DP-Degree Colorable Hypergraphs

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Abstract
In order to solve a question on list coloring of planar graphs, Dvořák and Postle [10] introduced the concept of DP-coloring, which shifts the problem of finding a coloring of a graph $G$ from a given list $L$ to finding an independent transversal in an auxiliary cover-graph $H$ with vertex set $\{(v, c) \mid v \in V(G), c \in L(v)\}$. This leads to a new graph parameter, called the DP-chromatic number $\chi_{DP}(G)$ of $G$, which is an upper bound for the list-chromatic number $\chi_{\ell}(G)$ of $G$. The DP-coloring concept was analyzed in detail by Bernshteyn, Kostochka, and Pron [4] for graphs and multigraphs; they characterized DP-degree colorable multigraphs and deduced a Brooks’ type result from this. In this paper, the concept of DP-colorings is extended to hypergraphs having multiple (hyper-)edges. We characterize the DP-degree colorable hypergraphs and, furthermore, the corresponding ’bad’ covers. This gives a Brooks’ type result for the DP-chromatic number of a hypergraph. In the last part, we examine DP-critical graphs and establish some basic facts on their structure as well as a Gallai-type bound on the minimum number of edges.

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1 Introduction

1.1 Hypergraph Basics
A hypergraph is a triple $G = (V, E, i)$, whereas $V$ and $E$ are two finite sets and $i : E \to 2^V$ is a function with $|i(e)| \geq 2$ for $e \in E$ (i.e., no loops

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are allowed). Then, \( V(G) = V \) is the \textbf{vertex set} of \( G \); its elements are the \textbf{vertices} of \( G \). Furthermore, \( E(G) = E \) is the \textbf{edge set} of \( H \); its elements are the \textbf{edges} of \( H \). Lastly, the mapping \( i_G = i \) is the \textbf{incidence function} and \( i_G(e) \) is the set of vertices that are \textbf{incident} to the edge \( e \) in \( G \). Two vertices \( u \neq v \) from \( G \) are \textbf{adjacent} if there is an edge \( e \in E(H) \) with \( \{u, v\} \subseteq i_G(e) \). The \textbf{empty hypergraph} is the hypergraph \( G \) with \( V(G) = E(G) = \emptyset \); we denote it by \( G = \emptyset \).

Let \( G \) be an arbitrary hypergraph. Then, \( |V(G)| = |G| \) is called \textbf{order} of \( G \). An edge \( e \) of \( G \) is a \textbf{hyperedge} if \( |i_G(e)| \geq 3 \) and an \textbf{ordinary edge}, otherwise. Thus, a \textbf{graph} is a hypergraph that contains only ordinary edges. Two edges \( e \neq e' \) are \textbf{parallel}, if \( i_G(e) = i_G(e') \). A \textbf{simple} hypergraph is a hypergraph without parallel edges. If \( G \) is a hypergraph such that there exists an edge \( e \in E(G) \) with \( V(G) = i_H(e) \) and \( E(G) = \{e\} \), we will briefly write \( G = <e> \).

A hypergraph \( G \) is a \textbf{subhypergraph} of \( G' \), written \( G' \subseteq G \), if \( V(G') \subseteq V(G), E(G') \subseteq E(G) \), and \( i_G = i_{G'}|_{E(G')} \). Furthermore, \( G' \) is a proper subhypergraph of \( G \) if \( G' \subseteq G \) and \( G' \neq G \). For two subhypergraphs \( G_1 \) and \( G_2 \) of \( G_U \), we define the \textbf{union} \( G_1 \cup G_2 \) and the \textbf{intersection} \( G_1 \cap G_2 \) as usual. Another important operation for the class of hypergraphs that will be needed in this paper is the so called \textbf{merging} operation. Given two \textbf{disjoint} hypergraphs \( G_1 \) and \( G_2 \), that is \( V(G_1) \cap V(G_2) = E(G_1) \cap E(G_2) = \emptyset \), two vertices \( v^j \in V(G^j) (j \in \{1, 2\}) \), and a vertex \( v^* \) that is not contained in \( V(G_1) \cup V(G_2) \), we define a new hypergraph \( G \) as follows. Let \( V(G) = ((V(G^1) \cup V(G^2)) \setminus \{v^1, v^2\}) \cup \{v^*\} \), \( E(G) = E(G^1) \cup E(G^2) \), and

\[
i_G(e) = \begin{cases} 
i_G^j(e) & \text{if } e \in E(G^j), v^j \notin i_G^j(e) \ (j \in \{1, 2\}), \\
i_G(e) \setminus \{v^j\} \cup \{v^*\} & \text{if } e \in E(G^j), v^j \in i_G^j(e) \ (j \in \{1, 2\}). \end{cases}
\]

In this case, we say that \( G \) is obtained from \( G^1 \) and \( G^2 \) by merging \( v^1 \) and \( v^2 \) to \( v^* \).

Let \( G \) be a hypergraph and let \( X \subseteq V(G) \) be a vertex set. Then, \( G[X] \) denotes the subhypergraph of \( G \) \textbf{induced by} \( X \), that is, \( V(G[X]) = X, E(G[X]) = \{e \in E(G) \mid i_G(e) \subseteq X\} \), and \( i_G|_{E(G[X])} = i_G|_{E(G[X])} \). Moreover, let \( G - X = G[V(G) \setminus X] \). The set \( X \) is called \textbf{independent} in \( G \) if \( E(G[X]) = \emptyset \). Finally, given a vertex \( v \in V(G) \), let \( G \div v \) be the hypergraph with \( V(G \div v) = V(G) \setminus \{v\}, E(G \div v) = \{e \in E(G) \mid |i_G(e) \setminus \{v\}| \geq 2\} \) and \( i_G(e) \div v = i_G(e) \setminus \{v\} \) for all \( e \in E(G \div v) \). We say that \( G \div v \) results from \( G \) by \textbf{shrinking} \( G \) at \( v \). Note that if \( G \) is a graph, then \( G \div v = G - \{v\} \).
For a hypergraph $G$ and a vertex $v$ from $V(G)$, let

$$E_G(v) = \{ e \in E(H) \mid v \in i_G(e) \}.$$ 

Then, $d_G(v) = |E_G(v)|$ is the degree of $v$ in $H$. As usual, we call $\delta(G) = \min_{v \in V(G)} d_G(v)$ the minimum degree of $G$ and $\Delta(G) = \max_{v \in V(G)} d_G(v)$ the maximum degree of $G$. If $G$ is empty, we set $\delta(G) = \Delta(G) = 0$. A hypergraph $G$ is $r$-regular or, briefly, regular if each vertex in $G$ has degree $r$.

Let $u, v$ be two distinct vertices of a hypergraph $G$. Then, $E_G(u, v) = E(G[\{u, v\}])$ is the set of ordinary edges that are incident to $u$ as well as $v$, and $\mu_G(u, v) = |E_G(u, v)|$ is the multiplicity of $u$ and $v$. Note that if $u$ and $v$ are distinct vertices from $G$, then it clearly holds

$$d_{G^2}(u) = d_G(u) - \mu_G(u, v)$$ (1.1)

The ordinary neighborhood of a vertex $v$ in a hypergraph $G$ is the set of all vertices $u \in V(G)$ such that there is an edge $e$ with $i_G(e) = \{u, v\}$, we denote it by $N_G(v)$.

A hypermatching of $G$, or, briefly, matching of $G$ is an edge set $M \subseteq E(G)$ such that each vertex $v \in V(G)$ is contained in at most one edge from the set $M$. A perfect (hyper-)matching is a matching $M$ such that for each $v \in V(G)$ there is a vertex $e \in M$ with $v \in i_G(e)$. A matching $M$ is called empty if $M = \emptyset$.

Let $G$ be a non-empty hypergraph. A hyperpath of $G$ is a sequence $(v_1, e_1, v_2, e_2, \ldots, v_q, e_q, v_{q+1})$ of distinct vertices $v_1, v_2, \ldots, v_{q+1}$ of $G$ and distinct edges $e_1, e_2, \ldots, e_q$ of $G$ such that $\{v_i, v_{i+1}\} \subseteq i_G(e_i)$ for $1 \leq i \leq q$ and $q \geq 0$. The hypergraph $G$ is connected if there is a hyperpath in $G$ between any two of its vertices. A (connected) component of $G$ is a maximal connected subhypergraph of $G$. A separating vertex of $G$ is a vertex $v \in V(G)$ such that $G$ is the union of two induced subhypergraph $G_1$ and $G_2$ with $V(G_1) \cap V(G_2) = \{v\}$ and $|G_i| \geq 2$ for $i \in \{1, 2\}$. Note that $v$ is a separating vertex of $G$ if and only if $G \setminus v$ has more components than $G$. Regarding edges, an edge $e$ is a bridge of a hypergraph $G$, if $G - e$ has $|i_G(e)| - 1$ more components than $G$. Finally, a block of $G$ is a maximal connected subhypergraph of $G$ that has no separating vertex. Thus, every block of $G$ is a connected induced subhypergraph of $G$. It is easy to see that two blocks of $G$ have at most one vertex in common, and that a vertex $v$ is a separating vertex of $G$ if and only if it is contained in more than one block.
By $B(G)$ we denote the set of all blocks of $G$. If $G$ does not contain any separating vertex, that is, $B(G) = \{G\}$, we will also say that $G$ is a block. In this paper we need some definitions only for graphs. As usual, by $C_n$ we denote the cycle with $n$ vertices and by $K_n$ we denote the complete graph on $n$ vertices. Moreover, for $n, t \geq 1$, by $K_{n,t}$ we denote the complete $n$-partite graph all of whose partite sets have $t$ vertices. In particular, $K_{2,t}$ is the complete bipartite graph $K_{t,t}$. If $G$ is a simple graph, let $tG$ denote the graph that results from $G$ by replacing each edge $e$ with $t$ parallel edges. In particular, $1G = G$. By $A(G)$ we denote the set of all two-subsets $\{u, v\} \in V(G)$ such that there in $G$ there is an edge $e$ with $i_G(e) = \{u, v\}$.

1.2 Hypergraph Colorings

A coloring of a hypergraph $G$ with color set $C$ is a mapping $\varphi : V(G) \to C$ such that for each edge $e$ there are vertices $u, v \in i_G(e)$ with $\varphi(u) \neq \varphi(v)$. If $C = \{1, 2, \ldots, k\}$ and if $G$ admits a coloring with color set $C$, we say that $G$ is $k$-colorable. The smallest $k \geq 0$ such that $G$ is $k$-colorable is called the chromatic number $\chi(G)$ of $G$. This coloring concept was introduced by Erdős and Hajnai[11] in the 1960s. Note that if $G$ is a graph, this coloring concept coincides with the usual coloring concept for graphs. In particular, it is possible to extend various well known theorems in the topic of graph colorings to hypergraph colorings. For example, Brooks’ well known theorem (see [6]) that a connected graph $G$ satisfies $\chi(G) \leq \Delta(G) + 1$ and equality holds if and only if $G$ is a complete graph or an odd cycle, was extended to hypergraphs by Jones [16]. He showed that if $G$ is a connected hypergraph, then $\chi(G) \leq \Delta(G) + 1$ and equality holds if and only if $G$ is a complete graph, an odd cycle, or $G = < e >$ for some edge $e$.

A more generalized coloring concept is the so called list-coloring concept. Let again $C$ be a color set and let $G$ be a hypergraph. A mapping $L : V(G) \to 2^C$ is called list-assignment. An $L$-coloring of $G$ is a coloring $\varphi$ of $G$ with color set $C$ such that $\varphi(v) \in L(v)$ for all $v \in V(G)$. The hypergraph $G$ is said to be $k$-list-colorable if $G$ admits an $L$-coloring for any list-assignment $L$ satisfying $|L(v)| \geq k$ for all $v \in V(G)$. The list-chromatic number or choice number $\chi_\ell(G)$ is the least integer $k$ such that $G$ is $k$-list-colorable. For graphs, list-colorings were introduced by Erdős, Rubin, and Taylor [12] and, independently, by Vizing [24]. They proved a Brooks type theorem, which was later extended to hypergraphs by Kostochka, Stieb-
itz, and Wirth [19]. In this paper, we will generalize their theorem to DP-colorings of hypergraphs.

2 DP-Colorings of Hypergraphs

This paper deals with **DP-colorings** of hypergraphs. For graphs, this concept was introduced by Dvořák and Postle [10]; they called it **correspondence coloring**. The original idea was taken from Plesnević and Vizing [22] who showed how to transform the problem of finding a \( k \)-coloring of a graph to the problem of finding an independent vertex set of size \(|V(G)|\) in the Cartesian product \( G \square K_k \). Later, a lot of work on the topic of DP-colorings was done by Bernsteyn, Kostochka et al (see [2], [4], [5]) who were the first to use the term DP-colorings. We will use an equivalent but slightly modified definition.

2.1 The DP-Chromatic Number

Let \( G \) be a hypergraph. A **cover** of \( G \) is a pair \((X, H)\) consisting of a map \( X \) and a hypergraph \( H \) such that the following conditions are fulfilled:

**(C1)** \( X : V(G) \rightarrow 2^{V(H)} \) is a function that assigns each vertex \( v \in V(G) \) a vertex set \( X_v = X(v) \subseteq V(H) \) such that the sets \( X_v \) with \( v \in V(G) \) are pairwise disjoint.

**(C2)** \( H \) is a hypergraph with \( V(H) = \bigcup_{v \in V(G)} X_v \) such that \( X_v \) is an independent set of \( H \), for each edge \( e \in E(G) \) there is a possibly empty (hyper-)matching \( M_e \) in \( H[\bigcup_{v \in \bar{G}(e)} X_v] \) with \(|i_H(\tilde{e})| = |i_G(e)|\) for all \( \tilde{e} \in M_e \). Moreover, \( E(H) = \bigcup_{e \in E(G)} M_e \).

Now let \((X, H)\) be a cover of \( G \). A vertex set \( T \subseteq V(H) \) is a **transversal** of \((X, H)\) if \(|T \cap X_v| = 1\) for each vertex \( v \in V(G) \). An **independent transversal** of \((X, H)\) is a transversal of \((X, H)\), which is an independent set of \( H \). An independent transversal of \((X, H)\) is also called an **\((X, H)\)-coloring** of \( G \); the vertices of \( H \) are called **colors**. We say that \( G \) is \((X, H)\)-colorable if \( G \) admits an \((X, H)\)-coloring. Let \( f : V(G) \rightarrow \mathbb{N}_0 \) be a function. Then, \( G \) is said to be **DP-\( f \)-colorable** if \( G \) is \((X, H)\)-colorable for any cover \((X, H)\) of \( G \) satisfying \(|X_v| \geq f(v)\) for all \( v \in V(G) \). When \( f(v) = k \) for all \( v \in V(G) \), the term becomes **DP-\( k \)-colorable**. The **DP-chromatic number** \( \chi_{DP}(G) \)
is the least integers $k \geq 0$ such that $G$ is DP-$k$-colorable. Recently, Bernsteyn and Kostochka [3] also introduced the DP-chromatic number of a hypergraph in an equivalent but slightly different way.

An especially interesting fact about DP-colorings is that one can reduce the list-coloring problem to DP-colorings. To see this, let $G$ be a hypergraph and let $L$ be a list-assignment for $G$. Let $(X, H)$ be a cover of $G$ as follows:

- For $v \in V(G)$, let $X_v = \{(v, x) \mid x \in L(v)\}$ and let $V(H) = \bigcup_{v \in V(G)} X_v$.

- For any set $S = \{(v_1, x_1), (v_2, x_2), \ldots, (v_\ell, x_\ell)\}$ of vertices from $H$, there is an edge $e' \in E(H)$ with $i_H(e') = S$ if and only if in $G$ there is an edge $e$ with $i_G(e) = \{v_1, v_2, \ldots, v_\ell\}$ and if $x_1 = x_2 = \ldots = x_\ell$.

It is easy to check that $(X, H)$ is indeed a cover of $G$. Furthermore, if $\varphi$ is an $L$-coloring of $G$, then $T = \{(v, \varphi(v)) \mid v \in V(G)\}$ clearly is an independent transversal of $H$ and so $G$ is $(X, H)$-colorable. If conversely $T$ is an independent transversal of $H$, then there is a mapping $\varphi$ from $V(G)$ into a color set such that $\varphi(v) \in L(v)$ for all $v \in V(G)$ and that $T = \{(v, \varphi(v)) \mid v \in V(G)\}$. Then, it is easy to see that $\varphi$ is an $L$-coloring of $G$. Thus, $G$ is $L$-colorable if and only if $G$ is $(X, H)$-colorable. Furthermore, we clearly have $|X_v| = |L(v)|$ for all $v \in V(G)$. Hence, if $k \geq 0$ is an integer, then $G$ is $k$-list-colorable if and only if $G$ is DP-$k$-colorable and, in particular, $\chi_e(G) \leq \chi_{DP}(G)$.

In order to obtain an upper bound for the DP-chromatic number we use a sequential coloring algorithm.

\begin{algorithm}
1: Input: hypergraph $G$ and cover $(X, H)$ of $G$.
2: Choose an arbitrary vertex order $(v_1, v_2, \ldots, v_n)$ of $G$.
3: Let $T = \emptyset$.
4: for all $i = 1, 2, \ldots, n$ do
5: \hspace{1em} Choose a vertex (color) $x_i$ from $X_{v_i}$ such that $E(H[S \cup \{x_i\}]) = \emptyset$.
6: \hspace{1em} Let $T = T \cup \{x_i\}$.
7: end for
8: Return: Independent transversal $T$.
\end{algorithm}

Clearly, if $|X_{v_i}| \geq d_{G[v_1, v_2, \ldots, v_i]}(v_i) + 1$ for all $i \in \{1, 2, \ldots, n\}$, in step 5 there is always a possible choice for $x_i$ and, thus, the algorithm terminates.
with an \((X, H)\)-coloring of \(G\). This is due to the fact that for each edge \(e \in E(G)\) with \(v_i \in i_H(e)\) and for any set of fixed colors
\[
\{x_k \mid v_k \in i_H(e), k = 1, 2, \ldots, i - 1\},
\]
at most one color from \(X_v\) is forbidden. Hence, in \(X_v\), at most \(d_G[v_1, v_2, \ldots, v_i](v_i)\) vertices are forbidden. As a consequence, if \(f(v) \geq d_G(v) + 1\) for all \(v \in V(G)\), then \(G\) is DP-\(f\)-colorable. As usual, the coloring number \(\text{col}(G)\) of a hypergraph \(G\) is the least integer \(k\) such that each non-empty subhypergraph contains a vertex of degree at most \(k\). Therefore, as a consequence of the above sequential coloring algorithm, we have \(\chi_{\text{DP}}(G) \leq \text{col}(G)\). Summarizing, we obtain
\[
\chi(G) \leq \chi_{\ell}(G) \leq \chi_{\text{DP}}(G) \leq \text{col}(G) \leq \Delta(G) + 1. \tag{2.1}
\]

Our aim is to characterize the hypergraphs \(G\) for which \(\chi_{\text{DP}}(G) = \Delta(G) + 1\) holds. Clearly, if \(G\) is an odd cycle, we have \(\chi(G) = \Delta(G) + 1 = 3\) and, thus, equality holds. To see that \(\chi_{\text{DP}}(G) = 3\) holds for even cycles, as well, we construct an appropriate cover of \(G\). Assume that \(V(G) = \{1, 2, \ldots, n\}\) with \(n \geq 2\) even and \(E(G) = \{uv \mid u, v \in V(G)\text{ and } u - v \equiv 1(\text{mod } n)\}\). Let \((X, H)\) be the cover of \(G\) with \(X_v = \{v\} \times \{1, 2\}\) for all \(v \in V(G)\) and \(E(H) = \{(u, i)(v, j) \mid |u - v| = 1\text{ and } i = j; \text{ or } \{u, v\} = \{1, n\}\text{ and } i - j \equiv 1\text{ (mod 1)}\}\). Then, \((X, H)\) is a cover of \(G\) with \(|X_v| = 2\) for all \(v \in V(G)\). Moreover, \(H = C_{2n}\) and \((X, H)\) has no independent transversal. As emphasized in [4], the fact that \(\chi_{\text{DP}}(C_n) = 3\) for all \(n \geq 2\) and not only for odd \(n \geq 3\) marks an important difference between the DP-chromatic number and the list-chromatic number.

### 2.2 DP-Degree Colorable Hypergraphs

We say that a hypergraph \(G\) is DP-degree colorable if \(G\) is \((X, H)\)-colorable whenever \((X, H)\) is a cover of \(G\) such that \(|X_v| \geq d_G(v)\) for all \(v \in V(G)\). In the following, we will give a characterization of DP-degree-colorable hypergraphs as well as a characterization of the corresponding 'bad' covers. Clearly, it suffices to do this only for connected hypergraphs. For graphs, a characterization was given by Kim and Ozeki [18] using an approach different from ours.

A feasible configuration is a triple \((G, X, H)\) consisting of a connected hypergraph and a cover \((X, H)\) of \(G\). A feasible configuration is said to
be degree-feasible if \(|X_v| \geq d_G(v)\) for each vertex \(v \in V(G)\). Furthermore, \((G, X, H)\) is colorable if \(G\) is \((X, H)\)-colorable, otherwise it is called uncolorable. The next proposition lists some basic properties of feasible configurations; the proofs are straightforward and left to the reader.

**Proposition 1** Let \((G, X, H)\) be a feasible configuration. Then, the following statements hold.

(a) For distinct vertices \(u, v\) of \(G\), the hypergraph \(H[X_v \cup X_w]\) is a bipartite graph with parts \(X_v\) and \(X_w\) whose maximum degree is at most \(\mu_G(v, w)\). Furthermore, for every vertex \(v \in V(G)\) and every vertex \(x \in X_v\), we have \(d_H(x) \leq d_G(v)\).

(b) Let \(H'\) be a spanning subhypergraph of \(H\). Then, \((G, X, H')\) is a feasible configuration. If \((G, X, H)\) is colorable, then \((G, X, H')\) is colorable, too. Furthermore, \((G, X, H)\) is degree-feasible if and only if \((G, X, H')\) is degree feasible.

The above proposition leads to the following concept. We say that a feasible configuration \((G, X, H)\) is minimal uncolorable if \((G, X, H)\) is uncolorable, but \((G, X, H - e)\) is colorable for each \(e \in E(H)\). As usual, \(H - e\) denotes the hypergraph obtained from \(H\) by deleting the edge \(e\). Clearly, if \(|G| \geq 2\) and if \(\bar{H}\) is the edgeless spanning hypergraph of \(H\), then \((G, X, \bar{H})\) is colorable. Thus, it follows from the above Proposition that if \((G, X, H)\) is an uncolorable feasible configuration, then there is a spanning subhypergraph \(H'\) of \(H\) such that \((G, X, H')\) is a minimal uncolorable feasible configuration. Furthermore, if \((G, X, H)\) is a minimal uncolorable feasible configuration, then \(H\) clearly is a simple hypergraph.

In order to characterize the class of minimal uncolorable degree-feasible configurations, we firstly need to introduce three basic types of degree-feasible configurations.

We say that \((G, X, H)\) is a **K-configuration** if \(G = tK_n\) for some integers \(t, n \geq 1\) and if \((X, H)\) is a cover of \(G\) such that for every vertex \(v \in V(G)\), there is a partition \((X^1_v, X^2_v, \ldots, X^{n-1}_v)\) of \(X_v\) satisfying the following conditions:

- For every \(i \in \{1, 2, \ldots, n-1\}\), the graph \(H^i = H[\bigcup_{v \in V(G)} X^i_v]\) is a \(K_{(m,v)}\) whose partite sets are the sets \(X^i_v\) with \(v \in V(G)\), and \(H = H^1 \cup H^2 \cup \ldots \cup H^{n-1}\).
It is an easy exercise to check that each $K$-configuration is a minimal uncolorable degree-feasible configuration. Note that for $n = 1$, we have $G = K_1$, $X = \emptyset$, and $H = \emptyset$.

Next we define the so called C-configurations. We say that $(G, X, H)$ is an odd C-configuration if $G = tC_n$ for some integers $t \geq 1$ and $n \geq 5$ odd and if $(X, H)$ is a cover of $G$ such that for every vertex $v \in V(G)$, there is a partition $(X^1_v, X^2_v)$ of $X_v$ satisfying the following conditions:

- For every $i \in \{1, 2\}$ and for every set $\{u, v\} \in A(G)$, the graph $H^i_{\{u,v\}} = H[X^i_u \cup X^i_v]$ is a $K_{t,t}$ whose partite sets are $X^i_u$ and $X^i_v$, and
- $H$ is the union of all graphs $H^i_{\{u,v\}}$ with $i \in \{1, 2\}$ and $\{u, v\} \in A(G)$.

It is easy to verify that any odd C-configuration is a minimal uncolorable degree-feasible configuration.

We call $(G, X, H)$ an even C-configuration if $G = tC_n$ for some integers $t \geq 1, n \geq 4$ even and if $(X, H)$ is a cover of $G$ such that for every vertex $v \in V(G)$, there is a partition $(X^1_v, X^2_v)$ of $X_v$ and a set $\{w, w'\} \in A(G)$ satisfying the following conditions:

- For every $i \in \{1, 2\}$ and for every set $\{v, w\} \in A(G)$ different from $\{w, w'\}$, the graph $H^i_{\{v,w\}} = H[X^i_v \cup X^i_w]$ is a $K_{t,t}$ whose partite sets are $X^i_v$ and $X^i_w$,
- $H^1_{\{w,w'\}} = H[X^1_w \cup X^2_{w'}]$ is a $K_{t,t}$ whose partite sets are $X^1_w$ and $X^2_{w'}$,
- $H^2_{\{w,w'\}} = H[X^2_w \cup X^1_{w'}]$ is a $K_{t,t}$ whose partite sets are $X^2_w$ and $X^1_{w'}$, and
- $H$ is the union of all graphs $H^i_{\{u,v\}}$ with $i \in \{1, 2\}$ and $\{u, v\} \in A(G)$.

Again, it is easy to check that any even C-configuration is a minimal uncolorable degree-feasible configuration. By a C-configuration we either mean an even or an odd C-configuration.

Finally, we say that $(G, X, H)$ is an E-configuration if $G = e \in E(H)$, if $|X_v| = 1$ for each $v \in V(G)$ and if $H \cong G$. Clearly, each E-configuration is a minimal uncolorable degree-feasible configuration.

We will show that we can construct any minimal uncolorable degree-feasible configuration from these three basic configurations using the following operation. Let $(G^1, X^1, H^1)$ and $(G^2, X^2, H^2)$ be two feasible configurations, which are disjoint, that is, $V(G^1) \cap V(G^2) = \emptyset$ and $V(H^1) \cap V(H^2) =$
Furthermore, let $G$ be the hypergraph obtained from $G_1$ and $G_2$ by merging two vertices $v^1 \in V(G_1)$ and $v^2 \in V(G_2)$ to a new vertex $v^\ast$. Finally, let $H = H_1 \cup H_2$ and let $X: V(G) \rightarrow 2^{V(H)}$ be the mapping such that

$$X_v = \begin{cases} X^1_v \cup X^2_v & \text{if } v = v^\ast, \\ X^i_v & \text{if } v \in V(G^i) \setminus \{v^i\} \text{ and } i \in \{1, 2\} \end{cases}$$

for $v \in V(H)$. Then, $(G, X, H)$ is a feasible configuration and we say that $(G, X, H)$ is obtained from $(G^1, X^1, H^1)$ and $(G^2, X^2, H^2)$ by merging $v^1$ and $v^2$ to $v^\ast$. Since $d_G(v^\ast) = d_{G^1}(v^1) + d_{G^2}(v^2)$, it follows that $(G, X, H)$ is degree-feasible if both $(G^1, X^1, H^1)$ and $(G^2, X^2, H^2)$ are degree-feasible.

Now we define the class of constructible configurations as the smallest class of feasible configurations that contains each K-configuration, each C-configuration and each M-configuration and that is closed under the merging operation. Thus, if $(G, X, H)$ is a constructible configuration, then each block of $G$ is a $tK_n$ for $t \geq 1, n \geq 1$, a $tC_n$ for $t \geq 1, n \geq 3$, or of the form $<e>$ for some edge $e$. We call a block $B \in \mathcal{B}(G)$ a DP-brick if $B = tK_n$ for some $t \geq 1, n \geq 1$ or if $B = tC_n$ for some $t \geq 1, n \geq 3$. Moreover, we say that $B \in \mathcal{B}(G)$ is a DP-hyperbrick, if $B$ is either a DP-brick or of the form $<e>$ for some edge $e$. The next proposition describes the block-configurations of constructible configurations, the proof can be done by induction on the number of blocks and is left to the reader.

**Proposition 2** Let $(G, X, H)$ be a constructible configuration. Then, for each block $B \in \mathcal{B}(G)$ there is a uniquely determined cover $(X^B, H^B)$ of $B$ such that the following statements hold:

(a) For each block $B \in \mathcal{B}(G)$, the triple $(B, X^B, H^B)$ is a K-configuration, a C-configuration, or an M-configuration.

(b) The hypergraphs $H^B$ with $B \in \mathcal{B}(G)$ are pairwise disjoint and $H = \cup_{B \in \mathcal{B}(G)} H^B$.

(c) For every vertex $v \in V(G)$ it holds $X_v = \bigcup_{B \in \mathcal{B}(G), v \in V(B)} X^B_v$.

As mentioned already, our aim is to show that the class of minimal uncolorable degree-feasible configurations is exactly the class of constructible configurations. To this end, we will frequently use the following reduction method. Similar propositions to the next two propositions were proven by Bernsteyn, Kostochka, and Pron for graphs in [4].
Proposition 3 Let \((G, X, H)\) be a feasible configuration with \(|G| \geq 2\), let \(v\) be a non-separating vertex of \(G\), and let \(x \in X_v\) be a color. We define a cover of the hypergraph \(G' = G \div v\) as follows. For \(u \in V(G')\) let

\[ X'_u = X_u \setminus N_H(x) \]

and let \(H'\) be the hypergraph with \(V(H') = \bigcup_{u \in V(G')} X'_u\),

\[ E(H') = \{e \mid e \in E(H), |i_H(e) \setminus \{x\}| \geq 2, \text{ and } (i_H(e) \setminus \{x\}) \subseteq V(H')\}, \]

and

\[ i_{H'}(e) = i_H(e) \setminus \{x\} \]

for all \(e \in E(H')\).

Then, \((G', X', H')\) is a feasible configuration, and in what follows we briefly write \((G', X', H') = (G, H, X)/(v, x)\). Moreover, the following statements hold:

(a) If \((G, X, H)\) is degree-feasible, then \((G', X', H')\) is degree-feasible, too.

(b) If \((G, X, H)\) is uncolorable, then \((G', X', H')\) is uncolorable, too.

Proof: Clearly, \((X', H')\) is a cover of \(G'\) and, hence, \((G', X', H')\) is a feasible configuration. Moreover, for \(u \in V(G')\) it holds \(d_{G'}(u) = d_G(u) - \mu_G(u, v)\) and \(|N_H(x) \cap X_u| \leq \mu_G(u, v)\) (see (1.1) and Proposition 1). Thus, we obtain

\[ |X'_u| = |X_u| - |N_H(x) \cap X_u| \geq |X_u| - \mu_G(u, v). \]

As \(|X_u| \geq d_G(u)|\), this leads to \(|X'_u| \geq d_{G'}(u)|\) and \((G', H', X')\) is degree-feasible. Furthermore, if \(T'\) is an independent transversal of \((X', H')\), then \(T = T' \cup \{x\}\) is an independent transversal of \((X, H)\). This proves (b).

The next proposition is the key proposition for the proof of our main result.

Proposition 4 Let \((G, X, H)\) be an uncolorable degree-feasible configuration. Then, the following statements hold:

(a) \(|X_v| = d_G(v)|\) for all \(v \in V(G)\).

(b) For each non-separating vertex \(z\) of \(G\) and each vertex \(v \neq z\) of \(G\), it holds \(|N_H(x) \cap X_v| = \mu_G(v, z)|\) for all \(x \in X_v\).
(c) Every hyperedge $e$ of $G$ is a bridge of $G$ and, therefore, $<e>$ is a block of $G$. As a consequence, there are no parallel hyperedges in $G$.

(d) If $G$ is a block, then $G$ is regular, and for distinct vertices $u, v$ of $G$, the hypergraph $H[X_u \cup X_v]$ is a $\mu_G(u,v)$-regular bipartite graph whose partite sets are $X_u$ and $X_v$.

(e) For each vertex $v \in V(G)$ there is an independent set $T$ in $H$ satisfying $|T \cap X_u| = 1$ for all $u \in V(G) \setminus \{v\}$.

**Proof:** We prove (a) by induction on the order of $G$. If $G$ consists of only one vertex $v$, then $X_v = \emptyset$ and $H = \emptyset$. Thus, (a) is fulfilled. Now assume $|G| \geq 2$ and choose an arbitrary vertex $v$ of $G$. As $G$ is connected, there is a non-separating vertex $z \neq v$ in $G$ and $X_u \neq \emptyset$. Let $x \in X_u$. Then, $(G', X', H') = (G, X, H)/(z, x)$ is an uncolorable degree-feasible configuration (by Proposition 3). Applying the induction hypothesis then leads to $|X_v'| = d_{G'}(v)$ and we conclude

$$d_{G'}(v) = |X_v'| = |X_v| - |N_H(x) \cap X_v| \geq |X_v| - \mu_G(v, z) \geq d_G(v) - \mu_G(v, z) = d_{G'}(v).$$

This implies $|X_v| = d_G(v)$ and $|N_H(x) \cap X_v| = \mu_G(v, z)$; thus, (a) is proven.

The same argument can be applied in order to prove (b).

For the proof of (c) assume that some hyperedge $e \in E(G)$ is not a bridge of $G$. Then, for some vertex $v \in i_H(e)$, the hypergraph $G' = (V(G), E(G) \setminus \{e\})$ is connected. Let $X' = X$ and let $H'$ be the hypergraph with vertex set $V(H)$ and edge set $(E(H) \setminus M_e) \cup M_{e-v}$, whereas $M_{e-v}$ denotes the restriction of $M_e$ to the vertices of $i_H(e) \setminus \{v\}$. Clearly, $(G', X', H')$ is a degree-feasible configuration. However, (a) implies that $|X'_v| = |X_v| = d_G(v) > d_{G'}(v)$ and so, again by (a), $(G', X', H')$ is colorable. Hence, there is an independent transversal $T'$ of $(G', X', H')$. We claim that $T'$ is also an independent transversal of $(G, X, H)$. Otherwise, by construction of $H'$ there would be an edge $\tilde{e} \in E(H)$ with $v \in i_H(\tilde{e}) \subseteq T'$. But then, $i_H(\tilde{e} - v) \subseteq T'$ and so $T'$ is not an independent transversal of $H'$, a contradiction. Hence, $T'$ is an independent transversal of $(G, X, H)$ and so $(G, X, H)$ is colorable, which is impossible. This settles the case (c).

In order to prove (d), assume that $G$ is a block. If $G = <e>$ for some hyperedge $e$, then $G$ is regular and the statement clearly holds. Thus, by (c),
we may assume that $G$ does not contain any hyperedge. Let $u, v$ be distinct vertices of $G$. Then, $H[X_u \cup X_v]$ is a $\mu_G(u, v)$-regular bipartite graph with parts $X_u$ and $X_v$ (by (b)). This is only possible if $|X_u| = |X_v|$. By (a), this leads to $d_G(u) = d_G(v)$ and (d) is proven.

Finally, for the proof of (e), let $v$ be an arbitrary vertex of $G$. Let $U$ be the vertex set of a component of $G - v$, let $X'$ be the restriction of $X$ to $U$, and let $H' = H[\bigcup_{u \in U} X_u]$. Then, $(G[U], X', H')$ is a degree-feasible configuration and, as $G$ is connected, it holds $|X_u| = d_G(u) > d_G[u](u)$ for at least one vertex $u \in U$. Hence, there is an independent transversal $T_u$ of $(X', H')$ (by (a)). Let $T$ be the union of the transversals $T_u$ over all components $G[U]$ of $G - v$. Clearly, $T$ is an independent set of $H$ such that $|T \cap X_w| = 1$ for all $w \in V(G) \setminus \{v\}$. This proves (e).

Before stating our main result, we connect the concept of being minimal uncolorable with the merging operation.

**Proposition 5** Let $(G, X, H)$ be obtained from two disjoint degree-feasible configurations $(G_1, X_1, H_1)$ and $(G_2, X_2, H_2)$ by merging $v^1 \in V(G_1)$ and $v^2 \in V(G_2)$ to a new vertex $v^*$. Then, $(G, X, H)$ is a degree-feasible configuration and the following conditions are equivalent:

(a) Both $(G_1, X_1, H_1)$ and $(G_2, X_2, H_2)$ are minimal uncolorable.

(b) $(G, X, H)$ is minimal uncolorable.

**Proof:** First we show that (a) implies (b). Assume that $(G, X, H)$ is colorable. Then, there is an independent transversal $T$ of $(X, H)$, that is, an independent set of $H$ such that $|T \cap X_v| = 1$ for all $v \in V(H)$. As $X_{v^*} = X_{v^1} \cup X_{v^2}$, this implies (by symmetry) that $|T \cap X_{v^1}| = 1$. As a consequence, $T_1 = T \cap V(H_1)$ is an independent transversal of $(X_1, H_1)$ and so $(G_1, X_1, H_1)$ is colorable, a contradiction to (a). Thus, $(G, X, H)$ is uncolorable. Let $e \in E(H)$ be an arbitrary (hyper)-edge. By the structure of $H = H_1 \cup H_2$, we may assume that $e \in E(H_1)$. Due to the fact that $(G_1, X_1, H_1)$ is minimal uncolorable, there is an independent transversal $T_1$ of the cover $(X_1, H_1)$. Since $(G_2, X_2, H_2)$ is also minimal uncolorable and as $G^2$ is connected, it follows from Proposition 4(e) that there is an independent set $T_2$ in $H_2$ satisfying $|T_2 \cap X^2_v| = 1$ for all $v \in V(H_2) \setminus \{v^2\}$. However, as $H = H_1 \cup H_2$ and $H_1 \cap H_2 = \emptyset$, the set $T = T_1 \cup T_2$ is an independent transversal of $(X, H - e)$ and so $(G, X, H - e)$ is colorable. Thus, (b) holds.
In order to prove that (a) can be deduced from (b), we only need to show that \((G^1, X^1, H^1)\) is minimal uncolorable (by symmetry). First assume that \((G^1, X^1, H^1)\) is colorable, that is, \((X^1, H^1)\) has an independent transversal \(T^1\). Since \((G, X, H)\) is minimal uncolorable and connected and as \(H^2 - v^2\) is a subhypergraph of \(H\), Proposition 4(d) implies that there is an independent set \(T^2\) in \(H^2 - v^2\) such that \(|T^2 \cap X^2_v| = 1\) for all \(v \in V(G^2) \setminus \{v^2\}\). Then again, \(T = T^1 \cup T^2\) is an independent transversal of \((X, H)\), contradicting (b). Thus, \((G^1, X^1, H^1)\) is uncolorable. Now let \(e \in E(H^1)\) be an arbitrary edge. Then, as \((G, X, H)\) is minimal uncolorable, there is an independent transversal \(T\) of \((X, H - e)\) and \(T^1 = T \cap V(H^1)\) clearly is an independent transversal of \(H^1\). Consequently, \((G^1, X^1, H^1 - e)\) is colorable and the proof is complete.

2.3 Main Result

**Theorem 6** Let \((G, X, H)\) be a degree-feasible configuration. Then, \((G, X, H)\) is minimal uncolorable if and only if \((G, X, H)\) is constructable.

**Proof:** If \((G, X, H)\) is constructable, then \((G, X, H)\) is minimal uncolorable (by Proposition 5 and as each K,C and M-configuration is a minimal uncolorable degree-feasible configuration). Let \((G, X, H)\) be a minimal uncolorable degree-feasible configuration. We prove that \((G, X, H)\) is constructable by induction on the order of \(G\). Clearly, if \(|G| = 1\), then \(X = \emptyset\), \(H = \emptyset\) and \((G, X, H)\) is a K-configuration. Assume that \(|G| \geq 2\). By Proposition 1(a), it holds

\[
|X_v| = d_G(v) \tag{2.2}
\]

for each vertex \(v \in V(G)\). We distinguish between two cases.

**Case 1:** \(G\) contains a separating vertex \(v^*\). Then, \(G\) is the union of two connected induced subhypergraphs \(G^1\) and \(G^2\) with \(V(G^1) \cap V(G^2) = \{v^*\}\) and \(|G^j| < |G|\) for \(j \in \{1, 2\}\). For \(j \in \{1, 2\}\), by \(T^j\) we denote the set of all independent sets \(T\) of \(H^j\) such that \(|T \cap X_{v^*}| = 1\) for all \(v \in V(G^j)\). By Proposition 4(e), both \(T^1\) and \(T^2\) are non-empty. For \(j \in \{1, 2\}\), let \(X_j\) be the set of all vertices of \(X_{v^*}\) that do not occur in any independent set from \(T^j\). Then, \(X_{v^*} = X_1 \cup X_2\). Suppose, to the contrary, that there is a vertex \(u \in X_{v^*} \setminus (X_1 \cup X_2)\). Then, \(u\) is contained in two independent sets \(T^j \in T^j(j = 1, 2)\) and \(T = T^1 \cup T^2\) would be an independent transversal of \((X, H)\). This is due to the fact that each hyperedge of \(G\) is contained in \(G^j\)
for some \( j \in \{1, 2 \} \) and that for \( u \in V(G^1) \setminus \{v^*\} \) and \( v \in V(G^2) \setminus \{v^*\} \) we have \( \mu_G(u, v) = 0 \) and so \( E_H(X_u, X_v) = \emptyset \) (by Proposition 1(a)). Thus, \((G, X, H)\) is colorable, a contradiction. Consequently, \( X_{v^*} = X_1 \cup X_2 \). For \( j \in \{1, 2\} \), let \((X^j, H^j)\) be a cover of \( G^j \) as follows. For \( v \in V(G^j) \), let

\[
X_v^j = \begin{cases} 
X_v & \text{if } v \neq v^* \\
X_j & \text{if } v = v^*,
\end{cases}
\]

and let \( H^j = H[\bigcup_{v \in V(G^j)} X_v^j] \). Then, \((G^j, X^j, H^j)\) is an uncolorable feasible configuration. Moreover, for each vertex \( v \in V(G^j) \setminus \{v^*\} \) it holds \( |X_v| = d_G(v) = d_{G^j}(v) \) (by (2.2)). As \((G^j, X^j, H^j)\) is uncolorable, it follows from Proposition 4(a) that \( |X_{v^j}| \leq d_{G^j}(v^j) \) for \( j \in \{1, 2\} \). Since \( X_{v^j} = X_1 \cup X_2 = X_{v^1} \cup X_{v^2} \), we conclude from (2.2) that

\[
|X_{v^1}| + |X_{v^2}| \geq |X_{v^1} \cup X_{v^2}| = |X_{v^*}| = d_G(v^*) = d_{G^1}(v^*) + d_{G^2}(v^*),
\]

and, thus, \( |X_{v^j}| = d_{G^j}(v^j) \) and \( X_{v^*} \cap X_{v^*} = \emptyset \). Hence, \((G^j, X^j, H^j)\) is a degree-feasible configuration. Moreover, \( H' = H^1 \cup H^2 \) is a spanning subhypergraph of \( H \) and \( V(H^1) \cap V(H^2) = \emptyset \). So, \((G, X, H')\) is a degree-feasible configuration (by Proposition 1(b)) that is obtained from two isomorphic copies of \((G^1, X^1, H^1)\) and \((G^2, X^2, H^2)\) by the merging operation. Clearly, \((G, X, H')\) is uncolorable. Otherwise, there would exist an independent transversal \( T \) of \((X, H')\) and, by symmetry, \( T \) would contain a vertex of \( X_{v^*} \). But then, \( T^1 = T \cap V(H^1) \) would be an independent transversal of \((X^1, H^1)\), which is impossible. As \((G, X, H)\) is minimal uncolorable and as \( H' \) is a spanning subhypergraph of \( H \), this implies that \( H = H' \) and \((G, X, H)\) is obtained from two isomorphic copies of \((G^1, X^1, H^1)\) and \((G^2, X^2, H^2)\) by the merging operation. By Proposition 5, both \((G^1, X^1, H^1)\) and \((G^2, X^2, H^2)\) are minimal uncolorable (and also degree-feasible). Applying the induction hypotheses leads to \((G^j, X^j, H^j)\) being constructible for \( j \in \{1, 2\} \), and so \((G, X, H)\) is constructible. Thus, the first case is complete.

**Case 2:** \( G \) is a block. If \( G \) contains any hyperedge \( e \), then it follows from Proposition 4(c) that \( G = < e > \) and \((G, X, H)\) is not colorable if and only if \((G, X, H)\) is an M-configuration. Thus, in the following we may assume that \( G \) does not contain any hyperedges. We prove that \((G, X, H)\) is either a K-configuration or a C-configuration. This is done via a sequence of claims.

**Claim 1** Let \( v \) be an arbitrary vertex of \( G \), let \( x \in X_v \) be an arbitrary color, and let \((G', X', H') = (G, X, H)/(v, x)\). Then, there is a spanning subhy-
pergraph \( \tilde{H} \) of \( H' \) such that \((G, X, \tilde{H})\) is minimal uncolorable. Moreover, \((G', X', \tilde{H})\) is constructible and so each block of \( G' = G - v \) is a DP-brick.

**Proof:** Since \(|G| \geq 2\) and \( G \) is connected, \( X_v \neq \emptyset \) (by (2.2)). Thus, \((G', X', H') = (G, X, H)/(v, x)\) is an uncolorable degree-feasible configuration (by Proposition 3) and, therefore, there is a spanning subhypergraph \( \tilde{H} \) of \( H' \) such that \((G', X', \tilde{H})\) is minimal uncolorable. Then, the induction hypothesis implies that \((G', X', \tilde{H})\) is constructible, and, as \( G' = G - v \), this particularly implies that each block of \( G' \) is a DP-brick (since \( G \) does not contain any hyperedge). \(\square\)

By a **multicycle** or **multipath** we mean a multigraph that can be obtained from a cycle, respectively a path, by replacing each edge \( e \) of the cycle or path by a set of \( t_e \) parallel edges, where \( t_e \geq 1 \). Given integers \( s, t \geq 1 \), we say that a graph \( H \) is an \((s, t)\)-**multicycle** if \( H \) can be obtained from an even cycle \( C \) by replacing each edge of \( C \) by a set of \( s \) parallel edges and each other edge of \( C \) by a set of \( t \) parallel edges. Clearly, each \((s, t)\)-multicycle is \( r \)-regular for \( r = s + t \). Moreover, if \( H \) is a regular multicycle, then either \( H = tC_n \) for some integers \( t \geq 1 \) and \( n \geq 3 \), or \( H \) is an \((s, t)\)-multicycle for some integers \( s, t \geq 1 \).

**Claim 2** The graph \( G \) is a DP-brick.

**Proof:** Since \( G \) is a block, Proposition 4(d) implies that \( G \) is \( r \)-regular for some integer \( r \geq 1 \). For any vertex \( v \) of \( G \), each block of \( G - v \) is a DP-brick (by Claim 1). Let \( S \) denote the set of all vertices \( v \) of \( G \) such that \( G - v \) is a block. If \( S \neq \emptyset \), then for every vertex \( v \in S \), \( G - v \) is a DP-brick and, therefore, regular. As \( G \) is regular, too, for \( v \in S \) there must be an integer \( t_v \geq 1 \) such that \( \mu_G(u, v) = t_v \) for all \( u \in V(G) \setminus \{v\} \). As a consequence, \( S = V(G) \) and it clearly holds \( t_v = t \) for all \( v \in V(G) \). Thus, \( G = tK_n \) with \( n = |G| \).

It remains to consider the case that \( S = \emptyset \). Let \( v \) be an arbitrary vertex of \( G \). Then, \( G - v \) has at least two end-blocks and each block of \( G - v \) is a DP-brick and therefore regular. Let \( B \) be an arbitrary end-block of \( G - v \). Then, \( B \) is \( t_B \)-regular for some \( t_B \geq 1 \) and \( B \) contains exactly one separating vertex \( v_B \) of \( G - v \). As \( G \) is \( r \)-regular, there is an integer \( s_B \) such that \( \mu_G(u, v) = s_B \) for all vertices \( u \in V(B) \setminus \{v_B\} \). As a consequence, \( |B| = 2 \), since otherwise every vertex of \( B - v_B \) belongs to \( S \) and so \( S \neq \emptyset \), which is impossible. Hence, \( B = t_BK_2 \), \( r = t_B + s_B \), \( V(B) = \{v', v_B\} \), and \( N_G(v') = \{v, v_B\} \).
We prove that $G$ multicycle. Since $G$ is regular, this implies that either $G = tC_n$ with $t \geq 1$ and $n \geq 3$, or $G$ is an $(s, t)$-multicycle with $s \neq t$. If $G = tC_n$, we are done. We prove that $G$ cannot be an $(s, t)$-multicycle by reducction ad absurdum. By symmetry, we may assume $1 \leq s < t$. By (2.2), for each vertex $v$ we have $|X_v| = s + t$. Let $v \in V(G)$. Then, $G - v$ is a multipath and one end-block of $G - v$, say $B$, is a $tK_2$. Then, $B$ consists of two vertices $u$ and $w$ with $d_{G - v}(u) = t$ and $d_{G - v}(w) = s + t$. Let $x \in X_v$ be an arbitrary color and set $(G', X', H') = (G, X, H)/\langle v, x \rangle$. Then, there is a spanning subgraph $\tilde{H}$ of $H'$ such that $(G', X', \tilde{H})$ is constructible (by Claim 1). Moreover, (2.2) together with Proposition 2 implies that $|X'_u| = t$, $|X'_w| = s + t$ and that there is a subset $X'_u$ of $X'_u$ such that $|X'_u| = t$ and $H_1 = H[X'_u \cup X'_w]$ is a $K_{t,t}$ with parts $X'_u$ and $X'_w$. The graph $H_1$ is a subgraph of $H^2 = H[X_u \cup X_w]$, and $H^2$ is a $t$-regular bipartite graph with parts $X_u$ and $X_w$ (by Proposition 4(d)). Since $|X'_u| = |X'_w| = s + t$ and $1 \leq s < t$, this is impossible and the claim is proven. □

By Claim 2, $G$ is either a $tK_n$ with $t \geq 1$ and $n \geq 2$, or $G = tC_n$ with $t \geq 1$ and $n \geq 4$. In order to complete the proof we show that in the first case, $(G, X, H)$ is a K-configuration, and, in the second case, $(G, X, H)$ is a C-configuration.

Claim 3 If $G = tK_n$ for integers $t \geq 1$ and $n \geq 2$, then $(G, X, H)$ is a K-configuration.

Proof: Since $(G, X, H)$ is minimal uncolorable, for each vertex $v$ of $G$ and each pair $u, w$ of distinct vertices of $G$, it holds

(a) $|X_v| = t(n - 1)$ and $H[X_u \cup X_w]$ is a $t$-regular bipartite graph with parts $X_u$ and $X_w$

(by (2.2) and by Proposition 4(d)). If $n = 2$, then $G$ has exactly two vertices, say $u$ and $w$, and $H[X_u \cup X_w]$ is a $K_{t,t}$ (by (a)), and so $(G, X, H)$ is a K-configuration as claimed.

Now assume that $n \geq 3$. Let $v$ be an arbitrary vertex of $G$, and let $x \in X_v$ be an arbitrary color. Moreover, let $(G', X', H') = (G, X, H)/\langle v, x \rangle$. Then, there is a spanning subgraph $\tilde{H}$ of $H' = H - (X_v \cup N_H(x))$ such that $(G', X', \tilde{H})$ is a constructible configuration (by Claim 1). As $G' = G - v = tK_{n - 1}$, $(G', X', \tilde{H})$ is a K-configuration. Consequently, for every vertex $u \in V(G)$, there is a partition $(X_u^1, X_u^2, \ldots, X_u^{n - 2})$ of $X'_u = X_u \setminus N_H(x)$ such that, for $i \in \{1, 2, \ldots, n - 2\}$,
(b) the graph $H_i = \tilde{H}[\bigcup_{u \in V(G')} X_u^i]$ is a $K_{(n-1,t)}$ whose partite sets are the sets $X_u^i$ with $w \in V(G')$, and $\tilde{H} = H^1 \cup H^2 \cup \ldots \cup H^{n-1}$.

For $u \in V(G')$ let $X_u^{n-1} = X_u \setminus X'_u$. Then, for every vertex $u \in V(G')$, $|X_u^{n-1}| = t$ and $(X'_u, X_1^u, \ldots, X_u^{n-1})$ is a partition of $X_u$. Since $\tilde{H}$ is a spanning subgraph of $H'$, it follows from (a) and (b) that $H^i$ is an induced subgraph of $\tilde{H}$ (for $i \in \{1,2,\ldots,n-2\}$), and the graph

$$H^{n-1} = H[\bigcup_{u \in V(G')} X_u^{n-1}]$$

is a $K_{(n-1,t)}$ whose partite sets are the sets $X_u^{n-1}$ with $u \in V(G')$. Moreover,

$$H - X_v = H^1 \cup H^2 \cup \ldots \cup H^{n-1},$$

and $N_H(x) = V(H^{n-1})$.

Since the color $x \in X_v$ was chosen arbitrarily, this implies that for each $x \in X_v$ there is an index $i \in \{1,2,\ldots,n-1\}$ such that $N_H(x) = V(H^i)$, and, by (a) and (b), for each index $i \in \{1,2,\ldots,n-1\}$ there are exactly $t$ colors $x$ from $X_v$ such that $N_H(x) = V(H^i)$. As a consequence, there is a partition $(X'_v, X_v^2, \ldots, X_v^{n-1})$ of $X_v$ such that $|X_v^i| = t$ and $N_H(x) = V(H^i)$ for $x \in X_v^i$ and for $i \in \{1,2,\ldots,n-1\}$. Hence, for $i \in \{1,2,\ldots,n\}$, the graph

$$H_i = H[\bigcup_{u \in V(G')} X_u^i]$$

is a $K_{(n,t)}$ whose partite sets are the sets $X_u^i$ with $u \in V(G)$, and, moreover,

$$H = H_1 \cup H_2 \cup \ldots \cup H_n.$$

Thus, $(G, X, H)$ is a K-configuration. \qed

**Claim 4** If $G = tC_n$ for integers $t \geq 1$ and $n \geq 4$, then $(G, X, H)$ is a C-configuration.

**Proof**: Since $(G, X, H)$ is minimal uncolorable, for each vertex $v \in V(G)$ and each 2-set $\{u, w\} \in A(G)$, it holds

(a) $|X_v| = 2t$ and $H[X_u \cup X_w]$ is a $t$-regular bipartite graph with parts $X_u$ and $X_w$

(by (2.2) and by Proposition 4(d)). Let $v$ be an arbitrary vertex of $G$, and let $x \in X_v$ be an arbitrary color. Moreover, let $(G', X', H') = (G, X, H)/(v, x)$. Then, there is a spanning subgraph $\tilde{H}$ of $H' = H - (X_v \cup N_H(x))$ such that

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$(G', X', \tilde{H})$ is a constructible configuration (by Claim 1). Since $G' = G - v = tP_{n-1}$, the vertices of $G'$ can be arranged in a sequence, say $v_1, v_2, \ldots, v_{n-1}$, such that two vertices are adjacent in $G'$ if and only if they are consecutive in the sequence. Note that $N_G(v) = \{v_1, v_{n-1}\}$ and each block of $G'$ is a $tK_2$.

We claim that for each vertex $u$ of $G'$ there is a partition $(X'_u, X^2_u)$ of $X_u$ such that the following conditions hold:

(b) For every $i \in \{1, 2\}$ and every $k \in \{1, 2, \ldots, n-2\}$, the graph $H^i_k = \tilde{H}[X^i_{vk} \cup X^i_{vk+1}]$ is a $K_{t,t}$ whose partite sets are $X^i_{vk}$ and $X^i_{vk+1}$.

(c) The graph $H - X_u$ is the union of all graphs $H^i_k$ with $i \in \{1, 2\}$ and $k \in \{1, 2, \ldots, n-2\}$.

(d) If $n$ is even, then $N_H(x) = X^1_{v_1} \cup X^2_{v_{n-1}}$, or $N_H(x) = X^2_v \cup X^1_{v_{n-1}}$.

(e) If $n$ is odd, then $N_H(x) = X^1_v \cup X^1_{v_{n-1}}$, or $N_H(x) = X^2_v \cup X^2_{v_{n-1}}$.

For $k \in \{1, 2, \ldots, n-2\}$, the graph $B^k = G[\{v_k, v_{k+1}\}]$ is a block of $G'$. Clearly, $B(G') = \{B^1, B^2, \ldots, B^{n-2}\}$ and the only end-blocks of $G'$ are $B^1$ and $B^{n-2}$. Since $(G', X', \tilde{H})$ is a constructible configuration and since each block of $G'$ is a $tK_2$, it follows from Proposition 2 that for each $k \in \{1, 2, \ldots, n-2\}$ there is a uniquely determined cover $(\tilde{X}^k, \tilde{H}^k)$ of $B^k$ such that

- $\tilde{H}^k$ is a $K_{t,t}$ with parts $\tilde{X}^k_{vk}$ and $\tilde{X}^k_{vk+1}$,
- $\tilde{H}$ is the disjoint union of the graphs $\tilde{H}^1, \tilde{H}^2, \ldots, \tilde{H}^{n-2}$,
- $X'_{v_1} = \tilde{X}^1_{v_1}, X'_{v_{n-1}} = \tilde{X}^{k-1}_{v_k} \cup \tilde{X}^k_{vk} (k \in \{1, 2, \ldots, n-2\})$, and $X'_{v_{n-1}} = \tilde{X}^{n-2}_{v_{n-1}}$.

Since $\{v_k, v_{k+1}\} \in A(G)$ for $k \in \{1, 2, \ldots, n-2\}$, it follows from (a) that $\tilde{H}^k$ is an induced subgraph of $H$. Let $X^0_{v_1} = X_{v_1} \setminus X'_{v_1}$ and $\tilde{X}^{n-1}_{v_{n-1}} = X^{n-1}_{v_{n-1}} \setminus X'_{v_{n-1}}$. Then, both sets $X^0_{v_1}$ and $\tilde{X}^{n-1}_{v_{n-1}}$ have exactly $t$ elements, and $N_H(x) = X^0_{v_1} \cup \tilde{X}^{n-1}_{v_{n-1}}$. Furthermore, we conclude from (a) that, for $k \in \{1, 2, \ldots, n-2\}$,

- the graph $H[\tilde{X}^{k-1}_{vk} \cup \tilde{X}^{k+1}_{vk+1}]$ is a $K_{t,t}$ with parts $\tilde{X}^{k-1}_{vk}$ and $\tilde{X}^{k+1}_{vk+1}$.

If $n$ is even, we set

$$(X^1_{v_1}, X^2_{v_2}, \ldots, X^1_{v_{n-1}}) = (\tilde{X}^1_{v_1}, \tilde{X}^1_{v_2}, \tilde{X}^3_{v_3}, \tilde{X}^3_{v_4}, \ldots, \tilde{X}^{n-3}_{v_{n-3}}, \tilde{X}^{n-3}_{v_{n-2}}, \tilde{X}^{n-1}_{v_{n-1}}).$$
and

\[(X^2_{v_1}, X^2_{v_2}, \ldots, X^2_{v_{n-1}}) = (\tilde{X}^0_{v_1}, \tilde{X}^2_{v_2}, \tilde{X}^2_{v_3}, \tilde{X}^4_{v_4}, \tilde{X}^4_{v_5}, \ldots, \tilde{X}^{n-2}_{v_{n-2}}, \tilde{X}^{n-2}_{v_{n-1}}).\]

If \(n\) is odd, let

\[(X^1_{v_1}, X^1_{v_2}, \ldots, X^1_{v_{n-1}}) = (\tilde{X}^1_{v_1}, \tilde{X}^1_{v_2}, \tilde{X}^3_{v_3}, \tilde{X}^3_{v_4}, \ldots, \tilde{X}^{n-2}_{v_{n-2}}, \tilde{X}^{n-2}_{v_{n-1}}),\]

and

\[(X^2_{v_1}, X^2_{v_2}, \ldots, X^2_{v_{n-1}}) = (\tilde{X}^0_{v_1}, \tilde{X}^2_{v_2}, \tilde{X}^2_{v_3}, \ldots, \tilde{X}^{n-3}_{v_{n-3}}, \tilde{X}^{n-3}_{v_{n-2}}, \tilde{X}^{n-1}_{v_{n-1}}).\]

By using (a) and Proposition 4(b), it is easy to check that, for every vertex \(u\) of \(G'\), \((X^1_{u}, X^2_{u})\) is a partition of \(X_u\) such that the conditions (b), (c), (d), and (e) are satisfied. Since the color \(x \in X_v\) was chosen arbitrarily, it follows from (a) and Proposition 4(b) that there is a partition \((X^1_{v}, X^2_{v})\) of \(X_v\) such that \(|X^1_v| = |X^2_v| = t\) and the following conditions hold:

- If \(n\) is even, then \(N_H(x) = X^1_{v_1} \cup X^2_{v_{n-1}}\) for all \(x \in X^1_v\) and \(N_H(x) = X^2_{v_1} \cup X^1_{v_{n-1}}\) for all \(x \in X^2_v\).
- If \(n\) is odd, then \(N_H(x) = X^1_{v_1} \cup X^1_{v_{n-1}}\) for all \(x \in X^1_v\) and \(N_H(x) = X^2_{v_1} \cup X^2_{v_{n-1}}\) for all \(x \in X^2_v\).

Clearly, this implies that \((G, X, H)\) is a C-configuration, and the claim is proven. \(\square\)

This settles Case 2. Hence, in both cases we showed that \((G, X, H)\) is a constructible configuration and the proof of the theorem is complete. \(\blacksquare\)

2.4 A Brooks’ Type Theorem for \(\chi_{DP}\)

The next two corollaries are direct consequences of Theorem 6 and Proposition 2.

**Corollary 7** Let \((G, X, H)\) be a degree-feasible configuration. If \((G, X, H)\) is minimal uncolorable, then for each block \(B \in B(G)\) there is a uniquely determined cover \((X^B, H^B)\) of \(B\) such that the following statements hold.

(a) For every block \(B \in B(G)\), the triple \((B, X^B, H^B)\) is a K-configuration, a C-configuration, or an M-configuration.
(b) The hypergraphs $H^B$ with $B \in \mathcal{B}(G)$ are pairwise disjoint and $H = \bigcup_{B \in \mathcal{B}(G)} H^B$.

(c) For each vertex $v \in V(G)$ it holds $X_v = \bigcup_{B \in \mathcal{B}(G), v \in V(B)} X_v^B$.

**Corollary 8** A connected hypergraph $G$ is not DP-degree-colorable if and only if each block of $G$ is a DP-hyperbrick.

To conclude this paper, we are now able to give a Brooks-type theorem for DP-colorings of hypergraphs. For graphs, the theorem was proven already by Bernshteyn, Kostochka, and Pron [4].

**Theorem 9** Let $G$ be a connected hypergraph. Then, $\chi_{DP}(G) \leq \Delta(G) + 1$ and equality holds if and only if $G$ is a DP-hyperbrick.

**Proof:** It follows from (2.1) that $\chi_{DP}(G) \leq \Delta(G) + 1$ always holds. Moreover, it is obvious that any DP-hyperbrick $G$ satisfies $\chi_{DP}(G) = \Delta(G) + 1$, just take a K-, C-, or M-configuration. Now assume that $\chi_{DP}(G) = \Delta(G) + 1$. Then, there is a cover $(X, H)$ of $G$ such that $|X_v| \geq \Delta(G)$ for all $v \in V(G)$ and $G$ is not $(X, H)$-colorable. Hence, $(G, X, H)$ is an uncolorable degree-feasible configuration and there is a spanning subhypergraph $H'$ of $H$ such that $(G, X, H')$ is minimal uncolorable. Then, $G$ is regular (by Proposition 4(a)) and each block of $G$ is a DP-hyperbrick (by Theorem 6). As any DP-hyperbrick is regular, this implies that $G$ has only one block and, therefore, is a DP-hyperbrick. This completes the proof.

3 DP-Critical Hypergraphs

In coloring theory of graphs and hypergraphs, it is often useful to consider critical graphs and hypergraphs. Following Dirac [7], [8], a graph $G$ is (vertex) $k$-critical if $\chi(G - v) < \chi(G) = k$ for every $v \in V(G)$. The hypergraph-equivalent was introduced by Lovász [20]. Note that critical graphs and hypergraphs are always simple. Regarding DP-colorings of graphs, Bernsteyn et al. introduced the term vertex DP-$k$-critical in [4]. Extending their definition to hypergraphs, we say that a hypergraph $G$ is (vertex) **DP-$k$-critical** if $\chi_{DP}(G) = k$, but $\chi_{DP}(G - v) < k$ for each $v \in V(G)$. Since we only regard vertex critical hypergraphs in this paper, we will omit the term vertex from now on. Now let $G$ be a DP-$k$-critical hypergraph. Then, there is a cover
(X, H) of G with |X_v| ≥ k − 1 for all v ∈ V(G) such that (G, X, H) is uncolorable. Let v ∈ V(G) be an arbitrary vertex. Then, we define a cover (X', H') of G' as follows. Let X'_u = X_u for u ∈ V(G') and let H' = H[∪_{u∈V(G')} X_u]. Since G is DP-k-critical, we have χ_{DP}(G') ≤ k − 1 and, as |X'_u| ≥ k − 1 for all u ∈ V(G'), (G', X', H') is colorable. This motivates the following definition. Given a hypergraph G and a cover (X, H) of G, we say that G is (X, H)-critical if (G, X, H) is uncolorable but (G − v, X^v, H^v) is colorable for each v ∈ V(G), whereas (X^v, H^v) denotes the restriction of (X, H) to G − v, that is, X^v = X_u for all u ∈ V(G − v) and H^v = H[∪_{u∈V(G−v)} X_u]. Moreover, we say that a connected hypergraph G is a Gallai-DP-forest or, briefly, GDP-tree if each block of G is a DP-hyperbrick. A Gallai-DP-forest (GDP-forest) is a hypergraph of which components all are GDP-trees. In the next proposition, we establish some basic facts on (X, H)-critical connected hypergraphs for which the following definitions are necessary. The next Proposition as well as most of the definitions were introduced for list-colorings of hypergraphs in [17], big parts of the proof can be carried over. For graphs, Bernsteyn, Kostochka, and Pron stated a similar proposition in [4]. More information on DP-critical graphs can also be found in [1].

Let G be a hypergraph and let U ⊆ V(G) be a set. By G(U) we denote the hypergraph that results from G by shrinking iteratively at the vertices of V(G) \ U. Note that for any vertices u ≠ v from V(G) it clearly holds G \ u \ v = G \ v \ u and, thus, G(U) is well defined. Furthermore, if G' is a component of G(U), it is trivial that G' = G(W) for some W ⊆ U. Now let W ⊆ V(G) be a set. By E_W(G) we denote the set of all edges e ∈ E(G) satisfying |i_G(e) \ W| = 1. If W = {w} is a singleton, we also write E_w(G).

A W-mapping is a function v that assigns to every edge e ∈ E_W(G) a vertex v(e) ∈ i_G(e) \ W. Given a W-mapping v and a vertex w ∈ W, let

\[ N^v_W(w : G) = \{ u ∈ V(G) \mid u = v(e) \text{ and } i_G(e) \cap W = \{w\} \text{ for some } e ∈ E_W(G)\}. \]

Then, it clearly holds N^v_W(w : G) ⊆ V(G) \ W and

\[ d_G(w) ≥ d_{G,W}(w) + |N^v_W(w : G)|. \]

**Proposition 10** Let G be an (X, H)-critical connected hypergraph and let U = \{v ∈ V(G) \mid d_G(v) ≤ |X_v|\}. Furthermore, let W be a non-empty subset of U, let v be an W-mapping of G, and let

\[ F = \{ e ∈ E(G) \mid |i_G(e) \cap W| ≥ 2 \text{ and } i_G(e) \setminus W ≠ \emptyset \}. \]
Then, the following statements hold:

(a) \( d_G(v) = |X_v| \) for all \( v \in U \),

(b) \( G(W) \) is a GDP-forest.

(c) \( d_G(w) = d_G(W)(w) + |N^w_W(w : G)| \) for each \( w \in W \).

(d) If \( w \in U \), then \( |i_G(e) \cap i_G(e')| = 1 \) for every two distinct edges \( e, e' \in E_w(G) \).

(e) If \( e, e' \in F \) and \( e \neq e' \), then \( i_G(e) \cap W \neq i_G(e') \cap W \).

(f) If \( e \in F \), then the corresponding edge \( e \in E(G(W)) \) with \( i_G(W)(e) = i_G(e) \cap W \) is a bridge of \( G(W) \).

**Proof:** Clearly, it is sufficient to consider the case that \( G(W) \) is connected (otherwise consider each component of \( G(W) \) individually). Let \( Z = V(G) \setminus W \) and let \( G = G_0 \). As \( W \) is a connected, we can choose a vertex order \((v_1, v_2, \ldots, v_\ell)\) of \( Z \) such that for \( i = 1, 2, \ldots, \ell \), \( v_i \) is not a separating vertex of \( G_{i-1} \), whereas \( G_i = G \setminus v_i \). Since \( G \) is \((X, H)\)-critical, there is an independent transversal \( T_Z = \{x_1, x_2, \ldots, x_\ell\} \) in \( H[\bigcup_{i=1}^{\ell} X_{v_i}] \) with \( T_Z \cap X_{v_i} = x_i \). Let \((G, X, H) = (G_0, X_0, H_0)\) and, for \( i = 1, 2, \ldots, \ell \), let \((G_i, X_i, H_i) = (G_{i-1}, X_{i-1}, H_{i-1})/(v_i, x_i)\). Since \( T \) is an independent transversal, there are no edges in \( H[\{x_1, x_2, \ldots, x_\ell\}] \) and so we have \( x_i \in V(H_{i-1}) \) for \( i = 1, 2, \ldots, \ell \). Thus, the reduction is well defined and, by Proposition 3(b), each triple \((G_i, X_i, H_i)\) is uncolorable. Let \((G', X', H') = (G_\ell, X_\ell, H_\ell)\). Then, \( G' = G(W) \) and it is easy to check that \( |X'_w| \geq |X_w| - |N^w_W(w : G)| \) for all \( w \in W \) (by (C2) and Proposition 3). Moreover, since \( d_G(w) \leq |X_w| \) for each \( w \in W \), we obtain

\[
|X'_w| \geq d_G(w) \geq d_{G'}(w) + |N^w_W(w : G)|
\]

and, hence,

\[
|X'_w| \geq |X_w| - |N^w_W(w : G)| \geq d_{G'}(w).
\]

Thus, \((G', X', H')\) is a degree feasible uncolorable configuration and there is a spanning subhypergraph \( \tilde{H} \) of \( H' \) such that \((G', X', \tilde{H})\) is a minimal uncolorable degree feasible configuration. Then, it follows from Proposition 4(a) that \|X'_w\| = d_{G'}(w) for all \( w \in W \) and so \|X_w\| = d_G(w) = d_{G'}(w) + |N^w_W(w : G)| for all \( w \in W \). Finally, we obtain from Corollary 8 that \( G(Y) \) is a GDP-tree. This proves (a)-(c).
In order to prove (d), assume that there is a vertex \( w \in W \) and two distinct edges \( e, e' \in E_w(G) \) such that \( |i_G(e) \cap i_G(e')| \geq 2 \). Then, for the set \( W' = \{ w \} \), there is a \( W' \)-mapping \( v' \) of \( G \) such that \( v'(e) = v'(e') \). But then, \( d_G(w) > d_{G(W)}(w) + |N_{W'}(w : G)| \), contradicting (c). Clearly, (e) is an immediate consequence of (d).

It remains to prove (f). Suppose that there is an edge \( e \in F \) such that the corresponding edge \( e \in E(G(W)) \) with \( i_{G(W)}(e) = i_G(e) \cap W \) is not a bridge of \( G' = G(W) \). Then, by (b), \( e \) is an ordinary edge of \( G' \), i.e. \( i'_G(e) = \{ w_1, w_2 \} \) for some \( w_1, w_2 \in W \), and \( \tilde{G} = G' - e \) is a connected hypergraph. Recall from the proof of (a) that there is an independent transversal \( T = \{ x_1, x_2, \ldots, x_\ell \} \) in \( \tilde{H} = \bigcup_{i=1}^\ell X_{v_i} \). As \( e \in F \), there is a vertex \( v_1 \in Z = \{ v_1, v_2, \ldots, v_\ell \} = V(G) \setminus W \) with \( v_i \in i_G(e) \), say \( v_1 \). Moreover, as \( (G, X, H) \) is uncolorable, there is a color \( x_{w_1} \in X_{w_1} \) and a color \( x_{w_2} \in X_{w_2} \) such that \( \{ x_{w_1}, x_{w_2} \} = i_H'(e') \) for some \( e' \in E(H') \) and that \( \{ x_{w_1}, x_{w_2}, x_1 \} \subseteq i_H(e) \) for some \( e \in E(H) \).

Let \( (\tilde{X}, \tilde{H}) \) be the cover of \( \tilde{G} \) satisfying \( \tilde{X}_w = X'_w \) for all \( w \in W \setminus \{ w_1 \} \), \( \tilde{X}_{w_1} = X'_{w_1} \setminus \{ x_{w_1} \} \) and \( \tilde{H} = H' \cup_{w \in W} \tilde{X}_w \). Then, \( |\tilde{X}_w| \geq |d_G(w)| \) for all \( w \in W \) and \( |\tilde{X}_{w_1}| > d_G(w_2) \). Thus, by Proposition 4(a), \( (G, \tilde{X}, \tilde{H}) \) admits an independent transversal \( \tilde{T} \). But then, \( \tilde{T} \cup T \) is an independent transversal of \( H \) implying that \( (G, X, H) \) is colorable, a contradiction. \( \blacksquare \)

The next corollary is a direct consequence of Proposition 10(b).

**Corollary 11** Let \( G \) be a DP-\( k \)-critical hypergraph with \( k \geq 1 \), and let \( W = \{ v \in V(G) \mid d_G(v) = k - 1 \} \). Then, \( \delta(G) \geq k - 1 \) and \( G(W) \) is a GDP-forest (possibly empty).

There are various well known lower bounds for the degree-sum \( d(G) \) over all vertices of a critical (hyper-)graph \( G \). The first such bound was established by Dirac [9]. He proved that if \( G \) is a \( k \)-critical graph distinct from \( K_k \), then

\[
d(G) \geq (k - 1)n + k - 3.
\]

Another famous bound is due to Gallai [13], [14]. He showed that any \( k \)-critical graph \( G \) of order \( n \geq k + 1 \) satisfies

\[
d(G) \geq (k - 1)n + \frac{k - 3}{k^2 - 3} n.
\]

In [17], Stiebitz and Kostochka provided an improvement of this bound for \( k \)-critical hypergraphs. That the Gallai bound also holds for DP-\( k \)-critical
simple graphs was proven by Bernsteyn, Kostochka, and Pron [4]. Furthermore, Bernsteyn and Kostochka proved that the Dirac bound holds for DP-
$k$-critical simple graphs in [1].

Regarding DP-$k$-critical simple hypergraphs, it directly follows from Corollary 11 that $d(G) \geq n(k - 1)$. Moreover, using the approach from Kostochka and Stiebitz in [17] as well as the approach in [23] (especially for cycles of even length) it is possible to extend their bound also to DP-$k$-critical simple hypergraphs. Note that if $G$ is a DP-$k$-critical simple hypergraph, then it follows from Proposition 10(f) that $G(W)$ is a simple hypergraph, as well, whereas $W$ denotes the set of vertices of $G$ with degree $k - 1$. Hence, the methods from [17] can be used. As it would take several pages to display their proof, we will only state the theorem here.

**Theorem 12** Let $G$ be a DP-$k$-critical simple hypergraph distinct from $K_k$ with $k \geq 4$. Then, it holds

$$d(G) \geq (k - 1 + \frac{k - 3}{k^2 - 3})n.$$  

4 Concluding Remarks

It is often of interest to determine the complexity of specific coloring problems. Clearly, a graph has chromatic number 2 if and only if it is bipartite. By Knig’s Theorem this is equivalent to having no cycles of odd length, which can easily be checked in polynomial time. However, Lovász [21] showed that for a fixed integer $k \geq 3$ it is an NP-complete decision problem to decide whether a graph admits a $k$-coloring. Moreover, he proved that it is NP-complete to decide whether a hypergraph is bipartite or not. This implies in particular that determining the chromatic number of a hypergraph is NP-hard. Regarding list-colorings, Erdős, Rubin, and Taylor [12] and independently Vizing [24] showed that one can check in polynomial time if a graph admits an $L$-coloring provided that each vertex gets assigned a list of at most 2 colors. Furthermore, Erdős, Rubin, and Taylor [12] observed that, given a fixed integer $k \geq 3$, the problem if a graph is $k$-list colorable is $\Pi^p_2$-complete whereas $\Pi^p_2$ is a complexity class in the polynomial hierarchy containing both NP and coNP. Since DP-colorings are an extension of (list-)colorings of hypergraphs we conclude that, given a cover $(X, H)$ of a hypergraph $G$, it is NP-hard to decide if $G$ admits an $(X, H)$ coloring. Nevertheless, it might be
an interesting topic to examine conditions under which a graph $G$ admits an $(X, H)$-coloring for some cover $(X, H)$. In order to get some ideas we recommend taking a look at a survey by Golovach, Johnson, Paulusma, and Song [15] that analyzes the complexity of coloring problems with respect to some forbidden subgraphs. Regarding list-colorings of (simple) hypergraphs with lists containing at least degree many colors it is easy to deduce a polynomial time algorithm from the proof of Kostochka, Stiebitz and Wirt [19] that, given a simple hypergraph $G$ and a list-assignment $L$ with $|L(v)| \geq d_G(v)$ for all $v \in V(H)$, either finds an $L$-coloring of $G$ or returns a 'bad' block. A similar algorithm for DP-degree colorability can be deduced from our proof.

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