Analytic Solution of the Starobinsky Model for Inflation

Andronikos Paliathanasis\textsuperscript{1,2,}\textsuperscript{*}

\textsuperscript{1} Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile, Valdivia, Chile
\textsuperscript{2} Institute of Systems Science, Durban University of Technology, PO Box 1334, Durban 4000, Republic of South Africa

We prove that the field equations of the Starobinsky model for inflation in a Friedmann-Lemaître-Robertson-Walker constitute an integrable system as the field equations pass the singularity test. The analytical solution in terms of a Painlevé Series for the Starobinsky model is presented for the case of zero and nonzero spatial curvature. In both cases the leading-order term describes the radiation era provided by the corresponding higher-order theory.

PACS numbers: 98.80.-k, 95.35.+d, 95.36.+x

Keywords: Cosmology; $f(R)$-gravity; Starobinsky model; Integrability

1. INTRODUCTION

In the so-called modified/extended theories of gravity\textsuperscript{1} new dynamical terms, of geometric origin, are introduced which force the evolution of the gravitational field equations in order to explain various phenomena which were raised by recent observations\textsuperscript{2,3}. However, in the modified gravitational theories the new terms increase the complexity of the field equations and even in the simplest models, such as that of an isotropic and homogeneous universe, the existence of an analytical solution is not obvious. Although numerical methods can be applied to approximate the evolution of the field equations that is not sufficient for the complete study of a theory; while the analysis of the critical points it is not sufficient to provide us with information for the evolution of a system far from the critical points. Consequently the existence of analytical solutions for the field equations has lead to the application of various techniques from the analysis of dynamical systems for the study of the integrability\textsuperscript{1}.

One of the simplest modifications of the Einstein-Hilbert Action which consider quantum corrections is the Starobinsky model of inflation\textsuperscript{9} with Action Integral

$$S = \int d^4x \sqrt{-g} (R + q R^2) + \int d^4x \sqrt{-g} L_m,$$

where $R^2$ describes the quantum-gravitational effects in the early universe and $L_m$ is the Lagrangian of the matter source. The latter Action Integral corresponds to the family of the so-called quadratic theories instance\textsuperscript{10–12}. The gravitational field equations are of fourth order and in the case of a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe with line element\textsuperscript{2}

$$ds^2 = -dt^2 + a^2(t) \left(dx^2 + dy^2 + dz^2\right)$$

are calculated to be

$$3 \left(\frac{\dot{a}}{a}\right)^2 - 54q \left(\frac{\dot{a}}{a}\right)^4 + 18q \left(2 \left(\frac{\dot{a}}{a}\right)^2 - \left(\frac{\dot{a}}{a}\right)^2 + 2 \left(\frac{\dot{a}}{a}\right)^2 a^{(3)}\right) = \rho_m$$

and

$$2 \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + 18q \left(\frac{\dot{a}}{a}^4 + \frac{\dot{a}}{a^2}\right) + 24q \left(\frac{\dot{a}}{a^2} a^{(3)} - 3 \left(\frac{\dot{a}}{a}\right)^2 \frac{\dot{a}}{a}\right) + 12q \frac{a^{(4)}}{a} = -p_m,$$

where $\rho_m, p_m$ are the energy density and the pressure of the matter source. In the case of the vacuum the field equations\textsuperscript{33-41} admit an unstable (special) nonsingular solution\textsuperscript{9}. Moreover it is important to mention that,

\textsuperscript{*}Electronic address: anpaliat@phys.uoa.gr

\textsuperscript{1} For the application of the invariant transformations in modified theories see for instance\textsuperscript{4–8} and references therein.

\textsuperscript{2} We have assumed that the lapse function in the FLRW line element is constant, i.e., $N(t) = 1$ and $a(t)$ denotes the scale factor.
when the equation of state parameter of the matter source, \( \rho_m, p_m \) is that of an ideal gas, i.e., \( p_m = (\gamma - 1) \rho_m \), then equation (4) can be integrated to (3), while in general the conservation law \( \dot{\rho}_m + 3H (\rho_m + p_m) = 0 \) exists.

The Action Integral (11) corresponds to the \( f(R) \)-theories of gravity [13], where \( f(R) = R + qR^2 \), while a more general consideration of the Starobinsky model is the\(^3\) \( f(R) = R + qR^n \) theory\(^4\) [28] or [29]. Exact solutions of power-law \( f(R) \) theories can be found in [30, 31]. The importance of the Starobinsky model (1) is that it provides an inflationary scenario which is favored by the observations [32]. Furthermore it has been shown that various models of inflation are identical with the Starobinsky model when the inflationary phase takes place [33] whereas the Mixmaster universe provides nonchaotic trajectories [34].

The introduction of a Lagrange Multiplier in \( f(R) \)-theories [22] can be used to reduce the order of the theory from a fourth-order to a second-order theory by increasing at the same time the number of degrees of freedom [14]. In particular a new field is introduced which is equivalent to that of a Brans-Dicke scalar field with zero Brans-Dicke parameter [15, 16] the so-called O’Hanlon theory [36]. Therefore the field equations in a FLRW background form a two-dimensional canonical Hamiltonian system which describes a particle moving in a flat space while the potential which forces the evolution of the system is related with the form of the \( f(R) \) function. Because the scalar field description is that of Brans-Dicke field the theory is defined in the Jordan frame. Hence under a conformal transformation a minimally coupled scalar field is defined and the theory is defined now in the Einstein frame. The two equations are not independent and derivation of (6) gives the fourth-order equation (7). There are various power-law \( f(R) \) theories can be found in [30, 31]. The importance of the Starobinsky model (1) is that it provides an inflationary phase of the universe, for a review see [40].

By using the property that the field equations describe a canonical Hamiltonian system various functions \( f(R) \) have been determined in which the field equations admit conservation laws which are linear or quadratic in the momentum \( p_a \) and independent variable \( u \).

A specific \( f(R) \) theory provides a de Sitter universe if there exists \( R = R_0 \) such that the Barrow-Ottowill [42] condition holds

\[
R_0 f’ (R_0) - 2f (R_0) = 0. \tag{5}
\]

It is straightforward to see that for arbitrary \( R_0 \), that is, \( R_0 \to R \), the latter condition can be seen as a first-order differential equation with solution the quadratic function \( f(R) = f_0 R^2 \), where \( f_0 \) is a constant of integration.

2. INTEGRABILITY OF THE FIELD EQUATIONS

In the case of the vacuum the field equations in \( f(R) = R^2 \) theory are

\[
2a^2 \dot{a} (3) - 3 (\dot{a})^4 - a \ddot{a} (a \ddot{a} - 2 \dot{a}^2) = 0 \tag{6}
\]

and

\[
2a^3 \dot{a} (4) + 4a^2 \dot{a} (3) + 3 (\dot{a})^4 + 3a \ddot{a} (a \ddot{a} - 4 (\dot{a})^2) = 0 \tag{7}
\]

The two equations are not independent and derivation of (6) gives the fourth-order equation (7). There are various ways in which equation (6) can be written as a first-order ordinary differential equation\(^6\). If we select the new dependent variable \( w = \frac{1}{a} \frac{du}{dv} \) and independent variable \( v \), where \( u = \dot{a}, \ v = a \) then equation (6) becomes the following Riccati equation

\[
2 \frac{dw}{dv} + 3w^2 + 2 \frac{w}{v} - 3 \frac{3}{v^2} = 0 \quad \tag{8}
\]

\(^3\) For reviews in \( f(R) \)-gravity see for instance [13, 17] while some observational constraints can be found in [18, 21].

\(^4\) There is a plethora of physical theories which has been inspired by the Starobinsky model of inflation such as in SUGRA or in other gravitational theories, for instance see [22, 27] and references therein.

\(^5\) For a discussion between these two frames see [37, 38] and references therein.

\(^6\) The field equations (6, 7) admit as point symmetries the \( \partial_t, \partial \theta \) and \( s \partial_s \) vector fields which form the \( \{2A_1 \otimes A_1 \} \) Lie algebra.
with solution \( w(v) = \frac{v^3 - v_0}{v(v^3 + v_0)} \), where \( v_0 \) is a constant of integration. Therefore it follows that \( \frac{H(t)}{H_0} = \left( \frac{v_0^2 a^{-\frac{3}{2}} + a^{\frac{3}{2}}}{} \right)^{\frac{1}{3}} \), where \( H(t) = \frac{\dot{a}}{a} \), and for initial conditions such that \( v_0 = 0 \) provides the closed-form solution \( a(t) \approx t^{-1} \). That is not the unique case. In order to see that consider now the new variables \( \{ x, y \} = \{ H(t), \frac{dH}{dt}(H(t)) \} \) equation (6) takes the form of the linear equation

\[
2 \frac{dy}{dx} - y + 6x^2 = 0
\]

with solution \( y(x) = -2x^2 + u_1 \sqrt{x} \), that is \( \int \frac{dH}{u_1 \sqrt{H - 2a}} = (t - t_0) \) where in the limit \( u_1 = 0 \), gives \( H(t) = \frac{1}{2(t - t_0)} \), that is, \( a(t) = a_0 \sqrt{(t - t_0)} \), which is an ideal gas solution which mimics radiation solution, while it is a singular (special) solution. This singular solution is used below in order to prove the integrability of the Starobinsky model. The existence of the radiation solution it is not a surprise in the sense that \( f(R) \)-gravity can provides always a radiation epoch in the evolution of the universe \[43\]. However the radiation solution have been investigated before in a higher-order theory which include the Starobinsky term as also other terms follows from the Gauss-Bonnet invariant in \[54\] \[56\]. Moreover the radiation solution in quadratic theories has been found that can describes a past isotropic singularity for the Bianchi I universe \[57\].

The method that we apply is that of the singularity analysis and specifically we follow the ARS algorithm \[14\] \[40\]. Singularity analysis is a powerful method which has been applied in cosmological studies for the reconstruction of the analytical solution of various models \[41\] \[47\] \[50\]. We omit the properties of the singularity analysis and we refer the reader to the extended review \[51\].

We continue by firstly applying the method for the quadratic theory \( f(R) = R^2 \) and consider now equation (7).

We find that the leading-order behavior is the power-law solution \( a(t) = a_0 \tau^{1/2} \), where \( \tau = t - t_0 \) and \( t_0 \) denotes the position of the singularity. The application of the ARS algorithm provides the resonances to be \( s_1 = -1 \), \( s_2 = 0 \), \( s_3 = \frac{1}{2} \) and \( s_4 = \frac{3}{2} \), which means that the analytic solution is expressed by the Right Painlevé Series \[52\]

\[
a(t) = a_0 \tau^{\frac{1}{2}} + a_1 \tau + a_2 \tau^{\frac{3}{2}} + a_3 \tau^2 + \sum_{i=4}^{\infty} a_i \tau^{\frac{i+1}{2}},
\]

where the constants of integration are the \( a_0 \), \( a_3 \), \( a_5 \) and the position of the singularity \( t_0 \). However, with the use of \[6\] we find that \( a_5 = 0 \), while the calculation of the first coefficient constants gives the solution to be

\[
a(t) = a_0 \tau^{\frac{1}{2}} + a_3 \tau^2 + \frac{19}{32} a_0 \tau^{\frac{5}{2}} + \frac{17}{264} (a_3)^3 \tau^5 + \sum_{j=10}^{\infty} a_j \tau^{\frac{j+1}{2}}.
\]

For the field equations of the Starobinsky model we apply the same algorithm and we find the same resonances as those of the quadratic model, which means that the analytic solution is given by expression (10) or specifically by calculation the first nine coefficient constants the solution is

\[
a(t) = a_0 \tau^{\frac{1}{2}} + a_3 \tau^2 - \frac{a_0}{72q} \tau^{\frac{5}{2}} + a_5 \tau^3 + \frac{19}{32} a_0 \tau^{\frac{7}{2}} - \frac{5 a_3}{252} \tau^4 + \left( \frac{41}{2592000q^2} + \frac{a_3 a_5}{16 a_0} \right) \tau^{\frac{9}{2}} + \left( \frac{17}{264} (a_3)^3 - \frac{a_5}{132q} \right) \tau^5 + \sum_{j=10}^{\infty} a_j \tau^{\frac{j+1}{2}},
\]

where the constants of integration are again the coefficients \( a_0 \), \( a_3 \), \( a_5 \) and the position of the singularity \( t_0 \), while the constraint equation \[4\] gives that \( a_5 = 0 \) or, if we assume the existence of a dust fluid, that is, \( p_m = 0 \) and \( \rho_m = \rho_m 0 a^{-3} \), it follows that \( \rho_m = \frac{315}{2} a_5 (a_0)^2 \). While in the latter scenario it is important to mention that the term \( a_0 t^{1/2} \) describes the leading-order behaviour.

From the values of the resonances it is easy to see that the radiation solution is an unstable solution\(^7\), while the field equations of the Starobinsky model for inflation in a spatially flat FLRW spacetime pass the singularity test and are integrable.

\(^7\) For a discussion on the relation between the values of the resonances and the stability of the leading-order behaviour see \[53\].
3. DISCUSSION

Singularity analysis is a powerful method to study the integrability of dynamical systems. However, it has a basic disadvantage in that it is coordinate dependent. That is the reason that the Starobinsky model did not pass the singularity analysis in the consideration of [41]. The reason is that in the space of variables \( \{a, R\} \), in which usually \( f(R) \)-gravity is referred, the leading-order behaviour \( a(t) = a_0t^2 \), provides a singular behaviour for only one of the dynamical variables while for the Ricci Scalar it is a constant. However, we overpassed that problem by working directly on the fourth-order differential equation and without using the Lagrange Multiplier.

We now consider the case of nonzero spatially curved spacetime. Hence for the Action Integral (1) the field equations are derived to be

\[
\rho_m a^3 = 3\left(\frac{\dot{a}}{a}\right)^2 - 54q\left(\frac{\dot{a}}{a}\right)^4 + 18q\left(2\left(\frac{\dot{a}}{a}\right)^2 - \left(\frac{\ddot{a}}{a}\right)^2 + 2\left(\frac{\dot{a}}{a}\right)^2 a^{(3)}\right) + \\
+ \frac{k}{2a^2} - 6qk\left(\frac{\dot{a}}{a}\right)^2 + \frac{qk^2}{a^4}
\]

\[
0 = 2\dot{a} + \left(\frac{\dot{a}}{a}\right)^2 + 18q\left(\frac{\dot{a}}{a}\right)^4 + 24q\left(\frac{\dot{a}}{a}\right)^2 a^{(3)} - 3\left(\frac{\ddot{a}}{a}\right)^2 + 12qa^{(4)} + \frac{k}{6a^2} + 2qk\left(\frac{\dot{a}}{a}\right)^2 - 2\frac{\dddot{a}}{a} - \frac{qk^2}{6a^4}.
\]

(13)

(14)

where for the matter source we assumed that of a dust fluid.

We apply the ARS algorithm and we find that the solution is expressed again by the Right Painlevé Series (11) where now the coefficient constants depend also upon the curvature \( k \). For instance the first terms of the solution are

\[
a(t) = a_0\tau^2 - \frac{k}{12a_0}\tau^3 + a_3\tau^3 + \left(\frac{a_0}{72q} + \frac{k^2}{288(a_0)^2}\right)\tau^3 + a_5\tau^3 + \sum_{r=6}^{\infty} \bar{a}_j\tau^{\frac{1+r}{2}}
\]

(15)

where from (13) follows \( \rho_{m0} = \frac{315}{4}qa_5(a_0)^2 + 30a_3qk. \)

We conclude that the Starobinsky model for inflation in a FLRW spacetime with or without spatial curvature it is an integrable system. Last but not least from the singularity analysis we found that the radiation era is described by an unstable point which is in agreement with the dynamical analysis for a higher-order theory [54, 55].

Acknowledgments

The author acknowledges financial support of FONDECYT grant no. 3160121 and thanks the Durban University of Technology for the hospitality provided while part of this work was performed.

[1] T. Clifton, P.G. Ferreira, A. Padilla and C. Skordis, Phys. Rep. 513, 1 (2012)
[2] S. Capozziello, Int. J Mod. Phys. D 11, 483 (2002)
[3] K. Koyama, Rept. Prog. Phys. 79, 046902 (2016)
[4] S. Capozziello, E. Piedipalumbo, C. Rubano and P. Scudellaro, Phys. Rev. D. 80, 104030 (2009)
[5] Y. Zhang, Y.-G. Gong, Z.-H. Zhu, Phys. Lett. B 688, 13 (2010)
[6] B. Vakili, Phys. Lett. B 664, 16 (2008)
[7] A. Paliathanasis, M. Tsamparlis and S. Basilakos, Phys. Rev. D 84, 123514 (2011)
[8] A. Paliathanasis, Class. Quantum Gravit. 33, 075012 (2016)
[9] A.A. Starobinsky, Phys. Lett. B 91, 99 (1980)
[10] H. Nariai and K. Tomita, Prog. Theor. Phys. 46, 776 (1971)
[11] G.V. Bicknell, J. Phys. A.: Math. Nucl. Gen. 7, 1061 (1974)
[12] J.D. Barrow, Nucl. Phys. B 296, 679 (1988)
[13] H.A. Buchdahl, Mon. Not. Roy. Astron. Soc. 150, 1 (1970)
[14] B. Whitt, Phys. Lett. B 145, 175 (1984)
