An Interpolated Galerkin Finite Element Method for the Poisson Equation

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Abstract
We develop a new approach to construct finite element methods to solve the Poisson equation. The idea is to use the pointwise Laplacian as a degree of freedom followed by interpolating the solution at the degree of freedom by the given right-hand side function in the partial differential equation. The finite element solution is then the Galerkin projection in a smaller vector space. This idea is similar to that of interpolating the boundary condition in the standard finite element method. Our approach results in a smaller system of equations and of a better condition number. The number of unknowns on each element is reduced significantly from \((k^2 + 3k + 2)/2\) to \(3k\) for the \(P_k\) \((k \geq 3)\) finite element. We construct bivariate \(P_2\) conforming and nonconforming, and \(P_k\) \((k \geq 3)\) conforming interpolated Galerkin finite elements on triangular grids; prove their optimal order of convergence; and confirm our findings by numerical tests.

Keywords  Finite element · Interpolated finite element · Triangular grid · Poisson equation

Mathematics Subject Classification  65N30 · 65N15

1 Introduction

This study began as a generalization of techniques developed in [14, 15] for Laplace equation to Poisson equation. The main advantage and the novelty of the finite elements in [14, 15] is the use of a proper subspace of the full space of polynomials of degree \(\leq k\) on each element to achieve full approximation order \(O(h^k)\) in the \(H^1\)-norm. While in [14, 15] we construct...
both conforming and non-conforming elements, neither construction yields an immediate adaptation to Poisson equation. In this paper, we show how to overcome this difficulty.

We recall that our construction in [14, 15] is based on using harmonic polynomials only on each element. In the conforming case, this yields a family of quadratic continuous piecewise harmonic splines. In the non-conforming case, we enrich the harmonic polynomials by a non-harmonic bubble function which vanishes on the 6 Gauss-Legendre points on the three edges, cf. [2, 4–9, 12, 18–20, 22].

We now turn our attention to the Poisson equation,

$$\begin{align*}
-\Delta u &= f, & \text{in } \Omega, \\
n u &= 0, & \text{on } \partial \Omega,
\end{align*}$$

(1.1)

where $\Omega$ is a bounded polygonal domain in $\mathbb{R}^2$. The sixth basis function of the $P_2$ nonconforming finite element must be added to the harmonic finite element method. This becomes the standard $P_2$ nonconforming element method, whose solution is

$$\begin{align*}
u_h &= \sum_{x_i \in \partial K \setminus \partial \Omega} u_i \phi_i + \sum_{x_j \in K^o} u_j \phi_j + \sum_{x_k \in \partial \Omega} c_k \phi_k,
\end{align*}$$

(1.2)

where $c_k$'s are the interpolated values on the boundary, and $u_i$'s and $u_j$'s are obtained from the Galerkin projection, i.e., from the solution of a discrete linear system of equations. The sixth basis function is local and is the only non-harmonic polynomial which can be obtained from the right hand side function $f$ in (1.1). Thus, the solution of the $P_2$ nonconforming interpolated Galerkin finite element is

$$\begin{align*}
u_h &= \sum_{x_i \in \partial K \setminus \partial \Omega} u_i \phi_i + \sum_{x_j \in K^o} \tilde{c}_j \phi_j + \sum_{x_k \in \partial \Omega} c_k \phi_k,
\end{align*}$$

(1.3)

where $\tilde{c}_j$'s are the interpolated values of the right hand side function $f$ (these could be simply $f(x_j)$ depending on the choice of $\phi_j$), $c_k$'s are the interpolated boundary values, and only $u_i$'s are obtained from the Galerkin projection. The new method does not only reduce the number of unknowns (from $O(k^2)$ to $O(k)$), but also improves the condition number. It is totally different from the traditional finite element static condensation which does Gaussian elimination from internal degrees of freedom first. The coefficients of the internal basis functions in the interpolated finite element method are not unknowns. They are simply the nodal values of the given right-hand side function. But in the (static condensation of) Lagrange finite element method, we have a large linear system of equations and these internal unknowns depend globally on all rest unknowns. The number of unknowns and the number of iterations are listed in our numerical tests. They would show further the interpolated finite element solution is different from the Lagrange finite element solution.

In this work, in addition to constructing special $P_2$ conforming and nonconforming interpolated finite elements, we redefine the basis functions of the $P_k$ ($k \geq 3$) Lagrange finite element. We keep the Lagrange nodal values on the boundary of each element, and replace the internal Lagrange nodal values by the internal Laplacian values at these internal Lagrange nodes. This way, the linear system of Galerkin projection equations involves only the unknowns on the inter-element boundary. Therefore, the number of unknowns on each element is reduced from $(k + 1)(k + 2)/2$ to $3k$ as all internal unknowns are interpolated by the given function $f$ directly. We show that the interpolated Galerkin finite element solution converges at the optimal order. Numerical tests are provided for comparison with the standard finite element method. The $P_2$ interpolated Galerkin conforming finite element is defined only on macro-
element grids, while the non-conforming $P_2$ and the higher order $P_k$ elements are defined on general triangular grids.

2 The $P_2$ Interpolated Galerkin Conforming Finite Element

In this section we construct a $P_2$ finite element that uses six basis functions on each element: $1$, $x$, $y$, $x^2 - y^2$, $xy$, and $x^2 + y^2$. Note that only the sixth basis function has a non-vanishing Laplacian.

The reference macro-element $\hat{K}_h$ is a square of size $h$, see Fig. 1(left). It is a union of four congruent triangles: $\hat{K}_h = \bigcup_{i=1}^{4} K_i$. The corresponding $P_2$ finite element space is given by

$$P_{\hat{K}_h} := \{ v_h \in L^2(\hat{K}_h) \mid v_h|_{K_i} \in P_2; \; v_h \in C^0(x_i), \; i = 1, 2, 3, 4; \; v_h \in C^1(x_0); \; \Delta v_h \in P_0 \},$$

where $v_h \in C^0(x_i)$ means that the adjoining at $x_i$ polynomial pieces of $v_h$ have the same values at $x_i$, $v_h \in C^1(x_0)$ means that the adjoining at $x_0$ polynomial pieces of $v_h$ and their first derivatives have matching values at $x_0$, and $\Delta v_h \in P_0$ means that the adjoining at $x_0$ polynomial pieces of $v_h$ have the same constant Laplacian. These conditions immediately imply that $v_h$ is continuous on $\hat{K}_h$. Next we show that the dimension of the space $P_{\hat{K}_h}$ is nine, and provide a nodal basis for $P_{\hat{K}_h}$.

**Lemma 2.1** The conforming $P_2$ finite element function $u_h \in P_{\hat{K}_h}$ is unisolvent by the eight nodal values, $u_h(x_i), \; i = 1, 2, \ldots, 8$, and the value $\Delta u_h(x_0)$.

**Proof** As in Lemma 2.1 in [15], we use the Bernstein-Bézier form of $u_h$ on $\hat{K}_h$ with the B-coefficients $c_1, c_2, \ldots, c_{13}$, associated with the domain points in $\hat{K}_h$ as depicted in Fig. 1(right). In Remark 2.1 we explain our choice of Bernstein-Bézier approach. For ease of understanding, we provide explicit expressions for the Bernstein-Bézier forms of each polynomial piece on $\hat{K}_h$. Let $b_{i,j}(x)$ be the barycentric coordinate associated with $x_i$ relative $K_j$. Each triangle in $\hat{K}_h$ defines three barycentric coordinates as follows:

| Triangle | B-coefficients |
|----------|----------------|
| $K_1$    | $b_{1,1}(x)$, $b_{2,1}(x)$, $b_{9,1}(x)$ |
| $K_2$    | $b_{2,2}(x)$, $b_{3,2}(x)$, $b_{9,2}(x)$ |
| $K_3$    | $b_{1,3}(x)$, $b_{4,3}(x)$, $b_{9,3}(x)$ |
| $K_4$    | $b_{3,4}(x)$, $b_{4,4}(x)$, $b_{9,4}(x)$ |

For example, $b_{9,4}(x)$ is the linear form that passes through $(x_0, 1)$, and vanishes on the edge $[x_3, x_4]$ opposite $x_0$ in $K_4$. We now write each polynomial piece using the Bernstein-Bézier basis as follows:
By Lemma 4.1 in [1], the following four conditions are necessary and sufficient for interpolation of the Laplacians $C$ be

$$u_h|_{K_1} = c_1 b_{1,1}^2 + c_2 b_{2,1}^2 + c_9 b_{9,1}^2 + 2 c_5 b_{1,1} b_{2,1} + 2 c_{10} b_{1,1} b_{9,1} + 2 c_{11} b_{2,1} b_{9,1},$$

$$u_h|_{K_2} = c_2 b_{2,2}^2 + c_3 b_{3,2}^2 + c_9 b_{9,2}^2 + 2 c_6 b_{2,2} b_{3,2} + 2 c_{11} b_{2,2} b_{9,2} + 2 c_{12} b_{3,2} b_{9,2},$$

$$u_h|_{K_3} = c_1 b_{1,3}^2 + c_4 b_{4,3}^2 + c_9 b_{9,3}^2 + 2 c_8 b_{1,3} b_{4,3} + 2 c_{10} b_{1,3} b_{9,3} + 2 c_{13} b_{4,3} b_{9,3},$$

$$u_h|_{K_4} = c_3 b_{3,4}^2 + c_4 b_{4,4}^2 + c_9 b_{9,4}^2 + 2 c_{7} b_{3,4} b_{4,4} + 2 c_{12} b_{3,4} b_{9,4} + 2 c_{13} b_{4,4} b_{9,4}.$$

Note that using (2.2), it is easy to see that $u_h$ is continuous on $\hat{K}_h$ for any choice of the B-coefficients $c_1, c_2, \ldots, c_{13}$. Applying interpolation of the nodal values, $u_h(x_i), i = 1, \ldots, 8$, to the B-forms above, we immediately compute eight of the thirteen unknown B-coefficients as follows:

$$c_i = u_h(x_i), \quad i = 1, \ldots, 4,$$

$$c_5 = 2 u_h(x_5) - (c_1 + c_2) / 2, \quad c_6 = 2 u_h(x_6) - (c_2 + c_3) / 2,$$

$$c_7 = 2 u_h(x_7) - (c_3 + c_4) / 2, \quad c_8 = 2 u_h(x_8) - (c_4 + c_1) / 2.$$

By Lemma 4.1 in [1], the following four conditions are necessary and sufficient for interpolation of the Laplacians $\Delta u_h = \Delta u_h(x_0)$ on each triangle $K_i, i = 1, \ldots, 4$, in the square $\hat{K}_h$:

$$2 c_9 + c_1 + c_2 - 2 c_{10} - 2 c_{11} = h^2 \Delta u_h(x_0) / 4,$$

$$2 c_9 + c_2 + c_3 - 2 c_{11} - 2 c_{12} = h^2 \Delta u_h(x_0) / 4,$$

$$2 c_9 + c_3 + c_4 - 2 c_{12} - 2 c_{13} = h^2 \Delta u_h(x_0) / 4,$$

$$2 c_9 + c_4 + c_1 - 2 c_{13} - 2 c_{10} = h^2 \Delta u_h(x_0) / 4. \quad (2.3)$$

By Theorem 2.28 in [10], the following two conditions are necessary and sufficient for $u_h$ to be $C^1$ at the center $x_0$ of the square $\hat{K}_h$:

$$2 c_9 - c_{10} - c_{12} = 0, \quad 2 c_9 - c_{11} - c_{13} = 0. \quad (2.4)$$

Using the null space of the matrix associated with (2.3) and (2.4) found in Lemma 2.1 in [15], we obtain the unique solution

$$c_9 = \frac{1}{4} (c_1 + c_2 + c_3 + c_4) - \frac{h^2}{8} \Delta u_h(x_0),$$

$$c_{10} = \frac{1}{2} c_1 + \frac{1}{4} (c_2 + c_4) - \frac{h^2}{8} \Delta u_h(x_0),$$

$$c_{11} = \frac{1}{2} c_2 + \frac{1}{4} (c_1 + c_3) - \frac{h^2}{8} \Delta u_h(x_0),$$

$$c_{12} = \frac{1}{2} c_3 + \frac{1}{4} (c_2 + c_4) - \frac{h^2}{8} \Delta u_h(x_0),$$

$$c_{13} = \frac{1}{2} c_4 + \frac{1}{4} (c_1 + c_3) - \frac{h^2}{8} \Delta u_h(x_0). \quad (2.5)$$

Thus, all thirteen B-coefficients $c_1, c_2, \ldots, c_{13}$ have been uniquely determined by the eight nodal values $u_h(x_i), i = 1, 2, \ldots, 8$, and by $\Delta u_h(x_0)$. □

**Remark 2.1** Note that due to affine invariance of barycentric coordinates, the formulae for the B-coefficients in Lemma 2.1 are independent of the coordinates of the vertices of the reference element $K_h$. Therefore, the construction in Lemma 2.1 yields a space of continuous quadratic splines with prescribed piecewise Laplacian on type-II partitions of uniform grids. In particular, this space can be used for interpolation and approximation of harmonic functions.
Let $\mathcal{M}_h$ be a subdivision of a suitable domain $\Omega$ into squares of equal size $h$. We subdivide each square $K$ into four triangles $K_i$ as in Fig. 1, and let $\mathcal{T}_h = \{K_i : K_i \subset K\}$ be the corresponding triangular grid of size $h$. The $P_2$ finite element space on the grid is defined by

$$V_h = \{ v_h \in H^1_0(\Omega) \mid v_h|_K = \sum_{i=1}^{8} c_i \phi_i + c_9 \phi_9 \in P_K, \quad \forall K \in \mathcal{M}_h\},$$

where $P_K$ is defined in (2.1) with the associated basis give by $\phi_i(x_j) = \delta_{ij}$ and $\Delta \phi_9(x_0) = \delta_{9}$. The interpolated Galerkin finite element problem is stated as follows: find $u_h = \sum_{K \in \mathcal{M}_h} \left( \sum_{i=1}^{8} u_i \phi_i - f(x_9) \phi_9 \right)$ such that

$$\left( \nabla u_h, \nabla v_h \right) = \left( f, v_h \right) \quad \forall v_h = \sum_{K \in \mathcal{M}_h} \sum_{i=1}^{8} v_i \phi_i.$$  \hspace{1cm} (2.6)

### 3 The $P_2$ Interpolated Nonconforming Finite Element

In this section we define a $P_2$ interpolated Galerkin nonconforming finite element on general triangular grids. On each triangle $K$, we enrich the conforming $P_2$ finite element space by one $P_2$ nonconforming bubble which vanishes at the 6 Gauss-Legendre points on the three edges of $K$, see Fig. 2 and [15]. Now, instead of solving for the coefficient of this last basis function from the discrete equations, we can interpolate the right hand side function to get this coefficient directly.

Let $\mathcal{T}_h$ be a shape-regular, quasi-uniform triangulation on $\Omega$. We will define the $P_2$ nonconforming interpolated finite element space by their basis functions. First, on a triangle $K$, $\phi_0(x) \in P_2(K)$ is defined as follows:

$$\Delta \phi_0 = -1 \quad \text{on} \ K,$$

$$\phi_0(x_i) = 0, \quad i = 1, \ldots, 6,$$

see Fig. 2. Let $\{\tilde{\phi}_i\}_{i=1}^{16}$ be the $P_2$ Lagrange nodal basis associated with the nodes $\{y_i\}_{i=1}^{16}$ shown in Fig. 2, i.e, $\tilde{\phi}_i(y_j) = \delta_{ij}$. The $P_2$ interpolated finite element basis functions are defined by the following harmonic polynomials:

$$\phi_i = \tilde{\phi}_i + (\Delta \tilde{\phi}_i) \phi_0, \quad i = 1, \ldots, 6.$$
The $P_2$ interpolated finite element space is defined by

$$V_h = \{ v_h \in L_2(\Omega) \mid v_h|_K = \sum_{i=1}^{6} c_i \phi_i + c_0 \phi_0 \in P_2(K), \quad \forall K \in T_h \}. \quad (3.1)$$

We note that while $\{ \phi_0, \phi_1, \cdots, \phi_6 \}$ is linearly dependent on one triangle $K$, the global basis is linearly independent.

The $P_2$-nonconforming, interpolated Galerkin finite element problem is stated as follows:

find $u_h = \sum_{K \in T_h} \left( \sum_{i=1}^{6} u_i \phi_i + f(x_0)\phi_0 \right)$ such that

$$(\nabla_h u_h, \nabla_h v_h) = (f, v_h) \quad \forall v_h = \sum_{K \in T_h} \sum_{i=1}^{6} v_i \phi_i. \quad (3.2)$$

### 4 $P_3$ and $P_k$ ($k \geq 4$) Interpolated Galerkin Finite Elements

There is clearly no $P_1$ interpolated finite element as the Laplacian of a $P_1$ polynomial is zero. The $P_2$ interpolated finite elements require special care, see the two previous sections. But for $P_3$ and higher degree interpolated finite elements, we can simply replace all internal Lagrange degrees of freedom by the Laplacian values, which are obtained from the given right hand side function $f$. However, we have not been able to prove the uni-solvence for arbitrary values of $k$. Instead, we treat the $P_3$ as a special case, and define another type of an interpolated finite element for $P_k$ ($k \geq 4$), where we use locally averaged Laplacian values instead of the pointwise ones.

Let $T_h$ be a shape-regular, general quasi-uniform triangulation of a polygonal domain $\Omega$. The $P_3$ interpolated finite element space is defined by

$$V_h = \{ v_h \in H^1_0(\Omega) \mid v_h|_K = \sum_{i=1}^{9} c_i \phi_i + c_0 \phi_0 \in P_3(K), \quad \forall K \in T_h \}, \quad (4.1)$$

where $\{ \phi_i \}_{i=1}^{9}$ are the boundary Lagrange basis functions which have vanishing Laplacian at the barycenter $x_0$ of $K$, while $\phi_0$ vanishes on the three edges of $K$ and $\Delta \phi_0(x_0) = -1$, see Fig. 3.

**Lemma 4.1** The nodal degrees of freedom in (4.1) uniquely define a cubic polynomial on each triangle $K$ in $T_h$.

**Fig. 3** Nodal points of $P_3$

interpolated finite elements
Fig. 4 Nodal points and moments of $P_5$ interpolated finite elements

\[
F_i(u) = u(x_i)
\]

\[
G_j(u) = \int_K p_j b \Delta u \, dx
\]

**Proof** We have a square system of 10 linear equations with 10 unknowns. The uniqueness guarantees existence. Let $u_h$ be a solution of the homogeneous system. Then $u_h$ vanishes on the three edges of $K$, see Fig. 3, and thus $u_h = Cb$, where $b$ is the $P_3$ bubble function on $K$, vanishing on the edges and assuming value 1 at the barycenter $x_0$. Since $\Delta u_h(x_0) = 0$ and $\Delta u_h$ is a linear function, we have, by symmetry,

\[
0 = -\Delta u_h(x_0) b = -\Delta u_h(x_0) \int_K b \, dx = - \int_K (\Delta u_h) b \, dx
\]

\[
= \int_K \nabla u_h \cdot \nabla b dx = C \int_K |\nabla b|^2 dx.
\]

Therefore $C = 0$, $u_h \equiv 0$, and the lemma is proved. $\blacksquare$

The $P_3$ interpolated Galerkin finite element problem reads:

Find $u_h = \sum_{K \in T_h} \left( \sum_{i=1}^9 u_i \phi_i + f(x_0) \phi_0 \right)$ such that

\[
(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h = \sum_{K \in T_h} \sum_{i=1}^9 v_i \phi_i.
\] (4.2)

For $k \geq 4$, to construct $P_k$ interpolated finite elements, we explicitly define two types degrees of freedom. Let the boundary nodal-value linear functionals be as follows:

\[
F_i(u) = u(x_i), \quad i = 1, \ldots, 3k, \text{ see Fig. 4.}
\] (4.3)

Let the Laplacian moment linear functionals be

\[
G_j(u) = \int_K p_j b \Delta u \, dx, \quad j = 1, \ldots, d_{k-3},
\] (4.4)

where $b$ is again the cubic bubble function on $K$, $d_{k-3} = \dim P_{k-3}$, and $\{p_j\}_{j=1}^{d_{k-3}}$ is an orthonormal basis obtained by the Gram-Schmidt process on $\{1, x, y, x^2, \ldots, y^{k-3}\}$ under a special semi-$H^1$ weighted inner product

\[
(u, v)_G = \int_K \nabla (bu) \cdot \nabla (bv) \, dx.
\] (4.5)

**Lemma 4.2** The $(3k + d_{k-3})$ linear functionals in (4.3) and (4.4) uniquely define a $P_k$ polynomial on each triangle $K$ in $T_h$. 

\[\text{Springer}\]
Proof Because $\dim P_k = 3k + d_{k-3}$, we have a square linear system of equations when applying the functionals to determine a $P_k$ polynomial. We only need to show the uniqueness.

Let $u \in P_k$ such that $F_i(u) = 0$ and $G_j(u) = 0$, $i = 1, ..., 3k$, $j = 1, ..., d_{k-3}$. Since $u = 0$ on the three edges (see Fig. 4) we have

$$u = bp$$

for some $p \in P_{k-3}$.

Let the combination of $p_j$ in $G_j$, defined in (4.4), be $p$. We get

$$0 = \sum_{j=0}^{d_{k-3}} c_j G_j(u) = \int_K \sum_{j=0}^{d_{k-3}} c_j p_j b \Delta u \, dx$$

$$= \int_K p b \Delta u \, dx = \int_K |\nabla u|^2 \, dx.$$  

Thus $\nabla u = 0$ and $u \equiv C$. Because $u = bp = 0$ on the boundary, $u \equiv 0$.  

The $P_k$ ($k \geq 4$) interpolated finite element space is defined by

$$V_h = \{ v_h \in H^1_0(\Omega) \mid v_h|_K = \sum_{i=1}^{3k} c_i \phi_i + \sum_{j=1}^{d_{k-3}} c_j \psi_j, \quad \forall K \in T_h \}, \quad (4.6)$$

where $\{ \phi_1, \ldots, \phi_{3k}, \psi_1, \ldots, \psi_{d_{k-3}} \}$ is the dual basis of $\{ F_1, \ldots, F_{3k}, G_1, \ldots, G_{d_{k-3}} \}$, cf. Lemma 4.2. That is, $\psi_j = -bp_j$ where semi-$H^1$-orthogonal polynomial $p_j$ is defined in (4.5), and

$$(\nabla \phi_i, \nabla \psi_j)_K = (\Delta \phi_i, b p_j)_K = 0, \quad i = 1, \ldots, 3k, \quad j = 1, \ldots, d_{k-3},$$

$$(\nabla \psi_i, \nabla \psi_j)_K = -(\Delta \psi_i, -bp_j)_K = \delta_{ij}, \quad i, j = 1, \ldots, d_{k-3}. \quad (4.7)$$

The $P_k$ ($k \geq 4$) interpolated Galerkin finite element problem reads:

Find $u_h = \sum_{K \in T_h} \left( \sum_{i=1}^{3k} u_i \phi_i - \sum_{j=1}^{d_{k-3}} (f, \psi_j)_K \psi_j \right)$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h = \sum_{K \in T_h} \sum_{i=1}^{3k} v_i \phi_i. \quad (4.9)$$

5 Convergence Theory

Theorem 5.1 Let $u$ and $u_h$ be the exact solution of (1.1) and the $P_k$ ($k \geq 4$) interpolated finite element solution of (4.9), respectively. Then

$$\|u - u_h\|_0 + h|u - u_h|_1 \leq C h^{k+1} |u|_{k+1},$$

where $\cdot |_k$ is the Sobolev (semi-)norm $H^k(\Omega)$.

Proof Testing (1.1) by $v_h = \phi_i \in H^1_0(\Omega)$, we have

$$(\nabla u, \nabla v_h) = (f, v_h). \quad (5.1)$$

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Subtracting (4.9) from (5.1), we obtain
\[ \nabla (u - u_h), \nabla v_h) = 0. \tag{5.2} \]

Testing (1.1) by \( v_h = \psi_j \in H_0^1(\Omega) \), by (4.5) and (4.8), we get
\[ (\nabla (u - u_h), \nabla \psi_j) = - \int_K \Delta u \psi_j dx - (f, \psi_j)_K \int_K |\nabla \psi_j|^2 dx \]
\[ = \int_K f \psi_j dx - (f, \psi_j)_K = 0. \tag{5.3} \]

Combining (5.2) and (5.3) implies
\[ |u - u_h|_1^2 = (\nabla (u - u_h), \nabla (u - I_h u)) \]
\[ \leq |u - u_h|_1 |u - I_h u|_1 \leq |u - u_h|_1 C h^k ||u||_{k+1}, \]

where \( I_h \) is the interpolation operator to \( V_h \). We note that the \( \psi_j \) coefficient of \( I_h u \) is that of \( u_h \) (both are same local averaging of \( f (\Delta u) \)). This is why the method is named an interpolated finite element method. Thus
\[ I_h u - u_h \in \text{span}\{\phi_1, \ldots, \phi_{3k}\} \] and \( (\nabla (u - u_h), \nabla (u_h - I_h u)) = 0. \]

As \( I_h \) is a finite element interpolation, by [3] and [13], we have
\[ |u - I_h u|_1 \leq C h^k ||u||_{k+1}. \]

This completes the \( H^1 \) estimate. Let \( w \in H^2(\Omega) \cap H_0^1(\Omega) \) solve
\[ (\nabla w, \nabla v) = (u - u_h, v) \quad \forall v \in V_h. \]

We assume \( H^2 \) regularity for the solution, i.e.,
\[ ||w||_2 \leq C ||u - u_h||_0. \]

Let \( w_h \) be \( P_k \) interpolated finite solution of \( w \). Then
\[ ||u - u_h||_0^2 = (\nabla w, \nabla (u - u_h)) = (\nabla (w - w_h), \nabla (u - u_h)) \]
\[ \leq |w - w_h|_1 |u - u_h|_1 \leq C ||w||_2 C h^k ||u||_{k+1} \]
\[ \leq ||u - u_h||_0 C h^{k+1} ||u||_{k+1}. \]

This gives the \( L^2 \) error estimate and completes the proof. \( \square \)

**Theorem 5.2** Let \( u \) be the exact solution of (1.1). Let \( u_h \) the \( P_2 \) conforming, or the \( P_2 \) nonconforming, or the \( P_3 \) finite element solution of (2.6), (3.2), or (4.2), respectively. Then
\[ ||u - u_h||_0 + h |u - u_h|_1 \leq C h^{k+1} ||u||_{k+1}, \]

where \( k = 2, \) or 3.

**Proof** As there is one local/internal Laplacian degree of freedom, the proof becomes very simple. Testing (1.1) by nodal value basis \( \tilde{v}_h = \phi_i \), we have
\[ (\nabla u, \nabla \tilde{v}_h) = (f, \tilde{v}_h). \tag{5.4} \]

Subtracting finite element equations from (5.4), we get
\[ (\nabla (u - \tilde{u}_h - u_0), \nabla \tilde{v}_h) = 0. \]
Here we separate the finite element solution $u_h$ into two parts: the nodal basis span $\tilde{u}_h$; and the interpolated Laplacian part $u_0$ (spanned by the last basis function $\phi_0$). Therefore, we have

$$|u - u_h|^2 = \langle \nabla (u - \tilde{u}_h - u_0), \nabla (u - \tilde{v}_h - u_0) \rangle$$

$$= \langle \nabla (u - \tilde{u}_h - u_0), \nabla (u - I_h u) \rangle$$

$$\leq |u - u_h| |u - I_h u| \leq |u - u_h| C h^k |u|_{k+1},$$

where $I_h$ is a special interpolation operator to $V_h$, defined by

$$I_h u(x_i) = u(x_i), \quad i = 1, \ldots, 8,$$

$$(\Delta (I_h u))(x_9) = (\Delta u)(x_9) = -f(x_9).$$

With a minor change, one can prove that this interpolation operator is also stable and provides quasi-optimal approximation:

$$|u - I_h u| \leq C h^k |u|_{k+1}.$$

This completes the $H^1$ estimate. The $L^2$ error estimate is identical to the one in the proof of Theorem 5.1. The treatment for the inconsistency by $P_2$ nonconforming element is standard, see e.g. [11, 15–17, 21].

6 Numerical Tests

Let the domain of the boundary value problem (1.1) be $\Omega = (0, 1)^2$, and let $f(x, y) = \pi^2 \sin \pi x \sin \pi y$. The exact solution is $u(x, y) = \sin \pi x \sin \pi y$. We choose a family of uniform triangular grids, shown in Fig. 5, in all numerical tests on $P_k$ interpolated Galerkin finite element methods.

We solve problem (1.1) first by the $P_2$ interpolated Galerkin conforming finite element method defined in (2.6) and by the $P_2$ Lagrange finite element method, on the same grids. The errors and the orders of convergence listed in Table 1 confirm the optimal order.

Next we solve the test problem (1.1) again, by the $P_2$ interpolated non-conforming finite element method (3.2) and by the standard $P_2$ nonconforming finite element method. The errors and the orders of convergence are listed in Table 2, which shows that both methods converge in the optimal order.

In Table 3 we list the results of $P_3$ interpolated finite elements (4.2) and the $P_3$ Lagrange finite elements.

In Table 4, we solve (1.1) by the $P_4$ interpolated finite element methods (4.9) and by the $P_4$ Lagrange finite element methods. The optimal order of convergence is achieved in both cases. Note that the $P_4$ interpolated finite element method has a fewer unknowns.
Table 1 The error $e_h = I_h u - u_h$ and the order of convergence, by the $P_2$ conforming interpolated finite element and by the $P_2$ Lagrange finite element

| grid | $\|e_h\|_0$ | $h^n$ | $|e_h|_1$ | $h^n$ | $\|e_h\|_0$ | $h^n$ | $|e_h|_1$ | $h^n$ |
|------|-------------|-------|-----------|-------|-------------|-------|-----------|-------|
|      | $P_2$ Interpolated conforming FE |       |           |       | $P_2$ Lagrange element |       |           |       |
| 4    | 0.614E-03   | 3.2   | 0.499E-01 | 2.0   | 0.615E-03   | 3.2   | 0.500E-01 | 2.0   |
| 5    | 0.723E-04   | 3.1   | 0.124E-01 | 2.0   | 0.723E-04   | 3.1   | 0.124E-01 | 2.0   |
| 6    | 0.887E-05   | 3.0   | 0.309E-02 | 2.0   | 0.887E-05   | 3.0   | 0.309E-02 | 2.0   |
| 7    | 0.110E-05   | 3.0   | 0.773E-03 | 2.0   | 0.110E-05   | 3.0   | 0.773E-03 | 2.0   |

Table 2 The error $e_h = I_h u - u_h$ and the order of convergence, by the $P_2$ interpolated nonconforming finite element and by the $P_2$ nonconforming finite element

| grid | $\|e_h\|_0$ | $h^n$ | $|e_h|_1$ | $h^n$ | $\|e_h\|_0$ | $h^n$ | $|e_h|_1$ | $h^n$ |
|------|-------------|-------|-----------|-------|-------------|-------|-----------|-------|
|      | $P_2$ Interpolated NC FE |       |           |       | $P_2$ nonconforming element |       |           |       |
| 4    | 0.181E-03   | 2.7   | 0.111E-01 | 1.7   | 0.208E-03   | 3.0   | 0.126E-01 | 2.0   |
| 5    | 0.244E-04   | 2.9   | 0.298E-02 | 1.9   | 0.260E-04   | 3.0   | 0.315E-02 | 2.0   |
| 6    | 0.316E-05   | 3.0   | 0.767E-03 | 2.0   | 0.325E-05   | 3.0   | 0.789E-03 | 2.0   |
| 7    | 0.406E-06   | 3.0   | 0.194E-03 | 2.0   | 0.407E-06   | 3.0   | 0.197E-03 | 2.0   |

Table 3 The error $e_h = I_h u - u_h$ and the order of convergence, by the $P_3$ interpolated finite element and by the $P_3$ Lagrange finite element

| grid | $\|e_h\|_0$ | $h^n$ | $|e_h|_1$ | $h^n$ | # unknowns |
|------|-------------|-------|-----------|-------|------------|
|      | $P_3$ Interpolated finite element |       |           |       |            |
| 1    | 0.335E-01   | 0.0   | 0.173E+00 | 0.0   | 6          |
| 2    | 0.287E-02   | 3.5   | 0.596E-01 | 1.5   | 29         |
| 3    | 0.189E-03   | 3.9   | 0.872E-02 | 2.8   | 141        |
| 4    | 0.119E-04   | 4.0   | 0.114E-02 | 2.9   | 629        |
| 5    | 0.742E-06   | 4.0   | 0.143E-03 | 3.0   | 2661       |
| 6    | 0.464E-07   | 4.0   | 0.180E-04 | 3.0   | 10949      |
| 7    | 0.290E-08   | 4.0   | 0.225E-05 | 3.0   | 44421      |
| 8    | 0.181E-09   | 4.0   | 0.281E-06 | 3.0   | 178949     |
|      | $P_3$ Lagrange finite element |       |           |       |            |
| 1    | 0.262E-01   | 0.0   | 0.311E+00 | 0.0   | 6          |
| 2    | 0.284E-02   | 3.2   | 0.656E-01 | 2.2   | 33         |
| 3    | 0.188E-03   | 3.9   | 0.894E-02 | 2.9   | 153        |
| 4    | 0.118E-04   | 4.0   | 0.114E-02 | 3.0   | 657        |
| 5    | 0.742E-06   | 4.0   | 0.143E-03 | 3.0   | 2721       |
| 6    | 0.464E-07   | 4.0   | 0.180E-04 | 3.0   | 11073      |
| 7    | 0.290E-08   | 4.0   | 0.225E-05 | 3.0   | 44673      |
| 8    | 0.181E-09   | 4.0   | 0.281E-06 | 3.0   | 179457     |
Table 4 The error $e_h = I_h u - u_h$ and the estimated order of convergence, by the $P_4$ interpolated finite element and by the $P_4$ Lagrange finite element

| grid | $|e_h|_0$ | $h^n$ | $|e_h|_1$ | $h^n$ | # unknowns |
|------|--------|-------|--------|-------|-----------|
|      |        |       |        |       |           |
| The $P_4$ interpolated finite element |
| 1    | 0.392E-02 | 0.0   | 0.837E-01 | 0.0   | 25        |
| 2    | 0.949E-04 | 5.4   | 0.304E-02 | 4.8   | 97        |
| 3    | 0.362E-05 | 4.7   | 0.199E-03 | 3.9   | 433       |
| 4    | 0.136E-06 | 4.7   | 0.132E-04 | 3.9   | 1873      |
| 5    | 0.464E-08 | 4.9   | 0.859E-06 | 3.9   | 7825      |
| 6    | 0.151E-09 | 4.9   | 0.547E-07 | 4.0   | 32017     |
|      |        |       |        |       |           |
| The $P_4$ Lagrange finite element |
| 1    | 0.494E-02 | 0.0   | 0.919E-01 | 0.0   | 25        |
| 2    | 0.144E-03 | 5.1   | 0.349E-02 | 4.7   | 113       |
| 3    | 0.499E-05 | 4.8   | 0.226E-03 | 3.9   | 481       |
| 4    | 0.159E-06 | 5.0   | 0.142E-04 | 4.0   | 1985      |
| 5    | 0.501E-08 | 5.0   | 0.891E-06 | 4.0   | 8065      |
| 6    | 0.157E-09 | 5.0   | 0.557E-07 | 4.0   | 32513     |

In Tables 5–6, we solve (1.1) by the $P_5$ and $P_6$ interpolated finite element methods (4.9) and by the corresponding Lagrange finite element methods, respectively. Although the optimal order of convergence is achieved in all cases, the interpolated finite element method has a substantially smaller number of conjugate gradient iterations.

Table 5 The error $e_h = I_h u - u_h$ and the estimated order of convergence, by the $P_5$ interpolated finite element and by the $P_5$ Lagrange finite element

| grid | $|e_h|_0$ | $h^n$ | $|e_h|_1$ | $h^n$ | # unknowns | # iter |
|------|--------|-------|--------|-------|-----------|-------|
|      |        |       |        |       |           |       |
| The $P_5$ interpolated finite element |
| 1    | 0.282E-03 | 0.0   | 0.329E-02 | 0.0   | 21        | 12    |
| 2    | 0.299E-04 | 3.2   | 0.152E-02 | 1.1   | 86        | 53    |
| 3    | 0.478E-06 | 6.0   | 0.499E-04 | 4.9   | 358       | 352   |
| 4    | 0.754E-08 | 6.0   | 0.158E-05 | 5.0   | 1478      | 1891  |
| 5    | 0.118E-09 | 6.0   | 0.495E-07 | 5.0   | 6022      | 3558  |
|      |        |       |        |       |           |       |
| The $P_5$ Lagrange finite element |
| 1    | 0.255E-03 | 0.0   | 0.409E-02 | 0.0   | 22        | 16    |
| 2    | 0.307E-04 | 3.1   | 0.154E-02 | 1.4   | 90        | 65    |
| 3    | 0.484E-06 | 6.0   | 0.501E-04 | 4.9   | 370       | 442   |
| 4    | 0.755E-08 | 6.0   | 0.158E-05 | 5.0   | 1506      | 2037  |
| 5    | 0.118E-09 | 6.0   | 0.495E-07 | 5.0   | 6082      | 3887  |
Table 6 The error $e_h = I_h u - u_h$ and the estimated order of convergence, by the $P_6$ interpolated finite element and by the $P_6$ Lagrange finite element.

| grid | $\|e_h\|_0$ | $h^n_1$ | $e_h$ | $h^n_2$ | # unknowns | # iter |
|------|-------------|---------|-------|---------|------------|--------|
| The $P_6$ interpolated finite element | | | | | | |
| 1    | 0.142E-03   | 0.0     | 0.302E-02 | 0.0     | 37         | 82     |
| 2    | 0.144E-05   | 6.6     | 0.639E-04 | 5.6     | 153        | 1331   |
| 3    | 0.122E-07   | 6.9     | 0.102E-05 | 6.0     | 625        | 10373  |
| 4    | 0.972E-10   | 7.0     | 0.160E-07 | 6.0     | 2529       | 24703  |
| The $P_6$ Lagrange finite element | | | | | | |
| 1    | 0.151E-03   | 0.0     | 0.321E-02 | 0.0     | 42         | 103    |
| 2    | 0.147E-05   | 6.7     | 0.661E-04 | 5.6     | 174        | 1635   |
| 3    | 0.123E-07   | 6.9     | 0.104E-05 | 6.0     | 714        | 14235  |
| 4    | 0.973E-10   | 7.0     | 0.162E-07 | 6.0     | 2898       | 33610  |

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