INDEX ESTIMATES FOR FREE BOUNDARY $f$-MINIMAL HYPERSURFACES

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Abstract. We prove that the index is bounded from below by a linear function of its first Betti number for any compact free boundary $f$-minimal hypersurface in certain positively curved weighted manifolds.

1. Introduction

Given a Riemannian manifold $(N^{n+1},g)$, the free boundary problem consists of finding critical points of the area functional among all compact hypersurfaces $M^n \subset N$ with $\partial M \subset \partial N$. Critical points for this problem are minimal hypersurfaces $M \subset N$ meeting $\partial N$ orthogonally along $\partial M$, called free boundary minimal hypersurfaces. There are many comparison results between the Morse index and the topology of free boundary minimal hypersurfaces (cf. [1, 3, 4, 5, 6, 8, 12], etc.).

Via embedding the ambient manifold (certain positively curved) in a Euclidean space and using the coordinates of $N \wedge \omega^\sharp$ as test functions (where $N$ is a unit normal vector field and $\omega$ is a harmonic 1-form of the hypersurface), Ambrozio, Carlotto and Sharp [2] proved that the index of any closed minimal hypersurface is bounded from below by a linear function of its first Betti number. In [3] they used this method to establish such index lower bound for free boundary minimal hypersurfaces in a general manifold with similar extrinsic assumptions (embedding the ambient manifold in a Euclidean space with some integral inequality assumption about curvatures). In [14], Impera, Rimoldi and Savo obtained such index bound for closed and noncompact complete $f$-minimal hypersurfaces in the weighted manifold $(\mathbb{R}^{n+1},g_{\text{can}},e^{-f}dV_{\mathbb{R}^{n+1}})$. Impera and Rimoldi [13] obtained the index of closed $f$-minimal hypersurfaces immersed in a general weighted manifold with similar extrinsic assumptions.

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In this paper, as a complement to the results above, we consider the index estimates for compact free boundary $f$-minimal hypersurfaces in general weighted manifolds with similar extrinsic assumptions.

**Theorem 1.1.** Let $M^n$ be a compact, orientable, free boundary $f$-minimal hypersurface of a weighted Riemannian manifold $(N^{n+1}, g, e^{-f} dV_N)$. Let $N^{n+1}$ be isometrically immersed in some Euclidean space $\mathbb{R}^d$.

(1) Assume that for any nonzero tangent $f$-harmonic 1-form $\omega \in \mathcal{H}^1_{Nf}(M)$,

$$\int_M \left( Ric^N_N(\omega^\sharp, \omega^\sharp) + Ric^N_f(N, N) |\omega|^2 - K^N(\omega^\sharp, N) \right) e^{-f} dV_M > \sum_{k=1}^n \left( |\Pi^N(e_k, \omega^\sharp)|^2 + |\Pi^N(e_k, N)|^2 |\omega|^2 \right) e^{-f} dV_M$$

Then

$$\text{Index}_{f}(M) \geq \frac{2}{d(d-1)} \dim H^1(M, \mathbb{R}).$$

(2) Assume that for any nonzero normal $f$-harmonic 1-form $\omega \in \mathcal{H}^1_{Nf}(M)$,

$$\int_M \left( Ric^N_f(\omega^\sharp, \omega^\sharp) + Ric^N_f(N, N) |\omega|^2 - K^N(\omega^\sharp, N) \right) e^{-f} dV_M > \sum_{k=1}^n \left( |\Pi^N(e_k, \omega^\sharp)|^2 + |\Pi^N(e_k, N)|^2 |\omega|^2 \right) e^{-f} dV_M$$

Then

$$\text{Index}_{f}(M) \geq \frac{2}{d(d-1)} \dim H^{n-1}(M, \mathbb{R}).$$

Here $\text{Ric}_f^N = \text{Ric}^N + \text{Hess}^N f$ denotes the Bakery-Emery Ricci tensor of $N$; $K^N$ denotes the sectional curvature of $N$; $\Pi^N$ denotes the second fundamental form of $N^{n+1}$ in $\mathbb{R}^d$; $\Pi^{\partial N}$ denotes the scalar second fundamental form of $\partial N$ in $N$ with respect to the inward unit normal vector field $\nu$; $H^{\partial N}_f = H^{\partial N} + \langle \nabla f, \nu \rangle$ is the $f$-mean curvature of $\partial N$ in $N$; $H^{\partial N}$ is the mean curvature of $\partial N$ in $N$; $N$ is a unit normal vector field of $M$ in $N$; $\omega^\sharp$ is the dual vector field of $\omega$; and $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on $M^n$.

Theorem 1.1 generalizes all the index estimates of compact hypersurfaces with or without boundary in [2, 3, 13, 14]. For example, we apply it to give the following.
Theorem 1.2. Let $N^{n+1}$ be a compact domain in $\mathbb{R}^{n+1}$. Let $M^n$ be a free boundary $f$-minimal hypersurface of the weighted manifold $(N^{n+1}, g_{can}, e^{-f}dV_N)$ with $\text{Hess}^N f \geq 0$.

1. If $\partial N$ is strictly two-convex in $N$, then
   $$\text{Index}_f(M) \geq \frac{2}{n(n+1)} \dim H^1(M, \mathbb{R}).$$

2. If $\partial N$ is strictly $f$-mean convex in $N$, then
   $$\text{Index}_f(M) \geq \frac{2}{n(n+1)} \dim H^{n-1}(M, \mathbb{R}).$$

When $f = 0$, Theorem 1.2 is a combination of Theorem A and Theorem F of [3].

2. Preliminary

In this section, we prepare the notations and some useful lemmas.

By Nash’s embedding theorem, any Riemannian manifold $(N^{n+1}, g)$ can be isometrically embedded in a sufficiently high-dimensional Euclidean space $\mathbb{R}^d$. For an immersed hypersurface $M^n$ of $N^{n+1}$, let $D, \nabla$ and $\nabla$ denote the Levi-Civita connection of the Euclidean space, $N$ and $M$ respectively. The relation between these connections is given by

\[
D_XY = \overline{\nabla}_XY + \Pi^N(X, Y), \\
\overline{\nabla}_XY = \nabla_XY + \Pi^M(X, Y),
\]

where $X, Y$ are vectors fields tangent to $M^n$; $\Pi^N$ is the second fundamental form of $N^{n+1}$ in $\mathbb{R}^d$; and $\Pi^M$ is the second fundamental form of $M^n$ in $N^{n+1}$.

A weighted Riemannian manifold $(N^{n+1}, g, e^{-f}dV_N)$ is a Riemannian manifold endowed with a measure with smooth positive density $e^{-f}$ with respect to the Riemannian volume measure $dV_N$. We are interested in orientable $f$-minimal hypersurfaces $M$ in $N$, namely the critical points of the weighted volume functional

\[
\text{Vol}_f(M) = \int_M e^{-f}dV_M.
\]

The first variation formula for a variation $M_t$ of $M$ with variation field $\xi$ is given by (see [11, Lemma 3.2])

\[
\frac{d}{dt} \text{Vol}_f(M_t) \Big|_{t=0} = -\int_M uH_f e^{-f}dV_M + \int_{\partial M} g(\eta, \xi) e^{-f}dV_{\partial M}.
\]

Here $u = g(N, \xi)$; $N$ is the fixed unit normal vector field of $M$ in $N$; $\eta$ is the outward unit normal vector field along $\partial M$ in $M$; and $H_f$ is the $f$-mean curvature of $M$ in $N$:

\[
H_f = H^M + \frac{\partial f}{\partial N},
\]
where $H^M = \operatorname{tr} g(\Pi^M, N)$ is the mean curvature of $M$ in $N$. Note that $\partial M$ varies inside $\partial N$ for the free boundary problem. Then $M$ is critical for the $f$-volume, called $f$-minimal with free boundary if and only if $H_f = 0$ identically on $M$ and $\eta = -\nu$, where $\nu$ is the inward unit normal vector field of $\partial N$ in $N$.

The quadratic form associated to the second variation of the $f$-volume of a free boundary $f$-minimal surface with variation field $\xi = uN$ is (see [11, Proposition 3.5])

$$Q_f(u, u) = \int_M \left( |\nabla u|^2 - (\Ric_f^N(N, N) + |\Pi^M|^2) u^2 \right) e^{-f} dV_M$$

$$- \int_{\partial M} \Pi^N(N, N) u^2 e^{-f} dV_{\partial M},$$

where $\Ric_f^N = \Ric^N + \operatorname{Hess}^N f$ denotes the Bakery-Emery Ricci tensor of the ambient manifold $N$ (see [11]) and $\Pi^N(X, Y) = g(\overline{\nabla}_X Y, \nu)$ denotes the scalar second fundamental form of $\partial N$ in $N$ with respect to $\nu$.

The $f$-index of $M$ is the maximal dimension of a linear subspace $V$ in $C^\infty(M)$ on which the quadratic form $Q_f$ is negative (cf. [13]). We can also write (2.1) making use of the divergence theorem by

$$Q_f(u, u) = \int_M \left( u \Delta_f u - \left( \Ric_f^N(N, N) + |\Pi^M|^2 \right) u^2 \right) e^{-f} dV_M$$

$$+ \int_{\partial M} u \left( \frac{\partial u}{\partial \eta} - \Pi^N(N, N) u \right) e^{-f} dV_{\partial M},$$

where $\Delta_f u = \Delta u + \langle \nabla f, \nabla u \rangle$, $\Delta u = -\operatorname{div}(\nabla u)$, and the bracket $\langle \cdot, \cdot \rangle$ always denotes the usual inner product between tensors induced by the metric. The elliptic operator $L_f = \Delta_f - (\Ric_f^N(N, N) + |\Pi^M|^2)$ is called the weighted Jacobi operator of $M$.

The boundary condition

$$\frac{\partial u}{\partial \eta} - \Pi^N(N, N) u = 0$$

makes the weighted Jacobi operator $L_f$ self-dual. Under this boundary condition, there exists a non-decreasing and diverging sequence of eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to \infty$, associated to a $L^2(M, e^{-f} dV_M)$-orthonormal basis $\{u_k\}_{k=1}^\infty$ of solutions to the eigenvalue problem

$$L_f u = \lambda u, \quad \text{in } M,$$

$$\frac{\partial u}{\partial \eta} - \Pi^N(N, N) u = 0, \quad \text{on } \partial M.$$

From the Courant-Hilbert variational characterization for solutions of (2.2) (see [9, 16]), if $V_k$ denotes the subspace spanned by the first $k$ eigenfunctions for the
above problem, then the next eigenvalue \( \lambda_{k+1}(L_f) \) equals the minimum of \( Q_f \) on the \( L^2(M, e^{-f}dV_M) \) orthogonal complement of \( V_k \), i.e.,

\[
\lambda_{k+1}(L_f) = \inf_{u \in V_k^1 \setminus \{0\}} \frac{Q_f(u, u)}{\int_M u^2 e^{-f}dV_M}
\]

Since we will use \( f \)-harmonic 1-forms of the \( f \)-minimal hypersurfaces rather than harmonic 1-forms to construct test functions for the quadratic form \( Q_f \), we introduce some basic facts about \( f \)-harmonic 1-forms on manifolds with boundary below.

Let \( \iota : \partial M \to M^n \) and \( \iota^* \) denote the natural inclusion and its pull-back map. A \( p \)-form \( \omega \in \Omega^p(M) \) is called \( f \)-harmonic if \( d\omega = 0 \) and \( \delta_f\omega = 0 \), where \( \delta_f\omega = \delta\omega + i_X\omega \); \( i_X \) is the contraction operator from the left hand by \( X \); \( \delta = (-1)^{(p+1)+1} * d* \) is the codifferential operator; and \( * \) is the Hodge operator with respect to the metric on \( M \) (see [14]). We denote the spaces of \( f \)-harmonic \( p \)-forms tangent and normal at the boundary respectively by

\[
\mathcal{H}^p_{Nf}(M) = \{ \omega \in \Omega^p(M) | d\omega = 0, \delta_f\omega = 0 \text{ on } M \text{ and } \iota^*\omega = 0 \text{ on } \partial M \},
\]

\[
\mathcal{H}^p_{Tf}(M) = \{ \omega \in \Omega^p(M) | d\omega = 0, \delta_f\omega = 0 \text{ on } M \text{ and } \iota^*\omega = 0 \text{ on } \partial M \}.
\]

For \( f \equiv 0 \), they are denoted as \( \mathcal{H}^p_N(M) \) and \( \mathcal{H}^p_T(M) \) respectively in [10]. By the Hodge decomposition, we know \( \mathcal{H}^p_N(M) \cong H^p(M, \mathbb{R}) \cong H_{n-p}(M, \partial M, \mathbb{R}) \) and \( \mathcal{H}^p_T(M) \cong H^p_{n-p}(M) \) which still hold in the weighted case (see [7, 17]). Hence, the dimension of \( \mathcal{H}^1_{Nf}(M) \) equals the first Betti number \( b_1(M) := \dim H^1(M, \mathbb{R}) \). In fact, the isomorphism \( \mathcal{H}^1_N(M) \cong \mathcal{H}^1_{Nf}(M) \) (resp. \( \mathcal{H}^1_T(M) \cong \mathcal{H}^1_{Tf}(M) \)) follows directly by setting \( \tilde{\omega} = \omega + du \) for \( \omega \in \mathcal{H}^1_N(M) \) (resp. \( \omega \in \mathcal{H}^1_T(M) \)), where \( u \in C^\infty(M) \) is a solution of the equation

\[
\begin{cases}
\Delta_f u = -i_X f \omega & \text{on } M, \\
\frac{\partial u}{\partial n} = 0 & (\text{resp. } \frac{\partial u}{\partial n} = -i^*\omega, u = 0) \quad \text{on } \partial M.
\end{cases}
\]

This equation is solvable because

\[
\int_M -i_X f \omega e^{-f}dV_M = \int_M -\delta(\omega e^{-f})dV_M = \int_{\partial M} i_\eta(\omega e^{-f})d\nu_M = 0.
\]

The definition of tangent and normal \( f \)-harmonic forms for compact manifolds with nonempty boundary also makes the action of the weighted Laplacian operator \( \Delta_f = d\delta_f + \delta_f d \) self-dual by the following lemma.

**Lemma 2.1.** For any two \( p \)-forms \( \omega_1, \omega_2 \in \Omega^p(M) \), we have

\[
(2.3) \quad \int_M \left( \langle \Delta_f \omega_1, \omega_2 \rangle - \langle d\omega_1, d\omega_2 \rangle - \langle \delta_f \omega_1, \delta_f \omega_2 \rangle \right) e^{-f}dV_M
\]

\[
= -\int_{\partial M} \left( \langle \iota_\eta d\omega_1, \iota^* \omega_2 \rangle - \langle \iota_\eta \omega_1, \iota^* \delta_f \omega_1 \rangle \right) e^{-f}d\nu_M.
\]
In particular, a $p$-form $\omega \in \Omega^p(M)$ is tangent $f$-harmonic if and only if $\Delta_f \omega = 0$ on $M$, $i_\eta d \omega = 0$ and $i_\eta \omega = 0$ on $\partial M$; $\omega \in \Omega^p(M)$ is normal $f$-harmonic if and only if $\Delta_f \omega = 0$ on $M$, $i^* \omega = 0$ and $i^* \delta_f \omega = 0$ on $\partial M$.

**Proof.** Using the notations of [17], let $\omega$ denote the contraction of tensors from the right hand. Then we have
\[
\langle \delta \alpha, \beta \rangle - \langle \alpha, d \beta \rangle = \delta (\alpha \wedge \beta), \quad \text{for } \alpha \in \Omega^{p+1}(M), \beta \in \Omega^p(M),
\]
and thus
\[
\langle \delta_f \alpha, \beta \rangle - \langle \alpha, d \beta \rangle = \langle i_{\nabla f} \alpha, \beta \rangle + \delta (\alpha \wedge \beta), \quad \text{for } \alpha \in \Omega^{p+1}(M), \beta \in \Omega^p(M).
\]
It follows that
\[
\langle \delta_f \omega_1, \omega_2 \rangle e^{-f} - \langle d \omega_1, d \omega_2 \rangle e^{-f} = \delta (d \omega_1 \wedge \omega_2 e^{-f}), \quad (2.3)
\]
\[
(\delta_f \omega_1, \omega_2) e^{-f} - \langle \delta_f \omega_1, \delta_f \omega_2 \rangle e^{-f} = -\delta (\omega_2 e^{-f} \delta_f \omega_1) \quad (2.4)
\]
Using the Stokes’ theorem
\[
\int_M \delta \theta dV_M = -\int_{\partial M} i_\eta \theta dV_{\partial M}, \quad \text{for } \theta \in \Omega^1(M),
\]
we obtain (2.3) by taking sum and integration of the above two formulae.

In this paper, we use the usual musical isomorphism to pass from 1-forms to vectors, i.e., for a 1-form $\omega \in \Omega^1(M)$, $\omega^\sharp$ is the unique vector field on $M$ such that $\omega(Y) = \langle \omega^\sharp, Y \rangle$ for all vector fields $Y$.

**Lemma 2.2.** Let $M^n$ be a free boundary $f$-minimal hypersurface of $N^{n+1}$.

(1) Let $\omega \in \mathcal{H}^f_{\text{tf}}(M)$ be a tangent $f$-harmonic 1-form. Then
\[
\int_M |\nabla \omega|^2 e^{-f} dV_M = -\int_M \left( \text{Ric}^N_f(\omega^\sharp, \omega^\sharp) - K^N(\omega^\sharp, N) - |H^M(\cdot, \omega^\sharp)|^2 \right) e^{-f} dV_M
\]
\[-\int_{\partial M} \Pi_{\partial N}(\omega^\sharp, \omega^\sharp) e^{-f} dV_{\partial M}.
\]
(2) Let $\omega \in \mathcal{H}^f_{\text{bf}}(M)$ be a normal $f$-harmonic 1-form. Then
\[
\int_M |\nabla \omega|^2 e^{-f} dV_M = -\int_M \left( \text{Ric}^N_f(\omega^\sharp, \omega^\sharp) - K^N(\omega^\sharp, N) - |H^M(\cdot, \omega^\sharp)|^2 \right) e^{-f} dV_M
\]
\[-\int_{\partial M} (H_{\partial M} + \langle \nabla f, \nu \rangle) |\omega|^2 e^{-f} dV_{\partial M}.
\]
**Proof.** (1) Since $\omega$ is $f$-harmonic, we have the $f$-Bochner-Weitzenbock formula (cf. [15])
\[
(2.4) \quad -\Delta_f \frac{|\omega|^2}{2} = |\nabla \omega|^2 + \text{Ric}^M_f(\omega^\sharp, \omega^\sharp).
\]
Computing the exterior derivative along $\partial M$, we get
\[ d\omega(\eta, \omega^f) = \langle \nabla_\eta \omega^f, \omega^f \rangle - \langle \nabla_\omega^f \omega^f, \eta \rangle. \]

Since $d\omega = 0$, we have
\[ \langle \nabla_\eta \omega^f, \omega^f \rangle = \langle \nabla_\omega^f \omega^f, \eta \rangle. \]

Since $\omega$ is tangent to the boundary, $\omega^f$ is a tangent vector field on $\partial M \subset \partial N$. Then it follows from the free boundary property and Lemma 2.1 that
\[
\int_M \left( \Delta f \frac{1}{2} |\omega|^2 \right) e^{-f} dV_M = -\int_{\partial M} \left( \iota_{\gamma} d\frac{|\omega|^2}{2} \right) e^{-f} dV_{\partial M} = -\int_{\partial M} \langle \nabla_\eta \omega^f, \omega^f \rangle e^{-f} dV_{\partial M}
\]
\[ = -\int_{\partial M} \langle \nabla_\omega^f \omega^f, \eta \rangle e^{-f} dV_{\partial M} = \int_{\partial M} \langle \nabla_\omega^f \omega^f, \nu \rangle e^{-f} dV_{\partial M} = \int_{\partial M} \Pi^{\partial N}(\omega^f, \omega^f) e^{-f} dV_{\partial M}. \]

Integrating (2.4) we get
\[
\int_M \left( |\nabla_\omega^f|^2 + \text{Ric}^M(\omega^f, \omega^f) \right) e^{-f} dV_M = -\int_{\partial M} \Pi^{\partial N}(\omega^f, \omega^f) e^{-f} dV_{\partial M}. \tag{2.5}
\]

The Gauss equation for $f$-minimal hypersurfaces implies
\[
\text{Ric}^N(\omega^f, \omega^f) = \text{Ric}^N(\omega, \omega) - K^N(\omega^f, N) - |\Pi(\cdot, \omega^f)|^2. \tag{2.6}
\]

Substituting (2.6) in (2.5) we obtain the required formula.

(2) Let $\{e_k\}_{k=1}^{n-1}$ be a local orthonormal frame of $\partial M$. Since $\omega$ is normal $f$-harmonic, $\delta_f \omega = 0$ and $\omega^f = \lambda \eta$ for some function $\lambda$ on $\partial M$. Then on $\partial M$,
\[
0 = \delta_f \omega = -\sum_{k=1}^{n-1} \langle \nabla_{e_k} \omega^f, e_k \rangle - \langle \nabla_\eta \omega^f, \eta \rangle + \langle \nabla f, \omega^f \rangle = -\lambda H^{\partial M} - \langle \nabla_\eta \omega^f, \eta \rangle + \lambda \langle \nabla f, \eta \rangle,
\]
where $H^{\partial M} = \text{tr} (\Pi^{\partial M}, -\eta)$ is the mean curvature of $\partial M$ in $M$.

Thus, since $\eta = -\nu$ on $\partial M$, we have
\[
\int_M \left( \Delta f \frac{1}{2} |\omega|^2 \right) e^{-f} dV_M = -\int_{\partial M} \langle \nabla_\eta \omega^f, \omega^f \rangle e^{-f} dV_{\partial M} = \int_{\partial M} (H^{\partial M} + \langle \nabla f, \nu \rangle) |\omega|^2 e^{-f} dV_{\partial M}.
\]

Then the required formula follows from (2.4) and (2.6). \qed

To estimate the index, we use the coordinates of $N \wedge \omega^f$ for $f$-harmonic 1-forms $\omega$ as the test functions.

**Lemma 2.3.** Let $M^n$ be a free boundary $f$-minimal hypersurface of a weighted manifold $(N^{n+1}, g, e^{-f} dV_N)$. Let $N^{n+1}$ be isometrically immersed in some Euclidean space $\mathbb{R}^d$. For a $f$-harmonic 1-form $\omega$ of $M$, let
\[
u_{ij} = \langle N \wedge \omega^f, \theta_i \wedge \theta_j \rangle, \quad 1 \leq i < j \leq d,
\]
be the coordinates of $N \wedge \omega^f$ under a fixed orthonormal basis $\{\theta_i \wedge \theta_j \mid 1 \leq i < j \leq d\}$ of $\Omega^2(\mathbb{R}^d)$. Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame of $M^n$. 

(1) Let $\omega \in \mathcal{H}_{N\mathcal{f}}^1(M)$ be a tangent $f$-harmonic 1-form. Then

$$\sum_{1 \leq i < j \leq d} Q_f(u_{ij}, u_{ij}) = \int_M \sum_{k=1}^n \left( |\Pi^N(e_k, \omega^t)|^2 + |\Pi^N(e_k, N)|^2 \right) e^{-f} dV_M$$

$$- \int_M \left( \text{Ric}^N(\omega^t, \omega^t) + \text{Ric}^N(N, N) |\omega|^2 - K^N(\omega^t, N) \right) e^{-f} dV_M$$

$$- \int_{\partial M} \left( \Pi^{\partial N}(\omega^t, \omega^t) + \Pi^{\partial N}(N, N) |\omega|^2 \right) e^{-f} dV_{\partial M}.$$ (2.7)

(2) Let $\omega \in \mathcal{H}_{N\mathcal{f}}^1(M)$ be a normal $f$-harmonic 1-form. Then

$$\sum_{1 \leq i < j \leq d} Q_f(u_{ij}, u_{ij}) = \int_M \sum_{k=1}^n \left( |\Pi^N(e_k, \omega^t)|^2 + |\Pi^N(e_k, N)|^2 \right) e^{-f} dV_M$$

$$- \int_M \left( \text{Ric}^N(\omega^t, \omega^t) + \text{Ric}^N(N, N) |\omega|^2 - K^N(\omega^t, N) \right) e^{-f} dV_M$$

$$- \int_{\partial M} (H^{\partial N} + (\nabla f, \nu)) |\omega|^2 e^{-f} dV_{\partial M}.$$ (2.7)

**Proof.** As $N$ is a unit normal vector field of $M^n$ in $\mathcal{N}^{n+1}$ and $\omega^t$ is a tangent vector field of $M^n$, we have

$$|N \wedge \omega^n|^2 = \sum_{i<j} u_{ij}^2 = |\omega|^2.$$ Substituting the test functions $u_{ij}$ in (2.7), we get

$$\sum_{i<j} Q_f(u_{ij}, u_{ij}) = \int_M \left( \sum_{i<j} |\nabla u_{ij}|^2 - (\text{Ric}^N(N, N) + \Pi^M|\omega|^2) \right) e^{-f} dV_M$$

$$- \int_{\partial M} \Pi^{\partial N}(N, N) |\omega|^2 e^{-f} dV_{\partial M}.$$ (2.7)

Note that

$$\sum_{i<j} |\nabla u_{ij}|^2 = \sum_{i<j} \sum_{k=1}^n (D_{e_k}(N \wedge \omega^t), \theta_i \wedge \theta_j)^2 = \sum_{k=1}^n |D_{e_k}(N \wedge \omega^t)|^2.$$

Using the orthogonal decompositions, we compute the squared norms as follows:

$$|D_{e_k}(N \wedge \omega^t)|^2 = |D_{e_k}N \wedge \omega^t + N \wedge D_{e_k} \omega^t|^2$$

$$= \left| (\nabla_{e_k} N + \Pi^N(e_k, N)) \wedge \omega^t + N \wedge (\nabla_{e_k} \omega^t + \Pi^N(e_k, \omega^t)) \right|^2$$

$$= \left| (-A_{e_k} + \Pi^N(e_k, N)) \wedge \omega^t + N \wedge (\nabla_{e_k} \omega^t + \Pi^N(e_k, \omega^t)) \right|^2$$

$$= |A_{e_k} \wedge \omega^t|^2 + |\Pi^N(e_k, N) \wedge \omega^t|^2 + |N \wedge \nabla_{e_k} \omega^t|^2 + |N \wedge \Pi^N(e_k, \omega^t)|^2$$

$$= |A_{e_k}|^2 |\omega|^2 - |A_{e_k} \wedge \omega^t|^2 + |\Pi^N(e_k, N)|^2 |\omega|^2 + |\nabla_{e_k} \omega|^2 + |\Pi^N(e_k, \omega^t)|^2,$$
where \( A \) is the shape operator of \( M^n \) such that \( \langle AX, Y \rangle = \langle \Pi^M(X, Y), N \rangle \). Thus
\[
\sum_{i<j} |\nabla u_{ij}|^2 = |\Pi^M|^2|\omega|^2 - |\Pi^M(\cdot, \omega^2)|^2 + \sum_{k=1}^n \left( |\Pi^N(e_k, N)|^2|\omega|^2 + |\Pi^N(e_k, \omega^2)|^2 \right) + |\nabla \omega|^2.
\]
By the free boundary property, i.e., \( \eta = -\nu \) along \( \partial M \subset \partial N \), \( N \) is also a unit normal vector field of \( \partial M \) in \( \partial N \). It follows that
\[
H^{\partial M} + \Pi^{\partial N}(N, N) = H^{\partial N}.
\]
Putting these in (2.7) and applying Lemma 2.2 we obtain both formulae for the two cases of the Lemma.

**3. Proof of the main theorems**

We are now ready to prove the main theorems.

**Proof of Theorem 1.1** Let \( k \) be the \( f \)-index of \( M \), that is the number of negative eigenvalues of the weighted Jacobi operator \( L_f \) of \( M \) in (2.2), and denote by \( \{\phi_p\}_{p=1}^\infty \) a \( L^2(M, e^{-f}dV_M) \)-orthonormal basis of the eigenfunctions corresponding to the eigenvalues \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k < 0 \leq \lambda_{k+1} \ldots \) of \( L_f \). Following the approach of Ambrozio-Carlotto-Sharp 2, 3, we define the following linear map
\[
\Phi : H^1_f(M) \longrightarrow \mathbb{R}^{d(d-1)k/2}
\]
\[
\omega \longmapsto \left[ \int_M u_{ij}\phi_p e^{-f}dV_M \right],
\]
where \( H^1_f(M) = H^1_{Nf}(M) \cong H^1(M, \mathbb{R}) \) in case (1) and \( H^1_f(M) = H^1_{Nf}(M) \cong H^{n-1}(M, \mathbb{R}) \) in case (2) of Theorem 1.1 respectively, \( u_{ij} = \langle N \wedge \omega^2, \theta_i \wedge \theta_j \rangle \) are the test functions as in Lemma 2.3, \( 1 \leq i < j \leq d, 1 \leq p \leq k \).

Assume by contradiction that \( \dim H^1_f(M) > d(d-1)k/2 \). Then there would exist a nonzero \( f \)-harmonic 1-form \( \omega \in H^1_f(M) \) such that \( \int_M u_{ij}\phi_p e^{-f}dV_M = 0 \) for all \( 1 \leq i < j \leq d \) and all \( p = 1, \ldots , k \). This means that each \( u_{ij} \) is \( L^2(M, e^{-f}dV_M) \)-orthogonal to all the first \( k \) eigenfunctions \( \phi_p \). Thus from the Courant-Hilbert variational characterization of eigenvalues it follows that
\[
\sum_{i<j} Q_f(u_{ij}, u_{ij}) \geq \lambda_{k+1} \sum_{i<j} \int_M u_{ij}^2 e^{-f}dV_M = \lambda_{k+1} \int_M |\omega|^2 e^{-f}dV_M \geq 0.
\]
In view of Lemma 2.3 this is a contradiction with either hypothesis of Theorem 1.1.

**Proof of Theorem 1.2** Now since \( N^{n+1} \) is totally geodesic in \( \mathbb{R}^{n+1} \), \( \text{Ric}^N = \text{Hess}^N f \geq 0, K^N = 0 \) and \( \Pi^N = 0 \). Hence, if \( \partial N \) is strictly two-convex in \( N \), or if \( \partial N \) is strictly \( f \)-mean convex in \( N \), then the corresponding inequality assumption of Theorem 1.1 is satisfied and thus the conclusion follows.
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