Quantum transport with strong scattering: beyond the nonlinear sigma model

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Abstract
Transport properties of a two-band system with spectral nodes are studied in the presence of random scattering. Starting from a Grassmann functional integral, we derive a bosonic representation that is based on random phase fluctuations. Averaging leads to a graphical representation of the correlation function with entangled random walks and four-vertices. In the strong scattering limit we derive a complex transition amplitude. For the example of two-dimensional Dirac fermions we obtain a localization length proportional to the inverse scattering rate.

Keywords: quantum diffusion, Anderson localization, strong scattering

1. Introduction
Diffusion, a concept known from classical physics, where the mean-square displacement of a particle position grows linearly with time, can also be observed in quantum systems. It provides our basic understanding for a large number of transport phenomena, such as the metallic behavior in electronic systems. The diffusion coefficient \( D(E) \) of quantum particles of energy \( E \) is defined by the correlation of two one-particle Green functions as \[ D(E) = \lim_{\epsilon \to 0} \epsilon^3 \sum_r \langle (G_{r0}(E+i\epsilon)G_{0r}(E-i\epsilon)) \rangle_d, \] (1)
where \( \langle \ldots \rangle_d \) is the average with respect to a distribution of random scatterers. The one-particle Green function is defined as the resolvent \( G(z) = (H - z^{-1}) \) of the Hamiltonian \( H \), and \( G_{r0}(E+i\epsilon) \) describes the propagation of a particle with energy \( E \) from the origin to a site \( r \). Scattering is introduced by the fact that the Hamiltonian \( H \) is random with mean \( \langle H \rangle_d = H_0 \). According to expression (1), a non-vanishing diffusion coefficient requires a long-range correlation for \( \epsilon \sim 0 \). This can be produced by a spontaneously broken symmetry, in analogy to the Goldstone modes in \( O(N) \) symmetric field theories. Indeed, \( \epsilon \) plays the role of a symmetry-breaking term, such that in the limit \( \epsilon \to 0 \) spontaneous symmetry breaking is indicated by a non-vanishing scattering rate [3, 4]. The corresponding symmetry depends on the Hamiltonian \( H \). A special case appears for a two-band Hamiltonian that is represented in
terms of Pauli matrices as $H = h_0\sigma_0 + \cdots + h_3\sigma_3$. Assuming that we can diagonalize all four matrices $h_0, \ldots, h_3$ by the same unitary transformation $h_j \rightarrow \lambda_j$ we get for the spectrum two-bands

$$E_{\pm} = \lambda_0 \pm \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}. \quad (2)$$

A spectral node, where the two-bands are degenerate, is characterized by $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0$, such that $\lambda_0$ moves us away from the node at zero energy.

In the presence of random scattering there is also Anderson localization due to strong quantum interference [5]. In one-band systems this effect is particularly strong in one or two dimensions [6–8]. However, the situation is different in two-band systems due to interband scattering. This causes special effects, such as Klein tunneling, which allows quantum particles to escape from localization. Transport measurements on graphene support the absence of Anderson localization at least in a weakly disordered two-dimensional electron gas with spectral nodes [9].

The correlation function of the expression (1) is invariant under non-Abelian chiral transformations if the Hamiltonian $H$ has a generalized particle–hole symmetry [10]

$$U H^T U^\dagger = -H, \quad U U^\dagger = 1 \quad (3)$$

with a unitary transformation $U$ and the transposition $^T$. This can be seen, following [10], by introducing the Green function $G_{\{i\varepsilon\}} = (\hat{H} - i\varepsilon)^{-1}$ for the extended Hamiltonian

$$\hat{H} = \begin{pmatrix} H_+ & 0 & 0 & 0 \\ 0 & H_- & 0 & 0 \\ 0 & 0 & H^T_+ & 0 \\ 0 & 0 & 0 & H^T_- \end{pmatrix}, \quad H_{\pm} = H \pm \mu \sigma_0. \quad (4)$$

where $\mu$ is a chemical potential that violates the relation (3) and corresponds to $\lambda_0$ in equation (2). Then, together with property (3), the matrix

$$\hat{S} = \begin{pmatrix} 0 & 0 & \varphi_+ U & 0 \\ 0 & 0 & 0 & \varphi_- U \\ \varphi_+ U^\dagger & 0 & 0 & 0 \\ 0 & \varphi_- U^\dagger & 0 & 0 \end{pmatrix} \quad (5)$$

for scalar variables $\varphi_+$, $\varphi_-$ anticommutes with $\hat{H}: \hat{S}\hat{H} = -\hat{H}\hat{S}$. This relation implies a non-Abelian chiral symmetry [10, 11]:

$$e^{\delta} \hat{H} e^{\delta} = \hat{H} \quad (6)$$

which is a symmetry relation for the extended Hamiltonian with respect to $\hat{U} = e^{\delta}$.

Since long-range correlations for the Green functions in (1) appear due to the spontaneously broken symmetry (6), it is sufficient to restrict the average $\langle \ldots \rangle_d$ to an integration with respect to the invariant measure of the symmetry transformation $\hat{U}$. The invariant measure is related to those degrees of freedom which leave the symmetry breaking order parameter invariant (i.e., the scattering rate), as discussed in [10]. This is similar to an $O(N)$-symmetric Heisenberg ferromagnet with fixed magnetization [14]. Another case is the random-matrix theory, where the invariant measure plays an essential role for universal properties such as level spacing [12]. For our model the situation is somewhat different from that of the zero-dimensional random-matrix model, though, due to spontaneous symmetry
breaking: the invariant measure of the random-matrix theory is defined by fixing the eigenvalues rather than an order parameter. On the other hand, the order parameter in a d-dimensional system is a spatially fluctuating quantity. Then the determination of the invariant measure requires the approximation by a constant order parameter.

The restriction of the integration with respect to the invariant measure creates a Jacobian, whose leading expansion terms represents the nonlinear sigma model. The latter provides a conventional approach to study diffusion and the breakdown of diffusion due to Anderson localization of quantum particles in a disordered environment [3, 4]. For the symmetry (6), however, this gives only diffusion [11, 13]. Therefore, to observe Anderson localization in this case we must go beyond the nonlinear sigma model by taking into account the full Jacobian.

It should be noticed that the Jacobian factorizes into the ± sectors as ~jj = +−jj, since the matrices are diagonal with respect to the index according to equations (4) and (5). In the following we consider only one factor and drop the index. Then the starting point is the integral with respect to the invariant measure of a non-Abelian chiral symmetry group in Bose–Fermi space of [10], where the integration variable is a (complex) Grassmann field ϕ with its conjugate field ϕ′ [15, 16]:

\[ K_{rr'} = \frac{1}{N} \int \phi_r \phi'_{r'} J D[\phi], \quad J = \det \left( 1 + \phi' \phi - \phi' h \phi \right)^{-1}, \quad N = \int JD[\phi], \]  

(7)

where

\[ h_{rr'} = \sigma_0 \delta_{rr'} + 2i\eta \left( H_0^T - i\eta \right)^{-1} \]  

(8)

with

\[ hh^\dagger = 1 - 4\epsilon (1 - \epsilon) \eta \left( H_0^2 + \eta^2 \right)^{-1}. \]  

(9)

Thus, h is unitary in the limit \( \epsilon \to 0 \). \( \eta \geq 0 \) is the scattering rate, which is also the order parameter for spontaneous breaking of the symmetry (6). In the absence of spontaneous symmetry breaking (i.e. for \( \eta = 0 \)), the Jacobian J becomes 1.

The correlation function \( K_{rr'} \) agrees for large distances \(|\vec{r} - \vec{r}'|\) with the correlation of the Green functions in (1)

\[ K_{rr'} \sim \langle G_{rr'}(E + i\epsilon)G_{r'r}(E - i\epsilon) \rangle_d \]

and can be used to calculate the diffusion coefficient of equation (1). The nonlinear sigma model is obtained from the expansion of the logarithm of the Jacobian log J up to second order in the Green function \( 2i\eta(H_0^2 \pm i\eta)^{-1} \). In the following, though, we will not use this approximation but treat the full Jacobian J of equation (7).

2. Summary of the results

Starting from a previously formulated fermionic functional integral (7) of scattered quantum particles in two-band systems with nodes [10], we derive a transformation from the fermionic (Grassmann) to a bosonic (complex) integral (section 3.1). This is used to express the scattering of the quantum states in terms of entangled random walks in sections 3.2 and 3.3. The transformation provides eventually the expression
\[
K_{rr'} = \langle C_{rr'}^{-1} \rangle \equiv \frac{\langle C_{rr'}^{-1} \det C \rangle}{\det(C)} \quad \text{with} \quad C_{rr'} = 2\delta_{rr'} - \sum_{j,j'} e^{i\alpha_{ij}} h_{jj'} e^{-i\alpha_{jj'}}
\]

where the brackets \( \langle \ldots \rangle \) mean averaging with respect to the angular variables \( \{\alpha_{ij}\} \) of \( C \), as defined in equation (15). The average \( \langle \ldots \rangle \) is the angular average with the additional complex weight \( \det C / \det(C) \). The result of the averaging process has a simple graphical interpretation by two types of random walks which are entangled by four-vertices, as illustrated in figures 1 and 2.

The correlation function is treated in section 3.5, which is based on the expansion in powers of \( C - \langle C \rangle \) around the unperturbed correlation function

\[
g = (1 - P)^{-1}
\]

**Figure 1.** Examples for the graphical representation of the correlation function \( K_{rr'} \) from equation (23): (a) before angular averaging, (b), (c) two special cases after angular averaging. The latter leads to a pairwise connection of the arrows and the circles in (a), which creates four-vertices.

**Figure 2.** Examples for the graphical representation of the correlation function \( K_{rr'} \) with an extra loop from the contribution of the determinant in equation (23): (a) before angular averaging, (b) a linked cluster after angular averaging.
with the transition probability

\[ P_{r,r'} = \delta_{r,r'} - \frac{\langle C_{r'} \rangle}{2} = \sum_{j,j'} h_{j,j'} \] with \( \lim_{\epsilon \to 0} \sum_{r,r'} P_{r,r'} = 1 \)

for a fixed time interval. \( P_{r,r'} \) describes a classical random walk, and from its Fourier transform \( \tilde{P}_{k} \), we get a diffusion pole for \( g \) at \( \epsilon = 0 \) and \( k = 0 \). Thus, the leading term of this expansion is a diffusion process. Including the next order term of the expansion (cf figure 3) there is a complex transition amplitude which reads

\[ \delta P_{r,r'} = \frac{1}{2} \text{Im} \sum_{j,j'} \left( h_{j,j'}^\dagger h_{j,j'} + h_{j,j'} h_{j,j'}^\dagger \right). \]

The imaginary correction indicates localization because the pole of the propagator \( K = (1 - P)^{-1} \) has moved away from the real axis. A special case exists for two-dimensional Dirac fermions, which is studied in section 4 for strong scattering. This requires a shift of the expansion point. In contrast to the diffusive case above, the correlation function (37) has two poles which are separated by a distance proportional to the scattering rate.

### 3. Transformation from fermionic to bosonic integrals

The starting point for our calculation is the Jacobian in equation (7) [11]

\[ J = \det \left( 1 + q' \varphi - q' h q h^\dagger \right)^{-1}, \] (10)

where \( h_{\pm,r,r'} \) is defined in equation (8). The calculation will be more transparent after a transformation from the Grassmann integral to a complex integral. This is based on the fact that the inverse determinant can also be written as an integral

\[ \det \left( 1 + q' \varphi - q' h q h^\dagger \right)^{-1} = \int \exp \left[ -\chi \cdot \left( 1 + q' \varphi - q' h q h^\dagger \right) \chi \right] \prod_{r,j} d\chi_{rj} \] (11)

with respect to a two-component complex (bosonic) field \( \chi_{rj} \) with spinor index \( j = 1, 2 \).
3.1. Integrating out fermions

After replacing the inverse determinant in equation (7) by the right-hand side of (11) and interchanging the bosonic integration with the fermionic integration, we can perform the latter because the fermion field appears only as a quadratic form in the exponent. First, however, we use the identity

$$
\int \prod \phi \chi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \ph
Then $C_{rr'}$ in equation (13) can be written, after replacing $\chi_{rj}$ by the phase factors, as
\[ C_{rr'} = 2\delta_{rr'} - \sum_{j,j'} e^{i\alpha r} h_{rj,rj'} \tilde{\chi}_{rj'}, \quad \tilde{\chi}_{rj'} = \sum_{j,r'} h_{rj',r} e^{-i\alpha_{rj'}} \] (16)
which implies for the correlation function in equation (14) the expression
\[ K_{rr'} = \frac{\partial}{\partial \alpha_{rj'}} \log \langle \det(C + \alpha) \rangle_{\alpha_{rj'}=0} = \frac{\langle C^{-1}_{rr'} \det(C) \rangle_{\alpha_{rj'}=0}}{\langle \det(C) \rangle_{\alpha_{rj'}=0}} \equiv \left\langle C^{-1}_{rr'} \right\rangle. \] (17)
This result can be understood as if we have a random walk with additional random phases $e^{i\alpha_{rj}}$, presented by $C_{rr'}^{-1}$, since $hh' \sim 1$ for $e \sim 0$. The random phases are averaged with respect to the complex weight $\langle \det(C) \rangle_{\alpha_{rj'}=0}$. If we ignore the random phases by setting $\alpha_{rj} = 0$, which corresponds to the saddle-point approximation ($\alpha_{rj} = \text{const.}$ is a saddle point due to the $U(1)$ symmetry), the result is
\[ C_{rr'} = 2\delta_{rr'} - \gamma \sum_{j,j'} h_{rj,rj'}, \quad \gamma = \tilde{h}_{j=0,\tilde{j}} = -1 + 2\epsilon/\tilde{j}, \] (18)
which describes a random walk with transition amplitude $T_{rr'} = (\gamma/2) \sum_{j,j'} h_{rj,rj'}$.

3.3. Linked cluster expansion

To calculate $\langle \det(C + \alpha) \rangle_{\alpha}$ in equation (17) we define
\[ C_{rr'} = 2\delta_{rr'} - C_{rr'} = \sum_{j,j'} e^{i\alpha r} h_{rj,rj'} \tilde{\chi}_{rj'}, \] (19)
and use the property of the determinant
\[ \det(C + \alpha) = \exp \{ \text{Tr} [ \log (C + \alpha) ] \} = 2^N \exp \{ \text{Tr} [ \log (1 + (C' - \alpha)/2) ] \}. \]
The exponent on the right-hand side is expanded as
\[ \det(C + \alpha) = 2^N \exp \left\{ -\sum_{l=1}^{2l} \frac{1}{2l} \text{Tr} \left[ (C' - \alpha)^l \right] \right\}, \] (20)
where the trace term in the exponential function
\[ A = -\sum_{l=1}^{2l} \frac{1}{2l} \text{Tr} \left[ (C' - \alpha)^l \right] = -\sum_{l=1}^{2l} \frac{1}{2l} \sum_{n_1, n_2, \ldots, n_l} (C' - \alpha)_{n_1} \cdots (C' - \alpha)_{n_l} \] can be understood as a summation of clusters: for a given cluster of sites $C_i = \{ n_1, n_2, \ldots, n_l \}$ we define $a_{C_i} = (C' - \alpha)_{n_1} \cdots (C' - \alpha)_{n_l}$, such that the series reads
\[ \sum_{l=1}^{2l} \frac{1}{2l} \text{Tr} \left[ (C' - \alpha)^l \right] = \sum_{l=1}^{2l} \frac{1}{2l} \sum_{C_i} a_{C_i}. \] In the next step we average the determinant with respect to the phase angles and write
\[ \left\langle e^A \right\rangle_{\alpha} = \exp \left( \sum_{m=1}^{2l} \beta_m \right), \] (21)
where \([p_m]\) are linked (or truncated) correlations \([17]\), which can be expressed as
\[
p_m = \frac{1}{m!} \frac{\partial^m}{\partial \lambda^m} \left( \log \left( e^{A \lambda} \right) \right) \bigg|_{\lambda=0} = \frac{1}{m!} \frac{\partial^{m-1}}{\partial \lambda^{m-1}} \left( e^{A \lambda} \right) \bigg|_{\lambda=0}.
\] (22)

The latter is implied by the Taylor expansion of \(f(A)\) around \(A = 0\):
\[
f(A) = \sum_{m \geq 0} \frac{A^m}{m!} \frac{\partial^m}{\partial A^m} f(a) \bigg|_{A=0} = \sum_{m \geq 0} \frac{A^m}{m!} \frac{\partial^m}{\partial (A \lambda)^m} f (\lambda A) \bigg|_{\lambda=0} = \sum_{m \geq 0} \frac{1}{m!} \frac{\partial^m}{\partial \lambda^m} f (\lambda A) \bigg|_{\lambda=0}.
\]

Examples are \(p_0 = \langle \lambda \rangle, p_1 = \langle \lambda \rangle, p_2 = (\langle \lambda^2 \rangle - \langle \lambda \rangle^2)/2\) etc. In simple words, factorizing products \(\langle \lambda \rangle\) do not contribute to \(p_m\).

From equation (17), together with the exponential representation (20) and the linked cluster relation (21), we obtain for the correlation function the linked cluster expansion
\[
K_{rr'} = \left\{ C_{rr'} \right\}^{-1} = \frac{\partial}{\partial \alpha_{r'}} \log \left( e^{A \lambda} \right) \bigg|_{\alpha_{r'}=0} = \sum_{n=1}^{m} \frac{\partial^n}{\partial \alpha_{r'}^n} \bigg|_{\alpha_{r'}=0},
\] (23)

where the right-hand side can be represented graphically by two types of entangled random walks forming loops. This is described in the next section.

3.4. Interpretation and graphical representation

A cluster contribution to the expression (23) describes either a loop connecting the cluster sites \(r, \ldots, \bar{r}\)
\[
a_i \bigg|_{\alpha_{r'}=0} = \frac{1}{2^l} \left[ C_{r1} \cdots C_{r\bar{l}} \right]
\] (24)
or a corresponding open walk which starts at \(r'\) and terminates at \(r\) if \(r_{\bar{n}} = r', r_{n+1} = r\):
\[
\frac{\partial a_i}{\partial \alpha_{r' r}} \bigg|_{\alpha_{r'}=0} = \frac{1}{2^l} \left[ C_{r1} \cdots C_{r\bar{l}} \cdots C_{r_{n\bar{r}}} \right]
\] (25)

where each step between sites \(r\) and \(r'\) contributes the (complex) weight \(C_{rr'}\) of (19). Then a linked cluster of equation (23) is a product of an open walk and \(n (n = 0, 1, \ldots)\) loops, which are connected by the angular integration. Both the loop and the open walk are graphically represented in figures 1(a) and 2(a). The circles represent the phase factors \(e^{i\alpha_i}\) and the arrows represent \(\chi = \sum_{h} \alpha \tilde{e}^h_j j r r\). Now we follow the sites with phase factors. The cancellation of the latter results in a random walk consisting of alternating steps \(h\) and \(h^\dagger\), which is visualized by connecting the arrows with the circles. Then we connect the intermediate sites without a phase factor with sites with a phase factor \(\alpha\) to take care of the site \(r_j\) in \(C_{r_j r} \cdots \alpha_{r_{n\bar{r}} r} \cdots C_{r_{n\bar{r}} r_{n\bar{r}}} \). This procedure connects all circles and arrows pairwise and creates four-vertices that are connected either by \(h\) or by \(h^\dagger\) steps (cf figures 1(b), (c) and 2(b)). The linked cluster property is reflected by the fact that the loops and the open walk are connected (linked) by \(h^\dagger\) steps (cf figure 2(b)). The formation of four-vertices can be seen as a self-crossing which is enforced by the angular integration.

There is an alternative interpretation of the graphs by two types of walks with either \(h\) or with \(h^\dagger\) steps only. This can be seen as that the \(h\) walk from \( \bar{r} \) to \( \bar{r} \) is decorated with the walk consisting of \(h^\dagger\) steps. Thus, the averaging of phase factors leads to a decoration of the walk with \(h\) steps. This decoration renormalizes the diffusion coefficient (i.e., the coefficient of the off-diagonal elements of \(h\)). Here it should be noticed that any additional crossing of these
walks does not contribute an extra weight, implying that the decorated walks are not interacting.

The number of graphs to each order of the four-vertex grows exponentially due to the random-walk nature of the expansion, connecting the two sites of the correlation function. Therefore, the expansion converges for a sufficiently small contribution to each connection between two four-vertices.

3.5. Three-vertex expansion

A shift of the expansion point in equation (20) by \( \langle C' \rangle_a \) means that we take into account the summation over terms from \( \langle C' \rangle_a \) in leading order of the expansion. An typical example of such terms is shown in the graph of figure 1(c). The shift can be achieved by using

\[
\det(C + \alpha) = \frac{2^N}{\det g} \exp \left\{ \text{Tr} \left[ \log \left( 1 + \frac{1}{2} g ( - C' + \langle C' \rangle_a ) \right) g \right] \right\}
\]

with

\[
g = \left( 1 + \frac{1}{2} \alpha - \frac{1}{2} \langle C' \rangle_a \right)^{-1}
\]

and employing the expansion in powers of \( \delta C = C' - \langle C' \rangle_a \)

\[
\text{Tr} \left[ \log \left( 1 - \frac{1}{2} g \delta C \right) \right] = -\sum_{l \geq 1} \frac{1}{2^l} \text{Tr} \left[ (g \delta C)^l \right].
\]

Then the leading order of the correlation function reads

\[
K \approx \left( 1 - P_c \right)^{-1}
\]

with the transition probability for a fixed time interval

\[
P_{c,rr'} = \langle C'_{rr'} \rangle_a / 2 = \sum_{j,j'} \left| h_{r,j} h_{r',j'} \right|^2, \quad \lim_{\epsilon \to 0} \sum_{r'} P_{c,rr'} = 1.
\]

This approximation agrees with the nonlinear sigma model of [10] and describes diffusion. The graphical representation of the expansion to higher orders is characterized by a walk with alternating steps of \( h \) and \( h^\dagger \) and by three-vertices, which are connecting pairwise points of the alternating walk with the diffusion propagators \( g \). An example is depicted in figure 3.

Including this correction yields for the correlation function

\[
K_{r'r} \sim g_{r'r} + \frac{1}{4} \left[ g \left( \delta C g \delta C \right)_a - \left\langle \left( \delta C \text{ Tr} (g \delta C) \right)_a \right\rangle g \right]_{r'r}
\]

\[
\sim \left( g^{-1} - \frac{1}{4} \left( \delta C g \delta C \right)_a - \left\langle \left( \delta C \text{ Tr} (g \delta C) \right)_a \right\rangle \right)^{-1}
\]

if the correction is small. The two terms inside the brackets are those in figure 3. This correction can be understood in terms of the expansion that the transition probability \( P_c \) in equation (28) is replaced by \( P = P_c + i \delta P \) with the correction term (details in the appendix)

\[
\delta P_{r'r} = \frac{1}{2} \text{ Im} \sum_{j,j'} \left( h_{j',j} \bar{h}_{r,j'j} \right) \bar{h}^\dagger_{r,j}. \text{ with } \bar{h}_{r,j'} = g_{r,j}. \]

Thus, we have replaced the classical random walk in equation (29) by a quantum random walk, where the transition probability \( P_c \) has been replaced by a complex transition amplitude \( P = P_c + i \delta P \). Since \( \sum_{r'} \delta P_{r'r} = 0 \), the completeness of the transition probability \( \sum_{r'} P_{c,rr'} = 1 \)
is preserved. However, the pole of the correlation function $K$ with respect to $\epsilon$ is not at $\epsilon = 0$ any more but has moved away from the real axis. It depends on the details of $h$ whether or not localization appears. An example is discussed in the next section.

4. Strong scattering limit for two-dimensional Dirac fermions

Starting from the expression (8), we can estimate the terms of the three-vertex expansion. First, it should be noticed that $\delta C$ is of order $1/\eta$. The properties of the expansion depend on the specific choice of the Hamiltonian $H_0$. In the case of two-dimensional Dirac fermions we have $H_0 = \vec{\sigma} \cdot \vec{k} = k_1 \sigma_1 + k_2 \sigma_2$ with the momentum cut-off $\lambda$: $k \leq \lambda$. This gives $g_k \sim \eta^2/(e/\eta + k^2/2\pi)$, which diverges like $\eta^2$. Thus, in this case the three-vertex expansion cannot be controlled by powers of $1/\eta$. However, this problem is fixed by shifting the expansion point $\langle C' \rangle_a \rightarrow \bar{C}$, where we replace the random phase factors in $\delta C$ of (19) by a constant angle $\alpha$:

$$C_{rr'} = -\sum_{\gamma\delta} h_{\gamma r',\delta r} e^{i(\phi_r - \phi_{r'})}$$

whose Fourier components read

$$\tilde{C}_k = -2 \left( 1 - \frac{2\eta \bar{\eta}}{k^2 + \bar{\eta}^2} + \frac{4i\eta \bar{\eta}}{k^2 + \bar{\eta}^2} \cdot \vec{s} \right)$$

with the two-component unit vector $\vec{s} = (\cos(\phi_1 - \phi_2), \sin(\phi_1 - \phi_2))$. For $\eta \gg \lambda$ we get

$$\tilde{C}_k \sim 2 \left( 1 - \frac{2e}{\eta} - \frac{4i}{\eta} \cdot \vec{s} - \frac{2k^2}{\eta^2} \right).$$

The determinant $\det(1 - \tilde{C}/2)$ does not depend on the global phase difference $\phi_1 - \phi_2$, which reflects the invariance of the angular integration under a global rotation.

A consequence of the shift is that we now have

$$\text{Tr} \left\{ \log \left[ 1 - \frac{1}{2} \gamma (\delta C + \langle C' \rangle_a - \bar{C}) \right] \right\} = -\sum_{i\neq j} \frac{1}{2\eta} \text{Tr} \left\{ \left[ \gamma (\delta C + \langle C' \rangle_a - \bar{C}) \right]^i \right\}$$

instead of the expansion (27) with

$$\gamma = \left( 1 + \frac{1}{2} \alpha - \frac{1}{2} \bar{C} \right)^{-1}. \quad (36)$$

The latter is of order $\eta$, since $\bar{C}$ is of order $\eta^{-1}$.

In comparison with the expansion (27) we have extra terms $\gamma (\langle C' \rangle_a - \bar{C})$ in the expansion (35). These terms can be collected as a geometric series and lead to factors $g$; i.e., they replace $\gamma \rightarrow g$. On the other hand, they must be combined with other terms of the expansion which are of the same order. For instance, the correction of figure 3 has additional terms, which are generated by replacing $\delta C$ with $\langle C' \rangle_a - \bar{C}$.

The leading term of the expansion

$$K_q \approx \frac{1}{1 - \frac{1}{2} \bar{C}_q} \sim \frac{\eta / 2}{e + 2i \bar{q} \cdot \vec{s} + \bar{q}^2 / \eta}$$

is not a diffusion propagator but it has the two separated poles $q = -i\eta \cos \varphi \pm i \sqrt{\eta e + \eta^2 \cos^2 \varphi}$, where $\varphi$ is the angle between $\vec{q}$ and $\vec{s}$. For a given $\vec{q}$
we can always choose \( \vec{s} \) such that \( q = 0 \). The diffusion propagator, on the other hand, has the poles \( q = \pm i \bar{\eta} \epsilon \). This difference in the pole structure is crucial when we apply a Fourier transformation to real space, since the diffusion propagator forces the path of the \( q \)-integration to 0 for \( \epsilon \to 0 \), whereas the propagator (37) allows the path of the \( q \)-integration to move away from the real axis by the distance \( 2\bar{\eta} \). This implies an intrinsic length scale \( \xi \propto \eta^{-1} \) which represents a localization length. Moreover, \( K_0 = \eta/2\epsilon \) diverges with \( \epsilon \sim 0 \), as it is required by the identity

\[
K_0 \approx \sum_r \left( G_{00}(E + i\epsilon)G_{00}(E - i\epsilon) \right)_d = \frac{i}{2\epsilon} \left( G_{00}(E + i\epsilon) - G_{00}(E - i\epsilon) \right)_d.
\]  

\( \text{(38)} \)

\textbf{Appendix. Perturbation theory}

Together with

\[
\left\{ \delta C_{\nu r} \delta C_{\mu \sigma} \right\}_u = \sum_{j_1, \ldots, j_4} h_{\nu j_1, j_2} h_{\mu j_3, j_4} h_{\sigma j_2, j_3} h_{\mu j_1, j_4}
\]

we obtain from (30) the expression

\[
g_{r' r} + \frac{1}{4} \sum_n g_{r' n} \sum_{n, j_1, \ldots, j_4} h_{\nu j_1, j_2} h_{\mu j_3, j_4} h_{\sigma j_2, j_3} h_{\mu j_1, j_4} g_{r n} \]

\[
- \frac{1}{4} \sum_n g_{r' n} \sum_{n, j_1, \ldots, j_4} h_{\nu j_1, j_2} h_{\mu j_3, j_4} h_{\sigma j_2, j_3} h_{\sigma j_1, j_4} g_{r n}
\]

The corresponding graphs are depicted in figure 3. This leads to a correction of the transition probability of the classical walk in equation (29):

\[
\delta P_{r r'} = \frac{-i}{4} \sum_n g_{r' n} \sum_{j_1, \ldots, j_4} h_{\nu j_1, j_2} h_{\mu j_3, j_4} h_{\sigma j_2, j_3} h_{\mu j_1, j_4}
\]

\[- \frac{1}{4} \sum_n g_{r' n} \sum_{j_1, \ldots, j_4} h_{\nu j_1, j_2} h_{\mu j_3, j_4} h_{\sigma j_2, j_3} h_{\sigma j_1, j_4} \]

This means that \( \sum_r P_r = 1 \) for \( \epsilon \to 0 \). With \( \tilde{h}_{r' r} = g_{r' r} h_{r, r'} \) this can also be written as

\[
\delta P_{r r'} = \frac{-i}{4} \sum_{j_1, j_2} \left[ (h h^\dagger)_{\nu j_1, j_2} h_{\mu j_3, j_4} - \left( (h h^\dagger)_{\nu j_1, j_2} h_{\mu j_3, j_4} \right)^\dagger \right].
\]

Inside the square brackets the matrix elements of the two terms are complex conjugate to each other, which implies imaginary matrix elements:

\[
\delta P_{r r'} = \frac{1}{2} \text{Im} \sum_{j_1, j_2} \left( h h^\dagger \right)_{\nu j_1, j_2} h_{\mu j_3, j_4}.
\]

\( \text{(A.1)} \)

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