ENTROPY SOLUTIONS TO THE DIRICHLET PROBLEM FOR NONLINEAR DIFFUSION EQUATIONS WITH CONSERVATIVE NOISE

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Abstract. Motivated by porous medium equations with randomly perturbed velocity field, this paper considers a class of nonlinear degenerate diffusion equations with nonlinear conservative noise in bounded domains. The existence, uniqueness and $L_1$-stability of non-negative entropy solutions under the homogeneous Dirichlet boundary condition are proved. The approach combines Kruzhkov’s doubling variables technique with a revised strong entropy condition that is automatically satisfied by the solutions of approximate equations.

1. Introduction

This paper is concerned with the Dirichlet problem for nonlinear diffusion equations

\begin{equation}
\begin{aligned}
du(t,x) &= \left( \Delta \Phi(u) + \nabla \cdot G(x,u) + F(x,u) \right) dt \\
&\quad + \left( \nabla \cdot \sigma^k(x,u) \right) \circ dW^k(t), \quad (t,x) \in (0,T) \times D; \\
u(0,x) &= \xi(x), \quad x \in D; \\
u(t,x) &= 0, \quad (t,x) \in [0,T] \times \partial D,
\end{aligned}
\end{equation}

where $D$ is a bounded domain in $\mathbb{R}^d$ with smooth boundary, and $\Phi : \mathbb{R} \to \mathbb{R}$ is a monotone function. The noise $\{W^k\}_{k \in \mathbb{N}}$ is a sequence of independent standard Brownian motions defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, and the stochastic integral is in the Stratonovich sense. The Einstein summation convention is used throughout this paper.

A typical example of (1.1) and also our primary motivation of this paper is the porous medium equation in random environment, which describes the flow of an ideal gas in a homogeneous porous medium: let $u$ be the gas density that satisfies

\begin{equation}
\epsilon \partial_t u + \nabla \cdot (uV) = 0,
\end{equation}

where $\epsilon \in (0,1)$ is the porosity of the medium, and $V$ is the randomly perturbed velocity field of the form

\begin{equation}
V = V_0 + \sigma^k \circ \dot{W}^k,
\end{equation}

where $V_0$ is derived from Darcy’s law

\begin{equation}
\mu V_0 = -k \nabla p = -k \nabla (p_0 u^{m-1}).
\end{equation}

Then, we can informally derive a stochastic porous medium equation

\begin{equation}
\begin{aligned}
du(t,x) &= c \Delta u^m dt + \epsilon^{-1} \nabla \cdot (u \sigma^k) \circ dW^k(t).
\end{aligned}
\end{equation}

For more details of the model, we refer to [V07] and references therein. Moreover, equations of type (1.1) also arise as limits of interacting particle systems driven by common noise from mean field models [LL06a, LL06b, LL07], and as simplified models of fluctuations in non-equilibrium

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statistical physics [DSZ16]; for more applications we refer to [FG19, DG20] and the references therein.

The well-posedness of the Cauchy problem for stochastic nonlinear diffusion equations with general noise has been investigated in various frameworks, for example, the variational approach in the space $H^{-1}$ (cf. [BDPR08, BDPR16, Cio20] etc.), the kinetic formulation (cf. [DHV16, GH18, FG21a] etc.), and the entropy formulation (cf. [BVW15, DGG19, DGT20] etc.). The case of linear gradient noise, say $\sigma(x,u) = h(x)u$ in (1.1), has been studied in [DG19, T20, Cio20], and the general case was discussed in [FG19] by a kinetic approach with rough path techniques (cf. [LPS13, LPS14, GS15, GS17a, GS17b]).

Stochastic nonlinear diffusion equations in a torus are considered in [LPT21]. The well-posedness of the Cauchy problem for stochastic nonlinear diffusion equations with general noise has been investigated in various frameworks, for example, the variational approach from [RRW07] with the monotone condition and affine noise, thus not covering our setting; [DGT20] considered entropy solutions of stochastic porous medium equations with the noise term $\sigma(x,u)dw$ and Lipschitz functions $\Phi$. Stochastic nonlinear diffusion equations in a torus are considered in [DCG19] for the noise term $\sigma(x,u)dw$ and in [DG20] for the conservative noise $\nabla \cdot \sigma(x,u)\circ dw$. The obstacle problem with the noise term $\sigma(x,u)dw$ in a torus is introduced in [LT21].

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The results on the Dirichlet problem for stochastic nonlinear diffusion equations are relatively few. The recent papers [BGV20, Hen21] adopted the variational approach from [RRW07] with the monotone condition and affine noise, thus not covering our setting; [DGT20] considered entropy solutions of stochastic porous medium equations with the noise term $\sigma(x,u)dw$. A recent paper [Cio23] is quite relevant to our paper, studying kinetic solutions to the Dirichlet problem for porous medium equations with nonlinear gradient noise driven by rough path. Thanks to the entropy approach, our result is built on the same regularity conditions with the existing work on the Cauchy problem in torus (cf. [DG20]), and do not require extra technical assumptions like [Cio23, Condition (2.4)] that prevents the space characteristics from escaping the domain. According to the recent work [FG21b], it is a very interesting question how to relax the regularity condition on $\gamma$ for the Dirichlet problem.

The strategy of the proof of our main result (Theorem 2.6) basically follows from [DCG19, DGT20, DG20], combining the method of strong entropy condition (called the $(\ast)$-property in this paper) and Kružkov’s doubling variables technique (cf. [Kru70]). The notion of strong entropy condition was introduced by [FN08] to tackle the uniqueness issue of stochastic scalar conservation laws. The strategy can be summarized to two steps:

1. $L_1$-estimates: to derive an estimate for $\mathbb{E}[\|u(t,\cdot) - \tilde{u}(t,\cdot)\|_{L_1(D)}^2]$, providing one of the entropy solutions $u$ and $\tilde{u}$ (with different initial data) satisfies the $(\ast)$-property.

2. Approximation: to construct a sequence of non-degenerate equations whose solutions have the $(\ast)$-property and converge to the entropy solution of (1.1).

Kružkov’s doubling variables technique plays a key role in the proof of the $L_1$-estimates.
The new difficulty arising in the Dirichlet problem, comparing to the Cauchy problem (cf. [DG20]), is how to deal with the boundary integral terms that may emerge when applying the divergence theorem in the proof of the $L_1$-estimates. In the paper [DG20] where the noise term is $\sigma(x,u)dw$, a weighted space with a weight function $w \in H^1_0$ satisfying $\Delta w = -1$ is introduced to eliminate the boundary terms. However, in the case of gradient noise, there will be a new “trouble” term $|(u - \bar{u})\nabla w|$ appearing in the estimate, which cannot be dominated by any “good” terms like $|u - \bar{u}|w$.

There are three key points in our approach to overcome the above difficulty: i) to expand the set of test functions in the definition of entropy solution (see Definition [2.2]), ii) to refine the strong entropy condition, and iii) to construct subtly a pair of the convex function $\eta$ and the test function $\phi$ to avoid the appearance of boundary terms. Specifically, when applying Kruzhkov’s doubling variables technique to our problem, we have to estimate both $(\tilde{u}(s,x) - u(t,y))^+$ and $(u(t,x) - \tilde{u}(s,y))^+$ rather than $|u(t,x) - \tilde{u}(s,y)|$ as in [DGG19], [DG20], [DG20]. Inspired by [Car99], we make use of a partition of unity and shifted mollifiers to keep the test function in $C_C^\infty(D)$ with respect to the variable $y$; and for the variable $x$, we choose a sequence of smooth convex functions to approach $\eta(r) = r^\gamma$. Those carefully chosen functions along our modified definition of entropy solution and the refined strong entropy condition let the boundary terms vanish. More specific details are given at the beginning of Section 3 and Remarks 3.5, 3.12 and 3.15. It is worth noting that our approach avoids the involvement of weight functions and the boundary terms vanish.

This paper is organized as follows. Section 2 describes the entropy formulation and presents the main theorem. Section 3, which is the main part of this paper, introduces a refined strong entropy condition and the approximate equations and proves the strong entropy condition of their solutions as well as their solvability. Section 4 constructs the approximate equations and proves the strong entropy condition of their solutions as well as their solvability. Section 5 completes the proof of the main theorem. Two auxiliary lemmas are proved in the final section.

We conclude the introduction with some notation. Fix $T > 0$. Define $\Omega_T := \Omega \times [0,T]$ and $D_T := [0,T] \times D$. Define $|D|$ and $|D|$ as the volume and closure of $D$, respectively. $L_p$ and $H^k_p$ are the usual Lebesgue spaces and Sobolev spaces. Denote by $H^k_{0,0}$ the closure of $C_C^\infty$ in $H^k_p$. When $p = 2$, we simplify the notation by $H^k := H^k_2$ and $H^k_{0,0} := H^k_{2,0}$. Moreover, if a function space is given on $\Omega$ or $\Omega_T$, we understand it to be defined with respect to $\mathcal{F}_T$ and the predictable $\sigma$-field, respectively. Let $E$ be a Banach space and $U = D$ or $\mathbb{R}$. For a function $f : U \to E$, we define

$$[f]_{C^\alpha(U,E)} := \sup_{x,y \in U, x \neq y} \frac{||f(x) - f(y)||_E}{|x - y|^\alpha}, \quad \alpha \in (0,1],$$

$$||f||_{C^\alpha(U,E)} := [f]_{C^\alpha(U,E)} + \sup_{x \in U} ||f(x)||_E.$$

We define a non-negative smooth mollifier $\rho : \mathbb{R} \to \mathbb{R}$, such that $\text{supp} \rho \subset (0,1)$, $\rho \leq 2$ and $\int_\mathbb{R} \rho(r)dr = 1$. For $\delta > 0$, we set $\rho_\delta(r) := \delta^{-1}\rho(\delta^{-1}r)$ as a sequence of mollifiers.

2. Entropy formulation and main results

First of all, we rewrite (1.1) into an Itô form (the notation follows from [DG20]):

$$du(t,x) = \left[\Delta \Phi(u) + \partial_x \left(a^j(x,u)\partial_x u + b^i(x,u) + f^i(x,u)\right) + F(x,u)\right]dt + (\nabla \cdot \sigma^k(x,u))dw^k(t), \quad (t,x) \in (0,T) \times D;$$

$$u(0,x) = \xi(x), \quad x \in D;$$

$$u(t,x) = 0, \quad (t,x) \in [0,T] \times \partial D,$$

(2.1)
where }i, j = 1, \ldots, d\text{ and }
\begin{align*}
a^{ij}(x, r) &= \frac{1}{2} \sigma_{ij}^{xx}(x, r) \sigma_{rr}^{xx}(x, r), \\
b^{i}(x, r) &= \sigma_{rr}^{xx}(x, r) \sigma_{rr}^{xx}(x, r), \\
f^{i}(x, r) &= G^{i}(x, r) - \frac{1}{2} b^{i}(x, r).
\end{align*}

We denote by }Π(Φ, ξ)\text{ the Dirichlet problem (2.1) with given }Φ\text{ and }ξ.

Throughout this paper, we denote
\begin{align*}
a(r) := \sqrt{Φ(r)};
\end{align*}
for a function }g : D \times \mathbb{R} \to \mathbb{R}\text{, we use the notation
\begin{align*}
\|g\|(x, r) := \int_{0}^{r} g(x, s)ds,
\end{align*}
and drop }x\text{ in the above if }g\text{ does not depend on }x \in D.\text{ Our condition on the nonlinearity }Φ\text{ is the same with [DGG19, DGT20, DG20].}

**Assumption 2.1.** }Φ : \mathbb{R} \to \mathbb{R}\text{ is differentiable, strictly increasing and satisfying }Φ(0) = 0. With }a(r) = \sqrt{Φ(r)},\text{ there exist constants }m > 1\text{ and }K > 0\text{ such that
\begin{align}
|a(0)| &\leq K, |a'(r)| \leq K|r|^{-1}1_{r \neq 0}, a(r) \geq K^{-1}1_{|r| \geq 1}, \\
|a(r) - a(s)| &\geq \begin{cases} 
K^{-1}|r - s|, & \text{if } |r| \lor |s| \geq 1, \\
K^{-1}|r - s|^{\frac{m+1}{m}}, & \text{if } |r| \lor |s| < 1.
\end{cases}
\end{align}

The following definition of entropy solution is based on the formulation in [DGT20, DG20] with a slight but significant modification inspired by [BVW14, Definition 1]. Define two sets
\begin{align*}
\mathcal{E} := \{ η \in C^{2}(\mathbb{R}) : η'' \geq 0, \text{ supp } η'' \text{ is compact} \}, \\
\mathcal{E}_0 := \{ η \in \mathcal{E} : η'(0) = 0 \}.
\end{align*}

**Definition 2.2.** An entropy solution of (2.1) is a predictable stochastic process }u : Ω_T \to L_1(D)\text{ such that
\begin{enumerate}
\item }u \in L_{m+1}(Ω_T; L_{m+1}(D));
\item For all }f \in C_b(\mathbb{R}),\text{ we have }\|a f\|(u) \in L_2(Ω_T; H_0^1(D))\text{ and }
\begin{align*}
\partial_x[a f](u) &= f(u) \partial_x[a](u);
\end{align*}
\item For all
\begin{align}
(η, φ, ϒ) \in (\mathcal{E} \times C_c^{\infty}[0, T) \times C_c^{\infty}(D)) \cup (\mathcal{E}_0 \times C_c^{\infty}[0, T) \times C^{\infty}(\overline{D}))
\end{align}
such that }φ := ϒ \times φ \geq 0,\text{ we have almost surely
\begin{align}
&-\int_{0}^{T} \int_{D} η(u) \partial_x φ dx dt \\
&\leq \int_{D} η(ξ)φ(0)dx + \int_{0}^{T} \int_{D} (\|a^{2}\|^2(u)\Delta φ + η''(u)\phi_{x,x})(x, u)dx dt \\
&+ \int_{0}^{T} \int_{D} (a^{ij}(x, u)\eta' - f^{i}(x, u))\phi_{x,i} dx dt \\
&+ \int_{0}^{T} \int_{D} (η'f^{x,i}(x, u) - \|f^{x,i}\|^2(x, u) + η''(u)F(x, u))\phi dx dt \\
&+ \int_{0}^{T} \int_{D} (\frac{1}{2} η''(u)\sum_{k=1}^{∞} |a^{ik}(x, u)|^2 - η''(u)|\nabla[a](u)|^2)\phi dx dt
\end{align}
\end{enumerate}
Proposition 2.7. 

\[ + \int_0^T \int_D \left( \eta'(u) \phi \sigma_{ik}^j(x, u) - \|\sigma_{ik}^j\|_{L^\infty(D)} \right) dx \, dW^k(t). \]

Remark 2.3. Comparing with [DG120] Definition 2.4, we expand the set of test functions \( (\eta, \varphi, \tilde{\varphi}) \) in order to give a more precise characterization of the behavior of solutions near the boundary. This is a critical point in our proof of the \( L^1 \)-estimates. Moreover, the Dirichlet boundary condition is satisfied implicitly according to Definition 2.2 (ii) and (iii).

The regularity assumption on coefficients coincides with [DG20] Assumption 2.3 for the Cauchy problem in a torus.

Assumption 2.4. Let \( G^i : D \times \mathbb{R} \to \mathbb{R} \) and \( \sigma^i : D \times \mathbb{R} \to l^2 \) for \( i \in \{1, \ldots, d\} \) and \( F : D \times \mathbb{R} \to \mathbb{R} \) are all continuous. For all \( i, l \in \{1, \ldots, d\} \), \( q \in \{1, 2\} \) and all multi-indices \( \gamma \in \mathbb{N}^d \) with \( q + |\gamma| \leq 3 \), the derivatives \( \partial_\gamma G^i, \partial_{x_i} G^i, \partial_{x_j} G^i \) and \( \partial_\gamma \partial_{x_i} \sigma^i \) exist and are continuous on \( D \times \mathbb{R} \). Moreover, there exist \( \kappa \in ((m + 2)^{-1}, 1) \), \( \beta \in ((2k)^{-1}, 1) \), \( N_0 > 0 \) and \( \beta \in (0, 1) \) such that for all \( i, j, l \in \{1, \ldots, d\} \) and \( r \in \mathbb{R} \), we have:

\[
\sup_{x \in D} \left| \partial_\gamma \sigma_{ik}^j(x, r) \right|_{C^\infty(\mathbb{R}^N)} + \left| \sigma_{ik}^j(x, r) \right|_{C^\infty(\mathbb{R}^N)} \leq N_0,
\]

\[
\sup_{x \in D} \left( \left| \partial_{x_i} \sigma_{ik}^j(x, r) \right|_{C^\infty(\mathbb{R}^N)} + \left| \sigma_{ik}^j(x, r) \right|_{C^\infty(\mathbb{R}^N)} \right) \leq N_0,
\]

\[
\sup_{x \in D} \left( \left| G^i_{x_i}(x, r) \right|_{C^\infty(\mathbb{R}^N)} + \left| \partial_{x_i} \sigma_{ik}^j(x, r) \right|_{C^\infty(\mathbb{R}^N)} \right) \leq N_0(1 + |r|),
\]

\[
\sup_{x \in D} \left( \left| \partial_{x_i} \partial_r \sigma_{ik}^j(x, r) \right|_{C^\infty(\mathbb{R}^N)} + \left| \partial_r \sigma_{ik}^j(x, r) \right|_{C^\infty(\mathbb{R}^N)} \right) \leq N_0(1 + |r|),
\]

\[
\sup_{x \in D} \left( \left| F(x, \cdot) \right|_{C^\infty(\mathbb{R}^N)} + \left| \partial_r F(x, \cdot) \right|_{C^\infty(\mathbb{R}^N)} \right) \leq N_0.
\]

Due to the physical background, we focus on non-negative solutions to the concerned problem, for which we need the following natural condition.

Assumption 2.5. The coefficients satisfy

\[
\nabla_x \cdot G(x, 0) + F(x, 0) \geq 0, \quad \nabla_x \cdot \sigma(x, 0) = \{0\}_{k=1}^\infty, \quad x \in D.
\]

The main result of this paper is stated as follows.

Theorem 2.6. Under Assumptions 2.4 and 2.5, for all non-negative initial function \( \xi \in L_{m+1}(\Omega, F_0; L_{m+1}(D)) \), we have that

(i) the problem \( \Pi(\phi, \xi) \) has a unique entropy solution \( u \); 
(ii) \( u \geq 0 \) for almost all \( (\omega, t, x) \in \Omega \times D \); 
(iii) if \( \tilde{u} \) is the entropy solution to \( \Pi(\phi, \tilde{\xi}) \) with \( 0 \leq \tilde{\xi} \in L_{m+1}(\Omega, F_0; L_{m+1}(D)) \), then

\[
es \sup_{t \in [0, T]} \mathbb{E} \| (u(t, \cdot) - \tilde{u}(t, \cdot))^+ \|_{L_1(D)} \leq C \mathbb{E} \| (\xi - \tilde{\xi})^+ \|_{L_1(D)},
\]

where the constant \( C \) depends only on \( N_0, K, d, T \) and \( |D| \).

We conclude this section by proving the non-negativity of entropy solutions under our assumptions.

Proposition 2.7. Under the condition of Theorem 2.6, each entropy solution to \( \Pi(\phi, \xi) \) with \( 0 \leq \xi \in L_{m+1}(\Omega, F_0; L_{m+1}(D)) \) is non-negative for almost all \( (\omega, t, x) \in \Omega \times D \).
Proof. For a small \( \delta > 0 \), we introduce a function \( \eta_\delta \in C^2(\mathbb{R}) \) defined by
\[
\eta_\delta(0) = \eta_\delta'(0) = 0, \quad \eta_\delta''(r) = \rho_\delta(r).
\]
Applying the entropy inequality \((2.3)\) with \( \eta(\cdot) = \eta_\delta(\cdot) \) and \( \phi \) independent of \( x \), using the non-negativity of \( \xi \), we have
\[
(2.13) \quad -E \int_0^T \int_D \eta_\delta(-u) \partial_t \phi \, dx \, dt \\
\leq E \int_0^T \int_D \left( \left[ f^\delta_r, \eta_\delta'(\cdot) \right](x,u) - \eta_\delta''(u) \left( f^\delta_r(x,u) - G^\delta_{x^i}(x,0) \right) \right) \phi \, dx \, dt \\
- E \int_0^T \int_D \eta_\delta(-u) \left( F(x,u) + G^\delta_{x^i}(x,0) \right) \phi \, dx \, dt \\
+ E \int_0^T \int_D \left( \frac{1}{2} \eta_\delta''(-u) \sum_{k=1}^\infty \left| \sigma^k_{x^i}(x,u) \right|^2 \phi - \eta_\delta''(-u) \left| \nabla \|u\| \right|^2 \phi \right) \, dx \, dt.
\]
With Assumption \((2.5)\) and \((2.10)\) in Assumption \((2.4)\) we have
\[
(2.14) \quad \sup_x \left( F(x,r) + G^\delta_{x^i}(x,0) \right) \geq \sup_x \left( F(x,0) + G^\delta_{x^i}(x,0) \right) - C|r| \geq -C|r|
\]
for a positive constant \( C \). Moreover, using Assumption \((2.5)\) the definition of \( f^i \) and \((2.5)\) and \((2.9)\) in Assumption \((2.4)\) we have
\[
(2.15) \quad \sup_x |\sigma^k_{x^i}(x,r)|_{L^2} + \sup_x |f^i_{x^i}(x,r) - G^\delta_{x^i}(x,0)| \leq C|r|.
\]
Combining \((2.13)-(2.15)\) with \((2.9)\) in Assumption \((2.4)\) we have
\[
- E \int_0^T \int_D \eta_\delta(-u) \partial_t \phi \, dx \, dt \leq CE \int_0^T \int_D (-u)^+ \phi \, dx \, dt + C\delta.
\]
Since \( |\eta_\delta(r)-r^-| \leq \delta \), taking \( \delta \to 0^+ \), we have
\[
(2.16) \quad -E \int_0^T \int_D (-u)^+ \partial_t \phi \, dx \, dt \leq CE \int_0^T \int_D (-u)^+ \phi \, dx \, dt.
\]
Let \( 0 < s < T < T \) be Lebesgue points of the function
\[
t \mapsto E \int_D (-u(t,x))^+ dx.
\]
Fix a constant \( \gamma \in (0,(T-s) \vee (T-t)) \). We choose a sequence of functions \( \{\phi_n\}_{n \in \mathbb{N}} \) satisfying \( \phi_n \in C^\infty_c((0,T)) \) and \( \|\phi_n\|_{L^\infty(0,T)} \leq 1 \), such that
\[
\lim_{n \to \infty} \|\phi_n - V(\gamma)\|_{H^1_T} = 0,
\]
where \( V(\gamma) : [0,T] \to \mathbb{R} \) satisfies \( V(\gamma)(0) = 0 \) and \( V(\gamma) = \gamma^{-1} 1_{[s,s+\gamma]} - \gamma^{-1} 1_{[\tau,\tau+\gamma]} \). Taking \( \phi = \phi_n \) in \((2.16)\) and passing to the limit \( n \to \infty \), we have
\[
\frac{1}{\gamma} E \int_0^{T+\gamma} \int_D (-u)^+ \, dx \, dt \leq CE \int_0^{T+\gamma} \int_D (-u)^+ \, dx \, dt + \gamma E \int_s^{s+\gamma} \int_D (-u)^+ \, dx \, dt.
\]
Let \( \gamma \to 0^+ \), we have
\[
E \int_0^{T} (-u(\tau,x))^+ dx \leq CE \int_0^{T} \int_D (-u)^+ dx \, dt + E \int_D (-u(s,x))^+ dx.
\]
holds for almost all $s \in (0, \tau)$. Then, for each $\tilde{\tau} \in (0, \tau)$, by averaging over $s \in (0, \tilde{\tau})$, we have
\[
E \int_D (-u(t, x))^+ \, dx \leq C E \int_0^\tau \int_D (-u)^+ \, dx \, dt + \frac{1}{\tilde{\tau}} E \int_0^\tilde{\tau} \int_D (-u)^+ \, dx \, ds.
\]
Taking the limit $\tilde{\tau} \to 0^+$ and using Lemma 6.1 the non-negativity of $\xi$ and Gronwall’s inequality, we have $u \geq 0$ for almost all $(\omega, t, x) \in \Omega_T \times D$. \hfill $\square$

### 3. Strong entropy condition and $L_1$-estimates

The uniqueness for entropy solutions of stochastic partial differential equations is usually a challenging problem. The seminal work [FNO08] introduced a notion of strong entropy condition, called the $(\ast)$-property in what follows, to deal with the stochastic integral in the $L_1$-estimates and proved the uniqueness of the strong entropy solution (namely, an entropy solution that satisfies the $(\ast)$-property) for the Cauchy problem of stochastic scalar conservation laws. Recently, a series of papers [DGG19, DGT20, DG20] improved this technique and proved the uniqueness of entropy solutions to stochastic porous medium equations. The basic idea is to estimate the $L_1$-difference between an entropy solution and a strong entropy solution, which leads to the uniqueness of entropy solutions by proving that the entropy solutions constructed from approximation always satisfy the $(\ast)$-property. The key technique in the proof of the $L_1$-estimates is Kruzhkov’s doubling variables method [Kru70], which is a classical method to study the uniqueness problem for deterministic conservation laws.

From the Cauchy problem to the Dirichlet problem, the new technical difficulties mainly lie in how to handle the boundary terms. Considering the noise form $\sigma(x, u) \, dW$, the paper [DGT20] introduced a weighted space to overcome the difficulties, where the weight function $w \in H^1_0$ satisfying $\Delta w = -1$ is used to remove the boundary terms in the estimate. However, in the case of \([1.1]\), there will be a new “trouble” term $|(u - \bar{u}) \nabla w|$ appearing in the $L_1$-estimates, which cannot be dominated by any “good” terms like $|u - \bar{u}|w$.

In order to deal with the boundary terms in our case, we do not introduce the weight function but modify the definition of the $(\ast)$-property by subtly choosing the test functions. Since the modified definition is highly intricate, we give a heuristic explanation before the exact formulation.

We begin by addressing the problem of obtaining the $L_1$-difference between entropy solutions $u$ and $\bar{u}$ of \([1.1]\). Using Kruzhkov’s doubling variables technique (see, for example, [Kru70, LWT2, DGG19, DGT20, DG20]), we wish to estimate the term
\[
E \int_{D_T} \int_{D_T} |u(t, x) - \bar{u}(s, y)| \rho_0(s - t) \phi_\varepsilon(x, y) \, dx \, dt \, dy \, ds,
\]
where $\rho_0(\cdot) := \theta^{-1} \rho(\theta^{-1} \cdot)$ defined before Section 2 is a time mollifier, and $\phi_\varepsilon \in C^\infty(\overline{D} \times \overline{D})$ is a spatial mollifier which satisfies $\lim_{\varepsilon \to 0^+} \phi_\varepsilon(x, y) = \delta_0(x - y)$ for all $(x, y) \in \overline{D} \times \overline{D}$. When $\theta, \varepsilon \to 0^+$, we have the estimate for $E \int_{D_T} |u(t, x) - \bar{u}(t, x)| \, dx \, dt$.

How to select a suitable spatial mollifier is important but quite subtle: this is standard when $x$ is an interior point (see, e.g., [DGT20 proof of Proposition 4.2]), but much more complicated when $x$ is on the boundary. Following the idea from [Car99, BVW14], we introduce a partition of unity $\sum_{i=1}^N \psi_i \equiv 1$ on $\overline{D}$ as the set of localization functions, and then choose corresponding mollifiers $\varphi_{\varepsilon, i} \in C^\infty(\mathbb{R}^d)$ such that $supp \varphi_{\varepsilon, i}(x - \cdot) \subset D$ for all $x \in supp \psi_i$ and sufficiently small $\varepsilon$. Then, the spatial mollifier $\varphi_\varepsilon(x, y) := \sum_{i=1}^N \psi_i(x) \varphi_{\varepsilon, i}(x - y)$ satisfies
\[
\lim_{\varepsilon \to 0^+} \varphi_\varepsilon(x, y) = \lim_{\varepsilon \to 0^+} \sum_{i=1}^N \psi_i(x) \varphi_{\varepsilon, i}(x - y) = \delta_0(x - y), \quad \forall x, y \in \overline{D}.
\]
It is worth noting that this mollifier is asymmetric in the spatial variables, specifically, for all sufficiently small $\varepsilon$, 

$$\varphi_\varepsilon(x, \cdot) \in C^\infty_c(\mathbb{D}), \quad \forall x \in \overline{\mathbb{D}}, \quad \text{but} \quad \varphi_\varepsilon(\cdot, y) \in C^\infty(\mathbb{D}), \quad \forall y \in \overline{\mathbb{D}}.$$  

Consequently, this asymmetry makes it difficult to estimate (5.1). Instead, we respectively estimate both 

$$\mathbb{E} \int_{D^T} \int_{D^T} (u(t, x) - \bar{u}(s, y))^+ \rho_\theta(s - t) \varphi_\varepsilon(x, y) dt dx dy, \quad \text{and} \quad \mathbb{E} \int_{D^T} \int_{D^T} (\bar{u}(t, x) - u(s, y))^+ \rho_\theta(s - t) \varphi_\varepsilon(x, y) dt dx dy.$$

Actually, when directly applying the methods in \textbf{DGG19, DGT20, DG20}, the estimate of (5.1) contains the following terms 

$$\mathbb{E} \int_0^T \int_D \int_D \partial_{x, x} \varphi_\varepsilon(x, y) \int_0^{u(t, x)} a^i_{ij}(x, r) \text{sgn}(r - \bar{u}(t, y)) dr dx dy dt$$

$$+ \mathbb{E} \int_0^T \int_D \int_D \partial_x \varphi_\varepsilon(x, y) \int_0^{u(t, x)} a^i_{ij}(x, r) \text{sgn}(r - \bar{u}(t, y)) dr dx dy dt,$$

which may not be equal to 

$$\mathbb{E} \int_0^T \int_D \int_D \partial_{x, x} \varphi_\varepsilon(x, y) \int_{\bar{u}(t, y)}^{u(t, x)} a^i_{ij}(x, r) \text{sgn}(r - \bar{u}(t, y)) dr dx dy dt$$

$$+ \mathbb{E} \int_0^T \int_D \int_D \partial_x \varphi_\varepsilon(x, y) \int_{\bar{u}(t, y)}^{u(t, x)} a^i_{ij}(x, r) \text{sgn}(r - \bar{u}(t, y)) dr dx dy dt.$$

The reason for this discrepancy is that when formally applying the divergence theorem in $x$ to the first term in (3.3), the boundary term does not vanish due to the asymmetry of the spatial mollifier, and estimating it when $\varepsilon \to 0^+$ is challenging due to the potential absence of a trace for $u$ caused by its low regularity. While for the case of (3.2), this equivalence is guaranteed by the support of $\eta(\cdot) = (\cdot - \bar{u}(s, y))^+$.

Note that the time mollifier is also asymmetric and always requires $s > t$. For this, we have an important observation: we only need the (s)-property of the function at the larger time variable $s$. However, when applying this observation to (3.2), it necessitates the (s)-property of both $u$ and $\bar{u}$, which is not what we desire. Therefore, we turn to estimate 

$$\mathbb{E} \int_{D^T} \int_{D^T} (u(t, x) - \bar{u}(s, y))^+ \rho_\theta(s - t) \varphi_\varepsilon(x, y) dt dx dy,$$

$$\mathbb{E} \int_{D^T} \int_{D^T} (\bar{u}(t, x) - u(s, y))^+ \rho_\theta(s - t) \varphi_\varepsilon(x, y) dt dx dy,$$

which only use $\bar{u}$ at time $s$ and thus require the (s)-property of $\bar{u}$. Since these two terms are asymmetric, we adapt the definition of the (s)-property for both formulations in (3.3) (see the definition of the test functions at the beginning of the Section 3.1). Fortunately, the entropy solutions constructed from the vanishing viscosity approximation satisfy this modified (s)-property, given a stronger integrability condition on the initial data in $\omega$.

**Construction of the spatial mollifier.** Now, we give the specific construction of the spatial mollifier. Define 

$$\text{dist}(A_1, A_2) := \inf_{x_1 \in A_1, x_2 \in A_2} |x_1 - x_2|, \quad A_1, A_2 \subset \mathbb{R}^d.$$
Fix an open covering of $\partial D$ by balls $\{B'_j\}_{j=1}^{N'}$ which satisfy that $B'_j \cap \partial D$ is part of a Lipschitz graph for each $j = 1, \ldots, N'$. Choose an open covering of $\overline{D}$ by $B_0$ and balls $\{B_i\}_{i=1}^{N}$ satisfying $\text{dist}(B_0, \partial D) > 0$, and for each $i > 0$, there exists $j \in \{1, \ldots, N'\}$ such that $B_i \subset B'_j$ and $\text{dist}(B_i, \partial B'_j) > 0$.

From [Bre11] Lemma 9.3, we know that there exist functions $\psi_0, \psi_1, \ldots, \psi_N \in C^\infty(\mathbb{R}^d)$ such that $0 \leq \psi_i \leq 1$, supp $\psi_i \subset B_i$ for $i = 0, 1, \ldots, N$, and

$$\sum_{i=0}^{N} \psi_i(x) \equiv 1, \quad x \in \overline{D}.$$  

Without loss of generality, we assume $\text{dist}(\text{supp} \psi_i, \partial B_i) > 0$ for all $i = 0, 1, \ldots, N$. Otherwise, we consider larger open domains, also denoted as $B_i$, for convenience, satisfying $\text{dist}(B_0, \partial D) > 0$ and $\text{dist}(B_i, \partial B'_i) > 0$ for at least one $j \in \{1, \ldots, N'\}$.

For $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, define the function $\tilde{g}(x) := \prod_{i=1}^{d} \rho(x_i - 1/2)$ and the mollifier $\tilde{g}_\varepsilon(\cdot) := \tilde{g}(\cdot/\varepsilon)/(\varepsilon^d)$. Similar to [BVW14] Section 3.2.1, for each $i = 1, \ldots, N$, there exist a constant $\varepsilon_i$ and a vector $\eta_i \in \mathbb{R}^d$, such that the translated sequence of mollifiers $g_{\varepsilon,i}(\cdot) := g_{\varepsilon,i}(-\eta_i)$ satisfying $y \mapsto g_{\varepsilon,i}(x - y) \in C^\infty_c(D)$ for all $(x, \varepsilon) \in (B_i \cap \overline{D}) \times (0, \varepsilon_i)$. The vector $\eta_i$ depends only on the local representation of the boundary of $D$ in $B'_i$ as the graph of a Lipschitz function. We also define $g_{\varepsilon,0}(\cdot) := \tilde{g}_\varepsilon(\cdot)$.

Remark 3.1. From the construction of the function $g_{\varepsilon,i}$, there exists a constant $\tilde{K}$ depending on the maximum norm of $\eta_i$, $i = 1, \ldots, N$, such that supp $g_{\varepsilon,i} \subset \{x \in \mathbb{R}^d : |x| < \tilde{K}\varepsilon\}$ holds for all $\varepsilon \in (0, \varepsilon_0)$ and $i = 0, 1, \ldots, N$.

Remark 3.2. For each $i = 1, \ldots, N$, since $\text{dist}(\text{supp} \psi_i, \partial B_i) > 0$, there exists a $\varepsilon_i \in (0, \varepsilon_0)$ such that supp $g_{\varepsilon,i}(x - \cdot) \subset B_i \cap D$ for all $(x, \varepsilon) \in (\text{supp} \psi_i \cap \overline{D}) \times (0, \varepsilon_i)$. Moreover, there exists a $\varepsilon_0 \in (0, 1)$ such that supp $g_{\varepsilon,0}(x - \cdot) \subset D$ for all $(x, \varepsilon) \in B_0 \times (0, \varepsilon_0)$. In the rest of this article, we define $\varepsilon := \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N, \varepsilon_N\}$.

3.1. $(\ast)$-property. For $i \in \{1, \ldots, N\}$, define the sets

$$C^- := \{f \in C^\infty_c(\mathbb{R}) : f' \in C_c(\mathbb{R}), \text{ supp } f \subset (-\infty, 0]\},$$

$$C^+ := \{f \in C^\infty_c(\mathbb{R}) : f' \in C_c(\mathbb{R}), \text{ supp } f \subset (0, \infty]\},$$

$$\Gamma_{B_i} := \{f \in C^\infty_c(D \times \overline{D}) : \text{ supp } f \subset (B_i \cap \overline{D}) \times (B_i \cap \overline{D}),$$

$$\Gamma'_{B_i} := \{f \in C^\infty_c(D \times \overline{D}) : \text{ supp } f \subset (B_i \cap \overline{D}) \times (\partial B_i \cap \overline{D}),$$

$$f(x, \cdot) \in C^\infty_c(D), \quad \forall x \in B_i \cap \overline{D},$$

$$f(\cdot, y) \in C^\infty_c(D), \quad \forall y \in B_i \cap \overline{D}.$$

Let $(g, \varphi, u, h) \in (\Gamma_{B_i} \cup \Gamma'_{B_i}) \times C^\infty_c((0, T)) \times L_{m+1}(\Omega_T; L_{m+1}(D)) \times (C^- \cup C^+)$. For $\theta > 0$, we introduce

$$\phi_\theta(t, s, x, y) := g(x, y)\rho_\theta(s-t)\varphi\left(\frac{t+s}{2}\right),$$

and

$$H_\theta(s, x, z) := \int_0^T \int_{\rho_\theta} \left(h(u(t, y) - z)\sigma_{\nu_t}^k(y, u(t, y))\phi_\theta(t, s, x, y) - \int_0^\rho_\theta h(r - z)\sigma_{\nu_t}^k(y, r)dr\phi_\theta(t, s, x, y)\right) dt.$$
\[\mathcal{E}(u, w; \theta) := -\mathbb{E} \int_{t,x,y} \partial_{x,y} \phi(t, s, x, y) \int_0^u \int_{\bar{\Gamma}} h'(r - \bar{r}) \sigma_{r,y}^k(y, \bar{r}) \sigma_{x,y}^{ik}(x, \bar{r}) \, d\bar{r} \, dr + \mathbb{E} \int_{t,x,y} \partial_{y} \phi(t, s, x, y) \int_0^u \int_{\bar{\Gamma}} h'(r - \bar{r}) \sigma_{r,y}^k(y, \bar{r}) \sigma_{x,y}^{ik}(x, \bar{r}) \, d\bar{r} \, dr - \mathbb{E} \int_{t,x,y} \partial_{x} \phi(t, s, x, y) \int_0^u \int_{\bar{\Gamma}} h'(r - \bar{r}) \sigma_{r,y}^k(y, \bar{r}) \sigma_{x,y}^{ik}(x, \bar{r}) \, d\bar{r} \, dr - \mathbb{E} \int_{t,x,y} \phi(t, s, x, y) \int_0^u \int_{\bar{\Gamma}} h'(r - \bar{r}) \sigma_{r,y}^k(y, \bar{r}) \sigma_{x,y}^{ik}(x, \bar{r}) \, d\bar{r} \, dr - \mathbb{E} \int_{t,x,y} \phi(t, s, x, y) h'(u - w) \sigma_{w,y}^k(y, \bar{r}) \sigma_{x,y}^{ik}(x, \bar{r}) \, d\bar{r} \, dw\]

where in the integrand \(u = u(t, y)\) and \(w = w(t, x)\). For simplicity, here and below we write \(\int_t^u \) in place of \(\int_t^u \) (and similarly for \(\int_s^t \), \(\int_s^w \) in place of \(\int_s^w \) (and similarly for \(\int_y^w \), and \(\int_x^y \) in place of \(\int_x^y \). However, to avoid confusion, we use the usual notation if the integral is taken on a different domain or is a stochastic integral.

**Remark 3.3.** Since \(\text{supp} \varphi \subset (0, T)\), for a sufficiently small \(\theta\), the function \(H_\theta\) has a similar form as [DGG19] (3.7). The definition of the sets \(\Gamma_{B_1}\) and \(\Gamma_{B_1}^c\) indicates \(\text{supp} H_\theta(s, z) \subset B_1 \cap \overline{D}\) for all \((s, z) \in [0, T] \times \mathbb{R}\). Furthermore, there exists a modification of \(H_\theta\) which is smooth in \((s, z, x, y)\) (see [Kum97] Exercise 3.15). Throughout this paper, we will use this smooth version and still denote it by \(H_\theta\).

Fix a constant \(\mu := (3m + 5)/(4m + 4)\), which is chosen so that \(\mu \in ((m + 3)/(2m + 2), 1)\).

**Definition 3.4.** We say that a function \(w \in L_{m+1}(\Omega_T; L_{m+1}(D))\) has the \((\ast)\)-property, if

(i) For each \(i \in \{0, 1, \ldots, N\}\) and all \((g, \varphi, u, h) \in \Gamma_{B_1}^c \times C^\infty_c((0, T)) \times L_{m+1}(\Omega_T; L_{m+1}(D)) \times C^-

satisfying \(u \geq 0\) for almost all \((\omega, t, x) \in \Omega_T \times D\), and for all sufficiently small \(\theta \in (0, 1)\), we have \(H_\theta(\cdot, \cdot, w(\cdot, \cdot)) \in L_1(\Omega_T \times D)\) and \(\mathbb{E} \int_{t,x} H_\theta(s, w(s, x)) \leq C \theta^{1-\mu} + \mathcal{E}(u, w, \theta)\)

for a constant \(C\) independent of \(\theta\).

(ii) For each \(i \in \{0, 1, \ldots, N\}\) and all \((g, \varphi, u, h) \in \Gamma_{B_1} \times C^\infty_c((0, T)) \times L_{m+1}(\Omega_T; L_{m+1}(D)) \times C^+

and sufficiently small $\theta \in (0, 1)$, we have $H_\theta(\cdot, \cdot, w(\cdot, \cdot)) \in L_1(\Omega_T \times D)$ and
\[
\mathbb{E} \int_{s,x} H_\theta(s, x, w(s, x)) \leq C\theta^{1-\mu} + \mathcal{E}(u, w, \theta)
\]
for a constant $C$ independent of $\theta$.

Remark 3.5. Differing from the $(\ast)$-property in [DGG19, DGT20, DG20], we adjust test functions $g$, $u$ and $h$ to apply the divergence theorem with the Dirichlet boundary condition (see the proof of Proposition 4.4). Remark 3.12 provides the justification for our distinct consideration of assertions (i) and (ii) in Definition 3.4.

Remark 3.6. If $g \in \Gamma_{B^+}^T$, define $\tilde{g}(y, x) := g(y, x)$, then we have $\tilde{g} \in \Gamma_{B^+}^T$. Moreover, if we relabel $x \leftrightarrow y$ in assertion (ii) of Definition 3.4, it becomes an estimate to
\[
\mathbb{E} \int_{s,y} H_\theta(s, y, w(s, y))
\]
\[
= \mathbb{E} \int_{s,y} \left[ \int_0^T \int_x \left( h(u(t, x) - z)\sigma^k_{x,t}(x, u(t, x))\tilde{g}(x, y)r\rho(s - t)\varphi(t + s) \right) 
- \int_0^{u(t,x)} h(r - z)\sigma^k_{r,t}(x, r)d\tilde{g}(x, y)r\rho(s - t)\varphi(t + s) \right] dz = \mathcal{E}(u, \tilde{g}, \theta).
\]
This is used in the proof of Lemma 3.11 for the case that $u$ has the $(\ast)$-property.

The following lemmas are introduced from [DGG19, DG20], and the proofs are similar no matter the space is $D$ or $\mathbb{T}^d$. We omit the proofs here.

**Lemma 3.7.** Under Assumption 2.4, for all $\lambda \in ((m + 3)/(2m + 2), 1)$, $k \in \mathbb{N}$ and sufficiently small $\theta \in (0, 1)$, we have
\[
\mathbb{E}\|\partial_z H_\theta\|^{m+1}_{L_\infty([0,T];H^k_{m+1}(\mathcal{D} \times \mathbb{R}))} \leq C\theta^{-\lambda(m+1)}\mathcal{N}_m(u),
\]
where
\[
\mathcal{N}_m(u) := \mathbb{E} \int_0^T \left( 1 + \|u(t)\|^{m+1}_{L_{m+1}(\mathcal{D})} + \|u(t)\|^2_{L_2(D)} \right) dt,
\]
and the constant $C = C(N_0, k, d, T, \lambda, |D|, m, h, \varphi)$ is independent of $\theta$. In particular, we have
\[
\mathbb{E}\|\partial_z H_\theta\|^{m+1}_{L_\infty([0,T];H^k_{m+1}(\mathcal{D} \times \mathbb{R}))} \leq C\theta^{-\lambda(m+1)} \left( 1 + \|u\|^2_{L_\infty([0,T];H^k_{m+1}(\mathcal{D} \times \mathbb{R}))} \right).
\]

**Lemma 3.8.** Let $w \in L_2(\Omega_T \times D)$. Then, for all sufficiently small $\theta \in (0, 1)$, we have
\[
\mathbb{E} \int_{s,x} H_\theta(s, x, w(s, x)) = \lim_{\lambda \to 0} \mathbb{E} \int_{s,x,z} H_\theta(s, x, z)\rho(x)(w(s, x) - z).
\]

**Lemma 3.9.** Let $\{w_n\}_{n \in \mathbb{N}}$ be a sequence bounded in $L_{m+1}(\Omega_T \times D)$ satisfying the $(\ast)$-property uniformly in $n$, which means the constant $C$ in Definition 3.4 (i) and (ii) are independent of $n$. Suppose that $w_n$ converges to a function $w$ for almost all $(\omega, t, x) \in \Omega_T \times D$. Then, $w$ has the $(\ast)$-property.
3.2. $L_1$-estimates. Now, we give the $L_1$-estimates of two entropy solutions $u$ and $\bar{u}$ using Kruzhkov’s doubling variables technique (cf. \cite{Kru70}), which can eliminate the coupling effect of $u - \bar{u}$. One key point is to select the appropriate function $\eta$ such that $\eta(\cdot - z) \in \mathcal{E}_0$ for any $z \geq 0$ to approximate the positive part function. Using the convolution, the non-negative $z$ can be replaced by another entropy solution under suitable conditions.

In this subsection, we fix $i \in \{0, 1, \ldots, N\}$. For the sake of brevity, we define $B \equiv B_i$, $\psi \equiv \psi_i$ and $q_i(x - y) := q_{i,i}(x - y)$ which are introduced in the definition of the spatial mollifier in Section 3. For a non-negative $\varphi \in C^\infty_c((0, T))$ satisfying $\|\varphi\|_{L^\infty((0, T))} \vee \|\partial_t \varphi\|_{L^1((0, T))} \leq 1$, we define the non-negative test functions

$$\phi_\varepsilon(t, x, y) := q_i(x - y)\varphi(t)\psi(x)\mathbf{1}_{\Gamma_\varepsilon}(x), \quad \phi_{\partial \varepsilon}(t, x, s, y) := \rho_\varepsilon(s - t)\phi_\varepsilon\left(\frac{s + t}{2}, x, y\right).$$

Remark 3.10. Compare to the test functions $\phi_\varepsilon$ in Section 3. We want to take $g(x, y) = \mathbf{1}_{\Gamma_{\varepsilon}}(x)\varphi(x)\psi(x)\mathbf{1}_{\Gamma_{\varepsilon}}(x) - y)$. In this case, with $\varepsilon \in (0, \varepsilon)$ (see Remark 3.2 for the definition of $\varepsilon$), we have $g \in \Gamma_{\varepsilon}$ and the function $\bar{g}(x, y) := g(y, x)$ is in $\Gamma_{\varepsilon}^\infty$.

To derive the $L_1$-estimates, we need the following lemma.

Lemma 3.11. Let $0 \leq \xi, \zeta \in L_{m+1}(\Omega, F_0; L_{m+1}(D))$. Suppose that $u$ and $\bar{u}$ are the entropy solutions to the Dirichlet problems $\Pi(\Phi, \xi)$ and $\Pi(\bar{\Phi}, \zeta)$, respectively. Let Assumptions \textbf{2.1}, \textbf{2.4} and \textbf{2.3} hold for both $\Phi$ and $\bar{\Phi}$. If $u$ or $\bar{u}$ has the $(\ast)$-property, for $\delta \in (0, 1)$, $\varepsilon \in (0, \varepsilon)$, $\lambda \in [0, 1]$, $\alpha \in [0, 1 \land (m/2)]$, and every non-negative $\varphi \in C^\infty_c((0, T))$ such that

$$\|\varphi\|_{L^\infty((0, T))} \vee \|\partial_t \varphi\|_{L^1((0, T))} \leq 1,$$

we have

$$\begin{align*}
- \mathbb{E} & \int_{t,x,y} \varphi(t)\Delta_x \psi(x)(\bar{u}(t, x)) - \Phi(u(t, x)))^+ \\
& \leq \mathbb{E} \int_{t,x,y} \varphi(t)\Delta_x \psi(x)(\bar{u}(t, x)) - \Phi(u(t, x)))^+ \\
& \quad + \mathbb{E} \sup_{\varepsilon \rightarrow 0} \|\nabla [a](u)\|_{L^2(D_{\varepsilon})}^2 \\
& \quad + C\mathbb{E}\left[\varepsilon^{\frac{\alpha}{\lambda}}\|u\|_{L^\infty(D_{\varepsilon})}^{\lambda(1 + \alpha)}\|\bar{u}\|_{L^\infty(D_{\varepsilon})}^{\lambda(1 + \alpha)} + \|u\|_{L^\infty(D_{\varepsilon})}^{\lambda(1 + \alpha)}\|\bar{u}\|_{L^\infty(D_{\varepsilon})}^{\lambda(1 + \alpha)}\right] \\
& \quad + C\mathbb{E} \int_{t,x,y} \mathbf{1}_{B_{\varepsilon}(x)}(x)\varphi(t)\left(\varepsilon^2 \sum_{i,j} |\partial_x \varphi_i\varphi_j(x - y)| + \varepsilon \sum_{i,j} |\partial_x \varphi_i\varphi_j(x - y)| \right)(\bar{u}(t, x) - u(t, y))^+.
\end{align*}$$

where

$$R_\lambda := \sup \left\{ R \in [0, \infty] : |a(r) - \bar{a}(r)| \leq \lambda, \forall |r| < R \right\},$$

$$\mathbb{E}(\varepsilon, \delta, \lambda, \alpha) := \varepsilon^{-2}\delta^{2\alpha} + \delta^{\beta} - 1 + \varepsilon^{\delta} + \varepsilon^{\delta} + \varepsilon^{\frac{\alpha}{\lambda}} + \varepsilon^{\frac{\alpha}{\lambda}} + \varepsilon^{-2}\delta^{\alpha} + \varepsilon^{-2}\lambda^2 + \varepsilon^{-1}\lambda,$$

and the constant $C$ depends only on $N_0$, $K$, $d$, $T$, $|D|$ and $\alpha$.

Remark 3.12. Estimate (3.6) is not symmetric for $u$ and $\bar{u}$ (See Remark 3.13 for the reason we focus on such terms) due to the $(\ast)$-property of $u$ or $\bar{u}$ and the different status of $x$ and $y$ in test function $\phi_\varepsilon(t, x, y)$. Therefore, it is necessary to consider the $(\ast)$-property separately as in Definition 3.1. This is a key point of our proof, which is different from \cite{DGG19, DGT20, DG20}.\]
Proof. We first prove the case that \( \tilde{u} \) has the \((\ast)\)-property. With Proposition \( \ref{prop} \) we have \( u, \tilde{u} \geq 0 \) for almost all \((\omega, t, x) \in \Omega_T \times D\). For each \( \delta > 0 \), define the function \( \eta_\delta \in C^2(\mathbb{R}) \) by
\[
\eta_\delta(0) = \eta_\delta'(0) = 0, \quad \eta_\delta''(r) = \rho_\delta(r).
\]
Thus, we have
\[
|\eta_\delta(r) - r^+| \leq \delta, \quad \text{supp} \eta_\delta'' \subset [0, \delta], \quad \int_\mathbb{R} |\eta_\delta''(r)| dr \leq 2, \quad |\eta_\delta''| \leq 2\delta^{-1}.
\]
Fix \((z, t, y) \in [0, \infty) \times D_T\). Since \( \tilde{u} \) is the entropy solution to \( \Pi(\hat{\Phi}, \hat{\xi}) \) and
\[
(\eta_\delta(-z), \rho_\delta(-t)\varphi\left(\frac{\cdot + t}{2}\right), \varphi(-y)\psi(\cdot)1_{\mathcal{F}_T}(\cdot)) \in \mathcal{E}_0 \times C^\infty((0, T)) \times C^\infty(\mathcal{F}_T)
\]
for \( \varepsilon \in (0, \varepsilon) \) and a sufficiently small \( \varepsilon \), using the entropy inequality \( \ref{entropy} \) of \( \tilde{u} \) with \((\eta_\delta(r - z), \phi_{\theta, \varepsilon}(t, \cdot, \cdot, y))\) instead of \((\eta(r), \phi)\), we have
\[
- \int_{s,x} \eta_\delta(\tilde{u} - z) \partial_s \phi_{\theta, \varepsilon}
\leq \int_{s,x} \left[ \tilde{\alpha}^2 \eta_\delta''(\cdot - z) \right] ||(\tilde{u})\Delta_x \phi_{\theta, \varepsilon} + \int_{s,x} [a_{ij} \eta_\delta''(\cdot - z)](x, \tilde{u}) \partial_{x,j} \phi_{\theta, \varepsilon}
+ \int_{s,x} \left[ a_{ij} \eta_\delta''(\cdot - z) - f_{ij} \eta_\delta''(\cdot - z) \right](x, \tilde{u}) \partial_{x,j} \phi_{\theta, \varepsilon}
- \int_{s,x} \eta_\delta'(\tilde{u} - z) b_i'(x, \tilde{u}) \partial_{x,i} \phi_{\theta, \varepsilon} + \int_{s,x} \eta_\delta'(\tilde{u} - z) F(x, \tilde{u}) \phi_{\theta, \varepsilon}
+ \int_{s,x} \eta_\delta'(\tilde{u} - z) f_{ij}'(x, \tilde{u}) \phi_{\theta, \varepsilon} - \int_{s,x} \left[ f_{ij} \eta_\delta''(\cdot - z) \right](x, \tilde{u}) \phi_{\theta, \varepsilon}
+ \int_{s,x} \frac{1}{2} \eta_\delta''(\tilde{u} - z) \sum_{k=1}^\infty |\sigma_{ik}^\varepsilon(x, \tilde{u})|^2 \phi_{\theta, \varepsilon} - \int_{s,x} \eta_\delta''(\tilde{u} - z) |\nabla_x [\tilde{\alpha}^2](\tilde{u})|^2 \phi_{\theta, \varepsilon}
+ \int_0^T \int_{s,x} \left( \eta_\delta'(\tilde{u} - z) \phi_{\theta, \varepsilon} \sigma_{ik}^\varepsilon(x, \tilde{u}) - |\sigma_{ik}^\varepsilon \eta_\delta''(\cdot - z)\right)(x, \tilde{u}) \phi_{\theta, \varepsilon}
- \left[ \sigma_{ik}^\varepsilon \eta_\delta''(\cdot - z) \right](x, \tilde{u}) \partial_{x,k} \phi_{\theta, \varepsilon} \right) dW^k(s),
\]
where \( \tilde{u} = \tilde{u}(s, x) \). Notice that all the expressions are continuous in \((z, t, y)\). We take \( z = u(t, y) \) by convolution and integrate over \((t, y) \in D_T\). By taking expectations, we have
\[
\text{E}(3.7)
\leq \text{E} \int_{t,x,y} \eta_\delta(\tilde{u} - u) \partial_s \phi_{\theta, \varepsilon}
\leq \text{E} \int_{t,x,y} \left[ \tilde{\alpha}^2 \eta_\delta''(\cdot - u) \right] ||(\tilde{u})\Delta_x \phi_{\theta, \varepsilon} + \text{E} \int_{t,x,y} [a_{ij} \eta_\delta''(\cdot - u)](x, \tilde{u}) \partial_{x,j} \phi_{\theta, \varepsilon}
+ \text{E} \int_{t,x,y} \left[ a_{ij} \eta_\delta''(\cdot - u) - f_{ij} \eta_\delta''(\cdot - u) \right](x, \tilde{u}) \partial_{x,j} \phi_{\theta, \varepsilon}
- \text{E} \int_{t,x,y} \eta_\delta'(\tilde{u} - u) b_i'(x, \tilde{u}) \partial_{x,i} \phi_{\theta, \varepsilon} + \text{E} \int_{t,x,y} \eta_\delta'(\tilde{u} - u) F(x, \tilde{u}) \phi_{\theta, \varepsilon}
+ \text{E} \int_{t,x,y} \eta_\delta'(\tilde{u} - u) f_{ij}'(x, \tilde{u}) \phi_{\theta, \varepsilon} - \text{E} \int_{t,x,y} \left[ f_{ij} \eta_\delta''(\cdot - u) \right](x, \tilde{u}) \phi_{\theta, \varepsilon}
+ \text{E} \int_{t,x,y} \frac{1}{2} \eta_\delta''(\tilde{u} - u) \sum_{k=1}^\infty |\sigma_{ik}^\varepsilon(x, \tilde{u})|^2 \phi_{\theta, \varepsilon} - \text{E} \int_{t,x,y} \eta_\delta''(\tilde{u} - u) |\nabla_x [\tilde{\alpha}^2](\tilde{u})|^2 \phi_{\theta, \varepsilon}
\]
inequalities (3.7)-(3.8) and taking the limit for $\phi_{\theta} = \phi_{\theta}(z - \cdot)$ and $\psi_{\theta}(x - \cdot)$, we have

$$\int_{t, y} \left[ \int_0^T \left( \eta_k^{*}(\tilde{u} - z) \psi_{\theta}(y, u) \phi_{\theta}(\tilde{u}) \phi_{\theta} + \right) \right] dW^k(s)$$

where $u = u(t, y)$ and $\tilde{u} = \tilde{u}(s, x)$. Similarly, for each $z \in [0, \infty) \times D_T$, since

$$\left( \eta_k^{*}(z - \cdot), \rho_\theta(s - \cdot) \varphi \left( \frac{s + h}{2} \right), g_\tau(x - \cdot, \psi(x) \mathbf{1}_{\mathbb{T}}(x) \right) \in \mathcal{E} \times C_\infty((0, T)) \times C_\infty(D)$$

for $\varepsilon \in (0, \delta)$ and a sufficiently small $\delta$, we apply the entropy inequality of $u$ with $\eta_k^{*}(z - r)$ and $\phi_{\theta}(t, y) := \phi_{\theta}(z - y)$. After substituting $z = \tilde{u}(s, x)$ by convolution, integrating over $(s, x) \in D_T$ and taking expectations, we have

$$(3.8) \quad - \mathbb{E} \int_{t, x, y} \eta_k^{*}(\tilde{u} - u) \partial_t \phi_{\theta}$$

$$\leq - \mathbb{E} \int_{t, x, y} \eta_k^{*}(\tilde{u} - u) \partial_t \phi_{\theta} - \mathbb{E} \int_{t, x, y} \eta_k^{*}(\tilde{u} - u) \partial_y \phi_{\theta}$$

where $u = u(t, y)$ and $\tilde{u} = \tilde{u}(s, x)$. Note that the stochastic integral term in (3.8) is zero due to the support of $\rho_\theta$. Meanwhile, we use the $(\cdot)$-property (i) in Definition 3.4 of $\eta_k^{*}$ for the last term of (3.8) with $h(r) = -\eta_k^{*}(r)$ (then $h \in C_\infty$) and $g(x, y) = \mathbf{1}_{\mathbb{T}}(x) \psi(x) g_\tau(x - y)$. Adding inequalities (3.7)-(3.8) and taking the limit $\theta \to 0^+$, we have

$$(3.9) \quad - \mathbb{E} \int_{t, x, y} \eta_k^{*}(\tilde{u} - u) \partial_t \phi_{\theta} \leq I + A + B + \mathcal{E},$$

where

$I := \mathbb{E} \int_{t, x, y} \eta_k^{*}(\tilde{u} - u) \partial_t \phi_{\theta}$

$A := \mathbb{E} \int_{t, x, y} \eta_k^{*}(\tilde{u} - u) \partial_y \phi_{\theta}$

and taking the limit $\theta \to 0^+$, we have

$$- \mathbb{E} \int_{t, x, y} \eta_k^{*}(\tilde{u} - u) \partial_t \phi_{\theta} \leq I + A + B + \mathcal{E},$$
\[ + \mathbb{E} \int_{t,x} \left[ a^{ij}_{x} \eta'_{b}(\cdot - u) \right](x, \tilde{u}) \partial_{x} \phi_{\varepsilon} - \mathbb{E} \int_{t,x} \left[ a^{ij}_{y} \eta'_{b}(\tilde{u} - \cdot) \right](y, u) \partial_{y} \phi_{\varepsilon} \\
- \mathbb{E} \int_{t,x} \eta''_{b}(\tilde{u} - u) b'_{x}(x, \tilde{u}) \partial_{x} \phi_{\varepsilon} + \mathbb{E} \int_{t,x} \eta''_{b}(\tilde{u} - u) b'(y, u) \partial_{y} \phi_{\varepsilon} \\
+ \mathbb{E} \int_{t,x} \frac{1}{2} \eta''_{b}(\tilde{u} - u) \left( \sum_{k=1}^{\infty} |\sigma^{ik}_{x}(x, \tilde{u})|^{2} + \sum_{k=1}^{\infty} |\sigma^{ik}_{y}(y, u)|^{2} \right) \phi_{\varepsilon}, \]

\[ B := - \mathbb{E} \int_{t,x} \left[ f^{i}_{x} \eta'_{b}(\cdot - u) \right](x, \tilde{u}) \partial_{x} \phi_{\varepsilon} + \mathbb{E} \int_{t,x} \left[ f^{i}_{y} \eta'_{b}(\tilde{u} - \cdot) \right](y, u) \partial_{y} \phi_{\varepsilon} \\
+ \mathbb{E} \int_{t,x} \eta''_{b}(\tilde{u} - u) \left( f^{i}_{x}(x, \tilde{u}) - f^{i}_{y}(y, u) \right) \phi_{\varepsilon} \\
- \mathbb{E} \int_{t,x} \left[ \left[ f^{i}_{x} \eta'_{b}(\cdot - u) \right](x, \tilde{u}) - \left[ f^{i}_{y} \eta'_{b}(\tilde{u} - \cdot) \right](y, u) \right] \phi_{\varepsilon} \\
+ \mathbb{E} \int_{t,x} \eta''_{b}(\tilde{u} - u) \left( F(x, \tilde{u}) - F(y, u) \right) \phi_{\varepsilon}, \]

and

\[ E := \sum_{i=1}^{9} E_{i} := - \mathbb{E} \int_{t,x} \partial_{x} \phi_{\varepsilon} \int_{u}^{\tilde{u}} \eta''_{b}(\tilde{r} - r) \sigma^{ik}_{r}(y, r) \sigma^{ik}_{x}(x, \tilde{r}) dr d\tilde{r} \\
- \mathbb{E} \int_{t,x} \partial_{y} \phi_{\varepsilon} \int_{u}^{\tilde{r}} \eta''_{b}(\tilde{r} - r) \sigma^{ik}_{r}(y, r) \sigma^{ik}_{y}(x, \tilde{u}) dr \\
+ \mathbb{E} \int_{t,x} \partial_{x} \phi_{\varepsilon} \int_{u}^{\tilde{r}} \eta''_{b}(\tilde{r} - r) \sigma^{ik}_{r}(y, r) \sigma^{ik}_{x}(x, \tilde{u}) dr \\
- \mathbb{E} \int_{t,x} \partial_{y} \phi_{\varepsilon} \int_{u}^{\tilde{r}} \eta''_{b}(\tilde{r} - r) \sigma^{ik}_{r}(y, r) \sigma^{ik}_{y}(x, \tilde{u}) dr \\
+ \mathbb{E} \int_{t,x} \phi_{\varepsilon} \int_{u}^{\tilde{r}} \eta''_{b}(\tilde{r} - r) \sigma^{ik}_{y}(y, r) \sigma^{ik}_{x}(x, \tilde{r}) dr \\
+ \mathbb{E} \int_{t,x} \phi_{\varepsilon} \int_{u}^{\tilde{r}} \eta''_{b}(\tilde{r} - r) \sigma^{ik}_{y}(y, r) \sigma^{ik}_{x}(x, \tilde{r}) dr \\
- \mathbb{E} \int_{t,x} \phi_{\varepsilon} \eta''_{b}(\tilde{u} - u) \sigma^{ik}_{y}(y, u) \sigma^{ik}_{x}(x, \tilde{u}), \]

and \( u = u(t, y) \) and \( \tilde{u} = \tilde{u}(t, x) \) in the integrand. The term \( I \) contains \( a \), the term \( A \) involves \( \sigma \), and the term \( B \) encompasses either \( f \) or \( F \). Term \( E \) is derived from the \((\ast)\)-property. Now, we estimate these terms. The following estimates are similar to the proof of [DGC19] Theorem 4.1 and [DG20] Theorem 4.1. The differences are caused by introducing the function \( \psi \) and the Dirichlet boundary condition. Therefore, we only focus on the application of the divergence theorem and the term about \( \psi \). First, we estimate the term \( I \). From

\[ \partial_{y_{i}} \int_{0}^{u} a^{2}(r) \eta''_{b}(\tilde{u} - r) dr = 0, \]
the support of $\eta'$ and $g_\varepsilon(x - \cdot) \in C^\infty_c(D)$ for all $(x, \varepsilon) \in (B \cap \overline{D}) \times (0, \varepsilon)$, we have

$$I = -\mathbb{E} \int_{t,x,y} 1_{u \leq \tilde{u}} \partial_{x_1} g_\varepsilon \int_u^{\tilde{u}} \int_0^r \tilde{a}^2(\tilde{r}) \eta''_\varepsilon(\tilde{r} - r) d\tilde{r} dr$$

$$- \mathbb{E} \int_{t,x,y} 1_{u \leq \tilde{u}} \partial_{x_1} g_\varepsilon \int_u^{\tilde{u}} \int_0^r \tilde{a}^2(\tilde{r}) \eta''_\varepsilon(\tilde{r} - r) d\tilde{r} dr$$

$$+ \mathbb{E} \int_{t,x,y} \varphi \partial_{x_1} (g_\varepsilon \partial_{x_1} \psi) \int_u^{\tilde{u}} \tilde{a}^2(\tilde{r}) \eta''_\varepsilon(\tilde{r} - u) d\tilde{r}$$

$$+ \mathbb{E} \int_{t,x,y} \varphi \partial_{x_1} (g_\varepsilon \partial_{x_1} \psi) \int_u^{\tilde{u}} \tilde{a}^2(\tilde{r}) \eta''_\varepsilon(\tilde{r} - u) dr$$

$$- \mathbb{E} \int_{t,x,y} \eta''_\varepsilon(\tilde{u} - u) (|\nabla_x [\tilde{a}](\tilde{u})|^2 + |\nabla_y [\tilde{a}](u)|^2) \phi_\varepsilon =: \sum_{i=1}^5 I_i.$$

Terms $I_3$ and $I_4$ are arisen from introducing $\psi$. The term $I_3$ can be written as

$$I_3 = \mathbb{E} \int_{t,x,y} \varphi(t) g_\varepsilon(x - y) \Delta_x \psi(x) \int_u^{\tilde{u}} \tilde{a}^2(\tilde{r}) \text{sgn}^+(\tilde{r} - u) d\tilde{r}$$

$$+ \mathbb{E} \int_{t,x,y} \varphi(t) g_\varepsilon(x - y) \Delta_x \psi(x) \int_u^{\tilde{u}} 1_{0 \leq \tilde{r} - u \leq \delta} \tilde{a}^2(\tilde{r}) (\eta''_\varepsilon(\tilde{r} - u) - \text{sgn}^+(\tilde{r} - u)) d\tilde{r}$$

$$+ \mathbb{E} \int_{t,x,y} \varphi(t) \partial_{x_1} g_\varepsilon(x - y) \partial_{x_1} \psi(x) \int_u^{\tilde{u}} \tilde{a}^2(\tilde{r}) \text{sgn}^+(\tilde{r} - u) d\tilde{r}$$

$$+ \mathbb{E} \int_{t,x,y} \varphi(t) \partial_{x_1} g_\varepsilon(x - y) \partial_{x_1} \psi(x) \int_u^{\tilde{u}} 1_{0 \leq \tilde{r} - u \leq \delta} \tilde{a}^2(\tilde{r}) (\eta''_\varepsilon(\tilde{r} - u) - \text{sgn}^+(\tilde{r} - u)) d\tilde{r}$$

$$= \sum_{i=1}^4 I_{3,i},$$

where

$$\text{sgn}^+(x) := \begin{cases} 1, & x > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Combining the boundness of $\Delta \psi$ and $\varphi$, the definition of $g_\varepsilon$ and Assumption 2.1, we obtain that

$$|I_{3,2}| + |I_{3,4}| \lesssim \delta \mathbb{E} \int_{t,x,y} (g_\varepsilon(x - y) + |\partial_{x_1} g_\varepsilon(x - y)|) \sup_{0 \leq \tilde{r} - u \leq \delta} \tilde{a}^2(\tilde{r})$$

$$\lesssim \delta (1 + \varepsilon^{-1}) \mathbb{E} (1 + ||u||_{L^m(D_\varepsilon)}^m).$$

Therefore, noticing $\varepsilon < 1$, we have

$$I_3 \leq I_{3,1} + \mathbb{E} \int_{t,x,y} \varphi(t) \partial_{x_1} g_\varepsilon(x - y) \partial_{x_1} \psi(x) (\Phi(\tilde{u}) - \Phi(u))^+$$

$$+ C \delta \varepsilon^{-1} \mathbb{E} (1 + ||u||_{L^m(D_\varepsilon)}^m).$$

With the same method, we also have

$$I_4 \leq -\mathbb{E} \int_{t,x,y} \varphi(t) \partial_{x_1} g_\varepsilon(x - y) \partial_{x_1} \psi(x) (\Phi(\tilde{u}) - \Phi(u))^+ + C \delta \varepsilon^{-1} \mathbb{E} (1 + ||\tilde{u}||_{L^m(D_\varepsilon)}^m).$$
Using the triangle inequality, the definition of $g_c$, the boundness of $\partial_x \psi$ and the fact

$$|\Phi(r) - \tilde{\Phi}(r)| \lesssim \lambda |r|^\frac{m+1}{2} + 1_{|r| \geq R_x} |r|^m, \quad \forall r \in \mathbb{R},$$

we have

$$E \int_{t,x,y} \varphi(t) \partial_x g_c(x-y) \psi(x) ((\tilde{\Phi}(\tilde{u}) - \Phi(u)) + (\Phi(u) - \tilde{\Phi}(\tilde{u})))$$

$$\leq E \int_{t,x,y} \varphi(t) \partial_x g_c(x-y) \psi(x) ((\Phi(u) - \tilde{\Phi}(\tilde{u})) + (\Phi(u) - \tilde{\Phi}(\tilde{u})))$$

$$\lesssim \varepsilon^{-1} \lambda E \left( \|u\|_{L^{m+1}(D_T)}^{m+1} + \|\tilde{u}\|_{L^{m+1}(D_T)}^{m+1} \right)$$

$$+ \varepsilon^{-1} E \left( \|1_{|u| \geq R_x} u\|_{L^{m}(D_T)}^m + \|1_{|\tilde{u}| \geq R_x \tilde{u}\|_{L^{m}(D_T)}^m \right).$$

Similarly, for $I_{3.1}$, we have

$$E \int_{t,x,y} \varphi(t) g_c(x-y) \Delta_x \psi(x) ((\tilde{\Phi}(\tilde{u}) - \Phi(u)) + (\Phi(u) - \tilde{\Phi}(\tilde{u})))$$

$$\lesssim \lambda |u|^{m+1}_{L^{m+1}(D_T)} + E \|1_{|u| \geq R_x} u\|_{L^{m}(D_T)}^m.$$ 

To estimate the new term in (3.10), using (6.8) in Lemma 6.2 we have

$$E \int_{t,x,y} \varphi(t) g_c(x-y) \Delta_x \psi(x) ((\tilde{\Phi}(\tilde{u}) - \Phi(u)) + (\Phi(u) - \tilde{\Phi}(\tilde{u})))$$

$$\leq E \int_{t,x} \varphi(t) \Delta_x \psi(x) ((\tilde{\Phi}(\tilde{u}(t,x)) - \Phi(u(t,x)))$$

$$+ C \varepsilon^{-1} E \left( 1 + \|u\|_{L^{m+1}(D_T)} + \|\nabla [a(u)]\|_{L^2(D_T)}^2 \right).$$

For $I_5$, using Definition 2.2 (ii) and [DGG19] Remark 3.1, we have

$$I_5 \leq -2 \int_{t,x,y} \eta''_0(\tilde{u} - u) \nabla_x [\tilde{a}(\tilde{u}) \cdot \nabla_y a(a) \psi \phi_c$$

$$= 2 \int_{t,x,y} \phi_c \partial_x [\tilde{a}(\tilde{u}) \cdot \partial_y \int_{\tilde{u}}^{\tilde{r}} \eta''_0(\tilde{u} - r)a(r) dr$$

$$= 2 \int_{t,x,y} \partial_x \phi_c \int_{\tilde{u}}^{\tilde{r}} \eta''_0(\tilde{r} - r) a(\tilde{r}) a(r) d\tilde{r}dr$$

$$+ 2 \int_{t,x,y} 1_{u \leq \tilde{u}} \partial_x \phi_c \int_{\tilde{u}}^{\tilde{r}} \eta''_0(\tilde{r} - r) a(\tilde{r}) a(r) d\tilde{r}dr.$$ 

Then, we have

$$I_1 + I_2 + I_5 \leq 2 \int_{t,x,y} \left| \tilde{a}(\tilde{r}) - a(\tilde{r}) \right|^2 d\tilde{r}dr.$$ 

Based on the estimates of $|a(r) - \tilde{a}(\tilde{r})|^2$ in the proof of [DGG19] Theorem 4.1] which using Assumption 2.1, combining the proceeding estimates and noticing $\varepsilon < 1$, we have

$$I \leq C(\delta \varepsilon^{-1} + \varepsilon^{-1} \lambda^2 + \varepsilon^{-2} \lambda + \varepsilon^{-2} \delta^2 \lambda) E \left( 1 + \|u\|_{L^{m+1}(D_T)}^{m+1} + \|\tilde{u}\|_{L^{m+1}(D_T)}^{m+1} \right)$$

$$+ C \varepsilon^{-3} E \left( \|1_{|u| \geq R_x} (1 + u)\|_{L^{m}(D_T)}^m + \|1_{|\tilde{u}| \geq R_x (1 + \tilde{u})\|_{L^{m}(D_T)}^m \right)$$

$$+ \varepsilon^{-2} E \|\nabla [a(u)]\|_{L^2(D_T)}^2.$$
where $\alpha \in (0, 1 \wedge (m/2))$. Now, we focus on the terms $\mathcal{A}$. With the fact that $\phi_2(t, x, \cdot) \in C^\infty_0(D)$ for any $(t, x, \varepsilon) \in D_T \times (0, \varepsilon)$, using the divergence theorem in $y$, we have

$$
\mathbb{E} \int_{t,x,y} \partial_{y_1} \phi_2 \int_0^{\tilde{u}} \eta_0' (\tilde{u} - r) a^{ij} (y, r) \, dr + \mathbb{E} \int_{t,x,y} \partial_{y_2} \phi_2 \int_0^{\tilde{u}} \eta_0' (\tilde{u} - r) a^{ij} (y, r) \, dr = 0.
$$

Then, from the support of $\eta_0'$, we have

$$
\mathcal{A} = - \mathbb{E} \int_{t,x,y} 1_{u \leq \tilde{u}} \partial_{x_i} \phi_2 \int_u^{\tilde{u}} \eta_0' (\tilde{u} - u) b^i (x, \tilde{u}) \, d\tilde{u} + \mathbb{E} \int_{t,x,y} \partial_{x_i} \phi_2 \eta_0' (u) b^i (x, u) + \mathbb{E} \int_{t,x,y} \partial_{x_i} \phi_2 \int_u^{\tilde{u}} \eta_0' (\tilde{u} - u) a^{ij} (x, \tilde{u}) \, d\tilde{u} - \mathbb{E} \int_{t,x,y} \partial_{y_1} \phi_2 \int_u^{\tilde{u}} \eta_0' (\tilde{u} - u) a^{ij} (y, r) \, dr + \mathbb{E} \int_{t,x,y} \partial_{y_2} \phi_2 \int_u^{\tilde{u}} \eta_0' (\tilde{u} - u) a^{ij} (y, r) \, dr
$$

$$
\leq \varepsilon \mathcal{E}_0 + \mathcal{E}_6 = \mathbb{E} \int_{t,x,y} \frac{1}{2} \eta_0' (\tilde{u} - u) \sum_{k=1}^\infty (\sigma_{x_k}^j (x, \tilde{u}) - \sigma_{y_k}^j (y, u))^2 \phi_2 x \leq \varepsilon 2^k \delta^{-1} \mathbb{E} (1 + \|u\|_{L^m+1 (D_T)}) + \delta^2.
$$

Next, with $\mathcal{A}_4, \mathcal{A}_5$ in Assumption 2.4 we have

$$
\mathcal{A}_4 + \mathcal{A}_5
$$

$$
\leq \mathbb{E} \int_{t,x,y} \int_{u \leq \tilde{u}} \phi_2 (\partial_{x_i} \eta_0' (\tilde{u} - u) a^{ij} (x, \tilde{u}) \, d\tilde{u}
$$

Moreover, if $|r - \tilde{r}| \leq \delta$, from the definition of $a^{ij}$ and $\mathcal{A}_6$ in Assumption 2.4 we have

$$
\partial_{y_1} \phi_2 (a^{ij} (x, \tilde{r}) + \sigma_{x_k}^j (x, \tilde{r}) - \sigma_{y_k}^j (y, r))\sigma_{r_k}^j (y, r))
$$

$$
\leq \frac{1}{2} \phi \psi \partial_{y_1} \phi_2 (a^{ij} (x, \tilde{r}) - \sigma_{y_k}^j (y, r)) (\sigma_{x_k}^j (x, \tilde{r}) - \sigma_{r_k}^j (y, r)) + \frac{1}{2} \phi \psi \partial_{y_2} \phi_2 (\sigma_{x_k}^j (x, \tilde{r}) (\sigma_{x_k}^j (x, \tilde{r}) - \sigma_{r_k}^j (y, r)) - \sigma_{r_k}^j (y, r)) (\sigma_{x_k}^j (x, \tilde{r}) - \sigma_{r_k}^j (y, r))
$$
\[ \sum_{i,j} |\partial_{x,y} \varphi_i(x-y)| + \varepsilon \sum_i |\partial_y \varphi_i(x-y)| \] 

Therefore,

(3.14) \[ A_1 + \mathcal{E}_1 \]

\[ \lesssim \mathbb{E} \int_{t,x,y} 1_{B^c \cap \mathcal{E}(x)} \varphi(t) \left( \varepsilon^2 + \delta^2 \sum_{i,j} |\partial_{x,y} \varphi_i(x-y)| + \varepsilon \sum_i |\partial_y \varphi_i(x-y)| \right) \right) (\tilde{u} - u)^+ \]

\[ + (\delta^2 \varepsilon^{-2} + \delta^2 \varepsilon^{-1}) \mathbb{E} \| \tilde{u} \|_{L^1(D_T)}. \]

Similarly, we have

(3.15) \[ A_3 + \mathcal{E}_2 + \mathcal{E}_4 \]

\[ \lesssim \delta^{-1} \mathbb{E} \left( \| u \|_{L^1(D_T)} + \| \tilde{u} \|_{L^1(D_T)} \right) \]

\[ + \mathbb{E} \int_{t,x,y} 1_{B^c \cap \mathcal{E}(x)} \varphi(t) \left( \varepsilon \sum_i |\partial_{x,y} \varphi_i(x-y)| + \varphi_i(x-y) \right) (\tilde{u} - u)^+. \]

To estimate \( A_2 + \mathcal{E}_3 + \mathcal{E}_7 \), define

\[ A_2 = -\mathbb{E} \int_{t,x,y} \partial_x \varphi \eta_j^r(\tilde{u} - u) b^j(x, \tilde{u}) \]

\[ + \mathbb{E} \int_{t,x,y} \partial_y \varphi \eta_j^r(\tilde{u} - u) b^j(y, u) =: A_{2.1} + A_{2.2}. \]

Using the definition of \( b^j \) and (2.4)-(2.5) in Assumption 2.4, and relabeling \( i \leftrightarrow j \) in \( \mathcal{E}_7 \), we have

\[ A_{2.2} + \mathcal{E}_7 = \mathbb{E} \int_{t,x,y} \varphi \partial_j \varphi \eta_j^r(\tilde{u} - u) \sigma^{jk}_{ij}(y, u) \left( \sigma^{ik}_{ij}(y, u) - \sigma^{ik}(x, u) \right) d\tilde{r} \]

\[ + \mathbb{E} \int_{t,x,y} \varphi \partial_j \varphi \eta_j^r(\tilde{u} - u) \sigma^{jk}_{ij}(y, u) \left( \sigma^{ik}_{ij}(x, u) - \sigma^{ik}(x, \tilde{r}) \right) d\tilde{r} \]

\[ + \mathbb{E} \int_{t,x,y} \varphi \partial_j \varphi \eta_j^r(\tilde{u} - u) \sigma^{jk}_{ij}(y, u) \sigma^{ik}_{ij}(y, u) d\tilde{r} \]

\[ - \mathbb{E} \int_{t,x,y} \varphi \partial_j \varphi \eta_j^r(\tilde{u} - u) \sigma^{jk}_{ij}(y, u) \sigma^{ik}_{ij}(y, u) - \sigma^{ik}(x, \tilde{r}) d\tilde{r} \]
Similarly, we have

\[ E \int_{t,x,y} \varphi \partial_{y_j} \psi \eta_0^\varepsilon (\tilde{u} - u) \sigma_{y_j}^{ik}(y, u)(y_i - x_i) \int_0^1 \sigma_{rx}^{ik}(x + \theta(y - x), u)d\theta \]
\[ + C\delta^\varepsilon \varepsilon^{-1}(1 + E\|u\|_{L_1(D_T)}) \]
\[ + E \int_{t,x,y} \varphi \partial_{\gamma_x} \partial_{y_i} \psi \eta_0^\varepsilon (\tilde{u} - u) b^i(y, u) + C(\delta^\varepsilon + \varepsilon)(1 + E\|u\|_{L_1(D_T)}). \]

Similarly, we have

\[ A_{2,1} + E_3 \]
\[ = -E \int_{t,x,y} \varphi \partial_{y_j} \psi \int_u^{\tilde{u}} \eta_0^\varepsilon (\tilde{u} - r) \sigma_{x_j}^{ik}(x, \tilde{u}) (\sigma_{r}^{ik}(y, r) - \sigma_{r}^{ik}(x, \tilde{u}))d\theta \]
\[ - E \int_{t,x,y} \varphi \partial_{\gamma_x} \psi \int_u^{\tilde{u}} \eta_0^\varepsilon (\tilde{u} - r) \sigma_{x_j}^{ik}(x, \tilde{u}) \sigma_{r}^{ik}(x, \tilde{u})d\theta \]
\[ \leq -E \int_{t,x,y} \varphi \partial_{y_j} \psi \int_u^{\tilde{u}} \eta_0^\varepsilon (\tilde{u} - u) \sigma_{x_j}^{ik}(x, \tilde{u})(y_i - x_i) \int_0^1 \sigma_{rx}^{ik}(x + \theta(y - x), \tilde{u})d\theta \]
\[ + C\delta^\varepsilon \varepsilon^{-1}(1 + E\|\tilde{u}\|_{L_1(D_T)}) - E \int_{t,x,y} \varphi \partial_{\gamma_x} \psi \eta_0^\varepsilon (\tilde{u} - u) b^i(x, \tilde{u}). \]

From (2.4) in Assumption 2.4 we have

\[ |\sigma_{y_j}^{ik}(y, u)| \int_0^1 \sigma_{rx}^{ik}(x + \theta(y - x), \tilde{u})d\theta = |\sigma_{x_j}^{ik}(x, \tilde{u})| \int_0^1 \sigma_{rx}^{ik}(x + \theta(y - x), \tilde{u})d\theta \]
\[ \lesssim |x - y|(1 + |u| + |\tilde{u}|) + |\sigma_{y_j}^{ik}(y, u)\sigma_{r}^{ik}(y, u) - \sigma_{x_j}^{ik}(x, \tilde{u}) \sigma_{r}^{ik}(x, \tilde{u})| \]
\[ \lesssim (|x - y| + |x - y|^\varepsilon)(1 + |u| + |\tilde{u}|) + |u - \tilde{u}|, \]

and

\[ |b^i(y, u) - b^i(x, \tilde{u})| \lesssim |u - \tilde{u}| + |x - y|^\varepsilon + |x - y|(1 + |u|). \]

Therefore, using the support of \( \eta_0^\varepsilon \), we have

(3.16) \quad A_2 + E_3 + E_7 \]
\[ \lesssim (\delta^\varepsilon \varepsilon^{-1} + \varepsilon^\kappa)E(1 + \|u\|_{L_1(D_T)} + \|\tilde{u}\|_{L_1(D_T)}) \]
\[ + E \int_{t,x,y} 1_{B_{2\eta}(\tilde{T})}(x) \left( \varepsilon \sum_i |\partial_{x_i} \partial_{y_j}(x - y)| + \partial_{x_i}(x - y) \right) \varphi(t)(\tilde{u} - u)^+. \]

The remaining terms in (3.9) are \( B, E_5, E_6 \) and \( E_8 \). Using (2.5) in Assumption 2.4 and the support of \( \eta_0^\varepsilon \), we have

\[ E_5 \leq E \int_{t,x,y} 1_{B_{2\eta}(\tilde{T})}(x) \varphi(t)(x - y)(\tilde{u} - u)^+, \]

and

(3.17) \quad E_6 + E_8 \]
\[ \leq E \int_{t,x,y} \varphi \int_u^{\tilde{u}} \eta_0^\varepsilon (\tilde{u} - r) \sigma_{y_j}^{ik}(y, \tilde{u}) \sigma_{x_j}^{ik}(x, \tilde{u})d\theta + C\delta E(1 + \|\tilde{u}\|_{L_1(D_T)}) \]
\[ + E \int_{t,x,y} \varphi \int_u^{\tilde{u}} \eta_0^\varepsilon (\tilde{u} - u) \sigma_{y_j}^{ik}(y, u) \sigma_{x_j}^{ik}(x, u)d\tilde{r} + C\delta E(1 + \|u\|_{L_1(D_T)}) \]
\[ \lesssim E \int_{t,x,y} \varphi \eta_0^\varepsilon (\tilde{u} - u) |\sigma_{y_j}^{ik}(y, u)\sigma_{x_j}^{ik}(x, u) - \sigma_{r}^{ik}(y, \tilde{u}) |. \]
+ δE(1 + ∥u∥_{L^1(D_T)} + ∥\tilde{u}\parallel_{L^1(D_T)}) \\
\lesssim (\delta + \epsilon^6)E(1 + ∥u∥_{L^1(D_T)} + ∥\tilde{u}\parallel_{L^1(D_T)}) + E \int_{t,x,y} 1_D (x) \varphi(t) \varphi(x-y)(\tilde{u} - u)^+.

From (3.18) in Assumption 2.4 we have

\[ (3.18) \quad B = E \int_{t,x,y} 1_{u \leq \varphi(t)} \partial_x \varphi(x-y) \psi(x) \int_u \int_u \eta_{\theta}^\varepsilon(y-r) \left( f^i(y,r) - f^i(x,r) \right) d\tilde{r} dr \\
- E \int_{t,x,y} \varphi(t) \varphi(x-y) \partial_x \varphi(x) \int_u \int_u \eta_{\theta}^\varepsilon(\tilde{r} - r) \left( f^i(y,r) - f^i(x,r) \right) d\tilde{r} dr \\
+ E \int_{t,x,y} \eta_{\theta}^\varepsilon(u - \tilde{u}) \left( f^i_x(y,u) - f^i_x(x,u) \right) \varphi \\
- E \int_{t,x,y} \eta_{\theta}^\varepsilon(\varphi(x) + \int_u \int_u \eta_{\theta}^\varepsilon(\tilde{r} - u) d\tilde{r} + \int_u \int_u \eta_{\theta}^\varepsilon(r - u) dr \\
+ E \int_{t,x,y} 1_D (x) \varphi(t) \left( \varepsilon |\partial_x \varphi(x-y)| + \varphi(x-y) \right) (\tilde{u} - u)^+.

Combining (3.18) with |\eta_{\theta}(r) - r^+| \leq \delta, and (3.12)-(3.18), one has (3.19).

For the case u has the \((\ast)\)-property, the proof is a similar procedure but using a different assertion of the \((\ast)\)-property. Specifically, for each (z, t, x) \in [0, \infty) \times D_T, since

\[ (\eta_{\theta}(\varepsilon \cdot), \rho_\varepsilon(\varepsilon \cdot - t), \varphi_h(\varepsilon \cdot \cdot - \cdot) \psi(x) 1_D (x) \in E \times C_c([0, T]) \times C_c^\infty (D) \]

for \(\varepsilon \in (0, \varepsilon)\) and a sufficiently small \(\theta\), we apply the entropy inequality (2.3) of (s, y) with \((\eta_{\theta}(r - z), \phi_{\theta,z}(\varepsilon \cdot \cdot - \cdot)) \psi(x)\) instead of \((\eta_{\theta}(r), \phi)\). Taking \(z = \tilde{u}(t, x)\) by convolution, integrating over \((t, x) \in D_T\) and taking expectations, we acquire an estimate of \(-E \int_{t,x,y} \eta_{\theta}(\tilde{u}(t, x) - u(s,y)) \partial_\theta \phi_{\theta,z}\).

Similarly, fix \((s, z, y) \in [0, \infty) \times D_T\). Since

\[ (\eta_{\theta}(\varepsilon \cdot - z), \rho_\varepsilon(\varepsilon \cdot - s), \varphi_h(\varepsilon \cdot \cdot - \cdot) \psi(x) 1_D (x) \in E \times C_c([0, T]) \times C_c^\infty (D) \]

for all sufficiently small \(\theta\), we use (2.3) of \(\tilde{u}(t, x)\) with \((\eta_{\theta}(r - z), \phi_{\theta,z}(\varepsilon \cdot \cdot - \cdot, s, y))\) instead of \((\eta_{\theta}(r), \phi)\). Then, taking \(z = u(s, y)\) by convolution, integrating over \((s, y) \in D_T\) and taking expectations, we obtain an estimate of \(-E \int_{t,x,y} \eta_{\theta}(\tilde{u}(t, x) - u(s,y)) \partial_\theta \phi_{\theta,z}\).

Note that \(\phi_{\theta,z}\) only acts on the set \(\{s, t \in [0, T] : s > t\}\), the stochastic integral term in the estimate of \(-E \int_{t,x,y} \eta_{\theta}(\tilde{u}(t, x) - u(s,y)) \partial_\theta \phi_{\theta,z}\) vanishes, while the one in the estimate of \(-E \int_{t,x,y} \eta_{\theta}(\tilde{u}(t, x) - u(s,y)) \partial_\theta \phi_{\theta,z}\) may not be zeros. Therefore, we apply assertion (ii) of the \((\ast)\)-property of \(u\) with \(h(r) = \eta_{\theta}(r)\) (then \(h \in C^\infty\)) \(, g(x, y) = 1_D (y) \psi(x) \varphi_h(x - y)\) (then \(g \in \Gamma^\infty\)) and relabel \(x \leftrightarrow y\) in the integral, then the stochastic integral terms is controlled by \(C \theta^{1-p} + E(\tilde{u}, u, \theta)\), in the integrand of which \(u = u(t, y)\) and \(\tilde{u} = \tilde{u}(t, x)\). After combining two estimates and taking the limit \(\theta \to 0^+\), there is only one time variable in the integrand. The remaining terms can be estimated as above. Hence, this lemma is proved.

\[ \square \]

Remark 3.13. In the whole proof of Lemma 3.11, the divergence theorem is applied mostly in \(y\) with \(\varphi_h(x - \cdot) \in C^\infty_c (D)\) for all \((x, \varepsilon) \in (B \cap \overline{D}) \times (0, \varepsilon)\), except (3.11), in which the divergence theorem in \(x\) is used with the zero boundary condition of \(\tilde{u}\).
Theorem 3.14. \((L_1\text{-estimates})\) Let \(0 \leq \xi, \tilde{\xi} \in L_{m+1}(\Omega, \mathcal{F}_0; L_{m+1}(D))\). Suppose \(u\) and \(\bar{u}\) are the entropy solutions to the Dirichlet problems \(\Pi(\Phi, \xi)\) and \(\Pi(\tilde{\Phi}, \tilde{\xi})\), respectively. Let Assumptions 2.1, 2.4 and 2.5 hold for both \(\Phi\) and \(\tilde{\Phi}\). If \(u\) or \(\bar{u}\) has the \((\ast)\)-property, then,

(i) if furthermore \(\tilde{\Phi} = \Phi\), then

\[
\text{ess sup}_{t \in [0,T]} E \| (\tilde{u}(t, \cdot) - u(t, \cdot))^+ \|_{L_1(D)} \leq C E \| (\tilde{\xi} - \xi)^+ \|_{L_1(D)},
\]

where the constant \(C\) depends only on \(N_0, K, d, T\) and \(|D|\).

(ii) for all \(\delta \in (0,1), \varepsilon \in (0, \varepsilon), \lambda \in [0, 1]\) and \(\alpha \in (0, 1 \wedge (m/2))\), we have

\[
E \int_0^T \int_x (\tilde{u}(\tau, x) - u(\tau, x))^+ \, d\tau \\
\leq C E \int_x (\tilde{\xi}(x) - \xi(x))^+ + C \text{ sup}_{|h| \leq 2\varepsilon} E \| \xi(\cdot) - \xi(\cdot + h) \|_{L_1(D)} + C \text{ sup}_{|h| \leq 2\varepsilon} E \| \tilde{\xi}(\cdot) - \tilde{\xi}(\cdot + h) \|_{L_1(D)} \\
+ C E \frac{1}{C_{\varepsilon}} E (\| \nabla [a] \|_{L_2(D_T)} + \| \nabla [\tilde{a}] \|_{L_2(D_T)}) + C E (\varepsilon, \delta, \lambda, \alpha) E (1 + \| u \|_{L_{m+1}(D_T)} + \| \tilde{u} \|_{L_{m+1}(D_T)}) \\
+ C E^{-2} E (\| 1_{u \geq R_{\lambda}}(1 + u) \|_{L_{m}(D_T)} + \| 1_{u \geq R_{\lambda}}(1 + \tilde{u}) \|_{L_{m}(D_T)})
\]

where \(E(\varepsilon, \delta, \lambda, \alpha)\) and \(R_{\lambda}\) are introduced in Lemma 3.13.

(3.21) \(\tilde{\xi}(x) := \begin{cases} \xi(x), & x \in D; \\ 0, & x \notin D, \end{cases}\)

and the constant \(C\) depends only on \(N_0, K, d, T\) and \(|D|\).

Remark 3.15. Since the positive part function is not an even function, it requires us to prove estimates with the \((\ast)\)-property of different entropy solutions \(u\) or \(\bar{u}\). Using this fact, we obtain the \(L_1\)-estimates with only one of the entropy solutions has the \((\ast)\)-property, which is a key point in proving the uniqueness in the proof of Theorem 2.8.

Proof. We first assume that \(\bar{u}\) has the \((\ast)\)-property. Let \(0 < s < \tau < T\) be Lebesgue points of the function

\[
t \rightarrow E \int_{x,y} (\bar{u}(t,x) - u(t,y))^+ \psi(x) \varphi_{t,y}(x-y),
\]

and fix a constant \(\gamma \in (0, (\tau - s) \wedge (T - \tau))\). Choose a sequence of functions \(\{\phi_n\}_{n \in \mathbb{N}}\) and the limit \(V(\gamma)\) as in the proof of Proposition 2.7, take \(\varphi = \phi_n \in \mathcal{K}(\gamma)\) (using the \((\ast)\)-property of \(\bar{u}\)) and pass to the limit \(n \to \infty\). Then, taking \(\gamma \to 0^+\) and using (4.3), we have

\[
- E \int_{x,y} (\bar{u}(s,x) - u(s,y))^+ \varphi_{t,y}(x-y) \psi(x) \\
+ E \int_{x,y} (\bar{u}(\tau,x) - u(\tau,y))^+ \varphi_{t,y}(x-y) \psi(x) \\
\leq E \int_s^T \int_x \Delta_x \psi(x) (\bar{\Phi}(\bar{u}(t,x)) - \Phi(u(t,x)))^+ \, dt + M + C E \int_0^T \int_{x,y} 1_{B(\tau)(x)} \\
\cdot (\varepsilon^2 \sum_{i} |\partial_{x,y} \varphi_{t,y}(x-y)| + \varepsilon \sum_{i} |\partial_{x} \varphi_{t,y}(x-y)| + \varphi_{t,y}(x-y)) (\bar{u}(t,x) - u(t,y))^+ \, dt
\]

where \(\Phi, \bar{\Phi}, \varphi_{t,y}\) are the solutions to the Dirichlet problems \(\Pi(\Phi, \xi)\) and \(\Pi(\tilde{\Phi}, \tilde{\xi})\) respectively.
holds for almost all \( s \in (0, \tau) \), where
\[
M := \frac{C\varepsilon^{m+1}}{m+1} E \|\nabla [a(u)]\|_{L^2(D_T)}^2 + C\mathcal{E}(\varepsilon, \delta, \lambda, \alpha) E(1 + \|u\|_{L^{m+1}(D_T)}^m + \|\tilde{u}\|_{L^{m+1}(D_T)}^m)
+ C\varepsilon^{-2} E \left( \|1_{|u| \geq R_3(1 + u)}\|_{L^m(D_T)} + \|1_{|\tilde{u}| \geq R_3(1 + \tilde{u})\|_{L^m(D_T)}^m \right).
\]

Then, for \( \tilde{\gamma} \in (0, \tau) \), by averaging over \( s \in (0, \tilde{\gamma}) \), setting \( \tilde{\gamma} \to 0^+ \) and using Lemma 6.1, we have
\[
(3.22) \quad E \int_{x,y} (\tilde{u}(\tau, x) - u(\tau, y))^+ \varrho_\epsilon(x - y) \psi(x)
\leq E \int_{x,y} (\tilde{\xi}(x) - \xi(y))^+ \varrho_\epsilon(x - y) \psi(x)
+ E \int_0^\tau \int_x \Delta_x \psi(x)(\bar{\Phi}(\tilde{u}(t, x)) - \Phi(u(t, x)))^+ dt + M + CE \int_0^\tau \int_{x,y} 1_{Br, \overline{\gamma}}(x)
\cdot \left( \varepsilon^2 \sum_{i,j} |\partial_{x,y} \varrho_\epsilon(x - y)| + \varepsilon \sum_i |\partial_x \varrho_\epsilon(x - y)| + \varrho_\epsilon(x - y) \right)(\tilde{u}(t, x) - u(t, y))^+ dt.
\]

To prove (3.19), taking \( \lambda = 0 \) and \( R_\lambda = \infty \), we have
\[
E \left( \|1_{|u| \geq R_3(1 + u)}\|_{L^m(D_T)} + \|1_{|\tilde{u}| \geq R_3(1 + \tilde{u})\|_{L^m(D_T)}^m \right) = 0,
\]
and \( \mathcal{E}(\varepsilon, \delta, \lambda, \alpha) \) becomes
\[
\mathcal{E}(\varepsilon, \delta, \alpha) := \varepsilon^{-2}\delta^{2\beta_3 + \beta_2\varepsilon^{-1} + \varepsilon^{\beta_3} + \varepsilon^{1/(m+1)} + \varepsilon^{2}\delta^{-1} + \varepsilon^{-2}\delta^{2\beta_3}.
\]

Since \( \beta \in ((2\beta_3)^{-1}, 1] \), we can choose \( \theta \in ((m + 2)^{-1} \vee (2\beta_3)^{-1}, \bar{\kappa}) \) and \( \alpha \in ((2\beta_3)^{-1}, 1 \wedge (m/2)). \)
Let \( \delta = \varepsilon^{2\beta_3} \) and \( \varepsilon \to 0^+ \), we have \( \mathcal{E}(\varepsilon, \delta, \alpha) \to 0^+ \). Notice that \( \varepsilon |\partial_{x,y} \varrho_\epsilon| \) and \( \varepsilon^2 |\partial_{x,y} \varrho_\epsilon| \) are all approximations of the identity up to a constant. Adding over \( \psi_i \) from partition of unity, with the continuity of translations in \( L_1 \), we have
\[
E \int_x (\tilde{u}(\tau, x) - u(\tau, x))^+ \leq E \int_x (\tilde{\xi}(x) - \xi(x))^+ + C \int_0^\tau E \int_x (\tilde{u}(t, x) - u(t, x))^+ dt
\]
holds for almost all \( \tau \in [0, T] \). Hence, (3.19) follows from Gronwall’s inequality.

Now, we prove (3.20). Notice that
\[
E \int_{x,y} (\xi(x) - \xi(y))^+ \varrho_\epsilon(x - y) \psi(x) \leq \int_{B^2} \varrho_\epsilon(h) \cdot E \int_{B^\kappa, D} (\xi(x) - \xi(x - h))^+ dx dh
\leq \sup_{|h| \leq \varepsilon} E \int_{B^\kappa, D} (\tilde{\xi}(x) - \tilde{\xi}(x - h))^+ dx.
\]

Fixing \( s_1 \in (0, T] \) and integrating (3.22) over \( \tau \in (0, s_1) \), we have
\[
E \int_0^{s_1} \int_{x,y} (\tilde{u}(\tau, x) - u(\tau, y))^+ \varrho_\epsilon(x - y) \psi(x) d\tau
\leq TE \int_x (\tilde{\xi}(x) - \xi(x))^+ \psi(x) + T \sup_{|h| \leq 2\varepsilon} E \int_{B^\kappa, D} (\tilde{\xi}(x) - \tilde{\xi}(x - h))^+ dx
\quad + E \int_0^{s_1} \int_{x,y} \Delta_x \psi(x)(\bar{\Phi}(\tilde{u}(t, x)) - \Phi(u(t, x)))^+ dt + TM
\quad + CE \int_0^{s_1} \int_{x,y} \left[ 1_{Br, \overline{\gamma}}(x) \cdot \left( \varepsilon^2 \sum_{i,j} |\partial_{x,y} \varrho_\epsilon(x - y)| + \varrho_\epsilon(x - y) \right)(\tilde{u}(t, x) - u(t, y))^+ \right] dt d\tau.
\]
Notice that $\varepsilon|\partial_n \varepsilon|$, $\varepsilon^2|\partial_{x_i} \varepsilon|$ are approximations of the identity up to a constant. Taking $\varepsilon \to 0$ and adding with different $\psi_i$, from the partition of unity, we have
\[
\mathbb{E} \int_0^t \int_x (\varepsilon(x) - u(\tau, x))^+ d\tau \\
\lesssim \mathbb{E} \int_0^t (\xi(x) - \xi(x))^+ + \sup_{|h| \leq 2\varepsilon} \mathbb{E} \int_D (\xi(x) - \xi(x - h))^+ dx \\
+ M + \mathbb{E} \int_0^t \int_0^t \int_x (\varepsilon(x) - u(\tau, x))^+ d\tau dt.
\]
Using Gronwall’s inequality, we acquire (3.20).

For the case that $u$ in $(\bar{u} - u)^+$ has the $(\ast)$-property, using (3.6) with the $(\ast)$-property of $u$ and following the proceeding method, we obtain the desired estimates. \qed

4. Approximation

We approximate the function $\Phi$ to make the approximate equations non-degenerate. The following proposition is taken from [DGG19, DG20] and we refer to [DGG19, Proposition 5.1] for the proof.

Proposition 4.1. Let $\Phi$ satisfy Assumption 2.7 with a constant $K > 1$. Then, for all $n$ there exists an increasing function $\Phi_n \in C^\infty(\mathbb{R})$ with bounded derivatives, satisfying Assumption 2.4 with constant $3K$, such that $a_n(r) \geq 2/n$, and
\begin{equation}
(4.1) \sup_{|r| \leq n} |a(r) - a_n(r)| \leq 4/n.
\end{equation}

Define $\xi_n := \xi \wedge n$. Denote by $(\cdot, \cdot)$ the inner product in $L_2(D)$.

Definition 4.2. An $L_2$-solution $u_n$ to $\Pi(\Phi_n, \xi_n)$ is a continuous $L_2(D)$-valued process, such that $u_n \in L_2(\Omega_T; H^1_0(D))$, $\nabla \Phi_n(u_n) \in L_2(\Omega_T; L_2(D))$, and the equality
\[
(u_n(t, \cdot), \phi) = (\xi_n, \phi) - \int_0^t \left( (\nabla \Phi_n(u_n), \nabla \phi) + (a^{ij}(\cdot, u_n) \partial_{x_j} u_n + b^i(\cdot, u_n) \\
+ f^i(\cdot, u_n, \partial_{x_i} \phi) + (F(\cdot, u_n), \phi) \right) ds - \int_0^t (\sigma^k(\cdot, u_n), \nabla \phi) dW^k(s)
\]
holds for all $\phi \in C^\infty_D$, almost surely for all $t \in [0, T]$.

Differing from Definition 2.2 with a strong regularity of the solution $u_n$, we can consider the Dirichlet boundary condition in the sense of trace. From (2.3), (2.8) and (2.10) in Assumption 2.4 we have the linear growth of $\sigma^{ij}_x$, $f^i_x$, and $F$ in $r$. Combining with (2.2) in Assumption 2.1 for all $p \geq 2$, the $L_2$-solution $u_n$ to $\Pi(\Phi_n, \xi_n)$ has the following a priori estimates
\begin{align}
(4.2) \quad & \mathbb{E} \sup_{t \leq T} \|u_n\|_{L^p_z(D)}^p + \mathbb{E} \|\nabla [a_n] (u_n)\|_{L^p(D_T)}^p \leq C(1 + \mathbb{E} \|\xi_n\|_{L^p_z(D)}^p), \\
(4.3) \quad & \mathbb{E} \sup_{t \leq T} \|u_n\|_{L^{p+1}_z(D)}^{p+1} + \mathbb{E} \|\nabla \Phi_n (u_n)\|_{L^p_z(D_T)}^{p+1} \leq C(1 + \mathbb{E} \|\xi_n\|_{L^{p+1}_z(D)}^{p+1}).
\end{align}

The proof of the above two estimates is almost the same as [DG20, Lemma A.1] with the Dirichlet boundary condition of $u_n$, and we omit it here. It is worth noting that the fact $a_n \geq 2/n > 0$ and (1.2) indicate
\begin{equation}
(4.4) \quad \mathbb{E} \|\nabla u_n\|_{L^p_z(D_T)}^p \leq C(n)(1 + \mathbb{E} \|\xi_n\|_{L^p_z(D)}^p).
\end{equation}
Remark 4.3. Note that for all \((\eta, \varphi, \varrho) \in \mathcal{E} \times C_c^\infty([0, T)) \times C_c^\infty(D)\) or \((\eta, \varphi, \varrho) \in \mathcal{E}_0 \times C_c^\infty([0, T)) \times C_c^\infty(D)\) and \(\phi := \varphi \times \varrho \geq 0\), applying Itô’s formula (see e.g. [Kry13]) to \(u_n \mapsto \int_D \eta(u_n) \rho dx\), with (4.2) and Assumption 2.1, we have that the \(L_2\)-solution \(u_n\) is also an entropy solution to \(\Pi(\Phi_n, \xi_n)\). Using Proposition 2.7, when \(0 \leq \xi \in L_{m+1}(\Omega, \mathcal{F}_0; L_{m+1}(D))\), we have \(u_n \geq 0\) for almost all \((\omega, t, x) \in \Omega \times T \times D\).

**Proposition 4.4.** Let \(0 \leq \xi \in L_{m+1}(\Omega, \mathcal{F}_0; L_{m+1}(D))\) and Assumptions 2.1, 2.4 and 2.5 hold. For each \(n \in \mathbb{N}\), let \(u_n\) be an \(L_2\)-solution of \(\Pi(\Phi_n, \xi_n)\). Then, \(u_n\) has the (*)-property. If in addition \(\mathbb{E}\|\xi\|^{(m+1)/m}_L < \infty\), the constants \(C\) in Definition 3.4 are independent of \(n\).

With the help of Remark 4.3 Proposition 4.4 and Theorem 3.14(i), following almost the same argument as [DG20] Proposition 5.4, we have the existence and uniqueness of the \(L_2\)-solution \(u_n\). Here, we omit the proof.

**Proposition 4.5.** Let \(0 \leq \xi \in L_{m+1}(\Omega, \mathcal{F}_0; L_{m+1}(D))\) and Assumptions 2.1, 2.4 and 2.5 hold. Then, for each \(n \in \mathbb{N}\), \(\Pi(\Phi_n, \xi_n)\) admits a unique \(L_2\)-solution \(u_n\).

Proof of Proposition 4.4. We first prove Definition 3.4(i). Fix \(i \in \{0, 1, \ldots, N\}\). For the sake of brevity, we define \(B := B_i, \psi := \psi_i\) and \(\varrho_i(x - y) := \varrho_{x, i}(x - y)\) which are introduced in the definition of the spatial mollifier in Section 3. Fix sufficiently small \(\gamma > 0\). Since \(y \mapsto \varrho_{x, i}(x - y) \in C_c^\infty(D)\) for all \((x, \gamma) \in (B \cap \overline{D}) \times (0, \varepsilon)\), for a function \(f \in L_2(D)\), let \(f^{(\gamma)}(x) := \int_D f(z) \varrho_{x, i}(x - z) dz\). Then, on \(B \cap \overline{D}\), \(u_n^{(\gamma)}\) satisfies (pointwise) the equation

\[
\frac{d}{dt} u_n^{(\gamma)} = \left[\Delta(\Phi_n(u_n^{(\gamma)}))^{(\gamma)} + \partial_x (a^{ij}(\cdot, u_n) \partial_x u_n + b^i(\cdot, u_n) + f^i(\cdot, u_n))^{(\gamma)} \right] + (F(\cdot, u_n))^{(\gamma)} \right] \frac{\partial_t}{\partial_x} (\sigma^{jk}(\cdot, u_n))^{(\gamma)} W_k(t).
\]

Note that

\[
\mathbb{E} \int_{s, x} H_\theta(s, x, u_n(s, x)) = \lim_{\lambda \to 0} \mathbb{E} \int_{s, x, z} H_\theta(s, x, z) 1_{B \cap \overline{D}}(x) \rho_{x, i}(u_n(s, x) - z),
\]

and

\[
\left| \mathbb{E} \int_{s, x, z} H_\theta(s, x, z) 1_{B \cap \overline{D}}(x) \rho_{x, i}(u_n(s, x) - z) - \rho_{x, i}(u_n^{(\gamma)}(s, x) - z) \right|
\leq C \left( \mathbb{E}\|u_n - u_n^{(\gamma)}\|_{L_2([0, T] \times (B \cap \overline{D}))}^2 \right)^{\frac{1}{2}} \mathbb{E}\|\partial_t H_\theta\|_{L_\infty(D)_x} \to 0.
\]

With Remark 3.3, we have \(\mathbb{E} H_\theta(s, x, z) X = 0\) for any \(X \rho_{x, i}\)-measurable bounded random variable \(X\). Then,

\[
\mathbb{E} \int_{s, x, z} H_\theta(s, x, z) 1_{B \cap \overline{D}}(x) \rho_{x, i}(u_n^{(\gamma)}(s, x) - z)
= \mathbb{E} \int_{s, x, z} H_\theta(s, x, z) 1_{B \cap \overline{D}}(x) \left( \rho_{x, i}(u_n^{(\gamma)}(s, x) - z) - \rho_{x, i}(u_n^{(\gamma)}(s, x - \theta, x) - z) \right)
\]

Using Itô’s formula, we have

\[
\int_{s, x, z} H_\theta(s, x, z) \rho_{x, i}(u_n^{(\gamma)}(s, x) - z) - \rho_{x, i}(u_n^{(\gamma)}(s, x - \theta, x) - z) = \sum_{i = 1}^{5} N_{\lambda, \gamma}^{(i)}
\]

where

\[
N_{\lambda, \gamma}^{(1)} := \int_{s, x, z} H_\theta(s, x, z) \int_{s - \theta}^s \rho^\lambda_{x, i}(u_n^{(\gamma)}(t, x) - z) \Delta_x(\Phi_n(u_n))^{(\gamma)} dt,
\]
combining with the definition of $a$

Similarly, integrating by parts twice in $lim$ which indicates

From the Dirichlet boundary condition of $\lambda$,

$$N_{\lambda,\gamma}^{(2)} := \int_{s,x,z} H_\theta(s,x,z) \int_{s-\theta}^s \rho_\lambda^{(1)}(u_n^{(1)}(t,x) - z) \partial_{x_i}(a^{ij} (\gamma, u_n)) \partial_{x_j} u_n + b^i(\gamma, u_n))^{(1)} dt,$$

$$N_{\lambda,\gamma}^{(3)} := \int_{s,x,z} H_\theta(s,x,z) \int_{s-\theta}^s \rho_\lambda^{(1)}(u_n^{(1)}(t,x) - z) (\partial_{x_i} f(\gamma, u_n) + F(\gamma, u_n))^{(1)} dt,$$

$$N_{\lambda,\gamma}^{(4)} := \int_{s,x,z} H_\theta(s,x,z) \int_{s-\theta}^s \rho_\lambda^{(1)}(u_n^{(1)}(t,x) - z) \partial_{x_i}(\sigma^{ik}(\gamma, u_n))^{(1)} dW^k(t),$$

$$N_{\lambda,\gamma}^{(5)} := \frac{1}{2} \int_{s,x,z} H_\theta(s,x,z) \int_{s-\theta}^s \rho_\lambda^{(1)}(u_n^{(1)}(t,x) - z) \sum_{k=1}^{\infty} \partial_{x_i}(\sigma^{ik}(\gamma, u_n))^{(1)}^2 dt.$$  

Using the divergence theorem in $x$, we have $N_{\lambda,\gamma}^{(1)} = \sum_{i=1}^3 N_{\lambda,\gamma}^{(i,1)}$, where

$$N_{\lambda,\gamma}^{(1,1)} := - \int_{s,x,z} \nabla_x H_\theta(s,x,z) \int_{s-\theta}^s \rho_\lambda^{(1)}(u_n^{(1)}(t,x) - z) \nabla_x(\Phi_n(u_n))^{(1)} dt,$$

$$N_{\lambda,\gamma}^{(1,2)} := \int_{s,x,z} H_\theta(s,x,z) \int_{s-\theta}^s \rho_\lambda^{(1)}(u_n^{(1)}(t,x) - z) \nabla_x u_n^{(1)}(t,x) \nabla_x(\Phi_n(u_n))^{(1)} dt,$$

$$N_{\lambda,\gamma}^{(1,3)} := \int_{s,z} \int_{\partial D} H_\theta(s,x,z) \int_{s-\theta}^s \rho_\lambda^{(1)}(u_n^{(1)}(t,x) - z) \nabla_x(\Phi_n(u_n))^{(1)} \cdot \nu dtdS,$$

and $\nu(x)$ is the unit normal vector of $\partial D$ at $x$. Using integration by parts in $z$ and $\int_{s,x,z}$, we have

$$E[N_{\lambda,\gamma}^{(1,1)}] = E \int_{s,x,z} 1_{\{s>0\}} \nabla_x H_\theta(s,x,z) \int_{s-\theta}^s \rho_\lambda^{(1)}(u_n^{(1)}(t,x) - z) \nabla_x(\Phi_n(u_n))^{(1)} dt$$

$$\leq C \theta(E(\nabla_x(\Phi_n(u_n)))^{(1)}||L_{c}(\theta))^{(2)} \leq C \theta^{1-\mu}.$$  

Similarly, integrating by parts twice in $z$ on $N_{\lambda,\gamma}^{(1,2)}$, we have

$$\lim_{\gamma \to 0} E[N_{\lambda,\gamma}^{(1,2)}] \leq C \theta^{1-\mu} \lim_{\gamma \to 0} (E(\nabla_x u_n^{(1)}(t,x) \nabla_x(\Phi_n(u_n))^{(1)}) ||\nabla_x(\Phi_n(u_n)))^{(1)}$$

$$\leq C \theta^{1-\mu} (E(\nabla_x u_n \nabla_x(\Phi_n(u_n))) ||\nabla_x(\Phi_n(u_n)))^{(1)} \leq C \theta^{1-\mu}.$$  

From the Dirichlet boundary condition of $u_n$, we have

$$\lim_{\gamma \to 0} N_{\lambda,\gamma}^{(1,3)} = \int_{s,z} \int_{\partial D} 1_{\{s>0\}} H_\theta(s,x,z) \int_{s-\theta}^s \rho_\lambda^{(1)}(0 - z) \nabla_x(\Phi_n(u_n))^{(1)} \cdot \nu dtdS$$

$$= \int_{s,z} 1_{\{z<0\}} \int_{\partial D} 1_{\{s>0\}} H_\theta(s,x,z) \int_{s-\theta}^s \rho_\lambda^{(1)}(0 - z) \nabla_x(\Phi_n(u_n))^{(1)} \cdot \nu dtdS.$$  

Since supp $\rho_\lambda^{(1)} \subset [0, \infty)$, the integrand only acts when $z \in (-\infty, 0]$. However, based on the non-negativity of $u$ and the definition of $\mathcal{C}$, we have for all $(h,z) \in \mathcal{C}^{(-\infty, 0)}$, $h(u(t,y) - z) = 0$, a.s. $(t,\omega, y) \in \Omega_T \times D$.

which indicates $\lim_{\gamma \to 0} N_{\lambda,\gamma}^{(1,3)} = 0$. Therefore,

$$\limsup_{\gamma \to 0} E[N_{\lambda,\gamma}^{(1)}] \leq C(n) \theta^{1-\mu}.$$  

Now we estimate $N_{\lambda,\gamma}^{(2)} + N_{\lambda,\gamma}^{(5)}$. As the estimate of $N_{\lambda,\gamma}^{(1,3)}$, using the divergence theorem in $x$ and combining with the definition of $a^\gamma$ and $b^\gamma$, we have

$$\limsup_{\gamma \to 0} E[N_{\lambda,\gamma}^{(2)} + N_{\lambda,\gamma}^{(5)}] \leq C(n) \theta^{1-\mu}.$$
with the linear growth of $\sigma$.

Using Itô’s isometry and taking $\gamma$ in (4.5) where

\[
\int_{s}^{t} \sigma_{x}^{i}(u_{n}(t, x) - z) \left( a^{ij}(\cdot, u_{n}) \partial_{x_{j}} u_{n}\right) dt
\]

and (2.7) in Assumption 2.4, we have

\[
E \left| \int_{s}^{t} \sigma_{x}^{i}(u_{n}(t, x) - z) \left( a^{ij}(\cdot, u_{n}) \partial_{x_{j}} u_{n}\right) dt \right| 
\]

and

\[
\left| \int_{s}^{t} \sigma_{x}^{i}(u_{n}(t, x) - z) \left( a^{ij}(\cdot, u_{n}) \partial_{x_{j}} u_{n}\right) dt \right| 
\]

\[
\leq C \theta \| \int_{s}^{t} \sigma_{x}^{i}(u_{n}(t, x) - z) \left( a^{ij}(\cdot, u_{n}) \partial_{x_{j}} u_{n}\right) dt \|
\]

with the linear growth of $\sigma_{x}^{i}$, $b^{i}$ and and the boundness of $a^{ij}_{x_{j}}$ and $a^{ij}$ derived from (2.4) and (2.11) in Assumption 2.4 we have

\[
\lim_{\gamma \to 0} \sup \left( E \left| N_{\gamma} \right|^{(2)} + N_{\gamma}^{(5)} \right) \leq C \theta^{1-\mu} \left( 1 + E \| u_{n}\|_{L^{2}(D_{T})}^{2m+1} \right) \leq C(\mu) \theta^{1-\mu}.
\]

For $N_{\gamma}$, with the linear growth of $f_{x_{i}}$, $F$ and the boundness of $f_{x_{i}}$ and $f_{x_{i}x_{i}}$ derived from 2.7 - 2.10 in Assumption 2.4 we similarly obtain

\[
\lim_{\gamma \to 0} \sup \left( E \left| N_{\gamma} \right|^{(3)} \right) \leq C \theta^{1-\mu} \left( 1 + E \| u_{n}\|_{L^{2}(D_{T})}^{2m+1} \right) \leq C(\mu) \theta^{1-\mu}.
\]

Using Itô’s isometry and taking $\gamma \to 0$, since

\[
\int_{s}^{t} \sigma_{x}^{i}(u_{n}(t, x) - z) \left( a^{ij}(\cdot, u_{n}) \partial_{x_{j}} u_{n}\right) dt 
\]

we have

\[
\lim_{\gamma \to 0} \left( E \left| N_{\gamma} \right|^{(4)} \right) = \sum_{i=1}^{6} I_{i},
\]

where

\[
I_{1} := \int_{s}^{t} \sigma_{x}^{i}(u_{n}(t, x) - z) \left( a^{ij}(\cdot, u_{n}) \partial_{x_{j}} u_{n}\right) dt,
\]

\[
I_{2} := \int_{s}^{t} \sigma_{x}^{i}(u_{n}(t, x) - z) \left( a^{ij}(\cdot, u_{n}) \partial_{x_{j}} u_{n}\right) dt,
\]

\[
I_{3} := \int_{s}^{t} \sigma_{x}^{i}(u_{n}(t, x) - z) \left( a^{ij}(\cdot, u_{n}) \partial_{x_{j}} u_{n}\right) dt,
\]
\[
\begin{aligned}
I_4 := & \int_{s, \bar{x}, z, y} \left( \int_{-\theta}^{s} \sigma_{r}^{ik}(y, r)h(r - z)dr \phi_{y} \rho_{\lambda}^{\prime}(u_{n}(t, x) - z) \right. \\
& \left. \cdot \sigma_{r}^{ik}(x, u_{n}(t, x)) \partial_{x_{j}} u_{n}(t, x) \right) dt,
\end{aligned}
\]
\[
I_5 := \int_{s, \bar{x}, z, y} \left( \int_{-\theta}^{s} h(u(t, y) - z) \phi_{r} \sigma_{y}^{ik}(y, u(t, y)) \rho_{\lambda}^{\prime}(u_{n}(t, x) - z) \sigma_{r}^{ik}(x, u_{n}(t, x)) dt, \right.
\]
\[
I_6 := \int_{s, \bar{x}, z, y} \left( \int_{-\theta}^{s} h(u(t, y) - z) \phi_{y} \sigma_{y}^{ik}(y, u(t, y)) \rho_{\lambda}^{\prime}(u_{n}(t, x) - z) \right.
\left. \cdot \sigma_{r}^{ik}(x, u_{n}(t, x)) \partial_{x_{j}} u_{n}(t, x) \right) dt.
\]

For \( I_2 + I_4 \), notice that
\[
\rho_{\lambda}^{\prime}(u_{n}(t, x) - z) \sigma_{r}^{ik}(x, u_{n}(t, x)) \partial_{x_{j}} u_{n}(t, x) = \partial_{x_{j}} \int_{0}^{u_{n}(t, x)} \rho_{\lambda}^{\prime}(\bar{r} - z) \sigma_{r}^{ik}(x, \bar{r}) d\bar{r}
\]
\[
- \int_{0}^{u_{n}(t, x)} \rho_{\lambda}^{\prime}(\bar{r} - z) \sigma_{r}^{ik}(x, \bar{r}) d\bar{r}.
\]

We can apply the divergence theorem in \( x \) and the Dirichlet boundary condition and integrate by parts in \( z \). Moreover, since \( h \in C^{-} \), the integrand is non-zero only when \( u(t, y) < z \). Using \( \text{supp} \rho_{\lambda} \subset \mathbb{R}_{+} \) and the non-negativity of \( u \), we have
\[
(4.6) \quad I_2 + I_4
\]
\[
= \int_{s, \bar{x}, z, y} \left( \int_{-\theta}^{s} \sigma_{y}^{ik}(y, r)h'(r - z)dr \phi_{r} \rho_{\lambda}^{\prime}(u_{n}(t, x) - z) \right. \\
\left. \cdot \sigma_{r}^{ik}(x, u_{n}(t, x)) \partial_{x_{j}} u_{n}(t, x) \right) dt,
\]
\[
= \int_{s, \bar{x}, z, y} \left( \int_{-\theta}^{s} \sigma_{y}^{ik}(y, r)h'(r - z)dr \phi_{r} \rho_{\lambda}^{\prime}(u_{n}(t, x) - z) \right. \\
\left. \cdot \sigma_{r}^{ik}(x, u_{n}(t, x)) \partial_{x_{j}} u_{n}(t, x) \right) dt.
\]

Similarly, we have
\[
I_6 = \int_{s, \bar{x}, z, y} \left( \int_{-\theta}^{s} h'(u(t, y) - z) \phi_{r} \sigma_{y}^{ik}(y, u(t, y)) \partial_{x_{j}} \phi_{y} \right. \\
\left. \cdot \sigma_{r}^{ik}(x, u_{n}(t, x)) \partial_{x_{j}} u_{n}(t, x) \right) dt,
\]
\[
= \int_{s, \bar{x}, z, y} \left( \int_{-\theta}^{s} h'(u(t, y) - z) \phi_{r} \sigma_{y}^{ik}(y, u(t, y)) \partial_{x_{j}} \phi_{y} \right. \\
\left. \cdot \sigma_{r}^{ik}(x, u_{n}(t, x)) \partial_{x_{j}} u_{n}(t, x) \right) dt.
\]

For \( I_1, I_3 \) and \( I_5 \), we integrate by parts in \( z \). Therefore, we have
\[
\lim_{\lambda \to 0} \lim_{\gamma \to 0} \mathbb{E}N_{\lambda}^{(4)}(u_{n}, \theta) = \mathcal{E}(u_{n}, \theta).
\]
Remark 4.3 shows that $u$ constructed in Section 4 is a Cauchy sequence.

Furthermore, inspired by (4.2), if $\|\xi\|_{L^2(\mathbb{R}^d)} < \infty$, we can choose $C$ independent of $n$.

To prove (ii) in Definition 3.4, note that for all $g \in \Gamma^*_B$, we have $g(\cdot, y) \in C_\mathbb{C}(D)$ for all $y \in B \cap \overline{D}$. It is easy to prove following the proceeding proof. The reason is that the boundary terms vanish using the divergence theorem in $x$, and other differences in the proof are (4.3), $I_2 + I_4$ and $I_6$. For (4.5), with $h \in C^+$, we have for all $(y,z) \in D \times [0,\infty)$,

$$\int_0^{\tau_n} \sigma_{t,y}(y,r) h(r-z) dr = \int_0^{\tau_n} \sigma_{r,y}(y,r) h(r-z) dr = 0.$$

For $z < 0$, using (2.3) in Assumption 2.3 (4.2), Remark 4.3 and the definition of $C^+$, the boundary term arising in the divergence theorem in $y$ will disappear when $\lambda \to 0$. Therefore, we have (4.6).

As for $I_2 + I_4$ and $I_6$, we can apply

$$\partial_x \int_0^{u(t,y)} \rho_\lambda(\tilde{r} - z) \sigma_{t,r}(x,\tilde{r}) d\tilde{r} dt = \int_0^{u(t,y)} \rho_\lambda(\tilde{r} - z) \sigma_{r,r}(x,\tilde{r}) d\tilde{r} dt = 0$$

instead of using the support of $h$. Therefore, this proposition is proved. \hfill \qed

5. Existence and Uniqueness

Now, we give the proof of our main theorem.

Proof of Theorem 2.1. In this part, we will use the $L^1$-estimates to prove that $\{u_n\}_{n \in \mathbb{N}}$ constructed in Section 3 is a Cauchy sequence.

For $n, n' \geq 1$, let $u_n$ and $u_{n'}$ be the $L^1$-solutions of $\Pi(\Phi_n, \xi_n)$ and $\Pi(\Phi_{n'}, \xi_{n'})$, respectively. Remark 4.3 shows that $u_n$ and $u_{n'}$ are also entropy solutions. Proposition 3.3 indicates that $\{u_n\}_{n \in \mathbb{N}}$ has the $(\ast)$-property. Without loss of generality, we assume that $n \leq n'$. Since $\beta \in ((2\kappa)^{-1}, 1]$, we can choose $\vartheta \in ((m + 2)^{-1} \vee (2\beta)^{-1}, \kappa)$ and $\alpha \in ((2\vartheta)^{-1}, 1 \wedge (m/2))$. Let $\delta = \varepsilon^{2\vartheta}$ and $\lambda = 8/n$. Using (4.1), we have $R_\lambda \geq n$. Applying Theorem 3.1 with $u_n$ and $u_{n'}$ and using (4.3) and the triangle inequality

$$E\|\xi_n(\cdot) - \xi_n(\cdot + h)\|_{L^1(D)} \leq E\|\xi(\cdot) - \xi(\cdot + h)\|_{L^1(D)} + 2E\|\xi - \xi_n\|_{L^1(D)}, \quad \forall n \in \mathbb{N},$$

we have

$$E \int_0^T \int_D |u_{n'}(\tau, x) - u_n(\tau, x)| dx d\tau \leq M(\varepsilon) + C\varepsilon \|\xi - \xi_n\|_{L^1(D)} + C\varepsilon \|\xi - \xi_n\|_{L^1(D)} + C\varepsilon^{-2} n^{-2} + C\varepsilon^{-1} n^{-1}$$

$$+ C\varepsilon^{-2} E \left( \|1_{\{u_{n'} \geq n(1 + u_n)\}}\|_{L^m(D_T)}^m \right. + \left. \|1_{\{u_{n'} \geq n(1 + u_{n'})\}}\|_{L^m(D_T)}^m \right),$$

where $M(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. For any $\varepsilon_0 > 0$, we select sufficiently small $\varepsilon \in (0, \varepsilon_0)$ such that $M(\varepsilon) \leq \varepsilon_0$. Then, using (4.3), we can choose $n_0$ sufficiently large so that for $n_0 \leq n \leq n'$, we have

$$C\varepsilon \|\xi_{n'} - \xi\|_{L^1(D)} + C\varepsilon \|\xi - \xi_n\|_{L^1(D)} + C\varepsilon^{-2} n^{-2} + C\varepsilon^{-1} n^{-1}$$

$$+ C\varepsilon^{-2} E \left( \|1_{\{u_{n'} \geq n(1 + u_n)\}}\|_{L^m(D_T)}^m \right. + \left. \|1_{\{u_{n'} \geq n(1 + u_{n'})\}}\|_{L^m(D_T)}^m \right) \leq \varepsilon_0.$$
Therefore, we have
\[
\lim_{n,n' \to \infty} \| u_{n'}(t,x) - u_n(t,x) \|_{L^1(\Omega_T \times D)} = 0.
\]
Moreover, by taking a subsequence, we may assume
\[
\lim_{n \to \infty} u_n = u, \quad \text{a.s.} \ (\omega, t, x) \in \Omega_T \times D.
\]
In addition, the sequence \( \{ |u_n(t,x)|^q \}_{n \in \mathbb{N}} \) is uniformly integrable on \( \Omega_T \times D \) for all \( q \in (0, m+1) \).

Now, we verify that \( u \) is an entropy solution to \( P(\Phi, \xi) \) under Definition 2.2.

Firstly, with the definition of \( \xi_n \), we have that \( \{ u_n \}_{n \in \mathbb{N}} \) is weak convergence in the Banach space \( L^{m+1}(\Omega_T; L^{m+1}(D)) \). Applying the Banach-Saks Theorem, taking a subsequence and using (4.3), we have
\[
\mathbb{E}\| u \|_{L^{m+1}(\Omega_T; L^{m+1}(D))} \leq \liminf_{n \to \infty} \| u_n \|_{L^{m+1}(\Omega_T; L^{m+1}(D))}
\leq C(1 + \liminf_{n \to \infty} \| \xi_n \|_{L^{m+1}(\Omega; L^{m+1}(D))})
\leq C(1 + \| \xi \|_{L^{m+1}(\Omega; L^{m+1}(D))}).
\]

To prove Definition 2.2 (ii) of \( u \), let \( f \in C_b(\mathbb{R}) \). From Assumption 2.1 and (4.3), we have
\[
\sup_n \mathbb{E} \int_0^T \int_D \| a_n f(u_n) \|^2 dx dt \leq \sup_n \mathbb{E} \int_0^T \int_D \left( |u_n| + |u_n|^{\frac{m+1}{2}} \right)^2 dx dt < \infty.
\]
Combining (4.2) with the fact that \( \{ a_n f(u_n) \} \subset L^2(\Omega_T; H^1_0(D)) \), we have
\[
\sup_n \mathbb{E} \int_0^T \int_D \| a_n f(u_n) \|^2_{H^1(D)} < \infty.
\]

With the pointwise convergence and uniform integrability of \( u_n \) and Proposition 4.1 by taking a subsequence, we obtain the weak convergence of \( \{ a_n f(u_n) \} \) and \( \{ a_n \} \) in \( L^2(\Omega_T; H^1_0(D)) \) as \( n \to \infty \), and the limits are \( \{ a f(u) \} \) and \( \{ a \} \), respectively. On the other hand, for all \( \phi \in C^\infty_c(0, T) \times D \) and \( A \in \mathcal{F} \), based on the strong convergence of \( f(u_n)\phi \) and weak convergence of \( \{ a_n \} \), we have
\[
\mathbb{E} \left[ \int_A \int_0^T \partial_x [a f(u_n)\phi] \right] = \lim_{n \to \infty} \mathbb{E} \left[ \int_A \int_0^T \partial_x [a_n f(u_n)\phi] \right]
= \lim_{n \to \infty} \mathbb{E} \left[ \int_A \int_0^T f(u_n) \partial_x [a_n\phi] \right]
= \mathbb{E} \left[ \int_A \int_0^T f(u) \partial_x [a\phi] \right].
\]

For Definition 2.2 (iii), let \( A \in \mathcal{F} \). Combining Remark 4.3 with It\'o’s product rule, we have
\[
\int_0^T \int_D \eta(x)n(u_n) \partial_t \phi dx dt
= \mathbb{E} 1_A \int_D \eta(x) \phi(0) dx + \int_0^T \int_D \left( \int a_n \phi''(u_n) \Delta \phi + \int a_n \phi' \right)(x,u_n) \partial_x \phi dx dt
+ \int_0^T \int_D \left( \int a_n \phi''(u_n) \Delta \phi + \int a_n \phi' \right)(x,u_n) \partial_x \phi dx dt
+ \int_0^T \int_D \left( - \int a_n \phi''(u_n) \Delta \phi + \int a_n \phi' \right)(x,u_n) \partial_x \phi dx dt
+ \int_0^T \int_D \left( \frac{1}{2} |\nabla a_n(u_n)|^2 \phi - \nabla (a_n(u_n)) \nabla \phi \right) dx dt.
\]
\[
\int_0^T \int_D \left( \eta'(u_n) \phi \sigma_{x,k}^{\eta}(x,u_n) - \| \sigma_{x,k}^{\eta}(x,u_n) \phi \| \right) dxdW^k(t).
\]

Since \( (\eta'')^{1/2} \in C_b(\mathbb{R}) \), as the proof in checking (ii), we have
\[
\partial_x \| (\eta'')^{1/2}a_n \| (u_n) = (\eta''(u_n))^{1/2} \partial_x \| a_n \| (u_n),
\]
can assume that \( \partial_x \| (\eta'')^{1/2}a_n \| (u_n) \) converges weakly to \( \partial_x \| (\eta'')^{1/2}a \| (u) \) in \( L_2(\Omega_T; L_2(D)) \). Then, we also have the weakly convergence in \( L_2(\Omega_T \times D, \mu) \), where \( d\mu := 1_B \phi d\mu \otimes dt \otimes dx \), which indicates
\[
E_1_B \int_{t,x} \phi \eta''(u) |\nabla [a_n](u)|^2 \leq \liminf_{n \to \infty} E_1_B \int_{t,x} \phi \eta''(u_n) |\nabla [a_n](u_n)|^2.
\]

Therefore, taking inferior limit on \( (6.2) \) and using Proposition 6.1 and Assumption 2.4 with the almost sure convergence and uniformly integrability of \( u_n \), we have that \( u \) satisfies \( (6.3) \) almost surely. Therefore, \( u \) is actually an entropy solution.

Now, we focus on the uniqueness. For \( \eta \in \mathbb{N} \), define \( \xi_n := \xi \wedge \eta \) and denote by \( u_n \) the entropy solution of \( \Pi(\Phi, \xi_n) \) constructed in the proof of existence. From the construction of \( u_n \), with Lemma 3.9 and Proposition 4.4, the entropy solution \( u_n \) has the \((*)\)-property. Let \( \tilde{u} \) be an entropy solution of \( \Pi(\Phi, \xi) \). Note that
\[
E[|\tilde{u}(t, \cdot) - u_n(t, \cdot)|_{L^1(D)}] = E[|\tilde{u}(t, \cdot) - u_n(t, \cdot)|_{L^1(D)}] + E[|u_n(t, \cdot) - \tilde{u}(t, \cdot)|_{L^1(D)}].
\]

The estimate of the first part on the right hand side is obtained using Theorem 3.14 in the case that \( u \) in \( (\tilde{u} - u)^+ \) has the \((*)\)-property. Similarly, the second part can be estimated using Theorem 3.14 in the case that \( \tilde{u} \) in \( (\tilde{u} - u)^+ \) has the \((*)\)-property. Combining these two estimates, we have
\[
\text{ess sup}_{t \in [0,T]} E[|\tilde{u}(t, \cdot) - u_n(t, \cdot)|_{L^1(D)}] \leq C \text{Ess sup}_{t \in [0,T]} \tilde{u}(t, \cdot),
\]
where the constant \( C \) depends only on \( N, K, d, T \) and \( |D| \). Taking the limit \( \eta \to \infty \), we obtain Theorem 2.6 (i). Following this method and using Theorem 4.1 (i), we complete the proof of Theorem 2.6 (iii).

6. Some auxiliary estimates

**Lemma 6.1.** Suppose \( u \) is an entropy solution to the Dirichlet problem \( \Pi(\Phi, \xi) \). Under Assumptions 2.1, 2.4, and 2.5 if \( \xi \in L_{m+1}(\Omega, F_0; L_{m+1}(D)) \), we have
\[
\lim_{\gamma \to 0^+} \frac{1}{\gamma} \mathbb{E} \int_{t,x} |u(t, x) - \xi(x)|^2 dt = 0.
\]

**Proof.** From the partition of unity in Section 3, we can fix \( i \in \{0, 1, \ldots, N\} \) and define \( B := B_i, \psi := \psi_i \) and \( \varphi := \varphi_{i,i} \). Notice that \( \text{dist}(\text{supp } \psi, \partial B) > 0 \). When \( \varepsilon \) is small enough, we have \( \text{supp } \varphi \subset B \) for all \( x \in \text{supp } \psi \). Then, from the definition of \( \varphi \), we have \( \psi(y)\varphi(y - x) \in C_c^\infty(D) \) for all \( y \in \overline{D} \) and sufficient small \( \varepsilon > 0 \). Now, we only need to prove
\[
\lim_{\gamma \to 0^+} \frac{1}{\gamma} \mathbb{E} \int_0^\gamma \int_{t,x} |u(t, x) - \xi(x)|^2 \psi(x) dt = 0.
\]

We split it into three parts
\[
\frac{1}{\gamma} \mathbb{E} \int_0^\gamma \int_{t,x} |u(t, x) - \xi(x)|^2 \psi(x) dt \leq 2 \mathbb{E} \int_D \int_{\mathbb{R}^d} |\xi(y) - \xi(x)|^2 \psi(x) \varphi(y - x) dy dx
\]
\[
\begin{align*}
&+ \frac{2}{\gamma} \mathbb{E} \int_0^\gamma \int_D \int_D |u(t,x) - \xi(y)|^2 \psi(x,\eta, y-x) dy dx dt \\
&+ \frac{2}{\gamma} \mathbb{E} \int_0^\gamma \int_D \int_{D_{y,x}^1} |u(t,x)|^2 \psi(x,\eta, y-x) dy dx dt,
\end{align*}
\]

where \( \gamma \in [0, T) \) and \( D_{y,x}^1 := \{ y \in \mathbb{R}^d : \eta(x, y) > 0, \exists x \in D \} \setminus D \). See (6.21) for the definition of \( \xi \). Notice that \( |D_{y,x}^1| = O(\varepsilon) \). We first estimate the third term on the right hand side as

(6.3)

\[
\frac{2}{\gamma} \mathbb{E} \int_0^\gamma \int_D \int_{D_{y,x}^1} |u(t,x)|^2 \psi(x,\eta, y-x) dy dx dt
\]

\[
= \frac{2}{\gamma} \mathbb{E} \int_0^\gamma \int_D \int_{D_{y,x}^1} |u(t,x)|^2 \psi(x,\eta, y-x) dy dx dt
\]

\[
\leq C \sup_{t \in (0, T)} \int_{D_{y,x}^2} |u(t,x)|^2 dx,
\]

where \( D_{y,x}^2 := \{ x \in D : \eta(x, y) > 0, \exists y \in D_{y,x}^1 \} \). We also have \( |D_{y,x}^2| = O(\varepsilon) \). Now, we choose a non-negative function \( w_\varepsilon \in C_0^\infty(\mathbb{R}^d) \) such that \( w_\varepsilon(x) = 1 \) for all \( x \in D_{y,x}^2 \) and \( |\nabla w_\varepsilon| \leq C \varepsilon \).

Suppose that \( s \in (0, \gamma) \) is a Lebesgue point of the function

\[
t \mapsto \mathbb{E} \int_x |u(t,x)|^2 w_\varepsilon(x),
\]

and \( \theta \in (0, T - \gamma) \). Define \( V(\theta) : [0, T] \to \mathbb{R}^+ \) by \( V(\theta)(0) := 1 \) and \( V'(\theta) := -\theta^{-1} \mathbf{1}_{[s,s+\theta]}(t) \). We take a sequence of non-negative functions \( \varphi_{\theta,n} \in C_0^\infty((0, T)) \) satisfying

\[
\| \varphi_{\theta,n} \|_{L_\infty(0,T)} \vee \| \nabla \varphi_{\theta,n} \|_{L_1(0,T)} \leq 1,
\]

such that

\[
\lim_{n \to \infty} \| \varphi_{\theta,n} - V(\theta) \|_{H^s(0,T)} = 0.
\]

For each \( \delta > 0 \), define \( \eta_\delta \in C^2(\mathbb{R}) \) by

(6.4)

\[
\eta_\delta(0) = \eta''_\delta(0) = 0, \quad \eta''_\delta(r) := 2 \cdot \mathbf{1}_{[0, \delta^{-1})}(|r|) + (-|r| + \delta^{-1} + 2) \cdot \mathbf{1}_{[\delta^{-1}, \delta^{-1}+2]}(|r|).
\]

It is easy to find that \( \eta_\delta(r) \to r^2 \) as \( \delta \to 0^+ \). Using the entropy inequality (2.3) for \( u \) with \( \eta = \eta_\delta \in \mathcal{E}_0 \) and \( \phi_n(t,x) = \varphi_{\theta,n}(t) w_\varepsilon(x) \), letting \( n \to \infty \) and taking expectations, with Assumption 2.1 and (2.4)-(2.10) in Assumption 2.4 we have

\[
\frac{1}{\theta} \int_0^{s+\theta} \mathbb{E} \int_D \eta_\delta(u) w_\varepsilon(x) dx dt
\]

\[
\leq \mathbb{E} \int_D \eta_\delta(x) w_\varepsilon(x) dx + \mathbb{E} \int_0^{s+\theta} \int_D \eta''_\delta(u) |\nabla \mathbf{a}(u)|^2 w_\varepsilon(x) dx dt
\]

\[
+ C \mathbb{E} \int_0^{s+\theta} \int_D (1 + |u|^{m+1}) \left( \sum_{i,j} |\partial_{x,x} w_\varepsilon| + \sum_i |\partial_x w_\varepsilon| + w_\varepsilon \right) dx dt.
\]

Taking \( \delta \to 0^+ \) and then letting \( \theta \to 0^+ \), we have

\[
\mathbb{E} \int_D |u(s,x)|^2 w_\varepsilon(x) dx dt
\]

\[
\leq \mathbb{E} \int_D |\xi(x)|^2 w_\varepsilon(x) dx + 2 \mathbb{E} \int_0^s |\nabla \mathbf{a}(u)|^2 w_\varepsilon(x) dx dt
\]

\[
+ C \mathbb{E} \int_0^s \int_D (1 + |u|^{m+1}) \left( \sum_{i,j} |\partial_{x,x} w_\varepsilon| + \sum_i |\partial_x w_\varepsilon| + w_\varepsilon \right) dx dt
\]
Remark 3.1. Let \( D \) be a non-negative function, integrating over \( D \) and for some \( \varepsilon > 0 \). For fixed \( (y, z) \in D \times \mathbb{R} \), using the entropy inequality (2.3) with \( \tilde{\gamma} \), integrating to \( t \), we have

\[
\limsup_{\gamma \to 0^+} - \int x, u \phi \epsilon \leq 2 \cdot 1_{[0, 2\gamma]}(x, u) \partial_x \phi \leq - \frac{1}{\gamma} \cdot 1_{[0, \gamma]}(x, u).
\]

Note that \( \psi(\cdot) \tilde{\gamma}(y - \cdot) \in C^\infty_c(D) \) for all \( y \in \overline{D} \) and sufficient small \( \varepsilon > 0 \). For fixed \( (y, z) \in D \times \mathbb{R} \), using the entropy inequality (2.3) with \( \phi(t,x) = \frac{1}{\gamma} \cdot 1_{[0, \gamma]}(x, u) \) and \( \eta_r = \eta_{\tilde{\gamma}}(r - z) \) defined in (6.4), with Assumptions 2.3 and 2.4 we have

\[
\frac{1}{\gamma} \int_0^\gamma \int_{D \times \mathbb{R}} |u(t,x) - \xi(y)|^2 \psi(\cdot) \tilde{\gamma}(y - x) dt dx.
\]

Combining (6.2) with (6.5) - (6.6), we have

\[
\limsup_{\gamma \to 0^+} - \int x, u \phi \epsilon \leq 2 \cdot 1_{[0, 2\gamma]}(x, u) \partial_x \phi \leq - \frac{1}{\gamma} \cdot 1_{[0, \gamma]}(x, u).
\]

Since \( \xi \in L_{m+1}(\Omega \times D) \), the right hand side goes to 0 as \( \varepsilon \to 0^+ \). The proof is complete. \( \square \)

**Lemma 6.2.** Let \( u \in L_{m+1}(\Omega \times D_T) \) and for some \( \varepsilon \in (0, 1) \). Denote by \( K_\varepsilon \) the constant in Remark 3.4. Let \( g : \mathbb{R}^d \to \mathbb{R} \) be a non-negative function, integrating to 1 and supported on a ball of radius \( K \varepsilon \) centered at the origin. Under Assumption 2.4, one has

\[
\varepsilon \int_{t,x,y} |u(t,x) - u(t,y)| g(x - y) \leq C \varepsilon \frac{1}{t+1} \varepsilon (1 + \|u\|_{L_{m+1}(D_T)}^m + \|\nabla[a](u)\|_{L^1(D_T)}),
\]
\begin{equation}
\int_{t,x,y} E |\Phi(u(t,x)) - \Phi(u(t,y))|\varrho(x-y) \\
\leq C \varepsilon \left( 1 + \|u\|_{L^{m+1}(D_T)}^{m+1} + \|\nabla [a](u)\|_{L^2(D_T)}^2 \right),
\end{equation}

where $C$ depends on $d$, $K$, $\overline{K}$ and $T$.

**Proof.** We first prove (6.8). Define a set

$$D_\varepsilon := \{ y \in D : y + z \in D, \text{ for all } z \in \mathbb{R}^d \text{ with } |z| \leq \overline{K}\varepsilon \}.$$ 

Notice that $|D \setminus D_\varepsilon| = O(\varepsilon)$, combining with Assumption 2.1 and Hölder’s inequality, we have

\begin{equation}
\int_{t,x,y} E |\Phi(u(t,x)) - \Phi(u(t,y))|\varrho(x-y) \\
\leq E \int_t \int_{D_\varepsilon} \int_{\mathbb{R}^d} |\Phi(u(t,y + z)) - \Phi(u(t,y))|\varrho(z)dzdy \\
+ E \int_t \int_{D_\varepsilon} \int_{\mathbb{R}^d} |\Phi(u(t,x)) - \Phi(u(t,y))|\varrho(x-y)dzdy \\
\leq C \varepsilon \int_t \int_{D_\varepsilon} \int_{\mathbb{R}^d} |a(u)\nabla[a](u)|(y + \lambda z)d\lambda dzdy \\
+ \left( E \int_t \int_{D_\varepsilon} |\Phi(u(t,x)) - \Phi(u(t,y))|^{\frac{m+1}{m}} \varrho(x-y)dy \right)^{\frac{m}{m+1}} \\
\leq C \varepsilon \left( 1 + \|u\|_{L^{m+1}(D_T)}^{m+1} + \|\nabla [a](u)\|_{L^2(D_T)}^2 \right).
\end{equation}

Moreover, using Assumption 2.1 as in the proof of [DGG10 Lemma 3.1], we have

\begin{equation}
\int_{t,x,y} |u(t,x) - u(t,y)|\varrho(x-y) \\
\leq C E \int_{t,x,y} 1_{|u(t,x)|+|u(t,y)| \geq 1} |a(u(t,x)) - [a](u(t,y))|\varrho(x-y) \\
+ C \left( E \int_{t,x,y} 1_{|u(t,x)|+|u(t,y)| \leq 1} |a(u(t,x)) - [a](u(t,y))|\varrho(x-y) \right)^{\frac{m}{m+1}}.
\end{equation}

Besides, as (6.9), we have

\begin{equation}
\int_{t,x,y} |a(u(t,x)) - [a](u(t,y))|\varrho(x-y) \\
\leq C \varepsilon \int_t \int_{D_\varepsilon} \int_{\mathbb{R}^d} |\nabla[a](u)(y + \lambda z)|d\lambda dzdy \\
+ \left( E \int_t \int_{D_\varepsilon} |a(u(t,x)) - [a](u(t,y))|^2 \varrho(x-y)dy \right)^{\frac{1}{2}} \\
\cdot \left( E \int_t \int_{D_\varepsilon} \varrho(x-y)dy \right)^{\frac{1}{2}} \\
\leq C \varepsilon \left( 1 + \|u\|_{L^{m+1}(D_T)}^{m+1} + \|\nabla [a](u)\|_{L^2(D_T)}^2 \right).
\end{equation}

Combining with (6.10), we obtain inequality (6.7). \qed
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