Abstract. We construct symplectomorphisms in dimension $d \geq 4$ having a semi-local robustly transitive partially hyperbolic set containing $C^2$-robust homoclinic tangencies of any codimension $c$ with $0 < c \leq d/2$.

1. Introduction

According to KAM theory, for symplectomorphisms close to an integrable one, an orbit, with large probability, belongs to an invariant torus and thus stays bounded for all time. Furthermore, orbits which are close, take a large time to escape from a neighborhood of these tori. However, in higher dimensional systems the action variables may, a priori, exhibit considerable change showing unbounded orbits. Such behavior was coined with the term of "Arnold diffusion" or instability. Diffusion orbits can be constructed by using normally hyperbolic invariant laminations. But, this kind of invariant laminations also allow the construction of robustly transitive sets which give a more complex mechanism of diffusion. In this paper we continue the work of [21] and the study of these large (semi-local) transitive sets and provide new examples in this context.

It was unknown if the examples constructed in [21] were robustly non-hyperbolic. One of the classical tools to create robustly non-hyperbolic dynamics is via a heterodimensional cycle. However in the symplectic case this idea fails since all hyperbolic periodic points have the same stability index and thus, there are no heterodimensional cycles. The other classical approach to destroy hyperbolicity is the construction of robust homoclinic tangencies. In the symplectic setting known results on persistence of tangencies are restricted to area-preserving diffeomorphisms. Motivated by this problem, we have developed in [7] a new method to construct robust homoclinic tangencies in higher dimensions. These techniques can be applied in the symplectic framework as we show in this paper. As a consequence, we extend the results of [21] showing that the semi-local transitive sets can be made robustly non-hyperbolic.

Assume that $N$ and $M$ are symplectic connected manifolds (not necessarily compact). Let $F : N \to N$ be a $C^r$-symplectomorphism having a hyperbolic set $\Lambda \subset N$ conjugated to a full shift with a big enough set of symbols (depending only on the dimension of $M$). In order to state our main theorem we also need the following notion:
A diffeomorphism $f$ has a homoclinic tangency of codimension $c > 0$ if there is a pair of points $P$ and $Q$ belonging to the same transitive hyperbolic set so that the unstable invariant manifold of $P$ and the stable invariant manifold of $Q$ have a non-transverse intersection $Y$ of codimension $c$. That is,

$$Y \in W^u(P) \cap W^s(Q) \quad \text{and} \quad c = \dim T_Y W^u(P) \cap T_Y W^s(Q).$$

Now we are ready to state the main result:

**Theorem A.** There is an arc $\{f_\varepsilon\}_{\varepsilon \geq 0}$ of $C^r$-symplectomorphisms of $N \times M$ such that $f_0 = F \times \text{id}$ and for $\varepsilon > 0$, any small enough $C^2$-perturbation $g$ of $f_\varepsilon$ has

- a transitive set $\Delta_g$ homeomorphic to $\Lambda \times M$,
- a homoclinic tangency (in $\Delta_g$) of codimension $c > 0$.

The codimension $c$ of the tangency can be chosen to be any integer $0 < c \leq \dim M/2$.

As far as we know, the theorem above gives the first direct construction (i.e., not based on a dimension reduction argument using normally-hyperbolic manifolds) of robust tangencies for symplectomorphisms in higher dimensions.

Notice that $f_0$ cannot be a complete integrable system since the fiber map on $M$ is the identity function. Even so, we construct nearby symplectomorphisms with a transitive set which projects onto $M$. Large transitive sets with this property had been previously constructed in [21] only close to integrable systems. The integrability could be a restriction on the manifold $M$ since, as far as we know, it is unknown if a given symplectic structure admits integrable systems. Thus, Theorem A covers, a priori, new examples where diffusion orbits can be obtained. To prove this result we will use a different method to get orbits drifting along $M$. The approach of [21] consists in constructing transversal invariant tori for the fiber dynamics on $M$. Also the geometrical mechanism of diffusion developed in [11] is based on a similar idea. Here we introduce a different mechanism of propagation (drift) called globalization, which will be obtained by means of small translations in Darboux local charts, compatible with the symplectic structure.

The paper is organized as follows. First, in §2 we introduce blenders in a framework of normally hyperbolic invariant laminations. These invariant laminations give rise to natural skew-shifts over the space of sequences in a finite number of symbols, called symbolic skew-products. These kind of systems have also been studied in [9] and are related to the formation of stochastic diffusive behaviour for the generalized Arnold example. Next, in §3 we provide a criterium to construct robustly transitive symbolic skew-products. Finally, in §4 the main result is proven by combining the criteria for robust transitivity in §3 and for robust tangencies from [7].

2. **Symbolic skew-products and blenders**

Let $\mathcal{A}$ be a finite set (with at least two points), that we call an alphabet of symbols, and fix $0 < \nu < 1$ and $0 < \alpha \leq 1$. Consider the product space $\Sigma \equiv \Sigma(\mathcal{A}, \nu) \equiv \mathcal{A}^\mathbb{Z}$ of the bi-sequences
\( \xi = (\xi_i)_{i \in \mathbb{Z}} \) of symbols in \( \mathcal{A} \) endowed with the metric
\[
d_{\xi}(\xi, \zeta) \overset{\text{def}}{=} \nu^\ell, \quad \ell = \min\{i \geq 0 : \xi_i \neq \zeta_i \text{ or } \xi_{-i} \neq \zeta_{-i}\}.
\]

In what follows \( M \) will denote a differentiable manifold (not necessarily compact and not necessarily boundaryless) of dimension \( c \geq 1 \).

2.1. Symbolic skew-products. Given a compact set \( K \) in \( M \), we consider the pseudometric in the set \( C^0(M) \) of continuous functions of \( M \) given by
\[
d_{C^0}(\phi, \psi)_K \overset{\text{def}}{=} \max_{x, y \in K} d(\phi(x), \psi(x)) \quad \text{for any } \phi, \psi \in C^0(M).
\]
Since \( M \) is \( \sigma \)-compact, there is a sequence of relatively compact subsets \( K_n \) whose union is \( M \) and then we can endow \( C^0(M) \) with the weak topology (also called compact-open topology) induced by the family of pseudometrics (1). That is,
\[
d_{C^0}(\phi, \psi) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_{C^0}(\phi, \psi)_{K_n}}{1 + d_{C^0}(\phi, \psi)_{K_n}}, \quad \text{for any } \phi, \psi \in C^0(M).
\]

2.1.1. The set of skew-products. We consider skew-product homeomorphisms of the form
\[
\Phi : \Sigma \times M \to \Sigma \times M, \quad \Phi(\xi, x) = (\tau(\xi), \phi_\xi(x))
\]
where the base map \( \tau : \Sigma \to \Sigma \) is the lateral shift map and the fiber maps \( \phi_\xi : M \to M \) are homeomorphisms of \( M \). In order to emphasize the role of the fiber maps we write \( \Phi = \tau \circ \phi_\xi \) and call it a symbolic skew-product. When no confusion arises we also write \( M = \Sigma \times M \).

For every \( n > 0 \) and \( (\xi, x) \in M \) set
\[
\phi_\xi^n(x) \overset{\text{def}}{=} \phi_{\tau^{n-1}(\xi)} \circ \cdots \circ \phi_\xi(x) \quad \text{and} \quad \phi_\xi^{-n}(x) \overset{\text{def}}{=} \phi_{\tau^{-n}(\xi)} \circ \cdots \circ \phi_{\tau^{-1}(\xi)}(x)
\]
and hence
\[
\Phi^n(\xi, x) = (\tau^n(\xi), \phi_\xi^n(x)) \quad \text{for all } n \in \mathbb{Z}.
\]

We introduce the set of symbolic skew-products with which we will work:

**Definition 2.1.** Denote by \( S(M) = S_{\alpha^H}^0(\Sigma \times M) \) the set of \( \alpha \)-Hölder continuous symbolic skew-products of \( M = \Sigma \times M \). This is, the set of symbolic skew-products \( \Phi = \tau \circ \phi_\xi \) as in (2) such that

- \( \phi_\xi \) are bi-Lipschitz homeomorphisms (uniform in \( \xi \)): there are positive constants \( \gamma \equiv \gamma(\Phi) > 0 \) and \( \gamma^{-1} \equiv \gamma^{-1}(\Phi) > 0 \) such that
  \[
  \gamma d(x, y) < d(\phi_\xi(x), \phi_\xi(y)) < \gamma^{-1} d(x, y), \quad \text{for all } x, y \in M \text{ and } \xi \in \Sigma,
  \]
- \( \phi_\xi \) depend \( \alpha \)-Hölder with respect to \( \xi \): there is a non-negative constant \( C_0 \equiv C_0(\Phi) \geq 0 \) such that
  \[
  d_{C^0}(\phi_\xi^+, \phi_\xi^{-}) \leq C_0 d_{\Sigma}(\xi, \xi)^\alpha \quad \text{for all } \xi, \zeta \in \Sigma \text{ with } \xi_0 = \zeta_0.
  \]

We will denote by \( S^0(M) \) the set \( S(M) \) with \( C^1 \)-diffeomorphisms for fiber maps.
We define in $\mathcal{S}(M)$ the metric
\[
d_\mathcal{S}(\Phi, \Psi) \overset{\text{def}}{=} d_0(\Phi, \Psi) + \text{Lip}_0(\Phi, \Psi) + \text{Hol}_0(\Phi, \Psi)
\] (5)
where the symbolic skew-products $\Phi = \tau \times \phi_\xi$ and $\Psi = \tau \times \psi_\xi$ belong to $\mathcal{S}(M)$ and
\[
\text{Lip}_0(\Phi, \Psi) \overset{\text{def}}{=} \max_{\xi \in \Sigma} |\text{Lip}(\phi_\xi^1) - \text{Lip}(\psi_\xi^1)|.
\]
\[
d_0(\Phi, \Psi) \overset{\text{def}}{=} \max_{\xi \in \Sigma} d_{CS}(\phi_\xi^1, \psi_\xi^1)
\]
and $\text{Hol}_0(\Phi, \Psi) \overset{\text{def}}{=} |C_0(\Phi) - C_0(\Psi)|$ with
\[
\text{Lip}(\phi) = \sup_{x \neq y} \frac{d(\phi(x), \phi(y))}{d(x, y)} \text{ being } \phi \text{ a bi-Lipschitz homeomorphism of } M.
\]

An important class of $\alpha$-Hölder continuous symbolic skew-products is the following:

**Definition 2.2.** A symbolic skew-product $\Phi = \tau \times \phi_\xi \in \mathcal{S}(M)$ is partially hyperbolic if
\[
v^\alpha < \gamma < 1 < \hat{\gamma}^{-1} < v^\alpha
\]
where $\gamma$ and $\hat{\gamma}$ are given in (3). We denote by $\mathcal{PHS}(M) \equiv \mathcal{PHS}_{\alpha, \hat{\gamma}}^s(M)$ the set of partially hyperbolic symbolic skew-products. In addition, $\mathcal{PHS}^0(M) = \mathcal{PHS}(M) \cap \mathcal{S}^0(M)$.

Finally we introduce a particular and important class of symbolic skew-products:

**Definition 2.3** (one-step maps). A symbolic skew-product $\Phi = \tau \times \phi_\xi$ is called one-step if the fiber maps $\phi_\xi$ only depend on the coordinate $\xi_0$ of the bi-sequences $\xi = (\xi_i)_{i \in \mathbb{Z}} \in \Sigma$. In this case we have $\phi_\xi = \phi_i$ if $\xi_0 = i$ and write $\Phi = \tau \times (\phi_1, \ldots, \phi_d)$.

The **iterated function system** (IFS for short) associated with a one-step map $\Phi = \tau \times (\phi_1, \ldots, \phi_d)$ is the semigroup action generated by $\phi_1, \ldots, \phi_d$. In what follows, $(\phi_1, \ldots, \phi_d)^\tau$ will denote the semigroup generated by the maps $\phi_1, \ldots, \phi_d$.

2.1.2. **Stable and unstable sets for skew-products.** We define the **local stable** and **unstable set** of the lateral shift map $\tau : \Sigma \to \Sigma$ at $\xi \in \Sigma$ respectively as
\[
W^s_{\text{loc}}(\xi) \equiv W^s_{\text{loc}}(\xi; \tau) \overset{\text{def}}{=} \{ \zeta \in \Sigma : \zeta_i = \xi_i, i \geq 0 \}, \quad W^u_{\text{loc}}(\xi) \equiv W^u_{\text{loc}}(\xi; \tau) \overset{\text{def}}{=} \{ \zeta \in \Sigma : \zeta_i = \xi_i, i < 0 \}.
\]
The (global) **stable set** of the skew-product map $\Phi : \mathcal{M} \to \mathcal{M}$ at $P \in \mathcal{M}$ is defined as
\[
W^s(P) \equiv W^s(P; \Phi) \overset{\text{def}}{=} \{ Q \in \mathcal{M} : \lim_{n \to \infty} d(\Phi^n(Q), \Phi^n(P)) = 0 \}.
\]
We define the (global) **stable set** of a compact $\Phi$-invariant set, i.e. so that $\Phi(\Gamma) = \Gamma$, by
\[
W^s(\Gamma) \equiv W^s(\Gamma; \Phi) \overset{\text{def}}{=} \{ P \in \mathcal{M} : \lim_{n \to \infty} d(\Phi^n(P), \Gamma) = 0 \}
\]
or equivalently as the set of the points of $\mathcal{M}$ so that its $\omega$-limit is contained in $\Gamma$. The set $\Gamma$ is called **isolated (or maximal invariant set)** if there is a compact neighborhood $U$ of $\Gamma$, called the **isolating neighborhood** for $\Gamma$, such that every invariant subset of $U$ lies in $\Gamma$. In such a
case, we introduce the local stable set of $\Gamma$ as the forward invariant set of $\Phi$ in the isolating neighborhood $\mathcal{U}$, that is,

$$W^s_{\text{loc}}(\Gamma) \equiv W^s_{\text{loc}}(\Gamma; \Phi) \overset{\text{def}}{=} \{ P \in \mathcal{M} : \Phi^n(P) \in \mathcal{U} \text{ for } n \geq 0 \} = \bigcap_{n \geq 0} \Phi^n(\mathcal{U}).$$

Similarly $W^u_{\text{loc}}(\Gamma) \equiv W^u_{\text{loc}}(\Gamma; \Phi)$ and $W^u(\Gamma) \equiv W^u(\Gamma; \Phi)$ are, respectively, the local unstable set and the global unstable set of $\Gamma$. We have that

$$W^s(\Gamma) = \bigcup_{n \geq 0} \Phi^{-n}(W^s_{\text{loc}}(\Gamma)) \quad \text{and} \quad W^u(\Gamma) = \bigcup_{n \geq 0} \Phi^n(W^u_{\text{loc}}(\Gamma)).$$

Finally, given an $\delta$-perturbation of $\Phi$, that is a symbolic skew-product $\Psi$ close to $\Phi$ in the metric given in (5), we denote by $\Gamma_{\Psi}$ the maximal invariant set in $\mathcal{U}$ of $\Psi$. Although isolated sets vary, a priori, just upper semicontinuously by an abuse of terminology, we call $\Gamma_{\Psi}$ the continuation of $\Gamma$ for $\Psi$.

2.1.3. Strong laminations for partially hyperbolic skew-products. Under the global assumption of domination introduced in Definition 2.2, the usual graph transform argument yields a local strong stable $W^{ss}$ and unstable $W^{uu}$ partitions:

**Proposition 2.4 ([5, 1]).** For every $\Phi \in \mathcal{P}(\mathcal{M})$ there exist unique partitions

$$W^{ss} = \{ W^{ss}_{\text{loc}}(\xi, x) : (\xi, x) \in \mathcal{M} \} \quad \text{and} \quad W^{uu} = \{ W^{uu}_{\text{loc}}(\xi, x) : (\xi, x) \in \mathcal{M} \}$$

of $\mathcal{M} = \Sigma \times \mathcal{M}$ such that

i) every leaf $W^{ss}_{\text{loc}}(\xi, x)$ is the graph of an $\alpha$-Hölder function $\gamma^s_{\xi, x} : W^s_{\text{loc}}(\xi) \to \mathcal{M}$,

ii) $W^{ss}_{\text{loc}}(\xi, x)$ varies continuously with respect to $(\xi, x)$ and depends continuously on $\Phi$,

iii) $\Phi(W^{ss}_{\text{loc}}(\xi, x)) \subset W^{ss}_{\text{loc}}(\Phi(\xi, x))$ for all $(\xi, x) \in \mathcal{M}$,

iv) $W^{ss}_{\text{loc}}(\xi, x) \subset W^s(\xi, x)$ for all $(\xi, x) \in \mathcal{M}$.

The partition $W^{uu}$ verifies analogous properties.

Each leaf of the partition $W^{ss}$ is called the local strong stable set. We define the (global) strong stable set of $\Phi$ at $P$ as

$$W^{ss}(P) \equiv W^{ss}(P; \Phi) \overset{\text{def}}{=} \bigcup_{n \geq 0} \Phi^{-n}(W^{ss}_{\text{loc}}(\Phi^n(P))) \subset W^s(P).$$

2.2. Blenders. In this subsection, we will first introduce the notion of hyperbolic set for symbolic skew-products homeomorphisms. After that we give the formal definition of blenders and finally we provide a criterion to obtain these local tools.

2.2.1. Hyperbolic sets. Fix $\epsilon > 0$ small enough. We introduce the local stable set (of size $\epsilon$) of $\Phi$ at $P = (\xi, x)$ as

$$W^s_\epsilon(P) \equiv W^s_\epsilon(P; \Phi) \overset{\text{def}}{=} \{ Q \in \mathcal{M} : d(\Phi^n(Q), \Phi^n(P)) \leq \epsilon, \ n \geq 0 \} \subset W^s_{\text{loc}}(\xi) \times \mathcal{M}.$$ 

The local unstable set (of size $\epsilon$), denoted by $W^u_\epsilon(P)$, is defined analogously.
Definition 2.5. A compact invariant set $\Gamma \subset M$ is hyperbolic (for $\Phi$) if there exist constants $\varepsilon > 0$, $K > 0$, $0 < \theta < 1$ such that
\[
\begin{align*}
    &d(\Phi^o(P), \Phi^o(Q)) \leq K\theta^n \text{ for all } P, Q \in W^s_\varepsilon(P) \text{ and } n \geq 0; \\
    &d(\Phi^{-n}(P), \Phi^{-n}(Q)) \leq K\theta^n \text{ for all } P, Q \in W^u_\varepsilon(P) \text{ and } n \geq 0;
\end{align*}
\]
and there exists $\delta > 0$ such that
\[
    \# W^s_\varepsilon(P) \cap W^u_\varepsilon(Q) = 1 \text{ for all } P, Q \in \Gamma \text{ with } d(P, Q) \leq \delta.
\]

Every isolated hyperbolic set $\Gamma$ for $\Phi$ is topologically stable [2]; i.e., there is an isolating neighborhood $U$ of $\Gamma$ such that for any homeomorphism $\Psi$ which is $C^0$ near $\Phi$, the restriction of $\Psi$ to the maximal invariant set in $U$, is semiconjugate to the restriction of $\Phi$ to $\Gamma$.

We will now introduce the notion of index of an isolated transitive hyperbolic set $\Gamma$ in our context. In the sequel we will assume that the topological dimension (in the sense of the Lebesgue covering dimension) of $M^s_\varepsilon(P) = W^s_\varepsilon(P) \cap ([\xi] \times M)$ and $M^u_\varepsilon(P) = W^u_\varepsilon(P) \cap ([\xi] \times M)$ depend continuously with respect to $P = (\xi, x) \in \Gamma$. From this assumption and being $\Gamma$ transitive, the dimensions of $M^s_\varepsilon(P)$ and $M^u_\varepsilon(P)$ remain constant for any $P \in \Gamma$. Thus, we may define the $cs$-index and $cu$-index of $\Gamma$, denoted by $\text{ind}^s(\Gamma)$ and $\text{ind}^u(\Gamma)$ as these dimensions respectively. Notice that $\dim M = \text{ind}^s(\Gamma) + \text{ind}^u(\Gamma)$ and from the topological stability, the $cs$-index remains constant under small $S$-perturbations of $\Phi$.

2.2.2. Symbolic blenders. In order to introduce the notion of a blender we need first to define families of $s$-discs and $u$-discs. To do this, we will consider a basic open set $B$ of $M$, i.e., a set of the form $V \times B$ where $V$ is an open set of $\Sigma$ and $B$ is an open set of $M$.

Definition 2.6 ($s$-discs). A set $D^s \subset M$ is called a $s$-disc in $B$ if there is $\xi \in V$ such that $D^s$ is a graph of an $\alpha$-Hölder function from $W^s_\text{loc}(\xi) \cap V$ to $B$.

We say that two $s$-discs, $D^s_1, D^s_2 \subset W^s_\text{loc}(\xi) \times M$ are close if they are the graphs of close $\alpha$-Hölder functions. This proximity between discs allows us to introduce the following:

Definition 2.7 (open set of $s$-discs). We say that a collection of discs $D^s$ is an open set of $s$-discs in $B$ if given $D^s_0 \in D^s$, every $s$-disc $D^s$ close enough to $D^s_0$ is a $s$-disc contained in $B$ and belongs to $D^s$.

Example of $s$-discs are the almost horizontal discs defined as follows: given $\delta > 0$ and a point $(\xi, x) \in B$, we say that a set $D^\delta \equiv D^\delta(\xi, x) \subset M$ is a $\delta$-horizontal disc in $B$ if
\[
    \begin{align*}
        &D^\delta \text{ is a graph of a } (\alpha, C)\text{-Hölder function } g : W^s_\text{loc}(\xi) \cap V \to B, \\
        &d(g(\zeta), x) < \delta \text{ for all } \zeta \in W^s_\text{loc}(\xi) \cap V, \\
        &C\nu^u < \delta.
    \end{align*}
\]

The set of all $\delta$-horizontal discs in $B$ is an open set of $s$-discs in $B$. Similarly we define $u$-discs in $B$, open set of $u$-discs in $B$ and we have that the set of almost vertical discs is an example of an open set of $u$-discs.

Following [21, 6, 7], we introduce symbolic $cs$, $cu$ and double-blenders.
**Definition 2.8** (blenders). Let $\Phi \in \mathcal{S}(M)$ be a symbolic skew-product. A transitive hyperbolic maximal invariant set $\Gamma$ in a relatively compact open set $\mathcal{U} \subset M = \Sigma \times M$ of $\Phi$ is called

1. **cs-blender** if $\text{ind}^{cs}(\Gamma) > 0$ and there exist a basic open set $\mathcal{B} \subset \mathcal{U}$ and an open set $\mathcal{D}^s$ of $s$-discs in $\mathcal{B}$ such that for every small enough $\mathcal{S}$-perturbation $\Psi$ of $\Phi$,

$$W^s_{\text{loc}}(\Gamma; \Psi) \cap \mathcal{D}^s \neq \emptyset \quad \text{for all } \mathcal{D}^s \in \mathcal{D}^s.$$

2. **cu-blender** if $\text{ind}^{cu}(\Gamma) > 0$ and there exist a basic open set $\mathcal{B} \subset \mathcal{U}$ and an open set $\mathcal{D}^u$ of $u$-discs in $\mathcal{B}$ such that for every small enough $\mathcal{S}$-perturbation $\Psi$ of $\Phi$,

$$W^u_{\text{loc}}(\Gamma; \Psi) \cap \mathcal{D}^u \neq \emptyset \quad \text{for all } \mathcal{D}^u \in \mathcal{D}^u.$$

3. **double-blender** if both (i) and (ii) hold (not necessarily for the same $\mathcal{B}$).

The open set $\mathcal{B}$ is called a superposition domain and the open sets of discs $\mathcal{D}^s$ and $\mathcal{D}^u$ are called the superposition regions of the blender. Finally, the cs-blender (resp. cu-blender) with cs-index (resp. cu-index) is equal to $\dim M$ is called a contracting-blender (resp. expanding-blender).

Blenders are actually a power tool in partially hyperbolic dynamics when the superposition region contains the local strong stable/unstable set in the superposition domain. For this reason, without loss of generality, we will assume the following blender properties:

**Scholium 2.9.** Consider $\Phi \in \mathcal{PHS}(M)$ and let $\Gamma$ be a cs-blender with superposition domain $\mathcal{B} = V \times B$ and superposition region $\mathcal{D}^s$. Then, there is a $\mathcal{S}$-neighborhood $\mathcal{V}$ of $\Phi$ such that for any $\Psi \in \mathcal{V}$,

- **(B1)** the open set of discs $\mathcal{D}^s$ contains the family of local strong stable sets of $\Psi$ in $\mathcal{B}$:

$$\text{if } W^s_{\text{loc}}(P; \Phi) \cap (V \times M) \subset \mathcal{B} \quad \text{then } W^s_{\text{loc}}(P; \Psi) \cap (V \times M) \in \mathcal{D}^s.$$

- **(B2)** if $W^s_{\text{loc}}(P; \Phi) \cap (V \times M) \subset \mathcal{B}$ then

$$W^u_{\text{loc}}(\Gamma; \Psi) \cap W^s_{\text{loc}}(P'; \Psi) \neq \emptyset \quad \text{for all } P' \text{ close enough to } P.$$

Similar conditions are also assumed for cu-blenders of partially hyperbolic skew-products.

We must show that property (B2) follows from the definition of a blender.

**Proof.** First of all, notice that the assumption (B1) and Definition 2.8 imply that

- if $W^s_{\text{loc}}(P; \Phi) \cap (V \times M) \subset \mathcal{B}$ then $W^u_{\text{loc}}(\Gamma; \Phi) \cap W^s_{\text{loc}}(P; \Phi) \neq \emptyset$ \quad $\mathcal{S}$-robustly. \hspace{1cm} (6)

A priori, the neighborhood of the $\mathcal{S}$-perturbation of $\Phi$ where (6) holds depends on the $s$-disc $W^s_{\text{loc}}(P; \Phi)$. However, this can be taken independent of the disc assuming that the disc belongs to a superposition subdomain $\mathcal{B}_0$ of $\mathcal{B} = V \times B$. That is if $W^s_{\text{loc}}(P; \Phi) \cap (V \times M) \subset \mathcal{B}_0$ where $\mathcal{B}_0 = V \times B_0, B_0$ is an open set whose closure is contained in $\mathcal{B}$. For this reason, without loss of generality, we can assume that (B2) holds. \hspace{1cm} $\Box$
2.2.3. Blenders from one-step maps. In order to provide a criterion to construct blenders we need the following definition.

**Definition 2.10** (blending region). Let \( \Phi = \tau \mapsto (\phi_1, \ldots, \phi_d) \in PHS(M) \). Consider bounded open sets \( B \) and \( D \) of \( M \) with \( \overline{B} \subset D \), a subset \( S \subset \mathcal{A} \), and a hyperbolic transitive set

\[
\Gamma \defeq \bigcap_{n \in \mathbb{Z}} \Phi^n(S^2 \times \overline{D}) = \bigcap_{n \in \mathbb{Z}} \Phi^n(V \times \overline{D}),
\]

where \( V \) denotes any isolating neighborhood of \( \Sigma^\pm \defeq \{ \xi \in \Sigma : \varepsilon_0 \in S \} \) and \( \Sigma^\pm \defeq \{ \xi \in \Sigma : \varepsilon_{-1} \in S \} \). We say that \( B \) is a cs/cu/double-blending region with respect to \( \{\phi_i : i \in S\} \) on \( D \) if there exists respectively a

\[\begin{align*}
&i) \text{ cs-cover: } \{\phi_i(B) : i \in S\} \text{ is an open cover of } \overline{B} \text{ and } \text{ind}^{cs}(\Gamma) > 0; \\
&ii) \text{ cu-cover: } \{\phi_i^{-1}(B) : i \in S\} \text{ is an open cover of } \overline{B} \text{ and } \text{ind}^{cu}(\Gamma) > 0; \\
&iii) \text{ double-cover: } \text{ both } (i) \text{ and } (ii) \text{ are true.}
\end{align*}\]

We call cs-index (resp. cu-index) of the blending region \( B \) the cs-index (resp. cu-index) of \( \Gamma \). As in the case of the blender, if its cs-index (resp. cu-index) is equal to dimension of \( M \) the blending region is called contracting (resp. expanding).

With the above terminology, the following result showed in [7, Corollary 5.3] gives a criterion to construct a blender.

**Theorem 2.11.** Let \( \Phi = \tau \mapsto (\phi_1, \ldots, \phi_d) \in PHS(M) \). Assume that there are a set \( S \subset \mathcal{A} \) and a

\[- \text{ cs/cu/double-blending region } B \text{ with respect to } \{\phi_i : i \in S\} \text{ on } D.\]

Then the maximal invariant set \( \Gamma \) in \( S^2 \times \overline{D} \) is a cs/cu/double-blender of \( \Phi \) whose superposition region contains the family of almost horizontal discs in \( \Sigma^+ \times B \) and almost vertical discs in \( \Sigma^- \times B \). Moreover, it also contains the family of local strong stable/unstable sets, i.e., (B1) holds.

Blending regions which cover an open ball \( B \) around a hyperbolic fixed point of a map \( \phi \) can be easily constructed from a sufficient number of sets of the form \( \phi_i(B) \), where \( \phi_i \) is a translation of \( \phi \). This idea was developed in [21] and [7, Proposition 5.6], obtaining a blending region in local coordinates:

**Proposition 2.12.** Consider a \( C^r \)-diffeomorphism \( \phi \) of \( \mathbb{R}^c \) with a hyperbolic attracting/repelling/saddle fixed point \( x \). Then, there exist an integer \( k \equiv k(\phi, c) \geq 2 \), arcs of \( C^r \)-diffeomorphisms of \( \mathbb{R}^c \), \( \phi_1 \equiv \phi_1(\varepsilon), \ldots, \phi_k \equiv \phi_k(\varepsilon) \) and bounded open sets \( D \equiv D(\varepsilon), \varepsilon \geq 0 \), such that

\[- \phi_i(0) = \phi \text{ for } i = 1, \ldots, k; \\
- \phi_i = T_i \circ \phi \text{ where } T_i \equiv T_i(\varepsilon) \text{ is a translation (moreover, one can take } \phi_1 = \phi); \\
- B \equiv B_\delta(x) \subset D \subset B_{2\delta}(x) \text{ for some } \delta \equiv \delta(\varepsilon) > 0; \\
- B \text{ is cs/cu/double-blending region with respect to } \{\phi_1, \ldots, \phi_k\} \text{ on } D \text{ for all } \varepsilon > 0.\]

Moreover, the cs-index of the blending region is equal to the s-index of the hyperbolic fixed point \( x \).
3. Robust transitivity

We explain how blenders can be used to yield a $S^0$-robust topologically mixing symbolic skew-product $\Phi$ of $M = \Sigma \times M$. That is, for any $S^0$-perturbation $\Psi$ of $\Phi$ and for every pair of open sets $U, V$ of $M$, there is $n_0 > 0$ such that $\Psi^n(U) \cap V \neq \emptyset$ for all $n \geq n_0$. In particular topologically mixing implies transitivity.

3.1. Criterion to yield robust transitivity. One of the classical ways to create robustly transitive diffeomorphisms is to construct a map that robustly has a hyperbolic periodic point with dense stable and unstable manifolds. Then, using the inclination lemma (or $\lambda$-lemma) one concludes that the diffeomorphism is topologically mixing. In the symbolic setting an analogous result was proved in [6, Theorem 5.7] for fiber attracting/repelling hyperbolic fixed points. Here we extend this criterion to any hyperbolic periodic point. Later on we will use this result to construct robustly topologically mixing skew-products.

**Theorem 3.1** (criterion for robust transitivity). Assume that $\Phi \in \mathcal{PHS}^0(M)$ has a cs-blender $\Gamma$ with superposition domain $\mathcal{B} = V \times B$ such that

$$(RT) \text{ there is an open set } B_0 \subset B \text{ such that } W^u(P; \Phi) \subset W^u(\Gamma; \Phi) \text{ and } W^s(\Gamma; \Phi) \subset W^s(P; \Phi).$$

Then $S^0$-robustly it holds that $M = \bigcup_n \Phi^n(B_0)$ $S^0$-robustly. \(^1\)

Then $S^0$-robustly it holds that $M = \bigcup_n \Phi^n(B_0)$ for any periodic point $P \in \Gamma$. If $\Gamma$ is a cu-blender satisfying (RT) for $\Phi^{-1}$, then the stable set of any periodic point of $\Gamma$ is $S^0$-robustly dense in $M$.

Moreover, if $\Gamma$ is either a double-blender, a contracting-blender or an expanding-blender satisfying (RT) for both $\Phi$ and $\Phi^{-1}$, then the global stable and unstable sets of any periodic point of $\Gamma$ are both $S^0$-robustly dense. In this case, some power $\Phi^k$ is $S^0$-robustly topologically mixing which in particular implies that $\Phi$ is $S^0$-robustly transitive.

Notice that the globalization property \(^1\) is a necessary condition for robust transitivity. In order to prove Theorem 3.1 we need the following lemmas.

**Lemma 3.2.** Consider a symbolic skew-product $\Phi$ and let $\Gamma$ be an isolated hyperbolic transitive set of $\Phi^\ell$ for some $\ell \in \mathbb{N}$. Then for every periodic point $P \in \Gamma$,

$$W^u(\Gamma; \Phi^\ell) \subset W^u(P; \Phi) \quad \text{and} \quad W^s(\Gamma; \Phi^\ell) \subset W^s(P; \Phi).$$

**Proof.** The transitivity of $\Gamma$ implies that $\Gamma \subset W^u(P; \Phi^\ell)$. Since $\Gamma$ is an isolated hyperbolic set of $\Phi^\ell$, from the in-phase result (c.f. [23, Prop. 10]), it holds that

$$W^u(\Gamma; \Phi^\ell) = \bigcup_{Q \in \Gamma} W^u(Q; \Phi^\ell).$$

\(^1\) For every compact set $K \subset M$ there is a $S^0$-neighborhood $\mathcal{U}$ of $\Phi$ such that $\Sigma \times K$ is contained in the closure of $\cup_n \Psi^n(B_0)$ for all $\Psi \in \mathcal{U}$.
Combining the above facts we get the density of the unstable set, and an analogous argument concludes the density of the stable set. □

**Lemma 3.3.** Consider \( \Phi \in S^0(M) \) and let \( P \) be a hyperbolic periodic point of period \( k > 0 \) such that \( W^u(P) \cap \mathcal{U} \neq \emptyset \) and \( W^s(P) \cap \mathcal{V} \neq \emptyset \) where \( \mathcal{U} \) and \( \mathcal{V} \) are open sets of \( M \). Then, there exists \( n_0 \in \mathbb{N} \) such that

\[
\Phi^n(\mathcal{V}) \cap \mathcal{U} \neq \emptyset \quad \text{for all } n \geq n_0.
\]

**Proof.** Without loss of generality, we can assume that \( P = (\xi, x) \) is a fixed point of \( \Phi \). Taking an integer \( m > 0 \) large enough, we get that \( \Phi^{-m}(\mathcal{U}) \supset (W^u_{loc}(\xi) \cap \mathcal{C}) \times \mathcal{U} \) and \( \Phi^m(\mathcal{V}) \supset (W^s_{loc}(\xi) \cap \mathcal{C}) \times \mathcal{V} \) where \( \mathcal{C} \) is a cylinder around \( \xi \) and \( \mathcal{U}, \mathcal{V} \) are open sets of \( M \) so that

\[
W^u_r(x; \phi_\xi) \cap \mathcal{U} \neq \emptyset \quad \text{and} \quad W^s_r(x; \phi_\xi) \cap \mathcal{V} \neq \emptyset
\]

for some \( \varepsilon > 0 \) sufficiently small. Thus, it suffices to prove the result for the hyperbolic fixed point \( x \in M \) of the \( C^1 \)-diffeomorphism \( \phi_\xi : M \to M \). Hence, as a consequence of the inclination lemma we get that \( \phi_\xi^{-n}(\mathcal{U}) \cap \mathcal{V} \neq \emptyset \) for any \( n > 0 \) large enough. □

**Proof of Theorem 3.1.** Consider any open set \( \mathcal{U} \subset M \). By (7), there is \( n \) arbitrarily large and \( R \in \mathcal{U} \) such that \( Q = \Phi^{-n}(R) \in \mathcal{B}_0 \). Hence, \( W^s_{loc}(Q) \subset \Phi^{-n}(W^s_{loc}(R) \cap \mathcal{U}) \). From (RT), it holds \( W^s(Q) \cap (V \times M) \subset \mathcal{B} \). Since \( \Gamma \) is a \( cs \)-blender with superposition domain \( \mathcal{B} \), by (B2),

\[
W^u_{loc}(\Gamma) \cap W^s_{loc}(Q) \neq \emptyset \quad S^0\text{-robustly. (8)}
\]

Hence, Lemma 3.2 implies \( W^u(P) \cap \Phi^{-n}(\mathcal{U}) \neq \emptyset \) for all periodic points \( P \in \Gamma \). Consequently, \( S^0 \)-robustly it holds \( W^u(P) \cap \mathcal{U} \neq \emptyset \). From the invariance of the local strong stable partition, we get that \( W^u(P) \) meets \( \mathcal{U} \), \( S^0 \)-robustly.

A similar argument holds assuming that \( \Gamma \) is a \( cu \)-blender and (RT) holds for \( \Phi^{-1} \). Thus according to Lemma 3.3 we conclude the theorem when \( \Gamma \) is a double-blender. If \( \Gamma \) is a contracting-blender (resp. expanding-blender), then the local stable (resp. unstable) set of any periodic point in \( \Gamma \) has dimension \( \dim \mathcal{M} \). Hence, any \( u \)-disc (resp. \( s \)-disc) in the superposition region transversally meets the local stable (resp. unstable) set of these periodic points and thus (8) immediately holds for \( \Phi^{-1} \) (resp. for \( \Phi \)). Therefore, in this case, the same argument works assuming the globalization property (RT) for \( \Phi \) and \( \Phi^{-1} \). This shows the density of both \( W^s(P) \) and \( W^u(P) \) and again, according to Lemma 3.3, we conclude the theorem. □

**Remark 3.4.** We actually get that \( \Phi \) is \( S^0 \)-robustly topologically mixing if \( \Gamma \) has fixed points.

3.2. Robust transitivity from one-step maps. Theorem 3.1 provides conditions to yield robustly transitive symbolic skew-products. In what follows, we will first translate these conditions to the particular case of one-step maps, \( \Phi = \tau \kappa (\phi_1, \ldots, \phi_d) \). Afterwards, we will construct arcs of IFSs unfolding from the identity and satisfying these conditions.
3.2.1. **Criterion for robustly transitive one-step maps.** Let \( \Phi = \tau \times (\phi_1, \ldots, \phi_d) \in \mathcal{P}\mathcal{H}(S(M)) \) be a one-step map and assume there exists a cs-blending region \( B \) with respect to \( \{\phi_1, \ldots, \phi_d\} \) on \( D \). Hence, the maximal invariant set \( \Gamma \) in the closure of \( \Sigma \times D \) for \( \Phi \) is a cs-blender with superposition domain \( \mathcal{B} = \Sigma \times B \).

It is not difficult to see that (7) is equivalent to

\[
M = \mathcal{P}\left( \bigcup_{n=1}^{\infty} \Phi^n(B_0) \right) \text{ S}^0\text{-robustly},
\]

where \( B_0 \) is an open set in \( \mathcal{B} \) such that

\[
B_0 = \Sigma \times B_0 \quad \text{with} \quad \overline{B_0} \subset B, \ W_{loc}^{cs}(\xi, x) \subset \mathcal{B} \quad \text{for all} \ (\xi, x) \in \mathcal{B}_0, \ S^0\text{-robustly}.
\]

The robustness in (9) means that for every compact set \( K \subset M \) there is a \( S^0 \)-neighborhood \( \mathcal{U} \) of \( \Phi \) such that \( K \) is contained in the projection on \( M \) of the closure of the union of forward \( \Psi \)-iterated of \( B_0 \) for all \( \Psi \in \mathcal{U} \). Observe that (9) holds if there exists \( n \in \mathbb{N} \) such that

\[
M = \mathcal{P}\left( \bigcup_{i=1}^{n} \Phi^n(B_0) \right).
\]

This finite cover requires the compactness of \( M \). In the non-compact case, (9) holds if there exists an increasing sequence of compact sets \( K_i \subset M \) such that their union is \( M \) and for all \( i \)

\[
K_i \subset \mathcal{P}\left( \bigcup_{n=1}^{\infty} \Phi^n(B_0) \right) = \{ h(x) : h \in \langle \phi_1, \ldots, \phi_d \rangle^+ \text{ and } x \in B_0 \} \overset{\text{def}}{=} \langle \phi_1, \ldots, \phi_d \rangle^+(B_0)
\]

where recall that \( \langle \phi_1, \ldots, \phi_d \rangle^+ \) denotes the semigroup generated by the maps \( \phi_1, \ldots, \phi_d \).

**Definition 3.5 (globalization).** Let \( B \) be an open set of \( M \). We say that \( B \) is forward and backward globalized (or simply globalized) by \( \langle \phi_1, \ldots, \phi_d \rangle^+ \) if for any compact set \( K \subset \text{int}(M), \)

\[
K \subset \langle \phi_1, \ldots, \phi_d \rangle^+(B) \quad \text{and} \quad K \subset \langle \phi_1^{-1}, \ldots, \phi_d^{-1} \rangle^+(B).
\]

With this terminology, as a consequence of Theorem 3.1 and Remark 3.4 we have

**Corollary 3.6 (robustly transitive one-step maps).** Let \( \Phi = \tau \times (\phi_1, \ldots, \phi_d) \in \mathcal{P}\mathcal{H}(S^0(M)) \).

Suppose that there are bounded open sets \( D, B_0 \) and \( B \) of \( M \) with \( \overline{B_0} \subset B \subset D \) such that

- \( B_0 \) is globalized by \( \langle \phi_1, \ldots, \phi_d \rangle^+ \),
- \( B \) is a contracting/expanding/double-blending region with respect to \( \{\phi_1, \ldots, \phi_d\} \) on \( D \).

Then \( \Phi \) is \( S^0 \)-robustly transitive. Moreover, \( \Phi \) is \( S^0 \)-robustly topologically mixing if some fiber map \( \phi_i \) has a hyperbolic fixed point in \( B \).

3.2.2. **Construction of globalized blending regions.** Consider a contracting/expanding/double-blending region \( B \) with respect to \( \{\phi_1, \ldots, \phi_d\} \) where \( \phi_i \) are diffeomorphisms of \( M \). According to [6, 15] one way to yield globalization is to add, if necessary, a pair of Morse-Smale diffeomorphisms without periodic points in common and having respectively an attracting/repelling periodic point in \( B \) with a dense stable/unstable manifold. The existence of
such a map can be constructed by perturbations of time-one maps of gradient-like vector fields. This explains the following proposition.

**Proposition 3.7** (globalized blending regions homotopic to the identity). Let $M$ be a connected manifold of dimension $c \geq 1$. Then there exists an integer $s \equiv s(c) \geq 3$ such that for every $x \in M$ there are arcs of $C^r$-diffeomorphisms $T_1(\epsilon), \ldots, T_s(\epsilon)$, $\epsilon \geq 0$, of $M$ such that

- $T_i(0) = \text{id}$ for $i = 1, \ldots, s$, and
- $B_i(x)$ is globalized by $(T_1(\epsilon), \ldots, T_s(\epsilon))^+$ for all $\epsilon > 0$ small enough.

Next, we will give another proof of the above statement, which will be useful later on for the symplectic setting. The following construction uses local tools in contrast to the global nature of Morse-Smale diffeomorphisms. The local perturbations will be translations of the identity map, compatible with symplectomorphisms. To show the result, we need the following lemma and notation. Given $\delta > 0$ and a subset $A$ of $\mathbb{R}^c$, we write

$$B_\delta(A) \overset{\text{def}}{=} \{y \in \mathbb{R}^c : d(y, A) < \delta\}.$$

**Lemma 3.8.** Let $U_0$ be a bounded connected open set of $\mathbb{R}^c$. Then, there are an integer $m \equiv m(c) \geq 2$ (that only depends on the dimension $c$) and arcs of $C^r$-diffeomorphisms $T_1(\epsilon), \ldots, T_m(\epsilon)$ of $U \equiv B_{2\varepsilon}(U_0)$ such that for every $x \in U_0$,

- $T_i(0) = \text{id}$ for $i = 1, \ldots, m$,
- $T_i(\epsilon) = \text{id}$ in $\partial U$ for all $i = 1, \ldots, m$ and $\epsilon > 0$,
- $T_i(\epsilon)$ is a small translation in $U_0$ for all $i = 1, \ldots, m$ and $\epsilon > 0$,
- $\overline{U_0} \subset \langle T_1(\epsilon), \ldots, T_m(\epsilon) \rangle^+(B_\epsilon(x))$ for all $\epsilon > 0$ small enough.

**Proof.** Fix $\epsilon > 0$. Let $B \equiv B(\epsilon)$ be an open ball of radius $\epsilon$ centered at a point $x$ in $U_0$. We can cover the closure of this ball by small translations of $B$. Notice that the number $m$ of translation required only depends on the dimension $c$. In fact, there exist unitary vectors $u_1, \ldots, u_m$ so that any ball in $\mathbb{R}^c$ can be covered by translations along the directions $u_1, \ldots, u_m$. Let $T_{v_1}, \ldots, T_{v_n}$ be these translations of small vectors $v_i = \delta u_i$, $0 < \delta < \epsilon$, such that $\overline{B} \subset T_{v_1}(B) \cup \cdots \cup T_{v_n}(B)$ with $T_{v_i}(U_1) \subset B_{\epsilon/2}(U_1)$ where $U_1 \equiv B_1(U_0)$. By the above observation, the same translations $T_{v_1}, \ldots, T_{v_n}$ cover any open ball of radius $\rho \geq \epsilon$.

By means of the extension lemma [17, Lemma 2.27], there exists a $C^r$-diffeomorphism $T_i \equiv T_i(\epsilon)$ of $U$ such that $T_i|_{U_1} = T_{v_i}$ and $T_i|_{\partial U} = \text{id}$. We will now show that $\overline{U_0} \subset \langle T_1, \ldots, T_m \rangle^+(B)$. Indeed, there is $\rho > \epsilon$ so that

$$\overline{B_\rho} \subset T_1(B) \cup \cdots \cup T_m(B) \quad \text{where } B_\rho \equiv B_\rho(x).$$

If $B_\rho \subset U_1$, as the translation directions $u_i$ only depend on the dimension $c$ then

$$\overline{B_{2\rho}} \subset T_1(B_\rho) \cup \cdots \cup T_m(B_\rho).$$

Repeating the above procedure, since the radius of the covered ball $B_\rho$ is strictly increasing, by changing the center we can reach the boundary of $U_1$ and thus cover $\partial U_0$. 

Since \( v_i(\epsilon) \) tends continuously to zero (i.e. \( \delta < \epsilon \) goes to zero continuously), we get that 
\( T_i(\epsilon) \) tends continuously to the identity and conclude the proposition. \( \square \)

**Proof of Proposition 3.7.** By means of the well-known procedure of Milnor (see [16, Proposition 2.1] and [18, Proposition 1.4.14]), there is an atlas 
\[
\mathcal{A} = \{ U_{ij} : i \in \mathbb{N}, j = 0, \ldots, \dim M \}
\]
of \( M \) such that \( \{ U_{ij}\}_{i \in \mathbb{N}} \) are pairwise disjoint for all \( j = 0, \ldots, \dim M \). Let 
\[
\mathcal{A}_0 = \{ U_{ij}^0 : i \in \mathbb{N}, j = 0, \ldots, \dim M \} \quad \text{with} \quad U_{ij} = B_{\epsilon_{ij}}(U_{ij}^0)
\]
be a refinement of \( \mathcal{A} \). Relabeling the atlas if necessary, we assume that \( x \in U_{10}^0 \). Lemma 3.8 provides (in local coordinates) \( C^r \)-diffeomorphisms \( T_1(\epsilon_{10}), \ldots, T_m(\epsilon_{10}) \) of \( U_{10} \), and an open ball \( B(\epsilon_{10}) \) centered at \( x \) such that \( T(\epsilon_{10})|_{\partial U_{10}} = \text{id} \) for all \( \ell = 1, \ldots, m \) and 
\[
\overline{U_{10}^0} \subset \langle T_1(\epsilon_{10}), \ldots, T_m(\epsilon_{10}) \rangle^+(B(\epsilon_{10})).
\]
For each \( U_{ij}^0 \) such that \( U_{ij}^0 \cap U_{ij}^{0} \neq \emptyset \), we take a point in this intersection and apply again Lemma 3.8 obtaining now new \( C^r \)-diffeomorphisms \( T_1(\epsilon_{ij}), \ldots, T_m(\epsilon_{ij}) \) of \( U_{ij} \) and an open ball \( B(\epsilon_{ij}) \subset U_{ij}^0 \cap U_{ij}^0 \) such that 
\[
T(\epsilon_{ij})|_{\partial U_{ij}} = \text{id} \quad \text{for all} \quad \ell = 1, \ldots, m \quad \text{and} \quad \overline{U_{ij}^0} \subset \langle T_1(\epsilon_{ij}), \ldots, T_m(\epsilon_{ij}) \rangle^+(B(\epsilon_{ij})). \tag{10}
\]
Let \( U_{ij}^0 \) be one of the open sets chosen from the preceding step. Consider the open sets \( U_{ij}^0 \) such that \( U_{ij}^0 \cap U_{ij}^0 \neq \emptyset \) and which were not considered in the previous step. Repeat the whole process by choosing a point in the intersection and applying Lemma 3.8. By induction, for all \( i \in \mathbb{N} \) and \( j = 0, \ldots, \dim M \), there exist \( C^r \)-diffeomorphisms \( T_1(\epsilon_{ij}), \ldots, T_m(\epsilon_{ij}) \) of \( U_{ij} \) and an open ball \( B(\epsilon_{ij}) \subset U_{ij}^0 \) satisfying (10).

For each \( \ell = 1, \ldots, m \) and \( j = 0, \ldots, \dim M \), we define the \( C^r \)-diffeomorphisms \( T_{\ell j} \) of \( M \) by 
\[
T_{\ell j}|_{U_{ij}^0} = T(\epsilon_{ij}) \quad \text{for all} \quad i \in \mathbb{N} \quad \text{and} \quad T_{\ell j} = \text{id} \quad \text{in} \quad M \setminus \bigcup_{i \in \mathbb{N}} U_{ij}.
\]
Observe that \( T_{\ell j} \) is well defined since for each \( j \), \( U_{ij} \), \( i \in \mathbb{N} \) are pairwise disjoint open sets and \( T(\epsilon_{ij}) \) restricted to \( \partial U_{ij} \) is equal to the identity for all \( \ell = 1, \ldots, m \). Moreover, \( \langle T_{\ell j} : \ell = 1, \ldots, m, j = 0, \ldots, \dim M \rangle^+ \) has forward globalization of the neighborhood \( B(\epsilon_{10}) \) of any \( x \). Similarly by the same procedure, we can assume that this semigroup also has backward globalization of \( B(\epsilon_{10}) \). Finally, if \( \epsilon_{ij} \to 0 \) then \( T_{\ell j} \) tends continuously to the identity and thus relabeling the maps, we have arcs of \( C^r \)-diffeomorphisms \( T_1 = T_1(\epsilon), \ldots, T_s(\epsilon), \epsilon \geq 0 \), \( s = (\dim M + 1)m \), satisfying the required properties. \( \square \)

### 3.2.3. Arcs of robustly transitive one-step maps

The previous proposition allows us to construct an arc of robustly transitive symbolic skew-products.

**Theorem 3.9.** Let \( M \) be a connected manifold of dimension \( c \geq 1 \). Then, there are an integer \( d \equiv d(c) \geq 3 \) and an arc of one-step maps \( \Phi_\epsilon = \tau \circ (\phi_1, \ldots, \phi_d) \) in \( \mathcal{PS}(M) \) isotopic to \( \Phi_0 = \tau \times \text{id} \) so that \( \Phi_\epsilon \) is \( S^1 \)-robustly topologically mixing for any \( \epsilon > 0 \).
Proof. Consider \( x \in M \). By means of an arbitrarily small perturbation of the identity map \([14, 15]\) we can create a map \( \phi \) for which \( x \) is a hyperbolic fixed point. Hence, applying Proposition 2.12 we can get arcs of \( C^r \)-diffeomorphisms \( \phi_1 \equiv \phi_1(\varepsilon), \ldots, \phi_k \equiv \phi_k(\varepsilon) \) homotopic to the identity as \( \varepsilon \to 0^+ \), where \( k \geq 2 \) and only depends on \( c \) and a \( cs \)-blending regions \( B \) in \( B_2(x) \). Without loss of generality, assume that \( B \) is a contracting/double-blending region. According to Proposition 3.7, there exist \( s \equiv s(c) \geq 3 \) arcs of \( C^r \)-diffeomorphisms \( T_1 \equiv T_1(\varepsilon), \ldots, T_s \equiv T_s(\varepsilon) \) homotopic to the identity as \( \varepsilon \to 0^+ \) so that \( \langle T_1, \ldots, T_s \rangle^+ \) has globalization of a small open ball \( B_0 \subset \mathcal{B} \). Adding these diffeomorphisms to the previous maps if necessary, \( B \) is a globalized contracting/double-blending region. Take \( d = s + k \), which only depends on \( c \), and set

\[
\Phi_k = \pi \circ (\phi_1, \ldots, \phi_k, T_1, \ldots, T_s) \in \mathcal{P}\mathcal{H}S(M).
\]

Hence \( \Phi_0 = \pi \circ \text{id} \) and according to Theorem 3.6, for any \( \varepsilon > 0 \), \( \Phi_\varepsilon \) is \( S^0 \)-robustly topologically mixing for any \( \varepsilon > 0 \). This completes the proof of the theorem. \( \square \)

4. Symplectic skew-products

By a symplectic manifold \( M \) we mean a manifold equipped with a closed non-degenerate differential two-form which is called the symplectic form. The nondegeneracy of the form implies that the space must be even-dimensional. We will create robust homoclinic tangencies inside semi-local transitive partially hyperbolic sets for symplectomorphisms, that is diffeomorphisms preserving the symplectic form. We will first show that the robust transitivity of Theorem 3.9 and robust tangencies constructed in [7] hold for symplectic symbolic skew-products, i.e., for symbolic skew-products where the fiber maps are symplectomorphisms. To do this, let us first recall the method to construct robust tangencies in symbolic skew-products.

4.1. Tangencies in symbolic skew-products. Following [7] we will introduce the notion of tangencies in symbolic skew-products and afterwards give a criterion for their construction.

4.1.1. The set of smooth symbolic skew-products. Since we will need to work with differentiable fiber maps, it will be useful to extend the set of symbolic skew-products to this setting.

Definition 4.1. For an integer \( r \geq 1 \), \( S^r(M) \equiv S^r_{x,\nu}(M) \) denotes the set of skew-products \( \Phi = \pi \circ \phi \) on \( M = \Sigma \times M \) such that there exist \( \gamma \equiv \gamma(\Phi) > 0 \), \( \gamma_0 \equiv \gamma(\Phi) > 0 \) and \( C_r \equiv C_r(\Phi) \geq 0 \) satisfying

- \( d\gamma(\phi_{\xi}^{\pm 1}, \phi_{\zeta}^{\pm 1}) \leq C_r d\gamma(\xi, \zeta)^\alpha \) for all \( \xi, \zeta \in \Sigma \) with \( \xi_0 = \zeta_0 \), and
- \( \phi_\xi \) are \( C^r \)-diffeomorphisms of \( M \) with \( D\phi_\xi^{\pm 1} \) Lipschitz (with uniform Lipschitz constant)

\[
\gamma < m(D\phi_\xi(x)) < ||D\phi_\xi(x)|| < \gamma^{-1} \text{ for all } (\xi, x) \in \Sigma \times X.
\]

In addition, \( \mathcal{P}\mathcal{H}S^r(M) = \mathcal{P}\mathcal{H}S^r_{x,\nu}(M) \equiv \mathcal{P}\mathcal{H}S(M) \cap S^r(M) \) for \( r \geq 1 \). Finally, a partially hyperbolic skew-product is said to be fiber bunched if \( \nu^\alpha < \gamma^\beta \).
We endow $S^r(M)$ with the metric
\[ d_{S^r}(\Phi, \Psi) \overset{\text{def}}{=} d_r(\Phi, \Psi) + \text{Lip}_r(\Phi, \Psi) + \text{Hol}_r(\Phi, \Psi) \]
where
\[ \text{Lip}_r(\Phi, \Psi) \overset{\text{def}}{=} \max_{\xi \in \Sigma} \left| \text{Lip}(D^r\phi_{\xi}^1) - \text{Lip}(D^r\psi_{\xi}^1) \right| \]
\[ d_r(\Phi, \Psi) \overset{\text{def}}{=} \max_{\xi \in \Sigma} d_C(\phi_{\xi}^1, \psi_{\xi}^1) \quad \text{and} \quad \text{Hol}_r(\Phi, \Psi) \overset{\text{def}}{=} \left| C_r(\Phi) - C_r(\Psi) \right|. \]

Hence $\mathcal{P}S^r(M)$ is an open set of $S^r(M)$ and $S^{r+1}(M) \subset S^r(M) \subset S(M)$ for $r \geq 0$.

4.1.2. Tangencies. To define the notion of a tangency for symbolic skew-products we first need to introduce the notion of a tangent direction.

**Definition 4.2** (tangent direction). Let $\Phi = \tau \times \phi_{\xi} \in S^r(M)$ be a symbolic skew-product with a pair of transitive hyperbolic sets $\Gamma^1$ and $\Gamma^2$ and suppose that $(\xi, x) \in W^u(\Gamma^1) \cap W^s(\Gamma^2)$. A unitary vector $v \in T_xM$ is called a tangent direction at $(\xi, x)$ if there are $C > 0$ and $0 < \lambda < 1$ such that
\[ \|D\phi_{\xi}^r(x)v\| \leq C\lambda^n \quad \text{for all} \ n \in \mathbb{Z}. \]
The maximum number of independent tangent directions at $(\xi, x)$ is denoted by $d_T \equiv d_T(\xi, x)$.

Now we are ready to give the definition of a tangency.

**Definition 4.3** (tangency). We say that $\Phi \in S^r(M)$ has a (bundle) tangency of dimension $\ell > 0$ between $W^u(\Gamma^1)$ and $W^s(\Gamma^2)$ if there exists $(\xi, x) \in W^u(\Gamma^1) \cap W^s(\Gamma^2)$ such that
\[ \ell = d_T(\xi, x) \quad \text{and} \quad \text{ind}^a(\Gamma^1) + \text{ind}^c(\Gamma^2) - \ell < c. \]
If $\Gamma^1 = \Gamma^2$, the tangency is called homoclinic, and otherwise heteroclinic. The tangency (of dimension $\ell$) is said to be $S^r$-robust if for any small enough $S^r$-perturbation $\Psi$ of $\Phi$ has a tangency (of dimension $\ell$) between the unstable set $W^u(\Gamma^1_\Psi)$ and the stable set $W^s(\Gamma^2_\Psi)$.

The codimension of the tangency is defined as $c_T = c - [\text{ind}^a(\Gamma^1) + \text{ind}^c(\Gamma^2) - \ell]$.

For the rest of this section, we will work in local coordinates and thus may assume that $M = \mathbb{R}^c$ with $c \geq 2$.

4.1.3. Cone fields in symbolic skew-products. Consider an integer $1 \leq \ell \leq c$. An $\ell$-dimensional vector subspace of $\mathbb{R}^c$ is called a $\ell$-plane. The Grassmannian manifold $G(\ell, c)$ is defined as the set of $\ell$-planes in $\mathbb{R}^c$. A standard $\ell$-cone in $\mathbb{R}^c$ is a set of the form
\[ \mathcal{C} = \{ (v, w) \in \mathbb{R}^c : v \in \mathbb{R}^\ell \text{ and } \|w\| \leq \rho \|v\| \text{ for some } \rho > 0 \}. \]
More generally, a $\ell$-cone is the image of a standard $\ell$-cone under an invertible linear map. In fact, any $\ell$-cone $\mathcal{C}$ in $\mathbb{R}^c$ induces an open set in $G(\ell, c)$, which we will continue denoting by $\mathcal{C}$.

**Definition 4.4** (stable and unstable cones). Let $\Phi = \tau \times \phi_{\xi} \in S^0(\mathbb{R}^c)$ and consider an open set $\mathcal{B}$ of $\mathcal{M} = \Sigma \times \mathbb{R}^c$. An $\ell$-cone $\mathcal{C}^{uu}$ in $\mathbb{R}^c$ is said to be unstable for $\Phi$ on $\mathcal{B}$ if there is $0 < \lambda < 1$ such that
\[ D\phi_{\xi}(x)\mathcal{C}^{uu} \subset \text{int}(\mathcal{C}^{uu}) \quad \text{and} \quad \|D\phi_{\xi}(x)v\| \geq \lambda^{-1}\|v\| \quad \text{for all} \ v \in \mathcal{C}^{uu}, (\xi, x) \in \mathcal{B} \cap \Phi^{-1}(\mathcal{B}). \]
Similarly we define the stable $\ell$-cone $\mathcal{C}^{ss}$ for $\Phi$ on $\mathcal{B}$. 
4.1.4. Tangencies in one-step maps. Let $\Phi = \tau \times (\phi_1, \ldots, \phi_d) \in \mathcal{PH}S^1(\mathbb{R}^c)$ be a fiber bunched one-step map. That is,

$$\nu^a < \gamma < m(D\phi_i) < \|D\phi_i\| < \gamma^{-1} < \nu^{-a} \quad \text{with} \quad \gamma > \nu^a \quad \text{for all} \quad i = 1, \ldots, d.$$

We fix $0 < \ell < c$ and denote

$$\hat{\phi}_i(x, E) = (\phi_i(x), D\phi_i(x)E), \quad (x, E) \in \hat{\mathbb{M}} \overset{\text{def}}{=} \mathbb{R}^c \times G(\ell, c), \quad \text{for} \quad i = 1, \ldots, d.$$

In order to provide a criterion to get robust tangencies we need the following definitions:

**Definition 4.5** (blending region with tangency). Let $B, D$ be bounded open sets of $\mathbb{R}^c$. We say that a cs-blending region, $B$, with respect to $\{\phi_1, \ldots, \phi_d\}$ on $D$ has a tangency of dimension $\ell$ if there exist bounded open sets $\hat{B}$ and $\hat{D}$ in $\mathbb{M}$ such that

$$\hat{B} \text{ is a cs-blending region with respect to } \{\hat{\phi}_1, \ldots, \hat{\phi}_d\} \text{ on } \hat{D} \text{ so that } \hat{B} \subset B \times c^{uu}.$$

Here $c^{uu}$ is an unstable $\ell$-cone for $\hat{\Phi} = \tau \times (\hat{\phi}_1, \ldots, \hat{\phi}_d)$ on $\Sigma \times B$. We say that the set $\hat{B}$ is a $\ell$-tangency (of the blending region $B$). Similarly, we define a cu-blending region with a tangency of dimension $\ell$.

Let $A_1$ and $A_2$ be two subsets of $\hat{\mathbb{M}}$.

**Definition 4.6** (transition). We say that the semigroup $\langle \hat{\phi}_1, \ldots, \hat{\phi}_d \rangle^+$ has a transition from $A_1$ to $A_2$ if there exist a point $x \in A_1$ and a map $T \in \langle \hat{\phi}_1, \ldots, \hat{\phi}_d \rangle^+$ such that $T(x) \in A_2$.

Using these terminologies, the following result from [7, Corollary 5.15] provides a criterion to construct robust tangencies:

**Theorem 4.7.** Let $\Phi = \tau \times (\phi_1, \ldots, \phi_d) \in \mathcal{PH}S^1(\mathbb{R}^c)$ be fiber bunched one-step map with small enough Lipschitz constant of $D\phi_i^\pm_1$. Consider integers

$$0 < i_1, i_2 < c \quad \text{and} \quad \max\{0, i_2 - i_1\} < \ell \leq \min\{c - i_1, i_2\}.$$

Suppose that there are bounded open sets $D_1, D_2, B_1, B_2$ of $\mathbb{R}^c$ such that with respect to $\{\phi_1, \ldots, \phi_d\}$,

- $B_1$ is a cs-blending region on $D_1$, with cs-index $i_1$ and a $\ell$-tangency $\hat{B}_1$;
- $B_2$ is a cu-blending region on $D_2$, with cs-index $i_2$ and a $\ell$-tangency $\hat{B}_2$;
- $\langle \hat{\phi}_1, \ldots, \hat{\phi}_d \rangle^+$ has a transition from $\hat{B}_1$ to $\hat{B}_2$.

Then $\Phi$ has a $S^1$-robust tangency of dimension $\ell$ between $W^u(\Gamma_1)$ and $W^s(\Gamma_2)$ where $\Gamma_1$ and $\Gamma_2$ are the maximal invariant sets in $\Sigma \times \overline{D_1}$ and $\Sigma \times \overline{D_2}$.

4.2. Symplectic perturbations. We would need the perturbative tools stated below. The following two remarks deal with local perturbations and are done in local Darboux charts, which are coordinates in which the symplectic form is written in the canonical way.

**Remark 4.8** (Pasting lemma). A symplectic pasting lemma [3, Lemma 3.9] states that given a $C^r$-symplectic map with a periodic point, one can locally around the periodic point, glue any other $C^r$-close symplectic map. Thus for example, one can modify the identity map to obtain locally any linear symplectic matrix.
The fact described in the next remark (see also [21, Proof of Theorem 3.16: perturbations]), says that one can locally perturb any symplectomorphism by translations.

**Remark 4.9** (Perturbations by translations). Let \( z \) be a point in \( M \). Using the Theorem of Darboux we select local coordinates in a neighborhood \( U \) of \( p \) of the form \((x_1, \ldots, x_n; y_1, \ldots, y_n)\) such that on \( U \) the symplectic form is \( dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n \). Consider a symplectic diffeomorphism \( \phi \) of \( M \) and let \( U_0 \) be a small bounded connected open set so that \( z \in U_0 \) and the closure of \( \phi(U_0) \) and \( U_0 \) are contained in \( U \). Given a sufficiently small vector \((u, v) = (u_1, \ldots, u_n; v_1, \ldots, v_n)\) in Darboux coordinates, let the Hamiltonian \( H_{(u, v)} \) on \( M \) be a bump function such that \( H_{(u, v)}(z) = 0 \) if \( z \in M \setminus U \) and if \( z \in U_0 \), \( H_{(u, v)} \) is expressed in the local coordinates as \( H_{(u, v)}(x, y) = vx - uy \). Observe that the time-one map \( T_{(u, v)} \) of the Hamiltonian flow \( H_{(u, v)} \) in \( U_0 \) is the translation in local coordinates \((x, y) \mapsto (x, y) + (u, v)\) and the identity in \( \partial U \). Then, we define the symplectic perturbation of \( \phi \) as \( \tilde{\phi} = T_{(u, v)} \circ \phi \).

Let \( \omega_N \) and \( \omega_M \) be the symplectic two-forms on the symplectic manifolds \( N \) and \( M \). Call \( \pi_N \) and \( \pi_M \) the projections of \( N \times M \) onto \( N \) and \( M \). Then a symplectic form on \( N \times M \) is given by \( \omega = \pi_N^* \omega_N + \pi_M^* \omega_M \) (\( \pi^* \) indicating the pull-back). The following lemma allows for semi-local perturbations, and will use the fact that we are dealing with skew-product symplectomorphisms defined on a product manifold \( N \times M \).

**Lemma 4.10** (semi-local perturbations). Consider \( \phi \), a \( C^r \)-symplectomorphism of \( M \), with the additional hypothesis that \( \phi \) is \( C^r \)-symplectically isotopic to the identity. Let \( F \) be a \( C^r \)-symplectomorphism of \( N \) and \( U_1, U_2 \) be small enough neighborhoods contained in some Darboux chart such that \( \overline{U_1} \subset U_2 \). Then there exists a \( C^r \)-symplectomorphism \( f \) of \( N \times M \) such that \( f|_{U_1 \times M} = F \times \phi \) and \( f|_{U_2 \times M} = F \times \text{id} \).

**Proof.** Let \( V \) be the Darboux chart of \( N \) so that \( U_1, U_2 \subset V \). We can suppose that \( V \) is, in fact, an open ball around the origin in \( \mathbb{R}^n \) where \( n \) is the dimension of \( N \). We take open balls of radius \( r_1 < r_2 < r_3 \) such that \( U_1 \subset B_{r_1}(0) \subset B_{r_2}(0) \subset U_2 \subset B_{r_3}(0) \subset V \).

Let \( \phi : M \to M \) be \( C^r \)-symplectically isotopic to the identity, and assume that the isotopy is defined in the interval \([0, r_3]\), writing \( \phi(s, y) \), for \( s \in [0, r_3] \). Moreover we can reparametrize so that \( \phi(s, \cdot) = \phi(\cdot) \) for \( s \in [0, r_1] \) and \( \phi(s, \cdot) = \text{id} \) for \( s \in [r_2, r_3] \). Consider a map \( f \) defined by
\[
f(x, y) = (F(x), \phi(|x|), y) \quad (x, y) \in U_2 \times M.
\]
Notice that \( f|_{U_1 \times M} = F \times \phi \) and \( f|_{U_2 \times M} = F \times \text{id} \). This map can be extended \( C^r \)-globally to \( N \times M \) simply by defining it as \( f = F \times \text{id} \) outside \( U_2 \times M \).

It is left to see that \( f \) preserves the form \( \omega \), and it is enough to show this locally in product Darboux coordinates \( V \times W \subset N \times M \). We can write the form \( \omega_N \) in \( V \) as \( \sum dx_i \wedge dy_i \) and the form \( \omega_M \) in \( W \) as \( \sum dy_i \wedge dx_j \), and then \( \omega = \sum dx_i \wedge dx_j + \sum dy_i \wedge dy_j \) where \((x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)\) are the coordinates in \( V \times W \). Then
\[
f^* \omega = \sum \det \left( \frac{\partial f_i}{\partial x_j} \right) dx_i \wedge dx_j + \sum \det \left( \frac{\partial f_i}{\partial y_j} \right) dy_i \wedge dy_j.
\]
Since $F$ is symplectic then $\det(\frac{\partial f}{\partial x}) = \det(\frac{\partial F}{\partial x}) = 1$. For the other hand,

$$\det(\frac{\partial f}{\partial y}) = \det(\frac{\partial \phi}{\partial y}),$$

where the index $k > n$.

Since for any $x$ fixed, the map $\phi(||x||, \cdot)$ preserves the form $\sum dy_k \wedge dy_\ell$, this means that writing $D\phi = (\partial_x \phi, \partial_y \phi)$, the submatrix $\partial_y \phi$ is symplectic. Going back this implies that $\det(\frac{\partial f}{\partial y}) = 1$, when $k > n$, and thus $f^* \omega = \omega$. 

4.3. Tangencies and transitivity for symplectic symbolic skew-products. In this subsection we want to get the following result:

**Theorem 4.11.** Let $M$ be a symplectic manifold of dimension $c \geq 2$ and consider a integer $0 < \ell \leq c/2$. Then, there are $d = d(c) \geq 2$ and an arc of symplectic one-step maps $\Phi_\epsilon = \tau \times (\phi_1, \ldots, \phi_d)$ in $\mathcal{P}\mathcal{S}'(M)$ isotopic to $\Phi_0 = \tau \times \text{id}$ so that for any $\epsilon > 0$

- $\Phi_\epsilon$ is $S^0$-robustly topologically mixing;
- $\Phi_\epsilon$ has a $S^1$-robust tangency of dimension $\ell$.

In order to prove the above theorem, according to Theorem 4.7 and Corollary 3.6 we need to create arcs of IFSs generated by symplectomorphisms having blending regions with tangency, transitions and globalization. Namely, fixing an integer $0 < \ell \leq c/2$, we need to prove the following result:

**Proposition 4.12.** There are $d = d(c) \geq 3$ and arcs of $C^\ell$-symplectomorphisms of $M$

$$\phi_1 \equiv \phi_1(\epsilon), \ldots, \phi_d \equiv \phi_d(\epsilon), \quad \epsilon \geq 0, \quad \text{with } \phi_i(0) = \text{id}$$

and having the following properties: for any $\epsilon > 0$, there are bounded open sets $B_1 \equiv B_1(\epsilon)$ and $B_2 \equiv B_2(\epsilon)$ of $M$ such that with respect to $\{\phi_1, \ldots, \phi_d\}$,

- $B_1$ is a double-blending region with cs-index $\ell$ and having a $\ell$-tangency $\hat{B}_1$;
- $B_2$ is a cu-blending region with cu-index $\ell$ and having a $\ell$-tangency $\hat{B}_2$;
- $\langle \phi_1, \ldots, \phi_d \rangle^+$ has transition from $\hat{B}_1$ to $\hat{B}_2$ and/or transition from $\hat{B}_1$ to $\hat{B}_1$;
- $B_1$ is globalized by $\langle \phi_1, \ldots, \phi_d \rangle^+$.

Now let us show the existence of such symplectic arcs. We will always work locally inside Darboux charts. To create local blending regions $B_1$ and $B_2$, one can take a symplectic map with a local fixed hyperbolic point and compose the map with the necessary local translations as described in the Remark 4.9 where the directions of the translations come from Proposition 2.12. The hardest work is to obtain a $\ell$-tangency for these local blending regions. First we need some basic facts from symplectic geometry:

A symplectic vector subspace is one on which the symplectic form still restricts to a non-degenerate two-form. By a coisotropic vector subspace is understood a subspace of $\mathbb{R}^c$ which contains its symplectic complement.
Lemma 4.13. Let $\phi$ be a hyperbolic linear map of $\mathbb{R}^c$ that preserves the symplectic form, i.e., a symplectic matrix, having a splitting in eigenspaces at the fixed point $p = 0$ of the form $E^s \oplus E^{cu} \oplus E^{uu}$ where the unstable direction is $E^{uu} = E^{cu} \oplus E^{uu}$ and $0 < \ell = \dim E^{uu} < c$. If $\ell$ is odd we ask that $E^{uu}$ is a cs-index of translations $\phi$ of $G$ is a cs-blending region with respect to $A$ and so is also a symplectic subspace. As symplectic matrices act transitively on symplectic $A$-tangency of translations $\phi$ of $G$ is a cs-blending region with respect to $A$ and so is also a symplectic subspace if $\ell$ is even.

Then, there are

i) an unstable $\ell$-cone $\mathcal{E}^{uu}$ around $E^{uu}$,

ii) neighborhoods $B$ of $p$ in $\mathbb{R}^c$ and $G \subset \mathcal{E}^{uu}$ of $E^{uu}$ in $G(\ell, c)$, and

iii) symplectic orthogonal matrices $A_1, \ldots, A_d$ and vectors $c_1, \ldots, c_d$ in $\mathbb{R}^c$ with $d = d(c) \geq 2$ such that

$$\hat{B} = B \subset \bigcup_{i=1}^{d} \hat{\phi}_i(B)$$

and $\hat{\phi}_i$ are the induced maps on $\hat{M} = \mathbb{R}^c \times G(\ell, c)$ by the symplectomorphisms $\phi_i = A_i \cdot \phi + c_i$. That is, $B$ is a cs-blending region with respect to $\{\phi_1, \ldots, \phi_d\}$ of cs-index $\dim E^s$ and a $\ell$-tangency. Moreover, the maps $\phi_1, \ldots, \phi_d$ can be constructed as an arc of symplectomorphisms homotopic to $\phi$.

Proof. As in Proposition 2.12, we can take a small neighborhood $B$ of $p$ in $\mathbb{R}^c$ and choose arcs of translations $f_i = \phi + c_i$ homotopic to $\phi$ with $c_i \in \mathbb{R}^c$ in order to have that $B$ is a cs-blending region with respect to $\{f_1, \ldots, f_k\}$. In particular, $\{f_i(B) : i = 1, \ldots, k_1\}$ is an open cover of $B$. Notice that there exists an $\epsilon > 0$, such that the above covering property holds for any $\epsilon$-perturbation $\phi_i$ of $f_i$.

On the other hand, $D\phi(p)$ induces a map $A$ on $G(\ell, c)$ given by $A(E) = D\phi(p)E$. Since $\phi$ is $C^2$, this map has a hyperbolic attracting fixed point $E^{uu}$. Then, as in Proposition 2.12, there are arcs $T_1 = T_1(\epsilon), \ldots, T_{k_2} = T_{k_2}(\epsilon)$ of translations on $G(\ell, c)$ homotopic to the identity and a neighborhood $G \equiv G(\epsilon)$ of $E^{uu}$ in $G(k, c)$, such that $\{T_j(G) : j = 1, \ldots, k_2\}$ is an open cover of $G$ where $F_j = T_j \circ \phi$. Each translation map $F_j$ in the Grassmannian corresponds to a map of the form $A_j \cdot D\phi(p)$ in the tangent bundle where $A_j$ is an orthogonal matrix and $\|A_j - \text{id}\| \to 0$ as $\epsilon \to 0$ for all $j = 1, \ldots, k_2$. Moreover, we can take $G$ small enough so that $G \subset \mathcal{E}^{uu}$ where $\mathcal{E}^{uu}$ is an unstable $\ell$-cone around $E^{uu}$ for $\phi$. Taking $\|A_j - \text{id}\| < \epsilon / \|\phi\|$ and $\phi_{ij} = A_j \cdot \phi + c_i$ then $\|\phi_{ij} - f_i\| < \epsilon$ and so

$$\hat{B} \subset \bigcup_{i=1}^{k_1} \bigcup_{j=1}^{k_2} \phi_{ij}(B).$$

The translations $c_i$ preserve the symplectic form, and so we only need to show the matrix $A_j$ can be taken symplectic. Suppose for now that $\ell$ is even and so $E^{uu}$ is a symplectic subspace. Observe that the symplectic subspaces form an open and dense subset in the set of $\ell$-planes. Then we can take the $\ell$-cone $\mathcal{E}^{uu}$ around $E^{uu}$ in $G(\ell, c)$ as an open set of symplectic subspaces. Since $A_j$ is arbitrarily close to the identity, we may assume that $E = A_j E^{uu}$ is in $\mathcal{E}^{uu}$ and so is also a symplectic subspace. As symplectic matrices act transitively on symplectic vector subspaces [24, Cap. V], we can take a symplectic matrix $\hat{A}_j$, close to the identity, such
that $\tilde{A}_j E^{uu} = E$. Without loss of generality we assume $\tilde{A}_j = A_j$. If $\ell$ is an odd number, we proceed similarly. Since coisotropic subspaces are an open and dense set of $G(\ell, c)$ where the symplectic matrices also act transitively [19, Exer. 2.33], then in the same manner as before, we can take $A_j$ as a symplectic matrix.

Finally, by construction, the maps $\phi_{ij}$ induce a set of maps $\hat{\phi}_{ij}$ on $\hat{M}$ so that $\{\hat{\phi}_{ij}(B)\}$ is an open cover of the closure of $\hat{B} = B \times G$. Moreover, since $A_j$ is an orthogonal matrix then the hyperbolicity of the blending regions still holds. That is, $B$ is also a $cs$-blending region with respect to $\{\phi_{ij} : i = 1, \ldots, k_1, j = 1, \ldots, k_2\}$ with $cs$-index equal to $\dim E^s$ and as well is a $\ell$-tangency (the $cs$-blending region $\hat{B}$ on $\hat{M}$). This completes the proof.

Notice that, by means of an arbitrarily small perturbation of the identity map, we can create a map $\phi$ for which $x$ is a hyperbolic fixed point. Thus the above construction can be done homotopic to the identity.

**Proof of Proposition 4.12.** To make the transition map from $\hat{B}_1$ to $\hat{B}_2$, the blending regions can be chosen inside the same Darboux chart in such a manner so that the transition map is simply a translation. On the other hand the transition map from $\hat{B}_1$ to $\hat{B}_1$, can be chosen as a local linear symplectic map that has a fixed point inside the double-blending region and takes the unstable cone into the stable one over several iterates, or for example maps $E^{uu}$ into $E^{ss}$. Recall that these subspaces can be chosen to be symplectic or coisotropic on which the symplectic matrices act transitively [24, Cap. V] and [19, Exer. 2.33].

To globalize the blending region $B_1$, observe that the $C^\infty$-diffeomorphisms constructed to prove Theorem A are locally small translations. Thus again using Darboux coordinates and a Hamiltonian bump function as in the Remark 4.9, one can glue the maps required for the Lemma 3.8 and Proposition 3.7. The Milnor atlas required for the globalization has to be taken subordinate to the Darboux atlas.

The proof of Theorem 4.11 is now complete.

### 4.4. Symplectic realization: proof of Theorem A.

Recall that our goal is to construct arcs of $C^\infty$-symplectomorphisms $f_\epsilon$ of $N \times M$ isotopic to $f_0 = F \times id$ having robust homoclinic (or equidimensional) tangencies of codimension $\ell \leq \dim(M)/2$. From Theorem 4.11, suppose $\Phi_\epsilon = \tau \cdot (\phi_{i_1}, \ldots, \phi_{i_d}) \in \text{PSH}^1(M)$ is an arc of robustly transitive symplectic symbolic skew-products with a robust homoclinic tangency of dimension $\ell$. It is desirable to obtain a one-parameter family of diffeomorphisms $f_\epsilon$ satisfying $f_\epsilon|_{R_i \times M} = F \times \phi_i$ for $i = 1, \ldots, d$ where $R_i$ is the Markov partition of the set $\Lambda$. The Markov partition must be small enough so that each rectangle $R_i$ is inside the Darboux chart $U_i$. If the initial set $\Lambda$ is conjugated to a shift of $d$ symbols, the pieces of the new partition would correspond to cylinders $C_i$ in the shift. A cylinder consists of all sequences with a given block of indices fixed around the zeroth coordinate, the length of the cylinder being the size of the given block. With each $U_i$ we associate the map $\phi_{k_i}$ where the index $k_i$ corresponds to the zeroth index of the cylinder $C_i$. 
Since $U_i$ are disjoint, we may apply Lemma 4.10 inductively with respect to the maps $\phi_k$, and thus obtain a symplectomorphism $f_\varepsilon$ in $N \times M$ satisfying

$$f_\varepsilon|_{R_i \times M} = F \times \phi_k, \quad f_\varepsilon = F \times \text{id outside of } \cup(U_i \times M).$$

Then by construction the dynamics over $\Lambda \times M$ will be conjugated to $\Phi_\varepsilon$. Using this conjugation one can obtain robust tangencies and transitive sets for the map $f_\varepsilon$, exactly as was proven in [7, Sec. 6.1]. This concludes the proof of Theorem A.

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