Tachyon Potential in KBc Subalgebra

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We evaluate the classical action and the effective tachyon potential of open string field theory within KBc subalgebra, which is extensively used in analytic solution for tachyon condensation recently found by Erler and Schnabl. It is found that the level expansion of the string field terminates at level 3. We find that the closed string vacuum is a saddle point of the classical action. We also evaluate the effective potential for tachyon field. The closed string vacuum becomes stable by integrating out an auxiliary field. It is found that the effective potential is bounded below hence has no runaway direction. We also argue validity of simple identity based solution.

Subject Index: 123, 126, 128

§1. Introduction

After Schnabl’s discovery of the analytic solution,1) open string field theory has established as a non-perturbative formulation of string theory. The solution successfully proves Sen’s conjecture2) for unstable D-branes. The value of the classical action exactly matches with the D-brane tension. Absence of open strings around closed string vacuum is also shown by the so-called homotopy operator.3) The paper1) also triggers broad interest in string field theory.*

Although basic features of Sen’s conjecture have already shown, analytic solution for tachyon condensation is still important to investigate various problems in string theory. The solution defines string field theory around closed string vacuum as

\[ S' [\Psi] = S [\Psi + \Psi_{Sc}] , \]

where \( \Psi_{Sc} \) denotes Schnabl’s solution and \( S \) is the action of Witten’s cubic SFT.5) In principle, many important problems could be investigated by this action. For example, one can show the absence of open string, as have been done in 3). Other D-branes such as lower dimensional D-branes or multiple D-branes are also expected to be constructed as classical solutions. More nontrivial issue is closed strings. Although the action (1.1) lacks physical excitation of open strings, it is expected that the action can describe closed string physics through open/closed duality on the world sheet. However, at our knowledge, such applications of Schnabl’s solution are quite few at present; the analysis of the homotopy operator, evaluation of the gauge invariant closed string operator6)–8) and the boundary state.9),10) One reason which prevents us from extensive research around closed string vacuum is the complexity

*1) A list of recent works is available in a review article.4)
of Schnabl’s solution. It is given by

$$\Psi_{Sc} = \lim_{N \to \infty} \left[ \psi_N - \sum_{n=0}^{N} \partial_n \psi_n \right],$$

(1.2)

where \( \psi_n \) is a string field composed by particular insertions of conformal ghost and anti ghost. The existence of isolated piece \( \psi_n \), called phantom piece, forces us to take delicate limit of large \( N \) in the evaluation of physical quantities such as classical action.

Recently, a very simplified version of the classical solution for tachyon condensation is given by Erler and Schnabl.\(^{11}\) Their solution is given by

$$\Psi_{ES} = \int_{0}^{\infty} dt (c + cKBc)e^{-t(K+1)},$$

(1.3)

where \( K, B \) and \( c \) are string fields which belong to a subspace of the star algebra\(^{12)-14}\) which is called ‘\( KBc \) subalgebra’. The phantom term is no more absent in their solution. The sum in Schnabl solution is replaced with an integral over the width of semi-infinite strip. Such simplification makes calculations easy therefore might be useful to investigate various problem in SFT.

In this paper, we apply the basis of string field used in 11) to evaluation of the classical and effective tachyon potential. We employ the same gauge condition as that of Erler-Schnabl solution (1.3) to string field. Within the \( KBc \) subalgebra, this leaves unique choice

$$\Psi = \int_{0}^{\infty} dt cf(K)Bce^{-t(K+1)},$$

(1.4)

where \( f(K) \) is a polynomial of \( K \). The power expansion of \( f(K) \) corresponds to level truncation with respect to the ‘dressed’ \( L_0 \) operator which is defined by an anticommutator of \( Q_B \) with the gauge condition. Then, following similar procedure shown in 11), it is straightforward to evaluate the classical action for \( \Psi \). Surprisingly, it turns out that the level truncation stops at finite level if we only allow a field configuration such that leaves the value of the action finite.

This paper is organized as follows. In §2, we give a brief review of the \( KBc \) subalgebra and the basis of string field in which we are working. Section 3 is devoted to evaluation of the tachyon potential. In §4, we give an example of identity based solution. Our results are summarized and discussion is given in §5.

§2. String field in the \( KBc \) subalgebra

2.1. The \( KBc \) subalgebra

The \( KBc \) subalgebra\(^{12)-14}\) is spanned by the string fields \( K, B \) and \( c \) which satisfy

$$\{c, B\} = 1, \quad [c, K] = cK - Kc, \quad [K, B] = 0,$$

(2.1)

where the (anti)commutator is taken with respect to star product. We omit the * symbol for star product as as in 11)–14). In addition, the BRST operator acts on
these string fields as
\[ Q_B c = c K c, \quad Q_B B = K, \quad Q_B K = 0. \] (2.2)

With the help of this algebra, the authors of Ref. 11) found a solution of equation of motion. The solution has a very simple expression as
\[ \Psi = c(1 + K) B c \frac{1}{1 + K}. \] (2.3)

Using (2.1) and (2.2), it is straightforward to show that this solution satisfies the equation of motion \( Q_B \Psi + \Psi^2 = 0 \). The authors\textsuperscript{11}) also have shown that the classical action correctly reproduce D-brane tension. In order to do this, they rewrote the \( 1/(1 + K) \) factor in (2.3) as
\[ \frac{1}{1 + K} = \int_0^\infty dt e^{-t} e^{-tK} = \int_0^\infty dt e^{-t} \Omega^t, \] (2.4)
where \( \Omega^t \) is the wedge state, which represents a semi-infinite strip of width \( \pi t/2 \) in the silver frame.

### 2.2. String field in dressed \( B_0 \) gauge

The authors\textsuperscript{11}) also argued the gauge condition which their solution obeys. It was found that their solution is in the ‘Dressed \( B_0 \) gauge’:
\[ \frac{1}{2} B_0^- [\Psi (1 + K)] \frac{1}{1 + K} = 0, \] (2.5)
where \( B_0^- = B_0 - B_0^\dagger \) is a derivation of the star product. It is not difficult to check (2.3) with the help of formulas
\[ \frac{1}{2} B_0^- c = 0, \quad \frac{1}{2} B_0^- B = 0, \quad B_0^- K = B. \] (2.6)

We would like to consider general form of string field in this gauge. It is soon realized that a ghost number 1 field in this gauge can be obtained by a slight modification of Erler-Schnabl solution, i.e.,
\[ \Psi = c f(K) B c \frac{1}{1 + K}, \] (2.7)
where \( f(K) \) is an arbitrary function of \( K \).\textsuperscript{*}) It is not difficult to see this is the unique choice if we restrict ourself within \( KBc \) subalgebra. For example, multiplying (2.7)

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\* Note that this string field does not satisfy even when \( f(K) \) is real valued function of \( K \). However, there always exists ‘real form’ of this string field given by\textsuperscript{11})
\[ \Psi = \frac{1}{\sqrt{1 + K}} c f(K) B c \frac{1}{\sqrt{1 + K}}. \]
As explained in 11), this string field is gauge equivalent to the non-real form. We use non-real form for convenience.
by a factor such as $cB\Phi(K)$ or $B\Phi(K)c$ still keeps the gauge condition. However, the string field can be reduced to the original form of (2.7) again by contractions of $B$ and $c$. In this paper, we consider a case in which $f(K)$ is given by a polynomial such as

$$f(K) = \sum_{n=0}^{N} t_n K^n.$$  

(2.8)

Erler-Schnabl solution (2.3) corresponds to a choice $t_0 = t_1 = 1$ and $t_n = 0$ for $n \geq 2$.

Before performing an expansion with respect to $K$, let us solve the equation of motion with ansatz (2.7). Each term in the equation motion can be brought into the form $c \cdots c \cdots Bc(1 + K)^{-1}$ by contractions with respect to $B$ and $c$. After some calculation we have

$$Q_B \Psi = [cKc f(K) - cf(K)cK] Bc \frac{1}{1 + K},$$  

(2.9)

$$\Psi^2 = \left[ \frac{c f(K)}{1 + K} - cf(K) \frac{f(K)}{1 + K} \right] Bc \frac{1}{1 + K}.$$  

(2.10)

Then the equation of motion is

$$Q_B \Psi + \Psi^2 = \left[ c \left( K + \frac{f(K)}{1 + K} \right) cf(K) - cf(K)c \left( K + \frac{f(K)}{1 + K} \right) \right] = 0.$$  

(2.11)

We find three solutions of (2.11). The first solution is $f(K) = 0$, which represents perturbative vacuum. The second one can be obtained by canceling two terms in (2.11) with each other. A solution in this case is given by

$$f(K) = 1 + K,$$  

(2.12)

which is nothing but Erler-Schnabl solution. The third one is obtained by setting $K + f(K)/(1 + K) = 0$.

$$f(K) = -K(1 + K).$$  

(2.13)

We give a discussion about this new solution in the end of §3.

§3. The classical and effective potential

In this section, we evaluate the classical action (or equivalently classical potential)

$$V = \frac{1}{2} \text{Tr}[\Psi Q_B \Psi] + \frac{1}{3} \text{Tr}[\Psi^3]$$  

(3.1)

in our setting of string field given by (2.7). All of the calculation can be performed by employing a procedure developed in Ref. 11). We evaluate the tachyon potential order by order with respect to the expansion (2.8). We assign ‘level’ $n$ to the $n$th order term in $f(K)$, since the corresponding string field is an eigenstate of ‘dressed’ $L_0$ operator with eigenvalue $n - 1$.\footnote{The ‘dressed’ $L_0$ is given by $L\Psi = \frac{1}{2} L_0^- \{ \Psi(1 + K) \}$, where $L_0^- = L_0 - L_0^+$.} Let us write the potential up to level $N$ as

$$V = \frac{1}{2} \sum_{m,n} t_m K_{mn} t_n + \frac{1}{3} \sum_{m,n,p} V_{mnp} t_m t_n t_p,$$  

(3.2)
where the sum over \( m, n, p \) is taken from 0 to \( N \). The coefficients \( K_{mn} \) and \( V_{mnp} \) can be obtained by plugging the level \( n \) field

\[
\psi_n = cK^n Bc \frac{1}{1 + K}
\]

\[
= \lim_{s \to 0} (-\partial_s)^n \int_0^\infty e^{-t} dt \ c\Omega^s Bc\Omega^t
\]

(3.3)

into the classical action. It is easily found that both \( K_{mn} \) and \( V_{mnp} \) can be reduced to the well-known trace\(^{12,13}\)

\[
g(r_1, r_2, r_3, r_4) \equiv \text{Tr} [Bc\Omega^2 c\Omega^2 c\Omega^3 c\Omega^4]
\]

\[
= \frac{(r_1 + r_2 + r_3 + r_4)^2}{4\pi^3} (r_1 + r_2 + r_4) \sin \left( \frac{2\pi r_1}{r_1 + r_2 + r_3 + r_4} \right)
\]

\[
+ r_4 \sin \left( \frac{2\pi r_2}{r_1 + r_2 + r_3 + r_4} \right) + r_2 \sin \left( \frac{2\pi r_4}{r_1 + r_2 + r_3 + r_4} \right)
\]

\[
- (r_1 + r_4) \sin \left( \frac{2\pi (r_1 + r_2)}{r_1 + r_2 + r_3 + r_4} \right) - (r_1 + r_2) \sin \left( \frac{2\pi (r_1 + r_4)}{r_1 + r_2 + r_3 + r_4} \right)
\]

\[
+ r_1 \sin \left( \frac{2\pi (r_1 + r_2 + r_3)}{r_1 + r_2 + r_3 + r_4} \right) \right). \quad (3.4)
\]

Then, \( K_{mn} \) and \( V_{mnp} \) is given by

\[
K_{mn} = \lim_{s_1 \to 0, s_2 \to 0} (-\partial_{s_1})^m (-\partial_{s_2})^n \int_0^\infty dt_1 \int_0^\infty dt_2 e^{-t_1 - t_2} h_2(t_1, t_2, s_1, s_2), \quad (3.5)
\]

\[
V_{mnp} = \lim_{s_1 \to 0, s_2 \to 0, s_3 \to 0} (-\partial_{s_1})^m (-\partial_{s_2})^n (-\partial_{s_3})^p \int_0^\infty dt_1 \int_0^\infty dt_2 \int_0^\infty dt_3
\]

\[
\times e^{-t_1 - t_2 - t_3} h_3(t_1, t_2, t_3, s_1, s_2, s_3), \quad (3.6)
\]

where

\[
h_2(t_1, t_2, s_1, s_2) = -\lim_{u \to 0} \partial_u \left[ g(t_2, s_1 + t_1, u, s_2) - g(t_2, s_1, t_1 + u, s_2)
\]

\[
+ g(t_2, s_1, t_1, u + s_2) - g(t_1, s_2 + u, t_2, s_1)
\]

\[
+ g(u, t_2, s_1 + t_1, s_2) - g(u, t_2, s_1, t_1 + s_2) \right], \quad (3.7)
\]

\[
h_3(t_1, t_2, t_3, s_1, s_2, s_3) = g(t_3, s_1 + t_1, s_2 + t_2, s_3) - g(t_3, s_1 + t_1, s_2, t_2 + s_3)
\]

\[
- g(t_3, s_1, t_1 + t_2 + s_2, s_3) + g(t_3, s_1, t_1 + s_2, t_2 + s_3). \quad (3.8)
\]

In (11), the integrals in the kinetic term (which corresponds to \( K_{mn} \) in our paper) is performed with the help of the reparametrization

\[
t_1 = u v, \quad t_2 = u(1 - v), \quad (3.9)
\]
where \( u = t_1 + t_2 \) parametrize the total width of the semi-infinite strip in the sliver frame, which corresponds to the total width of the world sheet. Under this reparametrization, the measure of the integral is transformed as

\[
\int_0^\infty dt_1 \int_0^\infty dt_2 \rightarrow \int_0^1 du \int_0^{1-v_1} dv \ u.
\]  

(3.10)

While the cubic term is not calculated in (11), we find that similar reparametrization also works. Thus the integrals in \( V_{mnp} \) can be performed with the replacement

\[
t_1 = uv_1, \quad t_2 = uv_2, \quad t_3 = u(1-v_1-v_2),
\]  

(3.11)

where \( u = t_1 + t_2 + t_3 \) also represents the width of the world sheet. The integration measure can also be rewritten into

\[
\int_0^\infty dt_1 \int_0^\infty dt_2 \int_0^\infty dt_3 \rightarrow \int_0^1 du \int_0^{1-v_1} dv_1 \int_0^{1-v_1} dv_2 \ u^2.
\]  

(3.12)

After integrating out \( v_1, v_2 \) and \( v_3 \), we obtain the potential as an integral with respect to the width \( u \) as

\[
V = \int_0^\infty du \ e^{-u} A(u, t_n),
\]  

(3.13)

where \( A(u, t_n) \) is finite order in \( u \), and possibly includes negative powers of \( u \).

3.1. The classical potential

We are now ready to evaluate the classical potential up to arbitrary level. In principle, all the calculation can be done by hand, but software such as Mathematica is useful to evaluate the derivatives and integrals which appear in \( K_{m,n} \) and \( V_{m,n,p} \). We find that the order of \( u \) in the integrand of (3.13) decreases as level increases. To illustrate this, we introduce a notation in which the \( u \) dependence of each coefficient in the potential is manifest.

\[
K_{m,n} = \int_0^\infty du \ e^{-u} k_{m,n}(u), \quad V_{m,n,p} = \int_0^\infty du \ e^{-u} v_{m,n,p}(u).
\]  

(3.14)

For example, \( k_{m,n}(u) \) is evaluated up to level 4 as

\[
k_{m,n}(u) = \begin{pmatrix}
-\frac{u^3}{2\pi^2} & 0 & \frac{3u}{\pi^2} & \frac{2(-3+\pi^2)}{\pi^2} & 0 \\
0 & 0 & 0 & 0 & 0 \\
\frac{3u}{\pi^2} & 0 & 0 & 0 & 0 \\
\frac{2(-3+\pi^2)}{\pi^2} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{48\pi^2}{u^4} & -\frac{48\pi^2}{u^4} & -\frac{1536\pi^2}{u^5}
\end{pmatrix}.
\]  

(3.15)

The first coefficient which includes negative power of \( u \) is

\[
k_{43}(u) = k_{34}(u) = -\frac{48\pi^2}{u^4}.
\]  

(3.16)
Table I. A complete list of $v_{m,n,p}(u)$ up to level 3. Other coefficients are zero.

| $(m, n, p)$ | $v_{m,n,p}(u)$ | $(m, n, p)$ | $v_{m,n,p}(u)$ |
|------------|----------------|------------|----------------|
| (0, 0, 0)  | $\frac{4u^6}{4\pi^4}$  | (0, 0, 1)  | $\frac{(-15+5\pi^2)}{6\pi^4}u^4$ |
| (0, 1, 1)  | $\frac{(-15+5\pi^2)}{3\pi^4}u^2$ | (0, 0, 2)  | $\frac{(-15+5\pi^2)}{3\pi^4}u^2$ |
| (1, 1, 2)  | $\frac{2(-15+5\pi^2)}{\pi^4}u$  | (0, 2, 2)  | $\frac{2(-15+5\pi^2)}{\pi^4}u$ |
| (1, 2, 2)  | $\frac{-4(-15+5\pi^2)}{\pi^4}$ | (0, 0, 3)  | $\frac{-2(6+5\pi^2)}{3\pi^4}u^2$ |
| (0, 1, 3)  | $\frac{2(-45-9\pi^2+9\pi^4)}{3\pi^4}u$ | (1, 1, 3)  | $\frac{-4(-45+6\pi^2+\pi^4)}{3\pi^4}$ |
| (0, 2, 3)  | $\frac{4(45-15\pi^2+2\pi^4)}{3\pi^4}$ |

This factor is divergent in the potential since an integral

$$-\int_0^\infty du e^{-u}u^{-4} \times (48\pi^2)$$

is proportional to Gamma function $\Gamma(-3)$, which is known to be divergent. Similarly, any terms with negative power of $u$ diverge due to poles of Gamma function at negative integer. The cubic coefficient $K_{m,n}$ also diverges as level increases. However, as seen in Table I, negative power of $u$ is absent up to level 3.* Therefore, if we restrict $f(K)$ in (2.7) to be polynomial in $K$, and also require finite coefficients in the classical action, its maximal level is three. In other words, level truncation terminates at level 3.** This is quite different from the situation in Siegel gauge or Schnabl’s $B_0$ gauge, where level expansion does not terminate.

The reason for appearance of negative power of $u$ can be understood by noting the fact that $K$ is replaced with a derivative on a CFT correlator. Typically, it takes form of

$$\lim_{s \to 0} \partial_s \sin \frac{z}{s+u} \cdots,$$

and then yields $u^{-2}$ factor.

According to (3.14), the final answer for the tachyon potential is obtained by performing integration with respect to $u$, together with the $e^{-u}$ factor. We present the result below for reference.

$$V = \frac{1}{\pi^2} \left( \frac{15}{64\pi^2}t_3^3 - \frac{15t_1}{16\pi^2}t_3^2 + \frac{1}{16}t_1t_3t_0 + \frac{15}{16\pi^2}t_2t_0 - \frac{1}{4}t_2^2t_0 - \frac{1}{12}\pi^2 t_3t_0^2 \right. \right.$$

$$+ \frac{1}{2}t_3t_0^2 - \frac{3}{32}t_0^3 + \frac{15}{16\pi^2}t_2t_0 - \frac{1}{16}t_2^2t_0 - \frac{15}{4\pi^2}t_3t_0^2 + \frac{3}{4}t_3^2t_0 - \frac{1}{6}\pi^2 t_1t_3t_0$$

$$- \frac{15}{2\pi^2}t_1t_3t_0 + \frac{3}{2}t_1t_2t_0 + \frac{4}{3}\pi^2 t_3t_3t_0 + \frac{30}{\pi^2}t_2t_2t_0 - 10t_2t_3t_0 + \pi^2 t_3t_0 - 3t_3t_0$$

$$+ \left. \frac{15}{\pi^2}t_1t_2^2 - t_1t_2^2 - \frac{15}{4\pi^2}t_2t_2^2 + \frac{1}{4}t_2^2t_2 - \frac{1}{3}\pi^2 t_1t_3 + \frac{15}{\pi^2}t_1t_3 - 2t_1^2t_3 \right).$$

* We first observe divergence at level 6.
** In (11), it is already shown that the classical solution terminates at level 1.
The instability of the classical potential

The stationary point of the classical potential, given by solutions of
\[ \frac{\partial V}{\partial t_i} = 0 \quad (i = 0, 1, 2, 3) \]  
(3.20)
can be obtained numerically. It turns out that there are eight branches. Among them, we found four nontrivial, real solutions, which are summarized in Table II. We discard other solutions since they are trivial (all fields are zero) or give rise to complex value of the classical action. The closed string vacuum is given by the configuration \((t_0, t_1, t_2, t_3) = (1, 1, 0, 0)\). The value of \(V\) coincides with D-brane tension \(-1/2\pi^2\) as expected. To see whether these stationary points are stable or not, we evaluate Hessian matrix at the closed string vacuum as have been done in 15) for \(B_0\) gauge,
\[ H_{ij} = \frac{\partial^2 V}{\partial t_i \partial t_j}. \]  
(3.21)
In particular, the closed string vacuum is stable (unstable) if all eigenvalues of \(H_{ij}\) is positive (negative). Otherwise, it is a saddle point. The Hessian matrix for the closed string vacuum is
\[
\begin{pmatrix}
180 + 6\pi^2 & -180 + 12\pi^2 & 180 - 30\pi^2 & -180 + 48\pi^2 \\
-180 + 12\pi^2 & 180 - 12\pi^2 & -180 + 12\pi^2 & 180 - 12\pi^2 - 12\pi^4 \\
180 - 30\pi^2 & -180 + 12\pi^2 & 180 + 24\pi^2 & 360 - 120\pi^2 + 16\pi^4 \\
-180 + 48\pi^2 & 180 - 12\pi^2 - 12\pi^4 & 360 - 120\pi^2 + 16\pi^4 & 0
\end{pmatrix},
\]  
(3.22)
and eigenvalues are found to be
\[ 1487.14, \quad -1261.59, \quad 412.919, \quad 79.1831. \]  
(3.23)
Thus, the Hessian matrix has a negative eigenvalue while others are positive. This concludes that the closed string vacuum of Erler and Schnabl is a saddle point of the classical action, as similar to the result of 15).

In addition to the above result, we have found the following facts.

- We have calculated eigenvalues of Hessian matrix for all solutions in Table II.
  It turns out that all of the four solutions are saddle points.
- We have checked that each solution satisfies equation of motion contracted with itself; i.e. \(\text{Tr}[\Psi Q_B \Psi + \Psi^3] = 0\) holds within accuracy of numerical evaluation.
3.3. Effective potential

The effective potential for tachyon can be obtained by eliminating fields other than tachyon field from the classical potential by solving equations of motion. In this paper, we identify \( t_0 \) as the tachyon mode, also such choice is ad-hoc. Therefore, in order to obtain an effective potential as a function of \( t_0 \), we have to solve

\[
\frac{\partial V}{\partial t_i} = 0 \quad (i = 0, 1, 2, 3) \tag{3.24}
\]

for other fields \( t_1, t_2, t_3 \). In general, there appear many branches since the equations of motion is cubic in each \( t_i \). However, one can see that the classical action (3.19) is linear in \( t_3 \). Therefore, \( t_3 \) is an auxiliary field and can be eliminated from the classical action by imposing a constraint

\[
\frac{\partial V}{\partial t_3} = 0. \tag{3.25}
\]

Furthermore, we find that the above constraint is again linear in \( t_2 \), because there are no \( t_2^2 t_3 \) term in the classical action. Therefore, by solving (3.25) for \( t_2 \) and plugging it to the classical potential, we obtain an unique potential which only depends on \( t_0 \) and \( t_1 \). We denote this potential \( V_2(t_0, t_1) \) and present a contour plot of this potential in Fig. 1.

We can still find stationary points of \( V_2 \) by numerical method. Again, we found four real, nontrivial stationary points among eight branches. We summarize the obtained solutions in Table III, together with the value of the potential and their stability deduced from the Hessian analysis. The expected closed string vacuum,
Table III. A summary of the four real stationary points of the potential $V_2$.

| $(t_0, t_1)$         | $V_2$      | stability |
|----------------------|------------|-----------|
| $(0.270333, -0.392821)$ | $-0.0306387$ | stable    |
| $(0.217487, 0.176975)$  | $0.00784555$ | saddle point |
| $(0.119477, 0.159294)$  | $0.00807677$ | unstable  |
| $(1.000000, 1.000000)$  | $-0.0506606$ | stable    |

Fig. 2. A plot of the potential $V_1$. We have chosen a branch which connects with the closed string vacuum.

stable in this case, is in the last row of Table III. Curiously, another stable vacuum shallower than the closed string vacuum appears in the first row. We will discuss possible interpretation of this vacuum later.

We can further integrate out $t_1$ from $V_2$ by solving equation of motion. In this case, $t_1$ cannot be solved uniquely as a function of $t_0$ since $V_2$ is no more linear in $t_1$. This yields four nontrivial branches. First, we would like to pick a branch which is connected to the closed string vacuum, which is our major concern. We can substitute $t_1$ to the solution of equation of motion to obtain effective potential $V_1(t_0)$. Again the analytic expression is quite long to show here. We show a plot of the potential in Fig. 2.

As seen in the plot, the closed vacuum at $t_0 = 1$ is stable and correctly reproduce the D-brane tension $1/(2\pi^2) \sim -0.0506$. Furthermore, we observed that the potential cannot be extended to negative value of $t_0$ due to singularity at $t_0 = 0$. For negative $t_0$, the potential becomes complex valued. In earlier results based on the level truncation in Siegel gauge or $B_0$ gauge, the potential is valid only in a compact region of tachyon field, typically starts from slightly before unstable point $t_0 = 0$ and terminates at some value of $t_0$. Our result seems to be more natural since potential exactly starts from the unstable vacuum. There is no roll off of the tachyon field towards negative value of the tachyon field. Such feature of the effective potential completely agrees with the physical picture of unstable D-brane that have been conjecture in — the closed string vacuum is the endpoint of the D-brane decay.

We can also solve $t_1$ in other branches in a similar way. A result is shown in Fig. 3. The branch 4 is the closed string vacuum branch which is already shown in...
Fig. 3. Four branches of the effective potential $V_1$. We find local minimum in branches 2 and 4. Branch 3 consists of two disconnected curves.

Fig. 2. It is seen that none of the branches are extended to the negative $t_0$ region beyond singularity at origin. We also found two nontrivial branch points $t_0 \sim 0.913$ and $t_0 \sim 1.177$ by numerical inspection. Such branch points are found by scanning a discontinuity of the imaginary part of $t_1$ as a function of $t_0$. Then we exclude a region in which $t_1$ is not real. One can see that the branches 3 and 2 both terminate at $t_0 \sim 0.913$, and also another part of branch 3 is connected with branch 1 at $t_0 \sim 1.177$ smoothly. All branches share common features already mentioned for the closed string vacuum branch; neither a runaway direction seen in 16), nor small deviation of tachyon field towards negative $t_0$ axis$^{15}$ is seen.

In closing this section, let us compare our result from level expansion with that of full equation of motion (2.11). Both methods share the string vacuum solution, $(t_0, t_1) = (1, 1)$. Remaining solutions in level expansion have no counterparts in the analysis of full equation of motion. It is not surprising since equation of motion in level truncated action does not necessarily coincide with full equation of motion. A rather surprising result is the alternative solution in full equation motion, $f(K) = -K(1 + K)$ in (2.13). It is not shown up as a stationary point of the level expanded action. Furthermore, it does not satisfy equation of motion contracted with itself, since

$$\text{Tr}[\Psi(Q_B\Psi + \Psi^2)] = 3 \left( -\frac{15}{\pi^4} + \frac{1}{\pi^2} \right),$$

(3.26)

which can be easily calculated from the classical potential (3.19). This indicates that this solution is ill-defined.
§4. Identity based solution

In our setting, there is an identity based solution\(^1\)

\[ \Psi = -cK. \] (4.1)

The equation of motion can be easily checked if one remember \( Q_B c = cKc \). As is well known, a regularization is needed to evaluate physical quantity such as classical action. We apply a naive regularization via narrow width limit.

\[ \Psi = -cK = \lim_{s \to 0} \partial_s c\Omega^s. \] (4.2)

With this regularization, it is very easy to evaluate the classical action. First, the quadratic term is given by

\[ \text{Tr} \Psi Q_B \Psi = - \lim_{s_1 \to 0} \lim_{s_2 \to 0} \lim_{u \to 0} \partial_{s_1} \partial_{s_2} \partial_u \text{Tr} [c\Omega^s c\Omega^s c\Omega^s]. \] (4.3)

The order of limits is important. The limit with respect to \( u \) is taken to be first since it originates from \( Q_B \), which does not change width of the world sheet. Therefore we take \( u \to 0 \) limit first. Next, we set \( s_1 = s_2 = s \) before sending them to zero. It turns out that the trace does not depend on \( s \), so without taking \( s \to 0 \) limit we have

\[ \text{Tr} \Psi Q_B \Psi = - \frac{1}{2} - \frac{2}{\pi^2}. \] (4.4)

Similarly, the cubic term

\[ \text{Tr} \Psi^3 = \lim_{s_1 \to 0} \lim_{s_2 \to 0} \lim_{s_3 \to 0} \partial_{s_1} \partial_{s_2} \partial_{s_3} \text{Tr} [c\Omega^s c\Omega^s c\Omega^s]. \] (4.5)

can be evaluated by setting \( s_1 = s_2 = s_3 \) without taking limits.

\[ \text{Tr} \Psi^3 = \frac{2}{9} + \frac{9\sqrt{3}}{4\pi^3} + \frac{\sqrt{3}}{2\pi}. \] (4.6)

The sum of the quadratic term with the cubic term must vanish since it is an equation of motion contracted with itself.

\[ \text{Tr} \left[ \Psi (Q_B \Psi + \Psi^2) \right]. \] (4.7)

However, as is clear from (4.2) and (4.4), it does not vanish. Therefore \( \Psi \) is not a classical solution in our regularization. The reason is easily understood from the fact that each trace has different width.

§5. Discussion

In this paper, we performed level expansion of string field within the \( KBc \) sub-algebra. We find that the level expansion terminates at level 3. It is found that the

\(^1\) While completing this paper, a similar solution, \( \Psi = c(1 - K) \), appears in 17).
closed string vacuum is a saddle point of the classical potential. As for the effective potential, we confirmed that it is bounded from below, and exactly starts from perturbative vacuum.

An expression of classical action in terms of total strip width $u$,

\[ S = \int_0^\infty du e^{-u} A(u, t_n), \]  

(5.1)

is important, since this tells us which width of world sheet is most dominant in the classical action. In principle, this kind of expression also appears in Schnabl solution\(^1\) (given as sums rather integrations) and marginal deformation\(^{18\text{–}21}\) but the multiple integrals or sums with respect to strip width is very difficult to perform completely. The simplicity of the $KBC$ subalgebra enables us to perform multiple integration.

Termination of level expansion is also impressive. Although we restrict $f(K)$ to be polynomial in this paper, we can also consider a case of certain series in $K$ such as $f(K) = 1/(1 + K)$. This example can be treated by introduction of Schwinger parameter. It is interesting to evaluate classical action for such string field.

We should note that our result for tachyon physics is only limited in very limited subspace spanned by $KBC$ subalgebra. Our results may change by inclusion of other modes outside $KBC$ subalgebra. However, we have done systematic analysis under certain gauge condition, so we believe that our analysis is useful to get insight about physics of tachyon condensation. Especially, we believe that the effective tachyon potential obtained in this paper will help an attempt to derive exact form of the effective potential, which is not yet available in CSFT. It will also be interesting to compare our potential with those derived from BSFT\(^{22\text{,}23}\) or $S$-matrix method\(^{24\text{,}25}\).

The $KBC$ subalgebra will be very useful for other proposes. Extension of this subalgebra to fields with nonzero momentum will be useful to investigate physics around closed string vacuum. Multiple D-branes or lump solutions will also be interesting. Application to the gauge invariant overlap is also important to understand closed string physics in terms of open string fields.

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