Origin of irregular X-ray mirage fringes from a bent, thin crystal

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The dynamical theory of diffraction is used to analyse irregular X-ray mirage interference fringes observed in Si220 X-ray reflection topography from a weakly bent, thin crystal due to gravity. The origin of the irregular fringes is attributed to the interference between mirage diffracted beams and a reflected beam from the back surface, which is a new type of interference fringe. The irregular fringes are reproduced by calculating the reflected intensities numerically. The effects of absorption and thermal vibration are quite important for the reproduction. The result shows that the interference fringes depend on the strain as well as the thickness of the crystal, which indicates that the fringes should be useful for analysing weak strain in a crystal as an application.

1. Introduction

In a weakly bent crystal with a constant strain gradient, the index of refraction is variable with respect to the depth. When the X-ray beam enters the crystal, it propagates along a hyperbolic trajectory and is reflected back to the entrance surface as shown by \( S_i \) in Fig. 1, which is called mirage diffraction (Authier, 2001). Jongsukswat et al. (2012) have observed interference fringes of mirage diffracted beams (IFMD) from a weakly bent plane-parallel crystal and measured the strain of the crystal. The IFMD are caused by the interference between two mirage diffracted beams such as \( S_1 \) and \( S_2 \) shown in Fig. 1(c). Fukamachi et al. (2011b) have observed another type of mirage interference fringe caused by interference between diffracted beam \( S_1 \) and reflected beam \( S_B \) from the back surface (IFMRB) when the strain gradient of a thin crystal is very small. The fringe spacing of IFMD decreases as a function of the distance between the incident point and the exit point (\( x \)), whereas that of IFMRB increases. It is easy to distinguish IFMD and IFMRB. However, IFMD and IFMRB can coexist depending on the strain gradient and the thickness of the crystal. We report on irregular mirage interference fringes when IFMD and IFMRB coexist and their origin using the dynamical theory of diffraction.

2. Experimental results

Fig. 1(a) shows the observed Si220 topography, which is a part of Fig. 3 of Jongsukswat et al. (2013). The sample was a plane-parallel Si single crystal 50 mm long, 15 mm wide and 0.28 mm thick. One end of the sample was pasted to an aluminium base with pine resin as shown in Fig. 1(b). The sample was bent due to the gravity force. The diffraction experiments were carried out at the bending-magnet beamline BL-15C, Photon Factory, KEK, Tsukuba, Japan. The incident X-rays were \( \sigma \) polarized.
and monochromated by a Si(111) double-crystal monochromator. The X-ray energy was 11100 ± 0.5 eV. The beam size was 0.02 mm long and 4 mm wide. The details of the experiment are described by Jongsukswat et al. (2013).

Fig. 1(a) shows the topography in the range of distance \(l\) between 16 and 20 mm from the free edge of the sample. The dark contrast around 1 mm from the left is the primary diffraction. The fringe contrasts observed on the right side of it are denoted as MIFMD (modified IFMD) to be studied in this paper. Fig. 1(c) shows schematically the trajectories of the refracted beams in the crystal for formation of IFMD and IFMRB. Here the X-ray beam propagating along the energy flow, that is the Poynting vector of the beam, is referred to as the refracted beam. \(P_0\) and \(P_h\) represent the incident and the reflected beams, respectively. \(P_{m}\) is the angle between \(S_1\) and the lattice plane at \(A_0\). (d) The real part of the dispersion surface (thick solid line). The abscissa represents the \(X\) axis and the ordinate the \(Z\) axis. \(L_0\) is the Lorentz point at \((k_{\alpha 0}, h/2)\). The part of the dispersion surface below \(d_2\) belongs to branch (1) and the part above \(d_1\) belongs to branch (2). The width of the shaded area corresponds to the divergent angle of the incident beam.

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Here \( \alpha \) is the incident glancing angle, \( C \) the polarization factor and \( \chi_h \) the \( h \)th Fourier coefficient of dielectric susceptibility of the crystal. According to Gronkowski & Malgrange (1984), the beam trajectory of the refracted beam in a weakly bent non-absorbing crystal is given by

\[
\left( \frac{\beta z}{\tan \theta_B} + W_z \right)^2 - \left[ \beta x + s(W_x)(W_x^2 - 1) \right]^{1/2} = 1, \tag{2}
\]

when the beam is incident outside the total reflection region (\( |W_x| \geq 1 \)). \( W_x \) is the value of \( W \) at the incident point on the surface. The parameter \( s(W_x) \) is 1 for \( W_x > 1 \) and \(-1 \) for \( W_x < 1 \). The trajectory shows a hyperbolic form as seen in Fig. 1(c). The origin of the coordinate is taken at the incident point \( A_0 \), the \( x \) axis is along the direction from \( A_0 \) to \( A_1 \) and the \( z \) axis along the inward normal to the surface. The strain gradient parameter \( \beta \) is defined as

\[
\beta = \frac{\lambda}{C(\chi_{h} \chi_{-h})^{-1/2}} \frac{\partial^2 (h \cdot u)}{\partial x \partial y}, \tag{3}
\]

where \( h \) is the reciprocal vector, \( u \) is the atomic displacement vector, \( \lambda \) the X-ray wavelength, \( x_0 \) and \( y_0 \) are the coordinates of the transmitted and diffracted beam directions, respectively.

When \( \beta W < 0 (\beta > 0) \), a beam incident on the crystal at \( A_0 \) propagates along the hyperbolic trajectory \( S \) in Fig. 1(c) and changes the direction from \(+z\) to \(-z\) at the vertex \( (x_a, z_a) \) to reach the point \( A_2 \). At \( A_2 \), a part of the beam is emitted from the surface as the mirage diffracted beam and the rest is reflected back to the crystal. The reflected beam propagates along a similar hyperbolic trajectory starting from \( A_2 \). This propagation process repeats. The electric field of the X-ray after \( n \) times of reflection is given by

\[
E^{(n)}_m \exp(-in\gamma_m) = E_m R^{(n)}_m \exp(-in\gamma_m) \tag{4}
\]

corrected according to Fukamachi et al. (2010). Here \( E_0 \) is the electric field of the incident X-ray, \( \gamma_m \) is the phase shift in vacuum and real. The phase shift of the refracted beam in the crystal \( \gamma_m \) is complex by taking the absorption effect into account and given by

\[
\gamma_m = \int \mathbf{k} \cdot d\mathbf{r} = \gamma_{mz} + i\gamma_{mx}. \tag{5}
\]

Here the integration is carried out along the trajectory, \( \mathbf{k} \) is the wavevector in the crystal and \( \mathbf{r} \) is the position vector. \( R^{(n)}_m \) is given as

\[
R^{(n)}_m = r^{n-1}(r^2 - 1), \quad n > 0, \tag{6}
\]

where \( r \) is expressed as

\[
r = \frac{D_h^{(1)}}{D_0^{(1)}}. \tag{7}
\]

\( D_h^{(1)} \) and \( D_0^{(1)} \) are the amplitudes of the 0 and \( h \)th Fourier coefficients of the electric displacement. The first number in the superscript represents the number of reflections and the second number (1) denotes the branch (1) of the dispersion surface.

We now need to calculate the phase shift \( \gamma_m \) by using the dispersion surface. When the absorption is weak \( (k_a < 0.1) \), the complex dispersion surface for the Bragg mode is given as

\[
Z = Z_r + iZ_i \approx \frac{G_B}{2\tan \theta_B} \left( (W + ig)^2 - (1 + i2k_a) \right)^{1/2}, \tag{8}
\]

to Fukamachi et al. (2002). \( k_a \) is expressed as

\[
k_a = \frac{|\chi_{hi}|}{|\chi_{h1}|}. \tag{9}
\]

\( \chi_{h1} \) and \( \chi_{hi} \) are the real and the imaginary parts of \( \chi_h \). \( G_B \) and \( g \) are given by

\[
G_B = \frac{K_0 C(\chi_{h} \chi_{-h})^{1/2}}{\cos \theta_B}, \tag{10}
\]

and

\[
g = \frac{\chi_{0k}}{C(\chi_{h1}^2 + |\chi_{hi}|^2)^{1/2}} \approx \frac{\chi_{0k}}{C|\chi_{hi}|}. \tag{11}
\]

By squaring both sides of equation (8) and ignoring small terms \( Z_i^2, k_a^2 \) and \( g^2 \), equation (8) becomes

\[
(Z_r + iZ_i)^2 \approx Z_r^2 + 2Z_rZ_i = \left( \frac{G_B}{2\tan \theta_B} \right)^2 \left[ W^2 - 1 + i2g(W - g') \right], \tag{12}
\]

where the parameter \( g' \) is given by

\[
g' = \frac{k_a}{g}. \tag{13}
\]

The real part of equation (12) becomes

\[
Z_r = \pm \frac{G_B}{2\tan \theta_B} (W^2 - 1)^{1/2} \tag{14}
\]

which agrees with the expression without absorption, indicating the validity of equation (2). The real part of the phase shift \( \gamma_{mx} \) can be written as

\[
\gamma_{mx} = \gamma_{mz} + \gamma_{mx} \approx \int k_{xz} dx + \int k_{xz} dz. \tag{15}
\]

Here \( k_{xz} \) and \( k_{xz} \) are \( x \) and \( z \) components of the real part \( \mathbf{k} \) of the wavevector \( \mathbf{k} \)

\[
\mathbf{k} = k_x + ik_z. \tag{16}
\]

with \( k_z \) its imaginary part. As the coordinate of the Lorentz point is \( (\kappa_{xx}, h/2) \) \( (h = |\mathbf{h}|) \) according to Fig. 1(d), \( \gamma_{mx} \) is rewritten as

\[
\gamma_{mx} = \int k_{xx} dx = \kappa_{xx} x + 2 \int X dx. \tag{17}
\]

\( \kappa_{xx} \) is the real part of the average wavenumber in the crystal. The first term in equation (17) \( \kappa_{xx} x \) is the order of \( \pi \), which is much smaller than the second term and can be neglected. As \( X \) in equation (17) is

\[
X = -\frac{G_B}{2} W \tag{18}
\]

and
\[ dx = \frac{1}{\beta} d(W^2 - 1)^{1/2} \]  
\( (19a) \)

\[ dz = \frac{\tan \theta_B}{\beta} dW, \]  
\( (19b) \)

by using equation (2), the second term of equation (17) becomes

\[ \int_0^{\pi/2} X \, dx = -\frac{G_B}{2\beta} \int_{W_s}^{-1} \frac{W^2}{(W^2 - 1)^{1/2}} \, dW. \]  
\( (20) \)

The phase shift \( \gamma_{mez} \) is obtained as

\[ \gamma_{mez} = \int k_x \, dz = -2 \int_0^{z_a} \left( \frac{h}{2} + Z_i \right) \, dz = -h_{za} - 2 \int_0^{z_a} Z_i \, dz \]  
\( (21) \)

by referring to Fig. 1(d). The first term on the right-hand side \( h_{za} \) is the order of \( \pi \) and much smaller than the second term. As the second term is given by

\[ \int_0^{z_a} Z_i \, dz = -\frac{G_B}{2\beta} \int_{W_s}^{-1} \frac{1}{(W^2 - 1)^{1/2}} dW = -\frac{G_B}{\beta} M(W_s) \]  
\( (22) \)

\[ \frac{M(W_s)}{W_s + (W^2 - 1)^{1/2}}. \]  
\( (23) \)

\[ (24) \]

From the relation of the imaginary part of equation (12),

\[ Z_i = \frac{1}{Z_i} \left( \frac{G_B}{2 \tan \theta_B} \right)^2 g(W - g') = \frac{g G_B}{2 \tan \theta_B} \frac{(W - g')}{(W^2 - 1)^{1/2}} \]  
\( (25) \)

is obtained.

For the imaginary part of the phase shift \( \gamma_m \), only the \( z \) component needs to be considered. By using the relation

\[ \frac{g G_B}{2 \tan \theta_B} \frac{K_0}{2 \sin \theta_B} = -\frac{\mu_a}{2 \sin \theta_B} \]  
\( (26) \)

and equation (25), \( \gamma_m \) is given by

\[ \gamma_m = \int k_x \, dz = 2 \int_0^{z_a} Z_i \, dz = \frac{\mu_a}{\beta \cos \theta_B} \int_{W_s}^{-1} \frac{(W - g')}{(W^2 - 1)^{1/2}} \, dW \]  
\( (27) \)

\[ = -\frac{\mu_a}{2 \cos \theta_B} \left[ 1 + \frac{2|g'|}{x |\beta|} M(W_s) \right]. \]  
\( (28) \)

Here \( \mu_a \) is the mean absorption coefficient. The exit point \( x \) of the mirage diffracted beam is given by

\[ x = \frac{2(W_s^2 - 1)^{1/2}}{|\beta|}, \]  
\( (29) \)

using the relation \( x = 2x_a \) due to the symmetry of the trajectory. The initial value of the deviation parameter \( W_s \) is obtained from the exit point \( x \) of the mirage diffracted beam. For a monatomic crystal with its atomic scattering factor being positive, the condition \( W < -1 \) corresponds to the anomalous transmission of the Borrmann effect (Fukamachi et al., 2002). In the following, we will study the fringes under this condition. As the X-rays are \( \sigma \) polarized in the experiment, the polarization factor \( C \) in equation (1) is 1.

### 3.2. Mirage interference fringes

In the first zone, the electric field of IFMD at \( A_2 \) is given by

\[ E_{0} R_{m}^{(0)} \exp(-i \gamma_{m}) = E_{0} R_{m}^{(0)} \exp(i \gamma_{m}) \exp(-i \gamma_{m}) \]  
\( (30) \)

with \( A_{m}^{(n)} \) as

\[ A_{m}^{(n)} = E_{0} R_{m}^{(n)} \exp(n \gamma_{m}). \]  
\( (31) \)

The electric field at \( A_2 \) is written by

\[ \exp(-i \gamma_{m}) [A_{m}^{(1)} + A_{m}^{(2)} \exp(-i \Delta \gamma_{m}) + \ldots] \]  
\( (32) \)

where the phase shift \( \Delta \gamma_{m} \) is defined by

\[ \Delta \gamma_{m} = n \gamma_{m} - \gamma_{m}^{(i)}. \]  
\( (33) \)

In the second zone, a similar equation is obtained without the first-order mirage diffracted beam.

IFMRB observed in the first zone has been explained by the interference between the first-order mirage diffracted beam \( S_1 \) and a reflected beam from the back surface \( S_2 \) in Fig. 1(c) as expressed by

\[ \exp(-i \gamma_{m}) [A_{m}^{(1)} + A_{B} \exp(-i \Delta \gamma_{B})]. \]  
\( (34) \)

The amplitude \( (A_{B} = |A_{B}|) \) is given by

\[ A_{B} = r_{B}(W_{B_{s}}) \left[ \frac{r_{B}(W_{B_{s}})}{r_{B}(W_{B_{s}})} - 1 \right] \exp(\gamma_{B_{s}}) |E_{0}|, \]  
\( (35) \)

where \( W_{B_{s}} \) is the value of \( W \) of the beam \( S_{B_{s}} \) at the incident point and \( W_{B_{s}} \) is that at the reflection point \( (x/2, H) \) on the back surface. There is a relation between \( W_{B_{s}} \) and \( W_{B_{s}} \) given by

\[ W_{B_{s}} = W_{B_{s}} + \frac{\beta H}{\tan \theta_B}. \]  
\( (36) \)

In equation (34), \( \Delta \gamma_{B_{s}} \) is given by

\[ \Delta \gamma_{B_{s}} = \frac{G_{B}}{\beta} \{ M(W_{s}^{(i)}) - [M(W_{B_{s}}) - M(W_{B_{s}})] \}, \]  
\( (37) \)

where \( W_{s}^{(i)} \) is the value of \( W \) at the incident point for the beam \( S_{1} \). \( \Delta \gamma_{B_{s}} \) in equation (35) is given by
between beam and the lattice plane; MIFMD in the first zone given by adding a back-surface observed in the experiment, it is necessary to introduce beams are excited within the Borrmann triangle. The divergent back-surface reflected beam is small enough to be neglected.

MIFMD in the second zone is composed of mirage diffracted beams higher than the first order, as the contribution of the back-surface reflected beam is small enough to be neglected.

3.3. Angular amplification

For observing MIFMD, it is necessary to have a certain width of $\Delta W$ to excite the first- as well as the higher-order mirage diffracted beams simultaneously. There is a relation

$$\tan \varphi = \frac{(W^2 - 1)^{1/2}}{|W|} \tan \theta_B$$

between $W$ and $\varphi$ which is the angle between the refracted beam and the lattice plane; $\varphi = 0$ when $W = -1$ and $\varphi \simeq \theta_B$ when $W = -3$ ($\Delta W = 2$). If $\Delta W \simeq 2$, the divergence angle of the refracted beam is approximately equal to $\theta_B$. The refracted beams are excited within the Borrmann triangle. The divergence angle of the incident X-rays corresponding to $\Delta W = 2$ can be derived as $\Delta \varphi = 8.8 \mu$rad by using equation (1). The angle amplification factor ($\Delta \varphi / \Delta \alpha$) is approximately $3.1 \times 10^4$.

As the topography was taken by fixing the crystal as shown in Fig. 1(c), the incident glancing angle $\alpha$ was fixed. In order to observe mirage interference fringes, a finite divergence angle $\Delta \alpha$ is needed. The divergence angle $\Delta \alpha$ is related to the divergence angle $\Delta \theta_B$ of the beam from the monochromator as

$$\Delta \theta_B = \sin \theta_B |\Delta \alpha|$$

(Fukamachi et al., 2014, 2015, 2019). By using the Bragg condition, the width $\Delta E$ of X-ray energy ($E$) and the angle width $\Delta \theta_B$ are related as

$$\frac{\Delta E}{E} = \frac{\Delta \theta_B}{\tan \theta_B}.$$ 

The refracted beams involved in the formation of mirage interference fringes have different wavelengths and different path lengths. Since the mirage interference fringes are observed in the experiment, the coherent condition is satisfied for the refracted beams. The details of the coherent condition have already been given in the previous papers by Fukamachi et al. (2014, 2015, 2019).

4. Results of calculation

4.1. Effects of absorption and thermal vibration

Fig. 2 shows the calculated results of MIFMD composed of mirage diffracted beams and a back-surface reflected beam. The used values of the strain gradient $\beta$ and the parameter $|g'|$ are 0.2 mm$^{-1}$ and 1.0, respectively. In the first zone ($0 < x < x_c$), the interference fringes are composed of mirage diffracted beams from the first to the tenth order and one back-surface reflected beam. In the second zone ($x > x_c$), the interference fringes are composed of mirage diffracted beams from the second to the tenth order.

The effect of the mean absorption $\mu_a$ is shown in Fig. 2. According to equation (26), we have

$$\mu_a = \cos \theta_B G_B k_a.$$ 

When the absorption is ignored, $k_a$ is zero. When the imaginary part of the anomalous scattering factor of Si is taken into account, $k_a$ is 0.02. The blue thin and the black thick curves show MIFMD for $k_a = 0$ and $k_a = 0.02$, respectively. Peaks indicated as $M_1$ to $M_7$ in the first zone and $M_1'$ and $M_2'$ in the second zone are eventually attributed to MIFMD. When $k_a = 0$, the peak $M_3$ is about twice higher than $M_1$, and the

**Figure 2**

The calculated Si220 reflected intensities of MIFMD for $k_a = 0.02$ (in black) and $k_a = 0$ (in blue) in the case of $\beta = 0.20$ mm$^{-1}$, $x_c = 6.3$ mm and $x_c = 8.4$ mm.

**Figure 3**

The calculated Si220 reflected intensities of MIFMD by taking the thermal vibration effect into account when $\beta = 0.20$ mm$^{-1}$ and $k_a = 0.02$. The orange, blue and black lines show the intensities for $|g'| = 1.0$, 0.98 and 0.96, respectively.
height of $M_1$ is comparable with that of $M_1$. When $k_g = 0.02$, all the peak heights are roughly a quarter of those for $k_g = 0$. The peak heights from $M_1$ to $M_7$ are not so much different, showing a similar trend of variations in the experiment. However, $M_1$ shows a similar peak height to $M_1$ while $M_1'$ in Fig. 1(a) shows a much lower peak than $M_1$. It is not possible to reproduce MIFMD observed in the experiment only by taking into account the effect of mean absorption.

The thermal vibration effect can be taken into account through $y_0g$ in equation (28), as it includes the term $g'$, which is given by

$$|g'| = \left| \frac{X_{hi}}{X_{hi}} \right| = \exp(-Bx^2),$$

where $B$ expresses the thermal vibration effect and $s = \sin \theta_0/\lambda$. The calculated MIFMD are shown in Fig. 3. The orange, blue and black lines show MIFMD for $|g'| = 1.0$, 0.98 and 0.96, respectively. When $|g'| = 1.0$, no thermal vibration is taken into account. When $|g'|$ becomes small, the overall peak heights become small. When $|g'| = 0.98$, the peak height of $M_1'$ is about 1/3 of $M_1$. The peak height of $M_1'$ becomes about 1/10 of $M_1$ for $|g'| = 0.96$. By comparing with the experimental results, we adopt 0.96 as the value of $|g'|$. The value of $B$ corresponding to $|g'| = 0.96$ is 0.60 Å², which is certainly larger than reported values such as 0.469 Å² by Flensburg & Stewart (1999) and 0.483 Å² by Sang et al. (2010). The large value may come from an anisotropic or anharmonic vibrational effect. A further study is needed to confirm such a vibrational effect quantitatively.

4.2. Comparison of MIFMD with IFMD and IFMRB

In Fig. 4 are shown the calculated intensities of IFMD (orange), IFMRB (black) and MIFMD (blue). The strain gradient $\beta$ is assumed to be 0.20 mm⁻¹. The intensity of IFMD shows a slow variation as a function of $x$. The width of a fringe and the interval between neighbouring fringes become small when $x$ increases. The peak height in $M_6$ decreases as $n(x)$ increases. The peak of $M_6$ is extremely small compared with the neighbouring peaks. IFMRB starts to appear from $x_{\text{min}} = H/\tan \theta_0$. The width of the fringe and the interval between neighbouring fringes become large as $x$ increases. The intervals between neighbouring fringes of IFMRB are smaller than those of IFMD when $x - x_{\text{min}}$ is small, but these two intervals are comparable when $x$ becomes close to $x_{\text{min}}$.

For $M_1$, MIFMD is approximately the sum of IFMD and IFMRB. For $M_2$ and $M_3$, two peaks appear in MIFMD by the influence of IFMRB. For $M_4$, the peak of IFMD and the valley of IFMRB appear at the same $x$ and the peak of MIFMD is lowered. For $M_5$ to $M_7$, the intervals of the fringes of IFMD and IFMRB are almost the same and the peak heights of MIFMD are enhanced more than twice those of IFMD.

4.3. Comparison of topographies

Fig. 5 shows the measured topography (a), the calculated topographies of MIFMD (b), IFMD (c) and IFMRB (d). The abscissa is the distance $l$ in (a) and the strain gradient $\beta$ in (b)–(d). The correspondence between $l$ and $\beta$ is described in Section 2.

In Fig. 5(a), there are three dark contrasts of $M_2$ on the left end ($l = 20$ mm). The uppermost dark contrast becomes the lowest fringe of dark contrast of $M_1$ on the right end ($l = 16$ mm). There are two dark contrasts of $M_3$ on the left end. The upper dark contrast is weaker than the lower one. The upper dark contrast is connected to the lowest dark contrast of $M_2$ on the right end after showing weak contrast around $l = 19$ mm. The lower dark contrast of $M_3$ on the left end becomes the dark contrast of $M_3$ on the right end after showing weak contrast around $l = 17$ mm.

In Fig. 5(b), there are three dark contrasts of $M_2$ on the left end. The dark uppermost contrast becomes weak around $\beta = 0.19$ mm⁻¹, then it is connected to the lowest dark contrast of $M_1$ on the right end. There are two dark contrasts of $M_3$ on the left end. The upper contrast is weaker than the lower one.

**Figure 4**

The calculated Si220 reflected intensities of IFMD (in orange), IFMRB (in black) and MIFMD (in blue) for $\beta = 0.20$ mm⁻¹, $k_g = 0.02$ and $|g'| = 0.96$.

**Figure 5**

(a) The observed Si220 topography. (b), (c) and (d) show the calculated topographies of MIFMD, IFMD and IFMRB, respectively, for $k_g = 0.02$ and $|g'| = 0.96$. Fringes in (d) are numbered as $B_n$ ($n = 1, 2, \ldots$) from $x_c$ to $x_{\text{min}}$, as the phase $\Delta Y_{B_n}$ is zero at $x_c$ and becomes large as $x$ decreases. The peak appears at a point $x$ corresponding to $\Delta Y_{B_n} = \pi(2n - 1)$ in equation (37).
contrast disappears around $\beta = 0.21 \text{ mm}^{-1}$, then becomes dark and the lowest dark contrast of $M_2$ on the right end. The lowest dark contrast of $M_1$ on the left end disappears around $\beta = 0.17 \text{ mm}^{-1}$, then becomes the dark contrast of $M_3$ on the right end. Similar behaviours of fringe contrasts of the measured topography in Fig. 5(a) are obtained in the calculated MIFMD topography in (b).

In the calculated IFMD topography in Fig. 5(c), the fringe $M_1$ shows a wide band of dark contrast. The higher-order fringes $M_2$ to $M_7$ show the narrower bands and the smaller interval between the neighbouring fringes than $M_1$. In the calculated IFMRB topography in Fig. 5(d), the width of the dark contrasts becomes monotonously large and the distance between the neighbouring fringes becomes large when $x$ increases from $x_{\text{min}}$. Around $x = 3.5 \text{ mm}$, the dark contrast of $M_2$ appears at the left end in (c) and the dark contrast of $B_{10}$ appears at the left end in (d). The shift of $M_2$ is 1.2 mm from the left to the right end, while that of $B_{10}$ is 0.3 mm. The shift of $M_2$ is four times larger than that of $B_{10}$. Similarly, the shift of $M_7$ is larger than that of $B_1$. The variations of interference fringes both in IFMD (c) and IFMRB (d) topographies are regular as a function of $x$. The irregular variations observed in experiment (a) are only reproduced in MIFMD topography (b). The irregular variations are caused by the different variations between IFMD and IFMRB as a function of $x$ as well as $\beta$.

5. Discussion and conclusion

The irregular X-ray mirage interference fringes reported by Jongsukswat et al. (2013) were analysed using the dynamical theory of diffraction. It is necessary to take the absorption as well as thermal vibration effects into account. The absorption effect reduces the peak intensities of the high-order fringes in the first zone. The thermal vibration effect reduces mainly the peak intensities of fringes in the second zone. The calculated MIFMD reproduces the observed irregular modulation of the fringes. The origin of the modulation is attributed to the interference of two or more mirage diffracted beams with a beam reflected from the back surface.

There are still two points that are unclear. (i) Fig. 6 shows (a) the line profile of the measured fringes along the dashed line in Fig. 1(a) and (b) that of the calculated reflected intensities of MIFMD. The peak positions of $M_1$ and $M_2$ in the calculated profile appear at $x$ closer to the incident point than in the measured one. One of the possible reasons is the dependence of the strain gradient ($\beta$) on the distance ($x$), which is difficult to estimate by using the deflection theory. More precise analysis of the MIFMD is necessary in future work. (ii) The other point is related to IFLSD observed in Fig. 1(a). The intensities of IFLSD are much higher than those of MIFMD in the second zone. Hirano et al. (2008, 2009a,b) have observed IFLSD from a plane-parallel crystal without distortion and pointed out that the fringes are caused by the interference between the beam directly reaching the lateral surface and the beam reflected once from the back surface. In the present geometry of a bent crystal, there is no beam reaching directly the lateral surface as shown in Fig. 1(c). It is necessary to apply the dynamical theory of diffraction for a distorted crystal for analysing the IFLSD and the strain gradient of the crystal. As an application, the X-ray beams of IFLSD can be used as an X-ray waveguide, since they propagate quite a long distance from the incident point to the exit point. Fukamachi et al. (2011a) carried out an experiment on an X-ray diffractometer by using X-rays of IFLSD from a plane-parallel crystal as a waveguide and beam splitter. Constructive interference between many beams is derived in the present analysis of MIFMD, which should be useful for developing an X-ray waveguide using IFLSD from a bent crystal.

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