LOW-RANK MATRIX APPROXIMATIONS DO NOT NEED A SINGULAR VALUE GAP*

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Abstract. This is a systematic investigation into the sensitivity of low-rank approximations of real matrices. We show that the low-rank approximation errors, in the two-norm, Frobenius norm and more generally, any Schatten $p$-norm, are insensitive to additive rank-preserving perturbations in the projector basis; and to matrix perturbations that are additive or change the number of columns (including multiplicative perturbations). Thus, low-rank matrix approximations are always well-defined and do not require a singular value gap. In the presence of a singular value gap, connections are established between low-rank approximations and subspace angles.

Key words. Singular value decomposition, principal angles, additive perturbations, multiplicative perturbations.

AMS subject classification. 15A12, 15A18, 15A42, 65F15, 65F35.

1. Introduction. An emerging problem in Theoretical Computer Science and Data Science is the low-rank approximation $ZZ^T A$ of a matrix $A \in \mathbb{R}^{m \times n}$ by means of an orthonormal basis $Z \in \mathbb{R}^{m \times k}$ [9, 27].

The ideal low-rank approximation consists of the left singular vectors $U_k$ associated with the $k$ dominant singular values $\sigma_1(A) \geq \cdots \geq \sigma_k(A)$ of $A$, because the low-rank approximation error in the two-norm is minimal and equal to the first neglected singular value, $\|(I - U_k U_k^T) A\|_2 = \sigma_{k+1}(A)$. Low-rank approximation $Z$ can be determined with subspace iteration or a Krylov space method [12, 18], with bounds for $\|(I - P) A\|_F$ that contain $\sigma_{k+1}(A)$ as an additive or multiplicative factor. Effort has been put into deriving bounds that do not depend on the existence of the singular value gap $\sigma_k(A) - \sigma_{k+1}(A) > 0$.

A closely related problem in numerical linear algebra is the approximation of the dominant subspace proper [20, 21], that is, computing an orthonormal basis $Z \in \mathbb{R}^{m \times k}$ whose space is close to the dominant subspace range($U_k$). Closeness here means that the sine of the largest principal angle between the two spaces, $\|\sin \Theta(Z, U_k)\|_2 = \|ZZ^T - U_k U_k^T\|_2$ is small. For the dominant subspace $U_k$ to be well-defined, the associated singular values must be separated from the remaining singular values, and there must be a gap $\sigma_k(A) - \sigma_{k+1}(A) > 0$, see [17, 22, 24, 28], which are all based on the perturbation results for invariant subspaces of Hermitian matrices [4, 5].

The purpose of our paper, following up on [9], is to establish a clear distinction between the mathematical problems of low-rank approximation, and approximation of dominant subspaces. In particular we show that low-rank approximations are well-defined and well-conditioned, by deriving bounds for the low-rank approximation error $(I - ZZ^T) A$ in the two-norm, Frobenius norm, and more generally, any Schatten $p$-norm. We establish relationships between the mathematical problems of dominant subspace computation and of low-rank approximation.

Overview. After setting the notation for the singular value decomposition (Section 1.1), and reviewing Schatten $p$-norms (Section 1.2) and angles between subspaces

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(Section 1.3), we highlight the main results (Section 1.4), followed by proofs for low-rank approximations (Section 2) and subspace angles (Section 3, Appendix A).

1.1. Singular Value Decomposition (SVD). Let the non-zero matrix $A \in \mathbb{R}^{m \times n}$ have a full SVD $A = U \Sigma V^T$, where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are orthogonal matrices, i.e.

$$UU^T = U^T U = I_m, \quad VV^T = V^T V = I_n,$$

and $\Sigma \in \mathbb{R}^{m \times n}$ a diagonal matrix with diagonal elements

$$\|A\|_2 = \sigma_1(A) \geq \cdots \geq \sigma_r(A) \geq 0, \quad r \equiv \min\{m, n\}. \quad (1.1)$$

For $1 \leq k \leq \text{rank}(A)$, the respective leading $k$ columns of $U$ and $V$ are $U_k \in \mathbb{R}^{m \times k}$ and $V_k \in \mathbb{R}^{n \times k}$. They are orthonormal, $U_k^T U_k = I_k = V_k^T V_k$, and are associated with the $k$ dominant singular values

$$\Sigma_k \equiv \text{diag} \left( \sigma_1(A) \cdots \sigma_k(A) \right) \in \mathbb{R}^{k \times k}.$$  

Then

$$A_k \equiv U_k \Sigma_k V_k^T = U_k U_k^T A \quad (1.2)$$

is a best rank-$k$ approximation of $A$, and satisfies in the two norm and in the Frobenius norm, respectively,

$$\| (I - U_k U_k^T) A \|_{2,F} = \| A - A_k \|_{2,F} = \min_{\text{rank}(B) = k} \| A - B \|_{2,F}.$$ 

Projectors. We construct orthogonal projectors to capture the target space, which is a dominant subspace of $A$.

**Definition 1.1.** A matrix $P \in \mathbb{R}^{m \times m}$ is an orthogonal projector, if it is idempotent and symmetric,

$$P^2 = P = P^T. \quad (1.3)$$

For $1 \leq k \leq \text{rank}(A)$, the matrix $U_k U_k^T = A_k A_k^T$ is the orthogonal projector onto the $k$-dimensional dominant subspace range($U_k$) = range($A_k$). Here the pseudo inverse is $A_k^\dagger = V_k \Sigma_k^{-1} U_k^T$.

1.2. Schatten $p$-norms. These are norms defined on the singular values of real and complex matrices, and thus special cases of symmetric gauge functions. We briefly review their properties, based on [3, Chapter IV] and [15, Sections 3.4-3.5].

**Definition 1.2.** For integers $p \geq 1$, the Schatten $p$ norms on $\mathbb{R}^{m \times n}$ are

$$\| A \|_p \equiv \sqrt[p]{\sigma_1(A)^p + \cdots + \sigma_r(A)^p}, \quad r \equiv \min\{m, n\}. \quad (1.4)$$

\footnote{The superscript $T$ denotes the transpose.}
Popular Schatten norms:

\[ p = 1 : \text{ Nuclear (trace) norm } \|A\|_1 = \sum_{j=1}^r \sigma_j(A) = \|A\|_1. \]
\[ p = 2 : \text{ Frobenius norm } \|A\|_F = \sqrt{\sum_{j=1}^r \sigma_j(A)^2} = \|A\|_2. \]
\[ p = \infty : \text{ Euclidean (operator) norm } \|A\|_\infty = \sigma_1(A) = \|A\|_\infty. \]

We will make ample use of the following properties.

**Lemma 1.3.** Let \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times \ell}, \) and \( C \in \mathbb{R}^{s \times m}. \)

- **Unitary invariance:** If \( Q_1 \in \mathbb{R}^{s \times m} \) with \( Q_1^T Q_1 = I_m \) and \( Q_2 \in \mathbb{R}^{\ell \times n} \) with \( Q_2^T Q_2 = I_n, \) then \( \|Q_1 A Q_2^T\|_p = \|A\|_p. \)

- **Submultiplicativity:** \( \|AB\|_p \leq \|A\|_p \|B\|_p. \)

- **Strong submultiplicativity (symmetric norm):** \( \|CA B\|_p \leq \sigma_1(C)\sigma_1(B) \|A\|_p = \|C\|_2 \|B\|_2 \|A\|_p. \)

- **Best rank-\( k \) approximation:** \( \|(I - U_k U_k^T) A\|_p = \|A - A_k\|_p = \min_{\text{rank}(B) = k} \|A - B\|_p \)

### 1.3. Principal Angles between Subspaces.

We review the definition of angles between subspaces, and the connections between angles and projectors.

**Definition 1.4** (Section 6.4.3 in [11] and Section 2 in [28]). Let \( Z \in \mathbb{R}^{m \times k} \) and \( \hat{Z} \in \mathbb{R}^{m \times \ell} \) with \( \ell \geq k \) have orthonormal columns so that \( Z^T Z = I_k \) and \( \hat{Z}^T \hat{Z} = I_\ell. \) The singular values of \( Z^T \hat{Z} \) are the diagonal elements of the \( k \times k \) diagonal matrix

\[ \cos \Theta(Z, \hat{Z}) \equiv \text{diag} \left( \cos \theta_1 \cdots \cos \theta_k \right), \]

where \( \theta_j \) are the principal (canonical) angles between \( \text{range}(Z) \) and \( \text{range}(\hat{Z}). \)

Next we show how to extract the principal angles between two subspaces of possibly different dimensions, we make use of projectors.

**Lemma 1.5.** Let \( P \equiv ZZ^T \) and \( \hat{P} \equiv \hat{Z} \hat{Z}^T \) be orthogonal projectors, where \( Z \in \mathbb{R}^{m \times k} \) and \( \hat{Z} \in \mathbb{R}^{m \times \ell} \) have orthonormal columns. Let \( \ell \geq k, \) and define

\[ \sin \Theta(P, \hat{P}) = \sin \Theta(Z, \hat{Z}) \equiv \text{diag} \left( \sin \theta_1 \cdots \sin \theta_k \right), \]

where \( \theta_j \) are the \( k \) principal angles between \( \text{range}(Z) \) and \( \text{range}(\hat{Z}). \)

1. If \( \text{rank}(\hat{Z}) = k = \text{rank}(Z), \) then

\[ \|\sin \Theta(Z, \hat{Z})\|_p = \|(I - P) \hat{P}\|_p = \|(I - \hat{P}) P\|_p. \]

In particular

\[ \|(I - P) \hat{P}\|_2 = \|P - \hat{P}\|_2 \leq 1 \]

represents the distance between the subspaces \( \text{range}(P) \) and \( \text{range}(\hat{P}). \)

2. If \( \text{rank}(\hat{Z}) > k = \text{rank}(Z), \) then

\[ \|\sin \Theta(Z, \hat{Z})\|_p = \|(I - \hat{P}) P\|_p \leq \|(I - P) \hat{P}\|_p. \]

**Proof.** The two-norm expressions follow from [11] Section 2.5.3 and [28] Section 2. The Schatten \( p \)-norm expressions follow from the CS decompositions in [19] Theorem 8.1, [28] Section 2, and Section [A] [D].
1.4. Highlights of the Main Results. We present a brief overview of the main results: The well-conditioning of low-rank approximations under additive perturbations in projector basis and the matrix (Section 1.4.1); the well-conditioning of low-rank approximations under perturbations that change the matrix dimension (Section 1.4.2); and the connection between low-rank approximation errors and angles between subspaces (Section 1.4.3).

Thus: Low-rank approximations are well-conditioned, and don’t need a gap.

1.4.1. Additive perturbations in the projector basis and the matrix. We show that the low-rank approximation error is insensitive to additive rank-preserving perturbations in the projector basis (Theorem 1 and Corollary 1), and to additive perturbations in the matrix (Theorem 2 and Corollary 2).

We start with perturbations in the projector basis.

**Theorem 1** (Additive rank-preserving perturbations in the projector basis). *Let \( A \in \mathbb{R}^{m \times n} \); let \( Z \in \mathbb{R}^{m \times \ell} \) be a projector basis with orthonormal columns so that \( Z^T Z = I_\ell \); and let \( \tilde{Z} \in \mathbb{R}^{m \times \ell} \) be its perturbation with\]

\[
\epsilon_Z \equiv \| \tilde{Z}^T \|_2 \| Z - \tilde{Z} \|_2 = \frac{\| \tilde{Z} - Z \|_2}{\| Z \|_2}.
\]

1. If rank(\( \tilde{Z} \)) = rank(\( Z \)) then

\[\|(I - ZZ^T)A\|_p - \epsilon_Z \|A\|_p \leq \|(I - \tilde{Z}\tilde{Z}^T)A\|_p \leq \|(I - ZZ^T)A\|_p + \epsilon_Z \|A\|_p.\]

2. If \( \|Z - \tilde{Z}\|_2 \leq 1/2 \), then rank(\( \tilde{Z} \)) = rank(\( Z \)) and \( \epsilon_Z \leq 2 \|Z - \tilde{Z}\|_2 \).

**Proof.** See Section 2 and in particular Theorem 2.2.

**Corollary 1** (Rank-preserving perturbation of dominant singular vectors). *Let \( U_k \in \mathbb{R}^{m \times k} \) in (1.2) be \( k \) dominant left singular vectors of \( A \). Let \( \tilde{U} \in \mathbb{R}^{m \times k} \) be a perturbation of \( U_k \) with rank(\( \tilde{U} \)) = \( k \) or \( \|U_k - \tilde{U}\|_2 \leq 1/2 \); and let \( \epsilon_U \equiv \|\tilde{U} - U_k\|_2 \). Then

\[\|(I - U_kU_k^T)A\|_p \leq \|(I - \tilde{U}\tilde{U}^T)A\|_p \leq \|(I - U_kU_k^T)A\|_p + \epsilon_U \|A\|_p.\]
Next we consider perturbations in the matrix, with a bound that is completely general and holds for any Schatten p-norm.

**Theorem 2** (Additive perturbations in the matrix). Let \( A, A + E \in \mathbb{R}^{m \times n} \); and let \( P \in \mathbb{R}^{m \times m} \) be an orthogonal projector as in \((1.6)\). Then

\[
\| (I - P) (A + E) \|_p - \| (I - P) A \|_p \leq \| E \|_p.
\]

*Proof.* See Section 2 and in particular Theorem 2.2. \( \square \)

Theorem 2 shows that the low-rank approximation error is well-conditioned, in the absolute sense, to additive perturbations in the matrix.

Theorem 2 also implies the following upper bound for a low-rank approximation of \( A \) by means of singular vectors of \( A + E \). Again, no singular value gap is required. We merely pick the leading \( k \) columns \( U_k \) obtained from some SVD of \( A \), and the leading \( k \) columns \( \tilde{U}_k \) obtained from some SVD of \( A + E \).

**Corollary 2** (Low-rank approximation from additive perturbation). Let \( U_k \in \mathbb{R}^{m \times k} \) in \((1.2)\) be \( k \) dominant left singular vectors of \( A \); and let \( \tilde{U}_k \in \mathbb{R}^{m \times k} \) be \( k \) dominant left singular vectors of \( A + E \). Then

\[
\| (I - U_k U_k^T) A \|_2 \leq \| (I - \tilde{U}_k \tilde{U}_k^T) A \|_2 \leq \| (I - U_k U_k^T) A \|_2 + 2 \| E \|_2.
\]

*Proof.* See Section 2 and in particular Corollary 2.3. \( \square \)

Bounds with an additive dependence on \( E \), like the two-norm bound above, can be derived for other Schatten p-norms as well, and can then be combined with bounds for \( E \) in \([1, 2, 10]\) where \( A + E \) is obtained from element-wise sampling from \( A \).

1.4.2. Perturbation that change the matrix dimension. We consider perturbations that can change the number of columns in \( A \) and include, among others, multiplicative perturbations of the form \( \hat{A} = AX \). However, our bounds are completely general and hold also in the absence of any relation between range(\( A \)) and range(\( \hat{A} \)).

Presented below are bounds for the two-norm (Theorem 3), the Frobenius norm (Theorem 4) and general Schatten p-norms (Theorem 5).

**Theorem 3** (Two-norm). Let \( A \in \mathbb{R}^{m \times n}; \hat{A} \in \mathbb{R}^{m \times c}; \) and \( P \in \mathbb{R}^{m \times m} \) an orthogonal projector as in \((1.3)\) with rank(\( P \)) = \( c \). Then

\[
\| (I - P) A \|_2^2 - \| (I - P) \hat{A} \|_2^2 \leq \| AA^T - \hat{A} \hat{A}^T \|_2. \tag{1.4}
\]

If also rank(\( \hat{A} \)) = \( c \) then

\[
\| (I - \hat{A} \hat{A}^T) A \|_2^2 \leq \| AA^T - \hat{A} \hat{A}^T \|_2. \tag{1.5}
\]

If \( \hat{A}_k \in \mathbb{R}^{m \times c} \) is a best rank-\( k \) approximation of \( \hat{A} \) with rank(\( \hat{A}_k \)) = \( k < c \) then

\[
\| (I - \hat{A}_k \hat{A}_k^T) A \|_2^2 \leq \| A - A_k \|_2^2 + 2 \| AA^T - \hat{A} \hat{A}^T \|_2. \tag{1.6}
\]

*Proof.* See Section 2 specifically Theorem 2.5 for \((1.4)\); Theorem 2.6 for \((1.5)\); and and Theorem 2.7 for \((1.6)\). \( \square \)
The bounds (1.6) are identical to [8], Theorem 3], while (1.4) and (1.5), though similar in spirit, are novel. The bound (1.4) holds for any orthogonal projector $P$, in contrast to prior work which was limited to multiplicative perturbations $\hat{A} = AX$ with bounds for $\|AA^T - AA^T\|_2$ for matrices $X$ that sample and rescale columns in a novel connection, and it should motivate further work into understanding the behaviour of the trace norm, thereby complementing prior investigations into the two-norm and Frobenius norm.

**Theorem 4 (Frobenius norm).** Let $A \in \mathbb{R}^{m \times n}; \hat{A} \in \mathbb{R}^{m \times c};$ and $P \in \mathbb{R}^{m \times m}$ an orthogonal projector as in (1.3) with $\|P\|_F$ bounds for $X$ and other constructions of $A$. Then

\[
\left\| (I - P) \hat{A} \right\|_F^2 - \left\| (I - P) A \right\|_F^2 \leq \min \left\{ \|AA^T - \hat{A}\hat{A}^T\|_*, \sqrt{m - c} \|AA^T - \hat{A}\hat{A}^T\|_F \right\}, \tag{1.7}
\]

If also $\text{rank}(\hat{A}) = c$ then

\[
\left\| (I - \hat{A}) A \right\|_F^2 \leq \min \left\{ \|AA^T - \hat{A}\hat{A}^T\|_*, \sqrt{m - c} \|AA^T - \hat{A}\hat{A}^T\|_F \right\}. \tag{1.8}
\]

**Proof.** See Section 2, specifically Theorem 2.5 for (1.7); Theorem 2.6 for (1.8); and Theorem 2.7 for (1.9). \]

The bound (1.7) holds for any $P$, and is the first one of its kind in this generality. The bound (1.9) is similar to [8], Theorem 2, and weaker for smaller $k$ but tighter for larger $k$.

More generally, Theorem 4 relates the low-rank approximation error in the Frobenius norm with the error $\|AA^T - \hat{A}\hat{A}^T\|$ in the trace norm, i.e. the Schatten-one norm. This is a novel connection, and it should motivate further work into understanding the behaviour of the trace norm, thereby complementing prior investigations into the two-norm and Frobenius norm.

**Theorem 5 (General Schatten p-norms).** Let $A \in \mathbb{R}^{m \times n}; \hat{A} \in \mathbb{R}^{m \times c};$ and $P \in \mathbb{R}^{m \times m}$ an orthogonal projector as in (1.3) with $\text{rank}(P) = c$. Then

\[
\left\| (I - P) A \right\|_p^2 - \left\| (I - P) \hat{A} \right\|_p^2 \leq \min \left\{ \|AA^T - \hat{A}\hat{A}^T\|_p, \sqrt[m]{m - c} \|AA^T - \hat{A}\hat{A}^T\|_p \right\}, \tag{1.10}
\]

If $\text{rank}(\hat{A}) = c$ then

\[
\left\| (I - \hat{A}) A \right\|_p^2 \leq \min \left\{ \|AA^T - \hat{A}\hat{A}^T\|_p, \sqrt[m]{m - c} \|AA^T - \hat{A}\hat{A}^T\|_p \right\}. \tag{1.11}
\]

If $\hat{A}_k \in \mathbb{R}^{m \times c}$ is a best rank-$k$ approximation of $\hat{A}$ with $\text{rank}(\hat{A}_k) = k < c$ then

\[
\left\| (I - \hat{A}_k A) \right\|_p^2 \leq \left\| A - A_k \right\|_p^2 + 2 \text{min} \left\{ \|AA^T - \hat{A}\hat{A}^T\|_p, \sqrt[m]{m - c} \|AA^T - \hat{A}\hat{A}^T\|_p \right\}. \tag{1.12}
\]
Proof. See Section 2, specifically Theorem 2.5 for (1.10); Theorem 2.6 for (1.11); and Theorem 2.7 for (1.12).

Theorem 5 is new. To our knowledge, non-trivial bounds for \(\|AA^T - \hat{A}\hat{A}^T\|_p\) for general \(p\) do not exist.

1.4.3. Relations between low-rank approximation error and subspace angle. For matrices \(A\) with a singular value gap, we bound the low-rank approximation error in terms of the subspace angle (Theorem 6) and discuss the tightness of the bounds (Remark 1). The singular value gap is required for the dominant subspace to be well-defined, but no assumptions on the accuracy of the low-rank approximation are required.

Assume that \(A \in \mathbb{R}^{m \times n}\) has a gap after the \(k\)th singular value,

\[
\|A\|_2 = \sigma_1(A) \geq \cdots \geq \sigma_k(A) > \sigma_{k+1}(A) \geq \cdots \geq \sigma_r(A) \geq 0, \quad r \equiv \min\{m, n\},
\]

Below, we highlight the bounds from Section 3 for low-dimensional approximations, compared to the dimension \(m\) of the host space for \(\text{range}(A)\).

**Theorem 6.** Let \(P_k \equiv A_kA_k^\dagger\) be the orthogonal projector onto the dominant \(k\)-dimensional subspace of \(A\); and let \(P \in \mathbb{R}^{m \times m}\) be some orthogonal projector as in (1.3) with \(k \leq \text{rank}(P) < m - k\). Then

\[
\sigma_k(A) \|\sin \Theta(P, P_k)\|_p \leq \|(I - P)A\|_p \leq \|A\|_2 \|\sin \Theta(P, P_k)\|_p + \|A - A_k\|_p.
\]

**Proof.** See Section 3 and in particular Theorems 3.1 and 3.2.

Theorem 6 shows that for dominant subspaces of sufficiently low dimension, the approximation error is bounded by the product of the subspace angle with a dominant singular value. The upper bound also contains the subdominant singular values.

**Remark 1 (Tightness of Theorem 6).**

- If \(\text{rank}(A) = k\), so that \(A - A_k = 0\), then the tightness depends on the spread of the non-zero singular values,

\[
\sigma_k(A) \|\sin \Theta(P, P_k)\|_p \leq \|(I - P)A\|_p \leq \|A\|_2 \|\sin \Theta(P, P_k)\|_p.
\]

- If \(\text{rank}(A) = k\) and \(\sigma_1(A) = \cdots = \sigma_k(A)\), then the bounds are tight, and they are equal to

\[
\|(I - P)A\|_p = \|A\|_2 \|\sin \Theta(P, P_k)\|_p.
\]

- If \(\text{range}(P) = \text{range}(P_k)\), so that \(\sin \Theta(P, P_k) = 0\), then the upper bound is tight and equal to

\[
\|(I - P)A\|_p = \|A - A_k\|_p.
\]

2. Well-conditioning of low-rank approximations. We investigate the effect of additive, rank-preserving perturbations in the projector basis on the orthogonal projector (Section 2.1) and on the low-rank approximation error (Section 2.2); and the effect on the low-rank approximation error of matrix perturbations, both additive and dimension changing (Section 2.3). We also relate low-rank approximation error and error matrix (Section 2.4).
2.1. Orthogonal projectors, and perturbations in the projector basis. We show that orthogonal projectors and subpace angles are insensitive to additive, rank-preserving perturbations in the projector basis (Theorem 2.1) if the perturbed projector basis is well-conditioned.

**Theorem 2.1.** Let \( A \in \mathbb{R}^{m \times n} \); let \( Z \in \mathbb{R}^{m \times s} \) be a projector basis with orthonormal columns so that \( Z^T Z = I_s \); and let \( \hat{Z} \in \mathbb{R}^{m \times s} \) be its perturbation.

1. If \( \text{rank}(\hat{Z}) = \text{rank}(Z) \), then the distance between \( \text{range}(Z) \) and \( \text{range}(\hat{Z}) \) is
   \[
   ||Z Z^T - \hat{Z} \hat{Z}^T||_2 = ||\sin \Theta(Z, \hat{Z})||_2 \leq \epsilon_Z \equiv ||\hat{Z}^\dagger||_2 ||Z - \hat{Z}||_2.
   \]

2. If \( ||Z - \hat{Z}||_2 \leq 1/2 \), then \( \text{rank}(\hat{Z}) = \text{rank}(Z) \) and \( \epsilon_Z \leq 2 ||Z - \hat{Z}||_2 \).

**Proof.**

1. The equality follows from Lemma 1.5. The upper bound follows from [23] Theorem 3.1 and [13] Lemma 20.12, but we provide a simpler proof for this context. Set \( \hat{Z} = Z + F \), and abbreviate \( P \equiv ZZ^T \) and \( \hat{P} \equiv \hat{Z} \hat{Z}^T \). Writing
   \[
   (I - P) \hat{P} = (I - ZZ^T) \hat{Z} \hat{Z}^T = (I - ZZ^T)(Z + \hat{Z} - Z) \hat{Z}^\dagger = (I - P) F \hat{Z}^\dagger
   \]
gives
   \[
   ||\sin \Theta(Z, \hat{Z})||_2 = ||(I - P) \hat{P}||_2 \leq ||\hat{Z}^\dagger||_2 ||F||_2. \tag{2.1}
   \]

2. To show \( \text{rank}(\hat{Z}) = \text{rank}(Z) \) in the special case \( ||Z - \hat{Z}||_2 \leq 1/2 \), consider the singular values \( \sigma_j(Z) = 1 \) and \( \sigma_j(\hat{Z}) \), \( 1 \leq j \leq s \). The well-conditioning of singular values [11] Corollary 8.6.2 implies
   \[
   |1 - \sigma_j(\hat{Z})| = |\sigma_j(Z) - \sigma_j(\hat{Z})| \leq ||F||_2 \leq 1/2, \quad 1 \leq j \leq s.
   \]
   Thus \( \min_{1 \leq j \leq s} \sigma_j(\hat{Z}) \geq 1/2 > 0 \) and \( \text{rank}(\hat{P}) = \text{rank}(\hat{Z}) = s = \text{rank}(P) \). Hence (2.1) holds with
   \[
   ||\sin \Theta(Z, \hat{Z})||_2 \leq ||\hat{Z}^\dagger||_2 ||F||_2 \leq 2 ||F||_2. \tag{2.2}
   \]

\[ \square \]

Note that \( \epsilon_Z \) represents both, an absolute and a relative perturbation as
\[
\epsilon_Z = ||\hat{Z}||_2 ||Z - \hat{Z}||_2 = ||\hat{Z}||_2 ||\hat{Z}^\dagger||_2 ||\hat{Z} - Z||_2 ||Z||_2.
\]

2.2. Approximation errors, and perturbations in the projector basis. We show that low-rank approximation errors are insensitive to additive, rank-preserving perturbations in the projector basis (Theorem 2.2), if the perturbed projector basis is well-conditioned.

**Theorem 2.2.** Let \( A \in \mathbb{R}^{m \times n} \); let \( Z \in \mathbb{R}^{m \times s} \) be a projector basis with orthonormal columns so that \( Z^T Z = I_s \); and let \( \hat{Z} \in \mathbb{R}^{m \times s} \) be its perturbation with \( \epsilon_Z \equiv ||\hat{Z}||_2 ||Z - \hat{Z}||_2 \).

1. If \( \text{rank}(\hat{Z}) = \text{rank}(Z) \) then
   \[
   ||(I - ZZ^*)A - \epsilon_Z ||_p ||A||_p \leq ||(I - \hat{Z}\hat{Z}^*)A||_p \leq ||(I - ZZ^*)A||_p + \epsilon_Z ||A||_p.
   \]

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2. If $\|Z - \hat{Z}\|_2 \leq 1/2$, then $\text{rank}(\hat{Z}) = \text{rank}(Z)$ and $\epsilon_Z \leq 2\|Z - \hat{Z}\|_2$.

Proof. This is a straightforward consequence of Theorem 2.1.

Abbreviate $P \equiv ZZ^T$ and $\tilde{P} \equiv \tilde{Z} \tilde{Z}^T$, and write

$$(I - \tilde{P})A = (I - P)A + (P - \tilde{P})A.$$ 

Apply the triangle and reverse triangle inequalities, strong submultiplicativity, and then bound the second summand with Theorem 2.1,

$$\|(P - \tilde{P})A\|_p \leq \|\hat{Z}\|_2 \|\tilde{Z} - Z\|_2 \|A\|_p = \epsilon_Z \|A\|_p.$$

$\blacksquare$

2.3. Approximation errors, and perturbations in the matrix. We show that low-rank approximation errors are insensitive to matrix perturbations that are additive (Theorem 2.3 and Corollary 2.4), and that are dimension changing (Theorem 2.5).

Theorem 2.3 (Additive perturbations). Let $A, E \in \mathbb{R}^{m \times n}$; and let $P \in \mathbb{R}^{m \times m}$ be an orthogonal projector with $P^2 = P = P^T$. Then

$$\|((I - P)A \|_p - \|E\|_p \leq \|((I - P)(A + E)\|_p \leq \|((I - P)A\|_p + \|E\|_p.$$

Proof. Apply the triangle and reverse triangle inequalities, and the fact that an orthogonal projector has at most unit norm, $\|I - P\|_2 \leq 1$.

Corollary 2.4 (Low-rank approximation from singular vectors of $A + E$). Let $U_k \in \mathbb{R}^{m \times k}$ in (1.2) be $k$ dominant left singular vectors of $A$; and let $\hat{U}_k \in \mathbb{R}^{m \times k}$ be $k$ dominant left singular vectors of $A + E$. Then

$$\|((I - U_k U_k^T)A\|_2 \leq \|((I - \hat{U}_k \hat{U}_k^T)A\|_2 \leq \|((I - U_k U_k^T)(A + E)\|_2 + \|E\|_2.$$

Proof. Setting $P = \hat{U}_k \hat{U}_k^T$ in the upper bound of Theorem 2.3 gives

$$\|((I - U_k U_k^T)A\|_2 \leq \|((I - \hat{U}_k \hat{U}_k^T)(A + E)\|_2 + \|E\|_2.$$

Express the approximation errors in terms of singular values,

$$\|((I - \hat{U}_k \hat{U}_k^T)(A + E)\|_2 = \sigma_{k+1}(A + E), \quad \|((I - U_k U_k^T)A\|_2 = \sigma_{k+1}(A),$$

apply Weyl’s theorem

$$|\sigma_{k+1}(A + E) - \sigma_{k+1}(A)| \leq \|E\|_2,$$

and combine the bounds. $\blacksquare$

Theorem 2.5 (Perturbations that change the number of columns). Let $A \in \mathbb{R}^{m \times n}$; $\hat{A} \in \mathbb{R}^{m \times c}$; let $P \in \mathbb{R}^{m \times m}$ be an orthogonal projector as in (1.3) and $\text{rank}(P) = s$; and let $p \geq 1$ an even integer. Then

1. Two norm ($p = \infty$)

$$\|((I - P)A\|_2^2 - \|((I - P)\hat{A}\|_2^2 \leq \|\hat{A}\hat{A}^T - AA^T\|_2.$$
2. Schatten $p$ norm ($p$ even)

\[ \left| \| (I - P)A \|_p^2 - \| (I - P)\hat{A} \|_p^2 \right| \leq \min \left\{ \| \hat{A}A^T - AA^T\|_{p/2}, \sqrt{m-s} \| \hat{A}A^T - AA^T\|_p \right\}. \]

3. Frobenius norm ($p = 2$)

\[ \left| \| (I - P)\hat{A} \|_F^2 - \| (I - P)A \|_F^2 \right| \leq \min \left\{ \| \hat{A}A^T - AA^T\|_*, \sqrt{m-s} \| \hat{A}A^T - AA^T\|_F \right\}. \]

**Proof.** The proof is motivated by that of [8 Theorems 2 and 3]. The bounds are obvious for $s = m$ where $P = I_m$, so assume $s < m$.

1. Two-norm. The invariance of the two norm under transposition and the triangle inequality imply

\[
\| (I - P)\hat{A} \|_2^2 = \| \hat{A}^T(I - P) \|_2^2 = \| (I - P)\hat{A}\hat{A}^T(I - P) \|_2 \\
= \| (I - P)AA^T(I - P) + (I - P) \left( \hat{A}\hat{A}^T - AA^T \right)(I - P) \|_2 \\
\leq \| (I - P)AA^T(I - P) \|_2 + \| (I - P) \left( \hat{A}\hat{A}^T - AA^T \right)(I - P) \|_2.
\]

The first summand equals

\[
\| (I - P)AA^T(I - P) \|_2 = \| (I - P)A \|_2^2,
\]

while the second one is bounded by submultiplicativity and $\| I - P \|_2 \leq 1$,

\[
\| (I - P) \left( \hat{A}\hat{A}^T - AA^T \right)(I - P) \|_2 \leq \| I - P \|_2^2 \| \hat{A}\hat{A}^T - AA^T \|_2 \\
\leq \| \hat{A}\hat{A}^T - AA^T \|_2.
\]

This gives the upper bound

\[
\| (I - P)\hat{A} \|_2^2 - \| (I - P)A \|_2^2 \leq \| \hat{A}\hat{A}^T - AA^T \|_2.
\]

Apply the inverse triangle inequality to show the lower bound,

\[
-\| \hat{A}\hat{A}^T - AA^T \|_2 \leq \| (I - P)\hat{A} \|_2^2 - \| (I - P)A \|_2^2.
\]

2. Schatten $p$-norm ($p$ even). The proof is similar to that of the two-norm, since an even Schatten $p$-norm is a $Q$-norm [3 Definition IV.2.9], that is, it represents a quadratic gauge function. This can be seen in terms of singular values,

\[
\| M \|_p = \sum_j (\sigma_j(M))^p = \sum_j \left( \sigma_j(MM^T) \right)^{p/2} = \| MM^T \|_{p/2}^{p/2}.
\]

Hence

\[
\| M \|_p^2 = \| MM^T \|_{p/2}.
\]
Abbreviate $M \equiv A^T A - A A^T$, and $B \equiv I - P$ where $B^T = B$ and $\|B\|_2 = 1$. Since singular values do not change under transposition, it follows from (2.3) and the triangle inequality that

$$\|B \hat{A}\|_p^2 = \|A^T B\|_p^2 = \|B \hat{A} A^T B\|_{p/2} = \|B A A^T B\|_{p/2} \leq \|B A A^T B\|_{p/2} + \|B M B\|_{p/2}. $$

Apply (2.3) to the first summand, $\|B A A^T B\|_{p/2} = \|B A\|_p^2$, and insert it into the above inequalities,

$$\|B \hat{A}\|_p^2 - \|B A\|_p^2 \leq \|B M B\|_{p/2}. \tag{2.4}$$

1. First term in the minimum: Bound (2.4) with strong submultiplicativity and $\|B\|_2 = 1$,

$$\|B M B\|_p \leq \|B\|_2 \|M\|_p \leq \|M\|_p,$$

which gives the upper bound

$$\|B \hat{A}\|_p^2 - \|B A\|_p^2 \leq \|M\|_p.$$

Apply the inverse triangle inequality to show the lower bound

$$- \|M\|_p \leq \|B \hat{A}\|_p^2 - \|B A\|_p^2.$$

2. Second term in the minimum: From

$$\text{rank}(B M B) \leq \text{rank}(B) = \text{rank}(I - P) = m - s > 0$$

follows $\sigma_j(B) = 1, \leq j \leq m - s$. With the non-descending singular value ordering in (1.1), the Schatten $p$-norm needs to sum over only the largest $m - s$ singular values. This, together with the singular value inequality [16] (7.3.14)

$$\sigma_j(B M B) \leq \sigma_1(B)^2 \sigma_j(M) = 1 \cdot \sigma_1(M), \quad 1 \leq j \leq m - s,$$

gives

$$\|B M B\|_{p/2} = \sum_{j=1}^{m-s} (\sigma_j(B M B))^{p/2} \leq \sum_{j=1}^{m-s} 1 \cdot (\sigma_j(M))^{p/2}.$$

At last apply the Cauchy-Schwartz inequality to the vectors of singular values

$$\sum_{j=1}^{m-s} 1 \cdot (\sigma_j(M))^{p/2} \leq \sqrt{m-s} \sqrt{\sum_{j=1}^{m-s} (\sigma_j(M))^p} \leq \sqrt{m-s} \|M\|_p^{p/2}.$$

Merging the last two sequences of inequalities gives

$$\|B M B\|_{p/2} \leq \sqrt{m-s} \|M\|_p^{p/2}.$$

Thus $\|B M B\|_{p/2} \leq \sqrt{m-s} \|M\|_p$, which can now be substituted into (2.3).

3. Frobenius norm. This is the special case $p = 2$ with $\|A\|_2 = \|A\|_F$ and $\|A\|_1 = \|A\|_*$. \[11]
2.4. Approximation error, and error matrix. We generalize [8, Theorems 2 and 3] to Schatten $p$-norms.

**Theorem 2.6.** Let $A \in \mathbb{R}^{m \times n}$ with rank($A$) $\geq k$; $C \in \mathbb{R}^{m \times c}$ with rank($C$) $= c \geq k$; and let $p \geq 1$ be an even integer. Then

1. Two-norm ($p = \infty$)
   \[
   \| (I - CC^\dagger) A \|_2^2 \leq \| AA^T - CC^T \|_2.
   \]

2. Schatten $p$-norm ($p$ even)
   \[
   \| (I - CC^\dagger) A \|_p^2 \leq \min \left\{ \| AA^T - CC^T \|_{p/2}, \sqrt{m-c} \| AA^T - CC^T \|_p \right\}.
   \]

3. Frobenius norm ($p = 2$)
   \[
   \| (I - CC^\dagger) A \|_F^2 \leq \min \left\{ \| AA^T - CC^T \|_*, \sqrt{m-c} \| AA^T - CC^T \|_F \right\}.
   \]

**Proof.** This follows from Theorem [2.5] with $P = CC^\dagger$, rank($P$) $= c$, $\hat{A} = C$, and

\[
(I - P) \hat{A} = (I - CC^\dagger) C = C = CC^\dagger C = 0.
\]

Recall Mirsky’s Theorem [16, Corollary 7.4.9.3], an extension of the Hoffman-Wielandt theorem to any unitarily invariant norm and, in particular, Schatten $p$-norms: for $A, H \in \mathbb{R}^{m \times n}$, the singular values $\sigma_j(AA^T)$ and $\sigma_j(HH^T)$, $1 \leq j \leq m$ are also eigenvalues and satisfy

\[
\sum_{j=1}^m |\sigma_j(AA^T) - \sigma_j(HH^T)|^p \leq \| AA^T - HH^T \|_p^p. \tag{2.5}
\]

**Theorem 2.7.** Let $A \in \mathbb{R}^{m \times n}$ with rank($A$) $\geq k$; let $C \in \mathbb{R}^{m \times c}$ with rank($C$) $= c \geq k$ and best rank-$k$ approximation $C_k$; and let $p \geq 1$ be an even integer. Then

1. Two-norm ($p = \infty$)
   \[
   \| (I - C_k C_k^\dagger) A \|_2^2 \leq \| A - A_k \|_2^2 + 2 \| AA^T - CC^T \|_2.
   \]

2. Schatten $p$-norm ($p$ even)
   \[
   \| (I - C_k C_k^\dagger) A \|_p^2 \leq \| A - A_k \|_p^2 + 2 \min \left\{ \| AA^T - CC^T \|_{p/2}, \sqrt{m-c} \| AA^T - CC^T \|_p \right\}.
   \]

3. Frobenius norm ($p = 2$)
   \[
   \| (I - C_k C_k^\dagger) A \|_F^2 \leq \| A - A_k \|_F^2 + 2 \min \left\{ \| AA^T - CC^T \|_*, \sqrt{m-c} \| AA^T - CC^T \|_F \right\}.
   \]

**Proof.** We first introduce some notation before proving the bounds.
0. Set up. Partition $\mathbf{A} = \mathbf{A}_k + \mathbf{A}_\perp$ and $\mathbf{C} = \mathbf{C}_k + \mathbf{C}_\perp$ to distinguish the respective best rank-$k$ approximations $\mathbf{A}_k$ and $\mathbf{C}_k$. From $\mathbf{A}_k \mathbf{A}_k^T = 0$ and $\mathbf{C}_k \mathbf{C}_k^T = 0$ follows

$$\mathbf{A} \mathbf{A}^T = \mathbf{A}_k \mathbf{A}_k^T + \mathbf{A}_\perp \mathbf{A}_\perp^T, \quad \mathbf{C} \mathbf{C}^T = \mathbf{C}_k \mathbf{C}_k^T + \mathbf{C}_\perp \mathbf{C}_\perp^T. \quad (2.6)$$

Since the relevant matrices are symmetric positive semi-definite, eigenvalues are equal to singular values. The dominant ones are

$$\sigma_j(\mathbf{A}_k \mathbf{A}_k^T) = \sigma_j(\mathbf{A} \mathbf{A}^T) = \sigma_j(\mathbf{A})^2, \quad \sigma_j(\mathbf{C}_k \mathbf{C}_k^T) = \sigma_j(\mathbf{C} \mathbf{C}^T) = \sigma_j(\mathbf{C})^2, \quad 1 \leq j \leq k,$$

and the subdominant ones are, with $j \geq 1$,

$$\sigma_j(\mathbf{A}_\perp \mathbf{A}_\perp^T) = \sigma_{k+j}(\mathbf{A} \mathbf{A}^T) = \sigma_{k+j}(\mathbf{A})^2, \quad \sigma_j(\mathbf{C}_\perp \mathbf{C}_\perp^T) = \sigma_{k+j}(\mathbf{C} \mathbf{C}^T) = \sigma_{k+j}(\mathbf{C})^2.$$

To apply Theorem 2.5 set $\hat{\mathbf{A}} = \mathbf{C}$, $\mathbf{P} = \mathbf{C}_k \mathbf{C}_k^T$, rank($\mathbf{P}_k$) = $k$, so that

$$(\mathbf{I} - \mathbf{P}) \hat{\mathbf{A}} = (\mathbf{I} - \mathbf{C}_k \mathbf{C}_k^T)(\mathbf{C}_k + \mathbf{C}_\perp) = \mathbf{C}_\perp.$$ Thus

$$\| (\mathbf{I} - \mathbf{P}) \hat{\mathbf{A}} \|^2 = \| \mathbf{C}_\perp \|^2. \quad (2.7)$$

Two-norm. Substituting (2.7) into the two norm bound in Theorem 2.5 gives

$$\| (\mathbf{I} - \mathbf{C}_k \mathbf{C}_k^T) \mathbf{A} \|^2 \leq \| \mathbf{C}_\perp \|^2 + \| \mathbf{A} \mathbf{A}^T - \mathbf{C} \mathbf{C}^T \|^2. \quad (2.8)$$

The above and Weyl’s theorem imply

$$\| \mathbf{C}_\perp \|^2 = \| \mathbf{C}_\perp \mathbf{C}_\perp^T \|^2 = \lambda_{k+1}(\mathbf{C} \mathbf{C}^T) \leq |\lambda_{k+1}(\mathbf{C} \mathbf{C}^T) - \lambda_{k+1}(\mathbf{A} \mathbf{A}^T)| + \lambda_1(\mathbf{A} \mathbf{A}^T) \leq \| \mathbf{A} \mathbf{A}^T - \mathbf{C} \mathbf{C}^T \|^2 + \| \mathbf{A} \mathbf{A}^T \|^2. \quad (2.9)$$

Substituting this into (2.8) gives

$$\| (\mathbf{I} - \mathbf{C}_k \mathbf{C}_k^T) \mathbf{A} \|^2 \leq \| \mathbf{A} - \mathbf{A}_k \|^2 + 2 \| \mathbf{A} \mathbf{A}^T - \mathbf{C} \mathbf{C}^T \|^2. \quad (2.10)$$

Schatten $p$-norm ($p$ even). Substituting (2.7) into the Schatten-$p$ norm bound in Theorem 2.5 gives

$$\| (\mathbf{I} - \mathbf{C}_k \mathbf{C}_k^T) \mathbf{A} \|^p \leq \| \mathbf{C}_\perp \|^p + \min \{ \| \mathbf{A} \mathbf{A}^T - \mathbf{C} \mathbf{C}^T \|^p, \sqrt[p]{m - c} \| \mathbf{A} \mathbf{A}^T - \mathbf{C} \mathbf{C}^T \|^p \}. \quad (2.11)$$

From $\mathbf{P}_k$ follows $\| \mathbf{C}_\perp \|^p = \| \mathbf{C}_\perp \mathbf{C}_\perp^T \|^p$. For a column vector $\mathbf{x}$, let

$$\| \mathbf{x} \|^p = \sqrt[p]{\sum_j |x_j|^p}$$

be the ordinary vector $p$-norm, and put the singular values of $\mathbf{C}_\perp \mathbf{C}_\perp^T$ into the vector

$$\mathbf{c}_\perp \equiv (\sigma_1(\mathbf{C}_\perp \mathbf{C}_\perp^T) \quad \cdots \quad \sigma_{m-k}(\mathbf{C}_\perp \mathbf{C}_\perp^T))^T.$$
Move from matrix norm to vector norm,
\[
\|C_\perp C_\perp^T\|_{p/2}^{p/2} = \sum_{j=1}^{m-k} \sigma_j (C_\perp C_\perp^T)^{p/2} = \sum_{j=1}^{m-k} c_j^{p/2} = \|c_\perp\|_{p/2}.
\]

Put the singular values of \(A_\perp A_\perp^T\) into the vector
\[
a_\perp \equiv (\sigma_1 (A_\perp A_\perp^T) \ldots \sigma_{m-k} (A_\perp A_\perp^T))^T,
\]
and apply the triangle inequality in the vector norm
\[
\|C_\perp C_\perp^T\|_{p/2} = \|c_\perp\|_{p/2} \leq \|c_\perp - a_\perp\|_{p/2} + \|a_\perp\|_{p/2}.
\]

Substituting the following expression
\[
\|a_\perp\|_{p/2} = \sum_{j=1}^{m-k} \sigma_j (A_\perp A_\perp^T)^{p/2} = \sum_{j=1}^{m-k} \sigma_j (A_\perp)^p = \|A_\perp\|_p^p
\]
into the previous bound and applying (2.11) again gives
\[
\|C_\perp\|_p^2 = \|C_\perp C_\perp^T\|_{p/2} \leq \|c_\perp - a_\perp\|_{p/2} + \|A_\perp\|_p^2.
\] (2.10)

1. First term in the minimum in (2.10): Apply Mirsky’s Theorem (2.5) to the first summand in (2.10)
\[
\|c_\perp - a_\perp\|_{p/2}^{p/2} = \sum_{j=1}^{m-k} |\sigma_{k+j} (CC^T) - \sigma_{k+j} (AA^T)|^{p/2}
\leq \sum_{j=1}^{m} |\sigma_j (CC^T) - \sigma_j (AA^T)|^{p/2} \leq \|CC^T - AA^T\|_{p/2}.
\]

Thus,
\[
\|c_\perp - a_\perp\|_{p/2} \leq \|CC^T - AA^T\|_{p/2}.
\]

Substitute this into (2.10), so that
\[
\|C_\perp\|_p^2 \leq \|A_\perp\|_p^2 + \|CC^T - AA^T\|_{p/2},
\]
and the result in turn into (2.9) to obtain the first term in the minimum,
\[
\|I - C_k C_k^T\|_p \leq \|A_\perp\|_p^2 + 2 \|CC^T - AA^T\|_{p/2}.
\]

2. Second term in the minimum in (2.9): Consider the first summand in (2.10), but apply the Cauchy Schwartz inequality before Mirsky’s Theorem (2.5),
\[
\|c_\perp - a_\perp\|_{p/2}^{p/2} = \sum_{j=1}^{m-k} |\sigma_{k+j} (CC^T) - \sigma_{k+j} (AA^T)|^{p/2}
\leq \sqrt{m-k} \sqrt{\sum_{j=1}^{m-k} |\sigma_{k+j} (CC^T) - \sigma_{k+j} (AA^T)|^p}
\leq \sqrt{m-k} \sqrt{\sum_{j=1}^{m} |\sigma_{k+j} (CC^T) - \sigma_{k+j} (AA^T)|^p}
\leq \sqrt{m-k} \||CC^T - AA^T\||_{p/2}.
\]
Thus,
\[ \|c_\perp - a_\perp\|_F^2 \leq \sqrt{m - k} \|CC^T - AA^T\|_F. \]

Substitute this into (2.10), so that
\[ \|C\|_F^2 \leq \|A\|_F^2 + \sqrt{m - k} \|CC^T - AA^T\|_F, \]
and the result in turn into (2.9) to obtain the second term in the minimum,
\[ \|(I - C_kC_k^\dagger)A\|_F^2 \leq \|A\|_F^2 + 2\sqrt{m - k} \|CC^T - AA^T\|_F. \]

3. Frobenius norm. This is the special case \( p = 2 \) with \( \|A\|_F = \|A\|_F \) and \( \|A\|_1 = \|A\|_\infty. \]

3. Approximation errors and angles between subspaces. We consider approximations where the rank of the orthogonal projector is at least as large as the dimension of the dominant subspace, and relate the low-rank approximation error to the subspace angle between projector and target space. After reviewing assumptions and notation (Section 3.1), we bound the low-rank approximation error in terms of the subspace angle from below (Section 3.2) and from above (Section 3.3).

3.1. Assumptions. Given \( A \in \mathbb{R}^{m \times n} \) with a gap after the \( k \)-th singular value,
\[ \|A\|_2 = \sigma_1(A) \geq \cdots \geq \sigma_k(A) > \sigma_{k+1}(A) \geq \cdots \geq \sigma_r(A) \geq 0, \quad r \equiv \min\{m, n\}. \]

Partition the full SVD \( A = U\Sigma V^T \) in Section 1.1 to distinguish between dominant and subdominant parts,
\[ U = (U_k \ U_\perp), \quad V = (V_k \ V_\perp), \quad \Sigma = \text{diag} (\Sigma_k \ \Sigma_\perp), \]
where the dominant parts are
\[ \Sigma_k = \text{diag} (\sigma_1(A) \ \cdots \ \sigma_k(A)) \in \mathbb{R}^{k \times k}, \quad U_k \in \mathbb{R}^{m \times k}, \quad V_k \in \mathbb{R}^{n \times k}, \]
and the subdominant ones
\[ \Sigma_\perp \in \mathbb{R}^{(m-k)\times(n-k)}, \quad U_\perp \in \mathbb{R}^{m \times (m-k)}, \quad V_\perp \in \mathbb{R}^{n \times (n-k)}. \]

Thus \( A \) is a "direct sum"
\[ A = A_k + A_\perp \quad \text{where} \quad A_k \equiv U_k\Sigma_kV_k^T, \quad A_\perp \equiv U_\perp\Sigma_\perpV_\perp \]
and
\[ A_\perp A_k^\dagger = 0 = A_\perp A_k^T. \quad (3.1) \]

The goal is to approximate the \( k \)-dimensional dominant left singular vector space,
\[ P_k \equiv U_kU_k^T = A_kA_k^\dagger. \quad (3.2) \]

To this end, let \( P \in \mathbb{R}^{m \times m} \) be an orthogonal projector as in (1.3), whose rank is at least as large as the dimension of the targeted subspace,
\[ \text{rank}(P) \geq \text{rank}(P_k). \]
3.2. Subspace angle as a lower bound for the approximation error. We bound the low-rank approximation error from below by the subspace angle and the kth singular value of $A$, in the two-norm and the Frobenius norm.

**Theorem 3.1.** With the assumptions in Section 3.1,

$$
\| (I - P)A \|_{2,F} \geq \sigma_k(A) \| \sin(\Theta(P, P_k)) \|_{2,F}.
$$

**Proof.** From Lemma 3.1, 3.2, and 3.1 follows

$$
\| \sin(\Theta(P, P_k)) \|_{2,F} = \| (I - P)P_k \|_{2,F} = \| (I - P)A_kA_k^\dagger \|_{2,F}
$$

$$
\text{=} \| (I - P)(A_k + A_\perp)A_k^\dagger \|_{2,F} \leq \| A_k^\dagger \|_2 \| (I - P)A \|_{2,F} = \| (I - P)A \|_{2,F}/\sigma_k(A).
$$

\[ \square \]

3.3. Subspace angle as upper bound for the approximation error. We present upper bounds for the low-rank approximation error in terms of the subspace angle, the two norm (Theorem 3.2) and Frobenius norm (Theorem 3.3).

The bounds are guided by the following observation. In the ideal case, where $P$ completely captures the target space, we have $\text{range}(P) = \text{range}(P_k) = \text{range}(A_k)$, and

$$
\| \sin(\Theta(P, P_k)) \|_{2,F} = 0, \quad \| (I - P)A \|_{2,F} = \| A_\perp \|_{2,F} = \| \Sigma_\perp \|_{2,F},
$$

thus suggesting an additive error in the general, non-ideal case.

**Theorem 3.2 (Two-norm).** With the assumptions in Section 3.1

$$
\| (I - P)A \|_{2} \leq \| A \|_{2} \| \sin(\Theta(P, P_k)) \|_{2} + \| A - A_k \|_{2} \| \cos(\Theta(I - P, I - P_k)) \|_{2}.
$$

If also $k < \text{rank}(P) + k < m$, then

$$
\| (I - P)A \|_{2} \leq \| A \|_{2} \| \sin(\Theta(P, P_k)) \|_{2} + \| A - A_k \|_{2}.
$$

**Proof.** From $A = A_k + A_\perp$ and the triangle inequality follows

$$
\| (I - P)A \|_{2} \leq \| (I - P)A_k \|_{2} + \| (I - P)A_\perp \|_{2}.
$$

**First summand.** Since $\text{rank}(P) \geq \text{rank}(P_k)$, Lemma 3.1 implies

$$
\| (I - P)A_k \|_{2} \leq \| (I - P)U_k \|_{2} \| \Sigma_k \|_{2} = \| A \|_{2} \| (I - P)P_k \|_{2}
$$

$$
= \| A \|_{2} \| \sin(\Theta(P, P_k)) \|_{2}
$$

Substitute this into the previous bound to obtain

$$
\| (I - P)A \|_{2} \leq \| A \|_{2} \| \sin(\Theta(P, P_k)) \|_{2} + \| (I - P)A_\perp \|_{2}.
$$

(3.3)

**Second summand.** Submultiplicativity implies

$$
\| (I - P)A_\perp \|_{2} \leq \| (I - P)U_\perp \|_{2} \| \Sigma_\perp \|_{2} = \| A - A_k \|_{2} \| (I - P)U_\perp \|_{2}.
$$

Regarding the last factor, the full SVD of $A$ in Section 3.1 implies

$$
\text{range}(U_\perp) = \text{range}(U_\perp U_\perp^T) = \text{range}(U_k U_k^T)^\perp = \text{range}(P_k)^\perp = \text{range}(I - P_k).
$$
so that
\[ \| (I - P) U \|_2 = \| (I - P) (I - P_k) \|_2 = \| \cos \Theta (I - P, I - P_k) \|_2. \]
Thus,
\[ \| (I - P) A \|_2 \leq \| A - A_k \|_2 \| \cos \Theta (I - P, I - P_k) \|_2. \]
Substitute this into (3.3) to obtain the first bound.

Special case rank(P) + k < m. From Corollary A.2 follows with \( \ell \equiv \text{rank}(P) \)
\[ \| \cos \Theta (I - P, I - P_k) \|_2 = \| \left( I - \frac{(k+\ell)}{m} \right) \cos \Theta (P, P_k) \|_2 = 1. \]

\[ \begin{proof} \]

With strong submultiplicativity, the analogue of (3.3) is
\[ \| (I - P) A \|_F \leq \| A \|_2 \| \sin \Theta (P, P_k) \|_F + \min \left\{ \| A - A_k \|_2 \| \Gamma \|_F, \| A - A_k \|_F \| \Gamma \|_2 \right\}, \]
where \( \Gamma \equiv \cos \Theta (I - P, I - P_k) \).

If also \( k < \text{rank}(P) + k < m \), then
\[ \| (I - P) A \|_F \leq \| A \|_2 \| \sin \Theta (P, P_k) \|_F + \| A - A_k \|_F. \]
\[ \end{proof} \]

\[ \begin{proof} \]

\[ \begin{align*}
\| (I - P) A \|_F &\leq \| A \|_2 \| \sin \Theta (P, P_k) \|_F + \| (I - P) A \|_2, \\
\| (I - P) A \|_2 &\leq \| A \|_2 \| \sin \Theta (P, P_k) \|_F + \| A - A_k \|_2, \\
\end{align*} \]
\[ \end{proof} \]

Appendix A. CS Decompositions. We review expressions for the CS decompositions from [15] Theorem 8.1 and [23] Section 2.

We consider a subspace \( \text{range}(Z) \) of dimension \( k \), and a subspace of \( \text{range}(\hat{Z}) \) of dimension \( \ell \geq k \), whose dimensions sum up to less than the dimension of the host space.

Let \( (Z \ Z_\perp), (\hat{Z} \ \hat{Z}_\perp) \in \mathbb{R}^{m \times m} \) be orthogonal matrices where \( Z \in \mathbb{R}^{m \times k} \) and \( \hat{Z} \in \mathbb{R}^{m \times \ell} \) with \( \ell < k \). Let
\[ (Z \ Z_\perp)^T (\hat{Z} \ \hat{Z}_\perp) = \begin{pmatrix} Z^T \hat{Z} & Z^T \hat{Z}_\perp \\ Z_\perp^T \hat{Z} & Z_\perp^T \hat{Z}_\perp \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \]
be a CS decomposition where $Q_{11} \in \mathbb{R}^{k \times k}$, $Q_{12} \in \mathbb{R}^{(m-k) \times (m-k)}$, $Q_{21} \in \mathbb{R}^{\ell \times \ell}$ and $Q_{22} \in \mathbb{R}^{(m-\ell) \times (m-\ell)}$ are all orthogonal matrices.

**Theorem A.1.** If $k < \ell < m - k$ then

$$ D = \begin{bmatrix}
I_r & s & \ell - (r + s) & m - (k + \ell) + r & s & k - (r + s) \\
C & 0 & 0 & S & I_{k-(r+s)} \\
0 & S & -S_{m-(k+\ell)+r} & -C & 0 \\
I_{\ell-(r+s)} & & & & \end{bmatrix} \begin{bmatrix}
r \\
s \\
m - (k + \ell) + r \\
s \\
\ell - (r + s) \\
k - (s + r)
\end{bmatrix} $$

Here $C^2 + S^2 = I_s$ with

$$ C = \text{diag} \left( \cos \theta_1, \cdots, \cos \theta_s \right), \quad S = \text{diag} \left( \sin \theta_1, \cdots, \sin \theta_s \right), \quad \text{and} $$

$$ r = \text{dim} \left( \text{range}(Z) \cap \text{range}(\hat{Z}) \right), \quad m - (k + \ell) + r = \text{dim} \left( \text{range}(Z_1) \cap \text{range}(\hat{Z}_1) \right) $$

$$ \ell - (r + s) = \text{dim} \left( \text{range}(Z_1) \cap \text{range}(\hat{Z}) \right), \quad k - (r + s) = \text{dim} \left( \text{range}(Z) \cap \text{range}(\hat{Z}_1) \right). $$

**Corollary A.2.** From Theorem A.1 follows

$$ \| \sin \Theta(Z, \hat{Z}) \|_{2,F} = \| Z^T \hat{Z}_1 \|_{2,F} = \left\| \begin{bmatrix} S & I_{k-(r+s)} \end{bmatrix} \right\|_{2,F} $$$$ \| \cos \Theta(Z, \hat{Z}) \|_{2,F} = \| Z^T \hat{Z} \|_{2,F} = \left\| \begin{bmatrix} I_r & C \end{bmatrix} \right\|_{2,F} $$$$ \| \cos \Theta(Z_1, \hat{Z}_1) \|_{2,F} = \| Z_1^T \hat{Z}_1 \|_{2,F} = \left\| \begin{bmatrix} I_{m-(k+\ell)} & \cos \Theta(Z, \hat{Z}) \end{bmatrix} \right\|_{2,F}. $$

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