Finite Dimensional Representations of Quadratic Algebras with Three Generators and Applications *

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Abstract

The finite dimensional representations of associative quadratic algebras with three generators are investigated by using a technique based on the deformed parafermionic oscillator algebra. One application on the calculation of the eigenvalues of the two-dimensional superintegrable systems is discussed.

1 Introduction

In classical mechanics, integrable system is a system possessing more constants of motion in addition to the energy. A comprehencive review of the two-dimensional integrable classical systems is given by Hietarinta[1], where the space was assumed to be flat. The case of non flat space is under current investigation[2].

An interesting subset of the totality of integrable systems is the set of systems, which possess a maximum number of integrals, these systems are

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termed as superintegrable ones. The Coulomb and the harmonic oscillator potentials are the most familiar classical superintegrable systems, whose quantum counterpart has nice symmetry properties, which are described by the \(so(N + 1)\) and \(su(N)\) Lie algebras.

The Hamiltonian of the classical systems is a quadratic function of the momenta. In the case of the flat space all the known two dimensional superintegrable systems with quadratic integrals of motion are simultaneously separable in more than two orthogonal coordinate systems\[3\]. The integrals of motion of a two dimensional superintegrable system in flat space close in a classical quadratic Poisson algebra\[1\]. The study of the quadratic Poisson algebras is a matter under investigation related to several branches of physics as: the solution of the classical Yang - Baxter equation \[4\], the two dimensional superintegrable systems in flat space or on the sphere \[4\], the statistics \[6\] or the case of ”exactly solvable” classical problems \[7\].

The quantization of classical integrable systems turns generally to quantum integrable systems, but sometime one has to add correction terms to the integrals of motion or to the Hamiltonian, these correction terms seem to be of order \(O(h^2)\) \[8\]. The classical Poisson algebra is shifted to some quantum polynomial algebra, the same thing is true in the case of quadratic Poisson algebra corresponding to the Yang - Baxter equation\[5\], which is turned to a quantum quadratic associative algebra\[9\]. The same idea was discussed in ref.\[7\], where the classical problems, which are expressed by a quadratic Poisson algebra are mapped to quantum ones described by the corresponding quantum operator quadratic algebra. The same shift is indeed true for the superintegrable systems, where the classical ones correspond to the quantum ones and the classical quadratic Poisson algebra is mapped to a quadratic associative algebra\[10\]–\[16\].

In this contribution we study the general form of the quadratic algebras, which are encountered in the case of the two dimensional quantum superintegrable systems, these algebras are called \(Qu(3)\). In references \[14\], \[10\], \[11\] was conjectured that, the energy eigenvalues correspond to finite dimensional representations of the latent quadratic algebras. Granovkii et al in \[7\] studied the representations of the quadratic Askey - Wilson algebras \(QAW\). Using there the proposed ladder representation, the finite dimensional representations are calculated and this method was applied to several superintegrable systems \[10\]–\[12\]. Another method\[13\], \[14\] for calculating the finite dimensional representations is the use of the deformed oscillator algebra \[17\] and their finite dimensional version which are termed as generalized deformed
parafermionic algebras[18].

2 The Qu(3) Algebra

Let consider the quadratic associative algebra generated by the generators \{A, B, C\}, which satisfy the commutation relations

\[
\begin{align*}
[A, B] &= C \\
[A, C] &= \alpha A^2 + \beta B^2 + \gamma \{A, B\} + \delta A + \epsilon B + \zeta \\
[B, C] &= aA^2 + bB^2 + c \{A, B\} + dA + eB + z
\end{align*}
\]

(1)

After rotating the generators A and B, we can always consider the case \(\beta = 0\).

The Jacobi equality for the commutator induces the relation

\[
[A, [B, C]] = [B, [A, C]]
\]

the following relations

\[
b = -\gamma, \quad c = -\alpha \quad \text{and} \quad e = -\delta
\]

must be satisfied, and consequently the general form of the quadratic algebra (1) can be explicitly written as follows:

\[
\begin{align*}
[A, B] &= C \\
[A, C] &= \alpha A^2 + \gamma \{A, B\} + \delta A + \epsilon B + \zeta \\
[B, C] &= aA^2 - \gamma B^2 - \alpha \{A, B\} + dA - \delta B + z
\end{align*}
\]

(2)

(3)

(4)

The Casimir of this algebra is given by:

\[
K = C^2 - \alpha \{A^2, B\} - \gamma \{A, B^2\} + (\alpha \gamma - \delta) \{A, B\} + \\
+ (\gamma^2 - \epsilon) B^2 + (\gamma \delta - 2\zeta) B + \\
+ \left(\frac{2\alpha}{3} A^3 + (d + \frac{2\gamma}{3}) + \alpha^2\right) A^2 + \left(\frac{a \epsilon}{3} + \alpha \delta + 2z\right) A^n
\]

(5)

another useful form of the Casimir of the algebra is given by:

\[
K = C^2 + \frac{2\alpha}{3} A^3 - \frac{\gamma}{3} \{A, A, B\} - \frac{\gamma}{3} \{A, B, B\} + \\
+ \left(\frac{2\alpha}{3} + d + \frac{2\gamma}{3}\right) A^2 + \left(-\epsilon + \frac{2\gamma}{3}\right) B^2 + \\
+ \left(-\delta + \frac{2\gamma}{3}\right) \{A, B\} + \left(\frac{2\alpha \delta}{3} + \frac{a \epsilon}{3} + \frac{d \gamma}{3} + 2z\right) A + \\
+ \left(-\frac{\alpha \epsilon}{3} + \frac{2\gamma}{3} - 2\zeta\right) B + \frac{2\gamma}{3} - \frac{\alpha \delta}{3}
\]

(6)
This quadratic algebra has many similarities to the Racah algebra $QR(3)$, which is a special case of the Askey - Wilson algebra $QAW(3)$. The algebra (2 - 4) does not coincide with the Racah algebra $QR(3)$, if $a \neq 0$ in the relation (2). We shall call this algebra $Qu(3)$ algebra. Unless this difference between $Qu(3)$ and $QR(3)$ algebra a representation theory can be constructed by following the same procedures as they were described by Granovskii, Lutzenko and Zhedanov in ref. [7, 10, 11]. In this paper we shall give a realization of this algebra using the deformed oscillator techniques [17]. The finite dimensional representations of the algebra $Qu(3)$ will be constructed by constructing a realization of the algebra $Qu(3)$ with the generalized parafermionic algebra introduced by Quesne [18].

3 Deformed Parafermionic Algebra

Let now consider a realization of the algebra $Qu(3)$, by using of the deformed oscillator technique, i.e. by using a deformed oscillator algebra [17] $\{b^\dagger, b, \mathcal{N}\}$, which satisfies the

$$\left[\mathcal{N}, b^\dagger\right] = b^\dagger, \quad \left[\mathcal{N}, b\right] = -b, \quad b^\dagger b = \Phi(\mathcal{N}), \quad bb^\dagger = \Phi(\mathcal{N} + 1)$$

(7)

where the function $\Phi(x)$ is a "well behaved" real function which satisfies the

$$\Phi(0) = 0, \quad \Phi(x), \quad \text{for} \quad x > 0$$

(8)

As it is well known [17] this constraint imposes the existence a Fock type representation of the deformed oscillator algebra, which is bounded by bellow, i.e. there is a Fock basis $|n>,$ $n = 0, 1, \ldots$ such that

$$\mathcal{N}|n> = n|n>$$

$$b^\dagger|n> = \sqrt{\Phi(n + 1)}|n + 1>,$$ $n = 0, 1, \ldots$

$$b|0> = 0$$

$$b|n> = \sqrt{\Phi(n)}|n - 1>,$$ $n = 1, 2, \ldots$

(9)

The Fock representation (9) is bounded by bellow.

In the case of nilpotent deformed oscillator algebras, there is a positive integer $p$, such that

$$b^{p+1} = 0, \quad (b^\dagger)^{p+1} = 0$$

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the above equations imply that
\[ \Phi(p + 1) = 0, \tag{10} \]

In that case the deformed oscillator (7) has a finite dimensional representation, with dimension equal to \( p + 1 \), this kind of oscillators are called deformed parafermion oscillators of order \( p \).

An interesting property of the deformed parafermionic algebra is that the existence of a faithfull finite dimensional representation of the algebra implies that:
\[ \mathcal{N} (\mathcal{N} - 1) (\mathcal{N} - 2) \cdots (\mathcal{N} - p) = 0 \tag{11} \]

The above restriction and the constraints (8) and (10) imply that the general form of the structure function \( \Phi(\mathcal{N}) \) has the general form [18]:
\[ \Phi(\mathcal{N}) = \mathcal{N} (p + 1 - \mathcal{N}) (a_0 + a_1 \mathcal{N} + a_2 \mathcal{N}^2 + \cdots a_{p-1} \mathcal{N}^{p-1}) \]

4 Oscillator realization of the algebra \( Qu(3) \)

We shall show, that there is a realization of the algebra \( Qu(3) \), such that
\[ A = A(\mathcal{N}) \tag{12} \]
\[ B = b(\mathcal{N}) + b^\dagger \rho(\mathcal{N}) + \rho(\mathcal{N}) b \tag{13} \]
where the \( A[x], b[x] \) and \( \rho(x) \) are functions, which will be determined. In that case (3) implies:
\[ C = [A, B] \Rightarrow C = b^\dagger \Delta A(\mathcal{N}) \rho(\mathcal{N}) - \rho(\mathcal{N}) \Delta A(\mathcal{N}) b \tag{14} \]
where
\[ \Delta A(\mathcal{N}) = A(\mathcal{N} + 1) - A(\mathcal{N}) \]

Using equations (12), (13) and (3) we find:
\[ [A, C] = [A(\mathcal{N}), b^\dagger \Delta A(\mathcal{N}) \rho(\mathcal{N}) - \rho(\mathcal{N}) \Delta A(\mathcal{N}) b] = \]
\[ = b^\dagger (\Delta A(\mathcal{N}))^2 \rho(\mathcal{N}) + \rho(\mathcal{N}) (\Delta A(\mathcal{N}))^2 b = \]
\[ = \alpha A^2 + \gamma \{A, B\} + \delta A + \epsilon B + \zeta = \]
\[ = b^\dagger (\gamma (A(\mathcal{N} + 1) + A(\mathcal{N})) + \epsilon) \rho(\mathcal{N}) + \]
\[ + \rho(\mathcal{N}) (\gamma (A(\mathcal{N} + 1) + A(\mathcal{N})) + \epsilon) b + \]
\[ + \alpha A(\mathcal{N})^2 + 2\gamma A(\mathcal{N}) b(\mathcal{N}) + \delta A(\mathcal{N}) + \epsilon B(\mathcal{N}) + \zeta \tag{15} \]
therefore we have the following relations:

\[(\Delta A(N))^2 = \gamma (A(N + 1) + A(N)) + \epsilon \quad (16)\]
\[\alpha A(N)^2 + 2\gamma A(N) b(N) + \delta A(N) + \epsilon b(N) + \zeta = 0 \quad (17)\]

while the function \(\rho(N)\) can be arbitrarily determined. In fact this function can be fixed, in order to have a polynomial structure function \(\Phi(x)\) for the deformed oscillator algebra \([3]\).

The solutions of equation (16) depend on the value of the parameter \(\gamma\), while the function \(b(N)\) is uniquely determined by equation (17) (provided that almost one among the parameters \(\gamma\) or \(\epsilon\) is not zero). At this stage, the cases \(\gamma \neq 0\) or \(\gamma = 0\), should be treated separately. We can see that:

**Case 1: \(\gamma \neq 0\)**

In that case the solutions of equations (16) and (17) are given by:

\[A(N) = \frac{\gamma}{2} \left( (N + u)^2 - 1/4 - \frac{\epsilon}{\gamma^2} \right) \quad (18)\]
\[b(N) = -\frac{\alpha((N+u)^{2-1/4})}{4\gamma^4} + \frac{\alpha\epsilon-\delta}{2\gamma^2} - \frac{\alpha^2-2\delta\epsilon+4\gamma^2\zeta}{4\gamma^4} \left( (N+u)^{2-1/4} \right) \quad (19)\]

**Case 2: \(\gamma = 0, \epsilon \neq 0\)**

The solutions of equations (16) and (17) are given by:

\[A(N) = \sqrt{\epsilon} (N + u) \quad (20)\]
\[b(N) = -\alpha (N + u)^2 - \frac{\delta}{\sqrt{\epsilon}} (N + u) - \frac{\zeta}{\epsilon} \quad (21)\]

The constant \(u\) will be determined later.

Using the above definitions of equations \(A(N)\) and \(b(N)\), the left hand side and right hand side of equation (4) gives the following equation:

\[2 \Phi(N + 1) \left( \Delta A(N) + \frac{\alpha}{2} \right) \rho(N) - 2 \Phi(N) \left( \Delta A(N - 1) - \frac{\alpha}{2} \right) \rho(N - 1) =
= aA^2(N) - \gamma b^2(N) - 2\alpha A(N)b(N) + dA(N) - \delta b(N) + z \quad (22)\]
Equation (5) gives the following relation:

\[ K = \Phi(N+1)(\gamma^2 - \epsilon - 2\gamma A(N) - \Delta A^2(N)) + \Phi(N) (\gamma^2 - \epsilon - 2\gamma A(N) - \Delta A^2(N-1)) - \rho(N) - 2\alpha A^2(N) b(N) + \gamma^2 - \epsilon - 2\gamma A(N) b^2(N) + 2(\alpha\gamma - \delta) A(N) b(N) + (\gamma - 2\zeta) b(N) \]

Equations (22) and (23) are linear functions of the expressions \( \Phi(N) \) and \( \Phi(N+1) \), then the function \( \Phi(N) \) can be determined, if the function \( \rho(N) \) is given. The solution of this system, i.e. the function \( \Phi(N) \) depends on two parameters \( u \) and \( K \) and it is given by the following formulae:

**Case 1:** \( \gamma \neq 0 \)

\[ \rho(N) = \frac{1}{3 \cdot 2^{12} \cdot \gamma^8(N+u)(1+N+u)(1+2(N+u))^2} \]

and

\[ \Phi(N) = -3072\gamma^6K(-1 + 2(N + u))^2 - 48\gamma^6(\alpha^2\epsilon - \alpha\delta\gamma + a\epsilon\gamma - d\gamma^2) \cdot (-3 + 2(N + u))(1 + 2(N + u))^4(1 + 2(N + u)) + \gamma^8(3\alpha^2 + 4a\gamma)(-3 + 2(N + u))^2(-1 + 2(N + u))^4(1 + 2(N + u))^2 + 768(\alpha\epsilon^2 - 2d\epsilon\gamma + 4\gamma^2\zeta)^2 + 32\gamma^4(-1 + 2(N + u))^2(-1 - 12(N + u) + 12(N + u)^2)^2 \cdot (3\alpha^2\epsilon^2 - 6\alpha\delta\epsilon\gamma + 2ae\epsilon^2\gamma + 2\delta^2\gamma^2 - 4d\epsilon\gamma^2 + 8\gamma^3z + 4\alpha\epsilon\gamma^2\zeta) - 256\gamma^2(-1 + 2(N + u))^2 \cdot (3\alpha^2\epsilon^3 - 9\alpha\delta\epsilon^2\gamma + a\epsilon^3\gamma + 6\delta^2\epsilon\gamma^2 - 3d\epsilon^2\gamma^2 + 2\delta^2\gamma^4 + 2d\epsilon\gamma^4 + 12e\gamma^3z - 4\gamma^5z + 12\alpha\epsilon\gamma^2\zeta - 12\delta\gamma^3\zeta + 4\alpha\gamma^4\zeta) \]

**Case 2:** \( \gamma = 0, \epsilon \neq 0 \)

\[ \rho(N) = 1 \]
5 \ Finite \ dimensional \ representations \ of \ the \ algebra \ Qu(3)

Let consider a representation of the algebra Qu(3), which is diagonal to the generator \(A\) and the Casimir \(K\). Using the parafermionic realization defined by equations (12) and (13), we see that this a representation diagonal to the parafermionic number operator \(N\) and the Casimir \(K\). The basis of a such representation corresponds to the Fock basis of the parafermionic oscillator, i.e. the vectors \(|k, n>\), \(n = 0, 1, \ldots\) of the carrier Fock space satisfy the equations

\[
N |k, n> = n |k, n>, \quad K |k, n> = k |k, n>
\]

The structure function (24) (or respectively (24)) depend on the eigenvalues of the of the parafermionic number operator \(N\) and the Casimir \(K\). The vectors \(|k, n>\) are also eigenvectors of the generator \(A\), i.e.

\[
A |k, n> = A(k, n) |k, n>
\]

In the case \(\gamma \neq 0\) we find from equation (18)

\[
A(k, n) = \frac{\gamma}{2} \left( (n + u)^2 - 1/4 - \frac{\epsilon}{\gamma^2} \right)
\]

In the case \(\gamma = 0, \ \epsilon \neq 0\) we find from equation (20)

\[
A(k, n) = \sqrt{\epsilon} (n + u)
\]

then the parameter \(u = u(k, p)\) is a solution of the system of equations (26).

If the deformed oscillator corresponds to a deformed Parafermionic oscillator of order \(p\) then the two parameters of the calculation \(k\) and \(u\) should
satisfy the constrints (8) and (10) the system:

\[
\Phi(0, u, k) = 0 \\
\Phi(p + 1, u, k) = 0
\] (26)

then the parameter \( u = u(k, p) \) is a solution of the system of equations (26).

Generally there are many solutions of the above system, but a unitary representation of the deformed parafermionic oscillator is restrained by the additional restriction

\[ \Phi(x) > 0, \quad \text{for} \quad 0 < x < p + 1 \]

We must point out that the system (26) corresponds to a representation with dimension equal to \( p + 1 \).

6 Application of the case \( \gamma = 0 \)

In this section, we shall give an example of the calculation of eigenvalues of a superintegrable two-dimensional system, by using the methods of the previous section. The calculation by an empirical method was performed in \([14]\) and the solution of the same problem by using separation of variables was studied in \([4]\). Here in order to show the effects of the quantization procedure we don’t use \( \bar{\hbar} = 1 \) as it was considered in references \([14]\) and \([4]\). That means that the following commutation relations are taken in consideration:

\[
[x, p_x] = i\hbar, \quad [x, p_x] = i\bar{\hbar}
\]

The superintegrable Holt system corresponds to the Hamiltonian:

\[
H = \frac{1}{2} \left( p_x^2 + p_y^2 \right) + \frac{\omega^2}{2} \left( 4x^2 + y^2 \right) + \frac{k^2 - \frac{1}{4}}{y^2}. \tag{27}
\]

This superintegrable system with two integrals:

\[
A = p_x^2 + 4\omega^2x^2, \quad \text{and} \quad B = \{p_y, xp_y - yp_x\} - 2\omega^2xy^2 - 2(1/4 - k^2)\frac{x}{y^2}
\]

From the above definitions we can verify that:

\[
[H, A] = 0, \quad [H, B] = 0,
\]

9
and

\[ [A, B] = C, \quad [A, C] = 16h^2 \omega^2 B, \]
\[ [B, C] = 24h^2 A^2 - 64h^2 HA + 8h^2 \left(4H^2 + \omega^2(1 - 4k^2 + 3h^2)\right) \]

The above algebra is a quadratic algebra \( Qu(3) \) of the form (2–4), corresponding to the following values of the coefficients:

\[ \alpha = 0, \quad \gamma = 0, \quad \delta = 0, \quad \epsilon = 16h^2 \omega^2, \quad \zeta = 0 \]
\[ a = 24h^2, \quad d = -64h^2 H, \quad z = 8h^2 \left(4H^2 + \omega^2(1 - 4k^2 + 3h^2)\right) \]

The value of the Casimir operator (5) is given by:

\[ K = C^2 + 16h^2 A^3 - 64h^2 HA^2 - 16h^2 \omega^2 B^2 + 16h^2 \left(4H^2 + \omega^2(1 - 4k^2 + 11h^2)\right) A = 256h^4 \omega^2 H \]

The representation of the above algebra, which is diagonal to the Hamiltonian \( H \) and to the integral of motion \( A \), corresponds to the eigenvalues of the energy \( E \) and the eigenvalue of the Casimir equal to \( 256h^4 \omega^2 E \). In that case, equations (22) and (23), which determine the function \( \Phi(x) \), are respectively:

\[ -32h^2 E^2 - 8h \omega \Phi(x) + 8h \omega \Phi(x + 1) - 8h^2 \omega^2 - \\
-24h^4 \omega^2 + 32h^2 k^2 \omega^2 + 256h^3 E \omega(x + u) - 384h^4 \omega^2(x + u)^2 = 0 \]

\[ -32 \left(\Phi(x) + \Phi(x + 1)\right) h^2 \omega^2 + \\
+64 h^3 \omega \left(4E^2 + \omega^2 + 11h^2 \omega^2 - 4k^2 \omega^2\right) (x + u) - \\
-1024 h^4 E^2 \omega^2(x + u)^2 + 1024 h^5 \omega^3(x + u)^3 = 0 \]

The above equations can be solved and we find that:

\[ \Phi(x) = -\frac{h \left(4E^2 + 8h E \omega + \omega^2 + 3h^2 \omega^2 - 4k^2 \omega^2\right)}{h \left(4E^2 + 16h E \omega + \omega^2 + 11h^2 \omega^2 - 4k^2 \omega^2\right) (x + u) - \\
-8h^2 \left(2E + 3h \omega\right)(x + u)^2 + 16h^3 \omega(x + u)^3} \]

The two parameters of this equation are the parameter \( u \) and the eigenvalue \( E \) of the energy \( H \); therefore we can solve the system:

\[ \Phi(0) = 0, \quad \Phi(p + 1) = 0 \]
and we find two solutions:

\[ u = \frac{1}{2}, \quad E = \frac{\omega}{2} \left( 4 h (1 + p) \pm \sqrt{-1 + h^2 + 4 k^2} \right) \]

7 Application of the case \( \gamma \neq 0 \)

Let consider the case of a potential on the two dimensional hyperboloid taken from ref [16] (case of the potential \( V_1 \)).

The two dimensional hyperboloid is characterized by the cartesian coordinates \( \omega_0, \omega_1, \omega_2 \), which obey to the restriction \( \omega_0^2 - (\omega_1^2 + \omega_2^2) = 1 \). The Hamiltonian is given by

\[ H = -\frac{1}{2} \Delta_{LB} + V \]

where \( \Delta_{LB} \) is the Laplace-Beltrami operator for the details see ref [16], where

\[ V = \frac{\alpha^2}{\omega_2^2} - \frac{\gamma^2}{(\omega_0 - \omega_1)^2} + \beta^2 \frac{\omega_0 + \omega_1}{\omega_0 - \omega_1} \]

The two integrals are

\[ A = (\omega_0 \partial_{\omega_1} + \omega_1 \partial_{\omega_0})^2 - 2\beta^2 \left( \frac{\omega_0 + \omega_1}{\omega_0 - \omega_1} \right)^2 + 2\gamma^2 \frac{\omega_0 + \omega_1}{\omega_0 - \omega_1} \]

and

\[ B = (\omega_0 \partial_{\omega_2} + \omega_2 \partial_{\omega_0} - \omega_1 \partial_{\omega_2} + \omega_2 \partial_{\omega_1})^2 - \frac{2\beta^2 \omega_2^2}{(\omega_0 - \omega_1)^2} - \frac{2\alpha^2(\omega_0 - \omega_1)^2}{\omega_2^2} \]

The operators \( H, A \) and \( B \) satisfy the following commutation relations:

\[ [H, A] = 0, \quad [H, A] = 0 \]

\[ [A, C] = 8\{A, B\} + 16\gamma^2 A - 16B - 16\gamma^2(1 - 2\alpha^2 - 2H) \]

\[ [B, C] = -8B^2 - 32\beta^2 A - 16\gamma^2 B - 16\beta^2(1 - 4\alpha^2 + 4H) \]

The above algebra is a quadratic algebra \( Qu(3) \) of the form \( \mathbb{H} \mathbb{H} \), corresponding to the following values of the coefficients:

\[ \alpha = 0, \quad \gamma = 8, \quad \delta = 16\gamma^2, \quad \epsilon = -16, \quad \zeta = -16\gamma^2(1 - 2\alpha^2 - 2H) \]

\[ a = 0, \quad d = -32\beta^2, \quad z = -16\beta^2(1 - 4\alpha^2 + 4H) \]
The value of the Casimir operator \( K \) is given by:

\[
K = C^2 - \frac{8}{3} \{A, B, B\} + \frac{176}{3} B^2 - 32 \beta^2 A^2 - 16 \gamma^2 \{A, B\} - \\
- \left( 64 \alpha^2 \gamma^2 + 64 \gamma^2 H - \frac{352}{3} \gamma^2 \right) B - \left( \frac{352}{3} - 128 \alpha^2 + 128 \beta^2 H \right) \beta^2 A - \\
- \frac{128}{3} \beta^2 (1 - 4 \alpha^2 + 4H) = \\
= 16 (-4 \beta^2 + 8 \alpha^2 \beta^2 + 8 \alpha^4 \beta^2 - 3 \gamma^4 + \\
+ 8 \alpha^2 \gamma^4 - 8 \beta^2 H + 16 \alpha^2 \beta^2 H + 8 \beta^2 H^2)
\]

The representation of the above algebra, which is diagonal to the Hamiltonian \( H \), to the integral of motion \( A \), corresponds to the eigenvalues of the energy \( E \) and the eigenvalue of the Casimir equal to above cited value.

In that case, equations (22) and (23), which determine the function \( \Phi(x) \), give the form (24):

\[
\Phi(x) = -3 \cdot 2^{34} \left( 2 \beta^2 - \gamma^4 - 8 \beta^2 (u + x) + 8 \beta^2 (u + x)^2 \right) \cdot \\
\cdot (-\alpha^2 + \alpha^4 + E + 2 \alpha^2 E + E^2 + (-1 + 4 \alpha^2 - 4E) (u + x) + \\
+ (5 - 4 \alpha^2 + 4E) (u + x)^2 - 8(u + x)^3 + 4(u + x)^4)
\]

The two parameters of this equation are the parameter \( u \) and the eigenvalue \( E \) of the energy \( H \), therefore we can solve the system:

\[
\Phi(0) = 0, \quad \Phi(p + 1) = 0
\]

and we find the solution:

\[
u = \frac{1}{2} \left( 1 - \frac{\gamma^2}{\sqrt{2} \beta} \right), \quad E = -\frac{1}{2} \left( 2p + 2 + \sqrt{2 \alpha^2 + 1/4 - \frac{\gamma^2}{\sqrt{2} \beta}} \right)^2 + \frac{1}{8}
\]

8 Discussion

From the above discussion, we have shown how to calculate finite dimensional representations of the \( Qu(3) \) algebra and we have given an application of this method in the calculation of the energy eigenvalues of the superintegrable systems. The systematic algebraic study of all the known superintegrable systems is under investigation.
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