Experimental Decoy State Quantum Key Distribution with Unconditional Security Incorporating Finite Statistics

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(Dated:)

We propose the improved decoy state quantum key distribution incorporating finite statistics due to the finite code length and report on its demonstration. In our experiment, four different intensities including the vacuum state for optimal pulses are used and the key generation rate of 200 bps is achieved in the 20 km telecom optical fiber transmission keeping the eavesdropper’s mutual information with the final key less than $2^{-9}$.

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Quantum key distribution (QKD) was originally proposed by Bennett and Brassard in 1984 \cite{1} as a protocol, by which two parties, Alice and Bob, share secret keys by using a quantum communication channel as well as a public classical channel \cite{2}. A remarkable feature is its unconditional security \cite{3}: it is guaranteed by the fundamental laws of quantum mechanics and thereby QKD provides the unconditionally secure communication system. In the practical setting of optical communication, however, it is the almost only option to substitute qubits in the original BB84 QKD protocol with heavily attenuated laser pulses because the perfect single photon emitting devices are not available in the current technology. Such laser pulses - the phase randomized weak coherent states - contain inevitably the multiphoton states at small but finite probability, which give a malicious eavesdropper (Eve) a chance to obtain some amount of information on the shared keys by a photon-number-splitting attack \cite{4}. Gottesman-Lo-Lütkenhaus-Preskill (GLLP) showed, however, that it is still possible to obtain unconditionally secret key by BB84 protocol with such imperfect light sources, although the key generation rate and distances are very limited \cite{5}.

The recently proposed decoy state method \cite{6, 7, 8, 9} is one of the promising practical solutions to BB84 QKD with coherent state light pulses, in which several coherent state light pulses with different intensities are used. Such optical pulses with different intensities have different photon number statistics. This simple fact equips Alice and Bob with a countermeasure against Eve. The original idea of the decoy state QKD is due to Hwang \cite{6}. So far, several experimental demonstrations of decoy state QKD have been reported \cite{10, 11, 12, 13, 14}. In most cases, the security analysis is based on the GLLP’s asymptotic arguments, whereas, in the practical setting, the code length is finite so that the asymptotic argument is no longer valid and the unconditonal security is actually not guaranteed any more. The security analysis of QKD with the finite code size must incorporate the statistical fluctuations of the observed quantities \cite{15}. Although several authors \cite{7, 8, 9, 11, 16} have considered the influence of statistical fluctuations on the decoy state QKD with finite code length, what all of them have done is limited to the re-adjustment of parameters of the asymptotic GLLP’s formula for the secure key generation rate. Such an \textit{ad hoc} treatment cannot be justified to claim the unconditional security.

In this paper, we propose a substantially improved decoy state QKD in the framework of finite coding length \cite{17} and report on its experimental results. In our experiment, we employ the decoy state method with three decoy pulses and demonstrate that in the 20 km optical fiber transmission, the final key was successfully generated at the rate 200 bps keeping the eavesdropper’s mutual information with the final key less than $2^{-9}$.

In our protocol, we use $k+1$ different intensities or mean photon numbers $\mu_0 = 0 < \mu_1 < \ldots < \mu_k$ including vacuum ($\mu_0$) for the optical pulses. We treat conjugate bases ($+$ and $\times$) separately so that $2k+1$ different pulses are involved in total. The vacuum state ($i = 0$) is sent at the probability $\overline{P}_0$ and the $\mu_k$ pulse with $\times$ ($+$) basis is sent at the probability $\overline{P}_k$ ($\overline{P}_{-k}$) ($i = 1, \ldots, k$). The pulse with intensity $\mu_{i_0}$ ($\mu_{i_0+k}$) (the signal pulse) is used to distill the final secret key and the remaining (decoy pulses) are used just for estimation of Eve’s attacks and/or the noise characteristics of quantum channel.

The code length (size of raw key) is denoted by $N$ and the time slot to generate a final secure key is denoted by $T$. We also fix the maximum number $N$ of the size of final key. Our
protocol is as follows. Within the time slot $T$, Alice randomly sends Bob a sequence of optical pulses of $k+1$ different intensities with randomly chosen basis. After that, Bob performs a measurement in one of the two bases and Alice and Bob compare bases and keep the pulses with a common basis by communicating via public channel. The number of sending pulses, received pulses, and pulses of a common basis are denoted by, respectively, $A_i$, $C_i$, and $E_i$ ($i = 0, \ldots, 2k$). The $E_i$ bit string of $i$th pulse contains error bits, which will be detected by comparing a portion of the bits (check bits). To prepare check bits, Alice and Bob firstly perform the random permutation on $E_{i_0}$ and $E_{i_0+k}$ bit strings by sharing common random numbers via public channel. Then, for $i = i_0$ and $i = i_0 + k$, the first $N$ bit string is used as the raw key and the remaining $E_{i_0} - N$ and $E_{i_0+k} - N$ bit string are used as the check bits, while the whole $E_i$ bits are used as check bits for $i \neq i_0, i_0 + k$. (If $E_{i_0} \leq N$ or $E_{i_0+k} \leq N$, then the protocol is aborted.) The number of detected errors of $i$th pulse is denoted by $H_i$ ($i = 1, \ldots, 2k$). From these quantities, Alice and Bob can evaluate the size of the final key guaranteeing the unconditional security. If the evaluated final key size is not positive, the protocol is aborted again. The final secret key $N_{\text{final}}$ is computed as $N\eta(H_{i_0+k}/(E_{i_0+k} - N)) - m_{\max}$ for + basis and $N_{\eta}(H_{i_0}/(E_{i_0} - N)) - m_{\max}$ for $\times$ basis, where $\eta(\cdot)$ denotes the error correcting coding rate and $m_{\max}$ ($m_{\max}$) represents the size of privacy amplification. If $N < N_{\text{final}}$ ($N < N_{\text{final}}$), they replace $N_{\text{final}}$ ($N_{\text{final}}$) by $N$.

The error correction (or reverse reconciliation in our protocol) is performed as follows. Suppose that Alice and Bob have, respectively, the random number sequences $X$ and $X'$ of $n$ bits, which contain some errors. The task is to distill the common random number sequence of $l$ bits with negligible errors. Let $G$ be the generator matrix of $[n, l]$ classical error correcting code. Bob generates the random sequence number $Z \in \{0, 1\}^l$ and sends the bit string $GZ + X'$ to Alice. Then Alice decodes $GZ + X' - X$ to extract $Z$. For the classical error correcting code, Low Density Parity Check (LDPC) code [18] is used. The advantage of LDPC code is that the decoding with $O(n)$ operations is possible by using Sum-Product decoding method [18], where $n$ is the coding length. Furthermore, the coding rate achieves the Shannon limit asymptotically. In the privacy amplification, we use the universal hash function. More specifically, we use $(l - m) \times l$ Toeplitz matrix $M_p$ [19] to subtract $m$ bit information from the original information of $l$ bits.

Now, let us describe the Eve’s possible attacks. To this end, we firstly define multiphoton states $\rho_i$ ($i = 2, \ldots, k + 1$) as

$$\rho_i = \Omega^{-1} \sum_{n=0}^{\infty} \left( \rho_{\mu_i}^{2} \right)^{n} | n \rangle \langle n |$$

with

$$\gamma_{i,n} = \sum_{j=1}^{l-1} \frac{\left( \mu_{j+1} - \mu_{j-1} \right) \cdots \left( \mu_{j+1} - \mu_{j} \right) \mu_{j+1} \mu_{j+2}}{(\mu_{j} - \mu_{j-1}) \cdots (\mu_{j} - \mu_{j+1}) (\mu_{j-1} - \mu_{j-2}) \cdots (\mu_{j+1} - \mu_{j+2})},$$

$\Omega$ being the normalization constant and $\mu_1 < \mu_2 < \cdots < \mu_k$.

The phase-randomized coherent state, $e^{\mu_n} \sum_{m=0}^{\infty} \left( \rho_{\mu_n}^{2} \right)^{n} | m \rangle \langle m |$ can be expressed as a convex combination of $|0\rangle \langle 0 |, |1\rangle \langle 1 |$, and $\rho_i$. Here, we adopt the worst case scenario. Namely, we assume that Eve can distinguish vacuum state ($j = 0$), single photon state ($j = 1$), multiphoton states $\rho_2, \ldots, \rho_{k+1}$ with $\times$ basis ($j = 2, \ldots, k + 1$) and those with $\times$ basis ($j = k + 2, \ldots, 2k + 1$). The number of $i$th state ($j = 0, \ldots, 2k + 1$) is denoted by $B^i$. According to the values of $B^i$, Eve can do the following attacks; She tricks Bob into detecting the $i$th state with probability $q_i^0$ and causes phase errors with probability $\rho_i$ for the $j$th state ($j = 1, 2, \ldots, k + 1$) and bit errors with probability $r_j$ for the $j$th state ($j = 1, k + 2, \ldots, 2k + 1$). In the following, we focus on the $+ \times$ basis case. The detection ratio $p_i = C_i/A_i$ ($i = 0, \ldots, 2k$) is written in terms of $q_i$ as

$$p_i = \sum_{j=0}^{2k+1} \mu_j q_i^j + p_D, \quad (1)$$

where $p_D$ is the detector dark count rate and $\mu_j$ is the generation probability of the $j$th state given that the $i$th pulse is emitted. The error probability $s_i = H_i/E_i$ ($i \neq i_0, 1 \leq i \leq k$) and $s_0 = H_{i_0}/(E_{i_0} - N)$ satisfies [20]

$$s_i p_i = \sum_{j=1}^{k+1} \mu_j q_i^j q_j^i + \frac{1}{2} \left( \mu_j q_i^0 + p_D \right). \quad (2)$$

In our decoy state method, Alice and Bob try to estimate parameters $q_i^j$ and $r_j$ ($\rho_i$) from the observed quantities $C_i$, $E_i$, and $H$ to the best of their ability. The complete determination is, however, beyond their ability so they put the safety standards most stringent. The computed size of privacy amplification is thus given by

$$m_{\max} = \max_{0\leq x \leq \sqrt{(1-p_D)} \leq \leq 1} m(x, y), \quad (3)$$

where

$$m(x, y) = \frac{Nq_1(x) \left[ H_1(y, x) - 1 \right]}{C_{i_0+k}} + \frac{1}{2} \left( \left( \Phi^1 \left( 2^{-y} \right) \right) \left( \nu_{x, y} \left( q_M, x, y \right) + \nu_{M, x} \left( q_M, x, y \right) \right) - \Phi^1 \left( 2^{-y} \right) \right) + \delta_3. \quad (4)$$

with $\delta_{1,2,3}$ being security parameters. Here,

$$\Phi(x) = \frac{1}{2\pi} \int_{x}^{\infty} e^{-\frac{y^2}{2}} dy$$

and

$$H(x) = \begin{cases} 1 & \text{if } 1/2 < x \leq 1, \\ -x \log_2(x - (1 - x) \log_2(1 - x)) & \text{if } 0 \leq x \leq 1/2. \end{cases}$$

In Eq. (4), $x = (q_1 + q_2)/\sqrt{2}$ and $y = \mu^{-1}$. $q_1(x)$ and $r_1(x)$ are direct solutions of Eqs. (1) and (2) as a function of $x$ and $y$, while $q_M(x, y)$ and $r_M(x, y)$ are values of maximal likelihood estimation. Furthermore, $m_{\max}$ and $\nu_{x, y} \left( q_M, x, y \right)$ are, respectively, the mean and variance of the stochastic variable.
incorporated in the computation of the respective hypergeometric distributions, which are fully information is less than 2 of the LDPC coding length 1 so that Eve’s mutual information with the final key is guaranteed to be less than 2−9. The detail will be published elsewhere [21].

Experiment.— The key generation experiment was done with a 20 km-long optical fiber in a common office environment. The setup, working on the 62.5 MHz clock, was based on the "plug-and-play" QKD system used in the 14-days-continuous quantum key distribution experiment [22]. The wavelength of the light was 1.55 µm. The intensity of the optical pulses were randomly chosen with a dual-drive Lithium Niobate Mach-Zehnder intensity modulator [23]. The modulator provides phase shifts simultaneously to define the four BB84 states, as the output wave is given by $E_{out} = E_{in} \cos[(\phi_1 - \phi_2)/2] \exp[i(\phi_1 + \phi_2)/2]$, where $\phi_1$ and $\phi_2$ are the phase shifts in the two arms of Mach-Zehnder interferometer. We employed "alternative-shifted-phase-modulation" to use a polarization dependent device [24]. A PC with Pentium(R)4 (3GHz) CPU and 2 GB memory was connected to the QKD apparatus to perform the error correction and privacy amplification. The random numbers used in the experiment were generated by a physical random number generation hardware.

Before running the protocol, the detector dark count rate $p_D$ must be determined in advance, which was measured as 3.00 × 10−4 per pulse. We used pulses with three different finite intensities $\mu_1, \mu_2$, and $\mu_3$ in addition to the vacuum state ($k = 3$) because at least four different intensities including vacuum are needed to achieve the almost optimal secure key generation rate asymptotically [25]. The final key was distilled from the $\mu_3$ pulses sequence. We set $\mu_3 = 0.5$. This is due to the observation that at the limit $\mu_1, \mu_2 \rightarrow 0$, the choice of $\mu_c = 0.5$ yields the best final key generation rate asymptotically [25]. The others are set to be $\mu_1 = 0.07$ and $\mu_2 = 0.35$, which are the smallest two values available in our system. The security parameters were set to be $\delta = 9$ so that Eve’s mutual information with the final key is guaranteed to be less than 2−9 [21].

The size of LDPC code was 1.0 × 104. The Sum-Product method is an approximate decoding one so that there is a finite probability of decoding without converging. In our case, the unconverging probability was about 3 × 10−3. If the decoding is unsuccessful, the bit string must be discarded, but after the successful decoding, the bit error rate was reduced to as small as 1.0 × 10−10. The coding length $N$ was set to be 10 times of the LDPC coding length 1.0 × 104 ($N = 1.0 \times 10^5$) and the error correction was performed on each 104 bit block of $N$.

We performed the above-described QKD experiment for 40 rounds. The sum of the key generation rate of + and × bases is shown in Fig 1. We set the time slot $T = 41.8$ sec and $A_0 : A_1(= A_4) : A_2(= A_3) : A_3(= A_6) = 0.125 : 0.1875 : 0.0625 : 0.1875$. We also fix $N = 2^{17}$ which maximizes the average of the key generation rates. The number of received pulses $C_i$ and the check bits for the coding length $N = 1.0\times10^5$ are listed in Table I. When the error rate on (+) basis was 5.2% (6.1%), the final secret key of 8.2×103 bit was generated and the generation rate was around 200 bps. In this case the raw key of $N = 1.0\times10^5$ bit was reduced to the size of 5.6×104 bit by error correction and further to the size of 4.1 × 102 bit by privacy amplification on each basis. If the error rates were more than 6.5%, then few final secret key was left. Although the key generation rates obtained in our experiments are not so large, the most important point is that the final keys are guaranteed to be unconditionally secure in our decoy method while previously reported ones [10, 11, 12, 13, 14] are not. The small values of the key generation rates are due to the statistical fluctuations of observed quantities which are never negligible even if the code length is 1.0 × 105.

Discussion on experimental parameters.— Many adjustable parameters are involved in our protocol and some of them are in the trade-off relation. In the following, we clarify, albeit qualitatively, the dependence of these parameters on our decoy state QKD experiment for further improvements.

Firstly, the sending probability of the signal pulse must be large enough to ensure $E_3, E_6 \geq N$. We also performed the same experiment but under the condition $A_0 : A_1 (= A_4) : A_2 (= A_3) : A_3 (= A_6) = 0.125 : 0.125 : 0.125$ and the resulting generation rate of the secret key was around 50 bps, which is less than that of Fig 1. However, the sending probability of pulses with intensities $\mu_1$ and $\mu_2$ also must not be
too small. Otherwise, the statistical error of estimating Eve’s parameters would become large reducing the size of final key. After all, our choice of sending probabilities that check bits for the intensity 0.07 are as much as those for the intensity 0.35 in Table [I] is quite favorable, although it is not optimized.

Secondly, the processing time of each step in our protocol affects directly the key generation rate. The most time-consuming process is the sharing of common random numbers via quantum channel. It took around 42 sec in the experiment to obtain the results of Fig [1]. The second one is the computation of the size of privacy amplification, which requires the nested numerical optimization and took around 10 sec. The generation of physical random numbers also took around 10 sec, but they did not degrade the total performance since they were performed in a parallel manner. The random permutation, error correction, and privacy amplification were the least time-consuming processes; they took only a few seconds.

Thirdly, the length of the time slot \( T \) is also one of the adjustable parameters. The numbers of received pulses \( C \), pulses with a common basis, and the check bits are proportional to \( T \). When these numbers are too small the contribution of statistical fluctuation of the right-hand side of Eq. [4] becomes large reducing the size of the final secret key. Thus, \( T \) must be large enough. However, too large \( T \) merely results in low key generation rate since the parameter estimation is not improved any longer for sufficiently large value of \( T \). Thus, we can expect there is some optimal value of \( T \).

Under the same conditions of our experiment, the largest key generation rate is expected to be achieved when the smallest size of check bits (for the intensity 0.07 in Fig [1]) is around 85% of the coding length \( N \approx 1.0 \times 10^5 \).

Finally, let us briefly discuss the size of random numbers used in our experiment. The random numbers, which are all physical random numbers, are used for random permutation, error correction, and the generation of Toeplitz matrix. Among them, the random permutation is most demanding; it requires the random numbers of the size \( O(N \log N) \approx 1.7 \times 10^6 \) bits since each bit in \( E_3 \) is randomly exchanged.

The overall performance of decoy state QKD could be improved if we used the unidirectional QKD system [26, 27] instead of ‘plug-and-play’ system here. By doing so, a larger bit string of common random numbers would be available so that the better performance of error correction and therefore higher secret key distillation rate would be expected. In our experiment, the bottleneck of the key generation time is the common random number sharing via quantum channel. This would be resolved if the transmission rate of optical pulses was improved, however, the computational time of the size of privacy amplification would emerge as the second difficulty. The time of random number sharing via quantum channel is proportional to the coding length \( N \), while the time of generating physical random numbers is proportional to \( O(N \log N) \). Therefore, if \( N \) is too large, the generation of random numbers may affect seriously the total performance.

In summary, we have proposed and demonstrated the improved decoy state QKD incorporating finite statistics due to the finite code length. We employed the decoy state method with four different intensities including vacuum state for optical pulses and achieved the final key generation rate of 200 bps in 20 km telecom optical fiber transmission keeping the eavesdropper’s mutual information with the final key as small as 2^{-9}. We also discussed the dependence of several parameters on our QKD experiment.

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