The strong chromatic index of 1-planar graphs

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The chromatic index $\chi'(G)$ of a graph $G$ is the smallest $k$ for which $G$ admits an edge $k$-coloring such that any two adjacent edges have distinct colors. The strong chromatic index $\chi''(G)$ of $G$ is the smallest $k$ such that $G$ has an edge $k$-coloring with the condition that any two edges at distance at most 2 receive distinct colors. A graph is 1-planar if it can be drawn in the plane so that each edge is crossed by at most one other edge.

In this paper, we show that every graph $G$ can be drawn in the plane so that each edge is crossed by at most one other edge.

As a corollary, we prove that every 1-planar graph $G$ with maximum degree $\Delta$ has $\chi''(G) \leq 14\Delta$, which improves a result, due to Bensmail et al., which says that $\chi''(G) \leq 24\Delta$ if $\Delta \geq 56$.

**Keywords:** Strong edge coloring, strong chromatic index, maximum average degree, 1-planar graph, matching.

1 Introduction

Only simple graphs are considered in this paper unless otherwise stated. Let $G$ be a graph with vertex set $V(G)$, edge set $E(G)$, minimum degree $\delta(G)$, and maximum degree $\Delta(G)$ (for short, $\Delta$), respectively. A vertex $v$ is called a $k$-vertex if the degree $d_G(v)$ of $v$ is $k$. The *girth* $g(G)$ of a graph $G$ is the length of a shortest cycle in $G$. The **maximum average degree** $\bar{d}(G)$ of a graph $G$ is defined as follows:

$$\bar{d}(G) = \max \{ \frac{2|E(H)|}{|V(H)|} \mid H \subseteq G \}.$$  

A *proper edge $k$-coloring* of a graph $G$ is a mapping $\phi : E(G) \to \{1, 2, \ldots, k\}$ such that $\phi(e) \neq \phi(e')$ for any two adjacent edges $e$ and $e'$. The **chromatic index** $\chi'(G)$ of $G$ is the smallest $k$ such that $G$ has a proper edge $k$-coloring. The coloring $\phi$ is called *strong* if any two edges at distance at most two get distinct colors. Equivalently, each color class is an induced matching. The **strong chromatic index**, denoted $\chi''(G)$, of $G$ is the smallest integer $k$ such that $G$ has a strong edge $k$-coloring.

Strong edge coloring of graphs was introduced by Fouquet and Jolivet [12]. It holds trivially that $\chi''(G) \geq \chi'(G) \geq \Delta$ for any graph $G$. In 1985, during a seminar in Prague, Erdős and Nešetřil put forward the following conjecture:

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Conjecture 1 For a simple graph $G$,

$$
\chi'_s(G) \leq \begin{cases} 
1.25\Delta^2, & \text{if } \Delta \text{ is even;} \\
1.25\Delta^2 - 0.5\Delta + 0.25, & \text{if } \Delta \text{ is odd.}
\end{cases}
$$

Erdős and Nešetřil provided a construction showing that Conjecture 1 is tight if it were true. Using probabilistic method, Molloy and Reed [23] showed that $\chi'_s(G) \leq 1.998\Delta^2$ for any graph $G$ with sufficiently large $\Delta$. This result was gradually improved to that $\chi'_s(G) \leq 1.93\Delta^2$ in [6], to that $\chi'_s(G) \leq 1.835\Delta^2$ in [4], and to that $\chi'_s(G) \leq 1.772\Delta^2$ in [17]. Andersen [1] and independently Horák et al. [14] confirmed Conjecture 1 for graphs with $\Delta = 3$. If $\Delta = 4$, then Conjecture 1 asserts that $\chi'_s(G) \leq 20$. However, the currently best known upper bound is 21 for this case, see [15].

A graph $G$ is $d$-degenerate if each subgraph of $G$ contains a vertex of degree at most $d$. Chang and Narayan [7] showed that $\chi'_s(G) \leq 10\Delta - 10$ for a 2-degenerate graph $G$. For a general $k$-degenerate graph $G$, it was shown that $\chi'_s(G) \leq (4k-2)\Delta - k(2k-1)+1$ in [31], $\chi'_s(G) \leq (4k-1)\Delta - k(2k+1)+1$ in [8], and $\chi'_s(G) \leq (4k-2)\Delta - 2k^2 + 1$ in [27].

Suppose that $G$ is a planar graph. Faudree et al. [11] first gave an elegant proof for the result that $\chi'_s(G) \leq 4\Delta + 4$, and constructed a class of planar graphs $G$ with $\Delta \geq 2$ such that $\chi'_s(G) = 4\Delta - 4$. For the class of special planar graphs, some better results have been obtained. It was shown in [16] that $\chi'_s(G) \leq 3\Delta$ if $g(G) \geq 7$, and in [2] that $\chi'_s(G) \leq 3\Delta + 1$ if $g(G) \geq 6$. Kostochka et al. [19] showed that if $\Delta = 3$ then $\chi'_s(G) \leq 9$. Hocquard et al. [13] showed that every outerplanar graph $G$ with $\Delta \geq 3$ has $\chi'_s(G) \leq 3\Delta - 3$. Wang et al. [29] showed that every $K_4$-minor-free graph $G$ with $\Delta \geq 3$ has $\chi'_s(G) \leq 3\Delta - 2$. Moreover, all upper bounds 9, $3\Delta - 3$, $3\Delta - 2$ given in the above results are best possible.

A 1-planar graph is a graph that can be drawn in the plane such that each edge crosses at most one other edge. A number of interesting results about structures and parameters of 1-planar graphs have been obtained in recent years. Fabrici and Madaras [10] proved that every 1-planar graph $G$ has $|E(G)| \leq 4|V(G)| - 8$, which implies that $\delta(G) \leq 7$, and constructed 7-regular 1-planar graphs. Borodin [5] showed that every 1-planar graph is vertex 6-colorable. Wang and Lih [28] proved that the vertex-face total graph of a plane graph, which is a class of special 1-planar graphs, is vertex 7-choosable. Zhang and Wu [33] studied the edge coloring of 1-planar graphs and showed that every 1-planar graph $G$ with $\Delta \geq 10$ satisfies $\chi'(G) = \Delta$.

Bensmail et al. [3] investigated the strong edge coloring of 1-planar graphs and proved that every 1-planar graph $G$ has $\chi'_s(G) \leq \max\{18\Delta + 330, 24\Delta - 6\}$. This implies that if $\Delta \geq 56$, then $\chi'_s(G) \leq 24\Delta - 6$. In this paper we will improve this result by showing that every 1-planar graph $G$ has $\chi'_s(G) \leq 14\Delta$. To obtain this result, we establish a connection between the strong chromatic index and maximum average degree of a graph. More precisely, we will show that $\chi'_s(G) \leq (2\delta(G) - 1)(\Delta + 1)$ for any simple graph $G$.

2 Preliminary

In this section, we summarize some known results, which will be used later.

A proper $k$-coloring of a graph $G$ is a mapping $\phi : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that $\phi(u) \neq \phi(v)$ for any two adjacent vertices $u$ and $v$. The chromatic number, denoted $\chi(G)$, of $G$ is the least $k$ such that $G$ has a proper $k$-coloring.

Using a greedy algorithm, the following conclusion holds automatically.
Lemma 1 If $G$ is a $d$-degenerate graph, then $\chi(G) \leq d + 1$.

As stated before, Borodin [5] showed the following sharp result:

Theorem 2 ([5]) Every 1-planar graph $G$ has $\chi(G) \leq 6$.

Given a graph $G$, it is trivial that $\chi'(G) \geq \Delta$. On the other hand, the celebrated Vizing Theorem [26] asserts:

Theorem 3 ([26]) Every simple graph $G$ has $\chi'(G) \leq \Delta + 1$.

A simple graph $G$ is of Class I if $\chi'(G) = \Delta$, and of Class II if $\chi'(G) = \Delta + 1$. As early as in 1916, König [18] showed that bipartite graphs are of Class I.

Theorem 4 ([18]) If $G$ is a bipartite graph, then $\chi'(G) = \Delta$.

Sanders and Zhao [24], and Zhang [32] independently, showed that planar graphs with maximum degree at least seven are of Class I.

Theorem 5 ([24, 32]) Every planar graph $G$ with $\Delta \geq 7$ has $\chi'(G) = \Delta$.

Another result, due to Zhang and Wu [33], claims that 1-planar graphs with maximum degree at least ten are of Class I.

Theorem 6 ([33]) Every 1-planar graph $G$ with $\Delta \geq 10$ has $\chi'(G) = \Delta$.

Zhou [34] observed an interesting relation between the degeneracy and chromatic index of a graph.

Theorem 7 ([34]) If $G$ is a $k$-degenerate graph with $\Delta \geq 2k$, then $\chi'(G) = \Delta$.

3 Contracting matchings in a graph

Let $G$ be a simple graph. An edge $e$ of $G$ is said to be contracted if it is deleted and its end-vertices are identified. An edge subset $M$ of $G$ is called a matching if no two edges in $M$ are adjacent in $G$. Specifically, a matching $M$ is called strong if no two edges in $M$ are adjacent to a common edge. This is equivalent to saying that $G[V(M)] = M$. Thus, a strong matching is also called an induced matching. Determining the chromatic index $\chi'(G)$ of a graph $G$ is certainly equivalent to finding the least $k$ such that $E(G)$ can be partitioned into $k$ edge-disjoint matchings, and determining the strong chromatic index $\chi'_s(G)$ of a graph $G$ is equivalent to finding the least $k$ such that $E(G)$ can be partitioned into $k$ edge-disjoint strong matchings. In what follows, an edge $k$-coloring of $G$ with the color classes $E_1, E_2, \ldots, E_k$ will be denoted by $(E_1, E_2, \ldots, E_k)$.

Given a graph $G$ and a matching $M$ of $G$, let $G_M$ denote the graph obtained from $G$ by contracting each edge in $M$. Note that $G_M$ may contain multi-edges, but no loops, even if $G$ is simple.

Let $a \geq 1$ and $b \geq 0$ be integers. A graph $G$ is said to be $(a, b)$-graph if every subgraph $G'$ of $G$ (including itself) has $|E(G')| \leq a|V(G')| - b$.

Theorem 8 Let $G$ be a $(a, b)$-graph with $a \geq 1$ and $b \geq 0$. Let $M$ be a matching of $G$. Then $G_M$ is a $(2a - 1, b)$-graph.
Proof: Let $H$ be any subgraph of $G_M$. Assume that $V(H) = V_1 \cup V_2$, where $V_1$ is the set of vertices in $G_M$ which are formed by contracting some edges in $M$, and $V_2 = V(H) \setminus V_1$, say, $V_1 = \{x_1, x_2, \ldots, x_{n_1}\}$, and $V_2 = \{y_1, y_2, \ldots, y_{n_2}\}$. Then $|V(H)| = n_1 + n_2$. Splitting each vertex $x_i \in V_1$ into two vertices $u_i$ and $v_i$ and restoring corresponding incident edges for $u_i$ and $v_i$ in $G$, we get a subgraph $G'$ of $G$ with

$$V(G') = \{u_1, u_2, \ldots, u_{n_1}; v_1, v_2, \ldots, v_{n_1}; y_1, y_2, \ldots, y_{n_2}\}$$

and

$$E(G') = E(H) \cup M',$$

where

$$M' = \{u_1 v_1, u_2 v_2, \ldots, u_{n_1} v_{n_1}\} \subseteq M.$$

It is easy to compute that $|V(G')| = 2n_1 + n_2$ and $|E(G')| = |E(H)| + n_1$. By the assumption, $|E(G')| \leq a|V(G')| - b$. Since $a \geq 1$, we have $2a - 1 \geq a$. Consequently,

$$|E(H)| = |E(G')| - n_1$$

$$\leq a|V(G')| - b - n_1$$

$$= a(2n_1 + n_2) - b - n_1$$

$$= (2a - 1)n_1 + an_2 - b$$

$$\leq (2a - 1)(n_1 + n_2) - b$$

$$= (2a - 1)|V(H)| - b.$$

This shows that $G_M$ is a $(2a - 1, b)$-graph. □

Corollary 9 Let $G$ be a $(a, b)$-graph with $a, b \geq 1$. Let $M$ be a matching of $G$. Then $G_M$ is $(4a - 3)$-degenerate.

Proof: It suffices to verify that $\delta(H) \leq 4a - 3$ for any $H \subseteq G_M$. Suppose to the contrary that $\delta(H) \geq 4a - 2$. Since $b \geq 1$, Theorem 8 and the Handshaking Theorem imply that $(4a - 2)|V(H)| \leq \delta(H)|V(H)| \leq \sum_{v \in V(H)} d_H(v) = 2|E(H)| \leq 2((2a - 1)|V(H)| - b) = (4a - 2)|V(H)| - 2b < (4a - 2)|V(H)|$. This leads to a contradiction. □

Similarly, we obtain the following consequence:

Corollary 10 Let $G$ be a $(a, 0)$-graph with $a \geq 1$. Let $M$ be a matching of $G$. Then $G_M$ is $(4a - 2)$-degenerate.

A matching $M$ of a graph $G$ is said to be partitioned into $q$ strong matchings of $G$ if $M = M_1 \cup M_2 \cup \cdots \cup M_q$ and $M_i \cap M_j = \emptyset$ for $i \neq j$ such that each $M_i$ is a strong matching of $G$. Let $\rho_G(M)$ denote the least $q$ such that $M$ is partitioned into $q$ strong matchings. By definition, $1 \leq \rho_G(M) \leq |M|$.

The following result is highly inspired from a result of [11] on the strong chromatic index of planar graphs. For the sake of completeness, we here give the detailed proof.

Lemma 11 Let $G$ be a graph and $M$ be a matching of $G$. Then $\rho_G(M) \leq \chi(G_M)$. 


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Proof: Let \( V(G_M) = S_1 \cup S_2 \), where \( S_1 \) is the set of vertices in \( G_M \) formed from \( G \) by contracting edges in \( M \) and \( S_2 = V(G) \setminus V(M) \). Set \( k = \chi(G_M) \). Then \( G_M \) admits a proper \( k \)-coloring \( \phi : V(G_M) \rightarrow \{1, 2, \ldots, k\} \). For \( 1 \leq i \leq k \), let \( V_i \) denote the set of vertices in \( G_M \) with the color \( i \). In \( G \), for \( 1 \leq i \leq k \), let

\[
E_i^* = \{ e \in M \mid e \text{ is contracted to some vertex } v_e \in S_1 \text{ with } \phi(v_e) = i \}.
\]

Let \( e_1, e_2 \in E_i^* \) be any two edges. Since \( e_1, e_2 \in M \), \( e_1 \) and \( e_2 \) are not adjacent in \( G \). We claim that no edge \( e \in E(G) \) is simultaneously adjacent to both \( e_1 \) and \( e_2 \). Assume to the contrary, there exists \( e = xy \in E(G) \) adjacent to \( e_1 \) and \( e_2 \). Without loss of generality, we may suppose that \( e_1 = xx' \) and \( e_2 = yy' \). Let \( v_{e_1} \) and \( v_{e_2} \) denote the corresponding vertices of \( e_1 \) and \( e_2 \) in \( S_1 \), respectively. Indeed, \( x \) is \( v_{e_1} \), and \( y \) is \( v_{e_2} \). Since \( xy \notin M \), it follows that \( xy \in E(G_M) \) and thus \( x \) is adjacent to \( y \) in \( G_M \). By the definition of \( \phi \), \( \phi(x) \neq \phi(y) \). Let \( \phi(x) = p \) and \( \phi(y) = q \). Then \( e_1 \in E_p^* \) and \( e_2 \in E_q^* \) with \( p \neq q \), which contradicts the assumption that \( e_1, e_2 \in E_i^* \). So, each of \( E_1^*, E_2^*, \ldots, E_k^* \) is a strong matching of \( G \). This confirms that \( \rho_G(M) \leq k = \chi(G_M) \). □

4 Strong chromatic index

In this section, we will discuss the strong edge coloring of some graphs by using the previous preliminary results.

4.1 An upper bound

We first establish an upper bound of strong chromatic index for a general graph \( G \), which reveals a relation between the strong chromatic index, chromatic index and maximum average degree of \( G \).

Lemma 12 Let \( H \) be a subgraph of a graph \( G \). Then \( |E(H)| \leq \frac{1}{2}d(G)\left|V(H)\right| \).

Proof: For any subgraph \( H \subseteq G \), it follows from the definition of \( \bar{d}(G) \) that \( \frac{2|E(H)|}{\left|V(H)\right|} \leq \bar{d}(G) \). Consequently, \( |E(H)| \leq \frac{1}{2}d(G) \left|V(H)\right| \). □

Theorem 13 Every graph \( G \) has \( \chi'_s(G) \leq (2\bar{d}(G) - 1)\chi'(G) \).

Proof: Let \( k = \chi'(G) \). Then \( G \) has an edge \( k \)-coloring \( (E_1, E_2, \ldots, E_k) \), where each \( E_i \) is a matching of \( G \). Let \( G_i \) be the graph obtained from \( G \) by contracting each of edges in \( E_i \). By Lemma 12 and Corollary 10, \( G_i \) is \((2\bar{d}(G) - 2)\)-degenerate. By Lemma 1, \( \chi(G_i) \leq 2\bar{d}(G) - 1 \). By Lemma 11, \( \chi'_s(G_i) \leq (2\bar{d}(G) - 1)k = (2\bar{d}(G) - 1)\chi'(G) \). □

By Theorems 3, 4 and 13, the following two corollaries hold automatically.

Corollary 14 Every graph \( G \) has \( \chi'_s(G) \leq (2\bar{d}(G) - 1)(\Delta + 1) \).

Corollary 15 Every bipartite graph \( G \) has \( \chi'_s(G) \leq (2\bar{d}(G) - 1)\Delta \).

Corollary 16 If \( G \) is a graph with \( \Delta \geq 2\bar{d}(G) \), then \( \chi'_s(G) \leq (2\bar{d}(G) - 1)\Delta \).

Proof: Since \( G \) is \( \bar{d}(G) \)-degenerate and \( \Delta \geq 2\bar{d}(G) \), Theorem 7 asserts that \( \chi'(G) = \Delta \). By Theorem 13, \( \chi'_s(G) \leq (2\bar{d}(G) - 1)\Delta \). □
4.2 1-planar graphs

Recently, Liu et al. [22] investigated the existence of light edges in a 1-planar graph with minimum degree at least three. For our purpose, we here list one of their results as follows:

**Theorem 17** ([22]) *Every 1-planar graph* $G$ *with* $\delta(G) = 7$ *contains two adjacent* $7$*-vertices.*

With a greedy coloring procedure, it can be constructively shown that the strong chromatic index of a simple graph $G$ is at most $2\Delta(\Delta - 1) + 1$.

**Theorem 18** *If* $G$ *is a 1-planar graph, then* $\chi'_s(G) \leq 14\Delta$.

**Proof:** The proof is split into the following cases, depending on the size of $\Delta$.

Case 1: $\Delta \leq 7$.

It is easy to check that $2\Delta(\Delta - 1) + 1 \leq 14\Delta$ and henceforth the result follows.

Case 2: $\Delta = 8$.

Since $G$ is 7-degenerate, it follows from the result of [27] that $\chi'_s(G) \leq (4 \times 7 - 2)\Delta - 2 \times 7^2 + 1 = 26\Delta - 97 = 111 < 112 = 14\Delta$.

Case 3: $\Delta = 9$.

The proof is given by induction on the number of edges in $G$. If $|E(G)| \leq 14\Delta = 126$, the result holds trivially, since we may color all edges of $G$ with distinct colors. Let $G$ be a 1-planar graph with $\Delta = 9$ and $|E(G)| > 126$. Without loss of generality, assume that $G$ is connected, hence $\delta(G) \geq 1$. We have to consider two subcases as follows.

Case 3.1: $\delta(G) \leq 6$.

Let $u \in V(G)$ with $d_G(u) = \delta(G) \geq 1$. Let $u_0, u_1, \ldots, u_{s-1}$ denote the neighbors of $u$ in a cyclic order, where $1 \leq s = \delta(G) \leq 6$. For $0 \leq i \leq s - 1$, let $x_1^i, x_2^i, \ldots, x_p^i$ denote the neighbors of $u_i$ other than $u$. Consider the graph $H = G - u$. Then $H$ is a 1-planar graph with $\Delta(H) \leq 9$ and $|E(H)| < |E(G)|$. By the induction hypothesis or Cases 1 and 2, $H$ admits a strong edge coloring $\phi$ using the color set $C = \{1, 2, \ldots, 126\}$. For a vertex $v \in V(H)$, let $C(v)$ denote the set of colors assigned to the edges incident with $v$. For $i = 0, 1, \ldots, s - 1$, define a list $L(uu_i)$ of available colors for the edge $uu_i$ as follows:

$$L(uu_i) = C - \bigcup_{0 \leq j \leq s-1; \ j \neq i} C(u_j) - \bigcup_{1 \leq t \leq p_i} C(x_t^i).$$

It is easy to calculate that

$$|L(uu_i)| \geq |C| - \left| \bigcup_{0 \leq j \leq s-1; \ j \neq i} C(u_j) \right| - \left| \bigcup_{1 \leq t \leq p_i} C(x_t^i) \right|$$

$$\geq 126 - (s - 1)(\Delta - 1) - (\Delta - 1)\Delta$$

$$\geq 126 - (6 - 1)(9 - 1) - (9 - 1) \times 9$$

$$= 14.$$
Case 4: \( \Delta \geq 10. \)

By Theorem 6, \( G \) is of Class I. Let \((E_1, E_2, \ldots, E_\Delta)\) be an edge \( \Delta \)-coloring of \( G \), where each \( E_i \) is a matching of \( G \). Let \( G_i \) be the graph obtained from \( G \) by contracting each edge in \( E_i \). Note that each subgraph \( H \) of \( G \) is 1-planar and therefore \(|E(H)| \leq 4|V(H)| - 8\). Taking \( a = 4 \) and \( b = 8 \) in Corollary 9, we deduce that \( G_i \) is 13-degenerate. By Lemma 1, \( \chi(G_i) \leq 14 \). Therefore \( \chi'(G) \leq 14\Delta. \)

\[ \square \]

4.3 Special 1-planar graphs

Suppose that \( G \) is a 1-planar graph which is drawn in the plane so that each edge is crossed by at most one other edge. Let \( E' \) and \( E'' \) denote the set of non-crossing edges and crossing edges of \( G \), respectively. Let \( H_1 = G[E'] \) and \( H_2 = G[E''] \). That is, \( H_1 \) and \( H_2 \) are the subgraphs of \( G \) induced by non-crossing edges and crossing edges, respectively.

**Theorem 19** Let \( G \) be a 1-planar graph. Then \( \chi'_s(G) \leq 6\chi'(H_1) + 14\chi'(H_2). \)
For each vertex $v$, let $\chi'(G) \leq k_1 + k_2$. Let $(E_1, E_2, \ldots, E_{k_1})$ be an edge $k_1$-coloring of $H_1$, and $(F_1, F_2, \ldots, F_{k_2})$ be an edge $k_2$-coloring of $H_2$. Then each of $E_i$’s and $F_j$’s is a matching in $G$. So, $(E_1, E_2, \ldots, E_{k_1}, F_1, F_2, \ldots, F_{k_2})$ is an edge $(k_1 + k_2)$-coloring of $G$. Similarly to the proof of Case 4 in Theorem 18, every $F_i$ can be partitioned into 14 strong matchings of $G$. Moreover, for each $1 \leq i \leq k_1$, because $G_{E_i}$ is a 1-planar graph, we derive that $\chi(G_{E_i}) \leq 6$ by Theorem 2. By Lemma 11, $E_i$ can be partitioned into 6 strong matchings. Consequently, $\chi'_s(G) \leq 6k_1 + 14k_2$. 

An IC-planar graph is a 1-planar graph such that two pairs of crossing edges have no common end-vertices. Equivalently, each vertex of this kind of 1-planar graph is incident with at most one crossing edge. It is easy to verify that every IC-planar graph is vertex 6-choosable. Král and Stacho [20] showed that every IC-planar graph is vertex 5-colorable. Yang et al. [30] showed that every IC-planar graph is vertex 6-choosable. Furthermore, Dvořák et al. [9] proved that every graph drawn in the plane so that the distance between every pair of crossings is at least 15 is 5-choosable.

Using Theorem 19, we can establish the smaller upper bound for the strong chromatic index of IC-planar graphs.

**Theorem 20** Every IC-planar graph $G$ has $\chi'_s(G) \leq 6\Delta + 20$.

**Proof:** If $\Delta \leq 5$, then it is easy to obtain that $\chi'_s(G) \leq 2\Delta(\Delta - 1) + 1 \leq 6\Delta + 20$ and therefore the theorem holds. So assume that $\Delta \geq 6$. Let $H_1$ and $H_2$ denote the graphs induced by non-crossing edges and crossing edges of $G$, respectively. Since no two crossing-edges of $G$ are adjacent, $H_2$ is a matching of $G$. Thus, $\chi'(H_2) \leq 1$. Note that $H_1$ is a planar graph with $\Delta(H_1) \leq \Delta$. If $\Delta(H_1) \geq 7$, then $\chi'(H_1) = \Delta(H_1) \leq \Delta$ by Theorem 5. So, by Theorem 19, $\chi'_s(G) \leq 6\chi'(H_1) + 14\chi'(H_2) \leq 6\Delta + 14$. Otherwise, we have to consider two cases as follows:

- $\Delta(H_1) = 6$. Then $6 \leq \Delta \leq 7$. By Theorem 3, $\chi'(H_1) \leq 7$. By Theorem 19, $\chi'_s(G) \leq 6\chi'(H_1) + 14\chi'(H_2) \leq 6 \times 7 + 14 = 56 \leq 6\Delta + 20$.

- $\Delta(H_1) = 5$. Then $\Delta = 6$ by the assumption. By Theorem 3, $\chi'(H_1) \leq 6$. By Theorem 19, $\chi'_s(G) \leq 6\chi'(H_1) + 14\chi'(H_2) \leq 6 \times 6 + 14 = 50 = 6\Delta + 20$. 

A 1-planar graph $G$ is called optimal if $|E(G)| = 4|V(G)| - 8$. A plane quadrangulation is a plane graph such that each face of $G$ is of degree 4. It is not hard to show that a 3-connected plane quadrangulation is a bipartite plane graph with minimum degree 3. Suzuki [25] showed that every simple optimal 1-planar graph $G$ can be obtained from a 3-connected plane quadrangulation by adding a pair of crossing edges to each face of $G$. So an optimal 1-planar graph is an Eulerian graph, i.e., each vertex is of even degree. It was shown in [21] that every optimal 1-planar graph $G$ can be edge-partitioned into two planar graphs $G_1$ and $G_2$ such that $\Delta(G_2) \leq 4$.

**Theorem 21** Every optimal 1-planar graph $G$ has $\chi'_s(G) \leq 10\Delta + 14$.

**Proof:** Let $G$ be an optimal 1-planar graph. Let $H_1$ and $H_2$ denote the graphs induced by non-crossing edges and crossing edges of $G$, respectively. Then $G = H_1 \cup H_2$, where $H_1$ is a bipartite plane graph. For each vertex $v \in V(G)$, it is easy to see that $d_{H_1}(v) = d_{H_2}(v) = \frac{1}{2}d_G(v)$; in particular, we have $\Delta(H_1) = \Delta(H_2) = \frac{\Delta}{2}$.

Since $H_1$ is bipartite, $\chi'(H_1) = \Delta(H_1) = \frac{\Delta}{2}$ by Theorem 4. By Theorem 3, $\chi'(H_2) \leq \Delta(H_2) + 1 = \frac{\Delta}{2} + 1$. By Theorem 19, $\chi'_s(G) \leq 6\chi'(H_1) + 14\chi'(H_2) \leq 6 \times \frac{\Delta}{2} + 14 \times \left(\frac{\Delta}{2} + 1\right) = 10\Delta + 14$. 

\qed
5 Concluding remarks

In this paper, we show that the strong chromatic index of every 1-planar graph is at most $14\Delta$. As for the lower bound of strong chromatic index, Bensmail et al. [3] showed that for each $\Delta \geq 5$, there exist 1-planar graphs with strong chromatic index $6\Delta - 12$. Based on these facts, we put forward the following:

**Question 1.** What is the least constant $c_1$ such that every 1-planar graph $G$ satisfies $\chi'_s(G) \leq c_1\Delta$?

The foregoing discussion asserts that $6 \leq c_1 \leq 14$. We think that it is very difficult to reduce further the value of $c_1$ by employing the method used in this paper.

This paper also involves the strong edge coloring of some special 1-planar graphs such as IC-planar graphs and optimal 1-planar graphs. In particular, we show that the strong chromatic index of every IC-planar graph is at most $6\Delta + 20$. For $\Delta \geq 4$, by attaching $\Delta - 4$ new pendant vertices to each vertex of the complete graph $K_5$, we get a graph $H_\Delta$. Since $K_5$ is an IC-planar graph, so is $H_\Delta$. It is easy to inspect that any two edges of $H_\Delta$ lie in a path of length 2 or 3. So it follows that $\chi'_s(H_\Delta) = |E(H_\Delta)| = 10 + 5(\Delta - 4) = 5\Delta - 10$.

**Question 2.** What is the least constant $c_2$ such that every IC-planar graph $G$ satisfies $\chi'_s(G) \leq c_2\Delta$?

Notice that $5 \leq c_2 \leq 6$. We conjecture that $c_2 = 5$.

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