SEMISIMPLE WEAK HOPF ALGEBRAS

DMITRI NIKSHEYCH

Abstract. We develop the theory of semisimple weak Hopf algebras and obtain analogues of a number of classical results for ordinary semisimple Hopf algebras. We prove a criterion for semisimplicity and analyze the square of the antipode \( S^2 \) of a semisimple weak Hopf algebra \( A \). We explain how the Frobenius-Perron dimensions of irreducible \( A \)-modules and eigenvalues of \( S^2 \) can be computed using the inclusion matrix associated to \( A \). A trace formula of Larson and Radford is extended to a relation between the categorical and Frobenius-Perron dimensions of \( A \). Finally, an analogue of the Class Equation of Kac and Zhu is established and properties of \( A \)-module algebras and their dimensions are studied.

1. Introduction

In [BSz1] G. Böhm and K. Szlachányi introduced weak Hopf algebras as a generalization of ordinary Hopf algebras. It was observed in [NV2] that these objects also extend the theory of groupoid algebras. A general theory of weak Hopf algebras was subsequently developed in [BNSz]. Briefly, a weak Hopf algebra is a vector space with structures of an algebra and coalgebra related to each other in a certain self-dual way. Weak Hopf algebras naturally appear in different areas of mathematics, including algebra, functional analysis, and representation theory [KN NV1 EN NTV]; we refer the reader to [NV2] for a survey of the subject.

In this paper we begin the study of semisimple weak Hopf algebras over an algebraically closed field of characteristic 0. These are the most accessible weak Hopf algebras that frequently appear in applications. In particular, the representation-theoretic importance of semisimple weak Hopf algebras can be seen from the result of V. Ostrik [O], who proved that every semisimple rigid monoidal category with finitely many classes of simple objects (such categories are called fusion categories) is equivalent to the representation category of a semisimple weak Hopf algebra. Weak Hopf algebra techniques were used to prove results about fusion categories in [N2 ENO]. In [ENO] it was shown that regular semisimple weak Hopf algebras and fusion categories do not admit continuous deformations (this extends the result of D. Stefan for semisimple Hopf algebras [S]). Here the regularity condition ensures the absence of “trivial” deformations [N2 Remark 3.7]. In particular, this implies that a classification of semisimple weak Hopf algebras is possible. Such a classification will include the classification of fusion categories and module categories over them and will be useful in operator algebras, quantum field theory, and representation theory of quantum groups. In order to proceed with this classification project it is first necessary to bring the theory of semisimple weak Hopf algebras and fusion categories to roughly the same level as that of semisimple Hopf algebras. The present paper is a step in this direction.

Date: September 18.
Semisimple Hopf algebras is one of the most established subjects of Hopf algebra theory. A number of structural and classification theorems is known, see surveys \[1, 2, 3\]. In this paper we extend several classical results from ordinary to weak Hopf algebras. Below we describe these results and organization of the paper.

In Section 2 we recall the definition and basic properties of weak Hopf algebras and their representation categories.

In Section 3 we first define a canonical integral of a weak Hopf algebra \( A \) and give a list of conditions equivalent to the semisimplicity of \( A \) in Proposition 3.1.5. Next, for a semisimple \( A \) we study two notions of dimensions of irreducible \( A \)-modules : the quantum and Frobenius-Perron dimensions introduced in [ENO]. We show that the vector of Frobenius-Perron dimensions of irreducible \( A \)-modules is a Frobenius-Perron eigenvector (i.e., an eigenvector with strictly positive entries) of the integer matrix \( \Lambda^t \Lambda \), where \( \Lambda \) is the matrix describing the inclusion of the base algebra \( A_s \) in \( A \) (Proposition 3.3.10). This extends a result of [BSz2] for weak Hopf \( C^* \)-algebras. In Proposition 3.4.2 we give in terms of \( \Lambda \) a sufficient condition of the triviality of a group-like element \( G \) implementing the square of the antipode of \( A \) (a pivotal element of \( A \)). In the case when \( A \) is pseudo-unitary we show in Corollary 3.4.5 that \( G = w S(w)^{-1} \) where \( w \) is a Frobenius-Perron element of the base \( A_s \). In particular, in this case the eigenvalues of \( S^2 \) are all positive and can easily be found.

In Section 4 we study the Grothendieck ring of a semisimple weak Hopf algebra and extend in Theorem 4.2.4 the Class Equation of G. Kac and Y. Zhu [K, Z1] (a version of this equation for pivotal fusion categories was proved in [ENO]).

In Section 5 we establish analogues of the trace formulas of Larson and Radford [LR1, LR2]. The first formula (Corollary 5.1.1) expresses \( Tr(S^2|_A) \) in terms of dual integrals in \( A \) and \( A^* \). The second formula shows that the categorical dimension \( \dim(A) \) of \( A \) divides its Frobenius-Perron dimension \( \text{FPdim}(A) \) in the ring of algebraic integers (Theorem 5.2.2). As a consequence we obtain that a weak Hopf algebra is pseudo-unitary if and only if all the eigenvalues of \( S^2 \) are positive.

Section 6 deals with properties of module algebras and coalgebras over a semisimple weak Hopf algebra \( A \). In Theorem 6.1.3 we extend the result of [Li] to prove that if \( A \) is a pseudo-unitary weak Hopf algebra then the Jacobson radical of any finite-dimensional \( A \)-module algebra is \( A \)-stable. In Theorem 6.2.2 we extend the result of [Z2] to Frobenius-Perron dimensions of comodule algebras over a semisimple weak Hopf algebra \( A \) : we show that if \( M \) is an indecomposable finite-dimensional semisimple \( A \)-comodule algebra, then for all irreducible \( M \)-modules \( M_1 \) and \( M_2 \) the number \( \text{FPdim}(A)\text{FPdim}(M_1)\text{FPdim}(M_2)/\text{FPdim}(M) \), where \( \text{FPdim} \) denotes the Frobenius-Perron dimension, is an algebraic integer. This generalizes the well-known fact that for a finite group \( G \) the cardinality of a transitive \( G \)-set divides the order of \( G \).

Acknowledgements. This research was supported by the NSF grant DMS-0200202. The author is grateful to Pavel Etingof for pointing out an error in an early version of this paper and to Victor Ostrik and Leonid Vainerman for valuable discussions. Thanks are also due to the referee whose comments helped to improve the presentation.
2. Preliminaries

2.1. Definition and basic properties of weak Hopf algebras. Let $k$ be an algebraically closed field of characteristic 0. All vector spaces and tensor products in this paper are over $k$.

One obtains a weak Hopf algebra by relaxing the axioms related to the unit and counit in the definition of an ordinary Hopf algebra.

Definition 2.1.1 ([BSz1, BNSz]). A weak Hopf algebra is a vector space $A$ with the structures of an associative algebra $(A, m, 1)$ with a multiplication $m : A \otimes A \to A$ and unit $1 \in A$ and a coassociative coalgebra $(A, \Delta, \epsilon)$ with a comultiplication $\Delta : A \to A \otimes A$ and counit $\epsilon : A \to k$ such that:

(i) The comultiplication $\Delta$ is a (not necessarily unit-preserving) homomorphism of algebras:

$$\Delta(gh) = \Delta(g)\Delta(h),$$

(ii) The unit and counit satisfy the following identities:

$$\epsilon(fgh) = \epsilon(fg(1))\epsilon(g(2)h) = \epsilon(fg(2))\epsilon(g(1)h),$$

(iii) There is a linear map $S : A \to A$, called an antipode, such that

$$m(id \otimes S)\Delta(h) = (\epsilon \otimes id)(\Delta(1)(h \otimes 1)),$$

$$m(S \otimes id)\Delta(h) = (id \otimes \epsilon)((1 \otimes h)\Delta(1)),$$

$$S(h) = S(h(1))h(2)S(h(3)),$$

for all $f, g, h \in A$.

Remark 2.1.2. We use Sweedler’s notation for a comultiplication in a coalgebra $C$, writing $\Delta(c) = c(1) \otimes c(2)$ for all $c \in C$.

Axioms (2) and (3) above weaken the usual bialgebra axioms requiring $\Delta$ to preserve the unit and $\epsilon$ to be an algebra homomorphism. Axioms (4) and (5) generalize the properties of the antipode in a Hopf algebra. The antipode $S$ of a weak Hopf algebra is an algebra and coalgebra antihomomorphism. If $A$ is finite-dimensional then $S$ is bijective [BNSz, 2.10].

Remark 2.1.3. A weak Hopf algebra is a Hopf algebra if and only if $\Delta(1) = 1 \otimes 1$ and if and only if $\epsilon$ is a homomorphism of algebras.

A morphism between weak Hopf algebras $A_1$ and $A_2$ is a map $\phi : A_1 \to A_2$ which is both algebra and coalgebra homomorphism preserving $1$ and $\epsilon$ and which intertwines the antipodes of $A_1$ and $A_2$, i.e., $\phi \circ S_1 = S_2 \circ \phi$. The image of a morphism is clearly a weak Hopf algebra.

For a finite-dimensional $A$ there is a natural weak Hopf algebra structure on the dual vector space $A^* = \text{Hom}_k(A, k)$ given by

$$\langle \phi \psi, h \rangle = \langle (\phi \otimes \psi), \Delta(h) \rangle,$$

$$\langle \Delta(\phi), g \otimes h \rangle = \langle \phi, gh \rangle,$$

$$\langle S(\phi), h \rangle = \langle \phi, S(h) \rangle,$$

for all $\phi, \psi \in A^*$, $g, h \in A$. The unit of $A^*$ is $\epsilon$ and the counit is $\phi \mapsto \langle \phi, 1 \rangle$. 
The linear maps defined by (4) and (5) are called target and source counital maps and denoted \( \varepsilon_t \) and \( \varepsilon_s \) respectively:
\[
\varepsilon_t(h) = \epsilon(1)(h)1(2), \quad \varepsilon_s(h) = 1(1)\epsilon(h)1(2),
\]
for all \( h \in A \).

An algebra \( M \) is a left \( A \)-comodule algebra if \( M \) is a left \( A \)-comodule via \( \delta : M \to A \otimes M : m \mapsto m^{(1)} \otimes m^{(2)} \) such that
\[
\delta(mn) = \delta(m)\delta(n), \quad \delta(1) = (\varepsilon_s \otimes \text{id})\delta(1),
\]
for all \( m, n \in M \).

An algebra \( M \) is a left \( A \)-module algebra if \( M \) is a left \( A \)-module via \( h \otimes m \mapsto h \cdot m \) such that
\[
h \cdot (mn) = (h^{(1)} \cdot m)(h^{(2)} \cdot n), \quad h \cdot 1 = \varepsilon_t(h),
\]
for all \( h \in A \) and \( m, n \in M \).

The definitions of right module and comodule algebras are similar.

**Remark 2.1.4.** If \( A \) is finite-dimensional then a left \( A \)-comodule algebra \( M \) is a right \( A^* \)-module algebra via
\[
m \cdot \phi = \langle \phi, m^{(1)} \rangle m^{(2)}, \quad \phi \in A^*, m \in M.
\]

### 2.2. Basic properties of weak Hopf algebras

The images of the counital maps defined in (10),
\[
A_t = \varepsilon_t(A), \quad A_s = \varepsilon_s(A)
\]
are semisimple subalgebras of \( A \), called target and source bases or counital subalgebras of \( A \). These subalgebras commute with each other; moreover
\[
A_t = \{ (\phi \otimes \text{id})\Delta(1) \mid \phi \in A^* \} = \{ h \in A \mid \Delta(h) = \Delta(1)(h \otimes 1) \},
\]
\[
A_s = \{ (\text{id} \otimes \phi)\Delta(1) \mid \phi \in A^* \} = \{ h \in A \mid \Delta(h) = (1 \otimes h)\Delta(1) \}.
\]

For any algebra \( B \) we denote by \( Z(B) \) the center of \( B \).

If \( p \neq 0 \) is an idempotent in \( A_t \cap A_s \cap Z(A) \), then \( A \) is the direct sum of weak Hopf algebras \( pA \) and \( (1 - p)A \). Consequently, we say that \( A \) is indecomposable if \( A_t \cap A_s \cap Z(A) = k1 \).

Every weak Hopf algebra \( A \) contains a canonical minimal weak Hopf subalgebra \( A_{\text{min}} \) generated, as an algebra, by \( A_s \) and \( A_t \) [N2, Section 3]. Obviously, \( A \) is an ordinary Hopf algebra if and only if \( A_{\text{min}} = k1 \). Minimal weak Hopf algebras over \( k \), i.e., those for which \( A = A_{\text{min}} \), were completely classified in [N2, Proposition 3.4].

The restriction of \( S^2 \) to \( A_{\text{min}} \) is always an inner automorphism of \( A_{\text{min}} \), see [N2].

**Note 2.2.1.** In what follows we will consider only weak Hopf algebras satisfying the following natural property:
\[
S^2|_{A_{\text{min}}} = \text{id}.
\]

**Definition 2.2.2.** We will call a weak Hopf algebra satisfying (15) regular.

**Remark 2.2.3.** It was shown in [NV2, 6.1] that every weak Hopf algebra can be obtained by twisting a regular weak Hopf algebra with the same algebra structure.
2.3. **Rigid monoidal categories and fusion categories.** Recall that a *monoidal category* consists of a category $\mathcal{C}$, a tensor product bifunctor $\boxtimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object $E$, and natural equivalences

\begin{equation}
\boxtimes(id \times \boxtimes) \cong \boxtimes(\boxtimes \times id), \quad ? \boxtimes E \cong id, \quad E\boxtimes? \cong id,
\end{equation}

satisfying the pentagon and triangle diagrams $[\text{Ma}] [\text{BK}].$

If $V$ is an object in $\mathcal{C}$ then a right dual of $V$ is an object $V^*$ together with two morphisms, $b_V: E \to V \boxtimes V^*$ and $d_V: V^* \boxtimes V \to E$, called coevaluation and evaluation, such that

\begin{align}
(id_V \boxtimes d_V )(b_V \boxtimes id_V ) &= id_V , \\
(d_V \boxtimes id_{V^*} )(id_{V^*} \boxtimes b_V ) &= id_{V^*} .
\end{align}

The definition of a left dual object is similar, see $[\text{BK}]$. A monoidal category $\mathcal{C}$ is called *rigid* if every object in $\mathcal{C}$ has right and left dual objects.

**Definition 2.3.1.** A *fusion category* is an Abelian semisimple rigid monoidal category $\mathcal{C}$ that has finitely many simple objects and is such that the unit object $E$ is simple and all Hom-spaces of $\mathcal{C}$ are finite-dimensional.

A left *module category* over a monoidal category $\mathcal{C}$ is an Abelian category $\mathcal{M}$ together with a bifunctor $\otimes: \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ and natural equivalences

\begin{equation}
\otimes(id \times \otimes) \cong \otimes(\otimes \times id), \quad E\otimes? \cong id,
\end{equation}

satisfying pentagon and triangle axioms; see $[\text{O}]$ for details.

2.4. **Representation category of a weak Hopf algebra and reconstruction of fusion categories.** The category $\text{Rep}(A)$ of finite-dimensional left $A$-modules has a natural structure of a rigid tensor category that we describe next, following $[\text{NTV}]$. The tensor product of two $A$-modules $V$ and $W$ is given by

\begin{equation}
V \boxtimes W := V \otimes_{A_t} W = \{ x \in V \otimes_k W | \Delta(1)x = x \},
\end{equation}

where the right action of $A_t$ on $V$ is by $vz := S(z)v$, $z \in A_t$, $v \in V$ and the $A$-module structure defined via $\Delta$. The tensor product of morphisms is defined in an obvious way. The unit object $E$ of $\text{Rep}(A)$ is the target counital subalgebra $A_t$ with the action $h \cdot z = \varepsilon_1(hz)$ for all $h \in A$, $z \in A_t$.

The unit object of $\text{Rep}(A)$ is irreducible if and only if bases of $A$ intersect trivially with the center of $A$, i.e., $Z(A) \cap A_t = k$.

**Definition 2.4.1.** If $Z(A) \cap A_t = k$ we will say that $A$ is *connected*. We will say that $A$ is *cocommected* if $A^*$ is connected and that $A$ is *biconnected* if it is both connected and cocommected.

**Remark 2.4.2.** $A$ is cocommected if and only if $A_s \cap A_t = k$ $[\text{NJ} 3.11]$.

If $V$ is an $A$-module then $V^* := \text{Hom}_k(V, k)$ is also an $A$-module via

\begin{equation}
\langle h \cdot \phi, v \rangle = \langle \phi, S(h) \cdot v \rangle,
\end{equation}

for all $h \in A$, $\phi \in A^*$, $v \in V$.

For any $V$ in $\text{Rep}(A)$, we define evaluation and coevaluation morphisms

\[ d_V: V^* \boxtimes V \to A_t, \quad b_V: A_t \to V \boxtimes V^*, \]
as follows. For \( \sum_j \phi^j \otimes v_j \in V^* \otimes V \), set

\[
d_V(\sum_j \phi^j \otimes v_j) = \sum_j \phi^j(1(1) \cdot v_j)1(2).
\]

Let \( \{f_i\} \) and \( \{\xi^i\} \) be bases of \( V \) and \( V^* \) dual to each other, then \( \sum_i f_i \otimes \xi^i \) does not depend on choice of these bases. Set

\[
b_V(z) = z \cdot (\sum_i f_i \otimes \xi^i).
\]

It was checked in [NTV] that \( d_V \) and \( b_V \) are well defined \( A \)-linear maps satisfying identities (17) and (18). Thus, \( \text{Rep}(A) \) becomes a rigid monoidal category. Clearly, \( \text{Rep}(A) \) is a fusion category if and only if \( A \) is semisimple and connected.

Let \( \mathcal{M} \) be a semisimple left module category over a fusion category \( \mathcal{C} \) such that \( \mathcal{M} \) has finitely many simple objects and all Hom-spaces of \( \mathcal{M} \) are finite-dimensional. It was shown in [Q] that there exists a biconnected semisimple weak Hopf algebra \( A \) such that \( \text{Rep}(A) \cong \mathcal{C} \) as fusion categories and \( \text{Rep}(A_t) \cong \mathcal{M} \) as module categories over \( \mathcal{C} \). Such a weak Hopf algebra \( A \) is not unique. In particular, the base \( A_t \) is defined up to a Morita equivalence, and one can always choose \( A_t \) to be a commutative algebra, in which case \( \mathcal{A} \) is regular in the sense of Definition (2.2).

Thus, one can study fusion categories and their module categories using weak Hopf algebra techniques.

2.5. Grothendieck rings and modules and Frobenius-Perron dimensions. Let \( A \) be a semisimple weak Hopf algebra. The Grothendieck ring \( K_0(A) \) is defined as follows. As an Abelian group it is generated by characters \( \chi_V \) of finite-dimensional \( A \)-modules \( V \) with the addition \( \chi_U + \chi_V = \chi_{U \oplus V} \) and multiplication \( \chi_U \chi_V = \chi_{U \otimes V} \) for all \( A \)-modules \( U \) and \( V \). The unit is the character of the trivial \( A \)-module. The map \( \chi_V \mapsto \chi_V \) extends to an anti-multiplicative involution of \( K_0(A) \) since \( S^2 \) is an inner automorphism of \( A \). Note that our definition does not differ from the usual one (see, e.g., [Lo]) since in the semisimple case we can identify finite-dimensional \( A \)-modules with their characters.

The ring \( K_0(A) \) has a \( \mathbb{Z} \)-basis consisting of characters \( \{\chi_1, \ldots, \chi_n\} \) of irreducible \( A \)-modules \( \{V_1, \ldots, V_n\} \). For every \( A \)-module \( V \) with the character \( \chi_V \) the matrix \( [\chi_V] \) of the left multiplication by \( \chi_V \) in the basis \( \{\chi_1, \ldots, \chi_n\} \) has non-negative entries and is not nilpotent. By the Frobenius-Perron theorem [Ga] the matrix \( [\chi_V] \) has a real non-negative eigenvalue. The largest among such eigenvalues we will call the Frobenius-Perron dimension of \( V \) and denote \( \text{FPdim}(V) \). It was shown in [ENO] that the map \( \phi : V \mapsto \text{FPdim}(V) \) defines a ring homomorphism \( K_0(A) \to \mathbb{R} \). This \( \phi \) is a unique homomorphism \( K_0(A) \to \mathbb{R} \) with the property \( \phi(\chi_i) > 0 \) for all \( i = 1, \ldots, n \).

Definition 2.5.1. Define the character algebra of \( A \) to be \( R(A) = K_0(A) \otimes_{\mathbb{Z}} \mathbb{C} \). If \( k = \mathbb{C} \), then \( R(A) \) may be regarded as a subalgebra of \( A^* \).

If \( M \) is a finite-dimensional semisimple \( A \)-comodule algebra via \( \delta : M \to A \otimes M \) then the Grothendieck group \( K_0(M) \) becomes a left \( K_0(A) \)-module (with non-negative integer structure constants) via

\[
\langle \chi \xi, m \rangle = \langle \chi \otimes \xi, \delta(m) \rangle,
\]

for all characters \( \chi \in K_0(A) \), \( \xi \in K_0(M) \) and \( m \in M \). Let \( R(M) = K_0(M) \otimes_{\mathbb{Z}} \mathbb{C} \).
Lemma 2.5.3. The above map \( \psi \) is a unique, up to a positive scalar multiple, \( K_0(A) \)-module homomorphism from \( K_0(M) \) to \( \mathbb{R} \) with the property \( \psi(\xi_i) > 0 \) for all \( i = 1, \ldots, t \). The following statement will be used in the sequel.

**Proof.** Let \([\chi_V]\) be the matrix of multiplication by \( \chi_V \) in the basis \( \{\xi_1, \ldots, \xi_t\} \) and let \( \tilde{m} \) be the column vector with entries \( \psi(\xi_i), i = 1, \ldots, t \). Then \([\chi_V]\)\(\tilde{m} = \psi(\tilde{m})\). Since \( \tilde{m} \) has strictly positive entries it belongs to the Frobenius-Perron eigenspace of \([\chi_V]\) for every \( A \)-module \( V \). Choosing \( V \) in such a way that \([\chi_V]\) has strictly positive entries we conclude that \( \tilde{m} \) is defined up to a scalar multiple (since in this case the Frobenius-Perron eigenspace of \([\chi_V]\) is 1-dimensional).

3. **Integrals, semisimplicity, and dimension theory**

3.1. **Integrals in weak Hopf algebras.** The following notion of an integral in a weak Hopf algebra is a generalization of that of an integral in a usual Hopf algebra.

**Definition 3.1.1** ([BNSZ]). A left (respectively, right) integral in a weak Hopf algebra \( A \) is an element \( \ell \in A \) (respectively, \( r \in A \)) such that

\[
(27) \quad h\ell = \varepsilon_\ell(h)\ell, \quad \text{(respectively, } rh = r\varepsilon_r(h)) \quad \text{for all } h \in A.
\]

The space of left (respectively, right) integrals in \( A \) is a right (respectively, left) ideal of \( A \) of dimension \( \dim_k(A) \). We will denote the spaces of left and right integrals of \( A \) by \( \int^l_A \) and \( \int^r_A \).

Any left integral \( \lambda \) in \( A^* \) satisfies the following invariance property:

\[
(28) \quad g(1)(\lambda, hg(2)) = S(h(1))\langle \lambda, h(2)g \rangle, \quad g, h \in A.
\]

In what follows we use the Sweedler arrows \( \rightarrow \) and \( \leftarrow \) for the dual actions:

\[
(29) \quad \langle h \rightarrow \phi, g \rangle = \langle \phi, gh \rangle \quad \text{and} \quad \langle \phi \leftarrow h, g \rangle = \langle \phi, hg \rangle.
\]

for all \( g, h \in A, \phi \in A^* \).

Recall that a functional \( \phi \in A^* \) is **non-degenerate** if its composition with the multiplication defines a non-degenerate bilinear form on \( A \). Equivalently, \( \phi \) is non-degenerate if the linear map \( h \mapsto (h \mapsto \phi) \) is injective. An integral (left or right)
in a weak Hopf algebra $A$ is called non-degenerate if it defines a non-degenerate functional on $A^*$. A left integral $\ell$ is called normalized if $\varepsilon(\ell) = 1$.

It was shown by P. Vecsernyes [V] that a finite-dimensional weak Hopf algebra always possesses a non-degenerate left integral. In particular, a finite-dimensional weak Hopf algebra is a Frobenius algebra (this extends the well-known result of Larson and Sweedler for usual Hopf algebras). It also follows that $\int_A^l$ is a free right $A_t$-module of rank 1 and a free right $A_s$-module of rank 1 for which any non-degenerate left integral can be taken as a basis.

Maschke’s theorem for weak Hopf algebras [BNSz, 3.13] states that a weak Hopf algebra $A$ is semisimple if and only if $A$ is separable, and if and only if there exists a normalized left integral in $A$. In particular, every semisimple weak Hopf algebra is finite-dimensional.

For a finite-dimensional $A$ there is a useful notion of duality between non-degenerate left integrals in $A$ and $A^*$ [BNSz, 3.18]. If $\ell$ is a non-degenerate left integral in $A$ then there exists a unique $\lambda \in A^*$ such that $\lambda \leftrightarrow \ell = 1$. This pair of non-degenerate left integrals $(\ell, \lambda)$ is called a pair of dual integrals.

A weak Hopf algebra $A$ is called unimodular if it has a 2-sided non-degenerate integral. A semisimple weak Hopf algebra is unimodular [BNSz].

Following Drinfeld, [D], we define the algebra of generalized characters of $A$ to be

$$ O(A) = \{ \phi \in A^* | \langle \phi, gh \rangle = \langle \phi, hS^2(g) \rangle, \ g, h \in A \}. $$

It was shown in [BNSz] that a left integral $\ell \in \int_A^l$ is $S$-invariant if and only if its dual left integral $\lambda \in \int_A^{l^*}$ is a generalized character.

**Proposition 3.1.2.** If $A$ is unimodular then $\{ \ell \in \int_A^l | \ell = S(\ell) \} \cong A_t \cap Z(A)$. In particular if $A$ is connected unimodular then there exist unique up to a scalar multiple non-degenerate $S$-invariant $\ell \in \int_A^l$ and non-degenerate left integral $\lambda \in O(A) \cap \int_{A^*}^l$ dual to each other.

**Proof.** Let $\ell \in A$ be a two-sided non-degenerate integral, then any other element of $\int_A^l$ is of the form $\ell' = \ell z$, $z \in A_t$. It is easy to see that $S(\ell') = \ell'$ if and only if $z \in A_t \cap Z(A)$. \qed

**Definition 3.1.3.** A canonical left integral in $A^*$ is a functional $\lambda$ defined by

$$ \langle \lambda, h \rangle = \text{Tr}(L_h \circ S^2|_A), $$

where $h \mapsto L_h$ is the left regular representation of $A$.

That the above $\lambda$ is an integral was shown, e.g., in [BNSz] and [ENO]. It follows that if $A$ is connected semisimple, then $\lambda$ is a basis of $O(A) \cap \int_{A^*}^l$.

The following definition was given in [BNSz].

**Definition 3.1.4.** A Haar integral in a weak Hopf algebra $A$ is a normalized 2-sided integral $\ell$ in $A$. Such an integral is necessarily unique and $S$-invariant.

Below we give a list of equivalent conditions characterizing the semisimplicity of a biconnected weak Hopf algebra in terms of the canonical and Haar integrals.

**Proposition 3.1.5.** Let $A$ be a biconnected weak Hopf algebra and let $\lambda \in A^*$ be the canonical integral defined by (31). Then the following conditions are equivalent.
Theorem 3.2. Since \( \text{Tr}(S^2|_A) = \text{Tr}(S^2|_{A^*}) \) we also get (i) ⇔ (iii).

To prove (iii) ⇔ (iv) observe that by [BNSz, Theorem 3.27] the existence of the Haar integral in \( A^* \) is equivalent to \( A^* \) being semisimple with \( S^2(\phi) = \gamma \phi \gamma^{-1} \) for \( \phi \in A^* \) and

\[
\text{Tr}_V(\gamma^{-1}) \neq 0
\]

for every irreducible \( A^* \)-module \( V \). But this condition is satisfied in every semisimple weak Hopf algebra, since the following sequence of \( A^* \)-module homomorphisms

\[
A^*_t \xrightarrow{b_{V^*}} V^* \otimes V^{**} \xrightarrow{id_{V^*} \otimes \epsilon^{-1}} V^* \otimes V \xrightarrow{dv} A^*_t
\]

is precisely the multiplication by \( \text{Tr}_V(\gamma^{-1}) \) and is non-zero by semisimplicity of \( A^* \).

The equivalence (iv) ⇔ (v) was proved in [BNSz, Proposition 3.26(i)]. □

Remark 3.1.6. When \( A \) is not biconnected there are refined criteria for semisimplicity, cf. [N2, 6.4] and [ENO, 4.10].

Let \( \{p_\alpha\} \) be the set of primitive idempotents in \( Z(A_s) \).

Proposition 3.1.7. For any biconnected weak Hopf algebra we have

\[
\varepsilon_t(\lambda) = \frac{\text{Tr}(S^2|_A)}{\dim_k(A_s)} \epsilon,
\]

\[
\varepsilon_s(\lambda) = \sum \frac{\text{Tr}(S^2|_{p_\alpha A_s})}{\dim_k(p_\alpha A_s)} (p_\alpha \rightarrow \epsilon).
\]

Proof. Let \( \lambda \) be the canonical integral defined by (31). The restriction of \( \lambda \) to \( A_{\min} = A_s A_t \) is an integral in \( (A_{\min})^* \) and hence there exists \( y \in A_s \) such that

\[
\langle \lambda, x \rangle = \text{Tr}(xy|_{A_{\min}}), \quad x \in A_{\min}.
\]

In particular, for \( x \in A_t \) we get \( \langle \lambda, x \rangle = \epsilon(x)\epsilon(y) \). On the other hand,

\[
\text{Tr}(S^2|_A) = \langle \lambda, 1 \rangle = \text{Tr}(y|_{A_{\min}}) = \epsilon(y) \dim_k(A_s),
\]

whence (33) follows.

For \( x \in A_s \) we have \( \langle \lambda, x \rangle = \epsilon(xy) \dim_k(A_s) \), so, \( \varepsilon_s(\lambda) = \dim_k(A_t)(y \rightarrow \epsilon) \).

Let \( y = \sum y_\alpha p_\alpha \) for some scalars \( y_\alpha \). To find these scalars, note that \( \langle \lambda, p_\alpha \rangle = \text{Tr}(S^2|_{p_\alpha A}) \) and also that

\[
\langle \lambda, p_\alpha \rangle = y_\alpha \text{Tr}(p_\alpha|_{A_{\min}}) = y_\alpha \dim_k(A_s) \dim_k(p_\alpha A_s).
\]

Thus,

\[
y = \sum \frac{\text{Tr}(S^2|_{p_\alpha A})}{\dim_k(A_s) \dim_k(p_\alpha A_s)} p_\alpha,
\]

which implies (34). □
3.2. Group-like and pivotal elements in a weak Hopf algebra. In [N2] a group-like element of \( A \) was defined as an invertible element \( g \in A \) such that 
\[
\Delta(g) = (g \otimes g)\Delta(1) = \Delta(1)(g \otimes g).
\]
Group-like elements of \( A \) form a group \( G(A) \) under multiplication. This group has a normal subgroup
\[
G_0(A) := G(A_{\text{min}}) = \{yS(y)^{-1} | y \in A_s\}
\]
of trivial group-like elements. If \( A \) is finite-dimensional, the quotient group \( \tilde{G}(A) = G(A)/G_0(A) \) is finite. It was shown in [N2] that if \( \ell \in A \) and \( \lambda \in A^* \) is a dual pair of left integrals, then there exist group-like elements \( \alpha \in G(A^*) \) and \( a \in G(A) \), called distinguished group-like elements, whose classes in \( \tilde{G}(A^*) \) and \( \tilde{G}(A) \) do not depend on the choice of \( \ell \) and \( \lambda \), such that
\[
S(\ell) = \alpha \rightarrow \ell \quad \text{and} \quad S(\lambda) = a \rightarrow \lambda.
\]
(Note that \( \alpha \) and \( \lambda \) themselves depend on the choice of \( \ell \) and \( \lambda \)).

The following result is an analogue of Radford’s formula [R] for usual Hopf algebras.

**Theorem 3.2.1.** [N2, Theorem 5.13] One has
\[
S^4(h) = a^{-1}(\alpha \rightarrow h \leftarrow \alpha^{-1})a.
\]
for all \( h \in A \).

**Remark 3.2.2.** \( A \) is unimodular if the coset of \( \alpha \) in \( G(A^*) \) is trivial.

Recall that a pivotal structure on a rigid monoidal category \( \mathcal{C} \) is an isomorphism between monoidal functors \( i : \text{Id} \rightarrow ** \). A weak Hopf algebra \( A \) is pivotal if there is a group-like element \( G \in A \) such that \( S^2(h) = GhG^{-1} \) for all \( h \in A \). In other words, \( A \) is pivotal if and only if \( \text{Rep}(A) \) is a pivotal category. The element \( G \) is called a pivotal element of \( A \).

3.3. Dimension theory for weak Hopf algebras. Let \( A \) be a semisimple weak Hopf algebra and let \( V \) be a finite-dimensional \( A \)-module. Below we define quantum and Frobenius-Perron dimensions of \( A \)-modules. These dimensions coincide with the \( k \)-vector space dimensions if and only if \( A \) is a usual Hopf algebra.

Let \( \mathcal{C} \) be a pivotal fusion category with an isomorphism between monoidal functors \( i : \text{Id} \rightarrow ** \). One can define the quantum dimension of any object \( V \) in \( \mathcal{C} \) by
\[
\dim(V) := \text{Tr}_V(i) = d_{V^*} \circ (i \otimes \text{id}_{V^*}) \circ b_V.
\]
Here the right hand side belongs to \( \text{End}_\mathcal{C}(E) = k \). Let \( \mathcal{C} = \text{Rep}(A) \) for a pivotal weak Hopf algebra \( A \) and let \( V \) be a finite-dimensional \( A \)-module. If \( G \) is a pivotal element of \( A \) then (10) becomes
\[
\dim(V) = \frac{\text{Tr}_V(G)}{\dim_k(A_t)}.
\]
Note that we use \( \dim \) for the quantum dimension and \( \dim_k \) for \( k \)-vector space dimension.

**Remark 3.3.1.** The squared norm \( |V|^2 \) of a simple object \( V \) of \( \mathcal{C} \) can be defined without assuming existence of a pivotal structure [Ma, ENO]. If \( \mathcal{C} = \text{Rep}(A) \) for
a semisimple \( A \), then \( \text{End}_k(V) \) can be identified with a minimal 2-sided ideal of \( A \) and

\[
|V|^2 = \frac{\text{Tr}(S^2|_{\text{End}_k(V)})}{\dim_k(A_t)^2}.
\]

This number is an algebraic integer, which is non-negative if \( k = \mathbb{C} \) [ENO, Theorem 2.3].

**Remark 3.3.2.** For a pivotal \( A \) one has \( |V|^2 = \dim(V) \dim(V^*) \) [Mu, ENO]. In general, \( \dim(V) \neq \dim(V^*) \), and so \( |V|^2 \neq \dim(V)^2 \). This is why we use the term “squared norm” rather than “squared dimension.”

When \( A \) is a semisimple weak Hopf algebra and \( \{V_j\}_{j=1}^n \) are all the irreducible \( A \)-modules, the **categorical dimension** of \( C = \text{Rep}(A) \) is defined as

\[
\dim(C) = \sum_{j=1}^n |V_j|^2.
\]

We will call \( \dim(A) = \dim(\text{Rep}(A)) \) the **categorical dimension** of \( A \). It follows from (42) that for \( k = \mathbb{C} \) one has

\[
\dim(A) = \frac{\text{Tr}(S^2|_A)}{\dim_k(A_t)^2}.
\]

It was shown in [ENO] that \( \dim(A) \) is an algebraic integer \( \geq 1 \).

**Note 3.3.3.** From now on let \( A \) be a semisimple biconnected weak Hopf algebra and let \( k = \mathbb{C} \) be the field of complex numbers (although some of the results below remain valid over an arbitrary algebraically closed field \( k \) of characteristic 0 we restrict our attention to \( \mathbb{C} \) since it is convenient to regard dimensions of \( A \)-modules as elements of the ground field).

Recall from Section 2.5 that the **Frobenius-Perron dimension** \( \text{FPdim}(V) \) of a finite-dimensional \( A \)-module \( V \) is defined as the largest positive eigenvalue of the matrix of multiplication by \( \chi_V \) in the basis \( \{\chi_1, \ldots, \chi_n\} \) of \( K_0(A) \), where \( \{\chi_1, \ldots, \chi_n\} \) are characters of irreducible \( A \)-modules \( \{V_1, \ldots, V_n\} \). Clearly, \( \text{FPdim}(V) \) is a positive algebraic integer.

The **Frobenius-Perron dimension** of \( A \) is then defined [ENO] as:

\[
\text{FPdim}(A) := \sum_{j=1}^n \text{FPdim}(V_j)^2.
\]

**Remark 3.3.4.** It follows from [ENO, Section 8.2] that \( \text{FPdim}(A) = \text{FPdim}(A^*) \).

If \( B \subset A \) is a biconnected semisimple weak Hopf subalgebra of \( A \) then \( \frac{\text{FPdim}(A)}{\text{FPdim}(B)} \) is an algebraic integer. We generalize this result in Section 6.2.

Let us define

\[
\rho := \sum_j \text{FPdim}(V_j)\text{Tr}_{V_j} \in R(A).
\]

This \( \rho \) formally replaces the character of the regular representation. We will call \( \rho \) the **Frobenius-Perron character** of \( A \). If \( V \) is an \( A \)-module with the character \( \chi \) then \( \chi\rho = \rho\chi = \text{FPdim}(V)\rho \). In particular, \( \text{FPdim}(A)^{-1}\rho \) is a minimal idempotent in \( R(A) \).
A relation between categorical and Frobenius-Perron dimensions was found in [ENO, Section 8.3]. For any simple module $V$ over a semisimple weak Hopf algebra $A$ one has $|V|^2 \leq \text{FPdim}(V)^2$, and hence, $\dim(A) \leq \text{FPdim}(A)$. The ratio $\dim(A)/\text{FPdim}(A)$ is an algebraic integer $\leq 1$. We explore this relation further when we derive the second trace formula in Theorem 5.2.2.

**Proposition 3.3.5.** Let $A$ be a semisimple weak Hopf algebra. There exists a unique, up to a scalar, element $w \in Z(A_s)$ that has strictly positive eigenvalues and satisfies

$$\text{Tr}_V(zwS(w)^{-1}) = \text{FPdim}(V)\epsilon(z)$$

for any $z \in A_t$ and any $A$-module $V$. In particular, $w$ satisfies

$$\text{FPdim}(V) = \frac{\text{Tr}_V(wS(w)^{-1})}{\dim_k(A_s)}.$$  

**Proof.** The target counital subalgebra $A_t$ is a left $A$-comodule algebra via the co-

multiplication. Since $A^*$ is connected, $A_t$ is indecomposable in the sense of Defini-
tion 2.5.2. As in Section 2.5, the Abelian group $K_0(A_t)$ is a left $K_0(A)$-module via

$$\langle \chi \xi, z \rangle = \langle \chi \otimes \xi, \Delta(z) \rangle, \quad z \in A_t,$$

for any characters $\chi$ of $A$ and $\xi$ of $A_t$. This module structure naturally extends to the algebra $R(A_t) = K_0(A_t) \otimes Z C$. It follows that there exists a unique, up to a scalar, $\xi_f \in R(A_t)$ with strictly positive coordinates in the basis of characters $\{\xi_1, \ldots, \xi_t\}$ of irreducible $A_t$-modules such that

$$\text{Tr}_V \xi_f = \text{FPdim}(V)\xi_f,$$

for any $A$-module $V$ with character $\text{Tr}_V$. Let $w = \xi_f \mapsto 1 \in Z(A_s)$, then

$$\text{Tr}_V \mapsto w = \text{FPdim}(V)w.$$  

Thus, $w$ is a Frobenius-Perron eigenvector of the linear transformation $\text{Tr}_V \mapsto$ of

$Z(A_s)$ with the eigenvalue $\text{FPdim}(V)$. Note that the corresponding eigenspace

is 1-dimensional by the Frobenius theorem [Ga]. The last equation, in turn, is equivalent to

$$1_{(1)}w^{-1}\text{Tr}_V(1_{(2)}w) = \text{FPdim}(V),$$

which is equivalent to (17) (this can be seen by applying $z \mapsto \epsilon$ to both sides). Therefore, $w$ is unique up to a scalar multiple and has positive eigenvalues. $\square$

**Definition 3.3.6.** A non-zero element $w \in Z(A_s)$ will be called a *Frobenius-Perron element* of $A_s$ if it has positive eigenvalues and satisfies

$$\text{Tr}_V \mapsto w = \text{FPdim}(V)w$$

for all finite-dimensional $A$-modules $V$. This element is defined up to a non-zero

scalar multiple.

**Corollary 3.3.7.** Every Frobenius-Perron element of $A_s$ is a multiple of $\rho \mapsto 1$.

Let $A$ be a semisimple weak Hopf algebra. To fix the notation in what follows we let

$$A \cong \bigoplus_{j=1}^n \text{End}_k(V_j), \quad A_s \cong \bigoplus_{n=1}^N M_n(C),$$
where \( M_n(\mathbb{C}) \) is the algebra of \((m \times m)\)-matrices over \( \mathbb{C} \). Let \( \Lambda = (\Lambda_{\alpha j}) \) be the \((l \times n)\) matrix of the inclusion \( A_s \subset A \), i.e., \( \Lambda_{\alpha j} \) is the multiplicity of the simple \( A_s \)-module corresponding to \( M_n(\mathbb{C}) \) in the restriction of the \( A \)-module \( V_j \) to \( A_s \).

**Definition 3.3.8.** The number

\[
(55) \quad \mu(A) = \sum_{j=1}^n \dim_k(V_j) \text{FPdim}(V_j)
\]

will be called an index of \( A \), cf. [BSZ2, 4.5].

In other words, \( \mu(A) \) is the Frobenius-Perron dimension of the regular representation of \( A \). The name index comes from the subfactor theory [GHJ]: if a subfactor comes from the crossed product with a weak Hopf \( C^* \)-algebra \( A \) then the Jones index of the subfactor is equal to the index of \( A \).

Define the Frobenius-Perron dimension vector of \( A \) as

\[
(56) \quad \vec{f} = (\text{FPdim}(V_j))_{j=1}^n.
\]

If \( A \) is pivotal with a pivotal element \( G \) define the quantum dimension vector of \( A \) as

\[
(57) \quad \vec{d} = (\dim(V_j))_{j=1}^n.
\]

Recall from (46) the canonical character \( \rho = \sum_j \text{FPdim}(V_j) \text{Tr}_{V_j} \) of \( A \). Let \( p_\alpha, \alpha = 1, \ldots, l \) be primitive central idempotents of \( A_s \).

Let \( w_\alpha = \frac{\langle \rho, p_\alpha \rangle}{n_\alpha \mu(A)} \) and \( v_\alpha = w_\alpha n_\alpha \). It follows from Corollary 3.3.7 that

\[
w = \sum_\alpha w_\alpha p_\alpha = \frac{1}{\mu(A)} \rho \rightarrow 1
\]

is a Frobenius-Perron element of \( A_s \). Let

\[
(58) \quad \vec{v} = (v_\alpha)_{\alpha=1}^l = \left( \frac{\langle \rho, p_\alpha \rangle}{n_\alpha \mu(A)} \right)_{\alpha=1}^l.
\]

Note that if \( S^2 = \text{id} \) then \( v_\alpha = n_\alpha \), i.e., \( \vec{v} \) is the dimension vector of \( A_s \).

**Remark 3.3.9.** The inclusion matrix \( \Lambda \), vector \( \vec{v} \), and index \( \mu(A) \) are not invariants of the representation category of \( A \), i.e., there exist semisimple weak Hopf algebras with monoidally equivalent representation categories but different \( \Lambda, \vec{v}, \) and \( \mu(A) \).

**Proposition 3.3.10.** Let \( A \) be a semisimple weak Hopf algebra. Then \( \Lambda \vec{f} = \mu(A) \vec{v} \) and \( \Lambda^t \vec{v} = \vec{f} \), where \( \Lambda^t \) is the transpose of \( \Lambda \). Consequently,

\[
(59) \quad \Lambda^t \Lambda \vec{f} = \mu(A) \vec{f} \quad \text{and} \quad \Lambda \Lambda^t \vec{v} = \mu(A) \vec{v}.
\]
Proof. We have $\text{FPdim}(V^*) = \text{FPdim}(V)$ for any simple $A$-module $V$. Let $f_j, j = 1, \ldots, n$ and $v\alpha, \alpha = 1, \ldots, l$ be the coordinates of vectors $\vec{f}$ and $\vec{v}$. We compute

\[
\sum_j \Lambda_{\alpha j} f_j = \sum_j \frac{\text{Tr}_{V_j}(p_{\alpha})}{n_{\alpha}} \text{FPdim}(V_j)
\]

\[
= \frac{\langle \rho, p_{\alpha} \rangle}{n_{\alpha}} = \mu(A)v_{\alpha},
\]

\[
\sum_{\alpha} \Lambda_{\alpha j} v_{\alpha} = \sum_{\alpha} \frac{\text{Tr}_{V_j}(p_{\alpha}) \langle \rho, p_{\alpha} \rangle}{n_{\alpha} n_{\alpha} \mu(A)}
\]

\[
= \mu(A)^{-1} \text{Tr}_{V_j^*}(1_{(2)}) \langle \rho, 1_{(1)} \rangle
\]

\[
= \mu(A)^{-1} \langle \rho \text{Tr}_{V_j^*}, 1 \rangle
\]

\[
= \text{FPdim}(V_j) \mu(A)^{-1} \langle \rho, 1 \rangle = f_j,
\]

whence the statement of the proposition follows. □

Remark 3.3.11. A version of Proposition 3.3.10 was proved in [BSz2, Theorem 4.5] for weak Hopf $C^*$-algebras.

Remark 3.3.12. It follows from Proposition 3.3.10 that $\rho$ is a Markov trace [GHJ] for the inclusion $A_s \subset A$.

The Frobenius-Perron vectors of $A_s$ and $A^*_s$ turn out to be colinear as we show in the next Proposition. In the language of [ENO] this means that the vectors of Frobenius-Perron dimensions of a finite semisimple module category $\mathcal{M}$ viewed as a module category over a fusion category $\mathcal{C}$ and over its dual $\mathcal{C}_\mathcal{M}$ are proportional.

Proposition 3.3.13. Let $w_A$ and $w_{A^*}$ be the Frobenius-Perron elements of $A_s$ and $A^*_s$ respectively. Then $w_{A^*}$ and $w_A \to \epsilon$ are scalar multiples of one another.

Proof. Recall that $w_A$ and $w_{A^*}$ are defined by (53) in terms of the left coregular actions of $R(A)$ and $R(A^*)$ on $Z(A_s)$ and $Z(A^*_s)$ respectively. If we identify $Z(A^*_s)$ with $Z(A_s)$ via $\phi \mapsto (\phi \to 1)$, $\phi \in Z(A^*_s)$, then the above actions make $Z(A_s)$ an $R(A) - R(A^*)$ bimodule with $w_A$ being an eigenvector of all $\chi \in R(A)$ and $x \in R(A^*)$. Since $w_{A^*}$ can be normalized to have positive entries it follows that it belongs to the Frobenius-Perron eigenspace of $R(A)$ that also contains $w_A \to \epsilon$. □
pivotal element of $A$. Below we prove that the canonical pivotal element is a trivial group-like element with positive eigenvalues. The next Proposition establishes a sufficient condition of the triviality of a pivotal element.

**Proposition 3.4.2.** Let $A$ be a semisimple weak Hopf algebra that has a pivotal element $G$. Let $\Lambda$ be the inclusion matrix of $A_\alpha \subset A$ and let $\vec{d} = (\dim(V_i))_{i=1}^n$ be the vector of quantum dimensions of irreducible $A$-modules. If $\Lambda \vec{d} \neq \vec{0}$ then $G$ is a trivial quantum group-like element.

**Proof.** The left coregular action

$$\phi \mapsto (G \mapsto \phi), \quad \phi \in A^*,$$

of a group-like element $G \in A$ on $A^*$ is an algebra automorphism of $A^*$. This automorphism preserves the minimal 2-sided ideal $I = \text{End}_k(A^*_i)$ of $A^*$ corresponding to the trivial $A^*$-module if and only if $G$ is trivial \([N2]\).

The canonical integral $\lambda \in A^*$ defined by (31) is a rank 1 element of $I$. Therefore, $G$ is a trivial group-like element if and only if $(G^{-1} \mapsto \lambda) \in I$, i.e., if $G^{-1} \mapsto \lambda$ acts as non-zero in the trivial representation of $A^*$. Since $G^{-1} \mapsto \lambda = \sum_i \text{Tr}_{V_i} (G^{-1}) \text{Tr}_{V_i}$, the last condition is equivalent to

$$\sum_i \text{Tr}_{V_i} (G) \text{Tr}_{V_i} |_{A_\alpha A_\beta} \neq 0.$$  

In particular, this condition follows if $\sum_i \text{Tr}_{V_i} (G) \text{Tr}_{V_i} (p_\alpha) \neq 0$ for some primitive idempotent $p_\alpha \in Z(A_\alpha)$. Since $\Lambda_{\alpha i} = \frac{\text{Tr}_{V_i}(p_\alpha)}{\dim(A_\alpha)}$ and $d_i = \frac{\text{Tr}_{V_i}(G)}{\dim(A_\alpha)}$, we get the result.

**Corollary 3.4.3.** Suppose that $A$ has a trivial pivotal element $G$ with respect to which all quantum dimensions of simple $A$-modules are non-negative. Then $G$ is a trivial group-like element.

**Corollary 3.4.4.** Let $A$ be a pseudo-unitary weak Hopf algebra and let $G \in A$ be the pivotal element with respect to which the quantum dimensions of all simple objects coincide with their Frobenius-Perron dimensions. Then $G$ is a trivial group-like element.

**Corollary 3.4.5.** If $A$ is pseudo-unitary, then $G = wS(w)^{-1}$, where $w \in Z(A_\alpha)$ is a Frobenius-Perron element of $A_\alpha$, is the canonical group-like element of $A$.

**Proof.** Up to a scalar multiple we have $w = \rho \rightarrow 1$. Since $A$ is pseudo-unitary, its canonical pivotal element $G$ is trivial by Corollary 3.4.4 i.e., $G = gS(g)^{-1}$ for some $g \in Z(A_\alpha)$. We also have $\rho = \dim_k(A_\alpha)^{-1}(G^{-1} \mapsto \lambda)$, where $\lambda$ is the canonical integral in $A^*$. We compute

$$\dim_k(A_\alpha) w = 1_{(1)} \langle \lambda, G^{-1}1_{(2)} \rangle = g1_{(1)} \langle \lambda, g^{-1}1_{(2)} \rangle = g(\lambda \rightarrow g^{-1}).$$

Since $\lambda$ is an integral and $A$ is biconnected, we have $\lambda \rightarrow g^{-1} \in A_\alpha \cap A_t = k1$ is a scalar, hence $w$ and $g$ are proportional and $wS(w)^{-1} = gS(g)^{-1} = G$. \qed

**Remark 3.4.6.** It follows that if $A$ is pseudo-unitary then the eigenvalues of $S^2$ belong to the set \(\{\frac{\alpha \beta \gamma \delta}{w^2 w'^2} | \alpha, \beta, \gamma, \delta = 1, \ldots, t\}\), where $w$ is a Frobenius-Perron element of $A_\alpha$. In particular, the eigenvalues of $S^2$ of a pseudo-unitary weak Hopf
algebra are strictly positive. Below we will see (Corollary \ref{cor:positive}) that the converse is also true.

4. **The class equation**

4.1. **The character algebra and the Grothendieck ring of a weak Hopf algebra.** Let $A$ be a semisimple weak Hopf algebra, $K_0(A)$ be the Grothendieck ring of $A$ defined in Section \ref{sec:grothendieck} and $R(A) \subset A^*$ be the character algebra.

**Remark 4.1.1.** Note that, in general, $\epsilon \not\in R(A)$. The identity element of $R(A)$ is the character of the trivial representation of $A$, $\chi_1 = \epsilon(1)\epsilon(2)$.

In what follows we will identify $K_0(A)$ with a subring of $R(A)$. For any finite-dimensional $A$-module $V$ let $\chi_V$ be its character. The operation of taking the dual module gives rise to an anti-isomorphism of $R(A)$:

$$\chi_V \mapsto \chi_{V^*} = \chi_V \circ S,$$

for any character $\chi_V$. The map $*$ extends to an involutive anti-linear algebra anti-homomorphism.

Let $V_1, \ldots, V_n$ be a complete set of simple $A$-modules, with $V_1 = A$ the trivial $A$-module. Let $\chi_1, \ldots, \chi_n$ be the corresponding characters and let $\chi_j^* = \chi_j \circ S$ be the character of $V_j^*$.

The following lemma is standard.

**Lemma 4.1.2.** The bilinear form $(\cdot, \cdot) : R(A) \times R(A) \to \mathbb{C}$ defined by

$$(\chi_V, \chi_W) := \dim_k \text{Hom}_A(V, W^*)$$

is non-degenerate, associative, $*$-invariant, and symmetric.

**Proof.** That the given form is non-degenerate follows from the fact that the bases $\{\chi_1, \ldots, \chi_n\}$ and $\{\chi_1^*, \ldots, \chi_n^*\}$ of $R(A)$ are dual to each other with respect to $(\cdot, \cdot)$, i.e., $(\chi_i, \chi_j^*) = \delta_{ij}$ for all $i$ and $j$. The associativity of $(\cdot, \cdot)$ follows from the isomorphism of vector spaces

$$\text{Hom}_A(V \otimes U, W^*) \cong \text{Hom}_A(V \otimes U \otimes W, V_1) \cong \text{Hom}_A(V, W^* \otimes U^*),$$

for all $A$-modules $V, U, W$. Clearly, the form is $*$-invariant and symmetric. \hfill \Box

**Lemma 4.1.3.** The character algebra $R(A)$ is semisimple.

**Proof.** We claim that $\sum_i \chi_i \chi_i^* = \sum_i \chi_i^* \chi_i$ for all $\chi \in R(A)$. Indeed, evaluating both sides of this equality against $\chi_k \otimes \chi_l$ we get $(\chi_k, \chi_l \chi_k)$, whence the claim follows by Lemma \ref{lem:associativity}.

Next, $(\sum_i \chi_i \chi_i^*) = \sum_i (\chi_i \chi_i^*) \geq 0$, where the equality occurs only for $\chi = 0$. This implies that $(\sum_i \chi_i \chi_i^*) \chi = 0$ for $\chi \not= 0$, and so $\sum_i \chi_i \chi_i^*$ is an invertible central element in $R(A)$. This means that $R(A)$ is separable, and therefore, semisimple. \hfill \Box

**Remark 4.1.4.** The element $\sum_i \chi_i \chi_i^* \in K_0(A)$ is the character of the adjoint representation of $A$ on the centralizer $C_A(A_t)$ of its target base, cf. \cite{NTV}.
4.2. The class equation for weak Hopf algebras. We extend the argument of [L0] to obtain an analogue of the class equation of Kac [K] and Zhu [Z1] showing that the categorical dimension \( \dim(A) \) of a semisimple weak Hopf algebra \( A \) is equal to a sum of its divisors in the ring of algebraic integers of \( \mathbb{C} \).

Let \( e \in R(A) \) be a primitive idempotent in \( R(A) \), and \( \hat{e} \) be its central support in \( R(A) \), i.e., the primitive idempotent in \( Z(R(A)) \) such that \( \hat{e} \varepsilon = e \). Let \( m \) be the dimension of the simple \( R(A) \)-module \( R(A)e \), then \( R(A)\hat{e} \cong (R(A)e)^m \) as \( R(A) \)-modules.

Let \( \mu \) be the character of \( R(A)\hat{e} \) and let \( \omega = \frac{1}{m} \mu \).

Lemma 4.2.1. We have \( \hat{e} = m\mu(\hat{e})^{-1} \omega(\hat{e})^{-1} e = e \), where \( \hat{e} = \sum_i \mu(\chi_i^*) \chi_i \) is an invertible element in \( Z(R(A)) \).

Proof. That \( \hat{e} e' = 0 \) for all primitive idempotents \( e' \in Z(R(A)) \) such that \( e' \neq \hat{e} \), therefore \( \hat{e} \) is proportional to \( e \) and \( \hat{e} = m \chi(\hat{e})^{-1} e \).

Let \( d = \dim_k(A) \).

Lemma 4.2.2. In any finite-dimensional weak Hopf algebra \( A \) we have

\[
\text{Tr}(S^2|_{1(1)1(2),A}) = \frac{1}{d} \text{Tr}(S^2|_A).
\]

Proof. Let \( \lambda \in A^* \) be the canonical left integral defined in \[31\]. Let \( y \in Z(A) \) be as in the proof of Proposition \[31\], i.e., such that

\[
\langle \lambda, x \rangle = \text{Tr}(xy|_{A_{\min}}), \quad \forall x \in A_{\min}.
\]

Observe that

\[
\text{Tr}(zy|_{A_{\min}}) = \epsilon(z) \epsilon(y), \quad \forall z \in A_L, y \in A_s.
\]

Taking \( x = 1(1)1(2) \) in \[35\] we get

\[
\text{Tr}(S^2|_{1(1)1(2),A}) = \text{Tr}(1(1)1(2)y|_{A_{\min}})
= \epsilon(1(1)y)\epsilon(1(2)) = \epsilon(y)
= \frac{1}{d} \text{Tr}(y|_{A_{\min}}) = \frac{1}{d} \text{Tr}(S^2|_A).
\]

Proposition 4.2.3. We have \( \text{Tr}(S^2|_{\varepsilon A^*}) \neq 0 \) and

\[
\frac{\text{Tr}(S^2|_A)}{d \text{Tr}(S^2|_{\varepsilon A^*})} = \omega(\hat{e}).
\]

Proof. Let \( \ell \) be the left canonical integral in \( A \), cf. \[31\], i.e., \( \langle \phi, \ell \rangle = \text{Tr}(L_\phi \circ S^2|_{A^*}), \quad \forall \phi \in A^* \), where \( \phi \rightarrow L_\phi \) is the left regular representation of \( A^* \). From Lemma 4.2.1 we get

\[
\text{Tr}(S^2|_{\varepsilon A^*}) = \langle \epsilon, \ell \rangle = \omega(\hat{e})^{-1} \langle \hat{e}, \ell \rangle.
\]

On the other hand, since \( \langle \chi_i, \ell \rangle = 0 \) if \( i \neq 1 \), we have

\[
\langle \hat{e}, \ell \rangle = \sum_i \mu(\chi_i^*) \langle \chi_i, \ell \rangle
= \mu(\chi_1) \langle \chi_1, \ell \rangle
= m \text{Tr}(S^2|_{\varepsilon(1)\varepsilon(2)A^*}).
\]
Combining two last formulas we get
\[ \omega(\hat{e}) \text{Tr}(S^2|_{\hat{e}A^*}) = m \text{Tr}(S^2|_{\epsilon(1)^*\epsilon(2)^*A^*}). \]

Next, since \( S^2|_{R(A)} = \text{id} \) the value \( \text{Tr}(S^2|_{\epsilon A^*}) \) is the same for all primitive idempotents \( e \in \hat{e}R(A) \), hence \( \text{Tr}(S^2|_{\hat{e}A^*}) = m \text{Tr}(S^2|_{\epsilon A^*}) \). Using this and Lemma \[ \text{Lem} \] we obtain the result.

Recall from equation \[ (4) \] that the categorical dimension of a semisimple weak Hopf algebra \( A \) is
\[ \dim(A) = \frac{\text{Tr}(S^2|_A)}{d^2} = \frac{\text{Tr}(S^2|_{A^*})}{d^2}. \]

**Theorem 4.2.4** (The Class Equation). Let \( e_1, \ldots, e_k \) be primitive idempotents of \( R(A) \) such that \( e_ie_j = \delta_{i,j}e_i \) and \( \sum_i e_i = \epsilon(1)^*\epsilon(2)^* \). Then
\[ \dim(A) = \sum_i \frac{\text{Tr}(S^2|_{e_iA^*})}{d}, \]
where the numbers \( n_i = \frac{\text{Tr}(S^2|_{e_iA^*})}{d}, i = 1, \ldots, k \) are such that the ratios \( \dim(A)/n_i \) are algebraic integers in \( \mathbb{C} \).

**Proof.** In view of Proposition \[ \text{Prop} \] it only remains to show that \( \omega(e) \) is an algebraic integer. We reproduce here the argument of \[ \text{Ref} \] for the sake of completeness. Note first that for any character \( \psi \) of \( R(A) \) we have \( \psi(K_0(A)) \subset \text{alg. int.} \mathbb{C} \). Indeed, the regular representation of \( K_0(A) \) on itself is faithful, and so each \( \chi_i \in K_0(A) \) satisfies a monic polynomial over \( \mathbb{Z} \), e.g., the characteristic polynomial of the matrix \( (N_i)_{jk} = (\chi_i\chi_j, \chi_k^*) \). Hence, so does the image \( \pi(\chi_i) \) under any representation \( \pi \) of \( R(H) \). Therefore, all eigenvalues of \( \pi(\chi_i) \) and \( \text{Tr}(\pi(\chi_i)) \) belong to \( \text{alg. int.} \mathbb{C} \).

This means that \( \hat{e} = \sum_i \mu(\chi_i^*)\chi_i \) is a linear combination of elements of \( K_0(A) \) with coefficients from \( \text{alg. int.} \mathbb{C} \). Since \( \omega : Z(R(A)) \to \mathbb{C} \) is an algebra homomorphism, \( \omega(\chi_i) \in \text{alg. int.} \mathbb{C}, i = 1, \ldots, n \). Thus, \( \omega(e) \in \text{alg. int.} \mathbb{C} \).

**Remark 4.2.5.** It was shown in \[ \text{Ref} \] that if \( A \) is pivotal then the numbers \( n_i \) are algebraic integers equal to the quantum dimensions of irreducible submodules of the \( D(A) \)-module induced from the trivial \( A \)-module, where \( D(A) \) is the Drinfeld double of \( A \) \[ \text{Ref} \].

5. Trace formulas

Recall that in the case of a usual semisimple Hopf algebra \( A \) at least one of the summands in the Class Equation \[ (69) \] equals 1. Namely, if \( \text{Tr} \) denotes the trace of the left regular representation of \( A \), then \( \rho = \dim_{\mathbb{C}}(A)^{-1} \text{Tr}^\rho \) is a minimal idempotent in \( R(A) \) and \( \text{Tr}(S^2|_{\rho A^*}) = 1 \). Below we extend this observation to weak Hopf algebras and also establish an analogue of the trace formula \[ \text{LR2 Formula (6)} \].

5.1. **The first trace formula.** A weak Hopf algebra analogue of the Larson-Radford formula for \( \text{Tr}(S^2) \) \[ LR2 \] was established in \[ N2 \]. As in the case of usual Hopf algebras, this formula is related to the semisimplicity of the corresponding weak Hopf algebra and its dual.
It follows from (28) that for a non-degenerate integral $\ell$ in $A$ with the dual integral $\lambda$, the element $(\ell(2) \rightarrow \lambda) \otimes S^{-1}(\ell(1)) \in A^* \otimes A$ is the dual bases tensor. This implies that for every $T \in \text{End}_C(A)$ we have

$$\text{Tr}(T) = \langle \lambda, T(S^{-1}(\ell(1)))\ell(2) \rangle.$$  

In particular, for $T = S^2$, we get the following analogue of [LR2, Theorem 2.5(a)] with counits replaced by counital maps.

**Corollary 5.1.1 (The first trace formula [N2]).** Let $\ell \in A$ be a non-degenerate integral in a finite-dimensional weak Hopf algebra $A$ and let $\lambda$ be the dual integral. Then

$$\text{Tr}(S^2|A) = \langle \epsilon_s(\lambda), \epsilon_s(\ell) \rangle.$$

**5.2. The second trace formula.** We will derive a relation between the categorical and Frobenius-Perron dimensions of a semisimple weak Hopf algebra $A$ that extends the result of [LR2].

Recall that $\chi_1 = \epsilon(1)\epsilon(2)$ is the character of the trivial left $A$-module $A$.  

**Lemma 5.2.1.** $\text{Tr}(S^2|_{\chi_1 A^*}) = \frac{1}{d} \text{Tr}(S^2|_{A^*})$. 

**Proof.** Follows from Lemma 4.2.2. \hfill \Box

Let $V_1, \ldots, V_n$ be irreducible $A$-modules with characters $\chi_1, \ldots, \chi_n$. Recall that if $\rho = \sum_j \text{FPdim}(V_j)\chi_j$ then $\text{FPdim}(A)^{-1}\rho$ is a minimal idempotent in $R(A)$.

**Theorem 5.2.2 (The second trace formula).** We have

$$\dim(A) = \frac{\text{Tr}(S^2|_{\rho A^*})}{d} \text{FPdim}(A).$$

**Proof.** Note that the canonical left integral $\ell$ defined by (68) belongs to the matrix block corresponding to the trivial representation of $A$ and $\text{Tr}(S^2|_{\chi_1 A^*})$ is a primitive idempotent. Hence, $\langle \rho, \ell \rangle = \langle \chi_1, \ell \rangle$. Using Lemma 5.2.1 we obtain

$$\text{Tr}(S^2|_{\rho A^*}) = \text{FPdim}(A)^{-1}\text{Tr}(\rho \circ S^2|_{A^*}) = \text{FPdim}(A)^{-1}\langle \rho, \ell \rangle = \text{FPdim}(A)^{-1}\text{Tr}(S^2|_{\chi_1 A^*}) = \text{FPdim}(A)^{-1}d^{-1}\text{Tr}(S^2|_{A^*}),$$

whence the result follows since $\dim(A) = d^{-2}\text{Tr}(S^2|_{A^*})$. \hfill \Box

**Remark 5.2.3.** It was shown in [ENO] that the ratio $\dim(A) / \text{FPdim}(A)$ is an algebraic integer.

**Corollary 5.2.4.** We have $\text{Tr}(S^2|_{\rho A^*}) \leq d$.

**Proof.** It follows from [ENO] that the categorical dimension of $A$ is not bigger than its Frobenius-Perron dimension, which implies the statement. \hfill \Box

**Corollary 5.2.5.** A finite-dimensional weak Hopf algebra $A$ is pseudo-unitary if and only if the eigenvalues of $S^2$ are positive.
Then we compute $T$ and similarly for $L$.

Lemma 6.1.1. If $A$ is a finite-dimensional weak Hopf algebra and let $\rho$ be an $A$-module ideal of $M$, we compute $\rho^A$. Let $A$ be a finite-dimensional weak Hopf algebra and let $\rho$ be an $A$-module ideal of $M$.

Proof. It was shown in Corollary 5.2.4 that for a pseudo-unitary $A$ the square of the antipode is the conjugation by $wS(w)^{-1}$, where $w \in Z(A)$ has positive eigenvalues. Since $w$ and $S(w)$ commute it follows that the eigenvalues of $S^2$ are positive.

To prove the converse, observe that the action of $\rho$ on the trivial $A^\ast$-module $V_1$ is not zero because $\langle \rho, 1 \rangle = \mu(A^\ast) \neq 0$, where $\mu(A)$ is the index of $A^\ast$ from Definition 3.3.8. Thus $\rho$ gives rise to a non-zero element in $\text{End}_C(V_1)$. Therefore, $\text{Tr}(S^2|_{\rho A^\ast}) \geq \text{Tr}(S^2|_{\rho \text{End}_C(V_1)}) = d$ and $\dim(A) = \text{FPdim}(A)$ by Corollary 5.2.4. □

Corollary 5.2.6. The dual of a pseudo-unitary weak Hopf algebra is pseudo-unitary. Weak Hopf subalgebras and quotients of a pseudo-unitary weak Hopf algebra are pseudo-unitary.

6. THE STRUCTURE OF MODULE ALGEBRAS OVER WEAK HOPF ALGEBRAS

6.1. $A$-module ideals and stability of the Jacobson radical. Let $A$ be a finite-dimensional weak Hopf algebra and let $M$ be a finite-dimensional left $A$-module algebra.

The following Lemma extends [Li, 2.2(a)].

Lemma 6.1.1. If $I \subset M$ is a two-sided ideal then

$$(73) \quad A \cdot I = \left\{ \sum_i h_i \cdot x_i \mid h_i \in A, x_i \in I \right\}$$

is an $A$-stable ideal of $M$.

Proof. It is clear that $A \cdot I$ is an $A$-stable subspace of $M$. For all $m \in M, h \in A, x \in I$ we compute

$$m(h \cdot x) = (1_{(1)} \cdot m)(1_{(2)}h \cdot x) = (h_{(2)}S^{-1}(h_{(1)}) \cdot m)(h_{(3)} \cdot x)$$
$$= h_{(2)} \cdot ((S^{-1}(h_{(1)}) \cdot m)x) \in A \cdot I;$$

$$(h \cdot x)a = (1_{(1)}h \cdot x)(1_{(2)} \cdot a) = (h_{(1)} \cdot x)(h_{(2)}S(h_{(3)}) \cdot a)$$
$$= h_{(1)} \cdot (x(S(h_{(2)}) \cdot a)) \in A \cdot I,$$

and hence, $A \cdot I$ is an ideal. □

Lemma 6.1.2. Let $L, R,$ and $T$ be linear endomorphisms of $M$ defined by

$$(74) \quad L(m)x = mx, \quad R(m)x = xm, \quad T(h)x = h \cdot x \text{ for all } m, x \in M, h \in A.$$

Then

(i) for all $y \in A_s$ and $z \in A_t$ we have $T(y) = R(y \cdot 1)$ and $T(z) = L(z \cdot 1),$

(ii) for all $h \in A$ and $m \in M$ we have $L(h \cdot m) = T(h_{(1)}) \circ L(m) \circ T(S(h_{(2)})).$

Proof. We compute

$$T(y)m = y \cdot (m1) = (1_{(1)} \cdot m)(1_{(2)}y \cdot 1) = m(y \cdot 1), \quad y \in A_s, m \in M,$$

and similarly for $T(z)$, which proves (i). Next, for $h \in A$ and $m, x \in M$ we have we compute

$$T(h_{(1)}) \circ L(m) \circ T(S(h_{(2)})x) = h_{(1)} \cdot (m(S(h_{(2)}) \cdot x))$$
$$= (h_{(1)} \cdot m)(h_{(2)}S(h_{(3)}) \cdot x)$$
$$= (1_{(1)}h \cdot m)(1_{(2)} \cdot x)$$
$$= (h \cdot m)x = L(h \cdot m)x,$$
which proves (ii). \[\square\]

In [Li] V. Linchenko proved that the Jacobson radical of a module algebra over a semisimple Hopf algebra \(A\) is an \(A\)-stable ideal. Below we extend this result to pseudo-unitary weak Hopf algebras, cf. Definition \ref{def:psunit} (note that a semisimple Hopf algebra is pseudo-unitary).

**Theorem 6.1.3.** Let \(A\) be a pseudo-unitary weak Hopf algebra and let \(M\) be a finite-dimensional \(A\)-module algebra. Then the Jacobson radical \(J(M)\) of \(M\) is an \(A\)-stable ideal of \(M\).

**Proof.** We will use notation from Lemma \ref{lem:jacobson}. Let \(G = wS(w^{-1}) \in A\) be the element from Corollary \ref{cor:jacobson} such that \(S^2(h) = GhG^{-1}\) for all \(h \in A\). Here \(w \in Z(A_s)\) is a Frobenius-Perron element of \(A_s\). In particular, all the eigenvalues of \(A_s\) are positive. Let \(\text{Tr}\) denote the trace of a linear endomorphism of \(M\). We compute

\[
\text{Tr}(T(G) \circ L(h \cdot m)) = \text{Tr}(T(S((h(2))Gh(1)) \circ L(m)))
\]

\[
= \text{Tr}(T(GS^{-1}(\epsilon_s(h))) \circ L(m))
\]

\[
= \text{Tr}(T(G) \circ L(\epsilon_s(h) \cdot 1m))
\]

\[
= \text{Tr}(L((S(w) \cdot 1)(\epsilon_s(h) \cdot 1)m) \circ R(w^{-1} \cdot 1)),
\]

for all \(h \in A, m \in M\). Now if \(m \in J(M)\), then \(L((S(w) \cdot 1)(\epsilon_s(h) \cdot 1)m)\) is a nilpotent endomorphism of \(M\) that commutes with \(R(w^{-1} \cdot 1)\), therefore,

(75) \[\text{Tr}(T(G) \circ L(h \cdot m)) = 0, \quad \text{for all } h \in A, m \in J(M).\]

Since \(T(G) = L(w) \circ R(w^{-1})\) and \(A \cdot J(M)\) is an \(A\)-stable ideal by Lemma \ref{lem:jacobson} it follows that

(76) \[\text{Tr}(L(h \cdot m) \circ R(w^{-1})) = 0, \quad \text{for all } h \in A, m \in J(M).\]

Consequently,

(77) \[\text{Tr}(L(x)^n \circ R(w^{-1})) = 0, \quad \text{for all } x \in A \cdot J(M) \text{ and } n = 1, 2, \ldots.\]

Since \(R(w^{-1})\) is a diagonalizable linear operator with positive eigenvalues it follows by a standard linear algebra argument that \(L(x)\) is a nilpotent operator for all \(x \in A \cdot J(M)\). Hence, any such \(x\) is a nilpotent element of \(M\) and, therefore, \(A \cdot J(M)\) is a nil ideal of \(M\). It is well known that every nil ideal is contained in the Jacobson radical, whence the result follows. \[\square\]

**6.2. Frobenius-Perron dimensions of module algebras.** A classical result in group theory states that if \(G\) is a finite group then the cardinality of any transitive \(G\)-set divides \(|G|\). This was extended to semisimple Hopf algebras by Y. Zhu who showed in [Zhu] that if \(M\) is an indecomposable semisimple module algebra over a semisimple Hopf algebra \(A\) and \(M_1, M_2\) are irreducible \(M\)-modules then \(\dim_C(M)\) divides \(\dim_C(M_1) \dim_C(M_2) \dim_C(H)\).

In this section we extend Zhu’s result to semisimple weak Hopf algebras and their (co)module algebras. In this case the vector space dimensions should be replaced by Frobenius-Perron dimension.

Let \(A\) be a semisimple weak Hopf algebra and let \(M\) be an indecomposable finite-dimensional semisimple left \(A\)-comodule algebra. Recall from Remark \ref{rem:fpdim} that in this case \(M\) is a right \(A^*\)-module algebra via

(78) \[m \cdot \phi := \langle m^{(1)}, \phi \rangle m^{(2)}, \quad m \in M, \phi \in A^*,\]
where \( \delta(m) = m^{(1)} \otimes m^{(2)} \) is the coaction of \( A \).

The Grothendieck group \( K_0(M) \) is a left \( K_0(A) \)-module via
\[
\langle \chi \xi, m \rangle := \langle \chi \otimes \xi, \delta(m) \rangle = \xi(m \cdot \chi),
\]
for all \( \chi \in K_0(A) \), \( \xi \in K_0(M) \), and \( m \in M \).

Let \( M_1, \ldots, M_t \) be irreducible \( M \)-modules and let \( \xi_1, \ldots, \xi_t \) be their characters.

Recall from Section 2.5 that there exists a unique up to a scalar multiple Frobenius-Perron character \( \xi \in R(M) = K_0(M) \otimes \mathbb{Z} \mathbb{C} \) of \( M \) defined by the property
\[
\text{Tr}_V \xi = \text{FPdim}(V) \xi,
\]
for all finite-dimensional \( A \)-modules \( V \). The coefficients of \( \xi \) in the basis \( \{ \xi_1, \ldots, \xi_t \} \) are Frobenius-Perron dimensions of irreducible \( M \)-modules.

The space of Frobenius-Perron characters of \( M \) is 1-dimensional and is equal to \( \rho R(M) \), where \( \rho \) is the Frobenius-Perron character of \( A \). Let \( w_{A^*} \in k \varepsilon_s(\rho) \) be any Frobenius-Perron element of \( A^*_s \) from Definition 3.3.6. The next Proposition expresses a Frobenius-Perron character of \( M \) in terms of this element.

**Proposition 6.2.1.** The element
\[
\xi_f = \sum_{k=1}^t \xi_k (1 \cdot w_{A^*}) \xi_k
\]
is a Frobenius-Perron character of \( M \).

**Proof.** Choose a normalization \( w_{A^*} = \varepsilon_s(\rho) \). Note that the map
\[
\Phi : K_0(A) \to \mathbb{R} : \xi \mapsto \xi(1 \cdot w_{A^*})
\]
is a \( K_0(A) \)-module homomorphism, where \( \mathbb{R} \) is a \( K_0(A) \)-module via \( \text{FPdim} \). Indeed, we have
\[
\Phi(\text{Tr}_V \xi) = \xi(1 \cdot w_{A^*} \cdot \text{Tr}_V)
= \xi(1 \cdot \rho \cdot \text{Tr}_V)
= \text{FPdim}(V) \xi(1 \cdot \rho) = \text{FPdim}(V) \Phi(\xi).
\]
Since \( \Phi(\xi_k) > 0 \) for all \( k = 1, \ldots, t \) it follows from Lemma 2.5.3 that \( \sum_k \Phi(\xi_k) \xi_k \) is a Frobenius-Perron character of \( M \). \( \square \)

It follows that we can choose the Frobenius-Perron dimensions of irreducible \( M \)-modules to be
\[
\text{FPdim}(M_k) = \xi_k (1 \cdot w_{A^*}), \quad k = 1, \ldots, t.
\]

For any \( M \)-module \( U \) with the character \( \xi = \sum_{k=1}^t a_k \xi_k \in K_0(M) \), \( a_k \in \mathbb{Z}_{\geq 0} \) its Frobenius-Perron dimension is
\[
\text{FPdim}(U) := \sum_{k=1}^t a_k \text{FPdim}(M_k).
\]

Define the **Frobenius-Perron dimension** of \( M \) by
\[
\text{FPdim}(M) := \sum_{k=1}^t \text{FPdim}(M_k)^2.
\]

Of course, the numbers \( \text{FPdim}(M_k) \), \( \text{FPdim}(M) \) depend (up to a scalar) on our choice, but the ratios \( \text{FPdim}(M_i) \text{FPdim}(M_k) / \text{FPdim}(M), i, k = 1, \ldots, t \) do not.

The following Theorem extends the main result of [Z2].
Theorem 6.2.2. Let $A$ be a semisimple weak Hopf algebra and $M$ be an indecomposable finite dimensional semisimple left $A$-comodule algebra. Let $M_1, \ldots, M_t$ be irreducible $M$-modules. Then the numbers

\begin{equation}
\frac{\text{FPdim}(A)\text{FPdim}(M_i)\text{FPdim}(M_k)}{\text{FPdim}(M)}, \quad i, k = 1, \ldots, t,
\end{equation}

are algebraic integers.

Proof. Let $\xi_k$ be the character of $M_k$, $k = 1, \ldots, t$ and let $e_k$ be the corresponding primitive central idempotent of $M$. Let $\xi_f = \sum_{k=1}^{t} \text{FPdim}(M_k)\xi_k$ be a Frobenius-Perron character of $M$ and let $\rho$ be the Frobenius-Perron character of $A$. Then

\[ \rho\xi_k = \sum_j \text{FPdim}(V_j)\text{Tr}_{V_j}\xi_k \]
\[ = \sum_{ji} N_{jk}^{i}\text{FPdim}(V_j)\xi_i, \]

where $N_{jk}^{i}$ are some non-negative integers. Hence, $\rho\xi_k$ is a linear combination of $\xi_i$, $i = 1 \ldots, t$ with algebraic integer coefficients. On the other hand, since both $\rho\xi_k$ and $\rho\xi_f$ are Frobenius-Perron characters of $M$, we have

\[ \rho\xi_k = \frac{\text{FPdim}(M_k)}{\text{FPdim}(M)} \rho\xi_f \]
\[ = \frac{\text{FPdim}(A)\text{FPdim}(M_k)}{\text{FPdim}(M)} \xi_f \]
\[ = \sum_i \frac{\text{FPdim}(A)\text{FPdim}(M_k)\text{FPdim}(M_i)}{\text{FPdim}(M)} \xi_i. \]

Comparing the coefficients of $\xi_i$ in the two above formulas for $\rho\xi_k$ we get the result. \qed

References

[A] N. Andruskiewitsch, *About finite-dimensional Hopf algebras*, in “Quantum symmetries in theoretical physics and mathematics,” Contemp. Math., 294, AMS, Providence (2002), 1-57.

[BK] B. Bakalov, A. Kirillov Jr., *Lectures on tensor categories and modular functors*, AMS, Providence, (2001).

[BNSz] G. Böhm, F. Nill, and K. Szlachányi, *Weak Hopf algebras I. Integral theory and C*-structure*, J. Algebra, 221 (1999), 385–438.

[BSz1] G. Böhm, K. Szlachányi, *A coassociative C*-quantum group with nonintegral dimensions*, Lett. in Math. Phys, 35 (1996), 437–456.

[BSz2] G. Böhm, K. Szlachányi, *Weak Hopf algebras II. Representation theory, dimensions and the Markov trace*, J. Algebra, 233 (2000), 156-212.

[D] V.G. Drinfeld, *On almost cocommutative Hopf algebras*, Leningrad Math. J. 1 (1990), 321-342.

[Ga] F. R. Gantmacher, *The theory of matrices*, AMS Chelsea Publishing, Providence, (1998).

[GHJ] F. Goodman, P. de la Harpe, and V.F.R. Jones, *Coxeter graphs and towers of algebras*, M.S.R.I. Publ. 14, Springer, Heidelberg, (1989).

[EN] P. Etingof, D. Nikshych, *Dynamical quantum groups at roots of 1*, Duke Math. J., 108 (2001), 135-168.

[ENO] P. Etingof, D. Nikshych, V. Ostrik, *On fusion categories*, math.QA/0010060 (2002).

[K] G. Kac, *Certain arithmetic properties of ring groups*, Functional Anal. Appl. 6 (1972), 158-160.

[KN] L. Kadison, D. Nikshych, *Frobenius extensions and weak Hopf algebras*, J. Algebra, 244 (2001), 312-342.
[LR1] R. Larson, D. Radford, \textit{Finite-dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple}, J. Algebra \textbf{117} (1988), 267–289.
[LR2] R. Larson, D. Radford, \textit{Semisimple cosemisimple Hopf algebras}, Amer. J. Math. \textbf{109} (1987), 187–195.
[Li] V. Linchenko, \textit{Nilpotent subsets of Hopf module algebras}, to appear in “Groups, rings, Lie and Hopf algebras”, Mathematics and its applications, \textbf{555} (2003).
[Lo] M. Lorenz, \textit{On the class equation for Hopf algebras}, Proc. AMS, \textbf{126} (1998), 2841-2844.
[Ma] S. MacLane, \textit{Categories for the working mathematician}, 2nd edition, Springer-Verlag (1998).
[Mo] S. Montgomery, \textit{Hopf algebras and their actions on rings}, CBMS Regional Conference Series in Mathematics, \textbf{82}, AMS, (1993).
[Mo1] S. Montgomery, \textit{Classifying finite-dimensional semisimple Hopf algebras}, in “Trends in the representation theory of finite-dimensional algebras” (Seattle, 1997), 256-279, Contemp. Math. \textbf{229}, AMS, (1998).
[Mo2] S. Montgomery, \textit{Representation theory of semisimple weak Hopf algebras}, Algebra–Representation theory (Constanta 2000), 189-218, NATO Sci. Ser. II Math. Phys. Chem., \textbf{28} (2001).
[Mu] M. M"uger, \textit{From subfactors to categories and topology I. Frobenius algebras in and Morita equivalence of tensor categories}, J. Pure Appl. Algebra \textbf{180} (2003), 81-157.
[N1] D. Nikshych, \textit{Duality for actions of weak Kac algebras and crossed product inclusions of II$_1$ factors} J. Operator Theory \textbf{46}, 635-655 (2001).
[N2] D. Nikshych, \textit{On the structure of weak Hopf algebras}, Adv. Math. \textbf{170}, 257-286 (2002).
[NTV] D. Nikshych, V. Turaev, and L. Vainerman, \textit{Quantum groupoids and invariants of knots and 3-manifolds}, Topology and Applications, \textbf{127} (2003), 91-123.
[NV1] D. Nikshych, L. Vainerman, \textit{A characterization of depth 2 subfactors of II$_1$ factors}, J. Func. Analysis, \textbf{171} (2000), 278-307.
[NV2] D. Nikshych, L. Vainerman, \textit{Finite quantum groupoids and their applications}, in “New Directions in Hopf Algebras,” MSRI Publications, \textbf{43} (2002), 211-262.
[O] V. Ostrik, \textit{Module categories, weak Hopf algebras and modular invariants}, Transform. Groups, \textbf{8} (2003), 177-206.
[R] D. Radford, \textit{The order of the antipode of a finite-dimensional Hopf algebra is finite}, Amer. J. Math., \textbf{98} (1976), 333–355.
[S] D. Stefan, \textit{The set of types of n-dimensional semisimple and cosemisimple Hopf algebras is finite}, J. Algebra \textbf{193} (1997), no. 2, 571–580.
[V] P. Vecsernyes, \textit{Larson-Sweedler theorem, grouplike elements, and invertible modules in weak Hopf algebras}, preprint (2001), math.QA/0111045.
[Z1] Y. Zhu, \textit{Hopf algebras of prime dimension}, Internat. Math. Res. Notices (1994), no. 1, 158-160.
[Z2] Y. Zhu, \textit{The dimension of irreducible modules for transitive Hopf algebras}, Comm. Alg., \textbf{29} (2001), 2877-2886.

Department of Mathematics and Statistics, University of New Hampshire, Durham, NH 03824, USA
E-mail address: nnikshych@math.unh.edu