Linearized Boltzmann Collision Operator: II. Polyatomic Molecules Modeled by a Continuous Internal Energy Variable

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Abstract: The linearized collision operator of the Boltzmann equation for single species can be written as a sum of a positive multiplication operator, the collision frequency, and a compact integral operator. This classical result has more recently been extended to multi-component mixtures and polyatomic single species with the polyatomicity modeled by a discrete internal energy variable. In this work we prove compactness of the integral operator for polyatomic single species, with the polyatomicity modeled by a continuous internal energy variable, and the number of internal degrees of freedom greater or equal to two. The terms of the integral operator are shown to be, or be the uniform limit of, Hilbert-Schmidt integral operators. Self-adjointness of the linearized collision operator follows. Coercivity of the collision frequency are shown for hard-sphere like and hard potential with cut-off like models, implying Fredholmness of the linearized collision operator.

Keywords: Boltzmann equation, Polyatomic gases, Linearized collision operator, Hilbert-Schmidt integral operator

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1 Introduction

The Boltzmann equation is a fundamental equation of kinetic theory of gases. Considering deviations of an equilibrium, or Maxwellian, distribution, a linearized collision operator is obtained. The linearized collision operator can in a natural way be written as a sum of a positive multiplication operator, the collision frequency, and an integral operator $-K$. Compact properties of the integral operator $K$ (for angular cut-off kernels) are extensively studied for monatomic single species, see e.g. [13, 11, 9, 16]. The integral operator can be written as the sum of two compact integral operators, in the form of a Hilbert-Schmidt integral operator and an approximately Hilbert-Schmidt integral operator, i.e. an operator, which is the uniform limit of Hilbert-Schmidt integral operators (cf. Lemma 4 in Section 4) [12], and so compactness of the integral operator $K$ follows. More recently, compactness results were also obtained for monatomic multi-component mixtures [6], see also [8] for a different approach, and for polyatomic single species, where the polyatomicity is modeled by a discrete internal energy variable [3]. In this work, we consider polyatomic single species, where the polyatomicity is modeled by a continuous internal energy variable [8, 1, 11]. We restrict ourselves to the physical case when the number of internal degrees of freedom is greater or equal to two. The compactness property in the case
when the molecules is restricted to undergo resonant collisions $^7$, i.e. collisions where the sum of the internal energies is conserved under the collision, is recently considered in $^5$.

Motivated by an approach by Kogan in $^{15}$ Sect. 2.8 for the monatomic single species case, a probabilistic formulation of the collision operator is considered as the starting point. With this approach, cf. $^3$, it is shown, based on modified arguments in the monatomic case, that the integral operator $K$ can be written as a sum of a Hilbert-Schmidt integral operator and an operator, which is the uniform limit of Hilbert-Schmidt integral operators (and even might be an Hilbert-Schmidt integral operator itself) - and so compactness of the integral operator $K$ follows. The operator $K$ is self-adjoint, as well as the collision frequency, why the linearized collision operator as the sum of two self-adjoint operators of which one is bounded, is also self-adjoint.

For hard sphere like models and hard potential with cut-off like models, bounds on the collision frequency are obtained. Then the collision frequency is coercive and becomes a Fredholm operator. The set of Fredholm operators is closed under addition with compact operators, why also the linearized collision operator becomes a Fredholm operator by the compactness of the integral operator $K$. For hard sphere like models, as well as, "super hard" potential like models, the linearized collision operator satisfies all the properties of the general linear operator in the abstract half-space problem considered in $^2$.

The rest of the paper is organized as follows. In Section 2 the model considered is presented. The probabilistic formulation of the collision operator considered and its relation to a more classical formulation $^8$, $^1$, $^11$ is accounted for in Section 2.1. Some classical results for the collision operator in Section 2.2 and the linearized collision operator in Section 2.3 are reviewed. Section 3 is devoted to the main results of this paper. A proof of compactness of the integral operator is presented in Section 4 while a proof of the bounds on the collision frequency appears in Section 5.

## 2 Kinetic model

In this section the model considered is presented. A probabilistic formulation of the collision operator, cf. $^{15}$, $^{18}$, $^4$, $^3$, is considered, whose relation to a more classical formulation is accounted for. Known properties of the model and corresponding linearized collision operator are also reviewed.

Consider a single species of polyatomic molecules with mass $m$, where the polyatomicity is modeled by an internal energy variable $I \in \mathbb{R}_+$. The distribution functions are nonnegative functions of the form $f = f(t,x,\xi,I)$, with $t \in \mathbb{R}_+$, $x = (x,y,z) \in \mathbb{R}^3$, and $\xi = (\xi_x,\xi_y,\xi_z) \in \mathbb{R}^3$.

Moreover, consider the real Hilbert space $\mathfrak{h} := L^2 (d\xi \, dI)$, with inner product

$$(f,g) = \int_{\mathbb{R}^3 \times \mathbb{R}_+} fg \, d\xi \, dI \quad \text{for} \quad f,g \in L^2 (d\xi \, dI).$$

The evolution of the distribution functions is (in the absence of external
forces) described by the Boltzmann equation

\[ \frac{\partial f}{\partial t} + (\mathbf{\xi} \cdot \nabla_x) f = Q(f,f), \]

where the collision operator \( Q = Q(f,f) \) is a quadratic bilinear operator that accounts for the change of velocities and internal energies of particles due to binary collisions (assuming that the gas is rarefied, such that other collisions are negligible).

### 2.1 Collision operator

The collision operator in the Boltzmann equation (1) for polyatomic molecules can be written in the form

\[ Q(f,f) = \int_{(\mathbb{R}^3 \times \mathbb{R}_+)^3} W(\mathbf{\xi},\mathbf{\xi}_*, I, I_* | \mathbf{\xi}', \mathbf{\xi}_*, I', I_*') \]

\[ \times \left( \frac{f'f_*'}{(II_*')^{\delta/2-1}} - \frac{ff_*}{(II_*)^{\delta/2-1}} \right) d\xi_* d\xi_*' dI_* dI_*'. \]

Here and below the abbreviations

\[ f_* = f(t, x, \mathbf{\xi}_*, I_*), \quad f' = f(t, x, \mathbf{\xi}', I'), \quad \text{and} \quad f'_* = f(t, x, \mathbf{\xi}_*, I_*') \]

are used, and \( \delta \), with \( \delta \geq 2 \), denote the number of internal degrees of freedom.

The transition probabilities \( W(\mathbf{\xi}, \mathbf{\xi}_*, I, I_* | \mathbf{\xi}', \mathbf{\xi}_*, I', I_*') \) are of the form, cf. [15, 18, 4] for the monatomic case,

\[ W(\mathbf{\xi}, \mathbf{\xi}_*, I, I_* | \mathbf{\xi}', \mathbf{\xi}_*, I', I_*') = 4m \left( \frac{m}{2} \left( |\mathbf{\xi}|^2 + |\mathbf{\xi}_*|^2 - |\mathbf{\xi}'|^2 - |\mathbf{\xi}'_*|^2 \right) - \Delta I \right) \]

\[ \times \delta_1 \left( \frac{m}{2} \left( |\mathbf{\xi}|^2 + |\mathbf{\xi}_*|^2 - |\mathbf{\xi}'|^2 - |\mathbf{\xi}'_*|^2 \right) - \Delta I \right), \]

where \( \Delta I = I' + I'_* - I - I_* \) and

\[ \sigma = \sigma(|\mathbf{g}|, |\mathbf{g}'|, I, I_*), \quad \text{a.e., with} \quad \cos \theta = \frac{\mathbf{g} \cdot \mathbf{g}'}{|\mathbf{g}| |\mathbf{g}'|}, \]

\[ \mathbf{g} = \mathbf{\xi} - \mathbf{\xi}_*, \quad \mathbf{g}' = \mathbf{\xi}' - \mathbf{\xi}_*, \quad \text{and} \quad \Delta I = I' + I'_* - I - I_* \],

where \( \delta_3 \) and \( \delta_1 \) denote the Dirac’s delta function in \( \mathbb{R}^3 \) and \( \mathbb{R} \), respectively; taking the conservation of momentum and total energy into account. Here it is assumed that the scattering cross sections \( \sigma \) satisfy the microreversibility
condition

\[(II_e)^{\delta/2-1} |g|^2 \sigma (|g|, |\cos \theta|, I, I_e, I_e', I_e') \]

\[= (I' I_e')^{\delta/2-1} |g'|^2 \sigma (|g'|, |\cos \theta|, I', I_e', I_e). \tag{4} \]

Furthermore, to have invariance of change of particles in a collision, it is assumed that the scattering cross sections \(\sigma\) satisfy the symmetry relations

\[\sigma (|g|, |\cos \theta|, I, I_e, I_e', I_e') = \sigma (|g|, |\cos \theta|, I, I_e, I_e', I') \]

\[= \sigma (|g|, |\cos \theta|, I, I_e, I', I_e). \tag{5} \]

The invariance under change of particles in a collision, which follows by the definition of the transition probability \(\mathcal{W}\) and the symmetry relations for the collision frequency, and the microreversibility of the collisions \(\mathcal{W}\), imply that the transition probabilities \(\mathcal{W}\) satisfy the relations

\[\mathcal{W}(\xi, \xi_e, I, I_e, |\xi', \xi_e', I', I_e') = \mathcal{W}(\xi, \xi_e, I, |\xi', \xi_e', I, I_e') \]

\[\mathcal{W}(\xi, \xi_e, I, I_e, |\xi', \xi_e', I', I_e) = \mathcal{W}(\xi, \xi_e, I, |\xi', \xi_e', I, I_e'). \tag{6} \]

Applying known properties of Dirac’s delta function, the transition probabilities may be transformed to

\[\mathcal{W}(\xi, \xi_e, I, I_e, |\xi', \xi_e', I', I_e) \]

\[= \frac{m}{2} (I' I_e')^{\delta/2-1} \sigma \frac{|g'|^2}{|g|} \delta_3 (G - G') \delta_1 \left( \frac{m}{4} \left( |g|^2 - |g'|^2 \right) - \Delta I \right) \]

\[= \frac{m}{2} (I' I_a')^{\delta/2-1} \sigma \frac{|g'|^2}{|g|} \delta_3 (G - G') \delta_1 (E - E') \]

\[= (I' I_e')^{\delta/2-1} \sigma \frac{|g'|^2}{|g|} \delta_3 (G - G') \delta_1 \left( \sqrt{|g|^2 - \frac{4m}{\Delta I}} - |g'| \right) 1_{m|g|^2 > 4\Delta I} \]

\[= (I_a e^{\delta/2-1}) \sigma \frac{|g|^2}{|g'|^2} \delta_3 (G - G') \delta_1 \left( \sqrt{|g|^2 - \frac{4m}{\Delta I}} - |g'| \right) 1_{m|g|^2 > 4\Delta I}, \text{ with} \]

\[G = \frac{\xi + \xi_e}{2}, \quad G' = \frac{\xi' + \xi_e}{2}, \quad E = \frac{m}{4} |g|^2 + I + I_e, \quad E' = \frac{m}{4} |g'|^2 + I' + I_e. \]

A series of change of variables: first \(\{\xi', \xi_e'\} \rightarrow \{g' = \xi' - \xi_e', G' = \frac{\xi' + \xi_e'}{2}\}\), followed by a change to spherical coordinates \(\{g'\} \rightarrow \{|g'|, \omega = \frac{g'}{|g'|}\}\), and then \(\{|g'|, I', I_e'\} \rightarrow \left\{ R = \frac{m|g|^2}{4E}, r = \frac{I'}{(1-R)E}, E' = \frac{m}{4} |g'|^2 + I' + I_e' \right\}\), ob-
serving that
\[ d\xi' d\xi_x' dI' dI_x' = dg' dG' dI' dI_x' = |g|^2 |g'| d\omega \ dG' dI' dI_x' \]
\[ = \frac{4}{m^{3/2}} E^{5/2} (1 - R) R^{1/2} dR d\omega \ dG' dr \ dE' \]
\[ = \frac{4 E^{\delta+1/2}}{m^{3/2} (I'I_x')^{\delta/2-1}} (r(1 - r))^{\delta/2-1} (1 - R)^{\delta-1} R^{1/2} \ dE' dG' \ d\omega \ dr \ dR, \]
where \( I' = r (1 - R) E \) and \( I_x' = (1 - r) (1 - R) E, \) (7)
results in a more familiar form of the Boltzmann collision operator for polyatomic molecules modeled with a continuous internal energy variable [8][1][11]

\[
Q(f, f) = \int_{(R^3 \times (R_+)^2) \times [0,1]^2 \times S^2} W(\xi, \xi_x, I, I_x | \xi', \xi_x', I', I_x') \]
\[ \times \frac{4 E^{\delta+1/2}}{m^{3/2} (I'I_x')^{\delta/2-1}} \left( \frac{f'f'_x}{(I'I_x')^{\delta/2-1}} - \frac{ff_x}{(I'I_x')^{\delta/2-1}} \right) \]
\[ \times (r(1 - r))^{\delta/2-1} (1 - R)^{\delta-1} R^{1/2} \ dE' dG' \ d\omega \ dr \ dR \ d\xi_x \ dI_x, \]

with the collision kernel
\[
B = \frac{2 \sigma |g| E^{\delta+1/2} I_m |g|^2 > 4 \Delta I}{\sqrt{m} \sqrt{|g|^2 - \frac{4}{m} \Delta I (I'I_x')^{\delta/2-1}}} = \frac{\sigma |g| E^2 1_{E>0}}{(1 - R)^{\delta-2} R^{1/2} (r(1 - r))^{\delta/2-1}}, \text{ where} \]
\[
E = \frac{m |g|^2}{4} - \Delta I \text{ and } \Delta I = I' + I_x' - I - I_x, \] (8)

and

\[
\frac{\xi}{2} = \frac{\xi + \xi_x}{2} + \sqrt{\frac{\xi - \xi_x}{2}} - \frac{4 \Delta I}{m} \omega = G + \frac{\sqrt{|g|^2 - \frac{4}{m} \Delta I}}{2} \omega, \quad \omega \in S^2. 
\]

Remark 1 By multiplying the transition probability [3] with an extra Dirac’s delta function in \( R \), namely
\[ \delta_1 (\Delta I) \]
we obtain the case where the molecules are assumed to undergo resonant collisions [7].
2.2 Collision invariants and Maxwellian distributions

The following lemma follows directly by the relations (6).

**Lemma 1** The measure

\[ dA = W(\xi, \xi', I, I') d\xi d\xi' dI dI' \]

is invariant under the interchanges of variables

\[ (\xi, \xi', I, I') \leftrightarrow (\xi', \xi, I, I'), \]
\[ (\xi, I) \leftrightarrow (\xi, I), \]
\[ (\xi', I') \leftrightarrow (\xi', I'), \]

respectively.

The weak form of the collision operator \( Q(f, f) \) reads

\[
(Q(f, f), g) = \int_{(\mathbb{R}^3 \times \mathbb{R}_+)^4} \left( \frac{f' f'_*}{(I' I'_*)^{3/2-1}} - \frac{f f_*}{(I I'_*)^{3/2-1}} \right) g \, dA
\]

for \( g = g(\xi, I) \), such that the first integral is defined, while the following integrals are obtained by applying Lemma 1.

We have the following proposition.

**Proposition 1** Let \( g = g(\xi, I) \) be such that

\[
\int_{(\mathbb{R}^3 \times \mathbb{R}_+)^4} \left( \frac{f' f'_*}{(I' I'_*)^{3/2-1}} - \frac{f f_*}{(I I'_*)^{3/2-1}} \right) g \, dA
\]

is defined. Then

\[
(Q(f, f), g) = \frac{1}{4} \int_{(\mathbb{R}^3 \times \mathbb{R}_+)^4} \left( \frac{f' f'_*}{(I' I'_*)^{3/2-1}} - \frac{f f_*}{(I I'_*)^{3/2-1}} \right) (g + g_* - g'_* + g'_*) \, dA.
\]

**Definition 1** A function \( g = g(\xi, I) \) is a collision invariant if

\[
(g + g_* - g'_* + g'_*) W(\xi, \xi', I, I' | \xi', \xi', I', I'_*) = 0 \ a.e.
\]
Then it is clear that $1, \xi_x, \xi_y, \xi_z$, and $m|\xi|^2 + 2I$ are collision invariants - corresponding to conservation of mass, momentum, and total energy - and, in fact we have the following proposition, cf. Proposition 1 in [8].

**Proposition 2** The vector space of collision invariants is generated by
\[
\{1, \xi_x, \xi_y, \xi_z, m|\xi|^2 + 2I\}.
\]

Define
\[
\mathcal{W}[f] := \left( Q(f, f), \log \left( I^{1-\delta/2} f \right) \right).
\]

It follows by Proposition 1 that
\[
\mathcal{W}[f] = -\frac{1}{4} \int_{(\mathbb{R}^3 \times \mathbb{R}_+)^4} \frac{f f_* (I I_*)^{\delta/2-1}}{(f f_* (I I_*)^{\delta/2-1})} (I I_*)^{\delta/2-1} f' f'_* \frac{dA}{(f f_* (I I_*)^{\delta/2-1})}.
\]

Since $(x - 1) \log (x) \geq 0$ for $x > 0$, with equality if and only if $x = 1$,
\[
\mathcal{W}[f] \leq 0,
\]
with equality if and only if
\[
\left( \frac{f f_*}{(I I_*)^{\delta/2-1}} - \frac{f' f'_*}{(I' I'_*)^{\delta/2-1}} \right) W(\xi, \xi_*, I, I_* | \xi', \xi'_*, I', I'_*) = 0 \text{ a.e.}, \quad (10)
\]
or, equivalently, if and only if
\[
Q(f, f) \equiv 0.
\]

For any equilibrium, or Maxwellian, distribution $M$, $Q(M, M) \equiv 0$, why it follows, by the relation (10), that
\[
\left( \log \frac{M}{I^{\delta/2-1}} + \log \frac{M_*}{I_*^{\delta/2-1}} - \log \frac{M'}{(I')^{\delta/2-1}} - \log \frac{M'_*}{(I'_*)^{\delta/2-1}} \right) \times W(\xi, \xi_*, I, I_* | \xi', \xi'_*, I', I'_*) = 0 \text{ a.e.}.
\]

Hence, $\log \frac{M}{I^{\delta/2-1}}$ is a collision invariant, and the Maxwellian distributions are of the form
\[
M = \frac{n I^{\delta/2-1} m^{3/2}}{(2\pi)^{3/2} T^{(\delta+3)/2} \Gamma \left( \frac{\delta}{2} \right)} e^{-\left( m |\xi - u|^2 + 2I \right)/(2T)},
\]
where $n = (M, 1)$, $u = \frac{1}{n} (M, \xi)$, and $T = \frac{m}{3n} \left( M, |\xi - u|^2 \right)$, while $\Gamma = \Gamma(s)$ denotes the Gamma function $\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx$. 

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Note that by equation (10) any Maxwellian distribution $M$ satisfies the relations
\[
\left( \frac{M' M_*}{(I_*')^{3/2-1}} - \frac{M M_*}{(I_*)^{3/2-1}} \right) W(\xi, \xi', I, I_* | \xi', \xi'_*, I', I'_*) = 0 \text{ a.e. .} \tag{11}
\]

**Remark 2** Introducing the $\mathcal{H}$-functional
\[
\mathcal{H}[f] = \left( f, \log \left( I - \frac{1}{2} W(\xi, \xi', I, I_* | \xi', \xi'_*, I', I'_*) \right) \right),
\]
an $\mathcal{H}$-theorem can be obtained, cf. [8, 1, 11].

### 2.3 Linearized collision operator

Considering deviations of a Maxwellian distribution $M = I_*^{d/2} e^{-\frac{|\xi|^2}{2}}$ of the form
\[
f = M + M^{1/2} h
\]
of the form
\[
\frac{\partial h}{\partial t} + (\xi \cdot \nabla_x) h + \mathcal{L} h = \Gamma (h, h), \tag{13}
\]
where the linearized collision operator $\mathcal{L}$ is given by
\[
\mathcal{L} h = -M^{-1/2} \left( Q(M, M^{1/2} h) + Q(M^{1/2} h, M) \right)
\]
\[
= M^{-1/2} \int_{(\mathbb{R}^3)^3} \left( \frac{M M_* M'_*}{(I_*')^{3/4-1/2}} W(\xi, \xi', I, I_* | \xi', \xi'_*, I', I'_*) \right)
\times \left( \frac{h}{M^{1/2}} + \frac{h_*}{M_*^{1/2}} - \frac{h'}{(M')^{1/2}} - \frac{h'_*}{(M'_*)^{1/2}} \right) d\xi d\xi' dI dI' dI'_*, \tag{14}
\]
with
\[
v = \int_{(\mathbb{R}^3)^3} \frac{M_*}{(I_*)^{3/2-1}} W(\xi, \xi', I, I_* | \xi', \xi'_*, I', I'_*) d\xi d\xi' dI dI' dI'_*,
\]
\[
\mathcal{K} (h) = M^{-1/2} \int_{(\mathbb{R}^3)^3} \left( \frac{M M_* M'_*}{(I_*')^{3/4-1/2}} W(\xi, \xi', I, I_* | \xi', \xi'_*, I', I'_*) \right)
\times \left( \frac{h'}{(M')^{1/2}} + \frac{h'_*}{(M'_*)^{1/2}} - \frac{h_*}{M_*^{1/2}} \right) d\xi d\xi' dI dI' dI'_*, \tag{15}
\]
while the quadratic term $\Gamma$ is given by
\[
\Gamma (h, h) = M^{-1/2} Q(M^{1/2} h, M^{1/2} h), \tag{16}
\]
The following lemma follows immediately by Lemma [1].
Lemma 2 The measure
\[ d\bar{A} = \frac{(MM_s M'_s)^{1/2}}{(II_s I'_s)^{3/4-1/2}} d\bar{A} \]
is invariant under the interchanges of variables \( \tilde{\mathcal{I}} \), respectively.

The weak form of the linearized collision operator \( \mathcal{L} \) reads
\[
(L h, g) = \int_{(\mathbb{R}^3 \times \mathbb{R}_+)} \left( \frac{h}{M^{1/2}} + \frac{h_s}{M_s^{1/2}} - \frac{h'_s}{(M'_s)^{1/2}} - \frac{h'_s}{(M^*_s)^{1/2}} \right) \frac{g}{M^{1/2}} d\bar{A}
\]
for \( g = g(\xi, I) \), such that the first integral is defined, while the following integrals are obtained by applying Lemma 2. Then we have the following lemma.

Lemma 3 Let \( g = g(\xi, I) \) be such that
\[
\int_{(\mathbb{R}^3 \times \mathbb{R}_+)} \left( \frac{h}{M^{1/2}} + \frac{h_s}{M_s^{1/2}} - \frac{h'_s}{(M'_s)^{1/2}} - \frac{h'_s}{(M^*_s)^{1/2}} \right) \frac{g}{M^{1/2}} d\bar{A}
\]
is defined. Then
\[
(L h, g) = \frac{1}{4} \int_{(\mathbb{R}^3 \times \mathbb{R}_+)} \left( \frac{h}{M^{1/2}} + \frac{h_s}{M_s^{1/2}} - \frac{h'_s}{(M'_s)^{1/2}} - \frac{h'_s}{(M^*_s)^{1/2}} \right) \frac{g}{M^{1/2}} d\bar{A}
\]
and the kernel of \( \mathcal{L} \), \( \text{ker} \mathcal{L} \), is generated by
\[
\left\{ M^{1/2}, \xi_\varepsilon M^{1/2}, \xi_\varepsilon M^{1/2}, \xi_\varepsilon M^{1/2}, (m |\xi|^2 + 2I) M^{1/2} \right\}.
\]

Proof. By Lemma 2 it is immediate that \( (L h, g) = (h, L g) \) and \( (L h, h) \geq 0 \).
Furthermore, \( h \in \text{ker} \mathcal{L} \) if and only if \( (L h, h) = 0 \), which will be fulfilled if and only if
\[
\left( \frac{h}{M^{1/2}} + \frac{h_s}{M_s^{1/2}} - \frac{h'_s}{(M'_s)^{1/2}} - \frac{h'_s}{(M^*_s)^{1/2}} \right) W(\xi, \xi_\varepsilon, I, I_\varepsilon |\xi|^2, I'_\varepsilon, I'_s) = 0 \text{ a.e.,}
\]
where
\[
W(\xi, \xi_\varepsilon, I, I_\varepsilon |\xi|^2, I'_\varepsilon, I'_s) = 0 \text{ a.e.,}
\]
i.e. if and only if \( \frac{h}{M^{1/2}} \) is a collision invariant. The last part of the lemma now follows by Proposition 2.

**Remark 3** Note also that the quadratic term is orthogonal to the kernel of \( \mathcal{L} \), i.e. \( \Gamma(h,h) \in (\ker \mathcal{L})^\perp \).

### 3 Main results

In this section the main results, concerning compactness properties in Theorem 1 and bounds on collision frequencies in Theorem 2, are presented. The proofs of the corollaries are essentially the same as the corresponding ones in [3], but are presented here for self-containment of the paper.

Assume that for some positive number \( \gamma \), such that \( 0 < \gamma < 1 \), there is a bound

\[
0 \leq \sigma(|g|, \cos \theta, I, I^*, I', I'') \leq C \left( \frac{\Psi + \Psi^{\gamma/2}}{|g|^2} \right) \left( \frac{1 + 1}{E} \right) \left( \frac{1}{m} \right) |g|^2 \geq 4 \Delta I^2
\]

with \( E = \frac{m}{4} |g|^2 + I + I_* \) and \( \Psi = |g| \sqrt{|g|^2 - \frac{4}{m} \Delta I} \),

on the scattering cross section \( \sigma \), or, equivalently, the bound

\[
0 \leq B(|g|, \cos \theta, I, I^*, I', I'') \leq CE \left( 1 + \frac{1}{\Psi^{\gamma/2}} \right) \left( \frac{1}{m} \right) |g|^2 \geq 4 \Delta I^2
\]

on the collision kernel [8], for some positive constant \( C > 0 \). Then the following result may be obtained.

**Theorem 1** Assume that the scattering cross section \( \sigma \), satisfy the bound [17] for some positive number \( \gamma \), such that \( 0 < \gamma < 1 \). Then the operator \( K \) given by [15] is a self-adjoint compact operator on \( L^2(d\xi \, dI) \).

The proof of Theorem 1 will be addressed in Section 4.

**Corollary 1** The linearized collision operator \( \mathcal{L} \), with scattering cross section satisfying [17], is a closed, densely defined, self-adjoint operator on \( L^2(d\xi \, dI) \).

**Proof.** The multiplication operator \( \Lambda \), where \( \Lambda f = vf \), is a closed, densely defined, self-adjoint operator. Hence, by Theorem 1 \( \mathcal{L} = \Lambda - K \) is closed, as the sum of a closed and a bounded operator, densely defined, since the domains of the linear operators \( \mathcal{L} \) and \( \Lambda \) are equal, \( D(\mathcal{L}) = D(\Lambda) \), and self-adjoint, since the set of self-adjoint operators is closed under addition of bounded self-adjoint operators, see Theorem 4.3 of Chapter V in [14].

**Remark 4** The collision kernels (cf. Model 1-3 in [77])

\[
B = bE^{\gamma/2}
\]
2) 

\[ B = b \left( R^{\alpha/2} |g|^\alpha + (1 - R)^{\alpha/2} \left( I + I^* \right) \frac{\alpha}{2} \right) \leq \frac{2^\alpha + 1}{m^{\alpha/2}} bE^{\alpha/2} \]

3) 

\[ B = b \left( R^{\alpha/2} |g|^\alpha + r (1 - R) \frac{I}{m} \right)^{\alpha/2} + \left( (1 - r) (1 - R) \frac{I}{m} \right)^{\alpha/2} \]

\[ \leq \frac{2^\alpha + 2}{m^{\alpha/2}} bE^{\alpha/2} \]

where \( \left\{ \begin{array}{l} R = \frac{m |g'|^2}{4E}, r = \frac{I'}{(1 - R)E} \end{array} \right\} \in [0, 1]^2 \), satisfy the bound (18) for positive numbers \( \alpha \) less or equal to 2, 0 < \( \alpha \leq 2 \), and bounded functions \( b = b(\cos \theta) \).

Remark 5 By slight modifications in the proof of Theorem 1 in Section 4 one may replace the bounds (17) and (18), with

\[ 0 \leq \sigma 1_{m|g|^2 > 4\Delta I} \leq C \frac{\Psi + \Psi^{\gamma/2} 1 + E^{-\eta-1} (I'')^{\delta/2} - 1}{|g|^2} 1_{m|g|^2 > 4\Delta I} \]

and

\[ 0 \leq B \leq C (E^{\eta} + E) \left( 1 + \frac{1}{\Psi^{1-\gamma/2}} \right) 1_{m|g|^2 > 4\Delta I} \]

for any \( 1 \leq \eta < \frac{\delta + 1}{2} \), respectively.

Now consider the scattering cross section

\[ \sigma = C \sqrt{\frac{|g|^2 - \frac{4}{m} \Delta I}{|g| E^{3+(\alpha-1)/2}}} (I'')^{\delta/2} - 1 \text{ if } |g|^2 > \frac{4}{m} \Delta I \]  

or, equivalently, the collision kernel (8)

\[ B = CE^{1-\alpha/2} 1_{E > 0} \]

for some positive constant \( C > 0 \) and nonnegative number \( \alpha \) less than 2, 0 \( \leq \alpha < 2 \) - cf. hard sphere models for \( \alpha = 1 \).

In fact, it would be enough with the bounds, if \( |g|^2 > \frac{4}{m} \Delta I \),

\[ C \sqrt{\frac{|g|^2 - \frac{4}{m} \Delta I}{|g| E^{3+(\alpha-1)/2}}} (I'')^{\delta/2} - 1 \leq \sigma \leq C \sqrt{\frac{|g|^2 - \frac{4}{m} \Delta I}{|g| E^{3+(\alpha-1)/2}}} (I'')^{\delta/2} - 1 \]
on the scattering cross sections, or, equivalently, the bounds

\[ C^- E^{1-\alpha/2} 1_{E>0} \leq B \leq C^+ E^{1-\alpha/2} 1_{E>0} \]  

(22)
on the collision kernel \([8]\), for some positive constants \(C_\pm > 0\) and nonnegative number \(\alpha\) less than 2, \(0 \leq \alpha < 2\) - cf. hard potential with cut-off models, with "super hard" potentials for \(0 \leq \alpha < 1\).

**Theorem 2** The linearized collision operator \(\mathcal{L}\), with scattering cross section \([19]\) (or \([21]\)), can be split into a positive multiplication operator \(\Lambda\), where \(\Lambda f = \nu f\), with \(\nu = \nu(|\xi|, I)\), minus a compact operator \(K\) on \(L^2(d\xi\,dI)\)

\[ \mathcal{L} = \Lambda - K, \]  

(23)
such that there exist positive numbers \(\nu_-\) and \(\nu_+\), \(0 < \nu_- < \nu_+\), such that for any positive number \(\varepsilon > 0\)

\[ \nu_- \left(1 + |\xi| + \sqrt{I}\right)^{2-\alpha} \leq \nu(|\xi|, I) \leq \nu_+ \left(1 + |\xi| + \sqrt{I}\right)^{2-\alpha+\varepsilon} \text{ for all } \xi \in \mathbb{R}^3. \]  

(24)
The decomposition \((23)\) follows by decomposition \((14), (15)\) and Theorem 1, while the bounds \((24)\) on the collision frequency will be proven in Section 5.

**Corollary 2** The linearized collision operator \(\mathcal{L}\), with scattering cross section \([19]\) (or \([21]\)), is a Fredholm operator.

**Proof.** By Theorem 2 the multiplication operator \(\Lambda\) is coercive, why it is a Fredholm operator. Furthermore, the set of Fredholm operators is closed under addition of compact operators, see Theorem 5.26 of Chapter IV in \([14]\) and its proof, so, by Theorem 2, \(\mathcal{L}\) is a Fredholm operator. \(\blacksquare\)

For hard sphere like models and "super hard" potential like models, we obtain the following result.

**Corollary 3** For the linearized collision operator \(\mathcal{L}\), with scattering cross section \([19]\) (or \([21]\)) with \(0 \leq \alpha \leq 1\), there exists a positive number \(\lambda\), \(0 < \lambda < 1\), such that

\[ (h, \mathcal{L} h) \geq \lambda (h, \nu(|\xi|, I) h) \geq \lambda \nu_- (h, (1 + |\xi|) h) \]
for all \(h \in D(\mathcal{L}) \cap \text{Im}\mathcal{L}\).

**Proof.** Let \(h \in D(\mathcal{L}) \cap (\ker\mathcal{L})^\perp = D(\mathcal{L}) \cap \text{Im}\mathcal{L}\). As a Fredholm operator, \(\mathcal{L}\) is closed with a closed range, and as a compact operator, \(K\) is bounded, and so there are positive constants \(\nu_0 > 0\) and \(c_K > 0\), such that

\[ (h, \mathcal{L} h) \geq \nu_0 (h, h) \text{ and } (h, Kh) \leq c_K (h, h). \]
Let $\lambda = \frac{\nu_0}{\nu_0 + c_K}$. Then the corollary follows, since

\begin{align*}
(h, Lh) &= (1 - \lambda)(h, Lh) + \lambda(h, (\nu(|\xi|, I) - K)h) \\
&\geq (1 - \lambda)\nu_0(h, h) + \lambda(h, \nu(|\xi|, I)h) - \lambda c_K(h, h) \\
&= (\nu_0 - \lambda(\nu_0 + c_K))(h, h) + \lambda(h, \nu(|\xi|, I)h) \\
&= \lambda(h, \nu(|\xi|, I)h).
\end{align*}

\[\square\]

**Remark 6**  For hard sphere like models, as well as, "super hard" potential like models, the linearized collision operator satisfies all the properties of the general linear operator in the abstract half-space problem considered in [2], under the assumption $B = \xi_x$ for the linear operator $B$ in [2], by Proposition 3 and Corollaries [7,8]. Indeed, by Proposition [3] and Corollaries [7,8] with $0 \leq \alpha \leq 1$, in the expressions 19] and [20], or, the bounds 21] and (22), for the scattering cross section and collision kernel, respectively, the linearized collision operator will be a nonnegative self-adjoint Fredholm operator on the real Hilbert space $\mathfrak{h} = L^2(d\xi dI)$, and moreover, there exists a positive number $\mu$, $\mu > 0$, such that

\[ (h, Lh) \geq \mu (h, (1 + |\xi_x|)h) \text{ for all } h \in D(L) \cap \text{Im}L. \]

\section{Compactness}

This section concerns the proof of Theorem 1. Note that in the proof the kernels are rewritten in such a way that $\xi_x$ and $I_x$ - and not $\xi'$ and $I'$, or, $\xi'$ and $I'$ - always will be the arguments of the distribution functions. Then there will be essentially two different types of kernels; either $(\xi_x, I_x)$ are arguments in the loss term (like $(\xi, I)$) or in the gain term (unlike $(\xi, I)$) of the collision operator. The kernel of the term from the loss part of the collision operator will be shown to be Hilbert-Schmidt in a quite direct way, while the kernel of the terms from the gain part of the collision operator will be shown to be approximately Hilbert-Schmidt, in the sense of Lemma 4 below, or, equivalently, a uniform limit of Hilbert-Schmidt integral operators. In fact, with similar arguments as in the proof below one may show that if the number of internal degrees of freedom $\delta$ is greater than $5/2$, $\delta > 5/2$, then, under the assumption 17], the terms from the gain part are Hilbert-Schmidt integral operators as well.

Denote, for any (nonzero) natural number $N$,

\[
\mathfrak{h}_N := \left\{ (\xi, \xi_x, I, I_x) \in (\mathbb{R}^3 \times \mathbb{R}_+)^2 : |\xi - \xi_x| \geq \frac{1}{N}; |\xi| \leq N \right\}, \text{ and}
\]

\[
b^{(N)} := b^{(N)}(\xi, \xi_x, I, I_x) := b(\xi, \xi_x, I, I_x)1_{\mathfrak{h}_N}.
\]

Then we have the following lemma, cf Glassey [12, Lemma 3.5.1] and Drange [10].
Lemma 4. Assume that $Tf(\xi, I) = \int_{\mathbb{R}^3} b(\xi, \xi_s, I, I_s) f(\xi_s, I_s) d\xi_s dI_s$, with $b(\xi, \xi_s, I, I_s) \geq 0$. Then $T$ is compact on $L^2(d\xi dI)$ if

(i) $\int_{\mathbb{R}^3 \times \mathbb{R}^+} b(\xi, \xi_s, I, I_s) d\xi dI$ is bounded in $(\xi_s, I_s)$;

(ii) $b^{(N)} \in L^2(d\xi d\xi_s dI_s)$ for any (nonzero) natural number $N$;

(iii) $\sup_{(\xi, I)} \int_{\mathbb{R}^3 \times \mathbb{R}^+} b(\xi, \xi_s, I, I_s) - b^{(N)}(\xi, \xi_s, I, I_s) d\xi_s dI_s \to 0$ as $N \to \infty$.

Then $T$ is the uniform limit of Hilbert-Schmidt integral operators [12, Lemma 3.5.1], and we say that the kernel $b(\xi, \xi_s, I, I_s)$ is approximately Hilbert-Schmidt, while $T$ is an approximately Hilbert-Schmidt integral operator. Note that by this definition a Hilbert-Schmidt integral operator, will also be an approximately Hilbert-Schmidt integral operator. The reader is referred to [12, Lemma 3.5.1] for a proof of Lemma 4.

Note that throughout the proof, $C$ will denote a generic positive constant.

Proof. Rewrite expression (15) as

$$Kh = M^{-1/2} \int_{(\mathbb{R}^3 \times \mathbb{R}^+)^3} w(\xi, \xi_s, I, I_s | \xi', \xi_s', I', I_s') \times \left( \frac{h_s}{M_s^{1/2}} - \frac{h'_s}{(M')^{1/2}} - \frac{h''_s}{(M'_s)^{1/2}} \right) d\xi_s d\xi' dI_s dI' dI'_s,$$

with

$$w(\xi, \xi_s, I, I_s | \xi', \xi_s', I', I_s') = \frac{(MM_sM'_s)^{1/2}}{(M'IM'I'_s)^{1/2}} W(\xi, \xi_s, I, I_s | \xi', \xi_s', I', I_s').$$

Due to relations (9), the relations

$$w(\xi, \xi_s, I, I_s | \xi', \xi_s', I', I_s') = w(\xi, \xi_s, I, I_s | \xi', \xi_s', I', I'_s)$$

$$w(\xi, \xi_s, I, I_s | \xi', \xi_s', I', I'_s) = w(\xi, \xi_s, I, I_s | \xi', \xi_s', I', I_s)$$

$$w(\xi, \xi_s, I, I_s | \xi', \xi_s', I', I'_s) = w(\xi, \xi_s, I, I_s | \xi', \xi_s', I', I'_s)$$

are satisfied.

By first renaming $\{\xi_s, I_s\} \Rightarrow \{\xi'_s, I'_s\}$, then $\{\xi_s, I_s\} \Rightarrow \{\xi'_s, I'_s\}$, followed by applying the last relation in (25),

$$\int_{(\mathbb{R}^3 \times \mathbb{R}^+)^3} w(\xi, \xi_s, I, I_s | \xi', \xi_s', I', I'_s) \frac{h'_s}{(M'_s)^{1/2}} d\xi_s d\xi'_s dI_s dI'_s$$

$$= \int_{(\mathbb{R}^3 \times \mathbb{R}^+)^3} w(\xi, \xi'_s, I, I'_s | \xi_s, \xi'_s, I_s, I'_s) \frac{h'_s}{(M'_s)^{1/2}} d\xi_s d\xi'_s dI_s dI'_s$$

$$= \int_{(\mathbb{R}^3 \times \mathbb{R}^+)^3} w(\xi, \xi'_s, I, I'_s | \xi_s, \xi'_s, I_s, I'_s) \frac{h_s}{M_s^{1/2}} d\xi_s d\xi'_s dI_s dI'_s$$

$$= \int_{(\mathbb{R}^3 \times \mathbb{R}^+)^3} w(\xi, \xi'_s, I, I'_s | \xi_s, \xi'_s, I_s, I'_s) \frac{h_s}{M_s^{1/2}} d\xi_s d\xi'_s dI_s dI'_s.$$
Moreover, by renaming \( \{ \xi_*, I_* \} \rightleftharpoons \{ \xi', I' \} \),

\[
\int_{(\mathbb{R}^3 \times \mathbb{R}^+)^3} w(\xi, \xi_*, I, I_* | \xi', \xi'_*, I', I'_*) \frac{h'}{(M')^{1/2}} d\xi d\xi' dI dI' dI'_* \\
= \int_{(\mathbb{R}^3 \times \mathbb{R}^+)^3} w(\xi, \xi', I, I' | \xi'_*, I_*, I'_*) \frac{h_*}{M_*^{1/2}} d\xi d\xi' dI dI' dI'_*. 
\]

It follows that

\[
K(h) = \int_{\mathbb{R}^3 \times \mathbb{R}^+} k(\xi, \xi_*, I, I_*) h_* d\xi dI_*, \text{ where} \\
k(\xi, \xi_*, I, I_*) = k_2(\xi, \xi_*, I, I_*) - k_1(\xi, \xi_*, I, I_*),
\]

with

\[
k_1(\xi, \xi_*, I, I_*) \\
= (M_*^{-1/2}) \int_{(\mathbb{R}^3 \times \mathbb{R}^+)^2} w(\xi, \xi_*, I, I_* | \xi', \xi'_*, I', I'_*) d\xi' dI' dI'_* \\
k_2(\xi, \xi_*, I, I_*) \\
= 2 (M_*^{-1/2}) \int_{(\mathbb{R}^3 \times \mathbb{R}^+)^2} w(\xi, \xi', I, I' | \xi'_*, I_*, I'_*) d\xi' dI' dI'_*.
\]

Note that

\[
k(\xi, \xi_*, I, I_*) = k_2(\xi, \xi_*, I, I_*) - k_1(\xi, \xi_*, I, I_*),
\]

since, by applying the first and the last relation in (25),

\[
k_1(\xi, \xi_*, I, I_*) \\
= (M_*^{-1/2}) \int_{(\mathbb{R}^3 \times \mathbb{R}^+)^2} w(\xi_*, \xi, I_* | \xi'_*, \xi'_*, I'_*, I'_*) d\xi' dI' dI'_* \\
= (M_*^{-1/2}) \int_{(\mathbb{R}^3 \times \mathbb{R}^+)^2} w(\xi_*, \xi, I_* | \xi'_*, \xi'_*, I'_*, I'_*) d\xi' dI' dI'_* \\
= k_1(\xi_*, \xi, I, I_*)
\]

and, by applying the second relation in (25) and renaming \( \{ \xi', I' \} \rightleftharpoons \{ \xi'_*, I'_* \} \),

\[
k_2(\xi, \xi_*, I, I_*) \\
= 2 (M_*^{-1/2}) \int_{(\mathbb{R}^3 \times \mathbb{R}^+)^2} w(\xi_*, \xi', I_* | \xi'_*, \xi'_*, I'_*, I'_*) d\xi' dI' dI'_* \\
= 2 (M_*^{-1/2}) \int_{(\mathbb{R}^3 \times \mathbb{R}^+)^2} w(\xi_*, \xi', I_* | \xi'_*, \xi'_*, I'_*, I'_*) d\xi' dI' dI'_* \\
= k_2(\xi_*, \xi, I, I_*).
\]

We now continue by proving the compactness for the two different types of collision kernel separately.
Figure 1: Typical collision of $K_1$. Classical representation of an inelastic collision.

I. Compactness of $K_1 = \int_{\mathbb{R}^3 \times \mathbb{R}^+} k_1(\xi; \xi_*, I, I_*) \, d\xi_* dI_*$. 

A change of variables \( \{\xi', \xi'_*\} \rightarrow \{ |g'| = |\xi' - \xi'_*|, \omega = \frac{g'}{|g'|}, G' = \frac{\xi' + \xi'_*}{2} \} \), cf. Figure 1, noting that (7) and using relation (11), expression (26) of $k_1$ may be transformed to

\[
k_1(\xi; \xi_*, I, I_*) = \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^*} \frac{(M'M'_*)^{1/2}}{1} W(\xi, \xi_*, I, I_* | \xi', \xi'_*, I', I'_*) \, d\xi_* dI_*/
\]

\[
= \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^*} \frac{(M'M'_*)^{1/2}}{1} |g'|^2 W(\xi, \xi_*, I, I_* | \xi', \xi'_*, I', I'_*)
\]

\[
dG' d|g'| d\omega \, dI' dI'_*
\]

\[
= (MM_*)^{1/4} (II_*)^{\delta/8-1/4} |g| \int_{\mathbb{S}^2 \times \mathbb{R}^2} \frac{(M'M'_*)^{1/4}}{1} \sigma 1_{m|g|^2 > 4\Delta t} \, d\omega \, dI' dI'_*.
\]

Since, $E \geq (I'I'_*)^{1/2}$ and $\Psi \leq (EE')^{1/2} = E$, it follows, by assumption (17),
that

\[
k^2_{m}(\xi, \xi, I, I_s) \\
\leq C \frac{(MM_s)^{1/2} (II_s)^{\delta/4-1/2}}{|g|^2} \left( \int_{S^2} \frac{(I'I_s)^{\delta/2-1}}{E^{5/2}} e^{-I'/4} e^{-I_s'/4} \left( \Psi + \Psi' \gamma / 2 \right)^2 1_{m |g|^2 > 4 \Delta t} d\omega dI'dI_s' \right)^2 \leq C \frac{(MM_s)^{1/2} (II_s)^{\delta/2-1}}{|g|^2} \left( E + E^\gamma / 2 \right)^2 \left( \int_{S^2} d\omega \right)^2 \left( \int_{0}^{\infty} e^{-1/4} t^{3/4} dI \right)^4.
\]

Now, noting that

\[
m \frac{|\xi|^2}{2} + m \frac{|\xi|^2}{2} + I + I_s = m |G|^2 + m \frac{|g|^2}{4} + I + I_s = m |G|^2 + E,
\]

the bound

\[
M^2_{\text{bol}}, \xi, I, I_s) \leq C e^{-m |G|^2/2 - E/2} \frac{(II_s)^{\delta/2-1}}{|g|^2} \left( E + E^\gamma / 2 \right)^2
\]

may be obtained. Then, by applying the bound \((29)\) and first changing variables of integration \(\{\xi, \xi_s\} \rightarrow \{g, G\}\), with unitary Jacobian, and then to spherical coordinates,

\[
\int_{(R^3 \times R^3)^2} k^2_{m}(\xi, \xi, I, I_s) d\xi d\xi_s dI dI_s
\]

\[
\leq C \int_{(R^3 \times R^3)^2} e^{-m |g|^2/2 - E/2} \frac{(II_s)^{\delta/2-1}}{|g|^2} \left( E + E^\gamma / 2 \right)^2 dG dI dI_s
\]

\[
= C \int_{R^3} e^{-m |g|^2/8} e^{-1/2} e^{-I_s/2} (II_s)^{\delta/2-1} \left( E + E^\gamma / 2 \right)^2 d|g| dI dI_s
\]

\[
\times \int_{0}^{\infty} R^2 e^{-R^2} dR \left( \int_{S^2} d\omega \right)^2 \leq C \int_{R^3} e^{-m |g|^2/8} e^{-1/2} e^{-I_s/2} (II_s)^{\delta/2-1} (1 + E^2) d|g| dI dI_s
\]

\[
\leq C \int_{0}^{\infty} e^{-m |g|^2/8} \left( 1 + \frac{m^2}{16} |g|^4 \right) d|g| \left( \int_{0}^{\infty} (1 + I)^2 e^{-1/2} I^{\delta/2-1} dI \right)^2 = C
\]

Hence,

\[
K_1 = \int_{R^3 \times R^3} \frac{k_1(\xi, \xi, I, I_s)}{h} d\xi_s dI_s
\]

is a Hilbert-Schmidt integral operator and as such compact on \(L^2(d\xi dI)\), see e.g. Theorem 7.83 in [17].
II. Compactness of $K_2 = \int_{\mathbb{R}^3 \times \mathbb{R}_+} k_2(\xi, \xi_*, I, I_*) h_* d\xi_* dI_*$. Denote, cf. Figure 2,

$$
\chi = (\xi_* - \xi') \cdot \frac{g}{|g|}, \text{ with } g = \xi - \xi_*;
$$

$$
g' = \xi' - \xi'_*, \quad \tilde{g} = \xi - \xi', \text{ and } g_* = \xi_* - \xi'_*.
$$

Note that, see Figure 2,

$$
W(\xi, \xi', I, I' | \xi_*, \xi'_*, I_*, I'_*) = 4m (II')^{3/2-1} \left| \frac{\tilde{g}}{|g_*|} \right| \delta_1 \left( \frac{m}{2} (|\xi|^2 - |\xi_*|^2 + |\xi'|^2 - |\xi'_*|^2) - \Delta I_*) \right) 
\times \delta_3 (\xi' - \xi'_* + \xi - \xi_*)
$$

$$
= 4m (II')^{3/2-1} \left| \frac{\tilde{g}}{|g_*|} \right| \delta_1 (m |g| \chi - \Delta I_*) \delta_3 (g' + g)
$$

$$
= 4 (II')^{3/2-1} \tilde{\sigma} \left| \frac{\tilde{g}}{|g_*|} \right| \delta_1 \left( \chi - \frac{\Delta I_*}{m |g|} \right) \delta_3 (g' + g), \text{ with }
$$

$$
\Delta I_* = I_* + I'_* - I - I', \text{ and } \tilde{\sigma} = \sigma \left( \frac{g}{|g|}, \frac{\tilde{g}}{|g_*|}, I, I', I_*' \right).
$$

By a change of variables $\{\xi', \xi_*'\} \rightarrow \{g' = \xi' - \xi'_*, \ h = \xi' - \xi_*\}$, where

$$
d\xi' d\xi_*' = dg' dh = dg' d\chi dw, \text{ with } w = \xi' - \xi_* + \chi n \text{ and } n = \frac{g}{|g|}.
$$
the expression (26) of \( k_2 \) may be transformed in the following way

\[
k_2(\xi, \xi_*, I, I_*) = \int_{(R^3 \times R_+)^2} 2^{(M'M'_*)^{1/2}} W(\xi, \xi', I, I'| \xi_*, \xi_*, I_*, I_*') \, dg \, dh \, dI' \, dI_*'
\]

Here, see Figure 2,

\[
\left\{
\begin{array}{l}
\xi' = \xi + w - \chi n \\
\xi_* = \xi + w - \chi n
\end{array}
\right.
\]

implying that

\[
\frac{||\xi'||^2}{2} + \frac{||\xi'_*||^2}{2} = \frac{||\xi + \xi_* - \chi n + w||^2 + \frac{||\xi - \xi_*||^2}{4}}{4}
\]

\[
= \frac{||\xi + \xi_*||_n + w||^2 + \left( \frac{||\xi + \xi_*||_n}{2} - \chi \right)^2 + \frac{||g'||^2}{4}}{4}
\]

\[
= \frac{||\xi + \xi_*||_n + w||^2 + \left( \frac{||\xi||^2 - ||\xi_*||^2 + 2\chi ||\xi - \xi_*||}{4 ||\xi - \xi_*||^2} \right)^2 + \frac{||g'||^2}{4}}{4},
\]

with

\[
(\xi + \xi_*)_n = (\xi + \xi_*) \cdot n = \frac{||\xi||^2 - ||\xi_*||^2}{||\xi - \xi_*||} \quad \text{and} \quad (\xi + \xi_*)_{\perp n} = \xi + \xi_* - (\xi + \xi_*)_n n.
\]

Note that for any number \( s \), such that \( s \geq -1/2 \),

\[
\frac{||II'||^{\delta/4 - 1/2}}{E^{\delta/2 + s - 1/2}} \leq \frac{I_*^{\delta/4 - 1/2 - \kappa/2}}{I_*^{\delta/4 + s - \kappa/2}} \text{ for } 0 \leq \kappa \leq \delta + 2s - 1, \tag{30}
\]

where \( E = m \frac{||\xi||^2}{2} / 4 + I + I' = m \frac{||\xi_*||^2}{2} / 4 + I_* + I_*' \).

By bound (30) and assumption (17), for any numbers \( s \) and \( \kappa \), such that
\[-1/2 \leq s \leq \delta/2 \text{ and } 0 \leq \kappa \leq \delta + 2s - 1,\]

\[
k^2_2(\xi, \xi_*, I, I_*)
\leq \frac{C}{|g|^2} I^2_*/I^{3/2+2s-\kappa}_* \left( \int_{\mathbb{R}_+^n} \exp \left( -m \frac{|\xi|^2}{8|g|^2} - m \frac{|g|^2}{|\xi|^2} \right) \right)
\times \int_{(\mathbb{R}^3)^{\perp n}} \left( 1 + \frac{1}{\Psi^{1-\gamma/2}} \right) \exp \left( -m \left| \frac{\xi + \xi_*}{2} + w \right|^2 \right) \, dw
\times e^{-(I'+I^*_*)/2} \left( I'I_*^{3/4-1/2} \right) \frac{dI'dI^*_*}{E^{5/2-s} - \gamma}
\leq \frac{C}{|g|^2} I^2_*/I^{3/2+2s-\kappa}_* \left( \int_{\mathbb{R}_+^n} \frac{|g|^2 + 2|\xi| \cos \varphi + 2\chi}{2} - m \frac{|g|^2}{|\xi|^2} \right)
\times e^{-(I'+I^*_*)/2} \left( I'I_*^{1/2-s/2} dI'dI^*_* \right), \text{ where } \Psi = |g|, |g_*| \text{ and } \cos \varphi = n \cdot \frac{\xi}{|\xi|} \quad (31)
\]

since

\[
\int_{(\mathbb{R}^3)^{\perp n}} \left( 1 + \frac{1}{\Psi^{1-\gamma/2}} \right) \exp \left( -m \left| \frac{\xi + \xi_*}{2} + w \right|^2 \right) \, dw
\leq C \int_{(\mathbb{R}^3)^{\perp n}} \left( 1 + |w|^{-\gamma/2} \right) \exp \left( -m \left| \frac{\xi + \xi_*}{2} + w \right|^2 \right) \, dw
\leq C \int_{|w| \leq 1} \left( 1 + |w|^{-\gamma/2} \right) \, dw \int_{|w| \geq 1} \exp \left( -m \left| \frac{\xi + \xi_*}{2} + w \right|^2 \right) \, dw
\leq C \left( \int_{|w| \leq 1} \left( 1 + |w|^{-\gamma/2} \right) \, dw + \int_{(\mathbb{R}^3)^{\perp n}} e^{-|\tilde{w}|^2} \, d\tilde{w} \right)
= C \left( \int_0^1 r + r^{-\gamma-1} \, dr + \int_0^\infty r e^{-r^2} \, dr \right) = C.
\]

For any numbers \( s \) and \( \kappa \), such that \(-1/2 \leq s \leq \delta/2 \) and \( 0 \leq \kappa \leq \delta + 2s - 1, \)
by the bound \([31]\) on \(k_2^2\) and the Cauchy-Schwarz inequality,

\[
k_2^2(\xi, \xi_*, I, I_*) \leq \frac{C}{|g|^2} \frac{I_{1/2}^{5/2-1-\kappa}}{I_{1/2}^{5/2+2s-\kappa}} \int_{\mathbb{R}^4_+} e^{-(I' + I_*)/2} (I'I_*^{1/2-s/2})^2 dI'dI_*' \\
\times \exp \left( -\frac{m}{4} (|g| + 2|\xi| \cos \varphi + 2\chi)^2 - \frac{m}{4} |g|^2 \right) \frac{e^{-(I' + I_*)/2}}{(I'I_*^{1/2-s/2})^2} dI'dI_*'
\]

\[
= \frac{C}{|g|^2} \frac{I_{1/2}^{5/2-1-\kappa}}{I_{1/2}^{5/2+2s-\kappa}} \int_{\mathbb{R}^4_+} \exp \left( -\frac{m}{4} (|g| + 2|\xi| \cos \varphi + 2\chi)^2 - \frac{m}{4} |g|^2 \right) \frac{e^{-(I' + I_*)/2}}{(I'I_*^{1/2-s/2})^2} dI'dI_*', \quad \text{with } \cos \varphi = \frac{\xi}{|\xi|},
\]

(32)

since,

\[
\int_{\mathbb{R}^4_+} \frac{e^{-(I' + I_*)/2}}{(I'I_*^{1/2-s/2})^2} dI'dI_*' = \left( \int_0^\infty \frac{e^{-t/2}}{t^{1/2-s/2}} dt \right)^2 = C, \tag{33}
\]

implying also

\[
k_2^2(\xi, \xi_*, I, I_*) \leq \frac{C}{|g|^2} \frac{I_{1/2}^{5/2-1-\kappa}}{I_{1/2}^{5/2+2s-\kappa}} \exp \left( -\frac{m}{4} |g|^2 \right). \tag{34}
\]

Note that by letting \(\kappa = (\delta - 1)/2 + s\)

\[
\frac{I_{1/2}^{5/2-1-\kappa}}{I_{1/2}^{5/2+2s-\kappa}} = (II_*)^{-1/2-s} = \begin{cases} (II_*)^{-5/4} & \text{for } s = 3/4 \\ 1 & \text{for } s = -1/2 \end{cases}, \tag{35}
\]

while by letting \(s = 1/8\)

\[
\frac{I_{1/2}^{5/2-1-\kappa}}{I_{1/2}^{5/2+2s-\kappa}} = \frac{I_{1/2}^{5/2-1-\kappa}}{I_{1/2}^{5/2+1/4-\kappa}} = \begin{cases} I_*^{-5/4} & \text{for } \kappa = \delta/2 - 1 \\ I_*^{-5/4} & \text{for } \kappa = \delta/2 + 1/4 \end{cases}. \tag{36}
\]

Moreover, by letting \(s = 3/4\)

\[
\frac{I_{1/2}^{5/2-1-\kappa}}{I_{1/2}^{5/2+2s-\kappa}} = \frac{I_{1/2}^{5/2-1-\kappa}}{I_{1/2}^{5/2+3/2-\kappa}} = \begin{cases} I_*^{-5/2} & \text{for } \kappa = \delta/2 - 1 \\ I_*^{-5/2} & \text{for } \kappa = \delta/2 + 3/2 \end{cases}. \tag{37}
\]

Hence, by the bound \([34]\) on \(k_2^2\), together with expressions \([35]\) and \([36]\),

\[
k_2^2(\xi, \xi_*, I, I_*) \leq \frac{C}{|g|^2} \exp \left( -\frac{m}{4} |g|^2 \right) \left( 1_{I_{\leq 1}} + I_*^{-5/4} 1_{I_{\geq 1}} \right) \left( 1_{I_{\leq 1}} + I_*^{-5/4} 1_{I_{\geq 1}} \right). \tag{38}
\]

To show that \(k_2(\xi, \xi_*, I, I_*) \in L^2(\mathbb{R}^3 d\xi \, dI)\) for any (non-zero) natural number \(N\), separate the integration domain of the integral of \(k_2^2(\xi, \xi_*, I, I_*)\) over \((\mathbb{R}^3 \times \mathbb{R}_+)^2\) into two separate domains

\[
\left\{ \left( \mathbb{R}^3 \times \mathbb{R}_+ \right)^2 ; \ |g| \geq |\xi| \right\} \quad \text{and} \quad \left\{ \left( \mathbb{R}^3 \times \mathbb{R}_+ \right)^2 ; \ |g| \leq |\xi| \right\}.
\]
The integral of $k_2^2$ over the domain \( \left\{ (\mathbb{R}^3 \times \mathbb{R}_+) \ ; \ |\mathbf{g}| \geq |\xi| \right\} \) will be bounded, since, by the bound (38),
\[
\int_0^\infty \int_0^\infty \int_{|\mathbf{g}| \geq |\xi|} k_2^2(\xi, \xi^*, I, I^*) \, d\xi dI dI^* \\
\leq C \int_{|\mathbf{g}| \geq |\xi|} \frac{e^{-m|\mathbf{g}|^2/4}}{|\mathbf{g}|^2} \, d\xi \left( \int_0^1 dI + \int_1^\infty I^{-5/4} dI \right)^2 \\
\leq C \int_{|\mathbf{g}| \geq |\xi|} \frac{e^{-m|\mathbf{g}|^2/4}}{|\mathbf{g}|^2} \, d\xi = C \int_0^\infty \int_0^\infty e^{-mR^2/4} r^2 dR dr \\
\leq C \int_0^\infty e^{-mR^2/8} dR \int_0^\infty e^{-mr^2/8} r^2 dr = C
\]

Regarding the second domain, consider the truncated domains \( \left\{ (\mathbb{R}^3 \times \mathbb{R}_+) \ ; \ |\mathbf{g}| \leq |\xi| \leq N \right\} \) for (non-zero) natural numbers \( N \). Then by the bound (32) on $k_2^2$, together with expressions (33), (35), (36),
\[
\int_{R^3_+} \int_{|\mathbf{g}| \leq |\xi| \leq N} k_2^2(\xi, \xi^*, I, I^*) \, d\xi dI dI^* \leq CN^2 \left( \int_0^1 dI + \int_1^\infty I^{-5/4} dI \right)^2 \\
= CN^2,
\]
since
\[
\int_{|\mathbf{g}| \leq |\xi| \leq N} \frac{C}{|\mathbf{g}|^2} \exp \left( -\frac{m}{4} \left( |\mathbf{g}| + 2 |\xi| \cos \varphi + 2\chi \right)^2 - \frac{m}{4} |\mathbf{g}|^2 \right) \, d\xi \, d\mathbf{g} \\
= C \int_0^N \int_0^r \int_0^r r^2 \exp \left( -\frac{m}{4} (R + 2r \cos \varphi + 2\chi) - \frac{m}{4} R^2 \right) \sin \varphi \, d\varphi \, dR \, dr \\
= C \int_0^N \int_0^r \int_0^{R+2\chi R+2r} re^{-m\eta^2/4} e^{-mR^2/4} d\eta \, dR \, dr \\
\leq C \int_0^N \int_0^r e^{-mR^2/4} dR \int_0^\infty e^{-m\eta^2/4} d\eta = CN^2, \text{ with } \chi_R = \frac{\Delta I_*}{mR}.
\]

Furthermore, the integral of $k_2(\xi, \xi^*, I, I^*)$ with respect to $(\xi, I)$ over $\mathbb{R}^3 \times \mathbb{R}_+$ is bounded in $(\xi^*, I^*)$. Indeed, by the bound (34) on $k_2^2$, together with expressions on $k_2^2$, together with expressions (35) and (37),
\[
k_2(\xi, \xi^*, I, I^*) \leq \frac{C}{|\mathbf{g}|} \exp \left( -\frac{m}{8} |\mathbf{g}|^2 \right) \left( 1_{I_* \leq 1} + I_*^{-5/4} 1_{I_* \geq 1} \right),
\]
why the following bound on the integral of $k_2$ with respect to $(\xi^*, I^*)$ over the
domain \( \{ \mathbb{R}^3 \times \mathbb{R}_+ ; |g| \geq |\xi| \} \) can be obtained for \( |\xi| \neq 0 \)

\[
\int_0^\infty \int_{|g| \geq |\xi|} k_2(\xi, \xi_*, I, I_*) d\xi_* dI_* \leq \frac{C}{|\xi|} \int_{|g| \geq |\xi|} e^{-m|g|^2/8} dg \left( \int_0^1 dI_* + \int_1^\infty \frac{dI_*}{I_*^{5/4}} \right)
\]

\[
\leq \frac{C}{|\xi|} \int_0^\infty e^{-m r^2/8} r^2 d\omega = \frac{C}{|\xi|}.
\]

Moreover, over the domain \( \{ \mathbb{R}^3 \times \mathbb{R}_+ ; |g| \leq |\xi| \} \), by the bound (31) on \( k_2^2 \), and expressions (33), (35), (37),

\[
\int_0^\infty \int_{|g| \leq |\xi|} k_2(\xi, \xi_*, I, I_*) d\xi_* dI_* \leq \frac{C}{|\xi|} \left( \int_0^1 dI_* + \int_1^\infty I_*^{-5/4} dI_* \right) = \frac{C}{|\xi|},
\]

since

\[
\int_{|g| \leq |\xi|} \frac{C}{|g|} \exp \left( -\frac{m}{8} (|g| + 2 |\xi| \cos \phi + 2 \chi)^2 - \frac{m}{8} |g|^2 \right) dg
\]

\[
= C \int_{|g| \leq |\xi|} \int_0^\pi e^{-m/8 (R + 2 |\xi| \cos \phi + 2 \chi^2 - m R^2)} \sin \phi d\phi dR
\]

\[
= C \int_{|g| \leq |\xi|} \int_0^{R + 2 \chi R + 2 |\xi|} e^{-m \eta^2/8} d\eta dR
\]

\[
\leq C \int_{|g| \leq |\xi|} \int_0^\infty e^{-m \eta^2/8} d\eta = \frac{C}{|\xi|}, \text{ with } \chi = \frac{\Delta I_*}{m R}.
\]

However, due to the symmetry \( k_2(\xi, \xi_*, I, I_*) = k_2(\xi_*, \xi, I_*, I) \) (28), also

\[
\int_0^\infty \int_{\mathbb{R}^3} k_2(\xi_*, I, I_*) d\xi dI \leq \frac{C}{|\xi_*|}.
\]

Therefore, if \( |\xi_*| \geq 1 \), then

\[
\int_0^\infty \int_{\mathbb{R}^3} k_2(\xi_*, I, I_*) d\xi dI \leq \frac{C}{|\xi_*|} \leq C.
\]

Otherwise, if \( |\xi_*| \leq 1 \), then, by the bounds bound (31) on \( k_2^2 \), and expressions (33), (35), (37),

\[
\int_0^\infty \int_{\mathbb{R}^3} k_2(\xi_*, I, I_*) d\xi dI = \int_0^\infty \int_{\mathbb{R}^3} k_2(\xi_*, I, I_*) d\xi dI
\]

\[
\leq C \left( \int_0^1 dI + \int_1^\infty I^{-5/4} dI \right) \leq C
\]
since

\[
\int_{\mathbb{R}^3} \frac{C}{|g|} \exp \left( -\frac{m}{8} (|g| + 2 |\xi_*| \cos \varphi_* - 2\chi)^2 - \frac{m}{8} |g|^2 \right) \, dg
\]

\[
= C \int_0^\infty \int_0^\pi R \exp \left( -\frac{m}{8} (R + 2 |\xi| \cos \varphi_* - 2\chi R)^2 - \frac{m}{8} R^2 \right) \sin \varphi_* \, d\varphi_* \, dR
\]

\[
\leq \frac{C}{|\xi_*|} \int_0^\infty R e^{-mR^2/8} \, dR \int_{R-2\chi_R-2|\xi_*|}^{R-2\chi_R+2|\xi_*|} dt = C, \text{ with } \chi_R = \frac{\Delta I_*}{mR}
\]

and

\[
\cos \varphi_* = \frac{n \cdot \xi_*}{|\xi_*|}.
\]

Furthermore,

\[
\sup_{(\xi,I) \in \mathbb{R}^3 \times \mathbb{R}^+} \int_{\mathbb{R}^3 \times \mathbb{R}^+} k_2(\xi,\xi_*,I,I_*) - k_2(\xi,\xi_*,I,I_*) 1_{\mathbb{R}^+} \, d\xi_* \, dI_*
\]

\[
\leq \sup_{(\xi,I) \in \mathbb{R}^3 \times \mathbb{R}^+} \int_0^\infty \int_{|\xi| \leq \frac{1}{N}} k_2(\xi,\xi_*,I,I_*) \, d\xi_* \, dI_*
\]

\[
+ \sup_{|\xi| \geq N} \int_0^\infty \int_{\mathbb{R}^3} k_2(\xi,\xi_*,I,I_*) \, d\xi_* \, dI_*
\]

\[
\leq \int_{|\xi| \leq \frac{1}{N}} \frac{C}{|g|} \, dg \left( \int_0^1 dI_* + \int_1^\infty \frac{dI_*}{I_*^{5/4}} \right) + \frac{C}{N} \leq C \left( \int_0^{\frac{1}{N}} R \, dR + \frac{1}{N} \right)
\]

\[
= C \left( \frac{1}{N^2} + \frac{1}{N} \right) \to 0 \text{ as } N \to \infty.
\]

Hence, by Lemma 4, the operator

\[
K_2 = \int_{\mathbb{R}^3 \times \mathbb{R}^+} k_2(\xi,\xi_*,I,I_*) h_* \, d\xi_* \, dI_*
\]

is compact on \(L^2(\,d\xi \, dI)\).

Concluding, the operator \(K = K_2 - K_1\) is a compact self-adjoint operator on \(L^2(\,d\xi \, dI)\). The self-adjointness is due to the symmetry relations \(27, 28\), cf. \(19\) p.198. 

5 **Bounds on the collision frequency**

This section concerns the proof of Theorem 2. Note that throughout the proof, \(C\) will denote a generic positive constant.
Consider two different cases separately:

Proof. Under assumption (19) the collision frequency $v$ equals

$$v = \int_{(\mathbb{R}^3 \times \mathbb{R}^+)^3} \frac{M_*}{(II_*)^{3/2-1} W(\xi, \xi_*, I, I_*, |\xi|, |\xi_*|, I, I_*)} d\xi d\xi_* dI dI_* dI' dI'_*$$

$$= C \int_{(\mathbb{R}^3 \times \mathbb{R}^+)^3} M_* \frac{|g|}{|g'|} |\xi| \delta_1 d\xi d\xi_* dI dI_* dI' dI'_*$$

$$= C \int_{(\mathbb{R}^3 \times \mathbb{R}^+)^3} \frac{1}{\delta^2} d\xi d\xi_* dI dI_* dI' dI'_*$$

$$= C \int_{(\mathbb{R}^3 \times \mathbb{R}^+)^3} e^{-I_* - I_* m|\xi|^2/2} \frac{1}{E^{3/2-1}} d\xi d\xi_* dI dI_* dI' dI'_*$$

$$\times \frac{E - (I + I_1)}{E^{3/2-1}} (I_* I'_*)^{\delta/2-1}$$

with

$$\delta_1 = \mathbf{1}_{m|g|^2 > 4\Delta I} \delta_1 \left( \sqrt{\frac{|g|^2 - \frac{4}{m} \Delta I - |g'|}{|g'|}} \right)$$

Then

$$v \geq C \int_{\mathbb{R}^3 \times \mathbb{R}^+} e^{-I_* - I_* m|\xi|^2/2} \frac{\sqrt{E - (I + I_1)}}{E^{3/2}(\alpha - 1)/2} (I_* I'_*)^{\delta/2-1}$$

$$\int_{I_*}^{E/2} e^{-I_* - I_* m|\xi|^2/2} \frac{\sqrt{E - (I + I_1)}}{E^{3/2}(\alpha - 1)/2} (I_* I'_*)^{\delta/2-1} d\xi_* dI_* dI' dI'_*$$

$$\geq C \int_{\mathbb{R}^3 \times \mathbb{R}^+} e^{-I_* - I_* m|\xi|^2/2} \frac{\sqrt{E - (I + I_1)}}{E^{3/2}(\alpha - 1)/2} (I_* I'_*)^{\delta/2-1} d\xi_* dI_*$$

$$\geq C \int_{\mathbb{R}^3} \left( |\xi|^2 + I \right)^{1/2} e^{-m|\xi|^2/2} d\xi_*$$

Consider two different cases separately: $|\xi| \leq 1$ and $|\xi| \geq 1$. Firstly, if $|\xi| \geq 1$, then

$$v \geq C \int_{|\xi| \leq 1/2} \left( |\xi|^2 + I \right)^{1/2} e^{-m|\xi|^2/2} d\xi_*$$

$$\geq C \left( |\xi|^2 + I \right)^{1/2} \int_{|\xi| \leq 1/2} e^{-m|\xi|^2/2} d\xi_* = C \left( |\xi|^2 + I \right)^{1/2}$$

$$\geq C \left( |\xi| + \sqrt{I} \right)^{2-\alpha} \geq C \left( 1 + |\xi| + \sqrt{I} \right)^{2-\alpha}.$$
Secondly, if $|\xi| \leq 1$, then

$$v \geq C \int_{|\xi| \geq 2} \left( (|\xi| - 1)^2 + I \right)^{1-\alpha / 2} e^{-m|\xi|^2 / 2} d\xi_* 
\geq C (1 + I)^{1-\alpha / 2} \int_{|\xi| \geq 2} e^{-m|\xi|^2 / 2} d\xi_* = C (1 + I)^{1-\alpha / 2} 
\geq C \left( 1 + \sqrt{T} \right)^{2-\alpha} \geq C \left( 1 + |\xi| + \sqrt{T} \right)^{2-\alpha} .$$

Hence, there is a positive constant $v_- > 0$, such that $v \geq v_- \left( 1 + |\xi| + \sqrt{T} \right)^{2-\alpha}$ for all $\xi \in \mathbb{R}^3$.

On the other hand, for any positive number $\varepsilon > 0$

$$v \leq C \int_{\mathbb{R}^3 \times \mathbb{R}_+} e^{-I_* - m|\xi|^2 / 2} \frac{(I_* I'_*)^{\delta/2-1}}{E^{\delta/2+\alpha/2-1}} d\xi_* dI_* dI'_* 
= C \int_{\mathbb{R}^3} e^{-m|\xi|^2 / 2} \int_0^\infty I_*^{\delta/2-1} e^{-I_*} \left( \int_0^1 \frac{1}{E^{\delta/2+\alpha/4-1/2}} dI' \right)^2 dI_* d\xi_* 
+ 2C \int_{\mathbb{R}^3} e^{-m|\xi|^2 / 2} \int_0^\infty I_*^{\delta/2-1} e^{-I_*} \int_0^\infty \frac{(I'_*)^{\delta/2-1}}{E^{(\delta/2+\alpha/2+1/4)}} dI'_* 
\times \int_0^1 \frac{1}{E^{\delta/2-\alpha/4}} dI'_* dI_* d\xi_* 
+ C \int_{\mathbb{R}^3 \times \mathbb{R}_+} E^{1+(\varepsilon-\alpha)/2} e^{-m|\xi|^2 / 2} I_*^{\delta/2-1} e^{-I_*} \left( \int_1^\infty \frac{1}{E^{\delta/2+\alpha/4+1/2}} dI' \right)^2 dI_* d\xi_* 
\leq C \int_{\mathbb{R}^3} e^{-m|\xi|^2 / 2} d\xi_* \int_0^\infty I_*^{\delta/2-1} e^{-I_*} dI_* \left( \int_1^1 \frac{1}{E^{\delta/2+\alpha/4+1/2}} dI' \right)^2 
+ C \int_{\mathbb{R}^3} e^{-m|\xi|^2 / 2} d\xi_* \int_0^\infty I_*^{\delta/2-1} e^{-I_*} dI_* \int_1^\infty \frac{1}{(I'_*)^{\delta/2+\alpha/2+5/4}} dI'_* \int_0^1 (I'_*)^{1/4} dI' 
+ C \int_{\mathbb{R}^3 \times \mathbb{R}_+} e^{1+(\varepsilon-\alpha)/2} e^{-m|\xi|^2 / 2} I_*^{\delta/2-1} e^{-I_*} d\xi_* dI_* \left( \int_1^\infty \frac{1}{(I'_*)^{1+\varepsilon/4}} dI'_* \right)^2 
\leq C \left( 1 + \int_{\mathbb{R}^3} \left( 1 + |\xi|^2 / 4 + I \right)^{1+(\varepsilon-\alpha)/2} \right) e^{-m|\xi|^2 / 2} d\xi_* 
\times \int_0^\infty (1 + I_*) I_0^{\delta/2-1} e^{-I_*} dI_* 
\leq C \left( 1 + (1 + |\xi|^2) \right)^{1+(\varepsilon-\alpha)/2} \int_0^\infty (1 + r^2)^{1+\varepsilon} r^2 e^{-2r^2} dr 
\leq C \left( 1 + |\xi| + \sqrt{T} \right)^{2-\alpha+\varepsilon} .$$
Hence, there is a positive constant $v_+ > 0$, such that $v \leq v_+(1 + |\xi| + \sqrt{I})^{2-\alpha+\varepsilon}$ for all $\xi \in \mathbb{R}^3$. ■

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