ON PSEUODOHOLONOMIC MAP BETWEEN ALMOST HERMITIAN MANIFOLDS

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ABSTRACT. In this paper, we use the canonical connection instead of Levi-Civita connection to study the smooth maps between almost Hermitian manifolds, especially, the pseudoholomorphic ones. By using the Bochner formulas, we obtain the $C^2$-estimate of canonical second fundamental form, Liouville type theorems of pseudoholomorphic maps, pseudoholomorphicity of pluriharmonic maps, and Simons integral inequality of pseudoholomorphic isometric immersion.

1. Introduction

In 1985, M.Gromov ([6]) introduced the beautiful theory of pseudoholomorphic curves in symplectic manifold, which has profoundly influenced the study of symplectic geometry and symplectic topology. One can refer to D.Macduff and D.Salamon’s book [9] and references therein. Moreover, Gromov’s compactness theorem of pseudoholomorphic curves was also an interesting problem from the analytic point of view. T.H.Parker and J.G.Wolfson ([12]), R.G.Ye ([24]) independently gave the analytic proofs of Gromov’s compactness theorem. This field has been extended to the case of domain manifold with higher dimension, i.e., the pseudoholomorphic map between almost Hermitian manifolds.

C.Y.Wang ([19]) studied the regularity and blow-up analysis of pseudoholomorphic maps by using the techniques developed in the theory of harmonic maps. T.Riviére and G.Tian ([13]) studied the almost complex four-manifold into algebraic varieties in connection with C.Taubes’ works. For the triholomorphic maps, Ch.Y.Wang ([20]), C.Bellettini and G.Tian ([11]) obtained the quantization of energy, compactness results respectively. In this paper, we wish to focus on the geometry of pseudoholomorphic maps by using the canonical connection on almost Hermitian manifold instead of the Levi-Civita connection.

The pseudoholomorphic map is closely related to the theory of harmonic maps. It is known that ([3], [21]) a holomorphic map between Kählerian manifolds is harmonic. Conversely, how to judge a harmonic map to be a holomorphic one is an important topic in the theory of harmonic map. Y.T.Siu ([15]) proved the holomorphicity result of harmonic maps between compact Kähler manifold under the target manifold has strongly negative curvature. J.Jost and S.T.Yau ([17]) used the Hermitian harmonic maps to study maps from Hermitian manifold to Riemannian manifolds, and they also obtained rigidity results in Hermitian geometry. K.F.Liu and X.K. Yang ([8]) extended Siu’s result to Hermitian manifolds, and they also studied several types of harmonic maps from Hermitian manifold. Y.X.Dong ([2]) has used the monotonicity formulae to derive holomorphicity and Liouville type results for pluriharmonic maps and harmonic maps between Kählerian manifolds. Y.Xin ([21]) found a sufficient condition of partial energy density to judge the holomorphicity of a harmonic maps from Riemannian surfaces into complex projective space. L.Ni ([10]) used the $\partial\bar{\partial}$-Bochner formulae to study the general Schwarz lemma and their applications on Kähler manifolds. K.Tang ([16]) generalized works of L.Ni to

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the case of Hermitian manifolds. It is clear that the Bochner technique, i.e., the Bochner
type formulae of different geometric quantity, plays an important role in studing holo-
morphicity and Liouville type results of harmonic maps.

The pseudoholomorphic isometry is a special class of pseudoholomorphic map that
preserves the metric. If we view from theory of submanifold, the geometry of pseudo-
holomorphic isometry between almost Hermitain manifolds is very similar to the minimal
one between Riemannian manifolds. For examples, one can choose adapted unitary frame
field along a pseudoholomorphic isometry, and the trace of canonical second fundamental
form is vanishing. In order to solve the Bernstein problem, J.Simons ([14]) calculated
the Laplacian of square norm of the second fundamental form of closed minimal sub-
manifolds in the unit sphere, and he obtained the famous Simons integral inequality.
K.Ogiue ([11]) derived the complex version of the formula for holomorphic isometry be-
tween Kähler manifolds. H.W.Xu ([22]) generalized Simons integral inequality to closed
minimal submanifold in pinched Riemannian manifolds. We wish to develop these works
to pseudoholomorphic isometry between almost Hermitian manifolds.

Notice that the canonical connection on almost Hermitian manifold is the unique one
preserves the metric, the almost complex structure and also has vanishing (1,1)-part
of torsion. So, it’s reasonable and natural to use the canonical connection to study the
properties of maps between almost Hermitian manifolds which are independent of Levi-
Civita connection, such as the Liouville type results, the pseudoholomorphicity and so
on. Inspired by the works of V.Tosatti ([17]), V.Tosatti, B.Wenkove and S.T.Yau ([18]),
X.Zhang ([25]), by using the canonical connection, one can find the Bochner formulae in
complex version is quite like the one of harmonic map. We utilize techniques and methods
of harmonic maps in the nice book ([21]) by Y.L.Xin to study the pseudoholomorphic
maps and pluriharmonic maps. These are the main motivation of our paper.

The paper organized as follows. In section 2, we recall the almost Hermitian geometry
with respect to the canonical connection. In section 3, we calculate the canonical Lapla-
cian of partial energy density of pluriharmonic maps, and Laplacian of the half norm of
canonical second fundamental form $S$ of pseudoholomorphic maps. In section 4 and 5, we
study the Liouville type results of pseudoholomorphic maps and pseudoholomorphicity
of pluriharmonic maps by using the Bochner type formulae obtained, respectively. In
section 6, we derivie Simons integral inequality for pseudoholomorphic isometry, and we
also give the bound of $S$ under the condition of parallel canonical second fundamental
form.

2. Geometries of almost Hermitian manifold

The purpose of this section is to establish notation, review some well known facts (see
[17], [18]) concerning almost Hermitian geometry, and to establish the basic identities of
smooth map between almost Hermitian manifolds related to the canonical connection.
For the curvature properties, one can refer to the detailed papers [4], [23] and references
therein.

2.1. Geometry of canonical connection. Let $(M, J, g)$ be a $2m$-dimensional almost
Hermitian manifold, which means that $J$ is an almost complex structure on $M$ and
Riemannian metric $g$ is $J$-invariant. The tangent bundle of $M$ denoted by $TM$, and its
complexification $TM \otimes \mathbb{C}$ denoted by $T^C M$. By extending the almost complex structure
$C$-linearly to $T^C M$, we obtain the decomposition

$$T^C M = T'M \oplus T''M,$$  \hspace{1cm} (2.1)
where $T'M$ and $T''M$ are the eigenspaces of $J$ corresponding to eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Similarly, by extending $J$ to forms, one can decompose $k$-forms into a sum of $(p, q)$-forms with $p + q = k$. Extending the almost Hermitian metric $\mathbb{C}$-linearly to $T\mathbb{C}M$, then it induces a Hermitian metric on $T'M$, still denoted by $g$. Locally, we can choose a unitary frame $\{e_1, \ldots, e_m\}$ with the dual coframe $\{\theta^1, \ldots, \theta^m\}$ so that

$$g = \theta^i \otimes \overline{\theta^i}; \quad dV_g = (\sqrt{-1})^m \theta^1 \wedge \overline{\theta^1} \wedge \cdots \wedge \theta^m \wedge \overline{\theta^m}. \quad (2.2)$$

The fundamental 2-form corresponding to the Hermitian metric $g$ is given by

$$\omega = \sqrt{-1} \theta^i \wedge \overline{\theta^i}. \quad (2.3)$$

Let $\nabla$ be an affine connection on $TM$, which is called \textit{almost-Hermitian} if it satisfies

$$\nabla J = 0, \quad \nabla g = 0. \quad (2.4)$$

It is easy to see that such connections always exist on any almost Hermitian manifold. One can refer to [4], [5] for details. Locally, it follows from (2.4) that the connection 1-forms $\{\theta^i_1\}$ are given by

$$\nabla e_i = \theta^j_1 e_j, \quad \theta^j_1 + \overline{\theta^j_1} = 0. \quad (2.5)$$

The first and second structure equations of $\nabla$ are determined by

$$d\theta^i = -\theta^i_j \wedge \theta^j + \Theta^i, \quad (2.6)$$

$$d\theta^j_1 = -\theta^k_1 \wedge \theta^j_1 + \Omega^j_1, \quad \Omega^j_1 + \overline{\Omega^j_1} = 0, \quad (2.7)$$

where the 2-forms $\Theta^i, \Omega^j_1$ are called \textit{torsion} and \textit{curvature} of $\nabla$, respectively. It is well known (see [5]) that there exists a unique almost Hermitian connection $\nabla$ on $T'M$ whose $(1, 1)$-part satisfies

$$(\Theta^i)^{(1,1)} = 0. \quad (2.8)$$

which is called \textit{canonical connection} on $(M, J, g)$. It is just the Chern connection when $J$ is integrable. We will always use $\nabla$ to stand for the canonical connection on almost Hermitian manifold.

There are several important families of almost Hermitian manifold defined by using the fundamental 2-form, Levi-Civita connection and almost complex structure. We say that $(M, J, g)$ is \textit{almost-Kählerian} if $d\omega = 0$, that it is \textit{quasi-Kählerian} if $(d\omega)^{(1,2)} = 0$, and that it is \textit{semi-Kählerian} if $d^*\omega = 0$, where $d^*$ is the codifferential operator of $d$ with respect to the metric $g$. To characterize them, we set

$$\Theta^i := L^i_{jk} \theta^j \wedge \theta^k + N^i_{jk} \overline{\theta^j} \wedge \overline{\theta^k} \quad (2.9)$$

with $L^i_{jk} = -L^i_{kj}, \quad N^i_{jk} = -N^i_{kj}$. Then, in terms of $L^i_{jk}$ and $N^i_{jk}$, it is easy to check that $(M, J, g)$ is almost-Kählerian if and only if

$$L^i_{jk} = 0 \quad \text{and} \quad N^i_{jk} + N^j_{ik} + N^k_{ij} = 0, \quad (2.10)$$

it is quasi-Kählerian if and only if

$$L^i_{jk} = 0, \quad (2.11)$$

and it is semi-Kählerian if and only if

$$L_j := \sum_i L^i_{ij} = 0. \quad (2.12)$$

On the other hand, by using the Levi-Civita connection, we call $(M, J, g)$ is \textit{nearly-Kählerian} if $(DXJ)X = 0$ for any tangent vector $X$ on $M$. This is equivalent to

$$L^i_{jk} = 0 \quad \text{and} \quad N^i_{jk} = N^j_{ik}. \quad (2.13)$$
We also set
\[ \Omega^i_j := R^i_{jk\ell} \theta^k \wedge \theta^\ell + R^i_{jk\ell} \omega^k \wedge \theta^\ell + R^i_{jk\ell} \theta^k \wedge \theta^\ell \] (2.14)
with \( R^i_{jk\ell} = -R^i_{j\ell k}, R^{i\ell}_{jk} = -R_{j\ell k}^i \). The skew-Hermitian of curvature forms \( \Omega^i_j \) implies
\[ R^i_{jk\ell} = \Omega^i_{jk\ell}, \quad R^{i\ell}_{jk} = \Omega^{i\ell}_{jk}, \] (2.15)
For a nonzero vectors \( X \in T'M, \) we call \( HS(X) := R^i_{jk\ell} X^i \overline{X^j} \overline{X^k} / \|X\|^4 \) the holomorphic sectional curvature of canonical connection in direction \( X, \) and for two nonzero vectors \( X, Y \in T'M \) we call \( HB(X, Y) := R^i_{jk\ell} X^i \overline{X^j} \overline{X^k} / (\|X\|^2 \|Y\|^2) \) the holomorphic bisectional curvature of canonical connection in the directions \( X, Y. \) We say the holomorphic bisectional curvature is bounded below by \( A \) and bounded above by \( B, \) if \( A \leq HB(X, Y) \leq B \) for all nonzero \((1,0)\)-type vectors \( X, Y. \) The tensors \( R^{i\ell}_{jk} := R^i_{j\ell k}, R^i_{j\ell k} := R^{i\ell}_{jk} \) are called the first Ricci curvature, second Ricci curvature of the canonical connection, which will be denoted by \( Ric'_M, Ric''_M, \) respectively. We say the first (resp. second) Ricci curvature is bounded below by \( A \) if \( R^{i\ell}_{jk} X^i \overline{X^j} \geq A \|X\|^2 \) (resp. \( R^i_{j\ell k} X^i \overline{X^j} \geq A \|X\|^2 \)) for all \((1,0)\)-type vector \( X. \) Similar bounds of torsion and \((2,0)\)-part of curvature can be defined. We will say \((M, J, g)\) has bounded geometry if its curvature, covariant derivatives of curvature are uniformly bounded.

Let \( u \) be a smooth function on \( M, \) its differential can be written as
\[ du = u_\theta \theta^i + u_\theta \bar{\theta}^i, \] (2.16)
where \( u_\theta = \overline{u_\theta}. \) Taking the exterior derivative of \((2.10)\) and using \((2.6),\) we obtain
\[ (u_{i,j} \theta^i + u_{i,j} \bar{\theta}^i) \wedge \theta^i + u_i \Theta^i + (u_{i,j} \theta^j + u_{i,j} \bar{\theta}^j) \wedge \bar{\theta}^i + u_\theta \bar{\theta}^i = 0, \] (2.17)
where \( u_{i,j}, u_{i,j}, u_{i,j}, u_{i,j} \) are defined by
\[ u_{i,j} \theta^i + u_{i,j} \bar{\theta}^i := du_i - u_j \theta_i^j, \] (2.18)
\[ u_{i,j} \bar{\theta}^i + u_{i,j} \bar{\theta}^i := du_i - u_j \bar{\theta}_i^j. \] (2.19)
Comparing the coefficients of \((2.17),\) we obtain
\[ u_{i,j} - u_{j,i} = 2 u_k L^k_{i,j} + 2 u_{i,k} \overline{L^k_{i,j}}, \] (2.20)
\[ u_{i,j} - u_{j,i} = 0. \] (2.21)
The canonical Hessian of a smooth function \( u \) on \( M \) is defined by \( \nabla du, \) which can be expressed as
\[ \nabla du = u_{i,j} \theta^i \otimes \theta^j + u_{i,j} \bar{\theta}^i \otimes \theta^j + u_{i,j} \theta^i \otimes \bar{\theta}^i + u_{i,j} \bar{\theta}^i \otimes \bar{\theta}^i. \] (2.22)
By taking the trace of \((2.22),\) the canonical Laplacian of \( u \) is defined by
\[ \Box u := \sum_i (u_{i,i} + u_{i,i}) = 2 \sum_i u_{i,i}. \] (2.23)
We call a smooth function \( u \) is strictly plurisubharmonic if \( \nabla du(X, \overline{X}) = u_{i,j} X^i \overline{X}^j > 0 \) for any nonzero vector \( X \in T'M. \)

Let \( \Delta \) be the usual Laplacian related to Levi-Civita connection of \( g. \) The difference (see \([17,18])\) between \( \Delta \) and \( \Box \) are given by
\[ \Delta u = \Box u - 2 \langle \nabla u, X_L \rangle, \] (2.24)
for a function \( u \in C^2(M), \) where \( \nabla u = u_\theta e_i + u_\theta \bar{\theta}, X_L = \overline{L_i} e_i + L_i \overline{e_i} \) and \( L_i \) defined in \((2.12).\) Similarly, for a vector field \( X = X^k e_k + \overline{X^k} \overline{e_k}, \) by using the canonical connection and Levi-Civita connection, one can define the canonical divergence and usual divergence
of $X$, denoted by $\text{div}^c(X)$ and $\text{div}(X)$ respectively. It is easy to check that the difference of such two divergences is given by

$$\text{div}(X) = \text{div}^c(X) - 2\langle X, X_L \rangle.$$  \hfill (2.25)

Thus, these two Laplacian and two divergences are equal respectively, when $(M, J, g)$ is semi-Kählerian, or quasi-Kählerian, or almost-Kählerian, or nearly-Kählerian.

### 2.2. Smooth map between almost Hermitian manifolds

Let $(M, J, g)$ and $(\overline{M}, \overline{J}, \overline{g})$ be two almost Hermitian manifolds of dimension $2m$ and $2n$, with the canonical connections $\nabla$, $\overline{\nabla}$, respectively. Let $f$ be a smooth map from $M$ into $\overline{M}$. Locally, choosing a unitary frame $\{e_i\}$ with the dual coframe $\{\theta^i\}$ on $M$, and choosing a unitary frame $\{\overline{e}_\alpha\}$ with the dual coframe $\{\overline{\theta}^\alpha\}$ on $\overline{M}$. Set

$$f^*\overline{\theta}^\alpha := a_i^\alpha \theta^i + a_i^\alpha \overline{\theta}^i, \quad f^*\theta^i = a_i^\alpha \theta^i + a_i^\alpha \overline{\theta}^i,$$

where $a_i^\alpha$, $a_i^\alpha$, $\overline{a}_i^\alpha$, $\overline{a}_i^\alpha$ satisfy

$$\overline{a}_i^\alpha = a_i^\alpha, \quad \overline{a}_i^\alpha = a_i^\alpha. \hfill (2.26)$$

Taking the exterior derivative of the first identity in (2.26), using the first structure equations of $\nabla$ and $\overline{\nabla}$, we obtain

$$(a_{i,j}^\alpha \theta^j + a_{i,j}^\alpha \overline{\theta}^j) \wedge \theta^i + (a_{i,j}^\alpha \theta^j + a_{i,j}^\alpha \overline{\theta}^j) \wedge \overline{\theta}^i + a_p^\alpha \Theta^p + a_p^\alpha \overline{\Theta}^p - \overline{\Theta}^\alpha = 0,$$

where $a_{i,j}^\alpha$, $a_{i,j}^\alpha$, $a_{i,j}^\alpha$, $a_{i,j}^\alpha$ are defined by

$$a_{i,j}^\alpha \theta^j + a_{i,j}^\alpha \overline{\theta}^j := da_i^\alpha - a_j^\alpha \theta^i + a_j^\beta \overline{\theta}^\beta, \hfill (2.27)$$

$$a_{i,j}^\alpha \theta^j + a_{i,j}^\alpha \overline{\theta}^j := da_i^\alpha - a_j^\alpha \theta^i + a_j^\beta \overline{\theta}^\beta. \hfill (2.28)$$

Here we have omitted the pull-back $f^*$ acts on $\overline{\Theta}^\alpha$, $\overline{\theta}^\alpha$, and similar conventions will be used in the sequel. Comparing the coefficients of (2.28), we obtain

$$a_{i,j}^\alpha - a_{i,j}^\alpha = 2a_p^\alpha P_{ij}^p + 2a_p^\alpha N_{ij}^p - 2a_i^\beta a_j^\gamma \tilde{L}_{\beta\gamma} - 2a_i^\beta a_j^\gamma \overline{N}_{\beta\gamma}$$

$$a_{i,j}^\alpha - a_{i,j}^\alpha = 2a_p^\alpha P_{ij}^p + 2a_p^\alpha N_{ij}^p - 2a_i^\beta a_j^\gamma \tilde{L}_{\beta\gamma} - 2a_i^\beta a_j^\gamma \overline{N}_{\beta\gamma}, \hfill (2.29)$$

$$(2.26)$$

Taking the exterior differential of (2.29), one have

$$(a_{i,j,k}^\alpha \theta^k + a_{i,j,k}^\alpha \overline{\theta}^k) \wedge \theta^i + (a_{i,j,k}^\alpha \theta^j + a_{i,j,k}^\alpha \overline{\theta}^j) \wedge \overline{\theta}^i + a_{i,j,k}^\alpha \Theta^p + a_{i,j,k}^\alpha \overline{\Theta}^p = -a_p^\alpha \Omega^p + a_i^\beta \overline{\Omega}^\beta,$$

which implies

$$a_{i,j,k}^\alpha - a_{i,k,j}^\alpha = 2a_p^\alpha R_{ijk} - 2a_i^\beta a_j^\gamma a_k^\delta \tilde{R}_{\beta\gamma\delta} - a_i^\beta (a_j^\gamma a_k^\delta - a_k^\gamma a_j^\delta) \tilde{R}_{\beta\gamma\delta}$$

$$- 2a_i^\beta a_j^\gamma a_k^\delta \tilde{R}_{\gamma\delta} + 2a_i^\beta a_j^\gamma a_k^\delta \overline{N}_{\beta\gamma\delta} + 2a_i^\beta a_j^\gamma a_k^\delta \overline{N}_{\beta\gamma\delta}$$

$$a_{i,j,k}^\alpha - a_{i,k,j}^\alpha = 2a_p^\alpha R_{ijk} - 2a_i^\beta a_j^\gamma a_k^\delta \tilde{R}_{\beta\gamma\delta} - a_i^\beta (a_j^\gamma a_k^\delta - a_k^\gamma a_j^\delta) \tilde{R}_{\beta\gamma\delta}$$

$$- 2a_i^\beta a_j^\gamma a_k^\delta \tilde{R}_{\gamma\delta} + 2a_i^\beta a_j^\gamma a_k^\delta \overline{N}_{\beta\gamma\delta} + 2a_i^\beta a_j^\gamma a_k^\delta \overline{N}_{\beta\gamma\delta}, \hfill (2.30)$$

Taking the exterior differential of (2.30), we obtain

$$a_{i,j,k}^\alpha \theta^k + a_{i,j,k}^\alpha \overline{\theta}^k := da_{i,j}^\alpha - a_{i,k}^\alpha \theta^i - a_{i,j}^\alpha \theta^i + a_{i,j}^\beta \overline{\theta}^\beta, \hfill (2.31)$$

$$a_{i,j,k}^\alpha \theta^k + a_{i,j,k}^\alpha \overline{\theta}^k := da_{i,j}^\alpha - a_{i,k}^\alpha \theta^i - a_{i,j}^\alpha \theta^i + a_{i,j}^\beta \overline{\theta}^\beta. \hfill (2.32)$$

$$(2.32)$$

where $a_{i,j,k}^\alpha$, $a_{i,j,k}^\alpha$, $a_{i,j,k}^\alpha$, $a_{i,j,k}^\alpha$ defined by

$$a_{i,j,k}^\alpha \theta^k + a_{i,j,k}^\alpha \overline{\theta}^k := da_{i,j}^\alpha - a_{i,p}^\alpha \theta^i - a_{i,j}^\alpha \theta^i + a_{i,j}^\beta \overline{\theta}^\beta, \hfill (2.33)$$

Taking the exterior differential of (2.33), we obtain

$$a_{i,j,k}^\alpha \theta^k + a_{i,j,k}^\alpha \overline{\theta}^k := da_{i,j}^\alpha - a_{i,p}^\alpha \theta^i - a_{i,j}^\alpha \theta^i + a_{i,j}^\beta \overline{\theta}^\beta.$$
\[ a^\alpha_{i,jk} \theta^k + a^\alpha_{i,jk} \overline{\theta}^k := da^\alpha_{i,j} - a^\alpha_{p,j} \theta^i - a^\alpha_{i,p} \overline{\theta}^i + a^\beta_{i,j} \overline{\theta}^\beta. \] (2.39)

Taking the exterior derivative of (2.30), we have
\[ (a^\alpha_{i,jk} \theta^k + a^\alpha_{i,jk} \overline{\theta}^k) \wedge \theta^j + (a^\alpha_{i,jk} \theta^k + a^\alpha_{i,jk} \overline{\theta}^k) \wedge \overline{\theta}^j + a^\alpha_{i,p} \Theta^p + a^\alpha_{i,p} \overline{\Theta}^p = -a^\alpha_{p,i} \overline{\theta}^p + a^\beta_{i,j} \overline{\Theta}^\beta, \] (2.40)
which implies
\[
\begin{align*}
\alpha_{i,jk}^\alpha - \alpha_{i,kj}^\alpha &= 2a^\alpha_{p,j} R_{i,jk}^p - 2a^\beta_{i,j} a^\gamma_k R_{j\beta\gamma\delta}^k - \alpha^\beta_i (a^\gamma_j a^\delta_k - a^\delta_j a^\gamma_k) R_{j\beta\gamma\delta}^k \\
&- 2a^\delta_{i,j} a^\gamma_k R_{j\beta\gamma\delta}^k + 2a^\alpha_{r,j} L_{r,j}^p + 2a^\alpha_{i,k} N_{i,k}^p, \\
\alpha_{i,jk}^\alpha - \alpha_{i,kj}^\alpha &= -a^\alpha_{p,j} R_{i,jk}^p - 2a^\beta_{i,j} a^\gamma_k R_{j\beta\gamma\delta}^k - \alpha^\beta_i (a^\gamma_j a^\delta_k - a^\delta_j a^\gamma_k) R_{j\beta\gamma\delta}^k \\
&- 2a^\delta_{i,j} a^\gamma_k R_{j\beta\gamma\delta}^k + 2a^\alpha_{r,j} L_{r,j}^p + 2a^\alpha_{i,k} N_{i,k}^p, \\
\alpha_{i,jk}^\alpha - \alpha_{i,kj}^\alpha &= 2a^\alpha_{p,j} R_{i,jk}^p - 2a^\beta_{i,j} a^\gamma_k R_{j\beta\gamma\delta}^k - \alpha^\beta_i (a^\gamma_j a^\delta_k - a^\delta_j a^\gamma_k) R_{j\beta\gamma\delta}^k \\
&- 2a^\delta_{i,j} a^\gamma_k R_{j\beta\gamma\delta}^k + 2a^\alpha_{r,j} L_{r,j}^p + 2a^\alpha_{i,k} N_{i,k}^p.
\end{align*}
\] (4.21)

where \( a^\alpha_{i,jk}, a^\alpha_{i,kj}, a^\alpha_{i,jk} \) and \( a^\alpha_{i,kj} \) defined by
\[
\begin{align*}
\alpha_{i,jk}^\alpha - \alpha_{i,kj}^\alpha &= 2a^\alpha_{p,j} R_{i,jk}^p - 2a^\beta_{i,j} a^\gamma_k R_{j\beta\gamma\delta}^k - \alpha^\beta_i (a^\gamma_j a^\delta_k - a^\delta_j a^\gamma_k) R_{j\beta\gamma\delta}^k \\
&- 2a^\delta_{i,j} a^\gamma_k R_{j\beta\gamma\delta}^k + 2a^\alpha_{r,j} L_{r,j}^p + 2a^\alpha_{i,k} N_{i,k}^p. \\
\alpha_{i,jk}^\alpha &= a^\alpha_{p,j} R_{i,jk}^p + a^\alpha_{p,j} R_{i,jk}^p - 2a^\beta_{i,j} a^\gamma_k R_{j\beta\gamma\delta}^k - \alpha^\beta_i (a^\gamma_j a^\delta_k - a^\delta_j a^\gamma_k) R_{j\beta\gamma\delta}^k \\
&- 2a^\delta_{i,j} a^\gamma_k R_{j\beta\gamma\delta}^k + 2a^\alpha_{r,j} L_{r,j}^p + 2a^\alpha_{i,k} N_{i,k}^p.
\end{align*}
\] (4.26)

where the covariant derivatives \( a^\alpha_{i,jk}, a^\alpha_{i,kj} \) defined in the natural way as before.

Similarly, taking the exterior derivative of \( (2.33) \), by comparing the (1,1)-part, one can get
\[
\begin{align*}
\alpha_{i,jk}^\alpha - \alpha_{i,kj}^\alpha &= a^\alpha_{p,j} R_{i,jk}^p + a^\alpha_{p,j} R_{i,jk}^p - 2a^\beta_{i,j} a^\gamma_k R_{j\beta\gamma\delta}^k - \alpha^\beta_i (a^\gamma_j a^\delta_k - a^\delta_j a^\gamma_k) R_{j\beta\gamma\delta}^k \\
&- 2a^\delta_{i,j} a^\gamma_k R_{j\beta\gamma\delta}^k + 2a^\alpha_{r,j} L_{r,j}^p + 2a^\alpha_{i,k} N_{i,k}^p.
\end{align*}
\] (4.26)

In terms of unitary frame, from (2.26), the differential of \( f \) can be expressed by
\[
df = a^\alpha_i \theta^i \otimes \hat{e}_\alpha + a^\alpha_i \overline{\theta}^i \otimes \hat{e}_\alpha + a^\alpha_i \theta^i \otimes \hat{e}_\alpha + a^\alpha_i \overline{\theta}^i \otimes \hat{e}_\alpha,
\] (4.27)
which is a smooth section of bundle \( T^*M \otimes f^{-1} T\hat{M} \) or \( (T^C M)^* \otimes f^{-1} T\hat{C} \hat{M} \). Taking the covariant differential of \( df \) by using the connection induced by the canonical connection \( \nabla \) and \( \hat{\nabla} \), one can get the canonical second fundamental form \( \nabla df \) of \( f \). More explicitly, we have
\[
\nabla df = a^\alpha_i \theta^i \otimes \hat{e}_\alpha + a^\alpha_i \overline{\theta}^i \otimes \hat{e}_\alpha + a^\alpha_i \theta^i \otimes \hat{e}_\alpha + a^\alpha_i \overline{\theta}^i \otimes \hat{e}_\alpha + a^\alpha_i \overline{\theta}^i \otimes \hat{e}_\alpha + a^\alpha_i \overline{\theta}^i \otimes \hat{e}_\alpha + a^\alpha_i \overline{\theta}^i \otimes \hat{e}_\alpha + a^\alpha_i \overline{\theta}^i \otimes \hat{e}_\alpha.
\] (4.28)

Such a map is called canonical geodesic if \( \nabla df = 0 \); and it is called pluriharmonic if \( (\nabla df)^{(1,1)} = 0 \), which means that \( a^\alpha_i = a^\alpha_{i,j} = a^\alpha_{i,j} = 0 \).

The partial energy densities \( e'(f), e''(f) \), energy density \( e(f) \) (see \( [3], [21] \)) of a smooth map \( f \) between almost Hermitian manifolds defined by
\[
e'(f) := \sum_{i,\alpha} |a^\alpha_i|^2, \quad e''(f) := \sum_{i,\alpha} |a^\alpha_i|^2, \quad e(f) := e'(f) + e''(f),
\] (4.29)
and the corresponding partial energies and energy of \( f \) defined by
\[
E'(f) := \int_M e'(f) dV_g, \quad E''(f) := \int_M e''(f) dV_g, \quad E(f) := \int_M e(f) dV_g.
\] (4.30)
respectively. A smooth map is called harmonic if it is a critical point of the energy functional $E$. For more details about harmonic map one can refer to the books [3], [21] and references therein.

Let $f$ be a smooth map between $(M, J, g)$ and $(\tilde{M}, \tilde{J}, \tilde{g})$, it is called pseudoholomorphic if its differential $df$ satisfies

$$df J = \tilde{J} df.$$  \hfill (2.51)

In terms of differential forms, one can check that the identity (2.51) is equivalent to

$$a^\alpha_i \alpha^\beta_j = 0,$$  \hfill (2.52)

which means that $f^*$ preserves the forms of $(1, 0)$-type. Thus, the definition (2.30) and the identity (2.32) imply

$$a^\alpha_{i,j} = a^\alpha_{i,j} = a^\alpha_{i,j} = 0.$$  \hfill (2.53)

This tell us that a pseudoholomorph map is naturally a pluriharmonic one.

### 3. Bochner Type Formulae

In this section we wish to calculate the canonical Laplacian of the partial energy densities of pluriharmonic maps, and the canonical Laplacian of half norm of canonical second fundamental form of pseudoholomorphic map.

**Proposition 3.1.** Let $f$ be a pluriharmonic map between almost Hermitian manifolds $(M, J, g)$ and $(\tilde{M}, \tilde{J}, \tilde{g})$, then we have

1. \( \frac{1}{2} \Box e' = |a^\alpha_{i,j}|^2 + a^\alpha_i \bar{a}^\beta_j R^\alpha_{i,j} - 4\text{Re}(a^\alpha_i \bar{a}^\beta_j a^\gamma_j \bar{a}^\delta_j \tilde{R}_j^\alpha_{i,j}) - a^\alpha_i \bar{a}^\beta_j (a^\gamma_j \bar{a}^\delta_j - \bar{a}^\gamma_j a^\delta_j) \tilde{R}_j^\alpha_{i,j}; \)

2. \( \frac{1}{2} \Box e'' = |a^\alpha_{i,j}|^2 + a^\alpha_i \bar{a}^\beta_j R^\alpha_{i,j} + 4\text{Re}(a^\alpha_i \bar{a}^\beta_j a^\gamma_j \bar{a}^\delta_j \tilde{R}_j^\alpha_{i,j}) + a^\alpha_i \bar{a}^\beta_j (a^\gamma_j \bar{a}^\delta_j - \bar{a}^\gamma_j a^\delta_j) \tilde{R}_j^\alpha_{i,j}. \)

**Proof.** Since $f$ is pluriharmonic, i.e., $a^\alpha_{i,j} = a^\alpha_{i,j} = 0$, we have

$$\frac{1}{2} \Box e' = (a^\alpha_i \bar{a}^\beta_j)_{i,j} = (a^\alpha_i \bar{a}^\beta_j + a^\alpha_i \bar{a}^\beta_j)_{i,j} = a^\alpha_{i,j} \bar{a}^\beta_j + a^\alpha_{i,j} \bar{a}^\beta_j. \quad (3.1)$$

By using (2.36) with $a^\alpha_{i,j} = 0$, we obtain

$$a^\alpha_{i,j} = a^\alpha_i \bar{a}^\beta_j - 2a^\alpha_i \bar{a}^\beta_j \bar{a}^\gamma_j \tilde{R}_j^\alpha_{i,j} - a^\beta_j (a^\gamma_j \bar{a}^\delta_j - \bar{a}^\gamma_j a^\delta_j) \tilde{R}_j^\alpha_{i,j} - 2a^\gamma_j \bar{a}^\delta_j \bar{a}^\alpha_i \tilde{R}_j^\alpha_{i,j}. \quad (3.2)$$

On the other hand, by (2.15), we have

$$a^\alpha_i \bar{a}^\beta_j \bar{a}^\gamma_j \tilde{R}_j^\alpha_{i,j} = a^\alpha_i \bar{a}^\beta_j \bar{a}^\gamma_j \tilde{R}_j^\alpha_{i,j} = a^\alpha_i \bar{a}^\beta_j \bar{a}^\gamma_j \tilde{R}_j^\alpha_{i,j}. \quad (3.3)$$

The first identity follows from (3.1), (3.2) and (3.3). Similar caculations do for the second identity.

q.e.d

The following Bochner formula belongs to V.Tosatti, which appeared in his work on Schwarz Lemma of pseudoholomorphic map between almost Hermitian manifolds.

**Corollary 3.2.** (17) Let $f$ be a pseudoholomorphic map between almost Hermitian manifolds $(M, J, g)$ and $(\tilde{M}, \tilde{J}, \tilde{g})$, we have

$$\frac{1}{2} \Box e = |a^\alpha_{i,j}|^2 + (a^\alpha_i \bar{a}^\beta_j - a^\gamma_j \bar{a}^\delta_j \tilde{R}_j^\alpha_{i,j}. \quad (7)$$
Proof. Since a pseudoholomorphic map is pluriharmonic, it follows from that \( a_i^\alpha = 0 \) and \( e = e' \).

q.e.d

For the square of the length second fundamental form of pseudoholomorphic maps, we have

**Proposition 3.3.** Let \( f \) be a pseudoholomorphic map between almost Hermitian manifolds \((M, J, g)\) and \((\tilde{M}, \tilde{J}, \tilde{g})\), and let \( S \) be the half norm of the second fundamental form of \( f \). Then

\[
\frac{1}{2} \Box S = |a_{i,jk}^\alpha|^2 + |a_{i,jk}^\alpha|^2 + a_{i,jk}^\alpha (a^{\alpha p}_{j,k} R_{i}^{\alpha p} + a^{\alpha p}_{i,j} R_{k}^{\alpha p} + 2 a^{\alpha}_{i,p,k} R_{j,k} - a^{\beta}_{i,j} a^{\alpha}_{k} \tilde{R}^{\alpha}_{\beta \gamma \delta})
\]

\[
+ 2 \text{Re}(a^{\alpha}_{i,j} a^{\alpha}_{p,k} R_{j,k} - a^{\alpha}_{i,j} a^{\alpha}_{p,k} \tilde{R}^{\alpha}_{\beta \gamma \delta}).
\]

**Proof.** Under the unitary frame field, we have \( S = |a_{i,j}^\alpha|^2 \), so

\[
\frac{1}{2} \Box S = (a_{i,jk}^\alpha a_{i,jk}^\alpha)_{,k} =
\]

\[
(a_{i,jk}^\alpha a_{i,jk}^\alpha + a_{i,jk}^\alpha a_{i,jk}^\alpha)_{,k}
\]

\[
|a_{i,jk}^\alpha|^2 + |a_{i,jk}^\alpha|^2 + (a_{i,jk}^\alpha)_{,k} a_{i,jk}^\alpha + a_{i,jk}^\alpha (a_{i,jk}^\alpha)_{,k}.
\]

(3.4)

By using the pseudoholomorphicity, the Ricci identities (2.36) and (2.46) give

\[
(a_{i,jk}^\alpha)_{,k} = (a_{i,jk}^\alpha)_{,k} + a_{i,p,k} R_{j,k} - a_{i,p,k} R_{j,k} - a_{i,jk}^\alpha a_{i,jk}^\alpha.
\]

(3.5)

Then the half of the canonical Laplacian of \( S \) is follows from (3.4), (3.5), (3.6) and the curvature properties (2.15).

q.e.d

Applying the Bochner formulae above, we can get a \( C^2 \)-estimate of pseudoholomorphic map. That is

**Theorem 3.4.** Let \( f \) be a pseudoholomorphic from closed almost Hermitian manifold \((M, J, g)\) into almost Hermitian manifold \((\tilde{M}, \tilde{J}, \tilde{g})\) with bounded geometry. If there is a constant \( \Lambda \) such that \( e(f) \leq \Lambda \), then

\[
S \leq C,
\]

where \( C \) is a constant only depend on \( \Lambda \), the curvatures and their covariant derivatives of \( g, \tilde{g} \).

**Proof.** It follows from Corollary 3.2 and Proposition 3.3, there exist constants \( C_1, C_2 \) and \( C_3 \) only depend on \( \Lambda \), the corresponding curvatures and their covariant derivatives such that

\[
\frac{1}{2} \Box e(f) \geq S - C_1,
\]

\[
\frac{1}{2} \Box S \geq -C_2 S - C_3,
\]

which imply

\[
\frac{1}{2} \Box (S + (1 + C_2) e(f)) \geq S - (C_1 + C_2 + C_3).
\]

By using the maximum principle to \( S + (1 + C_2) e(f) \) we can get the desired upper bound of \( S \).

q.e.d
4. Liouville type theorems

We will use the properties of pseudoholomorphic maps and the Bochner formulæ to get Liouville type results, which are similar to the theory of harmonic map. The main tools are the maximum principle and integral estimates.

**Theorem 4.1.** Let \( f \) be a pseudoholomorphic map from closed almost Hermitian manifold \((M, J, g)\) into almost Hermitian manifold \((\tilde{M}, \tilde{J}, \tilde{g})\), and \( f(M) \subset V \). If \( u \) is a strictly plurisubharmonic function on \( V \), then \( f \) is constant.

*Proof.* By using the pseudoholomorphicity, we have \( a_\alpha^\beta = 0 \) and \( a_{\alpha,j}^\beta = 0 \). So, the canonical Laplacian of \( u \circ f \) is given by

\[
\frac{1}{2} \Box (u \circ f) = (u \circ f)_{,k\bar{k}} \\
= (u_\alpha a_\alpha^\alpha + u_{,\bar{k}} \overline{a_\alpha^\alpha})_{,k\bar{k}} \\
= (u_\alpha a_\alpha^\alpha)_{,k} \\
= (u_\alpha,\beta a_\beta^\alpha + u_{,\alpha,\bar{k}} \overline{a_\alpha^\alpha}) a_k^\alpha + u_\alpha a_{\alpha,k}^\alpha \\
= u_\alpha a_\alpha^\alpha a_k^\alpha. 
\]

Notice that \( u \) is strictly plurisubharmonic, so the maximum principle implies that \( e(f) = 0 \), i.e., \( f \) is a constant.

*q.e.d*

**Theorem 4.2.** Let \( f \) be a pseudoholomorphic map from closed almost Hermitian manifold \((M, J, g)\) into almost Hermitian manifold \((\tilde{M}, \tilde{J}, \tilde{g})\) with \( \text{Ric}''_M \geq 0 \) and \( H B_{\tilde{M}} \leq 0 \). Then \( f \) is canonical totally geodesic. Furthermore, \( f \) is constant if \( \text{Ric}''_M \) is positive at a point in \( M \).

*Proof.* Recall the Bochner formulæ obtained in Corollary 3.2, that is

\[
\frac{1}{2} \Box e = |a_{i,j}^\alpha|^2 + a_i^\alpha a_j^\alpha R''_{ij} - a_i^\alpha a_j^\beta a_j^\alpha a_j^\beta R''_{ij}. \tag{4.1}
\]

The curvature conditions imply that \( \Box e \geq 0 \), so the maximum principle implies \( e \) is a constant. At this time, the terms in the right hand side of (4.1) are nonnegative and their summation is equal to zero, so we obtain \( |a_{i,j}^\alpha|^2 = 0 \), i.e., \( \nabla df = 0 \). If \( \text{Ric}''_M \) is positive at a point, the second term in the right hand side of (4.1) implies \( e = 0 \).

*q.e.d*

We can also obtain a similar result when the holomorphic bisectional curvature of target manifold might be positive.

**Theorem 4.3.** Let \((M, J, g), (\tilde{M}, \tilde{J}, \tilde{g})\) be two almost Hermitian manifolds with \( M \) is closed, and \( \text{Ric}''_M \geq A, HB_{\tilde{M}} \leq B \), where \( A, B \) are positive constants. If the pseudoholomorphic map \( f : M \to \tilde{M} \) satisfies \( e(f) \leq A \), then \( f \) is canonical totally geodesic and \( e(f) = 0 \) or \( e(f) = \frac{A}{B} \). Furthermore, \( f \) is constant if \( e(f) < \frac{A}{B} \).

*Proof.* By the curvature conditions, the Bochner formulæ (4.1) implies

\[
\frac{1}{2} \Box e(f) \geq |a_{i,j}^\alpha|^2 + (A - B e(f)) e(f). 
\]

So, if \( e(f) \leq \frac{A}{B} \), we have \( \Box e(f) \geq 0 \), then the maximum principle implies \( e(f) \) is a constant, \( |a_{i,j}^\alpha|^2 = 0 \) and \( (A - B e(f)) e(f) = 0 \). Thus, we have \( e(f) = 0 \) if \( e(f) < \frac{A}{B} \), which means that \( f \) is constant.
For the domain manifold is complete and non-compact, we can obtain the following Liouville type result provided the energy is finite.

**Theorem 4.4.** Let \((M, J, g)\) be a complete, non-compact almost Hermitian manifold with infinite volume and \(\text{Ric}_M^g \geq 0\), let \((\tilde{M}, \tilde{J}, \tilde{g})\) be an almost Hermitian manifold with \(\tilde{H}B_{\tilde{M}} \leq 0\). If the pseudoholomorphic \(f : M \to \tilde{M}\) has finite energy and

\[
\int_M \langle \nabla e(f), X_L \rangle dV_g = 0,
\]

then \(f\) is constant.

**Proof.** The Bochner formula (4.1) and Cauchy-Schwarz inequality imply

\[
\Box e(f) \geq 2S; \quad |\nabla e(f)|^2 \leq 2e(f) S;
\]

respectively. For any \(\epsilon > 0\), set \(u_\epsilon := \sqrt{e(f) + \epsilon}\), then we have

\[
\frac{1}{2} \Box u_\epsilon = (u_\epsilon)_{,\xi} \xi = \frac{1}{4} u_\epsilon^{-1} (\Box e(f) - \frac{|\nabla e(f)|^2}{2(e(f) + \epsilon)}).
\]

Together with (4.2), this implies

\[
\Box u_\epsilon \geq \frac{1}{2} u_\epsilon^{-1} S (1 - \frac{e(f)}{2(e(f) + \epsilon)}) \geq 0. \tag{4.3}
\]

Let \(x_0 \in M\) be a fixed point. We denote the geodesic balls centered at \(x_0\) with radii \(R, 2R\) by \(B_R, B_{2R}\), respectively. Choosing a cut-off function \(\eta\) such that \(0 \leq \eta \leq 1, \eta(x) = 1\) for \(x \in B_R, \eta(x) = 0\) for \(x \in M \setminus B_{2R}\). Moreover, we further assume that \(|\nabla \eta| \leq \frac{C}{R}\), where \(C\) is a positive constant. By using (2.24), the inequality (4.3) gives

\[
0 \leq \int_{B_{2R}} \eta^2 u_\epsilon \Box u_\epsilon dV_g
\]

\[
= \int_{B_{2R}} \eta^2 u_\epsilon (\Delta u_\epsilon + 2\langle \nabla u_\epsilon, X_L \rangle) dV_g
\]

\[
= -2 \int_{B_{2R}} \eta u_\epsilon \langle \nabla \eta, \nabla u_\epsilon \rangle dV_g - \int_{B_{2R}} \eta^2 |\nabla u_\epsilon|^2 + \int_{B_{2R}} \eta^2 \langle \nabla e(f), X_L \rangle dV_g
\]

\[
\leq 2 \left( \int_{B_{2R} \setminus B_R} \eta^2 |\nabla u_\epsilon|^2 dV_g \right)^{1/2} \left( \int_{B_{2R} \setminus B_R} u_\epsilon^2 |\nabla \eta|^2 dV_g \right)^{1/2} - \int_{B_{2R} \setminus B_R} \eta^2 |\nabla u_\epsilon|^2 dV_g
\]

\[
- \int_{B_R} |\nabla u_\epsilon|^2 dV_g + \int_{B_{2R}} \eta^2 \langle \nabla e(f), X_L \rangle dV_g. \tag{4.4}
\]

If we view (4.4) as a quadratic inequality of the term \(\left( \int_{B_{2R} \setminus B_R} \eta^2 |\nabla u_\epsilon|^2 dV_g \right)^{1/2}\), we get

\[
\int_{B_R} |\nabla u_\epsilon|^2 dV_g - \int_{B_{2R}} \eta^2 \langle \nabla e(f), X_L \rangle dV_g \leq \int_{B_{2R} \setminus B_R} u_\epsilon^2 |\nabla \eta|^2 dV_g
\]

\[
\leq \frac{C^2}{R^2} \int_{B_{2R}} u_\epsilon^2 dV_g. \tag{4.5}
\]
Notice that $|\nabla u_\epsilon|^2 = \frac{(|e(f)|^2)}{4(e(f)+\epsilon)}$, the inequality (1.5) implies
\[
\int_{\Sigma \cap B_R} \frac{|\nabla e(f)|^2}{4(e(f)+\epsilon)} \, dV_g - \int_{B_{2R}} \eta^2 \langle \nabla e(f), X_L \rangle \, dV_g \leq \frac{C^2}{R^2} \int_{B_{2R}} (e(f)+\epsilon) \, dV_g, \tag{4.6}
\]
where $\Sigma := \{x \in M \mid e(f)(x) \neq 0\}$. Letting $\epsilon \to 0$, $R \to +\infty$ and using $f$ has finite energy, we obtain
\[
\int_{\Sigma} \frac{|\nabla e(f)|^2}{4e(f)} \, dV_g \leq 0. \tag{4.7}
\]
This tells us that $e(f)$ is a constant. Hence, $e(f) = 0$ from the facts that $f$ has finite energy and $(M, g)$ has infinite volume.

Remark. The condition that $(M, g)$ has infinite volume can be replaced by a curvature condition. The integration condition is always satisfied when $(M, J, g)$ is semi-Kählerian, quasi-Kählerian, or almost-Kählerian, or nearly-Kählerian by the fact that $X_L = 0$.

5. PSEUDOHOLOMORPHICITY OF PLURIHARMONIC MAPS

We will investigate the pseudoholomorphicity of pluriharmonic maps by using the Bochner formulae and maximum principle. Two cases with respect to the topology of domain manifold will be considered.

Theorem 5.1. Let $(M, J, g)$ be a closed almost Hermitian manifold with $\text{Ric}_M' \geq A$, and let $(\tilde{M}, \tilde{J}, \tilde{g})$ be an almost Hermitian manifold with $\text{HB}_{\tilde{M}} \leq B$ and the $(2,0)$-part of the curvature is bounded by $C$. If a pluriharmonic map $f : M \to \tilde{M}$ satisfies $(B + 2C) e(f) < A$, then $f$ is pseudoholomorphic.

Proof. Under the curvature conditions, the second Bochner formlue in Proposition 3.1 gives
\[
\frac{1}{2} \Box''(f) \geq |a_{\tilde{J}J}|^2 + e''(f) \left( A - 4C \sqrt{e''(f)} \sqrt{e'(f) - B e(f)} \right) \\
\geq |a_{\tilde{J}J}|^2 + e''(f) \left( A - 2C e(f) - B e(f) \right). \tag{5.1}
\]
By using the maximum principle, the differential inequality (5.1) implies
\[
e''(f) \left( A - 2C e(f) - B e(f) \right) = 0.
\]
Thus, we have $e''(f) = 0$ by the fact that $(B + 2C) e(f) < A$. q.e.d

To proceed, we need the following maximum principle (Proposition 4.1 in [17]) due to V. Tosatti.

Lemma 5.2. ([17]) Let $(M, J, g)$ be a complete almost Hermitian manifold with second Ricci curvature bounded below, with torsion and $(2,0)$-part of the curvature bounded. Let $u$ be a real function that is bounded from below. Then given any $\epsilon > 0$ there exists a point $x_\epsilon \in M$ such that
\[
\liminf_{\epsilon \to 0} u(x_\epsilon) = \inf_{\tilde{M}} u, \quad |\nabla u|(x_\epsilon) \leq \epsilon, \quad \Box u(x_\epsilon) \geq -\epsilon.
\]

Theorem 5.3. Let $(M, J, g)$ be a complete almost Hermitian manifold with $\text{Ric}_M'' \geq A$ and with torsion and $(2,0)$-part of the curvature bounded by $C$. Let $(\tilde{M}, \tilde{J}, \tilde{g})$ be an almost Hermitian manifold with $B_1 \leq \text{HB}_{\tilde{M}} \leq B_2$. If a pluriharmonic map $f : M \to \tilde{M}$
satisfies \((B_2 + 2C)e''(f) \leq A + (B_1 - 2C)e'(f) - \delta\) for a positive constant \(\delta > 0\), then \(f\) is pseudoholomorphic.

**Proof.** For any \(\lambda > 0\), set \(u_\lambda := (e''(f) + \lambda)^{-1/2}\), through direct calculation, we have

\[
\Box u_\lambda = 3 u_\lambda^{-1} |\nabla u_\lambda|^2 - \frac{1}{2} u_\lambda^3 \Box e''(f). \tag{5.2}
\]

On the other hand, by using the second Bochner formula in Proposition 4.1, we obtain

\[
\frac{1}{2} \Box e''(f) \geq A e''(f) - 4C e'(f)^{1/2} (e''(f))^{3/2} + B_1 e'(f) e''(f) - B_2 (e''(f))^2
\]

\[
= e''(f) \left[ A - 4C e'(f)^{1/2} (e''(f))^{1/2} + B_1 e'(f) - B_2 e''(f) \right]
\]

\[
\geq e''(f) \left[ A + (B_1 - 2C) e'(f) - (B_2 + 2C) e''(f) \right]
\]

\[
\geq \delta e''(f). \tag{5.3}
\]

Applying the Lemma 5.2 to \(u_\lambda\), for any \(\epsilon > 0\), there exists a point \(x_\epsilon \in M\) such that

\[
u (x_\epsilon) \leq \inf_M u_\lambda + \epsilon, \quad |\nabla u_\lambda|(x_\epsilon) \leq \epsilon, \quad \Box u_\lambda(x_\epsilon) \geq -\epsilon.
\]

So, at point \(x_\epsilon\), the identity \((5.2)\) and the inequality \((5.3)\) imply

\[
\frac{\delta}{(e''(f) + \lambda)^2} = \delta e''(f) u_\lambda^3 \leq 3 \epsilon^2 + \epsilon u_\lambda
\]

\[
\leq 3 \epsilon^2 + \epsilon (\inf_M u_\lambda + \epsilon). \tag{5.4}
\]

Letting \(\epsilon \to 0\), then \(u_\lambda(x_\epsilon) \to \inf_M u_\lambda\) and hence \(e''(f)(x_\epsilon) \to \sup_M e''(f)\). Thus, the inequality \((5.4)\) implies that \(\sup_M e''(f) \leq 0\), i.e., \(f\) is pseudoholomorphic.

\[q.e.d\]

6. PSEUDOHOLONOMIC ISOMETRIC IMMERSION

The purpose of this section is to derive Simons integral inequality, to give bounds of the norm of parallel canonical second fundamental form, of the pseudoholomorphic isometry between the almost Hermitian manifolds. The idea is inspired from H.W.Xu’s work \([22]\) on closed minimal submanifold in pinched Riemannian manifolds. We will adopt the following range of indices:

\[1 \leq i, j, k, l \ldots \leq m; \quad m + 1 \leq \lambda, \mu, \nu, \sigma, \ldots \leq n; \quad 1 \leq \alpha, \beta, \gamma, \delta \ldots \leq n.\]

We first recall the structure equations of almost Hermitian manifold \((\widetilde{M}, \tilde{J}, \tilde{g})\). Locally, by choosing a unitary frame field \(\{\tilde{e}_\alpha\}\) with dual \(\{	ilde{\theta}^\alpha\}\), the first and second structure equations are given by

\[
d \tilde{\theta}^\alpha = -\tilde{\theta}_\beta^\alpha \wedge \tilde{\theta}^\beta + \tilde{\Theta}^\alpha, \quad \tilde{\theta}_\beta^\alpha + \tilde{\theta}_\alpha^\beta = 0, \tag{6.1}
\]

\[
d \tilde{\theta}_\beta^\alpha = -\tilde{\theta}_\gamma^\alpha \wedge \tilde{\theta}_\beta^\gamma + \tilde{\Omega}_\beta^\alpha, \quad \tilde{\Omega}_\beta^\alpha + \tilde{\Omega}_\alpha^\beta = 0, \tag{6.2}
\]

where \(\tilde{\theta}_\beta^\alpha, \tilde{\Theta}^\alpha, \tilde{\Omega}_\beta^\alpha\) are connection 1-forms, torsion and curvature forms respectively.

Let \(f\) be a pseudoholomorphic isometry from \((M, J, g)\) into \((\widetilde{M}, \tilde{J}, \tilde{g})\) with \(\dim M = 2m\) and \(\dim \widetilde{M} = 2n\). We choose an adapted unitary frame field \(\{\tilde{e}_\alpha\}\) along \(f\), which means that \(\{\tilde{e}_i\}\) are tangent to \(f(M)\) and \(\{\tilde{e}_\lambda\}\) are normal to \(f(M)\). So, we have

\[
f^* \tilde{\theta}^i = \theta^i, \quad f^* \tilde{\theta}^\lambda = 0. \tag{6.3}
\]
Similarly, taking $(\alpha, \beta)$ from (6.8), we have $
abla_j \alpha = 0$, $\nabla_i \theta^j = \alpha^j_i$ (6.5).

Taking $(\alpha, \beta) = (i, j)$ in (6.2), together with the first identities in (6.4), we obtain the Gauss equation

$$\Omega^i_j + (-\hat{\Omega}^i_j) \wedge \hat{\Omega}^j_i = 0,$$

which implies

$$R_{ijk}^i = a^\lambda_{j,k} \hat{\Gamma}^i_{jk} = \hat{R}_{ijk}^i. \quad (6.6)$$

and

$$R_{jk\ell}^i = \hat{R}_{jk\ell}^i, \quad R_{jk\ell}^i = \hat{R}_{jk\ell}^i \quad (6.7)$$

Similarly, taking $(\alpha, \beta) = (\lambda, \mu)$ in (6.2), we get the Ricci equation

$$\Omega^\lambda_{\mu} + \hat{\Omega}^\lambda_{\mu} \wedge (-\hat{\Omega}^\mu_{\lambda}) = 0,$$

where $\Omega^\lambda_{\mu}$ is the normal curvature forms defined by $\Omega^\lambda_{\mu} := d\hat{\Theta}^\lambda_{\mu} + \hat{\theta}^\lambda_{\mu} \wedge \hat{\Theta}^\mu_{\lambda}$. If we set

$$\Omega^\lambda_{\mu} := R^\lambda_{\mu jk} \theta^j \wedge \theta^k + R^\lambda_{\mu p k} \theta^j \wedge \theta^k + R^\lambda_{\mu j k} \theta^j \wedge \theta^k,$$

from (6.8), we have

$$R^\lambda_{\mu jk} = a^\lambda_{j,k} \hat{\Gamma}^i_{jk} = \hat{R}_{\mu jk}^\lambda. \quad (6.9)$$

$$R^\lambda_{\mu jk} = \hat{R}_{\mu jk}^\lambda, \quad R_{\mu jk}^\lambda = \hat{R}_{\mu jk}^\lambda. \quad (6.10)$$

We call $R^\lambda := \sum_{\mu, \lambda} R^\lambda_{\mu i}$ the normal scalar curvature of $M$ in $\tilde{M}$. The Codazzi equation is follows from (2.9)–(2.17) or by taking $(\alpha, \beta) = (\lambda, i)$ in (6.2), that is

$$a^\lambda_{i j k} = -\hat{R}_{i j k}^\lambda, \quad a^\lambda_{i j k} - a^\lambda_{i k j} = 2a^\lambda_{i, p} T^k_{p j k} - 2\hat{R}_{i j k}^\lambda, \quad a^\lambda_{i, p} N_{j k}^p = \hat{R}_{i j k}^\lambda. \quad (6.11)$$

There are some basic properties of pseudoholomorphic isometry.

**Proposition 6.1.** Let $(M, J, g)$, $(\tilde{M}, \tilde{J}, \tilde{g})$ be two almost Hermitian manifolds with $HB_{\tilde{M}} \leq B$. If $f : M \rightarrow \tilde{M}$ is a pseudoholomorphic isometry, then we have

1. $\text{Ric}_{\tilde{M}} - mB$ is negative semi-definite, $B$.
2. $R \leq m^2 B$.
3. $\text{HS}_M \leq B$.

**Proof.** For any $X = X^i e_i \in T'M$, by using the Gauss equation (6.6), we have

$$R_{jk}^i X^j X^k - mB |X|^2 \leq \sum_i R_{ijk}^i X^j X^k \leq \sum_i \tilde{R}_{ijk}^i X^j X^k \leq 0.$$

Similar proof for $\text{Ric}_{\tilde{M}} - mB$. The second and third inequalities are also follow from Gauss equation (6.6) and $HB_{\tilde{M}} \leq B$.

**Remark.** The Proposition 6.1 provides various curvature criterion to detect that whether an almost Hermitian manifold can be pseudoholomorphically immersed into another one.

q.e.d
Proposition 6.2. Let \((M, J, g), (\widetilde{M}, \widetilde{J}, \tilde{g})\) be two almost Hermitian manifolds with \(HB_{\tilde{M}} \leq B\). If \(f : M \to \widetilde{M}\) is a pseudoholomorphic isometry, then \(f\) is canonical totally geodesic if \(f\) satisfies one of the following conditions:

1. \(\text{Ric}_M = mBg\), or \(\text{Ric}''_M = mBg\).
2. \(R = m^2B\).
3. \(HS_M = B\).

Proof. By the Gauss equation (6.6), any one of the conditions implies that the canonical second fundamental form \(a_{ij}^k = 0\), i.e., \(\nabla df = 0\).

q.e.d

For the scalar curvature and normal scalar curvature, we have

Proposition 6.3. Let \((M, J, g), (\widetilde{M}, \widetilde{J}, \tilde{g})\) be two almost Hermitian manifolds with \(B_1 \leq HB_{\tilde{M}} \leq B_2\). For a pseudoholomorphic isometry \(f : M \to \widetilde{M}\), we have

\[ mnB_1 \leq R + R_\perp \leq mnB_2. \]

Proof. By the Gauss equation (6.6) and Ricci equation (6.9), we have

\[ R + R_\perp = \sum_{i,j} R_{ij}^i + \sum_{i,\lambda} \tilde{R}_{i\lambda}^\lambda. \]

So, the inequalities follow from the bounds of holomorphic bisectional curvature of \(\widetilde{M}\).

q.e.d

We will give an upper bound of the curvature tensor in terms of the bound of holomorphic bisectional curvature.

Lemma 6.4. Let \((M, J, g)\) be an almost Hermitian manifold with \(a \leq HB_M \leq b\). Under a unitary frame field, the components of \((1,1)\)-part of the curvature denoted by \(R_{i\kappa\tau}^i\), then for fixed indices \(i, j, k, \ell\), we have

1. \(|R_{i\kappa\tau}^i X^i\overline{X}^\tau|^2 \leq 2(b-a)^2|X|^4\) for \(k \neq \ell\) and any \(X \in TM\).
2. \(|R_{i\kappa\tau}^i X^k\overline{X}^\ell|^2 \leq 2(b-a)^2|X|^4\) for \(i \neq j\) and any \(X \in TM\).
3. \(|R_{i\kappa\tau}^i|^2 \leq 4(1+\sqrt{2}^2)(b-a)^2\) for \(i \neq j\) and \(k \neq \ell\).

Proof. For any \(X = X^i e_i, Y = Y^j e_j, Z = Z^k e_k \in TM\), we have

\[ R_{i\kappa\tau}^i X^i\overline{X}^\tau Y^k\overline{Z}^\ell + R_{i\kappa\tau}^i X^k\overline{X}^\ell Y^j\overline{Z}^\ell = R_{i\kappa\tau}^i X^i\overline{X}^\tau (Y^k + Z^k)(\overline{Y}^\ell + \overline{Z}^\ell) \]

\[ -R_{i\kappa\tau}^i X^i\overline{X}^\tau Y^k\overline{Z}^\ell - R_{i\kappa\tau}^i X^k\overline{X}^\ell Y^j\overline{Z}^\ell. \]  \hspace{1cm} (6.12)

Taking \(X = X^i e_i, Y = e_k, Z = e_\ell \) with \(k \neq \ell\) in (6.12) and together \(\overline{R}_{j\kappa\tau}^i = R_{i\kappa\tau}^j\), we obtain

\[ 2 \text{Re}(R_{i\kappa\tau}^i X^i\overline{X}^\tau) = R_{i\kappa\tau}^i X^k\overline{X}^\ell + R_{i\kappa\tau}^i X^i\overline{X}^\tau \]

\[ = R_{i\kappa\tau}^i X^k\overline{X}^\ell (\delta_{pk} + \delta_{p\ell})(\delta_{qk} + \delta_{q\ell}) - R_{i\kappa\tau}^i X^k\overline{X}^\ell - R_{i\kappa\tau}^i X^i\overline{X}^\tau, \] \hspace{1cm} (6.13)

which implies

\[ -(b-a)|X|^2 \leq \text{Re}(R_{i\kappa\tau}^i X^i\overline{X}^\tau) \leq (b-a)|X|^2. \] \hspace{1cm} (6.14)

By taking \(X = X^i e_i, Y = -\sqrt{-1}e_k, Z = e_\ell \) with \(k \neq \ell\) in (6.12), we can obtain

\[ -(b-a)|X|^2 \leq \text{Im}(R_{i\kappa\tau}^i X^i\overline{X}^\tau) \leq (b-a)|X|^2. \] \hspace{1cm} (6.15)

So, inequalities (6.14) and (6.15) give the desired upper bound of \(|R_{i\kappa\tau}^i X^i\overline{X}^\tau|^2\), and similar proof for \(|R_{j\kappa\tau}^i X^k\overline{X}^\ell|^2\).
Thus, from (6.17) and (6.18), we get
\[
R^i_{\ell k}(X^i + Y^i)(X^j + Y^j)Z^k \overline{W}^\ell - R^i_{\ell k}X^i \overline{X}^j Z^k \overline{W}^\ell - R^j_{\ell k}Y^j \overline{Y}^i Z^k \overline{W}^\ell.
\]

Taking \(X = e_i, Y = e_j, Z = e_k, W = e_\ell\) with \(i \neq j\) and \(k \neq \ell\) in (6.16), we obtain
\[
R^i_{\ell k} + R^i_{j k} = R^i_{p k \ell}(\delta_{p i} + \delta_{j p})(\delta_{q i} + \delta_{q j}) - R^i_{\ell k} - R^j_{j k}.
\]
and by taking \(X = \sqrt{-1}e_i, Y = e_j, Z = e_k, W = \sqrt{-1}e_\ell\) with \(i \neq j\) and \(k \neq \ell\) in (6.16), we obtain
\[
R^j_{\ell k} - R^i_{j k} = -\sqrt{-1}R^a_{p k \ell} (\sqrt{-1} \delta_{p i} + \delta_{j p})(\sqrt{-1} \delta_{q i} + \delta_{q j}) + \sqrt{-1}R^i_{\ell k} - \sqrt{-1}R^j_{j k}.
\]

Thus, from (6.17) and (6.18), we get
\[
2R^j_{i k \ell} = R^i_{p k \ell}(\delta_{p i} + \delta_{j p})(\delta_{q i} + \delta_{q j}) - \sqrt{-1}R^a_{p k \ell} (\sqrt{-1} \delta_{p i} + \delta_{j p})(\sqrt{-1} \delta_{q i} + \delta_{q j})
\]
\[- (1 - \sqrt{-1})R^j_{i k \ell} - (1 - \sqrt{-1})R^j_{j k \ell}.
\]

So, by using the estimates (1) and (2) have been obtained, we get
\[
2|R^j_{i k \ell}| \leq |R^i_{p k \ell}(\delta_{p i} + \delta_{j p})(\delta_{q i} + \delta_{q j})| + |R^a_{p k \ell} (\sqrt{-1} \delta_{p i} + \delta_{j p})(\sqrt{-1} \delta_{q i} + \delta_{q j})|
\]
\[+ \sqrt{2}|R^i_{p k \ell}| + \sqrt{2}|R^j_{i k \ell}|
\]
\[\leq \sqrt{2}(b - a)|e_i + e_j|^2 + \sqrt{2}(b - a)|\sqrt{-1}e_i + e_j|^2 + 4(b - a)
\]
\[= 4(1 + \sqrt{2})(b - a),
\]
which gives the desired estimate of \(|R^j_{i k \ell}|^2\) with \(i \neq j\) and \(k \neq \ell\).

We wish to give a lower bound of \(\frac{1}{2} \Delta S\) by using the bounds of holomorphic bisectional curvature of target manifold \(\widetilde{M}, \tilde{\mathcal{J}}, \tilde{g}\). In the sequel, we assume that the holomorphic bisectional curvature pointwisely satisfies \(a(x) \leq HB_M \leq b(x)\) for \(x \in \widetilde{M}\). According to the expression of \(\frac{1}{2} \Delta S\), for convenience, we define three terms as follows:

(I) := 2\text{Re}(\overline{a^0_{i,j}} a^0_{p \ell k} R^p_{\ell j k} ),

(II) := |a^0_{i,j} |^2 - 2\text{Re}(\overline{a^0_{i,j}} (a^\beta_{i,j} a^\alpha_{k} R^\alpha_{\ell \gamma \delta})_\ell k ),

(III) := \overline{a^0_{i,j}} (a^\alpha_{p \ell} R^p_{\ell j k} + a^\alpha_{i,p} R^p_{\ell j k} + 2 a^\alpha_{p,k} R^p_{i j k} - a^\beta_{i,j} a^\alpha_{k} R^\alpha_{\ell \gamma \delta} ).

**Lemma 6.5.** For the terms (I) and (II), we have

(1) (I) = 0.

(2) (II) \geq -\text{div}^c(X) - P(m,n)(b - a)^2 - m^2 \max\{a^2, b^2\}, where \(X\) defined in (6.22) and \(P(m,n)\) is a polynomial with respect to \(m, n\) defined in (6.25).

**Proof.** By choosing an adapted unitary frame field, we have \(a^k_{i,j} = 0\) and \(a^\lambda_{i,j} = 0\), which imply (I) = 0. Notice that
\[
2\text{Re}(\overline{a^0_{i,j}} (a^\beta_{i,j} a^\alpha_{k} R^\alpha_{\ell \gamma \delta})_\ell k ) = \overline{a^0_{i,j}} (a^\beta_{i,j} a^\alpha_{k} R^\alpha_{\ell \gamma \delta})_\ell k + a^0_{i,j} (a^\beta_{i,j} a^\alpha_{k} R^\alpha_{\ell \gamma \delta})_\ell k
\]
\[= X^k_{j k} + X^k_{j k} - \overline{a^0_{i,j}} R^\alpha_{i j k} - a^0_{i,j} R^\alpha_{i j k}.
\]
\[= \text{div}^c(X) - \overline{a^0_{i,j}} R^\alpha_{i j k} - a^0_{i,j} R^\alpha_{i j k}.
\]

(6.21)
Thus, by (6.21) and Cauchy-Schwarz inequality, we have

\[
(II) = |a_{i,j,k}^\alpha|^2 - \text{div}^\epsilon(X) + \alpha_{\tilde{i},\tilde{j},\tilde{k}} R_{\tilde{i},\tilde{j},\tilde{k}}^\alpha + a_{i,j,k}^\alpha \tilde{R}_{i,j,k}^\alpha
\geq |a_{i,j,k}^\alpha|^2 - |a_{i,j,k}^\alpha|^2 - |\tilde{R}_{i,j,k}^\alpha|^2
= -\text{div}^\epsilon(X) - |\tilde{R}_{i,j,k}^\alpha|^2. \tag{6.23}
\]

On the other hand, by using Lemma 6.4 repeatedly, we have

\[
\sum_{i,j,k,\alpha} |\tilde{R}_{i,j,k}^\alpha|^2 = \sum_{i,j,k,\alpha} |\tilde{R}_{i,j}^\alpha|^2 + \sum_{i,j,k,\alpha} |\tilde{R}_{i,k}^\alpha|^2 + \sum_{i,j,k,\alpha} |\tilde{R}_{j,k}^\alpha|^2
+ \sum_{i,j,k,\alpha} |\tilde{R}_{i,j}^\alpha|^2 + \sum_{i,j,k,\alpha} |\tilde{R}_{i,k}^\alpha|^2 + \sum_{i,j,k,\alpha} |\tilde{R}_{j,k}^\alpha|^2
\leq 2m^2(m - 1)(b - a)^2 + 2m^2(m - 1)(b - a)^2 + m^2\max\{a^2, b^2\}
+ 4(1 + \sqrt{2})^2 m^2(m - 1)^2(b - a)^2 + 2m^2(n - m)(b - a)^2
+ 4(1 + \sqrt{2})^2 m^2(m - 1)(b - a)^2
:= P(m, n)(b - a)^2 + m^2 \max\{a^2, b^2\}. \tag{6.24}
\]

where

\[
P(m, n) = 2m^2(m + n - 2) + 4(1 + \sqrt{2})^2 m^2(m - 1)(n - 1). \tag{6.25}
\]

**Lemma 6.6.** For the term (III), we have

\[
(III) \geq (m + 2)bS - P(m)(b - a)S - 4S^2,
\]

where \( P(m) = 2(m + 2) + 4(1 + \sqrt{2}). \)

**Proof.** Under an adapted unitary frame field, using the Gauss equation (6.6), we have

\[
(III) = \sum_{i,j,k} \tilde{a}_{i,j}^\alpha (a_{p,j}^\lambda \tilde{R}_{i,k}^p + a_{i,j}^\lambda \tilde{R}_{i,k}^p + 2a_{p,k}^\lambda \tilde{R}_{i,k}^p - a_{i,j}^\mu \tilde{R}_{i,k}^\mu)
- a_{i,j}^\lambda a_{i,k}^\mu a_{p,k}^\mu - a_{i,j}^\lambda a_{i,p}^\mu a_{j,k}^\mu a_{p,k}^\mu - 2a_{i,j}^\lambda a_{i,p,k} a_{i,j}^\alpha a_{p,k}^\alpha. \tag{6.26}
\]

By using the bounds of holomorphic bisectional curvature, we have

\[
\sum_{i,j,k,p,\lambda} \tilde{a}_{i,j}^\alpha a_{p,j}^\lambda \tilde{R}_{i,k}^p = \sum_{j,k,\lambda} \sum_{i,p} a_{i,j}^\alpha \tilde{R}_{i,k}^p
\geq a \sum_{k} \sum_{i,j,\lambda} |a_{i,j}^\alpha|^2 = maS. \tag{6.27}
\]

Similarly, we can get

\[
\sum_{i,j,k,p,\lambda} \tilde{a}_{i,j}^\alpha a_{i,j}^\lambda \tilde{R}_{i,k}^p \geq maS, \quad \sum_{i,j,k,\lambda,\mu} \tilde{a}_{i,j}^\alpha a_{i,j}^\lambda \tilde{R}_{i,k}^\mu \leq mbS. \tag{6.28}
\]
On the other hand, by Lemma 6.4, the term
\[
\sum_{i,j,k,p,\lambda} \alpha_{i,j}^{\lambda} a_{p,k}^{\mu} \bar{R}^{p}_{ij} = \sum_{i,j,k,\lambda} \alpha_{i,k}^{\lambda} a_{i,j}^{\lambda} \bar{R}^{i}_{ij} + \sum_{i,j,p,\lambda} \alpha_{i,j}^{\lambda} a_{p,j}^{\lambda} \bar{R}^{p}_{ij} - \sum_{i,j,\lambda} \alpha_{i,j}^{\lambda} \bar{R}^{i}_{ij} + \sum_{i\neq p,j,\lambda} \alpha_{i,j}^{\lambda} a_{p,k}^{\lambda} \bar{R}^{p}_{ij} \geq a \sum_{i,j,\lambda} |a_{i,j}^{\lambda}|^2 + a \sum_{j,\lambda} \sum_{i} |a_{i,j}^{\lambda}|^2 - b \sum_{i,j,\lambda} |a_{i,j}^{\lambda}|^2 - 2(1 + \sqrt{2})(b - a) \sum_{i\neq p,j,\lambda} |a_{i,j}^{\lambda}| a_{p,k}^{\lambda} \]
\[
\geq (2a - b)S - 2(1 + \sqrt{2})(b - a)S. \tag{6.29}
\]
To estimate the remaining terms in (III), we set
\[
S_{m}':= \sum_{j,\lambda} \alpha_{p,j}^{\lambda} a_{i,j}^{\lambda}, \quad S_{m}'' := \sum_{i,\lambda} \alpha_{i,j}^{\lambda} a_{i,j}^{\lambda}, \quad S_{m}''':= \sum_{i,j} \alpha_{i,j}^{\lambda} a_{i,j}^{\lambda}.
\]
It is clear that the matrices $S' = (S_{m}')$, $S'' = (S_{m}'')$, $S''' = (S_{m}''')$ are Hermitian and semi-positive definite. They are satisfy $S = tr(S') = tr(S'') = tr(S''')$. Locally, one can choose unitary frame to diagonalize $S'$, so we have
\[
\sum_{i,j,k,p,\lambda,\mu} \alpha_{i,j}^{\lambda} a_{p,j}^{\lambda} a_{i,k}^{\mu} \bar{a}_{p,k}^{\mu} = \sum_{i} S_{m}' S_{m}' \leq \left( \sum_{i} S_{m}' \right)^2 = S^2. \tag{6.30}
\]
Similarly, we also have
\[
\sum_{i,j,k,p,\lambda,\mu} \alpha_{i,j}^{\lambda} a_{p,k}^{\mu} a_{i,k}^{\mu} \bar{a}_{p,k}^{\mu} = \sum_{\lambda} S_{m}''' S_{m}''' \leq \left( \sum_{\lambda} S_{m}''' \right) = S^2. \tag{6.31}
\]
By the trick of diagonalization, the term
\[
\sum_{i,j,k,p,\lambda,\mu} \alpha_{i,j}^{\lambda} a_{i,p}^{\lambda} a_{j,k}^{\mu} a_{p,k}^{\mu} = \sum_{j} S_{m}'' S_{m}'' \leq \left( \sum_{j} S_{m}'' \right)^2 \leq S^2. \tag{6.32}
\]
Thus, the lower bound of (III) follows from (6.26)-(6.32).

Now we can prove the Simons integral inequality of pseudoholomorphic isometry.

**Theorem 6.7.** Let $(M, J, g)$ be a closed semi-Kähler manifold, and let $(\tilde{M}, \tilde{J}, \tilde{g})$ be an almost Hermitian manifold pointwisely satisfies $a(x) \leq HB_{\tilde{M}} \leq b(x)$ for $x \in \tilde{M}$. Let $f : M \to \tilde{M}$ be a pseudoholomorphic isometric immersion, then
\[
\int_{M} \left( -4S^2 + [(m + 2)b - P(m)(b - a)]S - P(m,n)(b - a)^2 - m^2 \max\{a^2, b^2\} \right) dV_{g} \leq 0,
\]
where $P(m) = 2(m+2)4(1 + \sqrt{2})$, $P(m,n) = 2m^2(m+n-2) + 4(1 + \sqrt{2})^2 m^2(m-1)(n-1)$.

**Proof.** By Proposition 3.3, Lemma 6.5 and Lemma 6.6, we have
\[
\frac{1}{2} \Box S \geq -\text{div}^c(X) - 4S^2 + [(m + 2)b - P(m)(b - a)]S - P(m,n)(b - a)^2 - m^2 \max\{a^2, b^2\}. \tag{6.33}
\]
q.e.d
Notic that \((M, J, g)\) is closed and semi-Kählerian, so \(\int_M \Box S \, dV_g = 0\) and \(\int_M \text{div}^c(X) \, dV_g = 0\), then the integral inequality follows from integrating on both sides of (6.33).

**Remark.** The semi-Kählerian condition is to ensure that the terms \(\Box S\) and \(\text{div}^c(X)\) integrated on \(M\) are zero. So, the same result holds when \((M, J, g)\) is quasi-Kählerian, or almost-Kählerian, or nearly-Kählerian.

**Theorem 6.8.** Let \(f : M \to \tilde{M}\) be a pseudoholomorphic isometric immersion from almost Hermitian manifold \((M, J, g)\) into \((\tilde{M}, \tilde{J}, \tilde{g})\) with parallel canonical second fundamental form. Suppose \((\tilde{M}, \tilde{J}, \tilde{g})\) pointwisely satisfies

\[
a(x) \leq HB_M \leq b(x)\]

for \(x \in \tilde{M}\), where \(a(x)\) and \(b(x)\) are bounded functions on \(\tilde{M}\). Then \(S = 0\), or

\[
\frac{1}{4} \left[ (m+2)b_0 - P(m)(b_0 - a_0) \right] \leq S \leq \frac{m(n-m)}{m+n} \left[ (m+2)a_0 + P(m)(b_0 - a_0) \right]
\]

provided \(P(m)(b_0 - ma_0) < (m+2)b_0\), where \(P(m) = 2(m+2)+4(1+\sqrt{2})\), \(a_0 := \inf_{x \in \tilde{M}} a(x)\) and \(b_0 := \sup_{x \in \tilde{M}} b(x)\).

**Proof.** Under the condition that \(f\) has parallel canonical fundamental form, we have

\[a^\alpha_{i,j,k} = a^\alpha_{i,j,k} = 0, \quad (6.34)\]

and hence the Codazzi equation (6.11) gives

\[\tilde{R}^\lambda_{ij,k} = 0. \quad (6.35)\]

The identity (3.4) and (6.34) imply that \(S\) is a constant. For \(X^k\) defined in (6.22), the identity (6.35) gives

\[X^k = a^\lambda_{i,j} \tilde{R}^\lambda_{ij,k} = 0. \quad (6.36)\]

Thus, the term (II) = 0 follows from (6.21), (6.35) and (6.36). By using that \(S\) is a constant, from Lemma 6.6, we have

\[0 \geq [(m+2)b_0 - P(m)(b_0 - a_0) - 4S]S, \quad (6.37)\]

where \(a_0 := \inf_{x \in \tilde{M}} a(x)\) and \(b_0 := \sup_{x \in \tilde{M}} b(x)\).

On the other hand, modify the proof of Lemma 6.6, we can get the following estimates:

\[
\sum_{i,j,k,p,\lambda} a^\lambda_{i,j} a^\lambda_{p,j} \tilde{R}^p_{ijk} \leq mb_0 S, \quad (6.38)
\]

\[
\sum_{i,j,k,p,\lambda} a^\lambda_{i,j} a^\lambda_{i,p} \tilde{R}^p_{jk} \leq mb_0 S, \quad (6.39)
\]

\[
\sum_{i,j,k,\lambda,\mu} a^\lambda_{i,j} a^\mu_{i,k} \tilde{R}^\lambda_{jk} \geq ma_0 S, \quad (6.40)
\]

\[
\sum_{i,j,k,p,\lambda} a^\lambda_{i,j} a^\lambda_{p,k} \tilde{R}^p_{ijk} \leq (2b_0 - a_0)S + 2(1+\sqrt{2})(b_0 - a_0)S, \quad (6.41)
\]

\[
\sum_{i,j,k,p,\lambda,\mu} a^\lambda_{i,j} a^\lambda_{p,j} a^\mu_{i,k} a^\mu_{p,k} \geq \frac{1}{m} S^2, \quad (6.42)
\]

\[
\sum_{i,j,k,p,\lambda,\mu} a^\lambda_{i,j} a^\lambda_{p,k} a^\mu_{i,k} a^\mu_{p,k} \geq \frac{1}{n-m} S^2, \quad (6.43)
\]
$$\sum_{i,j,k,p,\lambda,\mu} \overline{a}_{i,j}^{\lambda} a_{i,p}^{\mu} \overline{a}_{j,k}^{\mu} \geq 0. \quad (6.44)$$

From (6.38)-(6.44) and Proposition 3.3, together with $S$ is a constant, we have

$$0 \leq [(m + 2)a_0 + P(m)(b_0 - a_0) - \frac{n + m}{m(n - m)} S], \quad (6.45)$$

Then, the bounds of $S$ follows from identities (6.37) and (6.45).

q.e.d

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