Linear Langevin Equation of Critical Fluctuation in Chiral Phase Transition

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We derive the linear Langevin equation that describes the behavior of critical fluctuation above the critical temperature of the chiral phase transition in the Nambu-Jona-Lasinio model. The Langevin equation was attempted by Mori\textsuperscript{6}. Recently, the universality of phase transition is described by\textsuperscript{7,8} the Heisenberg equation of motion for the order parameter, that is, the chiral condensate $\langle \bar{q}q \rangle = (-250 \text{ MeV})^3$: $g = 5.01 \text{ GeV}^{-2}$ and $\Lambda = 650 \text{ MeV}$. The critical temperature of the chiral phase transition using this parameter set is $T_c = 185 \text{ MeV}$ in the mean field approximation.

In principle, the dynamics of the chiral phase transition is described by solving the Heisenberg equation of motion for the order parameter, that is, the chiral condensate $\langle \sigma \rangle = \langle \bar{q}q \rangle$. However, the Heisenberg equation incorporates motion on microscopic and macroscopic scales on an equal footing. To carry out coarse-graining of microscopic variables systematically, we introduce projection operators $P$ and $Q = 1 - P$. This technique was originally introduced by Nakajima\textsuperscript{8} to derive a master equation. The application to a Langevin equation was attempted by Morit\textsuperscript{8}. Recently, the unification and generalization of both treatments has been realized\textsuperscript{9}. It should be noted that the elimination of microscopic variables does not mean neglecting their effect. They give rise to dissipation terms and noise terms in the coarse-grained macroscopic equations.

Then, the time evolution of the chiral phase transition is described by\textsuperscript{8}

$$\frac{d}{dt} \delta \sigma(x, t) = e^{iLt} P_{1L} \delta \sigma(x, 0)$$

$$+ \int_0^t ds e^{iL(t-s)} P_{1L} e^{iQ \delta \sigma(x, 0) + iL_0 \delta \sigma(x, 0)} + \xi(x, t)$$

where $L$ and $L_0$ are Liouville operators of the Hamiltonian $H$ and $H_0$, respectively. Here, we calculate the fluctuation of the order parameter from its equilibrium...
value: $\delta \sigma(\mathbf{x}, t) \equiv \sigma(\mathbf{x}, t) - <\sigma(\mathbf{x})>_{eq}$. This equation is called the time-convolution (TC) equation.\[2, 3\] The first term on the r.h.s. of the equation expresses the term corresponding to a collective oscillation such as plasma wave, spin wave, etc. The second term is the memory term that causes dissipation. Here, we have already expanded the memory term up to lowest order in $H_1$, following Ref. \[3\]. The third term is the noise term and we do not give its concrete form here. This is defined through the fluctuation-dissipation theorem of second kind (2nd F-D theorem) as we will see later.

There are several possible projection operators that extract slowly varying parts from an operator $\hat{O} \equiv \hat{O}_1(\mathbf{x}, t)$, $\hat{O}_0$, $\hat{O}_0(\mathbf{x}, t)$, and $\hat{O}_0(\mathbf{x}, t)$. In this paper, we adopt the Mori projection operator (MPO) that projects any operators onto the space spanned by gross variables $\hat{O}_0$. To define it, we must realize the time scale of gross variables are extremely long compared to that of other microscopic variables. There are three candidates for gross variables $\hat{O}_0$: (i) order parameters, (ii) density variables associated with conserved quantities and (iii) their products. However, near critical points of the second order phase transition, the order parameters are probably much slower than the density variables because of critical slowing down (CSD). In this sense, we can ignore the time evolution of density variables and exclude (ii) from our gross variables. Furthermore, we ignore (iii) for simplicity. We shall discuss the importance of (iii) at the end of this paper. After all, the gross variable relevant in our calculation is the order parameter of the chiral phase transition $\sigma(\mathbf{x}, t)$. Then, the MPO in our calculation is defined by

$$PO = \int d^3x d^3x' \langle O, \delta \sigma(\mathbf{x}) \rangle \cdot (\delta \sigma(\mathbf{x}), \delta \sigma(\mathbf{x}'))^{-1} \cdot \delta \sigma(\mathbf{x}'),$$

where $O$ is an arbitrary operator. The inner product means the canonical correlation,

$$\langle X, Y \rangle = \int_0^\beta d\lambda \beta Tr[\rho e^{\lambda H_0} X e^{-\lambda H_0} Y], \tag{4}$$

where $\rho = \exp(-\beta H_0)/Tr[\exp(-\beta H_0)]$ and $\beta = 1/T$. The system attains the thermal equilibrium state with this temperature $T$. In the classical limit, the canonical correlation coincides with the classical correlation function $Tr[\rho XX^\dagger]$.

Substituting the MPO into the TC equation \[2\], we have

$$\frac{d}{dt} \delta \sigma(\mathbf{k}, t) = -\int_0^t d\tau \Gamma(\mathbf{k}, \tau) \delta \sigma(\mathbf{k}, t - \tau) + \xi(\mathbf{k}, t). \tag{5}$$

Here, the memory function $\Gamma(\mathbf{k}, \tau)$ is given by the inverse Laplace transformation of $\Gamma(L, s)$, defined by

$$\Gamma(L, s) = -s \chi_s^L(\mathbf{k})/(\chi_0(\mathbf{k}) + \chi_s^L(\mathbf{k})) \cdot (1 - 2\beta \chi_0(\mathbf{k})). \tag{6}$$

where the index $L$ means the Laplace transformation, $\chi_s^L(\mathbf{k}) = \int_0^\infty dt e^{-st} d\chi_t(\mathbf{k})/dt$ with

$$\chi_t(\mathbf{k}) = \frac{N_c N_f}{\beta V} \sum_p \{(1 - 2n(E_p)) E_p E_p + p(p + k)\}
\times \{e^{(E_p + p + k)t} + e^{-i(E_p + p + k)t}\}
\times \{e^{(E_p - p - k)t} + e^{-i(E_p - p - k)t}\}, \tag{7}$$

Here, $n(E) = [\exp(\beta E) + 1]^{-1}$, $E_k = |\mathbf{k}|$, $N_c = 3$ and $N_f = 2$. It should be noted that there is no chiral condensate above $T_c$. In the calculation of the memory term, we calculated only the term corresponding to a ring diagram in the scalar channel and employed the technique proposed in Ref. \[12\].

The memory function $\Gamma^L(0, s)$ vanishes at $T_c$. This is shown by using the self-consistency condition of the chiral condensate calculated in the mean field approximation,

$$1 - 2g\beta \chi_0(0)|_{T=T_c} = 0, \tag{8}$$

where $T_c = 185$ MeV. From Eqs. \[10\] and \[12\], one can see that $\Gamma^L(0, s)$ vanishes at $T_c$. The small $\Gamma^L(0, s)$ near $T_c$ means the slowing down of the relaxation, that is nothing but CSD. If we take account of the terms beyond the ring diagram approximation in the calculation of the memory function, the critical temperature deviates from that in the mean field approximation.

The TC equation \[5\] is still inadequate to be interpreted as a Langevin equation because the memory function $\Gamma(\mathbf{k}, t)$ converges to a finite value at $t \rightarrow \infty$, as is shown in Fig. 1. As a matter of fact, it is usually expected that the memory function tends to zero with the shorter time scale than the macroscopic one, because we can prove that the memory function is given by the time correlation of the noise. This is called the 2nd F-D theorem.\[2\] The anomalous behavior of the memory function gives rise to the oscillation in the time evolution

![FIG. 1: The time evolution of $\Gamma(0, t)$ in the unit of $\Lambda^2$. The solid, dashed and dotted lines are the $\Gamma(0, t)/\Lambda^2$ for the temperatures $\epsilon = (T - T_c)/T_c = 0.05, 0.1$ and 0.2, respectively.](image)
of $\delta\sigma(k,t)$, as we will see later in Fig. 2. This anomaly may arise because the MPO defined in Eq. (3) is incomplete such that the noise still incorporates gross variables other than $\delta\sigma(x)$. As an example, let us consider the Langevin equation with two gross variables $x$ and $p$. \[ \frac{d}{dt}x(t) = p(t), \]
\[ \frac{d}{dt}p(t) = -\omega^2 x(t) - \int_0^t ds \Xi(t-s)p(s) + f(t), \]
where $\Xi(t)$ is the memory function given by the time correlation of the noise $f(t)$ and has no long time correlation. Solving the first equation and substituting into the second one, we have
\[ \frac{d}{dt}p(t) = -\int_0^t ds (\omega^2 + \Xi(t-s))p(s) + f(t), \]
where we set the initial condition $x(0) = 0$. Although the equation has only one gross variable $p$, the memory function $\omega^2 + \Xi(t)$ has the long time correlation due to the oscillation. In this case, the true memory term given by the 2nd F-D theorem is not $\omega^2 + \Xi(t)$ but $\Xi(t)$.

Thus, to define a renormalized memory function without long time correlation, we separate the oscillation effect from the memory function. By carrying out the Fourier transformation, one can easily recognize that the correction to the frequency shift and the damping are given by the imaginary part and the real part of the Fourier transform of the memory function, respectively. Thus, the Langevin equation with the renormalized memory function $\Phi(k,t)$ is expressed as,
\[ \frac{d}{dt}\delta\sigma(k,t) = -\int_0^t d\tau \Omega_k^2(t-\tau)\delta\sigma(k,\tau) - \int_0^t d\tau \Phi(k,\tau)\delta\sigma(k,t-\tau) + \xi_k(t), \]
where $\Omega_k^2(t) = i \int d\omega \text{Im}[\Gamma^L(k,-i\omega + \epsilon)e^{-i\omega t}/2\pi]$ and $\Phi(k,t) = \int d\omega \text{Re}[\Gamma^L(k,-i\omega + \epsilon)e^{-i\omega t}/2\pi]$. This memory function does not have a long time correlation any more. Here, we artificially removed the oscillation effect from the memory term. However, a similar procedure should be automatically implemented by using the MPO defined by a complete set of gross variables.

Now, we can define the noise with short time correlation. The definition of the noise and the 2nd F-D theorem lead to the following correlation properties:
\[ \langle \xi(k,t), \delta\sigma(k',0) \rangle = \langle \xi(k,t) \rangle = 0, \]
\[ \langle \xi(k,t), \xi^*(k',t') \rangle = V \delta_k k' \xi(0,k), \]
where $\langle O \rangle = \text{Tr}[\rho O]$ and $V$ is the volume of the system. Here, we assume that the noise has the translational invariance in space and time. The first correlation indicates that the noise does not include components that vary with the gross time scale and it is possible to regard it as a random field for $\delta\sigma(k,t)$. The second correlation characterizes the noise as a random field. In the following, we solve Eq. (12) as a classical equation with a random noise. For this, we introduce the classical noise that reproduces the correlation properties defined above;
\[ \langle \xi(k,t) \rangle = 0, \]
\[ \langle \xi(k,t)\xi^*(k',t') \rangle = \langle \xi(k,t)\xi^*(k',t') \rangle, \]
where $\langle \gg \rangle$ means the average for noise with a suitable stochastic weight. It is worth emphasising that the correlations determined here coincide with the condition of thermalization for a Langevin equation. In this sense, the system described by the Langevin equation approaches a thermal equilibrium state with time.

The averaged time evolution of the critical fluctuation at vanishing momentum is shown in Fig. 2. One can see that the nonequilibrium fluctuation relaxes with oscillation and finally converges zero. This indicates that the critical dynamics of the chiral transition may not be described by a diffusion equation like the time dependent Ginzburg-Landau (TDGL) equation. The rate of the relaxation becomes slower as the temperature approaches $T_c$ because of CSD. The relaxation time is characterized by $\tau_r = 2/\gamma_0$ where $2\gamma_k = \int_0^\infty ds \Phi(k,s)$. At $\epsilon = (T - T_c)/T_c = 0.2$, $\tau_r$ is about 10 fm, that is the same order as the expected life time of QGP. Thus, we cannot ignore the fluctuation of the order parameter at a temperature lower than 222 MeV. On the other hand, at higher temperature, the fluctuation relaxes firstly and other gross variables become important.

As we have shown so far, the Langevin equation reveals CSD and thermalization. This means that the Langevin equation correctly describes the behavior near $T_c$. At the same time, as was pointed out in Ref. [14], a soft mode appears above $T_c$ in the correlation function of the fluctuation of the order parameter. To investigate the
correlation function, we define the power spectrum as

\[ I(k, \omega) = \lim_{T,V \to \infty} \frac{1}{T V} \ll |\delta \sigma(k, \omega)|^2 \gg, \quad (17) \]

where \( T \) is the time when we observe the system, and

\[ \ll |\delta \sigma(k, \omega)|^2 \gg = T V \frac{\text{Re} \left[ \omega^2 \Phi(k, \omega) \chi_0(k) \right]}{-\omega^2 + \Omega_k^2 - i \omega \Phi(k, \omega)^2} \quad (18) \]

The Wiener-Khinchin theorem tells us that the correlation function is given by the power spectrum

\[ C(x, t) = \lim_{t' \to \infty} \ll \delta \sigma(x + x', t + t') \delta \sigma(x', t') \gg = \int_{-\infty}^{\infty} \frac{d\omega d^3k}{(2\pi)^4} I(k, \omega) e^{-i\omega t} e^{ik \cdot x}. \quad (19) \]

The temperature dependence of the power spectrum at vanishing momentum is shown in Fig. 3. We can see that the peak moving toward origin becomes prominent as the temperature is lowered toward \( T_c \). The power spectrum characterizes the space-time correlation in energy-momentum space and hence can be interpreted as the spectral function in the thermal Green’s function. Then, the peak with narrow width reveals the existence of a collective mode whose energy tends to vanish as the temperature approaches \( T_c \). Such a mode is called a soft mode. The temperature dependence of the power spectrum is consistent with the previous result, where the spectral function is calculated in the linear response theory \[ [13] \].

We have derived the linear Langevin equation that describes the dynamics of the chiral phase transition in the NJL model using the projection operator method. The equation reveals CSD and shows thermalization. The order parameter relaxes exhibiting oscillation. This means that a simple diffusion-type equation like the TDGL equation may be inadequate to describe the dynamics of the chiral transition. The power spectrum was also calculated using the Langevin equation and we found that there exists a soft mode. As a result, we can conclude that the Langevin equation fulfills the requirements near \( T_c \): CSD, thermalization and soft mode.

We have discussed here only the linear Langevin equation and ignored the nonlinear effect. However, the mode coupling theory makes it clear that near critical points the nonlinear effect becomes important because of the large correlation length and leads to the deviation from the van Hove theory in calculating dynamical critical exponent \[ [3] \]. In order to take nonlinear terms into account, we must choose them as gross variables in defining the MPO. Another intriguing subject is to apply this formulation to finite density and the color superconducting phase transition. These are future projects.

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