eta, \]  

(2.11)

so that now \( \eta \) is not invariant under the holonomy group but transform in a particularly simple form. If one multiplies (2.11) by \( \gamma^j \) one gets

\[ R_{ij} \gamma^j \eta = 4(d-1) \tilde{\lambda}^2 \gamma_{ij} \eta \]  

where \( d \) is the dimension of the manifold \( K \), and one concludes that \( K \) is an Einstein space with

\[ R_{ij} = 4(d-1) \tilde{\lambda}^2 g_{ij}. \]  

(2.12)

We repeat the relevant formula for the case of present interest:

weak \( G_2 \) :

\[ D_j \eta = \frac{i}{8} \lambda \gamma_j \eta, \]

\[ d\phi = \lambda \ast \phi, \]

\[ R_{ij} = \frac{3}{8} \lambda^2 g_{ij}. \]  

(2.13)

3 \( G_2 \) holonomy from self-duality, and vice versa

In this section, we exclusively consider seven-dimensional manifolds. As before, \( a, b, \ldots = 1, \ldots 7 \) are flat indices while \( i, j, \ldots = 1, \ldots 7 \) are curved ones.
3.1 Self-duality equations

An antisymmetric tensor $A^{ab}$ transforms as the adjoint representation $21$ of SO(7). Under the embedding of $G_2$ into SO(7), we have

$$21 \rightarrow 14 + 7,$$

(3.1)

$14$ being the adjoint of $G_2$ and $7$ the fundamental representation. We will call $A^+_a$, resp. $A^-_a$ the part transforming as the $14$, resp. $7$ of $G_2$. The decomposition can be performed with projectors using the $G_2$ invariant 4-index tensor $\hat{\psi}^{abcd}$ which is the dual of the structure constants $\psi^{abc}$ of the imaginary octonions, $G_2$ being the automorphism group of the latter. Both $\psi^{abc}$ and $\hat{\psi}^{abcd}$, $a,b,c,d = 1,\ldots,7$, are completely antisymmetric and only take the values $\pm 1$ and $0$. They are given in the appendix together with many useful identities (see also e.g. references [21, 22, 23]). We only repeat here the most important for us:

$$\hat{\psi}_{abfg}\hat{\psi}^{fgcd} = -2\hat{\psi}^{abcd} + 4\delta_{ac}\delta_{bd} - 4\delta_{ad}\delta_{bc},$$

(3.2)

$$\hat{\psi}_{abde}\psi^{dec} = -4\psi^{abc},$$

(3.3)

$$\psi_{abcd}\psi^{bcd} = 6\delta_{ab}.$$  

(3.4)

Consider then the orthogonal projectors\footnote{Sub- and superscripts have the same significance.}

$$ (P_{14})_{ab}^{\;\;\;\;cd} = \frac{2}{3}\left(\delta_{ab}^{\;\;\;cd} + \frac{1}{4}\hat{\psi}_{ab}^{\;\;\;cd}\right), \quad (P_7)_{ab}^{\;\;\;\;cd} = \frac{1}{3}\left(\delta_{ab}^{\;\;\;cd} - \frac{1}{2}\hat{\psi}_{ab}^{\;\;\;cd}\right),$$

(3.5)

with $\delta_{ab}^{\;\;\;cd} = \frac{1}{2}(\delta_a^c\delta_b^d - \delta_a^d\delta_b^c)$. Since $(P_{14})_{ab}^{\;\;\;\;cd}\psi^{cde} = 0$ and $(P_7)_{ab}^{\;\;\;\;cd}\psi^{cde} = \psi^{abc}$, the representation $7$ can be written as $\psi^{abc}\zeta^c$ and the representation $14$ can be obtained by imposing the seven conditions $\psi^{abc}A^{ab} = 0$. The decomposition reads then

$$A^{ab} = A^+_a + A^-_a,$$

$$A^+_a = \frac{2}{3}(A^{ab} + \frac{1}{2}\hat{\psi}^{abcd}A^{cd}), \quad \psi^{abc}A^+_a = 0, \quad (14 \text{ of } G_2),$$

$$A^-_a = \frac{1}{3}(A^{ab} - \frac{1}{2}\hat{\psi}^{abcd}A^{cd}) \equiv \psi^{abc}\zeta^c, \quad \zeta^c = \frac{1}{6}\psi^{cab}A^{ab}, \quad (7 \text{ of } G_2).$$

(3.6)

In other words, the two $G_2$ irreducible components can be obtained by imposing on the adjoint of SO(7) the $G_2$ invariant self-duality conditions

$$A^+_a = \frac{1}{2}\hat{\psi}^{abcd}A^+_a,$$  

(14 of $G_2)$,  

$$A^-_a = -\frac{1}{4}\hat{\psi}^{abcd}A^-_a,$$  

(7 of $G_2)$,  

(3.7)
which respectively eliminate the 7 or the 14 component of $A_{ab}$. Alternatively, one might write out explicitly the relations (3.7) to see that they give 7 linear relations between the $A_{ab}^+$ (leaving 14 independent quantities) and 14 linear relations between the $A_{ab}^-$ (leaving 7). The relations (3.7) are analogous to the self-duality equations $F^{\mu \nu} = \pm \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} F_{\rho \sigma}$ in four dimensions realising the decomposition $6 \to 3 + 3$, except that now the coefficients $\pm \frac{1}{2}$ are replaced by $\frac{1}{2}$ and $-\frac{1}{4}$. We will refer to $A_{ab}^+$ as the self-dual part and to $A_{ab}^-$ as the anti self-dual part of $A_{ab}$. Since the projectors (3.5) are orthogonal and complementary, one has, of course, for any antisymmetric $A_{ab}$ and $B_{ab}$

$$A_{ab} B_{ab} = A_{ab}^+ B_{ab}^+ + A_{ab}^- B_{ab}^-.$$  

Each term is invariant under $G_2$ while the sum is invariant under $SO(7)$.

It is now evident that the 14 generators $G_{ab}^+$ of the $G_2$ algebra are the projection with $\mathcal{P}_{14}$ of the 21 generators $J^{ab}$ of $SO(7)$ [23],

$$G_{ab}^+ = J_{ab}^+.$$  

The component invariant under $G_2$ of an $SO(7)$ spinor will then verify

$$\gamma_{ab}^+ \eta \equiv \frac{2}{3} \left( \gamma_{ab} + \frac{1}{4} \hat{\psi}_{abcd} \gamma_{cd} \right) \eta = 0.$$  

With the explicit representation of the $\gamma$-matrices in terms of the octonionic structure constants given in the appendix, eq. (3.10) implies $\eta_\alpha = 0$ for $\alpha = 1, \ldots, 7$ and $\eta_8$ is the $G_2$ singlet direction. In particular, $G_2$ invariant spinors always exist (in seven dimensions) provided there is no global obstruction to the existence of spinors.

### 3.2 Self-dual spin connection, Killing spinors and $G_2$ holonomy

Assume now that the spin connection one form $\omega_{ab} = \omega_{ab}^j dx^j$ satisfies any of the three equivalent statements

$$\psi_{abc} \omega_{bc} = 0, \quad \omega_{ab}^- = 0, \quad \omega_{ab}^+ = \frac{1}{2} \hat{\psi}_{abcd} \omega_{cd},$$  

i.e. $\omega_{ab} = \omega_{ab}^+$. The covariant derivative of a spinor then is

$$D_j \eta = \left( \partial_j + \frac{1}{4} \omega_{ab}^+ \gamma_{ab} \right) \eta = \left( \partial_j + \frac{1}{4} \omega_{ab}^+ \gamma_{ab} \right) \eta.$$  

8
With (3.10), one deduces that a $G_2$ invariant ($\gamma^{ab} \eta = 0$) and constant ($\partial_j \eta = 0$) spinor is covariantly constant with respect to $\omega^{ab}$. As mentioned above, in the representation (3.13) of the $\gamma$-matrices, this is simply $\eta_\alpha = \text{const} \times \delta_\alpha 8$. Thus any of the three equations (3.11) implies the existence of (at least) one Killing spinor and hence the holonomy is contained in $G_2$.

The converse is also true: $G_2$ holonomy implies that we can find a self-dual spin connection $\omega^{ab} = \omega_+^{ab}$. This means that, given a $G_2$ manifold with an $\omega^{ab}$, it must be possible to make a local SO(7) transformation such that the new connection satisfies $\omega^{ab} = 0$. To show this, suppose that we have $G_2$ holonomy. Then there exists a single Killing spinor $\eta$ with $D_j \eta = 0$. This means that $\eta(x)$ and $\eta(x + dx)$ only differ by an SO(7) rotation with parameters $\omega^a_i dx^i$ and the “length” $\eta \eta = \eta^* \eta$ remains unchanged. But we may at any point rotate $\eta$ so that it only has an eighth component. As the “length” does not change as we vary $x^j$, we see that $\eta_\alpha = c \delta_\alpha 8$ with constant $c$, so that $\partial_j \eta = 0$. Then $D_j \eta = 0$ reduces to

$$0 = D_j \eta = \frac{1}{4} \omega^a_j \gamma^{ab} \eta.$$  \hfill (3.13)

Since $\eta_\alpha = c \delta_\alpha 8$, we see from the appendix that $(\gamma^{ab} \eta)_\alpha = c \psi_{\alpha a}$, so that (3.13) reads $\omega^a_j \psi_{\alpha a} = 0$. But by (3.10) this is the same as saying that $\omega^{ab} = 0$, which we wanted to show. This result is independent of the choice of a particular representation of $\gamma$-matrices and we conclude that

- The self-duality (3.11) of the spin connection $\omega^{ab}$ implies $G_2$ holonomy (or holonomy contained within $G_2$) and conversely, every $G_2$ holonomy metric can be derived from a self-dual connection satisfying (3.11).

By the general arguments of section 2 we then know that $d \phi = 0$, $d * \phi = 0$ and $R_{ij} = 0$, but it will be instructive to rederive these results directly from the self-duality (3.11). The relevant 3-form is (in the following the wedge product of forms will be implicitly understood)

$$\phi = \frac{1}{3!} \psi_{abc} e^a e^b e^c, \quad * \phi = \frac{1}{4!} \hat{\psi}_{abcd} e^a e^b e^c e^d,$$  \hfill (3.14)

where $e^a$ is the 7-bein one-form related to $\omega^{ab}$ by

$$d e^a + \omega^{ab} e^b = 0.$$  \hfill (3.15)

\footnote{On a single spinor direction, the action of SO(7) is due to the generators in the coset $SO(7)/G_2 \sim SO(8)/SO(7)$. This can also be verified using our choice of gamma matrices and the results given in the appendix.}
Note that using the explicit representation of the $\gamma$-matrices given in the appendix, as well as $\eta_\alpha = \delta_\alpha$ it is easy to see that (3.14) exactly coincides with the general definition (2.5). Using the self-duality (3.11) and the identity (A.15) of the appendix, one has

$$d \phi = - \frac{1}{2} \psi_{abc} e^a e^b \omega^{cd} e^d = - \frac{1}{4} \psi_{abc} \hat{\psi}_{cdef} e^a e^b \omega^{ef} e^d = \psi_{ade} e^a e^d \omega^{eb} e^b = - 2 d \phi .$$

(3.16)

Similarly, using (3.11) and the identity (A.16) leads to

$$d* \phi = - \frac{1}{6} \hat{\psi}_{abcd} e^a e^b e^c e^d = - \frac{1}{12} \hat{\psi}_{ae fg} \hat{\psi}_{abcd} \omega^{ef} g e^b e^c e^d$$

$$= - \frac{1}{3} \hat{\psi}_{fbcde} f e e^b e^c e^d = - 2 d* \phi .$$

(3.17)

We then obtain

$$d \phi = 0 ,$$

$$d* \phi = 0 ,$$

(3.18)

as required for $G_2$ holonomy.

Next, we show that self-duality of $\omega^{ab}$, eq. (3.11), implies self-duality of the curvature 2-form $R^{ab} = d\omega^{ab} + \omega^{ac} \omega^{cb} \equiv \frac{1}{2} R^{ab} \omega^{cd} e^d$. For the $d\omega^{ab}$ part this is obvious. For the remaining part we have, using again (3.11) and the identity (A.16)

$$\frac{1}{2} \hat{\psi}_{abcd} \omega^{ce} \omega^{ed} = \frac{1}{4} \hat{\psi}_{abcd} \hat{\psi}_{ce} \omega^{ef} g$$

$$= - \frac{1}{2} \hat{\psi}_{ab fc} \omega^{ce} \omega^{ed} + \frac{1}{4} \hat{\psi}_{ae fg} \omega^{be} \omega^{cf} g - \frac{1}{4} \hat{\psi}_{bc fg} \omega^{ae} \omega^{cf} g + \omega^{ac} \omega^{cb}$$

(3.19)

as desired. Hence

$$R^{ab} = \frac{1}{2} \hat{\psi}_{abcd} R^{cd} ,$$

(3.20)

which, as usual, immediately implies Ricci flatness:

$$R_{ab} = R_{acbc} = \frac{1}{2} \hat{\psi}_{acde} R_{debc} = \frac{1}{2} \hat{\psi}_{acde} R_{b[cde]} = 0 .$$

(3.21)

Note that self-duality of the curvature is the most direct statement that the generators of the holonomy group are in $G_2$ rather than $SO(7)$.

### 3.3 A rather general example

To illustrate the power of the self-duality equations, we consider an ansatz for a $G_2$ holonomy metric proposed in [5] and also independently in [6]. These authors have written a rather general cohomogeneity-one seven-dimensional metric ansatz with an
SU(2) × SU(2) × Z₂ symmetry acting on the manifold. It depends on six arbitrary functions of one coordinate \( r \):

\[
ds^2 = dr^2 + \sum_{i=1}^{3} a_i^2(r)(\sigma_i - \Sigma_i)^2 + \sum_{i=1}^{3} b_i^2(r)(\sigma_i + \Sigma_i)^2 ,
\]

where

\[
d\sigma_i = -\frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k , \quad d\Sigma_i = -\frac{1}{2}\epsilon_{ijk}\Sigma_j \wedge \Sigma_k
\]

are the left-invariant one forms for each of the SU(2), and the Z₂ acts by interchange \( \sigma_i \leftrightarrow \Sigma_i \). Note, that in this subsection \( i, j, k \) are flat indices and run over 1, 2 and 3 only (contrary to our usual conventions!). The obvious (though not unique) choice of seven-bein is

\[
e^i = a_i(\sigma_i - \Sigma_i) , \quad \hat{e}^i = b_i(\sigma_i + \Sigma_i) , \quad e^7 = dr .
\]

Then

\[
de^i = \frac{\dot{a}_i}{a_i} e^7 \wedge e^i - \frac{1}{2} \frac{a_i}{a_j b_k}\epsilon_{ijk} e^j \wedge e^k ,
\]

\[
de^\hat{i} = \frac{\dot{b}_i}{b_i} e^7 \wedge e^\hat{i} - \frac{1}{4} \frac{b_i}{b_j b_k}\epsilon_{ijk} \left( \frac{e^j \wedge e^k}{a_j a_k} + \frac{e^\hat{j} \wedge e^\hat{k}}{b_j b_k} \right),
\]

from which one reads the spin connection:

\[
\omega^7 = \frac{\dot{a}_i}{a_i} e^i , \quad \omega^{\hat{i}} = \frac{\dot{b}_i}{b_i} \hat{e}^i , \quad \omega^{\hat{i}i} = 0 \quad \text{(no sum over} \ i)\ ,
\]

\[
\omega^{ij} = \frac{1}{4} \epsilon_{ijk} \left( \frac{b_k}{a_i a_j} - \frac{a_i}{a_j b_k} - \frac{a_j}{a_i b_k} \right) e^k ,
\]

\[
\omega^{\hat{i}j} = \frac{1}{4} \epsilon_{ijk} \left( \frac{b_k}{b_i b_j} - \frac{b_i}{b_j b_k} - \frac{b_j}{b_i b_k} \right) e^k ,
\]

\[
\omega^{\hat{i}\hat{j}} = \frac{1}{4} \epsilon_{ijk} \left( \frac{a_k}{a_i a_j} - \frac{a_i}{a_k a_j} - \frac{b_j}{a_i a_k} \right) e^k \quad , \quad \omega^{ij} = -\omega^{\hat{i}j} .
\]

Now impose the self-duality equations (3.11). The most convenient form to use is \( \psi_{abc}\omega^{bc} = 0 \), since this directly only gives the 7 linearly independent equations. Using eq. (A.4), they can be written as

\[
0 = \psi_{iab}\omega^{ab} = \epsilon_{ijk}(\omega^{jk} - \omega^{\hat{j}\hat{k}}) - 2\omega^{7i} ,
\]

\[
0 = \psi_{iab}\omega^{ab} = -\epsilon_{ijk}(\omega^{jk} + \omega^{\hat{j}\hat{k}}) + 2\omega^{7i} = 2(\omega^{7i} - \epsilon_{ijk}\omega^{\hat{j}\hat{k}}) ,
\]

\[
0 = \psi_{7ab}\omega^{ab} = 2\omega^{7i} .
\]
The last equation is automatically satisfied while the other six immediately give the differential equations ($\dot{a}_i = \ldots, \dot{b}_i = \ldots$) of ref. [3]..

Similarly, one may consider other cohomogeneity-one metrics with principal orbits given by a coset, e.g. SU(3)/U(1)², and impose self-duality to get first order differential equations for the coefficient functions.

4 Weak $G_2$ holonomy and generalized self-duality

We now extend the results of the previous section to the case of weak holonomy. The self-duality condition on the spin connection is generalized by the addition of a particular inhomogeneous term, transforming in the 7 of $G_2$. We then show how this is equivalent to the existence of a conformal Killing spinor and hence to weak $G_2$ holonomy. We again rederive directly from the generalized self-duality that $d\phi = \lambda \ast \phi$ and that the metric is Einstein, as it should. Finally we discuss the relations to and implications for spin(7) holonomy metrics.

4.1 generalized self-duality and weak $G_2$ holonomy

We do not any longer assume that the Levi-Civita spin connection transforms as a pure 14 of $G_2$ but allow for a non-zero piece proportional to $\psi_{abc}\omega^{bc}$ transforming as a 7. The non-trivial condition we require is that this piece is proportional to the seven-bein $e^a$. We can write the condition in the following three equivalent ways (remember that $\psi_{abc}\omega^{bc} = 0$ is trivially true):

\[
\begin{align*}
\psi_{abc}\omega^{bc} &= -\frac{\lambda}{2} e^a, \\
\omega_{ab} &= -\frac{\lambda}{12}\psi_{abc} e^c, \\
\omega^{ab} &= \frac{1}{2}\psi_{abcd} \omega^{cd} - \frac{\lambda}{4}\psi_{abc} e^c.
\end{align*}
\]

Eqs. (4.1) now imply weak $G_2$ holonomy. To verify this, we recall from section 2 that we only need to show that there exists a spinor $\eta$ which satisfies

\[
D_j \eta = i\frac{\lambda}{8} \gamma_j \eta.
\]

We pick a $G_2$ invariant constant spinor $\eta$ obeying (3.10). As discussed before, such a spinor always exists. Then

\[
D_j \eta = \frac{1}{4} \omega_{ab} \gamma_{-j} \gamma^{ab} \eta = -\frac{\lambda}{48} e^c \psi_{cab} \gamma_{-} \eta.
\]
Using eqs. (A.5) and (A.6) and the corresponding $\eta = c\delta_{a8}$, this is easily seen to coincide with the right-hand side of (4.2), which is then verified as well as weak $G_2$ holonomy.

As in the previous section, we can also prove the converse: for every weak $G_2$ holonomy manifold, we can find a spin connection $\omega^{ab}$ verifying eqs. (4.1). Indeed, weak $G_2$ holonomy is equivalent to the existence of a spinor $\eta$ satisfying (4.2). By a suitable local $SO(7)$ rotation, we can choose $\eta$ such that $\gamma_{+}^{ab}\eta = 0$ and $\partial_i\eta = 0$, as in section 3.2. Within our representation of the gamma-matrices, which we will use to simplify the argument, this means $\eta = c\delta_{a8}$ after the $SO(7)$ transformation. But then (4.2) reduces to

$$\frac{1}{4}\omega_{-}^{ab}\gamma_{-}^{ab}\eta = i\frac{\lambda}{8}\gamma_j\eta$$

(4.4)

and, using (A.3) and (A.6), we get $\omega^{ab}\psi_{abc} = -\frac{\lambda}{2}\epsilon^e$ which is nothing but the first eq. (4.1). We conclude that

• The generalized self-duality (4.1) of the spin connection $\omega^{ab}$ implies weak $G_2$ holonomy and conversely, every weak $G_2$ holonomy metric can be derived from a self-dual connection satisfying (4.1).

Let us now show how the self-duality condition (4.1) directly leads to $d\phi = \lambda \ast \phi$, which is the way weak $G_2$ holonomy is usually stated in the mathematical literature. The three-form $\phi$ and the four-form $\ast \phi$ are still defined as in eq. (3.14) and $\omega^{ab}$ still obeys (3.15). Of course, eq. (3.16) is now modified:

$$d\phi = -\frac{1}{2}\psi_{abc} e^a \omega^{cd} e^d = -\frac{1}{4}\psi_{abc} \psi_{cd} e^a \omega^{cd} e^d + \frac{\lambda}{8}\psi_{abc} \psi_{cd} e^a e^c e^d = -2d\phi + 3\lambda \ast \phi.$$  

(4.5)

where we used (A.13) in the last step. So we indeed conclude that

$$d\phi = \lambda \ast \phi.$$  

(4.6)

Next we want to investigate the self-duality properties of $R^{ab}$ that follow from eq. (4.1). Clearly, $R^{ab}$ cannot be self-dual as it is not Ricci flat. However, eq. (2.11) or (2.12) suggests that the two-form

$$\hat{R}^{ab} = R^{ab} - \frac{\lambda^2}{16} e^a e^b$$

(4.7)

might be self-dual, \textit{i.e.} $\psi_{abc} \hat{R}^{bc} = 0$. This is indeed the case as we now show. Using
and (A.15), we have:

\[ \psi_{abc} \omega^{bd} \omega^{dc} = \psi_{cab} \left( \frac{1}{2} \psi_{bdef} \omega^{ef} - \frac{\lambda}{4} \psi_{bde} \right) \omega^{dc} = -\psi_{abc} \omega^{bd} \omega^{dc} - \lambda \omega^{ab} e^b + \frac{\lambda^2}{8} \psi_{abc} e^b e^c, \]

or

\[ \psi_{abc} \omega^{bd} \omega^{dc} = -\frac{\lambda}{2} \omega^{ab} e^b + \frac{\lambda^2}{16} \psi_{abc} e^b e^c. \]  

(4.8)

Also,

\[ \psi_{abc} d \omega^{bc} = -\frac{\lambda}{2} d e^a = \frac{\lambda}{2} \omega^{ab} e^b, \]

so that indeed

\[ \psi_{abc} R^{bc} \equiv \psi_{abc} \left( R^{bc} - \frac{\lambda^2}{16} e^b e^c \right) = 0. \]  

(4.9)

Equivalently we have

\[ \hat{R}^{ab} = \frac{1}{2} \hat{\psi}_{abcd} \hat{R}^{cd}, \]

(4.10)

where the components of \( \hat{R}^{ab} \) are \( \hat{R}_{cd}^{ab} = R_{cd}^{ab} - \frac{\lambda^2}{16} (\delta_c^a \delta_d^b - \delta_d^a \delta_c^b). \) This immediately implies as in eq. (3.21)

\[ 0 = \hat{R}_{abc} = R_{ab} - \frac{3}{8} \lambda^2 \delta_{ab} \quad \Rightarrow \quad R_{ij} = \frac{3}{8} \lambda^2 g_{ij}, \]

(4.11)

so that the manifold is Einstein, in agreement with eq. (2.13).

4.2 Connection with spin(7) holonomy

It is known that a given metric with weak holonomy \( G_2 \), respectively SU(3), can be associated to a metric with spin(7), respectively \( G_2 \), holonomy [18]. We will show that the self-duality (1.1) of the connection \( \omega^{ab} \) on the weak \( G_2 \) manifold induces a self-dual \( \omega^{ab} \) on the spin(7) manifold so that our results have a natural translation to the latter case as well.

Let \( e^a \) and \( \omega^{ab} \) be the seven-bein and connection and \( \tilde{e}^A \) and \( \tilde{\omega}^{AB} \), \( A, B = 1, \ldots, 8 \), be the eight-dimensional ones. We identify

\[ \tilde{e}^8 = dt, \quad \tilde{e}^a = -\frac{\lambda}{4} t e^a. \]  

(4.12)

Then, \( d_{(8)} = \frac{\partial}{\partial t} dt + d \), where \( d \) is just the 7-dimensional exterior derivative, so that \( d_{(8)} \tilde{e}^A + \tilde{\omega}^{AB} \tilde{e}^B = 0 \) implies

\[ \tilde{\omega}^{ab} = \omega^{ab}, \quad \tilde{\omega}^{8a} = \frac{\lambda}{4} e^a. \]  

(4.13)
Define as usual
\[ \Psi_{abcd} = \hat{\psi}_{abcd}, \quad \Psi_{abc8} = \hat{\psi}_{abc}. \] (4.16)

The eight-dimensional manifold has spin(7) holonomy if and only if
\[ d(8)\Phi = 0, \quad \Phi = \frac{1}{4!} \Psi_{ABCD} \tilde{e}^A \tilde{e}^B \tilde{e}^C \tilde{e}^D. \] (4.17)

Inserting the ansatz (4.14), this is
\[ \Phi = \left( \frac{\lambda t}{4} \right)^3 dt \wedge \phi + \left( \frac{\lambda t}{4} \right)^4 * \phi, \]
\[ d(8)\Phi = -\left( \frac{\lambda t}{4} \right)^3 dt \wedge (d\phi - \lambda * \phi) + \left( \frac{\lambda t}{4} \right)^4 d * \phi, \] (4.18)

where \( \phi \) and \( *\phi \) are the usual seven-dimensional 3- and 4-form constructed from the \( e^a \).

We see that spin(7) holonomy for the “cohomogeneity-one” ansatz (4.14) is equivalent to weak \( G_2 \) holonomy of the seven-dimensional level surfaces \( t = \text{const} \). We have shown that for the latter, one can always find a spin connection satisfying the self-duality relations (4.1). In fact, in view of eqs. (4.15), the latter are exactly equivalent to the following spin(7) self-duality condition
\[ \tilde{\omega}^{AB} = \frac{1}{2} \Psi_{ABCD} \tilde{\omega}^{CD}. \] (4.19)

This prompts the question whether any spin(7) holonomy metric can be derived from (4.19). Based on the preceding remarks, one would expect this to be true for cohomogeneity-one spin(7) metrics.

More generally, the same proof as for \( G_2 \) holonomy should work: using \( \Psi_{ABCD} \) one can again define projectors \( P_{21} \) and \( P_7 \) corresponding to the embedding of spin(7) \( \subset \) SO(8). Call again \( \tilde{\omega}_+^{AB} \), resp \( \tilde{\omega}_-^{AB} \) the part annihilated by \( P_7 \), resp \( P_{21} \). The self-duality condition (4.19) states that \( \tilde{\omega}_-^{AB} = 0 \) which is what we want to show. So assume spin(7) holonomy. Then proceeding as in section 3.2, since spin(7) holonomy implies the existence of a covariantly constant spinor, one shows that, after an appropriate local SO(8) transformation, \( D_J \eta = 0, J = 1, \ldots, 8 \), reduces to \( \tilde{\omega}_-^{AB} \gamma_-^{AB} \eta = 0 \) which then implies \( \tilde{\omega}_-^{AB} = 0 \), i.e. the self-duality equation (4.19).

### 4.3 Examples

A standard example for weak \( G_2 \) holonomy is the metric with principal orbits given by the Aloff-Walach spaces, see e.g. [17, 24]. It is easy to verify that the standard
choice of spin connection satisfies the generalized self-duality (4.1). Further examples and classifications can be found in [25, 26].

Here, we will consider the example of an eight-dimensional cohomogeneity-one spin(7) metric with principal orbits being triaxially squashed $S^3$’s over $S^4$ as first found by [6]. We will show that the differential equations imposed by spin(7) holonomy on the functions in the metric ansatz directly follow from the self-duality conditions (4.19). Reducing to seven dimensions in the way just described the equations for weak $G_2$ holonomy are just the generalized self-duality conditions (4.1).

The starting point is a slight generalization of the ansatz of [6]:

$$ds^2 = dt^2 + a_i^2 R_i^2 + a_i^2 P_i^2 + a_i^2 P_0^2 .$$  \hspace{1cm} (4.20)

Again, in this subsection, $i, j$ run over 1, 2 and 3 only, and $\hat{i} = i + 3$. Ref. [6] specialises to the case $a_i = a_i = a_7$ for reasons that will be clear soon. The $R_i$, $P_i$ and $P_0$ as well as an additional set of three $L_i$ are left-invariant one-forms of SO(5), and the ansatz (4.20) corresponds to the coset SO(5)/SU(2)$_L$. We define the eight-beins as

$$e^a = a_a E_a, \quad e^8 = dt \quad \text{with} \quad E_i = R_i , \quad E_i = P_i , \quad E_7 = P_0 . \hspace{1cm} (4.21)$$

The Maurer-Cartan equations of SO(5) for the generators in the coset SO(5)/SU(2)$_L$ yield (cf. [6])

$$dE_i = -\epsilon_{ijk} E_j \wedge E_k - \frac{1}{2} E_7 \wedge E_i - \frac{1}{4} \epsilon_{ijk} E_j \wedge E_k ,$$
$$dE_i = E_7 \wedge E_i - \epsilon_{ijk} E_j \wedge E_k + E_7 \wedge L_i + \epsilon_{ijk} L_j \wedge E_k ,$$
$$dE_7 = E_i \wedge E_i + L_i \wedge E_i . \hspace{1cm} (4.22)$$

This leads to

$$de^i = \frac{\dot{a}_i}{a_i} e^8 \wedge e^i - \epsilon_{ijk} \frac{a_i}{a_j a_k} e^j \wedge e^k - \frac{1}{2} \frac{a_i}{a_7 a_k} e^7 \wedge e^k - \frac{1}{4} \epsilon_{ijk} \frac{a_i}{a_j a_k} e^j \wedge e^k ,$$
$$de^i = \frac{\dot{a}_i}{a_i} e^8 \wedge e^i - \frac{a_i}{a_7 a_i} e^7 \wedge e^i - \epsilon_{ijk} \frac{a_i}{a_j a_k} e^j \wedge e^k + \frac{a_i}{a_7} e^7 \wedge L_i + \epsilon_{ijk} \frac{a_i}{a_k} L_j \wedge e^k ,$$
$$de^7 = \frac{\dot{a}_7}{a_7} e^8 \wedge e^7 + \frac{a_7}{a_i a_i} e^i \wedge e^i + \frac{a_7}{a_i} L_i \wedge e^i , \hspace{1cm} (4.23)$$
$$de^8 = 0 .$$

One cannot obtain a torsion-free spin connection (satisfying $\omega^{ab} = -\omega^{ba}$) from these equations in general, due to the terms involving the $L_i$. Indeed, one easily sees that
the terms involving $L_i$ are $\omega^7|_{\text{part}} = -(a_7/a_i)L_i$ and $\omega^i|_{\text{part}} = (a_i/a_7)L_i$ as well as $\omega^{ij}|_{\text{part}} = (a_i/a_j)\epsilon_{ijk}L_k$. So to be able to define a torsion-free $\omega^{ab} = -\omega^{ba}$ we must set

$$a_i = a_7 \quad \text{(4.24)}$$

Then we obtain for the spin connection

$$\omega^{sa} = \frac{\dot{a}_a}{a_a} e^a,$$
$$\omega^{ij} = -\epsilon_{ijk} \left( \frac{a_i}{a_j a_k} + \frac{a_j}{a_i a_k} - \frac{a_k}{a_i a_j} \right) e^k,$$
$$\omega^{ij} = -\epsilon_{ijk} \left( \frac{1}{a_k} - \frac{1}{4 a_7^2} \right) e^k + \epsilon_{ijk} L_k ,$$
$$\omega^{i7} = -\frac{1}{4 a_7^2} e^i ,$$
$$\omega^{i7} = \left( \frac{1}{a_i} - \frac{1}{4 a_7^2} \right) e^i + L_i \quad \text{(4.25)}$$

Now we impose the eight-dimensional self-duality conditions (4.19), which read with the present notation

$$\omega^{si} = -\frac{1}{2} \epsilon_{ijk} (\omega^{jk} - \omega^{jk}) + \omega^i ,$$
$$\omega^{si} = \epsilon_{ijk} \omega^{jk} - \omega^{7i} ,$$
$$\omega^{s7} = -\omega^{i7} \quad \text{(4.26)}$$

This yields

$$\dot{a}_1 = \frac{a_1^2 - (a_2 - a_3)^2}{a_2 a_3} - \frac{a_1^2}{2 a_7^2} \quad \text{(and cyclic)} , \quad \dot{a}_7 = \frac{1}{4} \sum_{i=1}^{3} \frac{a_i}{a_7} . \quad \text{(4.27)}$$

These are exactly the equation found in ref [6] for spin(7) holonomy by a very different method.

If we let

$$a_i(t) = -\frac{\lambda}{4} t A_i , \quad a_7(t) = -\frac{\lambda}{4} t A_7 \quad \text{(4.28)}$$

then we know from the previous subsection that the corresponding seven-metric has weak $G_2$ holonomy, $d\phi = \lambda \ast \phi$, provided the constants $A_i$ and $A_7$ satisfy the algebraic relations

$$\lambda A_1 = -4 \frac{A_1^2 - (A_2 - A_3)^2}{A_2 A_3} + \frac{2 A_1^2}{A_7^2} \quad \text{(and cyclic)} , \quad \lambda A_7^2 = -\sum_{i=1}^{3} A_i \quad \text{(4.29)}$$
5 Embedding in eleven-dimensional supergravity

The background geometries considered in the previous sections are thought to be relevant to \textit{M}-theory. It is then a natural task to consider their embedding in eleven-dimensional supergravity. The bosonic part of the Lagrangian density is [27]:

\[
\mathcal{L}_{bos.} = \frac{1}{2\kappa^2} \left[ eR - \frac{e_4}{48} F_{MNPQ} F^{MNPQ} \right] - \frac{1}{3456} \epsilon^{M_1 \ldots M_11} F_{M_1 \ldots M_4} F_{M_5 \ldots M_8} A_{M_9 M_{10} M_{11}} ,
\]

with \( F_{MNPQ} = 24 \partial_M A_{NPQ} \) and \( R = R_{AB} e^A e_B \). We take the space-time signature to be \((-1, 1, \ldots, 1)\). We consider this theory on \( M_4 \times M_7 \) with the eleven-bein

\[
e^A_M (X^N) = \begin{pmatrix} e^m_\mu (x^\nu) & 0 \\ 0 & e^a_i (x^j) \end{pmatrix}.
\]

In addition, we introduce bosonic backgrounds

\[
F_{\mu\nu\rho\sigma} = f e_4 \epsilon_{\mu\nu\rho\sigma}, \quad F_{ijkl} = \tilde{g} e^a_i e^b_j e^c_k \psi^{abcd},
\]

with two constants \( f \) and \( \tilde{g} \) and \( e_4 = \det \epsilon^m_\mu \). While the first component is clearly invariant under \( O(1, 3) \times O(7) \), the invariance of the second is \( O(1, 3) \times H, \) \( H \subset G_2 \), depending on the choice of the seven-bein \( e^a_i \). With this background, Einstein equations reduce to

\[
R_{\mu\nu} = -\frac{1}{3} \left( f^2 + \frac{7}{2} \tilde{g}^2 \right) g_{\mu\nu}, \quad R_{ij} = \frac{1}{6} \left( f^2 + 5 \tilde{g}^2 \right) g_{ij}.
\]

The four-dimensional space-time is an Einstein space with negative curvature whenever \( F_{MNPQ} \) has a non-trivial background. The field equation for \( F_{MNPQ} \) reduces to,

\[
\tilde{g} (d\phi - f \ast \phi) = 0,
\]

where \( \phi \) is the three-form

\[
\phi = \frac{1}{3!} \epsilon_{ijk} dx^i \wedge dx^j \wedge dx^k = \frac{1}{3! \epsilon^7} \epsilon_{ijklmnop} F^{lmpn} dx^i \wedge dx^j \wedge dx^k = \frac{1}{3! \epsilon^7} e_i^a e_j^b e_k^c \psi^{abc} dx^i \wedge dx^j \wedge dx^k
\]

with \( \epsilon^7 = \det \epsilon^a_i \). Notice at this stage that the appearance of \( \phi \) is a mere consequence of the \( G_2 \) structure, while field equation (5.3) only is non-empty if \( \tilde{g} \neq 0 \), \textit{i.e.} if \( \ast \phi \) is identified with a background for \( F_{MNPQ} \) on \( M_7 \). Of course, no information on \( \phi \) is obtained if \( \tilde{g} = 0 \), but weak holonomy is required by field equations if both \( f \) and \( \tilde{g} \) are not zero.
Supersymmetry breaking with a non-trivial background for $F_{MNPQ}$ has been studied long ago \[19, 20\]. The analysis is based upon the reduction on the background defined by eqs. (5.2) and (5.3) of the gravitino supersymmetry variation

$$\delta \psi_M = D_M \epsilon - \frac{1}{288} F_{PQRS} \left( \Gamma^{PQRS}_M - 8 \Gamma^{QRS}_M \delta_M^P \right) \epsilon,$$  
(5.7)

omitting all fermionic contributions which vanish on the background. This variation can be rewritten under $O(1,3) \times O(7)$ using the following decomposition of the gravitino field and gamma matrices $\Gamma^M, M = (\mu, i)$:

$$\psi_M \rightarrow \psi_\mu \otimes \hat{\psi}_\alpha, \quad \psi \otimes \hat{\psi}_{i,\alpha},$$

$$\Gamma^M \rightarrow \gamma^\mu \otimes I, \quad \gamma^5 \otimes \gamma^i,$$

where $\hat{\psi}_\alpha$ is a (real) $O(7)$ spinor, $\alpha = 1, \ldots, 8$, and $\hat{\psi}_{i,\alpha}$ is a vector-spinor. One obtains

$$\delta(\psi_\mu \otimes \hat{\psi}_\alpha) = D_\mu (\epsilon \otimes \hat{\epsilon}_\alpha) + \frac{1}{12} \tilde{g} (\gamma_\mu \epsilon) \otimes (\delta_{\alpha\beta} - 8 \delta_{\alpha8} \delta_{\beta8}) \hat{\epsilon}_\beta - \frac{i}{6} f (\gamma_\mu \gamma_5 \epsilon) \otimes \hat{\epsilon}_\alpha$$

$$\equiv D_\mu (\epsilon \otimes \hat{\epsilon}_\alpha),$$  
(5.8)

$$\delta(\psi \otimes \hat{\psi}_{i,\alpha}) = D_i (\epsilon \otimes \hat{\epsilon}_\alpha) + \frac{1}{12} (f - i \tilde{g} \gamma_5) \epsilon \otimes (i \gamma_i \epsilon) + \frac{i}{3} \tilde{g} (\gamma_5 \epsilon) \otimes (\gamma^a \epsilon \gamma^8 + 3 \delta_{a8} \gamma^i \epsilon \gamma^\beta)$$

$$\equiv D_i (\epsilon \otimes \hat{\epsilon}_\alpha).$$  
(5.9)

In principle, the first equation generates the supersymmetry variation of potential four-dimensional gravitino fields $\psi_\mu$. Unbroken supersymmetries require

$$0 = \delta[D_\mu, \psi_\nu \otimes \hat{\psi}_\alpha] = [D_\mu, D_\nu] \epsilon \otimes \hat{\epsilon}_\alpha$$  
(5.10)

for at least one direction of the $O(7)$ spinor $\hat{\epsilon}_\alpha$. With $[D_\mu, D_\nu] \epsilon = \frac{1}{4} R^{\mu \nu \gamma \delta}_{\mu \nu} \gamma_{\gamma \delta} \epsilon$, the condition leads to

$$R_{\mu \nu} (\epsilon \otimes \hat{\epsilon}_\alpha) = -\frac{1}{3} g_{\mu \nu} \left[ f^2 (\epsilon \otimes \hat{\epsilon}_\alpha) + \frac{1}{4} \tilde{g}^2 (\epsilon \otimes \hat{\epsilon}_\alpha + 48 \delta_{a8} \epsilon \otimes \hat{\epsilon}_8) \right].$$  
(5.11)

This is clearly compatible with Einstein equations (5.4) only if $\tilde{g} = 0$, in which case, as expected from four-dimensional supergravity,

$$\delta(\psi_\mu \otimes \hat{\psi}_\alpha) = D_\mu (\epsilon \otimes \hat{\epsilon}_\alpha) - \frac{1}{4} \sqrt{|R|} \left( i \gamma_\mu \gamma_5 \epsilon \otimes \hat{\epsilon}_\alpha \right),$$

with $R = R_{\mu \nu} g^{\mu \nu}$.  

19
Assuming then $\tilde{g} = 0$, the curvature of the compact space becomes $R = g^{ik} R_{ij} = \frac{7}{6} f^2$, and a conformal Killing spinor in this space verifies
\[
D_i \hat{\epsilon} = -\frac{1}{2} \sqrt{R} (i \gamma_i \hat{\epsilon}) = -\frac{1}{12} f (i \gamma_i \hat{\epsilon}),
\] (5.12)
which is precisely what is required to cancel the supersymmetry variation (5.9). Then, each solution of eq. (5.12) produces one four-dimensional supersymmetry. Comparing with eq. (2.8), one infers that with our background
\[
d\phi = -\frac{2}{3} f * \phi.
\] (5.13)
This result, which follows from the requirement of four-dimensional supersymmetry, is incompatible with the field equation (5.3) if $\tilde{g} \neq 0$. This again shows that supersymmetry breaks if $F_{ijkl}$ has a non-zero background value [19, 20, 28].

If one chooses a seven-bein $e_i^a$ leading to a metric for the seven-sphere, there are eight solutions and the four-dimensional theory has $N = 8$ supersymmetry if $\tilde{g} = 0$ and $N = 0$ if $g \neq 0$ [29, 30, 31, 19].

## 6 Conclusions

We have discussed compactifications of eleven-dimensional supergravity to four dimensions with background fluxes and rederived the known result that any flux in the internal space breaks all supersymmetry. On the other hand, a flux in space-time $\sim f \epsilon_{\mu\nu\rho\sigma}$ is allowed, and some supersymmetry remains unbroken provided the internal space admits at least one conformal Killing spinor satisfying $D_j \eta \sim i f \gamma_j \eta$. The flux parameter $f$ thus controls the internal geometry which must have weak $G_2$ holonomy for $f \neq 0$ and holonomy contained in $G_2$ for $f = 0$.

We have shown that weak $G_2$ holonomy is equivalent to the spin connection satisfying a generalized self-duality condition. Our argument is straightforward and is based on the equivalence between weak reduced holonomy and the existence of a conformal Killing spinor. The same argument applies to (non generalized) self-duality of the spin connection and holonomy contained in $G_2$. We also indicated how to transpose the proof to spin(7).

We have given some examples of how the self-duality conditions greatly simplify determining the first-order differential equations that ensure special (weak) holonomy.
Of course, explicitly solving these differential equations is another story, as always. Actually for all known examples the spin connections directly satisfy the appropriate self-duality conditions, without any need to first perform a local SO(7) (resp. SO(8)) transformation.

We expect the self-duality relations to be a rather powerful tool to generate new solutions with (weak) $G_2$ holonomy. This however, is left for future work.

Acknowledgements

This work has been supported by the Swiss National Science Foundation, by the European Union RTN program HPRN-CT-2000-00131 and by the Swiss Office for Education and Science.

7 Appendix: $SO(7)$ and $G_2$ algebras, conventions and identities

We review some useful relations and identities for octonions. Some useful references are [21, 22, 23, 32, 33, 34, 35]. The non-associative octonion algebra is given by (we write $o^a$ rather than the more standard $e^a$ to avoid confusion with the 7-beins)

$$o^a o^b + o^b o^a = -2\delta^{ab} I, \quad o^a o^b - o^b o^a = 2\psi_{abc} o^c, \quad (a, b, c = 1, \ldots, 7). \quad (A.1)$$

While the Clifford algebra is invariant under $SO(7)$, the octonion algebra and the structure constants $\psi_{abc}$ are invariant under $G_2$. A four-index invariant tensor can then also be defined:

$$\hat{\psi}_{abcd} = \frac{1}{3!} \epsilon^{abcdefg} \psi_{efg}. \quad (A.2)$$

In the standard basis, these tensors are

$$\psi_{123} = \psi_{516} = \psi_{624} = \psi_{435} = \psi_{471} = \psi_{673} = \psi_{572} = 1,$$

$$\hat{\psi}_{4567} = \hat{\psi}_{2374} = \hat{\psi}_{1357} = \hat{\psi}_{1276} = \hat{\psi}_{2356} = \hat{\psi}_{1245} = \hat{\psi}_{1346} = 1. \quad (A.3)$$

All other non-zero components follow from antisymmetry of the above values. A number of useful identities involving products of these tensors are listed in eqs. (A.13–A.21) below. A compact way to write the $\psi_{abc}$ is to observe that the indices split into 3 groups as $i = 1, 2, 3, \; \hat{i} \equiv i + 3 = 4, 5, 6,$ and 7. Then:

$$\psi_{ijk} = \epsilon_{ijk}, \; \psi_{ij\hat{k}} = \psi_{i\hat{j}k} = \psi_{i\hat{j}\hat{k}} = -\epsilon_{ijk}, \; \psi_{\hat{i}\hat{j}j} = \delta_{ij}. \quad (A.4)$$
The reduction of the \( SO(7) \) spinor into \( G_2, 8 = 7 + 1 \), is best performed using the antisymmetric and imaginary \( SO(7) \) gamma matrices

\[
(\gamma^a)_{\alpha\beta} = i(\psi_{\alpha\beta} + \delta_{\alpha\beta} - \delta_{a\beta}\delta_{a8}), \quad (A.5)
\]

where \( \alpha, \beta = 1, \ldots, 8 \) are indices in the spinor of \( SO(7) \) and \( \psi_{\alpha\beta} \) (and \( \hat{\psi}_{aba} \)) vanish if \( \alpha \) or \( \beta \) is 8. With this choice of gamma matrices, \( SO(7) \) generators for the spinor representation are real:

\[
\Sigma^{ab} = \frac{1}{2} \gamma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b],
\]

\[
(\gamma^{ab})_{\alpha\beta} = \hat{\psi}_{aba} + \psi_{aba}\delta_{\beta8} - \psi_{ab\beta}\delta_{a8} + \delta_{aa}\delta_{b3} - \delta_{ab}\delta_{ba}. \quad (A.6)
\]

The minimal spinor \( \eta \) is then also real, we may choose \( \bar{\eta} = \eta^\tau \) and the reduction of a Majorana \( SO(1,10) \) spinor \( \epsilon \) under \( SO(1,3) \times SO(7) \) is \( \epsilon = \epsilon_4 \otimes \epsilon_7 \), with a Majorana spinor \( \epsilon_4 \).

A \( G_2 \) transformation with parameters \( \omega_{ab} \) of a \( SO(7) \) spinor reads then

\[
\delta_{G_2} \eta_{\alpha} = \frac{1}{2} \omega_{ab}(\Sigma^{ab}_{\alpha})_{\alpha} = \frac{1}{2} \omega_{ab} \left( \frac{1}{3} \hat{\psi}_{aba} + \frac{2}{3} \delta_{aa} \eta_{b} - \frac{2}{3} \delta_{ba} \eta_{a} \right), \quad (A.7)
\]

with generators \( J^{ab} = \Sigma^{ab}_{\alpha} \) as defined in eq. \((3.9)\). Clearly, \( \eta_8 \) is invariant, \( \delta_{G_2} \eta_8 = 0 \), while \( \eta_a, a = 1, \ldots, 7 \), transforms in representation \( 7 \):

\[
\delta_{G_2} \eta_a = \frac{1}{2} \omega_{cd} \left( \frac{1}{3} \hat{\psi}_{cda} \eta_b + \frac{2}{3} \delta_{ca} \eta_d - \frac{2}{3} \delta_{da} \eta_c \right). \quad (A.8)
\]

Similarly, the action of the generators of the coset \( SO(7)/G_2 \) is

\[
\delta_{SO(7)/G_2} \eta_a = \frac{1}{2} \zeta^c \psi_{abc}(\Sigma^{ab}_{\alpha})_{\alpha} \eta_\beta = \frac{1}{2} \psi_{abc} \delta_{a8} + \frac{3}{2} \delta_{a8} \eta_8 \zeta^c, \quad (A.9)
\]

with \( \zeta_8 = 0 \). In particular, if one starts with \( \eta_\alpha = C\delta_{a8} \), then \( \delta_{SO(7)/G_2} \eta_a = \frac{3}{2} C \zeta_8 \), which is reminiscent from \( SO(7)/G_2 \sim SO(8)/SO(7) \).

The tensors \( \psi_{abc} \) and \( \hat{\psi}_{abcd} \) can be assembled into a single object \( \Psi^{\alpha\beta\gamma\delta} \), with \( \alpha, \beta, \ldots = 1, \ldots, 8 \), as in \((4.16)\), namely

\[
\Psi_{abc8} = \psi_{abc}, \quad \Psi_{abcd} = \hat{\psi}_{abcd}. \quad (A.10)
\]

Equation \((A.2)\) translates into the eight-dimensional self-duality of \( \Psi_{\alpha\beta\gamma\delta} \):

\[
\Psi_{\alpha\beta\gamma\delta} = \frac{1}{4!} \epsilon^{\alpha\beta\gamma\delta\eta\kappa\lambda} \Psi_{\eta\kappa\lambda}. \quad (A.11)
\]
The basic identity for the product of two $\Psi$'s is

$$\Psi_{\alpha\beta\gamma\delta} \Psi^{\xi\eta\rho\sigma} = 6\delta_\alpha^{[\xi} \delta_\beta^{\eta] \delta_\gamma^{\rho] \delta_\delta^{\sigma]} - 9\Psi_{[\alpha\beta}^{\xi\eta}[\eta]} \Psi_{\gamma\delta]}^{\rho\sigma}.$$  \hfill (A.12)

It can be used to derive most of the following useful identities for products of the tensors $\psi_{abc}$ and $\hat{\psi}_{abcd}$:

$$\psi_{abe} \psi_{cde} = -\hat{\psi}_{abcd} + \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc},$$  \hfill (A.13)

$$\psi_{acd} \psi_{bcd} = 6 \delta_{ab},$$  \hfill (A.14)

$$\psi_{abp} \hat{\psi}_{pcde} = 3\psi_{a[cde]}b - 3\psi_{b[cde]}a,$$  \hfill (A.15)

$$\hat{\psi}_{abcp} \hat{\psi}_{defp} = -3\psi_{[abc][def]}c - 2\psi_{[abc][d]ef]c} - 3\psi_{ab[d]e\{f]c} + 6\delta_{[a} \delta_{b]} \delta_{d]} \delta_{e]} \delta_{f]}$$  \hfill (A.16)

$$\hat{\psi}_{abpq} \psi_{pqc} = -4\psi_{abc},$$  \hfill (A.17)

$$\hat{\psi}_{abpq} \hat{\psi}_{pqcd} = -2\psi_{abcd} + 4(\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc})$$  \hfill (A.18)

$$\hat{\psi}_{apqr} \hat{\psi}_{bpqr} = 24 \delta_{ab},$$  \hfill (A.19)

$$\psi_{abp} \psi_{pcq} \psi_{qde} = \psi_{abd} \delta_{ce} - \psi_{abe} \delta_{cd} - \psi_{ade} \delta_{bc} + \psi_{bde} \delta_{ac}$$

$$- \psi_{acd} \delta_{be} + \psi_{ace} \delta_{bd} + \psi_{bcd} \delta_{ae} - \psi_{abe} \delta_{ad}$$  \hfill (A.20)

$$\psi_{paq} \psi_{qbs} \psi_{scp} = 3\psi_{abc}.$$  \hfill (A.21)
References

[1] A. Gray, Math. Z. 123 (1971) 290.

[2] R.L. Bryant and S.M. Salomon, Duke Math. J. 58 (1989) 829.

[3] G.W. Gibbons, D.N. Page and C.N. Pope, Commun. Math. Phys. 127 (1990) 529.

[4] M. Cvetic, G.W. Gibbons, H. Lu and C.N. Pope, Nucl. Phys. B617 (2001) 151; hep-th/0102185.

[5] A. Brandhuber, J. Gomis, S.S. Gubser and S. Gukov, Nucl. Phys. B611 (2001) 179; hep-th/0106034.

[6] M. Cvetic, G.W. Gibbons, H. Lu and C.N. Pope, Cohomogeneity one manifolds of Spin(7) and G(2) holonomy; hep-th/0108245.

[7] S. Gukov and J. Sparks, M-theory on Spin(7) manifolds. I; hep-th/0109025.

[8] I. Bakas, E.G. Floratos and A. Kehagias, Phys. Lett. B445 (1998) 69; hep-th/9810042.

[9] E.G. Floratos and A. Kehagias, Phys. Lett. B427 (1998) 283; hep-th/9802107.

[10] D. Brecher and M.J. Perry, Nucl. Phys. B566 (2000) 151; hep-th/9908018.

[11] B.S. Acharya and M. O'Loughlin, Phys. Rev. D55 (1997) 4521; hep-th/9612182.

[12] Y. Yasui and T. Ootsuka, Class. Quant. Grav. 18 (2001) 807, hep-th/0010055.

[13] Y. Konishi and M. Naka, Class. Quant. Grav. 18 (2001) 5521, hep-th/0104208.

[14] T. Eguchi and A.J. Hanson, Annals Phys. 120 (1979) 82.

[15] M.F. Atiyah, N.J. Hitchin and I.M. Singer, Proc. Roy. Soc. Lond. A362 (1978) 425.

[16] T. Eguchi, P. Gilkey and A.J. Hanson, Phys. Rep. 66 (1980) 213.

[17] H. Kanno and Y. Yasui, On spin(7) holonomy based on SU(3)/U(1); hep-th/0108226.
[18] N. Hitchin, *Stable forms and special metrics*, math.DG/0107101.

[19] B. Biran, F. Englert, B. de Wit and H. Nicolai, Phys. Lett. **124B** (1983) 45.

[20] P. Candelas and D.J. Raine, Nucl. Phys. **B248** (1984) 415.

[21] M. Günaydin and F. Güresey, J. Math. Phys. **14** (1973) 1651.

[22] F. Güresey and C.-H. Tze, Phys. Lett. **127B** (1983) 191.

[23] B. de Wit and H. Nicolai, Nucl. Phys. **B231** (1984) 506.

[24] F.M. Cabrera, M.D. Monar and A.F. Swann, J. London Math. Soc. **53** (1996) 407.

[25] T. Friedrich, I. Kath. A. Moroianu and U. Semmelmann, J. Geom. Phys. **23** (1997) 259.

[26] R. Cleyton and A. Swann, *Cohomogeneity-one $G_2$ structures*, math.DG/0111056.

[27] E. Cremmer, B. Julia and J. Scherk, Phys. Lett. **76B** (1978) 409.

[28] B.S. Acharya and B. Spence, *Flux, supersymmetry and $M$ theory on 7-manifolds*, hep-th/0007213.

[29] P.G.O. Freund and M.A. Rubin, Phys. Lett. **97B** (1980) 223.

[30] F. Englert, Phys. Lett. **119B** (1982) 339.

[31] M.J. Duff and C.N. Pope, in Supersymmetry and supergravity ’82, ed. S. Ferrara and J.G. Taylor (World Scientific, Singapore, 1983).

[32] S. Fubini and H. Nicolai, Phys. Lett. **155B** (1985) 369.

[33] D.B. Fairlie and J. Nuyts, J. Phys. A: Math. Gen. **17** (1984) 2867.

[34] R. Dündarer, F. Güresey and C.-H. Tze, Nucl. Phys. **B266** (1986) 440.

[35] M. Günaydin and H. Nicolai, Phys. Lett. **B351** (1995) 169; add. Phys. Lett. **B376** (1996) 329; hep-th/9502009.