APPLICATION OF A MODIFIED VES PRODUCTION FUNCTION MODEL

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Abstract. In the analyses on economic growth factors, researchers generally use the production function model to calculate the contribution rates of influencing factors to economic growth. The paper proposes a new modified VES production function model. As for the model’s parameter estimation, the conventional optimization methods are complicated, generally require information like the gradient of objective function, and have the poor convergence rate and precision. The paper gives a modern intelligent algorithm, i.e., the cuckoo search algorithm, which has the strong robustness, can be realized easily, has the fast convergence rate and can be used flexibly. To enhance the convergence rate and precision, the paper improves the conventional cuckoo search algorithm. Using the new model, the paper gives a method calculating the contribution rates of economic growth influencing factors scientifically. Finally, the paper calculates the contribution rates of influencing factors to economic growth in Shanghai City, China.

1. Introduction. Cobb-Douglas production function (C-D production function) was used earlier, and its generation can be traced back to von Thünen’s research in the late 1840s and Wicksell’s research in the early 1900s [14]. However, the C-D production function has many limitations of which the biggest one is it requires the elasticity of substitution of all input factors must be 1. To remedy the defect of C-D production function, Arrow, Chenery, Minhas and Solow proposed the CES (Constant Elasticity of Substitution) production function. In this case, the elasticity of substitution of factors in production function varies in different sectors and enterprises. However, in the CES production function, the elasticity of substitution can be any constant, but the constant is invariable for a specific production function, but, in fact, the elasticity of substitution may be variable rather than constant in different sample points. To reflect the characteristics of elasticity of substitution, Revankar, Sato and Hofman proposed the VES (Variable Elasticity of Substitution) production function. The function’s elasticity of substitution is no

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longer a constant but varies as the difficulty of substitution of factors or technological level changes. Therefore, the coefficient of variable elasticity of substitution of VES production function meets actual economic situation more relative to the coefficient of invariable elasticity of substitution of CES production function. However, in current economic studies, the C-D production function and the CES production function is applied more widely than the VES production function. Many scholars have made a great number of studies on the C-D production function and the CES production function and offered many research achievements on production function [3, 5, 7, 12, 15, 16, 18, 22].

However, there are few studies on the VES production function. For many reasons, the VES production function hasn’t been applied widely and the main reason is its form is more complicated than those of the C-D production function and the CES production function. Besides, researchers lack the corresponding understanding on the main properties of the VES production function. In the conventional VES production function, one function form is assuming the elasticity of substitution is the linear function of time. The paper generalizes the function of elasticity of substitution, i.e. assuming the elasticity of substitution is the quadratic function of time, and then proposes a new model of modified VES production function. Apparently, the modified VES production function has a more complicated form and its parameter estimation is more difficult. The conventional optimization methods generally require information like the gradient of subjective function, and have a poor rate of convergence and poor convergence precision. In recent years, with the rise of studies on intelligent optimization algorithms, many researchers have made a great amount of studies on the optimization problem using intelligent algorithm, such as the SA (Simulated Annealing) algorithm, the genetic algorithm and the firefly algorithm, and had some achievements [1, 2, 8, 17, 19, 21].

In intelligent optimization methods, the cuckoo search (CS) algorithm is very important which has strong robustness, can be realized easily, has a rapid rate of convergence and can be used flexibly. Many researchers have studied it [4, 11, 13, 20].

The paper proposes an improved CS algorithm for the parameter estimation of modified VES production function model. As for the application of model, conventional methods have defects, so the paper gives a method scientifically calculating the contribution rates of economic growth influencing factors [6, 9, 10, 23] and calculates the contribution rates of influencing factors to economic growth in Shanghai City, China in the final section.

2. The form of modified VES production function model.

2.1. The form of conventional VES production function model.

2.1.1. The Linear Function with K/L as the Elasticity of Substitution. We use $Y$, $K$ and $L$ to represent the output of economic growth, the capital input and the labor input, respectively.

Suppose the elasticity of substitution $\sigma = a + b \cdot \frac{K}{L}$, and then the general form of linear elasticity of substitution production function is

$$ z = A \exp \int \frac{dk}{k + c(\frac{k}{a + ek})^{1/a}}, $$

where $z = \frac{Y}{L}, k = \frac{K}{L}$. 

When $b = 0$, the equation above degrades into the form of CES production function; when $b = 0, a = 1$, and the elasticity of substitution $\sigma = 1$, the equation degrades into the form of C-D production function, when $a = 1, \sigma = 1 + bk$, the equation can be written into the following form:

$$Y = AK^{\frac{1}{\sigma + c}} \left( L + \frac{b}{1 + c} K \right)^{\frac{\sigma + c}{\sigma - 1}},$$

where $A, b$ and $c$ are the parameters to be estimated. The equation is in the case of the constant return to scale. When the return to scale is $\mu$, the equation is

$$Y = AK^{\frac{1}{\mu + c}} \left( L + \frac{b}{1 + c} K \right)^{\frac{\mu + c}{\mu - 1}},$$

2.1.2. The Linear Function with Time $t$ as the Elasticity of Substitution. Suppose the elasticity of substitution $\sigma = \sigma(t) = a + \rho \cdot t$ and the production function is in the CES form in any time point, considering $\sigma$ linearly varies over time because of technological progress, and then have

$$Y = B(\lambda L \frac{\sigma(t)-1}{\sigma(t)} + (1 - \lambda)K \frac{\sigma(t)-1}{\sigma(t)} \frac{\sigma(t)}{\mu(t)}),$$

where $B$ is technological progress, $\lambda$ is the distribution parameter, $L$ is the labor input, $K$ is the capital input and $Y$ is the output.

2.2. The form of modified VES production function model. Suppose a long-term VES production function is $Y = f(K, L)$, and then record $f_1 = \frac{\partial f}{\partial K}$, $f_2 = \frac{\partial f}{\partial L}$, $f_{11} = \frac{\partial^2 Y}{\partial K^2}$, $f_{22} = \frac{\partial^2 Y}{\partial L^2}$, $f_{12} = \frac{\partial^2 Y}{\partial K \partial L}$, $ff = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = f_{11}f_{22} - f_{12}f_{21}$.

A long-term VES production function $Y = f(K, L)$ should meet the following conditions:

- $Y = f(K, L)$ is continuous, has continuous first-order and second-order partial derivatives, and is a regular strictly concave function, and its first-order derivative is strictly positive. The regular strictly concave function indicates the principal minor from hessian determinant of $Y = f(K, L)$’s second-order derivative meets the following conditions in the whole interval:
  - $f_{11} < 0, f_{22} < 0$ and $ff > 0$.

We can see that, according to the definition of long-term VES production function $Y = f(K, L)$, the function’s second-order partial derivative should meet $f_{11} < 0, f_{22} < 0$ in the space of $(Y, K, L)$, i.e. each factor input should meet the law of diminishing marginal returns.

For the case that the VES production function’s elasticity of substitution is a time linear function, the paper proposes a modified VES production function, i.e. supposing the modified VES production function’s elasticity of substitution is a quadratic nonlinear function of time, that is $\rho = \rho(t) = a + bt + ct^2$. Therefore, the modified VES production function is

$$Y = A(t)\left(\delta_1 K \frac{\rho(t)-1}{\rho(t)} + \delta_2 L \frac{\rho(t)-1}{\rho(t)} \frac{\rho(t)}{\mu(t)} \right),$$

$$= A_0 e^{ct}\left(\delta_1 K \frac{\rho(t)-1}{\rho(t)} + \delta_2 L \frac{\rho(t)-1}{\rho(t)} \frac{\rho(t)}{\mu(t)} \right).$$

The equation is in the case of the constant return to scale. When the return to scale is $\mu$, the equation is

$$Y = A_0 e^{ct}\left(\delta_1 K \frac{\rho(t)-1}{\rho(t)} + \delta_2 L \frac{\rho(t)-1}{\rho(t)} \frac{\rho(t)}{\mu(t)} \right).$$
where \(A(t) = A_0 e^{\sigma t}\) is the technological progress level, \(Y\) is the output, \(K\) is the capital input, \(L\) is the labor input, \(\rho = \rho(t) = a + bt + ct^2\) is the elasticity of substitution, \(\delta_1, \delta_2\) are the parameters of distribution and \((A_0, \sigma, \delta_1, \delta_2, a, b, c, \mu)\) are the parameters to be estimated.

The paper verifies the conditions for modified VES production function to be a long-term production function through the example in Section 5.

3. The parameter estimation of production function model based on CS algorithm.

3.1. Fitness function. For known observed value \((K_t, L_t, Y_t)\) \((t = 1, 2, \ldots, n)\), there is

\[
Y_t = A_0 e^{\sigma t} (\delta_1 K_t^{\rho(t)-1} + \delta_2 L_t^{\rho(t)-1})^{\mu/(\mu-1)} + \varepsilon_t,
\]

where \(\varepsilon_t\) is the error between \(Y_t\)’s actual value and theoretical value.

Write

\[
g(\eta) = \sum_{t=1}^{n} \varepsilon_t^2 = \sum_{t=1}^{n} \left( Y_t - A_0 e^{\sigma t} (\delta_1 K_t^{\rho(t)-1} + \delta_2 L_t^{\rho(t)-1})^{\mu/(\mu-1)} \right)^2,
\]

and let it have the minimum value, and then get parameter estimate

\[
\eta = (A_0, \sigma, \delta_1, \delta_2, a, b, c, \mu).
\]

It, essentially, is a nonlinear optimization problem, for which the paper offers a CS algorithm.

The fitness function is

\[
g(\eta) = \sum_{t=1}^{n} \left( Y_t - A_0 e^{\sigma t} (\delta_1 K_t^{\rho(t)-1} + \delta_2 L_t^{\rho(t)-1})^{\mu/(\mu-1)} \right)^2 \rightarrow \text{min}.
\]

Because \(g(\eta)\) is a highly nonlinear function for which the conventional optimization method is relatively complicated, generally requires the value of partial derivative, falls into the local solution easily, and has the poor convergence rate and precision and thus is not suitable. In this paper, we use a modern intelligent optimization algorithm, i.e. the CS algorithm, for the parameter estimation. The algorithm has the strong robustness and can be realized easily. The paper improves the conventional CS algorithm to improve the convergence rate and precision.

3.2. CS algorithm. The CS algorithm is an emerging heuristic algorithm combining the cuckoo’s reproductive behaviors and flight characteristics. It has advantages of a limited number of parameters, the strong global search ability, good search paths and strong multiple-objective solving ability.

3.2.1. Idealized rules. To simplify the description of CS algorithm, we suppose

(1) one cuckoo lays only one egg at a time and selects a nest randomly for hatching and parasitism;

(2) in the nests selected randomly, the best nest shall be retained to the next generation;

(3) the number of available nests is \(n\) constantly, and the probability of being discovered by the host bird is \(p_a\), \(0 \leq p_a \leq 1\); when discovering the egg, the host bird shall throw the cuckoo egg out of nest or abandon the nest and nests again; the nest with a smaller \(p_a\) shall be retained to the next generation.
3.2.2. The Expressing Method of Solution. In the CS algorithm, let each egg in a nest represent a solution of problem and one cuckoo’s egg represent a new solution. The algorithm’s objective is replacing the poor solution in the nest with a new and potential good solution. Generally, suppose there is only one egg in the nest.

3.2.3. Nest Search Path & Position Update Formula. Suppose the dimensionality of problem to be optimized is \( N \), the number of nest is \( m \), and the current number of iteration is \( t \).

For nest \( i \) (\( 1 \leq i \leq m \)), in the \( N - D \) space, the position vector \( X_i = \{X_{i1}, X_{i2}, \cdots, X_{iN}\} \).

The cuckoo’s nest search path & position update formula is as follows:

\[
X_{i}^{t+1} = X_i^t + \alpha \odot \text{Levy}(\lambda),
\]

where \( X_i^t \) is nest \( i \)'s position vector in the \( t \)th iteration; \( \alpha \) is the control factor of step size (\( \alpha > 0 \)); \( \odot \) is the point-to-point multiplication; \( \text{Levy} \) is the Lévy random search path and its relationship with time follows the Lévy distribution.

The Lévy distribution is as follows:

\[
\text{Lévy}(\lambda) \sim \mu = t^{-\lambda} \quad (1 < \lambda \leq 3).
\]

Generally, the calculation formula of \( \text{Lévy}(\lambda) \) is

\[
s = \frac{\mu}{|v|^{\frac{1}{\beta}}},
\]

where \( s \) is the step size of Lévy random walk, i.e. \( \text{Lévy}(\lambda) \). \( \mu, v \) are normally distributed random numbers and satisfy

\[
\mu \sim N(0, \sigma_\mu^2), \quad v \sim N(0, \sigma_v^2).
\]

And,

\[
\sigma_\mu = \left\{ \frac{\Gamma(1+\beta)\sin(\pi\beta/2)}{\Gamma[(1+\beta)/2]2^{(\beta-1)/2}\beta} \right\}^{1/\beta},
\]

\[
\sigma_v = 1,
\]

where \( \beta \) is a parameter within [1, 2].

3.3. The improvement in CS algorithm. For a heuristic algorithm, balancing the globally high-efficiency random search strategy and the locally fine research strategy shall make the algorithm more efficient, so the selection of parameter must take the balance between global and local search abilities into consideration. The improved CS algorithm, on the basis of standard CS algorithm, balances global and local search abilities through parameter dynamics and then improves the algorithm’s performance.

3.3.1. Introduce an inertia weight. In the basic CS algorithm, the cuckoo’s nest search path and position are random. To improve the CS algorithm’s performance, the paper introduces the inertia weight \( w \) into the cuckoo’s nest search & position update formula as follows:

\[
X_{i}^{t+1} = wX_i^t + \alpha \odot \text{Levy}(\lambda).
\]

The introduction of inertia weight offers CS algorithm a trend to extend search space and an ability to search new area. Generally, in the global optimization algorithm, we always hope there is a strong global search ability in the early stage and a high development ability in the late stage to accelerate the rate of convergence. Experiments have proved that a big inertia weight \( w \) is good for jumping out of
the local optimal and making the global research of optimal; while, a small $w$ is good for the local search to accelerate the algorithm’s convergence. To balance the algorithm’s local and global search abilities, inertia weight $w$’s value should decrease as the number of iteration increases. However, the CS algorithm is nonlinear in the actual research process, in which case the linearly decreasing strategy of inertia weight can’t reflect the actual optimization search process, so the paper introduces an inertia weight nonlinear decreasing strategy:

$$w = \tau_t \cdot (w_{\text{max}} - w_{\text{min}}) + w_{\text{min}},$$

$$\tau_t = (\beta + 1) - \beta e^{\ln\left(\frac{\beta + 1}{\beta}\right)\left(\frac{t}{T_{\text{max}}}\right)^\gamma},$$

where $w_{\text{max}}$ is the maximum of inertia weight, $w_{\text{min}}$ is the minimum of inertia weight, $t$ is the current number of iteration, $T_{\text{max}}$ is the total number of iterations, and parameters $\gamma \geq 1$ and $\beta \geq 1$. $\tau_t$ is the control factor which is a Weibull function.

3.3.2. The Improvement in Parameter $p_a$. The improved algorithm adopts an adaptive strategy with $p_a$ changing dynamically with the algorithm process. Take a relatively small value of $p_a$ in the early stage of algorithm, in which case the number of nests updated is relatively great, to make the algorithm keep a strong global search ability; take a relatively big value of $p_a$ in the late stage of algorithm, in which case the number of nests updated is relatively small, to make the algorithm keep a strong local search ability.

$$p_a = p_{a,\text{min}} + \left(\frac{t}{T_{\text{max}}}\right)^\delta(p_{a,\text{max}} - p_{a,\text{min}}),$$

where $p_{a,\text{max}}, p_{a,\text{min}}$ are the constants set initially, $t$ is the current number of iteration, $T_{\text{max}}$ is the total number of iterations, and constant $0 \leq \delta \leq 1$. After a lot of experiments, we find that $p_a \in [0.1, 0.5]$ is reasonable.

3.3.3. Improvement in Parameter $\alpha$. Experiments have proved that a big $\alpha$ is good for jumping out of local optimal and making the global research; while, a small $\alpha$ is good for making the local search and accelerating the algorithm’s convergence. To balance the algorithm’s global and local search abilities, the value of $\alpha$ should decrease as the number of iteration increases.

$$\alpha = \tau_t \cdot (\alpha_{\text{max}} - \alpha_{\text{min}}) + \alpha_{\text{min}},$$

$$\tau_t = (\beta + 1) - \beta e^{\ln\left(\frac{\beta + 1}{\beta}\right)\left(\frac{t}{T_{\text{max}}}\right)^\gamma},$$

where $\alpha_{\text{max}}$ is the maximum of $\alpha$, $\alpha_{\text{min}}$ is the minimum of $\alpha$, $t$ is the current number of iteration, $T_{\text{max}}$ is the total number of iteration, and constants $\beta \geq 1$ and $\gamma \geq 1$. $\tau_t$ is the control factor, which is a Weibull function.

3.4. Steps of improved CS algorithm. The seven steps are as follows:

**Step1**: Set parameters: the maximum number of iteration $T_{\text{max}}$, population size $m$, inertia weight $w_{\text{max}}$ and $w_{\text{min}}$, and $p_{a,\text{max}}, p_{a,\text{min}}, \alpha_{\text{max}}, \alpha_{\text{min}}, \delta, \beta, \gamma$ and so on.

**Step2**: Generate the positions of $m$ nests randomly in the search space, and let $t = 1$.

**Step3**: Find the currently optimal position of nest and corresponding optimal fitness value.

**Step4**: Judge whether the algorithm has reached the satisfactory number of iteration or fitness precision; if yes, the algorithm stops; if not, turn to step 5.

**Step5**: Update the position of nest and get the new position.
4. The method calculating the contribution rates of economic growth influencing factors. Record the production function is $Y = F(X_1, X_2, \cdots, X_m)$. Suppose factor $X_i$ changes over time and point $M_1$ and point $M_n$ are the values of factor in the starting point and end point in corresponding analysis period.

Now suppose

1) Production function $Y = F(X_1, X_2, \cdots, X_m)$ is differentiable;
2) $H(t)$ is a curve connecting $M_1$ and $M_n$.

Record $\Delta Y_j$ as the influence of factor $j$ on output from period 1 to period $n$, and then

$$\Delta Y_j = \int_{H(t)} \frac{\partial F(X_1, X_2, \cdots, X_m)}{\partial X_j} dX_j,$$

so the sum of $\Delta Y_j$ ($j = 1, 2, \cdots, m$) compose all increments of the gross index.

Make a differential calculation to $Y = F(X_1, X_2, \cdots, X_m)$, and then get

$$dY = \frac{\partial F}{\partial X_1} dX_1 + \frac{\partial F}{\partial X_2} dX_2 + \cdots + \frac{\partial F}{\partial X_m} dX_m.$$

Get the integrals of both sides in $H(t)$, and then have

$$\Delta Y = \int_{H(t)} dY = \int_{H(t)} \frac{\partial F}{\partial X_1} dX_1 + \int_{H(t)} \frac{\partial F}{\partial X_2} dX_2 + \cdots + \int_{H(t)} \frac{\partial F}{\partial X_m} dX_m,$$

i.e.

$$\Delta Y = \Delta Y_1 + \Delta Y_2 + \cdots + \Delta Y_m.$$

And then, factor $j$’s contribution rate to economic growth from period 1 to period $n$ is

$$\frac{\Delta Y_j}{\Delta Y} \quad (j = 1, 2, \cdots, m).$$

The modified VES production function model is

$$Y = A(t)(\delta_1 K^{\frac{\mu(t)-1}{\rho(t)}} + \delta_2 L^{\frac{\mu(t)-1}{\rho(t)}}) \frac{\mu(t)}{\rho(t)-1}.$$ 

Then, get its partial derivative

$$\frac{\partial Y}{\partial A} = (\delta_1 K^{\frac{\mu(t)-1}{\rho(t)}} + \delta_2 L^{\frac{\mu(t)-1}{\rho(t)}}) \frac{\mu(t)}{\rho(t)-1} = \frac{Y}{A_0 e^{\sigma t}},$$

$$\frac{\partial Y}{\partial K} = A_0 e^{\sigma t} \left(\frac{\mu(t)}{\rho(t)}\right) \delta_1 (\frac{\mu(t)-1}{\rho(t)}) K^{\frac{\mu(t)-1}{\rho(t)}-1} + A_0 e^{\sigma t} \frac{\mu(t)}{\rho(t)} \left(\frac{\mu(t)-1}{\rho(t)}\right)^{-1} \delta_1 (\frac{\mu(t)}{\rho(t)}) K^{\frac{\mu(t)-1}{\rho(t)}-1}$$

$$= A_0 e^{\sigma t} \left(\frac{\mu(t)}{\rho(t)}\right) \frac{Y}{A_0 e^{\sigma t}} \frac{\mu(t)}{\rho(t)} \left(\frac{\mu(t)-1}{\rho(t)}\right)^{-1} \delta_1 (\frac{\mu(t)}{\rho(t)}) K^{\frac{\mu(t)-1}{\rho(t)}-1}$$

$$= A_0 Y^{- \frac{\mu(t)}{\rho(t)}} e^{- \frac{\mu(t)-1}{\rho(t)} t} \mu_1 K^{\frac{\mu(t)-1}{\rho(t)}-1}.$$
We can calculate $\Delta n$ period 1 to period $n$.

Suppose $H(t)$ is the curve connecting $(K_1, L_1, Y_1)$ to $(K_n, L_n, Y_n)$, then its parameter equation is

$$
\begin{align*}
Y_t &= a_0e^{a_1t}, \\
K_t &= b_0e^{b_1t}, \\
L_t &= c_0e^{c_1t}.
\end{align*}
$$

Therefore, the technological progress' absolute influence on economic growth from period 1 to period $n$ is

$$
\Delta Y_A = \int_H \frac{\partial Y}{\partial A} dA = \int_H \frac{\partial Y}{\partial K} dK = \int_H \frac{\partial Y}{\partial L} dL
$$

Factor $K$’s absolute influence on economic growth from period 1 to period $n$ is

$$
\Delta Y_K = \int_H \frac{\partial Y}{\partial K} dK
$$

Factor $L$’s absolute influence on economic growth from period 1 to period $n$ is

$$
\Delta Y_L = \int_H \frac{\partial Y}{\partial L} dL
$$

We can calculate $\Delta Y_K$ using the numerical integration.

We can calculate $\Delta Y_L$ using the numerical integration.

And then, the technological progress’s contribution rate to economic growth from period 1 to period $n$ is

$$
\frac{\Delta Y_A}{\Delta Y} = \frac{\Delta Y_A}{\Delta Y_A + \Delta Y_K + \Delta Y_L + \Delta Y_E}.
$$

The capital’s contribution rate to economic growth from period 1 to period $n$ is

$$
\frac{\Delta Y_K}{\Delta Y} = \frac{\Delta Y_K}{\Delta Y_A + \Delta Y_K + \Delta Y_L + \Delta Y_E}.
$$
The labor's contribution rate to economic growth from period 1 to period \( n \) is
\[
\frac{\Delta Y_L}{\Delta Y} = \frac{\Delta Y_L}{\Delta Y_A + \Delta Y_K + \Delta Y_L + \Delta Y_E}
\]

5. The calculation example of the contribution rates of influencing factors to economic growth in Shanghai City, China. To research the economic growth situation in Shanghai City, China, explore the growth way and calculate the contribution rates of input factors to economic growth, the paper selects the gross regional product \( Y \) (¥0.1 billion) of Shanghai City as the output, and fixed-asset investment \( K \) (¥0.1 billion) and the number of employees \( L \) (10,000 people) as economic influencing factors for the analysis. See Table 1 for the data.

| Year | \( Y \)  | \( K \)  | \( L \)  |
|------|--------|--------|--------|
| 1999 | 4222.30| 1856.72| 733.76 |
| 2000 | 4812.15| 1869.67| 745.24 |
| 2001 | 5257.66| 1994.73| 752.26 |
| 2002 | 5795.02| 2187.06| 792.04 |
| 2003 | 6762.38| 2452.11| 813.05 |
| 2004 | 8165.38| 3084.66| 836.87 |
| 2005 | 9365.54| 3542.55| 863.32 |
| 2006 | 10718.04| 3925.09| 885.51 |
| 2007 | 12668.12| 4458.61| 909.08 |
| 2008 | 14275.80| 4829.45| 1053.24|
| 2009 | 15285.58| 5273.33| 1064.42|
| 2010 | 17433.21| 5317.67| 1090.76|
| 2011 | 19533.84| 5067.09| 1104.33|
| 2012 | 20553.52| 5254.38| 1115.50|
| 2013 | 22257.66| 5647.79| 1137.35|
| 2014 | 24060.87| 6016.43| 1197.31|
| 2015 | 25643.47| 6352.70| 1361.51|
| 2016 | 28178.65| 6755.88| 1365.24|
| 2017 | 30632.09| 7246.60| 1372.65|
| 2018 | 32679.87| 7623.42| 1430.82|

Suppose the modified VES production function model is
\[
Y = A(t)(\delta_1K^{\frac{\rho(t)\mu}{\rho(t)}} + \delta_2L^{\frac{\rho(t)\mu}{\rho(t)}}) \rho(t)\mu
\]
\[
= A_0e^{\sigma t}(\delta_1K^{\frac{\rho(t)\mu}{\rho(t)}} + \delta_2L^{\frac{\rho(t)\mu}{\rho(t)}}) \rho(t)\mu,
\]
where \( A(t) = A_0e^{\sigma t} \) is the technological level, \( Y \) is the output, \( K \) is the capital input, \( L \) is the labor input, \( \rho = \rho(t) = a + b \cdot t + ct^2 \) is the elasticity of substitution, and \((A_0, \sigma, \delta_1, \delta_2, a, b, c, \mu)\) are the parameters to be estimated.

Use the improved CS algorithm given for the model's parameter estimation. Choose the following values for parameters: the maximum number of iteration \( T_{\text{max}} = 300 \), the population size \( m = 25 \), the inertia weight \( w_{\text{max}} = 1, w_{\text{min}} = 0.001 \), and \( p_{\alpha,\text{max}} = 0.5, p_{\alpha,\text{min}} = 0.1, \alpha_{\text{max}} = 1, \alpha_{\text{min}} = 0.001, \beta = 1, \gamma = 10 \) and so on.
Through calculation, get

\[ \eta = (A_0, \sigma, \delta_1, \delta_2, a, b, c, \mu) \]
\[ = (3.9414, 0.0433, 1.4430, 6.6868, 5.0012, 3.1064, 0.1497, 0.8293), \]

i.e.

\[ Y = 3.9414e^{0.0433} (1.4430K^{4.0012+3.1064t^{0.1497}} + 6.6868L^{5.0012+3.1064t^{0.1497}}) \times 0.8293. \]

The optimal value of objective function is \( G(\eta) = 9.9439e + 06. \)

The model's coefficient of determination is \( R^2 = 1 - \frac{\sum(Y_t - \hat{Y}_t)^2}{\sum(Y_t - \bar{Y})^2} = 0.9937. \)

It can be seen that the model’s fitting precision is high and coefficient of determination is close to 1.

To compare the conventional CS algorithm and the improved CS algorithm in terms of convergence rate and precision, the paper makes calculations. Table 2 shows calculation results. Figure 1 shows the variation curves of two algorithms' objective function value changing with the iteration number, from which we can see the improved CS algorithm has the convergence rate and precision higher than that of conventional CS algorithm.

### Table 2. The comparison of results of two CS algorithms.

| Method   | Conventional CS | Improved CS |
|----------|-----------------|-------------|
| \( A_0 \) | 4.9963          | 3.9414      |
| \( \sigma \) | 0.0450          | 0.0433      |
| \( \delta_1 \) | 1.1503          | 1.4430      |
| \( \delta_2 \) | 4.9788          | 6.6868      |
| \( a \)     | 4.9940          | 5.0012      |
| \( b \)     | 3.2084          | 3.1064      |
| \( c \)     | 0.1988          | 0.1497      |
| \( \mu \)   | 0.8231          | 0.8293      |

| Number of Iteration | 252 | 54 |
|---------------------|-----|----|
| \( G, \text{Optimal Value of Objective Function} \) | 1.0319e+07 | 9.9439e+06 |
| \( R^2, \text{Coefficient of Determination of Model} \) | 0.9935 | 0.9937 |

To verify whether the production function proposed in the paper meets the essential conditions of long-term production function, the paper does a calculation. Table 3 shows the calculation results.

From Table 3 we can see that for all \((K, L), f_1 = \frac{\partial Y}{\partial K} > 0, f_2 = \frac{\partial Y}{\partial L} > 0, f_{11} = \frac{\partial^2 Y}{\partial K^2} < 0, f_{22} = \frac{\partial^2 Y}{\partial L^2} < 0, \text{and } f f = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = f_{11}f_{22} - f_{12}f_{21} > 0. \)

Therefore, the production function model proposed in the paper is reasonable.

Through calculation, get: \( \Delta Y_A = 1.3021e + 04; \Delta Y_K = 1.2186e + 04; \Delta Y_L = 0.7325e + 04. \)

Therefore, from year 1999 to year 2018, the contribution rates of factors to economic growth are as follows:

- technological progress \( A \)'s contribution rate to economic growth is \( \frac{\Delta Y_A}{\Delta Y} = 40.03\%; \)
Figure 1. Two algorithms' objective function value variation curves with the changes of the number of iteration.

Table 3. Verification results of conditions of production function.

| Year | $f_1$   | $f_2$   | $f_{11}$ | $f_{22}$ | $f_{12}$ | $ff$     |
|------|---------|---------|----------|----------|----------|----------|
| 1999 | 1.2536  | 6.5002  | -9.275e-5| -13.7e-4 | -5.69e-5 | 1.2364e-7|
| 2000 | 1.2387  | 6.2050  | -7.511e-5| -11.8e-4 | -9.53e-5 | 7.9713e-8|
| 2001 | 1.2487  | 6.1583  | -6.335e-5| -10.9e-4 | -1.15e-4 | 5.5835e-8|
| 2002 | 1.2665  | 6.1774  | -5.439e-5| -9.92e-4 | -1.23e-4 | 3.8894e-8|
| 2003 | 1.2922  | 6.2664  | -4.787e-5| -9.33e-4 | -1.27e-4 | 2.8559e-8|
| 2004 | 1.3090  | 6.3446  | -3.959e-5| -8.48e-4 | -1.21e-4 | 1.8918e-8|
| 2005 | 1.3397  | 6.4705  | -3.565e-5| -7.93e-4 | -1.19e-4 | 1.4196e-8|
| 2006 | 1.3780  | 6.6312  | -3.337e-5| -7.56e-4 | -1.18e-4 | 1.1381e-8|
| 2007 | 1.4150  | 6.7923  | -3.086e-5| -7.15e-4 | -1.14e-4 | 8.9777e-9|
| 2008 | 1.4472  | 6.9094  | -2.805e-5| -6.34e-4 | -1.06e-4 | 6.5751e-9|
| 2009 | 1.4950  | 7.1239  | -2.709e-5| -6.20e-4 | -1.06e-4 | 5.6539e-9|
| 2010 | 1.5549  | 7.3862  | -2.754e-5| -6.24e-4 | -1.09e-4 | 5.2946e-9|
| 2011 | 1.6274  | 7.7058  | -2.929e-5| -6.53e-4 | -1.17e-4 | 5.4122e-9|
| 2012 | 1.6907  | 7.9924  | -2.949e-5| -6.59e-4 | -1.20e-4 | 5.0563e-9|
| 2013 | 1.7496  | 8.2616  | -2.888e-5| -6.48e-4 | -1.19e-4 | 4.5043e-9|
| 2014 | 1.8078  | 8.5239  | -2.803e-5| -6.28e-4 | -1.17e-4 | 3.9252e-9|
| 2015 | 1.8582  | 8.7431  | -2.631e-5| -5.82e-4 | -1.10e-4 | 3.1723e-9|
| 2016 | 1.9281  | 9.0661  | -2.625e-5| -5.85e-4 | -1.11e-4 | 2.9566e-9|
| 2017 | 1.9985  | 9.3922  | -2.600e-5| -5.81e-4 | -1.11e-4 | 2.7283e-9|
| 2018 | 2.0695  | 9.7171  | -2.563e-5| -5.71e-4 | -1.10e-4 | 2.4778e-9|

factor $K$’s contribution rate to economic growth is

$$\frac{\Delta Y_K}{\Delta Y} = 37.46\%;$$
factor $L$’s contribution rate to economic growth is

$$\frac{\Delta Y_L}{\Delta Y} = 22.51\%.$$  

\[\begin{array}{c}
\text{Contribution rate of technological progress} \\
\text{Contribution rate of capital} \\
\text{Contribution rate of labor}
\end{array}\]

\[\begin{array}{c}
40.03\% \\
37.46\% \\
22.51\%
\end{array}\]

\textbf{Figure 2.} The distribution diagram of contribution rates of influencing factors to economic growth in Shanghai City, China.

Figure 2 gives the distribution of contribution rates of influencing factors to economic growth in Shanghai City, China. It shows Shanghai’s economic growth mainly depends on technological progress, next on capital input and least on labor input. The results consist with the reality in Shanghai City, China.

6. **Conclusion.** The conclusion is as follows:

(1) The conventional VES production function model’s elasticity of substitution is the unary linear function of time $t$, but the paper supposes it is a quadric nonlinear function of time and gives a modified VES production function model which is more applicable.

(2) The paper gives an intelligent optimization solving method for the model’s parameters. With the improved CS algorithm, the mode’s parameter estimation shows the higher convergence rate and precision.

(3) As for actual applications, the paper gives a method scientifically calculating the contribution rates of influencing factors to economic growth. The algorithm has the higher calculation precision compared with that of conventional calculation methods.

(4) Finally, the paper calculates the contribution rates of influencing factors to economic growth in Shanghai City, China. Calculation results consist with the reality, indicating the effectiveness of method given.

There are few studies and applications of VES production function, so the method given in the paper has important significance for the in-depth research on VES production function model and promotes the practical applications of the model,
and thus can be used for reference in the studies and applications of other production function models.

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