ASYMMETRY PARAMETER ROLE IN DESCRIPTION OF PHASE STRUCTURE OF LATTICE GLUODYNAMICS AT FINITE TEMPERATURE.

L. A. Averchenkova, K. V. Petrov, V. K. Petrov, G. M. Zinovjev

Bogolyubov Institute for Theoretical Physics
National Academy of Sciences of Ukraine
Kiev 143, UKRAINE *

November 19, 2018

Abstract

The role of lattice asymmetry parameter $\xi$ in description of the $SU(2)$ gluodynamics phase structure at finite temperature is studied analytically. The fact that renormalization group relations which permit to remove the lattice asymmetry parameter from the thermodynamical quantities in the "naive" limit don’t do the same in the approximation $SU(N) \simeq Z(N)$ keeping the "naive" limit results at the same time was pointed out. An additional condition which would fix $\xi$ is needed for this dependence removal.

INTRODUCTION.

Following the heart point of the renormalization group theory after any regularization removal (in particular, lattice one) observed quantities should go to their physical values and should not depend on the renormalization scheme. So, although going to the continuum from a symmetrical lattice and from asymmetrical one are two different renormalization procedures in the gauge fields theory it is natural to expect that in the continuum limit no lattice asymmetry effects will be revealed. At the same time, strictly speaking, the renormalization group trajectories are universal (i.e., they don’t

*e-mail: hedpitg@gluk.apc.org, gezin@hrz.uni-bielefeld.de, lily@knb.kiev.ua
depend on the renormalization scheme) only near the critical point at zero coupling while at intermediate and strong couplings this statement is still questionable. It is quite probable the trajectories will depend on the degree of lattice asymmetry \[1\]. Moreover, today’s level of this problem understanding does not permit to exclude other possible revealings of the lattice asymmetry.

In this paper we consider the finite temperature $SU(N)$-gluodynamics on an asymmetrical lattice

\[ a_\mu \neq a_\nu; \quad \mu = 0, 1, 2, 3; \quad a_0N_0 = \beta; \quad a_kN_k = L_k; \quad k = 1, 2, 3. \]

$a_\mu$ – the lattice spacing along the direction $\mu$. The most wide-spread example of such asymmetry introducing is the Hamiltonian limit \[1, 2\] (where the lattice distance along the space directions is fixed and the spacing along the time direction is going to zero). The renormalization procedure independence actually means that renormalized couplings of both theories (for example, Hamiltonian and Euclidean $SU(N)$ lattice gauge theories) should be get equal. This naturally imposes the relation on the scales $\frac{\Lambda_E}{\Lambda_H}$. Following chosen approach \[1, 2\] this is enough for two renormalized theories to give the ”same physics”. Indeed, if in the standard Wilson’s action

\[ -S = \kappa_{\mu\nu} \sum_{x, \mu<\nu} \text{Re} \text{Tr}(1 - \frac{1}{N_c}U_\mu(x)U_\nu(x + \mu)U_\nu^+(x + \nu)U_\nu^+(x)) \quad (1) \]

we restrict ourselves to only ”naive” limit (which assumes the expansion of $SU(N)$ matrices around I-matrix, close to I-matrix or any fixed matrix)

\[ U_\mu(x) = e^{ia_\mu g_\mu A_\mu(x)} \simeq 1 + ia_\mu g_\mu A_\mu(x) \quad (2) \]

and we take

\[ \kappa_{\mu\nu} = \frac{2N}{g_\mu g_\nu} \frac{a_0 a_1 a_2 a_3}{a_\mu^2 a_\nu^2} \quad (3) \]

then at least in the Hamiltonian (H) and the Euclidean (E) \([a_\nu = a; \ g_\nu^2 \rightarrow g_E^2]\) limits the spectra of hamiltonians coincide. Moreover, not only in the classical case ($g_E^2 = g_H^2$), but in the quantum case ($g_E^2 \simeq g_H^2 + O(g_H^4)$) too.

However, investigation of the finite temperature $SU(N)$ lattice gauge theory in ”naive” limit is carried out using $SU(N)$ matrices expansion around fixed matrix thereby leaving the center $Z(N)$ out of the game. But the role of the gauge group center in the finite temperature QCD phase structure is well-known. Therefore, we will study the lattice asymmetry effects contributed by the center $Z(N)$. To do this we will use the approach $SU(N) \simeq Z(N)$, i.e., we will consider the finite temperature lattice gauge theory in which the
link variables are the center elements of the $SU(N)$ gauge group

$$\sigma_\mu(x) = e^{\frac{2\pi i q_\mu}{N}}; \quad q_\mu = 0, 1, 2 \ldots N - 1 \quad (4)$$

On the other hand, for theories with gauge groups which have a trivial center (for example, $SU(N)/Z(N)$) it may be quite reasonable to restrict ourselves to "naive" limit, whereas for the $SU(N)$ gauge theories such a restriction requires special substantiation because there is no naive limit for the center $Z(N)$ in common sense. Building the continuum quantum field theory becomes possible only if $Z(N)$ lattice gauge system undergoes the second order phase transition at zero of the renormalization group function $[3, 4, 5]$. To ensure agreement with the "naive" limit results we chose $\kappa_{\mu\nu}$ (3) to guarantee the coincidence of naive limits for the models with $SU(N)$ gauge group on the symmetrical lattice and asymmetrical one.

In the future we will calculate the corrections to our approximation. As a justification of our approach we should note that effect of the quantum fluctuations, which are taken into account besides the $Z(N)$ topological excitations, amounts to simple change in the coupling constant $[2, 3, 4]$

$$\kappa_{\text{new}} = \kappa_{\text{old}} - (N^2 - 1)/4 \quad (5)$$

Additional term (5) depends neither on $a_\mu$ nor on $\kappa_\mu$. Therefore, it does not depend on the lattice asymmetry. So, there is a reason to hope that including the quantum fluctuations besides of $Z(N)$ solutions in our theory on asymmetrical lattice will not change (5), i.e. will just result in the finite renormalization of couplings.

It will be enough for our purposes to consider the simplest case of lattice asymmetry, when $\kappa_{0n} \equiv \kappa_\tau$ and $\kappa_{nm} \equiv \kappa_\sigma$, i.e., we allow the couplings to be different for plaquettes lying in the temporal and in the spatial planes. Our results may be easily generalized to the case (3). We will also limit ourselves to the analyses of four limiting cases:

$$1) \ \kappa_\sigma \rightarrow 0; \quad 2) \ \kappa_\tau \rightarrow 0; \quad 3) \ \kappa_\sigma \rightarrow \infty; \quad 4) \ \kappa_\tau \rightarrow \infty; \quad (6)$$

(the estimations of phase structure in the whole plane $\kappa_\tau \otimes \kappa_\sigma$ we are planning to present in next paper).

As known our Wilson’s action is splitting into two parts:

$$- S(\sigma) = \kappa_{\sigma} \sum_{P_\sigma} \left(1 - \frac{1}{N} \text{Re} \text{Tr} \sigma_n(x) \sigma_n(x + n) \sigma_n^+(x + m) \sigma_n^+(x)\right) + \ldots$$
\[
\kappa_\tau \sum_{P_\tau} \left( 1 - \frac{1}{N} \text{Re} \text{Tr} \sigma_0(\vec{x}, \tau) \sigma_n(\vec{x}, \tau + 0) \sigma_n^+(\vec{x} + n, \tau) \sigma_n^+(\vec{x}, \tau) \right) 
\]

where \( \kappa_\sigma = 2N a_\sigma / g_\sigma^2 a_\sigma \) and \( \kappa_\tau = 2N a_\sigma / g_\tau^2 a_\tau \); \( a_\sigma(a_\tau) \) is the spatial (temporal) spacing and the sum \( P_\sigma(P_\tau) \) runs over all purely space-like (time-like) plaquettes.

Let’s first consider \( \gamma_\sigma \equiv \text{th} \kappa_\sigma \) and \( \gamma_\tau \equiv \text{th} \kappa_\tau \) independent temporary ignoring the dependence of \( g_\sigma(g_\tau) \) on \( a_\sigma(a_\tau) \) through the renormalization group relations, i.e., supposing that by varying \( a_\sigma \) and \( a_\tau \) we can get to any point of the square (\( 0 \leq \gamma_\tau \leq 1; \ 0 \leq \gamma_\sigma \leq 1 \)). We will investigate each of the limiting cases separately, joining the results for the couplings critical values through the same parameter.

**THE AREA OF SMALL \( \kappa_\sigma \).**

In the case \( \gamma_\sigma \simeq 0 \) we may throw off the magnetic part of action thereby leaving only the spin configurations with time-like plaquettes. In the static gauge we can sum over the space spin configurations \( \{ \sigma_n \} \), finally gaining the Ising model which is much known of.

Now let us dwell upon that. Fixing the static gauge \( \sigma_0(\vec{x}, \tau) = \omega_{\vec{x}} \) we consider the following gauge transitions

\[
\begin{align*}
\sigma_n(\vec{x}, \tau) & \rightarrow \omega_{\vec{x}}^\tau \sigma_n(\vec{x}, \tau) \omega_{\vec{x}+n}^{-\tau} \\
\sigma_n^+(\vec{x}, \tau) & \rightarrow \omega_{\vec{x}+n}^{-\tau} \sigma_n^+(\vec{x}, \tau) \omega_{\vec{x}}^\tau
\end{align*}
\]

Imposing the periodical boundary conditions in temporal direction upon \( \sigma \)-matrices: \( \sigma_n(\vec{x}, \tau = 1) \equiv \sigma_n(\vec{x}, \tau = N_\tau) \), all matrices \( \omega_{\vec{x}} \) are grouped on the last links resulting in the Polyakov’s loops

\[
\Omega_{\vec{x}} = \prod_{\tau=1}^{N_\tau} \omega_{\vec{x}} = \omega_{\vec{x}}^{N_\tau}
\]

We come to the following action

\[
-S_{P_\tau} = \kappa_\tau \sum_{\vec{x},n} \left[ \sum_{\tau=1}^{N_\tau-2} \left( 1 - \frac{1}{N} \text{Re} \text{Tr} \sigma_n(\vec{x}, \tau) \sigma_n^+(\vec{x}, \tau + 1) \right) + \left( 1 - \frac{1}{N} \text{Re} \text{Tr} \Omega_{\vec{x}} \sigma_n(\vec{x}, N_\tau - 1) \Omega_{\vec{x}+n}^+ \sigma_n^+(\vec{x}, 1) \right) \right]
\]

\[10\]
Summing over all configurations $\{\sigma_n\}$ in $Z(2)$ gauge system we get

$$-S_{P_\tau} = \kappa_\tau \sum_{\vec{x},n} \Omega_{\vec{x}} \Omega^*_{\vec{x}+n}$$

$$\text{th} \kappa_\tau = (\text{th} \kappa_\tau)^{N_\tau} = \tilde{\gamma}_\tau$$

As known, for the Ising model there exists the critical value of the coupling $\tilde{\gamma}_\tau = \gamma_c$ separating two phases. Consequently, the partition function (11) will have a critical point at $\gamma^c_\tau = \gamma^{1/N_\tau}$.

**THE AREA OF SMALL $\kappa_\tau$.**

In the other limiting case $\gamma_\tau \approx 0$ the functional integral is highly peaked about configurations with space-like plaquettes and we may throw off the electric part of action. In this case the partition function turns out to fall to $N_\tau$ equivalent disconnected contributions each of which is from a separate 3-dimensional layer. We get the set of standard 3-dimensional Wegner models

$$-S_{P_\sigma} \simeq \sum_{\tau=1}^{N_\tau} \sum_{x,n,m} \kappa_\sigma \sigma_n \sigma_m \sigma^*_n \sigma^*_m$$

There is no interaction between the layers, so summing over $\sigma_n(x, \tau)$ may be done independently for every $\tau = \text{const}$ layer.

The lattice dual to original hypercubical lattice can be constructed by shifting the lattice by half a lattice spacing in each direction (see, for example, [8]). Geometrical duality transforms $q$-dimensional manifolds into $(d-q)$-dimensional ones. "Dual coupling constant" $\tilde{\kappa}$ for the $Z(2)$ gauge theory $\tilde{\kappa}_\sigma = -\frac{1}{2} \ln \text{th} \kappa_\sigma$ is a monotonically decreasing function of the "original" coupling. The set of 3-dimensional Wegner models transforms into the set of 3-dimensional Ising models with spins in sites under duality transformations. These Ising models exhibit the transition at $\gamma^c_c$ simultaneously. Therefore, for the partition function (12) there is a critical point at $\gamma^c_\sigma = \frac{1-\gamma_c}{1+\gamma_c}$.

**TWO ANOTHER LIMITING CASES:** $\kappa_\tau \to \infty$ AND $\kappa_\sigma \to \infty$. 

When considering the duality transformation in 4-dimensional space-time it should be pointed out that just space-like plaquettes transform into time-like ones. In the other words, 

$$\kappa'_\sigma = -\frac{1}{2} \ln \tanh \kappa_\tau \quad \text{or} \quad \gamma'_\sigma = \frac{1}{1 + \gamma_{\tau}}$$

(13)

and vice versa, 

$$\kappa'_\sigma = -\frac{1}{2} \ln \tanh \kappa_\sigma.$$  

This statement becomes clear from the following. Let us rewrite the partition function of \( Z(2) \) system in the form

\[
Z = \sum_{\{\sigma\}} e^{\kappa_{\mu \nu} \sum_{x,\mu \nu} \square_{x,\mu \nu}} \\
= \sum_{\{\sigma\}} \prod_{x,\mu \nu} \text{ch}\kappa_{\mu \nu} (1 + \square_{x,\mu \nu} \tanh \kappa_{\mu \nu}) \\
= e^{-Nf} \sum_{\{q\}} \prod_{x,\mu \nu} \left( e^{\ln \tanh \kappa_{\mu \nu}} \right)^{q_{x,\mu \nu} + 1} 2 \delta_2 \left( \sum_{\mu=-3,\mu\neq\pm\nu}^{3} q_{x,\mu \nu} + 1 \right)
\]

where \( \square_{x,\mu \nu} = \sigma_\mu(x) \sigma_\nu(x + \mu) \sigma_\nu^*(x + \nu) \sigma_\mu^*(x) \) and \( f = N_{\tau}N_{\sigma}^3 \sum_{\mu,\nu} \ln \text{ch}\kappa_{\mu \nu} \).

We introduced a new set of variables \( \{q\} \) - one for each plaquette. The partition function is not equal to zero only if \( q_{x,\mu \nu} \) satisfies the following condition on the sum over six \( q_{x,\mu \nu} \) (associated with six plaquettes which adjoin the link \( x, \nu \), see fig.1):

\[
\frac{1}{2} \sum_{\mu=-3,\mu\neq\pm\nu}^{3} (q_{x,\mu \nu} + 1) = 0 \mod 2 \quad \text{or} \quad \sum_{\mu=-3,\mu\neq\pm\nu}^{3} q_{x,\mu \nu} = 2 \mod 4
\]

(15)

The solution of last equation can be found if we associate every \( q_{x,\mu \nu} \) with one of the cube plane and

\[
q_{x,\mu \nu} = s_\rho(x)s_\lambda(x + \rho)s_\lambda^*(x + \lambda)s_\rho^*(x) \\
\nu \neq \mu \neq \rho \neq \lambda
\]

(16)

where the dual link variable \( s_\rho(x) \) is the element of the \( Z(2) \) group. It becomes intuitively evident, if we consider the starting case when all \( s \) are equal 1. It dictates for \( \sum_{\mu\neq\pm\nu} q_{x,\mu \nu} \) to be equal \( 6 \mod 4 = 2 \mod 4 \). Every link enters the
solution twice (because the plaquettes form a cube) and changing the sign of a link to opposite results in changing \(\sum_{\mu=-3;\mu\neq\pm\nu} q_{x,\mu\nu}\) only by \(\pm 4\).

Consequently, in the plane \(\kappa_{\tau} \otimes \kappa_{\sigma}\) there is the self-duality line (see fig.2). R. Balian, J.M. Drouffe, C. Itzykson [9] pointed out the possibility of the critical behaviour at \(\kappa_c = 0.44\) for the 4-dimensional \(Z(2)\) pure gauge theory on symmetrical lattice supposing this critical point is single.

So, now we realize that under duality transformation our original 4-dimensional theory (7) transforms into the same one but with new coupling constants.

\[- S' = \kappa_{\sigma}' \Box_{\sigma} + \kappa_{\tau}' \Box_{\tau} \tag{17}\]

For this dual representation we can consider the two limiting cases in precisely the same way we have done for the original theory.

The case \(\gamma_{\sigma}' = \text{th} \kappa_{\sigma}' \simeq 0\) for the original theory means \(\gamma_{\sigma} \simeq 1\) in according to (13). Throwing off time-like part of the action we can go again from the set of 3-dimensional Wegner models to the set of standard 3-dimensional Ising models via the duality transformation. And after that we may come back to variables of the original theory.

\[
\kappa_{\sigma}' \sum_{P_{\sigma}} \Box_{\sigma} \rightarrow \kappa'_{\sigma} \sum_{\vec{x}n} s_{\vec{x}} s_{\vec{x}+n}^*)
\text{th} \kappa'_{\sigma} = \gamma'_{\sigma} = \frac{1 - \gamma'_{\sigma}}{1 + \gamma'_{\sigma}}
\gamma_{\sigma}^c = \frac{1 - \gamma_{\sigma}^c}{1 + \gamma_{\sigma}^c} \quad \gamma_{\tau}^c = \frac{1 - \gamma_{\tau}^c}{1 + \gamma_{\tau}^c} = \gamma_c
\tag{18}\]

Here tilde \(\tilde{\kappa}\) (prime \(\kappa'\)) means duality transformations in three (four) dimensions, respectively. The original theory undergoes the phase transition at \(\gamma_{\sigma} \simeq 1\) and \(\gamma_{\tau} = \gamma_c\).

Similarly, let’s consider the limit \(\gamma_{\sigma}' \simeq 0\) (it corresponds \(\gamma_{\tau} \simeq 1\) for the original theory) as we have done already for the limit \(\gamma_{\sigma} \simeq 0\). We can find the phase transition at \(\gamma_{\tau} \simeq 1\); \(\gamma_{\sigma} = \frac{1 - \gamma_{\sigma}^c/\kappa_{\tau}}{1 + \gamma_{\sigma}^c/\kappa_{\tau}}\). We may depict all these critical points in the plane \(\gamma_{\tau} \otimes \gamma_{\sigma}\) (fig.3).

To avoid arising infinite constant in the partition function at \(\gamma_{\sigma} = 1\) and at \(\gamma_{\tau} = 1\), we took \(\gamma_{\sigma}\) and \(\gamma_{\tau}\) close to 1 but not equal 1 exactly.
ANOTHER APPROACH TO THE CONSIDERATION OF LIMITS:
\( \kappa_\tau \to \infty \) AND \( \kappa_\sigma \to \infty \).

There is also another method to find critical points 3 and 4 (fig.3), different from the above duality transformation approach. Let us build up the effective action for the case \( \kappa_\sigma \to \infty \). It is obvious that space-like plaquette variables in this case \( \Box_{nm} \equiv \Box_\sigma = 1 \). The gauge is completely fixed if we have a maximal tree, i.e. tree which will have a closed loop after adding one more link to it.

We may choose the maximal tree in two dimensions like a snail as shown at the fig.4. By analogy we may obtain such a maximal tree for 3 dimensions. That choice of the maximal tree provides that the equation \( \Box_\sigma = 1 \) will have only single solution: \( \sigma^n_x = 1 \). Really, if we consider the first plaquette, which have three link variables equal 1 due to the gauge condition and provided with \( \Box_\sigma = 1 \), the fourth link is to be equal 1 also. Now we can apply these considerations to the second plaquette in the snail and so on, getting finally \( \sigma^n_x = 1 \). As a result, only temporal links will survive in the action.

\[
- S = \gamma_\tau \sum_\tau \sum_{\vec{x}n} \sigma_0(\vec{x}, \tau) \sigma_0(\vec{x} + n, \tau)
\]  

We’ve got the set of 3-dimensional independent Isings with critical point at \( \gamma_\sigma \simeq 1; \gamma_\tau = \gamma_c \) for our original system.

In the limit \( \gamma_\tau \simeq 1 \) the plaquette variables \( \Box_{0n} \equiv \Box_\tau = 1 \). In the Hamiltonian gauge \((\sigma_0 = 1) \sigma_n(t) = \sigma_n(t + 1) = \bar{\sigma}_n \) are time independent. We come to the static 3-dimensional Wegner system

\[
- S = N_\tau \kappa_\sigma \sum_{\vec{x}n} \bar{\sigma}_n(\vec{x}) \bar{\sigma}_m(\vec{x} + n) \bar{\sigma}_n(\vec{x} + n + m) \bar{\sigma}_m(\vec{x} + m)
\]  

which transforms into the Ising model under duality transformations.

\[
N_\tau \kappa_\sigma^c = -\frac{1}{2} \ln \gamma_\sigma^c \\
\kappa_\sigma^c = -\frac{1}{2} \ln \gamma_\sigma^{c1/N_\tau} \\
\gamma_\sigma^c = \frac{1 - \gamma_c^{1/N_\tau}}{1 + \gamma_c^{1/N_\tau}}
\]  

So, this alternative method confirms the results obtained previously.
We would like to estimate the connected correlation function \( \langle \sigma_x \sigma_{x+R} \rangle \) for the Ising model with different couplings in each direction within the spherical model. The crucial point is the following condition:

\[
\frac{1}{N^3} \sum_x \sigma_x^2 = 1 \tag{22}
\]

Then

\[
Z = \sum_{\{\sigma\}} \int_{c-i\infty}^{c+i\infty} \frac{d\alpha}{2\pi i} e^{\alpha N_\sigma^3 - \alpha \sum_x \sigma_x^2 + \frac{1}{2} \sum_{x,n} \kappa'_n \sigma_x \sigma_{x+n}} \tag{23}
\]

where \( \kappa'_n = \kappa_{mk} \) the constant \( c \) is chosen to ensure the legitimacy of interchanging the integration and summation order. It means that \( c \) is a line to the right of all \( \alpha \)-singularities.

We can rewrite the partition function as:

\[
Z = \int \frac{d\alpha}{2\pi i} e^{\alpha N_\sigma^3} e^{-\frac{1}{2} \sigma_x A_{x-x'} \sigma_{x'}} \tag{24}
\]

where

\[
A_{x-x'} = \alpha \delta_{x'}^x - \sum_{n=1}^3 \kappa'_n \delta_{x+n} = \int (\alpha - \sum_{n=1}^3 \kappa'_n \cos \phi_n) e^{i\phi(x-x')} d^3\phi \tag{25}
\]

The correlation function \( \langle \sigma_x \sigma_{x+R} \rangle \) can be calculated as the derivative of generation function over sources:

\[
\langle \sigma_x \sigma_{x+R} \rangle = \langle \sigma_0 \sigma_R \rangle = \frac{1}{Z} \frac{\partial}{\partial \eta_0} \frac{\partial}{\partial \eta_R} \int d\alpha \sum_{\{\sigma\}} e^{\alpha N_\sigma^3 - \sigma_x A_{x-x'} \sigma_{x'} + \eta_0 \sigma_x} \tag{26}
\]

and after shifting integration’s variables we have

\[
\langle \sigma_x \sigma_{x+R} \rangle = \frac{\partial}{\partial q_0} \frac{\partial}{\partial q_R} \int d\alpha e^{\frac{1}{2} q_x A_{x-x'}^{-1} q_{x'}} \tag{27}
\]

\[
= A_R^{-1} = \int \frac{e^{-i\phi R}}{\alpha_0 - \sum_{n=1}^3 \kappa'_n \cos \phi_n} d^3\phi
\]

\( \alpha_0 \) is the saddle point which is determined by the condition

\[
\int \frac{d^3\phi}{\alpha_0 - \sum_{n=1}^3 \kappa'_n \cos \phi_n} = \langle \sigma_0^2 \rangle = 1 \tag{28}
\]
At $R_n \to \infty \quad \phi_n \to 0$ and the pole will be determined by

$$\frac{1}{2} \sum_{n=1}^{3} \kappa'_n \phi_n^2 = -\alpha_0 + \sum_{n=1}^{3} \kappa'_n$$

(29)

Introducing ”symmetrical” variables

$$\phi_n = \frac{\zeta}{\sqrt{\kappa_n}}$$

(30)

$$\zeta = i \sqrt{2(\alpha_0 - \sum_{n=1}^{3} \kappa'_n)}$$

we obtain

$$\langle \sigma_0 \sigma_R \rangle = e^{\zeta \sum_{n=1}^{3} \frac{r_n}{\sqrt{\kappa_n} a_n}}$$

(31)

where $r = R_n a_n$

So, the potential between two probe sources depends on the choice of $n$.

**THE PHASE STRUCTURE AND LIMIT $a_{\tau,\sigma} \to 0$.**

We will make suggestions about the phase structure in the whole area of coupling constants and clarify the nature of the phases previously obtained. In the case $\kappa_\tau \simeq 0$ the 4-dimensional system transforms into the set of independent 3-dimensional subsystems with $t = t_j$ as mentioned already. The probe sources (the potential between them was calculated on dual lattice) correspond to the magnetic charges placed inside cubes of the original lattice.

The production over space-like cube’s plaquettes can be associated with magnetic field flux through the cube’s surface

$$\prod_{cube} \square_\sigma = \exp\{const \sum_{cube} \bar{B} \cdot \bar{n}\}$$

$$B_k = \frac{1}{2} \epsilon_{kmn} F_{mn}$$

(32)

and is not equal to zero when the probe source is placed in corresponding dual site. In the other words, the probe source of ”electric” charge in the site of the dual lattice corresponds to the monopol (”magnetic” charge) of the original one.
As known, if for Wilson’s loop $C^*$ in the plane $[tx]$ of 4-dimensional dual lattice we’d fix the time $t = t_0$ then the loop will pierce the plane $[zy]$ in two points (monopol - antimonopol). Dirak’s string which ties them together lies in the slice $t = t_0$.[10]

If the potential between probe sources in each slice will increase linearly with $R$ (in the region of coupling $\gamma_\tau < \gamma_\tau^c \equiv 1 - \gamma_\sigma^c$) then the average value of Wilson’s loop $\langle W \rangle = \prod_{t=0}^{T} \langle s_{x_0} s_{x_R} \rangle$ (when the slices are independent) will decrease exponentially according to area law. Average value of the corresponding t’Hooft’s loop $\langle t'H \rangle$ must behave in the same way in the region $\gamma_\sigma > \gamma_\sigma^c \equiv 1 - \gamma_\tau^c$.

$$\langle t'H \rangle_{\text{original}} = \prod_{t=1}^{T} \langle m_{x_0}, m_{x_R} \rangle = \langle W \rangle_{\text{dual}} = \prod_{t=1}^{T} \langle s_{x_0} s_{x_R} \rangle \sim e^{-\alpha TR} \quad (33)$$

It is obvious that parameters area ($\gamma_\tau$ and $\gamma_\sigma$) falls to four sectors depending on the behaviour of the average values of Wilson’s and t’Hooft’s loops.

|   | $\gamma_\tau > \gamma_\tau^c$ | $\gamma_\sigma < \gamma_\sigma^c$ | $\langle W \rangle \sim e^{-\alpha L_C}$ | $\langle t'H \rangle \sim e^{-\alpha' L_C^*}$ |
|---|-----------------|-----------------|-----------------|-----------------|
| I | $\gamma_\tau > \gamma_\tau^c$ | $\gamma_\sigma > \gamma_\sigma^c$ | $\langle W \rangle \sim e^{-\alpha L_C}$ | $\langle t'H \rangle \sim e^{-\alpha' L_C^*}$ |
| II | $\gamma_\tau > \gamma_\tau^c$ | $\gamma_\sigma > \gamma_\sigma^c$ | $\langle W \rangle \sim e^{-\alpha L_C}$ | $\langle t'H \rangle \sim e^{-\alpha' L_C^*}$ |
| III | $\gamma_\tau < \gamma_\tau^c$ | $\gamma_\sigma < \gamma_\sigma^c$ | $\langle W \rangle \sim e^{-\lambda L_C}$ | $\langle t'H \rangle \sim e^{-\alpha' L_C^*}$ |
| IV | $\gamma_\tau < \gamma_\tau^c$ | $\gamma_\sigma > \gamma_\sigma^c$ | $\langle W \rangle \sim e^{-\lambda L_C}$ | $\langle t'H \rangle \sim e^{-\alpha' L_C^*}$ |

This picture covers all four types of possible behaviour of the averages under consideration which were found by G.t’Hooft [11] from the commutation relations analysis. It seems impossible to ”see” all four phases on a lattice with fixed asymmetry ($\kappa_\tau = \text{const}\kappa_\sigma$) including the symmetrical one. Our results are in good agreement with [10] in the areas they studied on the symmetrical lattice (line $\gamma_\tau = \gamma_\sigma$).

Up to now we treated the couplings $\gamma_\sigma (\gamma_\tau)$ freely enough considering them independent. However, it is obvious that underlying constants $g_\sigma (g_\tau)$ should depend on the lattice spacings through the renormalization group relations. This connection may make some areas of the square $(0 \leq \gamma_\sigma \leq 1; \ 0 \leq \gamma_\tau \leq 1)$ inaccessible. To find exact borders of the accessible area we should build the renormalization group relations on an asymmetrical lattice and put $g_\sigma^2 (g_\tau^2)$ into $\gamma_\sigma (\gamma_\tau)$. In ”naive” limit $g_\sigma^2 \simeq g_\tau^2$. Introducing $\xi = \frac{\kappa_\tau}{a_\tau}$ we get $\frac{\kappa_\sigma}{\kappa_\tau} = \xi^2$. 

Investigating the phase structure of our theory we were dealing with effective couplings $\kappa_{\tau,\sigma}$. Now we are interested in clarifying whether critical values $\kappa_{\tau,\sigma}$ are within accessible area of temperatures $\beta^{-1} = (a_\tau N_\tau)^{-1}$ (i.e., $\beta \neq 0$ and $\beta \neq \infty$) or not. We should note that in the limit $(a_{\sigma,\tau} \to 0) \ g^2_\sigma \simeq g^2_\tau \simeq g^2 \ll 1$ [2]. So, $\kappa_{\tau,\sigma} = \frac{4N^2}{g^4} \gg 1$ (shaded region at fig.6). Say, at $\kappa_{\sigma} \to 0; \ \kappa_{\tau} \to \infty$ as $\frac{1}{g^4\kappa_{\sigma}}$ and $\xi = \frac{a_\sigma}{a_\tau}$ as $\frac{1}{g^2\kappa_{\sigma}}$. This narrows the accessible parameters area. Moreover, at $N_\tau \to \infty$ the points 1 and 4 move to point $(\gamma_\tau = 1; \ \gamma_\sigma = 0)$ (see fig.6).

Taking into account that in this area of parameters

$$\tilde{\gamma}_\tau \simeq \exp\{-2N_\tau e^{-2\kappa_\tau}\} \simeq \exp\{-2\beta e^{-2\kappa_\tau}\}$$

we may see the critical point $\gamma^{c}_\tau$ is accessible at finite temperature $\beta^{-1} = (a_\tau N_\tau)^{-1}$ when

$$g^2_\tau = \frac{2N_\tau \xi}{\ln \frac{1}{a_\tau \Lambda_\tau}}$$

where $\Lambda_\tau = e^{-2\kappa_\tau} a_\tau$.

As $\kappa_{\sigma,\tau}$ depend not only on $a_\tau$ but also on $a_\sigma$ (directly and via $g_{\sigma,\tau}$ too), finding functional relations between $\kappa_{\sigma,\tau}$ and $\beta$ seems impossible if we don’t fix the connection between $a_{\sigma}, a_\tau$ and $N_\tau$

$$\kappa_{\tau} = \frac{2N_\tau a_\sigma N_\tau}{g^2_\tau \beta} \quad \kappa_{\sigma} = \frac{2N_\tau \beta}{g^2_\sigma a_\sigma N_\tau}$$

In the ”naive” limit $g^2_\tau \simeq g^2_\sigma$ and

$$\frac{\kappa_{\tau}}{\kappa_{\sigma}} \simeq \frac{(a_\sigma N_\tau)^2}{\beta^2}$$

The investigation of the $SU(2)$ gluodynamics phase structure on asymmetrical lattice in the plane $\beta \times g^2$ is beyond the frames of this paper and will be done later.

The parameter $\xi = \sqrt{\frac{\kappa_\tau}{\kappa_\sigma}}$ usually is chosen arbitrarily ($\xi_{Hamilt} = \infty$ and $\xi_{Eucl} = 1$). So, if this parameter is not restricted with additional condition then changing it arbitrarily we may reach any point of the curve $\kappa_{\sigma,\tau} = const$ at any small $g^2$, thereby crossing at least one line between phases (II and IV). This says that thermodynamical quantities depend on $\xi$ and, moreover, the jump on this parameter is possible for some of them. It is commonly believed that changing the parameter $\xi$ should not result in any observable
effects. The renormalization group relations make possible excluding the dependence of observed quantities on $\xi$ in ”naive” limit, thereby making lattice regularization with different $\xi$ equivalent \cite{2,12,13}. To save the independence of observed quantities from $\xi$ in ”naive” limit, we chose $\kappa_\tau, \kappa_\sigma$ in precisely the same way as in \cite{2,12,13}. In the approximation $SU(N) \simeq Z(N)$ we failed to create the renormalization group relations with $\Lambda$ parameter being a function of $\xi$ which remove the lattice asymmetry parameter from the thermodynamical quantities keeping at the same time ”naive” limit results. So, we suggest that lattice gauge theories need some additional condition which fixes $\xi$.

Although all calculations were carried out for the $Z(2)$ group, they can be fulfilled for the $Z(3)$ group also, with small changes. There are reasons to hope the results for $SU(2)$ gauge group will be similar to those for the $Z(2)$ gauge group, at least within approximations of \cite{5}.

The authors are indebted for fruitful discussions with Prof. Adriano Di Giacomo. They are grateful to O.A.Borisenko for the critical notes.

References

[1] J. Shigemitsu, J.B. Kogut. Nucl.Phys. B190 (1981) 365
[2] A. Hasenfratz, P. Hasenfratz. Nucl.Phys. B193 (1981) 210
[3] M. Creutz. Quarks, gluons and lattice. Cambridge University Press, 1983, p.127
[4] E. Brezin and J.M. Drouffe. Nucl.Phys. B200 (1982) 93
[5] A. Pena, M. Socolovsky. DESY 83-003
[6] T. Yoneya. Nucl.Phys. B144 (1978) 195
[7] M. Luscher, P. Weisz. MPI-PhT/95-27; DESY 95-056; HEP-LAT-9504006
[8] R. Savit. Rev.Mod.Phys.52, 2 (1980) 453
[9] R. Balian, J.M. Drouffe and C. Itzykson. Phys.Rev. D11, 8 (1975) 2098
[10] A. Ukawa, P. Windey, A.H. Guth. Phys.Rev. D21 (1980) 1013
[11] G.’t Hooft In High Energy Physics Proceedings of the European Physical Society International Conference. Palermo. 1975.

[12] F. Karsch. Nucl.Phys. B205 (1982) 285

[13] M. Billo, M. Caselle, A. D’Adda, S. Panzeri hep-lat/9601020
Figure 1.
\( \kappa_\sigma = -\frac{1}{2} \ln \text{th} \kappa_\tau = \kappa_\tau \)
Figure 3.
Points 1-4 and 2-3 – dual symmetrical by pairs.
Figure 4.
Figure 5.

I  – deconfinement of electric and magnetic charges
II – magnetic confinement, electric deconfinement
III – electric confinement, magnetic deconfinement
IV – electric confinement, magnetic confinement
V  – we cannot investigate this region analytically
Figure 6.