DIVERGENT TORUS ORBITS IN HOMOGENEOUS SPACES OF Q-RANK TWO

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Abstract. Let $G$ be a semisimple algebraic $\mathbb{Q}$-group, let $\Gamma$ be an arithmetic subgroup of $G$, and let $T$ be an $\mathbb{R}$-split torus in $G$. We prove that if there is a divergent $T_{\mathbb{R}}$-orbit in $\Gamma \backslash G_{\mathbb{R}}$, and $\mathbb{Q}$-rank $G \leq 2$, then $\dim T \leq \mathbb{Q}$-rank $G$. This provides a partial answer to a question of G. Tomanov and B. Weiss.

1. Introduction

Let $G$ be a semisimple algebraic $\mathbb{Q}$-group, let $\Gamma$ be an arithmetic subgroup of $G$, and let $T$ be an $\mathbb{R}$-split torus in $G$. The $T_{\mathbb{R}}$-orbit of a point $\Gamma x_0$ in $X = \Gamma \backslash G_{\mathbb{R}}$ is divergent if the natural orbit map $T_{\mathbb{R}} \to X: t \mapsto \Gamma x_0 t$ is proper. G. Tomanov and B. Weiss [TW, p. 389] asked whether it is possible for there to be a divergent $T_{\mathbb{R}}$-orbit when $\dim T > \mathbb{Q}$-rank $G$. B. Weiss [W1, Conj. 4.11A] conjectured that the answer is negative.

1.1. Conjecture. Let

- $G$ be a semisimple algebraic group that is defined over $\mathbb{Q}$,
- $\Gamma$ be a subgroup of $G_{\mathbb{R}}$ that is commensurable with $G_{\mathbb{Z}}$,
- $T$ be a connected Lie subgroup of an $\mathbb{R}$-split torus in $G_{\mathbb{R}}$, and
- $x_0 \in G_{\mathbb{R}}$.

If the $T$-orbit of $\Gamma x_0$ is divergent in $\Gamma \backslash G_{\mathbb{R}}$, then $\dim T \leq \mathbb{Q}$-rank $G$.

The conjecture easily reduces to the case where $G$ is connected and $\mathbb{Q}$-simple. Furthermore, the desired conclusion is obvious if $\mathbb{Q}$-rank $G = 0$ (because this implies that $\Gamma \backslash G_{\mathbb{R}}$ is compact), and it is easy to prove if $\mathbb{Q}$-rank $G = 1$ (see [2]). Our main result is that the conjecture is also true in the first interesting case:

1.2. Theorem. Suppose $G$, $\Gamma$, $T$, and $x_0$ are as specified in Conj. 1.1 and assume $\mathbb{Q}$-rank $G \leq 2$. If the $T$-orbit of $\Gamma x_0$ is divergent in $\Gamma \backslash G_{\mathbb{R}}$, then $\dim T \leq \mathbb{Q}$-rank $G$.

For higher $\mathbb{Q}$-ranks, we prove only the upper bound $\dim T < 2(\mathbb{Q}$-rank $G)$ (see [6]). A result of G. Tomanov and B. Weiss [W] Thm. 1.4] asserts that if $\mathbb{Q}$-rank $G < \mathbb{R}$-rank $G$, then $\dim T < \mathbb{R}$-rank $G$. After seeing a preliminary version of our work, B. Weiss [W2] has recently proved the conjecture in all cases.

Geometric reformulation. We remark that, by using the well-known fact that flats in a symmetric space of noncompact type are orbits of $\mathbb{R}$-split tori in its isometry group [H, Prop. 6.1, p. 209], the conjecture and our theorem can also be stated in the following geometric terms.

Suppose $\bar{X}$ is a symmetric space, with no Euclidean (local) factors. Recall that a flat in $\bar{X}$ is a connected, totally geodesic, flat submanifold of $\bar{X}$. Up to isometry,
$\tilde{X} = G/K$, where $K$ is a compact subgroup of a connected, semisimple Lie group $G$ with finite center. Then $\mathbb{R}$-rank $G$ has the following geometric interpretation:

1.3. Fact. $\mathbb{R}$-rank $G$ is the largest natural number $r$, such that $\tilde{X}$ contains a topologically closed, simply connected, $r$-dimensional flat.

Now let $X = \Gamma\backslash\tilde{X}$ be a locally symmetric space modeled on $X$, and assume that $X$ has finite volume. Then $\mathbb{Q}$-rank $\Gamma$ is a certain algebraically defined invariant of $\Gamma$. It can be characterized by the following geometric property:

1.4. Proposition. $\mathbb{Q}$-rank $\Gamma$ is the smallest natural number $r$, for which there exists collection of finitely many $r$-dimensional flats in $X$, such that all of $X$ is within a bounded distance of the union of these flats.

It is clear from this that the $\mathbb{Q}$-rank does not change if $X$ is replaced by a finite cover, and that it satisfies $\mathbb{Q}$-rank $\Gamma \leq \mathbb{R}$-rank $G$. Furthermore, the algebraic definition easily implies that if $\mathbb{Q}$-rank $\Gamma = r$, then some finite cover of $X$ contains a topologically closed, simply connected flat of dimension $r$. If Conj. 1.1 is true, then there are no such flats of larger dimension. In other words, $\mathbb{Q}$-rank should have the following geometric interpretation, analogous to (1.3):

1.5. Conjecture. $\mathbb{Q}$-rank $\Gamma$ is the largest natural number $r$, such that some finite cover of $X$ contains a topologically closed, simply connected, $r$-dimensional flat $F$, for which the composition $F \hookrightarrow \tilde{X} \rightarrow X$ is a proper map.

More precisely, Conj. 1.1 is equivalent to the assertion that $\mathbb{Q}$-rank $\Gamma$ is the largest natural number $r$, such that $\tilde{X}$ contains a topologically closed, simply connected, $r$-dimensional flat $F$, for which the composition $F \hookrightarrow \tilde{X} \rightarrow X$ is a proper map.

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2. Example: A proof for $\mathbb{Q}$-rank 1

To illustrate the ideas in our proof of Thm. 1.2, we sketch a simple proof that applies when $\mathbb{Q}$-rank $G = 1$. (A similar proof appears in [W1 Prop. 4.12].)

Proof. Suppose $G$, $\Gamma$, $T$, and $x_0$ are as specified in Conj. 1.1. For convenience, let $\pi: G_R \rightarrow \Gamma\backslash G_R$ be the natural covering map. Assume that $\mathbb{Q}$-rank $G = 1$, that $\dim T = 2$, and that the $T$-orbit of $\pi(x_0)$ is divergent in $\Gamma\backslash G$. This will lead to a contradiction.

Let $E_1 = \Gamma\backslash G_R$. Because $\mathbb{Q}$-rank $G = 1$, reduction theory (the theory of Siegel sets) implies that there exist

- a compact subset $E_0$ of $\Gamma\backslash G_R$, and
- a $\mathbb{Q}$-representation $\rho: G \rightarrow \text{GL}_n$ (for some $n$),

such that, for each connected component $\mathcal{E}$ of $G_R \setminus \pi^{-1}(E_0)$, there is a nonzero vector $v \in \mathbb{Q}^n$, such that

\begin{equation}
(2.1) \quad \text{if } \Gamma g_n \rightarrow \infty \text{ in } \Gamma\backslash G_R, \text{ and } \{g_n\} \subset \mathcal{E}, \text{ then } \rho(g_n)v \rightarrow 0.
\end{equation}

(In geometric terms, this is the fact that, because $E_1 \setminus E_0$ consists of disjoint "cusps," $G_R \setminus \pi^{-1}(E_0)$ consists of disjoint "horoballs.")

Given $\epsilon > 0$, let $T_R$ be a large circle (1-sphere) in $T$, centered at the identity element. Because the $T$-orbit of $\pi(x_0)$ is divergent, we may assume $\pi(x_0 T_R)$ is
disjoint from $E_0$. Then, because $T_R \approx S^1$ is connected, the set $x_0 T_R$ must be contained in a single component of $G_{\mathbb{R}} \setminus \pi^{-1}(E_0)$. Thus, there is a vector $v \in \mathbb{Q}^n$, such that $\|\rho(t)v\| < \epsilon \|v\|$ for all $t \in T_R$.

Fix some $t \in T_R$. Then $t^{-1}$ also belongs to $T_R$, so $\|\rho(t)v\|$ and $\|\rho(t^{-1})v\|$ are both much smaller than $\|v\|$. This is impossible (see 3.2).

The above proof does not apply directly when $\mathbb{Q}$-rank $G = 2$, because, in this case, there are arbitrarily large compact subsets $C$ of $\Gamma \backslash G_{\mathbb{R}}$, such that $G_{\mathbb{R}} \setminus \pi^{-1}(C)$ is connected. Instead of only $E_0$ and $E_1$, we consider a more refined stratification $E_0 \subset E_1 \subset E_2$ of $\Gamma \backslash G$. (It is provided by the structure of Siegel sets in $\mathbb{Q}$-rank two. The set $E_0$ is compact, and, for $i \geq 1$, each component $E$ of $\pi^{-1}(E_1 \setminus E_{i-1})$ has a corresponding representation $\rho$ and vector $v$, such that 2.1 holds. Thus, it suffices to find a component of either $\pi^{-1}(E_1 \setminus E_0)$ or $\pi^{-1}(E_2 \setminus E_1)$ that contains two antipodal points of $T_R$. Actually, we replace $E_1$ with a slightly larger set that is open, so that we may apply the following property of $S^2$:

2.2. Proposition (see 3.1). Suppose $n \geq 2$, and that $\{V_1, V_2\}$ is an open cover of the $n$-sphere $S^n$ that consists of only 2 sets. Then there is a connected component $C$ of some $V_i$, such that $C$ contains two antipodal points of $S^n$.

2.3. Remark. In 3.1, we do not use the notation $E_0 \subset E_1 \subset E_2$. The role of $E_0$ is played by $\pi(QS_3^S)$, the role of an open set containing $E_1$ is played by $\pi(QS_\beta)$, and the role of $E_2 \setminus E_1$ is played by $\pi(QS_\delta)$.

3. Preliminaries

The classical Borsuk-Ulam Theorem implies that if $f : S^n \to \mathbb{R}^k$ is a continuous map, and $n \geq k$, then there exist two antipodal points $x$ and $y$ of $S^n$, such that $f(x) = f(y)$. We use this to prove the following stronger version of Prop. 2.2.

3.1. Proposition. Suppose $\mathcal{V}$ is an open cover of $S^n$, with $n \geq 2$, such that no point of $S^n$ is contained in more than 2 of the sets in $\mathcal{V}$. Then some $V \in \mathcal{V}$ contains two antipodal points of $S^n$.

Proof. Because $S^n$ is compact, we may assume $\mathcal{V}$ is finite. Let $\{\phi_V\}_{V \in \mathcal{V}}$ be a partition of unity subordinate to $\mathcal{V}$. This naturally defines a continuous function $\Phi$ from $S^n$ to the simplex

$$\Delta_\mathcal{V} = \left\{(x_V)_{V \in \mathcal{V}} \mid \sum_{V \in \mathcal{V}} x_V = 1\right\} \subset [0,1]^\mathcal{V}.$$ 

Namely, $\Phi(x) = (\phi_V(x))_{V \in \mathcal{V}}$. Our hypothesis on $\mathcal{V}$ implies that no more than 2 components of $\Phi(x)$ are nonzero, so the image of $\Phi$ is contained in the 1-skeleton $\Delta^{(1)}_\mathcal{V}$ of $\Delta_\mathcal{V}$. Because $S^n$ is simply connected, $\Phi$ lifts to a map from $S^n$ to the universal cover $\tilde{\Delta}^{(1)}_\mathcal{V}$ of $\Delta^{(1)}_\mathcal{V}$. The universal cover is a tree, which can be embedded in $\mathbb{R}^2$, so the Borsuk-Ulam Theorem implies that there exist two antipodal points $x$ and $y$ of $S^n$, such that $\tilde{\Phi}(x) = \tilde{\Phi}(y)$. Thus, there exists $V \in \mathcal{V}$, such that $\phi_V(x) = \phi_V(y) \neq 0$. So $x, y \in V$.

For completeness, we also provide a proof of the following simple observation.

3.2. Lemma. Let $T$ be any abelian group of diagonalizable $n \times n$ real matrices. There is a constant $\epsilon > 0$, such that if
• \( v \) is any vector in \( \mathbb{R}^n \), and
• \( t \) is any element of \( T \),

then either \( \|tv\| \geq \epsilon\|v\| \) or \( \|t^{-1}v\| \geq \epsilon\|v\| \).

Proof. The elements of \( T \) can be simultaneously diagonalized. Thus, after a change of basis (which affects norms by only a bounded factor), we may assume that each standard basis vector \( e_i \) is an eigenvector for every element of \( T \).

Let \( \epsilon = 1/n \), write \( v = (v_1, \ldots, v_n) \), and let \( t_i \) be the eigenvalue of \( t \) corresponding to the eigenvector \( e_i \). Some component \( v_j \) of \( v \) must be at least \( \|v\|/n \) in absolute value. We may assume \( |t_j| \geq 1 \), by replacing \( t \) with \( t - 1 \) if necessary. Then

\[
\|tv\| = \|(t_1 v_1, \ldots, t_n v_n)\| \geq |t_j v_j| \geq 1 \cdot \frac{\|v\|}{n} = \epsilon\|v\|,
\]
as desired. \( \square \)

4. Properties of Siegel sets

We present some basic results from reduction theory that follow easily from the fundamental work of A. Borel and Harish-Chandra [BH] (see also [B, §13–§15]). Most of what we need is essentially contained in [L, §2], but we are working in \( G, \) rather than in \( \tilde{X} = G/K \). We begin by setting up the standard notation.

4.1. Notation (cf. [L, §1]). Let
• \( G \) be a connected, almost simple \( \mathbb{Q} \)-group, with \( \mathbb{Q} \)-rank \( G = 2 \),
• \( G \) be the identity component of \( G_{\mathbb{R}} \),
• \( \Gamma \) be a finite-index subgroup of \( G_{\mathbb{Z}} \cap G \),
• \( P \) be a minimal parabolic \( \mathbb{Q} \)-subgroup of \( G \),
• \( A \) be a maximal \( \mathbb{Q} \)-split torus of \( G \),
• \( A \) be the identity component of \( A_{\mathbb{R}} \), and
• \( K \) be a maximal compact subgroup of \( G \).

We may assume \( A \subset P \). Then we have a Langlands decomposition \( P = U M A \), where \( U \) is unipotent and \( M \) is reductive. We remark that \( U \) and \( A \) are connected, but \( M \) is not connected (because \( P \) is not connected).

4.2. Notation (cf. [L, §1]). The choice of \( P \) determines an ordering of the \( \mathbb{Q} \)-roots of \( G \). Because \( \mathbb{Q} \)-rank of \( G = 2 \), there are precisely two simple \( \mathbb{Q} \)-roots \( \alpha \) and \( \beta \) (so the base \( \Delta \) is \( \{ \alpha, \beta \} \)). Then \( \alpha \) and \( \beta \) are homomorphisms from \( A \) to \( \mathbb{R}^+ \).

Any element \( g \) of \( G \) can be written in the form \( g = pak \), with \( p \in UM, a \in A \), and \( k \in K \). The element \( a \) is uniquely determined by \( g \), so we may use this decomposition to extend \( \alpha \) and \( \beta \) to continuous functions \( \tilde{\alpha} \) and \( \tilde{\beta} \) defined on all of \( G \):

\[
\tilde{\alpha}(g) = \alpha(a) \text{ if } g \in UMaK \text{ and } a \in A,
\tilde{\beta}(g) = \beta(a) \text{ if } g \in UMaK \text{ and } a \in A.
\]

4.3. Notation (cf. [L, §2]).
• Fix a subset \( Q \) of \( G_{\mathbb{Q}} \cap G \), such that

\( Q \) is a set of representatives of \( \Gamma \backslash (G_{\mathbb{Q}} \cap G)/(P_{\mathbb{Q}} \cap P) \).

Note that \( Q \) is finite.
• For \( \tau > 0 \), let \( A_{\tau} = \{ a \in A \mid \alpha(a) > \tau \text{ and } \beta(a) > \tau \} \).
Proof. It suffices to prove (1), for then (2) is immediate from the definition of $S_\gamma$ (and $D_{\alpha,\beta}^\circ$). Thus, let us suppose that $pS_\alpha \cap \gamma qS_\beta$ is not precompact. This will lead to a contradiction.

4.4. Lemma. For all $\gamma \in \Gamma$ and $p,q \in Q$, we have:

(1) $pS_\alpha \cap \gamma qS_\beta$ is precompact, and
(2) $pS_\alpha \cap \gamma qS_\beta \subset S_\alpha^+.$

Proof. It suffices to prove (1). for then (2) is immediate from the definition of $S_\alpha^+$ (and $D_{\alpha,\beta}^\circ$). Thus, let us suppose that $pS_\alpha \cap \gamma qS_\beta$ is not precompact. This will lead to a contradiction.
Because $\tilde{\alpha}$ is bounded on $S_\alpha$, but $S_\alpha \cap p^{-1}\gamma qS_\beta$ is not precompact, we know that $\tilde{\beta}$ is unbounded on $S_\alpha \cap p^{-1}\gamma qS_\beta$ (and, hence, on $S \cap p^{-1}\gamma qS$). Therefore, [L Prop. 2.3] implies that

$$p^{-1}\gamma q \in P_\alpha.$$  

Similarly (replacing $\gamma$ with $\gamma^{-1}$ and interchanging $p$ with $q$ and $\alpha$ with $\beta$), because $\gamma^{-1}pS_\alpha \cap qS_\beta = \gamma^{-1}(pS_\alpha \cap qS_\beta)$ is not precompact, we see that

$$q^{-1}\gamma^{-1}p \in P_\beta.$$  

Noting that $q^{-1}\gamma^{-1}p = (p^{-1}\gamma q)^{-1}$, we conclude that $p^{-1}\gamma q \in P_\alpha \cap P_\beta = P_\emptyset$, so [L Lem. 2.4(i)] tells us that $p = q$ and $p^{-1}\gamma q \in UM$. Therefore

$$\tilde{\alpha}(S_\alpha \cap p^{-1}\gamma qS_\beta) \subset \tilde{\alpha}(S_\alpha)$$  

and

$$\tilde{\beta}(S_\alpha \cap p^{-1}\gamma qS_\beta) \subset \tilde{\beta}(p^{-1}\gamma qS_\beta) \subset \tilde{\beta}(UMS_\beta) = \tilde{\beta}(S_\beta)$$

are precompact. So $S_\alpha \cap p^{-1}\gamma qS_\beta$ is precompact, which contradicts our assumption that $pS_\alpha \cap qS_\beta$ is not precompact. \qed

4.5. Lemma. If $\gamma \in \Gamma$ and $p, q \in Q$, such that $pS_\alpha \cap qS_\alpha \not\subset S_\Delta$, then $p = q$ and $p^{-1}\gamma q \in (UM)_Q$.

Proof. It suffices to show that both $\tilde{\alpha}$ and $\tilde{\beta}$ are unbounded on $S \cap p^{-1}\gamma qS$, for then the desired conclusion is obtained from [L Prop. 2.3 and Lem. 2.4(i)]. Thus, let us suppose (without loss of generality) that $\tilde{\alpha}$ is bounded on $S \cap p^{-1}\gamma qS$.

This will lead to a contradiction.

Case 1. Assume $\tilde{\beta}$ is also bounded on $S \cap p^{-1}\gamma qS$. Then $pS \cap qS = p(S \cap p^{-1}\gamma qS)$ is precompact, so, by definition, $pS \cap qS \subset S_\Delta^+$. Therefore

$$pS_\alpha \cap qS_\alpha \subset pS \cap qS \subset S_\Delta^+.$$  

This contradicts the hypothesis of the lemma.

Case 2. Assume $\tilde{\beta}$ is not bounded on $S \cap p^{-1}\gamma qS$. From [L Lem. 2.5], we see that $pS \cap qS \subset pS_\alpha$. Therefore

$$pS_\alpha \cap qS_\alpha \subset pS_\alpha \cap pS_\alpha = \emptyset \subset S_\Delta^+.$$  

This contradicts the hypothesis of the lemma. \qed

4.6. Corollary. If $x$ and $y$ are two points in the same connected component of $\Gamma Q S_\alpha \cap S_\Delta^+$, then there exist $\gamma_0, \gamma \in \Gamma$ and $q \in Q$, such that $x \in \gamma_0 qS_\alpha$, $y \in \gamma qS_\alpha$, and $q^{-1}\gamma q \in (UM)_Q$.

4.7. Lemma.

1. If $\gamma \in \Gamma$ and $p, q \in Q$, such that $pS_\alpha \cap qS_\alpha \not\subset S_\Delta^+$, then $p^{-1}\gamma q \in (P_\alpha)_Q$.

2. For each $p, q \in Q$, there exists $h_{p,q} \in (P_\alpha)_Q$, such that $p^{-1}\Gamma q \cap (P_\alpha)_Q \subset h_{p,q}(U_\alpha M_\alpha)_Q$.  

Proof. 1. Because \( pS_\alpha \cap \gamma qS_\alpha \not\subset S^+_{\Delta} \), we know, from the definition of \( S^+_{\Delta} \) (and \( D^p,q \)) that \( pS_\alpha \cap \gamma qS_\alpha \) is not precompact. Since \( \tilde{\alpha} \) is bounded on \( S_\alpha \), we conclude that \( \tilde{\beta} \) is not bounded on \( S_\alpha \cap p^{-1}qS_\alpha \) (and, hence, on \( S \cap p^{-1}qS \)). Then 1 Prop. 2.3 asserts that \( p^{-1}q \in (P_0)\), for \( \Theta = \{\alpha\} \) or \( \emptyset \). Because \( P_0 \subset P_\alpha \), we conclude that \( p^{-1}q \in (P_\alpha)\).

2. From 1 Lem. 2.4(ii), we see that the coset \( (p^{-1}q)(U_0 M_\alpha)\) does not depend on the choice of \( \gamma \), if we require \( \gamma \) to be an element of \( \Gamma \), such that \( p^{-1}q \in (P_\alpha)\).

4.8. Corollary. If \( x \) and \( y \) are two points in the same connected component of \( \Gamma Q S_\alpha \setminus \Gamma S^+_{\Delta} \), then there exist \( \gamma_0, \gamma \in \Gamma \) and \( p, q \in Q \), such that \( x \in \gamma_0 p S_\alpha \), \( y \in \gamma_0 q S_\alpha \), and \( p^{-1}q \in h_{p,q}(U_0 M_\alpha)\).

5. Proof of the Main Theorem

Let \( G, \Gamma, T \) and \( x_0 \) be as described in the hypotheses of Thm. 1.2 and assume \( \dim T \geq 3 \). (This will lead to a contradiction.) Let \( T_R \) be a large sphere (centered at the origin) in \( T \). Because \( S^+_{\Delta} \) is compact and the \( T \)-orbit of \( \Gamma x_0 \) is divergent in \( \Gamma \setminus G \), we may assume that

\[
(x_0 T_R) \cap (\Gamma S^+_{\Delta}) = \emptyset.
\]

Let \( V_1 = \{ t \in T_R \mid x_0 t \in \Gamma Q S_\alpha \} \) and

\[
V_2 = \{ t \in T_R \mid x_0 t \in \Gamma Q S_\alpha \cup \Gamma Q S_\beta \}.
\]

From Prop. 4.2, we know there exists \( t \in T_R \), and a connected component \( C \) of either \( V_1 \) or \( V_2 \), such that \( t \) and \( t^{-1} \) both belong to \( C \).

Case 1. Assume \( C \) is a component of \( V_1 \). From Cor. 4.6, we see that there exist \( \gamma_0, \gamma \in \Gamma \) and \( q \in Q \), such that \( x_0 t \in \gamma_0 q S_\alpha \), \( x_0 t^{-1} \in \gamma_0 q S_{\Delta} \), and \( q^{-1} \gamma q \in (U M)_{\emptyset} \).

Because \( \Gamma x_0 t \) and \( \Gamma x_0 t^{-1} \) are near infinity in \( \Gamma \setminus G \), we must have

1. either \( \tilde{\alpha}(q^{-1} \gamma_0^{-1} x_0 t) \gg 1 \) or \( \tilde{\beta}(q^{-1} \gamma_0^{-1} x_0 t^{-1}) \gg 1 \), and

2. either \( \tilde{\alpha}(q^{-1} \gamma_0^{-1} x_0 t^{-1}) \gg 1 \) or \( \tilde{\beta}(q^{-1} \gamma_0^{-1} x_0 t) \gg 1 \).

Since \( q^{-1} \gamma q \in (U M)_{\emptyset} \) is sent to the identity element by both \( \tilde{\alpha} \) and \( \tilde{\beta} \), we have 2 either \( \tilde{\alpha}(q^{-1} \gamma_0^{-1} x_0 t^{-1}) \gg 1 \) or \( \tilde{\beta}(q^{-1} \gamma_0^{-1} x_0 t) \gg 1 \).

Let

- \( V = \wedge^d g \), where \( d = \dim U \),
- \( \rho: G \to \text{GL}(V) \) be the natural adjoint representation of \( G \) on \( V \),
- \( v_u \) be a nonzero element of \( V_Z \) in the one-dimensional subspace \( \wedge^d u \), and
- \( v'_u = \rho(x_0^{-1} \gamma_0) v_u \).

It is important to note that \( ||v'_u|| \) is bounded away from 0, independent of the choice of \( q \) and \( \gamma_0 \). (There are only finitely many choices of \( q \), so \( q \) is not really an issue. The key point is that \( \rho(q)v_u \) is a \( Q \)-element of \( V \), so its \( G_Z \)-orbit is bounded away from 0.)

On the other hand, for any \( g \in P_0 \), we have \( \rho(g^{-1}) v_u = \tilde{\alpha}(g)^{\ell_1} \tilde{\beta}(g)^{\ell_2} v_u \), for some positive integers \( \ell_1 \) and \( \ell_2 \) (because the sum of the positive \( Q \)-roots of \( G \) is \( \ell_1 \alpha + \ell_2 \beta \)). Therefore, from 1 and 2, we see that

\[
\rho(t^{-1}) v'_u = \rho((q^{-1} \gamma_0^{-1} x_0 t^{-1})) v_u \approx 0,
\]
and
\[ \rho(t) v'_u = \rho((q^{-1} \gamma_0^{-1} x_0 t^{-1})^{-1}) v_u \approx 0. \]

This contradicts Lem. \[ 4.2 \]

Case 2. Assume \( C \) is a component of \( V_2 \). From Lem. \[ 1.4 \], we see that \( x_0 C \) is contained in either \( \Gamma Q S_\alpha \) or \( \Gamma Q S_\beta \). Assume, without loss of generality, that \( x_0 C \subset \Gamma QS_\alpha \). From Cor. \[ 4.8 \] we see that there exist \( \gamma_0, \gamma \in \Gamma \) and \( p, q \in Q \), such that
\[ x_0 t \in \gamma_0 \rho S_\alpha, \quad x_0 t^{-1} \in \gamma_0 \gamma q S_\alpha, \quad \text{and} \quad p^{-1} \gamma q \in h_{p,q}(U_\alpha M_\alpha) Q. \]

Let \( u_\alpha \) be the Lie algebra of \( U_\alpha \), and let \( \rho_\alpha : G \to GL(V_\alpha) \) be the natural adjoint representation of \( G \) on \( V_\alpha = \bigwedge^{d_\alpha} g \), where \( d_\alpha = \dim u_\alpha \).

We can obtain a contradiction by arguing as in Case 1, with the representation \( \rho_\alpha \) in the place of \( \rho \). To see this, note that:

- For \( a \in \ker \alpha \), we have \( \rho_\alpha(a^{-1}) v_{u_\alpha} = \beta(a)^{-\ell} v_{u_\alpha} \), for some positive integer \( \ell \). Since \( \rho_\alpha(UM) \subset \rho_\alpha(U_\alpha M_\alpha) \) fixes \( v_{u_\alpha} \), and \( \rho_\alpha(K) \) is compact, this implies that
  \[ \| \rho_\alpha(g^{-1}) v_{u_\alpha} \| \geq \beta(g)^{-\ell} \|
u_{u_\alpha}\| \quad \text{for} \quad g \in S_\alpha. \]

- Because \( \Gamma x_0 t \) and \( \Gamma x_0 t^{-1} \) are near infinity in \( \Gamma \setminus G \), and \( \bar{\alpha} \) is bounded on \( S_\alpha \), we must have
  \[ \beta(p^{-1} \gamma_0^{-1} x_0 t) \gg 1 \quad \text{and} \quad \beta(q^{-1} \gamma^{-1} \gamma_0^{-1} x_0 t^{-1}) \gg 1. \]

Therefore, letting \( v'_{u_\alpha} = \rho_\alpha(x_0^{-1} \gamma_{0p}) v_{u_\alpha} \), we have
  \[ \| \rho_\alpha(t^{-1}) v'_{u_\alpha} \| \approx 0. \]

Because \( h_{p,q} \in P_\alpha \) normalizes \( U_\alpha \), we have \( \rho_\alpha(h_{p,q} v_{u_\alpha}) = c_{p,q} v_{u_\alpha} \), for some scalar \( c_{p,q} \). Therefore, since \( (p^{-1} \gamma q) h_{p,q}^{-1} \in (U_\alpha M_\alpha) Q \) fixes \( v_{u_\alpha} \), and \( \{c_{p,q}\} \), being finite, is bounded away from 0, we have
  \[ \rho_\alpha(t) v'_{u_\alpha} = \rho_\alpha(t x_0^{-1} \gamma_{0p}) v_{u_\alpha} = \rho_\alpha(t x_0^{-1} \gamma_{0p} p^{-1} \gamma q) v_{u_\alpha} = c_{p,q}^{-1} \rho_\alpha(t x_0^{-1} \gamma_{0p} q) v_{u_\alpha} \approx 0. \]

This completes the proof of Thm. \[ 1.2 \]

6. Results for Higher \( \mathbb{Q} \)-rank

The proof of Thm. \[ 1.2 \] generalizes to establish the following result:

6.1. Theorem. Suppose \( G, \Gamma, T, \) and \( x_0 \) are as specified in Conj. \[ 1.1 \] and assume \( \mathbb{Q} \)-rank \( G \geq 1 \). If the \( T \)-orbit of \( \Gamma x_0 \) is divergent in \( \Gamma \backslash \mathbb{G}_\mathbb{R} \), then \( \dim T \leq 2(\mathbb{Q}\text{-rank } G) - 1 \).

Sketch of proof. As in \[ 1.4 \] §1 and §2, let \( \Delta \) be the set of simple \( \mathbb{Q} \)-roots, construct a fundamental set \( QS \), define the finite set \( \mathcal{D} \), and choose \( r > 0 \), such that, for \( q \in \mathcal{D} \) and \( \alpha \in \Delta \), we have
\[ \text{if } \tilde{\alpha} \text{ is bounded on } S \cap qS, \text{ then } \tilde{\alpha}(S \cap qS) < r. \]

Fix any \( r^* > r \). For each subset \( \Theta \) of \( \Delta \), let
\[ S_\Theta = \{ x \in S \mid \tilde{\alpha}(x) < r^*, \ \forall \alpha \in \Theta \} \quad \text{and} \quad S^-_\Theta = \{ x \in S \mid \tilde{\alpha}(x) \leq r, \ \forall \alpha \in \Theta \}. \]
and choose \( h_{p,q}^\Theta \) such that \( p^{-1}\Gamma q \cap (P_\Theta)_Q \subset h_{p,q}^\Theta(U_\Theta M_\Theta)_Q \) for \( p, q \in Q \). Set \( d = \mathbb{Q}\)-rank \( G \), and, for \( i = 0, \ldots, d \), let

\[
E_i \equiv \bigcup_{\Theta \subset \Delta} S^\Theta \quad \text{and} \quad E_i^\sim \equiv \bigcup_{\Theta \subset \Delta} S^\Theta_{\sim},
\]

Then \( \{ E_1, E_2 \setminus E_1, \ldots, E_d \setminus E_{d-1} \} \) is an open cover of \( \Gamma \backslash G \).

For \( p, q \in Q \) and \( \Theta_1, \Theta_2 \subset \Delta \), let

\[
D^{p,q}_{\Theta_1, \Theta_2} = \{ \gamma \in \Gamma \mid pS^\Theta_1 \cap \gamma qS^\Theta_2 \text{ is precompact and nonempty} \}.
\]

Define

\[
S^\Delta = \bigcup_{p, q \in Q, \Theta_1, \Theta_2 \subset \Delta, \gamma \in D^{p,q}_{\Theta_1, \Theta_2}} (pS^\Theta_1 \cap \gamma qS^\Theta_2).
\]

Suppose \( \dim T \geq 2d \). Then we may choose a large \((2d - 1)\)-sphere \( T_R \) in \( T \). Prop. 6.2 below implies that there exists \( t \in T_R \) and a component \( C \) of some \( E_i \setminus E_{i-1} \), such that \( x_0t \) and \( x_0t^{-1} \) belong to \( C \). If \( T_R \) is chosen large enough that \( x_0T_R \) is disjoint from \( \Gamma S^\Delta \), then there exist \( \Theta \subset \Delta \) (with \( \#\Theta = i \), \( \gamma_0, \gamma \in \Gamma \), and \( p, q \in Q \), such that \( x_0t \in \gamma_0 pS^\Theta_1 \), \( x_0t^{-1} \in \gamma_0 qS^\Theta_2 \), and \( \gamma^{-1}q \in h_{p,q}^\Theta(U_\Theta M_\Theta)_Q \). We obtain a contradiction as in Case 3 of 15 using \( u_\Theta \) in the place of \( u \).

The following result is obtained from the proof of Prop. 5.1 by using the fact that any simplicial complex of dimension \( d - 1 \) can be embedded in \( \mathbb{R}^{2d-1} \).

6.2. Proposition. Suppose \( n \geq 2d - 1 \), and that \( \{ V_1, V_2, \ldots, V_d \} \) is an open cover of the \( n \)-sphere \( S^n \) that consists of only \( d \) sets. Then there is a connected component \( C \) of some \( V_i \), such that \( C \) contains two antipodal points of \( S^n \).

6.3. Remark. For \( k \geq 1 \), it is known [3, 4] that there exist a simplicial complex \( \Sigma^k \) of dimension \( k \) and a continuous map \( f : S^{2k-1} \to \Sigma^k \), such that no two antipodal points of \( S^{2k-1} \) map to the same point of \( \Sigma^k \). This implies that the constant \( 2d - 1 \) in Prop. 6.2 cannot be improved to \( 2d - 3 \).

6.4. Remark. If \( \mathbb{Q}\)-rank \( G = 2 \), then the conclusion of Thm. 1.2 is stronger than that of Thm. 6.1. The improved bound in 1.2 results from the fact that if \( d = 2 \), then the universal cover of any \((d - 1)\)-dimensional simplicial complex embeds in \( \mathbb{R}^2 = \mathbb{R}^{2d-2} \). (See the proof of Prop. 5.1) When \( d > 2 \), there are examples of (simply connected) \((d - 1)\)-dimensional simplicial complexes that embed only in \( \mathbb{R}^{2d-1} \), not \( \mathbb{R}^{2d-2} \).

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