The analysis of the laminar-turbulent transition onset for the Couette flow in the flat channel with the velocity profile periodic background perturbation

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Abstract. To predict the laminar-turbulent transition, we adopted an approach, which is based on the transverse viscosity factor threshold addition hypothesis. Taking into account the conditions imposed on the corresponding dimensionless complexes within the framework of this approach, we analysed the transition onset for the Couette flow in the flat channel with consideration of the velocity field perturbation initial periodic background. The critical value for the Reynolds number was obtained, which is in satisfactory agreement with the known experimental data.

1. Introduction

At present, the main approach to predicting the laminar-turbulent transition onset is based on the hydrodynamic stability theory and involves the imposition of the periodic background of velocity and pressure perturbations on the initially laminar flow. In this case, the prediction of the transition onset is reduced to the analysis of the conditions under which such perturbations’ amplitudes or the perturbations’ kinetic energy will continue to rise without bound with increasing time [1-5]. As for the “mechanism” of the pulsating background origin, it does not become the subject of the discussion as a rule.

It should be noted that the turbulence initiation is facilitated by a large number of very different factors, which include the initial pulsations existence or the flow perturbations at the entrance to the considered flow region [6, 7], the presence of the roughness at the flow region's rigid boundaries [8], the fluid density fluctuations[9], acoustic perturbations [10-12].

One of the features of all incipient turbulent flows is the velocity components formation transverse to the pre-existing main laminar flow streamlines at the initial stage. In this case, the known experimental data indicate that the turbulence does not exist in all flow regimes, but arises as a “threshold” when some parameter characterizing the flow “intensity” degree exceeds a certain critical level.

We should note one more circumstance, concerning the transverse velocity components “generation”. Consideration of the boundary value problems about the steady flow in various cross-sections rectilinear channels for fluids, the rheology of which involves taking into account the transverse viscosity, leads to
the solutions demonstrating the secondary flows presence [13]. These flows are characterized by the transverse velocity components with respect to the streamlines, which could be obtained by solving the same boundary value problems, but without taking into account the transverse viscosity.

Based on these results, an approach was proposed in [14] to describe the laminar-turbulent transition onset, which is based on the transverse viscosity threshold “addition” hypothesis.

This hypothesis was implemented in the corresponding rheological model framework, in which the existence of the threshold value of the strain rate tensor second invariant modulus was factored, above which the transverse viscosity factor is “added” and, accordingly, the transverse velocity components are “generated” with respect to the initial laminar flow streamlines. At the same time, the possibility of their unlimited growth was taken into account. It was proposed to interpret such a situation as the laminar-turbulent transition onset. If the second invariant modulus does not exceed the threshold level factored in the rheological model, the transverse viscosity does not manifest itself and the classical Newtonian model describes the fluid behavior. In this case, it is assumed that the transverse velocity components “generation” does not occur.

Within this approach, based on the known experimental data processing results, the two empirical conditions were proposed: the condition for the transverse velocity components “generation” (with respect to the original laminar streamlines)

$$K_2(\overline{X}) > K_2G(\overline{X}) = k_0 + \frac{k_1}{K_3(\overline{X}) - k_2},$$

as well as the condition for an indefinite increase in the “generated” transverse velocity components

$$\max_{X \in G} [K_1(\overline{X})] > q_0 \left[ K_3(\overline{X}) \right]^{q_1},$$

where $G$ is the spatial zone inside the fluid flow region, in which condition (1) of transverse velocity components “generation” is satisfied; $k_0$, $k_1$, $k_2$, $q_0$, $q_1$ - dimensionless empirical parameters determined on the basis of the known experimental data processing [14, 15]; $K_1$, $K_2$, $K_3$ - are dimensionless complexes which are the functions of the considered spatial point coordinates $\overline{X}$ and which are determined exclusively through invariant quantities by means of the following relations

$$K_1 = \frac{E_s}{D_s}; \hspace{1cm} K_2 = \frac{\rho^2 \cdot U_s^2 \cdot I_2s}{E_s^2}; \hspace{1cm} K_3 = \frac{\rho^2 \cdot U_s^3}{\mu \cdot E_s};$$

$$\tilde{E} = \text{grad} \left\{ P + \frac{\rho \cdot U_s^2}{2} \right\}; \hspace{1cm} \tilde{D} = \text{grad} \left\{ 2 \cdot \mu \cdot \sqrt{I_2} \right\},$$

In these relations, the following notation was applied: $P$ is the pressure; $U$ is the fluid velocity vector modulus; $\rho$, $\mu$ are liquid density and liquid dynamic viscosity, respectively; $I_2$ is the strain rate tensor second invariant; $E$, $D$ are the vectors modules $\tilde{E}$, $\tilde{D}$, respectively. In this case, the subscript “s” indicates that the corresponding functions were calculated at the considered point $\overline{X}$.

In other words, within the framework of the proposed approach, it is assumed that in the case of simultaneous fulfillment of conditions (1), (2) at the corresponding point $\overline{X}$ in the flow region, it is proposed to consider such point and its small neighborhood as the laminar-turbulent transition initiator.
One of the simple examples of a shearing flow is the Couette flow in the flat channel with one movable boundary. In this case, the velocity profile has the simplest form and is described by a linear function.

The laminar-turbulent transition for the Couette flow in the flat channel as a test problem has been analysed many times on the basis of the hydrodynamic stability theory. However, the analysis of such a seemingly simple flow scheme for the laminar-turbulent transition onset turned out to be far from being simplistic. The complexity and the exceptional condition of the laminar-turbulent transition for the Couette flow in the flat channel from the standpoint of stability or instability was noted by the classics of hydrodynamics Rayleigh and Lamb. References and a review on the study of the Couette flow stability are given in [2]. Among other factors, the main conclusion, as noted in [2] with the reference to the theoretical results obtained in [16-20], is that the Couette flow is stable with respect to small perturbations. At the same time, the known experimental data clearly demonstrate the possibility of the turbulence development for the flows of this kind [21-24].

An analysis of the laminar-turbulent transition onset for the plane one-dimensional Couette flow was also considered in [25]. Due to the necessity to fulfill conditions (1), (2), it was shown that for such flow there is a threshold value of the Reynolds number, above which the formation of the laminar-turbulent transition takes place.

Using the approach proposed in [14], in this article we develop the results obtained in [25], and analyze the Couette flow in the flat channel with the initial periodic background of the velocity field perturbations in order to predict the laminar-turbulent transition onset.

2. The initial state of the channel velocity field and the main dimensionless complexes for characterizing the laminar-turbulent transition onset

Let us consider the plane Couette flow with a periodic background of the velocity field perturbation, for which, in the traditionally chosen coordinate system presented in figure 1, the distribution of the “main” longitudinal (excluding background perturbations) velocity component is described by the linear dependence

\[ V(y) = V_W \cdot y' \quad ; \quad y' = \frac{y}{h}, \]

where \( y \) is the transverse coordinate off the channel fixed boundary; \( h \) is the flat channel width; \( V_W \) is the channel movable boundary velocity.

Let us choose a spatial point \( O(x_0, y_0) \) in the channel and associate with it, as with a new origin, another, local coordinate system \( O_{x_1x_2} \).

Suppose that at some point in time, which is conditionally accepted as the start time, a plane flow with the velocity field perturbation periodic background has formed in the channel. Suppose that the components \( U_1 \) and \( U_2 \) of the velocity vector in the local Cartesian coordinate system point \( M \) around the particular point \( O \) in the dimensional record form are described by the following

\[ U_1 = V + u_1 = V + u; \quad U_2 = u_2 = -\delta \cdot u; \]

\[ V = V(x_{2M}) = V_W \cdot \left( \frac{y_0 + x_{2M}}{h} \right); \]

\[ u_1 = u = u(x_{1M}, x_{2M}) = U_A \cdot \sin \left( \alpha_1 \cdot x_{1M} + \alpha_2 \cdot x_{2M} \right); \]

\[ \alpha_1 = \frac{2 \cdot \pi}{\lambda_1}; \quad \alpha_2 = \frac{2 \cdot \pi}{\lambda_2}; \quad \delta = \frac{\alpha_1}{\alpha_2} = \frac{\lambda_2}{\lambda_1}, \]
where \( V(x_{2M}) \) is the fluid flow rate longitudinal component taking into account (4) as a function of the transverse coordinate \( x_{2M} \) in the local reference system; \( u_1, u_2 \) are the velocity field background perturbation components; \( u(x_{1M}, x_{2M}) \) is the periodic function that determines velocity field background perturbation components in the local coordinate system; \( U_A \) is the periodic background perturbation amplitude; \( \lambda_1, \lambda_2 \) are the periodic background perturbation components wavelengths in the longitudinal and transverse directions, respectively.

Note that the velocity vector components (5) identically satisfy the continuity condition. In this case, we especially note that they determine the velocity components at some spatial point, located around the particular point \( O \), merely at the start time.

![Figure 1. The problem geometry of the Couette flow with the velocity field perturbations periodic background.](image)

The initial stage of the flow evolution in the point's small neighborhood \( O \) from the declared starting state (5) will be determined by the dimensionless complexes (3). These complexes are calculated taking into account (5) the corresponding functions at the point \( O \) (the local coordinate origin) for \( x_{1M} = 0 \) and \( x_{2M} = 0 \) can eventually be worked out to

\[
K_1 \to \infty; \\
K_2 = \frac{4 \cdot \delta^2 \cdot K_{10}^2 \left[ \left( 1 + \delta^2 \right) + \left( 1 - \delta^2 \right) \cdot K_{10} \right]^2}{4 \cdot \left( 1 + \delta^2 \right) \cdot (1 + K_{10})^2}; \\
K_3 = \frac{Re \cdot y_0^2}{(1 + K_{10})}. 
\]

For brevity, in the latter relations, the following dimensionless parameters were applied

\[
Re = \frac{\rho \cdot V_W \cdot h}{\mu}; \\
K_{10} = \frac{\alpha_2 \cdot h \cdot U_A \cdot (1 + \delta^2)}{V_W}. 
\]

In (7) the dimensionless coordinate
is a parameter that determines the choice of the flow region point $O$ considered as the potential initiator of the laminar-turbulent transition onset.

Let us now, taking into account (6), (7), analyze possible variants of the hydrodynamic process development around the particular point of the flow region from the starting state (5) from standpoint of the conditions (1), (2).

3. The analysis of the laminar-turbulent transition conditions onset

First of all, note that the result (6) assumes that condition (2) is obviously fulfilled with an unlimited growth in the “generated” transverse velocity components relative to the initial laminar flow streamlines. The latter circumstance greatly simplifies the considered flow scheme analysis, since the condition for the laminar-turbulent transition onset is actually reduced to only one inequality satisfaction (1), which means the possibility of transverse velocity components “generation”.

We analyze the fulfillment of condition (1) using the example of the following rather simple special case, when $\delta = 1$ and the considered point in the flow region is located on the channel movable boundary ($y_0 = 1$).

For such special case, the main dimensionless complexes that appear in (1), taking into account (7), can be written as

$$K_2 = K_2(K_{10}) = \frac{(1 + K_{10}^2)}{4 \cdot (1 + K_{10})^2}; \quad K_3 = K_3(Re, K_{10}) = \frac{Re}{(1 + K_{10})};$$

$$K_{2G} = K_{2G}(Re, K_{10}) = k_0 + \frac{k_1}{K_3(Re, K_{10}) - k_2};$$

Here, obviously, it is assumed that the flow parameters $Re$ and $K_{10}$ must ensure the fulfillment of the condition

$$K_3(Re, K_{10}) > k_2,$$

which is reduced to the following restriction on the parameters’ possible values

$$K_{10} < K_{10,lim} = \frac{Re}{k_2} - 1;$$

where $K_{10,lim}$ is the maximum permissible (maximum) for the given $Re$ parameter value $K_{10}$, at which the fulfillment of (10) is permitted, and, consequently, the use of condition (1) is valid.

An example of a graphical interpretation of the fulfillment of condition (1) is given by the function graphs (8) and (9) shown in figure 2. At first glance an analysis of their interposition in the scale adopted in figure 2 indicates, allegedly, that condition (1) is not fulfilled. This means that the transverse velocity components “generation” for the considered initial velocity field around the channel movable boundary ($y_0 = 1$) does not occur and, accordingly, the necessary condition for the laminar-turbulent transition onset is not fulfilled.

This preliminary conclusion is in good agreement with the analytical results of studies from [16-20], which indicate the Couette flow hydrodynamic stability. In this case, note that the known experimental data, as mentioned in the Introduction section, on the contrary, demonstrate the possibility of the
turbulence initiation for such flow.

However, a more detailed consideration of the interposition of the function graphs (8) and (9) in a small neighborhood of the point $K_{10} = 0$ leads to somewhat different conclusions. For example, figure 3 on an enlarged scale shows function graphs fragments for various values of the Reynolds number.

![Figure 2](image)

**Figure 2.** The interposition of the function graphs (8) and (9) with the following main parameters values $Re = 420$; $\delta = 1$; from the point of view of the possibility of fulfilling condition (1) and, therefore, the velocity transverse components “generation” around the channel movable boundary for the plane Couette flow with the velocity perturbation background. 1: $K_2(K_{10})$; 2: $K_{2G}(K_{10})$; 3: vertical asymptote $K_{10} = K_{10,lim} = 3.999$.

Analyzing the data presented in this figure, one can trace the nature of the Reynolds number influence on the laminar – turbulent transition initiation.

At sufficiently small values of the Reynolds number (figure 3-a), condition (1) is not satisfied (transverse velocity components “generation” does not occur) and, despite the fulfillment of condition (2) for the dimensionless complex $K_1$ with regard to (6), the Couette flow in the flat channel remains laminar.

As the Reynolds number increases, the function graphs (8) and (9) approach each other and at about $Re \approx 380$ they join at the point $K_{10} = 0$ (figure 3-b).

The further increase in the Reynolds number (figure 3-c) already ensures the fulfillment of condition (1). In this case, the “generated” transverse velocity components, due to the fulfillment of conditions (2) taking into account (6), will immediately initiate the laminar-turbulent transition onset.

The obtained threshold value of the Reynolds number $Re_T \approx 380$ for the laminar-turbulent transition onset is in reasonable agreement with the known experimental data. For comparison, one can give some experimental values of the critical Reynolds number: $Re_{T,exp} \approx 280$ [21]; 325 [24]; 360 [23]; 370 [22, 26].

If we consider the interposition of the graphs presented in Figure 3, it follows that when the Reynolds number exceeds a certain critical level ($Re > Re_T \approx 380$), there is a range

$$0 < K_{10} < K_{10,T} < K_{10,lim}$$

(11)
of parameter value $K_{10}$ for which condition (1) is satisfied. For the considered Couette flow, this condition, taking into account the known fulfillment of (2) because of (6), is, at the same time, the condition for the laminar-turbulent transition initiation.

![Graphical interpretation of the fulfillment of condition (1) of the transverse velocity component “generation” in the region of sufficiently small values of the parameter $K_{10}$ for $\delta = 1$; $y'_0 = 1$; at $Re = 340$ (a); 380 (b); 420 (c).](attachment:graph.png)

Figure 3. The graphical interpretation of the fulfillment of condition (1) of the transverse velocity component “generation” in the region of sufficiently small values of the parameter $K_{10}$ for $\delta = 1$; $y'_0 = 1$; at $Re = 340$ (a); 380 (b); 420 (c). $1 - K_2(K_{10})$; $2 - K_{2G}(K_{10})$; $3 - K_{10} = K_{10,T}$.

The upper limit $K_{10,T}$ of the range (11) at the Reynolds number fixed value is the root of the equation, which follows from (1) taking into account (8), (9)

$$K_2(K_{10,T}) = K_{2G}(Re,K_{10,T})$$
The roots of this equation are the function $K_{10,T}(Re)$, the graph of which is shown in figure 4. This graph defines the boundary of a certain region (cross-hatched in figure 4) of the parameter values \{Re, $K_{10}$\} for which the process of the laminar-turbulent transition initiation takes place.

At the end of this section, let us note that the considered transition model has been constructed taking into account a number of simplifying assumptions.

Among other factors, the considered velocity field perturbation periodic background refers only to the start time. Therefore, the transition onset prognosis was carried out taking into account (1), (2), in fact, only on the basis of the velocity and pressure fields “starting” state. In this case, the flow evolution process (in the sense of its temporal development) from such starting state, naturally, was not considered. It should also be added that the velocity longitudinal component at the point under consideration had a profile inflection. In addition, note that in the course of consideration, it was simplified to assume the fulfillment of the condition $\delta = 1$.

The true picture of the laminar-turbulent transition onset, as one might expect, is, of course, much more complicated. At the same time, in spite of the simplifying assumptions taken into account, it turned out to be possible in the first approximation to trace some tendencies of the influence, in particular, of the Reynolds number and the characteristics of the velocity perturbation initial background on the laminar-turbulent transition onset within the framework of the proposed approach.

4. Conclusions

Using the approach proposed in [14], we analyzed the conditions for the laminar-turbulent transition onset for the Couette flow with the initial periodic background of the velocity field perturbation in the flat channel with one movable boundary. The minimum (critical) value of the Reynolds number has been determined, which is in satisfactory agreement with the known experimental data.

The obtained results indicate that the proposed empirical conditions (1), (2) in the first approximation can be used to predict the laminar-turbulent transition onset based on the velocity and pressure fields “starting” state analysis.
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