Casimir interaction between two concentric cylinders at nonzero temperature

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Abstract – We study the finite-temperature Casimir interaction between two concentric cylinders. When the separation between the cylinders is much smaller than the radii of the cylinders, the asymptotic expansions of the Casimir interaction are derived. Both the low-temperature and the high-temperature regions are considered. The leading terms are found to agree with the proximity force approximations. The low-temperature leading term of the temperature correction is also computed and it is found to be independent of the boundary conditions imposed on the larger cylinder.

Introduction. – In recent years, there is an intensive interest in studying the Casimir interactions between two objects due to the success in experimental verification of the Casimir effect [1,2] and the advent in the field-theoretical method for computing the Casimir interactions. To make a more accurate comparison between experimental measurements and theoretical results, the corrections to the proximity force approximations of the Casimir interactions have been extensively studied [3–10]. However, most of these studies only considered the zero-temperature interactions. Nevertheless, the finite-temperature corrections have started to gain more attention recently [11–16]. It was observed that there is an interesting interplay between geometry and temperature.

Recently, some experiments have been proposed to measure the Casimir interaction force between two eccentric cylinders [17,18] and the Casimir interaction force between a cylinder and a plate and the thermal correction [19–22]. Therefore, it is of impending importance to have more precise theoretical estimates about the strength of the Casimir force beyond those provided by the proximity force approximations. For the case of a cylinder in front of a plate, the first correction to the proximity force approximation has been computed analytically in [3,23]. Before one considers the problem for the configuration of two eccentric cylinders, it will be interesting to consider the limiting case of two concentric cylinders.

The zero-temperature Casimir interaction between two concentric cylinders has been considered in [9,17,24–29]. Of particular interest is the asymptotic behavior of the Casimir interaction when the separation between the cylinders is small, which have been considered in [9,17,29] for the case where the cylinders are perfectly conducting. In this work, we consider the finite-temperature Casimir interaction between two concentric cylinders and derive the small-separation asymptotic behaviors in the low- and high-temperature regions. We would consider both scalar fields and electromagnetic fields with different combinations of boundary conditions, which include Dirichlet-Dirichlet (DD), Neumann-Neumann (NN), Dirichlet-Neumann (DN —Dirichlet on the smaller cylinder and Neumann on the larger cylinder) and Neumann-Dirichlet (ND) for scalar fields; and perfectly conducting - perfectly conducting (PC-PC) and perfectly conducting - infinitely permeable (PC-IP) for electromagnetic fields.

Casimir free interaction energy between concentric cylinders. – Consider two concentric cylinders with radii \( a_1 < a_2 \). Let \( d = a_2 - a_1 \) be the separation between the cylinders. As was explained in [25], for the configuration of two concentric cylinders, the total Casimir energy is a sum of the self-action part and the interaction part. The self-action part is the sum of the Casimir energies of a single cylinder for the cylinders of radii \( a_1 \) and \( a_2 \), respectively. The Casimir energy of a single cylinder has been

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considered in [30–35]. In the following, we only consider the interaction part of the Casimir energy.

At zero temperature, the Casimir interaction energy between two concentric cylinders is given by [17,26–28,36]:

\[ E_{\text{Cas}}^{T=0} = \sum_{n=\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \ln(1 - M_n(\sqrt{q^2 + k^2})) \, dk \, dq, \]

where \( L \) is the length of the cylinders and \( M_n(\xi) = Z_n(\xi)/Z_n^2(\xi) \), with

\[ Z_n(\xi) = \frac{I_n(a,\xi)}{K_n(a,\xi)} \quad \text{or} \quad Z_n(\xi) = \frac{I_n^\prime(a,\xi)}{K_n^\prime(a,\xi)} \]

depending on whether Dirichlet or Neumann condition is imposed on the cylinder with radius \( a \). Here \( I_n(z) \) and \( K_n(z) \) are modified Bessel functions.

Using Matsubara representation, the finite-temperature Casimir free interaction energy is given by

\[ E_{\text{Cas}} = \sum_{n=\infty}^{\infty} \int_{0}^{\pi} \ln \left( 1 - M_n \left( \sqrt{\xi^2 + L^2} \right) \right) \, d\xi, \]

where \( \xi = 2\pi kT \) are the Matsubara frequencies. Using Poisson resummation formula, (2) can be rewritten as

\[ E_{\text{Cas}} = E_{\text{Cas}}^{T=0} + \sum_{n=\infty}^{\infty} \sum_{l=1}^{\infty} \int_{0}^{\infty} \xi J_0 \left( \frac{\xi}{\sqrt{\xi^2 + L^2}} \right) \ln(1 - M_n(\xi)) \, d\xi, \]

where \( J_0(z) \) is the Bessel function of first kind. Equation (2) is the so-called high-temperature expansion of the Casimir free energy and (3) is the so-called low-temperature expansion. The latter shows manifestly the thermal correction to the Casimir free energy, which is given by the second term on the right-hand side of (3).

For electromagnetic field with perfectly conducting (or infinitely permeable) condition on both cylinders, the Casimir free energy is the sum of the Casimir free energies for DD and NN boundary conditions. If one cylinder is perfectly conducting and one is infinitely permeable, then the Casimir free energy is the sum of the Casimir free energies for DN and ND boundary conditions.

In the following, we derive the asymptotic expansions of the Casimir free energy when the separation between the cylinders is small compared to the radii of the cylinders. We consider the low-temperature region where \( dT \ll a_2T \ll 1 \) and the high-temperature region where \( 1 \ll dT \ll a_2T \). In the low-temperature region, the dominating term of the Casimir free energy is the zeroth-order term (1), whereas in the high-temperature region, the dominating term is the classical term:

\[ E_{\text{Cas}}^{cl} = \frac{TL}{\pi} \sum_{n=0}^{\infty} \int_{0}^{\infty} \ln(1 - M_n(\xi)) \, d\xi, \]

whose Matsubara frequency is zero. Let \( \varepsilon = d/a_1 \) be a dimensionless parameter. Making a change of variables \( \omega = a_1\xi \) in (1) and (4), we have, respectively,

\[ E_{\text{Cas}}^{T=0} = \frac{L}{2\pi a_1^2} \sum_{n=0}^{\infty} \int_{0}^{\infty} \omega \ln(1 - A_n(\omega)) \, d\omega, \]

\[ E_{\text{Cas}}^{cl} = \frac{TL}{\pi a_1} \sum_{n=0}^{\infty} \int_{0}^{\infty} \ln(1 - A_n(\omega)) \, d\omega, \]

where \( A_n(\omega) = H_n^0(\omega)/H_n^2(\omega(1 + \varepsilon)) \),

\[ \frac{H_n^0(\omega)}{H_n^2(\omega(1 + \varepsilon))} \quad \text{or} \quad \frac{H_n^0(\omega)}{K_n^0(\omega)}, \]

depending on whether Dirichlet or Neumann condition is imposed on the cylinder with radius \( a_1 \). To unify the treatment for the zero-temperature Casimir energy and the classical term of the Casimir free energy, define

\[ E_{X} = \sum_{\omega \neq 0} \int_{0}^{\infty} \omega^X \ln(1 - A_n(\omega)) \, d\omega, \]

where \( X = 0 \) or 1. Separating the terms with \( n = 0 \) and the terms with \( n \geq 1 \), we have \( E_{X} = E_{X,0} + E_{X,r} \). Expanding the logarithm gives

\[ E_{X,0} = -\frac{1}{\pi} \sum_{k=1}^{\infty} \int_{0}^{\infty} \omega^X A_0(\omega)^k \, d\omega, \]

\[ E_{X,r} = -\sum_{n=1}^{\infty} \frac{1}{n^{X+1}} \sum_{s=1}^{\infty} \frac{1}{\pi} \int_{0}^{\infty} \omega^X A_0(n\omega)^s \, d\omega. \]

For the term with \( n = 0 \), we need to use asymptotic expansions of \( I_0(z) \) and \( K_0(z) \) when \( z \to \infty \), which give

\[ \frac{I_0(\omega)}{K_0(\omega)} \sim \frac{1}{\pi} \exp(2\omega + \cdots), \]

\[ \frac{I_0^\prime(\omega)}{K_0^\prime(\omega)} \sim \frac{1}{\pi} \exp(2\omega + \cdots). \]

For the term with \( n \neq 0 \), Debye asymptotic expansions for Bessel functions [37] show that

\[ \frac{I_n(n\omega)}{K_n(n\omega)} = \frac{1}{\pi} \exp \left( 2n\eta(\omega) + \frac{2D_1(t(\omega))}{n} + \cdots \right), \]

\[ \frac{I_n^\prime(n\omega)}{K_n^\prime(n\omega)} = -\frac{1}{\pi} \exp \left( 2n\eta(\omega) + \frac{2M_1(t(\omega))}{n} + \cdots \right), \]

where

\[ \eta(z) = \sqrt{1 + z^2} + \log \frac{1}{1 + \sqrt{1 + z^2}}, \quad t(z) = \frac{1}{\sqrt{1 + z^2}}, \]

\[ D_1(t) = \frac{t^3}{24}, \quad M_1(t) = -\frac{3t}{8} + \frac{7t^3}{24}. \]

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In the following, we consider the case of homogeneous Therefore, boundary conditions (DD or NN) and mixed boundary conditions (DN or ND) separately.

For homogeneous boundary conditions,

\[ A_0(\omega) \sim \exp(-2\varepsilon\omega + \cdots). \]

Therefore,

\[ \mathcal{E}_{\chi,0} \sim \begin{cases} \pi^2/24\varepsilon + o(1), & \chi = 0, \\ -\zeta_R(3)/8\varepsilon^2 + o(\varepsilon^{-1}), & \chi = 1. \end{cases} \tag{8} \]

On the other hand,

\[ A_n(n\omega) \sim \exp\left(-2n[(1+\varepsilon)^{-1} - \eta(\omega)]
-2P_1(t([1+\varepsilon]\omega) - P_1(t(\omega)))\right), \]

where

\[ P_1(t) = \lambda_0 t + \lambda_1 t^3 = \begin{cases} \mathcal{D}_1(t), & \text{for DD b.c.}, \\ \mathcal{M}_1(t), & \text{for NN b.c.}. \end{cases} \]

Applying the inverse Mellin transform formula

\[ e^{-v} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(z)v^{-z} \, dz \tag{9} \]

gives

\[ \mathcal{E}_{\chi,r} = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(z)\zeta_R(z+1)2^{-z-2}\varepsilon^{-z} \times(z-z - 1)A_\chi(z) - zA_\chi(z)B_\chi(z) \, dz, \tag{10} \]

where

\[ A_\chi(z) = \int_0^\infty \omega^{z-2}\eta'(\omega)^{-z}(1 - z - z\omega\eta'(\omega)^2/2) \, d\omega, \]

\[ B_\chi(z) = \int_0^\infty \omega^{z-2}\eta'(\omega)^{-z}P_1'(t(\omega))t'(\omega) \, d\omega. \]

Straightforward computations give

\[ A_\chi(z) = \frac{\Gamma(z+1)}{2} \frac{\Gamma(z+1)}{\Gamma(z)} \left( 1 + z - z - 1 \right) \frac{1 + z - z - 1}{2} \frac{z - 2z - 2z - 17 + 7z - 3z}{24(z + 2)}, \]

\[ B_\chi(z) = \frac{\Gamma(z+1)}{2} \frac{\Gamma(z+1)}{\Gamma(z+1)} \left( -\lambda_0 + (\lambda_0 - 3\lambda_1) \right) z - z + 1 z + 2 \frac{z - 2z - 2z - 17 + 7z - 3z}{24(z + 2)}. \]

The leading term

\[ \mathcal{E}_{\chi,r} = -\frac{\pi^2}{8\varepsilon^2} \zeta_R(3) \left( 1 + \varepsilon \right) + \pi^2 \frac{2}{24}\varepsilon 
+ \frac{\pi}{16} (4\lambda_0 + 3\lambda_1) \ln \varepsilon + O(1), \]

\[ \mathcal{E}_{1,r} = -\frac{\pi^4}{360\varepsilon^3} \left( 1 + \varepsilon \varepsilon^2 \frac{\varepsilon}{10} \right) + c \pi \varepsilon^2 \frac{2}{15} 20\lambda_0 + 12\lambda_1 + o(\varepsilon^2) \}

From these, we find that the asymptotic expansions of the zero-temperature Casimir energies for DD and NN boundary conditions are given, respectively, by

\[ E_{DD,T=0} \sim \frac{\pi^3}{720a_1^2\varepsilon^3} \left( 1 + \varepsilon \varepsilon^2 \frac{\varepsilon}{10} + o(\varepsilon^2) \right), \]

\[ E_{NN,T=0} \sim \frac{\pi^3}{720a_1^2\varepsilon^3} \left( 1 + \varepsilon \varepsilon^2 \frac{\varepsilon}{10} + o(\varepsilon^2) \right). \tag{11} \]

Again, the leading term

\[ -\frac{\pi^3 L}{8a_1^2\varepsilon^3} = -\frac{\pi^2}{1440d^3} \times 2\pi a_1 L \]

is what one would expect from the proximity force approximation. For the high-temperature asymptotic behavior of the Casimir free energies, we have

\[ E_{DD,c} \sim \frac{LT}{8a_1^2\varepsilon^3} \zeta_R(3) \left( 1 + \varepsilon \varepsilon^2 \frac{\varepsilon}{10} + o(\varepsilon^2) \right), \]

\[ E_{NN,c} \sim \frac{LT}{8a_1^2\varepsilon^3} \zeta_R(3) \left( 1 + \varepsilon \varepsilon^2 \frac{\varepsilon}{10} + o(\varepsilon^2) \right). \tag{12} \]
This agrees with the result obtained in [17]. On the other hand, the high-temperature asymptotic expansion is

\[ E_{\text{Cas}}^{\text{PC-PC,cl}} \sim -\frac{LT}{4\alpha t} e^{-\frac{z}{2}} \zeta_R(3) \left( 1 + \frac{\varepsilon}{2} + \frac{3}{16} \varepsilon^2 \ln \varepsilon + O(\varepsilon) \right). \]

For mixed boundary conditions,

\[ A_0(\omega) \sim -\exp(-2\varepsilon \omega + \cdots). \]

Therefore,

\[ \mathcal{E}_{\chi,0} \sim \left\{ \begin{array}{ll}
\frac{\pi^2}{48\varepsilon} + o(1), & \chi = 0, \\
\frac{3\zeta_R(3)}{2\varepsilon^2} + o(\varepsilon^{-1}), & \chi = 1.
\end{array} \right. \tag{13} \]

On the other hand,

\[ A_n(n\omega) \sim -\exp\left( -2n \left[ \eta[(1+\varepsilon)\omega] - \eta(\omega) \right] - 2Q_1(t((1+\varepsilon)\omega)) - P_1(t(\omega)) \right), \]

where

\[ \left( \begin{array}{c}
P_1(t) \\
Q_1(t)
\end{array} \right) = \left( \begin{array}{c}
\lambda_0 t + \lambda_1 t^3 \\
\omega_0 t + \omega_1 t^3
\end{array} \right) = \left( \begin{array}{c}
D_1(t) \\
M_1(t)
\end{array} \right), \text{ for DN b.c.}
\]

\[ \left( \begin{array}{c}
P_1(t) \\
Q_1(t)
\end{array} \right) = \left( \begin{array}{c}
M_1(t) \\
D_1(t)
\end{array} \right), \text{ for ND b.c.} \]

Applying the inverse Mellin transform formula (9) gives

\[ \mathcal{E}_{\chi,r} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(z)(1-2^{-z})\zeta_R(z+1)2^{-z}e^{-z} \times \left[ \zeta_R(z-\chi-1)A_\chi(z) - z\zeta_R(z-\chi+1)\zeta'_R(z) \right. \\
+ \frac{z(z+1)}{2\varepsilon^2} \zeta_R(z-\chi+3)G_\chi(z) \left. \right] \, dz, \tag{14} \]

where \( A_\chi(z) \) is as before,

\[ \mathcal{G}_\chi(z) = \int_0^\infty \omega^{x-z-2} \eta'(\omega) \left( \frac{Q_1(t(\omega))}{\eta'(\omega)} - \frac{z+1}{2} \frac{Q_1(t(\omega)) - P_1(t(\omega))}{\eta'(\omega)} \right) \, d\omega,
\]

\[ \frac{\Gamma \left( \frac{\chi+1}{2} \right) \Gamma \left( \frac{z+1}{2} \right)}{2 \Gamma \left( \frac{z+3}{2} \right)} \times \left( -\omega_0 + \left( \omega_0 - 3\omega_1 + \frac{z+1}{2} \kappa_0 \right) - \frac{z-\chi+1}{2} \\
+ \frac{3\omega_1 + \frac{z+1}{2} \kappa_1}{2} \right)(z-\chi+1)(z-\chi+3) \\
+ \frac{1}{\varepsilon} \left[ \kappa_0 + \kappa_1 \frac{z-\chi+1}{z+2} \right], \]

\[ G_\chi(z) = \int_0^\infty \omega^{x-z-2} \eta'(\omega) (-z-2)(Q_1(t(\omega)) - P_1(t(\omega))) \, d\omega, \]

\[ = \frac{\Gamma \left( \frac{\chi+1}{2} \right) \Gamma \left( \frac{z+1}{2} \right)}{2 \Gamma \left( \frac{z+4}{2} \right)} \left( \kappa_0^2 + 2\kappa_0 \kappa_1 z-\chi+3 \right) \\
+ \kappa_1(3-z-3)(z-\chi+5) \right) \right) \)

Here \( \kappa_i = \omega_i - \lambda_i \). Therefore,

\[ E_{0,r} = \frac{3\pi \zeta_R(3)}{32\varepsilon^2} (1 + \frac{\varepsilon}{2}) \frac{1}{\Gamma(\frac{1}{2})} + O(1), \]

\[ E_{1,r} = \frac{7\pi^2}{2880\varepsilon^3} \left( 1 + \frac{\varepsilon}{2} \right)^\frac{\varepsilon^2}{10} - \frac{\pi^2}{72\varepsilon}(3\kappa_0 + \kappa_1) \]

\[ - \frac{3\zeta_R(3)}{\varepsilon^3} + \frac{\pi^2}{24\varepsilon} \left( \frac{2\kappa_0 - \kappa_1}{3} + \frac{2\kappa_1 - \kappa_0}{2} \right) \]

\[ + \ln \frac{2}{\varepsilon} \kappa_0 + \frac{1}{\varepsilon} \left( \kappa_0^2 + \frac{2}{3} \kappa_0 \kappa_1 + \frac{1}{2} \kappa_1^2 \right) + O(\varepsilon^{-1}). \]

From these, we find that the asymptotic behaviors of the zero-temperature Casimir energies for DN and ND boundary conditions are given, respectively, by

\[ E_{\text{Cas}}^{\text{DN,T=0}} \sim \frac{7\pi^3 L}{5760a_1^2 \varepsilon^3} \left( 1 + \frac{\varepsilon}{2} + \frac{40}{7\pi^2} \right) + O(\varepsilon^2), \]

\[ E_{\text{Cas}}^{\text{ND,T=0}} \sim \frac{7\pi^3 L}{5760a_1^2 \varepsilon^3} \left( 1 + \frac{\varepsilon}{2} + \frac{40}{7\pi^2} \right) + O(\varepsilon^2). \]

The leading term

\[ \frac{7\pi^3 L}{5760a_1^2 \varepsilon^3} \times 2\pi a_1 L \]

agrees with the proximity force approximation. For the high-temperature asymptotic behaviors of the Casimir free energies, we have

\[ E_{\text{Cas}}^{\text{cl}} \sim \frac{3LT}{32a_1^2 \varepsilon^2} \zeta_R(3) \left( 1 + \frac{\varepsilon}{2} + \frac{4}{3\zeta_R(3)} \right) \]

\[ + \left[ \frac{7}{3} \zeta_R(3) - \frac{1}{4\zeta_R(3)} \right] \varepsilon^2 \ln \varepsilon + O(\varepsilon^2), \]

\[ E_{\text{Cas}}^{\text{ND,cl}} \sim \frac{3LT}{32a_1^2 \varepsilon^2} \zeta_R(3) \left( 1 + \frac{\varepsilon}{2} - \frac{4}{3\zeta_R(3)} \zeta_R(3) \right) \]

\[ - \left[ \frac{7}{3} \zeta_R(3) + \frac{1}{4\zeta_R(3)} \right] \varepsilon^2 \ln \varepsilon + O(\varepsilon^2). \]

Again, the leading term

\[ \frac{3LT}{32a_1^2 \varepsilon^2} \zeta_R(3) = \frac{3T}{64\pi d^2} \times 2\pi a_1 L \]

agrees with the proximity force approximation.
For electromagnetic field with perfect conductor condition on one cylinder and infinitely permeable condition on the other cylinder, we find that the asymptotic expansion of the zero-temperature Casimir energy is

\[ E_{\text{Cas}}^{\text{PC-IP,T}=0} \sim \frac{7n^2 L}{2880a_1^2 \varepsilon^2} \left( 1 + \frac{\varepsilon}{2} \right) + \varepsilon^2 \left[ -\frac{1}{10} - \frac{8}{7\pi^2} + \frac{192}{7\pi^4} \right] + o(\varepsilon^3) \]

whereas the high-temperature asymptotic behavior of the Casimir free energy is

\[ E_{\text{Cas}}^{\text{PC-IP,cl}} \sim \frac{3LT}{\varepsilon^2} \zeta_R(3) \left( 1 + \frac{\varepsilon}{2} - \frac{1}{4\zeta_R(3)} \varepsilon^2 \ln \varepsilon + O(\varepsilon^3) \right). \]

From the results above, we see that the leading terms always agree with the proximity force approximations. The correction terms are more complicated in the case of mixed boundary conditions, where the force is repulsive.

As we mentioned above, in the low-temperature region, the Casimir free energy is dominated by the zero-temperature term. However, it would be interesting to determine the order of magnitude of the temperature correction. For this, we need to apply the generalized Abel-Plana summation formula \[ [24,38,39] \], which states

\[ \int_0^{\infty} \frac{f(x)}{x} \, dx = \frac{1}{2} \left( f(0) + \lim_{T \to \infty} \int_{-T}^T \frac{f(x)}{x} \, dx \right) \]

for functions \( f(x) \) that can be analytically continued to the right half plane. Using this formula, we can express the zero-temperature term as

\[ \Delta E_{\text{Cas}} = \frac{L}{\pi^2 a_1^2} \sum_{n=0}^{\infty} \int_0^\infty \int_0^\infty \frac{\xi \, d\xi}{\sqrt{\xi^2 + k^2}} \times \frac{i[\ln(1-M_n(i\xi)) - \ln(1-M_n(-i\xi))]}{\sqrt{\xi^2 + 1}} \, d\xi + \text{exponentially decaying terms}. \]

From this, we see that in the low temperature region, the leading terms of the temperature correction come from the term

\[ \frac{L}{\pi^2 a_1^2} \sum_{n=0}^{\infty} \int_0^\infty \int_0^\infty \frac{iS_n(\omega)}{\sqrt{\omega^2 + k^2}} - 1 \frac{\omega \, d\omega}{\sqrt{\omega^2 + k^2}} \]

where \( S_n(\omega) = \ln(1-A_n(i\omega)) - \ln(1-A_n(-i\omega)) \). Using

\[ I_n(iz) = i^n J_n(z), \quad K_n(iz) = -i^n \frac{\pi}{2} i^n H_n^{(2)}(z), \]

one can show that for DD or DN boundary conditions,

\[ S_n(\omega) = \ln \left( 1 - i \frac{J_n(\omega)}{N_n(\omega)} \right) - \ln \left( 1 + i \frac{J_n(\omega)}{N_n(\omega)} \right) \]

whereas for ND or NN boundary conditions,

\[ S_n(\omega) = \ln \left( 1 - i \frac{J'_n(\omega)}{N'_n(\omega)} \right) - \ln \left( 1 + i \frac{J'_n(\omega)}{N'_n(\omega)} \right). \]

From here, we see that in the low-temperature region, the leading terms of the temperature correction do not depend on the boundary condition on the larger cylinder. As \( \omega \to 0 \),

\[ \frac{J_n(\omega)}{N_n(\omega)} = O(\omega^{2n}), \quad \frac{J'_n(\omega)}{N'_n(\omega)} = O(\omega^{2n}), \quad n \geq 1, \]

\[ \frac{J_0(\omega)}{N_0(\omega)} \sim \frac{\pi}{2 \ln \frac{1}{\omega} + \frac{\pi}{4} + \frac{\pi}{2 \ln \omega} + \cdots}, \quad \frac{J'_0(\omega)}{N'_0(\omega)} \sim \frac{\pi}{4 \omega^2 + \cdots}, \quad \frac{J_1(\omega)}{N'_1(\omega)} \sim \frac{\pi}{4 \omega^2 + \cdots}. \]

From these, we find that if the smaller cylinder is Dirichlet, the leading contribution to the thermal correction comes from the term with \( n = 0 \), which gives

\[ \Delta_T E_{\text{Cas}} \sim \frac{2L}{\pi a_1^2} \int_0^\infty \int_0^\infty \left( \frac{1}{\ln \omega \sqrt{\omega^2 + k^2}} - 1 \right) \omega \, d\omega \, d\omega \]

This is of order \( T^2/\ln T \). On the other hand, if the smaller cylinder is Neumann, then the leading contribution to the thermal correction comes from the terms with \( n = 0 \) and \( n = 1 \), which gives

\[ \Delta_T E_{\text{Cas}} \sim \frac{4\pi^2 L}{6 \ln(a_1)} \int_0^\infty u \, du \left( \frac{\omega^2}{\sqrt{1+\omega^2}} - 1 \right) \sqrt{\omega^2 + k^2} \]

This term is of order \( T^2/\ln T \).
This is of order $T^4$. Therefore, for electromagnetic cylinders with either perfectly conducting or infinitely permeable conditions, the leading term of the thermal correction is given by (22).

Conclusion. – We have derived the asymptotic expansions of the Casimir free interaction energy between two concentric cylinders when the separation between the cylinders is much smaller than the radii of the cylinders, in both the low-temperature and the high-temperature regions. We consider DD, NN, DN, ND boundary conditions for massless scalar fields and PC-PC, PC-IP boundary conditions for electromagnetic fields. In all cases, the leading terms are found to agree with the proximity force approximations. Our results give the next two correction terms.

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