Extending $\Lambda(t)$–CDM to the inflationary epoch using dynamical foliations and a pre-inflationary vacuum energy from 5D geometrical vacuum as a unifying mechanism

José Edgar Madriz Aguilar$^{1,a}$, J. Zamarripa$^{2,b}$, M. Montes$^{1,c}$, J. A. Licea$^{1,d}$, C. De Loza$^{2,e}$, A. Peraza$^{3,f}$

1 Departamento de Matemáticas, Centro Universitario de Ciencias Exactas e ingenierías (CUCEI), Universidad de Guadalajara (UdG), Av. Revolución 1500, S.R. 44430 Guadalajara, Jalisco, México
2 Centro Universitario de los Valles, Carretera Guadalajara-Ameca Km 45.5, C. P. 46600 Ameca, Jalisco, México
3 Departamento de Física, Centro Universitario de Ciencias Exactas e Ingenierías (CUCEI), Universidad de Guadalajara, Av. Revolución 1500, S. R. 44430 Guadalajara, Jalisco, México

Received: 30 August 2021 / Accepted: 4 November 2021
© The Author(s), under exclusive licence to Società Italiana di Fisica and Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract In this paper assuming a 5D quantum pre-inflationary vacuum energy, we propose a manner to extend some $\Lambda(t)$-CDM models to the inflationary period by using dynamical foliations of the five-dimensional (5D) Ricci-flat space-time manifold, regarding a non-compact extra space-like coordinate. In this formalism, we achieve also a geometrical unification of inflation and the present accelerating epoch. In this approach, inflation is generated by a pre-inflationary quantum vacuum energy that maintains the 5D classical vacuum on cosmological scales. We obtain from geometrical conditions that we can model the presence of the pre-inflationary vacuum energy in 4D as a dynamical cosmological constant. In this model, the 4D inflationary period is governed by a power law expansion and for certain values of some parameters of the model, we obtain a spectral index satisfying $0.9607 \lesssim n_s \lesssim 0.9691$ and a scalar-to-tensor ratio $r = 0.098$, values that fit well according to Planck 2018 results. The 4D inflationary potential is induced for the 5D geometry and the 4D pre-inflationary potential is determined by the model and its contribution is necessary so that $n_s$ and $r$ can fit the observational data. We also show that in this theoretical framework, the present acceleration in the expansion of the universe can be explained due to a remanent of this pre-inflationary vacuum energy scaled to the present epoch and that its description can be done with the same $\Lambda(t)$. In this period, we obtain a deceleration parameter in agreement with Planck 2018 data under certain restrictions of the parameters of the model. From the geometrical point of view, $\Lambda(t)$ is depending on the dynamical foliation of the 5D space-time manifold.

---

$^a$e-mails: madriz@mdp.edu.ar; jose.madriz@academicos.udg.mx (corresponding author)
$^b$e-mail: jose.zamarripa@academicos.udg.mx
$^c$e-mail: mariana.montes@academicos.udg.mx
$^d$e-mail: antonio.licea@academicos.udg.mx
$^e$e-mail: cynthia.deloza2536@alumnos.udg.mx
$^f$e-mail: americo.peraza@academicos.udg.mx

Published online: 18 January 2022
1 Introduction

The acceleration in the present expansion of the universe continues being one of greatest paradigms in cosmology [1,2]. Many different proposals have been put forward in trying to explain the origin and dynamics of that acceleration. One of the most popular ideas is the dark energy, which is assumed to drive the expansion of the universe on late times. However, until now we cannot understand accurately its origin and physical essence. Thus, alternative models coming from theories with extra dimensions have been also considered as possible routes. The simplest dark energy model is the known as cosmic concordance, in which the dark energy is described by a cosmological constant. Unfortunately, this feature is also its main trouble due to the cosmological constant problem [3]. The $10^{121}$ orders of magnitude of difference between the observational value of the cosmological constant and the one predicted by quantum field theory led to the emergence of dynamical vacuum decay models, also known as dynamical cosmological constant models or $\Lambda(t)$–CDM [4]. The main idea in these models is consider a vacuum energy driven by $\Lambda(t)$, whose decaying is a function of the Hubble parameter $H(t)$. Some of them are $\Lambda(t) = mH^2$ and $\Lambda(t) = \sigma H$ [5–8]. For the first one, the spectrum of gravitational waves during inflation results to be nearly scale invariant, in good agreement with CMB observations, whereas for the second one the same spectrum fails in this characteristic [9]. However, a model $\Lambda(t) = \sigma H$ seems to work in agreement with observations in the present, radiation and matter dominated epochs [7]. An interesting analysis of the evolution of the cosmological scale factor in the presence of a dynamical cosmological constant has been studied in [10].

The other period of accelerated expansion of the universe is the early inflation. It is normally believed that the expansion is governed in this period for a vacuum energy modeled by the inflation scalar field with a dominant potential over its kinetic part (the slow-roll regimen). However, depending on the model, there is an energy gap between the energy scale when inflation starts and the Planck scale of about 3 orders of magnitude. Thus, the universe could have passed for a pre-inflationary period. Pre-inflationary scenarios have been proposed in different frameworks, for example, in big bounce models in loop quantum cosmology (LQC) [11–14], in relativistic quantum geometry approaches [15] and in holographic cosmology [16]. Observational signatures of that period are being investigated [17].

On the other hand, physical models with extra dimensions have been a recourse to explain the acceleration of the cosmic expansion. Some of the most popular theories with extra dimensions are string theories, brane worlds and the induced matter theory. However, models in the framework of extra dimensions must comply some physical constraints, like, for example, the variation of the four-dimensional Newtonian constant [18]. In the induced matter theory, first proposed by Paul Wesson and collaborators, the Einstein field equations of general relativity are locally and isometrically embedded into a 5D Ricci flat space-time manifold. As a difference with respect other theories with extra dimensions, in the induced matter theory the four-dimensional Newton’s law of gravity is not modified [19,20]. Besides, matter appears in 4D induced by the 5D geometry. In particular, the idea of inducing a cosmological constant in the framework of the induced matter theory was first done in [21]. The idea of decaying dark energy was also explored in the context of the induced matter theory in [21,22]. Some other approaches like $\Lambda(t)$–CDM scenarios in the light of 5D Brans–Dicke theory can be found, for example, in [23].

The plan of this paper is as follows. Section 1 is devoted to an introduction. In Sect. 2, we develop the basic formalism of a 5D pre-inflationary epoch with a quantum scalar field with non-canonical kinetic energy under the condition of having a 5D classical vacuum. In Sect. 3, we show the general geometrical formalism for inducing a 4D dynamical cosmological
constant that can mimic the vacuum energy originated in the pre-inflationary epoch. In Sect. 3, we apply our formalism to the description of the inflationary epoch. Section 4 is left for the analysis in the present epoch of accelerated expansion. Finally, in Sect. 5 we give some final comments.

Our conventions are Latin indices run from 0 to 4, with exception of $i$ and $j$ that take values from 1 to 3. Greek indices run the range from 0 to 3. The 5D metric signature we use is $(+,-,-,-,-)$. Finally, we adopt units on which the speed of light $c = 1$.

2 The 5D pre-inflationary formalism

Let us start assuming that our 5D universe passed for a pre-inflationary epoch. Thus, inspired in the fact that some scalar–tensor theories in the Einstein frame can be obtained as the low-energy limit of string theories, we consider that the pre-inflationary period is governed by a quantum vacuum energy described by a scalar field $\zeta$ with non-canonical kinetic term, whose action reads

$$S = \int d^5y \sqrt{g_5} \left[ \frac{(5)}{2\kappa_5} R^{(5)} + \frac{1}{2} \Omega(\zeta) g^{ab} \xi_a \xi_b - V_{pi}(\zeta) \right],$$

where $V_{pi}(\zeta)$ is the potential associated with the pre-inflationary scalar field $\zeta$, $\kappa_5$ is the 5D gravitational coupling, $g_5$ denotes the determinant of the 5D metric

$$ds_5^2 = g_{ab} dy^a dy^b,$$

$(5) R$ is the 5D scalar curvature defined by $(5) R = g^{ab} (5) R_{ab}$ being $(5) R_{ab}$ the 5D Ricci tensor which satisfies $< (5) \hat{R}_{ab} > = (5) R_{ab}$, with $<>$ denoting the expectation value, and $\hat{R}_{ab}$ is the Ricci curvature operator related to the coordinate operator

$$\hat{y}^a(y^b) = \frac{1}{(2\pi)^{3/2}} \int d^3k_r d^3k_l \hat{E}^a \left[ b_{k_r,k_l} \Theta_{k_r,k_l}(y^b) + \delta_{k_r,k_l} \Theta^*_{k_r,k_l}(y^b) \right],$$

with $b^\dagger_{k_r,k_l}$ and $b_{k_r,k_l}$ being the creation and annihilation operators of the 5D space-time determined by the relation

$$\left\{ \Psi \left| \left[ b_{k_r,k_l}, b^\dagger_{k_r,k_l} \right] \right| \Psi \right\} = \delta^{(3)}(\vec{k}_r - \vec{k}_l') \delta(k_l - k_l'),$$

where $|\Psi >$ denotes a background quantum state and $\hat{E}^a = \varepsilon^a_{b c d f} E^b E^c E^d E^f$. With the help of (3), it can be shown that the eigenvalues obtained from applying the coordinate operator to a background state $|\Psi >$ satisfy

$$d\hat{y}^a |\Psi > = (5) U^a ds_5 |\Psi > = dy^a |\Psi >,$$

being $(5) U^a$ the 5-velocity. Thus, the 5D line element satisfies

$$\left\{ \Psi \left| d\hat{y}_a d\hat{y}^a \right| \Psi \right\} = (5) U_b (5) U^b ds_5^2 \left\{ \Psi |\Psi' > = ds_5^2 \delta_{\Psi,\Psi'} \right\}.$$

On the other hand, according to the Wesson’s induced matter theory of gravity the classical field equations are: $(5) R_{ab} = 0$ and therefore $\left\{ \hat{R}_{ab} \right\} = 0$. In this manner, the main idea in this paper is to consider that the quantum vacuum energy coming from a pre-inflationary stage generates the inflationary expansion maintaining at the same time the 5D vacuum on
large scales. This means that the contributions of the scalar field to the classical 5D Ricci curvature are null. Thus, we can interpret that as inflation goes on, the vacuum energy plays the role of a dynamical cosmological constant \( \Lambda(t) \). As we shall show, in this formalism \( \Lambda(t) \) can be expressed in terms of geometrical quantities or equivalently in terms of a physical scalar field induced on our 4D space-time by the 5D geometry. In fact, its geometrical origin allows us to postulate that a remanent of the same vacuum energy can be used to explain the present acceleration in the expansion of the universe.

With the idea that we have a 5D classical vacuum on cosmological scales, the field equation for \( \zeta \) derived from (1) reads

\[
\Omega(\zeta) \, (5) \Box^5 \zeta + \frac{1}{2} \Omega' g^{ab} \zeta,_{a} \zeta,_{b} + V_{pi}'(\zeta) = 0.
\]  

(7)

In general, the semiclassical approximation allows to write \( \zeta(x^\mu) = \zeta_c(t) + \delta \zeta \), with \( \zeta_c \) being the pre-inflationary scalar field on cosmological scales, \( \delta \zeta \) denoting quantum fluctuations on small non-cosmological scales such that \( < \delta \zeta > = 0 \). However, remembering that the pre-inflationary scalar field \( \zeta \) is in this approach considered only of quantum nature then \( \zeta_c = 0 \). Thus, we obtain the relations

\[
\Omega(\zeta) \simeq \Omega(0) + \Omega'(0) \zeta,
\]

(8)

\[
\Omega'(\zeta) \simeq \Omega'(0) + \Omega''(0) \zeta,
\]

(9)

where we have regarded that the quantum fluctuations are small deviations from the classical value of the field. Hence, considering these conditions it comes from (7) that

\[
(5) \Box^5 \zeta + \frac{V_{pi}'(0)}{\Omega(0)} + \frac{V_{pi}''(0)}{\Omega(0)} \zeta = 0.
\]

(10)

If we restrict our model to the case where \( V_{pi}'(0) = 0 \) and \( \Omega(0) \neq 0 \), the equation (10) reduces to

\[
(5) \Box^5 \zeta + \frac{V_{pi}''(0)}{\Omega(0)} \zeta = 0
\]

(11)

If we consider a 5D warped product manifold space-time with line element given by

\[
d s_5^2 = e^{2A(l)} g_{\alpha\beta}(x) dx^\alpha dx^\beta - dl^2,
\]

(12)

with \( A(l) \) being the warping factor, we obtain from (11) that the variation of the pre-inflationary field \( \zeta \) along the fifth non-compact extra coordinate \( l \), for a separable field \( \zeta(x^\alpha, l) = Q(x^\alpha) \Delta(l) \), is governed by the equation

\[
e^{-2A} \frac{d}{dl} \left( e^{4A} \frac{d \Delta}{dl} \right) + \left( \frac{V_{pi}''(0)}{\Omega(0)} e^{2A} + \alpha_l \right) \Delta = 0,
\]

(13)

where \( \alpha_l \) is a separation constant.

3 The geometrical dynamical cosmological constant

Now, to relate the pre-inflationary vacuum energy with a dynamical cosmological constant of geometrical origin, we proceed as follows. A well-known Ricci-flat solution of the field

\[
\Box^5 \zeta + \frac{1}{2} \Omega' g^{ab} \zeta,_{a} \zeta,_{b} + V_{pi}'(\zeta) = 0.
\]
equations: \(^{(5)}R_{ab} = 0\), is given by the 5D line element \([23,25]\)
\[
dS_5^2 = l^2 \frac{\Lambda(t)}{3} dt^2 - l^2 e^{2f} \sqrt{\frac{\Lambda(t)}{3}} dt \delta_{ij} dx^i dx^j - dl^2, \quad (14)
\]
where \(t\) is the cosmic time and \(\Lambda(t)\) is a metric function.

Now, to pass from 5D to 4D we assume that the 5D manifold can be dynamically foliated by a family of generic hypersurfaces given by: \(\Sigma : l = f(t)\). Dynamical foliations have been first introduced by J. Ponce de Leon in \([26,27]\). The differential line element induced on every leaf \(\Sigma\) reads
\[
dS_\Sigma^2 = \left[ \frac{1}{3} f^2 \Lambda(t) - f^2 \right] dt^2 - f^2 e^{2f} \sqrt{\frac{\Lambda(t)}{3}} dt \delta_{ij} dx^i dx^j, \quad (15)
\]
where the dot denotes time derivative. In order to every leaf to describe a FRW universe, the conditions
\[
\frac{1}{3} f^2 \Lambda(t) - f^2 = 1, \quad (16)
\]
\[
f^2 e^{2f} \sqrt{\frac{\Lambda(t)}{3}} dt = a^2(t), \quad (17)
\]
must be valid, where \(a(t)\) is the cosmic scale factor. The 4D Einstein equations induced on every \(\Sigma\) considering (15) and (16) can be written as \([28]\)
\[
3H^2 = 8\pi G \rho_{(IM)} + \Lambda(t), \quad (18)
\]
\[
2\frac{\ddot{a}}{a} + H^2 = -8\pi G p_{(IM)} + \Lambda(t). \quad (19)
\]
where \(H = \dot{a}/a\) denotes the Hubble parameter, and the energy density and pressure for induced matter are given, respectively, by
\[
\rho_{(IM)} = 3 \left( \frac{\dot{f}}{f} \right)^2 + 2 \frac{\ddot{f}}{f} \sqrt{3\Lambda}, \quad (20)
\]
\[
p_{(IM)} = -2 \frac{\ddot{f}}{f} - 2 \sqrt{3\Lambda} \frac{\dot{f}^2}{f} - \frac{\dot{\Lambda}}{\sqrt{3\Lambda}} - \left( \frac{\dot{f}}{f} \right)^2. \quad (21)
\]
The equations (18) and (19) indicate that the metric function \(\Lambda(t)\) can be interpreted as a dynamical cosmological constant describing a dynamical vacuum energy. It follows from (16) that \([28]\)
\[
\Lambda(t) = \frac{3}{f^2} \left( 1 + \dot{f}^2 \right), \quad (22)
\]
indicating that \(\Lambda(t)\) is, in the geometrical sense, depending on the dynamical foliation \(l = f(t)\).

On the other hand, as it was shown in \([29]\) the \(\Lambda(t)\)-CDM model introduced in \([30]\) in which \(\Lambda(t) = \sigma H(t)\), with \(\sigma^{1/3} \approx 150 \text{ MeV}\) being the energy scale of the chiral phase transitions of QCD, the spectral index for gravitational waves during inflation results to be \(n_{gw} \approx -0.5826\) which enters in contradiction with observations that indicate \(n = 0.9649 \pm 0.0042\) \([31]\). However, as suggested in \([7,32]\), this \(\Lambda(t)\)-CDM model exhibits observational viability during the radiation and matter dominated epochs as well as the present epoch. Thus, to include the inflationary epoch in a \(\Lambda(t)\)-CDM scenario we propose the more general form
\[
\Lambda(t) = (1 - \eta_{sr}) m H^2 + \eta_{sr} \sigma H, \quad (23)
\]
where \( m \) is a parameter and \( \eta_{sr} = -\dot{H}/H^2 \) is one of the slow-roll parameters employed in usual inflationary scenarios. As it is well known, \( \eta_{sr} \) is an increasing parameter varying from 0 to 1 during inflation, reaching 1 at the end of that period. Therefore, during inflation the \( H^2 \) term dominates and by the end of inflation and further the \( H \) term is dominant. By equating (22) with (23), we obtain

\[
(1 - \eta_{sr})mH^2 + \eta_{sr}\sigma H = \frac{3}{f^2} (1 + \dot{f}^2).
\]

(24)

Thus, it is not difficult to see that during inflation the equation (24) can be approximated by

\[
(1 - \eta_{sr})mH^2 = \frac{3}{f^2} (1 + \dot{f}^2),
\]

(25)

while in the rest of the epochs in the evolution of the universe we have

\[
\sigma H = \frac{3}{f^2} (1 + \dot{f}^2).
\]

(26)

According to the definition of \( \eta_{sr} \), the equation (25) can be explicitly written as

\[
\left(1 + \frac{\dot{H}}{H^2}\right)mH^2 = \frac{3}{f^2} (1 + \dot{f}^2).
\]

(27)

On the other hand, as \( l \) is a non-compact space-like coordinate, we will associate \( l \) with the size of the observable universe and thus we consider \( l(t) = \lambda H^{-1} \) where \( \lambda \) is a parameter taking values depending on the epoch of the universe. In this manner, we are requiring \( l > r_H \) with \( r_H \) being the Hubble radius, thus establishing that \( l \) cannot be directly observed.

## 4 Dynamical vacuum during inflation

Once we have introduced the general elements of the model we need, we are now in position to obtain solutions for the inflationary epoch.

It follows from (23) that during inflation the dynamical vacuum represented by \( \Lambda(t) \) can be approximated by \( \Lambda(t) = (1 - \eta_{sr})mH^2 \). The foliation function \( f \) that induces such \( \Lambda(t) \) is determined by (27), which given that \( f(t) = \lambda H^{-1} \) becomes

\[
\left(1 + \frac{\dot{H}}{H^2}\right)mH^2 = \frac{3H^2}{\lambda^2} \left[1 + \frac{\lambda^2\dot{H}^2}{H^4}\right].
\]

(28)

Solving (28), for \( \lambda^2 > (3/m) \) the physical solution reads

\[
H(t) = \frac{6\lambda}{\gamma \left(t - t_i + \frac{6\lambda}{\dot{H}_i}\right)},
\]

(29)

with \( \gamma = \sqrt{m^2\lambda + 12m\lambda - 36 - m\lambda} \) and where we have imposed the initial condition \( H(t_i) = H_i \), being \( t_i \) the time when inflation begins and \( H_i \) the value of the Hubble parameter at that time.

The scale factor corresponding to the Hubble parameter (29) is then

\[
a(t) = a_i \left[\frac{\gamma H_i}{6\lambda} (t - t_i) + 1\right]^{\frac{6\lambda}{\gamma}},
\]

(30)

where we have used the initial condition \( a(t_i) = a_i \).

\( \square \) Springer
It is not difficult to see that the Hubble parameter (29), which corresponds to a power law expansion, is obtained on the generic hypersurface determined by the dynamical foliation

\[ f(t) = \frac{\gamma}{6} \left( t - t_i + \frac{6\lambda}{\gamma H_i} \right). \]  

(31)

On the other hand, it follows from the action (1) that the 4D action induced on \( \Sigma : l = f(t) \) can be written as

\[ S = \int d^4x \sqrt{-h} \left[ \frac{4}{16\pi G} R^{(4)} + \frac{1}{2} h^{\mu\nu} \Omega(\Phi) \Phi,_{\mu} \Phi,_{\nu} - U_{\text{eff}}(\Phi) \right], \]  

(32)

where \( U_{\text{eff}}(\Phi) = U(\Phi) + U_{\text{pi}}(\Phi) \) with \( U_{\text{pi}}(\Phi) \) denoting the induced 4D pre-inflationary potential, \( h \) is the determinant of the 4D effective metric derived from (15),(16) and (17), \( \Omega(\Phi) \) is a well-behaved function of the inflation field \( \Phi \), and \( U(\Phi) \) is the 4D potential induced from the 5D part of the kinetic term of \( \zeta \), which is given by

\[ U(\Phi) = \frac{1}{2} \Omega(\zeta) \frac{\partial^2}{\partial \zeta^2} \mathcal{L}^2 \bigg|_{l = f(t)} = \frac{1}{2} \Omega(\Phi) \left( \frac{\Delta_i}{\Delta} \right)^2 \Phi^2, \]  

(33)

where we have considered that \( \Phi(x^a) = \zeta(x^a, l)|_{l = f(t)} \) is the 4D induced scalar field.

The dynamical equation for the scalar field \( \Phi \) derived from (32) is

\[ \Omega(\Phi) \Box \Phi + \frac{1}{2} \Omega'(\Phi) h^{\alpha\beta} \Phi,_{\alpha} \Phi,_{\beta} + U'_{\text{eff}}(\Phi) = 0, \]  

(34)

where the prime denotes derivative with respect to \( \Phi \). The corresponding Friedmann equation reads

\[ 3H^2 = \frac{8\pi G}{2} \left( \frac{1}{2} \Omega(\Phi) \Phi^2 + U_{\text{eff}}(\Phi) \right). \]  

(35)

The slow-roll parameters for (32) have the form

\[ \epsilon = M_p^2 \frac{U'_{\text{eff}}(\Phi)}{U_{\text{eff}}(\Phi)}, \]  

(36)

\[ \eta = M_p^2 \left( \frac{1}{2} \Omega(\Phi) \frac{U''_{\text{eff}}(\Phi)}{U_{\text{eff}}(\Phi)} - \Omega'(\Phi) \frac{U'_{\text{eff}}(\Phi)}{\Omega(\Phi) U_{\text{eff}}(\Phi)} \right), \]  

(37)

where \( M_p^2 = (16\pi G)^{-1} \). Now, implementing the field transformation

\[ \phi = \int \sqrt{\Omega(\Phi)} d\Phi, \]  

(38)

and using (18) and (35) we arrive to

\[ \frac{3H^2}{8\pi G} = \rho_{(IM)} + \frac{\Lambda(t)}{8\pi G} = \frac{1}{2} \phi^2 - \frac{1}{2a^2} (\nabla \phi)^2 + V(\phi), \]  

(39)

where \( V(\phi) = U_{\text{eff}}(\phi(\Phi)) \) and in order to be in agreement with inflationary scenarios, the both \( \rho_{(IM)} \) and \( \Lambda(t) \) describe in this period vacuum energy. This equation establishes the equivalence between the geometrical and physical vacuum energy. From one side, we have a physical vacuum energy modeled by the inflation field, and in the other, the same vacuum energy can be described geometrically by a dynamical cosmological constant.
Now, we employ the semiclassical approximation for the inflation field
\[ \phi(x^\nu) = \varphi(t) + \delta \phi(x^\nu), \] (40)
where \( \varphi(t) = \langle \phi(x^\nu) \rangle \) is the classical part of \( \phi \) while its quantum fluctuations are described by \( \delta \phi \), with \( \langle, \rangle \) denoting expectation value. Thus, with the help of (20) the classical part of (39) leads to
\[
3 \left( \frac{\dot{f}}{f} \right)^2 + 2 \frac{\dot{f}}{f} \sqrt{3 \Lambda_c(t)} + \frac{\Lambda_c(t)}{8\pi G} = \frac{1}{2} \dot{\varphi}^2 + V(\varphi),
\] (41)
where \( \Lambda_c(t) \) is the dynamical cosmological constant defined on cosmological scales given by (22). The corresponding quantum part of (39) then implies
\[
\frac{\dot{f}}{f} \delta \Lambda + \frac{\delta \Lambda}{8\pi G} = \dot{\varphi} \delta \dot{\varphi} + \frac{1}{2} \delta \dot{\varphi}^2 - \frac{1}{2} (\nabla \delta \varphi)^2 + \sum_{n=1}^{\infty} \frac{V^{(n)}(\varphi)}{n!} \delta \varphi^n.
\] (42)
It follows from (18), (19), (20) and (21) that the effective equation of state parameter is given by [28]
\[
\omega_{\text{eff}} = \frac{p_{\text{eff}}}{\rho_{\text{eff}}},
\] (43)
where
\[
\rho_{\text{eff}} = \rho(1M) + \frac{\Lambda_c}{8\pi G},
\] (44)
\[
p_{\text{eff}} = p(1M) - \frac{\Lambda_c}{8\pi G},
\] (45)
are the effective energy density and pressure, respectively. It is not difficult to see that \( \omega_{\text{eff}} \simeq -1 \) once the next condition is valid
\[
\frac{\dot{\Lambda}_c}{\sqrt{3} \Lambda_c} \ll 5 \left( \frac{\dot{f}}{f} \right)^2 + 2 \sqrt{3 \Lambda_c} \frac{\dot{f}}{f} + \Lambda_c.
\] (46)
With the help of (22) and (31), the condition (46) becomes
\[
8A^2 - 6A \sqrt{1 + A^2} + 3 \gg 2A \sqrt{1 + A^2},
\] (47)
where \( A = \gamma / 6 \). The inequality (47) is satisfied for the interval: \( 0 < \gamma \ll 9/2 \). Using the definition of \( \gamma \), it can be shown that \( m \gg (1/243)(36\epsilon - 63/4)^2 \), where in view that during inflation \( \lambda^2 > 3/m \), we have expressed this same condition as \( \lambda^2 = 3\epsilon/m \) with \( \epsilon > 1 \). These conditions are compatible with \( \lambda > 1 \). However, the former is in fact required in order to the fifth extra dimension to be longer than the Hubble radius, explaining in this manner why it remains directly unobservable in our 4D universe. Therefore, when these last conditions are valid, the equation of state results in \( \omega_{\text{eff}} \simeq -1 \), which is suitable to describe a slow-roll inflationary period.

Employing (41), (44) and (45), we obtain
\[
\dot{\varphi} = \sqrt{(1 + \omega_i)\rho_{\text{eff}}} = \frac{\alpha_{\text{eff}}}{t - B},
\] (48)
where \( B = t_i - 6\lambda/(\gamma H_i) \), \( \omega_i \) is the value of \( \omega_{\text{eff}} \) at the beginning of inflation, and

\[
\alpha_{\text{eff}} = M_p\tilde{\alpha} = M_p\sqrt{6(1 + \omega_i)(2A^2 - 2A\sqrt{1 + A^2} + 1)}.
\]  \( \text{(49)} \)

Solving (48), we arrive to

\[
\varphi(t) = \varphi_i + \alpha_{\text{eff}} \ln \left( \frac{t - B}{t_i - B} \right),
\]  \( \text{(50)} \)

where \( \varphi_i = \varphi(t_i) \), being \( t_i \) the time when inflation begins. Now, with the help of (39), (44) and (45) it is not difficult to show that

\[
V(\varphi) = \frac{1}{2}(1 - \omega_i)\rho_{\text{eff}}.
\]  \( \text{(51)} \)

By means of (44), the former expression finally becomes

\[
V(\varphi) = \left( \frac{1 - \omega_i}{1 + \omega_i} \right) \frac{2\alpha_{\text{eff}}^2}{(t_f - B)^2} \exp \left( \frac{1}{\alpha_{\text{eff}}} (\varphi - \varphi_f) \right),
\]  \( \text{(52)} \)

where \( \varphi_f = \varphi(t_f) \) with \( t_f \) being the time when inflation ends. The expression (52) is the potential for inflation induced from the 5D geometry in terms of the field \( \varphi \).

However, by using (38) the potential (52) becomes

\[
U_{\text{eff}}(\Phi_c) = U_0 \exp \left( \frac{1}{\alpha_{\text{eff}}} \int \sqrt{\Omega_1(\Phi_c)} d\Phi_c \right),
\]  \( \text{(53)} \)

where we have considered that \( \Phi(x^\mu) = \Phi_c(t) + \delta \Phi(x^\mu) \) and

\[
U_0 = \left( \frac{1 - \omega_i}{1 + \omega_i} \right) \frac{2\alpha_{\text{eff}}^2}{(t_f - B)^2} \exp \left( -\frac{1}{\alpha_{\text{eff}}} \Phi_f \right),
\]  \( \text{(54)} \)

with \( \Phi_f \) being the value of \( \Phi \) at the end of inflation.

Now, using the slow-roll parameters (36) and (37) the spectral index \( n_s = 1 - 6\epsilon + 2\eta \) and the scalar-to-tensor ratio \( r = 16\epsilon \) read, respectively,

\[
n_s = 1 - \frac{3M_p^2}{\Omega} \left( \frac{U'_{\text{eff}}}{U_{\text{eff}}} \right) + 2M_p^2 \left( \frac{1}{\Omega} \frac{U''_{\text{eff}}}{U_{\text{eff}}} - \frac{\Omega'}{\Omega^2} \frac{U'_{\text{eff}}}{U_{\text{eff}}} \right),
\]  \( \text{(55)} \)

\[
r = \frac{8M_p^2}{\Omega} \left( \frac{U'_{\text{eff}}}{U_{\text{eff}}} \right)^2.
\]  \( \text{(56)} \)

With the help of (53), the expressions (55) and (56) become

\[
n_s = 1 - \frac{M_p^2}{\alpha_{\text{eff}}^2} - \frac{M_p^2}{\alpha_{\text{eff}}} \frac{\Omega'}{\Omega^{3/2}},
\]  \( \text{(57)} \)

\[
r = \frac{8M_p^2}{\alpha_{\text{eff}}^2}.
\]  \( \text{(58)} \)
Now, considering that \( n_s \) is a constant, we can restrict ourselves to the case in which \( \Omega' / \Omega^{3/2} \) is also a constant. Thus, we obtain for \( \Omega(\Phi_c) \) the expression

\[
\Omega(\Phi_c) = \left( \frac{2}{2C - \theta_0 \Phi_c} \right)^2,
\]

where \( C \) is an integration constant and \( \theta_0 = \Omega' / \Omega^{3/2} \). Employing (29) and (59), the expressions (57) and (58) read

\[
\begin{align*}
    n_s &= 1 - \frac{1}{\tilde{\alpha}^2} - \frac{\theta_0}{\tilde{\alpha}} M_p, \\
    r &= \frac{8}{\tilde{\alpha}^2}.
\end{align*}
\]

According to the PLANCK 2018 results (Planck TT, TE, EE, lowE+lensing+BKIS+BAO): \( n_s = 0.9649 \pm 0.0042 \) and \( r < 0.10 \) [33]. Thus, it follows from (61) that \( \tilde{\alpha} > \sqrt{80} \approx 8.9443 \). Therefore, for a given \( \tilde{\alpha} \) we obtain an interval of values for \( \theta_0 \) that allows (60) to reproduce the observational values for \( n_s \). For example, for \( \tilde{\alpha} = 9 \), the interval \( 0.1669 M_p \leq \theta_0 \leq 0.2426 M_p \) corresponds to \( 0.9607 \leq n_s \leq 0.9691 \). For this value of \( \tilde{\alpha} \), the scalar-to-tensor ratio is \( r = 0.098 \).

Finally, by means of (53) and (59) we obtain

\[
U_{\text{eff}} (\Phi_c) = \frac{U_0}{(2C - \theta_0 \Phi_c)^{2/\theta_0 \tilde{\alpha}_{\text{eff}}}}.
\]

This is the effective inflationary potential induced by the 5D geometry. Hence, it follows from (33), (59) and the definition of \( U_{\text{eff}} \) that the pre-inflationary potential is given by

\[
U_{\text{pi}} (\Phi_c) = \frac{U_0}{(2C - \theta_0 \Phi_c)^{2/\theta_0 \tilde{\alpha}_{\text{eff}}}} - \frac{1}{2} \left( \frac{2\Phi_c}{2C - \theta_0 \Phi_c} \right)^2 \left( \frac{\Delta_{l=I}}{\Delta} \right)^2 l_{\text{f}(t)},
\]

where according to (13) and (14) the function \( \Delta(l) \) obeys the equation

\[
\frac{d^2 \Delta}{dl^2} + \frac{4}{l} \frac{d \Delta}{dl} + \left( \frac{V''_{\text{pi}}(0)}{\Omega(0)} + \frac{\alpha_l}{l^2} \right) \Delta = 0.
\]

The general solution for (64) reads

\[
\Delta(l) = B_1 l^{-3/2} J_{\mu} [Z(t)] + B_2 l^{-3/2} Y_{\mu} [Z(t)],
\]

with \( J_{\mu} [Z(t)] \) and \( Y_{\mu} [Z(t)] \) denoting the first and second kind Bessel functions, \( B_1 \) and \( B_2 \) being integration constants, \( \mu = (1/2) \sqrt{9 - 4\tilde{\alpha}^2} \) and where

\[
Z(t) = \left( \frac{V''_{\text{pi}}(0)}{\Omega(0)} \right)^{1/2} l. \]

Thus, with the help of (65) and (66) the pre-inflationary potential (63) can be specified. Notice that according to (33), the 4D potential for the inflation field \( U(\Phi) \) is induced by the 5D geometry, and as indicated in (53), the effective potential \( U_{\text{eff}} \) is determined by the form of the function \( \Omega(\Phi) \). However, the former has the form given in (59) to ensure that \( n_s \) and \( r \) fit in the PLANCK 2018 results. Thus, the pre-inflationary potential (63) can be fixed. Hence, there is the need of a pre-inflationary epoch in this model.
5 The present epoch

In order to study vacuum decay on the present epoch of accelerated expansion, we will obtain solutions of (26). Thus, as we used before, if we consider $f(t) = \lambda H^{-1}$ in (26), we obtain

$$\frac{\sigma \lambda^2}{3H} = 1 + \frac{\lambda^2 H^2}{H^4}.$$  \hfill (67)

Using the initial condition $H(t_0) = H_0$ with $t_0$ denoting the present time and $H_0$ the present value for the Hubble parameter, the physical solution for (67) has the form

$$H(t) = \frac{12\sigma \lambda^2}{\sigma^2 \lambda^2 t^2 - 2\sigma^2 \lambda \mu_0^2 t + \sigma \lambda \mu_0^2 + 36},$$  \hfill (68)

where $\sigma \lambda^2 > 3H_0$ and

$$\mu_0 = t_0 \pm \frac{6}{\sigma \lambda} \sqrt{\frac{\sigma \lambda^2}{3H_0} - 1}.$$  \hfill (69)

It follows from (69) that the scale factor is given by

$$a(t) = a_0 \frac{\text{arctanh} \left[ \frac{\sigma \lambda}{3t_0} (\lambda t - \mu_0^2) \right]}{\text{arctanh} \left[ \frac{\sigma \lambda}{3t_0} (\lambda t_0 - \mu_0^2) \right]}.$$  \hfill (70)

The Hubble parameter (69) corresponds to the generic dynamical hypersurface

$$f(t) = \frac{\sigma \lambda^2 t^2 - 2\sigma^2 \lambda \mu_0^2 t - \sigma \lambda \mu_0^2 + 36}{12\sigma \lambda}.$$  \hfill (71)

Now, with the help of (68) the deceleration parameter $q = -a\ddot{a}/\dot{a}^2$ reads

$$q = -\left(1 - \frac{2\sigma^2 \lambda^2 t - 2\sigma^2 \lambda \mu_0^2}{12\sigma \lambda^2}\right).$$  \hfill (72)

According to the Planck 2018 results, the present values for the deceleration parameter are $q_0 = -0.5581^{+0.0273}_{-0.0267}$ [33]. Evaluating (72) in the present time $t_0$ and using that $\lambda > 1$, we obtain

$$\lambda = \frac{\sigma \mu_0^2}{\sigma t_0 - 6(1 + q_0)} > 1.$$  \hfill (73)

This condition is valid for $\mu_0^2 > t_0 - \frac{6}{\sigma}(1 + q_0) > 0$. This last condition implies $\sigma > (6/t_0)(1 + q_0) > 0$. Thus, we have for $q_0$ the interval $-0.5848 < q_0 < -0.5308$, and in this manner for $q_0 = -0.5848$, the values of $\sigma$ satisfy $\sigma > 2.4912 t_0^{-1}$, while for $q_0 = -0.5308$ we obtain $\sigma > 2.8152 t_0^{-1}$. Therefore, (68) describes a scenario with an accelerating expansion determined by a deceleration parameter $q_0$ compatible with present observational values.

It is important to note that in this epoch, the 4D effective equation of state for induced matter can be explicitly obtained by inserting (71) in (43) and taking into account the restrictions for the $\sigma$, $\lambda$ and $\mu_0$ parameters established to have a present deceleration parameter compatible with observational data.
6 Final remarks

In this letter assuming a 5D quantum pre-inflationary vacuum energy, we have derived an inflationary and present accelerating expansion scenarios from a 5D vacuum employing dynamical foliations of the 5D metric. The 5D quantum pre-inflationary vacuum energy is modeled by a scalar field with non-canonical kinetic energy and a pre-inflationary potential. We show that the vacuum energy density coming from the 5D pre-inflationary period, induced on our 4D universe, can be geometrically modeled by a dynamical cosmological constant determined by a family of dynamical hypersurfaces. This is due to the condition that the pre-inflationary vacuum energy is characterized by maintaining the 5D classical vacuum on cosmological scales, i.e., such that \( R_{ab} = 0 \). Hence, we use the Wesson’s induced matter approach to obtain a geometrical equation of state on every leaf member of the foliation to the 5D space-time.

On the other hand, with the idea to incorporate an inflationary scenario in a \( \Lambda(t)-\)CDM scenario and inspired in the fact that the model \( \Lambda(t) = \sigma H(t) \) enters in contradiction with the value of the spectral index for gravitational waves, inferred by observations, during inflation [29], we have proposed the combined dynamical cosmological constant: \( \Lambda(t) = (1 - \eta_{sr}) m H^2 + \eta_{sr} \sigma H \) given in (23). As a result, we obtain that the term \( H^2 \) is dominant during inflation, and for the remaining epochs, the \( H \) term becomes relevant.

The 4D slow-roll inflationary scenario is driven by a scalar field with non-canonical kinetic term, whereas the inflationary potential is induced from the 5D geometry and results to be of the form shown in (62). We obtain that the dynamical foliation that generates the 4D inflationary model here developed (31) is linear with the cosmic time. The expansion of the universe during inflation is given by a power law. We obtain that for \( \tilde{\alpha} = 9 \), the scalar-to-tensor ratio is \( r = 0.098 \). Besides, the interval \( 0.1669 M_p^{-1} \leq \theta_0 \leq 0.2426 M_p^{-1} \) corresponds to \( 0.9607 \leq n_s \leq 0.9691 \). Something interesting that we would like to stress is that not only the inflationary potential was obtained, but also the model determines the pre-inflationary potential.

On the other hand, during the present epoch we found also an exponential law for the cosmic scale factor with a deceleration parameter compatible with Max Planck 2018 results when \( \sigma > (6/\theta_0)(1 + q_0) \). Therefore, for \( q_0 = -0.5848 \) the parameter \( \sigma \) satisfies \( \sigma > 2.4912 t_0^{-1} \), while for \( q_0 = -0.5308 \), we obtain \( \sigma > 2.8152 t_0^{-1} \). In this manner, we obtain an inflationary and a present accelerating expansion scenarios induced from a 5D vacuum generated by the same geometrical mechanism, and in this sense, we can say that it acts as a unifying mechanism.

Acknowledgements J.E. Madriz-Aguilar, M. Montes and J. A. Licea acknowledge CONACYT México and Departamento de Matemáticas of Centro Universitario de Ciencias Exactas e Ingenierías (CUCEI) of Universidad de Guadalajara for financial support. J. Zamarripa and C. De Loza acknowledge CONACYT México and Centro Universitario de los Valles of Universidad de Guadalajara for financial support. A. Peraza acknowledges Departamento de Física of CUCEI of Universidad de Guadalajara for financial support.

Data Availability Statement This manuscript has associated data in a data repository. [Authors’ comments: The experimental data used in this manuscript was published by PLANCK collaboration https://doi.org/10.1051/0004-6361/201833910.]

References

1. S. Perlmutter et al., Nature (London) 391, 51 (1998)
2. A. Riess et al., Astron. J. 116, 1009 (1998)
3. Eric V. Linder, Gen. Relativ. Gravit. 40, 329 (2008)
4. J.S. Alcaniz, Braz. J. Phys. 36, 1109 (2006)
5. R. Schitzhold, Phys. Rev. Lett. 89, 081302 (2002)
6. S. Banerjee et al., Phys. Lett. B 611, 27 (2005)
7. C. Piggozo, M.A. Dantas, S. Carneiro, J.S. Alcaniz, JCAP 08, 022 (2011)
8. F.R. Urban, A.R. Zhitnitsky, Nucl. Phys. B 835, 135 (2010)
9. L.M. Reyes, C. Moreno, J.E. Madriz-Aguilar, Eur. Phys. J. Plus 127, 142 (2012)
10. J.M. Overduin, F.I. Cooperstock, Phys. Rev. D 58, 43506 (1998)
11. A. Ashokkar, T. Pawlowski, P. Singh, Phys. Rev. Lett. 96, 141301 (2006)
12. T. Zhu, A. Wang, G. Cleaver, K. Kirsten, Q. Sheng, Phys. Rev. D 96, 083520 (2017)
13. Bao-Fei Li, P. Singh, A. Wang, Phys. Rev. D 100 (2019) 063513
14. Wei-Jian Jin, Y. Ma, T. Zhu, JCAP 02 (2019) 010
15. M. Bellini, Phys. Lett. B 771, 227–229 (2017)
16. H. Nastase, JHEP 12, 010 (2020)
17. A. Gruppuso, A. Sagnotti, Int. J. Mod. Phys. D24 (2015) 1544008, n012
18. J.M. Cline, J. Vinet, Phys. Rev. D 68, 025015 (2003)
19. P.S. Wesson, *Five-Dimensional Physics* (World Scientific, Singapore, 2006)
20. J.M. Overduin, P.S. Wesson, Phys. Rept. 283, 302 (1997)
21. J.M. Overduin, P.S. Wesson, B. Mashhoon, Astron. Astrophys. 473, 727 (2007)
22. P.S. Wesson, B. Mashhoon and J. M. Overduin, Int. J. Mod. Phys. D 17 (2008) n°13, 2527-2533
23. J.E. Madriz-Aguilar, J. Zamarripa, A. Peraza, J.A. Licea, Phys. Dark Univ. 18, 11–16 (2017)
24. M. Bellini, J.E. Madriz-Aguilar, M. Montes, P.A. Sánchez, Phys. Dark. Univ. 25, 100309 (2019)
25. M. Bellini, Phys. Lett. 632, 610 (2006). ArXiv: gr-qc/0510110
26. J. Ponce de Leon, Mod. Phys. Lett. A 21, 947–959 (2006)
27. J. Ponce de Leon, Int. J. Mod. Phys. D 15, 1237–1257 (2006)
28. J.E. Madriz-Aguilar, M. Bellini, M.A.S. Cruz, Grav. Cosmol. 14, 286–291 (2008)
29. L.M. Reyes, C. Moreno, J.E. Madriz-Aguilar, Eur. Phys. J. Plus 127, 142 (2012)
30. J.S. Alcaniz, JCAP 1108, 022 (2011)
31. Planck Collaboration (N. Aghanim et. al) (2018) 72pp. ArXiv:1807.06209 [astro-ph]
32. H.A. Borges, S. Carneiro, Gen. Rel. Grav. 37, 1385–1394 (2005)
33. N. Aghanim et al. (Planck Collaboration), (2018) ArXiv: 1807.06209