Paradoxes of time travel

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Abstract

Paradoxes that can supposedly occur if a time machine is created are discussed. It is shown that the existence of trajectories of “multiplicity zero” (i.e. trajectories that describe a ball hitting its younger self so that the latter cannot fall into the time machine) is not paradoxical by itself. This apparent paradox can be resolved (at least sometimes) without any harm to local physics or to the time machine. Also a simple model is adduced for which the absence of true paradoxes caused by self-interaction is proved.

1 Introduction

When one hears about time travel the first that comes to mind are paradoxes. In particular, a lot of paradoxes were described in science fiction. They all are essentially versions of one fundamental paradox, which can be formulated, for example, as follows.

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The paradox. Suppose a time traveller is going to kill his younger self some time before the latter used the time machine. His success would mean that he would never use the time machine and hence would never commit the murder. But as he would not be killed before using the time machine nothing would prevent his travelling and further killing his younger self, etc.

To see that this paradox is, in fact, a serious physical problem let us reformulate it. Consider an isolated system which is prepared in some initial state \( \eta(\tau_0) \) (\( \tau \) is the proper time of the system). In a causal world a state of the system at any \( \tau = \tau_1 > \tau_0 \) is a function of \( \eta(\tau_0) \). This can be expressed (not rigorously) as:

\[
\eta(\tau_1) = U_{\tau_0 \rightarrow \tau_1}(\eta(\tau_0)) \tag{1}
\]

Here \( U \) is determined by local physical laws. Assume, however, that we build a time machine and our system gets (at some \( \tau^* : \tau_0 < \tau^* < \tau_1 \)) into a region where closed timelike curves exist. Now it can interact with its older self. The system at \( \tau_1 \) may be influenced by itself at, say, \( \tau_2 : \tau_2 > \tau_1 \). Instead of \( (1) \) we would have to write

\[
\begin{align*}
\eta(\tau_1) &= U_{\tau_0 \rightarrow \tau_1}(\eta(\tau_0); \eta(\tau_2)) \\
\eta(\tau_2) &= U_{\tau_1 \rightarrow \tau_2}(\eta(\tau_1))
\end{align*} \tag{2}
\]

When a physical model we use is adequate, eq. \( (1) \) has a (unique) solution, but, in the general case, we do not know this for system \( (2) \). To make \( (2) \) consistent is beyond our power since \( U \) is already fixed by conventional causal physics (we consider only time machines like that from \cite{1, 2}, where causality violation is a manifestation of the global structure of otherwise “good” spacetime; that is why we write the same \( U \) in \( (1) \) and \( (2) \)). And suppose that for some \( \eta(\tau_0) \) system \( (2) \) has no solutions. How can this be interpreted? What is the actual evolution of the system at \( \tau > \tau^* \)? It is even hard to conceive where the answers to these questions may lie\footnote{Yet another reason to consider the paradoxes is that they can be somehow connected with a hypothetical mechanism “protecting causality”, which has long been looked for, but is not yet at hand.}. The above questions are somewhat academic until we find out whether the paradoxes can really occur. The answer is not evident at all. Apparent paradoxes like that with a traveller-suicide might be only due to our overlooking possible selfconsistent scenarios. For example, the traveller might be injured instead
of being killed. His wound might be of such a nature that he would not be able any more to shoot accurately enough and that is why later he would only injure his younger self. In Ref. [4] an attempt was made to find out whether paradoxes present in a specific model. A single perfectly elastic ball was considered in a spacetime with a wormhole-based time machine. The number of trajectories of the ball consistent with given initial data (the multiplicity of the given trajectory in terms of [4]) was evaluated. The model, however, turned out to be too complex and no definite answer was obtained to the question of whether there exist trajectories of multiplicity zero. Such a trajectory was found in [5], where a simpler time machine (the Politzer spacetime, see below) was considered. On the assumption that there are two perfectly elastic balls with the different masses in this spacetime, it was shown that their initial conditions in the past of the time machine cannot be arbitrary. This fact was interpreted in [5] as an argument against the existence of the time machine. In the present paper we consider a model so simple that the following becomes obvious:

1. Trajectories with multiplicity zero (or with multiplicity $> 1$) are not at all unique to spacetimes with time machines. Such trajectories appear whenever fixing the initial data on the partial Cauchy surface one considers the region beyond the Cauchy horizon.

2. The existence of a trajectory of multiplicity zero is not paradoxical by itself. An apparent paradox like that found in [4] or that pursued in [5] appears when one supplements the abovementioned data with some additional conditions.

Initial data inconsistent with any additional conditions permissible in a given model can be called the true paradox. We shall see that our model is free from true paradoxes, so the question of whether there is anything paradoxical in time travel remains unanswered.

\footnote{This term is meaningful in the context since it turned out that the multiplicity of a trajectory can be $> 1$ and even infinite (see Fig. 7 in [3]). We shall see below that the nonexistence and the nonuniqueness of trajectories may have the same source.}
2 Apparent paradox

In this section we construct a situation similar to that from [5]. We somewhat simplify it to make it more open to analysis. Let us consider a world containing a time machine and populated only by small elastic balls with the same masses. As a model of the time machine (Fig. 1) we take Politser’s spacetime [2]. It is the Minkowski plane where two cuts are made (say, along the segments $t = \pm 1, x \in (-1, 1)$) and after removing the points $t = \pm 1, x = \pm 1$

![Figure 1: Politzer’z spacetime](image)

the upper bank of each cut is glued to the lower bank of the other cut. The resulting manifold is a plane with handle and without two points (Fig. 1b). World lines of balls we describe by inextendible polygonal lines satisfying the following conditions

1. Each edge is a segment of a future-directed timelike straight line.
2. Edges do not intersect, but they can meet in vertices.

Now let us consider vertices, which are to describe collisions of balls. Since

![Figure 2: Same numbers correspond to same balls.](image)
our spacetime is time-orientable we can distinguish uniquely incoming and outgoing edges for any vertex. To make our model complete we must formulate the rule determining the outgoing edges from any set of the incoming ones. We adopt the following rule (in what follows we number both incoming and outgoing edges from left to right):

3. (a) Each vertex with \( n \) incoming edges has \( n \) outgoing edges.
   (b) The \( k \)th outgoing edge has the same direction as the \( (n - k + 1) \)th incoming edge. So the vertex looks like an intersection of \( n \) straight lines.
   (c) \( k \)th incoming and \( k \)th outgoing edges are deemed consequent parts of the same world line.

For \( n = 2 \) this rule is equivalent to the statement that our balls are perfectly elastic and distinguishable. For larger \( n \) we must adopt the rule 3 if we want the coordinates of the balls after collision to depend continuously on those before the collision. The rules 1–3 allow one to trace the history of balls if one fixes each ball by assigning at some \( t \) its coordinate and velocity or a part of its trajectory. Now we can construct a “trajectory of multiplicity zero”. Consider the two bold segments \((A_1, B_1)\) and \((A_2, B_2)\) in Fig. 3a. Let them be parts of the world lines of two balls.

![Figure 3: The apparent paradox and its resolutions.](image)

**Statement.** There is no evolution of two balls consistent with these initial conditions.
Proof. Suppose such evolution exists. It follows then from the rule 3 that
the world line of the left ball cannot lie to the right of the ray $FF'$. Similarly,
the world line of the right ball cannot lie to the left of the ray $DD'$. So the
balls cannot collide. Also neither of them can hit its younger self as they are
bounded away from the segment $\{ y = 1, \ -1 < x < 1 \}$ by $FF'$ and $DD'$. But
this implies that the balls will reach unobstructed the point $G$. Here they
must meet — a contradiction. Note that what we have obtained is actually
a version of the paradox with a traveller killing his younger self.

3 Resolution

The resolution of the paradox lies in the term “initial conditions”. Let us
introduce two new terms. We shall call conditions set on any partial Cauchy
surface (which is a connected achronal hypersurface without boundary) partial initial conditions, PIC. And anything that, taken together with the local
laws, determines uniquely evolution of balls in a spacetime under consider-
ation we shall call complete initial conditions, CIC. To begin with let us consider the Minkowski plane (Fig. 4) and the partial Cauchy surface
$S : t^2 - x^2 = 1$ in it. We can assign arbitrary conditions to balls on this

![Figure 4: The bold segment is a part of a single world line intersecting $S$. It can be used as PIC.](image)

surface and they will completely determine evolution of these balls inside the
domain $\mathcal{D}$ lying in the past cone of the origin of coordinates. ($\mathcal{D}$, which by
definition is a region where all past inextendible causal curves intersect $S$, is
called the Cauchy development of $S$.) However, to fix evolution of the balls
in the region $\mathcal{R}$, i.e. outside $\mathcal{D}$, we need some additional conditions. In other
words, the PIC set on $S$ are CIC in its Cauchy development, but cease to be CIC as soon as we intersect the Cauchy horizon (which is the boundary of $D$).

An important fact is that additional conditions we need for completing CIC affect the multiplicity of the trajectory determined by the PIC. Consider, for example, the situation depicted in Fig. 4 and assume we consider massless balls as well. As an additional condition one can require that there should be $m$ balls in $\mathcal{R}$. Then for $m = 0$ one obtains a trajectory of multiplicity zero, for $m = 1$, a unique trajectory, and for $m = 2$, a trajectory of the infinite multiplicity. The paradox cited in the previous section is just of the same nature. Conditions set on the partial Cauchy surface $t = t_0 < -1$ cease to be CIC as soon as we consider the region beyond the Cauchy horizon. The CIC here consist of the PIC and the requirement that there are only two balls in the spacetime. This requirement follows neither from PIC, nor from the local laws 1–3 (nor from common sense or intuition). It is an independent constraint and no wonder that it comes into conflict with PIC. If we abandon this constraint, we get infinitely many solutions satisfying both the equations of motions (1–3) and the PIC at $t < 0$. One of these solutions is shown in Fig. 3a, where the closed polygonal lines $(EFGE)$ and $(CDGC)$ are interpreted now as the world lines of two additional balls. We can see now the weak point in suggestion (2). A system isolated to the past of the time machine can cease to be isolated beyond the horizon and we cannot completely control its interaction there.

**Note.** The parallels between the causal and acausal cases discussed in this section can be made all the more evident if we “roll up” the part $x > t$ of the Minkowski plane into the Misner space (see [6] for details). The past cone of the origin of coordinates in this process will go into the causal part of the cylinder and the part of $\mathcal{R}$ will just form a time machine.

### 4 Discussion

The above solution of the paradox is typical for Politzer’s spacetime containing only pointlike perfectly elastic balls with the same masses. *There are no true paradoxes in such a world. For any initial conditions in the past of the time machine there exist infinitely many solutions satisfying both these conditions and the equations of motion.* To find these solutions, take a set of
straight lines satisfying the PIC. Mutual intersections break these lines into portions (segments and rays). Number these portions from left to right for any $t$. Interpreting each portion as a part of the world line of the ball with the same number we get a desired solution. Arbitrarily many other solutions can be obtained simply by drawing additional vertical lines (see Fig. 3b). Unfortunately, this method of finding a solution fits only the considered model, which, perhaps, is free from paradoxes only owing to its simplicity. To make the next step one should, possibly, consider various models available from this one by introducing different sorts of balls and arbitrary preassigned interaction instead of (3) (it is unlikely that our choice of the spacetime leads to noticeable loss of generality). It seems that the only reasonable restriction on the choice of interaction is the requirement that it must respect the symmetries of the spacetime. That is, if one admits the vertex Fig. 5a, one must admit also the vertices 5b,c. It should be particularly emphasized that the questions posed in the Introduction will face us as soon as we find a true paradox for any such model no matter how detailed and realistic it is. For, suppose that a paradox presents in some model, but disappears if one complements the model with, say, “back reaction” (i.e. adds Einstein equation to it). This would mean that there is some intimate connection between Einstein equations and the problem of existence of evolution for some quite abstract systems. Such a discovery would be every bit as surprising as the paradox itself. On the other hand the absence of true paradoxes in these generalized models would strongly suggest that they do not occur at all. If this is the case and any paradox can be (and must be) resolved in the manner shown above, one might obtain quite a bizarre picture. In science fiction terms it looks like the following. A potential suicide gets a wormhole. He

Figure 5: $V'_i$ are obtained from $V_i$ by some boost.
checks (for example, by travelling through it) that the throat of the wormhole is empty. By moving the mouths in the appropriate way [1] he transforms the wormhole into a time machine. Then he loads his gun and enters the mouth. And here, all of a sudden, he meets a policeman, who disarms him preventing the murder (and the paradox). We know that neither the existence of the policeman, nor his interaction with the traveller contradict any local physical laws. So, if we can find an appropriate world line for the policeman (it must not, in particular, intersect the Cauchy horizon), we will have to acknowledge that, however strange, this situation is not paradoxical.

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