The minimal geometric deformation approach extended

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Abstract

The minimal geometric deformation approach was introduced in order to study the exterior spacetime around spherically symmetric self-gravitating systems, such as stars or similar astrophysical objects, in the Randall–Sundrum braneworld framework. A consistent extension of this approach is developed here, which contains modifications of both the time component and the radial component of a spherically symmetric metric. A modified Schwarzschild geometry is obtained as an example of its simplest application, and a new solution that is potentially useful to describe stars in the braneworld is also presented.

Keywords: extra-dimensional gravity, black holes, braneworld

1. Introduction

The idea that the Universe we observe can be described by a sub-manifold (the brane) embedded in a larger space (the bulk) has attracted great interest in physics. From Kaluza and Klein’s early attempts to the more recent models of extra dimensions by Arkani-Hamed, Dimopoulos and Dvali (ADD) [1, 2] and Randall and Sundrum (RS) [3], what is ultimately...
sought after is a unified, hence simpler, description of fundamental interactions. While both the ADD and RS models could straightforwardly explain the hierarchy of fundamental interactions, only the RS model has a non-trivial bulk, which makes it more attractive for exploring gravity at high energies and generalisations of four-dimensional general relativity (GR). Specifically, the new terms in the effective four-dimensional Einstein field equations originating from the bulk, which can be viewed as corrections to GR, might be the key to solving some open issues in gravity, such as the dark matter problem [4]. In this respect, from the phenomenological point of view, the search for solutions to the four-dimensional effective Einstein field equations for self-gravitating systems is particularly relevant, especially the case of vacuum solutions beyond the Schwarzschild metric.

Unfortunately, even in the simplest case of the RS brane-world (BW), only a few candidate spacetimes of spherically symmetric self-gravitating systems are known exactly [5–11], mainly due to the complexity of the effective Einstein field equations. A useful guide in the search for such solutions is provided by the requirement that GR be recovered at low energies. Moreover, the RS model contains a free parameter, the brane tension $\sigma$, which allows one to restrain this prominent aspect by precisely setting the scale of high energy physics [12]. This fundamental requirement forms basis of the Minimal Geometric Deformation (MGD) approach [13], which has made it possible, among other things, to derive exact and physically acceptable solutions for both spherically symmetric [14] and non-uniform [15] stellar distributions, to generate other physically acceptable inner stellar solutions [12–16], to express the tidal charge in the metric found in [5] in terms of the Arnowitt–Deser–Misner (ADM) mass, to study microscopic black holes [17, 18], to clarify the role of exterior Weyl stresses (arising from bulk gravitons) acting on compact stellar distributions [19], as well as to extend the concept of variable tension introduced in [20] by analysing the behaviour of the black strings extending into the extra dimension [21] and to prove the existence of BW stellar distributions made of regular matter with a vacuum Schwarzschild exterior [22]. In the light of these results, it is interesting to study possible extensions of the MGD approach.

In addition, the field equations in the bulk and the brane were shown to be consistent perturbatively [23–28]. In this context, the four-dimensional solution is naturally embedded in the five-dimensional bulk, with a black string-like object associated to the extended MGD procedure. This result has already been established in the case of the standard MGD technique [21]. On a fluid brane with variable tension, the event horizon of the black string along the extra dimension is a particular case of the bulk metric near the brane, being based merely upon the brane metric. Numerical techniques will be employed to study Eötvös branes [29, 30] in the extended MGD framework, showing the embedding of the solution in the five-dimensional bulk. Such an embedding represents a widely employed method that has been previously investigated [23, 31, 32] and moreover applied to a variable tension brane [28], incorporating inflation [24, 25, 28], dark dust [26] and the realistic cases involving post-Newtonian approximations for the Casadio–Fabbri–Mazzacurati black string [27, 33].

In the next section, we shall review the standard MGD approach, which involves a modified radial component of the metric, and then extend it in section 3 to the case in which the time component of the metric is modified. In section 4, we shall describe the simplest application of the extended MGD approach to a Schwarzschild metric, and the relation between this MGD extended solution on the brane and the associated five-dimensional solution in the bulk is discussed in section 5. In section 6, two new exact black hole metrics are presented as a direct consequence of the extended MGD. Finally, in section 6.2, the possible extension of the MGD inside a self-gravitating system is discussed in detail. We summarise our work in section 7.
2. Minimal geometric deformation

In the generalised RS BW scenario, gravity acts in five dimensions and modifies the gravitational dynamics in the (3+1)-dimensional brane accessible to all other physical fields. The effective four-dimensional Einstein equations read

\[ G_{\mu\nu} = -k^2 T_{\mu\nu}^{\text{eff}} - \Lambda g_{\mu\nu}, \]  

(1)

and contain an effective energy–momentum tensor

\[ T_{\mu\nu}^{\text{eff}} = T_{\mu\nu} + \frac{6}{\sigma} S_{\mu\nu} + \frac{1}{8\pi} \mathcal{E}_{\mu\nu} + \frac{4}{\sigma} \mathcal{F}_{\mu\nu}, \]

(2)

where \( \sigma \) is the brane tension,

\[ S_{\mu\nu} = \frac{T}{12} - \frac{T_{\alpha\nu} T_{\nu}^{\alpha}}{4} + \frac{g_{\mu\nu}}{24} (3 T_{\alpha\beta} T^{\alpha\beta} - T^2) \]

(3)

represents a high-energy correction \((T = T^{\alpha\alpha})\), and

\[ k^2 \mathcal{E}_{\mu\nu} = \frac{6}{\sigma} \left[ \mathcal{U} \left( u_\mu u_\nu + \frac{1}{3} h_{\mu\nu} \right) + \mathcal{P}_{\mu\nu} + Q_{\mu\nu} u_\nu \right] \]

(4)

is a non-local source, arising from the five-dimensional Weyl curvature \((\mathcal{U}, \mathcal{P}, \mathcal{Q}, h)\) are the stress tensor and energy flux, respectively), and \(\mathcal{F}_{\mu\nu}\) contains contributions from all non-standard model fields possibly living in the bulk. For simplicity, we shall assume \(\mathcal{F}_{\mu\nu} = 0\) and \(\Lambda = 0\) throughout the paper.

Let us now restrict the problem to spherical symmetry, namely \(\mathcal{P}_{\mu\nu} = \mathcal{P} h_{\mu\nu}\) and \(Q_{\mu\nu} = 0\), and choose as the source term in equation (2) a perfect fluid,

\[ T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g_{\mu\nu}, \]

(5)

where \(\rho\) is the density, \(p\) the pressure, and \(u^\mu = e^{-\nu/2} \delta^\mu_0\) is the fluid four-velocity field in the Schwarzschild-like coordinates of the metric

\[ ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]

(6)

Here \(\nu = \nu(r)\) and \(\lambda = \lambda(r)\) are functions of the areal radius \(r\) only, ranging from \(r = 0\) (the centre) to some \(r = R\) (the surface) inside the star, and from \(r = R\) to some arbitrary \(r\) in the outer vacuum, where \(p = 0\).

The metric (6) must satisfy the field equations (1), namely [18]

\[ k^2 \left[ \rho + \frac{1}{\sigma} \left( \frac{\rho^2}{2} + \frac{6}{k^4} \mathcal{U} \right) \right] = \frac{1}{r^2} - e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right), \]

(7)

\[ k^2 \left[ \rho + \frac{1}{\sigma} \left( \frac{\rho^2}{2} + \rho \frac{P}{k^4 \mathcal{U}} \right) + \frac{4}{k^4} \mathcal{P} \right] = -\frac{1}{r^2} + e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right), \]

(8)

\[ k^2 \left[ \rho + \frac{1}{\sigma} \left( \frac{\rho^2}{2} + \rho \frac{P}{k^4 \mathcal{U}} \right) - \frac{2}{k^4} \mathcal{P} \right] = \frac{1}{4} e^{-\lambda} \left[ 2 \nu'' + \nu'^2 - \nu' \nu' + \frac{2 \nu' - \nu}{r} \right], \]

(9)

We use \(k^2 = 8\pi G\), where \(G\) is the four-dimensional Newton constant and \(\Lambda\) is the four-dimensional cosmological constant.
with primes denoting derivatives with respect to \( r \). Moreover, the conservation equation

\[
p' = -\frac{\nu'}{2}(\rho + p)
\]

holds and is unaffected. The four-dimensional GR equations are recovered for \( \sigma^{-1} \to 0 \), and the conservation equation (10) then becomes a linear combination of equations (7)–(9).

From the above field equations, one finds that the radial metric component is in general given by

\[
e^{-\lambda} = \mu(r) + e^{-l} \int^{r} \nu' + \frac{\nu''}{2} + \frac{2\nu'}{x} + \frac{1}{r^2} \left[ H(p, \rho, \nu) + \frac{1}{\sigma}(\rho^2 + 3\rho p) \right] dx + \beta e^{-l}
\]

\[
\equiv \mu(r) + f(r),
\]

where \( \beta = \beta(\sigma) \) is an integration constant, the function

\[
I(r, r_0) \equiv \int^{r} \frac{\nu'' + \frac{\nu'^2}{2} + \frac{2\nu'}{x} + \frac{1}{r^2} \mu(r)}{\nu' + \frac{1}{r}} dx,
\]

and \( \mu = \mu(r) \) is the standard GR expression of the radial metric component. In particular, by assuming that the space outside the star is empty, one has

\[
\mu(r) = \begin{cases} 
1 - \frac{2M}{r}, & \text{for } r > R \\
1 - \frac{k^2}{r} \int^{r} x^2 \rho dx \equiv 1 - \frac{2m(r)}{r}, & \text{for } r \leq R,
\end{cases}
\]

where \( R \) is the radius of the star and \( m = m(r) \) denotes the standard GR interior mass function for \( r < R \). The constant \( M \), in general, depends on the brane tension \( \sigma \) and must take the value of the GR mass \( M = m(R) \) in the absence of extra-dimensional effects.

A crucial role in the original MGD approach is played by the function \( H \) in equation (11), namely

\[
H \equiv -\left[ \mu'\left(\frac{\nu'}{2} + \frac{1}{r}\right) + \mu\left(\nu'' + \frac{\nu'^2}{2} + \frac{2\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}\right]
\]  

\[+ 3k^2p,\]

which vanishes for any time metric function \( \nu = \nu_{\text{GR}}(r) \) that corresponds to a standard GR solution. The geometric deformation in equation (11) is correspondingly ‘minimal’, that is simply given by contributions coming from the density and pressure of the source. One can thus start from a given GR solution \( \nu = \nu_{\text{GR}}(r) \), then obtain the corresponding deformed BW radial metric function \( \lambda = \lambda(r) \) by evaluating the integral in equation (11) with \( H = 0 \), and finally compute the BW time metric function \( \nu = \nu(r) \) from the remaining field equations. It is worth noting that the correction \( \nu - \nu_{\text{GR}} \) necessarily vanishes for \( \sigma^{-1} \to 0 \), and the starting GR solution is properly recovered in this low-energy limit.

Let us end this section by noting that the parameter \( \beta = \beta(\sigma) \) in equation (11) could also depend on the mass \( M \) of the self-gravitating system and must be zero in the GR limit. For interior solutions, the condition \( \beta = 0 \) has to be imposed in order to avoid singular solutions at the center \( r = 0 \). However, for vacuum solutions in the region \( r > R \), where there is a Weyl fluid filling the spacetime around the spherically symmetric stellar distribution, \( \beta \) does not
need to be zero, hence there can be a geometric deformation associated to the Schwarzschild solution. Overall, $\beta$ plays a crucial role in the search for BW exterior solutions, and in assessing their physical relevance. In fact, we were previously able to constrain $\beta$ in this MGD solution from the classical tests of GR in the solar system, and the strongest constraint is $|\beta| \lesssim (2.80 \pm 3.45) \times 10^{-11}$, from the perihelion precession of Mercury [42].

3. Extended geometric deformation

BW effects on spherically symmetric stellar systems have already been extensively studied (see e.g. [27, 34–39] for some recent results and [8, 40, 41] for earlier studies. In particular, the exterior $r > R$ (where $\rho = p = 0$) is filled with fields (Weyl fluid) arising from the bulk, whose effects on stellar structures are not clearly understood [19]. The MGD approach allows us to study this region by generating modifications to the GR Schwarzschild metric which, by construction, have the correct low-energy limit. The next natural step is thus to investigate a generalisation of the MGD for the exterior region filled with a Weyl fluid. This can be accomplished by considering, in addition to the geometric deformation on the radial metric component, given by the expression (11), a geometric deformation on the time metric component, that is

$$\nu(r) = \nu_0 + h(r),$$  \hspace{1cm} (15)

where $\nu_0$ is given by the Schwarzschild expression

$$e^{\nu_0} = 1 - \frac{2M}{r},$$ \hspace{1cm} (16)

and $h(r)$ is the time deformation produced by bulk gravitons, which should be proportional to $\sigma^{-1}$ to ensure the GR limit. Now, by using the expressions in equation (11) (with $H = \rho = p = 0$) and equation (15) in the vacuum equation

$$R_{\mu \nu} = e^{-\lambda} \left( \nu'' + \frac{\nu'}{r} + 2 \frac{\nu'}{2} + 2 \frac{\nu'}{r^2} \right) - \lambda' e^{-\lambda} \left( \frac{\nu'}{2} + \frac{2}{r} \right) - \frac{2}{r^2} = 0,$$ \hspace{1cm} (17)

we obtain the following first-order differential equation for the radial geometric deformation $f = f(r)$ in equation (11), in terms of the time geometric deformation $h = h(r)$,

$$\left( \frac{\nu'}{2} + \frac{2}{r} \right) f' + \left( \nu'' + \frac{\nu'}{2} + 2 \frac{\nu'}{r} + \frac{2}{r^2} \right) f + F(h) = 0,$$ \hspace{1cm} (18)

whose formal solution is given by

$$f(r) = e^{-I(r, R)} \left( \beta - \int_R^r \frac{e^{I(x, R)} F(h)}{\nu'} \frac{dx}{x} \right).$$ \hspace{1cm} (19)

The exponent $I = I(r, R)$ is again given by equation (12) and

$$F(h) = \mu \frac{h'}{2} + \mu \left( \frac{h''}{2} + \nu \frac{h'}{r} + \frac{h'^2}{2} + 2 \frac{h'}{r} \right).$$ \hspace{1cm} (20)
The exterior deformed radial metric component is thus expressed as
\[ e^{-\lambda(r)} = 1 - \frac{2M}{r} + e^{-1/r_R} \left( \beta - \int_R e^{f(x,R)} F(h) \, dx \right). \]  
\[ \text{(21)} \]

Finally, by using the expressions in equations (15) and (21) in the field equations (7) and (8), the Weyl functions \( U \) and \( P \) are written in terms of the radial deformation \( f(r) \) and time deformation \( h(r) \) as
\[ \frac{6U}{k^2\sigma} = \frac{-f}{r^2} - \frac{f'}{r}, \]
\[ \frac{12P}{k^2\sigma} = 4\frac{f}{r^2} + \frac{1}{r} \left[ f' + 3f\frac{\mu'}{\mu} + 3\mu h' + 3fh' \right]. \]  
\[ \text{(22)} \]
\[ \text{(23)} \]

To summarise, any given time deformation \( h = h(r) \) will induce a radial deformation \( f = f(r) \) according to equation (21). In particular, a vanishing time deformation \( h = 0 \) will produce \( F = 0 \) and the corresponding geometric deformation will be again minimal. For the Schwarzschild geometry, this procedure yields the deformed exterior solution previously studied in [21] (we note a constant \( h \) also produces \( F = 0 \) and corresponds to a time transformation \( dT = e^{h/2} \, dt \)).

4. Modified exterior solution

A more interesting case is provided by non-constant time deformations \( h = h(r) \) such that \( F(h) = 0 \), which will still produce a ‘minimal’ deformation (19). Let us therefore consider the nonlinear differential equation
\[ F(h) = 0, \]  
\[ \text{(24)} \]
whose solution is given by the simple expression
\[ e^{b/2} = a + \frac{b}{2M} \frac{1}{\sqrt{1 - 2M/r}}, \]  
\[ \text{(25)} \]
where the integration constants \( a \) and \( b \) are both functions of the brane tension \( \sigma \). Upon imposing the requirement that the spacetime is asymptotically flat, that is
\[ r \to \infty \Rightarrow e^\nu \to 1 \Rightarrow h \to 0, \]  
\[ \text{(26)} \]
one finds
\[ a = 1 - \frac{b}{2M}, \]  
\[ \text{(27)} \]
and the time metric component has the final form
\[ e^\nu = \left( 1 - \frac{2M}{r} \right) \left[ 1 + \frac{b(\sigma)}{2M} \left( \frac{1}{\sqrt{1 - \frac{2M}{r}}} - 1 \right) \right]^2. \]  
\[ \text{(28)} \]
for $r > 2M$, with the radial metric component given by

$$e^{-\lambda} = 1 - \frac{2M}{r} + \beta e^{-l},$$

where $l$ can be computed exactly but we omit it here for simplicity.

A particularly simple case is given when $\beta = 0$, which will produce no geometric deformation in the radial metric component, so that $\lambda = -\nu_0$ is exactly the Schwarzschild form in equation (16), and diverges for $r \to 2M$. However, due to the modified time component (28), this is now a real singularity, as can be seen by noting that the Kretschmann scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ diverges at $r = 2M$. Moreover, this singularity is soft likewise, since the higher order invariant $(\nabla_\mu R)(\nabla_\nu R^\mu\nu\sigma\tau)_{\mu\nu\sigma\tau}$ involving at least two derivatives of the curvature, further diverges at $r = 2M$. The above solution could therefore only represent the exterior geometry of a star with radius $R > 2M$. Since this solution has no deformation in its radial metric component, its dark radiation will be zero, $U = 0$, as shown by equation (22). However, its Weyl function is

$$\mathcal{P}(r) = \sigma = \frac{-bM^2k^2}{2\sqrt{1 - \frac{2M}{r} - b}\left(\sqrt{1 - \frac{2M}{r} - 1}\right) r^3},$$

which in fact diverges for $r \to 2M$.

Finally, equation (28) can be written for large $r$ as

$$e^\nu \approx 1 - \frac{2M - b}{r} + \frac{b(2M - b)}{4r^2},$$

from which one can read off the ADM mass

$$\mathcal{M} = M - \frac{b}{2}$$

and the tidal charge

$$Q = \frac{b(2M - b)}{4},$$

in terms of which the real singularity is located at

$$r_c = 2\mathcal{M}\left(1 - \frac{Q}{\mathcal{M}^2}\right).$$

It would now be interesting to probe whether one can change the nature of the singularity at $r = 2M$ if $\beta = \beta(\sigma, M) \neq 0$. In fact, the scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ might not diverge for $r \to 2M$ provided $\beta$ satisfies a very complicated algebraic equation that depends on $M$ and $b = b(\sigma)$. However, as we found that the classical solar system tests imply the very strong bound $|\beta| \lesssim (2.80 \pm 3.45) \times 10^{-11}$, it is very unlikely that $\beta$ meets this condition for arbitrary astrophysical masses $M$.

5. Five-dimensional solutions

Let $y$ denote the Gaussian coordinate, parameterising geodesics from the brane into the bulk, where $n_A dx^A = dy$, for $A = 0, 1, 2, 3, 5$ and $n_A$ the components of a vector field that is normal to the brane. The five-dimensional bulk metric $g_{AB}$ is related to the brane metric $g_{\mu\nu}$ by $g_{AB} dx^A dx^B = g_{\mu\nu}(x^\mu, y) dx^\mu dy^\nu + dy^2$. In addition, the effective brane four-dimensional...
cosmological constant $\Lambda$, the bulk cosmological constant $\Lambda_5$ and the brane tension $\sigma$ are fine-tuned by $[31] \Lambda = \frac{\kappa_5^2}{2} \left( \Lambda_5 + \frac{1}{6} \kappa_5^2 \sigma^2 \right)$, where $\kappa_5 = 8\pi G_5$, where $G_5$ denotes the five-dimensional Newton gravitational constant. The brane extrinsic curvature, due to the junction conditions, reads

$$K_{\mu\nu} = -\frac{1}{2} \kappa_5^2 \left[ T_{\mu\nu} + \frac{1}{3} (\sigma - T) g_{\mu\nu} \right].$$

(35)

The bulk metric near the brane can be expressed as the Taylor expansion along the extra dimension with respect to $y [28, 31, 32],

$$g_{\mu\nu}(x^\alpha, y) = \sum_j g^{(j)}_{\mu\nu}(x^\alpha, 0) \frac{|y|^j}{j!},$$

(36)

and one then finds that the expansion up to order $j = 4$ given in [28] is enough for a consistent analysis near the brane [28]. For the sake of conciseness, we only display the metric up to the third order here, that is

$$g_{\mu\nu}(x^\alpha, y) = g_{\mu\nu}(x^\alpha) - \kappa_5^2 \left[ \frac{1}{3} (\sigma - T) g_{\mu\nu} + T_{\mu\nu} \right] y^2 + \left[ \frac{1}{2} \kappa_5^2 \left( T_{\mu\nu} T_{\rho\sigma} - \frac{2}{3} (T - \sigma) T_{\mu\nu} \right) - 2\mathcal{E}_{\mu\nu} - \frac{1}{3} \left( \Lambda_5 - \frac{1}{6} \kappa_5^2 (T - \sigma)^2 \right) g_{\mu\nu} \right] y^3 + \left[ 2K_{\mu\rho}K_{\rho\nu} - \mathcal{E}_{(\mu|\rho|}K_{\nu)\rho} \right] y^4 + \frac{1}{6} g_{\mu\nu} K\Lambda_5 + R_{\rho\mu\nu\sigma}K^{\rho\sigma} - K\mathcal{E}_{\mu\nu} - \nabla_{\nu} B_{\rho(\mu|\sigma)} - K^2 K_{\mu\nu} + 3 K_{(\mu\rho\nu|\sigma)} + \mathcal{E}_{\rho|\nu|\sigma} + K_{\mu\rho}K^{\rho\nu} \right] y^5 + \cdots$$

where $g^{(j)}_{\mu\nu}(x^\alpha, 0) = 0$, $R_{\tau\rho\mu\nu}$ are the components of the brane Riemann tensor—here $R_{\mu\nu}$ and $R$ stand for the Ricci tensor and scalar curvature, respectively, and $B_{\rho\mu\nu\sigma}$ represents the trace-free bulk Weyl tensor. When the brane has variable tension, the black string event horizon along the extra dimension is affected and the above series contains additional terms [28]. In particular, terms of order $|y|^4$ in the expansion (37), involving derivatives of the variable brane tension, read [28]

$$g^{(4)\text{var}}_{\mu\nu} = \frac{2}{3} \kappa_5^2 \left[ \left( \nabla_{\mu} \nabla_{\nu} \sigma - \Box \sigma \right) g_{\mu\nu} \right].$$

(37)

Furthermore, terms of order $y^4$ can be obtained straightforwardly, although they are very cumbersome, and are given by equation (12) of [28]. In what follows, we shall consider only a time-dependent brane tension $\sigma = \sigma(t)$. More details about spherically symmetric or anisotropic branes can be found in [46].

Now, since the area of the five-dimensional horizon is determined by $g_{\mu\nu}(x^\alpha, y) [23, 28, 31], we consider the metric (6) with the coefficients (28) and (29). In fact, $g_{\mu\nu}(x^\alpha, y = 0) = R^{-2}_{H}$ is precisely the black hole event horizon squared, where $R_{H}$ denotes the coordinate singularity on the brane.

In BW models, we can take into account the huge variation of temperature in the Universe during the cosmological evolution. In this context, fluid membranes of Eötvös type [29] play a prominent role in phenomenological aspects. In fact, the so-called Eötvös law states that the tension of the fluid membrane does depend upon the temperature as $\sigma \sim (T_{\text{critical}} - T)$, where $T_{\text{critical}}$ denotes the highest temperature compatible with the
The four-dimensional coupling constant \( k \) and the brane tension \( \sigma \) are considered to be tiny in this context, which reinforces BW effects accordingly. In a suitable thermodynamic framework \([28, 43]\), we derived an expression for the time-dependent variable brane tension
\[
\sigma(t) = -\frac{1}{a(t)}
\]
where \( a(t) \) denotes the scale factor in a Friedmann–Robertson–Walker Universe, given in our particular case by
\[
a(t) = 1 - \exp(\alpha t) \quad [21, 28].
\]
Moreover, in this context we further have \( \Lambda \propto [1 - 1/a(t)]^2 \) \([28]\), which implies that \( \Lambda \) has negative values that become small positive ones as the Universe expands. Additional phenomenological analysis on this model was done in \([28]\). Hereupon, we consider \( \Lambda_5 = 1 = \kappa_5 \) and the brane tension, with a lower bound
\[
\sigma \sim 4.39 \times 10^8 \text{ MeV}^4,
\]
to be normalized as well.

**Figure 1.** Squared black string horizon \( g_{\infty} = 2M, y \) along the extra dimension for \( b(\sigma) = 1.9 \) (solid black line), \( b(\sigma) = 1.0 \) (dash-dotted line), \( b(\sigma) = 0.5 \) (thick gray line), \( b(\sigma) = 0.3 \) (dashed gray line) and \( b(\sigma) = 0.1 \) (dashed light gray line). Black hole mass \( M = 1 \) and \( \beta \sim 1/\sigma \).

**Figure 2.** Time-dependent black string event horizon along the extra dimension, for a variable brane tension.
Here we plot the black string event horizon \( r = 2M \) for different values of the brane tension \( \sigma \). In figure 1 the black string event horizon is shown to vary along the extra dimension as a function of \( \beta(\sigma) \sim \sigma^{-1} \) from the extended MGD technique. It is worth mentioning that there is a point of coordinate \( y_0 \) along the extra dimension where the horizon satisfies \( g_{00}(r = 2M, y_0) = 0 \). For instance, when \( \sigma = 0.5 \) then \( y_0 = 0.88 \), in figure 1.

The plot in figure 2 exhibits the time-dependent black string horizon along the extra dimension. As the cosmological time proceeds, the event horizon shrinks to a critical point along the extra dimension, whose constant time slices have already been depicted in figure 1 in a different range. These results are supported by the findings in [44, 45] and generalise those results to the extended MGD. From the perturbative analysis provided by the expansion (37), the black string horizon shrinks along the extra dimension. On the other hand, black strings are linearly unstable to long-wavelength perturbations. If we perform the Gregory–Laflamme perturbation analysis, we can show that, although in the range \( 0.4 \lesssim y \lesssim 0.7 \) the black string horizon associated with the extended MGD shrinks, in other regions along the extra dimension the event horizon’s size can increase again, as for instance is shown in [21]. Hence the total area is shown to increase, consistently with thermodynamic principles. A precursory numerical analysis reveals that the black string fragments along the extra dimension and forms five-dimensional black holes like those found in similar contexts [47]. However, a complete numerical analysis adapted to the MGD extended framework goes beyond the scope of the present paper and is thus left for future investigation.

6. New exact solution

We will next investigate a more general solution for equation (21) by considering a time deformation \( h(r) \) producing \( F(h) = 0 \). A reasonable choice for \( h(r) \) producing an analytical expression around the Schwarzschild solution in the simple form

\[
e^\nu = \left(1 - \frac{2M}{r}\right)^{1+k}
\]

is provided by

\[
h(r) = k \ln \left(1 - \frac{2M}{r}\right).
\]

We wish to emphasise that the parameter \( M \) in equation (38) is not the ADM mass \( \mathcal{M} \) of the self-gravitating system. Indeed they only coincide when there is no time deformation \( (k = 0) \). They are related, however, as is seen below in equation (49).

Many exact configurations may be generated by using the ‘deformation parameter’ \( k \) in equation (38), the simplest one being the minimal geometric deformation associated to the Schwarzschild solution, which corresponds to \( k = 0 \) (no time deformation). For \( k = 1 \), equations (38) and (21) respectively yield

\[
e^\nu = 1 - \frac{4M}{r} + \frac{4M^2}{r^2}
\]

and

\[
e^{-\lambda} = 1 - \frac{(2M - c_1)}{r} + \frac{(2M^2 - c_1 M)}{r^2},
\]
where
\[ c_1 \equiv \frac{R}{1 - M/R} \beta \] (42)
and \( r_0 = R \), the radius of the distribution, was used to evaluate the integral in equation (12).

In order to obtain the asymptotic Schwarzschild behaviour
\[ e^{-\lambda} \sim 1 - \frac{2M}{r} + O(r^{-2}) \] (43)
the constant \( c_1 \) must satisfy
\[ c_1 = -2M. \] (44)

Consequently, the expressions in equations (40) and (41) will reproduce the tidally charged solution given by [5]
\[ e^\nu = e^{-\lambda} = 1 - \frac{2M}{r} + \frac{Q}{r^2}, \] (45)
where the ADM mass \( M = 2M \) and the tidal charge \( Q = 4M^2 \). The black hole solution in equation (45) corresponds to an extremal black hole with degenerate horizons
\[ r_h = M \] (46)
which lies inside the Schwarzschild radius. Hence, according to this solution, extra-dimensional effects weaken the gravitational field.

6.1. New outer solution

Next we will show a new exact solution corresponding to \( k = 2 \). For \( k = 2 \), equations (38) and (21) lead to
\[ e^\nu = 1 - \frac{2M}{r} + \frac{Q}{r^2} - \frac{2M}{9} \frac{Q}{r^3}, \] (47)
and
\[ e^{-\lambda} = \left(1 - \frac{2M}{3r}\right)^{-1} \left[ 128 \frac{c_2}{r} \left(1 - \frac{M}{6r}\right)^7 + \frac{5}{224} \left(\frac{Q}{12r^2}\right) + \frac{5M}{16} \frac{Q}{3r} \left(\frac{Q}{12r^2}\right)^3 \right] \]
\[ + 5 \left(\frac{Q}{12r^2}\right)^3 \]
\[ - \frac{25M}{4} \frac{Q}{3r} \left(\frac{Q}{12r^2}\right)^2 + \frac{25}{2} \left(\frac{Q}{12r^2}\right)^2 - \frac{5M}{12} \frac{Q}{r^2} \]
\[ + \frac{10}{12} \frac{Q}{r^2} - \frac{4M}{3} + 1 \]. (48)
where
\[ M = 3M; \ Q = 12M^2; \ c_2 \equiv \frac{1 - 2M/R}{(2R - M)^7} R^8 \beta. \] (49)

From the Schwarzschild limit in equation (43), we obtain
\[ c_2 = -\frac{M}{32}. \] (50)
This solution displays two zeros of \( g_{rr} \), namely \( r_i = 0.095 \) and \( r_e = 1.124 \), and a surface \( r_{rc} = 2.3 \) where \( g_{rr}^{-1} \) diverges \((g_{tt} = 2/M)\). All shown in figure 3. These surfaces separate the spacetime into four regions, namely

- \( 0 < r < r_i \)
- \( r_i < r < r_c \)
- \( r_c < r < r_e \)
- \( r > r_e \)

An exterior observer at \( r > r_e \) will never see this singularity, as it is hidden behind the outer horizon \( r = r_e \). Nonetheless, we prefer to consider our solution as a candidate exterior of a self-gravitating distribution with radius \( R_{\text{e}} > r_e \), thus excluding the singular region \( r = r_e \). Finally, since the exterior horizon lies inside the Schwarzschild radius,
$r_e < r_0 = 2\mathcal{M}$, this solution also indicates that extra-dimensional effects weaken the gravitational field.

Both Weyl functions $\mathcal{U}$ and $\mathcal{P}$ are shown in figure 4. We see that $\mathcal{U}$ is always positive, which indicates a negative radial deformation, according to equation (22), and diverges at the singular surface $r = r_e$. On the other hand, $\mathcal{P}$ is always negative, showing thus a negative ‘pressure’ around the stellar distribution as a consequence of extra-dimensional effects. This function also diverges at the singular radius $r = r_e$.

### 6.2. The interior

So far we have successfully developed an extension of the MGD for the region $r > R_o$ surrounding a self-gravitating distribution. The next logical step is to consider this extension inside the distribution $r < R_o$, in order to investigate the consequences of bulk gravitons on physical variables such as the density and pressure. A critical point regarding the implementation of the MGD is the conservation equation (10), which, contrary to GR, is not a linear combination of the field equations (7)–(9). We know that in the MGD approach any chosen GR solution will automatically satisfy the conservation equation (10). However, when the time deformation in equation (15) is considered, the conservation equation (10) becomes

$$p' = -\frac{\nu_0}{2}(\rho + p) - \frac{h'}{2}(\rho + p)$$

where $\nu_0$ is the GR solution of the temporal metric component. The expression in equation (51) then yields

$$0 = -\frac{h'}{2}(\rho + p)$$

and therefore only a constant time deformation $h$, or equivalently a time transformation $dT = e^{h/2}dt$, may be implemented inside a self-gravitating system formed by a perfect fluid. The only way to overcome this problem is by considering a more complex interior structure, like an anisotropic distribution, which leads to a more complex form of the conservation equation. Furthermore, an exchange of energy between the bulk and the brane could be useful for implementing the extension of the MGD inside a stellar distribution.

### 7. Conclusions and outlook

The MGD deformation was consistently extended to the case when both gravitational potentials, namely the radial and time metric components, are affected by bulk gravitons. We showed that the deformation for the time metric component induces part of the deformation in the radial metric component, and the latter can be minimised by assuming that the former satisfies the differential equation (24).

Using this extension of the MGD approach, a new possible exterior geometry for a self-gravitating system was found. This new solution presents a physical singularity at the Schwarzschild radius $r = 2\mathcal{M}$. Since there is no horizon, the singularity is naked, albeit not point-like, and the geometry cannot be used to describe a BW black hole, but might still be used to describe the exterior of a self-gravitating star with size $R_o > 2\mathcal{M}$. From the currently available observational data in the solar system, we were able to put constraints on one of the BW parameters. We also argued that the four-dimensional solution can be embedded in the bulk, as is supported by the black string analysis described in section 5. We showed that the
black string event horizon associated to the extended MGD solution on the brane shrinks for the analysed values of the parameter that controls the extension of the MGD procedure. This is accomplished in the framework of a brane with variable tension.

Finally, a more general scenario was considered by allowing a time deformation \( h(r) \) in equation (15) to affect the geometric deformation \( f(r) \) in equation (19), in such a way that \( F(h) \neq 0 \). A corresponding deformation parameter \( k \) was thus introduced that allows new exact exterior solutions to be generated. In particular, the case \( k = 0 \) represents no time deformation, as is clearly seen throughout the expression in equation (39); \( k = 1 \) represents the tidally charged metric corresponding to an extremal black hole with degenerate horizons \( r_h = M \); the case \( k = 2 \) represents a new exterior solution for a self-gravitating system affected by the extra dimension in the form of a surrounding Weyl fluid. This Weyl fluid has a positive effective energy density \( \mathcal{U} \), which increases as we approach the stellar distribution. The effective pressure \( P \) is always negative, and decreases as we approach the distribution, thus inducing anisotropy on the exterior near the self-gravitating system. Both functions tend to disappear rapidly as we move away from the stellar distribution, and represent a ‘Weyl atmosphere’.

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