

Topologies on a triangulated category

So Okada∗

March 30, 2022

Abstract

On objects of a triangulated category with a stability condition, we construct a topology.

1 Introduction

For any triangulated category, Bridgeland [Brid] introduced the notions of stability conditions and stability manifolds, motivated by the Douglas’ work [Dou01a, Dou01b, Dou02] on II-stabilities of D-branes for Calabi-Yau manifolds.

In particular, for a K3 surface $X$ and the bounded derived category of coherent sheaves on $X$, denoted by $D(X)$, Bridgeland [Bric] proved that stability conditions on the stability manifold approximate Gieseker stabilities [Gie, Mar77, Mar78].

Roughly speaking in terms of string theory, for a Calabi-Yau manifold $X$, objects of $D(X)$ correspond to B-branes among D-branes, that are boundary conditions of open strings. For a stability condition, among B-branes, semistable objects correspond to BPS states, that recognize themselves as D-branes in the untwisted topological field theory. A physical quantity of each BPS state is called a central charge. The author recommends [Asp] for string theory related to this subject.

For a triangulated category $T$, stability conditions on the stability manifold describe variation of $t$-structures of $T$, in collaboration with other subjects. The author recommends [Bria], [Huy, Section 13], [Bri06] for introductions to this subject.

In some general settings, for a triangulated category, we have seen spectra [Ros, Section 12], [Bal] and moduli spaces of objects with vanishing conditions on extension groups [Ina02], [Lie], [Ina].

In this article, on objects of a triangulated category with a stability condition, we construct a topology induced from the central charge. Also, with a faithful or a numerically faithful stability condition, we find our topology compatible with the Grothendieck group or the numerical Grothendieck group. We

∗Address: Max-Planck-Institut für Mathematik, Vivatsgasse 7, Bonn Germany 53111, Email: okada@mpim-bonn.mpg.de
realize that objects of a triangulated category with a stability condition is a
connected space.

Acknowledgments
The author thanks I. Mirković and T. Mochizuki for their discussions. The
author is grateful to the Max-Planck-Institut für Mathematik for their excellent
support and working environment and to Bonner for their warm hospitality.

2 Stability conditions and stability manifolds
Throughout this paper, \( \mathcal{T} \) is a triangulated category such that for any objects
\( E \) and \( F \) of \( \mathcal{T} \), \( \oplus_{i \in \mathbb{Z}} \text{Hom}^i_{\mathcal{T}}(E, F) \) is a finite-dimensional vector space over \( \mathbb{C} \).
For example, \( \mathcal{T} \) can be \( D(X) \) of a smooth projective variety \( X \) over \( \mathbb{C} \).

In the Grothendieck group \( K(\mathcal{T}) \) of \( \mathcal{T} \) and for each object \( E \) in \( \mathcal{T} \), let \( [E] \) be
the class of \( E \). For objects \( E \) and \( F \) of \( \mathcal{T} \), the Euler paring \( \chi(E, F) \) is defined
to be \( \sum_{i \in \mathbb{Z}} (-1)^i \text{dim} \text{Hom}^i_{\mathcal{T}}(E, F) \). On this paring, the
numerical Grothendieck

2.1 Stability conditions

A stability condition \( \sigma = (Z, \mathcal{P}) \) on \( \mathcal{T} \) consists of a group homomorphism from
\( K(\mathcal{T}) \) to \( \mathbb{C} \), called a central charge \( Z \), and a family of full abelian subcate-
gories of \( \mathcal{T} \), called a slicing \( \mathcal{P}(k) \), indexed by real numbers \( k \), with the following
conditions.

For each real number \( k \), if \( E \) is an object of \( \mathcal{P}(k) \), then for some positive
real number \( m(E) \), called the mass of \( E \), we have \( Z(E) = m(E) \exp(i\pi k) \).
For each real number \( k \), we have \( \mathcal{P}(k + 1) = \mathcal{P}(k)[1] \). For any real numbers
\( k_1 > k_2 \) and any objects \( E_i \) of \( \mathcal{P}(k_i) \), \( \text{Hom}_{\mathcal{T}}(E_1, E_2) \) is the zero vector space.
For any nonzero object \( E \) of \( \mathcal{T} \), there exists a finite sequence of real numbers
\( k_1 > \cdots > k_n \) and objects \( H^k_{\mathcal{P}}(E) \) of \( \mathcal{P}(k_i) \) such that there exists a sequence of
exact triangles \( E_{i-1} \to E_i \to H^k_{\mathcal{P}}(E) \) with \( E_0 \) and \( E \) being the zero object and
\( E \).

The above sequence of the exact triangles is unique up to isomorphisms and
called the Harder-Narasimhan filtration of \( E \); also, we call a real number in
the above sequence of the real numbers a nontrivial phase of \( E \). For each real
number \( k \), any nonzero object of \( \mathcal{P}(k) \) is called semistable.

If the central charge \( Z \) factors through \( N(\mathcal{T}) \), then \( \sigma \) is called a numerical
stability condition.

2.2 Hearts of stability conditions

For an interval \( I \) in real numbers, \( \mathcal{P}(I) \) is defined to be the smallest full subcat-
egory of \( \mathcal{T} \) consisting of objects of \( \mathcal{P}(k) \) for each real number \( k \) in \( I \), it is closed
under extension; i.e., if $E \rightarrow G \rightarrow F$ is an exact triangle in $\mathcal{T}$ and both $E$ and $F$ are objects of $\mathcal{P}(I)$, then $G$ is an object of $\mathcal{P}(I)$. In particular, for each real number $j$, $\mathcal{P}((j-1, j])$ is a heart of a bounded $t$-structure of $\mathcal{T}$. We will call all $\mathcal{P}((j-1, j])$ for real numbers $j$, “hearts of $\sigma$”.

For each nonzero object $E$ of $\mathcal{P}((j-1, j])$, the phase of $E$ is defined to be $\phi(E) = (1/\pi) \arg Z(E) \in (j-1, j]$.

### 2.3 Stability manifolds

A subset of stability conditions on $\mathcal{T}$ makes the stability manifold $\text{Stab}(\mathcal{T})$, this has a natural topology induced from the central charges and each connected component is a manifold locally modeled on some topological vector subspace of $\text{Hom}_{\mathbb{Z}}(K(\mathcal{T}), \mathbb{C})$.

The subset of $\text{Stab}(\mathcal{T})$ consisting of numerical stability conditions makes a subspace, the numerical stability manifold $\text{Stab}_{\text{N}}(\mathcal{T})$, this is locally modeled on some topological vector subspace of $\text{Hom}_{\mathbb{Z}}(N(\mathcal{T}), \mathbb{C})$.

### 3 Faithful or numerically faithful stability conditions

Let $K(\mathcal{T})_\mathbb{Q}$ and $N(\mathcal{T})_\mathbb{Q}$ denote the tensor products $K(\mathcal{T}) \otimes \mathbb{Q}$ and $N(\mathcal{T}) \otimes \mathbb{Q}$.

**Definition 3.1.** Let $\sigma$ be a stability condition on $\mathcal{T}$. We call $\sigma$ faithful, if whenever nonzero objects $E$ and $F$ of a heart of $\sigma$ are linearly independent in $K(\mathcal{T})_\mathbb{Q}$, then objects $E$ and $F$ are with different phases. Likewise, we call $\sigma$ numerically faithful, if whenever nonzero objects $E$ and $F$ of a heart of $\sigma$ are linearly independent in $N(\mathcal{T})_\mathbb{Q}$, then $E$ and $F$ are with different phases.

In any stability manifolds that we are aware of, by the following lemma, there are faithful or numerically faithful stability conditions.

**Lemma 3.2.** If $K(\mathcal{T})_\mathbb{Q}$ has no more than countable dimension over $\mathbb{Q}$, and if $\text{Stab}(\mathcal{T})$ carries a connected component $M$ that is locally isomorphic to $\text{Hom}_{\mathbb{Z}}(K(\mathcal{T}), \mathbb{C})$, then the subset of $M$ consisting of faithful stability conditions is dense in $M$. Likewise, if $N(\mathcal{T})_{\mathbb{Q}}$ has no more than countable dimension over $\mathbb{Q}$, and if $\text{Stab}_{\text{N}}(\mathcal{T})$ carries a connected component $M$ that is locally isomorphic to $\text{Hom}_{\mathbb{Z}}(N(\mathcal{T}), \mathbb{C})$, then the subset of $M$ consisting of numerically faithful stability conditions is dense in $M$.

**Proof.** Let $T$ be the subset of $\text{Hom}_{\mathbb{Z}}(K(\mathcal{T}), \mathbb{C})$ consisting of $Z$ such that for some linearly independent classes $[E]$ and $[F]$ in $K(\mathcal{T})_\mathbb{Q}$, in the interval $(0, 2]$, real numbers $(1/\pi) \arg Z(E)$ and $(1/\pi) \arg Z(F)$ are the same. Then, $T$ is a countable union of codimension-one subspaces of $\text{Hom}_{\mathbb{Z}}(K(\mathcal{T}), \mathbb{C})$. Thus, the complement of $T$ in $\text{Hom}_{\mathbb{Z}}(K(\mathcal{T}), \mathbb{C})$ is dense in $\text{Hom}_{\mathbb{Z}}(K(\mathcal{T}), \mathbb{C})$. The same argument holds for the latter case. □

In particular, we have the following.
Corollary 3.3. For a stability condition \( \sigma \) of \( T \), let \( E \) and \( F \) be nonzero objects of a heart of \( \sigma \) with the same phases. If \( \sigma \) is a faithful stability condition, then for some positive rational number \( q \), \( [E] = q[F] \) in \( K(\mathcal{T}) \). If \( \sigma \) is a numerically faithful stability condition, then for some positive rational number \( q \), \( [E] = q[F] \) in \( N(\mathcal{T}) \).

Proof. Since \( \sigma = (Z, P) \) is faithful, \( [E] \) and \( [F] \) are linearly dependent in \( K(\mathcal{T}) \). So there exists a nonzero rational number \( q \) such that \( [E] = q[F] \) in \( K(\mathcal{T}) \). Here, \( q \) can not be zero, since \( E \) and \( F \) are nonzero objects of a heart of \( \sigma \), which implies \( Z(E) \) and \( Z(F) \) are not zero. Also, \( q \) can not be negative; otherwise, their phases would differ by an odd integer. The same argument holds for the latter case.

4 Topologies on a triangulated category

We notice that for a stability condition \( (Z, P) \) on \( T \) and each semistable object of \( T \), the central charge \( Z \) factors through \( \mathbb{P}^1 \); i.e., for each semistable object \( E \) of \( T \), its central charge is the exponential of the complex number \( \log(m(E)) + i\pi\phi(E) \) in \( \mathbb{P}^1 \). Now, we define the following.

Definition 4.1. Let \( \sigma = (Z, P) \) be a stability condition on \( T \). For a nonzero object \( E \) of \( T \), let \( \tilde{Z}(E) \) be the subset of \( \mathbb{P}^1 \) consisting of points \( \log(m(H^k_Z(E))) + i\pi\phi(H^k_Z(E)) \) for nontrivial phases \( k \) of \( E \). For the zero object, let the image of \( \tilde{Z} \) be the infinite point in \( \mathbb{P}^1 \). We call the function \( \tilde{Z} \) the extended central charge of \( \sigma \).

For our arguments here, let us assume the Euclidean topology on \( \mathbb{P}^1 \).

Proposition 4.2. For a triangulated category with a stability condition, there exists a unique topology on objects of the triangulated category such that the extended central charge is continuous.

Proof. Let \( \mathcal{T} \) be a triangulated category with a stability condition \( \sigma = (Z, P) \), and \( \tilde{Z} \) be the extended central charge. Then, finite unions of \( \tilde{Z}^{-1}(V) \) for closed subsets \( V \) of \( \mathbb{P}^1 \) are our closed subsets of objects of \( \mathcal{T} \).

On the topology in Proposition 4.2 we have the following corollaries.

Corollary 4.3. For a semistable object \( E \) of \( \mathcal{T} \), any object of the closed set \( \tilde{Z}^{-1}(\tilde{Z}(E)) \) is semistable.

Proof. The Harder-Narasimhan filtration of any object of \( \tilde{Z}^{-1}(\tilde{Z}(E)) \) is trivial, since its image under \( \tilde{Z} \) is a point in \( \mathbb{P}^1 \).

In particular, we have the following.

Corollary 4.4. For a faithful stability condition \( (Z, P) \) on \( \mathcal{T} \) and a semistable object \( E \) of \( \mathcal{T} \), any object of the closed set \( \tilde{Z}^{-1}(\tilde{Z}(E)) \) is semistable and has the class \( [E] \) in \( K(\mathcal{T}) \). For a numerically faithful stability condition \( (Z, P) \) on \( \mathcal{T} \) and a semistable object \( E \) of \( \mathcal{T} \), any object of the closed set \( \tilde{Z}^{-1}(\tilde{Z}(E)) \) is semistable and has the class \( [E] \) in \( N(\mathcal{T}) \).
4.1 Connectedness

Lemma 4.5. Let $\mathcal{T}$ be a triangulated category with a stability condition. On the topology in Proposition 4.2, if an open subset $U$ of objects of $\mathcal{T}$ contains the zero object of $\mathcal{T}$, then $U$ contains some semistable objects.

Proof. Let $(Z, \mathcal{P})$ denote the stability condition. Let $\tilde{Z}$ be the extended central charge. Now, the open subset $U$ is a complement of the union $\tilde{Z}^{-1}(V_1) \cup \cdots \cup \tilde{Z}^{-1}(V_n)$ for some closed subsets $V_i$ of $\mathbb{P}^1$. Since the zero object is not in $U$, there is an open subset $U'$ of $\mathbb{P}^1$ such that $\tilde{Z}^{-1}(U')$ is a subset of $U$ consisting of the infinite point of $\mathbb{P}^1$.

Here, since for any semistable object $E$ of $\mathcal{T}$, the direct sum $E \oplus E$ is again semistable with the doubled mass, so $\tilde{Z}^{-1}(U')$ contains some semistable objects.

Lemma 4.6. Let $\mathcal{T}$ be a triangulated category with a stability condition. On the topology constructed in Proposition 4.2 if a proper open subset $U$ of objects of $\mathcal{T}$ contains all semistable objects of $\mathcal{T}$, then the open set $U$ does not contain the zero object.

Proof. For closed subsets $V_i$ of $\mathbb{P}^1$ such that $U = \tilde{Z}^{-1}(V_1) \cup \cdots \cup \tilde{Z}^{-1}(V_n)$, by the assumption on $U$, there is no semistable object $E$ of $\mathcal{T}$ such that for some $V_i$, $\tilde{Z}(E)$ is in $V_i$. So, by Definition 4.1 either $\tilde{Z}^{-1}(V_i)$ is empty or of only the zero object.

Theorem 4.7. For any triangulated category with a stability condition, on the topology in Proposition 4.2 objects of the triangulated category is a connected space

Proof. Let $\mathcal{T}$ be a triangulated category with a stability condition $(Z, \mathcal{P})$, and $\tilde{Z}$ be the extended central charge. Let us prove the statement by a contradiction. So we suppose that some open subsets $U_1$ and $U_2$ of objects of $\mathcal{T}$ separate objects of $\mathcal{T}$.

For the former case, the direct sum $E_1 \oplus E_2$ is not a object of $U_1 \cup U_2$.

For the latter case, if the open set $U_1$ contains all semistable objects, then by Lemma 4.6, $U_2$ is an open subset consisting of the zero object. However, then by Lemma 4.5, $U_1$ and $U_2$ can not be disjoint.

Remark 4.8. For any triangulated category with a stability condition, the extended central charge naturally factors through the Ran’s space [BeiDri, 3.4] of $\mathbb{P}^1$ with respect to a topology on $\mathbb{P}^1$. 

Proof. By Corollaries 3.3 and 4.3 these statements hold. 

\[\square\]
4.2 Closed sets

For a faithful stability condition \( \sigma = (Z, \mathcal{P}) \) on \( \mathcal{T} \) and some semistable objects \( E \) of \( \mathcal{T} \), let us take a look at closed sets \( \tilde{Z}^{-1}(\tilde{Z}(E)) \) in Corollary 4.4. For a vector space \( V \), let \( V^* \) denote the dual of \( V \). From here, let the shift [2] be the Serre functor [BonKap] Definition 3.1] of \( \mathcal{T} \); i.e., for any objects \( E \) and \( F \) of \( \mathcal{T} \), there exist bifunctorial isomorphisms \( \Psi_{E,F} : \text{Hom}_\mathcal{T}(E,F) \cong \text{Hom}_\mathcal{T}(F,E[2])^* \) such that \((\Psi_{E,F}^{-1}|_{E[2]})^* \circ \Psi_{E,F} \) coincides with the isomorphism induced by [2]. In other words, \( \mathcal{T} \) is a 2-Calabi-Yau category [Kon].

If an object \( E \) of \( \mathcal{T} \) satisfies that for any integer \( i \) other than 0 or 2, \( \text{Hom}_\mathcal{T}^1(E,E) \) is the zero vector space and \( \text{Hom}_\mathcal{T}(E,E) \cong \text{Hom}_\mathcal{T}^2(E,E)^* = C \), then \( E \) is called spherical [SeiTho] Definition 1.1].

For semistable objects \( E \) and \( E' \) of \( \mathcal{T} \) with the same phases, since they are of a heart of \( \mathcal{T} \), for any negative integer \( i \), \( \text{Hom}_\mathcal{T}^1(E,E') \) is the zero vector space, and since \( \text{Hom}_\mathcal{T}^1(E,E') \cong \text{Hom}_\mathcal{T}^{-1}(E',E)^* \), for any integer \( i > 2 \), \( \text{Hom}_\mathcal{T}^i(E,E') \) is the zero vector space.

Let us recall that for a stability condition \( (Z, \mathcal{P}) \) on \( \mathcal{T} \), a semistable object \( E \) of \( \mathcal{P}(\phi(E)) \) is called stable if it has no nontrivial subobject of the abelian category \( \mathcal{P}(\phi(E)) \). In particular, any stable object \( E \) of \( \mathcal{T} \) satisfies \( \text{Hom}_\mathcal{T}(E,E) = C \). Since now \( \mathcal{T} \) is 2-Calabi-Yau, a simple case of the paring in [ReiVan] Proposition 1.1.4] tells us that \( \chi(E,E) \) is even [Oka] Remark 3.7].

Each stability condition in \( \text{Stab}(\mathcal{T}) \) has the property called locally-finiteness; in particular, for any locally-finite stability condition and a semistable object \( E \), in \( \mathcal{P}(\phi(E)) \), we have a Jordan-Hölder decomposition whose composition factors are stable objects, called stable factors of \( E \).

Then, in our formality, we realize a part of [Muk] Corollary 3.6].

**Proposition 4.9.** Let \( \sigma = (Z, \mathcal{P}) \) be a faithful or numerically faithful stability condition in \( \text{Stab}(\mathcal{T}) \) and \( E \) be a semistable object of \( \mathcal{T} \) such that \( \chi(E,E) \) is positive. Then, \( \tilde{Z}^{-1}(\tilde{Z}(E)) \) contains only \( E \), and \( E \) is a direct sum of a stable spherical object.

**Proof.** Any stable object \( S \) of \( \mathcal{P}(\phi(E)) \) is spherical; because, \( \chi(S,S) = 2 - \text{Hom}_\mathcal{T}^1(S,S) \) is even and by Corollary 3.3 for some positive rational number \( q \), \([S] = q[E] \), which implies \( \chi(S,S) = q^2 \chi(E,E) \) is positive. If we have nonisomorphic stable objects \( S_1 \) and \( S_2 \) in \( \mathcal{P}(\phi(E)) \), then \( \text{Hom}_\mathcal{T}(S_1,S_2) \) and \( \text{Hom}_\mathcal{T}^2(S_1,S_2) \cong \text{Hom}_\mathcal{T}(S_2,S_1)^* \) would be the zero vector spaces, and then, \( \chi(S_1,S_2) = -\dim \text{Hom}_\mathcal{T}^1(S_1,S_2) \) would not be positive. However, since by Corollary 3.3 for some positive rational numbers \( q_1 \) and \( q_2 \), we have \([E] = q_1[S_1] = q_2[S_2]\), \( \chi(S_1,S_2) = q_1q_2 \chi(E,E) \) is positive.

Extensions of a spherical object are direct sums of the spherical object, so the statement follows.

4.2.1 An example

Let \( X \) be the cotangent bundle of \( \mathbb{P}^1 \). From here, let \( \mathcal{T} \) be the full subcategory of \( D(X) \) consisting of objects supported over \( \mathbb{P}^1 \).
Since $K(T) \cong \mathbb{Z}[O_x] \oplus \mathbb{Z}[O_{y}]$ with $\chi(O_{y}, O_{y}) = 2$ and $[O_x]$ being zero in $N(T),$ for any object $E,$ $\chi(E, E)$ is not negative. So, by Proposition 4.9 we take semistable objects $E$ with $[E]$ being zero in $N(T)$ and look at $Z^{-1}(Z(E)).$

For a spherical object $E$ and an object $F$ of $\mathcal{T},$ the cone of the evaluation map $R\text{Hom}_T(E, F) \otimes E \to F$ is denoted by $T_F(E),$ the twist functor of $E.$ By [SeiTho Theorem 1.2], twist functors are autoequivalences of $\mathcal{T}.$ Any autoequivalence $\Phi$ of $\text{Stab} \left( \mathcal{T} \right)$, for any object $E$ with $[E]$ being zero in $N(T)$ and look at $Z^{-1}(Z(E)).$

Here, for an object $E$ of $\mathcal{T},$ if for any point $x$ in $\mathbb{P}^1$ and any integer $i,$ $\text{Hom}_{\mathcal{T}}^i(E, O_x)$ is the zero vector space, then $E$ is isomorphic to the zero object; for the largest integer $q_0$ such that the support of the cohomology sheaf $H^q_{\mathcal{T}}(E)$ has a point $x$ in $\mathbb{P}^1,$ the nonzero term $E^q_{\mathcal{T}}$ in the spectral sequence $E^q_{\mathcal{T}} = \text{Hom}_{\mathcal{T}}^q(H^q_{\mathcal{T}}(E), O_x) \Rightarrow \text{Hom}_{\mathcal{T}}^{q_{\mathcal{T}}}(E, O_x)$ survives at infinity (see [Bri99 Section 2]).

Let us recall that the connectedness of $\text{Stab} \left( \mathcal{T} \right)$ in [Oka Theorem 4.12] follows by proving that for any stability condition $\sigma$ in $\text{Stab} \left( \mathcal{T} \right)$ and for some integer $w,$ objects $O_{\mathcal{T}}(w)$ of $\mathcal{T}$ generate a heart of $\sigma$ and by using [Bri9] Lemmas 3.1 and 3.6, Theorem 1.3, as explained in [Oka Section 4.3].

Now, to prove Proposition 4.10 one way is to use [IshUeh] Proposition 18 and Corollary 20; for our conclusion on this case, we give a proof.

**Proposition 4.10.** Let $\sigma = (Z, \mathcal{P})$ be a faithful stability condition in $\text{Stab} \left( \mathcal{T} \right).$ For each semistable object $E$ of $\mathcal{T}$ with $[E] = [O_x],$ up to autoequivalences of $\mathcal{T},$ $Z^{-1}(Z(E)) = \{ O_x \mid x \in \mathbb{P}^1 \}.$

**Proof.** As mentioned above, for some integer $w,$ by extensions, objects $O_{\mathcal{T}}(w)$ generate a heart of $\sigma.$ For each point $x$ in $\mathbb{P}^1,$ by the exact triangle $O_{\mathcal{T}}(w) \to O_x \to O_{\mathcal{T}}(w - 1),$ the object $O_x$ is of the heart and, since $\sigma$ is faithful, phases of $O_{\mathcal{T}}(w)$ and $O_{\mathcal{T}}(w - 1)$ are distinct.

First, let us suppose $\phi(O_{\mathcal{T}}(w)) \supset \phi(O_{\mathcal{T}}(w - 1)),$ and $\phi(O_{\mathcal{T}}(w - 1)),$ and $\phi(O_{\mathcal{T}}(w))$ are distinct. With classes of objects of the heart, $[O_x]$ can be represented by only $[O_{\mathcal{T}}(w - 1)] + [O_{\mathcal{T}}(w)],$ if it is other than $[O_x].$ So, if $O_x$ were not semistable, by the assumption on phases, the Harder-Narasimhan filtration of $O_x$ must give the exact triangle $O_{\mathcal{T}}(w - 1) \to O_x \to O_{\mathcal{T}}(w);)$ however, $\text{Hom}_{\mathcal{T}}(O_{\mathcal{T}}(w - 1), O_x)$ is the zero vector space, since the objects $O_{\mathcal{T}}(w - 1)$ and $O_x$ are of a heart of $\mathcal{T}$ such as the category of the coherent sheaves of $X$ supported over $\mathbb{P}^1.$

Now, any object of $Z^{-1}(Z(E))$ is stable; otherwise, by Corollary 5.3 for some rational number $q > 1$ and a stable object $S,$ we would have $[O_x] = q[S].$

For a point $x$ in $\mathbb{P}^1$ and the integer $k = \phi(E) - \phi(O_x),$ we show that for each object $E'$ of $Z^{-1}(Z(E))$ and some point $y$ in $\mathbb{P}^1,$ $E'$ is isomorphic to the object $O_y.$ Here, for some object $E'$ in $Z^{-1}(Z(E)),$ we suppose otherwise and show that for any point $x$ in $\mathbb{P}^1$ and any integer $i,$ $\text{Hom}_{\mathcal{T}}^i(E', O_x)$ is the zero vector space. For any point $x$ in $\mathbb{P}^1,$ since the objects $E'$ and $O_x$ are stable with the same phases, $\text{Hom}_{\mathcal{T}}(E', O_x)$ and $\text{Hom}_{\mathcal{T}}(O_x, E')$ are the
zero vector spaces; so, $\text{Hom}_T^2(E', O_x) \cong \text{Hom}_T(\mathcal{O}_x, E')^*$ is also the zero vector space. Since the objects $E'$ and $\mathcal{O}_x$ are of the heart of $\sigma$, for any negative integer $i$, $\text{Hom}_T^i(E', \mathcal{O}_x)$ is the zero vector space, and also for any integer $i > 2$, $\text{Hom}_T^i(E', \mathcal{O}_x) \cong \text{Hom}_T^{2-i}(\mathcal{O}_x, E')^*$ is the zero vector space. Now, since $\sigma$ is faithful, $\chi(\mathcal{O}_x, \mathcal{O}_x) = \chi(E', \mathcal{O}_x) = -\dim \text{Hom}_T^1(E', \mathcal{O}_x)$ is zero.

For the other case, we apply the twist functor $T_{\mathcal{O}_{P^1}(w-1)}$ and use the above argument; by [IshUeh, Lemma 4.15 (i)(1)], $T_{\mathcal{O}_{P^1}(w-1)}(\mathcal{O}_{P^1}(w-1)[1]) = \mathcal{O}_{P^1}(w-1)$ and $T_{\mathcal{O}_{P^1}(w-1)}(\mathcal{O}_{P^1}(w)) = \mathcal{O}_{P^1}(w-2)[1]$. 

Before discussing the other cases, let us introduce some notions. For each semistable object $E$ and a Jordan-Hölder decomposition $E \supset E_1 \supset \cdots \supset E_n \supset 0$ in $\mathcal{P}(\phi(E))$, the cycle of simple components [Ses, Section 2] associated to the decomposition is defined to be $E_n \oplus E_{n-1}/E_{n-2} \oplus \cdots \oplus E/E_1$, which is by [Ses, Theorem 2.1], up to isomorphisms, independent of the choices of Jordan-Hölder decompositions. Semistable objects with isomorphic cycles of simple components are said to be S-equivalent [Ses, Remark 2.1], [Gie, Section 0].

Now, for some integer $|n| > 1$, a semistable object $E$ with $[E] = n[\mathcal{O}_x]$ in $K(T)$, the integer $k = \phi(E) - \phi(\mathcal{O}_x)$, and any object $E'$ of $\tilde{Z}^{-1}(\tilde{Z}(E[-k]))$, since for some point $x$ in $\mathbb{P}^1$, $\text{Hom}_T(E', \mathcal{O}_x)$ is not zero, $E'$ has a Jordan-Hölder decomposition whose composition factors are $\mathcal{O}_x$ for points $x$ in $\mathbb{P}^1$. So, up to S-equivalence on $\tilde{Z}^{-1}(\tilde{Z}(E))$ and autoequivalences on $\text{Stab}(T)$, for each object of $\tilde{Z}^{-1}(\tilde{Z}(E))$, we have the corresponding $n$-fold direct sum of points in $\mathbb{P}^1$.

References

[Asp] P. S. Aspinwall, D-branes on Calabi-Yau manifolds, DUKE-CGTP-04-04, arXiv:hep-th/0403166.

[Bal] P. Balmer. The spectrum of prime ideals in tensor triangulated categories, J. Reine Angew. Math., 588, 149–168, 2005.

[BeiDri] A. Beilinson and V. Drinfeld, Chiral algebras, 51, American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2004.

[BonKap] A. I. Bondal and M. M. Kapranov, Representable functors, Serre functors, and reconstructions, (Russian) Izv. Akad. Nauk SSSR Ser. Mat., 53 (6), 1183–1205, 1337, 1989; translation in Math. USSR-Izv., 35 (3), 519–541, 1990.

[Bria] T. Bridgeland, Spaces of stability conditions, arXiv:math.AG/0611510.

[Bri06] T. Bridgeland, Derived categories of coherent sheaves, Proceedings of the International Congress of Mathematicians, Vol II, (Madrid 2006), 563–582, Madrid, 2006, European Mathematical Society Publishing House, arXiv:math.AG/0602129.
[Brib] T. Bridgeland, *Stability conditions and Kleinian singularities*, arXiv:math.AG/0508257.

[Bric] T. Bridgeland, *Stability conditions on K3 surfaces*, arXiv:math.AG/0307164.

[Brid] T. Bridgeland, *Stability conditions on triangulated categories*, to appear in Ann. of Math. (2), arXiv:math.AG/0212237.

[Bri99] T. Bridgeland, *Equivalences of triangulated categories and Fourier-Mukai transforms*, Bull. London Math. Soc. 31 (1), 25–34, 1999.

[Dou01a] M. R. Douglas, *D-branes, categories and N = 1 supersymmetry*, J. Math. Phys., 42 (7), 2818–2843, 2001.

[Dou01b] M. R. Douglas, *D-branes on Calabi-Yau manifolds*, European Congress of Mathematics, Vol. II (Barcelona, 2000), volume 202 of Progr. Math., 449–466, Birkhäuser, Basel, 2001.

[Dou02] M. R. Douglas, *Dirichlet branes, homological mirror symmetry, and stability*, Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002), 395–408, Higher Ed. Press, Beijing, 2002.

[Gie] D. Gieseker, *On the moduli of vector bundles on an algebraic surface*, Ann. of Math. (2), 106 (1), 45–60, 1977.

[Huy] D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, Oxford, 2006.

[Ina] M. Inaba, *Moduli of stable objects in a triangulated category*, arXiv:math.AG/0612078.

[Ina02] M. Inaba, *Toward a definition of moduli of complexes of coherent sheaves on a projective scheme*, J. Math. Kyoto Univ., 42 (2), 317–329, 2002.

[IshUedUeh] A. Ishii, K. Ueda, and H. Uehara, *Stability conditions on $A_n$-singularities*, arXiv:math.AG/0609551.

[IshUeh] A. Ishii and H. Uehara, *Autoequivalences of derived categories on the minimal resolutions of $A_n$-singularities on surfaces*, J. Differential Geom., 71 (3), 385–435, 2005.

[Kon] M. Kontsevich, 1998 lectures at the École Normale Supérieure.

[Lie] M. Lieblich. *Moduli of complexes on a proper morphism*. J. Algebraic Geom., 15 (1), 175–206, 2006.
[Mar78] M. Maruyama, *Moduli of stable sheaves. II*, J. Math. Kyoto Univ., **18** (3), 557–614, 1978.

[Mar77] M. Maruyama, *Moduli of stable sheaves. I*, J. Math. Kyoto Univ., **17** (1), 91–126, 1977.

[Muk] S. Mukai, *On the moduli space of bundles on K3 surfaces. I*, Vector bundles on algebraic varieties (Bombay, 1984), Tata Inst. Fund. Res. Stud. Math., **11**, 341–413, Tata Inst. Fund. Res., Bombay, 1987.

[Oka] S. Okada, *On stability manifolds of Calabi-Yau surfaces*, Int. Math. Res. Not., **2006**, Article ID 58743, 16 pages, 2006.

[ReiVan] I. Reiten and M. Van den Bergh, *Noetherian hereditary abelian categories satisfying Serre duality*, J. Amer. Math. Soc., **15** (2), 295–366, 2002.

[Ros] A. Rosenberg, *Spectra related with localizations*, MPI-2003-112, Preprint of the Max-Planck-Institut für Mathematik.

[SeiTho] P. Seidel and R. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J., **108** (1), 37–108, 2001

[Ses] C. S. Seshadri, *Space of unitary vector bundles on a compact Riemann surface*, Ann. of Math. (2), **85**, 303–336, 1967.