On $\phi$-recurrent generalized Sasakian-space-forms

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Abstract

We show that there is no $\phi$-recurrent generalized Sasakian-space-forms, when $f_1 - f_3$ is a non-zero constant.

Keywords: Contact manifold, Generalized sasakian-space-forms, $\phi$-recurrent

1 Introduction

In differential geometry, the curvature of a Riemannian manifold $(M, g)$ plays a fundamental role, and, as is well known, the sectional curvatures of a manifold determine the curvature tensor $R$ completely. A Riemannian manifold with constant sectional curvature $c$ is called a real-space-form, and its curvature tensor satisfies the equation

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}. \quad (1)$$

Models for these spaces are the Euclidean spaces ($c = 0$), the spheres ($c > 0$) and the hyperbolic spaces ($c < 0$).

A Sasakian manifold $(M, \phi, \xi, \eta, g)$ is said to be a Sasakian-space-form, if all the $\phi$-sectional curvatures $K(X \wedge \phi X)$ are equal to a constant $c$, where $K(X \wedge \phi X)$ denotes the sectional curvature of the section spanned by the unit vector field $X$ orthogonal to $\xi$ and $\phi X$. In such a case, the Riemannian curvature tensor of $M$ is given by

$$R(X, Y)Z = \frac{c + 3}{4} \{g(Y, Z)X - g(X, Z)Y\} + \frac{c - 1}{4} \{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X\} + \frac{1}{4} \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\}. \quad (1)$$

As a natural generalization of these manifolds, P. Alegre, D. E. Blair and A. Carriazo [1] introduced the notion of generalized Sasakian-space-form. It is defined as almost contact metric manifold with Riemannian curvature tensor satisfying an equation similar to (1), in which the constant quantities $\frac{c + 3}{4}$ and $\frac{c - 1}{4}$ are replaced by differentiable functions.

Local symmetry is a very strong condition for the class of Sasakian manifolds. Indeed, such spaces must have constant curvature equal to 1 [5]. Thus Takahashi introduced the notion of a (locally) $\phi$-symmetric space in the context.

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of Sasakian geometry \cite{7}. Generalizing the notion of \(\phi\)-symmetry, De-Shaikh-Biswas introduced the notion of \(\phi\)-recurrent Sasakian manifold \cite{4}. Then Sarkar-Sen studied the notion of \(\phi\)-recurrent for Generalized Sasakian-Space-Forms \cite{6}. They deduce some results when \(f_1 - f_3\) is a nonzero constant and the dimension of the manifold is bigger than 3. For example the following theorem is one of them.

**Theorem 1.1.** \cite{6} A \(\phi\)-recurrent generalized-Sasakian-space-form \((M^{2n+1}, g)\) is an Einstein manifold provided \(f_1 - f_3\) is a non-zero constant.

In this paper we show that there is no \(\phi\)-recurrent generalized Sasakian-space-forms that the difference of \(f_1\) and \(f_3\) is a nonzero constant at all, when the dimension of the manifold is bigger than 3.

## 2 Contact Metric Manifolds

We start by collecting some fundamental material about contact metric geometry. We refer to \cite{2, 3} for further details.

A differentiable \((2n + 1)\)-dimensional manifold \(M^{2n+1}\) is called a contact manifold if it carries a global differential 1-form \(\eta\) such that \(\eta \wedge (d\eta)^n \neq 0\) everywhere on \(M^{2n+1}\). This form \(\eta\) is usually called the contact form of \(M^{2n+1}\). It is well known that a contact manifold admits an almost contact metric structure \((\phi, \xi, \eta, g)\), i.e., a global vector field \(\xi\), which will be called the characteristic vector field, a \((1, 1)\) tensor field \(\phi\) and a Riemannian metric \(g\) such that

\[
\begin{align*}
(i) \quad & \eta(\xi) = 1, \quad (ii) \quad \phi^2 = -\text{Id} + \eta \otimes \xi, \\
& g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\end{align*}
\]

for any vector fields on \(M^{2n+1}\). Moreover, \((\phi, \xi, \eta, g)\) can be chosen such that \(d\eta(X, Y) = g(\phi X, Y)\) and we then call the structure a contact metric structure and the manifold \(M^{2n+1}\) carrying such a structure is said to be a contact metric manifold. As a consequence of (2) and (3), we have

\[
\phi \xi = 0, \quad \eta \circ \phi = 0, \quad d\eta(\xi, X) = 0.
\]

Denoting by \(\mathcal{L}\), Lie differentiation, we define the operator \(h\) by following

\[
hX := \frac{1}{2}(\mathcal{L}_\xi \phi)X.
\]

The \((1, 1)\) tensor \(h\) is self-adjoint and satisfy

\[
(i) \quad h\xi = 0, \quad (ii) \quad h\phi = -\phi h, \quad (iii) \quad Tr h = Tr h = 0.
\]

Since the operator \(h\) anti-commutes with \(\phi\), if \(X\) is an eigenvector of \(h\) corresponding to the eigenvalue \(\lambda\), then \(\phi X\) is also an eigenvector of \(h\) corresponding to the eigenvalue \(-\lambda\).

If \(\nabla\) is the Riemannian connection of \(g\), then

\[
\nabla_X \xi = -\phi X - \phi h X, \quad \nabla_\xi \phi = 0.
\]

A contact structure on \(M^{2n+1}\) gives rise to an almost complex structure on the product \(M^{2n+1} \times \mathbb{R}\). If this structure is integrable, then the contact metric manifold is said to be Sasakian.
Definition 2.1. A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be locally $\phi$-symmetric if
\[
\phi^2((\nabla_W R)(X, Y)Z) = 0,
\]
for all vector fields $W, X, Y, Z$ orthogonal to $\xi$. If (6) holds for all vector fields $W, X, Y, Z$ (not necessarily orthogonal to $\xi$), then we call it $\phi$-symmetric.

The notion locally $\phi$-symmetric was introduced for Sasakian manifolds by Takahashi [7].

Definition 2.2. A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be $\phi$-recurrent if there exists a non-zero 1-form $A$ such that
\[
\phi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z,
\]
for all vector fields $X, Y, Z, W$.

This notation was introduced for Sasakian manifolds by De-Shahk-Biswas [4].

Given an almost contact metric manifold $M(\phi, \xi, \eta, g)$, we say that $M$ is generalized Sasakian-space-form if there exist three functions $f_1, f_2, f_3$ on $M$ such that the curvature tensor $R$ is given by
\[
R(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X\}
+ 2g(X, \phi Y)\phi Z + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\}
+ g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi,
\]
for any vector fields $X, Y, Z$ on $M$. In such a case we denote the manifold as $M(f_1, f_2, f_3)$. In [1] the authors cited several examples of such manifolds. If $f_1 = \frac{c+3}{4}, f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$ then a generalized Sasakian-space-form with Sasakian structure becomes Sasakian-space-form.

We also have for a generalized Sasakian-space-form (see [1], [9])
\[
R(X, Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y],
\]
\[
R(\xi, X)Y = (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X],
\]
\[
\eta(R(X, Y)Z) = (f_1 - f_3)(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)).
\]

3 Nonexistence of $\phi$-Recurrent Generalized Sasakian Space Form

In this section, we give the main result of the paper which implies some results of [6], for example Theorem 3.1, are incorrect.

Theorem 3.1. There is no $\phi$-recurrent generalized Sasakian space form $M^{2n+1}$, $n > 1$, with $f_1 - f_3 \neq 0$.

Proof. Let $M^{2n+1}$ ($n > 1)$, be a $\phi$-recurrent Sasakian manifold. Then using (ii) of (2) and (7), we get
\[
-(\nabla_W R)(X, Y)Z + \eta((\nabla_W R)(X, Y)Z)\xi = A(W)R(X, Y)Z,
\]
or
\[(\nabla_W R)(X, Y)Z = \eta((\nabla_W R)(X, Y)Z)\xi - A(W)R(X, Y)Z, \]  
(12)
where \(X, Y, Z, W\) are arbitrary vector fields on \(M\) and \(A\) is a non-zero 1-form on \(M\). Using Bianchi identity
\[(\nabla_W R)(X, Y)Z + (\nabla_X R)(Y, W)Z + (\nabla_Y R)(W, X)Z = 0, \]
in (12) implies that
\[A(W)R(X, Y)Z + A(X)R(Y, W)Z + A(Y)R(W, X)Z = 0. \]
Applying \(\eta\) to the above equation yields
\[A(W)\eta(R(X, Y)Z) + A(X)\eta(R(Y, W)Z) + A(Y)\eta(R(W, X)Z) = 0. \]  
(13)
Since \(f_1 - f_2\) is non-zero constant then plugging (11) in (13), it follows that
\[A(W)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] + A(X)[g(W, Z)\eta(Y) - g(Z, Y)\eta(W)] + A(Y)[g(W, Z)\eta(W) - g(W, Z)\eta(X)] = 0. \]  
(14)
Now, we choose the \(\phi\)-basis \(\{e_i, \phi e_i, \xi\}_{i=1}^n\) for \(M^{2n+1}\) \((n > 1)\). By setting \(Y = Z = e_i, W = e_j (j \neq i)\) and \(X = \xi\) in (14), we obtain
\[A(e_j) = 0. \]
Since \(j\) is arbitrary, then we deduce
\[A(e_k) = 0, \quad \forall k = 1, \ldots, n. \]  
(15)
Similarly, setting \(Y = Z = e_i, W = \phi e_j\) and \(X = \xi\) in (14) implies
\[A(\phi e_j) = 0. \]
Thus we deduce
\[A(\phi e_k) = 0, \quad \forall k = 1, \ldots, n. \]  
(16)
Let \(Y\) be a non-zero vector field orthogonal to \(\xi\). Then (9) gives us
\[(\nabla_\xi R)(\xi, Y)\xi = \nabla_\xi R(\xi, Y)\xi - R(\xi, \nabla_\xi Y)\xi = (f_3 - f_1)\eta(\nabla_\xi Y)\xi \]
From \(g(Y, \xi) = 0\) we obtain \(g(\nabla_\xi Y, \xi) = 0\) or \(\eta(\nabla_\xi Y) = 0\). Therefore using the above equation we deduce
\[(\nabla_\xi R)(\xi, Y)\xi = 0. \]  
(17)
Thus from (7) and (9) we derive that
\[0 = (f_3 - f_1)A(\xi)R(\xi, Y)\xi = (f_3 - f_1)A(\xi)Y. \]
The above equation give us \(A(\xi) = 0\). Thus by using (14) and (15), we deduce that \(A = 0\) on \(M\), which is a contradiction. \(\square\)
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