Two-leg fermionic Hubbard ladder system in presence of state-dependent hopping

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We study a two-leg fermionic Hubbard ladder model with a state-dependent hopping. We find that, contrarily to the case without a state-dependent hopping, for which the system has a superfluid nature regardless of the sign of the interaction at incommensurate filling, in the presence of such a hopping a spin-triplet superfluid, spin-density wave and charge-density wave phases emerge. We examine our results in the light of recent experiments on periodically-driven optical lattices in cold atoms. We give protocols allowing to realize the spin-triplet superfluid elusive in the cold atoms.

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I. INTRODUCTION

Strongly-correlated one dimensional systems have attracted a strong attention over the past decades. In general, in such systems the excitations differ strongly from their higher dimensional counterparts and for fermions are very different from the usual Landau quasiparticles occurring in a Fermi-liquid state [11]. Instead, many of the one dimensional systems belong to the universality class known to be the Tomonaga-Luttinger liquid [2].

In particular, the system made of two coupled fermionic chains, namely, two-leg ladder system has been intensively studied in the past [2–13]. This system has been shown to exhibit superconductivity not only for attractive interactions (s-wave superconductivity) but also quite remarkable for purely repulsive ones. In the later case the superconductivity is of d-wave symmetry. The d-wave superconductivity emerges by doping of a Mott insulating phase at half filling.

While the one dimensional system has been intensively studied as a first step towards other materials in higher dimensions, such as the high-Tc superconductors, nowadays it is a major subject in itself due to the relevance for some experiments, in particular in the field of cold atomic gases [14].

Indeed due to rapid advances in technology, cold atoms are a promising way to investigate the one dimensional systems with an unprecedented level of control on the interchain hopping and interactions. Most of the atoms utilized in experiments have internal degrees of freedom, which correspond to hyperfine states when we focus on alkali species, allowing already to reproduce models such as the Hubbard model [15][16]. More recently, ladder systems have also been produced, both for bosonic and fermionic states [17][21].

In addition to simulating systems directly existing in condensed matter physics, by using the unique manipulations available in experiments, cold atoms also allow to realize new quantum states of matter.

One of such extensions which is the focus of this paper is the time-modulation of optical lattices [22][26]. By applying such a modulation with sufficiently high frequencies, it is possible to tune the hopping matrix. This technique allows one to control the hopping not just in strength but also in sign since the renormalized hopping is essentially proportional to a Bessel function. In addition, by using the state-dependent optical lattice [27][28] or applying a magnetic field one can also control the hopping matrix element in a state dependent manner. In fact, such a setup has motivated several theoretical studies in a different context [29][34].

One may also expect the realization of an unconventional superfluid in cold atoms by means of such a unique technique. To realize a superfluid in cold atoms, so far, it is necessary to use a Feshbach resonance since the typical temperature in the experiments is of the order of tenth of the Fermi temperature [14]. A weak-coupling BCS transition temperature is extremely low compared to this temperature. A Feshbach resonance allows to boost enough the interactions so that s-wave superfluidity can be routinely realized for attractive interactions. However, other symmetries are not so easily attainable. A p-wave Feshbach resonance is unstable due to the atom-molecule and molecule-molecule inelastic collisions [35]. Therefore, the realization of an unconventional superfluid with cold atoms is a highly challenging issue.

In this paper, we show how one can realize a spin-triplet superfluid in a two-leg Hubbard ladder system. In the presence of a state-dependent hopping the d-wave pairing state in the normal ladder is replaced by a spin triplet superfluid and a spin density wave (SDW) state. We also discuss the case of an attractive interaction which would lead in the absence of state dependent hopping to s-wave superconductivity and which gives an incommensurate charge density wave (CDW) in the presence of state dependent hopping.

With a ladder system we thus show that we can obtain a spin-triplet state with purely local (s-wave) repulsive interactions, which is a situation attainable in experiments. In a single chain such a state would have demanded an extended Hubbard model with on-site repulsion and nearest-neighbor attraction of the same order of magnitude [36], something which is at the moment out of reach in cold atomic systems.

This paper is organized as follows. Section II discusses the Hamiltonian we propose and its low-energy description by means of the bosonization technique. In Sec. III the possible phases are determined by using a renormalization group analysis. In Sec. IV we discuss the properties of the strong-coupling limit in the system and experimental protocols toward its realization. Section V is the conclusion. Technical details can be found in the appendix.
II. HAMILTONIAN

We study two-component fermions confined in the two-leg ladder geometry. Our starting point is the following two-leg Hubbard ladder model:

\[
H = -t_\parallel \sum_{j=1}^{N} \sum_{\sigma=\uparrow,\downarrow} \sum_{p=1,2} (c_j^\dagger \sigma, p c_{j+1, \sigma, p} + h.c.) \\
- \sum_{j, \sigma} t_{\perp \sigma} (c_j^\dagger \sigma, 1 c_{j, \sigma, 2} + h.c.) + U \sum_{j, \sigma} n_{j, \uparrow, \sigma} n_{j, \downarrow, \sigma},
\]

where \(t_\parallel\) and \(t_{\perp \sigma}\) are respectively the hopping matrices along the chain and rung directions, and \(j\) and \(p\) indicate the chain and ladder indices. Here, the on-site Hubbard \(U\) can correspond to both repulsive and attractive interactions, which indeed can be realized experimentally. We focus on a system at incommensurate filling since we are interested in the stability of the superfluids in the presence of the state-dependent hopping, in particular, in the presence of such a hopping along the rung direction. The effect of the state-dependent chain hopping has been partially discussed in Refs. \[30, 31, 33, 35\]. In this section and section III we discuss the weak-coupling limit to analyze the possible phases using a field theory analysis. In our model, this condition implies \(t \gg |U|, t_{\perp \sigma}\).

To deal with the system in the weak-coupling limit correctly, we first move to the bonding and anti-bonding representation for the fermion operators:

\[
c_{j, \sigma, 0}(\varepsilon) = [c_{j, \sigma, 1} + (-)c_{j, \sigma, 2}] / \sqrt{2},
\]

which allows to diagonalize the hopping terms. While in the absence of the rung hopping, the bonding and anti-bonding bands are energetically degenerate, these are split in the presence of the the rung hopping. In the absence of the state-dependent rung hopping, the splitting is independent of the states (or spins), and therefore, there are four different points at the Fermi level as can be seen from Fig. 1. In the presence of the state-dependent rung hopping, however, the splitting starts to depend on the states and leads to eight different points at the Fermi level. At the same time, at \(t_{\parallel \uparrow} = -t_{\parallel \downarrow}\), the four point structure at the Fermi level is recovered even though in this case the degeneracies occur between \((\pi, \uparrow)\) and \((0, \downarrow)\) and between \((0, \uparrow)\) and \((\pi, \downarrow)\) (See Fig. 1). Then, the interaction term plays a role of an hybridization between the bonding and anti-bonding bands, which is essential to lead to nontrivial states of matter in the system.

We now consider the continuum limit to use the bosonization. The fermion in the continuum limit \(\psi\) can be expressed with conjugate phase fields \(\phi\) and \(\theta\) as

\[
\psi_{\mu \rho}(x) = \frac{1}{\sqrt{2 \pi \alpha}} \eta_{\mu \rho} e^{ikx} e^{-i[\phi_{\mu \rho}(x)-\theta_{\mu \rho}(x)]},
\]

with the Fermi momentum \(k_F\), index \(r = -1\) or \(1\) for the left or right mover, cut-off parameter \(\alpha\), and the phase fields \(\phi_{\mu \rho}\) and \(\theta_{\mu \rho}\) to be conjugate. Here, we explicitly introduce the Klein factor \(\eta\), which guarantees the correct anti-commutation relation of the fermions and is also important to obtain correct expressions for the bosonized Hamiltonian and correlation functions. By substituting (3) into (1), one may obtain the following low-energy effective Hamiltonian:

\[
H = \sum_{\mu=\rho,\sigma} \sum_{p=\pm \frac{1}{2}} \int \frac{dx}{2\pi} \left[ u_{\mu \rho} K_{\mu \rho} (\nabla \phi_{\mu \rho})^2 + \frac{u_{\mu \rho}}{K_{\mu \rho}} (\nabla \theta_{\mu \rho})^2 \right] + \int \frac{dx}{2(2\pi \alpha)^2} \left( \cos 2\phi_{\rho+} \{ g_1 \cos(2\phi_{\rho-} - \delta_{\rho-}) + g_2 \cos(2\phi_{\rho-} - \delta_{\rho+}) \} + \cos 2\phi_{\rho-} \{ g_3 \cos(2\phi_{\rho-} - \delta_{\rho+}) + g_4 \cos(2\phi_{\rho+}) \} - \cos 2\theta_{\rho-} \{ g_5 \cos(2\phi_{\rho-} - \delta_{\rho-}) + g_6 \cos(2\phi_{\rho+}) \} \right),
\]

where we introduced for \(\phi\) fields,

\[
\phi_{\rho+} = \frac{1}{2} (\phi_{0+} + \phi_{1+} + \phi_{2+} + \phi_{3+}),
\]

\[
\phi_{\rho-} = \frac{1}{2} (\phi_{0-} + \phi_{1-} - \phi_{2+} - \phi_{3+}),
\]

\[
\phi_{\sigma+} = \frac{1}{2} (\phi_{0+} - \phi_{1-} - \phi_{2+} + \phi_{3+}),
\]

\[
\phi_{\sigma-} = \frac{1}{2} (\phi_{0-} - \phi_{1+} - \phi_{2-} + \phi_{3-}),
\]

and similar relations for \(\theta\) fields. To obtain the above, we neglect the umklapp scatterings since we are at incommensurate filling. For our original Hamiltonian, we find \(\delta_{\rho-} = 2K_{\rho-}(t_{\parallel \uparrow} + t_{\parallel \downarrow})/u_{\rho-}, \delta_{\rho+} = 2K_{\rho+}(t_{\parallel \uparrow} - t_{\parallel \downarrow})/u_{\rho-}, g_i = U (i = 1, 2, \cdots, 6)\). In addition, \(u_{\rho \rho}\) and \(K_{\rho \rho}\) are the velocity and the Tomonaga-Luttinger parameter, respectively. We also note that to obtain the above bosonized Hamiltonian (4), we

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FIG. 1. Band structure of the two-leg fermionic Hubbard ladder of atoms with spin down (a) without the state-dependent hopping and (b) with state-dependent hopping as \(t_{\parallel \uparrow} = -t_{\parallel \downarrow}\). If the repulsive interaction is added, the latter leads to a spin-triplet superfluid while the former leads to a \(d\)-wave superfluid at incommensurate filling. The band structure of atoms with spin up does not change in the presence of the state-dependent hopping.
where the initial values are given as

\[ \eta_0 \eta_0 \eta_{\pi^+} \eta_{\pi^-} = 1. \]  

(9)

### III. RENORMALIZATION GROUP ANALYSIS

Based on the bosonized Hamiltonian \[ [4], \] we now determine the possible phases in this model. To this end, we employ the renormalization group (RG) approach in the bosonized Hamiltonian \[ [2]. \] By performing the scaling of the cut-off \( \rho \to \rho l \), one may obtain the set of the RG equations at the 1-loop level (quadratic with respect to the coupling constants), which is given by (See Appendix)

\[
\begin{align*}
\frac{dK_{\sigma\pi-}}{dl} &= -\frac{2}{2} J_0(\delta_{\sigma\pi-})[y_1^2 + y_2^2] + J_0(\delta_{\pi\pi-})y_3^2 + y_4^2, \\
\frac{dK_{\sigma\pi+}}{dl} &= -\frac{2}{2} J_0(\delta_{\sigma\pi+})y_3^2 + J_0(\delta_{\pi\pi+})y_4^2 + y_4^2, \\
\frac{dK_{\sigma\pi-}}{dl} &= -\frac{2}{2} J_0(\delta_{\sigma\pi-})[y_1^2 + y_2^2] + J_0(\delta_{\pi\pi-})y_3^2 + y_4^2.
\end{align*}
\]

(10)

(11)

(12)

\[
\begin{align*}
\frac{dy_1}{dl} &= (2 - K_{\sigma\pi-} - K_{\sigma\pi+})y_1 - y_3 y_4, \\
\frac{dy_2}{dl} &= (2 - K_{\sigma\pi-} - K_{\sigma\pi+})y_2 - y_3 y_6, \\
\frac{dy_3}{dl} &= (2 - K_{\sigma\pi-} - 1/K_{\sigma\pi-})y_3 - y_1 y_4, \\
\frac{dy_4}{dl} &= (2 - K_{\sigma\pi+} - 1/K_{\sigma\pi+})y_4 - y_1 y_3, \\
\frac{dy_5}{dl} &= (2 - K_{\sigma\pi-} - 1/K_{\sigma\pi-})y_5 - y_2 y_6, \\
\frac{dy_6}{dl} &= (2 - K_{\sigma\pi+} - 1/K_{\sigma\pi+})y_6 - y_2 y_5.
\end{align*}
\]

(13)

(14)

(15)

(16)

(17)

(18)

(19)

(20)

The initial values are given as \( y_i(0) = U/(2\pi v_F) \) \( i = 1, 2, \ldots, 6 \), \( K_{\sigma\pi-}(0) = K_{\sigma\pi-}(0) = 1, K_{\sigma\pi+} = 1/\sqrt{1 + U/(2\pi v_F)}, K_{\sigma\pi+}(0) = 1/\sqrt{1 - U/(2\pi v_F)} \) with the Fermi velocity \( v_F \). We note that since there is no cosine term with respect to \( \phi_{\sigma\pi-} \) and \( \theta_{\pi\pi-} \), which are decoupled from the other phase fields, \( K_{\sigma\pi+} \) does not flow up to this order of approximation. In addition, \( J_n \) \( n = 0, 1 \) is the \( n \)th order Bessel function, which plays a role in controlling the relevance of the corresponding cosine terms. Thus, one may classify the fixed points into the following three cases:

\[
\begin{align*}
& (a) \delta_{\pi\pi} \to \infty, \delta_{\sigma\pi} \to 0, \\
& (b) \delta_{\pi\pi} \to \infty, \delta_{\sigma\pi} \to \infty, \\
& (c) \delta_{\pi\pi} \to 0, \delta_{\sigma\pi} \to \infty.
\end{align*}
\]

First, let us consider the case (a), which corresponds to the limit \( t_{\uparrow \downarrow} \approx t_{\downarrow \downarrow} \). In this case, the terms proportional to \( g_2 \), \( g_3 \) can be dropped due to the rapid oscillation of the cosines. Thus, the RG equations reduce to ones without the state-dependent hopping \[ [2] \) since this limit also allows us to do the substitutions, \( J_0(\delta_{\sigma\pi-}) = 1 \) and \( J_0(\delta_{\pi\pi-}) = 0 \). The RG analysis shows the fixed point is given by \( g_1 \to -\infty, g_1 \to -\infty, g_1 \to -\infty, g_1 \to -\infty, g_1 \to -\infty, g_1 \to -\infty, g_1 \to -\infty \) for \( U > 0 \) and \( g_1 \to -\infty, g_1 \to -\infty, g_1 \to -\infty, g_1 \to -\infty \) for \( U < 0 \). While regardless of the sign of the interaction, \( \phi_{\pi\pi}, \phi_{\sigma\pi}, \) and \( \phi_{\pi\pi} \) are gapped, these minimums are different for opposite signs of the interaction. It turns out that the minimum can be determined by the fixed point. Then, the dominant correlations are the \( d \)-wave superfluid for \( U > 0 \) whose pairing occurs between the different chains and the \( s \)-wave superfluid for \( U < 0 \) whose pairing essentially occurs on-site. The corresponding operators are

\[
\begin{align*}
O_{\text{DSF}}(j) &= \sum_{\sigma, \rho} \alpha c_{j, \sigma, \rho} c_{j, \pi^- \sigma, \rho} \\
&\sim e^{-i\theta_{\pi^-}} (\cos \phi_{\rho^-} \sin \phi_{\pi+} \sin \phi_{\pi^-} - i \sin \phi_{\rho^-} \cos \phi_{\pi+} \cos \phi_{\pi^-}), \\
O_{\text{SSF}}(j) &= \sum_{\sigma, \rho} \alpha c_{j, \sigma, \rho} c_{j, \pi^- \sigma, \rho} \\
&\sim e^{-i\theta_{\pi^-}} (\cos \phi_{\rho^-} \cos \phi_{\pi+} \cos \phi_{\pi^-} + i \sin \phi_{\rho^-} \sin \phi_{\pi+} \sin \phi_{\pi^-}),
\end{align*}
\]

(21)

(22)

respectively \[ [2] \). In contrast to the single chain Hubbard model, we have for the ladder a superfluid regardless of sign of the interaction.

Let us next consider the case (b), where both of the rung hoppings \( t_{\uparrow \rho} \) and \( t_{\downarrow \rho} \) are relevant and the substitutions
and the Cooper pairs while the superfluid (c) occurs for the anti-bonding curves. The difference of the dashed curves is that the s-wave superfluid (a) occurs for the bonding band of the Cooper pairs while the superfluid (c) occurs for the anti-bonding band of the Cooper pairs. The CDW (b) has the alternate occupation on the two legs.

J₀(δp−α) = J₀(δσ−σ) = 0 are allowed. In this case, the effects of g₁, g₂, g₃, g₄ can be dropped due to the large oscillations. By solving the RG equations under these conditions, the fixed points are shown to be g₄ → ∞, g₆ → ∞ for U > 0 and g₄ → −∞, g₆ → −∞ for U < 0. Thus, we see θp−, φσ+, θσ− are going to be gapped. From the fixed point analysis, we find that the following SDW and CDW operators are relevant for U > 0 and U < 0, respectively:

\[ O_{SDW}(j) = \sum_{\sigma,p} p \sigma c_{j,\sigma,p}^\dagger c_{j−\sigma,p−} \]
\[ \sim e^{-i\phi_{\sigma+}} (\sin \theta_{p−} \cos \phi_{\sigma+} \cos \theta_{\sigma−} \sin \phi_{\sigma+} \sin \theta_{\sigma−}) , \]
(23)

\[ O_{CDW}(j) = \sum_{\sigma,p} p c_{j,\sigma,p}^\dagger c_{j−\sigma,p−} \]
\[ \sim e^{-i\phi_{\sigma+}} (\cos \theta_{p−} \cos \phi_{\sigma+} \sin \theta_{\sigma−} \sin \phi_{\sigma+} \cos \theta_{\sigma−}) , \]
(24)

where π indicates the difference of the densities on the two legs. Namely, the presence of the state-dependent rung hopping as \(|t_{1,1}/t_{1,1}| \neq 1\) tries to destroy the the superfluidity for the two-leg Hubbard ladder, and causes fluctuations toward crystalline orders such as the SDW or CDW. Here, one may notice the analogy with the single chain Hubbard system in the presence of a state-dependent hopping [30] where in the wide range of the parameters SDW and CDW are shown to be the dominant fluctuations for U > 0 and U < 0, respectively. Such emergences of the density wave states in the single chain system is natural since one of the spin components is reluctant to hop between different sites. However, now we impose the spin dependence only for the rung direction. Thus, the emergence of the SDW or CDW in our model is less trivial.

Let us finally consider the case (c), which can be realized when \(t_{1,1} \approx -t_{1,1}\) and therefore the substitutions \(J₀(\delta p−α) = 1\) and \(J₀(\delta σ−σ) = 0\) are justified. In this case, \(g₁, g₃\) can be dropped in a manner similar to the other cases. By solving the RG equations, we find the fixed points to be \(g₂ → −∞, g₄ → 0, g₅ → ∞, g₆ → ∞\) for \(U > 0\) and \(g₂ → −∞, g₄ → 0, g₅ → −∞, g₆ → −∞\) for \(U < 0\), and therefore, \(φ_{\sigma−}, φ_{\sigma+}, θ_{\sigma−}\) are gapped. In accordance with the fixed points, the dominant correlations are shown to be the spin-triplet superfluid along the z direction for \(U > 0\) and s-wave superfluid for \(U < 0\), where the corresponding operators are

\[ O_{TSF}(j) = \sum_{\sigma,p} c_{j,\sigma,p}^\dagger c_{j−\sigma,p−} \]
\[ \sim e^{-i\phi_{\sigma−}} (\cos \phi_{p−} \cos \phi_{\sigma+} \cos \theta_{\sigma−} − i \sin \phi_{p−} \sin \phi_{\sigma−} \sin \theta_{\sigma−}) , \]
(25)

\[ O_{SSF}(j) = \sum_{\sigma,p} p c_{j,\sigma,p}^\dagger c_{j−\sigma,p−} \]
\[ \sim e^{-i\phi_{\sigma−}} (\phi_{p−} \sin \phi_{\sigma+} \cos \theta_{\sigma−} + i \cos \phi_{p−} \cos \phi_{\sigma−} \sin \theta_{\sigma−}) , \]
(26)

respectively. We first focus on the emergence of the dominant fluctuation of the spin-triplet superfluid for \(U > 0\). Namely, the sign inversion in the rung hopping regarding only one of the spin components allows the change of nature of the pairings from the inter-chain spin-singlet to the inter-chain spin-triplet. In the bonding and anti-bonding representation, while the d-wave superfluid operator has the form as \(c_{j,\sigma,0} c_{j+1,\sigma,0} − c_{j,\sigma,π} c_{j+1,\sigma,π}\), the spin-triplet superfluid occurring is given as \(c_{j,\sigma,0} c_{j+1,\π} + c_{j,\sigma,π} c_{j+1,\sigma,π}\). To understand the mechanism, we first point out that such a sign inversion in the rung hopping can be achieved by introducing the Peierls phases both in charge and spin sectors by \(\pi/2\). Then, what is important for the pairing is the Peierls phase in the spin sector. In fact, it has been shown in Ref. [37] that such a Peierls phase causes the spin rotation of the fermions for one of the chains and transforms a spin-singlet into a spin-triplet pairing. For \(U < 0\), on the other hand, the difference between the s-wave superfluids in Eq. (26) and in Eq. (22) is that if we treat the Cooper pairs occurring in each chain as the bosons, the superfluid in the absence of the state-dependent hopping occurs for the bonding band of the bosons while the superfluid in the presence of it occurs for the anti-bonding band of the bosons. Compared with the situation from the spin-singlet to spin-triplet pairings for \(U > 0\), the important ingredient for this change of the s-wave superfluids for \(U < 0\) is the Peierls phase in charge sector. One may also accept this situation recalling that in a Bose-Einstein condensate on a double well potential, a BEC on the bonding band is normally the ground state while a BEC on the anti-bonding band becomes the ground state in the presence of the sign inversion hopping [38].

The possible phases are summarized in Fig. 2 and Fig. 3.
IV. DISCUSSION

A. Strong coupling limit

So far, we have discussed the weak-coupling limit by means of the bosonization and RG analysis, it is also interesting to see what happens in the strong-coupling limit in which naively a similar phase diagram may be expected.

For the $U > 0$ case, in fact, it may be difficult to depict a general phase diagram analytically since a faithful effective Hamiltonian has yet to be known except for commensurate filling such as half filling. In addition, the rung hopping is a relevant perturbation, which prevents one from starting at the single chain Hubbard model where the Bethe ansatz approach is available. At the same time, the previous numerical analyses in the absence of the state-dependent hopping show that the $d$-wave superfluid state emerges even in the strong-coupling limit [2, 9]. In addition, since the hybridization among the four different Fermi points by the on-site repulsive interaction shown in Fig. 1(a) is an essential ingredient of the bosonization and RG analysis, it is also interesting to approach is available. At the same time, the previous numerical calculations have been performed [10–13]. Therefore, the presence of the spin-triplet superfluid in the strong-coupling limit can also be shown with the argument in Sec. II. Namely, by using the canonical transformations $a_{iσ} \rightarrow c_{iσ} \rightarrow a_{iσ}$, the Hamiltonian with $t_{1f} = t_{1L}$ is mapped onto one with $t_{1f} = t_{1L}$, that is, a normal two-leg fermionic Hubbard ladder can be obtained. Accordingly, the operator of the spin-triplet superfluid is transformed into one of the $d$-wave superfluid. Therefore, once we confirm the emergence of the $d$-wave superfluid in the normal two-leg fermionic Hubbard ladder system, we see that the spin-triplet superfluid occurring in $t_{1f} \approx t_{1L}$ is robust. We also note that the essence of the spin-triplet superfluid is the manipulation on the rung hopping, and thus, nothing happens and the $d$-wave superfluid remains even if such a manipulation on the hopping is performed for the chain direction. Thus, to see the spin-triplet superfluid, the manipulation on the hopping along the rung direction is required. Another interesting but remaining issue may be a possibility of segregation in the limit $t_1 = 0$ or $t_1 \approx 0$.

On the other hand, for the $U < 0$ case, we can discuss the possible phases in the strong-coupling limit by means of an effective Hamiltonian approach. To see this, we first perform the so-called particle-hole transformation [30] in this model. Then, the original model is mapped onto the system with $U > 0$ and spin imbalance at half-filling, and therefore the effective Hamiltonian is shown to be

$$H = J_1 \sum_{j} \left( S_{j,1}^x S_{j+1,1}^x + S_{j,1}^y S_{j+1,1}^y + J_{1f}^x \sum_{j} (S_{j,1}^x S_{j+1,1}^y + S_{j,1}^y S_{j+1,1}^x) \right) + J_{1L}^x \sum_{j} S_{j,1}^x S_{j+1,1}^0 + J_{1L}^y \sum_{j} S_{j,1}^y S_{j+1,1}^0 \right) \right), \tag{27}
$$

where $J_1 = 4t_0^2/|U|$, $J_{1f}^x = 4t_1 t_1 / |U|$, $J_{1L}^x = 2(t_0^2 + t_1^2) / |U|$, and $H$ is a magnetic field corresponding to filling in the original attractive model. By performing bosonization for the above Hamiltonian [2], one may obtain

$$H^{\text{eff}} = \sum_{p=\pm} \int \frac{dx}{2π} \left( u_p K_p (\nabla \theta_p)^2 + \frac{1}{2(πa)^2} \right) \left( dx [J_{1L}^x \cos(\sqrt{2} \theta_p) + J_{1L}^y \cos(2 \sqrt{2} \theta_p)] \right), \tag{28}
$$

where $\theta_p = \theta_1 + (-\theta_2)/\sqrt{2}$ to be the phase field in chain $p$, $\theta_p (p = 1, 2)$, and similar relations for the $θ$ field.

The original spin fields and phase fields are related as $S_p^z(x) = -\nabla \theta_p(\pi / 2, x) \cos(2 \theta_p(x)) / (πa)$ and $S_p^z(x) = e^{-\theta_p(x)} \cos(2 \theta_p(x)) / \sqrt{2πa}$. Since $J_0 \approx J_{1L}$ is concerned, we can determine the Tomonaga-Luttinger parameters as

$$K_{x,a} = K \left( 1 + \frac{KF_{2L}}{2πa} \right). \tag{29}
$$

Here $K$ and $a$ are the Tomonaga-Luttinger parameter and velocity in the single chain Heisenberg model, respectively. The Tomonaga-Luttinger parameter $K$ can be determined by means of Bethe ansatz, and it is known that the possible range is $1/2 \leq K \leq 1$, where $K = 1/2$ corresponds to the no magnetization case and $K = 1$ to the fully polarized case [42]. Since the above consists of the linear combination of the simple cosine terms, one can determine the ground state with a simple scaling argument. In fact, $\cos \sqrt{2} \theta_p$ and $\cos \sqrt{2} \theta_p$, have the scaling dimensions of $(2K_{x,a})^{-1}$ and $2K_{x,a}$, respectively. Thus, we see that $\cos \sqrt{2} \theta_p$ is ordered for $K_{x,a} > 1/2$ and the situation is reversed for $K_{x,a} < 1/2$. As can be seen from Eqs. (28) and (29), $K_{x,a} > 1/2$, and we expect that $θ_p$ is ordered except for the limit $J_{1L}^0 \rightarrow 0$ where $θ_p$ is ordered. To specify the ground state in the spin language, let us introduce bonding and anti-bonding spin operators as $S_p^z = S_1^z + S_2^z$ and $S_p^x = S_1^x - S_2^x$, respectively. Then, one finds that the bonding (anti-bonding) transverse spin-spin correlation $\langle S_p^z(r) S_p^z(0) \rangle (\langle S_p^x(r) S_p^z(0) \rangle)$ is dominant for $θ_p$ to be gapped with $J_{1L}^0 > 0$ ($J_{1L}^0 < 0$), while the anti-bonding longitudinal spin-spin correlation $\langle S_p^z(r) S_p^z(0) \rangle$ is dominant for $θ_p$ to be gapped [2]. Now, we can determine the dominant correlation in the original model by using the particle-hole transformation again. Since by this transformation

$$S_p^- \rightarrow \sum_p p c^\dagger_{iρ} c_{iρ+p} = O_{\text{SSC}}, \tag{30}
$$

$$S_p^身份 \rightarrow \sum_p c^\dagger_{iρ} c_{iρ+p} = O_{\text{SSC}}, \tag{31}
$$

$$S_p^z \rightarrow \sum_{p,σ} pc^\dagger_{iρ} c_{iρ+p} = O_{\text{CDW}}, \tag{32}
$$

we conclude that the $s$-wave superfluid is dominant except for $t_{1L} \not\equiv 0$ where the CDW correlation is dominant. In particular, the bonding $s$-wave pairing state is realized for $t_{1L} > 0$ while the anti-bonding $s$-wave pairing state is realized for the opposite sign case. Thus, the phase structure is compatible with the weak-coupling analysis while in the weak-coupling limit the region of the $s$-wave superfluid is rather narrow but in the strong-coupling limit the situation
is reversed. This may be explained by the observation that the pairing gap becomes larger as increasing the attractive interaction and the pairing in the $s$-wave superfluid essentially occurs in a single site, and therefore the introduction of the small state-dependent rung hopping may not cause the disappearance of the superfluid correlation.

B. Experimental protocol

We now discuss the realization of our model and its ground states in cold atoms.

Experimentally, the two-leg ladder geometry can be naturally created by an optical superlattice [17, 18]. Then, to ensure the one-dimensional character of the system, the hoppingocoerently created by an optical superlattice [17, 18]. Then, to ensure the one-dimensional character of the system, the hopping is reversed. This may be explained by the observation that the pairing gap becomes larger as increasing the attractive interaction and the pairing in the $s$-wave superfluid essentially occurs in a single site, and therefore the introduction of the small state-dependent rung hopping may not cause the disappearance of the superfluid correlation.

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APPENDIX: RENORMALIZATION GROUP EQUATIONS

In this appendix, we wish to outline the derivation of the renormalization group equations in a similar way than Ref. [22]. We first consider the following correlation function:

$$R(r_1 - r_2) = \langle T e^{i\phi_{r_1} + (x_1, r_1)} e^{-i\phi_{r_2} + (x_2, r_2)} \rangle$$ (34)

where \( T \) denotes the time-ordered product. By expanding the above correlation function in terms of \( g_i \) up to third order, we obtain

$$R(r_1 - r_2) \approx e^{-\frac{\xi r}{\alpha}} F_1(r_1 - r_2) + (S) + (T),$$ (35)

where \( F_1(r) = \ln(r/\alpha) \),

\[
(S) = \frac{1}{2} \left( \frac{g_1}{8(\pi\alpha)^2 v_F} \right)^2 \sum_{\epsilon_1, \epsilon_2 = \pm 1} \int d^2 r' d^2 r'' \left[ \langle e^{i\phi_{r_1} + (x_1, r_1)} e^{-i\phi_{r_2} + (x_2, r_2)} \rangle_{0} \right]
\]

\[
+ \frac{1}{2} \left( \frac{g_2}{8(\pi\alpha)^2 v_F} \right)^2 \sum_{\epsilon_1, \epsilon_2 = \pm 1} \int d^2 r' d^2 r'' \left[ \langle e^{i\phi_{r_1} + (x_1, r_1)} e^{-i\phi_{r_2} + (x_2, r_2)} \rangle_{0} \right]
\]

\[
+ \frac{1}{2} \left( \frac{g_4}{8(\pi\alpha)^2 v_F} \right)^2 \sum_{\epsilon_1, \epsilon_2 = \pm 1} \int d^2 r' d^2 r'' \left[ \langle e^{i\phi_{r_1} + (x_1, r_1)} e^{-i\phi_{r_2} + (x_2, r_2)} \rangle_{0} \right]
\]

and

\[
(T) = -g_1 g_3 g_4 \left( \frac{1}{8(\pi\alpha)^2 v_F} \right)^3 \sum_{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 = \pm 1} \int d^2 r' d^2 r'' d^2 r''' \left[ \langle e^{i\phi_{r_1} + (x_1, r_1)} e^{-i\phi_{r_2} + (x_2, r_2)} \rangle_{0} \right]
\]

In the above, \( \langle \cdots \rangle_0 \) denotes the average without the cosine terms, that is, one with the Tomonaga-Luttinger Hamiltonian. When we focus on \( (T) \), the dominant contributions come from \( r''' = r' + \tilde{r} \) or \( r''' = r' - \tilde{r} \) for the term proportional to \( g_1 g_3 g_4 \), and from \( r''' = r' + \tilde{r} \) or \( r''' = r' - \tilde{r} \) for one proportional to \( g_2 g_3 g_6 \) with a small \( r \). Therefore, by expanding around \( \tilde{r} = 0 \), after a
straightforward calculation, we can obtain the following renormalization relations on the effective quantities:

\[
K_{\sigma+}^{\text{eff}} = K_{\sigma+} - \frac{K_{\sigma+}^2}{2} \int \frac{dr}{r} \left[ \frac{r}{a} \right]^{3-2(K_{\sigma+} + K_{\sigma-})} J_0(2\delta_{\sigma+} - r) + \frac{r}{a} \left[ \frac{r}{a} \right]^{3-2(K_{\sigma+} + K_{\sigma-})} J_0(2\delta_{\sigma-} - r)
\]

(38)

\[
+ y_1^2 \left( \frac{r}{a} \right)^{3-2(K_{\sigma+} + 1/K_{\sigma-})} J_0(2\delta_{\sigma+} - r) + y_2^2 \left( \frac{r}{a} \right)^{3-2(K_{\sigma+} + 1/K_{\sigma-})} J_0(2\delta_{\sigma-} - r).
\]

(39)

\[
(y_1^{\text{eff}})^2 = y_1^2 - 2y_1y_3y_4 \int \frac{r}{a} \left( \frac{r}{a} \right)^{1-2/K_{\sigma-}},
\]

(40)

\[
(y_2^{\text{eff}})^2 = y_2^2 - 2y_2y_5y_6 \int \frac{r}{a} \left( \frac{r}{a} \right)^{1-2/K_{\sigma+}},
\]

(41)

\[
(y_3^{\text{eff}})^2 = y_3^2 - 2y_3y_4y_5 \int \frac{r}{a} \left( \frac{r}{a} \right)^{1-2K_{\sigma+}} J_0(2\delta_{\sigma+} - r),
\]

(42)

By changing the cutoff \( \alpha \rightarrow e^{\epsilon} \alpha = \alpha + d\alpha \), we obtain Eqs. (10), (13), (14), (16), and (18). In a way similar to the above, the other RG equations can also be obtained.

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