Perfect state transfer over distance-regular spin networks

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Abstract

By considering distance-regular graphs as spin networks, first we introduce some particular spin Hamiltonians which are extended version of those of Refs.[1, 2]. Then, by using spectral analysis techniques and algebraic combinatoric structure of distance-regular graphs such as stratification introduced in [4, 5] and Bose-Mesner algebra, we give a method for finding a set of coupling constants in the Hamiltonians so that a particular state initially encoded on one site of a network will evolve freely to the opposite site without any dynamical controls, i.e., we show that how to derive the parameters of the system so that perfect state transfer (PST) can be achieved. As examples, the cycle networks with even number of vertices and $d$-dimensional hypercube networks are considered in details and the method is applied for some important distance-regular networks in appendix.

Keywords: Perfect state transfer, Spin networks, Association scheme, Stratification, Distance-regular network

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1 Introduction

The transfer of a quantum state from one part of a physical unit, e.g., a qubit, to another part is a crucial ingredient for many quantum information processing protocols [6]. Currently, there are several ways of moving data around in a quantum computer. While some methods transfer quantum states by moving them down a linear array of qubits, there are others which exploit the quantum property of entanglement for teleporting quantum states between distant qubits [7, 8]. In a quantum-communication scenario, the transfer of quantum states from one location $A$ to another location $B$, is rather explicit, since the goal is the communication between distant parties $A$ and $B$ (e.g., by means of photon transmission). Equally, in the interior of quantum computers good communication between different parts of the system is essential. The need is thus to transfer quantum states and generate entanglement between different regions contained within the system. There are various physical systems that can serve as quantum channels, one of them being a quantum spin system. This can be generally defined as a collection of interacting qubits (spin-1/2 particles) on a graph, whose dynamics is governed by a suitable Hamiltonian, e.g., the Heisenberg or $XY$ Hamiltonian.

Quantum communication over short distances through a spin chain, in which adjacent qubits are coupled by equal strength has been studied in detail, and an expression for the fidelity of quantum state transfer has been obtained [9, 10]. Similarly, in Ref. [11], near perfect state transfer was achieved for uniform couplings provided a spatially varying magnetic field was introduced. The propagation of quantum information in rings has been also investigated in [12]. In our work we focus on the situation in which state transference is perfect, i.e., the fidelity is unity, and in which we can design spin networks such that this can be achieved over arbitrarily long distances. We will also consider the case in which no external control is required during the state transference, i.e., we consider the case in which we have, after manufacturing the network, no further control over its dynamics. In general this will lead us to
think about more complicated spin networks than the linear chain or chains with preengineered nearest-neighbor interaction strengths. We provide two alternative methods for understanding how perfect state transfer is achieved with preengineered couplings. This paper expands and extends the work done in [1, 2]. We will consider distance-regular graphs as spin networks in the sense that with each vertex of a distance-regular graph a qubit or a spin is associated (although qubits represent generic two state systems, for convenience of exposition we will use the term spin as it provides a simple physical picture of the network). Then, due to the fact that distance-regular graphs are underlying graphs of association schemes (see for example [13, 20]), we use their algebraic properties in order to find suitable coupling constants in some particular spin Hamiltonians so that perfect transference of a quantum state between antipodes of the networks can be achieved. More clearly, for a given distance-regular network first we stratify the network with respect to an arbitrary chosen vertex of the network called reference vertex (for details about stratification of graphs, see [4, 5, 20]). Then, we consider coupling constants so that vertices belonging to the same stratum with respect to the reference vertex possess the same coupling strength with the reference vertex whereas vertices belonging to distinct strata possess different coupling strengths. Then we give a method for finding a suitable set of coupling constants so that PST over antipodes of the networks be possible. As examples we will consider the cycle networks with even number of vertices and $d$- dimensional hypercube networks in details and some important distance-regular networks in an appendix.

The organization of the paper is as follows: In section 2, we review some preliminary facts about association schemes, stratification, distance-regular graphs and spectral analysis techniques. Section 3 is devoted to perfect state transfer (PST) over antipodes of distance-regular networks, where a method for finding suitable coupling constants in particular spin Hamiltonians so that PST be possible, is given. The paper is ended with a brief conclusion and an appendix.
2 Preliminaries

In this section we will first review some preliminary facts about distance-regular graphs, corresponding stratification and spectral distribution associated with graphs.

2.1 Association schemes

First we recall the definition of association schemes. For further information on association schemes, the reader is referred to Ref. [13].

Definition. An association scheme with \(d\) associate classes on a finite set \(V\) is a set of matrices \(A_0, A_1, \ldots, A_d\) in \(\mathbb{R}^{V \times V}\), all of whose entries are equal to 0 or 1, such that

(i) \(A_0 = I_v\);

(ii) \(A_i\) is symmetric for \(i = 1, \ldots, d\);

(iii) for all \(i, j \in \{0, 1, \ldots, d\}\), the product \(A_i A_j\) is a linear combination of \(A_0, A_1, \ldots, A_d\);

(iv) none of the \(A_i\) is equal to \(O_v\), and \(\sum_{i=0}^{d} A_i = J_v\), where \(v := |V|\) and \(J_v\) is an \(v \times v\) all one matrix.

It should be noticed that, since \(A_i\) is symmetric with entries in \(\{0, 1\}\), the diagonal entries of \(A_i^2\) are the row-sums of \(A_i\). Condition (iii) implies that \(A_i^2\) has a constant element, say \(\kappa_i\), on its diagonal. Therefore every row and every column of \(A_i\) contains \(\kappa_i\) entries equal to 1. Hence \(A_i J_v = J_v A_i = \kappa_i J_v\). Moreover, \(A_0 A_i = A_i A_0 = A_i\).

From condition (iii), one can write

\[
A_i A_j = \sum_{k=0}^{d} P_{ij}^k A_k, \tag{2-1}
\]

which implies that the adjacency matrices \(A_0, A_1, \ldots, A_d\) form a basis for a commutative algebra known as Bose-Mesner algebra associated with the association scheme. This algebra has a second basis \(E_0, \ldots, E_d\) such that, \(E_i E_j = \delta_{ij} E_i\) and \(\sum_{i=0}^{d} E_i = I\) with \(E_0 = 1/v J_v\) [13]. The matrices \(E_i\) for \(0 \leq i \leq d\) are known as primitive idempotents of \(Y\). Furthermore, there are matrices \(P\) and \(Q\) such that the two bases of the Bose-Mesner algebras can be related to each
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other as follows

\[ A_i = \sum_{j=0}^{d} P_{ji} E_j, \quad 0 \leq j \leq d, \]

\[ E_i = \frac{1}{v} \sum_{j=0}^{d} Q_{ji} A_j, \quad 0 \leq j \leq d. \] (2-2)

Then, clearly we have

\[ PQ = QP = vI. \] (2-3)

It also follows that

\[ A_j E_i = P_{ij} E_i, \] (2-4)

which indicates that \( P_{ij} \) is the \( i \)-th eigenvalue of \( A_j \) and that the columns of \( E_i \) are corresponding eigenvectors. Also, \( m_i := tr E_i = v \langle \alpha | E_i | \alpha \rangle = Q_{i0} \) (where, we have used the fact that \( \langle \alpha | E_i | \alpha \rangle \) is independent of the choice of \( \alpha \in V \), see Eq.(2-3)) is the rank of the idempotent \( E_i \) which gives the multiplicity of the eigenvalue \( P_{ij} \) of \( A_j \) (provided that \( P_{ij} \neq P_{kj} \) for \( k \neq i \)).

Clearly, we have \( m_0 = 1 \) and \( \sum_{i=0}^{d} m_i = v \) since \( \sum_{i=0}^{d} E_i = I \).

The underlying network of an association scheme \( \Gamma = (V, E) \) is an undirected connected network with adjacency matrix \( A \equiv A_1 \). Obviously replacing \( A_1 \) with one of the other adjacency matrices \( A_i, i \neq 0, 1 \) will also gives us an underlying network \( \Gamma' = (V, E') \) (not necessarily a connected network) with the same set of vertices but a new set of edges.

As we will see in subsection 2.3, in the case of distance-regular networks, the adjacency matrices \( A_j \) are polynomials of the adjacency matrix \( A \equiv A_1 \), i.e., \( A_j = P_j(A) \), where \( P_j \) is a polynomial of degree \( j \), then the eigenvalues \( P_{ij} \) in (2-4) are polynomials of eigenvalues \( P_{i1} \equiv \lambda_i \) (eigenvalues of the adjacency matrix \( A \)). This indicates that in distance-regular graphs, the
matrix $P^t$ is a polynomial transformation [14] as

$$
P^t = \begin{pmatrix}
1 & 1 & \ldots & 1 \\
\lambda_0 & \lambda_1 & \ldots & \lambda_d \\
\lambda_0 & \lambda_1 & \ldots & \lambda_d \\
\vdots & \vdots & \ldots & \vdots \\
\lambda_0 & \lambda_1 & \ldots & \lambda_d \\
\end{pmatrix}
$$

or $P_{ji} = P_i(\lambda_j)$.

### 2.2 Stratifications

For a given vertex $\alpha \in V$, let $\Gamma_i(\alpha) := \{\beta \in V : (\alpha, \beta) \in R_i\}$ denotes the set of all vertices having the relation $R_i$ with $\alpha$. Then, the vertex set $V$ can be written as disjoint union of $\Gamma_i(\alpha)$ for $i = 0, 1, 2, \ldots, d$, i.e.,

$$
V = \bigcup_{i=0}^{d} \Gamma_i(\alpha).
$$

We fix a point $o \in V$ as an origin of the underlying graph of an association scheme, called reference vertex. Then, the relation (2-6) stratifies the graph into a disjoint union of associate classes $\Gamma_i(o)$ (called the $i$-th stratum with respect to $o$). Let $l^2(V)$ denote the Hilbert space of $C$-valued square-summable functions on $V$. With each associate class $\Gamma_i(o)$ we associate a unit vector in $l^2(V)$ defined by

$$
|\phi_i\rangle = \frac{1}{\sqrt{\kappa_i}} \sum_{\alpha \in \Gamma_i(o)} |\alpha\rangle,
$$

where, $|\alpha\rangle$ denotes the eigenket of $\alpha$-th vertex at the associate class $\Gamma_i(o)$ and $\kappa_i = |\Gamma_i(o)|$ is called the $i$-th valency of the graph. Now, let $A_i$ be the adjacency matrix of the graph $\Gamma = (V, R)$. Then, for the reference state $|\phi_0\rangle$ ($|\phi_0\rangle = |o\rangle$, with $o \in V$ as reference vertex), one can write

$$
A_i|\phi_0\rangle = \sum_{\beta \in \Gamma_i(o)} |\beta\rangle.
$$
Then, by using (2-7) and (2-8), we obtain

$$A_i|\phi_0\rangle = \sqrt{\kappa_i}|\phi_i\rangle.$$  \hfill (2-9)

One should notice that, in underlying networks of association schemes, stratification is reference state independent, namely one can choose any arbitrary vertex as a reference state.

### 2.3 Distance-regular networks and spectral techniques

Distance-regular graphs are underlying graphs of so called $P$-polynomial association schemes [13], where the adjacency matrices $A_i$ are defined based on shortest path distance. More clearly, if distance between nodes $\alpha, \beta \in V$ denoted by $\partial(\alpha, \beta)$ be the length of the shortest walk connecting $\alpha$ and $\beta$ (recall that a finite sequence $\alpha_0, \alpha_1, ..., \alpha_n \in V$ is called a walk of length $n$ if $\alpha_{k-1} \sim \alpha_k$ for all $k = 1, 2, ..., n$, where $\alpha_{k-1} \sim \alpha_k$ means that $\alpha_{k-1}$ is adjacent with $\alpha_k$), then the adjacency matrices $A_i$ for $i = 0, 1, ..., d$ in distance-regular graphs are defined as: $(A_i)_{\alpha,\beta} = 1$ if and only if $\partial(\alpha, \beta) = i$ and $(A_i)_{\alpha,\beta} = 0$ otherwise, where $d := \max\{\partial(\alpha, \beta) : \alpha, \beta \in V\}$ is diameter of the graph.

For distance-regular graphs, the non-zero intersection numbers are given by

$$a_i = p^i_{i1}, \quad b_i = p^i_{i+1,1}, \quad c_i = p^i_{i-1,1},$$  \hfill (2-10)

respectively. The intersection numbers (2-10) and the valencies $\kappa_i$ with $\kappa_1 \equiv \kappa(= \text{deg}(\alpha)$, for each vertex $\alpha$) satisfy the following obvious conditions

$$a_i + b_i + c_i = \kappa, \quad \kappa_{i-1}b_{i-1} = \kappa_ic_i, \quad i = 1, ..., d,$$

$$\kappa_0 = c_1 = 1, \quad b_0 = \kappa_1 = \kappa, \quad (c_0 = b_d = 0).$$  \hfill (2-11)

Thus all parameters of a distance-regular graph can be obtained from its intersection array $\{b_0, ..., b_{d-1}; c_1, ..., c_d\}$. Then, it can be shown that the following recursion relations are satisfied

$$A_1A_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1}, \quad i = 1, 2, ..., d - 1,$$
\[ A_1A_d = b_{d-1}A_{d-1} + (\kappa - c_d)A_d. \]  

The recursion relations (2-12) imply that

\[ A_i = P_i(A), \quad i = 0, 1, \ldots, d. \]  

Then, one can easily obtain the following three term recursion relations for the unit vectors \(|\phi_i\rangle, i = 0, 1, \ldots, d\)

\[ A|\phi_i\rangle = \beta_{i+1}|\phi_{i+1}\rangle + \alpha_i|\phi_i\rangle + \beta_i|\phi_{i-1}\rangle, \]  

where, the coefficients \(\alpha_i\) and \(\beta_i\) are defined as

\[ \alpha_0 = 0, \quad \alpha_k \equiv a_k = \kappa - b_k - c_k, \quad \omega_k \equiv \beta_k^2 = b_{k-1}c_k, \quad k = 1, \ldots, d, \]  

(see Ref. [5, 21, 22] for more details).

Now, we recall some preliminary facts about spectral techniques used in the paper, where more details have been given in Refs. [5, 21, 22].

For any pair \((A, |\phi_0\rangle)\) of a matrix \(A\) and a vector \(|\phi_0\rangle\), one can assign a measure \(\mu\) as follows

\[ \mu(x) = \langle \phi_0 | E(x) | \phi_0 \rangle, \]  

where \(E(x) = \sum_i |u_i\rangle\langle u_i|\) is the operator of projection onto the eigenspace of \(A\) corresponding to eigenvalue \(x\), i.e.,

\[ A = \int xE(x)dx. \]  

Then, for any polynomial \(P(A)\) we have

\[ P(A) = \int P(x)E(x)dx, \]  

where for discrete spectrum the above integrals are replaced by summation. Therefore, using the relations (2-16) and (2-18), the expectation value of powers of adjacency matrix \(A\) over reference vector \(|\phi_0\rangle\) can be written as

\[ \langle \phi_0 | A^m | \phi_0 \rangle = \int_R x^m \mu(dx), \quad m = 0, 1, 2, \ldots \]  

Obviously, the relation (2-19) implies an isomorphism from the Hilbert space of the stratification onto the closed linear span of the orthogonal polynomials with respect to the measure \( \mu \). More clearly, from orthonormality of the unit vectors \(|\phi_i\rangle\) (with \(|\phi_0\rangle\) as unit vector assigned to reference node) we have

\[
\delta_{ij} = \langle \phi_i | \phi_j \rangle = \frac{1}{\sqrt{K_i K_j}} \langle \phi_0 | A_i A_j | \phi_0 \rangle = \int_R P'_i(x) P'_j(x) \mu(dx),
\]

(2-20)

with \( P'_i(A) := \frac{1}{\sqrt{\kappa_i}} P_i(A) \) where, we have used the equations (2-9) and (2-13) to write

\[
|\phi_i\rangle = \frac{1}{\sqrt{K_i}} A_i |\phi_0\rangle = \frac{1}{\sqrt{K_i}} P_i(A) |\phi_0\rangle \equiv P'_i(A) |\phi_0\rangle. \tag{2-21}
\]

Now, by substituting (2-21) in (2-14) and rescaling \( P'_k \) as \( Q_k = \beta_1 \ldots \beta_k P'_k \), the spectral distribution \( \mu \) under question will be characterized by the property of orthonormal polynomials \( \{Q_k\} \) defined recurrently by

\[
xQ_k(x) = Q_{k+1}(x) + \alpha_k Q_k(x) + \beta_k^2 Q_{k-1}(x), \quad k \geq 1. \tag{2-22}
\]

where, the coefficients \( \alpha_i \) and \( \beta_i \) are defined as

\[
\alpha_k = \kappa - b_k - c_k, \quad \omega_k \equiv \beta_k^2 = b_{k-1} c_k, \quad k = 1, \ldots, d, \tag{2-23}
\]

If such a spectral distribution is unique, the spectral distribution \( \mu \) is determined by the identity

\[
G_\mu(x) = \int_R \frac{\mu(dy)}{x - y} = \frac{1}{x - \alpha_0 - \frac{\beta_1^2}{x - \alpha_1 - \frac{\beta_2^2}{x - \alpha_2 - \frac{\beta_3^2}{\gamma_3}}}} = \frac{Q^{(1)}_d(x)}{Q^{(1)}_{d+1}(x)} = \sum_{l=0}^{d} \frac{\gamma_l}{x - x_l}, \tag{2-24}
\]

where, \( x_l \) are the roots of polynomial \( Q_{d+1}(x) \). \( G_\mu(x) \) is called the Stieltjes/Hilbert transform of spectral distribution \( \mu \) or Stieltjes function and polynomials \( \{Q^{(1)}_k\} \) are defined recurrently as

\[
Q^{(1)}_0(x) = 1, \quad Q^{(1)}_1(x) = x - \alpha_1,
\]
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\[ xQ_k^{(1)}(x) = Q_{k+1}^{(1)}(x) + \alpha_{k+1}Q_k^{(1)}(x) + \beta_{k+1}^2 Q_{k-1}^{(1)}(x), \quad k \geq 1, \]  

(2-25)

respectively. The coefficients \( \gamma_l \) appearing in (2-24) are calculated as

\[ \gamma_l := \lim_{x \to x_l} [(x - x_l)G_{\mu}(x)] \]  

(2-26)

Now let \( G_{\mu}(z) \) is known, then the spectral distribution \( \mu \) can be recovered from \( G_{\mu}(z) \) by means of the Stieltjes/Hilbert inversion formula as

\[ \mu(y) - \mu(x) = -\frac{1}{\pi} \lim_{v \to 0^+} \int_x^y \text{Im}\{G_{\mu}(u + iv)\} du. \]  

(2-27)

Substituting the right hand side of (2-24) in (2-27), the spectral distribution can be determined in terms of \( x_l, l = 1, 2, ... \) and Guass quadrature constants \( \gamma_l, l = 1, 2, ... \) as

\[ \mu = \sum_l \gamma_l \delta(x - x_l) \]  

(2-28)

(for more details see Refs.[16, 17, 18, 19]).

3 Perfect State Transfer (PST) over antipodes of distance-regular networks

3.1 State Transfer in Quantum Spin Systems

The PST algorithm was proposed by Christandl et al. [1,2], and it can be implemented in the XY chain. The algorithm can transfer an arbitrary quantum state between the two ends of the chain in a fixed period time, only using XY interactions. For one-dimensional fermionic chains, the model of a system consisting of spinless fermions (or bosons) hopping freely in a network of \( N \) lattice sites can be mapped to spin chains in which spins are coupled through the XY Hamiltonian

\[ H = \frac{1}{2} \sum_{j=1}^{N-1} J_j (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) + \frac{1}{2} \sum_{j=1}^{N} \lambda_j (\sigma_j^z + 1), \]  

(3-29)
by the Jordan-Wigner transformation, where $J_j$ is the time-independent coupling constant between nearest-neighbor sites $j$ and $j+1$, and $\lambda_j$ represents the strength of the external static potential at site $j$.

A quantum spin system associated with a simple, connected, finite graph $G = (V, E)$ as a spin network is defined by attaching a spin-1/2 particle to each vertex of the graph so that to each vertex $i \in V$ one can associate a Hilbert space $\mathcal{H}_i \simeq \mathbb{C}^2$. The Hilbert space associated with $G$ is then given by

$$\mathcal{H}_G = \bigotimes_{i \in V} \mathcal{H}_i = (\mathbb{C}^2)^\otimes N, \quad (3-30)$$

where $N := |V|$ denotes the total number of vertices in $G$. On the other hand, quantum state transfer over a network is similar to the quantum random walk problem, where a variety of networks are equivalent to one-dimensional chains [1,22]. Therefore, it can be focused on a chain of $N$ sites. For $j = 1, 2, ..., N$, let $|j\rangle$ be the state where a single fermion (or boson) is at the site $j$ but is in the empty state $|0\rangle$ for all other sites and $|0\rangle$ be the vacuum state where all sites are empty. For spin chains, $|0\rangle$ corresponds to the state where all the spins are in the spin-down state $|↓\rangle$ and $|j\rangle$ corresponds to a spin-up state $|↑\rangle$ for the $j$th spin and spin-down for all other spins. The Hamiltonian in this single-particle subspace can be written in a tridiagonal form, which is real and symmetric:

$$H = \begin{pmatrix}
    \lambda_1 & J_1 & 0 & \ldots & 0 \\
    J_1 & \lambda_2 & J_2 & \ldots & 0 \\
    0 & J_2 & \lambda_3 & \ldots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & J_{N-1} & \lambda_N
\end{pmatrix} \quad (3-31)$$

The quantum state transfer protocol involves two steps: initialization and evolution. First, a quantum state $|\psi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A \in \mathcal{H}_A$ (with $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$) to be transmitted is created. The state of the entire spin system after this step is given by

$$|\psi(t = 0)\rangle = |\psi_A0\ldots0B\rangle = \alpha|0_A0\ldots0_B\rangle + \beta|1_A0\ldots0_B\rangle = \alpha|0\rangle_A + \beta|1\rangle_A, \quad (3-32)$$
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with $|0⟩ := |0_A...0_{0B}⟩$. Then, the network couplings are switched on and the whole system is allowed to evolve under $U(t) = e^{-iHt}$ for a fixed time interval, say $t_0$. The final state becomes

$$|ψ(t_0)⟩ = α|0⟩ + β \sum_{j=1}^{N} f_{jA}(t_0)|j⟩$$

where, $f_{jA}(t_0) := ⟨j|e^{-iHt_0}|A⟩$. Any site $B$ is in a mixed state if $|f_{AB}(t_0)| < 1$, which also implies that the state transfer from site $A$ to $B$ is imperfect. In this paper, we will focus only on perfect state transfer. This means that we consider the condition

$$|f_{AB}(t_0)| = 1 \text{ for some } 0 < t_0 < ∞$$

which can be interpreted as the signature of perfect communication (or perfect state transfer) between $A$ and $B$ in time $t_0$. The effect of the modulus in (3-34) is that the state at $B$, after transmission, will no longer be $|ψ⟩$, but will be of the form

$$α|0⟩ + e^{iϕ}β|1⟩.$$

The phase factor $e^{iϕ}$ is not a problem because $ϕ$ is independent of $α$ and $β$ and will thus be a known quantity for the graph, which we can correct for with an appropriate phase gate (for more details see for example [1, 2, 3]).

The model we will consider is a distance-regular network consisting of $N$ sites labeled by $\{1, 2, ..., N\}$ and diameter $d$. Then we stratify the network with respect to a chosen reference site, say 1, and assume that the network contains only the output site $N$ in its last stratum (i.e., $|ϕ_d⟩ = |N⟩$). At time $t = 0$, the qubit in the first (input) site of the network is prepared in the state $|ψ_{in}⟩$. We wish to transfer the state to the $N$th (output) site of the network with unit efficiency after a well-defined period of time. Although our qubits represent generic two state systems, for the convenience of exposition we will use the term spin as it provides a simple physical picture of the network. The standard basis for an individual qubit is chosen to be $\{|0⟩ = |↓⟩, \ |1⟩ = |↑⟩\}$, and we shall assume that initially all spins point “down” along
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a prescribed $z$ axis; i.e., the network is in the state $|0\rangle = |0_A00...00_B\rangle$. Then, we consider the dynamics of the system to be governed by the quantum-mechanical Hamiltonian

$$H_G = \frac{1}{2} \sum_{m=0}^{d} J_m \sum_{(i,j)\in R_m} H_{ij},$$

(3-36)

with $H_{ij}$ as

$$H_{ij} = \sigma_i \cdot \sigma_j,$$

(3-37)

where, $\sigma_i$ is a vector with familiar Pauli matrices $\sigma^x_i, \sigma^y_i$ and $\sigma^z_i$ as its components acting on the one-site Hilbert space $\mathcal{H}_i$, and $J_m$ is the coupling strength between the reference site 1 and all of the sites belonging to the $m$-th stratum with respect to 1.

The total spin of a quantum-mechanical system consisting of $N$ elementary spins $\vec{\sigma}_i$ on a one-dimensional lattice or better called chain is given by:

$$\vec{\sigma} = \sum_{i=1}^{N} \vec{\sigma}_i.$$  

(3-38)

One can easily see that, the Hamiltonian (3-36) commutes with the total Spin operator (conservation). That is, since the total $z$ component of the spin given by $\sigma^z_{tot} = \sum_{i\in V} \sigma^z_i$ is conserved, i.e., $[\sigma^z_{tot}, H_G] = 0$, hence the Hilbert space $\mathcal{H}_G$ decomposes into invariant subspaces, each of which is a distinct eigenspace of the operator $\sigma^z_{tot}$ (this property would be important to use its symmetry to diagonalize the Hamiltonian in the well known Bethe ansatz approach).

In order to consider perfect quantum state transfer, we write the hamiltonian (3-36) in terms of the adjacency matrices $A_i, i = 0, 1, ..., d$ of the underlying graph in order to use the techniques introduced in section 2 such as stratification and spectral distribution associated with the graph. To do so, we recall that the kets $|i_1, i_2, ..., i_N\rangle$ with $i_1, ..., i_N \in \{\uparrow, \downarrow\}$ form an orthonormal basis for Hilbert space $\mathcal{H}_G$. Then, one can easily obtain

$$H_{ij} |\ldots \uparrow_i \ldots \uparrow_j \ldots\rangle = |\ldots \uparrow_i \ldots \uparrow_j \ldots\rangle,$$

$$H_{ij} |\ldots \uparrow_i \ldots \downarrow_j \ldots\rangle = -|\ldots \uparrow_i \ldots \downarrow_j \ldots\rangle + 2|\ldots \downarrow_i \ldots \uparrow_j \ldots\rangle.$$  

(3-39)
where, we have used the facts that \( \sigma_z | \uparrow \rangle = | \uparrow \rangle, \sigma_z | \downarrow \rangle = -| \downarrow \rangle, \sigma_x | \uparrow \rangle = | \downarrow \rangle, \sigma_x | \downarrow \rangle = | \uparrow \rangle \) and \( \sigma_y | \uparrow \rangle = i | \downarrow \rangle, \sigma_y | \downarrow \rangle = -i | \uparrow \rangle \). The equation (3-39) implies that the action of \( H_{ij} \) on the basis vectors is equivalent to the action of the operator \( 2P_{ij} - I_N \), i.e., we have

\[
H_{ij} = 2P_{ij} - I_N, \tag{3-40}
\]

where, \( P_{ij} \) denotes the permutation operator which permutes \( i \)-th and \( j \)-th sites and \( I_N \) is \( N \times N \) identity matrix, where \( N \) is the number of vertices \( (N := |V|) \). Now, let \( | l \rangle \) denotes the vector state which its all components are \( \uparrow \) except for \( l \), i.e., \( | l \rangle = | \uparrow \ldots \downarrow \uparrow \ldots \rangle \). Then, we have

\[
\sum_{(i,j) \in R_m} P_{ij} | l \rangle = \frac{1}{2} \left( \sum_{i \in \Gamma_m(j), i, j \neq l} P_{ij} + 2 \sum_{i \in \Gamma_m(l)} P_{il} \right) | l \rangle = \left( \frac{N \kappa_m}{2} - \kappa_m \right) | l \rangle + \sum_{j \in \Gamma_m(l)} | j \rangle,
\]

which implies that

\[
\sum_{(i,j) \in R_m} P_{ij} = \left( \frac{\kappa_m(N-2)}{2} I + A_m \right). \tag{3-41}
\]

Then, by using (3-40) and (3-41), the hamiltonian in (3-36) can be written in terms of the adjacency matrices \( A_i, i = 0, 1, \ldots, d \) as follows

\[
H = \sum_{m=0}^{d} J_m \sum_{(i,j) \in R_m} (2P_{ij} - I_N) = 2 \sum_{m=0}^{d} J_mA_m + \frac{N - 4}{2} \sum_{m=0}^{d} J_m \kappa_m I. \tag{3-42}
\]

As it has been shown in [27], many known Hamiltonians suitable for PST are basically associated with permutations and can thus be obtained within the present unifying theoretical framework. For the purpose of the perfect transfer of state, we consider distance-regular graphs with \( \kappa_d = |\Gamma_d(o)| = 1 \), i.e., the last stratum of the graph contains only one site. Then, we impose the constraints that the amplitudes \( \langle \phi_i | e^{-iHt} | \phi_0 \rangle \) be zero for all \( i = 0, 1, \ldots, d - 1 \) and \( \langle \phi_d | e^{-iHt} | \phi_0 \rangle = e^{i\theta} \), where \( \theta \) is an arbitrary phase. Therefore, these amplitudes must be evaluated. To do so, we use the stratification and spectral distribution associated with distance-regular graphs to write

\[
\langle \phi_i | e^{-iHt} | \phi_0 \rangle = e^{-i\frac{N-4}{2}t} \sum_{m=0}^{d} J_m \kappa_m \langle \phi_i | e^{-2it} \sum_{m=0}^{d} J_mA_m | \phi_0 \rangle =
\]
\[ \frac{1}{\sqrt{P_t}} e^{-\frac{\sqrt{\kappa} \epsilon}{2} \sum_{m=0}^{d} J_m \kappa_m (\phi_0 | A_i e^{-2i\epsilon} \sum_{m=0}^{d} J_m P_m (A) ) | \phi_0 )} \]

Let the spectral distribution of the graph is \( \mu(x) = \sum_{k=0}^{d} \gamma_k \delta(x - x_k) \). Then, \( \langle \phi_i | e^{-iHt} | \phi_0 \rangle = 0 \) implies that

\[ \sum_{k=0}^{d} \gamma_k P_i(x_k) e^{-2i\epsilon \sum_{m=0}^{d} J_m P_m(x_k)} = 0, \quad i = 0, 1, \ldots, d - 1 \]

Denoting \( e^{-2i\epsilon \sum_{m=0}^{d} J_m P_m(x_k)} \) by \( \eta_k \), the above constraints are rewritten as follows

\[ \sum_{k=0}^{d} P_i(x_k) \eta_k \gamma_k = 0, \quad i = 0, 1, \ldots, d - 1, \]

\[ \sum_{k=0}^{d} P_d(x_k) \eta_k \gamma_k = e^{i\theta}. \] (3-43)

As it was discussed previously, \( P_i(x_k) \) are entries of the matrix \( P (P_{ki} = P_i(x_k)) \) which is invertible, i.e., the Eq.(3-43) can be written as

\[ \begin{pmatrix} \eta_0 \gamma_0 \\ \eta_1 \gamma_1 \\ \vdots \\ \eta_d \gamma_d \end{pmatrix} = (P_t)^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \] (3-44)

The above equation implies that \( \eta_k \gamma_k \) for \( k = 0, 1, \ldots, d \) are the same as the entries in the last column of the matrix \( (P_t)^{-1} = \frac{1}{v} Q_t^{(d)} \) multiplied with the phase \( e^{i\theta} \), i.e., for the purpose of PST, the following equations must be satisfied

\[ \eta_k \gamma_k = \gamma_k e^{-2it_0 \sum_{m=0}^{d} J_m P_m(x_k)} = \frac{e^{i\theta}}{v} (Q_t)^{kd}, \quad \text{for} \quad k = 0, 1, \ldots, d. \] (3-45)

In the following, we investigate PST between antipodes of some distance-regular networks such as cycle networks with even number of nodes and \( d \) dimensional hypercube networks.

### 3.2 Examples

1. **Cycle graph \( C_{2m} \)**

A well known example of distance-regular networks, is the cycle graph with \( N \) vertices denoted
by $C_N$ (see Fig. 1 for even $N = 2m$). For the purpose of perfect transfer of state, we consider the cycle graph with even number of vertices, since as it can be seen from Fig. 1, in this case the last stratum contains a single state corresponding to the $m$-th vertex. From Figure 1 it can be also seen that, for even number of vertices $N = 2m$, the adjacency matrices are given by

$$A_0 = I_{2m}, \quad A_i = S^i + S^{-i}, \quad i = 1, 2, ..., m - 1, \quad A_m = S^m, \quad (3-46)$$

where, $S$ is the $N \times N$ circulant matrix with period $N$ ($S^N = I_N$) defined as follows

$$S = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 1 & 0 \\
0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & 0
\end{pmatrix}. \quad (3-47)$$

By using (3-46), one can obtain the following recursion relations for $C_{2m}$

$$A_1A_i = A_{i-1} + A_{i+1}, \quad i = 0, 1, ..., m - 1; \quad A_1A_m = A_{m-1} \quad (3-48)$$

(the graph $C_{2m}$ consists of $m + 1$ strata). By comparing (3-48) with three term recursion relations (2-12), we obtain the intersection arrays for $C_{2m}$ as

$$\{b_0, ..., b_{m-1}; c_1, ..., c_m\} = \{2, 1, ..., 1; 1, ..., 1, 2\}. \quad (3-49)$$

Then, by using (2-23), the QD parameters are given by

$$\alpha_i = 0, \quad i = 0, 1, ..., m; \quad \omega_1 = \omega_m = 2, \quad \omega_i = 1, \quad i = 2, ..., m - 1, \quad (3-50)$$

By using the recursion relations (2-22), one can show that

$$Q_0(x) = P_0(x) = 1, \quad Q_i(x) = P_i(x) = 2T_i(x/2), \quad i = 1, ..., m - 1, \quad Q_m(x) = 2P_m(x) = 2T_m(x/2) \quad (3-51)$$
where $T_i$’s are Chebyshev polynomials of the first kind.

Then, the eigenvalues of the adjacency matrix $A \equiv A_1$ (roots of $Q_{m+1}(x) = 2T_{m+1}(x/2)$) are given by

$$x_i = \omega^i + \omega^{-i} = 2 \cos(2\pi i/N), \ i = 0, 1, ..., m$$

with $\omega := e^{2\pi i/N}$. Also, one can show that $\gamma_i$’s (degeneracies of eigenvalues $x_i$) are given by

$$\gamma_0 = \gamma_m = 1/2m, \ \gamma_i = 1/m, \ i = 1, 2, ..., m - 1. \quad (3-52)$$

Now, as regards the Eq. (2-5), the matrix $P^t$ associated with cycle graph $C_{2m}$ reads as

$$P^t = \begin{pmatrix}
1 & 1 & \cdots & 1 & 1 \\
2 & 2 \cos(2\pi/N) & \cdots & 2 \cos(2(m-1)\pi/N) & 2\omega^m \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
2 & 2 \cos(2(m-1)\pi/N) & \cdots & 2 \cos((m-1)^2\pi/N) & 2\omega^{m(m-1)} \\
1 & \omega^m & \cdots & \cdots & \omega^{m^2}
\end{pmatrix}. \quad (3-53)$$

One can see that $(P^t)^2 = NI$, so the inverse of $P^t$ is given by $(P^t)^{-1} = \frac{1}{N}P^t$. Therefore, by using (3-44) and (3-52), we obtain

$$\eta_i = e^{-it} \sum_{l=0}^{m} 2J_l T_l(\cos(2\pi i/N)) = (-1)^i e^{i\theta}, \ i = 0, 1, ..., m \quad (3-54)$$

For instance, for $N = 4$, we obtain

$$\eta_0 = e^{-it_0(J_0 + 2J_1 + J_2)} = e^{i\theta},$$

$$\eta_1 = e^{-it_0(J_0 - J_2)} = -e^{i\theta},$$

$$\eta_2 = e^{-it_0(J_0 - 2J_1 + J_2)} = e^{i\theta} \quad (3-55)$$

which gives us the following equations

$$-t(J_0 + 2J_1 + J_2) = \theta + 2l\pi,$$

$$-t(J_0 - J_2) = \theta + (2l' + 1)\pi,$$
\[-t(J_0 - 2J_1 + J_2) = \theta + 2l''\pi.\]  

(3-56)

For \(l = l' = l'' = 0\), one can obtain

\[J_0 = -\frac{2\theta + \pi}{4t_0}, \quad J_1 = 0, \quad J_2 = \frac{\pi}{4t_0},\]  

(3-57)

whereas by choosing \(l = l' = 0, l'' = 1\), the solution to (3-56) is given by

\[J_0 = -\frac{\theta + \pi}{2t_0}, \quad J_1 = \frac{\pi}{4t_0}, \quad J_2 = 0.\]  

(3-58)

In the first case, the time \(t_0\) at which the state \(|\phi_0\rangle = |0\rangle = |1000\rangle\) is perfectly transferred to the vertex \(|\phi_2\rangle = |2\rangle = |0010\rangle\) is given by

\[t_0 = \frac{2\theta + \pi}{4J_0} = \frac{\pi}{4J_2},\]  

(3-59)

whereas in the latter case \(t_0\) is given by

\[t_0 = -\frac{\theta + \pi}{2J_0} = \frac{\pi}{4J_1}.\]  

(3-60)

2. Hypercube network

The hypercube of dimension \(d\) (known also as binary Hamming scheme \(H(d, 2)\)) is a network with \(N = 2^d\) nodes, each of which can be labeled by an \(d\)-bit binary string. Two nodes on the hypercube described by bitstrings \(\vec{x}\) and \(\vec{y}\) are connected by an edge if \(|\vec{x} - \vec{y}| = 1\), where \(|\vec{x}|\) is the Hamming weight of \(\vec{x}\). In other words, if \(\vec{x}\) and \(\vec{y}\) differ by only a single bit flip, then the two corresponding nodes on the network are connected. Thus, each of \(2^d\) nodes on the hypercube has degree \(d\). For the hypercube network with dimension \(d\) we have \(d + 1\) strata with

\[\kappa_i = \frac{d!}{i!(d - i)!}, \quad 0 \leq i \leq d - 1.\]  

(3-61)

The intersection numbers are given by

\[b_i = d - i, \quad 0 \leq i \leq d - 1; \quad c_i = i, \quad 1 \leq i \leq d.\]  

(3-62)
Furthermore, the adjacency matrices of this network are given by
\[
A_i = \sum_{\text{perm.}} \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes I_2 \otimes \ldots \otimes I_2, \quad i = 0, 1, \ldots, n,
\] (3-63)
where, the summation is taken over all possible nontrivial permutations. In fact, the underlying network is the cartesian product of \(d\)-tuples of complete network \(K_2\). Also it can be shown that, the idempotents \(\{E_0, E_1, \ldots, E_d\}\) are symmetric product of \(d\)-tuples of corresponding idempotents of complete network \(K_2\). That is, we have
\[
E_i = \sum_{\text{perm.}} E_- \otimes E_- \otimes \ldots \otimes E_- \otimes E_+ \otimes \ldots \otimes E_+, \quad i = 0, 1, \ldots, d,
\] (3-64)
where
\[
E_\pm = \frac{1}{2}(I \pm \sigma_x).
\] (3-65)
It has been shown that the eigenmatrices \(P\) and \(Q\) for the Hamming scheme \(H(d, 2)\) are the same, i.e., this scheme is self dual [29]. Also, Delsarte [30] showed that the entries of the eigenmatrix \(P = Q\) for the Hamming scheme \(H(d, 2)\) can be found using the Krawtchouk polynomials as follows
\[
P_{il} = Q_{il} = K_i(l),
\] (3-66)
where \(K_i(x)\) are the Krawtchouk polynomials defined as
\[
K_i(x) = \sum_{i=0}^{l} \binom{x}{i} \binom{d-x}{l-i} (-1)^i.
\] (3-67)
Therefore, we have \(((P^t)^{-1})_{il} = \frac{1}{d!}K_i(l)\).

The eigenvalues \(x_l\) of the adjacency matrix \(A \equiv A_1\) and corresponding degeneracies \(\gamma_l\) are given by
\[
x_l = 2l - d;
\]
\[
\gamma_l = \frac{d!}{2^d l!(d - l)!}, \quad l = 0, 1, \ldots, d.
\] (3-68)
By using (3-66), we have

$$\eta_l = e^{-2it \sum_{m=0}^{d} J_m K_m(l)}, \quad l = 0, 1, \ldots, d.$$ \hspace{1cm} (3-69)

Now, in order to evaluate the time $t_0$ at which PST takes place, the following equations must be satisfied

$$\eta_l \gamma_l = \frac{e^{i\theta}}{2^d} Q dl = \frac{e^{i\theta}}{2^d} K_l(d), \quad \forall \; l = 0, 1, \ldots, d,$$

which are equivalent to

$$\frac{d!}{l!(d-l)!} e^{-2it_0 \sum_{m=0}^{d} J_m K_m(l)} = e^{i\theta} K_l(d), \quad \forall \; l = 0, 1, \ldots, d. \hspace{1cm} (3-70)$$

For instance, in the case of $d = 3$ (see Fig. 2), we must solve the following equations

$$e^{-2it_0(J_0+3J_1+3J_2+J_3)} = e^{i\theta},$$

$$e^{-2it_0(J_0+J_1-J_2-J_3)} = -e^{i\theta},$$

$$e^{-2it_0(J_0-J_1-J_2+J_3)} = e^{i\theta}$$

$$e^{-2it_0(J_0-3J_1+3J_2-J_3)} = -e^{i\theta}. \hspace{1cm} (3-71)$$

By solving Eqs. (3-71) one can obtain the following solution

$$J_0 = -\frac{2\theta + 3\pi}{4t_0}, \quad J_1 = -\frac{\pi J_0}{2\theta + 3\pi} = \frac{\pi}{4t_0}, \quad J_2 = J_3 = 0; \quad \theta \neq -3\pi/2 \hspace{1cm} (3-72)$$

that is the time $t_0$ at which PST takes place is given by

$$t_0 = \frac{2\theta + 3\pi}{4J_0} = \frac{\pi}{4J_1}. \hspace{1cm} (3-73)$$

In the appendix we consider PST over antipodes of some important finite distance-regular networks.
4 Conclusion

By using spectral analysis techniques and algebraic combinatoric structures of distance-regular graphs (as spin networks) such as stratification introduced in [4, 5] and Bose-Mesner algebra, a method for finding a set of coupling constants in some particular spin Hamiltonians associated with spin networks of distance-regular type was given so that perfect state transfer between antipodes of the networks can be achieved. As examples, the cycle networks with even number of vertices and $d$-dimensional hypercube networks were considered.
Appendix

In this appendix we consider some important finite distance-regular networks such that their last stratum contains only one node. Then by using the prescription of section 3, we investigate PST over antipodes of these networks.

1. Icosahedron [31]

Intersection array:
\[ \{b_0, b_1, b_2; c_1, c_2, c_3\} = \{5, 2, 1; 1, 2, 5\}. \]

Size of strata and QD parameters:
\[ \kappa_0 = 1, \quad \kappa_1 = 5, \quad \kappa_2 = 5, \quad \kappa_3 = 1, \]
\[ \alpha_0 = 0, \quad \alpha_1 = \alpha_2 = 2, \quad \alpha_3 = 0; \quad \omega_1 = 5, \quad \omega_2 = 4, \quad \omega_3 = 5. \]

Polynomials \( P_i(x) \):
\[ P_0 = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(x^2 - 2x - 5), \quad P_3(x) = \frac{1}{10}(x^3 - 4x^2 - 5x + 10). \]

Stieltjes function:
\[ G_\mu(x) = \frac{x^3 - 4x^2 - 5x + 10}{x^4 - 4x^3 - 10x^2 + 20x + 25}. \]

Spectral distribution ( \( \mu(x) = \sum_{i=0}^d \gamma_i \delta(x - x_i) \)):
\[ \mu(x) = \frac{1}{12} \{5\delta(x + 1) + \delta(x - 5) + 3\delta(x - \sqrt{5}) + 3\delta(x + \sqrt{5})\}. \]

Now, one can obtain the matrix \( P^t \) and its inverse. Then by solving the equations (3-45), the solution is obtained as follows
\[ J_0 = -\frac{6\theta + 7\pi}{12t_0}, \quad J_1 = -\frac{(5 - 3\sqrt{5})\pi}{60t_0}, \quad J_2 = -\frac{(5 + 3\sqrt{5})\pi}{60t_0}, \quad J_3 = \frac{5\pi}{12t_0}. \] (A-i)

Then, the time \( t_0 \) at which PST takes place is given by
\[ t_0 = -\frac{2\theta + \pi}{4J_0} = \frac{\pi}{4J_3}. \] (A-ii)
2. Desargues [31]

Intersection array:

\[ \{b_0, b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4, c_5\} = \{3, 2, 2, 1, 1; 1, 1, 2, 2, 3\}. \]

Size of strata and QD parameters:

\[ \kappa_0 = 1, \quad \kappa \equiv \kappa_1 = 3, \quad \kappa_2 = 6, \quad \kappa_3 = 6, \quad \kappa_4 = 3, \quad \kappa_5 = 1 \]

\[ \alpha_i = 0, \quad \omega_1 = 3, \quad \omega_2 = 2, \quad \omega_3 = 4, \quad \omega_4 = 2, \quad \omega_5 = 3. \]

Polynomials \( P_i(x) \):

\[ P_0 = 1, \quad P_1(x) = x, \quad P_2(x) = x^2 - 3, \quad P_3(x) = \frac{1}{2}(x^3 - 5x), \quad P_4(x) = \frac{1}{4}(x^4 - 9x^2 + 12), \quad P_5(x) = \frac{1}{12}(x^5 - 11x^3 + 22x). \]

Stieltjes function:

\[ G_{\mu}(x) = \frac{x^5 - 11x^3 + 22x}{x^6 - 14x^4 + 49x^2 - 36}. \]

Spectral distribution:

\[ \mu(x) = \frac{1}{20}\{5\delta(x + 1) + 5\delta(x - 1) + 4\delta(x + 2) + 4\delta(x - 2) + \delta(x + 3) + \delta(x - 3)\}. \]

The solution to Eq.(3-45) is given by:

\[ J_0 = -\frac{30\theta + 51\pi}{60t_0}, \quad J_1 = \frac{\pi}{10t_0}, \quad J_2 = -\frac{4\pi}{15t_0}, \quad J_3 = 0, \quad J_4 = \frac{\pi}{15t_0}, \quad J_5 = \frac{\pi}{4t_0}. \]

3. Dodecahedron [31]

Intersection array:

\[ \{b_0, b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4, c_5\} = \{3, 2, 1, 1, 1; 1, 1, 1, 2, 3\}. \]

Size of strata and QD parameters:

\[ \kappa_0 = 1, \quad \kappa \equiv \kappa_1 = 3, \quad \kappa_2 = 6, \quad \kappa_3 = 6, \quad \kappa_4 = 3, \quad \kappa_5 = 1 \]

\[ \alpha_0 = \alpha_1 = 0, \quad \alpha_2 = \alpha_3 = 1, \quad \alpha_4 = \alpha_5 = 0; \quad \omega_1 = 3, \quad \omega_2 = 2, \quad \omega_3 = 1, \quad \omega_4 = 2, \quad \omega_5 = 3. \]
Polynomials $P_i(x)$:

$$P_0 = 1, \quad P_1(x) = x, \quad P_2(x) = x^2 - 3, \quad P_3(x) = x^3 - 5x - x^2 + 3, \quad P_4(x) = \frac{1}{2}(x^4 - 5x^2 - 2x^3 + 8x), \quad P_5(x) = \frac{1}{6}(x^5 - 7x^3 - 2x^4 + 10x^2 + 10x - 6).$$

Stieltjes function:

$$G_\mu(x) = \frac{x^5 - 7x^3 - 2x^4 + 10x^2 + 10x - 6}{x^6 - 10x^4 + 16x^3 + 25x^2 - 30x}. $$

Spectral distribution:

$$\mu(x) = \frac{1}{20}\{4\delta(x) + 5\delta(x - 1) + 4\delta(x + 2) + \delta(x - 3) + 3\delta(x - \sqrt{5}) + 3\delta(x + \sqrt{5})\}. $$

The solution to Eq. (3-45) is given by:

$$ J_0 = -\frac{\theta + 2\pi}{2t_0}, \quad J_1 = \frac{(2 + 3\sqrt{5})\pi}{60t_0}, \quad J_2 = -\frac{17\pi}{60t_0}, \quad J_3 = \frac{\pi}{60t_0}, \quad J_4 = \frac{(2 - 3\sqrt{5})\pi}{60t_0}. $$

4. Taylor ($P(13)$) [32]

Intersection array:

$$\{b_0, b_1, b_2; c_1, c_2, c_3\} = \{13, 6, 1; 1, 6, 13\}. $$

Size of strata and QD parameters:

$$\kappa_0 = 1, \quad \kappa = \kappa_1 = 13, \quad \kappa_2 = 13, \quad \kappa_3 = 1,$$

$$\alpha_0 = 0, \quad \alpha_1 = \alpha_2 = 6, \quad \alpha_3 = 0; \quad \omega_1 = 13, \quad \omega_2 = 36, \quad \omega_3 = 13. $$

Polynomials $P_i(x)$:

$$P_0 = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{6}(x^2 - 6x - 13), \quad P_3(x) = \frac{1}{78}(x^3 - 12x^2 - 13x + 78). $$

Stieltjes function:

$$G_\mu(x) = \frac{x^3 - 12x^2 - 13x + 78}{x^4 - 12x^3 - 26x^2 + 156x + 169}. $$

Spectral distribution:

$$\mu(x) = \frac{1}{28}\{13\delta(x + 1) + \delta(x - 13) + 7\delta(x - \sqrt{13}) + 7\delta(x + \sqrt{13})\}. $$
The solution to Eq.(3-45) is given by:

\[ J_0 = -\frac{14\theta + 15\pi}{28t_0}, \quad J_1 = -\frac{(13 - 7\sqrt{13})\pi}{364t_0}, \quad J_2 = -\frac{(13 + 7\sqrt{13})\pi}{364t_0}, \quad J_3 = \frac{13\pi}{28t_0}. \]

5. Taylor(GQ(2, 2)) [32]

Intersection array:
\[ \{b_0, b_1, b_2; c_1, c_2, c_3\} = \{15, 8, 1; 1, 8, 15\}. \]

Size of strata and QD parameters:
\[ \kappa_0 = 1, \quad \kappa \equiv \kappa_1 = 15, \quad \kappa_2 = 15, \quad \kappa_3 = 1, \]
\[ \alpha_0 = 0, \quad \alpha_1 = \alpha_2 = 6, \quad \alpha_3 = 0; \quad \omega_1 = 15, \quad \omega_2 = 64, \quad \omega_3 = 15. \]

Polynomials \( P_i(x) \):
\[ P_0 = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{8}(x^2 - 6x - 15), \quad P_3(x) = \frac{1}{120}(x^3 - 12x^2 - 43x + 90). \]

Stieltjes function:
\[ G_\mu(x) = \frac{x^3 - 12x^2 - 43x + 90}{x^4 - 12x^3 - 58x^2 + 180x + 225}. \]

Spectral distribution:
\[ \mu(x) = \frac{1}{32}\{15\delta(x + 1) + 10\delta(x - 3) + 6\delta(x + 5) + \delta(x - 15)\}. \]

The solution to Eq.(3-45) is given by:
\[ J_0 = -\frac{16\theta + 15\pi}{32t_0}, \quad J_1 = \frac{\pi}{32t_0}, \quad J_2 = -\frac{3\pi}{32t_0}, \quad J_3 = \frac{13\pi}{32t_0}. \]

6. Taylor(T(6)) [32]

Intersection array:
\[ \{b_0, b_1, b_2; c_1, c_2, c_3\} = \{15, 6, 1; 1, 6, 15\}. \]

Size of strata and QD parameters:
\[ \kappa_0 = 1, \quad \kappa \equiv \kappa_1 = 15, \quad \kappa_2 = 15, \quad \kappa_3 = 1, \]
\( \alpha_0 = 0, \; \alpha_1 = \alpha_2 = 8, \; \alpha_3 = 0; \; \omega_1 = 15, \; \omega_2 = 36, \; \omega_3 = 15. \)

Polynomials \( P_i(x) \):

\[
P_0 = 1, \; P_1(x) = x, \; P_2(x) = \frac{1}{6}(x^2 - 8x - 15), \; P_3(x) = \frac{1}{90}(x^3 - 16x^2 + 13x + 120).
\]

Stieltjes function:

\[
G_\mu(x) = \frac{x^3 - 16x^2 + 13x + 120}{x^4 - 16x^3 - 2x^2 + 240x + 225}.
\]

Spectral distribution:

\[
\mu(x) = \frac{1}{32}\{15\delta(x + 1) + 10\delta(x + 3) + 6\delta(x - 5) + \delta(x - 15)\}.
\]

The solution to Eq.(3-45) is given by:

\[
J_0 = -\frac{16\theta + 15\pi}{32t_0}, \; J_1 = \frac{3\pi}{32t_0}, \; J_2 = \frac{\pi}{32t_0}, \; J_3 = \frac{13\pi}{32t_0}.
\]

7. Wells [33]

Intersection array:

\[
\{b_0, b_1, b_2, b_3; c_1, c_2, c_3, c_4\} = \{5, 4, 1, 1; 1, 1, 4, 5\}.
\]

Size of strata and QD parameters:

\[
\kappa_0 = 1, \; \kappa \equiv \kappa_1 = 5, \; \kappa_2 = 20, \; \kappa_3 = 5, \; \kappa_4 = 1,
\]

\[
\alpha_0 = \alpha_1 = 0, \; \alpha_2 = 3, \; \alpha_3 = \alpha_4 = 0; \; \omega_1 = 5, \; \omega_2 = \omega_3 = 4, \; \omega_4 = 5.
\]

Polynomials \( P_i(x) \):

\[
P_0 = 1, \; P_1(x) = x, \; P_2(x) = x^2 - 5, \; P_3(x) = \frac{1}{4}(x^3 - 9x - 3x^2 + 15), \; P_4(x) = \frac{1}{20}(x^4 - 13x^2 - 3x^3 + 15x + 20).
\]

Stieltjes function:

\[
G_\mu(x) = \frac{x^4 - 13x^2 - 3x^3 + 15x + 20}{x^5 - 18x^3 - 3x^4 + 30x^2 + 65x - 75}.
\]

Spectral distribution:

\[
\mu(x) = \frac{1}{32}\{10\delta(x - 1) + 5\delta(x + 3) + \delta(x - 5) + 8\delta(x + \sqrt{5}) + 8\delta(x - \sqrt{5})\}.
\]
The solution to Eq. (3-45) is given by:

\[ J_0 = -\frac{16\theta + 23\pi}{32t_0}, \quad J_1 = \frac{(5 - 8\sqrt{5})\pi}{160t_0}, \quad J_2 = -\frac{3\pi}{32t_0}, \quad J_3 = \frac{(5 + 8\sqrt{5})\pi}{160t_0}, \quad J_4 = \frac{9\pi}{32t_0}. \]

8. Hadamard network [34]

Intersection array:

\[ \{ b_0, b_1, b_2, c_1, c_2, c_3, c_4 \} = \{ 8, 7, 4, 1; 1, 4, 7, 8 \} \]

Size of strata and QD parameters:

\[ \kappa_0 = 1, \quad \kappa = \kappa_1 = 8, \quad \kappa_2 = 14, \quad \kappa_3 = 8, \quad \kappa_4 = 1, \]

\[ \alpha_i = 0, \quad i = 0, 1, \ldots, 4; \quad \omega_1 = 8, \quad \omega_2 = 28, \quad \omega_3 = 28, \quad \omega_4 = 8. \]

Polynomials \( P_i(x) \):

\[ P_0 = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{4}(x^2 - 8), \quad P_3(x) = \frac{1}{28}(x^3 - 36x), \quad P_4(x) = \frac{1}{224}(x^4 - 64x^2 + 224). \]

Stieltjes function:

\[ G_\mu(x) = \frac{x^4 - 64x^2 + 224}{x^5 - 72x^3 + 512x}. \]

Spectral distribution:

\[ \mu(x) = \frac{1}{32}\{14\delta(x) + \delta(x - 8) + \delta(x + 8) + 8\delta(x - 2\sqrt{2}) + 8\delta(x + 2\sqrt{2})\}. \]

The solution to Eq. (3-45) is given by:

\[ J_0 = -\frac{16\theta + 19\pi}{32t_0}, \quad J_1 = \frac{(1 + 2\sqrt{2})\pi}{32t_0}, \quad J_2 = -\frac{3\pi}{32t_0}, \quad J_3 = \frac{(1 - 2\sqrt{2})\pi}{32t_0}, \quad J_4 = \frac{13\pi}{32t_0}. \]

9. Taylor \( P(17) \) [32]

Intersection array:

\[ \{ b_0, b_1, b_2; c_1, c_2, c_3 \} = \{ 17, 8, 1; 1, 8, 17 \} \]

Size of strata and QD parameters:

\[ \kappa_0 = 1, \quad \kappa = \kappa_1 = 17, \quad \kappa_2 = 17, \quad \kappa_3 = 1, \]
Perfect state transfer

\[ \alpha_0 = 0, \ \alpha_1 = \alpha_2 = 8, \ \alpha_3 = 0; \ \omega_1 = 17, \ \omega_2 = 64, \ \omega_3 = 17. \]

Polynomials \( P_i(x) \):

\[ P_0 = 1, \ P_1(x) = x, \ P_2(x) = \frac{1}{8}(x^2 - 8x - 17), \ P_3(x) = \frac{1}{136}(x^3 - 16x^2 - 17x + 136). \]

Stieltjes function:

\[ G_\mu(x) = \frac{x^3 - 16x^2 - 17x + 136}{x^4 - 16x^3 - 34x^2 + 272x + 289}. \]

Spectral distribution:

\[ \mu(x) = \frac{1}{36}\{17\delta(x + 1) + \delta(x - 17) + 9\delta(x - \sqrt{17}) + 9\delta(x + \sqrt{17})\}. \]

The solution to Eq.(3-45) is given by:

\[ J_0 = -\frac{18\theta + 19\pi}{36t_0}, \quad J_1 = -\frac{(17 - 9\sqrt{17})\pi}{612t_0}, \quad J_2 = -\frac{(17 + 9\sqrt{17})\pi}{612t_0}, \quad J_3 = \frac{17\pi}{36t_0}. \]

10. Hadamard network [32]

Intersection array:

\[ \{b_0, b_1, b_2, b_3; c_1, c_2, c_3, c_4\} = \{12, 11, 6, 1; 1, 6, 11, 12\}. \]

Size of strata and QD parameters:

\[ \kappa_0 = 1, \ \kappa \equiv \kappa_1 = 12, \ \kappa_2 = 22, \ \kappa_3 = 12, \ \kappa_4 = 1, \]

\[ \alpha_i = 0, \ i = 0, 1, \ldots, 4; \ \omega_1 = 12, \ \omega_2 = \omega_3 = 66, \ \omega_4 = 12. \]

Polynomials \( P_i(x) \):

\[ P_0 = 1, \ P_1(x) = x, \ P_2(x) = \frac{1}{6}(x^2 - 12), \ P_3(x) = \frac{1}{66}(x^3 - 78x), \ P_4(x) = \frac{1}{792}(x^4 - 144x^2 + 792). \]

Stieltjes function:

\[ G_\mu(x) = \frac{x^4 - 144x^2 + 792}{x^5 - 156x^3 + 1728x}. \]
Spectral distribution:

\[ \mu(x) = \frac{1}{48}\{22\delta(x) + \delta(x + 12) + \delta(x - 12) + 12\delta(x - 2\sqrt{3}) + 12\delta(x + 2\sqrt{3})\}. \]

The solution to Eq.(3-45) is given by:

\[ J_0 = -\frac{24\theta + 27\pi}{48t_0}, \quad J_1 = -\frac{(1 - 2\sqrt{3})\pi}{48t_0}, \quad J_2 = -\frac{\pi}{16t_0}, \quad J_3 = -\frac{(1 + 2\sqrt{3})\pi}{48t_0}, \quad J_4 = \frac{21\pi}{48t_0}. \]

11. Taylor(SRG(25, 12)) [32]

Intersection array:

\[ \{b_0, b_1, b_2; c_1, c_2, c_3\} = \{25, 12, 1; 1, 12, 25\}. \]

Size of strata and QD parameters:

\[ \kappa_0 = 1, \quad \kappa_1 = 25, \quad \kappa_2 = 25, \quad \kappa_3 = 1, \]

\[ \alpha_0 = 0, \quad \alpha_1 = \alpha_2 = 12, \quad \alpha_3 = 0; \quad \omega_1 = 25, \quad \omega_2 = 144, \quad \omega_3 = 25. \]

Polynomials \( P_i(x) \):

\[ P_0 = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{12}(x^2 - 12x - 25), \quad P_3(x) = \frac{1}{300}(x^3 - 24x^2 - 25x + 300). \]

Stieltjes function:

\[ G_\mu(x) = \frac{x^3 - 24x^2 - 25x + 300}{x^3 - 24x^3 - 50x^2 + 600x + 625}. \]

Spectral distribution:

\[ \mu(x) = \frac{1}{52}\{25\delta(x + 1) + 13\delta(x - 5) + 13\delta(x + 5) + \delta(x - 25)\}. \]

The solution to Eq.(3-45) is given by:

\[ J_0 = -\frac{26\theta + 27\pi}{52t_0}, \quad J_1 = \frac{2\pi}{65t_0}, \quad J_2 = -\frac{9\pi}{130t_0}, \quad J_3 = \frac{25\pi}{52t_0}. \]

12. Gosset, Tayl(Schläfli) [32]

Intersection array:

\[ \{b_0, b_1, b_2; c_1, c_2, c_3\} = \{27, 10, 1; 1, 10, 27\}. \]
Size of strata and QD parameters:

\[ \kappa_0 = 1, \; \kappa \equiv \kappa_1 = 27, \; \kappa_2 = 27, \; \kappa_3 = 1, \]

\[ \alpha_0 = 0, \; \alpha_1 = \alpha_2 = 16, \; \alpha_3 = 0; \; \omega_1 = 27, \; \omega_2 = 100, \; \omega_3 = 27. \]

Polynomials \( P_i(x) \):

\[ P_0 = 1, \; P_1(x) = x, \; P_2(x) = \frac{1}{16}(x^2 - 10x - 27), \; P_3(x) = \frac{1}{432}(x^3 - 20x^2 - 183x + 270). \]

Stieltjes function:

\[ G_\mu(x) = \frac{x^3 - 32x^2 + 129x + 432}{x^4 - 32x^3 + 102x^2 + 864x + 729}. \]

Spectral distribution:

\[ \mu(x) = \frac{1}{56}\{27\delta(x + 1) + 21\delta(x + 3) + 7\delta(x - 9) + \delta(x - 27)\}. \]

The solution to Eq.(3-45) is given by:

\[ J_0 = -\frac{14\theta + 11\pi}{28t_0}, \; J_1 = -\frac{5\pi}{84t_0}, \; J_2 = \frac{\pi}{42t_0}, \; J_3 = \frac{5\pi}{14t_0}. \]

13. Taylor(Co-Schlafli) \[32]\]

Intersection array:

\[ \{b_0, b_1, b_2; c_1, c_2, c_3\} = \{27, 16, 1; 1, 16, 27\}. \]

Size of strata and QD parameters:

\[ \kappa_0 = 1, \; \kappa \equiv \kappa_1 = 27, \; \kappa_2 = 27, \; \kappa_3 = 1, \]

\[ \alpha_0 = 0, \; \alpha_1 = \alpha_2 = 10, \; \alpha_3 = 0; \; \omega_1 = 27, \; \omega_2 = 256, \; \omega_3 = 27. \]

Polynomials \( P_i(x) \):

\[ P_0 = 1, \; P_1(x) = x, \; P_2(x) = \frac{1}{16}(x^2 - 10x - 27), \; P_3(x) = \frac{1}{432}(x^3 - 20x^2 - 183x + 270). \]
Stieltjes function:
\[ G_\mu(x) = \frac{x^3 - 20x^2 - 183x + 270}{x^4 - 20x^3 - 210x^2 + 540x + 729}. \]

Spectral distribution:
\[ \mu(x) = \frac{1}{56} \{27\delta(x + 1) + 21\delta(x - 3) + 7\delta(x + 9) + \delta(x - 27)\}. \]

The solution to Eq.(3-45) is given by:
\[ J_0 = -\frac{14\theta + 11\pi}{28t_0}, \quad J_1 = -\frac{\pi}{42t_0}, \quad J_2 = -\frac{5\pi}{84t_0}, \quad J_3 = \frac{5\pi}{14t_0}, \]

14. Taylor(SRG(29, 14)) [32]

Intersection array:
\[ \{b_0, b_1, b_2; c_1, c_2, c_3\} = \{29, 14, 1; 1, 14, 29\}. \]

Size of strata and QD parameters:
\[ \kappa_0 = 1, \quad \kappa \equiv \kappa_1 = 29, \quad \kappa_2 = 29, \quad \kappa_3 = 1, \]
\[ \alpha_0 = 0, \quad \alpha_1 = \alpha_2 = 14, \quad \alpha_3 = 0; \quad \omega_1 = 29, \quad \omega_2 = 196, \quad \omega_3 = 29. \]

Polynomials \(P_i(x):\)
\[ P_0 = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{14}(x^2 - 14x - 29), \quad P_3(x) = \frac{1}{406}(x^3 - 28x^2 - 29x + 406). \]

Stieltjes function:
\[ G_\mu(x) = \frac{x^3 - 28x^2 - 29x + 406}{x^4 - 28x^3 - 58x^2 + 812x + 841}. \]

Spectral distribution:
\[ \mu(x) = \frac{1}{60}\{29\delta(x + 1) + \delta(x - 29) + 15\delta(x - \sqrt{29}) + 15\delta(x + \sqrt{29})\}. \]

The solution to Eq.(3-45) is given by:
\[ J_0 = -\frac{30\theta + 31\pi}{60t_0}, \quad J_1 = -\frac{(29 - 15\sqrt{29})\pi}{1740t_0}, \quad J_2 = -\frac{(29 + 15\sqrt{29})\pi}{1740t_0}, \quad J_3 = \frac{29\pi}{60t_0}. \]
15. Doubled odd(4) \[32\]

Intersection array:

\[\{b_0, b_1, b_2, b_3, b_4, b_5; c_1, c_2, c_3, c_4, c_5, c_6, c_7\} = \{4, 3, 3, 2, 2, 1, 1, 1, 2, 2, 3, 3, 4\}\]

Size of strata and QD parameters:

\[\kappa_0 = 1, \kappa \equiv \kappa_1 = 4, \kappa_2 = 12, \kappa_3 = 18, \kappa_4 = 18, \kappa_5 = 12, \kappa_6 = 4, \kappa_7 = 1,\]

\[\alpha_i = 0, i = 0, 1, \ldots, 7; \omega_1 = 4, \omega_2 = 3, \omega_3 = 6, \omega_4 = 4, \omega_5 = 6, \omega_6 = 3, \omega_7 = 4.\]

Polynomials \(P_i(x)\):

\[P_0 = 1, \ P_1(x) = x, \ P_2(x) = x^2 - 4, \ P_3(x) = \frac{1}{2}(x^3 - 7x), \ P_4(x) = \frac{1}{4}(x^4 - 13x^2 + 24), \]

\[P_5(x) = \frac{1}{12}(x^5 - 17x^3 + 52x), \ P_6(x) = \frac{1}{36}(x^6 - 23x^4 + 130x^2 - 144), \ P_7(x) = \frac{1}{144}(x^7 - 26x^5 + 181x^3 - 300x).\]

Stieltjes function:

\[G_\mu(x) = \frac{x^7 - 26x^5 + 181x^3 - 300x}{x^8 - 30x^6 + 273x^4 - 820x^2 + 576}.\]

Spectral distribution:

\[\mu(x) = \frac{1}{70} \{14[\delta(x-1) + \delta(x+1) + \delta(x-2) + \delta(x+2)] + 6[\delta(x-3) + \delta(x+3)] + \delta(x-4) + \delta(x+4)\}.\]

The solution to Eq.(3-45) is given by:

\[J_0 = -\frac{70\theta + 151\pi}{140t_0}, \ J_1 = 0, \ J_2 = -\frac{56\pi}{245t_0}, \ J_3 = 0, \ J_4 = \frac{4\pi}{105t_0}, \ J_5 = -\frac{\pi}{70t_0}, \ J_6 = -\frac{\pi}{35t_0}, \ J_7 = \frac{\pi}{4t_0}.\]

16. J(8,4) \[32\]

Intersection array:

\[\{b_0, b_1, b_2, b_3; c_1, c_2, c_3, c_4\} = \{16, 9, 4, 1; 1, 4, 9, 16\}.\]

Size of strata and QD parameters:

\[\kappa_0 = 1, \kappa \equiv \kappa_1 = 16, \kappa_2 = 36, \kappa_3 = 16, \kappa_4 = 1,\]
\( \alpha_0 = 0, \ \alpha_1 = 6, \ \alpha_2 = 8, \ \alpha_3 = 6, \ \alpha_4 = 0; \ \omega_1 = 16, \ \omega_2 = 36, \ \omega_3 = 36, \ \omega_4 = 16. \)

Polynomials \( P_i(x) \):

\[
P_0 = 1, \ \ P_1(x) = x, \ \ P_2(x) = \frac{1}{4}(x^2 - 6x - 16), \ \ P_3(x) = \frac{1}{36}(x^3 - 14x^2 - 4x + 128), \ \ P_4(x) = \frac{1}{576}(x^4 - 20x^3 + 44x^2 + 368x - 192).
\]

Stieltjes function:

\[
G_\mu(x) = \frac{x^4 - 20x^3 + 44x^2 + 368x - 192}{x^5 - 20x^4 + 28x^3 + 592x^2 - 128x - 2048}.
\]

Spectral distribution:

\[
\mu(x) = \frac{1}{70}\{\delta(x - 16) + 7\delta(x - 8) + 14\delta(x + 4) + 28\delta(x + 2) + 20\delta(x - 2)\}.
\]

The solution to Eq.(3-45) is given by:

\[
J_0 = -\frac{70\theta + 199\pi}{140t_0}, \quad J_1 = \frac{\pi}{35t_0}, \quad J_2 = \frac{13\pi}{210t_0}, \quad J_3 = -\frac{\pi}{14t_0}, \quad J_4 = -\frac{17\pi}{140t_0}.
\]

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Figure Captions

Figure 1: Denotes the cycle network $C_{2m}$, where the $m + 1$ vertical dashed lines show the $m + 1$ strata of the network.

Figure 2: Shows the cube or Hamming scheme $H(3, 2)$ with vertex set $V = \{(ijk) : i, j, k = 0, 1\}$, where the vertical dashed lines denote the four strata of the cube.