Domain wall Skyrmions

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Abstract

Skyrmions of different dimensions are related by domain walls. We obtain explicit full numerical solutions of various Skyrmion configurations trapped inside a domain wall. We find for the quadratic mass-term that multi-Skyrmions are ring-shaped, and conjecture for the linear mass-term, that the lowest-energy state of multi-Skyrmions will consist of charge-2 rings accommodated in a lattice.
1 Introduction

Skyrmions exist in the non-linear sigma model if it is amended by a higher-derivative term, which in the simplest case can be a fourth-order derivative term \[1\]. They are topological solitons supported by the homotopy group \(\pi_3(S^3) = \mathbb{Z}\) and are believed to describe baryons in the low-energy effective theory of quantum chromodynamics in which the elementary fields represent mesons (pions) \[2\]. The original Skyrme model is equivalent to the O(4) non-linear sigma model with the addition of the Skyrme term. The condition of having finite-energy configurations is tantamount to picking a direction of the field at infinity and thus spontaneously break O(4) down to O(3) yielding \(S^3\) which when mapped to the 3-dimensional space with spatial infinity identified as a point, forms the basis of the topological nature of the solitons, i.e. the Skyrmions.

Skyrmions exist also in other dimensions than 3+1; in 2+1 dimensions the O(3) baby-Skyrmion model \[3–6\] is analogous to its higher-dimensional cousin and it is described by the homotopy group \(\pi_2(S^2) = \mathbb{Z}\) along the same lines as mentioned above. Stepping down one dimension, formally, the sine-Gordon kink is the \((1+1)\)-dimensional Skyrmion as known already by Skyrme himself \[7\]. By virtue of Derrick’s theorem \[8\], the \((1+1)\)-dimensional soliton does not need higher-derivative terms in order to be stable and finite in size and energy as is the case in higher dimensions. So one might naively think that the sine-Gordon kink would be modified upon the presence of the Skyrme term. However, it so happens that along one spatial direction, the Skyrmion term leaves the kink unchanged.

Skyrmions in \(d\) spatial dimensions have been shown in refs. \[9–11\] to be related to those in \(d−1\) spatial dimensions via exactly a domain wall along one direction; yielding a full \(d\)-dimensional Skyrmion charge in the full \((d+1)\)-dimensional theory (see also refs. \[5, 12\]). In the low-energy effective theory living on the domain wall (the kink along one of the spatial directions), the effective soliton is a \((d−1)\)-dimensional Skyrmion enjoying a \((d−1)\)-dimensional Skyrmion charge as well. The dynamics of the lowest dimensional case \[9\] was studied numerically in a related model \[13\].

In this paper we will study static configurations in 3 + 1 dimensions in which we take one spatial direction (the \(x\)-direction) to be always a domain wall. Now we have two options for creating a full \((3 + 1)\)-dimensional Skyrmion trapped on the wall, which we will denote a domain wall Skyrmion. The first option is to make a domain line, say along the \(y\)-direction and then endow it with a sine-Gordon kink. The adequate potential giving rise to such configuration is of a hierarchical form, breaking O(4) \(\rightarrow\) O(3) \(\rightarrow\) O(2) and finally either completely, giving a Skyrmion, or down to \(\mathbb{Z}_2\) yielding a half-Skyrmion. For this to work out, we need the mass scales of the symmetry breakings to be of the form \(m_4 \gg m_3 \gg m_2\). This is the matryoshka construction of ref. \[11\] in which three consecutive domain walls form the full Skyrmion which is doubly trapped. An interesting fact about this configuration is that we do not need higher-derivative terms for creating this configuration and hence its applicability might be found even in condensed-matter systems.

The second option at hand is to start again with the domain wall in the \(x\)-direction and embed it with a baby-Skyrmion living on the two dimensional surface of the domain wall. Pictorially, we can imagine the domain line of the previous type of configurations to be rolled up. For this we do need the Skyrmion term as well as a mass term for the baby-Skyrmion, where the first acts as a pressure term and the latter as an attraction term. This option also does not allow for the half-Skyrmion which exists in the other branch of solutions. However, the effective baby-Skyrmion model living on the domain wall can easily host higher-charged Skyrmions which turn out to be ring-like configurations, depending on the potential. A note of caution is that although the theory living on the domain wall is effectively like that of the planar baby-Skyrmion, differences do persist. The two main differences are that the coefficients in the
effective Lagrangian are no longer canonical and the field feels the derivatives in the transverse direction (these two facts are related).

We find that for the linear potential in the baby-Skyrion theory induced on a domain wall, the Skyrmions of charge 1, 2 and 3 are stable while that with charge 4 is only meta-stable and will eventually decay to two charge-2 rings. The solution with charge 5 is unstable and decays into a bound state consisting of one 2-ring and the remaining \( B = 3 \) charge. For large \( B \), we conjecture that the lowest-energy solutions form lattices composed of 2-rings. For the quadratic potential in the baby-Skyrion theory all higher-charged configurations we have found are stable and ring-like in their energy distributions. For hierarchical quadratic and linear potentials in the baby-Skyrion theory, we find sine-Gordon kinks placed equidistantly on a ring inside the domain wall.

This paper is organized as follows. In section 2 we introduce the Skyrme model and the family of potentials, in which we find domain walls with domain lines with either full Skyrmions or half-Skyrmions in Sec. 3. In section 4 we study baby-Skyrmions trapped on a domain wall with both linear and quadratic induced potentials and we conclude with a summary and discussion in Sec. 5. We relegate additional technical information to the appendices, such as the numerical method in app. A, the domain wall with full and half-Skyrmions residing on domain lines, but without the Skyrme term in app. B and finally an asymmetric initial configuration relaxed down to an axially symmetric 2-ring in app. C.

2 The Skyrme model

Defining a four-vector of scalar fields \( n = \{ n^a(x) \} = \{ n^1(x), n^2(x), n^3(x), n^4(x) \} (a = 1, 2, 3, 4) \) subject to the constraint \( n^2 = 1 \), we consider the O(4) sigma model augmented by the Skyrme term \( L \)

\[
L = \frac{1}{2} \partial_\mu n \cdot \partial^\mu n + \frac{1}{4} (\partial_\mu n \cdot \partial_\nu n) (\partial^\mu n \cdot \partial^\nu n) - \frac{1}{4} (\partial_\mu n \cdot \partial^\mu n)^2 - V(n) , \tag{1}
\]

and a potential which we will choose according to the specific breed of matryoshka configuration in sight. The static equation of motion is given by

\[
\partial^2 n^a + (\partial_i \partial_j n \cdot \partial_j n) \partial_i n^a - (\partial^2 n \cdot \partial_i n^a) \partial_i n^a + (\partial_i n \cdot \partial_j n^a) \partial_j n^a - (\partial_i n \cdot \partial_j n^a) \partial_i n^a - \frac{\delta V}{\delta n^a} = 0 , \tag{2}
\]

where \( i, j = 1, 2, 3 \) are spatial indices and \( a = 1, 2, 3, 4 \) is the vector index. The Skyrmion or baryon number is given by the number of 3-cycles of the 3-sphere and reads

\[
B = -\frac{1}{12\pi^2} \int d^3 x \, \epsilon_{ijk} \epsilon^{abcd} \partial_i n^a \partial_j n^b \partial_k n^c n^d . \tag{3}
\]

The family of potentials we study is a subset of

\[
V = -\frac{1}{2} m_2^2 n_2^{a_2} - \frac{1}{2} m_3^2 n_3^{a_3} + \frac{1}{2} m_4^2 (1 - n_4^2) , \tag{4}
\]

where the powers \( a_{2,3} \) can be either 1 or 2. The squared component of the field allows for a domain wall configuration as both \( n_4 = \pm 1 \) are vacua \( \square \). The linear potential has no domain wall, but breaks \( \mathbb{Z}_2 \) allowing for a full trapped baryon, as we will show later.

\footnote{In the middle of the \( n_4 \)-domain wall, \( n_4 \approx 0 \) which allows for e.g. the domain line due to the “vacua” \( n_3 = \pm 1 \) and analogously for \( n_2 \).}
3 The Matryoshka-wall configurations

3.1 The wall

The simplest and exact solution on which we in the remainder of the paper will trap a series of configurations is the domain wall. If we set \( n = \{0, 0, \sin f(x), \cos f(x)\} \) and choose the potential

\[
V = \frac{1}{2} m_4^2 (1 - n_4^2),
\]

yielding

\[
\mathcal{L} = -\frac{1}{2} (\partial_x f)^2 - \frac{1}{2} m_3^2 \sin^2 f,
\]

which breaks the O(4)-symmetry down to O(3). This Lagrangian density gives rise to the equation of motion known as the sine-Gordon equation and it admits the following exact domain wall solutions

\[
f = 2 \tan^{-1} \exp(\pm m_4 x),
\]

or alternatively in the original variables

\[
n_4 = \mp \tanh(m_4 x), \quad n_3 = \text{sech}(m_4 x).
\]

This configuration is just a single point on the moduli space, \( S^2 \), which we can parametrize as follows

\[
n = \{a^1 \sin f, a^2 \sin f, a^3 \sin f, \cos f\},
\]

where \( a^2 = 1 \) and any constant vector \( a \) yields energetically a wall as the one described above.

All the different configurations will be trapped objects on this wall, by introducing potentials and/or twisting of the moduli \( a \); this will however induce some backreaction on the wall. In order to keep the host soliton (the wall) alive we need to choose a hierarchical order of the masses in the potential at hand: \( m_4 \gg m_3 \gg m_2 \). From an energetic point of view, it corresponds to a cascading symmetry breaking like for instance \( O(4) \to O(3) \to O(2) \to 1 \), depending on the specifics of the potential. Next, we will study a number of different entrapped configurations in turn.

3.2 The wall with a trapped Skyrmion on a domain line

The simplest entrapment we can deploy is a domain line on the domain wall. This can be illustrated by choosing \( n = \{0, \sin g(y) \sin f(x), \cos g(y) \sin f(x), \cos f(x)\} \) together with the potential

\[
V = -\frac{1}{2} m_3^2 n_3^2 + \frac{1}{2} m_4^2 (1 - n_4^2),
\]

which breaks O(4) down to O(3) by means of the wall and in turn down to O(2) due to the existence of the domain line. The O(2) symmetry resembles the rotational degree of freedom of the domain line inside the wall. Plugging into the Lagrangian we obtain

\[
-\mathcal{L} = \frac{1}{2} f_x^2 + \frac{1}{2} (m_4^2 - m_3^2) \sin^2 f + \frac{1}{2} \sin^2 f \left[ g_y^2 + m_3^2 \sin^2 g + f_x^2 g_y^2 \right],
\]

which teaches us two lessons; both \( f \) and \( g \) are roughly described by kinks and the effective mass-squared for the \( f \)-kink is \( (m_4^2 - m_3^2) \), which is positive only if we enforce the hierarchical ordering of mass scales as mentioned above. Finally, there is no need for the Skyrme term in this configuration as its only impact
lies in the derivative coupling being the last term in the bracket. The real solution is not simply the kink in $f$ and $g$, as the host wall receives a backreaction from the presence of the domain line, as we have mentioned already and $g$ is also a function of $x$ contrary to what we have shown here for simplicity. Let us stress that in our numerical solutions we do not impose any factorizing Ansätze.

We will solve the system numerically, however before doing so we will take the next step and inhabit the domain line with a kink. If we break the $O(2)$-symmetry completely by using a linear potential for $n_2$, then the configuration exhibits (a full) baryon charge. For illustrative purposes, let us consider

$$n = \{\sin h(z) \sin g(y) \sin f(x), \cos h(z) \sin g(y) \sin f(x), \cos g(y) \sin f(x), \cos f(x)\}, \quad (12)$$

along with the potential

$$V = -\frac{1}{2} m_2^2 n_2 - \frac{1}{2} m_3^2 n_3^2 + \frac{1}{2} m_4^2 (1 - n_4^2). \quad (13)$$

Plugging into the Lagrangian \([1]\) sums up to

$$-\mathcal{L} = \frac{1}{2} f_x^2 + \frac{1}{2} (m_1^2 - m_2^2) \sin^2 f + \frac{1}{2} \sin^2 f \left[ g_y^2 + m_3^2 \sin^2 g + f_x g_y^2 \right] \quad (14)$$

$$+ \frac{1}{2} \sin^2 f \sin^2 g \left[ h_z^2 + f_x h_z + \sin^2 (f) g_y^2 h_y^2 \right] - \frac{1}{2} m_2^2 \sin f \sin g \cos h,$$

from which we can again identify the three kink components of the system, albeit the terms for the $h$-kink are slightly deformed with respect to those of the $g$-kink and finally it also enjoys more mixing terms. The reason for breaking the symmetry completely (as opposed to leaving a $Z_2$ unbroken) is that it forces the configuration to wind $2\pi$ in the $h$ field yielding a full 3-cycle on the target space $S^3$ and in turn baryon (Skyrmion) charge unity.

We are now ready to find numerically the solutions that we have just described. We will not impose any such factorization in the numerical calculation; that was only for illustrative purposes in the above to get a feeling for what is going on. The field used is thus $n(x,y,z)$ subject to Dirichlet boundary conditions in the $x$-direction

$$n(\mp \infty, y, z) = \{0, 0, 0, \pm 1\}, \quad (15)$$

and Neumann boundary conditions on the remaining sides of the cube. The equations of motion that we will solve throughout the paper are those of eq. \([2]\). Feeding an appropriate initial configuration as a guess and relaxing the system numerically on a $129^3$ cubic lattice as described in more detail in app. \[A\] we obtain the configuration shown in fig. \[1\]. Notice that the energy density in the center slice of the $x$-domain wall, shown in fig. \[1b\], has a valley on the domain line on both sides of the Skyrmion peak. This is a manifestation of the solution not being an effective theory or factorized function, but a genuinely full-backreacted solution to the equations of motion.

### 3.3 The wall with a trapped half-Skyrmion on a domain line

With respect to the configuration in the previous (sub)section, we will now change the potential to

$$V = -\frac{1}{2} m_2^2 n_2 - \frac{1}{2} m_3^2 n_3^2 + \frac{1}{2} m_4^2 (1 - n_4^2), \quad (16)$$

i.e. the $n_2$-component goes from linear to a quadratic potential. In this case, we are not forced to make a full winding of the kink on the domain line, i.e. in terms of the illustrative vector \([12]\), $h$ needs only to wind from $0$ to $\pi$. This in turn halves the baryon charge of the configuration, leaving us with a “half
Skyrmion” trapped on the domain line residing in the wall. Plugging the Ansatz and the above potential into \((1)\), we obtain

\[
-L = \frac{1}{2} f_x^2 + \frac{1}{2} (m_4^2 - m_3^2) \sin^2 f + \frac{1}{2} \sin^2 f \left[ g_y^2 + (m_3^2 - m_2^2) \sin^2 g + f_x g_y^2 \right] \\
+ \frac{1}{2} \sin^2 f \sin^2 g \left[ h_z^2 + m_2^2 \sin^2 h + f_x h_z^2 + \sin^2 (f) g_y^2 h_y^2 \right],
\]  

(17)

where we can identify three consecutive domain walls inside each other with effective mass-squareds \((m_4^2 - m_3^2), (m_3^2 - m_2^2)\) and \(m_2^2\) for \(f\), \(g\) and \(h\), respectively.

We now solve this system numerically and again the field used is simply \(n(x, y, z)\) subject to Dirichlet boundary conditions \((15)\) and Neumann boundary conditions on the remaining sides of the cube, yielding the configuration shown in fig. 2. Notice again that there is a small valley in the energy density near the peak of the half-Skyrmion (with respect to that in fig. 1), see fig. 2).

4 The walls with trapped baby-Skyrmion configurations

4.1 Trapped Skyrmions in a linear potential

The next configuration we consider is a baby-Skyrmion living on the domain wall in the \(x\)-direction, which due to the “winding” of the wall enjoys a full Skyrmion charge. The baby-Skyrmion needs a mass term for compensating the pressure of the 2-dimensional Skyrme term (as the kinetic term is marginal in \(2 + 1\) dimensions). For illustrative purposes, we again choose a simplifying Ansatz, i.e. we take to be

\[
n = \left\{ \frac{\sin f(x)}{\rho} \sin g(\rho), \frac{\sin f(x)}{\rho} \sin g(\rho), \sin f(x) \cos g(\rho), \cos f(x) \right\},
\]

(18)
where \( \rho \equiv \sqrt{y^2 + z^2} \). Choosing the potential

\[
V = -\frac{1}{2} m_3^2 n_3 + \frac{1}{2} m_4^2 (1 - n_4^2),
\]

and plugging the above Ansatz and potential into the Lagrangian \([1]\) we get

\[
-\mathcal{L} = \frac{1}{2} f_x^2 + \frac{1}{2} m_4^2 \sin^2 f + \frac{1}{2} \sin^2 f (1 + f_x^2) \left[ g_\rho^2 + \frac{1}{\rho^2} \sin^2 g \right] + \frac{1}{2 \rho^2} \sin^4 f \sin^2 (g) g_\rho^2 - \frac{1}{2} m_3^2 \cos g \sin f.
\]

It is easy to recognize the wall in \( f \) spanned over the \( x \)-direction. Notice however, that by setting \( \sin f = 1 \) (which corresponds to being in the middle of the domain wall), this effective Lagrangian reduces to the baby-Skyrme model (for the baby-Skyrmion) with the one exception that the standard kinetic term is enhanced by a factor of \((1 + f_x^2)\). The kinetic term for \( f \) on the wall is in turn proportional to \( m_4^2 \).

Solving this system numerically with the generic field \( \mathbf{n}(x, y, z) \) subject to Dirichlet boundary conditions \([15]\) and Neumann boundary conditions on the remaining sides of the cube, we obtain the configuration shown in fig. 3.

We can trap more than one Skyrmion on the wall, by increasing the winding of the baby-Skyrmion residing in the wall. For axially symmetric solutions, we can consider the Ansatz

\[
\mathbf{n} = \{ \sin f(x) \sin k\phi \sin g(\rho), \sin f(x) \cos k\phi \sin g(\rho), \sin f(x) \cos g(\rho), \cos f(x) \},
\]

where \( \rho, \phi \) are polar coordinates on the wall. Keeping the potential unchanged, we can write the Lagrangian

\[
-\mathcal{L} = \frac{1}{2} f_x^2 + \frac{1}{2} m_4^2 \sin^2 f + \frac{1}{2} \sin^2 f (1 + f_x^2) \left[ g_\rho^2 + \frac{k^2}{\rho^2} \sin^2 g \right] + \frac{k^2}{2 \rho^2} \sin^4 f \sin^2 (g) g_\rho^2 - \frac{1}{2} m_3^2 \cos g \sin f.
\]

The axially symmetric solutions are not stable in the planar Faddeev-Skyrme model \([3, 14]\); i.e. the baby-Skyrmions with the potential \( \frac{1}{2} m_3^2 (1 - n_3) \) will form bound states resembling a baby-Skyrme lattice \([15]\).
Although the intuition of the baby-Skyrmions on $\mathbb{R}^2$ is useful, our trapped baby-Skyrmions experience the curvature of the wall which in turn induces terms and changes the coefficients of the Lagrangian, as we illustrated above and even more so without assuming factorization in the variables. We find that the baby-Skyrmion rings are stable for baryon numbers 1, 2 and 3 (see app. C for a sequence of the relaxation of an asymmetric configuration down to the 2-ring); in fig. 4 are shown numerical solutions of higher-winding Skyrmions trapped on the wall with charge $B = 2, 3, 4$, respectively.

Table 1: Baryon charge and energy of the baby-Skyrmion in the linear potential (that of the domain wall is subtracted from the total energy).

| $B$ | $B_{\text{numerical}}$ | $E/B$         |
|-----|------------------------|---------------|
| 1   | 0.998                  | 78.06 ± 0.78  |
| 2   | 1.997                  | 73.43 ± 0.12  |
| 3   | 2.996                  | 74.09 ± 0.10  |
| 4   | 3.995                  | 75.36 ± 0.08  |

In table 1 is shown the energy of the baby-Skyrmions, which is calculated by subtracting the wall contribution from the total energy. It is observed that the three lowest baryon-charged configurations are stable (minimum energy solutions for each $B = 1, 2, 3$); the $B = 2$ solution cannot decay into two $B = 1$ solutions and the $B = 3$ solution cannot decay into a sum of the $B = 1$ and $B = 2$ solutions, whereas the $B = 4$ solution is only meta-stable; although we could find the solution, it can decay into two $B = 2$ solutions. This can happen either by quantum tunneling or at finite temperature or if some kinematic perturbations impinge.

As observed in table 1, the charge 4 baby-Skyrmion is at best meta-stable. By starting with non-axially symmetric initial conditions we can still obtain the 4-ring, but by skewing the setup enough it falls into a lower-energy state, i.e. composed by two charge-2 rings. By starting with an asymmetric initial guess, we find the configuration shown in fig. 5. We notice that the energy of this configuration is lower than that of the ring, whose energy is stated in table 1.

With this knowledge at hand, we study the $B = 5$ solution with various different initial guesses in order to find the lowest energy solution (the globally stable one). An asymmetric initial guess upon numerical relaxation gives rise to the solution shown in fig. 6, which is some sort of bound state of a genuine 2-ring and another creature carrying charge 3. If on the other hand we start with an axially symmetric initial
\( B_{\text{numerical}} = 1.997: \)

\( B_{\text{numerical}} = 2.996: \)

\( B_{\text{numerical}} = 3.995: \)

(g) isosurfaces   (h) energy density   (i) baryon charge density

Figure 4: The domain wall with trapped \( B = 2, 3, 4 \) Skyrmions. For details see fig. [1] The calculations are done on a \( 129^3 \) cubic lattice and the potential used is \( \text{(19)} \) with \( m_4 = 4, m_3 = 2 \).
Figure 5: The domain wall with a minimal lattice structure of two trapped $B = 2$ Skyrmions giving the total of a $B = 4$ configuration. For details see fig. 1. The calculation is done on an $81^3$ cubic lattice, $B_{\text{numerical}} = 3.990$, $E = 4 \times 73.07$ and the potential used is (19) with $m_4 = 4, m_3 = 2$.

guess, we find a meta-stable solution which is shown in fig. 7. The latter solution has a higher energy per charge than the asymmetric solution, so eventually it will decay to that by means of tunneling or perturbations etc. We observe that the lowest energy solution found for $B = 5$ consists of one 2-ring and the remaining charge. With table 1 in mind, we conjecture that for large $B$, the stable configurations are formed of 2-rings situated in a lattice-like structure.

Figure 6: The domain wall with a bound state of a 2-ring and a deformed 3-ring forming a $B = 5$ configuration. For details see fig. 1. The calculation is done on an $81^3$ cubic lattice, $B_{\text{numerical}} = 4.988$, $E = 5 \times 73.61$ and the potential used is (19) with $m_4 = 4, m_3 = 2$. 
Figure 7: The domain wall with a symmetric initial condition for the 5-ring which collapses to the shown bound state of single Skyrmions forming a meta-stable $B = 5$ configuration. For details see fig. 1. The calculation is done on an $81^3$ cubic lattice, $B_{\text{numerical}} = 4.98$, $E = 5 \times 75.58$ and the potential used is \[ (19) \]

with $m_4 = 4, m_3 = 2$.

4.2 Trapped Skyrmions in a quadratic potential

In this section we switch to baby Skyrmions trapped on the wall and thus exhibiting full baryon charge in the modified potential \[ (5, 6), \] namely the quadratic one

\[ V = -\frac{1}{2} m_3^2 n_3^2 + \frac{1}{2} m_4^2 (1 - n_4^2). \]  

(23)

Using the Ansatz \[ (18) \] and the above given potential, the Lagrangian \[ (1) \] yields

\[ -\mathcal{L} = \frac{1}{2} f_x^2 + \frac{1}{2} (m_4^2 - m_3^2) \sin^2 f + \frac{1}{2} \sin^2 f (1 + f_x^2) \left[ g_p^2 + \frac{1}{\rho^2} \sin^2 g \right] + \frac{1}{2} \rho^2 \sin^4 f \sin^2(g) g_p^2 \]

\[ + \frac{1}{2} m_3^2 \sin^2 f \sin^2 g, \]

(24)

from which we can identify the domain wall in $f$ with effective mass-squared $(m_4^2 - m_3^2)$ and a baby-Skyrmion with a quadratic mass term as well as an enhanced factor of $(1 + f_x^2)$ in front of its kinetic term; $|f_x| \sim m_4$ on the domain wall.

In principle the above effective Lagrangian should allow for a ring-like baby-Skyrmion configuration; the problem however is this. The kinetic term is a marginal term, whereas the Skyrme term yields a pressure. With the enhanced prefactor of the kinetic term, the generic configuration is a baby-Skyrmion with a peak in both the energy distribution and the baryon charge distribution. Since the enhancement comes about due to the steepness of the domain wall in $f$, i.e. it is proportional to $m_4^2$, we can lower this mass and in turn increase the width of the domain wall. The condition however that $m_3 \ll m_4$ means that the mass term in the baby-Skyrmion effective theory is becoming very small. This mass term is needed for the baby-Skyrmion to exist on $\mathbb{R}^2$. Although we do not have a proof of non-existence for the ring-like structure at present, we have not been able to find any such configuration numerically, scanning over a range of parameters.

For completeness we obtain numerical solutions for the single Skyrmion trapped on the domain wall in the quadratic potential \[ (23) \] and it is shown in fig. 8. It is observed that the top of the energy density is more flattened out with respect to the Skyrmion in the linear potential (see fig. 3).
Since as explained above, the single charge Skyrmion configuration does not differ much from the that with a linear potential, we turn to the higher-winding ones. They are stable ring-like configurations similar in nature to the ones in the linear potential. One twist, however, is possible here as compared to the case with the linear potential, namely that we can add a mass term for \( n_2 \), breaking rotational symmetry on the wall ring. The potential is

\[
V = -\frac{1}{2} m_2^3 n_2 - \frac{1}{2} m_3^3 n_3 + \frac{1}{2} m_4^2 (1 - n_4^2),
\]

and if we plug this and the Ansatz (18) into the Lagrangian (1), we get

\[
-\mathcal{L} = \frac{1}{2} f_x^2 + \frac{1}{2} (n_1^2 - n_4^2) \sin^2 f + \frac{1}{2} \sin^2 f (1 + f_x^2) \left[ g_x^2 + \frac{k^2}{\rho^2} \sin^2 g \right] + \frac{k^2}{2\rho^2} \sin^4 f \sin^2(g) g_x^2
\]

\[
+ \frac{1}{2} m_3^3 \sin^2 f \sin^2 g - \frac{1}{2} m_2^3 \sin f \sin g \sum_{j=0}^{k} \left( \begin{array}{c} k \\ j \end{array} \right) \frac{y_j^2 z^{k-j}}{\rho^k} \cos \left( \frac{\pi}{2} (k - j) \right).
\]

Identifying the pieces of the above Lagrangian one-by-one we have the domain wall in \( f \) with effective mass-squared \((m_4^2 - m_3^2)\) and a baby-Skyrmion with mass \( m_3^2 \) and an enhanced kinetic term by a factor of \((1 + f_x^2)\). Finally, there is an angle-dependent mass term (with the angle being the usual one in polar coordinates on the wall) inducing \( k \) sine-Gordon kinks on top of the ring-like baby-Skyrmion structure. This type of object has been already realized in the planar case (i.e. in 2 + 1 dimensions) in the literature, see e.g. [16].

After being in the right mind-set, we can obtain the numerical solutions, which we again find using the generic field \( \mathbf{n}(x, y, z) \), subject to the boundary condition (15) and Neumann conditions in the other two directions. The solutions are shown in figs. 9 and 10.

For completeness we obtain also the single Skyrmion in the potential (25). The linear mass term for \( n_2 \) skews the Skyrmion configuration. For comparison we show the energy density with that of fig. 8 being a transparent shade in the background, see fig. 11.

In table 2 is shown the energy of the baby-Skyrmion, which is calculated by subtracting the wall contribution from the total energy. All the ring-like solutions we have found are stable (i.e. minimum energy configurations).
Figure 9: The domain wall with 2, 3, 4 sine-Gordon kinks living on trapped $B = 2, 3, 4$ Skyrmions. For details see fig. The calculations are done on $129^3, 129^3, 81^2$ cubic lattices and the potential used is \[ \text{with } m_4 = 4, m_3 = 2, m_2 = 1. \]
$B^{\text{numerical}} = 4.992$:

Figure 10: The domain wall with 5 sine-Gordon kinks living on a trapped $B = 5$ Skyrmion. For details see fig. 1. The calculation is done on an $81^3$ cubic lattice and the potential used is (25) with $m_4 = 4$, $m_3 = 2$, $m_2 = 1$.

Figure 11: The domain wall with a trapped Skyrmion in a quadratic potential (with a kink mass). The energy density and baryon charge density on the middle of the wall ($x = 0$) the Skyrmion in the potential (25) are shown with those of fig. 8 as a reference in a transparent shade. The calculation is done on an $81^3$ cubic lattice and $m_4 = 4$, $m_3 = 2$, $m_2 = 1$. 
Table 2: Baryon charge and energy of the baby-Skyrmion in the quadratic potential including the energy of the kinks living on the Skyrmion-ring (the energy of the domain wall is subtracted from the total energy).

\[
m_2 = 1 \text{ (with kinks)}
\]

| B     | 1    | 2    | 3    | 4    | 5    |
|-------|------|------|------|------|------|
| B^{\text{numerical}} | 0.985 | 1.996 | 2.991 | 3.992 | 4.992 |
| E/B   | 74.11 | 69.32 ± 0.06 | 68.05 ± 0.03 | 67.79 | 67.66 |

\[
m_2 = 0 \text{ (without kinks)}
\]

| B     | 1    | 2    | 3    | 4    | 5    |
|-------|------|------|------|------|------|
| B^{\text{numerical}} | 0.998 | 1.995 | 2.990 | 3.992 | 4.992 |
| E/B   | 76.18 ± 0.12 | 69.28 | 67.93 | 67.61 | 67.47 |

5 Summary and Discussion

In this paper we have studied domain walls with entrapped Skyrmions in various guises. The first ones are full and half-charged Skyrmions manifested as kinks on domain lines carrying 3-dimensional (full and half) Skyrmion charge. This line of systems is interesting because it is not dependent on having a Skyrme term and thus might find applications in condensed-matter systems. The next kind of configurations is a domain wall with endowed baby-Skyrmions carrying again full Skyrmion (baryon) charge. This type of configuration needs both the Skyrme term as well as a mass term. The mass term can be of any type, which from the point of view of the pion is unimportant. From a topological point of view, however, it matters. The two potentials under study in this paper are the traditional linear potential and the quadratic potential which allows for a domain wall. In the planar baby-Skyrmion model it is known that the linear potential does not allow for stable \( B > 1 \) configurations other than the lattice whilst the quadratic potential equips the model with stable ring-like configurations [14].

For the quadratic potential we find as in the planar case, that the higher-winding baby-Skyrmions resident on the domain wall are indeed stable for the baryon numbers we have investigated, i.e. \( B = 1 − 5 \). However, for the linear potential we find ourselves with a surprise; the stable multi-charged configuration is made of charge 2 and 3 ring-like configurations. We conjecture that the large charge configuration eventually becomes a baby-Skyrmion lattice composed of charge-2 ring-like object as depicted in fig. 5.

Another comment in store is due the backreaction of the soliton living on the domain wall. In the case of the Skyrmion and half-Skyrmion trapped on the domain line living on the domain wall, we observe a drop in the energy density on both sides of the peak. We interpret this as a binding energy of the Skyrmion.

We have seen in this paper that stable baby-Skyrmion configurations in baby-Skyrme theories induced on a domain wall are quite different for the linear potential \(-n_3\) compared to those of the corresponding planar theory, whereas for the quadratic potential \(-n_3^2\), the induced theories and their corresponding planar theories are qualitatively alike. Structures and dynamics of baby-Skyrmions in the planar \((2+1)\)-dimensional case with other choices of potentials have been studied thus far in e.g. refs. [17,19]. Studying stable configurations in these models induced on a domain wall will be interesting to explore. In particular, the potential \(-n_3^2\) will be the most interesting one because it is quite common in condensed-matter systems and admits molecule-type structures [17,18].

Another interesting possibility which we have left for future work is to consider a compactification of the domain wall to a sphere along the lines of ref. [20], which is possible by changing the sign of the Skyrme term and augment the theory by a sextic derivative term; in this case the domain line with sine-Gordon
kinks can still exist.

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A Numerical relaxation

We use a rather simple relaxation on a cubic lattice with an imaginary time that cools the configuration down to a state where the equations of motion are satisfied and the topological charge contained is measured as a check. The configurations are calculated on $129^3$ (or $81^3$) spatial lattices with a second order finite difference method and cooled in the imaginary time direction with a forward time step (FTS) method and time step of size $\lesssim (0.3 - 0.5)\min(h_x^4, h_y^4, h_z^4)$. This choice was made for the simplicity in the implementation. Our code is implemented in C++.

B Skyrmion and half-Skyrmion on a domain line on the domain wall without the Skyrme term

In figs. [12] and [13] are shown the configurations containing the domain wall with a domain line endowing a full Skyrmion or a half-Skyrmion, respectively. The figures are shown for the purpose of possible interest in the configurations not enjoying the Skyrme term, for instance in condensed-matter physics. We observe a pair of “lines” coming out from the Skyrmion in fig. [12] but we do not have an explanation for their presence.

![Figure 12: The domain wall with a domain line on which a Skyrmion resides in the theory without the Skyrme term. The calculation is done on a $129^4$ cubic lattice, $B_{\text{numerical}} = 0.9995$ and the potential used is (13) with $m_4 = 4, m_3 = 2, m_2 = 1.$](image)
Figure 13: The domain wall with a domain line on which half a Skyrmion resides. The calculation is done on a $129^3$ cubic lattice, $B^{\text{numerical}} = 0.49993$ and the potential used is (16) with $m_4 = 4, m_3 = 2, m_2 = 1$.

C Asymmetric initial condition for a double-winding Skyrmion in a linear potential

In fig. 14 we show a series of energy contours of the configuration in the middle of the domain wall (at $x = 0$) as function of relaxation time $\tau$ until the equations of motion are satisfied to the required accuracy level. We provide this series of configurations in order to show that the initial guess is not a symmetric state that relaxes to a symmetric configuration. We start off with a very asymmetric state and find a ring-like solution.

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Figure 14: Contours of constant energy in the $yz$-plane of the double-winding baby-Skyrmion on a wall (at $x = 0$) in a linear potential as function of relaxation time $\tau$. It is observed that the two separated baby-Skyrmions attract each other and find a minimum of the energy in a ring-like solution. This calculation was done on an $81^3$ cubic lattice and the potential used is (19) with $m_4 = 4, m_3 = 2$. 

17
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