Nikolskii-type inequalities
for rearrangement invariant spaces

OSTROVSKY E., SIROTA L.

Department of Mathematics and Statistics, Bar-Ilan University, 59200, Ramat Gan, Israel.

e-mail: galo@list.ru

Department of Mathematics and Statistics, Bar-Ilan University, 59200, Ramat Gan, Israel.

e-mail: sirota@zahav.net.il

Abstract. In this paper we generalize the classical Nikol’skii inequality on the many popular classes pairs of rearrangement invariant (r.i.) spaces and construct some examples in order to show the exactness of our estimations.

Key words: Nikol’skii inequality, moment rearrangement invariant spaces, polynomials, Orlicz, Lorentz, Marzinkiewitz spaces, slowly varying function, Fejer’s kernel, entire functions of finite order.

Mathematics Subject Classification (2000): primary 60G17; secondary 60E07; 60G70.

1. Introduction. Notations. Statement of problem.

First of all we recall the classical Nikol’skii inequality: there exist a values $p_{-}, p_{+}, q_{-}, q_{+}$, $1 \leq p_{-} < p_{+} < q_{-} < q_{+} \leq \infty$, such that for all the values $p \in (p_{-}, p_{+})$, $q \in (q_{-}, q_{+})$ the following inequality holds:

$$|t_{n}|_{q} \leq C \sigma^{1/p - 1/q} |t_{n}|_{p},$$  \hspace{1cm} (1)

and automatically $1 \leq p \leq q \leq \infty$.

Here $t_{n} \in A(n)$, where $A(n)$ is the (sub)space of trigonometrical or algebraic polynomial from several variables defined on the sets $T = [-\pi, \pi]$, $T = [0, 1]$ with general degree $\leq n$, $n = 1, 2, 3, \ldots$, or entire function of exponential type $\leq n$, $n \in [1, \infty)$ defined on the set $\mathbb{R}$, $\sigma = \sigma(n)$ is some increasing function; as a rule $\sigma = C_{1} n^{\gamma}$, $\gamma > 0$.

Recall that the entire exponential function $t_{n}(x)$, $x \in \mathbb{R}$ of finite type order $\leq n$ may be defined as follows:

$$t_{n}(x) = \int_{[-n,n]} \exp(ix\xi) g(\xi) \, d\xi, \; g(\cdot) \in L_{2}(\mathbb{R}).$$

Hereafter $C, C_{j}$ will denote any non-essential finite positive constants. As usually, for the measurable function $f : T \rightarrow \mathbb{R}$, where $(T, M, \mu)$ is a measurable space with non-trivial sigma-finite measure $\mu$, ...
\[ |f|_p(T, \mu) = |f|_p(\mu) = |f|_p = \left( \int_T |f(x)|^p \mu(dx) \right)^{1/p}, \quad p < \infty, \]

\[ L_p(T, \mu) = L_p(\mu) = \{ f : |f|_p < \infty \}; \quad m \text{ will denote usually Lebesque measure, and we will write } m(dx) = dx; \quad |f|_\infty \overset{\text{def}}{=} \sup_x |f(x)|. \]

There are many generalizations of inequality (1) on the case if \( T \) is convex polytop in the space \( \mathbb{R}^l \), \([2],[35],[36],[39] \); "weight" Nikol’skii inequalities for polynomials defined on the set \([-1, 1]\) with Jakobi weight; for polynomial defined on the set \( \mathbb{R}^l \) with exponential weight \( w = w(x) \) of a view \( w(x) = \exp(-|x|\theta), \theta > 0, |x| = |(x_1, x_2, \ldots, x_l)| = \sqrt{x_1^2 + x_2^2 + \ldots + x_l^2} \), see \([39],[40],[41]\) etc.

There exists a general definition of the Nikolskii class \( \{ A(n) \} \) over measurable triple \((T, M, \mu)\) with some function \( \sigma = \sigma(n) \) see \([2]\), where the inequality (1) is postulated under some increasing sequence (function) \( \sigma = \sigma(n) \) such that

\[ \lim_{n \to \infty} \sigma(n) = \infty, \quad \sigma(n) \geq 3. \]

In this definition instead the subspace of trigonometrical or algebraical polynomials with common degree \( \leq n \) used some monotonically decreased sequence (or set) \( A(n) \) of linear subspaces of a space

\[ L(p_-, q_+) = \bigcap_{p \in (p_-, q_+)} L_p(\mu); \quad m > n \Rightarrow \]

\[ A(n) \subset A(m); \quad \forall n A(n) \subset L(p_-, q_+). \]

**Further we will consider only the triples \((T, M, \mu)\) with some (non-trivial) Nikolskii class \( A(n) \).**

The Nikolskii inequality play a very important role in the theory of approximation, theory of function of several variables, functional analysis (imbedding theorems for Besov spaces). See, for example, \([4],[2],[35]\) etc.

The inequality (1) may be rewritten as follows. Let \((X, || \cdot ||_X)\) be any rearrangement invariant (r.i.) space on the set \( T \); denote by \( \phi(X, \delta) \) its fundamental function

\[ \phi(X, \delta) = \sup_{A, \mu(A) \leq \delta} ||I(A)||_X, \quad I(A) = I(A, x) = 1, x \in A, \]

\[ I(A) = I(A, x) = 0, x \notin A. \] We define also for two function r.i. spaces \((X, || \cdot ||_X)\) and \((Y, || \cdot ||_Y)\) over our set \( T \) and for arbitrary finite positive constants \( K_1, K_2 \) the so-called **Nikol’skii two-space functional, briefly: NF functional** between the spaces \( X \) and \( Y \)

\[ W_n(X, Y, K_1, K_2) \overset{\text{def}}{=} \sup_{t_n \neq 0, t_n \in A(n)} \frac{||t_n||_X}{\phi(X, K_1/\sigma)} : \frac{||t_n||_Y}{\phi(Y, K_2/\sigma)}, \]

\[ W_n(X, Y) = W_n(X, Y, 1, 1). \] Then (1) is equivalent to the following inequality:
\[ P \in (p_-, p_+), \quad q \in (q_-, q_+) \Rightarrow \sup_n W_n(L_q, L_p) < \infty. \quad (2) \]

**Definition 1.**

**By definition,** the pair of r.i. spaces \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) over some fixed Nikol’skii class is said to be a (strong) Nikol’skii pair, write: \((X, Y) \in \text{Nik},\) if the NF functional between \(X\) and \(Y\) is finite:

\[ \sup_n W_n(X, Y) < \infty, \quad (3) \]

and is called a weak Nikol’skii pair, write \((X, Y) \in \text{wNik},\) if for some non-trivial constants \(K_1, K_2\)

\[ \sup_n W_n(X, Y, K_1, K_2) < \infty, \quad (4) \]

**Our aim is description some pair of r.i. spaces with strong and weak Nikol’skii properties.**

Roughly speaking, we will prove that the most of popular pairs of r.i. spaces are strong, or at least weak Nikol’skii pairs.

The paper is organized as follows. In the next section we introduce and investigate a new class of r.i. spaces, so-called moment rearrangement invariant spaces, briefly, m.r.i. spaces. In the section 3 we formulate and prove the main result of paper for m.r.i. spaces.

In the pilcrow 4 we consider some examples in order to show the exactness of obtained estimations and investigate the low bounds in our inequalities. In the section 5 we will receive the Nikolskii inequality for (generalized) Zygmund spaces.

In the last section 6 we investigate the inverse Nikolskii inequality for Lorentz spaces in order to emphasize the precision of obtained results.

**2. Auxiliary facts. Moment rearrangement spaces.**

Let \((X, \| \cdot \|_X)\) be a r.i. space, where \(X\) is linear subset on the space of all measurable function \(T \to R\) over our measurable space \((T, M, \mu)\) with norm \(\| \cdot \|_X\) equipped with Nikol’skii class \(A(n)\).

**Definition 2.**

We will say that the space \(X\) with the norm \(\| \cdot \|_X\) is moment rearrangement invariant space, briefly: m.r.i. space, or \(X = (X, \| \cdot \|_X) \in \text{m.r.i.}\), if there exist a real constants \(a,b; 1 \leq a < b \leq \infty\), and some rearrangement invariant norm \(< \cdot >\) defined on the space of real functions defined on the interval \((a,b)\), non necessary to be finite on all the functions, such that

\[ \forall f \in X \Rightarrow \| f \|_X = < h(\cdot) >, \quad h(p) = |f|_p. \quad (5) \]
We will say that the space $X$ with the norm $\|\cdot\|_X$ is weak moment rearrangement space, briefly, w.m.r.i. space, or $X = (X, \|\cdot\|_X) \in w.m.r.i.$, if there exist a constants $a, b; 1 \leq a < b \leq \infty$, and some functional $F$, defined on the space of a real functions defined on the interval $(a, b)$, non necessary to be finite on all the functions, such that

$$\forall f \in X \Rightarrow \|f\|_X = F(h(\cdot)), \; h(p) = |f|_p.$$  \hspace{1cm} (6)

We will write for considered w.m.r.i. and m.r.i. spaces $(X, \|\cdot\|_X)$

$$(a, b) \overset{\text{def}}{=} \text{supp}(X),$$

(moment support; not necessary to be uniquely defined) and define for other such a space $Y = (Y, \|\cdot\|_Y)$ with $(c, d) = \text{supp}(Y)$

$$\text{supp}(X) \gg \text{supp}(Y),$$

iff $\min(a, b) > \max(c, d)$.

It is obvious that arbitrary m.r.i. space is r.i. space.

There are many r.i. spaces satisfied the condition (5): exponential Orlicz’s spaces, some Martzinkiewitz spaces, interpolation spaces (see [14], [42]).

In the article [13] are introduced the so-called $G(p, \alpha)$ spaces consisted on all the measurable function $f : T \to R$ with finite norm

$$\|f\|_{p, \alpha} = \left[ \int_1^\infty \left( \frac{|f|_x}{x^\alpha} \right)^p \nu(dx) \right]^{1/p},$$

where $\nu$ is some Borelian measure.

Astashkin in [42] proved that the space $G(p, \alpha)$ in the case $T = [0, 1]$ and $\nu = m$ coincides with the Lorentz $\Lambda_p(\log^{1-p\alpha}(2/s))$ space. Therefore, both this spaces are m.r.i. spaces.

Another examples. Recently (see [5], [7], [43], [18], [27] - [29], [31] - [34] etc.) appears the so-called Grand Lebesque Spaces $GLS = G(\psi) = G(\psi; a, b)$ spaces consisting on all the measurable functions $f : T \to R$ with finite norms

$$\|f\|G(\psi) = \sup_{p \in (a, b)} [|f|_p/\psi(p)].$$  \hspace{1cm} (7)

Here $\psi(\cdot)$ is some continuous positive on the open interval $(a, b)$ function such that

$$\inf_{p \in (a, b)} \psi(p) > 0, \; \sup_{p \in (a, b)} \psi(p) = \infty.$$

It is evident that $G(\psi; a, b)$ is m.r.i. space and $\text{supp}(G(\psi(a, b))) = (a, b)$.

This spaces are used, for example, in the theory of probability [5], [7], [8], [9], [10]; theory of PDE [27], [28], functional analysis [23], [43], theory of Fourier series, theory of martingales etc.
The spaces $G(\psi, a, b)$ are non-separable [43], but they satisfy the Fatou property. As long as its Boyds indices (see for detail definition [1, chapter 1]) $\gamma_-, \gamma_+$ are correspondingly

$$\gamma_- = 1/b, \ \gamma_+ = 1/a,$$

(see [43]), we conclude on the basis the main result of article [14] that the spaces $G(\psi; a, b)$ are interpolation spaces not only between the spaces $[L_1, L_\infty]$ but between the spaces $[L_v, L_s]$ for all values $(v, s); v < a, s > b.$

Since for arbitrary real-valued continuous function $f$ defined on the set $[0, 1]$ \[\|f\|C[0, 1] = \sup_{t \in [0, 1]} |f(t)| = \lim_{p \to \infty} \|f\|_p = \sup_{p \in [1, \infty)} |f|_p,\] the space $C[0, 1]$ is m.r.i. space with $supp(C[0, 1]) = [1, \infty)$ or equally, e.g., $supp(C[0, 1]) = (3, \infty).$

But all the Besov’s spaces $B_{p,s}^r(T)$ are w.m.r.i., but not are m.r.i. spaces.

Let us consider now the (generalized) Zygmund’s spaces $L_p \log^r L,$ which may be defined as an Orlicz’s spaces over some subset of the space $R^l$ with non-empty interior and with $N-$ Orliczs function of a view \[\Phi(u) = |u|^p \log^r (C + |u|), \ p \geq 1, \ r \neq 0.\]

**Lemma 1.**

1. All the spaces $L_p \log^r L$ over real line with measure $m$ with condition $r \neq 0$ are not m.r.i. spaces.
2. If $r$ is positive and integer, then the spaces $L_p \log^r L$ are w.m.r.i. space.

**Proof. 1.** It is sufficient to consider the case $T = R^l$ with the measure $m$ and the case $p > 1.$

There exists a function $f_0 = f_0(x)$ belonging to the space $L_p \log^r L:\n $$\int_T |f_0|^p \log^r (C + |f_0|) \, dx < \infty,$$ but such that for all sufficiently small values $\epsilon > 0$

$$\int_T |f_0|^{p+\epsilon} \, dx = \infty$$

in the case $p > 1$ and

$$\int_T |f_0|^{p+\epsilon} \, dx = \infty$$

In the case $p = 1.$

Therefore, the interval $(a, b)$ in the definition of m.r.i. spaces does not exists.

The assertion 2 it follows from the formulae

$$|f|^p \log |f| = d^k |f|^p / dp^k, \ k = 1, 2, \ldots.$$

**Lemma 2.** There exists an r.i. space without the w.m.r.i. property.
Proof. On the interval $T = [0, 1]$ with usual Lebesgue measure $m$ there exists a function $f$ with standard normal (Gaussian) distribution. This implies, for example, that

$$\int_T \exp(pf(x)) \, dx = \exp\left(0.5 \, p^2\right), \ p \in R.$$

There exist a functions $g : R \to R$ such that the function $h(x) = g(f(x))$ which distribution can not be uniquely defined by means of all positive moments, for instance, $h(x) = g(f(x)) = [f(x)]^3$ or $g(x) = \exp(f(x))$.

Let us consider a two such a functions $f_1$ and $f_2$ with different distributions, but with at the same moments, for example:

$$\int_T |f_1|^p \, dx = \int_T |f_2|^p \, dx = \int_T [\exp(f)]^p \, dx = \exp(p^2/2), \ p \in R.$$

We choose the (quasi)-concave positive strictly increasing continuous function $\theta(\cdot), \ \theta(0+) = 0$, for which

$$\int_0^\infty \theta(m\{x : |f_1(x)| > \lambda\}) \, d\lambda = \infty,$$

but

$$\int_0^\infty \theta(m\{x : |f_2(x)| > \lambda\}) \, d\lambda < \infty.$$

The Lorentz r.i. space $\Lambda(T, \theta)$ over $T = [0, 1]$ with the function $\theta(\cdot)$ and the classical norm (see [30], chapter 2, section 2)

$$||f||_{L(T, \theta)} = \int_0^\infty \theta(m\{x : |f(x)| > \lambda\}) \, d\lambda$$

is not w.m.r.i. space.

3. Main result. Nikolskii inequality for the pairs of m.r.i. spaces.

Theorem 1.

Let $(X, || \cdot || X)$ be any m.r.i. space with support $\text{supp}(X) = (c, d)$ relatively the auxiliary norm $<\cdot>$, and let $(Y, || \cdot || Y)$ be another m.r.i. space over at the same triple $(T, M, \mu)$ relatively the second auxiliary norm $<< \cdot >>$ and with $\text{supp}(Y) = (a, b)$, where $(c, d) >> (a, b)$ and suppose $1 \leq p_- = a, \ p_+ = b; \ q_- = c, \ q_+ = d \leq \infty$.

Then the pair of m.r.i. spaces $(X, || \cdot || X)$ and $(Y, || \cdot || Y)$ is the (strong) Nikolskii pair:

$$\sup_n W(X, Y) = C(X, Y) < \infty. \ (8)$$

Note that the restriction

$$1 \leq p_- = a, \ p_+ = b; \ q_- = c, \ q_+ = d \leq \infty$$

is not loss of generality and is necessary for the cases if $A(n)$ is the set of trigonometrical or algebraical polynomials.
Proof is very simple. It follows from the definition of Nikolskii Class $A(n)$ that for arbitrary non-zero element of $A(n) : t_n \in A(n)$

$$\frac{|t_n|_q}{(1/\sigma)^{1/p}} \leq C \frac{|t_n|_p}{(1/\sigma)^{1/q}},$$

$q \in (c,d), \ p \in (a,b)$. Taking from the bide-sides of the inequality (9) the norm $<< \cdot >>$ and taking into account the monotonocity of the norm $<< \cdot >>$ and the equality

$$<< \delta^{1/p} >> = ||I(A)||Y = \phi(Y, \delta), \ \mu(A) = \delta,$$

where $\delta = 1/\sigma$, we obtain:

$$|t_n|_q \phi(Y, 1/\sigma) \leq C||t_n||Y^{1/q}.$$

Calculate from the bide-side the norm $< \cdot >$, we find analogously

$$||t_n||X \cdot \phi(Y, 1/\sigma) \leq C||t_n||Y \cdot \phi(X, 1/\sigma),$$

wich is equivalent to (8).

4. Examples. Low bounds.

1. We consider now a very important for applications examples of $G(\psi)$ spaces. Let $a = const \geq 1, b = const \in (a, \infty); \alpha, \beta = const$. Assume also that at $b < \infty \ min(\alpha, \beta) \geq 0$ and denote by $h$ the (unique) root of equation

$$(h - a)^{\alpha} = (b - h)^{\beta}, \ a < h < b; \ \zeta(p) = \zeta(a, b; \alpha, \beta; p) =$$

$$(p - a)^{\alpha}, \ p \in (a, h); \ \zeta(a, b; \alpha, \beta; p) = (b - p)^{\beta}, \ p \in [h, b);$$

and in the case $b = \infty$ assume that $\alpha \geq 0, \beta < 0$; denote by $h$ the (unique) root of equation $(h - a)^{\alpha} = h^{\beta}, h > a$; define in this case

$$\zeta(p) = \zeta(a, b; \alpha, \beta; p) = (p - a)^{\alpha}, \ p \in (a, h); \ p \geq h \Rightarrow \zeta(p) = p^{\beta}.$$

Here and further $p \in (a, b) \Rightarrow \psi(p) \leq \nu(p)$ denotes that

$$0 < \inf_{p \in (a, b)} \psi(p)/\nu(p) \leq \sup_{p \in (a, b)} \psi(p)/\nu(p) < \infty.$$

The space $G = G_T = G_T(a, b; \alpha, \beta) = G(a, b; \alpha, \beta)$ consists by definition on all the measurable functions $f : T \rightarrow R$ with finite norm:

$$||f||G(a, b; \alpha, \beta) = \sup_{p \in (a, b)} |f|_p \cdot \zeta(p).$$

On the other words, $G(a, b; \alpha, \beta)$ is the $G(\psi; a, b)$ space with $\psi(p) = 1/\zeta(p)$.

These spaces was introduced in [5], [43]; and in this article was also calculated its fundamental functions.
We rewrite here only the asymptotical expression for \( \phi(G(a, b; \alpha, \beta, \delta)) \) as \( \delta \to 0^+ \):

\[
\phi(G(a, b; \alpha_1, \beta_1), \delta) \sim (\beta_1 b^2 / e)^{\beta_1} \cdot \delta^{1/b_1} |\log \delta|^{-\beta_1};
\]

\[
\phi(G(c, \infty; \alpha_2, -\beta_2), \delta) \sim (\beta_2)|\log \delta|^{-|\beta_2|},
\]

\( 1 \leq a < b < c < \infty; \alpha_1, \alpha_2, \beta_1, \beta_2 \in (0, \infty) \). We obtain using the theorem 1: if \( t_n \in A(n) \), then

\[
|t_n||G(c, \infty, \alpha_2, -\beta_2) \leq C(\alpha_1, \alpha_2, \beta_1, \beta_2, a, b, c) \times
\]

\[
 n^{1/b} (\log n)^{-\beta_1 - \beta_2} |t_n||G(a, b, \alpha_1, \beta_1). \]

Let now \( X = G(a_1, b_1, \alpha_1, \beta_1) \), \( Y = G(a_2, b_2, \alpha_2, \beta_2) \), where \( 1 \leq a_2 < b_2 < a_1 < b_1 < \infty \). We conclude on the basis of Theorem 1 and the expression for \( \phi(G(\psi), \delta) \) written above: \( t_n \in A(n), n \geq 3 \) \( \Rightarrow \)

\[
|t_n||G(a_1, b_1, \alpha_1, \beta_1) \leq C_1 n^{1/b_2 - 1/b_1} [\log n]^{\beta_1 - \beta_2} |t_n||G(a_2, b_2, \alpha_2, \beta_2),
\]

where

\[
 C_1 = C_1(a_1, a_2, b_1, b_2, \alpha_1, \alpha_2, \beta_1, \beta_2).
\]

2. In this subsections we will construct some examples in order to illustrate the exactness of result of section 3.

We consider here only the one-dimensional case \( l = 1 \) and only the trigonometrical case, i.e. \( T = [-\pi, \pi] \) and \( A(n) \) is the collection of trigonometrical polynomials with degree \( \leq n \) (discrete case) or \( T = R \) and \( A(n) \) is the set of all entire functions of order \( \leq n \); \( n \in [3, \infty) \) (continuous case).

**Theorem 2.** Let \( G(\psi) \) and \( G(\nu) \) be two arbitrary examples of \( G(\psi) \) spaces. We assert that

\[
\lim_{n \to \infty} \sup_{t_n \in A(n)} W_n(G(\psi), G(\nu)) = C > 0. \quad (10)
\]

**Note** that we do not suppose here

\[
\text{supp } G(\psi) \gg \text{supp } G(\nu).
\]

**Proof.** Let us consider the following functions:

\[
D_n(x) = \frac{\sin^2(nx/2)}{n^2 x^2}, x \neq 0; \quad D_n(0) = 1/4
\]

in the continuous case, i.e. \( T = R \) and

\[
D_n(x) = \frac{\sin^2(nx/2)}{n^2 \sin^2(x/2)}, x \neq 0; \quad D_n(0) = 1
\]
in the discrete case \( T = [-\pi, \pi] \) (renormed Fejer’s kernels). It is well-known that \( D_n(\cdot) \in A(n) \).

We restrict itself only by continuous case. Let us calculate all the moments of \( D_n \). We have:

\[
\int_{-\infty}^{\infty} |D_n(x)|^p \, dx = \int_{-\infty}^{\infty} \left| \frac{\sin(nx/2)}{nx} \right|^{2p} \, dx =
\]

\[
n^{-1} 2^{1-2p} \int_{-\infty}^{\infty} \left| \frac{\sin y}{y} \right|^{2p} \, dy = n^{-1} 2^{1-2p} I(2p),
\]

where

\[
I(s) \overset{\text{def}}{=} \int_{-\infty}^{\infty} \left| \frac{\sin y}{y} \right|^s \, dy,
\]

and \( s \in [2, \infty) \).

It is evident that at \( s \in [2, 10] \) \( I(s) \sim C_1 \). It follows from the saddle-point (or Laplace) method that as \( s \to \infty \) \( \Rightarrow \)

\[
I(s) \sim \frac{C}{s^{1/2}}.
\]

Since

\[
1 = \lim_{s \to \infty} s^{1/s} < \sup_{s \geq 1} s^{1/s} = e^{1/e} < \infty,
\]

we conclude that there exist two finite absolute positive constants \((C_-, C^+)\), \( C_- < C^+ \), such that for all values \( n \geq 1 \) and \( p \geq 1 \)

\[
C_- n^{-1/p} \leq |D_n|_p \leq C^+ n^{-1/p}.
\]

We receive calculating the \( G(\psi) \) norm of a function \( D_n(\cdot) \):

\[
\|D_n\|G(\psi) = \sup_{p \geq 1} \left[ |D_n|_p / \psi(p) \right] \geq C_- \sup_p \left[ n^{-1/p} / \psi(p) \right] =
\]

\[
C_- \phi(G(\psi), 1/n)
\]

and we find analogously

\[
\|D_n\|G(\psi) \leq C^+ \phi(G(\psi), 1/n).
\]

We have therefore for two spaces \( G(\psi) \) and \( G(\nu) \):

\[
W_n(G(\psi), G(\nu)) \geq C_- / C^+ \overset{\text{def}}{=} C_3,
\]

Q.E.D.

**Corollary.** A rouge estimation for the constant \( C_3 \) show that \( C_3 \geq 1/9 \). But in the source Nikolskii inequality (1) the exact value \( C \) is equal to 2.
Considering instead \( D_n(\cdot) \) a more general function \( D_n^{\alpha,\beta}(\cdot) \in A(n) \) of a view, for example,

\[
D_n^{\alpha,\beta}(x) = \int_{-n}^{n} \exp(itx) \left[ (1 - (|t|/n)^\alpha)^\beta \right] dt,
\]

where \( \alpha, \beta = \text{const} > 0 \), we get after the optimization over \( \alpha, \beta \) by means of computer computations that in the considered before trigonometrical and one-dimensional case

\[
1.4713 \ldots \leq \lim_{n \to \infty} W_n(G(\psi), G(\nu)) \leq \sup_n W_n(G(\psi), G(\nu)) \leq 2.
\]

3. Now we intend to generalize the low estimation on the Orlicz’s space \( L(\Phi) \), where \( \Phi(\cdot) \) is some Orlicz’s function, i.e. even, twice continuous differentiable, convex, \( \lim_{u \to 0} \Phi(u)/u = 0 \), \( \lim_{u \to \infty} \Phi(u)/u = \infty \) etc.

Recall that the Orlicz’s norm of a function \( f : T \to R \) may be calculated by equality:

\[
||f||_{L(\Phi)} = \inf_{v \geq 0} \left[ 1 + \int_T \Phi(vf(x)) dx \right]/v
\]

or up to norm equivalence

\[
||f||_{L(\Phi)} = \inf \left\{ v : v > 0, \int_T \Phi(|f(x)/v|) dx \leq 1 \right\}.
\]

see [19], [20], [21], [24].

**Theorem 3.** Let \( \Phi_1(\cdot) \), \( \Phi_2(\cdot) \) be a two Orlicz’s functions such that for some \( m = 1, 2, 3, \ldots \) and all the values \( v \geq 1 \)

\[
\int_1^\infty \Phi_i(v/y^{2m}) dy \leq \Phi_i(C_m v), \ i = 1, 2.
\]

Then there exist a two positive finite constants \( K_1, K_2, K_3 = K_{1,2,3}(\Phi_1, \Phi_2, T) \) such that for Nikolskii functional \( W_n(L(\Phi_1), L(\Phi_2)) \) there holds:

\[
\lim_{n \to \infty} W_n(L(\Phi_1), L(\Phi_2), K_1, K_2) \geq K_3.
\]

**Note** that the condition (11) is satisfies in the cases if \( \Phi(u) = |u|^p S(|u|), \ N(u) = \exp(|u|^p S(u)), \ u \geq 2, \) where \( S(u) \) is positive slowly varying as \( u \to \infty \) function (see for definition and properties of slowly varying functions [26], chapter 1, sections 1.3 - 1.5.)

**Proof.** It is enough to consider only the case \( l = 1 \) and \( T = R \). Let us choose the introduced function \( D_n(x) \).

In order to estimate the Orlicz’s \( ||D_n||_{L(\Phi)}, \ \Phi = \Phi_{1,2} \) norm of the function \( D_n \), we need to estimate the following integral:
\[ n \cdot J = n \cdot J(n, v, \Phi) \overset{\text{def}}{=} n \cdot \int_T \Phi(v D_n(x)) \, dx = \int_T \Phi \left( v \frac{\sin^2(y/2)}{y^2} \right) \, dy. \]

Note that at the values \( v : |v| \leq 1 \) the inequality (11) is satisfied. Further we consider only the values \( v \geq 1 \), as long as the function \( \Phi \) is even.

The low estimation for the integral \( J \)

\[ n \, J(n, v, \Phi) \geq \Phi(C_\neg v) \]

is evident; we must to obtain the upper bound at the same manner. We have:

\[ n \, J = \int_{-\pi}^{\pi} \Phi \left( v \frac{\sin^2(y/2)}{y^2} \right) \, dy + 2 \int_{\pi}^{\infty} \Phi \left( v \frac{\sin^2(y/2)}{y^2} \right) \, dy = I_1 + I_2. \]

The estimation \( I_1 \) is:

\[ I_1 = \int_{-\pi}^{\pi} \Phi \left( v \frac{\sin^2(y/2)}{y^2} \right) \, dy \leq 2\pi \Phi(0.25 \, v) \]

since the function \( v \to \Phi(v) \) is monotonic and \((\sin^2 y/2)/y^2 \leq 1/4\).

Further,

\[ I_2 \leq 2 \int_{\pi}^{\infty} \Phi \left( v \frac{1}{y^2} \right) \, dy \leq \Phi(C \, v) \]

by virtue of condition (11); the replacing \( 1 \to \pi \) in not essential.

Therefore,

\[ C^{-1} \, n \, ||D_n|| L(\Phi) \asymp \inf_{v>0} \left( \frac{1}{v} + \frac{\Phi(C_1 \, v)}{n \, v} \right) \quad (13) \]

The asymptotically optimal value \( v_0 \) in the right-side of equality (13) is attained at

\[ v_0 = C_2 \Phi^{-1}(C_3 \, n), \]

where \( \Phi^{-1} \) denotes the inverse function to the function \( \Phi \) on the left-part half-line.

We used here the known ([19],chapter 2, section 9; [20], chapter 2) the expression for the fundamental function of Orliczs spaces.

Thus,

\[ n \, ||D_n|| L(\Phi_i) \asymp \frac{C(1, i)}{\Phi_i^{-1}(C_{4,i}/n)} = C_{5,i} \, n \, \phi(L(\Phi_i), C_{6,i}/n). \]
Substituting into the expression for $W_n(X,Y)$, where $X = L(\Phi_1)$, $Y = L(\Phi_2)$, we get to the inequality (12).

The general case, if $m = 2, 3, 4, \ldots$ may be provided by the choice

$$t_n^m(x) = [D_n(x)]^m; [D_n(\cdot)]^m \in A(mn).$$

5. The case of (generalized) Zygmund spaces. Other method.

We will suppose in this section that the measure $\mu$ in the triple $(T,M,\mu)$ is atomless and that $n \geq 3$.

Recall that the (generalized) Zygmund space

$$X = L_q (\log L)^{\gamma}$$

over source triple is defined as an Orliczs space with the Orliczs function of a view:

$$\Phi(u) = \Phi(q, \gamma; u) = |u|^p \log(C(q, \gamma) + u)^\gamma,$$

where $C(q, \gamma)$ is sufficiently great constant.

Note that the fundamental functions for these spaces are as $n \to \infty$:

$$\phi(L_q (\log L)^{\gamma}, C_1/n) \sim C_2 n^{-1/q} (\log n)^{\gamma/q}.$$

Let $Y$ be another Zygmund space:

$$Y = L_p (\log L)^{-\beta},$$

where $q > p \geq 1$ (the alternative case is trivial).

Since the function $u \to |u|^p \log(C(q, \gamma) + u)^\gamma$ satisfies the $\Delta_2$ - condition, we can use the assertion of theorem 3:

$$\lim_{n \to \infty} \sup_{t_n \neq 0, t_n \in A(n)} \left[ \frac{||t_n||_{L_q [\log L]^\gamma}}{||t_n||_{L_p [\log L]^{-\beta}}} \right] \geq C_L(p, q, \gamma, \beta) \sigma^{1/p - 1/q} [\log \sigma]^{\gamma/q - \beta/p}. \tag{14}$$

The assertion (14) is the inverse inequality. We ground now the direct Nikolskii inequality for Zygmund spaces at the same manner as inequality (14), but only in the cases $1 < p < q, \gamma, \beta \geq 0$.

Theorem 4. Let

$$1 < p < q, \gamma \geq 0, \beta \geq 0.$$

We assert that for NF functional of considered spaces the following inequality is true:

$$\sup_{t_n \neq 0, t_n \in A(n)} \left[ \frac{||t_n||_{L_q [\log L]^\gamma}}{||t_n||_{L_p [\log L]^{-\beta}}} \right] \leq$$
\[ C_R(p, q, \gamma, \beta) \sigma^{1/p-1/q} [\log \sigma]^{\gamma/q-\beta/p}. \]  

**Proof.** Since the cases \( \gamma = 0 \) or \( \beta = 0 \) are simple, we investigate further the possibility \( \gamma > 0, \beta > 0 \).

It is proved the article [18] that at least for \( g \in \bigcup_n A(n) \) and for arbitrary values \( r > q \)

\[
||g||_{L_q[\log L]^{\gamma}} \leq C \left[ \frac{r}{r-q} \right]^{\gamma/r} |g|_r \quad (16a)
\]

and analogously may be proved the inverse inequality: for arbitrary \( s \in (1, p) \)

\[
||g||_{L_p[\log L]^{-\beta}} \geq C \left[ \frac{s}{p-s} \right]^{-\beta/s} |g|_s. \quad (16b)
\]

We have for the \( t_n \in A(n) \) combined the definition of Nikolskii class and inequalities (16a), (16b):

\[
||t_n||_{L_q[\log L]^{\gamma}} \leq C \left[ \frac{r}{r-q} \right]^{\gamma/r} \cdot |t_n|_r \leq \\
C \left[ \frac{r}{r-q} \right]^{\gamma/r} \cdot \sigma^{1/s-1/r} \cdot |t_n|_s \leq \\
C \left[ \frac{r}{r-q} \right]^{\gamma/r} \cdot \sigma^{1/s-1/r} \cdot ||t_n||_{L_p[\log L]^{-\beta}} \cdot \left[ \frac{s}{p-s} \right]^{-\beta/s} = \\
C \cdot ||t_n||_{L_p[\log L]^{-\beta}} \times Z,
\]

where

\[
Z = \sigma^{1/s-1/r} \left[ \frac{r}{r-q} \right]^{\gamma/r} \cdot \left[ \frac{s}{p-s} \right]^{-\beta/s}, \quad (17)
\]

\[
1 < s < p < q < r. \quad (18)
\]

Minimizing the variable \( Z \) over \((s, r)\) under the restrictions (18) for sufficiently greatest values \( \sigma; \sigma \geq 3 \), we prove the desired inequality of theorem 4.

More simple, we can choose in order to prove theorem 4 in the expression (17) for all sufficiently great values \( n \)

\[
r = r_0 = q + \frac{\gamma}{q \log \sigma}; \quad s = s_0 = p \frac{\beta}{p \log \sigma}.
\]

6. Lorentz spaces: inverse Nikolskii inequalities and regular r.i. spaces.
We know that the Lorentz spaces $\Lambda \left( \log^{-C_1}(C_2/s) \right)$ are m.r.i. spaces and, following, they satisfy the conclusion of theorem 1. We intend to construct in this section some examples of low bounds in the Nikol'skii inequalities for Lorentz’s spaces.

We consider here as before only the one-dimensional trigonometrical case $T = \mathbb{R}$ and consider some Lorentz’s spaces over $\mathbb{R}$, $\Lambda(\phi_i)$, $i = 1, 2$, $\delta \geq 0$, where $\phi$, $\phi_i = \phi_i(\delta)$ are continuous (quasi-)concave non-negative strictly increasing functions, $\phi_i(0+) = \phi_i(0) = 0$.

Denote by $G(\lambda)$ the following function of distribution:

$$G(\lambda) = m \left\{ y : \frac{\sin^2(y/2)}{y^2} > \lambda \right\}, \lambda \in (0, 1/4).$$

It is obvious that $G(\cdot)$ is continuous, strictly decreasing, $G(1/4 - 0) = 0$, $G(0+) = \infty$, $\delta \in (0, 1/8) \Rightarrow G(\delta) \asymp C\delta^{-1/2}$.

Let us introduce the following conditions on the functions $\phi$: $\phi(\cdot) \in Q$ iff

$$\forall \epsilon > 0 \Rightarrow \int_0^{1/4} \phi(\epsilon G(\lambda)) \, d\lambda \leq \phi(C \epsilon)).$$

(19)

Note that the converse inequality to the inequality (19):

$$\forall \epsilon > 0 \Rightarrow \int_0^{1/4} \phi(\epsilon G(\lambda)) \, d\lambda \geq \phi(C \epsilon))$$

is always true.

Theorem 5.

A. If $\phi_1 \in Q$, $\phi_2 \in Q$ then $\exists K_1, K_2, K_3 = \text{const} \in (0, \infty)$, such that

$$\lim_{n \to \infty} W_n(\Lambda(\phi_1), \Lambda(\phi_2), K_1, K_2) = K_3 > 0.$$ (20)

B. If $\phi \in Q$ and r.i. space $X$ over $\mathbb{R}$ is arbitrary $G(\psi)$ space or is the Orlicz’s space $L(\Phi)$, where the Orlicz’s function $\Phi$ satisfies the condition (11), then $\exists K_1, K_2 = \text{const} \in (0, \infty)$ $\Rightarrow$

$$\lim_{n \to \infty} W_n(\Lambda(\phi), X, K_1, K_2) = K_3 > 0.$$ (21)

Proof. 1. We consider as before the function $D_n(x)$. Let us estimate the distribution function for $D_n$. We get for the values $\lambda \in (0, 1/4)$:

$$m\{x : D_n(x) > \lambda\} = \int_R I(D_n(x) > \lambda) \, dx =$$

$$n^{-1} \int_R I \left( \frac{\sin^2(y/2)}{y^2} > \lambda \right) \, dy,$$

where $I$ denotes the indicator function.

It is easy to estimate
\[ \int_R I \left( \frac{\sin^2(y/2)}{y^2} > \lambda \right) \, dy = m \{ y : y^{-2} \sin^2(y/2) > \lambda \} \asymp G(\lambda), \ \lambda \in (0, 1/4). \] 
Hence

\[ m \{ x : D_n(x) > \lambda \} \asymp Cn^{-1} G(\lambda), \ \lambda \in (0, 1/4) \]

and

\[ m \{ x : D_n(x) > \lambda \} = 0 \]
in other case.

2. We estimate now the Lorentz norm of a function \( D_n \). We have based on the definition of the norm in the Lorentz space:

\[ ||D_n||_{A(\phi)} = \int_0^\infty \phi(m \{ x : D_n(x) > \lambda \}) \, d\lambda \asymp \int_0^\infty \phi(C\lambda/n) \, d\lambda. \]

The last integral is equivalent, by virtue of condition (19) to \( \phi(C/n) \).

3. Let now \( \phi_1 \in Q \) and \( \phi_2 \in Q \). We conclude repeating the consideration of section 4 for sufficiently great values \( n \) and taking into account that the function \( \phi(\cdot) \) is the fundamental function for \( A(\phi) \) space: \( W_n(A(\phi_1), A(\phi_2), K_1, K_2) \)

\[ \frac{||D_n||_{A(\phi_1)}}{\phi_1(K_1/n)} : \frac{||D_n||_{A(\phi_2)}}{\phi_2(K_2/n)} \geq \frac{\int_0^\infty \phi_1(G(\lambda)/n) \, d\lambda}{\phi_1(K_1/n)} : \frac{\int_0^\infty \phi_2(G(\lambda)/n) \, d\lambda}{\phi_2(K_2/n)} \geq C \]  \hspace{1cm} (22)

Therefore

\[ \lim_{n \to \infty} W_n(A(\phi), X, K_1, K_2) > 0. \]

4. The last assertion of theorem 5 provided analogously.

**Concluding remark.** The r.i. space \( (X, || \cdot ||_X) \) is said to be regular r.i. space, if

\[ \exists C \in (0, \infty) \Rightarrow ||D_n||_X \asymp \phi(X, C/n). \]  \hspace{1cm} (23)

For example, \( G(\psi) \) spaces, Zygmund, Lorentz spaces are regular r.i. spaces.

We can generalize for two regular r.i. spaces \( X, Y \) over \( R \) and \( A(n) = \text{collection of exponential type} \leq n \) there exist a pair of non-trivial constants \( K_1, K_2 \) such that

\[ \lim_{n \to \infty} W_n(X, Y, K_1, K_2) > 0. \]
REFERENCES

1. BENNET C., SHARPLEY R. Interpolation of operators. Orlando, Academic Press Inc., (1988).

2. DITZAN Z., TIKHONOV S. (2005). Ul’yanov and Nikolskii-type inequalities. Journal of Approximation Theory. 133(3) (2005), 100-133.

3. BELINSKY E., DAI F., DITZIAN Z. Multivariate approximating averages. Journal of Approximation Theory. 125(1) (2003), 85-1105.

4. NIKOL’SKII S.M. Inequalities for entire analytic functions of finite order and their applications to the theory of differentiable functions of several variables. Trudy Math. Inst. Steklov, 38 (1951), 244 - 278.

5. KOZACHENKO Yu. V., OSTROVSKY E.I. (1985). The Banach Spaces of random Variables of subgaussian type. Theory of Probab. and Math. Stat. (in Russian). Kiev, KSU, 32, 43 - 57.

6. LEDOUX M., TALAGRAND M. (1991) Probability in Banach Spaces. Springer, Berlin, MR 1102015.

7. OSTROVSKY E. Bide-side exponential and moment inequalities for tail of distribution of Polynomial Martingales. Electronic publication, arXiv: math.PR/0406532 v.1 Jun. 2004.

8. OSTROVSKY E.I. (1999). Exponential estimations for Random Fields and its applications (in Russian). Russia, OINPE.

9. OSTROVSKY E.I. (2002). Exact exponential estimations for random field maximum distribution. Theory Probab. Appl. 45 v.3, 281 - 286.

10. TALAGRAND M. (1996). Majorizing measure: The generic chaining. Ann. Probab. 24 1049 - 1103. MR1825156

11. TALAGRAND M. (2005). The Generic Chaining. Upper and Lower Bounds of Stochastic Processes. Springer, Berlin. MR2133757.

12. JAWERTH B., MILMAN M. Extrapolation Theory with Applications. Mem. Amer. Math. Soc., 440, (1991)

13. LUKOMSKY S.F. About convergence of Walsh series in the spaces nearest to $L_{\infty}$. Matem. Zametki, 2001, v.20 B.6,p. 882 - 889.(Russian).
14. Astashkin S.V. About interpolation spaces of sum spaces, generated by Rademacher system. RAEN, issue MIMI, 1997, v.1 N° 1, p. 8-35.

15. Nevaï, Totik V. Sharp Nikol’skii-type inequalities with exponential weight. Annal. Math., 13 (4),(1987), p. 261 - 267.

16. Grand R., Santicci P. Nikol’skii-type and maximal inequalities for generalized trigonometrical polynomials. Manuscripta Math., 99(4), 1999, p. 485 - 507.

17. Timan M.F. Orthonormal system satisfies an inequality of S.M.Nikol’skii. Annal. Math., 4(1), 1978, p. 75 - 82.

18. Capone C., Fiorenza A., Krbec M. On the Extrapolation Blowups in the $L_p$ Scale.

19. Krasnoeselsky M.A., Rutisky Ya.B. Convex functions and Orlicz’s Spaces. P. Noordhoff LTD, The Netherland, 1961, Groningen.

20. Rao M.M, Ren Z.D. Theory of Orlicz Spaces. - New York, Basel. Marcel Decker, 1991.

21. Davis H.W., Murray F.J., Weber J.K. Families of $L_p$ - spaces with inductive and projective topologies. Pacific J.Math. - 1970 - v. 34, p. 619 - 638.

22. Steigenwalt M.S. and While A.J. Some function spaces related to $L_p$. Proc. London Math. Soc. - 1971. - 22, p. 137 - 163.

23. Ostrovsky E., Sirota L. Fourier Transforms in Exponential Rearrangement Invariant Spaces. Electronic Publ., arXiv:Math., FA/040639, v.1, - 20.6.2004.

24. Rao M.M., Ren Z.D. Application of Orlicz Spaces. - New York, Basel. Marcel Decker, 2002.

25. Zygund A. Trigonometrical Series. Vol. 1. Cambridge University Press (1959)

26. Seneta E. Regularly Varying Functions. 1985, Moscow edition.

27. Bongioanni B., Forzani L., Harbourne E. Weak type and restricted weak type (p,p) operators in Orlicz spaces. (2002/2003). Real Analysis Exchange, Vol. 28(2), pp. 381 - 394.

28. Harbourne E., Salinas O. and Viviani B. Orlicz Roundedness for Certain Classical Operators. Colloquium Mathematicum 2002, 91 (No 2), pp. 263 - 282.
29. Cianchi A. Hardy inequalities in Orlicz Spaces. Trans. AMS, 351, No 6 (1999), 2456 - 2478.

30. Krein S.G., Petunin Yu., and Semenov E.M. Interpolation of linear operators. AMS, 1982.

31. A.Fiorenza. Duality and reflexivity in grand Lebesgue spaces. Collectanea Mathematica (electronic version), 51, 2, (2000), 131 - 148.

32. A. Fiorenza and G.E. Karadzhov. Grand and small Lebesgue spaces and their analogs. Consiglio Nazionale Delle Ricerche, Instituto per le Applicazioni del Calcoto Mauro Picine, Sezione di Napoli, Rapporto tecnico n. 272/03, (2005).

33. T.Iwaniec and C. Sbordone. On the integrability of the Jacobian under minimal hypotheses. Arch. Rat.Mech. Anal., 119, (1992), 129 - 143.

34. T.Iwaniec, P. Koskela and J. Onninen. Mapping of finite distortion: Monotonicity and Continuity. Invent. Math. 144 (2001), 507 - 531.

35. Dai F., Ditzian Z., Tikhonov S. Sharp Jackson inequalities. Journal of Appr. Theory, (2007), 04.015

36. Grand R., Santucci P. Nikol’skii-type and maximal inequality for generalized trigonometric polynomials. Manuscripta Math., 99(4), (1999), 485-507.

37. Levin E., Lubinsky D. Orthogonal polynomials for exponential weight. Canadian Math. Soc., v. 4 (2001) 44 - 51.

38. DeVore R.A., Lorentz G.G. Constructive Approximation. Springer, Berlin, (1993)

39. Mhaskar H.N. Introduction to the Theory of Weighted Polynomial Approximation. World Scientific, Singapore, (1996)

40. Nessel R., Totic V. Sharp Nikol’skii inequalities with exponential weight. Annal. Math., 13(4), (1987), 261 - 267.

41. Nessel R., Wilmes G. Nikol’skii inequalities for trigonometrical polynomials and entire functions of exponential type. J. Austr. Math. Soc. Ser. A, (1978), 7 - 18.

42. Astashkin S.V. Some new Extrapolation Estimates for the Scale of $L_p$ - Spaces. Funct. Anal. and Its Appl., v. 37 No 3 (2003), 73 - 77.

43. Ostrovsky E., Sirotk E. Moment Banach Spaces: Theory and Applica-
tions. HIAT Journal of Science and Engineering, Nolon, Israel, v. 4, Issue 1 - 2, (2007), 233 - 262.
Ostrovsky E.
Address: Ostrovsky E., ISRAEL, 76521, Rehovot, Shkolnik street. 5/8. Tel. (972)-8- 945-16-13.
e-mail: Galo@list.ru

Sirota L.
Address: Sirota L., ISRAEL, 84105, Ramat Gan.
e-mail: sirota@zahav@net.il