Synchronizing continuous-time neutrally stable linear systems via partial-state coupling

S. Emre Tuna
tuna@eee.metu.edu.tr

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Abstract

Synchronization of coupled continuous-time linear systems is studied in a general setting. For identical neutrally-stable linear systems that are detectable from their outputs, it is shown that a linear output feedback law exists under which the coupled systems globally asymptotically synchronize under all fixed (directed) connected network topologies. An algorithm is provided to compute one such feedback law based on individual system parameters. The dual case, where individual systems are neutrally stable and stabilizable from their inputs, is also considered and parallel results are established.

1 Introduction

In [20] we have shown, for identical discrete-time linear systems that are detectable (stabilizable) from their outputs (inputs) and neutrally stable, that a linear feedback law exists under which the coupled systems globally asymptotically synchronize for all fixed (directed) connected network topologies. There we have also provided an algorithm to compute such feedback law based on individual system parameters. In this companion paper we provide counterpart results for continuous-time linear systems.

1.1 Background

“The main issue in studying the synchronization of coupled dynamical systems is the stability of synchronization. As in all cases where stability is the issue, the question whose answer is sought is Under what conditions will the individual systems synchronize? In a simplified yet widely-studied scenario, where the individual system dynamics are identical and the coupling between them is linear, studies focus on two ingredients: the dynamics of an individual system and the network topology. Starting with the agreement algorithm in [19] a number of contributions [8, 10, 15, 1, 12] have gathered around the case where the weakest possible assumptions are made on the network topology at the
expense of restrictive individual system dynamics. It was established in those works on multi-agent systems \cite{11} that when the individual system is taken to be an integrator and the coupling is of full-state, synchronization (consensus) results for time-varying interconnections whose union over an interval are assumed to be connected instead of that each interconnection at every instant is connected.

Another school of research investigates networks with more complicated (nonlinear) individual system dynamics \cite{18,22}. When that is the case, the restrictions on the network topology have to be made stricter in order to ensure stability of synchronization. Generally speaking, more than mere connectedness of the network has been needed: coupling strength is required to be larger than some threshold and sometimes a symmetry or balancedness assumption is made on the connection graph. Different (though related) approaches have provided different insights over the years. The primary of such approaches is based on the calculations of the eigenvalues of the connection matrix and a parameter (e.g. the maximal Lyapunov exponent) depending on the individual system dynamics \cite{24,13,6}. In endeavor to better understand synchronization stability, tools from systems theory such as Lyapunov functions \cite{4,7}, passivity \cite{14,8,17,23}, contraction theory \cite{16}, and incremental input-to-state stability (δISS) theory \cite{5} have also proved useful.\footnote{By union of interconnections we actually mean the union of the graphs representing the interconnections.} \footnote{Borrowed from \cite{20}.}

1.2 Contribution

In this paper we study two dual problems. In the first case we consider the following individual system

\[
\dot{x}_i = Ax_i, \quad y_i = Cx_i, \quad (1)
\]

where \( A \) is assumed to be neutrally stable and pair \((C, A)\) detectable, and design a linear output feedback gain \( L \) that synchronizes any fixed connected network of any number of coupled replicas of \((1)\). Such \( L \) guarantees the synchronization of \( p \) individual systems when coupled as

\[
\dot{x}_i = Ax_i + L \sum_{j=1}^{p} \gamma_{ij} (y_j - y_i).
\]

As the dual problem we consider

\[
\dot{x}_i = Ax_i + Bu_i, \quad (2)
\]

where \( A \) is assumed to be neutrally stable and pair \((A, B)\) stabilizable, and design a linear feedback gain \( K \) that synchronizes any fixed connected network of any
number of coupled replicas of (2). Such $K$ guarantees the synchronization of $p$ individual systems when coupled as

$$\dot{x}_i = Ax_i + BK \sum_{j=1}^{p} \gamma_{ij} (x_j - x_i).$$

To the best of our knowledge, feedback design (in such a general setting) in order to guarantee synchronization under arbitrary (fixed) interconnections is a novelty of our work. It is worth noting that our main theorems make a compromise result between the two previously mentioned cases (i) where synchronization is established for very primitive individual system dynamics, such as that of an integrator, but under the weakest conditions on the network topology and (ii) where the network topology has to satisfy stronger conditions, such as that the coupling strength should be above a threshold, for want of achieving synchronization for nonlinear individual system dynamics.

### 1.3 Organization

The remainder of the paper is organized as follows. In the next section we provide notation and some preliminaries. Then we formally state our problems in Sections 3 and 4. Section 5 is where we establish our key result which we will later use to solve the problems we aim at. In Section 6 we provide an algorithm to design output feedback gain that we seek for synchronization and prove that it works. Then, in Section 7 we design a state feedback gain that solves the dual problem.

### 2 Notation and definitions

Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{R}_{\geq 0}$ set of nonnegative real numbers. Let $|\cdot|$ denote 2-norm. Identity matrix in $\mathbb{R}^{n \times n}$ is denoted by $I_n$. A matrix $A \in \mathbb{R}^{n \times n}$ is Hurwitz if all of its eigenvalues have strictly negative real parts. A matrix $S \in \mathbb{R}^{n \times n}$ is skew-symmetric if $S + S^T = 0$. Given $C \in \mathbb{R}^{m \times n}$ and $A \in \mathbb{R}^{n \times n}$, pair $(C, A)$ is observable if $[C^T \ A^T C^T \ A^{2T} C^T \ \ldots \ A^{(n-1)T} C^T]$ is full row rank. Pair $(C, A)$ is detectable (in the continuous-time sense) if that $Ce^{At}x = 0$ for some $x \in \mathbb{R}^n$ and for all $t \geq 0$ implies $\lim_{t \to \infty} e^{At}x = 0$. Given $B \in \mathbb{R}^{n \times m}$ and $A \in \mathbb{R}^{n \times n}$, pair $(A, B)$ is controllable (stabilizable) if $(B^T, A^T)$ is observable (detectable). Matrix $A \in \mathbb{R}^{n \times n}$ is neutrally stable (in the continuous-time sense) if it has no eigenvalue with positive real part and the Jordan block corresponding to any eigenvalue on the imaginary axis is of size one.\(^3\) Let $\mathbf{1} \in \mathbb{R}^p$ denote the vector with all entries equal to one.

\(^3\)Note that $A$ is neutrally stable if and only if there exists a symmetric positive definite matrix $P$ such that $A^T P + PA \leq 0$. [2].
Kronecker product of \( A \in \mathbb{R}^{m \times n} \) and \( B \in \mathbb{R}^{p \times q} \) is

\[
A \otimes B := \begin{bmatrix}
a_{11}B & \cdots & a_{1n}B \\
\vdots & \ddots & \vdots \\
a_{m1}B & \cdots & a_{mn}B
\end{bmatrix}
\]

Kronecker product comes with the properties \((A \otimes B)(C \otimes D) = (AC) \otimes (BD)\) (provided that products \(AC\) and \(BD\) are allowed) \(A \otimes B + A \otimes C = A \otimes (B + C)\) (for \(B\) and \(C\) that are of same size) and \((A \otimes B)^T = A^T \otimes B^T\).

A (directed) graph is a pair \((\mathcal{N}, \mathcal{A})\) where \(\mathcal{N}\) is a nonempty finite set (of nodes) and \(\mathcal{A}\) is a finite collection of pairs (arcs) \((n_i, n_j)\) with \(n_i, n_j \in \mathcal{N}\). A path from \(n_1\) to \(n_\ell\) is a sequence of nodes \(\{n_1, n_2, \ldots, n_\ell\}\) such that \((n_i, n_{i+1})\) is an arc for \(i \in \{1, 2, \ldots, \ell - 1\}\). A graph is connected if it has a node to which there exists a path from every other node.

The graph of a matrix \(\Gamma := [\gamma_{ij}] \in \mathbb{R}^{p \times p}\) is the pair \((\mathcal{N}, \mathcal{A})\) where \(\mathcal{N} = \{n_1, n_2, \ldots, n_p\}\) and \((n_i, n_j) \in \mathcal{A}\) iff \(\gamma_{ij} > 0\). Matrix \(\Gamma\) is said to be connected (in the continuous-time sense) if it satisfies:

(i) \(\gamma_{ij} \geq 0\) for \(i \neq j\);

(ii) each row sum equals 0;

(iii) its graph is connected.

For connected \(\Gamma\), it follows from definition that \(\lambda = 0\) is an eigenvalue with eigenvector \(1\) (i.e. \(\Gamma 1 = 0\)). Moreover, all the other eigenvalues have real parts strictly negative. Let \(r^T\) be the left eigenvector of eigenvalue \(\lambda = 0\) (i.e. \(r^T \Gamma = 0\)) with \(r^T 1 = 1\). Then \(\lim_{t \to \infty} e^{\Gamma t} = 1 r^T\).

Given maps \(\xi_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^n\) for \(i = 1, 2, \ldots, p\) and a map \(\bar{\xi} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n\), the elements of the set \(\{\xi_i(\cdot) : i = 1, 2, \ldots, p\}\) are said to synchronize to \(\bar{\xi}(\cdot)\) if \(|\xi_i(t) - \bar{\xi}(t)| \to 0\) as \(t \to \infty\) for all \(i\).

3 Problem I

We now formalize our first problem.

3.1 Systems under study

We consider \(p\) identical linear systems

\[
\dot{x}_i = Ax_i + u_i, \quad y_i = Cx_i, \quad i = 1, 2, \ldots, p
\]  

(3)

where \(x_i \in \mathbb{R}^n\) is the state, \(u_i \in \mathbb{R}^n\) is the input, and \(y_i \in \mathbb{R}^m\) is the output of the \(i\)th system. Matrices \(A\) and \(C\) are of proper dimensions. The solution of

\footnote{Note that this definition of connectedness for directed graphs is weaker than strong connectivity and stronger than weak connectivity.}
ith system at time $t \geq 0$ is denoted by $x_i(t)$. In this paper we consider the case where at each time instant only the following information

$$z_i = \sum_{j=1}^{p} \gamma_{ij}(y_j - y_i)$$

is available to $i$th system to determine an input value where $\gamma_{ij}$ are the entries of the matrix $\Gamma \in \mathbb{R}^{p \times p}$ describing the network topology. Nondiagonal entries of $\Gamma$ are nonnegative and each row sums up to zero. That is, the coupling between systems is diffusive.

3.2 Assumptions made

We make the following assumptions on systems (3) which will henceforth hold.

(A1) $A$ is neutrally stable.

(A2) $(C, A)$ is detectable.

3.3 Objectives

Our first objective is to show that there exists a linear feedback law $L \in \mathbb{R}^{n \times m}$ such that, for all $p$ and connected $\Gamma \in \mathbb{R}^{p \times p}$, solutions of systems (3) with $u_i = Lz_i$, where $z_i$ is as in (4), globally (i.e. for all initial conditions) synchronize to a bounded trajectory. Our second objective is to devise an algorithm to compute one such $L$.

4 Problem II

In this section we state the second problem, which, as noted earlier, is the dual of the first.

4.1 Systems under study

Consider $p$ identical linear systems

$$\dot{x}_i = Ax_i + Bu_i, \quad i = 1, 2, \ldots, p$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$. Matrices $A$ and $B$ are of proper dimensions. We consider the case where at each time instant the following information

$$z_i = \sum_{j=1}^{p} \gamma_{ij}(x_j - x_i)$$

is available to $i$th system to determine an input value.
4.2 Assumptions made

We make the following assumptions on systems (5) which will henceforth hold.

\( (B1) \) \(A\) is neutrally stable.
\( (B2) \) \((A, B)\) is stabilizable.

4.3 Objectives

Our first objective regarding the dual problem is to show that there exists a linear feedback law \(K \in \mathbb{R}^{m \times n}\) such that, for all \(p\) and connected \(\Gamma \in \mathbb{R}^{p \times p}\), solutions of systems (5) with \(u_i = Kz_i\), where \(z_i\) is as in (1), globally (i.e. for all initial conditions) synchronize to a bounded trajectory. Our second objective is to devise an algorithm to compute one such \(K\).

5 A special case

Before we attempt to solve Problems I and II, we first establish a preliminary result to be resorted later. Consider the following coupled systems

\[
\dot{\xi}_i = S\xi_i + H^T H \sum_{j=1}^{p} \gamma_{ij}(\xi_j - \xi_i), \quad i = 1, 2, \ldots, p
\]

where \(\xi_i \in \mathbb{R}^n\) is the state of the \(i\)th system, \(S \in \mathbb{R}^{n \times n}\), and \(H \in \mathbb{R}^{m \times n}\). We make the following assumptions on systems (7) which will henceforth hold.

\( (C1) \) \(S\) is skew-symmetric.
\( (C2) \) \((H, S)\) is observable.
\( (C3) \) \(\Gamma := [\gamma_{ij}]\) is connected.

Below we provide our first result.

**Theorem 1** Consider systems (7). Let \(r \in \mathbb{R}^p\) be such that \(r^T \Gamma = 0\) and \(r^T 1 = 1\). Then solutions \(\xi_i(\cdot)\), for \(i = 1, 2, \ldots, p\), synchronize to

\[
\bar{\xi}(t) := (r^T \otimes e^{St}) \begin{bmatrix} \xi_1(0) \\ \vdots \\ \xi_p(0) \end{bmatrix}
\]

**Proof.** Consider matrix \(\Gamma - 1r^T\). Observe that \((\Gamma - 1r^T)^k = \Gamma^k + (-1)^k 1r^T\) for \(k \in \mathbb{N}\). For \(t \in \mathbb{R}\) therefore we can write

\[
e^{(\Gamma - 1r^T)t} = I_p + t(\Gamma - 1r^T) + \frac{t^2}{2}(\Gamma - 1r^T)^2 + \ldots
\]

\[
= \left( I_p + t\Gamma + \frac{t^2}{2}\Gamma^2 + \ldots \right) - \left( t1r^T - \frac{t^2}{2}1r^T + \ldots \right)
\]

\[
= e^{rt} - (1 - e^{-t})1r^T.
\]
Consequently \( \lim_{t \to \infty} e^{(\Gamma - 1r^T)t} = 0 \). We deduce therefore that \( \Gamma - 1r^T \) is Hurwitz. Since \( \Gamma - 1r^T \) is Hurwitz, there exist symmetric positive definite matrices \( P, Q \in \mathbb{R}^{p \times p} \) such that

\[
-Q = (\Gamma - 1r^T)^T P + P(\Gamma - 1r^T).
\]

Define positive semidefinite matrices \( \tilde{P} := (I_p - 1r^T)^T P(I_p - 1r^T) \) and \( \tilde{Q} := (I_p - 1r^T)^T Q(I_p - 1r^T) \). Now pre- and post-multiply equation (8) by \( (I_p - 1r^T)^T \) and \( (I_p - 1r^T) \), respectively. We obtain

\[
-Q = (I_p - 1r^T)^T \Gamma (I_p - 1r^T) P(I_p - 1r^T) + (I_p - 1r^T)^T P(I_p - 1r^T) (I_p - 1r^T) \\
= \Gamma^T P(I_p - 1r^T) + (I_p - 1r^T)^T P \Gamma \\
= \Gamma^T (I_p - 1r^T)^T P(I_p - 1r^T) + (I_p - 1r^T)^T P(I_p - 1r^T) \Gamma \\
= \Gamma^T \tilde{P} + \tilde{P} \Gamma.
\]

We now stack the individual system states to obtain \( x := [\xi_1^T \xi_2^T \cdots \xi_p^T]^T \). We can then cast (7) into

\[
\dot{x} = (I_p \otimes S + \Gamma \otimes H^T H)x.
\]

Define \( V : \mathbb{R}^{pn} \to \mathbb{R}_{\geq 0} \) as \( V(x) := x^T(\tilde{P} \otimes I_n)x \). Differentiating \( V(x(t)) \) with respect to time we obtain

\[
\dot{V}(x) = x^T(I_p \otimes S^T + \Gamma^T \otimes H^T H)(\tilde{P} \otimes I_n)x \\
+ x^T(\tilde{P} \otimes I_n)(I_p \otimes S + \Gamma \otimes H^T H)x \\
= x^T(\tilde{P} \otimes (S^T + S) + (\Gamma^T \tilde{P} + \tilde{P} \Gamma) \otimes H^T H)x \\
= -x^T(\tilde{Q} \otimes H^T H)x.
\]

Thence \( \dot{V}(x) \leq 0 \) for both \( \tilde{Q} \) and \( H^T H \) (and consequently their Kronecker product) are positive semidefinite.

Given some \( \zeta \in \mathbb{R}^{pn} \), let \( \mathcal{X} \subset \mathbb{R}^{pn} \) be the closure of the set of all points \( \eta \) such that \( \eta = (1r^T \otimes e^{St})\zeta \) for some \( t \geq 0 \). Set \( \mathcal{X} \) is compact for it is closed by definition and bounded due to that \( \zeta \) is fixed and \( S \) is a neutrally-stable matrix. Having defined \( \mathcal{X} \), we now define

\[
\Omega := \{ \eta \in \mathbb{R}^{pn} : (1r^T \otimes I_n)\eta \in \mathcal{X}, V(\eta) \leq V(\zeta) \}.
\]

Let us show that \( \Omega \) is forward invariant. Observe that

\[
\frac{d}{dt} ((1r^T \otimes I_n)x(t)) = (1r^T \otimes I_n)(I_p \otimes S + \Gamma \otimes H^T H)x(t) \\
= (1r^T \otimes S + 1r^T \Gamma \otimes H^T H)x(t) \\
= (1r^T \otimes S)x(t) \\
= (I_p \otimes S)(1r^T \otimes I_n)x(t).
\]
We therefore have

\[(1r^T \otimes I_n)x(t) = (1r^T \otimes e^{St})x(0)\]  

which in turn implies that if \((1r^T \otimes I_n)x(0) \in \mathcal{X}\) then \((1r^T \otimes I_n)x(t) \in \mathcal{X}\) for all \(t \geq 0\). Likewise, if \(V(x(0)) \leq V(\zeta)\) then \(V(x(t)) \leq V(\zeta)\) for all \(t \geq 0\) thanks to (10). As a result, if \(x(0) \in \Omega\) then \(x(t) \in \Omega\) for all \(t \geq 0\), that is, \(\Omega\) is forward invariant with respect to (9).

Set \(\Omega\) is closed by construction. To show that it is compact therefore all we need to do is to establish its boundedness. Let

\[a := \sup_{V(\eta) \leq V(\zeta)} |\eta - (1r^T \otimes I_n)\eta| .\]

If we go back to the definition of \(V\) we immediately see that \(a < \infty\). Now let

\[b := \sup_{\omega \in \mathcal{X}} |\omega| .\]

Since \(\mathcal{X}\) is bounded, \(b < \infty\) as well. Now, given any \(\eta \in \Omega\) we have \(|\eta - (1r^T \otimes I_n)\eta| \leq a\). Hence we can write

\[|\eta| \leq a + |(1r^T \otimes I_n)\eta| \leq a + \sup_{\omega \in \mathcal{X}} |\omega| = a + b .\]

Therefore \(\Omega\) is bounded. Having shown that \(\Omega\) is forward invariant and compact, we can now invoke LaSalle’s invariance principle [9, Thm. 3.4] and claim that any solution starting in \(\Omega\) approaches to the largest invariant set \(W \subset \{\eta \in \Omega : V(\eta) = 0\}\).

Let now \(\eta(\cdot)\) be a solution of (9) such that \(\eta(t) \in W\) for all \(t \geq 0\). Given some \(\tau \geq 0\), since \(V(\eta(\tau)) = 0\), we can write

\[0 = \eta(\tau)^T (\tilde{Q} \otimes H^T H)\eta(\tau) = \eta(\tau)^T ((Ip - 1r^T)^T Q(I_p - 1r^T) \otimes H^T H)\eta(\tau)\]

which implies, since \(Q\) is positive definite, that either \(((Ip - 1r^T) \otimes I_n)\eta(\tau) = 0\) or \((Ip \otimes H)\eta(\tau) = 0\). Suppose now that

\[((Ip - 1r^T) \otimes I_n)\eta(\tau) \neq 0 .\]  

(12)

Continuity of \(\eta(\cdot)\) implies that there exists \(\delta > 0\) such that \(((Ip - 1r^T) \otimes I_n)\eta(t) \neq 0\) for \(t \in [\tau, \tau + \delta]\). Therefore we must have \((Ip \otimes H)\eta(t) = 0\) for \(t \in [\tau, \tau + \delta]\). However, observability of pair \((H, S)\) stipulates that \(\eta(t) = 0\) for \(t \in [\tau, \tau + \delta]\) which contradicts (12). We then deduce \(((Ip - 1r^T) \otimes I_n)\eta(t) = 0\) for all \(t \geq 0\). Therefore \(W \subset \{\omega \in \Omega : \omega = (1r^T \otimes I_n)\omega\} = \mathcal{X}\).

Let us now be given any solution \(x(\cdot)\) of (9). Since \(\zeta\) that we used to construct \(\Omega\) was arbitrary, without loss of generality, we can take \(x(0) = \zeta\).
That $x(0) \in \Omega$ implies that $x(t)$ approaches $\mathcal{X}$ as $t \to \infty$. Therefore we are allowed to write

$$0 = \lim_{t \to \infty} (x(t) - (1r^T \otimes I_n)x(t))$$

$$= \lim_{t \to \infty} (x(t) - (1r^T \otimes e^{St})x(0))$$

where we used (11).

The following result (cf. [7]) comes as a byproduct of Theorem 1.

Corollary 1 Consider coupled harmonic oscillators (in $\mathbb{R}^2$) described by

$$\dot{x}_i = y_i$$

$$\dot{y}_i = -x_i + \sum_{j=1}^{p} \gamma_{ij}(y_j - y_i), \quad i = 1, 2, \ldots, p.$$ 

Oscillators synchronize for all connected $\Gamma$.

6 Solution to Problem I

In this section we use Theorem 1 in order to reach our objectives stated in Section 3. We first give the following fact.

Fact 1 Let $F \in \mathbb{R}^{n \times n}$ be a neutrally-stable matrix with all its eigenvalues residing on the imaginary axis. Then

$$P := \lim_{t \to \infty} t^{-1} \int_0^t e^{F^T} e^{F} d\tau$$

is well-defined and symmetric positive definite. It also satisfies $PF + F^T P = 0$.

Proof. Matrix $F$ is similar to a skew-symmetric matrix. Therefore $e^{Ft}$ is (almost) periodic [21]. Periodicity directly yields that limit in (13) exists, that is, $P$ is well-defined. Similarity to a skew-symmetric matrix also brings that $\inf_{t \in \mathbb{R}} |e^{Ft}| > 0$ and $\sup_{t \in \mathbb{R}} |e^{Ft}| < \infty$. Same goes for $F^T$. Therefore there exist scalars $a, b > 0$ such that $aI_n \leq e^{F^T} e^{F} \leq bI_n$ for all $t \in \mathbb{R}$. We can then write

$$aI_n \leq t^{-1} \int_0^t e^{F^T} e^{F} d\tau \leq bI_n$$

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for all $t \geq 0$. Therefore $P$ is positive definite. Symmetricity of $P$ comes by construction. Finally, observe that

$$
|PF + F^{T}P| = \lim_{t \to \infty} t^{-1} \left| \int_{0}^{t} \left( e^{F^{T} \tau} e^{F \tau} F + F^{T} e^{F^{T} \tau} e^{F \tau} \right) d\tau \right|
$$

$$
= \lim_{t \to \infty} t^{-1} \left| \int_{0}^{t} d \left( e^{F^{T} \tau} e^{F \tau} \right) \right|
$$

$$
\leq \lim_{t \to \infty} t^{-1} \left( \left| e^{F^{T} \tau} e^{F \tau} \right| + \left| e^{F^{T} 0 e^{F 0}} \right| \right)
$$

$$
\leq \lim_{t \to \infty} t^{-1} (b + 1)
$$

$$
= 0
$$

whence the result follows.

\[\square\]

**Algorithm 1**

Given $A \in \mathbb{R}^{n \times n}$ that is neutrally stable and $C \in \mathbb{R}^{m \times n}$, we obtain $L \in \mathbb{R}^{n \times m}$ as follows. Let $n_{1} \leq n$ be the number of eigenvalues of $A$ that reside on the imaginary axis. Let $n_{2} := n - n_{1}$. If $n_{1} = 0$, then let $L := 0$; else construct $L$ according to the following steps.

Step 1: Choose $U \in \mathbb{R}^{n \times n_{1}}$ and $W \in \mathbb{R}^{n \times n_{2}}$ satisfying

$$
[U \ W]^{-1} A[U \ W] = \begin{bmatrix} F & 0 \\ 0 & G \end{bmatrix}
$$

where all the eigenvalues of $F \in \mathbb{R}^{n_{1} \times n_{1}}$ have zero real parts.

Step 2: Obtain $P \in \mathbb{R}^{n_{1} \times n_{1}}$ from $F$ by (13).

Step 3: Finally let $L := U P^{-1} (C U)^{T}$.

Below is our solution to Problem 1.

**Theorem 2**

Consider systems (3). Let $u_{i} = L z_{i}$ where $L \in \mathbb{R}^{n \times m}$ is constructed according to Algorithm 1 and $z_{i}$ is as in (4). Then for all network topologies described by connected $\Gamma$, solutions $x_{i}(\cdot)$ for $i = 1, 2, \ldots, p$ synchronize to

$$
\bar{x}(t) := (r^{T} \otimes e^{A t}) \begin{bmatrix} x_{1}(0) \\ \vdots \\ x_{p}(0) \end{bmatrix}
$$

where $r \in \mathbb{R}^{p}$ is such that $r^{T} \Gamma = 0$ and $r^{T} 1 = 1$.

**Proof.** Let the variables that are not introduced here be defined as in Algorithm 1. Let $H := C U P^{-1/2}$ and $S := P^{1/2} F P^{-1/2}$. Then $(H, S)$ is observable for $(C, A)$ is detectable. Also, note that $S$ is skew-symmetric due to $PF + F^{T}P = 0$. 

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We let $U^\dagger \in \mathbb{R}^{n_1 \times n}$ and $W^\dagger \in \mathbb{R}^{n_2 \times n}$ be such that

$$
\begin{bmatrix}
U^\dagger \\
W^\dagger
\end{bmatrix} = [U W]^{-1}.
$$

Note then that $U^\dagger U = I_{n_1}$, $W^\dagger W = I_{n_2}$, $U^\dagger W = 0$, and $W^\dagger U = 0$. Since $u_i = Lz_i$, we can combine (3) and (4) to obtain

$$
\dot{x}_i = Ax_i + LC \sum_{j=1}^{p} \gamma_{ij} (x_j - x_i) \quad (14)
$$

Let now $\xi_i \in \mathbb{R}^{n_1}$ and $\eta_i \in \mathbb{R}^{n_2}$ be

$$
\begin{bmatrix}
\xi_i \\
\eta_i
\end{bmatrix} :=
\begin{bmatrix}
P^{1\slash 2} / 2 \\
0
\end{bmatrix}
\begin{bmatrix}
U^\dagger \\
W^\dagger
\end{bmatrix} x_i \quad (15)
$$

Combining (14) and (15) we can write

$$
\begin{align*}
\dot{\xi}_i &= S \xi_i + H^T H \sum_{j=1}^{p} \gamma_{ij} (\xi_j - \xi_i) + H^T CW \sum_{j=1}^{p} \gamma_{ij} (\eta_j - \eta_i) \quad (16) \\
\dot{\eta}_i &= G \eta_i. \quad (17)
\end{align*}
$$

Let $\Gamma$ be connected and $r \in \mathbb{R}^p$ be such that $r^\top \Gamma = 0$. Then define $\omega_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_1}$ as $\omega_i(t) := e^{-St} \xi_i(t)$ for $i = 1, 2, \ldots, p$. Let $w := [\omega_1^\top \omega_2^\top \ldots \omega_p^\top]^\top$ and $v := [\eta_1^\top \eta_2^\top \ldots \eta_p^\top]^\top$. Starting from (16) and (17) we can write

$$
\dot{w}(t) = (\Gamma \otimes e^{-St}H^THe^{St})w(t) + (\Gamma \otimes e^{-St}H^TCWe^{Gt})v(0).
$$

Thence

$$
w(t) = \Phi(t, 0)w(0) + \left[ \int_0^t \Phi(t, \tau)(\Gamma \otimes e^{-S\tau}H^T CW e^{G\tau})d\tau \right] v(0) \quad (18)
$$

where

$$
\Phi(t, \tau) := \exp \left( \int_\tau^t (\Gamma \otimes e^{-S\alpha}H^THe^{S\alpha})d\alpha \right)
$$

is the state transition matrix [2]. From Theorem 1 we can deduce that $\Phi(t, \tau)$ is uniformly bounded for all $t$ and $\tau$. Also, for any fixed $\tau$ we have $\lim_{t \to \infty} \Phi(t, \tau) = \mathbf{1}_{n_1} \otimes I_{n_1}$. Moreover, $e^{St}$ is uniformly bounded for all $t$, and $e^{Gt}$ decays exponentially as $t \to \infty$ for $G$ is Hurwitz. Therefore we can write

$$
\begin{align*}
\lim_{t \to \infty} \int_0^t \Phi(t, \tau)(\Gamma \otimes e^{-S\tau}H^T CW e^{G\tau})d\tau &= \int_0^\infty \left( \lim_{t \to \infty} \Phi(t, \tau) \right)(\Gamma \otimes e^{-S\tau}H^T CW e^{G\tau})d\tau \\
&= \int_0^\infty (\mathbf{1}_{n_1} \otimes I_{n_1})(\Gamma \otimes e^{-S\tau}H^T CW e^{G\tau})d\tau \\
&= 0.
\end{align*}
$$
Then, by (18), we can write
\[
\lim_{t \to \infty} w(t) = (1^T \otimes I_{n_1})w(0).
\]
Therefore solutions \( \xi_i(\cdot) \) synchronize to \( (r^T \otimes e^{St})w(0) \). Moreover, \( \lim_{t \to \infty} v(t) = 0 \) for \( G \) is Hurwitz. Hence we can say that solutions \( \eta_i(\cdot) \) synchronize to \( (r^T \otimes e^{Gt})v(0) \). As a result, solutions \( x_i(\cdot) \) synchronize to
\[
(r^T \otimes [U P^{-1/2} W]) (e^{St} \begin{bmatrix} P^{1/2} U^\dagger \\ W^\dagger \end{bmatrix}) \begin{bmatrix} x_1(0) \\ \vdots \\ x_p(0) \end{bmatrix} = (r^T \otimes e^{At}) \begin{bmatrix} x_1(0) \\ \vdots \\ x_p(0) \end{bmatrix}
\]
Hence the result.

7 Solution to Problem II

This section, in which we provide a solution to Problem II, follows closely the previous one. We begin with the following algorithm.

**Algorithm 2** Given \( A \in \mathbb{R}^{n \times n} \) that is neutrally stable and \( B \in \mathbb{R}^{n \times m} \), we obtain \( K \in \mathbb{R}^{m \times n} \) as follows. Let \( n_1 \leq n \) be the number of eigenvalues of \( A \) that reside on the imaginary axis. Let \( n_2 := n - n_1 \). If \( n_1 = 0 \), then let \( K := 0 \); else construct \( K \) according to the following steps.

**Step 1:** Choose \( U \in \mathbb{R}^{n \times n_1} \) and \( W \in \mathbb{R}^{n \times n_2} \) satisfying
\[
[U W]^{-1} A [U W] = \begin{bmatrix} F & 0 \\ 0 & G \end{bmatrix}
\]
where all the eigenvalues of \( F \in \mathbb{R}^{n_1 \times n_1} \) have zero real parts. Let \( U^\dagger \in \mathbb{R}^{n_1 \times n} \) and \( W^\dagger \in \mathbb{R}^{n_2 \times n} \) be such that
\[
\begin{bmatrix} U^\dagger \\ W^\dagger \end{bmatrix} = [U W]^{-1}.
\]

**Step 2:** Obtain \( P \in \mathbb{R}^{n_1 \times n_1} \) from \( F \) by (13).

**Step 3:** Finally let \( K := (U^\dagger B)^T P U^\dagger \).

Below is our solution to Problem II.
Theorem 3 Consider systems (5). Let \( u_i = Kz_i \) where \( K \in \mathbb{R}^{m \times n} \) is constructed according to Algorithm 2 and \( z_i \) is as in (6). Then for all network topologies described by connected \( \Gamma \), solutions \( x_i(t) \) for \( i = 1, 2, \ldots, p \) synchronize to

\[
\bar{x}(t) := (r^T \otimes e^{At}) \begin{bmatrix} x_1(0) \\ \vdots \\ x_p(0) \end{bmatrix}
\]

where \( r \in \mathbb{R}^p \) is such that \( r^T \Gamma = 0 \) and \( r^T 1 = 1 \).

Proof. Let the variables that are not introduced here be defined as in Algorithm 2. Let \( H := (P^{1/2}U^+B)^T \) and \( S := P^{1/2}FP^{-1/2} \). Then \((S, H^T)\) is controllable for \((A, B)\) is stabilizable. Also, note that \( S \) is skew-symmetric due to \( PF + F^T P = 0 \).

Since \( u_i = Kz_i \), we can combine (5) and (6) to obtain

\[
\dot{x}_i = Ax_i + BK \sum_{j=1}^p \gamma_{ij}(x_j - x_i) \tag{19}
\]

Let now \( \xi_i \in \mathbb{R}^{n_1} \) and \( \eta_i \in \mathbb{R}^{n_2} \) be

\[
\begin{bmatrix} \xi_i \\ \eta_i \end{bmatrix} := \begin{bmatrix} P^{1/2} \\ 0 \end{bmatrix} \begin{bmatrix} 0 & I_{n_2} \\ U^+ & W^+ \end{bmatrix} x_i \tag{20}
\]

Combining (19) and (20) we can write

\[
\begin{align*}
\dot{\xi}_i &= S\xi_i + H^T H \sum_{j=1}^p \gamma_{ij}(\xi_j - \xi_i) \tag{21} \\
\dot{\eta}_i &= G\eta_i + W^+ BH \sum_{j=1}^p \gamma_{ij}(\xi_j - \xi_i). \tag{22}
\end{align*}
\]

Looking at (21), by Theorem 1 we assert that solutions \( \xi_i(\cdot) \) synchronize to

\[
(r^T \otimes e^{St}) \begin{bmatrix} \xi_1(0) \\ \vdots \\ \xi_p(0) \end{bmatrix}
\]

Now observe that \( |\xi_j(t) - \xi_i(t)| \to 0 \) exponentially as \( t \to \infty \) for all \((i, j)\) pairs. Also recall that \( G \) is Hurwitz. From (22) we can therefore deduce by input-to-state stability (ISS) arguments \([9]\) that \( \eta_i(t) \to 0 \) as \( t \to \infty \) for \( i = 1, 2, \ldots, p \). The remainder of the proof is same as that of proof of Theorem 2. \(\square\)
8 Conclusion

Let us now briefly discuss the generality of the assumptions in the paper. For linear time-invariant case with identical individual system dynamics, it should be evident that detectability (stabilizability) assumption is indispensable for synchronization. Regarding the neutral stability condition, it would be of great interest to study the synchronization of unstable systems. However, when neutral stability assumption on individual systems is relinquished, mere connectedness of the network should generally not be sufficient for individual systems to synchronize. The reason is that, due to unstable dynamics, the trajectories will tend to drift apart from each other when there is no (or very little) coupling. The coupling strength therefore should be above some threshold to overcome that tendency, which requires a stronger (than connectedness) condition on the network topology.

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