Confinement/Deconfinement Transition of Large $N$ Gauge Theories with $N_f$ Fundamentals: $N_f/N$ Finite

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Abstract

We consider large $N$ zero-coupling $d$-dimensional U($N$) gauge theories, with $N_f$ matter fields in the fundamental representation on a compact spatial manifold $S^{d-1} \times$ time, with $N_f/N$ finite. The Gauss’ law constraint induces interactions among the fields, in spite of the zero-coupling. This class of theories undergo a 3rd order deconfinement phase transition at a temperature $T_c$ proportional to the inverse length scale of the compact manifold.

The low-temperature phase has a free-energy of $\mathcal{O}(N_f^2)$, interpreted as that of a gas of (color singlet) mesons and glueballs. The high-temperature (deconfinement) phase has a free energy of order $N^2 f(N_f/N, T)$, which is interpreted as that of a gas of gluons and of fundamental and anti-fundamental matter states. This suggests the existence of a dual string theory, and a transition to a black hole at high temperature.

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1. Introduction

There has been steady interest in trying to find string duals to large $N$ gauge theories at weak-coupling [1]. Although this is a challenging problem, information keeps accumulating which keeps the subject progressing [2–8]. Large $N$ gauge theories are believed to undergo a deconfinement phase-transition at sufficiently high temperature, which might possibly be related to Hagedorn behavior in the dual string theory [9]. By considering gauge theories on a compact space [10,11], one obtains an additional parameter $R\Lambda$ to vary which may be tuned to weak coupling, where $R$ is the size of the compact space, and $\Lambda$ is the dynamical scale of the gauge theory. Because the gauge theory is on a compact manifold, one must impose a Gauss law constraint, which induces interactions among the gluons and the matter multiplets. Aharony, et al. [11], have provided a general framework, which we apply to the issues of our concern.

We are particularly interested in $U(N)$ gauge theories with $N_f$ matter multiplets in the fundamental representation of the gauge group, with $N_f/N$ finite in the large $N$ limit, as we have previously discussed such theories [12] and proposed [7] for $d=4$ a possible infinite spin representation, bulk/boundary correspondence, and $(\alpha')^{-1}$ expansion. In this paper we show that these theories on $S^{d-1}\times$ time have two phases, separated by a third-order phase transition at temperature $T_c$. The free energy in the low-temperature phase behaves as

$$F/T \sim N_f^2 f_1(T) \quad 0 \leq T \leq T_c$$

and in the high-energy phase as

$$F/T \sim N^2 f_2 \left( \frac{N_f}{N}, T \right) \quad T \geq T_c .$$

This is attributed to a gas of glueballs and (color singlet) mesons in the low-energy phase, and a phase-transition to a deconfinement transition, with the free-energy that of a gas of gluons and fundamental and anti-fundamental matter states. It is speculated that the low-temperature phase has a string dual, with a high-temperature transition to a black hole.

In sec. 2 we specialize the work of Aharony, et al. [11], to the models of our interest. Section 3 explores the phase-structure of these models. The $d=4$ gauged vector models [12] and their $\mathcal{N}=1$ supersymmetric cousins [7] are presented in Sec. 4 as concrete examples of the theories we are considering. Section 5 summarizes our principle findings, and argues for the possible existence of string duals for the class of theories studied throughout this paper.

2. $U(N)$ gauge theory on $S^{d-1}\times\mathbb{R}$ at large $N$

Aharony, et al. [11], showed that the partition function on a compact space for free $U(N)$ gauge theory, at large $N$, with free matter multiplets, subject only to Gauss’ law constraint is

$$\mathcal{Z}(x) = \int [dU] \exp \left\{ \sum_R \frac{1}{n} \left[ z_B^R(x^n) + (-1)^{n+1} z_F^R(x^n) \right] \cdot \chi_R(U^n) \right\}$$

(2.1)
In (2.1) \( U \) is a \( N \times N \) matrix, \( \chi_R(U) \) is the character for the representation \( R \), \( z_B^R(x) \) and \( z_F^R(x) \) are the single-particle bosonic and fermionic partition functions for each representation, and \( x = e^{-β} = e^{-1/T} \) at temperature \( T \). [See refs. [10,11] for details of the derivation, and for the explicit single-particle partition functions.] It is important to note that the Gauss’ law constraint induces interactions between the matter fields and the gauge sector, even though the theory is at zero-coupling.

2.1 U(N) gauge theory with \( N_f/N \) finite

Consider U(N) gauge theories with \( N_f \) matter multiplets in the fundamental representation of the gauge group, with \( N_f/N \) finite as \( N \to \infty \). Our motivation for discussing such theories is to explore the possible bulk/boundary duality for such field theories in the weak-coupling limit. The putative bulk string theory, considered in an \((α' \)-\(^{-1}\)) expansion, would correspond to the field theory at or near the UV fixed-point. It is speculated that the bulk theory has stringy behavior, and thus studying the free-field theory on a compact manifold may give additional insights for this issue.

With these considerations in mind, we restrict (2.1) to the adjoint and fundamental representations. Let \( z_B(x) \) and \( z_F(x) \) be the single-particle partition function for the adjoint representation, and \( Z_B(x) \) and \( Z_F(x) \) that for the fundamental representation. Then (2.1) becomes

\[
Z(x) = \int [dU] \exp \left\{ \sum_{n=1}^{∞} \frac{1}{n} \left[ Z_B(x^n) + (-1)^{n+1} Z_F(x^n) \right] \cdot \left[ tr(U^n) + tr(U^{n+1}) \right] \right\}.
\] (2.2)

It is to be emphasized that the same matrix \( U \) appears for the adjoint and fundamental matter, as the Gauss’ law constraint has induced interactions between these two sectors even at zero-coupling. One can write (2.2) in terms of the eigenvalues of \( U \); \( \{ \exp iα_i \} (-π < α_i \leq π) \) in the standard way. That is,

\[
Z(x) = \int [dα_i] \exp - \left[ \sum_{i \neq j} V_A(α_i - α_j) + \sum_i V_F(α_i) \right] \] (2.3)

where [11]

\[
V_A(θ) = -\ell n |\sin(θ/2)| - \sum_{n=1}^{∞} \frac{1}{n} \left[ z_B(x^n) + (-1)^{n+1} z_F(x^n) \right] \cos nθ \] (2.4a)

\[
= \ell n \ 2 + \sum_{n=1}^{∞} \frac{1}{n} \left[ 1 - z_B(x^n) - (-1)^{n+1} z_F(x^n) \right] \cos nθ \] (2.4b)

and

\[
V_F(θ) = -\sum_{n=1}^{∞} \frac{2}{n} \left[ Z_B(x^n) + (-1)^{n+1} Z_F(x^n) \right] \cos nθ. \] (2.5)
The second term in (2.4a), as well as (2.5) serve to bring the eigenvalues closer. Thus, the presence of matter in the fundamental representation increases the clustering of the eigenvalues.

Introduce the eigenvalue distribution $\rho(\theta)$ proportional to the density of eigenvalues $e^{i\theta}$ of $U$ at $\theta$, with $\rho$ everywhere non-negative subject to the choice of normalization

$$\int_{-\pi}^{\pi} d\theta \rho(\theta) = 1.$$  \hfill (2.6)

Define the moment

$$\rho_n = \int_{-\pi}^{\pi} d\theta \rho(\theta) \cos n\theta$$  \hfill (2.7)

where the eigenvalue distribution is assumed to be symmetric about $\theta = 0$, as $V_A(\theta) = V_A(-\theta)$ and $V_F(\theta) = V_F(-\theta)$. Then, for $N_f$ multiplets in the fundamental representation, the effective action for the eigenvalues becomes

$$S[\rho(\theta)] = \frac{N_f^2}{\pi} \sum_{n=1}^{\infty} \left[ (\rho_n)^2 V_n^A(T) + 2 \left( \frac{N_f}{N} \right) \rho_n V_n^F(T) \right]$$  \hfill (2.8)

where

$$V_n^A(T) = \frac{\pi}{n} \left[ 1 - z_B(x^n) - (-1)^{n+1} z_F(x^n) \right]$$  \hfill (2.9)

and

$$V_n^F(T) = -\frac{\pi}{n} \left[ Z_B(x^n) + (-1)^{n+1} Z_F(x^n) \right]$$  \hfill (2.10)

for the adjoint and fundamental representations respectively. That is

$$\mathcal{Z} = \sum_{n=1}^{\infty} \exp \left[ \frac{N_f^2}{\pi} \frac{(V_n^F)^2}{(V_n^A)} \right]$$

$$\cdot \int [d\rho_n] \sum_{n=1}^{\infty} \exp \left\{ -\frac{N_f^2}{\pi} \frac{(V_n^A)^2}{(V_n^A)} \left[ \rho_n + \left( \frac{N_f}{N} \right) \left( \frac{V_n^F}{V_n^A} \right) \right]^2 \right\},$$  \hfill (2.11)

which can be written as

$$\mathcal{Z} = \sum_{n=1}^{\infty} \exp \left[ \frac{N_f^2}{\pi} \frac{(V_n^F)^2}{(V_n^A)} \right]$$

$$\cdot \int [d\tilde{\rho}_n] \exp -\frac{N_f^2}{\pi} \sum_{n=1}^{\infty} (\tilde{\rho}_n)^2 V_n^A(T)$$  \hfill (2.12)

where

$$\tilde{\rho}_n = \rho_n + \left( \frac{N_f}{N} \right) \left( \frac{V_n^F}{V_n^A} \right).$$  \hfill (2.13)

The integral in (2.12) is analogous to that for purely adjoint matter, but with $\tilde{\rho}_n$ instead of $\rho_n$. 

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A saddle-point solution of (2.11) or (2.12) implies that
\[
\rho_n = - \left( \frac{N_f}{N} \right) \left( \frac{V_n^F}{V_n^A} \right).
\] (2.14)
or equivalently
\[
\tilde{\rho}_n = 0.
\] (2.15)

From (2.10) we observe that
\[
V_1^F(T) < 0
\] (2.16)
always, and
\[
V_1^A(T) > 0,
\] (2.17a)
if
\[
[z_B(x) + z_F(x)] < 1,
\] (2.17b)
so that \( \rho_n \geq 0 \) if (2.17) is satisfied. One might think that the solution \( \tilde{\rho}_n = 0 \) is valid as long as (2.17) is satisfied. However, (2.7) implies
\[
|\rho_n| \leq 1.
\] (2.18)

If we formally define \( T_H \) (the Hagedorn temperature) by
\[
[z_B(x_T) + z_F(x_T)] = 1,
\] (2.19)
then a necessary condition imposed by (2.14)–(2.18) is
\[
1 \geq \rho_1 > 0,
\] (2.20)
which occurs for \( 0 \leq T < T_H \). The Hagedorn temperature is never reached by the saddle-point solution, as (2.20) implies that
\[
[z_B(x) + z_F(x)] + \left( \frac{N_f}{N} \right) [Z_B(x) + Z_F(x)] \leq 1
\] (2.21)
so that (2.19) is not compatible with (2.21). We shall see in the next section that a third-order phase transition to a new phase occurs even before the upper-limit in (2.21) is attained.

Also noteworthy is that the free-energy
\[
F = -T \ln \mathcal{Z}
\sim N_f^2,
\] (2.22)
when the saddle-point solution is valid, as one observes from (2.12) with \( \tilde{\rho}_n = 0 \), contrary to the low-temperature phase for models with matter only in the adjoint representation, where \( F \sim \mathcal{O}(1) \).

3. Phase structure
Let us consider the phase-structure of this class of models in more detail. The solution for the eigenvalue distribution $\rho(\theta)$ can be obtained from the equilibrium conditions for a matrix model with action

$$S = N \sum_{n=1}^{\infty} \frac{1}{n} \left[ a_n \rho_n + \frac{N_f}{N} b_n \right] \left[ tr(U^n) + tr(U^{n+1}) \right]$$

(3.1)

which generalizes (5.22) of ref. [11], where we have defined

$$a_n = \left[ z_B(x^n) + (-1)^{n+1} z_F(x^n) \right]$$

(3.2)

and

$$b_n = \left[ Z_B(x^n) + (-1)^{n+1} Z_F(x^n) \right].$$

(3.3)

This model, with action

$$S = N \sum_{n=1}^{\infty} \frac{1}{2n} c_n \left[ tr(U^n) + tr(U^{n+1}) \right]$$

(3.4)

has been solved in ref. [13]), where here

$$c_n = 2 \left[ a_n \rho_n + \frac{N_f}{N} b_n \right].$$

(3.5)

[The notation in ref. [13] is $\beta_n$ instead of $c_n$.] The method used by Aharony, et al. [11], to treat $\rho_n$ and $\rho(\theta)$ independently, and solve for $\rho_n$ and $\rho(\theta)$ self-consistently, using

$$\rho_n = \int_{-\pi}^{\pi} d\theta \rho(\theta) \cos n\theta$$

(3.6)

works here as well.

There are two phases for the model under consideration, denoted by $A_0$ and $A_1$ in the notation of ref. [13]. The phase $A_0$ is characterized by a distribution of eigenvalues which covers the circle $-\pi < \theta \leq \pi$ without gaps, while in the phase $A_1$ the circle has a single gap with zero density.

3.1 The $A_0$ phase

The eigenvalue distribution in phase $A_0$ is not constant, but is

$$\rho_{A_0} = \frac{1}{2\pi} \left[ 1 + \sum_{n=1}^{\infty} c_n \cos n\theta \right]$$

(3.7)

for a density chosen to be symmetric about $\theta = 0$. Self-consistency requires

$$\rho_n = \frac{1}{2} c_n = \left[ a_n \rho_n + \frac{N_f}{N} b_n \right].$$

(3.8)

That is

$$\left( \rho_n \right)_{A_0} = \left( \frac{N_f}{N} \right) \frac{b_n}{1 - a_n}$$

(3.9a)
\[ = - \left( \frac{N_f}{N} \right) \left( \frac{V_F}{V_A} \right), \] (3.9b)

where we have used (2.9) and (2.10). The eigenvalue distribution in phase $A_0$ is identical to that obtained from the saddle-point solution, i.e., (2.14), so that $\bar{\rho}_n = 0$ in phase $A_0$. The free-energy for this phase is given by (2.11) and (2.22).

### 3.2 The $A_1$ phase

The $A_1$ phase can be described by a generalization of Aharony, et al. [11], eq’ns. (5.23)ff. Consider a solution for $\rho(\theta)$ where $\rho(\theta) \neq 0$ for $-\theta_0 < \theta < \theta_0$, and $\rho(\theta)$ otherwise, where $\theta_0 = \pi$ at the $(A_0, A_1)$ phase boundary. The solution for $\rho(\theta)$ in the $A_1$ phase is

\[
\rho(\theta) = \frac{1}{\pi} \sqrt{s^2 - \sin^2 \left( \frac{\theta}{2} \right)} \left[ \sum_{n=1}^{\infty} Q_n \cos \left( \left( n - \frac{1}{2} \right) \theta \right) \right] \] (3.10)

where $s = \sin(\theta_0/2)$, and

\[
Q_n = \sum_{\ell=0}^{\infty} c_{n+\ell} P_\ell(\cos \theta_0) \] (3.11)

where $P_\ell(x)$ is the Legendre polynomial. From eq’n. (5.15) of ref. [13], we have (where $M = 1$ for the $A_1$ phase)

\[
Q = Q_0 + 2 . \] (3.12)

That is (since $c_0 = 0$)

\[
\sum_{\ell=1}^{\infty} \left[ a_\ell \rho_\ell + \frac{N_f}{N} b_\ell \right] \left[ P_{\ell-1}(1 - 2s^2) - P_\ell(1 - 2s^2) \right] = 1 . \] (3.13)

Define

\[
G_\ell = a_\ell [P_{\ell-1}(1 - 2s^2) - P_\ell(1 - 2s^2)] \] (3.14)

and

\[
D_\ell = b_\ell [P_{\ell-1}(1 - 2s^2) - P_\ell(1 - 2s^2)] . \] (3.15)

Then (3.13) becomes

\[
\vec{G} \cdot \vec{\rho} = 1 - \frac{N_f}{N} \sum_{\ell=1}^{\infty} D_\ell \] (3.16)

Define

\[
B^{n-1/2}(s^2) = \frac{1}{\pi} \int_{-\theta_0}^{\theta_0} d\theta \left\{ \sqrt{s^2 - \sin^2 \left( \frac{\theta}{2} \right)} \left[ \cos \left( \left( n - \frac{1}{2} \right) \theta \right) \right] \right\} . \] (3.17)

Therefore

\[
\rho_n = \frac{1}{2} \sum_{k=1}^{\infty} Q_k \left[ B^{n-k+1/2}(s^2) + B^{n-k+1/2}(s^2) \right] \] (3.18)

\[
= \sum_{\ell=0}^{\infty} \sum_{k=1}^{\infty} \left[ a_\ell \rho_\ell + \frac{N_f}{N} b_\ell \right] \left[ B^{n-k+1/2}(s^2) + B^{n-k+1/2}(s^2) \right] \cdot P_{\ell-k}(1 - 2s) \] (3.19)
where the polynomials $B^{n+1/2}$ are defined by

$$
\sum_{n=0}^{\infty} B^{n+1/2}(x) z^n = \frac{1}{2z} \left[ \sqrt{(1-z)^2 + 4zx} + (z-1) \right].
$$

Let

$$
R_{nl} = a_l \sum_{k=1}^{l} \left[ B^{n+k-1/2}(s^2) + B^{n-k+1/2}(s^2) \right] P_{l-k}(1-2s^2)
$$

and

$$
C_{nl} = b_l \sum_{k=1}^{l} \left[ B^{n+k-1/2}(s^2) + B^{n-k+1/2}(s^2) \right] P_{l-k}(1-2s^2).
$$

Thus (3.18) becomes

$$
\rho_n = \sum_{\ell=1}^{\infty} \left[ R_{n\ell} \rho_n + \frac{N_f}{N} C_{n\ell} \right]
$$

That is

$$
R \vec{\rho} = \vec{\rho} - \frac{N_f}{N} \sum_{\ell=1}^{\infty} C_{n\ell}.
$$

The eigenvalue moments and angle $\theta_0$ in the $A_1$ phase are determined by (3.16) and (3.24), with solution

$$
\vec{\rho} = -\frac{N_f}{N} (R-I)^{-1} \vec{C}
$$

where $\vec{C} = \sum_{\ell=1}^{\infty} C_{n\ell}$.

It is difficult to solve these equations in general, so we consider the simplified model

with $a_n = b_n = 0$ for $n \geq 2$, which should give a good qualitative description of the model. In that case consider the pair of equations

$$
\rho_1 = R_{11} \rho_1 + \frac{N_f}{N} C_{11}
$$

and

$$
G_1 \rho_1 = 1 - \frac{N_f}{N} D_1.
$$

Explicitly

$$
R_{11} = a_1 (2s^2 - s^4)
$$

$$
C_{11} = b_1 (2s^2 - s^4)
$$

and

$$
D_1 = b_1 (2s^2)
$$

$$
G_1 = a_1 (2s^2)
$$
Combining (3.25)–(3.30) we find in the $A_1$ phase

$$\rho_1 = \left(1 - \frac{s^2}{2}\right)$$

$$= \left(1 - \frac{1}{2} \sin^2 \frac{\theta_0}{2}\right), \quad (3.32)$$

with

$$(\rho_1)_{A_1} = \frac{1}{2} \quad (3.33)$$

and

$$a_1 + \frac{2 N_f}{N} b_1 = 1 \quad (3.34)$$

at the $(A_0, A_1)$ boundary. However, in the $A_0$ phase

$$(\rho_1)_{A_0} = \left(\frac{N_f}{N}\right) \frac{b_1}{1 - a_1}. \quad (3.35)$$

Since the single-particle partition functions are continuous, (3.34) holds at the boundary of the $A_0$ phase, and thus

$$(\rho_1)_{A_0} = \frac{1}{2} \quad \text{at the boundary.} \quad (3.36)$$

Hence $\rho_1$ is continuous across the $(A_0, A_1)$ phase boundary, as it must. Equations (3.33) or (3.36) imply that the $(A_0, A_1)$ phase transition occurs before the bound (2.21) is reached. The temperature $T_c$ at which the phase transition takes place satisfies (3.34), i.e.,

$$\left[a_1(x_c) + \frac{2 N_f}{N} b_1(x_c)\right] = 1. \quad (3.37)$$

Since (3.37) implies $a_1(x_c) < 1$, so that $T_c < T_H$, as expected.

One can solve for $\rho(\theta)$, and $s = \sin^2 \frac{\theta_0}{2}$ for the $A_1$ phase in the simplified model. From (3.27) and (3.30)–(3.32) we obtain

$$s^2 = \left[\sin^2 \left(\frac{\theta_0}{2}\right)\right]_{A_1}$$

$$= \left(1 + \frac{N_f}{N} \frac{b_1}{a_1}\right) - \left[\left(1 + \frac{N_f}{N} \frac{b_1}{a_1}\right)^2 - \frac{1}{a_1}\right]^{1/2} \quad (3.38)$$

with $s^2 = 1$ at the $(A_0, A_1)$ boundary, and $0 \leq s^2 \leq 1$ throughout the $A_1$ phase. From (3.10), with

$$Q_1 = 2 \left[a_1 \rho_1 + \frac{N_f}{N} b_1\right] = \frac{1}{s^2} \quad (3.39)$$

and $Q_n = 0$ for $n \geq 2$, one obtains

$$\rho(\theta) = \frac{1}{\pi s^2} \left[s^2 - \sin^2 \left(\frac{\theta}{2}\right)\right]^{1/2} \cos \left(\frac{\theta}{2}\right) \quad (3.40)$$
\[ \rho_n = \int_{-b_0}^{b_0} d\theta \rho(\theta) \cos n\theta. \]  

(3.41)

For the simplified model, the free-energy in phase \( A_0 \), obtained from (2.19) and (3.9), is

\[ \frac{1}{T} (F)_{A_0} = \frac{N^2 b_1^2}{(1 - a_1)} \]  

(3.42)

\[ \xrightarrow{\text{bdy}} \frac{N^2}{4} (1 - a_1) \]  

(3.43a)

\[ = \frac{N^2}{2} \left( \frac{N f}{N} \right) b_1 \]  

(3.43b)

where we have used (3.34). On the other hand, in the \( A_1 \) phase

\[ \frac{1}{T} (F)_{A_1} = N^2 \left\{ - \left[ \frac{1}{2} \ln s^2 + \frac{1}{2} \frac{1}{s^2} - \frac{1}{2} \right] - \left( \frac{N f}{N} \right) b_1 \right\} \]  

(3.44a)

\[ = N^2 \left\{ - \left[ \frac{1}{2} \ln s^2 + \frac{3}{4} - s^2 + \frac{s^4}{4} \right] + (1 - a_1) \left( 1 - s^2 + \frac{s^4}{4} \right) \right\}, \]  

(3.44b)

where we used (3.32) and (3.39). Since \( s \to 1 \) at the boundary,

\[ \frac{1}{T} (F)_{A_1} \xrightarrow{\text{bdy}} \frac{N^2}{4} (1 - a_1) \]  

\[ = \frac{N^2}{2} \left( \frac{N f}{N} \right) b_1. \]  

(3.45)

Therefore, the free-energy is continuous across the boundary as required. One may verify from (3.42) and (3.44) that the phase-transition, of the Gross–Witten type [14], is third-order. Alternately, the arguments [and Fig. 1] of ref. [13] arrive at the same conclusion.

Consider (3.44b) in conjunction with (3.38). With \( \left( \frac{N f}{N} \right) \) fixed for a given model, the free-energy \( F_{A_1} \sim N^2 f_2 \left( \frac{N f}{N}, T \right) \), while recall that \( F_{A_0} \sim N^2 f_1(T) \), in the \( A_0 \) phase, with a smooth third-order transition at the \( (A_0, A_1) \) boundary as indicated by (3.43), due to the relation between \( a_1 \) and \( b_1 \) at the boundary.

3.2 Is there a Hagedorn transition?

Formally define the Hagedorn temperature \( T_H \) by (2.19), or equivalently by

\[ a_1(x_H) = 1, \]  

(3.46)

noting from (3.37), that \( T_H > T_c \) for our class of models. From (3.38), we observe that \( 0 < s^2 < 1 \) at \( x_H \). Further both (3.38) and (3.44) are non-singular at \( a_1 = 1 \). Recall that \( x = e^{-1/T} \), so that \( a_1 \) increases for \( T > T_H \), with \( a_1 \to \infty \) as \( x \to 1 \). But from (3.38) \( a_1 \to \infty \) implies \( s \to 0 \), with \( (F)_{A_1} < 0 \) in this limit.
From Fig. 1 of ref. [13], with $\beta_2 = 0$, corresponding to our simplified model, observe that there is no additional phase-transition for $\beta_1 > 1$. [The $\beta_n$ in that Figure is equivalent to our $c_n$, c.f. (3.5).] One might consider $\beta_2$ and $\beta_1 \neq 0$, i.e., $c_2$ and $c_1 \neq 0$, with

$$c_2 = 2 \left[ a_2(x^2) \rho_2 + \frac{N_f}{N} b_2(x^2) \right]. \quad (3.47)$$

We show that $c_2(x) << c_1(x)$ for the explicit models we shall consider. Then (7.4) of ref. [13] applies, and no further phase-transitions are anticipated, as is also seen in Fig. 1 of ref. [13].

The gaussian fluctuations to the saddle-point solution of (2.11)–(2.12) provide corrections which are $O(1)$, i.e., $1/N^2$ corrections to the $O(N^2_f)$ free-energy in phase $A_0$. These gaussian fluctuations to the free energy are of the form

$$\frac{1}{T} \delta F_1 \sim \ln V^A_1(T) \sim \ln(1 - a_1), \quad (3.48)$$

which becomes

$$\frac{1}{T} \delta F_1 \sim \ln b_1 \quad (3.49)$$

at the $(A_0, A_1)$ boundary. The restriction (3.37) means that these fluctuations do not diverge, contrary to models with adjoint matter only.

Therefore, there does not appear to be Hagedorn phase-transition in these models, by which we mean a divergent partition function, and accompanying phase-transition.

### 3.3 High-temperature behavior

From eq’n. (5.17) and (B.12) of Aharony, et al. [11], one has

$$z_i(x) \to 2N_i T^{d-1} + O(T^{d-2}) \quad (3.50)$$

at very high temperatures, where $N_i$ is the number of physical polarizations of the fields of the theory, with space-time dimension $d$. The potentials $V^F_n$ and $V^A_n$ become strongly attractive in the large temperature limit, and $\theta_0 \to 0$ in that limit, i.e., $s^2 \to 0$. Thus, at very high temperatures $\rho_n \to 1$ to leading order, and $\rho(\theta)$ approaches a delta-function. Therefore, for the class of models we are considering,

$$F(x \to 1) = -2N^2 T^d \zeta(d) \left\{ \left[ N_B^A + \left( \frac{2N_f}{N} \right) N_B^f \right] \right. \right.$$  
$$\left. + \left( 1 - \frac{1}{2d-1} \right) \left[ N_F^A + \left( \frac{2N_f}{N} \right) N_F^f \right] \right\} \quad (3.51)$$

where $N_B^A/N_B^f$ and $N_F^A/N_F^f$ are the bosonic and fermionic degrees of freedom for the adjoint (fundamental) fields respectively, so that free-energy behaves as $N^2$ for the adjoint and $NN_f$ for the fundamental representations.
4. Specific Models

One of our primary interests is to examine the thermodynamic structure of large $N$ gauged vector models in four-dimensions [12], as these theories are candidates [7] for an $(\alpha')^{-1}$ expansion in AdS$_5$, and a conjectured AdS$_5$/CFT correspondence. These theories have both UV and IR fixed points, with the considerations of this paper relevant to the UV fixed-point, while the IR fixed-points are also within the perturbative domain for appropriately chosen $(N_f/N)$.

The single-particle partition functions for $d=4$ are

\begin{align*}
  z_S &= \frac{x^2 + x}{(1 - x)^3} \quad (4.1) \\
  z_V &= \frac{6x^2 + 2x^3}{(1 - x)^3} \quad (4.2) \\
  &\text{and} \\
  z_F &= \frac{8x^{3/2}}{(1 - x)^3} \quad (4.3)
\end{align*}

for scalar, vector, and Dirac fermions respectively. Identical single-particle partition functions apply to $Z_S(x)$, $Z_V(x)$, and $Z_F(x)$ as well.

4.1 Gauged vector model

The gauged U(N) vector model [12] has a scalar and $N_f$ fermions in the fundamental representation, taken in the large $N$ limit, with $N_f/N$ finite. [That is, the gauged vector model is coupled to the Banks–Zaks model [15].] There is a window [12],

\begin{align*}
  3.6 \simeq \left( \frac{3\sqrt{3}}{2} + 1 \right) \leq \frac{N_f}{N} \leq \frac{11}{2} \quad (4.4)
\end{align*}

for which there is an IR fixed-point, which is more restrictive than that of the Banks–Zaks model.

For both the gauged vector-model, and the Banks–Zaks model

\begin{align*}
  a_n(x) &= z_V(x^n) \quad (4.5) \\
  &\text{and} \\
  b_n(x) &= (-1)^{n+1} z_F(x^n) \quad (4.6)
\end{align*}

in the large $N$ limit. The $(A_0, A_1)$ phase-transition occurs when

\begin{align*}
  a_1(x_c) + \frac{2N_f}{N} b_1(x_c) &= 1 \quad (4.7)
\end{align*}

according to (3.34), or

\begin{align*}
  z_V(x_c) + \frac{2N_f}{N} z_F(x_c) &= 1 \quad (4.8)
\end{align*}
Choosing \((N_f/N) = 5\), as in the figure of ref. [7], gives
\[ x_c \simeq 0.048 . \] (4.9)

Equation (4.9) justifies the use of the simplified model of (3.2).

4.2 Supersymmetric gauged vector model

Consider \(\mathcal{N}=1\) supersymmetric QCD with gauge group SU(N), \(N_f\) chiral multiplets \(Q^i\) in the fundamental representation, \(\tilde{Q}^i\) in the anti-fundamental representation \((i, \tilde{i} = 1\) to \(N_f)\), and a massless chiral superfield \(\sigma\), which is a color and flavor singlet [7]. The chiral superfields interact by means of the superpotential
\[ W = \sqrt{\frac{\lambda}{N}} \sigma \sum_{i=1}^{N_f} Q^i \tilde{Q}^i . \] (4.10)

The model with \(\lambda = 0\) was studied extensively by Seiberg [16]. For our discussion, we restrict consideration to the non-Abelian Coulomb phase with \(3N/2 < N_f < 3N\), which is the conformal window.

For this model the \((A_0, A_1)\) phase-transition occurs when (4.7) is satisfied. Here
\[ a_1(x) = z_V(x) + z_F(x) \] (4.11)
and
\[ b_1(x) = 2 z_S(x) + z_F(x) , \] (4.12)
as each chiral multiplet has a complex scalar and a Weyl fermion. For example, choosing \((N_f/N) = 2\), the \((A_0, A_1)\) phase-transition takes place when
\[ x_c \simeq 0.051 . \] (4.13)

Once again the simplified model is justified.

5. Discussion

We have considered the thermodynamic phase structure for free U(N) gauge theory together with \(N_f\) free matter multiplets, at large \(N\) on a compact manifold (in particular \(S^{d-1} \times \) time); where the Gauss’ law constraint induces interactions between the gluons and the matter multiplets. Two phases were found, with the low-temperature phase exhibiting a free energy which behaves as
\[ F/T \sim N_f^2 f_1(T) \quad 0 \leq T \leq T_c . \] (5.1)

There is a third-order phase transition to the second phase, for which
\[ F/T \sim N^2 f_2 \left( \frac{N_f}{N}, T \right) \quad T \geq T_c . \] (5.2)
for two different functions \( f_1 \) and \( f_2 \) of the indicated variables, subject to

\[
N_f^2 f_1(T_c) = N^2 f_2 \left( \frac{N_f}{N}, T_c \right). \tag{5.3}
\]

At very high temperatures, the limiting behavior of the free energy is given by (3.51), as appropriate for a gas of deconfined states.

The low-energy phase of this class of theories is that of a gas of glueballs and (color singlet) mesons \( M^b_a \) \((a, b = 1 \text{ to } N_f)\), since the glueball contribution gives an \( \mathcal{O}(1) \) contribution to the free energy, while the mesons have \( \mathcal{O}(N_f^2) \) degrees of freedom. The phase-transition at \( T_c \) is a (third-order) deconfining transition, where for \( T > T_c \) one has a gas of gluons, and fundamental and antifundamental matter states, since gluons have \( N^2 \) degrees of freedom, and \( N_f \) fundamental matter states contribute \( \mathcal{O}(N N_f) \) to the free-energy. Our computation of the free energies in phases \( A_0 \) and \( A_1 \) supports this picture.

The challenge is to find string duals for weakly coupled field theories. It has been argued that large \( N \) deconfined phases (our \( A_1 \) phase) should be associated with black holes [9], even for weakly coupled gauge theories [11]. That, for weakly coupled gauge theories in the high temperature phase, one may search for a bulk dual, with radius of curvature of the order of the string scale \( \ell_s \) where

\[
\left( \frac{\ell_p}{\ell_s} \right) << 1. \tag{5.4}
\]

Standard arguments [9,11] show that the entropy behaves as \( N^2 \) in the high temperature phase.

Is there evidence of stringy behavior in the low-temperature \((A_0)\) phase? In eq’n. (3.48) we argue that the \( \mathcal{O}(1) \) Gaussian fluctuations give a \( 1/N^2 \) correction to free-energy which behaves as

\[
\frac{a}{T} \delta F_1 \sim \ln V^A_1(T) \sim \ln (1 - a_1). \tag{5.5}
\]

We interpret this to mean that the glueballs contribute to the density of states which goes as

\[
\rho_{\text{adj}}(E) \sim \frac{1}{E} e^{E/T_H} \quad 0 \leq T \leq T_c < T_H \tag{5.6}
\]

which is “stringy”. However, the “limiting” temperature is never reached, as \( T_c < T_H \), since there is a deconfinement phase-transition at \( T_c \). We associate the glueball states with closed strings. On the other hand, one should associate the gas of meson states \( M^b_a \) in the low-temperature phase and the gas of fundamentals and anti-fundamentals in the high temperature phase, with open strings.

Notice that the specific models discussed in Sec. 4 all have IR fixed-points, which prevent extrapolation to arbitrary large \('t\) Hooft couplings, and thus exclude the limit of
$R_{\text{AdS}} \gg \ell_s$, which is the case of the better understood Maldacena limit [17]. For $d=4$ one might be tempted to consider a IIB model with $N$ D3 branes and $N_f$ D7 branes, with $N_f/N$ of $O(1)$ in the large $N$ limit, but this is not possible, as the number of allowed D7 branes is limited. In short, a specific string or brane picture eludes us. [See however ref. [7] for conjectured infinite spin-representations for examples in $d=4$.]

In conclusion, there is significant evidence that conformal theories at weak coupling have string duals, but there is a great deal that must be done to make this more concrete.

**Note added:**

Earlier work related to this paper is [18], where one considers a SU($N$) colored, quark-gluon gas partition function for $d=3$ in the large $N$ limit, with $N_f$ fundamentals, taking into account the colored-single constraint. It was shown that the first-order phase transition of the pure gluon gas changes to a third-order phase transition when $N_f/N$ is finite, but making use of Boltzmann statistics only. The general set-up is presented in [19]. Extensions to a conserved U(1) baryon number current in [20] may also be of interest. We thank Professor Skagerstam for bringing these papers to our attention.

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