DIHEDRAL BRANCHED COVERS OF FOUR-MANIFOLDS

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Abstract. Given a closed oriented PL four-manifold $X$ and a closed surface $B$ embedded in $X$ with isolated cone singularities, we give a formula for the signature of an irregular dihedral cover of $X$ branched along $B$. For $X$ simply-connected, we deduce a necessary condition on the intersection form of a simply-connected irregular dihedral branched cover of $(X, B)$. When the singularities on $B$ are two-bridge slice, we prove that the necessary condition on the intersection form of the cover is sharp. For $X$ a simply-connected PL four-manifold with non-zero second Betti number, we construct infinite families of simply-connected PL manifolds which are irregular dihedral branched coverings of $X$. Given two four-manifolds $X$ and $Y$ whose intersection forms are odd, we obtain a necessary and sufficient condition for $Y$ to be homeomorphic to an irregular dihedral $p$-fold cover of $X$, branched over a surface with a two-bridge slice singularity.

1. Introduction

The classification of all branched covers over a given base is a subject dating back to Alexander, who proved that every closed orientable PL $n$-manifold is a PL branched cover of $S^n$ [1]. Alexander’s branching sets are PL subcomplexes of the sphere; he concludes little else about them. Since 1920, the natural question of how complicated the branching set needs to be, and how many sheets are needed, in order to realize all manifolds in a given dimension as branched covers of the sphere, has received much interest – see, for instance, [3] and references therein. It is a famous theorem in dimension 3 that three-fold dihedral covers branched along knots suffice [18], [19], [26]. The question is considerably more subtle in dimension four. Piergallini and Iori, among others, have studied the minimal degree needed to realize all closed oriented PL four-manifolds as covers of the sphere. The branching sets they consider are either immersed PL submanifolds with transverse self-intersections or embedded and non-singular PL surfaces. Piergallini proved in [32] that every closed oriented PL four-manifold is a four-fold cover of $S^4$ branched over a over a transversally immersed PL surface. He and Iori later refined this result to show in [20] that singularities can be removed by stabilizing to a five-fold cover. In light of these universal realization theorems, one might wish for equally general methods for obtaining explicit descriptions of the branching sets needed to realize particular PL four-manifolds as a five-fold covers of $S^4$. It would also be of interest to better understand the trade-off between simplifying the branching set and increasing the degree of a cover. Most recently, Piergallini and Zuddas [33] showed that closed oriented topological four-manifolds are also five-fold covers of the sphere, if one allows for “wild” branching sets with potentially very pathological topology near isolated points. Still, the complexity of the branching sets near the wild points retains an air of mystery.

We assume a complementary approach, taking the point of view of studying all possible covers over a given base $X$ in terms of the branching set and its embedding into the base. As seen from the main theorem of [40], if $Y$ is a cover of $S^4$ branched over a closed oriented non-singular embedded surface, then the signature of $Y$ must be zero. Thus, for example, the existing results on five-fold covers of the four-sphere implicitly make use of nonorientable branching sets. In contrast, the constructions presented here make use of branching sets that are oriented surfaces, embedded in the base piecewise linearly except for finitely many cone singularities. These ideas have led to new examples of branched covers of $S^4$ and applications to the Slice–Ribbon Conjecture [5]. We work with irregular dihedral

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covers (Definition 1.1), which constitute the most direct generalization of the three-dimensional results of Hilden, Hirsch and Montesinos as well as four-dimensional results of Montesinos [27]. Dihedral covers are also the “simplest” three-fold covers which give rise to interesting examples where the branching sets are singularly embedded (see Remark 1.2).

Our results are not restricted to branched covers of the sphere but apply to any closed oriented four-manifold base. Given an irregular dihedral branched cover \( f : Y \to X \) between two simply-connected oriented four-manifolds \( X \) and \( Y \), we relate the intersection forms of \( X \) and \( Y \) via \( f \). Singularities for us play a central role, and we compute the signature of a branched cover in terms of data about the branching set and its singularity.

We begin by defining the type of covers and singularities considered. Throughout, \( D_p \) denotes the dihedral group of order \( 2p \) and \( p \) is odd.

**Definition 1.1.** Let \( f : Y \to X \) be a branched cover with branching set \( B \subset X \). If the unbranched cover \( f_1(X-B) \) corresponds under the classification of covering spaces to \( \phi^{-1}(\mathbb{Z}/2\mathbb{Z}) \) for some surjective homomorphism \( \phi : \pi_1(X-B,x_0) \to D_p \), we say that \( f \) is an irregular dihedral branched cover of \( X \).

Put differently, \( \phi \) is the monodromy representation of the unbranched cover associated to \( f \) and meridians of the branching set \( B \) map to reflections in the dihedral group \( D_p \) (thought of as a subgroup of the symmetric group \( S_p \)). In particular, the existence of a dihedral cover over a pair \((X,B)\) is a condition on the fundamental group of the complement of \( B \) in \( X \). When a (connected) dihedral cover over the pair \((S^3,\alpha)\) exists for some knot \( \alpha \), we say simply that \( \alpha \) admits a dihedral cover.

It is helpful to give a description of the pre-images of a point on the branching set \( B \) of an irregular dihedral cover \( f : Y \to X \). The covering space \( Y \) is a \( \mathbb{Z}/2\mathbb{Z} \) quotient of the \( 2p \)-fold regular dihedral cover \( Z \) corresponding to the kernel of the homomorphism \( \phi \) in Definition 1.1. For every locally flat point \( b \in B \) the pre-image \( f^{-1}(b) \) of a small neighborhood \( D_b \) of \( b \) in \( X \) contains \( \frac{p-1}{2} \) components of branching index 2 and one component of branching index 1. The index 1 component is the fixed set of the involution \( Z \to Z \).

**Definition 1.2.** Let \( X \) be a four-manifold and let \( B \) be a closed surface embedded in \( X \). Let \( \alpha \subset S^3 \) be a non-trivial knot. For a given point \( z \in B \), assume there exist a small open disk \( D_z \) about \( z \) in \( X \) such that there is a homeomorphism of pairs \((D_z-z,B-z) \cong (S^3 \times (0,1),\alpha \times (0,1))\). We say the embedding of \( B \) in \( X \) has a singularity of type \( \alpha \) at \( z \).

In other words, the knot \( \alpha \) is the link of the singularity of \( B \) at \( z \). For the covers considered we assume in addition that singularity is normal, meaning that the pre-image of \( z \) under the covering map is a single point. The presence of a singularity \( \alpha \) on the branching set \( B \) results in a defect, or correction term, to the signature of the covering manifold. While this defect depends only on \( \alpha \), it is computed with the help of an associated knot to \( \alpha \), defined below.

**Definition 1.3.** Let \( \alpha \subset S^3 \) and \( \beta \subset S^3 \) be two knot types. We say that \( \beta \) is a mod \( p \) characteristic knot for \( \alpha \) if there exists a Seifert surface \( V \) for \( \alpha \) with Seifert form \( L \) such that \( \beta \subset V^c \subset S^3 \) represents a non-zero primitive class in \( H_1(V;\mathbb{Z}) \) and \((L + L^T)\beta \equiv 0 \mod p \).

In [8] Cappell and Shaneson defined characteristic knots and proved that for \( p \) a positive odd square-free integer and \( \alpha \) a non-trivial knot, \( \alpha \) admits an irregular dihedral \( p \)-fold cover if and only if there exists a knot \( \beta \) which is a mod \( p \) characteristic knot for \( \alpha \). Furthermore, they gave an explicit construction of a cobordism, here denoted \( W(\alpha,\beta) \), between a dihedral \( p \)-fold branched cover of \( \alpha \) and a cyclic \( p \)-fold branched cover of \( \beta \). We recall this construction as needed in the proof of Proposition 2.6.

Throughout this article we adopt the following notation. Let \( \chi \) denote the Euler characteristic and \( c \) the signature of a manifold, and let \( e \) be the self-intersection number of an embedded closed submanifold. Given a positive odd integer \( p \) and a knot \( \alpha \) in \( S^3 \) which admits an irregular dihedral \( p \)-fold cover, denote by \( V \) a Seifert surface for \( \alpha \) with symmetrized Seifert pairing \( L_V := L + L^T \). Let
\( \beta \subset V^\circ \) be a mod \( p \) characteristic knot for \( \alpha \). Finally, denote by \( \sigma_\zeta \) the Tristram–Levine \( \zeta^i \)-signature of a knot \([39]\), where \( \zeta \) is a primitive \( p \)-th root of unity.

Our first theorem is a necessary condition for the existence of a \( p \)-fold irregular dihedral cover \( f : Y \to X \) between two four-manifolds \( X \) and \( Y \), with a specified embedded surface \( B \subset X \) as its branching set.

**Theorem 1.4** (Necessary condition). Let \( X \) and \( Y \) be closed oriented PL four-manifolds and let \( p \) be an odd prime. Let \( B \subset X \) be a closed connected surface, PL-embedded in \( X \) except for an isolated singularity \( z \) of type \( \alpha \). If there exists an irregular dihedral \( p \)-fold cover \( f : Y \to X \) branched along \( B \) with a normal singularity at \( z \), then the knot \( \alpha \) admits an irregular dihedral \( p \)-fold cover and this cover is \( S^3 \). Furthermore, given any corresponding (see footnote \([7]\) mod \( p \) characteristic knot \( \beta \) for \( \alpha \), the following formulas hold:

\[
\chi(Y) = p\chi(X) - \frac{p-1}{2}\chi(B) - \frac{p-1}{2},
\]

and

\[
\sigma(Y) = p\sigma(X) - \frac{p-1}{4}e(B) - \Xi_p(\alpha),
\]

where

\[
\Xi_p(\alpha) = \frac{p^2-1}{6p}L_V(\beta,\beta) + \sigma(W(\alpha,\beta)) + \sum_{i=1}^{p-1}\sigma_\zeta(\beta).
\]

**Remark 1.5.** The author believes that the above theorem as well as the rest of the results of this paper extend to the case where \( X \) and \( Y \) are topological four-manifolds.

The main substance of this theorem is finding an expression for \( \Xi_p(\alpha) \), the contribution to the signature of \( Y \) resulting from the presence of a singularity of type \( \alpha \) on the branching set. Note that it is straightforward to compute \( L_V(\beta,\beta) \) and \( \sum_{i=1}^{p-1}\sigma_\zeta(\beta) \) from diagrams of \( \alpha \) and \( \beta \). A less obvious but essential feature of this theorem is the fact that the term \( \sigma(W(\alpha,\beta)) \) can be expressed in terms of linking numbers of curves in a dihedral branched cover of \( \alpha \) (see Proposition \([2.6]\)). A combinatorial procedure for computing these linking numbers from a diagram of \( \alpha \) is described in Appendix \([3]\) using techniques of Perko \([30]\). This procedure was carried out in \([4]\) and implemented in Python.

It is clear from the definition of a characteristic knot that \( \beta \) is not uniquely determined by \( \alpha \). While each of the terms \( \frac{p^2-1}{6p}L_V(\beta,\beta) \), \( \sigma(W(\alpha,\beta)) \), and \( \sum_{i=1}^{p-1}\sigma_\zeta(\beta) \) depends on \( \beta \), we show in Proposition \([2.4]\) that their sum, \( \Xi_p(\alpha) \), is an invariant of \( \alpha \) and thus independent of the choice of characteristic knot \([1]\). The author and Cahn develop a combinatorial method for computing \( \Xi_p(\alpha) \) from a Fox \( p \)-colored diagram of \( \alpha \) and apply this method to specific examples of two-bridge singularities in \([5]\). They also show that, for \( \alpha \) a slice knot which arises as a singularity on a \( p \)-fold dihedral cover between four-manifolds, \( \Xi_p(\alpha) \) gives an obstruction to \( \alpha \) being homotopy ribbon. Precisely, if a slice singularity \( \alpha \) is in fact homotopy ribbon, then \( |\Xi_p(\alpha)| \leq \frac{p-1}{2} \) (Theorem 4 of \([5]\)). Since ribbon knots are homotopy ribbon, this means in particular that \( \Xi_p(\alpha) \) can be used to test potential counter-examples to the Slice Ribbon Conjecture such as those constructed in \([11]\).

In the case where the manifold \( Y \) are simply-connected, Equation \((1.1)\) is equivalent to determining the rank of its intersection form, which is why this (easy to obtain) equation is of interest. Lastly, we note that Theorem 1.4 generalizes in the obvious way to the situation where the branching set admits multiple cone singularities, the signature of the cover picking up a defect term for each singular point. That is, if the embedding of \( B \) in \( X \) has singularities \( \alpha_1, \ldots, \alpha_k \), then

\[
\chi(Y) = p\chi(X) - \frac{p-1}{2}\chi(B) - k\frac{p-1}{2} - \Xi_p(\alpha),
\]

where \( -\Xi_p(\alpha) \) is a sum of \( \Xi_p(\alpha_i) \) for each singularity \( \alpha_i \).

\[\text{Precisely, } \Xi_p(\alpha) \text{ is an invariant of } \alpha \text{ together with a representation of } \pi_1(S^3 - \alpha, x_0) \text{ onto } D_p. \text{ In a lot of cases, the latter is uniquely determined by } \alpha, \text{ up to the appropriate notion of equivalence. To each equivalence class of dihedral representations of } \pi_1(S^3 - \alpha, x_0) \text{ corresponds an equivalence class of mod } p \text{ characteristic knots for } \alpha, \beta \text{ can be chosen arbitrarily within this class. See } [3].\]
and
\[
\sigma(Y) = p\sigma(X) - \frac{p-1}{4}c(B) - \sum_{i=1}^{k} \mathbb{Z}_p(\alpha_i).
\]

The following theorem is a partial converse to Theorem 1.4.

**Theorem 1.6 (Sufficient condition).** Let \( X \) be a simply-connected closed oriented PL four-manifold. Let \( B \subset X \) be a closed connected surface PL-embedded in \( X \) and such that \( \pi_1(X - B, x_0) \cong \mathbb{Z}/2\mathbb{Z} \). Let \( p \) be an odd prime, and let \( \alpha \) be any two-bridge slice knot which admits a \( p \)-fold dihedral cover. If \( \sigma \) and \( \chi \) are two integers which satisfy Equations (1.1) and (1.2), respectively, with respect to \( X \), \( B \) and \( \alpha \), then there exists a simply-connected four-manifold \( Y \) such that \( \sigma(Y) = \sigma \), \( \chi(Y) = \chi \) and \( Y \) is an irregular dihedral \( p \)-fold cover of \( X \).

Note that Equations (1.1) and (1.2) make sense when \( B \) and \( \alpha \) are not related. The branching set of the covering map constructed in the proof of this theorem is a surface \( B_1 \cong B \), embedded in \( X \) with an isolated singularity \( z \) of type \( \alpha \) and such that \( e(B_1) = e(B) \). When restricted to the class of two-bridge slice singularities, Theorems 1.4 and 1.6 give a necessary and sufficient condition for a pair of integers \((\sigma, \chi)\) to arise as the signature and Euler characteristic of a simply-connected dihedral cover over a given base.

Next, we show that over any indefinite four-manifold \( X \), an infinite family of integer pairs \((\sigma, \chi)\) can be realized as the signatures and Euler characteristics of simply-connected \( p \)-fold dihedral covers over \( X \).

**Theorem 1.7.** Let \( X \) be a simply-connected closed oriented PL four-manifold whose second Betti number is positive. Let \( \alpha \) be a two-bridge slice knot which admits an irregular dihedral \( p \)-fold cover for \( p \) an odd prime. There exists an infinite family of pairwise non-homeomorphic simply-connected closed oriented four-manifolds \( \{Y_i\}_{i=1}^{\infty} \), each of which is an irregular \( p \)-fold cover of \( X \) branched over an oriented surface embedded in \( X \) with an isolated singularity of type \( \alpha \).

We remark that for any \( p \), infinitely many knots \( \alpha \) which satisfy the hypotheses of the theorem exist, as shown in Proposition 3.9. In [5], the author and Cahn used the construction in the proof of Theorem 1.7 to give an infinite family of three-fold dihedral covers \( \mathbb{C}P^2 \to S^4 \). Each of these covers is branched along a two-sphere embedded in \( S^4 \) with one two-bridge slice singularity, and the singularities used are pairwise distinct. Note that, as indicated previously, the signature obstructs the existence of a branched cover \( \mathbb{C}P^2 \to S^4 \) branched along a non-singular oriented surface.

The construction of infinite families of branched covers given in Theorem 1.7 relies on our ability to vary the branching set of a dihedral cover. It invites the question, under what conditions can a particular manifold \( Y \) be realized as a \( p \)-fold dihedral cover over a given base data \((X, B, \alpha)\)? In situations where the manifold \( Y \) is (nearly) determined by the rank and signature of its intersection form, we obtain a complete classification.

**Theorem 1.8.** Let \( X \) and \( Y \) be simply-connected closed oriented PL four-manifolds whose intersection forms are odd. Fix an odd square-free integer \( p \) and a two-bridge slice knot \( \alpha \) which admits a \( p \)-fold dihedral branched cover. Let \( B \subset X \) be a PL-embedded surface such that \( \pi_1(X - B, x_0) \cong \mathbb{Z}/2\mathbb{Z} \). Then, the Euler characteristic and signature of \( Y \) satisfy Equations (1.1) and (1.2) with respect to \( X \), \( B \) and \( \alpha \) if and only if \( Y \) is homeomorphic to an irregular \( p \)-fold dihedral cover of \( X \) branched along a surface \( B_1 \) embedded in \( X \) with a singularity \( \alpha \) and such that \( B_1 \cong B \) and \( e(B_1) = e(B) \).

The rest of this article is organized as follows. In Section 2 we prove Theorem 1.4 give a formula for \( \sigma(W(\alpha, \beta)) \) in terms of linking numbers in a branched cover of \( \alpha \), and show that the defect on the signature arising from a singularity on the branching set is an invariant of the singularity type. Section 3 is dedicated to the proofs of Theorems 1.6, 1.7 and 1.8. In Appendix 3 we study characteristic knots of two-bridge knots. Appendix 3 lays out a procedure for calculating linking numbers in a branched cover of a knot \( \alpha \) in terms of a diagram of \( \alpha \).
2. Signatures of dihedral covers

Our strategy in proving Theorem 1.4 is to resolve the singularity on the branching set and reduce the computation of the signature to the case of a PL embedded surface. Then, Novikov additivity [29] implies that the difference between the signatures of the smooth and singular covers is given by the signature of the manifold used to resolve the singularity. The final step is to compute the signature of this manifold and prove we can express it in terms of invariants of the singularity type.

Proof. Proof of Theorem 1.4 The assertion that $\alpha$ admits an irregular dihedral $p$-fold cover and this cover is the three-sphere is verified by considering the local picture around the singular point. The existence of a $p$-fold dihedral cover $f : Y \to X$ over the pair $(X, B)$ implies straight away that the knot $\alpha$ itself admits a $p$-fold dihedral cover $M$. Indeed, consider $f^{-1}(\partial N(z)) =: M$, where $z \in B \subset X$ is the singular point on the branching set and $N(z)$ denotes a small neighborhood. Since by assumption there is a homeomorphism of pairs

$$(\partial N(z), B \cap \partial N(z)) \cong (S^3, \alpha),$$

the restriction of $f$ to $M$ is a $p$-fold dihedral cover of $\alpha$, as claimed. It is connected since $z$ is normal. In particular, this means that the knot group of $\alpha$ surjects onto the dihedral group $D_p$. Furthermore, over $N(z)$ lies the cone on $M$. Since by assumption $Y$ is a manifold, $M$ is homeomorphic to the three-sphere.

We begin by verifying Equation (1.1), a straight-forward computation. Let $N(B)$ denote a regular neighborhood of $B$ in $X$. Then, we can write

$$X = (X - N(B)) \bigcup_{\partial N(B)} N(B).$$

Since $\partial N(B)$ is a closed oriented three-manifold, we have $\chi(\partial N(B)) = 0$. This gives:

$$\chi(X) = \chi(X - N(B)) + \chi(N(B)) = \chi(X - B) + \chi(B).$$

We can further break down this equation as:

$$\chi(X) = \chi(X - B) + \chi(B - z) + 1.$$

Similarly, denoting $B' := f^{-1}(B)$ and $z' := f^{-1}(z)$, we have,

$$\chi(Y) = \chi(Y - B') + \chi(B' - z') + 1.$$

We know that $f|_{Y - B'} : Y - B' \to X - B$ is a $p$-to-one covering map, $f|_{B' - z'} : B' - z' \to B - z$ is an $\frac{p+1}{2}$-to-one covering map, and, of course, $f|_{z'} : z' \to z$ is one-to-one. Therefore,

$$\chi(Y) = p\chi(X - B) + \frac{p + 1}{2} (\chi(B) - 1) + 1 = p\chi(X) - \frac{p - 1}{2} \chi(B) - \frac{p - 1}{2},$$

as claimed.

Now we turn to the computation of $\sigma(Y)$, a considerably harder task. We devise a geometric procedure for the resolution of the singularity on the branched cover. The singularity is resolved in two stages. At the start, the branching set has one singular point, in a neighborhood of which the branching set can be described in terms the knot $\alpha$. Our first step will be to replace this singularity by a circle’s worth of “standard” (that is, independent of the knot type $\alpha$) non-manifold points on the branching set. The second step will be to excise these “standard” singularities and construct a new cover whose branching set is a PL submanifold of the base. We carry out these two steps in detail below. In the last stage of the proof, we calculate the effect of the resolution of singularities on the signatures of the four-manifolds involved.

Step 1. Let $D_z \subset X$ be a neighborhood of the singular point $z$ such that $D_z \cap B$ is the cone on $\alpha$. As we already established, $\alpha$ admits a $p$-fold dihedral cover. Equivalently, if $V$ is any Seifert surface for $\alpha$, there exists a mod $p$ characteristic knot $\beta \subset V^5$ (see Definition 1.3). Let $W(\alpha, \beta)$ be the manifold constructed in [S] as a cobordism between a $p$-fold dihedral cover of $(S^3, \alpha)$ and a $p$-fold
cyclic cover of \((S^3, \beta)\). By construction of \(W(\alpha, \beta)\), which is recalled in the proof of Proposition 2.3, there is a \(p\)-fold branched covering map

\[ h_1 : W(\alpha, \beta) \to S^3 \times [0, 1], \]

which restricts to a \(p\)-fold dihedral cover of \((S^3 \times \{0\}, \alpha)\) and to a \(p\)-fold cyclic cover of \((S^3 \times \{1\}, \beta)\). Let

\[ h_2 : Q \to D^4 \]

be a \(p\)-fold cyclic cover of the closed four-ball branched over a pushed-in Seifert surface \(V'\) for \(\beta\), as constructed in Theorem 5 of [6]. Denote by \(\Sigma\) the \(p\)-fold cyclic cover of \((S^3, \beta)\). By construction, \(\partial Q \cong \Sigma\) and, similarly, \(W(\alpha, \beta)\) has one boundary component homeomorphic to \(\Sigma\). Moreover, for \(i = 1, 2\), the map

\[ h_i|_{\Sigma} : \Sigma \to S^3 \]

is the \(p\)-fold cyclic cover branched along \(\beta\). Therefore, we can construct a branched cover

\[ (2.1) \quad h_1 \cup h_2 : W(\alpha, \beta) \cup_{\Sigma} Q \to S^3 \times [0, 1] \bigcup_{S^3 \times \{1\}} D^4. \]

We denote \(W(\alpha, \beta) \cup_{\Sigma} Q\) by \(Z\) for short, and the map \(h_1 \cup h_2\) by \(h\). Thus, we can rewrite Equation (2.1) as

\[ h : Z \to D^4. \]

This map is a \(p\)-fold branched cover whose restriction to the boundary of \(Z\) a \(p\)-fold irregular dihedral cover of \((S^3, \alpha)\). So, denoting the branching set of \(h\) by \(T\), we have,

\[ (2.2) \quad T \cong \alpha \times [0, \frac{1}{2}] \bigcup_{\alpha \times \{\frac{1}{2}\}} V \times \{\frac{1}{2}\} \bigcup_{\beta \times \{\frac{1}{2}\}} \beta \times \{\frac{1}{2}, 1\} \bigcup \beta \times \{1\} \]

We see from this description that \(T\) is a two-dimensional PL subcomplex of \(D^4\) which is a manifold away from the curve \(\beta \times \{\frac{1}{2}\}\). Observe that \(\beta \times \{\frac{1}{2}\}\) is embedded in the interior of \(V \times \{\frac{1}{2}\}\). Therefore, in a small neighborhood of the curve \(\beta \times \{\frac{1}{2}\}\), the branching set is homeomorphic to the Cartesian product of \(S^1\) and the letter “\(\top\)”. (For more details on this construction we once again refer the reader to [8].)

The idea is to use the map \(h\) to construct a new cover of the manifold \(X\) which will differ from the original cover \(f\) only over a neighborhood of the singularity \(z \in B\). Specifically, let \(D'_z := f^{-1}(D_z)\) and observe that the restrictions of the maps \(f\) and \(h\) to the boundaries of \(Y - D'_z\) and \(Z\), respectively, are the \(p\)-fold irregular dihedral branched cover\(^{2}\) of \((S^3, \alpha)\), which is again \(S^3\). We thus obtain a new branched covering map

\[ (2.3) \quad f \cup h : (Y - D'_z) \bigcup_{S^3} Z \to (X - D'_z) \bigcup_{S^3} D^4. \]

Denote the covering manifold \((Y - D'_z) \bigcup_{S^3} Z\) above by \(Y_1\) and the map \(f \cup h\) by \(f_1\). Note that, by Novikov additivity [29], and since \(D'_z\) is a four-ball, \(\sigma(Y_1) = \sigma(Y) + \sigma(Z)\).

Now consider the base space of the branched covering map (2.3)

\[ X_1 := (X - D'_z) \bigcup_{S^3} D^4. \]

Since \(X_1 \cong X\), we will continue to denote this space by \(X\). Lastly, denote the branching set of \(f_1\) by \(B_1\) and remark that

\[ (2.4) \quad B_1 \cong B - N(z)^0 \bigcup_{\alpha \times \{0\}} T. \]

\[^{2}\]We use the phrase “the dihedral cover of \(\alpha\)” somewhat liberally throughout this paper. As noted previously, dihedral covers of \(\alpha\) are in bijective correspondence with equivalence classes of characteristic knots \(\beta\). Naturally, if \(\alpha\) admits multiple non-equivalent dihedral covers, we choose the one determined by \(f\) to construct \(Z\).
In other words, we replaced the cone on $\alpha$ by $T$. As prescribed, $B_1$ has a circle’s worth of non-manifold points – they are the points corresponding to $\beta \times \{1\}$ in Equation (2.2) – regardless of the choice of the knot $\alpha$.

Step 2. Denote by $\beta^*$ the curve of non-manifold points on $T$. We have, $\beta^* \subset T \subset D^4 \subset X$. Let $N(\beta^*)$ be a small open tubular neighborhood of $\beta^*$ in $X$. We give $N(\beta^*)$ the following trivialization. For every $b \in \beta^*$, let $\vec{n}_1(b)$ be the positive normal to $\beta$ in $V$ at the point $b$, $\vec{n}_2(b)$ the positive normal to $V$ in $S^3 \times \{1\}$, and $\vec{n}_3(b)$ the positive normal to $S^3$ in the product structure $S^3 \times I$. Clearly, $\{\vec{n}_1(b), \vec{n}_2(b), \vec{n}_3(b)\}$ are linearly independent for all $b \in \beta^*$, so they give a framing of $\beta^*$ in $X$.

We use the above framing to identify $N(\beta^*)$ with $S^1 \times B^3$ and $\partial N(\beta^*)$ with $\beta^* \times S^2$. Now, we construct a new closed oriented four-manifold, denoted $X_2$, as follows:

$$X_2 = (X - N(\beta^*)) \bigcup_{S^1 \times S^2} (X - N(\beta^*)).$$

The identification of the two copies of $\partial(X - N(\beta^*))$ is done by the homeomorphism

$$\phi : S^1 \times S^2 \to S^1 \times S^2$$

given by the formula

$$\phi(e^{i\theta}, y) = (e^{-i\theta}, y).$$

In particular, $\phi$ reverses orientation on $S^1 \times S^2$, so the manifold $X_2$ can be given an orientation which restricts to the original orientations on both copies of $X - N(\beta^*)$. Therefore, by Novikov additivity we obtain

$$\sigma(X_2) = 2\sigma(X - N(\beta^*)) = 2\sigma(X).$$

Note that, since $\phi$ restricts to the identity on the $S^2$ factor, it identifies the boundary of the branching set $T - N(\beta^*)$ in one copy of $X - N(\beta^*)$ with the boundary of branching set in the other copy of $X - N(\beta^*)$. Thus, the image of the branching set after this identification has the form

$$(B_1 - N(\beta^*)) \bigcup_{3S^1} (B_1 - N(\beta^*)) =: B_2.$$

Here the fact that the union is taken along three circles corresponds to the fact that the intersection of $\partial N(\beta^*)$ and $B_1$ consists of three closed curves, one for each “boundary point” of the letter “T”.

Note that, since $\phi$ reverses the orientation on each boundary circle, the orientations of the two copies of $(B_1 - N(\beta^*))$ can be combined to obtain a compatible orientation on $B_2$. Furthermore, the positive normal to the oriented surface $(V - N(\beta^*)) \cup_{\phi_1} (V - N(\beta^*))$ inside the three-manifold $(S^3 \times \{1\} - N(\beta^*)) \cup_{\phi_2} (S^3 \times \{1\} - N(\beta^*))$ restricts to the normals to $V$ in each corresponding copy of $S^3$. This observation will come into use shortly.

Recalling the definition of $B_1$, namely $B_1 = (B - D_2^0) \cup_{\alpha} (T - N(\beta^*))$, we can describe $B_2 = (B_1 - N(\beta^*)) \cup_{3S^1} (B_1 - N(\beta^*))$ in more detail as follows:

$$B_2 = ((B - D_2^0) \cup_{\alpha} (T - N(\beta^*))) \bigcup_{3S^1} ((B - D_2^0) \cup_{\alpha} (T - N(\beta^*) unborn)).$$

By construction, $B_2$ is embedded piecewise-linearly in $X_2$ — that is, all singularities have been resolved. In addition, $B_2$ has two connected components, since deleting a neighborhood of $\beta^*$ disconnects $T$. Thus, two copies of $(T - N(\beta^*))$ gives four disjoint surfaces with boundary. Attaching along the three curves in $(S^1 \times S^2) \cap (T - N(\beta^*))$ via $\phi_1$ has the effect of pairing off each of these four surfaces with boundary and its homeomorphic copy. This produces two closed surfaces which we denote $B'_2$ and $B''_2$. Here, $B'_2$ is the component of $B_2$ obtained by identifying two copies of $(B - D_2^0) \cup_{\alpha} (V - \beta)$ along $S^1 \cup S^1$, where $(V - \beta)$ denotes the complement in $V$ of a small open neighborhood of $\beta$. In turn, $B''_2$ is the component of $B_2$ obtained by identifying two copies of $V'$ along $S^1$. By construction, the cover over $B'_2$ is $p$-fold dihedral, whereas the cover over $B''_2$ is $p$-fold cyclic. That is, a point in $B''_2$ has

\[ \text{It would be more consistent with our earlier notation to say that } B''_2 \text{ is obtained from two copies of } \beta \times [\frac{1}{p}, 1] \cup_{\beta \times (1)} V', \text{ which, of course, is a surface homeomorphic to } V'. \]
Recall that both \( Y \) and \( \Sigma \) were constructed from copies of \((Y_1 - N')\) and \((X - N(\beta^*))\) by gluing via \( \phi \). Therefore, the restrictions of \( f_1 \) to the two copies of \((Y_1 - N')\),
\[
    f_1 | : (Y_1 - N') \rightarrow (X - N(\beta^*)) ,
\]
can be glued to obtain a map
\[
    f_2 : ((Y_1 - N') \cup_{S^1 \times S^2} (Y_1 - N')) \rightarrow (X - N(\beta^*)) \cup_{S^1 \times S^2} (X - N(\beta^*)) ,
\]
written for short as
\[
    f_2 : Y_2 \rightarrow X_2.
\]
This is the branched cover we will use in the final step of the proof to compute the signature of the original manifold \( Y \).

**Step 3.** To complete the proof, what remains is to compute the effect this construction has on the signatures of the base and covering manifolds. By Viro’s formula \[40\] for the signature of a branched cover, we have
\[
    \sigma(Y_2) = p \sigma(X_2) - \frac{p - 1}{4} e(B_2') - \frac{p^2 - 1}{3p} e(B_2'') ,
\]
Recall that from Equations \eqref{2.5} and \eqref{2.8} we have
\[
    \sigma(X_2) = 2 \sigma(X) ,
\]
and
\[
    \sigma(Y) = \frac{1}{2} \sigma(Y_2) - \sigma(Z) .
\]
Also, by Novikov additivity,
\[
    \sigma(Z, S^3) = \sigma(W(\alpha, \beta), S^3 \cup \Sigma) + \sigma(Q, \Sigma) = \sigma(W(\alpha, \beta)) + \sum_{i=1}^{p-1} \sigma_{\mathcal{C}_i}(\beta) .
\]
In the last step, we have expressed the signature of \( Q \) in terms of Tristram–Levine signatures of \( \beta \), using Theorem 5 of \[6\]. We have also shortened \( \sigma(W(\alpha, \beta), S^3 \cup \Sigma) \) to \( \sigma(W(\alpha, \beta)) \). Now we combine Equations \eqref{2.9}, \eqref{2.10}, \eqref{2.11} and \eqref{2.12} in order to express \( \sigma(Y) \) in terms of data about \( X \), the branching set, the singularity \( \alpha \) and its characteristic knot \( \beta \). After simplifying, we obtain,
\[
    \sigma(Y) = p \sigma(X) - \frac{1}{2} \left( \frac{p - 1}{4} e(B_2') + \frac{p^2 - 1}{3p} e(B_2'') \right) - \sigma(W(\alpha, \beta)) - \sum_{i=1}^{p-1} \sigma_{\mathcal{C}_i}(\beta) .
\]
To arrive at Equation (2.12), what remains is to compute the self-intersection numbers of $B'_2$ and $B''_2$ in $X_2$ and relate them to that of $B$ in $X$.

We denote the intersection number of two submanifolds by \("\), and the push-off of a submanifold $S$ along a normal $\mathbf{u}$ by $\mathbf{u}(S)$. For brevity, we also denote $B-D_\alpha$, the complement in $B$ of a small open neighborhood of the singularity $z$, by $B_z$.

Note that if $\tilde{v}$ is a continuous extension (not necessarily non-vanishing) to $B_z$ of the normal to $V$ in $S^3 = \partial D_z$ such that $B_z$ and $\tilde{v}(B_z)$ are transverse, then by definition

$$e(B) = (B_z \cup_\alpha V) \cdot \tilde{v}(B_z \cup_\alpha V).$$

(2.14)

Now, $V \subset D_z$ and $\tilde{v}(V) \subset D_z$, whereas $B_z \cap D_z = \alpha$ and $\tilde{v}$ can be chosen so that $\tilde{v}(B_z) \cap D_z = \tilde{v}(\alpha)$. In particular, $V$ is disjoint from both $\tilde{v}(V)$ and $\tilde{v}(B_z)$, and $B_z$ is disjoint from $\tilde{v}(V)$. Therefore, Equation (2.14) simplifies to

$$e(B) = B_z \cdot \tilde{v}(B_z),$$

(2.15)

where the right hand side represents the intersection number of transverse submanifolds with disjoint boundary in $X-D_z$. Recall that the surface $B'_2$ is obtained from two copies of $B_z \cup_\alpha (V - \beta)$ attached by a homeomorphism $\phi_1$ on their boundary $\beta_1 \cup \beta_2$, reversing orientation on each component. Recall also that the restriction to $S^3 \times \{ \frac{1}{2} \} \cap \partial N(\beta^*)$ of the positive normal to $V$ in $S^3 \times \{ \frac{1}{2} \}$ (and thus of $\tilde{v}$), is preserved by the gluing homeomorphism $\phi_1$. Therefore, the two copies of the normal $\tilde{v}$ to $B_z \cup_\alpha (V - \beta)$ can be combined obtain a normal, which we also denote $\tilde{v}$, to $B'_2$ in $X_2$. Then,

$$B'_2 = B_z \cup_\alpha (V - \beta) \cup_{\beta_1 \cup \beta_2} (V - \beta) \cup_\alpha B_z.$$

(2.16)

Since by the argument above $V - \beta$ and $\tilde{v}(V - \beta)$ contribute nothing to the self-intersection $B'_2 \cdot \tilde{v}(B'_2)$, we have

$$e(B'_2) = 2(B_z \cdot \tilde{v}(B_z)) = 2e(B).$$

(2.17)

Next, recall that $B''_2 = V' \cup_\beta V'$ and $\mathbf{n}_1$ is the normal to $\beta$ in $V$. Denote by $\tilde{v}'$ a continuous extension (not necessarily nowhere-zero) to $V'$ of the normal $\mathbf{n}_1$ so that $V'$ and $\tilde{v}'(V')$ are transverse. We have,

$$e(B''_2) = 2(V' \cdot \tilde{v}'(V')).$$

(2.18)

Recall that $L_V$ denotes the symmetrized linking form on $V$, the Seifert surface for $\alpha$. The last equality follows from the fact that $\mathbf{n}_1$ and $\mathbf{n}_2$, the normal to $\beta$ determined by $V$, are everywhere linearly independent, so $\text{lk}(\beta, \mathbf{n}_1(\beta)) = \text{lk}(\beta, \mathbf{n}_2(\beta))$.

Substituting for $e(B'_2)$ from Equation (2.17) and for $e(B''_2)$ from Equation (2.18), we can rewrite Equation (2.13) as

$$\sigma(Y) = p \sigma(X) - \frac{p-1}{4} e(B) - \frac{p^2-1}{6p} L_V(\beta, \beta) - \sigma(W(\alpha, \beta)) - \sum_{i=1}^{p-1} \sigma_{\xi_i}(\beta)$$

(2.19)

$$= p \sigma(X) - \frac{p-1}{4} e(B) - \Xi_p(\alpha).$$

With that, the proof is complete.

Remark 2.1. The property that a $p$-fold dihedral cover of a knot $\alpha$ is homeomorphic to the three-sphere can be regarded as a condition for $\alpha$ to be an allowable singularity on the branching set of an irregular $p$-fold dihedral cover between four-manifolds. The condition is satisfied, for example, for all two-bridge knots and any odd $p$ (see the proof of Lemma 3.3) and can be disregarded if one allows the covering space to be a stratified space. In this case, an analogous formula can be obtained, using intersection homology signature or Novikov signature of a singular space.

Remark 2.2. It is natural to consider computing signatures of cyclic branched covers using the same ideas. Indeed, our techniques would apply and the arguments would be considerably simpler: in the notation of the proof of the last theorem, only the manifold $Q$ would be needed to resolve the
singularity. However, it is a consequence of the Smith Conjecture [28] that no non-trivial knot can arise as a singularity on a cyclic cover between four-manifolds. This is why the case of cyclic covers is not considered in this work. Our methods, however, are applicable in a scenario where stratified spaces are allowed as the covers.

Although we have arrived at the desired equation, the formula we have obtained does not quite startle with its usefulness, as long as the term \( \sigma(W(\alpha, \beta)) \) remains obscure. As stated in the introduction, we aim to compute the defect to \( \sigma(Y) \) in terms of the singularity type \( \alpha \). That is, we need to express \( \sigma(W(\alpha, \beta)) \) explicitly in terms of some computable invariants of \( \alpha \). It turns out that we can give a formula for \( \sigma(W(\alpha, \beta)) \) using linking numbers of curves in the irregular dihedral \( p \)-fold branched cover of \( \alpha \). To this end, we first compute the second homology group of this manifold: this is the content of Proposition 2.3. In Corollary 2.4, we give a basis for this homology group in terms of lifts to a dihedral cover of \( \alpha \) of curves in the chosen Seifert surface \( V \). Finally, in Proposition 2.6 we give an explicit formula for the term \( \sigma(W(\alpha, \beta)) \) using linking numbers of the above curves.

**Proposition 2.3.** Let \( \alpha \subset S^3 \) be a knot which admits a \( p \)-fold irregular dihedral cover \( M \) for some odd prime \( p \). Let \( V \) be a Seifert surface for \( \alpha \) and let \( \beta \subset V \) be a mod \( p \) characteristic knot for \( \alpha \). Let \( \Sigma \) be the \( p \)-fold cyclic cover of \( \beta \). Let \( W(\alpha, \beta) \), here denoted \( W \), be the cobordism between \( M \) and \( \Sigma \) constructed in [8] and used in the proof of Theorem 1.4. Denote by \( V - \beta \) the surface \( V \) with a small open neighborhood of \( \beta \) removed, and by \( \beta_1 \) and \( \beta_2 \) the two boundary components of \( V - \beta \) that are parallel to \( \beta \). Then

\[
H_2(W, M; \mathbb{Z}) \cong \mathbb{Z}^{\mathbb{Z}^{\mathbb{Z}}_2} \oplus \langle H_1(V - \beta; \mathbb{Z})/([\beta_1], [\beta_2]) \rangle^{\mathbb{Z}^{\mathbb{Z}}_2}.
\]

**Proof.** Since Cappell and Shaneson’s construction of \( W \) is essential to our computation, we review it here. Let \( f: \Sigma \to S^3 \) be the cyclic \( p \)-fold cover of \( \beta \). Since \( p \) is prime, it is well known that \( \Sigma \) is a rational homology sphere [35]. Let

\[
f \times 1_I: \Sigma \times [0, 1] \to S^3 \times [0, 1]
\]

be the induced cyclic branched cover of \( S^3 \times [0, 1] \). Next, let

\[
J := f^{-1}(V \times [-\epsilon, \epsilon] \times \{1\})
\]

be the pre-image of a closed tubular neighborhood \( V \times [-\epsilon, \epsilon] \times \{1\} \) of \( V \times \{1\} \) in \( S^3 \times \{1\} \), and let \( T \) be its “core”, namely \( T := f^{-1}(V \times \{0\} \times \{1\}) \) with

\[
T \subset J \subset \Sigma \times \{1\}.
\]

Then \( J \) deformation-retracts to \( T \), and \( T \) consists of \( p \) copies of \( V \) identified along \( \beta \) and permuted cyclically by the group of covering transformations of \( f \).

Consider the involution \( \hat{h} \) of \( J \) defined in [8] as the lift of the map

\[
h: V \times [-\epsilon, \epsilon] \to V \times [-\epsilon, \epsilon],
\]

\[
h(u, t) \mapsto (u, -t)
\]

fixing a chosen copy of \( V \) in \( f^{-1}(V \times \{0\} \times \{1\}) \). Let \( q \) be the quotient map defined as

\[
q: \Sigma \to \Sigma/\{x \sim \hat{h}(x)|x \in J\} = \Sigma/\hat{h}.
\]

Similarly, let

\[
W := (\Sigma \times I)/\{x \sim \hat{h}(x)|x \in J \subset \Sigma \times \{1\}\}
\]

and let

\[
M := (\Sigma - J^\circ)/\hat{h}.
\]

By definition, \( \Sigma/\hat{h} = M \cup (J/\hat{h}) \). As shown in [8], \( M \) is the \( p \)-fold irregular dihedral cover of \( \alpha \) and \( W \) is a cobordism between \( M \) and the cyclic \( p \)-fold cover \( \Sigma = \Sigma \times \{0\} \) of \( \beta \).

This completes the description of the construction of the pair \((W, M)\) whose second homology group we are about to compute.
Note that $W$ is by definition the mapping cylinder of the quotient map $q$. Let $R := J/\bar{h}$. We have

$$H_2(W, M; \mathbb{Z}) \cong H_2(M \cup R, M; \mathbb{Z}) \cong H_2(R, M \cap R; \mathbb{Z}),$$

where the second isomorphism is excision, and the first follows from the fact that $W$ deformation-retracts onto $\Sigma/\bar{h} = M \cup R$. Since $M \cap R = \partial R - V_0$

(following the notation of [8], $V_0$ is the copy of $V$ in $T$ fixed by $\bar{h}$), we can rewrite the above isomorphism as

$$H_2(W, M; \mathbb{Z}) \cong H_2(R, \partial R - V_0; \mathbb{Z}).$$

(2.21)

The relevant portion of the long exact sequence of the pair $(R, \partial R - V_0)$ is

$$H_2(R; \mathbb{Z}) \to H_2(R, \partial R - V_0; \mathbb{Z}) \to H_1(\partial R - V_0; \mathbb{Z}) \to H_1(R; \mathbb{Z}).$$

(2.22)

We will shortly show that $H_2(R; \mathbb{Z}) = 0$ (see Equation (2.25)). Assuming this for the moment, the above exact sequence, combined with Equation (2.21), gives

$$H_2(W, M; \mathbb{Z}) \cong \ker(\iota_* : H_1(\partial R - V_0; \mathbb{Z}) \to H_1(R; \mathbb{Z})).$$

(2.23)

Our goal, therefore, is to compute this kernel. Note, furthermore, that we are not simply interested in its rank over $\mathbb{Z}$; we want to write down an explicit basis for $\ker(\iota_*)$ in terms of lifts to $M$ of curves in the complement of $\alpha \subset S^3$.

Recall that $V$ is a surface with boundary and that, by definition, $\beta$ represents a non-zero primitive class in $H_1(V; \mathbb{Z})$. Therefore, $\beta$ can be completed to a one-dimensional subcomplex $C \cup \beta$ which $V$ deformation-retracts to, where $C$ is the wedge of $\eta - 1$ circles, and $g$ the genus of $V$. Moreover, we can perform the deformation retraction of $V$ onto such a one-complex simultaneously on each copy of $V$ contained in $T$, fixing the curve of intersection $\beta$. Therefore, $T$ deformation-retracts to a one-complex containing $\beta$ wedged to $p$ copies of $C$, where

$$H_1(C; \mathbb{Z}) \cong H_1(V; \mathbb{Z})/\langle [\beta] \rangle.$$

It follows that

$$H_2(J; \mathbb{Z}) \cong H_2(T; \mathbb{Z}) \cong 0$$

and

$$H_1(J; \mathbb{Z}) \cong H_1(T; \mathbb{Z}) \cong \oplus_p (H_1(V; \mathbb{Z})/\langle [\beta] \rangle) \oplus \mathbb{Z},$$

where the singled-out copy of $\mathbb{Z}$ is generated by $\beta$.

Furthermore, since the deformation–retraction of $J$ onto $T$ can be chosen to commute with $\bar{h}$, $J/\bar{h} = R$ deformation-retracts to $T/\bar{h}$, which is isomorphic to $\frac{\mathbb{Z}^{i+1}}{\mathbb{Z}}$ copies of $V$ identified along $\beta$. (This isomorphism is seen from the fact that $V_0$ is fixed by $\bar{h}$, and the remaining $p - 1$ copies of $V$ in $T$ become pairwise identified in the quotient. All copies of $\beta$ are identified to a single circle in both $T$ and $T/\bar{h}$.) Therefore,

$$H_1(R; \mathbb{Z}) \cong (H_1(V; \mathbb{Z})/\langle [\beta] \rangle)^{\frac{i+1}{i+1}} \oplus \mathbb{Z}.$$  

(2.24)

By the same reasoning as above, we can also conclude that $T/\bar{h}$ deformation-retracts to a one-complex, so, as claimed,

$$H_2(R; \mathbb{Z}) = H_2(J/\bar{h}; \mathbb{Z}) \cong H_2(T/\bar{h}; \mathbb{Z}) \cong 0.$$  

(2.25)

Next, we examine $\partial(J)$ and $\partial R$. To start, $\partial(V \times [0, 1]) \cong V \cup_\alpha V$. Therefore, $\partial(J)$ can be thought of as the union of $p$ copies of $(V - \beta) \cup_\alpha (V - \beta)$, which we label $V_i^+ \cup_\alpha V_i^-$, $0 \leq i < p$, with further identifications we now describe. Denote the copy of $\beta$ lying in $V_i^+$ by $\beta^+_i$, and cut each $V_i^\pm$ along $\beta_i^\pm \subset V_i^\pm$. Now, $V_i^\pm - \eta(\beta_i^\pm)$ is a connected surface with three boundary components, $\alpha_i$, $\beta_i^\pm$, and $\beta_i^{\pm \pm}$, where the $\beta_i^{\pm \pm} \subset V_i^\pm$ are labeled in such a way that $\beta_i^+$ and $\beta_i^-$ correspond to $\beta_{i,j} \times \{1\}$ and $\beta_{i,j} \times \{0\}$ in $V_i \times [0, 1]$; that is, the projection map $V_i \times [0, 1] \to V_i$ sends $\beta_{i,j}^+$ and $\beta_{i,j}^-$ to the same boundary component of $V - \eta(\beta)$. 


Now, the covering translation \( \tau \) on \( J \) acts in a neighborhood of \( \beta \) as rotation by \( \frac{2\pi i}{p} \) degrees. With the labeling described above, \( \tau \) carries \( \beta_{i,j}^{\pm} \) to \( \beta_{i+1 \mod p,j}^{\pm} \). Moreover, in \( \partial(J) \), we have the following identifications: \( \beta_{ij}^{+} \sim \beta_{ij+1 \mod p,2}^{+} \) and \( \beta_{ij}^{-} \sim \beta_{ij+1 \mod p,2}^{-} \). Put differently, we can think of \( \partial(J) \) as obtained from \( 2p \) disjoint copies of \( V - \beta \), labeled \( V_{i}^{+} - \beta_{i}^{+} \), by gluing \( \alpha_{i}^{+} \) to \( \alpha_{i}^{-} \) and the \( \beta_{i,j}^{\pm} \)'s according to the identifications specified above. Thus, \( \partial(J) \) is a closed surface. By considering its Euler characteristic, we find that its genus is \((2g - 1)p + 1\). In addition, from the above decomposition of \( \partial(J) \) we find that

\[
H_{1}(\partial(J); \mathbb{Z}) \cong ((H_{1}(V - \beta; \mathbb{Z}))/([\beta_{1}], [\beta_{2}]))^{2p} \oplus \mathbb{Z}^{2p+2}.
\]

Recall that \( R \) is a \( \mathbb{Z}/2\mathbb{Z} \) quotient of \( J \), where the \( \mathbb{Z}/2\mathbb{Z} \) action fixes \( V_{0} \times I \) and pairs off \( V_{i}^{+} \) with \( V_{p-i}^{-} \) for \( 1 \leq i \leq \frac{2p}{p} \). It follows that \( \partial R - V_{0} \) is a surface of genus \( p(g - 1) + \frac{2p}{p} \) and we have,

\[
H_{1}(\partial R - V_{0}; \mathbb{Z}) \cong ((H_{1}(V - \beta; \mathbb{Z}))/([\beta_{1}], [\beta_{2}]))^{p} \oplus \mathbb{Z}^{p+1}.
\]

Recall that our aim is to compute \( \ker(i_{*} : H_{1}(\partial R - V_{0}; \mathbb{Z}) \to H_{1}(R; \mathbb{Z})) \).

Again, the idea behind writing \( H_{1}(\partial R - V_{0}; \mathbb{Z}) \) as in Equation (2.27) is to obtain a convenient basis for this kernel, and to relate this basis to a basis for the homology of \( V \). Specifically, a virtue of our expression for \( H_{1}(\partial R - V_{0}; \mathbb{Z}) \) is that a basis for \( H_{1}(V - \beta; \mathbb{Z}))/([\beta_{1}], [\beta_{2}]) \) can be extended to a basis for both \( H_{1}(V; \mathbb{Z}) \) and \( H_{1}(V^{+} - \beta; \mathbb{Z}) \). In particular, the inclusion \( i : \partial R - V_{0} \to R \) induces an injection \( i_{*} : H_{1}(V - \beta; \mathbb{Z}))/([\beta_{1}], [\beta_{2}]) \to H_{1}(R; \mathbb{Z}) \) for each copy of \( V - \beta \subset \partial R - V_{0} \). Furthermore, for each \( V_{j} \subset T/\hat{h} \simeq R \), the inclusion \( i : V_{j} \to R \) induces an injection on \( H_{1}(V - \beta; \mathbb{Z}))/([\beta_{1}], [\beta_{2}]) \).

With this in mind, using Equations (2.25) and (2.27), we rewrite

\[ i_{*} : H_{1}(\partial R - V_{0}; \mathbb{Z}) \to H_{1}(R; \mathbb{Z}) \]

as

\[ i_{*} : ((H_{1}(V - \beta; \mathbb{Z}))/([\beta_{1}], [\beta_{2}]))^{p} \oplus \mathbb{Z}^{p+1} \to (H_{1}(V^{+} - \beta; \mathbb{Z}))/[\beta]^{p} \oplus \mathbb{Z}.
\]

Note that \( i_{*} \) maps the copy of \( (H_{1}(V - \beta; \mathbb{Z}))/([\beta_{1}], [\beta_{2}]) \) coming from \( V_{0}^{+} \) isomorphically onto its image, and it “pairs off” the remaining \( p - 1 \) copies of \( (H_{1}(V - \beta; \mathbb{Z}))/([\beta_{1}], [\beta_{2}]) \), including each of them into one of the remaining \( \frac{2p}{p} \) copies of \( H_{1}(V^{+} - \beta; \mathbb{Z}) \) coming from \( V_{1}^{+} - \beta_{1}^{+} \) to \( \ker(i_{*}) \). In addition, the generators for the \( \mathbb{Z}^{p+1} \) summand in \( H_{1}(\partial R - V_{0}; \mathbb{Z}) \) can be chosen as follows. There are \( \mathbb{Z}^{p} \) generators which correspond to copies of \( \beta \) lying in the various copies of \( V - \beta \subset (\partial R - V_{0}) \); they all map to the single \( [\beta] \) in the image, contributing \( \mathbb{Z}^{p} \) to \( \ker(i_{*}) \). Finally, there are \( \mathbb{Z}^{p} \) additional \( \mathbb{Z}^{p} \) generators of the \( \mathbb{Z}^{p+1} \) summand in \( H_{1}(\partial R - V_{0}; \mathbb{Z}) \) which are mapped injectively by \( i_{*} \), onto classes in \( T/\hat{h} \) which are not in the image of \( i_{*}(H_{1}(V - \beta; \mathbb{Z})) \) for any copy of \( V \).

Consequently, as we claimed,

\[ \ker(i_{*}) \cong ((H_{1}(V - \beta; \mathbb{Z}))/([\beta_{1}], [\beta_{2}]))^{p+1} \oplus \mathbb{Z}^{p+1}.
\]

Furthermore, the above argument allows us to describe a basis for this kernel.

**Corollary 2.4.** Assume the notation of Proposition 2.3. Further, let \( w^{1}, w^{2}, \ldots, w^{r} \) be a basis for \( H_{1}(V - \beta; \mathbb{Z}))/([\beta_{1}], [\beta_{2}]) \), where \( r = 2g - 2 \) and \( g \) is the genus of \( V \). Denote by \( w_{j}^{i, \pm}, i \in \{1, \ldots, r\}, j \in \{1, \ldots, p\} \) the pre-images of the \( w^{i} \) lying in \( f^{-1}(V \times \{\pm 1\}) \) so that \( w_{j}^{i, \pm} \subset V_{j}^{\pm} \), \( f(w_{j}^{i, \pm}) = w_{j}^{i} \) and \( \tau w_{j}^{i, \pm} = w_{j+1 \mod p}^{i, \pm} \). Next, let \( h(w_{j}^{i, \pm}) := w_{j}^{i, \pm} \in M \). Finally, denote by \( \gamma_{k}, k = 0, 1, \ldots, p-1 \) the \( \mathbb{Z}^{p} \) generators of \( H_{1}(\partial R - V_{0}; \mathbb{Z}) \) which are represented by copies of \( \beta \). Then, a basis for \( \ker(i_{*}) \) is given by:

\[
\{w_{k}^{+} - w_{k}^{-}, [\beta_{k} - \beta_{k-1}]\}_{i=1, \ldots, r; k=1, \ldots, p-1}.
\]

**Proof.** The statement follows from the last paragraph of the proof of Proposition 2.3. \( \square \)
**Remark 2.5.** The heavy notation we have had to resort to here deserves a comment. Since $h$ identifies $V_j^+$ with $V_j^-$, we have $\overline{\nu}_j^+ = \overline{\nu}_j^-$. Secondly, there are many choices of $\beta_k^j$ curves $\beta_k$ so that the classes $[\beta_k^j]$ are independent generators of $H_1(\partial R - V_0; \mathbb{Z})$. We note that it is possible to impose the extra condition that $\overline{\beta}_k - \overline{\beta}_{k-1}$, $k = 1, 2, \ldots, 2^{\frac{p-1}{2}}$ bounds a cylinder $\beta \times [-1, 1]$ in $R$. We do this by choosing for the $\beta_k^j$’s “consecutive” copies of $\beta$ as we move counter-clockwise in $\partial R$, starting, for instance, with the copy of $\beta_1$ lying in $V_0^+$. This observation will allow us to simplify the proof of Proposition 2.6.

We are now ready to give a formula for the signature of $W$.

**Proposition 2.6.** Let $\alpha$ be a knot which admits an irregular dihedral $p$-fold cover for an odd prime $p$. In addition, assume that this cover is $S^3$. Using the notation of Corollary 2.4, let $A$ be the matrix of linking numbers in $S^3$ of the following set of links:

$$(2.29) S := \{ [\nu_j^+ - \overline{\nu}_j^k], [\overline{\beta}_k^j - \overline{\beta}_{k-1}] \}_{i=1, \ldots, r; k=1, \ldots, \frac{p-1}{2}},$$

where the orientation of each curve is compatible with a chosen orientation on the corresponding curve, $w^i$ or $\beta^j$, in $V$. Then, $\sigma(W) = \sigma(A)$.

**Proof.** We wish to compute the intersection form on the image $i_*(H_2(W; \mathbb{Z}))$ in $H_2(W, S^3 \cup \Sigma; \mathbb{Z})$. Since $p$ is prime, $\Sigma$ is a rational homology sphere $S^3$. It follows that

$$H_2(W, M; \mathbb{Z}) \cong i_*(H_2(W; \mathbb{Z})) \subset H_2(W, S^3 \cup \Sigma; \mathbb{Z}).$$

By Proposition 2.3, we already know that

$$H_2(W, M; \mathbb{Z}) \cong \ker(i_* : H_1(\partial R - V_0; \mathbb{Z}) \to H_1(R; \mathbb{Z})) =: K.$$ 

Furthermore, by Corollary 2.4, the set of classes in $S \subset H_1(\partial R - V_0; \mathbb{Z})$ defined above forms a basis for $K$. We use the isomorphism $H_2(W, M; \mathbb{Z}) \cong K$ to obtain an explicit basis for $H_2(W, M; \mathbb{Z})$ consisting of surfaces with boundary which are properly embedded in $W$.

Recall that the isomorphism between $H_2(W, M; \mathbb{Z})$ and $K$ is given by Equation (2.21), together with the boundary map $\partial$ in the long exact sequence (2.22). By our choice of basis for $K$, for any element $u \in S$, $u$ is the boundary of a cylinder $S^1 \times I =: U$ properly embedded in $(R, \partial R - V_0) \subset (W, M)$. We use this cylinder to represent the class $[U] \in H_2(W, M; \mathbb{Z})$ corresponding to $u$ under the above isomorphism $H_2(W, M; \mathbb{Z}) \cong K$. Next, given $u_1, u_2 \in S$, we can write $u_1 = a_1 - a_2$ and $u_2 = b_1 - b_2$, where $a_i, b_i$ are oriented curves in the dihedral cover $M$. Denote by $U_i, i = 1, 2$, the cylinder $S^1 \times I \subset V_j \times I$ with $\partial U_i = u_i$. (Here $V_j$ denotes the lift of $V$ for which $a_1 - a_2$, respectively $b_1 - b_2$, lies on the boundary of $V_j \times I$.) Now, if $F_i$ is any Seifert surface for $u_i$ in $M \cong S^3$, we have $U_i \cup_{u_i} F_i \subset H_2(W, \mathbb{Z})$ and $i_* : H_2(W; \mathbb{Z}) \to H_2(W, S^3 \cup \Sigma; \mathbb{Z})$ carries $U_i \cup_{u_i} F_i$ to $U_i$. We fix two Seifert surfaces, $F_1$ and $F_2$, for $u_1$ and $u_2$, respectively, and use the classes $U_1 \cup_{u_1} F_1$ and $U_2 \cup_{u_2} F_2$ to compute the intersection number $U_1 \cdot U_2$.

By giving $W$ a little collar, $M \times [0, \epsilon]$, and “pushing in” $U_2 \cup_{u_2} F_2$ ever so slightly, we can assume that $F_2$ lies in $M \times \{\epsilon\}$, and $U_2 \cup_{u_2} F_2$ is disjoint from $M \times [0, \epsilon)$. Since $F_1 \subset M \times \{\epsilon\}$, in order to compute $U_1 \cdot U_2$, it now suffices to consider the intersection of $U_1$ with $U_2 \cup_{u_2} F_2$. We consider several cases. If the curves $a_1$ and $b_1$ are disjoint, then so are $U_1 = a_1 \times I$ and $U_2 = b_2 \times I$, regardless of whether the $a_i, b_i$ live in the same lift of $V_j$ or in different lifts. In this case, the intersection is simply $U_1 \cup F_2 = lk(a_1 - a_2, b_1 - b_2)$, since $F_2 \subset M \times \{\epsilon\}, U_1 \cap (M \times \{\epsilon\}) = a_1 - a_2$ and $F_2$ is a Seifert surface for $b_1 - b_2$, so that, putting everything together, we have $U_1 \cdot F_2 = (a_1 - a_2) \cap F_2 = lk(a_1 - a_2, b_1 - b_2)$ by definition. Secondly, $U_1$ and $U_2$ can be distinct but intersecting cylinders. This can only happen if both live in the same lift of $V \times I$, which we again denote $V_j \times I$. In this case, we use the normal to $V_j \times I$ in $W$ to push off $U_1$ away from $V_j \times I$ and thus from $U_2$. Again, we find that $U_1 \cup F_2 = U_1 \cdot F_2 = lk(a_1 - a_2, b_1 - b_2)$. Lastly, we consider the case where $U_1 = U_2$. For some choice of $j$ we have $U_1 \subset V_j \times I$ with $\partial U_1 = (a_1 - a_2) \subset V_j \times \{0, 1\}$. We can push $a_1$ off itself using its (positive, say) normal in $V_j \times \{0\}$. This push-off extends across $U_1 = (a_1 \times I) \subset (V_j \times I)$, so the cylinder can be made disjoint from itself. Again, we conclude that $U_1 \cup U_2 = U_1 \cdot F_2 = lk(a_1 - a_2, a_1 - a_2)$, where the self-linking number is computed using the normal to $a_1 - a_2$ in $V_j \times \{0, 1\}$. Therefore,
the matrix of linking numbers between elements of our basis for \( K \) is also the intersection matrix for \( W = (W(\alpha, \beta)) \). This completes the proof. \( \square \)

**Remark 2.7.** We note that the self-linking with respect to the normal to \( a_1 - a_2 \) in \( V_j \times \{0, 1\} \) is equal to the self-linking with respect to the restriction to \( a_1 - a_2 \) of the normal to \( V_j \times \{0, 1\} \) in \( M \cong S^3 \), since the two vectors are everywhere linearly independent. This is useful for computations, since the normal to \( V_j \) in the dihedral cover is just the lift of the normal to \( V \) in \( S_3 \).

The Proof of Proposition 2.3 also allows us to compute the fundamental group of the manifold \( W(\alpha, \beta) \) for knots \( \alpha \) which can arise as singularities of dihedral branched covers between four-manifolds.

**Corollary 2.8.** Let \( p \) be an odd prime and let \( \alpha \) be a knot which admits a \( p \)-fold irregular dihedral cover. Assume moreover that this cover homeomorphic to \( S^3 \). Let \( \beta \) be a characteristic knot for \( \alpha \) and let \( W(\alpha, \beta) \) be the cobordism between \( S^3 \) and the \( p \)-fold cyclic cover of \( \beta \) constructed in [8]. Then \( W(\alpha, \beta) \) is simply-connected.

*Proof.* We assume the notation of the proof of Proposition 2.3 (In this notation, the additional assumption of this Corollary is that \( M \cong S^3 \)). We have seen that \( W(\alpha, \beta) \) is homotopy equivalent to \( M \cup R \) and that \( M \cap R = \partial R - V_0 \). We also know that \( i_* : \pi_1(\partial R - V_0, a_0) \to \pi_1(R, a_0) \) is surjective. On the other hand, any loop in \( \pi_1(\partial R - V_0, a_0) = \pi_1(M \cap R, a_0) \) is contractible in \( M \) since \( \pi_1(M; a_0) = 0 \). Therefore, by van Kampen’s Theorem, \( \pi_1(M \cup R, a_0) = 0 = \pi_1(W(\alpha, \beta), a_0) \). \( \square \)

Finally, we show that the defect to the signature of a branched cover arising from the presence of a singularity \( \alpha \) is an invariant of the knot type \( \alpha \).

**Proposition 2.9.** Let \( p \) be an odd square-free integer, and let \( \alpha \subset S^3 \) be knot which arises as the singularity of an irregular dihedral \( p \)-fold cover between four-manifolds. Assume that \( p^2 \) does not divide \( \Delta(-1) \), where \( \Delta(t) \) is the Alexander polynomial of \( \alpha \). In the notation of Theorem 1.4, the integer \( \Xi_p(\alpha) \), defined as

\[
(2.30) \quad \Xi_p(\alpha) = \frac{p^2 - 1}{6p} L_V(\beta, \beta) + \sigma(W(\alpha, \beta)) + \sum_{i=1}^{p-1} \sigma_{\zeta_i}(\beta)
\]

is an invariant of the knot type \( \alpha \).

*Proof.* Since \( \alpha \) arises as a singularity of an irregular dihedral \( p \)-fold cover, by Theorem 1.4 \( \alpha \) itself admits an irregular dihedral \( p \)-fold cover. Since \( p^2 \) does not divide \( \Delta(-1) \), this cover is unique (see footnote on p. 166 of [8] or, for a more thorough discussion, [14]).

When both \( \alpha \) and \( \beta \) are fixed, it is clear that each of the terms \( \frac{p^2 - 1}{6p} L_V(\beta, \beta) \), \( \sigma(W(\alpha, \beta)) \) and \( \sum_{i=1}^{p-1} \sigma_{\zeta_i}(\beta) \) is well-defined. We will show that their sum is in fact independent of the choice of \( \beta \).

Let \( f : Y \to X \) be an irregular dihedral \( p \)-fold cover, branched over an oriented surface \( B \subset X \), embedded in \( X \) with a unique singularity of type \( \alpha \). Such a cover exists by assumption. Then

\[
\Xi_p(\alpha) = p \sigma(X) - \frac{p - 1}{2} e(B) - \sigma(Y),
\]

a formula independent of the choice of \( \beta \).

A priori, however, it might be possible for another branched cover \( f' : Y' \to X' \), whose branching set also has a singularity of type \( \alpha \), to produce a different value of \( \Xi_p \). This does not occur. By the proof of Theorem 1.4 any choice of characteristic knot \( \beta \) can be used to compute the defect \( \Xi_p(\alpha) \) to the signature of \( Y \). Using the same \( \beta \) and Equation (2.30) to compute this signature defect for two different covers, for instance \( Y \) and \( Y' \), shows that \( \Xi_p(\alpha) \) does not vary with the choice of branched cover and indeed depends only on \( \alpha \).  

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4One could allow \( p^2 \) to divide \( \Delta(-1) \). In this case, \( \Xi_p \) would not necessarily be an invariant of the knot type \( \alpha \) but, rather, of \( \alpha \) together with a specified representation of \( \pi_1(S^3 - \alpha, x_0) \to D_p \).
3. Constructing dihedral covers

In this section, we describe a method for constructing an irregular $p$-fold dihedral cover of a simply-connected four-manifold $X$. We use this construction to prove Theorem 1.6 which is a partial converse to Theorem 1.4. Precisely, Theorem 1.6 establishes that, when two-bridge slice singularities are considered, all pairs of integers $(\sigma, \chi)$ afforded by the necessary condition (Theorem 1.4) as the signature and Euler characteristic of a $p$-fold irregular dihedral cover of a given base manifold $X$ with specified branching set $B$ are indeed realized as the signature and Euler characteristic of a $p$-fold irregular dihedral cover over $X$.

A main ingredient of the proof is constructing an irregular dihedral cover of $S^4$ branched over a singular two-sphere with a given singularity (Proposition 3.4). By taking a connected sum with this singular two-sphere, we can introduce a singularity to a PL embedded surface $B \subset X$ without changing its homeomorphism type or that of the ambient manifold. The dihedral cover of $X$ is constructed from the irregular dihedral cover of $(S^4, S^2)$, together with several copies of the double branched cover of $X$ over the locally flat surface $B$.

We begin with a simple lemma.

Lemma 3.1. Let $X$ be four-manifold and let $B \subset X$ be an embedded connected surface such that $\pi_1(X - B, x_0) \cong \mathbb{Z}/2\mathbb{Z}$. The double branched cover of $(X, B)$ is simply-connected.

Proof. Since $\pi_1(X - B, x_0) \cong \mathbb{Z}/2\mathbb{Z}$, a double cover of $X$ branched along $B$ exists. We denote this cover by $\hat{X}$ and denote by $\hat{B}$ the (homeomorphic) pre-image of $B$ under the covering map. We apply van Kampen’s theorem to $\hat{X} = (\hat{X} - \hat{B}) \cup_{\partial N(\hat{B})} N(\hat{B})$, where $N(\hat{B})$ denotes a small tubular neighborhood of $\hat{B}$. Being the universal cover of $(X - B)$, $(\hat{X} - \hat{B})$ is simply connected, so $i_* : \pi_1(\partial N(\hat{B}), b_0) \to \pi_1(\hat{X} - \hat{B}, b_0)$ is the zero homomorphism. In addition, $i_* : \pi_1(\partial N(\hat{B}), b_0) \to \pi_1(N(\hat{B}), b_0)$ is surjective, since every element in $\pi_1(N(\hat{B}), b_0)$ can be represented by a loop which is disjoint from the 0-section and which is therefore homotopic to a loop in $\partial N(\hat{B})$. It follows from van Kampen’s Theorem that $\hat{X}$ is simply-connected.

Next, we prove a couple of lemmas concerning the singularities which we will be introduced to the branching sets in the construction of dihedral covers. In Lemma 3.2 we recall a well-known fact about the fundamental groups of complements of ribbon disks. Lemma 3.3 allows us to extend a dihedral cover of a two-bridge slice knot to a cover of a disk it bounds.

Lemma 3.2. Let $K \subset S^3 = \partial B^4$ be a ribbon knot and let $D' \subset S^3$ be a ribbon disk for $K$. Then, there exists $D \subset B^4$, a slice disk for $K$, such that the map $i_* : \pi_1(S^3 - K, x_0) \to \pi_1(B^4 - D, x_0)$ induced by inclusion is surjective.

Proof. Since $D'$ is ribbon, we can push the interior of $D'$ into the interior of $B^4$ to obtain a slice disk $D$ with the property that $g$, the radial function on $B^4$, is Morse when restricted to $D$ and has no local maxima on the interior of $D$. Computing the fundamental group of the complement of $D$ in $B^4$ by cross-sections as outlined in [13], we start with $\pi_1(\partial B^4 - \partial D, x_0) = \pi_1(S^3 - K, x_0)$ and proceed to introduce new generators or relations at each critical point of $g$. Since $g$ has no maxima, no new generators are introduced, implying that $i_* : \pi_1(S^3 - K, x_0) \to \pi_1(B^4 - D, x_0)$ is a surjection.

In the notation of the above lemma, a disk $D$ with the property that $i_* : \pi_1(S^3 - K, x_0) \to \pi_1(B^4 - D, x_0)$ is a surjection is called a homotopy ribbon disk. Thus, the lemma can be rephrased by saying ribbon knots admit homotopy ribbon disks.

Lemma 3.3. Let $K \subset S^3 \subset \partial B^4$ be a slice knot and let $D \subset B^4$ be a slice disk for $K$. Let $p > 1$ be an odd square-free integer. If the pair $(S^3, K)$ admits an irregular $p$-fold dihedral cover, then the pair $(B^4, D)$ admits one as well. Furthermore, if $K$ is a two-bridge knot, $D$ can be chosen PL and such that the irregular dihedral cover of $B^4$ branched along $D$ is simply-connected.
Proof. Let $\Delta_K(t)$ denote the Alexander polynomial of $K$ and $\Delta_D(t)$ that of $D$. Denote by $\hat{S}$ the double branched cover of the pair $(S^3, K)$ and by $\hat{K}$ the pre-image of $K$ under the covering map. It is well known that $|\Delta_K(-1)| = |(H_1(\hat{S}; \mathbb{Z})|$. Similarly, denote by $\hat{D}$ the double cover of $B^4$ branched along $D$ and by $\hat{D}$ the pre-image of $D$. As above, we have $|\Delta_D(-1)| = |(H_1(\hat{D}; \mathbb{Z})|$, since $\pm \Delta_D(-1)$ is the determinant of a presentation matrix for the first homology of the double branched cover of $D$. (Denote by $B_\infty$ the infinite cyclic cover of the disk complement $B^4 - D$. Regard $H_1(B_\infty; \mathbb{Z})$ as a $\mathbb{Z}[\tau, \tau^{-1}]$-module, where the action of $\tau$ is that induced by a generator of the group of covering translations. Then $\Delta_D(t)$ is the characteristic polynomial of this action and $H_1(B_\infty; \mathbb{Z}) \cong \text{Coker} \{1 - \tau^2 : H_1(B_\infty; \mathbb{Z}) \to H_1(B_\infty; \mathbb{Z})\}$. For a thorough exposition on the homology of cyclic covers of a homology $S^1$, see [33].)

Since $K$ admits a dihedral cover, $H_1(\hat{S}; \mathbb{Z})$ surjects onto $\mathbb{Z}/p\mathbb{Z}$ [8]. It follows that $|\Delta_K(-1)| \equiv 0 \text{ mod } p$. Since $D$ is a slice disk for $K$, by results of Fox and Milnor [15] we have $|\Delta_K(-1)| = \pm (\Delta_D(-1))^2$, so $(\Delta_D(-1))^2 \equiv 0 \text{ mod } p$. Since $p$ is square-free by assumption, we conclude that $\Delta_D(-1) \equiv 0 \text{ mod } p$ as well. Then $H_1(\hat{D}; \mathbb{Z})$ surjects onto $\mathbb{Z}/p\mathbb{Z}$ and thus $\hat{D}$ admits a $p$-fold cyclic cover $T$ with $\partial T =: N$. This cover $T$ is the regular dihedral $2p$-fold branched cover of $(B^4, D)$. Let $Z$ be the quotient of $T$ by the action of any $\mathbb{Z}/2\mathbb{Z}$ subgroup of $D_p$. Then $Z$ is the desired irregular dihedral $p$-fold cover of $(B^4, D)$. Its boundary, which we denote by $U$, is the irregular dihedral $p$-fold cover of $K$.

Now assume in addition that $K$ is a two-bridge knot. In this case it is well-known that $U$ is in fact $S^3$. Indeed, the pre-image $S^*$ of a bridge sphere for $K$ is a dihedral cover of $S^2$ branched over four points, so $S^*$ has Euler characteristic

$$\chi(S^*) = p(\chi(S^2) - 4) + 4\frac{p + 1}{2} = 2.$$ 

A bridge sphere bounds a trivial tangle to either side, and the cover of a trivial tangle is a handlebody. Therefore, $S^*$ is a Heegaard surface for $U$, and, since the genus of $S^*$ is zero, $U \cong S^3$.

Since $K$ is two-bridge slice, it is ribbon. Hence, by Lemma 3.2 the slice disk $D$ for $K$ can be chosen to be PL and homotopy ribbon, i.e. such that $\pi_1(S^3 - K, x_0) \xrightarrow{i_*} \pi_1(B^4 - D, x_0)$ is a surjection. Therefore, given a homomorphism $\psi : \pi_1(B^4 - D, x_0) \to D_{2p}$, the pre-image $(\psi \circ i_*)^{-1}((\mathbb{Z}/2\mathbb{Z})$ surjects onto $\psi^{-1}(\mathbb{Z}/2\mathbb{Z})$ by $i_*$. This implies that the inclusion of the unbranched cover associated to $U$ into the unbranched cover associated to $Z$ induces a surjection on fundamental groups. Since the branching set of $U$ is a subset of the branching set of $Z$, it follows that $\pi_1(U, x_0) \xrightarrow{i_*} \pi_1(Z, x_0)$ is also a surjection. But $\pi_1(U, x_0) = 0$, and we conclude that the irregular dihedral cover of the pair $(B^4, D)$ is simply-connected. \qed

Proposition 3.4. Let $p > 1$ be an odd square-free integer and let $K \subset S^3$ be a slice knot such that the pair $(S^3, K)$ admits an irregular dihedral $p$-fold cover. Then there exists an embedded two-sphere $S^2 \subset S^4$ such that the pair $(S^4, S^2)$ admits an irregular dihedral $p$-fold cover $W$ and $S^2 \subset S^4$ is locally flat except at one point where it has a singularity of type $K$. Moreover, if $K$ a two-bridge knot, $W$ is a simply-connected manifold.

Proof. Let $D^2_1 \subset B^4_1$ be a PL slice disk for $K$. Denote the cone on the pair $(S^3, K)$ by $(B^4_1, D^2_2)$. The disk $D^2_2$ is a PL submanifold of $B^4_1$ except at the cone point $x$, where by construction $D^2_2$ has a singularity of type $K$. Identifying the two pairs $(B^4_1, D^2_1)$ and $(B^4_2, D^2_2)$ via the identity map along the two copies of $(S^3, K)$ lying on their boundaries, we obtain an embedding of a two-sphere $S := D^2_2 \cup_K D^2_2$ in $S^4 = B^4_1 \cup_{\partial B^4_1} B^4_2$ such that $S$ has a unique singularity of type $K$ at $x$.

By Lemma 3.3 the pair $(B^4_1, D^2_1)$ admits an irregular dihedral $p$-fold cover $W$, and its boundary $M$ is the irregular dihedral $p$-fold cover of the pair $(S^3, K)$. Since $(B^4_2, D^2_2)$ is a cone, its irregular dihedral $p$-fold cover is simply the cone on $M$. Thus, the pair $(S^4, S)$ admits a cover $Z := W \bigcup_{\partial W \sim M \times \{0\}} (M \times [0, 1]/M \times \{1\})$.
as claimed. If, in addition, \( K \) is a two-bridge knot, by Lemma 3.3 we know that \( M \) is the three-sphere and that we can pick the disk \( D^2_1 \) in the above construction to be homotopy ribbon so that \( W \) is simply-connected. Thus, for \( K \) two-bridge, \( Z \) is a simply-connected manifold.

**Proof of Theorem 1.10.** The proof is as follows: first, we modify the branching set \( B \) by introducing a singularity of type \( \alpha \) to the embedding of \( B \) in \( X \); next, we construct the desired covering space \( Y \) by gluing together several manifolds by homeomorphisms on their boundaries; we check that \( Y \) is indeed a \( p \)-fold irregular dihedral cover of \( X \) with the specified branching set; finally, we verify that \( Y \) is a simply-connected manifold.

We begin by modifying the branching set as outlined above. Let \( S^2 \subset S^4 \) be an embedded two-sphere with a unique singularity of type \( \alpha \), constructed as in the proof of Proposition 3.4. Let \( y \in S^2 \subset S^4 \) be any locally flat point with \( N(y) \) a neighborhood of \( y \) not containing the singular point \( x \). We use \( N(y) \) to form the connected sum of pairs \((X,B)\#(S^4,S^2)\) := \((X,B_1)\). By construction, \( B_1 \) is homeomorphic to \( B \), embedded in \( X \) with a unique singularity of type \( \alpha \) and satisfies \( e(B_1) = e(B) \). Furthermore, we see from the natural Mayer–Vietoris sequence that \( H_1(X - B; \mathbb{Z}) \cong H_1(X - B_1; \mathbb{Z}) \) and the latter group is \( \mathbb{Z}/2\mathbb{Z} \) by assumption. Hence, \( X \) admits a double cover \( f : \hat{X} \to X \) branched along \( B_1 \).

A useful way to visualize this cover is the following. Since \( y \in S^4 \) is a locally flat point, \( B \cap \partial N(y) \) is the unknot. Now viewing \( \partial N(y) \) as embedded in \( B_1 \), we note that the restriction of \( f \) to \( f^{-1}(\partial N(y)) \) is a double branched cover of the trivial knot, whose total space is again \( S^3 \). Furthermore, the pre-images under \( f \) of the connected summands \((X,B) - B^4 \) and \((S^4,S^2) - N(y) \) are the double branched covers of those summands. We can thus think of the double branched cover \( \hat{X} \) of the pair \((X,B_1)\) as the union (along \( S^3 \) viewed as a double cover of the unknot) of the double branched covers of a punctured \((X,B)\) and \((S^4,S^2) - N(y) \). For future use, we denote the restriction of \( f \) to the pre-image \( \hat{X}_0 \) of \( X - N(x) \) by \( f_0 \).

Next, consider the irregular dihedral \( p \)-fold cover \( g : Z \to S^4 \) of \((S^4,S^2) \) constructed as in Proposition 3.3. For \( y \) as above, the restriction of \( g \) to \( g^{-1}(\partial N(y)) \) is the irregular dihedral \( p \)-fold cover of the unknot, which consists of the disjoint union of \( \frac{p+1}{2} \) copies of \( S^3 \), \( \frac{p-1}{2} \) of which are double covers and one a single cover. Furthermore, \( g^{-1}(S^4 - N(y)) \) is the irregular dihedral \( p \)-fold cover of the pair \((B^4,D^2)\), where the two-disk is singular. The boundary of this dihedral cover consists of \( \frac{p+1}{2} \) copies of \( S^3 \). Of those, \( \frac{p-1}{2} \) double-cover the complement of the unknot and one is mapped homeomorphically by \( g \).

We now describe the manifold \( Y \) which we will show is homeomorphic to a dihedral cover of \( X \) along \( B_1 \). We attach to \( g^{-1}(S^4 - N(y)) \) a copy of \( \hat{X}_0 \) along each boundary component \( S^3 \) which double-covers the complement of the unknot. Naturally, the attachment identifies the boundary components by a homeomorphism of pairs \((S^3,S^3)\), where the second component is the (unknotted) branching set. In the same manner, we also attach a punctured copy of \( X \) along the boundary component \( S^3 \) which is a cover of index 1. The map

\[
h := g \cup f_0 \cup 1_{X - N(x)} : Y \to X
\]

is a branched cover of \((X,B_1)\). By construction, \( h \) satisfies the property that for all points \( z \in B - x \), if \( N(z) \) is a small neighborhood of \( z \) in \( X \) not containing \( x \), then \( h^{-1}(N(z)) \) has \( \frac{p-1}{2} \) components of index 2 and one component of index 1. So \( Y \) is the desired dihedral cover. By Theorem 1.14, the Euler characteristic and signature of \( Y \) are those determined by the prescribed triple \( X,B,\alpha \). (Here, we use the fact that the above construction does not change the homeomorphism type or self-intersection number of \( B \).)

Finally, we observe that \( Y \) consists of simply-connected manifolds joined together via homeomorphisms on their boundaries. Indeed, \( X \) is simply-connected by assumption, and \( \hat{X} \) is simply-connected by Proposition 3.1. The irregular dihedral cover \( Z \) of \( S^4 \) is simply-connected by Proposition 3.4 and,
therefore, so is \( g^{-1}(S^4 - N(y)) \). We conclude that \( Y \) is simply-connected, which completes the proof.  

**Remark 3.5.** One can obtain analogous results by varying the hypotheses on the branching set \( B \). For instance, if we do not require that our construction result in a simply-connected cover, we can relax the condition that \( \pi_1(X - B, x_0) \cong \mathbb{Z}/2\mathbb{Z} \) and use for our branching set any surface \( B \) which represents an even class in \( H_2(X; \mathbb{Z}) \). This allows us to produce, by introducing any two-bridge slice knot as the singularity and by varying the genus of \( B \) (see Lemma 3.6), infinitely many dihedral branched covers of \( S^4 \), which are easily distinguished by their Euler characteristic. Furthermore, if \( B \) is the boundary union of the cone on a two-bridge knot \( \alpha \) and a homotopy ribbon surface for \( \alpha \), one can construct simply-connected covers of \( S^4 \) by this method, as done in [5].

We can also use the techniques of Theorem 1.6 to construct over a given four-manifold \( X \) an infinite family of dihedral covers with the same singularity type on the branching set (Theorem 1.8). The first step is to establish the following Lemma.

**Lemma 3.6.** Let \( B \subset X^4 \) be an oriented surface of genus \( g \), PL embedded in \( X \) and such that \( \pi_1(X - B, x_0) \cong \mathbb{Z}/2\mathbb{Z} \). Then, there exists a PL embedded oriented surface \( C \) of genus \( g + 1 \) in \( X \) such that \( \pi_1(X - C, x_0) \cong \mathbb{Z}/2\mathbb{Z} \), and such that \( e(B) = e(C) \).

**Proof.** Let \( T \subset S^4 \) be the standard embedding of the two-torus in the four-sphere. We have \( \pi_1(S^4 - T, x_0) \cong \mathbb{Z} \), generated by any meridian of \( T \) in \( S^4 \).

Now consider the connected sum of pairs \((X, B)\#(S^4, T)\) and let \( C = B\#T \subset X\#S^4 \cong X \). Note that the genus of \( C \) is one higher than that of \( B \). Since a meridian \( m_1 \) of \( T \) in \( S^4 \) becomes identified under the connected sum with a meridian \( m_2 \) of \( B \) in \( X \), it follows that the fundamental group of \((X - C)\) is isomorphic to \( \langle m_1, m_2 | m_1 = m_2, m_2^2 = 0 \rangle \cong \mathbb{Z}/2\mathbb{Z} \).

Finally, under the isomorphism of pairs \((X, B)\#(S^4, T) \cong (X, C)\), the class \([C] \in H_2(X; \mathbb{Z})\) corresponds to the class \([B\#T] \in H_2(X\#S^4; \mathbb{Z})\). Since \([T] = 0 \in H_2(X\#S^4; \mathbb{Z})\), indeed \( e(B) = e(C) \). \( \square \)

We have now done most of the work needed to obtain an infinite family of covers over a given base.

**Proof of Theorem 1.7.** The first step of our construction is to find a closed surface \( B \subset X \), PL embedded in \( X \) and such that \( \pi_1(X - B, x_0) \cong \mathbb{Z}/2\mathbb{Z} \). Since \( X \) is simply-connected and its second Betti number is positive, such a surface exists, as we now show. Let \( F \) be a closed oriented surface, smoothly embedded in \( X \) and such that the maximum divisibility of \([F]\) in \( H_2(X; \mathbb{Z}) \) is 2. Then \( H_1(X - F; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \). By classical techniques, \( F \) can be modified to produce a new surface \( F' \), carrying the same homology class as \( F \), the fundamental group of whose complement is abelian, as follows. Since \( X \) is simply-connected, \( \pi_1(X - F) \) is normally generated by a meridian \( \mu \) of \( F \). For any \( g \in \pi_1(X - B, x_0) \), the commutator \([\mu, g\mu g^{-1}]\) can be killed by performing a finger move on the surface \( F \), as shown in Lemma 1 of [9]. After iterating this move finitely many times, the result is a self-transverse immersed surface \( F' \), the fundamental group of whose complement is generated by \( \mu \). Finally, self-intersections of \( F' \) can be removed by replacing, in a small neighborhood of any double point, the cone on the Hopf link by an annulus. This operation has no effect on the fundamental group of the complement and produces the desired surface \( B \).

The next step is to further modify the surface to introduce a singularity of the desired type. Following the procedure in the proof of Theorem 1.6, we use a two-sphere \( S \subset S^4 \), PL embedded in \( S^4 \) except for one singularity of type \( \alpha \); next, we construct a \( p \)-fold irregular dihedral cover of the pair \((X, B)\#(S^4, S) \cong (X, B)\), as in the proof of Theorem 1.6.

Fixing a knot \( \alpha \) as the singularity type, by Lemma 3.6, we can increase the genus of the branching set \( B \) to obtain an infinite family of such covers. These covers are pairwise non-homeomorphic and can be distinguished by their Euler characteristics. Using knots for which the values of \( \sum_p \) differ, it is possible to obtain covers distinguished by their signatures as well. \( \square \)

As an immediate consequence of our construction, we have the following.
Corollary 3.7. Let \((\sigma, \chi)\) be a pair of integers which satisfy Equations (1.1) and (1.2) for some given \(X, B, \alpha\) and \(p\), where \(p\) is an odd prime and \(\alpha\) a two-bridge slice knot. Then, if \(\chi' = \chi + (p-1)k\) for a natural number \(k\), there exists a manifold \(Y'\) which is homeomorphic to a \(p\)-fold irregular dihedral cover of \(X\) and satisfies \(\sigma(Y') = \sigma, \chi(Y') = \chi'\). Moreover, if \(\pi_1(X - B, x_0) \cong \mathbb{Z}/2\mathbb{Z}\), \(Y\) is simply-connected.

We conclude by proving Theorem 1.8.

Proof of Theorem 1.8

\(\Rightarrow\) If \(Y\) is homeomorphic to a dihedral \(p\)-fold cover of \(X\) with the specified branching data, by Theorem 1.4 the Euler characteristic and signature of \(Y\) satisfy Equations (1.1) and (1.2) with respect to \(B_1\) and thus, by assumption, with respect to \(B\).

\(\Leftarrow\) Assume the Euler characteristic and signature of \(Y\) satisfy Equations (1.1) and (1.2). We will construct a branched cover of \(X\) whose branching set has the specified properties, and we will prove that this cover is homeomorphic to \(Y\).

We follow the steps used in the proof of Theorem 1.6 to construct a \(p\)-fold irregular dihedral cover of \(X\) branched over a surface \(B_1 \cong B\) which is embedded in \(X\) with a singularity of type \(\alpha\) and so that \(e(B_1) = e(B)\). Call this cover \(Z\). Since \(\alpha\) is a two-bridge slice knot, by Theorem 1.6 \(Z\) is a simply-connected manifold. We will prove that the intersection form of \(Z\) is equivalent to that of \(Y\).

Being a dihedral cover of \(X\), \(Z\) satisfies the equations set forth in Theorem 1.4 where, again, \(B\) and \(B_1\) can be used interchangeably. By assumption, \(Y\) also satisfies these equations, so \(\sigma(Y) = \sigma(Z)\) and \(\chi(Y) = \chi(Z)\). Since \(Y\) is a simply-connected four-manifold, the rank of \(H_2(Y; \mathbb{Z})\) is \(\chi(Y) - 2\), and the analogous statement holds for \(Z\). In other words, the intersection forms of \(Y\) and \(Z\) have the same signature and rank. The intersection form of \(Y\) is odd by assumption. The intersection form of \(Z\) is also odd because by construction \(Z\) has a copy of \(X\) as a connected summand and \(X\) itself is odd. Therefore, the intersection forms of \(Y\) and \(Z\) have the same signature, rank and parity. In particular, both are definite or both are indefinite. If both forms are definite, since they arise as intersection forms of smooth four-manifolds, by Donaldson’s result [12], each diagonalizes to \(\pm n\), where \(n = \chi(Y) - 2 = \chi(Z) - 2\) and the sign determined by \(\sigma(Y) = \sigma(Z)\). If both are indefinite, we again conclude that they are isomorphic, this time using Serre’s classification [27] of indefinite unimodular integral bilinear forms. By Freedman’s classification of simply-connected four-manifolds [10], it follows that \(Y\) and \(Z\) are homeomorphic.

\(\square\)

Appendix A. Characteristic knots

Our construction of an infinite family of irregular dihedral \(p\)-fold covers of over a given four-manifold (Theorem 1.7) hinges on being able to find two-bridge slice knots which admit dihedral \(p\)-fold covers themselves. In this section we prove that, for any odd prime \(p\), infinitely many such knots exist. In particular, we exhibit for every \(p\) an infinite class of knots for which the necessary condition (Theorem 1.4) for the existence of a dihedral \(p\)-fold cover over a given base is sharp. As a biproduct, we also illustrate how to find characteristic knots in the two-bridge case.

Recall that Lisca [24] proved that, for two-bridge knots, being slice is equivalent to being ribbon. Previously, Casson and Gordon [10] gave a necessary condition for a two-bridge knot to be ribbon, and Lamme [21, 22] listed all knots satisfying this condition. He found that for all \(a \neq 0, b \neq 0\) the knots \(K_1(a, b) = C(2a, 2b, -2a, 2b)\) and \(K_2(a, b) = C(2a, 2b, 2a, 2b)\) are ribbon. Fig. 1 recalls the notation \(C(e_1, ..., e_6)\). In Fig. 2 we give a genus 3 Seifert surface \(V\) for the knot \(\alpha = C(e_1, e_2, e_3, e_4, e_5, e_6)\). We use the surface \(V\) for all subsequent computations.

Since two-bridge slice knots play a key role in our construction of dihedral covers of four-manifolds, we determine the values of the parameters \(a\) and \(b\) for which the knots \(K_1(a, b)\) admit three-fold dihedral covers.

Proposition 3.8. A knot of the type \(K_1(a, b)\) admits an irregular three-fold dihedral cover if and only if

\((1)\) \(a \equiv 0 \mod 3, b \equiv 2 \mod 3\) or
\((2)\) \(a \equiv 1 \mod 3, b \equiv 1 \mod 3\).
A knot of the type $K_2(a, b)$ admits an irregular 3-fold dihedral cover if and only if

1. $a \equiv 0 \mod 3, b \equiv 1 \mod 3$ or
2. $a \equiv 1 \mod 3, b \equiv 0 \mod 3$.

In these cases, a curve representing the class $\beta \in H_1(V; \mathbb{Z})$ is a mod 3 characteristic knot for the corresponding $K_1(a, b)$ if and only if there is a choice of orientation on $\beta$ such that, with respect to the basis \{0, $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6$\}, we have, respectively,

1. $[\beta] \equiv (1, 0, 1, 1, -1, 1) \mod 3,$
2. $[\beta] \equiv (-1, 1, 1, 0, 1, 1) \mod 3,$
3. $[\beta] \equiv (0, 1, -1, 1, 1) \mod 3,$
4. $[\beta] \equiv (-1, -1, 1, 0, 1) \mod 3.$

*Proof.* Let $V$ denote the Seifert surface for $C(e_1, e_2, e_3, e_4, e_5, e_6)$ depicted in Fig. 2. We think of the $e_i$ as being chosen so that the knot $C(e_1, e_2, e_3, e_4, e_5, e_6)$ is of type $K_1(a, b)$ or $K_2(a, b)$. Let $L$ denote the matrix of the linking form on $V$ with respect to the basis \{0, $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6$\}. The symmetrized linking form for $V$ in this basis is $L_V = L + L^T$. It equals:

$$
\begin{pmatrix}
-e_1 & 1 & 0 & 0 & 0 & 0 \\
1 & e_2 & -1 & 0 & 0 & 0 \\
0 & -1 & -e_3 & 1 & 0 & 0 \\
0 & 0 & 1 & e_4 & -1 & 0 \\
0 & 0 & 0 & -1 & -e_5 & 1 \\
0 & 0 & 0 & 0 & 1 & e_6 \\
\end{pmatrix}
$$

It is sufficient to check that $\det(L + L^T) \equiv 0 \mod 3$ precisely in situations (1), ..., (4). For instance, in the case $C(e_1, e_2, e_3, e_4, e_5, e_6) = K_1(a, b)$, we obtain $\det(L + L^T) = -(8ab + 2b - 1)^2$. So we need to solve the equation

$$8ab + 2b - 1 \equiv 0 \mod 3.$$

If $a \equiv 0 \mod 3$, the equation reduces to $2b - 1 \equiv 0 \mod 3$, so $b \equiv 2$. If $a \equiv 1 \mod 3$, then $b \equiv 1 \mod 3$. If $a \equiv 2 \mod 3$, there is no solution. The computations for $K_1(a, b)$ are equally trivial, so they are omitted.
To verify that the classes $[\beta] \in H_1(V; \mathbb{Z})$ listed represent all characteristic knots, it suffices to check that, for $a$, $b$ and $\beta$ as specified, we have $(L + L^T)\beta \equiv 0 \mod 3$ and moreover that the classes $\beta$ are the unique solutions mod 3 for each pair $(a, b)$. The arithmetic involved has been left out. 

More generally, we have the following:

**Proposition 3.9.** Let $p > 1$ be an odd prime. There exists an infinite family of integer pairs $(a, b)$ such that the two-bridge slice knot $K_1(a, b) \subset S^3$ admits an irregular dihedral $p$-fold cover, and similarly for $K_2(a, b)$.

**Proof.** The case $p = 3$ was treated in Proposition 3.8 so assume $p > 3$. The determinant $D_1(a, b)$ of the Seifert matrix of the knot $K_1(a, b)$ is equal to $-(8ab + 2b - 1)^2$. Setting $a \equiv 0 \mod p$, we find that $D_1(a, b) \equiv 0 \mod p$ if and only if $2b \equiv 1 \mod p$. Since $p$ is odd, a solution exists. Another pair of solutions is $a \equiv 8^{-1} \mod p$ and $b \equiv 3^{-1} \mod p$.

Similarly, we find that the determinant $D_2(a, b)$ of the Seifert matrix of the knot $K_2(a, b)$ is $(8ab + 2a + 2b + 1)^2$. Setting $b \equiv -1 \mod p$, we find that $a(6) \equiv 1 \mod p$. For $p > 3$, this gives a solution. 

For any given $p$ and any family of two-bridge slice knots $K_1(a, b)$ with $a$ and $b$ chosen so that $\det(L + L^T) \equiv 0 \mod p$, the classes in $H_1(V; \mathbb{Z})$ represented by characteristic knots are easily computed as in Proposition 3.8 by solving a system of equations mod $p$. One can see by direct examination that if $p = 3$ each of these homology classes can be realized by an unknot embedded in the interior of $V$. The same methods can be used to find knot types of characteristic knots for all $p$.

**Appendix B. Computing linking numbers in branched covers**

Let $\alpha \subset S^3$ be a knot, and let $f : M \to S^3$ be a cover branched along $\alpha$, arising from a presentation $\psi : \pi_1(S^3 - \alpha, x_0) \to S_n$. The linking numbers (when defined) between the various components of $f^{-1}(\alpha)$ constitute a subtle knot invariant studied extensively by Hartley and Murasugi [17], Bankwitz and Schumann [2], Laufer [29] and Perko [31], among others. Further applications of linking numbers in dihedral covers of knots were found by Cappell and Shaneson [7] and Litherland [25].

In his undergraduate thesis [30], Perko detailed a procedure, going back to Reidemeister [34], for computing linking numbers between branch curves. His method is, to this day, the most efficient and general algorithm known for computing these numbers. We give a very short summary of this classical method for computing linking numbers in a branched cover. We intend to provide just enough detail to be able to describe a generalization of these ideas which will allow us to calculate the linking numbers of other curves, as needed for evaluating the component of $\Xi_p(\alpha)$ which is expressed in terms of linking. Readers interested in the specifics needed to carry out the procedure can find them in [30] or [4].

**Perko’s procedure for computing linking numbers between branch curves in a branched cover $f : M \to S^3$ with branching set $\alpha$:**

1. Use a diagram for $\alpha$ to endow $S^3$ with a cell structure. The two-skeleton is the cone on $\alpha$, and there is a single three-cell.
2. Endow the cover $M$ with a cell structure as follows. The cells are the pre-images $f^{-1}(e_k^i)$ of the various cells in $S^3$. The attaching maps are determined by the action of the meridians of $\alpha$ on the interiors of the cells.
3. Compute the boundaries of all two-cells of $M$. This step is non-trivial for two-cells whose boundary contains one-cells corresponding to over-arcs in the knot diagram. Such two-cells will accrue additional boundary components determined by the action of meridians of $\alpha$ on the three-cells.
4. Solve a system of linear equations to determine, for each component $\alpha_i$ of $f^{-1}(\alpha)$, a two-chain with boundary $\alpha_i$, if such a two-chain exists.
(5) For each pair $(\alpha_i, \alpha_j)$, examine the signed intersection numbers of $\alpha_j$ with a two-chain, found in (4), whose boundary is $\alpha_i$. This gives $lk(\alpha_i, \alpha_j)$. We remark that, in practice, the intersection number of any one-cell with any two-cell is trivial to read off from the data examined in order to complete (3), so this final step of the computation poses no difficulty.

In order to compute the linking numbers of other curves in $M$, we introduce an appropriate subdivision of the cell structure described above. Consider a curve $\gamma \subset (S^3 - \alpha)$ whose lifts to $M$ are of interest. We use the cone on $\alpha \cup \gamma$ to form the two-skeleton of $S^3$. In order to lift this new cell structure to a cell structure on $M$, we treat $\gamma$ as a “pseudo-branch curve” of the map $f$. That is, we think of the homomorphism $\pi_1(S^3 - (\alpha \cup \gamma)) \to S_n$ as a homomorphism $\pi_1(S^3 - (\alpha \cup \gamma)) \to S_n$ in which meridians of $\gamma$ map to the trivial permutation. Naturally, this can be done for multiple curves $\gamma_i$ simultaneously. In this set up, linking numbers can be computed by following steps (3), (4) and (5) above. The above procedure is carried out in [4], and a computer algorithm for performing linking number calculations is provided.

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