Novel matter coupling in Einstein gravity

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Abstract. A novel type of matter coupling in the Einstein gravity is discovered. This type of matter coupling is theoretically self-consistent in the sense that all constraints in the Hamiltonian are preserved to be first class. At low energy scale it recovers the classic standard predictions of the Einstein gravity minimally coupled with matter content, including the gravitational potential and the equivalence principle, while at the high energy scale much richer phenomenology is granted. We predict a lower bound on the cross section between dark matter particles and Standard Model particles in this framework. The novel matter coupling may provide a resolution to the cosmological singularity problem.
1 Introduction

General relativity (GR) is an extremely simple and elegant theory that describes all known gravitational interactions in 4 dimensional space-time. Its underlying symmetry is the space-time diffeomorphism invariance, which leads to the equivalence principle in the low energy limit. It is precisely the equivalence principle that Einstein began with, embarking himself on what would be an eight-year search for a relativistic theory of gravity. Formulated in the framework of Riemann geometry, the space-time is viewed as a Riemannian manifold, and gravity is interpreted as a geometric property of space-time, rather than an external force known from the Newtonian theory of gravity. The space-time diffeomorphism invariance is, naively speaking, a local gauged version of space-time translation invariance under which all physical variables are transformed. This symmetry must be preserved as we introduce matter content to be coupled to gravity, and the Noether theorem warrants us a conserved energy momentum tensor. The minimal coupling and the BransDicke type of coupling are two well known examples of this kind which manifestly respects the space-time diffeomorphism. It is thus very intriguing to ask whether there exists any new type of matter coupling which also respects the diffeomorphism, but may not in a manifest manner? This is the question that we are trying to address in our current work.

At the action level, the space-time diffeomorphism invariance is manifest when all tensors and vectors contract with metric tensor \( g^{\mu\nu} \) and form scalars. While in the Hamiltonian, the manifestation of the space-time diffeomorphism invariance is exhibited in terms of 8 first
class constraints [1][2], which eliminate 16 degrees in the phase space and, as a consequence, graviton is massless and it has only two polarizations in its spectrum. A self-consistent matter coupling must maintain this appealing algebraic structure of the theory. However, the formulation of GR in the Hamiltonian language is not unique, and one form can be mapped onto another via canonical transformation [3][4]. Therefore, a more basic question that one should ask, prior to the one of novel matter coupling, is whether GR is the unique theory of which all constraints are first class? Or in other words, is there any other theory which is as good as GR in the sense that all constraints are first class, and therefore the structure of the theory is protected by these local gauge symmetries associated to the first class constraints? This question was initially formulated in Ref. [5], in which the first example that seemsly differs from GR, i.e. the so called square-root gravity, was discovered by solving the consistency condition that ensures a closed algebra and all constraints to be first class.

This framework was later extended to the whole class of theories of which the Hamiltonian constraint is written as an arbitrary function $f(H_g)$, where $H_g$ is the Hamiltonian constraint of GR [6] (see also Ref. [7]). On the other hand, the graviton scattering amplitude exhibits a perturbative equivalence between the square-root gravity and GR which holds up to 5 point function level for all possible helicity configurations in the Minkowskian vacuum. This perturbative equivalence delivers two important messages: (1) this whole class of theories is probably just GR in a different guise, which we will explicitly prove by two independent non-perturbative approaches in our current paper; (2) the minimal coupling of the square-root gravity renders the algebra unclosed [6], while a novel matter coupling is required to close it [8]. This implies the existence of a systematic and general framework to couple matter to gravity in a self-consistent manner, while it offers us much richer phenomenology. It is the main objective of our current work to find and study this general framework.

This paper is organized as follows: we will introduce a class of GR equivalents in the section 2. We will discuss how to couple matter to gravity in the self-consistent manner in the section 3. Phenomenology is discussed in the section 4. We conclude in the section 5.

2 A class of GR equivalents

In this section, we introduce a class of GR equivalents, whose equivalence to the Einstein gravity is proved by means of two non-perturbative approaches, the Hamiltonian analysis a la Dirac [1][2], and the equation of motion. We adopt the ADM decomposition, in which the space-time metric reads

$$ds^2 = -N^2 dt^2 + h_{ij} \left( dx^i + N^i dt \right) \left( dx^j + N^j dt \right),$$  

(2.1)

where $h_{ij}$ is the 3-dimensional induced metric, $N$ is the lapse and $N^i$ is the shift.

2.1 Hamiltonian analysis

This class of theories, whose first example was discovered in Ref. [5] and later was extended in Ref. [6] (see also [7]), contains 8 constraints in the Hamiltonian, where 4 of them are primary ones and another 4 of them are secondary ones. All these 8 constraints are first class, i.e. Poisson brackets between any two constraints vanish on the constraint surface. To determine the nature of this class of theories, one of the most direct methods is to find the local gauge symmetry generators associated with all first class constraints. To this end,
we adopt the Hamiltonian analysis approach introduced by Dirac \[1\][2], and write down the following Hamiltonian as our starting point,

\[ H = \int d^3x \left( N\mathcal{H}_0 + N^i\mathcal{H}_i + \lambda_N\pi_N + \lambda_i\pi_i \right), \quad (2.2) \]

where \( \pi_N \) is the conjugate momentum of the lapse \( N \), \( \pi_i \) is the conjugate momentum of the shift \( N^i \), \( \lambda_N \) and \( \lambda_i \) are Lagrangian multipliers that enforce the following 4 primary constraints,

\[ \pi_N \approx 0, \quad \pi_i \approx 0. \quad (2.3) \]

These 4 primary constraints must be preserved by time evolution of the system. The consistency conditions then give us another 4 secondary constraints, they are the Hamiltonian constraint and the momentum constraints,

\[ \mathcal{H}_0 \approx 0, \quad \mathcal{H}_i \approx 0, \quad (2.4) \]

where \( \mathcal{H}_i \approx 0 \) are the momentum constraints,

\[ \mathcal{H}_i \equiv -2\sqrt{h}\nabla_j \left( \frac{\pi_j}{\sqrt{h}} \right), \quad (2.5) \]

where \( h_{ij} \) is the induced 3-metric, \( \pi^{ij} \) is the conjugate momentum of \( h_{ij} \), and \( \nabla_i \) is the covariant derivative compatible to \( h_{ij} \), and the \( \mathcal{H}_0 \approx 0 \) is the Hamiltonian constraint, it can be written as a generic function \( f(H_g) \) of its argument,

\[ \mathcal{H}_0 = \sqrt{h}f(H_g), \quad H_g \equiv R + \frac{\lambda}{h} \left( \pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2 \right), \quad (2.6) \]

where \( R \) is the Ricci scalar of 3-d induced metric. Noted that we have only considered the pure gravity in a vacuum in this section, the inclusion of matter sector will be discussed later. The \( \lambda \) in the eq.(2.6) is an arbitrary constant, it varies if we rescale the space-time coordinates. In the case of GR, we have \( \mathcal{H}_0 = -H_g/2 \) and we set \( \lambda = -4 \) conventionally (so that the speed of light is unity). In the square-root gravity, we have \( \mathcal{H}_0 \sim \sqrt{H_g + \Lambda_1 + \Lambda_2} \), where \( \Lambda_1 \) and \( \Lambda_2 \) are constants. The spatial diffeomorphism invariance is manifest preserved, while the temporal diffeomorphism seems explicitly broken.

Now let us compute all Poisson brackets. In our paper, we adopt the following useful notations,

\[ \mathcal{O}[\alpha] \equiv \int d^3x \alpha \mathcal{O}, \quad \mathcal{O}_i[f^i] \equiv \int d^3x f^i \mathcal{O}_i. \quad (2.7) \]

According with the previous results [6], all brackets are vanishing weakly on the constraint surface,

\[ \{\mathcal{H}_0[\alpha], \mathcal{H}_0[\beta]\} \approx 0, \quad \{\mathcal{H}_0[\alpha], \mathcal{H}_i[f^i]\} \approx 0, \quad \{\mathcal{H}_0[\alpha], \pi_N[\beta]\} \approx 0, \]
\[ \{\mathcal{H}_0[\alpha], \pi_i[f^i]\} \approx 0, \quad \{\mathcal{H}_i[f^i], \mathcal{H}_i[g^i]\} \approx 0, \quad \{\mathcal{H}_i[f^i], \pi_N[\alpha]\} \approx 0, \]
\[ \{\mathcal{H}_i[f^i], \pi_i[g^i]\} \approx 0, \quad \{\pi_N[\alpha], \pi_N[\beta]\} \approx 0, \quad \{\pi_N[\alpha], \pi_i[f^i]\} \approx 0, \]
\[ \{\pi_i[f^i], \pi_N[g^i]\} \approx 0, \quad (2.8) \]
where $\alpha$, $\beta$, $f^i$ and $g^i$ are arbitrary functions that depend on space and time. The algebra closes and therefore all constraints are first class. It implies the existence of a mysterious local gauge symmetry, in addition to the spatial diffeomorphism invariance, prohibits the longitudinal mode of graviton. It turns out that this mysterious local gauge symmetry is nothing but temporal diffeomorphism, which we will prove it in the rest of this subsection.

It is indeed somewhat confusing since the action of this class of theories, which can be obtained via a Legendre transformation, does not look manifestly general covariant, but it is actually equivalent to a theory that fully respects all space-time diffeomorphism, i.e. the Einstein gravity. There is actually a similar example in the literature, where the Einstein gravity is reduced to the BSW action by integrating out the lapse \[10\], which also seemly breaks temporal diffeomorphism. Nevertheless, the Einstein gravity and the BSW action are completely equivalent. In our case, one of the important evidences of the equivalence is that the Hamiltonian constraint and the momentum constraints serve as generators of the space-time diffeomorphism. Noted that our theories are written in the manifestly spatial diffeomorphism invariant manner, thus the momentum constraints simply generate the spatial diffeomorphism,

$$\{h_{ij}, H_i[f^j]\} = \nabla_i f_j + \nabla_j f_i.$$ \hfill (2.9)

Now let’s check the Hamiltonian constraint. Firstly, the Hamiltonian’s equation of motion gives us

$$\dot{h}_{ij} = \{h_{ij}, H\} \approx \partial f \frac{2\lambda N}{\sqrt{h}} \left( \pi_{ij} - \frac{1}{2} \pi h_{ij} \right) + \nabla_i N_j + \nabla_j N_i,$$ \hfill (2.10)

the conjugate momentum evaluates to

$$\lambda \frac{\partial f}{\partial H} \pi_{ij} = K_{ij} - K h_{ij},$$ \hfill (2.11)

where $K_{ij}$ is the extrinsic curvature $K_{ij} \equiv \frac{1}{2\lambda N} (\partial_t h_{ij} - \nabla_j N_i - \nabla_i N_j)$. The conjugate momentum, and thus the gravity theory, become ill defined if $\frac{\partial f}{\partial H}$ vanishes weakly on the constraint surface. Therefore, throughout this paper, we only focus on the theories whose $\frac{\partial f}{\partial H}$ is not vanishing (not even weakly!)! The Hamiltonian constraint, as a first class constraint, generates the following local gauge transformation,

$$\{h_{ij}, H_0[\xi]\} \approx \partial f \frac{2\lambda \xi}{\partial H} \left( \pi_{ij} - \frac{1}{2} \pi h_{ij} \right) = \xi \partial_t h_{ij} - \xi \partial_j N_i - \xi \partial_i N_j,$$ \hfill (2.12)

where the eq. \((2.11)\) has been used, we have absorbed the lapse into the redefinition of $\xi$, and $N_i \equiv N^j h_{ij}$. The temporal diffeomorphism generator is the combination of the Hamiltonian constraint and the momentum constraints, i.e.

$$T[\xi] = \int \xi \left( H_0 + N^i H_i \right) d^3x.$$ \hfill (2.13)

We can check that it does generate the temporal diffeomorphism $t \rightarrow t + \xi(t, x)$,

$$\{h_{ij}, T[\xi]\} \approx \xi \partial_t h_{ij} + N_i \partial_j \xi + N_j \partial_i \xi = L_t h_{ij}.$$ \hfill (2.14)

Therefore, this class of theories whose Hamiltonian is written in the form of eq. \((2.2)\) is equivalent to the GR in a vacuum, in the sense that both of local gauge symmetries are the space-time diffeomorphism invariance and thus the physical observables are unaffected under the diffeomorphism. However, we have to emphasize that it is not yet clear which transformation maps the GR to the theories of eq. \((2.2)\).
2.2 The equation of motion

The equation of motion of graviton is another perspective from which we can see the equivalence between GR and the class of theories written in the eq. (2.2). In this subsection, we will derive the equation of motion and explicitly show this equivalence. Given the Hamiltonian eq. (2.2), the Hamiltonian’s equation of motion is given by the eq. (2.10), and the conjugate momentum of the induced metric $h_{ij}$ is given by the eq. (2.11). After a straightforward computation, we get the equation of motion for graviton,

$$\frac{d}{dt} \left[ \left( \frac{\partial f}{\partial H_g} \right)^{-1} K_{ij} \right] = \left\{ \frac{\lambda}{\sqrt{h}} \left( \pi_{ij} - \frac{1}{2} \pi h_{ij} \right), H \right\}$$

$$= \left( \frac{\partial f}{\partial H_g} \right)^{-1} \left[ 2NK^k_i K_{kj} - NKK_{ij} - \frac{1}{2} Nh_{ij} \left( K^{kl} K_{kl} - K^2 \right) + \mathcal{L}_N K_{ij} \right]$$

$$+ \lambda \frac{\partial f}{\partial H_g} \left( NG_{ij} - \nabla_i \nabla_j N \right)$$

$$- \lambda N \nabla_i \nabla_j \left( \frac{\partial f}{\partial H_g} \right)^{-1} + K_{ij} N^k \nabla_k \left( \frac{\partial f}{\partial H_g} \right)^{-1},$$

(2.15)

where the Lie derivative is directed along the shift $N^i$, i.e.

$$\mathcal{L}_N K_{ij} = N^i \nabla_i K_{ij} + K^l_i \nabla_i N_j + K^l_j \nabla_i N_i,$$

(2.16)

The Hamiltonian constraint is an algebraic equation. Assuming that it has at least one solution, we have then

$$f \left( H_g \right) \approx 0 \quad \rightarrow \quad H_g = \text{constant} \quad \rightarrow \quad \frac{\partial f}{\partial H_g} = \text{constant},$$

(2.17)

and therefore those two terms at the last line of eq. (2.15) vanish, and the equation of motion simplifies to

$$\frac{d}{dt} K_{ij} = 2NK^k_i K_{kj} - NKK_{ij} + \mathcal{L}_N K_{ij}$$

$$+ \lambda \left( \frac{\partial f}{\partial H_g} \right)^2 \left( NR_{ij} - \frac{1}{2} N \Lambda h_{ij} - \nabla_i \nabla_j N \right),$$

(2.18)

where we have used the Hamiltonian constraint $K^{ij} K_{ij} - K^2 = \lambda \left( \frac{\partial f}{\partial H_g} \right)^2 (\Lambda - R)$ to simplify the above equation. Noted that the factor $\left( \frac{\partial f}{\partial H_g} \right)^2$ at the last line of the above equation can be absorbed into a rescaling of $\lambda$ (which amounts to a space-time coordinate rescaling), the factor $\frac{\partial f}{\partial H_g}$ thus drops out of the equation of motion. At the end of the day, the equation of motion for graviton coincides with the one in GR, regardless of the form of $f(H_g)$ (as long as Hamiltonian constraint has at least one real solution). On the other hand, from eq. (2.17) it is easy to see that the Hamiltonian constraint and the momentum constraints coincide with the ones of GR too. We conclude that this class of theories is equivalent to the Einstein gravity also at the equation of motion level, as it should be since the equation of motion is invariant under the space-time diffeomorphism, which has shown to be the local gauge symmetry of the theories in the preceding subsection.
It is quite remarkable that the original motivation, which eventually leads to the discovery of this class of GR equivalents, is to look for the theories which are as good as GR in the sense that all constraints are first class [5]. However, it has turned out that in 4 dimensional space-time, the only theory we have found is the GR itself. Nevertheless, we would like to mention that this unique and distinctive role of GR is still challengeable, on the account of modifying the action principle [9].

3 The self-consistent matter coupling of the GR equivalents

In the last section, we have demonstrated that a class of theories whose Hamiltonian is written in the form of eq. (2.2) is equivalent to GR. Rewriting GR in terms of one of its equivalents, we actually define a new frame in which all physical laws are embedded. It is natural to ask how to couple matter to gravity in the new frame. The simplest one could be the minimal coupling between gravity and matter in the new frame. However, it has turned out that this type of coupling renders the algebra unclosed and the theory becomes inconsistent [6]. In this section, we will develop a systematic and general framework in which matter can couple to gravity in the theoretically self-consistent manner. We will start from a single scalar field, as it is the simplest, and then generalize it to the multi-field as well as the higher spins.

3.1 A consistency condition

Let’s start from the simplest single scalar field. Assuming that the spatial diffeomorphism is still manifestly invariant and thus we have the momentum constraints written as

\[ H_i = -2\sqrt{h} \nabla_j \left( \frac{\pi^j}{\sqrt{h}} \right) + \pi_\phi \nabla_i \phi \approx 0, \]

which also serves as the spatial diffeomorphism generators for both of graviton \( h_{ij} \) and the scalar \( \phi \). Inspired by Ref. [8], we adopt the ansatz that the Hamiltonian constraint is written as an algebraic function of its arguments

\[ H_0 = \sqrt{h} f (H_g, H_m) \approx 0, \]

where \( H_g \) is defined in the eq. (2.6) and \( H_m \) is the would-be Hamiltonian of the scalar field if minimally coupled with the Einstein gravity in the Einstein frame,

\[ H_m \equiv \zeta_1 \frac{\pi_\phi^2}{h} + \zeta_2 \nabla_i \phi \nabla_i \phi + \zeta_3, \quad \zeta_i \equiv \zeta_i(\phi), \]

where \( \zeta_1, \zeta_2 \) and \( \zeta_3 \) are three arbitrary functions of the scalar field \( \phi \). In our general setup, the Hamiltonian constraint is a generic function of its arguments \( H_g \) and \( H_m \). We work out all Poisson brackets as the following:

\[ \{ H_0[\alpha], H_i[f^i] \} \approx 0, \quad \{ H_0[\alpha], \pi_N[\beta] \} \approx 0, \quad \{ H_0[\alpha], \pi_i[f^i] \} \approx 0, \]
\[ \{ H_i[f^i], H_i[g^i] \} \approx 0, \quad \{ H_i[f^i], \pi_N[\alpha] \} \approx 0, \quad \{ H_i[f^i], \pi_i[g^i] \} \approx 0, \]
\[ \{ \pi_N[\alpha], \pi_N[\beta] \} \approx 0, \quad \{ \pi_N[\alpha], \pi_i[f^i] \} \approx 0, \quad \{ \pi_i[f^i], \pi_N[g^i] \} \approx 0, \]

and

\[ \{ H_0[\alpha], H_0[\beta] \} = -2 \int (\beta \nabla^i \alpha - \alpha \nabla^i \beta) \left[ \lambda \left( \frac{\partial f}{\partial H_g} \right)^2 \cdot \sqrt{h} \nabla^i \left( \frac{\pi_{ij}}{\sqrt{h}} \right) + 2\zeta_1 \zeta_2 \left( \frac{\partial f}{\partial H_m} \right)^2 \pi_\phi \nabla_i \phi \right]. \]

(3.5)
Therefore, the only non-trivial Poisson bracket is the one with Hamiltonian constraint commuting with itself, which vanishes weakly on the constraint surface if
\[ \lambda \left( \frac{\partial f}{\partial H_g} \right)^2 \approx -4 \cdot \zeta_1 \zeta_2 \cdot \left( \frac{\partial f}{\partial H_m} \right)^2. \] (3.6)

Noted that the product \( \zeta_1 \cdot \zeta_2 \) must be a constant, otherwise the above self-consistency condition can never be satisfied, given the Hamiltonian constraint in eq. (3.2). Throughout this paper, we fix their relation with \( \lambda \) so that \( \lambda = -4 \zeta_1 \zeta_2 \). This relation can always be realized via either the space-time coordinate rescaling or the variable redefinition \( f(H_m) \rightarrow \tilde{f}(\tilde{H}_m) \), where \( \tilde{H}_m \equiv \text{constant} \cdot H_m \). The condition eq. (3.6) thus acquires a dramatically simple form,
\[ \left( \frac{\partial f}{\partial H_g} \right)^2 \approx \left( \frac{\partial f}{\partial H_m} \right)^2. \] (3.7)

This is the consistency condition of the matter coupling in the GR equivalents, which is one of the main results of our current work.

In passing, we notice that for GR minimally couples to a canonical scalar field in the Einstein frame, we have
\[ -\frac{\partial f}{\partial H_g} = \frac{\partial f}{\partial H_m} = \frac{1}{2}, \quad \zeta_1 = \zeta_2 = 1, \quad \lambda = -4, \] (3.8)

provided the space-time coordinate is properly rescaled and the speed of light is unity. The consistency condition eq. (3.7) is trivially satisfied in this case.

### 3.2 The equivalents of minimal coupling

To find the novel matter coupling, we have to firstly sort out the equivalents of the minimal coupling in the Einstein frame. To simplify our analysis, let’s assume that both of \( \zeta_1 \) and \( \zeta_2 \) are constants and their relation with \( \lambda \) is fixed to be \( \lambda = -4 \zeta_1 \zeta_2 \). As its equivalence will be shown in this subsection, the Hamiltonian for the equivalents of minimal coupling can be written as
\[ H_0 = \sqrt{h} f (H_g + H_m) \approx 0, \] (3.9)

where the consistency condition eq. (3.7) is trivially satisfied. The Hamiltonian constraint, as an algebraic equation, implies that
\[ f (H_g + H_m) \approx 0 \quad \rightarrow \quad H_g + H_m \text{ constant} \quad \rightarrow \quad \frac{\partial f}{\partial H_g} = \frac{\partial f}{\partial H_m} = \text{constant}. \] (3.10)

We can show its equivalence with minimal coupling from the perspective of the equations of motion. The equation of motion for a graviton is derived as
\[
\frac{d}{dt} K_{ij} = 2NK^k K_{kj} - NKK_{ij} + \mathcal{L}_N K_{ij} - \frac{1}{2} Nh_{ij} \left( K^{ab} K_{ab} - K^2 \right) \\
+ \lambda \left( \frac{\partial f}{\partial H_g} \right)^2 \left[ NG_{ij} - \nabla_i \nabla_j N + N \zeta_2 \left( \nabla_i \phi \nabla_j \phi - \frac{1}{2} \nabla_a \phi \nabla^a \phi h_{ij} \right) \right] \\
+ \frac{1}{2N} \zeta_2 h_{ij} \left( \dot{\phi} - N^i \nabla_i \phi \right)^2. \] (3.11)
To reproduce the equation of motion of graviton that minimally couples to a scalar in the Einstein frame, we demand that \( \zeta_2 = 1 \) and rescale the space-time coordinate so that

\[
\lambda \left( \frac{\partial f}{\partial H_g} \right)^2 = -1. \tag{3.12}
\]

On the other hand, the equation of motion for scalar field reads,

\[
\frac{d}{dt} \left[ \frac{\sqrt{h}}{N} \left( \dot{\phi} - N^i \nabla_i \phi \right) \right] = \sqrt{h} \nabla_i \left( N \nabla^i \phi \right) - \frac{1}{2} N \frac{\partial \zeta_3}{\partial \phi} \sqrt{h} + \nabla_i \left[ \frac{N^i}{N} \left( \dot{\phi} - N^i \nabla_i \phi \right) \right] \sqrt{h}, \tag{3.13}
\]

which is exactly the same as the equation of motion of a canonical scalar field minimally coupled to the Einstein gravity in the Einstein frame\(^1\).

### 3.3 The novel non-minimal coupling

We obtain a novel type of non-minimal coupling if the Hamiltonian constraint can not be written in terms of the form of eq. (3.9). In this case, \( \partial f/\partial H_g \) is not a constant anymore, neither is \( \partial f/\partial H_m \). The equation of motion of a graviton is derived as the following,

\[
\frac{d}{dt} \left[ \left( \frac{\partial f}{\partial H_g} \right)^{-1} K_{ij} \right] = \left\{ \frac{\lambda}{\sqrt{h}} \left( \pi_{ij} - \frac{1}{2} \pi h_{ij} \right), H \right\} \nonumber \\
= \left( \frac{\partial f}{\partial H_g} \right)^{-1} \left[ 2NK^k_i K_{kj} - NKK_{ij} - \frac{1}{2} Nh_{ij} \left( K^{kl} K_{kl} - K^2 \right) + \mathcal{L}_N K_{ij} \right] \nonumber \\
+ \lambda \frac{\partial f}{\partial H_g} \left( NG_{ij} - \nabla_i \nabla_j N \right) - \lambda N \nabla_i \nabla_j \left( \frac{\partial f}{\partial H_g} \right) + K_{ij} N^k \nabla_k \left( \frac{\partial f}{\partial H_g} \right)^{-1} \nonumber \\
- \lambda \left[ \left( \frac{\partial f}{\partial H_m} \right)^2 h_{ij} \right] - \zeta_2 N \frac{\partial f}{\partial H_m} \left( \nabla_i \phi \nabla_j \phi - \frac{1}{2} \nabla_a \phi \nabla^a \phi h_{ij} \right) \tag{3.14}
\]

Neither \( \partial f/\partial H_g \) nor \( \partial f/\partial H_m \) are constants, it is thus impossible to recover the equation of motion of the minimal coupling by rescaling \( \lambda, \zeta_1 \) and \( \zeta_2 \).

Let’s give some examples of this kind,

- **Example I**: let’s adopt the ansatz that both of gravity Hamiltonian and matter Hamiltonian can be written in the following power law form,

\[
\mathcal{H}_0 = \xi \sqrt{h} \left[ (A_1 H_g + A_2)^n - (A_3 H_m + A_4)^p \right] \approx 0, \tag{3.15}
\]

where \( \xi = \pm 1 \) and \( \Lambda_i \)'s are constants. The consistency condition eq. (3.7) implies

\[
n = p, \quad \Lambda_1^2 = \Lambda_3^2. \tag{3.16}
\]

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\(^1\)We have \( \zeta_3 = 2V(\phi) \) according to our notations, where \( V(\phi) \) is the analog of the potential in Einstein frame.
- Example II: a variation of the example I,
\[ H_0 = \xi \sqrt{h} \left[ (\Lambda_1 H_g + \Lambda_2 H_m + \Lambda_3)^n - (\Lambda_4 H_m + \Lambda_5)^p \right] \approx 0, \] (3.17)
where \( \xi = \pm 1 \) and \( \Lambda_i' \)'s are constants. Plugging into the consistency condition eq. (3.7), we get
\[ n = p, \quad \Lambda_2^2 = (\Lambda_2 - \Lambda_4)^2. \] (3.18)
Noted that if \( \Lambda_2 = \Lambda_4 \), it leads to \( \Lambda_1 = 0 \), and the gravity theory is ill defined. Both of examples I and II give us the solution \( H_g = \pm H_m + \text{constant} \).

- Example III: the non-monotonous function
\[ H_0 = \xi \sqrt{h} \left[ \sin (\Lambda_1 H_g + \Lambda_2) - \sin (\Lambda_1 H_m + \Lambda_3) \right] \] (3.19)
is also allowed by our self-consistency condition eq. (3.7). The solution to this Hamiltonian constraint is
\[ H_g = H_m + \frac{\Lambda_3 - \Lambda_2 + 2\pi N}{\Lambda_1}, \] (3.20)
where \( N \) is an arbitrary integer. There is a discrete symmetry in both of the matter and the gravity sector, which may have some non-trivial implication on the UV physics. This topic is beyond the scope of our current work and we would like to leave it for future work.

3.4 Multi-scalar fields
The same analysis can be generalized to the case of multi scalar fields. We again assume that the theory is manifestly spatial diffeomorphism invariant, and the momentum constraint is written as
\[ H_i = -2\sqrt{h} \nabla_j \left( \frac{\pi^j_i}{\sqrt{h}} \right) + \sum_I \pi_I \nabla_i \phi^I \approx 0, \] (3.21)
where \( \pi_I \) is the conjugate momentum of the scalar \( \phi^I \). The consistency condition eq. (3.7) is accordingly extended to
\[ \left( \frac{\partial f}{\partial H_g} \right)^2 \approx \left( \frac{\partial f}{\partial H_m^I} \right)^2 \approx \left( \frac{\partial f}{\partial H_m^j} \right)^2, \] (3.22)
where
\[ H_m^I \equiv \zeta_1 \frac{\pi^2_I}{\sqrt{h}} + \zeta_2 I \nabla_i \phi^I \nabla_j \phi^I + \zeta_3 I, \quad -4\zeta_1^2 \zeta_2 = \lambda. \] (3.23)
We offer an example of the multi-field Hamiltonian
\[ H_0 = \Lambda_1 \sqrt{h} \left[ (H_g + \Lambda_2)^{n} - \left( \sum_I H_m^I + \Lambda_3 \right) \right] \approx 0, \] (3.24)
where \( \Lambda_i' \)'s are constants. The non-linear structure of the Hamiltonian implies the inevitable coupling among all components of the matter sector, which may have some important implications on dark matter physics. We will come back to this issue in the next section.
3.5 Scalar QED

As we have learned from the previous subsections, the novel matter coupling requires to modify the matter Hamiltonian (as well as Lagrangian). One of the important questions that we have to answer is whether the gauge symmetries of particle physics still holds. In quantum field theory, the local $U(1)$ gauge symmetry is the most established symmetry underlying the interaction between matter and the electromagnetic field. There are several profound physical implications including the current conservation and the absence of longitudinal mode of photon and so on. We would like to check whether at least the local $U(1)$ gauge symmetry is preserved, if matter couples to the Einstein gravity in the way similar to eq. (3.15). To this end, let’s check the scalar QED. Let’s firstly write down the full Hamiltonian,

$$H = \int d^3x \left( N\mathcal{H}_0 + N^i\mathcal{H}_i + \lambda_N\pi_N + \lambda^i\pi_i + \lambda_0\pi^0_A + A_0\mathcal{G} \right),$$

(3.25)

where $\pi^\mu_A$ is the conjugate momentum of the gauge field $A^\mu$, $\mathcal{G} = -\partial_i\pi^i_A + e(\pi_1\phi_2 - \pi_2\phi_1)$ is the Gauss law ($\mathcal{G}$ stands for Gauss), $\phi_1$ and $\phi_2$ are two components of a complex scalar that charged under $U(1)$, the Hamiltonian constraint reads

$$\mathcal{H}_0 = \Lambda_1 [(H_g + \Lambda_2)^n - (H_{\text{qed}} + \Lambda_3)^n] \approx 0,$$

(3.26)

where $H_g$ and $H_{\text{qed}}$ are the gravity Hamiltonian and the QED Hamiltonian evaluated in the Einstein frame respectively,

$$H_{\text{qed}} = \frac{\pi^i_A\pi^j_Ah_{ij}}{h} + \frac{1}{2}F_{ij}F^{ij} + \frac{\pi_1^2}{h} + \frac{\pi_2^2}{h} + 2eA_i(\phi_1\nabla_i\phi_2 - \phi_2\nabla_i\phi_1) + \nabla_i\phi_1\nabla^i\phi_1$$

$$+ \nabla_i\phi_2\nabla^i\phi_2 + (m^2 + e^2A_iA^i)(\phi_1^2 + \phi_2^2),$$

(3.27)

and the momentum constraint reads

$$\mathcal{H}_i = -2\sqrt{h}\nabla_j \left( \frac{\pi^j_A}{\sqrt{h}} \right) + \pi_1\partial_i\phi_1 + \pi_2\partial_i\phi_2 + \pi^j_AF_{ij} - eA_i(\pi_1\phi_2 - \pi_2\phi_1).$$

(3.28)

We have checked that the Poisson brackets among all constraints, i.e. $\mathcal{H}_0 \approx 0$, $\mathcal{H}_i \approx 0$, $\pi_N \approx 0$, $\pi_i \approx 0$, $\pi^0_A \approx 0$ and $\mathcal{G} \approx 0$, are all vanishing weakly on the constraint surface. Therefore, all constraints are first class, both of the space-time diffeomorphism and the $U(1)$ gauge symmetry are preserved. As we know that first class constraints are associated with the local gauge symmetries, and thus we expect our results hold also at the quantum level, and the Ward-Takahashi identity still protects the masslessness of photons against quantum loop corrections and thus a photon can find no rest. We expect the same is also true for the spinor QED, while the analysis is perhaps more technically involved and thus we are not going to cover it in our current work.

4 Phenomenologies

The novel matter coupling discovered in our current work opens up new possibilities for phenomenological studies. In this section, we will explore some aspects of its phenomenologies, including the spherical static solution, equivalence principle, implication to dark matter physics, and FLRW cosmologies.
4.1 Schwarzschild solution

We now derive the static spherically symmetric solution induced by a point mass, as it is one of the most basic phenomenologies. Let’s take a spherical static ansatz,

\[ ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \]  

(4.1)

The Hamiltonian of gravity is given in the eq. (2.2), and the Einstein tensor are derived in the appendix A. Under the spherical static ansatz, the Einstein tensor reduces to

\[
\begin{align*}
G^{0}_{0} &= -f, & G^{0}_{i} &= 0, & G^{i}_{0} &= 0, \\
G^{i}_{j} &= -\delta^{i}_{j}f + 2\frac{\partial f}{\partial H}R^{i}_{j} - \frac{2}{N}(\nabla^{i}\nabla_{j} - \delta^{i}_{j}\Delta)\left(N\frac{\partial f}{\partial H}\right).
\end{align*}
\]  

(4.2)

The 00 component of the Einstein equation leads to

\[ f (H_{g}) = 0. \]  

(4.3)

it implies that

\[ H_{g} \equiv R + \frac{\lambda}{\hbar} \left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^{2}\right) = \text{constant}. \]  

(4.4)

The constant in the above equation can be canceled out by a bare cosmological constant and we have a Minkowskian solution in the absence of mass. With a point mass included, the static ansatz eq. (4.1) implies the conjugate momentum \(\pi^{ij}\) of 3-d induced metric is vanishing and we have

\[ R = 0, \quad \rightarrow \quad B(r) = \left(1 - \frac{r_{S}}{r}\right)^{-1}, \]  

(4.5)

where the integral constant \(r_{S} = 2GM\) is fixed by the boundary condition at infinity large distance \(B - 1 \rightarrow \frac{2GM}{r}\) as \(r \rightarrow \infty\). Noted that we also have \(\frac{\partial f}{\partial H_{g}} = \text{constant}\) as its argument \(H_{g}\) is a constant, and thus the trace of the \(ij\) component of the Einstein tensor eq. (4.2) implies

\[ \partial_{r} (r^{2}\partial_{r} A) = 0, \]  

(4.6)

and again we have

\[ A(r) = 1 - \frac{r_{S}}{r}, \]  

(4.7)

and \(r_{S} = 2GM\) is fixed by the boundary condition at infinity large distance \(A - 1 \rightarrow -\frac{2GM}{r}\) as \(r \rightarrow \infty\).

It is not surprising that the spherical static vacuum solution coincides with the Schwarzschild solution, because our theory is equivalent to GR in the vacuum. However, new predictions would appear if matter couples to our theory in the novel manner introduced in the last section.
4.2 The equivalence principle

In this subsection, we follow the approach developed in Ref. [11] (see also Ref. [12]) and check whether the gravitational mass is equivalent to the inertial mass at Newtonian limit. Assuming that the gravity Hamiltonian and the matter Hamiltonian are separable, i.e. they can be formally written in terms of \( f(H_g, H_m) \sim f_1(H_g) + f_2(H_m) \), and the modified Einstein tensor and modified energy momentum tensor can be derived in the appendix A.

Here we adopt the original notation \( G_{\mu\nu} \) and \( T_{\mu\nu} \) for the modified Einstein tensor and energy momentum tensor, ones should not be confused. Rewriting the Einstein equation,

\[ G^{(1)}_{\mu\nu} = 8\pi G t^{(2)}_{\mu\nu}, \]  

(4.8)

where \( G^{(1)}_{\mu\nu} \) is the part of the modified Einstein tensor that is first order in metric perturbations, and \( t^{(2)}_{\mu\nu} \) is the pseudo energy-momentum tensor which is defined by

\[ t^{(2)}_{\mu\nu} = T_{\mu\nu} - \frac{1}{8\pi G} G^{(2)}_{\mu\nu}, \]  

(4.9)

and the definition of Newtonian constant \( 8\pi G \) will be clarified later. We consider the non-relativistic Newtonian limit, the metric in the Newtonian gauge is written as

\[ ds^2 = - (1 + 2\Phi) dt^2 + (1 - 2\Psi) \delta_{ij} dx^i dx^j. \]  

(4.10)

Let’s make a series expansion of the Hamiltonian,

\[ f(H_g) \simeq c_0 + c_1 R + \frac{1}{2!} c_2 R^2 + \frac{1}{3!} c_3 R^3 \ldots \]  

(4.11)

where \( c_0 \) is the bare cosmological constant in the gravity Hamiltonian, which is assumed to be cancelled out by the one in the matter Hamiltonian, \( c_1 = \frac{\partial f}{\partial H_g} \bigg|_{H_g=0} \), \( c_2 = \frac{\partial^2 f}{\partial H_g^2} \bigg|_{H_g=0} \) and \( c_3 = \frac{\partial^3 f}{\partial H_g^3} \bigg|_{H_g=0} \). Taking the static limit of eq. (A.7), the left hand side of eq. (4.8) evaluates to

\[ G^{(1)}_{i} = 2c_1 \left( R^i_i - \frac{1}{2} \delta^i_i R \right) - 2 \left( \partial_i \partial_j - \delta^i_j \partial^2 \right) \left( N \frac{\partial f}{\partial H_g} \right), \]  

(4.12)

where \( N = (1 + 2\Phi)^{1/2} \). It is easy to see that \( \partial_{\mu} G^{(1)}_{i} = 0 \) (we ignore all temporal derivatives acting on metric perturbation), and thus we have \( \partial_{i} t^{\mu}_{i} = 0 \). Let us now draw a sphere of radius \( r \) enclosing the extended object, and the momentum is defined as the integral of the pseudo energy-momentum density \( t^0_i \) over the interior of the sphere,

\[ P_i = \int d^3 x t^0_i, \]  

(4.13)

and the Gauss law tells us

\[ \frac{dP_i}{dt} = \int d^3 x \partial_t t^0_i = - \int d^3 x \partial_j t^j_i = - \oint dS_j t^j_i. \]  

(4.14)

Therefore, the gravitational force is given by the integrated momentum flux through the surface. Assuming the energy momentum \( T^\mu_{\nu} \) flux through the surface is negligible, we need
to consider only the contribution to \( t^j_i \) from \( G^{(2)}_{\mu} \), whose leading order in the perturbative expansion is given by

\[
G^{(2)}_{\mu} \simeq 2c_1 \cdot [\partial_i \Phi \partial_j \Phi - \partial_i \Phi \partial_j \Psi - \partial_j \Phi \partial_i \Psi + 3\partial_i \Psi \partial_j \Psi + 2\Phi \partial_i \partial_j \Phi + 2\Psi \partial_i \partial_j \Psi \\
- \partial_k \Phi \partial_i \partial_j \Phi - 2\partial_k \Psi \partial_i \partial_j \Psi - 2\partial^2 \Phi \delta_{ij} - 2\partial^2 \Psi \delta_{ij}]
\]

\[
+ c_2 \cdot [ -8\partial_i \partial_j \Phi \partial^2 \Psi - 24\partial_i \partial_j \Psi \partial^2 \Psi - 24\partial_i \partial_j \Psi \partial_i \partial_j \Psi - 8\partial_j \Phi \partial_i \partial^2 \Psi \\
- 8\partial_i \Phi \partial_j \partial^2 \Psi - 40\partial_i \Phi \partial_j \partial^2 \Psi - 40\partial_i \Psi \partial_j \partial^2 \Psi - 24\partial_i \Psi \partial_j \partial_i \partial_j \Psi \\
- 32\partial \partial_i \partial_j \partial^2 \Psi + \frac{45}{2} \partial_k \partial_i \Phi \partial_k \partial_j \Psi \delta_{ij} + 8\partial^2 \Phi \partial^2 \Psi \delta_{ij} + \frac{65}{2} \partial^2 \Psi \partial^2 \Psi \delta_{ij} \\
+ 16 \partial_i \Phi \partial_k \partial^2 \Psi \delta_{ij} + 88 \partial_i \Phi \partial_k \partial^2 \Psi \delta_{ij} + 32 \partial^4 \Psi \delta_{ij}]
\]

\[
+ c_3 \cdot [ -6\partial_i \partial_j \Phi \partial_j \partial_i \Psi - 30 \partial_i \partial^2 \Psi \partial_j \Psi - 6 \partial_k \partial_i \Phi \partial_j \partial_l \partial_k \partial_l \Psi \\
- 30 \partial^2 \Psi \partial_i \partial_j \partial^2 \Psi + 6 \partial_k \partial_i \partial_m \Psi \partial_k \partial_i \partial_m \Psi \delta_{ij} + 30 \partial^2 \Psi \partial_k \partial^2 \Psi \delta_{ij} \\
+ 30 \partial^2 \Psi \partial^2 \Psi \delta_{ij} + 6 \partial_k \partial_i \Phi \partial_k \partial_j \partial^2 \Psi \delta_{ij}]
\]

where \( \partial^2 \equiv \delta^{ij} \partial_i \partial_j \). Now let’s decompose \( \Phi \) and \( \Psi \) on this sphere,

\[
\Phi = \Phi_0 + \Phi_1(r), \quad \Psi = \Psi_0 + \Psi_1(r), \quad (4.16)
\]

where \( \Phi_1 \) and \( \Psi_1 \) are the gravitational field that generated by the extended object itself, and thus we have

\[
\Phi_1, \Psi_1 \simeq -G M/r \quad (4.17)
\]

as long as the size of sphere is much greater than the size of the extended object. On the other hand \( \Phi_0 \) and \( \Psi_0 \) are linear gravitational fields which can be added into the spherical solution (because the equation is at least second order in spatial derivatives),

\[
\Phi_0(x) = \Phi_0(0) + \partial_i \Phi_0(0)x^i, \\
\Psi_0(x) = \Psi_0(0) + \partial_i \Psi_0(0)x^i. \quad (4.18)
\]

We assume that \( \partial_i \Phi_0(0) \) and \( \partial_i \Psi_0(0) \) are constants on the scale of the sphere.

Given the eq. (4.17) and the eq. (4.18) we have \( R = 0 \) around the sphere and thus we have \( \partial f/\partial H_g = \text{constant} \). Therefore, the constant \( \partial f/\partial H_g \) in eq. (4.12) can be factored out and the effective Planck mass, or equivalently the inverse of the gravitational coupling constant in the Newtonian limit, is defined by

\[
M_p^2 \equiv \frac{1}{8\pi G} \equiv 2 \frac{\partial f}{\partial H_g} \bigg|_{H_g=0}. \quad (4.19)
\]

At the linear order in the series expansion of eq. (4.11), the rest of analysis goes exactly the same as the one in Ref.\[12\]. To be self-complete, let us finish rest of the analysis in the following. Plugging the eq. (4.16) into eq. (4.15), noted that only cross-terms between the background gravitational fields and the gravitational fields generated by object itself survive the surface integration, \( \oint dS_j \partial_i \Phi_0(0) \partial_j \Phi_0 \) vanishes because we have assumed that both of \( \partial_i \Phi_0(0) \) and \( \partial_i \Psi_0(0) \) are constants, \( \oint dS_j \partial_i \Phi_1 \cdots \partial_j \Psi_1 \) vanishes because both of \( \Phi_1 \) and \( \Psi_1 \) are spherical symmetric. A term contains \( \partial^2 \Phi \) (or \( \partial^2 \Psi \)) also vanishes because \( \partial^2 \Phi = \partial_r (r^2 \partial_r \Phi_1) = 0 \). At the leading order in the gradient expansion of the eq. (4.15), we recover the GR prediction in the Newtonian limit,

\[
\dot{P}_t = -M \partial_t \Phi_0. \quad (4.20)
\]
The center of mass of the object is defined by

\[ X^i \equiv - \int d^3x x^i t_0^0 / M, \]  

(4.21)

and the motion of the object (under the constant mass approximation) is given by

\[ M \dot{X}^i = \int d^3x \partial_j \left( x^i t_0^j \right) - t_0^i = - \int d^3x t_0^i = P^i \]  

(4.22)

Taking one more temporal derivative we get the Newtonian second law,

\[ M \ddot{X}^i = - M \partial_i \Phi_0, \]  

(4.23)

the inertial mass equals to the gravitational mass and the mass of the extended object drops out of the geodesic equation and the equivalence principle holds.

The eq. (4.23) only holds at the leading order in the gradient expansion. The corrections come from the higher order terms in the gradient expansion of the eq. (4.15). The gravitational force receives a correction from terms such as \( \oint dS_k \partial^k \Psi \partial^i \partial^j \partial^k \Psi \) and it gives the gravitational potential the following correction,

\[ M \ddot{X}^i = - M \left[ \partial_i \Phi_0 + \mathcal{O} \left( \frac{r^2_{NL}}{r^2} \right) \right], \]  

(4.24)

where \( r_{NL} \) is the length scale that non-linearity of the theory becomes important. If we equate \( \Psi \) and \( \Phi \) by completely fixing the gauge, the ratio between the inertial mass and the gravitational mass reads

\[ \frac{\text{inertial mass}}{\text{gravitational mass}} = 1 + \mathcal{O} \left( \frac{r^2_{NL}}{r^2} \right). \]  

(4.25)

Generally we would expect the \( r_{NL} \) is at atomic or even sub-atomic scale to suppress all non-standard predictions arising from the non-linear structure of the matter Hamiltonian \( f(H_m) \), and thus the violation of the equivalence principle is too small to be detected.

4.3 DM-SM cross section

One of the features is that dark matter inevitably couples to standard model particle due to the non-linear structure of the theory. Our matter coupling predicts a lower bound on the cross section between the dark matter (DM) particles and the standard model (SM) particles. Let’s consider a simplified toy model with two spin-0 fields \( \phi \) and \( \chi \), where \( \phi \) is a scalar field from the standard model of particle physics, and \( \chi \) is the scalar field from the dark sector. The would-be Hamiltonian in the case of minimal coupling is

\[ H_m = \frac{1}{2} \frac{\pi^2}{h} + \frac{1}{2} \nabla_i \phi \nabla^i \phi + \frac{1}{2} m_\phi^2 \phi^2 + \frac{1}{2} \frac{\pi^2}{h} + \frac{1}{2} \nabla_i \chi \nabla^i \chi + \frac{1}{2} m_\chi^2 \chi^2, \]  

(4.26)

where we have assumed that there is no direct coupling between the dark particle \( \chi \) and the standard model particle \( \phi \) at this level, since we are interested in the lower bound of the cross section. However, if the matter sector couples to the Einstein gravity in the novel manner, the analysis in the section (3.4) implies that the Hamiltonian can be an arbitrary function...
of $f(H_g, H_m)$, given the consistency condition eq. (3.22) is satisfied. A series expansion of the matter sector of the Hamiltonian

$$f(H_m) \approx f(0) + \frac{\partial f}{\partial H_m} \bigg|_{H_m=0} H_m + \frac{1}{2!} \frac{\partial^2 f}{\partial H^2_m} \bigg|_{H_m=0} H^2_m + \ldots$$  \hspace{1cm} (4.27)

implies that a direct coupling in the form of

$$\frac{m^2_m \phi^2}{\Lambda^4_{NL}} \chi^2$$  \hspace{1cm} (4.28)

is inevitably introduced due to the non-linear structure of the theory, where $\Lambda_{NL}$ is the energy scale above which the non-linear terms comes into play. One may argue that this mixing can be cancelled out by introducing a direct coupling in eq. (4.26), with an opposite sigh and a fine-tuned coupling constant. However, this fine-tuning is not protected by any symmetries and thus unnatural even in the technical sense. Without the fine-tuning and the accidental cancellation, the lower bound on the cross section between DM particle and SM particle is roughly estimated as

$$\sigma \sim \begin{cases} \frac{m^4_m \phi^2}{\Lambda^4_{NL}} & \text{if } m^2_\chi > m^2_\phi \\ \frac{m^2_\chi \phi^2}{\Lambda^4_{NL}} & \text{if } m^2_\chi < m^2_\phi \end{cases}$$  \hspace{1cm} (4.29)

This lower bound still exists even if one of the fields is massless. In this case, the leading effect comes from the kinetic and gradient mixing between DM and SM particles.

### 4.4 FLRW cosmology

We have learned that the vacuum solution coincides with the Schwarzschild solution, because the gravity theory is equivalent to the Einstein gravity in a vacuum. However, generally we would expect non-standard solution if matter couples to the Einstein gravity in the novel way. Cosmology provides us an important arena where the physical effects of the novel coupling come into play. Let’s take the theory

$$\mathcal{H}_0 = \xi [(\Lambda_1 H_g + \Lambda_2)^n - (\Lambda_1 H_m + \Lambda_2)^n] \approx 0,$$  \hspace{1cm} (4.30)

as an example, and adopt the FLRW ansatz,

$$ds^2 = -dt^2 + a^2 dx^2.$$  \hspace{1cm} (4.31)

The Hubble constant $H \equiv \dot{a}/a$ evaluates to

$$H^2 \sim H_m (\Lambda_1 H_m + \Lambda_2)^{2n-2}.$$  \hspace{1cm} (4.32)

At high energy limit $H_m \to \infty$, the Hubble constant $H$ approaches to a finite constant if $n = 1/2$, or zero if $n < 1/2$. The cosmological perturbation analysis in the square root gravity, where $n = 1/2$, reveals a weak coupling between the Einstein gravity and the matter sector [8], which may provide us a natural solution to the cosmological singularity problem [13].
5 Conclusion and discussion

We discover a novel matter coupling in the Einstein gravity, yet the path towards this discovery is somewhat tortuous. The original motivation was to look for a theory which is as good as the Einstein gravity in the sense that all constraints are first class. However, in 4 dimensional space-time the only theory that we found is just the Einstein gravity in different guises: replacing the Hamiltonian of the Einstein gravity \( H_g \) by an arbitrary function \( f(H_g) \), all constraints are still first class but the theory is equivalent to the Einstein gravity. The equivalence is demonstrated in our current work by means of two non-perturbative approaches. The first one is the Hamiltonian analysis, where we have found that all constraints are first class and they serve as the space-time diffeomorphism generators; the second one is the equation of motion, where we have found that the equation of motion for graviton in a vacuum coincides with the one of GR.

By rewriting the Einstein gravity in terms of one of its equivalents we actually define a new frame. The theoretical consistency requires all constraints must be preserved to be first class when matter couples to gravity in this new frame, which subjects to a self-consistency condition that derived in the eq. (3.7). This self-consistency condition implies the Hamiltonian of both gravity and matter can have very complicated non-linear structure, which grants very rich new phenomenologies. We have worked out some classical examples, including the spherical static solution, the equivalence principle, the DM-SM cross section, as well as the FLRW cosmology. The standard predictions are recovered at low energy scale, while new phenomenologies are granted at high energy scale. We find that the vacuum solution is just the Schwarzchild solution, as it should be since our theories are equivalent to GR. The equivalence principle holds in the Newtonian weak field limit, at the leading order in the gradient expansion. Dark matter particles inevitably couple to standard model particles, due to the non-linear structure of the matter Hamiltonian. A lower bound on the cross section between dark matter particles and standard model particles is derived. On cosmological background, the novel matter coupling may also provide a resolution to the cosmological singularity problem.

An interesting lesson we have learned is that sometimes the local gauge symmetries still exist, even the action is not gauge invariant. The essence of the local gauge symmetry is that physical observables are invariant under local gauge transformations, however, the action itself is not a physical observable. In some cases, for instance the examples demonstrated in our current work, the local gauge symmetries are hidden and we have to go through all Poisson brackets to find them.

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A Modified Einstein tensor

The energy momentum tensor and the Einstein tensor derivations are not that straightforward if the theory is written in terms of ADM variables, rather than metric tensor $g_{\mu\nu}$. We will discuss how to derive these tensors in this appendix. The metric tensor written in terms ADM variables reads

$$g^{00} = -\frac{1}{N^2}, \quad g^{0i} = \frac{N^i}{N^2}, \quad g^{ij} = h^{ij} - \frac{N^iN^j}{N^2}. \quad (A.1)$$

we have then

$$\frac{\partial S}{\partial N} = \frac{\partial S}{\partial g^{00}} \frac{\partial g^{00}}{\partial N} + 2 \frac{\partial S}{\partial g^{0i}} \frac{\partial g^{0i}}{\partial N} + \frac{\partial S}{\partial g^{ij}} \frac{\partial g^{ij}}{\partial N},$$

$$\frac{\partial S}{\partial N^i} = \frac{2}{\partial g^{0j}} \frac{\partial g^{0j}}{\partial N^i} + \frac{\partial S}{\partial g^{kl}} \frac{\partial g^{kl}}{\partial N^i},$$

$$\frac{\partial S}{\partial h^{ij}} = \frac{\partial S}{\partial g^{kl}} \frac{\partial g^{kl}}{\partial h^{ij}}. \quad (A.2)$$

Reversing eqs. (A.2) we get

$$\frac{\partial S}{\partial g^{00}} = \frac{N^3}{2} \left[ \frac{\partial S}{\partial N} + \frac{2N^i}{N} \frac{\partial S}{\partial N^i} + \frac{2N^iN^j}{N^3} \frac{\partial S}{\partial h^{ij}} \right],$$

$$\frac{\partial S}{\partial g^{0i}} = \frac{N^2}{2} \left( \frac{\partial S}{\partial N^i} + \frac{2N^j}{N^2} \frac{\partial S}{\partial h^{ij}} \right),$$

$$\frac{\partial S}{\partial g^{ij}} = \frac{\partial S}{\partial h^{ij}}. \quad (A.3)$$

The action of this class of GR equivalents can be written in the first order form,

$$S = \int \pi^{ij} \partial_t h^{ij} - N \sqrt{h} f (H_g) + 2N^i \sqrt{h} \nabla_j \left( \frac{\pi^i}{\sqrt{h}} \right), \quad (A.4)$$

where

$$H_g \equiv R + \frac{\lambda}{h} \left( \pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2 \right). \quad (A.5)$$

Taking the variation of the action with respect to the conjugate momentum $\pi^{ij}$, we get the relation between conjugate momentum and extrinsic curvature tensor,

$$\lambda \frac{\partial f}{\partial H_g} \pi^{ij} = K_{ij} - K h_{ij}. \quad (A.6)$$

According to the eq. (A.3), we get the modified Einstein tensor,

$$G^{00} = \frac{1}{\sqrt{h}} \frac{\partial S}{\partial N} + \frac{N^i}{N \sqrt{h}} \frac{\partial S}{\partial N^i} = -f + \frac{2N^i}{N} \nabla_j \left( \frac{\pi^i}{\sqrt{h}} \right), \quad (A.7)$$
\[ G^0_i = \frac{1}{N^2 \sqrt{h}} \frac{\partial S}{\partial N^i} = \frac{2}{N^2} \nabla_j \left( \frac{\pi^j}{\sqrt{h}} \right), \quad (A.8) \]

\[ G^i_0 = -\frac{N^i}{\sqrt{h}} - \frac{N h^i}{\sqrt{h}} \left( \frac{\partial S}{\partial N^j} + \frac{2 N^k}{N^2} \frac{\partial S}{\partial h^{jk}} \right), \]

\[ = -2N \nabla_j \left( \frac{\pi^j}{\sqrt{h}} \right) + 2N^j \frac{\partial f}{\partial H_g} R_{ij} - \frac{2N^j}{N} \nabla^i \nabla_j \left( \frac{N \frac{\partial f}{\partial H_g}}{N} \nabla^j \right) + \frac{2N^i}{N} \nabla^j \left( \frac{N \frac{\partial f}{\partial H_g}}{N} \nabla^j \right) \]

\[ + \frac{2N^i}{N} \frac{\partial f}{\partial H_g} \lambda \left( \pi^i \pi^j - \frac{1}{2} \pi^2 \right) + \frac{2N^k}{N} \frac{\partial f}{\partial H_g} \lambda \left( -2\pi^{ij} \pi_{jk} + \pi^i \pi_k \right) - \frac{2N^i}{N} \nabla^j \pi^j + \frac{2N^i}{N} \nabla^j \nabla^k N^i, \quad (A.9) \]

\[ G^i_j = -\frac{N^i}{N \sqrt{h} \frac{\partial S}{\partial N^j}} - \frac{2h^{ik}}{N \sqrt{h} \frac{\partial S}{\partial h^{kj}}} \]

\[ = -\frac{2N^i}{N} \nabla_k \left( \frac{\pi^k}{\sqrt{h}} \right) - \frac{2\pi^{ik} h_{kj}}{N \sqrt{h}} \delta^i_j f + \frac{2N^j}{N} \frac{\partial f}{\partial H_g} R^i_j - \frac{2N^i}{N} \left( \nabla^i \nabla_j - \delta^i_j \Delta \right) \left( \frac{N \frac{\partial f}{\partial H_g}}{N} \nabla^j \right) \]

\[ + \frac{2N^i}{N} \frac{\partial f}{\partial H_g} \lambda \left( \pi^i \pi^j - \frac{1}{2} \pi^2 \right) + \frac{2N^j}{N} \frac{\partial f}{\partial H_g} \lambda \left( -2\pi^{ik} \pi_{kj} + \pi^i \pi_k \right) \]

\[ - \frac{2N^i}{N \sqrt{h}} \left( \pi^i \nabla_k N^j + \pi^j \nabla_k N^i \right). \quad (A.10) \]

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