Square-Weighted Zero-Sum Constants

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Abstract

Let $A \subseteq \mathbb{Z}_n$ be a subset. A sequence $S = (x_1, \ldots, x_k)$ in $\mathbb{Z}_n$ is said to be an $A$-weighted zero-sum sequence if there exist $a_1, \ldots, a_k \in A$ such that $a_1 x_1 + \cdots + a_k x_k = 0$. By a square, we shall mean a non-zero square in $\mathbb{Z}_n$. We determine the smallest natural number $k$, such that every sequence in $\mathbb{Z}_n$ whose length is $k$, has a square-weighted zero-sum subsequence. We also determine the smallest natural number $k$, such that every sequence in $\mathbb{Z}_n$ whose length is $k$, has a square-weighted zero-sum subsequence whose terms are consecutive terms of the given sequence.

1 Introduction

For a finite set $A$, we let $|A|$ denote the number of elements of $A$. For $a, b \in \mathbb{Z}$, we let $[a, b]$ denote the set $\{x \in \mathbb{Z} : a \leq x \leq b\}$.

Let $R$ be a ring with unity, $M$ be an $R$-module, and $A \subseteq R$. A subsequence $T$ of a sequence $S = (x_1, x_2, \ldots, x_k)$ in $M$ is called an $A$-weighted zero-sum subsequence if the set $J = \{i : x_i \in T\}$ is non-empty and for every $i \in J$ there exists $a_i \in A$ such that $\sum_{i \in J} a_i x_i = 0$.

For a finite $R$-module $M$ and $A \subseteq R$, the $A$-weighted Davenport constant of $M$ denoted by $D_A$ is defined to be the least positive integer $k$, such that every sequence in $M$ whose length is $k$, has an $A$-weighted zero-sum subsequence.

Adhikari and Chen [1] introduced this constant for the ring $R = \mathbb{Z}$, i.e., for abelian groups. We define the constant $C_A$ to be the least positive integer $k$, such that every sequence in $M$ whose length is $k$, has an $A$-weighted zero-sum subsequence whose terms are consecutive terms.

Remark 1. Mondal, K. Paul, and S. Paul [2] have shown that $D_A \leq C_A \leq |M|$.

We also denote the ring $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}_n$. Let $U(n)$ denote the group of units in $\mathbb{Z}_n$ and $U(n)^2$ denote the set $\{x^2 : x \in U(n)\}$. For an odd prime $p$ we denote the set $U(p)^2$ by $Q_p$. 1
For a divisor $m$ of $n$, the homomorphism $f_{n,m} : \mathbb{Z}_n \to \mathbb{Z}_m$ is given by $f_{n,m}(a + n\mathbb{Z}) = a + m\mathbb{Z}$. Mondal et al. [6, Lem. 7] showed that the image of $U(n)$ under $f_{n,m}$ is $U(m)$.

Let $p$ be a prime divisor of $n$. We say that $v_p(n) = r$ if $p^r \mid n$ and $p^{r+1} \nmid n$. Suppose $r = v_p(n)$. For every $x \in \mathbb{Z}_n$ we denote the image of $x$ under $f_{n,p^r}$ by $x^{(p)}$. Given a sequence $S = (x_1, \ldots, x_l)$ in $\mathbb{Z}_n$, we get a sequence $S^{(p)} = (x_1^{(p)}, \ldots, x_l^{(p)})$ in $\mathbb{Z}_{p^r}$. From this point onwards, we will only consider the case when $M = R = \mathbb{Z}_n$.

Let $\Omega(n)$ denote the number of prime factors of $n$ counted with multiplicity. Grynkiewicz and Hennecart [3] showed that $D_{U(n)^2} \geq 2\Omega(n) + \min\{v_3(n), v_5(n)\} + 1$ when $n$ is odd, with equality if either $3 \mid n$ or $v_3(n) \geq v_5(n)$. This extends a result of Chintamani and Moriya [5], and another of Adhikari, David, and Urroz [4]. These results lead quite naturally to the question of the value of $D_{S(n)}$ where $S(n) = \{x^2 : x \in \mathbb{Z}_n\} \setminus \{0\}$.

We determine the value of $D_{S(n)}$ for every $n$ and show that it depends on the parity of $n$ when $n$ is a square, and on the parity of $v_2(n)$ when $n$ is not a square. We also investigate the value of $C_{S(n)}$.

We show that $C_{U(25)^2} = 9$, adding to the results which were obtained by Mondal et al. [2]. Using this fact, we get that $C_{S(n)} \leq 9$ when $n$ is an odd square. In this article, we have obtained the following results:

- We determine the size of $S(n)$ for every $n$.
- When $n$ is a square, we get that $D_{S(n)} = 4$ or 5 when $n$ is even or odd respectively.
- When $n$ is not a square, we get that $D_{S(n)} = 2$ or 3 when $v_2(n)$ is odd or even respectively.
- When $n$ is not a square of an odd number, we get that $C_{S(n)} = D_{S(n)}$.
- When $n$ is a square of an odd, squarefree number, we get that $C_{S(n)} = 9$.
- When $n$ is a square of an odd number $m$ such that $m$ is divisible by $p^2$ where $p$ is a prime which is at least seven, we get that $C_{S(n)} = D_{S(n)}$.

| $n$   | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $D_{S(n)}$ | 2  | 3  | 4  | 3  | 2  | 3  | 2  | 5  | 2  | 3  | 3  | 3  | 2  | 3  | 4  | 3  | 2  | 3  |

| $n$   | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $D_{S(n)}$ | 3  | 3  | 2  | 3  | 2  | 5  | 2  | 3  | 3  | 3  | 2  | 3  | 2  | 3  | 2  | 3  | 4  | 3  |

Table 1: Values of $D_{S(n)}$ for all $n \in [2,37]$.

The only values of $n$ in the set $[2,37]$ for which $C_{S(n)}$ differs from $D_{S(n)}$ are 9 and 25. The smallest $n$ for which we have not been able to determine $C_{S(n)}$ is 81.
2 The size of $S(n)$

If $n = p_1^{r_1} \cdots p_s^{r_s}$ where the $p_i$’s are distinct primes, then from the Chinese remainder theorem we get an isomorphism of rings

$$\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/p_1^{r_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_s^{r_s}\mathbb{Z}.$$ 

Hence, it follows that

$$|S(n)| = (|S(p_1^{r_1})| + 1) \cdots (|S(p_s^{r_s})| + 1) - 1.$$ 

Thus, it is enough to determine the size of $S(p^r)$ where $p$ is a prime and $r$ is a positive integer.

**Observation 2.** Let $p$ be a prime, $r$ be a positive integer, and $a \in \mathbb{Z}_{p^r} \setminus \{0\}$. Then there exists a unique $k \in [0, r-1]$ such that $a = p^k u$ where $u$ is a unit.

For a real number $x$, we let $|x|$ denote the greatest integer which is at most equal to $x$.

**Lemma 3.** Let $p$ be a prime, $r$ be a positive integer, and $l = [(r-1)/2]$. Then we have that $S(p^r) = \bigcup_{k \in [0, l]} p^{2k} U(p^r)^2$. Also, this is a disjoint union.

**Proof.** Let $a \in S(p^r)$. Then there exists $b \in \mathbb{Z}_{p^r} \setminus \{0\}$ such that $a = b^2$. Thus, there exists $u \in U(p^r)$ and $k \in [0, r-1]$ such that $b = p^k u$. As $a = p^{2k} u^2$ and $a \neq 0$, by Observation 2 we see that $2k \in [0, r-1]$ and so $k \in [0, l]$. Hence, we see that $S(p^r) \subseteq \bigcup_{k \in [0, l]} p^{2k} U(p^r)^2$. It is easy to see that the reverse inclusion holds. From Observation 2 it follows that this union is a disjoint union.

**Lemma 4.** Let $p$ be a prime, $r$ be a natural number, and $l = [(r-1)/2]$. Then for every $k \in [0, l]$ we have that $|p^{2k} U(p^r)^2| = |U(p^{r-2k})^2|.$

**Proof.** Let $k \in [0, l]$ and $f = f_{p^r, p^{r-2k}}$. For all $x, y \in \mathbb{Z}_{p^r}$ we see that

$$p^{2k} x = p^{2k} y \iff p^{2k} (x-y) = 0 \iff p^{r-2k} | (x-y) \iff x - y \in \ker f.$$

So we see that $\varphi : p^{2k} U(p^r)^2 \rightarrow U(p^{r-2k})^2$ defined as $\varphi(p^{2k} x) = f(x)$ is well-defined and injective. We claim that the map $\varphi$ is also surjective. Let $y \in U(p^{r-2k})^2$. As the image of $U(p^r)$ under $f$ is $U(p^{r-2k})$, it follows that there exists $x \in U(p^r)^2$ such that $f(x) = y$. Hence, we see that $\varphi(p^{2k} x) = y$. This proves our claim. So it follows that $\varphi$ is a bijection.

The next result follows from Lemmas 3 and 4.

**Theorem 5.** If $r$ is even

$$|S(p^r)| = |U(p^r)^2| + |U(p^{r-2})^2| + \cdots + |U(p)^2| + |U(p^2)^2|$$

and if $r$ is odd

$$|S(p^r)| = |U(p^r)^2| + |U(p^{r-2})^2| + \cdots + |U(p^3)^2| + |U(p)^2|.$$
It remains to determine the size of $U(n)^2$ when $n$ is a prime power. Let $n = p^r$ where $p$ is an odd prime and $r$ is a positive integer. Ireland and Rosen [7, Thm. 2, p. 43] have shown that $U(n)$ is a cyclic group. So there is exactly one element of order two in $U(n)$. Thus, the kernel of the onto map $U(n) \rightarrow U(n)^2$ given by $x \mapsto x^2$ has order two. Hence, we see that $U(n)^2$ has index two in $U(n)$. So it follows that $|U(n)^2| = |U(n)|/2 = p^{r-1}(p-1)/2$.

We have that $U(4)^2 = U(2)^2 = \{1\}$. Let $n = 2^r$ where $r$ is at least three. Ireland and Rosen [7, Thm. 2', p. 43] have shown that $U(n) \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2^{r-2}}$. So there are exactly three elements of order two in $U(n)$. Thus, the kernel of the onto map $U(n) \rightarrow U(n)^2$ given by $x \mapsto x^2$ has order four. Hence, we see that $U(n)^2$ has index four in $U(n)$. So it follows that

$$|U(n)^2| = |U(n)|/4 = 2^{r-1}/4 = 2^{r-3}.$$ 

### 3 Some general results

**Observation 6.** Let $n = p_1^{r_1}p_2^{r_2} \cdots p_k^{r_k}$ where the $p_i$’s are distinct primes. By the Chinese remainder theorem we get an isomorphism

$$\varphi : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/p_1^{r_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{r_k}\mathbb{Z}$$

given by $\varphi(a) = (a^{(p_1)}, \ldots, a^{(p_k)})$. As $\varphi$ is an isomorphism, we have that $a \in S(n)$ if and only if for every prime divisor $q$ of $n$, we have that $a^{(q)}$ is a square and there exists a prime divisor $p$ of $n$ such that $a^{(p)} \neq 0$.

**Lemma 7.** Let $S$ be a sequence in $\mathbb{Z}_n$ and $p$ be a prime divisor of $n$ such that $v_p(n) = r$. Suppose the sequence $S^{(p)}$ is an $S(p^r)$-weighted zero-sum sequence. Then the sequence $S$ is an $S(n)$-weighted zero-sum sequence.

**Proof.** Let $S = (x_1, \ldots, x_l)$. Then we have that $S^{(p)} = (x_1^{(p)}, \ldots, x_l^{(p)})$. There exist $b_1, \ldots, b_l \in S(p^r)$ such that $b_i x_1^{(p)} + \cdots + b_i x_l^{(p)} = 0$. By Observation 6 we see that for every $i \in [1, l]$ there exists $a_i \in S(n)$ such that $a_i^{(p)} = b_i$ and for each prime divisor $q$ of $n/p^r$ we have $a_i^{(q)} = 0$. Let $\varphi$ be the isomorphism given by the Chinese remainder theorem as in Observation 6. As we get that $\varphi(a_1 x_1 + \cdots + a_l x_l) = 0$, it follows that $a_1 x_1 + \cdots + a_l x_l = 0$. Hence, we see that $S$ is an $S(n)$-weighted zero-sum sequence in $\mathbb{Z}_n$.

**Corollary 8.** Let $p$ be a prime divisor of $n$ and $r = v_p(n)$. Then we have that $C_{S(n)} \leq C_{S(p^r)}$.

**Proof.** Let $m = p^r$. Suppose $S$ is a sequence in $\mathbb{Z}_n$ having length $C_{S(m)}$. As $S^{(p)}$ is a sequence in $\mathbb{Z}_m$ having length $C_{S(m)}$, it follows that there exists a subsequence $T$ of $S$ having consecutive terms such that $T^{(p)}$ is an $S(m)$-weighted zero-sum sequence. So from Lemma 7 we see that $T$ is an $S(n)$-weighted zero-sum sequence. Hence, it follows that $C_{S(n)} \leq C_{S(m)}$.

We will apply the next result later in the case when $p$ is a prime.
Lemma 9. Let \( p \) be an integer which is at least two, \( r \) be an odd number, and \( T \) be a sequence in \( \mathbb{Z}_p^r \). Suppose the image of \( T \) under \( f_{p^r, p} \) is a \( U(p)^2 \)-weighted zero-sum sequence. Then \( T \) is an \( S(p^r) \)-weighted zero-sum sequence.

Proof. Let \( T = (x_1, \ldots, x_k) \) be a sequence in \( \mathbb{Z}_p^r \) and \( T' = (x'_1, \ldots, x'_k) \) be the image of \( T \) under \( f_{p^r, p} \). For each \( i \in [1, k] \) there exist \( a'_i \in U(p)^2 \) such that \( a'_1x'_1 + \cdots + a'_kx'_k = 0 \). Mondal et al. [6, Lem. 7] showed that the image of \( U(p)^2 \) under \( f_{p^r, p} \) is \( U(p)^2 \). So for each \( i \in [1, k] \) there exists \( a_i \in U(p)^2 \) such that \( f_{p^r, p}(a_i) = a'_i \).

Let \( x = a_1x_1 + \cdots + a_kx_k \). As \( f_{p^r, p}(x) = a'_1x'_1 + \cdots + a'_kx'_k = 0 \), it follows that \( p \) divides \( x \). We see that \( c = p^{r-1} = (p^{(r-1)/2})^2 \in S(p^r) \). As \( p \) divides \( x \) we get that \( cx = 0 \). Thus, it follows that \((ca_1)x_1 + \cdots + (ca_k)x_k = 0\). For each \( i \in [1, k] \) we see that \( ca_i \in S(p^r) \). Hence, it follows that \( T \) is an \( S(p^r) \)-weighted zero-sum sequence. \( \square \)

4 When \( n \) is an even square

The next two results will be used to determine the value of \( D_{S(n)} \) when \( n \) is an even square.

Lemma 10. Let \( r \) be a non-zero even number. Let \( S = (x_1, x_2, x_3) \) be a sequence in \( U(2^r) \) whose image under \( f_{2^r, 4} \) is the sequence \((1, 1, 1)\). Then \( S \) does not have any \( (2^r) \)-weighted zero-sum subsequence.

Proof. As the image of \( U(2^r)^2 \) under \( f_{2^r, 4} \) is \( U(4)^2 = \{1\} \) and the sequence \((1, 1, 1)\) in \( \mathbb{Z}_4 \) does not have any zero-sum subsequence, it follows that \( S \) does not have any \( U(2^r)^2 \)-weighted zero-sum subsequence. However, as we want to show that \( S \) does not have any \( (2^r) \)-weighted zero-sum subsequence, we need to modify this argument. Suppose \( T \) is an \( (2^r) \)-weighted zero-sum subsequence of \( S \). Let \( I = \{i \in [1, 3] : x_i \text{ is a term of } T\} \).

For each \( i \in I \) we see that \( a_i \in (2^r) \) such that \( \sum_{i \in I} a_i x_i = 0 \). By Lemma 3 for each \( i \in I \) we see that \( a_i = 2^{r_i} u_i \) where \( r_i \) is an even number which is at most \( r - 2 \) and \( u_i \in U(2^r)^2 \). So we get that \( \sum_{i \in I} 2^{r_i} u_i, x_i = 0 \). Let \( r' \) be the minimum of the set \( \{r_i : i \in I\} \) and \( J = \{i \in I : r_i = r'\} \). Let \( f = f_{2^r, 4} \). As \( r' \leq r - 2 \) we see that four divides \( \sum_{i \in J} u_i, x_i \) and hence \( 0 = \sum_{i \in J} f(u_i) f(x_i) = \sum_{i \in J} 1 \). So we get the contradiction that the sequence \((1, 1, 1)\) in \( \mathbb{Z}_4 \) has a zero-sum subsequence. Hence, it follows that \( S \) does not have any \( (2^r) \)-weighted zero-sum subsequence. \( \square \)

Lemma 11. Let \( p \) be a prime and \( r \) be a non-zero even number. Let \((z_1, z_2)\) be a sequence in \( U(p^2) \) whose image under \( f_{p^r, p} \) is not a \( Q_p \)-weighted zero-sum sequence. Let \( S = (x_1, x_2, y_1) \) be a sequence in \( U(p^r) \) whose image under \( f_{p^r, p^2} \) is the sequence \((z_1, z_2, p)\). Then \( S \) does not have any \( (p^r) \)-weighted zero-sum subsequence.

Proof. Suppose the sequence \( S = (x_1, x_2, y_1) \) has an \( (p^r) \)-weighted zero-sum subsequence \( T \). Let \( I = \{i \in [1, 2] : x_i \text{ is a term of } T\} \). Let \( J = \{1\} \) if \( y_1 \) is a term of \( T \) and let \( J = \emptyset \) if \( y_1 \) is not a term of \( T \). Then for every \( i \in I \) and \( j \in J \) there exist \( a_i, b_j \in S(p^r) \) such that \( \sum_{i \in I} a_i, x_i + \sum_{j \in J} b_j y_j = 0 \). As \( y_1 \) maps to \( p \) under \( f_{p^r, p^2} \) there exists \( w_1 \in U(p^r) \) such that...
$y_1 = pw_1$. By Lemma 3, for each $i \in I$ and $j \in J$ we see that $a_i = p^{r_i}u_i$ and $b_j = p^{s_j}v_j$ where $r_i, s_j \in [0, r - 2]$ are even and $u_i, v_j \in U(p^r)^2$. So we get that

$$\sum_{i \in I} p^{r_i}u_i x_i + \sum_{j \in J} p^{s_j+1}v_j w_j = 0. \quad (1)$$

Consider the set $L = \{r_i : i \in I\} \cup \{s_j + 1 : j \in J\}$. Let $r'$ be the minimum of $L$. As $r \geq 2$ is even and $s_1$ is even, it follows that $r' \leq r - 1$. Suppose there exists $i \in I$ such that $r_i = r'$. We claim that $I = \{1, 2\}$ and $r_1 = r_2 = r'$. If not, from (1) we get that $p^{r'+1}$ divides $p^{r'} w$ where $w$ is a unit. As $r' \leq r - 1$ we get the contradiction that $p$ divides $w$. By a similar argument, we see that $s_1 + 1 \neq r'$.

As $r' \leq r - 1$, from (1) we see that $p$ divides $u_1 x_1 + u_2 x_2$. Let $f = f_{p^{r'}, p}$. We get that $f(u_1)f(x_1) + f(u_2)f(x_2) = 0$. As $u_1, u_2 \in U(p^r)^2$, it follows that $f(u_1), f(u_2) \in Q_p$. So the sequence $(f(x_1), f(x_2))$ is a $Q_p$-weighted zero-sum sequence. Thus, we get the contradiction that the image of the sequence $(z_1, z_2)$ under $f_{p^{r'}, p}$ is a $Q_p$-weighted zero-sum sequence.

Hence, it follows that $S$ does not have any $S(p^r)$-weighted zero-sum subsequence. $\square$

**Theorem 12.** Let $n$ be an even square. Then we have that $D_{S(n)} \geq 4$.

**Proof.** Mondal et al. [6, Cor. 2, Lem. 7] have shown that for every odd prime $p$ we can find a sequence $(u_p, v_p)$ in $U(p^2)$ whose image under $f_{p^r, p}$ is not a $Q_p$-weighted zero-sum sequence. Consider the sequence $(u_p, v_p, p)$ in $\mathbb{Z}_{p^2}$.

For each prime divisor $p$ of $n$, if $n_p = p^{v_p(n)}$ then the map $f_{n_p, p^2}$ is onto. So by the Chinese remainder theorem we can find a sequence $S = (x_1, x_2, x_3)$ in $\mathbb{Z}_n$ such that for every prime divisor $p$ of $n$ the image of $S$ under $f_{n_p, p^2}$ is $(u_p, v_p, p)$ if $p$ is odd, and under $f_{n, 4}$ is $(1, 1, 1)$.

For every prime divisor $p$ of $n$, we see that the sequence $S^{(p)}$ in $\mathbb{Z}_{p^r}$ has the form as in the statement of Lemma 10 if $p = 2$, or of Lemma 11 if $p$ is odd. So for every prime divisor $p$ of $n$, if $r = v_p(n)$, it follows that the sequence $S^{(p)}$ does not have any $S(p^r)$-weighted zero-sum subsequence.

Thus, it follows that the sequence $S$ does not have any $S(n)$-weighted zero-sum subsequence. Hence, we see that $D_{S(n)} \geq 4$. $\square$

**Lemma 13.** Let $r$ be a non-zero even number and $p$ be a positive integer. Suppose $T$ is a sequence in $\mathbb{Z}_{p^r}$ whose image under $f_{p^{r'}, p^2}$ is a $U(p^2)^2$-weighted zero-sum sequence. Then $T$ is an $S(p^r)$-weighted zero-sum sequence.

**Proof.** Let $T = (x_1, \ldots, x_k)$ be a sequence in $\mathbb{Z}_{p^r}$ and $T' = (x'_1, \ldots, x'_k)$ be the image of $T$ under $f_{p^{r'}, p^2}$. Suppose $T'$ is a $U(p^2)^2$-weighted zero-sum sequence. Then there exist $a'_1, \ldots, a'_k \in U(p^2)^2$ such that $a'_1x'_1 + \cdots + a'_kx'_k = 0$. Mondal et al. [6, Lem. 7] have shown
that $f_{p^r, p^2}(U(p^r)^2) = U(p^2)^2$. Thus, for each $i \in [1, k]$ there exists $a_i \in U(p^r)^2$ such that $f_{p^r, p^2}(a_i) = a_i'$.

Let $x = a_1x_1 + \cdots + a_kx_k$. As $f_{p^r, p^2}(x) = a_1'x_1' + \cdots + a_k'x_k' = 0$ it follows that $p^2$ divides $x$. As $r$ is an even number which is at least two, we see that $p^r - 2 = (p^{(r-2)/2})^2 \in S(p^r)$. As $p^2$ divides $x$ we get $p^{r-2}x = 0$ and so $(ca_1)x_1 + \cdots + (ca_k)x_k = 0$ where $c = p^{r-2}$. As the $a_i$'s are in $U(p^r)^2$ it follows that $T$ is an $S(p^r)$-weighted zero-sum sequence. \hfill \Box

The next result follows immediately from Lemma 13.

**Corollary 14.** Let $r$ be a non-zero even number and $p$ be a positive integer. Then we have that $D_{S(p^r)} \leq D_{U(p^2)^2}$ and $C_{S(p^r)} \leq C_{U(p^2)^2}$.

**Theorem 15.** Let $r$ be a non-zero even number. Then we have $C_{S(2^r)} \leq 4$.

**Proof.** Mondal et al. [2, Cor. 1] have shown that $C_{(1)} = 4$. As $U(4)^2 = \{1\}$, from Corollary 14 it follows that $C_{S(2^r)} \leq 4$. \hfill \Box

**Corollary 16.** Let $n$ be an even square. Then we have $D_{S(n)} = C_{S(n)} = 4$.

**Proof.** From Theorem 12 we have $D_{S(n)} \geq 4$. By Theorem 15 and Corollary 8 we have $C_{S(n)} \leq 4$. As $D_A(n) \leq C_A(n)$ for every $A \subseteq \mathbb{Z}_n$, it follows that $D_{S(n)} = C_{S(n)} = 4$. \hfill \Box

5 When $n$ is not a square

**Proposition 17.** Let $n$ be odd. We can find a sequence $S = (u, v)$ in $U(n)$ such that for each prime divisor $p$ of $n$, the image of $S$ under $f_{n, p}$ does not have any $Q_p$-weighted zero-sum subsequence.

**Proof.** Let $p$ be a prime divisor of $n$ and $v_p(n) = r$. By [6, Cor. 2] there exist $u_p, v_p \in U(p)$ such that the sequence $(u_p, v_p)$ does not have any $Q_p$-weighted zero-sum subsequence. As the image of $U(p^r)$ under $f_{p^r, p}$ is $U(p)$, there exist $u_p', v_p' \in U(p^r)$ such that the image of the sequence $(u_p', v_p')$ under $f_{p^r, p}$ is $(u_p, v_p)$.

By the Chinese remainder theorem, there exist $u, v \in U(n)$ such that for each prime divisor $p$ of $n$ if $n_p = p^{v_p(n)}$, then the image of the sequence $S = (u, v)$ under $f_{n, n_p}$ is $(u_p', v_p')$. It follows that the image of $S$ under $f_{n, p}$ is $(u_p, v_p)$ which is the same as the image of $(u_p', v_p')$ under the map $f_{n, n_p}$.

**Lemma 18.** Let $p$ be an odd prime and $r$ be a positive integer. Suppose $S = (v_1, v_2)$ is a sequence in $U(p^r)$ such that the image of $S$ under $f_{p^r, p}$ is not a $Q_p$-weighted zero-sum sequence. Then $S$ does not have any $S(p^r)$-weighted zero-sum subsequence.

**Proof.** Suppose $S$ is an $S(p^r)$-weighted zero-sum sequence. Then there exist $a_1, a_2 \in S(p^r)$ such that $a_1v_1 + a_2v_2 = 0$. By Lemma 3 we see that there exist $u_1, u_2 \in U(p^r)^2$ and even $r_1, r_2 \in [0, r - 1]$ such that $a_1 = p^{r_1}u_1$ and $a_2 = p^{r_2}u_2$. So we get that $p^{r_1}u_1v_1 + p^{r_2}u_2v_2 = 0$.
By Observation 2 we see that \( r_1 = r_2 \) and so \( p^{r_1}(u_1v_1 + u_2v_2) = 0 \). As \( r_1 < r \), it follows that \( p \) divides \( u_1v_1 + u_2v_2 \).

Let \( f \) be the map \( f_{p^r,p} \). We get that \( f(u_1)f(v_1) + f(u_2)f(v_2) = 0 \). As \( u_1, u_2 \in U(p^{r'}) \), it follows that \( f(u_1), f(u_2) \in Q_p \). Thus, we get the contradiction that the image of \( S \) under \( f_{p^r,p} \) is a \( Q_p \)-weighted zero-sum sequence. Hence, it follows that \( S \) is not an \( S(p^r) \)-weighted zero-sum sequence. As \( v_1, v_2 \in U(p^r) \), we see that \( S \) does not have any \( S(p^r) \)-weighted zero-sum subsequence of length one.

\[ \square \]

**Theorem 19.** Let \( n \) be an odd number. Then we have that \( D_{S(n)} \geq 3 \).

**Proof.** By Proposition 17 there exists a sequence \( S = (u,v) \) in \( U(n) \) such that for each prime divisor \( p \) of \( n \), the image of \( S \) under \( f_{n,p} \) does not have any \( Q_p \)-weighted zero-sum subsequence. Suppose \( T \) is an \( S(n) \)-weighted zero-sum subsequence of \( S \). As the terms of \( S \) are in \( U(n) \), we see that \( T \) must be \( S \). Thus, there exist \( a, b \in S(n) \) such that \( au + bv = 0 \). As \( a \neq 0 \), there exists a prime divisor \( p \) of \( n \) such that \( a^{(p)} \neq 0 \). Let \( k = v_p(n) \) and \( (u_p, v_p) \) be the image of \( S \) under \( f_{n,p^k} \).

It follows that the sequence \( (u_p, v_p) \) in \( \mathbb{Z}_{p^k} \) has an \( S(p^k) \)-weighted zero-sum subsequence. As the image of \( S = (u,v) \) under \( f_{n,p} \) is the same as the image of \( (u_p, v_p) \) under \( f_{p^k,p} \), it follows that \( (u_p, v_p) \) is a sequence in \( \mathbb{Z}_{p^k} \) whose image under \( f_{p^k,p} \) does not have any \( Q_p \)-weighted zero-sum subsequence. So by Lemma 18 we get the contradiction that the sequence \( (u_p, v_p) \) does not have any \( S(p^k) \)-weighted zero-sum subsequence.

Thus, we see that \( S \) does not have any \( S(n) \)-weighted zero-sum subsequence. Hence, it follows that \( D_{S(n)} \geq 3 \).

\[ \square \]

**Theorem 20.** We have that \( D_{S(n)} \geq 3 \) when \( v_2(n) \) is even and at least two.

**Proof.** By two results by Mondal et al. [6, Cor. 2, Lem. 7] and by the Chinese remainder theorem, we can find a sequence \( S = (v_1, v_2) \) in \( U(n) \) by a similar method as in Proposition 17 such that for each odd prime divisor \( p \) of \( n \) the sequence the image of \( S \) under \( f_{n,p} \) is not a \( Q_p \)-weighted zero-sum sequence and the image of \( S \) under \( f_{n,4} \) is \((1,1)\).

Suppose \( T \) is an \( S(n) \)-weighted zero-sum subsequence of \( S \). As the terms of \( S \) are in \( U(n) \), we see that \( T \) must be \( S \). Thus, there exists \( a, b \in S(n) \) such that \( au + bv = 0 \). As \( a \neq 0 \), there is a prime divisor \( q \) of \( n \) such that \( a^{(q)} \neq 0 \). We now use a similar argument as in the proof of Theorem 19, where we use Lemma 10 in addition to Lemma 18.

\[ \square \]

**Theorem 21.** Let \( p \) be an odd prime and \( r \) be odd. Then we have \( C_{S(p^r)} \leq 3 \).

**Proof.** Let \( S = (x, y, z) \) be a sequence in \( \mathbb{Z}_{p^r} \) and let \( S' \) be the image of \( S \) under \( f_{p^r,p} \). Mondal et al. [2, Thm. 5] showed that for an odd prime \( p \) we have \( C_{Q_p} = 3 \). Thus, we can find a subsequence \( T \) whose terms are consecutive terms of \( S \) such that the image of \( T \) under \( f_{p^r,p} \) is a \( Q_p \)-weighted zero-sum subsequence of \( S' \). So by Lemma 9 we see that \( T \) is an \( S(p^r) \)-weighted zero-sum sequence. Hence, it follows that \( C_{S(p^r)} \leq 3 \).

\[ \square \]

**Corollary 22.** Suppose \( n \) is not a square and \( v_2(n) \) is a non-negative, even integer. Then we have that \( D_{S(n)} = C_{S(n)} = 3 \).
Proof. By Theorem 20 we have $D_{S(n)} \geq 3$. From the assumptions on $n$, we see that there is an odd prime divisor $p$ of $n$ such that $v_p(n)$ is odd. Thus, by Corollary 8 and Theorem 21 we have that $C_{S(n)} \leq 3$. As $D_A(n) \leq C_A(n)$ for every $A \subseteq \mathbb{Z}_n$, it follows that $D_{S(n)} = C_{S(n)} = 3$. \qed

Theorem 23. We have that $C_{S(2^r)} \leq 2$ where $r$ is an odd number.

Proof. Let $S = (x, y)$ be a sequence in $\mathbb{Z}_{2^r}$ and $S' = (x', y')$ be the image of $S$ under $f_{2^r, 2}$. We can find a subsequence $T$ of $S$ such that the image of $T$ under $f_{2^r, 2}$ is a zero-sum sequence. So by Lemma 9 we see that $T$ is an $S(2^r)'$-weighted zero-sum sequence. Hence, it follows that $C_{S(2^r)} \leq 2$. \qed

Corollary 24. Suppose $n$ is an even positive integer such that $v_2(n)$ is odd. Then we have that $D_{S(n)} = C_{S(n)} = 2$.

Proof. It is easy to see that $D_{S(n)} \geq 2$. From Corollary 8 and Theorem 23, we get that $C_{S(n)} \leq 2$. For every $A \subseteq \mathbb{Z}_n$ as $D_A(n) \leq C_A(n)$, it follows that $D_{S(n)} = C_{S(n)} = 2$. \qed

6 $D_{S(n)}$ when $n$ is an odd square

Lemma 25. Let $p$ be a prime and $r$ be a non-zero even number. Suppose $(w_1, w_2)$ is a sequence in $U(p^r)$ whose image under $f_{p^r, p}$ is not a $Q_p$-weighted zero-sum sequence. Let $S = (u w_1, u w_2, p w_1, p w_2)$ where $u \in U(p^r)$. Then the sequence $S$ in $\mathbb{Z}_{p^r}$ does not have any $S(p^r)$-weighted zero-sum subsequence.

Proof. Suppose $T$ is an $S(p^r)$-weighted zero-sum subsequence of $S = (u w_1, u w_2, p w_1, p w_2)$. Let $I = \{i \in [1, 2] : u w_i$ is a term of $T\}$ and $J = \{j \in [1, 2] : p w_j$ is a term of $T\}$. As $T$ is an $S(p^r)$-weighted zero-sum sequence, for each $i \in I$ there exists $a_i \in S(p^r)$ and for each $j \in J$ there exists $b_j \in S(p^r)$ such that $\sum_{i \in I} a_i u w_i + \sum_{j \in J} b_j p w_j = 0$. From Lemma 3, for each $i \in I$ we have $a_i = p^{r'} u_i$ for some even $r' < r$ and $u_i \in U(p^r)^2$ and for each $j \in J$ we have $b_j = p^{s_j} v_j$ for some even $s_j < r$ and $v_j \in U(p^r)^2$. So we have

$$u \sum_{i \in I} p^{r'} u_i w_i + \sum_{j \in J} p^{s_j+1} v_j w_j = 0. \quad (2)$$

Consider the set $L = \{r_i : i \in I\} \cup \{s_j + 1 : j \in J\}$. Let $r'$ be the minimum of $L$. As $r$ is even and $s_j$ is even for each $j \in J$, it follows that $r' \leq r - 1$. Observe that $\{r_i : i \in I\} \cap \{s_j + 1 : j \in J\} = \emptyset$ as the $r_i$’s and $s_j$’s are even. Suppose there exists $i \in I$ such that $r_i = r'$. We claim that $I = \{1, 2\}$ and $r_1 = r_2 = r'$. If not, from (2) we get that $p^{r'+1}$ divides $p^r w$ where $w$ is a unit. As $r' \leq r - 1$ we get the contradiction that $p$ divides $w$. By a similar argument if there exists $j \in J$ such that $s_j + 1 = r'$, then $J = \{1, 2\}$ and $s_1 + 1 = s_2 + 1 = r'$.

Suppose $I = \{1, 2\}$ and $r_1 = r_2 = r'$. As $r' \leq r - 1$, from (2) we see that $p$ divides $u (u_1 w_1 + u_2 w_2)$. As $u \in U(p^r)$, it follows that $f(u) \in U(p)$ where $f = f_{p^r, p}$. So we get that
We get the contradiction that the image of the sequence \((u, v)\) in \(U(n)\) such that for each prime divisor \(p\) of \(n\), the image of the sequence \((u, v)\) under \(f_{n, p}\) does not have any \(Q_p\)-weighted zero-sum subsequence. Let 
\[ S = (u, v, mu, mv). \]
We claim that the sequence \(S\) in \(\mathbb{Z}_n\) does not have any \(S(n)\)-weighted zero-sum subsequence from which it follows that \(D_{S(n)} \geq 5\).

Suppose \(T\) is an \(S(n)\)-weighted zero-sum subsequence of \(S\). Let \(x\) be a term of \(T\) and \(a \in S(n)\) be the coefficient of \(x\) in an \(S(n)\)-weighted zero-sum which we obtain from \(T\). As \(a \neq 0\), there exists a prime divisor \(p\) of \(n\) such that \(a^{(p)} \neq 0\). Let \(r = v_p(n)\). It follows that the sequence \(S^{(p)}\) has an \(S(p^r)\)-weighted zero-sum subsequence.

The image of the sequence \((u^{(p)}, v^{(p)})\) under \(f_{p^r, p}\) does not have any \(Q_p\)-weighted zero-sum subsequence. As \(m\) is the largest squarefree divisor of \(n\), it follows that \(m^{(p)} = pw\) where \(w \in U(p^r)\). It follows that \(S^{(p)}\) is a sequence in \(\mathbb{Z}_{p^r}\) which has the form as in the statement of Lemma 25. As \(n\) is a square, we see that \(r\) is a non-zero even number. So by Lemma 25 we arrive at the contradiction that the sequence \(S^{(p)}\) does not have any \(S(p^r)\)-weighted zero-sum subsequence. Hence, our claim must be true.

We get the next result from the proof of [4, Thm. 7].

**Lemma 27.** Let \(p\) be an odd prime, \(r\) be a positive integer, and \(A = U(p^r)^2\). Suppose we are given \(y_1, y_2, y_3 \in U(p^r)\). Then we have that
\[
Ay_1 + (Ay_2 \cup \{0\}) + (Ay_3 \cup \{0\}) = \mathbb{Z}_{p^r}.
\]

**Theorem 28.** Let \(r\) be a non-zero even number and \(p\) be an odd prime. Then we have that \(D_{S^{(p^r)}} \leq 5\).

**Proof.** Suppose \(S = (x_1, \ldots, x_5)\) is a sequence in \(\mathbb{Z}_{p^r}\). If \(p^2\) divides some term \(x_i\) of \(S\), then \(p^{r-2}x_i = 0\). As \(r\) is even, we see that \(p^{r-2} = (p^{(r-2)/2})^2 \in S(n)\). So we see that \((x_i)\) is an \(S(p^r)\)-weighted zero-sum subsequence of \(S\) of length one. Thus, we may assume that \(p^2\) does not divide any term of \(S\).

It follows that each term of \(S\) is either a unit or a multiple of \(p\) by a unit. If at least three terms of \(S\) are units or at least three terms of \(S\) are of the form \(pu\) where \(u\) is a unit, by using Lemma 27 we get an \(S(p^r)\)-weighted zero-sum subsequence of \(S\). Thus, we see that \(D_{S(p^r)} \leq 5\).

**Corollary 29.** Let \(n\) be an odd square. Then we have that \(D_{S(n)} = 5\).

**Proof.** From Theorem 26 we see that \(D_{S(n)} \geq 5\) when \(n\) is an odd square. Since every prime divisor \(p\) of \(n\) is odd and \(v_p(n)\) is even, by using Corollary 8 and Theorem 28 we see that \(D_{S(n)} \leq 5\). Thus, it follows that \(D_{S(n)} = 5\).
7 \( C_{S(n)} \) when \( n \) is an odd square

Notation:
If \( T \) is a subsequence of \( S \), then \( S - T \) denotes the subsequence which is obtained by removing the terms of \( T \) from \( S \). The concatenation of the sequences \( S - T \) and \( T \) gives us a sequence whose terms are a permutation of the terms of the sequence \( S \).

If \( S \) is a sequence in \( \mathbb{Z}_n \) and \( d \in \mathbb{Z}_n \) such that all the terms of \( S \) are divisible by \( d \), then \( S/d \) denotes the sequence in \( \mathbb{Z}_n \) whose terms are obtained by dividing the corresponding terms of \( S \) by \( d \).

Theorem 30. We have that \( C_{U(25)^2} = 9 \).

Proof. Let \( S = (x_1, \ldots, x_9) \) be a sequence in \( \mathbb{Z}_{25} \). We may assume that all the terms of \( S \) are non-zero.

Suppose at least four terms of \( S \) are units. From [5, Lem. 2] it follows that \( S \) is a \( U(25)^2 \)-weighted zero-sum sequence. Let \( S_1 = (x_1, x_2, x_3) \), \( S_2 = (x_4, x_5, x_6) \), and \( S_3 = (x_7, x_8, x_9) \).

Suppose at most two terms of \( S \) are units. Then we see that there exists \( i \in [1, 3] \) such that all the terms of \( S_i \) are divisible by 5. Let \( S'_i \) denote the sequence in \( \mathbb{Z}_5 \) which is the image of \( S_i \) under \( f_{25,5} \). From [2, Thm. 4] we have that \( C_{Q_5} = 3 \). Thus, the sequence \( S'_i \) has a \( Q_5 \)-weighted zero-sum subsequence having consecutive terms. By [2, Lem. 5] it follows that the sequence \( S_i \) (and hence the sequence \( S \)) has a \( U(25)^2 \)-weighted zero-sum subsequence having consecutive terms.

So we may assume that exactly three terms of \( S \) are units. If at least three consecutive terms of \( S \) are non-units, by a similar argument as in the previous paragraph we get a \( U(25)^2 \)-weighted zero-sum subsequence of \( S \) having consecutive terms. So it follows that for each \( i \in [1, 3] \) there is exactly one term \( y_i \) in the sequence \( S_i \) which is a unit.

As \( C_{Q_5} = 3 \) we see that the sequence \( (y_1, y_2, y_3) \) has a subsequence \( S'_4 \) having consecutive terms whose image \( S'_4 \) under \( f_{25,5} \) is a \( Q_5 \)-weighted zero-sum sequence. As \( f_{25,5} \) is onto, it follows that there exists \( k \in \mathbb{Z}_{25} \) such that a \( U(25)^2 \)-weighted sum of the terms of \( S_4 \) is \(-5k\). We will use this observation a bit later in this proof.

Let \( J = \{i \in [1, 3] : y_i \text{ is a term of } S_4\} \). Let \( T \) be the subsequence of \( S \) which is the concatenation of the sequences \( S_i \) such that \( i \in J \). It follows that \( T \) is a subsequence of \( S \) having consecutive terms. We claim that \( T \) is a \( U(25)^2 \)-weighted zero-sum sequence. Let \( T_1 = T - S_4 \). As all the terms of \( S \) are non-zero, all the terms of \( T_1 \) are of the form \( 5u \) where \( u \in U(25) \).

Let \( T'_1 \) denote the image of \( T_1/5 \) under \( f_{25,5} \). Chintamani and Moriya [5, Lem. 2] showed that \( f_{25,5}(k) \in \mathbb{Z}_5 \) is a \( Q_5 \)-weighted sum of the terms of \( T'_1 \). Mondal et al. [2, Lem. 5] showed that \( 5k \) is a \( U(25)^2 \)-weighted sum of the terms of \( T_1 \). As we have seen that \(-5k\) is a \( U(25)^2 \)-weighted sum of the terms of \( S_4 \), it follows that \( T \) is a \( U(25)^2 \)-weighted zero-sum sequence.

Thus, every sequence in \( \mathbb{Z}_{25} \) having length nine has a \( U(25)^2 \)-weighted zero-sum subsequence whose terms are consecutive terms of the given sequence. So it follows that
$C_{U(25)^2} \leq 9$. Mondal et al. [2, Cor. 5] showed that $C_{U(25)^2} \geq 9$. Hence, it follows that $C_{U(25)^2} = 9$. □

**Theorem 31.** Let $n$ be an odd square. Then we have that $C_{S(n)} \leq 9$.

**Proof.** Mondal et al. [2, Cor. 6] showed that $C_{U(p^2)^2} = 9$ when $p$ is a prime which is at least seven. From Remark 1 we see that $C_{U(9)^2} \leq 9$ and from Theorem 30 we see that $C_{U(25)^2} \leq 9$. Thus, from Corollaries 8 and 14 it follows that $C_{S(n)} \leq 9$. □

Chintamani and Moriya [5, Lem. 1] showed the next result.

**Lemma 32.** Let $p$ be a prime which is at least seven and $A = U(p^r)^2$. Then for every $x_1, x_2, x_3 \in U(p^r)$ we have that $Ax_1 + Ax_2 + Ax_3 = Z_{p^r}$.

Mondal et al. [2, Lem. 2] showed the next result, which follows easily from Lemma 32.

**Lemma 33.** Let $p$ be a prime which is at least seven and $S = (x_1, \ldots, x_k)$ be a sequence in $Z_{p^r}$. Suppose at least three terms of $S$ are units. Then $S$ is a $U(p^r)^2$-weighted zero-sum sequence.

**Theorem 34.** Let $p$ be a prime which is at least seven and $r$ be an even number which is at least four. Then we have that $C_{S(p^r)} \leq 5$.

**Proof.** Let $S = (x_1, \ldots, x_5)$ be a sequence in $Z_{p^r}$. As $r$ is even, we see that $p^{r-2} = (p^{(r-2)/2})^2 \in S(p^r)$. If $p^2$ divides some term $x$ of $S$, then it follows that $p^{r-2}x = 0$ and so $S$ has an $S(p^r)$-weighted zero-sum subsequence of length one. Thus, we may assume that $p^2$ does not divide any term of $S$. So every term of $S$ is either a unit or of the form $pu$ where $u$ is a unit.

If at least three terms of $S$ are units, by Lemma 33 we see that $S$ is an $S(p^r)$-weighted zero-sum sequence. Thus, we may assume that at most two terms of $S$ are units. Then at least three terms of $S$ are of the form $pu$ where $u$ is a unit. We may assume that $x_1 = pu_1$, $x_2 = pu_2$, $x_3 = pu_3$ where $u_1, u_2, u_3 \in U(p^r)$.

Consider the sequence $T = (u_1, u_2, u_3, px_1, px_2)$. By Lemma 33 we see that $T$ is a $U(p^r)^2$-weighted zero-sum sequence. So there exist $a_i$’s in $U(p^r)^2$ such that $a_1u_1 + a_2u_2 + a_3u_3 + pa_4x_1 + pa_5x_2 = 0$. Thus, it follows that $a_1x_1 + a_2x_2 + a_3x_3 + p^2a_4x_4 + p^2a_5x_5 = 0$. As $r$ is at least four, we see that $p^2 \neq 0$. Hence, it follows that $S$ is an $S(p^r)$-weighted zero-sum sequence. □

**Corollary 35.** Let $n$ be an odd square which is divisible by $p^4$ where $p$ is a prime which is at least seven. Then we have that $C_{S(n)} = 5$.

**Proof.** As $n$ is a square, it follows that $v_p(n)$ is even. So from Corollary 8 and Theorem 34 we have $C_{S(n)} \leq 5$. As we have that $D_A(n) \leq C_A(n)$ for every $A \subseteq Z_n$, from Theorem 26 it follows that $C_{S(n)} = 5$. □

The next result follows easily from a result by Chintamani and Moriya [5, Lem. 2].
Lemma 36. Let $r$ be a positive integer and $S$ be a sequence in $\mathbb{Z}_5^r$. Suppose at least four terms of $S$ are units. Then $S$ is a $U(5^r)^2$-weighted zero-sum sequence.

Theorem 37. We have that $C_{S(5^r)} \leq 7$ when $r$ is an even number which is at least four.

Proof. We use a similar argument as in the proof of Theorem 34. The only change is that we replace Lemma 33 with Lemma 36.

Corollary 38. Let $n$ be a square which is divisible by $5^4$. Then we have that $C_{S(n)} \leq 7$.

Proof. As $n$ is a square, it follows that $v_5(n)$ is even. Also, we have that $v_5(n) \geq 4$. So from Corollary 8 and Theorem 37 we get that $C_{S(n)} \leq 7$.

Theorem 39. Let $n$ be a square of an odd squarefree number. Then we have that $C_{S(n)} = 9$.

Proof. By Theorem 31 we get that $C_{S(n)} \leq 9$. We will construct a sequence $S$ of length eight in $\mathbb{Z}_n$ which has no $S(n)$-weighted zero-sum subsequence having consecutive terms. Hence, it will follow that $C_{S(n)} = 9$.

By Proposition 17 there exists a sequence $S' = (u, v)$ in $U(n)$ such that for every prime divisor $p$ of $n$, the image $(u_p, v_p)$ of $S'$ under $f_{n, p}$ does not have any $Q_p$-weighted zero-sum subsequence. By the Chinese remainder theorem there exist $x, y \in \mathbb{Z}_n$ such that for each prime divisor $p$ of $n$ we have that $x^{(p)} = pu^{(p)}$ and $y^{(p)} = pv^{(p)}$. In this proof, for every $c \in \mathbb{Z}_n$ we will denote $f_{n, p}(c)$ by $c_p$. So it follows that $x_p = y_p = 0$. Consider the sequence $S$ in $\mathbb{Z}_n$ defined as follows:

$$S = (x, y, u, x, y, v, x, y).$$

Suppose there exists a subsequence $T$ of $S$ having consecutive terms which is an $S(n)$-weighted zero-sum sequence.

Case 1: Either $u$ or $v$ is a term of $T$.

Without loss of generality, we may assume that $u$ is a term of $T$.

Let $a \in S(n)$ be the coefficient of $u$ in the $S(n)$-weighted zero-sum which is obtained from $T$. As $a \neq 0$, there exists a prime divisor $p$ of $n$ such that $a^{(p)} \neq 0$. As $n$ is the square of a squarefree number, it follows that $v_p(n) = 2$. So we see that $a^{(p)} \in S(p^2)$. As every non-zero term of $\mathbb{Z}_{p^2}$ is either a unit or a unit multiple of $p$, we see that $S(p^2) = U(p^2)^2$. As $a^{(p)} \in U(p^2)^2$, it follows that $f_{p^2, p}(a^{(p)}) \in Q_p$ and so $a_p \in Q_p$.

We claim that the sequence $(u_p, v_p)$ has a $Q_p$-weighted zero-sum subsequence. Suppose $v$ is a term of $T$. Let $b \in S(n)$ be the coefficient of $v$ in the $S(n)$-weighted zero-sum which is obtained from $T$. As $b \in S(n)$, it follows that $b_p \in Q_p \cup \{0\}$. So we get that $a_p u_p + b_p v_p = 0$. If $v$ is not a term of $T$ then $a_p u_p = 0$. This proves our claim from which we get a contradiction.

Case 2: Neither $u$ nor $v$ is a term of $T$.

As $T$ is a subsequence of consecutive terms, it follows that $T$ is a subsequence of the sequence $(x, y)$. Suppose $x$ is a term of $T$. Let $a \in S(n)$ be the coefficient of $x$ in the $S(n)$-weighted zero-sum which is obtained from $T$. By a similar argument as in the previous case, we see that there is a prime divisor $p$ of $n$ such that $a_p \in Q_p$. We claim that the sequence $(u_p, v_p)$ has a $Q_p$-weighted zero-sum subsequence.
Suppose \( y \) is a term of \( T \). Then there exists \( b \in S(n) \) such that \( ax + by = 0 \). As \( b \in S(n) \), it follows that \( b_p \in Q_p \cup \{0\} \). As \( ax + by = 0 \), it follows that \( a(p)x(p) + b(p)y(p) = 0 \) in \( \mathbb{Z}_{p^2} \). Thus, we get that \( p \left( a(p)x(p) + b(p)y(p) \right) = 0 \) and so \( a(p)x(p) + b(p)y(p) \) is divisible by \( p \). Hence, it follows that \( a_p u_p + b_p v_p = 0 \). By a similar argument, we see that if \( y \) is not a term of \( T \) then \( a_p u_p = 0 \). This proves our claim from which we get a contradiction.

So we see that the sequence \( S \) does not have any \( S(n)-\)weighted zero-sum subsequence having consecutive terms.

\[ \square \]

8 Concluding remarks

We have been unable to determine the constants \( C_{S(3^r)} \) and \( C_{S(5^r)} \) where \( r \) is an even number which is at least four. For every such \( r \) we have shown that \( C_{S(3^r)} \in [5, 9] \) and \( C_{S(5^r)} \in [5, 7] \). If the values of these constants are known, we can determine the value of \( C_{S(n)} \) for every \( n \).

We can try to characterize the sequences in \( \mathbb{Z}_n \) of length \( C_{S(n)} - 1 \) which do not have any \( S(n)-\)weighted zero-sum subsequence having consecutive terms. We can also try to characterize sequences in \( \mathbb{Z}_n \) of length \( D_{S(n)} - 1 \) which do not have any \( S(n)-\)weighted zero-sum subsequence.

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References

[1] S. D. Adhikari and Y. G. Chen, Davenport constant with weights and some related questions II, *J. Combin. Theory Ser. A* 115 (2008), 178–184.

[2] S. Mondal, K. Paul, and S. Paul, On a different weighted zero-sum constant, *Discrete Math.* 346 (2023), 113350.

[3] D. J. Grynkiewicz and F. Hennecart, A weighted zero-sum problem with quadratic residues, *Unif. Distrib. Theory* 10 (2015), 69–105.

[4] S. D. Adhikari, C. David, and J. J. Urroz, Generalizations of some zero-sum theorems, *Integers* 8 (2008), #A52.

[5] M. N. Chintamani and B. K. Moriya, Generalizations of some zero-sum theorems, *Proc. Indian Acad. Sci. Math. Sci.* 122 (2012), 15–21.

[6] S. Mondal, K. Paul, and S. Paul, Extremal sequences for a weighted zero-sum constant, *Integers* 22 (2022), #A93.

[7] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer, New York, 1990.
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