Abstract. The purpose of this paper is to present a wide picture of asymptotic results of large powers of coefficients of power series, with non-negative coefficients, using as tools local central limit theorems for lattice random variables and for continuous families of lattice random variables combined with the theory of Khinchin families and of the Hayman class.

We also cast in this framework the asymptotic formula due to Otter and to Meir-Moon of the coefficients of the solutions of Lagrange equations when the data power series has non-negative coefficients. Asymptotics of Lagrangian probability distributions are also discussed.

1. Introduction

The purpose of this paper is to present a wide perspective of the asymptotic formulas for the coefficients of large powers of power series with non-negative coefficients.

The framework of the approach of this paper results from the combination of local central limit theorems for both lattice random variables and for continuous families of lattice random variables, on the one hand, with the theory of Khinchin families and, particularly, of the Hayman class on the other.

The saddle-point approximation, steepest descent or Laplace method, the usual tools to obtain these asymptotic formulas for large powers, are somehow built into the local central limit theorems and into the Hayman class.

We reserve $k$ to signify index of the coefficient and $n$ to denote the power to which a power series $\psi$ is raised. The power $n$ tends to infinity. Different results appear depending on how $k$ behaves with $n$, or actually, on how $k/n$ behaves as $n \to \infty$.

Gardy, in [11], gives a nice presentation of the results for $k/n \approx 1$ and $k/n \to 0$ as $n \to \infty$, from the point of view of saddle-point approximation. See also, De Angelis [7], for a quite direct approach.

If $\psi$ is the probability generating function of a random variable $X$ with values in $\{0, 1, 2, \ldots\}$, then the $k$-th coefficient of $\psi^n$ is the probability that the sum of $n$
independent copies of $X$ takes the value $k$. See Daniels [6], particularly [6, Section 8], and Good [13], particularly [13, Section 6], for saddle point approximations pertaining to this application of large powers.

**Plan of the paper.** Section 2 reviews the basic theory of Khinchin families. Section 3 is devoted to the local central theorem for continuous families of lattice variables.

The results about large powers, within the tools and framework described in the previous sections, are discussed in Section 4 and then in Sections 5, 6, 7 and 8.

We also cast in this framework the asymptotic formula due to Otter, [22] and Meir-Moon, [18] for the coefficients of the solutions of Lagrange equations when the data power series has nonnegative coefficients. The proof of the Otter-Meir-Moon Theorem is the content of Section 9.

Section 10 discusses applications to probability generating functions and to Lagrangian distributions.

Finally, the appendix 11 deals with the so-called uniformly Gaussian and uniformly Hayman Khinchin families which are just needed in Section 7.

**Notations.**

Expectation and variance of a random variable $X$ are generically denoted $E(X)$ and $V(X)$. Probability of an event $A$ is generically denoted by $P(A)$.

For random variables $X, Y$ the notation $X \overset{d}{=} Y$ signifies that $X$ and $Y$ have the same distribution: $P(X \in B) = P(Y \in B)$ for any Borel set $B \subset \mathbb{R}$.

For a sequence of random variables $(X_n)_{n \geq 1}$ and a random variable $Y$ we write

$$X_n \overset{d}{\to} Y$$

$n \to \infty$

to signify convergence in distribution.

A family $(Z_s)_{s \in I}$ indexed in a real interval $I$ is continuous in distribution if for any sequence $(s_n)_{n \geq 1}$ of points in $I$ which converges to a point $s^* \in I$ it holds that $Z_{s_n}$ converges in distribution to $Z_{s^*}$.

The abbreviations efg and ogf mean, respectively, exponential generating function of a labeled combinatorial class and ordinary generating function of a combinatorial class. For general background on Combinatorics pertaining to this paper we refer to the treatise [10].

The unit disk in the complex plane $\mathbb{C}$ is denoted by $\mathbb{D}$. The disk of center $a \in \mathbb{C}$ and radius $R$ is denoted $\mathbb{D}(a, R)$, while its closure is denoted $\text{cl} (\mathbb{D}(a, R))$.

For positive sequences $(a_n)_{n \geq 1}, (b_n)_{n \geq 1}$, by $a_n \sim b_n$, as $n \to \infty$, we mean that $\lim_{n \to \infty} a_n/b_n = 1$.

If $\psi$ and $f$ are probability generating functions, with $\mathcal{L}(\psi, f)$ we denote the Lagrangian probability distribution with generators $\psi$ and $f$, see Section 10.1.
2. Khinchin families

We denote by $\mathcal{K}$ the class of nonconstant power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with positive radius of convergence, which have nonnegative Taylor coefficients and such that $a_0 > 0$. Since $f \in \mathcal{K}$ is nonconstant, at least one coefficient other than $a_0$ is positive.

The Khinchin family of such a power series $f \in \mathcal{K}$ with radius of convergence $R > 0$ is the family of random variables $(X_t)_{t \in [0,R)}$ with values in $\{0, 1, \ldots\}$ and with mass functions given by

$$P(X_t = n) = \frac{a_n t^n}{f(t)}, \quad \text{for each } n \geq 0 \text{ and } t \in (0, R).$$

For $t = 0$, we define $X_0 \equiv 0$. Notice that $f(t) > 0$ for each $t \in [0, R)$.

Any Khinchin family is continuous in distribution in $[0, R)$. No hypothesis upon joint distribution of the variables $X_t$ is considered; they are families and not processes.

We refer to [3] and [4] for the basic theory of Khinchin families. Next we describe the specific aspects of the theory to be used in the present paper; for proofs, examples and applications we refer to [3].

2.0.1. Shifted Khinchin families. Sometimes it is convenient to consider power series $g(z) = \sum_{n=0}^{\infty} b_n z^n$ with radius of convergence $R > 0$ and nonnegative coefficients with at least two positive coefficients, but which may have $g(0) = 0$. We say then that $g$ is in the shifted class $\mathcal{K}^s$. Thus $g \in \mathcal{K}^s$ if there exists an integer $l \geq 0$ such that

$$g(z)/z^l \in \mathcal{K},$$

i.e., if there exists a power series $f(z) \in \mathcal{K}$ with radius of convergence $R > 0$ such that

$$g(z) = z^l f(z), \quad \text{for } z \in \mathbb{D}(0, R).$$

To $g \in \mathcal{K}^s$ we associate a Khinchin family $(Y_t)_{t \in (0,R)}$ as above:

$$P(Y_t = n) = \frac{b_n t^n}{g(t)}, \quad \text{for } n \geq 0 \text{ and } t \in (0, R).$$

If $(Z_t)_{t \in [0,R)}$ is the Khinchin family of $f \in \mathcal{K}$, then

$$Y_t \overset{d}{=} Z_t + l,$$

including, $Y_0 \equiv l$.

2.1. Basic properties. Along this section we let $f$ be a power series in $\mathcal{K}$ with radius of convergence $R > 0$ and Khinchin family $(X_t)_{t \in [0,R)}$.

2.1.1. Mean and variance functions. For the mean and variance of $X_t$ we reserve the notation $m(t) = \mathbb{E}(X_t)$ and $\sigma^2(t) = \mathbb{V}(X_t)$, for $t \in [0, R)$. In terms of $f$, the mean and the variance of $X_t$ may be written as

$$m(t) = tf'(t) / f(t), \quad \sigma^2(t) = tm'(t), \quad \text{for } t \in [0, R).$$
For each \( t \in (0, R) \), the variable \( X_t \) is not a constant, and so \( \sigma^2(t) > 0 \). Consequently, \( m(t) \) is strictly increasing in \([0, R)\), though, in general, \( \sigma(t) \) is not increasing. We denote

\[
M_f = \lim_{t \uparrow R} m(t).
\]

For \( g \in K^s \), with Khinchin family \( (Y_t) \), we also write \( m_g(t) = \mathbf{E}(Y_t) \) and \( \sigma^2(t) = \mathbf{V}(Y_t) \). If \( g(z) = z^l h(z) \), with \( l \geq 1 \) and \( h \in K \) we have that
\[
m_g(t) = l + m_h(t) \quad \text{and} \quad \sigma^2_g(t) = \sigma^2_h(t).
\]
The function \( m_g \) is an increasing diffeomorphism from \([0, R)\) to \([l, l + \infty)\).

For finite radius of convergence \( R > 0 \) and for any integer \( k \geq 1 \), the function \( f(t) \) and its derivatives \( f^{(k)}(t) \) are all increasing functions on the interval \([0, R)\). For \( k \geq 1 \), we denote with \( f^{(k)}(R) \) the limit

\[
f^{(k)}(R) = \lim_{t \uparrow R} f^{(k)}(t),
\]

including

\[
f(R) = \lim_{t \uparrow R} f(t);
\]
these limits exist, although they could be \(+\infty\).

2.1.2. Range of the mean. Since the mean function \( m(t) \) is increasing, its range is given by \([0, M_f)\).

- The case where \( M_f = \infty \) is particularly relevant and quite general. If \( M_f = \infty \), then \( m(t) \) is a diffeomorphism from \([0, R)\) onto \([0, +\infty)\) and, in particular, for each \( n \geq 1 \), there exists a unique \( t_n \in (0, R) \) such that \( m(t_n) = n \). These \( t_n \) play an important role in Hayman’s identity, Section 2.2 and in Hayman’s asymptotic formula, Section 2.6.

- The following Lemma A describes the quite specific cases where \( M_f < +\infty \). It is Lemma 2.2 of [3].

**Lemma A.** For \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) in \( K \) with radius of convergence \( R > 0 \), we have

\[ M_f < \infty \]

if and only if \((R < \infty \) and \( \sum_{n=0}^{\infty} n a_n R^n < \infty \)) or \( (R = \infty \) and \( f \) is a polynomial).

In the first case, we have that \( M_f = \sum_{n=0}^{\infty} n a_n R^n \), while for a polynomial \( f \in K \), the second case, we have that \( M_f = \deg(f) \).

2.1.3. Extension of the Khinchin family if \( M_f < \infty \) (and \( R < \infty \)). We assume now that \( M_f < \infty \) (and \( R < +\infty \)).

Thus we have that \( \sum_{n=0}^{\infty} n a_n R^n < \infty \) and also that \( f(R) = \sum_{n=0}^{\infty} a_n R^n < \infty \). The power series \( \sum_{n=0}^{\infty} a_n z^n \) defines a continuous (actually, \( C^1 \)) function on the whole
closed disk \( \text{cl}(\mathbb{D}(0,R)) \) which extends \( f \) from \( \mathbb{D}(0,R) \). We let \( f(R) \triangleq \sum_{n=0}^{\infty} a_n R^n = \lim_{t \uparrow R} f(t) \) and

\[
f'(R) \triangleq \sum_{n=1}^{\infty} na_n R^{n-1} = \lim_{t \uparrow R} f'(t),
\]

\[
f''(R) \triangleq \sum_{n=2}^{\infty} n(n-1)a_n R^{n-2} = \lim_{t \uparrow R} f''(t).
\]

In this case, we may extend the Khinchin family \( (X_t)_{t \in [0,R]} \) of \( f \) to \( t \in [0,R] \) by defining the variable \( X_R \) by

\[
P(X_R = n) = \frac{a_n R^n}{f(R)}, \quad \text{for each } n \geq 0.
\]

The extended family \( (X_t)_{t \in [0,R]} \) becomes continuous in distribution in the closed interval \( [0,R] \). Observe that \( X_R \) (like any other \( X_t \), with \( t \in (0,R) \)) is nonconstant.

The variable \( X_R \) has (finite) mean \( \mathbb{E}(X_R) = R f'(R)/f(R) = m(R) \) and variance

\[
\mathbb{V}(X_R) = \frac{\sum_{n=0}^{\infty} n^2 a_n R^n}{f(R)} - \mathbb{E}(X_R)^2.
\]

The variance \( \mathbb{V}(X_R) \) is nonzero since \( X_R \) is nonconstant, but it could be infinite. Actually, \( \mathbb{V}(X_R) \) is finite if and only if \( \sum_{n=0}^{\infty} n^2 a_n R^n < +\infty \) if and only if \( f''(R) < +\infty \), and in any case

\[
\mathbb{V}(X_R) = \lim_{t \uparrow R} \sigma^2(t).
\]

If \( \mathbb{V}(X_R) \) is finite, we write \( \sigma^2(R) \triangleq \mathbb{V}(X_R) \).

2.1.4. Normalization and characteristic functions. For each \( t \in (0,R) \), we denote by \( \tilde{X}_t \) the normalization of \( X_t \):

\[
\tilde{X}_t = \frac{X_t - m(t)}{\sigma(t)}, \quad \text{for } t \in (0,R).
\]

The characteristic function of \( X_t \) may be written in terms of the power series \( f \) as:

\[
\mathbb{E}(e^{it\theta X_t}) = \frac{f(\theta e^{it})}{f(t)}, \quad \text{for } t \in (0,R) \text{ and } \theta \in \mathbb{R},
\]

while for its normalized version \( \tilde{X}_t \) we have

\[
\mathbb{E}(e^{it\theta \tilde{X}_t}) = \mathbb{E}(e^{it\theta X_t/\sigma(t)}e^{-\theta m(t)/\sigma(t)}), \quad \text{for } t \in (0,R) \text{ and } \theta \in \mathbb{R},
\]

and so,

\[
|\mathbb{E}(e^{it\theta \tilde{X}_t})| = |\mathbb{E}(e^{it\theta X_t/\sigma(t)})|, \quad \text{for } t \in (0,R) \text{ and } \theta \in \mathbb{R}.
\]

If \( M_f < \infty \) (and \( R < \infty \)), we may define \( \tilde{X}_R = (X_R - m(R))/\sigma(R) \), with the understanding that \( \tilde{X}_R \equiv 0 \), if \( \sigma(R) = +\infty \). Recall that \( \sigma(R) > 0 \). The extended family \( (\tilde{X}_t)_{t \in [0,R]} \) of normalized variables is continuous (in distribution).
2.1.5. Scaling. For any power series \( g(z) = \sum_{n=0}^{\infty} a_n z^n \) in \( \mathcal{K} \) of radius \( R > 0 \), we denote
\[ Q_g \triangleq \gcd\{ n \geq 1 : a_n \neq 0 \} = \lim_{N \to \infty} \gcd\{ 1 \leq n \leq N : a_n \neq 0 \}. \]
If \( Q_g > 1 \), then we can write \( g(z) = f(z^{Q_g}) \) for a certain companion power series \( f \in \mathcal{K} \) with has radius of convergence \( R^{Q_g} \). Observe that \( Q_f = 1 \).
Let \((Y_t)_{t \in [0,R]}\) be the Khinchin family of \( g \) and \((X_t)_{t \in [0,R^{Q_g}]}\) be the Khinchin family of \( f \). We have that
\[ Y_t = Q_g \cdot X_{t^{Q_g}}, \quad \text{for any } t \in (0,R). \]

The mean and variance functions of \( g \) and \( f \) are related by
\[ m_g(t) = Q_g \cdot m_f(t^{Q_g}), \quad \sigma^2_g(t) = Q_g^2 \cdot \sigma_f^2(t^{Q_g}), \quad \text{for } t \in (0,R). \]

2.1.6. Fulcrum \( F \) of \( f \). A power series \( f \) in \( \mathcal{K} \) does not vanish on the real interval \([0,R]\), and so, it does not vanish in a simply connected region containing that interval. We may consider \( \ln f \), a branch of the logarithm of \( f \) which is real on \([0,R]\), and the function \( F \), called the fulcrum of \( f \), which is defined and holomorphic in a region containing \((-\infty, \ln R)\) and it is given by
\[ F(z) = \ln f(e^z). \]

If \( f \) does not vanish anywhere in the disk \( \mathbb{D}(0,R) \), then the fulcrum \( F(z) \) of \( f \) is defined in the whole half plane \( \Re z < \ln R \).

In general, the fulcrum \( F \) of \( f \) is defined and holomorphic in the band-like region \( \{ s + i\theta : s < \ln R, |\theta| < \sqrt{2/\sigma(e^s)} \} \). See Section 2.1.5 of [3].

For \( t = e^s \), we have \( m(t) = F'(s) \) and \( \sigma^2(t) = F''(s) \), for \( s < \ln R \).

2.2. Hayman’s identity. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be a power series in \( \mathcal{K} \). Cauchy’s formula for the coefficient \( a_n \) in terms of the characteristic function of its Khinchin family \((X_t)_{t \in [0,R]}\) reads
\[ a_n = \frac{f(t)}{2\pi i t^n} \int_{|\theta|<\pi} \mathbf{E}(e^{i\theta X_t}) e^{-i\theta n} \, d\theta, \quad \text{for each } t \in (0,R) \text{ and } n \geq 1. \]

In terms of the characteristic function of the normalized variable \( \hat{X}_t \), it becomes
\[ a_n = \frac{f(t)}{2\pi i t^n \sigma(t)} \int_{|\theta|<\pi \sigma(t)} \mathbf{E}(e^{i\theta \hat{X}_t}) e^{-\theta(n-m(t))/\sigma(t)} \, d\theta, \quad \text{for each } t \in (0,R) \text{ and } n \geq 1. \]

- If \( M_f = \infty \), we may take for each \( n \geq 1 \) the (unique) radius \( t_n \in (0,R) \) so that \( m(t_n) = n \), to write
\[ a_n = \frac{f(t_n)}{2\pi i t_n^n \sigma(t_n)} \int_{|\theta|<\pi \sigma(t_n)} \mathbf{E}(e^{i\theta \hat{X}_{t_n}}) \, d\theta, \quad \text{for each } n \geq 1, \]
which we call Hayman’s identity.

Although this identity (2.3) is just Cauchy’s formula with an appropriate choice of radius, it encapsulates the saddle point method.
If the Khinchin family \((X_t)\) may be extended to the closed interval \([0, R]\), then we have a similar identity with \(t = R\).

- If \(M_f < \infty\) and \((R < \infty)\) the power series \(f\) extends continuously to the closed disk \(\text{cl}(\mathbb{D}(0, R))\) and the Khinchin family \((X_t)_{t \in [0, R]}\) extends to the closed interval \([0, R]\). See Section 2.1.2 and the notations therein.

We may write
\[
a_n = \frac{f(R)}{2\pi R^n} \int_{|\theta| < \pi} E(e^{\theta X_R}) e^{-\theta n} d\theta, \quad \text{for } n \geq 1.
\]

If, moreover, \(\sigma^2(R) < \infty\), i.e., if \(\sum_{n=0}^{\infty} n^2 a_n R^n < \infty\), then \(X_R\) given by \(X_R = (X_R - m(R))/\sigma(R)\) is well defined and we also have that
\[
a_n = \frac{f(R)}{2\pi R^n \sigma(R)} \int_{|\theta| < \pi \sigma(R)} E(e^{\theta X_R}) e^{-\theta(n-m(R))/\sigma(R)} d\theta, \quad \text{for } n \geq 1.
\]

2.3. Basic moments of a Khinchin family near 0. Let \(f(z) = \sum_{n=0}^{\infty} a_n z^n\) be a power series in \(K\) with radius of convergence \(R > 0\). Let \((X_t)_{t \in [0, R]}\) be its Khinchin family and \(m(t)\) and \(\sigma^2(t)\) be, respectively, the mean and variance of \(X_t\), for \(t \in [0, R]\).

Assume first that \(a_1 > 0\). In this case, we have that
\[
m(t) = \frac{a_1}{a_0} t + O(t^2) \quad \text{and} \quad \sigma^2(t) = \frac{a_1}{a_0} t + O(t^2), \quad \text{as } t \downarrow 0,
\]
and, besides, that
\[
E(|X_t - m(t)|^3) = \frac{a_1}{a_0} t + O(t^2) \quad \text{and} \quad \sqrt{t} E(|\hat{X}_t|^3) = \sqrt{\frac{a_0}{a_1}} + O(t), \quad \text{as } t \downarrow 0.
\]

In general, if for \(k \geq 1\), we have \(a_k > 0\), but \(a_j = 0\), for \(1 \leq j < k\), then
\[
m(t) = \frac{k a_k}{a_0} t^k + O(t^{k+1}) \quad \text{and} \quad \sigma^2(t) = \frac{k^2 a_k}{a_0} t^k + O(t^{k+1}), \quad \text{as } t \downarrow 0,
\]
and, besides, that
\[
E(|X_t - m(t)|^3) = \frac{k^3 a_k}{a_0} t^k + O(t^{k+1}) \quad \text{and} \quad t^{k/2} E(|\hat{X}_t|^3) = \sqrt{\frac{a_0}{a_k}} + O(t), \quad \text{as } t \downarrow 0.
\]

2.4. Gaussian Khinchin families. We introduce next Gaussian and strongly Gaussian power series.

**Definition 1.** A power series \(f\) in \(K\) of radius of convergence \(R > 0\) and its Khinchin family \((X_t)_{t \in [0, R]}\) are termed Gaussian if \(\hat{X}_t\) converges in distribution to the standard normal, as \(t \uparrow R\), or, equivalently, if
\[
\lim_{t \uparrow R} E(e^{\theta \hat{X}_t}) = e^{-\theta^2/2}, \quad \text{for each } \theta \in \mathbb{R}.
\]

**Definition 2.** A power series \(f \in K\) and its Khinchin family \((X_t)_{t \in [0, R]}\) are termed strongly Gaussian if the following two conditions are satisfied
\[
\lim_{t \uparrow R} \sigma(t) = +\infty \quad \text{and} \quad \lim_{t \uparrow R} \int_{|\theta| < \pi \sigma(t)} \left| E(e^{\theta \hat{X}_t}) - e^{-\theta^2/2} \right| d\theta = 0.
\]
This notion of strongly Gaussian power series was introduced by Báez-Duarte in [1]. Strongly Gaussian power series are Gaussian, see Theorem A.

We refer to [3] and [4] for examples, for criteria to verify whether a power series is Gaussian or strongly Gaussian, and for a number of applications (asymptotic formulas of coefficients) of these concepts for set constructions in Analytic Combinatorics, including partitions.

We just mention (see [3]) that

- the exponential \( f(z) = e^z \) is strongly Gaussian and thus Gaussian,
- polynomials are not Gaussian. In fact, if \((X_t)_{t \in [0, \infty)}\) is the Khinchin family of a polynomial of degree \( N \), then, as \( t \uparrow \infty \), \( X_t \) tends in distribution to the constant \( N \) and \( \tilde{X}_t \) tends in distribution to 0.

**Lemma B.** If \( f \) is a Gaussian power series, then \( M_f = +\infty \)

*Proof.* Assume that \( M_f < +\infty \). We use the characterization of \( M_f < +\infty \) given in Lemma A. If the radius of convergence \( R \) of \( f \) is \( R = +\infty \), then \( f \) is a polynomial, which is not Gaussian as we just have mentioned.

If \( R < \infty \), then (see Section 2.1.2) we may extend the Khinchin family to include \( t = R \) with \( X_R \). Thus \( X_t \) tends in distribution to \( X_R \) as \( t \uparrow R \). If \( \sigma(R) < \infty \), then \( \tilde{X}_t \) tends in distribution to \( \tilde{X}_R = (X_R - m(R))/\sigma(R) \), while if \( \sigma(R) = \infty \), then \( \tilde{X}_t \) tends in distribution to the constant \( \tilde{X}_R \equiv 0 \). In both cases, the limit \( \tilde{X}_R \) is a discrete random variable. \( \square \)

2.5. **Hayman’s central limit Theorem.** Strongly Gaussian power series satisfy Theorem A, which is both a local and global Central Limit Theorem. This results, for Hayman’s admissible power series, comes from [16], but see [3] for a proof for the more general case of strongly Gaussian power series.

**Theorem A** (Hayman’s central limit theorem). If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) in \( K \) is a strongly Gaussian power series with radius of convergence \( R > 0 \), then

\[
\lim \sup_{t \uparrow R} \sup_{n \in \mathbb{Z}} \left| \frac{a_n t^n}{f(t)} \sqrt{2\pi \sigma(t)} - e^{-(n-m(t))^2/(2\sigma^2(t))} \right| = 0.
\]

(In this statement \( a_n = 0 \), for \( n < 0 \).)

Besides,

\[
\lim_{t \uparrow R} P(\tilde{X}_t \leq b) = \Phi(b), \quad \text{for every } b \in \mathbb{R}.
\]

And so, \( \tilde{X}_t \) converges in distribution towards the standard normal and \( f \) is Gaussian.

In other terms, Theorem A claims that if for each \( t \in [0, R) \) we define the set

\[
\mathcal{V}_t = \left\{ \frac{n - m(t)}{\sigma(t)} : n \in \mathbb{Z} \right\},
\]

then

\[
\lim \sup_{t \uparrow R} \sup_{x \in \mathcal{V}_t} \left| P(\tilde{X}_t = x) \sigma(t) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| = 0.
\]
2.6. Asymptotic formula of coefficients of strongly Gaussian power series.
The coefficients of strongly Gaussian power series obey a neat asymptotic formula:

**Theorem B** (Hayman’s asymptotic formula). If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) in \( \mathcal{K} \) is strongly Gaussian then

\[
a_n \sim \frac{1}{\sqrt{2\pi t_n^m \sigma(t_n)}} f(t_n), \quad \text{as } n \to \infty.
\]

In the asymptotic formula above, \( t_n \) is given by \( m(t_n) = n \), for each \( n \geq 1 \), which are uniquely defined because \( M_f = +\infty \), see Lemma [B]. This asymptotic formula follows readily from Theorem [A] or alternatively, from Hayman’s formula (2.3) and strong gaussianity.

2.7. Hayman class. The class of Hayman consists of power series \( f \) in \( \mathcal{K} \) which satisfy some concrete and verifiable conditions which imply that \( f \) is strongly Gaussian, see Theorem [C] below.

**Definition 3.** A power series \( f \in \mathcal{K} \) is in the Hayman class (or is Hayman-admissible or just \( H \)-admissible) if

\[
\text{variance condition: } \lim_{t \uparrow R} \sigma(t) = \infty,
\]

and for a certain function \( h : [0, R) \rightarrow (0, \pi] \), which we refer to as a cut between a major arc and a minor arc, the following conditions are satisfied.

\[
\text{major arc: } \lim_{t \uparrow R} \sup_{|\theta| \leq h(t) \sigma(t)} |E(e^{i\theta \hat{X}_t})e^{\theta^2/2} - 1| = 0,
\]

\[
\text{minor arc: } \lim_{t \uparrow R} \sigma(t) \sup_{h(t) \sigma(t) \leq |\theta| \leq \pi \sigma(t)} |E(e^{i\theta \hat{X}_t})| = 0.
\]

Some authors include in the Hayman class power series with a finite number of negative coefficients. This is not the case in the present paper.

For \( f \) in the Hayman class, the characteristic function of \( \hat{X}_t \) is uniformly approximated by \( e^{-\theta^2/2} \) in the major arc, while it is uniformly \( o(1/\sigma(t)) \) in the minor arc.

**Theorem C.** Power series in the Hayman class are strongly Gaussian.

See [B] for a proof of Theorem [C]. We may consider membership in Hayman’s class as a criterion for strong gaussianity.

3. Local central limit theorem and continuous families

To obtain asymptotic results for the coefficients of large powers of power series with nonnegative coefficients we will write down these coefficients in terms of Khinchin families and Hayman’s identities.

To handle the Hayman’s identities of a Khinchin family we shall use the local central limit Theorem for lattice variables in some integral form. This is Theorem [E] as applied to single lattice variable, and Theorem [3.3] when dealing with continuous families of lattice variables. We provide a proof of this last result; this requires some preliminary material.
3.1. Approximation and bound of characteristic functions. For a random variable $Y$ and integer $n \geq 1$, we have for $E(e^{i\theta Y/\sqrt{n}})^n$ the following well known approximation, Theorem 12 and bound, Theorem 13 involving the third moment of $Y$.

**Theorem D.** There exists a constant $C > 0$, such that for any random variable $Y$ verifying that $E(Y) = 0$, $E(Y^2) = 1$ and $E(|Y|^3) < \infty$, it holds that
\[
\left| E(e^{i\theta Y/\sqrt{n}})^n - e^{-\theta^2/2} \right| \leq C \frac{E(|Y|^3)}{\sqrt{n}}, \quad \text{if } |\theta| \leq \frac{\sqrt{n}}{E(|Y|^3)} \text{ and } n \geq 1.
\]

See, for instance, [14, Lemma 6.2, Chapter 7]. In particular,
\[
\lim_{n \to \infty} E(e^{i\theta Y/\sqrt{n}})^n = e^{-\theta^2/2}, \quad \text{for each } \theta \in \mathbb{R},
\]
which is the Central Limit Theorem for sums of independent identically distributed copies of the variable $Y$ with the extra assumption of finite third absolute moment.

**Theorem E.** For any random variable $Y$ such that $E(Y) = 0$, $E(Y^2) = 1$ and $E(|Y|^3) < \infty$, we have that
\[
\left| E(e^{i\theta Y/\sqrt{n}})^n \right| \leq e^{-\theta^2/3}, \quad \text{for any } \theta \text{ such that } |\theta| \leq \frac{\sqrt{n}}{4E(|Y|^3)} \text{ and } n \geq 1.
\]

See, for instance, step 1 in the proof of [14, Lemma 6.2, Chapter 7].

3.2. Lattice distributed variables. A random variable $Z$ is said to be lattice distributed or a lattice random variable if $Z$ is nonconstant and for a certain $a \in \mathbb{R}$ and $h > 0$ one has that $P(Z \in a + h\mathbb{Z}) = 1$. The largest $h$ such that $P(Z \in a + h\mathbb{Z}) = 1$ is called the gauge of the lattice variable $Z$. For gauge $h$, the variable $(Z - a)/h$ takes integer values and has gauge 1.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series in $\mathcal{K}$. Let $\mathcal{I} = \{n \geq 1 : a_n \neq 0\}$. Recall from Section 2.1.6 the notation $Q_f = \text{gcd}(\mathcal{I})$.

If $Q_f = 1$, then each $X_t$ of the Khinchin family of $f$ is a lattice variable of gauge 1 and each $\bar{X}_t$ is a lattice variable of gauge $1/\sigma(t)$. The condition $Q_f = 1$ is equivalent to the requirement that for each $d \geq 1$ there is $n \in \mathcal{I}$ such that $d \nmid n$.

In general, if $Q_f \geq 1$, then each $X_t$ of the Khinchin family of $f$ is a lattice variable of gauge $Q_f$ and each $\bar{X}_t$ is a lattice variable of gauge $Q_f/\sigma(t)$.

**Lemma C.** If $Z$ is a lattice random variable with gauge $h$, then
\[
|E(e^{i\theta Z})| < 1, \quad \text{for } 0 < \theta < 2\pi/h.
\]
In particular, for each $0 < \delta < \pi/h$, there exists $\omega = \omega_\delta < 1$ such that
\[
|E(e^{i\theta Z})| \leq \omega, \quad \text{for } |\theta - \pi/h| \leq \delta.
\]

See [8, Section 3.5]. The second part of the statement follows simply from continuity.

Next result is a standard inversion lemma: the mass function of a variable $U$ is expressed in terms of its characteristic function.
Lemma D. Let $U$ be a lattice random variable with gauge $h$, and so that for certain $a \in \mathbb{R}$ we have that $P(U \in a + h\mathbb{Z}) = 1$. Then

$$P(U = a + hk) = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} \mathbb{E}(e^{i\theta U}) e^{-i\theta(a + kh)} \, d\theta,$$

for each $k \in \mathbb{Z}$.

In other terms, for each possible value $x$ of the variable $U$ we have that

$$P(U = x) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} \mathbb{E}(e^{i\theta U}) e^{-i\theta x} \, d\theta.$$

Proof. We may assume that $a = 0$ and $h = 1$. For each $k \in \mathbb{Z}$, let $p_k = P(U = k)$. Then, $\sum_{k \in \mathbb{Z}} p_k = 1$ and

$$\mathbb{E}(e^{i\theta U}) = \sum_{k \in \mathbb{Z}} p_k e^{ik\theta},$$

for each $\theta \in \mathbb{R}$, and therefore, that

$$p_k = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} \mathbb{E}(e^{i\theta U}) e^{-ik\theta} \, d\theta,$$

for each $k \in \mathbb{Z}$. \qed

3.3. Local central limit Theorem for lattice variables: integral form.

Theorem F. Let $Z$ be a lattice random variable with gauge $h$ and such that $\mathbb{E}(Z) = 0$ and $\mathbb{E}(Z^2) = 1$. Then

$$\lim_{n \to \infty} \int_{|\theta| \leq \sqrt{n}/h} \left| \mathbb{E}(e^{i\theta Z/\sqrt{n}})^n - e^{-\theta^2/2} \right| \, d\theta = 0.$$

From the convergence of this integral, the usual statement of the Local Central Limit Theorem follows, see Section 3.3.1. In fact, Theorem F is what is actually shown (as an intermediate step) in the usual proofs of the Local Central Limit Theorem, see, for instance, [12, Section 43] and the proof of [8, Theorem 3.5.3].

We give a proof below with the extra assumption that $\mathbb{E}(|Y|^3) < \infty$ anticipating the corresponding result, Theorem C, for continuous families of lattice variables.

Proof. The central limit theorem, as in equation (3.1), gives us that, as $n \to \infty$, the integrand converges towards 0 for each $\theta \in \mathbb{R}$.

Denoting $\tau \triangleq 1/(2\mathbb{E}(|Z|^3))$, Theorem F gives us the bound

$$|\mathbb{E}(e^{i\theta Z/\sqrt{n}})^n| \leq e^{-\theta^2/6}, \quad \text{si } |\theta| \leq \tau \sqrt{n}.$$

Thus, dominated convergence gives that

$$\lim_{n \to \infty} \int_{|\theta| \leq \tau \sqrt{n}} \left| \mathbb{E}(e^{i\theta Z/\sqrt{n}})^n - e^{-\theta^2/2} \right| \, d\theta = 0.$$

Lemma C gives, in particular, a bound $\omega \in (0,1)$ such that

$$|\mathbb{E}(e^{i\theta Z})| \leq \omega, \quad \text{si } \tau \leq |\theta| \leq \pi/h,$$

and so that

$$|\mathbb{E}(e^{i\theta Z/\sqrt{n}})^n| \leq \omega^n, \quad \text{si } \tau \sqrt{n} \leq |\theta| \leq \pi \sqrt{n}/h.$$
The bound
\[ \int_{\tau \sqrt{n} \leq |\theta| \leq \pi \sqrt{n} / h} \left| \mathbf{E} \left( e^{i\theta Z / \sqrt{n}} \right)^n - e^{-\theta^2 / 2} \right| \ d\theta \leq \int_{\tau \sqrt{n} \leq |\theta| \leq \pi \sqrt{n} / h} \left( \omega^n + e^{-\theta^2 / 2} \right) d\theta, \]
tends to 0 as \( n \to \infty \). \( \square \)

3.3.1. **Local central limit theorem.** Let \( Y \) be a random variable such that \( \mathbf{E}(Y) = 0 \) and \( \mathbf{E}(Y^2) = 1 \). Let \( Y_1, Y_2, \ldots \) be a sequence of independent random variables all distributed like \( Y \).

Assume that \( Y \) is a lattice variable with gauge \( h \) and such that \( \mathbf{P}(Y \in a + h\mathbb{Z}) = 1 \), for some fixed \( a \in \mathbb{R} \).

For each \( n \geq 1 \), we denote \( S_n = (Y_1 + \cdots + Y_n) / \sqrt{n} \). The variable \( S_n \) is likewise a lattice variable, with gauge \( h / \sqrt{n} \).

Let \( \mathcal{V}_n \) denote the set of possible values of \( S_n \), i.e., \( \mathcal{V}_n = \{ a \sqrt{n} + kh / \sqrt{n} : k \in \mathbb{Z} \} \).

Applying Lemma [D] to the variable \( S_n \) and using that \( \mathbf{E}(e^{i\theta S_n}) = \mathbf{E}(e^{i\theta Y / \sqrt{n}})^n \), we have that
\[ \mathbf{P}(S_n = x) = \frac{h}{2\pi \sqrt{n}} \int_{-\pi \sqrt{n} / h}^{\pi \sqrt{n} / h} \mathbf{E}(e^{i\theta Y / \sqrt{n}})^n e^{-i\theta x} \ d\theta, \quad \text{for every } x \in \mathcal{V}_n. \]

Since
\[ \frac{1}{\sqrt{2\pi}} e^{-x^2 / 2} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\theta^2 / 2} e^{-i\theta x} \ d\theta, \quad \text{for every } x \in \mathbb{R}, \]
we deduce that
\[ \sup_{x \in \mathcal{V}_n} \left| \frac{\sqrt{n}}{h} \mathbf{P}(S_n = x) - \frac{1}{\sqrt{2\pi}} e^{-x^2 / 2} \right| \leq \frac{1}{2\pi} \int_{|\theta| \leq \pi \sqrt{n} / h} \left| \mathbf{E}(e^{i\theta Y / \sqrt{n}})^n - e^{-\theta^2 / 2} \right| \ d\theta \]
\[ \quad + \frac{1}{2\pi} \int_{|\theta| > \pi \sqrt{n} / h} e^{-\theta^2 / 2} \ d\theta. \]

Theorem [F] allows us to conclude then that

**Corollary A** (Local central limit theorem). With the notations above,
\[ \lim_{n \to \infty} \sup_{x \in \mathcal{V}_n} \left| \frac{\sqrt{n}}{h} \mathbf{P}(S_n = x) - \frac{1}{\sqrt{2\pi}} e^{-x^2 / 2} \right| = 0. \]

3.4. **Continuous families of lattice variables.** Recall that a family \( (Z_s)_{s \in [a, b]} \) of random variables indexed in the real interval \([a, b]\) is said to be **continuous** if it is continuous in distribution in the sense that if a sequence \( (s_n)_{n \geq 1} \) with \( s_n \in [a, b] \) converges to \( s \in [a, b] \) then \( Z_{s_n} \) converges to \( Z_s \) in distribution.

The family \( (Z_s)_{s \in [a, b]} \) is said to be **bounded** if \( \sup_{t \in [a, b]} \mathbf{E}(|Z_t|) < +\infty \).

**Lemma 3.1.** If the family \( (Z_s)_{s \in [a, b]} \) is continuous and bounded, then the function
\[ (s, \theta) \in [a, b] \times \mathbb{R} \to \mathbf{E}(e^{i\theta Z_s}) \in \mathbb{D} \]
is continuous.
We shall apply this lemma to the normalized version \( \tilde{X}_t \) of a Khinchin family of \( f \in \mathcal{K} \) when \( t \) is restricted to a closed interval \([a, b] \subset (0, R)\). Continuity is clear and regarding boundedness, observe that \( \mathbf{E}(|\tilde{X}_t|) \leq \mathbf{E}(|\tilde{X}_t|^2)^{1/2} = 1 \), for \( t \in [a, b] \). If \( M_f < +\infty \) (and \( R < +\infty \)), and if, besides, \( \sigma(R) < \infty \) (see Section 2.1.3), then \( (\tilde{X}_t) \) is continuous and bounded for \( t \in [a, R] \), (including \( t = R \)) for any \( a \in (0, R) \).

The (un-normalized) Khinchin family \( (X_t) \) of any \( f \in \mathcal{K} \) with radius of convergence \( R \) is continuous and bounded in any interval \([0, b] \) with \( 0 < b < R \). If \( M_f < +\infty \) (and \( R < +\infty \)) the family \( (X_t)_{t \in [0, R]} \) extended to the closed interval \([0, R] \) is continuous and bounded; observe that \( \mathbf{E}(|X_t|) = \mathbf{E}(X_t) \leq M_f \), for \( t \in [0, R] \).

**Proof.** Take two sequences \( (\theta_n)_{n \geq 1} \subseteq \mathbb{R} \) and \( (s_n)_{n \geq 1} \subseteq [a, b] \) which, respectively, converge to \( \theta \) and \( s \). First we write

\[
(3.2) \quad \mathbf{E}(e^{i\theta_n Z_{s_n}}) = \mathbf{E}(e^{i\theta Z_{s_n}}) + \mathbf{E}(e^{i\theta_n Z_{s_n}} - e^{i\theta Z_{s_n}}).
\]

By virtue of Levy’s convergence theorem, the first term in the right-hand side of (3.2) converges, as \( n \rightarrow \infty \), to \( \mathbf{E}(e^{i\theta Z_{s_n}}) \). We have the bound

\[
\left| \mathbf{E}(e^{i\theta_n Z_{s_n}} - e^{i\theta Z_{s_n}}) \right| \leq |\theta_n - \theta| \mathbf{E}(|Z_{s_n}|), \quad \text{for } n \geq 1,
\]

which follows from the numerical inequality: \( |e^x - e^y| \leq |x - y| \), for all \( x, y \in \mathbb{R} \).

Now using that \( (Z_s)_{s \in [a, b]} \) is bounded, we conclude that

\[
(s, \theta) \in [a, b] \times \mathbb{R} \rightarrow \mathbf{E}(e^{i\theta Z_s}) \in \mathbb{D},
\]

is a continuous function. \( \Box \)

Let \( (Z_s)_{s \in [a, b]} \) be a continuous family of lattice variables. For each \( s \in [a, b] \), let \( h(s) \) denote the gauge of \( Z_s \). The gauge function \( h \) is upper semicontinuous:

\[
\limsup_{u \rightarrow s} h(u) \leq h(s), \quad \text{for each } s \in [a, b].
\]

In general, the gauge of a continuous family is not a continuous function. For instance, for \( s \in [0,1/2] \), let \( Z_s \) take the values \( 0, 1/2, 1 \) with respective probabilities \( 1/2 - s, 2s, 1/2 - s \). The family \( (Z_s) \) is continuous, and \( h(s) = 1/2 \), for \( s \neq 0 \), but \( h(0) = 1 \).

The next lemma is a counterpart of Lemma [C] for continuous and bounded families; it follows by continuity from Lemma [C] and compactness.

**Lemma 3.2.** Let \( (Z_s)_{s \in [a, b]} \) be a continuous and bounded family of lattice random variables, with continuous gauge function \( h(s) \).

Let \( H > 0 \), be such that \( h(s) \leq H \), for each \( s \in [a, b] \).

Then,

\[
|\mathbf{E}(e^{i\theta Z_s})| < 1, \quad \text{if } s \in [a, b] \text{ and } 0 < |\theta| < 2\pi/h(s).
\]

In particular, for each \( 0 < \tau < \pi/H \), there exists \( \omega_\tau < 1 \) such that

\[
|\mathbf{E}(e^{i\theta Z_s})| \leq \omega_\tau, \quad \text{if } s \in [a, b] \text{ and } \tau \leq |\theta| \leq \pi/h(s).
\]

The next theorem is a uniform analogue of Theorem [E] for continuous families of lattice variables:
Theorem 3.3. Let \((Z_s)_{s \in [a, b]}\) be a continuous family of lattice variables with gauge function \(h(s)\) continuous in \([a, b]\) and such that for each \(s \in [a, b]\) we have that \(E(Z_s) = 0\), \(E(Z_s^2) = 1\) and \(E(|Z_s|^3) \leq \Gamma\), for a certain constant \(\Gamma > 0\).

Then,
\[
\lim_{n \to \infty} \sup_{s \in [a, b]} \int_{-\pi^{1/4}/h(s)}^{\pi^{1/4}/h(s)} \left| E(e^{i\theta Z_s}/\sqrt{n})^n - e^{-\theta^2/2} \right| \, d\theta = 0.
\]

Remark 3.4. The hypothesis \(\sup_{s \in [a, b]} E(|Z_s|^3) < \infty\) may be replaced by the assumption that \(\sup_{s \in [a, b]} E(|Z_s|^{2+\delta}) < \infty\), for some \(\delta > 0\), or even by assuming that the family \((|Z_s|^2)_{s \in [a, b]}\) is uniformly integrable.

Proof. First, we pay attention to the range of integration \(|\theta| \leq \sqrt{n}/(4\Gamma)\).

Theorem \(E\) gives us that
\[
\left| E(e^{i\theta Z_s}/\sqrt{n})^n \right| \leq e^{-\theta^2/3}, \quad \text{if } s \in [a, b] \text{ and } |\theta| \leq \frac{\sqrt{n}}{4\Gamma},
\]
while Theorem \(D\) gives us that
\[
\left| E(e^{i\theta Z_s}/\sqrt{n})^n - e^{-\theta^2/2} \right| \leq C \frac{\Gamma}{\sqrt{n}}, \quad \text{if } s \in [a, b] \text{ and } |\theta| \leq \frac{\sqrt{n}}{\Gamma}.
\]

Therefore, for any sequence \((s_n)_{n \geq 1}\) drawn from \([a, b]\) we deduce, from dominated convergence, that
\[
\lim_{n \to \infty} \int_{-\pi^{1/4}/(4\Gamma)}^{\pi^{1/4}/(4\Gamma)} \left| E(e^{i\theta Z_{s_n}}/\sqrt{n})^n - e^{-\theta^2/2} \right| \, d\theta = 0,
\]
and, therefore, that
\[
(*) \quad \lim_{n \to \infty} \sup_{s \in [a, b]} \int_{-\pi^{1/4}/(4\Gamma)}^{\pi^{1/4}/(4\Gamma)} \left| E(e^{i\theta Z_s}/\sqrt{n})^n - e^{-\theta^2/2} \right| \, d\theta = 0.
\]

Fix \(0 < J < H\) such that \(J \leq h(s) \leq H\), for each \(s \in [a, b]\). Take \(\tau > 0\) such that \(\tau < 1/(4\Gamma)\) and \(\tau < \pi/H\).

Since, for every \(s \in [a, b]\), we have that \(E(|Z_s|) \leq E(|Z_s|^2)^{1/2} = 1\), the family \((Z_s)_{s \in [a, b]}\) is bounded. Thus, Lemma 3.2 shows that there exists \(\omega_\tau > 1\) such that
\[
|E(e^{i\theta Z_s})| \leq \omega_\tau, \quad \text{if } s \in [a, b] \text{ and } \tau \leq |\theta| \leq \pi/h(s),
\]
and so such that
\[
|E(e^{i\theta Z_s}/\sqrt{n})^n| \leq \omega_\tau^{n}, \quad \text{if } s \in [a, b] \text{ and } \tau \sqrt{n} \leq |\theta| \leq \pi \sqrt{n}/h(s).
\]

Therefore, for each \(s \in [a, b]\) we have that
\[
\int_{\tau \sqrt{n} \leq |\theta| \leq \pi \sqrt{n}/h(s)} \left| E(e^{i\theta Z_s}/\sqrt{n})^n - e^{-\theta^2/2} \right| \, d\theta \leq \omega_\tau^n \pi \sqrt{n}/J + \int_{|\theta| \geq \tau \sqrt{n}} e^{-\theta^2/2} \, d\theta.
\]
Consequently,
\[
(**) \quad \lim_{n \to \infty} \sup_{s \in [a, b]} \int_{\tau \sqrt{n} \leq |\theta| \leq \pi \sqrt{n}/h(s)} \left| E(e^{i\theta Z_s}/\sqrt{n})^n - e^{-\theta^2/2} \right| \, d\theta = 0.
\]
Combining \((*)\) and \((**)\) we obtain the result since \(\tau < 1/(4\Gamma)\). \(\square\)
4. Coefficients of large powers

We now turn to the study of the asymptotic behavior of coefficients of large powers of functions $\psi$ in the class $\mathcal{K}$. For any $\psi \in \mathcal{K}$, we study the coefficient

$$\text{coeff}_{[k]}(\psi^n(z)), \quad \text{as } n \to \infty,$$

under a variety of assumptions upon the joint behaviour of the index $k$ and of the power $n$. At a later stage, we will consider the asymptotic behavior of the coefficients of $h(z)\psi^n(z)$ where $h(z) \in \mathcal{K}$.

Remark 4.1. The results which follow about large powers actually cover the general situation of $\psi$ with nonnegative coefficients, and not just $\psi \in \mathcal{K}$.

If $\psi$ has nonnegative coefficients and it is not in $\mathcal{K}$, then one of the two following possibilities occurs: 1) $\psi$ is a constant or a monomial like $\psi(z) = bz^m$, for some $b \neq 0$ and integer $m \geq 1$, which are trivial situations as coefficients and large powers are concerned, 2) $\psi(0) = 0$ and $\psi$ has at least two nonnegative coefficients, in which case, $\psi \in \mathcal{K}^n$ and for some integer $l \geq 1$ we have that $\varphi(z) \triangleq \psi(z)/z^l \in \mathcal{K}$, and thus $\text{coeff}_{[k]}(\psi^n) = \text{coeff}_{[k-nl]}(\varphi^n)$, for $k \geq nl$.

Henceforth, $\psi(z) = \sum_{j=0}^{\infty} b_j z^j$ will denote a power series in $\mathcal{K}$ with radius of convergence $R > 0$, and from now on we let $(Y_i)_{i \in [0,R]}$ denote its Khinchin family.

We reserve $k$ to denote index of coefficient and $n$ to denote power of $\psi$. In all asymptotic discussions below the power $n$ tends to $\infty$ (large powers), while the index $k$ could tend to $\infty$ in such a way that either $k \asymp n$ (including the case when $k/n$ tends to a finite nonzero limit) or $k = o(n)$ (including the possibility that $k$ could remain fixed) or $n = o(k)$ (which would require the power series $\psi$ to be uniformly Gaussian).

Some of these results are presented, for instance, in Gardy [11] in an elegant and systematic manner, using saddle point approximation methods. Our approach here is different: we will combine local central limit theorems for lattice random variables, with some Hayman’s formulae which express the coefficients of $\psi^n$ in terms of the characteristic function of the normalized Khinchin family $Y_t$.

If $\psi$ is the generating function of a combinatorial class, so that $b_n$ is the number of elements of size $n$ in the class, then $\text{coeff}_{[k]}(\psi^n(z))$ is the number of lists of length $n$ and total size $k$ formed with objects from the class.

If $\psi$ is the probability generating function of a random variable $Z$ so that $\sum_{n=0}^{\infty} b_n = 1$ and $Z_1, Z_2, \ldots$ are independent copies of $Z$, then

$$\text{coeff}_{[k]}(\psi^n(z)) = P(Z_1 + \cdots + Z_n = k),$$

4.1. Auxiliary function $\phi$. Recall the notation of Section 2.1.5

$$Q_f = \gcd\{n \geq 1 : a_n \neq 0\} = \lim_{N \to \infty} \gcd\{1 \leq n \leq N : a_n \neq 0\}$$

for any power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{K}$.

If $Q_\psi > 1$, then $\psi(z) = \phi(\psi(Q_\psi))$ for a certain auxiliary power series $\phi \in \mathcal{K}$ which has radius of convergence $R_Q^\psi$. Observe that $Q_\phi = 1$ and that

$$\text{coeff}_{[kQ_\psi]}(\psi^n(z)) = \text{coeff}_{[k]}(\phi^n(zQ_\psi)).$$
while for \( q \) not a multiple of \( Q_\psi \), we have that
\[
\text{COEFF}_{[q]}(\psi^n(z)) = 0.
\]

Denote by \((Z_t)_{t \in (0,R^Q)}\) the Khinchin family associated to this auxiliary function \( \phi \), then, see Section 2.1.5,
\[
Y_t \overset{d}{=} Q_\psi \cdot Z_t Q_\psi, \quad \text{for any } t \in (0,R).
\]
The mean and variance functions of \( \psi \) and \( \phi \) are related by
\[
m_\psi(t) = Q_\psi \cdot m_\phi(tQ_\psi) \quad \text{and} \quad \sigma^2_\psi(t) = Q^2_\psi \cdot \sigma^2_\phi(tQ_\psi).
\]
For each \( t \in (0,R) \), the variable \( Y_t \) is a lattice random variable with gauge \( Q_\psi \). Likewise, the normalized variable \( \tilde{Y}_t \) is a lattice random variable with gauge \( h(t) = Q_\psi / \sigma(t) \).

In subsequent analysis, we shall obtain asymptotic formulae for \( \text{COEFF}_{[k]}(\psi(z)^n) \) first under the assumption that \( Q_\psi = 1 \), and then in the general case when \( Q_\psi \geq 1 \), by considering the auxiliary power series \( \phi \), with \( Q_\phi = 1 \), and translating the results back to \( \psi \).

### 4.2. Hayman’s identities and large powers

We may express the coefficients \( \text{COEFF}_{[k]}(\psi(z)^n) \) in terms of the characteristic function of the normalized variables \((\tilde{Y}_t)_{t \in (0,R)}\) by Hayman’s identity as follows.

**Lemma 4.2.** With the notations above,
\[
\text{COEFF}_{[k]}(\psi(z)^n) = \frac{1}{2\pi} \psi^n(t) \frac{1}{t^k} \int_{|\theta| \leq \pi \sigma(t) \sqrt{n}} \mathbb{E}(e^{i\theta Y_t/\sqrt{n}})^n e^{i(nm(t)-k)\theta/(\sigma(t)\sqrt{n})} d\theta,
\]
for any index \( k \geq 1 \), power \( n \geq 1 \) and for all \( t \in (0,R) \).

**Proof.** For \( t \in (0,R) \), Cauchy’s integral formula gives that
\[
\text{COEFF}_{[k]}(\psi(z)^n) = \frac{1}{2\pi} \psi^n(t) \frac{1}{t^k} \int_{|\theta| < \pi} \frac{\psi(te^{i\theta})^n}{\psi(t)^n} e^{-i\theta k} d\theta
\]
\[
= \frac{1}{2\pi} \psi^n(t) \frac{1}{t^k} \int_{|\theta| < \pi} \mathbb{E}(e^{i\theta Y_t})^n e^{-i\theta k} d\theta
\]
\[
= \frac{1}{2\pi} \psi^n(t) \frac{1}{t^k} \frac{1}{\sqrt{n} \sigma(t)} \int_{|\theta| < \pi \sigma(t) \sqrt{n}} \mathbb{E}(e^{i\theta \tilde{Y}_t/\sqrt{n}})^n e^{i(nm(t)-k)\theta/(\sigma(t)\sqrt{n})} d\theta.
\]

Observe that the integral expression of (4.1) is greatly simplified if the radius \( t \) is such that \((*) \) \( nm(t) = k \). Whenever possible we will select and use that value of \( t \) such that \((*) \) holds exactly or at least approximately.

In the next sections, we shall use the formula (4.1) to study the asymptotic behaviour of \( \text{COEFF}_{[k]}(\psi^n(z)) \), as \( n \to \infty \), while \( k \asymp n \), \( k/n \to 0 \) or \( k/n \to \infty \).
A minor variation of the proof of Lemma 4.2 above gives that if $H(z)$ is a function holomorphic in $D(0, R)$, then, with the notations above, we have that

$$\text{coeff}_k |H(z)\psi(z)^n| = \frac{1}{2\pi} \frac{\psi^n(t)}{t^k} \frac{1}{\sqrt{n \sigma(t)}} \int |\theta| \leq \pi \sigma(t) \sqrt{n} \left( e^{i\theta Y_t / \sqrt{n}} e^{i(nm(t) - k)\theta / \pi} \right) d\theta,$$

for any index $k \geq 1$, power $n \geq 1$ and $t \in (0, R)$.

If $M_\psi < \infty$ (and $R < \infty$), then $\psi$ extends to be continuous in $\text{cl}(D(0, R))$ and the Khinchin family $(Y_t)_{t \in [0, R]}$ extends to the closed interval by adding $Y_R$ given by

$$P(Y_R = n) = b_n R^n / \psi(R), \quad n \geq 0.$$

See Section 2.1.2 and Section 2.2 and the notations therein. In this case, we have the following formula for the coefficients of $\psi$.

**Lemma 4.3.** With the notations above, if $R < \infty$ and $M_\psi = m_\psi(R) < \infty$ and $\sigma^2_\psi(R) < \infty$, then

$$\text{coeff}_k (\psi(z)^n) = \frac{1}{2\pi} \frac{\psi^n(R)}{R^k} \frac{1}{\sqrt{n \sigma(R)}} \int |\theta| \leq \pi \sigma(R) \sqrt{n} \left( e^{i\theta Y_R / \sqrt{n}} e^{i(nm(R) - k)\theta / \pi} \right) d\theta,$$

for any index $k \geq 1$ and power $n \geq 1$.

5. **Index $k$ and power $n$ are comparable**

In this section we discuss the asymptotic behaviour of $\text{coeff}_k (\psi^n(z))$ when the index $k$ and the power $n$ are such that $k \asymp n$, as $n \to \infty$.

Along this section we assume that, for certain $A, B$, fixed, $0 < A < B$, the index $k$ and the power $n$ are such that $A \leq k/n \leq B$.

To deal with this case, we assume additionally that $M_\psi > B$. This being the case, we denote $a = m^{-1}(A)$ and $b = m^{-1}(B)$.

Assume first that $Q_\psi = 1$; a restriction to be lifted shortly in Section 5.3. The family of random variables $\tilde{Y}_t$, where $t \in [a, b]$, is a continuous family of lattice random variables with gauge function $h(t) = 1/\sigma(t)$; see Section 3.4. In particular, $\max_{t \in [a, b]} E(|\tilde{Y}_t|^3) < +\infty$.

For each $n \geq 1$, take $\tau_n$ defined by $m(\tau_n) = k/n$. This choice is possible because $M_\psi > B$. For each $n \geq 1$, we have $\tau_n \in [a, b]$.

Taking $t = \tau_n$ in Hayman’s identity (4.1), the integral term in there simplifies to

$$I_n \equiv \int |\theta| \leq \pi \sigma(\tau_n) \sqrt{n} \left( e^{i\theta Y_{\tau_n} / \sqrt{n}} \right)^n d\theta.$$
Since $\sigma(\tau_n) = 1/b(\tau_n)$, Theorem 3.3 gives that

$$\lim_{n \to \infty} \int_{|\theta| \leq \pi \sigma(\tau_n) \sqrt{n}} |E(e^{i\theta Y_n}/\sqrt{n})^n - e^{-\theta^2/2}| d\theta = 0.$$  

In particular, since $\min_{t \in [a,b]} \sigma(t) > 0$, we conclude that $\lim_{n \to \infty} I_n = \sqrt{2\pi}$, and, thus, that

$$\text{COEFF}_k(\psi(z)^n) \sim \frac{1}{\sqrt{2\pi}} \frac{\phi(\tau_n')^n}{\tau_n'} \frac{1}{\sqrt{n} \sigma(\tau_n)} , \quad \text{as } n \to \infty.$$  

5.1. General $Q_\psi$. We now lift the assumption that $Q_\psi = 1$. To simplify notation, write $Q = Q_\psi \geq 1$. Consider the auxiliary function $\phi$, so that $\psi(z) = \phi(z^{Q_\psi})$.

Recall that $m_\phi(t) = Qm_\phi(t^{Q_\psi})$, let $A' = A/Q$ and $B' = B/Q$ and observe that $M_\phi = M_\psi/Q > B/Q = B'$.

Define $\tau_n'$ by $m_\phi(\tau_n') = k'/n$, for $k'$ such that $A' < k'/n < B'$.

Since $Q_\phi = 1$, we have that

$$\text{COEFF}_{k'}(\phi^n(z)) \sim \frac{1}{\sqrt{2\pi}} \frac{\phi(\tau_n')^n}{(\tau_n')^{k'}} \frac{1}{\sqrt{n} \sigma(\tau_n')} , \quad \text{as } n \to \infty.$$  

Let $\tau_n = (\tau_n')^{1/Q_\psi}$. Take $k = k'Q$, observe that $A < k/n < B$ and that

$$m_\psi(\tau_n) = Q \cdot m_\phi(\tau_n') = \frac{k'Q}{n} = \frac{k}{n},$$

and

$$\sigma(\tau_n) = Q \sigma(\tau_n').$$

Since $\text{COEFF}_k(\psi^n) = \text{COEFF}_{k'}(\phi^n)$, translating (z) into terms of $\psi$, we get that

$$\text{COEFF}_k(\psi^n(z)) \sim \frac{Q_\psi}{\sqrt{2\pi}} \frac{\psi(\tau_n)^n}{\tau_n^n} \frac{1}{\sqrt{n} \sigma(\tau_n)} , \quad \text{as } n \to \infty$$

while $A < k/n < B$ and $k$ is a multiple of $Q_\psi$.

Theorem 5.1. For $0 < A < B < M_\psi$, we have that

$$\text{COEFF}_k(\psi^n(z)) \sim \frac{Q_\psi}{\sqrt{2\pi}} \frac{\psi(\tau_n)^n}{\tau_n^n} \frac{1}{\sqrt{n} \sigma(\tau_n)},$$

as $n \to \infty$, while $A \leq k/n \leq B$ and $k$ is a multiple of $Q_\psi$, and where $\tau_n$ is given uniquely by $m_\psi(\tau_n) = k/n$.

5.2. With further information on $k/n$. We assume now that for some $L$ such that $0 < L < +\infty$ and $L \leq M_\psi$, we have that $k/n \to L$, as $n \to \infty$. With that information, we may obtain sharper asymptotic formula for $\text{COEFF}_k(\psi(z)^n)$.

Assume that $k/n \to L$, where $0 < L < +\infty$ and $L \leq M_\psi$, and, in fact, in such a way that for some $\omega \in \mathbb{R}$

$$\frac{k}{n} = L + \omega \frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right), \quad \text{as } n \to \infty.$$
or, equivalently, that
\[
(5.2) \quad \lim_{n \to \infty} \frac{nL - k}{\sqrt{n}} = -\omega.
\]

• Suppose that \( L < M_\psi \). Choose \( \tau \in (0, R) \) such that \( m(\tau) = L \). Observe that now we have a fixed value of \( \tau \), and no \( \tau_n \) varying with \( n \).

Assume first that \( Q_\psi = 1 \). If we choose \( t = \tau \) in formula (4.1) and invoke the Local Central Limit Theorem F we readily find that
\[
\text{COEFF}[k](\psi(z)^n) \sim Q_\psi \frac{e^{-\omega^2/(2\sigma^2(\tau))}}{\sqrt{2\pi \sigma(\tau)}} \frac{1}{\sqrt{n}} \frac{\psi^n(\tau)}{\tau^k}, \quad \text{as } n \to \infty.
\]

An argument much like the one in Section 5.1 allows us to deduce the general case \( Q_\psi \geq 1 \) from the case \( Q_\psi = 1 \).

**Theorem 5.2.** If \( L < M_\psi \) and
\[
\lim_{n \to \infty} \frac{nL - k}{\sqrt{n}} = -\omega,
\]
then
\[
\text{COEFF}[k](\psi(z)^n) \sim Q_\psi \frac{e^{-\omega^2/(2\sigma^2(\tau))}}{\sqrt{2\pi \sigma(\tau)}} \frac{1}{\sqrt{n}} \frac{\psi^n(\tau)}{\tau^k}, \quad \text{as } n \to \infty.
\]
where \( \tau \) is given by \( m_\psi(\tau) = L \) and \( k \) is a multiple of \( Q_\psi \).

The case where \( k = n-1 \) and thus \( L = 1 \) (and \( \omega = 0 \)) shall be used in the discussion of the Otter-Meir-Moon Theorem, Theorem G, on an asymptotic formula for the coefficients of the solutions of Lagrange’s equation.

• Suppose \( L = M_\psi \) and \( R < +\infty \). Observe that \( M_\psi < \infty \).

Now there is no \( \tau \in (0, R) \) such that \( m_\psi(\tau) = L \). But \( \psi \) extends to be continuous in \( \text{cl}(\mathbb{D}(0, R)) \) and the family \((Y_t)_{t \in [0, R]}\) may be completed with a variable \( Y_R \) which has \( m_\psi(R) = M_\psi = L \), see Section 2.1.3. If further \( \sigma^2_\psi(R) = V(Y_R) < \infty \), which occurs if \( \sum_{n=0}^{\infty} n^2 b_n R^n < \infty \), then we may use Lemma 4.3 instead of formula (4.1).

By the same argument above, we then have

**Theorem 5.3.** If \( M_\psi = L \) and \( R < +\infty \) and \( \sum_{n=0}^{\infty} n^2 b_n R^n < \infty \). If
\[
\lim_{n \to \infty} \frac{nL - k}{\sqrt{n}} = -\omega,
\]
then
\[
\text{COEFF}[k](\psi(z)^n) \sim Q_\psi \frac{e^{-\omega^2/(2\sigma^2(R))}}{\sqrt{2\pi \sigma(R)}} \frac{1}{\sqrt{n}} \frac{\psi^n(R)}{R^k},
\]
where \( k \) is a multiple of \( Q_\psi \).

**Remark 5.4.** In the remaining case, when \( L = M_f \) and \( R = +\infty \), then since \( M_f < +\infty \), we have that \( f \) is a polynomial of degree \( \deg(f) = L = M_f \). In particular \( L \) is an integer, and in the extreme case when \( k = L n \), we have that \( \text{COEFF}[k](\psi^n) = b_L^n \), where \( b_L \) is the \( L \)-th coefficient \( \psi(z) \).
5.2.1. **Binomial coefficients.** As an illustration, we apply next Theorem 5.2 to binomial coefficients. We take \( \psi(z) = 1 + z \), which belongs to \( \mathcal{K} \). In this case, we have
\[
m_\psi(t) = \frac{t}{1+t}, \quad \text{and} \quad \sigma^2_\psi(t) = \frac{t}{(1+t)^2}, \quad \text{for} \ t \in (0, +\infty).
\]
In particular, \( M_\psi = 1 \).

Observe that
\[
\text{COEFF}_k(\psi(z)^n) = \binom{n}{k}, \quad \text{for} \ n \geq 1 \text{ and } k \geq 1.
\]

- Let \( p \in (0, 1) \). For each \( n \geq 1/p \), let \( k = \lfloor pn \rfloor \). We have
\[
0 \leq p - \frac{k}{n} \leq \frac{1}{n},
\]
Thus we may apply Theorem 5.2 with \( L = p \) and \( \omega = 0 \), to deduce
\[
\binom{n}{\lfloor pn \rfloor} \sim \frac{1}{\sqrt{2\pi np(1-p)}} \frac{1}{(1-p)^{n-k}k^k}, \quad \text{as} \ n \to \infty.
\]
If \( p = 1/2 \), we have
\[
\binom{n}{\lfloor n/2 \rfloor} \sim \frac{2^{n+1}}{\sqrt{2\pi \sqrt{n}}}, \quad \text{as} \ n \to \infty.
\]

- Let \( p \in (0, 1) \) and \( \lambda \in \mathbb{R} \). For \( n \geq N \), let \( k = \lfloor pn + \lambda \sqrt{n} \rfloor \), where \( N \) is chosen so that \( pn + \lambda \sqrt{n} \geq 1 \), for \( n \geq N \). Then
\[
\frac{k}{n} = p + \frac{\lambda}{\sqrt{n}} + O\left(\frac{1}{n}\right), \quad \text{as} \ n \to \infty,
\]
and Theorem 5.2 with \( L = p \) and \( \omega = \lambda \), gives us that
\[
\binom{n}{\lfloor pn + \lambda \sqrt{n} \rfloor} \sim \frac{1}{\sqrt{2\pi np(1-p)}} \frac{1}{(1-p)^{n-k}k^k} e^{-\lambda^2/(2p(1-p))}, \quad \text{as} \ n \to \infty.
\]
For \( p = 1/2 \) and \( \lambda \in \mathbb{R} \) fixed, we have that
\[
\binom{n}{\lfloor n/2 + \lambda \sqrt{n} \rfloor} \sim \frac{2^{n+1}}{\sqrt{2\pi \sqrt{n}}} e^{-2\lambda^2}, \quad \text{as} \ n \to \infty.
\]

5.2.2. **Back to Local central limit theorem.** Let us assume that \( \psi \) is the probability generating function of a certain random variable \( X \) (with values in \( \{0, 1, 2 \ldots\} \)) which has mean \( \mu \) and standard deviation \( s \).

Assume further that \( \psi \) has radius of convergence \( R > 1 \) and that \( Q_\psi = 1 \).

We have \( M_\psi = \lim_{t \uparrow R} m_\psi(t) = m_\psi(1) = \mu \). For \( \tau = 1 \in (0, R) \), we have \( m_\psi(\tau) = \mu \) and \( \sigma_\psi(\tau) = s \).

Let \( Y \) be the random variable \( Y = (X - \mu)/s \). This variable \( Y \) is a lattice random variable with gauge \( h = 1/s \), since \( Q_\psi = 1 \). The variable \( Y \) has \( \mathbb{E}(Y) = 0 \) and \( \mathbb{E}(Y^2) = 1 \).

For each \( n \geq 1 \), let \( S_n \) denote the random variable
\[
S_n = \frac{Y_1 + \cdots + Y_n}{\sqrt{n}} = \frac{X_1 + \cdots + X_n}{s\sqrt{n}} - \left(\frac{\mu}{s}\right)\sqrt{n},
\]
where $X_1, X_2, \ldots$ are independent copies of $X$ and $Y_j = (X_j - \mu)/s$, for $j \geq 1$.

Let $\mathcal{V}_n$ denote the collection of possible values of $S_n$:

\[
\mathcal{V}_n = \left\{ v(k) = \frac{k}{s\sqrt{n}} - \left( \frac{\mu}{s} \right) \sqrt{n} : k \in \mathbb{Z} \right\}.
\]

For $x \in \mathcal{V}_n$, let $k_x = s\sqrt{n}x + \mu n$, so that $v(k_x) = x$. We have

\[
k_x = \mu + \frac{s}{\sqrt{n}}x.
\]

Moreover, with the notations of the hypothesis of Theorem 5.2, we have that $L = \mu$ and $\omega = sx$. Now we use that with $\tau = 1$, we have that $m_\psi(1) = \mu = L$, $\psi(1) = 1$, $\sigma_\psi(1) = s$ and $\omega^2/\sigma_\psi^2(1) = x^2$, and appealing to Theorem 5.2 we obtain that

\[
\text{COEFF}_{|k|}(\psi(z)^n) \sim \frac{1}{\sqrt{2\pi}} \frac{1}{s\sqrt{n}} e^{-x^2/2}, \text{ as } n \to \infty.
\]

Since

\[
P(S_n = x) = P(X_1 + \cdots + X_n = k_x) = \text{COEFF}_{|k_x|}(\psi(z)^n),
\]

we deduce that

\[
\lim_{n \to \infty} s\sqrt{n} P(S_n = x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \text{ as } n \to \infty,
\]

which is “consistent” with the local Central Limit Theorem as stated in Corollary A.

6. INDEX $k$ IS LITTLE ‘O’ OF $n$

Now we take care of the case where $k/n \to 0$ as $k, n \to \infty$.

For the discussion of this case and throughout this section we assume that $\psi'(0) \neq 0$. This hypothesis implies that $Q_\psi = 1$.

We will use formula (4.11) with an appropriate choice of $t$. To precise the limit value of the integral term as $n \to \infty$, we shall appeal to the following lemma, akin to Lemma 2 in [19].

**Lemma 6.1.** Suppose that $\psi \in K$ and $\psi'(0) \neq 0$, then for each $\theta_0 \in (0, \pi)$, there exists $c > 0$ and $r > 0$ such that

\[
|E(e^{i\theta \tilde{Y}_t})| = \left| \frac{\psi(te^{i\theta})}{\psi(t)} \right| \leq e^{-ct}, \text{ if } t \leq r \text{ and } \theta_0 \leq |\theta| \leq \pi.
\]

**Proof.** Without loss of generality we assume that $b_0 = 1$. We have

\[
|\psi(te^{i\theta})|^2 = 1 + 2b_1 t \cos \theta + O(t^2), \quad \text{as} \quad t \downarrow 0,
\]

and

\[
|\psi(t)|^2 = 1 + 2b_1 t + O(t^2), \quad \text{as} \quad t \downarrow 0.
\]

Therefore,

\[
|E(e^{i\theta \tilde{Y}_t})|^2 = \left| \frac{\psi(te^{i\theta})}{\psi(t)} \right|^2 = 1 + 2b_1 t (\cos \theta - 1) + O(t^2), \quad \text{as} \quad t \downarrow 0.
\]
For $\theta_0 \leq |\theta| \leq \pi$, we have $\cos \theta \leq 1 - \delta$, for certain $\delta > 0$, which may depend upon $\theta_0$. Since, by hypothesis, $b_1 > 0$, we then have, for certain $r \in (0, R)$, that

$$|\mathbf{E}(e^{i\theta\hat{Y}_t})|^2 = \left|\frac{\psi(te^{i\theta})}{\psi(t)}\right|^2 \leq 1 - 2b_1t\delta + O(t^2), \quad \text{for all } t \leq r.$$ 

Therefore, for $t \leq r$ we have that

$$|\mathbf{E}(e^{i\theta\hat{Y}_t})|^2 = \left|\frac{\psi(te^{i\theta})}{\psi(t)}\right|^2 \leq 1 - b_1t\delta \leq e^{-b_1\delta t},$$

as claimed. $\square$

For $n$ large enough, we have that $k/n < M\psi$ and thus we may define uniquely $\tau_n \in (0, R)$ such that $m(\tau_n) = k/n$. Observe that $\tau_n \to 0$, as $n \to \infty$.

We will use here the results from Section 2.3 about the moments of the Khinchin family $(Y_t)_{t \in [0, R)}$ when $t$ is close to 0. Since $b_1 > 0$, we have that

$$\lim_{t \downarrow 0} t^{1/2} \mathbf{E}(|\hat{Y}_t|^3) = \frac{\sqrt{b_0}}{b_1}.$$ 

As $m(t) = \frac{b_1}{b_0} t + O(t^2)$, we have that

$$\tau_n \sim \frac{b_0}{b_1} \frac{k}{n}, \quad \text{as } n \to \infty.$$ 

Besides,

$$\sigma^2(\tau_n) \sim \frac{k}{n}, \quad \text{as } n \to \infty.$$ 

For $t = \tau_n$ the expression (2) above tells us that

$$\mathbf{E}(|\hat{Y}_n|^3) \sim \sqrt{\frac{1}{k}}, \quad \text{as } k \to \infty.$$ 

If we choose $t = \tau_n$ in formula (4.1), the integral term simplifies to

$$I_n \triangleq \int_{|\theta| \leq \sqrt{k/5}} \mathbf{E}(e^{i\theta\hat{Y}_n/\sqrt{n}})^n d\theta.$$ 

Let us see that the integral $I_n$ converges to $\sqrt{2\pi}$ as $n \to \infty$.

Theorem [D] and (b) give that

$$\lim_{n \to \infty} \mathbf{E}(e^{i\theta_n/\sqrt{n}})^n = e^{-\theta^2/2}, \quad \text{for all } \theta \in \mathbb{R}.$$ 

Theorem [E] and (b) give that there exists $N \geq 1$ such that if $n \geq N$, then

$$|\mathbf{E}(e^{i\theta_n/\sqrt{n}})^n| \leq e^{-\theta^2/3}, \quad \text{for all } |\theta| \leq \sqrt{k}/5.$$ 

By dominated convergence we have

$$\lim_{n \to \infty} \int_{|\theta| \leq \sqrt{k}/5} |\mathbf{E}(e^{i\theta_n/\sqrt{n}})^n - e^{-\theta^2/2}| d\theta = 0.$$
We apply lemma 6.1 with $\theta_0 = 1/10$ to ascertain that there exists $N \geq 1$ and $C > 0$ such that for all $n \geq N$ we have that
\[
|E(e^{i\theta Y_{\tau_n}})| = \frac{|\psi(\tau_n e^{i\theta})|}{\psi(\tau_n)} \leq e^{-cn}, \quad \text{for } \frac{1}{10} \leq |\theta| \leq \pi,
\]
and, therefore, that
\[
|E(e^{i\theta Y_{\tau_n}/(\sigma(\tau_n)\sqrt{n})})|^n = \left|\frac{\psi(\tau_n e^{i\theta}/(\sigma(\tau_n)\sqrt{n}))}{\psi(\tau_n)}\right|^n \leq e^{-cn},
\]
for $\frac{1}{10}\sigma(\tau_n)\sqrt{n} \leq |\theta| \leq \pi\sigma(\tau_n)\sqrt{n}$.

Since for $n$ big enough we have that $\frac{1}{10}\sigma(\tau_n)\sqrt{n} \leq \sqrt{\kappa}/5$, we may bound
\[
\int_{\sqrt{\kappa}/5 \leq |\theta| \leq \pi\sigma(\tau_n)\sqrt{n}} \left|E(e^{i\theta Y_{\tau_n}/\sqrt{n}})^n - e^{-\theta^2/2}\right| d\theta \leq e^{-cn}(8\pi/3)\sqrt{\kappa} + \int_{|\theta| \geq \sqrt{\kappa}/3} e^{-\theta^2/2} d\theta,
\]
for $n$ large enough.

The first summand of this bound converges to 0, since $n\tau_n \sim (a_0/a_1)k$ and $k \to \infty$, while the second converges to 0 since $k \to \infty$. We conclude that
\[
\lim_{n \to \infty} \int_{\sqrt{\kappa}/5 \leq |\theta| \leq \pi\sigma(\tau_n)\sqrt{n}} \left|E(e^{i\theta Y_{\tau_n}/\sqrt{n}})^n - e^{-\theta^2/2}\right| d\theta = 0,
\]
and, therefore, that
\[
(6.1) \quad \lim_{n \to \infty} \int_{|\theta| \leq \pi\sigma(\tau_n)\sqrt{n}} \left|E(e^{i\theta Y_{\tau_n}/\sqrt{n}})^n - e^{-\theta^2/2}\right| d\theta = 0.
\]
In particular
\[
\lim_{n \to \infty} I_n = \lim_{n \to \infty} \int_{|\theta| \leq \pi\sigma(\tau_n)\sqrt{n}} E(e^{i\theta Y_{\tau_n}/\sqrt{n}})^n d\theta = \sqrt{2\pi}.
\]

In sum,

**Theorem 6.2.** If $\psi'(0) \neq 0$, then
\[
\text{COEFF}_k(\psi(z)^n) \sim \frac{1}{\sqrt{2\pi}} \frac{\psi^n(\tau_n)}{\tau_n^k} \frac{1}{\sqrt{n} \sigma(\tau_n)} \sim \frac{1}{\sqrt{2\pi}} \frac{\psi^n(\tau_n)}{\tau_n^k} \frac{1}{\sqrt{k}}, \quad \text{as } n \to \infty,
\]
where $k/n \to 0$ and $\tau_n$ is given by $m(\tau_n) = k/n$ (for $n$ large enough).

6.1. **With further information on $k/n$.** If some further information on how fast $k/n$ tends to 0 is available, then we may express the asymptotic formula (6.2) directly in terms of $k$ and $n$ and not on $\tau_n$.

We maintain the assumption that $\psi'(0) > 0$.

The function $m_\psi$ is holomorphic and injective near 0. Its inverse $m_\psi^{-1}$ is actually the solution of Lagrange’s equation with data $z/m_\psi(z) = \psi(z)/\psi'(z)$. Besides, $\ln \psi$
is holomorphic near 0, since as $\psi \in K$, we have $\psi(0) > 0$. Thus Lagrange’s inversion formula gives the expansion
\[
\ln \psi(m^{-1}_\psi(z)) = \ln b_0 + \sum_{j=1}^{\infty} B_j z^j,
\]
where
\[
B_j = \frac{1}{j} \text{COEFF}_{[j-1]} \left( \left( \frac{\psi'}{\psi} \right)^j \right) = \frac{1}{j} \text{COEFF}_{[j-1]} \left( \left( \frac{\psi}{\psi'} \right)^{j-1} \right), \quad \text{for } j \geq 1.
\]
For each $j \geq 1$, the coefficient $B_j$ depends only on $b_0, b_1, \ldots, b_j$.

Likewise,
\[
\ln \frac{m^{-1}_\psi(z)}{z} = \ln \left( \frac{\psi}{\psi'} \right) (m^{-1}_\psi(z)) = \ln b_0 + \sum_{j=1}^{\infty} C_j z^j,
\]
where
\[
C_j = \frac{1}{j} \text{COEFF}_{[j]} \left( \frac{\psi}{\psi'} \right)^j = \frac{j+1}{j} B_{j+1}, \quad \text{for } j \geq 1.
\]

Now, with the notations of the general discussion of the case $k/n \to 0$, we have that
\[
\ln \psi(\tau_n) = n \ln \psi(\tau_n) = n \ln \psi \left( m^{-1}_\psi \left( \frac{k}{n} \right) \right) = n \ln b_0 + \sum_{j=1}^{\infty} B_j \left( \frac{k^j}{n^{j-1}} \right),
\]
and also that,
\[
\ln \tau_n = k \ln \tau_n = k \ln m^{-1}_\psi \left( \frac{k}{n} \right) = k \ln b_0 + k \ln \frac{k}{n} + \sum_{j=1}^{\infty} C_j \frac{k^{j+1}}{n^j}.
\]

Using formula \((6.3)\), and that $B_1 = 1$, we may write
\[
\ln \psi(\tau_n)^n - \ln \tau_n^k = (n-k) \ln b_0 + k \ln b_1 + k \ln \frac{n}{k} + k - \sum_{j=2}^{\infty} \frac{B_j}{j-1} \frac{k^j}{n^{j-1}}.
\]

Let $\lambda = \limsup_{n \to \infty} \ln k / \ln n$ and let $J$ the smallest integer such that $J > \lambda/(1-\lambda)$; if $\lambda = 1$, we set $J = +\infty$. Thus
\[
\sum_{j=J+1}^{\infty} \frac{B_j}{j-1} \frac{k^j}{n^{j-1}} = nO \left( \frac{k}{n} \right)^{J+1} = o(1), \quad \text{as } n \to \infty,
\]
and then
\[
\frac{\psi(\tau_n)^n}{\tau_n^k} \sim b_0^{n-k} b_1^k \frac{n^k e^k}{k^k} \exp \left( - \sum_{j=2}^{J} \frac{B_j}{j-1} \frac{k^j}{n^{j-1}} \right), \quad \text{as } n \to \infty.
\]
Therefore

**Theorem 6.3.** If $\psi'(0) \neq 0$, and $\lambda$ and $J$ as above, then
\[
\text{COEFF}_{[k]}(\psi(z)^n) \sim \frac{1}{\sqrt{2\pi}} b_0^{n-k} b_1^k \frac{n^k e^k}{k^k \sqrt{k}} \exp \left( - \sum_{j=2}^{J} \frac{B_j}{j-1} \frac{k^j}{n^{j-1}} \right), \quad \text{as } n \to \infty.
\]
If $\lambda < 1/2$, then $J = 1$, and then

\[
(6.5) \quad \text{COEFF}_{[k]}(\psi(z)^n) \sim b_0^{n-k} b_1^n k^n / k! \sim \text{COEFF}_{[k]}((b_0 + b_1 z)^n), \quad \text{as } n \to \infty.
\]

For the particular case where, $k = \lfloor \sqrt{n} \rfloor$, (with $\lambda = 1/2$, and $J = 2$), we have that

\[
\text{COEFF}_{[k]}(\psi(z)^n) \sim b_0^{n-k} b_1^n k^n / k! e^{-B_2}, \quad \text{as } n \to \infty.
\]

where $B_2 = (1/2) - (b_2 b_0)/b_1^2$.

6.2. **Particular case: $k$ fixed, while $n \to \infty$.** In this particular case, where $k$ remains fixed, while $n \to \infty$, the analysis of the asymptotic behaviour of $\text{COEFF}_{[k]}(\psi(z)^n)$ just requires the multinomial theorem.

Observe that

\[
\text{COEFF}_{[k]}(\psi(z)^n) = \text{COEFF}_{[k]} \left( \sum_{j=0}^k b_j z^j \right)^n.
\]

The multinomial theorem gives then

\[
\text{COEFF}_{[k]}(\psi(z)^n) = \sum_{j_0! \ldots j_k!} n! / j_0! \ldots j_k! b_0^{j_0} \ldots b_k^{j_k},
\]

where the sum extends to all $k+1$-tuples $(j_0, \ldots, j_k)$ of nonnegative integers such that

\[
\begin{align*}
&j_0 + j_1 + \cdots + j_k = n, \\
&j_1 + 2j_2 + \cdots + k j_k = k.
\end{align*}
\]

For such a tuple, we have $n - k \leq j_0 \leq n$ and $j_i \leq k$, for $1 \leq i \leq k$. We write $j_0 = n - l$, with $0 \leq l \leq k$ and classify by $l$:

\[
\text{COEFF}_{[k]}(\psi(z)^n) = \sum_{l=0}^k \binom{n}{l} b_0^{n-l} \sum_{j_1 + \cdots + j_k = l} \binom{j_1 + \cdots + j_k}{j_1, \ldots, j_k} b_1^{j_1} \ldots b_k^{j_k}.
\]

For each integer $l, 0 \leq l \leq k$, the interior sum $C_l$ extends to all $k$-tuples with $j_1 + \cdots + j_k = l$ and $j_1 + 2j_2 + \cdots + k j_k = k$. Observe that $C_l$ does not depend upon $n$. Therefore

\[
\text{COEFF}_{[k]}(\psi(z)^n) = \sum_{l=0}^k \binom{n}{l} b_0^{n-l} C_l,
\]

is a polynomial in $n$ whose degree $\gamma$ is given by the largest $l \leq k$ such that $C_l \neq 0$, and

\[
\text{COEFF}_{[k]}(\psi(z)^n) \sim \frac{1}{\gamma!} b_0^{n-\gamma} C_\gamma n^\gamma, \quad \text{as } n \to \infty.
\]

- If $b_1 \neq 0$, then $C_k \neq 0$, and $\gamma = k$. In fact, the sum $C_k$ contains just one summand corresponding to $j_1 = k$ and $j_2 = \ldots = j_k = 0$, and thus $C_k = b_1^k$. We have in this case that

\[
\text{COEFF}_{[k]}(\psi(z)^n) \sim 1 / k! b_0^{n-k} b_1^k n^k \sim \binom{n}{k} b_0^{n-k} b_1^k, \quad \text{as } n \to \infty.
\]
In other terms and asymptotically speaking, the $k$-th coefficient of $\psi(z)^n$ behaves as the $k$-th coefficient of $(b_0 + b_1 z)^n$, as $n \to \infty$. Compare with the formula \((\ref{eq:unigau})\), where $k \to \infty$ with $n$, but slowly.

- If $b_1 = 0$, then the nonzero summands of $C_l$ must have $j_1 = 0$ and they must satisfy $j_2 + \ldots + j_k = l$ and $2j_2 + \ldots + k j_k = k$. Thus, if $C_l$ is not zero, then $2l \leq k$, and so, $\gamma \leq k/2$.

  Assume further that $b_2 \neq 0$. Distinguish: $k$ even and $k$ odd.

  If $k = 2q$, then $C_q$ has only one nonzero summand corresponding to $j_2 = q$, and all other $j$s equal to 0. Thus $C_q = b_2^q$, $\gamma = q = k/2$, and
  \[ \text{COEFF}_k(\psi(z)^n) \sim \frac{1}{(k/2)!} b_0^{n-k/2} b_2^{k/2} n^{k/2}, \quad \text{as } n \to \infty. \]

  If $k = 2q + 1$, then $\gamma \leq q$. The only nonzero summand of $C_q$ has $j_2 = q - 1, j_3 = 1$ and all other $j$s equal to 0. Thus $C_q = q b_2^q b_3$.

  If $b_3 \neq 0$, then $C_q \neq 0$, $\gamma = q$ and
  \[ \text{COEFF}_k(\psi(z)^n) \sim \frac{1}{((k-3)/2)!} b_0^{n-(k-1)/2} b_2^{(k-3)/2} b_3 n^{(k-1)/2}, \quad \text{as } n \to \infty. \]

  If $b_3 = 0$, then $C_q = 0$ and actually the degree $\gamma$ is at most $k/3$.

- In general, if $b_1 = \ldots = b_{m-1} = 0$, and $b_m \neq 0$, then for each $h, 0 \leq h \leq m-1$, the coefficients of index $k \equiv h \mod (m)$, satisfy an asymptotic formula
  \[ \text{COEFF}_k(\psi(z)^n) \sim \Omega_{k,h} n^{\gamma_{k,h}}, \quad \text{as } n \to \infty, \]
  where $\gamma_{k,h} \leq (k-h)/m$ and $\Omega_{k,h} > 0$.

  For $h = 0$, i.e., for those $k$ which are multiples of $m$, we have $\gamma_{k,0} = k/m$ and, actually, that
  \[ \text{COEFF}_k(\psi(z)^n) \sim \frac{1}{(k/m)!} b_0^{n-k/m} b_m^{k/m} n^{k/m}, \quad \text{as } n \to \infty. \]

  In particular, if $Q = Q_\psi > 1$ and $k$ is a multiple of $Q$, then
  \[ \text{COEFF}_k(\psi(z)^n) \sim \frac{1}{(k/Q)!} b_0^{n-k/Q} b_Q^{k/Q} n^{k/Q}, \quad \text{as } n \to \infty. \]

Observe the $b_0 > 0$. If $Q$ is not a divisor of $k$, then, of course, $\text{COEFF}_k(\psi(z)^n) = 0$. for every $n \geq 1$.

7. Power $n$ is little ‘o’ of index $k$

To obtain an asymptotic formula in this case where $k/n \to \infty$, as $n \to \infty$, we require more from $\psi$, namely, that $\psi$ is uniformly Gaussian. In an appendix of this paper, we describe at some length these so called uniformly Gaussian power series.

Since uniformly Gaussian power series are strongly Gaussian, Hayman’s asymptotic formula \((\ref{eq:hayman})\) is valid for $\psi$ and thus $Q_\psi = 1$. 
The assumption of uniform Gaussianity on $\psi$ is to be compared with those of Gardy in [11] Theorems 5 and 6, see also [11] Section 6.

The exponential generating function of set partitions, $e^{e^z - 1}$, and the ordinary generating function of partitions of numbers, $\prod_{j=1}^{\infty} 1/(1 - z^j)$ are examples of uniformly Gaussian power series.

Since $\psi$ is uniformly Gaussian, we have that $M_{\psi} = +\infty$, and thus we may define (uniquely) $\tau_n \in (0, R)$ as being such that $m_\psi(\tau_n) = k/n$. Observe that $\tau_n \uparrow R$, as $n \to \infty$.

Now, formula (4.1) of Lemma 4.2, with $t = \tau_n$, gives us that

$$\text{COEFF}[k](\psi(z)^n) = \frac{1}{2\pi} \psi^n(\tau_n) \frac{1}{\tau_n^k} \frac{1}{\sqrt{n} \sigma(\tau_n)} \int_{|\theta| \leq \pi \sigma(\tau_n) \sqrt{n}} E(e^{i\theta Y_{\tau_n}/\sqrt{n}})^n d\theta.$$  

Since $\tau_n \to R$, uniform Gaussianity of $\psi$ implies that

$$\lim_{n \to \infty} \int_{|\theta| \leq \pi \sigma(\tau_n) \sqrt{n}} E(e^{i\theta Y_{\tau_n}/\sqrt{n}})^n d\theta = \sqrt{2\pi},$$

and we conclude that

**Theorem 7.1.** If $\psi$ is uniformly Gaussian, then

$$\text{COEFF}[k](\psi(z)^n) \sim \frac{1}{\sqrt{2\pi}} \psi^n(\tau_n) \frac{1}{\tau_n^k} \frac{1}{\sqrt{n} \sigma(\tau_n)}, \quad \text{as } n \to \infty,$$

where $k/n \to \infty$ as $n \to \infty$ and $\tau_n$ is such that $m_\psi(\tau_n) = k/n$.

8. Coefficients of $h(z)\psi(z)^n$

We assume here from the start that $Q_\psi = 1$.

Let $h(z) = \sum_{j=0}^{\infty} c_j z^j$ be a power series in $K$. We assume throughout that the radius of convergence of $h$ is at least $R_\psi$, the radius of convergence of $\psi$.

We treat jointly the cases in which $k \asymp n$ and $k = o(n)$, with $k \to \infty$, by adjusting the previous arguments (whose notations we will use liberally) in which $h \equiv 1$.

Assume the hypothesis of Theorem 5.1 when $k \asymp n$, and of Theorem 6.2 when $k = o(n)$ and $k \to \infty$.

The $k$-th coefficient of $h(z)\psi(z)^n$ is given, see (4.2), by

$$\text{COEFF}[k](h(z)\psi(z)^n) = \frac{1}{2\pi} \psi^n(\tau_n) \frac{1}{\tau_n^k} \frac{1}{\sqrt{n} \sigma(\tau_n)} \int_{|\theta| \leq \pi \sigma(\tau_n) \sqrt{n}} h(\tau_n e^{i\theta/(\sigma(\tau_n) \sqrt{n})}) E(e^{i\theta Y_{\tau_n}/\sqrt{n}})^n d\theta,$$

where we are taking $\tau_n$ such that $m_\psi(\tau_n) = k/n$.

Denote by $I_n$ the integral on the right hand-side of the previous expressions.

For $0 \leq s \leq r < R$ and $\phi \in \mathbb{R}$ we have

$$|h(se^{i\phi}) - h(s)| \leq \max_{|z| \leq r} |h'(z)| s |\phi|.$$
For $k \gg n$, we have $\tau_n \leq n^{-1}B < R$, and for $k = o(n)$ we have $\tau_n \to 0$, so, in both cases, there exists a constant $K > 0$ such that
\[
\langle \nabla \rangle \left( h(\tau_ne^{\theta/\sigma(\tau_n)}\sqrt{n}) - h(\tau_n) \right) \leq K \frac{|\theta|}{\sigma(\tau_n)\sqrt{n}}.
\]
Denote by $D_n$ the interval $\{\theta \in \mathbb{R} : |\theta| \leq \pi\sigma(\tau_n)\sqrt{n}\}$.

Using the bound $\langle \nabla \rangle$ and writing
\[
E(e^{\theta Y_1/\sqrt{n}})^n = \left( E(e^{\theta Y_1/\sqrt{n}})^n - e^{-\theta^2/2} \right) + e^{-\theta^2/2},
\]
we conclude that
\[
I_n = h(\tau_n) \int_{D_n} e^{-\theta^2/2}d\theta + h(\tau_n) \int_{D_n} \left( E(e^{\theta Y_1/\sqrt{n}})^n - e^{-\theta^2/2} \right) d\theta
\]
\[
+ O\left( \int_{D_n} \left| E(e^{\theta Y_1/\sqrt{n}})^n - e^{-\theta^2/2} \right| \frac{|\theta|}{\sigma(\tau_n)\sqrt{n}}d\theta \right) + O\left( \int_{D_n} e^{-\theta^2/2} \frac{|\theta|}{\sigma(\tau_n)\sqrt{n}}d\theta \right).
\]

For $\theta \in D_n$, we have that
\[
\frac{|\theta|}{\sigma(\tau_n)\sqrt{n}} \leq \pi,
\]
and thus we see that the third term in the sum tends to 0, by virtue of (5.1) (if $k \gg n$) or (6.1) (if $k = o(n)$).

The second term in the sum tends to 0, since $h(\tau_n)$ is bounded and the integral converges to 0, by (5.1) and (6.1), respectively.

The fourth term in the sum tends to 0 since $\sigma(\tau_n)\sqrt{n} \approx \sqrt{k}$ and $\int_{\mathbb{R}} e^{-\theta^2/2}|\theta|d\theta = 2$.

The first term in the sum is $h(\tau_n)\sqrt{2\pi} + o(1)$, since $h(\tau_n)$ is bounded, and so
\[
I_n = h(\tau_n)\sqrt{2\pi} + o(1), \quad n \to \infty,
\]
but, given that $h(\tau_n)$ is bounded from below by $h(0)$ and recalling that $h \in K$, we conclude that
\[
I_n = h(\tau_n)\sqrt{2\pi}(1 + o(1)) , \quad n \to \infty.
\]

For the case $k \gg n$ we obtain
\[
\text{COEFF}_{[k]}(h(z)\psi(z)^n) \sim \frac{1}{\sqrt{2\pi}} \frac{h(\tau_n)\psi^n(\tau_n)}{\tau_n^k} \frac{1}{\sqrt{n} \sigma(\tau_n)} , \quad n \to \infty.
\]

For the case $k = o(n)$ and $k \to \infty$, we find that
\[
\text{COEFF}_{[k]}(h(z)\psi(z)^n) \sim \frac{h(0)\psi^n(\tau_n)}{\sqrt{2\pi}} \frac{1}{\tau_n^k} \frac{1}{\sqrt{k}} , \quad n \to \infty.
\]

Observe that, in this last case, $h(\tau_n) \to h(0)$ as $n \to \infty$.

If more information is available on how $k/n$ tends to 0, as discussed in Section 6.1, we could apply the argument and the conclusions obtained there to refine the above asymptotic formula.

Suppose that $k/n \to L > 0$, as $n \to \infty$ and that equation (5.2) is satisfied. Assume that $L < M\psi$. Then for $\tau = m^{-1}(L)$, we deduce, using the Hayman’s formula (1.2)
and arguing as in the case \( h \equiv 1 \), that

\[
(8.1) \quad \text{COEFF}_{[k]}(h(z)\psi(z)^n) \sim \frac{1}{\sqrt{2\pi}} \frac{h(\tau)\psi^n(\tau)}{\tau^k} \frac{1}{\sqrt{n} \sigma(\tau)}, \quad \text{as } n \to \infty.
\]

Recall that the power series \( h \) has radius of convergence \( R_\psi \) and that \( \tau \in (0, R_\psi) \).

To conclude, we consider the case in which \( k \) is fixed and \( n \to \infty \). Denote

\[
Q = Q_\psi = \{ n \geq 1 : b_n > 0 \}.
\]

Then

\[
\text{COEFF}_{[k]}(h(z)\psi(z)^n) = \sum_{j=0}^{k} c_j \text{COEFF}_{[k]}(z^j \psi(z)^n)
\]

The only \((k-j)\)-th coefficients of \( \psi(z)^n \) which are nonzero are the coefficients with indices \( j \equiv k \mod Q \). Appealing to the case \( h \equiv 1 \), we have that

\[
\text{COEFF}_{[k]}(h(z)\psi(z)^n) = \sum_{0 \leq j \leq k} c_j \text{COEFF}_{[k-j]}(\psi(z)^n).
\]

Denoting \( j_0 = \min\{0 \leq j \leq k : j \equiv k \mod Q \} \), we have that

\[
\text{COEFF}_{[k]}(h(z)\psi(z)^n) \sim c_{j_0} \frac{1}{((k-j_0)/Q)!} b_0^{n-(k-j_0)/Q} b_{Q}^{(k-j_0)/Q} n^{(k-j_0)/Q} (1 + o(1)).
\]

In particular, if \( h(0) = c_0 \neq 0 \) and if \( Q \) is a divisor of \( k \), then

\[
\text{COEFF}_{[k]}(h(z)\psi(z)^n) \sim c_0 \frac{1}{(k/Q)!} b_0^{n-k/Q} b_{Q}^{k/Q} n^{k/Q}, \quad \text{as } n \to \infty.
\]

Observe that in this case, \( j_0 = 0 \).

9. Coefficients of solutions of Lagrange’s equation

Next we shall apply the asymptotic results about large powers of Section 4 to obtain asymptotic formulæ for the coefficients of solutions of Lagrange’s equation when the data \( \psi \) is in \( K \).

9.1. Lagrange’s equation. We start with a function \( \psi(z) \) holomorphic in a neighborhood of \( z = 0 \), with radius of convergence \( R > 0 \) and such that \( \psi(0) \neq 0 \). Consider Lagrange’s equation:

\[
(\dagger) \quad g(w) = w\psi(g(w)).
\]

The solution \( g(w) \) of Lagrange’s equation with data \( \psi(z) \) is a holomorphic function \( g(w) = \sum_{n=1}^{\infty} A_n w^n \) which satisfies \((\dagger)\) in a neighborhood of \( w = 0 \).
Assume that the data $\psi(z)$ of Lagrange’s equation has power series expansion

$$
\psi(z) = \sum_{n=0}^{\infty} b_n z^n, \quad \text{for all } z \in \mathbb{D}(0, R).
$$

The holomorphic function $g(w)$, solution of Lagrange’s equation with data $\psi(z)$, is unique. In fact, the coefficients $A_n$ of the Taylor expansion of $g(w)$ around $w = 0$ are given by Lagrange’s inversion formula:

$$
A_n = \frac{1}{n} \text{coeff}_{[n-1]} (\psi(z)^n), \quad \text{for all } n \geq 1.
$$

For $n = 0$, we have $A_0 = g(0) = 0$.

This formula is exact for each coefficient $A_n$ of $g$ such that $n \geq 1$.

In a more general setting, consider the coefficients of $H(g(z))$, where $g$ is the solution of Lagrange’s equation with data $\psi$ and $H$ is a holomorphic function with nonnegative coefficients around $z = 0$, then

$$
\text{coeff}_{[n]} (H(g(z))) = \frac{1}{n} \text{coeff}_{[n-1]} (H'(z)\psi(z)^n), \quad \text{for all } n \geq 1,
$$

and

$$
\text{coeff}_{[0]} (H(g(z))) = H(0).
$$

A particular instance of this formula, that will be of interest later, is that given by $H(z) = z^q$ for an integer $q \geq 1$. In this case we have

$$
\text{coeff}_{[n]} (g(z)^q) = \frac{q}{n} \text{coeff}_{[n-q]} (\psi(z)^n), \quad \text{for all } n \geq 1,
$$

and

$$
\text{coeff}_{[0]} (g(z)^q) = H(0) = 0.
$$

If $Q_\psi > 1$, then the only nonzero coefficients $b_n$ are those where the index $n$ is a multiple of $Q_\psi$. Therefore, for the solution $g(w)$ of Lagrange’s equation with data $\psi$, the only nonzero coefficients $A_n$ are those where the index $n$ verifies that $n - 1$ is a multiple of $Q_\psi$.

If $\psi(0) = 0$, then the only solution of Lagrange’s equation is $g \equiv 0$, but, recall, nonetheless that $\psi \in K$ implies that $\psi(0) > 0$.

9.2. The Otter-Meir-Moon theorem and some extensions. The Otter-Meir-Moon Theorem, Theorem G below, comes from [22, Theorem 4] and [18, Theorem 3.1]. It gives an asymptotic formula for the coefficients $A_n$ of $g$, when $n-1$ is a multiple of $Q_\psi$ and $n \to \infty$, using minimal (but crucial) information about the function $\psi$.

We distinguish three cases, according as $M_\psi$ is $> 1, = 1$ or $< 1$.

♦ $M_\psi > 1$. This is the original assumption of both Otter and Meir-Moon; it means that there is a unique $\tau \in (0, R)$ such that $m_\psi(\tau) = 1$.

**Theorem G** (Otter-Meir-Moon theorem). Let $\psi(z)$ be a power series in $K$, with radius of convergence $R > 0$ and let $\psi(z) = \sum_{n=0}^{\infty} b_n z^n$ be its power series expansion.

Assume that $M_\psi > 1$ and let $\tau \in (0, R)$ be given by $m_\psi(\tau) = 1$.

Then the coefficients $A_n$ of the solution $g(w)$ of Lagrange’s equation verify that

- if $n \not\equiv 1 \mod Q_\psi$, then $A_n = 0$,
for the indices $n$ such that $n \equiv 1 \mod Q\psi$ we have the asymptotic formula

\begin{equation}
A_n \sim \frac{Q\psi}{\sqrt{2\pi}} \frac{\tau}{\sigma_\psi(\tau)} \frac{1}{n^{3/2}} \left( \frac{\psi(\tau)}{\tau} \right)^n, \text{ as } n \to \infty.
\end{equation}

Theorem \[\text{G}\] of Otter and Meir-Moon follows readily from Theorem \[5.2\] with $L = 1$ and $\omega = 0$, since $k = n - 1$.

Given that

\begin{equation}
\sigma_\psi^2(t) = m_\psi(t)(1 - m_\psi(t)) + t^2 \frac{\psi''(t)}{\psi(t)}, \text{ for all } t \in [0, R),
\end{equation}

we have

\begin{equation}
\frac{\psi''(\tau)}{\psi(\tau)} = \frac{\sigma_\psi^2(\tau)}{\tau^2}.
\end{equation}

We thus may rewrite the conclusion of Theorem \[\text{G}\] (in Laplace’s method or saddle point approximation style) as

\begin{equation}
A_n \sim \frac{Q\psi}{\sqrt{2\pi}} \frac{\sqrt{\psi(\tau)}}{\sigma_\psi(\tau)} \frac{1}{n^{3/2}} \left( \frac{\psi(\tau)}{\tau} \right)^n, \text{ when } n - 1 \text{ is a multiple of } Q\psi \text{ and } n \to \infty.
\end{equation}

\[\diamond \] $M_\psi = 1$ (and $R < +\infty$). For the case $M_\psi = 1$, there is an Otter-Meir-Moon like theorem under a certain non-degeneracy condition on $\psi$. Recall the notations of Section 2.1.2.

Theorem 9.1. Let $\psi(z)$ be a power series in $K$ with radius of convergence $R > 0$ and let $\psi(z) = \sum_{n=0}^{\infty} b_n z^n$ be its power series expansion.

Assume that $M_\psi = 1$ and that $R < \infty$. And besides that $\sum_{n=0}^{\infty} n^2 b_n R^n < +\infty$ or equivalently $\psi''(R) < +\infty$.

Then the coefficients $A_n$ of the solution $g(w)$ of Lagrange’s equation verify that

- if $n \not\equiv 1 \mod Q\psi$, then $A_n = 0$,
- for the indices $n$ such that $n \equiv 1 \mod Q\psi$ we have the asymptotic formula

\begin{equation}
A_n \sim \frac{Q\psi}{\sqrt{2\pi}} \frac{R}{\sigma_\psi(R)} \frac{1}{n^{3/2}} \left( \frac{\psi(R)}{R} \right)^n, \text{ as } n \to \infty.
\end{equation}

The power series $\psi$ extends to be continuous in $\text{cl}(\mathbb{D}(0, R))$ and the Khinchin family $(Y_t)_{t \in [0, R)}$, extends (continuously in distribution) to include a variable $Y_R$ with mean 1 and variance $\sigma_\psi^2(R) = \lim_{t \uparrow R} \sigma_\psi^2(t)$ which is finite because by hypothesis $\sum_{n=0}^{\infty} n^2 b_n R^n < +\infty$. See Section 2.1.2.

Theorem 9.1 follows readily from Theorem 5.3 with $L = 1$ and $\omega = 0$, since $k = n - 1$.

This limit case of the Otter-Meir-Moon Theorem appears as Remark 3.7 in [9, Chapter 3]; the proof suggested there, of a different nature than the one above, comes from the Appendix of [17]. See also [20, Section 2.3.1]. In there you may find the same asymptotic result under the assumption that $\psi$ is a probability generating function satisfying $\psi^{(4)}(R) = \lim_{t \uparrow R} \psi^{(4)}(t) < \infty$. 

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Observe that, in particular, it follows from Theorem \(\text{G} \) that when \( M_\psi > 1 \) and with \( m_\psi(\tau) = 1 \) the radius of convergence of the solution \( g \) is \( \tau/\psi(\tau) \). When \( M_\psi = 1 \) (and \( R < +\infty \)), the radius of convergence of \( g \) is \( R/\psi(\tau) \).

If \( M_\psi = 1 \) and \( R = +\infty \), then \( \psi \) is a polynomial of degree 1. Thus \( \psi(z) = a + bz \), with both \( a, b > 0 \). In this case the solution \( g(z) \) of Lagrange equation with data \( \psi \) is simply

\[
g(z) = \frac{az}{1 - bz} = \sum_{n=1}^{\infty} (ab^{n-1}) z^n.
\]

and thus

\[
A_n = ab^{n-1}, \quad \text{para } n \geq 1.
\]

\( \blacklozenge \) \( M_\psi < 1 \). In this case, we necessarily have that \( R < \infty \), since \( R = \infty \) and \( M_\psi < 1 \) would imply that \( \psi \) is a constant, which is not allowed. See Lemma \( \text{A} \).

We assume as above that \( \sum_{n=0}^{\infty} n^2b_n R^n < \infty \). As in the preceding case \( M_\psi = 1 \) above, the power series \( \psi \) is continuous in \( \text{cl}(D(0,R)) \) and the Khinchin family extends to include continuously a random variable \( Y_R \) with mean \( M_\psi < 1 \) and variance

\[
\lim_{t \uparrow R} \sigma^2_\psi(t) = \sigma^2_\psi(R) < \infty.
\]

Assume for simplicity that \( Q_\psi = 1 \). In this case, we do not obtain a proper asymptotic formula, just that

\[
(9.6) \quad \lim_{n \to \infty} A_n\frac{R^{n-1}}{\psi^n(R)}n^{3/2} = 0.
\]

For \( A_n \) we have that

\[
A_n = \frac{1}{n} \text{COEFF}_{[n-1]}(\psi^n) = \frac{1}{2\pi} \frac{\psi^n(R)}{R^{n-1}} \frac{1}{n^{3/2}} \frac{1}{\sigma_\psi(R)} I_n
\]

where

\[
I_n \equiv \int_{|\theta| \leq \sigma(R)\sqrt{n}} \mathbb{E}(e^{i\theta Y_R/\sqrt{n}})^n e^{i\theta \alpha(n)} d\theta, \quad \text{for } n \geq 1,
\]

and

\[
\alpha(n) = \frac{nM_\psi - (n-1)}{\sigma(R)\sqrt{n}}, \quad \text{for } n \geq 1.
\]

We shall now verify that \( I_n \) tends towards 0 as \( n \to \infty \). We split \( I_n \) as

\[
I_n = \int_{|\theta| \leq \sigma(R)\sqrt{n}} \left( \mathbb{E}(e^{i\theta Y_R/\sqrt{n}})^n - e^{-\theta^2/2} \right) e^{i\theta \alpha(n)} d\theta
\]

\[
+ \int_{\mathbb{R}} e^{-\theta^2/2} e^{i\theta \alpha(n)} d\theta
\]

\[
- \int_{|\theta| > \sigma(R)\sqrt{n}} e^{-\theta^2/2} e^{i\theta \alpha(n)} d\theta.
\]

The third summand obviously tends to 0, while the first summand tends to 0 on account of Theorem \( \text{E} \) the integral form of the Local Central Limit Theorem. The
second summand may be written as
\[
\int_{\mathbb{R}} e^{-\theta^2/2} e^{\theta \alpha(n)} d\theta = e^{-\alpha(n)^2/2}
\]
to observe that since \(M_\psi < 1\), we have that \(\lim_{n \to \infty} \alpha(n) = -\infty\), and thus that this second summand too converges to 0, as \(n \to \infty\).

Minami in [20, Section 2.3.2] obtains a faster rate of convergence (higher power of \(n\)) in (9.6) under the stronger assumption that \(\lim_{t \to R} \psi^{(k)}(t) < \infty\) for some \(k \geq 3\).

9.3. Coefficients of powers of solutions of Lagrange’s equation. We assume \(Q_\psi = 1\). We are interested now, see Meir-Moon [19], in asymptotic formulas for the coefficients of powers of the solution of Lagrange’s equation.

For \(q \geq 1\) and \(n \geq 1\), the \(n\)-th coefficient of \(g(w)^q\), which we will denote by \(B_{n,q}\), is given by the exact formula
\[
B_{n,q} = \frac{q}{n} \text{COEFF}_{[n,-q]}(\psi(z)^n).
\]

- For \(q \geq 1\) fixed, Theorem 5.2 gives, under the assumption \(M_\psi > 1\) and with \(\tau\) given by \(m_\psi(\tau) = 1\), that
\[
B_{n,q} \sim \frac{q}{\sqrt{2\pi \sigma_\psi(\tau)}} \frac{\tau^q}{\sigma_\psi(\tau)^{3/2}} \left( \frac{\psi(\tau)}{\tau} \right)^n, \quad \text{as } n \to \infty.
\]

- If \(q\) and \(n\) are related by \(q = \alpha n + \beta \sqrt{n} + o(\sqrt{n})\) as \(n \to \infty\), then for \(\tau_\alpha\) given by \(m_{\psi}(\tau_\alpha) = 1 - \alpha\), we obtain, analogously to (8.1), that
\[
B_{n,q} \sim e^{-\beta^2/(2\sigma_\psi^2(\tau_\alpha))} \frac{q}{\sqrt{2\pi \sigma_\psi(\tau_\alpha)^{3/2}}} \frac{\tau_{\alpha}^q}{\sigma_\psi(\tau_\alpha)^{3/2}} \left( \frac{\psi(\tau_\alpha)}{\tau_\alpha} \right)^n, \quad \text{as } n \to \infty.
\]

Observe that
\[
\sigma_\psi(\tau_\alpha) = \alpha(1 - \alpha) + \tau_\alpha^2 \frac{\psi''(\tau_\alpha)}{\psi'(\tau_\alpha)}.
\]

9.3.1. Coefficients of functions of solutions of Lagrange’s equation. In a more general setting, let \(H\) be a power series with nonnegative coefficients.

Assume further that the radius of convergence of \(H\) is at least the radius of convergence \(R\) of \(\psi\) and also that \(M_\psi > 1\). Let \(\tau\) be such that \(m_\psi(\tau) = 1\).

It follows from formulas (9.1) and (8.1) that the coefficients of \(H(g(z))\), where \(g(z)\) is the solution of Lagrange’s equation with data \(\psi\), satisfy:
\[
\text{COEFF}_{[n]}(H(g(z))) \sim \frac{1}{\sqrt{2\pi \sigma_\psi(\tau)}} \frac{H'(\tau)}{\tau^{3/2}} \left( \frac{\psi(\tau)}{\tau} \right)^n, \quad \text{as } n \to \infty.
\]

10. Probability generating functions

Let \(\psi\) be the probability generating function of a random variable \(X\).

We assume for convenience that \(\psi(0) \neq 0\) and that \(\psi'(0) \neq 0\), and further that \(Q_\psi = 1\), that \(M_\psi = +\infty\) and that the radius of convergence of \(\psi\) is \(R > 1\). This last assumption implies, in particular, that the variable \(X\) has finite moments of all orders.
Let \( F \) be the fulcrum of \( \psi \), see Section 2.1.6, given by \( F(z) = \ln \psi(e^z) \), in a region containing \([0, R_\psi)\). Recall that in terms of the fulcrum \( F \) we have that \( m_\psi(t) = F'(s) \) and \( \sigma^2_\psi(t) = F''(s) \), where \( s \) and \( t \) are related by \( e^s = t \).

The function \( e^{F(z)} = \psi(e^z) = \sum_{n=0}^{\infty} b_n e^{nz} \), which is holomorphic in a neighborhood of \( z = 0 \), is the moment generating function of \( X \).

If \( X_1, X_2, \ldots \) are independent copies of \( X \), then
\[
\text{COEFF}_k(\psi(z)^n) = P\left( \frac{X_1 + \cdots + X_n}{n} = \frac{k}{n} \right).
\]

Let \( m_\psi(\tau_n) = k/n \) and let \( s_n \) be given by \( e^{s_n} = \tau_n \). Observe that \( \psi(\tau_n) = e^{F(s_n)} \) and that \( \sigma^2_\psi(\tau_n) = F''(s_n) \).

In terms of the fulcrum \( F \) (or the \( \ln \) of the moment generating function), we have that if \( k \sim n \), see Theorem 5.1 or \( k/n \to 0 \), see Theorem 6.2, that
\[
P\left( \sum_{n=0}^{\infty} G_n = k \right) \sim \frac{1}{\sqrt{2\pi n \sigma_\psi(\tau_n)}} \frac{\psi(\tau_n)^n}{\tau_n^k} \times \frac{1}{\sqrt{2\pi n \sigma_\psi(\tau_n)}} e^{n[F(s_n) - \sigma_\psi(\tau_n)(k/n)]}, \quad \text{as } n \to \infty.
\]

If \( \psi \) is uniformly Gaussian, the asymptotic formula above holds also if \( k/n \to \infty \).

### 10.1. Lagrangian distributions.

We now introduce the so-called Lagrangian probability distributions. We refer to [26] for a neat presentation of the basic theory of these probability distributions. See also [5], for a comprehensive treatment, and [23].

We start with a probability generating function \( \phi \) of a variable \( Y \) taking values in \( \{0, 1, \ldots\} \). We assume that \( \phi \) is in \( K \).

This variable \( Y \) generates a cascade or Galton-Watson random process starting (initial stage) with a single individual. The variable \( Y \) gives the random number of immediate descendants, the offspring, of each individual in every generation.

The random number of individuals in generation \( n \) is denoted by \( G_n \); the generation 0 is the initial stage. We have \( G_0 = 1 \) and
\[
G_{n+1} = \sum_{j=1}^{G_n} Y_{n,j}, \quad \text{for } n \geq 0,
\]
where the \( Y_{n,j} \) are independent copies of \( Y \).

Let us denote \( \phi'(1) \triangleq \lim_{t \uparrow 1} \phi'(t) = E(Y) \). Observe that \( \phi'(1) = m_\phi(1) \).

The total progeny \( Z \) of the single individual of the generation 0 (including itself) is the random variable \( Z = \sum_{n=0}^{\infty} G_n \). This \( Z \) could take the value \( \infty \), but it is a proper random variable (i.e., \( P(Z < \infty) = 1 \)) if and only if \( \phi'(1) \leq 1 \).

Assume thus that \( \phi'(1) \leq 1 \). The probability generating function \( g \) of the total progeny \( Z \) is actually the solution of Lagrange’s equation with data \( \phi \).

Furthermore, let \( f \) be a power series with nonnegative coefficients which is the probability generating function of a random variable \( X \) taking values in \( \{0, 1, \ldots\} \).
We enhance the process by allowing the size of the initial stage (generation 0) to be randomly chosen following the distribution of $X$. Thus $G_0 = X$ and the evolution is determined by the recurrence (*). The total progeny $Z$ including the individuals of the initial stage has probability generating function $f \circ g$. This size of progeny $Z$ is said to follow a Lagrangian distribution $L(\phi, f)$; $\phi, f$ are called the generators of $L(\phi, f)$.

If $f(z) \equiv z$, then $X \equiv 1$, and there is (deterministically) a single individual in the initial generation.

We have, see (9.1), that

$$P(Z = n) = \text{coeff}[n](f(g(z))) = \frac{1}{n} \text{coeff}[n-1](f'(z)\phi^n(z)), \quad \text{for } n \geq 1,$$

and $P(Z = 0) = f(0)$.

10.2. Lagrangian distributions and Khinchin families. We will have two ingredients: a Khinchin family of offspring probability distributions and a Khinchin family of probability distribution for the initial distribution.

(a) Let $\psi(z) = \sum_{n=0}^{\infty} b_n z^n$ be a power series in $\mathcal{K}$ with radius of convergence $R$. We assume from the outset that $Q \psi = 1$. We let $(Y_t)_{t \in [0,R)}$ be such that $M \psi > 1$ and then we let $\tau \in (0, R)$ be such that $m \psi(\tau) = 1$ or $M \psi = 1$ and $R < \infty$ and $\lim_{t \uparrow R} \sigma \psi(t) = \sigma \psi(R) < \infty$ and then we let $\tau = R$.

For each $t \in (0, \tau]$, we let $\psi_t$ denote the power series $\psi_t(z) = (t z)/\psi(t)$. This $\psi_t$ is the probability generating function of $Y_t$, and besides, since $t \leq \tau$, we have that

$$\psi_t'(1) = m \psi(t) \leq 1.$$

For each $t \in (0, \tau]$, we let $g_t$ be the solution of Lagrange’s equation with data $\psi_t$:

$$(10.1)\quad g_t(z) = z \psi_t(g_t(z)).$$

Because of (b) and the discussion above in Section 10.1, $g_t$ is the probability generating function of the random distribution of the total progeny of a single individual with offspring distribution $Y_t$.

We let $g$ denote the solution of Lagrange’s equation with data $\psi$, then we may write each $g_t$ in terms of $g$ as follows

**Lemma 10.1.** With the notations above

$$g_t(z) = \frac{1}{t} g\left((t/\psi(t))z\right), \quad \text{for } t \leq \tau.$$

**Proof.** This is a consequence of the uniqueness of solution of Lagrange’s equation. Let $\tilde{g}_t(z) = \frac{1}{t} g\left((t/\psi(t))z\right)$. Now,

$$z \psi_t(\tilde{g}_t(z)) = \frac{z}{\psi(t)} \psi(t \tilde{g}_t(z)) = \frac{z}{\psi(t)} \psi(g(t/\psi(t))z) = \frac{1}{t} \frac{zt}{\psi(t)} \psi(g(t/\psi(t))z) = \frac{1}{t} g(t/\psi(t))z = \tilde{g}_t(z).$$

Uniqueness gives that $\tilde{g}_t(z) = g_t(z)$, as claimed. □
Remark 10.2. The $g_t$ are not the probability generating functions of a Khinchin family, but if we change parameters and substitute $t$ by $g(u)$ and let

$$
\tilde{g}_u(z) = g_{g(u)}(z)
$$

we have that $\tilde{g}_u(z) = g(uz)/g(u)$. The power series $g$ is not in $\mathcal{K}$ but $g(z)/z$ is in $\mathcal{K}$, since $g'(0) = \psi(0) > 0$. If $(W_u)$ is the Khinchin family of $g(z)/z$ then $\tilde{g}_u$ is the probability generating functions of $W_u + 1$.

(b) Next, we let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a nonconstant power series with nonnegative coefficients and radius of convergence $S > 0$. We do not require $f$ to be in $\mathcal{K}$; in fact $f(z) = z^m$, for integer $m \geq 1$, is particularly relevant.

For $s \in (0, S)$ we denote by $f_s$ the power series

$$
f_s(z) = \frac{f(sz)}{f(s)}
$$

which is the probability generating function of a random variable $X_s$, say.

Remark 10.3. If the power series $f$ were not in $\mathcal{K}$, then since it is not a constant, there should exist at least one index $m \geq 1$ so that $a_m \neq 0$. Thus only two cases could occur. In the first case, $f(z) = a_m z^m$, for some integer $m \geq 1$. And in the second; there is $\phi \in \mathcal{K}$ and an integer $l \geq 0$ so that $f(z) = z^l \phi(z)$.

In the first case $(X_s)$ is not a Khinchin family, but $X_s \equiv m$, for all $s \in (0, S)$, and therefore there are exactly $m \geq 1$ nodes in the first generation; this is the deterministic case $f_s(z) = z^m$.

In the second case $(X_s)_{s \in [0, S)}$ is a shifted Khinchin family. If $f(0) > 0$, then $(X_s)_{s \in (0, S)}$ is a proper Khinchin family.

Now, for $s \in (0, S)$ and $t \in (0, \tau]$, the composition $f_s(g_t(z))$ is the probability generating function of the total progeny $Z_{s,t}$ of a Galton-Watson process with initial distribution $f_s$ and offspring distribution $g_t$. The progeny $Z_{s,t}$ has Lagrange distribution $\mathcal{L}(\psi_t, f_s)$.

For radius $u > 0$ such that $us < S$ and $ut \leq \tau$ and appealing to (9.1) we may write

$$
P(Z_{s,t} = n) = \text{COEFF}[n](f_s(g_t(z))) = \frac{1}{n} \text{COEFF}[n-1](f'_s(z)\psi_t^n(z))
$$

$$
= \frac{s \psi(tu)^n}{f(s) \psi(t)^n} \frac{1}{u^{n-1}} \frac{1}{2\pi} \int_{|\theta| \leq \pi} f'(sue^{i\theta})\psi(tue^{i\theta})^n \psi(tu)^n e^{-n(n-1)\theta} d\theta.
$$

If the parameter $s$ is further restricted to $s\tau < tS$, we may take $u = \tau/t$ in the expression above and write

$$
P(Z_{s,t} = n) = \frac{1}{2\pi} \frac{s \psi(\tau)^n}{f(s) \psi(t)^n} \left( \frac{\tau}{\tau} \right)^{n-1} \frac{1}{n} \frac{1}{|\theta| \leq \pi} \int_{|\theta| \leq \pi} f'(s\tau e^{i\theta}) E(e^{(Y_{\tau}/\sqrt{\pi})})^n e^{-n(n-1)\theta} d\theta
$$

$$
= \frac{1}{2\pi} \frac{s \psi(\tau)^n}{f(s) \psi(t)^n} \left( \frac{\tau}{\tau} \right)^{n-1} \frac{1}{n^{3/2} \sigma_\psi(\tau)}
$$

$$
\int_{|\theta| \leq \pi} f'(s\tau e^{i\theta}/(\sigma_\psi(\tau)\sqrt{\pi})) E(e^{(Y_{\tau}/\sqrt{\pi})})^n e^{i\theta/(\sigma_\psi(\tau)\sqrt{\pi})} d\theta
$$
For \( \theta \) fixed, we have that
\[
\lim_{n \to \infty} f'(\frac{S_T}{t} e^{\theta/(\sigma_\psi(\tau) \sqrt{n})}) e^{\theta/(\sigma_\psi(\tau) \sqrt{n})} = f'(\frac{S_T}{t})
\]
and
\[
|f'(\frac{S_T}{t} e^{\theta/(\sigma_\psi(\tau) \sqrt{n})}) e^{\theta/(\sigma_\psi(\tau) \sqrt{n})}| \leq f'(\frac{S_T}{t}).
\]
And thus taking into account the integral form of the local central limit theorem, Theorem \( \text{[F]} \) we deduce that
\[
\lim_{n \to \infty} \int_{|\theta| \leq \pi \sigma_\psi(\tau) \sqrt{n}} f'(\frac{S_T}{t} e^{\theta/(\sigma_\psi(\tau) \sqrt{n})}) E(e^{i(Y - \sqrt{n})^2 e^{\theta/(\sigma_\psi(\tau) \sqrt{n})}}) d\theta = \sqrt{2\pi} f'(\frac{S_T}{t})
\]
and, consequently, that
\[
(10.2) \quad P(Z_{s,t} = n) \sim \frac{1}{\sqrt{2\pi}} s \psi(t)^n (\frac{t}{\tau})^{n-1} \frac{1}{n^{3/2}} \frac{1}{\pi \sqrt{n}} f'(\frac{S_T}{t}), \quad \text{as } n \to \infty,
\]
as long as \( st < tS \), which amounts to no restriction if \( S = +\infty \).

(c) As an illustration, consider the case where \( \psi(z) = e^z \) and \( f(z) = z^j \), for some integer \( j \geq 1 \).

In this case, \( R = S = \infty \), \( m_\psi(t) = t \) and \( \sigma_\psi^2(t) = t \). Also \( M_\psi = \infty \) and \( \tau = 1 \).

For \( t \leq 1 = \tau \), we have that \( \psi(t) = e^{t(z-1)} \) and for \( s < \infty \), we have that \( f_s(z) = z^j \).

Observe that for any \( s, f_s \) is the probability generating function of the constant \( j \).

For \( 0 < t \leq 1 \) and \( 0 < s < \infty \), the variable \( Z_{s,t} \) is the total progeny of a Galton-Watson process, where the initial generation consists of exactly \( j \) individuals and the offspring of each individual is given by a Poisson variable of parameter \( t \). This distribution, \( \mathcal{L}(e^{t(z-1)}, z^j) \), is the Borel-Tanner distribution with parameters \( t \) and \( j \). The case \( j = 1 \) is the Borel distribution. See \( \text{[5, 21 and 26]} \), and also the original sources \( \text{[2, 15 and 27]} \).

Using \( \text{[III.2]} \), we deduce that
\[
P(Z_{s,t} = n) \sim \frac{j}{\sqrt{2\pi}} \frac{1}{n^{3/2}} \frac{1}{n-j} e^{n(1-t)}, \quad \text{as } n \to \infty.
\]
In fact, for the Borel-Tanner distribution with parameters \( t \) and \( j \) we have the exact formula
\[
(10.3) \quad P(Z_{s,t} = n) = \frac{j e^{-ln(t)(n-j)}}{n (n-j)!}, \quad \text{for } n \geq j.
\]
The asymptotic formula above follows then from Stirling’s formula.

(d) Consider now the case \( \psi(z) = e^z \) and \( f(z) = e^z \), so that \( R = S = \infty \), \( m_\psi(t) = t \) and \( \sigma_\psi^2(t) = t \) and, also, \( M_\psi = \infty \) and \( \tau = 1 \).

For \( t \leq 1 = \tau \), we have that \( \psi(t) = e^{t(z-1)} \) and for \( s < \infty \), we have that \( f_s(z) = e^{s(z-1)} \).

For \( 0 < t \leq 1 \) and \( 0 < s < \infty \), the variable \( Z_{s,t} \) is the total progeny of a Galton-Watson process, where the size of the initial generation is drawn from a Poisson distribution of parameter \( s \) individuals and the offspring of each individual is given by a Poisson variable of parameter \( t \). This distribution, in Lagrangian distribution notation, is \( \mathcal{L}(e^{t(z-1)}, e^{s(z-1)}) \).
Remark 10.3. \( \text{sf} \)

where with a slight abuse of notation we have written \( R \).

observe that \( \) Watson process starting with a single individual and with offspring distribution \( f \) that \( a \) (proper) random variable.

we have \( \) Back to Galton-Watson. (e) See [26]. The asymptotic formula above follows then from Stirling’s formula.

Assume that \( M_\psi > 1 \) or \( M_\psi = 1 \) with \( R < \infty \) and \( \sigma_\psi(R) < \infty \). In the first case we take \( \tau \in (0,R) \) such that \( m_\psi(\tau) = 1 \). Since \( \psi'(1) \leq 1 \), we have that \( m_\psi(1) = 1 \), and since \( m_\psi \) is increasing we see that \( 1 \leq \tau < R \). In the second case we take \( \tau = R; \) observe that \( R \geq 1 \).

With \( t = 1 \) (so that \( \psi_1(z) \equiv \psi(z) \)) and \( s = 1 \) (an immaterial choice since \( f(z) = z \)), we have

\[
\mathbf{P}(Z = n) \sim \frac{1}{\sqrt{2\pi} \sigma_\psi(\tau)} \left( \frac{\psi(\tau)}{\tau} \right)^n \frac{1}{n^{3/2}}, \quad \text{as } n \to \infty.
\]

This is Theorem [G] applied to the solution of Lagrange’s equation with data \( \psi \). Recall that \( \sigma_\psi^2(\tau) = \tau^2 \psi''(\tau)/\psi(\tau) \), see formula [9.4].

10.2.1. Limit cases. By limit cases we mean \( t = \tau \) and \( s \to S \), with \( S < \infty \). See [21] for related results.

For \( t = \tau \) and \( s < S \) we have the exact formula

\[
\mathbf{P}(Z_{s,\tau} = n) = \frac{1}{2\pi f(s)} \frac{s}{n^{3/2} \sigma_\psi(\tau)} \int_{|\theta| \leq \pi \sigma_\psi(\tau) \sqrt{n}} \frac{f'(se^{i\theta}/(\sigma_\psi(\tau) \sqrt{n}))}{f'(s)} E(e^{iY_{n\theta}/\sqrt{n}})^n e^{i\theta}/(\sigma_\psi(\tau) \sqrt{n}) d\theta
\]

and the asymptotic formula.

\[
\mathbf{P}(Z_{s,\tau} = n) \sim \frac{1}{\sqrt{2\pi}} m_f(s) \frac{1}{\sigma_\psi(\tau)} \frac{1}{n^{3/2}}, \quad \text{as } n \to \infty,
\]

where with a slight abuse of notation we have written \( sf'(s)/f(s) = m_f(s) \), see Remark [10.4].

To consider \( s \to S \), we first rewrite \((b)\) as

\[
\mathbf{P}(Z_{s,\tau} = n) = \frac{1}{2\pi} \frac{m_f(s)}{n^{3/2}} \frac{1}{\sigma_\psi(\tau)} \int_{|\theta| \leq \pi \sigma_\psi(\tau) \sqrt{n}} E(e^{iY_{n\theta}/\sqrt{n}})^n e^{i\theta}/(\sigma_\psi(\tau) \sqrt{n}) d\theta
\]

(\(b)\)
If \( \sum_{n=0}^{\infty} n a_n S^n < \infty \), then \( f' \) extends continuously to \( \text{cl}(\mathbb{D}(0, S)) \), and by appealing to Theorem [\( \mathbb{F} \)] we readily see that

\[
\lim_{s \uparrow S, n \to \infty} \frac{n^{3/2}}{m_f(s)} P(Z_{\tau,s} = n) = \frac{1}{\sigma_\psi(\tau) \sqrt{2\pi}}.
\]

More generally, if \( s_n < S \) and \( n \) are such that

\[
(\sharp) \quad \lim_{n \to \infty} \frac{f'(s_n e^{i\phi/\sqrt{n}})}{f'(s_n)} = 1, \quad \text{for each } \phi \in \mathbb{R},
\]

then

\[
\lim_{n \to \infty} \frac{n^{3/2}}{m_f(s_n)} P(Z_{\tau,s_n} = n) = \frac{1}{\sigma_\psi(\tau) \sqrt{2\pi}}.
\]

For instance for \( f(z) = 1/(1-z) \), condition (\( \sharp \)) is satisfied if \( s_n \) and \( n \) are related so that

\[
\lim_{n \to \infty} (1 - s_n) \sqrt{n} = \infty.
\]

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11. Appendix. Uniformly Gaussian Khinchin families

Let \( f(z) = \sum_{n=0}^{\infty} b_n z^n \) be a power series in \( K \) with radius of convergence \( R > 0 \) and let \((X_t)_{t \in [0,R]}\) be its Khinchin family.

We have encountered two integral convergence results: the integral central limit Theorem \( \mathbb{C} \) and the notion of strongly gaussian power series, definition \( \mathbb{2} \)

- Assume that \( Q_f = \text{gcd}\{n \geq 1 : b_n > 0\} = 1 \). For each fixed \( t \in (0,R) \), the normalized variable \( \tilde{X}_t \) is a lattice random variable with gauge function \( 1/\sigma(t) \), since \( Q_f = 1 \). Because of Theorem \( \mathbb{C} \) we then have that

\[
\lim_{n \to \infty} \int_{|\theta| \leq \pi \sigma(t) \sqrt{n}} \left| \mathbb{E} \left( e^{i \theta \tilde{X}_t / \sqrt{n}} \right)^n - e^{-\theta^2 / 2} \right| d\theta = 0 .
\]

- If the Khinchin family \((X_t)_{t \in [0,R]}\) is strongly Gaussian, then we have for each fixed \( n \geq 1 \), that

\[
\lim_{t \uparrow R} \int_{|\theta| \leq \pi \sigma(t) \sqrt{n}} \left| \mathbb{E} \left( e^{i \theta \tilde{X}_t / \sqrt{n}} \right)^n - e^{-\theta^2 / 2} \right| d\theta = 0 .
\]

This fact follows from the bound

\[
\int_{|\theta| \leq \pi \sigma(t) \sqrt{n}} \left| \mathbb{E} \left( e^{i \theta \tilde{X}_t / \sqrt{n}} \right)^n - e^{-\theta^2 / 2} \right| d\theta = \sqrt{n} \int_{|\varphi| \leq \pi \sigma(t)} \left| \mathbb{E} \left( e^{i \varphi \tilde{X}_t} \right)^n - e^{-\varphi^2 n / 2} \right| d\varphi \\
\leq n^{3/2} \int_{|\varphi| \leq \pi \sigma(t)} \left| \mathbb{E} \left( e^{i \varphi \tilde{X}_t} \right) - e^{-\varphi^2 / 2} \right| d\varphi ,
\]

where, after the change of variables \( \theta = \varphi \sqrt{n} \), we have used that for complex numbers \( z, w \) such that \( |z|, |w| \leq 1 \) we have that \( |z^n - w^n| \leq n |z - w| \).
Thus, for a Gaussian power series the integral
\[ I_n(t) = \int_{|\theta| \leq \pi \sigma(t) \sqrt{n}} \left| E \left( e^{\theta \tilde{X}_t / \sqrt{n}} \right)^n - e^{-\theta^2 / 2} \right| d\theta \]
converges to 0 as \( n \to \infty \) with \( t \) fixed and as \( t \uparrow R \) with \( n \) fixed.

Power series \( f \) in \( K \) with Khinchin family \( (X_t)_{t \in [0,R]} \) are power series for which (†) and (‡) hold simultaneously in the sense that the involved integrals converges to 0, as \( n \to \infty \) or \( t \uparrow R \).

**Definition 4.** A power series \( f \) and its Khinchin family \( (X_t)_{t \in [0,R]} \) are called uniformly Gaussian if the following two conditions are satisfied:

\begin{align*}
\text{a) } & \lim_{t \uparrow R} \sigma(t) = \infty \quad \text{and} \quad \text{b) } \lim_{[n \to \infty \lor t \uparrow R]} \int_{|\theta| \leq \pi \sigma(t) \sqrt{n}} \left| E \left( e^{\theta \tilde{X}_t / \sqrt{n}} \right)^n - e^{-\theta^2 / 2} \right| d\theta = 0.
\end{align*}

By \( [n \to \infty \lor t \uparrow R] \) we mean that \( 1 \leq n \to \infty \) or (inclusive) \( 0 \leq t_0 \leq t \uparrow R \).

By fixing \( n = 1 \) and letting \( t \uparrow R \), we observe that uniformly Gaussian power series are strongly Gaussian. In particular, if \( f \) is a uniformly Gaussian power series then
\[ M_f = \lim_{t \uparrow R} m(t) = \infty. \]
Moreover, the coefficients \( b_n \) of \( f \) satisfy Hayman’s asymptotic formula and in particular \( Q_f = 1 \).

Let us verify that the exponential \( f(z) = e^z \) is uniformly Gaussian. We have \( \sigma(t) = \sqrt{t} \), for \( t > 0 \) and so \( \sigma(t) \sqrt{n} = \sigma(nt) \), for \( t > 0 \) and \( n \geq 1 \). For the characteristic function of the normalized variable \( \tilde{X}_t \), we have that
\[ E(e^{\theta \tilde{X}_t}) = \exp \left( t \left( e^{\theta / \sqrt{t}} - 1 - i\theta / \sqrt{t} \right) \right), \]
and thus that
\[ E(e^{\theta \tilde{X}_t / \sqrt{n}})^n = E(e^{\theta \tilde{X}_{nt}}). \]
Therefore,
\[ \int_{|\theta| \leq \pi \sigma(t) \sqrt{n}} \left| E \left( e^{\theta \tilde{X}_t / \sqrt{n}} \right)^n - e^{-\theta^2 / 2} \right| d\theta = \int_{|\theta| \leq \pi \sigma(nt)} \left| E \left( e^{\theta \tilde{X}_{nt}} \right) - e^{-\theta^2 / 2} \right| d\theta. \]
Since \( e^z \) is strongly Gaussian, this integral tends to 0 as \( (nt) \to \infty \), and thus as \( [n \to \infty \lor t \uparrow + \infty] \). Observe that in this case \( R = +\infty \).

The notion of uniformly Hayman power series which we are about to introduce generalizes that of Hayman power series, see Section 2.7. We will verify shortly that uniformly Hayman power series are uniformly Gaussian, much like power series in the Hayman class are strongly gaussian. Theorem 11.2 provides us with an ample class of power series which are uniformly Hayman.
Definition 5. Let \( f(z) \) be a power series in \( K \) with radius of convergence \( R > 0 \) and let \((X_t)_{t \in [0, R]}\) be its Khinchin family.

We say that \( f(z) \) and \((X_t)_{t \in [0, R]}\) are uniformly Hayman if for each \( n \geq 1 \) and \( t \in (0, r) \) there exists \( h(n, t) \in (0, \pi) \) (called cuts) such that the following requirements are satisfied:

\[
\sup_{|\theta| \leq h(n, t)\sigma(t)\sqrt{n}} \left| E(e^{i\theta X_t}/\sqrt{n})^n e^{i\theta^2/2} - 1 \right| \to 0, \quad \text{as } [n \to \infty \lor t \uparrow R],
\]

\[
\sqrt{n}\sigma(t) \sup_{h(n, t)\sigma(t)\sqrt{n} \leq |\theta| \leq \pi \sigma(t)\sqrt{n}} |E(e^{i\theta X_t}/\sqrt{n})^n| \to 0, \quad \text{as } [n \to \infty \lor t \uparrow R],
\]

\[
\lim_{t \to R} \sigma(t) = \infty.
\]

Condition (11.2) may be written equivalently as

\[
\sqrt{n}\sigma(t) \sup_{h(n, t)\sigma(t)\sqrt{n} \leq |\theta| \leq \pi \sigma(t)\sqrt{n}} |E(e^{i\theta X_t}/\sqrt{n})^n| \to 0, \quad \text{as } [n \to \infty \lor t \uparrow R].
\]

As announced,

**Theorem 11.1.** Uniformly Hayman power series are uniformly Gaussian.

The proof below is analogous to the proof of Theorem C of Section 2.7 which claims that Hayman power series are strongly Gaussian.

**Proof.** Denote \( \theta(n, t) = h(n, t)\sigma(t)\sqrt{n} \). First we show that

\[
\theta(n, t) \to \infty, \quad \text{as } [n \to \infty \lor t \uparrow R].
\]

Abbreviate \( \hat{\theta} = \theta(n, t) \). By (11.1), we have that

\[
E(e^{i\hat{\theta} X_t}/\sqrt{n}) e^{i\hat{\theta}^2/2} \to 1, \quad \text{as } [n \to \infty \lor t \uparrow R],
\]

while, from (11.2) we obtain that

\[
\sqrt{n}\sigma(t) E(e^{i\hat{\theta} X_t}/\sqrt{n})^n \to 0, \quad \text{as } [n \to \infty \lor t \uparrow R].
\]

From these two limits we deduce that

\[
\sqrt{n}\sigma(t) e^{-\hat{\theta}^2/2} \to 0, \quad \text{as } [n \to \infty \lor t \uparrow R],
\]

and thus, since \( \sqrt{n}\sigma(t) \to \infty \) as \([n \to \infty \lor t \uparrow R]\) we deduce that

\[
\hat{\theta} \to \infty, \quad \text{as } [n \to \infty \lor t \uparrow R].
\]

Denote by \( A(n, t), B(n, t) \), respectively, the supremums in (11.1) and (11.2).
We bound
\[\int_{|\theta| \leq h(n,t)\sigma(t)\sqrt{n}} \left| E(e^{j\theta X_t/\sqrt{n}})^n - e^{-\theta^2/2}\right| d\theta \]
\[= \int_{|\theta| \leq h(n,t)\sigma(t)\sqrt{n}} \left| E(e^{j\theta X_t/\sqrt{n}})^n e^{\theta^2/2} - 1\right| e^{-\theta^2/2} d\theta \]
\[\leq A(n,t)\sqrt{2\pi}, \]
and
\[\int_{h(n,t)\sigma(t)\sqrt{n} \leq |\theta| \leq \pi\sigma(t)\sqrt{n}} \left| E(e^{j\theta X_t/\sqrt{n}}) - e^{-\theta^2/2}\right| d\theta \]
\[\leq 2\pi\sigma(t)\sqrt{n}B(n,t) + \int_{|\theta| \geq h(n,t)\sigma(t)\sqrt{n}} e^{-\theta^2/2} d\theta. \]

These two bounds and conditions (11.1) and (11.2) combined with (11.5) give the result.

\[\square\]

11.1. Uniformly Hayman exponentials. Let \(g(z) = \sum_{n=0}^{\infty} b_n z^n\), with \(b_n \geq 0\), for \(n \geq 0\), and radius of convergence \(R > 0\). Let \(f \in K\) be given by \(f = e^g\).

Exponentials of power series with positive coefficients are very relevant, in particular, in Combinatorics since they codify (most) generating functions of the set construction, which includes among them generating functions of partitions of many sorts.

One of the main results of [4], see [3, Theorem 4.1] and [4, Theorem F] gives conditions on the powers series \(g\) that guarantees that \(f = e^g\) is in the Hayman class, and thus strongly Gaussian, and, therefore, amenable to the Hayman asymptotic formula, (2.6).

It turns out that these same conditions on \(g\) are enough for \(f\) being uniformly Hayman; this is the content of Theorem 11.2.

Denote

\[\omega_g(t) \triangleq \frac{1}{6}(b_1 t + 8b_2 t^2 + \frac{9}{2} t^3 g'''(t)), \quad \text{for } t \in (0, R). \]

\textbf{Theorem 11.2.} Let \(g\) be a nonconstant power series with radius of convergence \(R > 0\) and nonnegative coefficients.

Assume that the variance condition is satisfied

\[\lim_{t \uparrow R} (tg'(t) + t^2 g''(t)) = +\infty. \]

Assume further that there is a cut function \(h(t)\) satisfying

\[\lim_{t \uparrow R} \omega_g(t) h(t)^3 = 0. \]
and that there are positive functions $U,V$ defined in $(t_0,R)$, for some $t_0 \in (0,R)$, where $U$ takes values in $(0,\pi]$ and $V$ in $(0,\infty)$ and are such that
\begin{equation}
\label{eq:11.9}
\sup_{|\varphi| \geq \omega} (\Re g(te^{i\varphi}) - g(t)) \leq -V(t) \omega^2, \quad \text{for } \omega \leq U(t) \text{ and } t \in (t_0,R),
\end{equation}
and thus that the cut $h$ is such that
\begin{equation}
\label{eq:11.10}
h(t) \leq U(t), \quad \text{for } t \in (t_0,1) \quad \text{and} \quad \lim_{t \uparrow R} \sigma(t) e^{-V(t)h(t)^2} = 0.
\end{equation}
then the function $f = e^g$ is uniformly Hayman.

Proof. Condition (11.7) is directly condition (11.3).

We shall verify the conditions on the cuts of the definition of uniformly Hayman power series (11.1) and (11.4) with cuts $h(n,t)$ given by
\[ h(n,t) = h(t) n^{-\beta} \]
where the parameter $\beta$ satisfies $1/3 < \beta < 1/2$.

From the discussion of Section 4 of [3], we have that
\[ \left| \ln \mathbb{E}(e^{\theta \bar{X}_t}) + \frac{\theta^2}{2} \right| \leq \omega_g(t) \frac{\theta^3}{\sigma^3(t)}, \quad \text{for } t \in (0,R) \text{ and } \theta \in \mathbb{R}, \]
and, thus, that
\[ \left| n \ln \mathbb{E}(e^{\theta \bar{X}_t/\sqrt{n}}) + \frac{\theta^2}{2} \right| \leq \omega_g(t) \frac{\theta^3}{\sigma^3(t)\sqrt{n}}, \quad \text{for } t \in (0,R) \text{ and } \theta \in \mathbb{R}. \]

For $|\theta| \leq h(n,t)\sigma(t)\sqrt{n}$ we deduce that
\[ \left| n \ln \mathbb{E}(e^{\theta \bar{X}_t/\sqrt{n}}) + \frac{\theta^2}{2} \right| \leq \omega_g(t) h(t)^3 n^{1-3\beta} \]
Hypothesis (11.8) on $h(t)$ and the fact that $1 - 3\beta < 0$ gives us that
\[ \lim_{n \to \infty, t \uparrow R} \omega_g(t) h(t)^3 n^{1-3\beta} = 0, \]
and, thus, that condition (11.1) is satisfied.

Since $h(n,t) \leq h(t) \leq U(t)$, condition (11.9) gives us that
\[ n \sup_{h(n,t) \leq |\theta| \leq \pi} \left( \Re g(te^{i\theta}) - g(t) \right) \leq -V(t)h(t)^2 n^{1-2\beta}, \]
and so that
\[ \sup_{h(n,t) \leq |\theta| \leq \pi} \left| \mathbb{E}(e^{\theta \bar{X}_t}) \right| \leq \exp \left( -V(t)h(t)^2 n^{1-2\beta} \right). \]

Since $\lim_{t \uparrow R} V(t)h(t)^2 = \infty$, we have, for a certain $t_0 \in (0,R)$ that $V(t)h(t)^2 \geq 1$, for $t \in (t_0,R)$, and thus that
\[ V(t)h(t)^2 n^{1-2\beta} \geq V(t)h(t)^2 + n^{1-2\beta}. \]

We deduce that
\[ \sqrt{n} \sigma(t) \sup_{h(n,t) \leq |\theta| \leq \pi} \left| \mathbb{E}(e^{\theta \bar{X}_t}) \right| \leq \sqrt{n} e^{-n^{1-2\beta}} \sigma(t) e^{-V(t)h(t)^2}, \]
and, because of hypothesis (11.10) and since \( \beta < 1/2 \), that condition (11.4) of the definition of uniformly Hayman is satisfied. \( \square \)

In [4] a large number of exponentials \( f = e^g \) where \( g \) is a power series with non-negative coefficients which satisfy the conditions of Theorem 11.2 are exhibited. For instance, the egf of the Bell numbers, or the generating functions \( P \) of partitions or \( Q \) of partitions into distinct parts, are actually uniformly Hayman, and also are uniformly Hayman related examples like the egf of sets of pointed sets, the egf of sets of functions or the ogf of plane partitions or of some colored partitions.

11.2. Exponential of polynomials. Let \( g \) be a polynomial with nonnegative coefficients

\[
g(z) = \sum_{n=0}^{N} b_n z^n
\]

and of degree \( N \), so that \( b_N > 0 \).

Assume that \( Q_g = \gcd\{1 \leq n \leq N : b_n > 0\} = 1 \). Then \( f = e^g \) is in the Hayman. This is a particular case of a result of Hayman [16, Theorem X]. See [3, Proposition 5.1] for a simpler proof of this particular case.

We are going to show next that \( f = e^g \) is actually uniformly Hayman with an argument similar to the one used to show in [3, Proposition 5.1] that \( f = e^g \) is in the Hayman class.

Observe first that

\[
\sigma^2(t) = tg'(t) + t^2 g''(t) \sim N^2 b_N t^N, \quad \text{as} \quad t \to \infty.
\]

Thus, the variance condition (11.3) of being uniformly Hayman is satisfied.

We have

\[
\omega_g(t) = \frac{1}{6}(b_1 t + 8 b_2 t^2 + \frac{9}{2} t^3 g'''(t)) = O(t^N), \quad \text{as} \quad t \to \infty.
\]

For cuts we propose \( h(n, t) = h(t)n^{-\beta} = t^{-N\alpha} n^{-\beta} \), with \( \alpha, \beta \) in the interval \((1/3, 1/2)\). For concreteness, we take \( \alpha = \beta = 5/12 \).

From the proof of Theorem 11.2 we have that

\[
\left| n \ln E(e^{i\theta X_i/\sqrt{n}}) + \frac{\theta^2}{2} \right| \leq \omega_g(t) t^{-5N/4} n^{-1/4}
\]

\[
= O(t^{-N/4} n^{-1/4}), \quad \text{for} \quad |\theta| \leq h(n, t) \sigma(t) \sqrt{n},
\]

and thus, we see that condition (11.1) is satisfied.

Now, the proof that \( f = e^g \) is in the Hayman class of [3] gives \( \eta \in (0, \pi) \) and \( t_0 > 0 \), depending on \( g \), so that

\[
\sup_{|\theta| > \omega} (Rg(te^{i\theta}) - g(t)) \leq -C_g \min \left\{ t, t^N \omega^2 \right\}, \quad \text{for} \quad \omega \geq \eta \quad \text{and} \quad t > t_0,
\]

for some constant \( C_g \) depending on \( g \).
Thus for some \( t_1 > t_0 \), so that \( h(t) < \eta \), for \( t > t_1 \), we have that

\[
\sup_{\theta \leq \eta \leq \pi} (\Re(g(te^{i\theta}) - g(t))) \leq -C \min\{ t\, n, t^{N/6} n^{1 - 5/6} \} = -C \min\{ t\, n, t^{N/6} n^{1/6} \}.
\]

With \( \delta = \min\{ 1, N/6 \} \), we have, for \( t \geq 1 \), that

\[
\min\{ t\, n, t^{N/6} n^{1/6} \} \geq t^\delta n^{1/6} \geq (1/2)(t^\delta + n^{1/6}),
\]

and thus, since \( |\mathbb{E}(e^{i\theta X_t})| = e^{\Re(g(te^{i\theta}) - g(t))} \), that

\[
\sup_{\theta \leq \eta \leq \pi} |\mathbb{E}(e^{i\theta X_t})|^n = O\left( \sqrt{n}^{N/2} \exp\left( -C \min\{ t\, n, t^{N/6} n^{1/6} \} \right) \right)
\]

\[
= O\left( \sqrt{n} \exp\left( - (C/2)n^{1/6} \right) \right) O\left( t^{N/2} \exp\left( - (C/2)t^\delta \right) \right),
\]

and, therefore, condition (11.2) is satisfied.