Geometric Invariants of Spectrum of the Navier–Lamé Operator

Genqian Liu

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Abstract
For a compact connected Riemannian \( n \)-manifold \((\Omega, g)\) with smooth boundary, we explicitly calculate the first two coefficients \(a_0\) and \(a_1\) of the asymptotic expansion of

\[
\sum_{k=1}^{\infty} e^{-\tau_k t} = a_0 t^{-n/2} + a_1 t^{-(n-1)/2} + O(t^{1-n/2}) \quad \text{as} \quad t \to 0^+,
\]

where \(\tau_k^-\) (respectively, \(\tau_k^+\)) is the \(k\)-th Navier–Lamé eigenvalue on \(\Omega\) with Dirichlet (respectively, Neumann) boundary condition. These two coefficients provide precise information for the volume of the elastic body \(\Omega\) and the surface area of the boundary \(\partial \Omega\) in terms of the spectrum of the Navier–Lamé operator. This gives an answer to an interesting and open problem mentioned by Avramidi in (Non-Laplace type operators on manifolds with boundary, analysis, geometry and topology of elliptic operators. World Sci. Publ., Hackensack, pp. 107–140, 2006). As an application, we show that an \(n\)-dimensional ball is uniquely determined by its Navier–Lamé spectrum among all bounded elastic bodies with smooth boundary.

Keywords Navier–Lamé eigenvalues · Pseudodifferential operators · Navier–Lamé semigroup · Asymptotic expansion

Mathematics Subject Classification 74B05 · 35K50 · 35P20 · 35S05

1 Introduction

For the Navier–Lamé elastic wave equations, one of the most important problems is to study the shape of the elastic body from its vibrational frequencies, because this kind of geometric property reveals the essential behavior of the elastic body.
Let \((\Omega, g)\) be a Riemannian \(n\)-manifold with smooth boundary \(\partial \Omega\). Let \(P_g\) be the Navier–Lamé operator:

\[
P_g u := \mu \nabla^* \nabla u - (\mu + \lambda) \text{grad div } u - \mu \text{Ric}(u), \quad u = (u^1, \ldots, u^n),
\]

(1.1)

where \(\mu\) and \(\lambda\) are Lamé parameters satisfying \(\mu > 0\) and \(\mu + \lambda > 0\), \(\nabla^* \nabla\) is the Bochner Laplacian (see (2.11) in Section 2, or (2.12) of [50]), \(\text{div}\) and \(\text{grad}\) are the usual divergence and gradient operators, and

\[
\text{Ric}(u) = \left( \sum_{k,l=1}^{n} R^k_{lk} u^l, \sum_{k,l=1}^{n} R^k_{lk} u^l, \ldots, \sum_{k,l=1}^{n} R^k_{lk} u^l \right)
\]

(1.2)

denotes the action of Ricci tensor \(R^j_{lk} := \sum_{n} R^k_{lj} u^l\) on \(u\).

We denote by \(P_g^-\) and \(P_g^+\) the Navier–Lamé operators with the Dirichlet and Neumann boundary conditions, respectively. Since \(P_g^-\) (respectively, \(P_g^+\)) is an unbounded, self-adjoint, and positive (respectively, non-negative) operator in \([H^1_0(\Omega)]^n\) (respectively, \([H^1(\Omega)]^n\)) with discrete spectrum \(0 < \tau^-_1 < \tau^-_2 < \cdots \leq \tau^-_k < \cdots \rightarrow +\infty\) (respectively, \(0 \leq \tau^+_1 < \tau^+_2 < \cdots \leq \tau^+_k < \cdots \rightarrow +\infty\)), one has (see [10, 44, 59] or [32])

\[
P_g^\pm u_k^\pm = \tau_k^\pm u_k^\pm,
\]

(1.3)

where \(u_k^- \in [H^1_0(\Omega)]^n\) (respectively, \(u_k^+ \in [H^1(\Omega)]^n\)) is the eigenvector corresponding to eigenvalue \(\tau_k^-\) (respectively, \(\tau_k^+\)). (1.3) can be rewritten as

\[
\begin{align*}
\mu \nabla^* \nabla u_k^- - (\mu + \lambda) \text{grad div } u_k^- - \mu \text{Ric}(u_k^-) &= \tau_k^- u_k^- \quad \text{in } \Omega, \\
u_k^- &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

(1.4)

and

\[
\begin{align*}
\mu \nabla^* \nabla u_k^+ - (\mu + \lambda) \text{grad div } u_k^+ - \mu \text{Ric}(u_k^+) &= \tau_k^+ u_k^+ \quad \text{in } \Omega, \\
\frac{\partial u_k^+}{\partial v} &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(1.5)

where \(v\) is the unit outer normal to \(\partial \Omega\). Clearly, the eigenvalue problems (1.4) and (1.5) can be immediately obtained by considering the solutions of the form \(v(t, x) = T(t)u(x)\) in the following Navier–Lamé elastodynamic wave equations:

\[
\begin{align*}
\frac{\partial^2 v^-}{\partial t^2} + \mu \nabla^* \nabla v^- - (\mu + \lambda) \text{grad div } v^- - \mu \text{Ric}(v^-) &= 0 \quad \text{in } (0, +\infty) \times \Omega, \\
v^- &= 0 \quad \text{on } (0, +\infty) \times \partial \Omega, \\
v^-(0, x) = v_0, \quad \frac{\partial v^-}{\partial t}(0, x) = 0 \quad \text{on } [0] \times \Omega
\end{align*}
\]

(1.6)
$$\begin{align*}
\frac{\partial^2 v^+}{\partial t^2} + \mu \nabla^* \nabla v^+ - (\mu + \lambda) \text{grad div } v^+ - \mu \text{Ric}(v^+) = 0 \quad &\text{in } (0, +\infty) \times \Omega, \\
\frac{\partial v^+}{\partial \nu} = 0 \quad &\text{on } (0, +\infty) \times \partial \Omega, \\
v^+(0, x) = v_0 \quad &\text{on } [0] \times \Omega.
\end{align*}$$

(1.7)

In three spatial dimensions, the elastic wave equations (1.6) and (1.7) describe the propagation of waves in an isotropic homogeneous elastic medium. The elasticity of the material provides the restoring force of the wave. Most solid materials are elastic, so these two equations describe such phenomena as seismic waves in the Earth, ultrasonic waves used to detect flaws in materials and the deformations of thin elastic shells whose middle surface must stay inside a given surface in the three-dimensional Euclidean space. Equivalently, we can also rewrite the equation in (1.6) or (1.7) into another form (cf. [50] and [51]), as it must account for both longitudinal and transverse motion:

$$\frac{\partial^2 v^\mp}{\partial t^2} + \mu \text{curl curl } v^\mp - (2\mu + \lambda) \text{grad div } v^\mp - 2\mu \text{Ric}(v^\mp) = 0.$$  

(1.8)

In particular, if div $v^\mp$ are set to zero, (1.8) becomes (effectively) Maxwell’s equations for the propagation of the electric field $v^\mp$, which has only transverse waves (see [18,28,66]). In addition, if $\lambda \to -\mu$, then (1.6) and (1.7) reduce to the classical wave equations, which has only longitudinal waves. For the derivation of the Navier–Lamé elastic wave equations, its mechanical meaning, and the explanation of the Dirichlet and Neumann boundary conditions, we refer the reader to [50] for the case of Riemannian manifold and to [11,20,31,36,38,43,63] for the case of Euclidean space.

The Navier–Lamé eigenvalues are physical quantities because they just are the square of vibrational frequencies of an elastic body in two or three dimensions. And these basic physical quantities can be measured experimentally. An interesting question, which is similar to the famous Kac question for the Dirichlet–Laplacian (see [35,52] or [70]), is: “can one hear the shape of an elastic body by hearing the vibrational frequencies (or pitches) of the elastic body?” More precisely, does the spectrum of the Navier–Lamé operator determine the geometry of an elastic body (see [10])?

In the special case of $\mu + \lambda = 0$ (i.e., the Navier–Lamé operator reduces to the Laplace operator), a celebrated result of the spectral (geometric) invariants had been obtained by McKean and Singer [53]. They proved the famous Kac conjecture and gave an explicit expression to the first three coefficients of asymptotic expansion for the heat trace of the Laplacian on a bounded domain $\Omega$ of a Riemannian manifold:

$$\sum_{k=1}^{\infty} e^{-\beta_k t} = (4\pi t)^{-n/2} \left( \frac{\text{Vol}(\Omega)^{1/2}}{4^{1/2} \sqrt{\pi} t} \cdot \frac{1}{4} \text{Vol}(\partial \Omega) \right) \\
+ \frac{t}{3} \int_\Omega R - \frac{t}{6} \int_{\partial \Omega} J + O(t^{3/2}) \quad \text{as } t \to 0^+.
$$

(1.9)
where $\beta_k^-$ (respectively, $\beta_k^+$) is the $k$th Dirichlet–Laplacian (respectively, Neumann–Laplacian) eigenvalue on $\Omega$; $R$ and $J$ are the scalar curvature and the mean curvature of $\Omega$ and $\partial \Omega$, respectively; $O(t^{3/2})$ cannot be improved.

The symbolic approach by Seeley [61, 62] and Greiner [27] is a very powerful general analytical procedure for analyzing the structure of the asymptotic expansion based on the theory of pseudodifferential operators and the calculus of symbols of operators. This approach may be considered for calculation of the heat invariants explicitly in terms of the jets of the symbol of the operator; it provides an iterative procedure for such a calculation. However, as far as we know, because of the technical complexity and, most importantly, lack of the manifest covariance, such analytical tools have never been used for the actual calculation of the explicit form of the heat invariants in an invariant geometric form (see [10]). The systematic explicit calculation of heat kernel coefficients for Laplace type operators is now well understood due to the work of Gilkey [23] and many others (see [3, 6, 12, 24, 25, 29, 30, 40, 46, 69] and references therein) because the Riemannian structure on a manifold is determined by a Laplace type operator. For the classical boundary conditions, like Dirichlet, Neumann, Robin, and mixed combination thereof on vector bundles, the coefficients of the trace of heat kernel have been explicitly computed up to the first five terms (see, for example, [4, 13, 14, 39]). For other type operators (which originated from physics problems), the corresponding explicit form of the heat invariants has also been discussed. Liu in [47] explicitly calculated the first two coefficients of asymptotic expansion of the heat trace for the Stokes operator (i.e., incompressible slow flow operator), and in [48, 49] gave the first four coefficients of asymptotic expansion of the heat trace for the Dirichlet-to-Neumann map (We also refer the reader to [60] for the asymptotic expansion of the first three coefficients) as well as polyharmonic Steklov operator.

Contrary to the Laplace type operators, there are no systematic effective methods for an explicit calculation of the spectral invariants for second-order operators which are not of Laplace type. Such operators appear in so-called matrix geometry [7–9], when instead of a single Riemannian metric there is a matrix-valued symmetric 2-tensor. Let us point out that the Navier–Lamé operator is just a non-Laplace type operator. Four decades ago, Greiner [27], p. 164, indicated that “the problem of interpreting these coefficients geometrically remains open.” There has not been much progress in this direction. Thus, the geometric aspect of the spectral asymptotics of the Navier–Lamé operator remains an open problem (see [10]). In the celebrated paper [10], Avramidi considers general non-Laplace type operators on manifolds with boundary by introducing a “noncommutative” Dirac operator as a first-order elliptic partial differential operator such that its square is a second-order self-adjoint elliptic operator with positive definite leading symbol (not necessarily of Laplace type) and studies the spectral asymptotics of these operators with Dirichlet boundary conditions. However, Avramidi’s method is too complicated to be applied to the Navier–Lamé operator because of extremely technical difficulty to calculate the noncommutative Dirac operator and required symbol integral for such an operator.

In this paper, by combining the technique of calculus of symbols for the integral kernel and “method of images,” we obtain the following result:
Theorem 1.1 Let \((\Omega, g)\) be a compact Riemannian manifold of dimension \(n\) with smooth boundary \(\partial \Omega\), and let \(0 < \tau_1^- < \tau_2^- \leq \cdots \leq \tau_k^- \leq \cdots\) (respectively, \(0 \leq \tau_1^+ < \tau_2^+ \leq \cdots \leq \tau_k^+ \leq \cdots\)) be the eigenvalues of the Navier–Lamé operator \(P_g^-\) (respectively, \(P_g^+\)) with respect to the Dirichlet (respectively, Neumann) boundary condition. Then

\[
\sum_{k=1}^{\infty} e^{-\tau_k^+ t} = \text{Tr} \left( e^{-t P_g^+} \right) = \left[ \frac{(n-1)}{(4\pi \mu t)^{n/2}} + \frac{1}{(4\pi (2\mu + \lambda) t)^{n/2}} \right] \text{Vol} (\Omega)
\]

\[
\frac{1}{4} \left[ \frac{(n-1)}{(4\pi \mu t)^{(n-1)/2}} + \frac{1}{(4\pi (2\mu + \lambda) t)^{(n-1)/2}} \right] \text{Vol} (\partial \Omega)
\]

\[+ O(t^{1-n/2}) \quad \text{as} \quad t \to 0^+.
\]

(1.10)

Here \(\text{Vol} (\Omega)\) denotes the \(n\)-dimensional volume of \(\Omega\), \(\text{Vol} (\partial \Omega)\) denotes the \((n-1)\)-dimensional volume of \(\partial \Omega\).

Our result shows that not only the volume \(\text{Vol}(\Omega)\) but also the surface area \(\text{Vol}(\partial \Omega)\) can be obtained if we know all the Navier–Lamé eigenvalues with respect to the Dirichlet (respectively, Neumann) boundary condition. This gives an answer to an interesting and open problem mentioned by Avramidi in [10]. Roughly speaking, one can “hear” the volume of the domain and the surface area of its boundary by “hearing” all the pitches of the vibration of an elastic body.

The key ideas of this paper are as follows. We denote by \((e^{-t P_g^+})_{t \geq 0}\) the parabolic semigroups generated by \(P_g^+\). More precisely, \(w^+(t, x) = e^{-t P_g^+} w_0(x)\) solve the following initial-boundary problems:

\[
\begin{aligned}
\frac{\partial w^-}{\partial t} + \mu \nabla^* \nabla w^- - (\mu + \lambda) \text{grad div } w^- - \mu \text{Ric} (w^-) &= 0 \quad \text{in} \quad (0, +\infty) \times \Omega, \\
w^- &= 0 \quad \text{on} \quad (0, +\infty) \times \partial \Omega, \\
w^-(0, x) &= w_0
\end{aligned}
\]

(1.11)

and

\[
\begin{aligned}
\frac{\partial w^+}{\partial t} + \mu \nabla^* \nabla w^+ - (\mu + \lambda) \text{grad div } w^+ - \mu \text{Ric} (w^+) &= 0 \quad \text{in} \quad (0, +\infty) \times \Omega, \\
\frac{\partial w^+}{\partial \nu} &= 0 \quad \text{on} \quad (0, +\infty) \times \partial \Omega, \\
w^+(0, x) &= w_0
\end{aligned}
\]

(1.12)

If \(\{u_k^\pm\}_{k=1}^{\infty}\) are orthonormal eigenvectors of the Navier–Lamé problem corresponding to eigenvalues \(\{\tau_k^\pm\}_{k=1}^{\infty}\), then the integral kernels \(K^\pm(t, x, y) = e^{-t P_g^\pm} \delta(x - y)\) of the semigroups are given by

\[
K^\pm(t, x, y) = \sum_{k=1}^{\infty} e^{-\tau_k^\pm t} u_k^\pm(x) \otimes u_k^\pm(y).
\]

(1.13)
Thus the integrals of the traces of \( K^+(t, x, y) \) are actually spectral invariants:

\[
\int_{\Omega} (\text{Tr} (K^+(t, x, x))) \, dV = \sum_{k=1}^{\infty} e^{-t \tau_k^+}.
\]

(1.14)

To further analyze the geometric content of the spectrum, we calculate the same integral of the trace by another approach: let \( \mathcal{M} = \Omega \cup (\partial \Omega) \cup \Omega^* \) be the (closed) double of \( \Omega \), and \( \mathcal{P} \) the double to \( \mathcal{M} \) of the operator \( P_g \) on \( \Omega \). Then \( \mathcal{P} \) generates a strongly continuous semigroup \( (e^{-t \mathcal{P}})_{t \geq 0} \) on \( L^2(\mathcal{M}) \) with integral kernel \( K(t, x, y) \). Clearly, \( K^+(t, x, y) = K(t, x, y) \mathcal{P} K(t, y, x^*) \) for \( x, y \in \tilde{\Omega}, \) where \( * \) is the double of \( y \in \Omega \). This technique stems from McKean and Singer (see [53]), and is called “method of images.” Since \( e^{-t \mathcal{P}} f(x) = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t \tau} (\tau I - \mathcal{P})^{-1} f(x) \, d\tau \), we have

\[
e^{-t \mathcal{P}} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} \left( \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t \tau} \iota((\tau I - \mathcal{P})^{-1}) \hat{f}(\xi) \, d\tau \right) \, d\xi,
\]

so that

\[
K(t, x, y) = e^{-t \mathcal{P}} \delta(x - y)
\]

\[
= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} \left( \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t \tau} \iota((\tau I - \mathcal{P})^{-1}) \, d\tau \right) \, d\xi
\]

\[
= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} \left( \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t \tau} \left( \sum_{l \geq 0} q_{-2-l}(x, \xi, \tau) \right) \, d\tau \right) \, d\xi,
\]

where \( \mathcal{C} \) is a suitable curve in the complex plane in the positive direction around the spectrum of \( \mathcal{P} \), and \( \iota((\tau I - \mathcal{P})^{-1}) := \sum_{l \geq 0} q_{-2-l}(x, \xi, \tau) \) is the full symbol of the resolvent operator \((\tau I - \mathcal{P})^{-1}\). This implies that for any \( x \in \tilde{\Omega} \),

\[
\text{Tr} (K(t, x, x)) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left( \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t \tau} \sum_{l \geq 0} \text{Tr} (q_{-2-l}(x, \xi, \tau)) \, d\tau \right) \, d\xi,
\]

\[
\text{Tr} (K(t, x, x^*)) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-x^*) \cdot \xi} \left( \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t \tau} \sum_{l \geq 0} \text{Tr} (q_{-2-l}(x, \xi, \tau)) \, d\tau \right) \, d\xi.
\]

It is easy to show that for any \( x \in \Omega \),

\[
\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left( \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t \tau} \sum_{l \geq 1} \text{Tr} (q_{-2-l}(x, \xi, \tau)) \, d\tau \right) \, d\xi = O(t^{1-\frac{n}{2}}) \quad \text{as} \quad t \to 0^+,
\]

\[
\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-x^*) \cdot \xi} \left( \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t \tau} \sum_{l \geq 1} \text{Tr} (q_{-2-l}(x, \xi, \tau)) \, d\tau \right) \, d\xi = O(t^{1-\frac{n}{2}}) \quad \text{as} \quad t \to 0^+.
\]
In order to finally establish asymptotic estimate, we denote by $U_\epsilon(\partial\Omega)$ the $\epsilon$-neighborhood of $\partial\Omega$ in $\mathcal{M}$. We can also show that

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left( \frac{1}{2\pi i} \int_{\mathcal{C}} q_{-2}(x, \xi, \tau) \, d\tau \right) \, d\xi = \left( \frac{n-1}{(4\pi \mu)^{n/2}} + \frac{1}{(4\pi (2\mu + \lambda)^{n/2}} \right) \text{ for } x \in \Omega,$$

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-\hat{x}) \cdot \xi} \left( \frac{1}{2\pi i} \int_{\mathcal{C}} q_{-2}(x, \xi, \tau) \, d\tau \right) \, d\xi = O(t^{1-n}) \text{ as } t \to 0^+ \text{ for } x \in \Omega \setminus U_\epsilon(\partial\Omega)$$

and

$$\int_{\Omega \cap U_\epsilon(\partial\Omega)} \left\{ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-\hat{x}) \cdot \xi} \left( \frac{1}{2\pi i} \int_{\mathcal{C}} q_{-2}(x, \xi, \tau) \, d\tau \right) \, d\xi \right\} \, dV = \frac{1}{4} \left( \frac{n-1}{(4\pi \mu)^{(n-1)/2}} + \frac{1}{(4\pi (2\mu + \lambda))^{(n-1)/2}} \right) \text{ Vol}(\partial\Omega) + O(t^{1-n}) \text{ as } t \to 0^+. $$

Hence

$$\int_{\Omega} \text{Tr}(K^+(t, x, x)) \, dV = a_0 t^{-n/2} + a_1 t^{-(n-1)/2} + O(t^{1-n/2}) \text{ as } t \to 0^+, \quad (1.15)$$

where

$$a_0 = \left( \frac{n-1}{(4\pi \mu)^{n/2}} + \frac{1}{(4\pi (2\mu + \lambda))^{n/2}} \right) \text{ Vol}(\Omega),$$

$$a_1 = \frac{1}{4} \left( \frac{n-1}{(4\pi \mu)^{(n-1)/2}} + \frac{1}{(4\pi (2\mu + \lambda))^{(n-1)/2}} \right) \text{ Vol}(\partial\Omega).$$

As an application of theorem 1.1, we can prove the following spectral rigidity result:

**Theorem 1.2** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Suppose that the Navier–Lamé spectrum with respect to the Dirichlet (respectively, Neumann) boundary condition is equal to that of $B_r$, a ball of radius $r$, then $\Omega = B_r$.

Theorem 1.2 also shows that a ball is uniquely determined by its Navier–Lamé spectrum among all Euclidean bounded domains (elastic bodies) with smooth boundary.

### 2 Some Notations and Lemmas

Let $\Omega$ be an $n$-dimensional Riemannian manifold (possibly with boundary), and let $\Omega$ be equipped with a smooth metric tensor $g = \sum_{j,k=1}^n g_{jk} \, dx_j \otimes dx_k$. Denote by $[g_{jk}]_{n \times n}$ the inverse of the matrix $[g_{jk}]_{n \times n}$ and set $|g| := \det[g_{jk}]_{n \times n}$. In particular, $dV$, the volume element in $\Omega$ is locally given by $dV = \sqrt{|g|} \, dx_1 \cdots dx_n$. By $T\Omega$ and $T^*\Omega$ we denote, respectively, the tangent and cotangent bundle on $\Omega$. Throughout, we shall
also denote by \( T\Omega \) global \((C^\infty)\) sections in \( T\Omega \) (i.e., \( T\Omega \equiv C^\infty(\Omega, T\Omega) \)); similarly, \( T^*\Omega \equiv C^\infty(\Omega, T^*\Omega) \). A vector field on \( \Omega \) is a section of the map \( \pi : T\Omega \to \Omega \). More concretely, a vector field is a smooth map \( X : \Omega \to T\Omega \), usually written \( p \mapsto X_p \), with the property that
\[
\pi \circ X = \text{Id}_\Omega,
\]
or equivalently, \( X_p \in T_p\Omega \) for each \( p \in \Omega \). If \( (U; x_1, \cdots, x_n) \) is any smooth coordinate chart for \( \Omega \), we can write the value of \( X \) at any point \( p \in U \) in terms of the coordinate basis vectors \( \{ \frac{\partial}{\partial x_j} \}_{\mid p} \) of \( T_p\Omega \):
\[
X_p = \sum_{j=1}^n X^j(p) \frac{\partial}{\partial x_j} \bigg|_p.
\]
This defines \( n \) functions \( X^j : U \to \mathbb{R} \), called the component functions of \( X \) in the given chart. Recall first that
\[
\text{div} \ X := \sum_{j=1}^n \frac{1}{\sqrt{|g|}} \frac{\partial(\sqrt{|g|} X^j)}{\partial x_j} \quad \text{if} \quad X = \sum_{j=1}^n X^j \frac{\partial}{\partial x_j} \in T\Omega, \tag{2.1}
\]
and
\[
\text{grad} \ v := \sum_{j,k=1}^n \left( g^{jk} \frac{\partial v}{\partial x_k} \right) \frac{\partial}{\partial x_j} \quad \text{if} \quad v \in C^\infty(\Omega), \tag{2.2}
\]
are, respectively, the usual divergence and gradient operators. Accordingly, the Laplace–Beltrami operator \( \Delta_g \) is just given by
\[
\Delta_g := \text{div} \ \text{grad} = \frac{1}{\sqrt{|g|}} \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left( \sqrt{|g|} g^{jk} \frac{\partial}{\partial x_k} \right). \tag{2.3}
\]
Next, let \( \nabla \) be the associated Levi-Civita connection. For each \( X \in T\Omega \), \( \nabla X \) is the tensor of type \((0, 2)\) defined by
\[
(\nabla X)(Y, Z) := \langle \nabla_Y X, Z \rangle, \quad \forall \ Y, Z \in T\Omega. \tag{2.4}
\]
It is well known that in a local coordinate system with the naturally associated frame field on the tangent bundle,
\[
\nabla_{\frac{\partial}{\partial x_k}} X = \sum_{j=1}^n \left( \frac{\partial X^j}{\partial x_k} + \sum_{l=1}^n \Gamma^j_{lk} X^l \right) \frac{\partial}{\partial x_j} \quad \text{for} \quad X = \sum_{j=1}^n X^j \frac{\partial}{\partial x_j},
\]
where \( \Gamma^j_{lk} = \frac{1}{2} \sum_{m=1}^{n} g^{jm} (\frac{\partial g_{km}}{\partial x_l} + \frac{\partial g_{lm}}{\partial x_k} - \frac{\partial g_{kl}}{\partial x_m}) \) are the Christoffel symbols associated with the metric \( g \) (see, for example, p. 549 of [68]). If we denote

\[
X^j ; k = \frac{\partial X^j}{\partial x_k} + \sum_{l=1}^{n} \Gamma^j_{lk} X^l,
\]

then

\[
\nabla_Y X = \sum_{j,k=1}^{n} Y^k X^j ; k \frac{\partial}{\partial x_j} \quad \text{for} \quad Y = \sum_{k=1}^{n} Y^k \frac{\partial}{\partial x_k}.
\]

The symmetric part of \( \nabla X \) is \( \text{Def} X \), the deformation of \( X \), i.e.,

\[
(\text{Def} X)(Y, Z) = \frac{1}{2}(\langle \nabla_Y X, Z \rangle + \langle \nabla_Z X, Y \rangle), \quad \forall Y, Z \in T\Omega \quad (2.5)
\]

(whereas the antisymmetric part of \( \nabla X \) is simply \( dX \), i.e.,

\[
dX(Y, Z) = \frac{1}{2}(\langle \nabla_Y X, Z \rangle - \langle \nabla_Z X, Y \rangle), \quad \forall Y, Z \in T\Omega.
\]

Thus, \( \text{Def} X \) is a symmetric tensor field of type \((0, 2)\). In coordinate notation,

\[
(\text{Def} X)_{jk} = \frac{1}{2}(X_{j;k} + X_{k;j}), \quad \forall j, k, \quad (2.6)
\]

where, \( X_{k;j} := \frac{\partial X_k}{\partial x_j} - \sum_{l=1}^{n} \Gamma^l_{kj} X_l \) for a vector field \( X = \sum_{j=1}^{n} X^j \frac{\partial}{\partial x_j} \), and \( X_k = \sum_{l=1}^{n} g^{kl} X_l \). The adjoint \( \text{Def}^* \) of \( \text{Def} \) is defined in local coordinates by \( (\text{Def}^* w)^j = -\sum_{k=1}^{n} w^{jk} ; k \) for each symmetric tensor field \( w := w_{jk} \) of type \((0, 2)\). The Riemann curvature tensor \( R \) of \( \Omega \) is given by

\[
R(X, Y) Z = [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]}Z, \quad \forall X, Y, Z \in T\Omega, \quad (2.7)
\]

where \([X, Y] := XY - YX\) is the usual commutator bracket. It is convenient to change this into a \((0, 4)\)-tensor by setting

\[
R(X, Y, Z, W) := \langle R(X, Y) Z, W \rangle, \quad \forall X, Y, Z, W \in T\Omega.
\]

In other words, in a local coordinate system such as that discussed above,

\[
R_{jklm} = \left\langle R \left( \frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_m} \right), \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j} \right\rangle.
\]
The Ricci curvature \( \text{Ric} \) on \( \Omega \) is a \((0, 2)\)-tensor defined as a contraction of \( \mathcal{R} \):

\[
\text{Ric}(X, Y) := \sum_{j=1}^{n} \left( \mathcal{R} \left( \frac{\partial}{\partial x_j}, Y \right) X, \frac{\partial}{\partial x_j} \right) = \sum_{j=1}^{n} \left( \mathcal{R} \left( Y, \frac{\partial}{\partial x_j} \right) \frac{\partial}{\partial x_j} \right) X, \quad \forall X, Y \in T\Omega.
\]

That is,

\[
R_{jk} = \sum_{l=1}^{n} R_{jlk} = \sum_{l,m=1}^{n} g^{lm} R_{ljmk}.
\]  

Note that

\[
R_{jlk} = \frac{\partial}{\partial x_l} \Gamma_{jk} - \frac{\partial}{\partial x_k} \Gamma_{jl} + \Gamma_{sl} \Gamma_{jk} - \Gamma_{sk} \Gamma_{jl}.
\]  

In [50], by considering the equilibria states of elastic energy functional \( E(u) = \frac{1}{2} \int_{\Omega} \left( \lambda(\text{div } u)^2 + 2\mu \langle \text{Def } u, \text{Def } u \rangle \right) dV \), the author of this paper proved the following result, which generalizes the classical Navier–Lamé operator from the Euclidean space to a Riemannian manifold:

**Lemma 2.1** On a Riemannian manifold \((\Omega, g)\), modeling a homogeneous, linear, isotropic, elastic medium, the Navier–Lamé operator \( L_g \) is given by

\[
P_g u = \mu \nabla^* \nabla u - (\mu + \lambda) \text{grad div } u - \mu \text{Ric}(u) \quad \text{for } u = \sum_{k=1}^{n} u^k \frac{\partial}{\partial x_k} \in T\Omega,
\]

where \( \nabla^* \nabla u \) is the Bochner Laplacian of \( u \) defined by

\[
\nabla^* \nabla u = - \sum_{j=1}^{n} \left\{ \Delta_g u^j + 2 \sum_{m,k,l=1}^{n} g^{ml} \Gamma_{km}^{j} \frac{\partial u^k}{\partial x_l} + \sum_{m,k,l=1}^{n} \left( g^{ml} \frac{\partial \Gamma_{kl}^{j}}{\partial x_m} \right) \frac{\partial u^k}{\partial x_l} \right\} \frac{\partial}{\partial x_j},
\]

and

\[
\text{Ric}(u) = \sum_{j=1}^{n} \left( \sum_{k=1}^{n} R_j^k u^k \right) \frac{\partial}{\partial x_j}.
\]

In particular, \( P_g \) is strongly elliptic, formally self-adjoint.
We need the method of pseudodifferential operators. If $W$ is an open subset of $\mathbb{R}^n$, we denote by $S^m_{1,0} = S^m_{1,0}(W, \mathbb{R}^n)$ the set of all $p \in C^\infty(W, \mathbb{R}^n)$ such that for every compact set $O \subset W$ we have

$$|D_\alpha^x D_\xi^p p(x, \xi)| \leq C_O,\alpha,\beta (1 + |\xi|)^{m-|\alpha|}, \quad x \in O, \, \xi \in \mathbb{R}^n$$

(2.13)

for all $\alpha, \beta \in \mathbb{N}^n$, where $|\xi| = (\sum_{j=1}^n \xi_j^2)^{1/2}$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$, $\mathbb{N}^n$ is the set of $\gamma = (\gamma_1, \cdots, \gamma_n)$ with $\gamma_j$ integer $\geq 0$, and $|\gamma| = \gamma_1 + \cdots + \gamma_n$. The elements of $S^m_{1,0}$ are called symbols (or full symbols) of order $m$. It is clear that $S^m_{1,0}$ is a Fréchet space with semi-norms given by the smallest constants which can be used in (2.13) (i.e.,

$$\|p\|_{O,\alpha,\beta} = \sup_{x \in O} \left| \left( D_\alpha^x D_\xi^p p(x, \xi) \right) (1 + |\xi|)^{|\alpha|-m} \right|.$$  

Let $p(x, \xi) \in S^m_{1,0}$. A pseudodifferential operator in an open set $W \subset \mathbb{R}^n$ is essentially defined by a Fourier integral operator (cf. [29,34,41,68]):

$$p(x, D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) \, d\xi,$$  

(2.14)

and denoted by $OPS^m$. Here $u \in C^\infty_0(W)$ and $\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-iy \cdot \xi} u(y) \, dy$ is the Fourier transform of $u$. If there are smooth $p_{m-j}(x, \xi)$, homogeneous in $\xi$ of degree $m - j$ for $|\xi| \geq 1$, that is, $p_{m-j}(x, r\xi) = r^{m-j} p_{m-j}(x, \xi)$ for $r, \, |\xi| \geq 1$, and if

$$p(x, \xi) \sim \sum_{j \geq 0} p_{m-j}(x, \xi)$$  

(2.15)

in the sense that

$$p(x, \xi) - \sum_{j=0}^k p_{m-j}(x, \xi) \in S^{m-k-1}_{1,0}$$  

(2.16)

for all $k$, then we say $p(x, \xi) \in S^m_{cl}$, or just $p(x, \xi) \in S^m$. We call $p_m(x, \xi)$ the principal symbol of $p(x, D)$. Sometimes we denote by $\iota(p(x, D))$ the (full) symbol of $p(x, D)$.

Let $\Omega$ be a smooth $n$-dimensional Riemannian manifold (of class $C^\infty$). We denote by $C^\infty_0(\Omega)$ and $C^\infty(\Omega)$ the space of all smooth complex-valued functions on $\Omega$ and the subspace of all functions with compact support, respectively. Assume that we are given a linear operator

$$P : C^\infty_0(\Omega) \to C^\infty(\Omega).$$
If $G$ is some chart in $\Omega$ (not necessarily connected) and $\kappa : G \to U$ its diffeomorphism onto an open set $U \subset \mathbb{R}^n$, then let $\tilde{P}$ be defined by the diagram

$$
\begin{array}{ccc}
C_0^\infty (G) & \xrightarrow{p} & C_0^\infty (G) \\
\kappa^* & & \kappa^* \\
C_0^\infty (U) & \xrightarrow{\tilde{P}} & C_0^\infty (U)
\end{array}
$$

where $\kappa^*$ is the induced transformation from $C_0^\infty (U)$ into $C_0^\infty (G)$, taking a function $u$ to the function $u \circ \kappa$. (Note, in the upper row is the operator $r_G \circ P \circ i_G$, where $i_G$ is the natural embedding $i_G : C_0^\infty (G) \to C_0^\infty (\Omega)$ and $r_G$ is the natural restriction $r_G : C_0^\infty (\Omega) \to C_0^\infty (G)$; for brevity we denote this operator by the same letter $P$ as the original operator. An operator $P : C_0^\infty (\Omega) \to C_0^\infty (\Omega)$ is called a pseudodifferential operator on $\Omega$ if for any chart diffeomorphism $\kappa : G \to U$, the operator $\tilde{P}$ defined above is a pseudodifferential operator on $U$. We denote by $OPS^m$ the pseudodifferential operator $P$ of order $m$. We also write $OPS^{-\infty} = \bigcap_m OPS^m$.

It is well known (see [33,34] or p. 13 of [68]) that if $p_j(x, D) \in OPS^{m_j}, \ j = 1, 2$, then

$$
p_1(x, D)p_2(x, D) = q(x, D) \in OPS^{m_1+m_2},
$$

and

$$
q(x, \xi) = \sum_{\alpha \geq 0} \frac{i^{\vert \alpha \vert}}{\alpha!} D_\xi^\alpha p_1(x, \xi) D_x^\alpha p_2(x, \xi). \quad (2.17)
$$

An operator $p(x, D)$ is said to be an elliptic pseudodifferential operator of order $m$ if for every compact $O \subset \Omega$ there exists a positive constant $c = c(O)$ such that

$$
\vert p(x, \xi) \vert \geq c \vert \xi \vert^m, \ x \in O, \ \vert \xi \vert \geq 1.
$$

If $q(x, D) \in OPS^{-m}$ is a pseudodifferential operator of order $-m$ such that

$$
q(x, D)p(x, D) = I \mod OPS^{-\infty},
$$

$$
p(x, D)q(x, D) = I \mod OPS^{-\infty},
$$

then we say that $q(x, D)$ is a (two-sided) parametrix for $p(x, D)$. Furthermore, if $P$ is a non-negative elliptic pseudodifferential operator of order $m$, then the spectrum of $P$ lies in a right half-plane and has a finite lower bound $\rho (P) = \inf \{ \Re \tau \mid \tau \in \sigma (P) \}$, where $\sigma (P)$ denotes the spectrum of $P$. We can modify the principal symbol $h_m(x, \xi)$ for small $\xi$ such that $h_m(x, \xi)$ has a positive lower bound throughout and lies in $\{ \tau = r e^{i\theta} \mid r > 0, \vert \theta \vert \leq \theta_0 \}$, where $\theta_0 \in (0, \frac{\pi}{2})$. According to [29], the resolvent $(\tau - P)^{-1}$ exists and is holomorphic in $\tau$ on a neighborhood of a set

$$
W_{r_0, \epsilon} = \{ \tau \in \mathbb{C} \mid \vert \tau \vert \geq r_0, \ \arg \tau \in [\theta_0 + \epsilon, 2\pi - \theta_0 - \epsilon], \ \Re \tau \leq \rho (P) - \epsilon \}
$$
There exists a parametrix $Q'_\tau$ on a neighborhood of a possibly larger set (with $\delta > 0, \epsilon > 0$)

$$V_{b,\epsilon} = \{ \tau \in \mathbb{C} | |\tau| \geq \delta \text{ or } \arg \tau \in [\theta_0 + \epsilon, 2\pi - \theta_0 - \epsilon] \}$$

such that this parametrix coincides with $(\tau - P)^{-1}$ on the intersection. Its symbol $q(x, \xi, \tau)$ in local coordinates is holomorphic in $\tau$ there and has the form (cf. Section 3.3 of [29])

$$q(x, \xi, \tau) \sim \sum_{l \geq 0} q_{-m-l}(x, \xi, \tau),$$

(2.18)

where

$$q_{-m} = (\tau - p_m(x, \xi))^{-1}, \quad q_{-m-1} = b_{1,1}(x, \xi)q_{-m}^2,$$

$$\cdots, \quad q_{-m-l} = \sum_{k=1}^{2l} b_{l,k}(x, \xi)q_{-m}^{k+1}, \cdots, \quad l \geq 2$$

(2.19)

with symbols $b_{l,k}$ independent of $\tau$ and homogeneous of degree $mk-l$ in $\xi$ for $|\xi| \geq 1$. The semigroup $(e^{-tP})_{t \geq 0}$ can be defined from $P$ by the Cauchy integral formula (see p. 4 of [26]):

$$e^{-tP} = \frac{1}{2\pi i} \int_C e^{-t\tau}(\tau - P)^{-1}d\tau, \quad t \geq 0,$$

where $C$ is a suitable curve in the complex plane in the positive direction around the spectrum of $P$. Inserting (2.18)–(2.19) into above formula, we get the symbol $\frac{1}{2\pi i} \int_C e^{-t\tau}\left[ \sum_{l \geq 0} q_{-m-l}(x, \xi, \tau) \right]d\tau$ of the semigroup $(e^{-tP})_{t \geq 0}$, and furthermore, we can obtain the semigroup $(e^{-tP})_{t \geq 0}$ and its trace for any fixed $t \geq 0$.

### 3 Full Symbol of Resolvent Operator $(\tau I - P_g)^{-1}$

Let $(\Omega, g)$ be an $n$-dimensional Riemannian manifold with metric $g = (g_{ij})$. Note that for $u = \sum_{j=1}^n u^j \frac{\partial}{\partial x_j} \in T\Omega$,

$$\Delta_g u^j = \sum_{m,l=1}^n \left( g_{ml} \frac{\partial^2 u^j}{\partial x_m \partial x_l} - g_{ml} \Gamma^s_{ml} \frac{\partial u^j}{\partial x_s} \right)$$

and

$$\text{grad } \text{div } u = \sum_{j,k,m=1}^n \left( g^{jm} \left( \frac{\partial^2 u^k}{\partial x_m \partial x_k} + \sum_{l=1}^n \Gamma^l_{kl} \frac{\partial u^k}{\partial x_m} + \sum_{l=1}^n \frac{\partial \Gamma^l_{kl}}{\partial x_m} u^k \right) \right) \frac{\partial}{\partial x_j}.$$
By Lemma 2.1 we can write the Navier–Lamé operator $P_g$ in $\Omega$ as the form of components relative to coordinates:

$$
P_g \mathbf{u} = \begin{cases} -\mu \left( \sum_{m,l=1}^{n} g^{ml} \frac{\partial^2}{\partial x_m \partial x_l} \right) \mathbf{I}_n - (\mu + \lambda) \left[ \sum_{m=1}^{n} g^{1m} \frac{\partial^2}{\partial x_1 \partial x_m} \cdots \sum_{m=1}^{n} g^{1m} \frac{\partial^2}{\partial x_n \partial x_m} \right] \\
+ \mu \left( \sum_{m,l,s=1}^{n} g^{ml} \Gamma_{ms} \frac{\partial}{\partial x_s} \right) \mathbf{I}_n - (\mu + \lambda) \left[ \sum_{m,l=1}^{n} 2g^{ml} \Gamma_{1m} \frac{\partial}{\partial x_1} \cdots \sum_{m,l=1}^{n} 2g^{ml} \Gamma_{nm} \frac{\partial}{\partial x_n} \right] \\
- (\mu + \lambda) \left[ \sum_{m,l=1}^{n} g^{lm} \Gamma_{l1} \frac{\partial}{\partial x_m} \cdots \sum_{m,l=1}^{n} g^{lm} \Gamma_{nl} \frac{\partial}{\partial x_m} \right] \\
- \mu \left[ \sum_{l,m=1}^{n} g^{ml} \frac{\partial^2 \xi_l}{\partial x_m \partial x_m} + \Gamma_{lh} \Gamma_{lm} - \Gamma_{1h} \Gamma_{1m} \right] \cdots \sum_{l,m=1}^{n} g^{nl} \frac{\partial^2 \xi_n}{\partial x_m \partial x_m} + \Gamma_{nh} \Gamma_{nl} - \Gamma_{1h} \Gamma_{1n} \right] \\
- (\mu + \lambda) \left[ \sum_{m=1}^{n} g^{1m} \frac{\partial \xi_1}{\partial x_m} \cdots \sum_{m=1}^{n} g^{1m} \frac{\partial \xi_n}{\partial x_m} \right] - \mu \left[ R_1 \cdots R_n \right] \left\{ \begin{array}{c} u^1 \\ \vdots \\ u^n \end{array} \right\}.
\end{cases}
$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix. Furthermore, we have

$$
P_g \mathbf{u}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} (A_g(x, \xi)) \begin{pmatrix} \hat{u}_1(\xi) \\ \vdots \\ \hat{u}_n(\xi) \end{pmatrix} d\xi,
$$

where

$$
A_g(x, \xi) = \mu \left( \sum_{m,l=1}^{n} g^{ml} \frac{\partial^2 \xi_l}{\partial x_m \partial x_m} \right) \mathbf{I}_n + (\mu + \lambda) \left[ \sum_{m=1}^{n} g^{1m} \xi_m \xi_1 \cdots \sum_{m=1}^{n} g^{1m} \xi_m \xi_n \right] \\
- \mu \left[ \sum_{l,m=1}^{n} g^{ml} \frac{\partial \xi_l}{\partial x_m} \cdots \sum_{l,m=1}^{n} g^{nl} \frac{\partial \xi_n}{\partial x_m} \right] - \mu \left[ R_1 \cdots R_n \right] \left\{ \begin{array}{c} \hat{u}_1(\xi) \\ \vdots \\ \hat{u}_n(\xi) \end{array} \right\}.
$$
\[ +i \mu \left( \sum_{m,l,s=1}^{n} g^{m l} \Gamma_{m l}^{s} \xi_{s} \right) \mathbf{I}_{n} - i \mu \begin{bmatrix} \sum_{m,l=1}^{n} 2g^{m l} \Gamma_{1 m}^{l} \xi_{l} \cdots \sum_{m,l=1}^{n} 2g^{m l} \Gamma_{n m}^{l} \xi_{l} \\ \vdots \\ \sum_{m,l=1}^{n} 2g^{m l} \Gamma_{1 m}^{n} \xi_{l} \cdots \sum_{m,l=1}^{n} 2g^{m l} \Gamma_{n m}^{n} \xi_{l} \end{bmatrix} \]

\[ -i (\mu + \lambda) \begin{bmatrix} \sum_{m,l=1}^{n} g^{lm} \Gamma_{m l}^{l} \xi_{m} \cdots \sum_{m,l=1}^{n} g^{lm} \Gamma_{n m}^{l} \xi_{m} \\ \vdots \\ \sum_{m,l=1}^{n} g^{nm} \Gamma_{m l}^{l} \xi_{m} \cdots \sum_{m,l=1}^{n} g^{nm} \Gamma_{n m}^{l} \xi_{m} \end{bmatrix} \]

\[ - (\mu + \lambda) \begin{bmatrix} g^{lm} \frac{\partial \Gamma_{m l}^{l}}{\partial x_{m}} \cdots g^{lm} \frac{\partial \Gamma_{m l}^{l}}{\partial x_{m}} \\ \vdots \\ g^{nm} \frac{\partial \Gamma_{m l}^{l}}{\partial x_{m}} \cdots g^{nm} \frac{\partial \Gamma_{m l}^{l}}{\partial x_{m}} \end{bmatrix} \begin{bmatrix} R_{1}^{1} \\ \vdots \\ R_{n}^{n} \end{bmatrix} \]

For each \( \tau \in \mathbb{C} \), we denote

\[ \tau \mathbf{I}_{n} - A_{g} = \mathbf{a}_{2} + \mathbf{a}_{1} + \mathbf{a}_{0}, \]  

where

\[ \mathbf{a}_{2}(x, \xi) := (\tau - \mu \sum_{m,l=1}^{n} g^{m l} \xi_{s} \xi_{l}) \mathbf{I}_{n} - (\mu + \lambda) \begin{bmatrix} \sum_{m=1}^{n} g^{l m} \xi_{m} \xi_{1} \cdots \sum_{m=1}^{n} g^{l m} \xi_{m} \xi_{n} \\ \vdots \\ \sum_{m=1}^{n} g^{n m} \xi_{m} \xi_{1} \cdots \sum_{m=1}^{n} g^{n m} \xi_{m} \xi_{n} \end{bmatrix}, \]

\[ \mathbf{a}_{1}(x, \xi) := -i \mu \left( \sum_{m,l,s=1}^{n} g^{m l} \Gamma_{m l}^{s} \xi_{s} \right) \mathbf{I}_{n} + i \mu \begin{bmatrix} \sum_{m,l=1}^{n} 2g^{m l} \Gamma_{1 m}^{l} \xi_{l} \cdots \sum_{m,l=1}^{n} 2g^{m l} \Gamma_{n m}^{l} \xi_{l} \\ \vdots \\ \sum_{m,l=1}^{n} 2g^{m l} \Gamma_{1 m}^{n} \xi_{l} \cdots \sum_{m,l=1}^{n} 2g^{m l} \Gamma_{n m}^{n} \xi_{l} \end{bmatrix} \]

\[ + i (\mu + \lambda) \begin{bmatrix} \sum_{m,l=1}^{n} g^{lm} \Gamma_{m l}^{l} \xi_{m} \cdots \sum_{m,l=1}^{n} g^{lm} \Gamma_{m l}^{l} \xi_{m} \\ \vdots \\ \sum_{m,l=1}^{n} g^{nm} \Gamma_{m l}^{l} \xi_{m} \cdots \sum_{m,l=1}^{n} g^{nm} \Gamma_{m l}^{l} \xi_{m} \end{bmatrix} \]
Let 

\[
\mathbf{q}(x, \xi, \tau) \sim \mathbf{q}_{-2}(x, \xi, \tau) + \mathbf{q}_{-3}(x, \xi, \tau) + \cdots + \mathbf{q}_{-2-l}(x, \xi, \tau) + \cdots
\]  

be the expansion of the full symbol of \( Q \). Suppose that the complex parameter \( \tau \) have homogeneity 2 (This point of view stems from [61] or [23]). Let \( \mathbf{q}_{-2-l}(x, \xi, \tau) \) be homogeneous of order \(-2-l\) in the variables \((\xi, \tau^{1/2})\). This infinite sum defines \( \mathbf{q}(x, \xi, \tau) \) asymptotically. Our purpose is to determine \( \mathbf{q}(x, \xi, \tau) \) so that

\[
\iota((\tau I - P_g)Q) \sim \mathbf{I}_n,
\]  

where \( \iota(T) \) denotes the full symbol of pseudodifferential operator \( T \). By symbol formula (2.17) of the product of pseudodifferential operators, we can decompose the left-hand side of (3.7) into a sum of orders of homogeneity

\[
\sum_{\alpha \geq 0} \left( \partial^\alpha_{\xi} (\iota(\tau I - P_g)) \right) \cdot (D^\alpha_x \mathbf{q}) / \alpha! \sim \mathbf{I}_n,
\]  

where \( \partial^\alpha_{\xi} := \frac{\partial^{\left|\alpha\right|}}{\partial^{\left|\alpha\right|} \xi} \). Noticing that \( \iota(\tau I - P_g) = (\tau I - A_g) = \mathbf{a}_2 + \mathbf{a}_1 + \mathbf{a}_0 \), we find by (3.8) that

\[
\iota((\tau I - P_g)Q) \sim \sum_{l=0}^{\infty} \left( \sum_{l=j+2|\alpha| - 2 \to k} (\partial^\alpha_{\xi} \mathbf{a}_k) \cdot (D^\alpha_x \mathbf{q}_{-2-l}) / \alpha! \right).
\]
The sum is over terms which are homogeneous of order \(-l\). Thus (3.7) leads to the following equations

\[
I_n = \sum_{0 = j + |\alpha| + 2 - k} (\partial_{\xi}^\alpha a_k)(D_\alpha q_{-2-j})/\alpha! = a_2 q_{-2},
\]

\[
0 = \sum_{l = j + |\alpha| + 2 - k, l \geq 1} (\partial_{\xi}^\alpha a_k)(D_\alpha q_{-2-j})/\alpha!,
\]

\[
= a_2 q_{-2-l} + \sum_{j < l} (\partial_{\xi}^\alpha a_k)(D_\alpha q_{-2-j})/\alpha!, \quad l \geq 1.
\]

These equations determine the \(q_{-2-l}\) inductively. In other words, we have

\[
q_{-2} = a_2^{-1},
\]

\[
q_{-2-l} = -a_2^{-1} \left( \sum_{j < l} (\partial_{\xi}^\alpha a_k)(D_\alpha q_{-2-j})/\alpha! \right) \quad \text{for} \quad l = j + |\alpha| + 2 - k \geq 1.
\]

In order to calculate the \(a_2^{-1}\), by a direct calculation we find that

\[
\begin{bmatrix}
\sum_{r=1}^{n} g_{lr} \xi_r \xi_1 & \cdots & \sum_{r=1}^{n} g_{lr} \xi_r \xi_n \\
\vdots & \ddots & \vdots \\
\sum_{r=1}^{n} g_{nr} \xi_r \xi_1 & \cdots & \sum_{r=1}^{n} g_{nr} \xi_r \xi_n
\end{bmatrix}
= \left( \sum_{l,m=1}^{n} g_{lm} \xi_l \xi_m \right)
\begin{bmatrix}
\sum_{r=1}^{n} g_{lr} \xi_r \xi_1 & \cdots & \sum_{r=1}^{n} g_{lr} \xi_r \xi_n \\
\vdots & \ddots & \vdots \\
\sum_{r=1}^{n} g_{nr} \xi_r \xi_1 & \cdots & \sum_{r=1}^{n} g_{nr} \xi_r \xi_n
\end{bmatrix}
\]

Thus, the following two matrices play a key role:

\[
F := \left\{ I_n, \begin{bmatrix}
\sum_{r=1}^{n} g_{lr} \xi_r \xi_1 & \cdots & \sum_{r=1}^{n} g_{lr} \xi_r \xi_n \\
\vdots & \ddots & \vdots \\
\sum_{r=1}^{n} g_{nr} \xi_r \xi_1 & \cdots & \sum_{r=1}^{n} g_{nr} \xi_r \xi_n
\end{bmatrix}
\right\}.
\]

The set \(F\) of above two matrices can generate a matrix ring \(\mathfrak{F}\) according to the usual matrix addition and multiplication of \(\mathfrak{F}\) on the ring of functions. This implies that \(a_2^{-1}\)
must have the following form:

\[
a_2^{-1} = s_1 I_n + s_2 \begin{bmatrix}
\sum_{r=1}^{n} g^{1r} \xi_r \xi_1 & \ldots & \sum_{r=1}^{n} g^{1r} \xi_r \xi_n \\
\vdots & & \vdots \\
\sum_{r=1}^{n} g^{nr} \xi_r \xi_1 & \ldots & \sum_{r=1}^{n} g^{nr} \xi_r \xi_n
\end{bmatrix}, \tag{3.11}
\]

where \(s_1\) and \(s_2\) are unknown functions which will be determined later. This key idea is inspired by Galois group theory for solving the polynomial equation (see [2] or [19]). Substituting (3.11) into \(a_2a_2^{-1} = I_n\), we have

\[
\left\{ \begin{array}{c}
(\tau - \mu \sum_{m,l=1}^{n} g^{ml} \xi_m \xi_l) I_n - (\mu + \lambda) \\
+ s_2 \begin{bmatrix}
\sum_{r=1}^{n} g^{1r} \xi_r \xi_1 & \ldots & \sum_{r=1}^{n} g^{1r} \xi_r \xi_n \\
\vdots & & \vdots \\
\sum_{r=1}^{n} g^{nr} \xi_r \xi_1 & \ldots & \sum_{r=1}^{n} g^{nr} \xi_r \xi_n
\end{bmatrix}
\end{array} \right\} s_1 I_n
\]

\[
= I_n,
\]

i.e.,

\[
s_1 \left( \tau - \mu \sum_{l,m=1}^{n} g^{lm} \xi_l \xi_m \right) I_n + s_2 \left( \tau - \mu \sum_{l,m=1}^{n} g^{lm} \xi_l \xi_m \right) - s_1 (\mu + \lambda)
\]

\[
- s_2 (\mu + \lambda) \sum_{l,m=1}^{n} g^{lm} \xi_l \xi_m \right) \begin{bmatrix}
\sum_{r=1}^{n} g^{1r} \xi_r \xi_1 & \ldots & \sum_{r=1}^{n} g^{1r} \xi_r \xi_n \\
\vdots & & \vdots \\
\sum_{r=1}^{n} g^{nr} \xi_r \xi_1 & \ldots & \sum_{r=1}^{n} g^{nr} \xi_r \xi_n
\end{bmatrix}
\]

= \(I_n\).

Since the set \(F\) is a basis of the matrix ring \(\mathfrak{g}\), we get

\[
\left\{ \begin{array}{c}
s_1 \left( \tau - \mu \sum_{l,m=1}^{n} g^{lm} \xi_l \xi_m \right) = 1, \\
s_2 \left( \tau - \mu \sum_{l,m=1}^{n} g^{lm} \xi_l \xi_m \right) - s_1 (\mu + \lambda) - s_2 (\mu + \lambda) \sum_{l,m=1}^{n} g^{lm} \xi_l \xi_m = 0.
\end{array} \right\}
\]
It follows that

\[
\begin{aligned}
  s_1 &= \frac{1}{\tau - \mu \sum_{l,m=1}^n g^{lm} \xi_l \xi_m}, \\
  s_2 &= \frac{\mu + \lambda}{\left( \tau - \mu \sum_{l,m=1}^n g^{lm} \xi_l \xi_m \right) \left( \tau - (2\mu + \lambda) \sum_{l,m=1}^n g^{lm} \xi_l \xi_m \right)}.
\end{aligned}
\]

(3.12)

Therefore

\[
q_{-2} = a_2^{-1} = \left[ \frac{1}{\tau - \mu \sum_{l,m=1}^n g^{lm} \xi_l \xi_m} + \frac{\mu + \lambda}{\left( \tau - \mu \sum_{l,m=1}^n g^{lm} \xi_l \xi_m \right) \left( \tau - (2\mu + \lambda) \sum_{l,m=1}^n g^{lm} \xi_l \xi_m \right)} \right] \times
\]

\[
\begin{bmatrix}
  n \sum_{r=1}^n g^{1r} \xi_r \xi_1 & \cdots & n \sum_{r=1}^n g^{1r} \xi_r \xi_n \\
  \vdots & \ddots & \vdots \\
  n \sum_{r=1}^n g^{nr} \xi_r \xi_1 & \cdots & n \sum_{r=1}^n g^{nr} \xi_r \xi_n
\end{bmatrix}.
\]

(3.13)

Combining this and (3.10) we can get \( q_{-2-l} \) for all \( l \geq 1 \). For example, we can easily write out the first two terms \( q_{-2}, q_{-3} \):

\[
q_{-2}(x, \xi, \tau) = a_2^{-1},
\]

(3.14)

\[
q_{-3}(x, \xi, \tau) = -a_2^{-1} \left( a_1 a_2^{-1} - i \sum_{l=1}^n \frac{\partial a_2}{\partial \xi_l} \frac{\partial a_2^{-1}}{\partial x_l} \right).
\]

(3.15)

From (3.13) we immediately have the following:

**Lemma 3.1** Let \( Q \) be a pseudodifferential operator satisfy rm(3.7) and let \( q_{-2}(x, \xi, \tau) \) be the principal symbol of \( Q \). Then, for any \( n \geq 1 \),

\[
q_{-2}(x, \xi, \tau) = a_2^{-1},
\]

(3.16)

\[
\text{Tr} \left( q_{-2}(x, \xi, \tau) \right) = \frac{n}{\left( \tau - \mu \sum_{l,m=1}^n g^{lm} \xi_l \xi_m \right)} + \frac{\mu + \lambda}{\left( \tau - \mu \sum_{l,m=1}^n g^{lm} \xi_l \xi_m \right) \left( \tau - (2\mu + \lambda) \sum_{l,m=1}^n g^{lm} \xi_l \xi_m \right)},
\]

(3.17)

where \( a_2^{-1} \) is given by (3.13).
4 Asymptotic Expansion of Trace of the Integral Kernel

Proof of Theorem 1.1. From the theory of elliptic operators (see [55–58,65]), we see that the Navier–Lamé operator $P_g$ can generate strongly continuous semigroups $(e^{-tP_g})_{t \geq 0}$ with respect to the Dirichlet and Neumann boundary conditions, respectively, in suitable spaces of vector-valued functions (for example, in $[C_0(\Omega)]^n$ (see [65]) or in $[L^2(\Omega)]^n$ (see [15])). Furthermore, there exist matrix-valued functions $K^\mp(t, x, y)$, which are called the integral kernels, such that (see [15] or p. 4 of [22])

$$e^{-tP_g}w_0(x) = \int_\Omega K^\mp(t, x, y)w_0(y)dy, \quad w_0 \in [L^2(\Omega)]^n.$$

Let $\{u^\mp_k\}_{k=1}^\infty$ be the orthonormal eigenvectors of the elastic operators $P^\mp_g$ corresponding to the eigenvalues $\{\tau^\mp_k\}_{k=1}^\infty$, then the integral kernels $K^\mp(t, x, y) = e^{-tP_g} \delta(x - y)$ are given by

$$K^\mp(t, x, y) = \sum_{k=1}^\infty e^{-t\tau^\mp_k}u^\mp_k(x) \otimes u^\mp_k(y). \quad (4.1)$$

This implies that the integrals of the traces of $K^\mp(t, x, y)$ are actually spectral invariants:

$$\int_\Omega \text{Tr}(K^\mp(t, x, x))dV = \sum_{k=1}^\infty e^{-t\tau^\mp_k}. \quad (4.2)$$

We will combine calculus of symbols (see [62]) and “method of images” (which stems from McKean-Singer in §5 of [53]), to deal with asymptotic expansions for the integrals of traces of integral kernels. Let $M = \Omega \cup (\partial \Omega) \cup \Omega^*$ be the (closed) double of $\Omega$, and $P$ the double to $M$ of the operator $P_g$ on $\Omega$. The coefficients of operator $P$ jump as $x$ crosses $\partial \Omega$, but $\frac{\partial u}{\partial t} = Pu$ still has a nice fundamental solution (i.e., integral kernel) $K(t, x, y)$ of class $C^\infty[(0, \infty) \times (M \setminus \partial \Omega)^2] \cap C^1((0, \infty) \times M^2)$, approximable even on $\partial \Omega$, and the integral kernels $K^\mp(t, x, y)$ of $\frac{\partial u}{\partial t} = P^\mp_g u$ can be expressed on $(0, \infty) \times \Omega \times \Omega$ as

$$K^\mp(t, x, y) = K(t, x, y) \mp K(t, x, y^*), \quad (4.3)$$

$y^*$ being the double of $y \in \Omega$ (see, p. 53 of [53]). Since the strongly continuous semigroup $(e^{-tP})_{t \geq 0}$ can also be represented as

$$e^{-tP} = \frac{1}{2\pi i} \int_C e^{-t\tau} (\tau I - P)^{-1}d\tau,$$
where $C$ is a suitable curve in the complex plane in the positive direction around the spectrum of $P$ (i.e., a contour around the positive real axis). It follows that

$$
\begin{align*}
K(t, x, y) &= e^{-tP} \delta(x - y) \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} \left( \frac{1}{2\pi i} \int_C e^{-\tau t} i((\tau I - P)\tau)^{-1})d\tau \right) d\xi,
\end{align*}
$$

(4.4)

In particular, for every $x \in \Omega$,

$$
\begin{align*}
K(t, x, x) &= e^{-tP} \delta(x - x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-x) \cdot \xi} \left( \frac{1}{2\pi i} \int_C e^{-\tau t} i((\tau I - P_g)^{-1})d\tau \right) d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left( \frac{1}{2\pi i} \int_C e^{-\tau t} i((\tau I - P_g)^{-1})d\tau \right) d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\tau t} i((\tau I - P_g)^{-1})d\xi, \quad (4.5)
\end{align*}
$$

where $\sum_{l \geq 0} q_{-2-l}(x, \xi, \tau)$ is the full symbol of $(\tau I - P_g)^{-1}$.

Firstly, from the discussion of previous section, we know that

$$
q_{-2}(x, \xi, \tau) = \frac{1}{\tau - \mu} \sum_{l, m=1}^{n} g^{lm} \xi_l \xi_m
$$

$$
+ \frac{\mu + \lambda}{(\tau - \mu) \sum_{l, m=1}^{n} g^{lm} \xi_l \xi_m (\tau - (2\mu + \lambda) \sum_{l, m=1}^{n} g^{lm} \xi_l \xi_m)}
$$

$$
\times \left[ \begin{array}{c}
\sum_{r=1}^{n} g^{1r} \xi_r \xi_1 \\
\vdots \\
\sum_{r=1}^{n} g^{nr} \xi_r \xi_n
\end{array} \right] \\
\times \left[ \begin{array}{c}
\sum_{r=1}^{n} g^{1r} \xi_r \xi_1 \\
\vdots \\
\sum_{r=1}^{n} g^{nr} \xi_r \xi_n
\end{array} \right]
$$

and

$$
\text{Tr} \left( q_{-2}(x, \xi, \tau) \right) = \frac{n}{\tau - \mu \sum_{l, m=1}^{n} g^{lm} \xi_l \xi_m}
$$

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\[
\begin{align*}
\sum_{l,m=1}^{n} g^{lm} \xi_l \xi_m + \frac{(\mu + \lambda) \sum_{l,m=1}^{n} g^{lm} \xi_l \xi_m}{(\tau - \mu \sum_{l,m=1}^{n} g^{lm} \xi_l \xi_m)(\tau - (2\mu + \lambda) \sum_{l,m=1}^{n} g^{lm} \xi_l \xi_m)}.
\end{align*}
\]

(4.7)

For each \( x \in \Omega \), we use a geodesic normal coordinate system centered at this \( x \). It follows from §11 of Chap.1 in [67] that in such a coordinate system, \( g_{jk}(x) = \delta_{jk} \) and \( \Gamma^l_{jk}(x) = 0 \). Then (4.7) reduces to

\[
\begin{align*}
\text{Tr} (\mathbf{q}_{\mathbf{-2}}(x, \xi, \tau)) &= \frac{n}{(\tau - \mu |\xi|^2)} + \frac{(\mu + \lambda) |\xi|^2}{(\tau - \mu |\xi|^2)(\tau - (2\mu + \lambda) |\xi|^2)}, \\
&= (n - 1)e^{-\mu |\xi|^2} + e^{-t(2\mu + \lambda) |\xi|^2}.
\end{align*}
\]

(4.8)

where \( |\xi| = \sqrt{\sum_{k=1}^{n} \xi_k^2} \) for any \( \xi \in \mathbb{R}^n \). By applying the residue theorem (see, for example, Chap.4, §5 in [1]) we get

\[
\begin{align*}
\frac{1}{2\pi i} \int_{\mathcal{C}} e^{-\tau t} \left( \frac{n}{(\tau - \mu |\xi|^2)} + \frac{(\mu + \lambda) |\xi|^2}{(\tau - \mu |\xi|^2)(\tau - (2\mu + \lambda) |\xi|^2)} \right) d\tau \\
&= (n - 1)e^{-\mu t |\xi|^2} + e^{-t(2\mu + \lambda) |\xi|^2}.
\end{align*}
\]

(4.9)

It follows that

\[
\begin{align*}
\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left( \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-\tau t} \text{Tr} (\mathbf{q}_{\mathbf{-2}}(x, \xi, \tau)) d\tau \right) d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left( (n - 1)e^{-\mu t |\xi|^2} + e^{-t(2\mu + \lambda) |\xi|^2} \right) d\xi \\
&= \frac{n - 1}{(4\pi \mu t)^{n/2}} + \frac{1}{(4\pi (2\mu + \lambda) t)^{n/2}},
\end{align*}
\]

(4.10)

and hence

\[
\int_{\Omega} \left\{ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left( \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-\tau t} \text{Tr} (\mathbf{q}_{\mathbf{-2}}(x, \xi, \tau)) d\tau \right) d\xi \right\} dV \\
= \left( \frac{n - 1}{(4\pi \mu t)^{n/2}} + \frac{1}{(4\pi (2\mu + \lambda) t)^{n/2}} \right) \text{Vol}(\Omega).
\]

(4.11)

For given (small) \( \epsilon > 0 \), denote by \( U_\epsilon(\partial \Omega) = \{ z \in \mathcal{M} | \text{dist}(z, \partial \Omega) < \epsilon \} \) the \( \epsilon \)-neighborhood of \( \partial \Omega \) in \( \mathcal{M} \). When \( x \in \Omega \setminus U_\epsilon(\partial \Omega) \), we see by taking geodesic normal coordinate system at \( x \) that (4.8) still holds at this \( x \). From (4.9) we have that for any \( x \in \Omega \setminus U_\epsilon(\partial \Omega) \),

\[
\begin{align*}
\text{Tr} (\mathbf{q}_{\mathbf{-2}}(t, x, \xi)) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-x)^\cdot \xi} \left( (n - 1)e^{-\mu t |\xi|^2} + e^{-t(2\mu + \lambda) |\xi|^2} \right) d\xi \\
&= \frac{n - 1}{(4\pi \mu t)^{n/2}} e^{-\frac{|x-x|^2}{4\mu t}} + \frac{1}{(4\pi (2\mu + \lambda) t)^{n/2}} e^{-\frac{|x-x|^2}{4(2\mu + \lambda) t}},
\end{align*}
\]

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which exponentially tends to zero as $t \to 0^+$ because $|x - x^*| \geq \epsilon$. Hence

$$\int_{\Omega \setminus \cup_{x}(\partial \Omega)} \left( \text{Tr} \left( q_{-2}(t, x, x^*) \right) \right) \, dV = O(t^{1-\frac{n}{2}}) \quad \text{as} \quad t \to 0^+. \quad (4.12)$$

Secondly, for $l \geq 1$, it can be verified that $\text{Tr} \left( q_{-2-l}(x, \xi, \tau) \right)$ is a sum of finitely many terms, each of which has the following form:

$$r_k(x, \xi) = (\tau - \mu \sum_{l,m=1}^{n} g^{lm}_l \xi_l \xi_m) \cdot (\tau - (2\mu + \lambda) \sum_{l,m=1}^{n} g^{lm}_l \xi_l \xi_m)^{j},$$

where $k - 2s - 2j = -2 - l$, and $r_k(x, \xi)$ is the symbol independent of \tau and homogeneous of degree $k$. Again we take the geodesic normal coordinate systems center at $x$ (i.e., $g_{jk}(x) = \delta_{jk}$ and $\Gamma^i_{jk}(x) = 0$), by applying residue theorem we see that, for $l \geq 1$,

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left( \frac{1}{2\pi i} \int_{C} e^{-i\tau \text{Tr} \left( q_{-2-l}(x, \xi, \tau) \right)} d\tau \right) d\xi$$

$$= O(t^{1-\frac{n}{2}}) \quad \text{as} \quad t \to 0^+ \quad \text{uniformly for} \quad x \in \Omega,$$

and

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-x^*) \cdot \xi} \left( \frac{1}{2\pi i} \int_{C} e^{-i\tau \text{Tr} \left( q_{-2-l}(x, \xi, \tau) \right)} d\tau \right) d\xi$$

$$= O(t^{1-\frac{n}{2}}) \quad \text{as} \quad t \to 0^+ \quad \text{uniformly for} \quad x \in \Omega. \quad (4.13)$$

Therefore

$$\int_{\Omega} \left\{ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left( \frac{1}{2\pi i} \int_{C} e^{-i\tau \sum_{l \geq 1} \text{Tr} \left( q_{-2-l}(x, \xi, \tau) \right)} d\tau \right) d\xi \right\} dV$$

$$= O(t^{1-\frac{n}{2}}) \quad \text{as} \quad t \to 0^+, \quad (4.14)$$

and

$$\int_{\Omega} \left\{ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-x^*) \cdot \xi} \left( \frac{1}{2\pi i} \int_{C} e^{-i\tau \sum_{l \geq 1} \text{Tr} \left( q_{-2-l}(x, \xi, \tau) \right)} d\tau \right) d\xi \right\} dV$$

$$= O(t^{1-\frac{n}{2}}) \quad \text{as} \quad t \to 0^+. \quad (4.15)$$

Combining (4.5), (4.11) and (4.14), we have

$$\int_{\Omega} \text{Tr} \left( K(t, x, x) \right) dV = \left[ \frac{n-1}{(4\pi \mu t)^{n/2}} + \frac{1}{(4\pi(2\mu + \lambda)t)^{n/2}} \right] \text{Vol}(\Omega) + O(t^{1-\frac{n}{2}}) \quad \text{as} \quad t \to 0^+. \quad (4.16)$$
Finally, we will consider
\[
\int_{\Omega \cap U_\epsilon (\partial \Omega)} \left\{ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-x^*) \cdot \xi} \left( \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-i\tau \cdot \text{Tr} (q_{-2}(x, \xi, \tau))} d\xi \right) \right\} dV.
\]

We pick a self-double patch $W$ of $M$ (such that $W \subset U_\epsilon (\partial \Omega)$) covering a patch $W \cap \partial \Omega$ of $\partial \Omega$ endowed (see the diagram on p. 54 of [53]) with local coordinates $x$ such that $
abla x_n$ is constant on each geodesic segment in $\bar{\Omega}$ normal to $\partial \Omega$. This has the effect that (see, p. 54 of [53])

\[
g_{jk}(x) = g_{jk}(x_0) = \begin{cases} 
0 & \text{for } j < k = n \text{ or } j < j = n, \\
1 & \text{for } j, k < n \text{ or } j = k = n,
\end{cases}
\]

\[
\sqrt{|g|/g_{nn}} \, dx_1 \cdots dx_{n-1} = \text{the element of (Riemannian) surface area on } \partial \Omega.
\]

We choose coordinates $x' = (x_1, \ldots, x_{n-1})$ on an open set in $\partial \Omega$ and then coordinates $(x', x_n)$ on a neighborhood in $\bar{\Omega}$ such that $x_n = 0$ on $\partial \Omega$ and $|\nabla x_n| = 1$ near $\partial \Omega$ while $x_n > 0$ on $\Omega$ and such that $x'$ is constant on each geodesic segment in $\bar{\Omega}$ normal to $\partial \Omega$. Then the metric tensor on $\bar{\Omega}$ has the form (see [45] or p. 532 of [68])

\[
(g_{jk}(x', x_n))_{n \times n} = \begin{pmatrix} g_{jk}(x', x_n) & 0 \\
0 & 1 \end{pmatrix}.
\]

Furthermore, we can take a geodesic normal coordinate system for $(\partial \Omega, g)$ centered at $x_0 = 0$, with respect to $e_1, \ldots, e_{n-1}$, where $e_1, \ldots, e_{n-1}$ are the principal curvature vectors. As Riemann showed, one has (see p. 555 of [68])

\[
g_{jk}(x_0) = \delta_{jk}, \quad \frac{\partial g_{jk}}{\partial x_l}(x_0) = 0 \text{ for all } 1 \leq j, k, l \leq n - 1,
\]

\[
\frac{\partial g_{jk}}{\partial x_n}(x_0) = \kappa_k \delta_{jk} \text{ for all } 1 \leq j, k \leq n - 1.
\]
where \( \kappa_1 \cdots \kappa_{n-1} \) are the principal curvatures of \( \partial \Omega \) at point \( x_0 = 0 \). Due to the special geometric normal coordinate system and (4.21)–(4.20), we see that for any \( x \in \{ z \in \Omega | \text{dist}(z, \partial \Omega) < \varepsilon \} \),

\[
x - x^* = (0, \ldots, 0, x_n - (-x_n)) = (0, \ldots, 0, 2x_n).
\]

By (3.17), (4.21), (4.9) and (4.22), we find that

\[
\int_{\omega \cap \Omega} \left\{ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x, \xi)} \left( \frac{1}{2\pi i} \int_{C} e^{-i\tau} \text{Tr} \left( q_{-2}(x, \xi, \tau) \right) d\tau \right) d\xi \right\} dV \\
= \int_0^e dx_n \int_{\omega \cap \Omega} \frac{dx'}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(0, \xi') + i2x_n \xi_n} \\
\times \left[ \frac{1}{2\pi i} \int_{C} e^{-i\tau} \left( \frac{n}{(\tau - \mu |\xi|^2)} + \frac{(\mu + \lambda) |\xi|^2}{(\tau - \mu |\xi|^2)(\tau - (2\mu + \lambda) |\xi|^2)} \right) d\tau \right] d\xi' \\
= \int_0^e dx_n \int_{\omega \cap \Omega} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i2x_n \xi_n} \left( n - 1 \right) e^{-i\tau |\xi|^2} e^{i(\mu |\xi|^2 + \lambda t)} d\xi' \\
\times \int_{\mathbb{R}^n} \left( n - 1 \right) e^{-i\tau |\xi|^2} e^{i(\mu |\xi|^2 + \lambda t)} d\xi' \\
= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i2x_n \xi_n} \left( n - 1 \right) e^{-i\tau |\xi|^2} e^{i(\mu |\xi|^2 + \lambda t)} d\xi_n
\]

where \( \xi = (\xi', \xi_n) \in \mathbb{R}^n \), \( \xi' = (\xi_1, \ldots, \xi_{n-1}) \). A direct calculation shows that

\[
\frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} e^{i2x_n \xi_n} e^{-i\mu \xi_n^2} \left( n - 1 \right) e^{-i\tau \sum_{j=1}^{n-1} \xi_j^2} d\xi_n \\
= \frac{1}{(4\pi \mu t)^{n/2}} e^{-\frac{(2\pi \mu t)^2}{4\mu t}}.
\]

Hence

\[
\int_{\omega \cap \Omega} \left\{ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x, \xi)} \left( \frac{1}{2\pi i} \int_{C} e^{-i\tau} \text{Tr} \left( q_{-2}(x, \xi, \tau) \right) d\tau \right) d\xi \right\} dV \\
= \int_0^e dx_n \int_{\omega \cap \Omega} \frac{dx'}{(4\pi \mu t)^{n/2}} e^{-\frac{(2\pi \mu t)^2}{4\mu t}} + \frac{1}{(4\pi \mu t)^{n/2}} e^{-\frac{(2\pi \mu t)^2}{4\mu t}} d\xi'
\]

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- \int_{\epsilon}^{\infty} \, dx_n \int_{W \cap \partial \Omega} \left[ \frac{n - 1}{(4 \pi \mu t)^{n/2}} e^{-\left(\frac{2\pi n}{4\pi \mu t}\right)^2} + \frac{1}{(4 \pi (2 \mu + \lambda) t)^{n/2}} e^{-\left(\frac{2\pi n}{4\pi (2 \mu + \lambda) t}\right)^2} \right] \, dx' \\
= \frac{n - 1}{4} \cdot \frac{\text{Vol}(W \cap \partial \Omega)}{(4 \pi \mu t)^{(n-1)/2}} + \frac{1}{4} \cdot \frac{\text{Vol}(W \cap \partial \Omega)}{(4 \pi (2 \mu + \lambda) t)^{(n-1)/2}} \\
- \int_{W \cap \partial \Omega} \left\{ \int_{\epsilon}^{\infty} \left[ \frac{n - 1}{(4 \pi \mu t)^{n/2}} e^{-\left(\frac{2\pi n}{4\pi \mu t}\right)^2} + \frac{1}{(4 \pi (2 \mu + \lambda) t)^{n/2}} e^{-\left(\frac{2\pi n}{4\pi (2 \mu + \lambda) t}\right)^2} \right] \, dx_n \right\} \, dx'.

It is easy to verify that for any fixed \( \epsilon > 0 \),

\[
\int_{\epsilon}^{\infty} \frac{1}{(4 \pi \lambda t)^{n/2}} e^{-\left(\frac{2\pi n}{4\pi \lambda t}\right)^2} \, dx_n = O(t^{1-n/2}) \quad \text{as} \quad t \to 0^+, \tag{4.24}
\]

From (4.23) and (4.24), we get that

\[
\int_{W \cap \Omega} \left\{ \int_{\mathbb{R}^n} e^{i(x-x', \xi)} \left( \frac{1}{2 \pi i} \int_{C} e^{-i\tau} \text{Tr} \left( q_{-2}(x, \xi, \tau) \right) d\tau \right) \, d\xi \right\} dV \\
= \frac{n - 1}{4} \cdot \frac{\text{Vol}(W \cap \partial \Omega)}{(4 \pi \mu t)^{(n-1)/2}} \\
+ \frac{1}{4} \cdot \frac{\text{Vol}(W \cap \partial \Omega)}{(4 \pi (2 \mu + \lambda) t)^{(n-1)/2}} + O(t^{1-n/2}) \quad \text{as} \quad t \to 0^+. \tag{4.25}
\]

For any \( x \in \Omega \cap U_\epsilon(\partial \Omega) \), we have

\[
\text{Tr} (K(t, x, x')) \\
= \frac{1}{(2 \pi)^n} \int_{\mathbb{R}^n} e^{i(x-x', \xi)} \left( \frac{1}{2 \pi i} \int_{C} e^{-i\tau} \text{Tr} \left( q_{-2}(x, \xi, \tau) \right) d\tau \right) \, d\xi \\
+ \frac{1}{(2 \pi)^n} \int_{\mathbb{R}^n} e^{i(x-x', \xi)} \left( \sum_{l \geq 1} \frac{1}{2 \pi i} \int_{C} e^{-i\tau} \text{Tr} \left( q_{-2-l}(x, \xi, \tau) \right) d\tau \right) \, d\xi \\
= \frac{1}{(2 \pi)^n} \int_{\mathbb{R}^n} e^{i(x-x', \xi)} \left( \frac{1}{2 \pi i} \int_{C} e^{-i\tau} \text{Tr} \left( q_{-2}(x, \xi, \tau) \right) d\tau \right) \, d\xi + O(t^{1+\frac{2}{n}}) \quad \text{as} \quad t \to 0^+. \tag{4.26}
\]

where the second equality used (4.13). Combining (4.25) and (4.26), we have

\[
\int_{W \cap \Omega} \text{Tr}(K(t, x, x')) \, dx \\
= \frac{n - 1}{4} \cdot \frac{\text{Vol}(W \cap \partial \Omega)}{(4 \pi \mu t)^{(n-1)/2}} \\
+ \frac{1}{4} \cdot \frac{\text{Vol}(W \cap \partial \Omega)}{(4 \pi (2 \mu + \lambda) t)^{(n-1)/2}} + O(t^{1-n/2}) \quad \text{as} \quad t \to 0^+. \tag{4.27}
\]
It follows from (4.3), (4.6), (4.12), (4.15), (4.16) and (4.27) that

\[
\int_{W \cap \Omega} \text{Tr}(K^*(t, x, x)) \, dx = \int_{W \cap \Omega} \text{Tr}(K(t, x, x)) \, dx + \int_{W \cap \Omega} \text{Tr}(K(t, x, x^*)) \, dx
\]

\[
= \left[ \frac{n - 1}{(4\pi \mu t)^{n/2}} + \frac{1}{(4\pi (2\mu + \lambda)t)^{n/2}} \right] \text{Vol}(W \cap \Omega)
\]

\[
+ \frac{1}{4} \left( \frac{n - 1}{(4\pi \mu t)(n-1)/2} + \frac{\text{Vol}(W \cap \partial \Omega)}{(4\pi (2\mu + \lambda)t)^{(n-1)/2}} \right)
\]

\[+ O(t^{-n/2}) \quad \text{as} \; t \to 0^+, \quad (4.28)\]

and hence (1.10) holds. \(\square\)

**Remark 4.1** i) It is very clear that our method and results are still valid in the case of Euclidean space. In other words, if \(\Omega\) is a bounded domain in \(\mathbb{R}^n\), and if \(\{r^{-}_k\}\) and \(\{r^{+}_k\}\), respectively, be all the Navier–Lamé eigenvalues corresponding to the Navier–Lamé operator \(P \mathbf{u} = -\mu \Delta \mathbf{u} - (\mu + \lambda) \nabla (\nabla \cdot \mathbf{u})\), with the Dirichlet and Neumann boundary condition, then the asymptotic formula (1.10) still holds, where the \(\text{Vol}(\Omega)\) and \(\text{Vol}(\partial \Omega)\) are to be replaced by the \(n\)-dimensional Euclidean volume \(|\Omega|\) and \((n - 1)\)-dimensional Euclidean volume \(|\partial \Omega|\), respectively.

ii) Note that (see [5, 6, 10, 25, 54] or [29])

\[
\int_{\Omega} \text{Tr}(K^*(t, x, x)) \, dx
\]

\[= (4\pi t)^{-n/2} \left[ a_0 + a_1^+ t^{1/2} + a_2^+ t + \cdots + a_m^+ t^{m/2} + O(t^{(m+1)/2}) \right] \quad \text{as} \; t \to 0^+.\]

Besides the above obtained \(a_0\) and \(a_1^+\), we can also get all coefficients \(a_l^+\), \(2 \leq l \leq m\), for the asymptotic expansion of the integral of trace of integral kernel for the Navier–Lamé operator by our new method.

Now, we use the geometric invariants of the Navier–Lamé spectrum which have been obtained from Theorem 1.1 to finish the proof of Theorem 1.2.

**Proof of Theorem 1.2.** By Theorem 1.1, we know that the first two coefficients \(a_0\) and \(a_1\) of the asymptotic expansion in (1.10) are Navier–Lamé spectral invariants. From the expressions of \(a_0\) and \(a_1\), we can further know that \(|\Omega| = |B_r|\) and \(|\partial \Omega| = |\partial B_r|\). That is, \(|\partial \Omega|/|\Omega| = |\partial B_r|/|B_r|\). Note that for any \(r > 0\), \(|\partial B_r|/|B_r| = |\partial B_1|/|B_1|\). According to the classical isometric inequality (which states that for any bounded domain \(\Omega \subset \mathbb{R}^n\) with smooth boundary, the following inequality holds:

\[
|\partial \Omega|/|\Omega|^{(n-1)/n} \geq |\partial B_r|/|B_r|^{(n-1)/n}.
\]

Moreover, equality obtains if and only if \(\Omega\) is a ball, see [16] or p. 183 of [17]), we immediately get \(\Omega = B_r\). \(\square\)
Remark 4.2 By applying the Tauberian theorem (see, for example, Theorem 15.3 of p. 30 of [42] or p. 446 of [21]) for the first term on the right side of (1.10) (i.e., \( \sum_{k=1}^{\infty} e^{-t \tau_k^2} \int_0^\infty e^{-t \eta} dN^\pm(\eta) = \left[ \frac{n-1}{(4\pi \mu)^{n/2}} + \frac{1}{(4\pi (2\mu + \lambda))^{n/2}} \right] \text{Vol}(\Omega) + o(t^{n/2}) \text{ as } t \to 0^+ \)), we can easily obtain the Weyl-type law for the Navier–Lamé eigenvalues:

\[
N^\pm(\eta) = \max\{k \mid \tau_k^2 \leq \eta\} = \frac{\text{Vol}(\Omega)}{\Gamma\left(\frac{n}{4} + 1\right)} \left[ \frac{n-1}{(4\pi \mu)^{n/2}} + \frac{1}{(4\pi (2\mu + \lambda))^{n/2}} \right] \eta^{n/2} + o(\eta^{n/2}), \quad \text{as } \eta \to +\infty.
\]

Remark 4.3 Note that as \( \lambda \to -\mu \), the Navier–Lamé operator reduces to the classical Laplacian. Therefore, our results recover all corresponding results for the Laplacian by letting \( \lambda + \mu = 0 \) (cf. [60] and [49]).

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