Many body approach to quartet condensation in
nuclear matter

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Abstract. A self-consistent theory for quartet condensation in homogeneous infinite nuclear matter is proposed. It goes along similar lines as Gorkov theory for pairing. However, details and physical consequences are very different. For example in the quartet case no sharp quasi-particle pole develops in the single particle propagator. Also quartet condensation only exists in the BEC phase and no extended coherence length develops in weak coupling. The critical temperature for quartet condensation is also investigated in analogy to the Thouless criterion for pairing.

In attractive multi-component fermion systems there exists the possibility of bosonic multi-fermion cluster condensation: for example, in a four component fermion system, like nuclear matter, condensation of a four body bound state, the \( \alpha \) particle, can occur because four different nucleons can be bound only with \( s \)-wave coupling. On the other hand, a four body bound state in a two component system is not energetically favourable because \( p \)-wave coupling must be involved. In a finite nucleus, the Hoyle state, which is an excited state of \( ^{12}\text{C} \), is considered as the “condensation” state of three \( \alpha \) particles in the sense that three \( \alpha \)-particles occupy a single \( S \)-state in the mean field potential [1].

In this work we construct the quartet version in analogy to the BCS pairing theory. The BCS theory describes the condensation of the fermion pairs, where the order parameter is the expectation value of two fermion annihilation operators. We shall develop a BCS-like theory for quartet condensation where the order parameter is represented by the expectation value of four fermion annihilation operators. We apply it to \( \alpha \)-condensation in symmetric nuclear matter [2].

Let us first briefly recap the usual BCS gap equation. To derive the BCS gap equation, we need two equations. One is the single fermion mass operator: 

\[
M^\text{BCS}_1(\omega) = \sum_{22'} \frac{1}{2} \bar{v}_{1,2,2'} \langle c_3 c_2 \rangle \frac{1}{\omega + (\epsilon_1 - \mu)} \langle c_2^\dagger c_1^\dagger \rangle \frac{1}{2} \bar{v}_{1,2,2'},
\]

where \( c_i, c_i^\dagger \) are the fermion annihilation and creation operators with single particle quantum numbers \( i \), i.e. momentum, spin, and isospin in case of nuclear matter, and the single particle kinetic energy \( \epsilon_1 = \frac{k^2}{2m} \), the chemical potential \( \mu \), antisymmetrised two-body vertex \( \bar{v}_{12,34} \), and \( \langle ... \rangle \) being a thermal average. Mean field shifts are supposed to be incorporated in an approximate way into the chemical potential. In above expression, \( \bar{1} \) means the time reversal conjugate of the state 1. The other equation which needs to be derived for the BCS gap equation is the in-medium two body Schrödinger equation:
the imaginary part of the noninteracting three hole Green function: momentum conservation. This has strong physical consequences.

The middle of the quartet mass operator. The BCS mass operator for pairing has just one hole quartet mass operator (1) has a similar structure as the BCS mass operator. However, there with the Fermi distribution function

occupation number is derived from the one body Green function with the BCS mass operator. With these formulas and considerations, we can derive the BCS gap equation. We need analogous equations for the quartet case.

In analogy to BCS theory, one can show that the single fermion mass operator with quartet condensation has the following form [2]

\[ M_{\text{Quartet}}^{1}(\omega) = \sum_{2341'2'1'2'} \frac{1}{2} \bar{v}_{12,1'2'} \langle c_{4}c_{3}c_{2}c_{1'} \rangle \frac{f_{2}f_{3}f_{4} + f_{2}f_{3}f_{4}'}{\omega + \varepsilon_{2} + \varepsilon_{3} + \varepsilon_{4} - 3\mu} (c_{1'}^{+}c_{2'}^{+}c_{3'}^{+}) \frac{1}{2} \bar{v}_{12,1'2'} \] (1)

with the Fermi distribution function \( f_{1} = \frac{1}{e^{(\omega - \mu)/T} + 1} \) and \( \bar{f}_{1} = 1 - f_{1} \). It is seen that the quartet mass operator (1) has a similar structure as the BCS mass operator. However, there exist crucial differences. One is the Pauli blocking factor of the three hole Green function in the middle of the quartet mass operator. The BCS mass operator for pairing has just one hole propagator. This has as a consequence that in the quartet mass operator the indices in the three hole Green function are summed, while there is no sum in the BCS mass operator due to momentum conservation. This has strong physical consequences.

Let us consider in detail. Figure 1 shows the three hole level density, which represents the imaginary part of the noninteracting three hole Green function: \( g(\omega) = \sum_{2341'2'1'2'} \bar{v}_{12,1'2'} \langle c_{4}c_{3}c_{2}c_{1'} \rangle \frac{f_{2}f_{3}f_{4} + f_{2}f_{3}f_{4}'}{\omega + \varepsilon_{2} + \varepsilon_{3} + \varepsilon_{4} - 3\mu} (c_{1'}^{+}c_{2'}^{+}c_{3'}^{+}) \frac{1}{2} \bar{v}_{12,1'2'} \). Here we consider zero temperature \( T = 0 \), i.e. the Fermi distribution function becomes the step function, \( f_{1} = \theta(\mu - \varepsilon_{1}) \).

For positive chemical potential, see right and middle panels in fig. 1, the level densities become zero at \( \omega = 0 \), i.e. at the Fermi surface, what stems from the the Pauli blocking factor \( (f_{2}f_{3}f_{4} + f_{2}f_{3}f_{4}) \). This means that there are no correlations in the quartet case for positive chemical potential because the quartet correlations (as the pair correlations) arise at the Fermi surface. Therefore, we can predict that there is no quartet condensation for positive chemical potential. On the contrary, the level density of the one hole Green function in the BCS mass operator has no zero at the Fermi surface. Therefore, the BCS state occurs even for the very small attractive interaction and exists for positive chemical potential. This is a very special case which does not repeat itself for any cluster higher than two body. This is the crucial difference between BCS and quartet cases. For negative chemical potential, the three hole Pauli blocking factor is just one. So in the BEC phase, there is no qualitative difference between pairing and quartetting.

We also derived the in-medium four body Schrödinger equation in analogy to the pair case

\[ (\varepsilon_{1} + \varepsilon_{2} + \varepsilon_{3} + \varepsilon_{4}) \langle c_{1}c_{2}c_{3}c_{4} \rangle + (1 - \rho_{1} - \rho_{2}) \sum_{1'2'} \frac{1}{2} \bar{v}_{12,1'2'} \langle c_{1'}c_{2'}c_{3}c_{4} \rangle + \text{perm.} = 4\mu \langle c_{1}c_{2}c_{3}c_{4} \rangle. \] (2)
where occupation numbers again are obtained from the one body Green function with the quartet mass operator (1). In principle the quartet order parameter is computed iteratively by using eqs. (1) and (2) and the occupation numbers are obtained from the one body Green function in a self-consistent cycle quite similar to the BCS case described above. However, to solve in medium four body equations self consistently is out of reach for the moment. To simplify the numerics, we use the following ansatz for the quartet order parameter which was very successful for the calculation of the critical temperature for $\alpha$ particle condensation (see below):

$$\langle c_1 c_2 c_3 c_4 \rangle = \phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\chi_0(2\pi)^3\delta(k_1 + k_2 + k_3 + k_4),$$

i.e. it is the Hartree-Fock wave functions projected to zero c.o.m. momentum. $\chi_0$ is the spin-isospin singlet wave function. Besides, we use a separable two body interaction with the parameters adjusted to get the binding energy ($-28.3\text{MeV}$) and the rms radius of the free $\alpha$-particle (1.71fm). In this way the problem becomes much simpler because instead of the full four body correlation function, one only has to iterate one single particle S-state wave function.

Figure 2 shows the occupation numbers for various values of the chemical potential at zero temperature. The occupation numbers in fig. 2 are far from step functions but one also get this kind of occupation numbers in BCS theory for negative chemical potential. Note that the order parameter goes to zero above $\mu = 0.55\text{MeV}$, i.e. no condensation appears beyond that region where the density is $0.029\text{fm}^{-3}$, i.e. the mean $\alpha$ particle distance is $(0.029)^{-1/3} = 5.2\text{fm}$. Therefore, the $\alpha$ condensation breaks down at very low density compared with the rms radius of the free $\alpha$ particle implying that the $\alpha$’s are barely overlapping with their density tails. This demonstrates the dramatic difference with the pairing case. For quartets only the BEC phase exists whereas pairs (at rest) can be entangled as much as one can think of without losing their coherence.
Figure 4. The critical temperature for alpha condensation (solid line) and deuteron condensation (dashed line) as functions of the chemical potential (left) and free nucleon density (right). The density is calculated by $n^{(0)} = 4 \int d^3kf(k, T_c)/(2\pi)^3$ with $f(k, T_c) = [e^{(k^2/(2m)-\mu)/T_c} + 1]^{-1}$. Crosses are calculated with the Faddeev-Yakubovsky method using a realistic nuclear force.

Figure 3 shows the imaginary part of the mass operator at zero temperature as a function of energy for several momentum values. It is seen that there exists a considerable width of the single particle states which comes from the above mentioned integration over the three hole momenta. Therefore, the quasi-particle picture is not at all valid in the case of quartet condensation. On the other hand, the imaginary part of the BCS mass operator is sharp because the momentum in the one hole Green function in the BCS mass operator is fixed by momentum conservation.

For the calculation of the critical temperature, one has to linearise the in-medium four body Schrödinger equation (2) with respect to the order parameter, i.e. one makes the change $\rho_i \rightarrow f_i$ [3]. This is in analogy to the Thouless criterion for pairing.

Figure 4 shows the critical temperature in symmetric nuclear matter. The solid lines are for $\alpha$-condensation calculated with the product ansatz for the order parameter. Crosses are calculated by the Faddeev-Yakubovsky method with a realistic nuclear force. As seen in fig. 4, the agreement with the mean field values are very good for negative chemical potentials, the only domain where the numerical solution of the realistic four body problem was possible. Therefore, the product ansatz is a very valid approximation in the bound state domain. We compared with the case of deuteron condensation. The $\alpha$ condensation has higher critical temperature, but breaks down at much lower density than in the deuteron case. Note that we also investigated the critical temperature in asymmetric nuclear matter [4], where it is revealed that for highly imbalanced nuclear matter the $\alpha$-condensation has higher critical temperature and survives until higher density than in the case of deuteron BEC.

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