Further Remarks On Somewhere Dense Sets

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ABSTRACT

In this article, we prove that a topological space $X$ is strongly hyperconnected if and only if any somewhere dense set in $X$ is open, in addition we investigate some conditions that make sets somewhere dense in subspaces, finally, we show that any topological space defined on infinite set $X$ has SD-cover with no proper subcover.

Introduction:

Using the closure and the interior operators in topological space, different types of generalized open sets have been defined as; $\alpha$-set, semi-open set, preopen set, $\beta$-open set, $b$-open set and somewhere dense [1,2,3,4,5,6]. The concept of somewhere dense set was due to Pugh [6], where a set $A$ is somewhere dense if the interior of its closure is non-empty, clearly somewhere dense set is a generalization of both open set and dense set. In 2017, Alshami [7] provided the properties of somewhere dense sets, and he introduced the axiom of $SD_1$, then with Noiri they defined the notion of SD-cover and use it to introduced compactness and Lindelöfness via somewhere dense sets, see [8,9].

A space is hyperconnected [10], if every two non-empty open sets intersect; equivalently if any non-empty open set is dense, while a space is submaximal [11] if any dense set is open, and in 1994 Rose and Mahmoud [12] showed that a space is submaximal if and only if every preopen set is open, for more details see [13,14,15,16,17,18,19,20,21]. Recently, Alshami [7,8] defined strongly hyperconnected space as a hyperconnected submaximal space, and he charactrized this space using the notion of somewhere dense sets [8].

The main goal of this article, is to continue studying further properties on somewhere dense sets, and imporve some of the results given by Alshami and Noiri [7,8] regarding strongly hyperconnected space. Here we give solutions to the following questions, equiped with some examples:

Question 1. Find the necessary and sufficient condition under which every somewhere dense set is open?.

Question 2. If $(X,\tau)$ is a topological space and $Y \subseteq X$: find conditions under which set in the subspace $Y$ is somewhere dense in $X$?.

Question 3. If $(X,\tau)$ is a topological space where $X$ is infinite set: find a cover for $X$ by somewhere dense sets (SD-cover) which has no proper subcover.

The article is divided into four sections: somewhere dense sets in strongly hyperconnected space, somewhere dense sets in subspaces, SD-covers and conclusion.

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represents topological space and for a subset \( A \) of a space \( X, \overline{A} \) and \( A^{o} \) denote the closure and the interior of \( A \); respectively. Moreover, \( X/A \) (or \( A' \)), \( A/B \) and \( P(X) \) denote the complement of the set \( A \) in \( X \), the difference of \( A \) and \( B \), and the power set of \( X \); respectively, while \( \sim \) denotes the equivalence relation, and \( \alpha, \chi \) are the cardinality of the natural numbers \( \mathbb{N} \) and the real numbers \( \mathbb{R} \); respectively.

2. Somewhere Dense Sets In Strongly Hyperconnected Spaces

This section, consists basic definitions, theorems and some properties regarding somewhere dense sets needed in this work, and then we give a complete answer for question 1, by studying the statement when any somewhere dense set is open.

**Definition 2.1.** [7] A subset \( B \) of a topological space \((X, \tau)\) is called somewhere dense (briefly \( s\)-dense) if the interior of its closure is non-empty, i.e. \( \overline{B} \neq \emptyset \). The complement of \( s\)-dense set is called closed somewhere dense (briefly \( cs\)-dense), and the collection of all \( s\)-dense sets in \( X \) is denoted by \( S(\tau) \).

**Corollary 2.1.** [7] In a topological space \((X, \tau)\), we have:

\[ i. \text{ any open set is } s\text{-dense.} \]
\[ ii. \text{ any dense set is } s\text{-dense} \]
\[ iii. \text{ any set in } X \text{ that contains a } s\text{-dense set is } s\text{-dense.} \]

**Theorem 2.1.** [7] Every subset of a space \((X, \tau)\) is \( s\)-dense or \( cs\)-dense.

**Theorem 2.2.** For a non-discrete topological space \((X, \tau)\), the following are equivalent:

1. \( S(\tau) = \{ \emptyset \} \).
2. \( D(\tau) = \{ \emptyset \} \), where \( D(\tau) \) denotes the collection of all dense sets in \((X, \tau)\).

**Proof.**

1) \( \implies \) 2) Let \((X, \tau)\) be a non-discrete space that satisfies \( S(\tau) = \{ \emptyset \} \), and suppose that \( D(\tau) = \{ \emptyset \} \). From corollary 2.1.(i) any dense set is \( s\)-dense, then from assumption any dense set is open so \( D(\tau) = \tau \). Suppose that \( A \) is a non-empty open set but not dense in \( X \), hence \( \overline{A} \neq X \), therefore \( \overline{A} \neq \emptyset \). Now if \( x \in X \), then \( x \notin \overline{A} \) or \( x \notin \overline{A} \). In the case when \( x \in \overline{A} \), we have \( A \) is open so it is \( s\)-dense, and since \( \overline{A} \subseteq A \) from corollary 2.1.(iii) we obtain \( \overline{A} \) is also \( s\)-dense, so it is open. Since \( \overline{A} \) is open, so it is \( s\)-dense, then \( \overline{A} \cup \{ x \} \) is also \( s\)-dense, so it is open, and the intersection of two open sets is open, hence \( \{ x \} = \overline{A} \cap (\overline{A} \cup \{ x \}) \) is open. In the second case when \( x \notin \overline{A} \), and by similar reasons the sets \( \overline{A} \) and \( \overline{A} \cup \{ x \} \) are both open, then their intersection is also open, hence \( \{ x \} = \overline{A} \cap (\overline{A} \cup \{ x \}) \) is open. Thus \( \{ x \} \) is open for any \( x \in X \), hence \( \tau \) is the discrete topology on \( X \), which contradicts the assumption. Thus complete the prove, and \( D(\tau) = \{ \emptyset \} \).

2) \( \implies \) 1) Let \((X, \tau)\) be a non-discrete space that satisfies \( D(\tau) = \{ \emptyset \} \), and suppose \( A \) is a \( s\)-dense in \( X \), then \( \overline{A} \) is a non-empty open set in \( X \), therefore \( \overline{A} \) is dense in \( X \), but \( \overline{A} \) is open and \( \overline{A} \cap \overline{A} = \emptyset \), since \( \overline{A} \) is dense we have \( \overline{A} = \emptyset \), so \( \overline{A} \) is dense, as it is open, hence it is \( s\)-dense. Since, any open set is \( s\)-dense, we obtain \( S(\tau) = \{ \emptyset \} \).

**Definition 2.2.** [10,11] A topological space \((X, \tau)\) is called:

1) Submaximal if any dense set in \( X \) is open.
2) Hyperconnected if any non-empty open set in \( X \) is dense.

**Corollary 2.2.** If \((X, \tau)\) is submaximal and hyperconnected, then \( D(\tau) = \{ \emptyset \} \).

**Definition 2.3.** [7] A topological space \((X, \tau)\) is called strongly hyperconnected if non-empty open sets are coincide with dense sets, equivalently if \( D(\tau) = \{ \emptyset \} \).

**Corollary 2.3.** For a space \((X, \tau)\) the following are equivalent:

1) \( X \) is strongly hyperconnected space.
2) \( X \) is submaximal and hyperconnected space.
3) Any \( s\)-dense set in \( X \) is open.
4) \( D(\tau) = \{ \emptyset \} \).
5) \( S(\tau) = \{ \emptyset \} \).

**Examples 2.1.**

\[ i. \text{ If } (X, \tau) \text{ is a trivial topological space where } X \text{ has more than one element, then } X \text{ is not strongly hyperconnected since } S(\tau) = P(X) \text{ and } D(\tau) = \{ \emptyset \}. \]

\[ ii. \text{ If } (X, \tau) \text{ is a topological space where } \tau = \{ \emptyset \} \text{ and } X \text{ is a strongly hyperconnected, then } X \text{ is a strongly hyperconnected.} \]

\[ iii. \text{ The topological space } (X, \tau) \text{ where } \tau = \{ \emptyset \} \text{ is submaximal but not strongly hyperconnected space, since } S(\tau) = P(\emptyset) \text{ and } D(\tau) = \{ \emptyset \}. \]

3. Somewhere Dense Sets In Subspaces

Here we answer question 2 by investigating some conditions in topological space \( X \) that make a set in a subspace somewhere dense in \( X \).

**Corollary 3.1.** [11] Let \((X, \tau)\) be a topological space, \( Y \) be a subspace of \( X \) and \( A \subseteq Y \), then:

1) \( \overline{A} \subseteq \overline{A} \cap Y \) (where \( \overline{A} \) is the closure of \( A \) with respect to the subspace \( Y \)).
2) \( A^{o} \subseteq A^{o} \cap Y \) and \( A^{o} = A^{o} \cap Y \) (where \( A^{o} \) is the interior of \( A \) with respect to the subspace \( Y \)).

**Lemma 3.1.** Let \((X, \tau)\) be a topological space, \( Y \) be a subspace of \( X \), then:

1) If \( A \subseteq Y \) and \( Y \) is closed, then \( \overline{A} \cap Y = \overline{A} \cap Y \).
2) If \( Y \) is open and \( A \subseteq Y \), then \( A^{o} \cap Y = A^{o} \cap Y \).
3) If \( A \subseteq Y \) and \( Y \) is clopen, then \( \overline{A} \cap Y = \overline{A} \cap Y \) and \( A^{o} = A^{o} \cap Y \).

**Proof.**

1) Since \( A \subseteq Y \) and \( Y \) is closed, then \( \overline{A} \cap Y = \overline{A} \cap Y \). Since \( A \subseteq Y \) and \( Y \) is closed, then \( A = A^{o} \cap Y \).
2) Since \( A \subseteq Y \) and \( Y \) is open, then \( A^{o} \subseteq A^{o} \cap Y \) and \( A^{o} = A^{o} \cap Y \).
3) Since \( A \subseteq Y \) and \( Y \) is clopen, then \( \overline{A} \cap Y = \overline{A} \cap Y \) and \( A^{o} = A^{o} \cap Y \).

**Example 3.1.** In the space \( (\mathbb{R}, \tau) \) where \( \tau = \{ \emptyset \} \), we have \( S(\tau) = \{ \emptyset \} \) and \( D(\tau) = \{ \emptyset \} \).

**Remark 3.2.** If \( Y \) is a closed subset of \( X \) and \( A \subseteq Y \), then \( A \subseteq Y \). Note that, any \( s\)-closed set is closed.

**Theorem 3.2.** Let \((X, \tau)\) be a topological space, \( Y \) be a \( r\)-closed subspace of \( X \), then \( A \subseteq Y \) if \( A \subseteq Y \).

**Proof.** Suppose \( Y \) is a subspace of \( X \), then \( A \subseteq Y \). Suppose \( Y \) is a \( r\)-closed subspace of \( X \), then \( A \subseteq Y \). Since \( Y \) is a \( r\)-closed subspace of \( X \), then \( A \subseteq Y \). Suppose \( Y \) is a \( r\)-closed subspace of \( X \), then \( A \subseteq Y \). Hence, \( A \subseteq Y \).
In the present section, we answer question 3 by proving that any somewhere dense subsets is called SD dense subsets is called SD. However, in infinite sets, and we have obtained few results; as follows:

Theorem 4.1. If \( A \) is an SD cover, but the converse is not true.

**Proof:** By the mathematical induction, we have:

1. If \( \|A\|=\aleph_0 \), then \( Z \sim N \sim E \sim E^c \), where \( N \) and \( E \) are the natural numbers and the even numbers, respectively. Hence, there exists a bijection function \( F: \mathbb{N} \to \mathbb{N} \), so that \( |A| = |\mathbb{N}| = \aleph_0 \), and \( A = \mathbb{N} \) is a subset of \( Z \).

2. If \( \|A\|=\aleph_1 \), then \( Z \sim R \sim R^* \sim (R^*)^c \); where \( R \) and \( R^* \) are the real numbers and the positive real numbers, respectively. Hence, there exists a bijection function \( F: R \to Z \).

3. Suppose \( |A|=\aleph_2 \) and \( A \) is a subset of \( Z \) such that \( |A| = |\mathbb{N}| = \aleph_0 \). Let \( Y \) be a set with cardinal number \( \aleph_0+1 \), then \( P(Z) \sim Y \), and if \( P(Z) \sim P(A) \sim P(A^c) \), then \( F \) is a bijection function \( F: \mathbb{N} \to \mathbb{N} \). Then \( B = P(A) \sim Y \). Now since \( P(A^c) \subseteq P(A^c) \), and \( P(A^c) \subseteq B \), then \( F(P(A^c)) \subseteq B \). Hence, \( |A|=\aleph_2 \) is the SD cover for \( X \) with no proper subcover. Similarly, in the case when \( A^c \) is s-dense, we have \( |A \cup X| \) is SD cover for \( X \) with no proper subcover.

In the case when the subspace \( Y \) is open (closed) in \( X \), if \( A \) is somewhere dense in \( Y \), then \( A \) is somewhere dense in \( X \), while in the case when the subspace \( Y \) is regular closed in \( X \), \( A \) is somewhere dense in \( Y \) if and only if \( A \) is somewhere dense in \( X \).

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