It is known that for a conditional quasi-greedy basis $B$ in a Banach space $X$, the associated sequence $(k_m[B])_{m=1}^\infty$ of its conditionality constants verifies the estimate $k_m[B] = O(\log m)$ and that if the reverse inequality $\log m = O(k_m[B])$ holds then $X$ is non-superreflexive. However, in the existing literature one finds very few instances of non-superreflexive spaces possessing quasi-greedy basis with conditionality constants “as large as possible.” Our goal in this article is to fill this gap. To that end we enhance and exploit a combination of techniques developed independently, on the one hand by Garrigós and Wojtaszczyk in [17], and, on the other hand, by Dilworth et al. in [11], and craft a wealth of new examples of non-superreflexive classical Banach spaces having quasi-greedy bases $B$ with $k_m[B] = O(\log m)$.

1. Introduction

Let $B = (x_n)_{n=1}^\infty$ be a basis for a Banach space $X$ and let $(x_n^*)_{n=1}^\infty$ be its sequence of coordinate functionals. Given a subset $A$ of $\mathbb{N}$, the coordinate projection on $A$ is (when well defined) the linear operator

$$S_A[B] : X \to X, \quad f \mapsto \sum_{n \in A} x_n^*(f) x_n.$$

The basis $B$ is unconditional if and only if $\sup_{A \text{ finite}} \|S_A[B]\| < \infty$. Thus, in some sense, the conditionality of $B$ can be measured in terms of the growth of the sequence

$$k_m[B] := \sup_{|A| \leq m} \|S_A[B]\|, \quad m \in \mathbb{N}.$$

Recall that a basis $B = (x_j)_{j=1}^\infty$ is said to be quasi-greedy if it is semi-normalized (i.e., $0 < \inf_{j \in \mathbb{N}} \|x_j\| \leq \sup_{j \in \mathbb{N}} \|x_j\| < \infty$) and there is a
constant $C$ such that
\[
\| f - S_A[B](f) \| \leq C\| f \|
\] (1.1)
whenever $f \in X$ and $A \subseteq \mathbb{N}$ are such that $|x^*_j(f)| \leq |x^*_k(f)|$ for all $j \in \mathbb{N} \setminus A$ and all $k \in A$. The least constant $C$ such that (1.1) holds is known as the quasi-greedy constant of the basis (see [3, Remark 4.2]).

The next theorem summarizes the connection that exists between superreflexivity and the conditionality constants of quasi-greedy bases. Recall that a space $X$ is said to be superreflexive if every Banach space finitely representable in $X$ is reflexive.

**Theorem 1.1** (see [4, 5, 11, 17]). Let $X$ be a Banach space.

(a) If $B$ is a quasi-greedy basis for $X$ then $k_m[B] \lesssim \log m$ for $m \geq 2$.

(b) If $X$ is non-superreflexive then there is a quasi-greedy basis $B$ for a Banach space finitely representable in $X$ with $k_m[B] \gtrsim \log m$ for $m \geq 2$.

(c) If $X$ is superreflexive and $B$ is quasi-greedy then there is $0 < a < 1$ such that $k_m[B] \lesssim (\log m)^a$ for $m \geq 2$.

(d) For every $0 < a < 1$ there is a quasi-greedy basis $B$ for a Banach space (namely, $\ell_2$) finitely representable in $X$ with $k_m[B] \gtrsim (\log m)^a$ for $m \geq 2$.

Theorem 1.1 characterizes both the superreflexivity and the lack of superreflexivity of a Banach space $X$ in terms of the growth of the conditionality constants of quasi-greedy bases. It could be argued that the quasi-greedy bases whose existence is guaranteed in parts (b) and (d) lie outside the space $X$, and that, although this approach is consistent when dealing with “super” properties, in truth it does not tackle the question of the existence of a quasi-greedy basis with large conditionality constants in the space $X$ itself! Hence this discussion naturally leads to the following two questions:

**Question A.** Let $X$ be a non-superreflexive Banach space with a basis. Is there a quasi-greedy basis $B$ for $X$ with $k_m[B] \approx \log m$ for $m \geq 2$?

**Question B.** Let $X$ be a superreflexive Banach space with a basis. Given $0 < a < 1$, is there a quasi-greedy basis $B$ for $X$ with $k_m[B] \gtrsim (\log m)^a$ for $m \geq 2$?

Questions A and B can be regarded as a development of the query initiated by Konyagin and Telmyakov in 1999 [23] of finding conditional quasi-greedy bases in general Banach spaces, and which has evolved towards the more specific quest of finding quasi-greedy bases “as conditional as possible.” The reader will find a detailed account of this process in the papers [5, 11, 14, 16, 21, 32].
Let us outline the state of the art of those two questions. Garrigós and Wojtaszczyk proved in [17] that Question B has a positive answer for $X = \ell_p$, $1 < p < \infty$. As for Question A, it is known that Lindenstrauss’ basic sequence in $\ell_1$, the Haar system in $BV(\mathbb{R}^d)$ for $d \geq 2$, and the unit-vector system in the Konyagin-Telmyakov space $KT(\infty, p)$ for $1 < p < \infty$, are all quasi-greedy basic sequences with conditionality constants as large as possible (see [8, 16]). Moreover, in [17] it is proved that the answer to Question A is positive for $\ell_1 \oplus \ell_2 \oplus c_0$, and in [4] that the same holds true for mixed-norm spaces of the form $(\bigoplus_{n=1}^{\infty} \ell_1^n)_q$ ($1 < q < \infty$), providing this way the first-known examples of reflexive Banach spaces having quasi-greedy bases with conditionality constants as large as possible. More recently, the authors constructed in [5] the first-known examples of Banach spaces of nontrivial type and nontrivial cotype for which the answer to Question A is positive. These spaces are denoted by $W_{p,q}^0 \oplus W_{p,q}^0 \oplus \ell_2$, $1 < p, q < \infty$, where $W_{p,q}^0$ and $W_{p,q}'$ are the interpolation spaces defined from the space of sequences of bounded variation

$$v_1 = \{(a_j)_{j=1}^{\infty} : |a_1| + \sum_{j=2}^{\infty} |a_j - a_{j-1}| < \infty\},$$

and the subspace $v_1^0$ of $v_1$ resulting from the intersection of $v_1$ with $c_0$. Here and throughout this paper, $(X_0, X_1)_{\theta,q}$ denotes the Banach space obtained by applying the real interpolation method to the Banach couple $(X_0, X_1)$ with indices $\theta$ and $q$.

In this article we develop the necessary machinery that permits to extend the scant list of known Banach spaces for which the answer either to Question A or to Question B is positive. Moreover, in some cases, the examples of bases we provide are not only quasi-greedy but are almost greedy. Recall that a basis $B = (x_j)_{j=1}^{\infty}$ for a Banach space $X$ is almost greedy if there is a constant $C$ such that

$$\|f - S_A[B](f)\| \leq C\|f - S_B[B](f)\|$$

whenever $f \in X$, $|B| \leq |A| < \infty$, and $|x_j^*(f)| \leq |x_k^*(f)|$ for any $j \in \mathbb{N} \setminus A$ and $k \in A$. Almost greedy bases were characterized in [12] as those bases that are simultaneously quasi-greedy and democratic.

Our study includes, among other spaces, the finite direct sums

$$D_{p,q} := \begin{cases} \ell_p \oplus \ell_q & \text{if } 1 \leq p, q < \infty, \\ \ell_p \oplus c_0 & \text{if } 1 \leq p < \infty \text{ and } q = 0, \end{cases}$$
the matrix spaces

\[ Z_{p,q} := \begin{cases} \ell_q(\ell_p) & \text{if } 1 \leq p, q < \infty, \\ c_0(\ell_p) & \text{if } 1 \leq p < \infty \text{ and } q = 0, \\ \ell_q(c_0) & \text{if } p = 0 \text{ and } 1 \leq q < \infty, \end{cases} \]

and the mixed-norm spaces of the family

\[ B_{p,q} := (\bigoplus_{n=1}^\infty \ell_p^n)_q, \quad p \in [1, \infty], \ q \in \{0\} \cup [1, \infty), \ q \neq 0 \text{ when } p = \infty. \]

We use \((\bigoplus_{n=1}^\infty X_n)_q\) to denote the direct sum of the Banach spaces \(X_n\) in the \(\ell_q\)-sense (\(c_0\)-sense if \(q = 0\)).

In Section 3, roughly speaking, we prove that in all non-superreflexive spaces in the aforementioned list the answer to Question A is positive. We also show that, in turn, in all superreflexive spaces in the above list the answer to Question B is positive. Previously, in Section 2, we introduce the tools that we will use to achieve this goal.

Throughout this article we follow standard Banach space terminology and notation as can be found, e.g., in [6] but we would like to single out the notation that is more commonly employed. We deal with real or complex Banach spaces, and \(\mathbb{F}\) will denote the underlying scalar field. As it is customary, we put \(\delta_{j,k} = 1\) if \(j = k\) and \(\delta_{j,k} = 0\) otherwise. Given \(j \in \mathbb{N}\), the \(j\)-th unit vector is defined by \(e_j = (\delta_{j,k})_{k=1}^\infty\) and the unit-vector system will be sequence \((e_j)_{j=1}^\infty\). Given families of positive real numbers \((\alpha_i)_{i \in I}\) and \((\beta_i)_{i \in I}\), the symbol \(\alpha_i \lesssim \beta_i\) for \(i \in I\) means that \(\sup_{i \in I} \alpha_i/\beta_i < \infty\), while \(\alpha_i \approx \beta_i\) for \(i \in I\) means that \(\alpha_i \lesssim \beta_i\) and \(\beta_i \lesssim \alpha_i\) for \(i \in I\). Applied to Banach spaces, the symbol \(X \approx Y\) means that the spaces \(X\) and \(Y\) are isomorphic, while the symbol \(X \lesssim Y\) means that \(X\) is isomorphic to a complemented subspace of \(Y\). Given families of Banach spaces \((X_i)_{i \in I}\) and \((Y_i)_{i \in I}\), the symbol \(X_i \lesssim Y_i\) for \(i \in I\) means that the spaces \(X_i\) are uniformly isomorphic to complemented subspaces of \(Y_i\), i.e., there are linear operators \(L_i: X_i \to Y_i\), \(R_i: Y_i \to X_i\) such that \(R_i \circ L_i = \text{Id}_{X_i}\) and \(\sup_{i \in I} \|R_i L_i\| < \infty\). Similarly the symbol \(X_i \approx Y_i\) for \(i \in I\) means that Banach-Mazur distance from \(X_i\) to \(Y_i\) is uniformly bounded. We write \(X \oplus Y\) for the Cartesian product of the Banach spaces \(X\) and \(Y\) endowed with the norm

\[ \|(x, y)\| = \max\{\|x\|, \|y\|\}, \quad x \in X, \ y \in Y. \]

The closed linear span of a family \((x_i)_{i \in I}\) in a Banach space will be denoted by \([x_i: i \in I]\). A basis always will be a Schauder basis. Given a basis \(B = (x_j)_{j=1}^\infty\) for a Banach space \(X\) and \(m \in \mathbb{N}\) we denote by
\( S_m[\mathcal{B}] \) the \( m \)-th partial sum projection, i.e.,
\[
S_m[\mathcal{B}](f) = \sum_{j=1}^{m} x_j^*(f) x_j, \quad f \in \mathcal{X}.
\]

The fundamental function (or upper democracy function) of a basis \( \mathcal{B} \) is the sequence
\[
\phi_m[\mathcal{B}] = \sup_{|A| \leq m} \left\| \sum_{j \in A} x_j \right\|
\]
If there is a constant \( C \) such that \( \phi_m[\mathcal{B}] \leq C \sum_{j \in A} \| \sum_{j \in A} x_j \| \) whenever \( m \leq |A| \) the basis is said to be democratic.

For \( 1 \leq p \leq \infty \) and \( d \in \mathbb{N} \), the finite-dimensional space \( \ell_p^d \), will be the subspace of \( \ell_p \) consisting of all sequences \( (a_j)_{j=1}^\infty \) with \( a_j = 0 \) for \( j > d \). More generally, given a basis \( \mathcal{B} = (x_n)_{n=1}^\infty \) for a Banach space \( \mathcal{X} \) and \( d \in \mathbb{N} \) we will consider the closed linear span of the truncated finite sequence \( (x_j)_{j=1}^d \), i.e.,
\[
\mathcal{X}^d[\mathcal{B}] = [x_j : 1 \leq j \leq d].
\]

Other more specific notation will be introduced on the spot when needed.

2. Background and preliminary results

Most of the ideas behind the results we include in this preliminary section have appeared more or less explicitly in the literature before. Nonetheless, for the sake of clarity and completeness, we shall include the statements of the results we need in the form that best suits our purposes and the sketches of their proofs.

Definition 2.1. Given a basis \( \mathcal{B} \) for a Banach space \( \mathcal{X} \), we define the\( w \)-conditionality constants of \( \mathcal{B} \) by
\[
L_m[\mathcal{B}] = \sup \left\{ \frac{\| S_A[\mathcal{B}](f) \|}{\| f \|} : \max(\text{supp}(f)) \leq m, A \subseteq \mathbb{N} \right\}, \quad m \in \mathbb{N}.
\]

Notice that \( \mathcal{B} \) is unconditional if and only if \( \sup_m L_m[\mathcal{B}] < \infty \). Hence, the growth of the sequence \( (L_m[\mathcal{B}])_{m=1}^\infty \) provides also a measure of the conditionality of the basis. Since \( L_m[\mathcal{B}] \leq k_m[\mathcal{B}] \) for all \( m \in \mathbb{N} \), any result establishing that the size of the members of the sequence \( (L_m[\mathcal{B}])_{m=1}^\infty \) is large, is a stronger statement than the corresponding one enunciated in terms of \( (k_m[\mathcal{B}])_{m=1}^\infty \). The papers [5, 17] draw attention to the fact that, in some cases, the \( w \)-conditionality constants are more fit than the “usual” conditionality constants for transferring conditionality properties from a given basis to a basis constructed from it. This is the reason why we establish all the instrumental results of this section in terms of \( w \)-conditionality constants. Notice that, in contrast
to \((k_m[B])^\infty_{m=1}\), the sequence \((L_m[B])^\infty_{m=1}\) is not necessarily doubling. This fact leads us to use a doubling function in our statements. Recall that a function \(\delta: [0, \infty) \to [0, \infty)\) is said to be doubling if for some non-negative constant \(C\) one has \(\delta(2t) \leq C\delta(t)\) for all \(t \geq 0\).

Given sequences \(B_0 = (x_j)^\infty_{j=1}\) and \(B_1 = (y_j)^\infty_{j=1}\) in Banach spaces \(X\) and \(Y\), respectively, their direct sum \(B_0 \oplus B_1\) will be the sequence in \(X \times Y\) given by

\[
B_0 \oplus B_1 = ((x_1, 0), (0, y_1), (x_2, 0), (0, y_2), \ldots, ((x_j, 0), (0, y_j), \ldots).
\]

**Lemma 2.2** (cf. [16]). Let \(\delta: [0, \infty) \to [0, \infty)\) be a doubling increasing function. Suppose that \(B_0 = (x_j)^\infty_{j=1}\) and \(B_1 = (y_j)^\infty_{j=1}\) are bases in Banach spaces \(X\) and \(Y\), respectively, and assume that with \(L_m[B_0] \gtrsim \delta(m)\) for \(m \in \mathbb{N}\). Then \(B_0 \oplus B_1\) is a basis of \(X \oplus Y\) fulfilling \(L_m[B_0 \oplus B_1] \gtrsim \delta(m)\) for \(m \in \mathbb{N}\). Moreover we have:

(a) If \(B_0\) and \(B_1\) are quasi-greedy so is \(B_0 \oplus B_1\);

(b) If \(B_0\) and \(B_1\) are democratic bases with \(\phi_m[B_0] \approx \phi_m[B_1]\) for \(m \in \mathbb{N}\) then \(B_0 \oplus B_1\) is democratic with \(\phi_m[B_0 \oplus B_1] \approx \phi_m[B_0] \approx \phi_m[B_1]\) for \(m \in \mathbb{N}\).

**Proof.** It is similar to the proof of [16, Proposition 6.1], so we omit it. \(\square\)

Our next lemma follows an idea from [32] for constructing quasi-greedy bases. To state it properly it will be convenient to introduce some additional notation.

Let \(B = (x_j)^\infty_{j=1}\) be a basis in a Banach space \(X\). Given a sequence of positive integers \((d_n)^\infty_{n=1}\) we define a sequence \((z_k)^\infty_{k=1}\) that we denote \(\bigoplus_{n=1}^\infty B[d_n]\) in the space \(X^\mathbb{N}\) by

\[
z_k = (0, \ldots, 0, x_j, 0, \ldots, 0, \ldots),
\]

where, for a given \(k \in \mathbb{N}\), the integers \(r\) and \(j\) are univocally determined by the relations \(k = j + \sum_{n=1}^{r-1} d_n\) and \(1 \leq j \leq d_n\).

**Lemma 2.3.** Suppose \(p \in \{0\} \cup [1, \infty)\). Let \((d_n)^\infty_{n=1}\) be a sequence of positive integers, \(\delta: [0, \infty) \to [0, \infty)\) be a doubling increasing function, and \(B = (x_j)^\infty_{j=1}\) be a basis for a Banach space \(X\). Assume that \(L_m[B] \gtrsim \delta(m)\) for \(m \in \mathbb{N}\) and that there is \(D > 1\) such that \(d_n \approx D^n\) for \(n \in \mathbb{N}\). Then:

(a) \(B_0 = \bigoplus_{n=1}^\infty B[d_n]\) is a basis for the Banach space \((\bigoplus_{n=1}^\infty X^{d_n}[B])_p\) with \(L_m[B_0] \gtrsim \delta(m)\) for \(m \in \mathbb{N}\).

(b) If \(B\) is quasi-greedy so is \(B_0\).
(c) If \( p \neq 0 \) and \( B \) is democratic with \( \phi_m[B] \approx m^{1/p} \) for \( m \in \mathbb{N} \) then \( B_0 \) is democratic with \( \phi_m[B_0] \approx m^{1/p} \) for \( m \in \mathbb{N} \).

Lemma 2.3 is stated with sufficient generality to meet our goals in most cases. However, in one specific case we will need a more technical auxiliary result which includes Lemma 2.3 as a particular case.

Let \( (d_n)_{n=1}^{\infty} \) be a sequence of positive integers. Suppose that \( B = \langle x_j \rangle_{j=1}^{\infty} \) is a basis in a Banach space \( X \) and that, for each \( n \in \mathbb{N} \), \( P_n \) and \( Q_n \) are linear maps from \( X^{d_n}[B] \) into the (possibly zero) Banach spaces \( Y_n \) and \( Z_n \), respectively. We define a sequence \( \bigoplus_{n=1}^{\infty} B[P_n, Q_n] = (z_k)_{k=1}^{\infty} \) in \( \Pi_{n=1}^{\infty} Y_n \times \Pi_{n=1}^{\infty} Z_n \) by
\[
z_k = (0, \ldots, 0, P_n(x_j), 0, \ldots, 0, \ldots), (0, \ldots, 0, Q_n(x_j), 0, \ldots, 0, \ldots)),
\]
for \( n = r-1 \) times \( k = j + \sum_{n=1}^{r-1} d_n, 1 \leq j \leq d_n. \)

**Lemma 2.4.** Suppose \( p, q \in \{0\} \cup \{1, \infty\} \). Let \( (d_n)_{n=1}^{\infty} \) be a sequence of positive integers, \( \delta: [0, \infty) \to [0, \infty) \) be a doubling increasing function, \( (Y_n)_{n=1}^{\infty} \) and \( (Z_n)_{n=1}^{\infty} \) be sequences of Banach spaces, \( B = \langle x_n \rangle_{n=1}^{\infty} \) be a basis for a Banach space \( X \), and \( (P_n)_{n=1}^{\infty} \) and \( (Q_n)_{n=1}^{\infty} \) be sequences of linear maps. Assume that

- For each \( n \in \mathbb{N} \), \( (P_n, Q_n) \) is an isomorphism from \( X^{d_n}[B] \) onto \( Y_n \oplus Z_n \), and \( \sup_n \|P_n, Q_n\|\|P_n, Q_n\|^{-1} < \infty \),
- there is \( D > 0 \) such that \( d_n \approx D^n \) for \( n \in \mathbb{N} \), and
- \( L_m[B] \geq \delta(m) \) for \( m \in \mathbb{N} \).

Then \( B_0 = \bigoplus_{n=1}^{\infty} B[P_n, Q_n] \) is a basis for \( \bigoplus_{n=1}^{\infty} Y_n \oplus \bigoplus_{n=1}^{\infty} Z_n \) with \( L_m[B_0] \geq \delta(m) \). Moreover, if \( B \) is quasi-greedy so is \( B_0 \).

**Proof.** Let \( C_1 \) be such that \( C_1 L_m[B] \geq \delta(m) \) for all \( m \in \mathbb{N} \). Let \( C_2 \) and \( C_3 \) be such that \( d_n \leq C_2 D^n \) and \( D^n \leq C_3 d_n \) for all \( n \in \mathbb{N} \) and put \( C_4 = C_2 C_3 D^2/(D - 1) \). Since \( \delta \) is doubling there a constant \( C_5 \) such that \( C_5 \delta(m) \geq \delta(C_4 m) \) for all \( m \in \mathbb{N} \). Replacing \( \epsilon_0 \) with \( c_0 \) and \( \| \cdot \|_0 \) with \( \| \cdot \|_\infty \) when some of the indices involved is 0, we consider the Banach space
\[
\mathbb{V} = \{(f_n)_{n=1}^{\infty} \in \Pi_{n=1}^{\infty} X^{d_n}[B]: (\|P_n(f_n)\|)_{n=1}^{\infty} \in \ell_p, (\|Q_n(f_n)\|)_{n=1}^{\infty} \in \ell_q\},
\]
equipped with the norm
\[
\|(f_n)_{n=1}^{\infty}\| = \max\{\|(P_n(f_n))\|_{n=1}^{\infty}\|p, (\|P_n(f_n)\|)_{n=1}^{\infty}\|q\}.
\]
Since the mapping
\[
f = (f_n)_{n=1}^{\infty} \mapsto ((P_n(f_n))_{n=1}^{\infty}, (Q_n(f_n))_{n=1}^{\infty})
\]
is an isomorphism from $V$ onto $(\bigoplus_{n=1}^{\infty} Y_n)_p \oplus (\bigoplus_{n=1}^{\infty} Z_n)_q$, it suffices to show that $B_1 := \bigoplus_{n=1}^{\infty} B[d_n]$ is a quasi-greedy basis for $V$ verifying $L_m[B_1] \geq \delta(m)$ for $m \in \mathbb{N}$. It is clear that $B_1$ is a quasi-greedy basis with the same quasi-greedy constant as $B$, hence we must only take care of estimating its w-conditionality constants. Given $m \in \mathbb{N}$, put $r \in \mathbb{N}$ such that \[ \sum_{n=1}^{r} d_n \leq m < \sum_{n=1}^{r+1} d_n. \] Since \[ m \leq \sum_{n=1}^{r+1} d_n \leq C_2 \sum_{n=1}^{r+1} D^r = \frac{C_2 D^r}{D - 1} - 1 \leq \frac{C_2 C_3 D^2}{D - 1} - 1 \leq C_4 d_r, \] we have \[ C_1 C_5 L_m[B_1] \geq C_1 C_5 L_d[B] \geq C_5 \delta(d_r) \geq \delta(C_4 d_r) \geq \delta(m), \] as desired. \qed

Next, we appeal to a technique invented by Garrigós and Wojtaszczyk in [17] for tailoring a basis $O(B)$ for the direct sum $X \oplus \ell_2$ from a basis $B$ of a Banach space of $X$. This method, called for short the GOW-method in [5], can be summarized as follows.

**Theorem 2.5** ([17]). Let $\delta : [0, \infty) \rightarrow [0, \infty)$ be a doubling increasing function. Suppose that $B$ is a basis of a Banach space $X$ such that $L_m[B] \geq \delta(m)$ for $m \in \mathbb{N}$. Then

(i) The basis $B_0 := O(B)$ is an almost greedy basis of $Y := X \oplus \ell_2$,

(ii) $\phi_m[B_0] \approx m^{1/2}$, and

(iii) $L_m[B_0] \geq \delta(\log m)$ for all $m \in \mathbb{N}$.

Moreover \[ Y^{2n-2}[B_0] = X^n[B] \oplus \ell_2^{2n-n-2}, \quad n \in \mathbb{N}. \]

**Remark 2.6.** Notice that combining Proposition 4.4 and Theorem 5.4 from [12] with Theorem 2.5 yields that the dual basic sequence of the one obtained by the GOW-method is also almost greedy, with fundamental function of the order of $(m^{1/2})^{\infty}_{m=1}$.

Garrigós and Wojtaszczyk [17] combined Theorem 2.5 with the next theorem for building quasi-greedy bases as conditional as possible in $\ell_2$.

**Theorem 2.7** (cf. [17, Proposition 3.10]). For each $0 < a < 1$ there is a basis $B$ in $\ell_2$ with $L_m[B] \geq m^a$ for $m \in \mathbb{N}$.

Another technique for tailoring quasi-greedy bases was developed by Dilworth et al. in [11]. Among the many important results contained in that work, the authors proved the following two.
Theorem 2.8 (see [11], Corollary 6.3 and Theorem 7.1). Assume that a Banach space $X$ has a basis. Then $X \oplus \ell_1$ has an almost greedy basis $\mathcal{B}$ such that $\phi_m[\mathcal{B}] \approx m$ for $m \in \mathbb{N}$.

Theorem 2.9 (see [11], Theorem 7.2). Assume that a Banach space $X$ has a basis and that $1 < p < \infty$. Then $X \oplus \ell_p$ has a quasi-greedy basis.

Next we summarize the information we need on the Lindenstrauss sequence $L = (l_j)_{j=1}^{\infty}$, defined by

$$l_j = e_j - \frac{1}{2}(e_{2j} + e_{2j+1}), \quad j \in \mathbb{N}. $$

Theorem 2.10 (see [13, 16, 24, 29]). The Lindenstrauss sequence $L$ is an almost greedy basic sequence in $\ell_1$ verifying

(a) $L_m[L] \approx \log m$ for $m \geq 2$,
(b) $\ell_1^d[L] \approx \ell_1^d$ for $d \in \mathbb{N}$, and
(c) $\phi_m[L] \approx m$ for $m \in \mathbb{N}$.

Proof. That $L$ is a basic sequence is proved in [29, p. 455], and that is quasi-greedy is showed in [13]. The argument used in [16, Example 1] for computing the conditionality constants of $L$ gives (a). Finally, (b) is obtained in [24, Example 8.1], and (c) is a consequence of (b). □

Next, we consider the summing system $S = (s_j)_{j=1}^{\infty}$, given by

$$s_j = \sum_{k=1}^{j} e_j.$$  

It is known that $S$ is a conditional basis for $c_0$. Most proofs of this fact (see, e.g., [6, Example 3.1.2]) give the following.

Lemma 2.11. The summing system $S$ is a basis for $c_0$ with $L_m[S] \approx m$ for $m \in \mathbb{N}$, and $c_0^d[S] = \ell_1^d$ for all $d \in \mathbb{N}$.

By duality, the difference system $D = (d_j)_{j=1}^{\infty}$, defined by

$$d_j = e_j - e_{j-1} \quad \text{(with the convention $e_0 = 0$)},$$

is a conditional basis for $\ell_1$. Indeed, we have the following.

Lemma 2.12. The difference system $D$ is a basis of $\ell_1$ such that $L_m[D] \approx m$ for $m \in \mathbb{N}$, and $\ell_1^d[D] = \ell_1^d$ for all $d \in \mathbb{N}$.

A basis $(x_j)_{j=1}^{\infty}$ is said to be of type P if $\sup_k \|\sum_{j=1}^{k} x_j\| < \infty$ and $\inf_j \|x_j\| > 0$. Notice that both the unit vectors system in $c_0$ and the difference system in $\ell_1$ are bases of type P. The following lemma shows that Banach spaces with a basis of type P follow the pattern of $c_0$ and $\ell_1$ exhibited, respectively, in Lemmas 2.11 and 2.12.
Lemma 2.13 (see [5]). Let $\mathcal{B}$ be a basis of type $P$ of a Banach space $\mathbb{X}$. Then there is a basis $\mathcal{B}_0$ for $\mathbb{X}$ such that $L_m[\mathcal{B}_0] \approx m$ for $m \in \mathbb{N}$ and $\mathbb{X}^{2^n-2}[\mathcal{B}_0] = \mathbb{X}^{2^n-2}[\mathcal{B}]$ for all $n \geq 2$.

Proof. The proof of [5, Theorem 3.3] gives the result, although is not explicitly stated. \hfill \Box

The spaces $W^0_{p,q}$ and $W_{p,q}$ $(1 < p < \infty, 1 \leq q < \infty)$ defined in (1.2) were introduced and studied by Pisier and Xu [28]. It is verified that $W^0_{p,q} \approx W_{p,q}$. Moreover, when $q > 1$ these spaces have nontrivial type and nontrivial cotype and they are pseudo-reflexive.

Our next proposition is a new addition to the study of Pisier-Xu spaces, which will be used below. Recall that given $1 \leq q < \infty$ and a scalar sequence $w = (w_n)_{n=1}^{\infty}$, the Lorentz sequence space $d_q(w)$ consists of all sequences $f$ in $c_0$ whose non-increasing rearrangement $(a_n)_{n=1}^{\infty}$ verifies

$$
\|f\|_{d_q(w)} = \left( \sum_{n=1}^{\infty} a_n^q w_n \right)^{1/q} < \infty.
$$

In the case when $w = (n^{q/p-1})_{n=1}^{\infty}$ for some $1 \leq p < \infty$ we have that $\ell_{p,q} := d_q(w)$ is the classical sequence Lorentz space of indices $p$ and $q$.

Proposition 2.14. Let $1 < p < \infty$ and $1 \leq q < \infty$. Then $\ell_{p,q} \lesssim_c W^0_{p,q}$. Indeed, $(e_{2j-1})_{j=1}^{\infty}$ is a complemented basic sequence isometrically equivalent to the unit vector system in $\ell_{p,q}$.

Proof. Put $p = 1/(1 - \theta)$. Consider the linear maps $L, R : \ell^N \rightarrow \ell^N$ defined by

\begin{align*}
L((a_j)_{j=1}^{\infty}) & = (a_1, 0, a_2, 0, \ldots, a_j, 0, a_{j+1}, 0, \ldots), \quad (2.1) \\
R((a_j)_{j=1}^{\infty}) & = (a_1 - a_2, a_3 - a_4, \ldots, a_{2j-1} - a_{2j}, \ldots). \quad (2.2)
\end{align*}

We have $\|L : \ell_1 \rightarrow \ell_1\| \leq 1, \|L : c_0 \rightarrow c_0\| \leq 1, \|R : \ell_1 \rightarrow \ell_1\| \leq 1$, and $\|R : c_0 \rightarrow c_0\| \leq 2$. Taking into account that

$$
(\ell_1, c_0)_{\theta,q} = (\ell_1, \ell_{\infty})_{\theta,q} = \ell_{p,q}
$$

(see, e.g., [9, Theorem 1.9]), interpolation gives $\|L : \ell_{p,q} \rightarrow W^0_{p,q}\| \leq 1$ and $\|R : W^0_{p,q} \rightarrow \ell_{p,q}\| \leq 2^\theta$. Since $R(L(f)) = f$ for every $f \in \ell^N$ we are done. \hfill \Box

Corollary 2.15. Let $1 < p < \infty$ and $1 \leq q < \infty$. Then $\ell_q \lesssim_c W^0_{p,q}$.

Proof. In light of Proposition 2.14, it suffices to see that $\ell_q \lesssim_c \ell_{p,q}$. By [26, Proposition 4] we have $\ell_q \lesssim_c \ell_{p,q}$ if $q \leq p$ and $\ell_q \lesssim_c \ell_{p',q'}$ otherwise. We conclude the proof by dualizing (see [7, Theorem 1]). \hfill \Box
3. Banach spaces having quasi-greedy bases with large conditionality constants

The common thread running through this section is the search for results that will allow us to include the spaces $Z_{p,q}$, $B_{p,q}$, and $D_{p,q}$ (see Section 1) in the list of Banach spaces possessing highly conditional quasi-greedy bases. We recall that the matrix spaces $Z_{p,q}$ are isomorphic to Besov spaces over Euclidean spaces (see, e.g., [1]) and that the mixed-norm spaces $B_{p,q}$ are isomorphic to Besov spaces over the unit interval (see, e.g., [2, Appendix 4.2]).

Apart from the trivial cases, namely

$$D_{q,q} \approx Z_{q,q} \approx B_{q,q} \approx \ell^q, \quad 1 \leq q < \infty,$$

and the case

$$\ell^q \approx B_{2,q}, \quad 1 < q < \infty,$$

all the above-mentioned spaces are mutually non-isomorphic (see [2]).

The isomorphism in (3.1) was obtained by Pelczyński in [27] by combining the uniform complemented embeddings

$$\ell^q_n \lesssim \ell^q_p$$

for $n \in \mathbb{N}$, if $1 < p < \infty$,

$$(3.2)$$

(which can be obtained as a consequence of the boundedness of the Rademacher projections in $L_p$) with the Pelczyński decomposition technique (see, e.g., [6, Theorem 2.2.3]). Another well-known consequence of Pelczyński decomposition technique, is that for any unbounded sequence of integers $(d_n)_{n=1}^{\infty}$ we have

$$B_{p,q} \approx (\bigoplus_{n=1}^{\infty} \ell^q_{d_n}), \quad p \in [1, \infty], \quad q \in \{0\} \cup [1, \infty),$$

$$(3.3)$$

(see, e.g., [2, Appendix 4.1].)

First we deal with Banach spaces of trivial type. For that it is crucial to know how $\ell_1$ is positioned inside the spaces.

**Theorem 3.1.** Let $X$ be a Banach space with a basis. If $\ell_1 \lesssim_c X$ then $X$ has an almost greedy basis $B$ with $\phi_m[B] \approx m$ for $m \in \mathbb{N}$, and $L_m[B] \approx \log m$ for $m \geq 2$.

**Theorem 3.2.** Let $X$ be a Banach space with a quasi-greedy basis. If $B_{1,0} \lesssim_c X$ then there is a quasi-greedy basis $B$ for $X$ with $L_m[B] \approx \log m$ for $m \geq 2$.

**Proof of Theorems 3.1 and 3.2.** Let $p \in \{0, 1\}$. For $p = 0$, assume that the hypotheses of Theorem 3.2 hold (respectively, assume that for $p = 1$ the hypotheses of Theorem 3.1 hold).

By Lemma 2.3 and Theorem 2.10 $\mathcal{B}_0 = \bigoplus_{n=1}^{\infty} \mathcal{L}[2^n]$ is a quasi-greedy basis for $Y := (\bigoplus_{n=1}^{\infty} \ell^q_{2^n}([\mathcal{L}]))_p$ such that $L_m[\mathcal{B}_0] \approx \log m$ for $m \geq 2$. Moreover, in the case when $p = 1$, $\mathcal{B}_0$ is democratic with $\phi_m[\mathcal{B}_0] \approx m$. 
In the case when \( p = 0 \) the hypothesis give a quasi-greedy basis \( B_1 \) for \( S := X \) (respectively, when \( p = 1 \) Theorem 2.8 gives a quasi-greedy basis \( B_1 \) for \( S := X \oplus \ell_1 \) that is democratic with \( \phi_m[B_1] \approx m \)).

By Lemma 2.2, \( B_2 := B_1 \oplus B_0 \) is a quasi-greedy basis for \( Z := S \oplus Y \) such that \( L_m[B_2] \approx \log m \). Moreover, in the case when \( p = 1 \), \( B_2 \) is democratic with \( \phi_m[B_2] \approx m \). Combining Theorem 2.10 with the isomorphisms (3.3), \( \ell_1 \oplus B_{1,1} \approx B_{1,1} \) and \( B_{1,p} \oplus B_{1,p} \approx B_{1,p} \) give

\[
Z \approx S \oplus \left( \bigoplus_{n=1}^{\infty} \ell_1^n \right)_p \approx S \oplus B_{1,p} \approx X \oplus B_{1,p} \approx X,
\]

and, so, the proof is over. \( \square \)

**Example 3.3.**

(i) The list of Banach spaces for which Theorem 3.1 applies includes \( D_{1,p}, Z_{0,1} \) and \( Z_{1,p} \) for \( p \in \{0\} \cup (1, \infty) \), \( B_{p,1} \) for \( p \in (1, \infty) \), \( \ell_1, L_1[0,1] \), the Hardy space \( H_1 \), and the Lorentz sequence spaces \( d_1(w) \) for \( w \) decreasing.

(ii) By invoking [24, Proposition 7.3] and [20, Theorem 5.1], Theorem 3.1 applies to any separable \( L_1 \)-space. Indeed, since \( L_1 \)-spaces are GT-spaces (see [24, Theorem 4.1]), in light of [15, Theorem 4.2], the conclusion on democracy is redundant for such spaces.

(iii) Theorem 3.2 moves the space \( B_{1,0} \) to the list of Banach spaces possessing a quasi-greedy basis as conditional as possible.

**Remark 3.4.** We would like to point out that the argument used in the above proof also works for \( p \in (1, \infty) \). However, we will use an alternative method for including the spaces \( B_{1,p} \) is our list.

The next five theorems contain our study of the case of Banach spaces with trivial cotype. We emphasize that the lack of a conditional quasi-greedy basis in \( c_0 \) (see [11, Corollary 8.6]) constitutes an added difficulty when tackling this task.

**Theorem 3.5.** Let \( X \) be a Banach space with a basis of type \( P \). If \( \ell_2 \preceq_c X \), then \( X \) has an almost greedy \( B \) with \( \phi_m[B] \approx m^{1/2} \) for \( m \in \mathbb{N} \) and \( L_m[B] \approx \log m \) for \( m \geq 2 \).

**Theorem 3.6.** Let \( X \) be a Banach space with a basis. Assume that \( \ell_2 \preceq_c X \) and either \( c_0 \preceq_c X \) or \( \ell_1 \preceq X \). Then \( X \) has an almost greedy basis \( B \) with \( \phi_m[B] \approx m^{1/2} \) for \( m \in \mathbb{N} \) and \( L_m[B] \approx \log m \) for \( m \geq 2 \).

**Theorem 3.7.** Let \( X \) be a Banach space with a basis. Suppose that either \( B_{\ell_2,2} \preceq_c X \) or \( B_{1,2} \preceq X \). Then \( X \) has an almost greedy \( B \) with \( \phi_m[B] \approx m^{1/2} \) for \( m \in \mathbb{N} \) and \( L_m[B] \approx \log m \) for \( m \geq 2 \).
**Theorem 3.8.** Let $X$ be a Banach space with a basis and $1 < p < \infty$. Assume that either $B_{\infty,p} \lesssim X$ or $B_{1,p} \lesssim X$. Then $X$ has a quasi-greedy basis $B$ such that $L_m[B] \approx \log m$ for $m \geq 2$.

**Theorem 3.9.** Let $X$ be a Banach space with a quasi-greedy basis. If $B_{q,0} \lesssim X$ for some $1 < q < \infty$, then $X$ has a quasi-greedy basis $B$ with $L_m[B] \approx \log m$ for $m \geq 2$.

**Proof of Theorems 3.5, 3.6, 3.7, 3.8 and 3.9.** Let us consider Banach spaces $Y$ and $Z$ such that

- (A) either $Y = \{0\}$ and $Z$ has basis $B$ of type $P$,
- (B) or $Y$ is a Banach space with a basis and $Z$ is either $c_0$ or $\ell_1$.

By Invoking Lemmas 2.11, 2.12, 2.13 and 2.2 we claim that the space $Y \oplus Z$ has a basis $B_0$ with $L_m[B_0] \approx m$ for $m \in \mathbb{N}$. From Theorem 2.5 we deduce that $V := Y \oplus Z \oplus \ell_2$ has an almost greedy basis $B_1 = O(B_0)$ with $\phi_m[B_1] \approx m^{1/2}$ for $m \in \mathbb{N}$ and $L_m[B_1] \approx \log m$ for $m \geq 2$.

In order to prove Theorem 3.5 we consider the case (A) with $Z = X$. Notice that in this case, since $\ell_2 \oplus \ell_2 = \ell_2$, we have that $V = X \oplus \ell_2 \approx X$.

In order to prove Theorem 3.6 we consider the case (B) with $Y = X$. Now, since $Z \oplus Z \approx Z$ also holds, we have that $V = X \oplus Z \oplus \ell_2 \approx X$.

Assume that we are in case (A) and that $Z$ is either $c_0$ (in which case we put $r = \infty$) or $\ell_1$ (in which case we put $r = 1$). In this situation we choose $B_0$ to be the summing system when $r = \infty$ and the difference system when $r = 1$. Therefore

$$V^{2n-2}[B_1] = \ell_2^{n-2} \oplus \mathbb{Z}^n[B_0] = \ell_2^{n-2} \oplus \ell_r^n, \quad n \in \mathbb{N}.$$

Let $p \in \{0\} \cup (1, \infty)$ and assume that when $p = 2$ the hypotheses of Theorem 3.7 hold, that when $p = 0$ the hypotheses of Theorem 3.9 hold, and that when $p \in (1,2) \cup (2, \infty)$ the hypotheses of Theorem 3.8 hold. Applying Lemma 2.3 we get a quasi-greedy basis $B_2 := \bigoplus_{n=1}^{\infty} B_1[2^n - 2]$ for the Banach space

$$W = \left( \bigoplus_{n=1}^{\infty} (\ell_2^{n-2} \oplus \ell_r^n) \right)$$

with $L_m[B_2] \approx \log m$. Moreover, in the case when $p = 2$, $B_2$ is democratic with $\phi_m[B_2] \approx m^{1/2}$.

In the case when $p = 0$, we have that $S := X$ has a quasi-greedy basis $B_3$. In the case when $p \in (1,2) \cup (2, \infty)$ Theorem 2.9 yields the existence of a quasi-greedy basis $B_3$ for $S := X \oplus \ell_p$. In the case when $p = 2$ Theorem 2.5 gives that $S := X \oplus \ell_2$ has a quasi-greedy basis $B_3$ which is democratic with $\phi_m[B_2] \approx m^{1/2}$. Then, by Lemma 2.2, $B_4 := B_3 \oplus B_2$ is a quasi-greedy basis for the Banach space $U := S \oplus W$. 


with $L_m[\mathcal{B}_1] \approx \log m$. Moreover, in the case $p = 2$, $\mathcal{B}_1$ is democratic with $\phi_m[\mathcal{B}_1] \approx m^{1/2}$.

With the basis $\mathcal{B}_1$ and the isomorphisms (3.1) and (3.3) in hand, we are ready to complete the proof. Let $p \in (1, \infty)$. Taking into account that $B_{r,p} \oplus \ell_p \approx B_{r,p}$ and that $B_{r,p} \oplus B_{r,p} \approx B_{r,p}$, we obtain

$$U \approx X \oplus \ell_p \oplus \left( \bigoplus_{n=1}^{\infty} \ell_2^{2^n - n - 2} \right)_{p} \oplus \left( \bigoplus_{n=1}^{\infty} \ell_{n/r}^{n} \right)_{p}$$

$$\approx X \oplus \ell_p \oplus B_{2,p} \oplus \left( \bigoplus_{n=1}^{\infty} \ell_{n/r}^{n} \right)_{p}$$

$$= X \oplus \ell_p \oplus \ell_p \oplus B_{r,p} \approx X.$$  

This completes the proof of Theorems 3.7 and 3.8.

Let $p = 0$. Notice that the relations (3.2) and (3.3) give the isomorphisms $c_0 \oplus B_{2,0} \approx B_{2,0} \oplus B_{2,0} \approx B_{2,0}$ as well as the complemented embedding $B_{2,0} \lesssim_c B_{4,0}$. Consequently, $B_{2,0} \lesssim_c X$. The chain of isomorphisms

$$U \approx X \oplus \left( \bigoplus_{n=1}^{\infty} \ell_2^{2^n - n - 2} \right)_{0} \oplus \left( \bigoplus_{n=1}^{\infty} \ell_{n/r}^{n} \right)_{0} \approx X \oplus B_{2,0} \oplus c_0 = X,$$

completes the proof of Theorem 3.9. □

**Example 3.10.**

(i) The unit-vector system is a shrinking basis of type P for the James space $\mathcal{J}$ introduced in [18] (see, e.g., [6, Proposition 3.4.4 and Remark 3.4.5]). It is also known that $\ell_2 \lesssim_c \mathcal{J}$; indeed, the linear operator $L$ defined in (2.1) is bounded from $\ell_2$ into $\mathcal{J}$ and the linear operator $R$ defined in (2.2) is bounded from $\mathcal{J}$ into $\ell_2$ (see also [10, Corollary 11]). Hence Theorem 3.5 applies to $\mathcal{J}$ and to $\mathcal{J}^*$. By Proposition 2.14 and [5, Proposition 2.10] (which states that the unit-vector system is a basis of type P for Pisier-Xu spaces), Theorem 3.5 also applies to $\mathcal{W}^0_{p,2}$ for $1 < p < \infty$.

(ii) Theorem 3.6 applies to the spaces $D_{2,0}$, $D_{2,1}$, $Z_{1,2}$, $Z_{2,1}$, $Z_{2,0}$, $Z_{0,2}$, the Hardy space $H_1$ and its predual VMO.

(iii) Theorem 3.7 applies to the spaces $B_{\infty,2}$ and $B_{1,2}$.

(iv) Theorem 3.8 applies to the spaces $B_{\infty,p}$, $B_{1,p}$ and $Z_{0,p}$ for $1 < p < \infty$.

(v) Theorem 3.9 applies to $B_{p,0}$ and $Z_{p,0}$ for $1 < p < \infty$.

Let us mention that, for $p \neq 2$, $D_{p,0}$ is not in our list of Banach spaces with a quasi-greedy basis as conditional as possible yet.
**Theorem 3.11.** Let $X$ be a Banach space and $1 < p < \infty$. Assume that $\ell_p \leq_c X$, that $c_0 \leq_c X$ and that $X$ has a basis. Then $X$ has a quasi-greedy basis $B$ with $\|B\| \approx \log m$ for $m \geq 2$.

**Proof.** Since $\ell_p \oplus \ell_p \approx \ell_p$ and $c_0 \oplus c_0 \approx c_0$, taking into account Theorem 2.9 and Lemma 2.2, it suffices to prove the result for $X = D_{p,0}$. Let $S$ the summing basis if $c_0$ and let $P_n$ and $Q_n$ be the canonical projections from $\ell^m_\infty \oplus \ell^{2n-2}_2$ onto, respectively, $\ell^m_\infty$ and $\ell^{2n-2}_2$. From Lemma 2.11, Theorem 2.5, and Lemma 2.4 we infer that $B_0 = \bigoplus_{n=1}^\infty O(S)[P_n,Q_n]$ is a quasi-greedy basis for

$$
\left( \bigoplus_{n=1}^\infty \ell^n_\infty \right)_0 \oplus \left( \bigoplus_{n=1}^\infty \ell^{2n-2}_2 \right)_p \approx c_0 \oplus B_{2,p} \approx c_0 \oplus \ell_2 \approx D_{p,0}
$$

with $\|B_0\| \approx \log m$ for $m \geq 2$. \hfill \Box

**Example 3.12.** Theorem 3.11 applies to $D_{p,0}$ for $1 < p < \infty$.

To close this article, we revisit the advances in the superreflexive case carried out in [17].

**Theorem 3.13** (cf. [17, Theorem 1.2 and Corollary 3.13]). Let $X$ be a Banach space with a basis. If $\ell_2 \leq_c X$ then for any $0 < a < 1$ the space $X$ has an almost greedy basis $B$ with $\phi_m[B] \approx m^{1/2}$ and $\|B\| \gtrsim (\log m)^a$ for $m \in \mathbb{N}$.

**Proof.** Combining Theorem 2.5 with Theorem 2.7 we obtain an almost greedy basis $B_1$ for $\ell_2 \oplus \ell_2$ with $\phi_m[B_1] \approx m^{1/2}$ and $\|B_1\| \gtrsim (\log m)^a$. Applying again Theorem 2.5 we get an almost greedy basis $B_2$ for $X \oplus \ell_2$ with $\phi_m[B_2] \approx m^{1/2}$. Hence, by Lemma 2.2, $B_2 \oplus B_1$ is a basis as desired for $X \oplus \ell_2 \oplus \ell_2 \oplus \ell_2 \approx X$. \hfill \Box

**Example 3.14.**

(i) Theorem 3.13 applies to the spaces $Z_{p,2}$, $Z_{p,2}$, $D_{p,2}$, $D_{p,2}$, $L_p$ for $p \in (1, \infty)$, and the Lorentz sequence spaces $\ell_{p,2}$ for $1 < p < \infty$.

(ii) More generally (see [24, Proposition 7.3], [20, Theorem 5.1], [19, Corollary 1] and [22, Corollary 1]), Theorem 3.13 applies to any separable $L_p$-space that is non-isomorphic to $\ell_p$, for $p \in (1, \infty)$.

**Theorem 3.15** (cf. [17, Corollary 3.12]). Let $X$ be a Banach space with a basis. If $\ell_p \leq_c X$ for some $1 < p < \infty$, then for any $0 < a < 1$ the space $X$ has a quasi-greedy basis $B$ with $\|B\| \gtrsim (\log m)^a$ for $m \in \mathbb{N}$.

**Proof.** Taking into account Lemma 2.2, Theorem 2.9, and the fact that $\ell_p \oplus \ell_p \approx \ell_p$, it suffices to consider the case $X = \ell_p$. Use Theorem 3.13
to pick a quasi-greedy basis $B$ for $\ell_2$ with $L_m[B] \gtrsim (\log m)^a$ for all $m \in \mathbb{N}$. By Lemma 2.3, $B_0 = \bigoplus_{n=1}^{\infty} B[2^n]$ is a quasi-greedy basis for $Y = \left( \bigoplus \ell_2^n[B] \right)_p$ with $L_m[B_0] \gtrsim (\log m)^a$. Since any $d$-dimensional Hilbert space is isometric to $\ell_d^2$, the isomorphisms (3.1) and (3.3) yield $Y \approx \left( \bigoplus \ell_2^n[B] \right)_p \approx B_{2,p} \approx \ell_p$. □

Example 3.16. Theorem 3.15 applies to the spaces $\ell_p$, $D_{p,q}$, $B_{p,q}$ and $Z_{p,q}$ for $p, q \in (1, \infty) \setminus \{2\}$, and the Lorentz sequence spaces $\ell_{p,q}$ for $1 < p, q < \infty, q \neq 2$.

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BANACH SPACES WITH HIGHLY CONDITIONAL QUASI-GREEDY BASES

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