DETECTING FLATNESS OVER SMOOTH BASES

LUCHEZAR L. AVRAMOV AND SRIKANTH B. IYENGAR

Abstract. Given an essentially finite type morphism of schemes $f : X \to Y$ and a positive integer $d$, let $f^{(d)} : X^{(d)} \to Y$ denote the natural map from the $d$-fold fiber product $X^{(d)} = X \times_Y \cdots \times_Y X$ and $\pi_i : X^{(d)} \to X$ the $i$th canonical projection. When $Y$ smooth over a field and $F$ is a coherent sheaf on $X$, it is proved that $F$ is flat over $Y$ if (and only if) $f^{(d)}$ maps the associated points of $\bigotimes_{i=1}^d \pi_i^*F$ to generic points of $Y$, for some $d \geq \dim Y$. The equivalent statement in commutative algebra is an analog—but not a consequence—of a classical criterion of Auslander and Lichtenbaum for the freeness of finitely generated modules over regular local rings.

Introduction

Knowledge that a scheme can be fibered as a flat family over some regular base scheme represents fundamental structural information. For this reason—among others—it is desirable to have efficient methods for deciding the flatness of a morphism of noetherian schemes $f : X \to Y$ with $Y$ regular.

One necessary condition is that $f$ maps associated points of $X$ to generic points of $Y$. When $\dim Y = 1$, this condition is also sufficient; see [9, III.9.7].

When the morphism $f$ is finite, a criterion for freeness of finite modules over regular local rings, due to Auslander [2] and Lichtenbaum [10], translates into a similar criterion involving $d$-fold fiber products $X^{(d)} = X \times_Y \cdots \times_Y X$. If $d \geq \dim Y$ and the natural morphism $f^{(d)} : X^{(d)} \to Y$ sends associated points of $X^{(d)}$ to generic points of $Y$, then $f$ is flat. This was observed by Vasconcelos [13], who proved that the conclusion holds for all morphisms $f$ when $d = 2 = \dim Y$, and conjectured that it holds in all dimensions when $f$ is a morphism essentially of finite type.

We prove a criterion for flatness relative to $f$ of coherent sheaves $\mathcal{F}$ on $X$, using the canonical projections $\pi_i : X^{(d)} \to X$: Assume that $Y$ is essentially smooth over some field and $f$ is essentially of finite type. If $f^{(d)}$ maps the associated points of $\bigotimes_{i=1}^d \pi_i^*\mathcal{F}$ to generic points of $Y$ for some $d \geq \dim Y$, then $\mathcal{F}$ is flat over $Y$. Thus, flatness is detected by the values of $f^{(d)}$ at finitely many points. Setting $\mathcal{F} = \mathcal{O}_Y$ one obtains a proof of Vasconcelos’ conjecture in a case of prime interest in geometry.

The theorem above is an analog of Auslander’s criterion. It is equivalent to the following statement in commutative algebra, proved in Section 4.

Main Theorem. Let $K$ be a field, $R$ an essentially smooth $K$-algebra, $A$ an algebra essentially of finite type over $R$, and $M$ a finite $A$-module.

If $M^{\otimes d}$ is torsion-free over $R$ for some $d \geq \dim R$, then $M$ is flat over $R$.

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For \( K = \mathbb{C} \), the preceding result is proved by Adamus, Bierstone, and Milman in [1], subsequent to the special case \( M = A \) with \( A \) equidimensional and of finite type over \( \mathbb{C} \), settled by Galligo and Kwieciński [7]. In those papers the algebraic statements are deduced from analogous theorems about morphisms of complex-analytic spaces, proved by using transcendental-geometric tools.

Our proof is algebraic, and its architecture reflects Auslander’s design. The argument proceeds in four steps, summed up in the opening statements of the first four sections. Sections 1 and 4 are mostly homological in nature. They deal with vanishing of Tor modules—and thus, ultimately, with flatness—and allow for an adaptation of methods used by Auslander in [2].

In Sections 2 and 3 we relate vanishing of Tor modules to torsion-freeness. These sections form the core of the paper. The techniques applied in them are, by necessity, different from those that work for finite modules over local rings. More detailed comparisons are given in notes immediately following the first theorem in each section. In these notes we also explain why in the Main Theorem the base ring \( R \) is assumed to be an essentially smooth algebra over a field, rather than just a regular ring.

In Section 5 we focus on rings of dimension two. Assuming only that \( R \) is regular, we establish a criterion for flatness that covers a larger class of modules than those in the Main Theorem, and includes Vasconcelos’ result.

1. RIGIDITY

In this paper rings are assumed commutative.

An algebra \( B \) over a ring \( K \) is said to be \textit{essentially of finite type} if it is a localization of some finitely generated \( K \)-algebra. In case \( K \) is a field, \( B \) is \textit{essentially smooth} over \( K \) if, in addition, the ring \( B \otimes_K K' \) is regular for every field extension \( K \subseteq K' \); thus, when the field \( K \) is perfect, the \( K \)-algebra \( B \) is essentially smooth if and only if \( B \) is a regular ring.

For the rest of this section \( R \) denotes a noetherian ring.

For every prime ideal \( p \) of \( R \) we set \( k(p) = R_p / pR_p \).

We say that an \( R \)-module \( M \) is \textit{essentially of finite type} if there exists an \( R \)-algebra \( A \) essentially of finite type with the property that \( M \) is a finite \( A \)-module, and the \( R \)-module structure induced through \( A \) coincides with the original one. Any such algebra \( A \) will be called a \textit{witness} for \( M \).

It is clear that finite \( R \)-modules are essentially of finite type, and that the latter class is much larger than the former. Remark 3.3 describes an interesting family of modules that are \textit{not} essentially of finite type.

**Theorem 1.1.** Let \( R \) be an essentially smooth algebra over a field, and let \( M \) and \( N \) be \( R \)-modules essentially of finite type.

If \( \text{Tor}^R_i(M, N) = 0 \) for some \( i \geq 0 \), then \( \text{Tor}^R_j(M, N) = 0 \) for each \( j \geq i \).

**Notes.** When \( R \) is a regular local ring and \( M \) and \( N \) are finite \( R \)-modules, the conclusion above was proved by Auslander [2, 2.1] in case \( R \) is an algebra over some field, by using Koszul complexes, and extended to the general case by Lichtenbaum [10] Cor. 1], by applying different techniques.

Our proof of Theorem 1.1 also relies on Koszul complexes.

Given a finite sequence \( \mathbf{x} \) of elements of \( R \) and an \( R \)-module \( M \), set

\[
H_i(\mathbf{x}; M) = H_i(R(\mathbf{x}) \otimes_R M),
\]
where $R(x)$ is the Koszul complex on $x$. We recall an important fact:

1.2. **Koszul rigidity. I.** If $M$ is an $R$-module essentially of finite type, and $H_i(x; M) = 0$ for some integer $i \geq 0$, then $H_j(x; M) = 0$ holds for $j \geq i$.

Indeed, let $A$ be a witness for $M$ and $\alpha : R \to A$ the structure map. There is an isomorphism of complexes $R(x) \otimes_R M \cong A(\alpha(x)) \otimes_A M$, and since $M$ is finite over $A$, the result follows from the classical case; see [3, 2.6].

**Lemma 1.3.** Let $K$ be a field and $R$ an essentially smooth $K$-algebra.

Set $Q = R \otimes_K R$, let $\mu : Q \to R$ be the surjective homomorphism of rings, given by $\mu(r \otimes r') = rr'$, and set $I = \text{Ker}(\mu)$.

For every prime $q \in \text{Spec } Q$ with $q \supseteq I$, any minimal generating set $x$ for $I_q$, and each $j \in \mathbb{Z}$ there is an isomorphism of $R_q$-modules

$$\text{Tor}_j^R(M, N)_q \cong H_j(x; (M \otimes_K N)_q).$$

**Proof.** The first isomorphism is a localization of a formula from [3, IX.4.4]:

$$\text{Tor}_j^R(M, N)_q \cong \text{Tor}_j^Q(R, M \otimes_K N)_q \cong \text{Tor}_j^{Q_q}(R_q, (M \otimes_K N)_q).$$

For each prime ideal $p$ of $R$, set $k(p) = R_p/pR_p$. The map $r \mapsto r \otimes 1$ gives a flat homomorphism of rings $c : R \to Q$ with fibers $k(p) \otimes_K R$. Since $R$ is essentially smooth over $K$, both rings $R$ and $k(p) \otimes_K R$ are regular. By [11, 23.7], the ring $Q$ is regular as well. The induced map $Q_q \to R_q$ is a surjective homomorphism of regular local rings, so its kernel $I_q$ is generated by a regular sequence; see [11, 14.2]. By [11, 16.5], the Koszul complex $Q_q(x)$ is a free resolution of $R_q$ over $Q_q$; this gives an isomorphism

$$\text{Tor}_j^{Q_q}(R_q, (M \otimes_K N)_q) \cong H_j(x; (M \otimes_K N)_q).$$

Concatenating the displayed isomorphisms completes the proof. \hfill \Box

**Proof of Theorem 1.1.** By localization and Lemma 1.3 it suffices to show that $H_i(x; (M \otimes_K N)_q) = 0$ implies $H_j(x; (M \otimes_K N)_q) = 0$ for $j \geq i$ and for each $q$ in $\text{Spec } Q$ with $q \supseteq I$. This follows from 1.2 for the module $(M \otimes_K N)_q$ is essentially of finite type over $Q_q$, as witnessed by $(A \otimes_K B)_q$, where $A$ is a witness for $M$ and $B$ is one for $N$. \hfill \Box

**Remark.** The papers [11, 7] deal with almost finite modules over analytic algebras. This class is distinct from that of modules essentially of finite type.

2. **TORSION IN TENSOR PRODUCTS**

Let $R$ be a ring and $U$ its multiplicatively closed subset consisting of all the non-zero-divisors. The **torsion submodule**, $\mathcal{T}_RM$, of an $R$-module $M$ is the kernel of the localization map $M \to U^{-1}M$. There is an exact sequence

$$0 \to \mathcal{T}_RM \to M \to \nabla_RM \to 0 \tag{2.0}$$

of $R$-modules. The module $M$ is said to be **torsion** when $\mathcal{T}_RM = M$; it is called **torsion-free** when $\mathcal{T}_RM = 0$. Note that $\mathcal{T}_RM$ is a torsion module, while $\nabla_RM$ is a torsion-free one.

**Theorem 2.1.** Let $R$ be an essentially smooth algebra over a field, and let $M$ and $N$ be $R$-modules essentially of finite type.

If the $R$-module $M \otimes_R N$ is torsion-free, then the following statements hold:
(1) \( \text{Tor}^R_i(M, N) = 0 \) for each \( i \geq 1 \).
(2) \( \text{Tor}^R_1(M, \tau R N) = 0 = \text{Tor}^R_i(\tau R M, N) \) for all \( i \geq 0 \).

Notes. When \( R \) is a regular local ring, \( M \) and \( N \) are finite \( R \)-modules, and \( M \otimes_R N \) is torsion-free, the statements above hold by [2, 3.1(b)] (if \( R \) is unramified) and [10, Cor. 2(b)] (in general). The finiteness hypothesis is critical for the proofs of these results.

Our proof draws on different ideas. The hypothesis that \( R \) is an essentially smooth algebra is used at a crucial juncture of the argument, in order to replace certain Tor modules with appropriate Koszul homology modules.

The hypotheses of the theorem do not, in general, imply that \( M \) or \( N \) is a torsion-free \( R \)-module; see Example [2,10].

We start preparations for proving Theorem 2.1 with a standard calculation.

Lemma 2.2. Let \( R \) be a ring and let \( M \) and \( N \) be \( R \)-modules.
If \( M \otimes_R N \) is torsion-free, then there are natural isomorphisms
\[
(\tau_R M) \otimes_R N \cong M \otimes_R (\tau_R N) \cong (\tau_R M) \otimes_R (\tau_R N).
\]
If, in addition, \( \text{Tor}^R_1(\tau_R M, N) = 0 \), then \( (\tau_R M) \otimes_R N = 0 \).

Proof. Tensor the sequence (2.0) with \( N \) to get an exact sequence
\[
\text{Tor}^R_1(\tau_R M, N) \rightarrow (\tau_R M) \otimes_R N \rightarrow M \otimes_R N \rightarrow (\tau_R M) \otimes_R (\tau_R N) \rightarrow 0.
\]
As \( (\tau_R M) \otimes_R N \) is torsion and \( M \otimes_R N \) is torsion-free, this sequence shows that \( \tau \) is bijective. By symmetry, so is \( M \otimes_R N \rightarrow (\tau_R M) \otimes_R (\tau_R N) \). Thus, \( M \otimes_R (\tau_R N) \) is torsion-free, so the preceding argument shows that the homomorphism of \( R \)-modules \( M \otimes_R (\tau_R N) \rightarrow (\tau_R M) \otimes_R (\tau_R N) \) is bijective.

The final assertion of the lemma is clear from the exact sequence above.

Lemma 2.3. If \( M \) and \( N \) are \( R \)-modules essentially of finite type, with witnesses \( A \) and \( B \), respectively, then for each \( i \in \mathbb{Z} \) the \( R \)-module \( \text{Tor}^R_i(M, N) \) is essentially of finite type, with witness \( A \otimes_R B \).

Proof. By hypothesis, one has \( A \cong A'/I \), where \( A' \) is a localization of a polynomial ring over \( R \). As \( M \) is a finite \( A' \)-module, it has a resolution \( F \) by free \( A' \)-modules of finite rank. This also is a resolution by flat \( R \)-modules, so for each \( i \in \mathbb{Z} \) one has \( \text{Tor}^R_i(M, N) \cong H_i(F \otimes_R B) \). Since \( F \otimes_R B \) is a complex of finite \( (A' \otimes_R B) \)-modules, and the ring \( A' \otimes_R B \) is noetherian, \( H_i(F \otimes_R B) \) is a finite \( (A' \otimes_R B) \)-module. It is annihilated by \( I \otimes_R B \), so it has a structure of \( (A \otimes_R B) \)-module, which is necessarily finite.

We use elementary facts concerning depth. These are not well documented for not necessarily finite modules, so we collect the statements we need. As usual, local rings are assumed to be noetherian.

2.4. Depth. Let \( S \) be a local ring, \( n \) its maximal ideal, and \( L \) an \( S \)-module. Following Auslander and Buchsbaum [3], we define the depth of \( L \) by
\[
\text{depth}_S L = e - \sup \{ i \mid H_i(s; L) \neq 0 \},
\]
where \( s = s_1, \ldots, s_e \) is a generating set for \( n \). In particular, if \( H_i(s; L) = 0 \) for all \( i \), then depth \( L = \infty \). The definition of the Koszul complex \( S(s) \) gives
\[
H_e(s; L) = (0 : n)_L \cong \text{Hom}_S(S/n, L).
\]
The depth of $L$ equals the infimum of those integers $i$ with $\text{Ext}_S^i(S/\mathfrak{n}, L) \neq 0$; see [1] §1, Théorème 1. As usual, we set $\text{depth}_S S = 0$.

In applications, we need to track depth along ring homomorphisms.

**Lemma 2.5.** Let $S$ be a local ring, $\mathfrak{n}$ its maximal ideal, $\sigma: S \to S'$ a homomorphism of rings, and $L'$ an $S'$-module.

1. When $S'$ is local with maximal ideal $\mathfrak{n}'$, and $\sigma(\mathfrak{n}) \subseteq \mathfrak{n}'$, then $\text{depth}_{S'} L' = 0$ implies $\text{depth}_S L' = 0$; the converse holds when $\mathfrak{n}S' = \mathfrak{n}'$.

2. If $\tau: S' \to S''$ is a flat homomorphism, $\text{depth}_{S'} L' \leq \text{depth}_S (L' \otimes_{S'} S'')$, and equality holds in case $\tau$ is also faithful.

**Proof.** (1) Set $k = S/\mathfrak{n}$ and $k' = S'/\mathfrak{n}'$. In the string of $S$-linear maps

$$\text{Hom}_S(k, L') \cong \text{Hom}_{S'}(S' \otimes_S k, L') \cong \text{Hom}_{S'}(S'/\mathfrak{n}S', L') \cong \text{Hom}_{S'}(k', L')$$

the isomorphisms are standard and the inclusion is induced by the surjection $S'/\mathfrak{n}S' \to S'/\mathfrak{n}'$; it is an equality when $\mathfrak{n}S' = \mathfrak{n}'$. Now refer to (2).

(2) This follows from isomorphisms $H_i(s; L' \otimes_{S'} S'') \cong H_i(s; L') \otimes_{S'} S''$. □

When $M$ is an $R$-module $\text{Ass}_R M$ denotes the set of prime ideals $\mathfrak{p}$ of $R$ for which there is a monomorphism $R/\mathfrak{p} \to M$, and $\text{Ass}_R$ stands for $\text{Ass}_R R$. When $R$ is noetherian $\text{Ass}_R R$ is finite and contains every minimal prime of $R$.

Depth detects torsion through the following well-known observation:

**Lemma 2.6.** Let $R$ be a noetherian ring and $M$ an $R$-module.

The following condition implies that $M$ is torsion-free:

$$\text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} > 0 \text{ for every } \mathfrak{p} \in \text{Spec } R \setminus \text{Ass } R$$

The converse assertion holds if every associated prime ideal of $R$ is minimal.

**Proof.** Recall: the set $U$ of non-zero-divisors of $R$ is equal to $R \setminus \bigcup_{\mathfrak{q} \in \text{Ass } R} \mathfrak{q}$.

Assume that (2.6.1) holds, but $um = 0$ with $u \in U$ and $m \in M \setminus \{0\}$. Choose $\mathfrak{p} \in \text{Ass}_R(Rm)$ and a monomorphism $R/\mathfrak{p} \to Rm$. Composed with the inclusion $Rm \subseteq M$, it induces an injection $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \to M_{\mathfrak{p}}$, which gives $\text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = 0$. Now (2.6.1) implies $\mathfrak{p} \in \text{Ass } R$, which is impossible, since $\mathfrak{p}$ contains the non-zero-divisor $u$.

When primes ideals in $\text{Ass } R$ are minimal, prime avoidance shows that each $\mathfrak{p} \notin \text{Ass } R$ satisfies $\mathfrak{p} \cap U \neq \emptyset$. If $M$ is torsion-free, then for $u \in \mathfrak{p} \cap U$ one has $(0 : pR_{\mathfrak{p}})_{M_{\mathfrak{p}}} = ((0 : p)_{M})_{\mathfrak{p}} \subseteq ((0 : u)_{M})_{\mathfrak{p}} = 0$, so $\text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} > 0$, by (2.4). □

Regular rings are products of integral domains, see [11] 14.3 and Exercise 9.11], so their associated primes are minimal. Thus, the preceding lemma specializes to the following statement:

**Lemma 2.7.** When $R$ is a regular ring, an $R$-module $M$ is torsion-free if and only if $M_{\mathfrak{p}}$ is torsion-free over $R_{\mathfrak{p}}$ for each $\mathfrak{p} \in \text{Spec } R$. □

In the proof of Theorem 2.4 we use the following result, due to Huneke and Wiegand [8] 6.3. An extension to all modules $H_i(x; L)$ is given by Takahashi et. al [12] Thm. 2; it sharpens the classical rigidity theorem recalled in (1.2).

2.8. **Koszul Rigidity.** Let $(S, \mathfrak{n})$ be a local ring, $x$ a finite sequence of elements of $\mathfrak{n}$, and $L$ a finite $S$-module.

If $0 < \text{length}_S H_1(x; L) < \infty$ holds, then $\text{depth}_S H_0(x; L) = 0$. □
Proof of Theorem 2.11.} By hypothesis, $R$ is essentially smooth over a field, say $K$. Let $A$ and $B$ be witnesses for $M$ and $N$, respectively, and set
\[ Q = R \otimes_K R \quad \text{and} \quad C = A \otimes_K B. \]
The $Q$-algebra $C$ is essentially of finite type, and $\text{Tor}_i^R(M, N)$ is a finite $C$-module for each $i$; see Lemma 2.3. Recall the hypothesis: $\text{T}_R(M \otimes_R N) = 0$.

(1) In view of Theorem 1.1, it suffices to prove that $\text{Tor}_1^R(M, N)$ is zero.

Suppose it is not. Pick a prime ideal $m$ in $C$ such that $\text{length}_C \text{Tor}_1^R(M, N)_m$ is non-zero and finite. To get a contradiction, it suffices to show that for the ideal $q = m \cap Q$ one has
\[ \text{depth}_{Q_q}(M \otimes_R N)_q = 0. \]
Indeed, set $p = qR$. As $Q$ acts on $(M \otimes_R N)_q$ via the map $Q_q \to R_p$, Lemma 2.3 and formula (2.9.1) give $\text{depth}_{Q_q}(M \otimes_R N)_q = 0$. Note that $\text{Tor}_i^R(M_q, N_q)$ is non-zero, as it is isomorphic to the module $\text{Tor}_i^R(N_q)$, which localizes to $\text{Tor}_i^R(M, N)_m \neq 0$. In particular, $R_p$ is not a field. Since $R$ is reduced, this implies $p \notin \text{Ass } R$, so Lemma 2.7 gives the desired contradiction.

It remains to establish (2.9.1). Let $x$ be a minimal generating set for the kernel of the homomorphism $Q_q \to R_p$. Lemma 1.3 and localization give
\[ \text{Tor}_j^R(M, N)_m \cong H_j(x; (M \otimes_K N)_m) \]
as $R_p$-modules. It follows that $H_j(x; (M \otimes_K N)_m)$ has non-zero finite length over $C_m$. In view of (2.8) this gives the second equality in the string
\[ \text{depth}_{C_m}(M \otimes_R N)_m = \text{depth}_{C_m} H_0(x; (M \otimes_K N)_m) = 0. \]
The first one comes from (2.9.2) with $j = 0$. Lemma 2.3, applied to the local homomorphism $Q_q \to C_m$, gives $\text{depth}_{Q_q}(M \otimes_R N)_m = 0$. Now (2.9.1) results from Lemma 2.7 and the isomorphism $(M \otimes_R N)_m \cong (M \otimes_R N)_q \otimes_{C_q} C_m$.

(2) By symmetry, it suffices to prove $\text{Tor}_i^R(M \otimes_R N) = 0$ for each $i$.

The case $i = 0$ is settled by Lemma 2.2 in view of (1).

Observe that $\text{T}_R M$ is an $A$-submodule of $M$, so the exact sequence (2.1) is one of $A$-modules. In particular, $\text{T}_R M$ is a finite $A$-module, and hence is essentially of finite type over $R$. Our hypothesis and Lemma 2.2 imply that $(\text{T}_R M) \otimes_R N$ is torsion-free over $R$, so the already established part (1) yields
\[ \text{Tor}_i^R(M, N) = 0 = \text{Tor}_i^R(\text{T}_R M, N) \quad \text{for all } i \geq 1. \]
It now follows from (2.1) that $\text{Tor}_i^R(\text{T}_R M, N) = 0$ holds for $i \geq 1$. \qed

We show that when $M$ and $N$ are modules essentially of finite type over $R$, their torsion-freeness is not related to that of $M \otimes_R N$, in general.

Example 2.10. Let $R$ be a domain and let $p \neq 0$ and $q$ be prime ideals of $R$, with $p \nsubseteq q$. The $R$-module $M = R/p$ is finite and torsion, and the $R$-module $N = k(q)$ is essentially of finite type, as witnessed by $R_q$.

One has $M \otimes_R N = 0$, so $M \otimes_R N$ is torsion-free. On the other hand, the module $N$ is torsion when $q \neq 0$, and is torsion-free when $q = 0$. 

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3. Torsion in tensor powers

In Example 2.10 we noted that a tensor product of modules essentially of finite type may be torsion-free, while its factors need not have this property. Here we prove that such a situation does not occur for tensor powers:

**Theorem 3.1.** Let \( R \) be an essentially smooth algebra over a field, \( M \) an \( R \)-module essentially of finite type, and \( d \) a positive integer.

If the \( R \)-module \( M^\otimes n \) is torsion-free, then so is \( M^\otimes k \) for \( 1 \leq n \leq d \).

**Notes.** This step is barely visible in the proof of Auslander’s theorem: When \( R \) is a regular local ring and \( M \) is a finite \( R \)-module with \( M^\otimes d \) torsion-free, [2, 3.1(b)] and [10] Cor. 2(b)] give \( \tau_R(M^\otimes n) \otimes_R M^\otimes d-n = 0 \) for \( 1 \leq n \leq d \), whence \( \tau_R(M^\otimes d) = 0 \), by Nakayama’s Lemma.

The hypothesis on \( M \) cannot be weakened too much; see Remark 3.3.

The support of a module \( M \) over a commutative ring \( A \) is the set

\[
\operatorname{Supp}_A M = \{ m \in \operatorname{Spec} A \mid M_m \neq 0 \}.
\]

Note that one has \( \operatorname{Supp}_A M = \emptyset \) if and only if \( M = 0 \).

**Lemma 3.2.** Let \( R \) be a noetherian ring, \( M \) and \( N \) be \( R \)-modules essentially of finite type, and \( A \) an \( R \)-algebra that is a witness for \( M \) and \( N \).

If \( \operatorname{Supp}_A M \cap \operatorname{Supp}_A N \neq \emptyset \) holds, then \( M \otimes_R N \neq 0 \).

**Proof.** Since \( \operatorname{Supp}_A(M \otimes_A N) = \operatorname{Supp}_A M \cap \operatorname{Supp}_A N \), one has \( M \otimes_A N \neq 0 \), and hence \( M \otimes_R N \neq 0 \), due to the surjection \( M \otimes_R N \to M \otimes_A N \). \( \square \)

In the preceding lemma, the hypothesis that a common witness exists is necessary, because a tensor product of modules essentially of finite type may be zero otherwise, even when the ring \( R \) is local; see Example 2.10.

Recall that \( \tau_R N \) denotes the \( R \)-torsion submodule of \( N \).

**Proof of Theorem 3.1.** It suffices to prove that if \( \tau_R(M^\otimes n+1) = 0 \) holds, then \( \tau_R(M^\otimes n) = 0 \) holds as well.

Theorem 2.1(2), the isomorphism \( M^\otimes n \otimes_R M \cong M^\otimes n+1 \) and the hypothesis \( \tau_R(M^\otimes n+1) = 0 \) imply \( \tau_R(M^\otimes n) \otimes_R M = 0 \). Thus, we obtain

\[
\tau_R(M^\otimes n) \otimes_R M^\otimes n \cong \tau_R(M^\otimes n) \otimes_R (M \otimes_R M^\otimes n-1) \\
\cong \left( \tau_R(M^\otimes n) \otimes_R M \right) \otimes_R M = 0.
\]

Choose a witness \( A \) for \( M \) and set \( B = A^\otimes n \). The \( R \)-algebra \( B \) is a witness for \( M^\otimes n \), and hence also for \( \tau_R(M^\otimes n) \). In view of Lemma 3.2, the equality above implies the second one of the following equalities

\[
\operatorname{Supp}_B(\tau_R(M^\otimes n)) = \operatorname{Supp}_B(\tau_R(M^\otimes n)) \cap \operatorname{Supp}_B(M^\otimes n) = \emptyset.
\]

The first one holds as \( \tau_R(M^\otimes n) \) is a submodule of \( M^\otimes n \). So \( \tau_R(M^\otimes n) = 0 \). \( \square \)

**Remark 3.3.** Let \( M \) be an \( R \)-module. Recall that \( M \) is said to be divisible if the homothety \( m \mapsto um \) is surjective for every non-zero-divisor \( u \in R \).

If \( M \) is non-zero, torsion and divisible, then \( M \otimes_R M = 0 \). Such an \( M \) is not essentially of finite type; the conclusion of Theorem 3.1 fails for it.

As an example, let \( K \) be a field, set \( R = K[x] \), and take \( M = K(x)/K[x] \).
4. Flatness

Here we prove the Main Theorem, announced in the introduction:

**Theorem 4.1.** Let $R$ be an essentially smooth algebra over a field and let $M$ be an $R$-module essentially of finite type.

If $M^{\otimes d}$ is torsion-free for some integer $d \geq \dim R$, then $M$ is flat.

**Notes.** When $R$ is a regular local ring and $M$ is a finite $R$-module, the conclusion of Theorem 4.1 is established by Auslander [2, 3.2] when $R$ is unramified and by Lichtenbaum [10, Cor. 3] in general. It is deduced from the analogs of Theorems 2.1 and 3.1 by using the additivity of projective dimensions on Tor-independent modules.

We employ a similar technique to deduce Theorem 4.1 from Theorems 2.1 and 3.1. However, projective dimensions are not always additive for non-finite modules, so we replace them with an invariant that has the desired property.

When $(S, \mathfrak{n}, l)$ is a local ring, for each $S$-module $M$ one sets

$$\text{codepth}_S M = \sup \{ i \in \mathbb{Z} \mid \text{Tor}_i^S(l, M) \neq 0 \}.$$ 

In particular, when $\text{Tor}_i^S(l, M) = 0$ for all $i$, one has $\text{codepth}_S M = -\infty$.

The name codepth is motivated by the description of depth in terms of Ext, and by the well-known result below, whose proof is included for completeness.

**Lemma 4.2.** When $(S, \mathfrak{n}, l)$ is a regular local ring and $M$ an $S$-module,

\begin{align*}
(4.2.1) \quad \text{codepth}_S M &= \dim S - \text{depth}_S M.
\end{align*}

\begin{align*}
(4.2.2) \quad \text{codepth}_S (M \otimes_S N) &= \text{codepth}_S M + \text{codepth}_S N.
\end{align*}

**Proof.** Let $s$ be a minimal generating set for $\mathfrak{n}$. As $S$ is regular, the Koszul complex on $s$ is a free resolution of $l$, and hence $\text{Tor}_i^S(l, L) \cong H_i(s; L)$. A comparison of the definitions of depth (see 2.4) and codepth validates (4.2.1).

Let $F$ and $G$ be flat resolutions of $M$ and $N$, respectively. The hypothesis translates to the statement that the complex $F \otimes_S G$ of flat $S$-modules is a resolution of $M \otimes_S N$. This gives rise to the first isomorphism below:

$$\text{Tor}_i^S(l, M \otimes_S N) \cong H_i(l \otimes_S (F \otimes_S G))$$

$$\cong H_i((l \otimes S F) \otimes_l (l \otimes S G))$$

$$\cong H_i(l \otimes S F) \otimes_l H_i(l \otimes S G)$$

$$\cong \text{Tor}_i^S(l, M) \otimes_l \text{Tor}_i^S(l, N).$$

The second one is standard; the third one is the Künneth isomorphism. Now equate the highest degree in which a vector space on either side is non-zero. □

When a module $M$ over a noetherian ring $R$ has a finite flat resolution, and $\text{fd}_R M$ denotes the shortest length of such a resolution, one has

$$\text{fd}_R M = \sup \{ \text{codepth}_{R_p} M_p \mid p \in \text{Spec } R \}$$

by Chouinard [6, 1.2]. We use only the following special case of this result.

**Lemma 4.3.** Let $R$ be a regular ring and $M$ an $R$-module.

If $\text{codepth}_{R_p} M_p \leq 0$ holds for each $p \in \text{Spec } R$, then $M$ is flat.
Proof. Since $M$ is flat when $M_p$ is flat for each $p \in \text{Spec } R$, we may assume that $R$ is a regular local ring. Every $R$-module then has finite flat dimension, see [11, 19.2], so Chouinard’s formula, recalled above, gives the result. \hfill \Box

Proof of Theorem 4.1. By Lemma 4.3 it suffices to fix $p \in \text{Spec } R$ and prove $\text{codepth}_{R_p} M_p \leq 0$. It follows from the definitions that $R_p$ is essentially smooth over a field and $M_p$ is essentially of finite type over $R_p$. The $R_p$-module $(M_p)^{\otimes d}_{R_p}$ is isomorphic to $(M^{\otimes d})_p$, and hence it is torsion-free by Lemma 2.7. Thus, we assume that $R$ is local with maximal ideal $p$, and we set out to prove that if $M^{\otimes d}$ is torsion-free for some $d \geq \dim R$, then $\text{codepth}_R M \leq 0$.

If $\dim R = 0$, then $R$ is a field, and the assertion is obvious. Assume $\dim R \geq 1$. Theorem 3.1 shows that the $R$-module $M^{\otimes n}$ is torsion-free for $1 \leq n \leq d$. Thus Theorem 2.1(1), applied with $N = M^{\otimes n-1}$, yields

$$\text{Tor}_i^R(M, M^{\otimes n-1}) = 0 \quad \text{for each } n \text{ with } 2 \leq n \leq d \text{ and each } i \geq 1.$$ 

Repeated application of formula (4.2.2) then gives the equality below:

$$d \ \text{codepth}_R M = \text{codepth}_R (M^{\otimes d}) \leq \dim R - 1 < d.$$ 

The first inequality comes from (4.2.1) and Lemma 2.6, as $M^{\otimes d}$ is torsion-free; the second one holds by hypothesis. As a result, we get $\text{codepth}_R M \leq 0$. \hfill \Box

A reformulation of Theorem 4.1 gives the geometric criterion for flatness, stated in the introduction:

Remark 4.4. With $R$ and $M$ as in Theorem 4.1, let $B$ be a witness for $M^{\otimes d}$ and $\beta : R \to B$ the structure homomorphism; for example, set $B = A^{\otimes d}$, with $A$ a witness for $M$. For the induced map $^*\beta : \text{Spec } B \to \text{Spec } R$, one has:

If $^*\beta(\text{Ass}_R(M^{\otimes d}))$ is contained in $\text{Ass } R$, then $M$ is flat over $R$.

Indeed, [11, Ex. 6.7] gives $^*\beta(\text{Ass}_B(M^{\otimes d})) = \text{Ass}_R(M^{\otimes d})$, so the hypothesis yields $\text{Ass}_R(M^{\otimes d}) \subseteq \text{Ass } R$, hence $M^{\otimes d}$ is torsion-free over $R$; see Lemma 2.6.

5. Dimension Two

It is natural to ask whether the conclusion of Theorem 4.1 holds under the weaker assumptions that $R$ is a regular ring of finite Krull dimension and $M$ is module that is finite over a noetherian $R$-algebra. We give a positive answer when $d = \dim R \leq 2$, by extending the argument used by Vasconcelos to prove the case $M = A$; see [13, 6.1].

Proposition 5.1. Let $R$ be a regular ring with $\dim R \leq 2$, let $A$ be a noetherian $R$-algebra, and let $M$ be a finite $A$-module.

If the $R$-module $M \otimes_R M$ is torsion-free, then $M$ is flat over $R$.

Proof. It suffices to fix $p$ in $\text{Spec } R$ and prove $\text{codepth}_{R_p} M_p \leq 0$; see Lemma 4.3. Using Lemma 2.7 one can reduce to the case where $R$ is local, with $p$ its maximal ideal, so the desired result is that $\text{codepth}_R M \leq 0$. We may assume $\dim R \geq 1$.

It is enough to prove that $M$ is torsion-free and that there are equalities:

$$\text{Tor}_i^R(M, M) = 0 \quad \text{for } i = 1, 2.$$
Indeed, since $R$ is regular with $\dim R \leq 2$, they imply $\Tor_i^R(M, M) = 0$ for each $i \geq 1$. One has codepth $R(M \otimes R M) \leq 1$, by (4.2.1) and Lemma 2.6, so Lemma 4.2(2), applied with $N = M$, yields codepth $R M \leq 0$, as desired.

Recall that $\pi_R M$ is the $R$-torsion submodule of $M$ and $\perp_R M = M/\pi_R M$. As $\perp_R M$ is torsion-free and $R$ is a domain, there is an exact sequence

$$0 \rightarrow \perp_R M \rightarrow U^{-1}(\perp_R M) \rightarrow C \rightarrow 0$$

of $R$-modules, where $U = R \setminus \{0\}$. As $U^{-1}(\perp_R M)$ is flat over $R$, and one has $\text{fd}_R C \leq 2$, we obtain $\text{fd}_R (\perp_R M) \leq 1$. This gives the equalities below:

$$\Tor_1^R(\perp_R M, \perp_R M) \cong \Tor_2^R(\perp_R M, C) = 0 = \Tor_2^R(\perp_R M, \perp_R M).$$

The isomorphism is obtained by tensoring the sequence above with $\perp_R M$. To finish, we prove that $M$ is torsion-free; that is to say, $\pi_R M = 0$ holds.

By way of contradiction, assume $\pi_R M \neq 0$. As $\pi_R M$ is an $A$-module, we have $(\pi_R M)_m \neq 0$ for some $m$ in $\text{Spec } A$. There is a natural isomorphism of $A_m$-modules $(\pi_R M)_m \cong \pi_R(M_m)$, and $M_m \otimes R M_m$ is torsion-free over $R$, as it is a localization of $M \otimes R M$. Thus, we may also assume that $A$ is local.

Lemma 2.2 applied with $N = M$, shows that $M \otimes_R \perp_R M$ is torsion-free. As $\Tor_i^R(\perp_R M, \perp_R M) = 0$ holds, the last assertion in Lemma 2.2, now applied with $N = \perp_R M$, gives $\pi_R M \otimes_R \perp_R M = 0$. Note that $\pi_R M$, being an submodule of the finite $A$-module $M$, is itself finite, so $A$ is a witness for both $\pi_R M$ and $\perp_R M$.

As the maximal ideal of $A$ is contained in the support of every non-zero finite $A$-module, and $\pi_R M$ is non-zero, Lemma 3.2 implies $\perp_R M = 0$. Thus, $M$ is a torsion $R$-module, and then so is $M \otimes_R M$.

This contradicts our hypothesis. 

\hspace{1cm} \Box

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**References**

[1] J. Adamus, E. Bierstone, P. D. Milman, \textit{Geometric Auslander criterion for flatness of an analytic mapping}, preprint: \texttt{arXiv:0901.2744}

[2] M. Auslander, \textit{Modules over unramified regular local rings}, Illinois J. Math. \textbf{5} (1961), 631–647.

[3] M. Auslander, D. Buchsbaum, \textit{Codimension and multiplicity}, Annals of Math. \textbf{68} (1958), 625–657; \textit{Corrections}, ibid, \textbf{70} (1959), 395–397.

[4] N. Bourbaki, \textit{Élémets de mathématique, Algèbre commutative. Chapitre 10}, Springer-Verlag, Berlin, 2007.

[5] H. Cartan, S. Eilenberg, \textit{Homological algebra}, Princeton Univ. Press, Princeton, NJ, 1956.

[6] L. G. Chouinard, II, \textit{On finite weak and injective dimension}, Proc. Amer. Math. Soc. \textbf{60} (1976), 57–60.

[7] A. Galligo, M. Kwieciniński, \textit{Flatness and fibred powers over smooth varieties}, J. Algebra \textbf{232} (2000), 48–63.

[8] C. Huneke, R. Wiegand, \textit{Tensor products of modules, rigidity and local cohomology}, Math. Scand. \textbf{81} (1997), 161–183.

[9] R. Hartshorne, \textit{Algebraic Geometry}, Graduate Texts Math. \textbf{52}, Springer-Verlag, New York, 1977.

[10] S. Lichtenbaum, \textit{On the vanishing of Tor in regular local rings}, Illinois J. Math. \textbf{10} (1966), 220–226.
[11] H. Matsumura, *Commutative ring theory*, Cambridge Stud. Adv. Math. 8, Cambridge Univ. Press, Cambridge, 1986.

[12] K. Takahashi, H. Terakawa, K.-I. Kawasaki, Y. Hinohara, *A note on the new rigidity theorem for Koszul complexes*, Far East J. Math. Sci. 20 (2006), 269–281.

[13] W. V. Vasconcelos, *Flatness testing and torsionfree morphisms*, J. Pure Appl. Algebra 122 (1997), 313–321.

Department of Mathematics, University of Nebraska, Lincoln, NE 68588, U.S.A.

E-mail address: avramov@math.unl.edu

Department of Mathematics, University of Nebraska, Lincoln, NE 68588, U.S.A.

E-mail address: iyengar@math.unl.edu