HOLOMORPHIC DYNAMICS NEAR GERMS OF SINGULAR CURVES

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ABSTRACT. Let $M$ be a two dimensional complex manifold, $p \in M$ and $\mathcal{F}$ a germ of holomorphic foliation of $M$ at $p$. Let $S \subset M$ be a germ of an irreducible, possibly singular, curve at $p$ in $M$ which is a separatrix for $\mathcal{F}$. We prove that if the Camacho-Sad-Suwa index $\text{Ind}(\mathcal{F}, S, p) \notin \mathbb{Q}^+ \cup \{0\}$ then there exists another separatrix for $\mathcal{F}$ at $p$. A similar result is proved for the existence of parabolic curves for germs of holomorphic diffeomorphisms near a curve of fixed points.

1. INTRODUCTION

Let $M$ be a two dimensional complex manifold and $\mathcal{F}$ a germ of holomorphic foliation on $M$ near $p$. In local coordinates the foliation can be described by the vector field:

$$A(x, y) \frac{\partial}{\partial x} + B(x, y) \frac{\partial}{\partial y},$$

with $A, B$ suitable holomorphic functions. A separatrix for $\mathcal{F}$ is a non constant holomorphic solution of the system:

\[
\begin{align*}
\dot{x} &= A(x, y) \\
\dot{y} &= B(x, y)
\end{align*}
\]

with $x(0) = y(0) = 0$.

Obviously the interesting case is when $(0, 0)$ is a singularity for $\mathcal{F}$. In the singular case, in the well known paper [7], Camacho and Sad proved that there always exists (at least) one (possibly singular) irreducible separatrix - say $S$ - for $\mathcal{F}$ at $(0, 0)$. A natural question is whether the knowledge of this separatrix $S$ allows to infer the existence of another separatrix. There are essentially two types of results, one of local and the other of global flavour. The first kind of result is essentially a re-formulation of Camacho-Sad theorem (see the paper by J. Cano [8]) which says that if $S$ is non singular and $\text{Ind}(\mathcal{F}, S, p) \notin \mathbb{Q}^+ \cup \{0\}$ (where $\text{Ind}(\mathcal{F}, S, p)$ is the index introduced in [7]) then there exists another separatrix through $p$. The second type of result requires global conditions on $S$, like $S$ compact (but possibly singular), globally and locally irreducible and $S \cdot S < 0$ to provide the existence of another separatrix at some point of $S$ (see the paper by Sebastiani [13]).

One aim of this paper is to prove a result of local nature when $S$ is possibly singular, using the index defined by Suwa [11]. We prove:

**Theorem 1.1.** Let $M$ be a complex two dimensional manifold, $\mathcal{F}$ a holomorphic foliation on some open subset of $M$, $S \subset M$ a possibly singular curve locally irreducible at a point $p \in M$, such that it is a separatrix for $\mathcal{F}$ at $p$. If $\text{Ind}(\mathcal{F}, S, p) \notin \mathbb{Q}^+ \cup \{0\}$ then there exists (at least) another separatrix for $\mathcal{F}$ at $p$. 

1991 Mathematics Subject Classification. Primary 32H50, 37F99, 32S45, 32S65.
Abate, Bracci and Tovena have recently shown how to translate results about foliations to holomorphic diffeomorphisms with curves of fixed points. The proof of Theorem 1.1 respects their dictionary and so the results about the existence of separatrices for foliations can be translated into results about the existence of parabolic curves for diffeomorphisms. Using notations of we obtain:

Theorem 1.2. Let be a two dimensional complex manifold, a holomorphic map such that with a locally irreducible, possibly singular curve at a point . Assume that is tangential on and . Then there exists (at least) a parabolic curve for at .

Theorem 1.2 has been proved by Abate in case is non singular and by Bracci in case is a generalized cusp, i.e. of the form .

I want to sincerely thank professor Filippo Bracci without whose help this work would not have came to be.

2. Preliminary results

First of all we have to recall some basic notions about C.S.S. (Camacho-Sad-Suwa) index. This index was first introduced by Camacho and Sad in for a complex one codimension singular foliation defined in a neighborhood of a non singular compact curve embedded in a two dimensional complex manifold. Later Suwa generalized to a generic possibly singular compact invariant curve. The most interesting property of this is the following Index Theorem, that relates the dynamics of near a curve to the self intersection number of .

Theorem 2.1 (Index Theorem). Let be a compact curve in a two dimensional complex manifold invariant by a possibly singular foliation , then for every point there exists a complex number depending only on the local behaviour of and near such that:

We now recall the behaviour of this index under blow-up.

Proposition 2.2. Let be a two dimensional complex manifold, a holomorphic foliation, an -separatrix and a singularity of . We indicate by the blow-up of in , by the saturated foliation and by the exceptional divisor and the strict transform of . Then is an separatrix. Moreover if then

where is the multiplicity of in .

Cano in gives an algorithmic proof of Camacho-Sad result introducing a particular class of points that we will often use.

Definition 2.3. Let be a two dimensional complex manifold, a holomorphic foliation and a local separatrix for .
We say that a point \( p \in S \) is of type \((C_1)\) if \( S \) is nonsingular at \( p \) and
\[
\text{Ind}(\mathcal{F}, S, p) \notin \mathbb{Q}^+ \cup \{0\}.
\]

We say that a point \( p \in S \) is of type \((C_2)\) if \( S \) has two nonsingular branches \( S_0, S_1 \) at \( p \), intersecting transversally at \( p \), and there exists a real number \( r > 0 \) such that
\[
\text{Ind}(\mathcal{F}, S_0, p) \notin \mathbb{Q} \geq -\frac{1}{r} = \{ a \in \mathbb{Q} : a \geq -\frac{1}{r} \}
\]
\[
\text{Ind}(\mathcal{F}, S_1, p) \in \mathbb{Q} \leq -r = \{ a \in \mathbb{Q} : a \leq -r \}.
\]

According to Definition 7.6 of [3] and [8] we have:

**Definition 2.4.** A point \( p \in S \subset M \) where \( S \) is an \( \mathcal{F} \)-invariant curve is said to be an appropriate singularity for \( \mathcal{F} \) if after a finite number of blow-ups there exists a \((C_1)\) or \((C_2)\) point on the total transform.

The importance of this class of points is given by the following result:

**Proposition 2.5 ([3], [8]).** If \( p \in S \subset M \) is an appropriate singularity for a foliation \( \mathcal{F} \), then at least another separatrix trough \( p \) for \( \mathcal{F} \) exists.

### 3. Proof of the Result

In order to get Theorem 1.1 we will concentrate our attention on the particular class of points introduced in the previous section. The upshot is to show that under the hypotheses of Theorem 1.1 the point \( p \) is an appropriate singularity.

We know that the resolution of curves singularities theorem [10] ensures that after a finite number of blow-ups we have the geometric structure required for the existence of \((C_1)\) or \((C_2)\) points. To conclude we have to analyze the C.S.S. index under this process. The behaviour of the index is strongly related to the evolution of the geometric structure under blow-up. We can divide the proof in two steps:

1. study of the geometric structure under the resolution of singularities,
2. study of the C.S.S. index under this process.

#### 3.1. Geometric structure under blow-up.

In order to get step one we give the following definition:

**Definition 3.1.** Let \( M \) be a two dimensional complex manifold and \( S_1, \ldots, S_n \subset M \) given curves. We say that a point \( p \) is a double intersection point if \( p \) belongs to exactly two distinct curves among \( S_1, \ldots, S_n \). If instead \( p \) belongs to exactly three of them it is called a triple intersection point.

**Remark 3.2.** In the study of curve desingularization the set of curves we find is composed by the strict transform of the curve \( S \) and the several exceptional divisors obtained by successive blow-ups. Because of the structure of the blow-up process we can only have double and triple intersection points (see [10]). A triple intersection point belongs to the strict transform of \( S \) and to two exceptional divisors. To distinguish these two \( \subset \mathbb{P}^1 \) we will call old exceptional divisor the strict transform of a given exceptional divisor. Instead we will call new exceptional divisor the exceptional divisor produced by the last blow-up.
Now we can describe the geometric evolution under blow-up. Note that the only intersection point that can be triple is the one made up by the strict transform of $S$. We will prove the following behaviour.

**Proposition 3.3.** Let $S$ be a singular curve and let $p$ be a singularity of $S$. The resolution process of $S$ in $p$ is related to the behavior of the multiplicity of $S$ in $p$ in the following way:

- If we blow-up a singularity and the multiplicity does not reduce we have two cases:
  1. if we are in a double intersection point at the next blow-up we find another double intersection,
  2. if we are in a triple intersection point at the next blow-up we can find either a double intersection or a triple intersection point. More precisely we find a double intersection point if the tangent cone to the curve does not coincide with any exceptional divisor, while we find a triple intersection point if the tangent cone coincides with one of the two exceptional divisors and the new triple intersection point belongs to the strict transform of the old exceptional divisor.

- If we blow-up a singularity and the multiplicity reduces we have two cases:
  1. if we are in a double intersection point at the next blow-up we find a triple intersection point,
  2. if we are in a triple intersection point at the next blow-up we find a triple intersection point that belongs to the strict transform of the new exceptional divisor.

**Remark 3.4.** In the previous Proposition we have used improperly the expression “the tangent cone coincides with one of the two exceptional divisors” to mean that the tangent cone of $S$ in $p$ coincides with the tangent space of $D$ in $p$.

In order to get Proposition 3.3 we will prove some elementary Lemmas.

**Lemma 3.5.** Let $M$ be a two dimensional complex manifold, $S$ an analytic irreducible curve on $M$ and $p \in S$ a singularity of $S$. Blow-up $M$ in $p$ and let $\hat{S}$ be the strict transform of $S$, $D$ the exceptional divisor and $\hat{p} := \hat{S} \cap D$. The multiplicity of $\hat{S}$ in $\hat{p}$ is strictly smaller than the multiplicity of $S$ in $p$ if and only if $D$ coincides with the tangent cone of $\hat{S}$ in $\hat{p}$.

**Proof.** We can assume that $p = (0,0)$ and $S = \{l(x, y) = 0\}$ with $l(x, y) = y^m + l_{m+1}(x, y) + \cdots$. Blow-up in $p$ and using the chart such that the projection becomes $\pi(u, v) = (u, uv)$ we have: $\hat{S} = \{\hat{l}(u, v) = 0\}$, with $\hat{l}(u, v) = v^m + ul_{m+1}(1, v) + \cdots = v^m + uq_{k-1} + \cdots$ and $D = \{u = 0\}$. The multiplicity of $\hat{S}$ in $(0,0)$ is strictly less then $m$ if and only if $k < m$ and then if and only if the tangent cone is $\{uq_{k-1}(u, v) = 0\}$ and so if and only if $D$ is included in the tangent cone. Because $S$ is irreducible this can happen if and only if $q_{k-1}(u, v) = u^{k-1}$, i.e. if and only if $D$ is the tangent cone. \hfill \Box

**Lemma 3.6.** Let $M$ be a two dimensional complex manifold, $S$ an analytic irreducible curve on $M$ and $p \in S$ a singularity of $S$. Blow-up $M$ in $p$ and let $\hat{S}$ be the strict transform of $S$, $D$ the exceptional divisor and $\hat{p} := \hat{S} \cap D$. The exceptional divisor $D$ is the tangent cone of $\hat{S}$ in $\hat{p}$ if and only if blowing-up in $\hat{p}$ we get a triple intersection point.

**Proof.** Let $\hat{D}$ be the strict tranform of $D$ and $D_1$ the new exceptional divisor. Now $\hat{D}$ intersects $D_1$ in the point corresponding to the tangent of $D$ in $p$, so $\hat{D} \cap \hat{S} \neq \emptyset$ if and only if $D$ and $\hat{S}$ intersect.
have the same tangent in \( p \). So we get a triple intersection point if and only if the tangent cone of \( \hat{S} \) coincides with \( D \).

Using the previous two Lemmas we obtain the following:

**Lemma 3.7.** Let \( S \subset M \) be an analytic irreducible curve of multiplicity \( m \) in the singular point \( p \). Suppose that after a finite number of blows-up the strict transform of \( S \), \( \hat{S} \), intersects the exceptional divisor in a point \( \hat{p} \) and indicate with \( D \) the irreducible component of the exceptional divisor containing \( \hat{p} \), i.e. \( \hat{p} \) is a double intersection point. Blow-up in \( \hat{p} \) and let \( D_1 \) be the new exceptional divisor and \( \hat{S} \) the strict transform of \( S \). If the multiplicity of \( \hat{S} \) in \( \hat{p} := D_1 \cap \hat{S} \) is equal to the multiplicity of \( \hat{S} \) in \( \hat{p} \) then at the following blow-up we find again a double intersection point.

By Lemma 3.6 we also get:

**Lemma 3.8.** Let \( S \subset M \) be an analytic irreducible curve of multiplicity \( m \) in the singular point \( p \). Suppose that after a finite number of blows-up we have a triple intersection point. At the following blow-up we have two cases:

1. If the tangent cone in the singularity contains one of the two exceptional divisors then at the next blow-up we find again a triple intersection point,
2. If the tangent cone in the singularity does not contain any of the two exceptional divisors then at the next blow-up we find a double intersection point.

**Remark 3.9.** We observe that the demonstrative method used in Lemma 3.8 does not give informations on which of the exceptional divisors goes to create the new triple intersection. To get this information we need some more calculations. Let \( \hat{S} \) the strict transform of \( S \) after some blow-ups and suppose to have a triple intersection point. We can assume that \( p = (0,0) \) and \( \hat{S} = \{ \hat{l}(u,v) = 0 \} \) with \( \hat{l}(u,v) = v^m + u^{k_1}[q_{k_2-k_1}(u,v) + \cdots] \), and \( D_1 = \{ v = 0 \} \), \( \hat{D} = \{ u = 0 \} \) where \( D_1 \) is the new exceptional divisor and \( \hat{D} \) is the old one (according to Remark 3.2). Let examine the various cases:

1. If \( m > k_2 \) then the tangent cone is \( \{ u^{k_1}q_{k_2-k_1}(u,v) = 0 \} \) and by the irreducibility of \( S \) is \( \{ cu^{k_2} = 0 \} \) with \( c \neq 0 \) and so it contains an exceptional divisor, \( \hat{D} \). Blow-up again \( (0,0) \) and using the chart by which the projection is \( \pi(x,y) = (xy,y) \) we have:
   \[
   \hat{l}(xy,y) = y^m + cx^{k_2}y^{k_2} + x^{k_1}y^{k_2+1}[q_{k_2-k_1+1} + \cdots]
   \]
   and because \( m > k_2 \)
   \[
   \hat{l}(x,y) = y^{m_1-k_2} + cx^{k_2} + x^{k_1}y^{k_2-k_1+1}(x,1) + \cdots
   \]
   with \( D_2 = \{ y = 0 \} \) e \( \hat{\hat{D}} = \{ x = 0 \} \). So \( (0,0) \) is a triple intersection point made up by \( D_2, \hat{S}, \hat{D} \). If instead we use the other chart we find only a double intersection points.
2. If \( m_1 < k_2 \) we proceed in the same way obtaining a triple intersection point made by \( D_2, \hat{D}_1 \) and \( \hat{S} \).
3. If \( m_1 = k_2 \) the tangent cone is given by \( \{ u^{m_1} + u^{k_1}q_{k_2-k_1}(u,v) = 0 \} \) and by the irreducibility of the curve it is \( \{ v + cu^{m_1} = 0 \} \) with \( c \neq 0 \) and it does not contain any exceptional divisor. So by Lemma 3.8 at the next blow-up we find only double intersection points.
3.2. **C.S.S. index under blow-up.** Now we can proceed in order to get step two studying the behaviour of the index in a general resolution process via blow-up. The upshot is to prove that in the resolution process we necessarily find a \((C_1)\) or \((C_2)\) point, i.e., \(p\) is an appropriate singularity and then Theorem 3.1 holds.

The intent is to analyze the C.S.S. index in all possible geometric evolutions (see Proposition 3.3).

**Remark 3.10.** In the analysis we will omit the case in which at some blow-up we find a dicritical point (see Definition 3.2 in [3]). In fact in this case the goal is obtained by Proposition 7.8 [3] and by the proper mapping theorem [9].

We will consider resolution processes only at a combinatoric level in a sense that will be specified later.

Thanks to Proposition 3.3 the structure of a resolution process of a singular point \(p\) is completely described by the behaviour of the multiplicity of the strict transform at the intersection with the exceptional divisor. We can then consider a sequence of blow-ups only as a sequence of positive number (representing the evolution of the multiplicity) and forgetting any type of geometric obstruction.

**Definition 3.11.** A **process** is an ordinate list of the form:

\[
P = \{ (k, m), (\alpha_1, m_1), \ldots, (\alpha_n, m_n) \}
\]

where \(k, \alpha_i, m_i \in \mathbb{N}\) and \(m > m_1 \geq \ldots \geq m_n\). We associate to \(P\), from a purely formal point of view, a blow-up sequence for a curve \(S\) where the blows-up are made at the beginning at the point \(p\) and then at the intersection point of the strict transform of the curve and the exceptional divisor. The blow-up sequence satisfies the following rules:

- from the first to the \(k\)-th blow-up we find only double intersection points and the curve multiplicity is constantly equal to \(m\),
- from the \((k + 1)\)-th to the \((k + \alpha_1)\)-th blow-up we find a triple intersection point and the multiplicity of the strict transform of \(S\) is constantly equal to \(m_1 < m\),
- from the \((k + \alpha_1 + \cdots + \alpha_{n-1} + 1)\)-th to the \((k + \alpha_1 + \cdots + \alpha_n)\)-th blow-up we find a triple intersection point and the multiplicity is constantly equal to \(m_n \leq m_{n-1}\).

**Remark 3.12.** At the end of \(P\) the curve \(S\) is not desingularized, in fact we have triple points and this type of point are not admitted in the desingularized curve.

Now, according to Proposition 3.3 we start to analyze all the possible cases. For notations we refer to [3] and [6].

3.3. **Case of double intersection.** It corresponds to a process \(P = \{ (k, m) \}\), i.e. we start with multiplicity \(m\) and we remain with this multiplicity for \(k\) blows-up finding only double points. If we do not find \((C_1)\) or \((C_2)\) points in the total transform then (arguing as in Proposition 7.8(2) of [3]) at the \(k\)-th blow-up the indices are of type:

\[
\text{Ind}(\tilde{\mathcal{F}}, D, q) \in \mathbb{Q}_{\leq -\frac{1}{k}}
\]

\[
\text{Ind}(\tilde{\mathcal{F}}; \hat{S}, q) \notin \mathbb{Q}_{\geq -km^2}.
\]

where \(q := \hat{S} \cap D\).
3.4. **Case of triple intersection.** We consider now a slightly more complicated process, \( P = \{(k, m), (1, m_1), \ldots, (1, m_n)\} \). Let us suppose not to find \((C_1)\) or \((C_2)\) points during \( P \).

We indicate at the last blow-up with \( S \) the strict transform of the curve, \( \mathcal{F} \) the saturated foliation, \( D_1, D_2 \) the two exceptional divisors that intersect, with \( S \), in the last triple intersection point \( q \).

**Proposition 3.13.** In this situation at the last blow-up of \( P \), if we have not found \((C_1)\) or \((C_2)\) points, we can find \( x, y \in \mathbb{N} \) and \( a, b \in \mathbb{N} \cup \{0\} \) such that the indices are:

\[
\begin{align*}
\text{Ind}(\mathcal{F}, S, q) &\not\in \mathbb{Q}_{\geq-km^2-m_1^2-\cdots-m_n^2} \\
\text{Ind}(\mathcal{F}, D_1, q) &\in \mathbb{Q}_{\leq-\frac{x}{y}} \\
\text{Ind}(\mathcal{F}, D_2, q) &\in \mathbb{Q}_{\leq-\frac{yk+1}{ax+b}}.
\end{align*}
\]

**Proof.** At the \( k \)-th blow-up the indices are of type \((3.1)\). Let blow-up again. As \( P \) describes we have a multiplicity decrease and we find a triple point on the total transform. Then if some point of the new exceptional divisor \( D_1 \) is of type \((C_1)\), \( p \) is an appropriate singularity and we have the assertion. Otherwise \( \text{Ind}(\mathcal{F}, D_1, p) \in \mathbb{Q}_{\geq 0} \forall p \in D_1 \setminus \{q\} \) and then by Index Theorem:

\[
\text{Ind}(\mathcal{F}, D_1, q) \in \mathbb{Q}_{\leq -1}.
\]

Then by Proposition 2.2 and observing that \( D \) has multiplicity one:

\[
\begin{align*}
\text{Ind}(\mathcal{F}, \hat{S}, q) &\not\in \mathbb{Q}_{\geq-km^2-m_1^2} \\
\text{Ind}(\mathcal{F}, \hat{D}_2, q) &\in \mathbb{Q}_{\leq-\frac{k+1}{k}}.
\end{align*}
\]

Proceeding by induction on \( n \) we can assume the assertion true for \( n \) and we prove it for \( n + 1 \).

We have to analyze separately two different cases that can occur blowing-up:

1. The new triple point is made by \( \{\hat{S}, \hat{D}_2, D\} \);
2. The new triple point is made by \( \{\hat{S}, \hat{D}_1, D\} \),

where \( D \) is the new exceptional divisor and \( D_1 \) and \( D_2 \) are the ones of the \( n \) blow-up whose indices satisfy \((3.2)\) by inductive hypothesis. We consider only the case (1) because the other is similar. By Proposition 2.2 the indices are of type:

\[
\begin{align*}
\text{Ind}(\mathcal{F}, \hat{S}, q_1) &= \text{Ind}(\mathcal{F}, S, p) - m_1^2 - \cdots - m_n^2 - m_{n+1}^2 \\
\text{Ind}(\mathcal{F}, \hat{D}_2, q_1) &\in \mathbb{Q}_{\leq-(x+y)(k+(a+b))} \\
\text{Ind}(\mathcal{F}, \hat{D}_1, q_0) &\in \mathbb{Q}_{\leq-(\frac{x+y}{k})}.
\end{align*}
\]

where \( q_1 \) is the new triple point and \( q_0 := \hat{D}_1 \cap D \). If there are not \((C_1)\) points on \( D \setminus \{q_0, q_1\} \) then by Index Theorem \( q_0 \) is a \((C_2)\) point or \( \text{Ind}(\mathcal{F}, D, q_1) \in \mathbb{Q}_{\leq-\frac{x}{x+y}} \). In the last case the indices satisfy:

\[
\begin{align*}
\text{Ind}(\mathcal{F}, \hat{S}, q_1) &= \text{Ind}(\mathcal{F}, S, p) - m_1^2 - \cdots - m_n^2 - m_{n+1}^2 \\
\text{Ind}(\mathcal{F}, \hat{D}_2, q_1) &\in \mathbb{Q}_{\leq-(\frac{x+y)(k+(a+b))}{k+b}} \\
\text{Ind}(\mathcal{F}, D, q_1) &\in \mathbb{Q}_{\leq-\frac{x}{x+y}}.
\end{align*}
\]

and then the assertion follows putting \( y' = x + y, x' = x, a' = a + b, b' = b \). \( \square \)
Remark 3.14. A general process can always be written in the form \( P = \{(k, m), (\alpha_1, m_1), \ldots, (\alpha_n, m_n)\} \) with \( m_i \neq m_j \) if \( i \neq j \). The coefficients \((x, y, a, b)\) that occur in \( P \), by Proposition 3.13 depend only on the \( \alpha_i \) and to the order in which they appear but not to the multiplicities \( m_i \) and the coefficient \( k \).

We propose now some simple properties of the index under a process that will be useful later:

Lemma 3.15. In (3.2) it follows that \( xa - yb = 1 \).

Proof. We proceed by induction on the number of blows-up and argue as in the proof of Proposition 3.13.

With the same arguments we can also prove:

Lemma 3.16. Let consider a process \( P = \{(k, m), (1, m_1), \ldots, (1, m_n)\} \) and indicate with \( S, D_1, D_2 \) the curves that create the triple intersection point. Then if \((x, y, a, b)\) are the coefficients that appear in the indices (3.2) we have, according to Remark 3.2:

- if \( x > y \) then \( D_2 \) is the new exceptional divisor and \( D_1 \) is the old one,
- if \( x \leq y \) then \( D_1 \) is the new exceptional divisor and \( D_2 \) is the old one.

Using Lemma 3.16 and Remark 3.9 we can easily prove:

Lemma 3.17. If we blow-up a triple intersection point and we have a multiplicity decrease then the coefficients \((x', y', a', b')\) of the indices of the new triple are such that:

- if \( x > y \) then \( x' = x, y' = x + y \),
- if \( x \leq y \) then \( x' = x + y, y' = y \).

In the analysis of the C.S.S. index in the triple intersection case the knowledge of the index is equivalent to the knowledge of the coefficients \((x, y, a, b)\). According to Remark 3.9 the decrease or not of the multiplicity gives different coefficients. In the next subsections we are going to investigate these cases. To make clearer the possible evolutions of the coefficients we report below the coefficients \((x, y, a, b)\) that can appear in the first five blows-up in triple intersection. We indicate in black the coefficients related to a decrease of multiplicity and in grey the others.
3.5. Transition from triple intersection with multiplicity lowering to triple with constant multiplicity. We consider a process of type $P = \{(k, m), (\alpha_1, m_1), \cdots, (\alpha_{n-1}, m_{n-1}), (\alpha_n, m_n)\}$ with $m_i \neq m_j$ if $i \neq j$. We want to relate the coefficients of the last blow-up with the ones obtained at the first lowering of multiplicity $m_{n-1} \rightarrow m_n$, i.e., we want to relate the last indices of the process $\{(k, m), (\alpha_1, m_1), \cdots, (\alpha_{n-1}, m_{n-1}), (1, m_n)\}$ to the last ones of $P$.

**Proposition 3.18.** Suppose that, after $k$ blows-up with constant multiplicity we know that the new triple point is made up by the curve, the new exceptional divisor and the strict transform of the old one divisor is multiplicity.

\[
\text{Ind}(\mathcal{F}, S, p) \notin \mathbb{Q}_{\geq -km^2-\alpha_1m_1^2-\cdots-\alpha_{n-1}m_{n-1}^2-\alpha_nm_n^2}
\]

\[
\text{Ind}(\mathcal{F}, D_1, p) \in \mathbb{Q}_{\leq \frac{a}{y}}
\]

\[
\text{Ind}(\mathcal{F}, D_2, p) \in \mathbb{Q}_{\leq \frac{y^k+a}{(x+(\alpha_n-1)y)k+(\alpha_n-1)a+b}}
\]

with $m_i \neq m_j$ if $i \neq j$, i.e., $n$ is the number of multiplicity lowerings. The indices at the end of the process $P$ are of type:

If $x > y$

\[
\text{Ind}(\mathcal{F}, S, p) \notin \mathbb{Q}_{\geq -km^2-\alpha_1m_1^2-\cdots-\alpha_{n-1}m_{n-1}^2-\alpha_nm_n^2}
\]

\[
\text{Ind}(\mathcal{F}, D_1, p) \in \mathbb{Q}_{\leq \frac{a+(\alpha_n-1)y}{y}}
\]

\[
\text{Ind}(\mathcal{F}, D_2, p) \in \mathbb{Q}_{\leq \frac{y^k+a}{(x+(\alpha_n-1)y)k+(\alpha_n-1)a+b}}
\]

If $x \leq y$

\[
\text{Ind}(\mathcal{F}, S, p) \notin \mathbb{Q}_{\geq -km^2-\alpha_1m_1^2-\cdots-\alpha_{n-1}m_{n-1}^2-\alpha_nm_n^2}
\]

\[
\text{Ind}(\mathcal{F}, D_1, p) \in \mathbb{Q}_{\leq \frac{(\alpha_n-1)x+y}{y}}
\]

\[
\text{Ind}(\mathcal{F}, D_2, p) \in \mathbb{Q}_{\leq \frac{y^k+a}{(x+(\alpha_n-1)y)k+(\alpha_n-1)a+b}}
\]

**Proof.** By Proposition 3.3 blowing-up with constant multiplicity we know that the new triple point is made up by the curve, the new exceptional divisor and the strict transform of the old one (see Remark 3.2). We have to analyze separately the case in which the old exceptional divisor is $D_1$ or $D_2$. This distinction can be made in terms of $x > y$ or $x \leq y$ thanks to Lemma 3.16. Suppose, for instance, $x > y$ in the indices (3.4), then we conclude that the old exceptional divisor is $D_1$. Now blowing-up again and using Proposition 2.2 the Index theorem and the assumption of non existence of $(C_1)$ or $(C_2)$ points we can prove the result for $\alpha_n = 1, 2$. Then proceeding by induction and repeating the same argument for the case $x \geq y$ we have the assertion.

3.6. Transition from triple to double intersection. Suppose that, after $k$ blows-up in double intersection and a finite number of blows-up in triple intersection, we return to double intersection. Let consider the generic indices of the triple (3.2) and we write the index along $S$ in the form:

\[
\text{Ind}(\mathcal{F}, S, p) \notin \mathbb{Q}_{\geq -km^2-\alpha_1m_1^2-\cdots-\alpha_{n-1}m_{n-1}^2-\alpha_nm_n^2},
\]

with $m_i \neq m_j$ if $i \neq j$. Using Lemma 3.15 we obtain that the indices in the double point we find are:

\[
\text{Ind}(\mathcal{F}, S, q) \notin \mathbb{Q}_{\geq -km^2-\alpha_1m_1^2-\cdots-\alpha_{n-1}m_{n-1}^2-\alpha_nm_n^2}
\]

\[
\text{Ind}(\mathcal{F}, D, q) \in \mathbb{Q}_{\leq \frac{1}{(x+y)^2k+(x+y)(a+b)}}
\]
3.7. Estimate of the term $km^2$. We estimate the term $-km^2 - \alpha_1 m_2 - \ldots - \alpha_n m_n - m_2^2$, showing that, if the curve is resolved, then $q$ is a point of type $(C_2)$; otherwise we obtain indices of the form (3.1) and so we can utilize again the results found in the previous sections in order to get desingularization. In this subsection we estimate the term $km^2$.

**Proposition 3.19.** If we indicate with $(x_i^j, y_i^j, a_i^j, b_i^j)$ the coefficients that appear in the indices of the triple intersection point at the $j$-th blow-up with multiplicity $m_i$ then, if $n \geq 2$:

\[
\begin{align*}
\text{(3.8)} & \quad m = x_{n-1}^{\alpha n-1} m_{n-1} + y_{n-1}^{\alpha n-1} m_n \text{ if } x_{n-1}^{\alpha n-1} \geq y_{n-1}^{\alpha n-1}, \\
\text{(3.9)} & \quad m = y_{n-1}^{\alpha n-1} m_{n-1} + x_{n-1}^{\alpha n-1} m_n \text{ if } y_{n-1}^{\alpha n-1} \geq x_{n-1}^{\alpha n-1}.
\end{align*}
\]

**Proof.** We proceed by induction on the number $n$ of changes of multiplicity. For $n = 2$ the indices are of the form:

\[
\begin{align*}
& \text{Ind}(F, S, p) \notin \mathbb{Q}_{\geq-km^2-\alpha_2 m_2^2}, \\
& \text{Ind}(F, D_1, p) \in \mathbb{Q}_{-\alpha_2+1}, \\
& \text{Ind}(F, D_2, p) \in \mathbb{Q}_{-\frac{\alpha_2+1}{\alpha_1+1}}.
\end{align*}
\]

The indices we find at the $\alpha_1$-th blow-up with multiplicity $m_1$ are:

\[
\begin{align*}
& \text{Ind}(F, S, p) \notin \mathbb{Q}_{\geq-km^2-\alpha_1 m_1^2}, \\
& \text{Ind}(F, D_1, p) \in \mathbb{Q}_{-\frac{\alpha_1+1}{\alpha_1}}, \\
& \text{Ind}(F, D_2, p) \in \mathbb{Q}_{-\frac{1}{m_1}}.
\end{align*}
\]

Because we make $\alpha_1$ blows-up with multiplicity $m_1$ and because the curve is irreducible by Enriques-Chisini theorem (5 pag. 516) we have:

\[
m_2 = m - \alpha_1 m_1
\]

and then the assertion. We prove the inductive step. The index along $S$ is:

\[
\text{Ind}(F, S, p) \notin \mathbb{Q}_{\geq-km^2-\alpha_1 m_1^2 - \ldots - \alpha_n m_n^2 - \alpha_n m_{n+1}^2 - \alpha_n m_{n+1} - \alpha_n m_{n+1} + m_{n+1}}
\]

We consider the case $x_{n-1}^{\alpha n-1} \geq y_{n-1}^{\alpha n-1}$ (the other is similar). Because we make $\alpha_n$ blows-up with multiplicity $m_n$ we have:

\[
m_{n+1} = m_{n-1} - \alpha_n m_n \text{ and then } m_{n-1} = \alpha_n m_n + m_{n+1}.
\]

By inductive hypothesis and the above relation we find an expression of $m$ in terms of $m_n$ and $m_{n+1}$. Now we have to prove that this expression is the one of the statement. Using Lemma 3.17 we have that $x_n^1 \leq y_n^1$ and for Proposition 3.18 the indices at the $\alpha_n$-th blow-up with multiplicity $m_n$ are:

\[
\begin{align*}
& \text{Ind}(F, S, p) \notin \mathbb{Q}_{\geq-km^2-\alpha_1 m_1^2 - \ldots - \alpha_n m_n^2 - \alpha_n m_n^2 - \alpha_n m_n - \alpha_n m_{n+1}^2}, \\
& \text{Ind}(F, D_1, p) \in \mathbb{Q}_{-\frac{1}{m_1}}, \\
& \text{Ind}(F, D_2, p) \in \mathbb{Q}_{-\frac{1}{m_1}}.
\end{align*}
\]

Clearly $x_n^{\alpha n} \leq y_n^{\alpha n}$ and so computing the expression $y_n^{\alpha n} m_n + x_n^{\alpha n} m_{n+1}$, using the above form of the coefficients and Lemma 3.17 we get the assertion. \qed
Proposition 3.20. When in the resolution process we return in double intersection the indices:

\[
\begin{align*}
\text{Ind}(\mathcal{F}, S, p) & \notin \mathbb{Q}_{\geq -km^2 - \alpha_1 m_1^2 - \cdots - \alpha_n m_n^2}^\times, \\
\text{Ind}(\mathcal{F}, D, q_0) & \in \mathbb{Q}_{\leq -\frac{1}{(x_{n+1} + y_{n+1})^2 k + (x_{n+1} + y_{n+1})^2}}^\times,
\end{align*}
\]

(3.11)

Propositions 3.22. The indices at the return in double intersection (3.11), with \( n \geq 2 \), satisfy:

\[
\alpha_1 m_1^2 + \cdots + \alpha_n m_n^2 + m_n^2 \geq (x_n^{\alpha_n} + y_n^{\alpha_n})(a_n^{\alpha_n} + b_n^{\alpha_n})m_n^2.
\]

Before proving this statement we consider the following one:

Proposition 3.22. Let \( P = \{(k, m), (\alpha_1, m_1), \cdots, (\alpha_n, m_n)\} \) be a process and let indicate with \((x, y, a, b)\) the coefficients of the indices that appear at the last blow-up described by \( P \). We associate to \( P \) the process \( \tilde{P} = \{(k, m), (\alpha_2, m_2), \cdots, (\alpha_n, m_n)\} \) and we indicate with \((\bar{x}, \bar{y}, \bar{a}, \bar{b})\) the coefficients of the indices that appear at the last blow-up described by \( \tilde{P} \). Then:

\[
\begin{align*}
b &= \bar{y} \\
a &= \bar{x} \\
x &= \alpha_1 \bar{y} + \bar{a} \\
y &= \alpha_1 \bar{x} + \bar{b}
\end{align*}
\]

Proof. We proceed by induction on the number \( n \) of multiplicities decreases. By a direct calculation the Proposition is true for \( n = 2 \). Suppose the assertion true for \( n \) and let prove it for \( n + 1 \).

Let consider the two processes \( P' = \{(k, m), (\alpha_1, m_1), \cdots, (\alpha_n, m_n), (\alpha_{n+1}, m_{n+1})\} \) and \( \tilde{P}' = \{(k, m), (\alpha_2, m_2), \cdots, (\alpha_n, m_n), (\alpha_{n+1}, m_{n+1})\} \) with respectively end coefficients \((x', y', a', b')\) and \((\bar{x}', \bar{y}', \bar{a}', \bar{b}')\).

Let now construct the following two processes \( P = \{(k, m), (\alpha_1, m_1), \cdots, (\alpha_n, m_n)\} \), \( \tilde{P} = \{(k, m), (\alpha_2, m_2), \cdots, (\alpha_n, m_n)\} \) with end coefficients \((x, y, a, b)\) and \((\bar{x}, \bar{y}, \bar{a}, \bar{b})\). Starting by coefficients \((x, y, a, b)\) we get \((x', y', a', b')\) after one blow-up with multiplicity decrease and other \( \alpha_{n+1} - 1 \) blows-up with constant multiplicity \( m_{n+1} \). By Propositions 3.3 and 3.18:

\[
\begin{align*}
(x', y', a', b') &= (x, y + \alpha_n x, a + \alpha_n b, b) \text{ if } x > y, \\
(x', y', a', b') &= (x + \alpha_n y, y, a, \alpha_n a + b) \text{ if } x \leq y.
\end{align*}
\]

Similarly we get:

\[
\begin{align*}
(\bar{x}', \bar{y}', \bar{a}', \bar{b}') &= (\bar{x}, \bar{y} + \alpha_n \bar{x}, \bar{a} + \alpha_n \bar{b}, \bar{b}) \text{ if } \bar{x} > \bar{y}, \\
(\bar{x}', \bar{y}', \bar{a}', \bar{b}') &= (\bar{x} + \alpha_n \bar{y}, \bar{y}, \bar{a}, \alpha_n \bar{a} + \bar{b}) \text{ if } \bar{x} \leq \bar{y}.
\end{align*}
\]

The processes \( P \) and \( P' \) differs only on one multiplicity decrease. Propositions 3.3 and 3.18 say that \( x \) and \( y \) relations invert only when a multiplicity decrease occurs. Then we can conclude...
that $\bar{x} > \bar{y}$ if and only if $x \leq y$. If, for instance, $x > y$, by inductive hypothesis:

\[
\begin{align*}
\bar{b}' &= b = \bar{y}' = \bar{y}, \\
\bar{a}' &= a + \alpha_n b = \bar{x} + \alpha_n \bar{y} = \bar{x}', \\
x' &= x = \alpha_1 \bar{y} + \bar{a} = \alpha_1 \bar{y}' + \bar{a}', \\
y' &= y + \alpha_n x = \alpha_1 \bar{x} + \bar{b} + \alpha_n \bar{y} + \alpha_n \bar{a} = \alpha_1 (\bar{x} + \alpha_n \bar{y}) + (\bar{b} + \alpha_n \bar{a}) = \alpha_1 \bar{x}' + \bar{b}'.
\end{align*}
\]

and then the assertion. \qed

Now we can prove Proposition 3.21.

**Proof.** Let proceed by induction on the number of changes of multiplicity. If $n = 2$ the structure of the indices can be easily computed to obtain the assertion. Let prove the inductive step. Let $P = \{(k, m), (\alpha_1, m_1), \ldots, (\alpha_n, m_n), (\alpha_{n+1}, m_{n+1})\}$ be a generic process. Thanks to the inductive step applied on the process $\bar{P} = \{(k, m), (\alpha_2, m_2), \ldots, (\alpha_{n+1}, m_{n+1})\}$ we have:

\[
\alpha_1 m_1^2 + \cdots + \alpha_n m_n^2 + \alpha_{n+1} m_{n+1}^2 \geq \alpha_1 m_1^2 + (\bar{x} + \bar{y})(\bar{a} + \bar{b}) m_{n+1}^2.
\]

In order to estimate $\alpha_1 m_1^2$ we consider the process $P' = \{(\alpha_1, m_1), (\alpha_2, m_2), \ldots, (\alpha_{n+1}, m_{n+1})\}$ and thanks to Remark 3.14 and Proposition 3.20 we have:

\[
m_1^2 \geq (\bar{x} + \bar{y})^2 m_{n+1}^2.
\]

Then:

\[
\alpha_1 m_1^2 + \cdots + \alpha_n m_n^2 + \alpha_{n+1} m_{n+1}^2 \geq \alpha_1 (\bar{x} + \bar{y})^2 m_{n+1}^2 + (\bar{x} + \bar{y})(\bar{a} + \bar{b}) m_{n+1}^2
\]

\[
= (\bar{x} + \bar{y})(\alpha_1 \bar{x} + \alpha_1 \bar{y} + \bar{a} + \bar{b}) m_{n+1}^2.
\]

We conclude thanks to Proposition 3.22. \qed

**Remark 3.23.** The estimate of $km^2$ and of the remaining terms are valid only if $n \geq 2$. The case $n = 1$ can be easily proved using equation (3.10), Section 3.6 and observing that because of the $\alpha_1 + 1$ blows-up $m \geq (\alpha_1 + 1)m_1$.

3.9. **Proof of the Theorem.** All the previous separate particular cases can now be glued together to get Theorem 1.1. We have observed that in the resolution process we can have only double or triple intersection points and so we studied the index in these cases.

The triple point case presents two different subcases, linked to the multiplicity of the curve: it can decrease or not. This information is extremely useful for the study of the index evolution because it identifies the right exceptional divisor that will occur in the next triple point. Now we observe that if at the end of a process $P = \{(k, m), (\alpha_1, m_1), \ldots, (\alpha_n, m_n)\}$ we find a double point and the curve is desingularized we are in the geometric conditions of a $(C_2)$ point. The indices are the ones given by equation (3.7) and by Propositions 3.20 and 3.21 we can estimate them in such a way they became:

\[
\begin{align*}
\text{Ind}(\mathcal{F}, S, p) &\notin \mathbb{Q}_{\geq -[(x+y)k^2+(x+y)(a+b)]} \\
\text{Ind}(\mathcal{F}, D, p) &\in \mathbb{Q}_{\leq \frac{1}{(x+y)^2k+(x+y)(a+b)}}
\end{align*}
\]
and so $p$ is a $(C_2)$ point. Otherwise we are not in the right geometric conditions, i.e. the resolution is not just ended, but the same propositions gives indices:

$$\text{Ind}(\mathcal{F}, S, p) \notin \mathbb{Q}_{\geq -[(x+y)k^2+(x+y)(a+b)]m^2_n}$$
$$\text{Ind}(\mathcal{F}, D, p) \in \mathbb{Q}_{\leq (x+y)^2k+(x+y)(a+b)}$$

and so we have indices exactly of the form of the ones associated to a process $P = \{hm^2\}$ and then we can apply all the previous argument to the new process which is starting. Such process terminates after a finite number of blows-up by theorem of resolution of singularities [10].

### 4. Applications

The demonstrative method used to get Theorem [11] allows us to generalize to the case in which we start with more than one separatrix:

**Proposition 4.1.** Let $M$ be a two dimensional complex manifold, $\mathcal{F}$ an holomorphic foliation on $M$ and $S_0, S_1, \ldots, S_n$ separatrices of $\mathcal{F}$ passing through a point $p \in M$. Let assume that $S_1, \ldots, S_n$ are non singular and transverse each other and to $S_0$. If, besides, the indices are of the sequent form:

$$\text{Ind}(\mathcal{F}, S_0, p) \notin \mathbb{Q}_{\geq -m^2}$$
$$\text{Ind}(\mathcal{F}, S_i, p) \in \mathbb{Q}_{\leq -(2n-1)} \quad \forall i \geq 1,$$

then another separatrix through $p$ exists.

**Proof.** We prove that $p$ is an appropriate singularity. We observe that after the first blow-up, if we have not finished, we have the same indices found in the study made to prove Theorem [11] and so we conclude with the same argument. 

We show briefly that Theorem [11] includes as particular cases the classical results in discrete and continuous dynamics.

**Corollary 4.2 ([7]).** Let $M$ be a two dimensional complex manifold, $\mathcal{F}$ an holomorphic foliation on $M$ and $p \in M$ a singularity of $\mathcal{F}$. Then a separatrix of $\mathcal{F}$ for $p$ exists.

**Proof.** We blow-up $M$ in $p$. If the exceptional divisor is not a separatrix for the saturated foliation, Proposition 1 in [6] (pag.15) concludes. Otherwise using the index theorem (see [11]) and remembering that $D \cdot D = -1$, we obtain the existence of a singularity $\tilde{p}$ of the saturated foliation $\tilde{\mathcal{F}}$ such that $\text{Ind}(\tilde{\mathcal{F}}, D, \tilde{p}) \notin \mathbb{Q}^+ \cup \{0\}$ and then by Theorem [11] we have the existence of another separatrix for $\tilde{p}$ that projects in a separatrix for $\mathcal{F}$ in $p$. 

With similar arguments we also have:

**Corollary 4.3 ([13]).** Let $M$ be a two dimensional complex manifold, $\mathcal{F}$ an holomorphic foliation on $M$. Let $S \subset M$ be a compact, globally and locally irreducible curve with $S \cdot S < 0$. If $S$ is a separatrix for $\mathcal{F}$ then a point $p \in S$ for which passes another separatrix exists.

**Remark 4.4.** Analogously to what said for Theorem [12] we can obtain, in diffeomorphisms dynamics, a similar result to Proposition [4.1] and find as particular cases results of Abate [2] and Bracci [3].
Proposition 4.5. Let $M$ be a two dimensional complex manifold, $f : M \rightarrow M$ an holomorphic map on $M$ with $\text{Fix}(f) = S_0 \cup S_1, \ldots, \cup S_n$ with $S_0, \ldots, S_n$ analytic curves passing through the same point $p \in M$. Let suppose that $S_1, \ldots, S_n$ are non singular and transverse each other and to $S_0$. If, besides, the indices are of the sequent form:

$$\text{Ind}(f, S_0, p) \notin \mathbb{Q}_{\geq m^2}$$

$$\text{Ind}(f, S_i, p) \in \mathbb{Q}_{\leq -(2n-1)} \quad \forall i \geq 1,$$

then a parabolic curve through $p$ exists.

Corollary 4.6 ([2]). Let $M$ be a two dimensional complex manifold, $f : M \rightarrow M$ an holomorphic map on $M$ and $p \in M$ an isolated singularity of $f$ such that $df_p = Id$. Then a parabolic curve for $f$ through $p$ exists.

Corollary 4.7 ([13]). Let $M$ be a two dimensional complex manifold, $f : M \rightarrow M$ an holomorphic map on $M$. Let $S \subset M$ be a compact, globally and locally irreducible curve with $S \cdot S < 0$. If $f = S$ and $f$ is non degenerate along $S$ is then a point $p \in S$ for which passes a parabolic curve exists.

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