SO(4) Symmetry of the Transfer Matrix for the One-Dimensional Hubbard Model

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Abstract

The SO(4) invariance of the transfer matrix for the one-dimensional Hubbard model is clarified from the QISM (quantum inverse scattering method) point of view. We demonstrate the SO(4) symmetry by means of the fermionic \(L\)-operator and the fermionic \(R\)-matrix, which satisfy the graded Yang-Baxter relation. The transformation law of the fermionic \(L\)-operator under the SO(4) rotation is identified with a kind of gauge transformation, which determines the corresponding transformation of the fermionic creation and annihilation operators under the SO(4) rotation. The transfer matrix is confirmed to be invariant under the SO(4) rotation, which ensures the SO(4) invariance of the conserved currents including the Hamiltonian. Furthermore, we show that the representation of the higher conserved currents in terms of the Clifford algebra gives manifestly SO(4) invariant forms.

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§1. Introduction

The one-dimensional (1D) Hubbard model is one of the most important solvable models in condensed matter physics. The ground state energy of the 1D Hubbard model

\[ H = - \sum_{m=1}^{N} \sum_{s=\uparrow, \downarrow} \left( c_{ms}^\dagger c_{m+1s} + c_{m+1s}^\dagger c_{ms} \right) + U \sum_{m=1}^{N} (n_{m\uparrow} - \frac{1}{2})(n_{m\downarrow} - \frac{1}{2}), \]  

with the periodic boundary condition,

\[ c_{N+1s}^\dagger = c_{1s}^\dagger, \quad c_{N+1s} = c_{1s} \quad (s = \uparrow, \downarrow), \]

was given by Lieb and Wu \cite{1} by means of the coordinate Bethe ansatz method. Here \( c_{ms}^\dagger \) and \( c_{ms} \) are the fermionic creation and annihilation operators with spin \( s(=\uparrow, \downarrow) \) at site \( m(=1, 2, \ldots, N) \) satisfying the canonical anticommutation relations

\[ \{ c_{ms}^\dagger, c_{m's'} \} = \delta_{mm'} \delta_{ss'}, \quad \{ c_{ms}^\dagger, c_{m's'}^\dagger \} = \{ c_{ms}, c_{m's'} \} = 0, \]

and \( n_{ms} \) is the number density operator

\[ n_{ms} = c_{ms}^\dagger c_{ms} \quad (s = \uparrow, \downarrow). \]

The parameter \( U \) is the coupling constant describing the Coulomb interaction.

The Hamiltonian \((1.1)\) enjoys two \( su(2) \) symmetries \cite{2–4}. One is the spin-\( su(2) \) generated by

\[ S^+ = \sum_{m=1}^{N} c_{m\uparrow}^\dagger c_{m\downarrow}, \quad S^- = \sum_{m=1}^{N} c_{m\downarrow}^\dagger c_{m\uparrow}, \quad S^z = \frac{1}{2} \sum_{m=1}^{N} (n_{m\uparrow} - n_{m\downarrow}), \]

and the other is charge-\( su(2) \) \((\eta\text{-pairing } su(2)) \) generated by

\[ \eta^+ = \sum_{m=1}^{N} (-1)^m c_{m\uparrow}^\dagger c_{m\downarrow}^\dagger, \quad \eta^- = \sum_{m=1}^{N} (-1)^m c_{m\downarrow} c_{m\uparrow}, \quad \eta^z = \frac{1}{2} \sum_{m=1}^{N} (n_{m\uparrow} + n_{m\downarrow} - 1). \]

When we assume the periodic boundary condition \((1.2)\), the number of sites \( N \) should be even for the consistency of the charge-\( su(2) \). In this case, the spin-\( su(2) \) and the charge-\( su(2) \) are connected through the partial particle-hole transformation

\[ c_{m\uparrow} \rightarrow c_{m\uparrow}, \quad c_{m\downarrow} \rightarrow (-1)^m c_{m\downarrow}^\dagger, \quad U \rightarrow -U. \]

As is well known, these two \( su(2) \) are not independent and should be considered as elements of a bigger algebra \( so(4) \) \cite{4},

\[ so(4) = su(2) \oplus su(2). \]

The \( so(4) \) symmetry may be the most fundamental property of the 1D Hubbard model that characterizes the various physical features of the model \cite{4}. For example, it was proved by Ef\l{}er et al. \cite{8–10} that the Bethe ansatz states of the 1D Hubbard model are incomplete and have to be complemented by the \( so(4) \) symmetry. Ef\l{}er and Korepin \cite{11, 12} showed that the elementary excitations of the half-filled band constitute the multiplets of \( so(4) \).
Several authors have discussed the generalization of the Lie algebra symmetry \( so(4) \) to the group symmetry \( SO(4) \). Following Affleck et al. \[13\], we introduce the \( 2 \times 2 \) matrices

\[
\Psi_{2n-1} = \begin{pmatrix}
\beta_{2n-1\downarrow}^\dagger & i \alpha_{2n-1\uparrow}^- \\
i \alpha_{2n-1\uparrow}^+ & \beta_{2n-1\downarrow}^-
\end{pmatrix}, \quad \Psi_{2n} = \begin{pmatrix}
\beta_{2n\downarrow}^\dagger & -i \alpha_{2n\uparrow}^- \\
i \alpha_{2n\uparrow}^+ & \beta_{2n\downarrow}^-
\end{pmatrix}, \quad n = 1, \cdots, \frac{N}{2}.
\]

(1.9)

For convenience, the definition of \( \Psi_m \) in this paper is chosen to be different from the usual one \[6, 13\]. However, they are essentially equivalent.

The spin-\( SU(2) \) transformation can be realized by the left multiplication of an \( SU(2) \) matrix

\[\Psi_m \mapsto O_{\text{spin}} \Psi_m, \quad O_{\text{spin}} \in SU(2),\]

while the charge-\( SU(2) \) transformation corresponds to the right multiplication of another \( SU(2) \) matrix,

\[\Psi_m \mapsto \Psi_m O_{\text{charge}}, \quad O_{\text{charge}} \in SU(2).\]

Since the left and the right matrix multiplications are commutative, the transformation

\[\tilde{\Psi}_m = O_{\text{spin}} \Psi_m O_{\text{charge}}\]

(1.10)

gives the \( SU(2) \times SU(2) \) transformation among the fermion operators. To put it more precisely, the exact group symmetry is

\[SO(4) = [SU(2) \times SU(2)] / \mathbb{Z}_2,\]

because the choices \( O_{\text{spin}} = -1, O_{\text{charge}} = 1 \) and \( O_{\text{spin}} = 1, O_{\text{charge}} = -1 \) induce the same transformation. The infinitesimal transformation of (1.10) gives the Lie algebra symmetry (1.8).

The integrability of the 1D Hubbard model with periodic boundary condition was established by Shastry \[14–16\] and Olmedilla et al. \[17, 18\]. Shastry introduced a Jordan-Wigner transformation

\[c_m^\uparrow = (\sigma_1^z \cdots \sigma_{m-1}^z) \sigma_m^- , \quad c_m^\downarrow = (\sigma_1^z \cdots \sigma_N^z)(\tau_1^z \cdots \tau_{m-1}^z) \tau_m^- ,\]

(1.11)

which changes the fermionic Hamiltonian (1.1) into an equivalent coupled spin model

\[H = \sum_{m=1}^{N} (\sigma_{m+1}^+ \sigma_m^- + \sigma_m^+ \sigma_{m+1}^-) + \sum_{m=1}^{N} (\tau_{m+1}^+ \tau_m^- + \tau_m^+ \tau_{m+1}^-) + \frac{U}{4} \sum_{m=1}^{N} \sigma_m^z \tau_m^z ,\]

(1.12)

where \( \sigma \) and \( \tau \) are two species of the Pauli matrices commuting each other, and \( \sigma^\pm \) and \( \tau^\pm \) are defined by

\[\sigma_m^\pm = \frac{1}{2} (\sigma_m^x \pm i \sigma_m^y), \quad \tau_m^\pm = \frac{1}{2} (\tau_m^x \pm i \tau_m^y).\]

For this equivalent coupled spin model, Shastry constructed the \( L \)-operator and the \( R \)-matrix (see Appendix), which satisfy the Yang-Baxter relation

\[\hat{R}_{12}(\theta_1, \theta_2) [L_m(\theta_1) \otimes L_m(\theta_2)] = [L_m(\theta_2) \otimes L_m(\theta_1)] \hat{R}_{12}(\theta_1, \theta_2).\]

(1.13)

The Yang-Baxter equation for Shastry’s \( R \)-matrix was recently proved in \[19–21\].
The coupled spin model (1.12) is also referred to as the 1D Hubbard model, since they are related through the Jordan-Wigner transformation (1.1). However, there are differences between the coupled spin model (1.12) and the fermionic Hamiltonian (1.1). It is well known that the periodic boundary condition for the fermion model does not correspond to the periodic boundary condition for the equivalent spin model. Moreover, due to the non-locality of the Jordan-Wigner transformation (1.1), the generators of the \( so(4) \) symmetry, (1.7) and (1.8), become the non-local in terms of the spin operators \( \sigma \) and \( \tau \). Thus it is more transparent to employ the fermionic formulation of the Yang-Baxter relation developed by Olmedilla et al. [17], when we investigate the \( SO(4) \) or other symmetries of the 1D Hubbard model from the QISM (quantum inverse scattering method) point of view. Recently, Göhmann and Murakami [22] demonstrated that the transfer matrix constructed from the fermionic \( L \)-operators has the \( su(2) \oplus su(2) \) symmetry. The main purpose of this paper is to generalize their result to the finite symmetry, namely the \( SO(4) \) symmetry corresponding to (1.1).

The plan of this paper is as follows. In section 2, we give a brief summary of the fermionic formulation of the 1D Hubbard model. Some important properties of the fermionic R-matrix are explained. In section 3, we prove the \( SO(4) \) invariance of the fermionic transfer matrix. It is shown that the \( SO(4) \) rotation for the fermion operators is related to a kind of gauge transformation of the fermionic \( L \)-operator. When the number of sites is even, we can establish the \( SO(4) \) symmetry of the transfer matrix under the periodic boundary condition. When the number of sites is odd, we have to impose a twisted periodic boundary condition to establish the \( SO(4) \) symmetry of the transfer matrix. In section 4, we discuss the invariance of the transfer matrix under the partial particle-hole transformation. In section 5, we give a new representation of some higher conserved currents using the Clifford algebra. The \( SO(4) \) invariance of the conserved currents becomes obvious in this representation. The last section is devoted to discussions.

\[ \text{§2. Graded Yang-Baxter Relation for the 1D Hubbard Model} \]

As a preparation for later sections, we shall summarize the fermionic formulation of the 1D Hubbard model [17, 18, 23]. The fermionic \( L \)-operator is

\[
\mathcal{L}_m(\theta) = \begin{pmatrix}
-e^{i\theta} f_{m\uparrow}(\theta) f_{m\downarrow}(\theta) & -f_{m\uparrow}(\theta) c_{m\downarrow} & ic_{m\uparrow} f_{m\downarrow}(\theta) & ie^{i\theta} c_{m\uparrow} c_{m\downarrow} \\
-i f_{m\uparrow}(\theta) c_{m\downarrow}^{\dagger} & e^{-i\theta} f_{m\downarrow}(\theta) g_{m\downarrow}(\theta) & e^{-i\theta} c_{m\downarrow} c_{m\downarrow}^{\dagger} & ic_{m\uparrow} g_{m\downarrow}(\theta) \\
c_{m\uparrow}^{\dagger} f_{m\downarrow}(\theta) & e^{i\theta} c_{m\uparrow} c_{m\downarrow} & e^{-i\theta} g_{m\uparrow}(\theta) f_{m\downarrow}(\theta) & g_{m\uparrow}(\theta) c_{m\downarrow}^{\dagger} \\
-ie^{i\theta} c_{m\uparrow}^{\dagger} c_{m\downarrow}^{\dagger} & c_{m\uparrow}^{\dagger} g_{m\downarrow}(\theta) & ig_{m\uparrow}(\theta) c_{m\downarrow}^{\dagger} & -e^{i\theta} g_{m\uparrow}(\theta) g_{m\downarrow}(\theta)
\end{pmatrix},
\]

where

\[
f_{ms}(\theta) = \sin \theta - \{ \sin \theta - i \cos \theta \} n_{ms}, \quad g_{ms}(\theta) = \cos \theta - \{ \cos \theta + i \sin \theta \} n_{ms}.
\]

The parameter \( h \) is related to the spectral parameter \( \theta \) and the Coulomb coupling constant \( U \) through the relation

\[
\frac{\sinh 2h}{\sin 2\theta} = \frac{U}{4}.
\]
We express by $\otimes_s$ the Grassmann (graded) direct product

$$\begin{align*}
[A \otimes B]_{\alpha \gamma, \beta \delta} &= (-1)^{|P(\alpha)|+|P(\beta)|} A_{\alpha \beta} B_{\gamma \delta}, \\
P(1) &= P(4) = 0, \quad P(2) = P(3) = 1.
\end{align*}$$

(2.3)

The fermionic $L$-operator satisfies the graded Yang-Baxter relation \cite{[17]}

$$\hat{R}_{12}(\theta_1, \theta_2)[\mathcal{L}_m(\theta_1) \otimes \mathcal{L}_m(\theta_2)] = [\mathcal{L}_m(\theta_2) \otimes \mathcal{L}_m(\theta_1)]\hat{R}_{12}(\theta_1, \theta_2),$$

(2.4)

under the constraint of the spectral parameter

$$\frac{\sinh 2h_1}{\sin 2\theta_1} = \frac{\sinh 2h_2}{\sin 2\theta_2} = \frac{U}{4}. \quad (2.5)$$

The explicit form of $\hat{R}_{12}(\theta_1, \theta_2)$ is \cite{[17]}

$$\hat{R}_{12}(\theta_1, \theta_2) =
\begin{pmatrix}
  a^+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & e & 0 & 0 & ib^- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & d^+ & 0 & 0 & -if & 0 & 0 & if & 0 & 0 & -e^+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & -ib^+ & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & a^- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & if & 0 & 0 & d^- & 0 & 0 & -c^- & 0 & 0 & -if & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & -ib^+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & -ib^+ & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & -if & 0 & 0 & c^- & 0 & 0 & d^- & 0 & 0 & if & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^- & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & -c^+ & 0 & 0 & if & 0 & 0 & -if & 0 & 0 & d^+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ib^- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ib^- & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^+ & 0
\end{pmatrix}
\quad (2.6)$$

where

$$\begin{align*}
a^\pm &= \cos^2(\theta_1 - \theta_2) \left\{ 1 \pm \tanh(h_1 - h_2) \frac{\cos(\theta_1 + \theta_2)}{\cos(\theta_1 - \theta_2)} \right\}, \\
b^\pm &= \sin(\theta_1 - \theta_2) \cos(\theta_1 - \theta_2) \left\{ 1 \pm \tanh(h_1 - h_2) \frac{\sin(\theta_1 + \theta_2)}{\sin(\theta_1 - \theta_2)} \right\}, \\
c^\pm &= \sin^2(\theta_1 - \theta_2) \left\{ 1 \pm \tanh(h_1 + h_2) \frac{\sin(\theta_1 + \theta_2)}{\sin(\theta_1 - \theta_2)} \right\}, \\
d^\pm &= 1 \pm \tanh(h_1 - h_2) \frac{\cos(\theta_1 - \theta_2)}{\cos(\theta_1 + \theta_2)}, \\
f &= \frac{\cos(\theta_1 - \theta_2)}{\cosh(h_1 - h_2)}, \quad e = \frac{\sin(\theta_1 - \theta_2)}{\cosh(h_1 + h_2)}.
\end{align*}$$

(2.7)
The second equalities for the Boltzmann weights \(b^\pm\) and \(d^\pm\) are valid due to the constraints \[2.5\]. There are useful relations among the Boltzmann weights \[17\]

\[
d^\pm = a^\pm + c^\pm, \quad e^2 = a^+a^- + b^+b^-, \quad f^2 = b^+b^- + c^+c^-,
\]

\[
a^+c^+ + a^-c^- = (b^+)^2 + (b^-)^2, \quad a^+c^- + a^-c^+ = 2b^+b^-,
\]

\[
a^+ + a^- + c^+ + c^- = 2. \quad (2.8)
\]

For convenience, we introduce an equivalent fermionic \(R\)-matrix

\[
\mathcal{R}_{12}(\theta_1, \theta_2) \equiv \mathcal{P}_{12} \tilde{\mathcal{R}}_{12}(\theta_1, \theta_2),
\]

where \(\mathcal{P}_{12}\) is the graded permutation

\[
\mathcal{P}_{\alpha\gamma;\beta\delta} = (-1)^{P(\alpha)P(\gamma)}\delta_{\alpha\delta}\delta_{\gamma\beta}. \quad (2.10)
\]

In terms of \(\mathcal{R}_{12}(\theta_1, \theta_2) \quad (2.9)\), the graded Yang-Baxter relation \(2.4\) is cast into

\[
\mathcal{R}_{12}(\theta_1, \theta_2) \left( \mathcal{L}_m(\theta_1) \otimes I \right) \left( I \otimes \mathcal{L}_m(\theta_2) \right) = \left( I \otimes \mathcal{L}_m(\theta_2) \right) \left( \mathcal{L}_m(\theta_1) \otimes I \right) \mathcal{R}_{12}(\theta_1, \theta_2).
\]

Here \(I\) is the \(4 \times 4\) identity matrix.

The fundamental properties of the fermionic \(R\)-matrix \(\mathcal{R}_{12}(\theta_1, \theta_2)\) are summarized as follows \[24\].

1. Regularity (Initial condition):

\[
\mathcal{R}_{12}(\theta_0, \theta_0) = \mathcal{P}_{12}. \quad (2.11)
\]

2. Graded Yang-Baxter equation:

\[
\mathcal{R}_{12}(\theta_1, \theta_2)\mathcal{R}_{13}(\theta_1, \theta_3)\mathcal{R}_{23}(\theta_2, \theta_3) = \mathcal{R}_{23}(\theta_2, \theta_3)\mathcal{R}_{13}(\theta_1, \theta_3)\mathcal{R}_{12}(\theta_1, \theta_2), \quad (2.12)
\]

under the constraints

\[
\frac{\sinh 2h_1}{\sin 2\theta_1} = \frac{\sinh 2h_2}{\sin 2\theta_2} = \frac{\sinh 2h_3}{\sin 2\theta_3} = \frac{U}{4},
\]

3. Unitarity:

\[
\mathcal{R}_{12}(\theta_1, \theta_2)\mathcal{R}_{21}(\theta_2, \theta_1) = \rho(\theta_1, \theta_2) I, \quad (2.13)
\]

where

\[
\mathcal{R}_{21}(\theta_2, \theta_1) \equiv \mathcal{P}_{12} \mathcal{R}_{12}(\theta_2, \theta_1) \mathcal{P}_{12},
\]

and

\[
\rho(\theta_1, \theta_2) = \cos^2(\theta_1 - \theta_2) \left( \cos^2(\theta_1 - \theta_2) - \tanh^2(h_1 - h_2) \cos^2(\theta_1 + \theta_2) \right).
\]

Since the non-zero elements of the \(R\)-matrix \[2.9\] are even with respect to the parity \(P(\alpha) \quad (2.3)\), i.e.,

\[
P(\alpha) + P(\beta) + P(\alpha') + P(\beta') = 0 \quad (\text{mod} \ 2) \quad \text{for} \quad \mathcal{R}_{\alpha\beta:\alpha'\beta'}(\theta_1, \theta_2) \neq 0,
\]

the graded Yang-Baxter equation \[2.12\] can be expressed in terms of the matrix elements as \[25\]

\[
\mathcal{R}_{\alpha\beta:\alpha''\beta''}(\theta_1, \theta_2)\mathcal{R}_{\alpha''\gamma:\alpha':\gamma'}(\theta_1, \theta_3)\mathcal{R}_{\beta''\gamma':\beta':\gamma'}(\theta_2, \theta_3)(-)^{P(\beta'')(P(\alpha') + P(\alpha''))} = \mathcal{R}_{\beta\gamma:\beta':\gamma'}(\theta_2, \theta_3)\mathcal{R}_{\alpha'\gamma':\alpha''\gamma''}(\theta_1, \theta_3)\mathcal{R}_{\alpha'\beta':\alpha''\beta''}(\theta_1, \theta_2)(-)^{P(\beta')(P(\alpha) + P(\alpha''))}. \quad (2.14)
\]
Here the summations are taken over the repeated indices.

In our previous work [24], we found two important relations of the fermionic $R$-matrix with constant matrices $M$ and $V$. The first relation is the symmetry of the fermionic $R$-matrix

$$\left[\check{R}_{12}(\theta_1, \theta_2), M \otimes s M\right] = 0,$$

where the general form of the matrix $M$ is given by

$$M = \begin{pmatrix} M_{11} & 0 & 0 & M_{14} \\ 0 & M_{22} & M_{23} & 0 \\ 0 & M_{32} & M_{33} & 0 \\ M_{41} & 0 & 0 & M_{44} \end{pmatrix},$$

with the condition

$$\Delta M \equiv M_{11}M_{44} - M_{41}M_{14} = M_{22}M_{33} - M_{23}M_{32}. $$

We call the matrix $M$ symmetry matrix. For simplicity, we assume $\Delta M = 1$ throughout the paper.

The second relation is

$$\check{R}_{12}(\theta_1, \theta_2; U) \left[ V \otimes s V \right] = \left[ V \otimes s V \right] \check{R}_{12}(\theta_1, \theta_2; -U),$$

where the general form of the matrix $V$ is given by

$$V = \begin{pmatrix} 0 & V_{12} & V_{13} & 0 \\ V_{21} & 0 & 0 & V_{24} \\ V_{31} & 0 & 0 & V_{34} \\ 0 & V_{42} & V_{43} & 0 \end{pmatrix},$$

$V_{12}V_{43} - V_{13}V_{42} = V_{21}V_{34} - V_{31}V_{24}.$

In the relation (2.17), the $U$-dependence of the fermionic $R$-matrix is explicitly written. The coupling constant of the fermionic $R$-matrix in the RHS is $-U$, or equivalently, $h_1 \rightarrow -h_1$ and $h_2 \rightarrow -h_2$. The matrix $V$ is related to the partial particle-hole transformation (1.7). The constant matrices $M$ and $V$ play an important role in the consideration of the symmetry of the transfer matrix for the 1D Hubbard model (see §3 and §4). We remark that the symmetry matrix of Shastry’s $R$-matrix is not of the form (2.16) (see Appendix). This gives one of the reasons why the fermionic formulation employed in this paper is more appropriate for the investigation of the 1D Hubbard model (1.1).

The monodromy matrix is defined as the ordered product of the fermionic $L$-operators

$$T(\theta) = \prod_{m=1}^{N} L_m(\theta) = \mathcal{L}_N(\theta) \cdots \mathcal{L}_1(\theta).$$

From the (local) graded Yang-Baxter relation (2.4), we have the global relation for the monodromy matrix

$$\check{R}_{12}(\theta_1, \theta_2)[T(\theta_1) \otimes s T(\theta_2)] = [T(\theta_2) \otimes s T(\theta_1)]\check{R}_{12}(\theta_1, \theta_2),$$

(2.20)
or equivalently
\[ R_{12}(\theta_1, \theta_2) \hat{T}(\theta_1) \hat{T}(\theta_2) = \hat{T}(\theta_2) \hat{T}(\theta_1) R_{12}(\theta_1, \theta_2), \]  
\tag{2.21}

where
\[ \hat{T}(\theta) \equiv T(\theta_1) \otimes I, \quad \hat{T}(\theta) \equiv I \otimes T(\theta_2). \]

Define the (fermionic) transfer matrix by
\[ \text{str} K T(\theta) \equiv \text{tr} \{ (\sigma^z \otimes \sigma^z) K T(\theta) \}, \]
\tag{2.22}

where the constant matrix \( K \) assumes the form (2.16) and determines the boundary condition (2.24). In particular, \( K = I \) corresponds to the periodic boundary condition (1.2). Then from the global graded Yang-Baxter relation (2.20), we can prove that the transfer matrix (2.22) constitutes a commuting family
\[ [\text{str} K T(\theta_1), \text{str} K T(\theta_2)] = 0, \]
\tag{2.23}

which proves the integrability of the 1D Hubbard model with the (twisted) periodic boundary condition.

\section*{§3. SO(4) Symmetry of the Fermionic Transfer Matrix}

We shall discuss the \( SO(4) \) symmetry of the fermionic transfer matrix (2.22). Let us consider the following transformation of the fermionic \( L \)-operator
\[ \tilde{L}_m(\theta) = \tilde{M}^{-1} L_m(\theta) M, \]
\tag{3.1}

where the constant matrices \( M \) and \( \tilde{M} \) have the form of the symmetry matrix (2.16), i.e.,
\[
M = \begin{pmatrix}
M_{11} & 0 & 0 & M_{14} \\
0 & M_{22} & M_{23} & 0 \\
0 & M_{32} & M_{33} & 0 \\
M_{41} & 0 & 0 & M_{44}
\end{pmatrix},
\]
\[ M_{11}M_{44} - M_{41}M_{14} = M_{22}M_{33} - M_{23}M_{32} = 1, \]
\tag{3.2}

and
\[
\tilde{M} = \begin{pmatrix}
\tilde{M}_{11} & 0 & 0 & \tilde{M}_{14} \\
0 & \tilde{M}_{22} & \tilde{M}_{23} & 0 \\
0 & \tilde{M}_{32} & \tilde{M}_{33} & 0 \\
\tilde{M}_{41} & 0 & 0 & \tilde{M}_{44}
\end{pmatrix},
\]
\[ \tilde{M}_{11}\tilde{M}_{44} - \tilde{M}_{41}\tilde{M}_{14} = \tilde{M}_{22}\tilde{M}_{33} - \tilde{M}_{23}\tilde{M}_{32} = 1. \]
\tag{3.3}

Since the matrices \( M \) and \( \tilde{M} \) are the symmetry matrices, the transformed \( L \)-operator \( \tilde{L}_m(\theta) \) (3.1) also satisfies the graded Yang-Baxter relation with the \textit{same} fermionic \( R \)-matrix,
\[ \mathcal{R}_{12}(\theta_1, \theta_2) [\tilde{L}_m(\theta_1) \otimes \tilde{L}_m(\theta_2)] = [\tilde{L}_m(\theta_2) \otimes \tilde{L}_m(\theta_1)] \mathcal{R}_{12}(\theta_1, \theta_2). \]
\tag{3.4}
We now look for a special transformation of (3.1), which satisfies

$$\tilde{L}_m(\theta; c_{ms}) = L_m(\theta; \tilde{c}_{ms}).$$  \hspace{1cm} (3.5)$$

Here we explicitly write the dependence of the fermionic $L$-operator on the fermion operators. The fermion operators $c_{ms}$ and $\tilde{c}_{ms}$ are assumed to be connected through the transformation law \[(1.10).\] We discovered that the relation (3.5) is satisfied when the matrices $M$ and $\tilde{M}$ meet the following conditions

$$M_{44} = M_{11}^*, \quad M_{41} = -M_{14}^*, \quad M_{33} = M_{22}^*, \quad M_{32} = -M_{23}^*$$

$$\tilde{M}_{11} = M_{11}, \quad \tilde{M}_{44} = M_{44}, \quad \tilde{M}_{14} = -M_{14}, \quad \tilde{M}_{41} = -M_{41},$$

$$\tilde{M}_{22} = M_{22}, \quad \tilde{M}_{33} = M_{33}, \quad \tilde{M}_{23} = M_{23}, \quad \tilde{M}_{32} = M_{32}.$$  \hspace{1cm} (3.6)$$

The condition (3.2) now becomes

$$|M_{11}|^2 + |M_{14}|^2 = |M_{22}|^2 + |M_{23}|^2 = 1.$$  \hspace{1cm} (3.7)$$

It is useful to introduce the submatrices of the matrices $M$ and $\tilde{M}$ as

$$M_{\text{charge}} = \begin{pmatrix} M_{11} & M_{14} \\ M_{41} & M_{44} \end{pmatrix}, \quad M_{\text{spin}} = \begin{pmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{pmatrix},$$

$$\tilde{M}_{\text{charge}} = \begin{pmatrix} \tilde{M}_{11} & \tilde{M}_{14} \\ \tilde{M}_{41} & \tilde{M}_{44} \end{pmatrix}, \quad \tilde{M}_{\text{spin}} = \begin{pmatrix} \tilde{M}_{22} & \tilde{M}_{23} \\ \tilde{M}_{32} & \tilde{M}_{33} \end{pmatrix}.$$  \hspace{1cm} (3.8)$$

Then the conditions (3.6) and (3.7) are equivalent to the relations

$$\tilde{M}_{\text{charge}} = \sigma^z M_{\text{charge}} \sigma^z, \quad \tilde{M}_{\text{spin}} = M_{\text{spin}}, \quad M_{\text{charge}}, M_{\text{spin}} \in SU(2).$$  \hspace{1cm} (3.9)$$

The corresponding transformation law of the fermion operators is

$$\begin{pmatrix} \tilde{c}_{m\uparrow}^\dagger & i\tilde{c}_{m\downarrow}^\dagger \\ i\tilde{c}_{m\downarrow}^\dagger & \tilde{c}_{m\uparrow}^\dagger \end{pmatrix} = \begin{pmatrix} M_{22}^* & -M_{23} \\ M_{23} & M_{22} \end{pmatrix} \begin{pmatrix} c_{m\uparrow}^\dagger & i\tilde{c}_{m\downarrow} \\ i\tilde{c}_{m\downarrow} & c_{m\uparrow} \end{pmatrix} \begin{pmatrix} M_{11} & M_{14} \\ -M_{14}^* & M_{11}^* \end{pmatrix}.$$  \hspace{1cm} (3.10)$$

Hereafter, we implicitly assume the the conditions (3.9) for the matrices $M$ and $\tilde{M}$. Then the transformation (3.1) is not a gauge transformation in a strict sense, because $\tilde{M} \neq M$ (particularly $\tilde{M}_{\text{charge}} \neq M_{\text{charge}}$). So we try assigning the different transformation laws to the $L$-operators for odd sites and even sites as

$$\tilde{L}_{2n-1}(\theta) = M^{-1}L_{2n-1}(\theta)M, \quad \tilde{L}_{2n}(\theta) = M^{-1}L_{2n}(\theta)\tilde{M}.$$  \hspace{1cm} (3.11)$$

The corresponding transformation law of the fermion operators on odd sites is, of course, given by the formula (3.10),

$$\begin{pmatrix} \tilde{c}_{2n-1\uparrow}^\dagger & i\tilde{c}_{2n-1\downarrow}^\dagger \\ i\tilde{c}_{2n-1\downarrow}^\dagger & \tilde{c}_{2n-1\uparrow}^\dagger \end{pmatrix} = \begin{pmatrix} M_{22}^* & -M_{23} \\ M_{23} & M_{22} \end{pmatrix} \begin{pmatrix} c_{2n\downarrow}^\dagger & i\tilde{c}_{2n\uparrow} \\ i\tilde{c}_{2n\uparrow} & c_{2n\downarrow} \end{pmatrix} \begin{pmatrix} M_{11} & M_{14} \\ -M_{14}^* & M_{11}^* \end{pmatrix}. \hspace{1cm} (3.12)$$

Since the matrices $M$ and $\tilde{M}$ are related by the exchange $M_{14} \leftrightarrow -M_{14}$, the transformation law for even sites reads

$$\begin{pmatrix} c_{2n\downarrow}^\dagger & i\tilde{c}_{2n\uparrow} \\ i\tilde{c}_{2n\uparrow} & c_{2n\downarrow} \end{pmatrix} = \begin{pmatrix} M_{22}^* & -M_{23} \\ M_{23} & M_{22} \end{pmatrix} \begin{pmatrix} c_{2n\uparrow}^\dagger & i\tilde{c}_{2n\downarrow} \\ i\tilde{c}_{2n\downarrow} & c_{2n\uparrow} \end{pmatrix} \begin{pmatrix} M_{11} & -M_{14} \\ M_{14}^* & M_{11}^* \end{pmatrix}. \hspace{1cm} (3.13)$$

$$8$$
Multiplying the Pauli matrix \( \sigma^z \) from the right, we obtain
\[
\begin{pmatrix}
\tilde{c}^\dagger_{2n\downarrow} & -i\tilde{c}_{2n\uparrow} \\
i\tilde{c}^\dagger_{2n\uparrow} & -\tilde{c}_{2n\downarrow}
\end{pmatrix} =
\begin{pmatrix}
M_{22}^* & -M_{23} \\
M_{23}^* & M_{22}
\end{pmatrix}
\begin{pmatrix}
\tilde{c}^\dagger_{2n\downarrow} & -i\tilde{c}_{2n\uparrow} \\
i\tilde{c}^\dagger_{2n\uparrow} & -\tilde{c}_{2n\downarrow}
\end{pmatrix}
\begin{pmatrix}
M_{11} & M_{14} \\
-M_{14}^* & M_{11}^*
\end{pmatrix}.
\tag{3.13}
\]

Recalling the definition of the \( 2 \times 2 \) matrices \( \Psi_m \) \(^{(1.9)}\), we can summarize the transformation laws \((3.12)\) and \((3.13)\) as
\[
\tilde{\Psi}_m = M_{\text{spin}}^{-1}\Psi_m M_{\text{charge}},
\tag{3.14}
\]
which exactly coincides \((1.10)\) with the correspondences \( O_{\text{spin}} = M_{\text{spin}}^{-1} \) and \( O_{\text{charge}} = M_{\text{charge}} \).

As we have explained in \( \S 1 \), the transformation \((3.14)\) is the \( SO(4) \) rotation in the space of the fermion operators. Therefore we can conclude that a kind of gauge transformation \((3.14)\) induces the \( SO(4) \) rotations for the fermion operators. We call the transformation \((3.14)\) the \( SO(4) \) rotation for the fermionic \( L \)-operator \((2.1)\). Note that the canonical anticommutation relation \((1.3)\) is preserved under the transformation \((3.14)\)
\[
\{\tilde{c}^\dagger_{ms}, \tilde{c}_{m's'}\} = \delta_{mm'}\delta_{ss'}, \quad \{c^\dagger_{ms}, \tilde{c}^\dagger_{m's'}\} = \{\tilde{c}_{ms}, \tilde{c}_{m's'}\} = 0.
\]

Let us consider the \( SO(4) \) invariance of the fermionic transfer matrix \((2.22)\). First we assume that \( N \) is even and impose the periodic boundary condition. The local \( SO(4) \) rotation for the fermionic \( L \)-operators \((3.11)\) induces the \( SO(4) \) rotation for the monodromy matrix \((2.19)\)
\[
\tilde{T}(\theta) \equiv \prod_{m=1}^{N} \tilde{L}_m(\theta) = M^{-1}T(\theta)M.
\tag{3.15}
\]

Since the relation
\[
\text{str} \{X(\theta)M\} = \text{str} \{MX(\theta)\}
\tag{3.16}
\]
holds, the transfer matrix \((2.22)\) is invariant under the periodic boundary condition \((K = I)\)
\[
\text{str}\tilde{T}(\theta; c_{ms}) = \text{str}T(\theta; c_{ms}),
\tag{3.17}
\]
where we write the fermion operators explicitly. In the relation \((3.16)\), \( X(\theta) \) is any \( 4 \times 4 \) matrix, which may depend on the fermion operators. On the other hand, the transfer matrix \( \text{str}\tilde{T}(\theta; c_{ms}) \) can be expressed as
\[
\text{str}\tilde{T}(\theta; c_{ms}) = \text{str}T(\theta; \tilde{c}_{ms}),
\tag{3.18}
\]
due to the property \((3.5)\). Combining \((3.17)\) and \((3.18)\), we establish
\[
\text{str}T(\theta; c_{ms}) = \text{str}T(\theta; \tilde{c}_{ms}).
\tag{3.19}
\]

The relation \((3.19)\) shows that the fermionic transfer matrix is invariant under the \( SO(4) \) rotation for the fermion operators \((3.14)\). It indicates that all the higher conserved currents, which are embedded in the transfer matrix, also have the \( SO(4) \) symmetry (see \( \S 5 \)).

Now we shall write the transformation \((3.15)\) in terms of the submatrices \( M_{\text{charge}} \) and \( M_{\text{spin}} \) \((3.8)\). We introduce the following convenient notation for the monodromy matrix \((2.22)\)
\[
T(\theta) =
\begin{pmatrix}
D_{11}(\theta) & C_{11}(\theta) & C_{12}(\theta) & D_{12}(\theta) \\
B_{11}(\theta) & A_{11}(\theta) & A_{12}(\theta) & B_{12}(\theta) \\
B_{21}(\theta) & A_{21}(\theta) & A_{22}(\theta) & B_{22}(\theta) \\
D_{21}(\theta) & C_{21}(\theta) & C_{22}(\theta) & D_{22}(\theta)
\end{pmatrix},
\]
where we regard \( A(\theta) = (A_{ij}(\theta)) \), \( B(\theta) = (B_{ij}(\theta)) \), \( C(\theta) = (C_{ij}(\theta)) \) and \( D(\theta) = (D_{ij}(\theta)) \) as \( 2 \times 2 \) matrices. Then the transformation (3.14) can be expressed in terms of \( 2 \times 2 \) matrices \( A(\theta), \ldots, D(\theta) \) as

\[
\tilde{A}(\theta) = M^{-1}_{\text{spin}} A(\theta) M_{\text{spin}}, \\
\tilde{B}(\theta) = M^{-1}_{\text{charge}} B(\theta) M_{\text{spin}}, \\
\tilde{C}(\theta) = M^{-1}_{\text{spin}} C(\theta) M_{\text{charge}}, \\
\tilde{D}(\theta) = M^{-1}_{\text{charge}} D(\theta) M_{\text{charge}}.
\]

Because the transformed monodromy matrix also satisfies the graded Yang-Baxter relation with the fermionic \( R \)-matrix,

\[
\tilde{R}_{12}(\theta_1, \theta_2) \left[ \tilde{T}(\theta_1) \otimes \tilde{T}(\theta_2) \right] = \left[ \tilde{T}(\theta_2) \otimes \tilde{T}(\theta_1) \right] \tilde{R}_{12}(\theta_1, \theta_2),
\]

the associative algebra defined by the graded Yang-Baxter relation should be invariant under the transformation (3.20). From (3.20), we notice an interesting fact that the submatrix \( A(\theta) \) is transformed by the spin-\( SU(2) \) rotation and \( D(\theta) \) is transformed by the charge-\( SU(2) \) rotation. We believe this property plays a significant role in the application of the algebraic Bethe ansatz for the 1D Hubbard model [26].

Next we consider the case of \( N \) odd. The monodromy matrix transforms as

\[
\tilde{T}(\theta) = M^{-1} T(\theta) M.
\]

In this case, we have to twist the periodic boundary condition to make the transfer matrix \( SO(4) \) invariant. The condition for the matrix \( K \) in the transfer matrix is

\[
KM = MK.
\]

For example,

\[
K = \begin{pmatrix}
  i & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & -i
\end{pmatrix}
\]

solves the condition (3.22). From the formula in [24], we can see that the choice (3.23) corresponds to the twisted boundary condition

\[
c_{N+1\uparrow} = ic_{1\uparrow}, \quad c_{N+1\downarrow} = -ic_{1\downarrow}, \\
c_{N+1\uparrow} = ic_{1\downarrow}, \quad c_{N+1\downarrow} = -ic_{1\uparrow}.
\]

Assuming (3.23) and (3.24), we can prove the \( SO(4) \) invariance of the transfer matrix for \( N \) odd as

\[
\text{str}K T(\theta; c_{ms}) = \text{str}K T(\theta; \tilde{c}_{ms}),
\]

in a similar way to the even case.
§4. Partial Particle-Hole Transformation of the Fermionic Transfer Matrix

In [22], the transformation law of the fermionic $L$-operator corresponding to the partial particle-hole transformation (1.7) was found. We shall discuss the transformation law in connection with the relation (2.17). Consider the following transformations of the fermionic $L$-operators

$$\hat{\mathcal{L}}_{2n-1}(\theta) = V^{-1}\mathcal{L}_{2n-1}(\theta)V, \quad \hat{\mathcal{L}}_{2n}(\theta) = V^{-1}\mathcal{L}_{2n}(\theta)V,$$

where

$$V = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \bar{V} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$  \(4.1\)

Since the constant matrices $V$ and $\bar{V}$ are of the form (2.18), we have the following graded Yang-Baxter relation (4.3)

$$\mathcal{R}_{12}(\theta_1, \theta_2; -U)[\hat{\mathcal{L}}_m(\theta_1; U) \otimes \hat{\mathcal{L}}_m(\theta_2; U)] = [\hat{\mathcal{L}}_m(\theta_2; U) \otimes \hat{\mathcal{L}}_m(\theta_1; U)]\mathcal{R}_{12}(\theta_1, \theta_2; -U),$$  \(4.2\)

where we write the $U$-dependence explicitly. The graded Yang-Baxter relation (1.3) implies that the transformed $L$-operators $\hat{\mathcal{L}}_m(\theta; U)$ are related to the $L$-operators with the coupling constant $-U$. In fact the following relations hold,

$$\hat{\mathcal{L}}_{2n-1}(\theta; c_{2n-1s}, U) = i\mathcal{L}_{2n-1}(\theta; \hat{c}_{2n-1s}, -U), \quad \hat{\mathcal{L}}_{2n}(\theta; c_{2ns}, U) = i\mathcal{L}_{2n}(\theta; \hat{c}_{2ns}, -U),$$  \(4.4\)

where

$$\hat{c}_{2n-1\uparrow} = c_{2n-1\uparrow}, \quad \hat{c}_{2n-1\downarrow} = c_{2n-1\downarrow}^\dagger, \quad \hat{c}_{2n\uparrow} = c_{2n\uparrow}, \quad \hat{c}_{2n\downarrow} = c_{2n\downarrow}^\dagger.$$  \(4.5\)

The transformation (1.3) is nothing but the partial particle-hole transformation (1.7). Therefore we call (1.4) the partial particle-hole transformation of the fermionic $L$-operator (2.1).

It is quite interesting to note that the transformation (4.6) can be written in terms of the $2 \times 2$ matrix $\Psi_m$ (1.3) as

$$\hat{\Psi}_m = -\Psi_m^\dagger,$$  \(4.6\)

where $\dagger$ denotes the hermitian conjugation. Moreover, taking the hermitian conjugation of (1.4), we find

$$\bar{\Psi}_m = M_{\text{charge}}^{-1}\Psi_m M_{\text{spin}},$$  \(4.7\)

which shows that the spin-$SU(2)$ $M_{\text{spin}}$ and the charge-$SU(2)$ $M_{\text{charge}}$ are exchanged after the partial particle-hole transformation.

We are ready to verify the invariance of the transfer matrix of the 1D Hubbard model under the partial particle-hole transformation. First we assume that $N$ is even and impose the periodic boundary condition. Then the partial particle-hole transformation of the monodromy matrix induced by (1.1) is

$$\hat{T}(\theta; c_{ms}, U) = V^{-1}T(\theta; c_{ms}, U)V.$$  \(4.8\)
From the relations
\[
\text{str} \hat{T}(\theta; c_{ms}, U) = \text{str} \left\{ V^{-1} T(\theta; c_{ms}, U) V \right\} = -\text{str} T(\theta; c_{ms}, U),
\]
and
\[
\text{str} \hat{T}(\theta; c_{ms}, U) = i^N \text{str} T(\theta; \hat{c}_{ms}, -U),
\]
we obtain [22]
\[
\text{str} T(\theta; c_{ms}, U) = -i^N \text{str} T(\theta; \hat{c}_{ms}, -U). \tag{4.9}
\]
The last identity (4.9) proves the invariance of the fermionic transfer matrix (up to sign) under the the partial particle-hole transformation (1.7). Note a relation
\[
\text{str} \left\{ X(\theta)V \right\} = -\text{str} \left\{ VX(\theta) \right\}, \tag{4.10}
\]
which should be compared with (3.16).

We have a similar relation for \(N\) odd,
\[
\text{str} \left\{ KT(\theta; c_{ms}, U) \right\} = -i^{N-1} \text{str} \left\{ KT(\theta; \hat{c}_{ms}, -U) \right\}. \tag{4.11}
\]
where \(K\) is given by (3.23). In the derivation of (4.11), we have used the relation
\[
K \bar{V}^{-1} = i V^{-1} K.
\]
The formula (4.11) means that the fermionic transfer matrix for \(N\) odd is also invariant under the partial particle-hole transformation (1.7) when we assume the twisted periodic boundary condition (3.24).

§5. \textit{SO}(4) Symmetry of the Higher Conserved Currents

In §3, we have shown the \textit{SO}(4) symmetry of the transfer matrix, which means that all the conserved currents of the 1D Hubbard model also have the \textit{SO}(4) symmetry. As will be seen, the \textit{SO}(4) symmetry of the conserved currents can be manifestly read out in terms of the Clifford algebra. Hereafter, for simplicity of explanation, we assume the number of sites is always even and impose the periodic boundary condition.

Although the graded Yang-Baxter relation ensures the existence of infinitely many higher conserved currents in involution, it is not an easy task to obtain their explicit forms from the transfer matrix (see [13]). To construct the higher conserved currents, we often use the boost operator [27, 28] which recursively produces the higher conserved currents. However in the case of the 1D Hubbard model, the boost operator does not exist [29] and we have to resort to a more direct computation.

The first higher conserved current of the 1D Hubbard model was found by Shastry [14, 16] as
\[
I^{(2)} = i \sum_{m=1}^{N} \sum_{s=\uparrow,\downarrow} \left( c_{m+2s}^\dagger c_{ms} - c_{ms}^\dagger c_{m+2s} \right) -iU \sum_{m=1}^{N} \sum_{s=\uparrow,\downarrow} \left( c_{m+1s}^\dagger c_{ms} - c_{ms}^\dagger c_{m+1s} \right) (n_{m+1,-s} + n_{m,-s} - 1). \tag{5.1}
\]
Subsequently, some higher conserved currents were obtained in a similar fashion \[29\text{--}31\],

\[I^{(3)} = -\sum_{m=1}^{N} \sum_{s=\uparrow,\downarrow} \left( c_{m+3s}^\dagger c_{ms} + c_{ms}^\dagger c_{m+3s} \right) \]

\[
+ U \sum_{m=1}^{N} \sum_{s=\uparrow,\downarrow} \left\{ \left( c_{m+1s}^\dagger c_{m-1s} + c_{m-1s}^\dagger c_{m+1s} \right) \left( n_{m+1,-s} + n_{m,-s} + n_{m-1,-s} - \frac{3}{2} \right) \right. \\
- \left( c_{m+1s}^\dagger c_{ms} - c_{ms}^\dagger c_{m+1s} \right) \left( c_{m,-s}^\dagger c_{m-1s} - c_{m-1s}^\dagger c_{m,-s} \right) \\
- \left( n_{m+1s} - \frac{1}{2} \right) \left( n_{m,-s} - \frac{1}{2} \right) \left\} \right.
\]

\[
+ U \sum_{m=1}^{N} \left\{ \left( c_{m+1\uparrow}^\dagger c_{m\uparrow}^\dagger - c_{m\uparrow}^\dagger c_{m+1\uparrow} \right) \left( c_{m+1\downarrow}^\dagger c_{m\downarrow}^\dagger - c_{m\downarrow}^\dagger c_{m+1\downarrow} \right) \\
- \left( n_{m\uparrow} - \frac{1}{2} \right) \left( n_{m\downarrow} - \frac{1}{2} \right) \right\}
\]

\[- U^2 \sum_{m=1}^{N} \sum_{s=\uparrow,\downarrow} \left( c_{m+1s}^\dagger c_{ms} + c_{ms}^\dagger c_{m+1s} \right) \left( n_{m,-s} - \frac{1}{2} \right) \left( n_{m+1,-s} - \frac{1}{2} \right) \]

\[- \frac{U^3}{4} \sum_{m=1}^{N} \left( n_{m\uparrow} - \frac{1}{2} \right) \left( n_{m\downarrow} - \frac{1}{2} \right), \]

\[I^{(4)} = i \sum_{m=1}^{N} \sum_{s=\uparrow,\downarrow} \left( c_{m+4s}^\dagger c_{ms} - c_{ms}^\dagger c_{m+4s} \right) \]

\[-2iU \sum_{m=1}^{N} \sum_{s=\uparrow,\downarrow} \left\{ \left( c_{m+3s}^\dagger c_{ms} - c_{ms}^\dagger c_{m+3s} \right) \sum_{k=m}^{m+3} \left( n_{k,-s} - \frac{1}{2} \right) \\
- \left( c_{m+1s}^\dagger c_{ms} - c_{ms}^\dagger c_{m+1s} \right) \sum_{k=m-1}^{m+2} \left( n_{k,-s} - \frac{1}{2} \right) \\
+ \left( c_{m+2s}^\dagger c_{ms} + c_{ms}^\dagger c_{m+2s} \right) \sum_{k=m-1}^{m+2} \left( c_{k+1,-s}^\dagger c_{k,-s} - c_{k,-s}^\dagger c_{k+1,-s} \right) \right\} \]

\[+ 4iU^2 \sum_{m=1}^{N} \sum_{s=\uparrow,\downarrow} \left\{ \left( c_{m+2s}^\dagger c_{ms} - c_{ms}^\dagger c_{m+2s} \right) \left( n_{m,-s} - \frac{1}{2} \right) \left( n_{m+1,-s} - \frac{1}{2} \right) \\
+ \left( n_{m,-s} - \frac{1}{2} \right) \left( n_{m+2,-s} - \frac{1}{2} \right) + \left( n_{m+1,-s} - \frac{1}{2} \right) \left( n_{m+2,-s} - \frac{1}{2} \right) \right\} \]

\[
\left( c_{m+1s}^\dagger c_{ms} + c_{ms}^\dagger c_{m+1s} \right) \left( c_{m,-s}^\dagger c_{m-1s} - c_{m-1s}^\dagger c_{m,-s} \right) \left( n_{m+1,-s} - \frac{1}{2} \right) \]

\[
+ \left( c_{m+1,-s} c_{m+2,-s} - c_{m+2,-s} c_{m+1,-s} \right) \left( n_{m,-s} - \frac{1}{2} \right) \right\} \]

\[+ 2iU^3 \sum_{m=1}^{N} \sum_{s=\uparrow,\downarrow} \left( c_{m+1s}^\dagger c_{ms} - c_{ms}^\dagger c_{m+1s} \right) \left( n_{m+1,-s} + n_{m,-s} - 1 \right). \]

These currents are embedded in the fermionic transfer matrix \[\#22\#\] and should be \(SO(4)\) invariant from the result in the previous section. One can confirm the \(SO(4)\) invariance of
these currents using the transformation law of the fermion operators \([3.14]\). In the following, we present a different approach; we shall rewrite these currents in manifestly \(SO(4)\) invariant forms by use of the Clifford algebra.

Define \(\Gamma^a_j\) \((j = 1, \cdots, N, a = 1, \cdots, 4)\) by

\[
\begin{align*}
\Gamma^1_{2n-1} &= c^\dagger_{2n-1} + c_{2n-1}, \quad \Gamma^2_{2n-1} = i \left( c^\dagger_{2n-1} - c_{2n-1} \right), \\
\Gamma^3_{2n-1} &= c^\dagger_{2n-1} + c_{2n-1}, \quad \Gamma^4_{2n-1} = i \left( c^\dagger_{2n-1} - c_{2n-1} \right),
\end{align*}
\]

\[(5.4)\]

and

\[
\begin{align*}
\Gamma^1_{2n} &= i \left( c_{2n} - c^\dagger_{2n} \right), \quad \Gamma^2_{2n} = c_{2n} + c^\dagger_{2n}, \\
\Gamma^3_{2n} &= i \left( c_{2n} - c^\dagger_{2n} \right), \quad \Gamma^4_{2n} = c_{2n} + c^\dagger_{2n},
\end{align*}
\]

\[(5.5)\]

where \(n = 1, \cdots, \frac{N}{2}\). Then the operators \(\Gamma^a_j\) satisfy the defining relations of the Clifford algebra \([3.32]\)

\[
\{ \Gamma^a_j, \Gamma^b_k \} = 2\delta_{jk}\delta^{ab}, \quad j, k = 1, \cdots, N, \ a, b = 1, \cdots, 4.
\]

\[(5.6)\]

In terms of \(\Gamma^a_j\), the \(SO(4)\) rotation for the fermion operators \([3.14]\) can be expressed simply as

\[
\tilde{\Gamma}^a_j = \sum_{b=1}^4 G^{ab} \Gamma^b_j, \quad G = (G^{ab}) \in SO(4).
\]

\[(5.7)\]

The relation between the matrices \(G\) and \(M\) in \(\S 4\) is explicitly given by

\[
G = G^{(1)} G^{(2)},
\]

\[
G^{(1)} = \begin{pmatrix} \xi_0 & \xi_1 & \xi_2 & -\xi_3 \\ -\xi_1 & \xi_0 & -\xi_2 & -\xi_3 \\ -\xi_2 & \xi_1 & \xi_0 & -\xi_3 \\ \xi_3 & -\xi_2 & \xi_0 & \xi_1 \end{pmatrix}, \quad G^{(2)} = \begin{pmatrix} \xi_0 & \xi_1 & -\xi_3 & -\xi_2 \\ -\xi_1 & \xi_0 & \xi_2 & -\xi_3 \\ \xi_3 & -\xi_2 & \xi_0 & -\xi_1 \\ \xi_2 & \xi_3 & \xi_1 & \xi_0 \end{pmatrix},
\]

where \(\xi_i\) and \(\eta_i\) are real numbers given by

\[
\begin{align*}
\xi_0 &= \text{Re}(M_{11}), \quad \xi_1 = \text{Im}(M_{11}), \quad \xi_2 = \text{Re}(M_{14}), \quad \xi_3 = \text{Im}(M_{14}), \\
\zeta_0 &= \text{Re}(M_{22}), \quad \zeta_1 = \text{Im}(M_{22}), \quad \zeta_2 = \text{Re}(M_{23}), \quad \zeta_3 = \text{Im}(M_{23}), \\
\sum_{j=0}^3 \xi_j^2 &= \sum_{j=0}^3 \zeta_j^2 = 1.
\end{align*}
\]

\[(5.8)\]

Clearly the matrix \(G^{(1)}\) corresponds to the charge-\(SU(2)\) and the matrix \(G^{(2)}\) corresponds to the spin-\(SU(2)\). It is an interesting exercise to confirm that the matrices \(G^{(1)}\) and \(G^{(2)}\) commute each other

\[
G^{(1)} G^{(2)} = G^{(2)} G^{(1)}.
\]

Define the operator \(\Gamma^5_j\) by

\[
\Gamma^5_j = \Gamma^1_j \Gamma^2_j \Gamma^3_j \Gamma^4_j = \frac{1}{4!} \sum_{a,\cdots,d=1}^4 \epsilon_{abcd} \Gamma^a_j \Gamma^b_j \Gamma^c_j \Gamma^d_j.
\]

\[(5.9)\]
The operator \( \Gamma^5_j \) has the following properties

\[
\{ \Gamma^5_j, \Gamma^a_j \} = 0, \quad \left[ \Gamma^5_j, \Gamma^a_k \right] = 0 \quad (j \neq k), \quad a = 1, \ldots, 4.
\]  
(5.10)

It is clear that the operators such as

\[
\sum_{a=1}^{4} \Gamma^a_j \Gamma^a_k, \quad \Gamma^5_j,
\]
are invariant under the \( SO(4) \) rotation \( (5.7) \). We make use of this fact to rewrite the conserved currents. The Hamiltonian \( \mathcal{H} = I^{(1)} \) in terms of the operators \( \Gamma^a_j \) and \( \Gamma^5_j \) is \([6, 32]\)

\[
I^{(1)} = \sum_j \sum_a (-1)^j \Gamma^a_{j+1} \Gamma^a_j + u \sum_j \Gamma^5_j,
\]

(5.12)

where

\[
u = \frac{iU}{2}.
\]

Here and hereafter, we do not mind the difference of an overall factor. The formula \( (5.12) \) gives a manifestly \( SO(4) \) invariant representation of the Hamiltonian.

In the same way, we express the higher conserved currents in terms of the operators \( \Gamma^a_j \) and \( \Gamma^5_j \) as

\[
I^{(2)} = \sum_j \sum_a \Gamma^a_{j+2} \Gamma^a_j + u \sum_j \sum_a (-1)^j \Gamma^a_{j+1} \Gamma^a_j \left( \Gamma^5_{j+1} - \Gamma^5_j \right),
\]

(5.13)

\[
I^{(3)} = \sum_j \sum_a (-1)^j \Gamma^a_{j+3} \Gamma^a_j
\]

\[
- u \sum_j \left\{ \Gamma^5_j + \sum_a \Gamma^a_{j+2} \Gamma^a_j \left( \Gamma^5_{j+2} - \Gamma^5_j \right) \right. 
\]

\[
+ \sum_{a \neq b} \left( \Gamma^a_{j+2} \Gamma^b_{j+1} \Gamma^5_{j+1} - \frac{1}{2} \Gamma^a_{j+1} \Gamma^b_{j+1} \Gamma^b_j \Gamma^5_j \right) \}
\]

\[
+ u^2 \sum_j \sum_a (-1)^j \Gamma^a_{j+1} \Gamma^a_j \Gamma^5_{j+1} \Gamma^5_j
\]

\[
+ u^3 \sum_j \Gamma^5_j,
\]

(5.14)

\[
I^{(4)} = \sum_j \sum_a \Gamma^a_{j+4} \Gamma^a_j
\]

\[
+ u \sum_j (-1)^j \left\{ \sum_a \left\{ \Gamma^a_{j+3} \Gamma^a_j \left( \Gamma^5_{j+3} - \Gamma^5_j \right) - \Gamma^a_{j+1} \Gamma^a_j \left( \Gamma^5_{j+1} - \Gamma^5_j \right) \right\} 
\]

\[
+ \sum_{a \neq b} \left\{ \Gamma^a_{j+3} \Gamma^b_{j+2} \Gamma^b_j \Gamma^5_{j+2} + \Gamma^a_{j+3} \Gamma^b_{j+1} \Gamma^b_j \Gamma^5_{j+1} \right. 
\]

\[
- \Gamma^a_{j+2} \Gamma^b_{j+1} \Gamma^b_j \Gamma^5_{j+1} + \Gamma^a_{j+2} \Gamma^b_j \Gamma^b_j \Gamma^5_j \right\} 
\}
\]

\[
+ u^2 \sum_j \left\{ \sum_a \Gamma^a_{j+2} \Gamma^5_{j+2} \Gamma^5_j - \sum_{a \neq b} \Gamma^a_{j+2} \Gamma^a_{j+1} \Gamma^b_{j+1} \Gamma^5_{j+1} \Gamma^5_{j+1} \right\} 
\]

\[
+ u^3 \sum_j \sum_a (-1)^j \Gamma^a_{j+1} \Gamma^5_{j+1} \Gamma^5_{j+1} - \Gamma^5_j \right\}.
\]

(5.15)
Since the terms that constitute (5.13)–(5.15) are of the form (5.11), we can see that the higher conserved currents \( I^{(2)}, I^{(3)} \) and \( I^{(4)} \) are also manifestly \( SO(4) \) invariant. Note that the constraints \( a \neq b \) in the summations do not break the \( SO(4) \) symmetry. For example, we can write
\[
\sum_{a \neq b} \Gamma^a_{j+2} \Gamma^b_{j+1} \Gamma^c_j \Gamma^d_{j+1} = \sum_{a,b} \Gamma^a_{j+2} \Gamma^b_{j+1} \Gamma^c_j \Gamma^d_{j+1} - \sum_a \Gamma^a_{j+2} \Gamma^a_j \Gamma^5_{j+1}.
\]
Both terms in the RHS of (5.16) are clearly \( SO(4) \) invariant.

The infinitesimal generators of the \( SO(4) \) rotations (5.7) are given by \(^{33}\)
\[
Q^{ab} = -Q^{ba} = \frac{1}{4i} \sum_j \left[ \Gamma^a_j, \Gamma^b_j \right].
\]
In fact the generator (5.17) fulfills the defining relation of the Lie algebra \( so(4) \)
\[
\left[ Q^{ab}, Q^{cd} \right] = -i \left( \delta^{ac} Q^{bd} - \delta^{ad} Q^{bc} + \delta^{bd} Q^{ac} + \delta^{ad} Q^{bc} \right).
\]
We also have a relation
\[
\left[ Q^{ab}, \Gamma^c_j \right] = i \left( \delta^{ac} \Gamma^b_j - \delta^{bc} \Gamma^a_j \right),
\]
which is nothing but the infinitesimal transformation of (5.7). Using (5.19), one can confirm the commutativity
\[
\left[ Q^{ab}, I^{(n)} \right] = 0, \quad n = 1, \cdots, 4,
\]
which shows the Lie algebra \( so(4) \) symmetry of the conserved currents \( I^{(n)} \). Actually the generators of the spin-\( su(2) \) (1.3) and the charge-\( su(2) \) (1.6) are related to \( Q^{ab} \) as
\[
S^x = -\frac{1}{2} \left( Q^{14} - Q^{23} \right), \quad S^y = \frac{1}{2} \left( Q^{24} + Q^{13} \right), \quad S^z = -\frac{1}{2} \left( Q^{12} - Q^{34} \right),
\]
\[
\eta^x = -\frac{1}{2} \left( Q^{14} + Q^{23} \right), \quad \eta^y = \frac{1}{2} \left( Q^{24} - Q^{13} \right), \quad \eta^z = -\frac{1}{2} \left( Q^{12} + Q^{34} \right),
\]
where we introduced \( S^x, S^y, \eta^x \) and \( \eta^y \) through the relations
\[
S^\pm = S^x \pm i S^y, \quad \eta^\pm = \eta^x \pm i \eta^y.
\]
For the Clifford algebra (5.6), the partial particle-hole transformation (1.4) corresponds to
\[
\Gamma^3_j \rightarrow -\Gamma^3_j.
\]
Note that the transformation (5.23) exchanges the spin-\( su(2) \) (5.21) and the charge-\( su(2) \) (5.22).

As for the conserved currents, the transformation (5.23) preserves the operators like \( \sum_a \Gamma^a_j \Gamma^a_k \),
but changes the sign of \( \Gamma^5_j \). From the explicit formulas (5.12)–(5.15), one can immediately find that the conserved currents \( I^{(n)}(n = 1, \cdots, 4) \) are invariant under the partial particle-hole transformation
\[
\Gamma^5_j \rightarrow -\Gamma^5_j, \quad u \rightarrow -u.
\]
This is consistent with the result in §4.
§6. Discussions

We have investigated the $SO(4)$ symmetry of the 1D Hubbard model from the QISM point of view. Our approach is based on the fermionic formulation of the Yang-Baxter relation for the 1D Hubbard model found by Olmedilla et al. [17]. It consists of the fermionic $R$-matrix and the fermionic $L$-operator. We have discovered the transformation law (3.11) of the fermionic $L$-operator under the $SO(4)$ rotation. It is a kind of gauge transformation and induces the transformation of the monodromy matrix. The result is a fundamental property of the associative algebra defined by the fermionic $R$-matrix of the 1D Hubbard model. We believe that the property will also play an important role in the algebraic Bethe ansatz for the 1D Hubbard model, which was recently explored by Ramos and Martins [26]. We like to emphasize the advantage of the fermionic formulation of the Yang-Baxter relation. It is difficult to discuss the $SO(4)$ symmetry of the Hubbard model through Shastry's $R$-matrix and the related transfer matrix.

The $SO(4)$ invariance of the transfer matrix ensures the $SO(4)$ invariance of the conserved currents. We have demonstrated the $SO(4)$ symmetry of the higher conserved currents employing the Clifford algebra, which corresponds to the spinor representation of the rotation group. It should be interesting to explore a representation of the fermionic $L$-operator itself in terms of the Clifford algebra.

On the infinite lattice, the Lie algebra $so(4) = su(2) \oplus su(2)$ symmetry of the 1D Hubbard model is extended to the Yangian $Y(so(4)) = Y(su(2)) \oplus Y(su(2))$ symmetry

$$[Y(so(4)), I^{(1)}] = 0$$

as was discovered by Uglov and Korepin [34]. The generators of $Y(so(4))$ can be expressed in terms of the Clifford algebra $\Gamma_a^j$ as follows

$$Q_{ab}^{(0)} = -\frac{i}{4} \sum_j \left[ \Gamma^a_j, \Gamma^b_j \right],$$

$$Q_{ab}^{(1)} = -i \sum_j (-1)^j \left( \Gamma^a_{j+1} \Gamma^b_j + \Gamma^a_j \Gamma^b_{j+1} \right)$$

$$+ \frac{iu}{4} \left( \sum_{j<k} - \sum_{k>j} \right) \sum_{c\neq a,b} \Gamma^a_j \Gamma^c_k \Gamma^b_l \left( \Gamma^5_j + \Gamma^5_k \right).$$

By use of the fundamental properties of the Clifford algebra (5.9), (5.10) and (5.11), we have confirmed that the higher conserved currents $I^{(n)}(n = 2, 3, 4)$ also have the Yangian $Y(so(4))$ symmetry, i.e.,

$$[Q_{ab}^{(0)}, I^{(n)}] = [Q_{ab}^{(1)}, I^{(n)}] = 0, \quad n = 1, \ldots, 4.$$

All the conserved currents of the 1D Hubbard model on the infinite lattice are conjectured to have the $Y(so(4))$ symmetry. In fact Murakami and Gohmann [35] recently showed the existence of an infinite number of the conserved currents which have the Yangian symmetry on the infinite lattice. However, one of the two $Y(su(2))$ that constitute $Y(so(4))$ drops out [35]. It seems to be difficult to prove the full $Y(so(4))$ symmetry of the conserved currents simultaneously in their method.
Acknowledgements

The authors are grateful to F. Göhmann, K. Hikami, S. Murakami and T. Tsuchida for valuable discussions and comments. One of the authors (M. S.) also thank J. Suzuki and A. Kuniba for discussions. This work is supported by a Grant-in-Aid for JSPS Fellows from the Ministry of Education, Science, Sports and Culture of Japan.

Appendix: Symmetry Matrix of Shastry’s $R$-Matrix

The $L$-operator for the coupled spin model [12] [14] [17] is expressed as

$$L_m(\theta) = \begin{pmatrix} e^h p_m^+ (\theta) q_m^+ (\theta) & p_m^+ (\theta) \tau_m^- & \sigma_m^+ q_m^+ (\theta) & e^h \sigma_m^+ \tau_m^- \\ p_m^+ (\theta) \tau_m^+ & e^{-h} p_m^- (\theta) q_m^- (\theta) & e^{-h} \sigma_m^- \tau_m^- & p_m^- (\theta) \tau_m^+ \\ \sigma_m^+ q_m^- (\theta) & e^{-h} \sigma_m^- \tau_m^+ & e^{-h} p_m^- (\theta) q_m^- (\theta) & \sigma_m^+ \tau_m^- \\ e^h \sigma_m^- \tau_m^+ & \sigma_m^+ q_m^- (\theta) & p_m^- (\theta) \tau_m^+ & e^h p_m^- (\theta) q_m^- (\theta) \end{pmatrix}, \quad (A.1)$$

where

$$p_m^\pm (\theta) = \frac{1}{2} (\cos \theta \pm \sin \theta) \pm \frac{1}{2} (\cos \theta \mp \sin \theta) \sigma_m^\pm,$$

$$q_m^\pm (\theta) = \frac{1}{2} (\cos \theta \pm \sin \theta) \pm \frac{1}{2} (\cos \theta \mp \sin \theta) \tau_m^\pm. \quad (A.2)$$

Here the coupling constant $h$ is related to the spectral parameter $\theta$ through the formula [2.2]. The $R$-matrix $\tilde{R}_{12}(\theta_1, \theta_2)$, which satisfies the Yang-Baxter relation [13] with the $L$-operator [A.1], is connected to the fermionic $R$-matrix [2.6] through the formula [17]

$$\tilde{R}_{12}(\theta_1, \theta_2) = W_{12}^{-1} \tilde{R}_{12}(\theta_1, \theta_2) W_{12}, \quad (A.3)$$

where $W_{12}$ is a diagonal $16 \times 16$ matrix

$$W_{12} = \text{diag}(1, 1, -i, -i, -i, 1, 1, -1, -1, i, i, i, -1, -i, -1). \quad (A.4)$$

We consider the symmetry matrix $M$ of Shastry’s $R$-matrix $\tilde{R}_{12}(\theta_1, \theta_2)$ which is defined to be a constant matrix satisfying

$$[\tilde{R}_{12}(\theta_1, \theta_2), M \otimes M] = 0. \quad (A.5)$$

Here the matrix elements of $M$ are assumed to be commuting numbers. One might suppose the symmetry matrix of Shastry’s $R$-matrix is identical to that of the fermionic $R$-matrix (2.16). However, surprisingly enough, we notice they take different forms. In fact, solving the defining equation for the symmetry matrix (A.5) directly, we find that the followings are the symmetry matrix of Shastry’s $R$-matrix,

$$M = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & \gamma & 0 & 0 \\ \delta & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & \beta & 0 \\ 0 & \gamma & 0 & 0 \\ \delta & 0 & 0 & 0 \end{pmatrix} \quad (A.6)$$
where $\alpha, \beta, \gamma$ and $\delta$ are $c$-numbers obeying

$$\alpha \delta = \beta \gamma. \quad (A.7)$$

We ignore the difference of overall factors of the matrices. Then each matrix (A.6) depends only on two parameters. The result means that Shastry’s $R$-matrix does not reflect the $SO(4)$ symmetry of the fermionic Hamiltonian (I.1). Therefore the $SO(4)$ symmetry of the transfer matrix that we explored in this paper may not be discussed if we use Shastry’s $R$-matrix and $L$-operator (A.1).

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