Solving String Field Equations: 
New Uses for Old Tools

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Abstract: This is the contents of a talk by O. L. presented at the 35th International Symposium Ahrenshoop in Berlin, Germany, 26–30 August 2002. It is argued that the (NS-sector) superstring field equations are integrable, i. e. their solutions are obtainable from linear equations. We adapt the 25-year-old solution-generating “dressing” method and reduce the construction of nonperturbative superstring configurations to a specific cohomology problem. The application to vacuum superstring field theory is outlined.

1 Zero-curvature and linear equations (old tools)

The flatness of a gauge connection $A$,\(^1\)
\[
F(A) \equiv dA + A^2 = (d + A)^2 = 0
\]
may be seen as the compatibility condition for a linear system:
\[
\exists \Psi \text{ with } (d + A) \Psi = 0 \implies F(A) = 0.
\]
If we take the auxiliary function $\Psi$ to be Lie-group valued, solutions of the linear system yield flat connections $A = \Psi d\Psi^{-1}$ which are, however, pure gauge and hence trivial.

The situation changes when $d$ is just a partial differential and one has a second partial differential $\tilde{d}$ which anticommutes with the former. We may then combine the two and also their corresponding partial connections $A$ and $\tilde{A}$ to a family
\[
A(\lambda) = \tilde{A} + \lambda A \quad \text{and} \quad d(\lambda) = \tilde{d} + \lambda d \quad \text{with} \quad \tilde{d}^2 = d^2 = \tilde{d}d + d\tilde{d} = 0,
\]
where some parameter $\lambda \in \mathbb{C}P^1$ is introduced. The extended zero-curvature equation reads
\[
0 = F(A(\lambda)) = (d(\lambda) + A(\lambda))^2 = (\tilde{d}A + \tilde{A}^2) + \lambda(dA + d\tilde{A} + \{A, \tilde{A}\}) + \lambda^2(dA + A^2)
\]
\(^1\)We suppress the spacetime coordinate dependence throughout.
Exploiting the gauge freedom herein allows one to gauge away\(^2\) one of the two partial connections. We fix \(\tilde{A} = 0\) and remain with
\[
\tilde{d} A = 0 \quad \text{and} \quad dA + A^2 = 0 .
\] (5)
As long as the \(d\)- and \(\tilde{d}\)-cohomologies are empty, either one of these two equations is solved by the introduction of a prepotential, for which the remaining equation imposes a second-order relation:
\[
A = \tilde{d} \Upsilon \quad \implies \quad \tilde{d}d\Upsilon + (\tilde{d}\Upsilon)^2 = 0 ,
\] (6)
\[
A = e^{-\Phi} d\Phi \quad \implies \quad \tilde{d}(e^{-\Phi} d\Phi) = 0 .
\] (7)
The first and second of these reductions go back to [1] and [2], respectively. Despite appearance, \(A\) is not pure gauge unless \(\tilde{d}e\Phi = 0\) in which case \(e\Phi\) qualifies as a gauge parameter compatible with \(\tilde{A} = 0\).

The extended linear system associated with (5) is
\[
(\tilde{d} + \lambda d + \lambda A) \Psi(\lambda) = 0 .
\] (8)
Due to the \(\lambda\)-dependence, it gives rise to nontrivial solutions of (5). Indeed, if \(\Psi\) does not depend on \(\lambda\) the partial connection \(A\) must be pure gauge:
\[
\Psi(\lambda) = e^{-\lambda} \quad \implies \quad \tilde{d}e^{-\lambda} = 0 = (d + A)e^{-\lambda} .
\] (9)
The compactness of \(\mathbb{C}P^1\) excludes nontrivial holomorphic \(\Psi(\lambda)\); hence we consider meromorphic \(\Psi(\lambda)\). Not allowing for poles at \(\lambda=0\) or \(\lambda=\infty\), we fix the asymptotics
\[
\Psi(\lambda) \longrightarrow \begin{cases} 
1 - \lambda \Upsilon + O(\lambda^2) & \text{for } \lambda \to 0 \\
e^{-\Phi} + O(1/\lambda) & \text{for } \lambda \to \infty
\end{cases}
\] (10)
by a convenient residual gauge choice. The form of \(\Psi(\lambda)\) is constrained further by the antihermiticity of \(A\) up to a gauge transformation. Hermitian conjugation extends to an involution \(\Psi \mapsto \overline{\Psi}\) which sends \(d \mapsto -\tilde{d}\) and \(\tilde{d} \mapsto d\) but \(\lambda \mapsto \overline{\lambda}\). With this, an antihermitian connection requires that
\[
e^{-\Phi} = \Psi(\lambda) \overline{\Psi(-1/\lambda)} .
\] (11)
Finally, we may reconstruct \(A\) from a given solution \(\Psi(\lambda)\) via
\[
A = \Psi(\lambda)(d + \frac{1}{\lambda}\tilde{d})\Psi(\lambda)^{-1} .
\] (12)

2 Single-pole ansatz

The simplest non-constant meromorphic function possesses a single pole. The corresponding ansatz,\(^3\)
\[
\Psi(\lambda) = 1 - \frac{\lambda(1+\mu\bar{\mu})}{\lambda - \mu} P ,
\] (13)
\(^2\)on a topologically trivial manifold
\(^3\)This is an essential building block in the “dressing method”, a solution-generating technique invented by [3, 4] and developed by [5].
contains a moduli parameter \( \mu \) (the location of the pole) and a Lie-group valued \( \lambda \)-independent function \( P \) to be determined.

It turns out that all information resides in eqs. (11) and (12). Since their left hand sides are independent of \( \lambda \), the poles at \( \lambda = \mu \) and \( \lambda = -1/\bar{\mu} \) of their right hand sides must be removable. Putting to zero the residues in (11) yields algebraic relations,

\[
P^2 = P = \overline{P} \quad \iff \quad P \text{ is a hermitian projector} \quad .
\]

As such, \( P \) can be parametrized with a “column vector” \( T \) via

\[
P = T \frac{1}{T^T T} \quad .
\]

Similarly, the vanishing residues in (12) produce differential equations,

\[
P (\tilde{d} + \mu d) P = 0 = (1 - P)(d - \bar{\mu} \tilde{d}) P
\]

\[
\iff \quad (1 - P)(d - \bar{\mu} \tilde{d}) T = 0
\]

\[
\iff \quad (d - \bar{\mu} \tilde{d}) T = 0 ,
\]

whose solution requires the analysis of the cohomology of the operator \( d - \bar{\mu} \tilde{d} \).

Hence, the original nonlinear problem [5] has been reduced (“linearized”) to a linear homogeneous equation for \( T \). Any nonsingular element in the kernel of \( d - \bar{\mu} \tilde{d} \) yields a projector \( P \), through which the prepotentials and the connection are expressed as follows,

\[
\Upsilon = -\frac{1 + \mu \bar{\mu}}{\mu} P \quad \text{and} \quad e^{-\Phi} = 1 - (1 + \mu \bar{\mu}) P \quad ,
\]

\[
A = -\frac{1 + \mu \bar{\mu}}{\mu} \tilde{d} P = (1 + \bar{\mu} \mu) P d P - (1 + \frac{1}{\mu \bar{\mu}})(1 - P) d P \quad .
\]

3 String fields (new uses)

The complete structure presented so far carries over almost verbatim to string field theory. In addition to being Lie-group or Lie-algebra valued, our objects now are interpreted as string fields and are to be multiplied via Witten’s star product [6] (which we suppress). More precisely, the unextended situation [1] and [2] corresponds to the cubic open bosonic string [6], with \( d \mapsto Q \) (the BRST operator) and \( A \) having ghost number one. The linear system \( (Q + A)\Psi = 0 \) yields only trivial solutions to the string field equation of motion, \( QA + A^2 = 0 \).

Surprisingly, the extended case [5] and [8] can be mapped onto the NS sector of cubic open superstring field theory [7], by \( d \mapsto Q \) and \( \tilde{d} \mapsto \eta_0 \). Here, \( \eta_0 \) denotes the graded commutator action of the zero mode of \( \eta \), which emerges from bosonizing the worldsheet supersymmetry ghosts via \( \gamma = \eta e^\varphi \) and \( \beta = e^{-\varphi} \partial \xi \). The NS string field \( A \) carries ghost number one and picture number zero and lives in the “large Hilbert space” (including \( \xi_0 \)). Consequently, its equations of motion [8, 9] are

\[
\eta_0 A = 0 \quad \text{and} \quad QA + A^2 = 0 \quad .
\]

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4 More generally, it suffices to solve the ‘eigenvalue’ equation \((d - \bar{\mu} \tilde{d})T = T \alpha \) with a flat connection \( \alpha \), but we can gauge \( \alpha \) to zero locally.
Berkovits’ nonpolynomial open superstring (in the NS sector) \[10\] is also found:

\[ A = e^{-\Phi} Q e^\Phi \quad \Rightarrow \quad \eta_0 (e^{-\Phi} Q e^\Phi) = 0 \]. \hspace{1cm} (22)

The Berkovits string field \( \Phi \) has vanishing ghost and picture numbers.

It is known that \( \eta_0 \) and \( Q \) have zero cohomology in the large Hilbert space. Therefore, we can get all solutions to (21) from a “linear superstring system” \[11\],

\[ (Q + \frac{1}{\lambda} \eta_0 + A) \Psi(\lambda) = 0 \]. \hspace{1cm} (23)

This is the key equation for generating nonperturbative classical superstring field configurations. Repeating the earlier analysis, the analog of the ansatz \[13\] produces a hermitian projector string field \(^\footnote{The identity string field \( \mathcal{I} \) is the unit element in Witten’s star algebra.}^5\)

\[ P = T \frac{1}{\pi} \bar{T} \quad \text{subject to} \quad (\mathcal{I} - P)(Q - \bar{\mu} \eta_0)P = 0 \]. \hspace{1cm} (24)

A sufficient condition for the latter equation to hold is

\[ (Q - \bar{\mu} \eta_0) T = 0 \]. \hspace{1cm} (25)

From a solution \( T \) one builds \( P \) and finally (using star multiplication) one reconstructs \[11\] the string fields

\[ e^{-\Phi} = \mathcal{I} - (1 + \mu \bar{\mu}) P \quad \text{and} \quad A = -\frac{1 + \mu \bar{\mu}}{\mu} \eta_0 P \]. \hspace{1cm} (26)

4 \hspace{0.5cm} \textbf{Ghost-picture modification}

Our “master equation” (23) has a flaw: It is inhomogeneous in picture number because \( \eta_0 \) lowers the picture by one unit. Therefore, any solution \( \Psi \) is an infinite sum over all picture sectors (requiring an extension of standard superstring field theory), unless we modify our equation by introducing a picture-raising multiplier,

\[ \eta_0 \rightarrow X(i) \eta_0 \quad \text{with} \quad X(i) = \{Q, \xi(i)\} \], \hspace{1cm} (27)

to be inserted at the string midpoint (with worldsheet coordinate \( i \)). The master equation then changes to

\[ (Q + \frac{1}{\lambda} X(i) \eta_0 + A) \Psi(\lambda) = 0 \]. \hspace{1cm} (28)

and the string field equations of motion become

\[ QA + A^2 = 0 \quad \text{and} \quad X(i) \eta_0 A = X(i) \eta_0 (e^{-\Phi} Q e^\Phi) = 0 \]. \hspace{1cm} (29)

The modification may create extra solutions due to zero modes of \( X(i) \).
5 Shifting the background

We may view solving (28) as “dressing” the vacuum solution,

\[(\Psi_0, A_0) = (\mathcal{I}, 0) \quad \mapsto \quad (\Psi, A) , \]

via \( \Psi = \Psi(\lambda) \Psi_0 \) and \( A = \text{Ad}_{\Psi(\lambda)} A_0 \). This process can be iterated, and such transformations form a group. In fact, they generate all solutions and, hence, relate all classical backgrounds with one another. Clearly, shifting the vacuum background \((\Psi_0, A_0)\) to a new reference \((\Psi_1, A_1)\) is also a dressing transformation:

\[
\begin{align*}
\text{background:} & \quad \Psi_0 = \mathcal{I} \quad \mapsto \quad \Psi_1 \quad \mapsto \quad \Psi \\
\text{deviation:} & \quad A_0 = 0 \quad \mapsto \quad A_1 \\
& \quad \Psi \quad \mapsto \quad \Psi' \quad \mapsto \quad \tilde{\Psi} \\
& \quad A_0 + A \quad \mapsto \quad \tilde{A}
\end{align*}
\]

where vertical arrows turn on a deviation, again via dressing. Composing the two dressing transformations (defining \( \tilde{\Psi} = \Psi' \Psi_1 \) and \( \tilde{A} = A_1 + A' \)) one gets

\[
0 = (Q + \frac{1}{\lambda} X(i) \eta_0 + \tilde{A}) \tilde{\Psi} = [(Q' + \frac{1}{\lambda} X(i) \eta_0 + A') \Psi'] \Psi_1
\]

with \( Q' \Psi' := Q \Psi' + [A_1, \Psi] \). Measuring fields from the new reference field \( A_1 \) we obtain

\[
(Q' + \frac{1}{\lambda} X(i) \eta_0 + A') \Psi' = 0
\]

which takes the same form as (28), except for \( Q \mapsto Q' \).

6 Tachyon vacuum superstring fields

Of particular interest is the structure of (super)string field theory around the (NS) tachyon vacuum. Let us suppose the latter can be reached as \( A_1 \) within our ansatz. It happens to be useful to redefine the fluctuations around \( A_1 \) by a world-sheet parametrization, inducing

\[
A' \mapsto \mathcal{U} A' =: A \quad \text{and} \quad \Psi' \mapsto \mathcal{U} \Psi' =: \psi
\]

in such a way that \( Q := \mathcal{U} Q' \mathcal{U}^{-1} \) is the proper zero-cohomology pure-ghost “vacuum” BRST operator \([12, 13]\). Note that \( \eta_0 \) and \( \xi(i) \) remain unchanged. Thus, our key equation for vacuum superstring field theory reads

\[
(Q + \frac{1}{\chi} \mathcal{X}(i) \eta_0 + \mathcal{A}) \psi(\lambda) = 0 \quad \text{with} \quad \mathcal{X}(i) = \{Q, \xi(i)\} .
\]

Its solutions have the by now familiar form and fulfill the cubic equation

\[
QA + \mathcal{A}^2 = 0 \quad \text{with} \quad \mathcal{X}(i) \eta_0 \mathcal{A} = 0 . \quad (36)
\]

Assuming \([14]\) that the vacuum string fields describing D-branes factorize,

\[
\mathcal{A} = \mathcal{A}_g \otimes \mathcal{A}_m \quad \text{and} \quad \Phi = \Phi_g \otimes \Phi_m , \quad (37)
\]
the field equation (36) splits into
\[ A_m^2 = A_m \quad \text{and} \quad QA_g + A_g^2 = 0 \quad \text{with} \quad \chi^{(i)} \eta_0 A_g = 0. \] (38)
The matter part \( A_m \) is only a “spectator” in the linear system. Our previous analysis then reduces to the solution for the ghost part,
\[ A_g = -\frac{1+\mu\bar{\mu}}{\mu} \chi^{(i)} \eta_0 P_g \quad \text{with} \quad (\mathcal{I} - P_g)(Q - \bar{\mu} \chi^{(i)} \eta_0)P_g = 0 \] (39)
for a ghost projector \( P_g \). Three remarks are in order. First, neither \( \psi \) nor \( e^{\pm\Phi} \) factorize. Second, the form (39) is not compatible with the ansatz \( \Phi_m^2 = \Phi_m \) proposed by \[15\]. Third, nontrivial solutions to (39) are not obtained via \( QP_g = 0 \) but rather governed by the cohomology of the operator \( Q - \bar{\mu} \chi^{(i)} \eta_0 \) for a given moduli parameter \( \mu \).

Let us briefly expand on the last remark. From
\[ \eta_0 \psi(\lambda) = 0 \quad \Rightarrow \quad \eta_0 \Phi = 0 \quad \text{and} \quad \eta_0 P = 0 \] (40)
we infer that \((\mathcal{I} - P)QP = 0\) and indeed \( A = 0 \). Thus, the simplest situation of \( Q \)-closed projectors in the “small Hilbert space” cannot describe D-branes \[16\]. Such “supersliver states” can be expressed as squeezed (i.e. generalized coherent) one-string states over the first-quantized vacuum and have been constructed recently \[17\]. For a solution featuring a non-vanishing \( A \), however, one must construct a ghost projector which satisfies (39) in a less trivial manner, e.g., by intertwining its \((b,c)\) and \((\beta,\gamma)\) ghost contents in an appropriate fashion.

7 Outlook

The “linearization” of the superstring field equations provides us with a new window to nonperturbative superstring physics. The ideas presented here are just the beginning. Among future developments one may consider

- the construction of nontrivial classical string fields \((A \neq 0)\)
- the precise relation with noncommutative solitons (in the Moyal basis)
- the generalization to the multi-pole ansatz for \( \Psi(\lambda) \)
- the interpretation of solutions (as D-branes?) and their moduli \( \mu \)
- the computation of energy densities, tensions, etc. for a given solution
- the analysis of fluctuations around constructed classical configurations
- the extension to the Ramond sector

First steps have been made in \[18\].

Acknowledgement

O. L. would like to thank the organizers for a very stimulating conference in a pretty setting. He also acknowledges discussions with I. Aref’eva and L. Bonora.
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