**L^4- Norms of Hecke Newforms of Large Level**

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**Abstract.** We prove a new upper bound for the $L^4$-norm of a holomorphic Hecke newform of large fixed weight and prime level $q \to \infty$. This is achieved by proving a sharp mean value estimate for a related $L$-function on $GL(6)$.

1. Introduction

Modular forms, particularly cusp forms, are very special functions on the upper half complex plane which arise almost everywhere in mathematics. A natural way to understand a cusp form is to study its $L^p$-norms, for in principle a function can be recovered from the knowledge of its moments. In recent years, this has been an exciting topic for which even a 2010 Fields Medal was awarded, to Lindenstrauss. Settling an important case of the Quantum Unique Ergodicity conjecture, Lindenstrauss [15] and Soundararaman [21] showed that the $L^2$-norm of a Hecke Maass cusp form of large laplacian eigenvalue restricted to a finite region of the complex upper half plane depends only on the area of the region. In other words, the $L^2$-mass is equidistributed. The analogous problem for holomorphic Hecke cusp forms of large weight, the Rudnick-Sarnak conjecture, was solved by Holowinsky and Soundararaman [9]. Nelson [19] proved that a holomorphic Hecke newform of large square-free level has equidistributed $L^2$-mass. This version of the QUE conjecture in the level aspect was posed by Kowalski, Michel, and VanderKam.

What can be said about higher $L^p$-norms in the level aspect? Let $B^\text{new}_k(q)$ denote the set of $L^2$-normalized holomorphic Hecke newforms of level $q$, weight $k$, and trivial nebentypus, and let $f \in B^\text{new}_k(q)$. Blomer and Holowinsky [3] were the first to establish a non-trivial upper bound for the $L^\infty$-norm. This was later improved by Harcos and Templier [7], who showed that

$$\|f\|_\infty \ll q^{-1/6+\epsilon}$$

for any $\epsilon > 0$, where the implied constant depends on $\epsilon$ and $k$. (We adopt this convention throughout the paper, and also that $\epsilon$ may not be the same arbitrarily small positive constant from one occurrence to the next.) As a consequence we have that

$$\|f\|_4 \ll q^{-1/3+\epsilon},$$

which is an improvement over the “trivial” bound

$$\|f\|_4 \ll q^\epsilon.$$  

A more direct way to get a handle on the $L^4$-norm is through $L$-functions. Blomer [2, sec. 1] commented that “it seems to be very hard to improve” in this way, but nevertheless Liu, Masri, and Young [16] succeeded in the case that $q$ is prime and achieved the bound

$$\|f\|_4 \ll q^{-1/2+\epsilon}.$$

In this paper, we give a further improvement. Let $q$ denote a prime throughout the rest of the paper.

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1These authors actually proved their results for Hecke Maass forms, but very similar arguments would work for holomorphic Hecke newforms as well.
Theorem 1.1. Let \( \epsilon > 0 \). There exists \( k_\epsilon > 0 \) such that for even \( k > k_\epsilon \) the following result holds. Let \( q \) be prime and \( f \in \mathcal{B}^{new}_k(q) \). Suppose that the subconvexity bound

\[
L\left(\frac{1}{2}, g\right) \ll_{k, \epsilon} q^{1/4 - \delta + \epsilon}
\]

holds for all \( g \in \mathcal{B}^{new}_{2k}(q) \), where \( L(s, g) \) is defined in (2.9) and \( \delta > 0 \). Then we have that

\[
\|f\|_4^4 \ll_{k, \epsilon} q^{-3/4 - \delta + \epsilon}.
\]

We have concentrated only on arriving at a new bound in the level aspect, and have not tried to optimize \( k_\epsilon \). The best known subconvexity bound, due to Duke, Friedlander, and Iwaniec [5], is given by \( \delta = 1/192 \). It is conjectured of course that we can take \( \delta = 1/4 \); this would yield the best expected bound for \( \|f\|_4 \).

The proof of Theorem 1.1 rests on a new mean value estimate for an \( L \)-function on \( GL(6) \), which is of interest in its own right.

Theorem 1.2. Let \( \epsilon > 0 \). There exists \( k_\epsilon > 0 \) such that for even \( k > k_\epsilon \), \( q \) prime, and \( f \in \mathcal{B}^{new}_k(q) \), we have that

\[
\sum_{g \in \mathcal{B}^{new}_{2k}(q)} L\left(\frac{1}{2}, \text{sym}^2 f \times g\right) L\left(1, \text{sym}^2 g\right) \ll_{k, \epsilon} q^{1+\epsilon}.
\]

After noting (2.13), we see that the estimate above is consistent with the Lindelöf hypothesis. Further, using Lapid’s [13] result that

\[
L\left(\frac{1}{2}, \text{sym}^2 f \times g\right) \geq 0,
\]

dropping all but one term recovers the convexity bound for these central values.

The novelty of our paper which facilitates the proof of Theorem 1.2 is a type of \( GL(3) \) Voronoi summation formula for non-trivial level. The \( GL(3) \) Voronoi summation formula for level one, due to Miller and Schmid [18], is a valuable tool in analytic number theory which has been used extensively in the subject. Although Ichino and Templier [10] have established a more general formula, it does not cover the case of interest to us. We will need to analyze a sum of the type

\[
\sum_{n < N} A_f(n, 1) e\left(\frac{nh}{cq}\right)
\]

where \( A_f(n, 1) \) are the coefficients of \( L(s, \text{sym}^2 f) \) as given in (2.11), \( c \) is an integer less than \( q' \), and \( h \) is an integer coprime to \( cq \). It would be very desirable to obtain a practical formula for the above sum for general values of \( c \), but this seems to be a challenging task. Our idea to get around this problem in the current situation is to realize that it is enough to develop a formula for the telling case \( c = 1 \) but to not fully work out the contribution coming from \( c > 1 \). Since \( c < q' \), this approach is good enough.

Lemma 1.3. Let \( A_f(n, 1) \) be the coefficients of \( L(s, \text{sym}^2 f) \) as given in (2.11). Let \( \phi(x) \) be a smooth function compactly supported on the positive real numbers. For \( (h, q) = 1 \), we have that

\[
\sum_{n \geq 1 \atop (n, q) = 1} A_f(n, 1) e\left(\frac{nh}{q}\right) \phi(n) = \frac{q}{2} \sum_{\alpha = \pm 1} i^{\alpha + 1} \sum_{n \geq 1 \atop (n, q) = 1} A_f(n, 1) \left( S(-n\overline{\tau}, 1; q) + \alpha S(n\overline{\tau}, 1; q) \right) \Phi_{\alpha}\left(\frac{n}{q^2}\right)
\]

\[
- \sum_{n \geq 1 \atop (n, q) = 1} A_f(n, 1) \left( \phi(n) + \frac{iq}{n} \Phi_1\left(\frac{n}{q^2}\right) \right),
\]

where \( \Phi_1(x) = \frac{\Phi(x)}{1 - x} \).
where $\overline{h}$ is the multiplicative inverse of $h$ modulo $q$ and for $\tilde{\phi}(s)$ the Mellin transform of $\phi$ and $H_\alpha(s)$ as in (5.13), we define

$$\Phi_\alpha(x) = \frac{1}{2\pi i} \int_{(\sigma)} x^{-s} H_\alpha(1-s) - \overline{H_\alpha} \tilde{\phi}(s) \, ds$$

for any $x, \sigma > 0$.

It would be interesting to prove an analogue of Theorem 1.2 for Hecke Maass cusp forms of level $q$, and we expect that very similar arguments would work. This would lead to a new bound for the $L^4$-norm of Maass forms and an improvement of the subconvexity bound [13, Corollary 1.2]. That a result like Theorem 1.2 would improve their subconvexity bound was noted by Liu, Masri, and Young in [16, section 6].

1.1. Sketch. We now give a rough outline of the proof of Theorem 1.2.

By the method of approximate functional equations, we write

$$L(\frac{1}{2}, \text{sym}^2 f \times g) \approx \sum_{n<q^{2+\epsilon}} A_f(n,1) \lambda_g(n) / n^{1/2},$$

where $\lambda_g(n)$ are the coefficients of $L(s, g)$ as given in (2.9). Thus,

$$\sum_{g \in B_{2k}^\infty(q)} L(\frac{1}{2}, \text{sym}^2 f \times g) L(1, \text{sym}^2 g) \approx \sum_{n<q^{2+\epsilon}} A_f(n,1) / n^{1/2} \sum_{g \in B_{2k}^\infty(q)} \lambda_g(n) / L(1, \text{sym}^2 g).$$

By the Petersson trace formula, this is roughly

$$q \left( 1 + \sum_{n<q^{2+\epsilon}} A_f(n,1) / n^{1/2} \sum_{c \geq 1} J_{2k-1}(4\pi \sqrt{n/cq}) S(n, 1; cq) / cq \right).$$

Now for large enough $k$, the $J$-Bessel function $J_{2k-1}(x)$ decays very quickly as $x \to 0$. Therefore it suffices to consider only $n > q^{2-\epsilon}$ and $c < q^{\epsilon}$. To fix ideas, we restrict ourselves in this sketch to the case $q^2 < n < 2q^2$ and $c = 1$. In these ranges, $n^{1/2} \approx q$ and the $J$-Bessel function is roughly constant. We therefore have to show that

$$\sum_{q^2 < n < 2q^2} A_f(n,1) S(n, 1; q) \ll q^{2+\epsilon}.$$

Opening the Kloosterman sum, we have to show that

$$\sum_{h \mod q} e\left(\frac{h}{q}\right) \sum_{q^2 < n < 2q^2} A_f(n,1) e\left(\frac{nh}{q}\right) \ll q^{2+\epsilon},$$

where * restricts the sum to primitive residue classes. We develop a Voronoi summation formula with the following shape:

$$\sum_{q^2 < n < 2q^2} A_f(n,1) e\left(\frac{nh}{q}\right) \approx q \sum_{m<q^{1+\epsilon}} A_f(m,1) / m S(m, \overline{h}; q).$$

Inserting this into (1.16), we have to show that

$$\sum_{m<q^{1+\epsilon}} A_f(m,1) / m \sum_{h \mod q} e\left(\frac{h}{q}\right) S(m, \overline{h}; q) \ll q^{1+\epsilon}.$$

Evaluating the $h$-sum, we must show that

$$\sum_{m<q^{1+\epsilon}} A_f(m,1) / m \ll q^\epsilon.$$
The required bound now follows by estimating trivially. Proving any further cancelation in the sum above seems like a very difficult problem, as the length of the sum is the square root of the conductor of \( L(s, \text{sym}^2 f) \).

To obtain the Voronoi formula, we write the exponential in terms of Dirichlet characters to get

\[
\sum_{n \geq 1} A_f(n, 1) e\left(\frac{nh}{q}\right) \phi\left(\frac{n}{q^2}\right) \approx \frac{1}{q} \sum_{\chi \mod q, \chi \neq 1} \tau(\chi) \sum_{n \geq 1} A_f(n, 1) \overline{\chi(nh)} \phi\left(\frac{n}{q^2}\right),
\]

where \( \phi(x) \) is a smooth bump function supported on \( 1 < x < 2 \) and \( \tau(\chi) \) is the Gauss sum. In terms of \( L \)-functions, this equals

\[
\frac{1}{q} \sum_{\chi \mod q, \chi \neq 1} \tau(\chi) \overline{\chi(h)} \frac{1}{2\pi i} \int_{2-\infty}^{2+i\infty} L(s, \text{sym}^2 f \times \overline{\chi}) q^{2s} \tilde{\phi}(s) ds,
\]

where \( \tilde{\phi} \) is the Mellin transform of \( \phi \). The summation formula comes from an application of the functional equation of \( L(s, \text{sym}^2 f \times \chi) \), which is available from the work of Li [14].

2. Background

2.1. Cusp forms. Let \( \mathbb{H} \) denote the upper half complex plane and \( \Gamma_0(q) \) the usual congruence subgroup of \( SL_2(\mathbb{Z}) \). Let \( S_k(q) \) denote the space of cusp forms of weight \( k \) and trivial nebentypus for \( \Gamma_0(q) \). This space is equipped with the inner product

\[
\langle f_1, f_2 \rangle = \int_{\Gamma_0(q) \backslash \mathbb{H}} y^{k/2} f_1(z) y^{k/2} f_2(z) \frac{dx dy}{y^2}
\]

and the \( L^p \)-norms

\[
\|f\|_p = \left( \int_{\Gamma_0(q) \backslash \mathbb{H}} |y^{k/2} f(z)|^p \frac{dx dy}{y^2} \right)^{1/p},
\]

\[
\|f\|_{\infty} = \sup\{|y^{k/2} f(z)| : z \in \Gamma_0(q) \backslash \mathbb{H}\}.
\]

By [12 Proposition 2.6] and [2 sec. 2], there exists an orthonormal basis \( B_{k}^{\text{new}}(q) \cup B_{k}^{\text{old}}(q) \) for \( S_k(q) \), where every \( f \in B_{k}^{\text{old}}(q) \) is an oldform with

\[
\|f\|_{\infty} \ll q^{-1/2}.
\]

We have that

\[
\dim S_k(q) \sim |B_{k}^{\text{new}}(q)| \sim \frac{q(k-1)}{12}
\]

as \( q \to \infty \).

2.2. \( L \)-functions. Every \( f \in B_{k}^{\text{new}}(q) \) is an eigenfunction of the Hecke operators \( T_n \). Say

\[
T_n f = n^{k/2} \lambda_f(n) f
\]

for some real numbers \( \lambda_f(n) \) satisfying

\[
\lambda_f(q) \ll q^{-1/2},
\]

Deligne’s bound \( \lambda_f(n) \ll n^{\epsilon} \), and the multiplicative relation

\[
\lambda_f(n) \lambda(m) = \sum_{d | (n,m), (d,q) = 1} \lambda_f\left(\frac{nm}{d^2}\right).
\]
The $L$-function associated to $f$ equals

\[(2.9) \quad L(s, f) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s} \]

for $\Re(s) > 1$ with analytical continuation to the rest of the complex plane. By the work of Guo \[6\], we know that the central value is non-negative:

\[(2.10) \quad L(\frac{1}{2}, f) \geq 0.\]

We will also need to work with $GL(1)$ and $GL(2)$ twists of the symmetric-square $L$-function, given for $\Re(s) > 1$ by

\[(2.11) \quad L(s, \text{sym}^2 f) = \left(1 - \frac{1}{q^{s+1}}\right)^{-1} \sum_{n \geq 1} \frac{A_f(n, 1)}{n^s},\]

where

\[(2.12) \quad A_f(n, 1) = \sum_{d \mid (d_1, d_2, n)} \mu(d) \frac{A_f(d_1)}{d_1} \frac{A_f(d_2)}{d_2}\]

for $(n, q) = 1$. At the edge of the critical strip have the bounds

\[(2.13) \quad C \epsilon q^{-\epsilon} < L(1, \text{sym}^2 f) \ll q^\epsilon,\]

where $C_\epsilon$ is some positive constant depending on $\epsilon$. The lower bound is due to Goldfeld, Hoffstein, and Lieman \[8\]. The constant $C_\epsilon$ is ineffective if $f$ is dihedral, but in this case even better bounds for the $L^4$-norm may be available by arguments such as in \[17\].

We will consider for $(c, q) = 1$ and $\chi$ a primitive Dirichlet character of modulus $cq$, the twist

\[(2.14) \quad L(s, \text{sym}^2 f \times \chi) = \sum_{n \geq 1} \frac{A_f(n, 1) \chi(n)}{n^s}\]

for $\Re(s) > 1$. Its functional equation is given in section \[5.2\].

Let $g \in B_{2k}^{\text{new}}(q)$ and

\[(2.15) \quad A_f(n, m) = \sum_{d \mid (n, m)} \mu(d) A_f\left(\frac{n}{d}, 1\right) A_f\left(\frac{m}{d}, 1\right)\]

for $(nm, q) = 1$. By \[22\] section 3.1, the $GL(2)$ twist

\[(2.16) \quad L(s, \text{sym}^2 f \times g) = \left(1 - \frac{\lambda_g(q)}{q^s}\right)^{-1} \left(1 - \frac{\lambda_g(q)}{q^{s+1}}\right)^{-1} \sum_{n,m \geq 1} \frac{A_f(n, m) \lambda_g(n)}{n^s m^{2s}}\]

for $\Re(s) > 1$ continues to an entire function and has the functional equation

\[(2.17) \quad q^{2s} G(s) L(s, \text{sym}^2 f \times g) = q^{2(1-s)} G(1-s) L(1-s, \text{sym}^2 f \times g),\]

where

\[(2.18) \quad G(s) = \pi^{-3s} \Gamma(s + 3) \Gamma(s + k - \frac{3}{2}) \Gamma(s + k - \frac{1}{2}) \Gamma(s + \frac{3}{2}).\]

We will use an approximate functional equation to get a handle on the central values.

**Lemma 2.1.** We have

\[(2.19) \quad L(\frac{1}{2}, \text{sym}^2 f \times g) = 2 \sum_{n,m \geq 1} \frac{A_f(n, m) \lambda_g(n)}{n^{1/2} m^{1/2}} \sum_{r_1, r_2 \geq 0} \left(\frac{\lambda_g(n)}{q^{1/2}}\right)^{r_1} \left(\frac{\lambda_g(q)}{q^{3/2}}\right)^{r_2} V\left(\frac{nm^2}{q^{2-r_1-r_2}}\right),\]
where
\[ V(x) = \frac{1}{2\pi i} \int_{(\sigma)} x^{-s} \frac{G(\frac{1}{2} + s)}{G(\frac{1}{2})} ds \]
satisfies
\[ V^{(\ell)}(x) \ll \epsilon x^{-\ell-\sigma} \]
for \( x, \ell, \sigma > 0 \). Thus the sum in (2.19) is essentially supported on \( nm^2 < q^{2-r_1-r_2+\epsilon} \).

\[ \text{Proof.} \] By Cauchy’s theorem, we have that
\[ L(\frac{1}{2}, \text{sym}^2 f \times g) = \frac{1}{2\pi i} \int_{(\frac{1}{4})} L(\frac{1}{2} + s, \text{sym}^2 f \times g) q^{2s} \frac{G(\frac{1}{2} + s)}{G(\frac{1}{2})} ds. \]

Applying the functional equation to the integrand in the second line of (2.22), we get that
\[ L(\frac{1}{2}, \text{sym}^2 f \times g) = \frac{1}{\pi i} \int_{(\frac{1}{4})} L(\frac{1}{2} + s, \text{sym}^2 f \times g) q^{2s} \frac{G(\frac{1}{2} + s)}{G(\frac{1}{2})} ds. \]

On this line of integration we may use (2.16) and the Taylor expansion of the Euler factors at \( \sigma > 0 \).
Doing so, we arrive at (2.19) with \( \sigma = 3/4 \). Now the line of integration may be moved to any \( \sigma > 0 \). \( \square \)

2.3. Trace formula. We will need the following trace formula over newforms, implied by [12, Propositions 2.1 and 2.8]. For \( (n, q) = 1 \), we have that
\[ \frac{12\zeta(2)}{q(k-1)} \sum_{f \in B_{2k}^{\text{new}}(q)} \frac{\lambda_f(n)}{L(1, \text{sym}^2 f)} = \delta(n) + 2\pi i^k \sum_{c \geq 1} \frac{S(n, 1; cq)}{cq} J_{k-1}\left(4\pi \sqrt{n} \frac{1}{cq}\right) + O\left(\frac{n^c}{q}\right), \]
where \( \delta(n) = 1 \) if \( n = 1 \) and \( \delta(n) = 0 \) otherwise, \( S(n, 1; cq) \) is a Kloosterman sum, and \( J_{k-1}(x) \) is the \( J \)-Bessel function, which satisfies
\[ J_{k-1}(x) \ll \min\{x^{k-1}, x^{-1/2}\} \]
for \( x > 0 \). We have the very basic uniform bound
\[ J_{k-1}^{(\ell)}(x) \leq 1 \]
for all \( x > 0 \) and \( \ell \geq 0 \).

3. Reduction of Theorem 1.1

The goal of this section is to relate the \( L^4 \)-norm to \( L \)-functions and reduce Theorem 1.1 to Theorem 1.2. Note that if \( f \in B_{2k}^{\text{new}}(q) \) then \( f^2 \in S_{2k}(q) \). Thus by Parseval’s theorem and (2.4,2.5), we have
\[ \|f\|^4 = ||f^2||^2 = \sum_{g \in B_{2k}^{\text{new}}(q) \cup B_{2k}^{\text{old}}(q)} |\langle f^2, g \rangle|^2 = \sum_{g \in B_{2k}^{\text{new}}(q)} |\langle f^2, g \rangle|^2 + O(q^{-1}). \]

By Watson’s formula [22, section 4.1] (see also [2, sec. 5]) for \( |\langle f^2, g \rangle|^2 \), we have that
\[ \|f\|^4 \ll \frac{1}{q^2} \sum_{g \in B_{2k}^{\text{new}}(q)} \frac{L(\frac{1}{2}, f \times f \times g)}{L(1, \text{sym}^2 f)^2 L(1, \text{sym}^2 g)} + O(q^{-1}), \]
where
\[ L(s, f \times f \times g) = L(s, g)L(s, \text{sym}^2 f \times g) \]
is a triple product $L$-function (the complex conjugation can be dropped because $\lambda_q(n) \in \mathbb{R}$). Now by \((1.3)\), we have that

\[
\|f\|_4^4 \ll q^{-7/4-\delta+\epsilon} \sum_{g \in \mathcal{B}_{2\pi}((q))} \frac{L(\frac{1}{2}, \text{sym}^2 f \times g)}{L(1, \text{sym}^2 g)} + O(q^{-1}).
\]

Thus Theorem 1.1 follows from Theorem 1.2. The same strategy was used in [4] to study the $L^4$-norm in terms of the weight $k$. Notice how this differs from the approach in [16]: there, the Cauchy-Schwarz inequality is applied to \((3.2)\) and the problem is reduced to studying the second power moments of $L(\frac{1}{2}, g)$ and $L(\frac{1}{2}, \text{sym}^2 f \times g)$, for Maass forms.

We remark that the Lindelöf bound for the triple product $L$-function above would imply the best expected bound $\|f\|_4 \ll q^{-1/4+\epsilon}$.

4. Proof of Theorem 1.2

By \((2.19)\), we have that

\[
(4.1) \quad \frac{1}{q} \sum_{g \in \mathcal{B}_{2\pi}((q))} \frac{L(\frac{1}{2}, \text{sym}^2 f \times g)}{L(1, \text{sym}^2 g)} = \frac{2}{q} \sum_{g \in \mathcal{B}_{2\pi}((q))} \sum_{n,m \geq 1} \frac{A_f(n,m)}{n^{1/2}m} \frac{\lambda_q(n)}{L(1, \text{sym}^2 g)} \sum_{r_1,r_2 \geq 0} \left( \frac{\lambda_q(q)}{q^{1/2}} \right)^{r_1} \left( \frac{\lambda_q(q)}{q^{1/2}} \right)^{r_2} V \left( \frac{nm^2}{q^{2-r_1-r_2}} \right).
\]

Now by \((2.19)\), this equals

\[
(4.2) \quad \frac{2}{q} \sum_{n,m \geq 1} \frac{A_f(n,m)}{n^{1/2}m} V \left( \frac{nm^2}{q^2} \right) \frac{1}{q} \sum_{g \in \mathcal{B}_{2\pi}((q))} \frac{\lambda_q(n)}{L(1, \text{sym}^2 g)} + O(q^\epsilon).
\]

Thus on applying \((2.24)\), we have that

\[
(4.3) \quad \frac{1}{q} \sum_{g \in \mathcal{B}_{2\pi}((q))} \frac{L(\frac{1}{2}, \text{sym}^2 f \times g)}{L(1, \text{sym}^2 g)} \ll \sum_{c \geq 1} \sum_{n,m \geq 1} \frac{A_f(n,m)}{n^{1/2}m} \frac{S(n,1; cq)}{cq} J_{2k-1} \left( \frac{4\pi \sqrt{n}}{cq} \right) V \left( \frac{nm^2}{q^2} \right) + O(q^\epsilon).
\]

We make the following observation: the contribution to \((4.3)\) of the terms not satisfying

\[
(4.4) \quad 1 \leq c, m \leq q^\epsilon \quad \text{and} \quad q^{2-\epsilon} \leq n \leq q^{2+\epsilon}
\]

is certainly less than $q^\epsilon$. To see this, we may assume by \((2.21)\) that $n < q^{2+\epsilon}m^{-2}$. Now if \((4.4)\) is not satisfied then

\[
(4.5) \quad \frac{4\pi \sqrt{n}}{cq} < q^{-\epsilon},
\]

so that

\[
(4.6) \quad J_{2k-1} \left( \frac{4\pi \sqrt{n}}{cq} \right) \ll q^{10}
\]

by \((2.23)\) provided that $k$ is large enough. This implies the claim. For the terms that do satisfy \((4.4)\), we analyze the right hand side of \((4.3)\) in dyadic intervals of $n$. To this end, let $U(x)$ be a smooth function, compactly supported on $1 \leq x \leq 2$. By \((2.13)\) and the observation above, we see that Theorem 1.2 follows from

**Lemma 4.1.** For any integers

\[
(4.7) \quad 1 \leq c, d, m \leq q^\epsilon, \quad q^{2-\epsilon} \leq N \leq q^{2+\epsilon},
\]
we have that

\[ \sum_{n \geq 1 \atop (n,q)=1} A_f(n,1)S(nd, 1; cq)W\left(\frac{n}{N}\right) \ll q^{2+\epsilon}, \]

where

\[ W(x) = J_{2k-1}\left(\frac{4\pi \sqrt{xdN}}{cq}\right)V\left(\frac{xd^3m^2}{q^2}\right)U(xd) \]

is supported on \( q^{-\epsilon} \leq x \leq q^{\epsilon} \) and satisfies

\[ W(\ell x) \ll \ell q^{\ell \epsilon}. \]

5. Proofs of Lemmas 1.3 and 4.1

Writing \( d_1 = d/(d, c), \ c_1 = c/(d, c) \), and

\[ S(nd, 1; cq) = \sum_{h \mod cq}^* e\left(\frac{ndh + \overline{h}}{cq}\right), \]

where * indicates that the sum is restricted to primitive residue classes, we have that (4.8) is equivalent to

\[ \sum_{h \mod cq}^* e\left(\frac{\overline{h}}{cq}\right) \sum_{n \geq 1 \atop (n,q)=1} A_f(n,1)e\left(\frac{nd_1h}{c_1q}\right)W\left(\frac{n}{N}\right) \ll q^{2+\epsilon}. \]

We now concentrate our efforts on the inner \( n \)-sum. The proof of Lemma 1.3 can be gleaned from this analysis on taking \( d_1 = c_1 = 1 \). Right at the end, the outer \( h \)-sum will be executed to complete the proof of Lemma 4.1.

5.1. Dirichlet characters. Grouping the left hand side of (5.2) by the value of \( (n, c_1) \), say \( d_2 \), and writing \( c_2 = c_1/d_2 \), we have that it equals

\[ \sum_{h \mod cq}^* e\left(\frac{\overline{h}}{cq}\right) \sum_{d_2|c_1} \sum_{n \geq 1 \atop (n,c_2q)=1} A_f(nd_2, 1)e\left(\frac{nd_1h}{c_2q}\right)W\left(\frac{nd_2}{N}\right). \]

Note that \( (nd_1h, c_2q) = 1 \) so that the following identity holds:

\[ e\left(\frac{nd_1h}{c_2q}\right) = \frac{1}{\varphi(c_2q)} \sum_{\chi \mod c_2q} \tau(\chi)\overline{\chi}(nd_1h), \]

where \( \varphi \) is Euler’s totient function. Writing the characters above in terms of the primitive characters which induce them and using \( [11, \text{Lemma 3.1}] \), we have that

\[ e\left(\frac{nd_1h}{c_2q}\right) = \frac{1}{\varphi(c_2q)} \sum_{\chi_3 \mod c_3q} \mu(c_2/c_3)\chi(c_2/c_3)\tau(\chi)\overline{\chi}(nd_1h) \]

and

\[ + \sum_{\chi_3 \mod c_3} \mu(c_2q/c_3)\chi(c_2q/c_3)\tau(\chi)\overline{\chi}(nd_1h) \].

Thus we have shown that to prove Lemma 4.1 it is enough to establish that

\[ \sum_{h \mod cq}^* e\left(\frac{\overline{h}}{cq}\right) \sum_{\chi \mod c_3q}^{\star} \chi(\overline{nd_1h}/c_2/c_3)\tau(\chi) \sum_{n \geq 1 \atop (n,c_2/c_3)=1} A_f(nd_2, 1)\overline{\chi}(n)W\left(\frac{nd_2}{N}\right) \ll q^{3+\epsilon}, \]

where
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where all the new parameters $c_i$ and $d_i$ are less than $q'$. We simplify this a little more. We write

$$h = h_1c + h_2q,$$

where $h_1$ varies over the primitive residue classes modulo $q$ and $h_2$ varies over the primitive residue classes modulo $c$. We also write

$$\chi = \chi_1\chi_2$$

where $\chi_1$ varies over the primitive characters modulo $q$ and $\chi_2$ varies over the primitive characters modulo $c_3$. Thus, using \[11\] (3.16),(12.20), it suffices to prove that

$$\sum_{h_1 \text{ mod } q}^* \epsilon\left(\frac{h_1c}{q}\right) \sum_{\chi_1 \text{ mod } q}^* \chi_1(h_1d_1c_2)\tau_1(\chi_1) \sum_{n \geq 1}^{d_2} A_f(nd_2, 1)\chi_1\chi_2(n)W\left(\frac{nd_2}{N}\right) \ll q^{3+\varepsilon}.$$

5.2. **Functional equations.** To analyze the innermost sum in (5.9), we need the functional equation of $L(s, \text{sym}^2 f \times \chi_1\chi_2)$.

**Lemma 5.1.** Let $\chi_1$ and $\chi_2$ be primitive characters mod $q$ and mod $c_3$ respectively, for $c_3 < q'$. Let

$$\alpha = \chi_1\chi_2(-1).$$

For some complex number $\varepsilon$, depending on $q$ and $\chi_2$, and some some integers $b_1$ and $b_2$ satisfying

$$|\varepsilon| = 1, \quad 1 \leq b_1, b_2 < q',$$

we have that

$$H_\alpha(s)L(s, \text{sym}^2 f \times \chi_1\chi_2) = \frac{\varepsilon\Gamma(s + 1 - \frac{1 + \alpha}{2} + s)}{\prod_{p \mid q} (1 - \frac{1}{p})^2} \Gamma(s + k - 1)2^{-s-1}\pi^{-\frac{s}{2}}|b_2|^2,$$

where

$$H_\alpha(s) = \frac{\varepsilon}{\prod_{p \mid q} (1 - \frac{1}{p})^2} \Gamma(s + k - 1)2^{-s-1}\pi^{-\frac{s}{2}}|b_2|^2.$$

If $c_3 = 1$ (so that $\chi_2 = 1$), then $\varepsilon = b_1 = b_2 = 1$. The left hand side of (5.12) is an entire function.

**Proof.** The final statement of the lemma follows from the work of Shimura \[20\] Theorem 1, Theorem 2 and the following remarks. The functional equation is a result of the work of Li \[14\] and Atkin and Li \[11\]. Specifically, in \[14\] Theorem 2.2, we set

$$F_1 = \frac{f_{\chi_1\chi_2}}{\chi_1\chi_2},$$

(5.15)

and we read off the functional equation of

$$L_{F_1,F_2}(s) = L(2s, \chi_1\chi_2) \sum_{n \geq 1} \frac{\lambda_f(n)^2\chi_1\chi_2(n)}{n^s}.$$
By \cite[(0.4)]{20}, we have that
\begin{equation}
L(s, \text{sym}^2 f \times \chi_1 \chi_2^2) = \frac{L_{F_1, F_2}(s)}{L(s, \chi_1 \chi_2)}.
\end{equation}
Thus \eqref{5.12} follows from \eqref{5.10} and the functional equation of $L(s, \chi_1 \chi_2)$, which may be found in \cite[Theorem 4.15]{11} for example. \hfill \square

### 5.3. Summation

Let
\begin{equation}
\widetilde{W}(s) = \int_0^\infty W(x)x^{s-1} \, dx
\end{equation}
denote the Mellin transform of $W$, which satisfies
\begin{equation}
\widetilde{W}(s) \ll \Re(s) q^{\ell} (|s| + 1)^{-\ell}
\end{equation}
for any integer $\ell \geq 0$ by\cite[(4.16)]{10} and integration by parts $\ell$ times. For a prime $p$ and integer $r \geq 0$, let
\begin{equation}
R_{p,r}(s) = \sum_{j=r}^{\infty} \frac{A_f(p^j, 1)\chi_1 \chi_2(p^j)}{p^{js}}
\end{equation}
and
\begin{equation}
R(s) = \prod_{p \mid c} R_{p,0}(s)^{-1} \prod_{p \nmid d_2} R_{p,0}(s)^{-1} R_{p,r}(s).
\end{equation}

Note that
\begin{equation}
\prod_{p \mid d_2} R_{p,0}(s)^{-1} R_{p,r}(s) = \prod_{p \mid d_2} R_{p,0}(s)^{-1} \left( R_{p,0}(s) - \sum_{j=0}^{r-1} \frac{A_f(p^j, 1)\chi_1 \chi_2(p^j)}{p^{js}} \right)
\end{equation}
and
\begin{equation}
R_{p,0}(s)^{-1} = 1 - \frac{A_f(p, 1)\chi_1 \chi_2(p)}{p^s} + \frac{A_f(p, 1)\chi_1 \chi_2(p^2)}{p^{2s}} - \frac{\chi_1 \chi_2(p^3)}{p^{3s}}
\end{equation}
by \cite[(0.2)]{20}.
We have that
\begin{equation}
\sum_{n \geq 1} A_f(nd_2, 1)\chi_1 \chi_2(n) W\left(\frac{nd_2}{N}\right) = \frac{1}{2\pi i} \int_{(2)} \sum_{n \geq 1} A_f(nd_2, 1)\chi_1 \chi_2(n) \left( \frac{N}{d_2} \right)^s \tilde{W}(s) \, ds
\end{equation}
\begin{equation}
= \frac{1}{2\pi i} \int_{(2)} R(s) L(s, \text{sym}^2 f \times \chi_1 \chi_2) \left( \frac{N}{d_2} \right)^s \tilde{W}(s) \, ds.
\end{equation}

By \cite[(4.28)]{3}, we have that the integral in \eqref{5.28} equals
\begin{equation}
\epsilon^{\frac{i\pi}{4} - \frac{1}{2} \frac{\chi(b_1)}{q^2}} \int_{(2)} R(s) L(1-s, \text{sym}^2 f \times \chi_1 \chi_2) (q^{\frac{1}{2}})^{1-2s} \left( \frac{N}{d_2} \right)^s \frac{H_\alpha(1-s)}{H_\alpha(s)} \tilde{W}(s) \, ds.
\end{equation}
Keeping in mind the last statement of Lemma \ref{5.12}, moving the line of integration to $\Re(s) = -\sigma$ for any $\sigma > 0$, and using \cite[(2.14)]{13}, we have that \eqref{5.28} equals
\begin{equation}
\epsilon^{\frac{i\pi}{4} - \frac{1}{2} \frac{\chi(b_1)}{q^2}} \int_{(\sigma)} \left( \frac{nN}{q^2d_2} \right)^s R(s) \frac{H_\alpha(1-s)}{H_\alpha(s)} \tilde{W}(s) \, ds.
\end{equation}
Now, expanding out $R(s)$ and using (5.36, 2.12, 5.25, 5.26), we have that (5.30) equals
\begin{equation}
\sum_{a_1, a_2} \pm \chi_1 \chi_2(a_1) \lambda_f(a_2) \varepsilon_i \sum_{n \geq 1} \frac{A_f(n, 1) \chi_1 \chi_2(n)}{n} W_\alpha \left( \frac{n}{q^3 N^{-1} a_1^{-1} d_2} \right)
\end{equation}
for a sum over some integers $a_1, a_2 < q^\epsilon$, where
\begin{equation}
W_\alpha(x) = \frac{1}{2\pi i} \int_{(\sigma)} x^{-s} \frac{H_\alpha(1 - s)}{H_\alpha(s)} \tilde{W}(s) \, ds
\end{equation}
is defined for any $x, \sigma > 0$ and satisfies
\begin{equation}
W_\alpha(x) \ll x^{-\sigma}.
\end{equation}
Thus the $n$-sum in (5.31) is essentially supported on $n < q^{1+\epsilon}$.

5.4. **Character sums.** Putting (5.31), the evaluation of the left hand side of (5.27), back into (5.27) and exchanging the order of summation, we see that it is enough to prove that
\begin{equation}
\sum_\ast \sum_{\chi_1 \mod q} e\left(\frac{b_1 c_2}{q}\right) \sum_{n \geq 1} \frac{A_f(n, 1) \chi_2(n)}{n} W_\alpha \left( \frac{n}{q^3 N^{-1} a_1^{-1} d_2} \right) \sum_\ast \chi_1(b_1 d_1 a_1 b_2 n) \tau(\chi_1) \tau(\chi_2)^3
\end{equation}
for $\alpha = \pm 1$ and any $a_1, a_2 < q^\epsilon$. For $(n, q) = 1$, the innermost sum equals
\begin{equation}
\frac{q}{2} \sum_{\chi_1 \mod q} \chi_1(-h_1 d_1 a_1 b_2 n) \tau(\chi_1)^2 (\chi_1 \chi_2(-1) \alpha + 1)
\end{equation}
\begin{equation}
= \frac{q(q - 1)}{2} \left( S(-h_1 d_1 a_1 b_2 n, 1; q) + \alpha \chi_2(-1) S(h_1 d_1 a_1 b_2 n, 1; q) \right) - \frac{q(1 + \alpha \chi_2(-1))}{2},
\end{equation}
while it equals 0 if $q | n$. Thus, it remains to prove that
\begin{equation}
\sum_\ast \sum_{\chi_1 \mod q} e\left(\frac{b_1 c_2}{q}\right) \sum_{n \geq 1} \frac{A_f(n, 1) \chi_2(n)}{n} W_\alpha \left( \frac{n}{q^3 N^{-1} a_1^{-1} d_2} \right) S(\pm h_1 d_1 a_1 b_2 n, 1; q) \ll q^{1+\epsilon}.
\end{equation}
Exchanging summation again, it is easy to show that
\begin{equation}
\sum_\ast e\left(\frac{b_1 c_2}{q}\right) S(\pm h_1 d_1 a_1 b_2 n, 1; q) \ll q.
\end{equation}
Now (5.36) follows from the immediate bound
\begin{equation}
\sum_{n \geq 1} \left| \frac{A_f(n, 1) \chi_2(n)}{n} W_\alpha \left( \frac{n}{q^3 N^{-1} a_1^{-1} d_2} \right) \right| \ll q^\epsilon.
\end{equation}

**References**

1. A. O. L. Atkin and Wen Ch’ing Winnie Li, *Twists of newforms and pseudo-eigenvalues of W-operators*, Invent. Math. 48 (1987), no. 3, 221–243.
2. Valentin Blomer, *On the 4-norm of an automorphic form*, J. Eur. Math. Soc. (to appear).
3. Valentin Blomer and Roman Holowinsky, *Bounding sup-norms of cusp forms of large level*, Invent. Math. 179 (2010), no. 3, 645–681.
4. Valentin Blomer, Rizwanur Khan, and Matthew Young, *Mass distribution of holomorphic cusp forms*, Duke Math. J. (to appear).
5. W. Duke, J. B. Friedlander, and H. Iwaniec, *Bounds for automorphic L-functions. II*, Invent. Math. 115 (1994), no. 2, 219–239.
6. Jiandong Guo, *On the positivity of the central critical values of automorphic L-functions for GL(2)*, Duke Math. J. 83 (1996), no. 1, 157–190.
7. Gergely Harcos and Nicolas Templier, *On the sup-norm of maass cusp forms of large level, III*, preprint.
8. Jeffrey Hoffstein, Paul Lockhart, and Daniel Lieman, *An effective zero-free region*, Ann. of Math. (2) **140** (1994), no. 1, 177–181, Appendix to *Coefficients of Maass forms and the Siegel zero*.
9. Roman Holowinsky and Kannan Soundararajan, *Mass equidistribution for Hecke eigenforms*, Ann. of Math. (2) **172** (2010), no. 2, 1517–1528.
10. Atsushi Ichino and Nicolas Templier, *On the voronoi formula for GL(n)*, Amer. J. Math. **135** (2013), no. 1, 65–101.
11. Henryk Iwaniec and Emmanuel Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004.
12. Henryk Iwaniec, Wenzhi Luo, and Peter Sarnak, *Low lying zeros of families of L-functions*, Inst. Hautes Études Sci. Publ. Math. (2000), no. 91, 55–131 (2001).
13. Erez M. Lapid, *On the nonnegativity of Rankin-Selberg L-functions at the center of symmetry*, Int. Math. Res. Not. (2003), no. 2, 65–75.
14. Wen Ch'ing Winnie Li, *L-series of Rankin type and their functional equations*, Math. Ann. **244** (1979), no. 2, 135–166.
15. Elon Lindenstrauss, *Invariant measures and arithmetic quantum unique ergodicity*, Ann. of Math. (2) **163** (2006), no. 1, 165–219.
16. Sheng-Chi Liu, Riad Masri, and Matthew Young, *Subconvexity and equidistribution of Heegner points in the level aspect*, Compositio Math. (to appear).
17. Wenzhi Luo, *$L^4$-norms of the dihedral maass forms*, Int. Math. Res. Not. (to appear).
18. Stephen D. Miller and Wilfried Schmid, *Automorphic distributions, L-functions, and Voronoi summation for GL(3)*, Ann. of Math. (2) **164** (2006), no. 2, 423–488.
19. Paul D. Nelson, *Equidistribution of cusp forms in the level aspect*, Duke Math. J. **160** (2011), no. 3, 467–501.
20. Goro Shimura, *On the holomorphy of certain Dirichlet series*, Proc. London Math. Soc. (3) **31** (1975), no. 1, 79–98.
21. Kannan Soundararajan, *Quantum unique ergodicity for $SL_2(\mathbb{Z})\backslash \mathbb{H}$*, Ann. of Math. (2) **172** (2010), no. 2, 1529–1558.
22. Thomas Watson, *Rankin triple products and quantum chaos*, http://arxiv.org/abs/0810.0425.

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