Perturbative renormalization and BRST

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1 Main problems in the perturbative quantization of gauge theories

Gauge theories are field theories in which the basic fields are not directly observable. Field configurations yielding the same observables are connected by a gauge transformation. In the classical theory the Cauchy problem is well posed for the observables, but in general not for the nonobservable gauge variant basic fields, due to the existence of time dependent gauge transformations.

Attempts to quantize the gauge invariant objects directly have not yet been completely satisfactory. Instead one modifies the classical action by adding a gauge fixing term such that standard techniques of perturbative quantization can be applied and such that the dynamics of the gauge invariant classical fields is not changed. In perturbation theory this problem shows up already in the quantization of the free gauge fields (Sect. 3). In the final (interacting) theory the physical quantities should be independent on how the gauge fixing is done (‘gauge independence’).

Traditionally, the quantization of gauge theories is mostly analyzed in terms of path integrals (e.g. by Faddeev and Popov) where some parts of the arguments are only heuristic. In the original treatment of Becchi, Rouet and Stora (cf. also Tyutin) (which is called ‘BRST-quantization’), a restriction to purely massive theories was necessary; the generalization to the massless case by Lowenstein’s method is cumbersome.
The BRST-quantization is based on earlier work of Feynman, Faddeev and Popov (introduction of “ghost fields”), and of Slavnov. The basic idea is that after adding a term to the Lagrangian which makes the Cauchy problem well posed but which is not gauge invariant one enlarges the number of fields by infinitesimal gauge transformations (“ghosts”) and their duals (“anti-ghosts”). One then adds a further term to the Lagrangian which contains a coupling of the anti-ghosts and ghosts. The BRST transformation acts as an infinitesimal gauge transformation on the original fields and on the gauge transformations themselves and maps the anti-ghosts to the gauge fixing terms. This is done in such a way that the total Lagrangian is invariant and that the BRST transformation is nilpotent. The hard problem in the perturbative construction of gauge theories is to show that BRST-symmetry can be maintained during renormalization (Sect. 4). By means of the ’Quantum Action Principle’ of Lowenstein (1971) and Lam (1972-1973) a cohomological classification of anomalies was worked out (an overview is e.g. given in the book of Piguet and Sorella (1995)). For more details see → ‘BRST Quantization’.

The BRST-quantization can be carried out in a transparent way in the framework of Algebraic Quantum Field Theory (‘AQFT’, see → ‘Algebraic approach to quantum field theory’). The advantage of this formulation is that it allows to separate the three main problems of perturbative gauge theories:

• the elimination of unphysical degrees of freedom,
• positivity (or “unitarity”)
• and the problem of infrared divergences.

In AQFT, the procedure is the following: starting from an algebra of all local fields, including the unphysical ones, one shows that after perturbative quantization the algebra admits the BRST transformation as a graded nil-potent derivation. The algebra of observables is then defined as the cohomology of the BRST transformation. To solve the problem of positivity, one has to show that the algebra of observables, in contrast to the algebra of all fields, has a nontrivial representation on a Hilbert space. Finally, one can attack the infrared problem by investigating the asymptotic behavior of states. The latter problem is nontrivial even in quantum electrodynamics (since an electron is accompanied by a ‘cloud of soft photons’) and may be related to confinement in quantum chromodynamics.

The method of BRST quantization is by no means restricted to gauge theories, but applies to general constrained systems. In particular, massive vector fields, where the masses are usually generated by the Higgs mechanism, can alternatively be treated directly by the BRST formalism, in close analogy to the massless case, cf. Sect. 3.
2 Local operator BRST-formalism

In AQFT, the principal object is the family of operator algebras $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ (where $\mathcal{O}$ runs e.g. through all double cones in Minkowski space), which fulfills the Haag-Kastler axioms (cf. ‘Algebraic approach to quantum field theory’). To construct these algebras one considers the algebras $\mathcal{F}(\mathcal{O})$ which are generated by all local fields including ghosts $u$ and anti-ghosts $\tilde{u}$. Ghosts and anti-ghosts are scalar fermionic fields. The algebra gets a $\mathbb{Z}_2$ grading with respect to even and odd ghost numbers, where ghosts get ghost numbers $+1$ and anti-ghosts ghost number $-1$. The BRST-transformation $s$ acts on these algebras as a $\mathbb{Z}_2$-graded derivation with $s^2 = 0$, $s(\mathcal{F}(\mathcal{O})) \subset \mathcal{F}(\mathcal{O})$ and $s(F^*) = -(-1)^{\delta_F} s(F)^*$, $\delta_F$ denoting the ghost number of $F$.

The observables should be $s$-invariant and may be identified if they differ by a field in the range of $s$. Since the range $\mathcal{A}_{00}$ of $s$ is an ideal in the kernel $\mathcal{A}_0$ of $s$, the algebra of observables is defined as the quotient $\mathcal{A} := \mathcal{A}_0 / \mathcal{A}_{00}$, and the local algebras $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}$ are the images of $\mathcal{A}_0 \cap \mathcal{F}(\mathcal{O})$ under the quotient map $\mathcal{A}_0 \to \mathcal{A}$.

To prove that $\mathcal{A}$ admits a nontrivial representation by operators on a Hilbert space one may use the BRST-operator formalism (Kugo - Ojima (1979) and Dütsch - Fredenhagen (1999)): one starts from a representation of $\mathcal{F}$ on an inner product space $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ such that $\langle F^* \phi, \psi \rangle = \langle \phi, F \psi \rangle$ and that $s$ is implemented by an operator $Q$ on $\mathcal{K}$, i.e.

$$s(F) = [Q, F],$$

with $[\cdot, \cdot]$ denoting the graded commutator, such that $Q$ is symmetric and nil-potent. One may then construct the space of physical states as the cohomology of $Q$, $\mathcal{H} := \mathcal{K}_0 / \mathcal{K}_{00}$ where $\mathcal{K}_0$ is the kernel and $\mathcal{K}_{00}$ the range of $Q$. The algebra of observables now has a natural representation $\pi$ on $\mathcal{H}$:

$$\pi([A])[\phi] := [A\phi]$$

(2.3)

(2.3)

(where $A \in \mathcal{A}_0$, $\phi \in \mathcal{K}_0$, $[A] := A + \mathcal{A}_{00}$, $[\phi] := \phi + \mathcal{K}_{00}$). The crucial question is whether the scalar product on $\mathcal{H}$ inherited from $\mathcal{K}$ is positive definite.

In free quantum field theories $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ can be chosen in such a way that the positivity can directly be checked by identifying the physical degrees of freedom (Sect. 3). In interacting theories (Sect. 5) one may argue in terms of scattering states that the free BRST operator on the asymptotic fields coincides with the BRST operator of the interacting theory. This argument, however, is invalidated by infrared problems in massless gauge theories. Instead one may use a stability property of the construction.
Namely, let $\tilde{F}$ be the algebra of formal power series with values in $F$, and let $\tilde{K}$ be the vector space of formal power series with values in $K$. $\tilde{K}$ possesses a natural inner product with values in the ring of formal power series $\mathbb{C}[[\lambda]]$, as well as a representation of $\tilde{F}$ by operators. One also assumes that the BRST transformation $\tilde{s}$ is a formal power series $\tilde{s} = \sum_n \lambda^n s_n$ of operators $s_n$ on $F$ and that the BRST operator $\tilde{Q}$ is a formal power series $\tilde{Q} = \sum_n \lambda^n Q_n$ of operators on $K$. The algebraic construction can then be done in the same way as before yielding a representation $\tilde{\pi}$ of the algebra of observables $\tilde{A}$ by endomorphisms of a $\mathbb{C}[[\lambda]]$ module $\tilde{H}$, which has an inner product with values in $\mathbb{C}[[\lambda]]$.

One now assumes that at $\lambda = 0$ the inner product is positive, in the sense that

$$(\text{Positivity}) \quad (i) \quad \langle \phi, \phi \rangle \geq 0 \quad \forall \phi \in K \text{ with } Q_0 \phi = 0 ,$$

and

$$(ii) \quad Q_0 \phi = 0 \wedge \langle \phi, \phi \rangle = 0 \implies \phi \in Q_0 K \quad (2.4)$$

Then the inner product on $\tilde{H}$ is positive in the sense that for all $\tilde{\phi} \in \tilde{H}$ the inner product with itself, $\langle \tilde{\phi}, \tilde{\phi} \rangle$, is of the form $\tilde{c}^* \tilde{c}$ with some power series $\tilde{c} \in \mathbb{C}[[\lambda]]$, and $\tilde{c} = 0$ iff $\tilde{\phi} = 0$.

This result guarantees that, within perturbation theory, the interacting theory satisfies positivity, provided the unperturbed theory was positive and BRST symmetry is preserved.

### 3 Quantization of free gauge fields

The action of a classical free gauge field $A$,

$$S_0(A) = -\frac{1}{4} \int dx F^{\mu\nu}(x)F_{\mu\nu}(x) = \frac{1}{2} \int dk \hat{A}_\mu(k)^* M^{\mu\nu}(k) \hat{A}_\nu(k) \quad (3.1)$$

(where $F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu$ and $M^{\mu\nu}(k) := k^2 g^{\mu\nu} - k^\mu k^\nu$) is unsuited for quantization because $M^{\mu\nu}$ is not invertible: due to $M^{\mu\nu}k_\mu = 0$ it has an eigenvalue 0. Therefore, the action is usually modified by adding a Lorentz invariant gauge fixing term: $M^{\mu\nu}$ is replaced by $M^{\mu\nu}(k) + \lambda k^\mu k^\nu$ where $\lambda \in \mathbb{R} \setminus \{0\}$ is an arbitrary constant. The corresponding Euler-Lagrange equation reads

$$\Box A^\mu - (1 - \lambda) \partial^\mu \partial_\nu A^\nu = 0 . \quad (3.2)$$

For simplicity let us choose $\lambda = 1$, which is referred to as Feynman gauge. Then the algebra of the free gauge field is the unital $*$-algebra generated by elements $A^\mu(f), f \in \mathcal{D}(\mathbb{R}^4)$, which fulfill the relations:

- $f \mapsto A^\mu(f)$ is linear, \hspace{1cm} (3.3)
- $A^\mu(\Box f) = 0$ , \hspace{1cm} (3.4)
- $A^\mu(f)^* = A^\mu(\bar{f})$ , \hspace{1cm} (3.5)
- $[A^\mu(f), A^\nu(g)] = ig^{\mu\nu} \int dx dy f(x)D(x - y)g(y)$ \hspace{1cm} (3.6)
where $D$ is the massless Pauli-Jordan distribution.

This algebra does not possess Hilbert space representations which satisfy the microlocal spectrum condition, a condition which in particular requires the singularity of the 2-point function to be of the so-called Hadamard form. It possesses instead representations on vector spaces with a nondegenerate sequilinear form, e.g. the Fock space over a one particle space with scalar product

$$\langle \phi, \psi \rangle = (2\pi)^{-3} \int \frac{d^3p}{2|\vec{p}|} \phi^\mu(p) \psi_\mu(p)|_{p^0 = |\vec{p}|}.$$

(3.7)

Gupta and Bleuler characterized a subspace of the Fock space on which the scalar product is semidefinite; the space of physical states is then obtained by dividing out the space of vectors with vanishing norm.

After adding a mass term $\frac{m^2}{2} \int dx A_\mu(x) A^\mu(x)$ to the action (3.1), it seems to be no longer necessary to add also a gauge fixing term. The fields then satisfy the Proca equation

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0,$$

(3.8)

which is equivalent to $\Box A^\mu = 0$ together with the constraint $\partial_\mu A^\mu = 0$. The Cauchy problem is well posed, and the fields can be represented in a positive norm Fock space with only physical states (corresponding to the three physical polarizations of $A$). The problem, however, is that the corresponding propagator admits no power counting renormalizable perturbation series.

The latter problem can be circumvented in the following way: For the algebra of the free quantum field one takes only $\Box A^\mu = 0$ into account (or equivalently one adds the 'gauge fixing term' $\frac{1}{2}(\partial_\mu A^\mu)^2$ to the Lagrangian) and goes over from the physical field $A^\mu$ to

$$B^\mu := A^\mu + \frac{\partial^\mu \phi}{m},$$

(3.9)

where $\phi$ is a real scalar field to the same mass $m$ where the sign of the commutator is reversed (‘bosonic ghost field’ or ‘Stückelberg field’). The propagator of $B^\mu$ yields a power counting renormalizable perturbation series, however $B^\mu$ is an unphysical field. One obtains four independent components of $B$ which satisfy the Klein Gordon equation. The constraint $0 = \partial_\mu A^\mu = \partial_\mu B^\mu + m\phi$ is required for the expectation values in physical states only. So quantization in the case $m > 0$ can be treated in analogy to (3.4)-(3.6) by replacing $A^\mu$ by $B^\mu$, the wave operator by the Klein Gordon operator $\Box + m^2$ in (3.4) and $D$ by the corresponding massive commutator distribution $\Delta_m$ in (3.6). Again the algebra can be nontrivially represented on a space with indefinite metric, but not on a Hilbert space.

One can now use the method of BRST quantization in the massless as well as in the massive case. One introduces a pair of fermionic scalar fields
(‘ghost fields’) \((u, \bar{u})\). \(u, \bar{u}\) and (for \(m > 0\)) \(\phi\) fulfil the Klein Gordon equation to the same mass \(m \geq 0\) as the vector field \(B\). The free BRST-transformation reads

\[
s_0(B^\mu) = i \partial^\mu u, \quad s_0(\phi) = imu, \quad s_0(u) = 0 \quad s_0(\bar{u}) = -i(\partial_\nu B^\nu + m\phi),
\]

(3.10)

see e.g. the second book of G. Scharf in the list below. It is implemented by the free BRST-charge

\[
Q_0 = \int_{x^0 = \text{const.}} d^3x j_0^{(0)}(x^0, \bar{x}),
\]

(3.11)

where

\[
j_\mu^{(0)} := (\partial_\nu B^\nu + m\phi)\partial_\mu u - \partial_\mu(\partial_\nu B^\nu + m\phi)u
\]

(3.12)

is the free BRST-current, which is conserved. (The interpretation of the integral in (3.11) requires some care.) \(Q_0\) satisfies the assumptions of the (local) operator BRST-formalism (Sect. 2), in particular it is nil-potent and positive \((2.4)\). Distinguished representatives of the equivalence classes \([\phi] \in \text{Ke} Q_0 / \text{Ra} Q_0\) are the states built up from the three spatial (two transversal for \(m = 0\), respectively) polarizations of \(A\) only.

\section{4 Perturbative renormalization}

The starting point for a perturbative construction of an interacting quantum field theory is Dyson’s formula for the evolution operator in the interaction picture. To avoid conflicts with Haag’s Theorem on the nonexistence of the interaction picture in quantum field theory one multiplies the interaction Lagrangian \(L\) with a test function \(g\) and studies the local S-matrix

\[
S(gL) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int dx_1 \cdots dx_n g(x_1) \cdots g(x_n) T(L(x_1) \cdots L(x_n))
\]

(4.1)

where \(T\) denotes a time ordering prescription. In the limit \(g \to 1\) (adiabatic limit) \(S(gL)\) tends to the scattering matrix. This limit, however, is plagued by infrared divergences and does not always exist. Interacting fields \(F_{gL}\) are obtained by Bogoliubov’s formula

\[
F_{gL}(x) = \left. \delta \over \delta h(x) \right|_{h=0} S(gL)^{-1} S(gL + hF).
\]

(4.2)

The algebraic properties of the interacting fields within a region \(\mathcal{O}\) depend only on the interaction within a slightly larger region (Brunetti - Fredenhagen (2000)), hence the net of algebras in the sense of AQFT can be constructed in the adiabatic limit without infrared problems. (This is called the ‘algebraic adiabatic limit’.)
The construction of the interacting theory is thus reduced to a definition of time ordered products of fields. This is the program of causal perturbation theory ('CPT') which was developed by Epstein - Glaser (1973) on the basis of previous work by Stückelberg and Bogoliubov - Shirkov (1959). For simplicity we describe CPT for a real scalar field. Let $\phi$ be a classical real scalar field which is not restricted by any field equation. Let $P$ denote the algebra of polynomials in $\phi$ and all its partial derivatives $\partial^a \phi$ with multi-indices $a \in \mathbb{N}^4$ and. The time ordered products $(T_n)_{n \in \mathbb{N}}$, are linear and symmetric maps $T_n : (P \otimes D(\mathbb{R}^4))^\otimes_n \rightarrow L(D)$, where $L(D)$ is the space of operators on a dense invariant domain $D$ in the Fock space of the scalar free field. One often uses the informal notation

$$T_n(g_1 F_1 \otimes ... \otimes g_n F_n) = \int dx_1...dx_n T_n(F_1(x_1), ..., F_n(x_n))g_1(x_1)...g_n(x_n),$$

(4.3)

where $F_j \in P$, $g_j \in D(\mathbb{R}^4)$.

The sequence $(T_n)$ is constructed by induction on $n$, starting with the initial condition

$$T_1(\prod_j \partial^{a_j} \phi(x)) = : \prod_j \partial^{a_j} \phi(x) :,$$

(4.4)

where the r.h.s. is a Wick polynomial of the free field $\phi$. In the inductive step the requirement of causality plays the main role, i.e. the condition that

$$T_n(f_1 \otimes ... \otimes f_n) = T_k(f_1 \otimes ... \otimes f_k)T_{n-k}(f_{k+1} \otimes ... \otimes f_n)$$

(4.5)

if $(\text{supp } f_1 \cup ... \cup \text{supp } f_k) \cap ((\text{supp } f_{k+1} \cup ... \cup \text{supp } f_n) + \bar{V}_-) = \emptyset$ (where $\bar{V}_-$ is the closed backward light cone). This condition expresses the composition law for evolution operators in a relativistically invariant and local way. Causality determines $T_n$ as an operator valued distribution on $\mathbb{R}^{4n}$ in terms of the inductively known $T_l$, $l < n$ outside of the total diagonal $\Delta_n := \{(x_1, ..., x_n) | x_1 = ... = x_n\}$, i.e. on test functions from $D(\mathbb{R}^{4n} \setminus \Delta_n)$.

Perturbative renormalization is now the extension of $T_n$ to the full test function space $D(\mathbb{R}^{4n})$. Generally, this extension is non-unique. In contrast to other methods of renormalization no divergences appear, but the ambiguities correspond to the finite renormalizations which remain after removal of divergences by infinite counter terms. The ambiguities can be reduced by (re-)normalization conditions, which means that one requires that certain properties which hold by induction on $D(\mathbb{R}^{4n} \setminus \Delta_n)$ are maintained in the extension, namely:

- (N0) A bound on the degree of singularity near the total diagonal.
- (N1) Poincaré covariance.
- (N2) Unitarity of the local S-matrix.
- (N3) A relation to the time-ordered products of sub-polynomials.
• (N4) The field equation for the interacting field \( \varphi_{g\mathcal{L}} \) (4.2).

• (AWI) The Action Ward identity (Stora and Dütsch - Fredenhagen (2003)): \( \partial^{\mu}T(...F_{i}(x)...) = T(...)\partial^{\mu}F_{i}(x)...) \). This condition can be understood as the requirement that physics depends on the action only, so total derivatives in the interaction Lagrangian can be removed.

• Further symmetries, in particular in gauge theories Ward identities expressing BRST-invariance. A universal formulation of all symmetries which can be derived from the field equation in classical field theory is the Master Ward Identity (which presupposes (N3) and (N4)) (Boas - Dütsch - Fredenhagen (2002-2003)), see Sect. 5.

The problem of perturbative renormalization is to construct a solution of all these normalization conditions. Epstein and Glaser have constructed the solutions of (N0)-(N3). Recently, the conditions (N4) and (AWI) have been included. The Master Ward Identity cannot always be fulfilled, the obstructions are the famous 'anomalies' of perturbative Quantum Field Theory.

5 Perturbative construction of gauge theories

In the case of a purely massive theory the adiabatic limit \( S = \lim_{g \to 1} S(g\mathcal{L}) \) exists (Epstein - Glaser (1976)), and one may adopt a formalism due to Kugo and Ojima (1979) who use the fact that in these theories the BRST charge \( Q \) can be identified with the incoming (free) BRST charge \( Q_{0} \) (3.11).

For the scattering matrix \( S \) to be a well defined operator on the physical Hilbert space of the free theory, \( \mathcal{H} = K e^{Q_{0}/R a} Q_{0} \), one then has to require

\[
\lim_{g \to 1} [Q_{0}, T((g\mathcal{L})^{\otimes n})]|_{\ker Q_{0}} = 0 .
\]  

(5.1)

This is the motivation for introducing the condition of 'perturbative gauge invariance' (Dütsch - Hurth - Krahe - Scharf (1993-1996), see the second book of G. Scharf in the list below): According to this condition, there should exist a Lorentz vector \( \mathcal{L}^{\nu}_{1} \in \mathcal{P} \) associated to the interaction \( \mathcal{L} \), such that

\[
[Q_{0}, T_{n}(\mathcal{L}(x_{1})...\mathcal{L}(x_{n}))] = i \sum_{l=1}^{n} \partial_{\nu}^{l} T_{n}(\mathcal{L}(x_{1})...\mathcal{L}^{\nu}_{1}(x_{l})...\mathcal{L}(x_{n})) .
\]  

(5.2)

This is a somewhat stronger condition than (5.1) but has the advantage that it can be formulated independently of the adiabatic limit. The condition (5.1) (or perturbative gauge invariance) can be satisfied for tree diagrams (i.e. the corresponding requirement in classical field theory can be fulfilled). In the massive case this is impossible without a modification of
the model; the inclusion of additional physical scalar fields (corresponding to Higgs fields) yields a solution. It is gratifying that, making a polynomial ansatz for the interaction $L \in \mathcal{P}$, perturbative gauge invariance (5.2) for tree diagrams, renormalizability (i.e. the mass dimension of $L$ is $\leq 4$) and some obvious requirements (e.g. Lorentz invariance) determine $L$ to a fair extent. In particular, the Lie algebraic structure needs not to be put in, it can be derived in this way (Stora (1997)). Including loop diagrams (i.e. quantum effects), it has been proved that (N0)-(N2) and perturbative gauge invariance can be fulfilled to all orders for massless $SU(N)$-Yang-Mills theories.

Unfortunately, in the massless case, it is unlikely that the adiabatic limit exists and, hence, an $S$-matrix formalism is problematic. One should better rely on the construction of local observables in terms of couplings with compact support. But then the selection of the observables (2.1) has to be done in terms of the BRST-transformation $\tilde{s}$ of the interacting fields. For the corresponding BRST-charge one makes the ansatz

$$\tilde{Q} = \int d^4x \tilde{j}_\mu^g(x) b_\mu(x) , \quad L = \sum_{n \geq 1} \mathcal{L}_n \lambda^n , \quad (5.3)$$

where $(b_\mu)$ is a smooth version of the $\delta$-function characterizing a Cauchy surface$^1$ and $\tilde{j}_\mu^g$ is the interacting BRST-current (4.2) (where $\tilde{j}_\mu = \sum_n j_\mu^{(n)} \lambda^n$ ($j_\mu^{(n)} \in \mathcal{P}$) is a formal power series with $j_\mu^{(0)}$ given by (3.12)). A crucial requirement is that $\tilde{j}_\mu^g$ is conserved in a suitable sense. This condition is essentially equivalent to perturbative gauge invariance and hence its application to classical field theory determines the interaction $L$ in the same way, and in addition the deformation $j^{(0)} \rightarrow \tilde{j}_g L$. The latter gives also the interacting BRST charge and transformation, $\tilde{Q}$ and $\tilde{s}$, by (5.3) and (2.2). Mostly the so obtained $\tilde{Q}$ is nil-potent in classical field theory (and hence this holds also for $\tilde{s}$). However, in QFT conservation of $\tilde{j}_g L$ and $\tilde{Q}^2 = 0$ require the validity of additional Ward identities, beyond the condition of perturbative gauge invariance (5.2). All the necessary identities can be derived from the Master Ward Identity

$$T_{n+1}(A, F_1, ..., F_n) = -\sum_{k=1}^n T_n(F_1, ..., \delta_A F_k, ..., F_n) , \quad (5.4)$$

where $A = \delta_A S_0$ with a derivation $\delta_A$. The Master Ward Identity is closely related to the Quantum Action Principle which was formulated in the formalism of generating functionals of Green’s functions. In the latter framework the anomalies have been classified by cohomological methods. The

$^1$There is a volume divergence in this integral, which can be avoided by a spatial compactification. This does not change the abstract algebra $\mathcal{F}_\mathcal{L}(\mathcal{O})$. 

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vanishing of anomalies of the BRST symmetry is a selection criterion for physically acceptable models.

In the particular case of QED, the Ward identity

\[ \partial_\mu T \left( j^\mu(y) F_1(x_1) \ldots F_n(x_n) \right) = i \sum_{j=1}^{n} \delta(y - x_j) T \left( F_1(x_1) \ldots (\theta F_j)(x_j) \ldots F_n(x_n) \right) \]

(5.5)

for the Dirac current \( j^\mu := \bar{\psi} \gamma^\mu \psi \), is sufficient for the construction, where \( (\theta F) := i(r - s)F \) for \( F = \psi^r \bar{\psi}^s B_1 \ldots B_l \) (\( B_1, \ldots, B_l \) are non-spinorial fields) and \( F_1, \ldots, F_n \) run through all sub-polynomials of \( \mathcal{L} = j^\mu A_\mu \), \( \text{(N0)-(N4)} \) and \( (5.5) \) can be fulfilled to all orders (Dütsch - Fredenhagen (1999)).

Further Reading

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