Analytic Differential Operators on the Unit Disk

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June 13, 2018

Abstract

Formally symmetric differential operators on weighted Hardy-Hilbert spaces are analyzed, along with adjoint pairs of differential operators. Eigenvalue problems for such operators are rather special, but include many of the classical Riemann and Heun equations. Symmetric minimal operators are characterized. A regular class whose leading coefficients have no zeros on the unit circle are shown to be essentially self-adjoint. Eigenvalue asymptotics are established. Some extensions to non-self-adjoint operators are also considered.

Keywords: Analytic differential operators, weighted Hardy space, self-adjoint differential operators

AMS subject classification: 47E05, 47B25, 34L05, 34M03
Contents

1 Introduction 3

2 Adjoint pairs in weighted Hardy spaces 4
   2.1 Weighted Hardy spaces 4
   2.2 Adjoint differential operators 7
   2.3 First order expressions and weight restrictions 9
   2.4 Algebraic properties of adjoint pairs 12

3 $\mathbb{D}$ – regular operators 15
   3.1 $\mathcal{H}_\beta$ Sobolev spaces 16
   3.2 The domain of $\mathcal{L}_{\text{max}}$ 20
   3.3 Fredholm index 23

4 Eigenvalues 25
   4.1 Periodic expressions on $\mathbb{R}$ 26
   4.2 $\mathbb{D}$ – regular expressions 27
1 Introduction

If

\[ L = \sum_{k=0}^{N} p_k(z) D^k, \quad D = \frac{d}{dz}, \] (1.1)

is a differential expression whose coefficients \( p_k(z) \) are analytic on the unit disk \( \mathbb{D} \subset \mathbb{C} \), it is natural to ask about the action of \( L \) as an operator on a Hilbert space or Banach space of analytic functions on \( \mathbb{D} \). Because self adjoint operators play such a dominant role in applications, their identification and analysis is likely to be fundamental. A. Villone initiated just such a project in his dissertation, written under the direction of E.A. Coddington, and in subsequent papers [9] - [13]. The chosen Hilbert space was the Bergman space \( A^2 \) of analytic functions square integrable with respect to area measure on \( \mathbb{D} \). This project was extended in the dissertation of W. Stork, written under the direction of J. Weidman, as reported in [7, 8].

Both Villone and Stork begin with a minimal operator \( L_{\text{min}} \) on \( A^2 \) acting by \( L_{\text{min}} f = Lf \) on the domain \( \mathcal{D}_{\text{min}} \) consisting of polynomials in \( z \). Basic questions include whether \( L_{\text{min}} \) is symmetric, has self adjoint extensions, or is essentially self adjoint. While the framework appears natural, the class of symmetric operators \( L_{\text{min}} \) is quite small. The coefficients \( p_k(z) \) must be polynomials of degree at most \( N + k \). First order minimal operators are symmetric if and only if

\[ L = (a_2 z^2 + a_1 z + a_0) D + (b_1 z + b_0), \]

with constants \( a_j, b_j \) satisfying

\[ a_0 = \overline{a_2}, \quad a_1, b_0 \in \mathbb{R}, \quad b_1 = 2a_2. \]

This work starts by generalizing the earlier investigations in two ways: by introducing new Hilbert spaces, and by extending the focus on symmetric operators to include adjoint pairs of differential operators. The single space \( A^2 \) is extended to a more general class of weighted Hardy spaces [6] in which the functions \( z^n, n = 0, 1, 2, \ldots \) are orthogonal. The adjoint pairs of differential expressions \( L, L^+ \) on a weighted Hardy space is still highly constrained; again, the coefficients \( p_k(z) \) must be polynomials of degree at most \( N + k \). In turn, asking for adjoint pairs \( L, L^+ \) of the same order (a natural requirement when symmetry is a central concern) constrains the admissible weights,
which are essentially characterized by a single positive parameter. Adjoint pairs $L, L^+$ have a role in the formally symmetric constructions $L+L^+$, $L^+L$, and formally symmetric operator matrices such as

\[
\begin{pmatrix}
0 & L^+ \\
L & 0
\end{pmatrix}.
\]

Expressions $L^+L$ and their perturbations will appear as generators of semigroups $\mathcal{L}_{\text{def}}$. By exploiting the $L+L^+$ form, symmetric minimal operators $\mathcal{L}_{\text{min}}$ are given a straightforward characterization that was missing in the earlier work.

The next section introduces a class of $D-regular$ expressions. By focusing on symmetric differential expressions whose leading coefficients had no zeros on the unit circle, Stork [8] identified a class of essentially self-adjoint minimal operators $\mathcal{L}_{\text{min}}$. The self-adjoint extensions have discrete spectrum; a general description of the eigenvalue distribution was provided. Leaning rather heavily on the operator-theoretic methods of [4], new techniques are developed below to extend Stork’s result to symmetric $D-regular$ expressions on the weighted Hardy spaces, with related results for non-self-adjoint operators. A simple formula for the Fredholm index is established.

Continuing the perturbation analysis, the last section provides significantly improved eigenvalue estimates for self-adjoint $D-regular$ operators and their non-self-adjoint perturbations by lower order terms. Initial steps involve restricting the expressions to the unit circle, applying conventional reductions, and studying the eigenvalue problem for an extended operator in a conventional Hilbert space of $2\pi$-periodic functions on $\mathbb{R}$. Once the eigenvalues are constrained, final results are developed with an operator deformation argument in the weighted Hardy spaces.

2 Adjoint pairs in weighted Hardy spaces

2.1 Weighted Hardy spaces

A sequence $\beta = \{\beta_n, n = 0, 1, 2, \ldots\}$ of positive numbers can be used to define an inner product on the vector space $\mathcal{D}_{\text{min}}$ of polynomials $f : \mathbb{C} \rightarrow \mathbb{C}$. For polynomials $f_1 = \sum_{n=0}^{\infty} a_n z^n$ and $f_2 = \sum_{n=0}^{\infty} c_n z^n$ written as power series whose coefficients are eventually zero, an inner product and corresponding
norm are given by

$$\langle f_1, f_2 \rangle_{\beta} = \sum_{n=0}^{\infty} a_n \overline{c_n} \beta_n^2, \quad \|f_1\|^2_{\beta} = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2.$$  

The completion of this inner product space is the Hilbert space $\mathcal{H}_\beta$, consisting of the formal power series $g = \sum_{n=0}^{\infty} a_n z^n$ such that

$$\|g\|^2_{\beta} = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty.$$  

$\mathcal{H}_\beta$ has an orthonormal basis given by

$$e_n = z^n / \beta_n, \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (2.1)  

Standard examples of weighted spaces $\mathcal{H}_\beta$ include the classical Hardy space $\mathcal{H}^2$, where, if $z = |z| e^{i\theta}$, the inner product is

$$\langle f, g \rangle_{\mathcal{H}^2} = \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\theta, \quad \beta_n = 1,$$  \hspace{1cm} (2.2)  

and the Bergman space $\mathcal{A}^2$, with

$$\langle f, g \rangle_{\mathcal{A}^2} = \int_{\mathbb{D}} f(z) \overline{g(z)} \, dx \, dy, \quad \beta_n = \sqrt{\pi/(n+1)}.$$  \hspace{1cm} (2.3)  

If multiplication by $z$, and so polynomials in $z$, act as bounded operators on $\mathcal{H}_\beta$, then differential expressions $L$ with coefficients in $\mathcal{H}_\beta$ will satisfy $L f \in \mathcal{H}_\beta$ for all $f \in \mathcal{D}_{min}$. That is, $L_{min}$ will be densely defined. The needed weight condition is easily characterized.

**Proposition 2.1.** The operator $M_z f = zf$ is bounded on $\mathcal{H}_\beta$ if and only if the sequence $\{\beta_{n+1}/\beta_n\}$ is bounded, in which case $\|M_z\| \leq \sup_n \beta_{n+1}/\beta_n$.

**Proof.** If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\|zf\|^2_{\beta} = \| \sum_{n=0}^{\infty} a_n z^{n+1} \|^2_{\beta} = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 \beta_{n+1}^2 \beta_n^2.$$  

If

$$C_\beta = \sup_n \beta_{n+1}/\beta_n < \infty,$$  \hspace{1cm} (2.4)
then \( \|zf\|_\beta \leq C_\beta \|f\|_\beta \). If \( \beta_{n+1}/\beta_n \) is unbounded, then

\[
\|z^n\|_\beta = \beta_n, \quad \|z \cdot z^n\|_\beta = \left| \frac{\beta_{n+1}}{\beta_n} \right| \|\beta_n\|,
\]

and \( M_\beta \) is unbounded.

When \( C_\beta < \infty \) a simple induction shows that \( \beta_n \leq \beta_0 C_\beta^n \), implying that the power series of elements of \( H_\beta \) converge to an analytic function on some disk centered at \( z = 0 \). The next proposition shows that the condition

\[
\lim_{n \to \infty} \beta_n^{1/n} = r, \quad r > 0,
\]

implies that \( H_\beta \) has a natural interpretation as a space of analytic functions on \( \mathbb{D}_r \), the open disk of radius \( r \) centered at \( z = 0 \).

**Proposition 2.2.** If (2.5) holds, then every \( f \in H_\beta \) is analytic in the open disk \( \mathbb{D}_r \). For \( j = 0, 1, 2, \ldots \) and \( z_1 \in \mathbb{D}_r \), the linear functional \( f^{(j)}(z_1) \) given by derivative evaluation is uniformly bounded in the \( H_\beta \) norm on compact subsets of \( \mathbb{D}_r \). For any \( R > r \), every function analytic in \( \mathbb{D}_R \) belongs to \( H_\beta \), and some functions in \( H_\beta \) do not extend analytically to \( \mathbb{D}_R \).

**Proof.** Suppose \( f = \sum_{n=0}^\infty a_n z^n \in H_\beta \). For \( |z_1| < r \), the Cauchy-Schwarz inequality gives

\[
|f^{(j)}(z_1)| \leq \sum_{n=j}^\infty |a_n| n(n-1) \ldots (n-j) |z_1|^{n-j} \leq \left( \sum_{n=j}^\infty |a_n|^2 \beta_n^2 \right)^{1/2} \left( \sum_{n=j}^\infty n^{2j} |z_1|^{2n-2j} \right)^{1/2} \leq \|f\|_\beta \left( \sum_{n=j}^\infty n^{2j} |z_1|^{2n-2j} \beta_n^2 \right)^{1/2}.
\]

Since \( \lim_{n \to \infty} (n^{2j} |z_1|^{2n} \beta_n^2)^{1/2} = |z_1|^2/r^2 < 1 \), the root test shows that the derivative evaluation functionals are uniformly bounded on compact subsets of \( \mathbb{D}_r \). Since \( \sum_{n=0}^\infty a_n z^n \) converges uniformly on compact subsets of \( \mathbb{D}_r \), \( f(z) \) is analytic on \( \mathbb{D}_r \).

Suppose that \( R > r \) and \( g(z) = \sum_{n=0}^\infty c_n z^n \) is analytic on \( \mathbb{D}_R \). For \( r < R_1 < R \) this series converges absolutely for \( |z| \leq R_1 \), so \( |c_n| < R_1^{-n} \) for sufficiently large \( n \), and \( \sum |c_n|^2 \beta_n^2 \) converges by the root test. Finally, the function \( \sum_{n=1}^\infty z^n/(n \beta_n) \) is in \( H_\beta \), but does not extend analytically to \( \mathbb{D}_R \).

\( \square \)
2.2 Adjoint differential operators

Assume that the weight sequence $\beta$ satisfies (2.4) and (2.5), while $p_k(z) \in \mathcal{H}_\beta$ for $k = 0, \ldots, N$. A differential expression $L = \sum_{k=0}^{N} p_k(z) D^k$, also known as a formal differential operator, is said to have order at most $N$. $L$ will have order equal to $N$ if $p_N(z)$ is not the zero function. Since $L f \in \mathcal{H}_\beta$ for every $f \in \mathcal{D}_{\text{min}}$, the minimal operator $L_{\text{min}} : \mathcal{D}_{\text{min}} \to \mathcal{H}_\beta$ acting by $L_{\text{min}} f = L f$ is then a densely defined operator on $\mathcal{H}_\beta$. $L_{\text{min}}$ has a Hilbert space adjoint operator $L_{\text{min}}^*$, but expectations about the adjoint based on traditional integration by parts computations may be misleading. Typically, $L_{\text{min}}^*$ will not be a differential operator.

The concept of an adjoint pair of operators [4, p. 167] is useful for this discussion. Two differential operators $L_{\text{min}}$ and $L_{\text{min}}^+$ are adjoint to each other if

$$\langle L_{\text{min}} f, g \rangle_\beta = \langle f, L_{\text{min}}^+ g \rangle_\beta$$

for all $f, g \in \mathcal{D}_{\text{min}}$. $L_{\text{min}}^*$ will then be an extension of $L_{\text{min}}^+$. The corresponding expressions $L$ and $L^+$ will be called formal adjoints. $L$ is formally symmetric if $L = L^+$. Being part of an adjoint pair of differential expressions on a space $\mathcal{H}_\beta$ is rather restrictive. The next result is similar to one in [9], where it is assumed that $L = L^+$.

**Theorem 2.3.** Suppose $p_k(z)$ and $q_k(z)$ are in $\mathcal{H}_\beta$ for $k = 0, \ldots, N$, with $\beta$ satisfying (2.4) and (2.5). If $L = \sum_{k=0}^{N} p_k(z) D^k$ has a formal adjoint $L^+ = \sum_{k=0}^{N} q_k(z) D^k$, then $L$ has polynomial coefficients, with $\deg(p_k(z)) \leq N + k$.

**Proof.** With respect to the basis (2.1), the matrix elements of the differential expressions satisfy

$$\langle L e_m, e_n \rangle_\beta = \langle e_m, L^+ e_n \rangle_\beta = \langle L^+ e_n, e_m \rangle_\beta, \quad m, n = 0, 1, 2, \ldots.$$

Differentiation reduces the degree of $e_m$, while multiplication of $e_m$ by an analytic function $p_k(z)$ produces terms of equal or higher degree. Since $L$ has order at most $N$,

$$\langle L e_m, e_n \rangle_\beta = 0, \quad n < m - N.$$

$L^+$ has similar behaviour, so

$$\overline{\langle L e_m, e_n \rangle_\beta} = \langle L^+ e_n, e_m \rangle_\beta = 0, \quad m < n - N,$$
or
\[
\langle Le_m, e_n \rangle_\beta = 0, \quad N < |m - n|.
\] (2.6)

The proof now proceeds by induction on the coefficient index, starting
with \( k = 0 \). Write \( p_0(z) = \sum_{j=0}^{\infty} c_j z^j \) and notice that
\[
\langle Le_0, e_n \rangle_\beta = \beta_0^{-1} \langle p_0(z), e_n \rangle_\beta.
\]
The bound (2.6) means that \( c_j = 0 \) for \( j > N \), so \( p_0(z) \) is a polynomial
of degree at most \( N \). Suppose \( p_k(z) \) is a polynomial of degree at
most \( N + k \) for \( k < M \). Letting \( p_M(z) = \sum_{j=0}^{\infty} c_j z^j \), and using \( D^k e_M = 0 \) for \( K > M \),
\[
\langle Le_M, e_n \rangle_\beta = \langle p_M(z) D^M e_M, e_n \rangle_\beta + \sum_{k=0}^{M-1} \langle p_k(z) D^k e_M, e_n \rangle_\beta,
\]
with
\[
\langle p_M(z) D^M e_M, e_n \rangle_\beta = \beta_M^{-1} \langle p_M(z) D^M z^M, e_n \rangle_\beta = \beta_M^{-1} M! \langle p_M(z), e_n \rangle_\beta.
\]
The induction hypothesis means that
\[
\sum_{k=0}^{M-1} \langle p_k(z) D^k e_M, e_n \rangle_\beta = 0, \quad n > M + N,
\]
while \( \langle Le_M, e_n \rangle_\beta = 0 \) for \( n > M + N \) by (2.6), so \( c_j = 0 \) for \( j > M + N \), and
\( p_M(z) \) is a polynomial of degree at most \( M + N \).

**Proposition 2.4.** With the hypotheses of Proposition 2.3, for each \( L \) the
formal adjoint \( L^+ \) is unique. The set of expressions \( L \) with \( \mathcal{H}_\beta \) differential
expression adjoints is a complex algebra \( \mathbb{A}_{\beta} \) closed under the adjoint opera-
tion.

**Proof.** Suppose
\[
\langle Le_m, e_n \rangle_\beta = \langle e_m, L^+_1 e_n \rangle_\beta = \langle e_m, L^+_2 e_n \rangle_\beta, \quad m, n = 0, 1, 2, \ldots,
\]
with \( L^+_1 - L^+_2 = \sum_{k=0}^{N} q_k(z) D^k \). Starting with \( k = 0 \), an induction shows that
\[
0 = \langle e_m, (L^+_1 - L^+_2) e_k \rangle_\beta = \beta_k^{-1} k! \langle e_m, q_k(z) \rangle_\beta, \quad m = 0, 1, 2, \ldots,
\]
for each \( k \), so \( L^+_1 = L^+_2 \).

The formulas
\[
(L_1 + L_2)^+ = L^+_1 + L^+_2, \quad (L_1 L_2)^+ = L^+_2 L^+_1, \quad (cL)^+ = \overline{c} L^+, c \in \mathbb{C}
\]
are then easily verified.

\[\square\]
2.3 First order expressions and weight restrictions

It is easy to check that the expressions $L = zD$, and more generally polynomials in $L$, have formal adjoints. Since $L z^n = n z^n$, the space $\mathcal{H}_\beta$ has an orthonormal basis of eigenfunctions for $L_{\text{min}}$, which is thus essentially self adjoint. For $a_1, b_0 \in \mathbb{C}$, if $L = a_1 z D + b_0$, then $L^+ = \overline{a_1} z D + \overline{b_0}$ is adjoint to $L$ on $\mathcal{H}_\beta$ without restrictions on $\beta$. This example is unusual. For a more general expression of order 1 satisfying the degree conditions of Theorem 2.3, the existence of a formal adjoint of the same order imposes additional constraints on both the weights and the coefficients of the expression. The weight restrictions are quite severe.

Consider finding first order adjoint pairs,

\[ L_1 = p_1(z) D + p_0(z), \quad L_1^+ = q_1(z) D + q_0(z). \]

By Theorem 2.3, the expressions $L_1$ and $L_1^+$ have the form

\[ L_1 = [a_2 z^2 + a_1 z + a_0] D + [b_1 z + b_0], \quad L_1^+ = [c_2 z^2 + c_1 z + c_0] D + [d_1 z + d_0]. \]  

(2.7)

The coefficients are constrained by the requirement

\[ \langle L_1 z^m, z^n \rangle_\beta = \langle z^m, L_1^+ z^n \rangle_\beta \]

for $m, n = 0, 1, 2, \ldots$, or

\[ \langle m[a_2 z^{m+1} + a_1 z^m + a_0 z^{m-1}] + b_1 z^{m+1} + b_0 z^m, z^n \rangle_\beta \]

(2.8)

\[ = \langle z^m, n[c_2 z^{n+1} + c_1 z^n + c_0 z^{n-1}] + d_1 z^{n+1} + d_0 z^n \rangle_\beta. \]

The inner products are zero if $|m - n| > 1$. For the cases $m = 0, n = 0, 1$ and $n = 0, m = 0, 1$ the equations (2.8) are equivalent to

\[ \beta_0^2 b_0 = \beta_1^2 d_0, \quad \beta_1^2 b_1 = \beta_0^2 c_0, \quad \beta_0^2 a_0 = \beta_1^2 d_1. \]  

(2.9)

For $m \geq 1$ and $n \geq 1$, (2.8) is equivalent to the equations

\[ n = m + 1 : \beta_{m+1}^2 (ma_2 + b_1) = \beta_m^2 (m + 1)c_0, \]

(2.10)

\[ n = m : ma_1 + b_0 = m\overline{c_1} + \overline{d_0}, \]

\[ n = m - 1 : \beta_{m-1}^2 ma_0 = \beta_m^2 [(m - 1)\overline{c_2} + \overline{d_1}]. \]

The first and third equations in (2.10) can be recast in the same form,

\[ \beta_{m+1}^2 (ma_2 + b_1) = \beta_m^2 (m + 1)c_0, \quad \beta_{m+1}^2 (m\overline{c_2} + \overline{d_1}) = \beta_m^2 (m + 1)a_0. \]  

(2.11)
Eliminating the weights leaves

\[(m \overline{c_2} + \overline{d_1}) \overline{c_0} = (ma_2 + b_1)a_0. \quad (2.12)\]

The equations (2.9), (2.10), and (2.12) provide the following relations.

\[\overline{d_0} = b_0, \quad \beta_0^2 \overline{c_0} = \beta_1^2 b_1, \quad \beta_1^2 \overline{d_1} = \beta_0^2 a_0, \quad (2.13)\]
\[\overline{c_1} = a_1, \quad \overline{c_0 d_1} = a_0 b_1, \quad \overline{c_0 c_2} = a_0 a_2.\]

**Theorem 2.5.** Let \(\{\beta_n\}\) be a positive sequence. Suppose \(L_1 = [a_2 z^2 + a_0]D + b_1 z\) has a formal adjoint \(L_1^+ = [c_2 z^2 + c_1 z + c_0]D + [d_1 z + d_0]\) on \(H_\beta\). If \(a_2 \neq 0\), then \(b_1 = \sigma a_2\) for some \(\sigma > 0\). The weights satisfy

\[\beta_{m+1}^2 = \beta_m^2 \frac{\beta_1^2 (m+1)\sigma}{m + \sigma}, \quad (2.14)\]

and \(H_\beta\) is a Hilbert space of functions analytic on \(\mathbb{D}\), with \(r^2 = \beta_1^2 \sigma / \beta_0^2\).

**Proof.** If \(a_2 \neq 0\), then (2.11) implies \(c_0 \neq 0\), and then \(b_1 = \beta_0^2 \overline{c_0} / \beta_1^2 \neq 0\) from (2.13). Using \(\overline{c_0} = \beta_1^2 b_1 / \beta_0^2\) in (2.11) gives

\[\beta_{m+1}^2 = \beta_m^2 \frac{\beta_1^2 (m+1)b_1}{\beta_0^2 ma_2 + b_1}. \quad (2.15)\]

The weights \(\beta_m\) are positive, so

\[\lim_{m \to \infty} \frac{\beta_{m+1}^2}{\beta_m^2} = \frac{\beta_1^2 b_1}{\beta_0^2 a_2} > 0,\]

and \(b_1 = \sigma a_2\) for some \(\sigma > 0\).

The weight sequence now satisfies (2.14) so by induction

\[\beta_m^2 = \beta_1^2 \left(\frac{\beta_1^2 \sigma}{\beta_0^2}\right)^{m-1} \prod_{k=1}^{m-1} \frac{k+1}{k+\sigma} = \beta_1^2 \left(\frac{\beta_1^2 \sigma}{\beta_0^2}\right)^{m-1} \prod_{k=1}^{m-1} \frac{1 + 1/k}{1 + \sigma/k}, \quad m \geq 2.\]

A Taylor expansion gives

\[\lim_{m \to \infty} \left[\prod_{k=1}^{m-1} \frac{1 + 1/k}{1 + \sigma/k}\right]^{1/m} = \lim_{m \to \infty} \exp \left(\frac{1}{m} \sum_{k=1}^{m-1} \log(1 + 1/k) - \log(1 + \sigma/k)\right)\]
\[
\lim_{m \to \infty} \exp \left( \frac{1}{m} \sum_{k=1}^{m-1} \left( \frac{1}{k} - \sigma + O(k^{-2}) \right) \right) = 1,
\]
so
\[
\lim_{m \to \infty} \beta_{2/m}^2 = \frac{\beta_2^2 \sigma}{\beta_0^2}.
\]
As in Proposition 2.2, the natural domain for functions in \( H_\beta \) is \( \mathbb{D}_r \).

Define a collection of restricted weight sequences \( \beta \) with
\[
\sigma > 0, \quad \beta_0 = 1, \quad \beta_1^2 = 1/\sigma, \quad \beta_{m+1}^2 = \beta_m^2 \frac{m+1}{m+\sigma}, \quad m \geq 1.
\]
(2.16)
The proof of Theorem 2.5 and Proposition 2.2 show that \( \mathbb{D} \) is the natural domain for \( H_\beta \). With these weight restrictions the adjoint pairs \( L_1, L_1^+ \) on \( H_\beta \) have a simple description.

**Theorem 2.6.** Let \( \beta \) be a weight sequence satisfying (2.16). On \( H_\beta \) every differential expression
\[
L_1 = [a_2 z^2 + a_1 z + a_0]D + \sigma a_2 z + b_0
\]
has a formal adjoint
\[
L_1^+ = [\overline{a_0} z^2 + \overline{a_1} z + \overline{a_2}]D + \sigma \overline{a_0} z + \overline{b_0}.
\]

**Proof.** Using the notation of (2.7), notice first that the equations of (2.9) are satisfied. Since the terms \( \beta_m \) are given by (2.16), the equations of (2.10) become
\[
\begin{align*}
n &= m+1 : (m+1)(m+\sigma)a_2 = (m+\sigma)(m+1)\sigma a_2/\sigma, \\
n &= m : ma_1 + b_0 = ma_1 + b_0, \\
n &= m-1 : (m-1+\sigma)ma_0 = m[(m-1)+\sigma]a_0,
\end{align*}
\]
so the equations of (2.10) are satisfied. Since (2.9) and (2.10) are equivalent to the satisfaction of (2.8), \( L_1 \) and \( L_1^+ \) are an adjoint pair.

The leading coefficient \( p_1(z) \) of \( L_1 \) may be an arbitrary polynomial of degree at most 2. The roots of the leading coefficients of \( L_1 \) and \( L_1^+ \) are related by a simple transformation.
Corollary 2.7. Suppose $\beta$ satisfies (2.16), $p_1(z) = a_2 z^2 + a_1 z + a_0$ is the leading coefficient of $L_1$, and $L_1^+$ is the formal adjoint of $L_1$ on $H_\beta$, with leading coefficient $q_1(z)$. If $z_1 \neq 0$, then $p_1(z_1) = 0$ if and only if $q_1(1/z_1) = 0$.

Proof. By Theorem 2.6, the leading coefficient of $L_1^+$ is $q_1(z) = a_0 z^2 + a_1 z + a_2$. If $p_1(z_1) = 0$ then

$$0 = p_1(z_1) = z_1^2 [a_2 + a_1/z_1 + a_0/z_1^2] = z_1^2 q_1(1/z_1).$$

\[\square\]

2.4 Algebraic properties of adjoint pairs

Henceforth, the weight sequence $\beta$ is assumed to satisfy (2.16). In the proof of Theorem 2.5 it was noted that (2.5) holds with $r = 1$. It is easy to check that (2.4), and so Proposition 2.1 also hold. Let $A_\beta$ denote the complex algebra of differential expressions $L$ with a formal $H_\beta$ adjoint expression $L^+$.

Theorem 2.8. As an algebra, $A_\beta$ is generated by its expressions with order at most one. If $L = \sum_{k=0}^N p_k(z) D^k \in A_\beta$, with $p_N(z) = \sum_{j=0}^{2N} c_j z^j$, then the leading coefficient of $L^+$ is $q_N(z) = \sum_{j=0}^{2N} \overline{c_j} z^{2N-j}$. If $z_1 \neq 0$, then $p_N(z_1) = 0$ if and only if $q_N(1/z_1) = 0$.

Proof. The claim that $A_\beta$ is generated by its expressions with order at most one is proved by induction on the order, with the case of order at most 1 trivially valid. Suppose the result is true for order less than $N$ and assume $L$ has order $N$ with (nonzero) leading coefficient $p_N(z)$. By Theorem 2.3, $p_N(z)$ is a polynomial of degree $K$, with $K \leq 2N$, which may be written in factored form

$$p_N(z) = \alpha (z - z_1) \ldots (z - z_K).$$

By Theorem 2.6, the expressions in $A_\beta$ with order at most one may have any polynomial leading coefficient of degree at most 2. Recall that if $F_1, \ldots, F_N$ are differential expressions and the product $P = F_1 \cdots F_N$ is written in the standard form (1.1), then the leading coefficient of $P$ is the product of the leading coefficients of the $F_m$. Since $\deg(p_N(z)) \leq 2N$ there is a product $P$ of $N$ first order expressions $F_1, \ldots, F_N \in A_\beta$ whose leading coefficient matches that of $L$. $P$ has the formal adjoint $F_N^+ \ldots F_1^+$. The difference $L - P$ has strictly lower order than $N$. By the induction hypothesis
$L - P$ is in the algebra generated by expressions of order at most one, and so is $L$.

Suppose that for $m = 1, \ldots, N$ the expressions $F_m$ have leading coefficients $a_2(m)z^2 + a_1(m)z + a_0(m)$. Then

$$p_N(z) = \sum_{j=0}^{2N} c_j z^j = \prod_{m=1}^{N} (a_2(m)z^2 + a_1(m)z + a_0(m)),$$

and if $q_N(z)$ denotes the leading coefficient of $L^+$, then by Theorem 2.6

$$q_N(z) = \prod_{m=1}^{N} (\overline{a_0(m)}z^2 + \overline{a_1(m)}z + \overline{a_2(m)})$$

$$= z^{2N} \prod_{m=1}^{N} (\overline{a_0(m)} + \overline{a_1(m)}z^{-1} + \overline{a_2(m)}z^{-2}) = z^{2N} \sum_{j=0}^{2N} c_j z^{-j} = \sum_{j=0}^{2N} c_j z^{2N-j}.$$

Finally, if $z_1 \neq 0$ and $p_N(z_1) = 0$, then

$$0 = \sum_{j=0}^{2N} c_j z_1^j = z_1^{2N} \sum_{j=0}^{2N} \overline{c_j} z_1^{-j-2N} = z_1^{2N} q_N(1/z_1).$$

Notice that the mapping $z_1 \to 1/z_1$ taking roots of $p_N(z_1)$ to roots of $q_N$ extends to $z_1 = 0$ and $z_1 = \infty$ in the sense that if $p_N(z)$ has degree $K \leq 2N$ with $z = 0$ a root of order $m$, then $q_N(z)$ has degree $2N - m$ with $z = 0$ a root of order $2N - K$.

Of course $p_N(z) = q_N(z)$ when $L$ is formally symmetric, so Theorem 2.8 provides the following corollary.

**Corollary 2.9.** Suppose $L$ is formally symmetric with order $N$. Then the leading coefficient $p_N(z)$ has the form

$$p_N(z) = c_N z^N + \sum_{j=0}^{N-1} [c_j z^j + \overline{c_j} z^{2N-j}], \quad c_N = \overline{c_N}.$$

The nonzero roots $z_j$ of $p_N(z)$ are closed under the map $z_j \to 1/z_j$. 

13
Both [9] and [8] consider the problem of characterizing formally symmetric expressions \( L \) on \( \mathcal{A}^2 \). Villone [9] succeeds with a fairly complex recursive technique; a similar method appears again in [13]. Stork [8] shows that for \( c \neq 0 \) and \( n = 1, 2, 3, \ldots \) the examples
\[
 l_{n,r} = (cz^{n+r} + \overline{c}z^{n-r})D^n + \sum_{k=1}^{r} c\binom{r}{k} \frac{(n+1)!}{(n+1-k)!} z^{n+r-k}D^{n-k}, \quad 0 \leq r \leq n,
\]
are formally symmetric in \( \mathcal{A}^2 \) without providing a characterization of formal symmetry. By taking advantage of simple adjoint formulas, an explicit characterization for \( \mathcal{H}_\beta \) is possible.

For \( n = 0, 1, 2, \ldots \) and \( 0 \leq r \leq n \) define expressions
\[
 B_{n,r} = (zD)^{n-r} D^r.
\]
This is simply a product of first order expressions. The expression \( zD \) is formally symmetric, while the \( \mathcal{H}_\beta \) formal adjoint of \( D \) is \( z^2D + \sigma z \) by Theorem 2.6. Taking the adjoint factors in reverse order gives
\[
 B_{n,r}^+ = (z^2D + \sigma z)^r(zD)^{n-r}.
\]
If \( n = 0 \), then \( B_{0,0} \) is simply multiplication by 1. Formally symmetric expressions can be obtained by adding an expression in \( \mathcal{A}_\beta \) to its adjoint expression, leading to the following observation.

**Lemma 2.10.** For \( n = 0, 1, 2, \ldots, 0 \leq r \leq n \), and \( c_{n,r} \in \mathbb{C} \), the expressions \( c_{n,r}B_{n,r} + \overline{c}_{n,r}B_{n,r}^+ \) are formally symmetric, with highest order term \( (c_{n,r}z^{n-r} + \overline{c}_{n,r}z^{n+r})D^n \) when written in the standard form (1.1).

**Theorem 2.11.** An differential expression \( L \) is formally symmetric in \( \mathcal{H}_\beta \) if and only if it can be written in the form
\[
 L = \sum_{n=0}^{N} \sum_{r=0}^{n} [c_{n,r}B_{n,r} + \overline{c}_{n,r}B_{n,r}^+].
\]  

*Proof.* The given form is the sum of formally symmetric operators, so is formally symmetric.

Suppose \( L \) is formally symmetric and of order \( N \). If \( N = 0 \) then the expression is multiplication by a real constant. Proceeding by induction, assume a formally symmetric expression with order less than \( N \) has the given
form. By Corollary 2.9 and Lemma 2.10 the leading coefficient of $L$ can be matched by a formally symmetric expression $\sum_{r=0}^{N}[c_{N,r}B_{N,r} + c_{N,r}B_{N,r}^+]$. Since $L - \sum_{r=0}^{N}[c_{N,r}B_{N,r} + c_{N,r}B_{N,r}^+]$ has order less than $N$, it has the desired form by the induction hypothesis, and so $L$ has the prescribed form.

3 $\mathbb{D}$ - regular operators

Although the class of expressions $L \in A_{\beta}$ is rather restricted, there is an interesting overlap with equations of the Fuchsian class. When $L \in A_{\beta}$ has order two, eigenvalue equations $Ly = \lambda y$ include many of the classical Riemann and Heun equations, as well as problems with five regular singular points. As demonstrated below, the root symmetry in the leading coefficients of symmetric expressions can lead to existence theorems for analytic eigenfunctions, a manifestation of global features of the monodromy data at regular singular points.

Recall that the weight sequences $\beta$ satisfy (2.16). If the differential expression $L$ as in (1.1) has coefficients $p_k(z) \in H_{\beta}$, the minimal operator $L_{\text{min}}$ may be extended to a maximal operator $L_{\text{max}}$ with the domain,

$$D_{\text{max}} = \{f \in H_{\beta} \midLf \in H_{\beta}\}.$$  \hfill (3.1)

**Proposition 3.1.** The operator $L_{\text{max}}$ with domain $D_{\text{max}}$ is closed in $H_{\beta}$.

*Proof. The argument follows [9]. Suppose $f_j \in D_{\text{max}}$ for $j = 1, 2, 3, \ldots$, and that $\{f_j\}$ and $\{L_\lambda f_j\}$ are Cauchy sequences in $H_{\beta}$, converging respectively to $f$ and $g$. By Proposition 2.2 the sequences $\{f_j^{(k)}\}$ converge uniformly to $f^{(k)}$ on any compact subset $K$ of $\mathbb{D}$, so $\{L_\lambda f_j\}$ converges uniformly on $K$, with $Lf = g$.\qed

With additional hypotheses, $L$ may be interpreted as a more conventional operator on $\mathbb{R}$. If the coefficients $p_k(z)$ are analytic on the closed disk $\overline{\mathbb{D}}$, the change of variables $z = e^{-it}$ on the unit circle leads to a $2\pi$-periodic expression

$$L_t = \sum_{k=0}^{N} p_k(e^{it})(-ie^{-it} \frac{d}{dt})^k, \quad t \in \mathbb{R}.$$  

If the leading coefficient $p_N(z)$ has no zeros when $|z| = 1$, then the leading coefficient $p_N(e^{it})(-ie^{-it})^N$ will have no zeros for $t \in \mathbb{R}$. The expression $L_t$
will then fall into the periodic "regular case" [2, p. 188-194] [3, p. 1280]. Features such as closed range and compact resolvent are common for operators acting by $L_t$ on a variety of domains consisting of $2\pi$-periodic functions.

It seems reasonable to expect a class of such "regular" problems on $\mathcal{H}_\beta$. Say that $L$ is $\mathbb{D} - regular$ if the coefficients $p_i(z)$ are analytic on the closed unit disk $\overline{\mathbb{D}}$, and $p_N(z) \neq 0$ when $|z| = 1$; in particular the set $\{z_k \mid |z_k| < 1, p_N(z_k) = 0\}$ is finite. Stork [8] has shown that if $L$ is $\mathbb{D} - regular$ and formally symmetric on $\mathcal{A}^2$, then the closure of the minimal operator $L_{\text{min}}$ is self adjoint, with compact resolvent. The main goal of this section is to develop the techniques needed to extend Stork’s result to formally symmetric expressions on $\mathcal{H}_\beta$, with related results for more general $\mathbb{D} - regular$ expressions.

### 3.1 $\mathcal{H}_\beta$ Sobolev spaces

For weight sequences $\beta$ satisfy (2.16), and for $k = 0, 1, 2, \ldots$, define additional Hilbert spaces $\mathcal{H}_\beta^k$ of functions $f \in \mathcal{H}_\beta$ whose $k$-th derivative is also in $\mathcal{H}_\beta$. For $f = \sum_{n=0}^{\infty} c_n z^n$, and $g = \sum_{n=0}^{\infty} b_n z^n$, the $\mathcal{H}_\beta^k$ inner product is

$$\langle f, g \rangle_k = \sum_{n=0}^{\infty} (1 + n^{2k}) c_n \overline{b_n} \beta_n^2,$$

the norm is given by $\|f\|_k^2 = \langle f, f \rangle_k$, and the set of elements is

$$\mathcal{H}_\beta^k = \{ f = \sum_{n=0}^{\infty} c_n z^n \mid \sum_{n=0}^{\infty} (1 + n^{2k}) |c_n|^2 \beta_n^2 < \infty \}. \quad (3.2)$$

The fact that multiplication by $z$ acts as a bounded operator can be extended to a broader class of multiplication operators on the spaces $\mathcal{H}_\beta^k$. Let $C_\beta = \sup_{n} \beta_{n+1}/\beta_n$. The conditions (2.16) imply $1 \leq C_\beta < \infty$, and $C_\beta = 1$ for $\sigma \geq 1$.

**Proposition 3.2.** Suppose $\phi(z) = \sum_{n=0}^{\infty} \alpha_n z^n$, with $\sum_{n=0}^{\infty} |\alpha_n| 2^{nk} C_\beta^n < \infty$. Then the operator $M_\phi f = \phi(z) f(z)$ is bounded on $\mathcal{H}_\beta^k$.

**Proof.** First consider multiplication by $z$ on $\mathcal{H}_\beta^k$. With $f = \sum_{n=0}^{\infty} c_n z^n$, and the usual operator norm $\|M_z\| = \sup_{\|f\| \leq 1} \|zf\|_k$,

$$\|zf\|_k^2 = \sum_{n=0}^{\infty} (1 + n^{2k}) |c_n|^2 \beta_n^2 \frac{\beta_{n+1}^2}{\beta_n^2} \frac{(1 + [n + 1]^{2k})}{(1 + n^{2k})}.$$
so $M_\varphi$ is a bounded operator on $H_\beta^k$ with norm bounded by $2^k C_\beta$. The operator norm is submultiplicative, and absolutely convergent series in the Banach space of bounded operators are convergent, so $M_\varphi$ is bounded on $H_\beta^k$. \hfill \Box

The next lemma uses a standard Fourier series argument.

**Lemma 3.3.** Suppose $f \in H_\beta^k$ and $0 \leq j < k$. Then for any $\epsilon > 0$ there is a constant $C$, independent of $f$, such that

$$
\|f\|_j \leq \epsilon \|f\|_k + C \|f\|_\beta.
$$

The set $B = \{f \in H_\beta^k \mid \|f\|_k \leq 1\}$ has compact closure in $H_\beta^j$.

**Proof.** For $\epsilon > 0$ the inequality $(1 + n^{2j}) \leq \epsilon^2 (1 + n^{2k}) + C^2$ holds for $n = 0, 1, 2, \ldots$, and $C$ sufficiently large. Thus

$$
\|f\|_j^2 \leq \epsilon^2 \|f\|_k^2 + C^2 \|f\|_\beta^2 \leq (\epsilon \|f\|_k + C \|f\|_\beta)^2,
$$

establishing (3.3).

Suppose $\{f_i(z) = \sum_{n=0}^{\infty} c_n(l) z^n\}$ is a sequence in $B$. For each fixed $n$, the sequence $c_n(l)$ is bounded, so by the usual diagonalization argument [5, p. 167] the sequence $\{f_i(z)\}$ has a subsequence $\{f_m(z)\}$ such that $\{c_n(m), m = 1, 2, 3, \ldots\}$ is a Cauchy sequence in $\mathbb{C}$ for each $n$.

Since $\|f_m(z)\|_k \leq 1$, for any $\epsilon > 0$ there is an $N$ such that

$$
\sum_{n=N}^{\infty} (1 + n^{2j}) |c_n(m)|^2 \beta_n^2 < \epsilon, \quad m = 1, 2, 3, \ldots.
$$

Since the sequences $c_n(m) \in \mathbb{C}$ are convergent for $n < N$, the sequence $\{f_m(z)\}$ is a Cauchy sequence in $H_\beta^j$. \hfill \Box

**Lemma 3.4.** If $f(z)$ is analytic in $\mathbb{D}$ and $D^k f \in H_\beta^k$, then $f \in H_\beta^k$.

**Proof.** Since $f(z)$ is analytic in $\mathbb{D}$, both $f$ and $D^k f$ have power series which converge for every $z \in \mathbb{D}$,

$$
f = \sum_{n=0}^{\infty} c_n z^n, \quad D^k f = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) c_n z^{n-k}.
$$

17
The assumption that $D^k f \in H_\beta$ means that

$$\sum_{n=k}^{\infty} |n(n-1)\cdots(n-k+1)c_n\beta_{n-k}|^2 < \infty.$$  

Now $\beta_{n-k} = \beta_n \prod_{j=1}^{k} \beta_{n-j}/\beta_{n-j+1}$, and $\lim_{n \to \infty} \beta_{n-j}/\beta_{n-j+1} = 1$, so $f \in H_\beta$ since

$$\sum_{n=k}^{\infty} n^{2k}|c_n\beta_n|^2 < \infty.$$

\[\square\]

**Lemma 3.5.** Suppose $f(z) \in H_\beta^k$, $r(z) = (z - z_1)\cdots(z - z_M)$, and $j$ is an integer, with $1 \leq j \leq k$. Then there is a $g \in H_\beta^k$ such that $r(z)D^j f = D^j g$, with $\|g\|_k \leq C\|f\|_k$.

**Proof.** Since multiplication by $z$ is a bounded operator on $H_\beta^k$, the function

$$F(z) = \sum_{n=0}^{\infty} \frac{c_n}{n+1} z^{n+1} = z \sum_{n=0}^{\infty} \frac{c_n}{n+1} z^n$$

is in $H_\beta^k$, with $F'(z) = f(z)$. Begin an induction with the case $r(z) = (z - z_1)$. By the product rule,

$$(z - z_1)D^j f = D^j[(z - z_1)f(z) - jF(z)] = (z - z_1)D^j f + jf^{(j-1)} - jf^{(j-1)},$$

and $\|(z - z_1)f(z) - jF(z)\|_k \leq C\|f\|_k$.

Suppose the result holds when there are fewer than $M$ factors. Then $(z - z_1)\cdots(z - z_M)D^j f = (z - z_1)D^j g$, and the first case may be applied to finish the proof.

\[\square\]

The next result provides a kind of lower bounded for the operator of multiplication by a nonzero polynomial $r(z)$ with no roots on $\partial \mathbb{D} = \{|z| = 1\}$.

**Theorem 3.6.** Suppose $j$ and $k$ are nonnegative integers. If

$$r(z) = \alpha \prod_{m=1}^{M} (z - z_m), \quad |z_m| \notin \partial \mathbb{D}, \quad \alpha \neq 0,$$

then...
then there are positive constants $C_1, C_2$ such that
\[ \|D^j f\|_k \leq C_1\|r(z)D^j f\|_k + C_2\|f\|_\beta, \] (3.4)
for any $f \in H_{\beta}^{k+j}$.

**Proof.** With $f = \sum_{n=0}^{\infty} c_n z^n$,
\[ \|zD^j f\|_k^2 = \sum_{n=0}^{\infty} (1 + n^{2k})(n(n-1) \cdots (n-j+1))^2 |c_n|^2 \beta_{n-j+1}^2 \]
while
\[ \|D^j f\|_k^2 = \sum_{n=0}^{\infty} (1 + n^{2k})(n(n-1) \cdots (n-j+1))^2 |c_n|^2 \beta_{n-j+1}^2 \frac{\beta_{n-j}^2}{\beta_{n-j+1}^2}. \]

Since $\lim_{n \to \infty} \beta_{n+1}/\beta_n = 1$, for any $\epsilon$ with $0 < \epsilon < 1$ there is an $N$ such that
\[ (1 - \epsilon)^2 \sum_{n=N}^{\infty} (1 + n^{2k})(n(n-1) \cdots (n-j+1))^2 |c_n|^2 \beta_{n-j+1}^2 \beta_{n-j}^2 \]
\[ \leq \sum_{n=N}^{\infty} (1 + n^{2k})(n(n-1) \cdots (n-j+1))^2 |c_n|^2 \beta_{n-j+1}^2 \]
\[ \leq (1 + \epsilon)^2 \sum_{n=N}^{\infty} (1 + n^{2k})(n(n-1) \cdots (n-j+1))^2 |c_n|^2 \beta_{n-j+1}^2 \beta_{n-j}^2 \]
and so a $C > 0$, depending on $\epsilon$, such that
\[ (1 - \epsilon)^2 \|D^j f\|_k^2 \leq \|zD^j f\|_k^2 + C\|f\|_\beta^2 \leq (\|zD^j f\|_k + C\|f\|_\beta)^2 \] (3.5)
and
\[ \|zD^j f\|_k^2 \leq ((1 + \epsilon)\|D^j f\|_k + C\|f\|_\beta)^2. \] (3.6)

Suppose $r(z) = (z - z_1)$, with $|z_1| > 1$. By (3.6) the reverse triangle inequality gives
\[ \|(z - z_1)D^j f\|_k \geq |z_1|\|D^j f\|_k - \|zD^j f\|_k \]
\[ \geq |z_1|\|D^j f\|_k - (1 + \epsilon)\|D^j f\|_k - C\|f\|_\beta. \]
Since $|z_1| > 1$, $\epsilon$ may be chosen with $1 + \epsilon < |z_1|$, giving
\[
\|D^j f\|_k \leq \frac{1}{|z_1| - (1 + \epsilon)} \left(\|(z - z_1)D^j f\|_k + C\|f\|_\beta\right).
\]

In case $r(z) = (z - z_1)$ with $|z_1| < 1$, (3.5) gives
\[
\|(z - z_1)D^j f\|_k + C\|f\|_\beta \geq \|(z - z_1)D^j f\|_k \geq (1 - \epsilon)\|D^j f\|_k - |z_1|\|D^j f\|_k.
\]

Choose $\epsilon$ so that $1 - \epsilon > |z_1|$ to get
\[
\|D^j f\|_k \leq \frac{1}{(1 - \epsilon) - |z_1|} \left(\|(z - z_1)D^j f\|_k + C\|f\|_\beta\right).
\]

Using Lemma 3.5 if $r_1(z) = \alpha(z - z_1) \cdots (z - z_{M-1})$, there is a $g$ such that $D^j g = r_1(z)D^j f$, with $\|g\|_k \leq C\|f\|_k$. The proof of (3.4) concludes by induction on the number of factors $M$, with the first case established. Using the first case, and the assumed validity of the result with fewer than $M$ factors,
\[
\|r(z)D^j f\|_k = \|(z - z_M)r_1(z)D^j f\|_k = \|(z - z_M)D^j g\|_k \geq C_1\|D^j g\|_k - C_2\|g\|_\beta
\]  
\[
= C_1\|r_1(z)D^j f\|_k - C_3\|f\|_\beta \geq C_4\|D^j f\|_k - C_5\|f\|_\beta.
\]

3.2 The domain of $\mathcal{L}_{\text{max}}$

The discussion of operator domains for $\mathbb{D} - \text{regular}$ expressions begins with a lemma similar to one in [8]. The parameter $\sigma$ is from (2.16).

**Lemma 3.7.** Suppose $L$ is $\mathbb{D} - \text{regular}$ with order $N$. If $\sigma \geq 1$, then $\mathcal{H}^N_\beta$ is in the domain of the closure of $\mathcal{L}_{\text{min}}$. This result holds for all $\sigma > 0$ if $L$ has polynomial coefficients.

**Proof.** If $f(z) \in \mathcal{H}^N_\beta$, then the $m$-th order Taylor polynomials $t_m(z)$ for $f(z)$ centered at zero converge to $f$ in the $\mathcal{H}^N_\beta$ norm. Thus for $j = 0, 1, \ldots, N$ the derivatives $D^j t_m$ converge to $D^j f$ in the $\mathcal{H}_\beta$ norm. $C_\beta = 1$ in Proposition 3.2 since $\sigma \geq 1$. Also, the coefficients $p_k(z)$ of $L$ are analytic on the closed disk $\overline{\mathbb{D}}$, so the Taylor series for the coefficients converge absolutely for $|z| \leq 1$. By
Proposition 3.2 multiplication by \( p_k(z) \) acts as a bounded operator on \( \mathcal{H}_\beta \), and thus \( L_{t_m}(z) \) converges to \( Lf \), putting \( f \) in the domain of the closure of \( L_{min} \).

In case \( L \) has polynomial coefficients, multiplication by \( p_k(z) \) acts as a bounded operator for all \( \sigma > 0 \).

As a densely defined operator on a Hilbert space, \( L_{min} \) has an adjoint operator \( L_{\text{min}}^* \), whose graph is the set of all pairs \( (g_1, g_2) \in \mathcal{H}_\beta \oplus \mathcal{H}_\beta \) such that

\[
\langle f, g_2 \rangle \beta = \langle L_{\text{min}} f, g_1 \rangle \beta
\]

for all polynomials \( f \). Recall that \( \mathcal{A}_\beta \) is the set of differential expressions \( L \) with a formal adjoint expression \( L^+ \), and that all \( L \in \mathcal{A}_\beta \) have polynomial coefficients by Theorem 2.3. The next result is similar to one in [9], where \( L_{\text{min}} \) is assumed to be symmetric.

Theorem 3.8. If \( L \in \mathcal{A}_\beta \), then \( L_{\text{min}}^* = L_{\text{max}}^+ \).

Proof. For any \( f \in \mathcal{D}_{\text{min}} \), the function \( L_{\text{min}} f = \sum_{j=0}^{m} b_j z^j \) is also a polynomial. Suppose \( g = \sum_{j=0}^{\infty} c_j z^j \) is in the domain of \( L_{\text{min}}^* \). Since the powers \( z^j \) are an orthogonal basis for \( \mathcal{H}_\beta \), if \( M > m + N \) then the orthogonal projection \( g_M(z) = \sum_{j=0}^{M} c_j z^j \) of \( g \) onto the span of \( 1, \ldots, z^M \) is a polynomial which satisfies

\[
\langle L_{\text{min}} f, g \rangle = \langle L_{\text{min}} f, g_M \rangle = \langle f, L^+ g_M \rangle = \langle f, L^+ g \rangle.
\]

(3.7)

Since the polynomials are dense in \( \mathcal{H}_\beta \), \( L_{\text{min}}^* g = L^+ g \), and \( g \) is in the domain of \( L_{\text{max}}^+ \). In addition, (3.7) shows that any \( g \) in the domain of \( L_{\text{max}}^+ \) is in the domain of \( L_{\text{min}}^* \).

Theorem 3.9. Suppose \( L = \sum_{k=0}^{N} p_k(z) D^k \) is \( \mathbb{D} \)-regular of order \( N \) with polynomial coefficients. Then \( \mathcal{D}_{\text{max}} = \mathcal{H}_\beta^N \) and \( L_{\text{max}} \) is a closed operator on \( \mathcal{H}_\beta \).

Proof. Since multiplication by \( p_k(z) \) is bounded on \( \mathcal{H}_\beta \), \( \mathcal{H}_\beta^N \subset \mathcal{D}_{\text{max}} \) for \( L \).

The result holds trivially if \( N = 0 \), since \( f \in \mathcal{H}_\beta^0 \) if \( f \in \mathcal{D}_{\text{max}} \). The proof proceeds by induction on the order \( N \geq 1 \). If \( N = 1 \), then \( L = p_1(z) D + p_0(z) \). Since \( p_0(z) f \in \mathcal{H}_\beta \), we have \( p_1(z) Df \in \mathcal{H}_\beta \), and \( f \in \mathcal{H}_\beta^1 \) by Theorem 3.6.

Suppose \( N \geq 2 \) and the result holds for \( K < N \).
Using the product rule, the expression \( L \) can be written in the form 
\[
L = b_0(z) + D \sum_{k=1}^{N} b_k(z)D^{k-1},
\]
with \( b_N(z) = p_N(z) \). Since \( Lf - b_0(z)f = D \sum_{k=1}^{N} b_k(z)D^{k-1}f \in \mathcal{H}_\beta \), Lemma 3.4 implies \( \sum_{k=1}^{N} b_k(z)D^{k-1}f \in \mathcal{H}_\beta^1 \). By the induction hypothesis, \( f \in \mathcal{H}_\beta^{N-1} \). Finally, since \( Df \in \mathcal{H}_\beta^1 \), the induction hypothesis gives \( Df \in \mathcal{H}_\beta^{N-1} \), so \( f \in \mathcal{H}_\beta^{N} \).

By Lemma 3.7, \( L_{\text{max}} \) is the closure of \( L_{\text{min}} \).

A result similar to Theorem 3.9 holds for more general coefficients when \( \sigma \geq 1 \).

**Theorem 3.10.** If \( \sigma \geq 1 \) and \( L = \sum_{k=0}^{N} p_k(z)D^k \) is \( \mathbb{D} \)-regular of order \( N \), then \( D_{\text{max}} = \mathcal{H}_{\beta}^N \). \( L_{\text{max}} \) is a closed operator on \( \mathcal{H}_{\beta} \).

**Proof.** Since the coefficients \( p_k(z) \) are analytic on \( \mathbb{D} \), and \( C_{\beta} = 1 \) if \( \sigma \geq 1 \), multiplication by \( p_k(z) \) is bounded on \( \mathcal{H}_{\beta} \) and \( \mathcal{H}_{\beta}^N \subset D_{\text{max}} \) for \( L \).

Assume \( f \in \mathcal{H}_{\beta} \) and \( Lf \in \mathcal{H}_{\beta} \). If \( z_1, \ldots, z_K \) are the roots of \( p_N(z) \) in \( \mathbb{D} \), listed with multiplicity, let \( r(z) = (z-z_1)\cdots(z-z_K) \). The leading coefficient can be factored as \( p_N(z) = r(z)q(z) \), where \( q(z) \) is analytic with no zeros on \( \mathbb{D} \). Since \( C_{\beta} = 1 \) and \( 1/q(z) \) has an absolutely convergent Taylor series on \( \{ |z| \leq 1 \} \), multiplication by \( 1/q(z) \) is bounded on \( \mathcal{H}_{\beta} \) by Lemma 3.2. Thus \( (1/q(z))Lf \in \mathcal{H}_{\beta} \), and \( L \) can be assumed to have leading coefficient \( p_N(z) = r(z) \).

The induction argument from the proof of Theorem 3.9 may now be applied again.

Suppose \( L \) is a formally symmetric \( \mathbb{D} \)-regular expression. The next result shows that \( L_{\text{max}} \) is self-adjoint, with compact resolvent \( R(\lambda) = (L_{\text{max}} - \lambda I)^{-1} \).

**Theorem 3.11.** Suppose \( L = \sum_{k=0}^{N} p_k(z)D^k \) is \( \mathbb{D} \)-regular and formally symmetric on \( \mathcal{H}_{\beta} \), with order \( N \geq 1 \). Then \( L_{\text{max}} \) is the closure of \( L_{\text{min}} \), and \( L_{\text{max}} \) is self adjoint. The resolvent \( R(\lambda) : \mathcal{H}_{\beta} \to \mathcal{H}_{\beta}^N \) is uniformly bounded on compact subsets of the resolvent set, and \( R(\lambda) : \mathcal{H}_{\beta} \to \mathcal{H}_{\beta} \) is compact.
Proof. The minimal operator $\mathcal{L}_{\min}$ is symmetric on $\mathcal{H}_\beta$, with $\mathcal{L}_{\min}^* = \mathcal{L}_{\max}$ by Theorem 3.8. Let $\tilde{\mathcal{L}}_{\min}$ denote the closure of $\mathcal{L}_{\min}$. Then $\tilde{\mathcal{L}}_{\min}$ is symmetric [4, p. 269] with $\tilde{\mathcal{L}}_{\min}^* = \mathcal{L}_{\max}$ [4, p. 168]. Now $\mathcal{D}_{\max} = \mathcal{H}_\beta^N$ by Theorem 3.9 and $\mathcal{H}_\beta^N$ is in the domain of $\tilde{\mathcal{L}}_{\min}$ by Lemma 3.7. Thus $\tilde{\mathcal{L}}_{\min}$ is an extension of $\mathcal{L}_{\max}$, and $\tilde{\mathcal{L}}_{\min} = \mathcal{L}_{\max} = \tilde{\mathcal{L}}_{\min}^*$.

For $f \in \mathcal{H}_\beta^N$, the reverse triangle inequality gives
\[
(L - \lambda I)f \geq \|p_N(z)D^N f\| - \sum_{k=0}^{N-1} \|p_k(z)D^k f\|,
\]
so Theorem 3.6 implies the existence of constants $C_k$, which may be chosen uniformly for $\lambda$ in compact subsets of the resolvent set, such that
\[
\|D^N f\|_\beta \leq C_N\|p_N(z)D^N f\|_\beta + C_N\|f\|_\beta \leq C_N\|(L - \lambda I)f\|_\beta + \sum_{k=0}^{N-1} C_k\|D^k f\|_\beta.
\]
By Lemma 3.3 the terms $\|D^k f\|_\beta$ may be replaced by $\epsilon\|D^N f\|_\beta + C\|f\|_\beta$ for any $\epsilon > 0$, so with a new constant $C'$,
\[
\|D^N f\|_\beta \leq C\|(L - \lambda I)f\|_\beta + C\|f\|_\beta.
\]
Taking $f = R(\lambda)g$ for $g \in \mathcal{H}_\beta$,
\[
\|D^N R(\lambda)g\|_\beta \leq C\|g\|_\beta + C\|R(\lambda)g\|_\beta,
\]
so the resolvent $R(\lambda)$ of $\mathcal{L}_{\max}$ is a bounded operator from $\mathcal{H}_\beta$ to $\mathcal{H}_\beta^N$. By Lemma 3.3 the image of the unit ball has compact closure.

3.3 Fredholm index

Suppose $T$ is a closed operator on $\mathcal{H}_\beta$, while $A$ is another operator on $\mathcal{H}_\beta$ whose domain includes the domain of $T$. Recall [4, p. 194] that $A$ is relatively compact with respect to $T$ if for every bounded sequence $\{u_n\}$ in the domain of $T$, with $\{Tu_n\}$ also bounded, the sequence $\{Au_n\}$ has a convergent subsequence. The operator $T + A$ with the domain of $T$ will be closed.

Also recall, [4, p. 230] that a closed operator $T$ is Fredholm if $T$ has a finite dimensional null space and a closed range of finite codimension. The
index of a Fredholm operator is \( \text{ind}(T) = \dim(\text{null } T) - \text{codim}(\text{range } T) \). If \( A \) is relatively compact with respect to the Fredholm operator \( T \), then [4, p. 238] the operator \( T + A \) is Fredholm, with \( \text{ind}(T + A) = \text{ind}(T) \).

**Lemma 3.12.** Suppose \( \sigma \geq 1 \), \( L = p_N(z)D^N \) is \( \mathbb{D} \)-regular of order \( N \geq 1 \), and \( L_0 = \sum_{k=0}^{N-1} p_k(z)D^k \) has coefficients analytic on \( \mathbb{D} \). If \( L_0 \) acts by \( L_0 \) and \( \mathcal{L} = \mathcal{L}_{\text{max}} \) acts by \( L \) on \( \mathcal{H}_\beta^N \), then \( \mathcal{L}_0 \) is relatively compact with respect to \( \mathcal{L} \).

**Proof.** Assume that \( z_1, \ldots, z_K \) are the roots of \( p_N(z) \) in \( \mathbb{D} \), listed with multiplicity. As in Theorem 3.10, \( p_N(z) \) may be factored as \( p_N(z) = r(z)q(z) \), with \( r(z) = (z - z_1) \cdots (z - z_K) \), and with \( q(z) \) analytic with no zeros on \( \mathbb{D} \). Since multiplication by \( q(z) \) and \( 1/q(z) \) are bounded on \( \mathcal{H}_\beta \) by Proposition 3.2, it suffices to assume that \( p_N(z) = r(z) \).

Assume \( u_m \in \mathcal{H}_\beta^N \), and the sequences \( \{u_m\} \) and \( \{r(z)D^Nu_m\} \) are bounded. By Theorem 3.6, the sequence \( u_m \) is bounded in \( \mathcal{H}_\beta^N \). For \( k < N \) the terms \( D^ku_m \) thus have subsequences which are convergent in \( \mathcal{H}_\beta \) by Lemma 3.3, and the same holds for \( \{\mathcal{L}_0u_m\} \).

**Theorem 3.13.** Suppose \( \sigma \geq 1 \), and \( L = \sum_{k=0}^{N} p_k(z)D^k \) is \( \mathbb{D} \)-regular of order \( N \geq 1 \). If \( p_N(z) \) has \( K \) roots in \( \mathbb{D} \), counted with multiplicity, then \( \mathcal{L} \) with domain \( \mathcal{H}_\beta^N \) is Fredholm with index \( N - K \).

**Proof.** By Lemma 3.12 it suffices to prove the result when \( L = p_N(z)D^N \). The null space of \( \mathcal{L} \), being the polynomials with degree at most \( N - 1 \), has dimension \( N \).

For \( j = 1, \ldots, J \), let \( z_j \) be the distinct roots of \( p_N(z) \) in \( \mathbb{D} \) with multiplicities \( M_j \). Factor \( p_N(z) = r(z)q(z) \) with \( r(z) = (z - z_1)^{M_1} \cdots (z - z_J)^{M_J} \). Since multiplication by \( q(z) \) and \( q^{-1}(z) \) are bounded operators on \( \mathcal{H}_\beta \), it suffices to assume that \( p_N(z) = r(z) \).

To establish that the range of \( L = r(z)D^N \) is closed, suppose \( \{f_m\} \) is a sequence in \( \mathcal{H}_\beta^N \), \( h_m = r(z)D^Nf_m \), and the sequence \( \{h_m\} \) converges to \( h \) in \( \mathcal{H}_\beta \). By Lemma 3.5 there is a sequence \( \{g_m\} \) in \( \mathcal{H}_\beta^N \) such that \( h_m = D^Ng_m \).

The first \( N \) terms of the power series for \( g \) may be discarded,

\[
g_m = \sum_{k=0}^{\infty} a_k z^k, \quad \tilde{g}_m = \sum_{k=N}^{\infty} a_k z^k,
\]

giving \( h_m = D^N\tilde{g}_m \). The sequence \( \{\tilde{g}_m\} \) converges in \( \mathcal{H}_\beta^N \), and \( h \) is in the range of \( \mathcal{L}_{\text{max}} \) since \( \mathbb{D} \)-regular operators are closed by Theorem 3.10.
If $h$ is in the range of $\mathcal{L}$, then $h$ has a zero of order at least $M_j$ at $z_j$. Using Proposition 2.2, the $K$ functionals given by $f^{(l)}(z_j)$ for $l = 0, \ldots, M_j - 1$ are independent continuous linear functionals on $\mathcal{H}_\beta$. By the Riesz representation theorem there are $K$ independent elements $h_1, \ldots, h_K$ of $\mathcal{H}_\beta$ orthogonal to the range of $p(z)D^N$, which thus has codimension at least $K$.

If $r(z)$ is any polynomial, then $r(z) = p(z)s(z) + t(z)$ where $s(z)$ and $t(z)$ are polynomials and $\deg t(z) < K$. (For instance $t(z)$ could be in the span of the Lagrange basis for the roots of $p$). Since the polynomials are dense in $\mathcal{H}_\beta$, the polynomials $p(z)s(z)$ are dense in the range of $\mathcal{L}$. Thus the codimension of the range is at most $K$.

The assumption that $\sigma \geq 1$ may be dropped if $p_N(z)$ is a polynomial. The simplified proofs of Lemma 3.12 and Theorem 3.13 are omitted.

**Theorem 3.14.** Suppose $L = \sum_{k=0}^N p_k(z)D^k$ is $D$-regular of order $N \geq 1$. If $p_N(z)$ is a polynomial with $K$ roots in $\mathbb{D}$, counted with multiplicity, then $\mathcal{L}$ with domain $\mathcal{H}_\beta^N$ is Fredholm with index $N - K$.

## 4 Eigenvalues

Consider the first order expression $L = a_2(z - z_1)(z - z_2)D + \sigma a_2 z + b_0$, where $a_2 \neq 0$, $|z_1| < 1$, $|z_2| > 1$. Elementary computations show that the eigenvalues $\lambda_n$ of the maximal operator $\mathcal{L}$ are

$$\lambda_n = b_0 + a_2[\sigma z_1 - (z_2 - z_1)n], \quad n = 0, 1, 2, \ldots,$$

with eigenfunctions

$$y_n(z) = C \left( \frac{z - z_1}{z - z_2} \right)^n (z - z_2)^{-\sigma}.$$

Except for the restriction to $n \geq 0$, $\{\lambda_n\}$ is similar to an eigenvalue sequence for periodic eigenfunctions of a first order periodic expression on $\mathbb{R}$.

The connection linking expressions on $\mathcal{H}_\beta$ and periodic problems on $\mathbb{R}$ will be made explicit. These efforts start by reinterpreting $L$ as an expression on $\mathbb{R}$ with periodic coefficients. Introduce the Hilbert space $L^2_{\text{per}}$ of $2\pi$ periodic functions which are (Lebesgue) square integrable on $[0, 2\pi]$ with inner product $\langle f, g \rangle = \int_0^{2\pi} f(\theta)\overline{g(\theta)} d\theta$. When $L$ is $D$-regular and formally symmetric, the periodic eigenvalue problem is a perturbation of an eigenvalue problem for a self-adjoint operator on $L^2_{\text{per}}$. 

25
4.1 Periodic expressions on \( \mathbb{R} \)

Rather than considering \( H_\beta \) as a space of functions analytic on \( \mathbb{D} \), another interpretation is available. Begin with the complex vector space of trigonometric polynomials \( f : \mathbb{R} \rightarrow \mathbb{C} \) having the form

\[
f(\theta) = \sum_{n=0}^{\infty} c_n e^{in\theta},
\]

with only finitely many nonzero coefficients \( c_n \). As before, define the inner product

\[
\langle f_1, f_2 \rangle_\beta = \sum_{n=0}^{\infty} b_n c_n b_n^2,
\]

for polynomials \( f_1 = \sum_{n=0}^{\infty} b_n \exp(\imath n\theta) \) and \( f_2 = \sum_{n=0}^{\infty} c_n \exp(\imath n\theta) \). The completion of this inner product space is a Hilbert space, denoted \( H_{\beta, \mathbb{R}} \). The map \( H_{\beta, \mathbb{R}} \rightarrow H_\beta \) given by

\[
f(\theta) \mapsto f(z) = \sum_{n=0}^{\infty} c_n z^n,
\]

is an isometric bijection of Hilbert spaces. (Although function notation is used, elements of \( H_{\beta, \mathbb{R}} \) are typically periodic distributions on \( \mathbb{R} \).)

This mapping takes the expression \( \imath \frac{dz}{dz} \) from \( H_\beta \) to \( \frac{d}{d\theta} \) on \( H_{\beta, \mathbb{R}} \). The more general expressions \( L = \sum_{k=0}^{N} p_k(z) D^k \) become \( L_{\text{per}} = \sum_{k=0}^{N} p_k(e^{i\theta})(-\imath e^{-i\theta} \frac{d}{d\theta})^k \). This reinterpretation of differential expressions is particularly useful for locating the eigenvalues of \( D - \text{regular} \) formally symmetric expressions, whose eigenfunctions become \( 2\pi \)-periodic for \( L_{\text{per}} \). Eigenvalue estimates and can also be developed for certain nonselfadjoint operators on \( H_\beta \).

Corollary 2.9 showed that the highest order term of a formally symmetric expression on \( H_\beta \) has the form

\[
p_N(z) D^N = \left(a_N z^N + \sum_{j=0}^{N-1} [a_j z^j + \overline{a}_j z^{2N-j}] \right) d^N, \quad a_N = \overline{a}_N.
\]

Using the polar form \( a_j = |a_j| e^{i\phi_j} \), the corresponding expression on \( \mathbb{R} \) is

\[
\left(a_N e^{iN\theta} + \sum_{j=0}^{N-1} [a_j e^{i\theta} + \overline{a}_j e^{i(2N-j)\theta}] \right)(-\imath e^{-i\theta} \frac{d}{d\theta})^N.
\]

Moving the derivatives to the right and displaying the highest order terms gives

\[
(-\imath)^N P_N(\theta) \frac{d^N}{d\theta^N} + \cdots = (-\imath)^N \left(a_N + \sum_{j=0}^{N-1} 2|a_j| \cos([N-j]\theta - \phi_j) \right) \frac{d^N}{d\theta^N} + \cdots.
\]
Note that $P_N(\theta)$ is a real-valued function.

The periodic expressions $L_{\text{per}}$ coming from formally symmetric expressions $L$ on $H_\beta$ are typically not formally symmetric on $L^2_{\text{per}}$. In the first order case, formally symmetric expressions on $H_\beta$ have the form

$$L = [a_2 z^2 + a_1 z + \bar{a}_2]D + \sigma a_2 z + b_0, \quad a_1 = \bar{a}_1, \quad b_0 = \bar{b}_0.$$ 

With $a_2 = |a_2|e^{i\phi}$, the corresponding periodic expressions are

$$L_{\text{per}} = -2i\left[\frac{a_1}{2} + |a_2| \cos(\theta + \phi)\right] \frac{d}{d\theta} + \sigma a_2 e^{i\theta} + b_0.$$ 

The term $\sigma a_2 e^{i\theta}$ is typically not real-valued. Also note that the leading coefficient may have zeros. A simple second order example starts with

$$L = c_1(zD)^2 + c_2 zD^2 + \overline{c}_2(z^2D + \sigma z)zD, \quad c_1 = \overline{c}_1.$$ 

With $c_2 = |c_2|e^{i\phi}$, the corresponding periodic expression is

$$L_{\text{per}} = [-c_1 - 2|c_2| \cos(\theta - \phi)] \frac{d^2}{d\theta^2} + i(c_2 e^{-i\theta} - \overline{c}_2 e^{i\theta}) \frac{d}{d\theta}.$$ 

### 4.2 $\mathcal{D} -$-regular expressions

Eigenvalue estimates for perturbations of self-adjoint or normal operators often depend on estimates for the separation of the eigenvalues of the unperturbed operator. In anticipation of such an argument, an elementary number theoretic result is needed.

**Lemma 4.1.** Suppose $\tau > 0$, $C \in \mathbb{C}$, $N \geq 2$ is an integer, and

$$\gamma_n = (n/\tau + C)^N, \quad n = 0, \pm 1, \pm 2, \ldots.$$ 

There is a $K > 0$ such that for all $n$ with $|n|$ sufficiently large,

$$|\gamma_n - \gamma_m| \geq Kn^{N-1}, \quad \gamma_m \neq \gamma_n. \quad (4.1)$$

If $2\tau C$ is an integer, $N$ is even, and $|n|$ is sufficiently large, then there is a unique $m \neq n$ with $\gamma_m = \gamma_n$. If $|n|$ is sufficiently large and either $2\tau C$ is not an integer or $N$ is odd, there is no $m \neq n$ with $\gamma_m = \gamma_n$. 

27
Proof. The difference $\gamma_n - \gamma_m$ can be written

$$\tau^{-N}[(n + C_1)^N - (m + C_1)^N], \quad C_1 = \tau C,$$

so it suffices to consider the differences $(n + C_1)^N - (m + C_1)^N$. Notice that $2\tau C$ is an integer exactly when $2C_1$ is an integer.

When $|n|$ large, $|(n + C_1)^N|$ is a strictly increasing function of $|n|$. If $N$ is even and $2C_1$ is an integer, then $\gamma_n = \gamma_m$ for $m = -n - 2C_1$. The monotonicity means there is a unique $m \neq n$ with $\gamma_m = \gamma_n$. Also,

$$\lim_{n \to +\infty} \frac{\gamma_{n+1} - \gamma_n}{n^{N-1}} = \lim_{n \to +\infty} \frac{1}{n^{N-1}} \left[ (n + 1)^N(n + 1 + C_1)^N - n^N(n + C_1)^N \right]$$

$$= \lim_{n \to +\infty} \frac{1}{n^{N-1}} \left[ Nn^{N-1} + NC_1(n + 1)^{N-1} - NC_1n^{N-1} \right] = N,$$

establishing (4.1) when $N$ is even and $2C_1$ is an integer. Simple modifications of this argument also establish the lemma if $N$ is odd.

To handle the cases when $N$ is even and $2C_1$ is not an integer, assume now that $z = n+C_1$, $w = m+C_1$, and $|n|$ is large enough that $|n+C_1| \geq (|n|+1)/2$. If in addition $|z| - |w| \geq 1$, then

$$|z^N - w^N| \geq |z|^N - |w|^N = \left| \int_{|w|}^{|z|} N x^{N-1} \, dx \right| \geq N\left(\frac{|n| - 1}{2}\right)^{N-1}.$$ 

The remaining values of $m, n$ to consider must satisfy $-1 < |n + C_1| - |m + C_1| < 1$, implying $-1 - 2|C_1| < |n| - |m| < 1 + 2|C_1|$. In other words, there is a constant $K$ such that (4.1) is satisfied except possibly for

$$m = n - K, \ldots, n + K, \quad \text{or} \quad m = -n - K, \ldots, -n + K.$$ 

Use the identity

$$x^N - y^N = (x - y) \sum_{j=0}^{N-1} x^j y^{N-1-j},$$

to get

$$\gamma_n - \gamma_m = (n + C_1)^N - (m + C_1)^N = (n - m) \sum_{j=0}^{N-1} (n + C_1)^j(m + C_1)^{N-1-j}.$$ 

28
When \( m = n + k \) with \( k \neq 0 \),
\[
\gamma_n - \gamma_m = k \sum_{j=0}^{N-1} (n + C_1)^j (n + k + C_1)^{N-1-j}.
\]
The sum is a polynomial of degree \( N - 1 \) in \( n \) with highest order term \( kNn^{N-1} \).

Next take \( m = -n + k \), where
\[
\gamma_n - \gamma_m = (2n - k) \sum_{j=0}^{N-1} (n + C_1)^j (-n + k + C_1)^{N-1-j}.
\]

Since \( N \) is even, \( \gamma_n - \gamma_m \) is a polynomial in \( n \) of degree at most \( N - 1 \), with the degree \( N - 1 \) term,
\[
2n \sum_{j=0}^{N-1} (-1)^{N-1-j} j C_1 n^{N-2} + \sum_{j=0}^{N-1} (-1)^{N-2-j} (N - 1 - j)(k + C_1) n^{N-2} \]
\[
= 2n \sum_{j=0}^{N-1} (-1)^{N-1-j} j C_1 n^{N-2} + (k + C_1) \sum_{j=0}^{N-1} (-1)^{N-1-j} j n^{N-2} \]
\[
= 2(k + 2C_1) n^{N-1} \sum_{j=0}^{N-1} (-1)^{N-1-j} j = 2(k + 2C_1) n^{N-1} [(N - 1) - (N - 2)/2].
\]

Since \( k + 2C_1 \neq 0 \), the inequality (4.1) holds for large enough \( |n| \) in each of the cases \( m = n + k \) and \( m = -n + k \) for \(-K \leq k \leq K\), completing the proof.

The next theorem will provide detailed information about the eigenvalues of a class of \( D \) – regular maximal operators. Motivated by the self-adjoint case, the main hypothesis describes well-behaved polynomial leading coefficients \( p_N(z) \). By Corollary 2.9, if \( L \) is a formally symmetric expression of order \( N \), the nonzero roots \( z_j \) of \( p_N(z) \) are closed under the map \( z_j \to 1/z_j \). If in addition \( L \) is \( D \) – regular then the maximal operator must have Fredholm index 0, so by Theorem 3.13 there must be \( N \) roots \( z_j \in \mathbb{D} \).

Given \( N \geq 1 \), say that a polynomial \( p_N(z) \) is \( R \) – symmetric if \( p_N(z) \) has degree at most \( 2N \), and the following conditions are satisfied:
(i) $p_N(z)$ has $N$ roots $z_m \in \mathbb{D}$, with roots $z_1, \ldots, z_M$ not equal to zero and listed with multiplicity, and $z = 0$ a root of multiplicity $N - M$.

(ii) the remaining roots of $p_N(z)$ are $1/z_1, \ldots, 1/z_M$.

If $p_N(z)$ is $R$-symmetric then there is a nonzero constant $C_1 \in \mathbb{C}$ such that

$$p_N(z) = C_1 \rho_1(z) \cdots \rho_M(z) z^{N-M}, \quad \rho_m(z) = \bar{z}_m (z - z_m) (z - 1/z_m). \quad (4.2)$$

Note that if $z_m = |z_m|e^{i\phi_m}$, and $z = e^{i\theta}$ lies on the unit circle, then

$$\rho_m(e^{i\theta}) = |z_m|e^{-i\phi_m} e^{2i\theta} - (|z_m|^2 + 1)e^{i\theta} + |z_m|e^{i\phi_m}$$

$$= e^{i\theta} [2|z_m| \cos(\theta - \phi_m) - (|z_m|^2 + 1)].$$

That is, $e^{-i\theta} \rho_m(e^{i\theta})$ is a real-valued nonvanishing function of $\theta \in \mathbb{R}$. Similarly, $p_N(z)$ has no roots on the unit circle, and $C_1^{-1} e^{-iN\theta} p_N(e^{i\theta})$ is real-valued for $\theta \in \mathbb{R}$.

**Theorem 4.2.** Suppose $L = \sum_{k=0}^N p_k(z) D^k$ is $\mathbb{D}$-regular expression of order $N \geq 2$, whose leading coefficient $p_N(z)$ is $R$-symmetric. Let $\mathcal{L}$ be the corresponding maximal operator.

The eigenvalues of $\mathcal{L}$ are isolated points of finite algebraic multiplicity. For any $\epsilon$ satisfying $0 < \epsilon < 1$, each eigenvalue $\lambda$ is an element of a sequence \( \{\mu_n, n = 0, \pm 1, \pm 2, \ldots\} \) having the form

$$\mu_n/C_1 = (n/\tau + C_2)^N + O(n^{N-2+\epsilon}),$$

with constants $C_1 \neq 0$, $\tau > 0$, and $C_2$.

**Proof.** Since multiplication of $\mathcal{L}$ by a constant $C_1$ simply multiplies eigenvalues by $C_1$, it suffices to assume that $p_N(z)$ has the form \[12\] with $C_1 = 1$. The change of variables $z = e^{i\theta}$ with $\theta \in \mathbb{R}$ changes $L$ to $L_{\text{per}} = \sum_{k=0}^N p_k(e^{i\theta})(-ie^{-i\theta} \frac{d}{d\theta})^k$, with $L_{\text{per}}$ acting on $L^2_{\text{per}}$. If $\psi(z)$ is an eigenfunction of $\mathcal{L}$, then $\psi(e^{i\theta})$ is a $2\pi$-periodic eigenfunction for $L_{\text{per}}$. (The same remark applies to generalized eigenfunctions $\psi$ satisfying an equation $(\mathcal{L} - \lambda I)\psi = 0$.) The proof will proceed by showing that the sequence \( \{\mu_n\} \) describes the larger set of eigenvalues of $2\pi$-periodic eigenfunctions for $L_{\text{per}}$.

The leading coefficient $L_{\text{per}}$ is $(-i)^N p_N(\theta) = (-i)^N e^{-iN\theta} p_N(e^{i\theta})$. As noted above, $P_N(\theta)$ is real-valued with no zeros for $\theta \in \mathbb{R}$. Absorbing the sign in $C_1$ if necessary, assume that $P_N(\theta) > 0$. Conventional reductions [2, p. 308-9] are available. First use the change of real variables

$$t = \tau^{-1} \int_0^\theta P_N(s)^{-1/N} \, ds, \quad \tau = \frac{1}{2\pi} \int_0^{2\pi} P_N(s)^{-1/N} \, ds,$$
which carries $2\pi$-periodic functions of $\theta$ to $2\pi$-periodic functions of $t$. Since $P_N(\theta)^{1/N} d/d\theta = \tau^{-1} d/dt$, the new expression has the form

$$L_t = (-i\tau^{-1} d/dt)^N + \sum_{k=0}^{N-1} \tilde{p}_k(t)(-i\tau^{-1} d/dt)^k,$$

with $2\pi$-periodic coefficients $\tilde{p}_k(t)$.

Next, let

$$r(t) = \exp\left(\int_0^t -i\tau C + i\tau \frac{\tilde{p}(s)}{N} ds\right), \quad C = \frac{1}{2\pi} \int_0^{2\pi} \frac{\tilde{p}(s)}{N} ds.$$

The function $r(t)$ is periodic with period $2\pi$, and conjugation with $r(t)$ leaves an expression

$$L_r = r^{-1}(t)L_t r(t) = (-i\tau^{-1} d/dt + C)^N + \sum_{k=0}^{N-2} q_k(t)(-i\tau^{-1} d/dt)^k, \quad C = C/N.$$

The coefficients $q_k(t)$ are also periodic with period $2\pi$. The eigenvalues of $2\pi$-periodic eigenfunctions for $L_r$ are the same as those for $L_{per}$.

The expression $(-i\tau^{-1} d/dt)$ has $2\pi$-periodic eigenfunctions $\exp(i\nu t)$ with eigenvalues $n/\tau$, $n = 0, \pm 1, \pm 2, \ldots$. These eigenfunctions form an orthogonal basis for $L^2_{per}$. Let $L_0$ denote the normal operator given by the expression $(-i\tau^{-1} d/dt + C)^N$ with these eigenfunctions. Let $L_p$ be given by the expression $L_p = \sum_{k=0}^{N-2} q_k(t)(-i\tau^{-1} d/dt)^k$ on the domain of $L_0$, and take $L_r = L_0 + L_p$. The eigenvalues of $L_0$ are $\gamma_n = (n/\tau + C)^N$ whose behavior is described in Lemma 4.1. Suppose $0 < \epsilon < 1$, $R_\epsilon = n^{-2+\epsilon}$, and $S_\epsilon$ is the circle $S_\epsilon = \{ \zeta \in \mathbb{C} \mid |\zeta - \gamma_n| = R_\epsilon \}$. Then for $|n|$ sufficiently large, there is a $K > 0$ such that $|\zeta - \gamma_m| \geq Kn^{N-1}$ for all $\zeta \in S_\epsilon$ and $\gamma_m \neq \gamma_n$.

Let $R_r(\lambda) = (L_r - \lambda I)^{-1}$ be the resolvent of $L_r$, and let $R_0(\lambda)$ be the resolvent of $L_0$. Consider the formula

$$(L_r - \lambda I)^{-1} = (L_0 + L_p - \lambda I)^{-1} = (L_0 - \lambda I)^{-1}(I + L_p(L_0 - \lambda I)^{-1})^{-1} \quad \text{(4.3)}$$

with

$$L_p(L_0 - \lambda I)^{-1} = \sum_{k=0}^{N-2} q_k(-i\tau^{-1} d/dt)^k(L_0 - \lambda I)^{-1}.$$
Use an expansion $f = \sum_{j=-\infty}^{\infty} b_j \exp(i\pi t)$ for $f \in L^2_{\text{per}}$ to compute

$$(-i\pi^{-1} d/dt)^k (L_0 - \lambda I)^{-1} f = \sum_{j=0}^{\infty} b_j \frac{(n/\pi)^k}{\gamma_j - \lambda}.$$ 

If $\lambda \in S_n$ and $0 \leq k \leq N - 2$, then $\|(-i\pi^{-1} d/dt)^k (L_0 - \lambda I)^{-1}\| = O(n^{-1})$ as $|n| \to \infty$. Similarly, since multiplication by $q_k$ is a bounded operator on $L^2_{\text{per}}$, $\|L_p (L_0 - \lambda I)^{-1}\| = O(n^{-1})$ as $|n| \to \infty$.

Assume $\lambda \in S_n$. Since $L_0$ is normal [4, p. 277],

$$\|R_0(\lambda)\| = \max_{\gamma_n} d(\lambda, \gamma_n)^{-1} = O(n^{-N+2-\epsilon}).$$

By (4.3), the circles $S_n$ are in the resolvent set of $L_r$ for $|n|$ sufficiently large, and $\|R_r(\lambda) - R_0(\lambda)\| = O(n^{-N+1-\epsilon})$. Recall [4, p. 67] that the difference of the $L_r$ eigenprojections $P_{r,n}$ and the $L_0$ eigenprojections $P_{0,n}$ for eigenvalues inside $S_n$ is given by

$$P_{r,n} - P_{0,n} = -\frac{1}{2\pi i} \int_{S_n} R_r(\lambda) - R_0(\lambda) \, d\lambda.$$ 

Since $\|P_{r,n} - P_{0,n}\| = O(n^{-1})$, the algebraic multiplicity of the eigenvalues of $L_r$ contained in $S_n$ is the same as for $L_0$ when $|n|$ is large.

Theorem 4.2 does not distinguish between the eigenvalues coming from eigenfunctions of $L$ on $H_\beta$ and the larger set of eigenvalues from $L^2_{\text{per}}$. By working in $H_\beta$, the results of Theorem 4.2 can be refined.

Lemma 4.3. Suppose $N \geq 1$ and the polynomial $p_N(z)$ is $R -$symmetric. For each $\sigma > 0$ there is a constant $C \neq 0$ and a formally symmetric expression $L$ of order $N$ on $H_\beta$ whose leading coefficient is $Cp_N(z)$.

Proof. As in (4.2), the polynomial $p_N(z)$ has the form

$$p_N(z) = C_1 \rho_1(z) \cdots \rho_M(z) z^{N-M},$$

with each factor $\rho_m(z)$ of the form

$$\rho_m(z) = \overline{\rho_m}(z - z_m)(z - 1/\overline{z_m}) = \overline{\rho_m} z^2 - (|z_m|^2 + 1) z + z_m. \quad (4.4)$$
An induction proof will show that if \( q(z) = \rho_1(z) \cdots \rho_M(z) \), then
\[
q(z) = c_M z^M + \sum_{j=0}^{M-1} [c_j z^j + \overline{c_j} z^{2M-j}], \quad c_M = \overline{c_M},
\]
with the case of one factor established in (4.4). Suppose the result holds for \( q(z) \) with \( M \) factors. Then
\[
\rho_{M+1}(z)q(z) = \left[ z_{M+1}^2 - (|z_{M+1}|^2 + 1)z_{M+1} \right] \left[ c_M z^M + \sum_{j=0}^{M-1} [c_j z^j + \overline{c_j} z^{2M-j}] \right].
\]
The coefficient of \( z_{M+1}^{M+1} \) is \( z_{M+1} c_{M-1} - (|z_{M+1}|^2 + 1) c_M + z_{M+1} \overline{c_{M-1}} \), which is real. For \( j < M + 1 \) the coefficient of \( z^j \) is \( z_{M+1} c_{j-2} - (|z_{M+1}|^2 + 1) c_{j-1} + z_{M+1} \overline{c_j} \), while the coefficient of \( z^{2(M+1)-j} \) is \( z_{M+1} \overline{c_{j-2}} - (|z_{M+1}|^2 + 1) c_{j-1} + z_{M+1} \overline{c_j} \), preserving the desired symmetry.

Finally, the coefficients of \( p_N(z) = \rho_1(z) \cdots \rho_M(z) z^{N-M} \) are obtained from those of \( \rho_1(z) \cdots \rho_M(z) \) by an index shift. By Lemma 2.10 and Theorem 2.11, there is a formally symmetric expression \( L \) of order \( N \) on \( H_\beta \) whose leading coefficient is \( \rho_1(z) \cdots \rho_M(z) z^{N-M} \).

Recall that if \( L = \sum_{k=0}^{N} p_k(z) D^k \) is a \( \mathbb{D} \) – regular formally symmetric expression of order \( N \geq 1 \) on \( H_\beta \), then by Theorem 3.9 the domain of the maximal operator \( L \) is \( H_\beta^N \), and \( L \) is self-adjoint with compact resolvent by Theorem 3.11.

**Theorem 4.4.** Suppose \( L = \sum_{k=0}^{N} p_k(z) D^k \) is a \( \mathbb{D} \) – regular formally symmetric expression of order \( N \geq 2 \) on \( H_\beta \), with self-adjoint maximal operator \( L \). The eigenvalues of \( L \) can be enumerated as a sequence \( \{ \lambda_n, n = 0, 1, 2, \ldots \} \) with the following description: for \( 0 < \epsilon < 1 \) there are real nonzero constants \( C_1 \) and \( \tau > 0 \), and a \( C_2 \in \mathbb{C} \) such that
\[
\lambda_n/C_1 = (n/\tau + C_2)^N + O(n^{N-2+\epsilon}).
\]
In particular, with at most finitely many exceptions, the eigenvalues are either all positive or all negative, and have multiplicity 1.

**Proof.** Let \( z_1, \ldots, z_M \) be the nonzero roots of the leading coefficient \( p_N(z) \) inside \( \mathbb{D} \). By Corollary 2.9 there are \( M \) factors \( \rho_m(z) \) as in (4.4) such that \( p_N(z) = C_1 \rho_1(z) \cdots \rho_M(z) z^{N-M} \). Without loss of generality we may take
$C_1 = 1$. The main idea is to construct and study a one parameter family of symmetric $D$-regular differential expressions $L(t)$ connecting $L$ with the elementary operator $(zD)^N$.

For $0 \leq t \leq 1$ and each $\rho_m(z)$, define a one parameter family of polynomials by

$$
\rho_m(t, z) = t\frac{z_m(z - tz_m)(z - \frac{1}{t z_m})}{\gamma_m} = t z_m z^2 - (t |z_m|^2 + 1)z + tz_m.
$$

These polynomials maintain the form (4.4), deforming $\rho_m(z)$ to $-z$ while keeping exactly one root inside $D$. Now define

$$
p_N(t, z) = \rho_1(t, z) \cdots \rho_M(t, z) z^{N-M}.
$$

By Lemma 4.3 there is a the one parameter family of formally symmetric expressions $L(t)$ with $L(1) = L$ and with $L(t)$ having leading coefficient $p_N(t, z) = \rho_1(t, z) \cdots \rho_M(t, z) z^{N-M}$. Let $L(t)$ denote the self-adjoint maximal operator with expression $L(t)$.

For $0 \leq t_0 \leq 1$, let $L_0 = L(t_0)$ and $L_p(t) = L(t) + L(t_0)$. By Theorem 3.11, the resolvent $R_0(\lambda) = (L(t_0) - \lambda I)^{-1}$ is bounded from $\mathcal{H}_\beta$ to $\mathcal{H}_\beta^N$ uniformly on compact subsets of the resolvent set. The perturbation formula

$$
(L(t) - \lambda I)^{-1} = (L_0 + L_p(t) - \lambda I)^{-1} = (L_0 - \lambda I)^{-1}(I + L_p(t)(L_0 - \lambda I)^{-1})^{-1}
$$

shows that $L(t) : \mathcal{H}_\beta \to \mathcal{H}_\beta$ is a continuous operator valued function of $t$. In particular the eigenvalues of $L(t)$, vary continuously with $t$.

Taking advantage of Theorem 4.2 for each $0 \leq t \leq 1$ there is a sequence $\gamma_n(t) = (n/\tau(t) + C_2(t))^N$ and circles $S_n(t) = \{ \zeta \in \mathbb{C} \mid |\zeta - \gamma_n(t)| = n^{N-2+\epsilon}\}$ containing all eigenvalues $\lambda_n(t)$. The circles are pairwise disjoint for $n$ large. Since the eigenvalues $\lambda_n(t)$ and resolvents are continuous functions, there is a cover of $[0,1]$ by open intervals $I_t$ centered at $t$ such that for $t_1 \in I_t$ all eigenvalues of $L(t)$ are contained in some $S_n(t)$, and the number of eigenvalues in $S_n(t_1)$ agrees with the number for $S_n(t)$ if $n$ is sufficiently large. Taking a finite subcover, and using the fact that $L(0)$ has expression $(zD)^N$, with eigenvalues $n^N$ for $n = 0, 1, 2, \ldots$, it follows that for $n$ large there is one eigenvalue $\lambda_n$, counted with multiplicity, in $S_n$. 

\[\square\]
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