ON THE LEVEL OF A CALABI–YAU HYPERSURFACE

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Abstract. Boix–De Stefani–Vanzo defined the notion of level for a smooth projective hypersurface over a finite field in terms of the stabilisation of a chain of ideals previously considered by Álvarez-Montaner–Blickle–Lyubeznik, and showed that in the case of an elliptic curve the level is 1 if and only if it is ordinary and 2 otherwise. Here we extend their theorem to the case of Calabi–Yau hypersurfaces by relating their level to the $F$-jumping exponents of Blickle–Mustaţă–Smith and the Hartshorne–Speiser–Lyubeznik numbers of Mustaţă–Zhang.

1. Introduction

Let $R$ be a finitely-generated $k$-algebra where $k$ is a perfect field of characteristic $p > 2$ and let $F^e: R \to R$ denote the Frobenius morphism $a \mapsto a^{p^e}$. Let $F^e_* R$ denote $R$ with the $R$-module structure $r_1 \cdot r_2 := r_1^{p^e} r_2$, then $F^e: R \to F^e_* R$ is an $R$-module homomorphism. A ring $R$ is said to be $F$-finite if $F^e_* R$ is finitely-generated as an $R$-module.

Let $D^e_R$ denote the ring of $k$-linear differential operators on $R$ (see [15, IV, §16] or [23, Chp. 1] for definition and further details). Each element $\delta \in D^e_R$ of order $\leq p^e - 1$ is $F^e_* R$-linear, so in particular

$$D^e_R \subseteq \bigcup_{e \geq 1} D^e_R,$$

where $D^e_R = \text{End}_{R^{p^e}}(R)$, and since $R$ is a finitely-generated $k$-algebra with $k$ perfect, then $R$ is $F$-finite and thus (1) is actually an equality [23, Chp. 1, §4].

For an ideal $b \subset R$ and integer $e > 0$, write $b^{[1/p^e]}$ for the smallest ideal $J$ such that $b \subseteq J^{[p^e]}$ where $J^{[a]} := (x^a : x \in J)^1$. For an element $f \in R$, we have by [1, Lem. 3.4] a descending chain of ideals

$$R = (f^{p^e-1})^{[1/p^e]} \supseteq (f^{p^e-1})^{[1/p]} \supseteq (f^{p^e-1})^{[1/p^2]} \supseteq (f^{p^e-1})^{[1/p^3]} \supseteq \ldots,$$

that stabilises [1, Thm. 3.7], and it stabilises at $e$ (i.e. $(f^{p^e-1})^{[1/p^e]} = (f^{p^{e+1}-1})^{[1/p^{e+1}] = \ldots}$) if and only if there exists $\delta \in D^e_{R^{(e+1)}}$ such that $\delta(f^{p^e-1}) = f^{p^e-p}$ [1, Prop. 3.5].

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1The equivalent notation for $b^{[1/p^e]}$ used in [1] and [7] is $L_e(b)$—the notation used here is the one of [5] and [6].
Remark 1. Álvarez-Montaner–Blickle–Lyubeznik [1] considered the above chain of ideals in connection with the question of determining the minimum integer $i$ such that $1/f^i$ generates $R_f$ as a $D_R$-module which is known to be finite (where the $D_R$-structure on $R_f$ is ‘via the quotient-rule’). In the case of a field of characteristic 0 and $f$ a non-zero polynomial, then the integer $i$ is related in a non-trivial way to the Bernstein–Sato polynomial for $f$. In the case of a field of positive characteristic and $f \in R$ a non-zero polynomial, then one always has $i = 1$ [1, Thm. 1.1].

Definition 2. Let $R$ be a finitely-generated $k$-algebra where $k$ is a perfect field of characteristic $p > 2$. The level of $f \in R$ is defined to be $e + 1$, where $e$ is the integer where the chain (2) stabilises.

Given $f \in R$ a homogeneous polynomial, one can consider the projective hypersurface $X$ defined by the vanishing of $f$. The level of $f$ was shown to be connected with the geometry of $X$ in the case of an elliptic curve, in the following

Theorem 3 (Boix–De Stefani–Vanzo, [7, Thm. 1.1]). Let $R = k[x_0, x_1, x_2]$ where $k$ is a perfect field of characteristic $p > 2$, and let $X = \text{Proj}(R/fR)$ define an elliptic curve. Then the level of $f$ is 1 if and only if $X$ is ordinary, and 2 otherwise.

Remark 4. The above result has been partially generalised to the case of hyperelliptic curves in [4].

The objective here is to extend the above theorem to the case of Calabi–Yau hypersurfaces, and in the process to describe the relation between level and the $F$-jumping exponents and the Hartshorne–Speiser–Lyubeznik numbers.

2. $F$-jumEmail exponents and the level of a Calabi–Yau variety

Here we outline the relationship between the level of a polynomial $f$ and the $F$-jumping exponents of the ideal of $f$.

Let $R$ be a regular ring of characteristic $p > 0$. For an ideal $a$ in $R$ and positive real number $\lambda$, and $e \geq 1$ then by [5, Lem. 2.8] we have

$$(a^{\lceil \lambda r^e \rceil}/1/r^e) \subseteq (a^{\lceil \lambda r^{e+1} \rceil}/1/r^{e+1}),$$

where for a real number $r$, we set $[r]$ to be the smallest integer $\geq r$. The ring $R$ is Noetherian so the chain stabilises to a limit ideal for large $e$ and we set $\tau(a^\lambda)$ to be this limit called the generalised test ideal of $a$ with exponent $\lambda$. It is known that for every $\lambda$ there is $\varepsilon > 0$ such that $\tau(a^\lambda) = \tau(a^{\lambda'})$ for $\lambda' \in [\lambda, \lambda + \varepsilon)$ [5, Cor. 2.16]. A positive $\lambda$ is called an $F$-jumping exponent of $a$ if $\tau(a^\lambda) \neq \tau(a^{\lambda'})$ for all $\lambda' < \lambda$. A positive $\lambda$ is called an $F$-jumping exponent simple if it lies in the interval $(0, 1]$. If $a$ is a principal ideal then it follows from [5, Prop. 2.25] that $\tau(a^\lambda) = \tau(a^{\lambda+1})$ so in our situation it is enough to look at the simple $F$-jumping exponents.
Remark 5. These should be regarded as the characteristic-$p$ analogues of the related notions from birational geometry, e.g. the log canonical threshold, the jumping coefficients of Ein–Lazarsfeld–Smith–Varolin \cite{10} etc.

Proposition 6. Let $R$ be an $F$-finite ring of characteristic $p > 2$ and let $I = (f)$ be a principal ideal. Let $e$ be the largest simple $F$-jumping exponent of $I$. Then the level of $f$ is $[1 - \log_p(1 - e)]$.

Proof. Set $\lambda_k = 1 - \frac{1}{p^k}$, then $\tau(I^{\lambda_k}) = (f^{p^k-1})^{[1/p^k]}$ \cite[Lem. 2.1]{6}. Since the level is well-defined (i.e. $(f^{p^k-1})^{[1/p^k]} = (f^{p^{k+1}-1})^{[1/p^{k+1}]}$ for $k \gg 0$), thus $\tau(I^{\lambda_k}) = \tau(I^{\lambda_{k+1}})$ for $k \gg 0$ hence the number $e$ is well-defined. Let the level of $f$ be $m$, so $(f^{p^m-1})^{[1/p^m]} = (f^{p^m-1})^{[1/p^m]}$ and thus $\tau(a^{\lambda_{m-1}}) = \tau(a^{\lambda_m})$. Hence there is an $F$-jumping exponent in the interval $(\lambda_{m-2}, \lambda_{m-1})$ and none in $(\lambda_{m-1}, 1)$. It is straightforward to see that the function $f(x) := [1 - \log_p(1 - x)]$ is constant on intervals $(\lambda_{m-1}, \lambda_m]$ and equal to $m + 1$. The result follows. \qed

Remark 7. In particular, if the level of $f$ is 1 then there is no $F$-jumping exponent for $I = (f)$ in $(0, 1)$ and if the level of $f$ is $> 2$ then there is an $F$-jumping exponent for $I$ in $(1 - 1/p, 1)$.

2.1. Calabi–Yau hypersurfaces. A Calabi–Yau variety is a smooth projective variety $X$ of dimension $n$, over a field of characteristic $p > 0$ with trivial canonical bundle such that $\dim H^i(X, \mathcal{O}_X) = 0$ for $i = 1, \ldots, n - 1$. Following Artin–Mazur \cite{2}, one considers the functor $\text{AM}_X : \text{Art}_k \to \text{Ab}$ defined by

$$\text{AM}_X(S) := \ker \left( F_* : \Pi^a_0(X \times S, \mathbb{G}_m) \to \Pi^a_0(X, \mathbb{G}_m) \right),$$

on the category $\text{Art}_k$ of local Artinian $k$-algebras $S$ with residue field $k$. The functor is pro-representable by a smooth formal group of dimension 1 \cite[II, §2]{2}, which is characterised by a number called the height which can be any positive integer or infinity. We say that a Calabi–Yau variety is ordinary if the aforementioned formal group has height 1. In the case that $X$ is an elliptic curve then this agrees with the other definition.

Proposition 8. Let $X = \text{Proj}(R/fR)$ be a Calabi–Yau hypersurface over a perfect field of characteristic $p > 0$. Then $f$ has level 1 if and only if $X$ is ordinary.

Proof. If $f$ has level one then from remark 7 there is no $F$-jumping number in $(0, 1)$ hence by \cite[Thm. 1.1]{3} $X$ is ordinary and the reverse implication holds as well. \qed

3. The Hartshorne–Speiser–Lyubeznik number

Definition 9 (\cite{21}). Let $R$ be a Noetherian local ring of characteristic $p > 0$, let $M$ be an Artinian $R$-module and let $\varphi : M \to M$ be an additive
map satisfying \( \varphi(am) = a^p \varphi(m) \) for \( a \in R \) and \( m \in M \). For integers \( i \geq 1 \), define a chain of ascending submodules

\[ N_i = \{ z \in M : \varphi^i(z) = 0 \}, \]

then a theorem of Lyubeznik [18, Prop. 4.4] implies that the chain stabilises: \( N_\ell = N_{\ell+1} \) for \( \ell \gg 1 \). The Hartshorne–Speiser–Lyubeznik number of the pair \((M, \varphi)\) is the smallest positive integer \( \ell \) such that \( N_\ell = N_{\ell+j} \) for all \( j \geq 1 \).

Now let \( R = k[x_0, \ldots, x_n] \) where \( k \) is a perfect field of characteristic \( p > 0 \) and let \( f \in R \) be non-zero and \( m = (x_0, \ldots, x_n) \).

**Proposition 10** ([21, Prop. 5.7 and Cor. 5.8]). The Hartshorne–Speiser–Lyubeznik number of \((\text{H}^{n+1}_m(R/fR), \Theta)\) is the smallest positive integer \( \ell \) such that

\[ \tau((f)^{1-\frac{1}{p^\ell}}) = \tau((f)^{1-\frac{1}{p^\ell+i}}), \]

for all \( i \geq 1 \), where \( \Theta \) denotes the map induced on the top local cohomology of \( R/fR \) at the origin by the Frobenius morphism.

**Proposition 11.** Let \( X = \text{Proj}(R/fR) \) be a Calabi–Yau hypersurface and \( p > n^2 - n - 1 \). Then the Hartshorne–Speiser–Lyubeznik number of \( X \) is 1.

**Proof.** The Grothendieck–Serre correspondence [9, Thm. 20.4.4] yields

\[ \bigoplus_{i \in \mathbb{Z}} \text{H}^n(X, \mathcal{O}_X(i)) \cong \text{H}^{n+1}_m(R/fR), \]

which is a graded isomorphism and Frobenius compatible. Assume that \( p > n^2 - n - 1 \) then [3, Thm. 3.5] implies that Frobenius acts injectively on the negative graded part of the top local cohomology. Hence if one is interested in the Hartshorne–Speiser–Lyubeznik number of \( X \) then one only need consider the powers of Frobenius acting on \( \text{H}^n(X, \mathcal{O}_X) \).

If \( X \) is ordinary then since the \( F \)-pure threshold is 1 by [3, Thm. 1.1], the result follows. Otherwise suppose \( X \) has height \( h > 1 \). Then [12, Lem. 4.4] implies that the height of \( X \) is the smallest integer \( i \) such that the Frobenius map \( F: \text{H}^n(X, W_i \mathcal{O}_X) \to \text{H}^n(X, W_i \mathcal{O}_X) \) is non-zero (here \( W_i \mathcal{O}_X \) is the sheaf of truncated Witt vectors of length \( i \) of \( \mathcal{O}_X \))^2. But when this map is zero then the maps \( F: \text{H}^n(X, W_j \mathcal{O}_X) \to \text{H}^n(X, W_j \mathcal{O}_X) \) are zero for all \( 1 \leq j < i \). In particular the Frobenius action on \( \text{H}^n(X, \mathcal{O}_X) \to \text{H}^n(X, \mathcal{O}_X) \) is zero so the result follows in this case also.

**Corollary 12.** Let \( p > n^2 - n - 1 \) then the simple \( F \)-jumping exponents of a non-ordinary Calabi–Yau hypersurface \( X = \text{Proj}(R/fR) \) lie in the interval \([1 - h/p, 1 - 1/p]\), where \( h \) is the order of vanishing of the Hasse invariant on the versal deformation space of \( X \).

**Proof.** This is a combination of proposition 11 and [3, Thm. 1.1].

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2The result proved is for the case of K3 surfaces, but as noted in [13, Pg. 2], the same proof goes through in the higher-dimensional case.
Corollary 13. Let \( p > n^2 - n - 1 \) then the level of a Calabi–Yau hypersurface \( X = \text{Proj}(R/fR) \) is 1 if and only if it is ordinary, and 2 otherwise.

Proof. This is a combination of proposition 11 and proposition 8. \(\square\)

Remark 14. The connection between Hartshorne–Speiser–Lyubeznik numbers and the level described above can be exploited to give another proof of the main results of [4].

4. Example of Fermat Hypersurfaces

In this section we describe how to construct the differential operators guaranteed to exist by proposition 11 for the example of Fermat hypersurfaces.

In [7] there is described an algorithm to construct an operator of level 1 or level 2 for an elliptic curve. That algorithm works also in our case but invokes a script of Katzman–Schwede [17] that requires two computations of Gröbner bases (it is not exactly clear what the complexity of the algorithm in [7] is). The method to be described for the case of Fermat hypersurfaces can be shown to work also for a general elliptic curve using degree by degree arguments and has complexity \( O(p^2) \). A computer implementation seems to indicate that it works also for K3 surfaces defined by smooth quartics in \( \mathbb{P}^3 \).

Fix \( R = k[x_0, x_1, \ldots, x_n] \) with \( k \) a perfect field of characteristic \( p > n \geq 2 \). The ring \( \text{End}_{R_{p}}(R) \) is the ring extension of \( R \) generated by the operators \( D_{t,i} \) for \( i = 0, \ldots, n \) and \( t = 1, \ldots, p^e - 1 \) where (see [15, IV, §16])

\[
D_{t,i}(x_j^s) = \begin{cases} \binom{t}{s} x_i^{s-t} & \text{if } s \geq t \text{ and } i = j, \\ 0 & \text{otherwise}. \end{cases}
\]

Now let \( f_n(x_0, \ldots, x_n) := x_0^{n+1} + x_1^{n+1} + \cdots + x_n^{n+1} \), and denote by \( X_{n,p} \) the Fermat hypersurface of Calabi-Yau type

\[
X_{n,p} = \text{Proj}(R/f_nR) \subset \mathbb{P}^{n}_{k}. 
\]

Suppose that \( p \equiv 1 \pmod{n+1} \) then the monomial \( x_0^{p-1}x_1^{p-1} \cdots x_n^{p-1} \) appears in \( f_n^{p-1} \) with non-zero coefficient (by an application of Kummer’s theorem for binomial coefficients). Hence an operator of level 1 for \( f_n \) when \( p \equiv 1 \pmod{n+1} \) is (up to a non-zero constant)

\[
\Psi_1 = \prod_{i=0}^{n} D_{p-1,i}.
\]

Now we will construct an operator of level 2 for all \( f_n \). Set

\[
\alpha := (n+1)(p^2-1) - n(n+1)p.
\]

For a given \( j \in \{0, \ldots, n\} \) then the monomial

\[
m_j := x_0^{(n+1)p} x_1^{(n+1)p} x_2^{(n+1)p} \cdots x_{j-1}^{(n+1)p} x_j^{(n+1)p} x_{j+1}^{(n+1)p} \cdots x_n^{(n+1)p},
\]
appears in $f_n^{p^2-1}$ with non-zero coefficient (again by an application of Kummer’s theorem). Set $\beta := p^2 - 1 - (n + 1)p$ and define an operator

$$\delta_j := x_0^\beta x_1^\beta \cdots x_{j-1}^\beta x_j^{2p^2-1-\alpha} x_{j+1}^\beta \cdots x_n^\beta \circ \prod_{i=0}^{n} D_{p^2-1,i},$$

where the $\circ$ indicates precomposition. Then $\delta_j(m_j) = x_j^{p^2}$ (up to a non-zero constant). Suppose that the monomial

$$(3) \quad x_0^{p^2-p} x_1^{p^2-p} \cdots x_n^{p^2-p},$$

does not appear in $f_n^{p^2-p}$. Then by the pigeonhole principle and $f_n^{p^2-p} = (f_n^p)^{p-1}$, for every monomial appearing in $f_n^{p^2-p}$ there is a variable $x_k$ in that monomial whose power is $\geq p^2$. Hence we can use the operators $\{\delta_j\}_j$ to construct an operator of level 2 for $f_n$. Otherwise, suppose that the monomial (3) does appear in $f_n^{p^2-p}$ with a non-zero coefficient, then the monomial $x_0^{p^2-1} x_1^{p^2-1} \cdots x_{m-1}^{p^2-1}$ appears in $f_n^{p^2-1}$ with non-zero coefficient and $\Psi_1$ is an operator of level 1, and also of level 2:

$$\Psi_1(f_n^{p^2-1}) = \Psi_1(f_n^{p^2-p} \cdot f_n^{p^2-1}) = f_n^{p^2-p} \Psi_1(f_n^{p^2-1}) = f_n^{p^2-p}.$$ 

Explicitly, if we take the case $n = 2$ and $p = 5$ the above procedure obtains the following operator of level 2 for $f_2$:

$$\begin{align*}
&\left(x_0^{35} \delta_0 + x_1^{35} \delta_1 + x_2^{35} \delta_2\right) + \left(x_0^5 x_1^{30} \delta_0 + x_1^5 x_2^{30} \delta_1 + x_2^5 x_0^{30} \delta_2\right) \\
&- \left([x_0^{20} x_1^{15} + x_0^{20} x_2^{15}] \delta_0 + [x_1^{15} x_1^{20} + x_1^{15} x_2^{20}] \delta_1 + [x_0^{15} x_2^{20} + x_1^{15} x_2^{20}] \delta_2\right) \\
&+ 2\left(x_0^5 x_1^{15} x_2^{15} \delta_0 + x_0^{15} x_1^{5} x_2^{15} \delta_1 + x_0^{15} x_1^{15} x_2^{5} \delta_2\right).
\end{align*}$$

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