Bridging from Eulerian to Lagrangian statistics in 3D hydro- and magnetohydrodynamic turbulent flows

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Abstract. We present measurements of conditional probability density functions (PDFs) that allow one to systematically bridge from Eulerian to Lagrangian statistics in fully developed 3D turbulence. The transition is investigated for hydro- as well as magnetohydrodynamic flows and comparisons are drawn. Significant differences in the transition PDFs are observed for these flows and traced back to the differing coherent structures. In particular, we address the problem of an increasing degree of intermittency going from Eulerian to Lagrangian coordinates by means of the conditional PDFs involved in this transformation. First simple models of these PDFs are investigated in order to distinguish different contributions to the degree of Lagrangian intermittency.

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1. Introduction

Using the Lagrangian description of turbulence instead of the Eulerian one is more than a simple change of variables. While Eulerian increment statistics rely on spatial differences of observables of the flow, Lagrangian statistics include a temporal contribution.

The trapping of tracers in vortex filaments serves as an example. Their occurrence leads to very strong and long-lasting velocity fluctuations along the tracer trajectory. Despite the fact that these trapping events occur rarely, they have a strong signature on Lagrangian increment statistics [1, 2].

The additional temporal information influencing temporal statistics renders the prediction of Lagrangian quantities from pure Eulerian ones difficult. One of the first approaches relating Lagrangian and Eulerian velocity increments was presented by Borgas [3]. Continuing this line, there are several recent works predicting temporal scaling laws based on spatial ones [4, 5]. In addition experimental [6, 7] and numerical works [4, 8, 9] have been dedicated to the computation of inertial range Lagrangian statistics. However, a verification of the proposed models is still missing due to a drastically reduced scaling range of the Lagrangian data compared with the Eulerian data. Therefore, a precise measurement of the temporal scaling exponents is not yet available.

The relation between Eulerian and Lagrangian statistics remains up to now an important and unsolved problem as it reveals properties of the spatial and temporal flow structures. In this work, we shed light on this issue by investigating an exact relation connecting velocity increment probability density functions (PDFs). This relation allows one to bridge systematically from the Eulerian to the Lagrangian formulation. We are going to compare fully developed three-dimensional (3D) hydrodynamic and magnetohydrodynamic (MHD) turbulent flows by means of high-resolution direct numerical simulations (DNS).

This paper is organized as follows. First, we will briefly present the bridging relation between Eulerian and Lagrangian increment statistics. After a brief paragraph dedicated to the numerical method we will present the main ingredients of the bridging relation, namely two
conditional PDFs. After demonstrating the validity of this relation, we turn to the problem of Lagrangian intermittency by modeling these transition PDFs.

2. The bridging relation

In [10], an exact relation between the velocity increment PDFs in the Eulerian and Lagrangian picture was established. The derivation started with the definition of the Eulerian velocity increment

\[ u_c = v(y + x, t) - v(y, t), \]  

which is the velocity difference between the points \( y \) and \( y + x \) at time \( t \). Here, the velocities \( v \) denote the projection \( v \cdot \hat{e}_i \) of the velocity vector on one of the coordinate axes \( i = x, y, z \). To transform the Eulerian increment into the Lagrangian velocity increment, we replace the fixed distance \( x \) by the trajectory of a tracer that starts at \( y \) and travels the distance \( x(y, \tau, t) \) during the time lag \( \tau \). Doing this, we arrive at

\[ u_{el} = v(y + x(y, \tau, t), t) - v(y, t), \]  

which is a mixed Eulerian–Lagrangian quantity. The scaling properties of the latter have been investigated in [11].

In the next step, we account for the velocity change at the starting point of the tracer by introducing

\[ u_p = v(y, t) - v(y, t - \tau). \]  

The sum \( u_{el} + u_p \) leads to the Lagrangian velocity increment

\[ u_l = v(y + x(y, \tau, t), t) - v(y, t - \tau). \]  

Corresponding to the definitions of the increments we have to define the PDFs. This is done by

\[ f_e(v_c; x, y, t) = \langle \delta(u_c - v_c) \rangle, \]  

where \( u_c \) is the random variable and \( v_c \) is the independent sample-space variable. The brackets denote ensemble averaging. The quantity \( f_e \) is a function with respect to the variables \( x, y, t \) and a PDF with respect to the variable \( v_c \). In exactly the same way we can define the PDFs for all the other quantities of interest. Using these definitions we are now able to establish a connection between the increment PDFs. The step between (1) and (2) corresponds to

\[ f_{el}(v_c; y, \tau, t) = \int dx \ p_a(x|v_c; y, \tau, t) f_e(v_c; x, y, t) \]  

with \( p_a = f(x|v_c; y, \tau, t) \). The subscript in \( f_{el} \) denotes the fact that due to the integration we now have \( v_c = u_{el} \) instead of \( v_c = u_c \). The transition from (2) to (4) in terms of PDFs can be written as

\[ f_l(v_c; y, \tau, t) = \int dv_c p_b(v_l - v_c|v_c; y, \tau, t) f_{el}(v_c; y, \tau, t) \]  

with \( p_b = f_b(v_l - v_c|v_c; y, \tau, t) = f_b(v_c|v_c; y, \tau, t) \). Inserting (6) into (7) and assuming stationarity, homogeneity and isotropy of the flow, we finally arrive at

\[ f_l(v_c; \tau) = \int dv_c p_b(v_l - v_c|v_c; \tau) \int_0^\infty dr \ p_a(r|v_c; \tau) f_e(v_c; r) \]
with \( r = |x| \). For convenience we included the factor \( 4\pi \) stemming from the spherical integration in \( p_a \). We see that the transition from the Eulerian increment to the Lagrangian increment is described by the two transition PDFs \( p_a \) and \( p_b \). According to [10] these two PDFs characterize the turbulent transport of the tracers and the correlation of the velocity change along the trajectory and at the starting point of the tracer.

### 3. Numerics

In the case of plasma turbulence we integrate the MHD equations

\[
\begin{align*}
\partial_t \omega &= \nabla \times [v \times \omega - B \times (\nabla \times B)] + \nu \Delta \omega, \\
\partial_t B &= \nabla \times (v \times B) + \eta_d \Delta B, \\
\nabla \cdot v &= 0, \quad \nabla \cdot B &= 0,
\end{align*}
\]

in a periodic cube. Here, \( v \) denotes the velocity field, related to the vorticity \( \omega \) by \( \omega = \nabla \times v \), and \( \nu \) and \( \eta_d \) are the kinematic viscosity and magnetic diffusivity, respectively. We assumed an electric field \( E \) according to Ohm’s law:

\[
E = -v \times B + \eta_d j.
\]

For hydrodynamic turbulence we solve the Navier–Stokes equations, which are obtained by setting \( B = 0 \) in (9).

The simulations are carried out using a pseudo-spectral solver. The underlying equations are treated in Fourier space, while convolutions arising from nonlinear terms are computed in real space. A fast-Fourier transformation (FFT)\(^6\) is used to switch between these two spaces. The time scheme is a Runge–Kutta scheme of third order [12] and the interprocess communication uses the message passing interface (MPI). In order to obtain the velocity and magnetic field at the particle position from the grid values, we use a tricubic interpolation. This interpolation scheme parallelizes efficiently and allows for high particle numbers [13]. The main parameters of the simulations are given in table 1.

### 4. Characterization of conditional PDFs \( p_a \) and \( p_b \)

As described in section 2, the transformation from the Eulerian to the Lagrangian framework is hidden in the two conditional probability functions \( p_a \) and \( p_b \). In this section, we turn to the characterization of these functions for the two different flows under consideration.

First, we will address \( p_a \). This function quantifies the probability for a tracer to travel a distance \( r \) during a time \( \tau \). It is conditioned on the change of fluid velocity \( v_e \) over this separation. Looking at figure 1 we see that for both types of flows the basic structure of the PDF is similar. For increasing absolute values of \( v_e \) the mean of the function is shifted to larger \( r \), whereas the function gets broader. The second point reveals the statistical nature of \( p_a \). For a given velocity increment \( v_e \) tracers travel different separations with a certain probability. Nevertheless, looking at the details of \( p_a \), we see differences between Navier–Stokes turbulence and MHD that will be addressed later on. To reduce the amount of information in the PDF, we show in figure 2 cuts through \( p_a \) for fixed \( v_e \). The cuts can be fitted by functions of the

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\[^6\] Parallel 3D fast Fourier transforms (p3dfft) http://www.sdsc.edu/us/resources/p3dfft
Table 1. Parameters of the numerical simulations. NS: Navier–Stokes turbulence; MHD: magnetohydrodynamic turbulence; $R_\lambda$: Taylor–Reynolds number $\sqrt{\nu^3 L/v}$; $v_{\text{rms}}$: root-mean-square velocity $\sqrt{2/3 \epsilon}$; $\epsilon_k$: mean kinetic energy dissipation rate; $\nu$: kinematic viscosity; $\eta_d$: resistivity; $dx$: grid spacing; $\eta$: dissipation length scale $(\nu^3/\epsilon)^{1/4}$; $\tau_\eta$: Kolmogorov timescale $(\nu/\epsilon)^{1/2}$; $L = (2/3 E)^{1/2} / (\epsilon_k + \epsilon_m)$: integral scale; $T_L = L / v_{\text{rms}}$: large-eddy turnover time; $N^3$: number of collocation points; $N_p$: number of particles, $B_{\text{rms}} = 0.31$: root-mean-square magnetic field $\sqrt{2/3 E_m}$, $\epsilon_m = 1.5 \times 10^{-4}$: mean magnetic energy dissipation rate.

| Type | $R_\lambda$ | $v_{\text{rms}}$ | $\epsilon_k$ | $v = \eta_d$ | $dx$ | $\eta$ | $\tau_\eta$ | $L$ | $T_L$ | $N^3$ | $N_p$ |
|------|-------------|-----------------|-------------|--------------|-----|-------|------------|----|------|-------|-------|
| NS   | 320         | $3.6 \times 10^{-3}$ | $5 \times 10^{-4}$ | $0.61 \times 10^{-2}$ | $0.25 \times 10^{-2}$ | 0.12 | 1.83 | 10 | 1024 | $8 \times 10^6$ |
| NS   | 180         | $4 \times 10^{-3}$ | $5 \times 10^{-4}$ | $1.2 \times 10^{-2}$ | $0.54 \times 10^{-2}$ | 0.19 | 1.72 | 9 | 512$^3$ | $10^6$ |
| MHD  | 180         | $1 \times 10^{-3}$ | $3 \times 10^{-4}$ | $1.2 \times 10^{-2}$ | $0.72 \times 10^{-2}$ | 0.17 | 2.28 | 6 | 512$^3$ | $10^6$ |

Figure 1. Conditional PDF $p_a$ for $\tau = 0.3 \tau_\eta$: left: Navier–Stokes; right: MHD.

type $f(r) = ar^2 \exp(-b(r-c)^2)$. Apart from small deviations for $v_e = 0.15$ in the MHD case, this works quite well. It seems possible to split up the transport of the tracers in two parts. The deterministic part is characterized by the $v_e$ dependence of the center of the cuts. In a purely deterministic case, the $f(r)$ would be $\delta$ functions. The turbulent fluctuations widen these functions.

An important question is how the PDFs vary for different $\tau$ and whether there are significant differences for the two types of flows. In order to deal with these questions, we compute the mean $r_{\text{mean}} = \int r p_a(r|v_e; \tau) \, dr$ and the variance $\sigma_r^2 = \int (r - r_{\text{mean}})^2 p_a(r|v_e; \tau) \, dr$ of the PDFs. The mean indicates the average distance traveled by a tracer for a given time $\tau$ depending on the velocity change over this distance. We see from the left part of figure 3 that the larger the given spatial velocity difference, the larger the traveled distance of the tracer. Increasing the time lag renders the functional shape flatter and flatter. This implies that the condition on the velocity difference at the start and end point becomes weaker. When a tracer travels for a long time, the initial and ending velocities are unimportant for the traveled distance due to the velocity fluctuations picked up along the path and a relation $r_{\text{mean}} = v_{\text{rms}} \tau$ holds. A difference between hydrodynamics and MHD turbulence is that this dependence is larger in MHD than in Navier–Stokes for all time lags. An explanation might be the reduced degree of
Figure 2. Cuts through $p_a$ for different $v_e$. The lines denote the function $f(r) = ar^2 \exp[-b(r - c)^2]$. Left: Navier–Stokes with $a_0 = 19238$, $b_0 = 411.417$, $c_0 = 0.01$ and $d_0.375 = 2399$, $b_0.375 = 558$, $c_0.375 = 0.05$. Right: MHD with $a_0 = 3e + 09$, $b_0 = 7.12$, $c_0 = -1.38$ and $a_0.15 = 616$, $b_0.15 = 36$, $c_0.15 = -0.02$.

Figure 3. Mean $r_{\text{mean}}$ (left) and variance $\sigma_r^2$ (right) of $p_a$.

chaoticity of the corresponding tracer trajectories. In MHD, vortex sheets render the motion of tracer smoother than vortex filaments in Navier–Stokes turbulence [9] so that the picked-up fluctuations are less important in MHD.

Looking at the variance of the mean distance for a given fluctuation (see the right part of figure 3), one recognizes a similar difference in the functional form for hydrodynamics and MHD as discussed in the previous paragraph. The variance for Navier–Stokes is much flatter than that for MHD. Again, the condition on the velocity difference is weaker for the former than for the latter. For both flows the variance increases when increasing $\tau$ because of the random character of the fluctuations.

We now turn to the characterization of $p_b$ for Navier–Stokes and MHD turbulence (see figure 4). Again the functions show basically a similar structure. Both functions exhibit an anticorrelation of $v_p$ and $v_e$. This can be explained as the signature of random sweeping effects. Small scales are swept by large scales, which yield $v_p \sim -v_e$. This effect is clearly visible for small time lags in figures 6 and 7. We measured an angle of approximately $41^\circ$ and $39^\circ$ in the...
limit of small time lags for hydrodynamics and MHD flow, respectively. As for $p_a$ we show in figure 5 the PDF $p_b$ for different fixed $v_e$. For $v_e = 0$ the shape is similar to the shape of the acceleration PDF. This can be understood easily. For $v_e = 0$ we have $v_l = v_p$ and for small $\tau$ the acceleration can be approximated by $a \approx v_l / \tau$. For this reason the cut at $v_e = 0$ has in this case nearly the same shape as the acceleration PDF.

In figure 6, we look at the mean $u_{p_{\text{mean}}} = \int_{u_p} u_p p_b(u_p|v_e; \tau) du_p$ and the variance $\sigma_{u_p}^2 = \int_{u_p} (u_p - u_{p_{\text{mean}}})^2 p_b(u_p|v_e; \tau) du_p$ of $p_b$ to study its time lag dependence. From the left part of figure 6 one immediately observes the already mentioned anticorrelation. A difference between Navier–Stokes and MHD turbulence is that the sweeping effect persists to larger $\tau$ for the former than for the latter (see also figure 7). This can be traced back to the different coherent structures. In order to observe sweeping, one needs the tracer to be in a coherent structure during a certain time over which the complete structure is swept by a larger one. Tracers in Navier–Stokes are known to be trapped in vortex filaments. The latter are swept by large-scale eddies. Contrarily, no trapping of tracers occurs in MHD turbulence. Tracers in the vicinity of vortex sheets leave them easily at the edges. While these sheets might be swept by the large-scale flow, tracers are not attached to these sheets for a significant amount of time. Figure 7 also reveals a negligible Reynolds number dependence.
5. Reconstruction of Lagrangian PDFs

We will reconstruct Lagrangian statistics from Eulerian data by means of (8) and the numerically calculated transition PDFs presented in the previous section. We restrict ourselves to the hydrodynamic flow. We measured the Eulerian increment PDFs $f_e(v_e; r)$ and performed the integration in (8). The resulting Lagrangian increment PDFs are given in figure 8. The ones computed directly from the tracer data collapse well with the reconstructed ones.

Slight differences are observable for small time lags. An explanation might be the assumption of isotropy in the derivation of (8). Imagine a mean flow superimposed in one direction. While the Lagrangian statistics and Eulerian statistics remain unchanged, the transition PDF $p_a$ will be altered in such a way that one observes larger distances $r$ for a given velocity increment $u_{el}$. Using such a spread PDF for the reconstruction will mix up Eulerian PDFs from too large separations and will therefore overestimate the real velocity increment.
Figure 8. Lagrangian PDFs for Navier–Stokes turbulence, reconstructed (rec) from (8) and directly computed data (ref).

6. Splitting up the bridging relation

The transformation (8) can naturally be split up into two substeps. After applying $p_a$ on the Eulerian PDFs one ends up with the mixed Euler–Lagrangian PDFs. The application of $p_b$ yields the final Lagrangian ones.

This split-up allows one to shed light on an as yet unexplained observation [9]. While MHD turbulence is more intermittent than Navier–Stokes turbulence from the Eulerian point of view, the situation is reversed when regarding these two flows from the Lagrangian one.

Figure 9 illustrates these findings, where relative structure functions are shown. We computed the coordinate projected Eulerian structure functions

$$S_p^e(r) = \frac{1}{3} \sum_{i=1}^{3} \langle [(v(y + re_i) - v(y)) \cdot e_i]^3 \rangle$$

for Navier–Stokes and MHD turbulence (with $S_p \propto r^{\zeta_p}$ within the inertial range). Measuring velocity increments over spatial scales yields a lower exponent (see the left part of figure 9) for MHD, and hence more intermittency. This relation changes when considering temporal increments (see the right part of figure 9). Here Lagrangian structure functions

$$S_p^l(\tau) = \frac{1}{3} \sum_{i=1}^{3} \langle [(v(y + \tilde{x}(\tau), t) - v(y, t - \tau)) \cdot e_i]^3 \rangle$$

are shown. The difference in the Eulerian–Lagrangian scaling exponent is much smaller than for the Euler exponent, while the Lagrangian exponent of the Navier–Stokes flow is significantly smaller than the Lagrangian MHD exponent. Hence, the ratio of intermittency of hydrodynamics and MHD changes switching from the Eulerian to the Lagrangian point of view.
**Figure 9.** Relative structure functions: left: Eulerian structure functions (e); right: Eulerian–Lagrangian (el) and Lagrangian (l) structure functions.

**Figure 10.** Eulerian–Lagrangian and Lagrangian PDFs.

The application of $p_a$ introduces an intermediate step, which sheds light on the subtle change of this intermittency property. The scaling exponent of MHD turbulence is only slightly affected by $p_a$ (see the right part of figure 9), while it is decreased in the Navier–Stokes case. Indeed, the increment PDFs remain nearly unchanged for MHD turbulence (see figure 10), while the core is pronounced in the Navier–Stokes measurement, which leads to more intermittency. This is connected to the variance in the traveled distance. This distance determines the range of scales of Eulerian increment PDFs summed to build the mixed increment PDF. A larger variance results in a larger mixing of scales and therefore in a larger deviation from the Eulerian PDF.

**7. Modeling $p_a$ and $p_b$**

Looking at the transition probabilities $p_a$ and $p_b$ depicted in figures 1 and 4, one can see the complex structure underlying the translation process. Nevertheless we want to start with very simple models for $p_a$ and $p_b$ in order to get further insight on the influence of the transition probabilities. We will start with an analytical ansatz and then turn to numerical models.
7.1. Analytical modeling

In our framework the bridging relation \( r = v_e \tau \), introduced by Borgas [3] and used in [8] and [4] on dimensional grounds, between the time scale \( \tau \) and the length scale \( r \) corresponds to

\[
p_a(r | v_e; \tau) = \delta(r - v_e \tau).
\]

Using the additional assumption

\[
p_b(v_p | v_e; \tau) = \delta(v_1 - v_e)
\]

and equation (8), we arrive at the relation between \( f_e \) and \( f_l \):

\[
f_l(v; \tau) = f_e(v; v \tau).
\] (17)

In [10], it was shown that by using (17) and the Mellin transform, one can establish the relation

\[
\zeta_l(n - \zeta_e(n)) = \zeta_e(n)
\] (18)

between the exponents of the Eulerian structure functions \( \zeta_e(n) \) and the exponents of the Lagrangian structure functions \( \zeta_l(n) \). Equation (18) is identical to the formulae derived in Biferale et al [4]. But this equation contains a major problem since it obeys the so-called monotonicity property and thus does not solve the Navier–Stokes-MHD puzzle which was already mentioned in the previous paragraph: Eulerian Navier–Stokes turbulence is less intermittent than Eulerian MHD turbulence but Lagrangian Navier–Stokes turbulence is more intermittent than Lagrangian MHD turbulence. In order to show that equation (18) possesses the monotonicity property, we start with the property

\[
\zeta_{NS}^e(p) \geq \zeta_{MHD}^e(p), \text{ for } p \geq 3,
\]

which was numerically observed by Homann et al [9] for Navier–Stokes (NS) and MHD turbulence (NS). Using equation (18) we obtain

\[
\zeta_l^{NS}(p - \zeta_{NS}^e(p)) = \zeta_{NS}^e(p) \geq \zeta_{MHD}^e(p) = \zeta_l^{MHD}(p - \zeta_{MHD}^e(p)).
\]

It follows immediately that

\[
\zeta_l^{NS}(n) = \frac{\zeta_{NS}^e(n + \zeta_{NS}^e(j) - \zeta_{MHD}^e(j))}{\zeta_{NS}^e(n)}, \text{ for } n = i - \zeta_{NS}^e(j),
\]

since \( \zeta_{NS}^e(j) - \zeta_{MHD}^e(j) \geq 0 \). This clearly shows that we have to use more sophisticated model assumptions for \( p_a \) and \( p_b \) to explain the violation of monotonicity. Looking again at figure 4, we can see that in the preceding analysis we did not account for the fact that \( v_p = v_e \) is not valid. This can easily be done by using

\[
p_b(v_1 - v_e | v_e; \tau) = \frac{1}{\alpha} \delta(v_1/\alpha - v_e),
\]

where \( \alpha - 1 \) denotes the slope in the linear relation between \( u_p \) and \( u_e \). Using the relation between Eulerian and Lagrangian PDFs

\[
f_l(v; \tau) = \frac{1}{\alpha} \int dv_e \delta(v_1/\alpha - v_e) \int_0^\infty dr \delta(r - v_\tau \tau) \cdot f_e(v_e; r),
\]

we obtain directly

\[
f_l(v; \tau) = \frac{1}{\alpha} f_e \left( \frac{v_1 - v_\tau \tau}{\alpha} \right).
\] (19)
In order to calculate the Lagrangian structure function exponents, we make use of the Mellin transform as done in [14], where it was used to calculate general multi-fractal PDFs:

\[ f_e(v_e, r) = \frac{1}{v_e} \int_{-\infty}^{\infty} dn \, S_e(n)(v_e)^{-n} \]

with \( S_e(n) = A_e(n)r^{\xi_e(n)} \). Using equation (19) and the Mellin transform we obtain

\[ f_l(v_1; \tau) = \frac{1}{v_1} \int_{-\infty}^{\infty} dn \, A_e(n) r^{\xi_e(n)} v_1^{\xi_e(n)-n} \, d^n r^{\xi_e(n)}. \]

This Lagrangian PDF is now inserted into the inverse Mellin transform to obtain the Lagrangian structure functions

\[ S_l(n) = \int_0^\infty dv_1 \, v_1^n f_l(v_1; \tau) = \int_0^\infty dv_1 \, \frac{1}{v_1^n} \int_{-\infty}^{\infty} dj \, A_e(j) r^{\xi_e(j)} v_1^{\xi_e(j)-j} \, d^n j^{\xi_e(j)}. \]

As suggested by figure 7 we make the ansatz \( \alpha = c \tau^a \), which leads to

\[ S_l(n) = \int_0^\infty dv_1 \, \frac{1}{v_1^n} \int_{-\infty}^{\infty} dj \, A_e(j) r^{\xi_e(j)+a(j-\xi_e(j))} v_1^{\xi_e(j)-j}. \]

Now we substitute \( j'(j) = j - \xi_e(j), \, dj' = (1 - \partial_j \xi_e(j)) dj \) and denote the inverse function by \( j = j'(j') \). Thus we have

\[ S_l(n) = \int_0^\infty dv_1 \, \frac{1}{v_1^n} \int_{-\infty}^{\infty} dj' \, A_e(j'(j')) r^{\xi_e(j'(j'))+a(j'(j')-\xi_e(j'(j')))} v_1^{-j'} \]

\[ = \int_0^\infty dv_1 \, \frac{1}{v_1^n} \int_{-\infty}^{\infty} dj' \, S_l(j'(j')) v_1^{-j'} \]

with \( S_l(j'(j)) = \frac{A_e(j')}{1-\partial_j \xi_e(j')} r^{\xi_e(j') (j)+(j-\xi_e(j'))} \) and \( j' = j - \xi_e(j) \) and we obtain for the exponents

\[ \xi(j - \xi_e(j)) = \xi_e(j) + a[j - \xi_e(j)]. \quad (20) \]

In order to see the influence of the angle variation on the structure function exponents more easily, we evaluate the Lagrangian structure function exponents \( S_l(j) \) for the case where the Eulerian turbulence is given by the K41-scaling \( S_e(j) = j/3 \). Equation (20) reads for this case

\[ \xi(j) = (\frac{1}{3} + a) j. \quad (21) \]

Equation (21) shows that an angle variation of the form \( \tau^a \) has a smoothing effect and shifts the exponents to higher values. In an intermittent situation this corresponds to the effect of decreasing intermittency.

7.2. Numerical modeling of \( p_b \)

The analytic results of the previous paragraph will now be complemented by modeling the transition PDF \( p_b \) in a more realistic way. This transition PDF \( p_b \) and the PDF \( f_{oe} \) obtained from the numerical simulation will be used to evaluate the resulting structure functions numerically. Here we will choose different representations of \( p_b \) in order to test the impact of certain assumptions on the reconstructed Lagrangian statistics.

First, we start with the transition PDF (16). The reconstructed structure functions using this relation are given in figure 11. There are two main observations comparing the original to the
modeled structure functions. Firstly, the dip around $2\tau_\eta$ is strongly decreased. This breakdown of power-law scaling due to dissipative effects [15] was modeled in an extended multifractal approach in Arneodo et al [16]. Qualitatively, the dip can be associated with the presence of vortex trapping of tracers [1, 2]. This mechanism is apparently excluded using (16). Secondly, the theoretically predicted values [4] (horizontal line in figure 11) are well reproduced by the model of $p_b$ while the actually measured values are smaller. The degree of intermittency is therefore underestimated by the assumption leading to (16).

The transition probability $p_b$ exhibits two eye-catching characteristics: The change in the angle and variance with respect to $\tau$. In order to check the influence of these two features separately, we first apply a pure change in the angle of $p_b$ (analogous to the previous paragraph) and then a pure change in the variance.

We start with the $\tau$-dependent angle variations. Guided by the left part of figure 7 we choose a representation of the form

$$p_b = \frac{1}{\sqrt{2\pi \sigma}} \exp\left(-\frac{(v_p - (\beta - 1)v_c)^2}{2\sigma^2}\right), \quad (22)$$

with $\beta = (3\tau / (2T_L))^{0.4}$ and $\sigma = dx$. Reconstructing the Lagrangian structure functions makes the dip around $2\tau_\eta$ disappear (see ‘pure angle variation’ in figure 11) since this angle variation does not take into account dissipative effects. Furthermore, the degree of intermittency is significantly reduced as already stated in the previous section.

To study the change in the variance, we compute the impact of a change of the variance of $p_b$ without changing the angle. In order to mimic the right part of figure 6 we choose a representation of the form (22) but here with

$$\sigma = (0.156|v_c| + 0.577)(1 - \alpha) + 10\alpha, \quad (23)$$

Figure 11. Lagrangian ESS structure function of fifth order. Reference: directly measured; pure angle variation: model (22); pure variance variation: model (23); $\delta$-model for $p_b$: model (16); horizontal line corresponds to the value 2.0 theoretically predicted in [4].
$\beta = 0.12$ and $\alpha = \tau / T_L$. This modification increased the degree of intermittency around $10 \tau_\eta$ (see ‘pure variance variation’ in figure 11).

From these observations it becomes clear that the main source of increased intermittency is the $\tau$-dependent change of the variance going from a localized (close to a $\delta$ function) transition PDF at small times $\tau$ to a broad distribution mixing the increments of different timescales.

8. Summary and conclusions

In this work, an exact relation between Eulerian and Lagrangian increment statistics is analyzed for 3D hydrodynamic as well as MHD turbulence. By reconstructing the Lagrangian increment PDFs from the Eulerian ones, the validity of this relation has been shown.

The investigation of the specific shape of the two conditional PDFs ($p_a$ and $p_b$) involved in this relation reveals differences in hydrodynamics and MHD flows. A Navier–Stokes flow shows strong signatures of sweeping effects of tracers in smaller structures by larger ones. This sweeping is much less important in MHD turbulence, which can be traced back to the coherent turbulent structures in conducting (sheets) flows. In MHD flows, the tracer particles are mainly traveling parallel to current and vortex sheets. This type of motion also has an impact on the chaoticity of the flow and the fluctuations picked up by a tracer trajectory. Here, an MHD flow renders the motion of a small fluid element smoother than in Navier–Stokes turbulence. This results in a smaller variance of the corresponding separation PDF and a slower neglect of the initial conditions of the tracer path.

Concerning the puzzle of increased intermittency in Lagrangian coordinates compared with the Eulerian framework, the bridging relation allows for a detailed study of its origin. The relation naturally splits up the transformation process into two parts. In the first part described by the transition PDF $p_a$, Eulerian increment PDFs are summed up over a range of scales determined by the variance of the traveled distance. This variance is larger for hydrodynamic flows and results in accentuating stronger fluctuations and thus in an increase of intermittency. The second part described by the transition PDF $p_b$ takes into account the velocity fluctuation at the initial point of the tracer trajectory. Modeling these transition PDFs reveals that a sophisticated model is needed in order to extract correctly Lagrangian information from Eulerian information where the $\tau$-dependent change of the variance is a key parameter. As shown in [11] the first transition PDF $p_a$ does not reduce the size of the inertial scaling range. Thus, the observed reduction of the Lagrangian scaling range has to be attributed to the statistical nature of the second part of the transition.

We were able to show that a purely dimensional model [4] underestimates the real degree of Lagrangian intermittency. It does not seem to be possible to build up a model by using only delta functions. A key ingredient is the statistic nature of the tracer trajectory, which is reflected in the variance of transition PDFs.

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